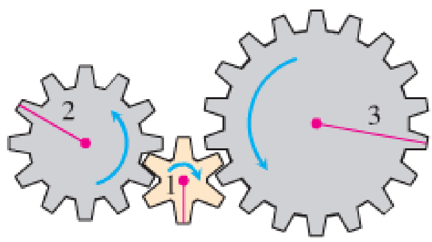
Section 2.6 – Chain Rule

Derivative of a Composite Function

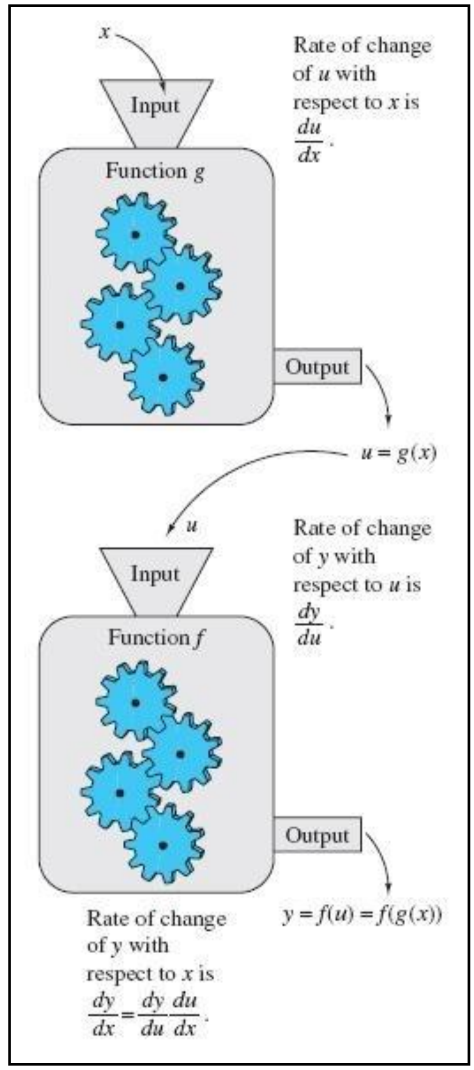
$$y = f\left(g(x)\right) = f(u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

$$\frac{dy}{dx} = f'[g(x)] \bullet g'(x)$$



C: y turns B: u turns A: x turns



Example

Find the derivative of $y = \left(3x^2 + 1\right)^2$

Solution

$$u = 3x^2 + 1 \quad \Rightarrow \quad (u)' = 6x$$

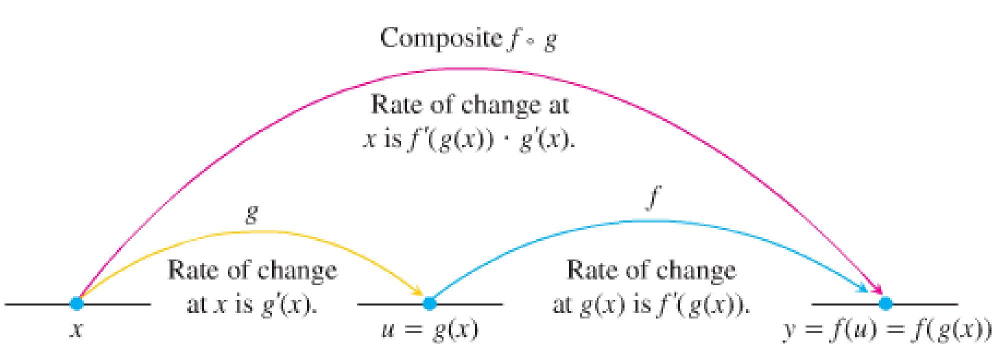
$$\frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 6x$$

$$= 2\left(3x^2 + 1\right) \cdot 6x$$

$$\underline{= 36x^3 + 12x}$$

Calculating from the expand formula: $y = \left(3x^2 + 1\right)^2 = 9x^4 + 6x^2 + 1$

$$\underline{y' = 36x^3 + 12x}$$



Intuitive "*Proof*" of the Chain Rule

Let Δu be the change in u when x changes by Δx, so that

$$\Delta u = g(x+\Delta x) - g(x)$$

Let Δy be the change in y when u changes by Δu, so that

$$\Delta y = f(u+\Delta u) - f(u)$$

If $\Delta u \neq 0 \Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \frac{dy}{du} \cdot \frac{du}{dx}
\end{aligned}$$

Example

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2+1)$. Find the velocity of the object as a function of t.

