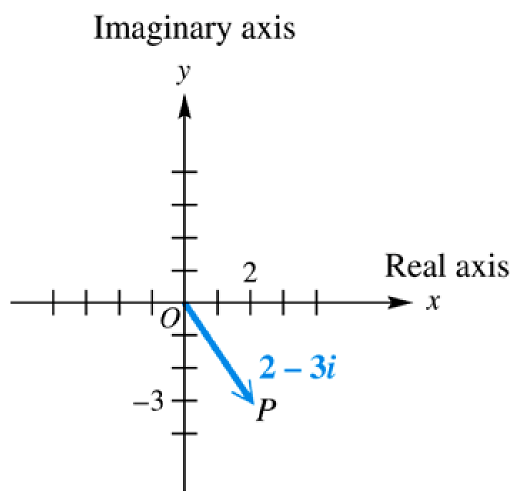
Section 6.7 – Trigonometric Form

$\sqrt{-1} = i$

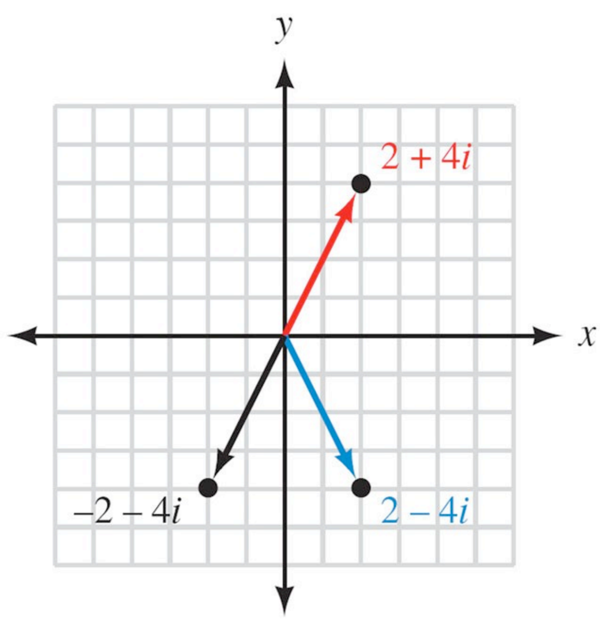
The graph of the complex number $x = yi$ is a vector (arrow) that extends from the origin out to the point (x, y)

- Horizontal axis: ***real axis***
- Vertical axis: ***imaginary axis***



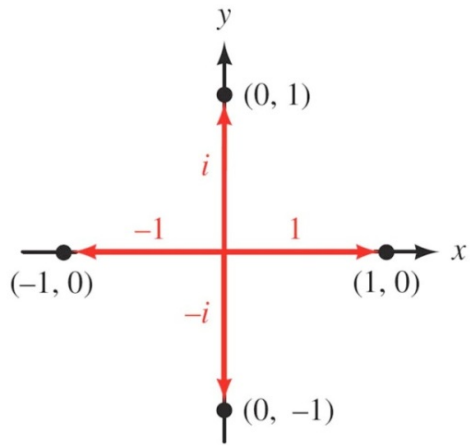
Example

Graph each complex number: $2+4i$, $-2-4i$, and $2-4i$



Example

Graph each complex number: $1,\ i,\ -1,\ and\ -i$



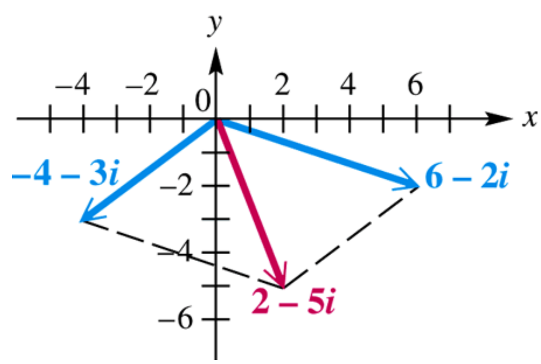
Example

Find the sum of $6 - 2i$ and $-4 - 3i$. Graph both complex numbers and their resultant.

