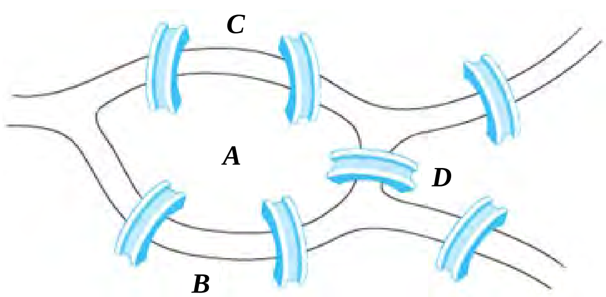
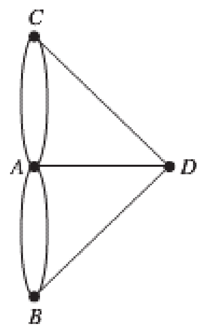
Section 4.9 – Euler and Hamilton Paths

The town of *Königsberg* was divided into 4 sections by the branches of the Pregel River. The town people took long walks through town. They wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing and bridge twice, and return to the starting point.



Königsberg Town



Multigraph

The Swiss mathematician Leonhard Euler solved this problem. His solution, published in 1736, may be the first use of graph theory. Euler studied this problem using the multigraph obtained when the four regions are represented by vertices and the bridges by edges.

Definition

Let G be a graph. An Euler Circuit for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit for G is a sequence of adjacent vertices and edges in G that has at least one edge, starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.

Theorem

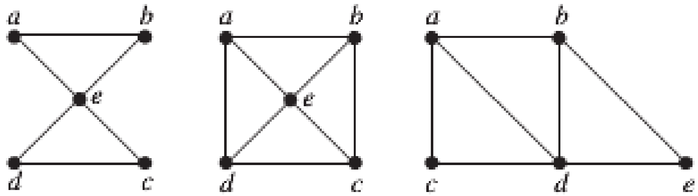
A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

Theorem

A connected multigraph has Euler path but not an Euler circuit if and only if it has exactly 2 vertices of odd degree.

Example

Which of the undirected graph have Euler circuit? Of those that do not, which have an Euler path?



G_1 G_2 G_3

<u>Solution</u>

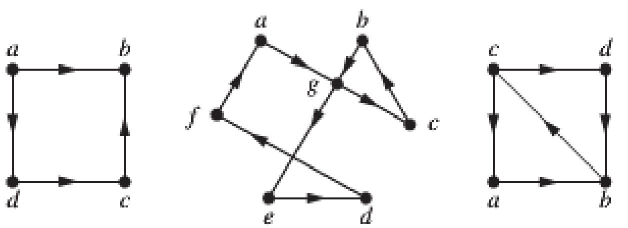
The graph G_1 has Euler circuit, for example, *a, e, c, d, e, b, a.*

G_2 & G_3 do not have Euler circuit. Because vertices *a, b, c, d* of G_2 & *a, b* of G_3 have degree 3

G_3 has an Euler path, *a, c, d, e, b, d, a, b*. G_2 does not have Euler path.

Example

Which of the undirected graph have Euler circuit? Of those that do not, which have an Euler path?



G_1 G_2 G_3

<u>Solution</u>

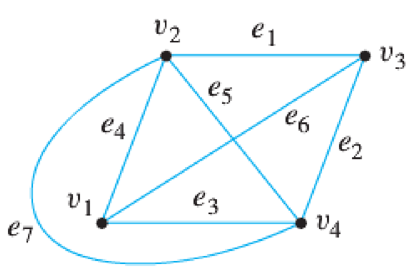
The graph G_2 has Euler circuit, for example, *a, g, c, b, g, e, d, f, a.*

G_1 & G_3 do not have Euler circuit. Because vertices *a, b, c, d* of G_1 have degree 1 (*odd*) & *c, b* of G_3 have degree 3

G_3 has an Euler path, *c, a, b, c, d, b*. but G_1 does not have Euler path.

Example

Show that the graph below does not have an Euler circuit



<u>Solution</u>

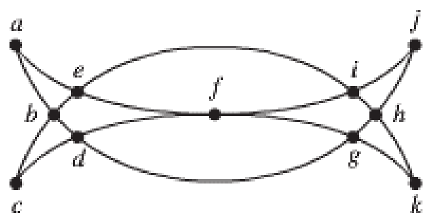
Vertices v_1 & v_3 both have degree 3, which is odd. Hence by the contrapositive form, this graph does not have an Euler circuit.

Definition

Let *G* be a graph, and let *v* and *w* be two distinct vertices of *G*. An Euler trail from *v* to *w* is a sequence of adjacent edges and vertices that starts at *v*, ends at *w*, passes through every vertex of *G* at least once, and traverses every edge of *G* exactly once.

Example

Many puzzles ask you to draw a picture in a continuous motion without lifting a pencil so that no part of the picture is retraced. We can solve such puzzles using Euler circuits and paths. For example, can Mohammed's scimitars, shown in Figure below, be drawn in this way, where the drawing begins and ends at the same point?



Solution

Let denote G for the graph. G has an Euler circuit, because all its vertices have even degree.

