Lecture 3. Infinite Sequences & Series

3.1 - Sequences.

formula

a seq. $a_1, a_2, \dots, a_n, \dots$ → ∞ seq

n-seq.

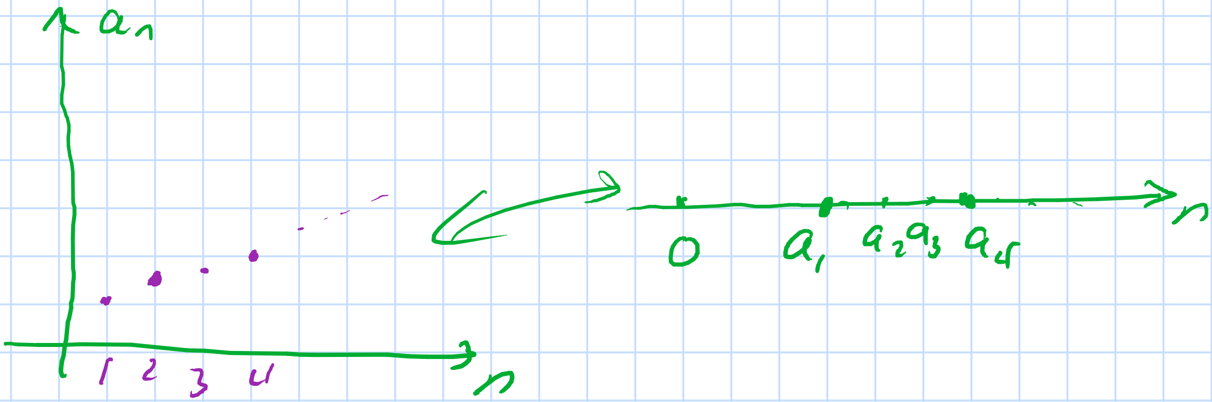
f(x) is continuous fct

a_n dots

$n \in \mathbb{N}$

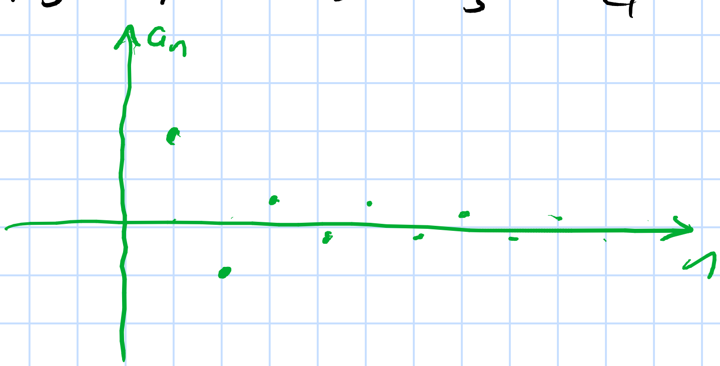
$a_n = \sqrt{n}$

$\{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots\}$

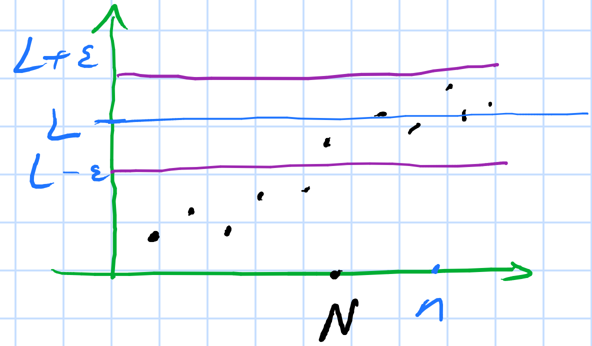


$a_n = (-1)^{n+1}\frac{1}{n}$ (alternating series)

$\{a_n\} = \{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, (-1)^{n+1}\frac{1}{n}, \dots\}$



