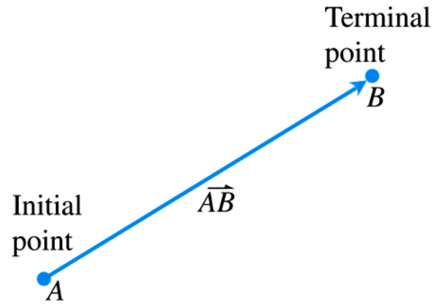
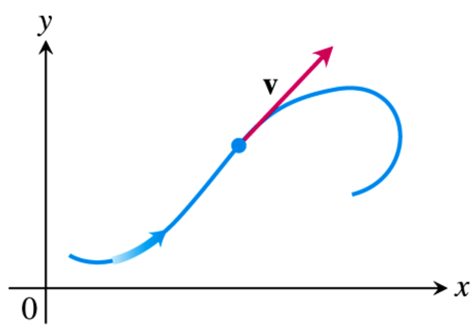
Lecture One – Vectors and Vector-Values Functions

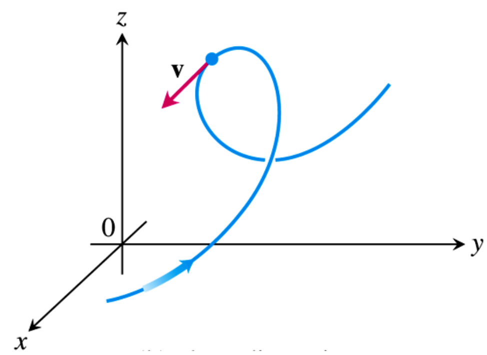
Section 1.1 – Vectors

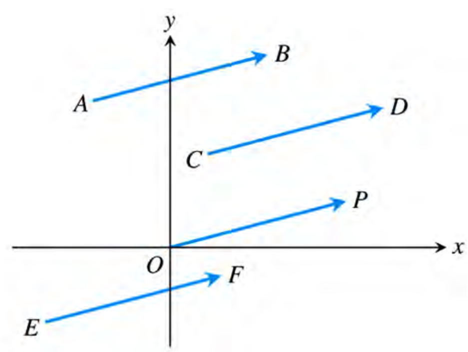


Component Form

A quantity such as force, or velocity is called a vector and is represented by a directed line segment.

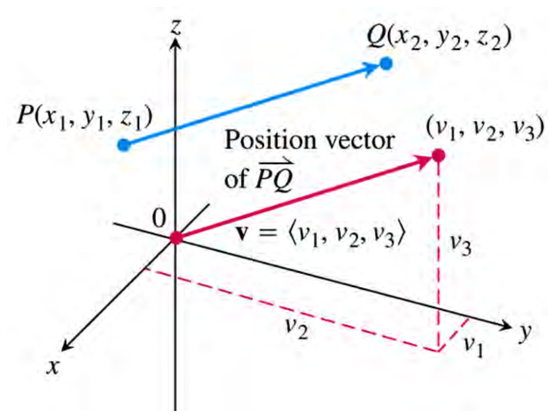






Definition

The vector represented by the direected line segment $\overrightarrow{PQ}$ has initial point P and terminal point Q and its length is denoted by $\left|\overrightarrow{PQ}\right|$



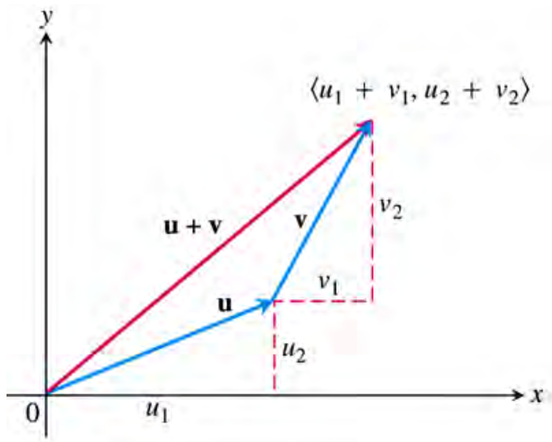
Vector Algebra Operations

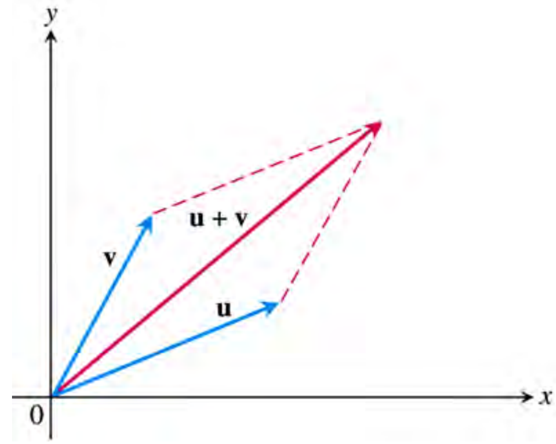
Definitions

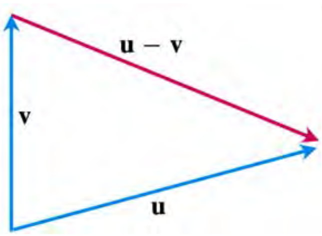
Let $\boldsymbol{u} = \langle u_1,\ u_2,\ u_3 \rangle$ and $\boldsymbol{v} = \langle v_1,\ v_2,\ v_3 \rangle$ be vectors with $\boldsymbol{k}$ a scalar

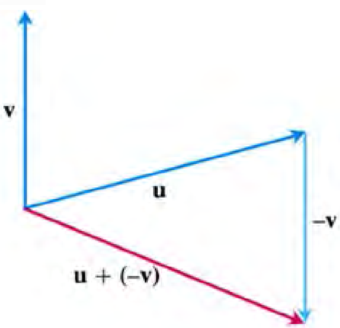
Addition: $\boldsymbol{u} + \boldsymbol{v} = \langle u_1 + v_1,\ u_2 + v_2,\ u_3 + v_3 \rangle$

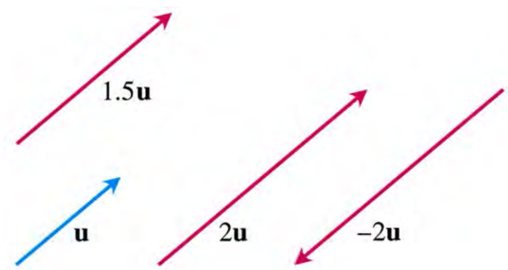
Scalar multiplication: $k\boldsymbol{u} = \langle ku_1,\ ku_2,\ ku_3 \rangle$











Example

Let $\boldsymbol{u} = \langle -1,\ 3,\ 1 \rangle$ and $\boldsymbol{v} = \langle 4,\ 7,\ 0 \rangle$. Find the components of

a) $2\boldsymbol{u} + 3\boldsymbol{v}$ *b)* $\boldsymbol{u} - \boldsymbol{v}$ *c)* $\left|\frac{1}{2}\boldsymbol{u}\right|$

Solution

a) $2\boldsymbol{u} + 3\boldsymbol{v} = 2\langle -1,\ 3,\ 1 \rangle + 3\langle 4,\ 7,\ 0 \rangle$

$= \langle -2,\ 6,\ 2 \rangle + \langle 12,\ 21,\ 0 \rangle$

$= \langle 10,\ 27,\ 2 \rangle$

b) $\boldsymbol{u} - \boldsymbol{v} = \langle -1,\ 3,\ 1 \rangle - \langle 4,\ 7,\ 0 \rangle$

$= \langle -5,\ -4,\ 1 \rangle$

c) $\left|\frac{1}{2}\boldsymbol{u}\right| = \left|\left\langle -\frac{1}{2},\ \frac{3}{2},\ \frac{1}{2}\right\rangle\right|$

$$= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{9}{4} + \frac{1}{4}}$$

$$= \sqrt{\frac{11}{4}}$$

$$\boxed{= \frac{\sqrt{11}}{2}}$$

Proporties of Vector Operations

Let ***u, v, w*** be vectors and *a, b* be scalars

1. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
2. $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$
3. $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$
4. $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$
5. $0\boldsymbol{u} = \mathbf{0}$
6. $1\boldsymbol{u} = \boldsymbol{u}$
7. $a(b\boldsymbol{u}) = (ab)\boldsymbol{u}$
8. $(a+b)\boldsymbol{u} = a\boldsymbol{u} + b\boldsymbol{u}$
9. $a(\boldsymbol{u} + \boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v}$

Definition

If ***v*** is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point $(v_1,\ v_2)$, then the ***component form*** of ***v*** is

$$\boldsymbol{v} = \langle v_1,\ v_2\rangle$$

If ***v*** is a **three-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point $(v_1,\ v_2,\ v_3)$, then the ***component form*** of ***v*** is

$$\boldsymbol{v} = \langle v_1,\ v_2,\ v_3\rangle$$

The magnitude or length of the vector $\boldsymbol{v} = \overrightarrow{PQ}$ is the nonegative number

$$|\boldsymbol{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The only vector with length 0 is the ***zero vector*** $\mathbf{0} = \langle 0,\ 0,\ 0\rangle$

Example

Find the component form and the length of the vector with initial point $P(-3,\ 4,\ 1)$ and terminal point $Q(-5,\ 2,\ 2)$

Solution

The component form of $\overrightarrow{PQ}$ is $\langle -5-(-3),\ 2-4,\ 2-1\rangle = \boxed{\langle -2,\ -2,\ 1\rangle}$

The length is $\left|\overrightarrow{PQ}\right| = \sqrt{(-2)^2+(-2)^2+1^2}\ \underline{=3}$

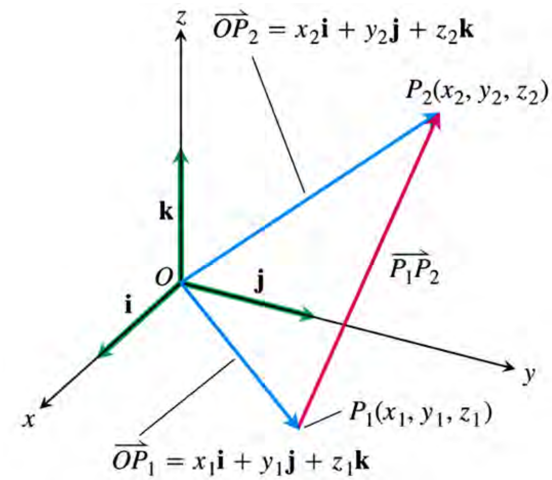
Unit Vectors

A vector $\boldsymbol{v}$ of length 1 is called a ***unit vector***. The ***standard unit vectors*** are

$$\hat{\boldsymbol{i}} = \langle 1,\ 0,\ 0\rangle, \quad \hat{\boldsymbol{j}} = \langle 0,\ 1,\ 0\rangle, \quad \textit{and} \quad \hat{\boldsymbol{k}} = \langle 0,\ 0,\ 1\rangle$$

Any vector $\boldsymbol{v} = \langle v_1,\ v_2,\ v_3\rangle$ can be written as a linear combination of the standard unit vectors as follows:

$$\begin{aligned}\boldsymbol{v} &= \langle v_1,\ v_2,\ v_3\rangle \\ &= \langle v_1,\ 0,\ 0\rangle + \langle 0,\ v_2,\ 0\rangle + \langle 0,\ 0,\ v_3\rangle \\ &= v_1\langle 1,\ 0,\ 0\rangle + v_2\langle 0,\ 1,\ 0\rangle + v_3\langle 0,\ 0,\ 1\rangle \\ &= v_1\hat{\boldsymbol{i}} + v_2\hat{\boldsymbol{j}} + v_3\hat{\boldsymbol{k}}\end{aligned}$$



Example

Find a unit vector $\boldsymbol{u}$ in the direction of the vector from $P_1(1,\ 0,\ 1)$ to $P_2(3,\ 2,\ 0)$.

Solution

$$\overrightarrow{P_1P_2} = (3-1)\boldsymbol{i} + (2-0)\boldsymbol{j} + (0-1)\boldsymbol{k} \ = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$$

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{2^2+2^2+(-1)^2} = \sqrt{9}\ \underline{=3}$$

$$\boldsymbol{u} = \frac{\overrightarrow{P_1P_2}}{\left|\overrightarrow{P_1P_2}\right|} = \frac{2\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}}{3} \quad \underline{=\frac{2}{3}\boldsymbol{i}+\frac{2}{3}\boldsymbol{j}-\frac{1}{3}\boldsymbol{k}}$$

Example

If $\boldsymbol{v} = 3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ is a velocity vector, express $\boldsymbol{v}$ as a product of its speed times a unit vector in the direction of motion.

Solution

Speed is the magnitude (length) of $\boldsymbol{v}$: $|\boldsymbol{v}| = \sqrt{3^2 + (-4)^2} = \underline{5|}$

The unit vector has the same direction as $\boldsymbol{v}$: $\dfrac{\boldsymbol{v}}{|\boldsymbol{v}|} = \dfrac{3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}}{5}$ $\underline{= \frac{3}{5}\hat{\boldsymbol{i}} - \frac{4}{5}\hat{\boldsymbol{j}}|}$

$$\boldsymbol{v} = 3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} = \underset{\text{Length (speed)}}{5}\underbrace{\left(\frac{3}{5}\hat{\boldsymbol{i}} - \frac{4}{5}\hat{\boldsymbol{j}}\right)}_{\text{Direction of motion}}$$

Note:

If $v \neq 0$, then

1. $\dfrac{\boldsymbol{v}}{|\boldsymbol{v}|}$ is a unit vector in the direction of $\boldsymbol{v}$;

2. The equation $\boldsymbol{v} = |\boldsymbol{v}|\dfrac{\boldsymbol{v}}{|\boldsymbol{v}|}$ expresses $\boldsymbol{v}$ as its length times its direction.

Example

A force of 6 Newton is applied in the direction of the vector $\boldsymbol{v} = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$. Express the force $\boldsymbol{F}$ as a product of its magnitude and direction.

Solution

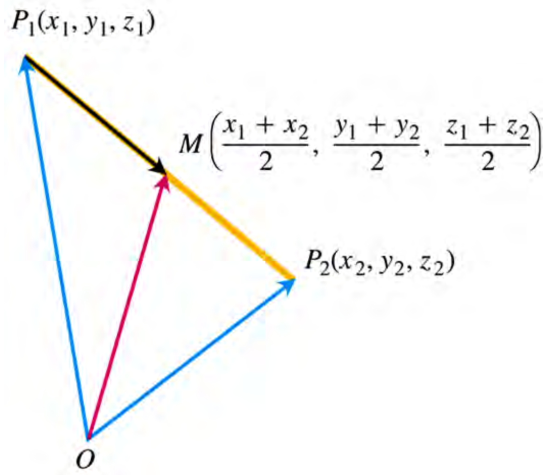
$$|v| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

$$\begin{aligned} \boldsymbol{F} &= |\boldsymbol{v}|\frac{\boldsymbol{v}}{|\boldsymbol{v}|} \\ &= 3\frac{2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}}{3} \\ &= 3\left(\frac{2}{3}\hat{\boldsymbol{i}} + \frac{2}{3}\hat{\boldsymbol{j}} - \frac{1}{3}\hat{\boldsymbol{k}}\right) \end{aligned}$$

Midpoint of a Line Segment

The midpoint M of the line segment joining points $P_1\left(x_1, y_1, z_1\right)$ and $P_2\left(x_2, y_2, z_2\right)$ is the point

$$\boxed{\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)}$$



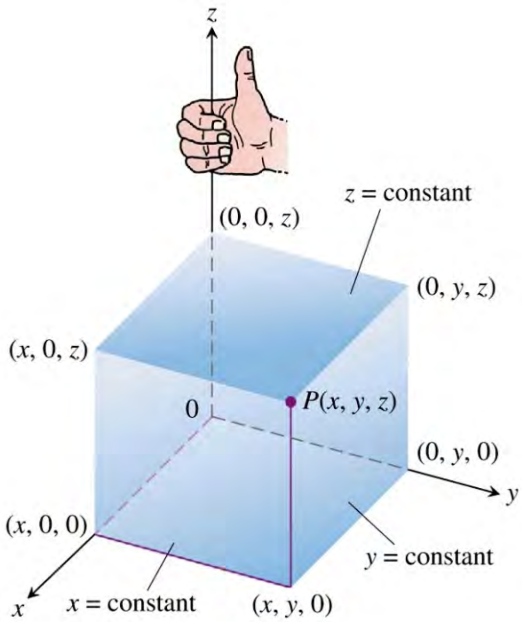
Example

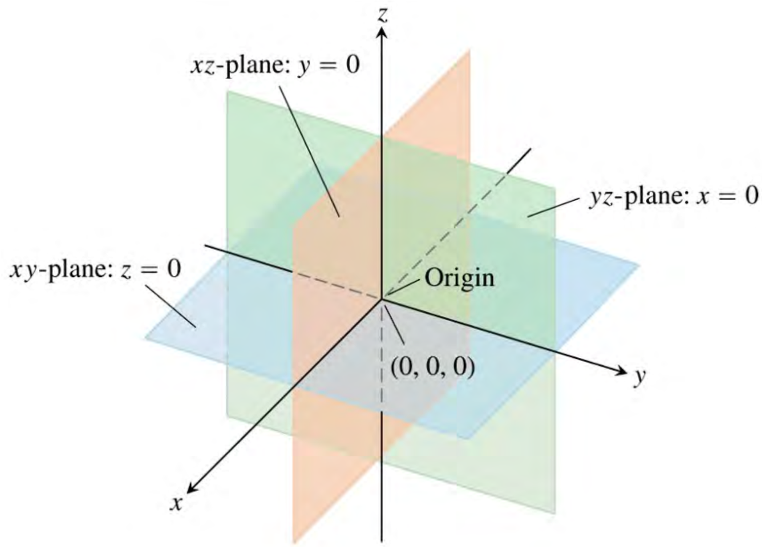
Find the midpoint of the segment $P_1(3, -2, 0)$ and $P_2(7, 4, 4)$

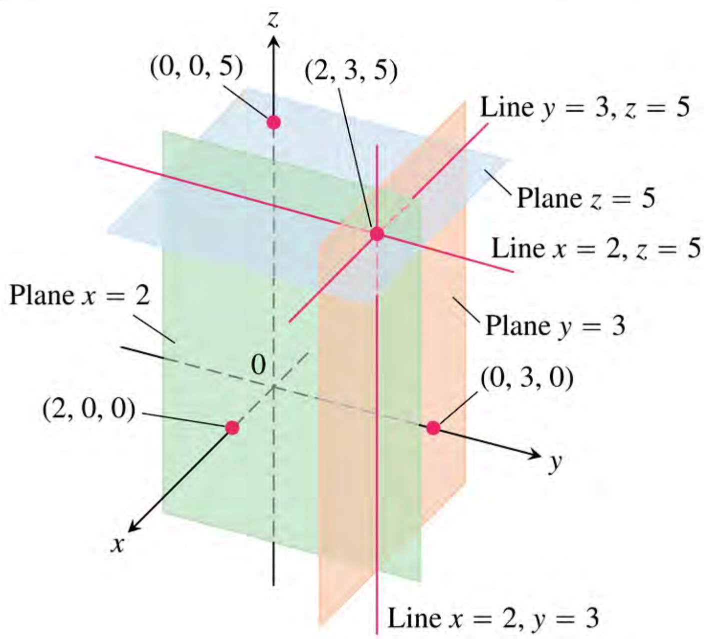
Solution

$$M = \left(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+4}{2}\right)$$

$$= (5, 1, 2)$$



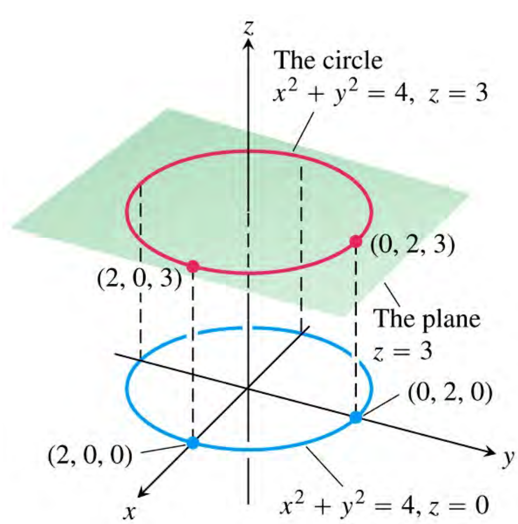




Example

What points $P(x, y, z)$ satisfy the equations

$$x^2 + y^2 = 4 \quad and \quad z = 3$$



Solution

The point lie in the horizontal plane $z = 3$ and the circle $x^2 + y^2 = 4$.

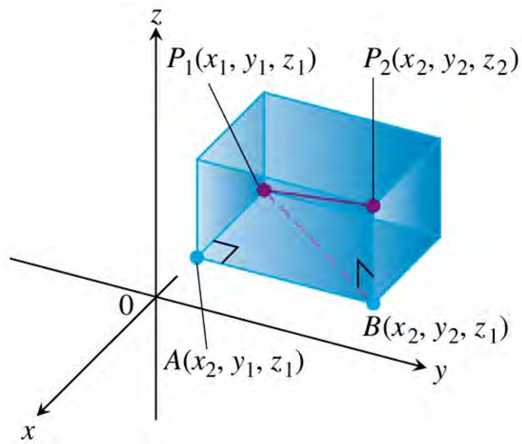
The solution is the set of points:

"the circle $x^2 + y^2 = 4$ in the plane $z = 3$"

Distance in Space

The distance between $P_1\left(x_1, y_1, z_1\right)$ and $P_2\left(x_2, y_2, z_2\right)$ is

$$\left|P_1P_2\right| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$$



Proof

$\left|P_1A\right| = \left|x_2 - x_1\right|, \quad \left|AB\right| = \left|y_2 - y_1\right|, \quad \left|BP_2\right| = \left|z_2 - z_1\right|$

From the right triangles P_1AB and P_1BP_2 :

$\left|P_1B\right|^2 = \left|P_1A\right|^2 + \left|AB\right|^2$ *and* $\left|P_1P_2\right|^2 = \left|P_1B\right|^2 + \left|BP_2\right|^2$

$$\begin{aligned}\left|P_1P_2\right|^2 &= \left|P_1B\right|^2 + \left|BP_2\right|^2 \\ &= \left|P_1A\right|^2 + \left|AB\right|^2 + \left|BP_2\right|^2 \\ &= \left|x_2 - x_1\right|^2 + \left|y_2 - y_1\right|^2 + \left|z_2 - z_1\right|^2 \\ &= \left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2 \quad \checkmark\end{aligned}$$

Example

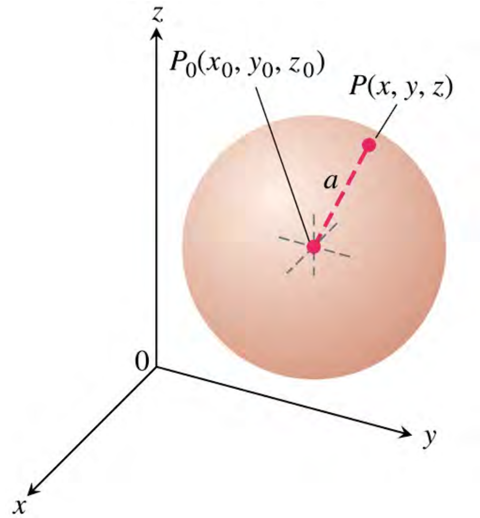
Find the distance between $P_1\left(2,\ 1,\ 5\right)$ and $P_2\left(-2,\ 3,\ 0\right)$

Solution

$$\begin{aligned}\left|P_1P_2\right| &= \sqrt{(-2-2)^2 + (3-1)^2 + (0-5)^2} \\ &= \sqrt{16+4+25} \\ &= \sqrt{45} \ \ \textit{or} \ \approx 6.708\end{aligned}$$

The Standard Equation for the Sphere of Radius a and Center (x_0, y_0, z_0)

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$



Example

Find the center and radius of the sphere $x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$

Solution

$$\left(x^2 + 3x\right) + y^2 + \left(z^2 - 4z\right) = -1$$

$$\left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) + y^2 + \left(z^2 - 4z + \left(\frac{-4}{2}\right)^2\right) = -1 + \left(\frac{3}{2}\right)^2 + \left(\frac{-4}{2}\right)^2$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = -1 + \frac{9}{4} + 4$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = \frac{21}{4}$$

Therefore; the center is $\boxed{\left(-\frac{3}{2},\ 0,\ 2\right)}$ and the radius is $\boxed{\frac{\sqrt{21}}{2}}$

Applications

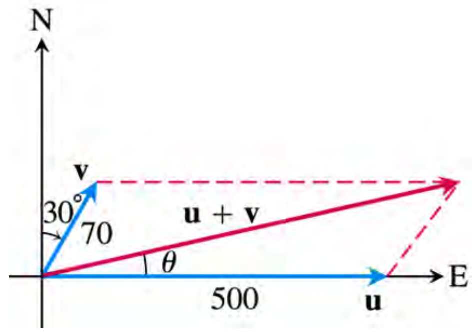
Example

A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?

