Solution Section 2.1 – Functions and Graphs

Exercise

$$f(x)=\begin{cases} 2+x & \text{if } x<-4 \\ -x & \text{if } -4\le x\le 2 \\ 3x & \text{if } x>2 \end{cases}$$

Find: $f(-5), f(-1), f(0),$ *and* $f(3)$

Solution

a) $f(-5)=2-5=-3$

b) $f(-1)=-(-1)=1$

c) $f(0)=-0=0$

d) $f(3)=3(3)=9$

Exercise

$$f(x)=\begin{cases} -2x & \text{if } x<-3 \\ 3x-1 & \text{if } -3\le x\le 2 \\ -4x & \text{if } x>2 \end{cases}$$

Find: $f(-5), f(-1), f(0),$ *and* $f(3)$

Solution

a) $f(-5)=-2(-5)=10$

b) $f(-1)=3(-1)-1=-4$

c) $f(0)=3(0)-1=-1$

d) $f(3)=-4(3)=-12$

Exercise

$$f(x)=\begin{cases} x^3+3 & \text{if } -2\le x\le 0 \\ x+3 & \text{if } 0<x<1 \\ 4+x-x^2 & \text{if } 1\le x\le 3 \end{cases}$$

Find: $f(-5),\ f(-1),\ f(0),$ *and* $f(3)$

Solution

a) $f(-5)=$ *doesn't exist*

b) $f(-1)=(-1)^3+3$

$\quad \underline{=2}$

c) $f(0)=(0)^3+3$

$\underline{=3}$

d) $f(3)=4+(3)-(3)^2$

$\underline{=-2}$

Exercise

$$h(x)=\begin{cases}\dfrac{x^2-9}{x-3} & if \quad x\neq 3\\ 6 & if \quad x=3\end{cases}$$

Find: $h(5)$, $h(0)$, *and* $h(3)$

Solution

a) $h(5)=\dfrac{5^2-9}{5-3}$

$\underline{=8}$

b) $h(0)=\dfrac{0^2-9}{0-3}$

$\underline{=3}$

c) $\underline{h(3)=6}$

Exercise

$$f(x)=\begin{cases}3x+5 & if \quad x<0\\ 4x+7 & if \quad x\geq 0\end{cases}$$

Find

a) $f(0)$ *b)* $f(-2)$ *c)* $f(1)$ *d)* $f(3)+f(-3)$ *e)* Graph $f(x)$

Solution

a) $f(0)=4(0)+7$

$\underline{=7}$

b) $f(-2)=3(-2)+5$

$\underline{=-1}$

c) $f(1)=4(1)+7$

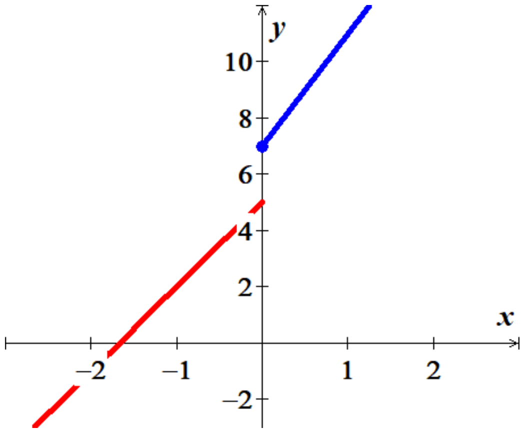
$\underline{=11}$

d) $f(3)+f(-3)=4(3)+7+3(-3)+5$

$=12+7-9+5$

$\underline{=15}$

e)



Exercise

$$f(x)=\begin{cases} 6x-1 & if \quad x<0 \\ 7x+3 & if \quad x\ge 0 \end{cases} \quad \text{Find}$$

