Solution *Section* R.2 – Integration

Exercise

Find each indefinite integral. $\displaystyle\int \frac{x+2}{\sqrt{x}}dx$

Solution

$$\int \frac{x+2}{\sqrt{x}}dx = \int\left[\frac{x}{x^{1/2}} + \frac{2}{x^{1/2}}\right]dx$$

$$= \int \frac{x}{x^{1/2}}dx + \int \frac{2}{x^{1/2}}dx$$

$$= \int x^{1/2}dx + 2\int x^{-1/2}dx$$

$$= \frac{x^{3/2}}{3/2} + 2\frac{x^{1/2}}{1/2} + C$$

$$= \frac{2}{3}x^{3/2} + 4x^{1/2} + C$$

Exercise

Find each indefinite integral $\displaystyle\int 4y^{-3}dy$

Solution

$$\int 4y^{-3}dy = 4\frac{y^{-2}}{-2} + C$$

$$= -\frac{2}{y^2} + C$$

Exercise

Find each indefinite integral $\displaystyle\int (x^3 - 4x + 2)dx$

Solution

$$\int \left(x^3 - 4x + 2\right)dx = \frac{1}{4}x^4 - 2x^2 + 2x + C$$

Exercise

Find each indefinite integral $\int \left(\sqrt[4]{x^3}+1\right)dx$

Solution

$$\int \left(x^{3/4}+1\right)dx = \boxed{\frac{4}{7}x^{7/4}+x+C}$$

Exercise

Find each indefinite integral $\int \sqrt{x}\,(x+1)dx$

Solution

$$\int x^{1/2}(x+1)dx = \int \left(x^{3/2}+x^{1/2}\right)dx$$

$$\boxed{=\frac{2}{5}x^{5/2}+\frac{2}{3}x^{3/2}+C} = x+5x^{-1}+C$$

Exercise

Find each indefinite integral $\int (1+3t)t^2 dt$

Solution

$$\int \left(t^2+3t^3\right)dt = \boxed{\frac{1}{3}t^3+\frac{3}{4}t^4+C}$$

Exercise

Find each indefinite integral $\int \frac{x^2-5}{x^2}dx$

Solution

$$\int \frac{x^2-5}{x^2}dx = \int \left(1-\frac{5}{x^2}\right)dx$$

$$= \int \left(1-5x^{-2}\right)dx$$

$$\boxed{= x+\frac{5}{x}+C}$$

Exercise

Find each indefinite integral $\int(-40x+250)\,dx$

Solution

$$\int(-40x+250)\,dx = \underline{-20x^2+250x+C}$$

Exercise

Find each indefinite integral $\int\left(7-3x-3x^2\right)(2x+1)\,dx$

Solution

$$\begin{aligned}\int\left(7-3x-3x^2\right)(2x+1)\,dx &= \int\left(14x+7-6x^2-3x-6x^3-3x^2\right)dx \\ &= \int\left(-6x^3-9x^2+11x+7\right)dx \\ &= \underline{-\frac{3}{2}x^4-3x^3+\frac{11}{2}x^2+7x+C}\end{aligned}$$

Exercise

Evaluate the integral $\int xe^{2x}dx$

Solution

Let: $u = x \Rightarrow du = dx$

$$dv = e^{2x}dx \Rightarrow v = \int dv = \int e^{2x}dx = \frac{1}{2}e^{2x}$$

$$\int u\,dv = uv - \int v\,du$$

$$\begin{aligned}\int xe^{2x}dx &= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx \\ &= \frac{1}{2}xe^{2x} - \frac{1}{2}\frac{1}{2}e^{2x} + C \\ &= \underline{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}\end{aligned}$$

