

Solution **Section 4.1 – Infinite Sequences and Summation Notation**

Exercise

Find the first four terms and the eight term of the sequence: $\{12 - 3n\}$

Solution

$$a_n = 12 - 3n$$

$$a_1 = 12 - 3(1) = 9, \quad a_2 = 12 - 3(2) = 6, \quad a_3 = 12 - 3(3) = 3, \quad a_4 = 12 - 3(4) = 0$$

$$a_8 = 12 - 3(8) = -12$$

$$\boxed{9, 6, 3, 0; -12}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{ \frac{3n-2}{n^2+1} \right\}$

Solution

$$a_n = \frac{3n-2}{n^2+1}$$

$$a_1 = \frac{3-2}{1^2+1} = \frac{1}{2}, \quad a_2 = \frac{3(2)-2}{2^2+1} = \frac{4}{5}, \quad a_3 = \frac{3(3)-2}{3^2+1} = \frac{7}{10}, \quad a_4 = \frac{3(4)-2}{4^2+1} = \frac{10}{17}$$

$$a_8 = \frac{3(8)-2}{8^2+1} = \frac{22}{65}$$

$$\boxed{\frac{1}{2}, \frac{4}{5}, \frac{7}{10}, \frac{10}{17}; \frac{22}{65}}$$

Exercise

Find the first four terms and the eight term of the sequence: $\{9\}$

Solution

$$\boxed{9, 9, 9, 9; 9}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{ (-1)^{n-1} \frac{n+7}{2n} \right\}$

Solution

$$a_1 = (-1)^{1-1} \frac{1+7}{2(1)} = 4, \quad a_2 = (-1)^{2-1} \frac{2+7}{2(2)} = -\frac{9}{4},$$

$$a_3 = (-1)^{3-1} \frac{3+7}{2(3)} = \frac{5}{3}, \quad a_4 = (-1)^{4-1} \frac{4+7}{2(4)} = -\frac{11}{8}$$

$$a_8 = (-1)^{8-1} \frac{8+7}{2(8)} = -\frac{15}{16}$$

$4, -\frac{9}{4}, \frac{5}{3}, -\frac{11}{8}, -\frac{15}{16}$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{ \frac{2^n}{n^2 + 2} \right\}$

Solution

$$a_1 = \frac{2^1}{1^2 + 2} = \frac{2}{3}, \quad a_2 = \frac{2^2}{2^2 + 2} = \frac{2}{3},$$

$$a_3 = \frac{2^3}{3^2 + 2} = \frac{8}{11}, \quad a_4 = \frac{2^4}{4^2 + 2} = \frac{8}{9}$$

$$a_8 = \frac{2^8}{8^2 + 2} = \frac{128}{33}$$

$\frac{2}{3}, \frac{2}{3}, \frac{8}{11}, \frac{8}{9}, \frac{128}{33}$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{ (-1)^{n-1} \frac{n}{2n-1} \right\}$

Solution

$$a_1 = (-1)^{0} \frac{1}{2-1} = \underline{1}; \quad a_2 = (-1)^1 \frac{2}{4-1} = \underline{-\frac{2}{3}}; \quad a_3 = (-1)^2 \frac{3}{6-1} = \underline{\frac{3}{5}}; \quad a_4 = (-1)^3 \frac{4}{8-1} = \underline{-\frac{4}{7}}$$

$$a_8 = (-1)^7 \frac{8}{16-1} = \underline{-\frac{8}{15}}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{ \frac{2^n}{3^n + 1} \right\}$

Solution

$$a_1 = \frac{2^1}{3^1 + 1} = \frac{2}{4} = \underline{\frac{1}{2}}; \quad a_2 = \frac{2^2}{3^2 + 1} = \frac{4}{10} = \underline{\frac{2}{5}}; \quad a_3 = \frac{2^3}{3^3 + 1} = \frac{8}{28} = \underline{\frac{2}{7}}; \quad a_4 = \frac{2^4}{3^4 + 1} = \frac{16}{82} = \underline{\frac{8}{41}}$$

$$a_8 = \frac{2^8}{3^8 + 1} = \frac{256}{6562} = \underline{\frac{128}{3281}}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{ \frac{n^2}{2^n} \right\}$

