Solution Section 4.6 – Orthogonal Diagonalization

Exercise

Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$

a) $A=\begin{pmatrix}3&1\\1&3\end{pmatrix}$

b) $A=\begin{pmatrix}6&2\sqrt{3}\\2\sqrt{3}&7\end{pmatrix}$

c) $A=\begin{pmatrix}-2&0&-36\\0&-3&0\\-36&0&-23\end{pmatrix}$

d) $A=\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}$

e) $A=\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}$

f) $A=\begin{pmatrix}-7&24&0&0\\24&7&0&0\\0&0&-7&27\\0&0&24&7\end{pmatrix}$

Solution

a) $\det(A-\lambda I)=\begin{vmatrix}3-\lambda&1\\1&3-\lambda\end{vmatrix}$

$$=(3-\lambda)^2-1$$

$$=\lambda^2-6\lambda+8=0$$

The eigenvalues are: $\boxed{\lambda=2 \quad and \quad \lambda=4}$

For $\lambda_1=2$, we have: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\rightarrow\quad\begin{cases}x+y=0\\x+y=0\end{cases}\Rightarrow x=-y$$

Therefore the eigenvector $V_1=\begin{pmatrix}-1\\1\end{pmatrix}$

For $\lambda_2=4$, we have: $(A-\lambda_2 I)V_2=0$

$$\begin{pmatrix}-1&1\\1&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\rightarrow\quad\begin{cases}-x+y=0\\x-y=0\end{cases}\Rightarrow x=y$$

Therefore the eigenvector $V_2=\begin{pmatrix}1\\1\end{pmatrix}$

$$u_1=\frac{v_1}{\|v_1\|}=\frac{(-1,1)}{\sqrt{(-1)^2+1^2}}=\frac{(-1,1)}{\sqrt{2}}=\boxed{\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)}$$

$$w_2=v_2-(v_2.u_1)u_1$$

$$=(1,1)-\left[(1,1)\cdot\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right]\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$$

$$=(1,1)-(0)\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$$

$$=(1,\ 1)$$

$$u_2=\frac{w_2}{\|w_2\|}=\frac{(1,\ 1)}{\sqrt{1^2+1^2}}=\boxed{\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)}$$

$$P=\begin{pmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$P^{-1}AP=P^TAP=\begin{pmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}2 & 0\\ 0 & 4\end{pmatrix}$$

b) $\det(A-\lambda I)=\begin{vmatrix}6-\lambda & 2\sqrt{3}\\ 2\sqrt{3} & 7-\lambda\end{vmatrix}$

$$=(6-\lambda)(7-\lambda)-12$$

$$=\lambda^2-13\lambda+30=0$$

The eigenvalues are: $\boxed{\lambda=3 \quad and \quad \lambda=10}$

For $\lambda_1=3$, we have: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}3 & 2\sqrt{3}\\ 2\sqrt{3} & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\xrightarrow{rref}\begin{pmatrix}1 & \frac{2}{\sqrt{3}}\\ 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow x=-\frac{2}{\sqrt{3}}y$$

Therefore the eigenvector $V_1=\begin{pmatrix}-\frac{2}{\sqrt{3}}\\ 1\end{pmatrix}$

For $\lambda_2=10$, we have: $(A-\lambda_2 I)V_2=0$

$$\begin{pmatrix}-4 & 2\sqrt{3}\\ 2\sqrt{3} & -3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\xrightarrow{rref}\begin{pmatrix}1 & -\frac{\sqrt{3}}{2}\\ 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow x=\frac{\sqrt{3}}{2}y$$

