

# Analytic Geometry

## Section 4.6 – Circles and Parabolas

### The *Distance* Formula

The distance,  $d$ , between the points  $(x_1, y_1)$  and  $(x_2, y_2)$  in the rectangular coordinate system is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

To complete the distance between two points. Find the square of the difference between the  $x$ -coordinate plus the square of the difference between the  $y$ -coordinates. The principal square root of this sum is the distance.

### *Midpoint* Formula

Consider a line segment whose endpoints are  $(x_1, y_1)$  and  $(x_2, y_2)$ . The coordinates of the segment's midpoint are

$$\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

To find the midpoint, take the average of the two  $x$ -coordinates and the average of the  $y$ -coordinates

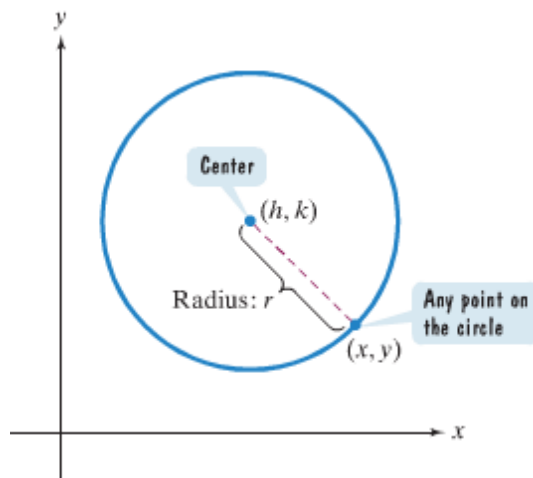
### *The Standard Form of the Equation of a Circle*

The *standard form of the equation of a circle* with center  $(h, k)$  and radius  $r$  is

$$(x - h)^2 + (y - k)^2 = r^2$$

A circle with center  $(0, 0)$  and radius  $r$  has equation:

$$x^2 + y^2 = r^2$$



### Example

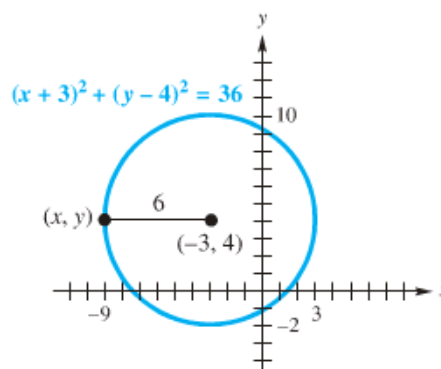
Find the center-radius form of the equation of each circle.

- a) Center at  $(-3, 4)$ , radius 6

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - (-3))^2 + (y - 4)^2 = 6^2$$

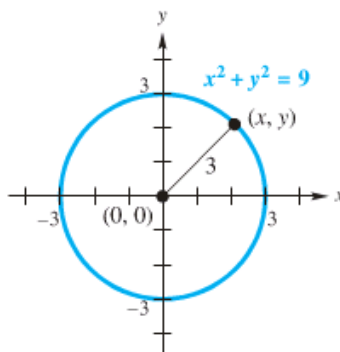
$$(x + 3)^2 + (y - 4)^2 = 36$$



- b) Center at  $(0, 0)$ , radius 3

$$(x - 0)^2 + (y - 0)^2 = 3^2$$

$$x^2 + y^2 = 9$$



### Example

Find the equation of a circle with center  $(-1, 4)$  that passes through  $(3, 7)$ .

#### Solution

$$\begin{aligned} r = d &= \sqrt{(-1 - 3)^2 + (4 - 7)^2} \\ &= \sqrt{(-4)^2 + (-3)^2} \\ &= \sqrt{16 + 9} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

