Solution Section 4.1 – Parameterizations of Plane Curves

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty$

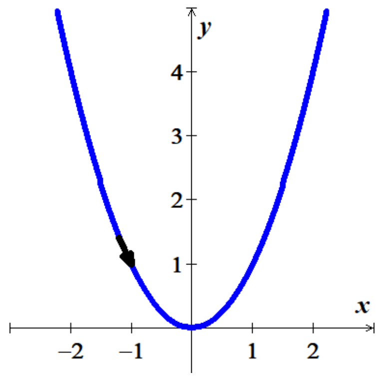
Solution

$$x = 3t \quad \Rightarrow \quad t = \frac{x}{3}$$

$$y = 9t^2$$

$$= 9\left(\frac{x}{3}\right)^2$$

$$\underline{= x^2} \qquad (Parbola)$$



Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

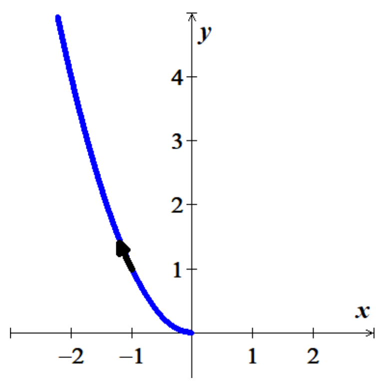
$x = -\sqrt{t}, \quad y = t, \quad t \ge 0$

Solution

$$x = -\sqrt{t}$$

$$= -\sqrt{y}$$

$$y = x^2, \quad x \le 0$$



Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

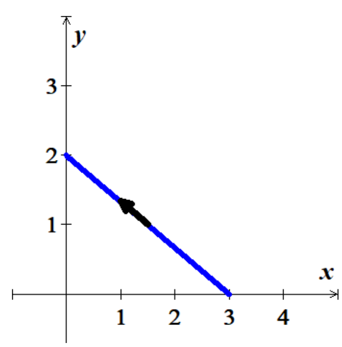
$x = 3 - 3t, \quad y = 2t, \quad 0 \le t \le 1$

Solution

$$x = 3 - 3t$$

$$3t = 3 - x$$

$$t = 1 - \frac{x}{3}$$



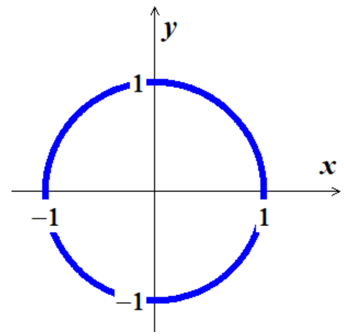
$$y = 2\left(1 - \frac{x}{3}\right)$$

$$\boxed{= 2 - \frac{2}{3}x} \qquad (Line)$$

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$x = \cos 2t, \quad y = \sin 2t, \quad 0 \le t \le \pi$



Solution

$$\cos^2 2t + \sin^2 2t = 1$$

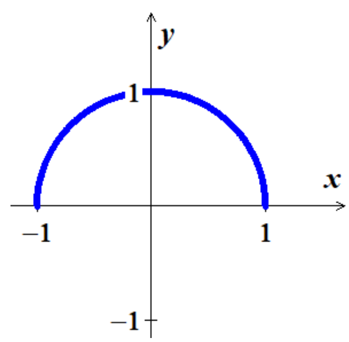
$$\boxed{x^2 + y^2 = 1}$$

∴ Unit circle centered at the origin.

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$x = \cos(\pi - t), \quad y = \sin(\pi - t), \quad 0 \le t \le \pi$



Solution

$$\cos^2(\pi - t) + \sin^2(\pi - t) = 1$$

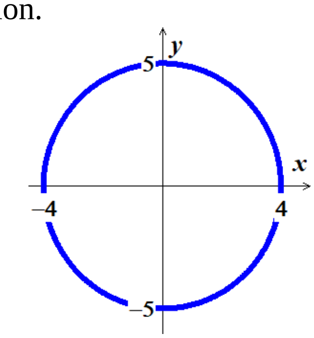
$$\boxed{x^2 + y^2 = 1; \quad y \ge 0}$$

∴ Semi–circle centered at the origin with radius 1.

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = 4\sin t, \quad y = 5\cos t, \quad 0 \le t \le 2\pi$$



Solution

$$\sin t = \frac{x}{4}, \quad \cos t = \frac{y}{5}$$

$$\sin^2 t + \cos^2 t = 1$$

$$\boxed{\frac{x^2}{16}+\frac{y^2}{25}=1} \qquad (Ellipse)$$

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x=1+\sin t, \quad y=\cos t-2, \quad 0\le t\le 2\pi$$

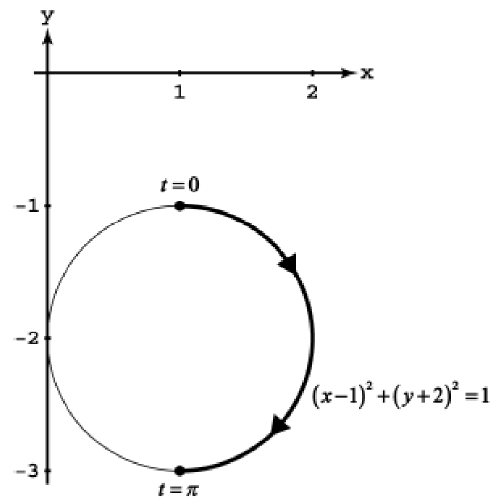
Solution

$\sin t = x-1, \quad \cos t = y+2$

$\sin^2 t+\cos^2 t=1$

$$\boxed{(x-1)^2+(y+2)^2=1}$$

$\therefore$ Circle centered at $(1, \ -2)$ with radius 1.



Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

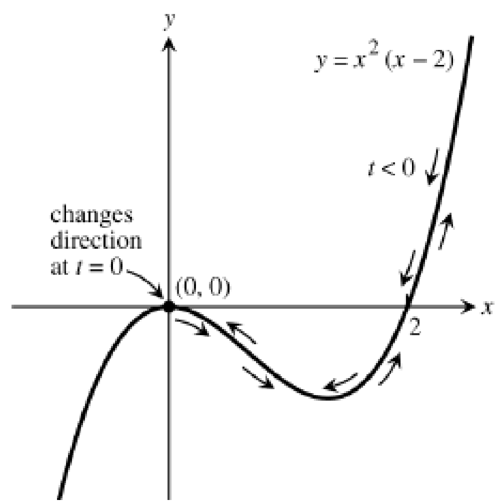
$$x=t^2, \quad y=t^6-2t^4, \quad -\infty<t<\infty$$

Solution

$y=t^6-2t^4$

$\quad =\left(t^2\right)^3-2\left(t^2\right)^2$

$\quad \boxed{=x^3-2x^2}$



Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = \frac{t}{t-1}, \quad y = \frac{t-2}{t+1}, \quad -1 < t < 1$$

Solution

$$x = \frac{t}{t-1}$$

$$xt - x = t$$

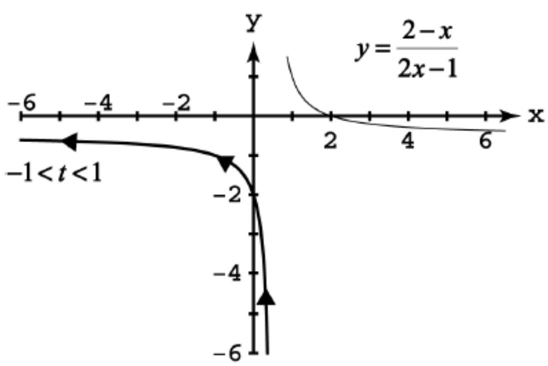
$$t(x-1) = x$$

$$\to \quad \boxed{t = \frac{x}{x-1}}$$

$$y = \frac{\frac{x}{x-1} - 2}{\frac{x}{x-1} + 1}$$

$$= \frac{x - 2x + 2}{x + x - 1}$$

$$= \boxed{\frac{-x+2}{2x-1}}$$



Exercise

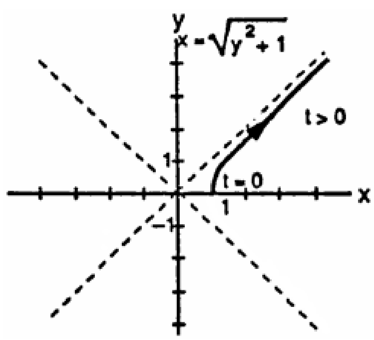
Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = \sqrt{t+1}, \quad y = \sqrt{t}, \quad t \ge 0$$

