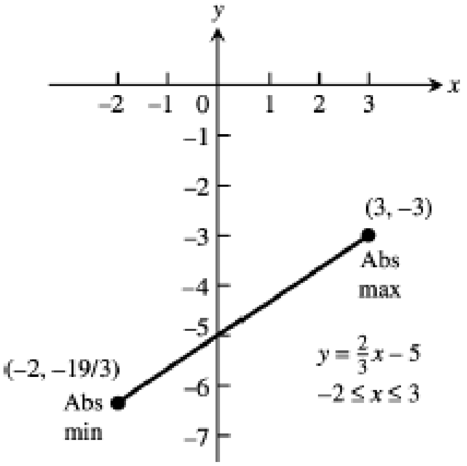
Solution Section 3.1 – Extreme Values of Functions

Exercise

Find the absolute maximum and minimum values of the function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$$f(x)=\frac{2}{3}x-5 \qquad -2\le x\le 3$$



Solution

$$f'(x)=\frac{2}{3} \qquad \text{No Critical Points } (CP) \text{ or } (CN).$$

$$f(-2)=\frac{2}{3}(-2)-5=-\frac{19}{3}$$

$$f(3)=\frac{2}{3}(3)-5=-3$$

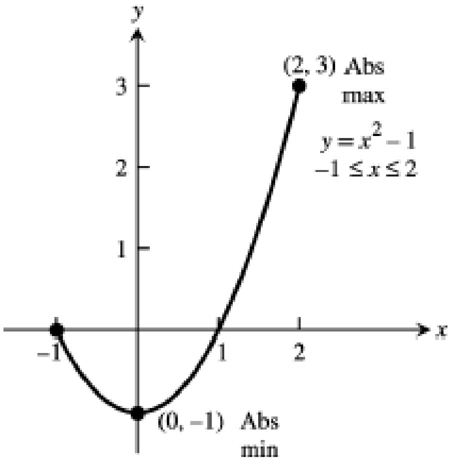
The ***absolute minimum***: $\left(-2,\ -\frac{19}{3}\right)$

The ***absolute maximum***: $(3,\ -3)$

Exercise

Find the absolute maximum and minimum values of the function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$$f(x)=x^2-1 \qquad -1\le x\le 2$$



Solution

$$f'(x)=2x=0 \quad\Rightarrow\quad \boxed{x=0} \qquad (CN)$$

$$f(-1)=(-1)^2-1=0$$

$$f(0)=(0)^2-1=-1$$

$$f(2)=(2)^2-1=3$$

The ***absolute maximum***: $(2,\ 3)$

The ***absolute minimum***: $(0,\ -1)$

Exercise

Find the absolute maximum and minimum values of the function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$$f(x) = -\frac{1}{x^2} \qquad 0.5 \le x \le 2$$

Solution

$$f'(x) = \frac{1}{2x^3} \qquad \boxed{x=0}$$ Which it is not in the domain

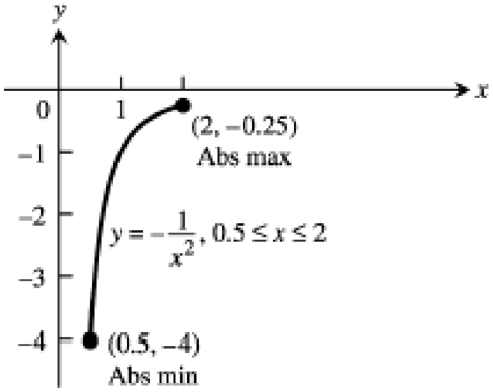
No critical point.

$$f(0.5) = -\frac{1}{(0.5)^2} = -4$$

$$f(2) = -\frac{1}{(2)^2} = -0.25$$

The ***absolute maximum***: $(2, \ -0.25)$

The ***absolute minimum***: $(0.5, \ -4)$



Exercise

Find the absolute maximum and minimum values of the function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$$f(x) = \sqrt{4-x^2} \qquad -2 \le x \le 1$$

Solution

$$f(x) = \left(4-x^2\right)^{1/2}$$

$$f'(x) = \frac{1}{2}\left(4-x^2\right)^{-1/2}(-2x)$$

$$= \frac{-x}{\sqrt{4-x^2}} = 0 \qquad \to \begin{cases} \boxed{x=0} \\ 4-x^2=0 \Rightarrow \boxed{x=\pm 2} \end{cases}$$

Critical points: $x = 0, \ -2$

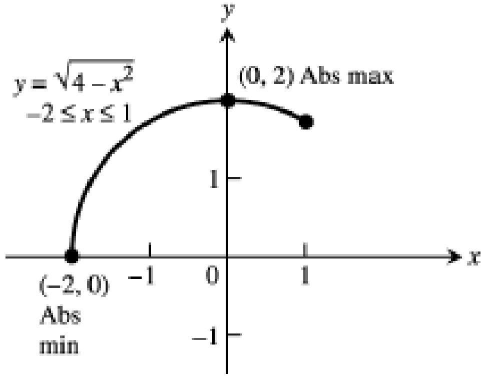
$$f(-2) = \sqrt{4-(-2)^2} = 0$$

$$f(0) = \sqrt{4-(0)^2} = 2$$

$$f(1) = \sqrt{4-(1)^2} = \sqrt{3}$$

The absolute maximum: $(0, \ 2)$

The absolute minimum: $(-2, \ 0)$



Exercise

Find the absolute maximum and minimum values of the function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$$f(\theta)=\sin\theta \qquad -\frac{\pi}{2}\le\theta\le\frac{5\pi}{6}$$

Solution

$$f'(\theta)=\cos\theta=0 \quad\Rightarrow\quad \theta=\frac{\pi}{2}\ \ (CN)$$

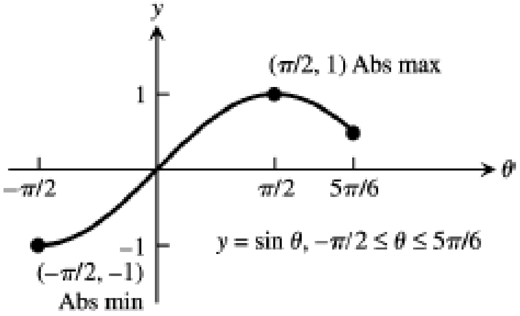
$$f\left(-\frac{\pi}{2}\right)=\sin\left(-\frac{\pi}{2}\right)=-1$$

$$f\left(\frac{\pi}{2}\right)=\sin\left(\frac{\pi}{2}\right)=1$$

$$f\left(\frac{5\pi}{6}\right)=\sin\left(\frac{5\pi}{6}\right)=\frac{1}{2}$$

Absolute Minimum: $\left(-\frac{\pi}{2},\ -1\right)$

Absolute Maximum: $\left(\frac{\pi}{2},\ 1\right)$



Exercise

Find the absolute maximum and minimum values of the function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$$g(x)=\sec x \qquad -\frac{\pi}{3}\le x\le\frac{\pi}{6}$$

Solution

$$g'(x)=\sec x\tan x=0 \quad\Rightarrow\quad \boxed{x=0}\ \ (CN)$$

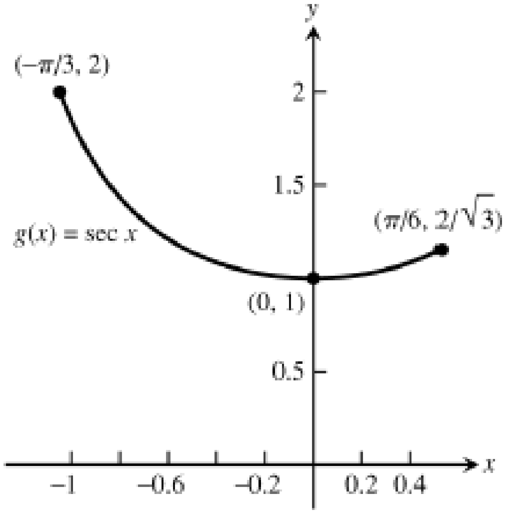
$$g\left(-\frac{\pi}{3}\right)=\sec\left(-\frac{\pi}{3}\right)=2$$

$$g(0)=\sec(0)=1$$

$$g\left(\frac{\pi}{6}\right)=\sec\left(\frac{\pi}{6}\right)=\frac{2}{\sqrt{3}}$$

Absolute Maximum: $\left(-\frac{\pi}{3},\ 2\right)$

Absolute Minimum: $(0,\ 1)$



Exercise

Find the absolute maximum and minimum values of $f(x) = x^{4/3}, \quad -1 \le x \le 8$

Solution

$$f'(x) = \frac{4}{3}x^{1/3} = 0 \quad \Rightarrow \quad \boxed{x = 0} \qquad (\boldsymbol{CN})$$

$$f(-1) = 1$$

$$f(0) = 0$$

$$f(8) = 16$$

Absolute Maximum: $(8,\ 16)$

Absolute Minimum: $(0,\ 0)$

Exercise

Find the absolute maximum and minimum values of $f(\theta) = \theta^{3/5}, \quad -32 \le \theta \le 1$

Solution

$$f'(\theta) = \frac{3}{5}\theta^{-2/5} = 0 \quad \Rightarrow \quad \boxed{\theta = 0} \qquad (\boldsymbol{CN})$$

$$f(-32) = -8$$

$$f(0) = 0$$

$$f(1) = 1$$

Absolute Maximum: $(1,\ 1)$

Absolute Minimum: $(-32,\ -8)$

Exercise

Determine all critical points of $y = x^2 - 6x + 7$

Solution

$$y' = 2x - 6 = 0 \quad \Rightarrow \quad \boxed{x = 3} \qquad (\boldsymbol{CN})$$

$$y\Big|_{x=3} = 3^2 - 6(3) + 7 = -2$$

Critical point: $(3,\ -2)$

Exercise

Determine all critical points of $g(x)=(x-1)^2(x-3)^2$

Solution

$$g'(x)=2(x-1)(x-3)^2+2(x-1)^2(x-3)$$

$$=2(x-1)(x-3)(x-3+x-1)$$

$$=2(x-1)(x-3)(2x-4)$$

The ***critical numbers*** are: $x = 1, 2, 3$

$$g(1)=0$$

$$g(2)=1$$

$$g(3)=0$$

The ***critical points*** are : (1, 0), (2, 1) and (3, 1)

Exercise

Determine all critical points of $f(x)=\dfrac{x^2}{x-2}$

Solution

$$f'(x)=\frac{2x(x-2)-x^2}{(x-2)^2}$$

$$=\frac{x^2-4x}{(x-2)^2}=0$$

$x = 2$ is not in the domain

The critical numbers are: $x = 0, 4$

$$f(0)=0$$

$$f(4)=8$$

The ***critical points*** are: $\underline{(3,-2),\ (4,8)}|$

Exercise

Determine all critical points of $g(x) = x^2 - 32\sqrt{x}$

Solution

$$g'(x) = 2x - \frac{16}{\sqrt{x}}$$

$$= \frac{2x^{3/2} - 16}{\sqrt{x}} = 0$$

$$\begin{cases} 2x^{3/2} - 16 = 0 \Rightarrow x^{3/2} = 8 \rightarrow \boxed{x = 4} \\ \sqrt{x} = 0 \Rightarrow \boxed{x = 0} \end{cases}$$

The critical numbers are: $x = 0,\ 4$

$$g(0) = 0$$

$$g(4) = 16 - 32\sqrt{4} = 48$$

The ***critical points*** are: $\boxed{(0,0),\ (4,48)}$

Exercise

Find the extreme values (absolute and local) of the function and where they occur $y = x^3 - 2x + 4$

Solution

$$y' = 3x^2 - 2 = 0 \quad \Rightarrow \quad \boxed{x = \pm\sqrt{\frac{2}{3}}}$$

$$x = -\sqrt{\frac{2}{3}} \quad \Rightarrow \quad y = \left(-\sqrt{\frac{2}{3}}\right)^3 - 2\left(-\sqrt{\frac{2}{3}}\right) + 4 = 5.089$$

$$x = \sqrt{\frac{2}{3}} \quad \Rightarrow \quad y = \left(\sqrt{\frac{2}{3}}\right)^3 - 2\left(\sqrt{\frac{2}{3}}\right) + 4 = 2.911$$

Local maximum: $(-.816,\ 5.089)$

Local minimum: $(.816,\ 2.911)$

Exercise

Find the extreme values (absolute and local) of the function and where they occur $y = \sqrt{x^2 - 1}$

Solution

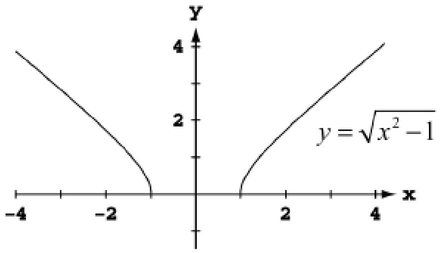
Domain for the function: $x \le -1 \quad x \ge 1$

$$y' = \frac{x}{\sqrt{x^2 - 1}} = 0 \quad \Rightarrow \quad x = \cancel{0},\ \pm 1$$

Critical points are: $x=\pm 1$

$$y=\sqrt{(\pm 1)^2-1}=0$$

Local minimum: $(-1,\ 0)\quad (1,\ 0)$



Exercise

Find the extreme values (absolute and local) of the function and where they occur $y=\dfrac{1}{\sqrt[3]{1-x^2}}$

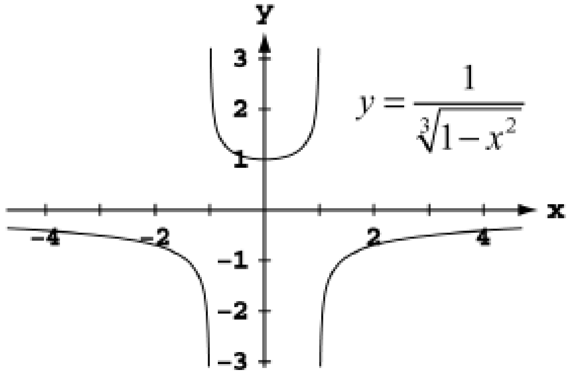
Solution

Asymptotes: $x=\pm 1$

$$y=\left(1-x^2\right)^{-1/3}$$

$$y'=-\frac{1}{3}\left(1-x^2\right)^{-4/3}(-2x)=\frac{2}{3}\frac{x}{\left(1-x^2\right)^{4/3}}=0$$

There is a ***local minimum*** at (0, 1)



Exercise

Find the extreme values (absolute and local) of the function and where they occur $y=\dfrac{x+1}{x^2+2x+2}$

Solution

$$y'=\frac{x^2+2x+2-(2x+2)(x+1)}{\left(x^2+2x+2\right)^2}$$

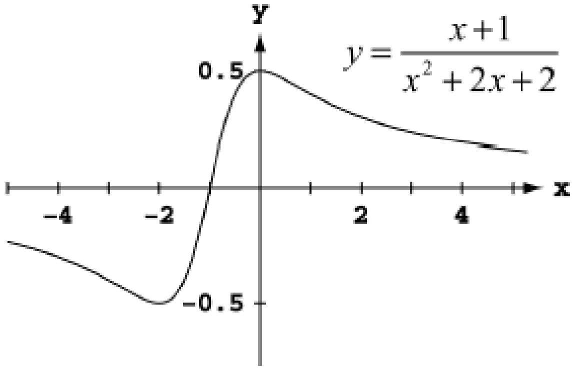
$$=\frac{x^2+2x+2-2x^2-2x-2x-2}{\left(x^2+2x+2\right)^2}$$

$$=\frac{-x^2-2x}{\left(x^2+2x+2\right)^2}=0$$

$\boxed{x=0,\ \frac{1}{2}}$ (***CN***)

$y\big|_{x=0}=\frac{1}{2}\qquad y\Big|_{x=\frac{1}{2}}=-2$

RMAX: $\left(0,\ \frac{1}{2}\right)\qquad$ ***RMIN***: $\left(-2,\ -\frac{1}{2}\right)$



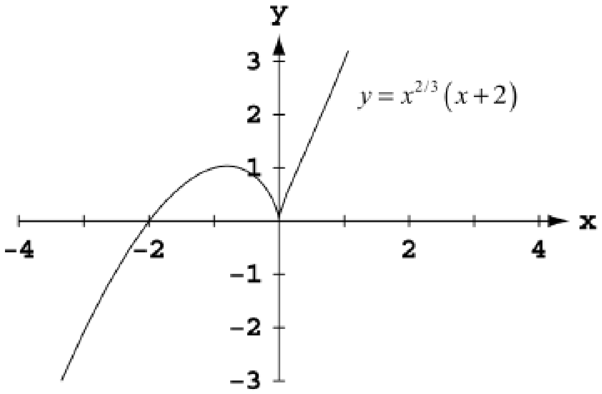
Exercise

Find the critical points, domain endpoints, and local extreme values (absolute and local)

$y = x^{2/3}(x+2)$

Solution

$$\begin{aligned} y' &= \frac{2}{3}x^{-1/3}(x+2) + x^{2/3} \\ &= \frac{2}{3}\frac{x+2}{x^{1/3}} + x^{2/3} \\ &= \frac{2x+4+3x}{x^{1/3}} \\ &= \frac{5x+4}{\sqrt[3]{x}} = 0 \end{aligned} \qquad \boxed{x = -\frac{4}{5},\ 0} \qquad (CN)$$



$$y\Big|_{x=-\frac{4}{5}} = \left(-\frac{4}{5}\right)^{2/3}\left(-\frac{4}{5}+2\right) = 1.034 \qquad y\Big|_{x=0} = 0$$

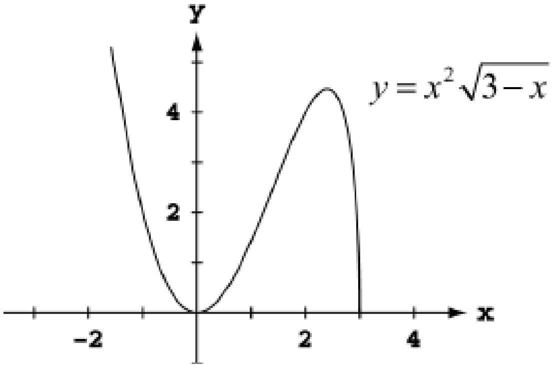
LMAX: $\left(-\frac{4}{5},\ 1.034\right)$ ***LMIN***: $(0,\ 0)$

Exercise

Find the critical points, domain endpoints, and local extreme values (absolute and local) $y = x^2\sqrt{3-x}$

Solution

$$\begin{aligned} y' &= 2x\sqrt{3-x} + \frac{1}{2}\left(\frac{-1}{\sqrt{3-x}}\right)x^2 \\ &= \frac{4x(3-x)-x^2}{2\sqrt{3-x}} \\ &= \frac{4x(3-x)-x^2}{2\sqrt{3-x}} \\ &= \frac{12x-5x^2}{2\sqrt{3-x}} = 0 \end{aligned} \qquad \boxed{x = \frac{5}{12},\ 0,\ 3} \qquad (CN)$$



$$y\Big|_{x=\frac{5}{12}} = 0.279 \qquad y\Big|_{x=0} = 0 \qquad y\Big|_{x=3} = 0$$

LMAX: $\left(\frac{5}{12},\ 0.279\right)$ ***LMIN***: $(0,\ 0)$ $(3,\ 0)$

Exercise

Find the critical points, domain endpoints, and local extreme values (absolute and local) $y = x\sqrt{4-x^2}$

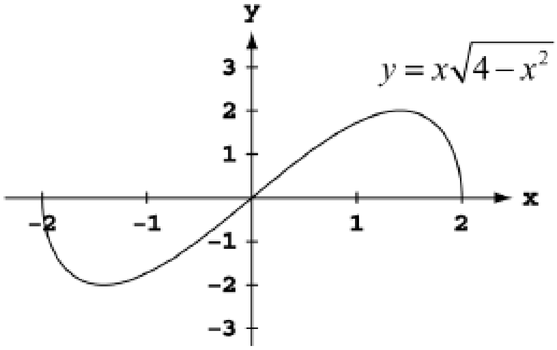
Solution

$$y' = \sqrt{4-x^2} + \left(\frac{1}{2}\frac{-2x}{\sqrt{4-x^2}}\right)(x)$$

$$= \frac{4-x^2-x^2}{\sqrt{4-x^2}}$$

$$= \frac{4-2x^2}{\sqrt{4-x^2}} = 0$$

$$\begin{cases} 4-2x^2 = 0 \\ 4-x^2 = 0 \end{cases} \rightarrow \boxed{x = \pm\sqrt{2},\ \pm 2}$$



$$y\Big|_{x=-\sqrt{2}} = -\sqrt{2}\sqrt{2} = -2, \quad y\Big|_{x=\sqrt{2}} = 2, \quad y\Big|_{x=\pm 2} = 0$$

LMAX: $(-2,\ 0)$ ***LMIN***: $(2,\ 0)$

MAX: $(\sqrt{2},\ 2)$ ***MIN***: $(-\sqrt{2},\ -2)$

Exercise

Let $f(x) = (x-2)^{2/3}$

a) Does $f'(2)$ exist?

b) Show the only local extreme value of f occurs at $x = 2$.

c) Does the result in part (*b*) contradict the Extreme Value Theorem?

