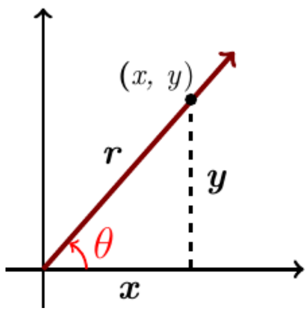
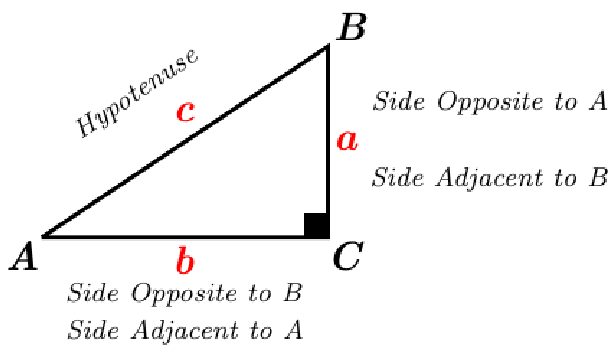
Section 6.3 – Trigonometric Functions

Let (x, y) be a point on the terminal side of an angle θ in standard position

The distance from the point to the origin is given by: $r=\sqrt{x^2+y^2}$

Six Trigonometry Functions





$\sin\theta=\frac{Opposite}{Hypotenuse}=\frac{opp}{hyp}=\frac{y}{r}$ $\qquad$ $\sin A=\frac{a}{c}=\cos B$

$\csc\theta=\frac{hyp}{opp}=\frac{1}{\sin\theta}=\frac{r}{y}$ $\qquad$ $\csc A=\frac{c}{a}=\sec B$

$\cos\theta=\frac{Adjacent}{Hypotenuse}=\frac{adj}{hyp}=\frac{x}{r}$ $\qquad$ $\cos A=\frac{b}{c}=\sin B$

$\sec\theta=\frac{hyp}{adj}=\frac{1}{\cos\theta}=\frac{r}{x}$ $\qquad$ $\sec A=\frac{c}{b}=\csc B$

$\tan\theta=\frac{opp}{adj}=\frac{\sin\theta}{\cos\theta}=\frac{y}{x}$ $\qquad$ $\tan A=\frac{a}{b}=\cot B$

$\cot\theta=\frac{adj}{opp}=\frac{\cos\theta}{\sin\theta}=\frac{1}{\tan\theta}=\frac{x}{y}$ $\qquad$ $\cot A=\frac{b}{a}=\tan B$

Example

Find the six trigonometry functions of θ if θ is in the standard position and the point (8, 15) is on the terminal side of θ.

Solution

$r=\sqrt{8^2+15^2}$ $\qquad$ $r=\sqrt{x^2+y^2}$

$=17$

$\sin\theta=\frac{y}{r}=\frac{15}{17}$ $\qquad$ $\cos\theta=\frac{x}{r}=\frac{8}{17}$ $\qquad$ $\tan\theta=\frac{y}{x}=\frac{15}{8}$

$\csc\theta=\frac{r}{y}=\frac{17}{15}$ $\qquad$ $\sec\theta=\frac{r}{x}=\frac{17}{8}$ $\qquad$ $\cot\theta=\frac{x}{y}=\frac{8}{15}$

Example

Which will be greater, tan 30° or tan 40°? How large could $\tan\theta$ be?

Solution

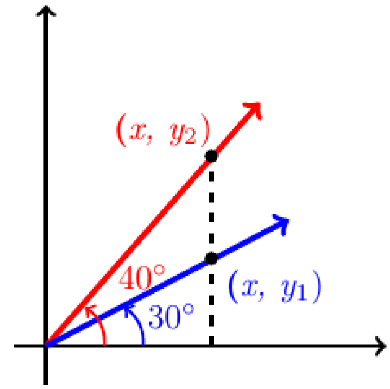
$\tan 30^\circ = \dfrac{y_1}{x}$

$\tan 40^\circ = \dfrac{y_2}{x}$

$Ratio: \dfrac{y_2}{x} > \dfrac{y_1}{x}$

$\rightarrow$ tan 40° > tan 30°

No limit as to how large $\tan\theta$ can be.



Example

If $\cos\theta = \dfrac{\sqrt{3}}{2}$, and θ is **Q**IV, find $\sin\theta$ and $\tan\theta$.