$$\underline{(x + 1)^2 + (y - 4)^2 = 25}$$

### Example

Write in standard form:  $x^2 + y^2 + 4x - 4y - 1 = 0$

#### Solution

$$x^2 + 4x + y^2 - 4y = 1$$

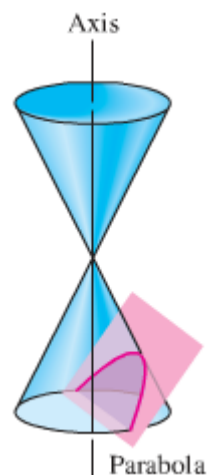
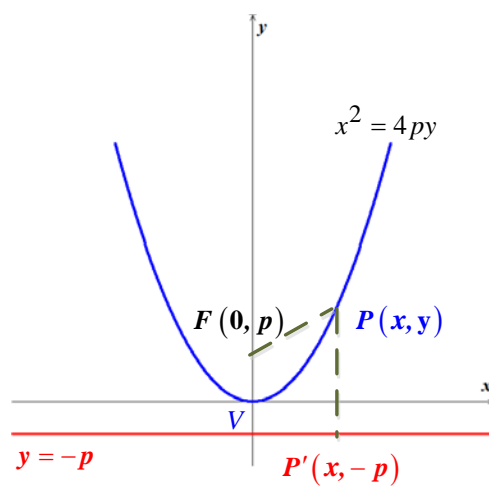
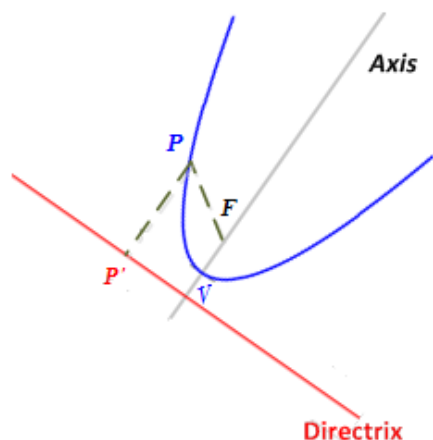
$$x^2 + 4x + \left(\frac{4}{2}\right)^2 + y^2 - 4y + \left(\frac{-4}{2}\right)^2 = 1 + \left(\frac{4}{2}\right)^2 + \left(\frac{-4}{2}\right)^2$$

$$x^2 + 4x + 2^2 + y^2 - 4y + 2^2 = 1 + 4 + 4$$

$$\underline{(x + 2)^2 + (y - 2)^2 = 9}$$

## Definition of a Parabola

A **parabola** is the set of all points in a plane equidistant from a fixed point  $F$  (the **focus**) and a fixed line  $l$  (the **directrix**) that lie in the plane



$$y = \frac{1}{4p}x^2 \quad \text{or} \quad x^2 = 4py \quad \rightarrow \begin{cases} \text{Focus: } F(0, p) \\ \text{Directrix: } y = -p \end{cases}$$

$$x = \frac{1}{4p}y^2 \quad \text{or} \quad y^2 = 4px \quad \rightarrow \begin{cases} \text{Focus: } F(p, 0) \\ \text{Directrix: } x = -p \end{cases}$$

The standard equation  $y = ax^2$  or  $x = ay^2$  is a parabola with vertex  $V(0, 0)$ . Moreover,

$$a = \frac{1}{4p} \quad \text{or} \quad p = \frac{1}{4a}$$

Equation, focus, Directrix	Graph for $p > 0$	Graph for $p < 0$
$x^2 = 4py$ or $y = \frac{1}{4p}x^2$  Focus: $F(0, p)$  Directrix: $y = -p$		
$y^2 = 4px$ or $x = \frac{1}{4p}y^2$  Focus: $F(p, 0)$  Directrix: $x = -p$		
$(y - k)^2 = 4p(x - h)$ $x = ay^2 + by + c$  Focus: $F(h, k + p)$  Directrix: $x = h - p$		
$(x - h)^2 = 4p(y - k)$ $y = ax^2 + bx + c$  Focus: $F(h + p, k)$  Directrix: $y = k - p$		

### ***Example***

Find the focus and directrix of the parabola  $y = -\frac{1}{6}x^2$ .

#### **Solution**

$$a = -\frac{1}{6} \Rightarrow \left| p = \frac{1}{4a} = \frac{1}{4\left(-\frac{1}{6}\right)} = -\frac{6}{4} = -\frac{3}{2} \right|$$

The parabola opens downward and has focus  $F\left(0, -\frac{3}{2}\right)$ .

The directrix is the horizontal line  $y = \frac{3}{2}$  which is a distance  $\frac{3}{2}$  above V.

### ***Example***

- a) Find an equation of a parabola that has vertex at the origin, open right, and passes through the point  $P(7, -3)$ .
- b) Find the focus of the parabola.

