Solution   *Section* 3.1 – Quadratic Functions

Exercise

Give the vertex, axis, domain, and range. Then, graph the function $f(x) = x^2 + 6x + 5$

Solution

Vertex: $x = -\frac{b}{2a}$

$= -\frac{6}{2(1)}$

$= -3$

$y = f(-3) = (-3)^2 + 6(-3) + 5$

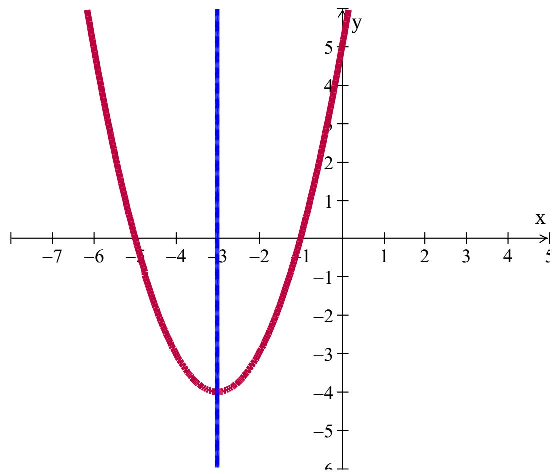
$= -4$

Vertex point: $(-3, -4)$

Axis of symmetry: $x = -3$

Domain: $(-\infty, \infty)$

Range: $[-4, \infty)$



Exercise

Give the vertex, axis, domain, and range. Then, graph the function $f(x) = -x^2 - 6x - 5$

Solution

Vertex: $x = -\frac{b}{2a}$

$= -\frac{-6}{2(-1)}$

$= -3$

$y = f(-3) = -(-3)^2 - 6(-3) - 5$

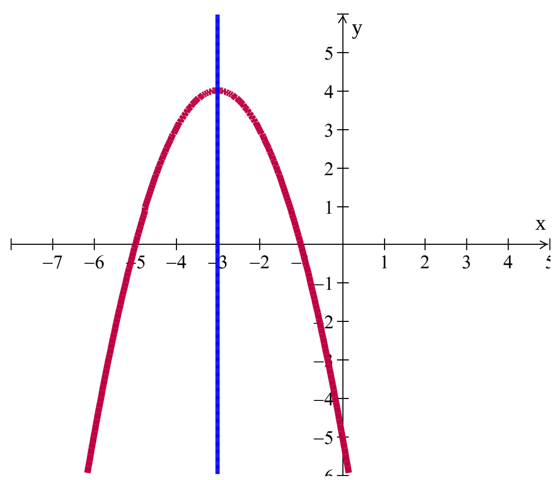
$= 4$

Vertex point: $(-3,\ 4)$

Axis of symmetry: $x = -3$

Domain: $(-\infty, \infty)$

Range: $(-\infty, 4]$



Exercise

Graph the quadratic. Give the vertex, axis of symmetry, domain, and range:

$f(x) = x^2 - 4x + 2$

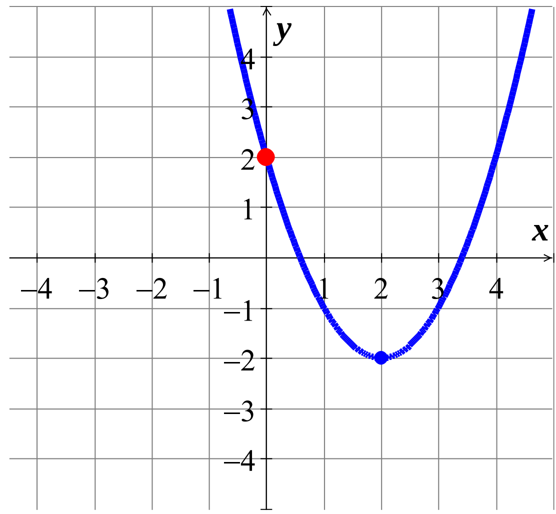
Solution

Vertex point:

$$x = -\frac{b}{2a} = -\frac{-4}{2(1)} = 2$$

$$f(2) = 2^2 - 4(2) + 2 = -2$$

The ***vertex point***: $(2,\ -2)$



Axis of symmetry is: $x = 2$

Domain: $(-\infty,\ \infty)$

Range: $[-2,\ \infty)$ ***(Since function has a minimum)***

To graph: find another point:

$x = 0 \quad \Rightarrow \quad y = f(0) = 2$

Exercise

Graph the quadratic. Give the vertex, axis of symmetry, domain, and range:

$f(x) = -2x^2 + 16x - 26$

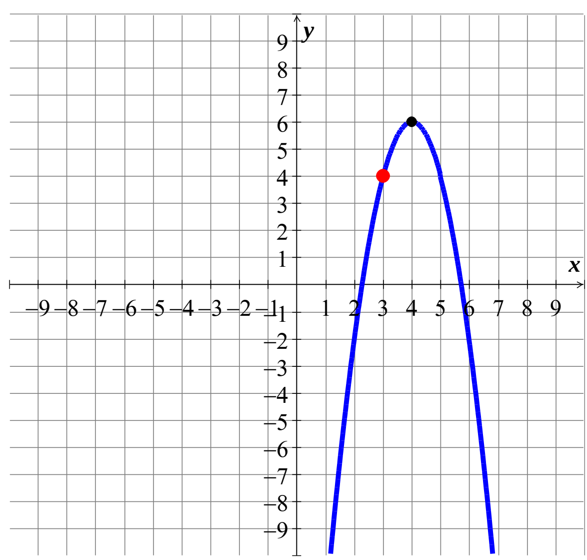
Solution

Vertex point:

$$x = -\frac{b}{2a} = -\frac{16}{2(-2)} = 4$$

$$f(4) = -2\left(4^2\right) + 16(4) - 26 = 6$$

The ***vertex point***: $(4,\ 6)$



Axis of symmetry is: $x = 4$

Domain: $(-\infty,\ \infty)$

Range: $(-\infty,\ 6]$ ***(Since function has a maximum)***

To graph: find another point:

$x = 3 \quad \Rightarrow \quad y = f(3) = 4$

Exercise

You have 600 *ft* of fencing to enclose a rectangular plot that borders on a river. If you do not fence the side along the river, find the length and width of the plot that will maximize the area. What is the largest area that can be enclosed?

