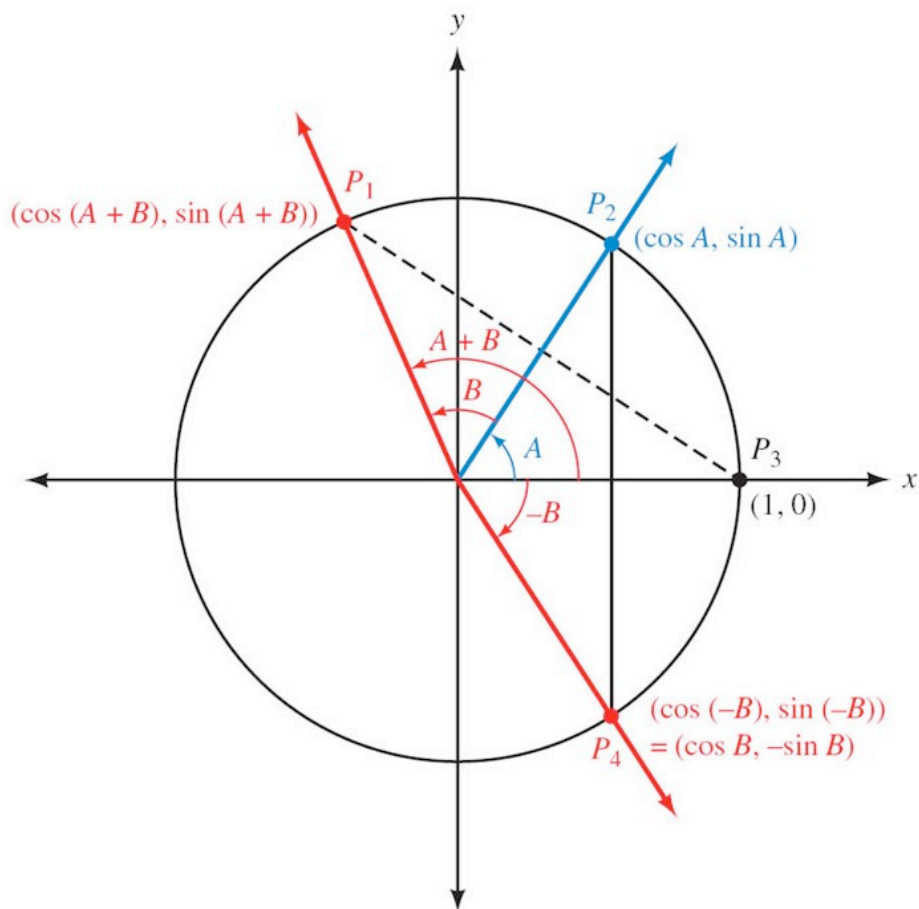


## Section 4.2 – Sum and Difference Formulas



$$P_1 P_3 = P_2 P_4$$

$$(P_1 P_3)^2 = (P_2 P_4)^2$$

*Distance between points*

$$[\cos(A+B) - 1]^2 + [\sin(A+B) - 0]^2 = (\cos A - \cos B)^2 + (\sin A + \sin B)^2$$

$$\cos^2(A+B) - 2\cos(A+B) + 1 + \sin^2(A+B) = (\cos A - \cos B)^2 + (\sin A + \sin B)^2$$

$$1 - 2\cos(A+B) + 1 = \cos^2 A - 2\cos B \cos A + \cos^2 B + \sin^2 A + 2\sin B \sin A + \sin^2 B$$

$$2 - 2\cos(A+B) = \cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B - 2\cos B \cos A + 2\sin B \sin A$$

$$2 - 2\cos(A+B) = 1 + 1 - 2\cos B \cos A + 2\sin B \sin A$$

$$2 - 2\cos(A+B) = 2 - 2\cos B \cos A + 2\sin B \sin A$$

$$-2\cos(A+B) = -2\cos B \cos A + 2\sin B \sin A$$

$$\boxed{\cos(A+B) = \cos B \cos A - \sin B \sin A}$$

$$\begin{aligned}\cos(A + B) &= \cos A \cos B - \sin A \sin B \\ \cos(A - B) &= \cos A \cos B + \sin A \sin B \\ \sin(A + B) &= \sin A \cos B + \cos A \sin B \\ \sin(A - B) &= \sin A \cos B - \cos A \sin B\end{aligned}$$

### ***Example***

Find the exact value for  $\cos 75^\circ$

### **Solution**

$$\begin{aligned}\cos 75^\circ &= \cos(45^\circ + 30^\circ) \\ &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \frac{1}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