Solution

$$a_1 = \frac{1^2}{2^1} = \frac{1}{2}; \quad a_2 = \frac{2^2}{2^2} = 1; \quad a_3 = \frac{3^2}{2^3} = \frac{9}{8}; \quad a_4 = \frac{4^2}{2^4} = \frac{16}{16} = 1$$

$$a_8 = \frac{8^2}{2^8} = \frac{64}{256} = \frac{1}{4}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{ \frac{n}{e^n} \right\}$

Solution

$$a_1 = \frac{1}{e^1} = \frac{1}{e}; \quad a_2 = \frac{2}{e^2}; \quad a_3 = \frac{3}{e^3}; \quad a_4 = \frac{4}{e^4}$$

$$a_8 = \frac{8}{e^8}$$

Exercise

Find the first four terms and the eight term of the sequence: $\{c_n\} = \{(-1)^{n+1} n^2\}$

Solution

$$c_1 = (-1)^2 1^2 = 1; \quad c_2 = (-1)^3 2^2 = -4; \quad c_3 = (-1)^4 3^2 = 9; \quad c_4 = (-1)^5 4^2 = -16$$

$$c_8 = (-1)^9 8^2 = -64$$

Exercise

Find the first four terms and the eight term of the sequence: $\{c_n\} = \left\{ \frac{(-1)^n}{(n+1)(n+2)} \right\}$

Solution

$$c_1 = \frac{(-1)^1}{2 \cdot 3} = -\frac{1}{6}; \quad c_2 = \frac{(-1)^2}{3 \cdot 4} = \frac{1}{12}; \quad c_3 = \frac{(-1)^3}{4 \cdot 5} = -\frac{1}{20}; \quad c_4 = \frac{(-1)^4}{5 \cdot 6} = \frac{1}{30}$$

$$c_8 = \frac{(-1)^8}{9 \cdot 10} = \frac{1}{90}$$

Exercise

Find the first four terms and the eight term of the sequence: $\{c_n\} = \left\{\left(\frac{4}{3}\right)^n\right\}$

Solution

$$c_1 = \left(\frac{4}{3}\right)^1 = \underline{\frac{4}{3}}; \quad c_2 = \left(\frac{4}{3}\right)^2 = \underline{\frac{16}{9}}; \quad c_3 = \left(\frac{4}{3}\right)^3 = \underline{\frac{64}{27}}; \quad c_4 = \left(\frac{4}{3}\right)^4 = \underline{\frac{256}{81}}$$

$$c_8 = \left(\frac{4}{3}\right)^8 = \underline{\frac{65,536}{6,561}}$$

Exercise

Find the first four terms and the eight term of the sequence: $\{b_n\} = \left\{\frac{3^n}{n}\right\}$

Solution

$$b_1 = \frac{3^1}{1} = \underline{3}; \quad b_2 = \frac{3^2}{2} = \underline{\frac{9}{2}}; \quad b_3 = \frac{3^3}{3} = \underline{9}; \quad b_4 = \frac{3^4}{4} = \underline{\frac{81}{4}}$$