Solution

$\boldsymbol{u}$ = the velocity of the airplane
$\boldsymbol{v}$ = the velocity of the tailwind

Given: $|u| = 500 \quad |v| = 70$



$$\boldsymbol{u} = \langle 500,\ 0\rangle$$

$$\boldsymbol{v} = \langle 70\cos 60°,\ 70\sin 60°\rangle$$
$$= \langle 35,\ 35\sqrt{3}\rangle$$

$$\boldsymbol{u} + \boldsymbol{v} = \langle 535,\ 35\sqrt{3}\rangle = 535\boldsymbol{i} + 35\sqrt{3}\boldsymbol{j}$$

$$|\boldsymbol{u} + \boldsymbol{v}| = \sqrt{535^2 + \left(35\sqrt{3}\right)^2}$$
$$\approx 538.4$$

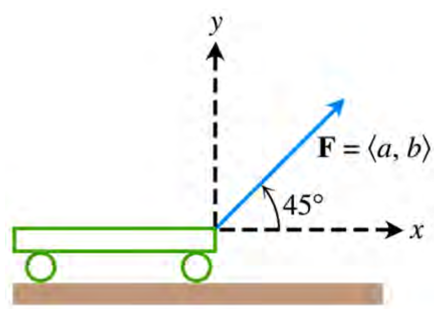
$$\underline{|\theta} = \tan^{-1}\frac{35\sqrt{3}}{535}\ \underline{\approx 6.5°|}$$

The ground speed of the airplane is about 538.4 mph, and its direction is about 6.5° north of east.

Example

A small cart is being pulled along a 20-*lb* smooth horizontal floor with a force **F** making a 45° angle to the floor. What is the effective force moving the cart forward?

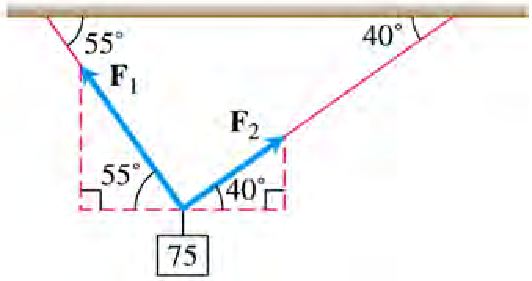
Solution



$$a = |F|\cos 45°$$
$$= (20)\left(\frac{\sqrt{2}}{2}\right)$$
$$\underline{\equiv 14.14\ \ lb|}$$

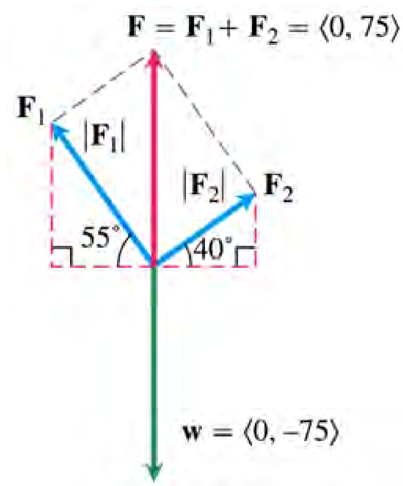
Example

A 75-N weight is suspended by two wires.



Find the forces F_1 and F_2 acting both wires

Solution



$$F_1 = \left\langle -\left|F_1\right|\cos 55°,\ \left|F_1\right|\sin 55°\right\rangle$$

$$F_2 = \left\langle \left|F_2\right|\cos 40°,\ \left|F_2\right|\sin 40°\right\rangle$$

$$F_1 + F_2 = \langle 0,\ 75\rangle$$

$$-\left|F_1\right|\cos 55° + \left|F_2\right|\cos 40° = 0 \quad \Rightarrow \quad \left|F_2\right| = \left|F_1\right|\frac{\cos 55°}{\cos 40°}$$

$$\left|F_1\right|\sin 55° + \left|F_2\right|\sin 40° = 75$$

$$\left|F_1\right|\sin 55° + \left|F_1\right|\frac{\cos 55°}{\cos 40°}\sin 40° = 75$$

$$\left|F_1\right|\left(\sin 55° + \cos 55° \tan 40°\right) = 75$$

$$\left|F_1\right| = \frac{75}{\sin 55° + \cos 55° \tan 40°} \ \underline{\approx 57.67\ N}$$

$$\left|F_2\right| = 57.67\frac{\cos 55°}{\cos 40°} \ \underline{\approx 43.18\ N}$$

The force vectors are then:

$$\begin{aligned} F_1 &= \left\langle -\left|F_1\right|\cos 55°,\ \left|F_1\right|\sin 55°\right\rangle \\ &= \langle -57.67\cos 55°,\ 57.67\sin 55°\rangle \\ &\underline{= \langle -33.08,\ 47.24\rangle} \end{aligned}$$

$$\begin{aligned} F_2 &= \left\langle \left|F_2\right|\cos 40°,\ \left|F_2\right|\sin 40°\right\rangle \\ &= \langle 43.18\cos 40°,\ 43.18\sin 40°\rangle \\ &\underline{= \langle 33.08,\ 27.76\rangle} \end{aligned}$$

Exercises *Section* 1.1 – Vectors

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations

1. $x^2+z^2=4, \quad y=0$

2. $x^2+y^2=4, \quad z=-2$

3. $x^2+y^2+z^2=1, \quad x=0$

4. $x^2+(y-1)^2+z^2=4, \quad y=0$

5. $x^2+y^2+z^2=4, \quad y=x$

Find the distance between points P_1 *and* P_2

6. $P_1(1,\ 1,\ 1), \quad P_2(3,\ 3,\ 0)$

7. $P_1(-1,\ 1,\ 5), \quad P_2(2,\ 5,\ 0)$

8. $P_1(1,\ 4,\ 5), \quad P_2(4,\ -2,\ 7)$

9. $P_1(3,\ 4,\ 5), \quad P_2(2,\ 3,\ 4)$

Find the center and radii of the spheres

10. $x^2+y^2+z^2+4x-4z=0$

11. $x^2+y^2+z^2-6y+8z=0$

12. $2x^2+2y^2+2z^2+x+y+z=9$

13. Find a formula for the distance from the point $P(x,\ y,\ z)$ to x-axis

14. Find a formula for the distance from the point $P(x,\ y,\ z)$ to xy-plane

15. Let $\boldsymbol{u}=\langle 3,\ -2\rangle$ *and* $\boldsymbol{v}=\langle -2,\ 5\rangle$. Find the component form and the magnitude if the vector

a) $3\boldsymbol{u}$

b) $\boldsymbol{u}-\boldsymbol{v}$

c) $2\boldsymbol{u}-3\boldsymbol{v}$

d) $-2\boldsymbol{u}+5\boldsymbol{v}$

e) $-\frac{5}{13}\boldsymbol{u}+\frac{12}{13}\boldsymbol{v}$

16. Find the component form of the vector: The sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ where

$A=(1,-1), \quad B=(2,0), \quad C=(-1,3), \quad and \quad D=(-2,2)$

17. Find the component form of the vector: The unit vector that makes an angle $\theta=\frac{2\pi}{3}$ with the positive x-axis

18. Find the component form of the vector: The unit vector obtained by rotating the vector $\langle 0,\ 1\rangle$ $120°$ counterclockwise about the origin

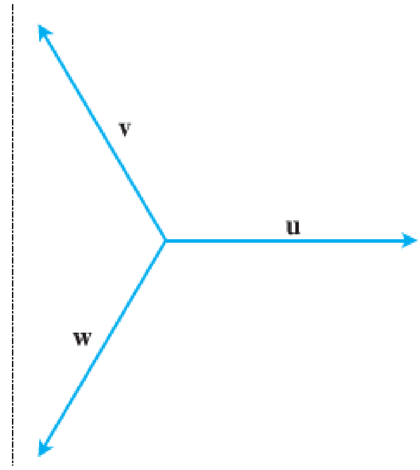
19. Find the component form of the vector: The unit vector obtained by rotating the vector $\langle 1,\ 0\rangle$ $135°$ counterclockwise about the origin

20. Sketch the indicated vector

a) $\boldsymbol{u}-\boldsymbol{v}$

b) $2\boldsymbol{u}-\boldsymbol{v}$

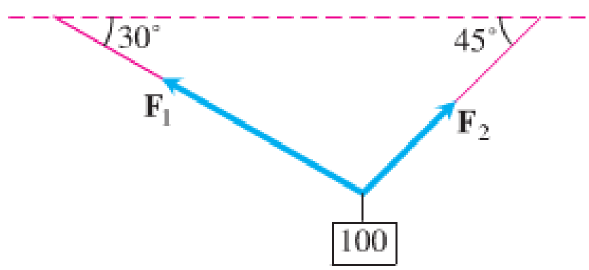
c) $\boldsymbol{u}-\boldsymbol{v}+\boldsymbol{w}$



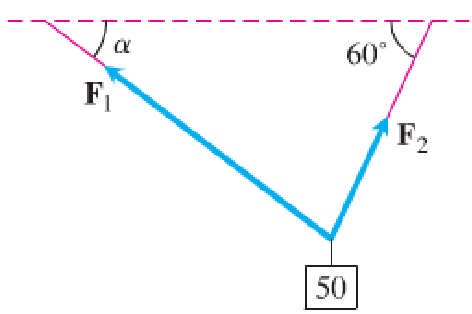
21. An Airplane is flying in the direction 25° west of north at 800 *km/h*. Find the component form of the velocity of the airplane, assuming that the positive *x*-axis represents due east and the positive *y*-axis represents due north.

22. A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What speed and direction should the jetliner have in order for the resultant vector to be 500 *mph* due east?

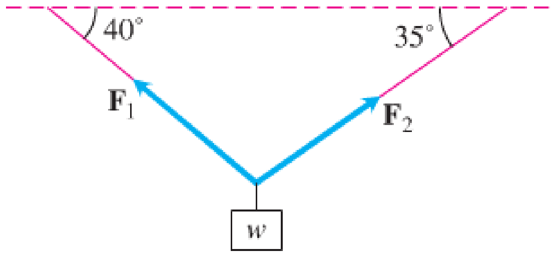
23. Consider a 100-N weight suspended by two wires. Find the magnitudes and components of the force vectors F_1 *and* F_2



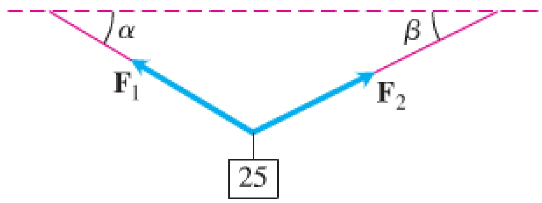
24. Consider a 50-N weight suspended by two wires, If the magnitude of vector $F_1 = 35\ N$, find the angle α and the magnitude of vector F_2



25. Consider a **w**-N weight suspended by two wires, If the magnitude of vector $F_2 = 100\ N$, find **w** and the magnitude of vector F_1



26. Consider a 25-N weight suspended by two wires, If the magnitude of vector F_1 *and* F_2 are both 75 N, then angles α and β are equal. Find α.

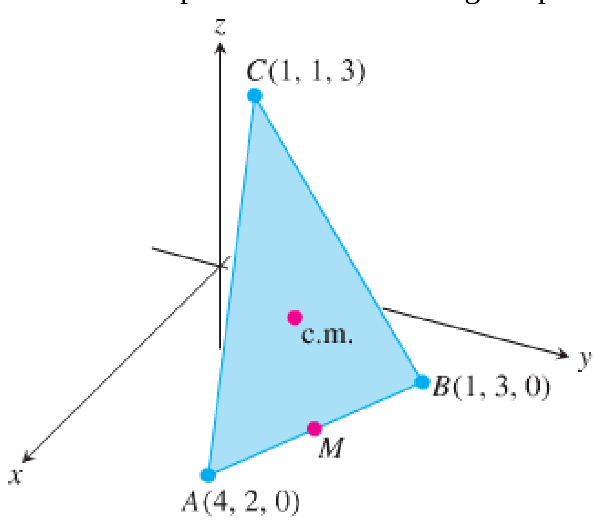


27. A bird flies from its nest 5 km in the direction 60° north east, where it stops to rest on a tree. It then flies 10 km in the direction due southeast and lands atop a telephone pole. Place an *xy*-coordinate system so that the origin is the bird's nest, the *x*-axis points east, and the *y*-axis points north.

a) At what point is the tree located?

b) At what point is the telephone pole?

28. Suppose that *A*, *B*, and *C* are the corner points of the thin triangular plate of constant density.



a) Find the vector from *C* to the midpoint *M* of side *AB*.

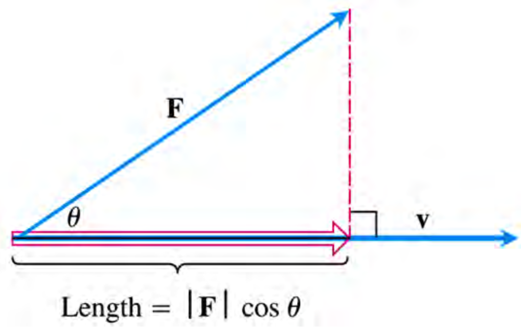
b) Find the vector from *C* to the point that lies two-thirds of the way from *C* to *M* on the median *CM*.

c) Find the coordinates of the point in which the medians of $\triangle ABC$ intersect (this point is the plate's center of mass).

29. Show that a unit vector in the plane can be expressed as $\boldsymbol{u} = (\cos\theta)\boldsymbol{i} + (\sin\theta)\boldsymbol{j}$, obtained by rotating $\boldsymbol{i}$ through an angle θ in the counterclockwise direction. Explain why this form gives ***every*** *unit vector* in the plane.

Section 1.2 – Dot Products

If a force $\boldsymbol{F}$ is applied to a particle moving along a path, we often need to know the magnitude of the force and the direction of motion.



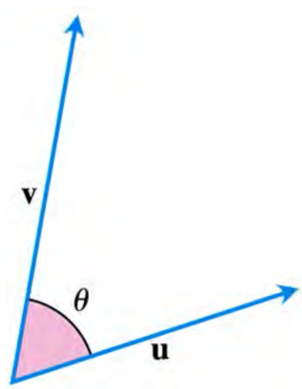
To calculate the angle between two vectors directly from their component, called the ***dot product***, also called *inner* or *scalar* products.

Angle between Vectors

Theorem

The angle θ between two nonzero vectors $\boldsymbol{u}=\langle u_1, u_2, u_3\rangle$ *and* $\boldsymbol{v}=\langle v_1, v_2, v_3\rangle$ is given by

$$\boxed{\theta=\cos^{-1}\left(\frac{u_1v_1+u_2v_2+u_3v_3}{|\boldsymbol{u}||\boldsymbol{v}|}\right)}$$



Definition

The dot product $\boldsymbol{u}\bullet\boldsymbol{v}$ of vector $\boldsymbol{u}=\langle u_1, u_2, u_3\rangle$ *and* $\boldsymbol{v}=\langle v_1, v_2, v_3\rangle$ is

$$\boldsymbol{u}\bullet\boldsymbol{v}=u_1v_1+u_2v_2+u_3v_3$$

Example

Find the dot product:

a) $\langle 1,-2,-1\rangle \bullet \langle -6,2,-3\rangle$ *b)* $\left(\frac{1}{2}\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}\right)\bullet\left(4\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}\right)$

Solution

a) $\langle 1,-2,-1\rangle \bullet \langle -6,2,-3\rangle = 1(-6)+(-2)(2)+(-1)(-3) \underline{=-7}$

b) $\left(\frac{1}{2}\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}\right)\bullet\left(4\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}\right) = \frac{1}{2}(4)+3(-1)+1(2) \underline{=1}$

Example

Find the angle between $\boldsymbol{u}=\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}-2\hat{\boldsymbol{k}}$ *and* $\boldsymbol{v}=6\hat{\boldsymbol{i}}+3\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$

Solution

$$\boldsymbol{u}\bullet\boldsymbol{v}=1(6)+(-2)(3)+(-2)(2) \underline{=-4}$$

$$\boldsymbol{u}=\sqrt{1^2+(-2)^2+(-2)^2} \underline{=3}$$

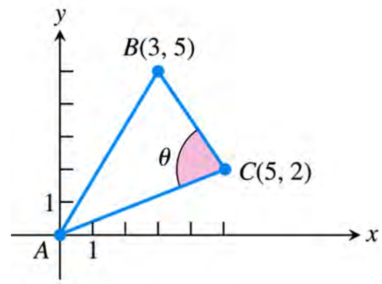
$$\boldsymbol{v}=\sqrt{6^2+3^2+2^2} \underline{=7}$$

$$\theta=\cos^{-1}\frac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} = \cos^{-1}\left(\frac{-4}{(3)(7)}\right) \underline{\approx 1.76\ rad}$$

Example

Find the angle θ of the triangle ABC determined by the vertices
$A=(0,\ 0),\quad B=(3,\ 5),\quad$ *and* $\quad C=(5,\ 2)$

Solution



$$\overrightarrow{CA}=\langle -5,\ -2\rangle \quad \overrightarrow{CB}=\langle -2,\ 3\rangle$$

$$\overrightarrow{CA}\bullet\overrightarrow{CB}=(-5)(-2)+(-2)(3)=4$$

$$\left|\overrightarrow{CA}\right|=\sqrt{(-5)^2+(-2)^2}=\sqrt{29}$$

$$\left|\overrightarrow{CB}\right|=\sqrt{(-2)^2+(3)^2}=\sqrt{13}$$

$$\theta=\cos^{-1}\frac{\overrightarrow{CA}\bullet\overrightarrow{CB}}{\left|\overrightarrow{CA}\right|\left|\overrightarrow{CB}\right|} = \cos^{-1}\left(\frac{4}{\sqrt{29}\sqrt{13}}\right)$$

$$\underline{\approx 1.36\ rad} \quad or \quad 78.1^\circ$$

Perpendicular (*Orthogonal*) Vectors

Definition

Vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal (or perpendicular) *iff* $\boldsymbol{u} \bullet \boldsymbol{v} = 0$

Example

Determine if the two vectors are orthogonal

a) $\boldsymbol{u} = \langle 3,\ -2 \rangle$ *and* $\boldsymbol{v} = \langle 4,\ 6 \rangle$

b) $\boldsymbol{u} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ *and* $\boldsymbol{v} = 2\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$

Solution

a) $\boldsymbol{u} \bullet \boldsymbol{v} = 3(4) + (-2)(6) = 0$ The two vectors are orthogonal

b) $\boldsymbol{u} \bullet \boldsymbol{v} = 3(0) + (-2)(2) + 1(4) = 0$ The two vectors are orthogonal

Dot Product Properties and Vector Projection

Properties of the Dot Product

If $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are any vectors and c is a scalar, then

a) $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$

b) $\boldsymbol{u} \cdot \boldsymbol{u} = |\boldsymbol{u}|^2$

c) $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$

d) $\boldsymbol{u} \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} - \boldsymbol{u} \cdot \boldsymbol{w}$

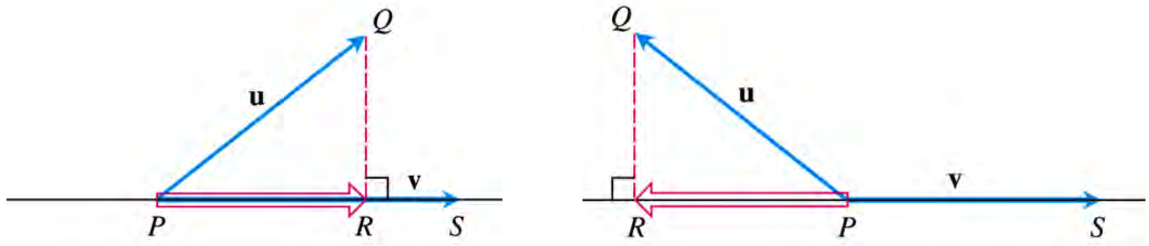
e) $(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}$

f) $(\boldsymbol{u} - \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} - \boldsymbol{v} \cdot \boldsymbol{w}$

g) $c(\boldsymbol{u} \cdot \boldsymbol{v}) = (c\boldsymbol{u}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (c\boldsymbol{v})$

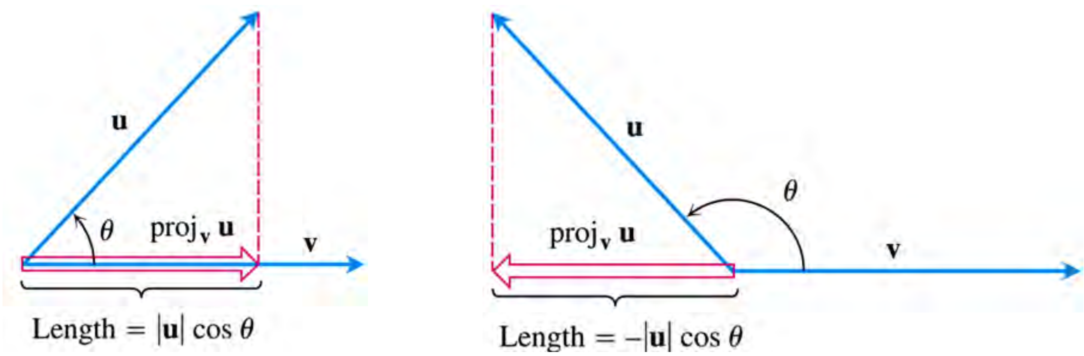
h) $0 \cdot \boldsymbol{v} = \boldsymbol{v} \cdot 0 = 0$

The vector projection of $\boldsymbol{u} = \overrightarrow{PQ}$ onto a nonzero vector $\boldsymbol{v} = \overrightarrow{PS}$ is the vector $\overrightarrow{PR}$ determined by dropping a perpendicular from Q to the line PS.



The notation for this vector is

$$proj_{\boldsymbol{v}}\boldsymbol{u} \qquad \text{(The vector projection of } \boldsymbol{u} \text{ onto } \boldsymbol{v}\text{)}$$



$$proj_{\boldsymbol{v}}\boldsymbol{u} = (|\boldsymbol{u}|\cos\theta)\frac{\boldsymbol{u}}{|\boldsymbol{v}|} = \left(\frac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$$

The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$ is the scalar: $|\boldsymbol{u}|\cos\theta = \frac{\boldsymbol{u}.\boldsymbol{v}}{|\boldsymbol{v}|} = \boldsymbol{u}.\frac{\boldsymbol{v}}{|\boldsymbol{v}|}$

Example

Find the vector projection of $\boldsymbol{u} = 6\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ *onto* $\boldsymbol{v} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ and the scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$.