Solution

$$y = \sqrt{t} \quad \to \quad y^2 = t$$

$$\boxed{x = \sqrt{y^2 + 1} \quad y \ge 0}$$

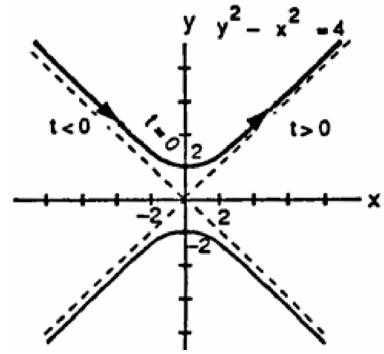


Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = 2\sinh t, \quad y = 2\cosh t, \quad -\infty < t < \infty$$

Solution



$$\sinh t = \frac{x}{2}, \quad \cosh t = \frac{y}{2}$$

$$\cosh^2 t - \sinh^2 t = 1$$

$$\frac{y^2}{4} - \frac{x^2}{4} = 1$$

$$\boxed{y^2 - x^2 = 4}$$

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = 4\cos 2\pi t, \quad y = 4\sin 2\pi t, \quad 0 \le t \le 1$$

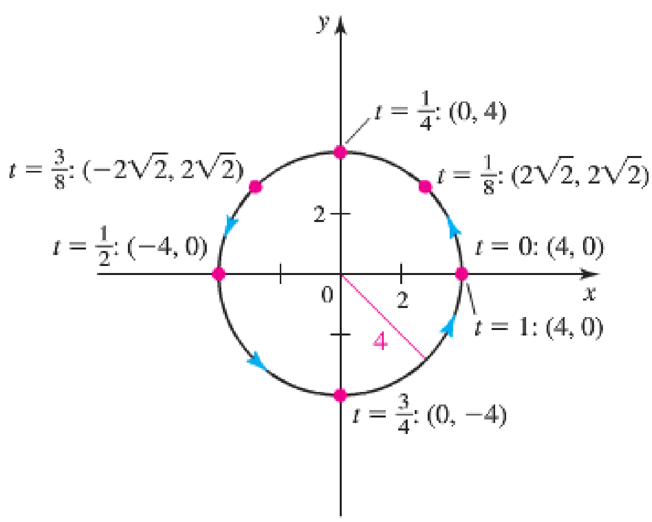
Solution

$$x^2 + y^2 = 16\cos^2 2\pi t + 16\sin^2 2\pi t$$

$$\boxed{= 16}$$

The equation represents a circle with a center at origin of radius 4.

t	$(x,\ y)$
0	$(4,\ 0)$
$\frac{1}{8}$	$(2\sqrt{2},\ 2\sqrt{2})$
0.25	$(0,\ 4)$
0.5	$(-4,\ 0)$
0.75	$(0,\ -4)$
1	$(4,\ 0)$



Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

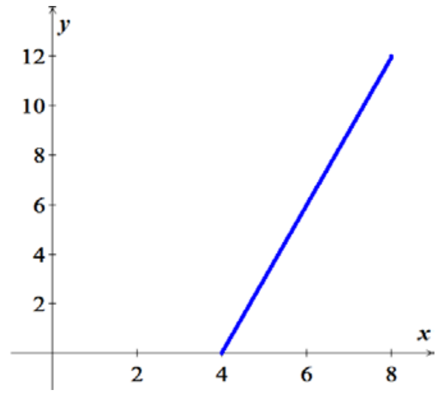
$$x = \sqrt{t} + 4, \quad y = 3\sqrt{t}; \quad 0 \le t \le 16$$

Solution

$\sqrt{t} = \frac{y}{3}$

$x = \sqrt{t} + 4 = \frac{1}{3}y + 4$

$\underline{y = 3(x-4)|}$ (***Line***)



Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

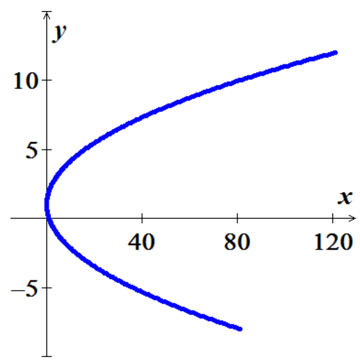
$$x = (t+1)^2, \quad y = t + 2; \quad -10 \le t \le 10$$

Solution

$t = y - 2$

$x = (y-1)^2$

$\underline{= y^2 - 2y + 1|}$



Exercise

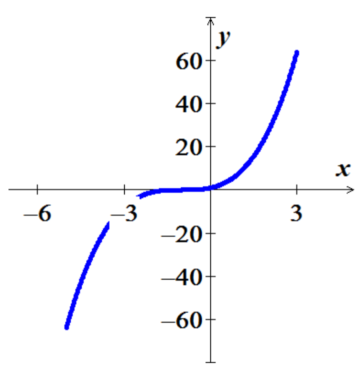
Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = t - 1, \quad y = t^3; \quad -4 \le t \le 4$$

Solution

$t = x + 1$

$\underline{y = (x+1)^3 |}$



Exercise

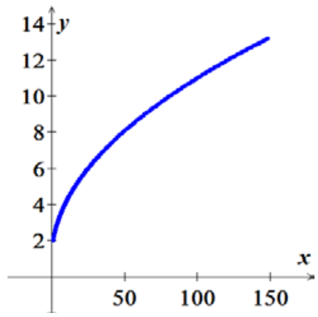
Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = e^{2t}, \quad y = e^t + 1; \quad 0 \le t \le 2.5$$

Solution

$e^t = \sqrt{x}$

$\boxed{y = \sqrt{x} + 1}$



Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = 3\cos t, \quad y = 3\sin t; \quad \pi \le t \le 2\pi$$

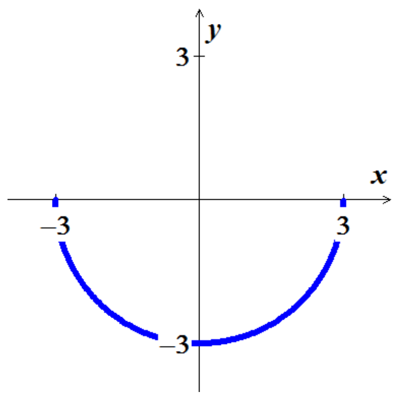
Solution

$\cos t = \frac{x}{3}, \quad \sin t = \frac{y}{3}$

$\cos^2 t + \sin^2 t = 1$

$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$\boxed{x^2 + y^2 = 9} \qquad -3 \le x \le 3 \quad 0 \le y \le 3$



$\therefore$ The equation represents a semi-circle with a center at origin of radius 3.

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

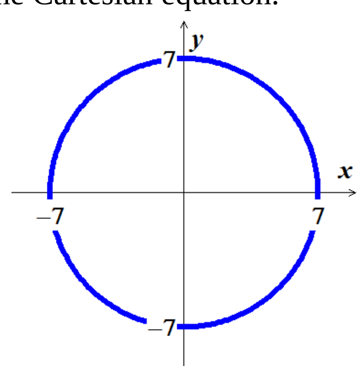
$$x = -7\cos 2t, \quad y = -7\sin 2t; \quad 0 \le t \le \pi$$

Solution

$\cos 2t = -\frac{x}{7}, \quad \sin 2t = -\frac{y}{7}$

$\cos^2 2t + \sin^2 2t = 1$

$\left(-\frac{x}{7}\right)^2 + \left(-\frac{y}{7}\right)^2 = 1$



$$x^2 + y^2 = 49$$

$\therefore$ The equation represents a circle with a center at origin of radius 7.

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = 1 - 3\sin 4\pi t, \quad y = 2 + 3\cos 4\pi t; \quad 0 \le t \le \frac{1}{2}$$

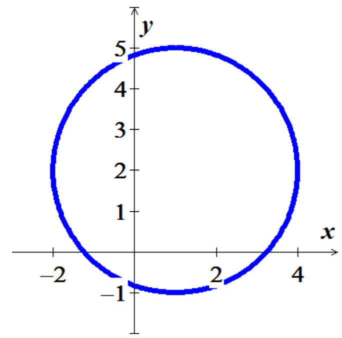
Solution

$$\sin 4\pi t = \frac{1-x}{3}, \quad \cos 4\pi t = \frac{y-2}{3}$$

$$\sin^2 4\pi t + \cos^2 4\pi t = 1$$

$$\left(\frac{1-x}{3}\right)^2 + \left(\frac{y-2}{3}\right)^2 = 1$$

$$(x-1)^2 + (y-2)^2 = 49$$



$\therefore$ The equation represents a circle with a center at $(1,\ 2)$ of radius 7.

