

Solution **Section 1.8 – Exponential Models**

Exercise

Find the derivative of $y = \ln\left(\frac{\sqrt{\sin \theta \cos \theta}}{1 + 2 \ln \theta}\right)$

Solution

$$\begin{aligned} y &= \ln(\sin \theta \cos \theta)^{1/2} - \ln(1 + 2 \ln \theta) \\ &= \frac{1}{2}(\ln(\sin \theta) + \ln(\cos \theta)) - \ln(1 + 2 \ln \theta) \\ y' &= \frac{1}{2} \left(\frac{(\sin \theta)'}{\sin \theta} + \frac{(\cos \theta)'}{\cos \theta} \right) - \frac{(1 + 2 \ln \theta)'}{1 + 2 \ln \theta} \\ &= \frac{1}{2} \left(\frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} \right) - \frac{\frac{2}{\theta}}{1 + 2 \ln \theta} \\ &= \frac{1}{2} (\cot \theta - \tan \theta) - \frac{2}{\theta(1 + 2 \ln \theta)} \end{aligned}$$

Exercise

Find the derivative of $f(x) = e^{(4\sqrt{x} + x^2)}$

Solution

$$\begin{aligned} \frac{d}{dx} e^{(4\sqrt{x} + x^2)} &= e^{(4\sqrt{x} + x^2)} \frac{d}{dx} (4\sqrt{x} + x^2) \\ &= \left(\frac{2}{\sqrt{x}} + 2x \right) e^{(4\sqrt{x} + x^2)} \end{aligned}$$

Exercise

Find the derivative of $f(t) = \ln(3te^{-t})$

Solution

$$\begin{aligned} \frac{d}{dt} \ln(3te^{-t}) &= \frac{(3te^{-t})'}{3te^{-t}} \\ &= 3 \frac{e^{-t} - te^{-t}}{3te^{-t}} \end{aligned}$$

$$\begin{aligned} \ln(3te^{-t}) &= \ln 3 + \ln t + \ln e^{-t} \\ &= \ln 3 + \ln t - t \end{aligned}$$

$$\left(\ln(3te^{-t}) \right)' = \frac{1}{t} - 1$$

$$\frac{e^{-t}(1-t)}{te^{-t}} = \frac{1-t}{t}$$

Exercise

Find the Derivative of $f(x) = \frac{e^{\sqrt{x}}}{\ln(\sqrt{x}+1)}$

Solution

$$f = e^{\sqrt{x}} \quad U = \sqrt{x} = x^{1/2} \Rightarrow U' = \frac{1}{2}x^{-1/2} \quad f' = \frac{1}{2}x^{-1/2}e^{\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

$$g = \ln(\sqrt{x}+1) \quad U = x^{1/2} + 1 \Rightarrow U' = \frac{1}{2}x^{-1/2} \quad g' = \frac{\frac{1}{2}x^{-1/2}}{\sqrt{x}+1} = \frac{1}{2x^{1/2}(\sqrt{x}+1)}$$

$$\begin{aligned} f'(x) &= \frac{\frac{e^{\sqrt{x}}}{2\sqrt{x}} \ln(\sqrt{x}+1) - \frac{1}{2\sqrt{x}(\sqrt{x}+1)} e^{\sqrt{x}}}{\left(\ln(\sqrt{x}+1)\right)^2} \\ &= \frac{\frac{(\sqrt{x}+1)e^{\sqrt{x}} \ln(\sqrt{x}+1) - e^{\sqrt{x}}}{2\sqrt{x}(\sqrt{x}+1)}}{\left(\ln(\sqrt{x}+1)\right)^2} \\ &= \frac{e^{\sqrt{x}} \left[(\sqrt{x}+1) \ln(\sqrt{x}+1) - 1 \right]}{2\sqrt{x}(\sqrt{x}+1) \left(\ln(\sqrt{x}+1)\right)^2} \end{aligned}$$

Exercise

Find the Derivative of $y = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}$

Solution

$$y = \left(\frac{(x+1)^{10}}{(2x+1)^5} \right)^{1/2}$$

$$\ln y = \ln \left(\frac{(x+1)^{10}}{(2x+1)^5} \right)^{1/2}$$

$$\begin{aligned}
 \ln y &= \frac{1}{2} \ln \left(\frac{(x+1)^{10}}{(2x+1)^5} \right) \\
 &= \frac{1}{2} \left(\ln(x+1)^{10} - \ln(2x+1)^5 \right) \\
 &= \frac{1}{2} (10 \ln(x+1) - 5 \ln(2x+1)) \\
 &= 5 \ln(x+1) - \frac{5}{2} \ln(2x+1)
 \end{aligned}$$

$$\frac{y'}{y} = 5 \frac{1}{x+1} - \frac{5}{2} \frac{2}{2x+1}$$

$$\frac{y'}{y} = \frac{5}{x+1} - \frac{5}{2x+1}$$

$$y' = y \left(\frac{5}{x+1} - \frac{5}{2x+1} \right)$$

$$y' = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5} \left(\frac{5}{x+1} - \frac{5}{2x+1} \right)}$$

Exercise

Find the derivative of $f(x) = (2x)^{4x}$

Solution

$$\ln f(x) = 4x \ln(2x)$$

$$\frac{f'}{f} = 4 \left(\ln 2x + x \frac{2}{2x} \right)$$

$$f'(x) = 4(\ln 2x + 1)(2x)^{4x}$$

Exercise

Find the derivative of $f(x) = 2^{x^2}$

Solution

$$f'(x) = 2x \cdot 2^{x^2} \ln 2$$

Exercise

Find the derivative of $h(y) = y^{\sin y}$

Solution

$$\ln h = \ln y^{\sin y} = \sin y \ln y$$

$$\frac{h'}{h} = \cos y \ln y + \frac{\sin y}{y}$$

$$\underline{h'(y) = y^{\sin y} \left(\cos y \ln y + \frac{\sin y}{y} \right)}$$

Exercise

Find the derivative of $f(x) = x^\pi$

Solution

$$\ln f = \pi \ln x$$

$$\frac{f'}{f} = \frac{\pi}{x}$$

$$\underline{f'(x) = \pi x^{\pi-1}}$$

Exercise

Find the derivative of $h(t) = (\sin t)^{\sqrt{t}}$

Solution

$$\ln h = \ln(\sin t)^{\sqrt{t}} = \sqrt{t} \ln(\sin t)$$

$$\frac{h'}{h} = \frac{1}{2\sqrt{t}} \ln \sin t + \sqrt{t} \frac{\cos t}{\sin t}$$

$$\underline{h'(t) = \frac{1}{2\sqrt{t}} (\ln \sin t + 2t \cot t) (\sin t)^{\sqrt{t}}}$$

Exercise

Find the derivative of $p(x) = x^{-\ln x}$

Solution

$$\ln p(x) = \ln x^{-\ln x}$$

$$= -(\ln x)^2$$

$$\frac{p'}{p} = -\frac{2 \ln x}{x}$$

$$p'(x) = -\frac{2 \ln x}{x} x^{-\ln x}$$

$$\underline{= -\frac{2 \ln x}{x^{1+\ln x}}}$$

Exercise

Find the derivative of $f(x) = x^{2x}$

Solution

$$\begin{aligned}\ln f &= \ln x^{2x} \\ &= 2x \ln x\end{aligned}$$

$$\frac{f'}{f} = 2 \ln x + 2 \frac{x}{x}$$

$$\underline{f'(x) = 2(1 + \ln x)x^{2x}}$$

Exercise

Find the derivative of $f(x) = x^{\tan x}$

Solution

$$\begin{aligned}\ln f(x) &= \ln x^{\tan x} \\ &= \tan x \ln x\end{aligned}$$

$$\frac{f'}{f} = \sec^2 x \ln x + \frac{\tan x}{x}$$

$$\underline{f'(x) = \left(\sec^2 x \ln x + \frac{\tan x}{x} \right) x^{\tan x}}$$