### ***Example***

Show that  $\cos(x + 2\pi) = \cos x$

### **Solution**

$$\begin{aligned}\cos(x + 2\pi) &= \cos x \cos 2\pi - \sin x \sin 2\pi \\ &= \cos x \cdot (1) - \sin x \cdot (0) \\ &= \cos x\end{aligned}$$



### ***Example***

Simplify:  $\cos 3x \cos 2x - \sin 3x \sin 2x$

### **Solution**

$$\begin{aligned}\cos 3x \cos 2x - \sin 3x \sin 2x &= \cos(3x + 2x) \\ &= \cos 5x\end{aligned}$$

### ***Example***

Show that  $\cos(90^\circ - A) = \sin A$

### **Solution**

$$\begin{aligned}\cos(90^\circ - A) &= \cos 90^\circ \cos A + \sin 90^\circ \sin A \\ &= 0 \cdot \cos A + 1 \cdot \sin A \\ &= \sin A \quad \checkmark\end{aligned}$$

### ***Example***

Find the exact value of  $\sin \frac{\pi}{12}$

### **Solution**

$$\begin{aligned}\sin \frac{\pi}{12} &= \sin \left( \frac{\pi}{3} - \frac{\pi}{4} \right) \\ &= \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4} \\ &= \frac{\sqrt{3}}{2} \frac{\sqrt{2}}{2} - \frac{1}{2} \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

### ***Example***

Find the exact value of  $\cos 15^\circ$

### **Solution**

$$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ) \\ &= \cos(45^\circ) \cos(30^\circ) + \sin(45^\circ) \sin(30^\circ) \\ &= \frac{\sqrt{2}}{2} \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \frac{1}{2} \\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \\ &= \frac{\sqrt{6} + \sqrt{2}}{4}\end{aligned}$$

### ***Example***

If  $\sin A = \frac{3}{5}$  with  $A$  in QI, and  $\cos B = -\frac{5}{13}$  with  $B$  in QIII, find  $\sin(A + B)$ ,  $\cos(A + B)$ , and  $\tan(A + B)$

### **Solution**

$$\sin A = \frac{3}{5} \rightarrow A \in QI$$

$$\cos B = -\frac{5}{13} \rightarrow B \in QIII$$

$$\cos A = \frac{4}{5}$$

$$\sin B = -\frac{12}{13}$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\begin{aligned} &= \frac{3}{5} \left( -\frac{5}{13} \right) + \frac{4}{5} \left( -\frac{12}{13} \right) \\ &= -\frac{15}{65} - \frac{48}{65} \\ &= -\frac{63}{65} \end{aligned}$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\begin{aligned} &= \frac{4}{5} \left( -\frac{5}{13} \right) - \frac{3}{5} \left( -\frac{12}{13} \right) \\ &= -\frac{20}{65} + \frac{36}{65} \\ &= \frac{16}{65} \end{aligned}$$

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

$$= -\frac{63}{16}$$

$$\begin{aligned}
\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\
&= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \\
&= \frac{\frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B - \sin A \sin B}{\cos A \cos B}} \\
&= \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}} \\
&= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A}{\cos A} \frac{\sin B}{\cos B}} \\
&= \frac{\tan A + \tan B}{1 - \tan A \tan B}
\end{aligned}$$

$$\boxed{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

$$\boxed{\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}}$$

### ***Example***

If  $\sin A = \frac{3}{5}$  with  $A$  in  $QI$ , and  $\cos B = -\frac{5}{13}$  with  $B$  in  $QIII$ , find  $\tan(A+B)$

### **Solution**

$$\begin{aligned}
\tan A &= \frac{3/5}{4/5} & \tan A &= \frac{\sin A}{\cos A} \\
&= \frac{3}{4}
\end{aligned}$$

$$\begin{aligned}
\tan B &= \frac{-12/13}{-5/13} & \tan B &= \frac{\sin B}{\cos B} \\
&= \frac{12}{5}
\end{aligned}$$

$$\begin{aligned}
\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
&= \frac{\frac{3}{4} + \frac{12}{5}}{1 - \frac{3}{4} \frac{12}{5}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\frac{15}{20} + \frac{48}{20}}{1 - \frac{36}{20}} \\
&= \frac{\frac{63}{20}}{-\frac{16}{20}} \\
&= -\frac{63}{16}
\end{aligned}$$

### Example

Establish the identity:  $\frac{\cos(x-y)}{\sin x \sin y} = \cot x \cot y + 1$

#### Solution

$$\begin{aligned}
\frac{\cos(x-y)}{\sin x \sin y} &= \frac{\cos x \cos y + \sin x \sin y}{\sin x \sin y} \\
&= \frac{\cos x \cos y}{\sin x \sin y} + \frac{\sin x \sin y}{\sin x \sin y} \\
&= \cot x \cot y + 1
\end{aligned}$$



