Solution   *Section* 2.7 – Coordinates, Basis and Dimension

Exercise

Suppose $\vec{v}_1$, ..., $\vec{v}_n$ is a basis for R^n and the n by n matrix A is invertible. Show that $A\vec{v}_1$, ..., $A\vec{v}_n$ is also a basis for R^n.

Solution

Put the basis vectors $\vec{v}_1$, ..., $\vec{v}_n$ in the columns of an invertible matrix **V**. then $A\vec{v}_1$, ..., $A\vec{v}_n$ are the columns of **AV**. Since **A** is invertible, so is **AV** and its column give a basis.

Suppose $c_1 A\vec{v}_1 + \ldots + c_n A\vec{v}_n = 0$. This is $A\vec{v} = 0$ with $\vec{v} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$. Multiply by A^{-1} to get $\vec{v} = 0$. By linear independence of $\vec{v}'s$, all $c_i = 0$. So, the $Av's$ are independent.

Exercise

Consider the matrix $A = \begin{pmatrix} 1 & 0 & a \\ 2 & -1 & b \\ 1 & 1 & c \\ -2 & 1 & d \end{pmatrix}$

a) Which vectors $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ will make the columns of A linearly dependent?

b) Which vectors $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ will make the columns of A a basis for $\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : y + w = 0 \right\}$?

c) For $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$, compute a basis for the four subspaces.

Solution

a) All linear combination of $\begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

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b) To satisfy $b + d = 0$. For example, $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$A\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + B\begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix} + C\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} \; ; \; A \neq 0$$

c) $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix} + 4\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & 5 \\ -2 & 1 & 2 \end{pmatrix} \quad \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -4 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{pmatrix} \quad \begin{matrix} \\ \\ R_3 + R_2 \\ R_4 + R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} x_1 + x_3 = 0 \\ -x_2 - 4x_3 = 0 \\ \\ \end{matrix}$$

The first 2 columns span the column space $C(A)$.

If $x_3 = 1$ that implies that the nullspace

$N(A):\; \left\{ \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} \right\}$

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Rank$(A) = 2$ and $\begin{bmatrix} -1 & -4 & 1 \end{bmatrix}^T$ is a basis for the one-dimensional $N(A)$.

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Exercise

Find a basis for $x - 2y + 3z = 0$ in $\mathbb{R}^3$.

Find a basis for the intersection of that plane with xy plane. Then find a basis for all vectors perpendicular to the plane.

Solution

This plane is the nullspace of the matrix

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The special solutions: $s_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ $s_2 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ give a basis for the nullspace, and for the plane.

The intersection of this plane with the xy-plane is a line $(x, \ -2x, \ 3x)$ and the vector $(1, \ -2, \ 3)^T$ lies in the xy-plane.

The vector $v_3 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ is perpendicular to both vectors s_1 *and* s_2 : the space vectors perpendicular

to a plane $\mathbb{R}^3$ is one-dimensional, it gives a basis.

Exercise

U comes from A by subtracting row 1 from row 3:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad and \quad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Find the bases for the two column spaces. Find the bases for the two row spaces. Find bases for the two nullspaces.

Solution

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 - 3R_2$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \end{matrix}$$

a) The pivots are in the first two columns, so one possible basis for $C(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}$ and for

$C(U)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}$

b) Both A and U have the same nullspace $N(A) = N(U)$,

with basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

c) Both A and U have the same row space

$$C\left(A^T\right) = C\left(U^T\right), \quad with \ basis \ \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Exercise

Write a 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives $c_1 P_1 + \ldots + c_5 P_5 = 0$, and check entries to prove c_i is zero.) The five permutation matrices are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

Solution

Assume:

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad P_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad P_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad P_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$P_1 + P_2 + P_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and $P_4 + P_5 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$$P_1 + P_2 + P_3 - P_4 - P_5 = I$$

$$c_1 P_1 + c_2 P_2 + c_3 P_3 + c_4 P_4 + c_5 P_5 = \begin{pmatrix} c_3 & c_1 + c_4 & c_2 + c_5 \\ c_1 + c_5 & c_2 & c_3 + c_4 \\ c_2 + c_4 & c_3 + c_5 & c_1 \end{pmatrix} = 0$$

$c_1 = c_2 = c_3 = 0$ *(diagonal)*

$$\begin{pmatrix} 0 & 0 + c_4 & 0 + c_5 \\ 0 + c_5 & 0 & 0 + c_4 \\ 0 + c_4 & 0 + c_5 & 0 \end{pmatrix} = 0$$

$\underline{c_4 = c_5 = 0}$

Exercise

Choose three independent columns of $A = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix}$. Then choose a different three independent

columns. Explain whether either of these choices forms a basis for $C(A)$.