Solution

$$P = l + 2w$$

$$600 = l + 2w \qquad \to l = 600 - 2w$$

$$\begin{aligned} A &= lw \\ &= (600 - 2w)w \\ &= 600w - 2w^2 \\ &= -2w^2 + 600w \end{aligned}$$

Vertex: $w = -\dfrac{600}{2(-2)} = 150$

$$\to l = 600 - 2w = 300 \; (x-8)(x-22) = 0$$

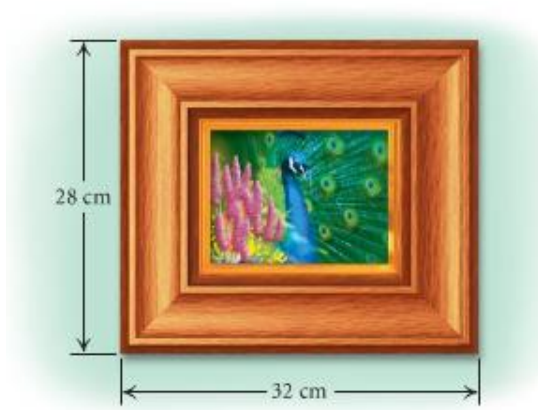
$$\begin{aligned} A &= lw = (300)(150) \\ &\boxed{= 45000 \ ft^2} \end{aligned}$$

Exercise

A picture frame measures 28 cm by 32 cm and is of uniform width. What is the width of the frame if 192 cm^2 of the picture shows?

Solution

Area of the picture = $(32-2x)(28-2x) = 192$



$$896 - 64x - 56x + 4x^2 = 192$$

$$896 - 120x + 4x^2 - 192 = 0$$

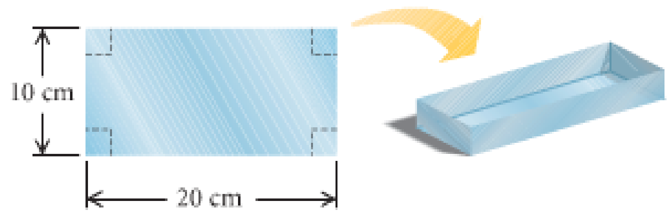
$$4x^2 - 120x + 704 = 0$$

$$x^2 - 30x + 176 = 0$$

$$\begin{cases} x - 8 = 0 \to \boxed{x = 8} \\ x - 22 = 0 \to \boxed{x = 22} \end{cases}$$

Exercise

An open box is made from a 10-cm by 20-cm of tin by cutting a square from each corner and folding up the edges. The area of the resulting base is 96 cm^2. What is the length of the sides of the squares?



Solution

Area of the base $=(20-2x)(10-2x)=96$

$$200-40x-20x+4x^2=96$$

$$4x^2-60x+200-96=0$$

$$4x^2-60x+104=0 \qquad \textit{Solve for x}$$

$$\boxed{x=2,\ \cancel{13}}$$

The length of the sides of the squares is 3-cm

Exercise

A fourth-grade class decides to enclose a rectangular garden, using the side of the school as one side of the rectangle. What is the maximum area that the class can enclose with 32 ft. of fence? What should the dimensions of the garden be in order to yield this area?

Solution

Perimeter: $P=l+2w=32$

$$l=32-2w$$

Area: $A=lw$

$$A=(32-2w)w$$

$$=32w-2w^2$$

$$=-2w^2+32w$$

Vertex: $w=-\frac{32}{2(-2)}=8$

$\rightarrow l=32-2(8)=16$

$$A=lw=(16)(8)$$

$$=128\ ft^2$$

Exercise

A rancher needs to enclose two adjacent rectangular corrals, one for cattle and one for sheep. If a river forms one side of the corrals and 240 yd of fencing is available, what is the largest total area that can be enclosed?

Solution

Perimeter: $P = l + 3w = 240$

$$l = 240 - 3w$$

Area: $A = lw$

$$A = (240 - 3w)w$$

$$= 240w - 3w^2$$

$$= -3w^2 + 240w$$

Vertex: $w = -\frac{240}{2(-3)} = 40$

$\rightarrow l = 240 - 3(40) = 120$

$$A = lw = (120)(40)$$

$$= 4800\ yd^2$$

Exercise

A Norman window is a rectangle with a semicircle on top. Sky Blue Windows is designing a Norman window that will require 24 *ft.* of trim on the outer edges. What dimensions will allow the maximum amount of light to enter a house?

Solution

Perimeter of the semi-circle $= \frac{1}{2}(2\pi x)$

Perimeter of the rectangle $= 2x + 2y$

Total perimeter: $\pi x + 2x + 2y = 24$

$$2y = 24 - \pi x - 2x$$

$$y = 12 - \frac{\pi}{2}x - x$$

A Norman window

Area $= \frac{1}{2}\left(\pi x^2\right) + (2x)y$

$$= \frac{\pi}{2}x^2 + 2x\left(12 - \frac{\pi}{2}x - x\right)$$

$$= \frac{\pi}{2}x^2 + 24x - \pi x^2 - 2x^2$$

$$= 24x - \left(\frac{\pi}{2} + 2\right)x^2$$

$$= -\left(\frac{\pi}{2}+2\right)x^2 + 24x$$

$$x = -\frac{b}{2a} = -\frac{24}{2\left(-\frac{\pi}{2}-2\right)} = -\frac{24}{-2\left(\frac{\pi+4}{2}\right)} = \boxed{\frac{24}{\pi+4}}$$

$$y = 12 - \frac{\pi}{2}\frac{24}{\pi+4} - \frac{24}{\pi+4}$$

$$= \frac{24\pi + 96 - 24\pi - 48}{2(\pi+4)}$$

$$\boxed{= \frac{24}{\pi+4}}$$

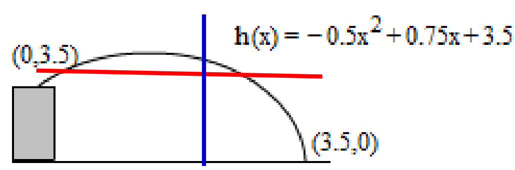
Exercise

A frog leaps from a stump 3.5 ft. high and lands 3.5 ft. from the base of the stump.
It is determined that the height of the frog as a function of its distance, *x*, from the base of the stump is given by the function $h(x) = -0.5x^2 + 0.75x + 3.5$ where *h* is in feet.

a) How high is the frog when its horizontal distance from the base of the stump is 2 ft.?
b) At what two distances from the base of the stump after is jumped was the frog 3.6 ft. above the ground?
c) At what distance from the base did the frog reach its highest point?
d) What was the maximum height reached by the frog?

