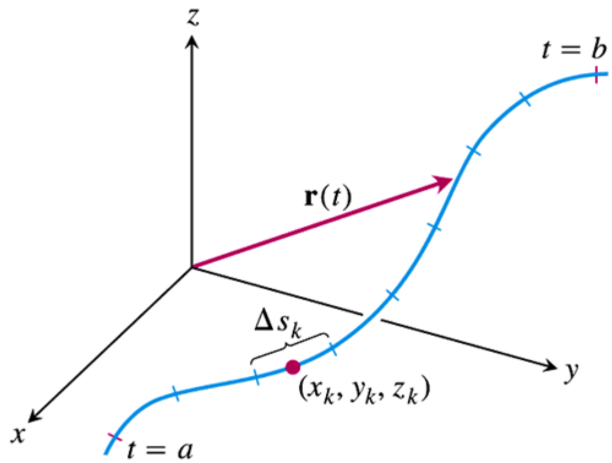
Section 4.2 – Line Integrals

Definition

If f is defined on a curve C given parametrically by $\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$, $a \le t \le b$, then the line integral of f over C is

$$\int_C f(x, y, z)\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k$$

Provided this limit exists.



How to Evaluate a Line Integral

1. Find a smooth parametrization of C, $\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$, $a \le t \le b$
2. aluate the integral as

$$\int_C f(x, y, z)\,ds = \int_a^b f(g(t), h(t), k(t))\ |\vec{v}(t)|\ dt$$

Example

Integrate $f(x, y, z) = x - 3y^2 + z$ over the line segment C joining the origin to the point (1, 1, 1).

Solution

Assume that:

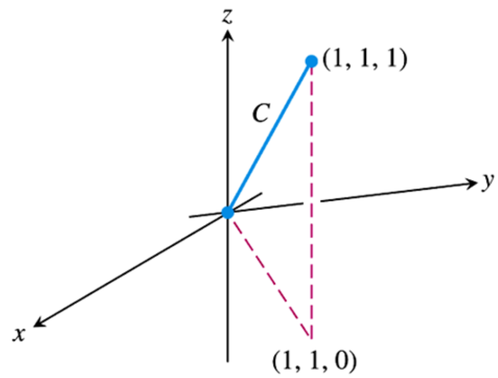
$\vec{r}(t) = t\hat{i} + t\hat{j} + t\hat{k}$, $0 \le t \le 1$

$$\left|\vec{v}(t)\right| = \left|\hat{i} + \hat{j} + \hat{k}\right|$$

$$= \sqrt{1^2 + 1^2 + 1^2}$$

$$= \sqrt{3} \neq 0$$ (The parameterization is smooth)

$$\int_C f(x,y,z)\,ds = \int_0^1 f(t,t,t)\,\left(\sqrt{3}\right)\,dt$$



$$= \sqrt{3}\int_0^1 \left(t - 3t^2 + t\right)\,dt$$

$$= \sqrt{3}\int_0^1 \left(2t - 3t^2\right)\,dt$$

$$= \sqrt{3}\left(t^2 - t^3\right.\Big|_0^1$$

$$= \sqrt{3}(1-1)$$

$$\underline{= 0\,|}$$

Example

Integrate $f(x,y,z) = x - 3y^2 + z$ over $C_1 \cup C_2$ using the path the origin to the point (1, 1, 1).

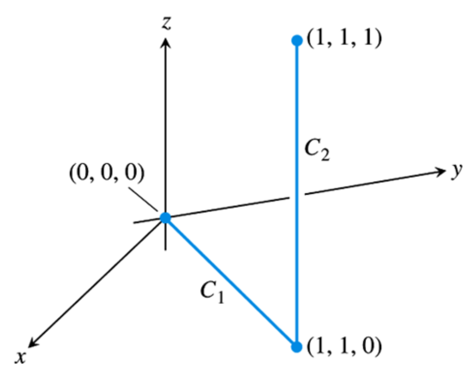
Solution

$C_1: \quad \vec{r}_1(t) = t\hat{i} + t\hat{j} \quad 0 \le t \le 1$

$$\left|\vec{v}_1\right| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$C_2: \quad \vec{r}(t) = \hat{i} + \hat{j} + t\hat{k} \quad 0 \le t \le 1$

$$\left|\vec{v}_2\right| = \sqrt{0^2 + 0^2 + 1^2} = 1$$



$$\int_{C_1 \cup C_2} f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds$$

$$= \int_0^1 f(t,t,0)\sqrt{2}\,dt + \int_0^1 f(1,1,t)(1)\,dt$$

$$= \sqrt{2}\int_0^1 \left(t - 3t^2 + 0\right) dt + \int_0^1 (1 - 3 + t)\, dt$$

$$= \sqrt{2}\left(\tfrac{1}{2}t^2 - t^3\right. \Big|_0^1 + \left(-2t + \tfrac{1}{2}t^2\right. \Big|_0^1$$