Solution

$\cos\theta = \dfrac{\sqrt{3}}{2} = \dfrac{x}{r} \quad \rightarrow x = \sqrt{3}, \quad r = 2$

$$
\begin{aligned}
y &= \sqrt{r^2 - x^2} \\
&= \sqrt{2^2 - \left(\sqrt{3}\right)^2} \\
&= \sqrt{4-3} \\
&= 1
\end{aligned}
$$

Since θ is Q IV $\Rightarrow$ $y = -1$

$\sin\theta = -\dfrac{1}{2}$ $\qquad\qquad$ $\sin\theta = \dfrac{y}{r}$

$\tan\theta = \dfrac{-1}{\sqrt{3}}$ $\qquad\qquad$ $\tan\theta = \dfrac{y}{x}$

$= -\dfrac{1}{\sqrt{3}}\dfrac{\sqrt{3}}{\sqrt{3}}$

$= -\dfrac{\sqrt{3}}{3}$

Reciprocal Identities

$\csc\theta = \dfrac{1}{\sin\theta}$ $\sin\theta = \dfrac{1}{\csc\theta}$ $\cot\theta = \dfrac{1}{\tan\theta}$

$\sec\theta = \dfrac{1}{\cos\theta}$ $\cos\theta = \dfrac{1}{\sec\theta}$ $\tan\theta = \dfrac{1}{\cot\theta}$

Ratio Identities

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Pythagorean Identities

$$x^2 + y^2 = r^2$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1 \quad \Rightarrow \quad \boxed{\cos^2\theta + \sin^2\theta = 1}$$

Solving for $\cos\theta$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$$

Solving for $\sin\theta$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\left(\frac{\cos\theta}{\cos\theta}\right)^2 + \left(\frac{\sin\theta}{\cos\theta}\right)^2 = \left(\frac{1}{\cos\theta}\right)^2$$

$$\boxed{1+\tan^2\theta = \sec^2\theta}$$

$\cos^2\theta + \sin^2\theta = 1$
$\cos\theta = \pm\sqrt{1-\sin^2\theta}$
$\sin\theta = \pm\sqrt{1-\cos^2\theta}$
$1+\tan^2\theta = \sec^2\theta$
$1+\cot^2\theta = \csc^2\theta$

Example

Write $\tan\theta$ in terms of $\sin\theta$.

Solution

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\sin\theta}{\pm\sqrt{1-\sin^2\theta}}$$

$$= \pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$$

Example

Simplify the expression $\sqrt{x^2+9}$ as much as possible after substituting $3\tan\theta$ for x

Solution

$$x = 3\tan\theta$$

$$\sqrt{x^2+9} = \sqrt{(3\tan\theta)^2+9}$$

$$= \sqrt{9\tan^2\theta+9}$$

$$= \sqrt{9\left(\tan^2\theta+1\right)}$$

$$= 3\sqrt{\sec^2\theta}$$

$$\boxed{= 3\sec\theta}$$

Example

Triangle *ABC* is a right triangle with *C* = 90°. If *a* = 6 and *c* = 10, find the six trigonometric functions of *A*.

Solution

$$b=\sqrt{c^2-a^2}=\sqrt{10^2-6^2}=8$$

$$6,\ b\ \to\ 10\quad \to\quad 2(3,\ b\ \to\ 5)$$

$$\underline{b=4\,|}$$

$\sin A=\frac{a}{c}=\frac{3}{5}$	$\cos A=\frac{b}{c}=\frac{8}{10}=\frac{4}{5}$	$\tan A=\frac{a}{b}=\frac{6}{8}=\frac{3}{4}$
$\csc A=\frac{5}{3}$	$\sec A=\frac{5}{4}$	$\cot A=\frac{4}{3}$

$$\textit{if}\quad A+B=90^\circ\quad\Rightarrow\quad\begin{cases}\sin A=\cos B\\ \sec A=\csc B\\ \tan A=\cot B\end{cases}$$

Cofunction Theorem

A trigonometric function of an angle is always equal to the cofunction of the complement of the angle.