Solution

$$\begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix} \quad R_2 - 2R_1$$

$$\begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix} \quad \begin{array}{l} 2R_2 - R_2 \\ \\ \\ R_4 - R_2 \end{array}$$

$$\begin{pmatrix} 4 & 0 & 1 & 2 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 0 & 1 & 2 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \tfrac{1}{9}R_3$$

$$\begin{pmatrix} 4 & 0 & 1 & 2 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_1 - 2R_3$$

$$\begin{pmatrix} 4 & 0 & 1 & 0 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \tfrac{1}{4}R_1 \\ \tfrac{1}{6}R_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{7}{6} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank(A) = 3, the columns space is 3 which form a basis of $C(A)$. The variable is x_3

If $x_3 = 1$

$$\begin{pmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{7}{6} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} x_1 + \frac{1}{4}x_3 &= 0 \\ x_2 + \frac{7}{6}x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

$$\underline{x_1 = -\frac{1}{4} \quad x_2 = -\frac{7}{6} \quad x_4 = 0}$$

$N(A)$ is spanned by $x_n = \begin{pmatrix} -\frac{1}{4} \\ -\frac{7}{6} \\ 1 \\ 0 \end{pmatrix}$, which gives the relation of the columns.

The special solution x_n gives a relation $-\frac{1}{4}\vec{v}_1 - \frac{7}{6}\vec{v}_2 + \vec{v}_3 = 0$. If we take any two columns from the first three columns and the column 4, they will span a three-dimensional space since there will be no relation among them. Hence, they form a basis of $C(A)$.

Exercise

Which of the following sets of vectors are bases for $\mathbb{R}^2$?

a) $\{(2, 1), (3, 0)\}$

b) $\{(0, 0), (1, 3)\}$

Solution

a) $k_1(2, 1) + k_2(3, 0) = (0, 0)$

$k_1(2, 1) + k_2(3, 0) = (b_1, b_2)$

$\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} = -3 \neq 0$

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Therefore, the vectors $\{(2, 1), (3, 0)\}$ are linearly independent and span $\mathbb{R}^2$, so they form a basis for $\mathbb{R}^2$.

b) $k_1 (0, 0) + k_2 (1, 3) = (0, 0)$

$k_1 (0, 0) + k_2 (1, 3) = (b_1, b_2)$

$\begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} = 0$

Therefore; the vectors $\{(0, 0), (1, 3)\}$ are linearly dependent, so they don't form a basis for $\mathbb{R}^2$.

Exercise

Which of the following sets of vectors are bases for $\mathbb{R}^3$?

a) $\{(1, 0, 0), (2, 2, 0), (3, 3, 3)\}$ *c)* $\{(2, -3, 1), (4, 1, 1), (0, -7, 1)\}$

b) $\{(3, 1, -4), (2, 5, 6), (1, 4, 8)\}$ *d)* $\{(1, 6, 4), (2, 4, -1), (-1, 2, 5)\}$

Solution

a) $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix} = 6 \neq 0$

Therefore, the set of vectors are linearly independent.

The set form a basis for $\mathbb{R}^3$.

b) $\begin{vmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{vmatrix} = 26 \neq 0$

Therefore, the set of vectors are linearly independent.

The set form a basis for $\mathbb{R}^3$.

c) $\begin{vmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{vmatrix} = 0$

Therefore, the set of vectors are linearly dependent.

The set don't form a basis for $\mathbb{R}^3$.

d) $\begin{vmatrix} 1 & 2 & -1 \\ 6 & 4 & 2 \\ 4 & -1 & 5 \end{vmatrix} = 0$

Therefore; the set of vectors are linearly dependent.

The set don't form a basis for $\mathbb{R}^3$.

Exercise

Let V be the space spanned by $\vec{v}_1 = \cos^2 x, \quad \vec{v}_2 = \sin^2 x, \quad \vec{v}_3 = \cos 2x$

a) Show that $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is not a basis for V.

b) Find a basis for V.

Solution

a) $\cos 2x = \cos^2 x - \sin^2 x$

$k_1 \cos^2 x + k_2 \sin^2 x + k_3 \cos 2x = 0$

$k_1 \cos^2 x + k_2 \sin^2 x + k_3 \left(\cos^2 x - \sin^2 x\right) = 0$

$\left(k_1 + k_3\right)\cos^2 x + \left(k_2 - k_3\right)\sin^2 x = 0$

$\begin{cases} k_1 + k_3 = 0 & \rightarrow k_1 = -k_3 \\ k_2 - k_3 = 0 & \rightarrow k_2 = k_3 \end{cases}$

If $k_3 = -1 \Rightarrow k_1 = 1, \quad k_2 = -1$

$(1)\cos^2 x + (-1)\sin^2 x + (-1)\cos 2x = 0$

This shows that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent, therefore it is **not** a basis for V.

b) For $c_1 \cos^2 x + c_2 \sin^2 x = 0$ to hold for all real x values, we must have $c_1 = 0 \quad (x=0)$ and

$c_2 = 0 \quad \left(x = \frac{\pi}{2}\right).$

Therefore, the vectors $\vec{v}_1 = \cos^2 x \quad \vec{v}_2 = \sin^2 x$ are linearly independent.

$v = k_1 \cos^2 x + k_2 \sin^2 x + k_3 \cos 2x$

$= \left(k_1 + k_3\right)\cos^2 x + \left(k_2 - k_3\right)\sin^2 x$

This proves that the vectors $\vec{v}_1 = \cos^2 x \quad and \quad \vec{v}_2 = \sin^2 x$ span V.

We can conclude that $\vec{v}_1 = \cos^2 x \quad and \quad \vec{v}_2 = \sin^2 x$ can form a basis for V.