Solution

a) $f'(x) = \frac{2}{3}(x-2)^{-1/3}$ is undefined at $x = 2$

b) $f(x=2) = (2-2)^{2/3} = 0$ and $f(x) > 0 \quad \forall x \neq 2$

c) No, $f(x)$ domain is all real numbers and doesn't need to have a global maximum. Any restriction of f to a closed interval of the form $[a, b]$ would have a maximum and minimum value on the interval.

Exercise

Find the absolute extrema of $f(x)=x^{8/3}-16x^{2/3}$ on the interval $[-1,\ 8]$.

Solution

$$f'(x)=\frac{8}{3}x^{5/3}-\frac{32}{3}x^{-1/3}$$

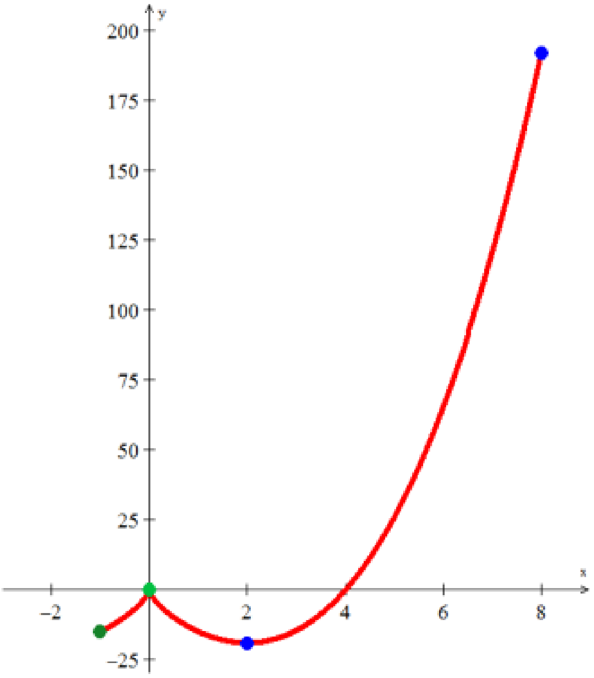
$$=\frac{8}{3}\left(x^{5/3}-\frac{4}{x^{1/3}}\right)$$

$$=\frac{8}{3}\left(\frac{x^2-4}{x^{1/3}}\right)=0$$

$$CN:\ \boxed{x=\pm 2}$$

$x\neq -2\notin[-1,\ 8]$

The derivative is undefined at $\boxed{x=0}$



x	$f(x)$
-1	-15
0	0
2	-19.05
8	192

Absolute Maximum (8, 192)

Absolute Minimum (2, -19.05)

Exercise

Find the minimum and maximum values of $f(x)=x^2-8x+10$ on the interval [0, 7].

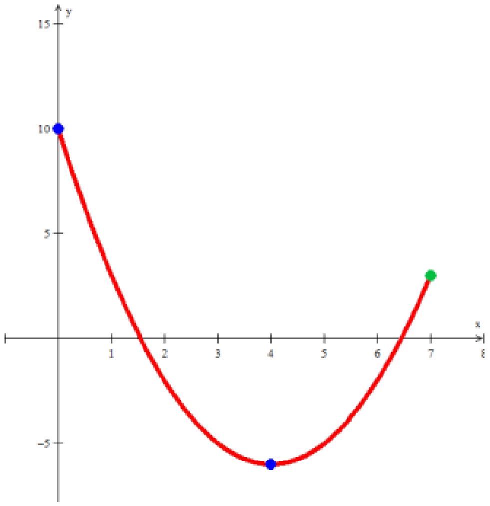
Solution

$$f'(x)=2x-8=0$$

$\Rightarrow x=4\ \ (CN)$

$\rightarrow y$ = 16 - 32 + 10 = -6

$$\begin{cases} x=0\rightarrow y=10 \\ x=7\rightarrow y=3 \end{cases}$$



Absolute Maximum (0, 10)

Absolute Minimum (4, -6)

Exercise

Find the absolute extrema of the function on the closed interval $f(x)=2(3-x), \ [-1,\ 2]$

Solution

$$f'=-2$$
$$f(-1)=2(3-(-1))=8$$
$$f(2)=2(3-2)=2$$

RMAX: $(-1,\ 8)$ ***RMIN***: $(2,\ 2)$

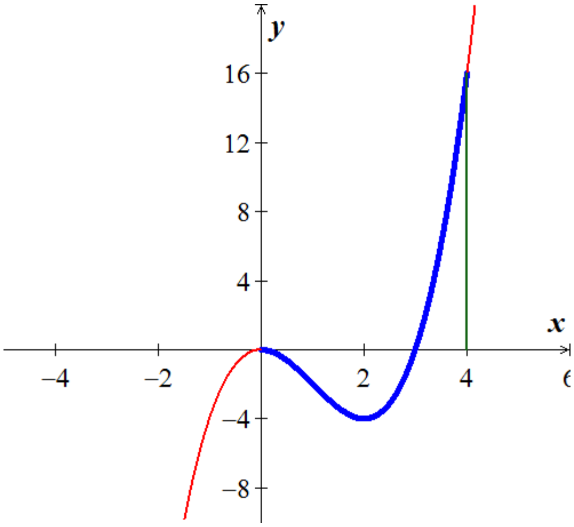
Exercise

Find the absolute extrema of the function on the closed interval $f(x)=x^3-3x^2, \ [0,\ 4]$

Solution

$$f'(x)=3x^2-6x=0$$
$$3x(x-2)=0 \ \rightarrow \begin{cases} \boxed{x=0} \\ x-2=0 \Rightarrow \boxed{x=2} \end{cases}$$
$$f(0)=0^3-3(0)^2=0$$
$$f(2)=2^3-3(2)^2=-4$$
$$f(4)=4^3-3(4)^2=16$$

RMAX: $(4,\ 16)$ ***RMIN***: $(2,\ -4)$



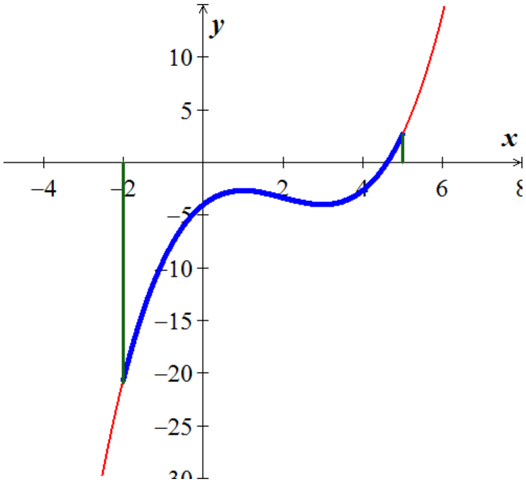
Exercise

Find the absolute extrema of the function on the closed interval $f(x)=\frac{1}{3}x^3-2x^2+3x-4, \ [-2,\ 5]$

Solution

$$f'(x)=x^2-4x+3=0 \ \rightarrow \begin{cases} \boxed{x=1} \\ \boxed{x=3} \end{cases}$$
$$f(-2)=\tfrac{1}{3}(-2)^3-2(-2)^2+3(-2)-4=-20.66$$
$$f(1)=\tfrac{1}{3}(1)^3-2(1)^2+3(1)-4=-2.66$$
$$f(3)=\tfrac{1}{3}(3)^3-2(3)^2+3(3)-4=-4$$
$$f(5)=\tfrac{1}{3}(5)^3-2(5)^2+3(5)-4=2.66$$

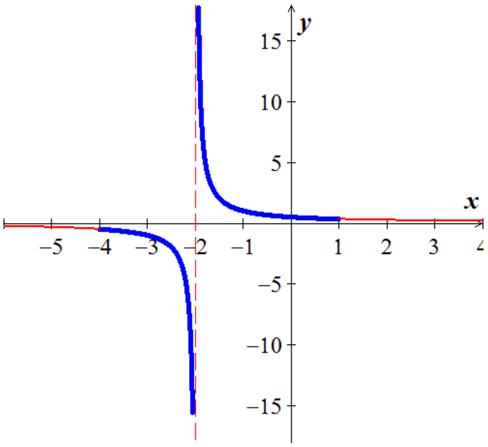
RMAX: $(5,\ 2.66)$ ***RMIN***: $(-2,\ -20.66)$



Exercise

Find the absolute extrema of the function on the closed interval

$f(x)=\frac{1}{x+2}, \quad [-4,\ 1]$



Solution

$x+2\neq 0 \rightarrow x\neq -2 \quad (Asymptote)$

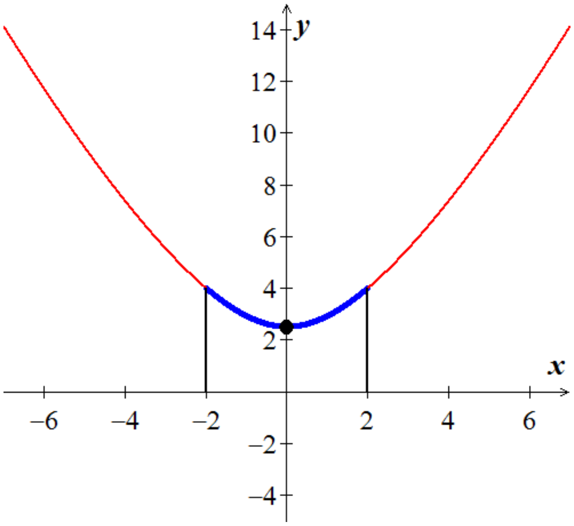
$f'(x)=-\frac{1}{(x+2)^2}\neq 0$

There is no Relative Extrema.

Exercise

Find the absolute extrema of the function on the closed interval $f(x)=\left(x^2+4\right)^{2/3}, \quad [-2,\ 2]$

Solution



$$f'(x)=\frac{2}{3}(2x)\left(x^2+4\right)^{2/3-1}$$

$$=\frac{4x}{3}\left(x^2+4\right)^{-1/3}$$

$$f'=\frac{4x}{3}\left(x^2+4\right)^{-1/3}=0; \quad x^2+4\neq 0$$

$$\boxed{x=0}$$

$$f(x=-2)=\left((-2)^2+4\right)^{2/3}=4$$

$$f(x=0)=\left((0)^2+4\right)^{2/3}=2.52$$

$$f(x=2)=\left((2)^2+4\right)^{2/3}=4$$

RMAX: $(-2,\ 4) \quad (2,\ 4)$ ***RMIN***: $(0,\ 2.52)$

Solution *Section* 3.3 – Monotonic Functions and the First Derivative Test

Exercise

Find the open intervals on which the function $f(x)=x^3+3x^2-9x+4$ is increasing or decreasing

Solution

$$f'(x)=3x^2+6x-9$$

$$3x^2+6x-9=0 \Rightarrow \boxed{x=-3,\ 1} \quad (CN)$$

-∞ -3	1	∞
$f'(-4)>0$	$f'(0)<0$	$f'(2)>0$
Increasing	*Decreasing*	*Increasing*

Increasing: (-∞, -3) and (1, ∞)

Decreasing: (-3, 1)

Exercise

Find the critical numbers and decide on which the function $f(x)=(x-1)^{2/3}$ is increasing or decreasing

Solution

$$f'(x)=\frac{2}{3}(x-1)^{-1/3}$$

$$=\frac{2}{3(x-1)^{1/3}}=0$$

$$f'(x)\neq 0$$

$x-1=0 \Rightarrow \boxed{x=1}$ is the only critical number

-∞ **1**	∞
$f'(0)<0$	$f'(2)>0$
Decreasing	*Increasing*

Decreasing: (−∞, 1)

Increasing: (1, ∞)

Exercise

Find the critical numbers and the open intervals on which the function is increasing or decreasing.

$$f(x) = x\sqrt{x+1}$$

Solution

$$\begin{aligned} f'(x) &= \sqrt{x+1} + \frac{1}{2}x(x+1)^{-1/2} \\ &= \sqrt{x+1} + \frac{x}{2\sqrt{x+1}} \\ &= \frac{2x+2+x}{2\sqrt{x+1}} \\ &= \frac{3x+2}{2\sqrt{x+1}} = 0 \end{aligned}$$

Critical numbers are $x = -\frac{2}{3}$ and $x = -1$, but the domain is $[-1, \infty)$.

Interval(s)	$(-1, -2/3)$	$(-2/3, \infty)$
Sign of f'	$f'(-0.9) < 0$	$f'(0) > 0$
Conclusion for f	*decreasing*	*increasing*

Decreasing on $\left(-1,\ -\frac{2}{3}\right)$ ***Increasing*** on $\left(-\frac{2}{3},\ \infty\right)$

Exercise

Find the critical numbers and the open intervals on which the function is increasing or decreasing.

$$f(x) = \frac{x}{x^2+4}$$

Solution

$$\begin{aligned} f'(x) &= \frac{(1)\left(x^2+4\right) - x(2x)}{\left(x^2+4\right)^2} \\ &= \frac{x^2+4-2x^2}{\left(x^2+4\right)^2} \\ &= \frac{-x^2+4}{\left(x^2+4\right)^2} \end{aligned}$$

$$-x^2+4=0 \Rightarrow x^2 = 4 \rightarrow x = \pm 2$$

Critical numbers are $x = \pm 2$.

Interval(s)	$(-\infty,-2)$	$(-2,\ 2)$	$(2,\ \infty)$
Sign of f'	$f'(-2)<0$	$f'(0)>0$	$f'(0)<0$
Conclusion for f	*decreasing*	*increasing*	*decreasing*

Decreasing: $(-\infty,-2)\cup(2,\infty)$ ***Increasing:*** $(-2,\ 2)$.

Exercise

Find the critical numbers and the open intervals on which the function is increasing or decreasing.

$$f(x)=\frac{x}{x^2+1}$$

Solution

$$f'(x)=-\frac{(x-1)(x+1)}{(x^2+1)^2}$$

Critical numbers are $x = 1$, and $x=-1$.

Interval(s)	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
Sign of f'	$f'(-2)<0$	$f'(0)>0$	$f'(0)<0$
Conclusion for f	*decreasing*	*increasing*	*decreasing*

Decreasing: $(-\infty,-1)\cup(1,\infty)$. ***Increasing***: $(-1,\ 1)$.

Exercise

Find the critical numbers and the open intervals on which the function is increasing or decreasing.

$$f(x)=x^3-12x$$

Solution

$$f'(x)=3x^2-12=0$$

$$\Rightarrow 3x^2=12$$

$$x^2=4$$

$$\Rightarrow \boxed{x=\pm 2}$$ (Critical Numbers - *CN*)

$-\infty$ **-2** **2** ∞

$f'(-3)>0$	$f'(1)<0$	$f'(3)>0$
Increasing	*Decreasing*	*Increasing*

Increasing: $(-\infty, -2)$ and $(2, \infty)$ ***Decreasing***: $(-2, 2)$

Exercise

Find the open intervals on which the function $f(x)=x^{2/3}$ is increasing or decreasing

Solution

$$f'(x)=\tfrac{2}{3}x^{-1/3}$$

$$=\frac{2}{3x^{1/3}}=0$$

$\Rightarrow$ *Undefined* $\boxed{x=0}$ (CN)

$-\infty \qquad\qquad \mathbf{0} \qquad\qquad \infty$

$f'(-1)<0$	$f'(1)>0$
Decreasing	*Increasing*

Decreasing: $(-\infty,\ 0)$

Increasing: $(0,\ \infty)$

Exercise

Find the open intervals on which the function is increasing and decreasing. Then, identify the function's local and absolute extreme values, if any, saying where they occur.

$g(t)=-t^2-3t+3$

Solution

$g'(t)=-2t-3=0 \quad\Rightarrow\quad \boxed{t=-\frac{3}{2}} \qquad$ (***CP***)

$-\infty \qquad\qquad -\frac{3}{2} \qquad\qquad \infty$

$f'(-2)>0$	$f'(2)<0$
Increasing	***Decreasing***

Decreasing: $\left(-\frac{3}{2},\ \infty\right)$

Increasing: $\left(-\infty,\ -\frac{3}{2}\right)$

Local maximum: $g\left(-\frac{3}{2}\right)=-\left(-\frac{3}{2}\right)^2-3\left(-\frac{3}{2}\right)+3=\frac{21}{4} \quad \boxed{\left(-\frac{3}{2},\ \frac{21}{4}\right)}$

Exercise

Find the open intervals on which the function is increasing and decreasing. Then, identify the function's local and absolute extreme values, if any, saying where they occur.

$h(x) = 2x^3 - 18x$

Solution

$$h'(x) = 6x^2 - 18 = 0 \quad \Rightarrow \quad x^2 = 3 \rightarrow \boxed{x = \pm\sqrt{3}} \quad (\boldsymbol{CN})$$

$$\begin{cases} x = -\sqrt{3} & \rightarrow h = -6\sqrt{3} + 18\sqrt{3} = 12\sqrt{3} \\ x = \sqrt{3} & \rightarrow h = 6\sqrt{3} - 18\sqrt{3} = -12\sqrt{3} \end{cases}$$

$$\begin{array}{ccccccc} -\infty & & -\sqrt{3} & & \sqrt{3} & & \infty \\ \hline & f'(-4) > 0 & \vert & f'(1) < 0 & \vert & f'(4) > 0 & \\ & \textit{Increasing} & \vert & \textit{Decreasing} & \vert & \textit{Increasing} & \end{array}$$

Decreasing: $\left(-\sqrt{3},\ \sqrt{3}\right)$ ***Increasing***: $\left(-\infty,\ -\sqrt{3}\right)$ *and* $\left(\sqrt{3},\ \infty\right)$

Local maximum: $\boxed{\left(-\sqrt{3},\ 12\sqrt{3}\right)}$ ***Local minimum***: $\boxed{\left(\sqrt{3},\ -12\sqrt{3}\right)}$

Exercise

Find the open intervals on which the function is increasing and decreasing. $f(\theta) = 3\theta^2 - 4\theta^3$
Then, identify the function's local and absolute extreme values, if any, saying where they occur.

Solution

$$f'(\theta) = 6\theta - 12\theta^2 = 6\theta(1 - 2\theta) = 0 \quad \Rightarrow \quad \boxed{\theta = 0,\ \frac{1}{2}}$$

$$\begin{cases} \theta = 0 & f(0) = 0 \\ \theta = \frac{1}{2} & f\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right)^3 = \frac{1}{4} \end{cases}$$

$$\begin{array}{ccccccc} -\infty & & 0 & & \frac{1}{2} & & \infty \\ \hline & f'(-1) < 0 & \vert & f'\left(\frac{1}{4}\right) > 0 & \vert & f'(1) < 0 & \\ & \textit{Decreasing} & \vert & \textit{Increasing} & \vert & \textit{Decreasing} & \end{array}$$

Decreasing: $(-\infty,\ 0)$ $\left(\frac{1}{2},\ \infty\right)$ ***Increasing***: $(-\infty,\ 0)$ *and* $\left(\frac{1}{2},\ \infty\right)$

Local maximum: $\boxed{\left(\frac{1}{2},\ \frac{1}{4}\right)}$ ***Local minimum***: $\boxed{(0,\ 0)}$

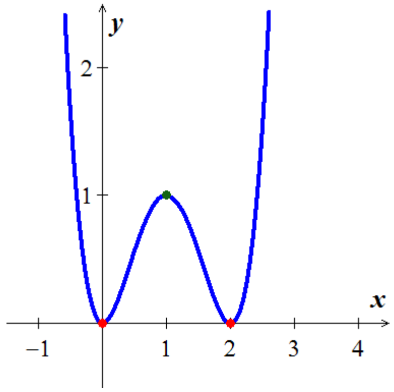
Exercise

Find the open intervals on which the function is increasing and decreasing $g(x) = x^4 - 4x^3 + 4x^2$. Then, identify the function's local and absolute extreme values, if any, saying where they occur.

Solution

$$g'(x) = 4x^3 - 12x^2 + 8x$$

$$= 4x\left(x^2 - 3x + 2\right) \quad \Rightarrow \quad \boxed{x = 0,\ 1,\ 2}$$



$-\infty \qquad 0 \qquad 1 \qquad 2 \qquad \infty$

$f'(-1) < 0$	$f'\left(\frac{1}{2}\right) > 0$	$f'\left(\frac{3}{2}\right) < 0$	$f'(3) > 0$
Decreasing	***Increasing***	***Decreasing***	***Increasing***

Decreasing: $(-\infty,\ 0) \cup (1,\ 2)$ ***Increasing***: $(0,\ 1) \cup (2,\ \infty)$

Local maximum: $\boxed{(1,\ 1)}$ ***Local minimum***: $\boxed{(0,\ 0),\ (2,0)}$

Absolute minimum: $\boxed{(0,\ 0),\ (2,0)}$

Exercise

Find the open intervals on which the function is increasing and decreasing. $f(x) = x - 6\sqrt{x-1}$ Then, identify the function's local and absolute extreme values, if any, saying where they occur.