### Example

Establish the identity:  $\cot(x+y) = \frac{\cot x \cot y - 1}{\cot x + \cot y}$

#### Solution

$$\begin{aligned}
\cot(x+y) &= \frac{\cos(x+y)}{\sin(x+y)} \\
&= \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y} \\
&= \frac{\frac{\cos x \cos y}{\sin x \cos y} - \frac{\sin x \sin y}{\sin x \cos y}}{\frac{\sin x \sin y}{\sin x \cos y} + \frac{\cos x \sin y}{\sin x \sin y}} \\
&= \frac{\cot x \cot y - 1}{\cot x + \cot y}
\end{aligned}$$



### ***Example***

Establish the identity:  $\sec(x - y) = \frac{\cos x \cos y - \sin x \sin y}{\cos^2 x - \sin^2 y}$

### **Solution**

$$\begin{aligned}\sec(x - y) &= \frac{1}{\cos(x - y)} \frac{\cos(x + y)}{\cos(x + y)} \\&= \frac{\cos x \cos y - \sin x \sin y}{(\cos x \cos y + \sin x \sin y)(\cos x \cos y - \sin x \sin y)} \\&= \frac{\cos x \cos y - \sin x \sin y}{\cos^2 x \cos^2 y - \sin^2 x \sin^2 y} \\&= \frac{\cos x \cos y - \sin x \sin y}{\cos^2 x \cos^2 y - \sin^2 x \sin^2 y} \\&= \frac{\cos x \cos y - \sin x \sin y}{\cos^2 x (1 - \sin^2 y) - (1 - \cos^2 x) \sin^2 y} \\&= \frac{\cos x \cos y - \sin x \sin y}{\cos^2 x - \cos^2 x \sin^2 y - \sin^2 y + \cos^2 x \sin^2 y} \\&= \frac{\cos x \cos y - \sin x \sin y}{\cos^2 x - \sin^2 y} \quad \checkmark\end{aligned}$$

## Exercises

## Section 4.2 – Sum and Difference Formulas

1. Write the expression as a single trigonometric function  $\sin 8x \cos x - \cos 8x \sin x$
2. Show that  $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$
3. If  $\sin A = \frac{4}{5}$  ( $A \in QII$ ), and  $\cos B = -\frac{5}{13}$  ( $B \in QIII$ ), find
  - a)  $\sin(A + B)$
  - b)  $\cos(A + B)$
  - c)  $\tan(A + B)$
  - d)  $\sin(A - B)$
  - e)  $\cos(A - B)$
  - f)  $\tan(A - B)$
4. If  $\sin A = \frac{3}{5}$  ( $A \in QII$ ), and  $\cos B = -\frac{12}{13}$  ( $B \in QIII$ ), find
  - a)  $\sin(A + B)$
  - b)  $\cos(A + B)$
  - c)  $\tan(A + B)$
  - d)  $\sin(A - B)$
  - e)  $\cos(A - B)$
  - f)  $\tan(A - B)$
5. If  $\sin A = \frac{1}{\sqrt{5}}$  ( $A \in QI$ ), and  $\tan B = \frac{3}{4}$  ( $B \in QI$ ), find
  - a)  $\sin(A + B)$
  - b)  $\cos(A + B)$
  - c)  $\tan(A + B)$
  - d)  $\sin(A - B)$
  - e)  $\cos(A - B)$
  - f)  $\tan(A - B)$
6. If  $\sin A = \frac{3}{5}$  ( $A \in QII$ ), and  $\cos B = \frac{12}{13}$  ( $B \in QIV$ ), find
  - a)  $\sin(A + B)$
  - b)  $\cos(A + B)$
  - c)  $\tan(A + B)$
  - d)  $\sin(A - B)$
  - e)  $\cos(A - B)$
  - f)  $\tan(A - B)$
7. If  $\sin A = \frac{7}{25}$  ( $A \in QII$ ), and  $\cos B = -\frac{8}{17}$  ( $B \in QIII$ ), find
  - a)  $\sin(A + B)$
  - b)  $\cos(A + B)$
  - c)  $\tan(A + B)$
  - d)  $\sin(A - B)$
  - e)  $\cos(A - B)$
  - f)  $\tan(A - B)$
8. If  $\cos A = -\frac{4}{5}$  ( $A \in QII$ ), and  $\sin B = \frac{24}{25}$  ( $B \in QII$ ), find
  - a)  $\sin(A + B)$
  - b)  $\cos(A + B)$
  - c)  $\tan(A + B)$
  - d)  $\sin(A - B)$
  - e)  $\cos(A - B)$
  - f)  $\tan(A - B)$
9. If  $\cos A = \frac{15}{17}$  ( $A \in QI$ ), and  $\cos B = -\frac{12}{13}$  ( $B \in QII$ ), find
  - a)  $\sin(A + B)$
  - b)  $\cos(A + B)$
  - c)  $\tan(A + B)$
  - d)  $\sin(A - B)$
  - e)  $\cos(A - B)$
  - f)  $\tan(A - B)$
10. If  $\sin A = -\frac{3}{5}$  ( $A \in QIV$ ), and  $\sin B = \frac{7}{25}$  ( $B \in QII$ ), find
  - a)  $\sin(A + B)$
  - b)  $\cos(A + B)$
  - c)  $\tan(A + B)$
  - d)  $\sin(A - B)$
  - e)  $\cos(A - B)$
  - f)  $\tan(A - B)$