Solution

a) At $x = 2\,ft.$ *Find* $h(x = 2)$



$h = 3.6$ $h(2) = -0.5\left(2^2\right) + 0.75(2) + 3.5 \boxed{= 3\,ft}$

$x = 2$

b) $h(x) = -0.5x^2 + 0.75x + 3.5 = 3.6$

$-0.5x^2 + 0.75x + 3.5 - 3.6 = 0$

$-0.5x^2 + 0.75x - .1 = 0$

Solve for *x*: $\boxed{x = 0.1,\ \ 1.4\,ft}$

c) The distance from the base for the frog to reach the highest point is

$$x = -\frac{b}{2a} = -\frac{.75}{2(-.5)} \boxed{= .75\,ft}$$

d) Maximum height:

$$h(x = .75) = -0.5(.75)^2 + 0.75(.75) + 3.5 \boxed{= 3.78\,ft}$$

Exercise

For the graph of the function $f(x) = x^2 + 6x + 3$

a) Find the vertex point

b) Find the line of symmetry

c) State whether there is a maximum or minimum value *and* find that value

d) Find the zeros of $f(x)$

e) Find the y-intercept

f) Find the range and the domain of the function.

g) Graph the function and label, show part *a thru d* on the plot below:

h) On what intervals is the function increasing? Decreasing?

Solution

a) $x = -\frac{6}{2(1)} = -3$

$y = f(-3) = (-3)^2 + 6(-3) + 3 = -6$ ***Vertex point*** $(-3, -6)$

b) Line of symmetry: $x = -3$

c) Minimum point, value $(-3, -6)$

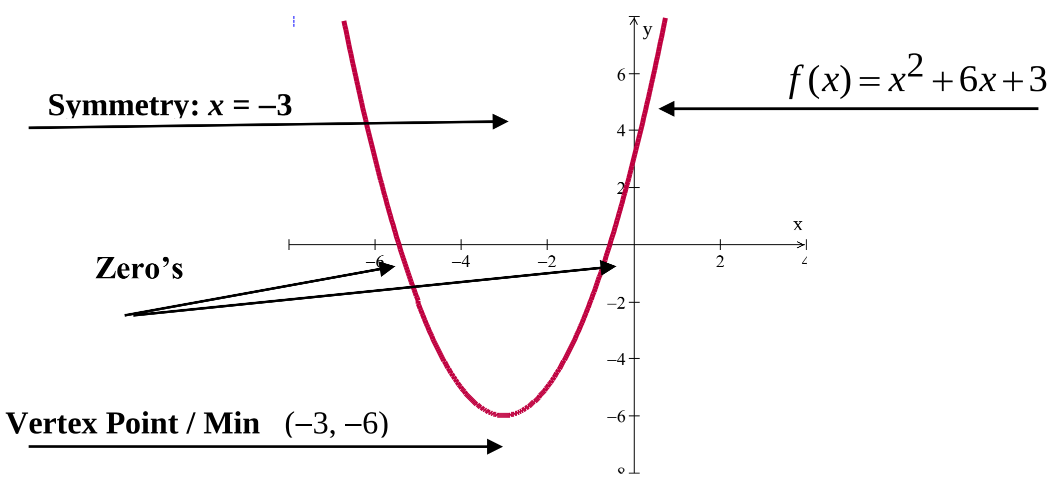
d) $x = \frac{-6 \pm \sqrt{6^2 - 4(1)(3)}}{2(1)} = \frac{-6 \pm \sqrt{36 - 12}}{2} = \frac{-6 \pm \sqrt{24}}{2} = \frac{-6 \pm 2\sqrt{6}}{2} = -3 \pm \sqrt{6}$

$$x = \begin{cases} -3 + \sqrt{6} = -0.5 \\ -3 - \sqrt{6} = -5.45 \end{cases}$$

e) y-intercept $y = 3$

f) *Range*: $[-6, \infty)$ *Domain*: $(-\infty, \infty)$

g)



h) *Decreasing*: $(-\infty, -3)$ *Increasing*: $(-3, \infty)$

Solution   Section 3.2 – Polynomial Functions

Exercise

$f(x)$ Determine the end behavior of the graph of the polynomial function $f(x)=5x^3+7x^2-x+9$

Solution

Leading term: $5x^3$ with 3rd degree (n is odd)

$x\to-\infty \ \Rightarrow\ f(x)\to-\infty$ $\quad f(x)$ falls left

$x\to\infty \ \Rightarrow\ f(x)\to\infty$ $\quad f(x)$ rises right

$11x^3$

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=11x^3-6x^2+x+3$

Solution

Leading term: with 3rd degree (n is odd)

$x\to-\infty \ \Rightarrow\ f(x)\to-\infty$ $\quad f(x)$ falls left

$x\to\infty \ \Rightarrow\ f(x)\to\infty$ $\quad$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=-11x^3-6x^2+x+3$

Solution

Leading term: $-11x^3$ with 3rd degree (n is odd)

$x\to-\infty \ \Rightarrow\ f(x)\to\infty$ $\quad f(x)$ rises left

$x\to\infty \ \Rightarrow\ f(x)\to-\infty$ $\quad f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=5x^4+7x^2-x+9$

Solution

Leading term: $5x^4$ with 4rd degree (n is even)

$x\to-\infty \ \Rightarrow\ f(x)\to\infty$ $\quad f(x)$ rises left

$x\to\infty \ \Rightarrow\ f(x)\to\infty$ $\quad f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=11x^4-6x^2+x+3$

Solution

Leading term: $11x^4$ with 4^{rd} degree (n is even)

$x\to-\infty \ \Rightarrow\ f(x)\to\infty$ $\quad f(x)$ rises left

$x\to\infty \ \Rightarrow\ f(x)\to\infty$ $\quad f(x)$ rises right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=-5x^4+7x^2-x+9$

Solution

Leading term: $-5x^4$ with 4^{rd} degree (n is even)

