# Solution 	*Section* 2.1 – Vectors in 2-Space, 3-Space, and *n*-Space

## *Exercise*

Sketch the following vectors with initial points located at the origin

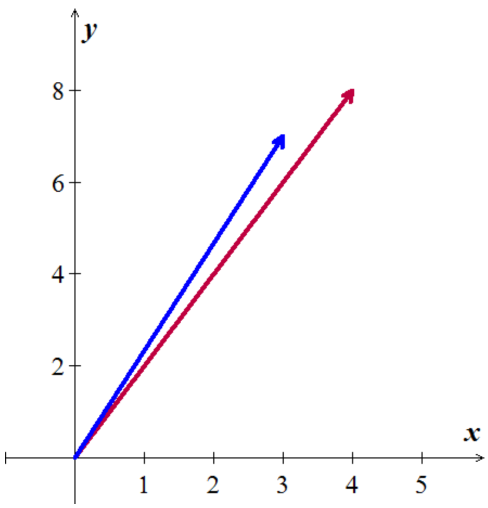
*a)* $P_1(4,\ 8)\quad P_2(3,\ 7)$

*b)* $P_1(-1,\ 0,\ 2)\quad P_2(0,\ -1,\ 0)$

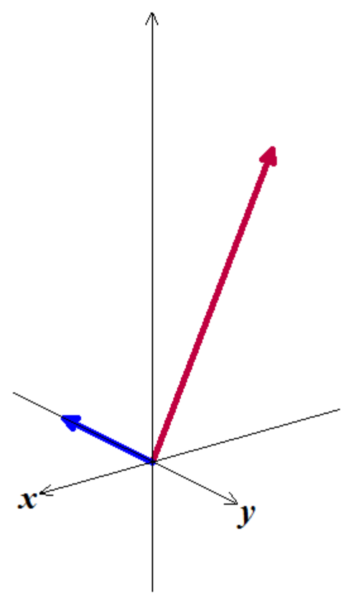
*c)* $P_1(3,\ -7,\ 2)\quad P_2(-2,\ 5,\ -4)$

***Solution***

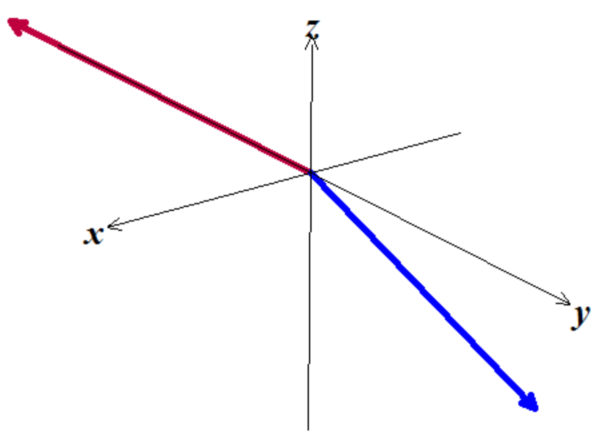
***a)***




***b)***




***c)***



## *Exercise*

Find the components of the vector $\overrightarrow{P_1P_2}$

*a)* $P_1(3,\ 5)\quad P_2(2,\ 8)$

*b)* $P_1(5,\ -2,\ 1)\quad P_2(2,\ 4,\ 2)$

*c)* $P_1(0,\ 0,\ 0)\quad P_2(-1,\ 6,\ 1)$

***Solution***

**a)** $\overrightarrow{P_1P_2} = (2-3,\ 8-5)\ \underline{=(-1,\ 3)|}$

**b)** $\overrightarrow{P_1P_2} = (2-5,\ 4-(-2),\ 2-1)\ \underline{=(-3,\ 6,\ 1)|}$

**c)** $\overrightarrow{P_1P_2} = (-1-0,\ 6-0,\ 1-0)\ \underline{=(-1,\ 6,\ 1)|}$

## *Exercise*

Find the terminal point of the vector that is equivalent to ***u*** = (1, 2) and whose initial point is $A(1,\ 1)$

***Solution***

The terminal point: $B(b_1,\ b_2)$

$(b_1-1,\ b_2-1)=(1,\ 2)$

$$\begin{cases} b_1-1=1 & \Rightarrow b_1=2 \\ b_2-1=2 & \Rightarrow b_2=3 \end{cases}$$

The terminal point: $\underline{B(2,\ 3)|}$

## *Exercise*

Find the initial point of the vector that is equivalent to $\vec{u}$ = (1, 1, 3) and whose terminal point is $B(-1,\ -1,\ 2)$

***Solution***

The initial point: $A(x,y,z)$

$(-1-x,\ -1-y,\ 2-z)=(1,\ 1,\ 3)$

$$\begin{cases} -1-x=1 & \Rightarrow x=-2 \\ -1-y=1 & \Rightarrow y=-2 \\ 2-z=3 & \Rightarrow z=-1 \end{cases}$$

The initial point: $\underline{A(-2,\ -2,\ -1)|}$

## Exercise

Find a nonzero vector $\boldsymbol{u}$ with initial point $P(-1, 3, -5)$ such that

*a)* $\boldsymbol{u}$ has the same direction as $\boldsymbol{v} = (6, 7, -3)$

*b)* $\boldsymbol{u}$ is oppositely directed as $\boldsymbol{v} = (6, 7, -3)$

**Solution**

**a)** $\boldsymbol{u}$ has the same direction as $\boldsymbol{v} \Rightarrow \boldsymbol{u} = \boldsymbol{v} = (6, 7, -3)$

The initial point $P(-1, 3, -5)$ then the terminal point :

$$(-1+6,\ 3+7,\ -5-3) = \underline{(5,\ 10,\ -8)}$$

**b)** $\boldsymbol{u}$ is oppositely as $\boldsymbol{v} \Rightarrow \boldsymbol{u} = -\boldsymbol{v} = (-6, -7, 3)$

The initial point $P(-1, 3, -5)$ then the terminal point :

$$(-1-6,\ 3-7,\ -5+3) = \underline{(-7,\ -4,\ -2)}$$

## Exercise

Let $\boldsymbol{u} = (-3, 1, 2)$, $\boldsymbol{v} = (4, 0, -8)$, and $\boldsymbol{w} = (6, -1, -4)$. Find the components

*a)* $\vec{v} - \vec{w}$

*b)* $6\vec{u} + 2\vec{v}$

*c)* $5(\vec{v} - 4\vec{u})$

*d)* $-3(\vec{v} - 8\vec{w})$

*e)* $(2\vec{u} - 7\vec{w}) - (8\vec{v} + \vec{u})$

*f)* $-\vec{u} + (\vec{v} - 4\vec{w})$

**Solution**

**a)** $$\vec{v} - \vec{w} = (4-6,\ 0-(-1),\ -8-(-4))$$
$$= \underline{(-2,\ 1,\ -4)}$$

**b)** $$6\vec{u} + 2\vec{v} = (-18,6,12) + (8,0,-16)$$
$$= \underline{(-10,\ 6,\ -4)}$$

**c)** $$5(\vec{v} - 4\vec{u}) = 5(4-(-12),\ 0-4,\ -8-8)$$
$$= 5(16,\ -4,\ -16)$$
$$= \underline{(80,\ -20,\ -80)}$$

**d)** $$-3(\vec{v} - 8\vec{w}) = -3(4-48,\ 0-(-8),\ -8-(-32))$$
$$= -3(-44,\ 8,\ 24)$$
$$= \underline{(32,\ -24,\ -72)}$$

**e)** $$(2\vec{u} - 7\vec{w}) - (8\vec{v} + \vec{u}) = \left[(-6, 2,\ 4) - (42,\ -7,\ -28)\right] - \left[(32,\ 0,\ -64) + (-3,\ 1,\ 2)\right]$$
$$= (-48,\ 9,\ 32) - (29,\ 1,\ -62)$$
$$= \underline{(-77,\ 8,\ 94)}$$

***f)*** $-u+(v-4w)=(3,-1,-2)+\left[(4,0,-8)-(24,-4,-16)\right]$
$$=(3,\ -1,\ -2)+(-20,\ 4,\ 8)$$
$$\boxed{=(-17,\ 3,\ 6)}$$

## Exercise

Let $\boldsymbol{u}$ = (2, 1, 0, 1, –1) and $\boldsymbol{v}$ = (–2, 3, 1, 0, 2). Find scalars $a$ and $b$ so that $a\boldsymbol{u}+b\boldsymbol{v}$ = (–8, 8, 3, –1, 7)

***Solution***

$$\begin{aligned} a\vec{u}+b\vec{v} &= a(2,\ 1,\ 0,\ 1,\ -1)+b(-2,\ 3,\ 1,\ 0,\ 2) \\ &= (a-2b,\ a+3b,\ b,\ a,\ -a+2b) \\ &\boxed{=(-8,\ 8,\ 3,\ -1,\ 7)} \end{aligned}$$

$$\begin{cases} a-2b=-8 \\ a+3b=8 \\ b=3 \\ a\quad=-1 \\ -a+2b=7 \end{cases} \quad \rightarrow \quad a=-1 \quad b=3 \ \textit{Unique solution}$$

## Exercise

Find all scalars $c_1,\ c_2,\ and\ c_3$ such that $c_1(1,\ 2,\ 0)+c_2(2,\ 1,\ 1)+c_3(0,\ 3,\ 1)=(0,\ 0,\ 0)$

***Solution***

$$\begin{aligned} c_1(1,\ 2,\ 0)+c_2(2,\ 1,\ 1)+c_3(0,\ 3,\ 1) &= (c_1+2c_2,\ 2c_1+c_2+3c_3,\ c_2+c_3) \\ &= (0,\ 0,\ 0) \end{aligned}$$

$$\begin{cases} c_1+2c_2 \qquad = 0 \\ 2c_1+c_2+3c_3=0 \\ \qquad c_2+c_3=0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\boxed{c_1=c_2=c_3=0}$$

## Exercise

Find the distance between the given points $\begin{bmatrix}5 & 1 & 8 & -1 & 2 & 9\end{bmatrix}$, $\begin{bmatrix}4 & 1 & 4 & 3 & 2 & 8\end{bmatrix}$

## Solution

$$\begin{aligned} d &= \sqrt{(4-5)^2+(1-1)^2+(4-8)^2+(3+1)^2+(2-2)^2+(8-9)^2} \\ &= \sqrt{1+0+16+16+0+1} \\ &= \sqrt{34} \end{aligned}$$

## Exercise

Let $V$ be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operation on $\boldsymbol{u}=(u_1,\ u_2)$ $\boldsymbol{v}=(v_1,\ v_2)$

$$\boldsymbol{u}+\boldsymbol{v}=(u_1+v_1+1,\ u_2+v_2+1) \qquad k\boldsymbol{u}=(ku_1,\ ku_2)$$

*a)* Compute $\boldsymbol{u}+\boldsymbol{v}$ and $k\boldsymbol{u}$ for $\boldsymbol{u}$ = (0, 4), $\boldsymbol{v}$ = (1, –3), and $k$ = 2.

*b)* Show that (0, 0) ≠ **0**.

*c)* Show that (–1, –1) = 0.

*d)* Show that $\boldsymbol{u}+(-\boldsymbol{u})=0$ for $\boldsymbol{u}=(u_1,\ u_2)$

*e)* Find two vector space axioms that fail to hold.

## Solution

***a)*** $\boldsymbol{u}+\boldsymbol{v}=(0+1+1,\ 4-3+1)=(2,\ 2)$

$k\boldsymbol{u}=(ku_1,\ ku_2)=(2(0),\ 2(4))=(0,\ 8)$

***b)***
$$\begin{aligned} (0,\ 0)+(u_1,\ u_2) &= (0+u_1+1,\ 0+u_2+1) \\ &= (u_1+1,\ u_2+1) \\ &\neq (u_1,\ u_2) \end{aligned}$$

Therefore (0, 0) is not the zero vector **0** required (by Axiom).

***c)***
$$\begin{aligned} (-1,\ -1)+(u_1,\ u_2) &= (-1+u_1+1,\ -1+u_2+1) \\ &= (u_1,\ u_2) \end{aligned}$$

$$\begin{aligned} (u_1,\ u_2)+(-1,\ -1) &= (u_1-1+1,\ u_2-1+1) \\ &= (u_1,\ u_2) \end{aligned}$$

Therefore (–1, –1) = **0** holds.

***d)*** Let $\boldsymbol{u}=\left(u_1,\ u_2\right)\ -\boldsymbol{u}=\left(-2-u_1,\ \ -2-u_2\right)$

$$\boldsymbol{u}+\left(-\boldsymbol{u}\right)=\left(u_1+\left(-2-u_1\right)+1,\ u_2+\left(-2-u_2\right)+1\right)$$

$$=\left(-1,\ -1\right)$$

$$\underline{=\mathbf{0}}$$

$\boldsymbol{u}+\left(-\boldsymbol{u}\right)=0$ holds

***e)*** Axiom 7: $k\left(\boldsymbol{u}+\boldsymbol{v}\right)\overset{?}{=}k\boldsymbol{u}+k\boldsymbol{v}$

$$k\left(\boldsymbol{u}+\boldsymbol{v}\right)=k\left(u_1+v_1+1,\ u_2+v_2+1\right)=\left(ku_1+kv_1+k,\ ku_2+kv_2+k\right)$$

$$k\boldsymbol{u}+k\boldsymbol{v}=\left(ku_1,\ ku_2\right)+\left(kv_1,\ kv_2\right)=\left(ku_1+kv_1+1,\ ku_2+kv_2+1\right)$$

Therefore, $k\left(\boldsymbol{u}+\boldsymbol{v}\right)\neq k\boldsymbol{u}+k\boldsymbol{v}$ ; Axiom 7 fails to hold

Axiom 8: $\left(k+m\right)\boldsymbol{u}\overset{?}{=}k\boldsymbol{u}+m\boldsymbol{u}$

$$\left(k+m\right)\boldsymbol{u}=\left(\left(k+m\right)u_1,\ \left(k+m\right)u_2\right)=\left(ku_1+mu_1,\ ku_2+mu_2\right)$$

$$k\boldsymbol{u}+m\boldsymbol{u}=\left(ku_1,\ ku_2\right)+\left(mu_1,\ mu_2\right)=\left(ku_1+mu_1+1,\ ku_2+mu_2+1\right)$$

Therefore, $\left(k+m\right)\boldsymbol{u}\neq k\boldsymbol{u}+m\boldsymbol{u}$ ; Axiom 8 fails to hold

## *Exercise*

Find $\vec{w}$ given that $10\vec{u}+3\vec{w}=4\vec{v}-2\vec{w}$ , $\vec{u}=\begin{pmatrix}1\\-6\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-20\\5\end{pmatrix}$.

***Solution***

$$-10\vec{u}+10\vec{u}+3\vec{w}+2\vec{w}=-10\vec{u}+4\vec{v}-2\vec{w}+2\vec{w}$$

$$5\vec{w}=-10\vec{u}+4\vec{v}$$

$$\vec{w}=-2\vec{u}+\frac{4}{5}\vec{v}$$

$$=-2\begin{pmatrix}1\\-6\end{pmatrix}+\frac{4}{5}\begin{pmatrix}-20\\5\end{pmatrix}$$

$$=\begin{pmatrix}-2\\12\end{pmatrix}+\begin{pmatrix}-16\\4\end{pmatrix}$$

$$\underline{=\begin{pmatrix}-18\\16\end{pmatrix}}$$

## *Exercise*

Find $\vec{w}$ given that $\vec{u}+3\vec{v}-2\vec{w}=5\vec{u}+\vec{v}-4\vec{w}$ , $\vec{u}=\begin{pmatrix}1\\-1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-2\\3\end{pmatrix}$

***Solution***

$$\vec{u}-\vec{u}+3\vec{v}-3\vec{v}-2\vec{w}+4\vec{w}=5\vec{u}-\vec{u}+\vec{v}-3\vec{v}-4\vec{w}+4\vec{w}$$

$$2\vec{w}=4\vec{u}-2\vec{v}$$

$$\vec{w}=2\vec{u}-\vec{v}$$

$$=2\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}-2\\3\end{pmatrix}$$

$$=\begin{pmatrix}2\\-2\end{pmatrix}+\begin{pmatrix}-2\\3\end{pmatrix}$$

$$=\begin{pmatrix}0\\1\end{pmatrix}$$

## *Exercise*

Find $\vec{w}$ given that $2\vec{u}+\vec{v}-3\vec{w}=5\vec{u}+7\vec{v}+3\vec{w}$ , $\vec{u}=\begin{pmatrix}1\\-1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-2\\3\end{pmatrix}$

***Solution***

$$2\vec{u}-2\vec{u}+\vec{v}-\vec{v}-3\vec{w}-3\vec{w}=5\vec{u}-2\vec{u}+7\vec{v}-\vec{v}+3\vec{w}-3\vec{w}$$

$$-6\vec{w}=3\vec{u}+6\vec{v}$$

$$\vec{w}=-\frac{1}{2}\vec{u}-\vec{v}$$

$$=-\frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}-\begin{pmatrix}-2\\3\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\end{pmatrix}-\begin{pmatrix}-2\\3\end{pmatrix}$$

$$=\begin{pmatrix}\frac{3}{2}\\-\frac{5}{2}\end{pmatrix}$$

### Exercise

Draw $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, and $\vec{u}+2\vec{v}$ $\quad\quad \vec{u}=\begin{pmatrix}1\\-1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-2\\3\end{pmatrix}$

***Solution***

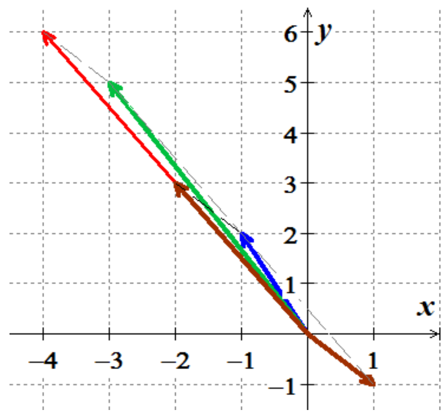
$$\vec{u}+\vec{v}=\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}-2\\3\end{pmatrix}$$

$$=\begin{pmatrix}-1\\2\end{pmatrix}$$

$$\vec{u}+2\vec{v}=\begin{pmatrix}1\\-1\end{pmatrix}+2\begin{pmatrix}-2\\3\end{pmatrix}$$

$$=\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}-4\\6\end{pmatrix}$$

$$=\begin{pmatrix}-3\\5\end{pmatrix}$$




### Exercise

Draw $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, and $\vec{u}+2\vec{v}$ $\quad\quad \vec{u}=\begin{pmatrix}-1\\1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}2\\1\end{pmatrix}$

***Solution***

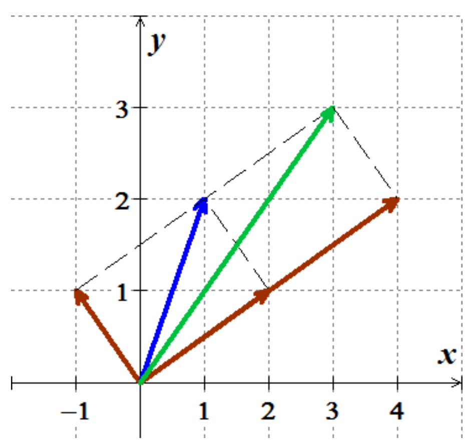
$$\vec{u}+\vec{v}=\begin{pmatrix}-1\\1\end{pmatrix}+\begin{pmatrix}2\\1\end{pmatrix}$$

$$=\begin{pmatrix}1\\2\end{pmatrix}$$

$$\vec{u}+2\vec{v}=\begin{pmatrix}-1\\1\end{pmatrix}+2\begin{pmatrix}2\\1\end{pmatrix}$$

$$=\begin{pmatrix}-1\\1\end{pmatrix}+\begin{pmatrix}4\\2\end{pmatrix}$$

$$=\begin{pmatrix}3\\3\end{pmatrix}$$



### Exercise

Draw $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, and $\vec{u}+2\vec{v}$ $\quad \vec{u}=\begin{pmatrix}1\\0\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}1\\2\end{pmatrix}$

***Solution***

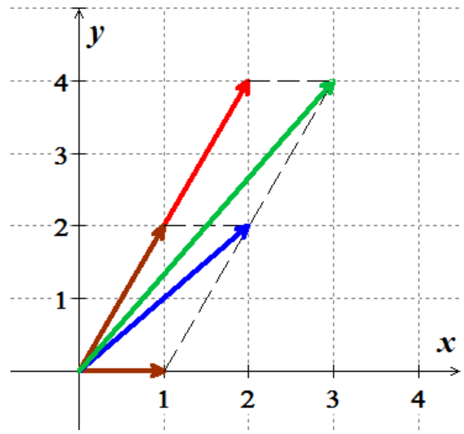
$$\vec{u}+\vec{v}=\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}1\\2\end{pmatrix}$$

$$=\begin{pmatrix}2\\2\end{pmatrix}$$

$$\vec{u}+2\vec{v}=\begin{pmatrix}1\\0\end{pmatrix}+2\begin{pmatrix}1\\2\end{pmatrix}$$

$$=\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}2\\4\end{pmatrix}$$

$$=\begin{pmatrix}3\\4\end{pmatrix}$$




### Exercise

Draw $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, and $\vec{u}+2\vec{v}$ $\quad \vec{u}=\begin{pmatrix}-1\\0\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}1\\-1\end{pmatrix}$

***Solution***

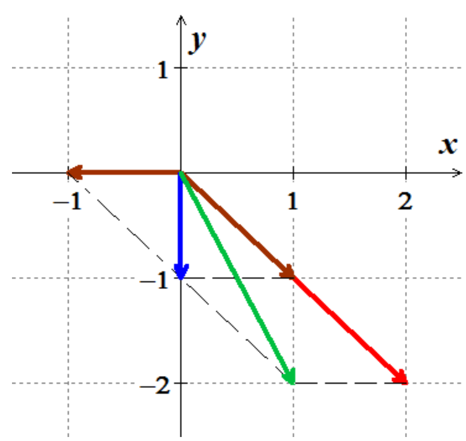
$$\vec{u}+\vec{v}=\begin{pmatrix}-1\\0\end{pmatrix}+\begin{pmatrix}1\\-1\end{pmatrix}$$

$$=\begin{pmatrix}0\\-1\end{pmatrix}$$

$$\vec{u}+2\vec{v}=\begin{pmatrix}-1\\0\end{pmatrix}+2\begin{pmatrix}1\\-1\end{pmatrix}$$

$$=\begin{pmatrix}-1\\0\end{pmatrix}+\begin{pmatrix}2\\-2\end{pmatrix}$$

$$=\begin{pmatrix}1\\-2\end{pmatrix}$$



# Solution  *Section* 2.2 – Norm, Dot product, and distance in $R^n$

### *Exercise*

If $\|\vec{v}\|=5$ and $\|\vec{w}\|=3$, what are the smallest and largest possible values of $\|\vec{v}-\vec{w}\|$ and $\vec{v}.\vec{w}$?

***Solution***

$$\|\vec{v}-\vec{w}\| \le \|\vec{v}\|+\|\vec{w}\| = 5+3=8$$

$$\|\vec{v}-\vec{w}\| \ge \|\vec{v}\|-\|\vec{w}\| = 5-3=2$$

$$|\vec{v}.\vec{w}| = \|\vec{v}\|.\|\vec{w}\|.cos\,\theta \le \|\vec{v}\|.\|\vec{w}\|$$

$$-\|\vec{v}\|.\|\vec{w}\| \le |\vec{v}.\vec{w}| \le \|\vec{v}\|.\|\vec{w}\|$$

$$-(3)(5) \le |\vec{v}.\vec{w}| \le (3)(5)$$

$$-15 \le |\vec{v}.\vec{w}| \le 15$$

The minimum value occurs when the dot product is a small as possible, $\vec{v}$ and $\vec{w}$ are parallel, but point in opposite directions. Thus, the smallest value is $-15$.

The maximum value occurs when the dot product is a large as possible, $\vec{v}$ and $\vec{w}$ are parallel and point in same direction. Thus, the largest value is 15.

### *Exercise*

If $\|\vec{v}\|=7$ and $\|\vec{w}\|=3$, what are the smallest and largest possible values of $\|\vec{v}+\vec{w}\|$ and $\vec{v}.\vec{w}$?

***Solution***

$$\|\vec{v}+\vec{w}\| \le \|\vec{v}\|+\|\vec{w}\| = 7+3=10$$

$$\|\vec{v}+\vec{w}\| \ge \|\vec{v}\|-\|\vec{w}\| = 7-3=4$$

$$|\vec{v}.\vec{w}| \le \|\vec{v}\|.\|\vec{w}\|$$

$$-\|\vec{v}\|.\|\vec{w}\| \le |\vec{v}.\vec{w}| \le \|\vec{v}\|.\|\vec{w}\|$$

$$-(7)(3) \le |\vec{v}.\vec{w}| \le (7)(3)$$

$$-21 \le |\vec{v}.\vec{w}| \le 21$$

The minimum value occurs when the dot product is a small as possible, $\vec{v}$ and $\vec{w}$ are parallel, but point in opposite directions. Thus, the smallest value is -21. $\vec{v}=(7,\ 0,\ 0,\ \cdots)$ and

$\vec{w}=(-3,\ 0,\ 0,\ \cdots)$

The maximum value occurs when the dot product is a large as possible, $\vec{v}$ and $\vec{w}$ are parallel and point in same direction. Thus, the largest value is 21. $\vec{v}=(7,\ 0,\ 0,\ \cdots)$ and $\vec{w}=(3,\ 0,\ 0,\ \cdots)$

### Exercise

Given that $cos(\alpha) = \frac{\vec{v}_1}{\|\vec{v}\|}$ and $sin(\alpha) = \frac{\vec{v}_2}{\|\vec{v}\|}$. Similarly, $cos(\beta) =$ ______ and $sin(\beta) =$ ______ . The angle $\theta$ is $\beta - \alpha$. Substitute into the trigonometry formula $\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ for $cos(\beta - \alpha)$ to find $\cos\theta = \frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\|.\|\vec{w}\|}$

***Solution***

$$\cos\beta = \frac{\vec{w}_1}{\|\vec{w}\|}$$

$$\sin\beta = \frac{\vec{w}_2}{\|\vec{w}\|}$$

$$\begin{aligned}\cos(\beta - \alpha) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ &= \frac{\vec{v}_1}{\|\vec{v}\|}\frac{\vec{w}_1}{\|\vec{w}\|} + \frac{\vec{v}_2}{\|\vec{v}\|}\frac{\vec{w}_2}{\|\vec{w}\|} \\ &= \frac{\vec{v}_1\vec{w}_1 + \vec{v}_2\vec{w}_2}{\|\vec{v}\|.\|\vec{w}\|} \\ &= \frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\|.\|\vec{w}\|}\end{aligned}$$

### Exercise

Can three vectors in the *xy* plane have $\vec{u} \bullet \vec{v} < 0$ and $\vec{v} \bullet \vec{w} < 0$ and $\vec{u} \bullet \vec{w} < 0$?

***Solution***

Let consider: $\vec{u} = (1,\ 0)$, $\vec{v} = \left(-\frac{1}{2},\ \frac{\sqrt{3}}{2}\right)$, $\vec{w} = \left(-\frac{1}{2},\ -\frac{\sqrt{3}}{2}\right)$

$$\vec{u} \bullet \vec{v} = (1)\left(-\frac{1}{2}\right) + 0 = -\frac{1}{2}$$

$$\begin{aligned}\vec{v} \bullet \vec{w} &= \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) \\ &= \frac{1}{4} - \frac{3}{4} \\ &= -\frac{1}{2}\end{aligned}$$

$$\vec{u} \bullet \vec{w} = (1)\left(-\frac{1}{2}\right) + (0)\left(-\frac{\sqrt{3}}{2}\right) = -\frac{1}{2} < 0$$

Yes, it is.

### *Exercise*

Find the norm of $\vec{v}$, a unit vector that has the same direction as $\vec{v}$, and a unit vector that is oppositely directed.

*a)* $\vec{v} = (4, -3)$

*b)* $\vec{v} = (1, -1, 2)$

*c)* $\vec{v} = (-2, 3, 3, -1)$

***Solution***

***a)*** $\|\vec{v}\| = \sqrt{4^2 + (-3)^2} = 5$

***Same direction unit vector***: $\vec{u}_1 = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{1}{5}(4, -3) = \left(\dfrac{4}{5}, -\dfrac{3}{5}\right)$

***Opposite direction unit vector***: $\vec{u}_2 = -\dfrac{\vec{v}}{\|\vec{v}\|} = -\dfrac{1}{5}(4, -3) = \left(-\dfrac{4}{5}, \dfrac{3}{5}\right)$

***b)*** $\|\vec{v}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$

***Same direction unit vector***:

$$\vec{u}_1 = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{6}}(1, -1, 2) = \left(\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)$$

***Opposite direction unit vector***:

$$\vec{u}_2 = -\frac{\vec{v}}{\|\vec{v}\|} = -\frac{1}{\sqrt{6}}(1, -1, 2) = \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right)$$

***c)*** $\|\vec{v}\| = \sqrt{(-2)^2 + (3)^2 + (3)^2 + (-1)^2} = \sqrt{23}$

***Same direction unit vector***:

$$\vec{u}_1 = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{23}}(-2, 3, 3, -1) = \left(\frac{-2}{\sqrt{23}},\ \frac{3}{\sqrt{23}},\ \frac{3}{\sqrt{23}},\ -\frac{1}{\sqrt{23}}\right)$$

***Opposite direction unit vector***:

$$\vec{u}_2 = -\frac{\vec{v}}{\|\vec{v}\|} = -\frac{1}{\sqrt{23}}(-2, 3, 3, -1) = \left(\frac{2}{\sqrt{23}},\ -\frac{3}{\sqrt{23}},\ -\frac{3}{\sqrt{23}},\ \frac{1}{\sqrt{23}}\right)$$

## *Exercise*

Evaluate the given expression with $\boldsymbol{u} = (2, -2, 3)$, $\boldsymbol{v} = (1, -3, 4)$, and $\boldsymbol{w} = (3, 6, -4)$

*a)* $\|\vec{u}+\vec{v}\|$ *b)* $\|-2\vec{u}+2\vec{v}\|$

c) $\|3\vec{u}-5\vec{v}+\vec{w}\|$ *d)* $\|3\vec{v}\|-3\|\vec{v}\|$

*e)* $\|\vec{u}\|+\|-2\vec{v}\|+\|-3\vec{w}\|$

***Solution***

***a)***
$$\begin{aligned}\|\vec{u}+\vec{v}\| &= \|(2,-2,3)+(1,-3,4)\| \\ &= \|(3,\ -5,\ 7)\| \\ &= \sqrt{3^2+(-5)^2+7^2} \\ &= \boxed{\sqrt{83}}\end{aligned}$$

***b)***
$$\begin{aligned}\|-2\vec{u}+2\vec{v}\| &= \|(-4,\ 4,\ -6)+(2,\ -6,\ 8)\| \\ &= \|(-2,\ -2,\ 2)\| \\ &= \sqrt{(-2)^2+(-2)^2+2^2} \\ &= \sqrt{12} \\ &= \boxed{2\sqrt{3}}\end{aligned}$$

***c)***
$$\begin{aligned}\|3\vec{u}-5\vec{v}+\vec{w}\| &= \|(6,\ -6,\ 9)-(5,\ -15,\ 20)+(3,\ 6,\ -4)\| \\ &= \|(4,\ 15,\ -15)\| \\ &= \sqrt{(4)^2+(15)^2+(-15)^2} \\ &= \boxed{\sqrt{466}}\end{aligned}$$

***d)***
$$\begin{aligned}\|3\vec{v}\|-3\|\vec{v}\| &= \|(3,\ -9,\ 12)\|-3\|(1,\ -3,\ 4)\| \\ &= \sqrt{3^2+(-9)^2+12^2}-3\sqrt{1^2+(-3)^2+4^2} \\ &= \sqrt{234}-3\sqrt{26} \\ &= 3\sqrt{26}-3\sqrt{26} \\ &= \boxed{0}\end{aligned}$$

$$\|3v\|-3\|v\| = 3\|v\|-3\|v\| = \boxed{0}$$

***e)***
$$\begin{aligned}\|\vec{u}\|+\|-2\vec{v}\|+\|-3\vec{w}\| &= \|\vec{u}\|-2\|\vec{v}\|-3\|\vec{w}\| \\ &= \sqrt{2^2+(-2)^2+3^2}-2\sqrt{1^2+(-3)^2+4^2}-3\sqrt{3^2+6^2+(-4)^2} \\ &= \boxed{\sqrt{17}-2\sqrt{26}-3\sqrt{61}}\end{aligned}$$

## Exercise

Let $\boldsymbol{v}$ = (1, 1, 2, –3, 1). Find all scalars $k$ such that $\|k\vec{v}\| = 5$

***Solution***

$$\|k\vec{v}\| = |k|\|\vec{v}\|$$
$$= |k| \ \|(1,\ 1,\ 2,\ -3,\ 1)\|$$
$$= |k| \ \sqrt{1^2 + 1^2 + 2^2 + (-3)^2 + 1^2}$$
$$= |k|\sqrt{49}$$
$$= 7|k|$$

$$7|k| = 5 \quad \rightarrow \quad |k| = \frac{5}{7} \Rightarrow \boxed{k = \pm\frac{5}{7}}$$

## Exercise

Find $\vec{u}\bullet\vec{v}, \quad \vec{u}\bullet\vec{u}, \quad and \quad \vec{v}\bullet\vec{v}$

*a)* $\vec{u}$ = (3, 1, 4), $\vec{v}$ = (2, 2, –4)

*b)* $\vec{u}$ = (1, 1, 4, 6), $\vec{v}$ = (2, –2, 3, –2)

*c)* $\vec{u}$ = (2, –1, 1, 0, –2), $\vec{v}$ = (1, 2, 2, 2, 1)

***Solution***

***a)***
$$\vec{u}\bullet\vec{v} = (3,\ 1,\ 4)\bullet(2,\ 2,\ -4)$$
$$= 3(2) + 1(2) + 4(-4)$$
$$\underline{= -8|}$$

$$\vec{u}\bullet\vec{u} = \|\vec{u}\|^2$$
$$= 3^2 + 1^2 + 4^2$$
$$\underline{= 26|}$$

$$\vec{v}\bullet\vec{v} = \|\vec{v}\|^2$$
$$= 2^2 + 2^2 + (-4)^2$$
$$\underline{= 24|}$$

***b)***
$$\vec{u}\bullet\vec{v} = (1,\ 1,\ 4,\ 6)\bullet(2,\ -2,\ 3,\ -2)$$
$$= 1(2) + 1(-2) + 4(3) + 6(-2)$$
$$\underline{= 0|}$$

$$\vec{u}\bullet\vec{u} = \|\vec{u}\|^2$$
$$= 1^2 + 1^2 + 4^2 + 6^2$$
$$\underline{= 54|}$$

$$\vec{v}\bullet\vec{v} = \|\vec{v}\|^2$$
$$= 2^2 + (-2)^2 + 3^2 + (-2)^2$$
$$\underline{= 21|}$$

***c)*** $\vec{u}\bullet\vec{v} = (2,\ -1,\ 1,\ 0,\ -2)\bullet(1,\ 2,\ 2,\ 2,\ 1)$
$$= 2(1) - 1(2) + 1(2) + 0(2) - 2(1)$$
$$\underline{= 0|}$$

$$\vec{u}\bullet\vec{u} = \|\vec{u}\|^2$$
$$= 2^2 + (-1)^2 + 1^2 + 0 + (-2)^2$$
$$\underline{= 10|}$$

$$\vec{v}\bullet\vec{v} = \|\vec{v}\|^2$$
$$= 1^2 + 2^2 + 2^2 + 2^2 + 1^2$$
$$\underline{= 14|}$$

## *Exercise*

Find the Euclidean distance between ***u*** and ***v***, then find the angle between them

*a)* $\vec{u}$ = (3, 3, 3), $\vec{v}$ = (1, 0, 4)

*b)* $\vec{u}$ = (1, 2, –3, 0), $\vec{v}$ = (5, 1, 2, –2)

*c)* $\vec{u}$ = (0, 1, 1, 1, 2), $\vec{v}$ = (2, 1, 0, –1, 3)

***Solution***

***a)*** $d = \|\vec{u} - \vec{v}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
$$= \sqrt{(-2)^2 + (-3)^2 + (1)^2}$$
$$\underline{= \sqrt{14}|}$$

$$\cos\theta = \frac{\vec{u}\bullet\vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$$
$$= \frac{3(1) + 3(0) + 3(4)}{\sqrt{3^2 + 3^2 + 3^2}\sqrt{1^2 + 0^2 + 4^2}}$$

$$= \frac{15}{\sqrt{27}\sqrt{17}}$$

$$\theta = \cos^{-1}\left(\frac{15}{\sqrt{27}\sqrt{17}}\right) = \boxed{45.56°}$$

***b)*** $d = \|\vec{u} - \vec{v}\| = \sqrt{(1-5)^2 + (-2-1)^2 + (-3-2)^2 + (-2-0)^2}$

$$= \sqrt{(-4)^2 + (-3)^2 + (-5)^2 + (-2)^2}$$

$$\boxed{= \sqrt{46}}$$

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\| \ \|\vec{v}\|}$$

$$= \frac{1(5) + 2(1) - 3(2) + 0(-2)}{\sqrt{1^2 + 2^2 + (-3)^2 + 0}\ \sqrt{5^2 + 1^2 + 2^2 + (-2)^2}}$$

$$= \frac{1}{\sqrt{14}\sqrt{34}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{14}\sqrt{34}}\right) = \boxed{87.37°}$$

***c)*** $d = \|\vec{u} - \vec{v}\| = \sqrt{(0-2)^2 + (1-1)^2 + (1-0)^2 + (1-(-1))^2 + (2-3)^2}$

$$\boxed{= \sqrt{10}}$$

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\| \ \|\vec{v}\|}$$

$$= \frac{0(2) + 1(1) + 1(0) + 1(-1) + 2(3)}{\sqrt{0 + 1^2 + 1^2 + 1^2 + 2^2}\ \sqrt{2^2 + 1^2 + 0 + (-1)^2 + (3)^2}}$$

$$= \frac{6}{\sqrt{7}\sqrt{15}}$$

$$\theta = \cos^{-1}\left(\frac{6}{\sqrt{7}\sqrt{15}}\right) = \boxed{54.16°}$$

## *Exercise*

Find a unit vector that has the same direction as the given vector

*a)* $(-4, -3)$ *b)* $\left(-3,\ 2,\ \sqrt{3}\right)$ *c)* $(1, 2, 3, 4, 5)$

***Solution***

**a)** $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{(-4,\ -3)}{\sqrt{(-4)^2 + (-3)^2}}$

$= \frac{(-4,\ -3)}{\sqrt{25}}$

$= \left(-\frac{4}{5},\ -\frac{3}{5}\right)$

**b)** $\vec{u} = \frac{1}{\sqrt{(-3)^2 + (2)^2 + (\sqrt{3})^2}}\left(-3,\ 2,\ \sqrt{3}\right)$

$= \frac{1}{\sqrt{17}}\left(-3,\ 2,\ \sqrt{3}\right)$

$= \left(-\frac{3}{\sqrt{17}},\ \frac{2}{\sqrt{17}},\ \frac{\sqrt{3}}{\sqrt{17}}\right)$

**c)** $\vec{u} = \frac{1}{\sqrt{1^2 + 2^2 + 3^2 + 4^2 + 5^2}}(1,\ 2,\ 3,\ 4,\ 5)$

$= \frac{1}{\sqrt{55}}(1,\ 2,\ 3,\ 4,\ 5)$

$= \left(\frac{1}{\sqrt{55}},\ \frac{2}{\sqrt{55}},\ \frac{3}{\sqrt{55}},\ \frac{4}{\sqrt{55}},\ \frac{5}{\sqrt{55}}\right)$

## *Exercise*

Find a unit vector that is oppositely to the given vector

*a)* $(-12, -5)$

*b)* $(3, -3, 3)$

*c)* $\left(-3,\ 1,\ \sqrt{6},\ 3\right)$

***Solution***

**a)** $\vec{u} = -\frac{1}{\sqrt{(-12)^2 + (-5)^2}}(-12,\ -5)$

$= -\frac{1}{\sqrt{169}}(-12,\ -5)$

$= \left(\frac{12}{13},\ \frac{5}{13}\right)$

**b)** $\vec{u} = -\frac{1}{\sqrt{(3)^2 + (-3)^2 + (3)^2}}(3,\ -3,\ 3)$

$$= -\frac{1}{\sqrt{27}}(3,\ -3,\ 3)$$

$$= -\frac{1}{3\sqrt{3}}(3,\ -3,\ 3)$$

$$= \left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)$$

***c)*** $\vec{u} = -\frac{1}{\sqrt{(-3)^2 + 1^2 + \left(\sqrt{6}\right)^2 + 3^2}}\left(-3,\ 1,\ \sqrt{6},\ 3\right)$

$$= -\frac{1}{\sqrt{25}}\left(-3,\ 1,\ \sqrt{6},\ 3\right)$$

$$= \left(\frac{3}{5},\ -\frac{1}{5},\ -\frac{\sqrt{6}}{5},\ -\frac{3}{5}\right)$$

## *Exercise*

Verify that the Cauchy-Schwarz inequality holds

*a)* $\vec{u} = (-3,\ 1,\ 0)$, $\vec{v} = (2, -1, 3)$

*b)* $\vec{u} = (0, 2, 2, 1)$, $\vec{v} = (1, 1, 1, 1)$

*c)* $\vec{u} = (1, 3, 5, 2, 0, 1)$, $\vec{v} = (0, 2, 4, 1, 3, 5)$

***Solution***

***a)*** $|\vec{u} \bullet \vec{v}| = |(-3,1,0) \bullet (2,-1,3)|$

$$= |-3(2) + 1(-1) + 0(3)|$$

$$= |-7|$$

$$= 7$$

$$\|\vec{u}\|\|\vec{v}\| = \sqrt{(-3)^2 + 1^2 + 0}\ \sqrt{(2)^2 + (-1)^2 + 3^2}$$

$$= \sqrt{10}\sqrt{14}$$

$$\approx 11.83$$

$|\vec{u} \bullet \vec{v}| \le \|\vec{u}\|\|\vec{v}\|$ Cauchy-Schwarz inequality holds

***b)*** $|\vec{u} \bullet \vec{v}| = |(0,\ 2,\ 2,\ 1) \bullet (1,\ 1,\ 1,\ 1)|$

$$= |0 + 2 + 2 + 1|$$

$$= 5$$

$$\|\vec{u}\|\|\vec{v}\| = \sqrt{0+2^2+2^2+1^2}\ \sqrt{1^2+1^2+1^2+1^2}$$

$$= \sqrt{9}\sqrt{4}$$

$$\underline{= 6}$$

$|\vec{u}\bullet\vec{v}| \le \|\vec{u}\|\|\vec{v}\|$ Cauchy-Schwarz inequality holds

***c)*** $|\vec{u}\bullet\vec{v}| = |(1,\ 3,\ 5,\ 2,\ 0,\ 1)\bullet(0,\ 2,\ 4,\ 1,\ 3,\ 5)|$

$$= |0+6+20+2+0+5|$$

$$\underline{= 23}$$

$$\|\vec{u}\|\|\vec{v}\| = \sqrt{1^2+3^2+5^2+2^2+0+1^2}\ \sqrt{0+2^2+4^2+1^2+3^2+5^2}$$

$$= \sqrt{40}\sqrt{55}$$

$$\underline{\approx 46}$$

$|\vec{u}\bullet\vec{v}| \le \|\vec{u}\|\|\vec{v}\|$ Cauchy-Schwarz inequality holds

## *Exercise*

Find $\boldsymbol{u}\bullet\boldsymbol{v}$ and then the angle $\theta$ between $\boldsymbol{u}$ and $\boldsymbol{v}$ $\boldsymbol{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$ $\boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$

***Solution***

$$\boldsymbol{u}\bullet\boldsymbol{v} = 3+0-2-1\ \underline{= 0}$$

$$\theta = \cos^{-1}\frac{0}{\sqrt{15}\sqrt{3}} = \cos^{-1}(0)\ \ \underline{= 90^\circ}$$

## *Exercise*

Find the norm: $\|\boldsymbol{u}\|+\|\boldsymbol{v}\|$, $\|\boldsymbol{u}+\boldsymbol{v}\|$ for $\boldsymbol{u} = (3,\ -1,\ -2,\ 1,\ 4)$ $\boldsymbol{v} = (1,\ 1,\ 1,\ 1,\ 1)$

***Solution***

$$\|\boldsymbol{u}\|+\|\boldsymbol{v}\| = \sqrt{3^2+(-1)^2+(-2)^2+1^2+4^2}+\sqrt{1+1+1+1+1}$$

$$\underline{= \sqrt{31}+\sqrt{5}}$$

$$\|\boldsymbol{u}+\boldsymbol{v}\| = \|(4,\ 0,\ -1,\ 2,\ 5)\|\ = \sqrt{16+0+1+4+25}$$

$$\underline{= \sqrt{46}}$$

### Exercise

Find all numbers $r$ such that: $\left\| r\left(1,\ 0,\ -3,\ -1,\ 4,\ 1\right)\right\| = 1$

**Solution**

$$r\sqrt{1+9+1+16+1} = \pm 1$$

$$r\sqrt{28} = \pm 1$$

$$r = \pm \frac{1}{2\sqrt{7}} = \pm \frac{\sqrt{7}}{14}$$

### Exercise

Find the distance between $P_1(7,-5,1)$ and $P_2(-7,-2,-1)$

**Solution**

$$\begin{aligned}\left\|P_1P_2\right\| &= \sqrt{(-7-7)^2+(-2+5)^2+(-1-1)^2}\\ &= \sqrt{14^2+3^2+(-2)^2}\\ &= \sqrt{196+9+4}\\ &= \sqrt{209}\end{aligned}$$

### Exercise

Given $\boldsymbol{u} = (1, -5, 4)$, $\boldsymbol{v} = (3, 3, 3)$

*a)* Find $\vec{u} \bullet \vec{v}$

*b)* Find the cosine of the angle $\theta$ between $\boldsymbol{u}$ and $\boldsymbol{v}$.