Exercise

Find the derivative of $f(x) = x^e + e^x$

Solution

$$\underline{f'(x) = ex^{e-1} + e^x}$$

Exercise

Find the derivative of $f(x) = x^{x^{10}}$

Solution

$$\ln f = x^{10} \ln x$$

$$\frac{f'}{f} = 10x^9 \ln x + \frac{x^{10}}{x}$$

$$f'(x) = x^{x^{10}} (10x^9 \ln x + x^9)$$

$$= x^{9+x^{10}} (10 \ln x + 1)$$

Exercise

Find the derivative of $f(x) = \left(1 + \frac{4}{x}\right)^x$

Solution

$$\ln f = x \ln \left(1 + \frac{4}{x}\right)$$

$$\frac{f'}{f} = \ln \left(1 + \frac{4}{x}\right) + x \frac{-\frac{4}{x^2}}{1 + \frac{4}{x}}$$

$$f'(x) = \left(1 + \frac{4}{x}\right)^x \left(\ln \left(1 + \frac{4}{x}\right) - \frac{4}{x+4}\right)$$

Exercise

Find the derivative of $f(x) = \cos(x^{2 \sin x})$

Solution

$$f' = -\left(x^{2 \sin x}\right)' \sin(x^{2 \sin x})$$

$$\text{Let : } y = x^{2 \sin x}$$

$$\ln y = (2 \sin x) \ln x$$

$$\frac{y'}{y} = 2 \cos x \ln x + \frac{2 \sin x}{x}$$

$$f' = -x^{2 \sin x} \left(2 \cos x \ln x + \frac{2 \sin x}{x}\right) \sin(x^{2 \sin x})$$

Exercise

Find the derivative of $f(x) = \ln(\ln x)$

Solution

$$f'(x) = \frac{\frac{1}{x}}{\ln x} = \frac{1}{x \ln x}$$

Exercise

Find the derivative of $f(x) = \ln(\cos^2 x)$

Solution

$$\begin{aligned} f'(x) &= \frac{-2 \cos x \sin x}{\cos^2 x} \\ &= -2 \tan x \end{aligned}$$

Exercise

Find the derivative of $f(x) = \frac{\ln x}{\ln x + 1}$

Solution

$$\begin{aligned} f'(x) &= \frac{\frac{\ln x + 1}{x} - \frac{\ln x}{x}}{(\ln x + 1)^2} \\ &= \frac{\ln x + 1 - \ln x}{x(\ln x + 1)^2} \\ &= \frac{1}{x(\ln x + 1)^2} \end{aligned}$$

Exercise

Find the derivative of $f(x) = \frac{\ln x}{x}$

Solution

$$f'(x) = \frac{1 - \ln x}{x^2}$$

Exercise

Find the derivative of $f(x) = \frac{\tan^{10} x}{(5x + 3)^6}$

Solution

$$f'(x) = \frac{\tan^9 x}{(5x + 3)^7} (10(5x + 3) \sec^2 x - 30 \tan x)$$

$$(U^m V^n)' = U^{m-1} V^{n-1} (mUV' + nUV')$$

Exercise

Find the derivative of $f(x) = \frac{(x+1)^{3/2}(x-4)^{5/2}}{(5x+3)^{2/3}}$

Solution

$$(U^m V^n W^p)' = U^{m-1} V^{n-1} W^{p-1} (mU'VW + nUV'W + pUVW')$$

$$\begin{aligned} f'(x) &= \frac{(x+1)^{1/2}(x-4)^{3/2}}{(5x+3)^{5/3}} \left(\frac{3}{2}(x-4)(5x+3) + \frac{5}{2}(x+1)(5x+3) - \frac{2}{3}(5)(x+1)(x-4) \right) \\ &= \frac{1}{6} \frac{(x+1)^{1/2}(x-4)^{3/2}}{(5x+3)^{5/3}} \left(9(5x^2 - 17x - 12) + 15(5x^2 + 8x + 3) - 20(x^2 - 3x - 4) \right) \\ &= \frac{1}{6} \frac{(x+1)^{1/2}(x-4)^{3/2}}{(5x+3)^{5/3}} \left(45x^2 - 153x - 108 + 75x^2 + 120x + 48 - 20x^2 + 60x + 80 \right) \\ &= \frac{1}{6} \frac{(x+1)^{1/2}(x-4)^{3/2}}{(5x+3)^{5/3}} (100x^2 + 27x + 20) \end{aligned}$$

Exercise

Find the derivative of $f(x) = \frac{x^8 \cos^3 x}{\sqrt{x-1}}$

Solution

$$(U^m V^n W^p)' = U^{m-1} V^{n-1} W^{p-1} (mU'VW + nUV'W + pUVW')$$

$$\begin{aligned} f'(x) &= \frac{x^7 \cos^2 x}{(x-1)^{3/2}} \left(7(x-1) \cos x - x(x-1) \sin x - \frac{1}{2} x \cos x \right) \\ &= \frac{x^7 \cos^2 x}{(x-1)^{3/2}} \left(7x \cos x - 7 \cos x - x(x-1) \sin x - \frac{1}{2} x \cos x \right) \\ &= \frac{x^7 \cos^2 x}{(x-1)^{3/2}} \left(\frac{13}{2} x \cos x - 7 \cos x - x(x-1) \sin x \right) \end{aligned}$$

Exercise

Find the derivative of $f(x) = (\sin x)^{\tan x}$

Solution

$$\begin{aligned}
 \ln f(x) &= \ln \left((\sin x)^{\tan x} \right) \\
 &= \tan x \ln(\sin x) \\
 \frac{f'}{f} &= \sec^2 x \ln(\sin x) + \frac{\tan x}{x} \\
 f'(x) &= (\sin x)^{\tan x} \left(\sec^2 x \ln(\sin x) + \frac{\tan x}{x} \right)
 \end{aligned}$$

Exercise

Find the derivative of $f(x) = \left(1 + \frac{1}{x}\right)^{2x}$

Solution

$$\begin{aligned}
 \ln f(x) &= \ln \left(1 + \frac{1}{x} \right)^{2x} \\
 &= 2x \ln \left(1 + \frac{1}{x} \right) \\
 \frac{f'}{f} &= 2 \left(\ln \left(1 + \frac{1}{x} \right) + x \frac{1}{1 + \frac{1}{x}} \left(-\frac{1}{x^2} \right) \right) \\
 &= 2 \left(\ln \left(1 + \frac{1}{x} \right) - \frac{x}{x+1} \left(\frac{1}{x} \right) \right) \\
 f'(x) &= 2 \left(1 + \frac{1}{x} \right)^{2x} \left(\ln \left(1 + \frac{1}{x} \right) - \frac{1}{x+1} \right)
 \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{2y dy}{y^2 - 25}$

Solution

$$\begin{aligned}
 \int \frac{2y dy}{y^2 - 25} &= \int \frac{d(y^2 - 25)}{y^2 - 25} \\
 &= \ln |y^2 - 25| + C
 \end{aligned}$$

$$d(y^2 - 25) = 2y dy$$

Exercise

Evaluate the integral $\int \frac{\sec y \tan y}{2 + \sec y} dy$

Solution

$$\int \frac{\sec y \tan y}{2 + \sec y} dy = \int \frac{d(2 + \sec y)}{2 + \sec y}$$

$$= \ln|2 + \sec y| + C$$

$$d(2 + \sec y) = \sec y \tan y dy$$

Exercise

Find the integral $\int \frac{5}{e^{-5x} + 7} dx$

Solution

$$\int \frac{5}{e^{-5x} + 7} \frac{e^{5x}}{e^{5x}} dx = \int \frac{5e^{5x}}{1 + 7e^{5x}} dx$$

$$= \frac{1}{7} \int \frac{1}{1 + 7e^{5x}} d(1 + 7e^{5x})$$

$$= \frac{1}{7} \ln|1 + 7e^{5x}| + C$$

$$d(1 + 7e^{5x}) = 35e^{5x} dx$$

Exercise

Find the integral $\int \frac{e^{2x}}{4 + e^{2x}} dx$

Solution

$$\int \frac{e^{2x}}{4 + e^{2x}} dx = \frac{1}{2} \int \frac{1}{4 + e^{2x}} d(4 + e^{2x})$$