$x\to-\infty \ \Rightarrow\ f(x)\to-\infty$ $\quad f(x)$ falls left

$x\to\infty \ \Rightarrow\ f(x)\to-\infty$ $\quad f(x)$ falls right

Exercise

Determine the end behavior of the graph of the polynomial function $f(x)=-11x^4-6x^2+x+3$

Solution

Leading term: $-11x^4$ with 4^{rd} degree (n is even)

$x\to-\infty \ \Rightarrow\ f(x)\to-\infty$ $\quad f(x)$ falls left

$x\to\infty \ \Rightarrow\ f(x)\to-\infty$ $\quad f(x)$ falls right

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x)=x^3-x-1;\quad between\ 1\ and\ 2$

Solution

$$f(1)=(1)^3-(1)-1=-1$$

$$f(2)=(2)^3-(2)-1=5$$

Since $f(1)\ and\ f(2)$ have opposite signs; therefore, the polynomial has a real zero between 1 and 2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^3 - 4x^2 + 2;\quad between\ \ 0\ and\ 1$

Solution

$$f(0) = (0)^3 - 4(0)^2 + 2 \underline{= 2}|$$

$$f(1) = (1)^3 - 4(1)^2 + 2 \underline{= -1}|$$

Since $f(0)$ *and* $f(1)$ have opposite signs; therefore, the polynomial has a real zero between 0 and 1.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = 2x^4 - 4x^2 + 1;\quad between\ \ -1\ and\ 0$

Solution

$$f(-1) = 2(-1)^4 - 4(-1)^2 + 1 \underline{= -1}|$$

$$f(0) = 2(0)^4 - 4(0)^2 + 1 \underline{= 1}|$$

Since $f(0)$ *and* $f(-1)$ have opposite signs; therefore, the polynomial has a real zero between -1 and 0.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^4 + 6x^3 - 18x^2;\quad between\ \ 2\ and\ 3$

Solution

$$f(2) = (2)^4 + 6(2)^3 - 18(2)^2 \ \underline{= -8}|$$

$$f(3) = (3)^4 + 6(3)^3 - 18(3)^2 \ \underline{= 81}|$$

Since $f(2)$ *and* $f(3)$ have opposite signs; therefore, the polynomial has a real zero between 2 and 3.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^3 + x^2 - 2x + 1;\quad between\ \ -3\ and\ -2$

Solution

$f(-3) = (-3)^3 + (-3)^2 - 2(-3) + 1 \underline{= -11|}$

$f(-2) = (-2)^3 + (-2)^2 - 2(-2) + 1 \underline{= 1|}$

Since $f(-3)$ *and* $f(-2)$ have opposite signs; therefore, the polynomial has a real zero between –2 and –3.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = x^5 - x^3 - 1;$ *between* 1 *and* 2

Solution

$f(1) = (1)^5 - (1)^3 - 1 \underline{= -1|}$

$f(2) = (2)^5 - (2)^3 - 1 \underline{= 23|}$

Since $f(1)$ *and* $f(2)$ have opposite signs; therefore, the polynomial has a real zero between 1 and 2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = 3x^3 - 10x + 9;$ *between* -3 *and* -2

Solution

$f(-3) = 3(-3)^3 - 10(-3) + 9 \underline{= -42|}$

$f(-2) = 3(-2)^3 - 10(-2) + 9 \underline{= 5|}$

Since $f(-3)$ *and* $f(-2)$ have opposite signs; therefore, the polynomial has a real zero between –3 and –2.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x) = 3x^3 - 8x^2 + x + 2;$ *between* 2 *and* 3

Solution

$f(2) = 3(2)^3 - 8(2)^2 + (2) + 2 \underline{= -4|}$

$f(3) = 3(3)^3 - 8(3)^2 + (3) + 2 \underline{= 14|}$

Since $f(2)$ *and* $f(3)$ have opposite signs; therefore, the polynomial has a real zero between 2 and 3.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x)=3x^3-8x^2+x+2;\quad between\ \ 1\ and\ \ 2$

Solution

$$f(1)=3(1)^3-8(1)^2+(1)+2 \underline{=-2}$$

$$f(2)=3(2)^3-8(2)^2+(2)+2 \underline{=-4}$$

Since $f(1)$ *and* $f(2)$ have same signs; therefore, cannot be determined.

Exercise

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers. $f(x)=x^5+2x^4-6x^3+2x-3;\quad between\ \ 0\ and\ 1$

Solution

$$f(0)=(0)^5+2(0)^4-6(0)^3+2(0)-3 \underline{=-3}$$

$$f(1)=(1)^5+2(1)^4-6(1)^3+2(1)-3 \underline{=-4}$$

Since $f(0)$ *and* $f(1)$ have same signs; therefore, cannot be determined.

Solution   *Section* 3.3 – Properties of Division

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=2x^4-x^3+7x-12; \quad p(x)=x^2-3$

Solution

$$\begin{array}{r|l}
 & \quad 2x^2-x+6 \\ \hline
x^2-3 & 2x^4-x^3+0x^2+7x-12 \\
 & \underline{2x^4 \qquad -6x^2} \\
 & \qquad -x^3+6x^2+7x \\
 & \qquad \underline{-x^3 \qquad +3x} \\
 & \qquad\quad 6x^2+4x-12 \\
 & \qquad\quad \underline{6x^2 \qquad -18} \\
 & \qquad\qquad 4x+6
\end{array}$$

$Q(x)=2x^2-x+6; \quad R(x)=4x+6$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=3x^3+2x-4; \quad p(x)=2x^2+1$

Solution

$$\begin{array}{r|l}
 & \qquad \frac{3}{2}x \\ \hline
2x^2+1 & 3x^3+0x^2+2x-4 \\
 & \underline{3x^3 \qquad +\frac{3}{2}x} \\
 & \qquad\quad \frac{1}{2}x-4
\end{array}$$

$Q(x)=\frac{3}{2}x; \quad R(x)=\frac{1}{2}x-4$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=7x+2; \quad p(x)=2x^2-x-4$

Solution

$Q(x)=0; \quad R(x)=7x+2$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=9x+4;\quad p(x)=2x-5$

Solution

$$\begin{array}{r} \frac{9}{2} \\ 2x-5\overline{\big)\,9x+4} \\ 9x-\frac{45}{2} \\ \hline -\frac{37}{2} \end{array}$$

$$Q(x)=\frac{9}{2};\quad R(x)=-\frac{37}{2}$$

Exercise

Use the remainder theorem to find $f(c)$: $f(x)=x^4-6x^2+4x-8;\quad c=-3$

Solution

$$f(-3)=(-3)^4-6(-3)^2+4(-3)-8\ \underline{=7}|$$

Exercise

Use the remainder theorem to find $f(c)$: $f(x)=x^4+3x^2-12;\quad c=-2$

Solution

$$f(-2)=(-2)^4+3(-2)^2-12\ \underline{=16}|$$

Exercise

Use the factor theorem to show that $x-c$ is a factor of $f(x)$: $f(x)=x^3+x^2-2x+12;\quad c=-3$

Solution

$$f(-3)=(-3)^3+(-3)^2-2(-3)+12\ \underline{=0}|$$

From the factor theorem; $x+3$ is a factor of $f(x)$.