Example

Write each function in terms of its cofunction

a) $\cos 52^\circ$

Solution

$$\cos 52^\circ=\sin(90^\circ-52^\circ)=\sin 38^\circ$$

b) $\tan 71^\circ$

Solution

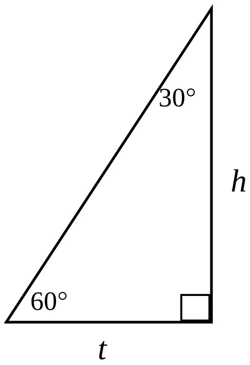
$$\tan 71^\circ=\cot(90^\circ-71^\circ)=\cot 19^\circ$$

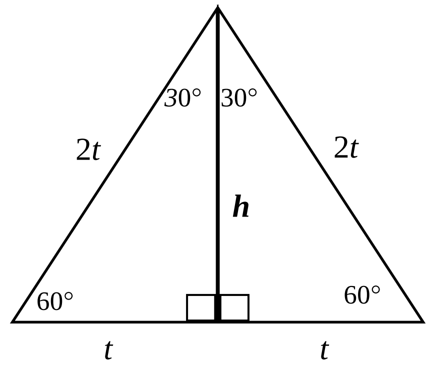
c) $\sec 24^\circ$

Solution

$$\sec 24^\circ=\csc(90^\circ-24^\circ)=\csc 66^\circ$$

30° - 60° - 90° *Triangle*





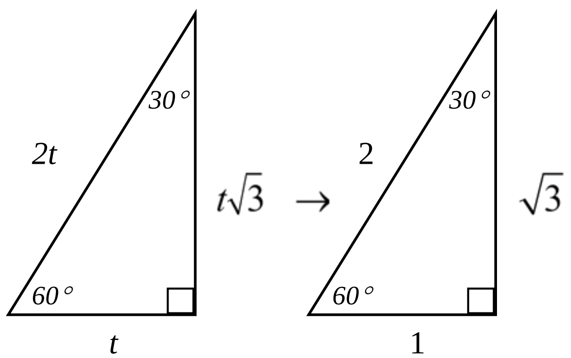
$$t^2 + h^2 = (2t)^2$$

$$t^2 + h^2 = 4t^2$$

$$h^2 = 4t^2 - t^2$$

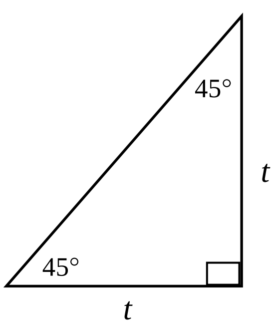