$$= \frac{1}{2} \ln(4 + e^{2x}) + C$$

$$d(4 + e^{2x}) = 2e^{2x} dx$$

Exercise

Find the integral $\int \frac{dx}{x \ln x \ln(\ln x)}$

Solution

$$\int \frac{dx}{x \ln x \ln(\ln x)} = \int \frac{1}{\ln(\ln x)} d(\ln(\ln x))$$

$$= \ln \ln(\ln x) + C$$

$$d(\ln(\ln x)) = \frac{1/x}{\ln x} = \frac{1}{x \ln x}$$

Exercise

Find the integral $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

Solution

$$\begin{aligned}\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx &= 2 \int e^{\sqrt{x}} d(\sqrt{x}) \\ &= \underline{2e^{\sqrt{x}} + C}\end{aligned}$$

$$d(\sqrt{x}) = \frac{1}{2\sqrt{x}} dx$$

Exercise

Find the integral $\int \frac{e^{\sin x}}{\sec x} dx$

Solution

$$\begin{aligned}\int \frac{e^{\sin x}}{\sec x} dx &= \int e^{\sin x} d(\sin x) \\ &= \underline{e^{\sin x} + C}\end{aligned}$$

$$d(\sin x) = \cos x dx = \frac{dx}{\sec x}$$

Exercise

Find the integral $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$

Solution

$$\begin{aligned}\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx &= \int \frac{1}{e^x - e^{-x}} d(e^x - e^{-x}) \\ &= \underline{\ln(e^x - e^{-x}) + C}\end{aligned}$$

$$d(e^x - e^{-x}) = (e^x + e^{-x}) dx$$

Exercise

Find the integral $\int \frac{4^{\cot x}}{\sin^2 x} dx$

Solution

$$\begin{aligned}\int \frac{4^{\cot x}}{\sin^2 x} dx &= - \int 4^{\cot x} d(\cot x) \\ &= \underline{\frac{4^{\cot x}}{\ln 4} + C}\end{aligned}$$

$$d(\cot x) = -\csc^2 x dx = -\frac{dx}{\sin^2 x}$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

Exercise

Find the integral $\int \frac{4x^2 + 2x + 4}{x+1} dx$

Solution

$$\begin{aligned}\int \frac{4x^2 + 2x + 4}{x+1} dx &= \int \left(4x + 2 + \frac{6}{x+1} \right) dx \\ &= \int (4x - 2) dx + \int \frac{6}{x+1} dx \\ &= \int (4x - 2) dx + 6 \int \frac{d(x+1)}{x+1} \\ &= \underline{2x^2 - 2x + 6 \ln|x+1| + C}\end{aligned}$$

$$\int \frac{d(U)}{U} = \ln|U|$$

Exercise

Find the integral $\int \frac{e^x}{4e^x + 6} dx$

Solution

$$\begin{aligned}\int \frac{e^x}{4e^x + 6} dx &= \frac{1}{4} \int \frac{1}{4e^x + 6} d(4e^x + 6) \\ &= \underline{\frac{1}{4} \ln(4e^x + 6) + C}\end{aligned}$$

Exercise

Find the integral $\int \frac{x+4}{x^2 + 8x + 25} dx$

Solution

$$\begin{aligned}\int \frac{x+4}{x^2 + 8x + 25} dx &= \frac{1}{2} \int \frac{1}{x^2 + 8x + 25} d(x^2 + 8x + 25) \\ &= \underline{\frac{1}{2} \ln|x^2 + 8x + 25| + C}\end{aligned}$$

Exercise

Find the integral $\int \frac{e^{2x}}{\sqrt{e^{2x} + 4}} dx$

Solution

$$\begin{aligned}\int \frac{e^{2x}}{\sqrt{e^{2x} + 4}} dx &= \frac{1}{2} \int (e^{2x} + 4)^{-1/2} d(e^{2x} + 4) \\ &= \sqrt{e^{2x} + 4} + C\end{aligned}$$

Exercise

Find the integral $\int_{e^2}^{e^8} \frac{dx}{x \ln x}$

Solution

$$\begin{aligned}\int_{e^2}^{e^8} \frac{dx}{x \ln x} &= \int_{e^2}^{e^8} \frac{d(\ln x)}{\ln x} \\ &= \ln(\ln x) \Big|_{e^2}^{e^8} \\ &= \ln(\ln e^8) - \ln(\ln e^2) \\ &= \ln 8 - \ln 2 \\ &= \ln \frac{8}{2} \\ &= \ln 4\end{aligned}$$

Exercise

Find the integral $\int \frac{x^2}{2x^3 + 1} dx$

Solution

$$\begin{aligned}\int \frac{x^2}{2x^3 + 1} dx &= \frac{1}{6} \int \frac{1}{2x^3 + 1} d(2x^3 + 1) \\ &= \frac{1}{6} \ln|2x^3 + 1| + C\end{aligned}$$

Exercise

Find the integral $\int \frac{\sec^2 x}{\tan x} dx$

Solution

$$\begin{aligned}\int \frac{\sec^2 x}{\tan x} dx &= \int \frac{1}{\tan x} d(\tan x) \\ &= \ln|\tan x| + C\end{aligned}$$

Exercise

Find the integral $\int_1^4 \frac{10^{\sqrt{x}}}{\sqrt{x}} dx$

Solution

$$\begin{aligned}\int_1^4 \frac{10^{\sqrt{x}}}{\sqrt{x}} dx &= 2 \int_1^4 10^{\sqrt{x}} d(\sqrt{x}) \\ &= \frac{10^{\sqrt{x}}}{\ln 10} \Big|_1^4 \\ &= \frac{1}{\ln 10} (10^2 - 10) \\ &= \frac{90}{\ln 10}\end{aligned}$$

Exercise

Evaluate the integral $\int_{\ln 4}^{\ln 9} e^{x/2} dx$

Solution

$$\begin{aligned}\int_{\ln 4}^{\ln 9} e^{x/2} dx &= 2e^{x/2} \Big|_{\ln 2^2}^{\ln 3^2} \\ &= 2 \left(e^{(2 \ln 3)/2} - e^{(2 \ln 2)/2} \right) \\ &= 2 \left(e^{\ln 3} - e^{\ln 2} \right) \\ &= 2(3 - 2) \\ &= 2\end{aligned}$$

Exercise

Evaluate the integral $\int_0^3 \frac{2x-1}{x+1} dx$

Solution

$$\begin{aligned}\int_0^3 \frac{2x-1}{x+1} dx &= \int_0^3 \left(2 - \frac{3}{x+1} \right) dx \\ &= 2x - 3 \ln|x+1| \Big|_0^3 \\ &= \underline{6 - 3 \ln 4} \end{aligned}$$
$$\begin{array}{r} \overline{) 2x-1} \\ \underline{-2x-2} \\ -3 \end{array}$$

Exercise

Evaluate the integral $\int_e^{e^2} \frac{dx}{x \ln^3 x}$

Solution

$$\begin{aligned}\int_e^{e^2} \frac{dx}{x \ln^3 x} &= \int_e^{e^2} \ln^{-3} x \, d(\ln x) \\ &= -\frac{1}{2} \ln^{-2} x \Big|_e^{e^2} \\ &= -\frac{1}{2} (2-1) \\ &= \underline{-\frac{1}{2}} \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/2} \frac{\sin x}{1+\cos x} dx$

Solution

$$\begin{aligned}\int_0^{\pi/2} \frac{\sin x}{1+\cos x} dx &= -\int_0^{\pi/2} \frac{1}{1+\cos x} d(1+\cos x) \\ &= -\ln|1+\cos x| \Big|_0^{\pi/2} \\ &= -(\ln 1 - \ln 2) \\ &= \underline{\ln 2} \end{aligned}$$