1. Form the circuit: *a, b, d, c, b, e, i, f, e, a*. We obtain the subgraph *H* by deleting the edges in this circuit and all vertices that become isolated when these edges are removed.
2. Form: *d, g, h, j, i, h, k, g, f, d* in H. After forming this circuit we have used all edges in G. Splicing this new circuit into the first circuit at the appropriate place produces the Euler circuit *a, b, d, g, h, j, i, h, k, g, f, d, c, b, e, i, f, e, a*. This circuit gives a way to draw the scimitars without lifting the pencil or retracting part of the picture.

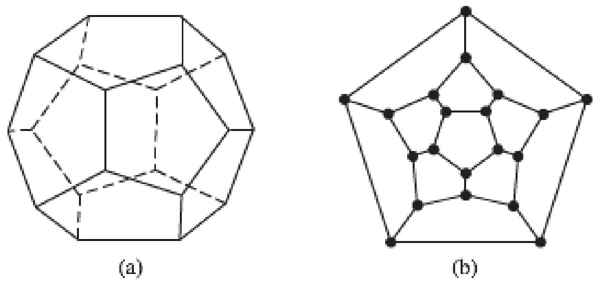
Hamilton Path and Circuits

A simple path in a graph G that passes through every vertex exactly once is called a ***Hamilton path***, and a simple circuit G in a graph G that passes through every vertex exactly once is called a *Hamilton circuit*.

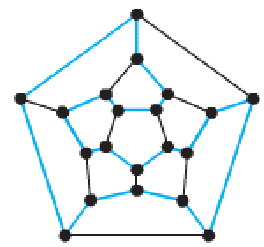
That is, the simple path $x_0, x_1, \ldots, x_{n-1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if

$V = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$, and the simple circuit

$x_0, x_1, \ldots, x_{n-1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, \ldots, x_{n-1}, x_n$ is a Hamilton path.

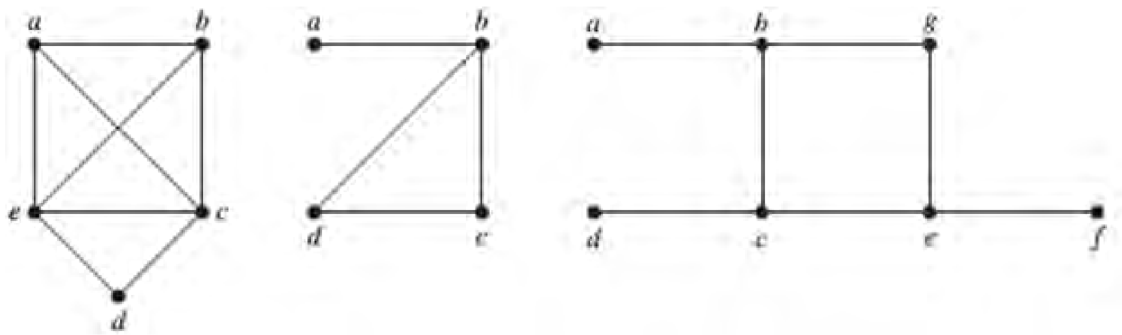


Hamilton's "A Voyage Round the World" Puzzle


Solution to the "A Voyage Round the World" Puzzle

Example

Which of the simple given graphs have a Hamilton circuit or, if not, a Hamilton path?



G_1 G_2 G_3

Solution

G_1 has a Hamilton circuit: *a, b, c, d, e, a*.

There is no Hamilton circuit in G_2, every vertex contain the edge {*a, b*} twice, but G_2 does have a Hamilton path, namely *a, b, c, d*.

G_3 has neither a Hamilton circuit nor a Hamilton path, because any path containing all vertices must contain one of the edges {*a, b*}, {*e, f*}, and {*c, d*} more than once.

Example

Show that K_n has a Hamilton circuit whenever $n \geq 3$.

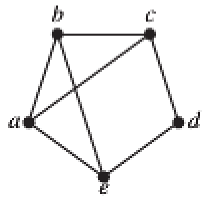
Solution

We can form a Hamilton circuit in K_n beginning at any vertex. Such a circuit can be built by visiting vertices in any order we choose, as long as the path begins and ends at the same vertex and visits each other vertex exactly once. This is possible because there are edges in K_n between any two vertices.

Exercises *Section* 4.9 – Euler and Hamilton Paths

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

1.