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = 2t, \quad y = 3t - 4; \quad -10 \le t \le 10$$

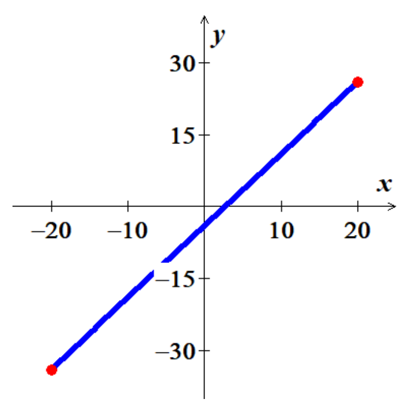
Solution

$$x = 2t \quad \Rightarrow \quad t = \frac{x}{2}$$

$$-10 \le t \le 10 \quad \rightarrow \quad -20 \le x \le 20$$

$$y = 3t - 4$$

$$= \frac{3}{2}x - 4$$



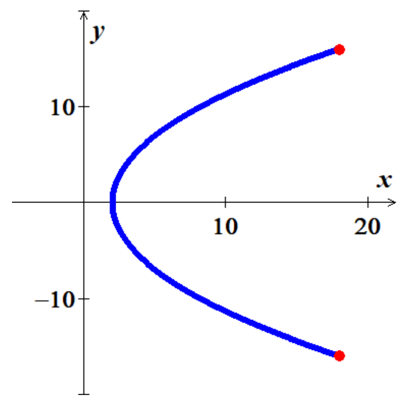
$\therefore$ The equation represents a segment line $(-20 \le x \le 20)$.

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = t^2 + 2, \quad y = 4t; \quad -4 \le t \le 4$$

Solution



$y = 4t \quad \Rightarrow \quad t = \dfrac{y}{4}$

$-4 \le t \le 4 \quad \to \quad -16 \le y \le 16$

$x = t^2 + 2$

$\quad \boxed{= \dfrac{1}{16} y^2 + 2}$

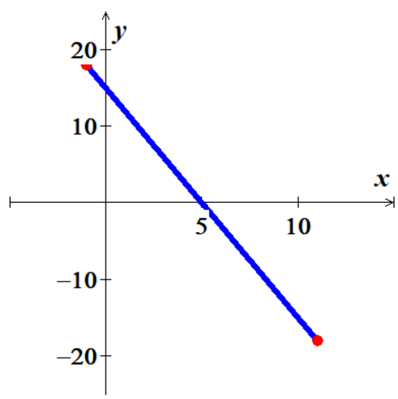
$\therefore$ The equation represents a parabola.

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = -t + 6, \quad y = 3t - 3; \quad -5 \le t \le 5$$

Solution



$x = -t + 6 \quad \Rightarrow \quad t = 6 - x$

$-5 \le t \le 5 \quad \to \quad -1 \le x \le 11$

$y = 3t - 3$

$\quad = 3(6 - x) - 3$

$\quad \boxed{= -3x + 15}$

$\therefore$ The equation represents a segment line $(-1 \le x \le 11)$.

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = t^3 - 1, \quad y = 5t + 1; \quad -3 \le t \le 3$$

Solution

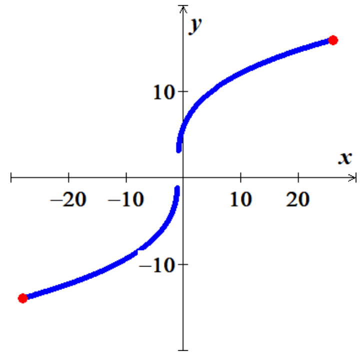
$y = 5t + 1 \quad \Rightarrow \quad t = \dfrac{y - 1}{5}$

$-3 \le t \le 3 \quad \to \quad -14 \le y \le 16$

$-28 \le x \le 26$

$x = t^3 - 1$

$\boxed{= \frac{1}{125}(y-1)^3 - 1}$



Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = \cos t, \quad y = \sin^2 t, \quad 0 \le t \le \pi$$

Solution

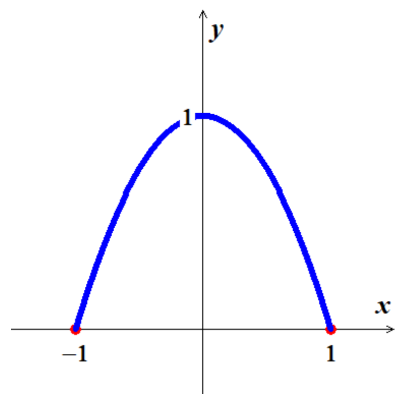
$0 \le t \le \pi \quad \to \quad -1 \le x \le 1$

$\qquad\qquad\qquad\quad 0 \le y \le 1$

$\cos^2 t + \sin^2 t = 1$

$x^2 + y = 1$

$\boxed{y = -x^2 + 1}$



$\therefore$ The equation represents a parabola.

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = 1 - \sin^2 t, \quad y = \cos t, \quad \pi \le t \le 2\pi$$

Solution

$x = 1 - \sin^2 t \quad \Rightarrow \quad \sin^2 t = 1 - x$

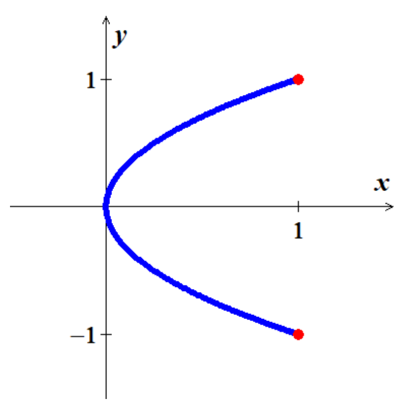
$\pi \le t \le 2\pi \quad \to \quad 0 \le x \le 1$

$\qquad\qquad\qquad\quad -1 \le y \le 1$

$\cos^2 t + \sin^2 t = 1$

$y^2 + 1 - x = 1$

$\boxed{x = y^2}$

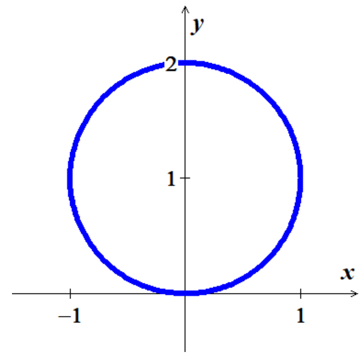


$\therefore$ The equation represents a parabola.

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = \cos t, \quad y = 1 + \sin t; \quad 0 \le t \le 2\pi$$



Solution

$y = 1 + \sin t \quad \Rightarrow \quad \sin t = y - 1$

$0 \le t \le 2\pi \quad \to \quad -1 \le x \le 1$

$\qquad\qquad\qquad 0 \le y \le 2$

$\cos^2 t + \sin^2 t = 1$

$\boxed{x^2 + (y-1)^2 = 1}$

$\therefore$ Circle centered at $(0,\ 1)$ with radius 1.

Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = 2\sin t - 3, \quad y = 5 + \cos 2t; \quad 0 \le t \le 2\pi$$

Solution

$x = 2\sin t - 3 \quad \to \quad \sin t = \frac{1}{2}(x+3)$

$y = 5 + \cos 2t$

$\quad = 5 + 2\cos^2 t - 1$

$\quad = 4 + 2\cos^2 t$

$\cos^2 t = \frac{1}{2}(y-4)$

$0 \le t \le 2\pi \quad \to \quad -5 \le x \le -1$

$\qquad\qquad\qquad 4 \le y \le 5$

$\cos^2 t + \sin^2 t = 1$

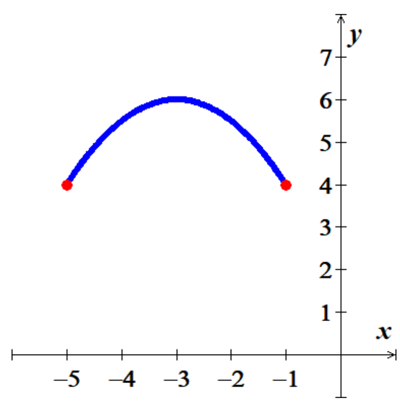
$\frac{1}{2}(y-4) + \frac{1}{4}(x+3)^2 = 1$

$2y - 8 + (x+3)^2 = 4$

$2y + x^2 + 6x + 9 = 12$

$2y = -x^2 - 6x + 3$

$\boxed{y = -\frac{1}{2}x^2 + -3x + \frac{3}{2}}$



Exercise

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

$$x = 4\cos t - 1, \quad y = 4\sin t + 2; \quad 0 \le t \le 2\pi$$

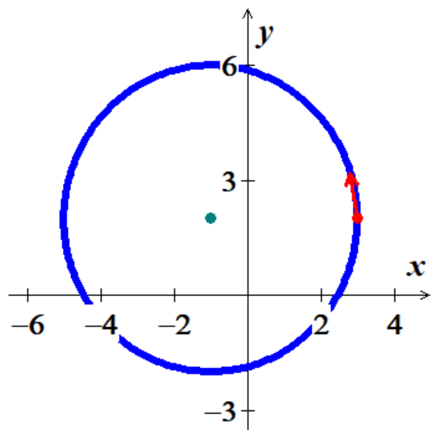
Solution

$x = 4\cos t - 1 \quad \Rightarrow \quad \cos t = \dfrac{x+1}{4}$

$y = 4\sin t + 2 \quad \Rightarrow \quad \sin t = \dfrac{y-2}{4}$

$0 \le t \le 2\pi \quad \to \quad -5 \le x \le 3$

$\qquad\qquad\qquad\quad -2 \le y \le 6$



$\cos^2 t + \sin^2 t = 1$

$\left(\dfrac{x+1}{4}\right)^2 + \left(\dfrac{y-2}{4}\right)^2 = 1$

$\dfrac{(x+1)^2}{16} + \dfrac{(y-2)^2}{16} = 1$

$\boxed{(x+1)^2 + (y-2)^2 = 16}$

$\therefore$ Circle centered at $(-1,\ 2)$ with radius 4.

Exercise

For the given parametric equations: $\quad x = t^2 + 4, \quad y = 6 - t; \quad -\infty \le t \le \infty \quad (5,\ 5)$

a) Plot the following curve, indicating the positive orientation.

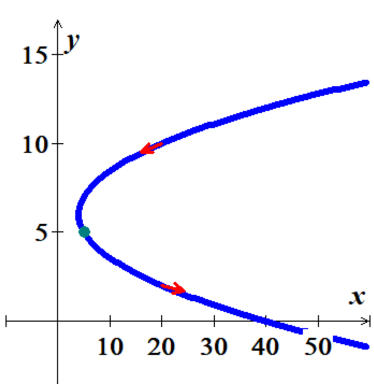
b) Eliminate the parameter to obtain an equation in x and y.

c) Identify or briefly describe the curve.

d) Evaluate $\dfrac{dy}{dx}$ at the specified point.

Solution

a)



b) $y = 6 - t \quad \Rightarrow \quad t = 6 - y$

$x = t^2 + 4$

$= (6 - y)^2 + 4$

$\boxed{= y^2 - 12y + 40}$

c) $x' = 2y - 12 = 0$

$y = 6 \quad \rightarrow \quad x = 4$

The curve is parabola with vertex at $(4,\ 6)$

d) $x = y^2 - 12y + 40$

$1 = 2yy' - 12y'$

$(2y - 12)\, y' = 1$

$\dfrac{dy}{dx} = \dfrac{1}{2y - 12} \Big|_{(5,\ 5)}$

$= \dfrac{1}{10 - 12}$

$\boxed{= -\dfrac{1}{2}}$

Exercise

For the given parametric equations: $x = e^t, \quad y = 3e^{-2t}; \quad -\infty \le t \le \infty \quad (1,\ 3)$

a) Plot the following curve, indicating the positive orientation.

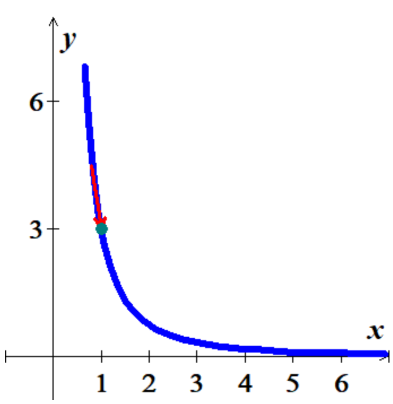
b) Eliminate the parameter to obtain an equation in x and y.

c) Identify or briefly describe the curve.

d) Evaluate $\dfrac{dy}{dx}$ at the specified point.

Solution

a)



b) $x = e^t \quad \Rightarrow \quad t = \ln x$

$$y = 3e^{-2t}$$
$$= 3e^{-2\ln x}$$
$$= 3e^{\ln x^{-2}}$$
$$= 3x^{-2}$$
$$\boxed{= \frac{3}{x^2}}$$

c) The curve represents the proportional of the function $f(x) = \frac{3}{x^2}$ with $x > 0$

d) $y = \frac{3}{x^2}$

$$\frac{dy}{dx} = -\frac{6}{x^3}\Big|_{(1,\ 3)}$$
$$\boxed{= -6}$$

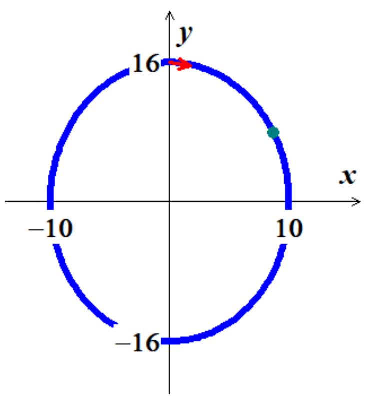
Exercise

For the given parametric equations: $x = 10\sin 2t, \quad y = 16\cos 2t; \quad 0 \le t \le \pi \quad \left(5\sqrt{3},\ 8\right)$

a) Plot the following curve, indicating the positive orientation.
b) Eliminate the parameter to obtain an equation in x and y.
c) Identify or briefly describe the curve.
d) Evaluate $\frac{dy}{dx}$ at the specified point.

Solution

a)



b) $x = 10\sin 2t \quad \Rightarrow \quad \sin 2t = \frac{x}{10}$

$$y = 16\cos 2t \quad \Rightarrow \quad \cos 2t = \frac{y}{16}$$

$$\cos^2 2t + \sin^2 2t = 1$$

$$\left(\frac{x}{10}\right)^2 + \left(\frac{y}{16}\right)^2 = 1$$

$$\boxed{\frac{x^2}{100} + \frac{y^2}{256} = 1}$$

c) The curve represents an ellipse traced clockwise.

d) $\frac{x^2}{100} + \frac{y^2}{256} = 1$

$$\frac{x}{50} + \frac{yy'}{128} = 0$$

$$\frac{yy'}{128} = -\frac{x}{50}$$

$$\frac{dy}{dx} = -\frac{64}{25}\frac{x}{y}\Big|_{(5\sqrt{3},\ 8)}$$

$$\boxed{= -\frac{8\sqrt{3}}{5}}$$

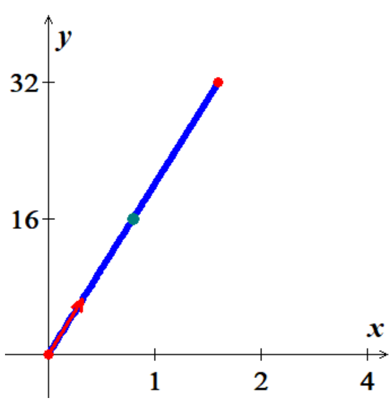
Exercise

For the given parametric equations: $x = \ln t, \quad y = 8\ln t^2; \quad 1 \le t \le e^2 \quad (1,\ 16)$

a) Plot the following curve, indicating the positive orientation.
b) Eliminate the parameter to obtain an equation in x and y.
c) Identify or briefly describe the curve.
d) Evaluate $\frac{dy}{dx}$ at the specified point.

