Solution   Section 1.8 – Curvature and Normal Vectors

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the plane curves: $\vec{r}(t) = t\,\hat{i} + (\ln\cos t)\,\hat{j}, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

Solution

$$\vec{v}(t) = \hat{i} - \frac{\sin t}{\cos t}\hat{j} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$= \hat{i} - \tan t\,\hat{j}$$

$$|\vec{v}| = \sqrt{1+\tan^2 t}$$

$$= \sqrt{\sec^2 t}$$

$$= \sec t$$

$$\vec{T} = \frac{1}{\sec t}\hat{i} - \frac{\tan t}{\sec t}\hat{j} \qquad\qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \cos t\,\hat{i} - \cos t\frac{\sin t}{\cos t}\hat{j}$$

$$= \cos t\,\hat{i} - \sin t\,\hat{j}$$

$$\frac{d\vec{T}}{dt} = -(\sin t)\hat{i} - (\cos t)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\vec{N} = \frac{-(\sin t)\hat{i} - (\cos t)\hat{j}}{1} \qquad\qquad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$= -(\sin t)\hat{i} - (\cos t)\hat{j}$$

$$\kappa = \frac{1}{\sec t}(1) \qquad\qquad \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$= \cos t$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the plane curves: $\vec{r}(t) = (\ln\sec t)\hat{i} + t\,\hat{j}, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

Solution

$$\vec{v}(t) = \frac{\sec t\tan t}{\sec t}\hat{i} + \hat{j} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$= \tan t\,\hat{i} + \hat{j}$$

$$|\vec{v}| = \sqrt{\tan^2 t + 1}$$
$$= \sqrt{\sec^2 t}$$
$$= \sec t$$

$$\vec{T} = \frac{\tan t}{\sec t}\hat{i} + \frac{1}{\sec t}\hat{j} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\underline{= (\sin t)\hat{i} + (\cos t)\hat{j}}$$

$$\frac{d\vec{T}}{dt} = (\cos t)\hat{i} - (\sin t)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t}$$
$$\underline{= 1}$$

$$\underline{\vec{N} = (\sin t)\hat{i} + (\cos t)\hat{j}} \qquad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\kappa = \frac{1}{\sec t}(1) \qquad \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$
$$\underline{= \cos t}$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the plane curves: $\vec{r}(t) = (2t+3)\hat{i} + \left(5 - t^2\right)\hat{j}$

Solution

$$\vec{v}(t) = 2\hat{i} - 2t\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{4 + 4t^2}$$
$$\underline{= 2\sqrt{1+t^2}}$$

$$\vec{T} = \frac{2}{2\sqrt{1+t^2}}\hat{i} - \frac{2t}{2\sqrt{1+t^2}}\hat{j} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$
$$\underline{= \frac{1}{\sqrt{1+t^2}}\hat{i} - \frac{t}{\sqrt{1+t^2}}\hat{j}}$$

$$\frac{d\vec{T}}{dt} = \frac{-2t}{2\left(1+t^2\right)^{3/2}}\hat{i} - \frac{\left(1+t^2\right)^{1/2} - \frac{1}{2}\left(1+t^2\right)^{-1/2}(2t)t}{\left(1+t^2\right)}\hat{j}$$

$$= \frac{-t}{\left(1+t^2\right)^{3/2}}\hat{i} - \frac{1+t^2-t^2}{\left(1+t^2\right)^{3/2}}\hat{j}$$

$$= \frac{-t}{\left(1+t^2\right)^{3/2}}\hat{i} - \frac{1}{\left(1+t^2\right)^{3/2}}\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{t^2}{\left(1+t^2\right)^3} + \frac{1}{\left(1+t^2\right)^3}}$$

$$= \sqrt{\frac{t^2+1}{\left(1+t^2\right)^3}}$$

$$= \sqrt{\frac{1}{\left(1+t^2\right)^2}}$$

$$= \frac{1}{\left(1+t^2\right)}$$

$$\vec{N} = \left(1+t^2\right)\left(\frac{-t}{\left(1+t^2\right)^{3/2}}\hat{i} - \frac{1}{\left(1+t^2\right)^{3/2}}\hat{j}\right) \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= \frac{-t}{\sqrt{1+t^2}}\hat{i} - \frac{1}{\sqrt{1+t^2}}\hat{j}$$

$$\kappa = \frac{1}{2\sqrt{1+t^2}}\frac{1}{\left(1+t^2\right)} \qquad \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$= \frac{1}{2\left(1+t^2\right)^{3/2}}$$

Exercise

Find ***T***, ***N***, and κ for the plane curves: $\vec{r}(t) = (\cos t + t\sin t)\hat{i} + (\sin t - t\cos t)\hat{j}, \quad t > 0$

Solution

$$\vec{v} = (-\sin t + \sin t + t\cos t)\hat{i} + (\cos t - \cos t + t\sin t)\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$= (t\cos t)\hat{i} + (t\sin t)\hat{j}$$

$$|\vec{v}| = \sqrt{t^2\cos^2 t + t^2\sin^2 t}$$

$$= \sqrt{t^2\left(\cos^2 t + \sin^2 t\right)}$$

$$= \sqrt{t^2}$$

$$= |t|$$

$$= t$$

$$\vec{T} = \left(\frac{t\cos t}{t}\right)\hat{i} + \left(\frac{t\sin t}{t}\right)\hat{j} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\boxed{= (\cos t)\hat{i} + (\sin t)\hat{j}}$$

$$\frac{d\vec{T}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t} \quad \boxed{=1}$$

$$\boxed{\vec{N} = (-\sin t)\hat{i} + (\cos t)\hat{j}} \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$\boxed{\kappa = \frac{1}{t}} \qquad \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

Exercise

Find ***T***, ***N***, and κ for the space curves: $\vec{r}(t) = (3\sin t)\hat{i} + (3\cos t)\hat{j} + 4t\,\hat{k}$

Solution

$$\vec{v} = (3\cos t)\hat{i} - (3\sin t)\hat{j} + 4\,\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{9\cos^2 t + 9\sin^2 t + 16}$$

$$= \sqrt{9+16}$$

$$\boxed{=5}$$

$$\boxed{\vec{T} = \frac{3}{5}\cos t\;\hat{i} - \frac{3}{5}\sin t\;\hat{j} + \frac{4}{5}\;\hat{k}} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\frac{d\vec{T}}{dt} = -\frac{3}{5}\sin t\;\hat{i} - \frac{3}{5}\cos t\;\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{9}{25}\sin^2 t + \frac{9}{25}\cos^2 t}$$

$$= \sqrt{\frac{9}{25}}$$

$$= \frac{3}{5}$$

$$\vec{N}=\frac{5}{3}\left(-\frac{3}{5}\sin t\,\hat{i}-\frac{3}{5}\cos t\,\hat{j}\right) \qquad \vec{N}=\frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$=(-\sin t)\hat{i}-(\cos t)\hat{j}$$

$$\kappa=\frac{1}{5}\frac{3}{5} \qquad \kappa=\frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$=\frac{3}{25}$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the space curves: $\boldsymbol{r}(t)=\left(e^t\cos t\right)\boldsymbol{i}+\left(e^t\sin t\right)\boldsymbol{j}+2\boldsymbol{k}$

Solution

$$\vec{v}(t)=\left(e^t\cos t-e^t\sin t\right)\hat{i}+\left(e^t\sin t+e^t\cos t\right)\hat{j} \qquad \vec{v}(t)=\frac{d\vec{r}}{dt}$$

$$|\vec{v}|=\sqrt{\left(e^t\cos t-e^t\sin t\right)^2+\left(e^t\sin t+e^t\cos t\right)^2}$$

$$=\sqrt{e^{2t}\cos^2 t-e^{2t}\cos t\sin t+e^{2t}\sin^2 t+e^{2t}\sin^2 t+e^{2t}\cos t\sin t+e^{2t}\cos^2 t}$$

$$=\sqrt{2e^{2t}\left(\cos^2 t+\sin^2 t\right)}$$

$$=e^t\sqrt{2}$$

$$\vec{T}=\left(\frac{e^t\cos t-e^t\sin t}{\sqrt{2}e^t}\right)\hat{i}+\left(\frac{e^t\sin t+e^t\cos t}{\sqrt{2}e^t}\right)\hat{j} \qquad \vec{T}=\frac{\vec{v}}{|\vec{v}|}$$

$$=\left(\frac{\cos t-\sin t}{\sqrt{2}}\right)\hat{i}+\left(\frac{\sin t+\cos t}{\sqrt{2}}\right)\hat{j}$$

$$\frac{d\vec{T}}{dt}=\left(\frac{-\sin t-\cos t}{\sqrt{2}}\right)\hat{i}+\left(\frac{\cos t-\sin t}{\sqrt{2}}\right)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right|=\sqrt{\frac{(-\sin t-\cos t)^2}{2}+\frac{(\cos t-\sin t)^2}{2}}$$

$$=\frac{1}{\sqrt{2}}\sqrt{\sin^2 t+2\sin t\cos t+\cos^2 t+\sin^2 t-2\sin t\cos t+\cos^2 t}$$

$$=\frac{1}{\sqrt{2}}\sqrt{2\sin^2 t+2\cos^2 t}$$

$$=\frac{1}{\sqrt{2}}\sqrt{2}$$

$$\boxed{=1}$$

$$\boxed{\vec{N}=\left(\frac{-\sin t-\cos t}{\sqrt{2}}\right)\hat{i}+\left(\frac{\cos t-\sin t}{\sqrt{2}}\right)\hat{j}}\qquad \vec{N}=\frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$\boxed{\kappa=\frac{1}{\sqrt{2}e^{t}}}\qquad \kappa=\frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

Exercise

Find ***T***, ***N***, and κ for the space curves: $\vec{r}(t)=\frac{t^3}{3}\hat{i}+\frac{t^2}{2}\hat{j},\quad t>0$

Solution

$$\vec{v}=\left(t^2\right)\hat{i}+t\,\hat{j}\qquad \vec{v}(t)=\frac{d\vec{r}}{dt}$$

$$|\vec{v}|=\sqrt{t^4+t^2}$$

$$=|t|\sqrt{t^2+1}$$

$$\boxed{=t\sqrt{t^2+1}}\quad (t>0)$$

$$\vec{T}=\left(\frac{t^2}{t\sqrt{t^2+1}}\right)\hat{i}+\left(\frac{t}{t\sqrt{t^2+1}}\right)\hat{j}\qquad \vec{T}=\frac{\vec{v}}{|\vec{v}|}$$

$$\boxed{=\left(\frac{t}{\sqrt{t^2+1}}\right)\hat{i}+\left(\frac{1}{\sqrt{t^2+1}}\right)\hat{j}}$$

$$\frac{d\vec{T}}{dt}=\frac{\left(1+t^2\right)^{1/2}-\frac{1}{2}\left(1+t^2\right)^{-1/2}(2t)t}{\left(1+t^2\right)}\hat{i}+\frac{-2t}{2\left(1+t^2\right)^{3/2}}\hat{j}$$

$$=\frac{1+t^2-t^2}{\left(1+t^2\right)^{3/2}}\hat{i}-\frac{t}{\left(1+t^2\right)^{3/2}}\hat{j}$$

$$=\frac{1}{\left(1+t^2\right)^{3/2}}\hat{i}-\frac{t}{\left(1+t^2\right)^{3/2}}\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right|=\sqrt{\frac{1}{\left(1+t^2\right)^3}+\frac{t^2}{\left(1+t^2\right)^3}}$$

$$=\sqrt{\frac{t^2+1}{\left(1+t^2\right)^3}}$$

$$=\sqrt{\frac{1}{\left(1+t^2\right)^2}}$$

$$=\frac{1}{1+t^2}$$

$$\vec{N}=\left(1+t^2\right)\left(\frac{1}{\left(1+t^2\right)^{3/2}}\hat{i}-\frac{t}{\left(1+t^2\right)^{3/2}}\hat{j}\right) \qquad \vec{N}=\frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$\underline{=\frac{1}{\sqrt{1+t^2}}\hat{i}-\frac{t}{\sqrt{1+t^2}}\hat{j}}$$

$$\kappa=\frac{1}{t\sqrt{t^2+1}}\frac{1}{1+t^2} \qquad \kappa=\frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$\underline{=\frac{t}{t\left(t^2+1\right)^{3/2}}}$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the space curves: $\vec{r}(t)=\left(\cos^3 t\right)\hat{i}+\left(\sin^3 t\right)\hat{j}, \quad 0<t<\frac{\pi}{2}$

Solution

$$\vec{v}=-\left(3\cos^2 t\sin t\right)\hat{i}+\left(3\sin^2 t\cos t\right)\hat{j} \qquad \vec{v}(t)=\frac{d\vec{r}}{dt}$$

$$|\vec{v}|=\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}$$

$$=3\sqrt{\cos^2 t\sin^2 t\left(\cos^2 t+\sin^2 t\right)}$$

$$=3\sqrt{\cos^2 t\sin^2 t}$$

$$=3|\cos t\sin t|$$

$$=\underline{3\cos t\sin t}$$

$$\vec{T}=-\left(\frac{3\cos^2 t\sin t}{3|\cos t\sin t|}\right)\hat{j}+\left(\frac{3\sin^2 t\cos t}{3|\cos t\sin t|}\right)\hat{j} \qquad \vec{T}=\frac{\vec{v}}{|\vec{v}|}$$

$$\underline{=-(\cos t)\hat{i}+(\sin t)\hat{j}}$$

$$\frac{d\vec{T}}{dt} = (\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\boxed{\vec{N} = (\sin t)\hat{i} + (\cos t)\hat{j}} \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$\kappa = \frac{1}{3\cos t \sin t}(1) \qquad \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$\boxed{= \frac{t}{3\cos t \sin t}}$$

Exercise

Find ***T***, ***N***, and κ for the space curves: $\vec{r}(t) = (\cosh t)\hat{i} - (\sinh t)\hat{j} + t\,\hat{k}$

Solution

$$\vec{v} = (\sinh t)\hat{j} - (\cosh t)\hat{j} + \hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{\sinh^2 t + \cosh^2 t + 1} \qquad \cosh^2 t - \sinh^2 t = 1 \quad \Rightarrow \quad \cosh^2 t = 1 + \sinh^2 t$$

$$= \sqrt{\cosh^2 t + \cosh^2 t}$$

$$\boxed{= \sqrt{2}\cosh t}$$

$$\vec{T} = \left(\frac{1}{\sqrt{2}\cosh t}\sinh t\right)\hat{i} - \left(\frac{1}{\sqrt{2}\cosh t}\cosh t\right)\hat{j} + \frac{1}{\sqrt{2}\cosh t}\hat{k} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\boxed{= \left(\frac{1}{\sqrt{2}}\tanh t\right)\hat{i} - \frac{1}{\sqrt{2}}\hat{j} + \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right)\hat{k}}$$

$$\frac{d\vec{T}}{dt} = \left(\frac{1}{\sqrt{2}}\operatorname{sech}^2 t\right)\hat{i} - \left(\frac{1}{\sqrt{2}}\operatorname{sech} t \tanh t\right)\hat{k}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{1}{2}\operatorname{sech}^4 t + \frac{1}{2}\operatorname{sech}^2 t \tanh^2 t}$$

$$= \frac{1}{\sqrt{2}}\operatorname{sech} t\sqrt{\operatorname{sech}^2 t + \tanh^2 t}$$

$$= \frac{1}{\sqrt{2}}\operatorname{sech} t\sqrt{1}$$

$$\boxed{= \frac{1}{\sqrt{2}}\operatorname{sech} t}$$

$$\vec{N} = \frac{\sqrt{2}}{\operatorname{sech} t}\left(\left(\frac{1}{\sqrt{2}}\operatorname{sech}^2 t\right)\hat{i} - \left(\frac{1}{\sqrt{2}}\operatorname{sech} t \tanh t\right)\hat{k}\right) \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= (\operatorname{sech} t)\hat{i} - (\tanh t)\hat{k}$$

$$\kappa = \frac{1}{\sqrt{2}\cosh t}\left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right) \qquad \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$= \frac{1}{2}\operatorname{sech}^2 t$$

Exercise

Find an equation for the circle of curvature of the curve $\vec{r}(t) = t\,\hat{i} + (\sin t)\,\hat{j}$, at the point $\left(\frac{\pi}{2}, 1\right)$. (The curve parametrizes the graph $y = \sin x$ in the *xy*-plane.)