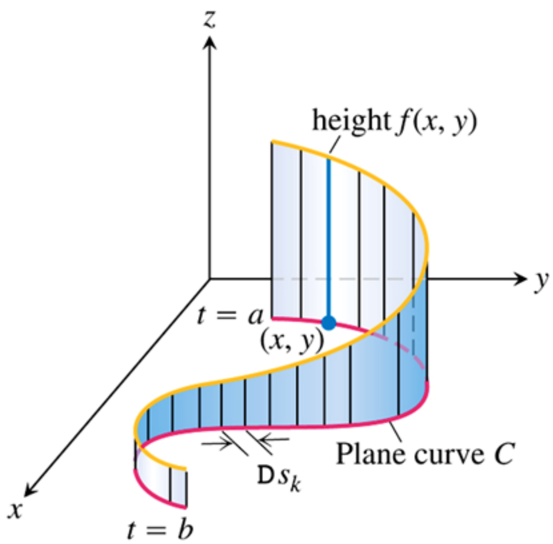
$$= \sqrt{2}\left(\tfrac{1}{2} - 1\right) + \left(-2 + \tfrac{1}{2}\right)$$

$$= -\frac{\sqrt{2}}{2} - \frac{3}{2}$$

- The value of the line integral along a path joining two points can change if you change the path between them.

Line Integrals in the Plane

There is an interesting geometric interpretation for line integrals in the plane. If C is a smooth curve in the xy-plane parametrized by $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}, \quad a \le t \le b$, we generate a cylindrical surface by moving a straight line along C orthogonal to the plane, holding the line parallel to the z-axis.



The cylinder cuts through the surface, forming a curve on it. The part of the cylindrical surface that lies beneath the surface curve and above the xy-plane is like a ***winding wall*** or ***fence*** standing on the curve C and orthogonal to the plane.

$$\int_C f\ ds = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(x_k, y_k\right)\Delta s_k$$

Where $\Delta s_k \to 0 \ \ as \ \ n \to \infty$, we see that he line integral $\int_C f\ ds$ is the area of the wall.

Line Integrals with Respect to the *xyz* Coordinates

$$\int_C M(x,y,z)\,dx = \int_a^b M\big(g(t),h(t),k(t)\big)\ g'(t)\ dt$$

$$\int_C N(x,y,z)\,dy = \int_a^b N\big(g(t),h(t),k(t)\big)\ h'(t)\ dt$$

$$\int_C P(x,y,z)\,dz = \int_a^b P\big(g(t),h(t),k(t)\big)\ k'(t)\ dt$$

Example

Evaluate the line integral $\int_C -ydx + zdy + 2xdz$, where C is the helix

$$\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k} \quad 0 \le t \le 2\pi$$

Solution

$$x = \cos t, \quad y = \sin t, \quad z = t$$

$$dx = (-\sin t)dt, \quad dy = (\cos t)dt, \quad dz = dt$$

$$\int_C -ydx + zdy + 2xdz = \int_0^{2\pi} \left[(-\sin t)(-\sin t) + t\cos t + 2\cos t\right]dt$$

$$= \int_0^{2\pi} \left(\sin^2 t + t\cos t + 2\cos t\right) dt$$

$$= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2t + t\cos t + 2\cos t\right) dt$$

		$\cos t$
+	t	$\sin t$
−	1	$-\cos t$

$$= \frac{1}{2}t - \frac{1}{4}\sin 2t + (t\sin t + \cos t) + 2\sin t \Big|_0^{2\pi}$$

$$= \left(\frac{1}{2}(2\pi) + 1\right) - (1)$$

$$= \pi + 1 - 1$$

$$= \pi$$

Exercises *Section* 4.2 – Line Integrals

1. Evaluate $\int_C (x+y)\,ds$ where C is the straight-line segment $x=t,\ y=(1-t),\ z=0$ from (0, 1, 0) to (1, 0, 0).

2. Evaluate $\int_C (x-y+z-2)\,ds$ where C is the straight-line segment $x=t,\ y=(1-t),\ z=1$ from $(0,\ 1,\ 1)$ to $(1,\ 0,\ 1)$.

