16.2 Line Integrals

$$\begin{cases} x(t) \\ y(t) \\ z(t) \end{cases}$$

speed

$$L = \int_a^b f(g(t), h(t), k(t)) \, \underbrace{|v(t)|}_{} \, dt$$

Ex. $f(x, y, z) = x - 3y^2 + z$

$O(0,0,0) \longrightarrow (1, 1, 1)$

$$\left.\begin{aligned} &x = (1-0)t \\ &\vec{r}(t) = t\hat{i} + t\hat{j} + t\hat{k} \end{aligned}\right\} \quad 0 \le t \le 1$$

$$f(t) = t - 3t^2 + t$$
$$= 2t - 3t^2$$

$$\vec{v}(t) = r'(t) = \hat{i} + \hat{j} + \hat{k}$$

$$|\vec{v}| = \sqrt{1+1+1} = \sqrt{3}$$

$$\int_c f \, ds = \sqrt{3} \int_0^1 (2t - 3t^2) \, dt$$

$$= \sqrt{3} \left(t^2 - t^3 \right|_0^1$$

$$\underline{= 0 \ ?}$$

$L = 0$??

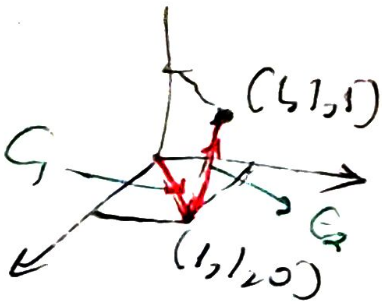
Ex

$C_1: \ \vec{r}(t) = t\hat{i} + t\hat{j} \quad 0 \le t \le 1$

$|\vec{v}| = \sqrt{1+1} = \sqrt{2}$

$f(t,t,0) = t - 3t^2$

$C_2: \ \vec{r}(t) = \hat{i} + \hat{j} + t\hat{k} \quad 0 \le t \le 1$



$|\vec{v}| = 1 \qquad f_2(1,1,t) = -2 + t$

$$\sqrt{2}\int_0^1 (t - 3t^2)\,dt + \int_0^1 (t-2)\,dt$$

$$= \sqrt{2}\left(\frac{1}{2}t^2 - t^3\right|_0^1 + \left(\frac{1}{2}t^2 - 2t\right|_0^1$$

$$\boxed{= -\frac{\sqrt{2}}{2} - \frac{3}{2}}$$

Q8 $\int_C -y\,dx + z\,dy + 2x\,dz$

$C: \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k} \quad 0 \le t \le 2\pi$

$x = \cos t \rightarrow dx = -\sin t\,dt$

$y = \sin t \rightarrow dy = \cos t\,dt$

$z = t \rightarrow dz = dt$

$$\int_C -y\,dx + z\,dy + 2x\,dz = \int_0^{2\pi} (\sin^2 t + t\cos t + 2\cos t)\,dt$$

$$= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2t + \underline{t\cos t} + 2\cos t\right) dt$$

	$\int \cos t$
$+t$	$\sin t$
-1	$-\cos t$

$$= \frac{1}{2}t - \frac{1}{4}\sin 2t + t\sin t + \cos t + 2\sin t \Big|_0^{2\pi}$$