a) $f(0)$ *b)* $f(-1)$ *c)* $f(4)$ *d)* $f(2)+f(-2)$ *e)* Graph $f(x)$

Solution

a) $f(0)=7(0)+3$

$=3$

b) $f(-2)=6(-1)-1$

$=-7$

c) $f(4)=7(4)+3$

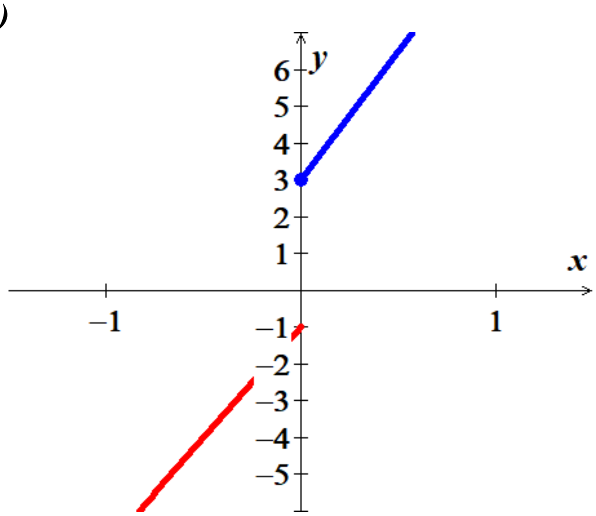
$=31$

d) $f(2)+f(-2)=7(2)+3+6(-2)-1$

$=14+3-12-1$

$=4$

e)



Exercise

$f(x)=\begin{cases} 2x+1 & if \quad x\le 1 \\ 3x-2 & if \quad x>1 \end{cases}$ Find

a) $f(0)$ *b)* $f(2)$ *c)* $f(-2)$ *d)* $f(1)+f(-1)$ *e)* Graph $f(x)$

Solution

a) $f(0)=2(0)+1$

$\underline{=1}$

b) $f(2)=3(2)-2$

$\underline{=4}$

c) $f(-2)=2(-2)+1$

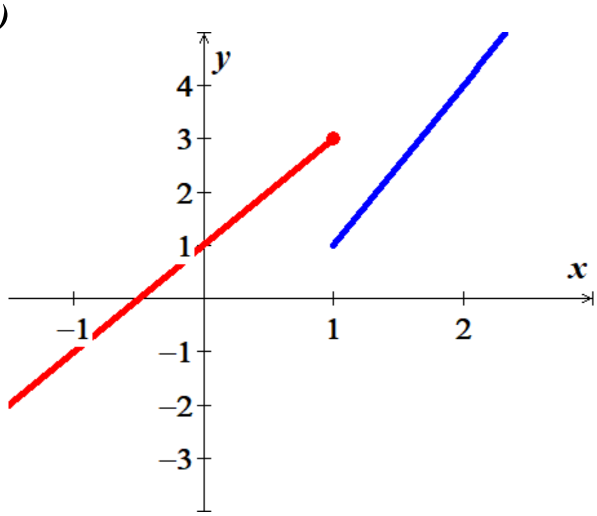
$\underline{=-3}$

d) $f(1)+f(-1)=(2(1)+1)+(2(-1)+1)$

$=2+1-2+1$

$\underline{=2}$

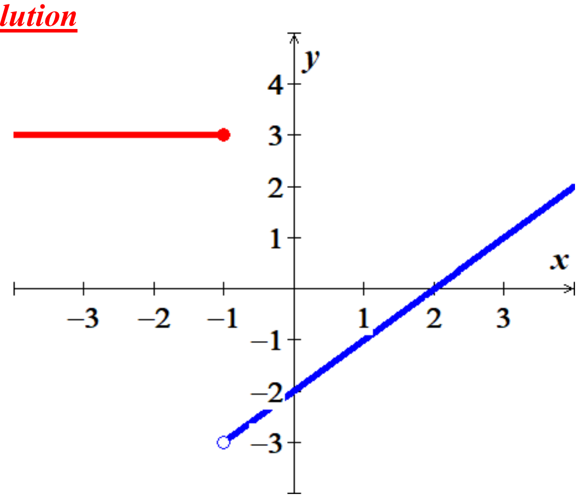
e)



Exercise

Graph the piecewise function defined by $f(x)=\begin{cases} 3 & \text{if} \quad x \le -1 \\ x-2 & \text{if} \quad x > -1 \end{cases}$

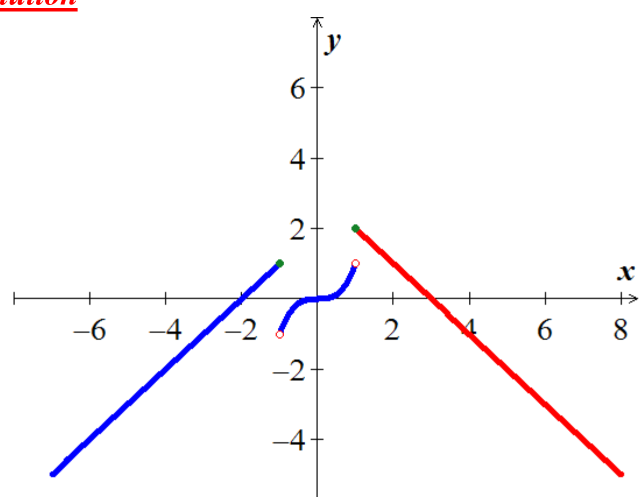
Solution



Exercise

Sketch the graph $f(x)=\begin{cases} x+2 & \text{if} \quad x \le -1 \\ x^3 & \text{if} \quad -1 < x < 1 \\ -x+3 & \text{if} \quad x \ge 1 \end{cases}$