$$b_8 = \frac{3^8}{8} = \underline{\frac{6,561}{8}}$$

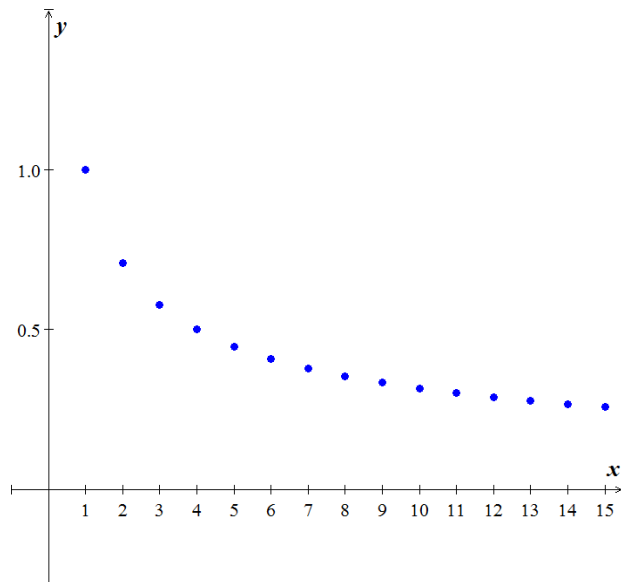
Exercise

Graph the sequence $\left\{\frac{1}{\sqrt{n}}\right\}$

Solution

$$\left\{\frac{1}{\sqrt{n}}\right\} = \frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, \dots$$

$$\approx 1, 0.71, 0.58, 0.5, 0.45$$



Exercise

Find the first four terms of the sequence of partial sums for the given sequence. $\left\{3 + \frac{1}{2}n\right\}$

Solution

$$S_1 = a_1 = 3 + \frac{1}{2}(1) = \underline{\frac{7}{2}}$$

$$S_2 = S_1 + a_2 = \frac{7}{2} + 3 + \frac{1}{2}(2) = \underline{\frac{15}{2}}$$

$$S_3 = S_2 + a_3 = \frac{15}{2} + 3 + \frac{1}{2}(3) = \underline{12}$$

$$S_4 = S_3 + a_4 = 12 + 3 + \frac{1}{2}(4) = \underline{17}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2$, $a_{k+1} = 3a_k - 5$

Solution

$$k = 1 \rightarrow \boxed{a_2 = 3a_1 - 5 = 3(2) - 5 = \underline{1}} \qquad k = 2 \rightarrow \boxed{a_3 = 3a_2 - 5 = 3(1) - 5 = \underline{-2}}$$

$$k = 3 \rightarrow \boxed{a_4 = 3a_3 - 5 = 3(-2) - 5 = \underline{-11}} \qquad k = 4 \rightarrow \boxed{a_5 = 3a_4 - 5 = 3(-11) - 5 = \underline{-38}}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = -3$, $a_{k+1} = a_k^2$

Solution

$$k = 1 \rightarrow \boxed{a_2 = a_1^2 = (-3)^2 = \underline{9}} \qquad k = 2 \rightarrow \boxed{a_3 = a_2^2 = (9)^2 = \underline{81}}$$

$$k = 3 \rightarrow \boxed{a_4 = a_3^2 = (81)^2 = \underline{6561}} \qquad k = 4 \rightarrow \boxed{a_5 = a_4^2 = (6561)^2 = \underline{43046721}}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 5$, $a_{k+1} = ka_k$

Solution

$$k = 1 \rightarrow \boxed{a_2 = 1a_1 = \underline{5}} \qquad k = 2 \rightarrow \boxed{a_3 = 2a_2 = 2(5) = \underline{10}}$$

$$k = 3 \rightarrow \boxed{a_4 = 3a_3 = 3(10) = \underline{30}} \qquad k = 4 \rightarrow \boxed{a_5 = 4a_4 = 4(30) = \underline{120}}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2$, $a_n = 3 + a_{n-1}$

Solution

$$a_2 = 3 + a_1 = 3 + 2 = \underline{5} \qquad a_3 = 3 + a_2 = 3 + 5 = \underline{8}$$

$$a_4 = 3 + a_3 = 3 + 8 = \underline{11} \qquad a_5 = 3 + a_4 = 3 + 11 = \underline{14}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 5$, $a_n = 2a_{n-1}$

Solution

$$a_2 = 2a_1 = 2(5) = \underline{10} \qquad a_3 = 2a_2 = 2(10) = \underline{20}$$

$$a_4 = 2a_3 = 2(20) = \underline{40} \qquad a_5 = 2a_4 = 2(40) = \underline{80}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = \sqrt{2}$, $a_n = \sqrt{2 + a_{n-1}}$

