

Section 3.2 – Angle and Orthogonality in Inner Product Spaces

Cosine Formula

If \vec{u} and \vec{v} are nonzero vectors that implies

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$\theta = \cos^{-1} \left(\frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|} \right)$$

$$-1 \leq \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \leq 1$$

Example

Let \mathbb{R}^4 have the Euclidean inner product. Find the cosine angle θ between the vectors $\vec{u} = (4, 3, 1, -2)$ and $\vec{v} = (-2, 1, 2, 3)$.

Solution

$$\begin{aligned} \|\vec{u}\| &= \sqrt{4^2 + 3^2 + 1^2 + (-2)^2} \\ &= \sqrt{30} \end{aligned}$$

$$\begin{aligned} \|\vec{v}\| &= \sqrt{4 + 1 + 4 + 9} \\ &= \sqrt{18} \\ &= 3\sqrt{2} \end{aligned}$$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= 4(-2) + 3(1) + 1(2) - 2(3) \\ &= -9 \end{aligned}$$

$$\begin{aligned} \cos \theta &= -\frac{9}{3\sqrt{30}\sqrt{2}} \\ &= -\frac{3}{\sqrt{60}} \\ &= -\frac{3}{2\sqrt{15}} \end{aligned}$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

Theorem – Cauchy-Schwarz Inequality

If \vec{u} and \vec{v} are vectors in a real inner product space V , then

$$\|\langle \vec{u}, \vec{v} \rangle\| \leq \|\vec{u}\| \|\vec{v}\|$$

Proof

If either \vec{u} or \vec{v} is equal to zero, then both sides equal to zero
Inequality holds.

Suppose that $\vec{u}, \vec{v} \neq \mathbf{0}$ and if \vec{w} any vector

$$\|\vec{w}\| = \vec{w} \cdot \vec{w} \geq 0$$

Let $\vec{w} = \vec{u} - t\vec{v}$, then:

$$\begin{aligned} 0 &\leq \vec{w} \cdot \vec{w} \\ &= (\vec{u} - t\vec{v}) \cdot (\vec{u} - t\vec{v}) \\ &= \vec{u} \cdot \vec{u} - t(\vec{u} \cdot \vec{v}) - t(\vec{v} \cdot \vec{u}) + t^2(\vec{v} \cdot \vec{v}) \\ &= \vec{u} \cdot \vec{u} - 2t(\vec{u} \cdot \vec{v}) + t^2(\vec{v} \cdot \vec{v}) \quad \text{Let } t = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \\ &= \vec{u} \cdot \vec{u} - 2\left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right)(\vec{u} \cdot \vec{v}) + \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right)^2(\vec{v} \cdot \vec{v}) \\ &= \vec{u} \cdot \vec{u} - 2\frac{(\vec{u} \cdot \vec{v})^2}{\vec{v} \cdot \vec{v}} + \frac{(\vec{u} \cdot \vec{v})^2}{\vec{v} \cdot \vec{v}} \\ &= \vec{u} \cdot \vec{u} - \frac{(\vec{u} \cdot \vec{v})^2}{\vec{v} \cdot \vec{v}} \\ &= \frac{(\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}) - (\vec{u} \cdot \vec{v})^2}{\vec{v} \cdot \vec{v}} \quad \text{Since } \vec{v} \cdot \vec{v} > 0 \\ &\leq (\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}) - (\vec{u} \cdot \vec{v})^2 \\ (\vec{u} \cdot \vec{v})^2 &\leq (\vec{u} \cdot \vec{u})(\vec{v} \cdot \vec{v}) \\ \|\langle \vec{u}, \vec{v} \rangle\| &\leq \|\vec{u}\| \|\vec{v}\| \end{aligned}$$

The following two alternative forms of the Cauchy-Schwarz inequality are useful to know:

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle^2 &\leq \langle \vec{u}, \vec{u} \rangle \langle \vec{v}, \vec{v} \rangle \\ \langle \vec{u}, \vec{v} \rangle^2 &\leq \|\vec{u}\|^2 \|\vec{v}\|^2 \end{aligned}$$