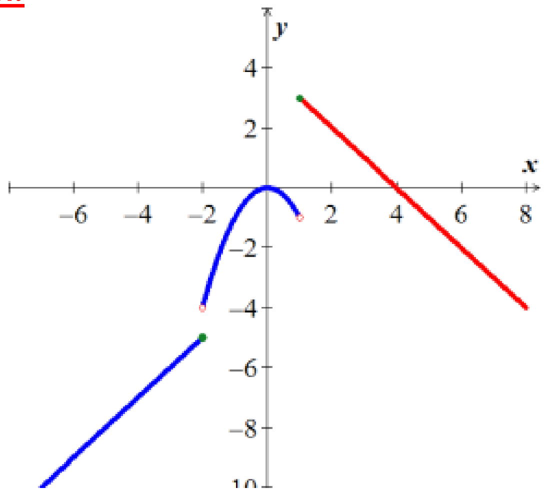
Solution



Exercise

Sketch the graph $f(x)=\begin{cases} x-3 & \text{if} \quad x \le -2 \\ -x^2 & \text{if} \quad -2 < x < 1 \\ -x+4 & \text{if} \quad x \ge 1 \end{cases}$

Solution



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x)=x^2-2x+3$$

Solution

Relative Maximum: *None*

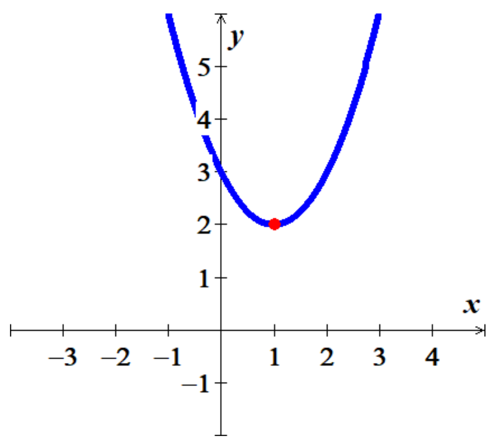
Minimum Point: $(1,\ 2)$

Increasing: $(1,\ \infty)$

Decreasing: $(-\infty,\ 1)$

Domain: $\mathbb{R}$

Range: $[2,\ \infty)$



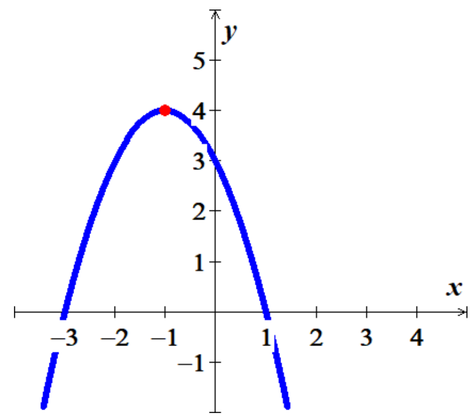
Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = -x^2 - 2x + 3$$

Solution

Maximum Point:	$(-1,\ 4)$
Relative Minimum:	*None*
Increasing:	$(-\infty,\ -1)$
Decreasing:	$(-1,\ \infty)$
Domain:	$\mathbb{R}$
Range:	$(-\infty,\ 4]$



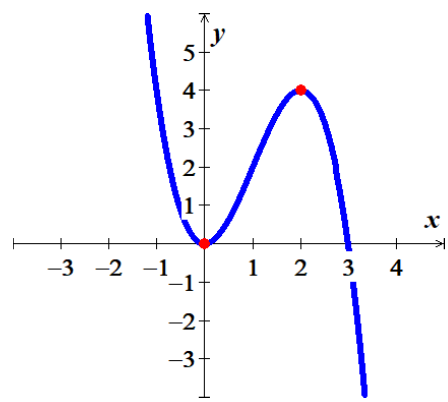
Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = -x^3 + 3x^2$$

Solution

Relative Maximum:	$(2,\ 4)$
Relative Minimum:	$(0,\ 0)$
Increasing:	$(0,\ 2)$
Decreasing:	$(-\infty,\ 0)\quad(2,\ \infty)$
Domain:	$\mathbb{R}$
Range:	$\mathbb{R}$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = x^3 - 3x^2$$