Solution

$$\vec{v} = \hat{j} + (\cos t)\,\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{1+\cos^2 t}$$

$$\left|\vec{v}\left(\frac{\pi}{2}\right)\right| = \sqrt{1+\cos^2\left(\frac{\pi}{2}\right)}$$

$$= \sqrt{1+0}$$

$$= 1$$

$$\vec{T} = \frac{1}{\sqrt{1+\cos^2 t}}\hat{i} + \left(\frac{\cos t}{\sqrt{1+\cos^2 t}}\right)\hat{j} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\frac{d\vec{T}}{dt} = -\frac{1}{2}\frac{2\cos t(-\sin t)}{\left(1+\cos^2 t\right)^{3/2}}\hat{i} + \frac{-\sin t\left(1+\cos^2 t\right)^{1/2} - \cos t\left(\left(\frac{1}{2}\right)2\cos t(-\sin t)\right)\left(1+\cos^2 t\right)^{-1/2}}{\left(1+\cos^2 t\right)}\hat{j}$$

$$= \frac{\cos t \sin t}{\left(1+\cos^2 t\right)^{3/2}}\hat{i} + \frac{-\sin t\left(1+\cos^2 t\right) + \sin t\cos^2 t}{\left(1+\cos^2 t\right)^{3/2}}\hat{j}$$

$$= \frac{\cos t \sin t}{\left(1+\cos^2 t\right)^{3/2}}\hat{i} + \frac{-\sin t\left(1+\cos^2 t - \cos^2 t\right)}{\left(1+\cos^2 t\right)^{3/2}}\hat{j}$$

$$= \frac{\sin t \cos t}{\left(1+\cos^2 t\right)^{3/2}}\hat{i} - \frac{\sin t}{\left(1+\cos^2 t\right)^{3/2}}\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \left|\frac{(\sin t\cos t)\hat{i} - (\sin t)\hat{j}}{\left(1+\cos^2 t\right)^{3/2}}\right|$$

$$= \frac{\sqrt{\sin^2 t\cos^2 t + \sin^2 t}}{\left(1+\cos^2 t\right)^{3/2}}$$

$$= \frac{|\sin t|\sqrt{\cos^2 t + 1}}{\left(1+\cos^2 t\right)^{3/2}}$$

$$= \frac{|\sin t|}{1+\cos^2 t}$$

$$\left|\frac{d\vec{T}}{dt}\right|_{t=\frac{\pi}{2}} = \frac{\left|\sin\frac{\pi}{2}\right|}{1+\cos^2\frac{\pi}{2}}$$

$$\underline{=1\ |}$$

$\underline{\kappa = 1\ |}$ $\qquad$ $\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$

The radius of curvature is: $\rho = \frac{1}{\kappa} = \frac{1}{1} = 1$

The center of the circle is $\left(\frac{\pi}{2},\ 0\right)$

The equation of the osculating circle is: $\left(x-\frac{\pi}{2}\right)^2 + y^2 = 1$

Exercise

Write $\vec{a}$ of the motion $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$ without finding $\boldsymbol{T}$ and $\boldsymbol{N}$. $\vec{r}(t) = (a\cos t)\hat{i} + (a\sin t)\hat{i} + bt\,\hat{k}$

Solution

$\vec{v} = (-a\sin t)\hat{i} + (a\cos t)\hat{j} + (b)\hat{k}$ $\qquad$ $\vec{v}(t) = \frac{d\vec{r}}{dt}$

$$|\vec{v}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}$$

$$= \sqrt{a^2 + b^2}$$

$$a_T = \frac{d}{dt}|\vec{v}| = 0$$

$$\vec{a} = \vec{v}' = (-a\cos t)\hat{i} - (a\sin t)\hat{j}$$

$$|\vec{a}| = \sqrt{a^2\cos^2 t + a^2\sin^2 t}$$

$= |a|$

$a_N = \sqrt{a^2 + 0}$ $\qquad a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$

$= |a|$

$\vec{a} = (0)\vec{T} + |a|\vec{N}$ $\qquad \vec{a} = a_T\vec{T} + a_N\vec{N}$

$\underline{= |a|\vec{N}\,|}$

Exercise

Write $\vec{a}$ of the motion $\vec{a} = a_T\vec{T} + a_N\vec{N}$ without finding ***T*** and ***N***. $\vec{r}(t) = (1+3t)\hat{i} + (t-2)\hat{j} - 3t\,\hat{k}$

Solution

$\vec{v} = 3\hat{i} + \hat{j} - 3\hat{k}$ $\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$|\vec{v}| = \sqrt{9+1+9}$

$= \sqrt{19}$

$a_T = \frac{d}{dt}|\vec{v}| = 0$

$\vec{a} = \vec{v}' = 0$

$a_N = 0$ $\qquad a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$

$\vec{a} = (0)\vec{T} + 0\vec{N}$ $\qquad \vec{a} = a_T\vec{T} + a_N\vec{N}$

$\underline{= \vec{0}\,|}$

Exercise

Write $\vec{a}$ of the motion $\vec{a} = a_T\vec{T} + a_N\vec{N}$ at the given value of *t* without finding ***T*** and ***N***.

$$\vec{r}(t) = (t+1)\hat{i} + 2t\,\hat{j} + t^2\hat{k}, \quad t = 1$$

Solution

$\vec{v} = \hat{i} + 2\hat{j} + 2t\,\hat{k}$ $\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$|\vec{v}| = \sqrt{1+4+4t^2}$

$= \sqrt{5+4t^2}$

$a_T = \frac{1}{2}(8t)\left(5+4t^2\right)^{-1/2}$ $\qquad a_T = \frac{d}{dt}|\vec{v}|$

$$= 4t\left(5+4t^2\right)^{-1/2}$$

$$a_T\Big|_{t=1} = 4(5+4)^{-1/2}$$

$$= 4(9)^{-1/2}$$

$$= \frac{4}{3}$$

$$\vec{a} = \vec{v}' = 2\hat{k}$$

$$|\vec{a}| = \sqrt{4} = 2$$

$$a_N = \sqrt{4 - \frac{16}{9}} \qquad\qquad a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$$

$$= \sqrt{\frac{20}{9}}$$

$$= \frac{2\sqrt{5}}{3}$$

$$\boxed{\vec{a} = \frac{4}{3}\vec{T} + \frac{2\sqrt{5}}{3}\vec{N}} \qquad\qquad \vec{a} = a_T\vec{T} + a_N\vec{N}$$

Exercise

Write $\vec{a}$ of the motion $\vec{a} = a_T\vec{T} + a_N\vec{N}$ at the given value of t without finding ***T*** and ***N***.

$\vec{r}(t) = (t\cos t)\hat{i} + (t\sin t)\hat{j} + t^2\hat{k}, \quad t = 0$

Solution

$$\vec{v} = (\cos t - t\sin t)\hat{i} + (\sin t + t\cos t)\hat{j} + 2t\,\hat{k} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 4t^2}$$

$$= \sqrt{\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + 4t^2}$$

$$= \sqrt{\cos^2 t + \sin^2 t + t^2\left(\sin^2 t + \cos^2 t\right) + 4t^2}$$

$$= \sqrt{1 + 5t^2}$$

$$a_T = \frac{d}{dt}|\vec{v}|$$

$$= \frac{1}{2}(10t)\left(1+5t^2\right)^{-1/2}$$

$$= 5t\left(1+5t^2\right)^{-1/2}$$

$$a_T\Big|_{t=0} = 0$$

$$\vec{a} = \vec{v}' = (-\sin t - \sin t - t\cos t)\hat{i} + (\cos t + \cos t - t\sin t)\hat{j} + 2\hat{k}$$

$$= (-2\sin t - t\cos t)\hat{i} + (2\cos t - t\sin t)\hat{j} + 2\hat{k}$$

$$\vec{a}\Big|_{t=0} = 2\hat{j} + 2\hat{k}$$

$$|\vec{a}|\Big|_{t=0} = \sqrt{4+4}$$

$$= 2\sqrt{2}$$

$$a_N = \sqrt{8-0} \qquad\qquad a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$$

$$= 2\sqrt{2}$$

$$\underline{\vec{a} = 2\sqrt{2}\vec{N}} \qquad\qquad \vec{a} = a_T\vec{T} + a_N\vec{N}$$

Exercise

Write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$ at the given value of t without finding $\boldsymbol{T}$ and $\boldsymbol{N}$.

$$\vec{r}(t) = \left(e^t \cos t\right)\hat{i} + \left(e^t \sin t\right)\hat{j} + \sqrt{2}e^t\hat{k}, \quad t = 0$$

Solution

$$\vec{v} = \left(e^t \cos t - e^t \sin t\right)\hat{i} + \left(e^t \sin t + e^t \cos t\right)\hat{j} + \sqrt{2}e^t\hat{k} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{\left(e^t \cos t - e^t \sin t\right)^2 + \left(e^t \sin t + e^t \cos t\right)^2 + 2e^{2t}}$$

$$= \sqrt{e^{2t}\left(\cos^2 t - 2\cos t\sin t + \sin^2 t\right) + e^{2t}\left(\cos^2 t + 2\cos t\sin t + \sin^2 t\right) + 2e^{2t}}$$

$$= e^t\sqrt{1 - 2\cos t\sin t + 1 + 2\cos t\sin t + 2}$$

$$= e^t\sqrt{4}$$

$$= 2e^t$$

$$a_T = \frac{d}{dt}|\vec{v}| = 2e^t$$

$$a_T\Big|_{t=0} = 2$$

$$\vec{a} = \vec{v}' = \left(e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t\right)\hat{i} + \left(e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t\right)\hat{j} + \sqrt{2}e^t\hat{k}$$

$$= \left(-2e^t \sin t\right)\hat{i} + \left(2e^t \cos t\right)\hat{j} + \sqrt{2}e^t\hat{k}$$

$$\vec{a}\Big|_{t=0} = 2\hat{j} + \sqrt{2}\hat{k}$$

$$\left|\vec{a}\right|\Big|_{t=0} = \sqrt{4+2}$$

$$= \sqrt{6}$$

$$a_N = \sqrt{6-4} \qquad\qquad a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$$

$$\underline{= \sqrt{2}|}$$

$$\underline{\vec{a} = 2\vec{T} + \sqrt{2}\vec{N}\ |} \qquad\qquad \vec{a} = a_T\vec{T} + a_N\vec{N}$$

Exercise

Write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$ without finding $\boldsymbol{T}$ and $\boldsymbol{N}$.

$$\vec{r}(t) = \left(2 + 3t + 3t^2\right)\hat{i} + \left(4t + 4t^2\right)\hat{j} - (6\cos t)\hat{k} \quad t = 0$$

Solution

$$\vec{v} = (3+6t)\hat{i} + (4+8t)\hat{j} + 6\sin t\,\hat{k} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{(3+6t)^2 + (4+8t)^2 + 36\sin^2 t}$$

$$= \sqrt{9(1+2t)^2 + 16(1+2t)^2 + 36\sin^2 t}$$

$$= \sqrt{(9+16)(1+2t)^2 + 36\sin^2 t}$$

$$= \sqrt{25(1+2t)^2 + 36\sin^2 t}$$

$$a_T = \frac{1}{2}\left(100(1+2t) + 72\sin t\cos t\right)\left(25(1+2t)^2 + 36\sin^2 t\right)^{-1/2} \qquad\qquad a_T = \frac{d}{dt}|\vec{v}|$$

$$= \frac{1}{2}\frac{100(1+2t) + 72\sin t\cos t}{\sqrt{25(1+2t)^2 + 36\sin^2 t}}$$

$$a_T\Big|_{t=0} = \frac{1}{2}\frac{100}{\sqrt{25}}$$

$$\underline{= 10\ |}$$

$$\vec{a} = 6\hat{i} + 8\hat{j} + 6\cos t\,\hat{k} \qquad\qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$\vec{a}\Big|_{t=0} = 6\hat{i} + 8\hat{j} + 6\hat{k}$$

$$\left|\vec{a}\right|\Big|_{t=0} = \sqrt{36+64+36}$$

$$= \sqrt{136}$$

$$a_N = \sqrt{133 - 100} \qquad\qquad a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$$

$$\underline{= 6|}$$

$$\underline{\vec{a} = 10\vec{T} + 6\vec{N}\,|} \qquad\qquad \vec{a} = a_T\vec{T} + a_N\vec{N}$$

Exercise

Write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$ without finding $\boldsymbol{T}$ and $\boldsymbol{N}$.

$$\vec{r}(t) = (2+t)\hat{i} + \left(t + 2t^2\right)\hat{j} + \left(1 + t^2\right)\hat{k} \quad t = 0$$

Solution

$$\vec{v} = \hat{i} + (1+4t)\hat{j} + 2t\,\hat{k} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{1 + (1+4t)^2 + 4t^2}$$

$$= \sqrt{1 + 1 + 8t + 16t^2 + 4t^2}$$

$$= \sqrt{2 + 8t + 20t^2}$$

$$a_T = \frac{8 + 40t}{2\sqrt{2 + 8t + 20t^2}} \qquad\qquad a_T = \frac{d}{dt}|\vec{v}|$$

$$= \frac{4 + 20t}{\sqrt{2 + 8t + 20t^2}}$$

$$a_T\Big|_{t=0} = \frac{4}{\sqrt{2}}$$

$$\underline{= 2\sqrt{2}\,|}$$

$$\vec{a} = 4\hat{j} + 2\hat{k} \qquad\qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$|\vec{a}|\Big|_{t=0} = \sqrt{16 + 4}$$

$$= 2\sqrt{5}$$

$$a_N = \sqrt{20 - 8} \qquad\qquad a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$$

$$\underline{= 2\sqrt{3}\,|}$$

$$\underline{\vec{a} = 2\sqrt{2}\vec{T} + 2\sqrt{3}\vec{N}\,|} \qquad\qquad \vec{a} = a_T\vec{T} + a_N\vec{N}$$

Exercise

Graph the curves and sketch their velocity and acceleration vectors at the given values of t. Then write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$ without finding $\boldsymbol{T}$ and $\boldsymbol{N}$, and find the value of κ at the given values of t.

$$\vec{r}(t) = (4\cos t)\hat{i} + \left(\sqrt{2}\sin t\right)\hat{j}, \quad t = 0 \text{ and } \frac{\pi}{4}$$

Solution

$$\begin{cases} x = 4\cos t & \rightarrow \cos t = \frac{x}{4} \\ y = \sqrt{2}\sin t & \rightarrow \sin t = \frac{y}{\sqrt{2}} \end{cases}$$

$$\cos^2 t + \sin^2 t = 1$$

$$\frac{x^2}{16} + \frac{y^2}{2} = 1 \quad \rightarrow \quad \textit{Ellipse}$$

$$\vec{v} = -4\sin t\,\hat{i} + \sqrt{2}\cos t\,\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\vec{a} = -4\cos t\,\hat{i} - \sqrt{2}\sin t\,\hat{j} \qquad \vec{a} = \frac{d\vec{v}}{dt}$$

$t = 0$

$$\vec{r}(0) = 4\hat{i}$$

$$\vec{v}(0) = \sqrt{2}\hat{j}$$

$$\vec{a}(0) = -4\hat{i}$$

$t = \frac{\pi}{4}$

$$\vec{r}\left(\frac{\pi}{4}\right) = 2\sqrt{2}\,\hat{i} + \hat{j}$$

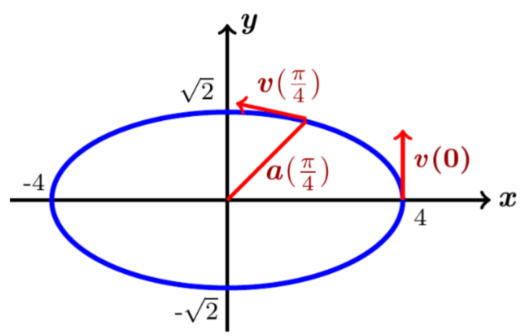
$$\vec{v}\left(\frac{\pi}{4}\right) = -2\sqrt{2}\,\hat{i} + \hat{j}$$

$$\vec{a}\left(\frac{\pi}{4}\right) = -2\sqrt{2}\,\hat{i} - \hat{j}$$

$$|\vec{v}| = \sqrt{16\sin^2 t + 2\cos^2 t}$$

$$a_T = \frac{32\sin t\cos t - 4\cos t\sin t}{2\sqrt{16\sin^2 t + 2\cos^2 t}} \qquad a_T = \frac{d}{dt}|\vec{v}|$$

$$= \frac{14\cos t\sin t}{\sqrt{16\sin^2 t + 2\cos^2 t}}$$



$t = 0$

$$a_T = 0$$

$$|\vec{a}| = \sqrt{16 + 0} = 4$$

$$a_N = \sqrt{4 - 0} = 4$$

$$\vec{a} = 0\vec{T} + 4\vec{N}$$

$$|\vec{v}(0)| = \sqrt{2}$$

$$\kappa = \frac{4}{2} = 2 \qquad \kappa = \frac{a_N}{|\vec{v}|^2}$$

$t = \frac{\pi}{4}$

$$a_T = \frac{14\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}}{\sqrt{8+1}} = \frac{7}{3}$$

$$|\vec{a}| = \sqrt{8+1} = 3$$

$$a_N = \sqrt{9 - \frac{49}{9}}$$

$$= \frac{4\sqrt{2}}{3}$$

$$\vec{a} = \frac{7}{3}\vec{T} + \frac{4\sqrt{2}}{3}\vec{N}$$

$$\left|\vec{v}\left(\frac{\pi}{4}\right)\right| = \sqrt{8+1} = 3$$

$$\kappa = \frac{4\sqrt{2}}{3}\frac{1}{9} \qquad \kappa = \frac{a_N}{|\vec{v}|^2}$$

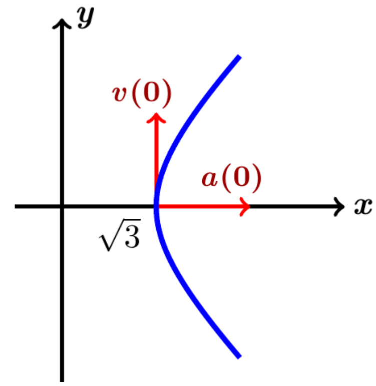
$$= \frac{4\sqrt{2}}{27}$$

Exercise

Graph the curves and sketch their velocity and acceleration vectors at the given values of t. Then write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$ without finding $\boldsymbol{T}$ and $\boldsymbol{N}$, and find the value of κ at the given values of t.

$$\vec{r}(t) = \left(\sqrt{3}\sec t\right)\hat{i} + \left(\sqrt{3}\tan t\right)\hat{j}, \quad t = 0$$