Converges / diverges



Defn

$\{a_n\}$ converges to the nb L $\forall \varepsilon \in \mathbb{R}^+$

$\exists N \in \mathbb{N} \; (\mathbb{Z}^+) \ni$

$$n > N \Rightarrow |a_n - L| < \varepsilon$$

$f(x)$

if $L \nexists$, $\{a_n\}$ diverges

doesn't exist

$\begin{cases} L = \infty \\ L = \pm \end{cases}$

$\{a_n\}$ converges to L

$$\lim_{n \to \infty} a_n = L \Leftrightarrow a_n \longrightarrow L$$

Ex $\lim_{n \to \infty} \frac{1}{n} = 0$

let $\varepsilon > 0$, $\exists N \in \mathbb{Z}^+ \ni \forall n$

$$n > N \Rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon$$

$$\left|\frac{1}{n}\right| < \varepsilon \Rightarrow \underbrace{-\frac{1}{n} < \varepsilon < \frac{1}{n}}$$

$$\varepsilon < \frac{1}{n} \Rightarrow n < \frac{1}{\varepsilon}$$

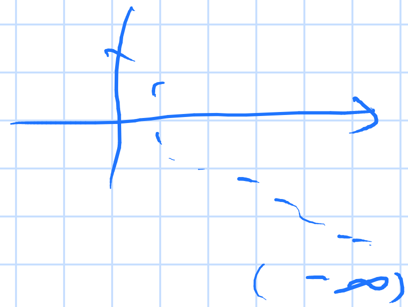
$$\boxed{\frac{1}{n} > \varepsilon \Rightarrow n > \frac{1}{\varepsilon}}$$

$n > N$

$N > \frac{1}{\varepsilon}$

Defn $\{a_n\}$ diverges to ∞, $\forall M, \exists N \in \mathbb{N}$ $\ni$ $\forall n$ larger than N $(n > N)$

$$\lim_{n \to \infty} a_n = \infty$$



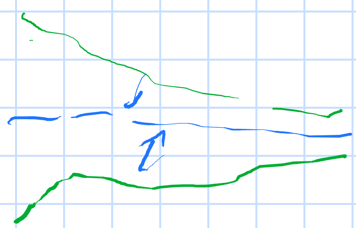
$$\lim_{n \to \infty} \left(-\frac{1}{n}\right) = 0$$

$$\lim_{n \to \infty} \left(\frac{n-1}{n}\right) = 1$$

$$\lim_{n \to \infty} \left(\frac{4 - 7n^6}{n^6 + 3}\right) = -7$$

Theorem Sandwich

$$-1 \le \begin{matrix} \text{sine} \\ \text{cosine} \end{matrix} \le 1$$



$$\frac{\cos n}{n} \to 0$$

$$-1 \le \cos n \le 1$$

$$-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$$

$$\pm \frac{1}{n} \to 0$$

$$0 \le \frac{\cos n}{n} \le 0$$

$$\frac{\cos n}{n} \to 0$$

Ex $\frac{1}{2^n} \to 0$ $\qquad 0 \le \frac{1}{2^n} \le \frac{1}{n}$

$(-1)^n \frac{1}{n} \to 0$ $\qquad -\frac{1}{n} \le (-1)^n \le \frac{1}{n}$

$a_n \Rightarrow \not f(x)$

Ex $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$\underline{= 0}$$

$a_n = \left(\frac{n+1}{n-1}\right)^n$ converges

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

$$\ln a_n = \ln \left(\frac{n+1}{n-1}\right)^n$$

$$= n \ln\left(\frac{n+1}{n-1}\right) \qquad \infty \cdot 0$$

$$= \frac{\ln\left(\frac{n+1}{n-1}\right)}{\frac{1}{n}} = \frac{0}{0} \qquad \left(\frac{n+1}{n-1}\right)'$$

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\frac{-2}{(n-1)^2}}{\frac{-1}{n^2}}$$