Solution

a)



b) $x = \ln t \quad \Rightarrow \quad t = e^x$

$y = 8\ln t^2$

$$= 8\ln\left(e^{x}\right)^{2}$$

$$\underline{= 16x}$$

c) The curve represents a line from point $(0,\ 0)$ to $(2,\ 32)$.

d) $y = 16x$

$$\frac{dy}{dx} = 16 \Big|_{(1,\ 16)}$$

$$\underline{= 16} \qquad (\forall x)$$

Exercise

Find parametric equation for the left half of the parabola $y = x^2 + 1$, originating at $(0,\ 1)$

Solution

Let $x(t) = -t \quad \to y(t) = t^2 + 1 \qquad$ for $0 \le t \le \infty$

Exercise

Find parametric equation for the line that passes through the points $(1,\ 1)$ and $(3,\ 5)$, oriented in the direction of increasing *x*.

Solution

$$y = \frac{5-1}{3-1}(x-1) + 1 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$\underline{= 2x - 1}$$

$$x = 1 + (3-1)t = 1 + 2t$$

$$y = 1 + (5-1)t \ \underline{= 1 + 4t}$$

$$\begin{cases} x(t) = 1 + 2t \\ y(t) = 1 + 4t \end{cases} \quad -\infty < t < \infty$$

Exercise

Find parametric equation for the lower half of the circle centered at $(-2,\ 2)$ with radius 6, oriented in the counterclockwise direction.

Solution

$$(x+2)^2+(y-2)^2=36$$

$$\begin{cases} x+2=-6\cos t \\ y-2=-6\sin t \end{cases}$$

$(-)$ since it oriented in *ccw* direction and lower half, therefore, *y*-value has to be negative.

$$\begin{cases} x(t)=-2-6\cos t \\ y(t)=2-6\sin t \end{cases} \quad 0 \le t \le \pi$$

Exercise

Find parametric equation for the upper half of the parabola $x=y^2$, originating at $(0,\ 0)$

Solution

Let $y=t \quad \Rightarrow \ x=t^2 \quad 0 \le t < \infty$

Exercise

Find parametric equation for a circle centered at the origin with radius 4, generated counterclockwise.

Solution

$$x^2+y^2=4^2$$

$$\begin{cases} x(t)=4\cos t \\ y(t)=4\sin t \end{cases} \quad 0 \le t \le 2\pi$$

Exercise

Find parametric equation for a circle centered at the origin with radius 12, generated clockwise with initial point $(0,\ 12)$

Solution

$$x^2+y^2=144$$

$$\begin{cases} x(t)=12\cos t \\ y(t)=12\sin t \end{cases} \quad \frac{\pi}{2} \le t \le \frac{5\pi}{2}$$

Exercise

Find parametric equation for a circle centered at $(2,\ 3)$ with radius 1, generated counterclockwise.

Solution

$$(x-2)^2+(y-3)^2=1$$

$$\begin{cases} x-2=\cos t \\ y-3=\sin t \end{cases}$$

$$\begin{cases} x(t)=\cos t+2 \\ y(t)=\sin t+3 \end{cases} \quad 0\le t\le 2\pi$$

Exercise

Find parametric equation for a circle centered at $(2,\ 0)$ with radius 3, generated clockwise.

Solution

$$(x-2)^2+y^2=9$$

$$\begin{cases} x-2=-3\cos t \\ \quad y=-3\sin t \end{cases}$$

$$\begin{cases} x(t)=-3\cos t+2 \\ \quad y(t)=-3\sin t \end{cases} \quad 0\le t\le 2\pi$$

Exercise

Find parametric equation for a circle centered at $(-2,\ -3)$ with radius 8, generated clockwise.

Solution

$$(x+2)^2+(y+3)^2=64$$

$$\begin{cases} x+2=-8\cos t \\ y+3=-8\sin t \end{cases}$$

$$\begin{cases} x(t)=-8\cos t-2 \\ y(t)=-8\sin t-3 \end{cases} \quad 0\le t\le 2\pi$$

Exercise

Find parametric equation for the circle $x^2 + y^2 = 9$, generated clockwise.

Solution

$$\begin{cases} x(t) = 3\cos t \\ y(t) = -3\sin t \end{cases} \quad 0 \le t \le 2\pi$$

Exercise

Find parametric equation for the upper half of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, generated counterclockwise.

Solution

$$\begin{cases} x(t) = 3\cos t \\ y(t) = 2\sin t \end{cases} \quad 0 \le t \le \pi$$

Exercise

Find parametric equation for the right side of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, generated counterclockwise.

Solution

$$\begin{cases} x(t) = 3\cos t \\ y(t) = 2\sin t \end{cases} \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

Exercise

Find parametric equation for the line $y = 4x + 11$

Solution

Let $x = t$

$y = 4t + 11$

$$\begin{cases} x(t) = t \\ y(t) = 4t + 11 \end{cases} \quad -\infty \le t \le \infty$$

Exercise

Find parametric equation for the line segment from $P(-1,\ 0)$ to $Q(1,\ 1)$ and the line segment from Q to P.

Solution

Line segment from $P(-1,\ 0)$ to $Q(1,\ 1)$

$$y=\frac{1-0}{1+1}(x+1)$$

$$=\frac{1}{2}x+\frac{1}{2}$$

$$\underline{2y-x=1}$$

Let $y=t$

$x=2t-1$

$$\begin{cases} x(t)=2t-1 \\ y(t)=t \end{cases} \quad 0\le t\le 1$$

Line segment from $Q(1,\ 1)$ to $P(-1,\ 0)$

$$\begin{aligned}(x(t),\ y(t)) &= tP+(1-t)Q \\ &= t(-1,\ 0)+(1-t)(1,\ 1) \\ &= (-t,\ 0)+(1-t,\ 1-t) \\ &= \underline{(1-2t,\ 1-t)}\end{aligned}$$

$$\begin{cases} x(t)=1-2t \\ y(t)=1-t \end{cases} \quad 0\le t\le 1$$

Exercise

Find parametric equation for the segment of the curve $f(x)=x^3+2x$ from $(0,\ 0)$ to $(2,\ 12)$

Solution

Let $x=t$

$$\begin{cases} x=0=t \\ x=2=t \end{cases} \rightarrow \quad 0\le t\le 2$$

$$y=x^3+2x$$

$$\underline{=t^3+2t}$$

$$\begin{cases} x(t)=t \\ y(t)=t^3+2t \end{cases} \quad 0\le t\le 2$$

Exercise

What is the relationship among a, b, c, and d such that the equations

$$\begin{cases} x(t) = a\cos t + b\sin t \\ y(t) = c\cos t + d\sin t \end{cases}$$ describe a circle?

What is the radius of the circle?

Solution

$$x^2 = (a\cos t + b\sin t)^2$$
$$= a^2\cos^2 t + 2ab\cos t\sin t + b^2\sin^2 t$$
$$y^2 = (c\cos t + d\sin t)^2$$
$$= c^2\cos^2 t + 2cd\cos t\sin t + d^2\sin^2 t$$

$$x^2 + y^2 = a^2\cos^2 t + 2ab\cos t\sin t + b^2\sin^2 t + c^2\cos^2 t + 2cd\cos t\sin t + d^2\sin^2 t$$
$$= \left(a^2 + c^2\right)\cos^2 t + 2(ab + cd)\cos t\sin t + \left(b^2 + d^2\right)\sin^2 t$$
$$xy = (a\cos t + b\sin t)(c\cos t + d\sin t)$$
$$= ac\cos^2 t + (ad + bc)\cos t\sin t + bd\sin^2 t$$

If $ad + bc = 0$

$bc = -ad$

$\frac{b}{a} = -\frac{c}{d}$

$$x(t) = a\cos t + b\sin t = A\cos(t - \varphi)$$

Where $A = \sqrt{a^2 + b^2}$

$\varphi = \tan^{-1}\left(\frac{b}{a}\right)$

$$y(t) = c\cos t + d\sin t = A\sin(t - \varphi)$$

Where $A = \sqrt{c^2 + d^2}$

$\varphi = \tan^{-1}\left(-\frac{c}{d}\right) = \tan^{-1}\left(\frac{b}{a}\right)$

$$x^2 = A^2\cos^2(t - \varphi) \quad \rightarrow \quad \cos^2\left(t - \frac{b}{a}\right) = \frac{x^2}{A^2}$$

$$y^2 = A^2\sin^2(t - \varphi) \quad \rightarrow \quad \sin^2\left(t - \frac{b}{a}\right) = \frac{y^2}{A^2}$$

$$\cos^2\left(t - \frac{b}{a}\right) + \sin^2\left(t - \frac{b}{a}\right) = 1$$

$$\frac{x^2}{A^2}+\frac{y^2}{A^2}=1$$

$$x^2+y^2=A^2$$

The radius of the circle:

$$r=A$$

$$\boxed{=\sqrt{a^2+b^2}}$$

Exercise

Find parametric equations (not unique) of an ellipse centered at the origin with major axis of length 6 on the *x-axis* and minor axis of length 3 on the *y-axis*, generated counterclockwise. Graph the ellipse and find a description in terms of *x* and *y*.