Solution

Domain: $x > 1$

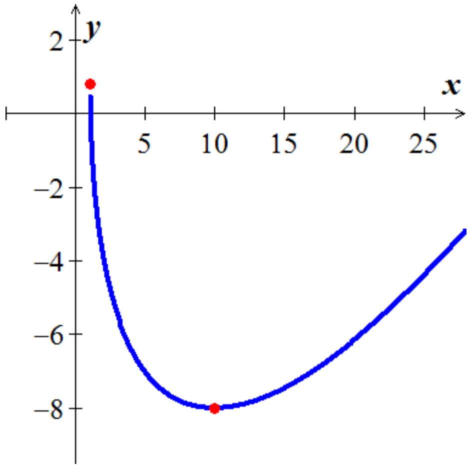
$$f'(x) = 1 - 6\frac{\frac{1}{2}}{\sqrt{x-1}}$$

$$= \frac{\sqrt{x-1} - 3}{\sqrt{x-1}} = 0$$

$$\sqrt{x-1} = 3 \quad \rightarrow \quad x - 1 = 3^2$$

$$\boxed{x = 9 + 1 = 10}$$

Critical points: $\boxed{x = 1,\ 10}$



$1 \qquad 10 \qquad \infty$

$f'(2) < 0$	$f'(11) > 0$
Decreasing	***Increasing***

Decreasing: $(1,\ 10)$ ***Increasing***: $(10,\ \infty)$

Local minimum: $\boxed{(10,\ -8)}$ ***Local maximum***: $\boxed{(1,\ 1)}$

Absolute minimum: $\boxed{(10,\ -8)}$ ***Absolute maximum:*** $\boxed{(1,\ 1)}$

Exercise

Find the open intervals on which the function is increasing and decreasing. $f(x)=\dfrac{x^3}{3x^2+1}$

Then, identify the function's local and absolute extreme values, if any, saying where they occur.

Solution

$$f'(x)=\frac{3x^2\left(3x^2+1\right)-6x\left(x^3\right)}{\left(3x^2+1\right)2}$$

$$=\frac{9x^4+3x^2-6x^4}{\left(3x^2+1\right)2}$$

$$=\frac{3x^4+3x^2}{\left(3x^2+1\right)2}$$

$$=\frac{3x^2\left(x^2+1\right)}{\left(3x^2+1\right)2}=0 \quad\Rightarrow\quad \boxed{x=0} \quad (\boldsymbol{CP})$$

$-\infty$	0	∞
$f'(-1)>0$		$f'(1)>0$
Increasing		***Increasing***

Increasing: $\boxed{(-\infty,\ 0)\cup(0,\ \infty)}$

No local extrema, no absolute extrema

Exercise

Find the open intervals on which the function is increasing and decreasing. Then, identify the function's local and absolute extreme values, if any, saying where they occur.

$f(x)=x^{1/3}(x+8)$

Solution

$$\begin{aligned} f'(x) &= \tfrac{1}{3}x^{-2/3}(x+8)+x^{1/3} \\ &= \frac{1}{3}x^{1/3}+\frac{8}{3}x^{-2/3}+x^{1/3} \\ &= \frac{4}{3}x^{1/3}+\frac{8}{3x^{2/3}} \\ &= \frac{4x+8}{3x^{2/3}} = 0 \end{aligned}$$

$$\to \begin{cases} 4x+8=0 & \Rightarrow x=-2 \\ x^{2/3}=0 & \Rightarrow x=0 \end{cases} \quad \boxed{x=-2,\ 0} \quad (CP)$$

$-\infty \qquad -2$	0	∞
$f'(-3)<0$	$f'(-1)>0$	$f'(1)>0$
Decreasing	***Increasing***	***Increasing***

Decreasing: $(-\infty,\ 0)$ ***Increasing***: $(-2,\ 0)\cup(0,\ \infty)$

Local minimum: $\boxed{\left(-2,\ -6\sqrt[3]{2}\right)}$ ***Local maximum***: None

Absolute minimum: $\boxed{\left(-2,\ -6\sqrt[3]{2}\right)}$ ***Absolute maximum:*** None

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing $f(x) = 2x^3 - 6x + 1$

Solution

$$f'(x) = 6x^2 - 6 = 0$$

$$\Rightarrow 6x^2 = 6$$

$$\Rightarrow x^2 = 1 \quad \to x = \pm 1 \ \ (CN)$$

$$\begin{cases} x = 1 \to y = f(1) = -3 \\ x = -1 \to y = f(-1) = 5 \end{cases} \quad (-1, 5), \ \ (1, -3)$$

$-\infty$	-1	1	∞
$f'(-2) > 0$ ↗ *Increasing*	$f'(0) < 0$ ↘ *Decreasing*	$f'(2) > 0$ ↗ *Increasing*	

RMAX: (–1, 5); ***RMIN***: (1, –3)

Increasing:(–∞, –1) and (1, ∞); ***Decreasing***: (–1, 1)

Exercise

Find all relative Extrema of $f(x) = 6x^{2/3} - 4x$ and Find the open intervals on which is increasing or decreasing

Solution

$$f'(x) = 4x^{-1/3} - 4$$

$$= 4\left(\frac{1}{x^{1/3}} - 1\right)$$

$$f'(x) = 4\left(\frac{1}{x^{1/3}} - 1\right) = 0 \qquad \boxed{x \neq 0}$$

$$\frac{1}{x^{1/3}} - 1 = 0$$

$$\frac{1}{x^{1/3}} = 1 \qquad \textit{Multiply both sides by } x^{1/3}$$

$$1 = x^{1/3}$$

$$\boxed{x = 1^3 = 1}$$

$$CN: x = 0, \ 1$$

$$\begin{cases} x=0 \rightarrow y=0 \\ x=1 \rightarrow y=2 \end{cases} \qquad (0,\ 0) \text{ and } (1,\ 2)$$

$-\infty$	**0**	**1**	∞
$f'(-1) > 0$	$f'(\frac{1}{2}) < 0$	$f'(2) > 0$	
Decreasing	*Increasing*	*Decreasing*	

RMIN: (0, 0) ***RMAX***: (1, 1)

Increasing: (0, 1) ***Decreasing***:(–∞, 0) and (1, ∞)

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing

$f(x) = x^4 - 4x^3$

Solution

$$f'(x) = 4x^3 - 12x^2$$

$$= 4x^2(x-3) = 0$$

$$\Rightarrow \boxed{x = 0,\ 3} \quad (CN)$$

$$x = 3 \rightarrow y = f(3) = -27$$

$-\infty$	**0**	**3**	∞
$f'(-1) < 0$ ↘	$f'(1) < 0$ ↘	$f'(4) > 0$ ↗	
Decreasing	Decreasing	Increasing	

RMIN: (3, –27);

Decreasing: (–∞, 3); ***Increasing***: (3, ∞)

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing $f(x) = 3x^{2/3} - 2x$

Solution

$$f'(x) = 2x^{-1/3} - 2$$

$$= 2\left(\frac{1}{x^{1/3}} - 1\right)$$

$$f'(x) = 2\left(\frac{1-x^{1/3}}{x^{1/3}}\right) = 0$$

$$\Rightarrow \begin{cases} x^{1/3} = 0 \rightarrow x = 0 \\ 1 - x^{1/3} = 0 \rightarrow x^{1/3} = 1 \Rightarrow x = 1 \end{cases}$$

$$\begin{cases} x = 0 \rightarrow y = 0 \\ x = 1 \rightarrow y = 1 \end{cases}$$

(0, 0) and (1, 1)

$-\infty$ **0**	**1**	∞
$f'(-1) > 0$	$f'\left(\frac{1}{2}\right) < 0$	$f'(2) > 0$
Increasing	*Decreasing*	*Increasing*

RMAX: (0, 0); ***RMIN***: (1, 1);

Decreasing: (0, 1) ***Increasing***:(−∞, 0) and (1, ∞);

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing $y = \sqrt{4 - x^2}$

Solution

$$f'(x) = \frac{-x}{\sqrt{4 - x^2}}$$

The critical values are $x = 0, \ \pm 2$, but the domain of the function is [–2,2]. We can't go outside of that interval to test.

Interval(s)	$(-2, 0)$	$(0, 2)$
Sign of f'	$f'(-1) > 0$	$f'(1) < 0$
Conclusion for f	increasing	decreasing

The function has a RMAX of $f(0) = 2$ @ $x = 0$. Some texts also consider $f(-2) = 0$ and $f(2) = 0$ as RMIN

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing $f(x) = x\sqrt{x+1}$

Solution

$$\begin{aligned} f'(x) &= \sqrt{x+1} + \frac{1}{2}x\left(x+1\right)^{-1/2} \\ &= \sqrt{x+1} + \frac{x}{2\sqrt{x+1}} \\ &= \frac{2x+2+x}{2\sqrt{x+1}} \\ &= \frac{3x+2}{2\sqrt{x+1}} \end{aligned}$$

Critical points are $x = -\frac{2}{3}$ and $x = -1$, but the domain is $[-1, \infty)$.

Interval(s)	$(-1, -2/3)$	$(-2/3, \infty)$
Sign of f'	$f'(-0.9) < 0$	$f'(0) > 0$
Conclusion for f	*decreasing*	*increasing*

Decreasing $\left(-1,\ -\frac{2}{3}\right)$ ***Increasing*** $\left(-\frac{2}{3},\ \infty\right)$

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing $f(x) = \dfrac{x}{x^2+1}$

Solution

$$\begin{aligned} f'(x) &= \frac{(1)\left(x^2+1\right) - x\left(2x\right)}{\left(x^2+1\right)^2} \\ &= \frac{x^2+1-2x^2}{\left(x^2+1\right)^2} \\ &= \frac{-x^2+1}{\left(x^2+1\right)^2} \end{aligned} \qquad -x^2+1=0 \Rightarrow x^2=1 \to x=\pm 1$$

Critical numbers are $\boxed{x = \pm 1}$.

Interval(s)	$(-\infty, -1)$	$(-1,\ 1)$	$(1, \infty)$
Sign of f'	$f'(-2) < 0$	$f'(0) > 0$	$f'(0) < 0$
Conclusion for f	*decreasing*	*increasing*	*decreasing*

Decreasing: $(-\infty, -1) \cup (1, \infty)$. **Increasing:** $(-1,\ 1)$.

The function has a RMIN of $f(-1) = -\frac{1}{2}$ @ $x = -1$.

The function has a RMAX of $f(1) = \frac{1}{2}$ @ $x = 1$.

Exercise

Find all relative Extrema as well as where the function is increasing and decreasing $f(x) = x^4 - 8x^2 + 9$

Solution

$$f'(x) = 4x^3 - 16x$$

$$= 4x\left(x^2 - 4\right) = 0$$

$$\boxed{x = 0} \qquad x^2 - 4 = 0$$

$$x^2 = 4 \Rightarrow \boxed{x = \pm 2}$$

$$CP: \quad \boxed{x = -2,\ 0,\ 2}$$

$-\infty$ -2 0 2 ∞

$f'(-3) < 0$	$f'(-1) > 0$	$f'(1) < 0$	$f'(3) > 0$
decreasing	*increasing*	*decreasing*	*increasing*

$x = -2 \quad \rightarrow f(-2) = -7$

$x = 0 \quad \rightarrow f(0) = 9$

$x = 2 \quad \rightarrow f(2) = -7$

Increasing: $(-2,\ 0) \cup (2,\ \infty)$ **Decreasing:** $(-\infty,\ -2) \cup (0,\ 2)$

RMIN: $(-2,\ -7)$ *and* $(2,\ -7)$ **RMAX:** $(0,\ 9)$

Exercise

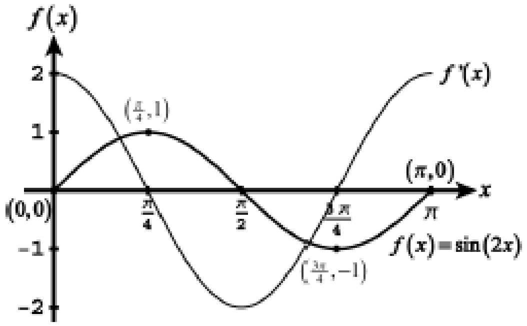
Find the local extrema of the function on the given interval, and say where they occur

$f(x) = \sin 2x \quad 0 \le x \le \pi$

Solution

$$f'(x) = 2\cos 2x = \underline{0}| \quad \rightarrow \begin{cases} 2x = \frac{\pi}{2} & \Rightarrow x = \frac{\pi}{4} \\ 2x = \frac{3\pi}{2} & \Rightarrow x = \frac{3\pi}{4} \end{cases} \rightarrow \boxed{x = \frac{\pi}{4}, \frac{3\pi}{4}} \quad (CN)$$

$$\begin{cases} x = 0 & \Rightarrow f(x) = 0 \\ x = \frac{\pi}{4} & \Rightarrow f(x) = 1 \\ x = \frac{3\pi}{4} & \Rightarrow f(x) = -1 \\ x = \pi & \Rightarrow f(x) = 0 \end{cases}$$



$0 \qquad \frac{\pi}{4} \qquad \frac{3\pi}{4} \qquad \pi$

$f'\left(\frac{\pi}{6}\right) > 0$	$f'\left(\frac{\pi}{3}\right) < 0$	$f'\left(\frac{5\pi}{6}\right) > 0$
Increasing	***Decreasing***	***Increasing***

Increasing: $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right)$ ***Decreasing:*** $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

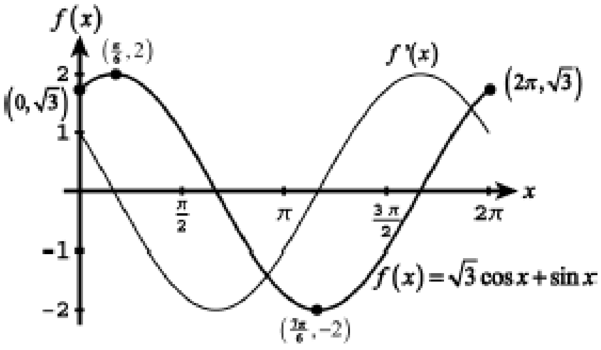
LMIN: $\left(\frac{3\pi}{4}, -1\right) \quad (0, 0)$ ***LMAX:*** $\left(\frac{\pi}{4}, 1\right) \quad (\pi, 0)$

Exercise

Find the local extrema of the function on the given interval, and say where they occur

$f(x) = \sqrt{3}\cos x + \sin x \quad 0 \le x \le 2\pi$

Solution



$$f'(x) = -\sqrt{3}\sin x + \cos x = \underline{0}|$$

$$\sqrt{3}\sin x = \cos x \quad \rightarrow \boxed{x = \frac{\pi}{6}, \frac{7\pi}{6}} \quad (CN)$$

$$\begin{cases} x = 0 & \Rightarrow f(x) = \sqrt{3} \\ x = \frac{\pi}{6} & \Rightarrow f(x) = 2 \\ x = \frac{7\pi}{6} & \Rightarrow f(x) = -2 \\ x = 2\pi & \Rightarrow f(x) = \sqrt{3} \end{cases}$$

$0 \qquad \frac{\pi}{6} \qquad \frac{7\pi}{6} \qquad 2\pi$

$f'\left(\frac{\pi}{12}\right) > 0$	$f'\left(\frac{\pi}{2}\right) < 0$	$f'\left(\frac{3\pi}{2}\right) > 0$
Increasing	***Decreasing***	***Increasing***

Increasing: $\left(0,\ \frac{\pi}{6}\right) \cup \left(\frac{7\pi}{6},\ 2\pi\right)$ ***Decreasing:*** $\left(\frac{\pi}{6},\ \frac{7\pi}{6}\right)$

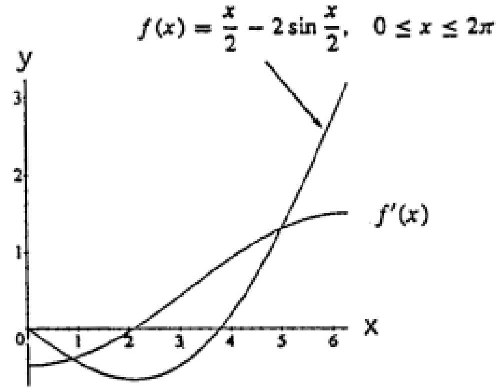
LMIN: $\left(\frac{7\pi}{6},\ -2\right)$ ***LMAX:*** $\left(\frac{\pi}{6},\ 2\right)$

Exercise

Find the local extrema of the function on the given interval, and say where they occur

$f(x) = \frac{x}{2} - 2\sin\frac{x}{2} \quad 0 \le x \le 2\pi$

Solution



$$f'(x) = \frac{1}{2} - 2\left(\frac{1}{2}\right)\cos\frac{x}{2} = 0$$

$$\cos\frac{x}{2} = \frac{1}{2} \quad \rightarrow \begin{cases} \frac{x}{2} = \frac{\pi}{3} \\ \frac{x}{2} = \frac{5\pi}{3} \end{cases} \qquad \boxed{x = \frac{2\pi}{3},\ \cancel{\frac{10\pi}{3}}\ (>2\pi)}$$

$$\begin{cases} x = 0 & \Rightarrow f(x) = 0 \\ x = \frac{2\pi}{3} & \Rightarrow f(x) = \frac{\pi}{3} - \sqrt{3} \\ x = 2\pi & \Rightarrow f(x) = \pi \end{cases}$$

$0 \qquad \frac{2\pi}{3} \qquad 2\pi$

$f'\left(\frac{\pi}{2}\right) < 0$	$f'(\pi) > 0$
decreasing	***Increasing***

Increasing: $\left(\frac{2\pi}{3},\ 2\pi\right)$ ***Decreasing:*** $\left(0,\ \frac{2\pi}{3}\right)$

LMIN: $\left(\frac{2\pi}{3},\ \frac{\pi}{3} - \sqrt{3}\right)$ ***LMAX:*** $(2\pi,\ \pi)$

Exercise

Find the local extrema of the function on the given interval, and say where they occur

$f(x)=\sec^2 x-2\tan x \quad -\frac{\pi}{2}\le x\le \frac{\pi}{2}$

Solution

$$f'(x)=2\sec x\cdot\sec x\cdot\tan x-2\sec^2 x$$

$$=2\sec^2 x(\tan x-1)=0$$

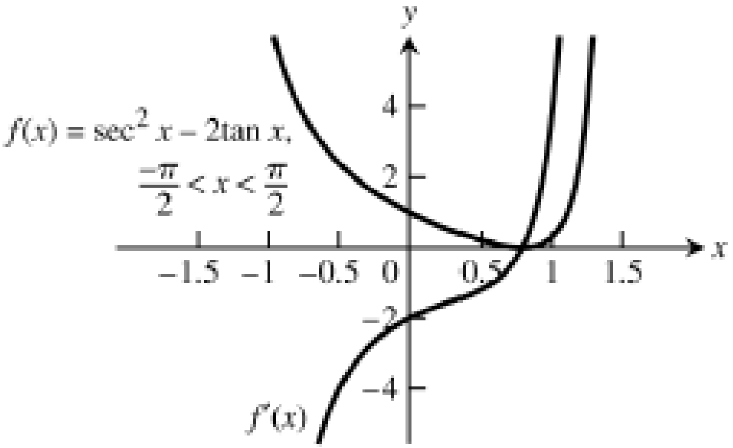
$$\begin{cases}\sec 2x\ne 0\\ \tan x-1=0 \quad \Rightarrow \tan x=1\to \boxed{x=\frac{\pi}{4}} \quad (CN)\end{cases}$$

$$\begin{cases}x=-\frac{\pi}{2}\\ x=\frac{\pi}{4} \quad \Rightarrow f(x)=0\\ x=\frac{\pi}{2}\end{cases}$$

$-\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$f'\left(\frac{\pi}{6}\right)<0$		$f'\left(\frac{\pi}{3}\right)>0$
decreasing		***Increasing***

Increasing: $\left(\frac{\pi}{4},\ \frac{\pi}{2}\right)$ ***Decreasing:*** $\left(-\frac{\pi}{2},\ \frac{\pi}{4}\right)$

LMIN: $\left(\frac{\pi}{4},\ 0\right)$ ***LMAX:*** None



Exercise

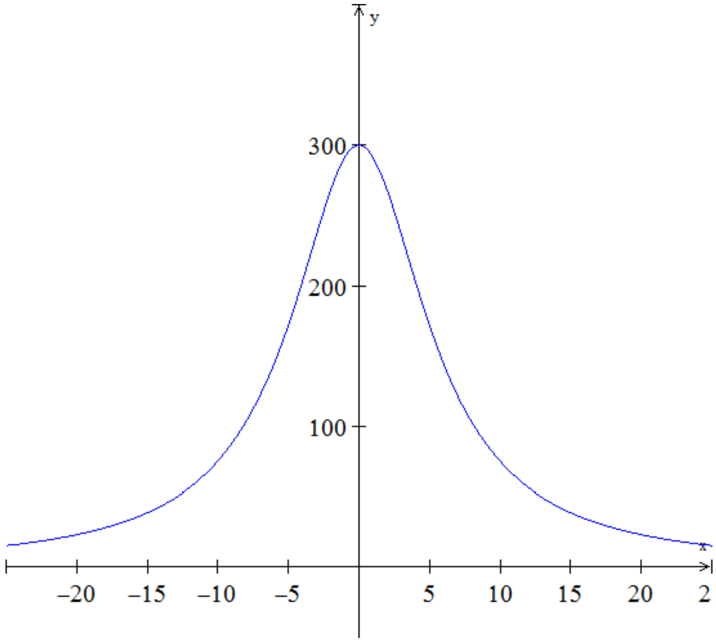
A county realty group estimates that the number of housing starts per year over the next three years will be

$$H(r)=\frac{300}{1+0.03r^2}$$

Where r is the mortgage rate (in percent).

a) Where is $H(r)$ increasing?

b) Where is $H(r)$ decreasing?