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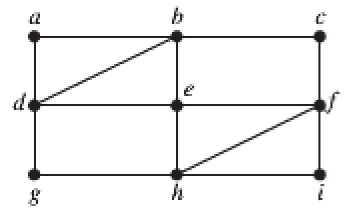

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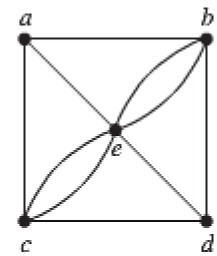

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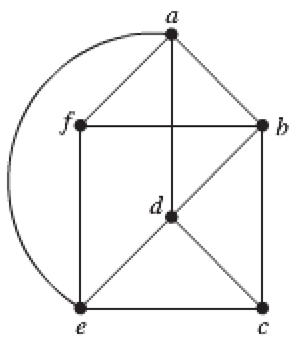

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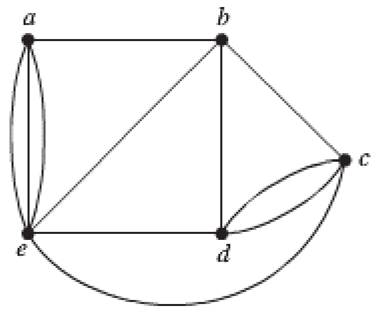

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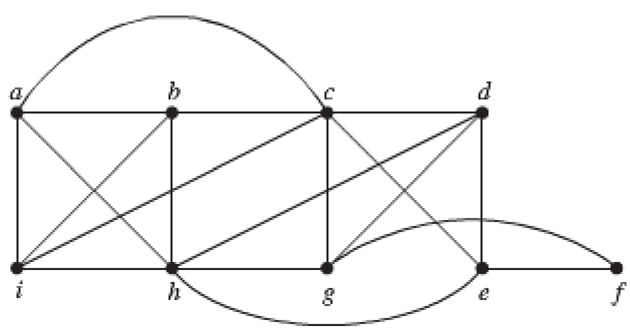

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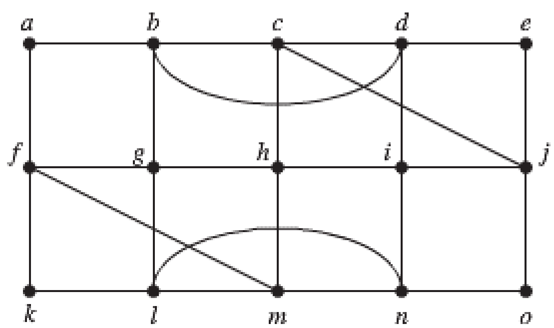
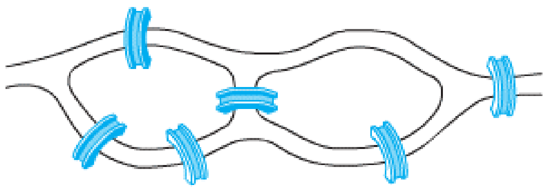
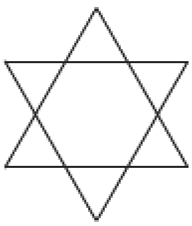


8. Can someone cross all the bridges shown in this map exactly once and return to the starting point?



Determine whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture

9.



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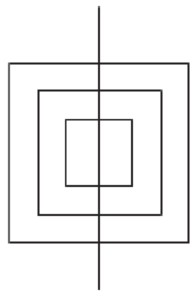

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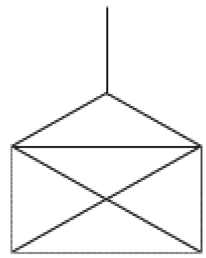
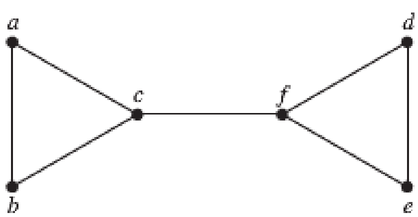


Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

12.



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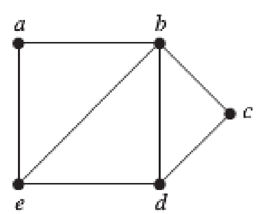

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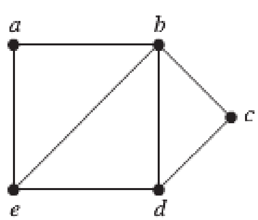

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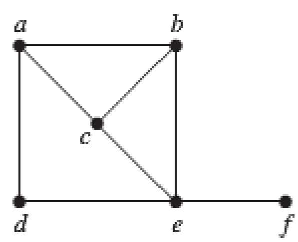

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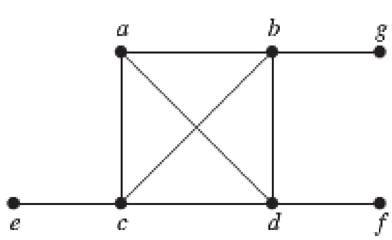

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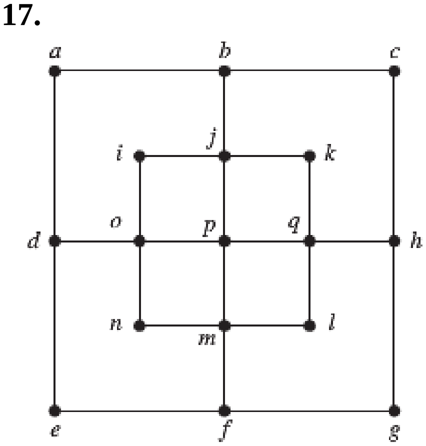

18. Imagine that the drawing below is a map showing 4 cities and the distances in kilometers between them. Suppose that a salesman must travel to each city exactly once, starting and ending in city *A*. Which route from city to city will minimize the total distance that must be traveled?