Solution

Relative Maximum: $(0,\ 0)$

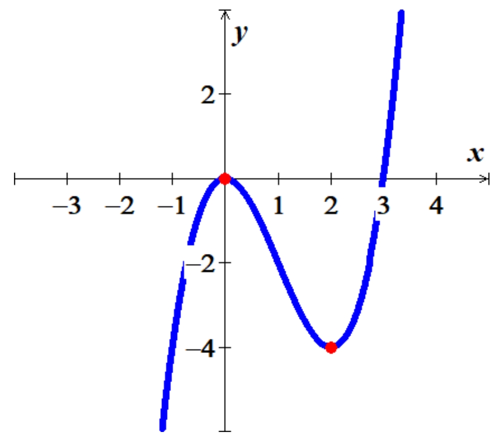
Relative Minimum: $(2,\ -4)$

Increasing: $(-\infty,\ 0)\ (2,\ \infty)$

Decreasing: $(0,\ 2)$

Domain: $\mathbb{R}$

Range: $\mathbb{R}$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = \frac{1}{4}x^4 - 2x^2$$

Solution

Relative Maximum: $(0,\ 0)$

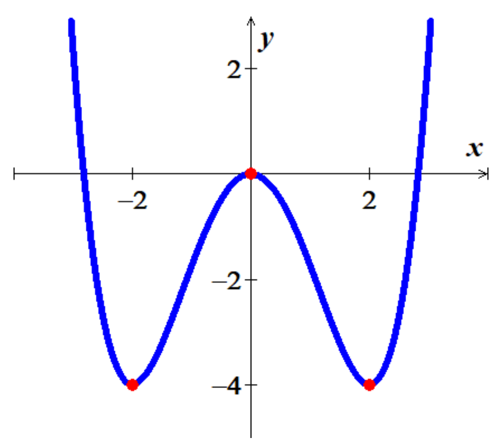
Minimum Points: $(-2,\ -4)\ \&\ (2,\ -4)$

Increasing: $(-2,\ 0) \cup (2,\ \infty)$

Decreasing: $(-\infty,\ -2) \cup (0,\ 2)$

Domain: $\mathbb{R}$

Range: $[-4,\ \infty)$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x)=\frac{4}{81}x^4-\frac{8}{9}x^2+4$$

Solution

Relative Maximum: $(0,\ 4)$

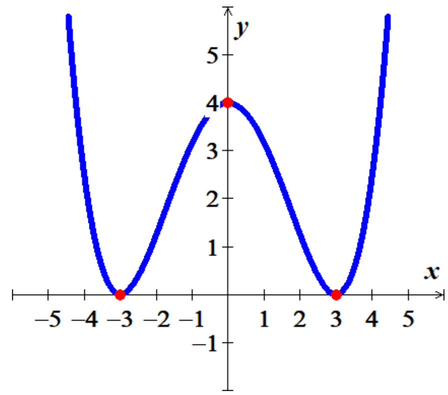
Minimum Points: $(-3,\ 0)\ \&\ (3,\ 0)$

Increasing: $(-3,\ 0)\cup(3,\ \infty)$

Decreasing: $(-\infty,\ -3)\cup(0,\ 3)$

Domain: $\mathbb{R}$

Range: $[0,\ \infty)$



Exercise

The elevation H, in *meters*, above sea level at which the boiling point of water is in t degrees Celsius is given by the function

$$H(t)=1000(100-t)+580(100-t)^2$$

At what elevation is the boiling point 99.5°.

Solution

$$H(99.5)=1000(100-99.5)+580(100-99.5)^2$$

$$\underline{=645\ m}$$

Exercise

A hot-air balloon rises straight up from the ground at a rate of 120 *ft./min*. The balloon is tracked from a rangefinder on the ground at point P, which is 400 *ft*. from the release point Q of the balloon. Let ***d*** = the distance from the balloon to the rangefinder and ***t*** – the time, in minutes, since the balloon was released. Express ***d*** as a function of ***t***.

