Section 4.2 – Matrices and Linear Systems

Let $a_{11}(t),\ a_{12}(t),\ \ldots,\ a_{nn}(t)$ and $b_1(t),\ b_2(t),\ \ldots,\ b_n(t)$ be continuous functions on the interval *I*.

The system of n 1st-order differential equations:

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \ \ldots + a_{1n}(t)x_n + b_1(t) \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \ \ldots + a_{2n}(t)x_n + b_2(t) \\ &\ \ \vdots \\ x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ \ldots + a_{nn}(t)x_n + b_n(t) \end{aligned}$$

Is called a 1st-order linear differential system.

The system is ***homogeneous*** if $b_1(t) \equiv b_2(t) \equiv \ \ldots \equiv b_m(t) \equiv 0$ on *I*, otherwise, the system is ***nonhomogeous*** if the functions $b_i(t)$ are not all identically zero on *I*.

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{pmatrix} \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad b(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

The system can be written in the vector-matrix form $\quad X' = A(t)X + b(t) \qquad (S)$

$A(t)$: Coefficient matrix

$b(t)$: Constant matrix

A solution of the linear differential system (S) is a differentiable vector function

$$\vec{v} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

Satisfies (S) on the interval *I*.

The derivative of *A*: $\quad A'(t) = \dfrac{dA}{dt} = \left[\dfrac{da_{ij}}{dt}\right]$

Example

Find the derivative if $x(t)=\begin{pmatrix} t \\ t^2 \\ e^{-t} \end{pmatrix}$ $\quad A(t)=\begin{pmatrix} \sin t & 1 \\ t & \cos t \end{pmatrix}$

Solution

$$x'(t)=\begin{pmatrix} 1 \\ 2t \\ -e^{-t} \end{pmatrix} \qquad A(t)=\begin{pmatrix} \cos t & 0 \\ 1 & -\sin t \end{pmatrix}$$

Example

The 1st-order system $\begin{cases} x_1' = 4x_1 - 3x_2 \\ x_2' = 6x_1 - 7x_2 \end{cases}$

$$X' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4x_1 - 3x_2 \\ 6x_1 - 7x_2 \end{bmatrix}$$

$$= \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix} X$$

$$\frac{dX}{dt} = P(t)X + f(t) \quad with \quad P(t)=\begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix} \quad f(t)=\begin{pmatrix} 0 \\ 0 \end{pmatrix}=0$$

To verify that the vector functions:

$$x_1(t)=\begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix} \quad x_2(t)=\begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix}$$

Are both solutions of the matrix differential equations with coefficient matrix P.

$$Px_1 = \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix}\begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix} = \begin{pmatrix} 6e^{2t} \\ 4e^{2t} \end{pmatrix} = x_1'$$

$$Px_2 = \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix}\begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix} = \begin{pmatrix} -5e^{-5t} \\ -15e^{-5t} \end{pmatrix} = x_2'$$

When $f(t)=0 \;\Rightarrow\; \frac{dX}{dt}=P(t)X$ is a homogeneous equation

A ***homogeneous*** system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ \dots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ \dots + a_{mn}x_n &= 0 \end{aligned}$$

Always has at least one solution namely $x_1 = x_2 = \ \dots = x_n = 0$ called the ***trivial solution***

That is, homogeneous systems are always ***consistent***

Theorem

If $\vec{v}$ is a solution of (*H*) and α is any $\mathbb{R}$, then $\vec{u} = \alpha\vec{v}$ is also a solution of (*H*); any constant multiple of a solution of (*H*) is a solution of (*H*).

Theorem

If $\vec{v}_1$ and $\vec{v}_2$ are solutions of (*H*), then $\vec{u} = \vec{v}_1 + \vec{v}_2$ is also a solution of (*H*); the sum of any 2 solutions of (*H*) is a solution of (*H*).

$$\vec{v}_1' = A(t)\vec{v}_1 \qquad \vec{v}_1' + \vec{v}_2' = A(t)\vec{v}_1 + A(t)\vec{v}_2$$

$$\vec{v}_2' = A(t)\vec{v}_2 \qquad \left(\vec{v}_1 + \vec{v}_2\right)' = A(t)\left(\vec{v}_1 + \vec{v}_2\right)$$

$$\vec{u}' = A(t)\vec{u}' \qquad \text{Since } \vec{u} = \vec{v}_1 + \vec{v}_2$$

In general,

Theorem

If $\vec{v}_1, \vec{v}_2, \ \dots, \vec{v}_n$ are solutions of (*H*), and is $c_1, c_2, \ \dots, c_n$ are $\mathbb{R}$ then

$$c_1\vec{v}_1, c_2\vec{v}_2, \ \dots, c_n\vec{v}_n$$

Is a solution of (*H*); any linear combination of solutions of (*H*) is also a solution of (*H*).

