Lecture Two – Functions

Section 2.1 – Functions and Graphs

Relations

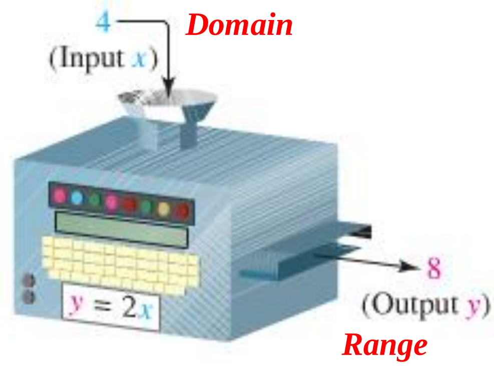
A ***relation*** is any set of ordered pairs. The set of all first components of ordered pairs is called the domain of the relation and the set of second components is called the range of the relation.

Definition of a Function

A ***function*** is a relation between two variables such that to matches each element of a first set (called ***domain***) to an element of a second set (called ***range***) in such way that no element in the first set is assigned to two different elements in the second set.

The ***domain*** of the function is the set of all values of the independent variable for which the function is defined.

The ***range*** of the function is the set of all values taken on by the dependent variable.



Function machine

Example

Determine whether each relation is a function and *find the domain and the range*.

a) $F=\{(1,2),(-2,4),(3,-1)\}$

Function: Yes

Domain: $\{-2,\ 1,\ 3\}$

Range: $\{-1,\ 2,\ 4\}$

b) $G=\{(1,1),(1,2),(1,3),(2,3)\}$

Function: No

Domain: {1, 2}

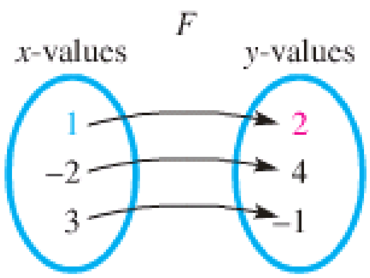
Range: {1, 2, 3}

c) $H=\{(-4,1),(-2,1),(-2,0)\}$

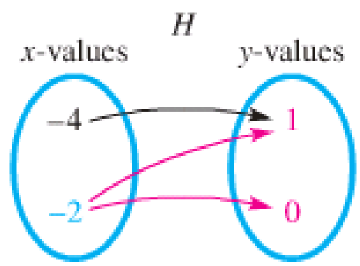
Function: No

Domain: $\{-4,\ -2\}$

Range: {0, 1}



F is a function.



H is not a function.

Example

Give the domain and range of each relation

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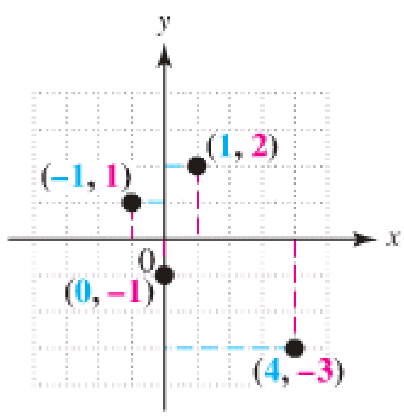
<td>
</td>
<td>Domain: {-1, 0, 1, 4}

Range: {-3, -1, 1, 2}</td>
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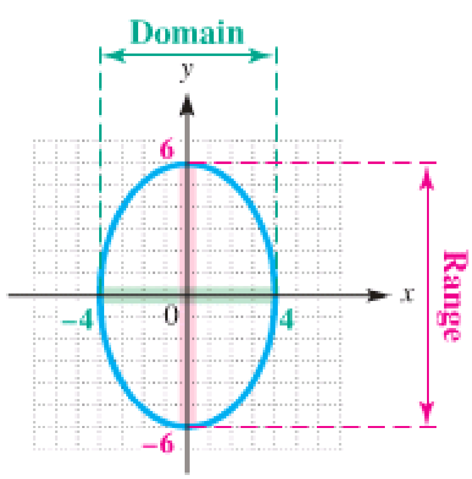
<td>
</td>
<td>Domain: $[-4, 4]$

Range: $[-6, 6]$</td>
</tr>
</table>

Functions as Equations $y = -0.016x^2 + 0.93x + 8.5$

x: independent
y: depend on x

Notation for Functions

***f*(*x*)** read "f of x" or "f at x" represents the value of the function at the number x.

Example

Let $f(x) = -x^2 + 5x - 3$

a) $f(2)$

$$f(x) = -x^2 + 5x - 3$$
$$f(\text{--}) = -(\text{--})^2 + 5(\text{--}) - 3$$
$$f(2) = -(2)^2 + 5(2) - 3$$
$$\underline{= 3}$$

b) $f(q)$

$$f(q) = -(q)^2 + 5(q) - 3$$
$$\underline{= -q^2 + 5q - 3}$$

Example

If $f(x) = x^2 - 2x + 7$, evaluate each of the following:

a) $f(-5)$
b) $f(x+4)$

<u>Solution</u>

a) $f(-5) = ?$

$$f(\text{--}) = (\text{--})^2 - 2(\text{--}) + 7$$
$$f(-5) = (-5)^2 - 2(-5) + 7$$
$$= 25 + 10 + 7$$
$$\underline{= 42}$$

b) $f(x+4)=?$

$$f(\text{—})=(\text{—})^2-2(\text{—})+7$$

$$f(x+4)=(x+4)^2-2(x+4)+7 \qquad (a+b)^2=a^2+2ab+b^2$$

$$=x^2+2(4)x+4^2-2x-8+7$$

$$=x^2+8x+16-2x-1$$

$$\underline{=x^2+6x+15}$$

Example

Let $g(x)=2x+3$, find $g(a+1)$

Solution

$$g(x)=2x+3$$

$$g(a+1)=2(a+1)+3$$

$$=2a+2+3$$

$$\underline{=2a+5}$$

Example

Given: $f(x)=2x^2-x+3$, find the following.

a) $f(0)$

b) $f(-7)$

c) $f(5a)$

Solution

a) $f(x=0)=2(0)^2-(0)+3$

$$\underline{=3}$$

b) $f(-7)=2(-7)^2-(-7)+3$

$$\underline{=108}$$

c) $f(5a)=2(5a)^2-(5a)+3$

$$\underline{=50a^2-5a+3}$$

Increasing and Decreasing Functions

- A function *rises from left to right (x-coordinate)*, the function *f* is said to be ***increasing*** on an open interval I (a, b) (x-coordinate)

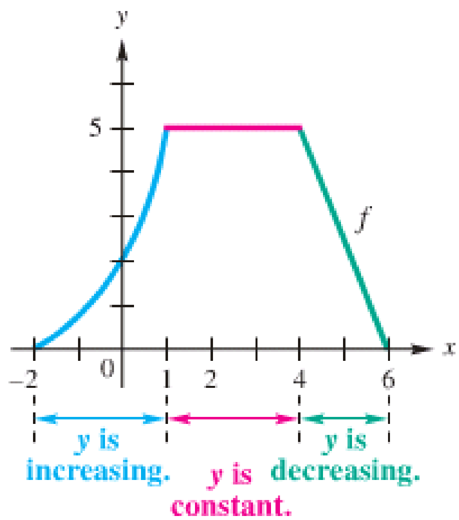
$$a < b \quad \Rightarrow \quad f(a) < f(b)$$

- A function *f* is said to be ***decreasing*** on an open interval I

$$a < b \quad \Rightarrow \quad f(a) > f(b)$$

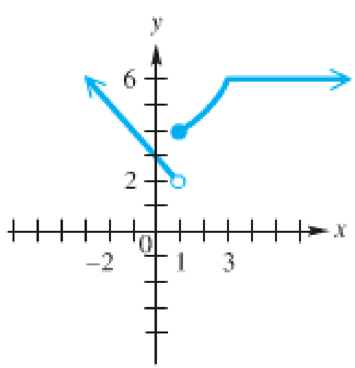
- A function *f* is said to be ***constant*** on an open interval I

$$a < b \quad \Rightarrow \quad f(a) = f(b)$$



Example

Determine the intervals over which the function is increasing, decreasing, or constant



Increasing: [1, 3]

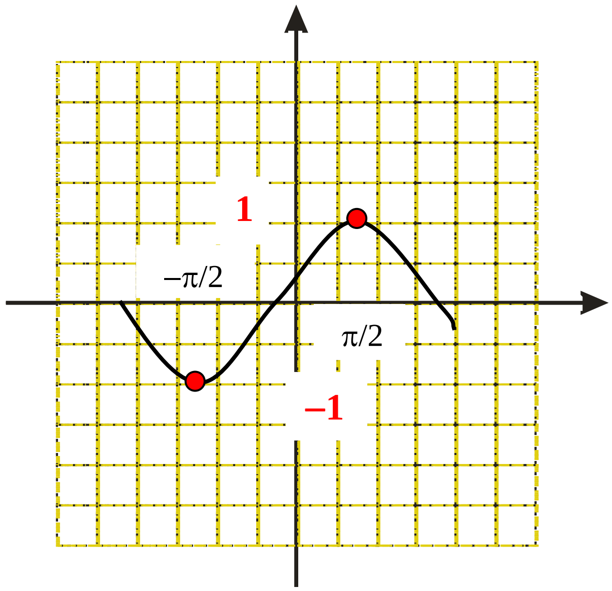
Decreasing: $(-\infty, 1)$

Constant: $[3, \infty)$

Relative *Maxima* *(um)* and *Minima* *(um)*

f(a) is a relative maximum if there exists an open interval I about *a* such that $f(a) > f(x)$, for all *x* in I.

f(a) is a relative minimum if there exists an open interval I about *a* such that $f(a) < f(x)$, for all *x* in I.

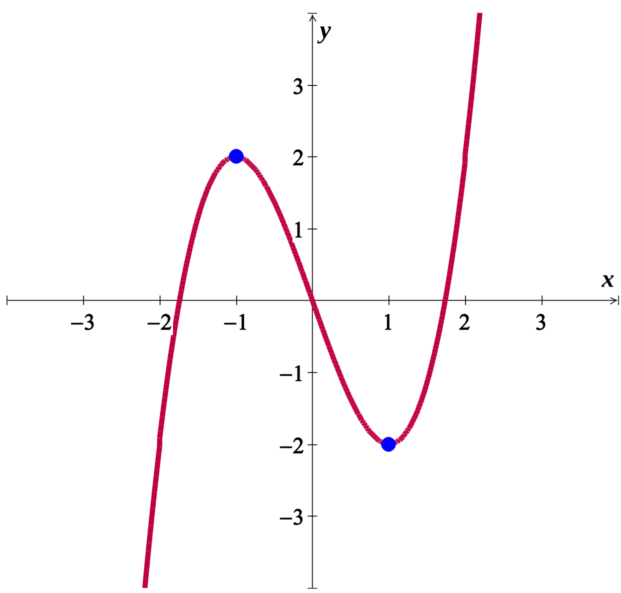


The relative minimum value of the function is −1 @ $x = -\pi/2$

The relative maximum value of the function is 1 @ $x = \pi/2$

Example

State the intervals on which the given function $f(x) = x^3 - 3x$ is increasing, decreasing, or constant, and determine the extreme values



Increasing $(-\infty, -1), (1, \infty)$ ***RMIN*** $(1, -2)$

Decreasing $(-1, 1)$ ***RMAX*** $(-1, 2)$

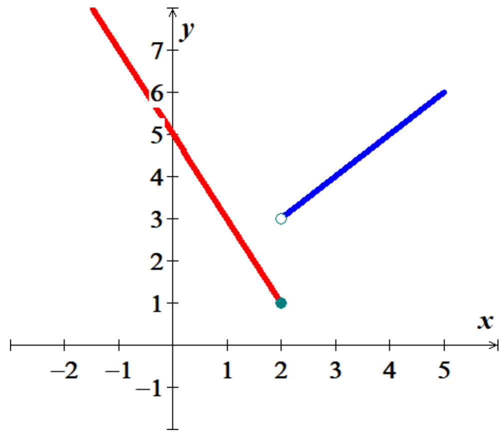
Piecewise-Defined Functions

Function are sometimes described by more than one expression, we call such functions ***piecewise-defined functions***.

Example

Graph function

$$f(x)=\begin{cases} -2x+5 & \text{if } x\le 2 \\ x+1 & \text{if } x>2 \end{cases}$$



Find:

$$f(2)=-2(2)+5=1$$

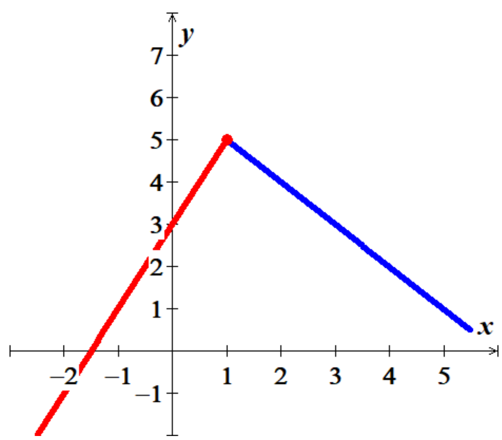
$$f(0)=-2(0)+5=5$$

$$f(4)=4+1=5$$

Example

Graph function

$$f(x)=\begin{cases} 2x+3 & \text{if } x\le 1 \\ -x+6 & \text{if } x>1 \end{cases}$$



Example

$$C(t)=\begin{cases} 20 & \text{if } 0\le t\le 60 \\ 20+0.40(t-60) & \text{if } t>60 \end{cases}$$

Find $C(40)$, $C(80)$, and $C(60)$

Solution

a) $C(40) = 20$

b) $C(80) = 20 + 0.40(80 - 60) = 28$

c) $C(60) = 20$

Exercise *Section* 2.1 – Functions and Graphs

(1 – 7) Determine whether each relation is a function and *find the domain and the range*.

1. $\{(1,\ 2),\ (3,\ 4),\ (5,\ 6),\ (5,\ 8)\}$

2. $\{(1,\ 2),\ (3,\ 4),\ (6,\ 5),\ (8,\ 5)\}$

3. $\{(9,\ -5),\ (9,\ 5),\ (2,\ 4)\}$

4. $\{(-2,\ 5),\ (5,\ 7),\ (0,\ 1),\ (4,\ -2)\}$

5. $\{(-5,\ 3),\ (0,\ 3),\ (6,\ 3)\}$

6. $\{(1,\ 2),\ (3,\ 4),\ (6,\ 5),\ (8,\ 5),\ (1,\ 5)\}$

7. $\{(-1,\ 3),\ (3,\ 4),\ (6,\ 5),\ (8,\ 5),\ (1,\ 5)\}$

8. Let $f(x) = -3x + 4$, find $f(0)$, $f(-1)$, $f(h)$, and $f(a-1)$

9. Let $g(x) = -x^2 + 4x - 1$, find $g(-x)$, $g(2)$, and $g(-2)$

10. Let $f(x) = -3x + 4$, find $f(a+4)$

11. Given: $f(x) = 2|x| + 3x$, find $f(2-h)$.

12. Given: $g(x) = \frac{x-4}{x+3}$, find $g(x+h)$

13. Given: $g(x) = \frac{x}{\sqrt{1-x^2}}$, find $g(0)$ *and* $g(-1)$

14. Given that $g(x) = 2x^2 + 2x + 3$. Find $g(p+3)$

15. If $f(x) = x^2 - 2x + 7$, evaluate each of the following: $f(-5)$, $f(x+4)$, $f(-x)$

16. Find $g(0)$, $g(-4)$, $g(7)$, *and* $g\left(\frac{3}{2}\right)$ *for* $g(x) = \frac{x}{\sqrt{16-x^2}}$

17. For $f(x) = 3x - 4$, determine

a) $f(0)$ *b)* $f\left(\frac{5}{3}\right)$ *c)* $f(-2a)$ *d)* $f(x+h)$

18. For $f(x) = 3x^2 + 3x - 1$, determine

a) $f(0)$ *b)* $f(x+h)$ *c)* $f(2)$ *d)* $f(h)$

19. For $f(x) = 2x^2 - 4$, determine

a) $f(0)$ *b)* $f(x+h)$ *c)* $f(2)$ *d)* $f(2) - f(-3)$

20. For $f(x) = 3x^2 + 4x - 2$, determine

a) $f(0)$ *b)* $f(x+h)$ *c)* $f(3)$ *d)* $f(-5)$

21. For $f(x) = -x^3 - x^2 - x + 10$, determine

a) $f(0)$ *b)* $f(-1)$ *c)* $f(2)$ *d)* $f(1) - f(-2)$

22. For $\frac{1}{10}x^{10} - \frac{1}{2}x^6 + \frac{2}{3}x^3 - 10x$, determine

a) $f(2) - f(-2)$ *b)* $f(1) - f(-1)$ *c)* $f(2) - f(0)$

23. For $f(x) = 3x^4 + x^2 - 4$, determine

a) $f(2) - f(-2)$ *b)* $f(1) - f(-1)$ *c)* $f(2) - f(0)$

24. For $f(x) = -\frac{2}{3}x^3 + 4x$, determine

a) $f(2) - f(-2)$ *b)* $f(1) - f(-1)$ *c)* $f(2) - f(0)$

25. For $f(x) = \dfrac{2x-3}{x-4}$, determine

a) $f(0)$ *b)* $f(3)$ *c)* $f(x+h)$ *d)* $f(-4)$

26. For $f(x) = \dfrac{3x-1}{x-5}$, determine

a) $f(0)$ *b)* $f(3)$ *c)* $f(x+h)$ *d)* $f(-5)$

27. $f(x) = \begin{cases} 2+x & \text{if } x < -4 \\ -x & \text{if } -4 \le x \le 2 \\ 3x & \text{if } x > 2 \end{cases}$ Find: $f(-5),\ f(-1),\ f(0),\ \text{and } f(3)$