Therefore the eigenvector $V_2=\begin{pmatrix}\frac{\sqrt{3}}{2}\\ 1\end{pmatrix}$

$$u_1=\frac{v_1}{\|v_1\|}=\frac{\left(-\frac{2}{\sqrt{3}},1\right)}{\sqrt{\left(-\frac{2}{\sqrt{3}}\right)^2+1^2}}=\frac{\sqrt{3}}{\sqrt{7}}\left(-\frac{2}{\sqrt{3}},1\right)=\boxed{\left(-\frac{2}{\sqrt{7}},\frac{\sqrt{3}}{\sqrt{7}}\right)}$$

$$u_2=\frac{v_2}{\|v_2\|}=\frac{1}{\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+(1)^2}}\left(\frac{\sqrt{3}}{2},1\right)=\frac{2}{\sqrt{7}}\left(\frac{\sqrt{3}}{2},1\right)=\boxed{\left(\frac{\sqrt{3}}{\sqrt{7}},\ \frac{2}{\sqrt{7}}\right)}$$

$$P=\begin{pmatrix}-\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}}\\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}}\end{pmatrix}$$

$$P^{-1}AP=P^TAP=\begin{pmatrix}-\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}}\\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}}\end{pmatrix}\begin{pmatrix}6 & 2\sqrt{3}\\ 2\sqrt{3} & 7\end{pmatrix}\begin{pmatrix}-\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}}\\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}}\end{pmatrix}=\begin{pmatrix}3 & 0\\ 0 & 10\end{pmatrix}$$

c) $$\det(A-\lambda I)=\begin{vmatrix}-2-\lambda & 0 & -36\\ 0 & -3-\lambda & 0\\ -36 & 0 & -23-\lambda\end{vmatrix}$$

$$=(-2-\lambda)(-3-\lambda)(-23-\lambda)-(36)(-3-\lambda)(36)$$

$$=-\left(6+5\lambda+\lambda^2\right)(23+\lambda)+3888+1296\lambda$$

$$=-138-115\lambda-23\lambda-6\lambda-5\lambda^2-\lambda^3+3888+1296\lambda$$

$$=-\lambda^3-28\lambda^2+1175\lambda+3750=0$$

The eigenvalues are: $\boxed{\lambda_1=25,\quad \lambda_2=-3,\quad and\quad \lambda_3=-50}$

For $\lambda_1=25$, we have: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}-27 & 0 & -36\\ 0 & -28 & 0\\ -36 & 0 & -48\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\ \to\ \begin{cases}-27x-36z=0\\ \boxed{y=0}\\ -36x-48z=0\end{cases}\Rightarrow 27x=36z\Rightarrow\boxed{x=-\frac{4}{3}z}$$

Therefore the eigenvector $V_1=\begin{pmatrix}-4\\ 0\\ 3\end{pmatrix}$

For $\lambda_2=-3$, we have: $(A-\lambda_2 I)V_2=0$

$$\begin{pmatrix}1 & 0 & -36\\ 0 & 0 & 0\\ -36 & 0 & -20\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\xrightarrow{rref}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\to\begin{cases}\boxed{x=0}\\ \\ \boxed{z=0}\end{cases}\to\boxed{y=1}$$

Therefore the eigenvector $V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_3 = -25$, we have: $\left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} 48 & 0 & -36 \\ 0 & 47 & 0 \\ -36 & 0 & 27 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & -\frac{3}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} \boxed{x = \frac{3}{4}z} \\ \boxed{y = 0} \end{cases}$$

Therefore the eigenvector $V_3 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(-4,0,3)}{\sqrt{16+9}} = \frac{(-4,0,3)}{5} = \underline{\left(-\frac{4}{5},\ 0,\ \frac{3}{5}\right)}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{(0,1,0)}{\sqrt{1^2}} = \underline{(0,\ 1,\ 0)}$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{(3,0,4)}{\sqrt{3^2+4^2}} = \frac{(3,0,4)}{5} = \underline{\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)}$$

$$P = \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

$$P^{-1}AP = P^T AP = \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}\begin{pmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{pmatrix}\begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 25 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -50 \end{pmatrix}$$

d) $\det\left(A - \lambda I\right) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix}$

$$= -\lambda\left[(1-\lambda)^2 - 1\right]$$

$$= -\lambda\left(\lambda^2 - 2\lambda\right) = 0$$

The eigenvalues are: $\boxed{\lambda_{1,2} = 0 \quad and \quad \lambda_3 = 2}$

For $\lambda_{1,2} = 0$, we have: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad x + y = 0 \Rightarrow \boxed{x = -y}$$