Theorem

If \vec{u} , \vec{v} and \vec{w} are vectors in a real inner product space V , and if k is any scalar, then

$$a) \quad \|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\| \quad (\text{Triangle inequality for vectors})$$

$$b) \quad d(\vec{u}, \vec{v}) \leq d(\vec{u}, \vec{w}) + d(\vec{w}, \vec{v}) \quad (\text{Triangle inequality for distances})$$

Proof (a)

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= \langle \vec{u} + \vec{v}, \vec{u} + \vec{v} \rangle \\ &= \langle \vec{u}, \vec{u} \rangle + 2\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{v} \rangle \\ &\leq \langle \vec{u}, \vec{u} \rangle + 2|\langle \vec{u}, \vec{v} \rangle| + \langle \vec{v}, \vec{v} \rangle \\ &\leq \langle \vec{u}, \vec{u} \rangle + 2\|\vec{u}\| \|\vec{v}\| + \langle \vec{v}, \vec{v} \rangle \\ &= \|\vec{u}\|^2 + 2\|\vec{u}\| \|\vec{v}\| + \|\vec{v}\|^2 \\ &= (\|\vec{u}\| + \|\vec{v}\|)^2 \end{aligned}$$

$$\|\vec{u} + \vec{v}\|^2 \leq (\|\vec{u}\| + \|\vec{v}\|)^2$$

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

Definition

Two vectors \vec{u} and \vec{v} in an inner product space are called orthogonal if $\langle \vec{u}, \vec{v} \rangle = 0$

Example

The vectors $\vec{u} = (1, 1)$ and $\vec{v} = (1, -1)$ are orthogonal with respect to the Euclidean inner product on \mathbb{R}^2 , since

$$\begin{aligned} \vec{u} \cdot \vec{v} &= 1(1) + 1(-1) \\ &= 0 \end{aligned}$$

They are not orthogonal with the respect to the weighted Euclidean inner product

$\langle \vec{u}, \vec{v} \rangle = 3u_1v_1 + 2u_2v_2$, since

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= 3(1)(1) + 2(1)(-1) \\ &= 1 \neq 0 \end{aligned}$$

Example

$U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ are orthogonal, since

$$\begin{aligned} U \cdot V &= 1(0) + 0(2) + 1(0) + 1(0) \\ &= 0 \end{aligned}$$

Definition

If W is a subspace of an inner product space V , then the set of all vectors are orthogonal to every vector in W is called the **orthogonal complement** of W and is denoted by the symbol W^\perp

Theorem

If W is a subspace of an inner product space V , then:

- a) W^\perp is a subspace of V .
- b) $W \cap W^\perp = \{0\}$

Proof

- a) Let set W^\perp contains at least the zero vector, since $\langle \vec{0}, \vec{w} \rangle = 0$ for every vector \vec{w} in W . We need to show that W^\perp is closed under addition and scalar multiplication.

Suppose that \vec{u} and \vec{v} are vectors in W^\perp , so every vector \vec{w} in W we have $\langle \vec{u}, \vec{w} \rangle = 0$ and $\langle \vec{v}, \vec{w} \rangle = 0$

$$\begin{aligned} \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Closed under addition

$$\begin{aligned} \langle k\vec{u}, \vec{w} \rangle &= k \langle \vec{u}, \vec{w} \rangle \\ &= k(0) \\ &= 0 \end{aligned}$$

Closed under scalar multiplication

Which proves that $\vec{u} + \vec{w}$ and $k\vec{u}$ are in W^\perp

- b) If \vec{v} is any vector in both W and W^\perp , then \vec{v} is orthogonal to itself; that is, $\langle \vec{v}, \vec{v} \rangle = 0$. It follows from the positivity axiom for inner products that $\vec{v} = \vec{0}$