Exercise

Evaluate the integral $\int x\ln x\,dx$

Solution

Let: $u=\ln x \Rightarrow du=\frac{1}{x}dx$

$$dv = x\,dx \Rightarrow v=\int dv=\int x\,dx=\frac{1}{2}x^2$$

$$\int x\ln x\,dx=\frac{1}{2}x^2\ln x-\frac{1}{2}\int x^2\frac{dx}{x}$$

$$=\frac{1}{2}x^2\ln x-\frac{1}{2}\int x\,dx$$

$$\boxed{=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C}$$

Exercise

Evaluate the integral $\int x^2\sin x\,dx$

Solution

		$\int \sin x$
x^2	$(+)$	$-\cos x$
$2x$	$(-)$	$-\sin x$
2	$(+)$	$\cos x$

$$\int x^2\sin x\,dx=\boxed{-x^2\cos x-2x\sin x+2\cos x+C}$$

Exercise

Evaluate the integral $\int\left(x^2-2x+1\right)e^{2x}dx$

Solution

		$\int e^{2x}$
$+$	x^2-2x+1	$\frac{1}{2}e^{2x}$
$-$	$2x-2$	$\frac{1}{4}e^{2x}$
$+$	2	$\frac{1}{8}e^{2x}$

$$\int\left(x^2-2x+1\right)e^{2x}dx=\frac{1}{2}\left(x^2-2x+1\right)e^{2x}-\frac{1}{4}(2x-2)e^{2x}+\frac{1}{8}(2)e^{2x}+C$$

$$=\left(\frac{1}{2}x^2-x+\frac{1}{2}-\frac{1}{2}x+\frac{1}{2}+\frac{1}{4}\right)e^{2x}+C$$

$$\boxed{=\left(\frac{1}{2}x^2-\frac{3}{2}x+\frac{5}{5}\right)e^{2x}+C}$$

Exercise

Evaluate the integral $\int e^{2x}\cos 3x\,dx$

Solution

		$\int \cos 3x$
+	e^{2x}	$\frac{1}{3}\sin 3x$
-	$\frac{1}{2}e^{2x}$	$-\frac{1}{9}\cos 3x$
+	$\frac{1}{4}e^{2x}$	

$$\int e^{2x}\cos 3x\,dx = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}\int e^{2x}\cos 3x\,dx$$

$$\int e^{2x}\cos 3x\,dx + \frac{9}{4}\int e^{2x}\cos 3x\,dx = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x + C_1$$

$$\frac{13}{4}\int e^{2x}\cos 3x\,dx = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x + C_1$$

$$\frac{4}{13}\frac{13}{4}\int e^{2x}\cos 3x\,dx = \frac{4}{13}\frac{1}{2}e^{2x}\cos 3x + \frac{4}{13}\frac{3}{4}e^{2x}\sin 3x + \frac{4}{13}C_1$$

$$\int e^{2x}\cos 3x\,dx = \frac{e^{2x}}{13}\left(2\cos 3x + 3\sin 3x\right) + C$$

Exercise

Find the general solution of the differential equation $y' = 2t + 3$

Solution

$$\int dy = \int (2t+3)\,dt$$

$$y(t) = t^2 + 3t + C$$

Exercise

Find the general solution of the differential equation $y' = 3t^2 + 2t + 3$

Solution

$$\int dy = \int \left(3t^2 + 2t + 3\right)dt$$

$$y(t) = t^3 + t^2 + 3t + C$$

Exercise

Find the general solution of the differential equation $y' = \sin 2t + 2\cos 3t$

Solution

$$\int dy = \int (\sin 2t + 2\cos 3t)\,dt$$

$$\boxed{y(t) = -\frac{1}{2}\cos 2t + \frac{2}{3}\sin 3t + C}$$

Exercise

Find the general solution of the differential equation: $y' = x^3(3x^4+1)^2$

Solution

$$\int x^3\left(3x^4+1\right)^2\,dx$$

$$u = 3x^4 + 1 \Rightarrow du = 12x^3 dx$$

$$\int x^3(3x^4+1)^2\,dx = \int \frac{1}{12}u^2 du$$

$$= \frac{1}{12}\frac{(3x^4+1)^3}{3} + C$$

$$= \frac{1}{36}(3x^4+1)^3 + C$$

$$\boxed{y(x) = \frac{1}{36}\left(3x^4+1\right)^3 + C}$$

Exercise

Find the general solution of the differential equation: $y' = 5x\sqrt{x^2-1}$

Solution

$$\int 5x\sqrt{x^2-1}\,dx$$

$$u = x^2 - 1 \quad \Rightarrow du = 2xdx$$

$$\int 5x\left(x^2-1\right)^{1/2}\,dx$$

$$= 5\int u^{1/2}\ \frac{1}{2}du$$

Substitute for x and dx

$$= \frac{5}{2}\int u^{1/2}du$$

$$= \frac{5}{2}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{5}{3}u^{3/2} + C$$