Exercise

Find the coordinate vector of $\vec{w}$ relative to the basis $S = \{\vec{u}_1, \vec{u}_2\}$ for $\mathbb{R}^2$

a) $\vec{u}_1 = (1,\ 0),\quad \vec{u}_2 = (0,\ 1),\quad \vec{w} = (3,\ -7)$

b) $\vec{u}_1 = (2,\ -4),\quad \vec{u}_2 = (3,\ 8),\quad \vec{w} = (1,\ 1)$

c) $\vec{u}_1 = (1,\ 1),\quad \vec{u}_2 = (0,\ 2),\quad \vec{w} = (a,\ b)$

d) $\vec{u}_1 = (1,\ -1),\quad \vec{u}_2 = (1,\ 1),\quad \vec{w} = (0,\ 1)$

e) $\vec{u}_1 = (1,\ -1),\quad \vec{u}_2 = (1,\ 1),\quad \vec{w} = (1,\ 1)$

Solution

a) $\vec{u}_1 = (1,\ 0),\quad \vec{u}_2 = (0,\ 1),\quad \vec{w} = (3,\ -7)$

We must first express $\vec{w}$ as a linear combination of the vectors in S; $\vec{w} = c_1 \vec{u}_1 + c_2 \vec{u}_2$

$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -7 \end{array}\right) \qquad \begin{array}{l} c_1 = 3 \\ c_2 = -7 \end{array}$$

$$(3,\ -7) = 3(1,\ 0) - 7(0,\ 1)$$
$$= 3u_1 - 7u_2$$

Therefore, $(\vec{w})_S = (3, -7)$

b) $\vec{u}_1 = (2,\ -4),\quad \vec{u}_2 = (3,\ 8),\quad \vec{w} = (1,\ 1)$

Solve: $\vec{w} = c_1 \vec{u}_1 + c_2 \vec{u}_2$

$$c_1(2,\ -4) + c_2(3,\ 8) = (1,\ 1)$$

$$\begin{cases} 2c_1 + 3c_2 = 1 \\ -4c_1 + 8c_2 = 1 \end{cases}$$

$$\begin{bmatrix} 2 & 3 & | & 1 \\ -4 & 8 & | & 1 \end{bmatrix} \qquad R_2 + 2R_1$$

$$\begin{bmatrix} 2 & 3 & | & 1 \\ 0 & 14 & | & 3 \end{bmatrix} \qquad \tfrac{1}{14}R_2$$

$$\begin{bmatrix} 2 & 3 & | & 1 \\ 0 & 1 & | & \tfrac{3}{14} \end{bmatrix} \qquad R_1 - 3R_2$$

$$\begin{bmatrix} 2 & 0 & | & \tfrac{5}{14} \\ 0 & 1 & | & \tfrac{3}{14} \end{bmatrix} \qquad \tfrac{1}{2}R_1$$

$$\begin{bmatrix} 1 & 0 & \bigg| & \dfrac{5}{28} \\ 0 & 1 & \bigg| & \dfrac{3}{14} \end{bmatrix}$$

$$\frac{5}{28}(2,\ -4)+\frac{3}{14}(3,\ 8)=(1,\ 1)$$

Therefore, $(\vec{w})_S = \left(\dfrac{5}{28},\ \dfrac{3}{14}\right)$

c) $\vec{u}_1 = (1,\ 1),\quad \vec{u}_2 = (0,\ 2),\quad \vec{w} = (a,\ b)$

Solve: $\vec{w} = c_1\vec{u}_1 + c_2\vec{u}_2$

$c_1(1,\ 1)+c_2(0,\ 2)=(a,\ b)$

$$\begin{cases} c_1 = a \\ c_1 + 2c_2 = b \end{cases} \Rightarrow c_2 = \frac{b-a}{2}$$

$a(1,\ 1)+\dfrac{b-a}{2}(0,\ 2)=(a,\ b)$

Therefore, $(\vec{w})_S = \left(a,\ \dfrac{b-a}{2}\right)$

d) $\vec{u}_1 = (1,\ -1),\quad \vec{u}_2 = (1,\ 1),\quad \vec{w} = (0,\ 1)$

Solve: $\vec{w} = c_1\vec{u}_1 + c_2\vec{u}_2$

$c_1(1,\ -1)+c_2(1,\ 1)=(0,\ 1)$

$$\begin{cases} c_1 + c_2 = 0 \\ -c_1 + c_2 = 1 \end{cases}$$

$$c_1 = \frac{\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}} = -\frac{1}{2} \qquad\qquad c_2 = \frac{\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}}{2} = \frac{1}{2}$$

$-\dfrac{1}{2}(1,\ -1)+\dfrac{1}{2}(1,\ 1)=(0,\ 1)$

Therefore, $(\vec{w})_S = \left(-\dfrac{1}{2},\ \dfrac{1}{2}\right)$

e) $\vec{u}_1 = (1,\ -1),\quad \vec{u}_2 = (1,\ 1),\quad \vec{w} = (1,\ 1)$

Solve: $\vec{w} = c_1\vec{u}_1 + c_2\vec{u}_2$

$c_1(1,\ -1)+c_2(1,\ 1)=(1,\ 1)$

$$\begin{cases} c_1 + c_2 = 1 \\ -c_1 + c_2 = 1 \end{cases}$$

$$c_1 = \dfrac{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}} = 0 \mid \qquad\qquad c_2 = \dfrac{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}}{2} = 1 \mid$$

$$0(1,\ -1) + 1(1,\ 1) = (1,\ 1)$$

Therefore, $(\vec{w})_S = (0,\ 1) \mid$

Exercise

Find the coordinate vector of $\vec{v}$ relative to the basis $S = \{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\}$

a) $\vec{v} = (2,\ -1,\ 3),\quad \vec{v}_1 = (1,\ 0,\ 0),\quad \vec{v}_2 = (2,\ 2,\ 0),\quad \vec{v}_3 = (3,\ 3,\ 3)$

b) $\vec{v} = (5,\ -12,\ 3),\quad \vec{v}_1 = (1,\ 2,\ 3),\quad \vec{v}_2 = (-4,\ 5,\ 6),\quad \vec{v}_3 = (7,\ -8,\ 9)$