Theorem

If W is a subspace of a finite-dimensional inner product space V , then the orthogonal complement of W^\perp is W ; that is

$$(W^\perp)^\perp = W$$

Example

Let W be the subspace of \mathbb{R}^6 spanned by the vectors

$$\begin{aligned}\bar{w}_1 &= (1, 3, -2, 0, 2, 0), & \bar{w}_2 &= (2, 6, -5, -2, 4, -3) \\ \bar{w}_3 &= (0, 0, 5, 10, 0, 15), & \bar{w}_4 &= (2, 6, 0, 8, 4, 18)\end{aligned}$$

Find a basis for the orthogonal complement of W .

Solution

The Space W is the same as the row space of the matrix

$$A = \begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix} \quad \begin{array}{l} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{array}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 0 & 0 & 4 & 8 & 0 & 18 \end{pmatrix} \quad \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 5R_2 \\ R_4 + 4R_2 \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 \\ 0 & 0 & -1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix} \quad \begin{array}{l} -R_2 \\ \\ \frac{1}{6}R_3 \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} R_1 - 6R_3 \\ R_2 - 3R_3 \\ \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution

$$\begin{aligned}\begin{pmatrix} x_1, x_2, x_3, x_4, x_5, x_6 \end{pmatrix} &= \begin{pmatrix} -3x_2 - 4x_4 - 2x_5, x_2, -2x_4, x_4, x_5, 0 \end{pmatrix} \\ &= x_2 \begin{pmatrix} -3, 1, 0, 0, 0, 0 \end{pmatrix} + x_4 \begin{pmatrix} -4, 0, -2, 1, 0, 0 \end{pmatrix} + x_5 \begin{pmatrix} -2, 0, 0, 0, 1, 0 \end{pmatrix} \\ \vec{v}_1 &= \begin{pmatrix} -3, 1, 0, 0, 0, 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -4, 0, -2, 1, 0, 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -2, 0, 0, 0, 1, 0 \end{pmatrix}\end{aligned}$$

Definition

A collection of vectors in \mathbb{R}^n (or inner space) is called orthogonal if any 2 are perpendicular.

$$\vec{v}_i \cdot \vec{v}_j = \vec{v}_i^T \vec{v}_j = \begin{cases} 0 & \text{for } i \neq j \quad (\text{orthogonal vectors}) \\ 1 & \text{for } i = j \quad (\text{unit vectors}) \end{cases}$$

Theorem

If $\vec{v}_1, \dots, \vec{v}_m$ are nonzero orthogonal vectors, then they are linearly independent.

Definition

A vector \vec{v} is called normal if $\|\vec{v}\| = 1$

A collection of vectors $\vec{v}_1, \dots, \vec{v}_m$ is called orthonormal if they are orthogonal and each $\|\vec{v}_i\| = 1$.

An orthonormal basis is a basis made up of orthonormal vectors.

Example

Q rotates every vector in the plane through the angle θ .

$$Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$Q^{-1} = \frac{1}{\cos^2 \theta + \sin^2 \theta} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \quad \cos^2 \theta + \sin^2 \theta = 1$$

$$= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$= \underline{Q^T}$$

The dot product $(\cos \theta \sin \theta - \sin \theta \cos \theta = 0)$, the columns are orthogonal.

They are unit vectors because $\cos^2 \theta + \sin^2 \theta = 1$. Those columns give an orthonormal basis for the plane \mathbb{R}^2 .