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $2x^3-3x^2+4x-5;\quad x-2$

Solution

$$\begin{array}{r|rrrr} 2 & 2 & -3 & 4 & -5 \\ & & 4 & 2 & 12 \\ \hline & 2 & 1 & 6 & \boxed{7} \end{array}$$

$Q(x) = 2x^2 + x + 6 \qquad R(x) = 7$

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $5x^3 - 6x^2 + 15; \quad x - 4$

Solution

$$\begin{array}{r|rrrr} 4 & 5 & -6 & 0 & 15 \\ & & 20 & 56 & 224 \\ \hline & 5 & 14 & 56 & \boxed{239} \end{array}$$

$Q(x) = 5x^2 + 14x + 56 \qquad R(x) = 239$

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $9x^3 - 6x^2 + 3x - 4; \quad x - \frac{1}{3}$

Solution

$$\begin{array}{r|rrrr} \frac{1}{3} & 9 & -6 & 3 & -4 \\ & & 3 & -1 & \frac{2}{3} \\ \hline & 9 & -3 & 2 & \boxed{-\frac{10}{3}} \end{array}$$

$Q(x) = 9x^2 - 3x + 2 \qquad R(x) = -\frac{10}{3}$

Exercise

Use the synthetic division to find $f(c)$: $f(x) = 2x^3 + 3x^2 - 4x + 4; \quad c = 3$

Solution

$$\begin{array}{r|rrrr} 3 & 2 & 3 & -4 & 4 \\ & & 9 & 36 & 93 \\ \hline & 3 & 12 & 32 & \boxed{97} \end{array}$$

$\boxed{f(3) = 97}$

Exercise

Use the synthetic division to find $f(c)$: $f(x)=8x^5-3x^2+7;\quad c=\frac{1}{2}$

Solution

$$\begin{array}{c|cccccc} \frac{1}{2} & 8 & 0 & 0 & -3 & 0 & 7 \\ & & 4 & 2 & 1 & -1 & -\frac{1}{2} \\ \hline & 8 & 4 & 2 & -2 & -1 & \boxed{\frac{13}{2}} \end{array}$$

$$\boxed{f\left(\frac{1}{2}\right)=\frac{13}{2}}$$

Exercise

Use the synthetic division to find $f(c)$: $f(x)=x^3-3x^2-8;\quad c=1+\sqrt{2}$

Solution

$$\begin{array}{c|cccc} 1+\sqrt{2} & 3 & -3 & 0 & -8 \\ & & 3+3\sqrt{2} & 6+3\sqrt{2} & 12+9\sqrt{2} \\ \hline & 3 & 3\sqrt{2} & 6+3\sqrt{2} & \boxed{4+9\sqrt{2}} \end{array}$$

$$\boxed{f\left(1+\sqrt{2}\right)=4+9\sqrt{2}}$$

Exercise

Use the synthetic division to show that c is a zero of $f(x)$: $f(x)=3x^4+8x^3-2x^2-10x+4;\quad c=-2$

Solution

$$\begin{array}{c|ccccc} -2 & 3 & 8 & -2 & -10 & 4 \\ & & -6 & -4 & 12 & -4 \\ \hline & 3 & 2 & -6 & 2 & \boxed{0} \end{array}$$

$$\boxed{f(-2)=0}$$

Exercise

Use the synthetic division to show that c is a zero of $f(x)$: $f(x)=27x^4-9x^3+3x^2+6x+1;\quad c=-\frac{1}{3}$

Solution

$$\begin{array}{c|ccccc} -\frac{1}{3} & 27 & -9 & 3 & 6 & 1 \\ & & -9 & 6 & -3 & -1 \\ \hline & 27 & -18 & 9 & 3 & \boxed{0} \end{array}$$

$$\boxed{f\left(-\frac{1}{3}\right)=0}$$

Exercise

Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

$f(x) = kx^3 + x^2 + k^2x + 3k^2 + 11; \quad x+2$

Solution

$$\begin{array}{r|cccc} -2 & k & 1 & k^2 & 3k^2+11 \\ & & -2k & 4k-2 & -2k^2-8k+4 \\ \hline & k & 1-2k & k^2+4k-2 & k^2-8k+15 \end{array}$$

$k^2 - 8k + 15 = 0 \Rightarrow \boxed{k = 3,\ 5}$

Exercise

Find all solutions of the equation: $x^3 - x^2 - 10x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \frac{\pm 1,\ \pm 2,\ \pm 4,\ \pm 8}{\pm 1}$

Using the calculator, the result will show that the solutions are: $x = -1,\ -2,\ 4$

Exercise

Find all solutions of the equation: $x^3 + x^2 - 14x - 24 = 0$

Solution

possibilities for $\frac{c}{d} : \frac{\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 8,\ \pm 12,\ \pm 24}{\pm 1}$

Using the calculator, the result will show that the solutions are: $\boxed{x = -2}$

$$\begin{array}{r|cccc} -2 & 1 & 1 & -14 & -24 \\ & & -2 & 2 & 24 \\ \hline & 1 & -1 & -12 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)\left(x^2 - x - 12\right) = 0$

$x^2 - x - 12 = 0 \Rightarrow \boxed{x = -3,\ 4}$

Exercise

Find all solutions of the equation: $2x^3 - 3x^2 - 17x + 30 = 0$

Solution

possibilities for $\frac{c}{d}: \frac{\pm 1,\ \pm 2,\ \pm 3,\ \pm 5,\ \pm 6,\ \pm 10,\ \pm 15,\ \pm 30}{\pm 1,\ \pm 2}$