Solution

$$a_2 = \sqrt{2 + a_1} = \sqrt{2 + \sqrt{2}}$$

$$a_3 = \sqrt{2 + a_2} = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$$

$$a_4 = \sqrt{2 + a_3} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$

$$a_5 = \sqrt{2 + a_4} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2$, $a_{n+1} = 7 - 2a_n$

Solution

$$a_2 = 7 - 2a_1 = 7 - 4 = 3$$

$$a_3 = 7 - 2a_2 = 7 - 6 = 1$$

$$a_4 = 7 - 2a_3 = 7 - 2 = 5$$

$$a_5 = 7 - 2a_4 = 7 - 10 = -5$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 128$, $a_{n+1} = \frac{1}{4}a_n$

Solution

$$a_2 = \frac{1}{4}a_1 = \frac{1}{4}128 = 32$$

$$a_3 = \frac{1}{4}a_2 = \frac{32}{4} = 8$$

$$a_4 = \frac{1}{4}a_3 = \frac{8}{4} = 2$$

$$a_5 = \frac{1}{4}a_4 = \frac{2}{4} = \frac{1}{2}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2$, $a_{n+1} = (a_n)^n$

Solution

$$a_2 = (a_1)^1 = 2$$

$$a_3 = (a_2)^2 = 2^2 = 4$$

$$a_4 = (a_3)^3 = 4^3 = 64$$

$$a_5 = (a_4)^4 = 64^4$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = A$, $a_n = a_{n-1} + d$

Solution

$$a_2 = a_1 + d = A + d$$

$$a_3 = a_2 + d = A + d + d = A + 2d$$

$$a_4 = a_3 + d = A + 3d$$

$$a_5 = a_4 + d = A + 4d$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = A$, $a_n = ra_{n-1}$, $r \neq 0$

Solution

$$a_2 = ra_1 = \underline{rA} \quad a_3 = ra_2 = \underline{Ar^2} \quad a_4 = ra_3 = \underline{Ar^3} \quad a_5 = ra_4 = \underline{Ar^4}$$

Exercise

Find the first 5 terms of the recursively defined infinite sequence: $a_1 = 2$, $a_2 = 2$; $a_n = a_{n-1} \cdot a_{n-2}$

Solution

$$a_3 = a_2 \cdot a_1 = 2 \cdot 2 = \underline{4} \quad a_4 = a_3 \cdot a_2 = 4 \cdot 2 = \underline{8}$$
$$a_5 = a_4 \cdot a_3 = 8 \cdot 4 = \underline{32} \quad a_6 = a_5 \cdot a_4 = 32 \cdot 8 = \underline{256}$$

Exercise

Express each sum using summation notation $1 + 2 + 3 + \dots + 20$

Solution

$$1 + 2 + 3 + 4 + \dots + 20 = \sum_{k=1}^{20} k$$

Exercise

Express each sum using summation notation $1 + 2 + 3 + \dots + 40$

Solution

$$1 + 2 + 3 + \dots + 40 = \sum_{k=1}^{40} k$$

Exercise

Express each sum using summation notation $1^3 + 2^3 + 3^3 + \dots + 8^3$

Solution

$$1^3 + 2^3 + 3^3 + \dots + 8^3 = \sum_{k=1}^8 k^3$$

Exercise

Express each sum using summation notation

$$1^2 + 2^2 + 3^2 + \dots + 15^2$$

Solution

$$1^2 + 2^2 + 3^2 + \dots + 15^2 = \sum_{k=1}^{15} k^2$$

Exercise

Express each sum using summation notation

$$2^2 + 2^3 + 2^4 + \dots + 2^{11}$$

Solution

$$2^2 + 2^3 + 2^4 + \dots + 2^{11} = \sum_{k=2}^{11} 2^k$$

Exercise

Express each sum using summation notation

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots + \frac{13}{14}$$

Solution

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots + \frac{13}{14} = \sum_{k=1}^{13} \frac{k}{k+1}$$