#### **Solution**

- a) An equation of a parabola with vertex at the origin that opens right has the form  $x = ay^2$

$$7 = a(-3)^2$$

$$a = \frac{7}{9}$$

The equation is:  $x = \frac{7}{9}y^2$

$$b) \left| p = \frac{1}{4a} = \frac{1}{4\left(\frac{7}{9}\right)} = \frac{9}{28} \right|$$

Thus, the focus has coordinate  $\left(\frac{9}{28}, 0\right)$

### Example

Sketch the graph of  $2x = y^2 + 8y + 22$

### Solution

$$2x - 22 = y^2 + 8y$$

$$y^2 + 8y + \left(\frac{1}{2}8\right)^2 = 2x - 22 + \left(\frac{1}{2}8\right)^2$$

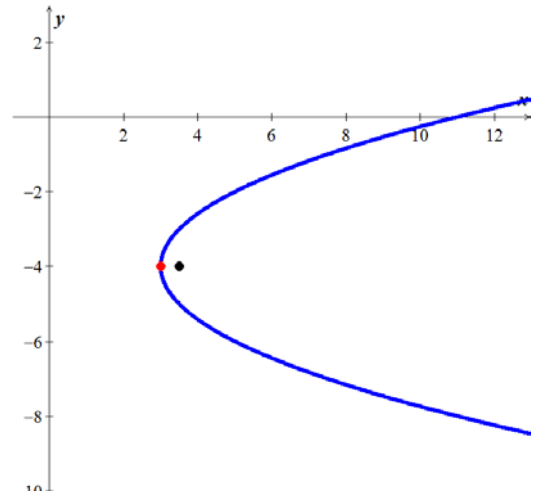
$$(y + 4)^2 = 2x - 6$$

$$(y + 4)^2 = 2(x - 3)$$

The vertex is  $V(h, k) = V(3, -4)$

The focus is  $F(h + p, k) = F\left(3 + \frac{1}{2}, -4\right) = F\left(\frac{7}{2}, -4\right)$

The directrix is  $x = h - p = 3 - \frac{1}{2} = \underline{\underline{\frac{5}{2}}}$



### Example

A parabola has vertex  $V(-4, 2)$  and directrix  $y = 5$ . Express the equation of the parabola in the form

$$y = ax^2 + bx + c$$

### Solution

$$\text{Directrix: } y = k - p \Rightarrow \underline{p = k - y = 2 - 5 = -3}$$

$$(x - h)^2 = 4p(y - k)$$

$$(x + 4)^2 = 4(-3)(y - 2)$$

$$(x + 4)^2 = -12(y - 2)$$

$$x^2 + 8x + 16 = -12y + 24$$

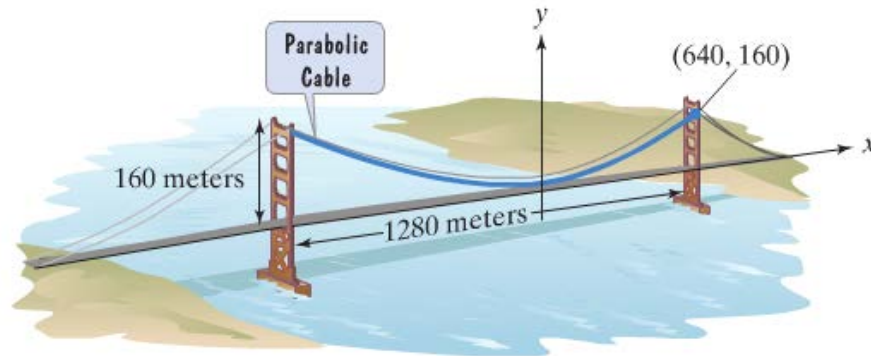
$$x^2 + 8x + 16 - 24 = -12y + 24 - 24$$

$$-12y = x^2 + 8x - 8$$

$$\underline{y = -\frac{1}{12}x^2 - \frac{2}{3}x + \frac{2}{3}}$$

### Example

The towers of the Golden Gate Bridge connecting San Francisco to Marin County are 1280 *meters* apart and rise 160 *meters* above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 200 *meters* from a tower?



### Solution

**Given** the point: (640, 160)

$$(640)^2 = 4p(160)$$

$$x^2 = 4py$$

$$p = \frac{640^2}{640} = 640$$

$$x = 640 - 200 = 440$$

$$(440)^2 = 4(640)y$$

$$x^2 = 4py$$

$$y = \frac{440^2}{4(640)} \approx 75.625$$

The height is 76 *meters*.