Solution

$$H'(r)=\frac{-300(0.06r)}{\left(1+0.03r^2\right)^2}$$

$$H'(r)=\frac{-18r}{\left(1+0.03r^2\right)^2}$$

$$-18r=0\Rightarrow\boxed{r=0}\quad (CN)$$

a) $H(r)$ is ***increasing*** on the interval $(-\infty,\ 0)$

b) $H(r)$ is ***decreasing*** on the interval $(0,\ \infty)$

Exercise

Suppose the total cost $C(x)$ to manufacture a quantity x of insecticide (in hundreds of liters) is given by $C(x)=x^3-27x^2+240x+750$. Where is $C(x)$ decreasing?

Solution

$$C'(x)=3x^2-54x+240=0$$

$$\Rightarrow x=8,\ 10$$

0	**8**	**10**
$C'(1)=189>0$	$C'<0$	$C'>0$
Increasing	***Decreasing***	***Increasing***

$C(x)$ is ***decreasing*** (8, 10)

Exercise

The cost of a computer system increases with increased processor speeds. The cost C of a system as a function of processor speed is estimated as $C(x) = 14x^2 - 4x + 1200$, where x is the processor speed in MHz. Determine the intervals where the cost function $C(x)$ is decreasing.

Solution

$$C'(x) = 28x - 4 = 0$$

$$\Rightarrow x = \frac{4}{28} = \frac{1}{7}$$

$\frac{1}{7}$

$C'(0) = -4 < 0$	$C' > 0$
Decreasing	***Increasing***

The cost function $C(x)$ is decreasing $\left(0,\ \frac{1}{7}\right)$

Exercise

The percent of concentration of a drug in the bloodstream t hours after the drug is administered is given by $K(t) = \frac{t}{t^2 + 36}$. On what time interval is the concentration of the drug increasing?

Solution

$$f = t \qquad f' = 1$$

$$g = t^2 + 36 \quad g' = 2t$$

$$K'(t) = \frac{1\left(t^2+36\right) - 2t(t)}{\left(t^2+36\right)^2} \qquad K = \frac{f}{g} \Rightarrow K' = \frac{f'g + g'f}{g^2}$$

$$= \frac{t^2 + 36 - 2t^2}{\left(t^2+36\right)^2}$$

$$= \frac{36 - t^2}{\left(t^2+36\right)^2}$$

$$K'(t) = 0$$

$$\frac{36 - t^2}{\left(t^2+36\right)^2} = 0 \Rightarrow 36 - t^2 = 0$$

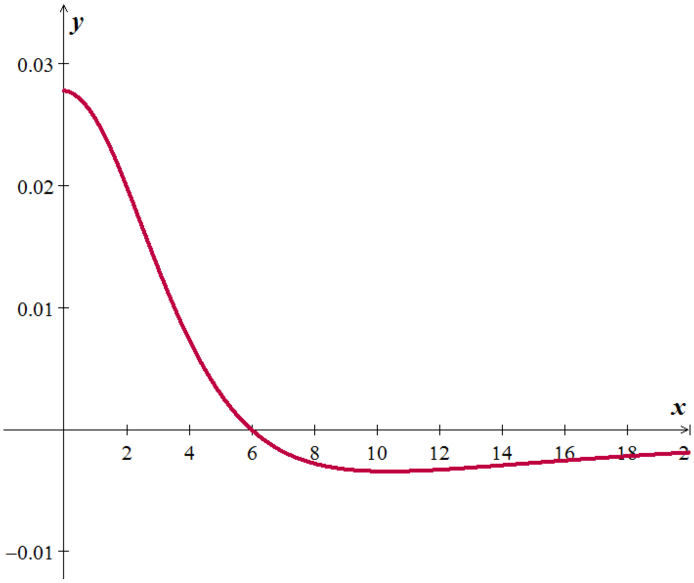
$$t^2 = 36$$

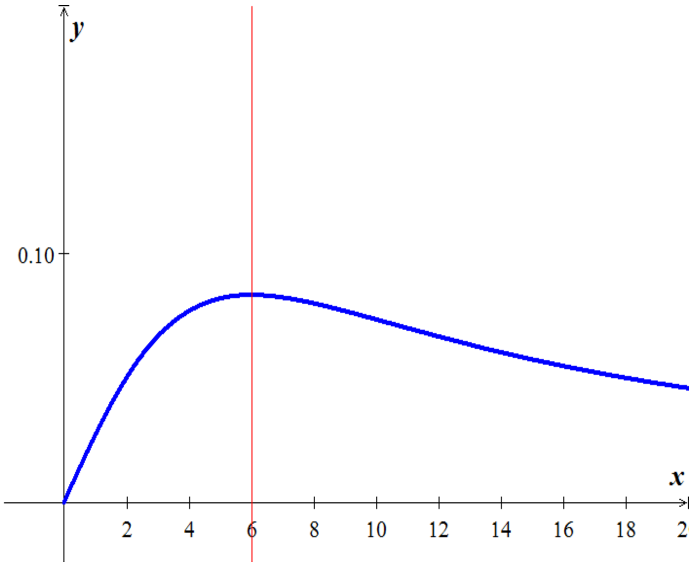
$$t = \pm\sqrt{36} = \pm 6 \quad \Rightarrow \boxed{t = 6}$$

0 **6**

$K'(1) = \frac{35}{37^2} > 0$	$K'(7) < 0$
Increasing	***Decreasing***

The concentration of the drug is increasing over $\boxed{(0,\ 6)}$





Exercise

Coughing forces the trachea to contract, this in turn affects the velocity of the air through the trachea. The velocity of the air during coughing can be modeled by: $v = k(R - r)r^2, \quad 0 \le r < R$ where k is a constant, R is the normal radius of the trachea (also a constant) and r is the radius of the trachea during coughing. What radius r will produce the maximum air velocity?

Solution

$$v = k\left(Rr^2 - r^3\right)$$

$$\begin{aligned} v' &= k\left(2Rr - 3r^2\right) \\ &= kr(2R - 3r) = 0 \end{aligned}$$

$$r = 0 \quad or \quad 2R - 3r = 0$$

$$r = 0 \quad or \quad r = (2/3)R$$

A trachea radius of zero minimizes air velocity (duh!). And a radius of 2/3 its normal size maximizes air flow.

Exercise

$P(x) = -x^3 + 15x^2 - 48x + 450, \quad x \ge 3$ is an approximation to the total profit (in thousands of dollars) from the sale of x hundred thousand tires. Find the number of hundred thousands of tires that must be sold to maximize profit.

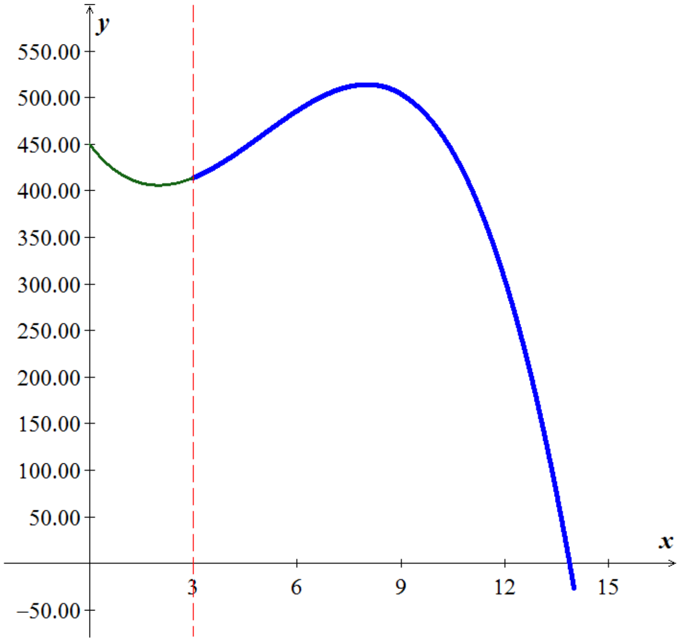
Solution

$$\begin{aligned} P'(x) &= -3x^2 + 30x - 48 = 0 \\ &\Rightarrow x = 2,\ 8 \end{aligned}$$

$$Since\ x \ge 3 \quad \Rightarrow \boxed{x = 8}$$

$$\begin{aligned} P(x = 8) &= -(8)^3 + 15(8)^2 - 48(8) + 450 \\ &= 541 \end{aligned}$$

The number of tires that must be sold to maximize profit is 800,000 tires



Exercise

$P(x) = -x^3 + 3x^2 + 360x + 5000; \quad 6 \le x \le 20$ is an approximation to the number of salmon swimming upstream to spawn, where *x* represents the water temperature in degrees Celsius. Find the temperature that produces the maximum number of salmon.

Solution

$$P'(x) = -3x^2 + 6x + 360 = 0$$

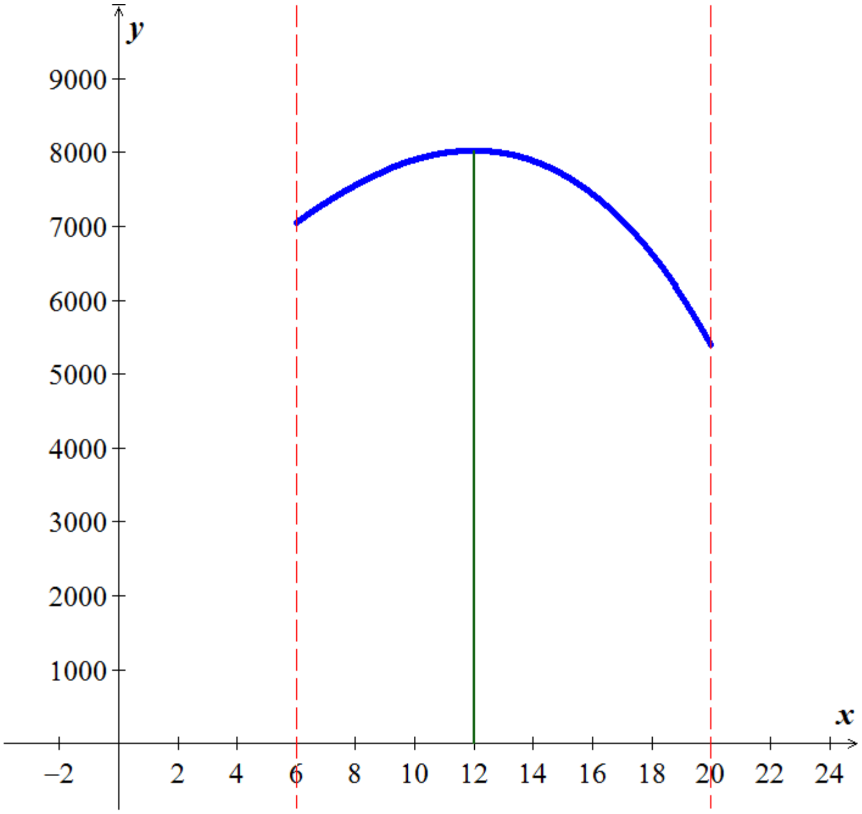
$$\Rightarrow x = 12, \quad -10\left(\textit{not in the interval}\right)$$

$$P(x=6) = -(6)^3 + 3(6)^2 + 360(6) + 5000 \underline{= 7052|}$$

$$P(x=20) = -(20)^3 + 3(20)^2 + 360(20) + 5000 \underline{= 5400|}$$

$$P(x=12) = -(12)^3 + 3(12)^2 + 360(12) + 5000 \underline{= 8024|}$$

12° is the temperature that produces the maximum number of salmon



Solution *Section* 3.4 – Concavity and Curve Sketching

Exercise

Determine the intervals on which the graph of the function is concave upward or concave downward.

$$f(x)=\frac{x^2-1}{2x+1}$$

Solution

$$f'(x)=\frac{(2x+1)(2x)-(x^2-1)(2)}{(2x+1)^2}$$

$$=\frac{4x^2+2x-2x^2+2}{(2x+1)^2}$$

$$=\frac{2x^2+2x+2}{(2x+1)^2}$$

$$=\frac{2(x^2+x+1)}{(2x+1)^2}$$

$$f''(x)=2\frac{(2x+1)^2(2x+1)-(x^2+x+1)(2)(2x+1)(2)}{(2x+1)^4}$$

$$=2\frac{(2x+1)^3-4\left(x^2+x+1\right)(2x+1)}{(2x+1)^4}$$

$$=2\frac{(2x+1)[(2x+1)^2-4(x^2+x+1)]}{(2x+1)^4}$$

$$=2\frac{4x^2+4x+1-4x^2-4x-4}{(2x+1)^3}$$

$$=2\frac{-3}{(2x+1)^3}$$

$$=-\frac{6}{(2x+1)^3}$$

$2x=1=0 \;\Rightarrow\; x=-\frac{1}{2}$

$-\infty \qquad -\frac{1}{2} \qquad \infty$

$f''(-1)>0$	$f''(0)<0$
Upward	***Downward***

f is concave upward on $\left(-\infty,-\frac{1}{2}\right)$

f is concave downward on $\left(-\frac{1}{2},\infty\right)$

Exercise

Determine the intervals on which the graph of the function is concave upward or concave downward.

$f(x) = -4x^3 - 8x^2 + 32$

Solution

$$f'(x) = -12x^2 - 16x$$
$$f''(x) = -24x - 16$$
$$f''(x) = -24x - 16 = 0$$
$$\Rightarrow -24x = 16$$
$$\Rightarrow x = \frac{16}{-24} = -\frac{2}{3}$$

$-\infty \qquad\qquad -\frac{2}{3} \qquad\qquad \infty$

$f''(-1) > 0$	$f''(0) < 0$
Upward	***Downward***

Concave up on $(-\infty, -2/3)$ and concave down on $(-2/3, \infty)$

Exercise

Find the points of inflection. $f(x) = x^3 - 9x^2 + 24x - 18$

Solution

$$f'(x) = 3x^2 - 18x + 24$$
$$f''(x) = 6x - 18 = 0 \Rightarrow x = 3$$
$$x = 3 \Rightarrow f(3) = 0$$

→ Point of inflection (3, 0)

Exercise

Find the second derivative of $f(x) = -2\sqrt{x}$ and discuss the concavity of the graph

Solution

$$f'(x) = -x^{-1/2}$$
$$\Rightarrow f''(x) = \frac{1}{2}x^{-3/2}$$
$$= \frac{1}{2x^{3/2}} > 0 \text{ for all } x > 0$$

f is concave up for all $x > 0$.

Exercise

Determine the intervals on which the graph of the function is concave upward or concave downward.

$f(x)=\dfrac{12}{x^2+4}$

Solution

$$f(x)=12(x^2+4)^{-1}$$

$$f'(x)=-12(x^2+4)^{-2}(2x)=-\frac{12x}{(x^2+4)^2}$$

$$f''(x)=-\frac{12(x^2+4)^2-12x(2)(x^2+4)(2x)}{(x^2+4)^4}$$

$$=-\frac{12(x^2+4)^2-48x^2(x^2+4)}{(x^2+4)^4}$$

$$=-\frac{12(x^2+4)\left[(x^2+4)-4x^2\right]}{(x^2+4)^4}$$

$$=-\frac{12(x^2+4)\left[x^2+4-4x^2\right]}{(x^2+4)^4}$$

$$=-\frac{12(x^2+4)(-3x^2+4)}{(x^2+4)^4}$$

$$=-\frac{12(-3x^2+4)}{(x^2+4)^3}$$

Solve for x:

$$f''(x)=-\frac{12(-3x^2+4)}{(x^2+4)^3}=0$$

$$\Rightarrow -3x^2+4=0$$

$$\rightarrow -3x^2=-4$$

$$\rightarrow x^2=\frac{4}{3}$$

$$\Rightarrow x=\pm\sqrt{\frac{4}{3}}$$

$$=\pm\frac{\sqrt{4}}{\sqrt{3}}$$

$$=\pm\frac{2}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}}$$

$$=\pm\frac{2\sqrt{3}}{3}$$

$-\infty \qquad -\frac{2\sqrt{3}}{3} \qquad \frac{2\sqrt{3}}{3} \qquad \infty$

$f''(-2) > 0$	$f''(0) < 0$	$f''(2) > 0$
upward	***downward***	***upward***

f is ***concave up*** on $\left(-\infty, -\frac{2\sqrt{3}}{3}\right)$ *and* $\left(\frac{2\sqrt{3}}{3}, \infty\right)$

f is ***concave down*** on $\left(-\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right)$

Exercise

Find the extrema using the second derivative test $f(x) = \frac{4}{x^2+1}$

Solution

$$f'(x) = \frac{-8x}{\left(x^2+1\right)^2} \qquad \textit{CN} \text{ is } x = 0$$

$$f''(x) = \frac{8(3x^2-1)}{\left(x^2+1\right)^3}$$

$f''(0) = -8 < 0 \Rightarrow f(0) = 4$ is a ***local maximum*** *(LMAX)*

Exercise

Discuss the concavity of the graph of f and find its points of inflection. $f(x) = x^4 - 2x^3 + 1$

Solution

$$f'(x) = 4x^3 - 12x^2$$

$$f'(x) = 4x^2(x-3) = 0 \rightarrow \boxed{x = 0,\ 3}$$

$$f''(x) = 12x^2 - 12x$$

Points:	(0, 1)	$f''(0) = 0$ Test fails
	(3, –26)	$f''(3) > 0 \Rightarrow$ ***local Minimum*** *(LMIN)*

Exercise

Find all relative extrema of $f(x) = x^4 - 4x^3 + 1$

Solution

$$f'(x) = 4x^3 - 6x^2$$

$$f''(x) = 12x^2 - 12x = 0$$

$$12x(x-1) = 0 \Rightarrow x = 0,1$$

For $x = 0 \Rightarrow f(0) = 0^4 - 2(0)^3 + 1 = 1 \to (0,1)$

For $x = 0 \Rightarrow f(1) = 1^4 - 2(1)^3 + 1 = 0 \to (1,0)$

$$-\infty \qquad \mathbf{0} \qquad \mathbf{1} \qquad \infty$$

$f''(-1) > 0$ ***upward***	$f''(1/2) < 0$ ***downward***	$f''(2) > 0$ ***upward***

Concave up on $(-\infty, 0)$ *and* $(1, \infty)$ ***concave down*** on $(0,1)$

Points of inflection: (0, 1), (1, 0)

Exercise

Sketch the graph $f(x) = x^4 - 4x^3 + 5$

Solution

$$f'(x) = 4x^3 - 12x^2 = 0$$

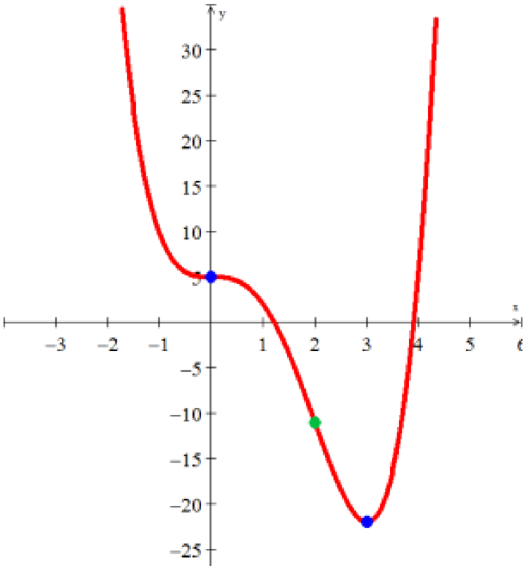
$$4x^2(x-3) = 0$$

$$\Rightarrow x = 0,\ 0,\ 3$$

$$f''(x) = 12x^2 - 24x = 0$$

$$12x(x-2) = 0$$

$$\Rightarrow x = 0,\ 2$$



	f	f'	f''	
(–∞, 0)		–	+	Decreasing, Concave up
$x = 0$	5	0	0	RMAX
(0, 2)		–	–	Decreasing, Concave down
$x = 2$	–11	–	0	Point of Inflection
(2, 3)		v	+	Decreasing, Concave up
$x = 3$	–22	0	+	RMIN
(3, ∞)		+	+	Increasing, Concave up