Exercise

Express each sum using summation notation

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \dots + (-1)^6 \frac{1}{3^6}$$

Solution

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \dots + (-1)^6 \frac{1}{3^6} = \sum_{k=0}^6 (-1)^k \frac{1}{3^k}$$

Exercise

Express each sum using summation notation

$$\frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \dots + (-1)^{12} \left(\frac{2}{3}\right)^{11}$$

Solution

$$\frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \dots + (-1)^{12} \left(\frac{2}{3}\right)^{11} = \sum_{k=1}^{11} (-1)^{k+1} \left(\frac{2}{3}\right)^k$$

Exercise

Express each sum using summation notation

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{14}{14+1}$$

Solution

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{14}{14+1} = \sum_{k=1}^{14} \frac{k}{k+1}$$

Exercise

Express each sum using summation notation

$$\frac{1}{e} + \frac{2}{e^2} + \frac{3}{e^3} + \cdots + \frac{n}{e^n}$$

Solution

$$\frac{1}{e} + \frac{2}{e^2} + \frac{3}{e^3} + \cdots + \frac{n}{e^n} = \sum_{k=1}^n \frac{k}{e^k}$$

ExerciseFind the sum: $\sum_{k=1}^5 (2k - 7)$ **Solution**

$$\sum_{k=1}^5 (2k - 7) = (-5) + (-3) + (-1) + 1 + 3 = -5$$

ExerciseFind the sum: $\sum_{k=0}^5 k(k - 2)$ **Solution**

$$\sum_{k=0}^5 k(k - 2) = 0 + (-1) + 0 + 3 + 8 + 15 = 25$$

ExerciseFind the sum: $\sum_{k=1}^5 (-3)^{k-1}$ **Solution**

$$\sum_{k=1}^5 (-3)^{k-1} = 1 + (-3) + 9 + (-27) + 81 = \underline{61}$$

Exercise

Find the sum: $\sum_{k=253}^{571} \left(\frac{1}{3}\right)$

Solution

$$\sum_{k=253}^{571} \left(\frac{1}{3}\right) = (571 - 253 + 1) \left(\frac{1}{3}\right) = \underline{\frac{319}{3}}$$

$$\sum_{k=m}^n c = (n - m + 1)c$$

Exercise

Find the sum: $\sum_{k=1}^{50} 8$

Solution

$$\sum_{k=1}^{50} 8 = (50 - 1 + 1)8 = \underline{400}$$

$$\sum_{k=m}^n c = (n - m + 1)c$$

Exercise

Find the sum: $\sum_{k=1}^{40} k$

Solution

$$\sum_{k=1}^{40} k = \frac{40(41)}{2} = \underline{820}$$

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

Exercise

Find the sum: $\sum_{k=1}^5 (3k)$

Solution

$$\sum_{k=1}^5 3k = 3(1) + 3(2) + 3(3) + 3(4) + 3(5) = \underline{45}$$

Exercise

Find the sum: $\sum_{k=1}^{10} (k^3 + 1)$

Solution

$$\begin{aligned}\sum_{k=1}^{10} (k^3 + 1) &= \sum_{k=1}^{10} k^3 + \sum_{k=1}^{10} 1 \\ &= \frac{10^2(10+1)^2}{4} + 10(1) \\ &= \frac{12100}{4} + 10 \\ &= 3025 + 10 \\ &= \underline{3035}\end{aligned}$$

$$\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$$

Exercise

Find the sum: $\sum_{k=1}^{24} (k^2 - 7k + 2)$

Solution

$$\begin{aligned}\sum_{k=1}^{24} (k^2 - 7k + 2) &= \frac{24(24+1)(2 \cdot 24 + 1)}{6} - 7 \frac{24(24+1)}{2} + 2(24) \\ &= \underline{2848}\end{aligned}$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