**Solution**

***a)*** $\vec{u} \bullet \vec{v} = 3 - 15 + 12 = 0$

***b)*** $\cos\theta = \dfrac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\ \|\boldsymbol{v}\|} = 0$

### Exercise

Let $\vec{u} = \begin{pmatrix}2\\1\end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix}1\\4\end{pmatrix}$. Find $\left\| \dfrac{1}{\|2\vec{u}+\vec{v}\|}\left(2\vec{u}+\vec{v}\right)\right\|$

**Solution**

Since, the unit vector equals to a vector $\left(2\vec{u}+\vec{v}\right)$ divided by its magnitude.

Therefore, $\left\| \frac{1}{\|2\vec{u}+\vec{v}\|}\left(2\vec{u}+\vec{v}\right) \right\| = \frac{1}{\|2\vec{u}+\vec{v}\|}\|2\vec{u}+\vec{v}\| \; \underline{=1}$

### Exercise

Let $\vec{u}=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix} 3 \\ 3 \end{pmatrix}$. Find $\left\| \frac{1}{\|\vec{u}-\vec{v}\|}\left(\vec{u}-\vec{v}\right) \right\|$

**Solution**

$$\left\| \frac{1}{\|\vec{u}-\vec{v}\|}\left(\vec{u}-\vec{v}\right) \right\| = \frac{1}{\|\vec{u}-\vec{v}\|}\|\vec{u}-\vec{v}\| \; \underline{=1}$$

### Exercise

Let $\vec{u}=\begin{pmatrix} 18 \\ 6 \end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix} -11 \\ 12 \end{pmatrix}$. Find $\left\| \frac{1}{\|5\vec{u}+3\vec{v}\|}\left(5\vec{u}+3\vec{v}\right) \right\|$

**Solution**

$$\left\| \frac{1}{\|5\vec{u}+3\vec{v}\|}\left(5\vec{u}+3\vec{v}\right) \right\| = \frac{1}{\|5\vec{u}+3\vec{v}\|}\|5\vec{u}+3\vec{v}\| \; \underline{=1}$$

### Exercise

Let $\vec{u}=\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ and $\vec{v}=\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$. Calculate the following:

*a)* $\vec{u}+\vec{v}$ *b)* $2\vec{u}+3\vec{v}$ *c)* $\vec{v}+\left(2\vec{u}-3\vec{v}\right)$ *d)* $\|\vec{u}\|$ *e)* $\|\vec{v}\|$ *f)* *unit vector of* $\vec{v}$

**Solution**

***a)*** $\vec{u}+\vec{v}=\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}+\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

$$=\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

***b)*** $2\vec{u}+3\vec{v}=3\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}+3\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} 9 \\ 3 \\ -3 \end{pmatrix} + \begin{pmatrix} -6 \\ 3 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$$

***c)*** $\vec{v} + \left(2\vec{u} - 3\vec{v}\right) = \vec{v} + 2\vec{u} - 3\vec{v}$

$$= 2\vec{u} - 2\vec{v}$$

$$= 2\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} - 2\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 2 \\ -2 \end{pmatrix} - \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 10 \\ 0 \\ -6 \end{pmatrix}$$

***d)*** $\|\vec{u}\| = \sqrt{3^2 + 1^2 + (-1)^2}$

$$= \sqrt{9+1+1}$$

$$\underline{= \sqrt{11}\,}|$$

***e)*** $\|\vec{v}\| = \sqrt{(-2)^2 + 1^2 + 2^2}$

$$= \sqrt{4+1+4}$$

$$\underline{= 3\,}|$$

***f)*** $\textit{unit vector of } \vec{v} = \dfrac{\vec{v}}{\|\vec{v}\|}$

$$= \frac{(-2,\ 1,\ 2)}{3}$$

$$\underline{= \left(-\tfrac{2}{3},\ \tfrac{1}{3},\ \tfrac{2}{3}\right)\,}|$$

## *Exercise*

Let $\vec{u} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 4 \\ 3 \\ 1 \end{pmatrix}$. Calculate the following:

*a)* $\vec{u} - \vec{v}$ *b)* $3\vec{u} - 2\vec{v}$ *c)* $2(\vec{u} - \vec{v}) + 3\vec{u}$ *d)* $\|\vec{u}\|$ *e)* *unit vector of* $\vec{v}$

**Solution**

***a)*** $$\vec{u} - \vec{v} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -5 \\ -3 \\ 0 \end{pmatrix}$$

***b)*** $$3\vec{u} - 2\vec{v} = 3\begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 4 \\ 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ -3 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 8 \\ 6 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -11 \\ -6 \\ 1 \end{pmatrix}$$

***c)*** $$2(\vec{u} - \vec{v}) + 3\vec{u} = 2\vec{u} - 2\vec{v} + 3\vec{u}$$

$$= 5\vec{u} - 2\vec{v}$$

$$= 5\begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 4 \\ 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 10 \\ -5 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 8 \\ 6 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ -13 \\ -6 \\ 3 \end{pmatrix}$$

***d)*** $\|\vec{u}\| = \sqrt{2^2 + (-1)^2 + 0 + 1^2}$

$= \sqrt{4+1+1}$

$= \sqrt{6}$

***e)*** *unit vector of* $\vec{v} = \dfrac{\vec{v}}{\|\vec{v}\|}$

$$= \frac{(1,\ 4,\ 3,\ 1)}{\sqrt{1+16+9+1}}$$

$$= \frac{(1,\ 4,\ 3,\ 1)}{\sqrt{27}}$$

$$= \left( \frac{1}{3\sqrt{3}},\ \frac{4}{3\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{3\sqrt{3}} \right)$$

### *Exercise*

Let $\vec{u} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \\ -1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ -1 \end{pmatrix}$. Calculate the following:

*a)* $\vec{v} - \vec{u}$ *b)* $\vec{u} + 3\vec{v}$ *c)* $3(\vec{u} + \vec{v}) - 3\vec{u}$ *d)* $\|\vec{v}\|$ *e)* *unit vector of* $\vec{v}$

**Solution**

***a)*** $\vec{v} - \vec{u} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \\ -1 \end{pmatrix}$

$$= \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

***b)*** $\vec{u} + 3\vec{v} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ -1 \end{pmatrix}$

$$= \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 6 \\ 3 \\ -3 \end{pmatrix}$$

$$= (5,\ 1,\ 9,\ 3,\ -4)$$

***c)*** $3(\vec{u} + \vec{v}) - 3\vec{u} = 3\vec{u} + 3\vec{v} - 3\vec{u}$

$$= 3\vec{v}$$

$$= 3(1,\ 0,\ 2,\ 1,\ -1)$$

$$= (3,\ 0,\ 6,\ 3,\ -3)$$

***d)*** $\|\vec{v}\| = \sqrt{1^2 + 0 + 2^2 + 1^2 + (-1)^2}$

$$= \sqrt{1+4+1+1}$$

$$= \sqrt{7}$$

***e)*** *unit vector of* $\vec{v} = \dfrac{\vec{v}}{\|\vec{v}\|}$

$$= \frac{(1,\ 0,\ 2,\ 1,\ -1)}{\sqrt{7}}$$

$$= \left(\frac{1}{\sqrt{7}},\ 0,\ \frac{2}{\sqrt{7}},\ \frac{1}{\sqrt{7}},\ -\frac{1}{\sqrt{7}}\right)$$

## *Exercise*

Let $\vec{u} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$. Calculate the following:

*a)* $\vec{u} \bullet \vec{v}$ *b)* $\vec{u} \bullet (\vec{v} + \vec{w})$ *c)* $(\vec{u} + 2\vec{v}) \bullet \vec{w}$ *d)* $\|(\vec{w} \bullet \vec{v})\vec{u}\|$

**Solution**

***a)*** $\vec{u} \bullet \vec{v} = (2, \ -1, \ 1) \bullet (1, \ 2, \ -2)$

$= 2 - 2 - 2$

$= -2$

***b)*** $\vec{u} \bullet (\vec{v} + \vec{w}) = (2, \ -1, \ 1) \bullet \left[ (1, \ 2, \ -2) + (3, \ 2, \ 1) \right]$

$= (2, \ -1, \ 1) \bullet (4, \ 4, \ -1)$

$= 8 - 4 - 1$

$= 3$

***c)*** $(\vec{u} + 2\vec{v}) \bullet \vec{w} = \left[ (2, \ -1, \ 1) + 2(1, \ 2, \ -2) \right] \bullet (3, \ 2, \ 1)$

$= (4, \ 3, \ -3) \bullet (3, \ 2, \ 1)$

$= 12 + 6 - 3$

$= 15$

***d)*** $\|(\vec{w} \bullet \vec{v})\vec{u}\| = |\vec{w} \bullet \vec{v}| \ \|\vec{u}\|$

$= |(3, \ 2, \ 1) \bullet (1, \ 2, \ -2)| \sqrt{2^2 + (-1)^2 + 1^2}$

$= |3 + 4 - 2| \sqrt{4 + 1 + 1}$

$= 5\sqrt{6}$

## *Exercise*

Let $\vec{u} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -2 \\ 5 \\ 2 \\ -6 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 4 \\ -1 \\ 0 \\ -2 \end{pmatrix}$. Calculate the following:

*a)* $\vec{u} \bullet \vec{v}$ *b)* $\vec{u} \bullet (\vec{v} + \vec{w})$ *c)* $(\vec{u} + \vec{v}) \bullet (\vec{u} - \vec{v})$ *d)* $\|(\vec{w} \bullet \vec{v})\vec{u}\|$

**Solution**

***a)*** $\vec{u} \bullet \vec{v} = (1, \ 3, \ 2, \ 1) \bullet (-2, \ 5, \ 2, \ -6)$

$= -2 + 15 + 4 - 6$

$$\boxed{=11}$$

***b)*** $\vec{u}\cdot(\vec{v}+\vec{w})=(1,\ 3,\ 2,\ 1)\cdot\left[(-2,\ 5,\ 2,\ -6)+(4,\ -1,\ 0,\ -2)\right]$

$$=(1,\ 3,\ 2,\ 1)\cdot(2,\ 4,\ 2,\ -8)$$

$$=2+12+4-8$$

$$\boxed{=10}$$

***c)*** $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v})=\left[(1,\ 3,\ 2,\ 1)+(-2,\ 5,\ 2,\ -6)\right]\cdot\left[(1,\ 3,\ 2,\ 1)-(-2,\ 5,\ 2,\ -6)\right]$

$$=(-1,\ 8,\ 4,\ -5)\cdot(3,\ -2,\ 0,\ 7)$$

$$=-3-16+0-35$$

$$\boxed{=-54}$$

***Or*** $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v})=\vec{u}\cdot\vec{u}-\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}-\vec{v}\cdot\vec{v}$

$$=\vec{u}\cdot\vec{u}-\vec{v}\cdot\vec{v}$$

$$=(1+9+4+1)-(4+25+4+36)$$

$$=15-69$$

$$\boxed{=-54}$$

***d)*** $\left\|(\vec{w}\cdot\vec{v})\vec{u}\right\|=\left|\vec{w}\cdot\vec{v}\right|\ \left\|\vec{u}\right\|$

$$=\left|(4,\ -1,\ 0,\ -2)\cdot(-2,\ 5,\ 2,\ -6)\right|\sqrt{1+9+4+1}$$

$$=\left|-8-5+12\right|\sqrt{15}$$

$$\boxed{=\sqrt{15}}$$

## ***Exercise***

Let $\vec{u}=\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}$, $\vec{v}=\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$ and $\vec{w}=\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$. Calculate the following:

*a)* $\vec{u}\cdot(\vec{v}+\vec{w})$ *b)* $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v})$ *c)* $(\vec{u}\cdot\vec{w})\vec{v}+(\vec{v}\cdot\vec{w})\vec{u}$ *d)* $(\vec{u}+2\vec{v})\cdot(\vec{u}-\vec{v})$

***Solution***

***a)*** $\vec{u}\cdot(\vec{v}+\vec{w})=(1,\ 0,\ -2,\ 1)\cdot\left[(0,\ 1,\ 1,\ 0)+(1,\ -1,\ -1,\ 1)\right]$

$$=(1,\ 0,\ -2,\ 1)\cdot(1,\ 0,\ 0,\ 1)$$

$$\boxed{=2}$$

***b)*** $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v})=\left[(1,\ 0,\ -2,\ 1)+(0,\ 1,\ 1,\ 0)\right]\cdot\left[(1,\ 0,\ -2,\ 1)-(0,\ 1,\ 1,\ 0)\right]$

$$=(1,\ 1,\ -1,\ 1)\cdot(1,\ -1,\ -3,\ 1)$$

$=1-1+3+1$

$\underline{=4}$

***c)*** $(\vec{u}\bullet\vec{w})\vec{v}+(\vec{v}\bullet\vec{w})\vec{u}=\left[(1,\ 0,\ -2,\ 1)\bullet(1,\ -1,\ -1,\ 1)\right](0,\ 1,\ 1,\ 0)$

$+\left[(0,\ 1,\ 1,\ 0)\bullet(1,\ -1,\ -1,\ 1)\right](1,\ 0,\ -2,\ 1)$

$=4(0,\ 1,\ 1,\ 0)-2(1,\ 0,\ -2,\ 1)$

$\underline{=(-2,\ 4,\ 8,\ 2)}$

***d)*** $(\vec{u}+2\vec{v})\bullet(\vec{u}-\vec{v})=\left[(1,\ 0,\ -2,\ 1)+2(0,\ 1,\ 1,\ 0)\right]\bullet\left[(1,\ 0,\ -2,\ 1)-(0,\ 1,\ 1,\ 0)\right]$

$=(1,\ 2,\ 0,\ 1)\bullet(1,\ -1,\ -3,\ 1)$

$=1-2+1$

$\underline{=0}$

## *Exercise*

Suppose $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$ such that $\vec{u}\bullet\vec{v}=2$, $\vec{u}\bullet\vec{w}=-3$, and $\vec{v}\bullet\vec{w}=5$. If possible, calculate the following values:

*a)* $\vec{u}\bullet(\vec{v}+\vec{w})$

*b)* $(\vec{u}+\vec{v})\bullet\vec{w}$

*c)* $\vec{u}\bullet(2\vec{v}-\vec{w})$

*d)* $\vec{w}\bullet(2\vec{v}-4\vec{u})$

*e)* $(\vec{u}+\vec{v})\bullet(\vec{v}+\vec{w})$

*f)* $\vec{w}\bullet(5\vec{v}+\pi\vec{u})$

*g)* $\vec{w}\bullet((\vec{u}\bullet\vec{w})\vec{u})$

*h)* $\vec{u}\bullet((\vec{u}\bullet\vec{v})\vec{v}+(\vec{u}\bullet\vec{w})\vec{w})$

***Solution***

***a)*** $\vec{u}\bullet(\vec{v}+\vec{w})=\vec{u}\bullet\vec{v}+\vec{u}\bullet\vec{w}$

$=2+(-3)$

$\underline{=-1}$

***b)*** $(\vec{u}+\vec{v})\bullet\vec{w}=\vec{u}\bullet\vec{w}+\vec{v}\bullet\vec{w}$

$=-3+5$

$\underline{=2}$

***c)*** $\vec{u}\bullet(2\vec{v}-\vec{w})=\vec{u}\bullet(2\vec{v})-\vec{u}\bullet\vec{w}$

$=2\vec{u}\bullet\vec{v}-\vec{u}\bullet\vec{w}$

$=2(2)-(-3)$

$\underline{=7}$

***d)*** $\vec{w}\bullet(2\vec{v}-4\vec{u})=\vec{w}\bullet(2\vec{v})-\vec{w}\bullet(4\vec{u})$

$=2\vec{w}\bullet\vec{v}-4\vec{w}\bullet\vec{u}$

$=2(\vec{v}\bullet\vec{w})-4(\vec{u}\bullet\vec{w})$

$=2(5)-4(-3)$

$$=22$$

***e)*** $(\vec{u}+\vec{v})\bullet(\vec{v}+\vec{w}) = \vec{u}\bullet\vec{v}+\vec{u}\bullet\vec{w}+\vec{v}\bullet\vec{v}+\vec{v}\bullet\vec{w}$

$$= 2+(-3)+\vec{v}\bullet\vec{v}+5$$

$$= 4+\vec{v}^2$$

***f)*** $\vec{w}\bullet(5\vec{v}+\pi\vec{u}) = \vec{w}\bullet(5\vec{v})+\vec{w}\bullet(\pi\vec{u})$

$$= 5(\vec{w}\bullet\vec{v})+\pi(\vec{w}\bullet\vec{u})$$

$$= 5(\vec{v}\bullet\vec{w})+\pi(\vec{u}\bullet\vec{w})$$

$$= 5(5)+\pi(-3)$$

$$= 25-3\pi$$

***g)*** $\vec{w}\bullet((\vec{u}\bullet\vec{w})\vec{u}) = \vec{w}\bullet((-3)\vec{u})$

$$= -3(\vec{w}\bullet\vec{u})$$

$$= -3(\vec{u}\bullet\vec{w})$$

$$= -3(-3)$$

$$= 9$$

***h)*** $\vec{u}\bullet((\vec{u}\bullet\vec{v})\vec{v}+(\vec{u}\bullet\vec{w})\vec{w}) = \vec{u}\bullet(2\vec{v}+5\vec{w})$

$$= \vec{u}\bullet(2\vec{v})+\vec{u}\bullet(5\vec{w})$$

$$= 2\vec{u}\bullet\vec{v}+5\vec{u}\bullet\vec{w}$$

$$= 2(2)+5(-3)$$

$$= -11$$

## ***Exercise***

You are in an airplane flying from Chicago to Boston for a job interview. The compass in the cockpit of the plane shows that your plane is pointed due East, and the airspeed indicator on the plane shows that the plane is traveling through the air at 400 *mph*. there is a crosswind that affects your plane however, and the crosswind is blowing due South at 40 *mph*.

Given the crosswind you wonder; relative to the ground, in what direction are you really flying and how fast are you really traveling?

***Solution***

Let the air velocity of the plane be: $\vec{a} = \begin{pmatrix} 400 \\ 0 \end{pmatrix}$

The wind velocity be: $\vec{w} = \begin{pmatrix} 0 \\ -40 \end{pmatrix}$

The ground the velocity is:

$$\vec{g} = \vec{a} + \vec{w}$$
$$= \begin{pmatrix} 400 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -40 \end{pmatrix}$$
$$= \begin{pmatrix} 400 \\ -40 \end{pmatrix}$$


The magnitude: $\sqrt{400^2 + (-40)^2} = 402 \ mph$

The direction: $\theta = \tan^{-1} \frac{-40}{400} \approx 5.71°$

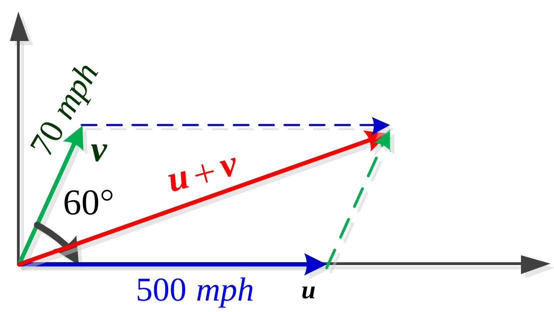
## *Exercise*

A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What speed and direction should the jetliner have in order for the resultant vector to be 500 *mph* due east?

***Solution***

$\boldsymbol{u} = (x,\ y) =$ the velocity of the airplane

$\boldsymbol{v}$ = the velocity of the tailwind




$$\vec{v} = (70\cos 60°,\ 70\sin 60°)$$
$$= \left(35,\ 35\sqrt{3}\right)$$
$$\boldsymbol{u} + \boldsymbol{v} = (500,\ 0)$$
$$(x,\ y) + \left(35,\ 35\sqrt{3}\right) = (500,\ 0)$$
$$(x,\ y) = (500,\ 0) - \left(35,\ 35\sqrt{3}\right)$$
$$= \left(765,\ -35\sqrt{3}\right)$$
$$\boldsymbol{u} = \left(765,\ -35\sqrt{3}\right)$$
$$|\boldsymbol{u}| = \sqrt{465^2 + \left(-35\sqrt{3}\right)^2}$$
$$\approx 468.9\ mph$$
$$\theta = \tan^{-1} \frac{-35\sqrt{3}}{465} \approx -7.4°$$

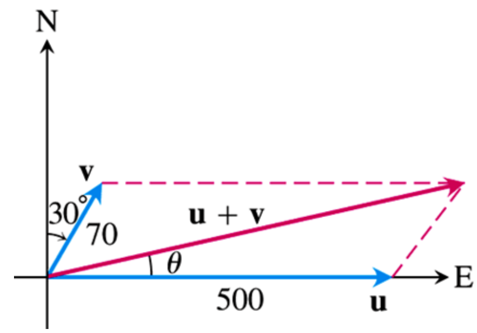
∴ The direction is 7.4° south of east

### *Example*

A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?

***Solution***

$\boldsymbol{u}$ = the velocity of the airplane
$\boldsymbol{v}$ = the velocity of the tailwind




***Given***: $|u| = 500 \quad |v| = 70$

$$\boldsymbol{u} = \langle 500,\ 0\rangle$$

$$\boldsymbol{v} = \langle 70\cos 60°,\ 70\sin 60°\rangle$$

$$= \langle 35,\ 35\sqrt{3}\rangle$$

$$\boldsymbol{u}+\boldsymbol{v} = \langle 535,\ 35\sqrt{3}\rangle = 535\boldsymbol{i} + 35\sqrt{3}\boldsymbol{j}$$

$$|\boldsymbol{u}+\boldsymbol{v}| = \sqrt{535^2 + \left(35\sqrt{3}\right)^2}$$

$$\approx 538.4\ mph$$

$$\theta = \tan^{-1}\frac{35\sqrt{3}}{535} \approx 6.5°$$

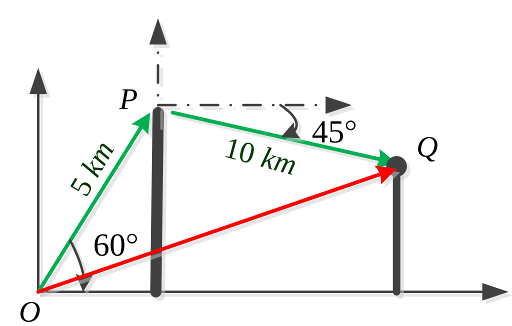
The ground speed of the airplane is about 538.4 *mph* 538.4 *mph*, and its direction is about 6.5° north of east.

### *Exercise*

A bird flies from its nest 5 *km* in the direction 60° north east, where it stops to rest on a tree. It then flies 10 *km* in the direction due southeast and lands atop a telephone pole. Place an *xy*-coordinate system so that the origin is the bird's nest, the *x*-axis points east, and the *y*-axis points north.

*a)* At what point is the tree located?
*b)* At what point is the telephone pole?

***Solution***




**a)** $\overrightarrow{OP} = (5\cos 60°)\boldsymbol{i} + (5\sin 60°)\boldsymbol{j}$

$$= \frac{5}{2}\boldsymbol{i} + \frac{5\sqrt{3}}{2}\boldsymbol{j}$$

The tree is located at the point

$$\boxed{P=\left(\frac{5}{2},\ \frac{5\sqrt{3}}{2}\right)}$$

***b)*** $\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{PQ}$

$$=\frac{5}{2}\boldsymbol{i}+\frac{5\sqrt{3}}{2}\boldsymbol{j}+(10\cos 315°)\boldsymbol{i}+(10\sin 315°)\boldsymbol{j}$$

$$=\frac{5}{2}\boldsymbol{i}+\frac{5\sqrt{3}}{2}\boldsymbol{j}+\left(10\frac{\sqrt{2}}{2}\right)\boldsymbol{i}+\left(10\left(-\frac{\sqrt{2}}{2}\right)\right)\boldsymbol{j}$$

$$=\left(\frac{5}{2}+5\sqrt{2}\right)\boldsymbol{i}+\left(\frac{5\sqrt{3}}{2}-\frac{10\sqrt{2}}{2}\right)\boldsymbol{j}$$

$$=\left(\frac{5+10\sqrt{2}}{2}\right)\boldsymbol{i}+\left(\frac{5\sqrt{3}-10\sqrt{2}}{2}\right)\boldsymbol{j}$$

The pole is located at the point $\boxed{Q=\left(\frac{5+10\sqrt{2}}{2},\ \frac{5\sqrt{3}-10\sqrt{2}}{2}\right)}$

## *Exercise*

Prove $\vec{u}\bullet\vec{u}=\|\vec{u}\|^2\ \geq 0$

***Solution***

Let $\vec{u}=(u_1,\ u_2,\ \dots,\ u_n)$

$$\vec{u}\bullet\vec{u}=(u_1,\ u_2,\ \dots,\ u_n)\bullet(u_1,\ u_2,\ \dots,\ u_n)$$

$$=u_1u_1+u_2u_2+\dots+u_nu_n$$

$$=u_1^2+u_2^2+\dots+u_n^2$$

$$\|\vec{u}\|^2=\left(\sqrt{u_1^2+u_2^2+\dots+u_n^2}\right)^2$$

$$=u_1^2+u_2^2+\dots+u_n^2$$

Thus, $\vec{u}\bullet\vec{u}=\|\vec{u}\|^2$

Each $u_i\in\mathbb{R}$, then $u_i^2\geq 0$ for each $1\leq i\leq n$, thus $u_1^2+u_2^2+\dots+u_n^2\geq 0$.

Hence, $\|\vec{u}\|^2\geq 0$

## Exercise

Prove, for any vectors and $\vec{v}$ in $\mathbb{R}^2$ and any scalars $c$ and $d$,

$\left(c\vec{u}+d\vec{v}\right)\bullet\left(c\vec{u}+d\vec{v}\right)=c^2\|\vec{u}\|^2+2cd\vec{u}\bullet\vec{v}+d^2\|\vec{v}\|^2$

***Solution***

$$\begin{aligned}\left(c\vec{u}+d\vec{v}\right)\bullet\left(c\vec{u}+d\vec{v}\right)&=\left(c\vec{u}+d\vec{v}\right)\bullet c\vec{u}+\left(c\vec{u}+d\vec{v}\right)\bullet d\vec{v}\\&=c\vec{u}\bullet c\vec{u}+d\vec{v}\bullet c\vec{u}+c\vec{u}\bullet d\vec{v}+d\vec{v}\bullet d\vec{v}\\&=c^2\left(\vec{u}\bullet\vec{u}\right)+cd\left(\vec{u}\bullet\vec{v}\right)+cd\left(\vec{u}\bullet\vec{v}\right)+d^2\left(\vec{v}\bullet\vec{v}\right)\\&=c^2\|\vec{u}\|^2+2cd\left(\vec{u}\bullet\vec{v}\right)+d^2\|\vec{v}\|^2 \quad \checkmark\end{aligned}$$

## Exercise

Prove $\vec{u}\bullet\left(\vec{v}+\vec{w}\right)=\vec{u}\bullet\vec{v}+\vec{u}\bullet\vec{w}$

***Solution***

Let $\vec{u}=\left(u_1,\ u_2,\ \ldots,\ u_n\right)$, $\vec{v}=\left(v_1,\ v_2,\ \ldots,\ v_n\right)$, and $\vec{w}=\left(w_1,\ w_2,\ \ldots,\ w_n\right)$

$$\begin{aligned}\vec{u}\bullet\left(\vec{v}+\vec{w}\right)&=\left(u_1,\ u_2,\ \ldots,\ u_n\right)\bullet\left(\left(v_1,\ v_2,\ \ldots,\ v_n\right)+\left(w_1,\ w_2,\ \ldots,\ w_n\right)\right)\\&=\left(u_1,\ u_2,\ \ldots,\ u_n\right)\bullet\left(v_1+w_1,\ v_2+w_2,\ \ldots,\ v_n+w_n\right)\\&=u_1\left(v_1+w_1\right)+u_2\left(v_2+w_2\right)+\cdots\ +u_n\left(v_n+w_n\right)\\&=u_1v_1+u_1w_1+u_2v_2+u_2w_2+\cdots\ +u_nv_n+u_nw_n\\&=\left(u_1v_1+u_2v_2+\cdots++u_nv_n\right)+\left(u_1w_1+u_2w_2+\cdots\ +u_nw_n\right)\\&=\left(u_1,\ u_2,\ \ldots,\ u_n\right)\bullet\left(v_1,\ v_2,\ \ldots,\ v_n\right)+\left(u_1,\ u_2,\ \ldots,\ u_n\right)\bullet\left(w_1,\ w_2,\ \ldots,\ w_n\right)\\&=\vec{u}\bullet\vec{v}+\vec{u}\bullet\vec{w} \quad \checkmark\end{aligned}$$

## Exercise

Prove Minkowski theorem: $\|\vec{u}+\vec{v}\|\le\|\vec{u}\|+\|\vec{v}\|$

***Solution***

$$\begin{aligned}\|\vec{u}+\vec{v}\|^2&=\left(\vec{u}+\vec{v}\right)\bullet\left(\vec{u}+\vec{v}\right)\\&=\vec{u}\bullet\vec{u}+2\vec{u}\bullet\vec{v}+\vec{v}\bullet\vec{v}\\&\le\|\vec{u}\|^2+2\|\vec{u}\|\ \|\vec{v}\|+\|\vec{v}\|^2\end{aligned}$$

$$= \left( \|\vec{u}\| + \|\vec{v}\| \right)^2$$

$$\sqrt{\|\vec{u} + \vec{v}\|^2} \le \sqrt{\left( \|\vec{u}\| + \|\vec{v}\| \right)^2}$$

$$\|\vec{u} + \vec{v}\| \le \|\vec{u}\| + \|\vec{v}\| \qquad \checkmark$$

# Solution Section 2.3 – Orthogonality

### Exercise

Determine whether $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

*a)* $\boldsymbol{u}=(-6,\ -2),\quad \boldsymbol{v}=(5,\ -7)$

*b)* $\boldsymbol{u}=(6,\ 1,\ 4),\quad \boldsymbol{v}=(2,\ 0,\ -3)$

*c)* $\boldsymbol{u}=(1,\ -5,\ 4),\quad \boldsymbol{v}=(3,\ 3,\ 3)$

*d)* $\boldsymbol{u}=(-2,\ 2,\ 3),\quad \boldsymbol{v}=(1,\ 7,\ -4)$

**Solution**

***a)*** $\boldsymbol{u}\cdot\boldsymbol{v}=(-6)(5)+(-2)(-7)$

$=-30+14$

$\underline{=-16\neq 0}$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are not orthogonal

***b)*** $\boldsymbol{u}\cdot\boldsymbol{v}=6(2)+1(0)+4(-3)$

$\underline{=0}$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

***c)*** $\boldsymbol{u}\cdot\boldsymbol{v}=1(3)-5(3)+4(3)$

$\underline{=0}$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

***d)*** $\boldsymbol{u}\cdot\boldsymbol{v}=-2(1)+2(7)+3(-4)$

$\underline{=0}$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

### Exercise

Determine whether the vectors form an orthogonal set

*a)* $\vec{v}_1=(2,\ 3),\quad \vec{v}_2=(3,\ 2)$

*b)* $\vec{v}_1=(1,\ -2),\quad \vec{v}_2=(-2,\ 1)$

*c)* $\vec{u}=(-4,\ 6,\ -10,\ 1)\quad \vec{v}=(2,\ 1,\ -2,\ 9)$

*d)* $\vec{u}=(a,\ b)\quad \vec{v}=(-b,\ a)$

*e)* $\vec{v}_1=(-2,\ 1,\ 1),\quad \vec{v}_2=(1,\ 0,\ 2),\quad \vec{v}_3=(-2,\ -5,\ 1)$

*f)* $\boldsymbol{v}_1=(1,\ 0,\ 1),\quad \boldsymbol{v}_2=(1,\ 1,\ 1),\quad \boldsymbol{v}_3=(-1,\ 0,\ 1)$

*g)* $\boldsymbol{v}_1=(2,-2,1),\quad \boldsymbol{v}_2=(2,1,-2),\quad \boldsymbol{v}_3=(1,2,2)$

***Solution***

***a)*** $\vec{v}_1 \bullet \vec{v}_2 = 2(3)+3(2)\ \underline{=12 \neq 0}|$

$\therefore$ Vectors don't form an orthogonal set

***b)*** $\vec{v}_1 \bullet \vec{v}_2 = 1(-2)-2(1)\ \underline{=-4 \neq 0}|$

$\therefore$ Vectors don't form an orthogonal set

***c)*** $\boldsymbol{u} \bullet \boldsymbol{v} = -8+6+20+9\ \underline{=27 \neq 0}|$; These vectors are not orthogonal

***d)*** $\boldsymbol{u} \bullet \boldsymbol{v} = -ab+ab\ \underline{=0}|$; These vectors are orthogonal

***e)*** $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = -2(1)+1(0)+1(2)\underline{=0}|$

$\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = -2(-2)+1(-5)+1(1)\underline{=0}|$

$\boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 1(-2)+0(-5)+2(1)\underline{=0}|$

$\therefore$ Vectors form an orthogonal set

***f)*** $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 1(1)+0(1)+1(1)\ \underline{=2 \neq 0}|$

$\therefore$ Vectors don't form an orthogonal set

***g)*** $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 2(2)-2(1)+1(-2)\underline{=0}|$

$\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = 2(1)-2(2)+1(2)\underline{=0}|$

$\boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 2(1)+1(2)-2(2)\underline{=0}|$

$\therefore$ Vectors form an orthogonal set

## *Exercise*

Find a unit vector that is orthogonal to both $\boldsymbol{u} = (1, 0, 1)$ and $\boldsymbol{v} = (0, 1, 1)$

***Solution***

Let $\boldsymbol{w} = (w_1, w_2, w_3)$ be the unit vector that is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$.

$\boldsymbol{u} \cdot \boldsymbol{w} = 1(w_1)+0(w_2)+1(w_3) = \underline{w_1 + w_3 = 0}|$

$$\boxed{w_3 = -w_1}$$

$\boldsymbol{v} \cdot \boldsymbol{w} = 0(w_1)+1(w_2)+1(w_3) = \underline{w_2 + w_3 = 0}|$

$$\boxed{w_3 = -w_2}$$

$w_1 = w_2 = -w_3$

The orthogonal vector to both $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{w} = (1,\ 1,\ -1)$, therefore the unit vector is

$$\frac{\mathbf{w}}{\|\mathbf{w}\|}=\frac{1}{\sqrt{1^2+1^2+(-1)^2}}(1,\ 1,\ -1)$$

$$=\frac{1}{\sqrt{3}}(1,\ 1,\ -1)$$

$$=\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)$$

The possible vectors are: $\boxed{\pm\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)}$

## Exercise

*a*) Show that $\mathbf{v}=(a,\ b)$ and $\mathbf{w}=(-b,\ a)$ are orthogonal vectors.

*b*) Use the result to find two vectors that are orthogonal to $\mathbf{v}=(2,\ -3)$.

*c*) Find two unit vectors that are orthogonal to $(-3,\ 4)$

**Solution**

***a)*** $\mathbf{v}\cdot\mathbf{w}=a(-b)+b(a)=-ab+ab=0$ are orthogonal vectors.

***b)*** (2, 3) and (–2, 3).

***c)*** $\vec{u}_1=\frac{1}{\sqrt{4^2+3^2}}(4,\ 3)=\boxed{\left(\frac{4}{5},\ \frac{3}{5}\right)}$

$\vec{u}_2=-\frac{1}{\sqrt{4^2+3^2}}(4,\ 3)=\boxed{\left(-\frac{4}{5},\ -\frac{3}{5}\right)}$

## Exercise

Find the vector component of $\mathbf{u}$ along $\mathbf{a}$ and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$.

*a)* $\mathbf{u}=(6,\ 2),\quad \mathbf{a}=(3,\ -9)$

*b)* $\mathbf{u}=(3,\ 1,\ -7),\quad \mathbf{a}=(1,\ 0,\ 5)$

*c)* $\mathbf{u}=(1,\ 0,\ 0),\quad \mathbf{a}=(4,\ 3,\ 8)$

*d)* $\mathbf{u}=(1,\ 1,\ 1),\quad \mathbf{a}=(0,\ 2,\ -1)$

*e)* $\mathbf{u}=(2,\ 1,\ 1,\ 2),\quad \mathbf{a}=(4,\ -4,\ 2,\ -2)$

*f)* $\mathbf{u}=(5,0,-3,7),\quad \mathbf{a}=(2,1,-1,-1)$

**Solution**

***a)*** $proj_{\mathbf{a}}\mathbf{u}=\frac{\mathbf{u}\bullet\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$

$$= \frac{6(3)+2(-9)}{3^2+(-9)^2}(3,\ -9)$$

$$= \frac{0}{90}(3,\ -9)$$

$$\boxed{= (0,\ 0)}$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = (6,\ 2) - (0,\ 0)$$

$$\boxed{= (6,\ 2)}$$

***b)*** $$proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} = \frac{3(1)+0-7(5)}{1^2+0+5^2}(1,\ 0,\ 5)$$

$$= \frac{-32}{26}(1,\ 0,\ 5)$$

$$\boxed{= \left(-\frac{16}{13},\ 0,\ -\frac{80}{13}\right)}$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = (1, 0, 5) - \left(-\frac{16}{13},\ 0,\ -\frac{80}{13}\right)$$

$$\boxed{= \left(\frac{55}{13},\ 1,\ -\frac{11}{13}\right)}$$

***c)*** $$proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$= \frac{1(4)+0+0}{4^2+3^2+8^2}(4,\ 3,\ 8)$$

$$= \frac{4}{89}(4,\ 3,\ 8)$$

$$\boxed{= \left(\frac{16}{89},\ \frac{12}{89},\ \frac{32}{89}\right)}$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = (1,\ 0,\ 0) - \left(\frac{16}{89},\ \frac{12}{89},\ \frac{32}{89}\right)$$

$$\boxed{= \left(\frac{73}{89},\ -\frac{12}{89},\ -\frac{32}{89}\right)}$$

***d)*** $$proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$= \frac{1(0)+1(2)+1(-1)}{0^2+2^2+(-1)^2}(0,\ 2,\ -1)$$

$$= \frac{1}{5}(0,\ 2,\ -1)$$

$$\boxed{= \left(0,\ \frac{2}{5},\ -\frac{1}{5}\right)}$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = (1,1,1) - \left(0,\ \frac{2}{5},\ -\frac{2}{5}\right)$$

$$= \boxed{\left(1,\ \frac{3}{5},\ \frac{6}{5}\right)}$$

***e)*** $proj_{\boldsymbol{a}}\boldsymbol{u} = \dfrac{\boldsymbol{u} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$

$$= \frac{2(4)+1(-4)+1(2)+2(-2)}{4^2+(-4)^2+2^2+(-2)^2}(4,\ -4,\ 2,\ -2)$$

$$= \frac{2}{40}(4,\ -4,\ 2,\ -2)$$

$$= \boxed{\left(\frac{1}{5},\ -\frac{1}{5},\ \frac{1}{10},\ -\frac{1}{10}\right)}$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = (2,\ 1,\ 1,\ 2) - \left(\frac{1}{5},\ -\frac{1}{5},\ \frac{1}{10},\ -\frac{1}{10}\right)$$

$$= \boxed{\left(\frac{9}{5},\ \frac{6}{5},\ \frac{9}{10},\ \frac{21}{10}\right)}$$

***f)*** $proj_{\boldsymbol{a}}\boldsymbol{u} = \dfrac{\boldsymbol{u} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$

$$= \frac{5(2)+0(1)-3(-1)+7(-1)}{2^2+1^2+(-1)^2+(-1)^2}(2,\ 1,\ -1,\ -1)$$

$$= \frac{6}{7}(2,\ 1,\ -1,\ -1)$$

$$= \boxed{\left(\frac{12}{7},\ \frac{6}{7},\ -\frac{6}{7},\ -\frac{6}{7}\right)}$$

$$\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} = (5,\ 0,\ -3,\ 7) - \left(\frac{12}{7},\ \frac{6}{7},\ -\frac{6}{7},\ -\frac{6}{7}\right)$$

$$= \boxed{\left(\frac{23}{7},\ -\frac{6}{7},\ -\frac{15}{7},\ \frac{55}{7}\right)}$$

## *Exercise*

Project the vector $\boldsymbol{v}$ onto the line through $\boldsymbol{a}$, Check that $\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$:

*a)* $v = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ *and* $a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

*b)* $v = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ *and* $a = \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}$

$$c)\quad v=\begin{pmatrix}1\\1\\1\end{pmatrix}\quad and \quad a=\begin{pmatrix}1\\2\\2\end{pmatrix}$$

***Solution***

***a)***
$$proj_{\boldsymbol{a}}\boldsymbol{v}=\frac{\boldsymbol{v}\bullet\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$
$$=\frac{1(1)+2(1)+2(1)}{1^2+1^2+1^2}(1,\ 1,\ 1)$$
$$=\frac{5}{3}(1,\ 1,\ 1)$$
$$=\left(\frac{5}{3},\ \frac{5}{3},\ \frac{5}{3}\right)$$

$$\boldsymbol{e}=\boldsymbol{v}-proj_{\boldsymbol{a}}\boldsymbol{v}=(1,\ 2,\ 2)-\left(\frac{5}{3},\ \frac{5}{3},\ \frac{5}{3}\right)$$
$$=\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$\boldsymbol{e}\cdot\boldsymbol{a}=\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)\bullet(1,\ 1,\ 1)$$
$$=-\frac{2}{3}+\frac{1}{3}+\frac{1}{3}$$
$$=0$$

$\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$

***b)***
$$proj_{\boldsymbol{a}}\boldsymbol{v}=\frac{\boldsymbol{v}\bullet\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$
$$=\frac{1(-1)+3(-3)+1(-1)}{(-1)^2+(-3)^2+(-1)^2}(-1,\ -3,\ -1)$$
$$=\frac{-11}{11}(-1,\ -3,\ -1)$$
$$=(1,\ 3,\ 1)$$

$$\boldsymbol{e}=\boldsymbol{v}-proj_{\boldsymbol{a}}\boldsymbol{v}=(1,\ 3,\ 1)-(1,\ 3,\ 1)$$
$$=(0,\ 0,\ 0)$$

$$\boldsymbol{e}\cdot\boldsymbol{a}=(0,\ 0,\ 0)\bullet(-1,\ -3,\ -1)$$
$$=0$$

$\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$

$$\textit{c)}\quad proj_{\boldsymbol{a}}\boldsymbol{v} = \frac{\boldsymbol{v} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$= \frac{1(1)+1(2)+1(2)}{(1)^2+(2)^2+(2)^2}(1,\ 2,\ 2)$$

$$= \frac{5}{9}(1,\ 2,\ 2)$$

$$= \left(\frac{5}{9},\ \frac{10}{9},\ \frac{10}{9}\right)$$

$$\boldsymbol{e} = \boldsymbol{v} - proj_{\boldsymbol{a}}\boldsymbol{v} = (1,\ 1,\ 1) - \left(\frac{5}{9},\ \frac{10}{9},\ \frac{10}{9}\right)$$

$$= \left(\frac{4}{9},\ -\frac{1}{9},\ -\frac{1}{9}\right)$$

$$\boldsymbol{e}\cdot\boldsymbol{a} = \left(\frac{4}{9},\ -\frac{1}{9},\ -\frac{1}{9}\right)\bullet(1,\ 2,\ 2)$$

$$= \frac{4}{9} - \frac{2}{9} - \frac{2}{9}$$

$$= 0$$

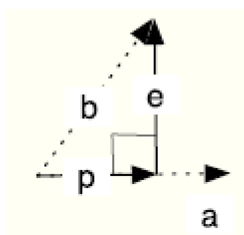
$\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$

## *Exercise*

Draw the projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ and also compute it $\quad b = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad and \quad a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

***Solution***

$$proj_{\boldsymbol{a}}\boldsymbol{b} = \frac{\boldsymbol{b} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$




$$= \frac{\cos\theta(1)+\sin\theta(0)}{(1)^2+0}(1,0)$$

$$= \cos\theta(1,\ 0)$$

$$= (\cos\theta,\ 0)$$

$$\boldsymbol{e} = \boldsymbol{b} - proj_{\boldsymbol{a}}\boldsymbol{b} = (\cos\theta, \sin\theta) - (\cos\theta, 0)$$

$$= (0,\ \sin\theta)$$

## *Exercise*

Draw the projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ and also compute it $\quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad and \quad a = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

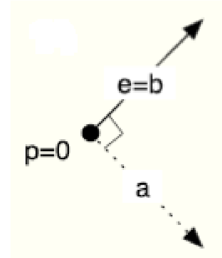
***Solution***

$$proj_{\boldsymbol{a}}\boldsymbol{b} = \frac{\boldsymbol{b} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$= \frac{1(1)+1(-1)}{1^2+(-1)^2}(1, \ -1)$$

$$= \frac{0}{2}(1,-1)$$

$$\boxed{=(0, \ 0)}$$




$$\boldsymbol{e} = \boldsymbol{b} - proj_{\boldsymbol{a}}\boldsymbol{b}$$

$$=(1, \ 1)-(0, \ 0)$$

$$\boxed{=(1, \ 1)}$$

## Exercise

Find the projection matrix $proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}$ onto the line through $a = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

***Solution***

$$a^T a = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 9$$

$$P = \frac{1}{a^T a}a.a^T$$

$$= \frac{1}{9}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix}$$

## Exercise

Show that if $\boldsymbol{v}$ is orthogonal to both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, then $\boldsymbol{v}$ is orthogonal to $k_1\boldsymbol{w}_1 + k_2\boldsymbol{w}_2$ for all scalars $k_1$ and $k_2$.