28. $f(x) = \begin{cases} -2x & \text{if } x < -3 \\ 3x-1 & \text{if } -3 \le x \le 2 \\ -4x & \text{if } x > 2 \end{cases}$ Find: $f(-5),\ f(-1),\ f(0),\ \text{and } f(3)$

29. $f(x)=\begin{cases} x^3+3 & if \quad -2\le x\le 0 \\ x+3 & if \quad 0<x<1 \\ 4+x-x^2 & if \quad 1\le x\le 3 \end{cases}$ Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

30. $h(x)=\begin{cases} \dfrac{x^2-9}{x-3} & if \quad x\ne 3 \\ 6 & if \quad x=3 \end{cases}$ Find: $h(5),\ h(0),\ and\ h(3)$

31. $f(x)=\begin{cases} 3x+5 & if \quad x<0 \\ 4x+7 & if \quad x\ge 0 \end{cases}$ Find

a) $f(0)$ *b)* $f(-2)$ *c)* $f(1)$ *d)* $f(3)+f(-3)$ *e)* Graph $f(x)$

32. $f(x)=\begin{cases} 6x-1 & if \quad x<0 \\ 7x+3 & if \quad x\ge 0 \end{cases}$ Find

a) $f(0)$ *b)* $f(-1)$ *c)* $f(4)$ *d)* $f(2)+f(-2)$ *e)* Graph $f(x)$

33. $f(x)=\begin{cases} 2x+1 & if \quad x\le 1 \\ 3x-2 & if \quad x>1 \end{cases}$ Find

a) $f(0)$ *b)* $f(2)$ *c)* $f(-2)$ *d)* $f(1)+f(-1)$ *e)* Graph $f(x)$

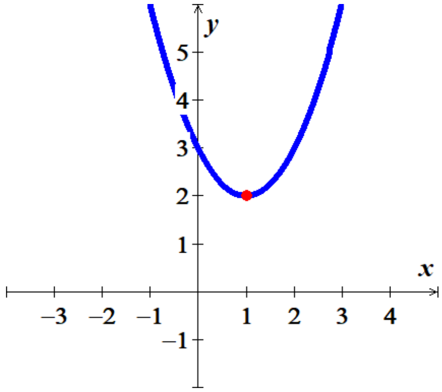
34. Graph the piecewise function defined by $f(x)=\begin{cases} 3 & if \quad x\le -1 \\ x-2 & if \quad x>-1 \end{cases}$

35. Sketch the graph $f(x)=\begin{cases} x+2 & if \quad x\le -1 \\ x^3 & if \quad -1<x<1 \\ -x+3 & if \quad x\ge 1 \end{cases}$

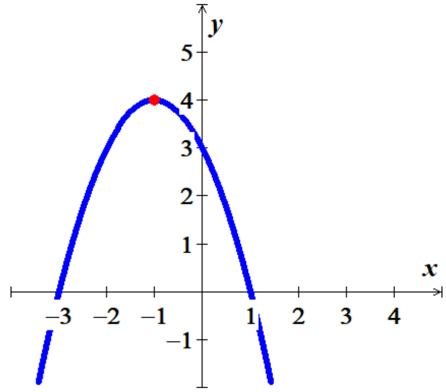
36. Sketch the graph $f(x)=\begin{cases} x-3 & if \quad x\le -2 \\ -x^2 & if \quad -2<x<1 \\ -x+4 & if \quad x\ge 1 \end{cases}$

(**37 – 42**) Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

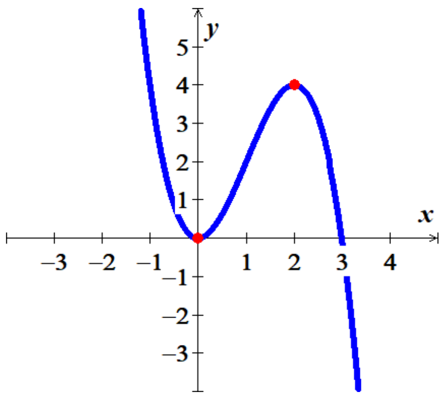
37. $f(x) = x^2 - 2x + 3$



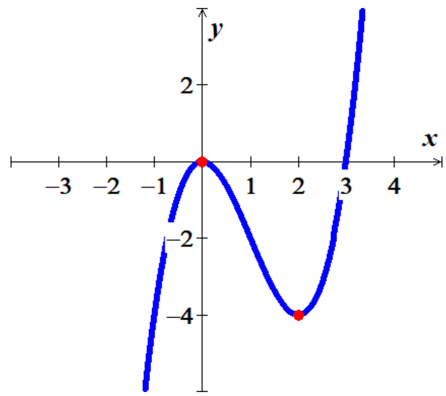
38. $f(x) = -x^2 - 2x + 3$



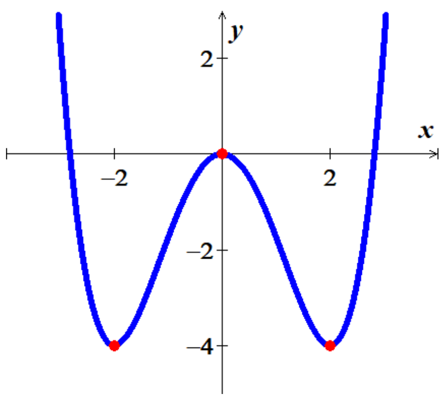
39. $f(x) = -x^3 + 3x^2$



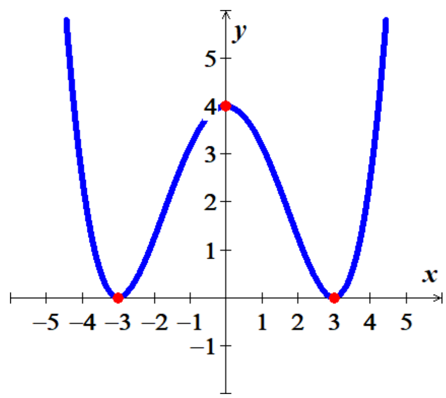
40. $f(x) = x^3 - 3x^2$



41. $f(x) = \frac{1}{4}x^4 - 2x^2$



42. $f(x) = \frac{4}{81}x^4 - \frac{8}{9}x^2 + 4$

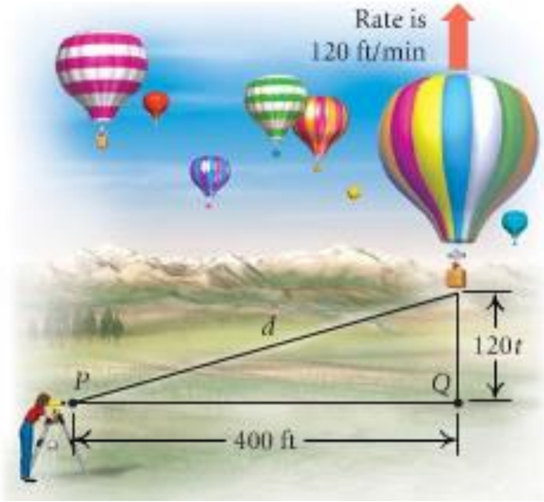


43. The elevation *H*, in *meters*, above sea level at which the boiling point of water is in ***t*** *degrees Celsius* is given by the function

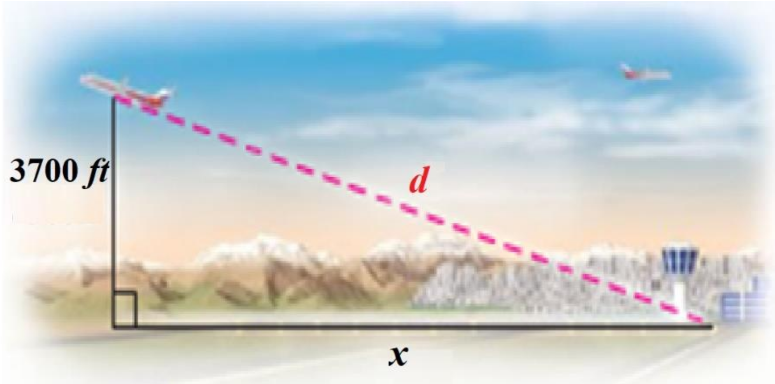
$$H(t) = 1000(100 - t) + 580(100 - t)^2$$

At what elevation is the boiling point 99.5°.

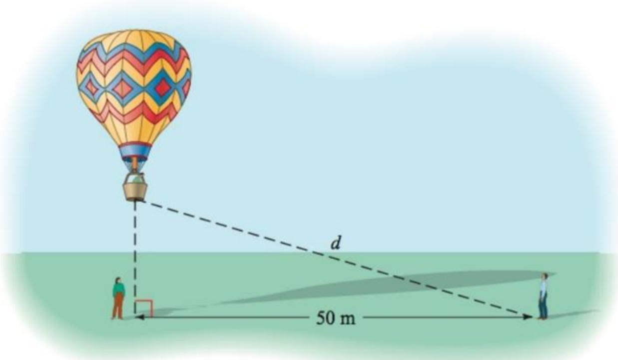
44. A hot-air balloon rises straight up from the ground at a rate of 120 *ft./min*. The balloon is tracked from a rangefinder on the ground at point P, which is 400 *feet*. from the release point Q of the balloon. Let d be the distance from the balloon to the rangefinder and t – the time, in *minutes*, since the balloon was released. Express d as a function of t.



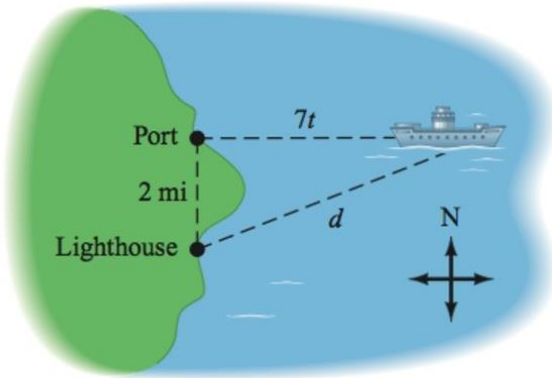
45. An airplane is flying at an altitude of 3700 *feet*. The slanted distance directly to the airport is d *feet*. Express the horizontal distance x as a function of d.



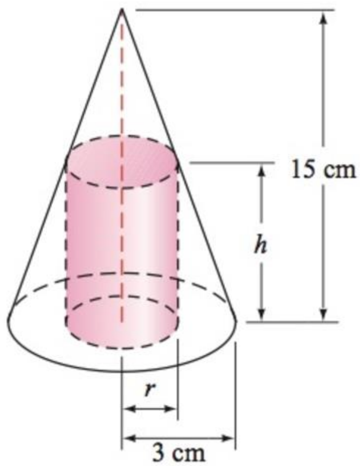
46. For the first minute of flight, a hot air balloon rises vertically at a rate of 3 *m/sec*. If t is the time in *seconds* that the balloon has been airborne, write the distance d between the balloon and a point on the ground 50 *meters* from the point to lift off as a function of t.



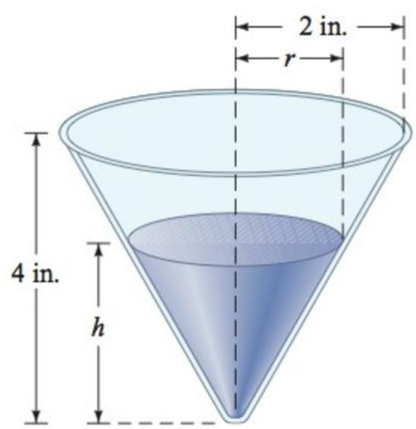
47. A light house is 2 *miles* south of a port. A ship leaves port and sails east at a rate of 7 *miles* per *hour*. Express the distance d between the ship and the lighthouse as a function of time, given that the ship has been sailing for t *hours*.



48. A cone has an altitude of 15 *cm* and a radius of 3 *cm*. A right circular cylinder of radius r and height h is inscribed in the cone. Use similar triangles to write h as a function of r.



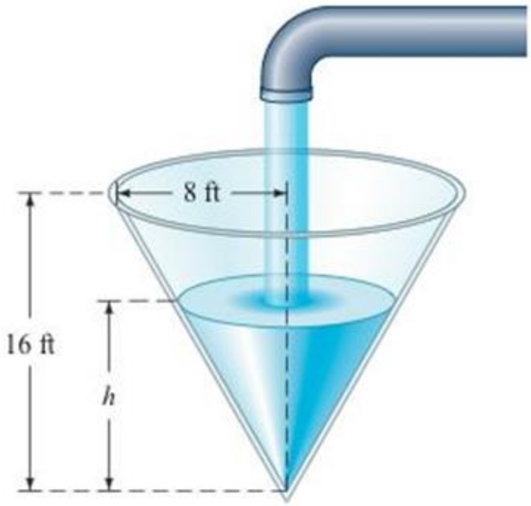
49. Water is flowing into a conical drinking cup with an altitude of 4 *inches* am a radius of 2 *inches*.



a) Write the radius r of the surface of the water as a function of its depth h.

b) Write the volume V of the water as a function of its depth h.

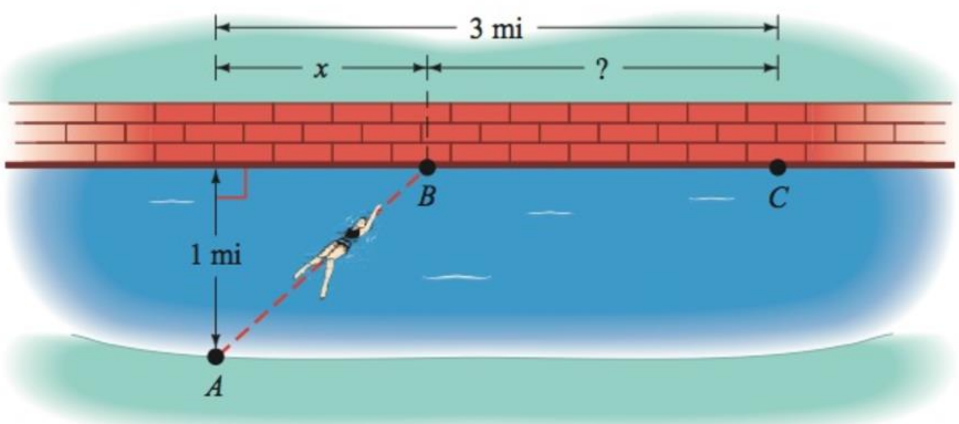
50. A water tank has the shape of a right circular cone with height 16 *feet* and radius 8 *feet*. Water is running into the tank so that the radius r (in *feet*) of the surface of the water is given by $r = 1.5t$, where t is the time (in *minutes*) that the water has been running.



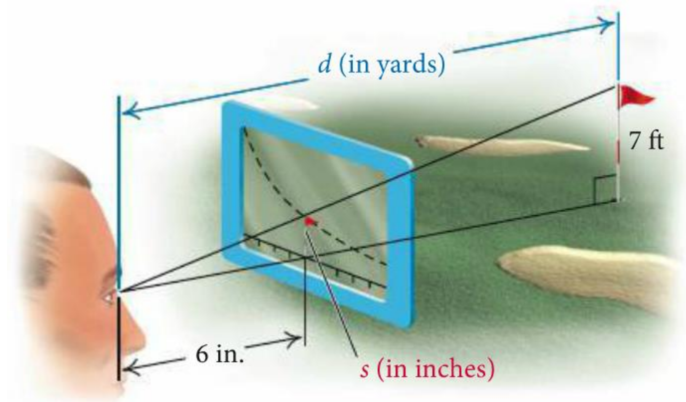
a) The area A of the surface of the water is $A = \pi r^2$. Find $A(t)$ and use it to determine the area of the surface of the water when $t = 2$ *minutes*.

b) The volume V of the water is given by $V = \frac{1}{3}\pi r^2 h$. Find $V(t)$ and use it to determine the volume of the water when $t = 3$ *minutes*

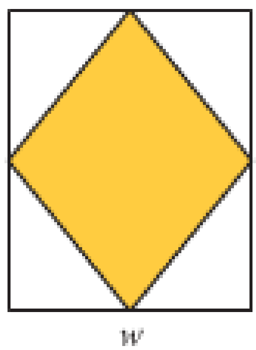
51. An athlete swims from point ***A*** to point ***B*** at a rate of 2 *miles* per *hour* and runs from point ***B*** to point ***C*** at a rate of 8 *miles* per *hour*. Use the dimensions in the figure to write the time t required to reach point ***C*** as a function of x.



52. A device used in golf to estimate the distance ***d***, in *yards*, to a hole measures the size ***s***, in *inches*, that the 7-*feet* pin appears to be in a viewfinder. Express the distance ***d*** as a function of ***s***.



53. A rhombus is inscribed in a rectangle that is ***w*** *meters* wide with a perimeter of 40 *m*. Each vertex of the rhombus is a midpoint of a side of the rectangle. Express the area of the rhombus as a function of the rectangle's width.

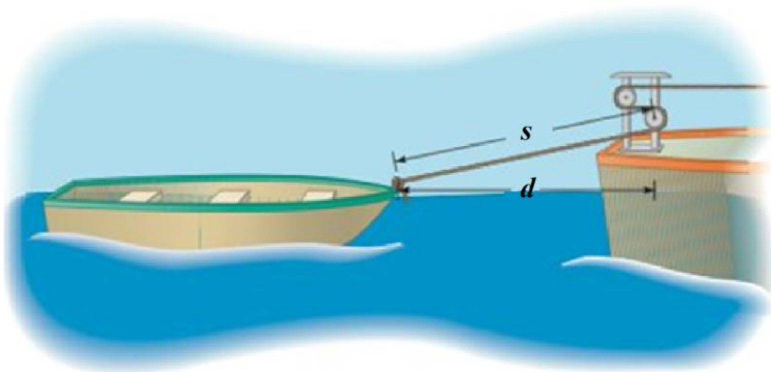


54. The surface area *S* of a right circular cylinder is given by the formula $S = 2\pi rh + 2\pi r^2$. if the height is twice the radius, find each of the following.

a) A function $S(r)$ for the surface area as a function of r.

b) A function $S(h)$ for the surface area as a function of h.