Exercise

Given $f(x)=\frac{x^2+1}{x^2-1}$

Solution

VA: $x=\pm 1$

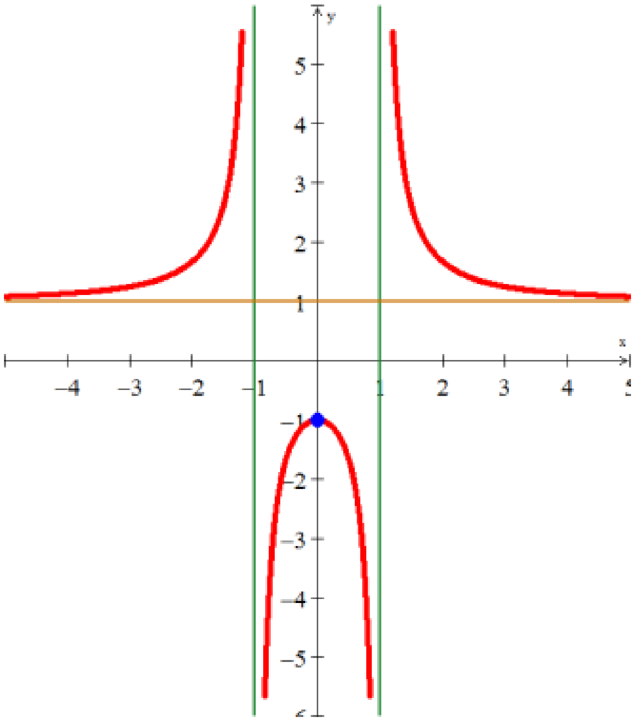
HA: $y=1$

$$f'(x)=\frac{(2x)(x^2-1)-(x^2+1)(2x)}{(x^2-1)^2}$$

$$=\frac{2x^3-2x-2x^3-2x}{(x^2-1)^2}$$

$$=\frac{-4x}{(x^2-1)^2}=0$$

$$\Rightarrow x=0$$



$$f''=\left(\frac{-4x}{(x^2-1)^2}\right)'$$

$$=\frac{-4(x^2-1)^2-(-4x)(x^2-1)(2x)}{(x^2-1)^4}$$

$$=\frac{(x^2-1)\left[-4(x^2-1)-(-4x)(2x)\right]}{(x^2-1)^4}$$

$$=\frac{-4x^2+4+8x^2}{(x^2-1)^3}$$

$$=\frac{4x^2+4}{(x^2-1)^3}$$

$$=\frac{4(x^2+1)}{(x^2-1)^3}=0$$

$\Rightarrow x^2+1=0 \;\Rightarrow x^2=-1$ (*no zeros*)

	f	f'	f''	
$(-\infty,-1)$		+	–	Increasing, Concave up
$x=-1$	*Undef.*	*Undef.*	*Undef.*	Vertical Asymptote
$(-1,0)$		+	–	Increasing, Concave down
$x=0$	−1	0	–	RMAX
$(0,1)$		–	–	Decreasing, Concave down
$x=1$	*Undef.*	*Undef.*	*Undef.*	Vertical Asymptote
$(1,\infty)$		–	+	Decreasing, Concave up

Exercise

Given $f(x)=2x^{3/2}-6x^{1/2}$

Solution

$$f'(x)=3x^{1/2}-3x^{-1/2}=0$$

$$x^{1/2}\left(3x^{1/2}-3x^{-1/2}\right)=0$$

$$3x-3=0$$

$$\Rightarrow x=1$$

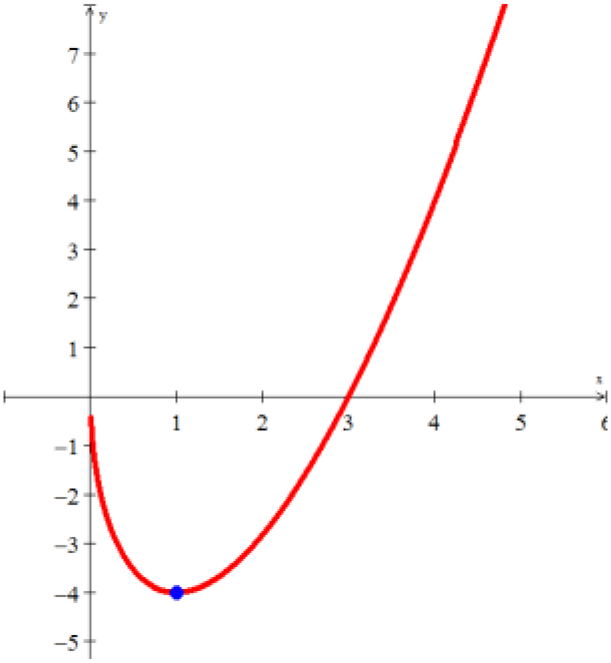
$$f''(x)=\frac{3}{2}x^{-1/2}+\frac{3}{2}x^{-3/2}=0$$

$$\frac{2}{3}x^{3/2}\left(\frac{3}{2}x^{-1/2}+\frac{3}{2}x^{-3/2}\right)=0$$

$$x+1=0$$

$$\rightarrow x=-1<0$$

x	f	f'	f''	
(0, 1)		–	+	Decreasing, Concave up
$x = 1$	– 4	0	+	RMIN
(1, ∞)		+	+	Increasing, Concave up



Exercise

Sketch the graph $y = x^3 - 3x + 3$

Solution

$$y' = 3x^2 - 3 = \underline{0|}$$

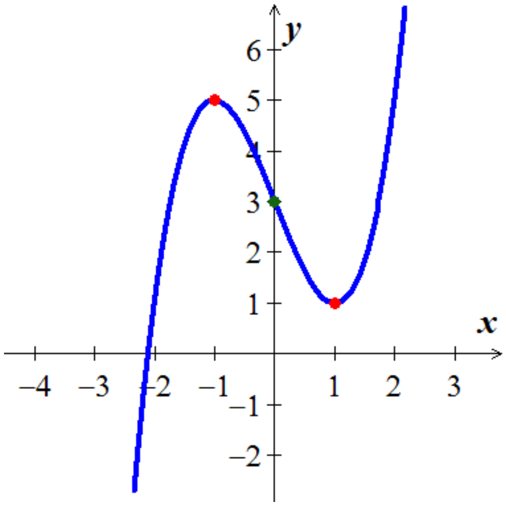
$$x^2 = 1 \Rightarrow \boxed{x = \pm 1} \quad (CP)$$

$$\begin{cases} x = -1 & \rightarrow y = 5 \\ x = 1 & \rightarrow y = 1 \end{cases}$$

$$y'' = 6x = \underline{0|} \quad \Rightarrow \boxed{x = 0}$$

$$(x = 0 \quad \rightarrow y = 3$$

x	***f***	***f'***	***f''***	
$(-\infty, -1)$		+	+	Increasing, Concave Up
$x = -1$	5	0	+	Concave Up
$(-1, 0)$		–	+	Decreasing, Concave Up
$x = 0$	3	–	0	Decreasing, Pt. of Inflection
$(0, 1)$		–	–	Decreasing, Concave Down
$x = 1$	1	0	–	Concave Down
$(1, \infty)$		+	–	Increasing, Concave Down



Decreasing: $(-1,\ 1)$

Increasing: $(-\infty, -1) \cup (1, \infty)$

Concave Down: $(0,\ \infty)$

Concave Up: $(-\infty,\ 0)$

Local Minimum: $(-1,\ 5)$

Local Maximum: $(1,\ 5)$

Points of inflection: $(0,\ 3)$

Exercise

Sketch the graph $y = -x^4 + 6x^2 - 4$

Solution

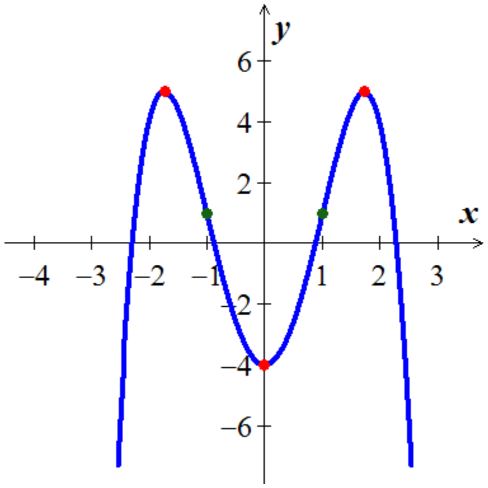
$$y' = -4x^3 + 12x$$

$$= -4x\left(x^2 - 3\right) = 0$$

$$\begin{cases} x = 0 \\ x^2 = 3 \quad \rightarrow x = \pm\sqrt{3} \end{cases}$$

$$\boxed{x = 0, \ \pm\sqrt{3}} \quad (CP)$$

$$\begin{cases} x = -\sqrt{3} & \rightarrow y = 5 \\ x = 0 & \rightarrow y = -4 \\ x = \sqrt{3} & \rightarrow y = 5 \end{cases}$$



$$y'' = -12x^2 + 12 = 0$$

$$x^2 = 1 \quad \rightarrow \boxed{x = \pm 1} \quad (\textbf{\textit{Points of Inflection}})$$

$$\begin{cases} x = -1 & \rightarrow y = 1 \\ x = 1 & \rightarrow y = 1 \end{cases}$$

x	***f***	***f′***	***f″***	
$\left(-\infty, \ -\sqrt{3}\right)$		+	–	Increasing, Concave Down
$x = -\sqrt{3}$	5	0	–	Concave Down
$\left(-\sqrt{3}, \ -1\right)$		–	–	Decreasing, Concave Down
$x = -1$	1	–	0	Decreasing, Pt. of Inflection
$(-1, 0)$		–	+	Decreasing, Concave Up
$x = 0$	–4	0	+	Concave Up
$(0, 1)$		+	+	Increasing, Concave Up
$x = 1$	1	+	0	Increasing, Pt. of Inflection
$\left(1, \ \sqrt{3}\right)$		+	–	Increasing, Concave Down
$x = \sqrt{3}$	5	0	–	Concave Down
$\left(\sqrt{3}, \ \infty\right)$		–	–	Decreasing, Concave Down

Decreasing: $\left(-\sqrt{3}, \ 0\right) \cup \left(\sqrt{3}, \ \infty\right)$ ***Increasing:*** $\left(-\infty, \ -\sqrt{3}\right) \cup \left(0, \ \sqrt{3}\right)$

Concave Down: $(-1, \ 1)$ ***Concave Up:*** $(-\infty, \ -1) \quad (1, \ \infty)$

Local Minimum: $(0, \ -4)$ ***Local Maximum***: $\left(-\sqrt{3}, \ 5\right) \quad \left(\sqrt{3}, \ 5\right)$

Points of inflection: $(-1, \ 1) \quad (1, \ 1)$

Exercise

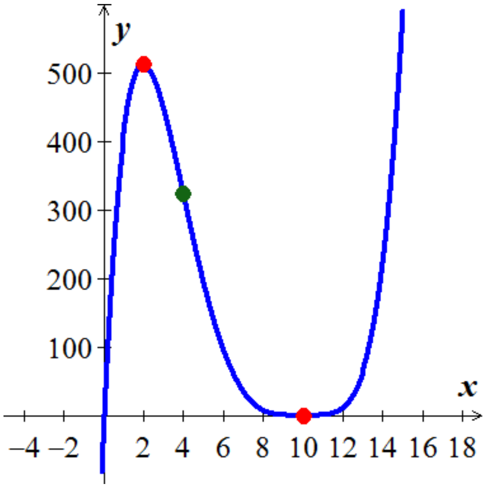
Sketch the graph $y = x\left(\frac{x}{2} - 5\right)^4$

Solution

$$\begin{aligned} y' &= \left(\frac{x}{2}-5\right)^4 + 4x\left(\frac{1}{2}\right)\left(\frac{x}{2}-5\right)^3 \\ &= \left(\frac{x}{2}-5\right)^3\left(\frac{x}{2}-5+2x\right) \\ &= \left(\frac{x}{2}-5\right)^3\left(\frac{5x}{2}-5\right) = 0 \end{aligned}$$

$$\begin{cases} \frac{x}{2}-5=0 & \rightarrow \boxed{x=10} \quad (CP) \\ \frac{5x}{2}-5=0 & \rightarrow \boxed{x=2} \quad (CP) \end{cases} \Rightarrow \begin{cases} x=2 & \rightarrow y=512 \\ x=10 & \rightarrow y=0 \end{cases}$$

$$\begin{aligned} y'' &= 3\left(\frac{1}{2}\right)\left(\frac{x}{2}-5\right)^2\left(\frac{5x}{2}-5\right) + \frac{5}{2}\left(\frac{x}{2}-5\right)^3 \\ &= \frac{1}{2}\left(\frac{x}{2}-5\right)^2\left(3\left(\frac{5x}{2}-5\right)+5\left(\frac{x}{2}-5\right)\right) \\ &= \frac{1}{2}\left(\frac{x}{2}-5\right)^2\left(\frac{15x}{2}-15+\frac{5x}{2}-25\right) \\ &= \frac{1}{2}\left(\frac{x}{2}-5\right)^2\left(\frac{20x}{2}-40\right) \\ &= \frac{1}{2}\left(\frac{x}{2}-5\right)^2(10x-40) \\ &= \frac{1}{2}\left(\frac{x}{2}-5\right)^2(10)(x-4) \\ &= 5\left(\frac{x}{2}-5\right)^2(x-4) = 0 \end{aligned}$$



$$\begin{cases} \frac{x}{2}-5=0 & \rightarrow x=10 \\ x-4=0 & \rightarrow x=4 \end{cases} \Rightarrow \begin{cases} x=10 & \rightarrow y=0 \\ x=4 & \rightarrow y=324 \end{cases}$$

x	***f***	***f′***	***f″***	
$(-\infty,\ 2)$		+	–	Increasing, Concave Down
$x=2$	512	0	–	Concave Down
$(2,\ 4)$		–	–	Decreasing, Concave Down
$x = 4$	324	–	0	Decreasing, Pt. of Inflection
(1, 10)		–	+	Decreasing, Concave Up
$x = 10$	0	0	0	Pt. of Inflection
$(10,\ \infty)$		+	+	Increasing, Concave Up

Exercise

Sketch the graph $y = x + \sin x \quad 0 \le x \le 2\pi$

Solution

$$y' = 1 + \cos x = 0$$

$$\cos x = -1 \quad \to \quad \boxed{x = \pi} \quad (CP)$$

$$\begin{cases} x = 0 & \to y = 0 \\ x = \pi & \to y = \pi \\ x = 2\pi & \to y = 2\pi \end{cases}$$

$$y'' = -\sin x = 0 \quad \to \boxed{x = 0,\ \pi,\ 2\pi}$$

x	***f***	***f′***	***f″***	
$x = 0$	0	+	0	
$(0,\ \pi)$		+	–	Increasing, Concave Down
$x = \pi$	π	0	0	Pt. of Inflection
$(\pi,\ 2\pi)$		+	+	Increasing, Concave Up
$x = 2\pi$	2π	+	0	

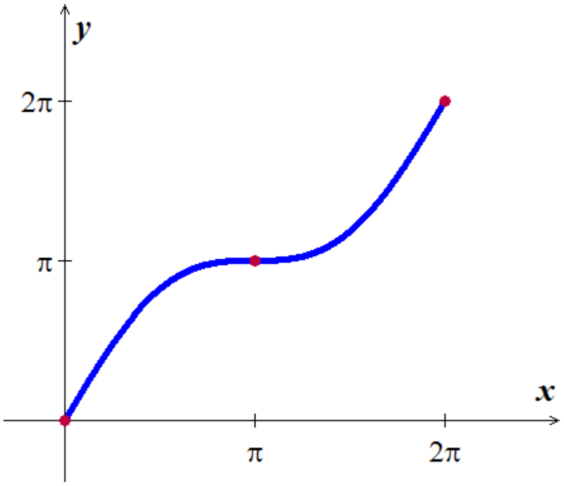
Decreasing: ***Increasing:*** $(0,\ 2\pi)$

Concave Down: $(0,\ \pi)$ ***Concave Up:*** $(\pi,\ 2\pi)$

Local and Absolute Minimum: $(0,\ 0)$

Local and Absolute Maximum: $(2\pi,\ 2\pi)$

Points of inflection: $x = \pi$



Exercise

Sketch the graph $\quad y=\cos x+\sqrt{3}\sin x \quad 0\le x\le 2\pi$

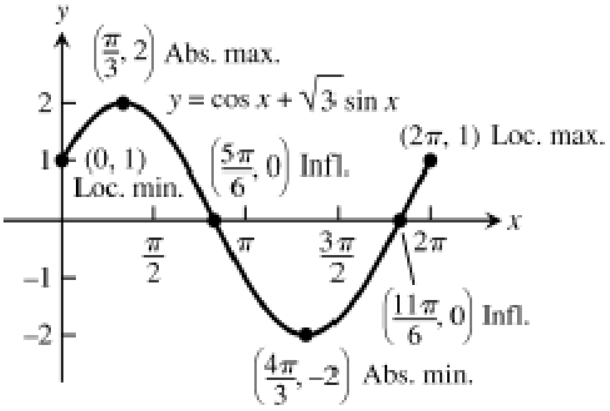
Solution

$$y'=-\sin x+\sqrt{3}\cos x=0$$

$$\sin x=\sqrt{3}\cos x$$

$$\frac{\sin x}{\cos x}=\sqrt{3}=\tan x \;\rightarrow \boxed{x=\frac{\pi}{3},\ \frac{4\pi}{3}} \quad (CN)$$

$$\rightarrow \begin{cases} x=0 & \rightarrow y=1 \\ x=\frac{\pi}{3} & \rightarrow y=2 \\ x=\frac{4\pi}{3} & \rightarrow y=-2 \\ x=2\pi & \rightarrow y=1 \end{cases}$$



$$y''=-\cos x-\sqrt{3}\sin x=0$$

$$\sqrt{3}\sin x=-\cos x$$

$$\frac{\sin x}{\cos x}=-\frac{1}{\sqrt{3}}=\tan x$$

$$\rightarrow \boxed{x=\frac{5\pi}{6},\ \frac{11\pi}{6}} \quad (\textbf{Points of Inflection})$$

x	***f***	***f′***	***f″***	
$x=0$	1			***Absolute Min.***
$\left(0,\ \frac{\pi}{3}\right)$		+	–	Increasing, Concave Down
$x=\frac{\pi}{3}$	2	0	–	***LMAX***, Concave Down
$\left(\frac{\pi}{3},\ \frac{5\pi}{6}\right)$		–	–	Decreasing, Concave Down
$x=\frac{5\pi}{6}$	0	–	0	Decreasing, ***Pt. of Inflection***
$\left(\frac{5\pi}{6},\ \frac{4\pi}{3}\right)$		–	+	Decreasing, Concave Up
$x=\frac{4\pi}{3}$	–2	0	+	***LMIN***, Concave Up
$\left(\frac{4\pi}{3},\ \frac{11\pi}{6}\right)$		+	+	Increasing, Concave Up
$x=\frac{11\pi}{6}$	0	+	0	***Pt. of Inflection***
$\left(\frac{11\pi}{6},\ 2\pi\right)$		+	–	Increasing, Concave Down
$x=2\pi$	1			***Absolute Max.***

Exercise

Sketch the graph $y=\dfrac{x}{\sqrt{x^2+1}}$

Solution

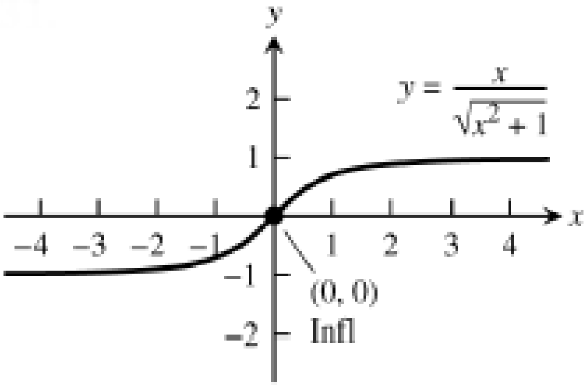
$$y'=\frac{\left(x^2+1\right)^{1/2}-x(x)\left(x^2+1\right)^{-1/2}}{\left(\sqrt{x^2+1}\right)^2}\frac{\left(x^2+1\right)^{1/2}}{\left(x^2+1\right)^{1/2}}$$