$$\begin{aligned} \vec{v}_1' &= A(t)\vec{v}_1 + c_1 \\ \vec{v}_2' &= A(t)\vec{v}_2 + c_2 \\ \vdots \quad & \quad \vdots \qquad \vdots \\ \vec{v}_n' &= A(t)\vec{v}_n + c_n \end{aligned}$$

Linear Dependent and Independent

Let

$$\vec{x}_1(t)=\begin{pmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{m1} \end{pmatrix}, \quad \vec{x}_2(t)=\begin{pmatrix} v_{12} \\ v_{22} \\ \vdots \\ v_{m2} \end{pmatrix}, \quad ,\cdots \quad \vec{x}_m(t)=\begin{pmatrix} v_{1n} \\ v_{2n} \\ \vdots \\ v_{mn} \end{pmatrix}$$

Be vector functions defined on some interval *I*.

The vectors are linearly dependent on *I* if exist *n* real numbers $c_1,\ c_2,\ \ldots,\ c_n$ not all zero such that

$$c_1\vec{v}_1(t)+\ c_2\vec{v}_2(t)+\ \ldots+\ c_n\vec{v}_n(t)=0 \qquad on\ \ I$$

Otherwise the vectors are linearly independent on I.

Wronskian of solutions

Theorem

Let $x_1,\ x_2,\ \ldots,\ x_n$ are *n* solutions of the homogeneous linear equation $x'=P(t)x$ on an interval *I*.

Let $W=W\left(x_1,\ x_2,\ \ldots,\ x_n\right)$

$$W=\begin{vmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ v_{n1} & \cdots & \cdots & v_{nn} \end{vmatrix}=0 \qquad on\ \mathrm{I}$$

Called the Wronskian of the vector functions $\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n$

Special Case *n* solutions of (*H*)

Theorem

Let $\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n$ be solution of (*H*). Exactly one of the following holds.

1. $W\left(\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n\right)(t)\equiv 0$ on *I* and the solutions are Linearly Dependent.
2. $W\left(\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n\right)(t)\neq 0$ for all $t\in I$ and the solutions are Linearly Independent.

Theorem

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be n L.I solutions of (H) $(W \neq 0)$

Let $\vec{u}$ be any solution of (H). Then there exists a unique set of constants $c_1, c_2, \ldots, c_n$ such that

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$$

That is, every solution of (H) can be written as a unique linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$

A set of n L.I solutions $(W \neq 0)$ of (H) is called a ***fundamental set of solutions***.

A fundamental set is also called a ***solution basis*** for (H).

Example

Determine if the solutions are linearly dependent or independent using Wronskian.

$$\vec{x}_1(t) = \begin{pmatrix} 2e^t \\ 2e^t \\ e^t \end{pmatrix}, \quad \vec{x}_2(t) = \begin{pmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{pmatrix}, \quad \vec{x}_3(t) = \begin{pmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{pmatrix}$$

Solution

$$W = \begin{vmatrix} 2e^t & 2e^{3t} & 2e^{5t} \\ 2e^t & 0 & -2e^{5t} \\ e^t & -e^{3t} & e^{5t} \end{vmatrix} = -4e^{9t} - 4e^{9t} - 4e^{9t} - 4e^{9t} \underline{= -16e^{9t} \neq 0}$$

$$or \quad W = \begin{vmatrix} 2e^t & 2e^{3t} & 2e^{5t} \\ 2e^t & 0 & -2e^{5t} \\ e^t & -e^{3t} & e^{5t} \end{vmatrix} = e^{9t} \begin{vmatrix} 2 & 2 & 2 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{vmatrix} \underline{= -16e^{9t} \neq 0}$$

The solutions $x_1, x_2,$ *and* x_3 are linearly independent.