Solution

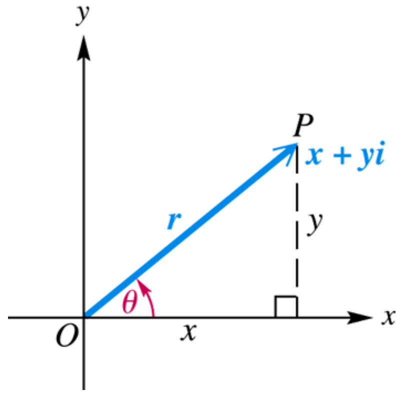
$(6 - 2i) + (-4 - 3i) = 6 - 4 - 2i - 3i$

$= 2 - 5i$



Definition

The *absolute value* or ***modulus*** of the complex number $z = x + yi$ is the distance from the origin to the point (x, y). If this distance is denoted by r, then



$$r = |z| = |x + yi|$$
$$= \sqrt{x^2 + y^2}$$

The ***argument*** of the complex number $z = x + yi$ denoted $\arg(z)$ is the smallest possible angle θ from the positive real axis to the graph of z.

$$\cos\theta = \frac{x}{r} \qquad \Rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \qquad \Rightarrow y = r\sin\theta$$

$$z = x + yi$$
$$= r\cos\theta + (r\sin\theta)\, i$$
$$= r(\cos\theta + i\ \sin\theta) \qquad \rightarrow \text{is called the } \textit{trigonometric} \text{ from of } z.$$

Definition

If $z = x + y\ i$ is a complex number in standard form then the ***trigonometric form*** for z is given by

$$z = r(\cos\theta + i\ \sin\theta) \ = \ r\ cis\theta$$

Where $\boldsymbol{r}$ is the modulus or absolute value of z and

$\boldsymbol{\theta}$ is the argument of z.

We can convert back and forth between standard form and trigonometric form by using the relationships that follow

For $z = x + y\ i = r(\cos\theta + i\ \sin\theta) = r\ cis\theta$

$$r = \sqrt{x^2 + y^2}$$

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}, \quad and \quad \tan\theta = \frac{y}{x}$$

Example

Write $z=-1+i$ in trigonometric form

Solution

The modulus *r*:

$$r=\sqrt{(-1)^2+1^2}=\sqrt{2}$$

$$\cos\theta=\frac{x}{r}=\frac{-1}{\sqrt{2}}$$

$$\sin\theta=\frac{y}{r}=\frac{1}{\sqrt{2}}$$

$$\rightarrow \theta=135^\circ$$

$$z=x+y\,i$$

$$=\sqrt{2}(\cos 135^\circ+i\ \sin 135^\circ)$$

$$=\sqrt{2}\ cis 135^\circ$$

In radians: $\underline{z=\sqrt{2}\ cis\left(\frac{3\pi}{4}\right)}$

Example

Write $z=2\ cis 60^\circ$ in rectangular form.

Solution

$$z=2\ cis 60^\circ$$

$$=2(\cos 60^\circ+i\ \sin 60^\circ)$$

$$=2\left(\frac{1}{2}+i\ \frac{\sqrt{3}}{2}\right)$$

$$\underline{=1+i\ \sqrt{3}}$$

Example

Express $2(\cos 300^\circ+i\sin 300^\circ)$ in rectangular form.

Solution

$$2(\cos 300^\circ+i\sin 300^\circ)=2\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$$

$$\underline{=1-i\sqrt{3}}$$

Example

Find the modulus of each of the complex numbers $5i$, 7, and $3 + 4i$

Solution

For $z = 5i$

$= 0 + 5i$

$r = |z|$

$= \sqrt{0^2 + 5^2}$

$= 5$

For $z = 7$

$= 7 + 0i$

$r = |z|$

$= \sqrt{7^2 + 0^2}$

$= 7$

For $3 + 4i$

$\Rightarrow r = \sqrt{3^2 + 4^2}$

$= 5$

Product Theorem

If $r_1\left(\cos\theta_1 + i\sin\theta_1\right)$ and $r_2\left(\cos\theta_2 + i\sin\theta_2\right)$ are any two complex numbers, then

$$\left[r_1\left(\cos\theta_1 + i\sin\theta_1\right)\right]\left[r_2\left(\cos\theta_2 + i\sin\theta_2\right)\right] = r_1 r_2\left[\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right]$$

$$\left(r_1 cis\theta_1\right)\left(r_2 cis\theta_2\right) = r_1 r_2 cis\left(\theta_1 + \theta_2\right)$$

$$\boxed{(a+bi)(a-bi) = a^2 + b^2}$$

$$\boxed{\left(\sqrt{a}+\sqrt{b}i\right)\left(\sqrt{a}-\sqrt{b}i\right) = a+b}$$

Example

Find the product of $3(\cos 45° + i\sin 45°)$ and $2(\cos 135° + i\sin 135°)$. Write the result in rectangular form.