$$= 2 \lim_{n\to\infty} \frac{n^2}{(n-1)^2} \qquad \frac{n^2}{n^2}$$

$$= 2$$

$$\lim_{n\to\infty} \ln a_n = 2$$

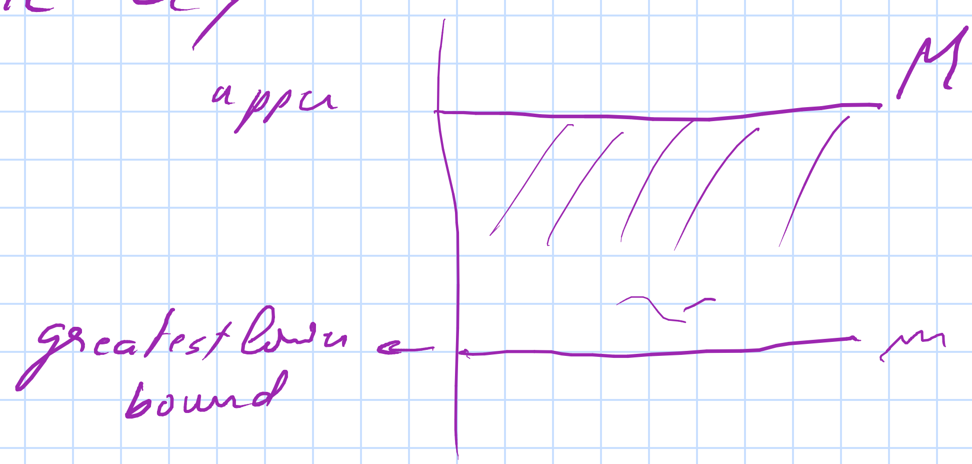
$$a_n \longrightarrow e^2$$

pg 7

$$\left.\begin{array}{lll} \frac{\ln n}{n} \longrightarrow 0 & & \\ \sqrt[n]{n} \longrightarrow 1 & & \\ x^n \longrightarrow 1 & |x| < 1 & \\ x^{1/n} \longrightarrow 0 & x > 0 & \\ \left(1+\frac{x}{n}\right)^n \longrightarrow e^x & & \end{array}\right\} n \longrightarrow \infty$$

$$\frac{x^n}{n!} \longrightarrow 0$$

Monotonic Seq



3.2 Infinite Series

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L$$

$$S_n = a_1 + \cdots + a_n$$

$$a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

$L \nexists$ $\{a_n\}$ series diverges ??

Geometric Series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} a r^{n-1}$$

$$a_{k+1} = a_k r \qquad r: \text{common ratio}$$

$$a_n = a_1 r^{n-1}$$

Defn if $|r| < 1 \Rightarrow \sum_{n=1}^{\infty} a r^{n-1} \boxed{= \frac{a}{1-r}}$

series converges to $\frac{a}{1-r}$

if $|r| \geq 1$, the series diverges

✓ seq may converge to a single value, $\lim a_n$

✓ seq terms increases $\Rightarrow$ seq diverges

✓ seq remains bounded into an oscillating pattern (2 or more values) diverges

✓ seq. terms bounded, but wander chaotic w/o pattern $\rightarrow$ seq diverges

Ex

$$\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n} = \sum_{n=0}^{\infty} 5\left(-\frac{1}{4}\right)^n$$

$$= \sum_{n=1}^{\infty} 5\left(-\frac{1}{4}\right)^{n-1}$$

$$|r| = \left|-\frac{1}{4}\right| = \frac{1}{4} < 1$$

$$S = \frac{5}{1+\frac{1}{4}}$$

$$= \boxed{4}$$

$\therefore$ By the Geometric series, the given series converges w/ sum equal to 4

Ex $a = 6$ $r = \frac{2}{3}$

Total distance

$$S = a + \overset{\frown}{2ar} + 2ar^2 + \dots$$

$$= a + 2a(r + r^2 + \dots)$$

$$= a + \frac{2ar}{1-r}$$

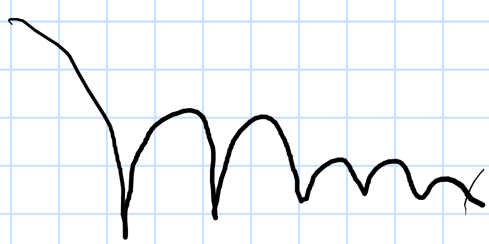
$$= \frac{a + ar}{1-r}$$

$$= \boxed{a\,\frac{1+r}{1-r}}$$

$$= 6\,\frac{1+\frac{2}{3}}{1-\frac{2}{3}}$$

$$= 6\left(\frac{5}{3}\cdot 3\right)$$

$$= \boxed{30}$$



Ex "Telescoping series" $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$S_k = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{k}} - \frac{1}{k+1}\right)$$

$$= 1 - \frac{1}{k+1} \longrightarrow 1$$

$$\frac{1}{k+1} \to 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Diverges Test Series

If $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow a_n \longrightarrow 0$

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or $\neq 0$

Ex $\sum_{n=1}^{\infty} \frac{n+1}{n}$ $\qquad \frac{n+1}{n} \longrightarrow 1$

The given series diverges by divergent series Test.