Solution



$$\begin{cases} x = \sqrt{3}\sec t & \to \sec t = \frac{x}{\sqrt{3}} \\ y = \sqrt{3}\tan t & \to \tan t = \frac{y}{\sqrt{3}} \end{cases}$$

$$\sec^2 t - \tan^2 t = 1$$

$$\frac{x^2}{3} - \frac{y^2}{3} = 1 \quad \to \quad \textit{Hyperbolic}$$

$$\vec{v} = \sqrt{3}\sec t\tan t\,\hat{i} + \sqrt{3}\sec^2 t\,\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\vec{a} = \sqrt{3}\left(\sec t\tan^2 t + \sec^3 t\right)\hat{i} + 2\sqrt{3}\sec^2 t\tan t\,\hat{j} \qquad \vec{a} = \frac{d\vec{v}}{dt}$$

At $t = 0$

$$\vec{r}(0) = \sqrt{3}\hat{i}, \quad \vec{v}(0) = \sqrt{3}\hat{j}, \quad \vec{a}(0) = \sqrt{3}\hat{i}$$

$$\left|\vec{v}(0)\right| = \sqrt{3}$$

$$\left|\vec{v}\right| = \sqrt{3\sec^2 t\tan^2 t\ + 3\sec^4 t}$$

$$= \sqrt{3}\ \sqrt{\sec^2 t\left(\sec^2 t - 1\right) + \sec^4 t}$$

$$= \sqrt{3}\sqrt{2\sec^4 t - \sec^2 t}$$

$$a_T = \frac{\sqrt{3}}{2}\frac{8\sec^4 t\tan t + 2\sec^2 t\tan t}{\sqrt{2\sec^4 t - \sec^2 t}} \qquad a_T = \frac{d}{dt}\left|\vec{v}\right|$$

$$= \sqrt{3}\,\frac{\sec^2 t\tan t\left(4\sec^2 t + 1\right)}{\sec t\sqrt{2\sec^2 t - 1}}$$

$$= \frac{\sqrt{3}\ \sec t\tan t\left(4\sec^2 t + 1\right)}{\sqrt{2\sec^2 t - 1}}.$$

$$a_T\Big|_{t=0} = 0$$

$$a_N = \sqrt{3 - 0} \qquad a_N = \sqrt{\left|\boldsymbol{a}\right|^2 - a_T^2}$$

$$\underline{= \sqrt{3}\ |}$$

$$\underline{\vec{a} = 0\ \vec{T} + \sqrt{3}\ \vec{N}\ |} \qquad \vec{a} = a_T\vec{T} + a_N\vec{N}$$

$$\underline{\vec{a} = \sqrt{3}\ \vec{N}\ |}$$

$$\underline{\kappa = \frac{\sqrt{3}}{3}\ |} \qquad \kappa = \frac{a_N}{\left|\vec{v}\right|^2}$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$, τ, and κ at the given value of t for the plane curves

$$\vec{r}(t) = \tfrac{4}{9}(1+t)^{3/2}\,\hat{i} + \tfrac{4}{9}(1-t)^{3/2}\,\hat{j} + \tfrac{1}{3}t\,\hat{k}; \quad t = 0$$

Solution

$$\vec{v} = \tfrac{2}{3}(1+t)^{1/2}\,\hat{i} - \tfrac{2}{3}(1-t)^{1/2}\,\hat{j} + \tfrac{1}{3}\,\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\left|\vec{v}\right| = \sqrt{\tfrac{4}{9}(1+t) + \tfrac{4}{9}(1-t) + \tfrac{1}{9}}$$

$$= \sqrt{\tfrac{4}{9}(1+t+1-t) + \tfrac{1}{9}}$$

$$= \sqrt{\tfrac{8}{9} + \tfrac{1}{9}}$$

$$\boxed{=1}$$

$$\vec{T}=\tfrac{2}{3}(1+t)^{1/2}\,\hat{i}-\tfrac{2}{3}(1-t)^{1/2}\,\hat{j}+\tfrac{1}{3}\,\hat{k} \qquad\qquad \vec{T}=\frac{\vec{v}}{|\vec{v}|}$$

$$\boxed{\vec{T}(0)=\tfrac{2}{3}\hat{i}-\tfrac{2}{3}\hat{j}+\tfrac{1}{3}\,\hat{k}}$$

$$\frac{d\vec{T}}{dt}=\tfrac{1}{3}(1+t)^{-1/2}\,\hat{i}+\tfrac{1}{3}(1-t)^{-1/2}\,\hat{j}$$

$$\left.\frac{d\vec{T}}{dt}\right|_{t=0}=\tfrac{1}{3}\hat{i}+\tfrac{1}{3}\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}(0)\right|=\sqrt{\frac{1}{9}+\frac{1}{9}}$$

$$=\frac{\sqrt{2}}{3}$$

$$\vec{N}(0)=\frac{3}{\sqrt{2}}\left(\tfrac{1}{3}\hat{i}+\tfrac{1}{3}\hat{j}\right) \qquad\qquad \vec{N}=\frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\boxed{=\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}}$$

$$\vec{B}(0)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{vmatrix} \qquad\qquad \vec{B}=\vec{T}\times\vec{N}$$

$$\boxed{=-\frac{1}{3\sqrt{2}}\hat{i}+\frac{1}{3\sqrt{2}}\hat{j}+\frac{4}{3\sqrt{2}}\hat{k}}$$

$$\vec{a}(t)=\tfrac{1}{3}(1+t)^{-1/2}\,\hat{i}+\tfrac{1}{3}(1-t)^{-1/2}\,\hat{j} \qquad\qquad \vec{a}(t)=\frac{d\vec{v}}{dt}$$

$$\vec{a}(0)=\tfrac{1}{3}\hat{i}+\tfrac{1}{3}\hat{j}$$

$$\vec{v}(0)\times\vec{a}(0)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & 0\end{vmatrix}$$

$$=-\tfrac{1}{9}\hat{i}+\tfrac{1}{9}\hat{j}+\tfrac{4}{9}\hat{k}$$

$$\vec{v}(0)\times\vec{a}(0)=\sqrt{\frac{1}{81}+\frac{1}{81}+\frac{16}{81}}$$

$$=\sqrt{\frac{18}{81}}$$

$$=\frac{3\sqrt{2}}{9}$$

$$= \frac{\sqrt{2}}{3}$$

$$\boxed{\kappa(0) = \frac{\sqrt{2}}{3}} \qquad\qquad \kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$$

$$\vec{a}' = -\tfrac{1}{6}(1+t)^{-3/2}\,\hat{i} + \tfrac{1}{6}(1-t)^{-3/2}\,\hat{j}$$

$$\vec{a}'(0) = -\tfrac{1}{6}\hat{i} + \tfrac{1}{6}\hat{j}$$

$$\tau(0) = \frac{1}{\left(\frac{\sqrt{2}}{3}\right)^2}\begin{vmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 \end{vmatrix} \qquad\qquad \tau = \frac{1}{|\vec{v} \times \vec{a}|^2}\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}$$

$$= \frac{\frac{1}{3}\left(2\frac{1}{18}\right)}{\frac{2}{9}}$$

$$\boxed{= \frac{1}{6}}$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$, τ, and κ at the given value of t for the plane curves

$$\vec{r}(t) = \left(e^t \sin 2t\right)\hat{i} + \left(e^t \cos 2t\right)\hat{j} + 2e^t\,\hat{k}; \quad t = 0$$

Solution

$$\vec{v} = e^t\left(\sin 2t + 2\cos 2t\right)\hat{i} + e^t\left(\cos 2t - 2\sin 2t\right)\hat{j} + 2e^t\hat{k} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = e^t\sqrt{\left(\sin 2t + 2\cos 2t\right)^2 + \left(\cos 2t - 2\sin 2t\right)^2 + 4}$$

$$= e^t\sqrt{\sin^2 2t + 4\sin 2t\cos 2t + 4\cos^2 2t + \cos^2 2t - 4\sin 2t\cos 2t + 4\sin^2 2t + 4}$$

$$= e^t\sqrt{5\cos^2 2t + 5\sin^2 2t + 4}$$

$$= e^t\sqrt{5+4}$$

$$\boxed{= 3e^t}$$

$$\vec{T} = \frac{e^t\left(\sin 2t + 2\cos 2t\right)\hat{i} + e^t\left(\cos 2t - 2\sin 2t\right)\hat{j} + 2e^t\hat{k}}{3e^t} \qquad\qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \tfrac{1}{3}\left(\left(\sin 2t + 2\cos 2t\right)\hat{i} + \left(\cos 2t - 2\sin 2t\right)\hat{j} + 2\hat{k}\right)$$

$$\boxed{\vec{T}(0) = \tfrac{2}{3}\hat{i} + \tfrac{1}{3}\hat{j} + \tfrac{2}{3}\,\hat{k}}$$

$$\frac{d\vec{T}}{dt} = \frac{1}{3}(2\cos 2t - 4\sin 2t)\hat{i} + \frac{1}{3}(-2\sin 2t - 4\sin 2t)\hat{j}$$

$$\left.\frac{d\vec{T}}{dt}\right|_{t=0} = \frac{2}{3}\hat{i} - \frac{4}{3}\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}(0)\right| = \sqrt{\frac{4}{9} + \frac{16}{9}}$$

$$= \frac{2}{3}\sqrt{5}$$

$$\vec{N}(0) = \frac{3}{2\sqrt{5}}\left(\frac{2}{3}\hat{i} - \frac{4}{3}\hat{j}\right) \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= \frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}$$

$$\vec{B}(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \end{vmatrix} \qquad \vec{B} = \vec{T} \times \vec{N}$$

$$= \frac{4}{3\sqrt{5}}\hat{i} + \frac{2}{3\sqrt{5}}\hat{j} - \frac{5}{3\sqrt{5}}\hat{k}$$

$$\vec{a}(t) = e^t(4\cos 2t - 3\sin 2t)\hat{i} + e^t(-4\sin 2t - 3\cos 2t)\hat{j} + 2e^t\hat{k} \qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$\vec{a}(0) = 4\hat{i} - 3\hat{j} + 2\hat{k}$$

$$\vec{v}(0) \times \vec{a}(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 2 \\ 4 & -3 & 2 \end{vmatrix}$$

$$= 8\hat{i} + 4\hat{j} - 10\hat{k}$$

$$\vec{v}(0) \times \vec{a}(0) = \sqrt{64 + 16 + 100}$$

$$= \sqrt{180}$$

$$= 6\sqrt{5}$$

$$\kappa(0) = \frac{6\sqrt{5}}{3^2} \qquad \kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$$

$$= \frac{2\sqrt{5}}{3}$$

$$\vec{a}' = e^t(4\cos 2t - 3\sin 2t - 8\sin 2t - 6\cos 2t)\hat{i}$$

$$+ e^t(-4\sin 2t - 3\cos 2t - 8\cos 2t + 6\sin 2t)\hat{j} + 2e^t\hat{k}$$

$$= e^t(-2\cos 2t - 11\sin 2t)\hat{i} + e^t(2\sin 2t - 11\cos 2t)\hat{j} + 2e^t\hat{k}$$

$$\vec{a}'(0) = -2\hat{i} - 11\hat{j} + 2\hat{k}$$

$$\tau(0) = \frac{1}{180}\begin{vmatrix} 2 & 1 & 2 \\ 4 & -3 & 2 \\ 2 & -11 & 2 \end{vmatrix} \qquad \tau = \frac{1}{|\vec{v}\times\vec{a}|^2}\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}$$

$$= \frac{-80}{180}$$

$$= -\frac{4}{9}$$

Exercise

Find ***T***, ***N***, ***B***, τ, and κ at the given value of t for the plane curves $\vec{r}(t) = t\,\hat{i} + \left(\frac{1}{2}e^{2t}\right)\hat{j};\quad t = \ln 2$

Solution

$$\vec{v} = \hat{i} + e^{2t}\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\vec{v}(\ln 2) = \hat{i} + e^{2\ln 2}\hat{j}$$

$$= \hat{i} + e^{\ln 4}\hat{j}$$

$$= \hat{i} + 4\hat{j}$$

$$|\vec{v}| = \sqrt{1+e^{4t}}\,\Big|_{t=\ln 2}$$

$$= \sqrt{1+e^{4\ln 2}}$$

$$= \sqrt{1+e^{\ln 2^4}}$$

$$= \sqrt{1+16}$$

$$= \sqrt{17}$$

$$\vec{T} = \frac{1}{\sqrt{1+e^{4t}}}\hat{i} + \frac{e^{2t}}{\sqrt{1+e^{4t}}}\hat{j} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{T}(\ln 2) = \frac{1}{\sqrt{17}}\hat{i} + \frac{4}{\sqrt{17}}\hat{j}$$

$$\frac{d\vec{T}}{dt} = \frac{-2e^{4t}}{\left(1+e^{4t}\right)^{3/2}}\hat{i} + \frac{2e^{2t}\left(1+e^{4t}\right) - \left(2e^{4t}\right)e^{2t}}{\left(1+e^{4t}\right)^{3/2}}\hat{j}$$

$$= \frac{-2e^{4t}}{\left(1+e^{4t}\right)^{3/2}}\hat{i} + \frac{2e^{2t}}{\left(1+e^{4t}\right)^{3/2}}\hat{j}$$

$$\left.\frac{d\vec{T}}{dt}\right|_{t=\ln 2} = -\frac{2e^{4\ln 2}}{(1+16)^{3/2}}\hat{i} + \frac{2(4)}{17\sqrt{17}}\hat{j}$$

$$= -\frac{32}{17\sqrt{17}}\hat{i} + \frac{8}{17\sqrt{17}}\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}(0)\right| = \sqrt{\frac{32^2}{17^3} + \frac{64}{17^3}}$$

$$= \frac{8\sqrt{17}}{17\sqrt{17}}$$

$$\boxed{= \frac{8}{17}}$$

$$\vec{N}(\ln 2) = \frac{17}{8}\left(-\frac{32}{17\sqrt{17}}\hat{i} + \frac{8}{17\sqrt{17}}\hat{j}\right) \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$\boxed{= -\frac{4}{\sqrt{17}}\hat{i} + \frac{1}{\sqrt{17}}\hat{j}}$$

$$\vec{B}(\ln 2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} & 0 \\ \frac{-4}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 \end{vmatrix} \qquad \vec{B} = \vec{T} \times \vec{N}$$

$$\boxed{= \hat{k}}$$

$$\vec{a}(t) = 2e^{2t}\hat{j} \qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$\vec{a}(\ln 2) = 8\hat{j}$$

$$(\vec{v} \times \vec{a})(\ln 2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 0 \\ 0 & 8 & 0 \end{vmatrix}$$

$$= 8\hat{k}$$

$$\left|(\vec{v} \times \vec{a})(\ln 2)\right| = 8$$

$$\boxed{\kappa(\ln 2) = \frac{8}{\sqrt{17}}} \qquad \kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$$

$$\vec{a}' = 4e^{2t}\hat{j}$$

$$\vec{a}'(\ln 2) = 16\hat{j}$$

$$\tau(\ln 2) = \frac{1}{180}\begin{vmatrix} 1 & 4 & 0 \\ 0 & 8 & 0 \\ 0 & 16 & 0 \end{vmatrix} \qquad \tau = \frac{1}{|\vec{v}\times\vec{a}|^2}\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}$$

$$= 0$$

Exercise

Find ***r, T, N***, and ***B*** at the given value of t. Then find equations for the osculating, normal, and rectifying planes at that value of t. $\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} - \boldsymbol{k}, \quad t = \frac{\pi}{4}$

Solution

$$\vec{r}\left(t = \frac{\pi}{4}\right) = \left(\cos\frac{\pi}{4}\right)\hat{i} + \left(\sin\frac{\pi}{4}\right)\hat{j} - \hat{k}$$

$$= \frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j} - \hat{k}$$

$$\vec{v} = (-\sin t)\hat{i} + (\cos t)\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{\sin^2 t + \cos^2 t} \;\; = 1$$

$$\vec{T} = (-\sin t)\hat{i} + (\cos t)\hat{j} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{T}\left(t = \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j}$$

$$\frac{d\vec{T}}{dt} = (-\cos t)\hat{i} + (-\sin t)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t} \;\; = 1$$

$$\vec{N} = (-\cos t)\hat{i} + (-\sin t)\hat{j} \qquad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\vec{N}\left(t = \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\hat{i} - \frac{\sqrt{2}}{2}\hat{j}$$

$$\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} \;\; = \hat{k}$$

$$\vec{B}\left(t = \frac{\pi}{4}\right) = \hat{k}$$

The normal to the osculating plane $\boldsymbol{r} = \frac{\sqrt{2}}{2}\boldsymbol{i} + \frac{\sqrt{2}}{2}\boldsymbol{j} - \boldsymbol{k} \;\Rightarrow\; P\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2},\ -1\right)$ lies on the osculating plane (using ***B***):

$$0\left(x-\frac{\sqrt{2}}{2}\right)+0\left(y-\frac{\sqrt{2}}{2}\right)+\left(z-(-1)\right)=0$$

$z=-1$ is the osculating plane.

T is normal to the normal plane

$$-\frac{\sqrt{2}}{2}\left(x-\frac{\sqrt{2}}{2}\right)+\frac{\sqrt{2}}{2}y\left(x-\frac{\sqrt{2}}{2}\right)+0\left(z-(-1)\right)=0$$

$$-\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y=0$$

$-x+y=0$ is the normal plane

N is normal to the rectifying plane:

$$-\frac{\sqrt{2}}{2}\left(x-\frac{\sqrt{2}}{2}\right)-\frac{\sqrt{2}}{2}\left(y-\frac{\sqrt{2}}{2}\right)+0\left(z-(-1)\right)=0$$

$$-\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}y+\frac{1}{2}+\frac{1}{2}=0$$

$x+y=\sqrt{2}$ is the rectifying plane.