## Exercises      Section 4.6 – Circles and Parabolas

Find the center and the radius of

1.  $x^2 + y^2 + 6x + 2y + 6 = 0$

2.  $x^2 + y^2 + 8x + 4y + 16 = 0$

3.  $x^2 + y^2 - 10x - 6y - 30 = 0$

4.  $x^2 - 6x + y^2 + 10y + 25 = 0$

Find the vertex, focus, and directrix of the parabola. Sketch its graph.

5. ..

6.  $2y^2 = -3x$

7.  $(x+2)^2 = -8(y-1)$

8.  $(x-3)^2 = \frac{1}{2}(y+1)$

9.  $(y+1)^2 = -12(x+2)$

10.  $y = x^2 - 4x + 2$

11.  $y^2 + 14y + 4x + 45 = 0$

12.  $x^2 + 20y = 10$

13.  $x^2 = 16y$

14.  $x^2 = -\frac{1}{2}y$

15.  $(y+1)^2 = -4(x-2)$

16.  $x^2 + 6x - 4y + 1 = 0$

17.  $y^2 + 2y - x = 0$

18.  $y^2 - 4y + 4x + 4 = 0$

19.  $x^2 - 4x - 4y = 4$

Find an equation of the parabola that satisfies the given conditions

20. Focus:  $F(2,0)$  directrix:  $x = -2$

21. Focus:  $F(0,-40)$  directrix:  $y = 4$

22. Focus:  $F(-3,-2)$  directrix:  $y = 1$

23. Vertex:  $V(3,-5)$  directrix:  $x = 2$

24. Vertex:  $V(-2,3)$  directrix:  $y = 5$

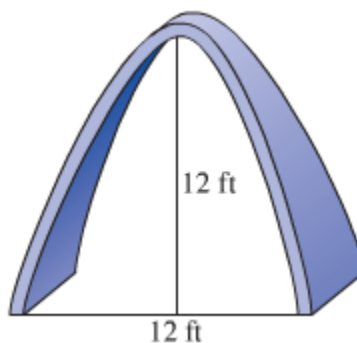
25. Vertex:  $V(-1,0)$  focus:  $F(-4,0)$

26. Vertex:  $V(1,-2)$  focus:  $F(1,0)$

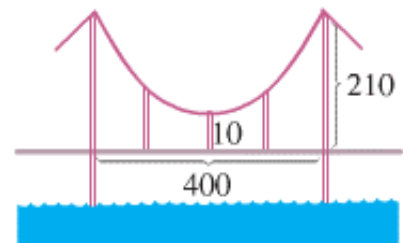
27. Vertex:  $V(0,1)$  focus:  $F(0,2)$

28. Vertex:  $V(3,2)$  focus:  $F(-1,2)$

29. An arch in the shape of a parabola has the dimensions shown in the figure. How wide is the arch 9 feet up?

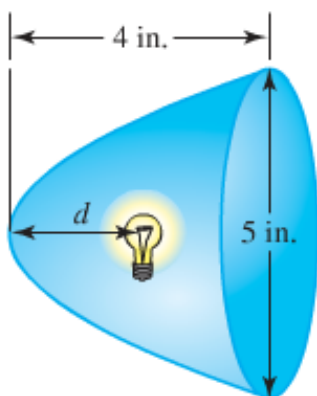


30. The cable in the center portion of a bridge is supported as shown in the figure to form a parabola. The center support is 10 feet high, the tallest supports are 210 feet high, and the distance between the two tallest supports is 400 feet. Find the height of the remaining supports if the supports are evenly spaced.