$$= \pi + 1 - 1$$

$$= \boxed{\pi}$$

Integration By part

$t^n \,\Big|\, e^{at}$, $\cos bt$, $\sin bt$

$e^{at} \,\big|\, \cos bt$, $\sin bt$

$$\int_1^2 \int_0^4 2xy\,dy\,dx = \int_1^2 2x\,dx \int_0^4 y\,dy$$

$$= x^2\Big|_1^2 \quad \frac{1}{2}y^2\Big|_0^4$$

$$= (3)\quad(8)$$

$$= \boxed{24}$$

$$\int_1^3 \int_1^{e^x} \frac{x}{y}\,dy\,dx = \int_1^3 x \ln y\Big|_1^{e^x} dx$$

$$= \int_1^3 x^2 \, dx$$

$$= \frac{1}{3}x^3\Big|_1^3$$

$$= \boxed{\frac{26}{3}}$$

$V?$ $z = x^2 + y^2$, $-1 \le x \le 1$ $-1 \le y \le 1$

$$V = \int_{-1}^{1}\int_{-1}^{1} (x^2+y^2)\,dx\,dy$$

$$= \int_{-1}^{1} \left(\frac{1}{3}x^3 + y^2 x \Big|_{-1}^{1}\right) dy$$

$$= \int_{-1}^{1} \left(\frac{1}{3} + y^2 + \frac{1}{3} + y^2\right) dy$$

$$= 2\int_{-1}^{1} \left(\frac{1}{3} + y^2\right) dy$$

$$= 2\left(\frac{1}{3}y + \frac{1}{3}y^3 \Big|_{-1}^{1}\right)$$

$$= 2\left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right)$$

$$= \frac{8}{3} \text{ unit}^3$$

Area? $x = y - y^2$ $y = -x$

$x = -y$

$x = y - y^2 = -y$

$y^2 - 2y = 0 \rightarrow y = 0, 2$

$$\text{Area} = \int_0^2 \int_{-y}^{y-y^2} dx\,dy$$

$$= \int_0^2 (y - y^2 + y)\,dy$$

$$= \int_0^2 (-y^2 + 2y)\,dy$$

$$= -\frac{1}{3}y^3 + y^2 \Big|_0^2$$

$$= -\frac{8}{3} + 4$$

unit^2

$$= \frac{4}{3}$$

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,dy\,dx$$

r^2

$0 \le r \le 1$

$\pi \le \theta \le \frac{3\pi}{2}$

$$= \int_{\pi}^{3\pi/2} d\theta \int_0^1 \frac{2}{1+r}\, r\,dr$$

$$= 2\left(\frac{3\pi}{2}-\pi\right)\int_0^1 \frac{r}{1+r}\,dr$$

$$= \pi \int_0^1 \left(1-\frac{1}{1+r}\right)dr$$

$$\frac{dx}{a+x} = \ln(x+a)$$

$$= \pi\left(r - \ln(1+r)\right)\Big|_0^1$$

$$= \pi(1-\ln 2)$$

$$\iint_R 2xy\,dA \qquad R = \{(r,\theta) \mid 1 \le r \le 3 \quad 0 \le \theta \le \pi/2\}$$

$$\iint_R 2xy\,dA = 2\int_0^{\pi/2}\int_1^3 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta$$

$$= \cancel{\int_0^{\pi/2} 2\cos\theta\sin\theta\,d\theta}$$

$$= \int_0^{\pi/2} \sin 2\theta\,d\theta \int_1^3 r^3\,dr$$

$$= -\frac{1}{2}\cos 2\theta \Big|_0^{\pi/2} \frac{1}{4} r^4 \Big|_1^3$$

$$= -\frac{1}{2}(-1-1)\ \frac{1}{4}(81-1)$$

$$= \underline{20}$$

$$\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx$$

$$= \int_0^1\int_0^1 \left(x^2z+y^2z+\frac{1}{3}z^3\right|_0^1 dy\,dx$$

$$= \int_0^1\int_0^1 \left(x^2+y^2+\frac{1}{3}\right) dy\,dx$$

$$= \int_0^1 \left(x^2y+\frac{1}{3}y^3+\frac{1}{3}y\right|_0^1 dx$$

$$= \int_0^1 \left(x^2+\frac{2}{3}\right) dx$$

$$= \frac{1}{3}x^3+\frac{2}{3}x \Big|_0^1$$

$$= \frac{1}{3}+\frac{2}{3}$$

$$= \underline{1}$$

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^3 (x^2+y^2)^{3/2} \, dx \, dy \, dz$$

$0 \le z \le 3$

$0 \le y \le \sqrt{9-x^2}$ $\qquad 0 \le r \le 3$

$0 \le x \le 3$ $\qquad 0 \le \theta \le \frac{\pi}{2} \rightarrow$ $y(+)$ $x(+)$

$$\int_0^{\pi/2} \int_0^3 \int_0^3 r^3 \, dz \, r \, dr \, d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^3 r^4 \, dr \int_0^3 dz$$

$$= \frac{\pi}{2} \quad \frac{1}{5}\left(r^5\right|_0^3 \quad (3)$$

$$= \frac{3\pi}{10}(243)$$

$$= \boxed{\frac{729\pi}{10}}$$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin\phi\,\rho^3\Big|_0^{2\cos\phi}d\phi$$

$$=\frac{2\pi}{3}(8)\int_0^{\pi/2}\sin\phi\cos^3\phi\,d\phi$$

$$=-\frac{16\pi}{3}\int_0^{\pi/2}\cos^3\phi\,d(\cos\phi)$$

$$=-\frac{4\pi}{3}\cos^4\phi\Big|_0^{\pi/2}$$

$$=\boxed{\frac{4\pi}{3}}$$

$$\rho(x) = 2 - \frac{x^2}{16} \qquad 0 \le x \le 4$$

$\bar{x}$?

$$m = \int_0^4 \left(2 - \frac{1}{16}x^2\right) dx$$

M.

$$= 2x - \frac{1}{48}x^3 \Big|_0^4$$

$$= 8 - \frac{4}{3}$$

$$= \frac{20}{3}$$

$$\bar{x} = \frac{3}{20} \int_0^4 \left(2x - \frac{x^3}{16}\right) dx$$

$$= \frac{3}{20} \left(x^2 - \frac{1}{64}x^4 \Big|_0^4\right.$$

$$= \frac{3}{20}(16 - 4)$$

$$= \boxed{\frac{9}{5}}$$

$$\iiint_D yz\,dV \qquad \begin{cases} x+2y=1 \\ x+2y=2 \end{cases} \qquad \begin{cases} x-z=0 \\ x-z=2 \end{cases} \qquad \begin{cases} 2y-z=0 \\ 2y-z=3 \end{cases}$$