Exercise

Find ***r, T, N***, and ***B*** at the given value of t. Then find equations for the osculating, normal, and rectifying planes at that value of t. $\vec{r}(t)=(\cos t)\hat{i}+(\sin t)\hat{j}+t\,\hat{k}, \quad t=0$

Solution

$$\vec{r}(t=0)=(\cos 0)\hat{i}+(\sin 0)\hat{j}+0\hat{k}$$

$$=\hat{i}$$

$$\vec{v}=(-\sin t)\hat{i}+(\cos t)\hat{j}+\hat{k} \qquad \vec{v}(t)=\frac{d\vec{r}}{dt}$$

$$|\vec{v}|=\sqrt{\sin^2 t+\cos^2 t+1}$$

$$=\sqrt{2}$$

$$\vec{T}=-\left(\frac{\sin t}{\sqrt{2}}\right)\hat{i}+\left(\frac{\cos t}{\sqrt{2}}\right)\hat{j}+\frac{1}{\sqrt{2}}\hat{k} \qquad \vec{T}=\frac{\vec{v}}{|\vec{v}|}$$

$$\vec{T}(t=0)=\frac{1}{\sqrt{2}}\hat{j}+\frac{1}{\sqrt{2}}\hat{k}$$

$$\frac{d\vec{T}}{dt}=\left(-\frac{1}{\sqrt{2}}\cos t\right)\hat{i}-\left(\frac{1}{\sqrt{2}}\sin t\right)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right|=\sqrt{\frac{1}{2}\cos^2 t+\frac{1}{2}\sin^2 t}$$

$$= \frac{1}{\sqrt{2}}$$

$$\vec{N} = \sqrt{2}\left(\left(-\frac{1}{\sqrt{2}}\cos t\right)\hat{i} - \left(\frac{1}{\sqrt{2}}\sin t\right)\hat{j}\right) \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= (-\cos t)\hat{i} - (\sin t)\hat{j}$$

$$\vec{N}(t=0) = -\hat{i}$$

$$\vec{B}(t=0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \end{vmatrix} \qquad \vec{B} = \vec{T} \times \vec{N}$$

$$= -\frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$$

The normal to the osculating plane $\vec{r}(t) = \hat{i} \quad \Rightarrow \quad P(1,\ 0,\ 0)$ lies on the osculating plane (using ***B***):

$$0(x-1) - \frac{1}{\sqrt{2}}(y-0) + \frac{1}{\sqrt{2}}(z-0) = 0$$

$$-\frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}z = 0$$

$y - z = 0$ is the osculating plane.

T is normal to the normal plane

$$0(x-1) + \frac{1}{\sqrt{2}}(y-0) + \frac{1}{\sqrt{2}}(z-0) = 0$$

$$\frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}z = 0$$

$y + z = 0$ is the normal plane

N is normal to the rectifying plane:

$$-(x-1) + 0(y-0) + 0(z-0) = 0$$

$$-x + 1 = 0$$

$x = 1$ is the rectifying plane.

Exercise

Find ***B*** and τ for: $\vec{r}(t) = (3\sin t)\hat{i} + (3\cos t)\hat{j} + 4t\,\hat{k}$

Solution

$$\vec{v} = (3\cos t)\hat{i} - (3\sin t)\hat{j} + 4\,\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{9\cos^2 t + 9\sin^2 t + 16}$$
$$= \sqrt{9+16}$$
$$= 5$$

$$\vec{T} = \frac{3}{5}\cos t\ \hat{i} - \frac{3}{5}\sin t\ \hat{j} + \frac{4}{5}\ \hat{k}$$ $$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\frac{d\vec{T}}{dt} = -\frac{3}{5}\sin t\ \hat{i} - \frac{3}{5}\cos t\ \hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{9}{25}\sin^2 t + \frac{9}{25}\cos^2 t}$$
$$= \sqrt{\frac{9}{25}}$$
$$= \frac{3}{5}$$

$$\vec{N} = \frac{5}{3}\left(-\frac{3}{5}\sin t\ \hat{i} - \frac{3}{5}\cos t\ \hat{j}\right)$$ $$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$
$$= (-\sin t)\hat{i} - (\cos t)\hat{j}$$

$$\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{3}{5}\cos t & -\frac{3}{5}\sin t & \frac{4}{5} \\ -\sin t & -\cos t & 0 \end{vmatrix}$$ $$\vec{B} = \vec{T} \times \vec{N}$$
$$= \left(\frac{4}{5}\cos t\right)\hat{i} - \left(\frac{4}{5}\sin t\right)\hat{j} - \frac{3}{5}\hat{k}$$

$$\vec{a} = (-3\sin t)\hat{i} - (3\cos t)\hat{j}$$ $$\vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \end{vmatrix}$$
$$= (12\cos t)\hat{i} - (12\sin t)\hat{j} - 9\hat{k}$$

$$|\vec{v} \times \vec{a}|^2 = 144\cos^2 t + 144\sin^2 t + 81$$
$$= 144 + 81$$
$$= 225$$

$$\tau = \frac{\begin{vmatrix} 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \\ -3\cos t & 3\sin t & 0 \end{vmatrix}}{225}$$ $$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2}$$

$$= \frac{4\left(-9\sin^2 t - 9\cos^2 t\right)}{225}$$

$$= -\frac{36}{225}$$

$$\boxed{= -\frac{4}{25}}$$

Exercise

Find ***B*** and τ for: $\vec{r}(t) = (\cos t + t\sin t)\hat{i} + (\sin t - t\cos t)\hat{j} + 3\hat{k}$

Solution

$$\vec{v} = (-\sin t + \sin t + t\cos t)\hat{i} + (\cos t - \cos t + t\sin t)\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$= (t\cos t)\hat{i} + (t\sin t)\hat{j}$$

$$|\vec{v}| = \sqrt{t^2\cos^2 t + t^2\sin^2 t}$$

$$= \sqrt{t^2\left(\cos^2 t + \sin^2 t\right)}$$

$$= \sqrt{t^2}$$

$$= |t|$$

$$= t$$

$$\vec{T} = \left(\frac{t\cos t}{t}\right)\hat{i} + \left(\frac{t\sin t}{t}\right)\hat{j} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\boxed{= (\cos t)\hat{i} + (\sin t)\hat{j}}$$

$$\frac{d\vec{T}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t} \quad = 1$$

$$\boxed{\vec{N} = (-\sin t)\hat{j} + (\cos t)\hat{j}} \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$\boldsymbol{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & \sin t & 0 \\ -\sin t & -\cos t & 0 \end{vmatrix} \qquad \vec{B} = \vec{T} \times \vec{N}$$

$$= \left(\cos^2 t + \sin^2 t\right)\hat{k}$$

$$\boxed{= \hat{k}}$$

$$\vec{a}\left(\cos t - t\sin t\right)\hat{i} + \left(\sin t + t\cos t\right)\hat{j} \qquad \vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{v}\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t\cos t & t\sin t & 0 \\ \cos t - t\sin t & \sin t + t\cos t & 0 \end{vmatrix}$$

$$= \left(t\cos t\sin t + t^2\cos^2 t - t\sin t\cos t + t^2\sin^2 t\right)\hat{k}$$

$$= t^2\hat{k}$$

$$\left|\vec{v}\times\vec{a}\right|^2 = \left(t^2\right)^2 = t^4$$

$$\tau = \frac{\begin{vmatrix} t\cos t & t\sin t & 0 \\ \cos t - t\sin t & \sin t + t\cos t & 0 \\ -2\sin t - t\cos t & 2\cos t - t\sin t & 0 \end{vmatrix}}{t^4} \qquad \tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{\left|\vec{v}\times\vec{a}\right|^2}$$

$$= \frac{0}{t^4}$$

$$\underline{= 0}$$

Exercise

Find ***B*** and τ for: $\vec{r}(t) = (6\sin 2t)\hat{i} + (6\cos 2t)\hat{j} + 5t\,\hat{k}$

Solution

$$\vec{v} = (12\cos 2t)\hat{i} - (12\sin 2t)\hat{j} + 5\,\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{144\cos^2 t + 144\sin^2 t + 25}$$

$$= \sqrt{144 + 25}$$

$$\underline{= 13}$$

$$\underline{\vec{T} = \frac{12}{13}\cos 2t\;\hat{i} - \frac{12}{13}\sin 2t\,\hat{j} + \frac{5}{13}\;\hat{k}} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\frac{d\vec{T}}{dt} = -\frac{24}{13}\sin 2t\;\hat{i} - \frac{24}{13}\cos 2t\,\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{576}{169}\sin^2 t + \frac{576}{169}\cos^2 t}$$

$$= \sqrt{\frac{576}{169}}$$

$$= \frac{24}{13}$$

$$\vec{N} = \frac{13}{24}\left(-\frac{24}{13}\sin 2t\,\hat{i} - \frac{24}{13}\cos 2t\,\hat{j}\right) \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= \boxed{(-\sin 2t)\hat{i} - (\cos 2t)\,\hat{j}}$$

$$\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{12}{13}\cos 2t & -\frac{12}{13}\sin 2t & \frac{5}{13} \\ -\sin 2t & -\cos 2t & 0 \end{vmatrix}$$

$$= \boxed{\left(\frac{5}{13}\cos 2t\right)\hat{i} - \left(\frac{5}{13}\sin 2t\right)\hat{j} - \frac{12}{13}\hat{k}}$$

$$\vec{a} = (-24\sin 2t)\hat{i} - (24\cos 2t)\,\hat{j} \qquad \vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{v}\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 12\cos 2t & -12\sin 2t & 5 \\ -24\sin 2t & -24\cos 2t & 0 \end{vmatrix}$$

$$= (120\cos 2t)\hat{i} - (120\sin 2t)\,\hat{j} - 288\hat{k}$$

$$\left|\vec{v}\times\vec{a}\right|^2 = 14400\cos^2 2t + 14400\sin^2 2t + 288^2$$

$$= 14400 + 82944 = 97344$$

$$\tau \frac{\begin{vmatrix} 12\cos 2t & -12\sin 2t & 5 \\ -24\sin 2t & -24\cos 2t & 0 \\ -48\cos 2t & 48\sin 2t & 0 \end{vmatrix}}{97344} \qquad \tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{\left|\vec{v}\times\vec{a}\right|^2}$$

$$= \frac{5\left(-1152\sin^2 2t - 1152\cos^2 2t\right)}{97344}$$

$$= -\frac{5760}{97344}$$

$$= \boxed{-\frac{10}{169}}$$

Exercise

The speedometer on your car reads a steady 35 *mph*. Could you be accelerating? Explain.

Solution

Yes.

If the car is moving along a curved path, then $\kappa \neq 0$ and $a_N = \kappa\left|\vec{v}\right|^2 \neq 0$

$\Rightarrow \vec{a} = a_T\vec{T} + a_N\vec{N} \neq 0$

Exercise

Can anything be said about the acceleration of a particle that is moving at a constant speed? Give reasons for your answer.

Solution

$|v|$ is constant $\Rightarrow \vec{a}_T = \frac{d\vec{v}}{dt} = 0$

$\vec{a} = a_T\vec{T} + a_N\vec{N}$

$\quad = a_N\vec{N}$ is orthogonal to $\vec{T}$.

$\therefore$ The acceleration is normal to the path.

Exercise

Find ***T***, ***N***, ***B***, τ and κ as functions of t for the plane curves: $\vec{r}(t) = (\sin t)\hat{i} + (\sqrt{2}\cos t)\hat{j} + (\sin t)\hat{k}$, then write $\vec{a}$ of the motion $\vec{a} = a_T\vec{T} + a_N\vec{N}$

Solution

$\vec{v} = (\cos t)\hat{i} - (\sqrt{2}\sin t)\hat{j} + (\cos t)\hat{k}$ $\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$|\vec{v}| = \sqrt{\cos^2 t + 2\sin^2 t + \cos^2 t}$

$\quad = \sqrt{2\cos^2 t + 2\sin^2 t}$

$\quad = \sqrt{2}$

$\vec{T} = \left(\frac{\cos t}{\sqrt{2}}\right)\hat{i} - (\sin t)\hat{j} + \left(\frac{\cos t}{\sqrt{2}}\right)\hat{k}$

$\frac{d\vec{T}}{dt} = -\frac{\sin t}{\sqrt{2}}\hat{i} - (\cos t)\hat{j} - \frac{\sin t}{\sqrt{2}}\hat{k}$

$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{1}{2}\sin^2 t + \cos^2 t + \frac{1}{2}\sin^2 t}$

$\quad = \sqrt{\sin^2 t + \cos^2 t}$

$\quad = 1$

$\vec{N} = -\frac{\sin t}{\sqrt{2}}\hat{i} - (\cos t)\hat{j} - \frac{\sin t}{\sqrt{2}}\hat{k}$ $\qquad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$

$$\boldsymbol{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\cos t}{\sqrt{2}} & -\sin t & \frac{\cos t}{\sqrt{2}} \\ -\frac{\sin t}{\sqrt{2}} & -\cos t & -\frac{\sin t}{\sqrt{2}} \end{vmatrix}$$

$$= \left(\frac{\sin^2 t}{\sqrt{2}} + \frac{\cos^2 t}{\sqrt{2}} \right)\hat{i} - \left(-\frac{\sin t \cos t}{\sqrt{2}} + \frac{\sin t \cos t}{\sqrt{2}} \right)\hat{j} + \left(-\frac{\cos^2 t}{\sqrt{2}} - \frac{\sin^2 t}{\sqrt{2}} \right)\hat{k}$$

$$\boxed{= \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{k}}$$

$$\boxed{\kappa = \frac{1}{\sqrt{2}}} \qquad \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$\boxed{\vec{a} = (-\sin t)\hat{i} - \left(\sqrt{2}\cos t\right)\hat{j} - (\sin t)\,\hat{k}}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & -\sqrt{2}\sin t & \cos t \\ -\sin t & -\sqrt{2}\cos t & -\sin t \end{vmatrix}$$

$$= \left(\sqrt{2}\sin^2 t + \sqrt{2}\cos^2 t\right)\hat{i} + \left(-\sqrt{2}\cos^2 t - \sqrt{2}\sin^2 t\right)\hat{k}$$

$$= \left(\sqrt{2}\right)\hat{i} - \left(\sqrt{2}\right)\hat{k}$$

$$|\vec{v} \times \vec{a}| = \sqrt{2+2} = 2$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2} = \frac{\begin{vmatrix} \cos t & -\sqrt{2}\sin t & \cos t \\ -\sin t & -\sqrt{2}\cos t & -\sin t \\ -\cos t & \sqrt{2}\sin t & -\cos t \end{vmatrix}}{4}$$

$$= \frac{\sqrt{2}\cos^3 t - \sqrt{2}\sin^2 t\cos t - \sqrt{2}\sin^2 t\cos t - \sqrt{2}\cos^3 t + \sqrt{2}\sin^2 t\cos t + \sqrt{2}\sin^2 t\cos t}{4}$$

$$\boxed{= 0}$$

Exercise

Consider the ellipse $\vec{r}(t) = \langle 3\cos t,\ 4\sin t\rangle$ *for* $0 \le t \le 2\pi$

a) Find the tangent vector $\vec{r}'$, the unit vector $\vec{T}$, and the principal unit normal vector $\vec{N}$ at all points on the curve.

b) At what points does $|\vec{r}'|$ have maximum and minimum values?

c) At what points does the curvature have maximum and minimum values? Interpret this result in light of part (*b*).

d) Find the points (if any) at which $\vec{r}$ and $\vec{N}$ are parallel.