Using the calculator, the result will show that the solutions are: $\boxed{x = 2}$

$$\begin{array}{r|rrrr} 2 & 2 & -3 & -17 & 30 \\ & & 4 & 2 & -30 \\ \hline & 2 & 1 & -15 & \boxed{0} \end{array}$$

We have the factorization of: $(x-2)\left(2x^2 + x - 15\right) = 0$

$2x^2 + x - 15 = 0 \Rightarrow \boxed{x = -3,\ \frac{5}{2}}$

Exercise

Find all solutions of the equation: $12x^3 + 8x^2 - 3x - 2 = 0$

Solution

possibilities for $\frac{c}{d}: \frac{\pm 1,\ \pm 2}{\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 12}$

Using the calculator, the result will show that the solutions are: $\boxed{x = \frac{1}{2}}$

$$\begin{array}{r|rrrr} \frac{1}{2} & 12 & 8 & -3 & -2 \\ & & 6 & 7 & 2 \\ \hline & 12 & 14 & 4 & \boxed{0} \end{array}$$

We have the factorization of: $\left(x - \frac{1}{2}\right)\left(12x^2 + 14x + 4\right) = 0$

$12x^2 + 14x + 4 = 0 \Rightarrow \boxed{x = -\frac{2}{3},\ -\frac{1}{2}}$

Exercise

Find all solutions of the equation: $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$

Solution

possibilities for $\frac{c}{d}: \frac{\pm 1,\ \pm 2,\ \pm 4,\ \pm 7,\ \pm 8,\ \pm 14, \cdots, \pm 56}{\pm 1}$

Using the calculator, the result will show that the solutions are: $\boxed{x = 4}$

$$\begin{array}{r|rrrrr} 4 & 1 & 3 & -30 & -6 & 56 \\ & & 4 & 28 & -8 & -56 \\ \hline & 1 & 7 & -2 & -14 & \boxed{0} \end{array}$$

We have the factorization of: $(x-4)\left(x^3+7x^2-2x-14\right)=0$

For $x^3+7x^2-2x-14 \quad \Rightarrow \quad \frac{c}{d}=\frac{\pm 1,\ \pm 2,\ \pm 7,\ \pm 14}{\pm 1}$

$\boxed{x=-7}$ is another solution.

$$\begin{array}{r|rrrr} -7 & 1 & 7 & -2 & -14 \\ & & -7 & 0 & 14 \\ \hline & 1 & 0 & -2 & \boxed{0} \end{array}$$

We have the factorization of: $(x+4)(x+7)\left(x^2-2\right)=0$

By applying quadratic formula to solve: $x^2-2=0 \ \Rightarrow \boxed{x=\pm\sqrt{2}}$

Exercise

Find all solutions of the equation: $3x^5-10x^4-6x^3+24x^2+11x-6=0$

Solution

$$\boxed{x=-1,\ \ -1,\ \ \frac{1}{3},\ \ 2,\ \ 3}$$

Exercise

Find all solutions of the equation: $6x^5+19x^4+x^3-6x^2=0$

Solution

$$x^2\left(6x^3+19x^2+x-6\right)=0$$

$$\boxed{x=0,\ \ 0,\ \ -\frac{2}{3},\ \ -3,\ \ \frac{1}{2}}$$

Exercise

Find the vertical and horizontal asymptotes (if any) of: $y=\dfrac{3x}{1-x}$

Solution

$1-x=0 \Rightarrow x=1$

$y=\dfrac{3x}{-x}=\dfrac{3}{-1}=-3$

VA	$x=1$
HA	$y=-3$

Exercise

Find the vertical and horizontal asymptotes (if any) of: $y=\dfrac{x^2}{x^2+9}$

Solution

VA: n/a

HA: $y=1$

Exercise

Find the vertical and horizontal asymptotes (if any) of: $y=\dfrac{x-2}{x^2-4x+3}$

Solution

$x^2-4x+3=0 \Rightarrow x=1,\ 3$

$y=\dfrac{x}{x^2}\rightarrow 0$

VA: $x=1,\ x=3$ ***HA***: $y=0$

Exercise

Find the vertical and horizontal asymptotes (if any) of: $y=\dfrac{3}{x-5}$

Solution

VA: $x=5$ ***HA***: $y=0$

Exercise

Find the vertical and horizontal asymptotes (if any) of: $y=\dfrac{x^3-1}{x^2+1}$

Solution

VA: *none* /***HA***: *none*

Exercise

Find the vertical and horizontal asymptotes (if any) of: $y=\dfrac{3x^2-27}{(x+3)(2x+1)}$

Solution

$$x+3=0 \to x=-3$$

$$2x+1=0 \to x=-\frac{1}{2}$$

$$y=\frac{3x^2}{(x)(2x)}=\frac{3x^2}{2x^2}=\frac{3}{2}$$

VA: $x=-3,\ -\frac{1}{2}$ ***HA***: $y=\frac{3}{2}$

Exercise

Find the vertical and horizontal asymptotes (if any) of: $y=\dfrac{x^3+3x^2-2}{x^2-4}$

Solution

VA: $x=\pm 2$ ***HA***: n/a

Exercise

Find the vertical and horizontal asymptotes (if any) of: $y=\dfrac{x-3}{x^2-9}$

Solution

$$x^2-9=0 \to \boxed{x=\pm 3}$$

$$y=\frac{x-3}{(x-3)(x+3)}=\frac{1}{x+3}$$

VA: $x=3$ ***HA***: $y=0$ ***Hole***: $x=3 \to y=\frac{1}{6}$

Exercise

Find the vertical and horizontal asymptotes (if any) of: $y=\dfrac{6}{\sqrt{x^2-4x}}$

Solution

$$x^2-4x=0 \Rightarrow x(x-4)=0 \rightarrow \boxed{x=0,4}$$

VA: $x=0,\ x=4$ ***HA***: $y=0$

Exercise

Find the vertical and horizontal asymptotes (if any) of: $y=\dfrac{5x-1}{1-3x}$

Solution

VA: $x=\frac{1}{3}$ ***HA***: $y=-\frac{5}{3}$

Exercise

Find the vertical and horizontal asymptotes (if any) of: $f(x)=\dfrac{2x-11}{x^2+2x-8}$