Solution

$$\begin{aligned}
proj_{\boldsymbol{v}}\boldsymbol{u} &= \frac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{v}|^2}\boldsymbol{v} \\
&= \frac{6(1)+3(-2)+2(-2)}{1^2+(-2)^2+(-2)^2}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}\right) \\
&= \frac{-4}{9}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}\right) \\
&= -\frac{4}{9}\hat{\boldsymbol{i}} + \frac{8}{9}\hat{\boldsymbol{j}} + \frac{8}{9}\hat{\boldsymbol{k}}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{u}\cos\theta &= \boldsymbol{u}\bullet\frac{\boldsymbol{v}}{|\boldsymbol{v}|} \\
&= \left(6\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}\right)\bullet\frac{\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}}{\sqrt{1^2+(-2)^2+(-2)^2}} \\
&= \left(6\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}\right)\bullet\left(\frac{1}{3}\hat{\boldsymbol{i}} - \frac{2}{3}\hat{\boldsymbol{j}} - \frac{2}{3}\hat{\boldsymbol{k}}\right) \\
&= 6\left(\frac{1}{3}\right) + 3\left(-\frac{2}{3}\right) + 2\left(-\frac{2}{3}\right) \\
&= 2 - 2 - \frac{4}{3} \\
&= -\frac{4}{3}
\end{aligned}$$

Example

Find the vector projection of a force $\boldsymbol{F} = 5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ *onto* $\boldsymbol{v} = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ and the scalar component of F in the direction of $\boldsymbol{v}$.

Solution

$$\begin{aligned}
proj_{\boldsymbol{v}}\boldsymbol{F} &= \frac{\boldsymbol{F}\bullet\boldsymbol{v}}{|\boldsymbol{v}|^2}\boldsymbol{v} \\
&= \frac{5(1)+2(-3)}{1^2+(-3)^2}\left(\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\right) \\
&= -\frac{1}{10}\left(\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\right) \\
&= -\frac{1}{10}\hat{\boldsymbol{i}} + \frac{3}{10}\hat{\boldsymbol{j}}
\end{aligned}$$

$$F\cos\theta = \boldsymbol{F}\bullet\frac{\boldsymbol{v}}{|\boldsymbol{v}|}$$

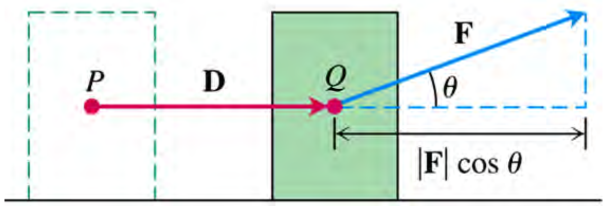
$$= \frac{\left(5\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}\right)\bullet\left(\hat{\boldsymbol{i}}-3\hat{\boldsymbol{j}}\right)}{\sqrt{1^2+(-3)^2}}$$

$$= \frac{5-6}{\sqrt{10}}$$

$$\boxed{= -\frac{1}{\sqrt{10}}}$$

Work

The work is done by a constant force of magnitude ***F*** in moving an object through a distance ***d*** as $W = Fd$.



$$Work = \begin{pmatrix} scalar\ \ component\ \ of\ \ \boldsymbol{F} \\ \text{in}\ \ the\ direction\ of\ \ \boldsymbol{D} \end{pmatrix}\left(length\ \ of\ \ \boldsymbol{D}\right)$$

$$= \left(|\boldsymbol{F}|\cos\theta\right)|\boldsymbol{D}|$$

$$= \boldsymbol{F}\bullet\boldsymbol{D}$$

Definition

The work done by a constant force ***F*** acting through a displacement $D = \overrightarrow{PQ}$ is

$$W = \boldsymbol{F}\bullet\boldsymbol{D}$$

Example

If $|\boldsymbol{F}| = 40\ N,\quad |\boldsymbol{D}| = 3\ m,\quad and\quad \theta = 60°$ find the work done by ***F*** in acting from ***P*** to ***Q***.

Solution

$$Work = \boldsymbol{F}\bullet\boldsymbol{D}$$

$$= |\boldsymbol{F}||\boldsymbol{D}|\cos\theta$$

$$= (40)(3)\cos 60°$$

$$\boxed{= 60\ \ J\ \ (joules)}$$

Exercises *Section* 1.2 – Dot Products

(Exercises 1–5) Find

a) $\boldsymbol{v} \bullet \boldsymbol{u},\ |\boldsymbol{v}|,\ |\boldsymbol{u}|$

b) The cosine of the angle between v and u

c) The scalar component of u in the direction of v

d) The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

1. $\boldsymbol{v} = 2\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + \sqrt{5}\hat{\boldsymbol{k}}, \quad \boldsymbol{u} = -2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - \sqrt{5}\hat{\boldsymbol{k}}$

2. $\boldsymbol{v} = \frac{3}{5}\hat{\boldsymbol{i}} + \frac{4}{5}\hat{\boldsymbol{k}}, \quad \boldsymbol{u} = 5\hat{\boldsymbol{i}} + 12\hat{\boldsymbol{j}}$

3. $\boldsymbol{v} = 2\hat{\boldsymbol{i}} + 10\hat{\boldsymbol{j}} - 11\hat{\boldsymbol{k}}, \quad \boldsymbol{u} = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$

4. $\boldsymbol{v} = -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}, \quad \boldsymbol{u} = 2\hat{\boldsymbol{i}} + \sqrt{17}\hat{\boldsymbol{j}}$

5. $\boldsymbol{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right\rangle, \quad \boldsymbol{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}} \right\rangle$

6. Find the angles between the vectors $\boldsymbol{u} = 2\boldsymbol{i} + \boldsymbol{j}, \quad \boldsymbol{v} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

7. Find the angles between the vectors $\boldsymbol{u} = \sqrt{3}\boldsymbol{i} - 7\boldsymbol{j}, \quad \boldsymbol{v} = \sqrt{3}\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

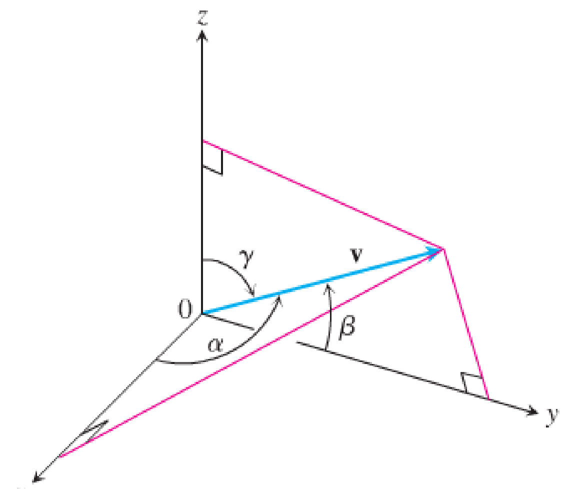
8. Find the angles between the vectors $\boldsymbol{u} = \boldsymbol{i} + \sqrt{2}\boldsymbol{j} - \sqrt{2}\boldsymbol{k}, \quad \boldsymbol{v} = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

9. The direction angles α, β, and γ of a vector $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ are defined as follows:

α is the angle between $\boldsymbol{v}$ and the positive *x*-axis $(0 \le \alpha \le \pi)$

β is the angle between $\boldsymbol{v}$ and the positive *y*-axis $(0 \le \beta \le \pi)$

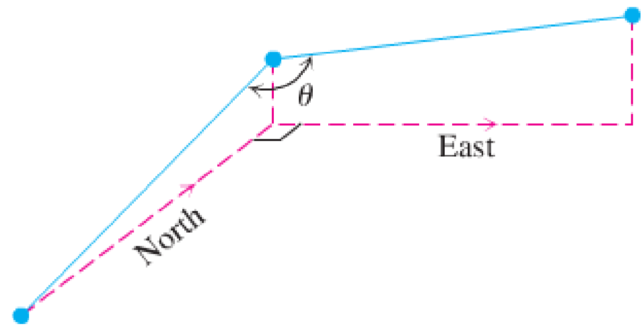
γ is the angle between $\boldsymbol{v}$ and the positive *z*-axis $(0 \le \gamma \le \pi)$



a) Show that $\cos\alpha = \frac{a}{|\boldsymbol{v}|}, \quad \cos\beta = \frac{b}{|\boldsymbol{v}|}, \quad \cos\gamma = \frac{c}{|\boldsymbol{v}|}$, and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. These cosines are called the direction cosines of $\boldsymbol{v}$.

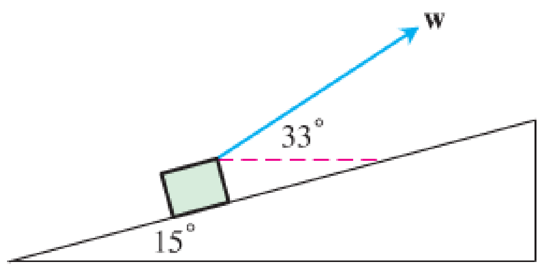
b) Show that if $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ is a unit vector, then *a*, *b*, and *c* are the direction cosines of $\boldsymbol{v}$.

10. A water main is to be constructed with 20% grade in the north direction and a 10% grade in the east direction. Determine the angle θ required in the water main for the turn from north to east.



11. A gun with muzzle velocity of 1200 ft/sec is fired at an angle of 8° above the horizontal. Find the horizontal and vertical components of the velocity.

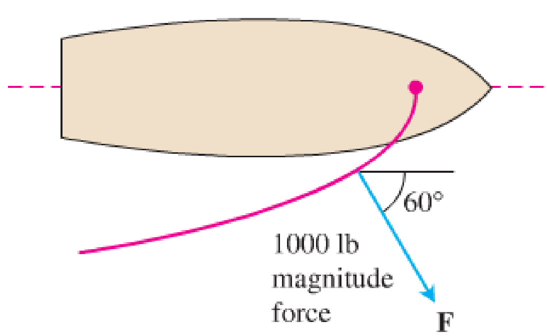
12. Suppose that a box is being towed up an inclined plane. Find the force ***w*** needed to make the component of the force parallel to the indicated plane equal to 2.5 lb.



13. Find the work done by a force $\boldsymbol{F} = 5\boldsymbol{i}$ (magnitude 5 N) in moving an object along the line from the origin to the point (1, 1) (distance in meters)

14. How much work does it take to slide a crate 20 m along a loading dock by pulling on it with a 200 N force at an angle of 30° from the horizontal?

15. The wind passing over a boat's sail exerted a 100-lb magnitude force F. How much work did the wind perform in moving the boat forward 1 mi? Answer in foot-pounds.

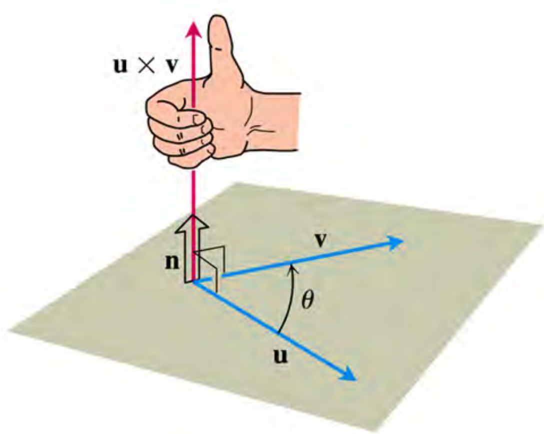


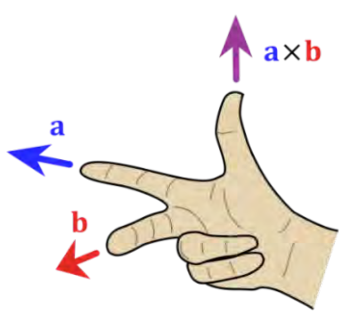
Section 1.3 – Cross Products

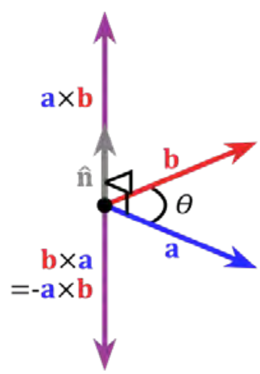
The *Cross* Product

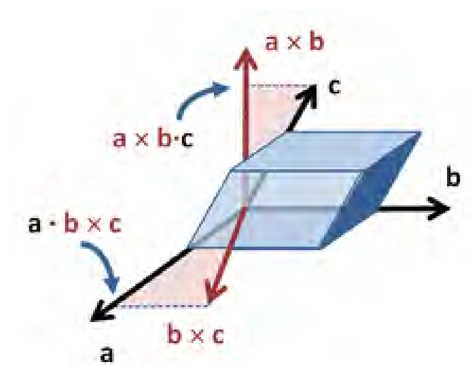
To find a vector in 3-space that is perpendicular to two vectors; the type of vector multiplication that facilities this construction is the cross product.

We start with two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in space. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are no parallel, they determine a plane. We select a unit vector $\boldsymbol{n}$ perpendicular to the plane by the ***right-hand rule***. Then the cross product $\boldsymbol{u}\times\boldsymbol{v}$ ("$\boldsymbol{u}$ cross $\boldsymbol{v}$") is the vector defined as follows









Definition

$$\boldsymbol{u}\times\boldsymbol{v} = \left(|\boldsymbol{u}||\boldsymbol{v}|\sin\theta\right)\boldsymbol{n}$$

Parallel Vectors

Nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel iff $\boldsymbol{u}\times\boldsymbol{v} = \mathbf{0}$

Properties of the Cross Product

If $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are any vectors and r, s are scalars, then

a) $\boldsymbol{u}\times\boldsymbol{v} = -(\boldsymbol{v}\times\boldsymbol{u})$

b) $\boldsymbol{u}\times(\boldsymbol{v}+\boldsymbol{w}) = (\boldsymbol{u}\times\boldsymbol{v})+(\boldsymbol{u}\times\boldsymbol{w})$

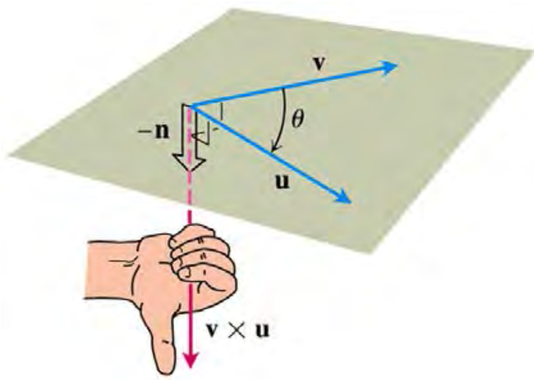
c) $(\boldsymbol{u}+\boldsymbol{v})\times\boldsymbol{w} = (\boldsymbol{u}\times\boldsymbol{w})+(\boldsymbol{v}\times\boldsymbol{w})$

d) $r(\boldsymbol{u}\times\boldsymbol{v}) = (r\boldsymbol{u})\times\boldsymbol{v} = \boldsymbol{u}\times(r\boldsymbol{v})$

e) $(r\boldsymbol{u})\times(s\boldsymbol{v}) = (rs)(\boldsymbol{u}\times\boldsymbol{v})$

f) $\boldsymbol{u}\times 0 = 0\times\boldsymbol{u} = 0$

g) $\boldsymbol{u}\times\boldsymbol{u} = 0$



Note:

✓ $\boldsymbol{i}\times\boldsymbol{i} = \boldsymbol{j}\times\boldsymbol{j} = \boldsymbol{k}\times\boldsymbol{k} = \mathbf{0}$

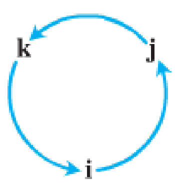
✓ $\boldsymbol{i}\times\boldsymbol{j} = \boldsymbol{k}, \quad \boldsymbol{j}\times\boldsymbol{k} = \boldsymbol{i}, \quad \boldsymbol{k}\times\boldsymbol{i} = \boldsymbol{j}$

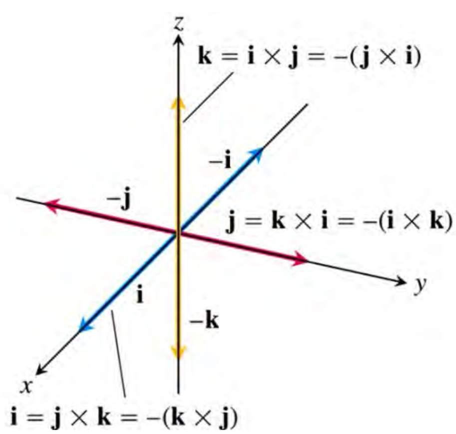
✓ $\boldsymbol{j}\times\boldsymbol{i} = -\boldsymbol{k}, \quad \boldsymbol{k}\times\boldsymbol{j} = -\boldsymbol{i}, \quad \boldsymbol{i}\times\boldsymbol{k} = -\boldsymbol{j}$

✓ $\boldsymbol{i}\times\boldsymbol{j} = -(\boldsymbol{j}\times\boldsymbol{i}) = \boldsymbol{k}$

✓ $\boldsymbol{j}\times\boldsymbol{k} = -(\boldsymbol{k}\times\boldsymbol{j}) = \boldsymbol{i}$

✓ $\boldsymbol{k}\times\boldsymbol{i} = -(\boldsymbol{i}\times\boldsymbol{k}) = \boldsymbol{j}$

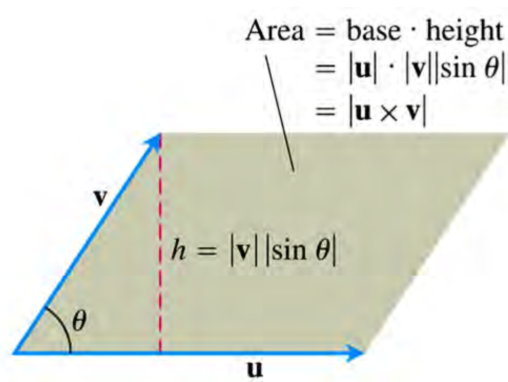




$|\boldsymbol{u}\times\boldsymbol{v}|$ Is the *Area* of the Parallelogram

Because $\boldsymbol{n}$ is a unit vector, the magnitude of $\boldsymbol{u}\times\boldsymbol{v}$ is

$$|\boldsymbol{u}\times\boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}||\sin\theta||n| = |\boldsymbol{u}||\boldsymbol{v}||\sin\theta|$$



Determinant Formula for $\boldsymbol{u}\times\boldsymbol{v}$

Definition

The cross product of $\boldsymbol{u}=\left(u_1,\ u_2,\ u_3\right)$ and $\boldsymbol{v}=\left(v_1,\ v_2,\ v_3\right)$ is the vector

$$\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$=\begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix}\boldsymbol{i}-\begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix}\boldsymbol{j}+\begin{vmatrix} u_1 & v_2 \\ u_2 & v_1 \end{vmatrix}\boldsymbol{k}$$

$$=\left(u_2v_3-u_3v_2\right)\boldsymbol{i}-\left(u_1v_3-u_3v_1\right)\boldsymbol{j}+\left(u_1v_2-u_2v_1\right)\boldsymbol{k}$$

$$\boxed{=\left(u_2v_3-u_3v_2,\ \ u_3v_1-u_1v_3,\ \ u_1v_2-u_2v_1\right)}$$

Example

Find $\boldsymbol{u}\times\boldsymbol{v}$ and $\boldsymbol{v}\times\boldsymbol{u}$ if $\boldsymbol{u}=2\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{v}=-4\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$

Solution

$$\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}\boldsymbol{i}-\begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}\boldsymbol{j}+\begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}\boldsymbol{k}$$

$$=-2\boldsymbol{i}-6\boldsymbol{j}+10\boldsymbol{k}$$

$$\boldsymbol{v}\times\boldsymbol{u}=-\boldsymbol{u}\times\boldsymbol{v}=2\boldsymbol{i}+6\boldsymbol{j}-10\boldsymbol{k}$$

Example

a) Find a vector perpendicular to the plane of $P(1,\ -1,\ 0),\quad Q(2,\ 1,\ -1),\quad and\ R(-1,\ 1,\ 2)$

b) Find the area of the triangle with vertices *P*, *Q*, and *R*.

c) Find a unit vector perpendicular to the *P*, *Q*, and *R*

Solution

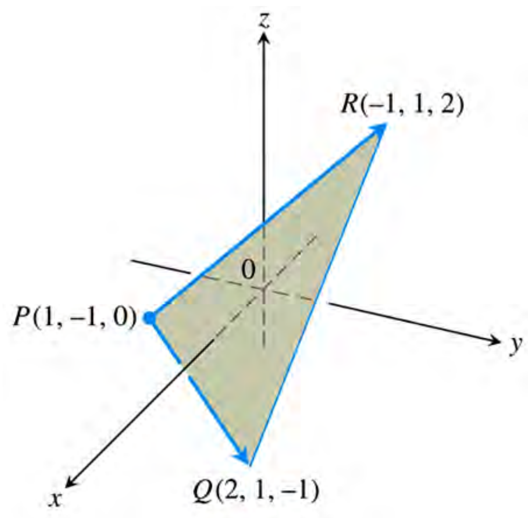
a) The vector $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to the plane.

$$\overrightarrow{PQ}=(2-1)\boldsymbol{i}+(1+1)\boldsymbol{j}+(-1-0)\boldsymbol{k}=\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$$

$$\overrightarrow{PR}=-2\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k}$$

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 2 & -1\\ -2 & 2 & 2\end{vmatrix}$$

$$=6\boldsymbol{i}+6\boldsymbol{k}$$



b) The area of the triangle is equal half the parallelogram determined by *P*, *Q*, and *R*.

$$\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|=\sqrt{6^2+6^2}=\sqrt{2\cdot 36}=6\sqrt{2}$$

Area of the triangle: $\frac{1}{2}\left(6\sqrt{2}\right)=\boxed{3\sqrt{2}}$

c) Since $\overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to the plane, its direction *n* is a unit vector ⊥ to the plane

$$\boldsymbol{n}=\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|}$$

$$=\frac{6\boldsymbol{i}+6\boldsymbol{k}}{6\sqrt{2}}$$

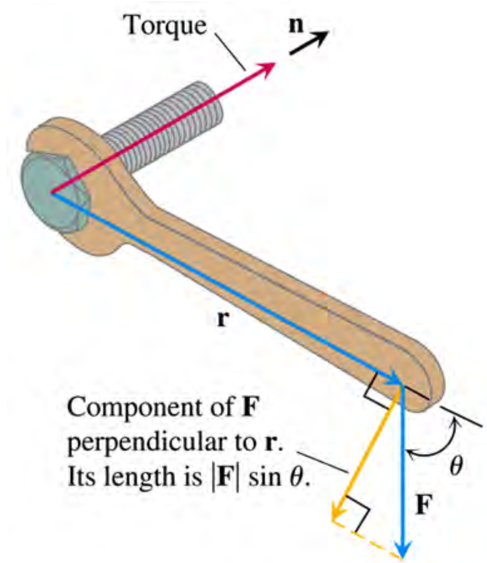
$$=\frac{1}{\sqrt{2}}\boldsymbol{i}+\frac{1}{\sqrt{2}}\boldsymbol{k}$$

Torque

When we turn a bolt by applying a force $\boldsymbol{F}$ to a wrench, we produce a torque that causes the bolt to rotate. The ***torque vector*** points in the direction of the axis of the bolt according to the right-hand rule (*ccw*).