Solution

a) $\underline{\vec{r}'(t) = \langle -3\sin t,\ 4\cos t\rangle}$

$$\vec{T} = \frac{\langle -3\sin t,\ 4\cos t\rangle}{\sqrt{9\sin^2 t + 16\cos^2 t}} \qquad \vec{T} = \frac{r'(t)}{|r'(t)|}$$

$$= \frac{1}{\sqrt{9\sin^2 t + 9\cos^2 t + 7\cos^2 t}}\langle -3\sin t,\ 4\cos t\rangle$$

$$= \left\langle -\frac{3\sin t}{\sqrt{9+7\cos^2 t}},\ \frac{4\cos t}{\sqrt{9+7\cos^2 t}}\right\rangle$$

$$\frac{d\vec{T}}{dt} = \left\langle -3\frac{\cos t\left(9+7\cos^2 t\right) + 7\cos t\sin^2 t}{\left(9+7\cos^2 t\right)^{3/2}},\ 4\frac{-\sin t\left(9+7\cos^2 t\right) + 7\cos^2 t\sin t}{\left(9+7\cos^2 t\right)^{3/2}}\right\rangle$$

$$= \left\langle -3\frac{9\cos t + 7\cos^3 t + 7\cos t\sin^2 t}{\left(9+7\cos^2 t\right)^{3/2}},\ 4\frac{-9\sin t - 7\cos^2 t\sin t + 7\cos^2 t\sin t}{\left(9+7\cos^2 t\right)^{3/2}}\right\rangle$$

$$= \left\langle -3\cos t\frac{9+7\cos^2 t + 7\sin^2 t}{\left(9+7\cos^2 t\right)^{3/2}},\ -\frac{36\sin t}{\left(9+7\cos^2 t\right)^{3/2}}\right\rangle$$

$$= \left\langle -\frac{48\cos t}{\left(9+7\cos^2 t\right)^{3/2}},\ -\frac{36\sin t}{\left(9+7\cos^2 t\right)^{3/2}}\right\rangle$$

$$= -\frac{12}{\left(9+7\cos^2 t\right)^{3/2}}\langle 4\cos t,\ 3\sin t\rangle$$

$$\left|\frac{d\vec{T}}{dt}\right| = \frac{12}{\left(9+7\cos^2 t\right)^{3/2}}\sqrt{16\cos^2 t + 9\sin^2 t}$$

$$= \frac{12}{\left(9+7\cos^2 t\right)^{3/2}}\sqrt{9+7\cos^2 t}$$

$$= \frac{12}{\sqrt{9+7\cos^2 t}}$$

$$\vec{N} = \frac{\sqrt{9+7\cos^2 t}}{12}\frac{-12}{\left(9+7\cos^2 t\right)^{3/2}}\langle 4\cos t,\ \ 3\sin t\rangle \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= \left\langle -\frac{4\cos t}{\sqrt{9+7\cos^2 t}},\ \ -\frac{3\sin t}{\sqrt{9+7\cos^2 t}} \right\rangle$$

b) $\left|\vec{r}'(t)\right| = \sqrt{9\sin^2 t + 16\cos^2 t}$

$$\frac{d}{dt}\left|\vec{r}'(t)\right| = \frac{18\sin t\cos t - 32\cos t\sin t}{2\sqrt{9\sin^2 t + 16\cos^2 t}}$$

$$= \frac{-7\cos t\sin t}{\sqrt{9\sin^2 t + 16\cos^2 t}} = 0$$

$$\begin{cases} \cos t = 0 & \rightarrow\ t = \frac{\pi}{2},\ \frac{3\pi}{2} & \left|\vec{r}'(t)\right| = 3 & \textit{Minimum} \\ \sin t = 0 & \rightarrow\ t = 0,\ \pi & \left|\vec{r}'(t)\right| = 4 & \textit{Maximum} \end{cases}$$

c) $r''(t)\times\vec{r}'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\cos t & -4\sin t & 0 \\ -3\sin t & 4\cos t & 0 \end{vmatrix}$

$$= \left\langle 0,\ 0,\ -12\cos^2 t - 12\sin^2 t \right\rangle$$

$$= \langle 0,\ 0,\ -12\rangle$$

$$\tau = \frac{\left|\langle 0,\ 0,\ -12\rangle\right|}{\left(9+7\cos^2 t\right)^{3/2}} \qquad \kappa = \frac{\left|r''(t)\times\vec{r}'(t)\right|}{\left(\left|r'(t)\right|\right)^3}$$

$$\boxed{= \frac{12}{\left(9+7\cos^2 t\right)^{3/2}}}$$

For τ to be maximum the denominator has to be the smallest

$$\cos^2 t = 0 \ \rightarrow\ t = \frac{\pi}{2},\ \frac{3\pi}{2}$$

From part (*b*) result of *t*-values

$$\begin{cases} t = \frac{\pi}{2}, \ \frac{3\pi}{2} & \tau = \frac{12}{27} = \frac{4}{9} \quad \textit{Maximum} \\ t = 0, \ \pi & \tau = \frac{12}{64} = \frac{3}{16} \quad \textit{Minimum} \end{cases}$$

Velocity is maximized where curvature is minimal.
Velocity is maximized where curvature is minimal.

d) $\vec{r}(t) = \langle 3\cos t, \ 4\sin t \rangle \quad // \quad \vec{N} = \left\langle -\frac{4\cos t}{\sqrt{9+7\cos^2 t}}, \ -\frac{3\sin t}{\sqrt{9+7\cos^2 t}} \right\rangle$

$$3\cos t = -\frac{4\cos t}{\sqrt{9+7\cos^2 t}} \cdot m \ \rightarrow \ \cos t = 0$$

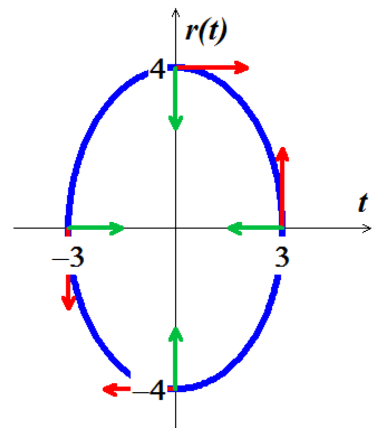
$$4\sin t = -\frac{3\sin t}{\sqrt{9+7\cos^2 t}} \cdot m \ \rightarrow \ \sin t = 0$$

$\cos t = 0 \ \rightarrow \ \vec{r}(t) = \langle 0, \ 4\sin t \rangle$

Points are: $(0, \ 4) \quad (0, \ -4)$

$\sin t = 0 \ \rightarrow \ \vec{r}(t) = \langle 3\cos t, \ 0 \rangle$

Points are: $(3, \ 0) \quad (-3, \ 0)$



Exercise

Find the following for all values of *t* for which the given curve is defined by

$$\vec{r}(t) = \langle 6\cos t, \ 3\sin t \rangle, \quad 0 \le t \le 2\pi$$

a) Find the tangent vector and the unit tangent vector

b) Find the curvature.

c) Find the principal unit normal vector.

d) Verify that $|\vec{N}| = 1 \quad and \quad \vec{T} \bullet \vec{N} = 0$

e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

Solution

a) $\vec{v}(t) = \langle -6\sin t, \ 3\cos t \rangle \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$$\begin{aligned} |\vec{v}(t)| &= \sqrt{36\sin^2 t + 9\cos^2 t} \\ &= 3\sqrt{4\sin^2 t + \cos^2 t} \\ &= 3\sqrt{1 + 3\sin^2 t} \end{aligned}$$

$$\vec{T} = \frac{\langle -6\sin t,\ 3\cos t\rangle}{3\sqrt{1+3\sin^2 t}} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\boxed{= \frac{1}{\sqrt{1+3\sin^2 t}}\langle -2\sin t,\ \cos t\rangle}$$

b) $$r''(t)\times \vec{r}'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6\cos t & -3\sin t & 0 \\ -6\sin t & 3\cos t & 0 \end{vmatrix}$$

$$= \left\langle 0,\ 0,\ -18\cos^2 t - 18\sin^2 t \right\rangle$$

$$= \langle 0,\ 0,\ -18\rangle$$

$$\tau = \frac{|\langle 0,\ 0,\ -18\rangle|}{\left(3\sqrt{1+3\sin^2 t}\right)^3} \qquad \kappa = \frac{|r''(t)\times \vec{r}'(t)|}{\left(|r'(t)|\right)^3}$$

$$= \frac{18}{27\left(1+3\sin^2 t\right)^{3/2}}$$

$$\boxed{= \frac{2}{3\left(1+3\sin^2 t\right)^{3/2}}}$$

c) $$\frac{d\vec{T}}{dt} = \left\langle -2\frac{\cos t\left(1+3\sin^2 t\right) - 3\cos t\sin^2 t}{\left(1+3\sin^2 t\right)^{3/2}},\ \frac{-\sin t\left(1+3\sin^2 t\right) - 3\cos^2 t\sin t}{\left(1+3\sin^2 t\right)^{3/2}} \right\rangle$$

$$= \left\langle \frac{-2\cos t}{\left(1+3\sin^2 t\right)^{3/2}},\ -\frac{\sin t + 3\sin^3 t - 3\cos^2 t\sin t}{\left(1+3\sin^2 t\right)^{3/2}} \right\rangle$$

$$= \left\langle \frac{-2\cos t}{\left(1+3\sin^2 t\right)^{3/2}},\ -\sin t\frac{1+3\left(\sin^2 t + \cos^2 t\right)}{\left(1+3\sin^2 t\right)^{3/2}} \right\rangle$$

$$= \left\langle \frac{-2\cos t}{\left(1+3\sin^2 t\right)^{3/2}},\ \frac{-4\sin t}{\left(1+3\sin^2 t\right)^{3/2}} \right\rangle$$

$$= \frac{-2}{\left(1+3\sin^2 t\right)^{3/2}}\langle \cos t,\ 2\sin t\rangle$$

$$\left|\frac{d\vec{T}}{dt}\right| = \frac{2}{\left(1+3\sin^2 t\right)^{3/2}}\sqrt{\cos^2 t + 4\sin^2 t}$$

$$= \frac{2}{\left(1+3\sin^2 t\right)^{3/2}}\sqrt{1+3\sin^2 t}$$

$$= \frac{2}{\sqrt{1+3\sin^2 t}}$$

$$\vec{N} = \frac{\sqrt{1+3\sin^2 t}}{2} = \frac{-2}{\left(1+3\sin^2 t\right)^{3/2}}\langle \cos t,\ \ 2\sin t\rangle \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= \frac{1}{\sqrt{1+3\sin^2 t}}\langle -\cos t,\ -2\sin t\rangle$$

d) $$\left|\vec{N}\right| = \frac{1}{\sqrt{1+3\sin^2 t}}\sqrt{\cos^2 t + 4\sin^2 t}$$

$$= \frac{1}{\sqrt{1+3\sin^2 t}}\sqrt{1+3\sin^2 t}$$

$$= 1 \quad \checkmark$$

$$\vec{T}\bullet\vec{N} = \frac{1}{\sqrt{1+3\sin^2 t}}\langle -2\sin t,\ \cos t\rangle \bullet \frac{1}{\sqrt{1+3\sin^2 t}}\langle -\cos t,\ -2\sin t\rangle$$

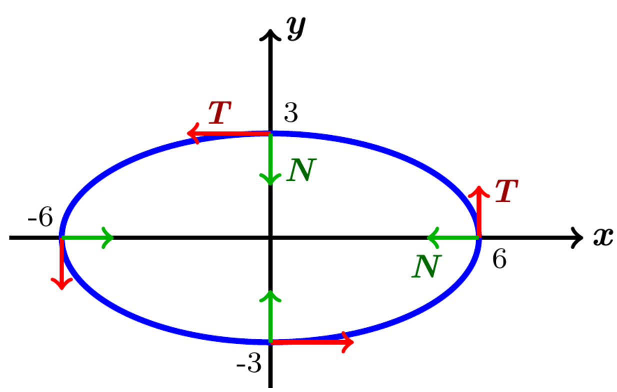
$$= \frac{2\sin t\cos t - 2\sin t\cos t}{1+3\sin^2 t}$$

$$= 0 \quad \checkmark$$

e) $x = 6\cos t \quad y = 3\sin t$

$$\cos^2 t + \sin^2 t = 1$$

$$\frac{x^2}{36} + \frac{y^2}{9} = 1$$



Exercise

Find the following for all values of *t* for which the given curve is defined by

$$\vec{r}(t) = (\cos t)\hat{i} + (2\sin t)\hat{j} + \hat{k}, \quad 0 \le t \le 2\pi$$

a) Find the tangent vector and the unit tangent vector

b) Find the curvature.

c) Find the principal unit normal vector.

d) Verify that $|\vec{N}| = 1$ *and* $\vec{T} \bullet \vec{N} = 0$

e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

Solution

a) $\vec{v}(t) = \langle -\sin t,\ 2\cos t,\ 0\rangle \qquad \vec{v}(t) = \dfrac{d\vec{r}}{dt}$

$$|\vec{v}(t)| = \sqrt{\sin^2 t + 4\cos^2 t}$$

$$= \sqrt{1 + 3\cos^2 t}$$

$$\boxed{\vec{T} = \frac{\langle -\sin t,\ 2\cos t,\ 0\rangle}{\sqrt{1+3\cos^2 t}}} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

b) $r''(t) \times \vec{r}'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\cos t & -2\sin t & 0 \\ -\sin t & 2\cos t & 0 \end{vmatrix}$

$$= \left\langle 0,\ 0,\ -2\cos^2 t - 2\sin^2 t \right\rangle$$

$$= \langle 0,\ 0,\ -2\rangle$$

$$\kappa = \frac{|\langle 0,\ 0,\ -2\rangle|}{\left(\sqrt{1+3\cos^2 t}\right)^3} \qquad \kappa = \frac{|r''(t) \times \vec{r}'(t)|}{\left(|r'(t)|\right)^3}$$

$$\boxed{= \frac{2}{\left(1+3\cos^2 t\right)^{3/2}}}$$

c) $\dfrac{d\vec{T}}{dt} = \left\langle -\dfrac{\cos t\left(1+3\cos^2 t\right) + 3\cos t\sin^2 t}{\left(1+3\cos^2 t\right)^{3/2}},\ 2\dfrac{-\sin t\left(1+3\cos^2 t\right) + 3\cos^2 t\sin t}{\left(1+3\cos^2 t\right)^{3/2}},\ 0\right\rangle$

$$= \frac{-1}{\left(1+3\cos^2 t\right)^{3/2}}\left\langle \cos t + 3\cos^3 t + 3\cos t\sin^2 t,\ \ 2\sin t,\ 0\right\rangle$$

$$= \frac{-2}{\left(1+3\cos^2 t\right)^{3/2}} \langle 2\cos t,\ \ \sin t,\ 0\rangle$$

$$\left|\frac{d\vec{T}}{dt}\right| = \frac{2}{\left(1+3\cos^2 t\right)^{3/2}} \sqrt{4\cos^2 t + \sin^2 t}$$

$$= \frac{2}{\left(1+3\cos^2 t\right)^{3/2}} \sqrt{3\cos^2 t + 1}$$

$$= \frac{2}{\sqrt{1+3\cos^2 t}}$$

$$\vec{N} = \frac{\sqrt{1+3\cos^2 t}}{2} \cdot \frac{-2}{\left(1+3\cos^2 t\right)^{3/2}} \langle 2\cos t,\ \ \sin t,\ 0\rangle \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= \frac{1}{\sqrt{1+3\cos^2 t}} \langle -2\cos t,\ -\sin t,\ 0\rangle$$

d) $$\left|\vec{N}\right| = \frac{1}{\sqrt{1+3\cos^2 t}} \sqrt{4\cos^2 t + \sin^2 t}$$

$$= \frac{1}{\sqrt{1+3\cos^2 t}} \sqrt{1+3\cos^2 t}$$

$$= 1 \quad \checkmark$$

$$\vec{T} \bullet \vec{N} = \frac{\langle -\sin t,\ 2\cos t,\ 0\rangle}{\sqrt{1+3\cos^2 t}} \bullet \frac{\langle -2\cos t,\ -\sin t,\ 0\rangle}{\sqrt{1+3\cos^2 t}}$$

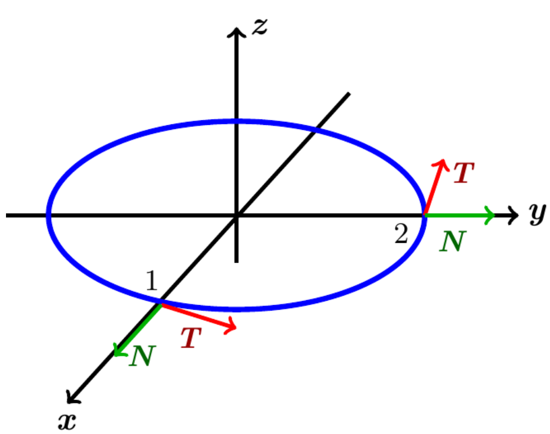
$$= \frac{2\sin t\cos t - 2\sin t\cos t}{1+3\cos^2 t}$$

$$= 0 \quad \checkmark$$

e) $x = \cos t \quad y = 2\sin t$

$\cos^2 t + \sin^2 t = 1$

$x^2 + \frac{y^2}{4} = 1$



Exercise

Find the following for all values of t for which the given curve is defined by

$$\vec{r}(t) = t\hat{i} + (2\cos t)\hat{j} + (2\sin t)\hat{k}, \quad 0 \le t \le 2\pi$$

a) Find the tangent vector and the unit tangent vector

b) Find the curvature.

c) Find the principal unit normal vector.

d) Verify that $|\vec{N}| = 1 \quad \text{and} \quad \vec{T} \bullet \vec{N} = 0$

e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

Solution

a) $\vec{v}(t) = \langle 1,\ -2\sin t,\ 2\cos t\rangle \qquad \vec{v}(t) = \dfrac{d\vec{r}}{dt}$

$$|\vec{v}(t)| = \sqrt{1 + 4\sin^2 t + 4\cos^2 t}$$

$$= \sqrt{5}$$

$$\boxed{\vec{T} = \frac{1}{\sqrt{5}}\langle 1,\ -2\sin t,\ 2\cos t\rangle} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

b) $r''(t) \times \vec{r}'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2\cos t & -2\sin t \\ 1 & -2\sin t & 2\cos t \end{vmatrix}$

$$= \langle -4,\ -2\sin t,\ 2\cos t\rangle$$

$$\tau = \frac{|\langle -4,\ -2\sin t,\ 2\cos t\rangle|}{(\sqrt{5})^3} \qquad \kappa = \frac{|r''(t) \times \vec{r}'(t)|}{(|r'(t)|)^3}$$

$$= \frac{\sqrt{16 + 4\sin^2 t + 4\cos^2 t}}{5\sqrt{5}}$$

$$= \frac{\sqrt{20}}{5\sqrt{5}}$$

$$\boxed{= \frac{2}{5}}$$

c) $\dfrac{d\vec{T}}{dt} = \dfrac{1}{\sqrt{5}}\langle 0,\ -2\cos t,\ -2\sin t\rangle$

$$\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{\sqrt{5}}\sqrt{4\cos^2 t + 4\sin^2 t}$$

$$= \frac{2}{\sqrt{5}}$$

$$\vec{N} = \frac{\sqrt{5}}{2} \cdot \frac{1}{\sqrt{5}}\langle 0,\ -2\cos t,\ -2\sin t\rangle \qquad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$= \langle 0,\ -\cos t,\ -\sin t \rangle$$

d) $\left|\vec{N}\right| = \sqrt{\cos^2 t + \sin^2 t}$

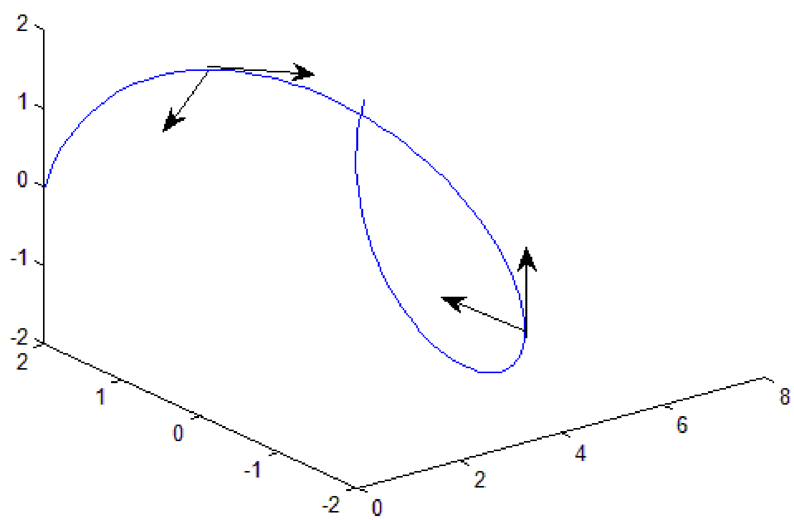
$$= 1 \quad \checkmark$$

$$\vec{T} \bullet \vec{N} = \frac{1}{\sqrt{5}} \langle 1,\ -2\sin t,\ 2\cos t \rangle \bullet \langle 0,\ -\cos t,\ -\sin t \rangle$$

$$= \frac{1}{\sqrt{5}} (2\sin t \cos t - 2\sin t \cos t)$$

$$= 0 \quad \checkmark$$

e)