$$\boxed{= \frac{5}{3}(x^2 - 1)^{3/2} + C}$$

Exercise

Find the general solution of the differential equation: $y' = x\sqrt{x^2 + 4}$

Solution

$$u = x^2 + 4 \quad \Rightarrow du = 2xdx$$

$$xdx = \frac{1}{2}du$$

$$\int \sqrt{x^2 + 4}\; xdx = \int u^{1/2}\; \frac{1}{2}du$$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{3}(x^2 + 4)^{3/2} + C$$

$$\boxed{y(x) = \frac{1}{3}\left(x^2 + 4\right)^{3/2} + C}$$

Exercise

Find the general solution of the differential equation: $y' = (2x + 1)e^{x^2 + x}$

Solution

$$\int dy = \int (2x + 1)e^{x^2 + x}\; dx \qquad\qquad u = x^2 + x \quad \Rightarrow \; du = (2x + 1)dx$$

$$\int dy = \int e^u\; du$$

$$y = e^u + C$$

$$\boxed{y(x) = e^{x^2 + x} + C}$$

Exercise

Find the general solution of the differential equation: $y' = \dfrac{1}{6x-5}$

Solution

$$\int dy = \int \frac{1}{6x-5}dx$$

$$\int dy = \frac{1}{6}\int \frac{1}{6x-5}d(6x-5)$$

$$\boxed{y(x) = \frac{1}{6}\ln|6x-5| + C}$$

Exercise

Find the general solution of the differential equation: $y' = \dfrac{x^2+2x+3}{x^3+3x^2+9x+1}$

Solution

$$\int dy = \int \frac{x^2+2x+3}{x^3+3x^2+9x+1}dx \qquad u = x^3+3x^2+9x+1 \quad du = 3\left(x^2+2x+3\right)dx$$

$$\int dy = \frac{1}{3}\int \frac{du}{u}$$

$$y(x) = \frac{1}{3}\ln|u| + C$$

$$\boxed{y(x) = \frac{1}{3}\ln\left|x^3+3x^2+9x+1\right| + C}$$

Exercise

Find the general solution of the differential equation: $y' = \dfrac{1}{x(\ln x)^2}$

Solution

$$\int dy = \int \frac{1}{x(\ln x)^2}dx \qquad u = \ln x \quad du = \frac{dx}{x}$$

$$\int dy = \int \frac{1}{u^2}du$$

$$y = -\frac{1}{u} + C$$

$$\boxed{y(x) = -\frac{1}{\ln x} + C}$$

Exercise

Evaluate the integrals $\int_{-2}^{2}\left(x^3-2x+3\right)dx$

Solution

$$\int_{-2}^{2}\left(x^3-2x+3\right)dx=\left[\frac{x^4}{4}-x^2+3x\right]_{-2}^{2}$$

$$=\left(\frac{(2)^4}{4}-(2)^2+3(2)\right)-\left(\frac{(-2)^4}{4}-(-2)^2+3(-2)\right)$$

$$=12$$

Exercise

Evaluate the integrals $\int_{0}^{1}\left(x^2+\sqrt{x}\right)dx$

Solution

$$\int_{0}^{1}\left(x^2+\sqrt{x}\right)dx=\left[\frac{x^3}{3}+\frac{2}{3}x^{3/2}\right]_{0}^{1}$$

$$=\left(\frac{(1)^3}{3}+\frac{2}{3}(1)^{3/2}\right)-0$$

$$=1$$

Exercise

Evaluate the integrals $\int_{0}^{\pi/3} 4\sec u\tan u\ du$

Solution

$$\int_{0}^{\pi/3} 4\sec u\tan u\ du=4\sec u\Big|_{0}^{\pi/3}$$

$$=4\left(\sec\frac{\pi}{3}-\sec 0\right)$$

$$=4(2-1)$$

$$=4$$

Exercise

Evaluate the integrals $\displaystyle\int_{\pi/4}^{3\pi/4} \csc\theta \cot\theta\, d\theta$