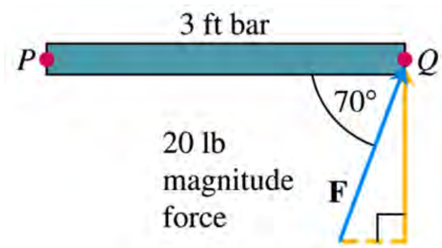
$$\textbf{\textit{Magnitude of torque vector}} = |\boldsymbol{r}||\boldsymbol{F}|\sin\theta$$

$$\textbf{\textit{Torque vector}} = \left(|\boldsymbol{r}||\boldsymbol{F}|\sin\theta\right)\boldsymbol{n}$$



Example

Find the magnitude of the torque generated by force F at the pivot point P.



Solution

$$\left|\overrightarrow{PQ}\times\boldsymbol{F}\right| = \left|\overrightarrow{PQ}\right||\boldsymbol{F}|\sin 70^\circ$$

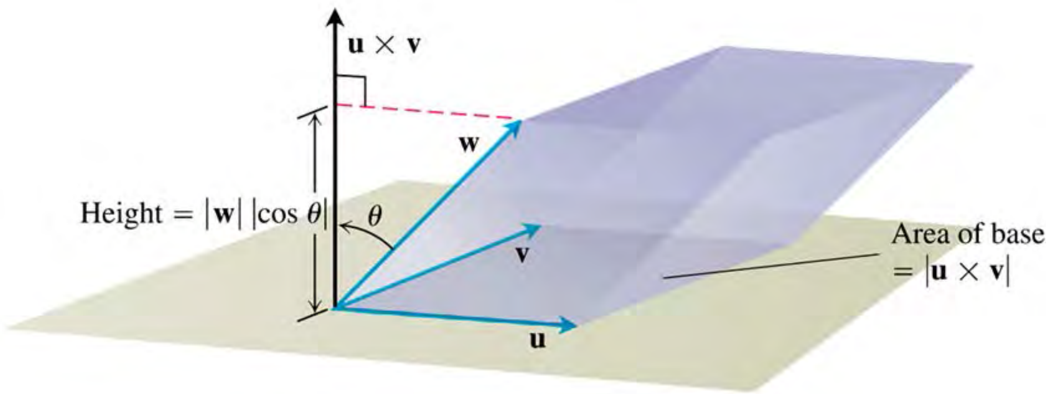
$$\approx (3)(20)(0.94)$$

$$\approx 56.4\ ft-lb$$

Triple Scalar or Box Product

The product $(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}$ is called the triple scalar product of ***u, v***, and ***w*** (in that order).

$$(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}=|\boldsymbol{u}\times\boldsymbol{v}||\boldsymbol{w}||\cos\theta|$$



Volume

The Volume of the Parallelepiped is

$$\begin{aligned}V&=(\text{area of base}).(\text{height})\\&=|\boldsymbol{u}\times\boldsymbol{v}||\boldsymbol{w}||\cos\theta|\\&=|\boldsymbol{u}\times\boldsymbol{v}|\frac{|(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}|}{|\boldsymbol{u}\times\boldsymbol{v}|}\\&=|(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}|\end{aligned}$$

$$V=(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}=\left|\det\begin{bmatrix}u_1&u_2&u_3\\v_1&v_2&v_3\\w_1&w_2&w_3\end{bmatrix}\right|$$

Example

Find the volume of the box (parallelepiped) determined by $\boldsymbol{u}=\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$, $\boldsymbol{v}=-2\boldsymbol{i}+3\boldsymbol{k}$, and $\boldsymbol{w}=7\boldsymbol{j}-4\boldsymbol{k}$.

Solution

$$V=(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}=\left|\det\begin{bmatrix}1&2&-1\\-2&0&3\\0&7&-4\end{bmatrix}\right|=|-23|\ \underline{=23|}$$

The volume is 23 units cubed.

Exercises *Section* 1.3 – Cross Products

1. Find the length and direction of $\boldsymbol{u}\times\boldsymbol{v}$ and $\boldsymbol{v}\times\boldsymbol{u}$: $\boldsymbol{u}=2\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}, \quad \boldsymbol{v}=\boldsymbol{i}-\boldsymbol{k}$

2. Find the length and direction of $\boldsymbol{u}\times\boldsymbol{v}$ and $\boldsymbol{v}\times\boldsymbol{u}$: $\boldsymbol{u}=\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}, \quad \boldsymbol{v}=0$

3. Find the length and direction of $\boldsymbol{u}\times\boldsymbol{v}$ and $\boldsymbol{v}\times\boldsymbol{u}$: $\boldsymbol{u}=\boldsymbol{i}\times\boldsymbol{j}, \quad \boldsymbol{v}=\boldsymbol{j}\times\boldsymbol{k}$

4. Find the length and direction of $\boldsymbol{u}\times\boldsymbol{v}$ and $\boldsymbol{v}\times\boldsymbol{u}$: $\boldsymbol{u}=-8\boldsymbol{i}-2\boldsymbol{j}-4\boldsymbol{k}, \quad \boldsymbol{v}=2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$

5. Sketch the coordinate axes and then include the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{u}\times\boldsymbol{v}$ as vectors starting origin for $\boldsymbol{u}=\boldsymbol{i}-\boldsymbol{k}, \quad \boldsymbol{v}=\boldsymbol{j}$

6. Sketch the coordinate axes and then include the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{u}\times\boldsymbol{v}$ as vectors starting origin for $\boldsymbol{u}=\boldsymbol{i}-\boldsymbol{k}, \quad \boldsymbol{v}=\boldsymbol{j}+\boldsymbol{k}$

7. Find the area of the triangle determined by the points *P*, *Q*, and *R*, and then find a unit vector perpendicular to plane *PQR*. $P(1,\ -1,\ 2), \quad Q(2,\ 0,\ -1), \quad and \quad R(0,\ 2,\ 1)$

8. Find the area of the triangle determined by the points *P*, *Q*, and *R*, and then find a unit vector perpendicular to plane *PQR*. $P(1,\ 1,\ 1), \quad Q(2,\ 1,\ 3), \quad and \quad R(3,\ -1,\ 1)$

9. Find the area of the triangle determined by the points *P*, *Q*, and *R*, and then find a unit vector perpendicular to plane *PQR*. $P(-2,\ 2,\ 0), \quad Q(0,\ 1,\ -1), \quad and \quad R(-1,\ 2,\ -2)$

10. Verify that $(\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{w}=(\boldsymbol{v}\times\boldsymbol{w})\cdot\boldsymbol{u}=(\boldsymbol{w}\times\boldsymbol{u})\cdot\boldsymbol{v}$ and find the volume of the parallelepiped determined by $\boldsymbol{u}=2\boldsymbol{i}, \quad \boldsymbol{v}=2\boldsymbol{j}, \quad and \quad \boldsymbol{w}=2\boldsymbol{k}$

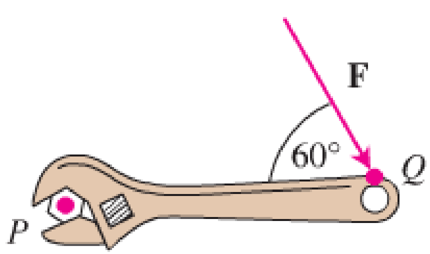
11. Find the volume of the parallelepiped determined by

$\boldsymbol{u}=\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}, \quad \boldsymbol{v}=2\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}, \quad and \quad \boldsymbol{w}=-\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$

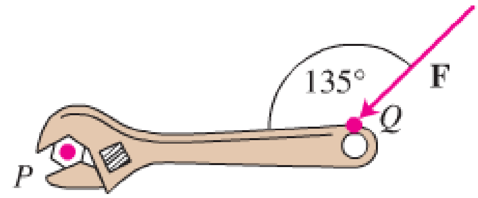
12. Find the volume of the parallelepiped determined by

$\boldsymbol{u}=\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}, \quad \boldsymbol{v}=-\boldsymbol{i}-\boldsymbol{k}, \quad and \quad \boldsymbol{w}=2\boldsymbol{i}+4\boldsymbol{j}-2\boldsymbol{k}$

13. Find the magnitude of the torque force exerted by $\boldsymbol{F}$ on the bolt at *P* if $\left|\overrightarrow{PQ}\right|=8\ in.$ and $\left|\boldsymbol{F}\right|=30\ lb.$



14. Find the magnitude of the torque force exerted by $\boldsymbol{F}$ on the bolt at P if $\left|\overrightarrow{PQ}\right| = 8\ in.$ and $\left|\boldsymbol{F}\right| = 30\ lb.$



15. Find the area of the parallelogram whose vertices are:

$$A(1,\ 0),\quad B(0,\ 1),\quad C(-1,\ 0),\quad D(0,\ -1)$$

16. Find the area of the parallelogram whose vertices are:

$$A(0,\ 0),\quad B(7,\ 3),\quad C(9,\ 8),\quad D(2,\ 5)$$

17. Find the area of the parallelogram whose vertices are:

$$A(-1,\ 2),\quad B(2,\ 0),\quad C(7,\ 1),\quad D(4,\ 3)$$

18. Find the area of the parallelogram whose vertices are:

$$A(0,\ 0,\ 0),\quad B(3,\ 2,\ 4),\quad C(5,\ 1,\ 4),\quad D(2,\ -1,\ 0)$$

19. Find the area of the parallelogram whose vertices are:

$$A(1,\ 0,\ -1),\quad B(1,\ 7,\ 2),\quad C(2,\ 4,\ -1),\quad D(0,\ 3,\ 2)$$

20. Find the area of the triangle whose vertices are:

$$A(0,\ 0),\quad B(-2,\ 3),\quad C(3,\ 1)$$

21. Find the area of the triangle whose vertices are:

$$A(-1,\ -1),\quad B(3,\ 3),\quad C(2,\ 1)$$

22. Find the area of the triangle whose vertices are:

$$A(1,\ 0,\ 0),\quad B(0,\ 0,\ 2),\quad C(0,\ 0,\ -1)$$

23. Find the area of the triangle whose vertices are:

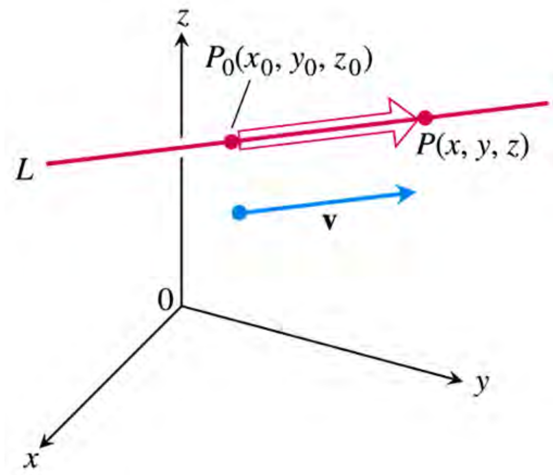
$$A(0,\ 0,\ 0),\quad B(-1,\ 1,\ -1),\quad C(3,\ 0,\ 3)$$

24. Find the volume of the parallelepiped if four of its eight vertices are:

$$A(0,\ 0,\ 0),\quad B(1,\ 2,\ 0),\quad C(0,\ -3,\ 2),\quad D(3,\ -4,\ 5)$$

Section 1.4 – Lines and Curves in Space

Lines and Line Segments in Space



The expanded form of the equation $\overrightarrow{P_0P} = tv$ is

$$\left(x - x_0\right)\boldsymbol{i} + \left(y - y_0\right)\boldsymbol{j} + \left(z - z_0\right)\boldsymbol{k} = t\left(v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}\right)$$

Vector Equation for a Line

A **vector equation for the line *L*** through $P_0\left(x_0, y_0, z_0\right)$ parallel to ***v*** is

$$r(t) = r_0 + tv, \quad -\infty < t < \infty$$

Where *r* is the position vector of a point $P(x, y, z)$ on L and r_0 is the position vector of $P_0\left(x_0, y_0, z_0\right)$.

Parametric Equations for a Line

A **standard parametrization** of the line through $P_0\left(x_0, y_0, z_0\right)$ parallel to $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$ is

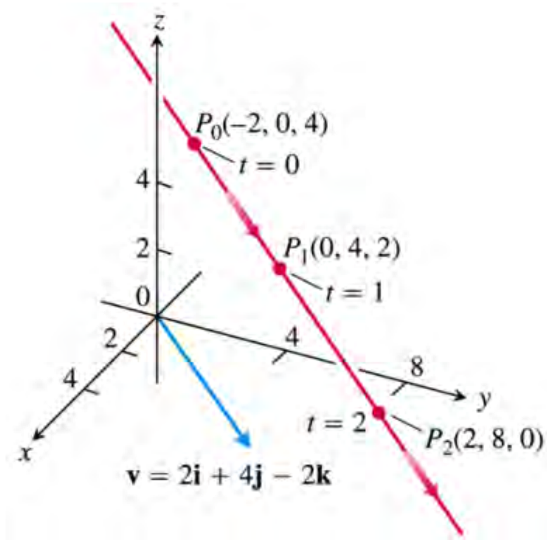
$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty$$

Example

Find the parametric equations for the line through (–2, 0, 4) parallel to $\boldsymbol{v} = 2\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$

Solution

$$x = -2 + 2t, \quad y = 4t, \quad z = 4 - 2t$$



Example

Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$

Solution

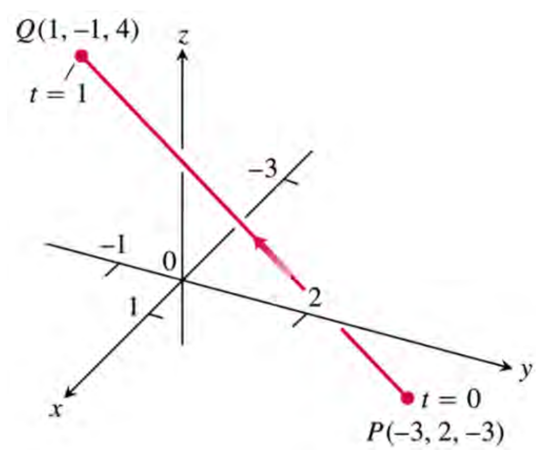
$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t$$

The point $(x,\ y,\ z) = (-3 + 4t,\ 2 - 3t,\ -3 + 7t)$

On the line passes through P at $t = 0$ and Q at $t = 1$.

That implies the restriction $0 \le t \le 1$ to parameterize the segment

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t, \quad 0 \le t \le 1$$



The position of a particle at time t is written:

$$r(t) = r_0 + tv$$

$$= r_0 + t|v|\frac{v}{|v|}$$

Example

A helicopter is to fly directly from a helipad at the origin in the direction of the point (1, 1, 1) at a speed of 60 *ft/sec*. What is the position of the helicopter after 10 *sec*?

Solution

The unit vector: $\dfrac{(1,\ 1,\ 1)}{\sqrt{1^2+1^2+1^2}} = \dfrac{1}{\sqrt{3}}\boldsymbol{i}+\dfrac{1}{\sqrt{3}}\boldsymbol{j}+\dfrac{1}{\sqrt{3}}\boldsymbol{k}$

Therefore; the position of the helicopter at any time t is

$$\begin{aligned} r(t) &= r_0 + t\boldsymbol{u} \\ &= 0 + t(60)\left(\frac{1}{\sqrt{3}}\boldsymbol{i}+\frac{1}{\sqrt{3}}\boldsymbol{j}+\frac{1}{\sqrt{3}}\boldsymbol{k}\right) \\ &= 20\sqrt{3}\ t(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}) \end{aligned}$$

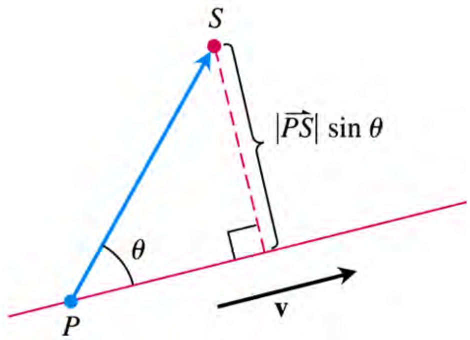
The position after 10 sec:

$$\begin{aligned} r(10) &= 20\sqrt{3}\ (10)(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}) \\ &= 200\sqrt{3}(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}) \end{aligned}$$

The distance is traveled: $|r(10)| = 200\sqrt{3}\sqrt{1^2+1^2+1^2} = \underline{600\ ft}$

Distance from a Point *S* to a Line through *P* parallel to *v*

$$d = \frac{\left|\overrightarrow{PS}\times\boldsymbol{v}\right|}{|\boldsymbol{v}|}$$



Example

Find the distance from the point $S\ (1, 1, 5)$ to the line

$L:\quad x = 1+t,\quad y = 3-t,\quad z = 2t$

Solution

At t = 0, the equations for L passes through P(1, 3, 0) parallel to $\boldsymbol{v} = \boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$

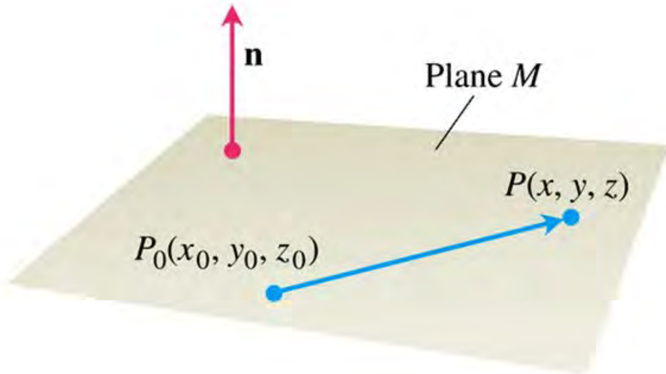
$\overrightarrow{PS} = (1-1)\boldsymbol{i}+(1-3)\boldsymbol{j}+(5-0)\boldsymbol{k} = -2\boldsymbol{j}+5\boldsymbol{k}$

$$\overrightarrow{PS}\times\boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \boldsymbol{i}+5\boldsymbol{j}+2\boldsymbol{k}$$

$$d = \frac{\left|\overrightarrow{PS}\times\boldsymbol{v}\right|}{|\boldsymbol{v}|} = \frac{\sqrt{1+25+4}}{\sqrt{1+1+4}} = \frac{\sqrt{30}}{\sqrt{6}} \underline{= \sqrt{5}}$$

An Equation for a Plane in Space

A plane in space is determined by knowing a point on the plane and its "tilt" or orientation. This "tilt" is defined by specifying a vector that is perpendicular or normal to the plane.



The dot product $\boldsymbol{n} \bullet \overrightarrow{P_0 P} = 0$, since $\overrightarrow{P_0 P}$ is orthogonal to $\boldsymbol{n} = A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}$.

$$\boldsymbol{n} \bullet \overrightarrow{P_0 P} = 0 \quad \Leftrightarrow \quad (A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}) \cdot \left(\left(x - x_0\right)\boldsymbol{i} + \left(y - y_0\right)\boldsymbol{j} + \left(z - z_0\right)\boldsymbol{k} \right) = 0$$

$$A\left(x - x_0\right) + B\left(y - y_0\right) + C\left(z - z_0\right) = 0$$

Equation for a Plane

The plane through $P_0\left(x_0, y_0, z_0\right)$ normal to $\boldsymbol{n} = A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}$ has

Vector equation: $\boldsymbol{n} \bullet \overrightarrow{P_0 P} = 0$

Component equation: $A\left(x - x_0\right) + B\left(y - y_0\right) + C\left(z - z_0\right) = 0$

Component equation simplified: $Ax + By + Cz = D \quad \text{where} \quad D = Ax_0 + By_0 + Cz_0$

Example

Find an equation for the plane through $P_0(-3,\ 0,\ 7)$ perpendicular to $\boldsymbol{n} = 5\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

Solution

The component equation is

$$5\left(x - (-3)\right) + 2(y - 0) + (-1)(z - 7) = 0$$

$$5(x + 3) + 2y - z + 7 = 0$$

$$5x + 15 + 2y - z + 7 = 0$$

$$\boxed{5x + 2y - z = -22}$$

Example

Find an equation for the plane through $A(0,\ 0,\ 1),\quad B(2,\ 0,\ 0),\quad C(0,\ 3,\ 0)$.

Solution

The cross product

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 2 & 0 & -1\\ 0 & 3 & -1\end{vmatrix}=3\boldsymbol{i}+2\boldsymbol{j}+6\boldsymbol{k}$$ Is the normal to the plane.

We substitute the components of this vector and the coordinates of A(0, 0, 1) into the component form of the equation to obtain

$$3(x-0)+2(y-0)+6(z-1)=0$$

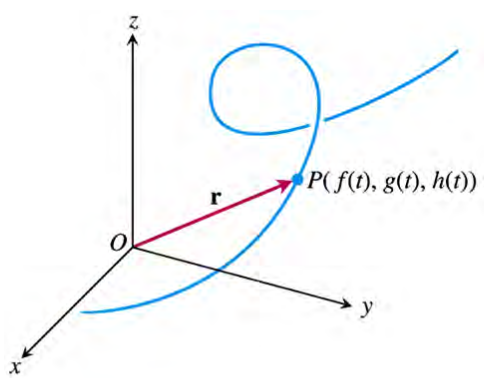
$$\boxed{3x+2y+6z-6=0} \quad \textit{or} \quad \boxed{3x+2y+6z=6}$$

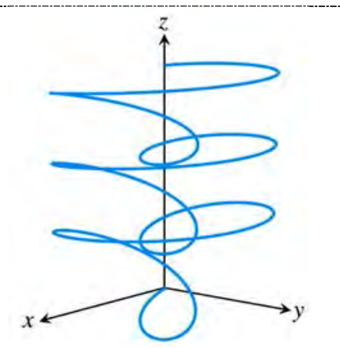
Curves

The coordinates for a particle moving through space during a time interval *I*, are defined as function on *I*:

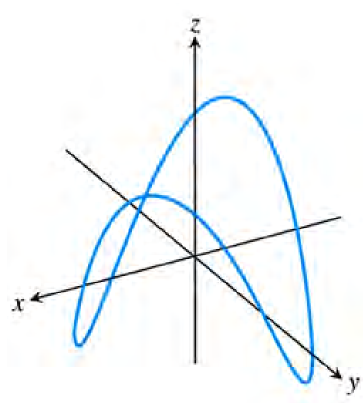
$$x=f(t),\quad y=g(t),\quad z=h(t),\quad t\in I.$$

The points $(x,\ y,\ z)=\left(f(t),\ g(t),\ h(t)\right),\quad t\in I$, make up the curve in space that we call the particle's path.