$$h^2 = 3t^2$$

$$h = t\sqrt{3}$$



$$\Rightarrow \boxed{\sin 60° = \frac{\sqrt{3}}{2}}$$

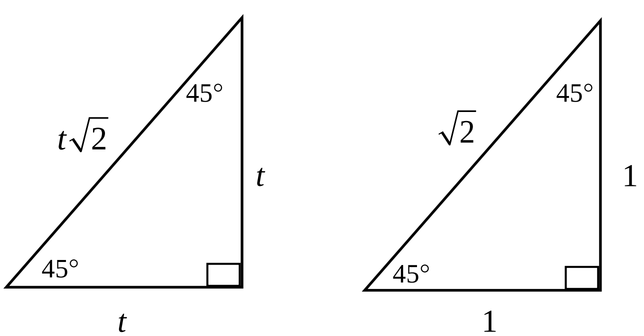
45° - 45° - 90° *Triangle*



$$hypotenuse^2 = t^2 + t^2$$

$$hypotenuse = \sqrt{2t^2}$$

$$hypotenuse = t\sqrt{2}$$

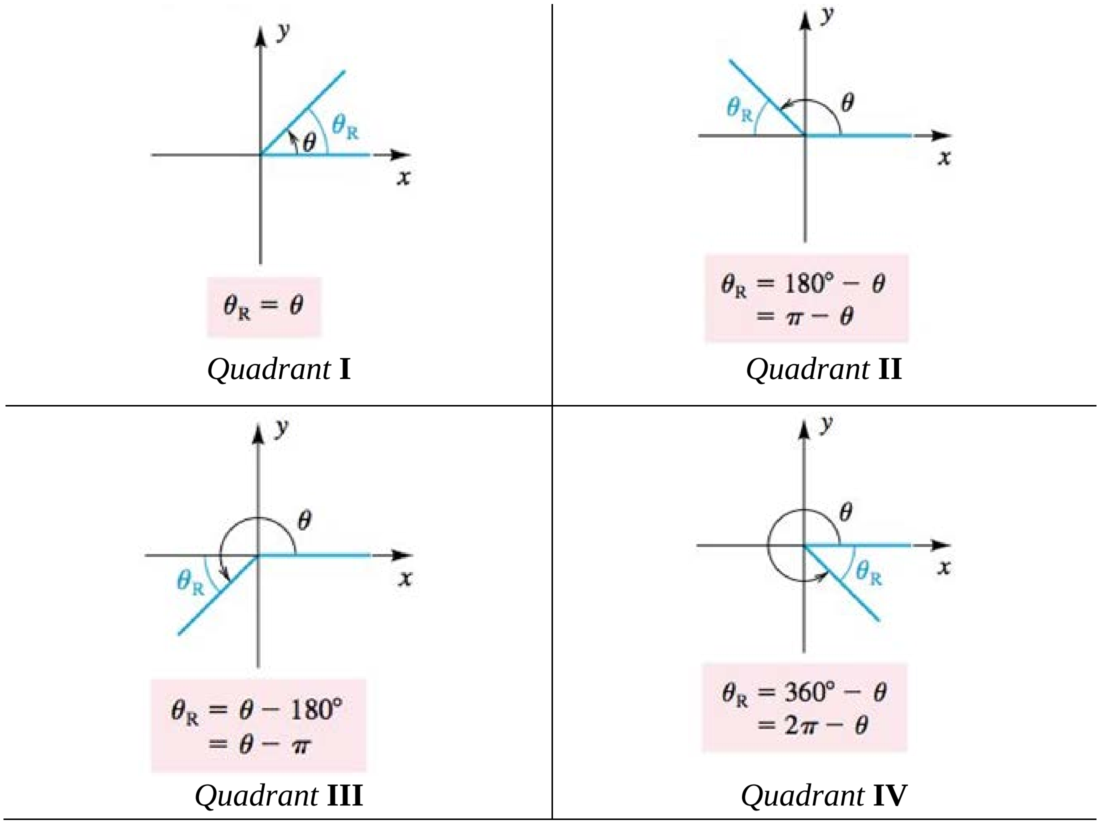


$$\Rightarrow \boxed{\cos 45° = \frac{1}{\sqrt{2}}}$$

Reference Angle

Definition

The reference angle or related angle for any angle θ in standard position ifs the positive acute angle between the terminal side of θ and the *x*-axis, and it is denoted $\hat{\theta}$