Solution

$$\left[3(\cos 45° + i\sin 45°)\right]\left[2(\cos 135° + i\sin 135°)\right]$$

$$= (3)(2)\left[\cos(45° + 135°) + i\sin(45° + 135°)\right]$$

$$= 6(\cos 180° + i\sin 180°)$$

$$= 6(-1 + i.0)$$

$$= -6$$

Quotient Theorem

If $r_1\left(\cos\theta_1 + i\sin\theta_1\right)$ and $r_2\left(\cos\theta_2 + i\sin\theta_2\right)$ are any two complex numbers, then

$$\frac{r_1\left(\cos\theta_1 + i\sin\theta_1\right)}{r_2\left(\cos\theta_2 + i\sin\theta_2\right)} = \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right]$$

$$\frac{r_1 cis\theta_1}{r_2 cis\theta_2} = \frac{r_1}{r_2} cis\left(\theta_1 - \theta_2\right)$$

Example

Find the quotient $\frac{10cis\left(-60°\right)}{5cis\left(150°\right)}$. Write the result in rectangular form.

Solution

$$\begin{aligned}\frac{10cis\left(-60°\right)}{5cis\left(150°\right)} &= \frac{10}{5}cis\left(-60° - 150°\right)\\ &= 2cis\left(-210°\right)\\ &= 2\left[\cos\left(-210°\right) + i\sin\left(-210°\right)\right]\\ &= 2\left[-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right]\\ &= \underline{-\sqrt{3} + i}\end{aligned}$$

De Moivre's *Theorem*

If $r(\cos\theta + i\sin\theta)$ is a complex number, then

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$$

$$\boxed{(rcis\theta)^n = r^n(cisn\theta)}$$

Example

Find $\left(1+i\sqrt{3}\right)^8$ and express the result in rectangular form.

Solution

$$1+i\sqrt{3} \Rightarrow \begin{cases} x = 1 \\ y = \sqrt{3} \end{cases}$$

$$r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$$

$$\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$$

θ is in QI, that implies: $\theta = 60°$

$$1+i\sqrt{3} = 2cis60°$$

Apply De Moivre's theorem:

$$\begin{aligned}\left(1+i\sqrt{3}\right)^8 &= (2cis60°)^8 \\ &= 2^8\left[cis(8.60°)\right] \\ &= 256\left[cis(480°)\right] && 480° - 360° = 120° \\ &= 256\left[cis(120°)\right] \\ &= 256\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= -128 + 128i\sqrt{3}\end{aligned}$$

n^{th} Root Theorem

For a positive integer n, the complex number $a + bi$ is an $\boldsymbol{n^{th}}$ **root** of the complex number $x + iy$ if

$$(a+bi)^n = x + yi$$

If n is any positive integer, r is a positive real number, and θ is in degrees, then the nonzero complex number $r(\cos\theta + i\sin\theta)$ has exactly n distinct nth roots, given by

$$\sqrt[n]{r}\left(\cos\alpha + i\sin\alpha\right) \quad \textit{or} \quad \sqrt[n]{r}\ cis\alpha$$

Where $\alpha = \dfrac{\theta + 360°k}{n}, \quad k = 0,1,2,\cdots,n-1 \qquad \boxed{\alpha = \dfrac{\theta}{n} + \dfrac{360°k}{n}}$

$$\alpha = \frac{\theta + 2\pi k}{n}, \quad k = 0,1,2,\cdots,n-1 \qquad \boxed{\alpha = \frac{\theta}{n} + \frac{2\pi k}{n}}$$

Example

Find the two square root of $\mathbf{4}\boldsymbol{i}$. Write the roots in rectangular form.