Solution

VA: $x=2,\ x=-4$ ***HA***: $y=0$

Exercise

Find the vertical and horizontal asymptotes (if any) of: $f(x)=\dfrac{x^2-4x}{x^3-x}$

Solution

$$f(x)=\frac{x(x-4)}{x(x^2-1)}=\frac{x-4}{x^2-1}$$

VA: $x=-1,\ x=1$ ***HA***: $y=0$ ***Hole***: $x=0 \rightarrow y=4$

Exercise

Find the vertical and horizontal asymptotes (if any) of: $f(x)=\dfrac{x-2}{x^3-5x}$

Solution

VA: $x=0,\ x=\pm\sqrt{5}$ ***HA***: $y=0$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{4x}{x^2+10x}$

Solution

$x^2+10x=0 \quad \to \quad x=0, \ -10$ **Domain**: $(-\infty, \ -10)\cup(-10, \ 0)\cup(0, \ \infty)$

$$f(x)=\frac{4x}{x(x+10)}=\frac{4}{x+10}$$

VA: $x=-10$ **HA**: $y=0$

Hole: $x=0 \quad \to \quad y=\frac{4}{10} \Rightarrow$ *hole* $\left(0, \ \frac{2}{5}\right)$

Oblique asymptote: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{3-x}{(x-4)(x+6)}$

Solution

Domain: $(-\infty, \ -6)\cup(-6, \ 4)\cup(4, \ \infty)$

VA: $x=-6$ *and* $x=4$ **HA**: $y=0$

Hole: n/a **Oblique asymptote**: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3}{2x^3-x^2-3x}$

Solution

$2x^3-x^2-3x=x\left(2x^2-x-3\right)=0 \quad \to \quad x=0, \ -1, \ \frac{3}{2}$

Domain: $(-\infty, \ -1)\cup(-1, \ 0)\cup\left(0, \ \frac{3}{2}\right)\cup\left(\frac{3}{2}, \ \infty\right)$

$$f(x)=\frac{x^3}{2x^3-x^2-3x}=\frac{x^3}{x\left(2x^2-x-3\right)}=\frac{x^2}{2x^2-x-3}$$

VA: $x=-1$ *and* $x=\frac{3}{2}$ **HA**: $y=\frac{1}{2}$

Hole: $x=0 \quad \to \quad y=0 \Rightarrow$ *hole* $(0, \ 0)$

Oblique asymptote: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{3x^2+5}{4x^2-3}$

Solution

$4x^2-3=0 \quad \to \quad x=\pm\dfrac{\sqrt{3}}{2}$ **Domain**: $\left(-\infty,\ -\dfrac{\sqrt{3}}{2}\right)\cup\left(-\dfrac{\sqrt{3}}{2},\ \dfrac{\sqrt{3}}{2}\right)\cup\left(\dfrac{\sqrt{3}}{2},\ \infty\right)$

VA: $x=-\dfrac{\sqrt{3}}{2}$ *and* $x=\dfrac{\sqrt{3}}{2}$ ***HA***: $y=\dfrac{3}{4}$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x+6}{x^3+2x^2}$

Solution

$x^3+2x^2=x^2(x+2)=0 \quad \to \quad x=0,-2$ **Domain**: $(-\infty,\ -2)\cup(-2,\ 0)\cup(0,\ \infty)$

VA: $x=0$ *and* $x=2$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2+4x-1}{x+3}$

Solution

$x+3=0 \quad \to \quad x=-3$ **Domain**: $(-\infty,\ -3)\cup(-3,\ \infty)$

$$\begin{array}{r|l} & x+1 \\ x+3 & x^2+4x-1 \\ & \underline{-x^2-3x} \\ & \quad x-1 \\ & \quad \underline{-x-3} \\ & \quad\ -4 \end{array}$$

$$f(x)=\frac{x^2+4x-1}{x+3}=x+1-\frac{4}{x+3}$$

VA: $x=-3$ ***HA***: n/a

Hole: n/a ***Oblique asymptote***: $y=x+1$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2-6x}{x-5}$

Solution

$x-5=0 \quad \to \quad x=5$

Domain: $(-\infty,\ 5)\cup(5,\ \infty)$

$$\begin{array}{r}
x-1 \\
x-5\overline{)\,x^2-6x} \\
\underline{-x^2+5x} \\
-x \\
\underline{x-5} \\
-5
\end{array}$$

VA: $x=5$

HA: N/A

Hole: N/A

Oblique asymptote: $y=x-1$

$$f(x)=\frac{x^2-6x}{x-5}=x-1-\frac{5}{x-5}$$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3-x^2+x-4}{x^2+2x-1}$

Solution

$x^2+2x-1=0 \quad \to \quad x=-1\pm\sqrt{2}$

Domain: $\left(-\infty,\ -1-\sqrt{2}\right)\cup\left(-1-\sqrt{2},\ -1+\sqrt{2}\right)\cup\left(-1+\sqrt{2},\ \infty\right)$

$$\begin{array}{r}
x-3 \\
x^2+2x-1\overline{)\,x^3-x^2+x-4} \\
\underline{-x^3-2x^2+x} \\
-3x^2+2x-4 \\
\underline{3x^2+6x-3} \\
8x-7
\end{array}$$

VA: $x=-1\pm\sqrt{2}$

HA: N/A

Hole: N/A

Oblique asymptote: $y=x-3$

$$f(x)=\frac{x^3-x^2+x-4}{x^2+2x-1}=x-3+\frac{8x-7}{x^2+2x-1}$$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{-3x}{x+2}$

Solution

VA: $x=-2$ ***HA***: $y=-3$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x+1}{x^2+2x-3}$

Solution

VA: $x=-3,\ 1$ ***HA***: $y=0$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{2x^2-2x-4}{x^2+x-12}$

Solution

VA: $x=-4,\ 3$ ***HA***: $y=2$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{-2x^2+10x-12}{x^2+x}$

Solution

VA: $x=-1,\ 0$ ***HA***: $y=-2$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2-x-6}{x+1}$

Solution

$$
\begin{array}{r|l}
 & x-2 \\ \hline
x+1 & x^2-x-6 \\
 & \underline{x^2+x} \\
 & -2x-6 \\
 & \underline{-2x-2} \\
 & \boxed{-4}
\end{array}
$$

$$f(x)=\frac{x^2-x-6}{x+1}=x-2-\frac{4}{x+1}$$

The oblique asymptote is: $y=x-2$

The vertical asymptote is: $x=-1$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3+1}{x-2}$