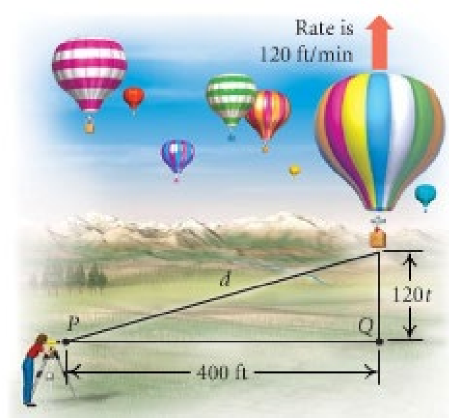
Solution

$$d^2=(120t)^2+400^2$$

$$d=\sqrt{14400t^2+160000}$$

$$d=\sqrt{1600\left(9t^2+100\right)}$$

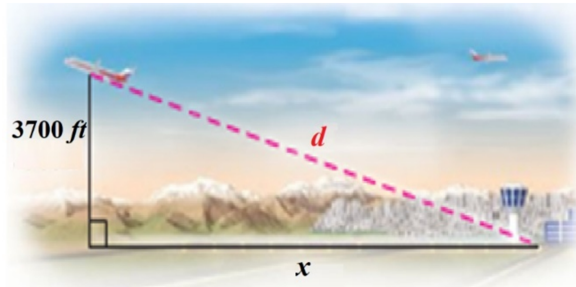
$$\underline{d(t)=40\sqrt{9t^2+100}}$$



Exercise

An airplane is flying at an altitude of 3700 *feet*. The slanted distance directly to the airport is d *feet*. Express the horizontal distance x as a function of d.

Solution



$$d^2 = (3,700)^2 + x^2$$

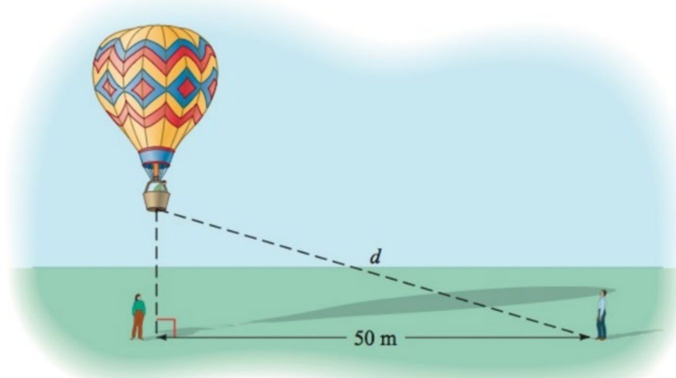
$$h^2 = d^2 - (3700)^2$$

$$\boxed{h(t) = \sqrt{d^2 - (3,700)^2}}$$

Exercise

For the first minute of flight, a hot air balloon rises vertically at a rate of 3 *m/sec*. If t is the time in *seconds* that the balloon has been airborne, write the distance d between the balloon and a point on the ground 50 *meters* from the point to lift off as a function of t.

Solution



$\boxed{h = 3t}$ $\qquad v = \dfrac{h}{t}$

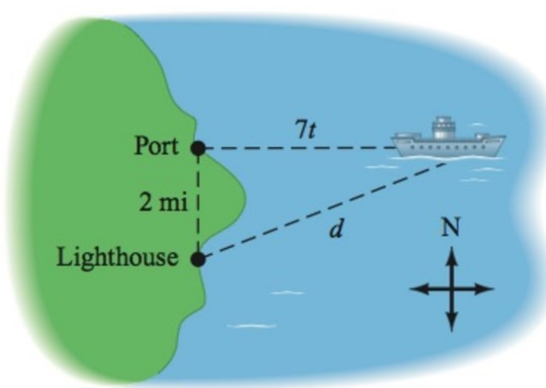
$$d^2 = h^2 + 50^2$$

$$\boxed{d(t) = \sqrt{9t^2 + 2,500}}$$

Exercise

A light house is 2 *miles* south of a port. A ship leaves port and sails east at a rate of 7 *miles* per *hour*. Express the distance d between the ship and the lighthouse as a function of time, given that the ship has been sailing for t hours.

Solution

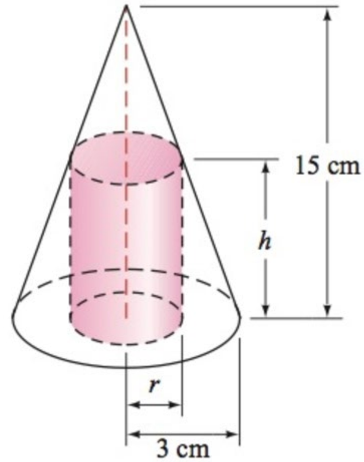


$$d^2 = 4^2 + (7t)^2$$

$$\boxed{d(t) = \sqrt{16 + 49t^2}}$$

Exercise

A cone has an altitude of 15 *cm* and a radius of 3 *cm*. A right circular cylinder of radius r and height h is inscribed in the cone. Use similar triangles to write h as a function of r.



