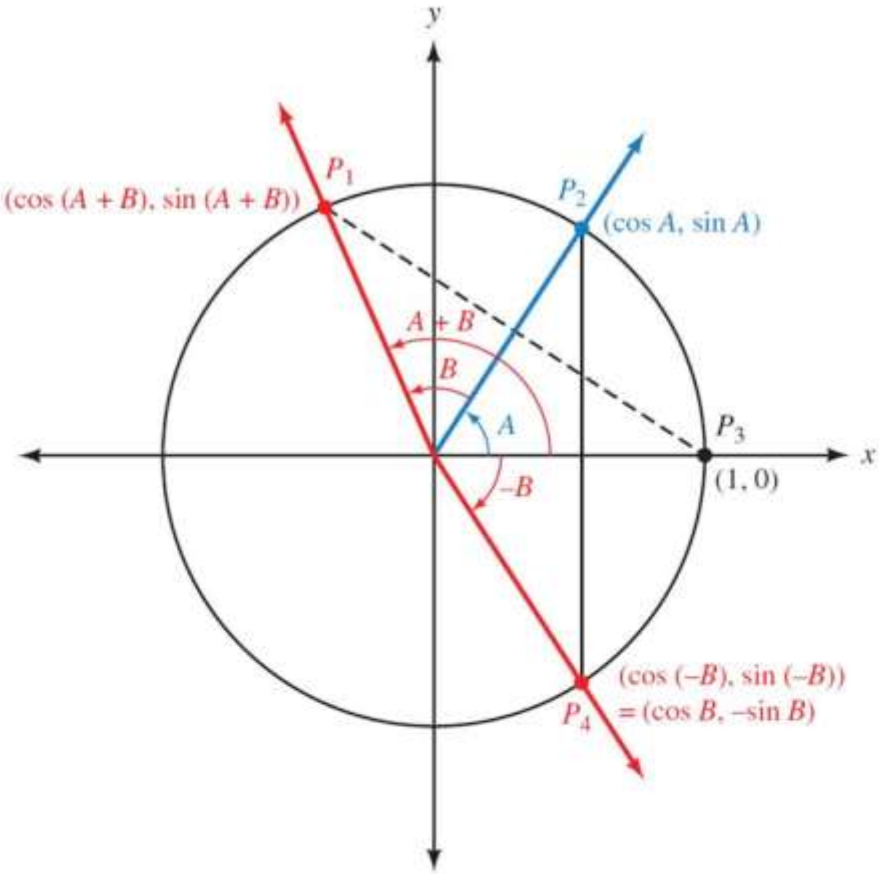
Section 3.2 – Sum and Difference Formulas



$$P_1P_3 = P_2P_4$$

$$\left(P_1P_3\right)^2 = \left(P_2P_4\right)^2$$ *Distance between points*

$$[\cos(A+B)-1]^2 + [\sin(A+B)-0]^2 = (\cos A - \cos B)^2 + (\sin A + \sin B)^2$$

$$\cos^2(A+B) - 2\cos(A+B) + 1 + \sin^2(A+B) = (\cos A - \cos B)^2 + (\sin A + \sin B)^2$$

$$1 - 2\cos(A+B) + 1 = \cos^2 A - 2\cos B\cos A + \cos^2 B + \sin^2 A + 2\sin B\sin A + \sin^2 B$$

$$2 - 2\cos(A+B) = \cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B - 2\cos B\cos A + 2\sin B\sin A$$

$$2 - 2\cos(A+B) = 1 + 1 - 2\cos B\cos A + 2\sin B\sin A$$

$$2 - 2\cos(A+B) = 2 - 2\cos B\cos A + 2\sin B\sin A$$

$$-2\cos(A+B) = -2\cos B\cos A + 2\sin B\sin A$$

$$\boxed{\cos(A+B) = \cos B\cos A - \sin B\sin A}$$

Example

Find the exact value for $\cos 75°$

Solution

$$\begin{aligned}\cos 75° &= \cos(45° + 30°)\\ &= \cos 45° \cos 30° - \sin 45° \sin 30°\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Example

Show that $\cos(x + 2\pi) = \cos x$

Solution

$$\begin{aligned}\cos(x + 2\pi) &= \cos x \cos 2\pi - \sin x \sin 2\pi\\ &= \cos x \cdot (1) - \sin x \cdot (0)\\ &= \cos x\end{aligned}$$

Example

Simplify: $\cos 3x \cos 2x - \sin 3x \sin 2x$

Solution

$$\begin{aligned}\cos 3x \cos 2x - \sin 3x \sin 2x &= \cos(3x + 2x)\\ &= \cos 5x\end{aligned}$$