Solution

a) $\vec{v} = (2,\ -1,\ 3),\quad \vec{v}_1 = (1,\ 0,\ 0),\quad \vec{v}_2 = (2,\ 2,\ 0),\quad \vec{v}_3 = (3,\ 3,\ 3)$

Solve: $c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{v}$

$c_1(1,\ 0,\ 0) + c_2(2,\ 2,\ 0) + c_2(3,\ 3,\ 3) = (2,\ -1,\ 3)$

$$\begin{cases} c_1 + 2c_2 + 3c_3 = 2 \\ 2c_2 + 3c_3 = -1 \\ 3c_3 = 3 \end{cases}$$

$\rightarrow\ c_1 = 2 - 2c_2 - 3c_3 = 3 \mid$

$\rightarrow\ c_2 = \dfrac{-3c_3 - 1}{2} = -2 \mid$

$\rightarrow\ c_3 = 1 \mid$

$3(1,\ 0,\ 0) - 2(2,\ 2,\ 0) + 1(3,\ 3,\ 3) = (2,\ -1,\ 3)$

Therefore, $(\vec{v})_S = (3,\ -2,\ 1) \mid$

b) $\vec{v} = (5,\ -12,\ 3),\quad \vec{v}_1 = (1,\ 2,\ 3),\quad \vec{v}_2 = (-4,\ 5,\ 6),\quad \vec{v}_3 = (7,\ -8,\ 9)$

Solve: $c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{v}$

$c_1(1,\ 2,\ 3) + c_2(-4,\ 5,\ 6) + c_2(7,\ -8,\ 9) = (5,\ -12,\ 3)$

$$\begin{cases} c_1 - 4c_2 + 7c_3 = 5 \\ 2c_1 + 5c_2 - 8c_3 = -12 \\ 3c_1 + 6c_2 + 9c_3 = 3 \end{cases}$$

$$c_1 = \frac{\begin{vmatrix} 5 & -4 & 7 \\ -12 & 5 & -8 \\ 3 & 6 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & -4 & 7 \\ 2 & 5 & -8 \\ 3 & 6 & 9 \end{vmatrix}} = \frac{-480}{240} = -2 \underline{\quad}$$

$$c_2 = \frac{\begin{vmatrix} 1 & 5 & 7 \\ 2 & -12 & -8 \\ 3 & 3 & 9 \end{vmatrix}}{240} = \frac{0}{240} = 0 \underline{\quad}$$

$$c_2 = \frac{\begin{vmatrix} 1 & -4 & 5 \\ 2 & 5 & -12 \\ 3 & 6 & 3 \end{vmatrix}}{240} = \frac{240}{240} = 1 \underline{\quad}$$

$$-2(1,\ 2,\ 3) + 0(-4,\ 5,\ 6) + 1(7,\ -8,\ 9) = (5,\ -12,\ 3)$$

Therefore, $(\vec{v})_S = (-2,\ 0,\ 1)$

Exercise

Show that $\{A_1, A_2, A_3, A_4\}$ is a basis for M_{22}, and express A as a linear combination of the basis vectors

a) $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $A = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$

b) $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

c) $A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$, $A_1 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Solution

a) Matrices $\{A_1, A_2, A_3, A_4\}$ are linearly independent if the equation

$$k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4 = 0$$

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$$k_1 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + k_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k_4 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \mathbf{0}$$

Has only the trivial solution.

For these matrices to span M_{22}, it must be expressed every matrix $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ as

$$k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4 = \mathbf{A}$$

$$k_1 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + k_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k_4 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$$

The 2 equations can be written as linear systems

$$
\begin{aligned}
k_1 + k_2 + k_3 \quad &= 0 \\
k_2 \quad &= 0 \\
k_1 \qquad + k_4 &= 0 \\
k_3 \quad &= 0
\end{aligned}
\quad \text{and} \quad
\begin{aligned}
k_1 + k_2 + k_3 \quad &= a_1 \\
k_2 \quad &= a_2 \\
k_1 \qquad + k_4 &= a_3 \\
k_3 \quad &= a_4
\end{aligned}
$$

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -1 \neq 0,$$

That the homogeneous system has only the trivial solution.