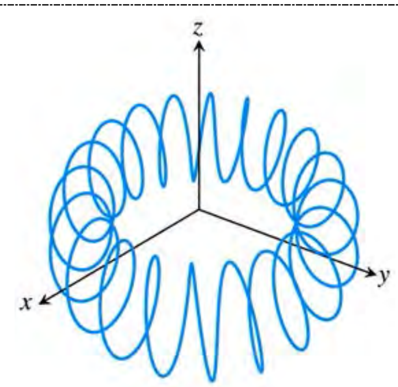




$\boldsymbol{r}(t)=(\sin 3t)(\cos t)\boldsymbol{i}+(\sin 3t)(\sin t)\boldsymbol{j}+t\boldsymbol{k}$



$\boldsymbol{r}(t)=(\cos t)\boldsymbol{i}+(\sin t)\boldsymbol{j}+(\sin 2t)\boldsymbol{k}$



$$\begin{aligned}\boldsymbol{r}(t)=&\,(4+\sin 20t)(\cos t)\boldsymbol{i}\\&+(4+\sin 20t)(\sin t)\boldsymbol{j}+(\cos 20t)\boldsymbol{k}\end{aligned}$$

Example

Graph the vector function $\boldsymbol{r}(t)=(\cos t)\boldsymbol{i}+(\sin t)\boldsymbol{j}+t\boldsymbol{k}$

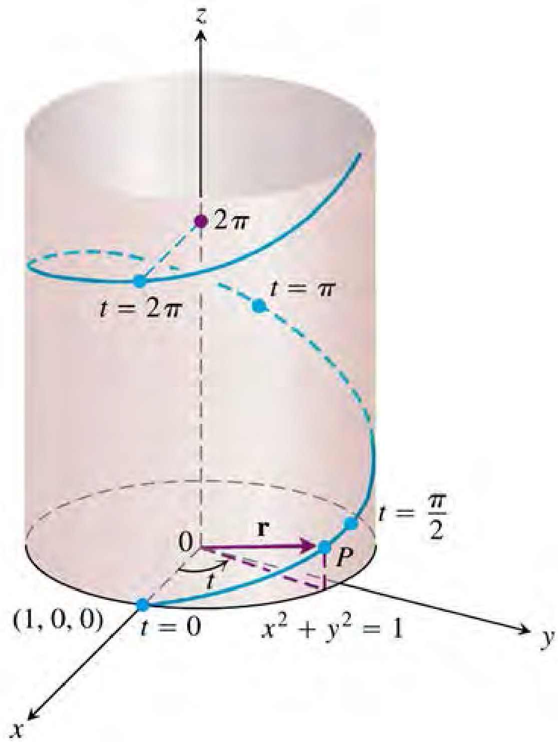
Solution

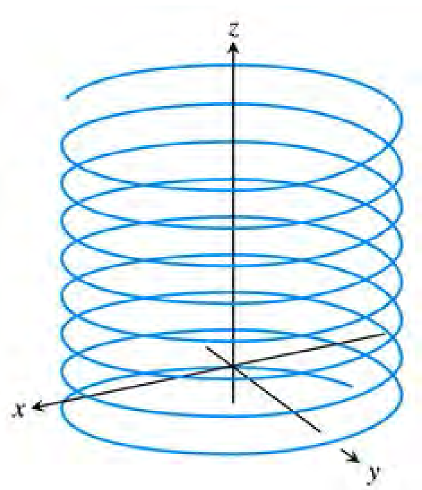
$x^2+y^2=(\cos t)^2+(\sin t)^2=1$

The curves traced by $\boldsymbol{r}$ winds around a circular cylinder, satisfies the equation.

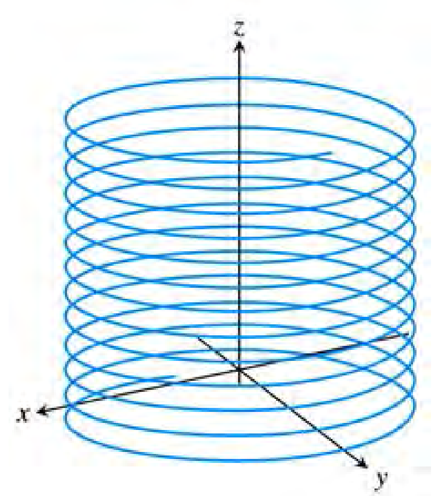
The curve rises as the $\boldsymbol{k}$-components $z = t$ increases. Each time t increases by 2π, the curve completes one turn around the cylinder. The curve is called a helix (from an old Greek word for "spiral"). The equations

$$x=\cos t, \quad y=\sin t, \quad z=t$$





$\boldsymbol{r}(t)=(\cos t)\boldsymbol{i}+(\sin t)\boldsymbol{j}+0.3t\boldsymbol{k}$



$\boldsymbol{r}(t)=(\cos 5t)\boldsymbol{i}+(\sin 5t)\boldsymbol{j}+t\boldsymbol{k}$

Limits and Continuity

Definition

Let $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$ be a vector function with domain *D*, and ***L*** a vector. We say that ***r*** has limit ***L*** as *t* approaches t_0 and write

$$\lim_{t \to t_0} \boldsymbol{r}(t) = \boldsymbol{L}$$

If, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$

$$\left|\boldsymbol{r}(t) - \boldsymbol{L}\right| < \varepsilon \quad \textit{whenever} \quad 0 < \left|t - t_0\right| < \delta$$

Example

Find the limit of $\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} + t\boldsymbol{k}$ as *t* approaches $\frac{\pi}{4}$

Solution

$$\lim_{t \to \pi/4} \boldsymbol{r}(t) = \left(\lim_{t \to \pi/4} \cos t\right)\boldsymbol{i} + \left(\lim_{t \to \pi/4} \sin t\right)\boldsymbol{j} + \left(\lim_{t \to \pi/4} t\right)\boldsymbol{k}$$

$$= \frac{\sqrt{2}}{2}\boldsymbol{i} + \frac{\sqrt{2}}{2}\boldsymbol{j} + \frac{\pi}{4}\boldsymbol{k}$$

Definition

A vector fumction $\boldsymbol{r}(t)$ is ***continuous at a point*** $t = t_0$ in its domain if $\lim\limits_{t \to t_0} \boldsymbol{r}(t) = \boldsymbol{r}(t_0)$. The function is continuous if it is continuous at every point in its domain.

Lines of Intersection

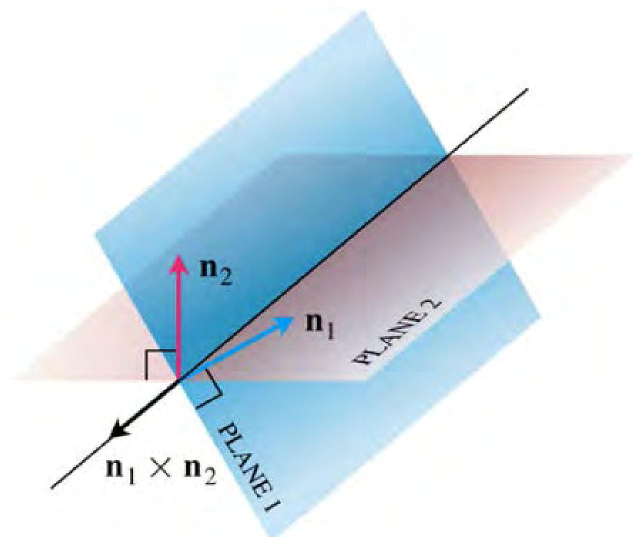
Example

Find a vector parallel to the line of intersection of the planes $3x-6y-2z=15$ *and* $2x+y-2z=5$

Solution

The line of intersection of two planes is perpendicular to both planes' normal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ and therefore parallel to $\boldsymbol{n}_1 \times \boldsymbol{n}_2$.

$$\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = \underline{14\boldsymbol{i}+2\boldsymbol{j}+15\boldsymbol{k}}$$



Example

Find the point where the line $x=\frac{8}{3}+2t, \quad y=-2t, \quad z=1+t$ intersects the plane $3x+2y+6z=6$.

Solution

The point: $\left(\frac{8}{3}+2t,\ -2t,\ 1+t\right)$

lies in the plane if its coordinates satisfy the equation of the plane, that is, if

$$3\left(\frac{8}{3}+2t\right)+2(-2t)+6(1+t)=6$$

$$8+6t-4t+6+6t=6$$

$$8t=-8$$

$$\boxed{t=-1}$$

The point of intersection is:

$$\left.\left(\frac{8}{3}+2t,\ -2t,\ 1+t\right)\right|_{t=-1} = \underline{\left(\frac{2}{3},\ 2,\ 0\right)}$$

The distance from a Point to a Plane

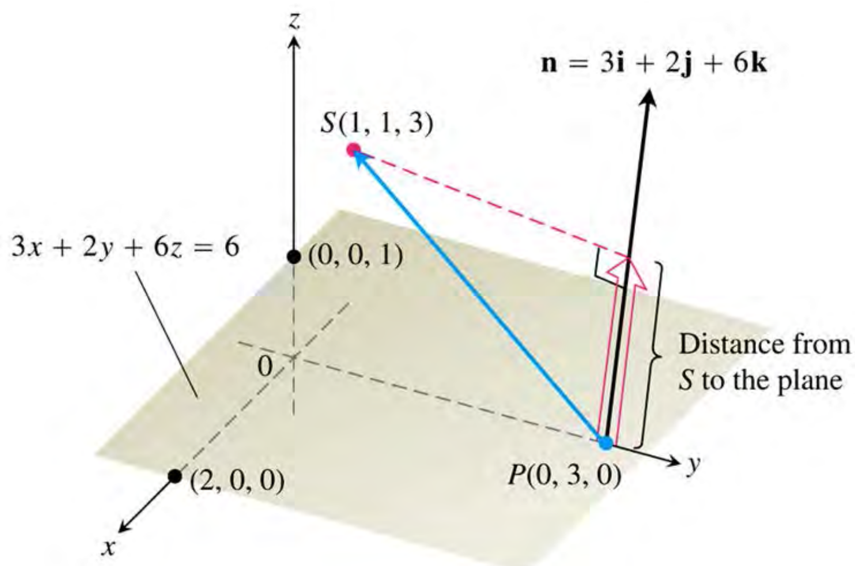
$$d = \left| \overrightarrow{PS} \cdot \frac{\boldsymbol{n}}{|\boldsymbol{n}|} \right|$$

Example

Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$

Solution

The coefficients in the equation $3x + 2y + 6z = 6$ give $\boldsymbol{n} = 3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$



$\overrightarrow{PS} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$

$|\boldsymbol{n}| = \sqrt{3^2 + 2^2 + 6^2} = 7$

The distance from S to the plane is

$$\begin{aligned} d &= \left| \overrightarrow{PS} \cdot \frac{\boldsymbol{n}}{|\boldsymbol{n}|} \right| \\ &= \left| (\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}) \cdot \left(\frac{3}{7}\boldsymbol{i} + \frac{2}{7}\boldsymbol{j} + \frac{6}{7}\boldsymbol{k} \right) \right| \\ &= \left| \frac{3}{7} - \frac{4}{7} + \frac{18}{7} \right| \\ &= \frac{17}{7} \end{aligned}$$

Angles Between Planes

Example

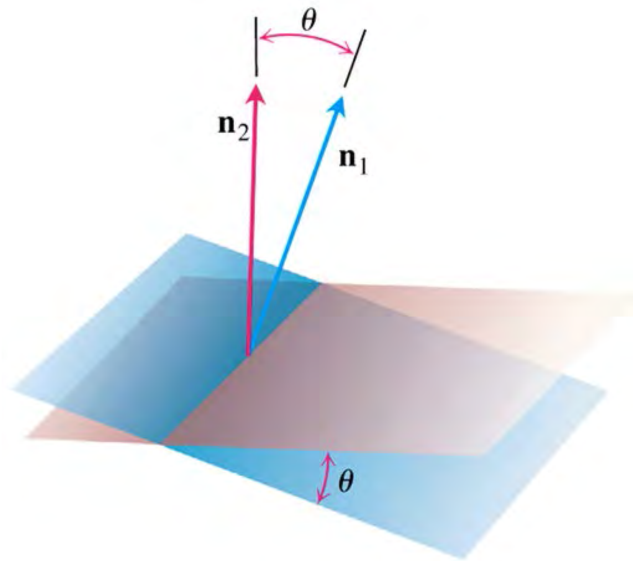
Find the angle between the planes $3x-6y-2z=15$ and $2x+y-2z=5$

Solution

The vectors: $\boldsymbol{n}_1 = 3\boldsymbol{i} - 6\boldsymbol{j} - 2\boldsymbol{k}, \quad \boldsymbol{n}_2 = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$ are normal to the planes.

The angle between them is:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\boldsymbol{n}_1 \bullet \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}\right)\\ &= \cos^{-1}\left(\frac{6-6+4}{\sqrt{9+36+4}\sqrt{4+1+4}}\right)\\ &= \cos^{-1}\left(\frac{4}{21}\right)\\ &\approx 1.38 \ \ rad\end{aligned}$$



Exercises *Section* 1.4 – Lines and Curves in Space

1. Find the parametric equation for the line through the point $P(3,-4,-1)$ parallel to the vector $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$

2. Find the parametric equation for the line through the points $P(1,2,-1)$ and $Q(-1,0,1)$

3. Find the parametric equation for the line through the points $P(-2,0,3)$ and $Q(3,5,-2)$

4. Find the parametric equation for the line through the origin parallel to the vector $2\boldsymbol{j}+\boldsymbol{k}$

5. Find the parametric equation for the line through the point $P(3,-2,1)$ parallel to the line $x=1+2t, \quad y=2-t, \quad z=3t$

6. Find the parametric equation for the line through $(2,4,5)$ perpendicular to the plane $3x+7y-5z=21$

7. Find the parametric equation for the line through $(2,3,0)$ perpendicular to the vectors $\boldsymbol{u}=\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}$ and $\boldsymbol{v}=3\boldsymbol{i}+4\boldsymbol{j}+5\boldsymbol{k}$

8. Find the parameterization for the line segment joining the points $(0,0,0)$, $\left(1,1,\frac{3}{2}\right)$. Draw coordinate axes and sketch the segment, indicate the direction on increasing *t* for the parametrization.

9. Find the parameterization for the line segment joining the points $(1,0,-1)$, $(0,3,0)$. Draw coordinate axes and sketch the segment, indicate the direction on increasing *t* for the parametrization.

10. Find equation for the plane through $P_0(0,2,-1)$ normal to $\boldsymbol{n}=3\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}$

11. Find equation for the plane through $(1,-1,3)$ parallel to the plane $3x+y+z=7$

12. Find equation for the plane through $(1,1,-1)$, $(2,0,2)$ and $(0,-2,1)$

13. Find equation for the plane through $P_0(2,4,5)$ perpendicular to the line $x=5+t, \quad y=1+3t, \quad z=4t$

14. Find equation for the plane through $A(1,-2,1)$ perpendicular to the vector from the origin to *A*.

15. Find the point of intersection of the lines $x=2t+1, \quad y=3t+2, \quad z=4t+3$ and $x=s+2, \quad y=2s+4, \quad z=-4s-1$, and find the plane determined by these lines.

16. Find the plane determined by the intersecting lines:

$L_1 :\ x = -1 + t, \quad y = 2 + t, \quad z = 1 - t; \quad -\infty < t < \infty$

$L_2 :\ x = 1 - 4s, \quad y = 1 + 2s, \quad z = 2 - 2s; \quad -\infty < s < \infty$

17. Find a plane through $P_0\,(2,1,-1)$ and perpendicular to the line of intersection of the planes $2x + y - z = 3, \quad x + 2y + z = 2$

18. Find the distance from the point to the plane $(0,0,12), \quad x = 4t, \quad y = -2t, \quad z = 2t$

19. Find the distance from the point to the plane $(2,1,-1), \quad x = 2t, \quad y = 1 + 2t, \quad z = 2t$

20. Find the distance from the point to the plane $(3,-1,4), \quad x = 4 - t, \quad y = 3 + 2t, \quad z = -5 + 3t$

21. Find the distance from the point to the plane $(2,-3,4), \quad x + 2y + 2z = 13$

22. Find the distance from the point to the plane $(0,0,0), \quad 3x + 2y + 6z = 6$

23. Find the distance from the point to the plane $(0,1,1), \quad 4y + 3z = -12$

24. Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$

25. Find the angle between the planes $x + y = 1, \quad 2x + y - 2z = 2$

26. Find the angle between the planes $5x + y - z = 10, \quad x - 2y + 3z = -1$

27. Find the point in which the line meets the plane $x = 1 - t, \quad y = 3t, \quad z = 1 + t; \quad 2x - y + 3z = 6$

28. Find the point in which the line meets the plane

$x = 2, \quad y = 3 + 2t, \quad z = -2 - 2t; \quad 6x + 3y - 4z = -12$

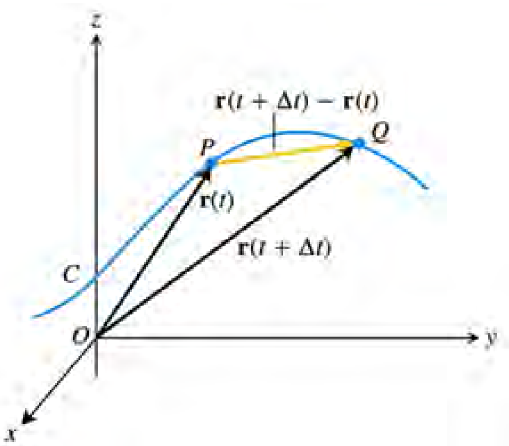
Section 1.5 – Calculus of Vector-Valued Functions

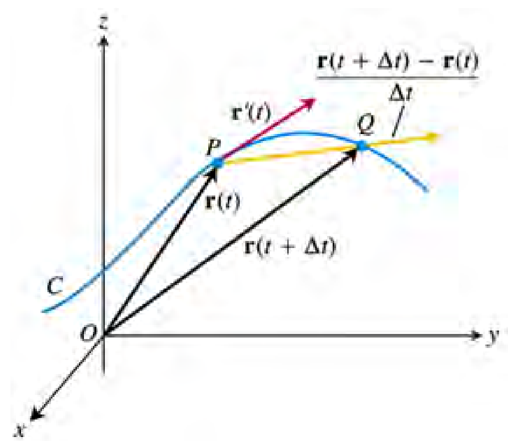
Derivative

Definition

The vector function $\boldsymbol{r}(t)=f(t)\boldsymbol{i}+g(t)\boldsymbol{j}+h(t)\boldsymbol{k}$ has a derivative (is differentiable) at *t* if *f*, *g*, and *h* have derivatives at *t*. The derivative is the vector function

$$\boldsymbol{r}'(t)=\frac{d\boldsymbol{r}}{dt}=\lim_{\Delta t\to 0}\frac{\boldsymbol{r}(t+\Delta t)-\boldsymbol{r}(t)}{\Delta t}=\frac{df}{dt}\boldsymbol{i}+\frac{dg}{dt}\boldsymbol{j}+\frac{dh}{dt}\boldsymbol{k}$$





Definitions

If $\boldsymbol{r}$ is the position vector of a particle moving along a smooth curve in space, then

$$\boldsymbol{v}(t)=\frac{d\boldsymbol{r}}{dt}$$

is the particle's ***velocity vector***, tangent to the curve. At any time t, the direction of v is the ***direction of motion***, the magnitude of v is the particle's ***speed***, and the derivative $\boldsymbol{a}=\frac{d\boldsymbol{v}}{dt}$, when it exists, is the particle's ***acceleration vector***. In summary,

1. Velocity is the derivative of position: $\boldsymbol{v}(t)=\frac{d\boldsymbol{r}}{dt}$
2. Speed is the magnitude of velocity: $Speed=|\boldsymbol{v}|$
3. Acceleration is the derivative of velocity: $\boldsymbol{a}=\frac{d\boldsymbol{v}}{dt}=\frac{d^2\boldsymbol{r}}{dt^2}$
4. The unit vector $\frac{\boldsymbol{v}}{|\boldsymbol{v}|}$ is the direction of motion at time *t*.

Example

Find the velocity, speed, and acceleration of a particle whose motion in space is given by the position vector $\boldsymbol{r}(t) = 2\cos t\ \boldsymbol{i} + 2\sin t\ \boldsymbol{j} + 5\cos^2 t\ \boldsymbol{k}$. Sketch the velocity vector $\boldsymbol{v}\left(\frac{7\pi}{4}\right)$

Solution

The velocity vector at time *t* is:

$$\begin{aligned}\boldsymbol{v}(t) = \boldsymbol{r}'(t) &= -2\sin t\ \boldsymbol{i} + 2\cos t\ \boldsymbol{j} - 10\cos t \sin t\ \boldsymbol{k} \\ &= -2\sin t\ \boldsymbol{i} + 2\cos t\ \boldsymbol{j} - 5\sin 2t\ \boldsymbol{k}\end{aligned}$$

The acceleration vector at time *t* is:

$$\boldsymbol{a}(t) = \boldsymbol{r}''(t) = -2\cos t\ \boldsymbol{i} - 2\sin t\ \boldsymbol{j} - 10\cos 2t\ \boldsymbol{k}$$

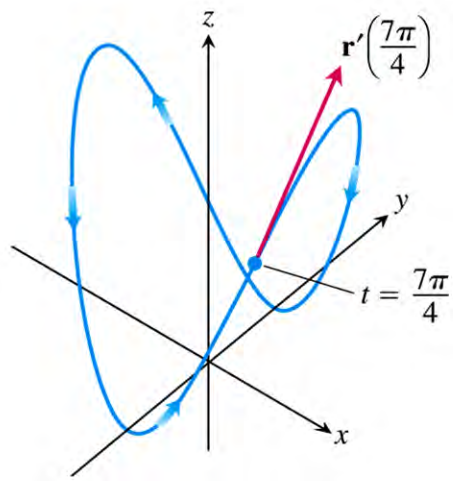
The speed is:

$$\begin{aligned}|\boldsymbol{v}(t)| &= \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-5\sin 2t)^2} \\ &= \sqrt{4\sin^2 t + 4\cos^2 t + 25\sin^2 2t} \\ &= \sqrt{4\left(\sin^2 t + \cos^2 t\right) + 25\sin^2 2t} \qquad \sin^2 t + \cos^2 t = 1 \\ &= \sqrt{4 + 25\sin^2 2t}\end{aligned}$$

$$\begin{aligned}\boldsymbol{v}\left(\frac{7\pi}{4}\right) &= -2\sin\left(\frac{7\pi}{4}\right)\boldsymbol{i} + 2\cos\left(\frac{7\pi}{4}\right)\boldsymbol{j} - 5\sin\left(\frac{7\pi}{2}\right)\boldsymbol{k} \\ &= \underline{\sqrt{2}\boldsymbol{i} + \sqrt{2}\boldsymbol{j} + 5\ \boldsymbol{k}}\end{aligned}$$

$$\begin{aligned}\boldsymbol{a}\left(\frac{7\pi}{4}\right) &= -2\cos\left(\frac{7\pi}{4}\right)\boldsymbol{i} - 2\sin\left(\frac{7\pi}{4}\right)\boldsymbol{j} - 10\cos\left(\frac{7\pi}{2}\right)\boldsymbol{k} \\ &= \underline{-\sqrt{2}\boldsymbol{i} + \sqrt{2}\boldsymbol{j}}\end{aligned}$$

$$\left|\boldsymbol{v}\left(\frac{7\pi}{4}\right)\right| = \sqrt{4 + 25\sin^2\left(\frac{7\pi}{2}\right)} = \underline{\sqrt{29}}$$



Differentiation Rules for vector Functions

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be differentiable vector functions of t, C a constant vector, c any scalar and f any differentiable scalar function.