Exercise

Find the sum: $\sum_{k=6}^{20} (4k^2)$

Solution

$$\begin{aligned}\sum_{k=6}^{20} (4k^2) &= 4 \left(\sum_{k=1}^{20} k^2 - \sum_{k=1}^5 k^2 \right) \\ &= 4 \left(\frac{20(20+1)(2 \cdot 20 + 1)}{6} - \frac{5(5+1)(2 \cdot 5 + 1)}{6} \right) \\ &= 4 \left(\frac{20(21)(41)}{6} - \frac{5(6)(11)}{6} \right) \\ &= 4(2870 - 55) \\ &= \underline{11,260}\end{aligned}$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

Exercise

Find the sum: $\sum_{k=1}^{16} (k^2 - 4)$

Solution

$$\begin{aligned}\sum_{k=1}^{16} (k^2 - 4) &= \sum_{k=1}^{16} k^2 - \sum_{k=1}^{16} 4 \\ &= \frac{16(16+1)(2 \cdot 16 + 1)}{6} - 4(16) \\ &= 1496 - 64 \\ &= \underline{1432}\end{aligned}$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

Exercise

Find the sum: $\sum_{k=1}^6 (10 - 3k)$

Solution

$$\sum_{k=1}^6 (10 - 3k) = 7 + 4 + 1 - 2 - 5 - 8 = \underline{-3}$$

Exercise

Find the sum: $\sum_{k=1}^{10} [1 + (-1)^k]$

Solution

$$\sum_{k=1}^{10} [1 + (-1)^k] = 0 + 2 + 0 + 2 + 0 + 2 + 0 + 2 + 0 + 2 = \underline{10}$$

Exercise

Find the sum: $\sum_{k=1}^6 \frac{3}{k+1}$

Solution

$$\sum_{k=1}^6 \frac{3}{k+1} = \frac{3}{2} + 1 + \frac{3}{4} + \frac{3}{5} + 2 + \frac{3}{7} = \underline{\frac{879}{140}}$$

Exercise

Find the sum: $\sum_{k=137}^{428} 2.1$

Solution

$$\sum_{k=137}^{428} 2.1 = (428 - 137 + 1)2.1 = (292)2.1 = \underline{613.2}$$

$$\sum_{k=m}^n c = (n - m + 1)c$$

Exercise

Write out each sum $\sum_{k=1}^n (k + 2)$

Solution

$$\sum_{k=1}^n (k + 2) = \underline{3 + 5 + 7 + 9 + \cdots + (n + 2)}$$

Exercise

Write out each sum $\sum_{k=1}^n k^2$

Solution

$$\sum_{k=1}^n k^2 = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \underline{1 + 4 + 9 + 16 + \cdots + n^2}$$

Exercise

Write out each sum $\sum_{k=2}^n (-1)^k \ln k$

Solution

$$\begin{aligned} \sum_{k=2}^n (-1)^k \ln k &= (-1)^2 \ln 2 + (-1)^3 \ln 3 + (-1)^4 \ln 4 + (-1)^5 \ln 5 + \cdots + (-1)^n \ln n \\ &= \underline{\ln 2 - \ln 3 + \ln 4 - \ln 5 + \cdots + (-1)^n \ln n} \end{aligned}$$

Exercise

Write out each sum $\sum_{k=3}^n (-1)^{k+1} 2^k$

Solution

$$\begin{aligned}\sum_{k=3}^n (-1)^{k+1} 2^k &= (-1)^4 2^3 + (-1)^5 2^4 + (-1)^6 2^5 + (-1)^7 2^6 + \cdots + (-1)^{n+1} 2^n \\ &= \underline{8 - 16 + 32 - 64 + \cdots + (-1)^{n+1} 2^n}\end{aligned}$$

Exercise

Write out each sum $\sum_{k=0}^n \frac{1}{3^k}$

Solution

$$\sum_{k=0}^n \frac{1}{3^k} = \underline{1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^n}}$$