$u=x \qquad v=\left(x^2+1\right)^{1/2}$

$u'=1 \quad v'=\frac{1}{2}\left(x^2+1\right)^{-1/2}(2x)=x\left(x^2+1\right)^{-1/2}$

$$=\frac{\left(x^2+1\right)-x^2}{\left(x^2+1\right)^{3/2}}$$

$$=\frac{1}{\left(x^2+1\right)^{3/2}}\neq 0$$



$$y''=\frac{-2x\left(\frac{3}{2}\right)\left(x^2+1\right)^{1/2}}{\left(\left(x^2+1\right)^{3/2}\right)^2}$$

$$=\frac{-3x\left(x^2+1\right)^{1/2}}{\left(x^2+1\right)^3}$$

$$=\frac{-3x}{\left(x^2+1\right)^{5/2}}=\underline{0} \quad \to \boxed{x=0}$$

$-\infty \qquad\qquad 0 \qquad\qquad \infty$

$f''(-1)>0$	$f'(1)<0$
Concave Up	***Concave Down***

Concave Down: $(0,\ \infty)$ ***Concave Up:*** $(-\infty,\ 0)$

No Local or Absolute Extrema

Points of inflection: $x=0$

Exercise

Sketch the graph $y=x^2+\frac{2}{x}$

Solution

Vertical Asymptote: $x=0$

$$y'=2x-\frac{2}{x^2}=\frac{2x^3-2}{x^2}=0$$

$$2x^3-2=0\Rightarrow x^3=1\quad \boxed{x=1}\quad (CP)$$

$$\{x=1\ \rightarrow y=3$$

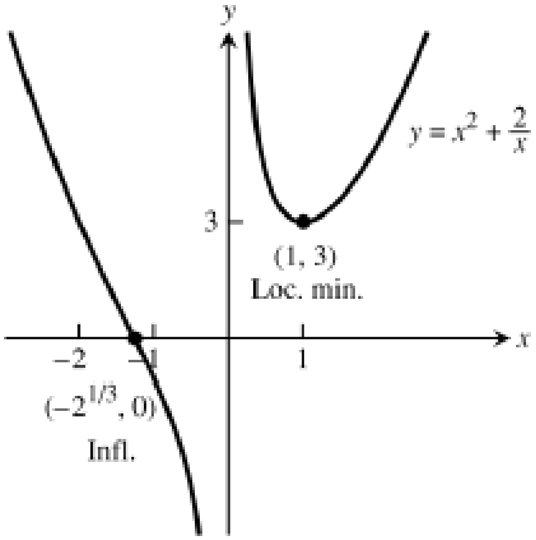
$$y''=2\cdot\frac{3x^2\left(x^2\right)-(2x)\left(x^3-1\right)}{x^4}$$

$$=2\cdot\frac{3x^4-2x^4+2x}{x^4}$$

$$=2\cdot\frac{x^4+2x}{x^4}$$

$$=2\cdot\frac{x^3+2}{x^3}=0$$

$$x^3+2=0\quad \boxed{x=-\sqrt[3]{2}}$$



x	***f***	***f′***	***f″***	
$\left(-\infty,\ -\sqrt[3]{2}\right)$		–	+	Decreasing, Concave Up
$x=-\sqrt[2]{3}$	0	–	0	Decreasing, ***Pt. of Inflection***
$\left(-\sqrt[2]{3},\ 0\right)$		–	–	Decreasing, Concave Down
$x = 0$				***V.A.***
(0, 1)		–	+	Decreasing, Concave Up
$x = 1$	3	0	+	***LMIN***
$(1,\ \infty)$		+	+	Increasing, Concave Up

Exercise

Sketch the graph $y = \dfrac{x^2-3}{x-2}$

Solution

Vertical Asymptote: $x = 2$

$$y' = \frac{2x(x-2)-\left(x^2-3\right)}{(x-2)^2}$$

$$= \frac{2x^2-4x-x^2+3}{(x-2)^2}$$

$$= \frac{x^2-4x+3}{(x-2)^2} = \underline{0|} \quad \Rightarrow \boxed{x=1,\ 3} \quad (CP)$$

$$\to \begin{cases} x=1 & \to y=2 \\ x=3 & \to y=6 \end{cases}$$

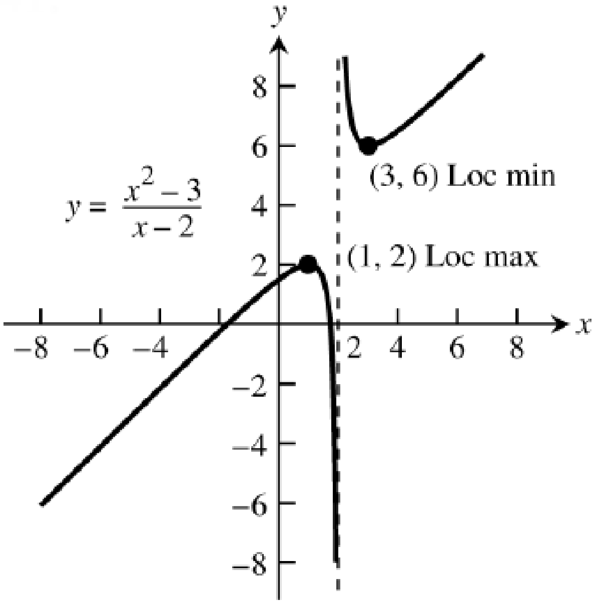
$$y'' = \frac{(2x-4)(x-2)^2-(2x-4)\left(x^2-4x+3\right)}{(x-2)^4}$$

$$= (x-2)\frac{2(x-2)(x-2)-2\left(x^2-4x+3\right)}{(x-2)^4}$$

$$= \frac{2\left(x^2-4x+4\right)-2x^2+8x-6}{(x-2)^3}$$

$$= \frac{2x^2-8x+8-2x^2+8x-6}{(x-2)^3}$$

$$= \frac{2}{(x-2)^3} \quad \underline{\neq 0|}$$



x	***f***	***f ′***	***f ″***	
$(-\infty,\ 1)$		+	-	Increasing, Concave Up
$x=1$	2	0		***LMAX***
$(1,\ 2)$		–	–	Decreasing, Concave Down
$x = 2$				***V.A.***
(2, 3)		–	+	Decreasing, Concave Up
$x = 3$	6	0	+	***LMIN***
$(3,\ \infty)$		+	+	Increasing, Concave Up

Exercise

Sketch the graph $y=\dfrac{5}{x^4+5}$

Solution

Horizontal Asymptote: $y=0$

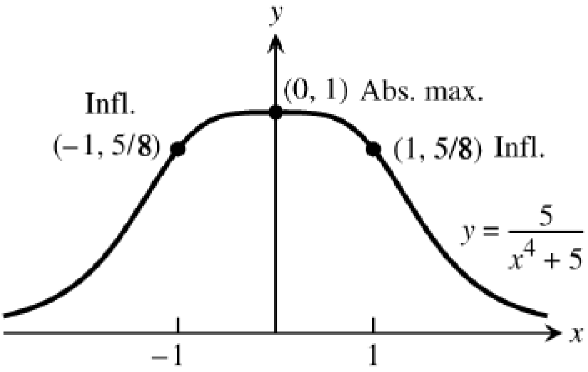
$$y'=\frac{5\left(-4x^3\right)}{\left(x^4+5\right)^2}=\frac{-20x^3}{\left(x^4+5\right)^2}=0 \quad \to x^3=0 \Rightarrow \boxed{x=0} \quad (CP)$$

$$\to \{x=0 \quad \to y=1$$

$$y''=-20\frac{3x^2\left(x^4+5\right)^2-2\left(4x^3\right)\left(x^4+5\right)\left(x^3\right)}{\left(x^4+5\right)^4}$$

$$=-20x^2\left(x^4+5\right)\frac{3\left(x^4+5\right)-8x^4}{\left(x^4+5\right)^4}$$

$$=-20x^2\,\frac{15-5x^4}{\left(x^4+5\right)^3}=\frac{100x^2\left(x^4-3\right)}{\left(x^4+5\right)^3}=0$$



$$x^2\left(x^4-3\right)=0 \to \begin{cases} x^2=0 & x=0 \\ x^4-3=0 & \boxed{x=\pm\sqrt[4]{3}} \end{cases} \qquad \to \begin{cases} x=-\sqrt[4]{3} & y=\frac{5}{8} \\ x=\sqrt[4]{3} & y=\frac{5}{8} \end{cases}$$

x	***f***	***f'***	***f''***	
$\left(-\infty,\ -\sqrt[4]{3}\right)$		+	+	Increasing, Concave Up
$x=-\sqrt[4]{3}$	2	+	0	Increasing, ***Pt. of Inflection***
$\left(-\sqrt[4]{3},\ 0\right)$		+	–	Increasing, Concave Down
$x=0$		0	0	***Abs. maximum, HA***
$\left(0,\ \sqrt[4]{3}\right)$		–	–	Decreasing, Concave Down
$x=\sqrt[4]{3}$	6	–	0	Decreasing, ***Pt. of Inflection***
$\left(\sqrt[4]{3},\ \infty\right)$		–	+	Decreasing, Concave Up

Exercise

Sketch the graph $y = \dfrac{x^2 - 49}{x^2 + 5x - 14}$

Solution

$$y = \frac{x^2 - 49}{x^2 + 5x - 14} = \frac{(x-7)(x+7)}{(x-2)(x+7)} = \frac{x-7}{x-2} = 1 - \frac{5}{x-2}$$

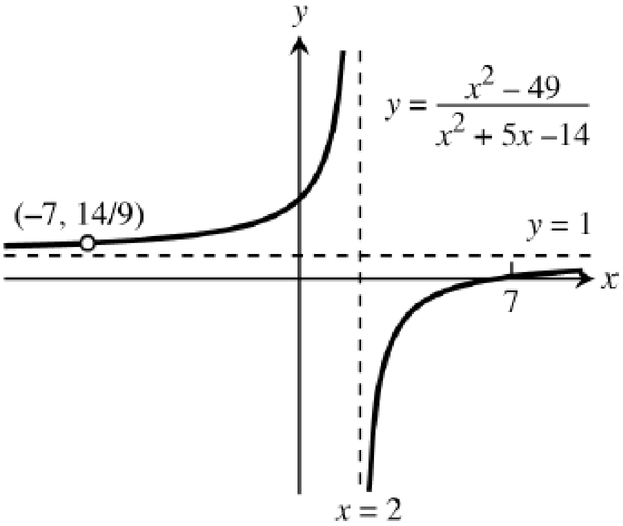
Hole: $x = -7$

Oblique Asymptote: $y = 1$

Vertical Asymptote: $x = 2$

$$y' = -5\frac{-1}{(x-2)^2} = \frac{5}{(x-2)^2} \neq 0$$

$$y'' = \frac{5(-2)(x-2)}{(x-4)^4} = \frac{-10}{(x-2)^3} \neq 0$$



Exercise

Sketch the graph $y=\dfrac{x^4+1}{x^2}$

Solution

$$y=\frac{x^4+1}{x^2}=\frac{x^4}{x^2}+\frac{1}{x^2}=x^2+\frac{1}{x^2}$$

Vertical Asymptote: $x = 0$

Oblique Asymptote: $y=x^2$

$$y'=\frac{4x^3x^2-2x\left(x^4+1\right)}{x^4}$$

$$y'=\left(x^2+\frac{1}{x^2}\right)'$$

$$=\frac{2x\left(2x^4-x^4-1\right)}{x^4}$$

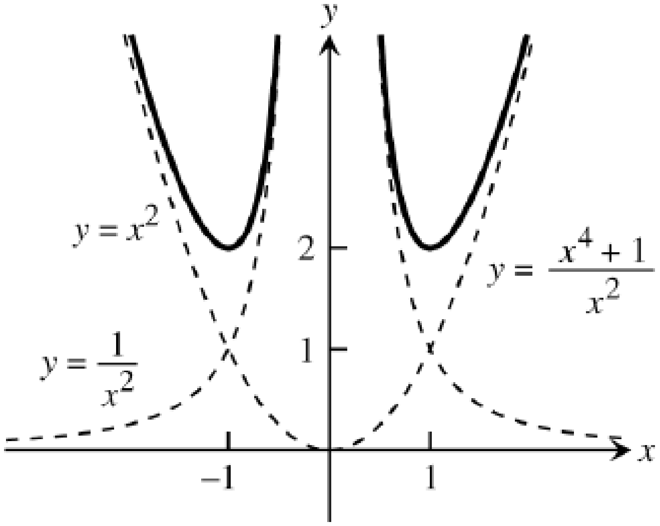
$$=2x-\frac{2x}{x^4}=2x-\frac{2}{x^3}$$

$$=\frac{2\left(x^4-1\right)}{x^3}=\underline{0}| \quad \to x^4-1=0 \quad \boxed{x=\pm 1} \quad (CN)$$

$$\to\begin{cases} x=-1 & \to y=2 \\ x=1 & \to y=2 \end{cases}$$

$-\infty$ -1 0 1 ∞

$f'(-2)<0$	$f'(-0.5)>0$	$f'(0.5)<0$	$f'(2)>0$
Decreasing	***Increasing***	***Decreasing***	***Increasing***



Exercise

Sketch the graph $y = \dfrac{x^2-4}{x^2-2}$

Solution

$x^2 - 2 = 0 \;\Rightarrow\; x = \pm\sqrt{2}$

Vertical Asymptote: $x = \pm\sqrt{2}$

***Horizontal Asymptote*:** $y = 1$

$$y' = \frac{2x\left(x^2-2\right) - 2x\left(x^2-4\right)}{\left(x^2-2\right)^2}$$

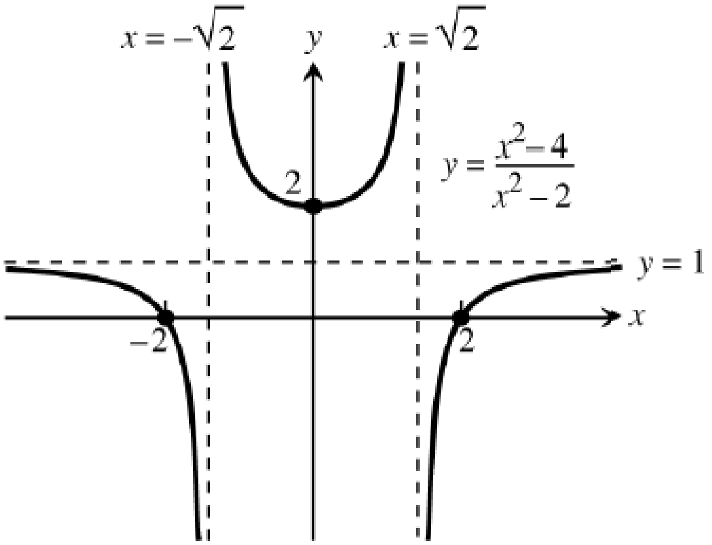
$$= \frac{2x^3 - 4x - 2x^3 + 8x}{\left(x^2-2\right)^2}$$

$$= \frac{4x}{\left(x^2-2\right)^2} = \underline{0} \;\rightarrow\; \boxed{x = 0,\ \pm\sqrt{2}} \quad (CN)$$

$$\rightarrow \begin{cases} x = -\sqrt{2} \;\Rightarrow\; y = 0 & \rightarrow \left(-\sqrt{2}, 0\right) \\ x = 0 \;\Rightarrow\; y = 2 & \rightarrow (0, 2) \\ x = \sqrt{2} \;\Rightarrow\; y = 0 & \rightarrow \left(\sqrt{2}, 0\right) \end{cases}$$

$-\infty$ $\quad -\sqrt{2}$ $\quad 0$ $\quad \sqrt{2}$ $\quad \infty$

$f'(-2) < 0$	$f'(-1) < 0$	$f'(1) > 0$	$f'(2) > 0$
Decreasing	***Decreasing***	***Increasing***	***Increasing***



Exercise

Sketch the graph $y=-\dfrac{x^2-x+1}{x-1}$

Solution

$$\begin{array}{r} x \\ x-1\overline{)\,x^2-x+1} \\ x^2-x \\ 1 \end{array}$$

$$y=-\frac{x^2-x+1}{x-1}=-\left(x+\frac{1}{x-1}\right)$$

Vertical Asymptote: $x=1$

Oblique Asymptote: $y=-x$

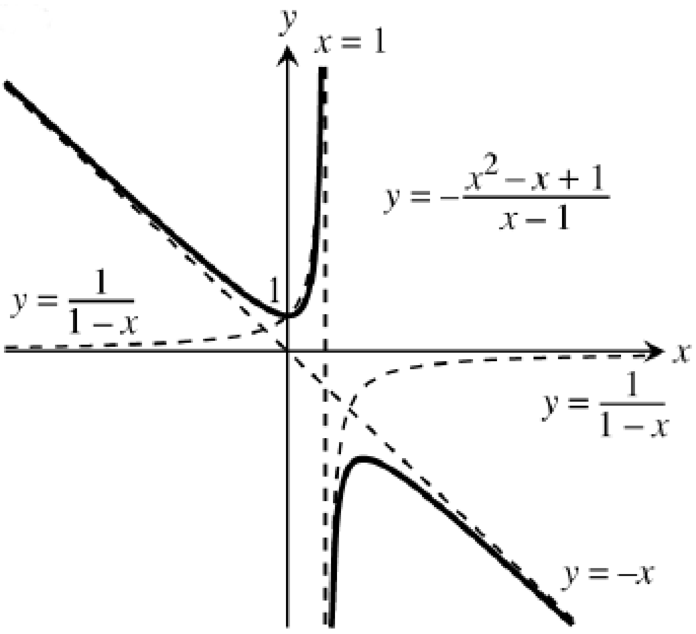
$$y'=-\left(1-\frac{1}{(x-1)^2}\right)$$

$$=\frac{1}{(x-1)^2}-1$$

$$=\frac{-x^2+2x}{(x-1)^2}=\underline{0|}\quad\rightarrow\quad\boxed{x=0,\ 1,\ 2}\quad(CN)$$

$$\rightarrow\begin{cases} x=0\ \Rightarrow\ y=1 & \rightarrow(0,1) \\ x=2\ \Rightarrow\ y=3 & \rightarrow(2,3)\end{cases}$$

$-\infty$	0	1	2	∞
$f'(-2)<0$	$f'(0.5)>0$	$f'(1.5)>0$	$f'(3)<0$	
Decreasing	***Increasing***	***Increasing***	***Decreasing***	



Exercise

Sketch the graph $y = \dfrac{x^3 - 3x^2 + 3x - 1}{x^2 + x - 2}$

Solution

$$\begin{aligned} y &= \frac{x^3 - 3x^2 + 3x - 1}{x^2 + x - 2} \\ &= \frac{(x-1)(x-1)(x-1)}{(x-1)(x+2)} \\ &= \frac{x^2 - 2x + 1}{x + 2} \\ &= x - 4 + \frac{9}{x+2} \end{aligned}$$

$$\begin{array}{r} x - 4 \\ x+2 \overline{\smash{\big)}\, x^2 - 2x + 1} \\ \underline{x^2 + 2x} \quad\quad \\ -4x + 1 \\ \underline{-4x - 8} \\ 9 \end{array}$$

Vertical Asymptote: $x = -2$

Hole: $x = 1 \Rightarrow y = 0$

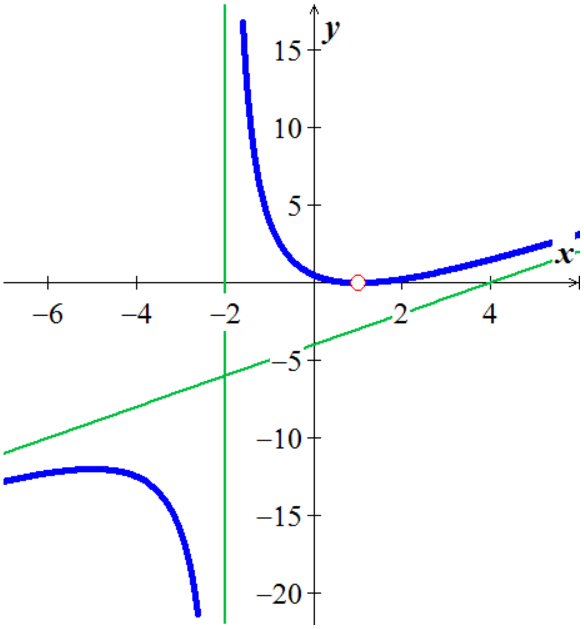
Oblique Asymptote: $y = x - 4$

$$y' = 1 - \frac{9}{(x+2)^2} \quad = \frac{(x+2)^2 - 9}{(x+2)^2} = \underline{0|}$$

$$(x+2)^2 = 9 \to x + 2 = \pm 3 \Rightarrow x = -2 \pm 3 \to (x = -5,\ 1)$$

$$\to \begin{cases} x = -5 \Rightarrow y = 1 & \to (-5, 1) \\ x = 1 \Rightarrow y = 0 & \to (1, 0) \end{cases}$$

$-\infty$	-5	-2	1	∞
$f'(-6) > 0$	$f'(-3) < 0$	$f'(0) < 0$	$f'(2) > 0$	
Increasing	***Decreasing***	***Decreasing***	***Increasing***	



Exercise

Sketch the graph $y = \dfrac{4x}{x^2+4}$

Solution

Vertical Asymptote: N/A

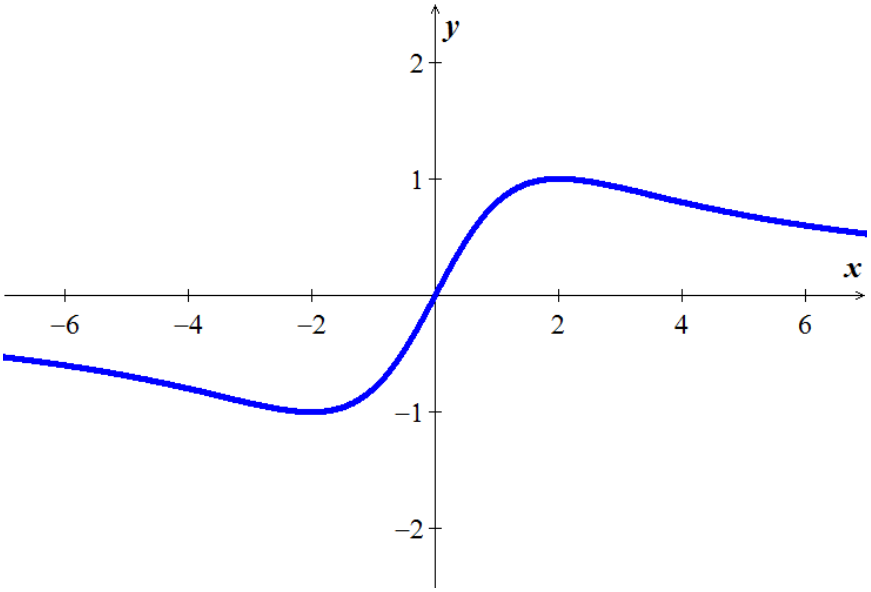
Horizontal Asymptote: $y=0$

$$y' = \frac{4\left(x^2+4\right)-(4x)(2x)}{\left(x^2+4\right)^2}$$

$$= \frac{16-4x^2}{\left(x^2+4\right)^2} \underline{=0|}$$

$$16-4x^2=0 \to x^2=4 \Rightarrow \boxed{x=\pm 2} \quad (CN)$$

$$\to \begin{cases} x=-2 & y=-1 \\ x=2 & y=1 \end{cases}$$



Exercise

The revenue R generated from sales of a certain product is related to the amount x spent on advertising by

$$R(x)=\frac{1}{15,000}\left(600x^2-x^3\right), \qquad 0\le x\le 600$$

Where x and R are in thousands of dollars. Is there a point of diminishing returns for this function?