We have: $QQ^T = I = Q^T Q$ (This type is called *rotation*)

Exercises Section 3.2 – Angle and Orthogonality in Inner Product Spaces

1. Which of the following form orthonormal sets?

- a) $(1, 0), (0, 2)$ in \mathbb{R}^2
- b) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in \mathbb{R}^2
- c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in \mathbb{R}^2
- d) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ in \mathbb{R}^3
- e) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ in \mathbb{R}^3
- f) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ in \mathbb{R}^3

2. Find the cosine of the angle between \vec{u} and \vec{v} .

- a) $\vec{u} = (1, -3), \vec{v} = (2, 4)$
- b) $\vec{u} = (-1, 0), \vec{v} = (3, 8)$
- c) $\vec{u} = (-1, 5, 2), \vec{v} = (2, 4, -9)$
- d) $\vec{u} = (4, 1, 8), \vec{v} = (1, 0, -3)$
- e) $\vec{u} = (1, 0, 1, 0), \vec{v} = (-3, -3, -3, -3)$
- f) $\vec{u} = (2, 1, 7, -1), \vec{v} = (4, 0, 0, 0)$
- g) $\vec{u} = (1, 3, -5, 4), \vec{v} = (2, -4, 4, 1)$
- h) $\vec{u} = (1, 2, 3, 4), \vec{v} = (-1, -2, -3, -4)$

3. Find the cosine of the angle between A and B .

- a) $A = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix}, B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$
- b) $A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}, B = \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$
- c) $A = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$
- d) $A = \begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix}, B = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$

4. Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

- a) $\vec{u} = (-1, 3, 2), \vec{v} = (4, 2, -1)$
- b) $\vec{u} = (a, b), \vec{v} = (-b, a)$
- c) $\vec{u} = (-2, -2, -2), \vec{v} = (1, 1, 1)$
- d) $\vec{u} = (-4, 6, -10, 1), \vec{v} = (2, 1, -2, 9)$
- e) $\vec{u} = (-4, 6, -10, 1), \vec{v} = (2, 1, -2, 9)$

5. Do there exist scalars k and l such that the vectors

$\vec{u} = (2, k, 6), \vec{v} = (l, 5, 3),$ and $\vec{w} = (1, 2, 3)$ are mutually orthogonal with respect to the Euclidean inner product?

6. Let \mathbb{R}^3 have the Euclidean inner product. For which values of k are \vec{u} and \vec{v} orthogonal?
- a) $\vec{u} = (2, 1, 3)$, $\vec{v} = (1, 7, k)$ b) $\vec{u} = (k, k, 1)$, $\vec{v} = (k, 5, 6)$
7. Let V be an inner product space. Show that if \vec{u} and \vec{v} are orthogonal unit vectors in V , then $\|\vec{u} - \vec{v}\| = \sqrt{2}$
8. Let \mathcal{S} be a subspace of \mathbb{R}^n . Explain what $(\mathcal{S}^\perp)^\perp = \mathcal{S}$ means and why it is true.
9. The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ – (**note** that all six edges have length $\sqrt{2}$, so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ to the vertices?
10. Determine if the given vectors are orthogonal.
- $\vec{x}_1 = (1, 0, 1, 0)$, $\vec{x}_2 = (0, 1, 0, 1)$, $\vec{x}_3 = (1, 0, -1, 0)$, $\vec{x}_4 = (1, 1, -1, -1)$
11. Which of the following sets of vectors are orthogonal with respect to the Euclidean inner
- a) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ $\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$
- b) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$ $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$
12. Consider vectors $\vec{u} = (2, 3, 5)$ $\vec{v} = (1, -4, 3)$ in \mathbb{R}^3
- a) $\langle \vec{u}, \vec{v} \rangle$ b) $\|\vec{u}\|$ c) $\|\vec{v}\|$ d) Cosine between \vec{u} and \vec{v}
13. Consider vectors $\vec{u} = (1, 1, 1)$ $\vec{v} = (1, 2, -3)$ in \mathbb{R}^3
- a) $\langle \vec{u}, \vec{v} \rangle$ b) $\|\vec{u}\|$ c) $\|\vec{v}\|$ d) Cosine θ between \vec{u} and \vec{v}
14. Consider vectors $\vec{u} = (1, 2, 5)$ $\vec{v} = (2, -3, 5)$ $\vec{w} = (4, 2, -3)$ in \mathbb{R}^3
- a) $\langle \vec{u}, \vec{v} \rangle$ d) $\|\vec{u}\|$ g) Cosine α between \vec{u} and \vec{v}
- b) $\langle \vec{u}, \vec{w} \rangle$ e) $\|\vec{v}\|$ h) Cosine β between \vec{u} and \vec{w}
- c) $\langle \vec{v}, \vec{w} \rangle$ f) $\|\vec{w}\|$ i) Cosine θ between \vec{v} and \vec{w}
- j) $(\vec{u} + \vec{v}) \cdot \vec{w}$