Solution

$$\frac{15-h}{15}=\frac{r}{3}$$

$$15-h=5r$$

$$\boxed{h(r)=15-5r}$$

Exercise

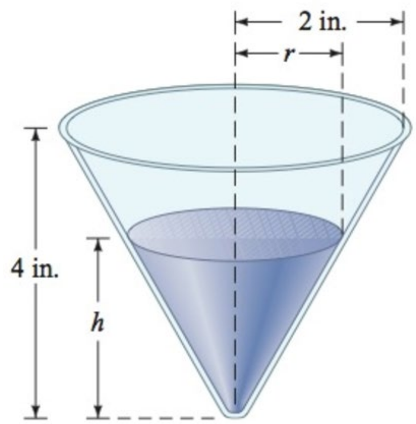
Water is flowing into a conical drinking cup with an altitude of 4 *inches* am a radius of 2 *inches*.

a) Write the radius r of the surface of the water as a function of its depth h.
b) Write the volume V of the water as a function of its depth h.

Solution

a) $\frac{h}{4}=\frac{r}{2}$

$$\boxed{r(h)=\frac{1}{2}h}$$



b) $Area=\pi r^2$

$$V=\frac{1}{3}\pi r^2 h$$

$$=\frac{1}{3}\pi\left(\frac{h^2}{4}\right)h$$

$$\boxed{=\frac{1}{12}\pi h^3}$$

Exercise

A water tank has the shape of a right circular cone with height 16 *feet* and radius 8 *feet*. Water is running into the tank so that the radius r (in *feet*) of the surface of the water is given by $r=1.5t$, where t is the time (in *minutes*) that the water has been running.

a) The area A of the surface of the water is $A=\pi r^2$. Find $A(t)$ and use it to determine the area of the surface of the water when $t=2$ *minutes*.

b) The volume V of the water is given by $V=\frac{1}{3}\pi r^2 h$. Find $V(t)$ and use it to determine the volume of the water when $t=3$ *minutes*

Solution

c) $Area = \pi r^2$

$$A(t) = \pi\left(\frac{3}{2}t\right)^2$$

$$= \frac{9\pi}{4}t^2$$

d) $\frac{h}{16} = \frac{r}{8}$

$h = 2r$

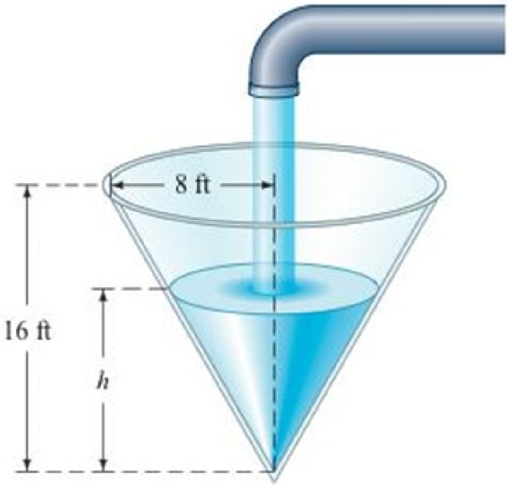
$$V(t) = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi r^2 (2r)$$

$$= \frac{2}{3}\pi r^3$$

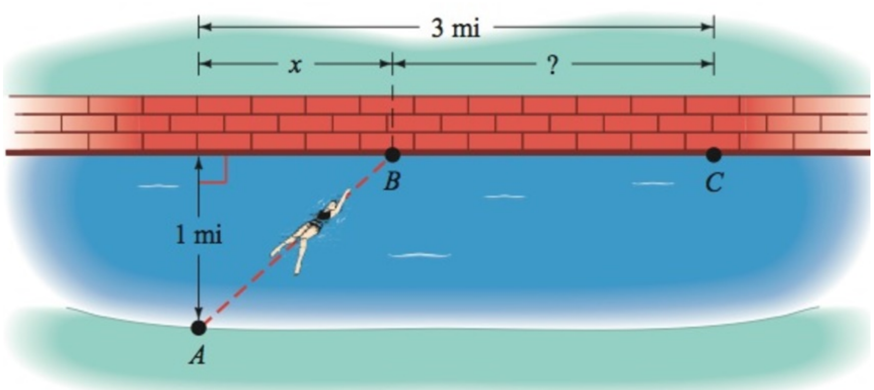
$$= \frac{2}{3}\pi\left(\frac{3}{2}t\right)^3$$

$$= \frac{9}{4}\pi t^3$$



Exercise

An athlete swims from point ***A*** to point ***B*** at a rate of 2 *miles* per *hour* and runs from point ***B*** to point ***C*** at a rate of 8 *miles* per *hour*. Use the dimensions in the figure to write the time *t* required to reach point ***C*** as a function of *x*.