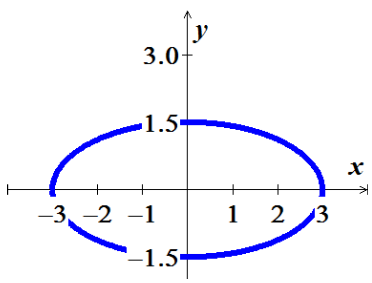
Solution

$$a=3 \quad b=\frac{3}{2}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (Ellipse)$$

$$\frac{x^2}{9}+\frac{4y^2}{9}=1$$

$$\begin{cases} x=3\cos t \\ y=\frac{3}{2}\sin t \end{cases} \quad 0\le t\le 2\pi$$



Exercise

Find parametric equations (not unique) of an ellipse centered at the origin with major axis of length 12 on the *x-axis* and minor axis of length 2 on the *y-axis*, generated clockwise. Graph the ellipse and find a description in terms of *x* and *y*.

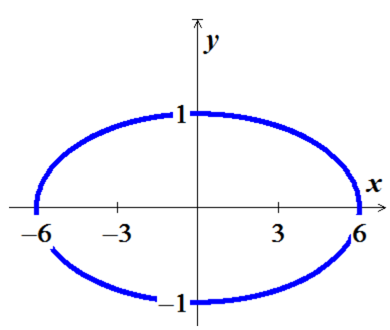
Solution

$$a=6 \quad b=1$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (Ellipse)$$

$$\frac{x^2}{36}+y^2=1$$

$$\begin{cases} x=6\cos t \\ y=-\sin t \end{cases} \quad 0\le t\le 2\pi \quad cw$$

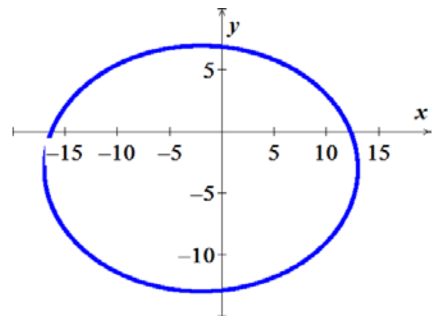


Exercise

Find parametric equations (not unique) of an ellipse centered at $(-2,\ -3)$ with major and minor axes of lengths 30 and 20, parallel to the *x-axis* and *y-axis*, respectively. Graph the ellipse and find a description in terms of *x* and *y*.

Solution

$a = 15 \quad b = 10$



$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x+2)^2}{15^2} + \frac{(y+3)^2}{100} = 1$$

$$\begin{cases} x = -2 + 15\cos t \\ y = -3 + 10\sin t \end{cases} \quad 0 \le t \le 2\pi \quad cw$$

Exercise

Find a parametric equations and a parameter interval for the motion of a particle starting at the point (2, 0) and tracing the top half of the circle $x^2 + y^2 = 4$ four times.

Solution

The top half of the circle: $y \ge 0$

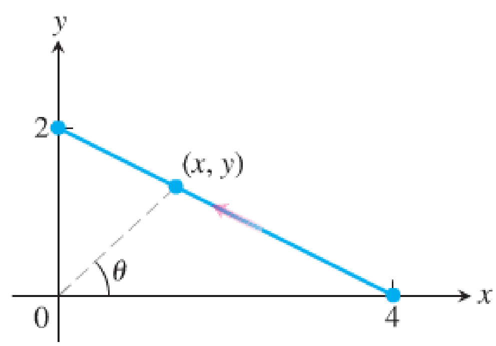
$$x = 2\cos t, \quad y = 2|\sin t|, \quad 0 \le t \le \frac{\pi}{4}$$

Exercise

Find a parametrization for the line segment joining points (0,2) and (4, 0) using the angle θ in the accompanying figure as the parameter.

Solution

$$\tan\theta = \frac{y}{x} \quad \Rightarrow \quad y = x\tan\theta$$



Slope: $m = \dfrac{0-2}{4-0}$

$$= -\frac{1}{2}$$

The equation of the line passing thru (0,2) and (4, 0):

$$y = -\tfrac{1}{2}(x-4)$$

$$y = -\frac{1}{2}x + 2$$

$$x\tan\theta = -\frac{1}{2}x + 2$$

$$x\tan\theta + \frac{1}{2}x = 2$$

$$x(2\tan\theta + 1) = 4$$

$$x = \frac{4}{1 + 2\tan\theta}$$

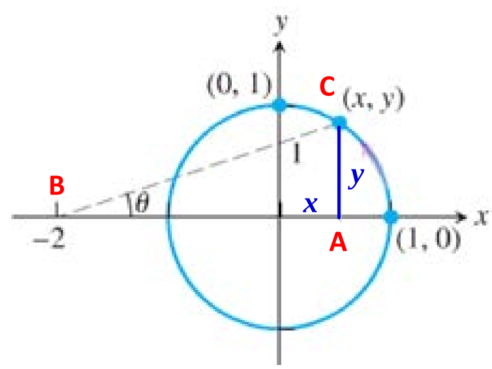
$$y = x\tan\theta$$

$$= \frac{4\tan\theta}{1 + 2\tan\theta} \qquad 0 \le \theta < \frac{\pi}{2}$$

Exercise

Find a parametrization for the circle $x^2 + y^2 = 1$ starting at (1, 0) and moving counterclockwise to the terminal point (0, 1), using the angle θ in the accompanying figure as the parameter.

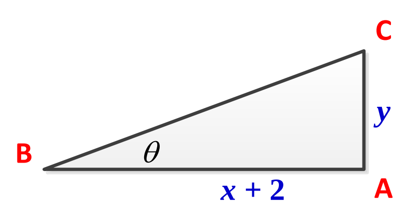
Solution



$$\tan\theta = \frac{y}{x + 2}$$

$$y = (x + 2)\tan\theta$$

The equation of the circle is given by: $x^2 + y^2 = 1$



$$x^2 + (x + 2)^2\tan^2\theta = 1$$

$$x^2 + \left(x^2 + 4x + 4\right)\tan^2\theta = 1$$

$$\left(1 + \tan^2\theta\right)x^2 + 4\tan^2\theta x + 4\tan^2\theta - 1 = 0$$

$$\sec^2\theta x^2 + 4\tan^2\theta x + 4\tan^2\theta - 1 = 0$$

$$x = \frac{-4\tan^2\theta \pm \sqrt{16\tan^4\theta - 4\sec^2\theta\left(4\tan^2\theta - 1\right)}}{2\sec^2\theta}$$

$$= \frac{-4\tan^2\theta \pm \sqrt{16\tan^4\theta - 4\left(1 + \tan^2\theta\right)\left(4\tan^2\theta - 1\right)}}{2\sec^2\theta}$$

$$= \frac{-4\tan^2\theta \pm \sqrt{16\tan^4\theta - 16\tan^2\theta + 4 - 16\tan^4\theta + 4\tan^2\theta}}{2\sec^2\theta}$$

$$= \frac{-4\tan^2\theta \pm \sqrt{4 - 12\tan^2\theta}}{2\sec^2\theta}$$

$$= \frac{-4\tan^2\theta \pm 2\sqrt{1 - 3\tan^2\theta}}{2\sec^2\theta}$$

$$= \frac{-2\tan^2\theta \pm \sqrt{1 - 3\tan^2\theta}}{\sec^2\theta}$$

$$= -2\frac{\tan^2\theta}{\sec^2\theta} \pm \cos^2\theta\,\sqrt{1 - 3\tan^2\theta}$$

$$= -2\sin^2\theta \pm \cos\theta\,\sqrt{\cos^2\theta - 3\sin^2\theta}$$

$$= -2\left(1 - \cos^2\theta\right) \pm \cos\theta\,\sqrt{\cos^2\theta - 3\left(1 - \cos^2\theta\right)}$$