$\{A_1, A_2, A_3, A_4\}$ span M_{22}

$$\rightarrow \begin{cases} k_1 + k_2 + k_3 \quad = 6 \\ k_2 \quad = 2 \\ k_1 \qquad + k_4 = 5 \\ k_3 \quad = 3 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & | & 6 \\ 0 & 1 & 0 & 0 & | & 2 \\ 1 & 0 & 0 & 1 & | & 5 \\ 0 & 0 & 1 & 0 & | & 3 \end{bmatrix} \quad R_3 - R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & | & 6 \\ 0 & 1 & 0 & 0 & | & 2 \\ 0 & -1 & -1 & 1 & | & -1 \\ 0 & 0 & 1 & 0 & | & 3 \end{bmatrix} \quad \begin{matrix} R_1 - R_2 \\ \\ R_3 + R_2 \end{matrix}$$

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$$\begin{bmatrix} 1 & 0 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & 0 & | & 2 \\ 0 & 0 & -1 & 1 & | & 1 \\ 0 & 0 & 1 & 0 & | & 3 \end{bmatrix} \quad \begin{array}{l} R_1 + R_3 \\ \\ \\ R_4 + R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & | & 5 \\ 0 & 1 & 0 & 0 & | & 2 \\ 0 & 0 & -1 & 1 & | & 1 \\ 0 & 0 & 0 & 1 & | & 4 \end{bmatrix} \quad \begin{array}{l} \\ \\ -R_3 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & | & 5 \\ 0 & 1 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & -1 & | & -1 \\ 0 & 0 & 0 & 1 & | & 4 \end{bmatrix} \quad \begin{array}{l} R_1 - R_4 \\ \\ R_3 + R_4 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & 0 & | & 3 \\ 0 & 0 & 0 & 1 & | & 4 \end{bmatrix} \quad \begin{array}{l} k_1 = 1 \\ k_2 = 2 \\ k_3 = 3 \\ k_4 = 4 \end{array}$$

$$\underline{\mathbf{A} = A_1 + 2A_2 + 3A_3 + 4A_4}$$

b) Matrices $\{A_1, A_2, A_3, A_4\}$ are linearly independent if the equation

$$k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4 = \mathbf{0}$$

$$k_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + k_3 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + k_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{0}$$

Has only the trivial solution.

For these matrices to span M_{22}, it must be expressed every matrix $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ as

$$k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4 = \mathbf{A}$$

$$k_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + k_3 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + k_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

The 2 equations can be written as linear systems

$$\begin{aligned}
k_1 &= 0 \\
k_1 + k_2 &= 0 \\
k_1 + k_2 + k_3 &= 0 \\
k_1 + k_2 + k_3 + k_4 &= 0
\end{aligned} \quad and \quad
\begin{aligned}
k_1 &= a_1 \\
k_1 + k_2 &= a_2 \\
k_1 + k_2 + k_3 &= a_3 \\
k_1 + k_2 + k_3 + k_4 &= a_4
\end{aligned}$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1 \neq 0 \text{, that the homogeneous system has only the trivial solution.}$$

$\{A_1, A_2, A_3, A_4\}$ span M_{22}

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 \end{array}\right] \quad \begin{array}{l} \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} \\ \\ \\ R_4 - R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \quad \begin{array}{l} k_1 = 1 \\ k_2 = -1 \\ k_3 = 1 \\ k_4 = -1 \end{array}$$

$\underline{\mathbf{A} = A_1 - A_2 + A_3 - A_4}$

c) $k_1 \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + k_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + k_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \quad \begin{array}{l} \\ R_2 + R_1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & | & 2 \\ 0 & 2 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & 0 & | & -1 \\ 0 & 0 & 0 & 1 & | & 3 \end{bmatrix} \qquad \frac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & 0 & | & 1 \\ 0 & 0 & 1 & 0 & | & -1 \\ 0 & 0 & 0 & 1 & | & 3 \end{bmatrix} \qquad \begin{aligned} k_1 &= 1 \\ k_2 &= 1 \\ k_3 &= -1 \\ k_4 &= 3 \end{aligned}$$

$$\mathbf{A} = A_1 + A_2 - A_3 + 3A_4$$

Exercise

Consider the eight vectors

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

a) List all of the one-element. Linearly dependent sets formed from these.
b) What are the two-element, linearly dependent sets?
c) Find a three-element set spanning a subspace of dimension three, and dimension of two? One? Zero?
d) Which four-element sets are linearly dependent? Explain why.

Solution

a) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ zero vector is the only linearly dependent.

b) The set that contains zero vector and any other vector.

c) 2-dimension:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad or \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad or \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

1-dimensional subspace if we allow duplicates (zero vector) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

d) All four-element sets are linearly dependent in three-dimensional space.

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Exercise

Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space

a)
$$\begin{cases} x_1 + x_2 - x_3 = 0 \\ -2x_1 - x_2 + 2x_3 = 0 \\ -x_1 \qquad + x_3 = 0 \end{cases}$$

d)
$$\begin{cases} x + y + z = 0 \\ 3x + 2y - 2z = 0 \\ 4x + 3y - z = 0 \\ 6x + 5y + z = 0 \end{cases}$$

b)
$$\begin{cases} 3x_1 + x_2 + x_3 + x_4 = 0 \\ 5x_1 - x_2 + x_3 - x_4 = 0 \end{cases}$$

e)
$$\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 \qquad + 5x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

c)
$$\begin{cases} x_1 - 3x_2 + x_3 = 0 \\ 2x_1 - 6x_2 + 2x_3 = 0 \\ 3x_1 - 9x_2 + 3x_3 = 0 \end{cases}$$