Solution

$$\textit{Swimming distance } = \sqrt{x^2+1}$$

$$t_{swim} = \frac{\sqrt{x^2+1}}{2} \qquad t = \frac{d}{v}$$

$$\textit{Running distance } = 3 - x$$

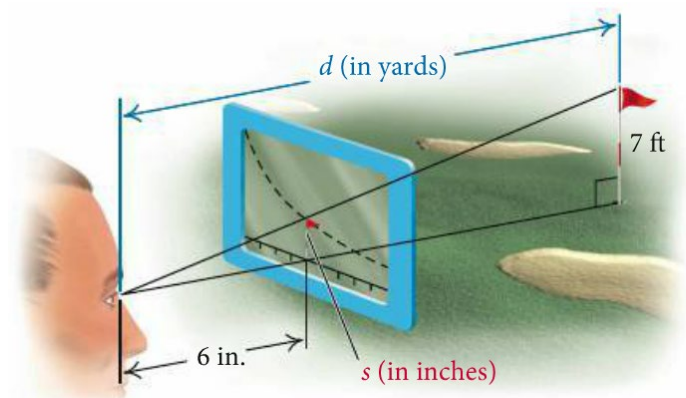
$$t_{run} = \frac{3-x}{8} \qquad\qquad t = \frac{d}{v}$$

$$\boxed{t_{total} = \frac{\sqrt{x^2+1}}{2} + \frac{3-x}{8}}$$

Exercise

A device used in golf to estimate the distance d, in *yards*, to a hole measures the size $\boldsymbol{s}$, in *inches*, that the 7-*foot* pin appears to be in a viewfinder. Express the distance d as a function of $\boldsymbol{s}$.

Solution



$$\frac{d}{6} = \frac{7}{s}\frac{ft}{in}$$

$$d = \frac{7}{s}\frac{ft}{in}6in$$

$$d = \frac{42}{s}ft\frac{1yd}{3ft}$$

$$\boxed{d(s) = \frac{14}{s}}$$

Exercise

A *rhombus* is inscribed in a rectangle that is w *meters* wide with a perimeter of 40 m. Each vertex of the rhombus is a midpoint of a side of the rectangle. Express the area of the *rhombus* as a function of the rectangle's width.

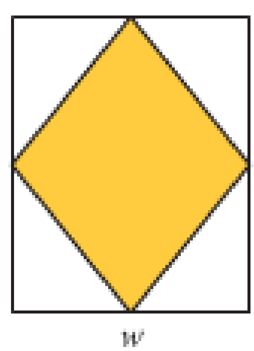
Solution

The area of the rhombus = $\frac{1}{2}$ area of the rectangle, since each vertex of the rhombus is a midpoint of a side of the rectangle.

$$Perimter: \quad 2l + 2w = 40 \qquad \textit{Divide both sides by 2}$$

$$l + w = 20$$

$$l = 20 - w$$



$$Area\ of\ the\ rectangle = lw = (20-w)w$$

$$Area\ of\ the\ rhombus = \frac{1}{2}\left(20w - w^2\right)$$

$$\boxed{= -\frac{1}{2}w^2 + 10w}$$

Exercise

The surface area *S* of a right circular cylinder is given by the formula $S = 2\pi rh + 2\pi r^2$. if the height is twice the radius, find each of the following.

a) A function $S(r)$ for the surface area as a function of *r*.

b) A function $S(h)$ for the surface area as a function of *h*.