11. If  $\sec A = \sqrt{5}$  with  $A$  in  $QI$ , and  $\sec B = \sqrt{10}$  with  $B$  in  $QI$ , find  $\sec(A + B)$

(12–30) Prove the identity

$$12. \frac{\sin(A - B)}{\cos A \cos B} = \tan A - \tan B$$

$$13. \sec(A + B) = \frac{\cos(A - B)}{\cos^2 A - \sin^2 B}$$

$$14. \frac{\cos 4\alpha}{\sin \alpha} - \frac{\sin 4\alpha}{\cos \alpha} = \frac{\cos 5\alpha}{\sin \alpha \cos \alpha}$$

$$15. \frac{\cos(x + y)}{\cos(x - y)} = \frac{\cot y - \tan x}{\cot y + \tan x}$$

$$16. \frac{\sin(x + y)}{\sin(x - y)} = \frac{\cot y + \cot x}{\cot y - \cot x}$$

$$17. \frac{\sin(x - y)}{\sin x \cos y} = 1 - \cot x \tan y$$

$$18. \frac{\sin(x - y)}{\sin x \sin y} = \cot y - \cot x$$

$$19. \frac{\cos(x + y)}{\cos x \sin y} = \cot y - \tan x$$

$$20. \frac{\sin(x + y)}{\cos(x - y)} = \frac{1 + \cot x \tan y}{\cot x + \tan y}$$

$$21. \sin\left(\frac{\pi}{4} + x\right) + \sin\left(\frac{\pi}{4} - x\right) = \sqrt{2} \cos x$$

$$22. \cos(A + B) + \cos(A - B) = 2 \cos A \cos B$$

$$23. \sin(x - y) - \sin(y - x) = 2 \sin x \cos y - 2 \cos x \sin y$$

$$24. \cos(x - y) + \cos(y - x) = 2 \cos x \cos y + 2 \sin x \sin y$$

$$25. \tan(x + y) \tan(x - y) = \frac{\tan^2 x - \tan^2 y}{1 - \tan^2 x \tan^2 y}$$

$$26. \frac{\cos(\alpha - \beta)}{\sin(\alpha + \beta)} = \frac{1 + \tan \alpha \tan \beta}{\tan \alpha + \tan \beta}$$

$$27. \sec(x + y) = \frac{\cos x \cos y + \sin x \sin y}{\cos^2 x - \sin^2 y}$$

$$28. \csc(x - y) = \frac{\sin x \cos y + \cos x \sin y}{\sin^2 x - \sin^2 y}$$

$$29. \tan(x + y) + \tan(x - y) = \frac{2 \tan x}{\cos^2 y (1 - \tan^2 x \tan^2 y)}$$

$$30. \frac{\cos(x - y)}{\cos(x + y)} = \frac{1 + \tan x \tan y}{1 - \tan x \tan y}$$

31. Common household current is called **alternating current** because the current alternates direction within the wires. The voltage  $V$  in a typical 115-volt outlet can be expressed by the function

$V(t) = 163 \sin \omega t$  where  $\omega$  is the angular speed (in *radians per second*) of the rotating generator at the electrical plant, and  $t$  is time measured in seconds.

a) It is essential for electric generators to rotate at precisely 60 cycles per second so household appliances and computers will function properly. Determine  $\omega$  for these electric generators.

b) Determine a value of  $\phi$  so that the graph of  $V(t) = 163 \cos(\omega t - \phi)$  is the same as the graph of

$$V(t) = 163 \sin \omega t$$