***Solution***

$$\mathbf{v}\cdot\left(k_1\mathbf{w}_1+k_2\mathbf{w}_2\right)=\mathbf{v}\cdot\left(k_1\mathbf{w}_1\right)+\mathbf{v}\cdot\left(k_2\mathbf{w}_2\right)$$

$$=k_1\left(\mathbf{v}\cdot\mathbf{w}_1\right)+k_2\left(\mathbf{v}\cdot\mathbf{w}_2\right)$$

***If v is orthogonal to*** $\mathbf{w}_1$ & $\mathbf{w}_2$

$\rightarrow \mathbf{v}\cdot\mathbf{w}_1=\mathbf{v}\cdot\mathbf{w}_2=0$

$$=k_1(0)+k_2(0)$$

$$=0$$

### *Exercise*

*a)* Project the vector $\boldsymbol{v}$ = (3, 4, 4) onto the line through $\boldsymbol{a}$ = (2, 2, 1) and then onto the plane that also contains $\boldsymbol{a}^*=(1,\ 0,\ 0)$.

*b)* Check that the first error vector $\boldsymbol{v}-\boldsymbol{p}$ is perpendicular to $\boldsymbol{a}$, and the second error vector $\boldsymbol{v}-\boldsymbol{p}^*$ is also perpendicular to $\boldsymbol{a}^*$.

***Solution***

***a)*** $$proj_{\boldsymbol{a}}\boldsymbol{v}=\frac{\boldsymbol{v}\bullet\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$=\frac{3(2)+4(2)+4(1)}{(2)^2+(2)^2+(1)^2}(2,\ 2,\ 1)$$

$$=\frac{18}{9}(2,\ 2,\ 1)$$

$$=(4,\ 4,\ 2)$$

The plane contains the vectors $\boldsymbol{a}$ and $\boldsymbol{a}^*$ and is the column space of $\boldsymbol{A}$.

$$A=\begin{bmatrix}2&1\\2&0\\1&0\end{bmatrix}$$

$$A^TA=\begin{bmatrix}2&2&1\\1&0&0\end{bmatrix}\begin{bmatrix}2&1\\2&0\\1&0\end{bmatrix}=\begin{bmatrix}9&2\\2&1\end{bmatrix}\qquad \left(A^TA\right)^{-1}=\begin{bmatrix}9&2\\2&1\end{bmatrix}^{-1}=\frac{1}{5}\begin{bmatrix}1&-2\\-2&9\end{bmatrix}$$

$$P=A\left(A^TA\right)^{-1}A^T$$

$$=\frac{1}{5}\begin{bmatrix}2&1\\2&0\\1&0\end{bmatrix}\begin{bmatrix}1&-2\\-2&9\end{bmatrix}\begin{bmatrix}2&2&1\\1&0&0\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\0&.8&.4\\0&.4&.2\end{bmatrix}$$

***b)*** The error vector: $e = v - p = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

$$ae = \begin{pmatrix} 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = 2(-1) + 2(0) + 1(2) = 0.$$

Therefore, ***e*** is perpendicular to ***a***.

$$p^* = Pv = \begin{pmatrix} 1 & 0 & 0 \\ 0 & .8 & .4 \\ 0 & .4 & .2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4.8 \\ 2.4 \end{pmatrix}$$

The error vector: $e^* = v - p^* = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 4.8 \\ 2.4 \end{pmatrix} = \begin{pmatrix} 0 \\ -.8 \\ 1.6 \end{pmatrix}$

$$a^* e^* = \begin{pmatrix} 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & -.8 & 1.6 \end{pmatrix} = 2(0) + 2(-.8) + 1(1.6) = 0.$$

Therefore, ***e**** is perpendicular to ***a****.

### *Exercise*

Compute the projection matrices $\boldsymbol{aa}^T / \boldsymbol{a}^T\boldsymbol{a}$ onto the lines through $\boldsymbol{a}_1 = (-1,\ 2,\ 2)$ and $\boldsymbol{a}_2 = (2,\ 2,\ -1)$. Multiply those projection matrices and explain why their product $P_1P_2$ is what it is. Project $\boldsymbol{v} = (1,\ 0,\ 0)$ onto the lines $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and also onto $\boldsymbol{a}_3 = (2,\ -1,\ 2)$.Add up the three projections $p_1 + p_2 + p_3$.

***Solution***

For $\boldsymbol{a}_1 = (-1,\ 2,\ 2)$

$$a_1 a_1^T = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}\begin{pmatrix} -1 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}$$

$$a_1^T a_1 = \begin{pmatrix} -1 & 2 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

$$\underline{= 9|}$$

$$P_1 = \frac{aa^T}{a^T a}$$

$$= \frac{1}{9}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}$$

For $\boldsymbol{a}_2 = (2,\ 2,\ -1)$

$$a_2 a_2^T = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}\begin{pmatrix} 2 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$a_2^T a_2 = \begin{pmatrix} 2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$\underline{= 9}$$

$$P_2 = \frac{aa^T}{a^T a}$$

$$= \frac{1}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$P_1 P_2 = \frac{1}{9}\left(\frac{1}{9}\right)\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{81}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{= 0}$$

This because $a_1$ *and* $a_2$ are perpendicular.

For $\boldsymbol{a}_3 = (2,\ -1,\ 2)$

$$a_3 a_3^T = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}$$

$$a_3{}^T a_3 = \begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$$\underline{= 9}$$

$$P_3 = \frac{a_3 a_3{}^T}{a_3{}^T a_3}$$

$$= \frac{1}{9} \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}$$

$$p_3 = P_3 v$$

$$= \frac{1}{9} \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ \frac{4}{9} \end{pmatrix}$$

$$p_1 = P_1 v$$

$$= \frac{1}{9} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{9} \\ -\frac{2}{9} \\ -\frac{2}{9} \end{pmatrix}$$

$$p_2 = P_2 v$$

$$= \frac{1}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{2}{9} \end{pmatrix}$$

$$p_1 + p_2 + p_3 = \begin{pmatrix} \frac{1}{9} \\ -\frac{2}{9} \\ -\frac{2}{9} \end{pmatrix} + \begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{2}{9} \end{pmatrix} + \begin{pmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ \frac{4}{9} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{= \boldsymbol{v}}$$

The reason is that $a_3$ is perpendicular to $a_1$ *and* $a_2$.

Hence, when you compute the three projections of a vector and add them up you get back to the vector you start with.

### *Exercise*

If $P^2 = P$ show that $(I - P)^2 = I - P$. When $P$ projects onto the column space of $A$, $I - P$ projects onto the ____.

***Solution***

$$(I - P)^2 v = (I - P)(I - P)v$$

$$=(I-P)(Iv-Pv)$$
$$=I^2v-IPv-PIv+P^2v$$
$$=v-Pv-Pv+P^2v \qquad\qquad P^2v=Pv \quad \textbf{\textit{By definition}}$$
$$=v-Pv-Pv+Pv$$
$$=v-Pv$$

$$(I-P)^2\vec{v}=(I-P)\vec{v} \;\Rightarrow\; \underline{(I-P)^2=(I-P)}$$

When $P$ projects onto the column space of $A$, then $I-P$ projects onto the left nullspace.

Because $(I-P)^2v=(I-P)v$; if $P\boldsymbol{v}$ is in the column space of $A$, then $v-Pv$ is a vector perpendicular to $C(A)$.

## *Exercise*

What linear combination of $(1,\ 2,\ -1)$ and $(1,\ 0,\ 1)$ is closest to $\vec{v}=(2,\ 1,\ 1)$?

***Solution***

$$\tfrac{1}{2}(1,\ 2,\ -1)+\tfrac{3}{2}(1,\ 0,\ 1)=(2,\ 1,\ 1)$$

So, this $\boldsymbol{v}$ is actually in the span of the two given vectors.

## *Exercise*

Show that $\vec{u}-\vec{v}$ is orthogonal to $\vec{u}+\vec{v}$ if and only if $\|\vec{u}\|=\|\vec{v}\|$

***Solution***

Suppose that $\vec{u}-\vec{v}$ is orthogonal to $\vec{u}+\vec{v}$. Then

$$0=\langle\vec{u}-\vec{v},\ \vec{u}+\vec{v}\rangle$$
$$=(\vec{u}-\vec{v})^T(\vec{u}+\vec{v})$$
$$=\left(\vec{u}^T-\vec{v}^T\right)(\vec{u}+\vec{v})$$
$$=\vec{u}^T\vec{u}+\vec{u}^T\vec{v}-\vec{v}^T\vec{u}-\vec{v}^T\vec{v}$$
$$=\langle\vec{u},\ \vec{u}\rangle+\langle\vec{u},\ \vec{v}\rangle-\langle\vec{v},\ \vec{u}\rangle-\langle\vec{v},\ \vec{v}\rangle \qquad\qquad \langle\vec{u},\ \vec{v}\rangle=\langle\vec{v},\ \vec{u}\rangle$$
$$=\langle\vec{u},\ \vec{u}\rangle-\langle\vec{v},\ \vec{v}\rangle$$

So $\langle\vec{u},\ \vec{u}\rangle=\langle\vec{v},\ \vec{v}\rangle$. Therefore, $\|\vec{u}\|^2=\|\vec{v}\|^2 \;\Rightarrow\; \|\vec{u}\|=\|\vec{v}\|$.

Suppose $\|\vec{u}\|=\|\vec{v}\|$. Then

$$\langle\vec{u}-\vec{v},\ \vec{u}+\vec{v}\rangle=(\vec{u}-\vec{v})^T(\vec{u}+\vec{v})$$
$$=\left(\vec{u}^T-\vec{v}^T\right)(\vec{u}+\vec{v})$$

$$= \vec{u}^T\vec{u} + \vec{u}^T\vec{v} - \vec{v}^T\vec{u} - \vec{v}^T\vec{v}$$

$$= \langle \vec{u},\ \vec{u} \rangle + \langle \vec{u},\ \vec{v} \rangle - \langle \vec{v},\ \vec{u} \rangle - \langle \vec{v},\ \vec{v} \rangle \qquad \langle \vec{u},\ \vec{v} \rangle = \langle \vec{v},\ \vec{u} \rangle$$

$$= \langle \vec{u},\ \vec{u} \rangle - \langle \vec{v},\ \vec{v} \rangle$$

$$= \|\vec{u}\|^2 - \|\vec{v}\|^2$$

$$= 0$$

So we can see that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$

We conclude that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$ if and only if $\|\vec{u}\| = \|\vec{v}\|$, as desired.

## *Exercise*

Given $\boldsymbol{u} = (3,\ -1,\ 2)$ $\boldsymbol{v} = (4,\ -1,\ 5)$ *and* $\boldsymbol{w} = (8,\ -7,\ -6)$

*a)* Find $3\boldsymbol{v} - 4(5\boldsymbol{u} - 6\boldsymbol{w})$

*b)* Find $\boldsymbol{u}\bullet\boldsymbol{v}$ and then the angle $\theta$ between $\boldsymbol{u}$ and $\boldsymbol{v}$.

***Solution***

***a)*** $3\boldsymbol{v} - 4(5\boldsymbol{u} - 6\boldsymbol{w}) = 3(4,\ -1,\ 5) - 4(5(3,\ -1,\ 2) - 6(8,\ -7,\ -6))$

$$= (12,\ -3,\ 15) - 4((15,\ -5,\ 10) - (48,\ -42,\ -36))$$

$$= (12,\ -3,\ 15) - 4(-33,\ 37,\ 46)$$

$$= (12,\ -3,\ 15) - (-132,\ 148,\ 184)$$

$$= (144,\ -151,\ -169)$$

***b)*** $\boldsymbol{u}\bullet\boldsymbol{v} = (3,\ -1,\ 2)\bullet(1,\ 1,\ -1)$

$$= 3 - 1 - 2$$

$$= 0$$

$$\theta = 90^\circ$$

## *Exercise*

Given: $\boldsymbol{u} = (3,\ 1,\ 3)$ $\boldsymbol{v} = (4,\ 1,\ -2)$

*a)* Compute the projection $\boldsymbol{w}$ of $\boldsymbol{u}$ on $\boldsymbol{v}$

*b)* Find $\boldsymbol{p} = \boldsymbol{u} - \boldsymbol{w}$ and show that $\boldsymbol{p}$ is perpendicular to $\boldsymbol{v}$.

***Solution***

***a)*** $\boldsymbol{w} = proj_{\boldsymbol{v}}\boldsymbol{u} = \dfrac{\boldsymbol{u}\bullet\boldsymbol{v}}{\|\boldsymbol{v}\|^2}\boldsymbol{v}$

$$= \frac{(3,\ 1,\ 3)\bullet(4,\ 1,\ -2)}{4^2 + 1^2 + (-2)^2}(4,\ 1,\ -2)$$

$$= \frac{12+1-6}{21}(4,\ 1,\ -2)$$

$$= \frac{7}{21}(4,\ 1,\ -2)$$

$$= \frac{1}{3}(4,\ 1,\ -2)$$

$$= \left(\frac{4}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)$$

***b)*** $\boldsymbol{p} = (3,\ 1,\ 3) - \left(\frac{4}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)$

$$= \left(\frac{5}{3},\ \frac{2}{3},\ \frac{11}{3}\right)$$

$$\boldsymbol{p} \bullet \boldsymbol{u} = \left(\frac{5}{3},\ \frac{2}{3},\ \frac{11}{3}\right) \bullet (4,\ 1,\ -2)$$

$$= \frac{20}{3} + \frac{2}{3} - \frac{22}{3}$$

$$= 0$$

***p*** is perpendicular to ***v***.

## *Exercise*

*a*) Show that $\boldsymbol{v} = (a,\ b)$ and $\boldsymbol{w} = (-b,\ a)$ are orthogonal vectors

*b*) Use the result in part (*a*) to find two vectors that are orthogonal to $\boldsymbol{v} = (2,\ -3)$

*c*) Find two unit vectors that are orthogonal to $(-3,\ 4)$

***Solution***

***a)*** $\boldsymbol{u} \cdot \boldsymbol{v} = -ab + ba = 0$; 2 vectors are orthogonal vectors.

***b)*** $\boldsymbol{v} = (2,\ -3) \Rightarrow \boldsymbol{w} = (-3,\ -2)$ and $\boldsymbol{w} = (3,\ 2)$

***c)*** $(-3,\ 4) \Rightarrow u = \frac{(-3,\ 4)}{\sqrt{9+16}} = \left(-\frac{3}{5},\ \frac{4}{5}\right)$

$$\boldsymbol{u}_1 = \left(\frac{4}{5},\ \frac{3}{5}\right) \quad \textit{and} \quad \boldsymbol{u}_2 = \left(-\frac{4}{5},\ -\frac{3}{5}\right)$$

## *Exercise*

Show that $A(3,\ 0,\ 2)$, .$B(4,\ 3,\ 0)$, and $C(8,\ 1,\ -1)$ are vertices of a right triangle. At which vertex is the right angle?

***Solution***

$$AB = (4-3,\ 3-0,\ 0-2) = (1,\ 3,\ -2)$$

$$AC = (5,\ 1,\ -3)$$

$BC = (4,\ -2,\ -1)$

$AB \bullet AC = 5 + 3 + 6 = 14$
$AB \bullet BC = 4 - 6 + 2 = 0$
$AC \bullet BC = 20 - 2 + 3 = 21$

The right triangle at point $B$

## *Exercise*

Establish the identity: $\boldsymbol{u}\bullet\boldsymbol{v} = \frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2$

***Solution***

Let $\boldsymbol{u}(u_1,\ u_2,\ \ldots,\ u_n)$ *and* $\boldsymbol{v} = (v_1,\ v_2,\ \ldots,\ v_n)$

$$\boxed{\boldsymbol{u}\bullet\boldsymbol{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n}$$

$$\boldsymbol{u}+\boldsymbol{v} = (u_1+v_1,\ u_2+v_2,\ \ldots, u_n+v_n)$$

$$\begin{aligned}\|\boldsymbol{u}+\boldsymbol{v}\|^2 &= (u_1+v_1)^2 + (u_2+v_2)^2 + \cdots + (u_n+v_n)^2 \\ &= u_1^2 + v_1^2 + 2u_1v_1 + u_2^2 + v_2^2 + 2u_2v_2 + \cdots + u_2^2 + v_n^2 + 2u_nv_n\end{aligned}$$

$$\boldsymbol{u}-\boldsymbol{v} = (u_1-v_1,\ u_2-v_2,\ \ldots, u_n-v_n)$$

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{v}\|^2 &= (u_1-v_1)^2 + (u_2-v_2)^2 + \cdots + (u_n-v_n)^2 \\ &= u_1^2 + v_1^2 - 2u_1v_1 + u_2^2 + v_2^2 - 2u_2v_2 + \cdots + u_2^2 + v_n^2 - 2u_nv_n\end{aligned}$$

$$\begin{aligned}\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2 &= u_1^2 + v_1^2 + 2u_1v_1 + u_2^2 + v_2^2 + 2u_2v_2 + \cdots + u_2^2 + v_n^2 + 2u_nv_n \\ &\quad - \left(u_1^2 + v_1^2 - 2u_1v_1 + u_2^2 + v_2^2 - 2u_2v_2 + \cdots + u_2^2 + v_n^2 - 2u_nv_n\right) \\ &= u_1^2 + v_1^2 + 2u_1v_1 + u_2^2 + v_2^2 + 2u_2v_2 + \cdots + u_2^2 + v_n^2 + 2u_nv_n \\ &\quad - u_1^2 - v_1^2 + 2u_1v_1 - u_2^2 - v_2^2 + 2u_2v_2 - \cdots - u_2^2 - v_n^2 + 2u_nv_n \\ &= 4u_1v_1 + 4u_2v_2 + \cdots + 4u_nv_n\end{aligned}$$

$$\frac{1}{4}\left(\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2\right) = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Therefore; $\boldsymbol{u}\bullet\boldsymbol{v} = \frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2$ is true.

**2nd *method*:**

$$\frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2 = \frac{1}{4}\left[(\boldsymbol{u}+\boldsymbol{v})(\boldsymbol{u}+\boldsymbol{v}) - (\boldsymbol{u}-\boldsymbol{v})(\boldsymbol{u}-\boldsymbol{v})\right]$$

$$=\frac{1}{4}\left[\boldsymbol{uu}+2\boldsymbol{uv}+\boldsymbol{vv}-\left(\boldsymbol{uu}-2\boldsymbol{uv}+\boldsymbol{vv}\right)\right]$$

$$=\frac{1}{4}\left[\boldsymbol{uu}+2\boldsymbol{uv}+\boldsymbol{vv}-\boldsymbol{uu}+2\boldsymbol{uv}-\boldsymbol{vv}\right]$$

$$=\frac{1}{4}\left(4\boldsymbol{uv}\right)$$

$$=\boldsymbol{u}\bullet\boldsymbol{v}$$

## Exercise

Find the Euclidean inner product $\boldsymbol{u}\bullet\boldsymbol{v}$: $\boldsymbol{u}=\left(-1,\ 1,\ 0,\ 4,\ -3\right)$ $\boldsymbol{v}=\left(-2,\ -2,\ 0,\ 2,\ -1\right)$

**Solution**

$$\boldsymbol{u}\bullet\boldsymbol{v}=2-2+0+8+3\ \underline{=11|}$$

## Exercise

Find the Euclidean distance between $\boldsymbol{u}$ and $\boldsymbol{v}$: $\boldsymbol{u}=\left(3,\ -3,\ -2,\ 0,\ -3\right)$ $\boldsymbol{v}=\left(-4,\ 1,\ -1,\ 5,\ 0\right)$

**Solution**

$$d\left(\boldsymbol{u},\ \boldsymbol{v}\right)=\left\|\boldsymbol{u}-\boldsymbol{v}\right\|\ =\sqrt{\left(u_1-v_1\right)^2+\left(u_2-v_2\right)^2+\ldots+\left(u_n-v_n\right)^2}$$

$$=\sqrt{(3+4)^2+(-3-1)^2+(-2+1)^2+(0-5)^2+(-3-0)^2}$$

$$=\sqrt{49+16+1+25+9}$$

$$=\sqrt{100}$$

$$\underline{=10|}$$

## Exercise

Find for $\vec{v}=2\hat{\boldsymbol{i}}-4\hat{\boldsymbol{j}}+\sqrt{5}\hat{\boldsymbol{k}},\quad \vec{u}=-2\hat{\boldsymbol{i}}+4\hat{\boldsymbol{j}}-\sqrt{5}\hat{\boldsymbol{k}}$

*a)* $\boldsymbol{v}\bullet\boldsymbol{u},\ |\boldsymbol{v}|,\ |\boldsymbol{u}|$

*b)* The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

*c)* The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

*d)* The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

**Solution**

***a)*** $\boldsymbol{v}\bullet\boldsymbol{u}=\left(2\boldsymbol{i}-4\boldsymbol{j}+\sqrt{5}\boldsymbol{k}\right)\cdot\left(-2\boldsymbol{i}+4\boldsymbol{j}-\sqrt{5}\boldsymbol{k}\right)$

$$=-4-16-5$$

$$\underline{=-25|}$$

$$\begin{aligned}|\boldsymbol{v}| &= \sqrt{2^2+(-4)^2+\left(\sqrt{5}\right)^2}\\ &= \sqrt{4+16+5}\\ &= \sqrt{25}\\ &\underline{=5|}\end{aligned}$$

$$\begin{aligned}|\boldsymbol{u}| &= \sqrt{(-2)^2+4^2+\left(-\sqrt{5}\right)^2}\\ &= \sqrt{25}\\ &\underline{=5|}\end{aligned}$$

***b)*** $\cos\theta = \dfrac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$

$$= \frac{-25}{(5)(5)}$$

$$\underline{=-1|}$$

***c)*** $|\boldsymbol{u}|\cos\theta = (5)(-1)$

$$\underline{=-5|}$$

***d)*** $proj_{\boldsymbol{v}}\boldsymbol{u} = \left(\dfrac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$

$$= \left(\frac{-25}{5^2}\right)\left(2\boldsymbol{i}-4\boldsymbol{j}+\sqrt{5}\boldsymbol{k}\right)$$

$$= -\left(2\boldsymbol{i}-4\boldsymbol{j}+\sqrt{5}\boldsymbol{k}\right)$$

$$\underline{=-2\boldsymbol{i}+4\boldsymbol{j}-\sqrt{5}\boldsymbol{k}|}$$

## *Exercise*

Find for $\boldsymbol{v} = \frac{3}{5}\boldsymbol{i}+\frac{4}{5}\boldsymbol{k},\quad \boldsymbol{u} = 5\boldsymbol{i}+12\boldsymbol{j}$

*a)* $\boldsymbol{v}\bullet\boldsymbol{u},\ |\boldsymbol{v}|,\ |\boldsymbol{u}|$

*b)* The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

*c)* The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

*d)* The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

***Solution***

***a)*** $\boldsymbol{v}\bullet\boldsymbol{u} = \left(\frac{3}{5}\boldsymbol{i}+\frac{4}{5}\boldsymbol{k}\right)\cdot(5\boldsymbol{i}+12\boldsymbol{j})$

$$\underline{=3|}$$

$$|\boldsymbol{v}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$
$$= \sqrt{\frac{9}{25} + \frac{16}{25}}$$
$$= \sqrt{\frac{25}{25}}$$
$$\underline{= 1}$$

$$|\boldsymbol{u}| = \sqrt{5^2 + 12^2}$$
$$\underline{= 13}$$

***b)*** $\cos\theta = \dfrac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$

$$= \frac{3}{(1)(13)}$$
$$\underline{= \frac{3}{13}}$$

***c)*** $|\boldsymbol{u}|\cos\theta = (13)\left(\frac{3}{13}\right) = \underline{3}$

***d)*** $proj_{\boldsymbol{v}}\boldsymbol{u} = \left(\dfrac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$

$$= \left(\frac{3}{1^2}\right)\left(\frac{3}{5}\boldsymbol{i} + \frac{4}{5}\boldsymbol{k}\right)$$
$$\underline{= \frac{9}{5}\hat{i} + \frac{12}{5}\hat{k}}$$

## *Exercise*

Find for $\boldsymbol{v} = 2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k}, \quad \boldsymbol{u} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

*a)* $\boldsymbol{v}\bullet\boldsymbol{u}, \ |\boldsymbol{v}|, \ |\boldsymbol{u}|$

*b)* The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

*c)* The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

*d)* The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

***Solution***

***a)*** $\boldsymbol{v}\bullet\boldsymbol{u} = \left(2\hat{i} + 10\hat{j} - 11\hat{k}\right)\cdot\left(2\hat{i} + 2\hat{j} + \hat{k}\right)$

$$= 4 + 20 - 11$$
$$\underline{= 13}$$

$$|\boldsymbol{v}| = \sqrt{2^2 + 10^2 + (-11)^2}$$
$$= \sqrt{4 + 100 + 121}$$
$$= \sqrt{225}$$
$$\boxed{= 15}$$

$$|\boldsymbol{u}| = \sqrt{2^2 + 2^2 + 1^2}$$
$$\boxed{= 3}$$

***b)*** $\cos\theta = \dfrac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$

$$= \frac{13}{(3)(15)}$$
$$= \boxed{\frac{13}{45}}$$

***c)*** $|\boldsymbol{u}|\cos\theta = (3)\left(\dfrac{13}{45}\right) = \boxed{\dfrac{13}{15}}$

***d)*** $proj_{\boldsymbol{v}}\boldsymbol{u} = \left(\dfrac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$

$$= \left(\frac{13}{15^2}\right)(2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k})$$
$$\boxed{= \frac{13}{225}(2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k})}$$

## ***Exercise***

Find for $\boldsymbol{v} = 5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}, \quad \boldsymbol{u} = 2\hat{\boldsymbol{i}} + \sqrt{17}\hat{\boldsymbol{j}}$

*a)* $\boldsymbol{v}\bullet\boldsymbol{u},\ |\boldsymbol{v}|,\ |\boldsymbol{u}|$

*b)* The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

*c)* The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

*d)* The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

***Solution***

***a)*** $\boldsymbol{v}\bullet\boldsymbol{u} = (5\hat{i} + \hat{j})\cdot(2\hat{i} + \sqrt{17}\hat{j})$

$$\boldsymbol{v}\bullet\boldsymbol{u} = (5\boldsymbol{i} + \boldsymbol{j})\cdot(2\boldsymbol{i} + \sqrt{17}\boldsymbol{j}) \ \boxed{= 10 + \sqrt{17}}$$

$$|\boldsymbol{v}| = \sqrt{25 + 1} \ \boxed{= \sqrt{26}}$$

$$|\boldsymbol{u}| = \sqrt{4 + 17} \ \boxed{= \sqrt{21}}$$

***b)*** $\cos\theta = \dfrac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$

$$= \frac{10+\sqrt{17}}{\sqrt{21}\sqrt{26}}$$

$$\boxed{= \frac{10+\sqrt{17}}{\sqrt{546}}}$$

***c)*** $|\boldsymbol{u}|\cos\theta = \left(\sqrt{21}\right)\left(\dfrac{10+\sqrt{17}}{\sqrt{546}}\right)$

$$\boxed{= \frac{10+\sqrt{17}}{\sqrt{26}}}$$

***d)*** $proj_{\boldsymbol{v}}\,\boldsymbol{u} = \left(\dfrac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$

$$\boxed{= \left(\frac{10+\sqrt{17}}{26}\right)(5\boldsymbol{i}+\boldsymbol{j})}$$

## *Exercise*

Find for $\vec{v} = \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}}\right)$, $\vec{u} = \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{3}}\right)$

*a)* $\boldsymbol{v}\bullet\boldsymbol{u},\ |\boldsymbol{v}|,\ |\boldsymbol{u}|$

*b)* The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

*c)* The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

*d)* The vector $proj_{\boldsymbol{v}}\,\boldsymbol{u}$

***Solution***

***a)*** $\boldsymbol{v}\bullet\boldsymbol{u} = \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}}\right)\bullet\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{3}}\right) = \frac{1}{2} - \frac{1}{3}\ \boxed{= \frac{1}{6}}$

$$|\boldsymbol{v}| = \sqrt{\frac{1}{2}+\frac{1}{3}} = \frac{\sqrt{5}}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}}\ \boxed{= \frac{\sqrt{30}}{6}}$$

$$|\boldsymbol{u}| = \sqrt{\frac{1}{2}+\frac{1}{3}} = \frac{\sqrt{5}}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}}\ \boxed{= \frac{\sqrt{30}}{6}}$$

***b)*** $\cos\theta = \dfrac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$

$$=\frac{\frac{1}{6}}{\frac{\sqrt{30}}{6}\frac{\sqrt{30}}{6}}$$

$$=\frac{1}{6}\left(\frac{36}{30}\right)$$

$$\boxed{=\frac{1}{5}}$$

***c)*** $|\boldsymbol{u}|\cos\theta=\left(\frac{\sqrt{30}}{6}\right)\left(\frac{1}{5}\right)$

$$=\frac{\sqrt{30}}{30}$$

$$\boxed{=\frac{1}{\sqrt{30}}}$$

***d)*** $proj_{\boldsymbol{v}}\boldsymbol{u}=\left(\frac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$

$$=\frac{1}{6}\left(\frac{36}{30}\right)\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}}\right)$$

$$\boxed{=\frac{1}{5}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}}\right)}$$

## *Exercise*

Suppose Ted weighs 180 *lb*. and he is sitting on an inclined plane that drops 3 *units* for every 4 horizontal units. The gravitational force vector is $\vec{F}_g=\begin{pmatrix}0\\-180\end{pmatrix}$.

*a)* Find the force pushing Ted down the slope.
*b)* Find the force acting to hold Ted against the slope

***Solution***

A vector parallel to the slope of the inclined plane is $\vec{v}=\begin{pmatrix}4\\-3\end{pmatrix}$.

***a)*** The vector of the force acting to push Ted down the slope is

$$\vec{F}_s=\frac{\vec{v}\bullet\vec{F}_g}{|\vec{v}|^2}\vec{v}$$

$$=\frac{(4,\ -3)\bullet(0,\ -180)}{16+9}(4,\ -3)$$

$$=\frac{540}{25}(4,\ -3)$$

$$= \boxed{\left( \frac{432}{5},\ -\frac{324}{5} \right)}$$

The magnitude of the force pushing Ted down the slope is

$$\left\| \vec{F}_s \right\| = \sqrt{\left(\frac{432}{5}\right)^2 + \left(\frac{324}{5}\right)^2}$$

$$= \frac{540}{5}$$

$$= \boxed{108\ \ lb}$$

***b)*** The vector of the force acting to hold Ted against the slope is

$$\vec{F}_p = \vec{F}_g - \vec{F}_s$$

$$= \begin{pmatrix} 0 \\ -180 \end{pmatrix} - \begin{pmatrix} \frac{432}{5} \\ -\frac{324}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{432}{5} \\ -\frac{576}{5} \end{pmatrix}$$

$$\left\| \vec{F}_p \right\| = \sqrt{\left(\frac{432}{5}\right)^2 + \left(\frac{576}{5}\right)^2}$$

$$= \frac{720}{5}$$

$$= \boxed{144\ \ lb}$$

## *Exercise*

Prove that is two vectors $\vec{u}$ and $\vec{v}$ in $R^2$ are orthogonal to nonzero vector $\vec{w}$ in $R^2$, then $\vec{u}$ and $\vec{v}$ are scalar multiples of each other.

***Solution***

Since $\vec{u}$ is orthogonal to $\vec{w} \rightarrow \vec{u} \bullet \vec{w} = 0$

$\vec{v}$ is orthogonal to $\vec{w} \rightarrow \vec{v} \bullet \vec{w} = 0$

$\Rightarrow \vec{u} \bullet \vec{w} = \vec{v} \bullet \vec{w} = 0$

There exist $a \in \mathbb{R}$ such that $(a\vec{v}) \bullet \vec{w} = a(\vec{v} \bullet \vec{w}) = 0$

$\vec{u} = a\vec{v} \quad \vec{u} \bullet \vec{w} = \vec{v} \bullet \vec{w} = 0 = (a\vec{v}) \bullet \vec{w}$

Therefore, $\vec{u}$ and $\vec{v}$ are scalar multiples of each other

# Solution   *Section* 2.4 – Cross Product

### *Exercise*

Prove when the cross product $\boldsymbol{u}\times\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$, then $\boldsymbol{u}.(\boldsymbol{u}\times\boldsymbol{v})=0$

***Solution***

Let $\boldsymbol{u}=(u_1,\ u_2,\ u_3)$ and $\boldsymbol{v}=(v_1,\ v_2,\ v_3)$

$$\begin{aligned}\boldsymbol{u}.(\boldsymbol{u}\times\boldsymbol{v}) &= (u_1,\ u_2,\ u_3).(u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1)\\ &= u_1(u_2v_3-u_3v_2)+u_2(u_3v_1-u_1v_3)+u_3(u_1v_2-u_2v_1)\\ &= u_1u_2v_3-u_1u_3v_2+u_2u_3v_1-u_2u_1v_3+u_3u_1v_2-u_3u_2v_1\\ &= 0\end{aligned}$$

### *Exercise*

Find $\boldsymbol{u}\times\boldsymbol{v}$, where $\boldsymbol{u}=(1,\ 2,\ -2)$ *and* $\boldsymbol{v}=(3,\ 0,\ 1)$ and show that $\boldsymbol{u}\times\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ and to $\boldsymbol{v}$.

***Solution***

$$\begin{aligned}u\times v &= \left(\begin{vmatrix}2 & -2\\ 0 & 1\end{vmatrix},\ -\begin{vmatrix}1 & -2\\ 3 & 1\end{vmatrix},\ \begin{vmatrix}1 & 2\\ 3 & 0\end{vmatrix}\right)\\ &= (2,\ -7,\ -6)\end{aligned}$$

$$\begin{aligned}u\bullet(u\times v) &= (1,\ 2,\ -2)\bullet(2,\ -7,\ -6)\\ &= 2-14+12\\ &= 0\end{aligned}$$

$$\begin{aligned}v\bullet(u\times v) &= (3,\ 0,\ 1)\bullet(2,\ -7,\ -6)\\ &= 6-0-6\\ &= 0\end{aligned}$$

$\boldsymbol{u}\times\boldsymbol{v}$ is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$.

### *Exercise*

Given $\boldsymbol{u}=(3,\ 2,\ -1),\quad \boldsymbol{v}=(0,\ 2,\ -3),\ \textit{and}\quad \boldsymbol{w}=(2,\ 6,\ 7)$ Compute the vectors

*a)* $\boldsymbol{u} \times \boldsymbol{v}$

*b)* $\boldsymbol{v} \times \boldsymbol{w}$

*c)* $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w})$

*d)* $(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w}$

*e)* $\boldsymbol{u} \times (\boldsymbol{v} - 2\,\boldsymbol{w})$

***Solution***

**a)** $u \times v = \left( \begin{vmatrix} 2 & -1 \\ 2 & -3 \end{vmatrix},\ -\begin{vmatrix} 3 & -1 \\ 0 & -3 \end{vmatrix},\ \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} \right)$

$= (-4,\ 9,\ 6)$

**b)** $v \times w = \left( \begin{vmatrix} 2 & -3 \\ 6 & 7 \end{vmatrix},\ -\begin{vmatrix} 0 & -3 \\ 2 & 7 \end{vmatrix},\ \begin{vmatrix} 0 & 2 \\ 2 & 6 \end{vmatrix} \right)$

$= (32,\ -6,\ -4)$

**c)** $u \times (v \times w) = (3,\ 2,\ -1) \times (32,\ -6,\ -4)$

$= \left( \begin{vmatrix} 2 & -1 \\ -6 & -4 \end{vmatrix},\ -\begin{vmatrix} 3 & -1 \\ 32 & -4 \end{vmatrix},\ \begin{vmatrix} 3 & 2 \\ 32 & -6 \end{vmatrix} \right)$

$= (-14,\ -20,\ -82)$

**d)** $(u \times v) \times w = (-4,\ 9,\ 6) \times (2,\ 6,\ 7)$

$= \left( \begin{vmatrix} 9 & 6 \\ 6 & 7 \end{vmatrix},\ -\begin{vmatrix} -4 & 6 \\ 2 & 7 \end{vmatrix},\ \begin{vmatrix} -4 & 9 \\ 2 & 6 \end{vmatrix} \right)$

$= (27,\ 40,\ -42)$

**e)** $u \times (v - 2w) = (3,\ 2,\ -1) \times \left[ (0,\ 2,\ -3) - 2(2,\ 6,\ 7) \right]$

$= (3,\ 2,\ -1) \times (-4,\ -10,\ -17)$

$= \left( \begin{vmatrix} 2 & -1 \\ -10 & -17 \end{vmatrix},\ -\begin{vmatrix} 3 & -1 \\ -4 & -17 \end{vmatrix},\ \begin{vmatrix} 3 & 2 \\ -4 & -10 \end{vmatrix} \right)$

$= (-44,\ 47,\ -22)$

## Exercise

Use the cross product to find a vector that is orthogonal to both

*a)* $\boldsymbol{u}=(-6,\ 4,\ 2), \quad \boldsymbol{v}=(3,\ 1,\ 5)$

*b)* $\boldsymbol{u}=(1,\ 1,\ -2), \quad \boldsymbol{v}=(2,\ -1,\ 2)$

*c)* $\boldsymbol{u}=(-2,\ 1,\ 5), \quad \boldsymbol{v}=(3,\ 0,\ -3)$

## Solution

**a)** $\boldsymbol{u}\times\boldsymbol{v}=(-6,\ 4,\ 2)\times(3,\ 1,\ 5)$

$$=\left(\begin{vmatrix}4 & 2\\1 & 5\end{vmatrix},\ -\begin{vmatrix}-6 & 2\\3 & 5\end{vmatrix},\ \begin{vmatrix}-6 & 4\\3 & 1\end{vmatrix}\right)$$

$$\underline{=(18,\ 36,\ -18)}$$

**b)** $\boldsymbol{u}\times\boldsymbol{v}=(1,\ 1,\ -2)\times(2,\ -1,\ 2)$

$$=\left(\begin{vmatrix}1 & -2\\-1 & 2\end{vmatrix},\ -\begin{vmatrix}1 & -2\\2 & 2\end{vmatrix},\ \begin{vmatrix}1 & 1\\2 & -1\end{vmatrix}\right)$$

$$\underline{=(0,\ -6,\ -3)}$$

**c)** $\boldsymbol{u}\times\boldsymbol{v}=(-2,\ 1,\ 5)\times(3,\ 0,\ -3)$

$$=\left(\begin{vmatrix}1 & 5\\0 & -3\end{vmatrix},\ -\begin{vmatrix}-2 & 5\\3 & -3\end{vmatrix},\ \begin{vmatrix}-2 & 1\\3 & 0\end{vmatrix}\right)$$

$$\underline{=(-3,\ 9,\ -3)}$$

## Exercise

Find the area of the parallelogram determined by the given vectors

*a)* $\boldsymbol{u}=(1,\ -1,\ 2)$ *and* $\boldsymbol{v}=(0,\ 3,\ 1)$

*b)* $\boldsymbol{u}=(3,\ -1,\ 4)$ *and* $\boldsymbol{v}=(6,\ -2,\ 8)$

**c)** $\boldsymbol{u}=(2,\ 3,\ 0)$ *and* $\boldsymbol{v}=(-1,\ 2,\ -2)$

## Solution

**a)** $Area=\|u\times v\|$

$$=\left\|\left(\begin{vmatrix}-1 & 2\\3 & 1\end{vmatrix},\ -\begin{vmatrix}1 & 2\\0 & 1\end{vmatrix},\ \begin{vmatrix}1 & -1\\0 & 3\end{vmatrix}\right)\right\|$$

$$=\|(-7,\ -1,\ 3)\|$$

$$=\sqrt{7^2+1^2+3^2}$$

$$= \boxed{\sqrt{59}} \quad (Area)$$

**b)** $Area = \|u \times v\|$

$$= \left\| \left( \begin{vmatrix} -1 & 4 \\ -2 & 8 \end{vmatrix}, \ -\begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix}, \ \begin{vmatrix} 3 & -1 \\ 6 & -2 \end{vmatrix} \right) \right\|$$

$$= \|(0, \ 0, \ 0)\|$$

$$= \boxed{0}$$

**c)** $Area = \|u \times v\| = (2, \ 3, \ 0) \times (-1, \ 2, \ -2)$

$$= \left\| \left( \begin{vmatrix} 3 & 0 \\ 2 & -2 \end{vmatrix}, \ -\begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix}, \ \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} \right) \right\|$$

$$= \|(-6, \ 4, \ 7)\|$$

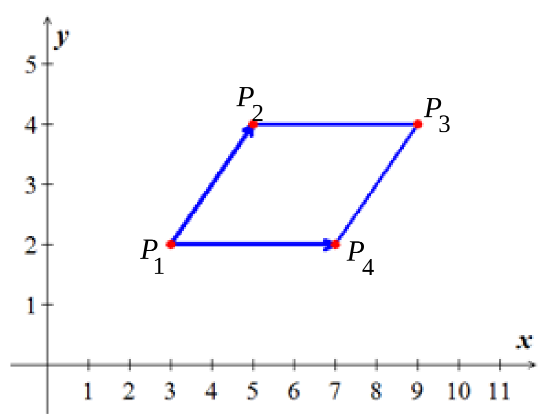
$$= \sqrt{(-6)^2 + 4^2 + 7^2}$$

$$= \boxed{\sqrt{101}} \quad (Area)$$

## *Exercise*

Find the area of the parallelogram with the given vertices $P_1(3,2), \quad P_2(5,4), \quad P_3(9,4), \quad P_4(7,2)$

***Solution***




$$\overrightarrow{P_1P_2} = (5-3, 4-2) = (2, \ 2)$$

$$\overrightarrow{P_4P_3} = (9-7, 4-2) = (2, \ 2)$$

$$\overrightarrow{P_1P_4} = (7-3, 2-2) = (4, \ 0)$$

$$\overrightarrow{P_2P_3} = (9-5, 4-4) = (4, \ 0)$$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_2} = (2,2) \times (4,0)$$

$$= \left( \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix}, \ -\begin{vmatrix} 2 & 0 \\ 4 & 0 \end{vmatrix}, \ \begin{vmatrix} 2 & 2 \\ 4 & 0 \end{vmatrix} \right)$$

$$= (0, \ 0, \ -8)$$

The area of the parallelogram is

$$\left\| \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_2} \right\| = \sqrt{0 + 0 + (-8)^2} = \boxed{8}$$

### *Exercise*

Find the area of the triangle with the given vertices:

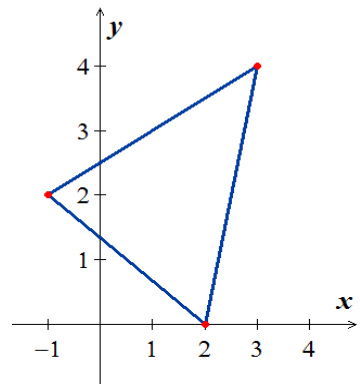
*a)* $A(2,\ 0)\quad B(3,\ 4)\quad C(-1,\ 2)$

*b)* $A(1,\ 1)\quad B(2,\ 2)\quad C(3,\ -3)$

*c)* $P(2,\ 6,\ -1)\quad Q(1,\ 1,\ 1)\quad R=(4,\ 6,\ 2)$

### Solution

**a)** $\overrightarrow{AB}=(1,\ 4)\qquad \overrightarrow{AC}=(-3,\ 2)$




$$\overrightarrow{AB}\times\overrightarrow{AC}=(1,\ 4,\ 0)\times(-3,\ 2,\ 0)$$

$$=\left(\begin{vmatrix}4 & 0\\ 2 & 0\end{vmatrix},\ -\begin{vmatrix}1 & 0\\ -3 & 0\end{vmatrix},\ \begin{vmatrix}1 & 4\\ -3 & 2\end{vmatrix}\right)$$

$$\underline{=(0,\ 0,\ 14)}|$$

$$\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\sqrt{0+0+14^2}$$

$$\underline{=14}|$$

The area of the triangle is

$$\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\frac{1}{2}14$$

$$\underline{=7}|$$

**b)** $\overrightarrow{AB}=(1,\ 1)\qquad \overrightarrow{AC}=(2,\ -4)$

$$\overrightarrow{AB}\times\overrightarrow{AC}=(1,\ 1,\ 0)\times(2,\ -4,\ 0)$$

$$=\left(\begin{vmatrix}1 & 0\\ -4 & 0\end{vmatrix},\ -\begin{vmatrix}1 & 0\\ 2 & 0\end{vmatrix},\ \begin{vmatrix}1 & 1\\ 2 & -4\end{vmatrix}\right)$$

$$\underline{=(0,\ 0,\ -6)}|$$

$$\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\sqrt{0+0+(-6)^2}$$

$$\underline{=6}|$$

The area of the triangle is

$$\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\frac{1}{2}(6)$$

$$\underline{=3}|$$

**c)** $\overrightarrow{PQ}=(-1,\ -5,\ 2)\qquad \overrightarrow{PR}=(2,\ 0,\ 3)$

$$\overrightarrow{PQ}\times\overrightarrow{PR}=(-1,\ -5,\ 2)\times(2,\ 0,\ 3)$$

$$\begin{array}{ccccc}-1 & -5 & 2 & -1 & -5\\ 2 & 0 & 3 & 2 & 0\end{array}$$

$$= (-15,\ 7,\ 10)$$

$$\left\| \overrightarrow{PQ} \times \overrightarrow{PR} \right\| = \sqrt{(-15)^2 + 7^2 + 10^2}$$

$$= \sqrt{374}$$

The area of the triangle is

$$\frac{1}{2}\left\| \overrightarrow{PQ} \times \overrightarrow{PR} \right\| \underline{= \frac{1}{2}\sqrt{374}|}$$

## *Exercise*

*a)* Find the area of the parallelogram with edges $v = (3,\ 2)$ and $w = (1,\ 4)$

*b)* Find the area of the triangle with sides ***v*, *w***, and ***v* + *w***. Draw it.