1. *Constant Function Rule*: $\frac{d}{dt}\boldsymbol{C} = \mathbf{0}$
2. *Scalar Multiple Rules*: $\frac{d}{dt}\left[c\boldsymbol{u}(t)\right] = c\boldsymbol{u}'(t)$

   $\frac{d}{dt}\left[f(t)\boldsymbol{u}(t)\right] = f'(t)\boldsymbol{u}(t) + f(t)\boldsymbol{u}'(t)$
3. *Sum Rule*: $\frac{d}{dt}\left[\boldsymbol{u}(t) + \boldsymbol{v}(t)\right] = \boldsymbol{u}'(t) + \boldsymbol{v}'(t)$
4. *Difference Rule*: $\frac{d}{dt}\left[\boldsymbol{u}(t) - \boldsymbol{v}(t)\right] = \boldsymbol{u}'(t) - \boldsymbol{v}'(t)$
5. *Dot Product Rule*: $\frac{d}{dt}\left[\boldsymbol{u}(t)\bullet\boldsymbol{v}(t)\right] = \boldsymbol{u}'(t)\bullet\boldsymbol{v}(t) + \boldsymbol{u}(t)\bullet\boldsymbol{v}'(t)$
6. *Cross Product Rule*: $\frac{d}{dt}\left[\boldsymbol{u}(t)\times\boldsymbol{v}(t)\right] = \boldsymbol{u}'(t)\times\boldsymbol{v}(t) + \boldsymbol{u}(t)\times\boldsymbol{v}'(t)$
7. *Chain Rule*: $\frac{d}{dt}\left[\boldsymbol{u}(f(t))\right] = f'(t)\boldsymbol{u}'(f(t))$

Vector Functions of Constant Length

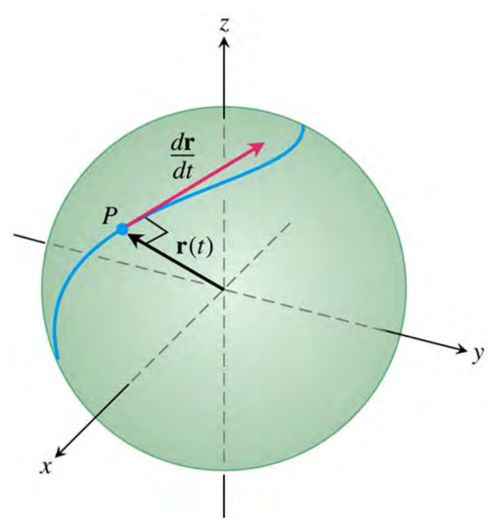
The position vector, of a particle that is moving on a sphere, has a constant length equal to the radius of the sphere. The velocity vector $\frac{d\boldsymbol{r}}{dt}$, tangent to the path of motion, is tangent to the sphere and hence perpendicular to $\boldsymbol{r}$. the vector and its first derivative are orthogonal.

$\boldsymbol{r}(t)\bullet\boldsymbol{r}(t) = c^2$ $\qquad$ $|\boldsymbol{r}(t)| = c$ *is constant*

$\frac{d}{dt}\left[\boldsymbol{r}(t)\bullet\boldsymbol{r}(t)\right] = 0$ $\qquad$ ***Differentiate both sides***

$\boldsymbol{r}'(t)\bullet\boldsymbol{r}(t) + \boldsymbol{r}(t)\bullet\boldsymbol{r}'(t) = 0$

$2\boldsymbol{r}'(t)\bullet\boldsymbol{r}(t) = 0$



If $\boldsymbol{r}$ is a differentiable vector function of t of constant length, then

$$\boldsymbol{r}\bullet\frac{d\boldsymbol{r}}{dt} = 0$$

Exercises *Section* 1.5 – Calculus of Vector-Valued Functions

(*Exercises* 1 - 4) $\boldsymbol{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find an equation in *x* and *y* whose is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of *t*.

1. $\boldsymbol{r}(t)=(t+1)\boldsymbol{i}+\left(t^2-1\right)\boldsymbol{j}, \quad t=1$

2. $\boldsymbol{r}(t)=\frac{t}{t+1}\boldsymbol{i}+\frac{1}{t}\boldsymbol{j}, \quad t=-\frac{1}{2}$

3. $\boldsymbol{r}(t)=e^t\boldsymbol{i}+\frac{2}{9}e^{2t}\boldsymbol{j}, \quad t=\ln 3$

4. $\boldsymbol{r}(t)=(\cos 2t)\boldsymbol{i}+(3\sin 2t)\boldsymbol{j}, \quad t=0$

Give the position vectors of particles moving along various curves in the *xy*-plane. Find the particle's velocity and acceleration vectors at the stated times and sketch them as vectors on the curve

5. Motion on the circle $x^2+y^2=1$ $\quad \boldsymbol{r}(t)=(\sin t)\boldsymbol{i}+(\cos t)\boldsymbol{j}, \quad t=\frac{\pi}{4}$ *and* $\frac{\pi}{2}$

6. Motion on the cycloid $x=t-\sin t, \ \ y=1-\cos t; \quad \boldsymbol{r}(t)=(1-\sin t)\boldsymbol{i}+(1-\cos t)\boldsymbol{j}; \quad t=\pi\ \&\ \frac{3\pi}{2}$

$\boldsymbol{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of *t*. Write the particle's velocity at that time as the product of its speed and direction.

7. $\boldsymbol{r}(t)=(t+1)\boldsymbol{i}+\left(t^2-1\right)\boldsymbol{j}+2t\boldsymbol{k}, \quad t=1$

8. $\boldsymbol{r}(t)=(t+1)\boldsymbol{i}+\frac{t^2}{\sqrt{2}}\boldsymbol{j}+\frac{t^3}{3}\boldsymbol{k}, \quad t=1$

9. $\boldsymbol{r}(t)=(2\cos t)\boldsymbol{i}+(3\sin t)\boldsymbol{j}+4t\boldsymbol{k}, \quad t=\frac{\pi}{2}$

10. $\boldsymbol{r}(t)=(2\ln(t+1))\boldsymbol{i}+t^2\boldsymbol{j}+\frac{t^2}{2}\boldsymbol{k}, \quad t=1$

11. $\boldsymbol{r}(t)=\left(e^{-t}\right)\boldsymbol{i}+(2\cos 3t)\boldsymbol{j}+(2\sin 3t)\boldsymbol{k}, \quad t=0$

Section 1.6 – Motion in Space

Integrals of Vector Functions

A differentiable vector function $\boldsymbol{R}(t)$ is an antiderivative of a vector function $\boldsymbol{r}(t)$ on interval *I* if $\frac{d\boldsymbol{R}}{dt}=\boldsymbol{r}$ at each point on *I*.

Definition

The indefinite integral of $\boldsymbol{r}$ with respect to *t* is the set of all antiderivatives of $\boldsymbol{r}$, denoted by $\int \boldsymbol{r}(t)dt$. If $\boldsymbol{R}$ is any antiderivative of $\boldsymbol{r}$, then

$$\int \boldsymbol{r}(t)dt=\boldsymbol{R}(t)+C$$

Example

Integrate: $\int \left((\cos t)\boldsymbol{i}+\boldsymbol{j}-2t\boldsymbol{k}\right)dt$

Solution

$$\begin{aligned}\int \left((\cos t)\boldsymbol{i}+\boldsymbol{j}-2t\boldsymbol{k}\right)dt &= \left(\int \cos t\ dt\right)\boldsymbol{i}+\left(\int dt\right)\boldsymbol{j}-\left(\int 2tdt\right)\boldsymbol{k}\\ &= \left(\sin t+C_1\right)\boldsymbol{i}+\left(t+C_2\right)\boldsymbol{j}-\left(t^2+C_3\right)\boldsymbol{k}\\ &= (\sin t)\boldsymbol{i}+t\boldsymbol{j}-t^2\boldsymbol{k}+\boldsymbol{C} \qquad \boldsymbol{C}=C_1\boldsymbol{i}+C_2\boldsymbol{j}+C_3\boldsymbol{k}\end{aligned}$$

Definition

If the components of $\boldsymbol{r}(t)=f(t)\boldsymbol{i}+g(t)\boldsymbol{j}+h(t)\boldsymbol{k}$ are integrable over [*a*, *b*], then so is r, and the ***definite integral*** of $\boldsymbol{r}$ from *a* to *b* is

$$\int_a^b \boldsymbol{r}(t)dt=\left(\int_a^b f(t)dt\right)\boldsymbol{i}+\left(\int_a^b g(t)dt\right)\boldsymbol{j}+\left(\int_a^b h(t)dt\right)\boldsymbol{k}$$

Example

Evaluate the integral: $\displaystyle\int_0^{\pi}\left((\cos t)\boldsymbol{i}+\boldsymbol{j}-2t\boldsymbol{k}\right)dt$

Solution

$$\int_0^{\pi}\left((\cos t)\boldsymbol{i}+\boldsymbol{j}-2t\boldsymbol{k}\right)dt=\left[\sin t\right]_0^{\pi}\boldsymbol{i}+\left[t\right]_0^{\pi}\boldsymbol{j}-\left[t^2\right]_0^{\pi}\boldsymbol{k}$$

$$=\left[0-0\right]\boldsymbol{i}+\left[\pi-0\right]\boldsymbol{j}-\left[\pi^2-0\right]\boldsymbol{k}$$

$$=\pi\boldsymbol{j}-\pi^2\boldsymbol{k}$$

Example

Suppose the acceleration vector of the path of a hang glider is given by $\boldsymbol{a}(t)=-(3\cos t)\boldsymbol{i}-(3\sin t)\boldsymbol{j}+2\boldsymbol{k}$. At time $t = 0$, the glider departed from the point (3, 0, 0) with velocity $\boldsymbol{v}(0)=3\boldsymbol{j}$. Find the glider's position as a function of t.

Solution

$$\boldsymbol{v}(t)=\int\boldsymbol{a}(t)dt=\int\left(-(3\cos t)\boldsymbol{i}-(3\sin t)\boldsymbol{j}+2\boldsymbol{k}\right)dt$$

$$=-(3\sin t)\boldsymbol{i}+(3\cos t)\boldsymbol{j}+2t\boldsymbol{k}+\boldsymbol{C}_1$$

Given: $\boldsymbol{v}(0)=3\boldsymbol{j}$

$$3\boldsymbol{j}=-(3\sin 0)\boldsymbol{i}+(3\cos 0)\boldsymbol{j}+2(0)\boldsymbol{k}+\boldsymbol{C}_1$$

$$3\boldsymbol{j}=3\boldsymbol{j}+\boldsymbol{C}_1 \quad\Rightarrow\quad \boxed{\boldsymbol{C}_1=\boldsymbol{0}}$$

$$\boldsymbol{v}(t)=-(3\sin t)\boldsymbol{i}+(3\cos t)\boldsymbol{j}+2t\boldsymbol{k}$$

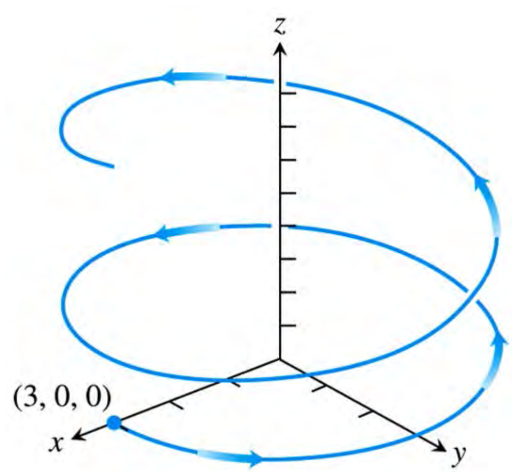
$$\boldsymbol{r}(t)=\int\boldsymbol{v}(t)dt=\int\left(-(3\sin t)\boldsymbol{i}+(3\cos t)\boldsymbol{j}+2t\boldsymbol{k}\right)dt$$

$$=(3\cos t)\boldsymbol{i}+(3\sin t)\boldsymbol{j}+t^2\boldsymbol{k}+\boldsymbol{C}_2$$

Given: $\boldsymbol{r}(0)=3\boldsymbol{i}$

$$3\boldsymbol{i}=(3\cos 0)\boldsymbol{i}+(3\sin 0)\boldsymbol{j}+(0)^2\boldsymbol{k}+\boldsymbol{C}_2$$

$$3\boldsymbol{i}=3\boldsymbol{i}+\boldsymbol{C}_2 \quad\Rightarrow\quad \boxed{\boldsymbol{C}_2=\boldsymbol{0}}$$

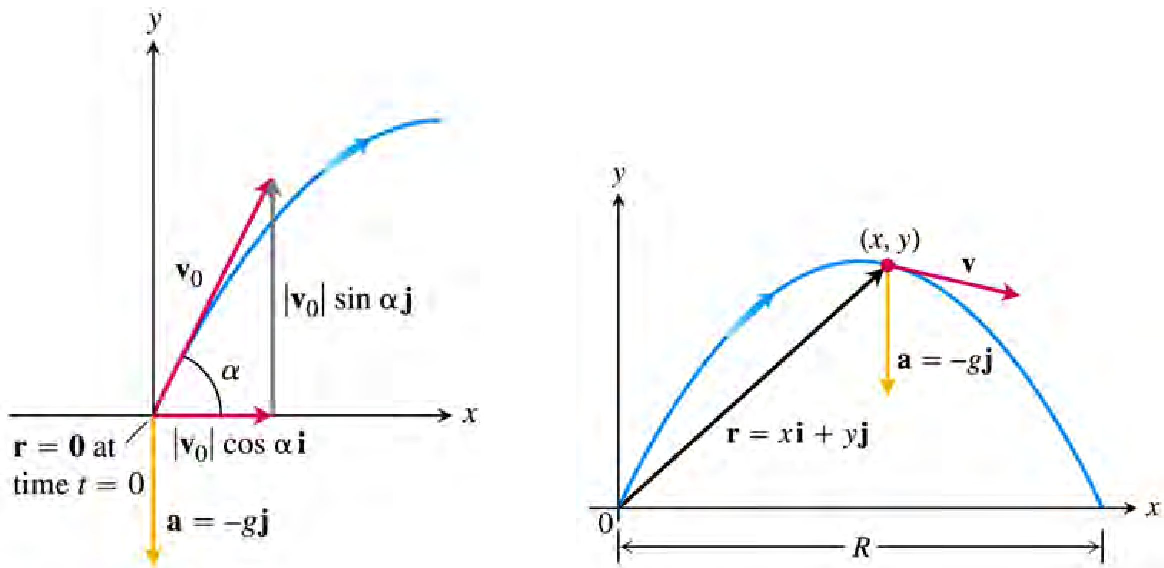


$$\boxed{\boldsymbol{r}(t)=(3\cos t)\boldsymbol{i}+(3\sin t)\boldsymbol{j}+t^2\boldsymbol{k}}$$

The vector and Parametric Equations for Ideal Projectile Motion

A projectile motion describes how an object fired at some angle from an initial position, and acted upon by only the force of gravity, moves in a vertical coordinate plane. We ignore the effects of any frictional drag on the object, which may vary with its speed and altitude, and also the fact the force of gravity changes slightly with the projectile's changing height.

To derive equations for projectile motion, we assume that the projectile behaves like a particle moving in a vertical coordinate plane and the only force acting on the projectile during its flight is the constant force of gravity, which always points straight down.



Ideal Projectile Motion Equation

$$\boldsymbol{r} = \left(v_0 \cos\alpha\right)t\,\boldsymbol{i} + \left(\left(v_0 \sin\alpha\right)t - \tfrac{1}{2}gt^2\right)\boldsymbol{j}$$

This is the vector equation for ideal projectile motion. The angle α is the projectile's ***launch angle*** (***firing angle, angle of elevation***), and v_0 is the projectile's ***initial speed***. The components of ***r*** give the parametric equations

$$x = \left(v_0 \cos\alpha\right)t \quad \textit{and} \quad y = \left(v_0 \sin\alpha\right)t - \tfrac{1}{2}gt^2$$

Where x is the distance downrange and y is the height of the projectile at time $t \geq 0$.

Example

A projectile is fired from the origin over horizontal ground at an initial speed of 500 m/sec and a launch angle of 60°. Where will the projectile be 10 sec later?

Solution

Given: $v_0 = 500, \quad \alpha = 60°, \quad g = 9.8, \quad t = 10$

$$\boldsymbol{r} = \left(v_0 \cos\alpha\right)t\boldsymbol{i} + \left(\left(v_0 \sin\alpha\right)t - \tfrac{1}{2}gt^2\right)\boldsymbol{j}$$

$$= \left(500\cos 60°\right)(10)\boldsymbol{i} + \left(\left(500\sin 60°\right)(10) - \tfrac{1}{2}9.8(10)^2\right)\boldsymbol{j}$$

$$\approx 2500\boldsymbol{i} + 3840\boldsymbol{j}$$

After 10 *sec*, the projectile is about 3840 *m* above the ground and 2500 *m* downrange from the origin.

Height, Flight Time, and Range for Ideal Projectile Motion

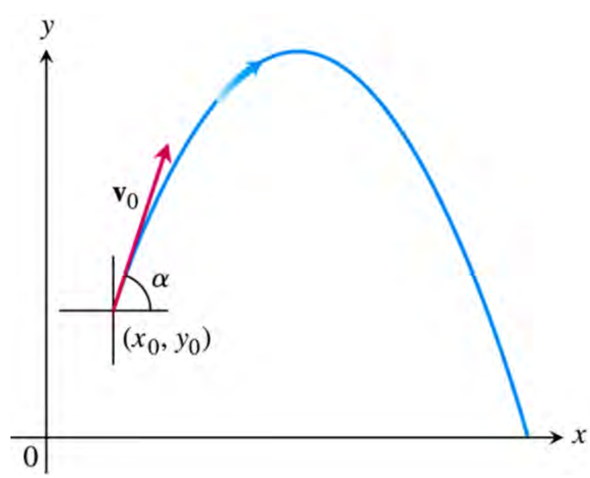
For ideal projectile motion when an object is launched from the origin over a horizontal surface with initial speed v_0 and launch angle α:

Maximum height: $y_{\max} = \dfrac{\left(v_0 \sin\alpha\right)^2}{2g}$

Maximum time: $t_{\max} = \dfrac{v_0 \sin\alpha}{g}$

Flight time: $t = \dfrac{2v_0 \sin\alpha}{g}$

Range: $R = \dfrac{v_0^2}{g}\sin 2\alpha$



If we fire a projectile from a point $\left(x_0, y_0\right)$ instead of the origin, then the position vector for the path of motion is

$$\boldsymbol{r} = \left(x_0 + \left(v_0 \cos\alpha\right)t\right)\boldsymbol{i} + \left(y_0 + \left(v_0 \sin\alpha\right)t - \tfrac{1}{2}gt^2\right)\boldsymbol{j}$$

Projectile Motion with Wind Gusts

Example

A baseball is hit when it is 3 ft above the ground. It leaves the bat with initial speed of 152 ft/sec, making an angle of 20° with the horizontal. At the instant the ball hit, an instantaneous gust of wind blows in the horizontal direction directly opposite the direction the ball is taking toward the outfield, adding a component of $-8.8\boldsymbol{i}$ $(ft/\sec)$ to the ball's initial velocity $(8.8\ ft/\sec = 6\ mph)$.

a) Find a vector equation (position vector) for the path of the baseball.
b) How high does the baseball go, and when does it reach maximum height?
c) Assuming that the ball is not caught, find its range and flight time.

Solution

a) The initial velocity of the baseball is:

$$\boldsymbol{v} = \left(v_0 \cos\alpha\right)\boldsymbol{i} + \left(v_0 \sin\alpha\right)\boldsymbol{j} - 8.8\,\boldsymbol{i}$$

$$= \left(152\cos 20°\right)\boldsymbol{i} + \left(152\sin 20°\right)\boldsymbol{j} - 8.8\,\boldsymbol{i}$$

The initial position is $r_0 = 0\boldsymbol{i} + 3\boldsymbol{j}$.

$$\boldsymbol{r} = -\tfrac{1}{2}gt^2\boldsymbol{j} + \boldsymbol{v}_0 t + \boldsymbol{r}_0$$

$$= -16t^2\boldsymbol{j} + \left(\left(152\cos 20°\right)\boldsymbol{i} + \left(152\sin 20°\right)\boldsymbol{j} - 8.8\,\boldsymbol{i}\right)t + 3\boldsymbol{j}$$

$$= -16t^2\boldsymbol{j} + \left(152\cos 20°\right)t\,\boldsymbol{i} + \left(152\sin 20°\right)t\boldsymbol{j} - 8.8t\,\boldsymbol{i} + 3\boldsymbol{j}$$

$$= \left(152\cos 20° - 8.8\right)t\,\boldsymbol{i} + \left(3 + \left(152\sin 20°\right)t - 16t^2\right)\boldsymbol{j}$$

$$= 134.033t\,\boldsymbol{i} + \left(3 + 51.987t - 16t^2\right)\boldsymbol{j}$$

b) The baseball reaches its highest point when the vertical component of velocity is zero:

$$\frac{dy}{dt} = \frac{d}{dt}\left(3 + \left(152\sin 20°\right)t - 16t^2\right)$$

$$= 152\sin 20° - 32t = 0$$

$$t = \frac{152\sin 20°}{32} \approx 1.62\ \sec$$

$$y_{Max} = 3 + \left(152\sin 20°\right)(1.62) - 16(1.62)^2$$

$$\approx 45.2\ ft$$

The maximum height of the baseball is about 45.2 *ft*, reached about 1.6 *sec* after leaving the bat.

c) The vertical component for ***r*** equal to 0:

$$3+51.987t-16t^2=0$$

$$-16t^2+51.987t+3=0 \qquad \textbf{\textit{Solve for t.}}$$

$$\Rightarrow \quad \boxed{t=3.3\ \text{sec}} \quad \textit{and} \quad t=-0.06\ \text{sec}$$

$$R=134.033(3.3)$$

$$\approx 442\ \ ft$$

The horizontal range is about 442 *ft*, and the flight time is about 3.3 *sec*.

Exercises *Section* 1.6 – Motion in Space

Evaluate the integral

1. $\int_0^1 \left(t^3\boldsymbol{i} + 7\boldsymbol{j} + (t+1)\boldsymbol{k}\right)dt$

2. $\int_1^2 \left((6-6t)\boldsymbol{i} + 3\sqrt{t}\,\boldsymbol{j} + \frac{4}{t^2}\boldsymbol{k}\right)dt$

3. $\int_{-\pi/4}^{\pi/4} \left((\sin t)\boldsymbol{i} + (1+\cos t)\boldsymbol{j} + \left(\sec^2 t\right)\boldsymbol{k}\right)dt$

4. $\int_0^{\pi/3} \left((\sec t \tan t)\boldsymbol{i} + (\tan t)\boldsymbol{j} + (2\sin t\cos t)\boldsymbol{k}\right)dt$

5. $\int_0^1 \left(\frac{2}{\sqrt{1-t^2}}\boldsymbol{i} + \frac{\sqrt{3}}{1+t^2}\boldsymbol{k}\right)dt$

6. $\int_1^{\ln 3} \left(te^t\boldsymbol{i} + e^t\boldsymbol{j} + (\ln t)\boldsymbol{k}\right)dt$

7. $\int_0^{\pi/2} \left(\cos t\ \boldsymbol{i} - \sin 2t\ \boldsymbol{j} + \sin^2 t\ \boldsymbol{k}\right)dt$

8. Solve the initial value problem for $\boldsymbol{r}$ as a vector function of t.

$$\begin{cases} \textit{Differential equation}: & \dfrac{d\boldsymbol{r}}{dt} = -t\boldsymbol{i} - t\boldsymbol{j} - t\boldsymbol{k} \\ \textit{Initial condition}: & \boldsymbol{r}(0) = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k} \end{cases}$$

9. Solve the initial value problem for $\boldsymbol{r}$ as a vector function of t.

$$\begin{cases} \textit{Differential equation}: & \dfrac{d\boldsymbol{r}}{dt} = (180t)\boldsymbol{i} + \left(180t - 16t^2\right)\boldsymbol{j} \\ \textit{Initial condition}: & \boldsymbol{r}(0) = 100\boldsymbol{j} \end{cases}$$

10. Solve the initial value problem for $\boldsymbol{r}$ as a vector function of t.

$$\begin{cases} \textit{Differential equation}: & \dfrac{d\boldsymbol{r}}{dt} = \frac{3}{2}(t+1)^{1/2}\boldsymbol{i} + e^{-t}\boldsymbol{j} + \frac{1}{t+1}\boldsymbol{k} \\ \textit{Initial condition}: & \boldsymbol{r}(0) = \boldsymbol{k} \end{cases}$$

11. Solve the initial value problem for $\boldsymbol{r}$ as a vector function of t.

$$\textit{Differential equation}: \quad \frac{d^2\boldsymbol{r}}{dt^2} = -32\boldsymbol{k}$$

$$\textit{Initial condition}: \quad \boldsymbol{r}(0) = 100\boldsymbol{k}; \quad \left.\frac{d\boldsymbol{r}}{dt}\right|_{t=0} = 8\boldsymbol{i} + 8\boldsymbol{j}$$

12. Solve the initial value problem for $\boldsymbol{r}$ as a vector function of t.

$$\textit{Differential equation}: \quad \frac{d^2\boldsymbol{r}}{dt^2} = -(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$$

$$\textit{Initial condition}: \quad \boldsymbol{r}(0) = 10\boldsymbol{i} + 10\boldsymbol{j} + 10\boldsymbol{k}; \quad \left.\frac{d\boldsymbol{r}}{dt}\right|_{t=0} = \boldsymbol{0}$$

13. At time $t = 0$, a particle is located at the point $(1,\ 2,\ 3)$. It travels in a straight line to the point $(4,\ 1,\ 4)$, has speed 2 at $(1,\ 2,\ 3)$ and constant acceleration $3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$. Find an equation for the position vector $\boldsymbol{r}(t)$ of the particle at time t.