Exercise

Fred has a balance of \$3,000 on his card which charges 1% interest per month on any unpaid balance. Fred can afford to pay \$100 toward the balance each month. His balance each month after making a \$100 payment is given by the recursively defined sequence

$$B_0 = \$3,000 \quad B_n = 1.01B_{n-1} - 100$$

Determine Fred's balance after making the first payment. That is, determine B_1

Solution

$$\begin{aligned}B_1 &= 1.01B_0 - 100 \\ &= 1.01(3,000) - 100 \\ &= \underline{\$2,930}\end{aligned}$$

Fred's balance is \$2,930 after making the first payment.

Exercise

A pond currently has 2,000 trout in it. A fish hatchery decides to add an additional 20 trout each month. Is it also known that the trout population is growing at a rate of 3% per month. The size of the population after n months is given by the recursively defined sequence

$$P_0 = 2,000 \quad P_n = 1.03P_{n-1} + 20$$

How many trout are in the pond after 2 months? That is, what is P_2 ?

Solution

$$P_1 = 1.03P_0 + 20 = 1.03(2,000) + 20 = \underline{2,080}$$

$$P_2 = 1.03P_1 + 20 = 1.03(2,080) + 20 = \underline{2,162.4}$$

There are approximately 2162 trout in the pond after 2 months.

Exercise

Fred bought a car by taking out a loan for \$18,500 at 0.5% interest per month. Fred's normal monthly payment is \$434.47 per month, but he decides that he can afford to pay \$100 extra toward the balance each month. His balance each month is given by the recursively defined sequence

$$B_0 = \$18,500 \quad B_n = 1.005B_{n-1} - 534.47$$

Determine Fred's balance after making the first payment. That is, determine B_1

Solution

$$\begin{aligned} B_1 &= 1.005B_0 - 534.47 \\ &= 1.005(18,500) - 534.47 \\ &= \underline{\$18,058.03} \end{aligned}$$

Fred's balance is \$18,058.03 after making the first payment.

Exercise

The Environmental Protection Agency (EPA) determines that Maple Lake has 250 *tons* of pollutant as a result of industrial waste and that 10% of the pollutant present is neutralized by solar oxidation every year. The EPA imposes new pollution control laws that result in 15 *tons* of new pollutant entering the lake each year. The amount of pollutant in the lake after n years is given by the recursively defined sequence

$$P_0 = 250 \quad P_n = 0.9P_{n-1} + 15$$

Determine the amount of pollutant in the lake after 2 years? That is, what is P_2 ?

Solution

$$P_1 = 0.9P_0 + 15 = 0.9(250) + 15 = \underline{240}$$

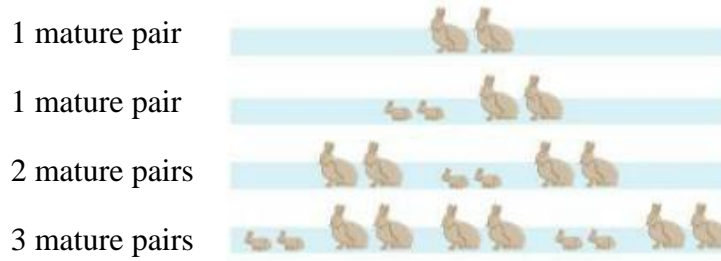
$$P_2 = 0.9P_1 + 15 = 0.9(240) + 15 = \underline{231}$$

There are 231 tons of pollutants after 2 years.

Exercise

A colony of rabbits begins with one pair of mature rabbits, which will produce a pair of offspring (one male, one female) each month. Assume that all rabbits mature in 1 month and produce a pair of offspring

(one male, one female) after 2 months. If no rabbits ever die, how many pairs of mature rabbits are there after 7 months?



Solution

$$a_1 = 1, \quad a_2 = 1, \quad a_3 = 2, \quad a_4 = 3, \quad a_5 = 5$$

$$a_6 = 8, \quad a_7 = 13, \quad a_8 = 21, \quad \dots \quad a_n = a_{n-1} + a_{n-2}$$

After 7 months there are 21 mature pairs of rabbits.