*c)* Find the area of the triangle with sides ***v*, *w***, and ***v*** – ***w***. Draw it.

***Solution***

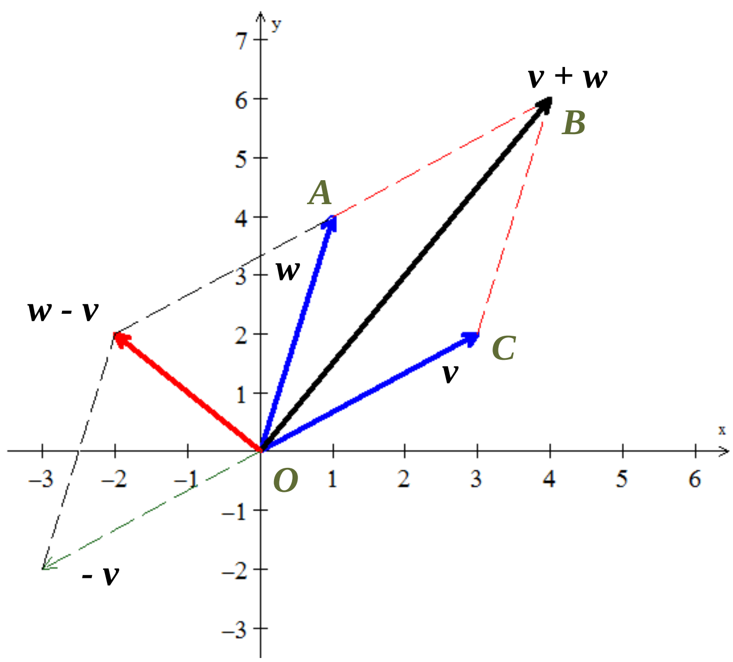
***a)*** $Area = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 10$ (which is the parallelogram *OABC*)

***b)*** The area of the triangle with sides ***v*, *w*,** and ***v* + *w*** is the triangle OCB or OAB which it is half the parallelogram (by definition).

$$Area = 5$$

$$v + w = (3,\ 2) + (1,\ 4) = (4,\ 6)$$

$$Area = \frac{1}{2}\begin{vmatrix} 3 & 4 \\ 2 & 6 \end{vmatrix} = \frac{1}{2}(10) \underline{= 5|}$$



***c)*** The area of the triangle with sides ***v, w,*** and ***v - w*** is equivalent to the triangle OAC which it is half the parallelogram (by definition).

$$Area = 5$$

$$Area = \frac{1}{2}\begin{vmatrix} 2 & -2 \\ -3 & -2 \end{vmatrix} = \frac{1}{2}\left|-10\right| \underline{= 5|}$$

## *Exercise*

Find the volume of the parallelepiped with sides $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

*a)* $\boldsymbol{u} = (2, -6, 2), \quad \boldsymbol{v} = (0, 4, -2), \quad \boldsymbol{w} = (2, 2, -4)$

*b)* $\boldsymbol{u} = (3, 1, 2), \quad \boldsymbol{v} = (4, 5, 1), \quad \boldsymbol{w} = (1, 2, 4)$

***Solution***

***a)*** $\boldsymbol{u}.(\boldsymbol{v} \times \boldsymbol{w}) = \begin{vmatrix} 2 & -6 & 2 \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{vmatrix} = -16$

The volume of the parallelepiped is $\left|-16\right| = \underline{16|}$

***b)*** $\boldsymbol{u}.(\boldsymbol{v} \times \boldsymbol{w}) = \begin{vmatrix} 3 & 1 & 2 \\ 4 & 5 & 1 \\ 1 & 2 & 4 \end{vmatrix} = 45$

The volume of the parallelepiped is $\boxed{45}$

## *Exercise*

Compute the scalar triple product $\boldsymbol{u}.(\boldsymbol{v} \times \boldsymbol{w})$

*a)* $\boldsymbol{u} = (-2, 0, 6), \quad \boldsymbol{v} = (1, -3, 1), \quad \boldsymbol{w} = (-5, -1, 1)$

*b)* $\boldsymbol{u} = (-1, 2, 4), \quad \boldsymbol{v} = (3, 4, -2), \quad \boldsymbol{w} = (-1, 2, 5)$

*c)* $\boldsymbol{u} = (a, 0, 0), \quad \boldsymbol{v} = (0, b, 0), \quad \boldsymbol{w} = (0, 0, c)$

*d)* $\boldsymbol{u} = 3\boldsymbol{i} - 2\boldsymbol{j} - 5\boldsymbol{k}, \quad \boldsymbol{v} = \boldsymbol{i} + 4\boldsymbol{j} - 4\boldsymbol{k}, \quad \boldsymbol{w} = 3\boldsymbol{j} + 2\boldsymbol{k}$

*e)* $\boldsymbol{u} = (3,\ -1,\ 6) \quad \boldsymbol{v} = (2,\ 4,\ 3) \quad \boldsymbol{w} = (5,\ -1,\ 2)$

***Solution***

***a)*** $\boldsymbol{u}.(\boldsymbol{v} \times \boldsymbol{w}) = \begin{vmatrix} -2 & 0 & 6 \\ 1 & -3 & 1 \\ -5 & -1 & 1 \end{vmatrix} = \underline{-92|}$

**b)** $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w})=\begin{vmatrix}-1 & 2 & 4\\ 3 & 4 & -2\\ -1 & 2 & 5\end{vmatrix}=\underline{-10}$

**c)** $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w})=\begin{vmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{vmatrix}=\underline{abc}$

**d)** $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w})=\begin{vmatrix}3 & -2 & -5\\ 1 & 4 & -4\\ 0 & 3 & 2\end{vmatrix}\underline{=49}$

**e)** $\boldsymbol{u}\bullet(\boldsymbol{v}\times\boldsymbol{w})=\begin{vmatrix}3 & -1 & 6\\ 2 & 4 & 3\\ 5 & -1 & 2\end{vmatrix}\underline{=-110}$

## *Exercise*

Use the cross product to find the sine of the angle between the vectors $\boldsymbol{u}=(2,3,-6),\quad \boldsymbol{v}=(2,3,6)$

### ***Solution***

$$\begin{aligned}\boldsymbol{u}\times\boldsymbol{v}&=(2,3,-6)\times(2,3,6)\\ &=\left(\begin{vmatrix}3 & -6\\ 3 & 6\end{vmatrix},\ -\begin{vmatrix}2 & -6\\ 2 & 6\end{vmatrix},\ \begin{vmatrix}2 & 3\\ 2 & 3\end{vmatrix}\right)\\ &=\underline{(36,\ -24,\ 0)}\end{aligned}$$

$$\|\boldsymbol{u}\times\boldsymbol{v}\|=\sqrt{36^2+(-24)^2+0}=\sqrt{1872}=12\sqrt{13}$$

$$\begin{aligned}\sin\theta&=\left(\frac{\|\boldsymbol{u}\times\boldsymbol{v}\|}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right)\\ &=\frac{12\sqrt{13}}{\sqrt{2^2+3^2+(-6)^2}\sqrt{2^2+3^2+6^2}}\\ &=\frac{12\sqrt{13}}{(7)(7)}\\ &=\underline{\frac{12}{49}\sqrt{13}}\end{aligned}$$

### *Exercise*

Simplify $(\boldsymbol{u}+\boldsymbol{v})\times(\boldsymbol{u}-\boldsymbol{v})$

***Solution***

$$
\begin{aligned}
(\boldsymbol{u}+\boldsymbol{v})\times(\boldsymbol{u}-\boldsymbol{v}) &= (\boldsymbol{u}+\boldsymbol{v})\times\boldsymbol{u}-(\boldsymbol{u}+\boldsymbol{v})\times\boldsymbol{v}\\
&= (\boldsymbol{u}\times\boldsymbol{u})+(\boldsymbol{v}\times\boldsymbol{u})-\left[(\boldsymbol{u}\times\boldsymbol{v})+(\boldsymbol{v}\times\boldsymbol{v})\right]\\
&= 0+(\boldsymbol{v}\times\boldsymbol{u})-\left[(\boldsymbol{u}\times\boldsymbol{v})+0\right]\\
&= (\boldsymbol{v}\times\boldsymbol{u})-(\boldsymbol{u}\times\boldsymbol{v})\\
&= (\boldsymbol{v}\times\boldsymbol{u})-(-(\boldsymbol{v}\times\boldsymbol{u}))\\
&= (\boldsymbol{v}\times\boldsymbol{u})+(\boldsymbol{v}\times\boldsymbol{u})\\
&= 2(\boldsymbol{v}\times\boldsymbol{u})
\end{aligned}
$$

### *Exercise*

Prove Lagrange's identity: $\|\boldsymbol{u}\times\boldsymbol{v}\|^2=\|\boldsymbol{u}\|^2\ \|\boldsymbol{v}\|^2-(\boldsymbol{u}.\boldsymbol{v})^2$

***Solution***

Let $\boldsymbol{u}=(u_1,\ u_2,\ u_3)$ and $\boldsymbol{v}=(v_1,\ v_2,\ v_3)$

$\|\boldsymbol{u}\|^2=u_1^2+u_2^2+u_3^2$

$\|\boldsymbol{v}\|^2=v_1^2+v_2^2+v_3^2$

$(\boldsymbol{u}.\boldsymbol{v})^2=(u_1v_1+u_2v_2+u_3v_3)^2$

$$
\begin{aligned}
\|\boldsymbol{u}\times\boldsymbol{v}\|^2 &= (u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2\\
&= u_2^2v_3^2-2u_2v_3u_3v_2+u_3^2v_2^2+u_3^2v_1^2-2u_3v_1u_1v_3+u_1^2v_3^2+u_1^2v_2^2-2u_2v_1u_2v_1+u_2^2v_1^2
\end{aligned}
$$

$$
\begin{aligned}
\|\boldsymbol{u}\|^2\ \|\boldsymbol{v}\|^2-(\boldsymbol{u}.\boldsymbol{v})^2 &= \left(u_1^2+u_2^2+u_3^2\right)\left(v_1^2+v_2^2+v_3^2\right)-(u_1v_1+u_2v_2+u_3v_3)^2\\
&= u_1^2v_1^2+u_1^2v_2^2+u_1^2v_3^2+u_2^2v_1^2+u_2^2v_2^2+u_2^2v_3^2+u_3^2v_1^2+u_3^2v_2^2+u_3^2v_3^2\\
&\qquad -u_1^2v_1^2-u_1v_1u_2v_2-u_1v_1u_3v_3\\
&\qquad -u_2v_2u_1v_1-u_2^2v_2^2-u_2v_2u_3v_3\\
&\qquad -u_1v_1u_3v_3-u_2v_2u_3v_3-u_3^2v_3^2
\end{aligned}
$$

$$= u_2^2 v_3^2 - 2u_2 v_2 u_3 v_3 + u_3^2 v_2^2$$
$$+ u_3^2 v_1^2 - 2u_1 v_1 u_3 v_3 + u_1^2 v_3^2$$
$$+ u_1^2 v_2^2 - 2u_1 v_1 u_2 v_2 + u_2^2 v_1^2$$

$$\Rightarrow \boxed{\|\boldsymbol{u}\times\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2\ \|\boldsymbol{v}\|^2 - (\boldsymbol{u}.\boldsymbol{v})^2}$$

### *Exercise*

Polar coordinates satisfy $x = r\cos\theta$ *and* $y = \sin\theta$. Polar area $J\ dr\ d\theta$ includes $J$:

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

The two columns are orthogonal. Their lengths are ______. Thus $J$ = ______.

### ***Solution***

The length of the first column is: $= \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$

The length of the second column is: $= \sqrt{r^2\sin^2\theta + r^2\cos^2\theta}$

$$= \sqrt{r^2\left(\sin^2\theta + \cos^2\theta\right)}$$
$$= \sqrt{r^2}$$
$$\boxed{= r}$$

So $J$ is the product $1.\ r = r$.

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$
$$= r\left(\cos^2\theta + \sin^2\theta\right)$$
$$\boxed{= r}$$

### *Exercise*

Prove that $\|\vec{u}+\vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$ if and only if $\vec{u}$ and $\vec{v}$ are parallel vectors.

### ***Solution***

If $\vec{u}$ and $\vec{v}$ are parallel vectors, then $\vec{u}\times\vec{v} = 0$

Which the two vectors are collinear, which implies that $\vec{u} = a\vec{v}$

$$\|\vec{u}+\vec{v}\| = \|\vec{u}+a\vec{u}\|$$
$$= \|(1+a)\vec{u}\|$$
$$= (1+a)\|\vec{u}\|$$
$$= \|\vec{u}\| + a\|\vec{u}\|$$
$$= \|\vec{u}\| + \|a\vec{u}\|$$
$$= \|\vec{u}\| + \|\vec{v}\| \quad \checkmark$$

### *Exercise*

State the following statements as True or False

*a)* The cross product of two nonzero vectors $\vec{u}$ and $\vec{v}$ is a nonzero vector if and only if $\vec{u}$ and $\vec{v}$ are not parallel.

*b)* A normal vector to a plane can be obtained by taking the cross product of two nonzero and noncollinear vectors lying in the plane.

*c)* The scalar triple product of $\vec{u}$, $\vec{v}$, and $\vec{w}$ determines a vector whose length is equal to the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$.

*d)* If $\vec{u}$ and $\vec{v}$ are vectors in 3-space, then $\|\vec{u}\times\vec{v}\|$ is equal to the area of the parallelogram determine by $\vec{u}$ and $\vec{v}$.

*e)* For all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $R^3$, the vectors $(\vec{u}\times\vec{v})\times\vec{w}$ and $\vec{u}\times(\vec{v}\times\vec{w})$ are the same.

*f)* If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $R^3$, where $\vec{u}$ is nonzero and $\vec{u}\times\vec{v}=\vec{u}\times\vec{w}$, then $\vec{v}=\vec{w}$

***Solution***

***a)*** True,

$\vec{u}\times\vec{v} = \|\vec{u}\|\ \|\vec{v}\|\sin\theta = 0$ if $\theta = 0$ which the two vectors are parallel.

***b)*** True;

The cross product of two nonzero and non collinear vectors will be perpendicular to both vectors, hence normal to the plane containing the vectors.

***c)*** False;

The scalar triple product is a scalar, not a vector.

***d)*** True;

***e)*** False;

Let $\vec{u}=\hat{i} \quad \vec{v}=\vec{w}=\hat{j}$

$(\vec{u}\times\vec{v})\times\vec{w} = (\hat{i}\times\hat{j})\times\hat{j} = \hat{k}\times\hat{j} = -\hat{i}$

$\vec{u}\times(\vec{v}\times\vec{w}) = \hat{i}\times(\hat{j}\times\hat{j}) = \hat{i}\times\vec{0} = \vec{0}$

Hence, $(\vec{u}\times\vec{v})\times\vec{w} \neq (\vec{u}\times\vec{v})\times\vec{w}$

***f)*** False;

Let $\vec{u} = \hat{i} + \hat{j} \quad \vec{v} = \hat{i} + \hat{j} + \hat{k} \quad \vec{w} = -\hat{i} - \hat{j} + \hat{k}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \hat{i} - \hat{j}$$

$$\vec{u} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{vmatrix} = \hat{i} - \hat{j}$$

$\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, but $\vec{v} \neq \vec{w}$

# Solution Section 2.5 – Subspaces, Span and Null Space

## Exercise

Suppose $S$ and $T$ are two subspaces of a vector space **V**.

*a)* The sum $S+T$ contains all sums $s+t$ of a vector $s$ in $S$ and a vector $t$ in $T$. Show that $S+T$ satisfies the requirements (addition and scalar multiplication) for a vector space.

*b)* If $S$ and $T$ are lines in $\boldsymbol{R}^m$, what is the difference between $S+T$ and $S\cup T$? That union contains all vectors from $S$ and $T$ or both. Explain this statement: The span of $S\cup T$ is $S+T$.

**Solution**

***a)*** Let $s,\ s'$ be vectors in $S$, Let $t,\ t'$ be vectors in $T$, and let $c$ be a scalar. Then

$$(s+t)+(s'+t')=(s+s')+(t+t') \text{ and}$$

$$c(s+t)=cs+ct$$

Thus $S+T$ is closed under addition and scalar multiplication, it satisfies the two requirements for a vector space.

***b)*** If $S$ and $T$ are distinct lines, then $S$ and $T$ is a plane, whereas $S\cup T$ is not even closed under addition. The span of $S\cup T$ is the set of all combinations of vectors in this union. In particular, it contains all sums $s+t$ of a vector $\boldsymbol{s}$ in $S$ and a vector $\boldsymbol{t}$ in $T$, and these sums form $S+T$.
$S+T$ contains both $S$ and $T$; so it contains $S\cup T$. $S+T$ is a vector space.

***c)*** So, it contains all combinations of vectors in itself; in particular, it contains the span of $S\cup T$. Thus, the span of $S\cup T$ is $S+T$.

## Exercise

Determine which of the following are subspaces of $\boldsymbol{R}^3$?

*a)* All vectors of the form $(a, 0, 0)$

*b)* All vectors of the form $(a, 1, 1)$

*c)* All vectors of the form $(a, b, c)$, where $b = a + c$

*d)* All vectors of the form $(a, b, c)$, where $b = a + c + 1$

*e)* All vectors of the form $(a, b, 0)$

**Solution**

***a)*** $(a_1,\ 0,\ 0)+(a_2,\ 0,\ 0)=(a_1+a_2,\ 0,\ 0)$

$k(a,\ 0,\ 0)=(ka,\ 0,\ 0)$

This is a subspace of $\boldsymbol{R}^3$

**b)** $(a_1, 1, 1)+(a_2, 1, 1)=(a_1+a_2, 2, 2)$ which is not in the set.

Therefore, this is not a subspace of $\boldsymbol{R}^3$

**c)** $(a_1, b_1, c_1)+(a_2, b_2, c_2)=(a_1+a_2, b_1+b_2, c_1+c_2)$

$$=(a_1+a_2, a_1+c_1+a_2+c_2, c_1+c_2)$$

$$=(a_1+a_2, (a_1+a_2)+(c_1+c_2), c_1+c_2)$$

$$=(a_1, a_1+c_1, c_1)+(a_2, a_2+c_2, c_2)$$

$k(a, b, c)=(ka, kb, kc)$

$$=(ka, k(a+c), kc)$$

$$=k(a, (a+c), c)$$

This is a subspace of $\boldsymbol{R}^3$

**d)** $k(a+c+1)\neq ka+kc+1$ so $k(a,b,c)$ is not in the set.

Therefore, this is not a subspace of $\boldsymbol{R}^3$

**e)** $(a_1, b_1, 0)+(a_2, b_2, 0)=(a_1+a_2, b_1+b_2, 0)$

$k(a, b, 0)=(ka, kb, 0)$

This is a subspace of $\boldsymbol{R}^3$

## *Exercise*

Determine which of the following are subspaces of $\boldsymbol{R}^\infty$?

*a)* All sequences $\boldsymbol{v}$ in $\boldsymbol{R}^\infty$ of the form $v = (v, 0, v, 0, \ldots)=(kv, k, kv, k, \cdots)$

*b)* All sequences $\boldsymbol{v}$ in $\boldsymbol{R}^\infty$ of the form $v = (v, 1, v, 1, \ldots)$

*c)* All sequences $\boldsymbol{v}$ in $\boldsymbol{R}^\infty$ of the form $v = (v, 2v, 4v, 8v, 16v, \ldots)$

***Solution***

**a)** $(v_1,0,v_1,0,\ldots)+(v_2,0,v_2,0,\ldots)=(v_1+v_2, 0, v_1+v_2, 0, \ldots)$

$kv=k(v,0,v,0,\ldots)=(kv,0,kv,0,\ldots)$

This is a subspace of $\boldsymbol{R}^\infty$

**b)** $kv=k(v,1,v,1,\ldots)$

$kv$ is not in the set

Since $k \neq 1$, then is not a subspace of $\boldsymbol{R}^{\infty}$

***c)*** $\left(v_1, 2v_1, 4v_1, 8v_1, \ldots\right) + \left(v_2, 2v_2, 4v_2, 8v_2, \ldots\right) = \left(v_1 + v_2, 2v_1 + 2v_2, 4v_1 + 4v_2, 8v_1 + 8v_2, \ldots\right)$

$$= \left(v_1 + v_2, 2\left(v_1 + v_2\right), 4\left(v_1 + v_2\right), 8\left(v_1 + v_2\right), \ldots\right)$$

$k\left(v, 2v, 4v, 8v, \ldots\right) = \left(kv,\ 2kv,\ 4kv,\ 8kv,\ \ldots\right)$

This is a subspace of $\boldsymbol{R}^{\infty}$

## *Exercise*

Which of the following are linear combinations of $\boldsymbol{u} = (0, -2, 2)$ and $\boldsymbol{v} = (1, 3, -1)$?

*a)* (2, 2, 2) *b)* (3, 1, 5) *c)* (0, 4, 5) *d)* (0, 0 ,0)

***Solution***

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \\ 2 & -1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

***a)*** $b = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \to \left[\begin{array}{cc|c} 0 & 1 & 2 \\ -2 & 3 & 2 \\ 2 & -1 & 2 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$

(2, 2, 2) = $2\boldsymbol{u} + 2\boldsymbol{v}$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

***b)*** $b = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \to \left[\begin{array}{cc|c} 0 & 1 & 3 \\ -2 & 3 & 1 \\ 2 & -1 & 5 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$

(3, 1, 5) = $4\boldsymbol{u} + 3\boldsymbol{v}$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

***c)*** $b = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} \to \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & 3 & 4 \\ 2 & -1 & 5 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$

(0, 4, 5) is not a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

***d)*** $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & 3 & 0 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

(0, 0, 0) = $0\boldsymbol{u} + 0\boldsymbol{v}$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

## Exercise

Which of the following are linear combinations of $\boldsymbol{u} = (2, 1, 4)$, $\boldsymbol{v} = (1, -1, 3)$ and $\boldsymbol{w} = (3, 2, 5)$?

*a)* $(-9, -7, -15)$

*b)* $(6, 11, 6)$

*c)* $(0, 0, 0)$

## Solution

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{pmatrix}$$

***a)*** $$\left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Therefore, $(-9,\ -7,\ -15) = -2\boldsymbol{u} + 1\boldsymbol{v} - 2\boldsymbol{w}$

***b)*** $$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore, $(6,\ 11,\ 6) = 4\boldsymbol{u} - 5\boldsymbol{v} + 1\boldsymbol{w}$

***c)*** $$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, $(0,\ 0,\ 0) = 0\boldsymbol{u} + 0\boldsymbol{v} + 0\boldsymbol{w}$

## Exercise

Determine whether the given vectors span $\boldsymbol{R}^3$

*a)* $\vec{v}_1 = (2,\ 2,\ 2),\quad \vec{v}_2 = (0,\ 0,\ 3),\quad \vec{v}_3 = (0,\ 1,\ 1)$

*b)* $\vec{v}_1 = (2,\ -1,\ 3),\quad \vec{v}_2 = (4,\ 1,\ 2),\quad \vec{v}_3 = (8,\ -1,\ 8)$

*c)* $\vec{v}_1 = (3,\ 1,\ 4),\quad \vec{v}_2 = (2,\ -3,\ 5),\quad \vec{v}_3 = (5,\ -2,\ 9),\quad \vec{v}_4 = (1,\ 4,\ -1)$

## Solution

***a)*** $$det\begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{pmatrix} = -6 \neq 0$$

The system is consistent for all values so the given vectors span $\boldsymbol{R}^3$.

***b)*** $det\begin{pmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{pmatrix} = 0$

The system is not consistent for all values so the given vectors do not span $\boldsymbol{R}^3$.

***c)*** $\left[\begin{array}{cccc|c} 3 & 2 & 5 & 1 & b_1 \\ 1 & -3 & -2 & 4 & b_2 \\ 4 & 5 & 9 & -1 & b_3 \end{array}\right] \xrightarrow{\textit{leads to}} \left[\begin{array}{cccc|c} 1 & -3 & -2 & 4 & b_2 \\ 0 & 1 & 1 & -1 & \frac{1}{11}b_1 - \frac{3}{11}b_2 \\ 0 & 0 & 0 & 0 & -\frac{17}{11}b_1 + \frac{7}{11}b_2 + b_3 \end{array}\right]$

The system has a solution only if $-\frac{17}{11}b_1 + \frac{7}{11}b_2 + b_3 = 0$. But since this is a restriction that the given vectors don't span on all of $\boldsymbol{R}^3$. So the given vectors do not span $\boldsymbol{R}^3$.

## *Exercise*

Which of the following are linear combinations of $A = \begin{pmatrix} 4 & 0 \\ -2 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$

*a)* $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ *b)* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *c)* $\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$

***Solution***

$$\begin{pmatrix} 4 & 1 & 0 \\ 0 & -1 & 2 \\ -2 & 2 & 1 \\ -2 & 3 & 4 \end{pmatrix}$$

***a)*** $\left[\begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -5 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = 1A + 2B - 3C$ is a linear combinations of *A*, *B*, and *C*.

***b)*** $\left[\begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ -2 & 2 & 1 & 0 \\ -2 & 3 & 4 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0A + 0B + 0C$$ is a linear combinations of *A*, *B*, and *C*.

***c)*** $$\left[\begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & 0 \\ -2 & 2 & 1 & 3 \\ -2 & 3 & 4 & 8 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix} = 1A + 2B + 1C$$ is a linear combination of *A*, *B*, and *C*.

## *Exercise*

Suppose that $\vec{v}_1 = (2,\ 1,\ 0,\ 3)$, $\vec{v}_2 = (3,\ -1,\ 5,\ 2)$, $\vec{v}_3 = (-1,\ 0,\ 2,\ 1)$. Which of the following vectors are in span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

*a)* (2, 3, –7, 3) *b)* (0, 0, 0, 0) *c)* (1, 1, 1, 1) *d)* (–4, 6, –13, 4)

***Solution***

In order to be span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, there must exists scalars *a*, *b*, *c* that $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \boldsymbol{w}$

$$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

***a)*** $$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system is consistent, it has only solution is *a* = 2, *b* = –1, *c* = –1

$$2\vec{v}_1 - 1\vec{v}_2 - 1\vec{v}_3 = (2,\ 3,\ -7,\ 3)$$

Therefore, (2, 3, –7, 3) is in span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

***b)*** The vector (0, 0 , 0, 0) is obviously in span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

Since $0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = (0,\ 0,\ 0,\ 0)$

***c)*** $\left[\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 5 & 2 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

This system is inconsistent, therefore (1, 1, 1, 1) is *not* in span $\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\}$

***d)*** $\left[\begin{array}{ccc|c} 2 & 3 & -1 & -4 \\ 1 & -1 & 0 & 6 \\ 0 & 5 & 2 & -13 \\ 3 & 2 & 1 & 4 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

This system is consistent, it has only solution is $a = 3$, $b = -3$, $c = 1$

$3\vec{v}_1 - 3\vec{v}_2 + 1\vec{v}_3 = (-4,\ 6,\ -13,\ 4)$

Therefore, (–4, 6, –13, 4) is in span $\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\}$

## *Exercise*

Let $\boldsymbol{f} = \cos^2 x$ and $\boldsymbol{g} = \sin^2 x$. Which of the following lie in the space spanned by $\boldsymbol{f}$ and $\boldsymbol{g}$

*a)* $\cos 2x$ *b)* $3 + x^2$ *c)* $\sin x$ *d)* $0$

***Solution***

***a)*** $\cos 2x = \cos^2 x - \sin^2 x$, therefore $\cos 2x$ is in span $\{\boldsymbol{f}, \boldsymbol{g}\}$

***b)*** In order for $3 + x^2$ to be in span $\{\boldsymbol{f}, \boldsymbol{g}\}$, there must exist scalars *a* and *b* such that

$a\cos^2 x + b\sin^2 x = 3 + x^2$

When $\left.\begin{array}{l} x = 0 \quad \Rightarrow \quad a = 3 \\ x = \pi \quad \Rightarrow \quad a = 3 + \pi^2 \end{array}\right\} \rightarrow \textit{contradiction}$

Therefore $3 + x^2$ is *not* in span $\{\boldsymbol{f}, \boldsymbol{g}\}$

***c)*** In order for $\sin x$ to be in span $\{\boldsymbol{f}, \boldsymbol{g}\}$, there must exist scalars *a* and *b* such that

$a\cos^2 x + b\sin^2 x = \sin x$

When $\left.\begin{array}{l} x = \frac{\pi}{2} \quad \Rightarrow \quad b = 1 \\ x = -\frac{\pi}{2} \quad \Rightarrow \quad b = -1 \end{array}\right\} \rightarrow \textit{contradiction}$

Therefore $\sin x$ is *not* in span $\{\boldsymbol{f}, \boldsymbol{g}\}$

***d)*** In order for $0$ to be in span $\{\boldsymbol{f}, \boldsymbol{g}\}$, there must exist scalars *a* and *b* such that

$$0\cos^2 x + 0\sin^2 x = 0$$

Therefore $\mathbf{0}$ is in span $\{\boldsymbol{f}, \boldsymbol{g}\}$

## Exercise

$V = \mathbb{R}^3, \quad S = \{(0,\ s,\ t) \mid s,\ t \text{ are real numbers}\}$ where *V* is a vector space and *S* is subset of *V*

*a)* Is *S* closed under addition?

*b)* Is *S* closed under scalar multiplication?

*c)* Is *S* a subspace of *V*?

**Solution**

***a)*** Let $\boldsymbol{u} = (0,\ s_1,\ t_1) \quad and \quad \boldsymbol{v} = (0,\ s_2,\ t_2)$

$$\begin{aligned}\boldsymbol{u}+\boldsymbol{v} &= (0,\ s_1+s_2,\ t_1+t_2)\\ &= (0,\ s,\ t)\end{aligned}$$

Yes, *S* is closed under addition

***b)*** $k\boldsymbol{u} = (0,\ ks_1,\ kt_1) = (0,\ s,\ t)$

Yes, *S* is closed under scalar multiplication

***c)*** Since *S* is closed under addition and scalar multiplication, then *S* is a subspace of *V*.

## Exercise

$V = \mathbb{R}^3, \quad S = \{(x,\ y,\ z) \mid x,\ y\ ,\ z\ \ge 0\}$ where *V* is a vector space and *S* is subset of *V*

*a)* Is *S* closed under addition?

*b)* Is *S* closed under scalar multiplication?

*c)* Is *S* a subspace of *V*?

**Solution**

***a)*** Let $\boldsymbol{u} = (x_1, y_1, z_1) \quad and \quad \boldsymbol{v} = (x_2, y_2, z_2)$

$$\begin{aligned}\boldsymbol{u}+\boldsymbol{v} &= (x_1+x_2, y_1+y_2, z_1+z_2) \qquad \textit{where } x = x_1+x_2,\ y = y_1+y_2,\ z = z_1+z_2\\ &= (x,\ y,\ z)\end{aligned}$$

Yes, *S* is closed under addition

***b)*** $(-1)\boldsymbol{u} = (-x_1,\ -y_1,\ -z_1)$

*S* is ***not*** closed under scalar multiplication since $x_1 \ge 0 \ \Rightarrow\ -x_1 \le 0$

***c)*** *S* is ***not*** a subspace of *V*.

### *Exercise*

$V=\mathbb{R}^3,\quad S=\{(x,\ y,\ z)|\ z=x+y+1\}$ where $V$ is a vector space and $S$ is subset of $V$

*a)* Is $S$ closed under addition?

*b)* Is $S$ closed under scalar multiplication?

*c)* Is $S$ a subspace of $V$?

***Solution***

***a)*** Let $\boldsymbol{u}=(0,\ 1,\ 2)\quad and \quad \boldsymbol{v}=(1,\ 2,\ 4)$

$$\boldsymbol{u}+\boldsymbol{v}=(1,\ 3,\ 6)$$

$$\neq(1,\ 3,\ 1+3+1)$$

No, $S$ is *not* closed under addition

***b)*** $2\boldsymbol{u}=(2x_1,\ 2y_1,\ 2z_1)$

$$=(2x_1,\ 2y_1,\ 2(x_1+y_1+1))$$

$$=(2x_1,\ 2y_1,\ 2x_1+2y_1+2) \qquad \textit{Where}\quad x=2x_1,\quad y=2y_1,\quad 2z=2(x_1+y_1+1)$$

$$=(x,\ y,\ z)$$

Yes, $S$ is closed under scalar multiplication

***c)*** $S$ is ***not*** a subspace of $V$.

### *Exercise*

Let $S=\left\{(a_1,\ a_2,\ a_3)\in\mathbb{R}^3:\ a_1=3a_2\ \ and\ \ a_3=-a_2\right\}$, Determine:

*a)* Is $S$ closed under addition?

*b)* Is $S$ closed under scalar multiplication?

*c)* Is $S$ a subspace of $\boldsymbol{R}^3$?

***Solution***

***a)*** Let $\vec{u}=(a_1,a_2,a_3)\quad and \quad \vec{v}=(b_1,b_2,b_3)$

$$\vec{u}+\vec{v}=(a_1,a_2,a_3)+(b_1,b_2,b_3)$$

$$=(3a_2,a_2,-a_2)+(3b_2,b_2,-b_2)$$

$$=(3a_2+3b_2,a_2+b_2,-a_2-b_2)$$

$$=(3(a_2+b_2),a_2+b_2,-(a_2+b_2)) \qquad c_2=a_2+b_2$$

$$=(3c_2,c_2,-c_2)$$

$$=\left(c_1, c_2, c_3\right): \quad c_1 = 3c_2 \quad c_3 = -c_2$$

$S$ is closed under addition

***b)*** $k\vec{u} = k\left(a_1, a_2, a_3\right)$

$$= k\left(3a_2, a_2, -a_2\right)$$

$$= \left(3ka_2, ka_2, -ka_2\right) \qquad c_2 = ka_2$$

$$= \left(3c_2, c_2, -c_2\right)$$

$$= \left(c_1, c_2, c_3\right): \quad c_1 = 3c_2 \quad c_3 = -c_2$$

$S$ is closed under scalar multiplication.

***c)*** $S$ is a subspace of $V$.

## *Exercise*

Let $S = \left\{\left(a_1, a_2, a_3\right) \in \mathbb{R}^3 : a_1 = a_3 + 2\right\}$, Determine:

*a)* Is $S$ closed under addition?

*b)* Is $S$ closed under scalar multiplication?

*c)* Is $S$ a subspace of $\boldsymbol{R}^3$?

***Solution***

***a)*** Let $\vec{u} = (2, 1, 0)$ *and* $\vec{v} = (3, 0, 1)$ $\qquad a_1 = a_3 + 2$

$$\vec{u} + \vec{v} = (2, 1, 0) + (3, 0, 1)$$

$$= (5, 1, 1) \qquad 5 \overset{?}{=} 1 + 2$$

$$\neq (3, 1, 1)$$

$S$ is *not* closed under addition

***b)*** $k\vec{u} = k\left(a_1, a_2, a_3\right)$

$$= k\left(a_3 + 2, a_2, a_3\right)$$

$$= \left(ka_3 + 2k, ka_2, ka_3\right)$$

$$a_1 = a_3 + 2 \quad \to \quad ka_3 + 2k = a_3 + 2$$

$$2k \neq 2 \quad (\forall k)$$

$S$ is *not* closed under scalar multiplication.

***c)*** $S$ is ***not*** a subspace of $V$.

## *Exercise*

Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ 2a_1-7a_2+a_3=0\right\}$, Determine:

*a)* Is *S* closed under addition?
*b)* Is *S* closed under scalar multiplication?
*c)* Is *S* a subspace of $\boldsymbol{R}^3$?

**Solution**

***a)*** Let $\vec{u}=\left(a_1,a_2,a_3\right)$ *and* $\vec{v}=\left(b_1,b_2,b_3\right)$

$$2a_1-7a_2+a_3=0\ \rightarrow\ a_3=7a_2-2a_1$$

$$\begin{aligned}\vec{u}+\vec{v}&=\left(a_1,a_2,a_3\right)+\left(b_1,b_2,b_3\right)\\&=\left(a_1,a_2,7a_2-2a_1\right)+\left(b_1,b_2,7b_2-2b_1\right)\\&=\left(a_1+b_1,a_2+b_2,7a_2-2a_1+7b_2-2b_1\right)\\&=\left(a_1+b_1,a_2+b_2,7\left(a_2+b_2\right)-2\left(a_1+b_1\right)\right) &&\text{Let } c_1=a_1+b_1\quad c_2=a_2+b_2\\&=\left(c_1,c_2,7c_2-2c_1\right) && c_3=7c_2-2c_1\ \rightarrow\ 2c_1-7c_2+c_3=0\\&=\left(c_1,c_2,c_3\right)\end{aligned}$$

*S* is closed under addition

***b)*** 
$$\begin{aligned}k\vec{u}&=k\left(a_1,a_2,a_3\right)\\&=k\left(a_1,a_2,7a_2-2a_1\right)\\&=\left(ka_1,ka_2,7ka_2-2ka_1\right) &&\text{Let } c_1=ka_1\quad c_2=ka_2\\&=\left(c_1,c_2,7c_2-2c_1\right) && c_3=7c_2-2c_1\ \rightarrow\ 2c_1-7c_2+c_3=0\\&=\left(c_1,c_2,c_3\right)\end{aligned}$$

*S* is closed under scalar multiplication.

***c)*** *S* is a subspace of *V*.

## *Exercise*

Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ a_1-4a_2-a_3=0\right\}$, Determine:

*a)* Is *S* closed under addition?
*b)* Is *S* closed under scalar multiplication?

*c)* Is $S$ a subspace of $\boldsymbol{R}^3$?

**Solution**

$$a_1 - 4a_2 - a_3 = 0 \quad \rightarrow \quad a_1 = 4a_2 + a_3$$

***a)*** Let $\vec{u} = \left(a_1, a_2, a_3\right)$ *and* $\vec{v} = \left(b_1, b_2, b_3\right)$

$$\begin{aligned}
\vec{u} + \vec{v} &= \left(a_1, a_2, a_3\right) + \left(b_1, b_2, b_3\right) \\
&= \left(4a_2 + a_3, a_2, a_3\right) + \left(4b_2 + b_3, b_2, b_3\right) \\
&= \left(4a_2 + a_3 + 4b_2 + b_3, a_2 + b_2, a_3 + b_3\right) \\
&= \left(4\left(a_2 + b_2\right) + \left(a_3 + b_3\right), a_2 + b_2, a_3 + b_3\right) \qquad \text{Let } c_2 = a_2 + b_2 \quad c_3 = a_3 + b_3 \\
&= \left(4c_2 + c_3, c_2, c_3\right) \qquad c_1 - 4c_2 - c_3 = 0 \rightarrow c_1 = 4c_2 + c_3 \\
&= \left(c_1, c_2, c_3\right)
\end{aligned}$$

$S$ is closed under addition

***b)***

$$\begin{aligned}
k\vec{u} &= k\left(a_1, a_2, a_3\right) \\
&= k\left(4a_2 + a_3, a_2, a_3\right) \\
&= \left(4ka_2 + ka_3, ka_2, ka_3\right) \qquad \text{Let } c_2 = ka_2 \quad c_3 = ka_3 \\
&= \left(4c_2 + c_3, c_2, c_3\right) \qquad c_1 = 4c_2 + c_3 \rightarrow c_1 - 4c_2 - c_3 = 0 \\
&= \left(c_1, c_2, c_3\right)
\end{aligned}$$

$S$ is closed under scalar multiplication.

***c)*** $S$ is a subspace of $V$.

## *Exercise*

Let $S = \left\{\left(a_1, a_2, a_3\right) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 0\right\}$, Determine:

*a)* Is $S$ closed under addition?
*b)* Is $S$ closed under scalar multiplication?
*c)* Is $S$ a subspace of $\boldsymbol{R}^3$?

**Solution**

$$a_1 + 2a_2 - 3a_3 = 0 \quad \rightarrow \quad a_1 = -2a_2 + 3a_3$$

***a)*** Let $\vec{u}=\left(a_1, a_2, a_3\right)$ *and* $\vec{v}=\left(b_1, b_2, b_3\right)$

$$\vec{u}+\vec{v}=\left(a_1, a_2, a_3\right)+\left(b_1, b_2, b_3\right)$$

$$=\left(-2a_2+3a_3, a_2, a_3\right)+\left(-2b_2+3b_3, b_2, b_3\right)$$

$$=\left(-2a_2+3a_3-2b_2+3b_3, a_2+b_2, a_3+b_3\right)$$

$$=\left(-2\left(a_2+b_2\right)+3\left(a_3+b_3\right), a_2+b_2, a_3+b_3\right) \qquad \text{Let } c_2=a_2+b_2 \quad c_3=a_3+b_3$$

$$=\left(-2c_2+3c_3, c_2, c_3\right) \qquad c_1+2c_2-3c_3=0 \to c_1=-2c_2+3c_3$$

$$=\left(c_1, c_2, c_3\right)$$

*S* is closed under addition

***b)*** $k\vec{u}=k\left(a_1, a_2, a_3\right)$

$$=k\left(4a_2+a_3, a_2, a_3\right)$$

$$=\left(-2ka_2+3ka_3, ka_2, ka_3\right) \qquad \text{Let } c_2=ka_2 \quad c_3=ka_3$$

$$=\left(-2c_2+3c_3, c_2, c_3\right) \qquad c_1=-2c_2+3c_3 \to c_1-2c_2+3c_3=0$$

$$=\left(c_1, c_2, c_3\right)$$

*S* is closed under scalar multiplication.

***c)*** *S* is a subspace of *V*.

## *Exercise*

Let $S=\left\{\left(a_1, a_2, a_3\right)\in\mathbb{R}^3 : a_1+2a_2-3a_3=1\right\}$, Determine:

*a)* Is *S* closed under addition?