Example

Show that $\cos(90° - A) = \sin A$

Solution

$$\begin{aligned}\cos(90° - A) &= \cos 90° \cos A + \sin 90° \sin A\\ &= 0 \cdot \cos A + 1 \cdot \sin A\\ &= \sin A\end{aligned}$$

$$\boxed{\begin{aligned}&\cos(A+B)=\cos A\cos B-\sin A\sin B\\&\cos(A-B)=\cos A\cos B+\sin A\sin B\\&\sin(A+B)=\sin A\cos B+\cos A\sin B\\&\sin(A-B)=\sin A\cos B-\cos A\sin B\end{aligned}}$$

Example

Find the exact value of $\sin\frac{\pi}{12}$

<u>*Solution*</u>

$$\begin{aligned}\sin\frac{\pi}{12}&=\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\\&=\sin\frac{\pi}{3}\cos\frac{\pi}{4}-\cos\frac{\pi}{3}\sin\frac{\pi}{4}\\&=\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}-\frac{1}{2}\frac{\sqrt{2}}{2}\\&=\frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Example

Find the exact value of $\cos 15°$

<u>*Solution*</u>

$$\begin{aligned}\cos 15°&=\cos\left(45°-30°\right)\\&=\cos\left(45°\right)\cos\left(30°\right)+\sin\left(45°\right)\sin\left(30°\right)\\&=\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\frac{1}{2}\\&=\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}\\&=\frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

Example

If $\sin A = \frac{3}{5}$ with A in QI, and $\cos B = -\frac{5}{13}$ with B in QIII, find $\sin(A+B)$, $\cos(A+B)$, and $\tan(A+B)$

<u>*Solution*</u>

$$\sin A = \frac{3}{5} \to A \in QI$$

$$\begin{aligned}
\to \cos A &= \sqrt{1-\sin^2 A} \\
&= \sqrt{1-\left(\frac{3}{5}\right)^2} \\
&= \sqrt{1-\frac{9}{25}} \\
&= \sqrt{\frac{25-9}{25}} \\
&= \sqrt{\frac{16}{25}} \\
&= \frac{4}{5}
\end{aligned}$$

$$\cos B = -\frac{5}{13} \to B \in QIII$$

$$\begin{aligned}
\to \sin B &= -\sqrt{1-\cos^2 B} \\
&= -\sqrt{1-\left(\frac{5}{13}\right)^2} \\
&= -\frac{12}{13}
\end{aligned}$$

$$\begin{aligned}
\sin(A+B) &= \sin A \cos B + \cos A \sin B \\
&= \frac{3}{5}\left(-\frac{5}{13}\right) + \frac{4}{5}\left(-\frac{12}{13}\right) \\
&= -\frac{15}{65} - \frac{48}{65} \\
&= -\frac{63}{65}
\end{aligned}$$

$$\begin{aligned}
\cos(A+B) &= \cos A \cos B - \sin A \sin B \\
&= \frac{4}{5}\left(-\frac{5}{13}\right) - \frac{3}{5}\left(-\frac{12}{13}\right) \\
&= -\frac{20}{65} + \frac{36}{65} \\
&= \frac{16}{65}
\end{aligned}$$

$$\begin{aligned}
\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\
&= \frac{-\frac{63}{65}}{\frac{16}{65}} \\
&= -\frac{63}{16}
\end{aligned}$$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$= \frac{\frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B - \sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A}{\cos A}\frac{\sin B}{\cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\boxed{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

$$\boxed{\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}}$$

Example

If $\sin A = \frac{3}{5}$ with A in QI, and $\cos B = -\frac{5}{13}$ with B in QIII, find $\tan(A+B)$

Solution

$$\tan A = \frac{\sin A}{\cos A} = \frac{3/5}{4/5} = \frac{3}{4}$$

$$\tan B = \frac{\sin B}{\cos B} = \frac{-12/13}{-5/13} = \frac{12}{5}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{3}{4} + \frac{12}{5}}{1 - \frac{3}{4}\frac{12}{5}}$$

$$= \frac{\frac{15}{20} + \frac{48}{20}}{1 - \frac{36}{20}}$$

$$= \frac{\frac{63}{20}}{-\frac{16}{20}}$$

$$= -\frac{63}{16}$$

Example

Common household current is called ***alternating current*** because the current alternates direction within the wires. The voltage V in a typical 115-volt outlet can be expressed by the function $V(t) = 163\sin \omega t$ where ω is the angular speed (in radians per second) of the rotating generator at the electrical plant, and t is time measured in seconds.