Therefore the eigenvector $V_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad V_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda_3 = 2$, we have: $\left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} -x + y = 0 \Rightarrow \boxed{x = y} \\ \boxed{z = 0} \end{cases}$$

Therefore the eigenvector $V_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(-1,1,0)}{\sqrt{1^2+1^2}} = \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{(0,0,1)}{\sqrt{1^2}} = (0,\ 0,\ 1)$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{(1,1,0)}{\sqrt{1^2+1^2}} = \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$P^{-1}AP = P^T AP = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

e) $\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix}$

$$=(2-\lambda)^3-1-1-(2-\lambda)-(2-\lambda)-(2-\lambda)$$
$$=8-12\lambda+6\lambda^2-\lambda^3-2-6+3\lambda$$
$$=-\lambda^3+6\lambda^2-9\lambda=0$$

The eigenvalues are: $\lambda_1=0$ *and* $\lambda_{2,3}=3$

For $\lambda_1=0$, we have: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1-z_1=0 \\ y_1-z_1=0 \\ \end{cases}$$

Therefore the eigenvector $V_1=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

For $\lambda_{2,3}=3$, we have: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow -x_1-y_1-z_1=0 \Rightarrow x_1=-y_1-z_1$$

Therefore the eigenvector $V_2=\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $V_3=\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$u_1=\frac{V_1}{\|V_1\|}=\frac{(1,1,1)}{\sqrt{3}}=\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$w_2=V_2-(V_2\bullet u_1)u_1$$

$$=(-1,\ 1,\ 0)-\left[(-1,\ 1,\ 0)\bullet\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$=(-1,\ 1,\ 0)-0\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$=(-1,\ 1,\ 0)$$

$$u_2=\frac{w_2}{\|w_2\|}=\frac{(-1,\ 1,\ 0)}{\sqrt{2}}=\boxed{\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)}$$

$$w_3=V_3-\left(V_3\bullet u_2\right)u_2$$

$$=(-1,\ 0,\ 1)-\left[(-1,\ 0,\ 1)\bullet\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)\right]\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$=(-1,\ 0,\ 1)-\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$=(-1,\ 0,\ 1)-\left(-\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$=\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)$$

$$u_3=\frac{w_3}{\|w_3\|}=\frac{\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}=\frac{1}{\frac{\sqrt{6}}{2}}\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)=\boxed{\left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)}$$

$$P=\begin{pmatrix}\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}\end{pmatrix}$$

$$P^TAP=\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 2\end{pmatrix}$$

f) $\det(A-\lambda I)=\begin{vmatrix} -7-\lambda & 24 & 0 & 0 \\ 24 & 7-\lambda & 0 & 0 \\ 0 & 0 & -7-\lambda & 24 \\ 0 & 0 & 24 & 7-\lambda \end{vmatrix}$

$$=(-7-\lambda)(7-\lambda)\left[(-7-\lambda)(7-\lambda)-24^2\right]-24^2\left[(-7-\lambda)(7-\lambda)-24^2\right]$$

$$=\left(\lambda^2-49\right)\left(\lambda^2-625\right)-576\left(\lambda^2-625\right)$$

$$=\left(\lambda^2-625\right)\left(\lambda^2-49-576\right)$$

$$=\left(\lambda^2-625\right)^2=0$$

The eigenvalues are: $\boxed{\lambda_{1,2}=25 \quad and \quad \lambda_{3,4}=-25}$