Solution

$$\begin{array}{r}
x^2+2x+4 \\
x-2\,\overline{)\,x^3-1} \\
\underline{x^3-2x^2} \\
2x^2 \\
\underline{2x^2-4x} \\
4x-1 \\
\underline{4x-8} \\
7
\end{array}$$

$$f(x)=x^2+2x+4+\frac{7}{x-2}$$

The oblique asymptote is:

$y=x^2+2x+4$

The vertical asymptote is: $x=2$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{2x^2+x-6}{x^2+3x+2}$

Solution

$$f(x)=\frac{(2x-3)(x+2)}{(x+1)(x+2)}$$

$$=\frac{2x-3}{x+1}$$

VA	$x=-1$
HA	$y=2$
Hole	$x=-2$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x-1}{1-x^2}$

Solution

$$f(x)=\frac{x-1}{(1-x)(1+x)}$$

$$f(x)=-\frac{1}{1+x}$$

VA	$x=-1$
HA	$y=0$
Hole	$x=1$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2+x-2}{x+2}$

Solution

$$\begin{aligned} f(x) &= \frac{(x+2)(x-1)}{x+2} \\ &= x-1 \end{aligned}$$

VA	*na*
HA	*na*
Hole	$x=-2$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3-2x^2-4x+8}{x-2}$

Solution

$$f(x)=\frac{\left(x^2-4\right)(x-2)}{x-2}=x^2-4$$

Hole	$x=2$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{2x^2-3x-1}{x-2}$

Solution

$$\begin{array}{r} 2x+1 \\ x-2\overline{\big)\,2x^2-3x-1} \\ \underline{-2x^2+4x} \\ x-1 \\ \underline{-x+2} \\ 1 \end{array}$$

$$f(x)=\frac{2x^2-3x-1}{x-2}=(2x+1)+\frac{1}{x-2}$$

The ***oblique asymptote*** is the line $y=2x+1$

VA: $x=2$ ***HA***: $y=1$ ***Hole***: n/a

Exercise

Determine all asymptotes of the function $f(x) = \dfrac{2x+3}{3x^2+7x-6}$

Solution

$3x^2+7x-6=0 \quad \Rightarrow \quad x=-3,\ \frac{2}{3}$

VA: $x=-3$ *and* $x=\frac{2}{3}$ ***HA***: $y=0$

Hole: n/a ***OA:*** n/a

Exercise

Determine all asymptotes of the function $f(x) = \dfrac{x^2-1}{x^2+x-6}$

Solution

$x^2+x-6=0 \quad \Rightarrow \quad x=-3,\ 2$

$$1=\frac{x^2-1}{x^2+x-6} \Rightarrow x^2+x-6=x^2-1$$

$$\boxed{x=5}$$

VA: $x=-3$ *and* $x=2$ ***HA***: $y=1$

Hole: n/a ***OA:*** n/a

Exercise

Determine all asymptotes of the function $f(x) = \dfrac{-2x^2-x+15}{x^2-x-12}$

Solution

$x^2-x-12=0 \quad \Rightarrow \quad x=-3,\ 4$

Domain: $(-\infty,-3)\cup(-3,4)\cup(4,\infty)$

$$f(x)=\frac{(-2x+5)(x+3)}{(x-4)(x+3)}=\frac{-2x+5}{x-4}$$

VA: $x=4$ ***HA***: $y=-2$

Hole: $x=-3 \quad \rightarrow \quad y=-\frac{11}{7}$ ***OA:*** n/a

hole $\left(-3,\ -\frac{11}{7}\right)$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \text{vertical asymptote}: \ x = 4 \\ \text{horizontal asymptote}: \ y = -1 \\ x - \text{intercept}: \ 3 \end{cases}$$

Solution

Vertical Asymptote: $f(x) = \dfrac{}{x-4}$

Horizontal Asymptote: $f(x) = \dfrac{-x+a}{x-4}$

x-intercept: $f(x=3) = \dfrac{-3+a}{3-4}$

$$0 = -3 + a$$

$$a = 3$$

$$f(x) = \frac{-x+3}{x-4}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \text{vertical asymptote}: \ x = -3, x = 1 \\ \text{horizontal asymptote}: \ y = 0 \\ x - \text{intercept}: \ -1, \ f(0) = -2 \\ \text{hole at } \ x = 2 \end{cases}$$

Solution

Vertical Asymptote: $f(x) = \dfrac{}{(x+3)(x-1)}$

Horizontal Asymptote: $f(x) = \dfrac{ax+b}{(x+3)(x-1)}$

x-intercept: $f(x=-1) = \dfrac{a(-1)+b}{(-1+3)(-1-1)} = \dfrac{-a+b}{-4} = 0$

$$-a + b = 0 \ \Rightarrow \ a = b$$

$$f(x=0) = \frac{a(0)+b}{(0+3)(0-1)} = \frac{b}{-3} = -2$$

$$\boxed{b = 6 = a}$$

$$f(x) = \frac{6x+6}{(x+3)(x-1)}$$

Hole at $x = 2$: $f(x) = \dfrac{6x+6}{(x+3)(x-1)} \dfrac{x-2}{x-2}$

$$=\frac{6(x+1)(x-2)}{\left(x^2+2x-3\right)(x-2)}$$

$$\boxed{=\frac{6\left(x^2-x-2\right)}{x^3-7x+6}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote:\ x=-4, x=5 \\ horizontal\ asymptote:\ y=\frac{3}{2} \\ x-intercept:\ -2 \end{cases}$$

Solution

Vertical Asymptote: $f(x)=\frac{}{(x+4)(x-5)}$

Horizontal Asymptote: $f(x)=\frac{3}{2}\frac{(x+a)(x+b)}{(x+4)(x-5)}$

x-intercept: $f(x=-2)=\frac{3}{2}\frac{(-2+a)(-2+b)}{}$

$$0=(-2+a)(-2+b)$$

$$\boxed{a=b=2}$$

$$f(x)=\frac{3}{2}\frac{(x-2)^2}{x^2-x-20}$$

$$\boxed{=\frac{3x^2-12x+12}{2x^2-2x-40}}$$