Exercise

Evaluate the integral $\int_3^4 \frac{dx}{2x \ln x \ln^3(\ln x)}$

Solution

$$\begin{aligned}\int_3^4 \frac{dx}{2x \ln x \ln^3(\ln x)} &= \frac{1}{2} \int_3^4 (\ln(\ln x))^{-3} d(\ln(\ln x)) \\ &= -\frac{1}{4} \frac{1}{(\ln(\ln x))^2} \Big|_3^4 \\ &= -\frac{1}{4} \frac{1}{\ln(\ln 4)^2} + \frac{1}{4} \frac{1}{\ln(\ln 3)} \\ &= \frac{1}{4} \frac{1}{\ln(\ln 3)} - \frac{1}{4} \frac{1}{\ln(\ln 4)^2}\end{aligned}$$

$$d(\ln(\ln x)) = \frac{1/x}{\ln x} = \frac{1}{x \ln x}$$

Exercise

Evaluate the integral $\int_{e^2}^{e^3} \frac{dx}{x \ln x \ln^2(\ln x)}$

Solution

$$\begin{aligned}\int_{e^2}^{e^3} \frac{dx}{x \ln x \ln^2(\ln x)} &= \int_{e^2}^{e^3} (\ln(\ln x))^{-2} d(\ln(\ln x)) \\ &= -\frac{1}{\ln(\ln x)} \Big|_{e^2}^{e^3} \\ &= -\frac{1}{\ln(\ln e^3)} + \frac{1}{\ln(\ln e^2)} \\ &= -\frac{1}{\ln 3} + \frac{1}{\ln 2}\end{aligned}$$

$$d(\ln(\ln x)) = \frac{1/x}{\ln x} = \frac{1}{x \ln x}$$

Exercise

Evaluate the integral $\int_0^1 \frac{y \ln^4(y^2 + 1)}{y^2 + 1} dy$

Solution

$$\begin{aligned}
 \int_0^1 \frac{y \ln^4(y^2 + 1)}{y^2 + 1} dy &= \frac{1}{2} \int_0^1 \ln^4(y^2 + 1) d(\ln(y^2 + 1)) \\
 &= \frac{1}{10} \ln^5(y^2 + 1) \Big|_0^1 \\
 &= \frac{1}{10} (\ln 2)^5
 \end{aligned}$$

$$d(\ln(y^2 + 1)) = \frac{2y}{y^2 + 1} dy$$

Exercise

Evaluate the integral $\int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^{2x} - 2 + e^{-2x}} dx$

Solution

$$\begin{aligned}
 \int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^{2x} - 2 + e^{-2x}} dx &= \int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{(e^x - e^{-x})^2} dx \\
 &= \int_{\ln 2}^{\ln 3} \frac{1}{(e^x - e^{-x})^2} d(e^x - e^{-x}) \\
 &= -\frac{1}{e^x - e^{-x}} \Big|_{\ln 2}^{\ln 3} \\
 &= -\frac{1}{e^{\ln 3} - e^{-\ln 3}} + \frac{1}{e^{\ln 2} - e^{-\ln 2}} \\
 &= \frac{1}{2 - \frac{1}{2}} - \frac{1}{3 - \frac{1}{3}} \\
 &= \frac{2}{3} - \frac{3}{8} \\
 &= \frac{7}{24}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_{-2}^2 \frac{e^{z/2}}{e^{z/2} + 1} dz$

Solution

$$\int_{-2}^2 \frac{e^{z/2}}{e^{z/2} + 1} dz = 2 \int_{-2}^2 \frac{1}{e^{z/2} + 1} d(e^{z/2} + 1)$$

$$d(e^{z/2} + 1) = \frac{1}{2} e^{z/2} dz$$

$$\begin{aligned}
 &= 2 \ln \left(e^{z/2} + 1 \right) \Big|_{-2}^2 \\
 &= 2 \ln(e+1) - 2 \ln(e^{-1} + 1) \Big|
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/2} 4^{\sin x} \cos x \, dx$

Solution

$$\begin{aligned}
 \int_0^{\pi/2} 4^{\sin x} \cos x \, dx &= \int_0^{\pi/2} 4^{\sin x} d(\sin x) \\
 &= \frac{1}{\ln 4} \left(4^{\sin x} \right) \Big|_0^{\pi/2} \\
 &= \frac{1}{\ln 4} (4 - 1) \\
 &= \frac{3}{\ln 4} \Big|
 \end{aligned}$$

$$\int a^x dx = \frac{a^x}{\ln a}$$

Exercise

Evaluate the integral $\int_{1/3}^{1/2} \frac{10^{1/p}}{p^2} dp$

Solution

$$\begin{aligned}
 \int_{1/3}^{1/2} \frac{10^{1/p}}{p^2} dp &= - \int_{1/3}^{1/2} 10^{1/p} d\left(\frac{1}{p}\right) \\
 &= - \frac{1}{\ln 10} \left(10^{1/p} \right) \Big|_{1/3}^{1/2} \\
 &= - \frac{1}{\ln 10} (10^2 - 10^3) \\
 &= \frac{900}{\ln 10} \Big|
 \end{aligned}$$

Exercise

Evaluate the integral $\int_1^2 (1 + \ln x) x^x \, dx$

Solution

$$y = x^x \rightarrow \ln y = x \ln x$$

$$\frac{y'}{y} = 1 + \ln x \Rightarrow (x^x)' = x^x (1 + \ln x)$$

$$\begin{aligned} \int_1^2 (1 + \ln x) x^x dx &= \int_1^2 d(x^x) \\ &= x^x \Big|_1^2 \\ &= 2^2 - 1 \\ &= 3 \end{aligned}$$

Exercise

Find a curve through the origin in the xy -plane whose length from $x = 0$ to $x = 1$ is $L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x} dx$

Solution

$$L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x} dx$$

$$dy = \frac{e^{x/2}}{2} dx$$

$$\begin{aligned} y &= \int \frac{e^{x/2}}{2} dx \\ &= e^{x/2} + C \end{aligned}$$

$$0 = e^{0/2} + C$$

$$0 = 1 + C \rightarrow C = -1$$

$$y = e^{x/2} - 1$$

Exercise

Find the length of the curve $y = \ln(e^x - 1) - \ln(e^x + 1)$ from $x = \ln 2$ to $x = \ln 3$

Solution

$$y = \ln(e^x - 1) - \ln(e^x + 1)$$

$$\frac{dy}{dx} = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1}$$

$$\begin{aligned}
&= \frac{e^{2x} + e^x - e^{2x} + e^x}{e^{2x} - 1} \\
&= \frac{2e^x}{e^{2x} - 1} \\
L &= \int_{\ln 2}^{\ln 3} \sqrt{1 + \left(\frac{2e^x}{e^{2x} - 1} \right)^2} dx \\
&= \int_{\ln 2}^{\ln 3} \sqrt{1 + \frac{4e^{2x}}{e^{4x} - 2e^{2x} + 1}} dx \\
&= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} - 2e^{2x} + 1 + 4e^{2x}}{(e^{2x} - 1)^2}} dx \\
&= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} + 2e^{2x} + 1}{(e^{2x} - 1)^2}} dx \\
&= \int_{\ln 2}^{\ln 3} \sqrt{\frac{(e^{2x} + 1)^2}{(e^{2x} - 1)^2}} dx \\
&= \int_{\ln 2}^{\ln 3} \frac{e^{2x} + 1}{e^{2x} - 1} dx \\
&= \int_{\ln 2}^{\ln 3} \frac{\overset{\text{red}}{e^{-x}}}{\underset{\text{red}}{e^x} \frac{e^{2x} - 1}{e^x}} dx \\
&= \int_{\ln 2}^{\ln 3} \frac{\overset{\text{red}}{e^{-x}}}{\underset{\text{red}}{e^x} \frac{e^{2x}}{e^x} - \underset{\text{red}}{e^x} \frac{1}{e^x}} dx \\
&= \int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^x - e^{-x}} dx \\
&= \int_{\ln 2}^{\ln 3} \frac{1}{e^x - e^{-x}} d(e^x - e^{-x}) \\
&= \ln(e^x - e^{-x}) \Big|_{\ln 2}^{\ln 3}
\end{aligned}$$

$$\begin{aligned}
&= \ln\left(3 - \frac{1}{3}\right) - \ln\left(2 - \frac{1}{2}\right) \\
&= \ln\frac{8}{3} - \ln\frac{3}{2} \\
&= \ln\frac{8/3}{3/2} \\
&= \ln\left(\frac{16}{9}\right) \Big|
\end{aligned}$$

Exercise

Find the length of the curve $y = \ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{4}$

Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-\sin x}{\cos x} = -\tan x \\
L &= \int_0^{\pi/4} \sqrt{1 + \tan^2 x} \, dx \\
&= \int_0^{\pi/4} \sqrt{\sec^2 x} \, dx \\
&= \int_0^{\pi/4} \sec x \, dx \\
&= \ln|\sec x + \tan x| \Big|_0^{\pi/4} \\
&= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0| \\
&= \ln|\sqrt{2} + 1| - \ln|1 + 0| \\
&= \ln|\sqrt{2} + 1| - 0 \\
&= \ln(\sqrt{2} + 1) \Big|
\end{aligned}$$

Exercise

Find the area of the surface generated by revolving the curve $x = \frac{1}{2}(e^y + e^{-y})$, $0 \leq y \leq \ln 2$, about y-axis

Solution

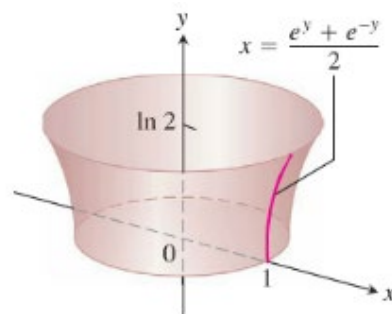
$$a = \frac{1}{2}, \quad m = 1, \quad b = \frac{1}{2}, \quad n = -1 \qquad f(x) = ae^{mx} + be^{nx}$$

$$1. \quad m = -n \quad \checkmark$$

$$2. \quad abmn = \frac{1}{2} \left(\frac{1}{2} \right) (1)(-1) = -\frac{1}{4} \quad \checkmark$$

$$\begin{aligned} S &= 2\pi \int_0^{\ln 2} \frac{1}{2} (e^y + e^{-y}) \frac{1}{2} (e^y + e^{-y}) dy \\ &= \frac{\pi}{2} \int_0^{\ln 2} (e^{2y} + e^{-2y} + 2) dy \\ &= \frac{\pi}{2} \left(\frac{1}{2} e^{2y} - \frac{1}{2} e^{-2y} + 2y \right) \Big|_0^{\ln 2} \\ &= \frac{\pi}{2} \left(\frac{1}{2} e^{2 \ln 2} - \frac{1}{2} e^{-2 \ln 2} + 2 \ln 2 - \frac{1}{2} e + \frac{1}{2} \right) \\ &= \frac{\pi}{2} \left(\frac{1}{2} \cdot 4 - \frac{1}{2} \cdot \frac{1}{4} + 2 \ln 2 \right) \\ &= \frac{\pi}{2} \left(\frac{15}{8} + 2 \ln 2 \right) \quad \text{unit}^2 \end{aligned}$$

$$S = 2\pi \int_a^b f(x) \overline{f'(x)} dx$$



Exercise

The population of a town with a 2010 population of 90,000 grows at a rate of 2.4% /yr. In what year will the population double its initial value (to 180,000)?

Solution

$$\begin{aligned} k &= \frac{\ln \frac{1.024(90,000)}{90,000}}{1} \\ &= \ln(1.024) \end{aligned}$$

$$\begin{aligned} T_2 &= \frac{\ln 2}{\ln 1.024} \\ &\approx 29.226 \text{ yrs} \end{aligned}$$

It reaches 180,000 around the year 2039.

Exercise

How long will it take an initial deposit of \$1500 to increase in value to \$2500 in a saving account with an APY of 3.1%? Assume the interest rate remains constant and no additional deposits or withdrawals are made.

Solution

$$y(t) = 1500 e^{kt}$$

$$k = \frac{\ln 1.031}{1} = \ln(1.031)$$

$$T = \frac{\ln\left(\frac{2500}{1500}\right)}{\ln 1.031}$$

$$\approx 16.7 \text{ yrs}$$

$$kT = \ln\left(\frac{y}{y_0}\right)$$

Exercise

The number of cells in a tumor doubles every 6 weeks starting with 8 cells. After how many weeks does the tumor have 1500 cells?

Solution

$$k = \frac{\ln 2}{T_2} = \frac{\ln 2}{6}$$

$$y(t) = 8 e^{(t \ln 2)/6}$$

$$t = 6 \frac{\ln\left(\frac{1500}{8}\right)}{\ln 2}$$

$$\approx 45.3 \text{ weeks}$$

$$kT = \ln\left(\frac{y}{y_0}\right)$$

Exercise

According to the 2010 census, the U.S. population was 309 million with an estimated growth rate of 0.8% /yr.

- Based on these figures, find the doubling time and project the population in 2050.
- Suppose the actual growth rate is just 0.2 percentage point lower than 0.8% /yr (0.6%). What are the resulting doubling time and projected 2050 population? Repeat these calculations assuming the growth rate is 0.2 percentage point higher than 0.8% /yr.
- Comment on the sensitivity of these projections to the growth rate.

Solution

$$a) \quad T_2 = \frac{\ln 2}{\ln 1.008}$$

$$\approx 87 \text{ yrs}$$

The population in 2050:

$$P(50) = 309 e^{40 \ln 1.008}$$

$$\approx 425 \text{ million}$$

- If the growth rate is 0.6%:

$$T_2 = \frac{\ln 2}{\ln 1.006}$$

$$\approx 116 \text{ yrs}$$

The population in 2050:

$$P(50) = 309e^{40 \ln 1.006}$$

$$\approx 392.5 \text{ million}$$

If the growth rate is 1%:

$$T_2 = \frac{\ln 2}{\ln 1.01}$$

$$\approx 69.7 \text{ yrs}$$

The population in 2050:

$$P(50) = 309e^{40 \ln 1.01}$$

$$\approx 460.1 \text{ million}$$

c) A growth rate of just 0.2% produces large differences in population growth.

Exercise

The homicide rate decreases at a rate of 3%/yr in a city that had 800 homicides /yr in 2010. At this rate, when will the homicide rate reach 600 homicides/yr?

Solution

The homicide rate is modeled by: $H(t) = 800e^{-kt}$

$$k = \ln(1 - .03) \approx -0.03$$

$$H(t) = 800e^{-0.03t}$$

$$t = \frac{\ln(6/8)}{-0.03}$$

$$\approx 9.6 \text{ yrs}$$

$$kT = \ln\left(\frac{y}{y_0}\right)$$

So it should achieve this rate in 2019.

Exercise

A drug is eliminated from the body at a rate of 15% /hr. after how many hours does the amount of drug reach 10% of the initial dose?

Solution

$$k = \ln(1 - .15)$$

$$\approx -\ln(.85)$$

$$t = \frac{\ln(.1)}{\ln(.85)}$$

$$\approx 14.17 \text{ hrs}$$

$$kT = \ln\left(\frac{y}{y_0}\right)$$

Exercise

The mass of radioactive material in a sample has decreased by 30% since the decay began. Assuming a half-life of 1500 years, how long ago did the decay begin?

Solution

$$k = \frac{\ln \frac{1}{2}}{1500}$$

$$= -\frac{\ln 2}{1,500}$$

$$kT = \ln(70\%)$$

$$-\frac{\ln 2}{1,500} T = \ln(.7)$$

$$T = -1,500 \frac{\ln(.7)}{\ln 2}$$

$$\approx 772 \text{ years}$$

Exercise

Growing from an initial population of 150,000 at a constant annual growth rate of 4%/yr., how long will it take a city to reach a population of 1 million?

Solution

$$t = \frac{\ln\left(\frac{150,000}{10^6}\right)}{\ln(1+.04)}$$

$$= \frac{\ln(0.15)}{\ln(1.04)}$$

$$\approx 48.37 \text{ years}$$

$$kT = \ln\left(\frac{y}{y_0}\right)$$

Exercise

A savings account advertises an annual percentage yield (APY) of 5.4%, which means that the balance in the account increases at an annual growth rate of 5.4%/yr.

- a) Find the balance in the account for $t \geq 0$ with an initial deposit of \$1500, assuming the APY remains fixed and no additional deposits or withdrawals are made.

- b) What is the doubling time of the balance?
 c) After how many years does the balance reach \$5,000?