$$\underline{= 2\cos^2\theta - 2 \pm \cos\theta\,\sqrt{4\cos^2\theta - 3}}$$

$$y = \left(2\cos^2\theta - 2 \pm \cos\theta\,\sqrt{4\cos^2\theta - 3} + 2\right)\tan\theta$$

$$= \left(2\cos^2\theta \pm \cos\theta\,\sqrt{4\cos^2\theta - 3}\right)\tan\theta$$

$$= 2\cos^2\theta\frac{\sin\theta}{\cos\theta} \pm \cos\theta\frac{\sin\theta}{\cos\theta}\,\sqrt{4\cos^2\theta - 3}$$

$$\underline{= 2\cos\theta\sin\theta \pm \sin\theta\,\sqrt{4\cos^2\theta - 3}}$$

At the point (0, 1):

$$y = (x + 2)\tan\theta$$

$$1 = 2\tan\theta$$

$$\tan\theta = \frac{1}{2}$$

$$\underline{\theta = \tan^{-1}\frac{1}{2}}$$

Exercise

A common task is to parameterize curves given either by either Cartesian equations or by graphs. Find a parametric representation of the following curves.

a) The segment of the parabola $y = 9 - x^2$, *for* $-1 \le x \le 3$

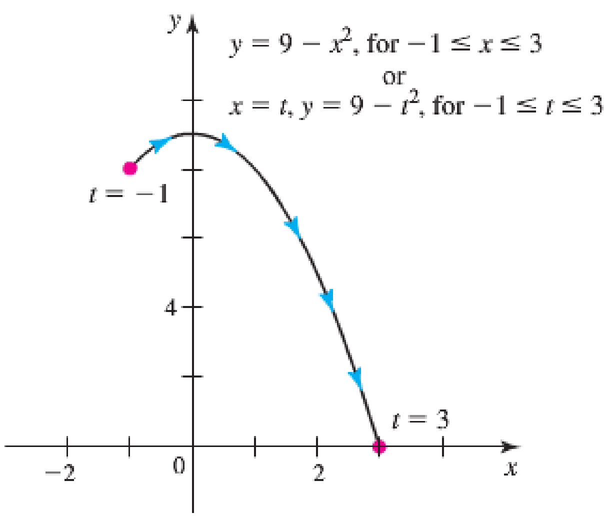
b) The complete curve $x = (y-5)^2 + \sqrt{y}$

c) The piecewise linear path that connects $P(-2,\ 0)$ to $Q(0,\ 3)$ to $R(4,\ 0)$ (in that order), where the parameter varies over the interval $0 \le t \le 2$

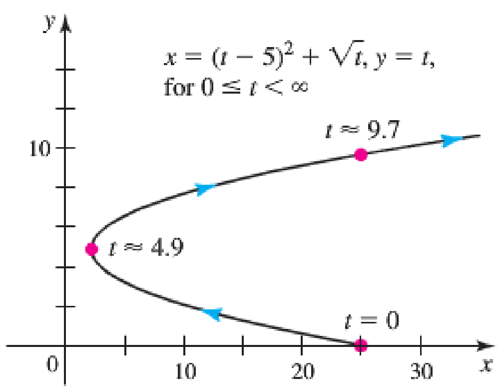
Solution

a) Let $x = t \quad \Rightarrow \quad y = 9 - t^2 \qquad$ for $-1 \le t \le 3$

Which represents a parabola



b) Let $y = t \quad \Rightarrow \quad x = (t-5)^2 + \sqrt{t}$



c) The path consists of 2–line segments that can be parameterized separately. $y = m(x - x_0) + y_0$

The line segment *PQ*:

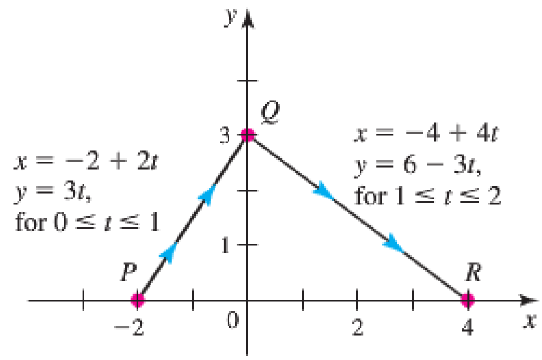
$P(-2,\ 0)\ \ Q(0,\ 3)$

$$y = \frac{3}{2}(x+2)$$

$$= \frac{3}{2}x + 3$$

$$2y - 3x = 6$$

$$\begin{cases} x = 2t - 2 \\ y = 3t \end{cases} \quad for \;\; 0 \le t \le 1$$



The line segment *QR*:

$Q(0,\ 3)\ \ R(4,\ 0)$

$$y = \frac{-3}{4}(x - 4)$$

$$= -\frac{3}{4}x + 4$$

$$4y + 3x = 16$$

$$\begin{cases} x = 4t - 4 \\ y = -3t + 6 \end{cases} \qquad for \;\; 1 \le t \le 2$$

Exercise

A projectile launched from the ground with an initial speed of 20 *m/s* and a launch angle θ follows a trajectory approximated by

$$x = (20\cos\theta)t, \quad y = -4.9t^2 + (20\sin\theta)t$$

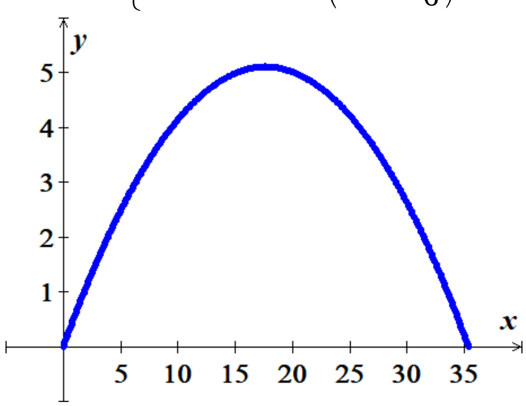
Where *x* and *y* are the horizontal and vertical positions of the projectile relative to the launch point $(0,\ 0)$.

a) Graph the trajectory for various of θ in the range $0 < \theta < \frac{\pi}{2}$.

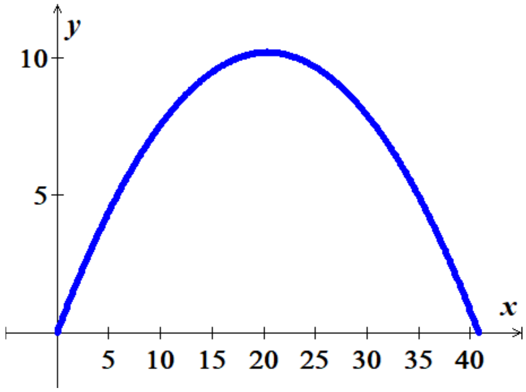
b) Based on your observations, what value of θ gives the greatest range (the horizontal distance between the launch and landing points)?

Solution

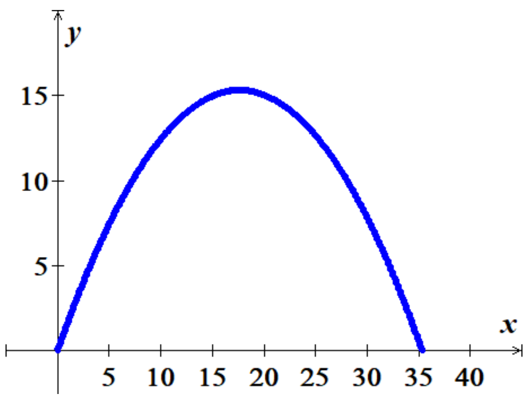
a) *At* $\theta = \frac{\pi}{6} \quad \to \quad \begin{cases} x = \left(20\cos\frac{\pi}{6}\right)t \ \underline{= 10\sqrt{3}\ t} \\ y = -4.9t^2 + \left(20\sin\frac{\pi}{6}\right)t \ \underline{= -4.9t^2 + 10t} \end{cases}$



$$At\ \theta = \frac{\pi}{4} \quad \rightarrow \quad \begin{cases} x = 10\sqrt{2}\ t \\ y = -4.9t^2 + 10\sqrt{2}\ t \end{cases}$$



$$At\ \theta = \frac{\pi}{3} \quad \rightarrow \quad \begin{cases} x = 10\ t \\ y = -4.9t^2 + 10\sqrt{3}\ t \end{cases}$$



b) The greatest range occurs when $\theta = \frac{\pi}{4}$

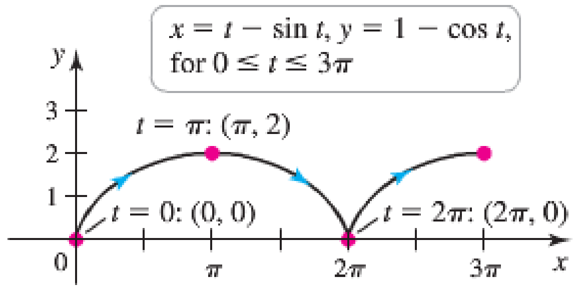
Exercise

Many fascinating curves are generated by points on rolling wheels. The path of a light on the rim of a rolling when is a cycloid, which has the parametric equations

$$x = a(t - \sin t), \quad y = a(1 - \cos t), \quad for \quad t \geq 0$$

Where $a > 0$. Graph the cycloid with $a = 1$. On what interval does the parameter generate one arch of the cycloid?