For $\lambda_{1,2}=25$, we have:

$$\begin{pmatrix} -32 & 24 & 0 & 0 \\ 24 & -18 & 0 & 0 \\ 0 & 0 & -32 & 24 \\ 0 & 0 & 24 & -18 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & -\frac{3}{4} & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1-\frac{3}{4}x_2=0 \\ x_3-\frac{3}{4}x_4=0 \end{cases}$$

Therefore the eigenvector $V_1=\begin{pmatrix} 0 \\ 0 \\ 3 \\ 4 \end{pmatrix} \quad V_2=\begin{pmatrix} 3 \\ 4 \\ 0 \\ 0 \end{pmatrix}$

For $\lambda_{3,4}=-25$, we have:

$$\begin{pmatrix} 18 & 24 & 0 & 0 \\ 24 & 32 & 0 & 0 \\ 0 & 0 & 18 & 24 \\ 0 & 0 & 24 & 32 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1+\frac{4}{3}x_2=0 \\ x_3+\frac{4}{3}x_4=0 \end{cases}$$

Therefore the eigenvector $V_3 = \begin{pmatrix} 0 \\ 0 \\ -4 \\ 3 \end{pmatrix}$ $V_4 = \begin{pmatrix} -4 \\ 3 \\ 0 \\ 0 \end{pmatrix}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(0,0,3,4)}{\sqrt{3^2+4^2}} = \underline{\left(0,\ 0,\ \frac{3}{5},\ \frac{4}{5}\right)}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{(3,4,0,0)}{\sqrt{3^2+4^2}} = \underline{\left(\frac{3}{5},\ \frac{4}{5},\ 0,\ 0\right)}$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{(0,0,-4,3)}{\sqrt{(-4)^2+3^2}} = \underline{\left(0,\ 0,\ -\frac{4}{5},\ \frac{3}{5}\right)}$$

$$u_4 = \frac{v_4}{\|v_4\|} = \frac{(-4,3,0,0)}{\sqrt{25}} = \underline{\left(-\frac{4}{5},\ \frac{3}{5},\ 0,\ 0\right)}$$

$$P = \begin{pmatrix} 0 & \frac{3}{5} & 0 & -\frac{4}{5} \\ 0 & \frac{4}{5} & 0 & \frac{3}{5} \\ \frac{3}{5} & 0 & -\frac{4}{5} & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} & 0 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} & 0 & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

$$P^T AP = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} & 0 & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} -7 & 24 & 0 & 0 \\ 24 & 7 & 0 & 0 \\ 0 & 0 & -7 & 24 \\ 0 & 0 & 24 & 7 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} & 0 & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -25 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & -25 & 0 \\ 0 & 0 & 0 & 25 \end{pmatrix}$$

g) For $A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\left|A - \lambda I\right| = \begin{vmatrix} 3-\lambda & 1 & 0 & 0 \\ 1 & 3-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix}$$

$$= \lambda^2 (3-\lambda)^2 - \lambda^2$$

$$= \lambda^2 \left(9 - 6\lambda + \lambda^2 - 1\right)$$

$$= \lambda^2 \left(\lambda^2 - 6\lambda + 8\right)$$

$$= \lambda^2 (\lambda - 2)(\lambda - 4)$$

Therefore the matrix has eigenvalues $\underline{\lambda_{1,2,3,4} = 0,\ 2,\ 4|}$

For $\lambda_{1,2} = 0$, then $(A - 0)V_1 = 0$

$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 3x_1 + y_1 = 0 \\ x_1 + 3y_1 = 0 \end{cases} \quad \Rightarrow x_1 = y_1 = 0$$

The eigenvectors are: $V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda_3 = 2$, then $(A - 2I)V_3 = 0$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} x_3 + y_3 = 0 \\ x_3 + y_3 = 0 \\ 2z_3 = 0 \\ 2w_3 = 0 \end{cases} \quad \Rightarrow \begin{cases} x_3 = -y_3 \\ z_3 = w_3 = 0 \end{cases}$$

The eigenvectors are: $V_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ or $V_3 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}$