Exercise

Find the following for all values of *t* for which the given curve is defined by

$\vec{r}(t) = (\cos t)\hat{i} + (2\cos t)\hat{j} + \left(\sqrt{5}\sin t\right)\hat{k}, \quad 0 \le t \le 2\pi$

a) Find the tangent vector and the unit tangent vector

b) Find the curvature.

c) Find the principal unit normal vector.

d) Verify that $\left|\vec{N}\right| = 1 \quad and \quad \vec{T} \bullet \vec{N} = 0$

e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

Solution

a) $\vec{v}(t) = \left\langle -\sin t,\ -2\sin t,\ \sqrt{5}\cos t \right\rangle \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$$\left|\vec{v}(t)\right| = \sqrt{\sin^2 t + 4\sin^2 t + 5\cos^2 t}$$

$$= \sqrt{5\sin^2 t + 5\cos^2 t}$$

$$= \sqrt{5}$$

$$\boxed{\vec{T} = \frac{1}{\sqrt{5}} \left\langle -\sin t,\ -2\sin t,\ \sqrt{5}\cos t \right\rangle}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

b) $\vec{r}''(t) = \left\langle -\cos t,\ -2\cos t,\ -\sqrt{5}\sin t \right\rangle$

$$r''(t) \times \vec{r}'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\cos t & -2\cos t & -\sqrt{5}\sin t \\ -\sin t & -2\sin t & \sqrt{5}\cos t \end{vmatrix}$$

$$= \left(-2\sqrt{5}\cos^2 t - 2\sqrt{5}\sin^2 t\right)\hat{i} - \left(-\sqrt{5}\cos^2 t - \sqrt{5}\sin^2 t\right)\hat{j}$$

$$+ \left(2\sin t\cos t - 2\sin t\cos t\right)\hat{k}$$

$$= -2\sqrt{5}\hat{i} + \sqrt{5}\hat{j}$$

$$\left| r''(t) \times \vec{r}'(t) \right| = \sqrt{20+5}$$

$$\boxed{= 5}$$

$$\kappa = \frac{5}{\left(\sqrt{5}\right)^3}$$

$$\kappa = \frac{\left| r''(t) \times \vec{r}'(t) \right|}{\left(\left| r'(t) \right|\right)^3}$$

$$\boxed{= \frac{1}{\sqrt{5}}}$$

c) $\dfrac{d\vec{T}}{dt} = \dfrac{1}{\sqrt{5}} \left\langle -\cos t,\ -2\cos t,\ -\sqrt{5}\sin t \right\rangle$

$$\left| \frac{d\vec{T}}{dt} \right| = \frac{1}{\sqrt{5}} \sqrt{\cos^2 t + 4\cos^2 t + 5\sin^2 t}$$

$$= \frac{1}{\sqrt{5}} \sqrt{5\cos^2 t + 5\sin^2 t}$$

$$= \frac{\sqrt{5}}{\sqrt{5}}$$

$$\boxed{= 1}$$

$$\boxed{\vec{N} = \frac{1}{\sqrt{5}} \left\langle -\cos t,\ -2\cos t,\ -\sqrt{5}\sin t \right\rangle}$$

$$\vec{N} = \frac{d\vec{T}/dt}{\left| d\vec{T}/dt \right|}$$

d) $\left| \vec{N} \right| = \dfrac{1}{\sqrt{5}} \sqrt{\cos^2 t + 4\cos^2 t + 5\sin^2 t}$

$$= \frac{1}{\sqrt{5}} \sqrt{5}$$

$$\boxed{= 1} \quad \checkmark$$

$$\vec{T} \cdot \vec{N} = \frac{\left\langle -\sin t,\ -2\sin t,\ \sqrt{5}\cos t \right\rangle}{\sqrt{5}} \cdot \frac{\left\langle -\cos t,\ -2\cos t,\ -\sqrt{5}\sin t \right\rangle}{\sqrt{5}}$$

$$= \frac{\sin t \cos t + 4\sin t \cos t - 5\cos t \sin t}{5}$$

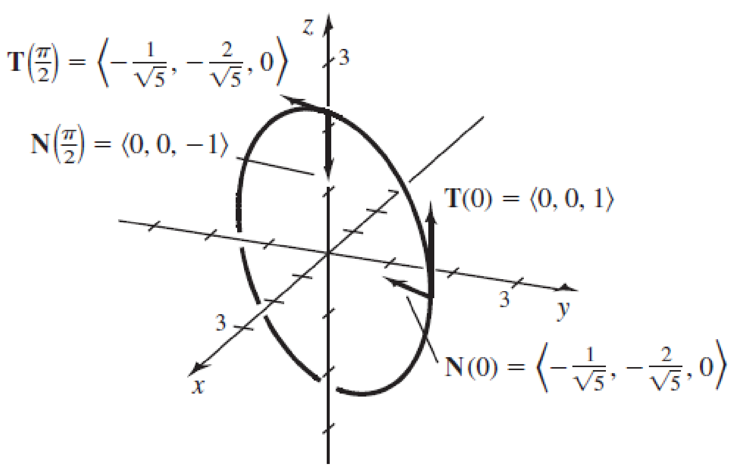
$$= 0 \quad \checkmark$$

e) $\vec{r}(t) = (\cos t)\hat{i} + (2\cos t)\hat{j} + \left(\sqrt{5}\sin t\right)\hat{k}$

$\vec{T} = \frac{1}{\sqrt{5}}\left\langle -\sin t,\ -2\sin t,\ \sqrt{5}\cos t \right\rangle$

$\vec{N} = \frac{1}{\sqrt{5}}\left\langle -\cos t,\ -2\cos t,\ -\sqrt{5}\sin t \right\rangle$

t	$\vec{r}$	$\vec{T}$	$\vec{N}$
0	$(1,\ 1,\ 0)$	$(0,\ 0,\ 1)$	$\left(-\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}},\ 0\right)$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2},\ \sqrt{2},\ \sqrt{\frac{5}{2}}\right)$		
$\frac{\pi}{2}$	$\left(0,\ 0,\ \sqrt{5}\right)$	$\left(-\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}},\ 0\right)$	$(0,\ 0,\ -1)$
π	$(-1,\ -1,\ 0)$		
$\frac{3\pi}{2}$	$\left(0,\ 0,\ -\sqrt{5}\right)$		
2π	$(1,\ 1,\ 0)$		



Exercise

Find equations for the osculating, normal and rectifying planes of the curve $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$ at the point (1, 1, 1).

Solution

$$\vec{v}(t) = \hat{i} + 2t\hat{j} + 3t^2\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

$$|\vec{v}(1)| = \sqrt{1+4+9}$$

$$= \sqrt{14}$$

$$\vec{T}(t) = \frac{1}{\sqrt{14}}\left(\hat{i} + 2t\hat{j} + 3t^2\hat{k}\right) \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{T}(1) = \frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$$ (Normal to the normal plane).

$$\frac{1}{\sqrt{14}}(x-1) + \frac{2}{\sqrt{14}}(y-1) + \frac{3}{\sqrt{14}}(z-1) = 0$$

$$x - 1 + 2(y-1) + 3(z-1) = 0$$

$\boxed{x + 2y + 3z = 6}$ (equation of the normal plane).

$$\vec{a}(t) = 2\hat{j} + 6t\hat{k} \qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$\vec{a}(1) = 2\hat{j} + 6\hat{k}$$

$$(\vec{v}\times\vec{a})(1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix}$$

$$= 6\hat{i} - 6\hat{j} + 2\hat{k}$$

$$|\vec{v}\times\vec{a}| = \sqrt{36+36+4}$$

$$= \sqrt{76}$$

$$\kappa = \frac{\sqrt{76}}{\left(\sqrt{14}\right)^3} \qquad \kappa = \frac{|\vec{v}\times\vec{a}|}{\left(|\vec{v}|\right)^3}$$

$$= \frac{2\sqrt{19}}{14\sqrt{14}}$$

$$\boxed{= \frac{\sqrt{19}}{7\sqrt{14}}}$$

$$\frac{ds}{dt} = |\vec{v}(t)| = \sqrt{1+4t^2+9t^4}$$

$$\frac{ds}{dt}(1) = \sqrt{14}$$

$$\frac{d^2s}{dt^2} = \left.\frac{4t + 18t^3}{\sqrt{1+4t^2+9t^4}}\right|_{t=1}$$

$$= \frac{22}{\sqrt{14}}$$

$$\vec{a} = \frac{d^2s}{dt^2}\vec{T} + \kappa\left(\frac{ds}{dt}\right)^2 \vec{N}$$

$$2\hat{j} + 6\hat{k} = \frac{22}{\sqrt{14}}\left(\frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}\right) + \frac{\sqrt{19}}{7\sqrt{14}}\left(\sqrt{14}\right)^2 \vec{N}$$

$$2\hat{j} + 6\hat{k} = \frac{11}{7}\left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \frac{2\sqrt{19}}{\sqrt{14}}\vec{N}$$

$$\frac{2\sqrt{19}}{\sqrt{14}}\vec{N} = 2\hat{j} + 6\hat{k} - \frac{11}{7}\hat{i} - \frac{22}{7}\hat{j} - \frac{33}{7}\hat{k}$$

$$\frac{2\sqrt{19}}{\sqrt{14}}\vec{N} = -\frac{11}{7}\hat{i} - \frac{8}{7}\hat{j} + \frac{9}{7}\hat{k}$$

$$\vec{N} = \frac{\sqrt{14}}{2\sqrt{19}}\left(-\frac{11}{7}\hat{i} - \frac{8}{7}\hat{j} + \frac{9}{7}\hat{k}\right)$$

$$-\frac{11}{7}(x-1) - \frac{8}{7}(y-1) + \frac{9}{7}(z-1) = 0$$

$$-11x + 11 - 8y + 8 + 9z - 9 = 0$$

$$11x + 8y - 9z = 10$$

$$\vec{B}(1) = \vec{T}(1) \times \vec{N}(1)$$

$$= \frac{1}{\sqrt{14}} \cdot \frac{\sqrt{14}}{2\sqrt{19}} \cdot \frac{1}{7}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -11 & -8 & 9 \end{vmatrix}$$

$$= \frac{1}{14\sqrt{19}}\left(42\hat{i} - 42\hat{j} + 14\hat{k}\right)$$

$$= \frac{1}{\sqrt{19}}\left(3\hat{i} - 3\hat{j} + \hat{k}\right)$$

$$3(x-1) - 3(y-1) + (z-1) = 0$$

$$3x - 3 - 3y + 3 + z - 1 = 0$$

$$3x - 3y + z = 1$$

Exercise

Consider the position vector $\vec{r}(t) = \left(t^2 + 1\right)\hat{i} + (2t)\hat{j}, \quad t \geq 0$ of the moving objects

a) Find the normal and tangential components of the acceleration.

b) Graph the trajectory and sketch the normal and tangential components of the acceleration at two points on the trajectory. Show that their sum gives the total acceleration.

Solution

a) $\vec{v}(t) = 2t\hat{i} + 2\hat{j}$ $\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$$\left|\vec{v}(t)\right| = \sqrt{4t^2+4}$$

$$= 2\sqrt{t^2+1}$$

$$\vec{T}(t) = \frac{2t\hat{i}+2\hat{j}}{2\sqrt{t^2+1}} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{t}{\sqrt{t^2+1}}\hat{i} + \frac{1}{\sqrt{t^2+1}}\hat{j}$$

$$\frac{d\vec{T}}{dt} = \frac{t^2+1-t^2}{\left(t^2+1\right)^{3/2}}\hat{i} - \frac{t}{\left(t^2+1\right)^{3/2}}\hat{j}$$

$$= \frac{1}{\left(t^2+1\right)^{3/2}}\hat{i} - \frac{t}{\left(t^2+1\right)^{3/2}}\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{1}{\left(t^2+1\right)^3} + \frac{t^2}{\left(t^2+1\right)^3}}$$

$$= \sqrt{\frac{1+t^2}{\left(t^2+1\right)^3}}$$

$$= \frac{1}{t^2+1}$$

$$\vec{N} = \left(t^2+1\right)\left(\frac{1}{\left(t^2+1\right)^{3/2}}\hat{i} - \frac{t}{\left(t^2+1\right)^{3/2}}\hat{j}\right)$$

$$= \frac{1}{\sqrt{t^2+1}}\hat{i} - \frac{t}{\sqrt{t^2+1}}\hat{j}$$

$$\vec{a}(t) = 2\hat{i} \qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$a_T = \frac{2t}{\sqrt{t^2+1}} \qquad a_T = \frac{d}{dt}|\vec{v}|$$

$$\vec{v}\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t & 2 & 0 \\ 2 & 0 & 0 \end{vmatrix}$$

$$= -4\hat{k}$$

$$\left|\vec{v}\times\vec{a}\right| = 4$$

$$\kappa = \frac{4}{8\left(\sqrt{t^2+1}\right)^3} \qquad\qquad \kappa = \frac{|\vec v \times \vec a|}{\left(|\vec v|\right)^3}$$

$$= \frac{1}{2\left(t^2+1\right)^{3/2}}$$

$$a_N = \frac{4}{2\sqrt{t^2+1}} \qquad\qquad a_N = \kappa|\vec v|^2 = \frac{|\vec v \times \vec a|}{|\vec v|}$$

$$= \frac{2}{\sqrt{t^2+1}}$$

$$\boxed{\vec a = \frac{2t}{\sqrt{t^2+1}}\vec T + \frac{2}{\sqrt{t^2+1}}\vec N}$$

b) $x = t^2 + 1 \quad y = 2t$

$t = \frac{1}{2}y \quad \rightarrow \quad \boxed{x = \frac{1}{4}y^2 + 1}$

At $t = 1$

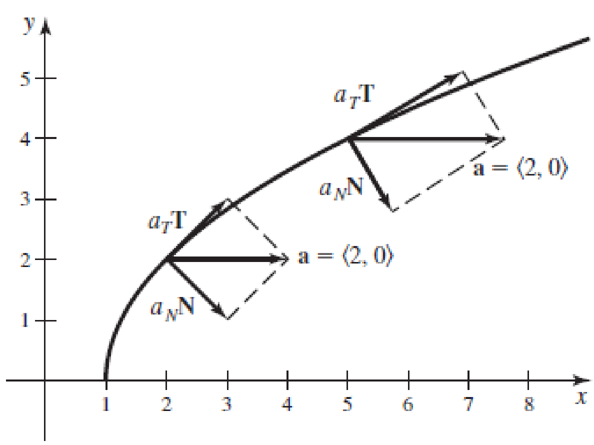
$$\vec a = \frac{2}{\sqrt 2}\vec T + \frac{2}{\sqrt 2}\vec N$$

$$= \frac{2}{\sqrt 2}\left(\frac{1}{\sqrt 2}\hat i + \frac{1}{\sqrt 2}\hat j\right) + \frac{2}{\sqrt 2}\left(\frac{1}{\sqrt 2}\hat i - \frac{1}{\sqrt 2}\hat j\right)$$

$$= \hat i + \hat j + \hat i - \hat j$$

$$\boxed{= 2\hat i}$$

$$= \langle 2,\ 0\rangle$$



At $t = 2$

$$\vec a = \frac{4}{\sqrt 5}\left(\frac{2}{\sqrt 5}\hat i + \frac{1}{\sqrt 5}\hat j\right) + \frac{2}{\sqrt 5}\left(\frac{1}{\sqrt 5}\hat i - \frac{2}{\sqrt 5}\hat j\right)$$

$$= \frac{8}{5}\hat i + \frac{4}{5}\hat j + \frac{2}{5}\hat i - \frac{4}{5}\hat j$$

$$\boxed{= 2\hat i} \quad = \langle 2,\ 0\rangle$$

At $t = 5$

$$\vec a = \frac{10}{\sqrt{26}}\left(\frac{5}{\sqrt{26}}\hat i + \frac{1}{\sqrt{26}}\hat j\right) + \frac{2}{\sqrt{26}}\left(\frac{1}{\sqrt{26}}\hat i - \frac{5}{\sqrt{26}}\hat j\right)$$

$$= \frac{50}{26}\hat i + \frac{10}{26}\hat j + \frac{2}{26}\hat i - \frac{10}{26}\hat j$$

$$= \frac{52}{26}\hat i$$

$$\boxed{= 2\hat i} \quad = \langle 2,\ 0\rangle$$

Exercise

Consider the position vector $\vec{r}(t)=(2\cos t)\hat{i}+(2\sin t)\hat{j}, \quad 0\le t\le 2\pi$ of the moving objects

c) Find the normal and tangential components of the acceleration.

d) Graph the trajectory and sketch the normal and tangential components of the acceleration at two points on the trajectory. Show that their sum gives the total acceleration.