14. A projectile is fired at a speed of 840 *m/sec* at an angle of 60°. How long will it take to get 21 *km* downrange?

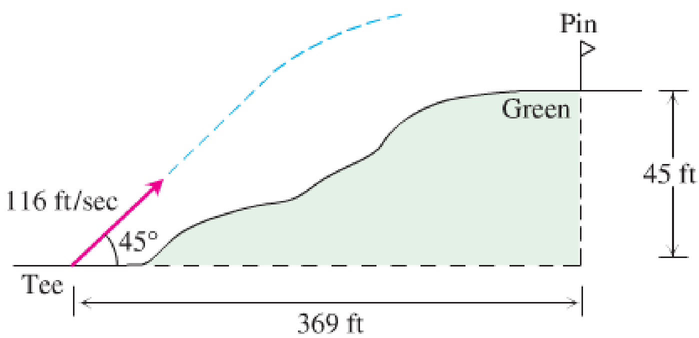
15. Find the muzzle speed of a gun whose maximum range is 24.5 *km*.

16. A spring gun at ground level fires a golf ball at an angle of 45°. The ball lands 10 *m* away.

a) What was the ball's initial speed?
b) For the same initial speed, find the two firing angles that make the range 6 *m*.

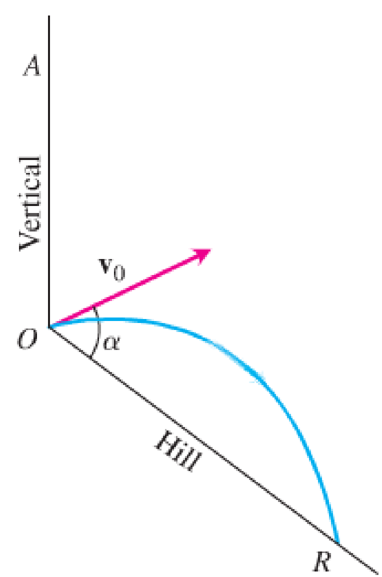
17. An electron in a TV tube is beamed horizontally at a speed of $5 \times 10^6\ m/\sec$ toward the face of the tube 40 *cm* away. About how far will the electron drop before it hits?

18. A golf ball is hit with an initial speed of 116 ft/sec at an angle of elevation of 45° from the tee to a green that is elevated 45 ft above the tee. Assuming that the pin, 369 ft downrange, does not get in the way, where will the ball land in relation to the pin?



19. An ideal projectile is launched straight down an inclined plane.

a) Show that the greatest downhill range is achieved when the initial velocity vector bisects angle AOR.
b) If the projectile were fired uphill instead of down, what launch angle would maximize its range?



20. A volleyball is hit when it is 4 *ft* above the ground and 12 *ft* from a 6-*ft*-high net. It leaves the point of impact with an initial velocity of 35 *ft/sec* at an angle of 27° and slips by the opposing team untouched.

a) Find a vector equation for the path of the volleyball.

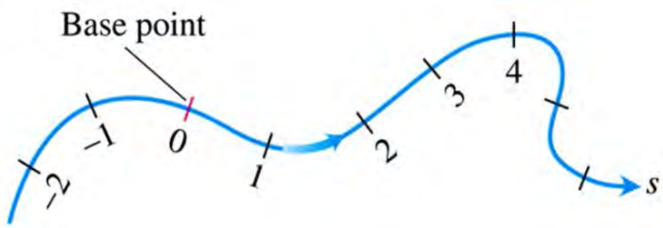
b) How high does the volleyball go, and when does it reach maximum height?

c) Find its range and flight time.

d) When is the volleyball 7 ft above the ground? How far (ground distance) is the volleyball from where it will land?

e) Suppose that the net is raised to 8 ft. Does this changes things? Explain.

Section 1.7 – Length of Curves



Arc Length along a Space Curve

Definition

The ***length*** of a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

Arc Length Formula $\qquad L = \int_a^b |\mathbf{v}|\ dt$

Example

A glider is soaring upward along the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\,\mathbf{k}$. How long is the glider's path from $t = 0$ to $t = 2\pi$?

Solution

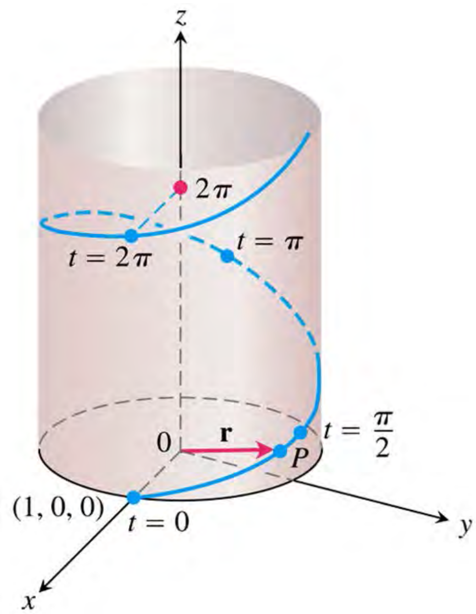
The path segment during this time corresponds to one full turn of the helix. The length of this portion of the curve is

$$\begin{aligned} L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt \\ &= \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\ dt \\ &= \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1^2}\ dt \\ &= \int_0^{2\pi} \sqrt{2}\ dt \end{aligned}$$

$$= \sqrt{2}\,[t]_0^{2\pi}$$

$$\underline{= 2\pi\sqrt{2}}$$

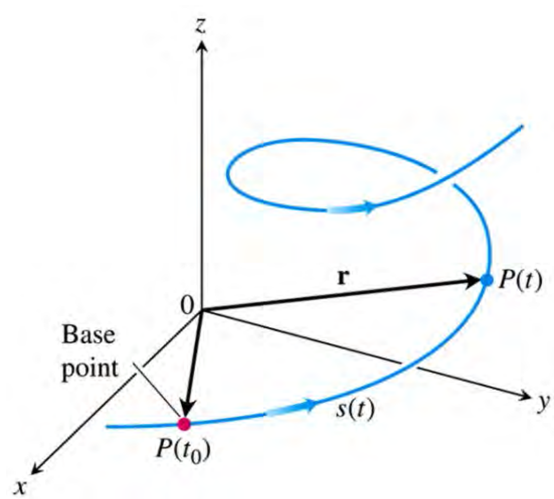
$\therefore$ This is $\sqrt{2}$ times the circumference of the circle in the *xy*-plane over which the helix stands.



Arc Length Parameter with Base Point $P(t_0)$

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\; d\tau$$

$$= \int_{t_0}^{t} |\mathbf{v}(\tau)|\; d\tau$$

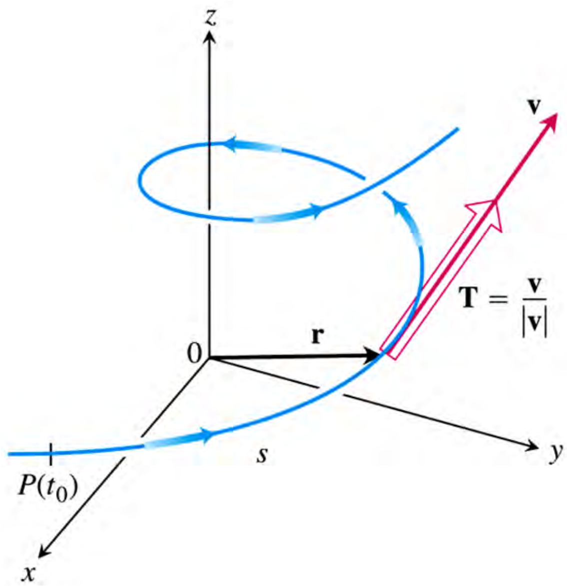


Unit Tangent Vector

The velocity vector $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$ is tangent to the curve $\boldsymbol{r}(t)$ and that the vector

$$\boldsymbol{T} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$$

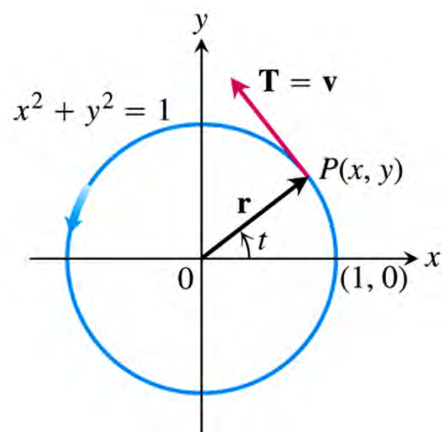
A unit vector tangent to the (*smooth*) curve, called the ***unit tangent vector***.



Example

Find the unit tangent vector of the curve $\boldsymbol{r}(t) = (3\cos t)\boldsymbol{i} + (3\sin t)\boldsymbol{j} + t^2\,\boldsymbol{k}$ representing the path of the glider.

Solution



$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt} = -(3\sin t)\boldsymbol{i} + (3\cos t)\boldsymbol{j} + 2t\,\boldsymbol{k}$$

$$|\boldsymbol{v}| = \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2}$$

$$= \sqrt{9 + 4t^2}$$

$$\boldsymbol{T} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$$

$$= -\frac{3\sin t}{\sqrt{9+4t^2}}\boldsymbol{i} + \frac{3\cos t}{\sqrt{9+4t^2}}\boldsymbol{j} + \frac{2t}{\sqrt{9+4t^2}}\boldsymbol{k}$$

Exercises   *Section* 1.7 – Length of Curves

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

1. $\boldsymbol{r}(t)=(2\cos t)\boldsymbol{i}+(2\sin t)\boldsymbol{j}+\sqrt{5}t\,\boldsymbol{k};\quad 0\le t\le \pi$

2. $\boldsymbol{r}(t)=t\boldsymbol{i}+\frac{2}{3}t^{3/2}\,\boldsymbol{k};\quad 0\le t\le 8$

3. $\boldsymbol{r}(t)=(2+t)\boldsymbol{i}-(t+1)\boldsymbol{j}+t\,\boldsymbol{k};\quad 0\le t\le 3$

4. $\boldsymbol{r}(t)=\left(\cos^3 t\right)\boldsymbol{i}+\left(\sin^3 t\right)\boldsymbol{k};\quad 0\le t\le \frac{\pi}{2}$

5. $\boldsymbol{r}(t)=(t\sin t+\cos t)\boldsymbol{i}+(t\cos t-\sin t)\boldsymbol{j};\quad \sqrt{2}\le t\le 2$

6. $\boldsymbol{r}(t)=(t\cos t)\boldsymbol{i}+(t\sin t)\boldsymbol{j}+\left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\boldsymbol{k};\quad 0\le t\le \pi$

7. Find the point on the curve $\boldsymbol{r}(t)=(5\sin t)\boldsymbol{i}+(5\cos t)\boldsymbol{j}+12t\,\boldsymbol{k}$ at a distance 26π units along the curve from the point (0, 5, 0) in the direction of increasing arc length.

Find the arc length parameter along the curve from the point where $t = 0$. Also, find the length of the indicated portion of the curve.

8. $\boldsymbol{r}(t)=(4\cos t)\boldsymbol{i}+(4\sin t)\boldsymbol{j}+3t\,\boldsymbol{k};\quad 0\le t\le \frac{\pi}{2}$

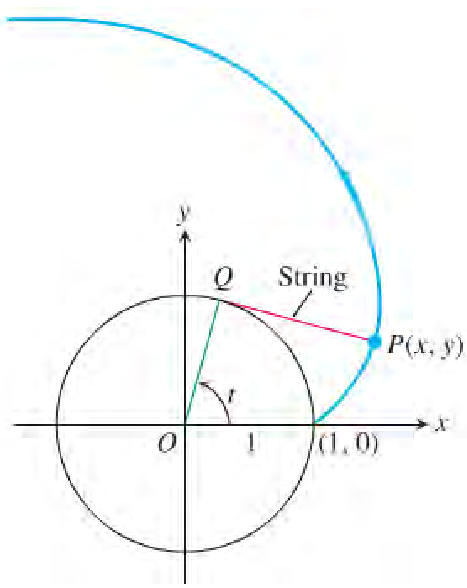
9. $\boldsymbol{r}(t)=\left(e^t\cos t\right)\boldsymbol{i}+\left(e^t\sin t\right)\boldsymbol{j}+e^t\,\boldsymbol{k};\quad -\ln 4\le t\le 0$

10. $\boldsymbol{r}(t)=(1+2t)\boldsymbol{i}+(1+3t)\boldsymbol{j}+(6-6t)\boldsymbol{k};\quad -1\le t\le 0$

11. If a string wound around a fixed circle in unwound while held taut in the plane of the circle, its end P traces an involute of the circle. The circle in question is the circle $x^2+y^2=1$ and the tracing point starts at (1, 0). The unwound portion of the string is tangent to the circle at Q, and t is the radian measure of the angle from the position x-axis to segment OQ. Derive the parametric equations

$$x=\cos t+t\sin t,\quad y=\sin t-t\cos t,\quad t>0$$

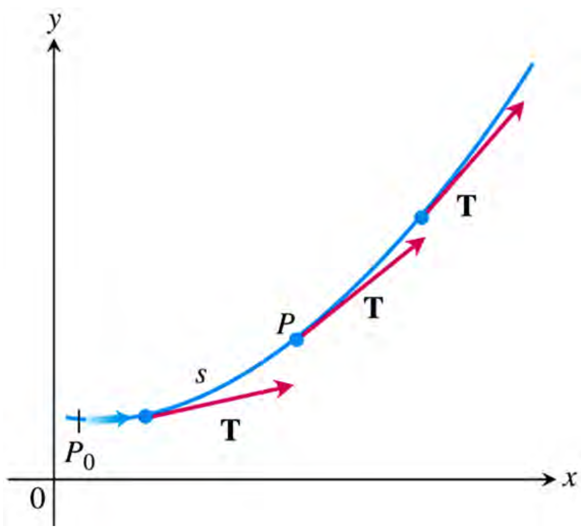
of the point $P(x,\, y)$ for the involute.



Section 1.8 – Curvature and Normal Vectors

Curvature of a Plane Curve

As a particle moves along a smooth curve in the plane, $\boldsymbol{T} = \frac{d\boldsymbol{r}}{ds}$ turns as the curve bends. Since $\boldsymbol{T}$ is a unit vector, its length remains constant and only its direction changes as particle moves along the curve. The rate at which $\boldsymbol{T}$ turns per unit of length along the curve is called the ***curvature***.



Definition

If $\boldsymbol{T}$ is the unit vector of a smooth curve, the ***curvature*** function of the curve is

$$\kappa = \left| \frac{d\boldsymbol{T}}{ds} \right|$$

Formula for Calculating Curvature

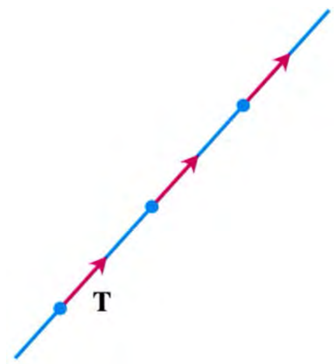
If $\boldsymbol{r}(t)$ is a smooth curve, then the curvature is

$$\kappa = \frac{1}{|v|} \left| \frac{d\boldsymbol{T}}{dt} \right|$$

Where $\boldsymbol{T} = \frac{v}{|v|}$ is the unit tangent vector.

Example

A straight line is parametrized by $\boldsymbol{r}(t) = \boldsymbol{C} + t\boldsymbol{v}$ for constant vectors $\boldsymbol{C}$ and $\boldsymbol{v}$. Thus $\boldsymbol{r}'(t) = \boldsymbol{v}$, and the unit tangent vector $\boldsymbol{T} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$ is a constant vector that always points in the same direction and has derivative **0**. It follows that, for any value of the parameter *t*, the curvature of the straight line is



$$\kappa = \frac{1}{|v|}\left|\frac{d\boldsymbol{T}}{dt}\right| = \frac{1}{|v|}|\boldsymbol{0}| = 0$$

Example

Find the curvature of a circle $\boldsymbol{r}(t) = (a\cos t)\boldsymbol{i} + (a\sin t)\boldsymbol{j}$ of radius *a*.

Solution

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt} = -(a\sin t)\boldsymbol{i} + (a\cos t)\boldsymbol{j}$$

$$\begin{aligned} |\boldsymbol{v}| &= \sqrt{(-a\sin t)^2 + (a\cos t)^2} \\ &= \sqrt{a^2\sin^2 t + a^2\cos^2 t} \\ &= |a|\sqrt{\sin^2 + \cos^2 t} \\ &= a \end{aligned}$$

Since a > 0

$$\boldsymbol{T} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = -(\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j}$$

$$\frac{d\boldsymbol{T}}{dt} = -(\cos t)\boldsymbol{i} - (\sin t)\boldsymbol{j}$$

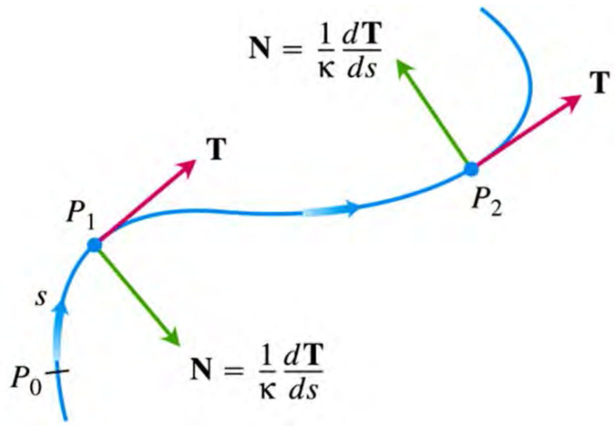
$$\left|\frac{d\boldsymbol{T}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

$$\begin{aligned} \kappa &= \frac{1}{|v|}\left|\frac{d\boldsymbol{T}}{dt}\right| \\ &= \frac{1}{a}(1) \\ &= \frac{1}{a} = \frac{1}{radius} \end{aligned}$$

Definition

At a point where $\kappa \neq 0$, the **principal unit normal** vector for a smooth curve in the plane is

$$\boxed{\boldsymbol{N} = \frac{1}{\kappa}\frac{d\boldsymbol{T}}{dt}}$$



Formula for Calculating N

If $\boldsymbol{r}(t)$ is a smooth curve, then the principal unit normal is

$$\boldsymbol{N} = \frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|}$$

Where $\boldsymbol{T} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$ is the unit tangent vector.

Example

Find $\boldsymbol{T}$ and $\boldsymbol{N}$ for the circular motion $\boldsymbol{r}(t) = (\cos 2t)\boldsymbol{i} + (\sin 2t)\boldsymbol{j}$

Solution

$$\boldsymbol{v} = \boldsymbol{r}' = -(2\sin 2t)\boldsymbol{i} + (2\cos 2t)\boldsymbol{j}$$

$$|\boldsymbol{v}| = \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2$$

$$\boldsymbol{T} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = -(\sin 2t)\boldsymbol{i} + (\cos 2t)\boldsymbol{j}$$

$$\frac{d\boldsymbol{T}}{dt} = -(2\cos 2t)\boldsymbol{i} - (2\sin 2t)\boldsymbol{j}$$

$$\left|\frac{d\boldsymbol{T}}{dt}\right| = \sqrt{4\cos^2 2t + 4\sin^2 2t} = 2$$

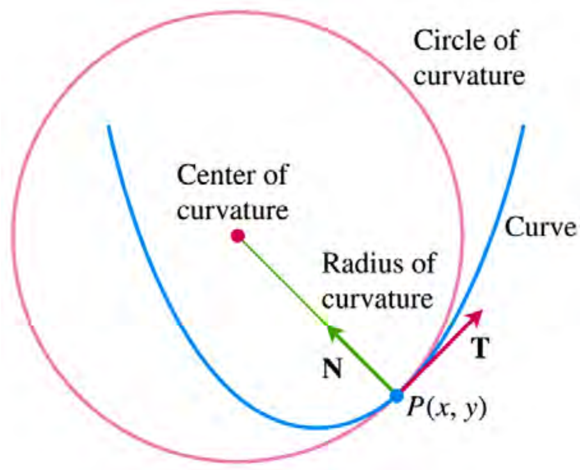
$$\boldsymbol{N} = \frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|} = \frac{-(2\cos 2t)\boldsymbol{i} - (2\sin 2t)\boldsymbol{j}}{2}$$

$$= -(\cos 2t)\boldsymbol{i} - (\sin 2t)\boldsymbol{j}$$

Circle of Curvature for plane Curves

The ***circle of curvature*** or ***osculating circle*** at a point P on a plane where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is tangent to the curve at P (has the same tangent line the curve has)
2. has the same curvature the curve has at P
3. lies toward the concave or inner side of the curve



The ***radius of curvature*** of the curve at P is the radius of the circle of curvature, which is

$$\textit{Radius of curvature} = \rho = \frac{1}{\kappa}$$

To find ρ, we find κ and take the reciprocal. The ***center of curvature*** of the curve at P is the center of the circle of curvature.