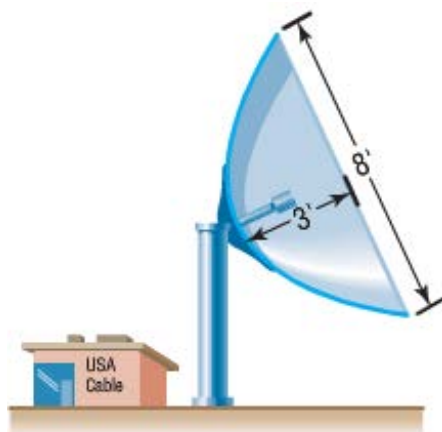




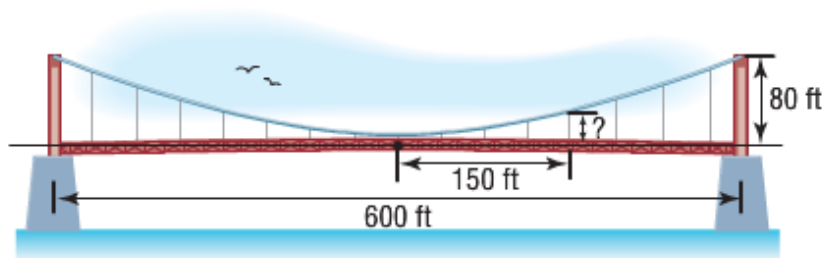
31. A headlight is being constructed in the shape of a paraboloid with depth 4 *inches* and diameter 5 *inches*. Determine the distance  $d$  that the bulb should be from the vertex in order to have the beam of light shine straight ahead.



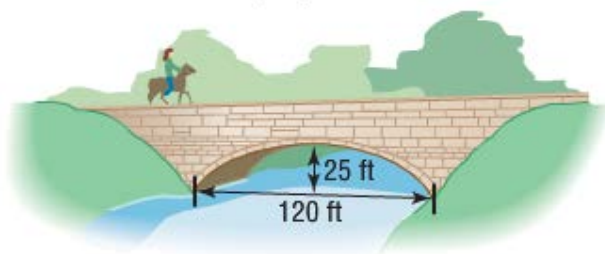
32. A satellite dish is shaped like a paraboloid of revolution. The signals that emanate from a satellite strike the surface of the dish and are reflected to a single point, where the receiver is located. If the dish is 8 *feet* across at its opening and 3 *feet* deep at its center, at what position should the receiver be placed? That is, where is the focus?



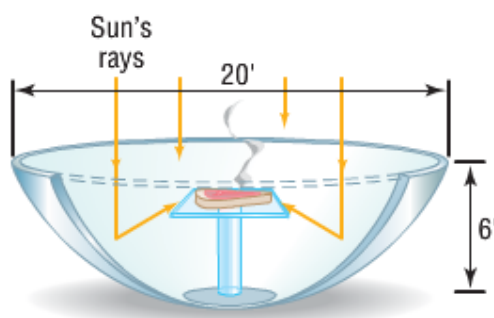
33. A cable TV receiving dish is in the shape of a paraboloid of revolution. Find the location of the receiver, which is placed at the focus, if the dish is 6 *feet* across at its opening and 2 *feet* deep.
34. The cables of a suspension bridge are in the shape of a parabola, as shown below. The towers supporting the cable are 600 *feet* apart and 80 *feet* high. If the cables touch the road surface midway between the towers, what is the height of the cable from the road at a point 150 feet from the center of the bridge?



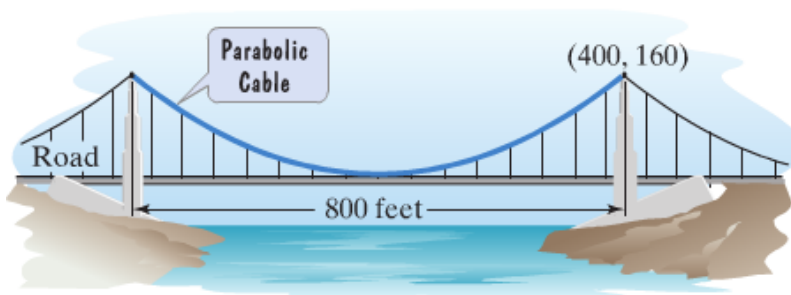
35. A bridge is built in the shape of a parabolic arch. The bridge has a span of 120 *feet* and a maximum height of 25 *feet*. Choose a suitable rectangular coordinate system and find the height of the arch at distances of 10, 30, and 50 *feet* from the center.



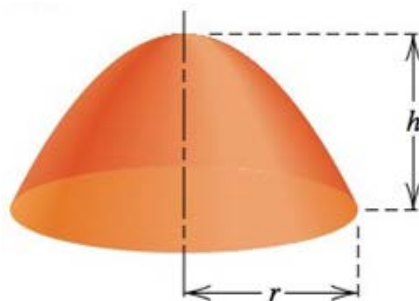
36. A mirror is shaped like a paraboloid of revolution and will be used to concentrate the rays of the sun at its focus, creating a heat source. If the mirror is 20 *feet* across at its opening and is 6 *feet* deep, where will the heat source be concentrated?