For $\lambda_4 = 4$, then $(A-4I)V_4 = 0$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x_4 \\ y_4 \\ z_4 \\ w_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -x_4 + y_4 = 0 \\ x_4 - y_4 = 0 \\ -4z_4 = 0 \\ -4w_4 = 0 \end{cases} \Rightarrow \begin{cases} x_4 = y_4 \\ z_4 = w_4 = 0 \end{cases}$$

The eigenvectors are: $V_4 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}$

$$P = \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \rightarrow P^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Exercise

Find the eigenvalues of A and B and check the Orthogonality of their first two eigenvectors. Graph these eigenvectors to see the discrete sines and cosines:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The $-1, 2, -1$ pattern in both matrices is a "second derivative. Then $Ax = \lambda x$ and $Bx = \lambda x$ are like $\frac{d^2x}{dt^2} = \lambda x$. This has eigenvectors $x = \sin kt$ and $x = \cos kt$ that are the bases for Fourier series. The matrices lead to "discrete sines" and "discrete cosines" that are the bases for the discrete Fourier Transform. This DFT is absolutely central to all areas of digital signal processing.

Solution

The eigenvalues of A are $\lambda = 2 \pm \sqrt{2}$ *and* 2.

Their sum is 6 (the trace of A) and their product is 4 (the determinant).

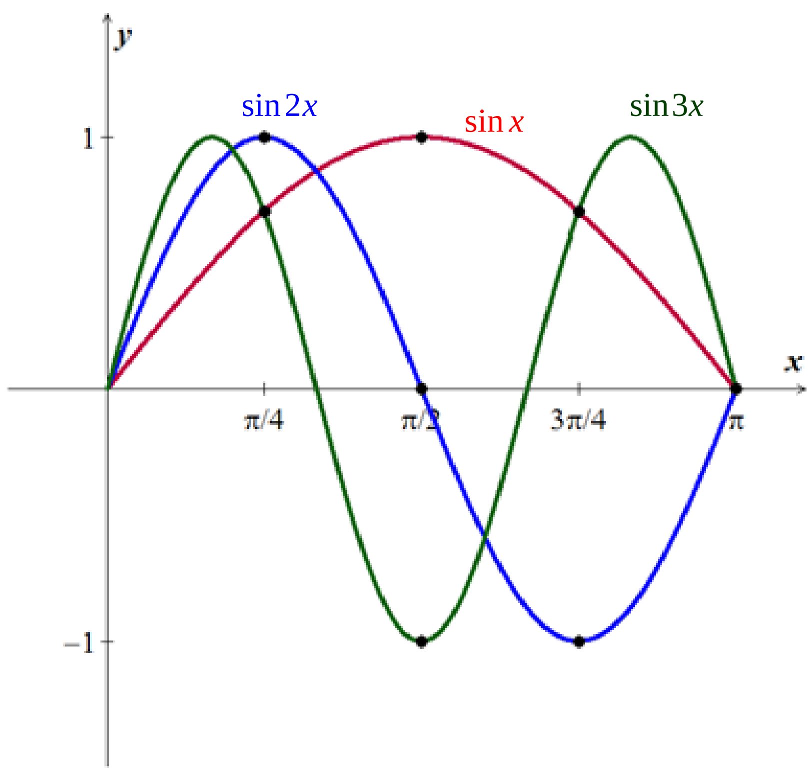
The eigenvector matrix S gives the "Discrete Sine Transform".

$$S = \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

$$V_1 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} \frac{\sqrt{2}}{2} \\ 1 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