Example

Find the exact value of $\sin 240°$

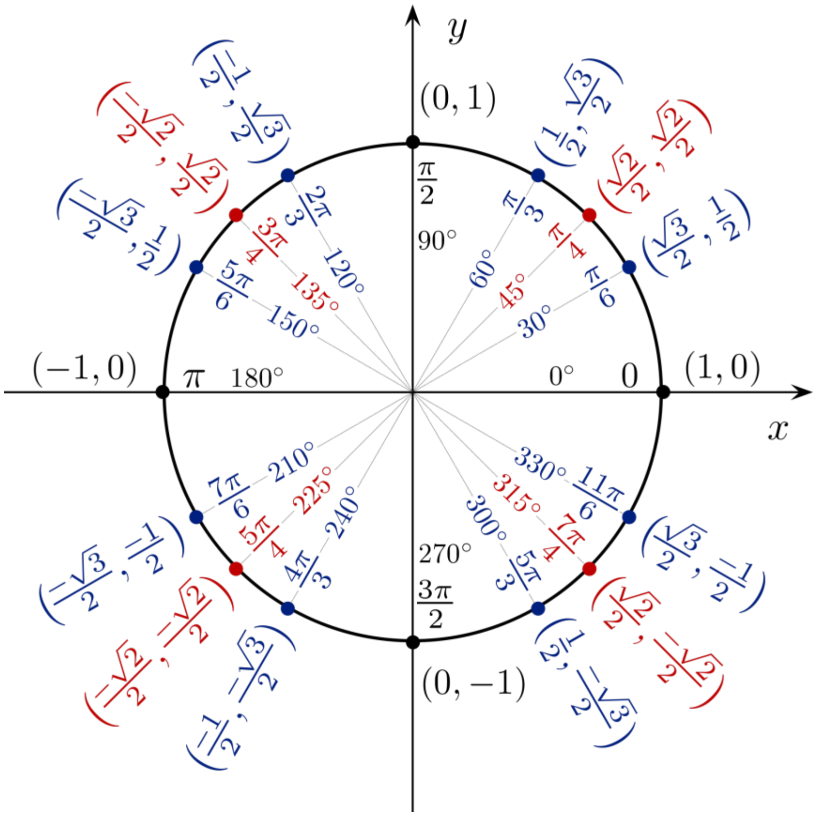
Solution

$\hat{\theta} = 240° - 180° = 60°$

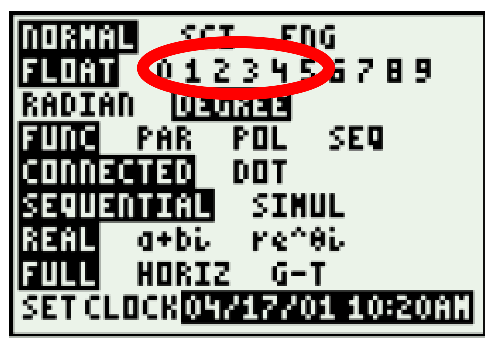
$\rightarrow 240° \in QIII$

$\sin 240° = -\sin 60°$

$= -\frac{\sqrt{3}}{2}$



Approximation- Simply using calculator



$\sin 250° \approx -0.9397$

$\cos 250° \approx -0.3420$

$\tan 250° \approx 2.7475$

$\csc 250° = \dfrac{1}{\sin 250°} \approx -1.0642$

To find the angle by using the inverse trigonometry functions, always enter a ***positive*** *value.*

Example

Find θ if $\sin\theta = -0.5592$ and θ terminates in QIII with $0° \le \theta < 360°$.

Solution

$\hat{\theta} = \sin^{-1} 0.5592 \approx 34°$

$\theta \in$ QIII

$\Rightarrow \theta = 180° + 34° = 214°$

Example

Find θ to the nearest degree if $\cot\theta = -1.6003$ and θ terminates in QII with $0° \le \theta < 360°$.

Solution

$$\cot\theta = -1.6003 = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{1}{-1.6003}$$

$$\hat{\theta} = \tan^{-1}\frac{1}{1.6003} \ \underline{= 32°}|$$

$$\theta \in \text{QII} \quad \Rightarrow \theta = 180° - 32° = \underline{148°}|$$

Exercise *Section* 6.3 – Trigonometric Functions

(**1 – 16**) Find the ***six*** trigonometry functions of θ if θ is in the standard position and the given point is on the terminal side of θ.

1. $(-2,\ 3)$
2. $(-3,\ -4)$
3. $(-3,\ 0)$
4. $(12,\ -5)$
5. $(5,\ -12)$
6. $(9,\ -12)$
7. $(16,\ -12)$
8. $(15,\ -8)$
9. $(-6,\ 8)$
10. $(-15,\ 8)$
11. $(-7,\ 24)$
12. $(10,\ -24)$
13. $(7,\ 24)$
14. $(-7,\ -24)$
15. $(-24,\ -7)$
16. $(24,\ -10)$

17. Find the values of the six trigonometric functions for an angle of $90°$.

18. Indicate the two quadrants θ could terminate in if $\cos\theta = \frac{1}{2}$

19. Indicate the two quadrants θ could terminate in if $\csc\theta = -2.45$

(**20 – 38**) Find the remaining trigonometric function of θ if

20. $\sin\theta = \frac{12}{13}$ and θ terminates in QI.

21. $\cot\theta = -2$ and θ terminates in QII.

22. $\tan\theta = \frac{3}{4}$ and θ terminates in $QIII$.

23. $\cos\theta = \frac{24}{25}$ and θ terminates in QIV.

24. $\cos\theta = \frac{\sqrt{3}}{2}$ and θ is terminates in QIV.