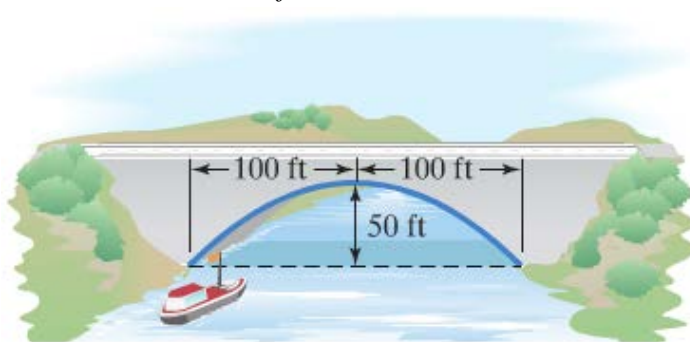
37. A reflecting telescope contains a mirror shaped a paraboloid of revolution. If the mirror is 4 *inches* across at its opening and is 3 *inches* deep, where will the collected light be concentrated?
38. Show that the graph of an equation of the form  $Ax^2 + Dx + Ey + F = 0$   $A \neq 0$
- Is a parabola if  $E \neq 0$
  - Is a vertical line if  $E = 0$  and  $D^2 - 4AF = 0$
  - Is two vertical lines if  $E = 0$  and  $D^2 - 4AF > 0$
  - Contains no points if  $E = 0$  and  $D^2 - 4AF < 0$
39. The towers of a suspension bridge are 800 *feet* apart and rise 160 *feet* above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 100 *feet* from a tower?



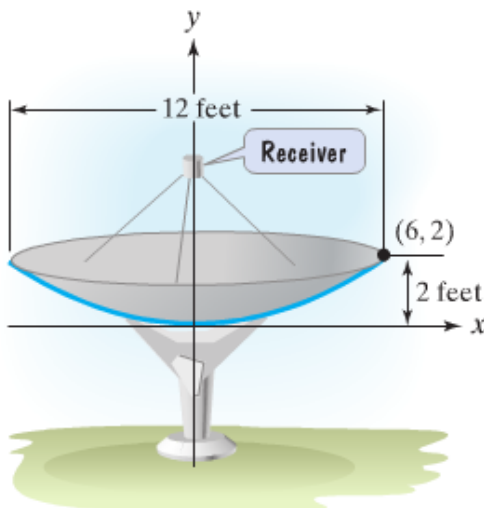
40. The cables of a suspension bridge are in the shape of a parabola. The towers supporting the cable are 400 *feet* apart and 100 *feet* high. If the cables are at a height of 10 *feet* midway between the towers, what is the height of the cable at a point 50 *feet* from the center of the bridge?
41. The focal length of the (finite) paraboloid is the distance  $p$  between its vertex and focus



- a) Express  $p$  in terms of  $r$  and  $h$ .
- b) A reflector is to be constructed with a focal length of 10 *feet* and a depth of 5 *feet*. Find the radius of the reflector.
42. The parabolic arch is 50 *feet* above the water at the center and 200 *feet* wide at the base. Will a boat that is 30 *feet* tall clear the arch 30 *feet* from the center?



43. A satellite dish, as shown below, is in the shape of a parabolic surface. Signals coming from a satellite strike the surface of the dish and are reflected to the focus, where the receiver is located. The satellite dish shown has a diameter of 12 *feet* and a depth of 2 *feet*. How far from the base of the dish should the receiver be placed?



44. A searchlight is shaped like a paraboloid of revolution. If the light source is located 2 *feet* from the base along the axis of symmetry and the opening is 5 *feet* across, how deep should the searchlight be?
45. A searchlight is shaped like a paraboloid of revolution. If the light source is located 2 *feet* from the base along the axis of symmetry and the depth of the searchlight is 4 *feet* across, how deep should the opening be?
46. A searchlight is shaped like a paraboloid, with the light source at the focus. If the reflector is 3 *feet* across at the opening and 1 *foot* deep, where is the focus?
47. A mirror for a reflecting telescope has the shape of a (finite) paraboloid of diameter 8 *inches* and depth 1 *inch*. How far from the center the mirror will the incoming light collect?