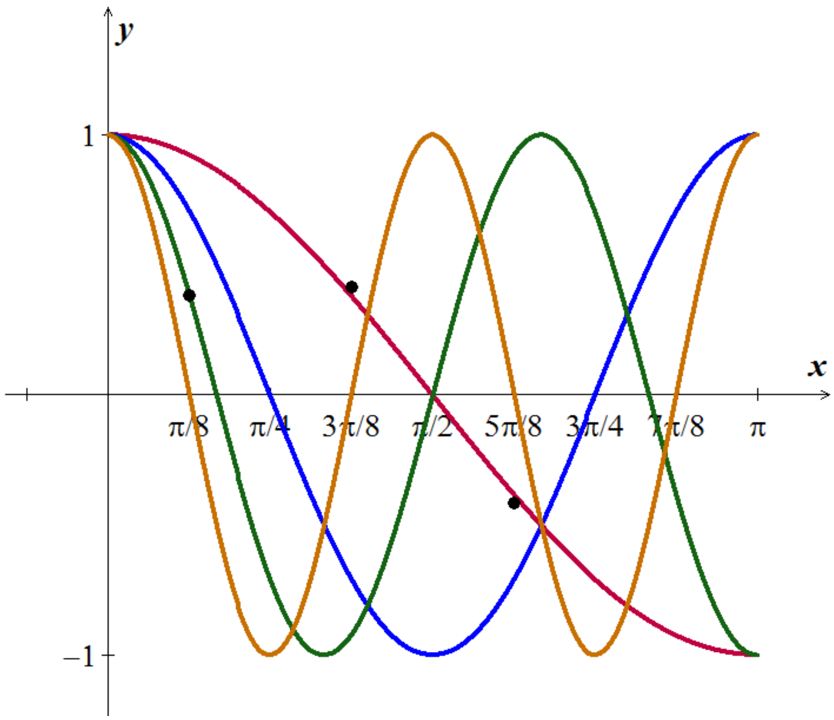
$$V_2 = \begin{pmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} \frac{\sqrt{2}}{2} \\ -1 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$



The eigenvalues of B are $\lambda = 2 \pm \sqrt{2},\ \ 2,\ \ 0$.

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \sqrt{2}-1 & -1 & 1-\sqrt{2} \\ 1 & 1-\sqrt{2} & -1 & \sqrt{2}-1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$



Exercise

Suppose $Ax = \lambda x$ and $Ay = 0y$ and $\lambda \neq 0$. Then y is in the nullspace and x is in the column space. They are perpendicular because _______. (why are these subspaces orthogonal?) If the second eigenvalue is a nonzero number β, apply this argument to $A - \beta I$. The eigenvalue moves to zero and the eigenvectors stay the same – so they are perpendicular.

Solution

Suppose that $A = A^T$ and $Ax = \lambda x$, $Ay = 0y$, and $\lambda \neq 0$. Then x is in the column space of A, and y is in the left nullspace of A since $N(A) = N\left(A^T\right)$. But $C(A)$ and $N\left(A^T\right)$ are orthogonal complements, so x and y are perpendicular.

If $Ay = \beta y$ with $\beta \neq \lambda$ then $(A - \beta I)x = (\lambda - \beta)x$ and $(A - \beta I)y = 0$. Since $\lambda - \beta \neq 0$ it follows that x is in the column space $A - \beta I$ and y is in the nullspace of $A - \beta I$, and $(A - \beta I)^T = A^T - \beta I^T = A - \beta I$, Therefore we can replace A with $A - \beta I$ in the argument of previous paragraph and it follows that x and y are perpendicular.

Exercise

True or false. Give a reason or a counterexample.

a) A matrix with real eigenvalues and eigenvectors is symmetric.

b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.

c) The inverse of a symmetric matrix is symmetric

d) The eigenvector matrix S of a symmetric matrix is symmetric.

e) A complex symmetric matrix has real eigenvalues.

f) If A is symmetric, then e^{iA} is symmetric.

g) If A is Hermitian, then e^{iA} is Hermitian.