Solution

$$4i \quad \rightarrow \quad \begin{cases} x = 0 \\ y = 4 \end{cases}$$

$$r = \sqrt{0^2 + 4^2}$$
$$= 4$$

$$\tan\theta = \frac{4}{0} = \infty$$

$$\theta = \frac{\pi}{2}$$

$$4i = 4cis\frac{\pi}{2}$$

The absolute value: $\sqrt{4} = 2$

Argument: $\alpha = \dfrac{\frac{\pi}{2} + 2\pi k}{2}$

$$= \frac{\frac{\pi}{2}}{2} + \frac{2\pi k}{2}$$

$$= \frac{\pi}{4} + \pi k$$

Since there are ***two*** square root, then $k = 0$ and 1.

If $k = 0$

$$\Rightarrow \quad \alpha = \frac{\pi}{4} + \pi(0) = \frac{\pi}{4}$$

If $k = 1$

$$\Rightarrow \quad \alpha = \frac{\pi}{4} + \pi(1) = \frac{5\pi}{4}$$

The square roots are: $2cis\frac{\pi}{4}$ *and* $2cis\frac{5\pi}{4}$

$$2cis\frac{\pi}{4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$= \sqrt{2} + i\sqrt{2}$$

$$2cis\frac{5\pi}{4} = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

$$= 2\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$= -\sqrt{2} - i\sqrt{2}$$

Example

Find all fourth roots of $-8+8i\sqrt{3}$. Write the roots in rectangular form.

Solution

$$-8+8i\sqrt{3} \quad \Rightarrow \quad \begin{cases} x=-8 \\ y=8\sqrt{3} \end{cases}$$

$$r=\sqrt{(-8)^2+\left(8\sqrt{3}\right)^2}$$

$$\underline{=16|}$$

$$\tan\theta = \frac{8\sqrt{3}}{-8}$$

$$=-\sqrt{3}$$

$$\underline{\theta = 120°|}$$

$$-8+8i\sqrt{3}=16cis120°$$

The fourth roots have absolute value: $\sqrt[4]{16}=2$

$$\alpha = \frac{120°}{4}+\frac{360°k}{4}$$

$$\underline{=30°+90°k|}$$

Since there are ***four*** roots, then $k = 0, 1, 2,$ and 3.

$$\text{If } k=0 \Rightarrow \alpha = 30°+90°(0)=30°$$

$$\text{If } k=1 \Rightarrow \alpha = 30°+90°(1)=120°$$

$$\text{If } k=2 \Rightarrow \alpha = 30°+90°(2)=210°$$

$$\text{If } k=3 \Rightarrow \alpha = 30°+90°(3)=300°$$

The fourth roots are: $2cis30°,\ \ 2cis120°,\ \ 2cis210°,\ and\ \ 2cis300°$

$$2cis30° = 2\left(\cos 30° + i\sin 30°\right)$$

$$=2\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)$$

$$\underline{=\sqrt{3}+i\ |}$$

$$2cis120° = 2\left(\cos 120° + i\sin 120°\right)$$

$$=2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$\underline{=-1+i\sqrt{3}\ |}$$

$$2cis210° = 2(\cos 210° + i \sin 210°)$$

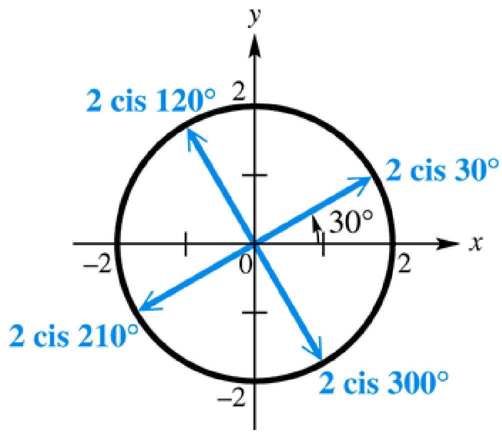
$$= 2\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$$

$$\underline{= -\sqrt{3} - i}$$

$$2cis300° = 2(\cos 300° + i \sin 300°)$$

$$= 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$\underline{= 1 - i\sqrt{3}}$$



Example

Find all complex number solutions of $x^5 - 1 = 0$. Graph them as vectors in the complex plane.

Solution

$x^5 - 1 = 0 \Rightarrow x^5 = 1$

There is one real solution, 1, while there are five complex solutions.