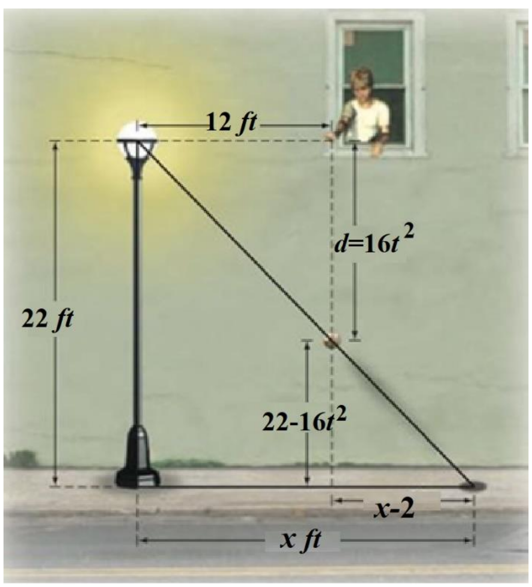
55. A boat is towed by a rope that runs through a pulley that is 4 *feet* above the point where the rope is tied to the boat. The length (in *feet*) of the rope from the boat to the pulley is given by $s = 48 - t$, where t is the time in *seconds* that the boat has been in tow. The horizontal distance from the pulley to the boat is *d*.



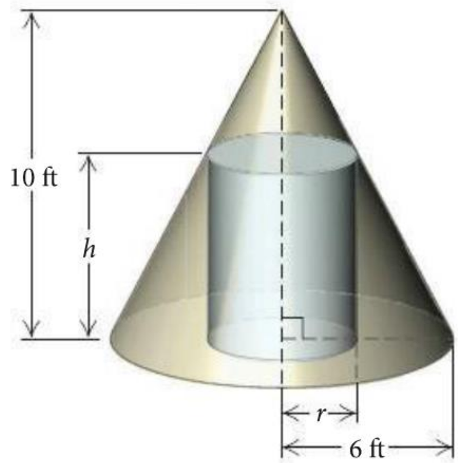
a) Find $d(t)$

b) Evaluate $s(35)$ and $d(35)$

56. The light from a lamppost casts a shadow from a ball that was dropped from a height of 22 *feet* above the ground. The distance *d*, in *feet*, the ball has dropped *t seconds* after it is released is given by $d(t) = 16t^2$. Find the distance *x*, in *feet*, of the shadow from the base of the lamppost as a function of time *t*.



57. A right circular cylinder of height h and a radius r is inscribed in a right circular cone with a height of 10 *feet* and a base with radius 6 *feet*.



a) Express the height h of the cylinder as a function of r.

b) Express the volume V of the cylinder as a function of r.

c) Express the volume V of the cylinder as a function of h.

Section 2.2 – Function Operations

The *Domain* of a Function

1. ***Rational*** function: $\dfrac{f(x)}{h(x)}$ $\Rightarrow$ ***Domain***: $\boxed{h(x) \neq 0}$

Example: $f(x) = \frac{1}{x-3}$

Domain: $x \neq 3$ $\{x \mid x \neq 3\}$

Or $(-\infty, 3) \cup (3, \infty)$ *Interval Notation*

Or $\mathbb{R} - \{3\}$

2. ***Irrational*** function: $\sqrt{g(x)}$ $\Rightarrow$ ***Domain***: $\boxed{g(x) \geq 0}$

Example: $g(x) = \sqrt{3-x} + 5$

$3 - x \geq 0$

$-x \geq -3$

Domain: $x < 3$ $(-\infty,\ 3]$

3. ***Otherwise***: Domain all real numbers $(-\infty,\ \infty)$

Example: $f(x) = x^3 + |x|$

Domain: All real numbers $\mathbb{R}$

$(-\infty,\ \infty)$

(1) & (2) $\rightarrow$ Find the domain: $f(x) = \frac{x+1}{\sqrt{x-3}}$

$x > 3$

Domain: $(3,\ \infty)$

Example

Find the domain

a) $f(x) = x^2 + 3x - 17$

Domain: $\mathbb{R}$

b) $g(x) = \dfrac{5x}{x^2 - 49}$

$x^2 \neq 49$

$x \neq \pm 7$

Domain: $\begin{cases} \{x \mid x \neq \pm 7\} & \text{or} \\ (-\infty, -7) \cup (-7, 7) \cup (7, \infty) \end{cases}$

c) $h(x) = \sqrt{9x - 27}$

$9x - 27 \geq 0$

$9x \geq 27$

Domain: $x \geq 3$ $\quad [3, \infty)$

The *Algebra* of Functions

$$(f+g)(x)=f(x)+g(x)$$

$$(f-g)(x)=f(x)-g(x)$$

$$(f\cdot g)(x)=f(x)\cdot g(x)$$

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$$

Example

Let $f(x)=x^2+1$ and $g(x)=3x+5$. Find each of the following $(f+g)(1)$, $(f-g)(-3)$, $(fg)(5)$, and $\left(\frac{f}{g}\right)(0)$

Solution

$$\begin{aligned}(f+g)(1)&=f(1)+g(1)\\&=1^2+1+3(1)+5\\&=1+1+3+5\\&\underline{=10}\end{aligned}$$

$$\begin{aligned}(f-g)(-3)&=f(-3)-g(-3)\\&=(-3)^2+1-(3(-3)+5)\\&\underline{=14}\end{aligned}$$

$$\begin{aligned}(fg)(5)&=f(5)\cdot g(5)\\&=\left(5^2+1\right)\cdot(3(5)+5)\\&=(26)\cdot(20)\\&\underline{=520}\end{aligned}$$

$$\begin{aligned}\left(\frac{f}{g}\right)(0)&=\frac{f(0)}{g(0)}\\&=\frac{0^2+1}{3(0)+5}\\&\underline{=\frac{1}{5}}\end{aligned}$$

Example

Let $f(x) = 8x - 9$ and $g(x) = \sqrt{2x-1}$. Find each of the following and give the domain

$(f+g)(x), \quad (f-g)(x), \quad (fg)(x), \left(\frac{f}{g}\right)(x)$

Solution

Domain of f: $\quad (-\infty, \infty)$

Domain of g: $\quad \left[\frac{1}{2}, \infty\right) \qquad\qquad \sqrt{2x-1 \geq 0} \rightarrow 2x \geq 1 \Rightarrow x \geq \frac{1}{2}$

a) $(f+g)(x) = 8x - 9 + \sqrt{2x-1}$

Domain: $x \geq \frac{1}{2} \qquad \left[\frac{1}{2}, \infty\right)$

b) $(f-g)(x) = 8x - 9 - \sqrt{2x-1}$

Domain: $x \geq \frac{1}{2} \qquad \left[\frac{1}{2}, \infty\right)$

c) $(fg)(x) = (8x-9)\sqrt{2x-1}$

Domain: $x \geq \frac{1}{2} \qquad \left[\frac{1}{2}, \infty\right)$

d) $\left(\frac{f}{g}\right)(x) = \frac{8x-9}{\sqrt{2x-1}}$

Domain: $x > \frac{1}{2} \qquad \left(\frac{1}{2}, \infty\right)$

Example

Let $f(x) = \sqrt{x-3}$ and $g(x) = \sqrt{x+1}$

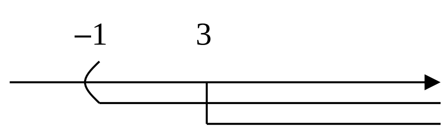
Find $(f+g)(x)$ and its domain, $\left(\frac{f}{g}\right)(x)$ and its domain

Solution

Domain $f(x)$: $x \geq 3$ *and* ***Domain*** $g(x)$: $x \geq -1$

a) $(f+g)(x) = \sqrt{x-3} + \sqrt{x+1}$

b) $x \geq 3$ *and* $x \geq -1 \Rightarrow$ ***Domain***: $x \geq 3$



c) $\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x-3}}{\sqrt{x+1}}$

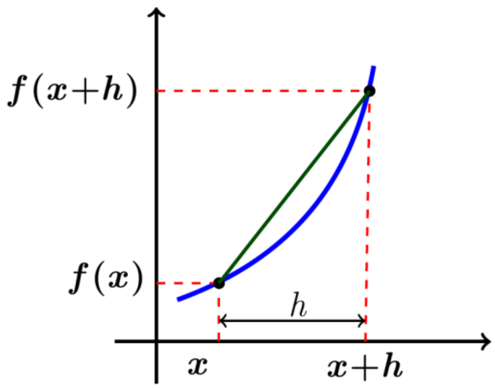
$$\to \begin{cases} x-3\ge 0 & \Rightarrow \quad \underline{x\ge 3|} \\ x+1>0 & \Rightarrow \quad \underline{x>-1|} \end{cases}$$

Domain: $x \ge 3 \qquad [3,\infty)$

Difference Quotients

$$\frac{f(x+h)-f(x)}{(x+h)-x}$$

The difference quotient is given by: $\dfrac{f(x+h)-f(x)}{h}$



Example

For the function f given by $f(x)=2x-3$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned} f(x+h) &= 2(\text{—})-3 \\ &= 2(x+h)-3 \\ &= 2x+2h-3 \end{aligned}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{\overbrace{2x+2h-3}^{f(x+h)} - \overbrace{(2x-3)}^{f(x)}}{h}$$

$$= \frac{2x+2h-3-2x+3}{h}$$

$$= \frac{2h}{h}$$

$$\underline{=2|}$$

Example

For the function *f* given by $f(x) = -2x^2 + x + 5$, find the difference quotient $\frac{f(x+h)-f(x)}{h}$

Solution

$$f(x+h) = -2(\text{——})^2 + (\text{——}) + 5$$

$$f(x+h) = -2(x+h)^2 + (x+h) + 5 \qquad (a+b)^2 = a^2 + 2ab + b^2$$

$$f(x+h) = -2\left(x^2 + 2hx + h^2\right) + x + h + 5$$

$$f(x+h) = -2x^2 - 4hx - 2h^2 + x + h + 5$$

$$\frac{f(x+h) - f(x)}{h} = \frac{-2x^2 - 4hx - 2h^2 + x + h + 5 - (-2x^2 + x + 5)}{h}$$

$$= \frac{-2x^2 - 4hx - 2h^2 + x + h + 5 + 2x^2 - x - 5}{h}$$

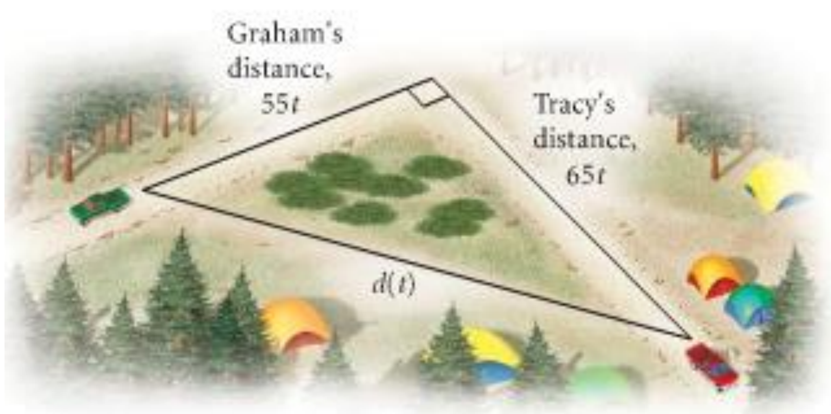
$$= \frac{-4hx - 2h^2 + h}{h}$$

$$= \frac{-4hx}{h} - \frac{2h^2}{h} + \frac{h}{h}$$

$$\boxed{= -4x - 2h + 1}$$

Example

Tracy and Graham drive away from a camp-ground at right angles to each other. Tracy's speed is 65 mph and Graham's is 55 *mph*.



a) Express the distance between the cars as a function of time.

b) Find the domain of the function.

Solution

a) *Distance = velocity * time*

Use Pythagorean Theorem:

$$d^2(t) = (65t)^2 + (55t)^2$$

$$d^2 = 4225t^2 + 3025t^2$$

$$= 7250t^2$$

$$d(t) = \sqrt{7250t^2}$$

$$= \sqrt{7250}\sqrt{t^2}$$

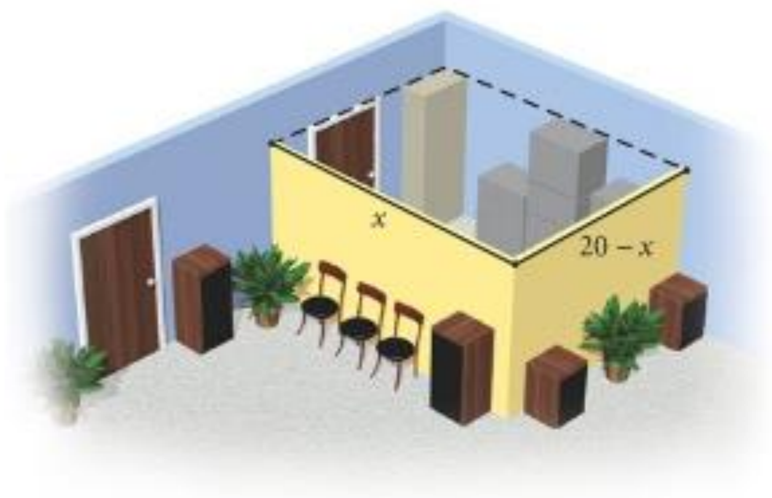
$$\approx 85.15|t|$$

$$\underline{= 85.15\ t}$$

b) ***Domain***: $t \ge 0$

Example: (storage area)

The sound Shop has 20 *feet*. of dividers with which to set off a rectangular area for the storage of overstock. If a corner of the store is used for the storage area, the partition need only form two sides of a rectangle.



a) Express the floor area of the storage space as a function of the length of the partition.

b) Find the domain of the function.

Solution

Let x = the length

width + *length* = 20

width = 20 − *length*

a) ***Area*** = *length* * *width*

$$= x(20 - x)$$

$$\underline{= 20x - x^2}$$

b) ***Domain***: x value varies from 0 to 20 $\Rightarrow$ (0, 20)

Exercises *Section* 2.2 – Function Operations

(**1 – 80**) Find the Domain

1. $f(x)=7x+4$

2. $f(x)=|3x-2|$

3. $f(x)=3x+\pi$

4. $f(x)=\sqrt{7}x+\frac{1}{2}$

5. $f(x)=-2x^2+3x-5$

6. $f(x)=x^3-2x^2+x-3$

7. $f(x)=x^2-2x-15$

8. $f(x)=4-\frac{2}{x}$

9. $f(x)=\frac{1}{x^4}$

10. $g(x)=\frac{3}{x-4}$

11. $y=\frac{2}{x-3}$

12. $y=\frac{-7}{x-5}$

13. $f(x)=\frac{x+5}{2-x}$

14. $f(x)=\frac{8}{x+4}$

15. $f(x)=\frac{1}{x+4}$

16. $f(x)=\frac{1}{x-4}$

17. $f(x)=\frac{3x}{x+2}$

18. $f(x)=x-\frac{2}{x-3}$

19. $f(x)=x+\frac{3}{x-5}$

20. $f(x)=\frac{1}{2}x-\frac{8}{x+7}$

21. $f(x)=\frac{1}{x-3}-\frac{8}{x+7}$

22. $f(x)=\frac{1}{x+4}-\frac{2x}{x-4}$

23. $f(x)=\frac{3x^2}{x+3}-\frac{4x}{x-2}$

24. $f(x)=\frac{1}{x^2-2x+1}$

25. $f(x)=\frac{x}{x^2+3x+2}$

26. $f(x)=\frac{x^2}{x^2-5x+4}$

27. $f(x)=\frac{1}{x^2-4x-5}$

28. $g(x)=\frac{2}{x^2+x-12}$

29. $h(x)=\frac{5}{\frac{4}{x}-1}$

30. $y=\sqrt{x}$

31. $f(x)=\sqrt{8-3x}$

32. $y=\sqrt{4x+1}$

33. $y=\sqrt{7-2x}$

34. $f(x)=\sqrt{8-x}$

35. $f(x)=\sqrt{3-2x}$

36. $f(x)=\sqrt{3+2x}$

37. $f(x)=\sqrt{5-x}$

38. $f(x)=\sqrt{x-5}$

39. $f(x)=\sqrt{6-3x}$

40. $f(x)=\sqrt{3x-6}$

41. $f(x)=\sqrt{2x+7}$

42. $f(x)=\sqrt{x^2-16}$

43. $f(x)=\sqrt{16-x^2}$

44. $f(x)=\sqrt{9-x^2}$

45. $f(x)=\sqrt{x^2-25}$

46. $f(x)=\sqrt{x^2-5x+4}$

47. $f(x)=\sqrt{x^2+5x+4}$

48. $f(x)=\sqrt{x^2+3x+2}$

49. $f(x)=\sqrt{x^2-3x+2}$

50. $f(x)=\sqrt{x-4}+\sqrt{x+1}$

51. $f(x)=\sqrt{3-x}+\sqrt{x-2}$

52. $f(x)=\sqrt{1-x}+\sqrt{4-x}$

53. $f(x)=\sqrt{1-x}-\sqrt{x-3}$

54. $f(x)=\sqrt{x+4}-\sqrt{x-1}$

55. $f(x)=\frac{\sqrt{x+1}}{x}$

56. $g(x)=\frac{\sqrt{x-3}}{x-6}$

57. $f(x)=\frac{\sqrt{x+4}}{\sqrt{x-1}}$

58. $f(x)=\frac{\sqrt{5-x}}{x}$

59. $f(x)=\frac{x}{\sqrt{5-x}}$

60. $f(x)=\dfrac{1}{x\sqrt{5-x}}$

61. $f(x)=\dfrac{x+1}{x^3-4x}$

62. $f(x)=\dfrac{\sqrt{x+5}}{x}$

63. $f(x)=\dfrac{x}{\sqrt{x+5}}$

64. $f(x)=\dfrac{1}{x\sqrt{x+5}}$

65. $f(x)=\dfrac{x+3}{\sqrt{x-3}}$

66. $f(x)=\dfrac{\sqrt{x+3}}{\sqrt{x-3}}$

67. $f(x)=\dfrac{\sqrt{x-2}}{\sqrt{x+2}}$

68. $f(x)=\dfrac{\sqrt{2-x}}{\sqrt{x+2}}$

69. $f(x)=\dfrac{x-4}{\sqrt{x-2}}$

70. $f(x)=\dfrac{1}{(x-3)\sqrt{x+3}}$

71. $f(x)=\sqrt{x+2}+\sqrt{2-x}$

72. $f(x)=\sqrt{(x-2)(x-6)}$

73. $f(x)=\sqrt{x+3}-\sqrt{4-x}$

74. $f(x)=\dfrac{\sqrt{4x-3}}{x^2-4}$

75. $f(x)=\dfrac{4x}{6x^2+13x-5}$

76. $f(x)=\dfrac{\sqrt{2x-3}}{x^2-5x+4}$

77. $f(x)=\dfrac{x^2}{\sqrt{x^2-5x+4}}$

78. $f(x)=\dfrac{x+2}{\sqrt{x^2+5x+4}}$

79. $f(x)=\dfrac{\sqrt{x+2}}{\sqrt{x^2+3x+2}}$

80. $f(x)=\dfrac{\sqrt{2x+3}}{x^2-6x+5}$

81. Let $f(x)=4x-3$ and $g(x)=5x+7$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\dfrac{f}{g}\right)(x)$

82. Let $f(x)=2x^2+3$ and $g(x)=3x-4$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\dfrac{f}{g}\right)(x)$

83. Let $f(x)=x^2-2x-3$ and $g(x)=x^2+3x-2$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\dfrac{f}{g}\right)(x)$

84. Let $f(x)=\sqrt{4x-1}$ and $g(x)=\dfrac{1}{x}$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\dfrac{f}{g}\right)(x)$