15. Consider polynomial $f(t) = 3t - 5$; $g(t) = t^2$ in $\mathbb{P}(t)$
- a) $\langle f, g \rangle$ b) $\|f\|$ c) $\|g\|$ d) Cosine between f and g
16. Consider polynomial $f(t) = t + 2$; $g(t) = 3t - 2$; $h(t) = t^2 - 2t - 3$ in $\mathbb{P}(t)$
- a) $\langle f, g \rangle$ d) $\|f\|$ g) Cosine α between f and g
b) $\langle f, h \rangle$ e) $\|g\|$ h) Cosine β between f and h
c) $\langle g, h \rangle$ f) $\|h\|$ i) Cosine θ between g and h
17. Suppose $\langle \vec{u}, \vec{v} \rangle = 3 + 2i$ in a complex inner product space V . Find:
- a) $\langle (2 - 4i)\vec{u}, \vec{v} \rangle$ b) $\langle \vec{u}, (4 + 3i)\vec{v} \rangle$ c) $\langle (3 - 6i)\vec{u}, (5 - 2i)\vec{v} \rangle$ d) $\|\vec{u}, \vec{v}\|$
18. Find the Fourier coefficient c and the projection $c\vec{v}$ of $\vec{u} = (3 + 4i, 2 - 3i)$ along $\vec{v} = (5 + i, 2i)$ in \mathbb{C}^2
19. Suppose $\vec{v} = (1, 3, 5, 7)$. Find the projection of \vec{v} onto W or find $\vec{w} \in W$ that minimizes $\|\vec{v} - \vec{w}\|$, where W is the subspace of \mathbb{R}^4 spanned by:
- a) $\vec{u}_1 = (1, 1, 1, 1)$ and $\vec{u}_2 = (1, -3, 4, -2)$
b) $\vec{v}_1 = (1, 1, 1, 1)$ and $\vec{v}_2 = (1, 2, 3, 2)$
20. Suppose $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ is an orthogonal set of vectors. Prove that (Pythagoras)
- $$\|\vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n\|^2 = \|\vec{u}_1\|^2 + \|\vec{u}_2\|^2 + \dots + \|\vec{u}_n\|^2$$
21. Suppose A is an orthogonal matrix. Show that $\langle \vec{u}A, \vec{v}A \rangle = \langle \vec{u}, \vec{v} \rangle$ for any $\vec{u}, \vec{v} \in V$
22. Suppose A is an orthogonal matrix. Show that $\|\vec{u}A\| = \|\vec{u}\|$ for every $\vec{u} \in V$
23. Let V be an inner product space over \mathbb{R} or \mathbb{C} . Show that
- $$\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$
- If and only if
- $$\|s\vec{u} + t\vec{v}\| = s\|\vec{u}\| + t\|\vec{v}\| \quad \text{for all } s, t \geq 0$$
24. Let V be an inner product vector space over \mathbb{R} .
- a) If e_1, e_2, e_3 are three vectors in V with pairwise product negative, that is,
- $$\langle e_i, e_j \rangle < 0, \quad i, j = 1, 2, 3, \quad i \neq j$$

Show that e_1, e_2, e_3 are linearly independent.

- b) Is it possible for three vectors on the xy -plane to have pairwise negative products?
- c) Does part (a) remain valid when the word “negative: is replaced with positive?
- d) Suppose \vec{u}, \vec{v} , and \vec{w} are three unit vectors in the xy -plane. What are the maximum and minimum values that

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle$$

Can attain? And when?