*b)* Is *S* closed under scalar multiplication?

*c)* Is *S* a subspace of $\boldsymbol{R}^3$?

***Solution***

$$a_1+2a_2-3a_3=1 \quad \to \quad a_1=1-2a_2+3a_3$$

***a)*** Let $\vec{u}=\left(a_1, a_2, a_3\right)$ *and* $\vec{v}=\left(b_1, b_2, b_3\right)$

$$\vec{u}+\vec{v}=\left(a_1, a_2, a_3\right)+\left(b_1, b_2, b_3\right)$$

$$=\left(1-2a_2+3a_3, a_2, a_3\right)+\left(1-2b_2+3b_3, b_2, b_3\right)$$

$$=\left(1-2a_2+3a_3+1-2b_2+3b_3,\ a_2+b_2,\ a_3+b_3\right)$$

$$=\left(2-2\left(a_2+b_2\right)+3\left(a_3+b_3\right),\ a_2+b_2,\ a_3+b_3\right) \qquad \text{Let } c_2=a_2+b_2 \quad c_3=a_3+b_3$$

$$=\left(2-2c_2+3c_3,\ c_2,\ c_3\right) \qquad c_1+2c_2-3c_3=1 \rightarrow c_1=1-2c_2+3c_3$$

$$\neq\left(1-2c_2+3c_3,\ c_2,\ c_3\right)$$

*S* is *not* closed under addition

***b)*** $\vec{u}=(2,\ 1,\ 1)$

$k\vec{u}=k(2,\ 1,\ 1)$

$$=(2k,\ k,\ k) \qquad a_1+2a_2-3a_3=1 \quad\rightarrow\quad 2k+2k-3k\overset{?}{=}1$$

$$k\neq 1 \quad (\forall k)$$

*S* is *not* closed under scalar multiplication.

***c)*** *S* is ***not*** a subspace of *V*.

## *Exercise*

Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ 5a_1^2-3a_2^2+6a_3^2=0\right\}$, Determine:

*a)* Is *S* closed under addition?

*b)* Is *S* closed under scalar multiplication?

*c)* Is *S* a subspace of $\boldsymbol{R}^3$?

***Solution***

$$5a_1^2-3a_2^2+6a_3^2=0 \quad\rightarrow\quad a_2^2=\frac{5}{3}a_1^2+2a_3^2$$

***a)*** Let $\vec{u}=\left(0,\ \sqrt{2},\ 1\right)$ *and* $\vec{v}=\left(3,\ \sqrt{17},\ 1\right)$

$$\vec{u}+\vec{v}=\left(0,\ \sqrt{2},\ 1\right)+\left(3,\ \sqrt{17},\ 1\right)$$

$$=\left(3,\ \sqrt{2}+\sqrt{17},\ 2\right)$$

$$a_2^2=\frac{5}{3}a_1^2+2a_3^2 \quad\rightarrow\quad \left(\sqrt{2}+\sqrt{17}\right)^2\neq 15+8$$

*S* is *not* closed under addition

***b)*** $k\vec{u}=k\left(a_1,\ a_2,\ a_3\right)$

$$=\left(ka_1,\ ka_2,\ ka_3\right)$$

$5\left(ka_1\right)^2 - 3\left(ka_2\right)^2 + 6\left(ka_3\right)^2 = 0$

$5k^2a_1^2 - 3k^2a_2^2 + 6k^2a_3^2 = 0$

$5a_1^2 - 3a_2^2 + 6a_3^2 = 0$

$S$ is closed under scalar multiplication.

***c)*** $S$ is ***not*** a subspace of $V$.

## *Exercise*

Let $S = \left\{\left(a_1,\ a_2,\ a_3\right) \in \mathbb{R}^3:\ a_3 = a_1 + a_2\right\}$, Determine:

*a)* Is $S$ closed under addition?

*b)* Is $S$ closed under scalar multiplication?

*c)* Is $S$ a subspace of $\boldsymbol{R}^3$?

***Solution***

***a)*** Let $\vec{u} = \left(a_1, a_2, a_3\right)$ *and* $\vec{v} = \left(b_1, b_2, b_3\right)$

$\vec{u} + \vec{v} = \left(a_1, a_2, a_3\right) + \left(b_1, b_2, b_3\right)$

$= \left(a_1, a_2, a_1 + a_2\right) + \left(b_1, b_2, b_1 + b_2\right)$

$= \left(a_1 + b_1, a_2 + b_2, a_1 + a_2 + b_1 + b_2\right)$ Let $c_1 = a_1 + b_1 \quad c_2 = a_2 + b_2$

$= \left(c_1, c_2, c_1 + c_2\right)$ Then, $c_3 = c_1 + c_2$

$= \left(c_1, c_2, c_3\right)$

$S$ is closed under addition

***b)*** $k\vec{u} = k\left(a_1, a_2, a_3\right)$

$= k\left(a_1, a_2, a_1 + a_2\right)$

$= \left(ka_1, ka_2, k\left(a_1 + a_2\right)\right)$

$= \left(ka_1, ka_2, ka_1 + ka_2\right)$ *Where* $c_1 = ka_1,\ c_2 = ka_2,\ c_3 = ka_1 + ka_2$

$= \left(c_1, c_2, c_3\right)$ $c_3 = ka_1 + ka_2 = c_1 + c_2$

$S$ is closed under scalar multiplication.

***c)*** $S$ is a subspace of $V$.

### ***Exercise***

Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ a_1+a_2+a_3=0\right\}$, Determine:

*a)* Is *S* closed under addition?

*b)* Is *S* closed under scalar multiplication?

*c)* Is *S* a subspace of $\boldsymbol{R}^3$?

***Solution***

***a)*** Let $\vec{u}=\left(a_1,\,a_2,\,a_3\right)\ \ni\ \ a_1+a_2+a_3=0$

$\vec{v}=\left(b_1,\,b_2,\,b_3\right)\ \ni\ \ b_1+b_2+b_3=0$

$$\vec{u}+\vec{v}=\left(a_1,\,a_2,\,a_3\right)+\left(b_1,\,b_2,\,b_3\right)$$

$$=\left(a_1+b_1,\,a_2+b_2,\,a_3+b_3\right)$$

Since $a_1+a_2+a_3=0\quad \&\quad b_1+b_2+b_3=0$

Then, $\rightarrow\ \left(a_1+b_1\right)+\left(a_2+b_2\right)+\left(a_3+b_3\right)=0$

*S* is closed under addition

***b)*** $k\vec{u}=k\left(a_1,\,a_2,\,a_3\right)$

$$=\left(ka_1,\,ka_2,\,ka_3\right)$$

$$ka_1+ka_2+ka_3=k\left(a_1+a_2+a_3\right)=k(0)=0$$

*S* is closed under scalar multiplication.

***c)*** *S* is a subspace of *V*.

### ***Exercise***

Let $S=\left\{\left(x_1,\ x_2,\ 1\right):\ x_1\ \textit{and}\ x_2\ \textit{are real numbers}\right\}$, Determine:

*a)* Is *S* closed under addition?

*b)* Is *S* closed under scalar multiplication?

*c)* Is *S* a subspace of $\boldsymbol{R}^3$?

***Solution***

***a)*** Let $\vec{u}=\left(x_1,\,x_2,\,1\right)\quad \&\quad \vec{v}=\left(y_1,\,y_2,\,1\right)$

$$\vec{u}+\vec{v}=\left(x_1,\,x_2,\,1\right)+\left(y_1,\,y_2,\,1\right)$$

$$=\left(x_1+y_1,\, x_2+y_2,\, 2\right) \qquad \text{If we let } z_1=x_1+y_1 \quad z_2=x_2+y_2$$

$$=\left(z_1,\, z_2,\, 2\right)$$

$$\neq\left(z_1,\, z_2,\, 1\right)$$

*S* is ***not*** closed under addition

***b)*** $k\vec{u}=k\left(x_1,\, x_2,\, 1\right)$

$$=\left(kx_1,\, kx_2,\, k\right) \qquad \text{If we let } z_1=kx_1 \quad z_2=kx_2$$

$$\neq\left(z_1,\, z_2,\, 1\right) \qquad k\neq 1 \quad (\forall k)$$

*S* is ***not*** closed under scalar multiplication.

***c)*** *S* is ***not*** a subspace of *V*.

## *Exercise*

Let $S=\left\{\left(x_1,\, x_2,\, x_3\right)\in\mathbb{R}^3:\ x_2=x_1+2x_3\right\}$, Determine:

*a)* Is *S* closed under addition?
*b)* Is *S* closed under scalar multiplication?
*c)* Is *S* a subspace of $\boldsymbol{R}^3$?

***Solution***

***a)*** Let $\vec{u}=\left(x_1,\, x_2,\, x_3\right)$ & $\vec{v}=\left(y_1,\, y_2,\, y_3\right)$

$$\vec{u}+\vec{v}=\left(x_1,\, x_2,\, x_3\right)+\left(y_1,\, y_2,\, y_3\right)$$

$$=\left(x_1+y_1,\, x_2+y_2,\, x_3+y_3\right) \qquad S=\left\{\left(x_1,\, x_2,\, x_3\right)\in\mathbb{R}^3:\ x_2=x_1+2x_3\right\}$$

$$x_2+y_2=x_1+2x_3+y_1+2y_3$$

$$=x_1+y_1+2\left(x_3+y_3\right)$$

*S* is closed under addition

***b)*** $k\vec{u}=k\left(x_1,\, x_2,\, x_3\right)$

$$=\left(kx_1,\, kx_2,\, kx_3\right) \qquad \text{If we let } z_1=kx_1 \quad z_2=kx_2$$

$$kx_2=kx_1+2kx_3$$

$$kx_2=k\left(x_1+2x_3\right)$$

$$x_2=x_1+2x_3$$

$S$ is closed under scalar multiplication.

***c)*** $S$ is a subspace of $V$.

## *Exercise*

Let $S=\left\{\begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \in M_{2\times 2} \mid a,\ c,\ d \in \mathbb{R}\right\}$ and $V=M_{2,2}$, Determine:

*a)* Is $S$ closed under addition?
*b)* Is $S$ closed under scalar multiplication?
*c)* Is $S$ a subspace of $V$ ?

***Solution***

***a)*** Let $A=\begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$ & $B=\begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix}$

$$A+B=\begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}+\begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix}$$

$$=\begin{pmatrix} a_1+a_2 & 2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix}$$ If we let $a=a_1+a_2 \quad c=c_1+c_2 \quad d=d_1+d_2$

$$=\begin{pmatrix} a & 2 \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$$

$S$ is ***not*** closed under addition

***b)*** $$kA=k\begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$$

$$=\begin{pmatrix} ka_1 & k \\ kc_1 & kd_1 \end{pmatrix}$$ If we let $a=ka_1 \quad c=kc_1 \quad d=kd_1$

$$=\begin{pmatrix} a & k \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$$ $k \neq 1 \quad (\forall k)$

$S$ is ***not*** closed under scalar multiplication.

***c)*** $S$ is ***not*** a subspace of $V$.

### *Exercise*

Let $S = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in M_{2\times2} \mid a,\ c,\ d \in \mathbb{R} \right\}$ and $V = M_{2,2}$, Determine:

*a)* Is $S$ closed under addition?

*b)* Is $S$ closed under scalar multiplication?

*c)* Is $S$ a subspace of $V$ ?

***Solution***

***a)*** Let $A = \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$ & $B = \begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix}$

$$A + B = \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + a_2 & 2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

If we let $a = a_1 + a_2 \quad c = c_1 + c_2 \quad d = d_1 + d_2$

$$= \begin{pmatrix} a & 2 \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$$

$S$ is ***not*** closed under addition

***b)*** $kA = k\begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$

$$= \begin{pmatrix} ka_1 & k \\ kc_1 & kd_1 \end{pmatrix}$$

If we let $a = ka_1 \quad c = kc_1 \quad d = kd_1$

$$= \begin{pmatrix} a & k \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$$

$k \neq 1 \quad (\forall k)$

$S$ is ***not*** closed under scalar multiplication.

***c)*** $S$ is ***not*** a subspace of $V$.

### *Exercise*

Let $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in M_{2\times2} \mid a,\ d \in \mathbb{R}\ \ \&\ \ ad \geq 0 \right\}$ and $V = M_{2,2}$, Determine:

*a)* Is $S$ closed under addition?

*b)* Is $S$ closed under scalar multiplication?

*c)* Is $S$ a subspace of $V$ ?

***Solution***

***a)*** Let $A=\begin{pmatrix}1&0\\0&2\end{pmatrix}\ \to 1(2)>0\quad \&\quad B=\begin{pmatrix}-2&0\\0&-1\end{pmatrix}\to(-2)(-1)>0$

$$A+B=\begin{pmatrix}1&0\\0&2\end{pmatrix}+\begin{pmatrix}-2&0\\0&-1\end{pmatrix}$$

$$=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$$

$ad\ge 0 \quad\to\quad (-1)(1)=-1<0$

$S$ is ***not*** closed under addition

***b)*** $kA=k\begin{pmatrix}a&0\\0&d\end{pmatrix}$

$$=\begin{pmatrix}ka&0\\0&kd\end{pmatrix}$$

$(ka)(kd)=k^2(ad)$

Since, $ad\ge 0\quad \&\quad k^2\ge 0$

$k^2(ad)\ge 0$

$S$ is closed under scalar multiplication.

***c)*** $S$ is ***not*** a subspace of $V$.

## *Exercise*

$V=M_{33},\quad S=\{A|\ A\ \textit{is invertible}\}$ where $V$ is a vector space and $S$ is subset of $V$

*a)* Is $S$ closed under addition?

*b)* Is $S$ closed under scalar multiplication?

*c)* Is $S$ a subspace of $V$?

***Solution***

***a)*** Let assume: $A=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ *and* $B=\begin{pmatrix}1&1\\0&-1\end{pmatrix}$ are invertible

But $A+B=\begin{pmatrix}2&1\\0&0\end{pmatrix}$ is not invertible.

$S$ is *not* closed under addition

***b)*** $S$ is *not* closed under scalar multiplication if $k=0$

***c)*** $S$ is ***not*** a subspace of $V$.

## Exercise

Let $S = \left\{ p(t) = a + 2at + 3at^3 \ \mid a \in \mathbb{R} \ \ \& \ \ p(t) \in P_2 \right\}$ and $V = P_2$, Determine:

*a)* Is $S$ closed under addition?
*b)* Is $S$ closed under scalar multiplication?
*c)* Is $S$ a subspace of $V$ ?

**Solution**

**a)** Let $p_1(t) = a + 2at + 3at^3 \quad \& \quad p_2(t) = b + 2bt + 3bt^3$

$$p_1(t) + p_2(t) = a + 2at + 3at^3 + b + 2bt + 3bt^3$$

$$= (a+b) + 2(a+b)t + 3(a+b)t^3 \qquad\qquad \textit{Let } c = a + b \ \in \mathbb{R}$$

$$= c + 2ct + 3ct^3$$

$S$ is closed under addition

**b)** $kp_1(t) = k\left(a + 2at + 3at^3\right)$

$$= ka + 2kat + 3kat^3 \qquad\qquad \textit{Let } c = ka \ \in \mathbb{R}$$

$$= c + 2ct + 3ct^3$$

$S$ is closed under scalar multiplication.

**c)** $S$ is a subspace of $V$.

## Exercise

Given: $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix}$

*a)* Find $NS(A)$
*b)* For which $n$ is $NS(A)$ a subspace of $\mathbb{R}^n$
*c)* Sketch $NS(A)$ in $\mathbb{R}^2$ or $\mathbb{R}^3$

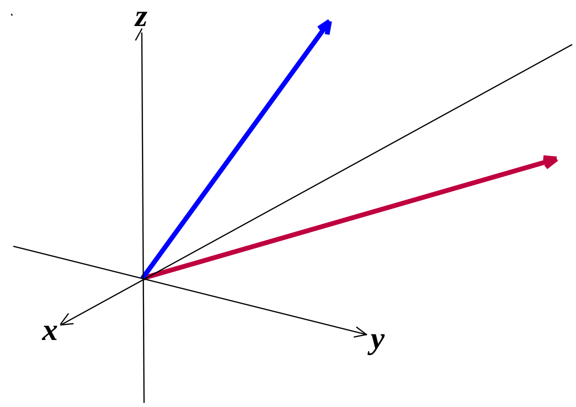
**Solution**

**a)** $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix} \xrightarrow[R_2 - 2R_1]{rref} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad x = -3y - 2z$

$$\left\{ y\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \middle| \, y, z \in \mathbb{R} \right\}$$

**b)** $n = 3$

*c)*




### *Exercise*

Determine which of the following are subspaces of $M_{22}$

*a)* All $2 \times 2$ matrices with integer entries

*b)* All matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a+b+c+d=0$

***Solution***

***a)*** Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are integers

$A+B = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_{3+b_3} & a_4+b_4 \end{bmatrix}$ where $a_1+b_1,\ a_2+b_2,\ a_3+b_3,\ a_4+b_4$ are integers too.

Then, it is closed under addition.

$$\frac{1}{2}A = \frac{1}{2}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix}$$

It is not closed under multiplication if the scalar is a real number.

Therefore; it is ***not*** a subspace of $M_{22}$

***b)*** Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ $a_1+a_2+a_3+a_4=0$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ $b_1+b_2+b_3+b_4=0$

$A+B = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_{3+b_3} & a_4+b_4 \end{bmatrix}$

$$a_1+a_2+a_3+a_4+b_1+b_2+b_3+b_4=0$$

$$\left(a_1+b_1\right)+\left(a_2+b_2\right)+\left(a_3+b_3\right)+\left(a_4+b_4\right)=0$$

Then, it is closed under addition.

$$kA = \begin{bmatrix} ka_1 & ka_2 \\ ka_3 & ka_4 \end{bmatrix} \quad ka_1 + ka_2 + ka_3 + ka_4 = k\left(a_1 + a_2 + a_3 + a_4\right) = k(0) = 0$$

It is closed under multiplication

Therefore; it is a subspace of $M_{22}$

### *Exercise*

Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}: \ ad - bc = 1 \right\}$. Is $V$ a vector space?

***Solution***

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

$$\begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k^2 ad - k^2 bc$$

$$= k^2(ad - bc)$$

$$= k^2 \neq k$$

$\therefore$ $V$ is *not* a vector space

### *Exercise*

Let $V = \left\{ (x, 0, y): \ x \,\&\, y \text{ are arbitrary } \mathbb{R} \right\}$. Define addition and scalar multiplication as follows:

$$\begin{cases} \left(x_1,\ 0,\ y_1\right) + \left(x_2,\ 0,\ y_2\right) = \left(x_1 + x_2,\ \ y_1 + y_2\right) \\ \quad c(x,\ 0,\ y) = (cx,\ \ cy) \end{cases}$$

Is $V$ a vector space?

***Solution***

Let $\vec{V}_1\left(x_1,\ 0,\ y_1\right)$ & $\vec{V}_2\left(x_2,\ 0,\ y_2\right)$

$$\vec{V}_1 + \vec{V}_2 = \left(x_1,\ 0,\ y_1\right) + \left(x_2,\ 0,\ y_2\right)$$

$$= \left(x_1 + x_2,\ \ y_1 + y_2\right)$$

$$\neq \left(x_1 + x_2,\ \ 0,\ \ y_1 + y_2\right) = \vec{V}_1 + \vec{V}_2$$

$\therefore$ $V$ is *not* a vector space

## Exercise

Construct a matrix whose column space contains $(1,\ 1,\ 0)$, $(0,\ 1,\ 1)$, and whose nullspace contains $(1,\ 0,\ 1)$ and $(0,\ 0,\ 1)$

**Solution**

It is *not* possible.
Since a matrix ($A$) must be $3\times 3$.
Since the nullspace contains 2 independent vectors, then $A$ can have at most $3-2=1$ pivot.
But the column space contains 2 independent vectors, $A$ must have at least 2 pivots.
These 2 conditions can't both be met.

## Exercise

How is the nullspace $N(C)$ related to the spaces $N(A)$ and $N(B)$, is $C=\begin{bmatrix} A \\ B \end{bmatrix}$ ?

**Solution**

$N(C)=N(A)\cap N(B)$

$Cx=\begin{bmatrix} Ax \\ Bx \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

*Iff* $Ax=0$ & $Bx=0$

## Exercise

True or False (check addition or give a counterexample)

*a)* If $V$ is a vector space and $W$ is a subset of $V$ that is a vector space, then $W$ is a subspace of $V$.
*b)* The empty set is a subspace of every vector space.
*c)* If $V$ is a vector space other than the zero vector space, then $V$ contains a subspace $W$ such that $W\neq V$.
*d)* The intersection of any two subsets of $V$ is a subspace of $V$.
*e)* Let $W$ be the $xy$-plane in $\boldsymbol{R}^3$; that is, $W=\left\{\left(a_1,\ a_2,\ 0\right):\ a_1,\ a_2\in \boldsymbol{R}\right\}$. Then $W=\boldsymbol{R}^2$

**Solution**

***a)*** False
$W$ is a subset of $V$, but not necessary that the scalar of a vector in $W$ is in $V$.
Therefore, $W$ is *not* a subspace of $V$

***b)*** False

***c)*** True

***d)*** False

***e)*** False

# Solution    *Section* 2.6 – Linear Independence

### Exercise

State the following statements as true or false

*a)* If $S$ is a linearly dependent set, then each vector in $S$ is a linear combination of other vectors in $S$.

*b)* Any set containing the zero vector is linearly dependent.

*c)* The empty set is linearly dependent.

*d)* Subsets of linearly dependent sets are linearly dependent.

*e)* Subsets of linearly independent sets are linearly independent.

*f)* If $a_1x_1 + a_2x_2 + \ \ldots + a_nx_n = \vec{0}$ and $x_1, x_2, \ldots, x_n$ are linearly independent, the nall the scalars $a_i$ are zero

**Solution**

***a)*** False

***b)*** True

***c)*** False

***d)*** False

***e)*** True

***f)*** True

### Exercise

Given three independent vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3$. Take combinations of those vectors to produce $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Write the combinations in a matrix form as $V = WM$.

$$\begin{aligned} v_1 &= w_1 + \ w_2 \\ v_2 &= w_1 + 2w_2 + w_3 \\ v_1 &= \qquad\ w_2 + cw_3 \end{aligned} \quad \text{which is } \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix}$$

What is the test on a matrix **V** to see if its columns are linearly independent?

If $c \neq 1$ show that $v_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

If $c = 1$ show that $v's$ are linearly *dependent*.

**Solution**

The nullspace of **V** must contain only the *zero* vector. Then $x = (0,\ 0,\ 0)$ is the only combination of the columns that gives **V**$x$ = zero vector.

$$M=\begin{bmatrix}1&1&0\\1&2&1\\0&1&c\end{bmatrix}\xrightarrow{R_2-R_1}\begin{bmatrix}1&1&0\\0&1&1\\0&1&c\end{bmatrix}\xrightarrow[R_3-R_2]{R_1-R_2}\begin{bmatrix}1&0&-1\\0&1&1\\0&0&\boxed{c-1}\end{bmatrix}$$

If $c \neq 1$, then the matrix $M$ is invertible. So if $x$ is any nonzero vector we know that $Mx$ is nonzero. Since $\boldsymbol{w}$'s are given as independent and $WMx$ is nonzero. Since $V=WM$ , this says that $x$ is not in the nullspace of **V**. therefore; $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent.

If $c=1$, that implies $\begin{cases} v_1 = w_1 + w_2 \\ v_2 = w_1 + w_2 + w_2 + w_3 \\ v_3 = w_2 + w_3 \end{cases} \Rightarrow \begin{matrix} v_1 = w_1 + w_2 \\ \boxed{v_2 = v_1 + v_3} \\ v_3 = w_2 + w_3 \end{matrix}$

$-v_1 + v_2 - v_3 = 0$, which means that $v's$ are linearly *dependent*.

The other way, the vector $x=(1,\ -1,\ 1)$ is in that nullspace, and $Mx=0$. Then certainly $WMx=0$ which is the same as $Vx=0$. So the $v's$ are dependent.

## *Exercise*

Find the largest possible number of independent vectors among

$$\vec{v}_1=\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}\quad \vec{v}_2=\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}\quad \vec{v}_3=\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\quad \vec{v}_4=\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\quad \vec{v}_5=\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\quad \vec{v}_6=\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$$

***Solution***

Since $\vec{v}_4=\vec{v}_2-\vec{v}_1,\ \vec{v}_5=\vec{v}_3-\vec{v}_1,\ and\ \vec{v}_6=\vec{v}_3-\vec{v}_2$, there are at most three independent vectors among these: furthermore, applying row reduction to the matrix $\begin{bmatrix}v_1 & v_2 & v_3\end{bmatrix}$ gives three pivots, showing that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent.

## *Exercise*

Show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent but $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are dependent:

$$\vec{v}_1=\begin{pmatrix}1\\0\\0\end{pmatrix}\quad \vec{v}_2=\begin{pmatrix}1\\1\\0\end{pmatrix}\quad \vec{v}_3=\begin{pmatrix}1\\1\\1\end{pmatrix}\quad \vec{v}_4=\begin{pmatrix}2\\3\\4\end{pmatrix}$$

Solve either $c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3=0$ *or* $Ax=0$. The $v$'s go in the columns of $\boldsymbol{A}$.

***Solution***

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

This matrix has 3 pivots with rank of 3 equals to rows that implies the $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent.

$\vec{v}_4 = \vec{v}_1 + \vec{v}_2 - 4\vec{v}_3$ *or* $\vec{v}_1 + \vec{v}_2 - 4\vec{v}_3 - \vec{v}_4 = 0$ that shows that $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are dependent.

## *Exercise*

Decide the dependence or independence of

*a)* The vectors (1, 3, 2), (2, 1, 3), and (3, 2, 1).

*b)* The vectors $(1, -3, 2)$, $(2, 1, -3)$, and $(-3, 2, 1)$.

***Solution***

***a)*** These are linearly independent. $x_1(1, 3, 2) + x_2(2, 1, 3) + x_3(3, 2, 1) = (0, 0, 0)$ only if $x_1 = x_2 = x_3 = 0$

***b)*** These are linearly dependent: $1(1, -3, 2) + 1(2, 1, -3) + 1(-3, 2, 1) = (0, 0, 0)$

## *Exercise*

Find two independent vectors on the plane $x + 2y - 3z - t = 0$ in $\boldsymbol{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

***Solution***

This plane is the nullspace of the matrix $A = \begin{pmatrix} 1 & 2 & -3 & -1 \end{pmatrix}$

$x_1 + 2x_2 - 3x_3 - x_4 = 0$

The pivot is 1st column, and the rest are 3 variables.

If $x_2 = -1 \quad x_3 = x_4 = 0 \Rightarrow x_1 = 2$. The vector is $(2, -1, 0, 0)$

If $x_3 = 1 \quad x_1 = x_4 = 0 \Rightarrow x_1 = 3$. The vector is $(3, 0, 1, 0)$

If $x_4 = 1 \quad x_1 = x_3 = 0 \Rightarrow x_1 = 1$. The vector is $(1, 0, 0, 1)$

The 3 vectors $(2, -1, 0, 0)$, $(3, 0, 1, 0)$, $(1, 0, 0, 1)$ are linearly independent.

We can't find 4 independent vectors because the nullspace only has dimension 3 (have 3 variables).

## *Exercise*

Determine whether the vectors are linearly dependent or linearly independent in $\boldsymbol{R}^3$

*a)* (4, –1, 2), (–4, 10, 2)

*b)* (8, –1, 3), (4, 0, 1)

*c)* (–3, 0, 4), (5, –1, 2), (1, 1, 3)

*d)* (–2, 0, 1), (3, 2, 5), (6, –1, 1), (7, 0, –2)

***Solution***

***a)*** The vector equation $a(4, -1, 2) + b(-4, 10, 2) = (0, 0, 0)$

$$\left[\begin{array}{cc|c} 4 & -4 & 0 \\ -1 & 10 & 0 \\ 2 & 2 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system has only the trivial solution $a = b = 0$.
We conclude that the given set of vectors is linearly independent.

***b)*** $a(8, -1, 3) + b(4, 0, 1) = (0, 0, 0)$

$$\left[\begin{array}{cc|c} 8 & 4 & 0 \\ -1 & 0 & 0 \\ 3 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, the system has only one trivial solution $a = b = 0$.
We conclude that the given set of vectors is linearly independent

***c)*** The vector equation $a(-3, 0, 4) + b(5, -1, 2) + c(1, 1, 3) = (0, 0, 0)$

$$\left[\begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore the system has only the trivial solution $a = b = c = 0$.
We conclude that the given set of vectors is linearly independent.

***d)*** The vector equation $a(-2, 0, 1) + b(3, 2, 5) + c(6, -1, 1) + d(7, 0, -2) = (0, 0, 0)$

$$\left[\begin{array}{cccc|c} -2 & 3 & 6 & 7 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 1 & 5 & 1 & -2 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{79}{29} & 0 \\ 0 & 1 & 0 & \frac{3}{29} & 0 \\ 0 & 0 & 1 & \frac{6}{29} & 0 \end{array}\right]$$

Therefore, the system has nontrivial solutions $a = \frac{79}{29}t, \quad b = -\frac{3}{29}t, \quad c = -\frac{6}{29}t, \quad d = t$

We conclude that the given set of vectors is linearly dependent.

### *Exercise*

Determine whether the vectors are linearly dependent or linearly independent in $\boldsymbol{R}^4$

*a)* $\{(3,\ 8,\ 7,\ -3),\ (1,\ 5,\ 3,\ -1),\ (2,\ -1,\ 2,\ 6),\ (1,\ 4,\ 0,\ 3)\}$

*b)* $\{(0,\ 0,\ 2,\ 2),\ (3,\ 3,\ 0,\ 0),\ (1,\ 1,\ 0,-1)\}$

*c)* $\{(0,\ 3,\ -3,\ -6),\ (-2,\ 0,\ 0,\ -6),\ (0,\ -4,\ -2,\ -2),\ (0,\ -8,\ 4,\ -4)\}$

*d)* $\{(3,\ 0,\ -3,\ 6),\ (0,\ 2,\ 3,\ 1),\ (0,\ -2,\ -2,\ 0),\ (-2,\ 1,\ 2,\ 1)\}$

*e)* $\{(1,\ 3,\ -4,\ 2),\ (2,\ 2,\ -4,\ 0),\ (2,\ 3,\ 2,\ -4),\ (-1,\ 0,\ 1,\ 0)\}$

*f)* $\{(1,\ 3,\ -4,\ 2),\ (2,\ 2,\ -4,\ 0),\ (1,\ -3,\ 2,\ -4),\ (-1,\ 0,\ 1,\ 0)\}$

*g)* $\{(1,\ 0,\ 0,\ -1),\ (0,\ 1,\ 0,\ -1),\ (0,\ 1,\ 0,\ -1),\ (0,\ 0,\ 0,\ 1)\}$

***Solution***

***a)*** $\det\begin{pmatrix} 3 & 1 & 2 & 1 \\ 8 & 5 & -1 & 4 \\ 7 & 3 & 2 & 0 \\ -3 & -1 & 6 & 3 \end{pmatrix} = 128 \neq 0$

The system has only the trivial solution and the vectors are linearly independent.

***b)*** $k_1(0,0,2,2)+k_2(3,3,0,0)+k_3(1,1,0,-1)=(0,0,0,0)$

$$\left[\begin{array}{ccc|c} 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_1 = k_2 = k_3 = 0$

The system has only the trivial solution and the vectors are linearly independent.

***c)*** $\det\begin{pmatrix} 0 & -2 & 0 & 0 \\ 3 & 0 & -4 & -8 \\ -3 & 0 & -2 & 4 \\ -6 & -6 & -2 & -4 \end{pmatrix} = 480 \neq 0$

The system has only the trivial solution and the vectors are linearly independent.

***d)*** $a(3,\ 0,\ -3,\ 6)+b(0,\ 2,\ 3,\ 1)+c(0,\ -2,\ -2,\ 0)+d(-2,\ 1,\ 2,\ 1)=(0,\ 0,\ 0,\ 0)$

$$\left[\begin{array}{cccc|c} 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & 1 & 0 \\ -3 & 3 & -2 & 2 & 0 \\ 6 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, the system has only one trivial solution $a = b = c = d = 0$.
The given set of vectors is linearly independent

***e)*** $\{(1,\ 3,\ -4,\ 2),\ (2,\ 2,\ -4,\ 0),\ (2,\ 3,\ 2,\ -4),\ (-1,\ 0,\ 1,\ 0)\}$

$$\begin{vmatrix} 1 & 2 & 2 & -1 \\ 3 & 2 & 3 & 0 \\ -4 & -4 & 2 & 1 \\ 2 & 0 & -4 & 0 \end{vmatrix} \underline{= 28 \neq 0 |}$$

The system has only the trivial solution and the vectors are linearly independent.

***f)*** $\{(1,\ 3,\ -4,\ 2),\ (2,\ 2,\ -4,\ 0),\ (1,\ -3,\ 2,\ -4),\ (-1,\ 0,\ 1,\ 0)\}$

$$\begin{vmatrix} 1 & 2 & 1 & -1 \\ 3 & 2 & -3 & 0 \\ -4 & -4 & 2 & 1 \\ 2 & 0 & -4 & 0 \end{vmatrix} \underline{= 0 |}$$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & 2 & -3 & 0 \\ -4 & -4 & 2 & 1 \\ 2 & 0 & -4 & 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underline{x_4 = 0, \quad x_2 = -\frac{3}{2}x_3, \quad x_1 = 2x_3 |}$$

***g)*** $\{(1,\ 0,\ 0,\ -1),\ (0,\ 1,\ 0,\ -1),\ (0,\ 1,\ 0,\ -1),\ (0,\ 0,\ 0,\ 1)\}$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 \end{vmatrix} \underline{= 0 |}$$

$\therefore$ The set is linearly independent.

## *Exercise*

*a* ) Show that the three vectors $\vec{v}_1 = (1,\ 2,\ 3,\ 4)$ $\vec{v}_2 = (0,\ 1,\ 0,\ -1)$ $\vec{v}_3 = (1,\ 3,\ 3,\ 3)$ form a linearly dependent set in $\boldsymbol{R}^4$.

*b*) Express each vector in part (*a*) as a linear combination of the other two.

***Solution***

***a)*** The vector equation $k_1(1,\ 2,\ 3,\ 4) + k_2 = (0,\ 1,\ 0,\ -1) + k_3(1,\ 3,\ 3,\ 3) = (0,\ 0,\ 0,\ 0)$

$$\begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 2 & 1 & 3 & | & 0 \\ 3 & 0 & 3 & | & 0 \\ 4 & -1 & 3 & | & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The solution: $k_1 = -t,\ k_2 = -t,\ k_3 = t$

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

***b)*** Since $k_1 = -t,\ k_2 = -t,\ k_3 = t$ and if we let $t = 1$, then $-\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = 0$

$$\vec{v}_1 = -\vec{v}_2 + \vec{v}_3, \quad \vec{v}_2 = -\vec{v}_1 + \vec{v}_3, \quad \vec{v}_3 = \vec{v}_1 + \vec{v}_2$$

## *Exercise*

For which real values of λ do the following vectors form a linearly dependent set in $\boldsymbol{R}^3$

$$\vec{v}_1 = \left(\lambda,\ -\tfrac{1}{2},\ -\tfrac{1}{2}\right) \quad \vec{v}_2 = \left(-\tfrac{1}{2},\ \lambda,\ -\tfrac{1}{2}\right) \quad \vec{v}_3 = \left(-\tfrac{1}{2},\ -\tfrac{1}{2},\ \lambda\right)$$

***Solution***

$$k_1\left(\lambda,\ -\tfrac{1}{2},\ -\tfrac{1}{2}\right) + k_2\left(-\tfrac{1}{2},\ \lambda,\ -\tfrac{1}{2}\right) + k_3\left(-\tfrac{1}{2},\ -\tfrac{1}{2},\ \lambda\right) = (0,\ 0,\ 0,\ 0)$$

$$\det\begin{pmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{pmatrix} = \frac{1}{4}\left(4\lambda^3 - 3\lambda - 1\right)$$

For $\lambda = 1 \quad \lambda = -\frac{1}{2}$, the determinant is zero and the vectors form a linearly dependent set.

## *Exercise*

Show that if $S = \left\{\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n\right\}$ is a linearly independent set of vectors, then so is every nonempty subset of S.

***Solution***

Let $\left\{\vec{v}_a,\ \vec{v}_b,\ \ldots,\ \vec{v}_r\right\}$ be a nonempty subset of $S$.

If this set is linearly dependent, then there would be a nonzero solution $\left(k_a,\ k_b,\ \ldots,\ k_r\right)$ to $k_a\vec{v}_a + k_b\vec{v}_b + \ \ldots + k_r\ \vec{v}_r = 0$. This can be expanded to a nonzero solution of

$k_1\vec{v}_1 + k_2\vec{v}_2 + \ \dots + k_n\ \vec{v}_n = 0$ by taking all other coefficients as 0. This contradicts the linear independence of *S*, so the subset must be linearly independent.

### Exercise

Show that if $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ is a linearly dependent set of vectors in a vector space *V*, and if $\vec{v}_{r+1}, \dots, \vec{v}_n$ are vectors in *V* that are not in *S*, then $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r, \vec{v}_{r+1}, \dots, \vec{v}_n\}$ is also linearly dependent.

**Solution**

If *S* is linearly dependent, then there is a nonzero solution $(k_1, k_2, \dots, k_r)$ to $k_1\vec{v}_1 + k_2\vec{v}_2 + \ \dots + k_r\vec{v}_r = 0$. Thus $(k_1, k_2, \dots, k_r, 0, 0, \dots, 0)$ is a nonzero solution to $k_1\vec{v}_1 + k_2\vec{v}_2 + \ \dots + k_r\ \vec{v}_r + k_{r+1}\ \vec{v}_{r+1}\ \dots + k_n\ \vec{v}_n = 0$ so the set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r, \vec{v}_{r+1}, \dots, \vec{v}_n\}$ is linearly dependent.

### Exercise

Show that $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent and $\vec{v}_3$ does not lie in span $\{\vec{v}_1, \vec{v}_2\}$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent.

**Solution**

If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly dependent, there exist a nonzero solution to $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = 0$ with $k_3 \neq 0$ (since $\vec{v}_1$ and $\vec{v}_2$ are linearly independent).

$k_3\vec{v}_3 = -k_1\vec{v}_1 - k_2\vec{v}_2 \quad \Rightarrow \quad \vec{v}_3 = -\frac{k_1}{k_3}\vec{v}_1 - \frac{k_2}{k_3}\vec{v}_2$ which contradicts that $\vec{v}_3$ is not in span $\{\vec{v}_1, \vec{v}_2\}$. Thus $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent.

### Exercise

By using the appropriate identities, where required, determine $F(-\infty, \infty)$ are linearly dependent.

*a)* $6,\ 3\sin^2 x,\ 2\cos^2 x$

*b)* $x,\ \cos x$

*c)* $1,\ \sin x,\ \sin 2x$

*d)* $(3-x)^2,\ x^2 - 6x,\ 5$

*e)* $\cos 2x,\ \sin^2 x,\ \cos^2 x$

**Solution**

***a)*** From the identity $\sin^2 x + \cos^2 x = 1$

$$(-1)(6)+(2)\left(3\sin^2 x\right)+(3)\left(2\cos^2 x\right)=-6+6\left(\sin^2 x+\cos^2 x\right) \underline{=0}$$

Therefore, the set is linearly dependent.

***b)*** $ax+b\cos x=0$

$x=0 \quad \Rightarrow \quad b=0$

$x=\frac{\pi}{2} \quad \Rightarrow \quad a=0$

Therefore, the set is linearly independent.

***c)*** $a(1)+b\sin x+c\sin 2x=0$

$x=0 \quad \Rightarrow \quad a=0$

$x=\frac{\pi}{2} \quad \Rightarrow \quad b=0$

$x=\frac{\pi}{4} \quad \Rightarrow \quad c=0$

Therefore, the set is linearly independent.

***d)*** $(3-x)^2=9-6x+x^2$

$(3-x)^2-\left(9-6x+x^2\right)=0$

$(3-x)^2-\left(x^2-6x\right)-9=0$

$(1)(3-x)^2+(-1)\left(x^2-6x\right)+\left(-\frac{9}{5}\right)5=0$

Therefore, the set is linearly dependent.

***e)*** By using the double angle:

$\cos 2x=\cos^2 x-\sin^2 x$ are linearly dependent.

## *Exercise*

$f_1(x)=\sin x,\quad f_2(x)=\cos x$ are linearly independent in $F(-\infty,\ \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using Wroński's test.

***Solution***

The Wronskian: $W(x)=\begin{vmatrix}\sin x & \cos x\\ \cos x & -\sin x\end{vmatrix}$

$$=-\sin^2 x-\cos^2 x$$

$$=-\left(\sin^2 x+\cos^2 x\right)$$

$$\underline{=-1\neq 0}$$

$\sin x \quad$ *and* $\quad \cos x$ are linearly independent

## Exercise

Show $f_1(x)=e^x, \quad f_2(x)=xe^x \quad f_3(x)=x^2e^x$ are linearly independent in $F(-\infty, \infty)$

***Solution***

$$W=\begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & e^x+xe^x & 2xe^x+x^2e^x \\ e^x & 2e^x+xe^x & 2e^x+4xe^x+x^2e^x \end{vmatrix} \qquad \begin{matrix} e^x \\ factor \;\; e^x \\ e^x \end{matrix}$$

$$=e^{3x}\begin{vmatrix} 1 & x & x^2 \\ 1 & 1+x & 2x+x^2 \\ 1 & 2+x & 2+4x+x^2 \end{vmatrix}$$

$$=e^{3x}\left[(1+x)\left(2+4x+x^2\right)+2x^2+x^3+2x^2+x^3-x^2-x^3-\left(2x+x^2\right)(2+x)-2x-4x^2-x^3\right]$$

$$=e^{3x}\left[2+4x+x^2+2x+4x^2+x^3-4x-2x^2-2x^2-x^3-2x-x^2\right]$$

$$=2e^{3x}\neq 0$$

$\left\{e^x, \; xe^x, \; x^2e^x\right\}$ are linearly independent

## Exercise

Use the Wronskian to show that $f_1(x)=\sin x, \quad f_2(x)=\cos x, \quad f_3(x)=x\cos x$ span a three-dimensional subspace of $F(-\infty, \infty)$

***Solution***

The Wronskian: $W(x)=\begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x - x\sin x \\ -\sin x & -\cos x & -2\sin x - x\cos x \end{vmatrix}$

$$=2\sin^3 x+x\sin^2 x\cos x-\sin x\cos^2 x+x\sin^2 x\cos x-x\cos^3 x$$
$$-x\sin^2 x\cos x+\sin x\cos^2 x-x\sin^2 x\cos x+2\sin x\cos^2 x+x\cos^3 x$$

$$=2\sin^3 x+2\sin x\cos^2 x$$

$$=2\sin x\left(\sin^2 x+\cos^2 x\right)$$

$$=2\sin x$$

Since $\sin x\neq 0$ for all real $x$ values, the vectors are linearly independent.