Solution

Given: $h = 2r$

a) $S = 2\pi rh + 2\pi r^2$

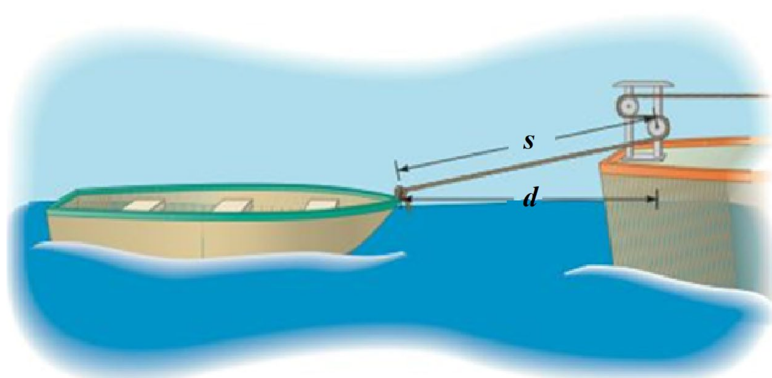
$$S(r) = 2\pi r(2r) + 2\pi r^2$$
$$= 4\pi r^2 + 2\pi r^2$$
$$\boxed{= 6\pi r^2}$$

b) $r = \frac{1}{2}h$

$$S(h) = 2\pi\left(\frac{1}{2}h\right)h + 2\pi\left(\frac{1}{2}h\right)^2$$
$$= \pi h^2 + \frac{1}{2}\pi h^2$$
$$\boxed{= \frac{3}{2}\pi h^2}$$

Exercise

A boat is towed by a rope that runs through a pulley that is 4 *feet* above the point where the rope is tied to the boat. The length (in *feet*) of the rope from the boat to the pulley is given by $s = 48 - t$, where *t* is the time in *seconds* that the boat has been in tow. The horizontal distance from the pulley to the boat is *d*.



a) Find $d(t)$

b) Evaluate $s(35)$ and $d(35)$

Solution

a) $s^2 = d^2 + 4^2$

$d^2 = (48-t)^2 - 16$

$d(t) = \sqrt{2,304 - 96t + t^2 - 16}$

$= \sqrt{t^2 - 96t + 2,288}$

b) $s(35) = 48 - 35$

$= 13 \ \textit{feet}$

$d(35) = \sqrt{(48-35)^2 - 16}$

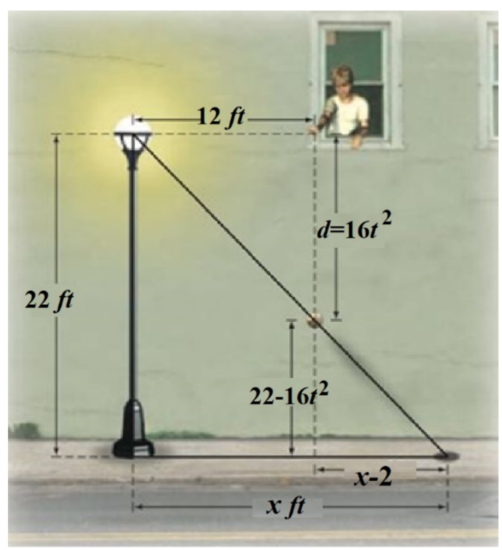
$= \sqrt{13^2 - 16}$

$= \sqrt{153} \ \textit{feet}$

Exercise

The light from a lamppost casts a shadow from a ball that was dropped from a height of 22 *feet* above the ground. The distance *d*, in *feet*, the ball has dropped *t seconds* after it is released is given by $d(t) = 16t^2$. Find the distance *x*, in *feet*, of the shadow from the base of the lamppost as a function of time *t*.

Solution



$\dfrac{22-16t^2}{22} = \dfrac{x-12}{x}$

$\left(22-16t^2\right)x = 22(x-12)$

$\left(22-16t^2\right)x = 22x - 264$

$\left(22-16t^2-22\right)x = -264$

$-16t^2 x = -264$

$x(t) = \dfrac{33}{2t^2}$

Exercise

A right circular cylinder of height *h* and a radius *r* is inscribed in a right circular cone with a height of 10 *feet* and a base with radius 6 *feet*.

a) Express the height *h* of the cylinder as a function of *r*.

b) Express the volume *V* of the cylinder as a function of *r*.

c) Express the volume *V* of the cylinder as a function of *h*.

Solution

a) $\dfrac{h}{10} = \dfrac{6-r}{6}$

$$\boxed{h(r) = \frac{5}{3}(6-r)}$$

b) $V = \pi r^2 h$

$$V(r) = \frac{5}{3}\pi r^2 (6-r)$$

$$\boxed{= \frac{5}{3}\pi\left(6r^2 - r^3\right)}$$

c) $\dfrac{3}{5}h = 6 - r$

$$r = 6 - \frac{3}{5}h$$

$$V = \pi r^2 h$$

$$V(h) = \pi\left(\frac{30-3h}{5}\right)^2 h$$

$$\boxed{= \frac{1}{25}\pi h(30-3h)^2}$$