Ex $\sum_{n=1}^{\infty} n^2$ diverges $n^2 \longrightarrow \infty$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \quad \text{diverges} \qquad \text{limit doesn't exist } +1, -1$$

Ex

$$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{3^{n-1}}{6^{n-1}} - \frac{1}{6^{n-1}} \right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{n-1}$$

$|r_1| = \frac{1}{2} < 1 \qquad r_2 = \frac{1}{6} < 1$

$$= \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{6}}$$

$$= 2 - \frac{6}{5}$$

$$= \boxed{\frac{4}{5}}$$

By the Geometric series, the given series converges w/ sum equals $\frac{4}{5}$

Ex

$$\sum_{n=0}^{\infty} \frac{4}{2^n} = \sum_{n=0}^{\infty} 4 \left(\frac{1}{2}\right)^n \qquad r = \frac{1}{2} < 1$$

$$= 4 \frac{1}{1-\frac{1}{2}}$$

$$= \boxed{8}$$

By the Geometric Series, the given series converges w/ sum $= 8$

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=0}^{\infty} \frac{1}{2^n}$$

$$\sum_{n=a} (\ )^{n-a}$$

$$= \sum_{n=5}^{\infty} \frac{1}{2^{n-5}}$$

$$= \sum_{n=-4}^{\infty} \frac{1}{2^{n+4}}$$

3.2 # 47

Converges or diverges

$$\sum_{n=0}^{\infty} e^{-2n} = \sum_{n=0}^{\infty} \frac{1}{e^{2n}}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{e^2}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{e^2}\right)^{n-1}$$

$$|r| = \frac{1}{e^2} < 1$$

$$S = \frac{1}{1-\frac{1}{e^2}}$$

$$= \frac{e^2}{e^2-1}$$

$\therefore$ By the Geometric series, the given series converges w/ sum $= \dfrac{e^2}{e^2-1}$

46

$$\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n}$$

$$\cos n\pi = (-1)^n$$

$$\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n}$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{5}\right)^n$$

$$|r| = \left|-\frac{1}{5}\right| < 1$$

$$S = \frac{1}{1+\frac{1}{5}}$$

$$= \frac{5}{6}$$

$\therefore$ By the Geometric series, the given series converges at sum $= \frac{5}{6}$

3.3 Integral Test

Ex

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

$$\int_1^{\infty} \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^{\infty}$$

$$= -(0-1)$$

$$= \boxed{1}$$

$\boxed{p\text{- Series}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} p>1 & \text{converges} \\ p\le 1 & \text{diverges} \end{cases}$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad p=2>1$$

By p-series $(p=2>1)$, the given series converges

Ex

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

$$\int_1^{\infty} \frac{dx}{x^2+1} = \arctan x\Big|_1^{\infty}$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \boxed{\frac{\pi}{4}}$$

By the Integral Test, the given series converges

#1/ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{0.2}}$ $\quad p = 0.2 \le 1$

By p-series ($p = .2 \le 1$), the given series diverges

#3/ $\displaystyle\sum_{n=1}^{\infty} e^{-2n}$

$\dfrac{1}{n^p}$ $\quad \cancel{\dfrac{1}{p^n}}$

$$\int_1^{\infty} e^{-2x}\,dx = -\frac{1}{2}e^{-2x}\Big|_1^{\infty}$$

$$= -\frac{1}{2}\left(0 - \frac{1}{e^2}\right)$$

$$= \frac{1}{2e^2}$$

By the integral Test, the given series converges

#11 $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)\ln(\ln k)}$ $\qquad (\ln(\ln x))' = \frac{\frac{1}{x}}{\ln x}$

$$\int_2^{\infty} \frac{dx}{x\ln x \, \ln(\ln x)} = \int_2^{\infty} \frac{d(\ln(\ln x))}{\ln(\ln x)}$$

$$= \ln(\ln(\ln x))\Big|_2^{\infty}$$

$$= \infty - \ln(\ln(\ln 2))$$

$$= +\infty$$

By the Integral test, the given series diverges.

#23 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^9}$ $\qquad p = 9 > 1$

By the p-series (>1), the given series converges

25 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{16k^2}} = \sum_{n=1}^{\infty} \frac{1}{2k^{1/2}}$

$$p = \frac{1}{2} \leq 1$$

By the p-series, the given series diverges.

#32 $\sum_{n=1}^{\infty} n e^{-n^2}$

$\frac{1}{n^p}$ p-

Geom $\left(\frac{1}{a}\right)^n$

$$\int_1^{\infty} x e^{-x^2} dx = -\frac{1}{2}\int_1^{\infty} e^{-x^2} d(-x^2)$$

$$= -\frac{1}{2} \left. \frac{1}{e^{x^2}} \right|_1^{\infty}$$

$$= -\frac{1}{2}\left(0 - \frac{1}{e}\right)$$

$$= \boxed{\frac{1}{2e}}$$

By the integral test, the given series converges.

Geometric series $\left(\frac{1}{a}\right)^n$

P-series $\frac{1}{n^p}$