### *Exercise*

Show by inspection that the vectors are linearly dependent.

$$\boldsymbol{v}_1(4,\ -1,\ 3),\quad \boldsymbol{v}_2(2,\ 3,\ -1),\quad \boldsymbol{v}_3(-1,\ 2,\ -1),\quad \boldsymbol{v}_4(5,\ 2,\ 3),\quad in\ \mathbb{R}^3$$

***Solution***

$$\begin{bmatrix} 4 & 2 & -1 & 5 \\ -1 & 3 & 2 & 2 \\ 3 & -1 & -1 & 3 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & \frac{11}{7} \\ 0 & 1 & 0 & \frac{1}{7} \\ 0 & 0 & 1 & \frac{11}{7} \end{bmatrix}$$

$$7\boldsymbol{v}_4 = 11\boldsymbol{v}_1 + \boldsymbol{v}_2 + 11\boldsymbol{v}_2$$

### *Exercise*

Determine if the given vectors are linearly dependent or independent, (any method)

$$(2,\ -1,\ 3),\quad (3,\ 4,\ 1),\quad (2,\ -3,\ 4),\quad in\ \mathbb{R}^3$$

***Solution***

$$a(2,\ -1,\ 3) + b(3,\ 4,\ 1) + c(2,\ -3,\ 4) = (0,\ 0,\ 0)$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 0 \\ -1 & 4 & -3 & 0 \\ 3 & 1 & 4 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The system has only he trivial solution $a = b = c = 0$.

$$\begin{vmatrix} 2 & 3 & 2 \\ -1 & 4 & -3 \\ 3 & 1 & 4 \end{vmatrix} = 32 - 27 - 2 - 24 + 6 + 12 \neq 0$$

The system has only the trivial solution and the vectors are linearly independent

### *Exercise*

Determine if the given vectors are linearly dependent or independent, (any method)

$$(1,\ 0,\ 0,\ 0),\quad (1,\ 1,\ 0,\ 0),\quad (1,\ 1,\ 1,\ 0),\quad (1,\ 1,\ 1,\ 1),\quad in\ \mathbb{R}^4$$

***Solution***

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

The system has only the trivial solution and the vectors are linearly independent

## Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$A_1\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix},\quad A_2\begin{bmatrix}-2 & 4\\ 1 & 0\end{bmatrix},\quad A_3\begin{bmatrix}3 & -1\\ 2 & 0\end{bmatrix},\quad in\ M_{22}$$

**Solution**

$$\begin{bmatrix}1 & -2 & 3 & 0\\ 2 & 4 & -1 & 0\\ 3 & 1 & 2 & 0\end{bmatrix}\xrightarrow{rref}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$$

The vectors are linearly independent

## Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{\begin{pmatrix}1 & -3 & 2\\ -4 & 0 & 5\end{pmatrix}, \begin{pmatrix}-3 & 7 & 4\\ 6 & -2 & -7\end{pmatrix}, \begin{pmatrix}-2 & 3 & 11\\ -1 & -3 & 2\end{pmatrix}\right\} \text{ in } M_{2\times 3}(\mathbb{R})$$

**Solution**

$$\left[\begin{array}{ccc|c}1 & -3 & -2 & 0\\ -3 & 7 & 3 & 0\\ 2 & 4 & 11 & 0\\ -4 & 6 & -1 & 0\\ 0 & -2 & -3 & 0\\ 5 & -7 & 2 & 0\end{array}\right]\begin{array}{l} \\ R_2+3R_1\\ R_3-2R_1\\ R_4+4R_1\\ \\ R_6-5R_1\end{array}$$

$$\left[\begin{array}{ccc|c}1 & -3 & -2 & 0\\ 0 & -2 & -3 & 0\\ 0 & 10 & 15 & 0\\ 0 & -6 & -9 & 0\\ 0 & -2 & -3 & 0\\ 0 & 8 & 12 & 0\end{array}\right]\begin{array}{l} \\ \\ R_3+5R_2\\ R_4-3R_2\\ R_5-R_2\\ R_6+4R_2\end{array}\quad \left[\begin{array}{ccc|c}1 & -3 & -2 & 0\\ 0 & -2 & -3 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]\begin{array}{l}\to a_1=3a_2+2a_3\\ \to 2a_2=-3a_3\\ \\ \\ \\ \\ \end{array}$$

$$\underline{a_2=-\tfrac{3}{2}a_3}\qquad a_1=-\tfrac{9}{2}a_3+2a_3\ \underline{=-\tfrac{5}{2}a_3}$$

It is linearly dependent.

$if\ a_3=-2\quad a_2=3\quad a_1=5$

$$5\begin{pmatrix}1 & -3 & 2\\ -4 & 0 & 5\end{pmatrix}+3\begin{pmatrix}-3 & 7 & 4\\ 6 & -2 & -7\end{pmatrix}-2\begin{pmatrix}-2 & 3 & 11\\ -1 & -3 & 2\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

### *Exercise*

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix}\right\} \text{ in } M_{2\times 2}(\mathbb{R})$$

***Solution***

$$\begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} = -2\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}$$

$$2\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\therefore$ Linearly dependent

### *Exercise*

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}\right\} \text{ in } M_{2\times 2}(\mathbb{R})$$

***Solution***

$$\begin{bmatrix} 1 & -1 \\ -2 & 1 \\ -1 & 2 \\ 4 & -4 \end{bmatrix} \begin{matrix} \\ R_2 + 2R_1 \\ R_3 + R_1 \\ R_4 - 4R_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$a_2 = 0 \quad \rightarrow a_1 = a_2 = 0$

$\therefore$ Linearly independent

### *Exercise*

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix}\right\} \quad in \quad M_{2\times 2}(\mathbb{R})$$

***Solution***

$$\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ -2 & 1 & 1 & -4 \\ 1 & 1 & 0 & 4 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 1 \\ 1 & 1 & -4 \\ 1 & 0 & 4 \end{vmatrix} - \begin{vmatrix} 0 & -1 & 1 \\ -2 & 1 & -4 \\ 1 & 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 0 & -1 & 2 \\ -2 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= -21 + 7 + 14$$

$= 0$

$\therefore$ Linearly dependent

### *Exercise*

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & -2 \end{pmatrix} \right\} \quad in \quad M_{2\times 2}(\mathbb{R})$$

***Solution***

$$W = \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & 1 \\ -2 & 1 & 1 & 2 \\ 1 & 1 & 0 & -2 \end{vmatrix} = 24 \neq 0$$

$\therefore$ Linearly independent

### *Exercise*

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ e^x, \ \ln x \right\} \quad in \quad \mathbb{R}$$

***Solution***

$$W = \begin{vmatrix} e^x & \ln x \\ e^x & \frac{1}{x} \end{vmatrix}$$

$$= e^x \left( \frac{1}{x} - \ln x \right) \neq 0$$

$\therefore$ Linearly independent

### *Exercise*

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{ x, \ \frac{1}{x} \right\} \quad in \quad \mathbb{R}$$

***Solution***

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

$$= -\frac{1}{x} - \frac{1}{x}$$

$$\boxed{= -\frac{2}{x} \neq 0}$$

$\therefore$ Linearly independent

## *Exercise*

Determine if the given vectors are linearly dependent or independent, (any method)

$\{1+x, \ 1-x\} \quad in \quad P_2(\mathbb{R})$

***Solution***

$$W = \begin{vmatrix} 1+x & 1-x \\ 1 & -1 \end{vmatrix}$$

$$= -1 - x - 1 + x$$

$$\boxed{= -2 \neq 0}$$

$\therefore$ Linearly independent

## *Exercise*

Determine if the given vectors are linearly dependent or independent, (any method)

$\{9x^2 - x + 3, \ 3x^2 - 6x + 5, \ -5x^2 + x + 1\} \quad in \quad P_3(\mathbb{R})$

***Solution***

$$W = \begin{vmatrix} 9 & -1 & 3 \\ 3 & -6 & 5 \\ -5 & 1 & 1 \end{vmatrix} \ \boxed{= -152 \neq 0}$$

$\therefore$ Linearly independent

$$W = \begin{vmatrix} 9x^2 - x + 3 & 3x^2 - 6x + 5 & -5x^2 + x + 1 \\ 18x - 1 & 6x - 6 & -10x + 1 \\ 18 & 6 & -10 \end{vmatrix}$$

### *Exercise*

Determine if the given vectors are linearly dependent or independent, (any method)

$\left\{-x^2,\ 1+4x^2\right\}$ *in* $P_3(\mathbb{R})$

***Solution***

$$W = \begin{vmatrix} -x^2 & 4x^2+1 \\ -2x & 8x \end{vmatrix} \boxed{= 2x \neq 0}$$

$\therefore$ Linearly independent

### *Exercise*

Determine if the given vectors are linearly dependent or independent, (any method)

$\left\{7x^2+x+2,\ 2x^2-x+3,\ -3x^2+4\right\}$ *in* $P_3(\mathbb{R})$

***Solution***

$$W = \begin{vmatrix} 7 & 1 & 2 \\ 2 & -1 & 3 \\ -3 & 0 & 4 \end{vmatrix} \boxed{= -51 \neq 0}$$

$\therefore$ Linearly independent

### *Exercise*

Determine if the given vectors are linearly dependent or independent, (any method)

$\left\{3x^2+3x+8,\ 2x^2+x,\ 2x^2+2x+2,\ 5x^2-2x+8\right\}$ *in* $P_3(\mathbb{R})$

***Solution***

$$\begin{bmatrix} 3 & 2 & 2 & 5 \\ 3 & 1 & 2 & -2 \\ 8 & 0 & 2 & 8 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ 3R_3 - 8R_1 \end{matrix} \rightarrow \begin{bmatrix} 3 & 2 & 2 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & -16 & -10 & -16 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 16R_2 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 3 & 2 & 2 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & -10 & 96 \end{bmatrix}$$

$$3a_1 = -2a_2 - 2a_3 - 5a_4$$

$$\boxed{a_2 = -7a_4}$$

$$\boxed{a_3 = \frac{48}{5}a_4}$$

$$3a_1 = 14a_4 - \frac{96}{5}a_4 - 5a_4$$

$$\boxed{a_1 = \frac{14}{3}a_4 - \frac{32}{5}a_4 - \frac{5}{3}a_4}$$

$\therefore$ Linearly dependent

## *Exercise*

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{x^3 + 2x^2, \quad -x^2 + 3x + 1, \quad x^3 - x^2 + 2x - 1\right\} \quad in \quad P_3(\mathbb{R})$$

**Solution**

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & -1 \\ 0 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -3 \\ 0 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 3R_2 \\ R_4 + R_2 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & -7 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\begin{cases} a_3 = 0 \\ a_1 = -3a_3 = 0 \\ a_1 = -a_3 = 0 \end{cases}$$

$\therefore$ Linearly independent

## *Exercise*

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{x^3 - x, \quad 2x^2 4, \quad -2x^3 + 3x^2 + 2x + 6\right\} \quad in \quad P_3(\mathbb{R})$$

**Solution**

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & 4 & 6 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_1 \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 4 & 6 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - 2R_2 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} 2a_2 = -3a_3 \\ a_1 = 2a_3 \end{cases}$$

If $a_3 = 2 \quad \rightarrow \quad a_1 = 4 \quad a_2 = -3$

$$\rightarrow \quad 4\left(x^3 - x\right) - 3\left(2x^2 + 4\right) + 2\left(-2x^3 + 3x^2 + 2x + 6\right) = 0$$

$\therefore$ Linearly dependent

## Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{\begin{array}{c} x^4-x^3+5x^2-8x+6, \quad -x^4+x^3-5x^2+5x-3, \quad x^4+3x^2-3x+5, \\ 2x^4+3x^3+4x^2-x+1, \quad x^3-x+2 \end{array}\right\} \quad in \quad P_4(\mathbb{R})$$

**Solution**

$$\begin{vmatrix} 1 & -1 & 1 & 2 & 0 \\ -1 & 1 & 0 & 3 & 1 \\ 5 & -5 & 3 & 4 & 0 \\ -8 & 5 & -3 & -1 & -1 \\ 6 & -3 & 5 & 1 & 2 \end{vmatrix} = -60 \neq 0$$

$\therefore$ Linearly dependent

## Exercise

Determine if the given vectors are linearly dependent or independent, (any method)

$$\left\{\begin{array}{c} x^4-x^3+5x^2-8x+6, \quad -x^4+x^3-5x^2+5x-3, \\ x^4+3x^2-3x+5, \quad 2x^4+x^3+4x^2+8x \end{array}\right\} \quad in \quad P_4(\mathbb{R})$$

**Solution**

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ -1 & 1 & 0 & 1 \\ 5 & -5 & 3 & 4 \\ -8 & 5 & -3 & 8 \\ 6 & -3 & 5 & 0 \end{bmatrix} \begin{array}{l} \\ R_2+R_1 \\ R_3-5R_1 \\ R_4+8R_1 \\ R_5-6R_1 \end{array} \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -6 \\ 0 & -3 & 5 & 24 \\ 0 & 3 & -1 & -12 \end{bmatrix} R_5+R_4$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -6 \\ 0 & -3 & 5 & 24 \\ 0 & 0 & 4 & 12 \end{bmatrix} \begin{array}{l} \\ \\ R_3+2R_2 \\ \\ R_5-4R_2 \end{array} \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 5 & 24 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \to a_1 = a_2 - a_3 - 2a_4 \\ \to a_3 = -3a_4 \\ \\ \to 3a_2 = 5a_3 + 24a_4 \\ \\ \end{array}$$

If $a_4 = -1 \quad \to a_3 = 3 \quad a_2 = -3 \quad a_1 = -4$

$$-4\left(x^4-x^3+5x^2-8x+6\right)-3\left(-x^4+x^3-5x^2+5x-3\right)$$
$$+3\left(x^4+3x^2-3x+5\right)-\left(2x^4+x^3+4x^2+8x\right)=0$$

$\therefore$ Linearly dependent

## *Exercise*

Suppose that the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \text{ and } \boldsymbol{u}_3$ are linearly dependent. Are the vectors $\boldsymbol{v}_1 = \boldsymbol{u}_1 + \boldsymbol{u}_2$, $\boldsymbol{v}_2 = \boldsymbol{u}_1 + \boldsymbol{u}_3$, and $\boldsymbol{v}_3 = \boldsymbol{u}_2 + \boldsymbol{u}_3$ also linearly dependent?

(***Hint***: Assume that $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 = 0$, and see what the $a_i's$ can be.)

***Solution***

Given: $\boldsymbol{u}_1, \boldsymbol{u}_2, \text{ and } \boldsymbol{u}_3$ are linearly dependent, then there are scalar $b_1, b_2, \text{ and } b_3$ such that $b_1\boldsymbol{u}_1 + b_2\boldsymbol{u}_2 + b_3\boldsymbol{u}_3 = 0$.

Assume that $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 = 0$

$$a_1(\boldsymbol{u}_1 + \boldsymbol{u}_2) + a_2(\boldsymbol{u}_1 + \boldsymbol{u}_3) + a_3(\boldsymbol{u}_2 + \boldsymbol{u}_3) = 0$$

$$a_1\boldsymbol{u}_1 + a_1\boldsymbol{u}_2 + a_2\boldsymbol{u}_1 + a_2\boldsymbol{u}_3 + a_3\boldsymbol{u}_2 + a_3\boldsymbol{u}_3 = 0$$

$$(a_1 + a_2)\boldsymbol{u}_1 + (a_1 + a_3)\boldsymbol{u}_2 + (a_2 + a_3)\boldsymbol{u}_3 = 0$$

If $a_1 + a_2 = b_1 \quad a_1 + a_3 = b_2 \quad a_2 + a_3 = b_3$ and since $\boldsymbol{u}_1, \boldsymbol{u}_2, \text{ and } \boldsymbol{u}_3$ are linearly dependent, therefore, $\boldsymbol{v}_1, \boldsymbol{v}_2, \text{ and } \boldsymbol{v}_3$ are linearly dependent.

## *Exercise*

Show that the set $F = \{1+t,\ t^2,\ t-2\}$ is a linearly independent subset of $P_2$.

***Solution***

$$W = \begin{vmatrix} 1+t & t^2 & t-2 \\ 1 & 2t & 1 \\ 0 & 2 & 0 \end{vmatrix}$$

$= 2t - 4 - 2 - 2t$

$\boxed{= -6 \ne 0}$ $\quad$ ∴ Linearly Independent.

$\exists\ c_1, c_2, c_3 \text{ constants } \ni\ 0 = c_1(1+t) + c_2 t^2 + c_3(t-2)$

$$\Rightarrow \begin{cases} t^0 & c_1 - 2c_3 = 0 \\ t & c_1 + c_3 = 0 \qquad \rightarrow \boxed{c_1 = c_3 = 0} \\ t^2 & \boxed{c_2 = 0} \end{cases}$$

Since the only solution to this system is the trivial one. $F$ is Linearly Independent subset of $P_2$

### *Exercise*

Suppose that *A* is linearly dependent set of vectors and *B* is any set containing *A*. Show that *B* must be linearly dependent.

***Solution***

If *A* is linearly dependent, then there are vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ in *A* and $\mathbb{R}$, $c_1, c_2, \ldots c_n$ with all not $c_i \neq 0$ and $c_1\vec{x}_1 + c_2\vec{x}_2 + \ldots + c_n\vec{x}_n = \vec{0}$

If *B* any set that contains *A*, then this same relation holds in *B* set.

*B* is also dependent.

### *Exercise*

Show that $\{\sin t,\ \sin 2t,\ \cos t\}$ is a linearly independent, subset of $C[0,\ 1]$. Does it span $C[0,\ 1]$

***Solution***

$$W = \begin{vmatrix} \sin t & \sin 2t & \cos t \\ \cos t & 2\cos 2t & -\sin t \\ -\sin t & -4\sin 2t & -\cos t \end{vmatrix}$$

$$= -2\sin t\cos t\cos 2t + \sin^2 t\sin 2t - 4\cos^2 t\sin 2t + 2\sin t\cos t\cos 2t - 4\sin^2 t\sin 2t + \cos^2 t\sin 2t$$

$$= \sin 2t - 4\sin 2t$$

$$\underline{= -3\sin 2t \neq 0}$$

$\therefore$ *L*inearly *I*ndependent.

$a\sin t + b\sin 2t + d\cos t = 0$

$$\text{If } \begin{cases} t = 0 & \rightarrow \quad d = 0 \\ t = \frac{\pi}{2} & \rightarrow \quad a = 0 \\ t = \frac{\pi}{4} & \rightarrow \quad b = 0 \end{cases}$$

Since all the polynomials are in $C[0,\ 1]$ and there is no other way that we can write them as linear combinations of $\sin t,\ \sin 2t,\ and\ \cos t$.

The set can't possible span $C[0,\ 1]$

### *Exercise*

Show that the set $\{\sin(t+a),\ \sin(t+b),\ \sin(t+c)\}$ is linearly dependent on $C[0,\ 1]$.

***Solution***

$$W = \begin{vmatrix} \sin(t+a) & \sin(t+b) & \sin(t+c) \\ \cos(t+a) & \cos(t+b) & \cos(t+c) \\ -\sin(t+a) & -\sin(t+b) & -\sin(t+c) \end{vmatrix}$$

$$= -\sin(t+a)\cos(t+b)\sin(t+c) - \sin(t+a)\cos(t+c)\sin(t+b) - \sin(t+b)\cos(t+a)\sin(t+c)$$
$$+ \sin(t+a)\cos(t+b)\sin(t+c) + \sin(t+a)\cos(t+c)\sin(t+b) + \sin(t+b)\cos(t+a)\sin(t+c)$$

$$= 0$$

$\therefore$ The set is linearly dependent on $C[0,\ 1]$

$$k_1 \sin(t+a) + k_2 \sin(t+b) + k_3 \sin(t+c) = 0$$

$$\text{If } \begin{cases} t = -a & \rightarrow \quad k_2 + k_3 = 0 \\ t = -b & \rightarrow \quad k_1 + k_3 = 0 \\ t = -c & \rightarrow \quad k_1 + k_2 = 0 \end{cases}$$

$$t = 0 \quad \rightarrow \quad k_1 \sin a + k_2 \sin b + k_3 \sin c = 0$$

$$t = \frac{\pi}{2} \quad \rightarrow \quad k_1 \cos a + k_2 \cos b + k_3 \cos c = 0$$

$$t = \pi \quad \rightarrow \quad -\left(k_1 \sin a + k_2 \sin b + k_3 \sin c\right) = 0$$

# Solution Section 2.7 – Coordinates, Basis and Dimension

### Exercise

Suppose $v_1, \ldots, v_n$ is a basis for $R^n$ and the $n$ by $n$ matrix $A$ is invertible. Show that $Av_1, \ldots, Av_n$ is also a basis for $R^n$.

**Solution**

Put the basis vectors $v_1, \ldots, v_n$ in the columns of an invertible matrix **V**. then $Av_1, \ldots, Av_n$ are the columns of ***A*V**. Since ***A*** is invertible, so is ***A*V** and its column give a basis.

Suppose $c_1Av_1 + \ldots + c_nAv_n = 0$. This is $Av = 0$ with $v = c_1v_1 + \ldots + c_nv_n$. Multiply by $A^{-1}$ to get $v = 0$. By linear independence of $v's$, all $c_i = 0$. So, the $Av's$ are independent.

### Exercise

Consider the matrix $A = \begin{pmatrix} 1 & 0 & a \\ 2 & -1 & b \\ 1 & 1 & c \\ -2 & 1 & d \end{pmatrix}$

*a)* Which vectors $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ will make the columns of ***A*** linearly dependent?

*b)* Which vectors $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ will make the columns of ***A*** a basis for $\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : y + w = 0 \right\}$?

*c)* For $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$, compute a basis for the four subspaces.

**Solution**

***a)*** All linear combination of $\begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

***b)*** To satisfy $b + d = 0$. For example, $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$A\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + B\begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix} + C\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} \;\; ; \;\; A \neq 0$$

***c)*** $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix} + 4\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & 5 \\ -2 & 1 & 2 \end{pmatrix} \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{array} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -4 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{pmatrix} \begin{array}{c} \\ \\ R_3 + R_2 \\ R_4 + R_2 \end{array}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \begin{cases} x_1 + x_3 = 0 \\ -x_2 - 4x_3 = 0 \\ \\ \end{cases}$$

The first 2 columns span the column space $C(\boldsymbol{A})$.

If $x_3 = 1$ that implies that the nullspace $N(\boldsymbol{A})$: $\left\{ \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} \right\}$

Rank($\boldsymbol{A}$) = 2 and $\begin{bmatrix} -1 & -4 & 1 \end{bmatrix}^T$ is a basis for the one-dimensional $N(\boldsymbol{A})$.

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

## *Exercise*

Find a basis for $x-2y+3z=0$ in $\boldsymbol{R}^3$.

Find a basis for the intersection of that plane with *xy* plane. Then find a basis for all vectors perpendicular to the plane.

***Solution***

This plane is the nullspace of the matrix

$$A=\begin{pmatrix}1 & -2 & 3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

The special solutions: $s_1=\begin{pmatrix}2\\1\\0\end{pmatrix}$ $\quad s_2=\begin{pmatrix}-3\\0\\1\end{pmatrix}$ give a basis for the nullspace, and for the plane.

The intersection of this plane with the *xy*-plane is a line $(x,\ -2x,\ 3x)$ and the vector $(1,\ -2,\ 3)^T$ lies in the *xy*-plane.

The vector $v_3=\begin{pmatrix}1\\-2\\3\end{pmatrix}$ is perpendicular to both vectors $s_1$ *and* $s_2$: the space vectors perpendicular to a plane $\mathbb{R}^3$ is one-dimensional, it gives a basis.

## *Exercise*

**U** comes from ***A*** by subtracting row 1 from row 3:

$$A=\begin{bmatrix}1 & 3 & 2\\ 0 & 1 & 1\\ 1 & 3 & 2\end{bmatrix} \quad and \quad U=\begin{bmatrix}1 & 3 & 2\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$$

Find the bases for the two column spaces. Find the bases for the two row spaces. Find bases for the two nullspaces.

***Solution***

$$\begin{bmatrix}1 & 3 & 2\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\xrightarrow{R_1-3R_2}\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\rightarrow\begin{cases}x_1-x_3=0\\ x_2+x_3=0\end{cases}$$

***a)*** The pivots are in the first two columns, so one possible basis for $C(\boldsymbol{A})$ is $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}3\\1\\3\end{bmatrix}\right\}$ and for $C(\boldsymbol{U})$ is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}3\\1\\0\end{bmatrix}\right\}$.

***b)*** Both $\boldsymbol{A}$ and $\boldsymbol{U}$ have the same nullspace $N(\boldsymbol{A}) = N(\boldsymbol{U})$, with basis $\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$

***c)*** Both $\boldsymbol{A}$ and $\boldsymbol{U}$ have the same row space $C\left(A^T\right)=C\left(U^T\right)$, *with basis* $\left\{\begin{bmatrix}1\\3\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$

## *Exercise*

Write a 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives $c_1P_1+\ldots+c_5P_5=0$, and check entries to prove $c_i$ is zero.) The five permutation matrices are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

***Solution***

$$P_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\quad P_2=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\quad P_3=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\quad P_4=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\quad P_5=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

$$P_1+P_2+P_3=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} \text{ and } P_4+P_5=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$$

$$P_1+P_2+P_3-P_4-P_5=I$$

$$c_1P_1+c_2P_2+c_3P_3+c_4P_4+c_5P_5=\begin{pmatrix}c_3&c_1+c_4&c_2+c_5\\c_1+c_5&c_2&c_3+c_4\\c_2+c_4&c_3+c_5&c_1\end{pmatrix}=0$$

$$c_1=c_2=c_3=0\ (diagonal)\Rightarrow\begin{pmatrix}0&0+c_4&0+c_5\\0+c_5&0&0+c_4\\0+c_4&0+c_5&0\end{pmatrix}=0\Rightarrow c_4=c_5=0$$

### *Exercise*

Choose three independent columns of $A = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix}$. Then choose a different three independent columns. Explain whether either of these choices forms a basis for $C(A)$.

***Solution***

$$\begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ \\ \\ \end{matrix} \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix} \begin{matrix} \\ \\ \\ R_4 - R_2 \end{matrix} \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & \frac{1}{2} & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \frac{1}{2}R_1 \\ \frac{1}{6}R_2 \\ \frac{1}{9}R_3 \\ \\ \end{matrix} \begin{pmatrix} 1 & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 1 & \frac{7}{6} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} R_1 - \frac{1}{2}R_3 \\ \\ \\ \\ \end{matrix} \begin{pmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{7}{6} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank(***A***) = 3, the columns space is 3 which form a basis of $C(A)$. The variable is $x_3$

If $x_3 = 1 \Rightarrow \begin{pmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{7}{6} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 + \frac{1}{4}x_3 = 0 \\ x_2 + \frac{7}{6}x_3 = 0 \\ \quad x_4 = 0 \end{cases} \rightarrow x_1 = -\frac{1}{4} \quad x_2 = -\frac{7}{6}$

$N(A)$ is spanned by $x_n = \begin{pmatrix} -\frac{1}{4} \\ -\frac{7}{6} \\ 1 \\ 0 \end{pmatrix}$, which gives the relation of the columns. The special solution

$x_n$ gives a relation $-\frac{1}{4}v_1 - \frac{7}{6}v_2 + v_3 = 0$. If we take any two columns from the first three columns and the column 4, they will span a three-dimensional space since there will be no relation among them. Hence, they form a basis of $C(A)$.

## Exercise

Which of the following sets of vectors are bases for $\boldsymbol{R}^2$?

*a)* $\{(2,1),\ (3,0)\}$

*b)* $\{(0,0),\ (1,3)\}$

**Solution**

***a)*** $k_1(2,1)+k_2(3,0)=(0,0)$

$k_1(2,1)+k_2(3,0)=\left(b_1,b_2\right)$

$\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} = -3 \neq 0$

Therefore, the vectors $\{(2,1),\ (3,0)\}$ are linearly independent and span $\boldsymbol{R}^2$ , so they form a basis for $\boldsymbol{R}^2$.

***b)*** $k_1(0,0)+k_2(1,3)=(0,0)$

$k_1(0,0)+k_2(1,3)=\left(b_1,b_2\right)$

$\begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} = 0$

Therefore; the vectors $\{(0,0),\ (1,3)\}$ are linearly dependent, so they don't form a basis for $\boldsymbol{R}^2$.

## Exercise

Which of the following sets of vectors are bases for $\boldsymbol{R}^3$?

*a)* $\{(1,0,0),\ (2,2,0),\ (3,3,3)\}$

*b)* $\{(3,1,-4),\ (2,5,6),\ (1,4,8)\}$

*c)* $\{(2,-3,1),\ (4,1,1),\ (0,-7,1)\}$

*d)* $\{(1,\ 6,\ 4),\ (2,\ 4,-1),\ (-1,\ 2,\ 5)\}$

**Solution**

***a)*** $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix} = 6 \neq 0$ Therefore, the set of vectors are linearly independent.

The set form a basis for $\boldsymbol{R}^3$.

***b)*** $\begin{vmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{vmatrix} = 26 \neq 0$ Therefore, the set of vectors are linearly independent.

The set form a basis for $\boldsymbol{R}^3$.

***c)*** $\begin{vmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{vmatrix} = 0$ Therefore, the set of vectors are linearly dependent.

The set don't form a basis for $\boldsymbol{R}^3$.

***d)*** $\begin{vmatrix} 1 & 2 & -1 \\ 6 & 4 & 2 \\ 4 & -1 & 5 \end{vmatrix} = 0$

Therefore; the set of vectors are linearly dependent.

The set don't form a basis for $\boldsymbol{R}^3$.

## *Exercise*

Let *V* be the space spanned by $v_1 = \cos^2 x, \quad v_2 = \sin^2 x, \quad v_3 = \cos 2x$

*a)* Show that $S = \{v_1, v_2, v_3\}$ is not a basis for *V*.

*b)* Find a basis for *V*.

***Solution***

***a)*** $\cos 2x = \cos^2 x - \sin^2 x$

$k_1 \cos^2 x + k_2 \sin^2 x + k_3 \cos 2x = 0$

$k_1 \cos^2 x + k_2 \sin^2 x + k_3 \left(\cos^2 x - \sin^2 x\right) = 0$

$\left(k_1 + k_3\right)\cos^2 x + \left(k_2 - k_3\right)\sin^2 x = 0 \quad \Rightarrow \begin{cases} k_1 + k_3 = 0 & \rightarrow k_1 = -k_3 \\ k_2 - k_3 = 0 & \rightarrow k_2 = k_3 \end{cases}$

If $k_3 = -1 \quad \Rightarrow k_1 = 1, \quad k_2 = -1$

$(1)\cos^2 x + (-1)\sin^2 x + (-1)\cos 2x = 0$

This shows that $\{v_1, v_2, v_3\}$ is linearly dependent, therefore it is not a basis for *V*.

***b)*** For $c_1 \cos^2 x + c_2 \sin^2 x = 0$ to hold for all real *x* values, we must have $c_1 = 0 \quad (x = 0)$ and $c_2 = 0 \quad \left(x = \frac{\pi}{2}\right)$. Therefore, the vectors $v_1 = \cos^2 x \quad v_2 = \sin^2 x$ are linearly independent.

$\boldsymbol{v} = k_1 \cos^2 x + k_2 \sin^2 x + k_3 \cos 2x$

$\quad = \left(k_1 + k_3\right)\cos^2 x + \left(k_2 - k_3\right)\sin^2 x$

This proves that the vectors $v_1 = \cos^2 x \quad and \quad v_2 = \sin^2 x$ span *V*. We can conclude that $v_1 = \cos^2 x \quad and \quad v_2 = \sin^2 x$ can form a basis for *V*.

## *Exercise*

Find the coordinate vector of $\boldsymbol{w}$ relative to the basis $S = \{u_1, u_2\}$ for $\boldsymbol{R}^2$

*a)* $u_1 = (1,\ 0),\quad u_2 = (0,\ 1),\quad w = (3,\ -7)$

*b)* $u_1 = (2,\ -4),\quad u_2 = (3,\ 8),\quad w = (1,\ 1)$

*c)* $u_1 = (1,\ 1),\quad u_2 = (0,\ 2),\quad w = (a,\ b)$

*d)* $u_1 = (1,\ -1),\quad u_2 = (1,\ 1),\quad w = (0,\ 1)$

*e)* $u_1 = (1,\ -1),\quad u_2 = (1,\ 1),\quad w = (1,\ 1)$

## *Solution*

***a)*** We must first express $\boldsymbol{w}$ as a linear combination of the vectors in *S;* $\boldsymbol{w} = c_1u_1 + c_2u_2$

$$(3,-7) = 3(1,0) - 7(0,1)$$

$$= 3u_1 - 7u_2$$

Therefore, $(w)_S = \boxed{(3,-7)}$

***b)*** Solve $c_1u_1 + c_2u_2 = \boldsymbol{w} \quad \Rightarrow \quad c_1(2,-4) + c_2(3,8) = (1,1)$

$$\to \begin{cases} 2c_1 + 3c_2 = 1 \\ -4c_1 + 8c_2 = 1 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ -4 & 8 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & \frac{5}{28} \\ 0 & 1 & \frac{3}{14} \end{array}\right]$$

Therefore, $(w)_S = \boxed{\left(\frac{5}{28},\ \frac{3}{14}\right)}$

***c)*** Solve $c_1u_1 + c_2u_2 = \boldsymbol{w} \quad \Rightarrow \quad c_1(1,1) + c_2(0,2) = (a,b)$

$$\to \begin{cases} \boxed{c_1 = a} \\ c_1 + 2c_2 = b \end{cases} \Rightarrow \boxed{c_2 = \frac{b-a}{2}}$$

Therefore, $(w)_S = \boxed{\left(a,\ \frac{b-a}{2}\right)}$

***d)*** Solve $c_1u_1 + c_2u_2 = \boldsymbol{w} \quad \Rightarrow \quad c_1(1,-1) + c_2(1,1) = (0,1)$

$$\to \begin{cases} c_1 + c_2 = 0 \\ -c_1 + c_2 = 1 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & 1 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{array}\right] \to \boxed{c_1 = -\frac{1}{2}} \quad \boxed{c_2 = \frac{1}{2}}$$

Therefore, $(w)_S = \boxed{\left(-\frac{1}{2},\ \frac{1}{2}\right)}$

***e)*** Solve $c_1 u_1 + c_2 u_2 = \boldsymbol{w} \quad \Rightarrow \quad c_1(1,-1) + c_2(1,1) = (1,1)$

$$\rightarrow \begin{cases} c_1 + c_2 = 1 \\ -c_1 + c_2 = 1 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right] \rightarrow \boxed{c_1 = 0} \quad \boxed{c_2 = 1}$$

Therefore, $(w)_S = (0,\ 1)$

## *Exercise*

Find the coordinate vector of $v$ relative to the basis $S = \{v_1, v_2, v_3\}$

*a)* $v = (2,-1,3), \quad v_1 = (1,0,0), \quad v_2 = (2,2,0), \quad v_3 = (3,3,3)$

*b)* $v = (5,-12,3), \quad v_1 = (1,2,3), \quad v_2 = (-4,5,6), \quad v_3 = (7,-8,9)$

***Solution***

***a)*** Solve $c_1 v_1 + c_2 v_2 + c_3 v_3 = \boldsymbol{v} \quad \Rightarrow \quad c_1(1,0,0) + c_2(2,2,0) + c_2(3,3,3) = (2,-1,3)$

$$\rightarrow \begin{cases} c_1 + 2c_2 + 3c_3 = 2 \\ \quad 2c_2 + 3c_3 = -1 \\ \quad\quad 3c_3 = 3 \end{cases} \qquad \begin{array}{l} \rightarrow c_1 = 2 - 2c_2 - 3c_3 = 3 \\ \rightarrow c_2 = \dfrac{-3c_3 - 1}{2} = -2 \\ \rightarrow c_3 = 1 \end{array}$$

Therefore, $(v)_S = (3,\ -2,\ 1)$

***b)*** Solve $c_1 v_1 + c_2 v_2 + c_3 v_3 = \boldsymbol{v} \quad \Rightarrow \quad c_1(1,2,3) + c_2(-4,5,6) + c_2(7,-8,9) = (5,-12,3)$

$$\rightarrow \begin{cases} c_1 - 4c_2 + 7c_3 = 5 \\ 2c_1 + 5c_2 - 8c_3 = -12 \\ 3c_1 + 6c_2 + 9c_3 = 3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -4 & 7 & 5 \\ 2 & 5 & -8 & -12 \\ 3 & 6 & 9 & 3 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{cases} c_1 = -2 \\ c_2 = 0 \\ c_3 = 1 \end{cases}$$

Therefore, $(v)_S = (-2,\ 0,\ 1)$

### *Exercise*

Show that $\{A_1, A_2, A_3, A_4\}$ is a basis for $M_{22}$, and express $A$ as a linear combination of the basis vectors

*a)* $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$

*b)* $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

*c)* $A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

***Solution***

***a)*** Matrices $\{A_1, A_2, A_3, A_4\}$ are linearly independent if the equation

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{0}$$

$$k_1\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + k_2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + k_3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k_4\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \mathbf{0}$$

Has only the trivial solution.

For these matrices to span $M_{22}$, it must be expressed every matrix $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ as

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{A}$$

$$k_1\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + k_2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + k_3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k_4\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$$

The 2 equations can be written as linear systems

$$\begin{cases} k_1 + k_2 + k_3 = 0 \\ k_2 = 0 \\ k_1 + k_4 = 0 \\ k_3 = 0 \end{cases} \quad \text{and} \quad \begin{cases} k_1 + k_2 + k_3 = a_1 \\ k_2 = a_2 \\ k_1 + k_4 = a_3 \\ k_3 = a_4 \end{cases}$$

$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -1 \neq 0$, that the homogeneous system has only the trivial solution.

$\{A_1, A_2, A_3, A_4\}$ span $M_{22}$

$$\rightarrow \begin{cases} k_1 + k_2 + k_3 & = 6 \\ k_2 & = 2 \\ k_1 + k_4 & = 5 \\ k_3 & = 3 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad \begin{matrix} k_1 = 1 \\ k_2 = 2 \\ k_3 = 3 \\ k_4 = 4 \end{matrix}$$

$$\mathbf{A} = \underline{A_1 + 2A_2 + 3A_3 + 4A_4}$$

***b)*** Matrices $\{A_1, A_2, A_3, A_4\}$ are linearly independent if the equation

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{0}$$

$$k_1\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + k_2\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + k_3\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + k_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{0}$$

Has only the trivial solution.

For these matrices to span $M_{22}$, it must be expressed every matrix $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ as

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{A}$$

$$k_1\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + k_2\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + k_3\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + k_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

The 2 equations can be written as linear systems

$$\begin{aligned} k_1 &= 0 \\ k_1 + k_2 &= 0 \\ k_1 + k_2 + k_3 &= 0 \\ k_1 + k_2 + k_3 + k_4 &= 0 \end{aligned} \quad \textit{and} \quad \begin{aligned} k_1 &= a_1 \\ k_1 + k_2 &= a_2 \\ k_1 + k_2 + k_3 &= a_3 \\ k_1 + k_2 + k_3 + k_4 &= a_4 \end{aligned}$$

$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$, that the homogeneous system has only the trivial solution.

$\{A_1, A_2, A_3, A_4\}$ span $M_{22}$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \quad \begin{array}{l} k_1 = 1 \\ k_2 = -1 \\ k_3 = 1 \\ k_4 = -1 \end{array}$$

$$\mathbf{A} = \underline{A_1 - A_2 + A_3 - A_4}$$

***c)*** $k_1\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + k_2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + k_3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + k_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \quad \begin{array}{l} k_1 = 1 \\ k_2 = 1 \\ k_3 = -1 \\ k_4 = 3 \end{array}$$

$$\mathbf{A} = \underline{A_1 + A_2 - A_3 + 3A_4}$$

## *Exercise*

Consider the eight vectors

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

*a)* List all of the one-element. Linearly dependent sets formed from these.
*b)* What are the two-element, linearly dependent sets?
*c)* Find a three-element set spanning a subspace of dimension three, and dimension of two? One? Zero?
*d)* Which four-element sets are linearly dependent? Explain why.

***Solution***

***a)*** $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ zero vector is the only linearly dependent.

***b)*** The set that contains zero vector and any other vector.

***c)*** 2-dimension:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad or \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad or \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

1-dimensional subspace if we allow duplicates (zero vector) $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$

***d)*** All four-element sets are linearly dependent in three-dimensional space.

## *Exercise*

Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space

*a)* $\begin{cases} x_1 + x_2 - x_3 = 0 \\ -2x_1 - x_2 + 2x_3 = 0 \\ -x_1 \qquad + x_3 = 0 \end{cases}$

*b)* $\begin{cases} 3x_1 + x_2 + x_3 + x_4 = 0 \\ 5x_1 - x_2 + x_3 - x_4 = 0 \end{cases}$

*c)* $\begin{cases} x_1 - 3x_2 + x_3 = 0 \\ 2x_1 - 6x_2 + 2x_3 = 0 \\ 3x_1 - 9x_2 + 3x_3 = 0 \end{cases}$

*d)* $\begin{cases} x + y + z = 0 \\ 3x + 2y - 2z = 0 \\ 4x + 3y - z = 0 \\ 6x + 5y + z = 0 \end{cases}$

*e)* $\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 \qquad + 5x_3 = 0 \\ \qquad x_2 + x_3 = 0 \end{cases}$

***Solution***

***a)*** $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad x_1 - x_3 = 0 \rightarrow x_1 = x_3$
$x_2 = 0$

The solution: $(x_1,\ 0,\ x_1) = x_1(1,\ 0,\ 1)$

The solution space has dimension 1 and a basis $\underline{(1,\ 0,\ 1)}$

***b)*** $\left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{array}\right]$ $\quad x_3 = s \qquad x_1 = -\frac{1}{4}x_3 = s$
$x_4 = t \quad x_2 = -\frac{1}{4}x_3 - x_4 = -\frac{1}{4}s - t$

The solution: $(x_1,\ x_2,\ x_3,\ x_4) = \left(-\frac{1}{4}s,\ -\frac{1}{4}s - t,\ s,\ t\right)$

$= s\left(-\frac{1}{4},\ -\frac{1}{4},\ 1,\ 0\right) + t(0,\ -1,\ 0,\ 1)$

The solution space has dimension 2 and a basis $\underline{\left(-\frac{1}{4},\ -\frac{1}{4},\ 1,\ 0\right),\ \ (0,\ -1,\ 0,\ 1)}$

**c)** $\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 - 3x_2 + x_3 = 0 \rightarrow x_1 = 3x_2 - x_3$

The solution: $(x_1, x_2, x_3) = (3x_2 - x_3, x_2, x_3)$

$$= x_2(3, 1, 0) + x_3(-1, 0, 1)$$

The solution space has dimension 2 and a basis $\boxed{(3, 1, 0) \ \ and \ \ (-1, 0, 1)}$

**d)** $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 2 & -2 & 0 \\ 4 & 3 & -1 & 0 \\ 6 & 5 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{array}{l} x = 4z \\ y = -5z \end{array}$

The solution: $(x, y, z) = (4z, \ -5z, \ z) = z(4, \ -5, \ 1)$

The solution space has dimension 1 and a basis $(4, \ -5, \ 1)$

**e)** $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ No basis and dimension = 0

## *Exercise*

If $AS = SA$ for the shift matrix $S$. Show that $\boldsymbol{A}$ must have this special form:

$$If \ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$then \ \ A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

"The subspace of matrices that commute with the shift $S$ has dimension ______."

***Solution***

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow d = g = h = 0 \quad b = f \quad a = e = i$$

$$A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

The subspace of matrices that commute with the shift $S$ has dimension 3, because the matrix has only three variables.

## Exercise

Find bases for the following subspaces of $\boldsymbol{R}^3$

*a)* All vectors of the form $(a, b, c, 0)$

*b)* All vectors of the form $(a, b, c, d)$, where $d = a + b$ and $c = a - b$.

*c)* All vectors of the form $(a, b, c, d)$, where $a = b = c = d$.

**Solution**

***a)*** The subspace can be expressed as span $S = \{(1,0,0,0),(0,1,0,0),(0,0,1,0)\}$ is a set of linearly independent vectors. Therefore; $S$ forms a basis for the subspace, so its dimension is 3.

***b)*** The subspace contains all vectors $(a,\ b,\ a+b,\ a-b) = a(1,0,1,1) + b(0,1,1,-1)$, the set $S = \{(1,0,1,1),(0,1,1,-1)\}$ is linearly independent vectors. Therefore; $S$ forms a basis for the subspace, so its dimension is 2.

***c)*** The subspace contains all vectors $(a,\ a,\ a,\ a) = a(1,1,1,1)$, we can express the set $S = \{(1,1,1,1)\}$ as span $S$ and it is linearly independent. Therefore, $S$ forms a basis for the subspace, so its dimension is 1.

## Exercise

Find a basis for the null space of A.

*a)* $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

*b)* $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

*c)* $A=\begin{bmatrix}1&4&5&6&9\\3&-2&1&4&-1\\-1&0&-1&-2&-1\\2&3&5&7&8\end{bmatrix}$

***Solution***

***a)*** $A=\begin{bmatrix}1&-1&3\\5&-4&-4\\7&-6&2\end{bmatrix}\xrightarrow{rref}\begin{bmatrix}1&0&-16\\0&1&-19\\0&0&0\end{bmatrix}\begin{array}{l}\to x_1=16x_3=16t\\\to x_2=19x_3=19t\\ \end{array}$

The general form of the solution of A$\boldsymbol{x}$ = **0** is $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=t\begin{bmatrix}16\\19\\1\end{bmatrix}$, therefore the vector $\begin{bmatrix}16\\19\\1\end{bmatrix}$ forms a basis for the null space of *A*.

***b)*** $A=\begin{bmatrix}1&4&5&2\\2&1&3&0\\-1&3&2&2\end{bmatrix}\xrightarrow{rref}\begin{bmatrix}1&0&1&-\frac{2}{7}\\0&1&1&\frac{4}{7}\\0&0&0&0\end{bmatrix}\begin{array}{l}\to x_1=-x_3+\frac{2}{7}x_4=-t+\frac{2}{7}s\\\to x_2=-x_3-\frac{4}{7}x_4=-t-\frac{4}{7}s\\ \end{array}$

The general form of the solution of A$\boldsymbol{x}$ = **0** is $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=t\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}+s\begin{bmatrix}\frac{2}{7}\\-\frac{4}{7}\\0\\1\end{bmatrix}$, therefore the vectors

$\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}$ *and* $\begin{bmatrix}\frac{2}{7}\\-\frac{4}{7}\\0\\1\end{bmatrix}$ form a basis for the null space of *A*.