a) It is essential for electric generators to rotate at precisely 60 cycles per second so household appliances and computers will function properly. Determine ω for these electric generators.

b) Determine a value of ϕ so that the graph of $V(t) = 163\cos(\omega t - \phi)$ is the same as the graph of $V(t) = 163\sin \omega t$

<u>*Solution*</u>

a) Each cycle is 2π radians at 60 cycles per second, so the angular speed is $\omega = 60(2\pi) = 120\pi$ radians per second.

b) $\cos\left(x - \frac{\pi}{2}\right) = \cos x \cos \frac{\pi}{2} + \sin x \sin \frac{\pi}{2}$

$$= \cos x(0) + \sin x(1)$$

$$= \sin x$$

If $\phi = \frac{\pi}{2} \rightarrow V(t) = 163\cos\left(\omega t - \frac{\pi}{2}\right) = 163\sin(\omega t)$

Exercises *Section* 3.2 – Sum and Difference Formulas

1. Prove the identity $\cos(A+B)+\cos(A-B)=2\cos A\cos B$

2. Prove the identity $\sin\left(\frac{\pi}{4}+x\right)+\sin\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$

3. Prove the identity $\dfrac{\sin(A-B)}{\cos A\cos B}=\tan A-\tan B$

4. Prove the identity $\sec(A+B)=\dfrac{\cos(A-B)}{\cos^2 A-\sin^2 B}$

5. Prove the identity $\dfrac{\cos 4\alpha}{\sin\alpha}-\dfrac{\sin 4\alpha}{\cos\alpha}=\dfrac{\cos 5\alpha}{\sin\alpha\cos\alpha}$

6. Write the expression as a single trigonometric function $\sin 8x\cos x-\cos 8x\sin x$

7. Show that $\sin\left(x-\frac{\pi}{2}\right)=-\cos x$

8. If $\sin A=\frac{4}{5}$ with A in QII, and $\cos B=-\frac{5}{13}$ with B in QIII, find $\sin(A+B)$, $\cos(A+B)$, and $\tan(A+B)$

9. If $\sin A=\frac{1}{\sqrt{5}}$ with A in QI, and $\tan B=\frac{3}{4}$ with B in QI, find $\sin(A+B)$, $\cos(A+B)$, and $\tan(A+B)$

10. If $\sec A=\sqrt{5}$ with A in QI, and $\sec B=\sqrt{10}$ with B in QI, find $\sec(A+B)$

11. Prove the following equation is an identity: $\sin(x-y)-\sin(y-x)=2\sin x\cos y-2\cos x\sin y$

12. Prove the following equation is an identity: $\cos(x-y)+\cos(y-x)=2\cos x\cos y+2\sin x\sin y$

13. Prove the following equation is an identity: $\tan(x+y)\tan(x-y)=\dfrac{\tan^2 x-\tan^2 y}{1-\tan^2 x\tan^2 y}$

14. Prove the following equation is an identity: $\dfrac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)}=\dfrac{1-\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$

15. Prove the following equation is an identity: $\sec(x+y)=\dfrac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\sin^2 y}$

16. Prove the following equation is an identity: $\csc(x-y)=\dfrac{\sin x\cos y+\cos x\sin y}{\sin^2 x-\sin^2 y}$

17. Prove the following equation is an identity: $\dfrac{\cos(x+y)}{\cos(x-y)}=\dfrac{\cot y-\tan x}{\cot y+\tan x}$

18. Prove the following equation is an identity: $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\cot y + \cot x}{\cot y - \cot x}$

19. Prove the following equation is an identity: $\dfrac{\sin(x-y)}{\sin x \cos y} = 1 - \cot x \tan y$

20. Prove the following equation is an identity: $\dfrac{\sin(x-y)}{\sin x \sin y} = \cot y - \cot x$

21. Prove the following equation is an identity: $\dfrac{\cos(x+y)}{\cos x \sin y} = \cot y - \tan x$

22. Prove the following equation is an identity: $\tan(x+y) + \tan(x-y) = \dfrac{2\tan x}{\cos^2 y\left(1 - \tan^2 x \tan^2 y\right)}$

23. Prove the following equation is an identity: $\dfrac{\sin(x+y)}{\cos(x-y)} = \dfrac{1 + \cot x \tan y}{\cot x + \tan y}$

24. Prove the following equation is an identity: $\dfrac{\cos(x-y)}{\cos(x+y)} = \dfrac{1 + \tan x \tan y}{1 - \tan x \tan y}$