Solution

a) ***False***. Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

Then $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = 0$

So A has eigenvalues $\lambda_1 = -1 \quad \lambda_2 = 2$

The eigenvectors are: $V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ so both the eigenvalues and eigenvectors are real but A is not symmetric.

b) ***True***. If the matrix A has orthogonal eigenvectors $x_1, x_2, \cdots, x_n$ with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, we can define $s_i = \dfrac{x_i}{\|x_i\|}$ for all i; then $As_i = \lambda_i s_i$ for all i and the s_i are orthonormal. Then we can diagonalize A as: $A = S\Lambda S^{-1}$ where the i^{th} column of S is s_i, and Λ is the diagonal matrix, so $S^T = S^{-1}$ and $A = S\Lambda S^T$.

$$A^T = \left(S^T\right)^T \Lambda^T S^T = S\Lambda S^T = A$$

So A is symmetric.

c) ***True***. If A is symmetric then it can be diagonalized by an orthogonal matrix Q, $A = QDQ^{-1}$, and then $A^{-1} = QD^{-1}Q^{-1} = QD^{-1}Q^T$. Since D^{-1} is still a diagonal matrix, it follows:

$$\left(A^{-1}\right)^T = QD^{-1}Q^T = A^{-1}$$

d) ***False***. The eigenvalues of $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ are: $\lambda_1 = 0 \quad \lambda_2 = 5$ and the eigenvectors are:

$V_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. We can diagonalize A with eigenvector matrix $S = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ which is not symmetric.

e) ***False***. For example $A = (i)$, the 1 by 1 matrix. The eigenvalue is $\boldsymbol{i}$, it is not a real number.

f) ***True***. $\left(e^{iA}\right)^T = e^{(iA)^T} = e^{iA}$

g) ***False***. $\left(e^{iA}\right)^H = e^{(iA)^H} = e^{-iA^H} = e^{-iA}$. It is typically not the same as e^{iA}. Taking $A = (1)$, the 1 by 1 matrix, would be a enough example because $e^{iA} = e^i$ which is not a real number.

Exercise

Find a symmetric matrix $\begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$ that has a negative eigenvalue.

a) How do you know it must have a negative pivot?

b) How do you know it can't have two negative eigenvalues?

Solution

a) The eigenvalues of that matrix are $1 \pm b /$ so take any $b < -1 \quad b > 1$. In this case, the determinant is $1 - b^2 < 0$.

b) The signs of the pivots coincide with the signs of the eigenvalues. Alternatively, the product of the pivots is the determinant, which is negative in this case. So, one of the two pivots must be negative.

c) The product of the eigenvalues equals the determinant, which is negative in this case. So, precisely one numbers cannot have a negative product.

Exercise

Which of these classes of matrices do A and B belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad B = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Which of these factorizations are possible for A and B: LU, QR, ADP^{-1}, QDQ^T?

Solution

Matrix A is invertible, orthogonal, a permutation matrix, diagonalizable, and Markov! (Everything but a projection).

Matrix A satisfies $A^2 = I$, $A = A^T$, and also $AA^T = I$, This means it is invertible, symmetric, and orthogonal. Since it is symmetric, it is diagonalizable (with real eigenvalues!). It is a permutation matrix by just looking at it. It is Markov since the columns add to 1. It is not a projection since $A^2 = I \neq A$.

All of the factorization are possible for it: LU and QR are always possible. PDP^{-1} is possible since it is diagonalizable, and QDQ^T is possible since it is symmetric.

Matrix B is a projection, diagonalizable, and Markov. It is not invertible, not orthogonal, and not a permutation.

B is a projection since $B^2 = B$, it is symmetric and thus diagonalizable, and it is Markov since the columns add to 1. It is not invertible since the columns are visibly linearly dependent, it is not orthogonal since the columns are far from orthonormal, and it's clearly not a permutation.

All the factorizations are possible for it: LU and QR are always possible. PDP^{-1} is possible since it is diagonalizable, and QDQ^T is possible since it is symmetric.

Exercise

Prove that A is any $m \times n$ matrix, then $A^T A$ has an orthonormal set of n eigenvectors

Solution

$\left(A^T A\right)^T = A^T \left(A^T\right)^T = A^T A$, then $A^T A$ is symmetric, therefore there is an eigenvector $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ for $A^T A$.