85. Given that $f(x)=x+1$ and $g(x)=\sqrt{x+3}$

a) Find $(f+g)(x)$

b) Find the domain of $(f+g)(x)$

c) Find: $(f+g)(6)$

86. Given that $f(x)=x^2-4$ and $g(x)=x+2$

a) Find $(f+g)(x)$ and its domain

b) Find $(f/g)(x)$ and its domain

87. Let $f(x)=x^2+1$ and $g(x)=3x+5$. Find $(f+g)(1)$, $(f-g)(-3)$, $(fg)(5)$, and $\left(\frac{f}{g}\right)(0)$

88. Find $(f+g)(x)$, $(f-g)(x)$, $(f\cdot g)(x)$, *and* $(f/g)(x)$ and the domain of $f(x)=\sqrt{3-2x}$, $g(x)=\sqrt{x+4}$

89. Find $(f+g)(x)$, $(f-g)(x)$, $(f\cdot g)(x)$, *and* $(f/g)(x)$ and the domain of $f(x)=\frac{2x}{x-4}$, $g(x)=\frac{x}{x+5}$

90. Find $(f+g)(x)$, $(f-g)(x)$, $(f\cdot g)(x)$, *and* $(f/g)(x)$ of $f(x)=x-5$ and $g(x)=x^2-1$

(**88 – 103**) Find and simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$ for the given function

91. $f(x)=9x+5$

92. $f(x)=6x+2$

93. $f(x)=4x+11$

94. $f(x)=3x-5$

95. $f(x)=-2x-3$

96. $f(x)=-4x+3$

97. $f(x)=3x-6$

98. $f(x)=-5x-7$

99. $f(x)=2x^2$

100. $f(x)=5x^2$

101. $f(x)=3x^2-4x$

102. $f(x)=2x^2-3x$

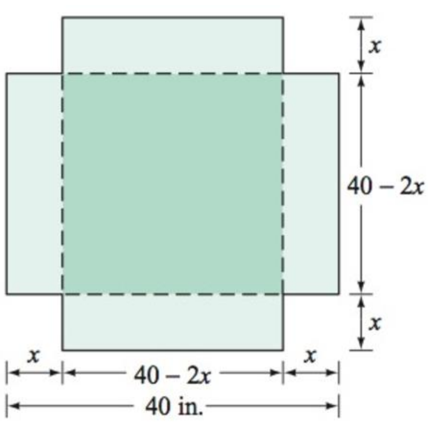
103. $f(x)=2x^2-x-3$

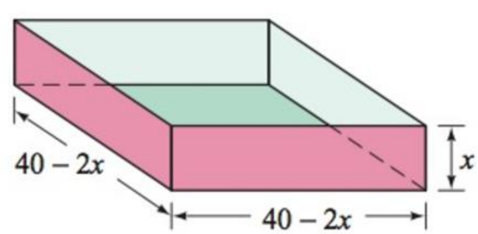
104. $f(x)=x^2-2x+5$

105. $f(x)=3x^2-2x+5$

106. $f(x)=-2x^2-3x+7$

107. An open box is to be made from a square piece of cardboard that measures 40 *inches* on each side, to construct the box, squares that measure *x inches* on each side are cut from each corner of the cardboard.

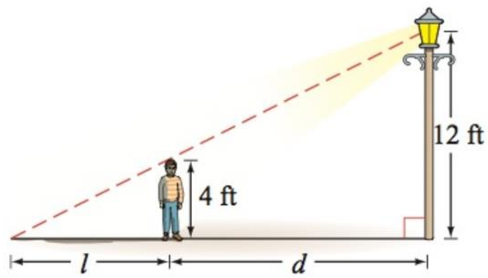




a) Express the volume *V* of the box as a function of *x*.

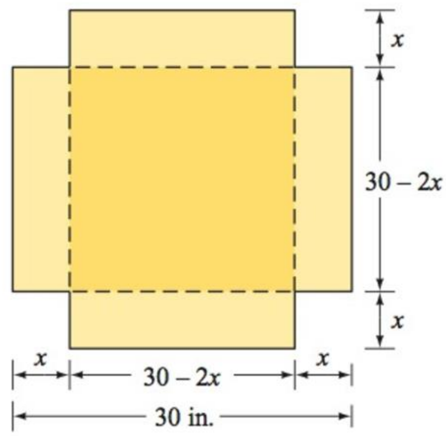
b) Determine the domain of *V*.

108. A child 4 *feet* tall is standing near a street lamp that is 12 *feet* high. The light from the lamp casts a shadow.



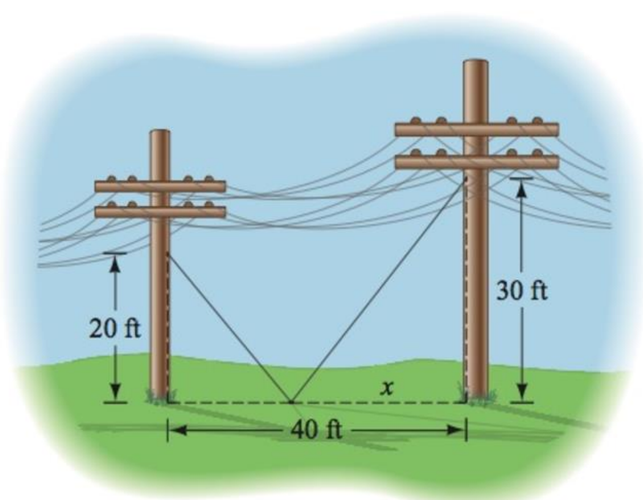
a) Find the length l of the shadow as a function of the distance d of the child from the lamppost.
b) What is the domain of the function?
c) What is the length of the shadow when the child is 8 *feet* from the base of the lamppost?

109. An open box is to be made from a square piece of cardboard with the dimensions 30 *inches* by 30 *inches* by cutting out squares of area x^2 from each corner.



a) Express the volume V of the box as a function of x.
b) Determine the domain of V.

110. Two guy wires are attached to utility poles that are 40 *feet* apart.



a) Find the total length of the two guy wires as a function of *x*.

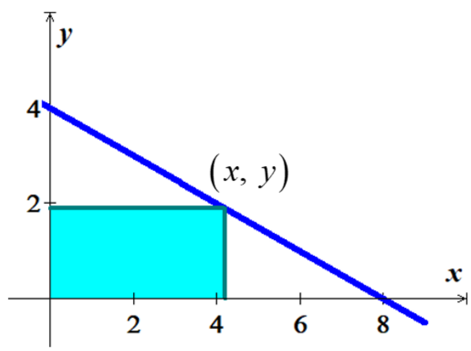
b) What is the domain of this function?

111. A rancher has 360 *yard*s. of fencing with which to enclose two adjacent rectangular corrals, one for sheep and one for cattle. A river forms one side of the corrals. Suppose the width of each corral is *x yards*.

a) Express the total area of the two corrals as a function of *x*.

b) Find the domain of the function.

112. A rectangle is bounded by the *x*- and *y-axis* of $y = -\frac{1}{2}x + 4$



a) Find the area of the rectangle as a function of *x*.

b) What is the domain of this function.

Section 2.3 – Composition Functions

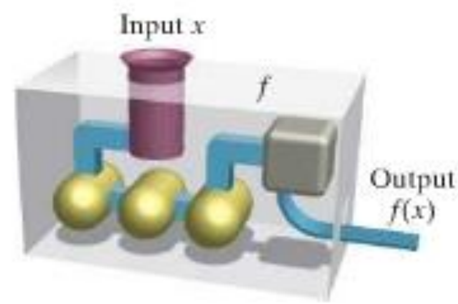
Composition of Functions

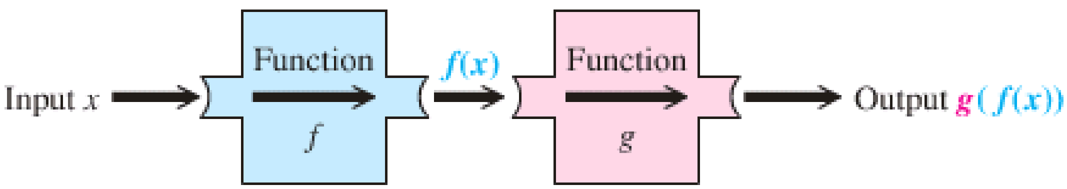
The composite function $g \circ f$, the composite of f and g, is defined as

$$\left(g \circ f\right)(x) = g\left(f\left(x\right)\right)$$

Where x is in the domain of f

and $g\left(x\right)$ is in the domain of f





Example

Given that $f(x) = 5x + 6$ and $g(x) = 2x^2 - x - 1$, find $\left(f \circ g\right)(x)$ and $\left(g \circ f\right)(x)$

Solution

$\left(f \circ g\right)(x) = f\left(g\left(x\right)\right)$ $\qquad = f(2x^2 - x - 1)$ $\qquad$ ***Domain:*** All real numbers

$= 5(\text{---------}) + 6$

$= 5(2x^2 - x - 1) + 6$

$= 10x^2 - 5x - 5 + 6$

$\underline{= 10x^2 - 5x + 1}$ $\qquad$ ***Domain:*** All real numbers

$\left(g \circ f\right)(x) = g\left(f\left(x\right)\right)$

$= g(5x + 6)$ $\qquad$ ***Domain:*** All real numbers

$= 2(\qquad)^2 - (\qquad) - 1$

$= 2(5x + 6)^2 - (5x + 6) - 1$

$= 2(25x^2 + 60x + 36) - 5x - 6 - 1$

$= 50x^2 + 120x + 72 - 5x - 7$

$\underline{= 50x^2 + 115x + 65}$ $\qquad$ ***Domain:*** All real numbers

Example

Let $f(x)=\sqrt{x}$ and $g(x)=4x+2$, find each of the following and its domain.

a) $(f\circ g)(x)$

b) $(g\circ f)(x)$

Solution

a) $(f\circ g)(x)=f(g(x))$

$= f(4x+2) \qquad (-\infty,\infty)$

$=\sqrt{4x+2}$

$$4x+2\ge 0$$
$$4x\ge -2$$
$$x\ge -\frac{2}{4}$$

Domain: $x\ge -\frac{1}{2}$ $\quad \left[-\frac{1}{2},\infty\right)$

b) $(g\circ f)(x)=g(f(x))$

$=g(\sqrt{x}) \qquad x\ge 0$

$=4\sqrt{x}+2 \qquad x\ge 0$

Domain: $x\ge 0$ $\quad [0,\infty)$

Example

Let $f(x)=2x-1$ and $g(x)=\dfrac{4}{x-1}$ Find:

a) $(f\circ g)(2)$

b) $(g\circ f)(-3)$

Solution

a) $(f\circ g)(2)=f(g(2))$

$= f\left(\frac{4}{2-1}\right)$

$= f(4)$

$=2(4)-1$

$=7$

b) $(g\circ f)(-3)=g(f(-3))$

$=g(2(-3)-1)$

$$= g(-7)$$

$$= \frac{4}{-7-1}$$

$$= \frac{4}{-8}$$

$$\boxed{= -\frac{1}{2}}$$

Example

Given that $f(x) = \dfrac{4}{x+2}$ and $g(x) = \dfrac{1}{x}$, find

a) $(f \circ g)(x)$

b) Domain of $(f \circ g)(x)$

Solution

a) $(f \circ g)(x) = f(g(x))$

$$= f\left(\frac{1}{x}\right) \qquad \textbf{\textit{Domain:}}: x \neq 0$$

$$= \frac{4}{\frac{1}{x}+2}$$

$$= \frac{4}{\frac{1+2x}{x}}$$

$$= 4 \div \frac{1+2x}{x}$$

$$= 4\frac{x}{1+2x}$$

$$\boxed{= \frac{4x}{1+2x}} \qquad \textbf{\textit{Domain:}}: x \neq -\frac{1}{2}$$

b) Domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right) \cup (0, \infty)$

Exercises Section 2.3 – Composition Functions

1. Given that $f(x)=2x-5$ and $g(x)=x^2-3x+8$, find $(f\circ g)(x)$, $(g\circ f)(x)$ and their domain then find $(f\circ g)(7)$

2. Given that $f(x)=\sqrt{x}$ and $g(x)=x-1$, find

a) $(f\circ g)(x)=f(g(x))$

b) $(g\circ f)(x)=g(f(x))$

c) $(f\circ g)(2)=f(g(2))$

3. Given that $f(x)=\dfrac{x}{x+5}$ and $g(x)=\dfrac{6}{x}$, find

a) $(f\circ g)(x)=f(g(x))$

b) $(g\circ f)(x)=g(f(x))$

c) $(f\circ g)(2)=f(g(2))$

4. Find $(f\circ g)(x)$, $(g\circ f)(x)$, $f(g(-2))$ *and* $g(f(3))$: $f(x)=2x^2+3x-4$, $g(x)=2x-1$

5. Find $(f\circ g)(x)$, $(g\circ f)(x)$, $f(g(-2))$ *and* $g(f(3))$: $f(x)=x^3+2x^2$, $g(x)=3x$

6. Find $(f\circ g)(x)$, $(g\circ f)(x)$, $f(g(-2))$ *and* $g(f(3))$: $f(x)=|x|$, $g(x)=-7$

(**7 – 36**) For the given function; find:

a) Find $(f\circ g)(x)$ and the ***domain*** of $f\circ g$

b) Find $(g\circ f)(x)$ and the ***domain*** of $g\circ f$

7. $f(x)=x-3$ *and* $g(x)=x+3$

8. $f(x)=\frac{2}{3}x$ *and* $g(x)=\frac{3}{2}x$

9. $f(x)=x-1$ *and* $g(x)=3x^2-2x-1$

10. $f(x)=3x-2$ *and* $g(x)=x^2-5$

11. $f(x)=x^2-2$ *and* $g(x)=4x-3$

12. $f(x)=4x^2-x+10$ *and* $g(x)=2x-7$

13. $f(x)=\sqrt{x}$ *and* $g(x)=x+3$

14. $f(x)=\sqrt{x}$ *and* $g(x)=2-3x$

15. $f(x)=3x+2$ *and* $g(x)=\sqrt{x}$

16. $f(x)=x^4$ *and* $g(x)=\sqrt[4]{x}$

17. $f(x)=x^n$ *and* $g(x)=\sqrt[n]{x}$

18. $f(x)=x^2-3x$ *and* $g(x)=\sqrt{x+2}$

19. $f(x)=\sqrt{x-2}$ *and* $g(x)=\sqrt{x+5}$

20. $f(x)=x^2+2$ *and* $g(x)=\sqrt{3-x}$

21. $f(x)=x^5-2$ *and* $g(x)=\sqrt[5]{x+2}$

22. $f(x)=1-x^2$ *and* $g(x)=\sqrt{x^2-25}$

23. $f(x)=2x+3$ *and* $g(x)=\frac{x-3}{2}$

24. $f(x)=4x-5$ *and* $g(x)=\frac{x+5}{4}$

25. $f(x)=\frac{4}{1-5x}$ *and* $g(x)=\frac{1}{x}$

26. $f(x)=\frac{1}{x-2}$ *and* $g(x)=\frac{x+2}{x}$

27. $f(x)=\frac{1}{1+x}$ *and* $g(x)=\frac{1-x}{x}$

28. $f(x)=\frac{3x+5}{2}$ *and* $g(x)=\frac{2x-5}{3}$

29. $f(x)=\frac{x-1}{x-2}$ *and* $g(x)=\frac{x-3}{x-4}$

30. $f(x)=\frac{6}{x-3}$ *and* $g(x)=\frac{1}{x}$

31. $f(x)=\frac{6}{x}$ *and* $g(x)=\frac{1}{2x+1}$

32. $f(x)=3x-7$ *and* $g(x)=\frac{x+7}{3}$

33. $f(x)=\frac{2x+3}{x-4}$ *and* $g(x)=\frac{4x+3}{x-2}$

34. $f(x)=\frac{2x+3}{x+4}$ *and* $g(x)=\frac{-4x+3}{x-2}$

35. $f(x)=x+1$ *and* $g(x)=x^3-5x^2+3x+7$

36. $f(x)=x-1$ *and* $g(x)=x^3+2x^2-3x-9$

(**37 – 48**) Evaluate each composite function, where $f(x)=2x-3$ *and* $g(x)=x^2-5x$