Example

Find the general solution of: $y''' - 3y'' - 4y' + 12y = 6e^t$

Solution

$$\lambda^3 - 3\lambda^2 - 4\lambda + 12 = 0$$

$$\lambda^2(\lambda - 3) - 4(\lambda - 3) = 0$$

$$\left(\lambda^2 - 4\right)(\lambda - 3) = 0 \qquad \Rightarrow \lambda_1 = 3,\ \lambda_2 = 2,\ \lambda_3 = -2$$

The Fundamental set: $\left\{y_1 = e^{3t},\quad y_2 = e^{2t},\quad y_3 = e^{-2t}\right\}$

$$y_h = C_1 e^{3t} + C_2 e^{2t} + C_3 e^{-2t}$$

Particular solution: $z = e^t \quad \Rightarrow \quad z(t) = Ae^t$

$$z' = Ae^t \quad z'' = Ae^t \quad z''' = Ae^t$$

$$Ae^t - 3Ae^t - 4Ae^t + 12Ae^t = 6e^t$$

$$6Ae^t = 6e^t \quad \Rightarrow \quad \boxed{A = 1}$$

$$y_p = e^t$$

General solution: $y(t) = C_1 e^{3t} + C_2 e^{2t} + C_3 e^{-2t} + e^t$

$$y''' = 3y'' + 4y' - 12y + 6e^t$$

$$y = x_1 \quad y' = x_2 \quad y'' = x_3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ 3x_3 + 4x_2 - 12x_1 + 6e^t \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 6e^t \end{pmatrix}$$

$y = e^{3t} + e^t$ is a solution of the equation

Proof: $y' = 3e^{3t} + e^t \quad y'' = 9e^{3t} + e^t \quad y''' = 27e^{3t} + e^t$

$$y''' = 3\left(9e^{3t} + e^t\right) + 4\left(3e^{3t} + e^t\right) - 12\left(e^{3t} + e^t\right) + 6e^t$$

$$= 27e^{3t} + 3e^t + 12e^{3t} + 4e^t - 12e^{3t} - 12e^t + 6e^t$$

$$= 27e^{3t} + e^t \quad \checkmark$$

Therefore; $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} e^{3t} + e^t \\ 3e^{3t} + e^t \\ 9e^{3t} + e^t \end{pmatrix}$

For $y_1 = e^{3t}$ $\qquad x_1(t) = \begin{pmatrix} y_1 \\ y_1' \\ y_1'' \end{pmatrix} = \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix}$

For $y_2 = e^{2t}$ $\qquad x_2(t) = \begin{pmatrix} y_2 \\ y_2' \\ y_2'' \end{pmatrix} = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix}$

For $y_1 = e^{-2t}$ $\qquad x_3(t) = \begin{pmatrix} y_3 \\ y_3' \\ y_3'' \end{pmatrix} = \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \\ 4e^{-2t} \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$W = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{vmatrix} = -12 \neq 0$$

Exercises Section 4.2 – Matrices and Linear Systems

Write the given system in the form $\boldsymbol{x}' = \boldsymbol{P}(t)x + \boldsymbol{f}(t)$

1. $x' = -3y, \quad y' = 3x$

2. $x' = 3x - 2y, \quad y' = 2x + y$

3. $x' = tx - e^t y + \cos t, \quad y' = e^{-t}x + t^2 y - \sin t$

4. $x' = y + z, \quad y' = z + x, \quad z' = x + y$

5. $x' = 2x - 3y, \quad y' = x + y + 2z, \quad z' = 5y - 7z$

6. $x' = 3x - 4y + z + t, \quad y' = x - 3z + t^2, \quad z' = 6y - 7z + t^3$

7. $x_1' = x_2, \quad x_2' = 2x_3, \quad x_3' = 3x_4, \quad x_4' = 4x_1$

8. $x_1' = x_2 + x_3 + 1, \quad x_2' = x_3 + x_4 + t, \quad x_3' = x_1 + x_4 + t^2, \quad x_4' = 4x_1 + x_2 + t^3$

For the systems below:

a) Verify that the given vectors are solutions of the given system.
b) Use the Wronskian to show that they are linearly independent.
c) Write the general solution of the system.
d) Find the particular solution that satisfies the given initial conditions

9. $\boldsymbol{x}' = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$

10. $\boldsymbol{x}' = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 5 \end{cases}$

11. $\boldsymbol{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}, \quad \begin{cases} x_1(0) = 5 \\ x_2(0) = -3 \end{cases}$

12. $\boldsymbol{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 8 \\ x_2(0) = 0 \end{cases}$

13. $\boldsymbol{x}' = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 0 \\ x_3(0) = 4 \end{cases}$

14. $\boldsymbol{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 10 \\ x_2(0) = 12 \\ x_3(0) = -1 \end{cases}$

15. $\boldsymbol{x}' = \begin{bmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{bmatrix}\boldsymbol{x}$; $\vec{x}_1 = \begin{bmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{bmatrix}$, $\vec{x}_4 = \begin{bmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{bmatrix}$, $\begin{cases} x_1(0) = 1 \\ x_2(0) = 3 \\ x_3(0) = 4 \\ x_4(0) = 7 \end{cases}$