3. Evaluate $\int_C (xy+y+z)\,ds$ along the curve $\vec{r}(t)=2t\hat{i}+t\hat{j}+(2-2t)\hat{k}, \quad 0\le t\le 1$

4. Evaluate $\int_C \left(xz-y^2\right)ds$ C: is the line segment from $(0,\ 1,\ 2)$ to $(-3,\ 7,\ -1)$.

5. Evaluate $\int_C xy\ ds$ C: is the unit circle $\vec{r}(s)=\langle\cos s,\ \sin s\rangle; \quad 0\le s\le 2\pi$

6. Evaluate $\int_C (x+y)\,ds$ C: is the circle of radius 1 centered at $(0,\ 0)$

7. Evaluate $\int_C \left(x^2-2y^2\right)ds$ C: is the line $\vec{r}(s)=\left\langle \frac{s}{\sqrt{2}},\ \frac{s}{\sqrt{2}}\right\rangle; \quad 0\le s\le 4$

8. Evaluate $\int_C x^2y\ ds$ C: is the line $\vec{r}(s)=\left\langle \frac{s}{\sqrt{2}},\ 1-\frac{s}{\sqrt{2}}\right\rangle; \quad 0\le s\le 4$

9. Evaluate $\int_C \left(x^2+y^2\right)ds$ C: is the circle of radius 4 centered at $(0,\ 0)$

10. Evaluate $\int_C \left(x^2+y^2\right)ds$ C: is the line segment from $(0,\ 0)$ to $(5,\ 5)$

11. Evaluate $\int_C \frac{x}{x^2+y^2}\,ds$ C: is the line segment from $(1,\ 1)$ to $(10,\ 10)$

12. Evaluate $\int_C (xy)^{1/3}\,ds$ C: is the curve $y=x^2, \quad 0\le x\le 1$

13. Evaluate $\int_C xy\ ds$ C: is a portion of the ellipse $\frac{x^2}{4}+\frac{y^2}{16}=1$ in the first quadrant, oriented counterclockwise.

14. Evaluate $\int_C (2x-3y)ds$ C: is the line segment from $(-1,\ 0)$ to $(0,\ 1)$ followed by the line segment from $(0,\ 1)$ to $(1,\ 0)$

15. Evaluate $\int_C (x+y+z)ds$; C is the circle $\vec{r}(t)=\langle 2\cos t,\ 0,\ 2\sin t\rangle \quad 0\le t\le 2\pi$

16. Evaluate $\int_C (x-y+2z)\ ds$; C is the circle $\vec{r}(t)=\langle 1,\ 3\cos t,\ 3\sin t\rangle \quad 0\le t\le 2\pi$

17. Evaluate $\int_C xyz\ ds$; C is the circle $\vec{r}(t)=\langle 1,\ 3\cos t,\ 3\sin t\rangle \quad 0\le t\le 2\pi$

18. Evaluate $\int_C xyz\ ds$; C is the line segment from $(0,\ 0,\ 0)$ to $(1,\ 2,\ 3)$

19. Evaluate $\int_C \frac{xy}{z}\ ds$; C is the line segment from $(1,\ 4,\ 1)$ to $(3,\ 6,\ 3)$

20. Evaluate $\int_C (y-z)ds$; C is the helix $\vec{r}(t)=\langle 3\cos t,\ 3\sin t,\ t\rangle \quad 0\le t\le 2\pi$

21. Evaluate $\int_C xe^{yz}ds$; C is $\vec{r}(t)=\langle t,\ 2t,\ -4t\rangle \quad 1\le t\le 2$

22. Find the integral of $f(x,y,z)=x+y+z$ over the straight–line segment from $(1,\ 2,\ 3)$ to $(0,\ -1,\ 1)$

23. Find the integral of $f(x,y,z)=\frac{\sqrt{3}}{x^2+y^2+z^2}$ over the curve $\vec{r}(t)=t\hat{i}+t\hat{j}+t\hat{k}, \quad 1\le t\le \infty$

24. Evaluate $\int_C x\ ds$ where C is

a) The straight-line segment $x=t,\ \ y=\frac{t}{2}$, from (0, 0) to (4, 2).

b) The parabolic curve $x=t,\ \ y=t^2$, from (0, 0) to (2, 4).