Solution

a) $\vec{v}(t)=-2\sin t\,\hat{i}+\ 2\cos t\,\hat{j}$ $\qquad \vec{v}(t)=\frac{d\vec{r}}{dt}$

$\vec{a}(t)=-2\cos t\,\hat{i}-\ 2\sin t\,\hat{j}$ $\qquad \vec{a}(t)=\frac{d\vec{v}}{dt}$

$$|\vec{v}|=\sqrt{4\cos^2 t+4\sin^2 t}$$

$$=\sqrt{2}$$

$$a_T=\frac{d|\vec{v}|}{dt}\ \underline{=0}$$

$$\vec{v}\times\vec{a}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2\sin t & 2\cos t & 0\\ -2\cos t & -2\sin t & 0\end{vmatrix}$$

$$=\left(4\sin^2 t+4\cos^2 t\right)\hat{k}$$

$$\underline{=4\,\hat{k}}$$

$a_N=\frac{4}{2}\ \underline{=2}$ $\qquad a_N=\frac{|\vec{v}\times\vec{a}|}{|\vec{v}|}$

$$\underline{\vec{a}=0\vec{T}+2\vec{N}}$$

b) $t=0 \quad\to\quad \vec{a}=-2\,\hat{i}$

$$=2\langle -1,\ 0\rangle$$

$$=2\vec{N}$$

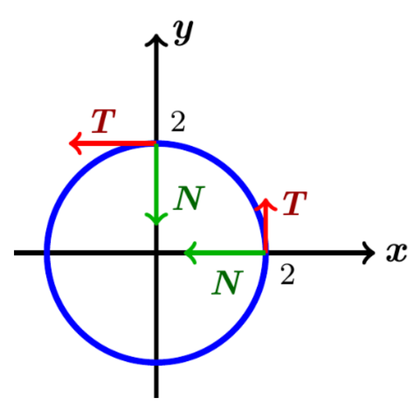
$t=\frac{\pi}{2} \quad\to\quad \vec{a}=\langle 0,\ -2\rangle$

$$=2\langle 0,\ -1\rangle$$

$$=2\vec{N}$$

$x=2\cos t \quad y=2\sin t$

$x^2+y^2=4$



Exercise

Consider the position vector $\vec{r}(t)=3t\,\hat{i}+(4-t)\,\hat{j}+t\,\hat{k}, \quad t\geq 0$ of the moving objects

Find the normal and tangential components of the acceleration.

Solution

a) $\vec{v}(t)=3\hat{i}-\hat{j}+\hat{k}$ $\qquad \vec{v}(t)=\frac{d\vec{r}}{dt}$

$\vec{a}(t)=\vec{0}$ $\qquad \vec{a}(t)=\frac{d\vec{v}}{dt}$

$$|\vec{v}|=\sqrt{9+1+1}$$

$$=\sqrt{11}$$

$$a_T=\frac{d|\vec{v}|}{dt}\ \underline{=0\,|}$$

$$a_N\ \underline{=0\,|}$$

$$\underline{\vec{a}=0\vec{T}+0\vec{N}\,|}$$

Exercise

Consider the position vector $\vec{r}(t)=(2\cos t)\hat{i}+(2\sin t)\hat{j}+(10t)\hat{k}, \quad 0\leq t\leq 2\pi$ of the moving objects

a) Find the normal and tangential components of the acceleration.

b) Graph the trajectory and sketch the normal and tangential components of the acceleration at two points on the trajectory. Show that their sum gives the total acceleration.

Solution

a) $\vec{v}(t)=-2\sin t\,\hat{i}+\ 2\cos t\,\hat{j}+10\hat{k}$ $\qquad \vec{v}(t)=\frac{d\vec{r}}{dt}$

$\vec{a}(t)=-2\cos t\,\hat{i}-\ 2\sin t\,\hat{j}$ $\qquad \vec{a}(t)=\frac{d\vec{v}}{dt}$

$$|\vec{v}|=\sqrt{4\sin^2 t+4\cos^2 t+100}$$

$$=2\sqrt{26}$$

$$a_T=\frac{d|\vec{v}|}{dt}\ \underline{=0\,|}$$

$$\vec{v}\times\vec{a}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2\sin t & 2\cos t & 10\\ -2\cos t & -2\sin t & 0\end{vmatrix}$$

$$=-20\sin t\,\hat{i}+20\cos t\,\hat{j}-4\,\hat{k}$$

$$a_N=\frac{\sqrt{400\sin^2 t+400\cos^2 t+16}}{2\sqrt{26}} \qquad a_N=\frac{|\vec{v}\times\vec{a}|}{|\vec{v}|}$$

$$= \frac{\sqrt{416}}{2\sqrt{26}}$$

$$= \frac{4\sqrt{26}}{2\sqrt{26}}$$

$$= 2$$

$$\vec{a} = 0\vec{T} + 2\vec{N}$$

b) $t = 0 \quad \to \quad \vec{a} = \langle -2,\ 0,\ 0 \rangle$

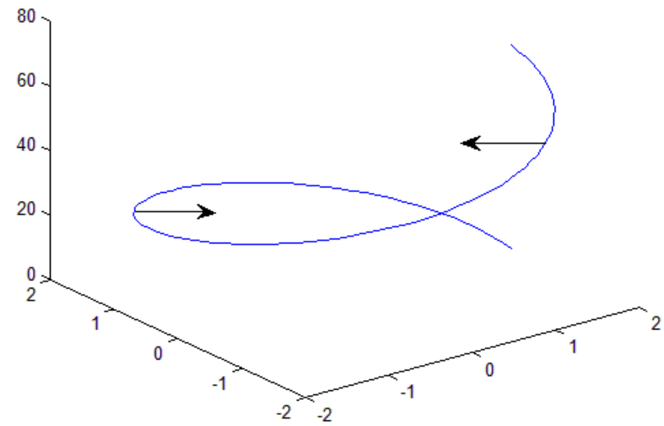
$$= 2\langle -1,\ 0,\ 0 \rangle$$

$$2\vec{N}$$

$t = \frac{\pi}{2} \quad \to \quad \vec{a} = \langle 0,\ -2,\ 0 \rangle$

$$= 2\langle 0,\ -1,\ 0 \rangle$$

$$2\vec{N}$$



Exercise

Compute the unit binormal vector ***B*** and the torsion of the curve $\vec{r}(t) = \langle t,\ t^2,\ t^3 \rangle$, *at* $t = 1$

Solution

$$\vec{v}(t) = \langle 1,\ 2t,\ 3t^2 \rangle \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\vec{v}(t=1) = \langle 1,\ 2,\ 3 \rangle$$

$$|\vec{v}(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

$$|\vec{v}(t=1)| = \sqrt{1+4+9}$$

$$= \sqrt{14}$$

$$\vec{T} = \frac{1}{\sqrt{1+4t^2+9t^4}}\langle 1,\ 2t,\ 3t^2 \rangle \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{T}(t=1) = \frac{1}{\sqrt{14}}\langle 1,\ 2,\ 3 \rangle$$

$$\frac{d\vec{T}}{dt} = \frac{1}{\left(1+4t^2+9t^4\right)^{3/2}}\left\langle -4t - 18t^3,\ 2 + 8t^2 + 18t^4 - 8t^2 - 36t^4,\ 3t\left(2 + 8t^2 + 18t^4 - 4t^2 - 18t^4\right)\right\rangle$$

$$= \frac{1}{\left(1+4t^2+9t^4\right)^{3/2}}\left\langle -4t - 18t^3,\ 2 - 18t^4,\ 6t + 12t^3 \right\rangle$$

$$= \frac{2}{\left(1+4t^2+9t^4\right)^{3/2}} \left\langle -2t-9t^3,\ 1-9t^4,\ 3t+6t^3 \right\rangle$$

$$\left|\frac{d\vec{T}}{dt}\right| = \frac{2}{\left(1+4t^2+9t^4\right)^{3/2}} \sqrt{\left(-2t-9t^3\right)^2 + \left(1-9t^4\right)^2 + \left(3t+6t^3\right)^2}$$

$$= \frac{2}{\left(1+4t^2+9t^4\right)^{3/2}} \sqrt{4t^2+36t^4+81t^6+1-18t^4+81t^8+9t^2+36t^4+36t^6}$$

$$= \frac{2}{\left(1+4t^2+9t^4\right)^{3/2}} \sqrt{1+13t^2+54t^4+117t^6+81t^8}$$

$$\vec{N} = \frac{\left(1+4t^2+9t^4\right)^{3/2}}{2\sqrt{1+13t^2+54t^4+117t^6+81t^8}} \frac{2}{\left(1+4t^2+9t^4\right)^{3/2}} \left\langle -2t-9t^3,\ 1-9t^4,\ 3t+6t^3 \right\rangle \quad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= \frac{1}{\sqrt{1+13t^2+54t^4+117t^6+81t^8}} \left\langle -2t-9t^3,\ 1-9t^4,\ 3t+6t^3 \right\rangle$$

$$\vec{N}(t=1) = \frac{1}{\sqrt{1+13+54+117+81}} \langle -11,\ -8,\ 9 \rangle$$

$$= \frac{1}{\sqrt{266}} \langle -11,\ -8,\ 9 \rangle$$

$$\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ \frac{-11}{\sqrt{266}} & \frac{-8}{\sqrt{266}} & \frac{9}{\sqrt{266}} \end{vmatrix} \qquad \vec{B} = \vec{T} \times \vec{N}$$

$$= \left\langle \frac{42}{14\sqrt{19}},\ \frac{-42}{14\sqrt{19}},\ \frac{14}{14\sqrt{19}} \right\rangle$$

$$= \left\langle \frac{3}{\sqrt{19}},\ \frac{-3}{\sqrt{19}},\ \frac{1}{\sqrt{19}} \right\rangle$$

$$\vec{r}''(t) = \langle 0,\ 2,\ 6t \rangle$$

$$\vec{r}'' \times \vec{r}' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 6t \\ 1 & 2t & 3t^2 \end{vmatrix}$$

$$= \left\langle -6t^2,\ 6t,\ -2 \right\rangle$$

$$\left|\vec{r}''\times\vec{r}'\right| = \sqrt{36t^4+36t^2+4}\,\Big|_{t=1}$$

$$= \sqrt{76}$$

$$\vec{r}'''(t) = \langle 0,\ 0,\ 6\rangle$$

$$\tau = \frac{1}{76}\begin{vmatrix} 1 & 2t & 3t^2 \\ 0 & 2 & 6t \\ 0 & 0 & 6 \end{vmatrix} \qquad\qquad \tau = \frac{1}{\left|\vec{r}'\times\vec{r}''\right|^2}\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}$$

$$= \frac{12}{76}$$

$$\boxed{= \frac{3}{19}}$$

Exercise

At point *P*, the velocity and acceleration of a particle moving in the plane are $\vec{v} = 3\hat{i}+4\hat{j}$ and $\vec{a} = 5\hat{i}+15\hat{j}$. Find the curvature of the particle's path at *P*.

Solution

$$\vec{v}\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ 5 & 15 & 0 \end{vmatrix}$$

$$\boxed{= 25\hat{k}}$$

$$\left|\vec{v}\times\vec{a}\right| = 25$$

$$\left|\vec{v}\right| = \sqrt{9+16}$$

$$\boxed{= 5}$$

$$\kappa = \frac{25}{5^3}$$

$$\boxed{= \frac{1}{5}}$$

Exercise

Consider the curve $C:\ \vec{r}(t) = \langle 3\sin t,\ 4\sin t,\ 5\cos t\rangle, \quad \mathit{for}\ 0\le t\le 2\pi$

a) Find $\boldsymbol{T}(t)$ at all points of *C*.

b) Find $\boldsymbol{N}(t)$ and the curvature at all points of *C*.

c) Sketch the curve and show $\boldsymbol{T}(t)$ and $\boldsymbol{N}(t)$ at the points of *C* corresponding to $t=0$ and $t=\frac{\pi}{2}$.

d) Are the results of parts (a) and (b) consistent with the graph?

e) Find $\boldsymbol{B}(t)$ at all points of C.

f) Describe three calculations that serve to check the accuracy of your results in part (a) – (f).

g) Compute the torsion at all points of C. Interpret this result.

Solution

a) $\vec{v}(t)=\langle 3\cos t,\ 4\cos t,\ -5\sin t\rangle$ $\qquad \vec{v}(t)=\frac{d\vec{r}}{dt}$

$$|\vec{v}(t)|=\sqrt{9\cos^2 t+16\cos^2 t+25\sin^2 t}$$

$$=\sqrt{25\cos^2 t+25\sin^2 t}$$

$$=5$$

$$\boxed{\vec{T}=\frac{1}{5}\langle 3\cos t,\ 4\cos t,\ -5\sin t\rangle} \qquad \vec{T}=\frac{\vec{v}}{|\vec{v}|}$$

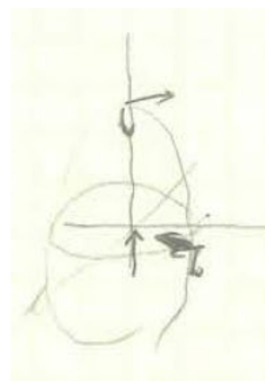
b) $\frac{d\vec{T}}{dt}=\frac{1}{5}\langle -3\sin t,\ -4\sin t,\ -5\cos t\rangle$

$$\left|\frac{d\vec{T}}{dt}\right|=\frac{1}{5}\sqrt{9\sin^2 t+16\sin^2 t+25\cos^2 t}$$

$$=1$$

$$\boxed{\vec{N}=\frac{1}{5}\langle -3\sin t,\ -4\sin t,\ -5\cos t\rangle} \qquad \vec{N}=\frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\boxed{\kappa=\frac{1}{25}\sqrt{9\sin^2 t+16\sin^2 t+25\cos^2 t}} \qquad \kappa=\frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$



c) At $t=0 \ \rightarrow\ \vec{T}=\left\langle \frac{3}{5},\ \frac{4}{5},\ 0\right\rangle \quad \vec{N}=\langle 0,\ 0,\ -1\rangle$

$$t=\frac{\pi}{2}\ \rightarrow\ \vec{T}=\langle 0,\ 0,\ -1\rangle \quad \vec{N}=\left\langle -\frac{3}{5},\ -\frac{4}{5},\ 0\right\rangle$$

$$9\sin^2 t+16\sin^2 t+25\cos^2 t=25$$

$$x^2+y^2+z^2=25$$

d) Yes; the results of parts (a) and (b) consistent with the graph

e)
$$\vec{B}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{3}{5}\cos t & \frac{4}{5}\cos t & -\sin t \\ -\frac{3}{5}\sin t & -\frac{4}{5}\sin t & -\cos t \end{vmatrix} \qquad \vec{B}=\vec{T}\times\vec{N}$$

$$=\left\langle -\frac{4}{5}\cos^2 t-\frac{4}{5}\sin^2 t,\ \frac{3}{5}\sin^2 t+\frac{3}{5}\cos^2 t,\ 0\right\rangle$$

$$\boxed{=\left\langle -\frac{4}{5},\ \frac{3}{5},\ 0\right\rangle}$$

f) $|\vec{T}|=\frac{1}{5}\sqrt{9\cos^2 t+16\cos^2 t+25\sin^2 t}$

$$=\frac{1}{5}\sqrt{25\cos^2 t+25\sin^2 t}$$

$$=\frac{1}{5}\sqrt{25}$$

$$\underline{=1}\rfloor \quad \checkmark$$

$$\left|\vec{N}\right|=\frac{1}{5}\sqrt{9\sin^2 t+16\sin^2 t+25\cos^2 t}$$

$$=\frac{1}{5}\sqrt{25\cos^2 t+25\sin^2 t}$$

$$\underline{=1}\rfloor \quad \checkmark$$

$$\left|\vec{B}\right|=\sqrt{\frac{16}{25}+\frac{9}{25}}$$

$$=\sqrt{\frac{25}{25}}$$

$$\underline{=1}\rfloor \quad \checkmark$$

$$\vec{T}\bullet\vec{N}=\frac{1}{5}\langle 3\cos t,\ 4\cos t,\ -5\sin t\rangle\bullet\frac{1}{5}\langle -3\sin t,\ -4\sin t,\ -5\cos t\rangle$$

$$=\frac{1}{25}\left(-9\cos t\sin t-16\cos t\sin t+25\cos t\sin t\right)$$

$$\underline{=0}\rfloor \quad \checkmark$$

$$\vec{T}\bullet\vec{B}=\frac{1}{5}\langle 3\cos t,\ 4\cos t,\ -5\sin t\rangle\bullet\left\langle -\frac{4}{5},\ \frac{3}{5},\ 0\right\rangle$$

$$=\frac{1}{25}\left(-12\cos t+12\cos t\right)$$

$$\underline{=0}\rfloor \quad \checkmark$$

$$\vec{B}\bullet\vec{N}=\left\langle -\frac{4}{5},\ \frac{3}{5},\ 0\right\rangle\bullet\frac{1}{5}\langle -3\sin t,\ -4\sin t,\ -5\cos t\rangle$$

$$=\frac{1}{25}\left(12\sin t-12\sin t+0\right)$$

$$\underline{=0}\rfloor \quad \checkmark$$

g) Since $\vec{B}$ is constant, then $\underline{\tau=0}\rfloor$

Exercise

Consider the curve $C:\ \vec{r}(t)=\langle 3\sin t,\ 3\cos t,\ 4t\rangle,\quad for\ 0\le t\le 2\pi$

a) Find $\boldsymbol{T}(t)$ at all points of C.

b) Find $\boldsymbol{N}(t)$ and the curvature at all points of C.

c) Sketch the curve and show $\boldsymbol{T}(t)$ and $\boldsymbol{N}(t)$ at the points of C corresponding to $t=0$ and $t=\frac{\pi}{2}$.

d) Are the results of parts (a) and (b) consistent with the graph?

e) Find $\boldsymbol{B}(t)$ at all points of *C*.

f) Describe three calculations that serve to check the accuracy of your results in part (*a*) – (*f*).

g) Compute the torsion at all points of *C*. Interpret this result.