Solution

$$\begin{aligned} a) \text{ Balance} &= 1,500(1 + .054)^t \\ &= 1,500(1.054)^t \end{aligned}$$

$$\begin{aligned} b) \quad t &= \frac{\ln 2}{\ln 1.054} \\ &\approx 13.18 \text{ years} \end{aligned}$$

$$\begin{aligned} c) \quad t &= \frac{\ln \frac{5,000}{1,500}}{\ln 1.054} \\ &\approx 22.89 \text{ years} \end{aligned}$$

Exercise

A large die-casting machine used to make automobile engine blocks is purchased for \$2.5 *million*. For tax purposes, the value of the machine can be depreciated by 6.8% of its current value each year.

- a) What is the value of the machine after 10 *years*?
 b) After how many years is the value of the machine 10% of its original value?

Solution

$$a) \quad V(t) = 2.5e^{-kt}$$

$$k = \frac{\ln(1 - .068)}{1}$$

$$\approx \ln(.932)$$

$$V(t) = 2.5e^{-t \ln .932}$$

$$V(10) = 2.5e^{-10 \cdot \ln .932}$$

$$\approx \$1.2 \text{ million}$$

$$kT = \ln\left(\frac{y}{y_0}\right)$$

$$\begin{aligned} b) \quad t &= \frac{\ln(.1)}{\ln(.932)} \\ &\approx 32.7 \text{ yrs} \end{aligned}$$

Exercise

Roughly 12,000 Americans are diagnosed with thyroid cancer every year, which accounts for 1% of all cancer cases. It occurs in women three times as frequency as in men. Fortunately, thyroid cancer can be treated successfully in many cases with radioactive iodine, or I-131. This unstable form of iodine has a half-life of 8 days and is given in small doses measured in millicuries.

- a) Suppose a patient is given an initial dose of 100 millicuries. Find the function that gives the amount of I-131 in the body after $t \geq 0$ days.
- b) How long does it take the amount of I-131 to reach 10% of the initial dose?
- c) Finding the initial dose to give a particular patient is a critical calculation. How does the time reach 10% of the initial dose change if the initial dose is increased by 5%?

Solution

$$a) \quad k = \frac{\ln 2}{8}$$

$$kT = \ln(y / y_0)$$

After t days would be: $y = 100e^{-(t \ln 2)/8}$ millicuries.

$$b) \quad t = \frac{-8 \ln\left(\frac{10}{100}\right)}{\ln(2)}$$
$$\approx 26.58 \text{ days}$$

$$c) \quad t = \frac{-8 \ln\left(\frac{10}{105}\right)}{\ln(2)}$$
$$\approx 27.14 \text{ days}$$

Exercise

City **A** has a current population of 500,000 people and grows at a rate of 3% /yr. City **B** has a current population of 300,000 and grows at a rate of 5%/yr.

- a) When will the cities have the same population?
- b) Suppose City **C** has a current population of $y_0 < 500,000$ and a growth rate of $p > 3\% / \text{yr}$. What is the relationship between y_0 and p such that the Cities **A** and **C** have the same population in 10 years?

Solution

$$a) \quad 500,000e^{\ln(1.03)t} = 300,000e^{\ln(1.05)t}$$

$$5e^{\ln(1.03)t} = 3e^{\ln(1.05)t}$$

$$\frac{5}{3} = e^{(\ln(1.05) - \ln(1.03))t}$$

$$\ln \frac{5}{3} = \left(\ln \frac{1.05}{1.03} \right) t$$

$$t = \frac{\ln(5/3)}{\ln(1.05/1.03)}$$

$$\approx 26.56 \text{ yrs} \quad |$$

$$b) \quad 500,000e^{\ln(1.03)(10)} = y_0 e^{\ln(1+p)(10)}$$

$$y_0 = 500,000e^{10(\ln(1.03)-\ln(1+p))}$$

$$= 500,000e^{\ln\left(\frac{1.03}{1+p}\right)^{10}}$$

$$= 500,000\left(\frac{1.03}{1+p}\right)^{10} \quad |$$

Exercise

Suppose the acceleration of an object moving along a line is given by $a(t) = -kv(t)$, where k is a positive constant and v is the object's velocity. Assume that the initial velocity and position are given by $v(0) = 10$ and $s(0) = 0$, respectively.

- Use $a(t) = v'(t)$ to find the velocity of the object as a function of time.
- Use $v(t) = s'(t)$ to find the position of the object as a function of time.
- Use the fact that $\frac{dv}{dt} = \frac{dv}{ds} \frac{ds}{dt}$ (by the *Chain Rule*) to find the velocity as a function of position.

Solution

$$a) \quad \text{If } a(t) = \frac{dv}{dt} = -kv \rightarrow \frac{dv}{v} = -kdt$$

$$\int \frac{dv}{v} = -k \int dt$$

$$\ln v = -kt + C \quad \text{Since } v(0) = 10$$

$$\ln 10 = C \quad |$$

$$\ln v = -kt + \ln 10$$

$$v = e^{-kt + \ln 10} = e^{-kt} e^{\ln 10}$$

$$v(t) = 10e^{-kt} \quad |$$

$$b) \quad v(t) = \frac{ds}{dt} = 10e^{-kt}$$

$$\int ds = 10 \int e^{-kt} dt$$

$$s(t) = -\frac{10}{k} e^{-kt} + C \quad \text{Since } s(0) = 0$$

$$0 = -\frac{10}{k} + C \rightarrow \underline{C = \frac{10}{k}}$$

$$\underline{s(t) = -\frac{10}{k}e^{-kt} + \frac{10}{k}}$$

$$c) \frac{dv}{dt} = \frac{dv}{ds} \frac{ds}{dt}$$

$$-10ke^{-kt} = \frac{dv}{ds}(10e^{-kt})$$

$$-k = \frac{dv}{ds}$$

$$\int dv = -k \int ds$$

$$v = -ks + C$$

$$\text{Since } v(0) = 10$$

$$\underline{v = 10 - ks}$$

Exercise

On the first day of the year ($t = 0$), a city uses electricity at a rate of 2000 MW. That rate is projected to increase at a rate of 1.3% per year.

- Based on these figures, find an exponential growth function for the power (rate of electricity use) for the city.
- Find the total energy (in MW-yr) used by the city over four full years beginning at $t = 0$
- Find a function that gives the total energy used (in MW-yr) between $t = 0$ and any future time $t > 0$

Solution

$$a) P(t) = 2000e^{kt}$$

At a rate of 1.3% per year:

$$k = \ln(1.013)$$

$$\underline{P(t) = 2000e^{t \ln 1.013}}$$

$$b) \int_0^4 P(t) dt = 2000 \int_0^4 e^{t \ln 1.013} dt$$

$$= \frac{2000}{\ln 1.013} e^{t \ln 1.013} \bigg|_0^4$$

$$\underline{\approx 8210.3}$$

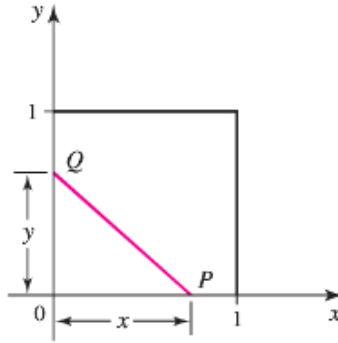
$$c) \int_0^t P(s) ds = 2000 \int_0^t e^{s \ln 1.013} ds$$

$$= \frac{2000}{\ln 1.013} e^{s \ln 1.013} \Big|_0^t$$

$$= -154,844 \left(1 + e^{t \ln(1.013)} \right) \Big|$$

Exercise

Two points P and Q are chosen randomly, one on each of two adjacent sides of a unit square.



What is the probability that the area of the triangle formed by the sides of the square and the line segment PQ is less than one-fourth the area of the square? Begin by showing that x and y must satisfy $xy < \frac{1}{2}$ in order for

the area condition to be met. Then argue that the required probability is $\frac{1}{2} + \int_{1/2}^1 \frac{dx}{2x}$ and evaluate the integral.

Solution

The area of the triangle is $\frac{1}{2}xy$

If $xy < \frac{1}{2}$, then if we let $0 < x < \frac{1}{2}$ we have $0 < y < 1$

Because there is a probability of $\frac{1}{2}$ of choosing $0 < x < \frac{1}{2}$, the probability we seek is at least $\frac{1}{2}$.