Let $A\vec{v}_1 = \lambda_1 \vec{v}_1 \quad and \quad A\vec{v}_2 = \lambda_2 \vec{v}_2$

$A\vec{v}_1 \bullet \vec{v}_2 = \vec{v}_1 \bullet A^T \vec{v}_2$ $\qquad A\vec{x} \bullet \vec{y} = \vec{x} \bullet A^T \vec{y}$

$A\vec{v}_1 \bullet \vec{v}_2 = \vec{v}_1 \bullet A\vec{v}_2$ $\qquad A^T = A \ \ (Since\ A\ is\ a\ symmetric)$

$\lambda_1 \vec{v}_1 \bullet \vec{v}_2 = \vec{v}_1 \bullet \lambda_2 \vec{v}_2$

$\lambda_1 \left(\vec{v}_1 \bullet \vec{v}_2\right) = \lambda_2 \left(\vec{v}_1 \bullet \vec{v}_2\right)$

$\left(\lambda_1 - \lambda_2\right)\left(\vec{v}_1 \bullet \vec{v}_2\right) = 0$ $\qquad$ Since $\lambda_1 \neq \lambda_2$

Therefore; $\boxed{\vec{v}_1 \bullet \vec{v}_2 = 0}$

Then the vectors $A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_n$ are orthogonal

$A\vec{v}_1 \bullet A\vec{v}_2 = \left(A\vec{v}_i\right)^T A\vec{v}_j = \vec{v}_i^T A^T A\vec{v}_j = \vec{v}_i \cdot \left(A^T A\vec{v}_j\right) = \vec{v}_i \cdot \left(\lambda_j \vec{v}_j\right) = \lambda_j \left(\vec{v}_i \cdot \vec{v}_j\right) = 0$

Example

Construct a 3 by 3 matrix A with no zero entries whose columns are mutually perpendicular. Compute $A^T A$. Why is it a diagonal matrix?

Solution

Consider the matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ to be columns mutual perpendicular

Let assume $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{pmatrix}$ *then* $A^T = \begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \\ 2 & 1 & 2 \end{pmatrix}$ *or* $A = \begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$\left(A^T A\right)_{ij} = (\text{column } i \text{ of } A)(\text{column } j \text{ of } A)$$

Exercise

Assuming that $b \neq 0$, find a matrix that orthogonally diagonalizes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$

Solution

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} \\ &= (a-\lambda)^2 - b^2 \\ &= (a-\lambda-b)(a-\lambda+b) \\ &= (a-b-\lambda)(a+b-\lambda) = 0 \end{aligned}$$

Therefore the eigenvalues are: $\lambda_1 = a - b$ *and* $\lambda_2 = a + b$

Assume that $b \neq 0$.

For $\lambda_1 = a - b$, then $\left(A - (a-b)I\right)V_1 = 0$

$$\begin{pmatrix} b & b \\ b & b \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad bx_1 = -by_1$$

The eigenvectors are: $V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For $\lambda_2 = a + b$, then $\left(A - (a+b)I\right)V_2 = 0$

$$\begin{pmatrix} -b & b \\ b & -b \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad bx_2 = by_2$$

The eigenvectors are: $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Applying the Gram Schmidt process.

$$\boldsymbol{u}_1 = \frac{V_1}{\|V_1\|} = \frac{(-1,\ 1)}{\sqrt{2}} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\boldsymbol{w}_2 = V_2 - \frac{\langle \boldsymbol{u}_1, V_2 \rangle}{\|V_2\|^2} = (1,\ 1) - \frac{(-1,\ 1)\bullet(1,\ 1)}{2}(1,\ 1)$$

$$= (1,\ 1)$$

$$\boldsymbol{u}_2 = \frac{\boldsymbol{w}_2}{\|\boldsymbol{w}_2\|} = \frac{(1,\ 1)}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$