***c)*** $\begin{bmatrix}1&4&5&6&9\\3&-2&1&4&-1\\-1&0&-1&-2&-1\\2&3&5&7&8\end{bmatrix}\xrightarrow{rref}\begin{bmatrix}1&0&1&2&1\\0&1&1&1&2\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$

$$\to\begin{cases}x_1=-x_3-2x_4-x_5=-r-2s-t\\x_2=-x_3-x_4-2x_5=-r-s-2t\end{cases}$$

The general form of the solution of A$\boldsymbol{x}$ = $\mathbf{0}$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, therefore the vectors $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \textit{ and } \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of *A*.

## *Exercise*

Find a basis for the subspace of $\boldsymbol{R}^4$ spanned by the given vectors

*a)* $(1,1,-4,-3), \quad (2,0,2,-2), \quad (2,-1,3,2)$

*b)* $(-1,1,-2,0), \quad (3,3,6,0), \quad (9,0,0,3)$

**Solution**

***a)*** $\begin{pmatrix} 1 & 1 & -4 & -3 \\ 2 & 0 & 2 & -2 \\ 2 & -1 & 3 & 2 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 1 & -4 & -3 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 1 & -\frac{1}{2} \end{pmatrix}$

A basis for the subspace is $(1,1,-4,-3), \quad (0,1,-5,-2), \quad \left(0,0,1,-\frac{1}{2}\right)$

***b)*** $\begin{pmatrix} -1 & 1 & -2 & 0 \\ 3 & 3 & 6 & 0 \\ 9 & 0 & 0 & 3 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{6} \end{pmatrix}$

A basis for the subspace is $(1,-1,2,0), \quad (0,1,0,0), \quad \left(0,0,1,-\frac{1}{6}\right)$

## *Exercise*

Determine whether the given vectors form a basis for the given vector space

*a)* $\boldsymbol{v}_1(3,\ -2,\ 1), \quad \boldsymbol{v}_2(2,\ 3,\ 1), \quad \boldsymbol{v}_3(2,\ 1,\ -3), \quad in\ \mathbb{R}^3$

*b)* $\boldsymbol{v}_1 = (1,\ 1,\ 0,\ 0), \quad \boldsymbol{v}_2 = (0,\ 1,\ 1,\ 0), \quad \boldsymbol{v}_3 = (0,\ 0,\ 1,\ 1), \quad \boldsymbol{v}_4 = (1,\ 0,\ 0,\ 1), \quad for\ \mathbb{R}^4$

*c)* $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad M_{22}$

***Solution***

***a)*** $\begin{vmatrix} 3 & 2 & 2 \\ -2 & 3 & 1 \\ 1 & 1 & -3 \end{vmatrix} = 14 \neq 0$

The given vectors are linearly independent and span $\boldsymbol{R}^3$, so they form a basis for $\boldsymbol{R}^3$.

***b)*** $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 2 \neq 0$

The given vectors are linearly independent and span $\boldsymbol{R}^4$, so they form a basis for $\boldsymbol{R}^4$.

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad M_{22}$$

***c)*** $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$

They form a basis for $M_{22}$.

## *Exercise*

Find a basis for, and the dimension of, the null space of the given matrix $\begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & -2 & -2 & 1 \end{bmatrix}$

***Solution***

$$\begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & -2 & -2 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -\frac{1}{2} & \frac{3}{8} \\ 0 & 1 & 0 & \frac{1}{4} \end{bmatrix} \quad \begin{aligned} x_1 - \tfrac{1}{2}x_3 + \tfrac{3}{8}x_4 &= 0 \\ x_2 + \tfrac{1}{4}x_4 &= 0 \end{aligned}$$

$$x_1 = \tfrac{1}{2}x_3 - \tfrac{3}{8}x_4$$

$$x_2 = -\tfrac{1}{4}x_4$$

$$\boldsymbol{x} = x_3 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{3}{8} \\ \frac{1}{4} \\ 0 \\ 1 \end{bmatrix}$$

The *bases* are: $\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} -\frac{3}{8} \\ \frac{1}{4} \\ 0 \\ 1 \end{bmatrix}$

*Dimension*: 2

# *Solution* *Section* 2.8 – Row and Column Spaces

### *Exercise*

List the row vectors and column vectors of the matrix

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ 3 & 5 & 7 & -1 \\ 1 & 4 & 2 & 7 \end{bmatrix}$$

***Solution***

Row vectors: $r_1 = [2 \quad -1 \quad 0 \quad 1], \quad r_2 = [3 \quad 5 \quad 7 \quad -1], \quad r_3 = [1 \quad 4 \quad 2 \quad 7]$

Column vectors: $c_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}, \quad c_4 = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$

### *Exercise*

Express the product A***x*** as a linear combination of the column vectors of *A*.

*a)* $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

*b)* $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$

*c)* $\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$

***Solution***

***a)*** $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

***b)*** $\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = -1\begin{bmatrix} -3 \\ 5 \\ 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 6 \\ -4 \\ 3 \\ 8 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}$

***c)*** $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = -2\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 6 \\ -1 \end{bmatrix} + 5\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$

### Exercise

Determine whether $\boldsymbol{b}$ is in the column space of $A$, and if so, express $\boldsymbol{b}$ as a linear combination of the column vectors of $A$.

*a)* $A=\begin{bmatrix}1 & 3\\ 4 & -6\end{bmatrix}, \quad b=\begin{bmatrix}-2\\ 10\end{bmatrix}$

*b)* $A=\begin{bmatrix}1 & -1 & 1\\ 9 & 3 & 1\\ 1 & 1 & 1\end{bmatrix}, \quad b=\begin{bmatrix}5\\ 1\\ -1\end{bmatrix}$

*c)* $A=\begin{bmatrix}1 & 1 & 2\\ 1 & 0 & 1\\ 2 & 1 & 3\end{bmatrix} \quad \boldsymbol{b}=\begin{bmatrix}-1\\ 0\\ 2\end{bmatrix}$

*d)* $A=\begin{bmatrix}1 & 2 & 0 & 1\\ 0 & 1 & 2 & 1\\ 1 & 2 & 1 & 3\\ 0 & 1 & 2 & 2\end{bmatrix}, \quad b=\begin{bmatrix}4\\ 3\\ 5\\ 7\end{bmatrix}$

**Solution**

***a)*** $\left[\begin{array}{cc|c}1 & 3 & -2\\ 4 & -6 & 10\end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c}1 & 0 & 1\\ 0 & 1 & -1\end{array}\right]$

$$\begin{bmatrix}-2\\ 10\end{bmatrix}=\begin{bmatrix}1\\ 4\end{bmatrix}-\begin{bmatrix}3\\ -6\end{bmatrix}$$

***b)*** $\left[\begin{array}{ccc|c}1 & -1 & 1 & 5\\ 9 & 3 & 1 & 1\\ 1 & 1 & 1 & -1\end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right]$

The system $A\boldsymbol{x}=\boldsymbol{b}$ is inconsistent and $\boldsymbol{b}$ is not in the column space of $A$.

***c)*** $\left[\begin{array}{ccc|c}1 & 1 & 2 & -1\\ 1 & 0 & 1 & 0\\ 2 & 1 & 3 & 2\end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right]$

The system $A\boldsymbol{x}=\boldsymbol{b}$ is inconsistent and $\boldsymbol{b}$ is not in the column space of $A$.

***d)*** $\left[\begin{array}{cccc|c}1 & 2 & 0 & 1 & 4\\ 0 & 1 & 2 & 1 & 3\\ 1 & 2 & 1 & 3 & 5\\ 0 & 1 & 2 & 2 & 7\end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c}1 & 0 & 0 & 0 & -26\\ 0 & 1 & 0 & 0 & 13\\ 0 & 0 & 1 & 0 & -7\\ 0 & 0 & 0 & 1 & 4\end{array}\right]$

$$\begin{bmatrix}4\\ 3\\ 5\\ 7\end{bmatrix}=-26\begin{bmatrix}1\\ 0\\ 1\\ 0\end{bmatrix}+13\begin{bmatrix}2\\ 1\\ 2\\ 1\end{bmatrix}-7\begin{bmatrix}0\\ 2\\ 1\\ 2\end{bmatrix}+4\begin{bmatrix}1\\ 1\\ 3\\ 2\end{bmatrix}$$

### Exercise

Suppose that $x_1 = -1, \quad x_2 = 2, \quad x_3 = 4, \quad x_4 = -3$ is a solution of a nonhomogeneous linear system $A\boldsymbol{x} = \boldsymbol{b}$ and that the solution set of the homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$ is given by the formulas

$$x_1 = -3r + 4s, \quad x_2 = r - s, \quad x_3 = r, \quad x_4 = s$$

*a)* Find a vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{0}$

*b)* Find a vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{b}$

***Solution***

***a)***
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

***b)***
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 4 \\ -3 \end{bmatrix} + r\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

### Exercise

Find the vector form of the general solution of the given linear system $A\boldsymbol{x} = \boldsymbol{b}$; then use that result to find the vector form of the general solution of $A\boldsymbol{x} = \boldsymbol{0}$.

*a)* $\begin{cases} x_1 - 3x_2 = 1 \\ 2x_1 - 6x_2 = 2 \end{cases}$

*b)* $\begin{cases} x_1 + x_2 + 2x_3 = 5 \\ x_1 \qquad + x_3 = -2 \\ 2x_1 + x_2 + 3x_3 = 3 \end{cases}$

*c)* $\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 = 4 \\ -2x_1 + x_2 + 2x_3 + x_4 = -1 \\ -x_1 + 3x_2 - x_3 + 2x_4 = 3 \\ 4x_1 - 7x_2 \qquad - 5x_4 = -5 \end{cases}$

*d)* $\begin{cases} x_1 - 2x_2 + x_3 + 2x_4 = -1 \\ 2x_1 - 4x_2 + 2x_3 + 4x_4 = -2 \\ -x_1 + 2x_2 - x_3 - 2x_4 = 1 \\ 3x_1 - 6x_2 + 3x_3 + 6x_4 = -3 \end{cases}$

***Solution***

***a)***
$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & -6 & 2 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = 1 + 3x_2$$

The solution of A$\boldsymbol{x}$ = $\boldsymbol{b}$ is $x_1 = 1+3t, \quad x_2 = t$ or $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

The general form of the solution of A$\boldsymbol{x}$ = **0** is $\boldsymbol{x} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

***b)*** $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 1 & 0 & 1 & -2 \\ 2 & 1 & 3 & 3 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{array}{l} \Rightarrow x_1 = -2 - x_3 \\ \Rightarrow x_2 = 7 - x_3 \end{array}$

The solution of A$\boldsymbol{x}$ = $\boldsymbol{b}$ is $x_1 = -2-t, \quad x_2 = 7-t, \quad x_3 = t$ or $\boldsymbol{x} = \begin{bmatrix} -2 \\ 7 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

The general form of the solution of A$\boldsymbol{x}$ = **0** is $\boldsymbol{x} = t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

***c)*** $\left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 4 \\ -2 & 1 & 2 & 1 & -1 \\ -1 & 3 & -1 & 2 & 3 \\ 4 & -7 & 0 & -5 & -5 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & -\frac{7}{5} & -\frac{1}{5} & \frac{6}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{array}{l} \Rightarrow x_1 = \frac{6}{5} + \frac{7}{5}x_3 + \frac{1}{5}x_4 \\ \Rightarrow x_2 = \frac{7}{5} + \frac{4}{5}x_3 - \frac{3}{5}x_4 \end{array}$

The solution of A$\boldsymbol{x}$ = $\boldsymbol{b}$ is $\boldsymbol{x} = \begin{bmatrix} \frac{6}{5} \\ \frac{7}{5} \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} \frac{7}{5} \\ \frac{4}{5} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{5} \\ -\frac{3}{5} \\ 0 \\ 1 \end{bmatrix}$

The general form of the solution of A$\boldsymbol{x}$ = **0** is $\boldsymbol{x} = s\begin{bmatrix} \frac{7}{5} \\ \frac{4}{5} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{5} \\ -\frac{3}{5} \\ 0 \\ 1 \end{bmatrix}$

***d)*** $\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & -1 \\ 2 & -4 & 2 & 4 & -2 \\ -1 & 2 & -1 & -2 & 1 \\ 3 & -6 & 3 & 6 & -3 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = -1 + 2x_2 - x_3 - 2x_4$

Let $x_2 = s \quad x_3 = t \quad x_4 = r$

The solution of $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

The general form of the solution of $A\boldsymbol{x} = \boldsymbol{0}$ is $\boldsymbol{x} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

## *Exercise*

Given the vectors $v_1 = (1,\ 2,\ 0)$ and $v_2 = (2,\ 3,\ 0)$

*a)* Are they linearly independent?
*b)* Are they a basis for any space?
*c)* What space **V** do they span?
*d)* What is the dimension of that space?
*e)* What matrices ***A*** have **V** as their column space?
*f)* Which matrices have **V** as their nullspace?
*g)* Describe all vectors $v_3$ that complete a basis $v_1, v_2, v_3$ for $\boldsymbol{R}^3$.

***Solution***

***a)*** $v_1, v_2$ are independent – the only combination to give **0** is $0.v_1 + 0.v_2$.

***b)*** Yes, they are a basis for whatever space **V** they span.

***c)*** That space **V** contains all vectors $(x,\ y,\ 0)$. It is the *xy* plane in $\boldsymbol{R}^3$.

***d)*** The dimension of **V** is 2 since the basis contains 2 vectors.

***e)*** This **V** is the column space of any 3 by *n* matrix ***A*** of rank 2, if every column is a combination of $v_1$ *and* $v_2$. In particular ***A*** could just have columns $v_1$ *and* $v_2$.

***f)*** This **V** is the nullspace of any *m* by 3 matrix ***B*** of rank 1, if every row is a multiple of $(0,\ 0,\ 1)$. In particular, take $B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. Then $Bv_1 = 0$ *and* $Bv_2 = 0$.

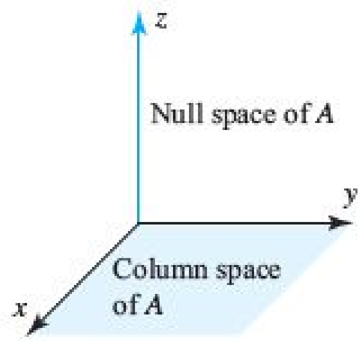
***g)*** Any third vector $v_3 = (a,\ b,\ c)$ will complete a basis for $\boldsymbol{R}^3$ provided $c \neq 0$.

## *Exercise*

*a*) Let $A=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}$

Show that relative to an *xyz*-coordinate system in 3-space the null space of *A* consists of all points on the *z*-axis and that the column space consists of all points in the *xy*-plane.

*b*) Find a 3 x 3 matrix whose null space is the *x*-axis and whose column space is the *yz*-plane.




***Solution***

***a)*** $A=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix} \to \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} \quad \begin{aligned}x&=0\\y&=0\\z&=t\end{aligned}$

The general form of the solution of A$\boldsymbol{x}$ = **0** is, $t\begin{bmatrix}0\\0\\1\end{bmatrix}$ therefore

the null space of *A* is the *z*-axis, and the column space is the span of $c_1=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $c_2=\begin{bmatrix}1\\0\\0\end{bmatrix}$

which is all linear combinations of *y* and *x* (*xy*-plane)

***b)*** $\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$

## *Exercise*

If we add an extra column b to a matrix *A*, then the column space gets larger unless _____. Give an example where the column space gets larger and an example where it doesn't. Why is $Ax=b$ solvable exactly when the column space doesn't get larger – it is the same for *A* and $\begin{bmatrix}A & b\end{bmatrix}$?

***Solution***

If we add an extra column ***b*** to a matrix ***A***, then the column space gets larger unless ***it contains b*** that is a linear combination of the columns of ***A***.

Let $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$; then the column space gets larger if $b=\begin{pmatrix}0\\1\end{pmatrix}$ and it doesn't if $b=\begin{pmatrix}1\\0\end{pmatrix}$.

The equation $Ax=b$ is solvable exactly when ***b*** is a (nontrivial) linear combination of the column of ***A***.

The equation $Ax = b$ is solvable exactly when $\boldsymbol{b}$ lies in the column space, when the column space doesn't get larger.

## Exercise

For which right sides (find a condition on $b_1$, $b_2$, $b_3$) are these solvable. (Use the column space $C(A)$ and the equation $Ax = b$)

*a)* $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

*b)* $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

***Solution***

***a)*** The column space consists of the vectors for $\begin{pmatrix} x_1 + 4x_2 + 2x_3 \\ 2x_1 + 8x_2 + 4x_3 \\ -x_1 - 4x_2 - 2x_3 \end{pmatrix}$ is $\begin{pmatrix} z \\ 2z \\ -z \end{pmatrix} = z\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

They are scalar multiples of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

***b)*** By substituting $x_1 + 4x_2$ with new variable $z$, then the column space consists of the vectors

$$\begin{pmatrix} x_1 + 4x_2 \\ 2x_1 + 9x_2 \\ -x_1 - 4x_2 \end{pmatrix} = z\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

They are linear combinations of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

## Exercise

Show that the matrices $A$ and $\left[A \ \ AB\right]$ (with extra columns) have the same column space. But find a square matrix with $C\left(A^2\right)$ smaller than $C(A)$. Important point: An $n$ by $n$ matrix has $C(A) = \boldsymbol{R}^n$ exactly when $A$ is an ______ matrix.

***Solution***

Each column of ***AB*** is a combination of the columns of ***A*** (the combining coefficients are the entries in the corresponding column of B). So, any combination of the columns of $\begin{bmatrix} A & AB \end{bmatrix}$ is a combination of the columns of ***A*** alone. Thus, ***A*** and $\begin{bmatrix} A & AB \end{bmatrix}$ have the same column space.

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; then $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so $C\left(A^2\right) = Z$.

$C(A)$ is the line through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Any *n* by *n* matrix has $C(A) = \boldsymbol{R}^n$ exactly when ***A*** is an ***invertible*** matrix, because $Ax = b$ is solvable for any given ***b*** when ***A*** is invertible.

### Exercise

The column of *AB* are combinations of the columns of *A*. This means: The column space of *AB* is contained in (possibly equal to) to the column space of *A*. Give an example where the column spaces *A* and *AB* are not equal.

***Solution***

The column space of ***AB*** is contained in (possibly equal to) to the column space of ***A***.
$B = 0 \;\; and \;\; A \neq 0$ is a case when $AB = 0$ has a smaller column space than ***A***.

### Exercise

Find a square matrix *A* where $C\left(A^2\right)$ (the column space of $A^2$ is smaller than $C(A)$.

***Solution***

For example, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; then $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Thus $C(A)$ is generated by vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is of one dimensional, but $C\left(A^2\right)$ is a zero space.

Hence $C\left(A^2\right)$ is strictly smaller than $C(A)$.

### Exercise

Suppose $Ax = b$ and $Cx = b$ have the same (complete) solutions for every ***b***. Is true that $A = C$?

***Solution***

Yes, if $A = C$, let ***y*** be any vector of the correct size, and set $b = Ay$. Then ***y*** is a solution to $Ax = b$ and it is also a solution to $Cx = b$; $b = Ay = Cy$

## *Exercise*

Apply Gauss-Jordan elimination to $Ux = 0$ and $Ux = c$. Reach $Rx = 0$ and $Rx = d$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \qquad \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

Solve $Rx = 0$ to find $x_n$ (its free variable is $x_2 = 1$).

Solve $Rx = d$ to find $x_p$ (its free variable is $x_2 = 0$).

***Solution***

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \xrightarrow[\frac{1}{4}R_2]{} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The free variable is $x_2$, since it is the only one. We have to let $x_2 = 1$

$$\begin{cases} x_1 + 2x_2 = 0 \\ x_3 = 0 \end{cases} \rightarrow x_1 = -2x_2$$

The special solution is $s_1(-2, 1, 0) \Rightarrow x_n = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix} \xrightarrow[\frac{1}{4}R_2]{} \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The free variable is $x_2$ that implies to $x_2 = 0$

$$\begin{cases} x_1 + 2x_2 = -1 \\ x_3 = 2 \end{cases} \rightarrow x_1 = -1$$

The particular solution is $x_p = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

## *Exercise*

Which of the following subsets of $\boldsymbol{R}^3$ are actually subspaces?

*a)* The plane of vectors $(b_1, b_2, b_3)$ with $b_1 = b_2$

*b)* The plane of vectors with $b_1 = 1$.

*c)* The vectors with $b_1 b_2 b_3 = 0$.

*d)* All linear combinations of $v = (1, 4, 0)$ and $w = (2, 2, 2)$.

*e)* All vectors that satisfies $b_1 + b_2 + b_3 = 0$

*f)* All vectors with $b_1 \le b_2 \le b_3$.

***Solution***

***a)*** This is subspace

- For $v = (b_1,\ b_2,\ b_3)$ with $b_1 = b_2$ and $w = (c_1,\ c_2,\ c_3)$ with $c_1 = c_2$ the sum $v + w = (b_1 + c_1,\ b_2 + c_2,\ b_3 + c_3)$ is in the same set as $b_1 + c_1 = b_2 + c_2$
- For an element $v = (b_1,\ b_2,\ b_3)$ with $b_1 = b_2$, $cv = (cb_1,\ cb_2,\ cb_3)$ and $cb_1 = cb_2$, thus it is in the same set.

***b)*** This is not a subspace. For example, for $v = (1,\ 0,\ 0)$ and $cv = -v = (-1,\ 0,\ 0)$ is not in the set.

***c)*** This is not a subspace. For example, for $v = (1,\ 1,\ 0)$ and $w = (1,\ 0,\ 1)$ are in the set, but their sum $v + w = (2,\ 1,\ 1)$ is not in the set.

***d)*** This is subspace, by definition of linear combination.

- For 2 vectors $v_1 = \alpha_1 v + \beta_1 w$ and $v_2 = \alpha_2 v + \beta_2 w$ the sum
$$\begin{aligned} v_1 + v_2 &= \alpha_1 v + \beta_1 w + \alpha_2 v + \beta_2 w \\ &= (\alpha_1 + \alpha_2) v + (\beta_1 + \beta_2) w \end{aligned}$$
is still the linear combination of $v$ and $w$.
- For an element $v_1 = \alpha_1 v + \beta_1 w$, $cv_1 = c\alpha_1 v + c\beta_1 w$ is still the linear combination of $v$ and $w$, thus it is the same set

***e)*** This is subspace, these are the vectors orthogonal to $(1,\ 1,\ 1)$

- For $v = (b_1,\ b_2,\ b_3)$ with $b_1 + b_2 + b_3 = 0$ and $w = (c_1,\ c_2,\ c_3)$ with $c_1 + c_2 + c_3 = 0$ the sum $v + w = (b_1 + c_1,\ b_2 + c_2,\ b_3 + c_3)$ is in the same set as
$b_1 + c_1 + b_2 + c_2 + b_3 + c_3 = 0$
- For an element $v = (b_1,\ b_2,\ b_3)$ with $b_1 + b_2 + b_3 = 0$, $cv = (cb_1,\ cb_2,\ cb_3)$ and $cb_1 + cb_2 + cb_3 = 0$, thus it is in the same set.

***f)*** This is not a subspace. For example, for $v = (1,\ 2,\ 3)$ and $-v = (-1,\ -2,\ -3)$ is not in the set.

## *Exercise*

We are given three different vectors $b_1,\ b_2,\ b_3$. Construct a matrix so that the equations $Ax = b_1$ and $Ax = b_2$ are solvable, but $Ax = b_3$ is not solvable.

***a)*** How can you decide if this possible?

***b)*** How could you construct A?

## *Solution*

The equations $Ax = b_1$ and $Ax = b_2$ will be solvable.

$$Ax = \begin{bmatrix} b_1 & b_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_3 \ \text{(solvable?)}$$

If $Ax = b_3$ is not solvable, we have the desired matrix $A$.

If $Ax = b_3$ is solvable, then it is not possible to construct $A$.

When the column space contains $b_1$ and $b_2$, it will have to contain their linear combinations.

So $b_3$ would necessarily be in that column space and $Ax = b_3$ would necessarily be solvable.

## *Exercise*

For which vectors $(b_1, b_2, b_3)$ do these systems have a solution?

*a)* $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

*b)* $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

*c)* $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

## *Solution*

***a)*** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \begin{array}{r} \to x_1 + x_2 + x_3 = b_1 \\ \to x_2 + x_3 = b_2 \\ \to x_3 = b_3 \end{array}$

$$\begin{cases} x_1 = b_1 - b_2 - b_3 \\ x_2 = b_2 - b_3 \\ x_3 = b_3 \end{cases} \Rightarrow \left(b_1 - b_2 - b_3,\ b_2 - b_3,\ b_3\right)$$

Solution for every $b$.

***b)*** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \begin{array}{r} \to x_1 + x_2 + x_3 = b_1 \\ \to x_2 + x_3 = b_2 \\ \to 0x_3 = b_3 \end{array}$

$$\begin{cases} x_1 = b_1 - b_2 \\ x_2 = b_2 \\ 0 = b_3 \end{cases} \Rightarrow \left(b_1 - b_2,\ b_2,\ 0\right)$$

Solvable only if $b_3 = 0$

***c)*** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \xrightarrow{R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 0 & b_3 - b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right]$

$$\Rightarrow b_3 - b_2 = 0 \Rightarrow \boxed{b_3 = b_2}$$

$$\begin{cases} x_1 = b_1 - 2b_3 \\ 0 = 0 \\ x_3 = b_3 \end{cases} \Rightarrow \left(b_1 - 2b_3,\ 0,\ b_3\right)$$

Solvable only if $b_3 = b_2$

## *Exercise*

Find a basis for the null space of $A$. $\quad A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

***Solution***

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 2 & \frac{4}{3} \\ 0 & 1 & 0 & 0 & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & -\frac{5}{12} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $x_4 = s \quad x_5 = t \quad \rightarrow \begin{cases} x_1 = -2x_4 - \frac{4}{3}x_5 = 2s - \frac{4}{3}t \\ x_2 = \frac{1}{6}x_5 = \frac{1}{6}t \\ x_3 = \frac{5}{12}x_5 = \frac{5}{12}t \end{cases}$

The general form of the solution of A$\boldsymbol{x}$ = **0** is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{4}{3} \\ \frac{1}{6} \\ \frac{5}{12} \\ 0 \\ 1 \end{bmatrix}$

Therefore, the vectors $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -\frac{4}{3} \\ \frac{1}{6} \\ \frac{5}{12} \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of *A*.

## *Exercise*

Is it true that is $m = n$ then the row space of *A* equals the column space.

***Solution***

False

Counterexample, let $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$

We have $m = n = 2$, but the row space of *A* contains multiple of $(1,\ 2)$ while the column space of A contains multiples of $(1,\ 3)$.

## *Exercise*

If the row space equals the column space the $A^T = A$

***Solution***

False,

Counterexample, let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Here, the row space and column space are both equal to all of $\boldsymbol{R}^2$ (since *A* is invertible).

But $A \neq A^T$

## *Exercise*

If $A^T = -A$, then the row space of *A* equals the column space.

***Solution***

True,

The row space of *A* equals to the column space of $A^T$, which for this particular *A* equals the column space of $-A$.
Since *A* and $-A$ have the same fundamental subsequences. We conclude that the row space of *A* equals the column space of *A*.

## *Exercise*

Does the matrices *A* and $-A$ share the same 4 subspaces?

***Solution***

True.

The nullspaces are identical because $A\vec{x} = 0 \quad \Leftrightarrow \quad -A\vec{x} = 0$

The column spaces are identical because any vector $\vec{v}$ that can be expressed as $\vec{v} = A\vec{x}$ for some $\vec{x}$ can also be expressed as $\vec{v} = (-A)(-\vec{x})$

## *Exercise*

Is *A* and *B* share the same 4 subspaces then *A* is multiple of *B*.

***Solution***

False

Any invertible $2 \times 2$ matrix will have $R^2$ as its column space and row space and zero vector as its (left and right) nullspace.
However, it is easy to produce 2 invertible $2 \times 2$ matrices that are not multiples of each other, as example

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

## *Exercise*

Suppose $A\vec{x} = b \quad \& \quad C\vec{x} = b$ have the same (complete) solutions for every *b*. Is it true that $A = C$

***Solution***

If $A\vec{x} = C\vec{x} = b$ for all vectors $\vec{x}$ of the correct size.

Then, it is true that $A = C$

### *Exercise*

$A$ and $A^T$ have the same left nullspace?

***Solution***

False,

Counterexample, take any a $1\times 2$ matrix, such as $A=\begin{bmatrix}1 & 1\end{bmatrix}$.

The left nullspace of $A$ contains vectors in $\boldsymbol{R}$ while the left nullspace of $A^T$, which is the right nullspace of $A$, contains vectors in $\boldsymbol{R}^2$.

So, they can't be the same.

# Solution Section 2.9 – Rank and the Fundamental Matrix Spaces

### Exercise

Verify that $rank(A) = rank\left(A^T\right)$

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

**Solution**

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -\frac{6}{7} & -\frac{4}{7} \\ 0 & 1 & \frac{17}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad rank(A) = 2$$

$$A^T = \begin{bmatrix} 1 & -3 & -2 \\ 2 & 1 & 3 \\ 4 & 5 & 6 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad rank\left(A^T\right) = 2$$

$$rank(A) = rank\left(A^T\right) = \underline{2|}$$

### Exercise

Find the rank and nullity of the matrix; then verify that the values obtained satisfy $rank(A) + N(A) = n$

*a)* $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

*b)* $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

*c)* $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

*d)* $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

**Solution**

***a)*** $A=\begin{bmatrix}1 & -1 & 3\\ 5 & -4 & -4\\ 7 & -6 & 2\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & -16\\ 0 & 1 & -19\\ 0 & 0 & 0\end{bmatrix}$

$rank(A)=2;\quad nullity(A)=1;\quad rank(A)+nullity(A)=2+1=3=n \quad \leftarrow number\ \ of\ \ columns$

***b)*** $A=\begin{bmatrix}1 & 4 & 5 & 2\\ 2 & 1 & 3 & 0\\ -1 & 3 & 2 & 2\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & 1 & -\frac{2}{7}\\ 0 & 1 & 1 & \frac{4}{7}\\ 0 & 0 & 0 & 0\end{bmatrix}$

$rank(A)=2;\quad nullity(A)=1;\quad rank(A)+nullity(A)=2+1=3=n$

***c)*** $\begin{bmatrix}1 & 4 & 5 & 6 & 9\\ 3 & -2 & 1 & 4 & -1\\ -1 & 0 & -1 & -2 & -1\\ 2 & 3 & 5 & 7 & 8\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & 1 & 2 & 1\\ 0 & 1 & 1 & 1 & 2\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$

$rank(A)=2;\quad nullity(A)=2;\quad rank(A)+nullity(A)=2+2=4=n$

***d)*** $A=\begin{bmatrix}1 & -3 & 2 & 2 & 1\\ 0 & 3 & 6 & 0 & -3\\ 2 & -3 & -2 & 4 & 4\\ 3 & -6 & 0 & 6 & 5\\ -2 & 9 & 2 & -4 & -5\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & 0 & 2 & \frac{4}{3}\\ 0 & 1 & 0 & 0 & -\frac{1}{6}\\ 0 & 0 & 1 & 0 & -\frac{5}{12}\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$

$rank(A)=3$; $NS(A)=2$; Number of column = 5; $\qquad rank(A)+NS(A)=3+2=5=n$

## *Exercise*

If *A* is an $m\times n$ matrix, what is the largest possible value for its rank and the smallest possible value of the nullity of *A*.

***Solution***

The largest possible value for the rank of an $m\times n$ matrix:

- $n \quad if \quad m\ge n$ (when every column of the *rref* (A) contains a leading 1)
- $m \quad if \quad m<n$ (when every row of the *rref* (A) contains a leading 1)

The smallest possible value for the nullity of an $m\times n$ matrix:

- $0 \quad if \quad m\ge n$ (when every column of the *rref* (A) contains a leading 1)
- $n-m \quad if \quad m<n$ (when every row of the *rref* (A) contains a leading 1)

### *Exercise*

Discuss how the rank of $A$ varies with $t$.

$a)\ A=\begin{bmatrix}1 & 1 & t\\ 1 & t & 1\\ t & 1 & 1\end{bmatrix}$ $\qquad$ $b)\ A=\begin{bmatrix}t & 3 & -1\\ 3 & 6 & -2\\ -1 & -3 & t\end{bmatrix}$

**Solution**

**a)** $\begin{vmatrix}1 & 1 & t\\ 1 & t & 1\\ t & 1 & 1\end{vmatrix}=t+t+t-t^3-1-1$

$$=-t^3+3t-2=0$$

Solve for $t$: $\boxed{t=1,-2,-2}$

Therefore, $rank(A)=3$ for $\forall t-\{1,-2,-2\}$

If $t=1$, $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ $rank(A)=1$

If $t=-2$, $A=\begin{bmatrix}1 & 1 & -2\\ 1 & -2 & 1\\ -2 & 1 & 1\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$ $rank(A)=2$

**b)** $\begin{vmatrix}t & 3 & -1\\ 3 & 6 & -2\\ -1 & -3 & t\end{vmatrix}=6t^2+6+9-6-6t-9t$

$$=6t^2-15t+9=0$$

Solve for $t$: $\boxed{t=1,\ \frac{3}{2}}$

Therefore, $rank(A)=3$ for $\forall t-\left\{1,\ \frac{3}{2}\right\}$

If $t=1$, $A=\begin{bmatrix}1 & 3 & -1\\ 3 & 6 & -2\\ -1 & -3 & 1\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -\frac{1}{3}\\ 0 & 0 & 0\end{bmatrix}$ $rank(A)=2$

If $t=\frac{3}{2}$, $A=\begin{bmatrix}\frac{3}{2} & 3 & -1\\ 3 & 6 & -2\\ -1 & -3 & \frac{3}{2}\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -\frac{5}{6}\\ 0 & 0 & 0\end{bmatrix}$ $rank(A)=2$

### Exercise

Are there values of $r$ and $s$ for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

Has rank 1? Has rank 2? If so, find those values.

***Solution***

Since the third column will always have a nonzero entry, the *rank* will never be 1. (row 1 and row 4 never have a nonzero entry).

If $r = 2$ and $s = 1$, that implies to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 3 \end{bmatrix} \to rank = 2$$

### Exercise

Find the row reduced form ***R*** and the rank $r$ of ***A*** (those depend on $c$).
Which are the pivot columns of ***A***? Which variables are free? What are the special solutions and the nullspace matrix ***N*** (always depending on $c$)?

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & c \end{bmatrix} \quad and \quad A = \begin{bmatrix} c & c \\ c & c \end{bmatrix}$$

***Solution***

***a)*** $c \neq 4 \quad R = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$

$rank(A) = 2$, the pivot columns are 1 and 3, the second variable $x_2$ is free.

The special solution: $x_2 = 1$ which yields to $N = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$c = 4 \quad R = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$

$rank(A) = 1$, the pivot column is column 1, the second and third variables $x_2, x_3$ are free.

The special solution goes into $N = \begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

***b)*** $c \neq 0 \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$

$rank(A) = 1$, the pivot column is the first column, the second variable $x_2$ is free.

The special solution: $x_2 = 1$ which yields to $N = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$c = 0 \quad R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$rank(A) = 0$, the matrix has no pivot column, and both variables are free.

The special solution is: $N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

## *Exercise*

Find the row reduced form $R$ and the rank $r$ of $A$ (those depend on $c$).
Which are the pivot columns of $A$? Which variables are free? What are the special solutions and the nullspace matrix $N$ (always depending on $c$)?

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad \textit{and} \quad A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$$

***Solution***

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & c-1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

***a)*** If $c = 1$, then

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This has only one pivot (first column) and 3 free variables $x_2, x_3, x_4$.

The nullspace matrix: $\begin{pmatrix} -1 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

***b)*** If $c \neq 1$, then

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{c-1}R_2} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are two pivots $\left(C_1,\ C_2\right)$ and 2 free variables $x_3,\ x_4$

The nullspace matrix: $\begin{pmatrix} -2 & -2 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$$

***a)*** If $c = 1 \Rightarrow A = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to 0a + b = 0 \Rightarrow b = 0$$

This has a single pivot in the second column and one free variable with the nullspace matrix $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

***b)*** If $c = 2 \Rightarrow A = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

$$\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to a - 2b = 0 \Rightarrow \mathit{if}\ \ b = 1 \qquad a = 2$$

This has a single pivot in the first column with the nullspace matrix $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

***c)*** Otherwise $c \neq 1,\ 2 \ \Rightarrow\ A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix} \xrightarrow{\frac{1}{1-c}R_1} \begin{bmatrix} 1 & \frac{2}{1-c} \\ 0 & 2-c \end{bmatrix}$

$$\begin{bmatrix} 1 & \frac{2}{1-c} \\ 0 & 2-c \end{bmatrix} \xrightarrow[\frac{1}{2-c}R_2]{} \begin{bmatrix} 1 & \frac{2}{1-c} \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - \frac{2}{1-c}R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The result is the identity matrix with 2 pivots, which has (2 – 2) 0 null space.

### *Exercise*

If $A$ has a rank $r$, then it has an $r$ by $r$ sub-matrix $S$ that is invertible. Remove $m-r$ rows and $n-r$ columns to find an invertible sub-matrix $S$ inside each $A$ (you could keep the pivot rows and pivot columns of $A$).

$$A=\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 4\end{pmatrix} \quad A=\begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 6\end{pmatrix} \quad A=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

**Solution**

If a matrix $\boldsymbol{A}$ has rank $r$, then the

***(dimension of the column space) = (dimension of the row space) = r***

For the invertible sub-matrix S, we need to find $r$ linearly independent rows and $r$ linearly independent columns.

For matrix $\boldsymbol{A}$:

$$\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 4\end{pmatrix} \xrightarrow[R_2-R_1]{} \begin{pmatrix}1 & 2 & 3\\ 0 & 0 & 1\end{pmatrix} \xrightarrow{R_1-3R_2} \begin{pmatrix}1 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}$$

The 1st and 3rd columns are linearly independent, and the 1st and 2nd rows are also linearly independent.

Rank ($\boldsymbol{A}$) = 2.

The sub matrices are: $S_A=\begin{pmatrix}1 & 3\\ 1 & 4\end{pmatrix}$ $S_A=\begin{pmatrix}2 & 3\\ 2 & 4\end{pmatrix}$ $S_A=\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}$

For matrix $\boldsymbol{B}$:

$$\begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 6\end{pmatrix} \xrightarrow[R_2-2R_1]{} \begin{pmatrix}1 & 2 & 3\\ 0 & 0 & 0\end{pmatrix}$$

Rank ($\boldsymbol{B}$) = 1.

The submatrix is: $S_A=(1)$

For matrix $\boldsymbol{C}$:

$$C=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Rank ($\boldsymbol{C}$) = 2.

The submatrix is by disregarding (deleting) 1st column and 2nd row: $S_A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

### *Exercise*

Suppose that column 3 of 4 x 6 matrix is all zero. Then $x_3$ must be a ______ variable. Give one special solution for this matrix.

***Solution***

The $x_3$ must be a ***free variable***.

A special solution for this variable can be taken to be.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

### *Exercise*

Fill in the missing numbers to make $A$ rank1, rank 2, rank 3. (your solution should be 3 matrices)

$$A = \begin{pmatrix} & -3 & \\ 1 & 3 & -1 \\ & 9 & -3 \end{pmatrix}$$

***Solution***

$$A = \begin{pmatrix} a & -3 & b \\ 1 & 3 & -1 \\ c & 9 & -3 \end{pmatrix}$$

If rank ($A$) = 1, then we need the 1st and 3rd to be multiple of the 2nd row to get zero in these rows.

$$A = \begin{pmatrix} a & -3 & b \\ 1 & 3 & -1 \\ c & 9 & -3 \end{pmatrix} \begin{matrix} R_1 + R_2 \\ \\ R_3 - 3R_2 \end{matrix} \begin{pmatrix} a+1 & 0 & b-1 \\ 1 & 3 & -1 \\ c-3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} a+1=0 \\ b-1=0 \\ c-3=0 \end{cases} \rightarrow \begin{cases} a=-1 \\ b=1 \\ c=3 \end{cases}$$

$$A = \begin{pmatrix} -1 & -3 & 1 \\ 1 & 3 & -1 \\ 3 & 9 & -3 \end{pmatrix}$$

If rank ($A$) = 2, then we need the 1st ***or*** 3rd to be multiple of the 2nd row to get zero row.

$$A=\begin{pmatrix} a & -3 & b \\ 1 & 3 & -1 \\ c & 9 & -3 \end{pmatrix} \begin{matrix} R_1+R_2 \\ \\ R_3-3R_2 \end{matrix} \begin{pmatrix} a+1 & 0 & b-1 \\ 1 & 3 & -1 \\ c-3 & 0 & 0 \end{pmatrix} \quad \boxed{c \neq 3}$$

$$A=\begin{pmatrix} -1 & -3 & 1 \\ 1 & 3 & -1 \\ 2 & 9 & -3 \end{pmatrix}$$

If rank ($A$) = 3 (full rank), then the appropriate to start using 0's or 1's to fill the blank.

$$A=\begin{pmatrix} 0 & -3 & 0 \\ 1 & 3 & -1 \\ 1 & 9 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 \\ 0 & -3 & 0 \\ 1 & 9 & -3 \end{pmatrix} \xrightarrow[R_3-R_1]{} \begin{pmatrix} 1 & 3 & -1 \\ 0 & -3 & 0 \\ 0 & 6 & -2 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{3}R_2} \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 6 & -2 \end{pmatrix} \xrightarrow[R_1-6R_2]{R_1-3R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\xrightarrow[-\frac{1}{2}R_3]{} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1+R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, it has rank 3.

## *Exercise*

Fill out these matrices so that they have rank 1:

$$A=\begin{pmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{pmatrix} \qquad B=\begin{pmatrix} 2 & & \\ 1 & & \\ 2 & 6 & -3 \end{pmatrix} \qquad M=\begin{pmatrix} a & b \\ c & \end{pmatrix}$$

***Solution***

Rank = 1 means that all the rows are multiples of each other.

$$A=\begin{pmatrix}1 & 2 & 4\\ 2 & a & b\\ 4 & c & d\end{pmatrix}\xrightarrow[R_3=4R_1]{R_2=2R_1}\begin{matrix}a=2(2) & b=2(4)\\ c=4(2) & d=4(4)\end{matrix}$$

$$A=\begin{pmatrix}1 & 2 & 4\\ 2 & 4 & 8\\ 4 & 8 & 16\end{pmatrix}$$

$$B=\begin{pmatrix}2 & a & b\\ 1 & c & d\\ 2 & 6 & -3\end{pmatrix}\xrightarrow[R_2=\frac{1}{2}R_3]{R_1=R_3}\begin{matrix}a=6 & b=-3\\ c=3 & d=-\frac{3}{2}\end{matrix}$$

$$B=\begin{pmatrix}2 & 6 & -3\\ 1 & 3 & -\frac{3}{2}\\ 2 & 6 & -3\end{pmatrix}$$

$$M=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\xrightarrow[R_2=\frac{c}{a}R_1]{}d=\frac{c}{a}b\to M=\begin{pmatrix}a & b\\ c & \frac{bc}{a}\end{pmatrix}$$

## Exercise

Suppose $A$ and $B$ are $n$ by $n$ matrices, and $AB = I$. Prove from $rank(AB)\le rank(A)$ that the $rank(A)=n$. So, $A$ is invertible and $B$ must be its two-sided inverse. Therefore $BA = I$ (which is not so obvious!).

***Solution***

Since $A$ is $n$ by $n$ $\Rightarrow rank(A)\le n$

$$n=rank\left(I_n\right)=rank(AB)\le rank(A)$$

## Exercise

Every $m$ by $n$ matrix of rank $r$ reduces to ($m$ by $r$) times ($r$ by $n$):

$$A = \text{(pivot columns of } A\text{) (first } r \text{ rows of } R) =(COL)(ROW)^T$$

Write the 3 by 4 matrix $A=\begin{pmatrix}1 & 1 & 2 & 4\\ 1 & 2 & 2 & 5\\ 1 & 3 & 2 & 6\end{pmatrix}$ as the product of the 3 by 2 from the pivot columns and the 2 by 4 matrix from $R$.