In addition, for $\frac{1}{2} < x < 1$, if $y < \frac{1}{2x}$,

$$\int_{1/2}^1 \frac{dx}{2x} = \frac{1}{2} \ln x \Big|_{1/2}^1$$

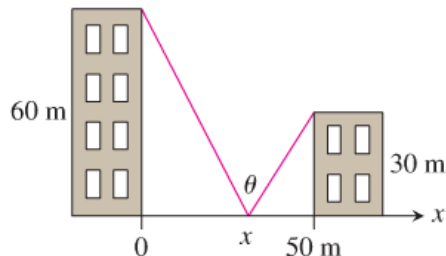
$$= \frac{\ln 2}{2} \Big|$$

$$\frac{1}{2} + \int_{1/2}^1 \frac{dx}{2x} = \frac{1}{2}(1 + \ln 2) \Big|$$

Exercise

You are under contract to build a solar station at ground level on the east-west line between the two buildings. How far from the taller building should you place the station to maximize the number of hours it will be in the sun on a day when passes directly overhead? Begin by observing that

$$\theta = \pi - \cot^{-1}\left(\frac{x}{60}\right) - \cot^{-1}\left(\frac{50-x}{30}\right)$$



Then find the value of x that maximizes θ .

Solution

$$\theta = \pi - \cot^{-1}\left(\frac{x}{60}\right) - \cot^{-1}\left(\frac{50-x}{30}\right)$$

$$\begin{aligned}\theta' &= -\frac{\frac{1}{60}}{1 + \left(\frac{x}{60}\right)^2} + \frac{-\frac{1}{30}}{1 + \left(\frac{50-x}{30}\right)^2} \\ &= \frac{60}{3,600 + x^2} - \frac{30}{900 + (50-x)^2} \\ &= 30 \frac{2(900 + 2500 - 100x + x^2) - 3,600 - x^2}{(3,600 + x^2)(900 + (50-x)^2)} = 0\end{aligned}$$

$$6,800 - 200x + 2x^2 - 3,600 - x^2 = 0$$

$$x^2 - 200x + 3,200 = 0$$

$$\begin{aligned}x &= \frac{200 \pm \sqrt{40,000 - 12,800}}{2} \\ &= \frac{200 \pm 10\sqrt{272}}{2} \\ &= 100 \pm 20\sqrt{17}\end{aligned}$$

$$\begin{cases} x = 100 + 20\sqrt{17} \approx 80.46 > 50 \\ x = 100 - 20\sqrt{17} \approx 17.54 \end{cases}$$

To maximize angle θ the distance $x = 100 - 20\sqrt{17} \approx 17.54 \text{ m}$

Exercise

A round underwater transmission cable consists of a core of copper wires surrounded by nonconducting insulation. If x denotes the ratio of the radius of the core to the thickness of the insulation, it is known that the speed of the transmission signal is given by the equation $v = x^2 \ln\left(\frac{1}{x}\right)$. If the radius of the core is 1 cm, what insulation thickness h will allow the greatest transmission speed?

Solution

$$v = x^2 \ln\left(\frac{1}{x}\right)$$

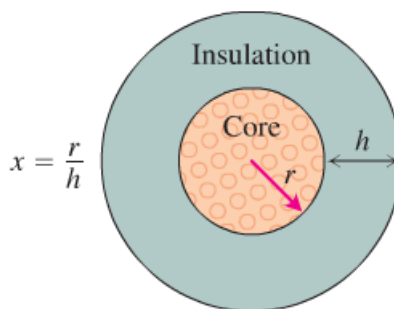
$$= -x^2 \ln x$$

$$v' = -(2x \ln x + x)$$

$$= -x(2 \ln x + 1) = 0$$

$$\begin{cases} x = 0 \\ \ln x = -\frac{1}{2} \rightarrow x = e^{-1/2} \end{cases}$$

0	$e^{-1/2}$
$v' < 0$	$v' > 0$



The greatest transmission speed at $x = e^{-1/2}$.

$$x = \frac{r}{h}$$

$$h = \frac{1}{e^{-1/2}}$$

$$= \sqrt{e} \text{ cm}$$

Exercise

A commonly used distribution in probability and statistics is the log-normal distribution. (If the logarithm of a variable has a normal distribution, then the variable itself has a log-normal distribution.) the distribution function is

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{\ln^2 x}{2\sigma^2}}, \quad \text{for } x > 0$$

Where $\ln x$ has zero mean and standard deviation $\sigma > 0$.

a) Graph f for $\sigma = \frac{1}{2}$, 1, and 2. Based on your graphs, does $\lim_{x \rightarrow 0^+} f(x)$ appear to exist?

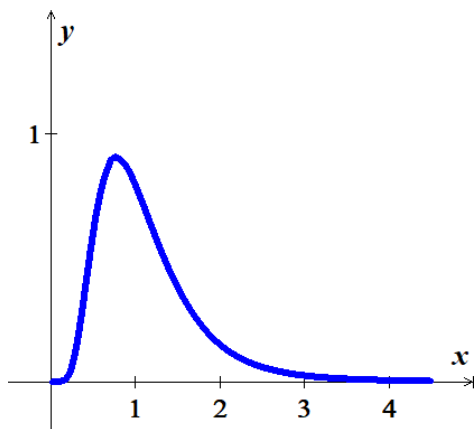
b) Evaluate $\lim_{x \rightarrow 0^+} f(x)$. (Hint: Let $x = e^y$)

c) Show that f has a single local maximum at $x^* = e^{-\sigma^2}$

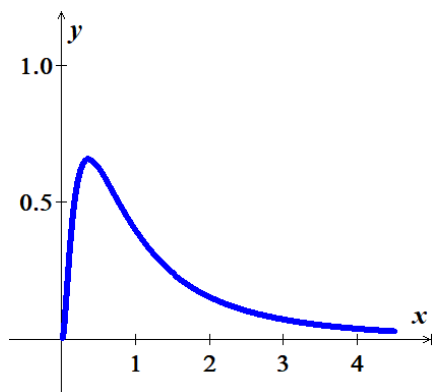
- d) Evaluate $f(x^*)$ and express the result as a function of σ .
- e) For what value of $\sigma > 0$ in part (d) does $f(x^*)$ have a minimum?

Solution

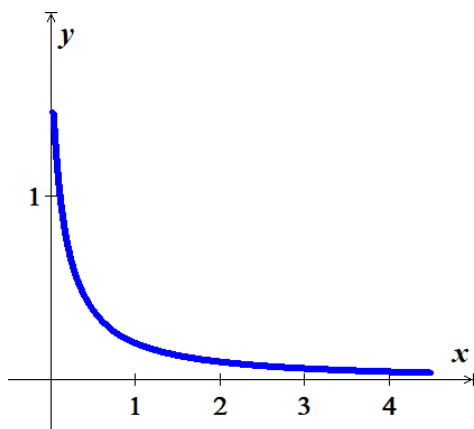
a) For $\sigma = \frac{1}{2} \rightarrow f(x) = \frac{2}{x\sqrt{2\pi}} e^{-2\ln^2 x}$



For $\sigma = 1 \rightarrow f(x) = \frac{1}{x\sqrt{2\pi}} e^{-\frac{\ln^2 x}{2}}$



For $\sigma = 2 \rightarrow f(x) = \frac{1}{2x\sqrt{2\pi}} e^{-\frac{\ln^2 x}{8}}$



$$\lim_{x \rightarrow 0^+} f(x) = \infty$$

b) Let $x = e^y \rightarrow y = \ln x$
 As $x \rightarrow 0 \Rightarrow y \rightarrow -\infty$