37. $(f\circ g)(4)$

38. $(g\circ f)(4)$

39. $(f\circ g)(-2)$

40. $(g\circ f)(-2)$

41. $(f\circ f)(-3)$

42. $(g\circ g)(7)$

43. $(f\circ g)\left(\sqrt{2}\right)$

44. $(g\circ f)\left(\sqrt{3}\right)$

45. $(f\circ g)(2a)$

46. $(g\circ f)(3b)$

47. $(f\circ g)(k+1)$

48. $(g\circ f)(k-1)$

Section 2.4 – Quadratic Functions and Models

Quadratic Function

A function f is a ***quadratic function*** if $f(x) = ax^2 + bx + c$

Vertex of a Parabola

The ***vertex*** of the graph of $f(x)$ is

$$V_x \text{ or } x_v = -\frac{b}{2a}$$

$$V_y \text{ or } y_v = f\left(-\frac{b}{2a}\right)$$

$$\textit{Vertex}\text{ Point } \left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right)$$

$f(x) = x^2 - 4x - 2$

$$x = -\frac{b}{2a} = -\frac{-4}{2(1)} = 2$$

$$y = f\left(-\frac{b}{2a}\right) = f(2)$$

$$= (2)^2 - 4(2) - 2$$

$$= -6$$

Vertex point: $\underline{(2, -6)}$

Axis of Symmetry: $x = V_x = -\frac{b}{2a}$

Axis of Symmetry: $x = 2$

Minimum or *Maximum* Point

If $a > 0 \Rightarrow f(x)$ has a ***minimum*** point

If $a < 0 \Rightarrow f(x)$ has a ***maximum*** point

@ vertex point (V_x, V_y)

Minimum point @ $(2, -6)$

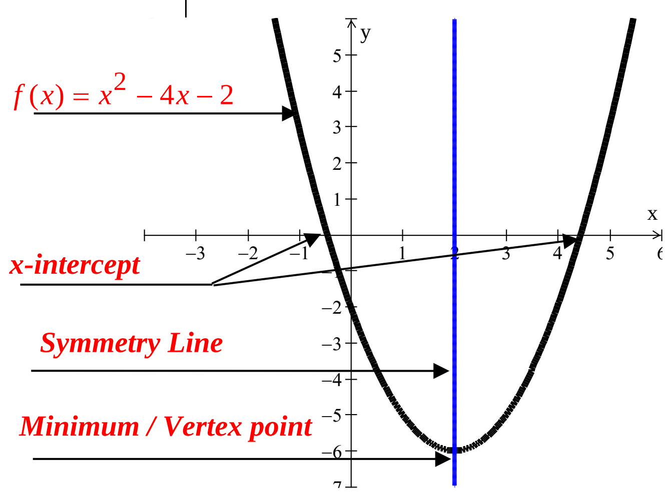
Range

If $a > 0 \Rightarrow [V_y,\ \infty)$

If $a < 0 \Rightarrow (-\infty,\ V_y]$

$[-6, \infty)$

Domain: $(-\infty, \infty)$



Example

For the graph of the function $f(x) = -x^2 - 2x + 8$

a. **Find the vertex point**

$$x = -\frac{-2}{2(-1)} = -1$$

$$y = f(-1) = -(-1)^2 - 2(-1) + 8 = 9$$

Vertex point $(-1, 9)$

b. **Find the line of symmetry:** $x = -1$

c. **State whether there is a maximum or minimum value *and* find that value**

Minimum point, value $(-1, 9)$

d. **Find the *x*-intercept**

$x = -4,\ 2$

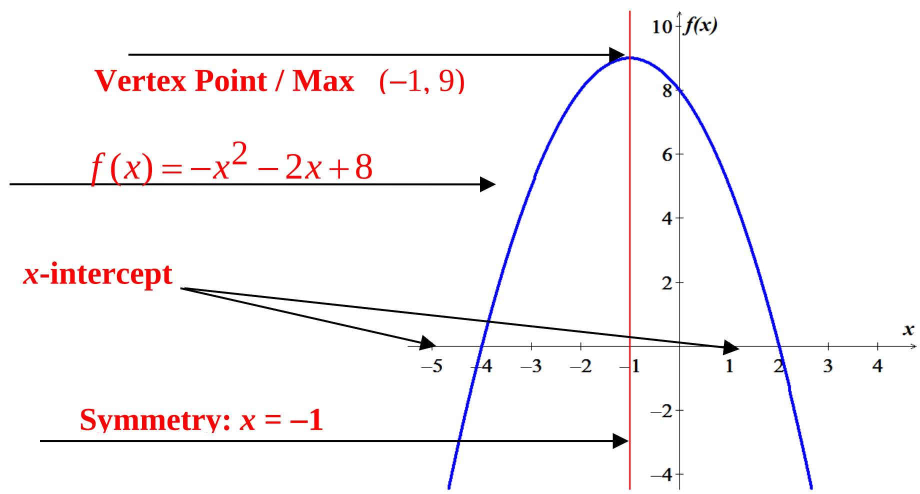
e. **Find the y-intercept**

$y = 8$

f. **Find the range and the domain of the function.**

Range: $(-\infty, 9]$ *Domain*: $(-\infty, \infty)$

g. **Graph the function and label, show part *a thru d* on the plot below**



h. **On what intervals is the function increasing? Decreasing?**

Increasing: $(-\infty, -1)$ *Decreasing*: $(-1, \infty)$

Example

Find the axis and vertex of the parabola having equation $f(x) = 2x^2 + 4x + 5$

Solution

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{4}{2(2)} \\ &= -1 \end{aligned}$$

Axis of the parabola: $x = -1$

$$\begin{aligned} y &= f(-1) \\ &= 2(-1)^2 + 4(-1) + 5 \\ &= 3 \end{aligned}$$

Vertex point: $(-1,\ 3)$

Maximizing Area

You have 120 *feet* of fencing to enclose a rectangular region. Find the dimensions of the rectangle that maximize the enclosed area. What is the maximum area?

Solution

$$\begin{aligned} P &= 2l + 2w \\ 120 &= 2l + 2w \\ 60 &= l + w \qquad \to \boxed{l = 60 - w} \end{aligned}$$

$$\begin{aligned} A &= lw \\ &= (60 - w)w \\ &= 60w - w^2 \\ &= -w^2 + 60w \end{aligned}$$

Vertex: $w = -\frac{60}{2(-1)} = 30$

$\to l = 60 - w = 30$

$$\begin{aligned} A &= lw = (30)(30) \\ &= 900\ ft^2 \end{aligned}$$

Example

A stone mason has enough stones to enclose a rectangular patio with 60 *feet* of stone wall. If the house forms one side of the rectangle, what is the maximum area that the mason can enclose? What should the dimensions of the patio be in order to yield this area?

Solution

$$P = l + 2w = 60$$

$$\boxed{l = 60 - 2w}$$

$$\begin{aligned} A &= lw \\ &= (60 - 2w)w \\ &= 60w - 2w^2 \\ &= -2w^2 + 60w \end{aligned}$$

$$\begin{aligned} w &= -\frac{b}{2a} \\ &= -\frac{60}{2(-2)} \\ &= 15\ ft \end{aligned}$$

$$\begin{aligned} l &= 60 - 2w = 60 - 2(15) \\ &\boxed{= 30\ ft} \end{aligned}$$

$$Area = (15)(30) = 450\ ft^2$$

Position Function (*Projectile Motion*)

Example

A model rocket is launched with an initial velocity of 100 *ft/sec* from the top of a hill that is 20 *feet* high. Its height t seconds after it has been launched is given by the function $s(t) = -16t^2 + 100t + 20$. Determine the time at which the rocket reaches its maximum height and find the maximum height.

Solution

$$\begin{aligned} t &= -\frac{b}{2a} \\ &= -\frac{100}{2(-16)} \\ &\boxed{= 3.125\ \sec} \end{aligned}$$

$$\begin{aligned} s(t = 3.125) &= -16(3.125)^2 + 100(3.125) + 20 \\ &\boxed{= 176.25\ ft} \end{aligned}$$

Exercises *Section* 2.4 – Quadratic Functions and Models

(1 – 21) For the Given functions

a) Find the vertex point
b) Find the line of symmetry
c) State whether there is a *maximum* or *minimum* value *and* find that value
d) Find the zeros of $f(x)$
e) Find the *y*-intercept
f) Find the *range* and the *domain* of the function.
g) Graph the function and label, show part *a thru d*
h) On what intervals is the function *increasing*? *decreasing*?

1. $f(x)=x^2+6x+3$

2. $f(x)=x^2+6x+5$

3. $f(x)=-x^2-6x-5$

4. $f(x)=x^2-4x+2$

5. $f(x)=-2x^2+16x-26$

6. $f(x)=x^2+4x+1$

7. $f(x)=x^2-8x+5$

8. $f(x)=x^2+6x-1$

9. $f(x)=x^2+6x+3$

10. $f(x)=x^2-10x+3$

11. $f(x)=x^2-3x+4$

12. $f(x)=x^2-3x-4$

13. $f(x)=x^2-4x-5$

14. $f(x)=2x^2-3x+1$

15. $f(x)=-x^2-3x+4$

16. $f(x)=-2x^2+3x-1$

17. $f(x)=-2x^2-3x-1$

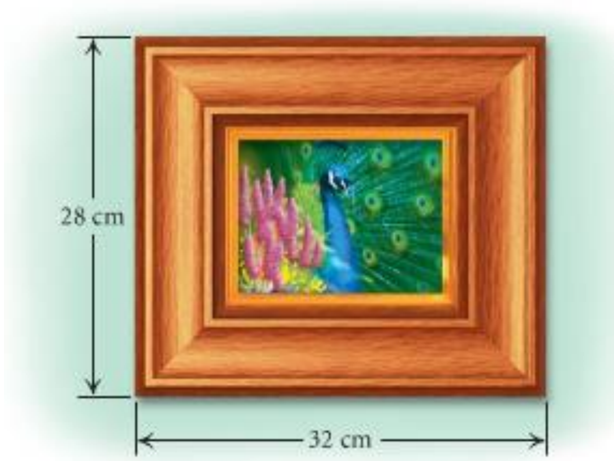
18. $f(x)=-x^2-4x+5$

19. $f(x)=-x^2+4x+2$

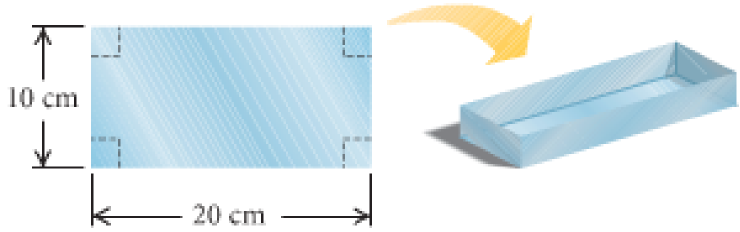
20. $f(x)=-3x^2+3x+7$

21. $f(x)=-x^2+2x-2$

22. A picture frame measures 28 *cm* by 32 *cm* and is of uniform width. What is the width of the frame if 192 cm^2 of the picture shows?



23. An open box is made from a 10-*cm* by 20-*cm* of tin by cutting a square from each corner and folding up the edges. The area of the resulting base is 96 cm^2. What is the length of the sides of the squares?



24. You have 600 *feet.* of fencing to enclose a rectangular plot that borders on a river. If you do not fence the side along the river.

a) Find the length and width of the plot that will maximize the area.

b) What is the largest area that can be enclosed?

25. You have 60 *yards* of fencing to enclosed a rectangular region.

a) Find the dimensions of the rectangle that maximize the enclosed area.

b) What is the maximum area?

26. You have 80 *yards* of fencing to enclosed a rectangular region.

a) Find the dimensions of the rectangle that maximize the enclosed area.

b) What is the maximum area?

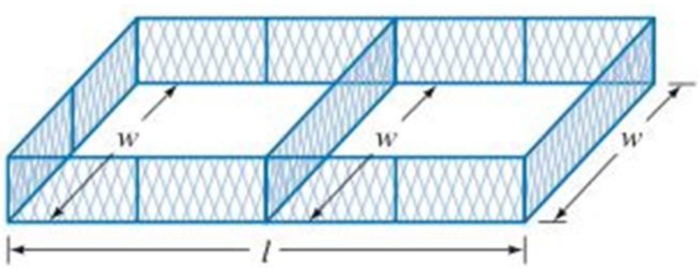
27. The sum of the length l and the width w of a rectangle tangular area is 240 *meters*.

a) Write w as a function of l.

b) Write the area A as a function of l.

c) Find the dimensions that produce the greatest area.

28. You use 600 *feet* of chainlink fencing to enclose a rectangular region and to subdivide the region into two smallerrectangular regions by placing a fence parallel to one of the sides.

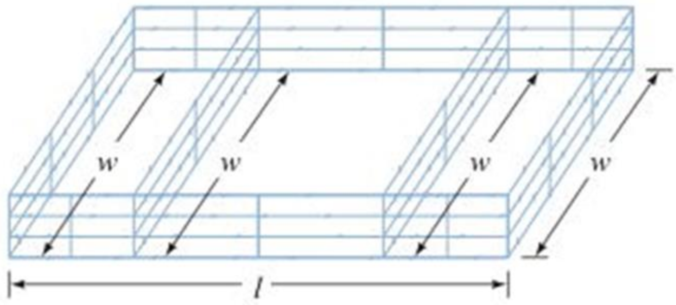


a) Write w as a function of l.

b) Write the area A as a function of l.

c) Find the dimensions that produce the greatest area.

29. You use 1,200 *feet* of chainlink fencing to enclose a rectangular region and to subdivide the region into three smallerrectangular regions by placing a fence parallel to one of the sides.



a) Write w as a function of l.

b) Write the area A as a function of l.

c) Find the dimensions that produce the greatest area.

30. A landscaper has enough stone to enclose a rectangular pond next to exiting garden wall of the house with 24 *feet* of stone wall. If the garden wall forms one side of the rectangle.

a) What is the maximum area that the landscaper can enclose?

b) What dimensions of the pond will yield this area?

31. A berry former needs to separate and enclose two adjacent rectangular fields, one for strawberries and one for blueberries. If a lake forms one side of the fields and 1,000 *feet* of fencing is available, what is the largest total area that can be enclosed?

32. A fourth-grade class decides to enclose a rectangular garden, using the side of the school as one side of the rectangle. What is the maximum area that the class can enclose with 32 *feet* of fence? What should the dimensions of the garden be in order to yield this area?

33. A rancher needs to enclose two adjacent rectangular corrals, one for cattle and one for sheep. If a river forms one side of the corrals and 240 *yard* of fencing is available, what is the largest total area that can be enclosed?

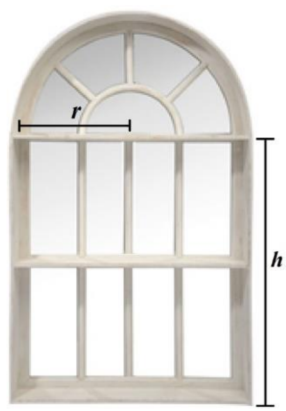
34. A Norman window is a rectangle with a semicircle on top. Sky Blue Windows is designing a Norman window that will require 24 *feet* of trim on the outer edges. What dimensions will allow the maximum amount of light to enter a house?

A Norman window

35. A Norman window has the shape of a rectangle surmounted by a semicircle. The exterior perimeter of the window is 48 *feet*.

Find the height h and the radius r that will allow the maximum amount of light to enter the window?

36. A Norman window has the shape of a rectangle surmounted by a semicircle. It requires 24 *feet* of trim on the outer edges.



What dimensions will allow the maximum amount of light to enter a house?

37. The temperature $T(t)$, in degrees Fahrenheit, during the day can be modeled by the equation $T(t) = -0.7t^2 + 9.4t + 59.3$, where t is the number of hours after 6:00 AM.

a) At what time the temperature a maximum?

b) What is the maximum temperature?

38. When a softball player swings a bat, the amount of energy $E(t)$, in *joules*, that is transferred to the bat can be approximated by the function

$$E(t) = -279.67t^2 + 82.86t$$

Where $0 \le t \le 0.3$ and t is measured in *seconds*. According to this model, what is the maximum energy of the bat?

39. Some softball fields are built in a parabolic mound shape so that water will drain off the field. A model for the shape of a certain field is given by

$$h(x) = -0.0002348x^2 + 0.0375x$$

Where $h(x)$ is the height, in *feet*, of the field at a distance of x *feet* from one sideline. Find the maximum height of the field.

40. The fuel efficiency for a certain midsize car is given by

$$E(v) = -0.018v^2 + 1.476v + 3.4$$

Where $E(v)$ is the fuel efficiency in *miles* per *gallon* for a car traveling v in *miles* per *hour*.

a) What speed will yield the maximum fuel efficiency?
b) What is the maximum fuel efficiency for this car?

41. If the initial velocity of a projectile is 128 *feet* per *second*, then the height *h*, in *feet*, is a function of time t, in *seconds*, given by the equation

$$h(t) = -16t^2 + 128t$$

a) Find the time t when the projectile achieves its maximum height.
b) Find the maximum height of the projectile.
c) Find the time t when the projectile hits the ground.

42. If the initial velocity of a projectile is 64 *feet* per *second* and an initial height of 80 *feet*, then the height *h*, in *feet*, is a function of time t, in *seconds*, given by the equation

$$h(t) = -16t^2 + 64t + 80$$

a) Find the time t when the projectile achieves its maximum height.
b) Find the maximum height of the projectile.
c) Find the time t when the projectile hits the ground.