Solution

a) $\vec{v}(t)=\langle 3\cos t,\ -3\sin t,\ 4\rangle \qquad \vec{v}(t)=\frac{d\vec{r}}{dt}$

$$\begin{aligned}|\vec{v}(t)| &= \sqrt{9\cos^2 t+9\sin^2 t+16}\\ &= \sqrt{9+16}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

$\vec{T}=\frac{1}{5}\langle 3\cos t,\ -3\sin t,\ 4\rangle \qquad \vec{T}=\frac{\vec{v}}{|\vec{v}|}$

b) $\frac{d\vec{T}}{dt}=\frac{1}{5}\langle -3\sin t,\ -3\cos t,\ 0\rangle$

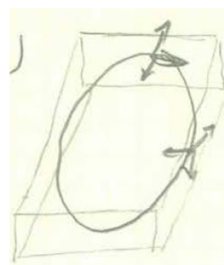
$$\begin{aligned}\left|\frac{d\vec{T}}{dt}\right| &= \frac{1}{5}\sqrt{9\sin^2 t+9\cos^2 t}\\ &= \frac{3}{5}\end{aligned}$$

$\vec{N}=\langle -\sin t,\ -\cos t,\ 0\rangle \qquad \vec{N}=\frac{d\vec{T}/dt}{|d\vec{T}/dt|}$

$\kappa=\frac{3}{25} \qquad \kappa=\frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$

c) At $t=0 \ \rightarrow\ \vec{T}=\left\langle \frac{3}{5},\ 0,\ \frac{4}{5}\right\rangle \quad \vec{N}=\langle 0,\ -1,\ .\rangle$

$t=\frac{\pi}{2} \ \rightarrow\ \vec{T}=\left\langle 0,\ -\frac{3}{5},\ \frac{4}{5}\right\rangle \quad \vec{N}=\langle -1,\ 0,\ 0\rangle$



d) Yes; the results of parts (*a*) and (*b*) consistent with the graph

e) $\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{3}{5}\cos t & -\frac{3}{5}\sin t & \frac{4}{5}\\ -\sin t & -\cos t & 0\end{vmatrix} \qquad \vec{B}=\vec{T}\times\vec{N}$

$=\left\langle \frac{4}{5}\cos t,\ -\frac{4}{5}\sin t,\ -\frac{3}{5}\right\rangle$

f)
$$\begin{aligned}|\vec{T}| &= \frac{1}{5}\sqrt{9\cos^2 t+9\sin^2 t+16}\\ &= \frac{1}{5}\sqrt{9+16}\\ &= \frac{1}{5}\sqrt{25}\end{aligned}$$

$$\underline{=1} \quad \checkmark$$

$$\left|\vec{N}\right| = \sqrt{\sin^2 t + \cos^2 t}$$

$$\underline{=1} \quad \checkmark$$

$$\left|\vec{B}\right| = \sqrt{\frac{16}{25}\cos^2 t + \frac{16}{25}\sin^2 t + \frac{9}{25}}$$

$$= \sqrt{\frac{16}{25} + \frac{9}{25}}$$

$$\underline{=1} \quad \checkmark$$

$$\vec{T} \bullet \vec{N} = \frac{1}{5}\langle 3\cos t,\ -3\sin t,\ 4\rangle \bullet \langle -\sin t,\ -\cos t,\ 0\rangle$$

$$= \frac{1}{5}\left(-3\cos t \sin t + 3\cos t \sin t + 0\right)$$

$$\underline{=0} \quad \checkmark$$

$$\vec{T} \bullet \vec{B} = \frac{1}{25}\langle 3\cos t,\ -3\sin t,\ 4\rangle \bullet \langle 4\cos t,\ -4\sin t,\ -3\rangle$$

$$= \frac{1}{25}\left(12\cos^2 t + 12\sin^2 t - 12\right)$$

$$= \frac{1}{25}(12 - 12)$$

$$\underline{=0} \quad \checkmark$$

$$\vec{B} \bullet \vec{N} = \frac{1}{5}\langle 4\cos t,\ -4\sin t,\ -3\rangle \bullet \langle -\sin t,\ -\cos t,\ 0\rangle$$

$$= \frac{1}{5}\left(-4\sin t\cos t + 4\sin t\cos t + 0\right)$$

$$\underline{=0} \quad \checkmark$$

g) $\dfrac{d\vec{B}}{dt} = \dfrac{1}{5}\langle -4\sin t,\ -4\cos t,\ 0\rangle$

$$\tau = \frac{1}{5}\langle -4\sin t,\ -4\cos t,\ 0\rangle \bullet \langle -\sin t,\ -\cos t,\ 0\rangle \qquad \tau = \frac{d\vec{B}}{dt}\bullet\vec{N}$$

$$= \frac{1}{5}\left(4\sin^2 t + 4\cos^2 t + 0\right)$$

$$\underline{= \frac{4}{5}}$$

Exercise

Suppose $\boldsymbol{r}(t)=\langle f(t),\ g(t),\ h(t)\rangle$, where *f*, *g*, and *h* are the quadratic functions $f(t)=a_1t^2+b_1t+c_1$, $g(t)=a_2t^2+b_2t+c_2$, and $h(t)=a_3t^2+b_3t+c_3$, and where at least one of the leading coefficients $a_1,\ a_2,\ or\ a_3$ is nonzero. Apart from a set of degenerate cases (for example $\boldsymbol{r}(t)=\langle t^2,\ t^2,\ t^2\rangle$, whose graph is a line), it can be shown that the graph of $\boldsymbol{r}(t)$ is a parabola that lies in a plane

a) Show by direct computation that $\boldsymbol{v}\times\boldsymbol{a}$ is constant. Then explain why the unit binormal vector is constant at all points on the curve. What does this result say about the torsion of the curve?

b) Compute $\boldsymbol{a}'(t)$ and explain why the torsion is zero at all points on the curve for which the torsion is defined.

Solution

a)

$$\vec{r}(t)=\left\langle a_1t^2+b_1t+c_1,\quad a_2t^2+b_2t+c_2,\quad a_3t^2+b_3t+c_3\right\rangle$$

$$\vec{v}(t)=\left\langle 2a_1t+b_1,\quad 2a_2t+b_2,\quad 2a_3t+b_3\right\rangle \qquad \vec{v}(t)=\frac{d\vec{r}}{dt}$$

$$\vec{a}(t)=\left\langle 2a_1,\quad 2a_2,\quad 2a_3\right\rangle \qquad \vec{a}(t)=\frac{d\vec{v}}{dt}$$

$$\vec{v}\times\vec{a}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2a_1t+b_1 & 2a_2t+b_2 & 2a_3t+b_3\\ 2a_1 & 2a_2 & 2a_3\end{vmatrix}$$

$$=2\left\langle 2a_2a_3t+a_3b_2-2a_2a_3t-a_2b_3,\ \ 2a_1a_3t+a_1b_3-2a_1a_3t-a_3b_1,\ \ 2a_1a_2t+a_2b_1-2a_1a_2t-a_1b_2\right\rangle$$

$$=2\left\langle a_3b_2-a_2b_3,\quad a_1b_3-a_3b_1,\quad a_2b_1-a_1b_2\right\rangle$$

$= \textbf{\textit{Constant}}$

$$|\vec{v}\times\vec{a}|=2\sqrt{\left(a_3b_2-a_2b_3\right)^2+\left(a_1b_3-a_3b_1\right)^2+\left(a_2b_1-a_1b_2\right)^2}$$

$$\vec{B}=\frac{\vec{v}\times\vec{a}}{|\vec{v}\times\vec{a}|}=constant$$

$\Rightarrow\ \underline{\tau=0}$

b) $\vec{a}'=\langle 0,\ 0,\ 0\rangle$

$$\tau=\frac{(\vec{v}\times\vec{a})\times\vec{a}'}{|\vec{v}\times\vec{a}|^2}$$

$\underline{=0}$

Exercise

Let *f* and *g* be continuous on an interval *I*. consider the curve

$$C: \boldsymbol{r}(t) = \left\langle a_1 f(t) + a_2 g(t) + a_3,\ b_1 f(t) + b_2 g(t) + b_3,\ c_1 f(t) + c_2 g(t) + c_3 \right\rangle$$

For *t* in *I*, and where $a_i,\ b_i,\ and\ c_i$, for $i = 1,\ 2,\ and\ 3$, are real numbers

a) Show that, in general, *C* lies in a plane.

b) Explain why the torsion is zero at all points of *C* for which the torsion is defined.

Solution

a) $\vec{r}(t) = \left\langle a_1 f(t) + a_2 g(t) + a_3,\ b_1 f(t) + b_2 g(t) + b_3,\ c_1 f(t) + c_2 g(t) + c_3 \right\rangle$

$\vec{r}(s) = \left\langle a_1 f(s) + a_2 g(s) + a_3,\ b_1 f(s) + b_2 g(s) + b_3,\ c_1 f(s) + c_2 g(s) + c_3 \right\rangle$

$$\vec{r}(t) \times \vec{r}(s) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 f(t) + a_2 g(t) + a_3 & b_1 f(t) + b_2 g(t) + b_3 & c_1 f(t) + c_2 g(t) + c_3 \\ a_1 f(s) + a_2 g(s) + a_3 & b_1 f(s) + b_2 g(s) + b_3 & c_1 f(s) + c_2 g(s) + c_3 \end{vmatrix}$$

$$\begin{aligned} &= \Big[\big(b_1 f(t) + b_2 g(t) + b_3\big)\big(c_1 f(s) + c_2 g(s) + c_3\big) - \\ &\qquad \big(c_1 f(t) + c_2 g(t) + c_3\big)\big(b_1 f(s) + b_2 g(s) + b_3\big)\Big]\hat{\boldsymbol{i}} \\ &\quad + \Big[\big(c_1 f(t) + c_2 g(t) + c_3\big)\big(a_1 f(s) + a_2 g(s) + a_3\big) - \\ &\qquad \big(a_1 f(t) + a_2 g(t) + a_3\big)\big(c_1 f(s) + c_2 g(s) + c_3\big)\Big]\hat{\boldsymbol{j}} \\ &\quad + \Big[\big(a_1 f(t) + a_2 g(t) + a_3\big)\big(b_1 f(s) + b_2 g(s) + b_3\big) - \\ &\qquad \big(b_1 f(t) + b_2 g(t) + b_3\big)\big(a_1 f(s) + a_2 g(s) + a_3\big)\Big]\hat{\boldsymbol{k}} \end{aligned}$$

$$
\begin{aligned}
&= \Big[b_1c_1 f(t)f(s) + b_1c_2 f(t)g(s) + b_1c_3 f(t) + b_2c_1 g(t)f(s) \\
&\quad + b_2c_2 g(t)g(s) + b_2c_3 g(t) + b_3c_1 f(s) + b_3c_2 g(s) + b_3c_3 \\
&\quad - b_1c_1 f(t)f(s) - b_2c_1 f(t)g(s) - b_3c_1 f(t) - b_1c_2 g(t)f(s) \\
&\quad - b_2c_2 g(t)g(s) - b_3c_2 g(t) - b_1c_3 f(s) - b_2c_3 g(s) - b_3c_3 \Big] \hat{\boldsymbol{i}} \\
&+ \Big[a_1c_1 f(t)f(s) + a_2c_1 f(t)g(s) + a_3\, c_1 f(t) + a_1c_2 g(t)f(s) \\
&\quad + a_2c_2 g(t)g(s) + a_3c_2 g(t) + a_1c_3 f(s) + a_2c_3 g(s) + a_3c_3 \\
&\quad - a_1c_1 f(s)f(t) - a_2c_1 f(s)g(t) - a_3c_1 f(s) - a_1c_2 g(s)f(t) \\
&\quad - a_2c_2 g(s)g(t) - a_3c_2 g(s) - a_1c_3 f(t) - a_2c_3 g(t) - a_3c_3 \Big] \hat{\boldsymbol{j}} \\
&+ \Big[a_1b_1 f(s)f(t) + a_2b_1 f(s)g(t) + a_3b_1 f(s) + a_1b_2 g(s)f(t) \\
&\quad + a_2b_2 g(s)g(t) + a_3b_2 g(s) + a_1b_3 f(t) + a_2b_3 g(t) + a_3b_3 \\
&\quad - a_1b_1 f(t)f(s) - a_2b_1 f(t)g(s) - a_3b_1 f(t) - a_1b_2 g(t)f(s) \\
&\quad - a_2b_2 g(t)g(s) - a_3b_2 g(t) - a_1b_3 f(s) - a_2b_3 g(s) - a_3\, b_3 \Big] \hat{\boldsymbol{k}}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big[\big(b_1c_2 - b_2c_1\big) f(t)g(s) + \big(b_2c_1 - b_1c_2\big) f(s)g(t) + \big(b_1c_3 - b_3c_1\big) f(t) \\
&\qquad + \big(b_2c_3 - b_3c_2\big) g(t) + \big(b_3c_1 - b_1c_3\big) f(s) + \big(b_3c_2 - b_2c_3\big) g(s) \Big] \hat{\boldsymbol{i}} \\
&+ \Big[\big(a_2c_1 - a_1c_2\big) f(t)g(s) + \big(a_1c_2 - a_2c_1\big) f(s)g(t) + \big(a_3c_1 - a_1c_3\big) f(t) \\
&\qquad + \big(a_3c_2 - a_2c_3\big) g(t) + \big(a_1c_3 - a_3c_1\big) f(s) + \big(a_2c_3 - a_3c_2\big) g(s) \Big] \hat{\boldsymbol{j}} \\
&+ \Big[\big(a_2b_1 - a_1b_2\big) f(s)g(t) + \big(a_1b_2 - a_2b_1\big) f(t)g(s) + \big(a_1b_3 - a_3b_1\big) f(t) \\
&\qquad + \big(a_2b_3 - a_3b_2\big) g(t) \big(+a_3b_1 - a_1b_3\big) f(s) + \big(a_3b_2 - a_2b_3\big) g(s) \Big] \hat{\boldsymbol{k}}
\end{aligned}
$$

If $a_3 = b_3 = c_3 = 0$

$$
\begin{aligned}
\vec{r}(t) \times \vec{r}(s) &= \Big[\big(b_1c_2 - b_2c_1\big) f(t)g(s) + \big(b_2c_1 - b_1c_2\big) f(s)g(t) \Big] \hat{\boldsymbol{i}} \\
&\quad + \Big[\big(a_2c_1 - a_1c_2\big) f(t)g(s) + \big(a_1c_2 - a_2c_1\big) f(s)g(t) \Big] \hat{\boldsymbol{j}} \\
&\quad + \Big[\big(a_1b_2 - a_2b_1\big) f(t)g(s) + \big(a_2b_1 - a_1b_2\big) f(s)g(t) \Big] \hat{\boldsymbol{k}} \\
&= \Big[\big(b_1c_2 - b_2c_1\big)\big(f(t)g(s) - f(s)g(t)\big) \Big] \hat{\boldsymbol{i}} \\
&\quad + \Big[\big(a_2c_1 - a_1c_2\big)\big(f(t)g(s) - f(s)g(t)\big) \Big] \hat{\boldsymbol{j}} \\
&\quad + \Big[\big(a_1b_2 - a_2b_1\big)\big(f(t)g(s) - f(s)g(t)\big) \Big] \hat{\boldsymbol{k}} \\
&= \big(f(t)g(s) - f(s)g(t)\big) \Big\langle b_1c_2 - b_2c_1,\ \ a_2c_1 - a_1c_2,\ \ a_1b_2 - a_2b_1 \Big\rangle
\end{aligned}
$$

Which is orthogonal to the same vector, the vectors $\vec{r}(t)$ must all be in the same plane.

If a_3, b_3, & $c_3 \neq 0$

Consider $\vec{p}(t) = \vec{r}(t) - \langle a_3, b_3, c_3 \rangle$

$\vec{p}(t)$ is the form of $\vec{r}(t) \times \vec{r}(s)$ with $a_3 = b_3 = c_3 = 0$.

$\therefore$ $\vec{p}(t)$ lies in a plane where $\vec{r}(t) = \vec{p}(t) + \langle a_3, b_3, c_3 \rangle$ lies in a plane too.

b) If the curve lies in a plane, $\vec{B}$ is always normal to the plane with $|\vec{B}| = 1$.

Hence, $\vec{B}$ is constant, so $\tau = \frac{d\vec{B}}{dt} \bullet \vec{N}$ $\underline{= 0}$