Solution

a)
$$\begin{bmatrix} 1 & 1 & -1 & | & 0 \\ -2 & -1 & 2 & | & 0 \\ -1 & 0 & 1 & | & 0 \end{bmatrix} \quad \begin{matrix} R_2 + 2R_1 \\ \\ R_3 + R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \end{bmatrix} \quad \begin{matrix} R_2 - R_2 \\ \\ R_3 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad \begin{matrix} x_1 - x_3 = 0 & \rightarrow & \underline{x_1 = x_3} \\ x_2 = 0 \end{matrix}$$

The solution: $\left(x_1,\ 0,\ x_1 \right) = x_1 \left(1,\ 0,\ 1 \right)$

The solution space has dimension 1 and a basis $\left(1,\ 0,\ 1 \right)$

b)
$$\begin{bmatrix} 3 & 1 & 1 & 1 & | & 0 \\ 5 & -1 & 1 & -1 & | & 0 \end{bmatrix} \quad 3R_2 - 5R_1$$

$$\begin{bmatrix} 3 & 1 & 1 & 1 & | & 0 \\ 0 & -8 & -2 & -8 & | & 0 \end{bmatrix} \quad 8R_1 + R_2$$

$$\begin{bmatrix} 24 & 0 & 6 & 0 & | & 0 \\ 0 & -8 & -2 & -8 & | & 0 \end{bmatrix} \quad \begin{matrix} \frac{1}{24}R_1 \\ -\frac{1}{8}R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & | & 0 \\ 0 & 1 & \frac{1}{4} & 1 & | & 0 \end{bmatrix} \quad \begin{array}{l} x_3 = s \\ x_4 = t \end{array} \quad \begin{array}{l} x_1 = -\frac{1}{4}x_3 = s \\ x_2 = -\frac{1}{4}x_3 - x_4 = -\frac{1}{4}s - t \end{array}$$

The solution:

$$\left(x_1, x_2, x_3, x_4\right) = \left(-\frac{1}{4}s, \; -\frac{1}{4}s - t, \; s, \; t\right)$$
$$= s\left(-\frac{1}{4}, \; -\frac{1}{4}, \; 1, \; 0\right) + t\left(0, \; -1, \; 0, \; 1\right)$$

The solution space has dimension 2 and a basis $\left(-\frac{1}{4}, \; -\frac{1}{4}, \; 1, \; 0\right), \; \left(0, \; -1, \; 0, \; 1\right)$

c)
$$\begin{bmatrix} 1 & -3 & 1 & | & 0 \\ 2 & -6 & 2 & | & 0 \\ 3 & -9 & 3 & | & 0 \end{bmatrix} \quad \begin{array}{l} R_2 - 2R_1 \\ \\ R_3 - 3R_1 \end{array}$$

$$\begin{bmatrix} 1 & -3 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad x_1 - 3x_2 + x_3 = 0 \; \rightarrow \; x_1 = 3x_2 - x_3$$

The solution:

$$\left(x_1, x_2, x_3\right) = \left(3x_2 - x_3, \; x_2, \; x_3\right)$$
$$= x_2\left(3, \; 1, \; 0\right) + x_3\left(-1, \; 0, \; 1\right)$$

The solution space has dimension 2 and a basis $\left(3, \; 1, \; 0\right) \; and \; \left(-1, \; 0, \; 1\right)$

d)
$$\begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 3 & 2 & -2 & | & 0 \\ 4 & 3 & -1 & | & 0 \\ 6 & 5 & 1 & | & 0 \end{bmatrix} \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - 4R_1 \\ R_4 - 6R_1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 0 & -1 & -5 & | & 0 \\ 0 & -1 & -5 & | & 0 \\ 0 & -1 & -5 & | & 0 \end{bmatrix} \quad \begin{array}{l} R_1 + R_2 \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & -1 & -5 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad -R_2$$

$$\begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & 1 & 5 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad \begin{matrix} x = 4z \\ y = -5z \end{matrix}$$

The solution: $(x,\ y,\ z) = (4z,\ -5z,\ z) = z(4,\ -5,\ 1)$

The solution space has dimension 1 and a basis $(4,\ -5,\ 1)$

e)
$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix} \qquad 2R_2 - R_1$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 7 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{matrix} R_1 + R_2 \\[6pt] R_3 + R_2 \end{matrix}$$

$$\begin{bmatrix} 2 & 0 & 10 \\ 0 & -1 & 7 \\ 0 & 0 & 8 \end{bmatrix} \qquad \begin{matrix} \frac{1}{2}R_1 \\[4pt] -R_2 \\[4pt] \frac{1}{8}R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{matrix} R_1 - 5R_3 \\[4pt] R_2 + 7R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

No basis and dimension $= 0$

Exercise

If $AS = SA$ for the shift matrix S. Show that A must have this special form:

$$\text{If } \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\text{then } A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

"The subspace of matrices that commute with the shift S has dimension _____."

Solution

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad d = g = h = 0 \quad b = f \quad a = e = i$$

$$A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

The subspace of matrices that commute with the shift S has dimension 3, because the matrix has only three variables.

Exercise

Find bases for the following subspaces of $\mathbb{R}^3$

a) All vectors of the form $(a, b, c, 0)$
b) All vectors of the form (a, b, c, d), where $d = a + b$ and $c = a - b$.
c) All vectors of the form (a, b, c, d), where $a = b = c = d$.