Solution

Let: $u = t^2+1 \quad \Rightarrow \quad u' = 2t$

$x = \cos(u) \;\Rightarrow\; x' = -\sin(u)$

By the Chain Rule:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{dx}{du} \cdot \frac{du}{dt} \\
&= -\sin(u) \cdot 2t \\
&= -2t\sin(t^2+1)
\end{aligned}$$

The General Power Rule

$$\frac{dy}{dx} = \frac{d}{dx}\left[u(x)^n\right]$$

$$= n\, u^{n\text{-}1}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[u^n\right] = \boxed{n\, u^{n\text{-}1}u'}$$

Example

Find the derivative of $\frac{d}{dx}\left(5x^3 - x^4\right)^7$

Solution

$$\frac{d}{dx}\overbrace{\left(5x^3 - x^4\right)^7}^{nu^n} = 7\overbrace{\left(5x^3 - x^4\right)^6}^{nu^{n-1}}\overbrace{\left(15x^2 - 4x^3\right)}^{u'}$$

Example

Find the derivative of $\frac{d}{dx}\left(\frac{1}{3x-2}\right)$

Solution

$$\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1}$$

$$= -3(3x-2)^{-2}$$

$$\boxed{= -\frac{3}{(3x-2)^2}}$$

Example

Find the derivative of $\frac{d}{dx}\left(\sin^5 x\right)$

Solution

$$\frac{d}{dx}\left(\sin^5 x\right) = 5\sin^4 x(\sin x)'$$

$$\boxed{= 5\sin^4 x\cos x}$$

Example

Find the derivative of $g(t)=\tan(5-\sin 2t)$

Solution

$$g'(t)=\sec^2(5-\sin 2t)\cdot(5-\sin 2t)' \qquad u=5-\sin 2t \quad (\tan u)'=\sec^2 u\cdot(u')$$

$$=\sec^2(5-\sin 2t)\cdot\left(0-(\cos 2t)(2t)'\right)$$

$$=\sec^2(5-\sin 2t)\cdot(-2\cos 2t)$$

$$\boxed{=-2(\cos 2t)\sec^2(5-\sin 2t)}$$

Example

Show that the slope of every line tangent to the curve $y=\dfrac{1}{(1-2x)^3}$ is positive.

Solution

$$y=(1-2x)^{-3}$$

$$y'=-3(1-2x)^{-4}(-2)$$

$$\boxed{=\frac{6}{(1-2x)^4}}$$

At any point except $\left(x\neq\frac{1}{2}\right)$, the slope is $\dfrac{6}{(1-2x)^4}$ which is positive.

Formula $\quad \left(U^mV^nW^p\right)'=U^{m-1}V^{n-1}W^{p-1}\left(mU'VW+nUV'W+pUVW'\right)$

Proof

$$\left(U^mV^nW^p\right)'=\left(U^m\right)'V^nW^p+U^m\left(V^n\right)'W^p+U^mV^n\left(W^p\right)'$$

$$=mU^{m-1}U'V^nW^p+nU^mV^{n-1}V'W^p+pU^mV^nW^{p-1}W' \qquad \text{factor } U^{m-1}V^{n-1}W^{p-1}$$