Solution

$$\int_{\pi/4}^{3\pi/4} \csc\theta \cot\theta\, d\theta = -\csc\theta \Big|_{\pi/4}^{3\pi/4}$$

$$= -\left(\csc\frac{3\pi}{4} - \csc\frac{\pi}{4}\right)$$

$$= -\left(\sqrt{2} - \sqrt{2}\right)$$

$$\underline{= 0}$$

Exercise

Evaluate the integrals $\displaystyle\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt$

Solution

$$\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt = \int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \pi t^{-2}\right) dt$$

$$= \left[4\tan t - \pi t^{-1}\right]_{-\pi/3}^{-\pi/4}$$

$$= \left(4\tan\left(-\frac{\pi}{4}\right) - \pi\left(-\frac{4}{\pi}\right)\right) - \left(4\tan\left(-\frac{\pi}{3}\right) - \pi\left(-\frac{3}{\pi}\right)\right)$$

$$= \left(4(-1) + 4\right) - \left(4\left(-\sqrt{3}\right) + 3\right)$$

$$= -\left(-4\sqrt{3} + 3\right)$$

$$\underline{= 4\sqrt{3} - 3}$$

Exercise

Evaluate the integrals $\displaystyle\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy$

Solution

$$\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy = \int_{-3}^{-1} \left(\frac{y^5}{y^3} - \frac{2y}{y^3}\right) dy$$

$$= \int_{-3}^{-1} \left(y^2 - 2y^{-2}\right) dy$$

$$=\left[\frac{1}{3}y^3+2y^{-1}\right]_{-3}^{-1}$$

$$=\left(\frac{1}{3}(-1)^3+\frac{2}{-1}\right)-\left(\frac{1}{3}(-3)^3+\frac{2}{-3}\right)$$

$$=\boxed{\frac{22}{3}}$$

Exercise

Evaluate the integrals $\displaystyle\int_1^8 \frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}}dx$

Solution

$$\int_1^8 \frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}}dx=\int_1^8 \frac{2x^{1/3}-x+2-x^{2/3}}{x^{1/3}}dx$$

$$=\int_1^8\left(2-x^{2/3}+2x^{-1/3}-x^{1/3}\right)dx$$

$$=\left[2x-\frac{3}{5}x^{5/3}+3x^{2/3}-\frac{3}{4}x^{4/3}\right]_1^8$$

$$=\left(2(8)-\frac{3}{5}(8)^{5/3}+3(8)^{2/3}-\frac{3}{4}(8)^{4/3}\right)-\left(2(1)-\frac{3}{5}(1)^{5/3}+3(1)^{2/3}-\frac{3}{4}(1)^{4/3}\right)$$

$$=\left(-\frac{16}{5}\right)-\left(\frac{73}{20}\right)$$

$$=\boxed{-\frac{137}{20}}$$

Exercise

Evaluate: $\displaystyle\int_0^1 (2t+3)^3dt$

Solution

$$\int_0^1 (2t+3)^3dt=\int_0^1 u^3\frac{1}{2}du \qquad u=2t+3\Rightarrow du=2dt\to\frac{du}{2}=dt$$

$$=\frac{1}{2}\int_0^1 u^3du$$

$$=\frac{1}{2}\frac{u^4}{4}\Big|_0^1$$

$$=\frac{1}{8}(2t+3)^4\Big|_0^1$$

$$= \frac{1}{8}\left[(2(1)+3)^4 - (2(0)+3)^4 \right]$$

$$= \frac{1}{8}\left[5^4 - 3^4 \right]$$

$$\underline{= 68}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{1} r\sqrt{1-r^2}\, dr$