Solution

a) The subspace can be expressed as span $S = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)\}$ is a set of linearly independent vectors. Therefore; S forms a basis for the subspace, so its dimension is 3.

b) The subspace contains all vectors $(a, b, a+b, a-b) = a(1, 0, 1, 1) + b(0, 1, 1, -1)$, the set $S = \{(1, 0, 1, 1), (0, 1, 1, -1)\}$ is linearly independent vectors. Therefore; S forms a basis for the subspace, so its dimension is 2.

c) The subspace contains all vectors $(a, a, a, a) = a(1, 1, 1, 1)$, we can express the set $S = \{(1, 1, 1, 1)\}$ as span S and it is linearly independent. Therefore, S forms a basis for the subspace, so its dimension is 1.

Exercise

Find a basis for the null space of A.

a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

Solution

a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$ $\quad \begin{array}{l} R_2 - 5R_1 \\ \\ R_2 - 7R_1 \end{array}$

$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix}$ $\quad \begin{array}{l} R_1 + R_2 \\ \\ R_3 - R_2 \end{array}$

$\begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \begin{array}{l} \rightarrow x_1 = 16x_3 = 16t \\ \rightarrow x_2 = 19x_3 = 19t \end{array}$

The general form of the solution of $A\vec{x} = \vec{0}$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$

Therefore, the vector $\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$ forms a basis for the null space of A.

b) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$ $\quad \begin{array}{l} R_2 - 2R_1 \\ \\ R_3 + R_1 \end{array}$

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$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{bmatrix} \quad R_3 + R_2$$

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad -\frac{1}{7}R_2$$

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 - 4R_2$$

$$\begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \rightarrow x_1 = -x_3 + \frac{2}{7}x_4 = -t + \frac{2}{7}s \\ \rightarrow x_2 = -x_3 - \frac{4}{7}x_4 = -t - \frac{4}{7}s \end{matrix}$$

The general form of the solution of $A\vec{x} = \vec{0}$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$

Therefore, the vectors $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

c) $\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix} \quad \begin{matrix} R_2 - 3R_1 \\ R_3 + R_1 \\ R_4 - 2R_1 \end{matrix}$

$\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & -14 & -14 & -14 & -28 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -5 & -5 & -5 & -10 \end{bmatrix} \quad \begin{matrix} -\frac{1}{14}R_2 \\ \frac{1}{4}R_3 \\ -\frac{1}{5}R_4 \end{matrix}$

$$\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix} \quad \begin{matrix} R_1 - 4R_2 \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{cases} x_1 = -x_3 - 2x_4 - x_5 = -r - 2s - t \\ x_2 = -x_3 - x_4 - 2x_5 = -r - s - 2t \end{cases}$$

The general form of the solution of $A\vec{x} = \vec{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the vectors $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, *and* $\begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

Exercise

Find a basis for the subspace of $\mathbb{R}^4$ spanned by the given vectors

a) $(1,\ 1,\ -4,\ -3),\ \ (2,\ 0,\ 2,\ -2),\ \ (2,\ -1,\ 3,\ 2)$

b) $(-1,\ 1,\ -2,\ 0),\ \ (3,\ 3,\ 6,\ 0),\ \ (9,\ 0,\ 0,\ 3)$

Solution

a) $(1,\ 1,\ -4,\ -3),\ \ (2,\ 0,\ 2,\ -2),\ \ (2,\ -1,\ 3,\ 2)$

$$\begin{pmatrix} 1 & 1 & -4 & -3 \\ 2 & 0 & 2 & -2 \\ 2 & -1 & 3 & 2 \end{pmatrix} \quad \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & -4 & -3 \\ 0 & -2 & 10 & 4 \\ 0 & -3 & 11 & 8 \end{pmatrix} \quad -\frac{1}{2}R_2$$

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$$\begin{pmatrix} 1 & 1 & -4 & -3 \\ 0 & 1 & -5 & -2 \\ 0 & -3 & 11 & 8 \end{pmatrix} \quad \begin{matrix} R_1 - R_2 \\ \\ R_3 + 3R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & -4 & 2 \end{pmatrix} \quad -\frac{1}{4}R_3$$

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 1 & -\frac{1}{2} \end{pmatrix} \quad \begin{matrix} R_1 - R_3 \\ R_2 + 5R_3 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{9}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{pmatrix}$$

A basis for the subspace is $\left(1,\ 0,\ 0,\ -\frac{1}{2}\right),\ \left(0,\ 1,\ 0,\ -\frac{9}{2}\right),\ \left(0,\ 0,\ 1,\ -\frac{1}{2}\right)$

b) $(-1,\ 1,\ -2,\ 0),\ (3,\ 3,\ 6,\ 0),\ (9,\ 0,\ 0,\ 3)$

$$\begin{pmatrix} -1 & 1 & -2 & 0 \\ 3 & 3 & 6 & 0 \\ 9 & 0 & 0 & 3 \end{pmatrix} \quad -R_1$$

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 3 & 3 & 6 & 0 \\ 9 & 0 & 0 & 3 \end{pmatrix} \quad \begin{matrix} R_2 - 3R_1 \\ R_2 - 9R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 9 & -18 & 3 \end{pmatrix} \quad \frac{1}{6}R_2$$

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 9 & -18 & 3 \end{pmatrix} \quad \begin{matrix} R_1 + R_2 \\ \\ R_3 - 9R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -18 & 3 \end{pmatrix} \quad -\frac{1}{18}R_3$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\dfrac{1}{6} \end{pmatrix} \qquad R_1 - 2R_3$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\dfrac{1}{6} \end{pmatrix}$$