43. If the initial velocity of a projectile is 100 *feet* per *second* and an initial height of 20 *feet*, then the height *h*, in *feet*, is a function of time t, in *seconds*, given by the equation

$$h(t) = -16t^2 + 100t + 20$$

a) Find the time t when the projectile achieves its maximum height.
b) Find the maximum height of the projectile.
c) Find the time t when the projectile hits the ground.

44. A frog leaps from a stump 3.5 *feet* high and lands 3.5 *feet* from the base of the stump.

It is determined that the height of the frog as a function of its distance, x, from the base of the stump is given by the function $h(x) = -0.5x^2 + 0.75x + 3.5$ where h is in feet.

a) How high is the frog when its horizontal distance from the base of the stump is 2 *feet*?

b) At what two distances from the base of the stump after is jumped was the frog 3.6 *feet* above the ground?

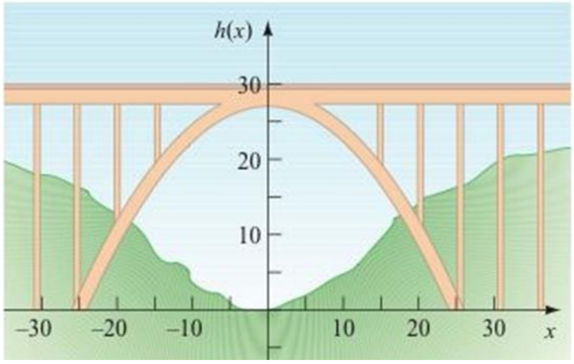
c) At what distance from the base did the frog reach its highest point?

d) What was the maximum height reached by the frog?

45. The height of an arch is given by

$$h(x) = -\frac{3}{64}x^2 + 27, \quad -24 \le x \le 24$$

Where $|x|$ is the horizontal distance in *feet* from the center of the arch to the ground.



a) What is the maximum height of the arch?

b) What is the height of the arch 10 *feet* to the right of center?

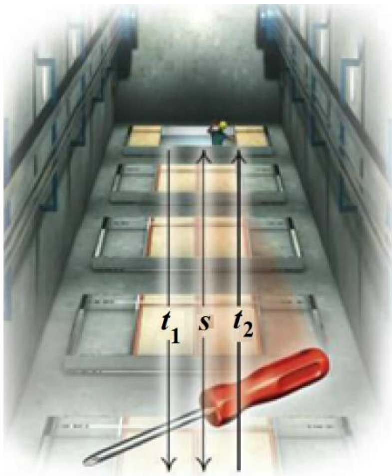
c) How far from the center is the arch 8 *feet* tall?

46. A weightless environment can be created in an airplane by flying in a series of parabolic paths. This is one method that NASA uses to train astronauts for the experience of weightlessness. Suppose the height *h*, in *feet*, of NASA's airplane is modeled by

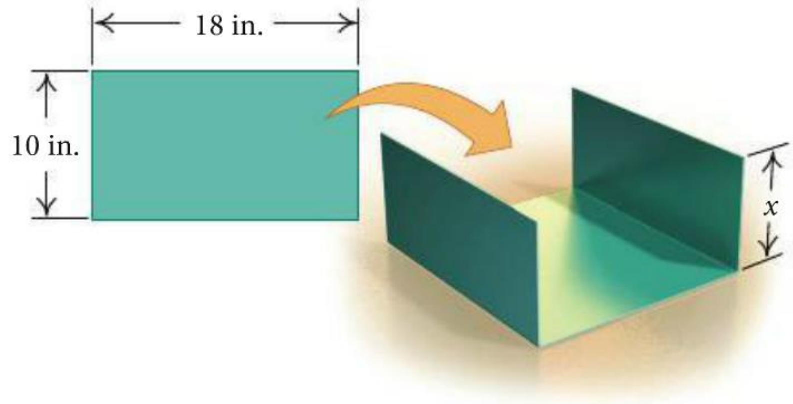
$$h(t) = -6.6t^2 + 430t + 28,000$$

Where t is the time, in *seconds*, after the plane enters its parabolic path.
Find the maximum height of the plane.

47. You drop a screwdriver from the top of an elevator shaft. Exactly 5 *seconds* later, you hear the sound of the screwdriver hitting the bottom of the shaft. The speed of sound is 1,100 *ft/sec*. How tall is the elevator shaft?



48. A company plans to produce a one- compartment vertical file by bending the long side of a 10-*in*. by 18-*in*. sheet of metal along two lines to form a $\lfloor\_\rfloor-$ shape. How tall should the file be in order to maximize the volume that it can hold?



49. The sum of the base and the height of a triangle is 20 *cm*. Find the dimensions for which the area is a maximum.

50. The sum of the base and the height of a parallelogram is 14 *inches*. Find the dimensions for which the area is a maximum.

Section 2.5 – Polynomial Functions

Polynomial Function

A *Polynomial function* $P(x)$ in x is a sum of the form is given by:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

Where the coefficients $a_n,\ a_{n-1},\ \ldots, a_2,\ a_1, a_0$ are real numbers and the exponents are whole numbers.

$a_n x^n$

Degree

Leading Term

Leading Coefficient

Non-polynomial Functions: $\frac{1}{x} + 2x;\quad \sqrt{x^2 - 3} + x;\quad \frac{x-5}{x^2+2}$

Degree of f	***Form of f(x)***	***Graph of f(x)***
0	$f(x) = a_0$	A horizontal line
1	$f(x) = a_1 x + a_0$	A line with slope a_1
2	$f(x) = a_2 x^2 + a_1 x + a_0$	A parabola with a vertical axis

All polynomial functions are ***continuous functions***.

End Behavior $\left(a_n x^n\right)$

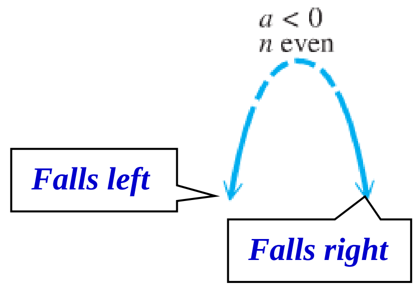
If n (degree) is ***even***:

If $a_n < 0$ (in front x^n is negative).

Then the function falls from the left and right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to -\infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty$$

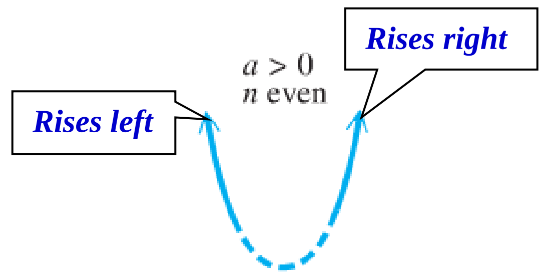


If $a_n > 0$ (in front x^n is positive).

Then the function rises from the left and right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to \infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to \infty$$



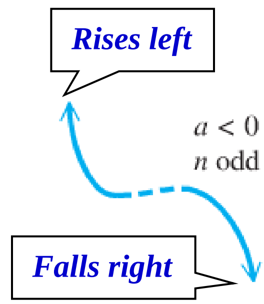
If n (degree) is ***odd***:

If $a_n < 0$ (negative).

Then the function rises from the left side and falls from the right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to \infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty$$

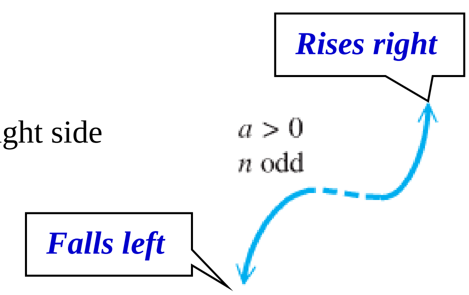


If $a_n > 0$ (positive).

Then the function falls from the left side and rises from the right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to -\infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to \infty$$



Example

Determine the end behavior of the graph of the polynomial function $f(x) = -4x^5 + 7x^2 - x + 9$

Solution

Leading term: $-4x^5$ with 5th degree (n is odd)

$x \to -\infty \quad \Rightarrow \quad f(x) = -(-)^5 = (-)(-) = + \to \infty \qquad f(x)$ rises left

$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty \qquad f(x)$ falls right

The Intermediate Value *Theorem*

For any polynomial function $f(x)$ with real coefficients and $f(a) \neq f(b)$ for $a < b$, then f takes on every value between $f(a)$ and $f(b)$ in the interval $[a,\ b]$.

$\therefore$ $f(a)$ and $f(b)$ are the ***opposite signs***. Then the function has a real zero between a and b.

Example

Using the intermediate value theorem, determine, if possible, whether the function has a real zero between *a* and *b*.

a) $f(x) = x^3 + x^2 - 6x; \quad a = -4,\ b = -2$

b) $f(x) = x^3 + x^2 - 6x; \quad a = -1,\ b = 3$

Solution

a) $f(x) = x^3 + x^2 - 6x; \quad a = -4,\ b = -2$

$f(-4) = (-4)^3 + (-4)^2 - 6(-4)$

$= -24$

$f(-2) = (-2)^3 + (-2)^2 - 6(-2)$

$= 8$

$\therefore$ $f(x)$ has a zero between -4 and -2

b) $f(x) = x^3 + x^2 - 6x; \quad a = -1,\ b = 3$

$f(-1) = (-1)^3 + (-1)^2 - 6(-1)$

$= 6$

$f(3) = (3)^3 + (3)^2 - 6(3) = 18$

$= 18$

$\therefore$ $f(x)$ *zeros* ***can't be determined***

Example

Show that $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3$ has a zero between 1 and 2.

Solution

$f(1) = 1 + 2 - 6 + 2 - 3$

$= -4$

$f(2) = (2)^5 + 2(2)^4 - 6(2)^3 + 2(2) - 3$

$= 17$

Since $f(1)$ *and* $f(2)$ have opposite signs.

Therefore, $f(c) = 0$ for at least one real number c between 1 and 2.

Exercises *Section* 2.5 – Polynomial Functions

Determine the end behavior of the graph of the polynomial function

1. $f(x) = 5x^3 + 7x^2 - x + 9$

2. $f(x) = 11x^3 - 6x^2 + x + 3$

3. $f(x) = -11x^3 - 6x^2 + x + 3$

4. $f(x) = 2x^3 + 3x^2 - 23x - 42$

5. $f(x) = 5x^4 + 7x^2 - x + 9$

6. $f(x) = 11x^4 - 6x^2 + x + 3$

7. $f(x) = -5x^4 + 7x^2 - x + 9$

8. $f(x) = -11x^4 - 6x^2 + x + 3$

9. $f(x) = 5x^5 - 16x^2 - 20x + 64$

10. $f(x) = -5x^5 - 16x^2 - 20x + 64$

11. $f(x) = -3x^6 - 16x^3 + 64$

12. $f(x) = 3x^6 - 16x^3 + 4$

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers.

13. $f(x) = x^3 - x - 1;$ *between* 1 *and* 2

14. $f(x) = x^3 - 4x^2 + 2;$ *between* 0 *and* 1

15. $f(x) = 2x^4 - 4x^2 + 1;$ *between* -1 *and* 0

16. $f(x) = x^4 + 6x^3 - 18x^2;$ *between* 2 *and* 3

17. $f(x) = x^3 + x^2 - 2x + 1;$ *between* -3 *and* -2

18. $f(x) = x^5 - x^3 - 1;$ *between* 1 *and* 2

19. $f(x) = 3x^3 - 10x + 9;$ *between* -3 *and* -2

20. $f(x) = 3x^3 - 8x^2 + x + 2;$ *between* 2 *and* 3

21. $f(x) = 3x^3 - 8x^2 + x + 2;$ *between* 1 *and* 2

22. $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3;$ *between* 0 *and* 1

23. $P(x) = 2x^3 + 3x^2 - 23x - 42, \quad a = 3, \quad b = 4$

24. $P(x) = 4x^3 - x^2 - 6x + 1, \quad a = 0, \quad b = 1$

25. $P(x) = 3x^3 + 7x^2 + 3x + 7, \quad a = -3, \quad b = -2$

26. $P(x) = 2x^3 - 21x^2 - 2x + 25, \quad a = 1, \quad b = 2$

27. $P(x) = 4x^4 + 7x^3 - 11x^2 + 7x - 15, \quad a = 1, \quad b = \frac{3}{2}$

28. $P(x) = 5x^3 - 16x^2 - 20x + 64, \quad a = 3, \quad b = \frac{7}{2}$

29. $P(x) = x^4 - x^2 - x - 4, \quad a = 1, \quad b = 2$

30. $P(x) = x^3 - x - 8,\quad a = 2,\quad b = 3$

31. $P(x) = x^3 - x - 8,\quad a = 0,\quad b = 1$

32. $P(x) = x^3 - x - 8,\quad a = 2.1,\quad b = 2.2$

Section 2.6 – Properties of Division

Long Division

Divide $\left(x^3+2x^2-5x-6\right)\ \div\ (x+1)$

$$
\begin{array}{r l}
 & \overbrace{x^2+x-6}^{\text{Quotient}} \\
\underset{\text{Divisor}}{x+1} & \overline{\big)\,x^3+2x^2-5x-6} \leftarrow \text{Dividend} \\
 & \underline{x^3+x^2} \\
 & \quad x^2-5x \\
 & \quad \underline{x^2-x} \\
 & \qquad -6x-6 \\
 & \qquad \underline{-6x-6} \\
 & \qquad\qquad 0 \leftarrow \text{Remainder}
\end{array}
$$

$\boxed{Q(x)=x^2+x-6}$

$\boxed{R(x)=0}$

Example

Use the long division to find the quotient and the remainder: $\left(x^4-16\right)\div\left(x^2+3x+1\right)$

Solution

$$
\begin{array}{r l}
 & \qquad\qquad\qquad x^2-3x+8 \\
x^2+3x+1 & \overline{\big)\,x^4+0x^3+0x^2+0x-16} \\
 & \underline{x^4+3x^3+x^2} \\
 & \quad -3x^3-x^2 \\
 & \quad \underline{-3x^3-9x^2-3x} \\
 & \qquad\qquad 8x^2+\ \ 3x-16 \\
 & \qquad\qquad \underline{8x^2+24x+\ 8} \\
 & \qquad\qquad\quad -21x-24
\end{array}
$$

$$\frac{x^4-16}{x^2+3x+1}=x^2-3x+8+\frac{-21x-24}{x^2+3x+1}$$

$$x^4-16=\boxed{\left(x^2+3x+1\right)\left(x^2-3x+8\right)+(-21x-24)}$$

Remainder *Theorem*

If a number c is substituted for x in the polynomial $f(x)$, then the result $f(c)$ is the remainder that would be obtained by dividing $f(x)$ by $x-c$.

That is, if $f(x)=(x-c)Q(x)+R(x)$ *then* $f(c)=R$

Example

If $f(x)=x^3-3x^2+x+5$, use the remainder theorem to find $f(2)$

Solution

$$
\begin{array}{r|l}
 & x^2 - x - 1 \\
\hline
x-2 & x^3 - 3x^2 + x + 5 \\
 & \underline{x^3 - 2x^2} \\
 & \quad - x^2 + x \\
 & \quad \underline{-x^2 + 2x} \\
 & \qquad\quad - x + 5 \\
 & \qquad\quad \underline{-x + 2} \\
 & \qquad\qquad\quad \boxed{3}
\end{array}
$$

$f(2)=3$

Factor *Theorem*

A polynomial $f(x)$ has a factor $x-c$ if and only if $f(c)=0$

Example

Show that $x-2$ is a factor of $f(x)=x^3-4x^2+3x+2$.

Solution

Since $f(2)=(2)^3-4(2)^2+3(2)=0$

From the factor theorem; $x-2$ is a factor of $f(x)$.

Synthetic Division

Use synthetic division to find the quotient and the remainder of $\left(4x^3 - 3x^2 + x + 7\right) \div (x-2)$

$$\begin{array}{c|cccc} & x^3 & x^2 & x^1 & x^0 \\ 2 & 4 & -3 & 1 & 7 \\ & & 8 & 10 & 22 \\ \hline & 4 & 5 & 11 & \boxed{29} \\ & x^2 & x^1 & x^0 & \end{array}$$

Quotient: $Q(x) = 4x^2 + 5x + 11$

Remainder: $R(x) = 29$

Example

If $f(x) = 3x^5 - 38x^3 + 5x^2 - 1$, use the synthetic division to find $f(4)$.

Solution

$$\begin{array}{c|cccccc} 4 & 3 & 0 & -38 & 5 & 0 & -1 \\ & & 12 & 48 & 40 & 180 & 720 \\ \hline & 3 & 12 & 10 & 45 & 180 & \boxed{719} \end{array}$$

$f(4) = 719$

Example

Show that -11 is a zero of the polynomial $f(x) = x^3 + 8x^2 - 29x + 44$

Solution

$$\begin{array}{c|cccc} -11 & 1 & 8 & -29 & 44 \\ & & -11 & 33 & -44 \\ \hline & 1 & -3 & 4 & \boxed{0} \end{array}$$

Thus, $f(-11) = 0$, and -11 is a zero of f.