$$=U^{m-1}V^{n-1}W^{p-1}\left(mU'VW+nUV'W+pUVW'\right)$$

$$\left(U^mV^n\right)'=U^{m-1}V^{n-1}\left(mU'V+nUV'\right)$$

Exercises *Section* 2.6 – Chain Rule

Find the derivative of

1. $y=\left(3x^4+1\right)^4\left(x^3+4\right)$

2. $p(t)=\dfrac{(2t+3)^3}{4t^2-1}$

3. $y=\left(x^3+1\right)^2$

4. $y=\left(x^2+3x\right)^4$

5. $y=\dfrac{4}{2x+1}$

6. $y=\dfrac{2}{(x-1)^3}$

7. $y=x^2\sqrt{x^2+1}$

8. $y=\left(\dfrac{x+1}{x-5}\right)^2$

9. $s(t)=\sqrt{2t^2+5t+2}$

10. $f(x)=\dfrac{1}{\left(x^2-3x\right)^2}$

11. $y=t^2\sqrt{t-2}$

12. $y=\left(\dfrac{6-5x}{x^2-1}\right)^2$

13. $y=4x(3x+5)^5$

14. $y=\left(3x^2-5x\right)^{1/2}$

15. $D_x\left(x^2+5x\right)^8$

16. $y=\dfrac{(3x+2)^7}{x-1}$

17. $y=\left(\dfrac{x^2}{8}+x-\dfrac{1}{x}\right)^4$

18. $y=\sqrt{3x^2-4x+6}$

19. $y=\cot\left(\pi-\dfrac{1}{x}\right)$

20. $y=5\cos^{-4}x$

21. $y=\sin\left(\dfrac{3\pi t}{2}\right)+\cos\left(\dfrac{3\pi t}{2}\right)$

22. $r=6(\sec\theta-\tan\theta)^{3/2}$

23. $g(x)=\dfrac{\tan 3x}{(x+7)^4}$

24. $f(\theta)=\left(\dfrac{\sin\theta}{1+\cos\theta}\right)^2$

25. $y=\sin^2(\pi t-2)$

26. $y=(t\tan t)^{10}$

27. $y=\cos\left(5\sin\left(\dfrac{t}{3}\right)\right)$

28. $y=4\sin\left(\sqrt{1+\sqrt{t}}\right)$

29. $y=\tan^2\left(\sin^3 x\right)$

30. $f(x)=\left(\left(x^2+3\right)^5+x\right)^2$

31. $y=\left(\dfrac{3x-1}{x^2+3}\right)^2$

32. $y=\cos\sqrt{\sin(\tan\pi x)}$

33. $f(x)=\dfrac{x}{\sqrt{x^2+1}}$

34. $y=\cos(1-2x)^2$

35. $f(x)=(4x-3)^2$

36. $f(x)=\dfrac{x}{\sqrt[3]{x^2+4}}$

37. $f(x)=\left(\dfrac{x^2}{x^3+2}\right)^2$

38. $y=\sin\sqrt[3]{x}+\sqrt[3]{\sin x}$

39. $f(\theta)=4\tan\left(\theta^2+3\theta+2\right)$

40. $f(\theta)=\tan(\sin\theta)$

41. $y=5x+\sin^3 x+\sin x^3$

42. $y=\csc^5 3x$

43. $y=2x\sqrt{x^2-2x+2}$

44. $\dfrac{d}{du}\left(\dfrac{4u^2+u}{8u+1}\right)^3$

45. $y=\frac{1}{2}x^2\sqrt{16-x^2}$

46. $y=\left(\dfrac{x-3}{2x+5}\right)^4$

47. $y=\left(\dfrac{5x-3}{2x+5}\right)^5$

48. $y=\left(\dfrac{6x-8}{2x-3}\right)^6$

49. $y=\left(\dfrac{3x^2-4}{2x^2-1}\right)^3$

50. $y=\left(\dfrac{3x^2+4}{2x^2+1}\right)^{-3}$

51. $y=\left(\dfrac{2x^2-3}{x^2+1}\right)^{1/3}$

52. $y=\sqrt{\dfrac{2x^3-3}{2x^3+1}}$

54. $y=\left(\dfrac{x^2-4x+1}{5x^2-2x-1}\right)^3$

55. $y=\left(\dfrac{3x^2-4x+2}{2x^2+x-1}\right)^{2/3}$

56. $f(x)=\left(\dfrac{3t^2-1}{3t^2+1}\right)^{-3}$

57. $f(x)=\left(\dfrac{x}{3x^2+2x+1}\right)^{1/3}$

58. $f(x)=\left(x^2+2x-3\right)^5(2x+3)^6$

59. $f(x)=\left(2x^2-4x+3\right)^4(3x-5)^5$

60. $f(x)=\left(x^2+2x-3\right)^4\left(x^2+3x+5\right)^6$

61. $f(x)=\left(2x^3-5x\right)^3\left(x^2+2x+1\right)^4(2x-3)^5$

62. $f(x)=\left(x^4+3x\right)^4\left(x^3+2x\right)^5(2x-3)^6$

63. $f(x)=\dfrac{\left(x^2-6x\right)^5}{\left(3x^2+5x-2\right)^4}$

64. $f(x)=\dfrac{\left(2x^2+3x1\right)^4}{\left(x^2+5x-6\right)^5}$

65. $f(x)=\dfrac{\left(x^3-3x\right)^3\left(x^2+4x\right)^4}{\left(x^2+4x+1\right)^2}$

66. $f(x)=\dfrac{x^2+3}{(2x-1)^3(3x+1)^4}$

67. $f(x)=\dfrac{\left(x^2-3x\right)^3\left(x^2+3x-3\right)^4}{\left(x^2-3x+2\right)^2}$

53. $y = \left(\dfrac{2x^4 - 3}{2x^4 + 1}\right)^5$

68. Find the ***second*** derivative $y = \dfrac{x^2 + 3}{(x-1)^3 + (x+1)^3}$

69. Find the ***second*** derivative of $y = \left(1 + \dfrac{1}{x}\right)^3$

70. Find the ***second*** derivative of $y = 9\tan\left(\dfrac{x}{3}\right)$

71. Find the tangent line to the graph of $y = \sqrt[3]{(x+4)^2}$ when $x = 4$

Evaluate the limit

72. $\lim\limits_{h \to 0} \dfrac{\sin^2\left(\frac{\pi}{4} + h\right) - \frac{1}{2}}{h}$

73. $\lim\limits_{x \to 5} \dfrac{\tan\left(\pi\sqrt{3x - 11}\right)}{x - 5}$