25. Evaluate $\displaystyle\int_C \sqrt{x+2y}\, ds$ where C is

a) The straight-line segment $x=t,\ \ y=4t$, from (0, 0) to (1, 4).

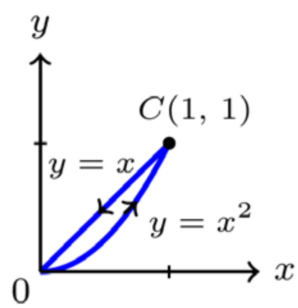
b) $C_1 \cup C_2:\ C_1$ is the line segment (0, 0) to (1, 0) and C_2 is the line segment (1, 0) to (1, 2).

26. Find the line integral of $f(x,y)=\dfrac{\sqrt{y}}{x}$ along the curve $\mathbf{r}(t)=t^3\mathbf{i}+t^4\mathbf{j},\ \ \frac{1}{2}\le t\le 1$

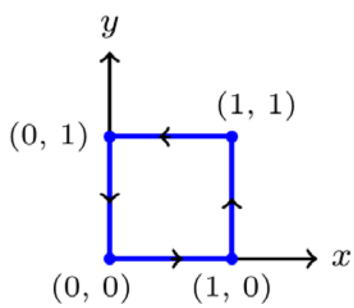
27. Find the line integral of $f(x,y)=\dfrac{x^3}{y}$ over the curve $C:\ \ y=\dfrac{x^2}{2},\ \ 0\le x\le 2$

28. Find the line integral of $f(x,y)=x^2-y$ over the curve $C:\ \ x^2+y^2=4$ in the first quadrant from (0, 2) to $\left(\sqrt{2},\ \sqrt{2}\right)$

29. Evaluate $\displaystyle\int_C \left(x+\sqrt{y}\right) ds$ where C is



30. Evaluate $\displaystyle\int_C \frac{1}{x^2+y^2+1}\, ds$ where C is



31. Find the line integral of $f(x,y)=\dfrac{x^3}{y}$ over the curve $C:\ \ y=\dfrac{x^2}{2},\ \ 0\le x\le 2$

32. Find the line integral of $f(x,y)=x^2-y$ over the curve $C:\ \ x^2+y^2=4$ in the first quadrant from (0, 2) to $\left(\sqrt{2},\ \sqrt{2}\right)$

33. Evaluate the line integral $\int_C \left(x^2 - 2xy + y^2\right) ds$; C is the upper half of a circle

$\vec{r}(t) = \langle 5\cos t,\ 5\sin t \rangle, \quad 0 \le t \le \pi \quad (ccw)$

34. Evaluate the line integral $\int_C ye^{-xz} ds$; C is the path $\vec{r}(t) = \langle t,\ 3t,\ -6t \rangle, \quad 0 \le t \le \ln 8$

35. Integrate $f(x, y, z) = \sqrt{x^2 + z^2}$ over the circle $\vec{r}(t) = (a\cos t)\hat{j} + (a\sin t)\hat{k}, \quad 0 \le t \le 2\pi$

36. Integrate $f(x, y, z) = \sqrt{x^2 + y^2}$ over the involute curve

$\vec{r}(t) = \langle \cos t + t\sin t, \quad \sin t - t\cos t \rangle, \quad 0 \le t \le \sqrt{3}$

(**37 – 40**) Find the average of the function on the given curves

37. $f(x,\ y) = x + 2y$ on the line segment from $(1,\ 1)$ to $(2,\ 5)$

38. $f(x,\ y) = x^2 + 4y^2$ on the circle of radius 9 centered at the origin.

39. $f(x,\ y) = xe^y$ on the circle of radius 1 centered at the origin.

40. $f(x,\ y) = \sqrt{4 + 9y^{2/3}}$ on the curve $y = x^{3/2}$, for $0 \le x \le 5$

(**41 – 42**) Find the length of the curve

41. $\vec{r}(t) = \left\langle 20\sin\frac{t}{4},\ 20\cos\frac{t}{4},\ \frac{t}{2} \right\rangle \quad 0 \le t \le 2$

42. $\vec{r}(t) = \langle 30\sin t,\ 40\sin t,\ 50\cos t \rangle \quad 0 \le t \le 2\pi$