The Rational Zeros *Theorem*

If the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ *has integer* coefficients and if $\frac{c}{d}$ is a rational zero of $f(x)$ such that ***c*** and ***d*** have no common prime factor, then

1. The numerator c of the zero is a factor of the constant term a_0
2. The denominator d of the zero is a factor of the leading coefficient a_n

$$\textit{possible rational zeros} = \frac{\textit{factors of the constant term } a_0}{\textit{factors of the leading coefficient } a_n} = \frac{\textit{possibilities for } a_0}{\textit{possibilities for } a_n}$$

Example

Find all rational solutions of the equation: $3x^4 + 14x^3 + 14x^2 - 8x - 8 = 0$

Solution

possibilities for a_0	$\pm 1, \ \pm 2, \ \pm 4, \ \pm 8$
possibilities for a_n	$\pm 1, \ \pm 3$
possibilities for c / d	$\pm 1, \ \pm 2, \ \pm 4, \ \pm 8, \ \pm \frac{1}{3}, \ \pm \frac{2}{3}, \ \pm \frac{4}{3}, \ \pm \frac{8}{3}$

Using the calculator, the result will show that –2 is a zero.

$$\begin{array}{r|rrrrr} -2 & 3 & 14 & 14 & -8 & -8 \\ & & -6 & -16 & 4 & 8 \\ \hline & 3 & 8 & -2 & -4 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)\left(3x^3 + 8x^2 - 2x - 4\right) = 0$

For $3x^3 + 8x^2 - 2x - 4 \quad \Rightarrow \quad \frac{c}{d} = \frac{\pm 1, \ \pm 2, \ \pm 4}{\pm 1, \ \pm 3}$

$x = -\frac{2}{3}$ is another solution.

$$\begin{array}{r|rrrr} -\frac{2}{3} & 3 & 8 & -2 & -4 \\ & & -2 & -4 & 4 \\ \hline & 3 & 6 & -6 & \boxed{0} \end{array}$$

We have the factorization of: $(x+2)\left(x + \frac{2}{3}\right)\left(3x^2 + 6x - 6\right) = 0$

By applying quadratic formula to solve: $3x^2 + 6x - 6 = 0 \Rightarrow x = -1 \pm \sqrt{3}$

Hence, the polynomial has two rational roots $x = -2$ *and* $-\frac{2}{3}$ and two irrational roots $x = -1 \pm \sqrt{3}$.

Exercises *Section* 2.6 – Properties of Division

1. Find the quotient and remainder if $f(x)$ is divided by $p(x)$:
$f(x)=2x^4-x^3+7x-12;\quad p(x)=x^2-3$

Find the quotient and remainder if $f(x)$ is divided by $p(x)$

2. $f(x)=3x^3+2x-4;\quad p(x)=2x^2+1$

3. $f(x)=7x+2;\quad p(x)=2x^2-x-4$

4. $f(x)=9x+4;\quad p(x)=2x-5$

5. Use the remainder theorem to find $f(c)$: $f(x)=x^4-6x^2+4x-8;\quad c=-3$

6. Use the remainder theorem to find $f(c)$: $f(x)=x^4+3x^2-12;\quad c=-2$

7. Use the factor theorem to show that $x-c$ is a factor of $f(x)$: $f(x)=x^3+x^2-2x+12;\quad c=-3$

8. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $2x^3-3x^2+4x-5;\quad x-2$

9. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $5x^3-6x^2+15;\quad x-4$

10. Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $9x^3-6x^2+3x-4;\quad x-\frac{1}{3}$

Use the synthetic division to find $f(c)$:

11. $f(x)=2x^3+3x^2-4x+4;\quad c=3$

12. $f(x)=8x^5-3x^2+7;\quad c=\frac{1}{2}$

13. $f(x)=x^3-3x^2-8;\quad c=1+\sqrt{2}$

14. Use the synthetic division to show that c is a zero of $f(x)$: $f(x)=3x^4+8x^3-2x^2-10x+4;\quad c=-2$

15. Use the synthetic division to show that c is a zero of $f(x)$:
$f(x)=27x^4-9x^3+3x^2+6x+1;\quad c=-\frac{1}{3}$

16. Find all values of k such that $f(x)$ is divisible by the given linear polynomial:
$f(x)=kx^3+x^2+k^2x+3k^2+11;\quad x+2$

(**17 – 62**) Find all solutions of the equation

17. $x^3 - x^2 - 10x - 8 = 0$

18. $x^3 + x^2 - 14x - 24 = 0$

19. $2x^3 - 3x^2 - 17x + 30 = 0$

20. $12x^3 + 8x^2 - 3x - 2 = 0$

21. $x^3 + x^2 - 6x - 8 = 0$

22. $x^3 - 19x - 30 = 0$

23. $2x^3 + x^2 - 25x + 12 = 0$

24. $3x^3 + 11x^2 - 6x - 8 = 0$

25. $2x^3 + 9x^2 - 2x - 9 = 0$

26. $x^3 + 3x^2 - 6x - 8 = 0$

27. $3x^3 - x^2 - 6x + 2 = 0$

28. $x^3 - 8x^2 + 8x + 24 = 0$

29. $x^3 - 7x^2 - 7x + 69 = 0$

30. $x^3 - 3x - 2 = 0$

31. $x^3 - 2x + 1 = 0$

32. $x^3 - 2x^2 - 11x + 12 = 0$

33. $x^3 - 2x^2 - 7x - 4 = 0$

34. $x^3 - 10x - 12 = 0$

35. $x^3 - 5x^2 + 17x - 13 = 0$

36. $6x^3 + 25x^2 - 24x + 5 = 0$

37. $8x^3 + 18x^2 + 45x + 27 = 0$

38. $3x^3 - x^2 + 11x - 20 = 0$

39. $x^4 - x^3 - 9x^2 + 3x + 18 = 0$

40. $2x^4 - 9x^3 + 9x^2 + x - 3 = 0$

41. $6x^4 + 5x^3 - 17x^2 - 6x = 0$

42. $x^4 - 2x^2 - 16x - 15 = 0$

43. $x^4 - 2x^3 - 5x^2 + 8x + 4 = 0$

44. $2x^4 - 17x^3 + 4x^2 + 35x - 24 = 0$

45. $x^4 + x^3 - 3x^2 - 5x - 2 = 0$

46. $6x^4 - 17x^3 - 11x^2 + 42x = 0$

47. $x^4 - 5x^2 - 2x = 0$

48. $3x^4 - 4x^3 - 11x^2 + 16x - 4 = 0$

49. $6x^4 + 23x^3 + 19x^2 - 8x - 4 = 0$

50. $4x^4 - 12x^3 + 3x^2 + 12x - 7 = 0$

51. $2x^4 - 9x^3 - 2x^2 + 27x - 12 = 0$

52. $2x^4 - 19x^3 + 51x^2 - 31x + 5 = 0$

53. $4x^4 - 35x^3 + 71x^2 - 4x - 6 = 0$

54. $2x^4 + 3x^3 - 4x^2 - 3x + 2 = 0$

55. $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$

56. $6x^5 + 19x^4 + x^3 - 6x^2 = 0$

57. $3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6 = 0$

58. $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = 0$

59. $x^5 - x^4 - 7x^3 + 7x^2 + 12x - 12 = 0$

60. $x^5 - 2x^3 - 8x = 0$

61. $x^5 - 32 = 0$

62. $3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24 = 0$

63. Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k) = \frac{1}{6}\left(k^3 + 3k^2 + 2k\right)$$

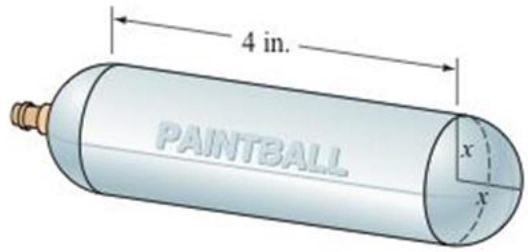
Where *k* is the number of levels in the pyramid. If 220 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

64. Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k) = \frac{1}{6}\left(2k^3 + 3k^2 + k\right)$$

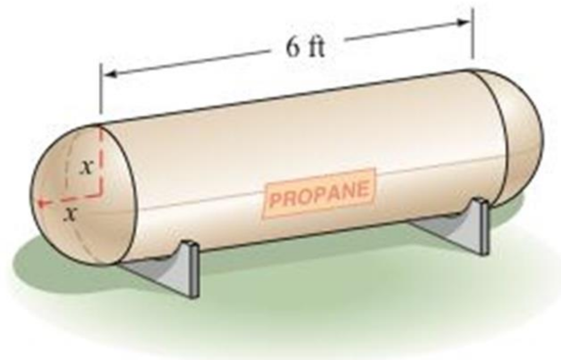
Where *k* is the number of levels in the pyramid. If 140 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

65. A carbon dioxide cartridge for a paintball rifle has the shape of a right circular cylinder with a hemisphere at each end. The cylinder is 4 *inches* long, and the volume of the cartridge is $2\pi \ \ in^3$.

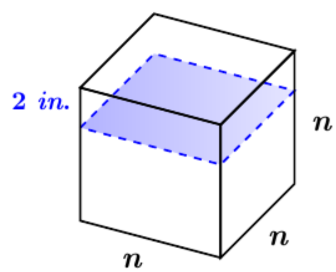


The common interior radius of the cylinder and the hemispheres is denoted by *x*. Estimate the length of the radius *x*.

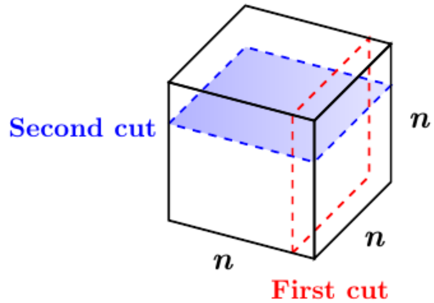
66. A propane tank has the shape of a circular cylinder with a hemisphere at each end. The cylinder is 6 *feet* long and the volume of the tank is $9\pi \ \ ft^3$. Find the length of the radius *x*.



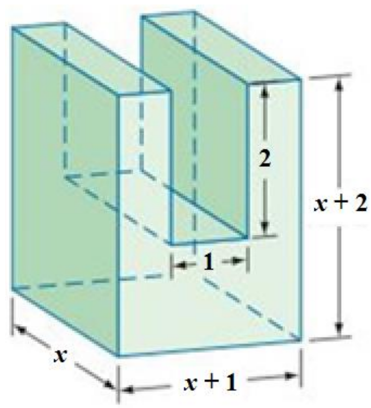
67. A cube measures *n* inches on each edge. If a slice 2 *inches* thick is cut from one face of the cube, the resulting solid has a volume of $567 \ \ in^3$. Find *n*.



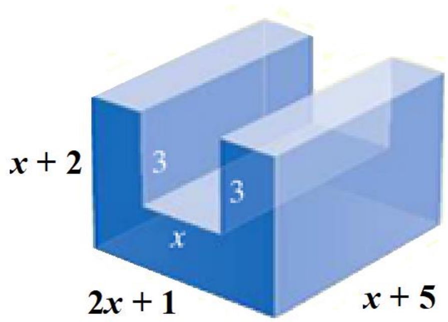
68. A cube measures *n* inches on each edge. If a slice 1 *inch* thick is cut from one face of the cube and then a slice 3 inches thick is cut from another face of the cube, the resulting solid has a volume of $1560 \ \ in^3$. Find the dimensions of the original cube.



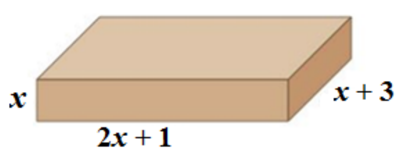
69. For what value of x will the volume of the following solid be $112\ in^3$



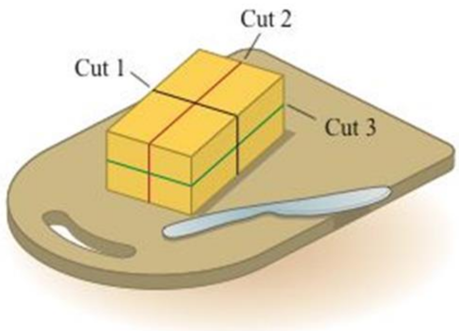
70. For what value of x will the volume of the following solid be $208\ in^3$



71. The length of rectangular box is 1 *inch* more than twice the height of the box, and the width is 3 *inches* more than the height. If the volume of the box is $126\ in^3$, find the dimensions of the box.



72. One straight cut through a thick piece of cheese produces two pieces. Two straight cuts can produce a maximum of four pieces. Two straight cuts can produce a maximum of four pieces. Three straight cuts can produce a maximum of eight pieces.



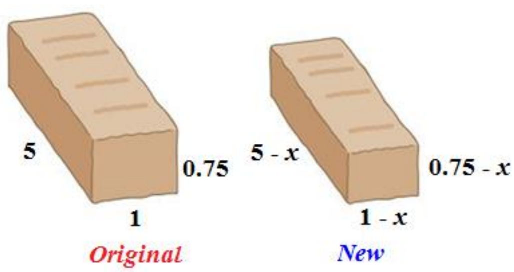
You might be inclined to think that every additional cut doubles number of pieces. However, for four straight cuts, you get a maximum of 15 pieces. The maximum number of pieces P that can be produced by n straight cuts is given by

$$P(n) = \frac{n^3 + 5n + 6}{6}$$

a) Determine number of pieces that can be produces by five straight cuts.
b) What is the fewest number of straight cuts that are needed to produce 64 pieces?

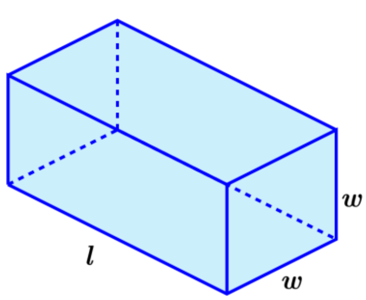
73. The number of ways one can select three cards from a group of n cards (the order of the selection matters), where $n \geq 3$, is given by $P(n) = n^3 - 3n^2 + 2n$. For a certain card trick, a magician has determined that there are exactly 504 *ways* to choose three cards from a given group. How many cards are in the group?

74. A nutrition bar in the shape of a rectangular solid measure 0.75 *in*. by 1 *in*. by 5 *inches*.



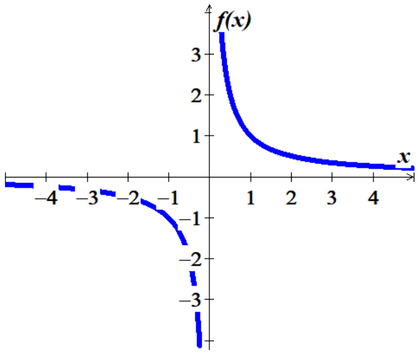
To reduce costs, the manufacturer has decided to decrease each of the dimensions of the nutrition bar by x *inches*, what value of x will produce a new bar with a volume that is $0.75 \ \ in^3$ less than the present bar's volume.

75. A rectangular box is square on two ends and has length plus girth of 81 *inches*. (Girth: distance *around* the box). Determine the possible lengths l $(l > w)$ of the box if its volume is $4900 \ \ in^3$.

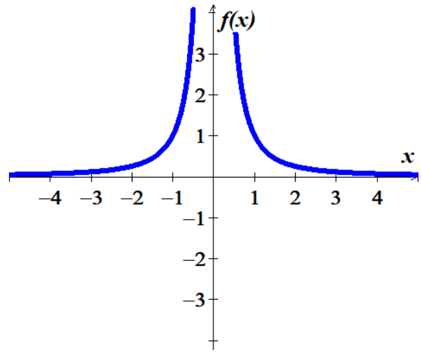


Section 2.7 – Rational Functions

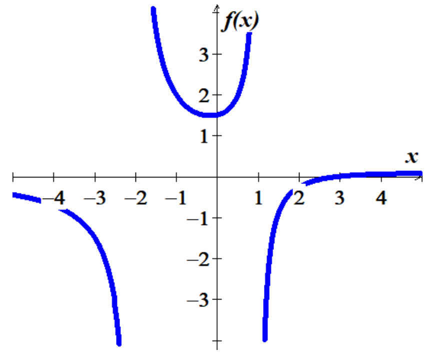
$$f(x) = \frac{1}{x}$$



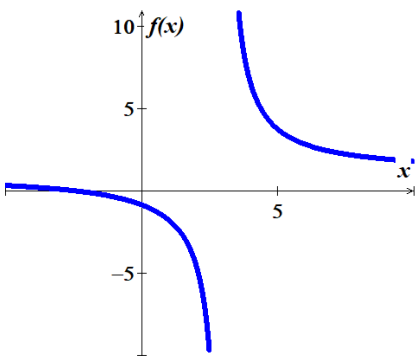
$$f(x) = \frac{1}{x^2}$$



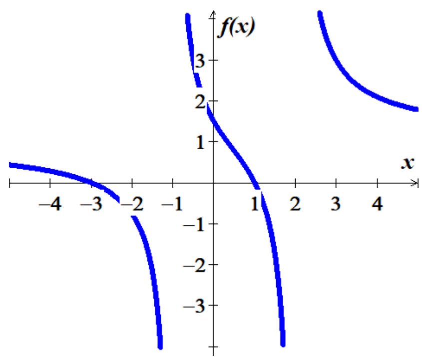
$$f(x) = \frac{x-3}{x^2+x-2}$$



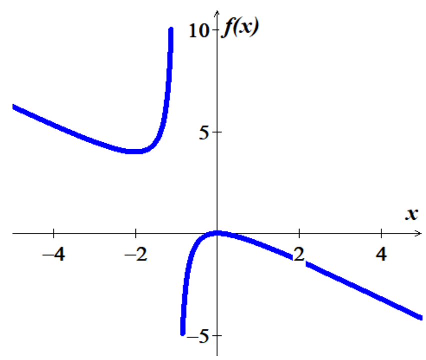
$$f(x) = \frac{2x+5}{2x-6}$$



$$f(x) = \frac{x^2+2x-3}{x^2-x-2}$$



$$f(x) = -\frac{x^2}{x+1}$$



Rational Function

A rational function is a function f that is a quotient of two polynomials, that is,

$$f(x) = \frac{g(x)}{h(x)}$$

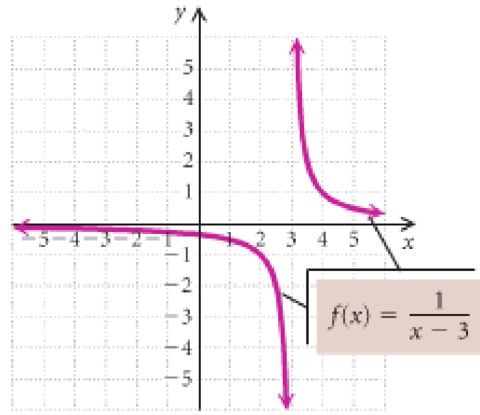
Where $g(x)$ and $h(x)$ are polynomials. The domain of f consists of all real numbers ***except*** the zeros of the denominator $h(x)$.