Exercise

Let
$$u_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}$$

Define the n th term of a sequence

a) Show that $u_1 = 1$ and $u_2 = 1$

b) Show that $u_{n+2} = u_{n+1} + u_n$

c) Draw the conclusion that $\{u_n\}$ is a Fibonacci sequence

d) Find the first ten terms of the sequence from part (c)

Solution

$$a) \quad u_1 = \frac{(1 + \sqrt{5})^1 - (1 - \sqrt{5})^1}{2^1 \sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = \underline{1}$$

$$\begin{aligned} u_2 &= \frac{(1 + \sqrt{5})^2 - (1 - \sqrt{5})^2}{2^2 \sqrt{5}} \\ &= \frac{(1 + \sqrt{5} - 1 + \sqrt{5}) - (1 - \sqrt{5} + 1 - \sqrt{5})}{2^2 \sqrt{5}} \\ &= \frac{4\sqrt{5}}{4\sqrt{5}} \\ &= \underline{1} \end{aligned}$$

$$a^2 - b^2 = (a - b)(a + b)$$

$$b) \quad u_{n+1} + u_n = \frac{(1 + \sqrt{5})^{n+1} - (1 - \sqrt{5})^{n+1}}{2^{n+1} \sqrt{5}} + \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}$$

$$\begin{aligned}
&= \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1} + 2(1+\sqrt{5})^n - 2(1-\sqrt{5})^n}{2^{n+1} \sqrt{5}} \\
&= \frac{(1+\sqrt{5})^n (1+\sqrt{5}+2) - (1-\sqrt{5})^n (1-\sqrt{5}+2)}{2^{n+1} \sqrt{5}} \\
&= \frac{(1+\sqrt{5})^n (3+\sqrt{5}) - (1-\sqrt{5})^n (3-\sqrt{5})}{2^{n+1} \sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n+2} \frac{3+\sqrt{5}}{(1+\sqrt{5})^2} - (1-\sqrt{5})^{n+2} \frac{3-\sqrt{5}}{(1-\sqrt{5})^2}}{2^{n+1} \sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n+2} \frac{3+\sqrt{5}}{1+2\sqrt{5}+5} - (1-\sqrt{5})^{n+2} \frac{3-\sqrt{5}}{1-2\sqrt{5}+5}}{2^{n+1} \sqrt{5}} \\
&= \frac{\frac{1}{2}(1+\sqrt{5})^{n+2} \frac{3+\sqrt{5}}{3+\sqrt{5}} - \frac{1}{2}(1-\sqrt{5})^{n+2} \frac{3-\sqrt{5}}{3-\sqrt{5}}}{2^{n+1} \sqrt{5}} \\
&= \frac{\frac{1}{2} \left((1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2} \right)}{2^{n+1} \sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2}}{2^{n+2} \sqrt{5}} \\
&= \underline{u_{n+2}} \quad \checkmark
\end{aligned}$$

c) Since $u_1 = 1$ and $u_2 = 1$ and $u_{n+2} = u_{n+1} + u_n$

$\therefore \{u_n\}$ is a Fibonacci sequence

e) $u_1 = 1, u_2 = 1$

$$u_3 = u_2 + u_1 = 1 + 1 = \underline{2} \quad u_4 = u_3 + u_2 = 2 + 1 = \underline{3} \quad u_5 = u_4 + u_3 = 3 + 2 = \underline{5}$$

$$u_6 = u_5 + u_4 = 5 + 3 = \underline{8} \quad u_7 = u_6 + u_5 = 8 + 5 = \underline{13} \quad u_8 = u_7 + u_6 = 13 + 8 = \underline{21}$$

$$u_9 = u_8 + u_7 = 21 + 13 = \underline{34} \quad u_{10} = u_9 + u_8 = 34 + 21 = \underline{55}$$