Solution

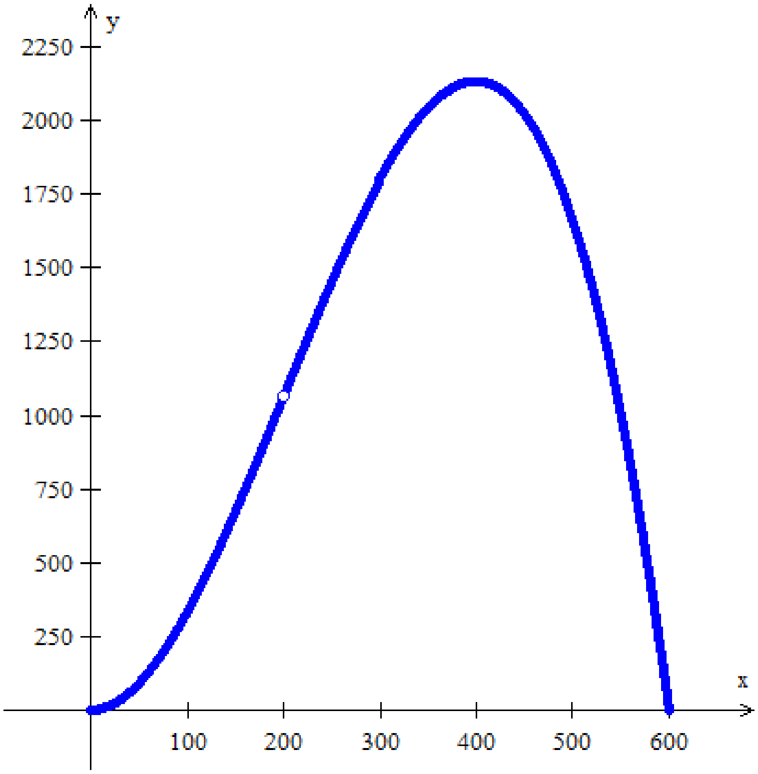
$$R'=\frac{1}{15,000}\left(1200x-3x^2\right)$$

$$R'=\frac{1}{15,000}(1200-6x)=0$$

$$\Rightarrow\ x=\frac{1200}{6}=200$$

x = 200 (or \$200,000) is a ***diminishing point***

An increased investment beyond this point is usually considered a poor use of capital



Exercise

Find the point of diminishing returns (x, y) for the function

$$R(x) = -x^3 + 45x^2 + 400x + 8000, \quad 0 \le x \le 20$$

where $R(x)$ represents revenue in thousands of dollars and x represents the amount spent on advertising in tens of thousands of dollars.

Solution

$$R'(x) = -3x^2 + 90x + 400$$

$$R''(x) = -6x + 90 = 0$$

$$-6x = -90$$

$$\boxed{x = \frac{-90}{-6} = 15}$$

$$R(x = 15) = -(15)^3 + 45(15)^2 + 400(15) + 8000$$

$$\boxed{= 20,750}$$

The point of diminishing returns is $\boxed{(15, \ \ 20,750)}$

Solution *Section* 3.5 – Applied Optimization

Exercise

Find two nonnegative numbers x and y for which $2x+y=30$, such that xy^2 is maximized.

Solution

$$2x+y=30$$
$$y=30-2x$$

$$
\begin{aligned}
M &= xy^2 \\
&= x(30-2x)^2 \\
&= x\left(900-120x+4x^2\right) \\
&= 900x-120x^2+4x^3
\end{aligned}
$$

$$M'=900-240x+12x^2$$

$$x=5 \quad \Rightarrow y=30-2(5)=20$$
$$x=15 \quad \Rightarrow y=30-2(15)=0$$

$$(0,\ 0) \quad M=xy^2=0\left(0^2\right)=0$$
$$(5,\ 20) \quad M=xy^2=5\left(20^2\right)=2000$$
$$(15,\ 0) \quad M=xy^2=15\left(0^2\right)=0$$

The values that maximize xy^2 are $\boxed{x=5 \quad and \quad y=20}$

Exercise

A rectangular page will contain 54 in^2 of print. The margins at the top and bottom of the page are 1.5 *inches* wide. The margins on each side are 1 *inch* wide. What should the dimensions of the page be to minimize the amount of paper used?

Solution

$$xy=54 \qquad y=\frac{54}{x}$$
$$A=(x+3)(y+2)$$
$$
\begin{aligned}
A &= (x+3)\left(\frac{54}{x}+2\right) \\
&= 54+2x+\frac{162}{x}+6
\end{aligned}
$$

$$= 60 + 2x + \frac{162}{x}$$

$$\frac{dA}{dx} = A' = 2 - \frac{162}{x^2}$$

$$2 - \frac{162}{x^2} = 0$$

$$-\frac{162}{x^2} = -2$$

$$162 = 2x^2$$

$$x^2 = 81$$

$$\rightarrow x = \pm 9 \Rightarrow x = 9 \ (only)$$

$$\rightarrow x + 3 = 12$$

$$y = \frac{54}{9} = 6$$

$$\rightarrow y + 2 = 8$$

Dimension: 8 by 12

Exercise

The product of two numbers is 72. Minimize the sum of the second number and twice the first number

Solution

$$xy = 72 \qquad \Rightarrow y = \frac{72}{x}$$

$$S = 2x + y$$

$$S = 2x + \frac{72}{x}$$

$$\frac{dS}{dx} = S' = 2 - \frac{72}{x^2}$$

$$2 - \frac{72}{x^2} = 0$$

$$-\frac{72}{x^2} = -2$$

$$72 = 2x^2$$

$$\Rightarrow x^2 = 36$$

$$x = \pm 6 \rightarrow x = 6$$

$$y = \frac{72}{x} = \frac{72}{6} = 12$$

Dimension: 6 by 12

Exercise

Verify the function $V = 27x - \frac{1}{4}x^3$ has an absolute maximum when $x = 6$. What is the maximum volume?

Solution

$$V' = 27 - \frac{3}{4}x^2 = 0$$

$$-\frac{3}{4}x^2 = -27$$

$$\Rightarrow x^2 = 27\frac{4}{3} = 36 \qquad \rightarrow x = \pm 6$$

$$\Rightarrow \boxed{x = 6} \ (only)$$

$$27x - \frac{1}{4}x^3 = 0$$

$$108x - x^3 = 0$$

$$x(108 - x^2) = 0$$

$$\Rightarrow x = 0, \ \sqrt{108}$$

$$V(6) = 27(6) - \frac{1}{4}(6)^3 \ \underline{= 108|}$$ is the maximum volume

Exercise

A net enclosure for golf practice is open at one end. The volume of the enclosure is $83\frac{1}{3}$ cubic meters. Find the dimensions that require the least amount of netting.

Solution

$$V = x^2 y = 83\frac{1}{3} = \frac{250}{3}$$

$$y = \frac{250}{3x^2}$$

$$S = x^2 + 3xy$$

$$= x^2 + 3x\frac{250}{3x^2}$$

$$= x^2 + \frac{250}{x}$$

$$= x^2 + 250x^{-1}$$

$$S' = 2x - 250x^{-2} = 2x - \frac{250}{x^2}$$

$$S' = 2x - \frac{250}{x^2} = 0$$

$$x^2\left(2x - \frac{250}{x^2}\right) = 0.x^2$$

$$\Rightarrow 2x^3 - 250 = 0$$

$$2x^3 = 250$$
$$\rightarrow x = 5m$$

$$y = \frac{250}{3x^2}$$
$$= \frac{250}{3(5)^2}$$
$$\approx 3.33m$$

Dimension: 5 by 3.33 *m*

Exercise

Find two numbers *x* and *y* such that their sum is 480 and x^2y is maximized.

Solution

Given: $x + y = 480; \Rightarrow y = 480 - x$

$$f(x) = x^2y = x^2(480 - x) = 480x^2 - x^3$$

$$f'(x) = 960x - 3x^2 = 0$$

$$\Rightarrow 3x(320 - x) = 0$$
$$\Rightarrow x = 0, x = 320$$
$$\Rightarrow y = 480, y = 160$$

$x = 0$, $y = 320$ actually gives a minimum for the function, while the solution that maximizes the function is **$x = 320$, $y = 160$**.

Exercise

If the price charged for a candy bar is *p*(*x*) cents, then *x* thousand candy bars will be sold in a certain city, where $p(x) = 82 - \frac{x}{20}$. How many candy bars must be sold to maximize revenue?

Solution

Recall that: $R = xp = 82x - \frac{x^2}{20}$

$$R'(x) = 82 - \frac{x}{10} = 0$$

$$\Rightarrow x = 820 \ (thousand)$$

x= $820,000 candy bars gives a maximum

Exercise

$S(x) = -x^3 + 6x^2 + 288x + 4000;\ \ 4 \le x \le 20$ is an approximation to the number of salmon swimming upstream to spawn, where x represents the water temperature in degrees Celsius. Find the temperature that produces the maximum number of salmon.

Solution

$$S'(x) = -3x^2 + 12x + 288 = 0$$

$$-3\left(x^2 - 4x - 96\right) = 0$$

$$-3(x-12)(x+8) = 0$$

$$x = 12, x = -8$$

Temperature that maximizes the number of salmon is 12 degrees C.

Exercise

A company wishes to manufacture a box with a volume of 52 cubic feet that is open on top and is twice as long as it is wide. Find the width of the box that can be produced using the minimum amount of material.

Solution

Given: $l = 2w,\ V = lwh = 52$

Substituting: $(2w)wh = 52, \Rightarrow 2w^2h = 52\ \Rightarrow h = \dfrac{26}{w^2}$

Surface Area (2 long sides, 2 short sides, bottom):

$$A = 2lh + 2wh + lw$$

$$A = 2(2w)\left(\frac{26}{w^2}\right) + 2w\left(\frac{26}{w^2}\right) + (2w)w$$

$$A = \frac{104}{w} + \frac{52}{w} + 2w^2 = 156w^{-1} + 2w^2$$

$$A'(w) = -156w^{-2} + 4w = 0$$

$$\left(-156w^{-2} + 4w\right)w^2 = 0(w^2)$$

$$\Rightarrow -156 + 4w^3 = 0$$

$$-4(39 - w^3) = 0$$

$$39 - w^3 = 0$$

$$w^3 = 39$$

$$\boxed{w = \sqrt[3]{39}}$$

Exercise

A rectangular field is to be enclosed on four sides with a fence. Fencing costs \$3 per foot for two opposite sides, and \$4 per foot for the other two sides. Find the dimensions of the field of area 730 square feet that would be the cheapest to enclose.

Solution

$$A = lw = 730 \Rightarrow l = \frac{730}{w}$$

It makes the most sense to let the short sides (*w*) cost \$4 per foot.

$$C = 2l(3) + 2w(4)$$

$$C = 6\frac{730}{w} + 8w$$

$$= 4380w^{-1} + 8w$$

$$C'(w) = -4380w^{-2} + 8 = 0$$

$$w^2\left(-4380w^{-2} + 8\right) = w^2(0)$$

$$-4380 + 8w^2 = 0$$

$$8w^2 = 4380 \Rightarrow w^2 = \frac{4380}{8} = 547.5$$

$$\boxed{w = \sqrt{547.5} \approx 23.4\ ft} \quad @\ \ \$4\ \text{ per } ft$$

$$l = \frac{730}{23.4} \approx \boxed{31.2\ ft} \quad @\ \ \$3\ \text{ per } ft$$

Exercise

A page is to contain 30 square inches of print. The margins at the top and bottom of the page are 2 inches wide. The margins on the sides are 1 inch wide. What dimensions will minimize the amount of paper used?

Solution

Let the dimensions of the original sheet have width x and height y.

The dimensions of the print area would then be, (x - 2) and (y - 4) respectively.

Area of the print space: $A_1 = (x-2)(y-4) = 30$

$$y-4 = \frac{30}{x-2} \Rightarrow y = \frac{30}{x-2} + 4$$

Area of the entire page: $A = xy = x\left(\frac{30}{x-2} + 4\right)$

$$A = \frac{30x}{x-2} + 4x$$

$$A' = \frac{(x-2)(30)-(30x)}{(x-2)^2} + 4$$

$$A' = \frac{-60}{(x-2)^2} + 4 = 0$$

$$\frac{4}{1} = \frac{60}{(x-2)^2}$$

$$\Rightarrow 4(x-2)^2 = 60$$

$$(x-2)^2 = 15$$

$$x = 2 - \sqrt{15} < 0$$

$$x = 2 + \sqrt{15} \approx 5.9 \ in.$$

$$y = \frac{30}{x-2} + 4 \approx \frac{30}{5.9-2} + 4 \approx 4.8 \ in.$$

Dimension: 5.9 in. x 4.8 in.

Exercise

Find the points of $y = 4 - x^2$ that are closet to (0, 3)

Solution

$$d = \sqrt{(x-0)^2 + (y-3)^2}$$

$$= \sqrt{x^2 + \left(4 - x^2 - 3\right)^2}$$

$$= \sqrt{x^2 + \left(1 - x^2\right)^2}$$

$$= \sqrt{x^2 + 1 - 2x^2 + x^4}$$

$$= \sqrt{x^4 - x^2 + 1}$$

$$f(x) = x^4 - x^2 + 1$$

$$f'(x) = 4x^3 - 2x = 2x(2x^2 - 1)$$

$$2x(2x^2 - 1) = 0 \rightarrow x = 0, \pm\sqrt{\frac{1}{2}}$$

$$\left(\sqrt{\frac{1}{2}}, \frac{7}{2}\right) \quad \left(-\sqrt{\frac{1}{2}}, \frac{7}{2}\right)$$

Exercise

A manufacturer wants to design an open box that has a square base and a surface area of 108 in^2. What dimensions will produce a box with a maximum volume?

Solution

$$V = x^2 h$$

Surface Area (S) = area of base + 4 (area of each side)

$$108 = x^2 + 4xh$$

$$108 - x^2 = 4xh$$

$$h = \frac{108 - x^2}{4x}$$

$$V = x^2 h$$

$$= x^2 \frac{108 - x^2}{4x}$$

$$= x \frac{108 - x^2}{4}$$

$$= \frac{108x}{4} - \frac{x^3}{4}$$

$$= 27x - \frac{x^3}{4}$$

$$27x - \frac{1}{4}x^3 = 0 \qquad \Rightarrow 108x - x^3 = 0$$

$$x(108 - x^2) = 0$$

$$x = 0, \ \sqrt{108}$$

$$V' = 27 - \frac{3}{4}x^2 = 0$$

$$-\frac{3}{4}x^2 = -27$$

$$\Rightarrow x^2 = 27\left(\frac{4}{3}\right) = 36$$

$$\rightarrow x = \pm 6 \ \Rightarrow x = 6 \ (only)$$

$$h = \frac{108 - x^2}{4x}$$

$$= \frac{108 - 6^2}{46}$$

$$\underline{= 3|}$$

Exercise

A company wants to manufacture cylinder aluminum can with a volume $1000cm^3$. What should the radius and height of the can be to minimize the amount of aluminum used?

Solution

$$V = \pi r^2 h = 1000 \quad \Rightarrow \quad h = \frac{1000}{\pi r^2}$$

Surface Area $S = 2\pi r^2 + 2\pi rh$

$$S = 2\pi r^2 + 2\pi r \frac{1000}{\pi r^2}$$

$$= 2\pi r^2 + \frac{2000}{r}$$

$$S' = 4\pi r - \frac{2000}{r^2} = 0$$

$$4\pi r = \frac{2000}{r^2}$$

$$r^3 = \frac{2000}{4\pi}$$

$$r = \left(\frac{2000}{4\pi}\right)^{1/3} \approx 5.419$$

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi \ 5.149^2} \approx \underline{10.84|}$$

Exercise

What is the smallest perimeter possible for a rectangle whose area is 16 in^2, and what are its dimensions?

Solution

Area: $A = \ell \cdot w = 16$

$$w = 16 \cdot \ell^{-1}$$

Perimeter: $P = 2\ell + 2w$

$$P = 2\ell + 2\left(16 \cdot \ell^{-1}\right)$$

$$P = 2\ell + 32 \cdot \ell^{-1}$$

$$P' = 2 - 32 \cdot \ell^{-2} = 0$$

$$2 = 32 \cdot \ell^{-2}$$

$$2 \cdot \frac{\ell^2}{2} = 32 \cdot \ell^{-2} \cdot \frac{\ell^2}{2}$$

$$\ell^2 = 16 \quad \Rightarrow \quad \ell = \pm 4$$

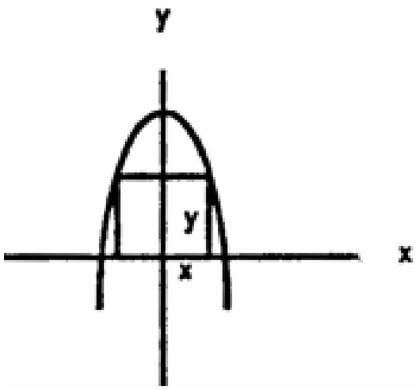
Since $\ell > 0 \quad \Rightarrow \boxed{\ell = 4}$

$$w = 16 \cdot 4^{-1} = 4$$

Exercise

A rectangle has its base on the x-axis and its upper vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

Solution



Area of the rectangle: $A = 2xy = 2x\left(12 - x^2\right)$

$$A = 24x - 2x^3$$

$$A = 24 - 6x^2 = 0$$

$$6x^2 = 24 \quad \rightarrow x^2 = 4 \Rightarrow x = \pm 2 \quad \boxed{x = 2}$$

$$y = 12 - 2^2 = 8$$

$$A = 2(2)(8) = 32$$

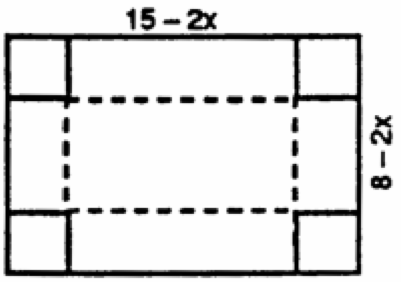
The largest area is 32 square units, and the dimensions are 4 units by 8 units.

Exercise

You are planning to make an open rectangular box from an 8-in. by 15-in. piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way, and what is its volume?

Solution

Volume of the box is: $V(x)=x(15-2x)(8-2x)$ $\quad 2x<8 \quad \Rightarrow \quad 0<x<4$



$$V(x)=x\left(120-46x+4x^2\right)$$

$$=120x-46x^2+4x^3$$

$$V'(x)=120-92x+12x^2=0$$

$$\boxed{x=\frac{5}{3}} \qquad \cancel{x=6} \quad (not\ in\ domain)$$

$$V\left(\frac{5}{3}\right)=120\left(\frac{5}{3}\right)-46\left(\frac{5}{3}\right)^2+4\left(\frac{5}{3}\right)^3 \approx 91\ in^3$$

$$\begin{cases} 15-2\left(\frac{5}{3}\right)=\frac{35}{3} \\ 8-2\left(\frac{5}{3}\right)=\frac{14}{3} \end{cases}$$

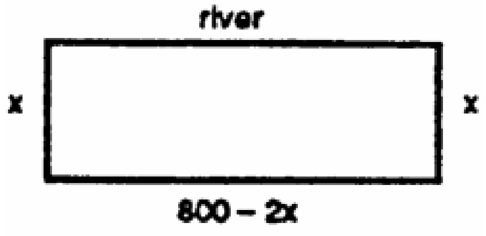
Box dimensions: $\frac{35}{3}\times\frac{14}{3}\times\frac{5}{3}$ *inches*

Exercise

A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 *m* of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?