Solution

Let $u = 1 - r^2 \quad \Rightarrow \quad du = -2rdr \quad \to \quad -\frac{1}{2}du = rdr$

$$\int_{-1}^{1} r\sqrt{1-r^2}\, dr = \int_{-1}^{1} u^{1/2}\left(-\tfrac{1}{2}du\right)$$

$$= -\frac{1}{2}\frac{2}{3}u^{3/2}\Big|_{-1}^{1}$$

$$= -\frac{1}{3}\left[\left(1-r^2\right)^{3/2}\right]_{-1}^{1}$$

$$= -\frac{1}{3}\left[\left(1-(1)^2\right)^{3/2} - \left(1-(-1)^2\right)^{3/2}\right]$$

$$= -\frac{1}{3}[0-0]$$

$$\underline{= 0}$$

Exercise

Find the general solution of $F'(x) = 4x + 2$, and find the particular solution that satisfies the initial condition $F(1) = 8$.

Solution

$$F(x) = \int (4x+2)dx$$

$$= 2x^2 + 2x + C$$

$$F(x) = 2(1)^2 + 2(1) + C = 8$$

$$2 + 2 + C = 8$$

$$C = 4$$

$$\Rightarrow \boxed{F(x) = 2x^2 + 2x + 4}$$

Exercise

Find the general solution of the differential equation: $y' = t\cos 3t$

Solution

$$u = t \to du = dt$$

$$dv = \cos 3t \to v = \frac{1}{3}\sin 3t$$

$$y = \frac{1}{3}t\sin 3t - \frac{1}{3}\int \sin 3t dt$$

$$= \frac{1}{3}t\sin 3t - \frac{1}{3}\frac{1}{3}\cos 3t + C$$

$$\boxed{y(t) = \frac{1}{3}t\sin 3t - \frac{1}{9}\cos 3t + C}$$

Exercise

A ball is thrown into the air from an initial height of 6 *m* with an initial velocity of 120 *m/s*. What will be the maximum height of the ball and at what time will this event occur?

Solution

$$\frac{dv}{dt} = -g \Rightarrow dv = -gdt$$

$$v(t) = -gt + C_1$$

$$v(t=0) = -g(0) + C_1 = 120$$

$$C_1 = 120$$

$$v(t) = -9.8t + 120$$

$$\frac{dx}{dt} = v \Rightarrow dx = vdt$$

$$x(t) = \int (-9.8t + 120)dt$$

$$= -4.9t^2 + 120t + C_2$$

$$x(0) = -4.9(0)^2 + 120(0) + C_2 = 6$$

$$C_2 = 6$$

$$x(t) = -4.9t^2 + 120t + 6$$

$$v(t) = -9.8t + 120 = 0 \quad \to t = \frac{120}{9.8} = 12.24\ sec$$

$$x(t=12.24) = -4.9(12.24)^2 + 120(12.24) + 6$$

$$\boxed{x(t) = 740.69\ m}$$

Exercise

Derive the position function if a ball is thrown upward with initial velocity of 32 *ft* per *second* from an initial height of 48 *ft*. When does the ball hit the ground? With what velocity does the ball hit the ground?

Solution

$$s(t) = -16t^2 + 32t + 48$$

$$s(0) = 48$$

$$s''(t) = -32$$

$$s'(t) = \int -32dt \quad s'(0) = 32$$

$$= -32t + C_1$$

$$s'(0) = -32(0) + C_1 = 32$$

$$\Rightarrow C_1 = 32$$

$$s'(t) = -32t + 32$$

$$s(t) = \int (-32t + 32)dt$$

$$= -32\frac{t^2}{2} + 32t + C_2$$

$$s(0) = -32\frac{0^2}{2} + 32(0) + C_2 = 48 \qquad \Rightarrow C_2 = 48$$

$$s(t) = -16t^2 + 32t + 48$$

$$s(t) = -16t^2 + 32t + 48 = 0$$

$$-t^2 + 2t + 3 = 0 \ \Rightarrow t = -1, \ t = 3$$

The ball hits the ground in **3** seconds

The velocity: $v(t) = s'(t) = -32t + 32$

$$v(t = 3) = -32(3) + 32 = \underline{-64\ ft / \sec^2}$$