Solution



The wheel completes one full revolution on the interval $0 \le t \le 3\pi$, which gives one arch of the cycloid.

Exercise

Find parametric equations that describe the circular path of the objects. Assume (x, y) denotes the position of the object relative to the origin at the center of the circle.
A go-cart moves counterclockwise with constant speed around a circular track of radius 400 *m*, completing in 1.5 *min*.

Solution

Let t be time in minute, so $0 \le t \le 1.5$

$$1.5\theta = \frac{3}{2}\theta = 2\pi t \quad \to \theta = \frac{4\pi}{3}t$$

$$\text{Since } \begin{cases} x = r\cos\theta & \to x = 400\cos\left(\frac{4\pi}{3}t\right) \\ y = r\sin\theta & \quad\ y = 400\sin\left(\frac{4\pi}{3}t\right) \end{cases}$$

$$\Rightarrow x^2 + y^2 = 400^2$$

The path is a circle of radius 400, center at origin and the circle is traversed counterclockwise.

Exercise

Find parametric equations that describe the circular path of the objects. Assume (x, y) denotes the position of the object relative to the origin at the center of the circle.
The tip of the 15-*in* second hand of a clock completes one revolution in 60 *sec*.

Solution

Let t be time in second, so $0 \le t \le 60$

$$60\theta = 2\pi t \quad \to \theta = \frac{\pi}{30}t$$

$$\text{Since } \begin{cases} x = r\cos\theta & \to x = 15\cos\left(\frac{\pi}{30}t\right) \\ y = r\sin\theta & \quad\ y = 15\sin\left(\frac{\pi}{30}t\right) \end{cases}$$

$$\cos^2\left(\frac{\pi}{30}t\right) + \sin^2\left(\frac{\pi}{30}t\right) = 1$$

$$\left(\frac{x}{15}\right)^2 + \left(\frac{y}{15}\right)^2 = 1$$

$$x^2 + y^2 = 15^2$$

The path is a circle of radius 15, center at origin and the circle is traversed clockwise.

Exercise

Find parametric equations that describe the circular path of the objects. Assume $(x,\ y)$ denotes the position of the object relative to the origin at the center of the circle.

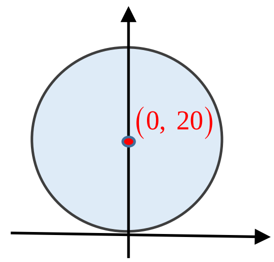
A Ferris wheel has a radius of 20 *m* and completes a revolution in the clockwise direction at constant speed in 3 *min*. Assume that *x* and *y* measure the horizontal and vertical positions of a seat on the Ferris wheel relative to a coordinate system whose origin is at the low point of the wheel. Assume the seat begins moving at the origin.

Solution

Let *t* be time in minute, so $0 \le t \le 3$

Since the low point is the origin, the circle has its center at $(0,\ 20)$ and a radius of 20.

$$3\theta = 2\pi t \quad \to \theta = \frac{2\pi}{3}t$$



$$\text{Since } \begin{cases} x = r\cos\theta & \to x = -20\cos\left(\frac{2\pi}{3}t\right) \\ y = r\sin\theta & y = 20 - 20\sin\left(\frac{2\pi}{3}t\right) \end{cases}$$

$$\Rightarrow x^2 + (y-20)^2 = 20^2$$

The path is a circle of radius 20, center at $(0,\ 20)$.

Exercise

A plane traveling horizontally at 80 *m/s* over flat ground at an elevation of 3000 *m* releases an emergency packet. The trajectory of the packet is given by

$$x = 80t, \quad y = -4.9t^2 + 3000, \quad for \ \ t \ge 0$$

Where the origin is the point on the ground directly beneath the plane at the moment of the release. Graph the trajectory of the packet and find the coordinates of the point where the packet lands.

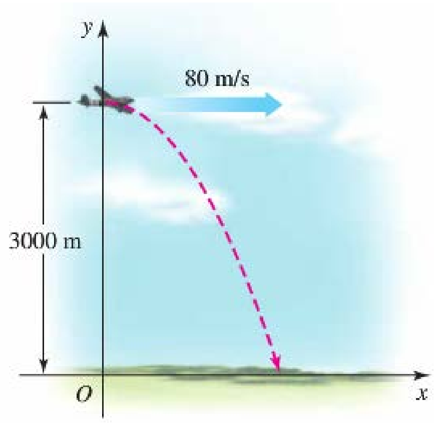
Solution

The packet hits ground when:

$$y = 0 = -4.9t^2 + 3000$$

$$t = \sqrt{\frac{3000}{4.9}}$$

$$\approx 24.744 \ \text{sec}$$



And $x = 80t$

$$\approx 80(24.744)$$

$$\approx 1979.487 \ m$$

Exercise

A plane traveling horizontally at 100 *m/s* over flat ground at an elevation of 4,000 *m* must drop an emergency packet on a target on the ground. The trajectory of the packet is given by

$$x(t) = 100t, \quad y(t) = -4.9t^2 + 4{,}000, \quad for \;\; t \ge 0$$

Where the origin is the point on the ground directly beneath the plane at the moment of the release. How many horizontal meters before the target should the packet be released in order to hit the target?

Solution

$y(t) = 0$ when the packet hits the ground

$$y(t) = -4.9t^2 + 4{,}000 = 0$$

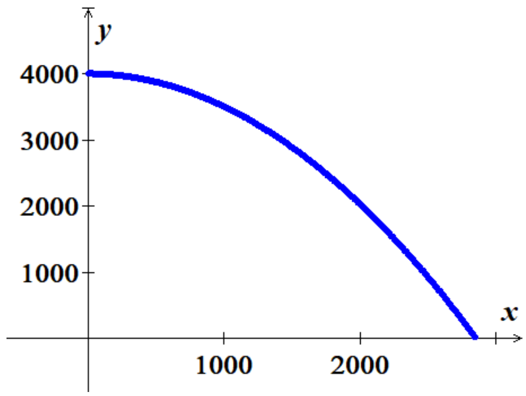
$$-4.9t^2 = -4{,}000$$

$$t^2 = \frac{40{,}000}{49}$$

$$t = \sqrt{\frac{40{,}000}{49}}$$

$$= \frac{200}{7}$$

$$\approx 28.57 \;\; sec$$



$$x\left(\frac{200}{7}\right) = 100\left(\frac{200}{7}\right)$$

$$\approx 2857 \;\; m$$

Exercise

The path of a point on circle *A* with radius $\frac{a}{4}$ that rolls on the inside of circle *B* with a radius *a* is an asteroid or hypocycloid. Its parametric equations are

$$x = a\cos^3 t, \quad y = a\sin^3 t, \quad 0 \le t \le 2\pi$$

Graph the asteroid with $a = 1$ and find its equation in terms of *x* and *y*.

Solution

$$x = a\cos^3 t \quad \Rightarrow \quad \cos t = \left(\frac{x}{a}\right)^{1/3}$$

$$y = a\sin^3 t \quad \Rightarrow \quad \sin t = \left(\frac{y}{a}\right)^{1/3}$$

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{a}\right)^{2/3}=1$$

$$x^{2/3}+y^{2/3}=1 \qquad (a=1)$$