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{y \rightarrow -\infty} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sigma\sqrt{2\pi} e^y} \\ &= \frac{1}{\sigma\sqrt{2\pi}} \lim_{y \rightarrow -\infty} \frac{1}{e^{y + \frac{y^2}{2\sigma^2}}} \\ &= 0 \end{aligned}$$

$$\begin{aligned} c) \quad f'(x) &= \frac{1}{\sigma\sqrt{2\pi}} \left(-\frac{2\ln x}{2\sigma^2 x^2} e^{-\frac{\ln^2 x}{2\sigma^2}} - \frac{1}{x^2} e^{-\frac{\ln^2 x}{2\sigma^2}} \right) \\ &= \frac{-1}{\sigma x^2 \sqrt{2\pi}} \left(\frac{\ln x}{\sigma^2} + 1 \right) e^{-\frac{\ln^2 x}{2\sigma^2}} = 0 \end{aligned}$$

$$\frac{\ln x}{\sigma^2} + 1 = 0$$

$$\ln x = -\sigma^2$$

$$x = e^{-\sigma^2} \quad (CN) \text{ Yields to a maximum point.}$$

$$\begin{aligned} d) \quad f\left(e^{-\sigma^2}\right) &= \frac{1}{\sigma\sqrt{2\pi} e^{-\sigma^2}} e^{-\frac{\ln^2 e^{-\sigma^2}}{2\sigma^2}} \\ &= \frac{1}{\sigma\sqrt{2\pi} e^{-\sigma^2}} e^{-\frac{(-\sigma^2)^2}{2\sigma^2}} \\ &= \frac{1}{\sigma\sqrt{2\pi}} \frac{e^{-\frac{1}{2}\sigma^2}}{e^{-\sigma^2}} \\ &= \frac{e^{\sigma^2/2}}{\sigma\sqrt{2\pi}} \end{aligned}$$

$$e) \quad g(\sigma) = \frac{e^{\sigma^2/2}}{\sigma\sqrt{2\pi}}$$

$$g'(\sigma) = \frac{1}{\sqrt{2\pi}} \left(\frac{\sigma^2 - 1}{\sigma^2} \right) e^{\sigma^2/2} = 0$$

$$\frac{\sigma^2 - 1}{\sigma^2} = 0 \rightarrow \underline{\sigma = \pm 1}$$

Since $\sigma > 0 \rightarrow \underline{\sigma = 1}$ (CN)

0	1	∞
-		+

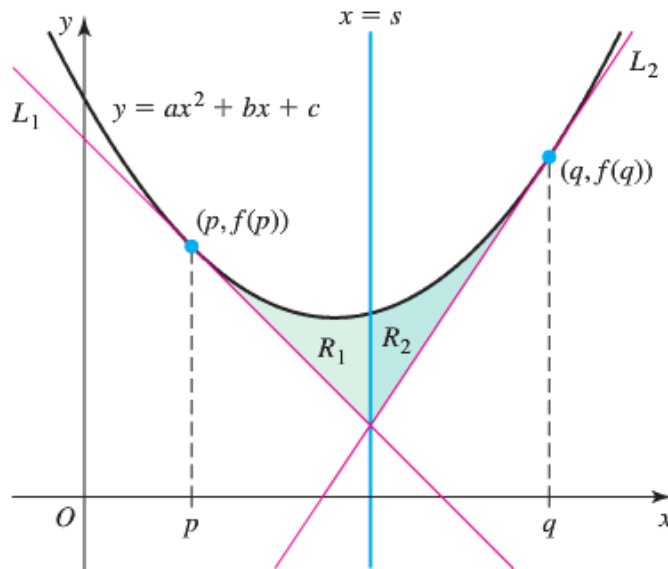
$\therefore g$ is decreasing on $(0, 1)$

g is increasing on $(1, \infty)$

So, g has a *minimum* at $\sigma = 1$

Exercise

Let $f(x) = ax^2 + bx + c$ be an arbitrary quadratic function and choose two points $x = p$ and $x = q$. Let L_1 be the line tangent to the graph of f at the point $(p, f(p))$ and let L_2 be the line tangent to the graph at the point $(q, f(q))$. Let $x = s$ be the vertical line through the intersection point of L_1 and L_2 . Finally, let R_1 be the region bounded by $y = f(x)$, L_1 , and the vertical line $x = s$, and let R_2 be the region bounded by $y = f(x)$, L_2 , and the vertical line $x = s$.



Prove that the area of R_1 equals the area of R_2

Solution

$$f(x) = ax^2 + bx + c$$

$$f' = 2ax + b$$

At $x = p$

$$\text{slope: } m = 2ap + b$$

Line L_1 :

$$\begin{aligned} y &= (2ap + b)(x - p) + f(p) \\ &= (2ap + b)x - 2ap^2 - bp + ap^2 + bp + c \\ &= \underline{(2ap + b)x - ap^2 + c} \end{aligned}$$

Line L_2 :

At $x = q$

$$\text{slope: } m = 2aq + b$$

$$\begin{aligned} y &= (2aq + b)(x - q) + f(q) \\ &= (2aq + b)x - 2aq^2 - bq + aq^2 + bq + c \\ &= \underline{(2aq + b)x - aq^2 + c} \end{aligned}$$

If $R_1 \text{ Area} = R_2 \text{ Area}$, then $s = \frac{p+q}{2}$

$$\begin{aligned} A_1 &= \int_p^s \left(f(x) - (2ap + b)x + ap^2 - c \right) dx \\ &= \int_p^s \left(ax^2 + bx + c - 2apx - bx + ap^2 - c \right) dx \\ &= \int_p^s \left(ax^2 - 2apx + ap^2 \right) dx \\ &= \frac{1}{3}ax^3 - apx^2 + ap^2x \Bigg|_p^{\frac{p+q}{2}} \\ &= \frac{1}{3}a\left(\frac{p+q}{2}\right)^3 - ap\left(\frac{p+q}{2}\right)^2 + ap^2\left(\frac{p+q}{2}\right) - \frac{1}{3}ap^3 + ap^3 - ap^3 \\ &= \frac{1}{24}a\left(p^3 + 3p^2q + 3pq^2 + q^3\right) - \frac{1}{4}ap\left(p^2 + 2pq + q^2\right) + \frac{1}{2}ap^3 + \frac{1}{2}ap^2q - \frac{1}{3}ap^3 \\ &= \frac{1}{24}ap^3 + \frac{1}{8}ap^2q + \frac{1}{8}apq^2 + \frac{1}{24}aq^3 - \frac{1}{4}ap^3 - \frac{1}{2}ap^2q - \frac{1}{4}apq^2 + \frac{1}{6}ap^3 + \frac{1}{2}ap^2q \\ &= -\frac{1}{24}ap^3 + \frac{1}{8}ap^2q - \frac{1}{8}apq^2 + \frac{1}{24}aq^3 \\ &= \frac{a}{24}\left(-p^3 + 3p^2q - 3pq^2 + q^3\right) \end{aligned}$$

$$= \frac{a}{24} (q^3 - 3q^2 p + 3qp^2 - p^3)$$

$$= \frac{a}{24} (q - p)^3 \Big|$$

$$A_2 = \int_s^p \left(f(x) - (2aq + b)x + aq^2 - c \right) dx$$

$$= \int_s^q \left(ax^2 + bx + c - 2aqx - bx + aq^2 - c \right) dx$$

$$= \int_s^q \left(ax^2 - 2aqx + aq^2 \right) dx$$

$$= \frac{1}{3} ax^3 - aqx^2 + aq^2 x \Big|_{\frac{p+q}{2}}^q$$

$$= a \left(\frac{1}{3} q^3 - q^3 + q^3 - \frac{1}{3} \left(\frac{p+q}{2} \right)^3 + q \left(\frac{p+q}{2} \right)^2 - q^2 \left(\frac{p+q}{2} \right) \right)$$

$$= a \left(\frac{1}{3} q^3 - \frac{1}{24} (p^3 + 3p^2 q + 3pq^2 + q^3) + \frac{1}{4} q (p^2 + 2qp + q^2) - \frac{1}{2} pq^2 - \frac{1}{2} q^3 \right)$$

$$= \frac{a}{24} (8q^3 - p^3 - 3p^2 q - 3pq^2 - q^3 + 6qp^2 + 12q^2 p + 6q^3 - 12pq^2 - 12q^3)$$

$$= \frac{a}{24} (q^3 - 3pq^2 + 3qp^2 - p^3)$$

$$= \frac{a}{24} (q - p)^3 \Big|$$