25. $\tan\theta = -\frac{1}{2}$ and $\cos\theta > 0$

26. $\cos\theta = \frac{3}{5}$ & $\theta \in QI$

27. $\cos\theta = -\frac{4}{5}$ & $\theta \in QII$

28. $\sin\theta = -\frac{3}{5}$ & $\theta \in QIII$

29. $\sin\theta = -\frac{3}{5}$ & $\theta \in QIV$

30. $\cos\theta = -\frac{12}{13}$ & $\theta \in QIII$

31. $\cos\theta = -\frac{5}{13}$ & $\theta \in QII$

32. $\cos\theta = \frac{12}{13}$ & $\theta \in QIV$

33. $\sin\theta = -\frac{8}{17}$ & $\theta \in QIII$

34. $\cos\theta = -\frac{15}{17}$ & $\theta \in QII$

35. $\cos\theta = -\frac{8}{17}$ & $\theta \in QII$

36. $\cos\theta = -\frac{7}{25}$ & $\theta \in QII$

37. $\sin\theta = -\frac{7}{25}$ & $\theta \in QIII$

38. $\sin\theta = -\frac{24}{25}$ & $\theta \in QIV$

39. If $\sin\theta = -\frac{5}{13}$, and θ is QIII, find $\cos\theta$ and $\tan\theta$.

40. If $\cos\theta = \frac{3}{5}$, and θ is QIV, find $\sin\theta$ and $\tan\theta$.

41. Use the reciprocal identities if $\cos\theta = \frac{\sqrt{3}}{2}$ find $\sec\theta$

42. Find $\cos\theta$, given that $\sec\theta = \frac{5}{3}$

43. Find $\sin\theta$, given that $\csc\theta = -\frac{\sqrt{12}}{2}$

44. Use a ratio identity to find $\tan\theta$ if $\sin\theta = \frac{3}{5}$ and $\cos\theta = -\frac{4}{5}$

45. If $\cos\theta = -\frac{1}{2}$ and θ terminates in QII, find $\sin\theta$

46. If $\sin\theta = \frac{3}{5}$ and θ terminated in QII, find $\cos\theta$ and $\tan\theta$.

47. Find $\tan\theta$ if $\sin\theta = \frac{1}{3}$ and θ terminates in QI

48. Find the remaining trigonometric ratios of θ, if $\sec\theta = -3$ and $\theta \in QIII$

49. Using the calculator and rounding your answer to the nearest hundredth, find the remaining trigonometric ratios of θ if $\csc\theta = -2.45$ and $\theta \in QIII$.

50. Write $\frac{\sec\theta}{\csc\theta}$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

51. Write $\cot\theta - \csc\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

52. Write $\frac{\sin\theta}{\cos\theta} + \frac{1}{\sin\theta}$ in terms of $\sin\theta$ and/or $\cos\theta$, and then simplify if possible.

53. Write $\sin\theta\cot\theta + \cos\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

54. Multiply $(1-\cos\theta)(1+\cos\theta)$

55. Multiply $(\sin\theta + 2)(\sin\theta - 5)$

56. Simplify the expression $\sqrt{25 - x^2}$ as much as possible after substituting $5\sin\theta$ for x.

57. Simplify the expression $\sqrt{4x^2 + 16}$ as much as possible after substituting $2\tan\theta$ for x

(**58 – 60**) Simplify by using the table

58. $5\sin^2 30°$ **59.** $\sin^2 60° + \cos^2 60°$ **60.** $(\tan 45° + \tan 60°)^2$

61. Find θ if $\sin\theta = -\frac{1}{2}$ and θ terminates in QIII with $0° \le \theta \le 360°$.

62. Find θ to the nearest degree if $\sec\theta = 3.8637$ and θ terminates in ***Q***IV with $0° \le \theta < 360°$.

(**63 – 67**) Find the exact value of

63. $\cos 225°$ **65.** $\tan 315°$ **67.** $\cot 480°$

64. $\csc 300°$ **66.** $\cos 420°$

(**68 – 70**) Use the calculator to find the value of

68. $\csc 166.7^\circ$ **69.** $\sec 590.9^\circ$ **70.** $\tan 195^\circ\, 10'$

71. Use the calculator to find θ to the nearest degree if $\sin\theta = -0.3090$ with $\theta \in \boldsymbol{Q}\text{IV}$ with $0^\circ \le \theta < 360^\circ$

72. Use the calculator to find θ to the nearest degree if $\cos\theta = -0.7660$ with $\theta \in \boldsymbol{Q}\text{III}$ with $0^\circ \le \theta < 360^\circ$

73. Use the calculator to find θ to the nearest degree if $\sec\theta = -3.4159$ with $\theta \in \boldsymbol{Q}\text{II}$ with $0^\circ \le \theta < 360^\circ$

74. Find θ to the nearest tenth of a degree if $\tan\theta = -0.8541$ and θ terminates in $\boldsymbol{Q}$IV with $0^\circ \le \theta < 360^\circ$

75. Use the calculator to find θ to the nearest degree if $\sin\theta = 0.49368329$ with $\theta \in \boldsymbol{Q}\text{II}$ with $0^\circ \le \theta < 360^\circ$