A basis for the subspace is $(1,\ 0,\ 0,\ 0),\ \ (0,\ 1,\ 0,\ 0),\ \ \left(0,\ 0,\ 1,\ -\dfrac{1}{6}\right)$

Exercise

Determine whether the given vectors form a basis for the given vector space

a) $\vec{v}_1(3,\ -2,\ 1),\ \ \vec{v}_2(2,\ 3,\ 1),\ \ \vec{v}_3(2,\ 1,\ -3),\ \ in\ \mathbb{R}^3$

b) $\vec{v}_1 = (1,\ 1,\ 0,\ 0),\ \ \vec{v}_2 = (0,\ 1,\ 1,\ 0),\ \ \vec{v}_3 = (0,\ 0,\ 1,\ 1),\ \ \vec{v}_4 = (1,\ 0,\ 0,\ 1),\ \ for\ \mathbb{R}^4$

c) $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ \ M_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ \ M_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},\ \ M_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\ \ M_{22}$

Solution

a) $\vec{v}_1(3,\ -2,\ 1),\ \ \vec{v}_2(2,\ 3,\ 1),\ \ \vec{v}_3(2,\ 1,\ -3),\ \ in\ \mathbb{R}^3$

$$\begin{vmatrix} 3 & 2 & 2 \\ -2 & 3 & 1 \\ 1 & 1 & -3 \end{vmatrix} = 14 \neq 0$$

The given vectors are linearly independent and span $\mathbb{R}^3$, so they form a basis for $\mathbb{R}^3$.

b) $\vec{v}_1 = (1,\ 1,\ 0,\ 0),\ \ \vec{v}_2 = (0,\ 1,\ 1,\ 0),\ \ \vec{v}_3 = (0,\ 0,\ 1,\ 1),\ \ \vec{v}_4 = (1,\ 0,\ 0,\ 1),\ \ for\ \mathbb{R}^4$

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 2 \neq 0$$

The given vectors are linearly independent and span R^4, so they form a basis for R^4.

c) $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $M_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ M_{22}

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

They form a basis for M_{22}.

Exercise

Find a basis for, and the dimension of, the null space of the given matrix $\begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & -2 & -2 & 1 \end{bmatrix}$

Solution

$$\begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & -2 & -2 & 1 \end{bmatrix} \quad R_2 - 2R_1$$

$$\begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & -4 & 0 & -1 \end{bmatrix} \quad 4R_1 + R_2$$

$$\begin{bmatrix} 8 & 0 & -4 & 3 \\ 0 & -4 & 0 & -1 \end{bmatrix} \quad \begin{array}{l} \frac{1}{8}R_1 \\ -\frac{1}{4}R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & \frac{3}{8} \\ 0 & 1 & 0 & \frac{1}{4} \end{bmatrix} \quad \begin{array}{l} x_1 = -\frac{1}{2}x_3 - \frac{3}{8}x_4 \\ x_2 = -\frac{1}{4}x_4 \end{array}$$

$$x = x_3 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{3}{8} \\ \frac{1}{4} \\ 0 \\ 1 \end{bmatrix}$$

The *bases* are: $\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} -\frac{3}{8} \\ \frac{1}{4} \\ 0 \\ 1 \end{bmatrix}$

Dimension: 2

Exercise

Let $\mathbb{R}$ be the set of all real numbers and let $\mathbb{R}^+$ be the set of all positive real numbers. Show that $\mathbb{R}^+$ is a vector space over $\mathbb{R}$ under the addition

$$\alpha \oplus \beta = \alpha\beta \quad \alpha, \beta \in \mathbb{R}^+$$

And the scalar multiplication

$$a \odot \alpha = \alpha^a \quad \alpha \in \mathbb{R}^+, \ a \in \mathbb{R}$$

Find the dimension of the vector space. Is $\mathbb{R}^+$ also a vector space if the scalar multiplication is instead defined as

$$a \otimes \alpha = a^\alpha \quad \alpha \in \mathbb{R}^+, \ a \in \mathbb{R}?$$

Solution

$$ab \odot \alpha = \alpha^{ab} \qquad \alpha \in \mathbb{R}^+, \ a, b \in \mathbb{R}$$

$$= \left(\alpha^b\right)^a$$

$$= a \odot \left(\alpha^b\right)$$

$$= a \odot (b \odot \alpha)$$

Since for $\alpha \in \mathbb{R}^+$, then

$$\alpha = (\log \alpha) \odot 10$$

Thus $\{10\}$ is a basis, therefore the dimension of the vector space is **1**.

$\mathbb{R}^+$ is not a vector space over $\mathbb{R}$ with respect to $\otimes$.

Since,

$$2 \otimes (1 \oplus 1) = 2 \otimes \big((1)(1)\big)$$

$$= 2 \otimes 1$$

$$= 2^1$$

$$= \underline{2 \ |}$$

$$(2 \otimes 1) \oplus (2 \otimes 1) = \left(2^1\right) \oplus \left(2^1\right)$$

$$= 2 \oplus 2$$

$$= (2)(2)$$

$$= \underline{4 \ |}$$

$2 \neq 4$

$$2 \otimes (1 \oplus 1) \neq (2 \otimes 1) \oplus (2 \otimes 1)$$