$$u = x+2y \qquad v = x-z \qquad w = 2y-z$$

$$1 \le u \le 2 \qquad 0 \le v \le 2 \qquad 0 \le w \le 3$$

$$\begin{aligned} x + 2y \quad\quad &= u \\ x \quad\quad - z &= v \\ 2y - z &= w \end{aligned} \Big] - \Rightarrow \quad \begin{aligned} x+2y &= u \\ \underline{x-2y} &= \underline{v-w} \\ 2x \quad &= u+v-w \end{aligned}$$

$$\begin{cases} x = \frac{1}{2}u + \frac{1}{2}v - \frac{1}{2}w \\ y = \frac{1}{4}u - \frac{1}{4}v + \frac{1}{4}w \\ z = \frac{1}{2}u - \frac{1}{2}v - \frac{1}{2}w \end{cases} \qquad \begin{aligned} 2y &= u - \tfrac{1}{2}u - \tfrac{1}{2}v + \tfrac{1}{2}w \\ z &= \tfrac{1}{2}u + \tfrac{1}{2}v - \tfrac{1}{2}w - v \end{aligned}$$

$$J = \begin{vmatrix} 1/2 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \frac{3}{16} - \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$$

$$= \frac{1}{4}$$

$$(u-v)^2 - w^2$$

$$(u-v+w)(u-v-w)$$

$$\iiint_D yz\,dV = \frac{1}{4}\int_0^3\int_0^2\int_1^2 \left(\frac{1}{4}u - \frac{1}{4}v + \frac{w}{4}\right)\left(\frac{u}{2} - \frac{v}{2} - \frac{w}{2}\right) du\,dv\,dw$$

$$= \frac{1}{32}\int_0^3\int_0^2\int_1^2 (u^2 - 2uv + v^2 - w^2)\,du\,dv\,dw$$

$$= \frac{1}{32}\int_0^3\int_0^2 \left(\frac{1}{3}u^3 - vu^2 + v^2u - w^2u \Big|_1^2\right) dv\,dw$$

$$= \frac{1}{32}\int_0^3\int_0^2 \left(\frac{8}{3} - 4v + 2v^2 - 2w^2 - \frac{1}{3} + v - v^2 + w^2\right) dv\,dw$$

$$= \frac{1}{32}\int_0^3\int_0^2 \left(\frac{7}{3} - 3v + v^2 - w^2\right) dv\, dw$$

$$= \frac{1}{32}\int_0^3 \left(\frac{7}{3}v - \frac{3}{2}v^2 + \frac{1}{3}v^3 - w^2 v \right|_0^2 dw$$

$$= \frac{1}{32}\int_0^3 \left(\frac{14}{3} - 6 + \frac{8}{3} - 2w^2\right) dw$$

$$= \frac{1}{32}\int_0^3 \left(\frac{4}{3} - 2w^2\right) dw$$

$$= \frac{1}{32}\left(\frac{4}{3}w - \frac{2}{3}w^3 \right|_0^3$$

$$= \frac{1}{32}\left(4 - 18\right)$$

$$-14$$

$$= -\frac{7}{16}$$

$x+2y=u$ $\quad$ $\Delta = \begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & -1 \end{vmatrix} = 4$

$x-z=v$

$2y-z=w$ $\quad$ $\Delta_x = \begin{vmatrix} u & 2 & 0 \\ v & 0 & -1 \\ w & 2 & -1 \end{vmatrix} = -2w+2u+2v$

$x=\frac{1}{2}(u+v-w)$ $\quad$ $\Delta_y = \begin{vmatrix} 1 & u & 0 \\ 1 & v & -1 \\ 0 & w & -1 \end{vmatrix} = -v+w+u$

$y=\frac{1}{4}(u-v+w)$

$z=\frac{1}{2}(u-v-w)$ $\quad$ $\Delta z = \begin{vmatrix} 1 & 2 & u \\ 1 & 0 & v \\ 0 & 2 & w \end{vmatrix} = 2u-2v-2w$

$$J = \frac{1}{16}\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{vmatrix} = \frac{1}{4}$$

$$\iiint_D yz\,dv = \frac{1}{4}\frac{1}{8}\int_1^2\int_0^2\int_0^3 ((u-v)^2-w^2)\,dw\,dv\,du$$

$$= \frac{1}{32}\int_1^2\int_0^2 \left((u^2-2uv+v^2)w-\frac{1}{3}w^3\right)\Big|_0^3 dv\,du$$

$$= \frac{1}{32}\int_1^2\int_0^2 (3u^2-6uv+3v^2-9)\,dv\,du$$

$$= \frac{1}{32}\int_1^2 (3u^2v-3uv^2+v^3-9v\Big|_0^2 du$$

$$= \frac{1}{32}\int_1^2 (6u^2-12u+8-18)\,du$$

$$= \frac{1}{32}\int_1^2 (6u^2-12u-10)\,du$$

$$= \frac{1}{32}(2u^3-6u^2-10u\Big|_1^2$$

$\frac{1}{2}\left(16 - 2d - 20 - 2 + 6 + 10\right)$

$32 - 4d$