$1 = 1 + 0i$

$$r = \sqrt{1^2 + 0^2}$$

$$= 1$$

$$\tan\theta = \frac{0}{1} = 0$$

$$\theta = 0°$$

$1 = 1cis0°$

The fifth roots have absolute value: $\sqrt[5]{1} = 1$

$$\alpha = \frac{0°}{5} + \frac{360°k}{5}$$

$$= 0° + 72°k$$

$$= 72°k$$

Since there are ***fifth*** roots, then k = 0, 1, 2, 3, and 4.

$If\ \ k = 0 \Rightarrow \alpha = 72°(0) = 0°$

$If\ \ k = 1 \Rightarrow \alpha = 72°(0) = 72°$

$If\ \ k = 2 \Rightarrow \alpha = 72°(2) = 144°$

$If\ \ k = 3 \Rightarrow \alpha = 72°(3) = 216°$

$If\ \ k = 4 \Rightarrow \alpha = 72°(4) = 288°$

$Solution:\ \ cis0°,\ \ cis72°,\ \ cis144°,\ cis216°,\ and\ \ cis288°$

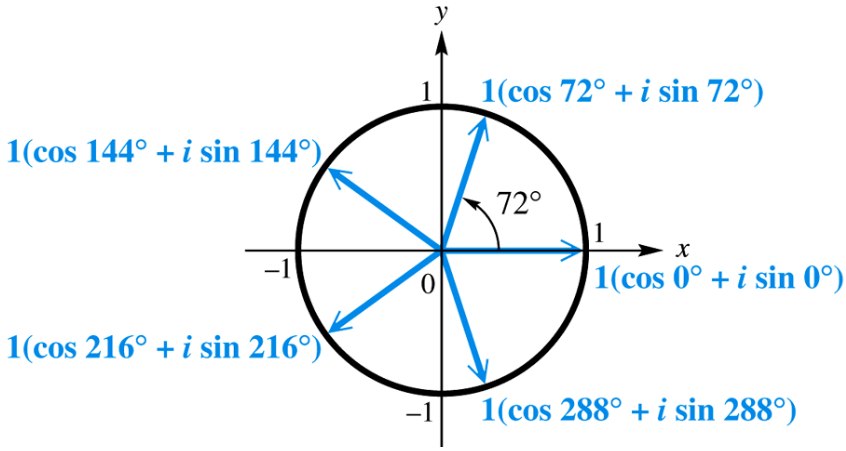
$cis0° = 1$

$cis72° = \cos 72° + i\sin 72°$

$cis144° = \cos 144° + i\sin 144°$

$cis216° = \cos 216° + i\sin 216°$

$cis288° = \cos 288° + i\sin 288°$



The graphs of the roots lie on a unit circle. The roots are equally spaced about the circle, 72° apart.

Exercises *Section* 6.7 – Trigonometric Form

(1 – 8) Write complex form in trigonometric form

1. $-\sqrt{3}+i$

2. $3-4i$

3. $-21-20i$

4. $11+2i$

5. $\sqrt{3}-i$

6. $1-\sqrt{3}i$

7. $9\sqrt{3}+9i$

8. $-2+3i$

(9 – 13) Write in standard form

9. $4\left(\cos 30°+i\sin 30°\right)$

10. $\sqrt{2}\ cis\frac{7\pi}{4}$

11. $3cis210°$

12. $4\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$

13. $4cis\frac{\pi}{2}$

14. Find the quotient $\frac{20cis(75°)}{4cis(40°)}$. Write the result in rectangular form.

15. Divide $z_1=1+i\sqrt{3}$ *by* $z_2=\sqrt{3}+i$. Write the result in rectangular form.

(16 – 25) Find and express the result in rectangular form

16. $(1+i)^8$

17. $(1+i)^{10}$

18. $(1-i)^5$

19. $\left(1-\sqrt{5}i\right)^8$

20. $(3cis80°)^3$

21. $\left(\sqrt{3}cis10°\right)^6$

22. $\left(\sqrt{2}-i\right)^6$

23. $(4cis40°)^6$

24. $(2cis30°)^5$

25. $\left(\frac{1}{2}cis72°\right)^5$

26. Find fifth complex roots of $z=1+i\sqrt{3}$ and express the result in rectangular form.

(27 – 30) Find the fourth roots of

27. $z=16cis60°$

28. $\sqrt{3}-i$

29. $4-4\sqrt{3}i$

30. $-16i$

(31 – 33) Find the cube roots of

31. 27

32. $8-8i$

33. -8

34. Find all complex number solutions of $x^3+1=0$.