The Domain of a Rational Function

Example

Consider: $f(x) = \dfrac{1}{x-3}$

Find the domain and graph *f*.



Solution

$x - 3 = 0 \quad \Rightarrow \quad \boxed{x = 3}$

Thus the domain is: $\{x | x \neq 3\}$ *or* $(-\infty,\ 3) \cup (3,\ \infty)$

Function	***Domain***	
$f(x) = \dfrac{1}{x}$	$\{x \mid x \neq 0\}$	$(-\infty,\ 0) \cup (0,\ \infty)$
$f(x) = \dfrac{1}{x^2}$	$\{x \mid x \neq 0\}$	$(-\infty,\ 0) \cup (0,\ \infty)$
$f(x) = \dfrac{x-3}{x^2 + x - 2}$	$\{x \mid x \neq -2 \text{ and } x \neq 1\}$	$(-\infty,\ -2) \cup (-2,\ 1) \cup (1,\ \infty)$
$f(x) = \dfrac{2x+5}{2x-6} = \dfrac{2x+5}{2(x-3)}$	$\{x \mid x \neq 3\}$	$(-\infty,\ 3) \cup (3,\ \infty)$

Asymptotes

Vertical Asymptote (*VA*) - ***Think Domain***

The line $x = a$ is a ***vertical asymptote*** for the graph of a function f if

$$f(x) \to \infty \quad or \quad f(x) \to -\infty$$

As ***x*** approaches ***a*** from either the left or the right

Example

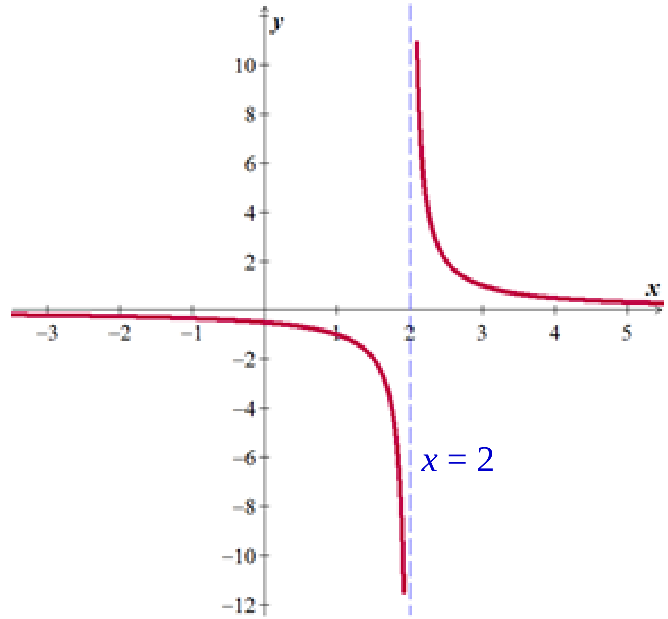
Find the vertical asymptote of $f(x) = \dfrac{1}{x-2}$, and sketch the graph.

Solution

VA: $x = 2$

$f(x) \to \infty \quad as \quad x \to 2^+$

$f(x) \to -\infty \quad as \quad x \to 2^-$



Horizontal Asymptote (*HA*)

The line $y = c$ is a ***horizontal asymptote*** for the graph of a function f if

$$f(x) \to c \quad as \quad x \to -\infty \quad or \quad x \to -\infty$$

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0} = \dfrac{a_n x^n}{b_m x^m}$ be a rational function.

1. If the degree of numerator is less than of denominator $(n < m) \Rightarrow$ y = 0

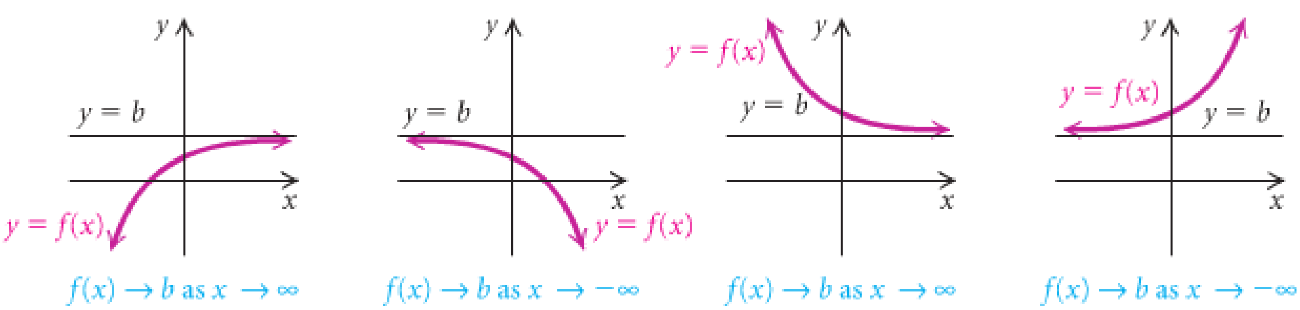
$$y = \frac{2x+1}{4x^2+5} \quad \Rightarrow \boxed{y = 0}$$

2. If the degree of numerator is equal of denominator $(n = m) \Rightarrow y = \dfrac{a_n}{b_m}$

$$y = \frac{2x^2+1}{4x^2+5} \quad \Rightarrow \boxed{y = \frac{2}{4} = \frac{1}{2}}$$

3. If the degree of numerator is greater than of denominator $(n > m) \Rightarrow$ No horizontal asymptote

$$y = \frac{2x^3+1}{4x^2+5} \quad \Rightarrow No \;\; HA$$



Example

Determine the horizontal asymptote of $f(x) = \dfrac{-7x^4 - 10x^2 + 1}{11x^4 + x - 2}$.

Solution

$$f(x) = \frac{-7x^4 - 10x^2 + 1}{11x^4 + x - 2} \to \frac{-7x^4}{11x^4} = -\frac{7}{11}$$

Therefore, the horizontal asymptote (*HA*) is: $\boxed{y = -\dfrac{7}{11}}$

Example

Find the vertical and the horizontal asymptote for the graph of f , if it exists

a) $f(x)=\frac{3x-1}{x^2-x-6}$

b) $f(x)=\frac{5x^2+1}{3x^2-4}$

c) $f(x)=\frac{2x^4-3x^2+5}{x^2+1}$

Solution

a) $f(x)=\frac{3x-1}{x^2-x-6}$

$$x^2-x-6=0 \quad \rightarrow \quad x=-2,\ 3$$

VA: $x=-2, \quad x=3$

HA: $y=0$

b) $f(x)=\frac{5x^2+1}{3x^2-4}$

$$3x^2-4=0 \quad \rightarrow \quad 3x^2=4 \quad \rightarrow \quad x^2=\frac{4}{3} \quad \rightarrow \quad \boxed{x=\pm\frac{2}{\sqrt{3}}}$$

VA: $x=-\frac{2}{\sqrt{3}}, \quad x=\frac{2}{\sqrt{3}}$

HA: $y=\frac{5}{3}$

c) $f(x)=\frac{2x^4-3x^2+5}{x^2+1}$

$$x^2+1=0 \quad \rightarrow \quad x^2=-1$$

VA: *n/a*

HA: *n/a*

Slant or *Oblique* Asymptotes

When the degree of the numerator is one greater than the degree of the numerator, the graph has a slant or oblique asymptote and it is a line $y = ax + b, \quad a \neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y = \frac{3x^2 - 1}{x + 2}$$

$$\begin{array}{r|l} & 3x - 6 \\ \hline x + 2 & 3x^2 + 0x - 1 \\ & \underline{3x^2 + 6x} \\ & \quad -6x - 1 \\ & \quad \underline{-6x - 12} \\ & \qquad R = 11 \end{array}$$

$$y = \frac{3x^2 - 1}{x + 2} = 3x - 6 + \frac{11}{x + 2}$$

The ***oblique asymptote*** is the line ***y* = 3*x* – 6**

Example

Find all the asymptotes of $f(x) = \dfrac{2x^2 - 3x - 1}{x - 2}$

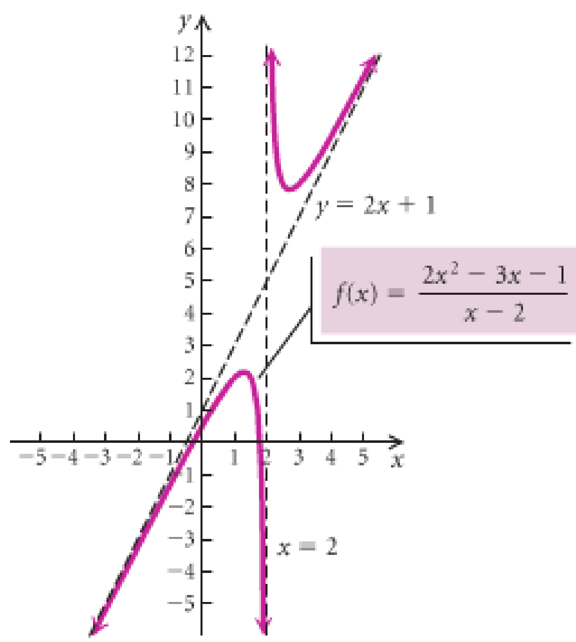
Solution

$$\begin{array}{r|l} & 2x + 1 \\ \hline x - 2 & 2x^2 - 3x - 1 \\ & \underline{-2x^2 + 4x} \\ & \qquad x - 1 \\ & \qquad \underline{-x + 2} \\ & \qquad\quad 1 \end{array}$$

$$f(x) = \frac{2x^2 - 3x - 1}{x - 2} = (2x + 1) + \frac{1}{x - 2}$$

The ***oblique asymptote*** is the line $y = 2x + 1$

VA:: $x = 2$



Graph That Has a *Hole*

Example

Sketch the graph of g if $g(x)=\dfrac{(3x+4)(x-1)}{(2x-5)(x-1)}$

Solution

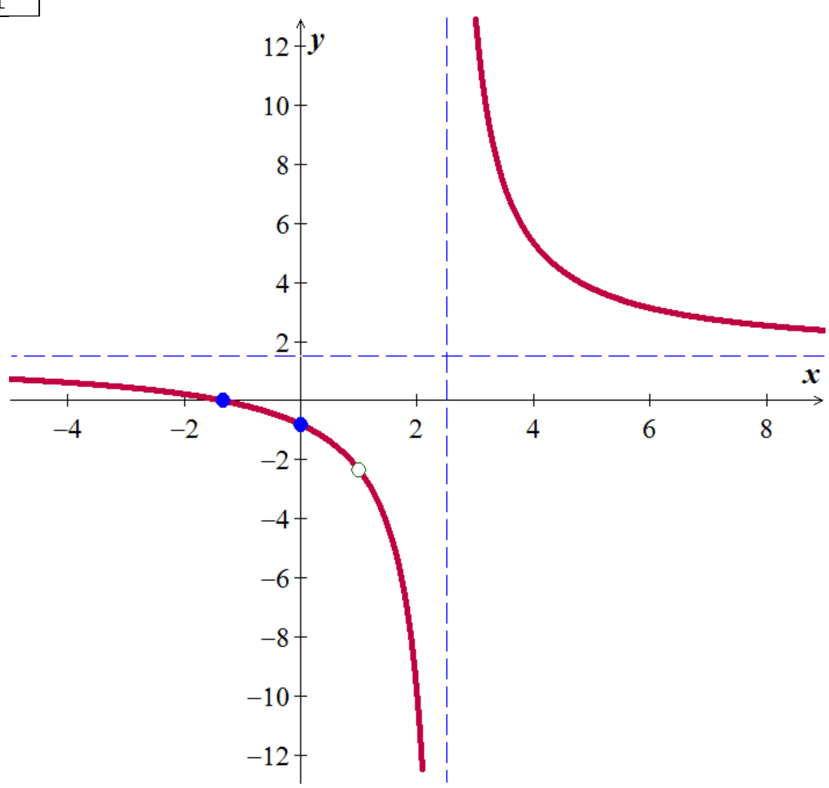
$$g(x)=\frac{(3x+4)(x-1)}{(2x-5)(x-1)}=\frac{3x+4}{2x-5}=f(x)$$

VA: $x=\frac{5}{2}$

HA: $y=\frac{3}{2}$

The only different between the graphs that g has a ***hole*** at $x=1 \to f(1)=-\frac{7}{3}$

x	***y***
–4	.6
1.3	0
0	–.8
4	5.3
6	3.1



Exercises *Section* 2.7 – Rational Functions

(**1 – 21**) Determine all asymptotes of the function

1. $y=\dfrac{3x}{1-x}$

2. $y=\dfrac{x^2}{x^2+9}$

3. $y=\dfrac{x-2}{x^2-4x+3}$

4. $y=\dfrac{3}{x-5}$

5. $y=\dfrac{x^3-1}{x^2+1}$

6. $y=\dfrac{3x^2-27}{(x+3)(2x+1)}$

7. $y=\dfrac{x^3+3x^2-2}{x^2-4}$

8. $y=\dfrac{x-3}{x^2-9}$

9. $y=\dfrac{6}{\sqrt{x^2-4x}}$

10. $y=\dfrac{5x-1}{1-3x}$

11. $f(x)=\dfrac{2x-11}{x^2+2x-8}$

12. $f(x)=\dfrac{x^2-4x}{x^3-x}$

13. $f(x)=\dfrac{x-2}{x^3-5x}$

14. $f(x)=\dfrac{4x}{x^2+10x}$

15. $f(x)=\dfrac{3-x}{(x-4)(x+6)}$

16. $f(x)=\dfrac{x^3}{2x^3-x^2-3x}$

17. $f(x)=\dfrac{3x^2+5}{4x^2-3}$

18. $f(x)=\dfrac{x+6}{x^3+2x^2}$

19. $f(x)=\dfrac{x^2+4x-1}{x+3}$

20. $f(x)=\dfrac{x^2-6x}{x-5}$

21. $f(x)=\dfrac{x^3-x^2+x-4}{x^2+2x-1}$

(**22 – 53**) Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

22. $f(x)=\dfrac{-3x}{x+2}$

23. $f(x)=\dfrac{x+1}{x^2+2x-3}$

24. $f(x)=\dfrac{2x^2-2x-4}{x^2+x-12}$

25. $f(x)=\dfrac{-2x^2+10x-12}{x^2+x}$

26. $f(x)=\dfrac{x^2-x-6}{x+1}$

27. $f(x)=\dfrac{x^3+1}{x-2}$

28. $f(x)=\dfrac{2x^2+x-6}{x^2+3x+2}$

29. $f(x)=\dfrac{x-1}{1-x^2}$

30. $f(x)=\dfrac{x^2+x-2}{x+2}$

31. $f(x)=\dfrac{x^3-2x^2-4x+8}{x-2}$

32. $f(x)=\dfrac{2x^2-3x-1}{x-2}$

33. $f(x)=\dfrac{2x+3}{3x^2+7x-6}$

34. $f(x)=\dfrac{x^2-1}{x^2+x-6}$

35. $f(x)=\dfrac{-2x^2-x+15}{x^2-x-12}$

36. $f(x)=\dfrac{1}{x-3}$

37. $f(x)=\dfrac{-2}{x+3}$

38. $f(x)=\dfrac{x}{x+2}$

39. $f(x)=\dfrac{x-5}{x+4}$

40. $f(x)=\dfrac{2x^2-2}{x^2-9}$

41. $f(x)=\dfrac{x^2-3}{x^2+4}$

42. $f(x)=\dfrac{x^2+4}{x^2-3}$

43. $f(x)=\dfrac{x^2}{x^2-6x+9}$

44. $f(x)=\dfrac{x^2+x+4}{x^2+2x-1}$

45. $f(x)=\dfrac{2x^2+14}{x^2-6x+5}$

46. $f(x)=\dfrac{x^2-4x-5}{2x+5}$

47. $f(x)=\dfrac{x-3}{x^2-3x+2}$

48. $f(x)=\dfrac{x^2+2}{x^2+3x+2}$

49. $f(x)=\dfrac{x-2}{x^2-3x+2}$

50. $f(x)=\dfrac{x^2+x}{x+1}$

51. $f(x)=\dfrac{x^2-2x}{x-2}$

52. $f(x)=\dfrac{x^2-3x}{x+3}$

53. $f(x)=\dfrac{x^3+3x^2-4x+6}{x+2}$

(**54 – 59**) Find an equation of a rational function f that satisfies the given conditions

54. $\begin{cases} \text{vertical asymptote}: \ x=4 \\ \text{horizontal asymptote}: \ y=-1 \\ x-\text{intercept}: \ 3 \end{cases}$

55. $\begin{cases} \text{vertical asymptote}: \ x=-4, x=5 \\ \text{horizontal asymptote}: \ y=\dfrac{3}{2} \\ x-\text{intercept}: \ -2 \end{cases}$

56. $\begin{cases} \text{vertical asymptote}: \ x=5 \\ \text{horizontal asymptote}: \ y=-1 \\ x-\text{intercept}: \ 2 \end{cases}$

57. $\begin{cases} \text{vertical asymptote}: \ x=-2, \ x=0 \\ \text{horizontal asymptote}: \ y=0 \\ x-\text{intercept}: \ 2, \quad f(3)=1 \end{cases}$

58. $\begin{cases} \text{vertical asymptote}: \ x=-3, \ x=1 \\ \text{horizontal asymptote}: \ y=0 \\ x-\text{intercept}: \ -1, \ f(0)=-2 \\ \text{hole at } \ x=2 \end{cases}$

59. $\begin{cases} \text{vertical asymptote}: \ x=-1, \ x=3 \\ \text{horizontal asymptote}: \ y=2 \\ x-\text{intercept}: \ -2, \ 1 \\ \text{hole}: \ x=0 \end{cases}$