***Solution***

$$A=\begin{pmatrix}1&1&2&4\\1&2&2&5\\1&3&2&6\end{pmatrix}\begin{matrix}\\R_2-R_1\\R_3-R_1\end{matrix}\begin{pmatrix}1&1&2&4\\0&1&0&1\\0&2&0&2\end{pmatrix}$$

$$\xrightarrow[R_3-2R_2]{}\begin{pmatrix}1&1&2&4\\0&1&0&1\\0&0&0&0\end{pmatrix}$$

$$\xrightarrow{R_1-R_2}\begin{pmatrix}1&0&2&3\\0&1&0&1\\0&0&0&0\end{pmatrix}$$

The pivots columns are the $1^{st}$ and $2^{nd}$ column.

$$A=\begin{pmatrix}1&1&2&4\\1&2&2&5\\1&3&2&6\end{pmatrix}=\begin{pmatrix}1&1\\1&2\\1&3\end{pmatrix}\begin{pmatrix}1&0&2&3\\0&1&0&1\end{pmatrix}$$

## *Exercise*

Suppose $R$ is $m$ by $n$ matrix of rank $r$, with pivot columns first: $\begin{bmatrix}I&F\\0&0\end{bmatrix}$

*a)* What are the shapes of those 4 blocks?

*b)* Find the right-inverse $B$ with $RB = I$ if $r = m$.

*c)* Find the right-inverse $C$ with $CR = I$ if $r = n$.

*d)* What is the reduced row echelon form of $R^T$ (with shapes)?

*e)* What is the reduced row echelon form of $R^T R$ (with shapes)?

Prove that $R^T R$ has the same nullspace as R. Then show that $A^T A$ always has the same nullspace as $A$ (a value fact).

*f)* Suppose you allow elementary column operations on $\boldsymbol{A}$ as well as elementary row operations (which get to $\boldsymbol{R}$). What is the "row-and-column reduced form" for an $m$ by $n$ matrix of rank $\boldsymbol{r}$?

***Solution***

**a)** $\begin{bmatrix}I&F\\0&0\end{bmatrix}:\begin{bmatrix}r\times r & r\times(n-r)\\(m-r)\times r & (m-r)\times(n-r)\end{bmatrix}$

**b)** $R=\begin{bmatrix}I & F\end{bmatrix}$

$RB=I\Rightarrow\begin{bmatrix}I & F\end{bmatrix}B=I$

$$\begin{bmatrix}I & F\end{bmatrix}\begin{pmatrix}M\\N\end{pmatrix}=I$$

$$IM + FN = I \Rightarrow \begin{cases} M = I \\ N = 0 \end{cases} \rightarrow \quad F : r \times (n-r)$$

$$B = \begin{bmatrix} I_{r\times r} \\ 0_{(n-r)\times r} \end{bmatrix}$$

***c)*** $R = \begin{bmatrix} I & 0 \end{bmatrix}$

$$CR = I \Rightarrow C\begin{bmatrix} I & 0 \end{bmatrix} = I$$

$$C = \begin{bmatrix} I_{r\times r} & 0_{r\times(m-r)} \end{bmatrix}$$

***d)*** $R^T = \begin{bmatrix} I_{r\times r} & 0_{(m-r)\times r} \\ F_{r\times(n-r)} & 0_{(m-r)\times(n-r)} \end{bmatrix} \Rightarrow rref\left(R^T\right) = \begin{bmatrix} I_{r\times r} & 0_{(m-r)\times r} \\ 0_{(n-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}$

***e)*** $R^T R = \begin{bmatrix} I & 0 \\ F & 0 \end{bmatrix}\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ FI & 0 \end{bmatrix}$

$FI : r \times (n-r) \quad r \times r$, the inner is not equal but to make work, we can use the $F$ transpose.

$(n-r)\times r \quad r \times r \Rightarrow F^T I = F^T$

$$R^T R = \begin{bmatrix} I & 0 \\ F & 0 \end{bmatrix}\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I & F \\ F^T & 0 \end{bmatrix}$$

$$rref\left(R^T R\right) = \begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix} \underline{= R}$$

So, that $N(A) = N\left(rref(A)\right)$ for any matrix ***A***. So, $N(A) = N\left(R^T R\right)$

***f)*** After getting to $R$ we can use the column operations to get rid of $F$.

$$\begin{bmatrix} I_{r\times r} & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}$$

### *Exercise*

True or False (check addition or give a counterexample)

*a)* The symmetric matrices in $M\left(with\ A^T = A\right)$ from a subspace.

*b)* The skew-symmetric matrices in $M\left(with\ A^T = -A\right)$ from a subspace.

*c)* The un-symmetric matrices in $M\left(with\ A^T \neq A\right)$ from a subspace.

*d)* Invertible matrices

*e)* Singular matrices

***Solution***

**a)** True: $A^T = A$ and $B^T = B$ lead to $(A+B)^T = A^T + B^T = A + B$

**b)** True: $A^T = -A$ and $B^T = -B$ lead to $(A+B)^T = A^T + B^T = -A - B = -(A+B)$

**c)** False: $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

**d)** False: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ are invertible matrices but $A + B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ is not invertible. $\therefore$ The zero matrix is not invertible but any linear subspace should contain the zero matrix. So, invertible matrices do not form a linear subspace.

**e)** False: $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are singular matrices but $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ is not singular.

### *Exercise*

Let $A = \begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 2 & 4 & -3 & 7 & 0 \\ 3 & 6 & -5 & 10 & -2 \\ 5 & 10 & -9 & 16 & 0 \end{pmatrix}$

*a)* Reduce $A$ to row-reduced echelon from.

*b)* What is the rank of $A$?

*c)* What are the pivots?

*d)* What are the free variables?

*e)* Find the special solutions. What is the nullspace $N(A)$?

*f)* Exhibit an $r$ x $r$ submatrix of $A$ which is invertible, where $r = rank(A)$. (An $r$ x $r$ submatrix of $A$ is obtained by keeping $r$ rows and $r$ columns of $A$)

***Solution***

***a)*** $A=\begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 2 & 4 & -3 & 7 & 0 \\ 3 & 6 & -5 & 10 & -2 \\ 5 & 10 & -9 & 16 & 0 \end{pmatrix} \begin{array}{l} \\ R_2-2R_1 \\ R_3-3R_1 \\ R_4-5R_1 \end{array} \begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$

$$\begin{array}{l} \\ \\ R_3-R_2 \\ R_4-R_2 \end{array} \begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

***b)*** Rank(***A***) = 3

***c)*** The pivots are $x_1, x_3, x_5$

***d)*** The free variables are $x_2, x_4$

***e)*** $\begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} -\frac{1}{2}R3 \begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} R_1+2R_2 \begin{pmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Let $x = x_2 s_2 + x_4 s_4$

$$Rx = \begin{pmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} x_1 + 2x_2 + 5x_4 = 0 \\ x_3 + x_4 = 0 \\ x_5 = 0 \end{cases}$$

**1.** Set $x_2 = 1, \quad x_4 = 0 \to \begin{cases} x_1 + 2 = 0 \Rightarrow x_1 = -2 \\ x_3 = 0 \\ x_5 = 0 \end{cases}$

The special solution: $s_2 = (-2,\ 1,\ 0,\ 0,\ 0)$

**2.** Set $x_2 = 0, \quad x_4 = 1 \to \begin{cases} x_1 + 5 = 0 \Rightarrow x_1 = -5 \\ x_3 + 1 = 0 \Rightarrow x_3 = -1 \\ x_5 = 0 \end{cases}$

The special solution: $s_3 = (-5,\ 0,\ -1,\ 1,\ 0)$

The nullspace is the set $\left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

***f)*** The pivot rows and columns must be included in a submatrix. To do that, just take the rows and columns of ***A*** containing pivots, which are columns 1, 3, 5 and rows 1, 2, 3. That will give us a 3 by 3 submatrix. Therefore, this submatrix of ***A*** will be invertible.

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -3 & 0 \\ 3 & -5 & -2 \end{pmatrix}$$

## *Exercise*

Let $A = \begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 2 & 1 & 0 & 0 & -15 \\ 6 & -1 & -8 & -1 & -47 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix}$

*a)* Reduce ***A*** to (ordinary) echelon from.

*b)* What the pivots?

*c)* What are the free variables?

*d)* Reduce ***A*** to row-reduced echelon form.

*e)* Find the special solutions. What is the nullspace $N(A)$?

*f)* What is the rank of ***A***?

*g)* Give the complete solution to $Ax = b,\ where\ \ b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

***Solution***

***a)*** $A = \begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 2 & 1 & 0 & 0 & -15 \\ 6 & -1 & -8 & -1 & -47 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix} \xrightarrow[R_3+6R_1]{R_2+2R_1} \begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 0 & 5 & 10 & 0 & -5 \\ 0 & 11 & 22 & -1 & -17 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix}$

$$\begin{array}{l} \\ \\ 5R_3-11R_2 \\ 5R_4-2R_2 \end{array}\begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 0 & 5 & 10 & 0 & -5 \\ 0 & 0 & 0 & -5 & -30 \\ 0 & 0 & 0 & 15 & 90 \end{pmatrix} \xrightarrow{R_4+3R_3} \begin{pmatrix} \boxed{-1} & 2 & 5 & 0 & 5 \\ 0 & \boxed{5} & 10 & 0 & -5 \\ 0 & 0 & 0 & \boxed{-5} & -30 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

***b)*** The pivots are –1, 5, and –5 (Columns 1, 2, 4)

***c)*** The free variables are 3rd and 5th $(x_3, x_5)$

***d)*** $$\begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 0 & 5 & 10 & 0 & -5 \\ 0 & 0 & 0 & -5 & -30 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{c} -R_1 \\ \frac{1}{5}R_2 \\ -\frac{1}{5}R_3 \\ \\ \end{array} \begin{pmatrix} 1 & -2 & -5 & 0 & -5 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1+2R_2} \begin{pmatrix} 1 & 0 & -1 & 0 & -7 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 & -1 & 0 & -7 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

***e)*** Let $x = x_3 s_1 + x_5 s_2$

$$Rx = \begin{pmatrix} 1 & 0 & -1 & 0 & -7 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} x_1 - x_3 - 7x_5 = 0 \\ x_2 + 2x_3 - x_5 = 0 \\ x_4 + 6x_5 = 0 \end{cases}$$

**1.** Set $x_3 = 1, \quad x_5 = 0 \to \begin{cases} x_1 - 1 = 0 \\ x_2 + 2 = 0 \\ x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = -2 \\ x_4 = 0 \end{cases}$

The special solution: $s_1 = (1,\ -2,\ 1,\ 0,\ 0)$

**2.** Set $x_3 = 0, \quad x_5 = 1 \to \begin{cases} x_1 - 7 = 0 \\ x_2 - 1 = 0 \\ x_4 + 6 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 7 \\ x_2 = 1 \\ x_4 = -6 \end{cases}$

The special solution: $s_2 = (7,\ 1,\ 0,\ -6,\ 1)$

The nullspace is the set $\left\{ x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 7 \\ 1 \\ 0 \\ -6 \\ 1 \end{pmatrix} \right\}$

***f)*** Rank($\boldsymbol{A}$) = 3

***g)*** $Ax = b,\ where\ \ b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

The complete solution = (the particular solution) + (special solution)

$$x = x_p + x_n$$

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 7 \\ 1 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

## *Exercise*

Let $A = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

*a)* Reduce $A$ to row-reduced echelon form.
*b)* What is the rank of $A$?
*c)* What the pivots variables?
*d)* What are the free variables?
*e)* Find the special solutions.
*f)* What is the nullspace $N(A)$?

***Solution***

***a)*** $\begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} \\ R_2 - R_1 \\ \\ R_3 - 2R_1 \\ \\ \end{matrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & -1 & -4 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} \\ -R_2 \\ R_3 - R_2 \\ R_4 - 2R_2 \\ \\ \end{matrix}$

$\begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ \\ R_5 - R_4 \end{matrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$R = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{matrix} x_1 = -2x_2 - 3x_4 \\ x_3 = -4x_4 \\ x_5 = 0 \end{matrix}$

***b)*** Rank($\boldsymbol{A}$) = 3

***c)*** The pivots variables are: $x_1,\ x_3,\ x_5$

***d)*** The free variables are: $x_2,\ x_4$

***e)*** Let $x = x_2 s_1 + x_4 s_2$

$$\begin{cases} x_1 = -2x_2 - 3x_4 \\ x_3 = \qquad\ \ -4x_4 \end{cases}$$

Set $x_2 = 1, \quad x_4 = 0$

The special solution: $s_1 = (-2,\ 1,\ 0,\ 0,\ 0)$

Set $x_2 = 0, \quad x_4 = 1$ ;

The special solution: $s_2 = (-3,\ 0,\ -4,\ 1,\ 0)$

***f)*** The nullspace is the set $\left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix} \right\}$

$$N(A)=\begin{pmatrix} -2 & -3 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

## *Exercise*

Let $A=\begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}$

*a)* Reduce $A$ to row-reduced echelon form.

*b)* What is the rank of $A$?

*c)* What the pivots?

*d)* What are the free variables?

*e)* Find the special solutions.

*f)* What is the nullspace $N(A)$?

***Solution***

***a)***

$$\begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix} \begin{array}{l} \\ 3R_2-R_1 \\ 3R_3-2R_1 \\ R_4-2R_1 \end{array} \to \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 0 & 0 & -3 & -15 & -3 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & -1 & -5 & 0 \end{pmatrix} \begin{array}{l} \\ -R_2 \\ \\ 3R_4+R_2 \end{array}$$

$$\begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 0 & 0 & 3 & 15 & 3 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \begin{array}{l} \frac{1}{3}R_1 \\ \frac{1}{3}R_2 \\ \frac{1}{3}R_3 \\ R_4-R_3 \end{array} \to \begin{pmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \\ R_2-R_3 \\ \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{aligned} x_1 &= -7x_2 - 3x_4 \\ x_3 &= \qquad\ \ -5x_4 \\ x_5 &= 0 \end{aligned}$$

***b)*** Rank($\boldsymbol{A}$) = 3

***c)*** The pivots variables are: $x_1,\ x_3,\ x_5$

***d)*** The free variables are: $x_2,\ x_4$

***e)*** Let $x = x_2 s_1 + x_4 s_2$

$$\begin{cases} x_1 = -7x_2 - 3x_4 \\ x_3 = \qquad\quad -5x_4 \end{cases}$$

Set $x_2 = 1, \quad x_4 = 0$

The special solution: $s_1 = (-7,\ 1,\ 0,\ 0,\ 0)$

Set $x_2 = 0, \quad x_4 = 1$ ;

The special solution: $s_2 = (-3,\ 0,\ -5,\ 1,\ 0)$

***f)*** The nullspace is the set $\left\{ x_2 \begin{pmatrix} -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix} \right\}$

$$N(A) = \begin{pmatrix} -7 & -3 \\ 1 & 0 \\ 0 & -5 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

## ***Exercise***

The 3 by 3 matrix $A$ has rank 2.

$$Ax = b \quad \text{is} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 + 5x_4 &= b_1 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 &= b_2 \\ 3x_1 + 6x_2 + 7x_3 + 13x_4 &= b_3 \end{aligned}$$

*a)* Reduce $[A \quad b]$ to $[U \quad c]$, so that $Ax = b$ becomes triangular system $Ux = c$.

*b)* Find the condition on $(b_1,\ b_2,\ b_3)$ for $Ax = b$ to have a solution

*c)* Describe the column space of $A$. Which plane in $\boldsymbol{R}^3$?

*d)* Describe the nullspace of $A$. Which special solutions in $\boldsymbol{R}^4$?

*e)* Find a particular solution to $Ax = (0,\ 6,\ -6)$ and then complete solution.

***Solution***

***a)*** $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2-2b_1 \\ 0 & 0 & -2 & -2 & b_3-3b_1 \end{array}\right]$

$$\xrightarrow[R_3+R_2]{} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3+b_2-5b_1 \end{array}\right]$$

***b)*** The last equation $b_3+b_2-5b_1=0$ shows the solvability condition.

***c)*** (***i***) The column space is the plane containing all combinations of the pivot columns: 1st (1, 2, 3) and 3rd (3, 8, 7).

(***ii***) The column space contains all vectors with $b_3+b_2-5b_1=0$. That makes $Ax=b$ solvable, so ***b*** is in the column space. All columns of $A$ pass this test $b_3+b_2-5b_1=0$. This is the equation for the plane in (***i***).

***d)*** The special solutions have free variables:

$$\begin{cases} x_1+2x_2+3x_3+5x_4=0 \\ 2x_3+2x_4=0 \end{cases} \Rightarrow \begin{cases} x_1=-2x_2-2x_4 \\ x_3=-x_4 \end{cases}$$

$$x_2=1,\ x_4=0 \Rightarrow \begin{cases} x_1=-2 \\ x_3=0 \end{cases} \rightarrow s_1=\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$x_2=0,\ x_4=1 \Rightarrow \begin{cases} x_1=-2 \\ x_3=-1 \end{cases} \rightarrow s_2=\begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

The nullspace $N(A)$ in $\boldsymbol{R}^4$ contains all $x_n=x_2\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}+x_4\begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

***e)*** One particular solution $x_p$ has free variables = zero.

$$\begin{cases} x_1+2x_2+3x_3+5x_4=0 \\ 2x_3+2x_4=6 \end{cases} \Rightarrow \begin{cases} x_1=-2x_2-3x_3-5x_4 \\ x_3=3-x_4 \end{cases} \rightarrow\Rightarrow \begin{cases} x_1=-2x_2-9-2x_4 \\ x_3=3-x_4 \end{cases}$$

$$x_2=x_4=0 \Rightarrow \begin{cases} x_1=-9 \\ x_3=3 \end{cases}$$

$$x_p = \begin{pmatrix} -9 \\ 0 \\ -3 \\ 0 \end{pmatrix}$$

The complete solution to $Ax = (0,\ 6,\ -6)$ is $x = x_p + all\ x_n$

### *Exercise*

Find the special solutions and describe the complete solution to $Ax = 0$ for

$A_1$ = 3 by 4 zero matrix $\qquad A_2 = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \qquad A_3 = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$

Which are the pivot columns? Which are the free variables? What is the $R$ (Reduced Row Echelon matrix) in each case?

***Solution***

$A_1 x = 0$ has 4 solutions. They are the columns $s_1,\ s_2,\ s_3,\ s_4$ of the identity matrix (4 by 4).

The Nullspace is of $\boldsymbol{R}^4$.

The complete solution: $x = c_1 s_1 + c_2 s_2 + c_3 s_3 + c_4 s_4$ in $\boldsymbol{R}^4$.

There are no pivot columns; all variables are free; the reduced $R$ is the same zero matrix as $A_1$.

$A_2 x = 0$

$$\Rightarrow A_2 x = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 + 2x_2 = 0$$

The vector solution: $s = (-2,\ 1)$, The first column of $A_2$ is its pivot column, and $x_2$ is the free variable.

$$\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \frac{1}{3}R_1 \rightarrow \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \xrightarrow[R_2 - R_1]{} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

$$R_3 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

All variables are free. There are three special solutions to $A_3 x = 0$

$$s_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad s_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad s_3 = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The complete solution: $x = c_1 s_1 + c_2 s_2 + c_3 s_3$ .

## *Exercise*

Create a 3 by 4 matrix whose special solutions to $Ax = 0$ are $s_1$ *and* $s_2$ :

$$s_1 = \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} -2 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

You could create the matrix A in row reduced form R. Then describe all possible matrices A with the required Nullspace $N(A) =$ all combinations of $s_1$ *and* $s_2$ .

***Solution***

We can write the solution:

$x = x_2 s_1 + x_4 s_2$

$$x_2 \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -6 \\ 1 \end{pmatrix} = \begin{pmatrix} -3x_2 - 2x_4 \\ 2x_2 \\ -6x_4 \\ x_4 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3x_2 - 2x_4 \\ 2x_2 \\ -6x_4 \\ x_4 \end{pmatrix} \rightarrow \begin{cases} x_1 = -3x_2 - 2x_4 \\ \\ x_3 = -6x_4 \end{cases}$$

$$\rightarrow \begin{cases} x_1 + 3x_2 + 2x_4 = 0 \\ \\ x_3 + 6x_4 = 0 \end{cases} \Rightarrow \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The entries 3, 2, 6 are the negatives of -3, -2, -6 in the special solutions.

Every 3 by 4 matrix has at least one special solution. These $A$'s have two.

### *Exercise*

The plane $x-3y-z=12$ is parallel to the plane $x-3y-z=0$. One particular point on this plane is $(12,\ 0,\ 0)$. All points on the plane have the form (fill the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}$$

***Solution***

$$x-3y-z=12 \Rightarrow x=3y+z+12$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 12+3y+z \\ y \\ z \end{pmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} z \\ 0 \\ 1 \end{bmatrix}$$

### *Exercise*

Construct a matrix whose column space contains (1, 1, 5) and (0, 3, 1) and whose Nullspace contains $(1,\ 1,\ 2)$.

***Solution***

$$A = \begin{pmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+2a \\ 1+3+2b \\ 5+1+2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 1+2a=0 \\ 4+2b=0 \\ 6+2c=0 \end{cases} \rightarrow \begin{cases} 2a=-1 \\ 2b=-4 \\ 2c=-6 \end{cases} \rightarrow \begin{cases} a=-\frac{1}{2} \\ b=-2 \\ c=-3 \end{cases}$$

$$A = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{pmatrix}$$

### *Exercise*

Construct a matrix whose column space contains (1, 1, 0) and (0, 1, 1) and whose Nullspace contains (1, 0, 1) and (0, 0, 1).

***Solution***

It is impossible. Matrix $\boldsymbol{A}$ must be 3 by 3. Since the nullspace is supposed to contain two independent vectors, $\boldsymbol{A}$ can have at most $3-2=1$ pivots. Since the column space supposes to contain two independent vectors. $A$ must has at least 2 pivots. These conditions can't both be met.

### *Exercise*

Construct a matrix whose column space contains (1, 1, 1) and whose Nullspace contains $(1,\ 1,\ 1,\ 1)$.

***Solution***

The matrix needs to be 3 by 4 matrix.

$$\begin{pmatrix} 1 & a & b & c \\ 1 & d & e & f \\ 1 & g & h & i \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 1+a+b+c=0 \\ 1+d+e+f=0 \\ 1+g+h+i=0 \end{cases} \Rightarrow \begin{cases} a+b+c=-1 \\ d+e+f=-1 \\ g+h+i=-1 \end{cases} \to \begin{cases} if \quad b=c=0 \quad a=-1 \\ if \quad d=f=0 \quad e=-1 \\ if \quad g=h=0 \quad i=-1 \end{cases}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

### *Exercise*

How is the Nullspace $N(C)$ related to the spaces N($A$) and N($B$), if $C=\begin{bmatrix} A \\ B \end{bmatrix}$?

***Solution***

$$Cx=\begin{bmatrix} Ax \\ Bx \end{bmatrix}=0=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If and only if $Ax$ =0 and $Bx$ = 0.

$N(C)=N(A)\bigcap N(B)$

### *Exercise*

Why does no 3 by 3 matrix have a nullspace that equals its column space?

***Solution***

If nullspace = column space, then $n - r = r$ (there are $r$ pivots).
For $n = 3 \Rightarrow 3 = 2\boldsymbol{r}$ is impossible.

### *Exercise*

If $AB = 0$ then the column space $B$ is contained in the _______ of $A$. Give an example of $A$ and $B$.

***Solution***

If $AB = 0$ then the column space $B$ is contained in the ***nullspace*** of $A$.

Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

### *Exercise*

True or false (with reason if true or example to show it is false)

*a)* A square matrix has no free variables.

*b)* An invertible matrix has no free variables.

*c)* An $m$ by $n$ matrix has no more than $\boldsymbol{n}$ pivot variables.

*d)* An $m$ by $n$ matrix has no more than $\boldsymbol{m}$ pivot variables.

***Solution***

***a)*** False. Any matrix with fewer than full number of pivots will. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

***b)*** True. Since it is invertible, we will get the full number of pivots. The nullspace has dimension, so we have 0 free variables.

***c)*** True, the number of pivot variables is the dimension of the nullspace, which is at most the number of columns. The nullspace dimension + column space dimension = number of columns.

***d)*** True, in reduced echelon matrix the pivot columns are all 0 except for a single 1, and there are only up to $m$ vectors of this type.

### *Exercise*

Suppose an $m$ by $n$ matrix has $r$ pivots. The number of special solutions is ______.

The Nullspace contains only $x = 0$ when $r =$ _______.

The column space is all of $\boldsymbol{R}^m$ when $r =$ ______.

***Solution***

Suppose an $m$ by $n$ matrix has $r$ pivots. The number of special solutions is _ $\boldsymbol{n-r}$_.

The Nullspace contains only $x = 0$ when $r =$ _ $\boldsymbol{n}$ _.

The column space is all of $\boldsymbol{R}^m$ when $r =$ _$\boldsymbol{m}$ _.

### *Exercise*

Find the complete solution in the form $x_p + x_n$ to these full rank system:

$a)\ \ x+y+z=4 \qquad\qquad b)\ \begin{matrix} x+y+z=4 \\ x-y+z=4 \end{matrix}$

***Solution***

$\boldsymbol{a})\ \ x+y+z=4$

The equivalent matrix is given by: $\begin{cases} \quad Ax=4 \\ A=\begin{pmatrix}1 & 1 & 1\end{pmatrix} \end{cases}$

The complete solution in the form $x = x_p + x_n$

$x_n$ is the homogeneous solution to $Ax_n = 0$

Size of $\boldsymbol{A}$ is $m = 1$ and $n = 3$, rank($\boldsymbol{A}$) = $r = 1$

$$Ax_n = 0 \Rightarrow \begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow x_1+x_2+x_3=0 \Rightarrow \boxed{x_1=-x_2-x_3}$$

Set $x_2 = 1, \quad x_3 = 0$ The special solution: $s_1 = (-1,\ 1,\ 0)$

Set $x_2 = 0, \quad x_3 = 1$ The special solution: $s_2 = (-1,\ 0,\ 1)$

The nullspace is the set $\left\{ x_2\begin{pmatrix}-1\\1\\0\end{pmatrix} + x_3\begin{pmatrix}-1\\0\\1\end{pmatrix} \right\}$

$x = 4-y-z \Rightarrow x_1 = 4-x_2-x_3$

Set $x_2 = 0, \quad x_3 = 0$ that implies to the particular solution: $x_p = \begin{pmatrix}4\\0\\0\end{pmatrix}$

The complete solution in the form $x = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Note: that the null space of A is spanned by the two linearly independent vectors $(-1,\ 1,\ 0)^T$ *and* $(-1,\ 0,\ 1)^T$

***b***) $\begin{matrix} x + y + z = 4 \\ x - y + z = 4 \end{matrix}$

The equivalent matrix is given by: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ and $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & 4 \end{array}\right] \xrightarrow[R_2 - R_1]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & 0 \end{array}\right] \xrightarrow[-\frac{1}{2}R_2]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

The pivots are $x_1, x_2$ ; The free variable is $x_3$

Rank $r = 2$, $n = 2$, $m = 3$. The nullspace has dimension $m - r = 1$.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + x_3 = 0 \to x_1 = -x_3 \\ \qquad x_2 = 0 \end{cases}$$

If $x_3 = 1 \Rightarrow x_1 = -1$ The special solution: $s_1 = (-1,\ 0,\ 1)$

The nullspace is the set $\left\{ x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Set $x_3 = 0$ that implies $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = 4 \\ x_2 = 0 \end{cases}$

Then the particular solution: $x_p = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$

The complete solution in the form $x = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

## *Exercise*

Find the complete solution in the form $x_p + x_n$ to the system: $\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

***Solution***

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{cccc|c} 1 & 3 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The pivots are $x_1, x_3$; The free variables are $x_2, x_4$

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + 3x_2 = 0 \\ x_3 + 2x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -3x_2 \\ x_3 = -2x_4 \end{cases}$$

1. Set $x_2 = 1, \quad x_4 = 0$ The special solution: $s_1 = (-3,\ 1,\ 0,\ 0)$
2. Set $x_2 = 0, \quad x_4 = 1$ The special solution: $s_2 = (0,\ 0,\ -2,\ 1)$

The special solution: $x_n = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + 3(0) = \frac{1}{2} \\ x_3 + 2(0) = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{2} \\ x_3 = \frac{1}{2} \end{cases}$$

Then the particular solution: $x_p = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$

The complete solution in the form $x = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$

### *Exercise*

If ***A*** is 3 x 7 matrix, its largest possible rank is ________. In this case, there is a pivot in every ________ of ***U*** and ***R***, the solution to $Ax = b$ _________ (always exists or is unique), and the column space of ***A*** is _________. Construct an example of such a matrix ***A***.

***Solution***

If ***A*** is 3 x 7 matrix, its largest possible rank is **3**. In this case, there is a pivot in every ***row*** of ***U*** and *R*, the solution to $Ax = b$ ***always exists***, and the column space of ***A*** is $\boldsymbol{R}^3$.

$$A = \begin{pmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{pmatrix}$$

$rank(A) \le 3$, that implies that you have 3 pivots (1 each row)

$$A = \begin{pmatrix} 1 & 0 & 0 & * & * & * & * \\ 0 & 1 & 0 & * & * & * & * \\ 0 & 0 & 1 & * & * & * & * \end{pmatrix} \rightarrow A = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 5 & 6 & 7 & 8 \\ 0 & 0 & 1 & 9 & 10 & 11 & 12 \end{pmatrix}$$

### *Exercise*

If ***A*** is 6 x 3 matrix, its largest possible rank is ________. In this case, there is a pivot in every ________ of ***U*** and *R*, the solution to $Ax = b$ _________ (always exists or is unique), and the nullspace of ***A*** is _________. Construct an example of such a matrix ***A***.

***Solution***

If ***A*** is 6 x 3 matrix, its largest possible rank is **3**. In this case, there is a pivot in every ***column*** of ***U*** and *R*, the solution to $Ax = b$ ***is unique***, and the column space of ***A*** is $\{\mathbf{0}\}$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

## *Exercise*

Find the rank of $A, A^T A$ and $AA^T$ for $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix}$

**Solution**

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix}_{R_3 + R_1} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{pmatrix} \tfrac{1}{2} R_2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 3 \end{pmatrix}_{R_3 - 3R_2} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \boxed{rank(A) = 2}$$

$$A^T A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 9 \end{pmatrix} \xrightarrow[2R_2 + R_1]{} \begin{pmatrix} 2 & -1 \\ 0 & 17 \end{pmatrix} \quad \Rightarrow \quad \boxed{rank\left(A^T A\right) = 2}$$

$$AA^T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 4 \\ 1 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 4 \\ 1 & 4 & 5 \end{pmatrix} \xrightarrow[2R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 6 & 9 \end{pmatrix} \xrightarrow[R_3 - 3R_2]{} \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \boxed{rank\left(AA^T\right) = 2}$$

$\therefore rank(A) = rank\left(A^T A\right) = rank\left(AA^T\right)$ for any matrix, $\boldsymbol{A}$.

## *Exercise*

Explain why these are all false:

*a)* The complete solution is any linear combination of $x_p$ *and* $x_n$.

*b)* A system $Ax = b$ has at most one particular solution.

*c)* The solution $x_p$ with all free variables zero is the shortest solution (minimum length $\|x\|$). Find a 2 by 2 counterexample.

*d)* If $A$ is invertible there is no solution $x_n$ in the null space.

**Solution**

***a)*** The coefficient of $x_p$ must be one.

**b)** If $x_n \in N(A)$ is the nullspace of $\boldsymbol{A}$ and $x_p$ is one particular solution, then $x_p$ *and* $x_n$ is also a particular solution.

**c)** If $\boldsymbol{A}$ is a 2 by 2 matrix of rank 1, then the solution to $Ax = b$ form a line parallel to the line that the nullspace. The line $x + y = 1$ gives such an example.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad x_p = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Then $\left\| x_p \right\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{2\frac{1}{4}} = \frac{1}{\sqrt{2}} < 1$ while the particular solutions having some coordinate equal to zero are (1, 0) and (0, 1) and they both have $\left\| x_p \right\| = 1$

**d)** There is always $x_n = 0$

## Exercise

Write down all known relation between $r$ and $m$ and $n$ if $Ax = b$ has

*a)* No solution for some $\boldsymbol{b}$.

*b)* Infinitely many solutions for every $\boldsymbol{b}$.

*c)* Exactly one solution for some $\boldsymbol{b}$, no solution for another $\boldsymbol{b}$.

*d)* Exactly one solution for every $\boldsymbol{b}$.

***Solution***

**a)** The system has less than full row rank: $r < m$.

**b)** The system has full row rank and less than full column rank: $m = r < n$.

**c)** The system has full column rank and less than full row rank: $n = r < m$.

**d)** The system has full row and column rank (it is invertible): $m = r = n$.

## Exercise

Find a basis for its row space, find a basis for its column space, and determine its rank

$$a)\ \begin{bmatrix} 0 & 2 & -3 & 4 & 1 & 2 & 1 & 7 \\ 0 & 0 & 3 & -2 & 0 & 4 & -5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 6 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad b)\ \begin{bmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{bmatrix}$$

***Solution***

**a)** Row Space: every row

Column Space: $\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\3\\0\\0\\0\end{bmatrix}, \begin{bmatrix}2\\4\\6\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-5\\-2\\2\\0\end{bmatrix}$

Rank = 4

***b)*** $\begin{bmatrix}3 & 2 & -1\\6 & 3 & 5\\-3 & -1 & -6\\0 & -1 & 7\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & \frac{13}{3}\\0 & 1 & -7\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$

Row Space: $\begin{bmatrix}3 & 2 & -1\end{bmatrix}, \begin{bmatrix}6 & 3 & 5\end{bmatrix}$

Column Space: $\begin{bmatrix}3\\6\\-3\\0\end{bmatrix}, \begin{bmatrix}2\\3\\-1\\-1\end{bmatrix}$

Rank = 2

## *Exercise*

Find a basis for the row space, find a basis for the null space, find $\dim RS$, find $\dim NS$, and verify $\dim RS + \dim NS = n$

$$\begin{bmatrix}1 & -2 & 4 & 1\\3 & 1 & -3 & -1\\5 & -3 & 5 & 1\end{bmatrix}$$

***Solution***

$$\begin{bmatrix}1 & -2 & 4 & 1\\3 & 1 & -3 & -1\\5 & -3 & 5 & 1\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & -\frac{2}{7} & -\frac{1}{7}\\0 & 1 & -\frac{15}{7} & -\frac{4}{7}\\0 & 0 & 0 & 0\end{bmatrix}$$

Row Space: $\begin{bmatrix}1 & -2 & 4 & 1\end{bmatrix}, \begin{bmatrix}3 & 1 & -3 & -1\end{bmatrix}$

Column Space: $\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}-2\\1\\-3\end{bmatrix}$

$\dim RS = 2$

$\dim NS = 2$

$2 + 2 = 2 \quad \Rightarrow \quad \dim RS + \dim NS = n$

## Exercise

Determine if $\boldsymbol{b}$ lies in the column space of the given matrix. If it does, express $\boldsymbol{b}$ as linear combination of the column.

$$\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

***Solution***

$$\left[\begin{array}{cc|c} 2 & -3 & 4 \\ -4 & 6 & -6 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$\boldsymbol{b}$ does not lie in the column space

## Exercise

Find the transition matrix from $B$ to $C$ and find $[\boldsymbol{x}]_c$

*a)* $B = \{(3, 1), (-1, -2)\}, \quad C = \{(1, -3), (5, 0)\}, \quad [x]_B = [-1 \quad -2]^T$

*b)* $B = \{(1, 1, 1), (-2, -1, 0), (2, 1, 2)\}, \quad C = \{(-6, -2, 1), (-1, 1, 5), (-1, -1, 1)\},$

$[\boldsymbol{x}]_B = [-3 \quad 2 \quad 4]^T$

***Solution***

***a)***
$$\left[\begin{array}{cc|cc} 1 & 5 & 3 & -1 \\ -3 & 0 & 1 & -2 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right]$$

$$[\boldsymbol{x}]_c = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

***b)***
$$\left[\begin{array}{ccc|ccc} -6 & -1 & -1 & 1 & -2 & 2 \\ -2 & 1 & -1 & 1 & -1 & 1 \\ 1 & 5 & 1 & 1 & 0 & 2 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{13} & \frac{4}{13} & -\frac{6}{13} \\ 0 & 1 & 0 & \frac{4}{13} & -\frac{3}{26} & \frac{11}{26} \\ 0 & 0 & 1 & -\frac{5}{13} & \frac{7}{26} & \frac{9}{26} \end{array}\right]$$

$$[\boldsymbol{x}]_c = \begin{bmatrix} -\frac{2}{13} & \frac{4}{13} & -\frac{6}{13} \\ \frac{4}{13} & -\frac{3}{26} & \frac{11}{26} \\ -\frac{5}{13} & \frac{7}{26} & \frac{9}{26} \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{10}{13} \\ \frac{17}{13} \\ \frac{35}{13} \end{bmatrix}$$

## *Exercise*

Does $A$ and $A^T$ have the same number of pivots.

***Solution***

True

The number of pivots of $A$ is its column rank, $r$. We know that the column rank of $A$ equals the row rank of $A$, which is the column rank of $A^T$.

Hence, $A^T$ must have the same number of pivots as $A$.

## *Exercise*

Let $\quad A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad where \;\; b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

*a)* What is the rank of $A$?

*b)* What is the dimension of $A$?

*c)* What are the pivots variables?

*d)* What are the free variables?

*e)* Find the special (homogeneous) solutions.

*f)* What is the nullspace $N(A)$?

*g)* Find the particular solution to $Ax = b$

*h)* Give the complete solution.

***Solution***

***a)*** Rank(***A***) = 2

***b)*** Dimension of $A$ = 2

***c)*** The pivots variables are: $x_1, x_3$

***d)*** The free variables are: $x_2, x_4$

***e)*** Let $x = x_2 s_1 + x_4 s_2$

$$A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = -3x_2 - 2x_4 \\ x_3 = \qquad\quad -4x_4 \end{cases}$$

Set $x_2 = 1, \quad x_4 = 0$

The special solution: $s_1 = (-3,\ 1,\ 0,\ 0)$

Set $x_2 = 0, \quad x_4 = 1$ ;

The special solution: $s_2 = (-2,\ 0,\ -4,\ 1)$

***f)*** The nullspace is the set $\left\{ x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix} \right\}$

$$N(A) = \begin{pmatrix} -3 & -2 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \end{pmatrix}$$

***g)*** $x_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

***h)*** $x = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

## *Exercise*

Let $A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *where* $b = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

*a)* What is the rank of $A$?

*b)* What is the dimension of $A$?

*c)* What are the pivots variables?

*d)* What are the free variables?

*e)* Find the special (homogeneous) solutions.

*f)* What is the nullspace $N(A)$?

*g)* Find the particular solution to $Ax = b$

*h)* Give the complete solution.

***Solution***

***a)*** Rank(***A***) = 3

***b)*** Dimension of $A$ = 1

***c)*** The pivots variables are: $x_1, x_2, x_4$

***d)*** The free variables are: $x_3$

***e)*** Let $x = x_3 s_1$

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{cases} x_1 = -2x_3 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

The special solution: $s_1 = (-2,\ 0,\ 1,\ 0)$

***f)*** $N(A) = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

***g)*** $x_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

***h)*** $x = x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

## *Exercise*

Let $A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *where* $b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

*a)* What is the rank of $A$?

*b)* What is the dimension of $A$?

*c)* What are the pivots variables?

*d)* What are the free variables?

*e)* Find the special (homogeneous) solutions.

*f)* What is the nullspace $N(A)$?

*g)* Find the particular solution to $Ax = b$

*h)* Give the complete solution.

***Solution***

$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ -\tfrac{1}{3}R_2 \ \rightarrow \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -\frac{1}{3} & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ R_1 - 2R_2$$

$$\begin{pmatrix} 1 & 0 & \frac{2}{3} & -4 \\ 0 & 1 & -\frac{1}{3} & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \begin{cases} x_1 = -\frac{2}{3}x_3 + 4x_4 \\ x_2 = \frac{1}{3}x_3 - 4x_4 \end{cases}$$

***a)*** Rank($\boldsymbol{A}$) = 2

***b)*** Dimension of $A$ = 2

***c)*** The pivots variables are: $x_1,\ x_2$

***d)*** The free variables are: $x_3,\ x_4$

***e)*** Let $x = x_3 s_1 + x_4 s_2$

Set $x_3 = 1, \quad x_4 = 0$

The special solution: $s_1 = \left(-\frac{2}{3},\ \frac{1}{3},\ 1,\ 0\right)$

Set $x_3 = 0, \quad x_4 = 1$ ;

The special solution: $s_2 = (4,\ -4,\ 0,\ 1)$

***f)*** The nullspace is the set $\left\{ x_3 \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ -4 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$N(A) = \begin{pmatrix} -\frac{2}{3} & 4 \\ \frac{1}{3} & -4 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

***g)*** $x_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

***h)*** $x = x_3\begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 4 \\ -4 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

## *Exercise*

Let $A = \begin{pmatrix} 1 & 0 & 0 & \frac{13}{11} \\ 0 & 1 & 0 & -\frac{17}{11} \\ 0 & 0 & 1 & \frac{6}{11} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *where* $b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

*a)* What is the rank of $A$?
*b)* What is the dimension of $A$?
*c)* What are the pivots variables?
*d)* What are the free variables?
*e)* Find the special (homogeneous) solutions.
*f)* What is the nullspace $N(A)$?
*g)* Find the particular solution to $Ax = b$
*h)* Give the complete solution.

**Solution**

***a)*** Rank($\boldsymbol{A}$) = 3

***b)*** Dimension of $A$ = 1

***c)*** The pivots variables are: $x_1, x_2, x_3$

***d)*** The free variables are: $x_4$

***e)*** Let $x = x_4 s_1$

$$A = \begin{pmatrix} 1 & 0 & 0 & \frac{13}{11} \\ 0 & 1 & 0 & -\frac{17}{11} \\ 0 & 0 & 1 & \frac{6}{11} \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = -\frac{13}{11} \\ x_2 = \frac{17}{11} \\ x_3 = -\frac{6}{11} \end{cases}$$

Set $x_4 = 1$

The special solution: $s_1 = \left(-\frac{13}{11}, \frac{17}{11}, -\frac{6}{11}, 1\right)$

***f)*** $N(A)=\begin{pmatrix} -\frac{13}{11} \\ \frac{17}{11} \\ -\frac{6}{11} \\ 1 \end{pmatrix}$

***g)*** $x_p=\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

***h)*** $x=x_4\begin{pmatrix} -\frac{13}{11} \\ \frac{17}{11} \\ -\frac{6}{11} \\ 1 \end{pmatrix}+\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

## *Exercise*

Let $A=\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *where* $b=A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

*a)* What is the rank of $A$?

*b)* What is the dimension of $A$?

*c)* What are the pivots variables?

*d)* What are the free variables?

*e)* Find the special (homogeneous) solutions.

*f)* What is the nullspace $N(A)$?

*g)* Find the particular solution to $Ax=b$

*h)* Give the complete solution.

***Solution***

***a)*** Rank(***A***) = 3

***b)*** Dimension of $A$ = 1

***c)*** The pivots variables are: $x_1, x_2, x_3$

***d)*** The free variables are: $x_4$

***e)*** Let $x = x_4 s_1$

$$A = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = \frac{1}{3} \\ x_2 = 1 \\ x_3 = \frac{1}{3} \end{cases}$$

Set $x_4 = 1$

The special solution: $s_1 = \left(\frac{1}{3}, 1, \frac{1}{3}, 1\right)$

***f)*** $N(A) = \begin{pmatrix} \frac{1}{3} \\ 1 \\ \frac{1}{3} \\ 1 \end{pmatrix}$

***g)*** $x_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

***h)*** $x = x_4 \begin{pmatrix} \frac{1}{3} \\ 1 \\ \frac{1}{3} \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

## *Exercise*

Let $A = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ *where* $b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

*a)* What is the rank of $A$?

*b)* What is the dimension of $A$?

*c)* What are the pivots variables?

*d)* What are the free variables?

*e)* Find the special (homogeneous) solutions.

*f)* What is the nullspace $N(A)$?

*g)* Find the particular solution to $Ax = b$

*h)* Give the complete solution.

**Solution**

***a)*** Rank(***A***) = 3

***b)*** Dimension of $A$ = 2

***c)*** The pivots variables are: $x_1, x_3, x_5$

***d)*** The free variables are: $x_2, x_4$

***e)*** Let $x = x_2 s_1 + x_4 s_2$

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = -2x_2 + x_4 \\ x_3 = \qquad -x_4 \\ x_5 = 0 \end{cases}$$

Set $x_2 = 1 \quad x_4 = 0$

The special solution: $s_1 = (-2, 1, 0, 0, 0)$

Set $x_2 = 0 \quad x_4 = 1$

The special solution: $s_2 = (1, 0, -1, 1, 0)$

***f)*** $N(A) = \begin{pmatrix} -2 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

***g)*** $x_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

***h)*** $x = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$