Example

Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

Solution

Assume: $t = x$

$$\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j} = t\,\boldsymbol{i} + t^2\boldsymbol{j}$$

$$\boldsymbol{v} = \boldsymbol{r}' = \boldsymbol{i} + 2t\boldsymbol{j}$$

$$|\boldsymbol{v}| = \sqrt{1+4t^2}$$

$$\boldsymbol{T} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{1}{\sqrt{1+4t^2}}\boldsymbol{i} + \frac{2t}{\sqrt{1+4t^2}}\boldsymbol{j}$$

$$\frac{d\boldsymbol{T}}{dt} = -\frac{4t}{\left(1+4t^2\right)^{3/2}}\boldsymbol{i} + \frac{2\left(1+4t^2\right)^{1/2} - 8t^2\left(1+4t^2\right)^{-1/2}}{\left(1+4t^2\right)}\boldsymbol{j}$$

$$= -\frac{4t}{\left(1+4t^2\right)^{3/2}}\boldsymbol{i} + \frac{2\left(1+4t^2\right) - 8t^2}{\left(1+4t^2\right)^{3/2}}\boldsymbol{j}$$

$$= -\frac{4t}{\left(1+4t^2\right)^{3/2}}\boldsymbol{i} + \frac{2}{\left(1+4t^2\right)^{3/2}}\boldsymbol{j}$$

At the origin, $t = 0$, so the curvature is

$$\left.\frac{d\boldsymbol{T}}{dt}\right|_{t=0} = 0\boldsymbol{i} + 2\boldsymbol{j} = 2\boldsymbol{j}$$

$$\kappa(0) = \frac{1}{|\boldsymbol{v}(0)|}\left|\frac{d\boldsymbol{T}}{dt}(0)\right|$$

$$= \frac{1}{\sqrt{1}}|2\boldsymbol{j}|$$

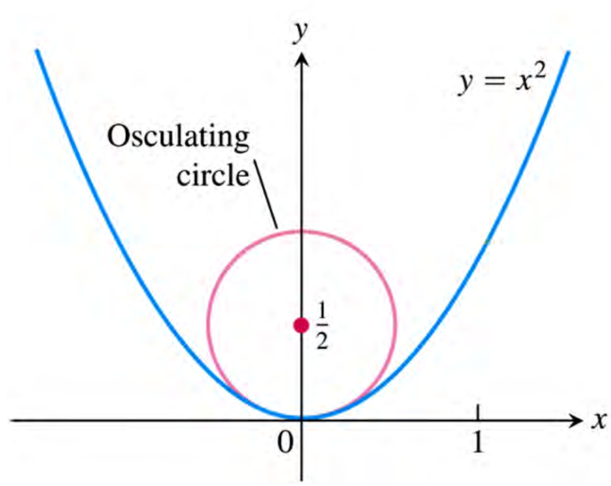
$$\underline{= 2}|$$

The radius of curvature is: $\rho = \frac{1}{\kappa} = \frac{1}{2}$

At the origin, t = 0, $\boldsymbol{T} = \boldsymbol{i} \quad \boldsymbol{N} = \boldsymbol{j}$

The center of the circle is $\left(0,\ \frac{1}{2}\right)$

The equation of the osculating circle is: $\underline{x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}}|$



Curvature and Normal Vectors for Space Curves

If a smooth curve in space is specified by the position $\boldsymbol{r}(t)$ as a function of somw parameter t, and if s is the arc length parameter of the curve, then the unit tangernt vector $\boldsymbol{T}$ is $\frac{d\boldsymbol{r}}{ds}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}$. The curvature in space is then defiined to be

$$\kappa=\left|\frac{d\boldsymbol{T}}{ds}\right|=\frac{1}{|\boldsymbol{v}|}\left|\frac{d\boldsymbol{T}}{dt}\right|$$

Just as for plane curves. The vector $\frac{d\boldsymbol{T}}{ds}$ is orthogonal to $\boldsymbol{T}$, and we define the ***principal unit normal*** to be

$$\boldsymbol{N}=\frac{1}{\kappa}\frac{d\boldsymbol{T}}{ds}=\frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|}$$

Example

Find the curvature for the helix $\boldsymbol{r}(t)=(a\cos t)\boldsymbol{i}+(a\sin t)\boldsymbol{j}+bt\,\boldsymbol{k},\quad a,b\ge 0,\quad a^2+b^2\ne 0$

Solution

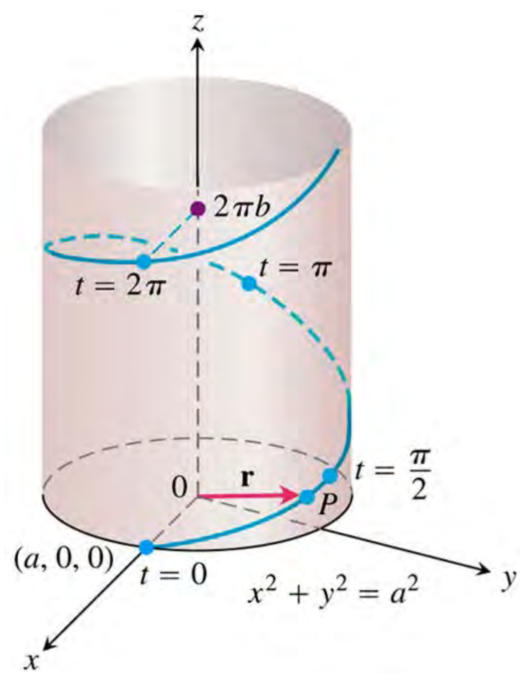
$$\boldsymbol{v}=-(a\sin t)\boldsymbol{i}+(a\cos t)\boldsymbol{j}+b\,\boldsymbol{k}$$

$$|\boldsymbol{v}|=\sqrt{a^2\sin^2 t+a^2\cos^2 t+b^2}=\sqrt{a^2+b^2}$$

$$\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=\frac{1}{\sqrt{a^2+b^2}}\left(-(a\sin t)\boldsymbol{i}+(a\cos t)\boldsymbol{j}+b\,\boldsymbol{k}\right)$$

$$\frac{d\boldsymbol{T}}{dt}=\frac{1}{\sqrt{a^2+b^2}}\left(-(a\cos t)\boldsymbol{i}-(a\sin t)\boldsymbol{j}\right)$$

$$=\frac{-a}{\sqrt{a^2+b^2}}\left((\cos t)\boldsymbol{i}+(\sin t)\boldsymbol{j}\right)$$



$$\kappa=\frac{1}{|\boldsymbol{v}|}\left|\frac{d\boldsymbol{T}}{dt}\right|$$

$$=\frac{1}{\sqrt{a^2+b^2}}\left|\frac{-a}{\sqrt{a^2+b^2}}\left((\cos t)\boldsymbol{i}+(\sin t)\boldsymbol{j}\right)\right|$$

$$=\frac{1}{\sqrt{a^2+b^2}}\frac{a}{\sqrt{a^2+b^2}}\sqrt{\sin^2 t+\cos^2 t}$$

$$\boxed{=\frac{a}{a^2+b^2}}$$

Example

Find $\boldsymbol{N}$ for the helix $\boldsymbol{r}(t)=(a\cos t)\boldsymbol{i}+(a\sin t)\boldsymbol{j}+bt\,\boldsymbol{k}$

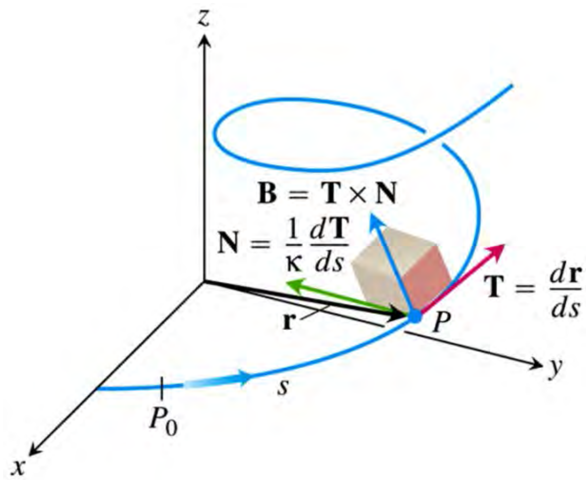
Solution

$$\frac{d\boldsymbol{T}}{dt}=\frac{-a}{\sqrt{a^2+b^2}}\big((\cos t)\boldsymbol{i}+(\sin t)\boldsymbol{j}\big) \qquad \left|\frac{d\boldsymbol{T}}{dt}\right|=\frac{a}{\sqrt{a^2+b^2}}$$

$$\begin{aligned}\boldsymbol{N}&=\frac{d\boldsymbol{T}/dt}{|d\boldsymbol{T}/dt|}=\frac{-a}{\sqrt{a^2+b^2}}\big((\cos t)\boldsymbol{i}+(\sin t)\boldsymbol{j}\big)\bullet\frac{\sqrt{a^2+b^2}}{a}\\&=-\big((\cos t)\boldsymbol{i}+(\sin t)\boldsymbol{j}\big)\\&=-(\cos t)\boldsymbol{i}-(\sin t)\boldsymbol{j}\end{aligned}$$

TNB Frame

The ***binormal vector*** of a curve in space $\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}$, a unit vector orthogonal to both $\boldsymbol{T}$ and $\boldsymbol{N}$. Together $\boldsymbol{T}$, $\boldsymbol{N}$, and $\boldsymbol{B}$ define a moving right-handed vector frame that play a significant role in calculating the paths of particles moving through space. It is called the ***Frenet frame*** or ***TNB frame***.

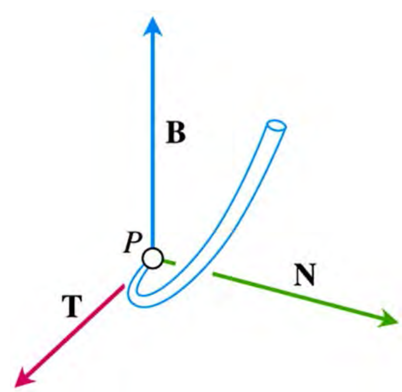


Tangential and Normal Components of Acceleration

When an object is accelerated by gravity, brakes, or a combination of rocket motors, how much of the acceleration acts in the direction of motion, in the tangential direction $\boldsymbol{T}$.

$$\boldsymbol{v}=\frac{d\boldsymbol{r}}{dt}=\frac{d\boldsymbol{r}}{ds}\frac{ds}{dt}=\boldsymbol{T}\frac{ds}{dt}$$

$$\begin{aligned}\boldsymbol{a}&=\frac{d\boldsymbol{v}}{dt}=\frac{d}{dt}\left(\boldsymbol{T}\frac{ds}{dt}\right)\\&=\frac{d^2s}{dt^2}\boldsymbol{T}+\frac{ds}{dt}\frac{d\boldsymbol{T}}{dt}\\&=\frac{d^2s}{dt^2}\boldsymbol{T}+\frac{ds}{dt}\left(\frac{d\boldsymbol{T}}{ds}\frac{ds}{dt}\right)\\&=\frac{d^2s}{dt^2}\boldsymbol{T}+\frac{ds}{dt}\left(\kappa\boldsymbol{N}\frac{ds}{dt}\right)\\&=\frac{d^2s}{dt^2}\boldsymbol{T}+\kappa\left(\frac{ds}{dt}\right)^2\boldsymbol{N}\end{aligned}$$



Definition

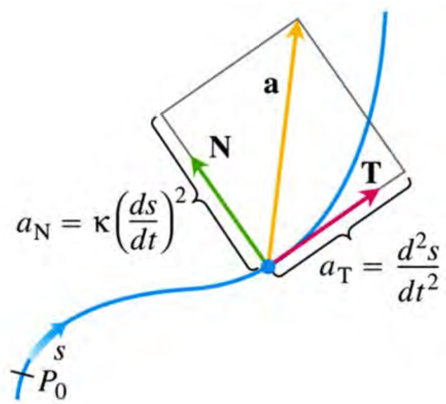
If the acceleration vector is written as

$$\boldsymbol{a} = a_T \boldsymbol{T} + a_N \boldsymbol{N}$$

then

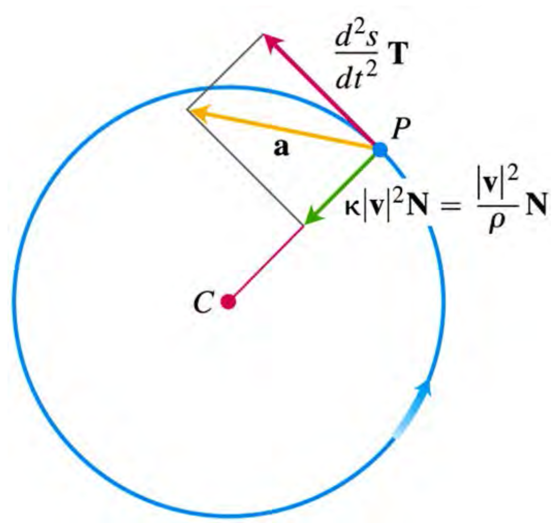
$$a_T = \frac{d^2 s}{dt^2} = \frac{d}{dt}|\boldsymbol{v}| \quad and \quad a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\boldsymbol{v}|^2$$

are the ***tangential*** and ***normal*** scalar components of acceleration.



Formula for Calculating the Normal Component of Acceleration

$$a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$$



Example

Without finding $\boldsymbol{T}$ and $\boldsymbol{N}$, write the acceleration of the motion

$$\boldsymbol{r}(t) = (\cos t + t\sin t)\boldsymbol{i} + (\sin t - t\cos t)\boldsymbol{j}, \quad t > 0$$

In the form $\boldsymbol{a} = a_T \boldsymbol{T} + a_N \boldsymbol{N}$

Solution

$$\boldsymbol{v} = \boldsymbol{r}' = (-\sin t + \sin t + t\cos t)\boldsymbol{i} + (\cos t - \cos t + t\sin t)\boldsymbol{j}$$
$$= (t\cos t)\boldsymbol{i} + (t\sin t)\boldsymbol{j}$$

$$|\boldsymbol{v}| = \sqrt{t^2\cos^2 t + t^2\sin^2 t}$$
$$= \sqrt{t^2\left(\cos^2 t + \sin^2 t\right)}$$
$$= |t| \qquad \boldsymbol{t > 0}$$
$$= t$$

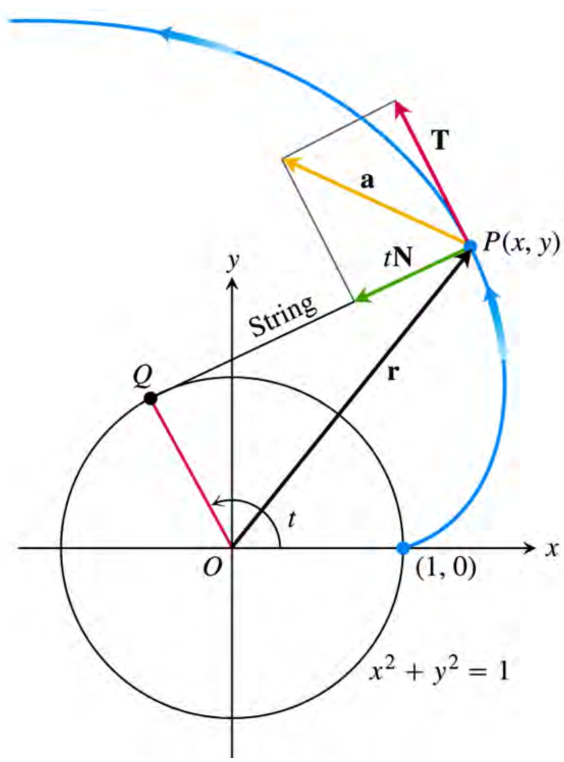
$$a_T = \frac{d}{dt}|\boldsymbol{v}|$$
$$= \frac{d}{dt}(t)$$
$$\underline{= 1}$$

$$\boldsymbol{a} = \boldsymbol{v}' = (\cos t - t\sin t)\boldsymbol{i} + (\sin t + t\cos t)\boldsymbol{j}$$

$$|\boldsymbol{a}|^2 = (\cos t - t\sin t)^2 + (\sin t + t\cos t)^2$$
$$= \cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\cos t\sin t + t^2\cos^2 t$$
$$= 1 + t^2\left(\sin^2 t + \cos^2 t\right)$$
$$= 1 + t^2$$

$$a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$$
$$= \sqrt{1 + t^2 - 1}$$
$$\underline{= t}$$

$$\boldsymbol{a} = a_T \boldsymbol{T} + a_N \boldsymbol{N}$$
$$\underline{= \boldsymbol{T} + t\boldsymbol{N}}$$

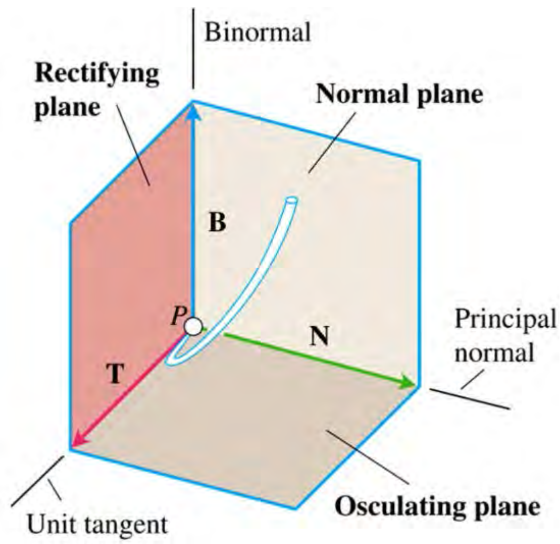


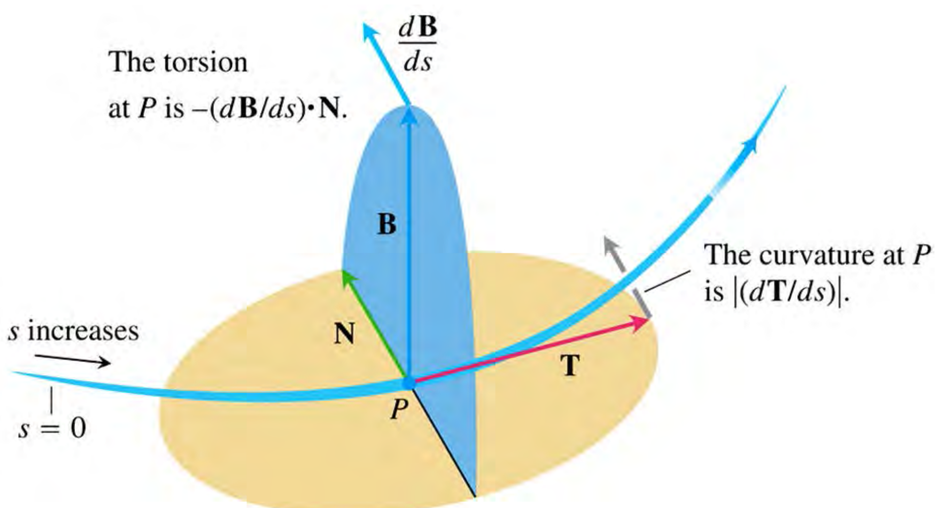
Torsion

Definition

Let $\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$. The torsion function of a smooth curve is

$$\tau = -\frac{d\boldsymbol{B}}{ds} \bullet \boldsymbol{N}$$





Computation Formulas for Curves in Space

Unit tangent vector: $\boldsymbol{T} = \dfrac{\boldsymbol{v}}{|\boldsymbol{v}|}$

Principal unit normal vector: $\boldsymbol{N} = \dfrac{d\boldsymbol{T}\,/\,dt}{|d\boldsymbol{T}\,/\,dt|} = \dfrac{1}{\kappa}\dfrac{d\boldsymbol{T}}{ds}$

Binormal vector: $\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N} = \dfrac{1}{v}\left|\dfrac{d\boldsymbol{T}}{dt}\right|$

Curvature: $\kappa = \left|\dfrac{d\boldsymbol{T}}{ds}\right| = \dfrac{|\boldsymbol{v} \times \boldsymbol{a}|}{|\boldsymbol{v}|^3} = \dfrac{1}{|v|}\left|\dfrac{d\boldsymbol{T}}{dt}\right|$

Torsion: $\tau = -\dfrac{d\boldsymbol{B}}{ds} \bullet \boldsymbol{N} = \dfrac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\boldsymbol{v} \times \boldsymbol{a}|^2}$

Tangential and normal scalar components of acceleration:

$$\boldsymbol{a} = a_T \boldsymbol{T} + a_N \boldsymbol{N}$$

$$a_T = \frac{d}{dt}|\boldsymbol{v}|$$

$$a_N = \kappa|\boldsymbol{v}|^2 = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$$

$$= |\boldsymbol{v}|\left|\frac{dT}{dt}\right|$$

Exercises *Section* 1.8 – Curvature and Normal Vectors

1. Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the plane curves: $\boldsymbol{r}(t) = t\boldsymbol{i} + (\ln\cos t)\boldsymbol{j}, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

2. Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the plane curves: $\boldsymbol{r}(t) = (\ln\sec t)\boldsymbol{i} + t\boldsymbol{j}, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

3. Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the plane curves: $\boldsymbol{r}(t) = (2t+3)\boldsymbol{i} + \left(5-t^2\right)\boldsymbol{j}$

4. Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the plane curves: $\boldsymbol{r}(t) = (\cos t + t\sin t)\boldsymbol{i} + (\sin t - t\cos t)\boldsymbol{j}, \quad t > 0$

5. Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the space curves: $\boldsymbol{r}(t) = (3\sin t)\boldsymbol{i} + (3\cos t)\boldsymbol{j} + 4t\boldsymbol{k}$

6. Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the space curves: $\boldsymbol{r}(t) = \left(e^t\cos t\right)\boldsymbol{i} + \left(e^t\sin t\right)\boldsymbol{j} + 2\boldsymbol{k}$

7. Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the space curves: $\boldsymbol{r}(t) = \frac{t^3}{3}\boldsymbol{i} + \frac{t^2}{2}\boldsymbol{j}, \quad t > 0$

8. Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the space curves: $\boldsymbol{r}(t) = \left(\cos^3 t\right)\boldsymbol{i} + \left(\sin^3 t\right)\boldsymbol{j}, \quad 0 < t < \frac{\pi}{2}$

9. Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the space curves: $\boldsymbol{r}(t) = (\cosh t)\boldsymbol{i} - (\sinh t)\boldsymbol{j} + t\boldsymbol{k}$

10. Find an equation for the circle of curvature of the curve $\boldsymbol{r}(t) = t\boldsymbol{i} + (\sin t)\boldsymbol{j}$, at the point $\left(\frac{\pi}{2}, 1\right)$.
(The curve parametrizes the graph $y = \sin x$ in the *xy*-plane.)

Write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$ without finding $\boldsymbol{T}$ and $\boldsymbol{N}$.

11. $\boldsymbol{r}(t) = (a\cos t)\boldsymbol{i} + (a\sin t)\boldsymbol{j} + bt\boldsymbol{k}$

12. $\boldsymbol{r}(t) = (1+3t)\boldsymbol{i} + (t-2)\boldsymbol{j} - 3t\boldsymbol{k}$

Write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$ at the given value of t without finding $\boldsymbol{T}$ and $\boldsymbol{N}$.

13. $\boldsymbol{r}(t) = (t+1)\boldsymbol{i} + 2t\boldsymbol{j} + t^2\boldsymbol{k}, \quad t = 1$

14. $\boldsymbol{r}(t) = (t\cos t)\boldsymbol{i} + (t\sin t)\boldsymbol{j} + t^2\boldsymbol{k}, \quad t = 0$

15. $\boldsymbol{r}(t) = \left(e^t\cos t\right)\boldsymbol{i} + \left(e^t\sin t\right)\boldsymbol{j} + \sqrt{2}e^t\boldsymbol{k}, \quad t = 0$

Find $\boldsymbol{r}$, $\boldsymbol{T}$, $\boldsymbol{N}$, and $\boldsymbol{B}$ at the given value of t. Then find equations for the osculating, normal, and rectifying planes at that value of t.

16. $\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} - \boldsymbol{k}, \quad t = \frac{\pi}{4}$

17. $\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} + t\boldsymbol{k}, \quad t = 0$

18. Find $\boldsymbol{B}$ and τ for: $\boldsymbol{r}(t)=(3\sin t)\boldsymbol{i}+(3\cos t)\boldsymbol{j}+4t\boldsymbol{k}$

19. Find $\boldsymbol{B}$ and τ for: $\boldsymbol{r}(t)=(\cos t+t\sin t)\boldsymbol{i}+(\sin t-t\cos t)\boldsymbol{j}+3\boldsymbol{k}$

20. Find $\boldsymbol{B}$ and τ for: $\boldsymbol{r}(t)=(6\sin 2t)\boldsymbol{i}+(6\cos 2t)\boldsymbol{j}+5t\boldsymbol{k}$

21. The speedometer on your car reads a steady 35 mph, could you be accelerating? Explain.

22. Can anything be said about the acceleration of a particle that is moving at a constant speed? Give reasons for your answer.

23. Find $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$, τ and κ as functions of $\boldsymbol{t}$ for the plane curves: $\boldsymbol{r}(t)=(\sin t)\boldsymbol{i}+\left(\sqrt{2}\cos t\right)\boldsymbol{j}+(\sin t)\boldsymbol{k}$, then write $\boldsymbol{a}$ of the motion $\boldsymbol{a}=a_T\boldsymbol{T}+a_N\boldsymbol{N}$