Solution

Area: $A(x)=x(800-2x) \quad 0\le x\le 400$



$$A(x)=800x-2x^2$$

$$A'(x)=800-4x=0 \quad \Rightarrow \quad \boxed{x=200}$$

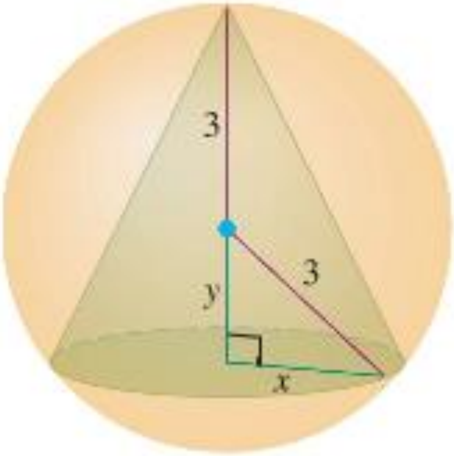
$$A(200)=800(200)-2(200)^2=80,000\ m^2$$

Dimensions: 200 *m* by **(800-2(200))** = 400 *m*.

Exercise

Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.

Solution



Volume of the cone is: $V = \frac{1}{3}\pi r^2 h$

where $r = x = \sqrt{9-y^2} \quad h = y+3$

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\sqrt{9-y^2}\right)^2 (y+3)$$

$$= \frac{1}{3}\pi\left(9-y^2\right)(y+3)$$

$$= \frac{1}{3}\pi\left(-y^3-3y^2+9y+27\right)$$

$$V'(y) = \frac{1}{3}\pi\left(-3y^2-6y+9\right)$$

$$V'(y) = \pi\left(-y^2-2y+3\right) = 0$$

Critical points are: $\boxed{y=1}$ $\quad \cancel{y=-3} < 0$

$V''(y) = \pi(-2y-2)\Big|_{y=1} < 0$ ***Using the 2nd derivative test***

$$V(y=1) = \frac{1}{3}\pi\left(-1^3-3(1)^2+9(1)+27\right) = \boxed{\frac{32\pi}{3} \ \text{cubic units}}$$

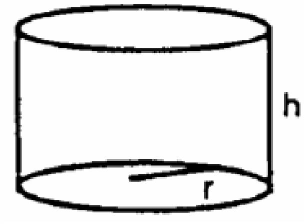
Exercise

What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of $1000\ cm^3$?

Solution

$$V = \pi r^2 h = 1000 \quad \Rightarrow \quad h = \frac{1000}{\pi \cdot r^2}$$

The amount of the material is the surface area by the sides and bottom of the can.



$$S = 2\pi rh + \pi r^2$$

$$= 2\pi r \frac{1000}{\pi r^2} + \pi r^2$$

$$= \frac{2000}{r} + \pi r^2, \quad r > 0$$

$$\frac{dS}{dr} = -\frac{2000}{r^2} + 2\pi r = \frac{-2000+2\pi r^3}{r^2} = 0$$

$$-2000 + 2\pi r^3 = 0$$

$$2\pi r^3 = 2000$$

$$r^3 = \frac{1000}{\pi}$$

$$\Rightarrow \boxed{r = \frac{10}{\sqrt[3]{\pi}}}$$

$S'' = \frac{4000}{r^3} + 2\pi > 0$, the surface has a minimum area at $r = \frac{10}{\sqrt[3]{\pi}}$

$$h = \frac{1000}{\pi \cdot r^2}$$

$$= \frac{1000}{\pi \cdot \left(\frac{10}{\sqrt[3]{\pi}}\right)^2}$$

$$= \frac{1000}{100\pi \cdot \pi^{-2/3}}$$

$$\boxed{= \frac{10}{\pi^{1/3}}}$$

Exercise

A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.

Solution

Perimeter = Perimeter of the rectangle + Perimeter of the semicircle

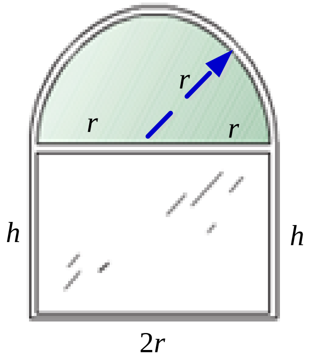
$$P = 2r + 2h + \pi r \quad \rightarrow \quad h = \tfrac{1}{2}(P - 2r - \pi r)$$

$$A = 2rh + \tfrac{1}{4}\pi r^2$$

$$= 2r\tfrac{1}{2}(P - 2r - \pi r) + \tfrac{1}{4}\pi r^2$$

$$= rP - 2r^2 - \pi r^2 + \tfrac{1}{4}\pi r^2$$

$$= rP - 2r^2 - \tfrac{3}{4}\pi r^2$$



$$\frac{dA}{dr} = P - 4r - \tfrac{3}{2}\pi r = 0$$

$$\left(4 + \tfrac{3}{2}\pi\right)r = P \quad \rightarrow r = \frac{2P}{8 + 3\pi}$$

$$h = \tfrac{1}{2}\left(P - 2\frac{2P}{8 + 3\pi} - \pi\frac{2P}{8 + 3\pi}\right)$$

$$=\frac{1}{2}\left(\frac{8P+3\pi P-4P-2\pi P}{8+3\pi}\right)$$

$$=\frac{P}{2}\left(\frac{4+\pi}{8+3\pi}\right)$$

The proportions that admit the most light:

$$\frac{2r}{h}=2\cdot\frac{2P}{8+3\pi}\bullet\frac{2}{P}\left(\frac{8+3\pi}{4+\pi}\right)$$

$$\boxed{=\frac{8}{4+\pi}}$$

Exercise

The cost per hour for fuel to run a train is $\frac{v^2}{4}$ dollars, where *v* is the speed of the train in miles per hour. (Note that the cost goes up as the square of the speed.) Other costs, including labor are $300 per hour. How fast should the train travel on a 360-mile trip to minimize the total cost for the trip?

Solution

x = number of hours it take the train to travel 360 miles.

Then $360=xv \quad\rightarrow\quad x=\frac{360}{v}$

Cost: $C=\left(300+\frac{v^2}{4}\right)\left(\frac{360}{v}\right)$

$$C(v)=\frac{108,000}{v}+90v$$

$$C'(v)=-\frac{108,000}{v^2}+90=0$$

$$\frac{108,000}{v^2}=90\quad\rightarrow\quad v^2=\frac{108,000}{90}=1,200$$

$$\boxed{v=\sqrt{1200}\approx 34.64}$$

$$C''(v)=\frac{216,000}{v^3}>0$$

The cost has an absolute minimum at $x=34.64$

$$C(34.64)=\frac{108,000}{34.64}+90(34.64)\ \boxed{=6,235.38}$$

Exercise

A piece of cardboard measures 10-in. by 15-in. Two equal squares are removed from the corners of 10-in. side. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.

a) Write a formula $V(x)$ for the volume of the box

b) Find the domain of V for the problem situation and graph V over this domain

c) Use the graphical or analytically method to find the maximum volume and the value of x that gives it.

Solution

a) $V(x) = x(10-2x)\left(\frac{15-2x}{2}\right)$

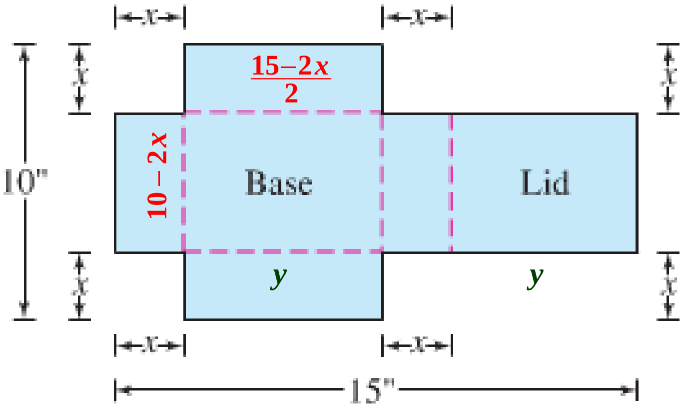
$$= \left(5x - x^2\right)(15-2x)$$

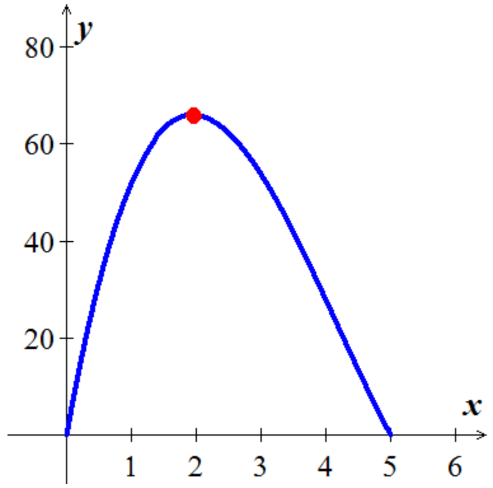
$$= 2x^3 - 25x^2 + 75x$$

b) $x > 0, \quad 2x < 10, \quad 2x < 15$

$$x < 5, \quad x < \frac{15}{2}$$

Domain: $(0,\ 5)$



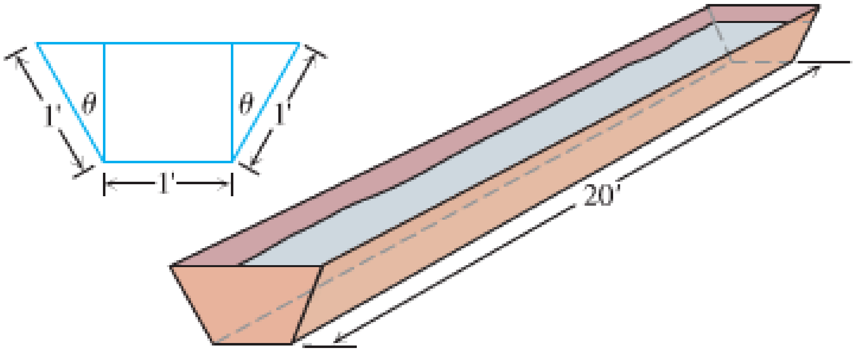


c) $V' = 6x^2 - 50x + 75 \underline{=0} \quad \Rightarrow \quad \underline{x \approx 1.96}, \ \xcancel{6.37}$

$$V(1.96) = 2(1.96)^3 - 25(1.96)^2 + 75(1.96) = \underline{66.02\ in^3}$$

Exercise

The trough in the figure is to be made to the dimensions shown. Only the angle θ can be varied. What value of θ will maximize the trough's volume?



Solution

The area of the cross section:

$$A(\theta) = \cos\theta + \sin\theta\cos\theta \qquad 0 < \theta < \frac{\pi}{2}$$

$$A'(\theta) = -\sin\theta + \cos^2\theta - \sin^2\theta$$

$$= -\sin\theta + 1 - \sin^2\theta - \sin^2\theta$$

$$= -2\sin^2\theta - \sin\theta + 1 = 0$$

$$\rightarrow \begin{cases} \sin\theta = \frac{1}{2} & \Rightarrow \theta = \frac{\pi}{6} \\ \xcancel{\sin\theta = -1} & 0<\theta<\frac{\pi}{2} \end{cases}$$

0	$\frac{\pi}{6}$	$\frac{\pi}{2}$
$A'(\theta) > 0$		$A'(\theta) < 0$
Increasing		***Decreasing***

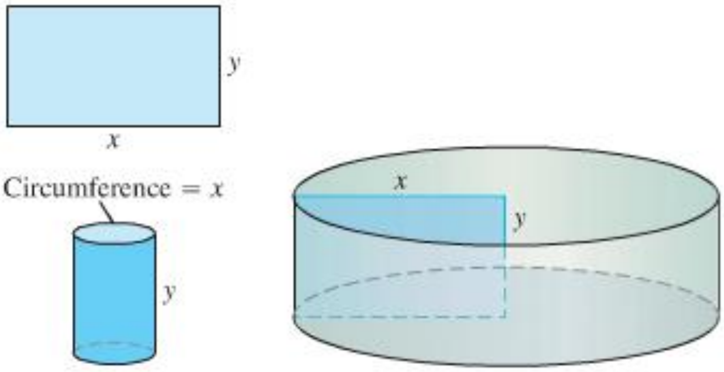
Therefore, there is a maximum value at $\boxed{\theta = \frac{\pi}{6}}$

Exercise

Compare the answers to the following two construction problems.

a) A rectangular sheet of perimeter 36 *cm* and dimensions *x cm* and *y cm* is to be rolled into a cylinder as shown in part (*a*) of the figure. What values of *x* and *y* give the largest volume?

b) The same sheet is to be revolved about one of the sides of length *y* to sweep out the cylinder as shown in part (*b*) of the figure. What values of *x* and *y* give the largest volume?



(a) (b)

Solution

a) $P = 2x + 2y \quad \Rightarrow \quad y = \frac{P}{2} - x$

$P = 36 \quad \Rightarrow \quad y = 18 - x$

When the cylinder is formed: $x = 2\pi r$ and $h = y = 18 - x$

Volume of the cylinder: $V = \pi r^2 h$

$$V(x) = \pi\left(\frac{x}{2\pi}\right)^2 (18 - x)$$

$$= \pi \frac{x^2}{4\pi^2}(18 - x)$$

$$= \frac{1}{4\pi}\left(18x^2 - x^3\right)$$

$$V'(x) = \frac{1}{4\pi}\left(36x - 3x^2\right) = 0 \qquad 3x(12 - x) = 0 \to \begin{cases} \cancel{x = 0} & \textit{no cylinder} \\ x = 12 \end{cases}$$

$$V''(x) = \frac{3}{4\pi}(12 - 2x) \quad \to \quad V''(12) < 0$$

There is a maximum at $x = 12$ cm, and $y = 18 - 12 = 6$ cm.

b) $V(x) = \pi x^2 (18 - x)$

$$V'(x) = 2\pi x(18 - x) - \pi x^2 = 36\pi x - 2\pi x^2 - \pi x^2 = 36\pi x - 3\pi x^2 = 0$$

$$3\pi x(12 - x) = 0 \to \begin{cases} \cancel{x = 0} & \textit{no cylinder} \\ x = 12 \end{cases}$$

$$V''(x) = 36\pi - 6\pi x \quad \to \quad V''(12) < 0$$

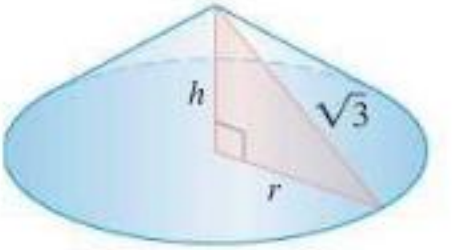
There is a ***maximum*** at $x = 12$ cm, and $y = 18 - 12 = 6$ cm.

Exercise

A right triangle whose hypotenuse is $\sqrt{3}\ m$ long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.

Solution

From the figure, the right triangle:



$$h^2 + r^2 = 3$$

Volume: $V = \frac{\pi}{3} r^2 h$

$$= \frac{\pi}{3}\left(3 - h^2\right)h \qquad\qquad 0 < h < \sqrt{3}$$

$$= \pi h - \frac{\pi}{3} h^3$$

$$\frac{dV}{dh} = \pi - \pi h^2$$

$$= \pi\left(1 - h^2\right) = 0$$

$$h^2 = 1 \quad \Rightarrow \quad \boxed{h = 1} \quad (\boldsymbol{CP})$$

0	1	$\sqrt{3}$
$V'(.5) > 0$		$V'(1.1) < 0$
Increasing		***Decreasing***

The volume has a maximum at the critical point.

$$\boxed{r = \sqrt{3 - h^2} = \sqrt{2}}$$

$$\boxed{V = \frac{\pi}{3} r^2 h = \frac{\pi}{3}\left(\sqrt{2}\right)^2 (1) = \frac{2\pi}{3}}$$

Therefore, the cone, with a maximum volume, has a radius $\sqrt{2}$ m, height 1 m, and volume $V = \frac{2\pi}{3}\ m^3$

Exercise

The height above the ground of an object moving vertically is given by

$$s(t) = -16t^2 + 96t + 112$$

With *s* in feet and *t* in seconds. Find

a) The object’s velocity when $t = 0$

b) Its maximum height and when it occurs

c) Its velocity when $s = 0$

Solution

a) $v(t) = s'(t) = -32t + 96$

$$v(t=0) = \boxed{96\ ft / \sec}$$

b) The maximum height occurs when $v(t) = -32t + 96 = 0$

$$\boxed{t = \frac{96}{32} = 3\ \sec}$$

$$s(t=3) = -16(3)^2 + 96(3) + 112$$

$$= \boxed{256\ ft}$$

c) $s = -16t^2 + 96t + 112 = 0 \quad \Rightarrow \quad \begin{cases} t = -1 \\ t = 7 \end{cases}$

$$v(t=7) = -32(7) + 96$$

$$= \boxed{-128\ ft / \sec}$$

Exercise

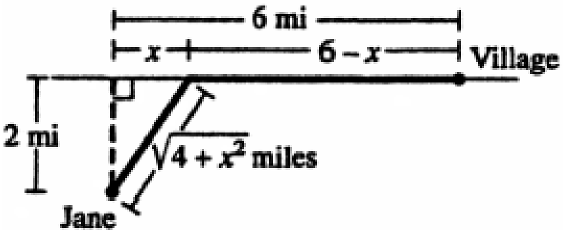
Jane is 2 mi offshore in a boat and wishes to reach a coastal village 6 mi down a straight shoreline from the point nearest the boat. She can row 2 mph and can walk 5 mph. Where should she land her boat to reach the village in the least amount of time?

Solution

x: distance from shoreline to boat.

$$row\ \sqrt{4+x^2}\ @\ 2\ mph \quad \to t_1 = \frac{d}{v} = \frac{\sqrt{4+x^2}}{2}$$

$$walk:\ 6-x\ @\ 5\ mph \quad \to t_2 = \frac{d}{v} = \frac{6-x}{5}$$



The total amount of time to reach the village:

$$f(x) = \frac{\sqrt{4+x^2}}{2} + \frac{6-x}{5}$$

$$f'(x) = \frac{1}{2}\frac{2x}{2\sqrt{4+x^2}} + \frac{-1}{5}$$

$$= \frac{x}{2\sqrt{4+x^2}} - \frac{1}{5} = 0$$

$$\frac{x}{2\sqrt{4+x^2}} = \frac{1}{5}$$

$$5x = 2\sqrt{4+x^2}$$

$$(5x)^2 = \left(2\sqrt{4+x^2}\right)^2$$

$$25x^2 = 4\left(4+x^2\right)$$

$$25x^2 = 16 + 4x^2$$

$$21x^2 = 16$$

$$x^2 = \frac{16}{21} \quad \to \quad x = \frac{4}{\sqrt{21}}$$

Distance can't be negative

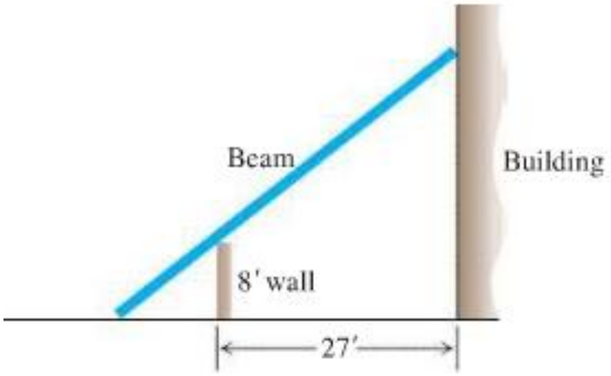
$$\to \begin{cases} x = 0 & \to f = 2.2 \\ x = \frac{4}{\sqrt{21}} & \to f \approx 2.12 \\ x = 6 & \to f \approx 3.16 \end{cases}$$

Jane should land her boat $\frac{4}{\sqrt{21}} \approx .87$ miles down the shoreline from the point nearest her boat.

Exercise

The 8-ft wall stands 27 ft from the building. Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.

Solution



$$\frac{h}{8}=\frac{x+27}{x}$$

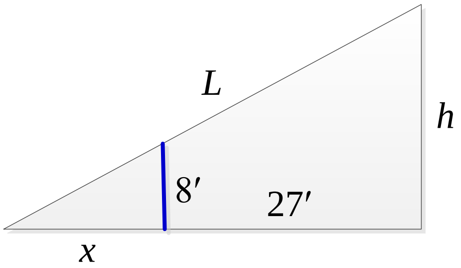
$$\frac{h}{8}=1+\frac{27}{x}$$

$$h=8+\frac{216}{x}$$

Using Pythagorean Theorem:

$$L^2=h^2+(x+27)^2$$

$$L(x)=\sqrt{\left(8+\frac{216}{x}\right)^2+(x+27)^2}$$



If we let:

$$f(x)=\left(8+\frac{216}{x}\right)^2+(x+27)^2$$

$$f'(x)=2\left(8+\frac{216}{x}\right)\left(-\frac{216}{x^2}\right)+2(x+27)(1)$$

$$=2(8)\left(1+\frac{27}{x}\right)\left(-\frac{216}{x^2}\right)+2(x+27)$$

$$=2(x+27)\left(-\frac{1728}{x^3}+1\right)$$

$$=2(x+27)\left(1-\frac{1728}{x^3}\right)=0$$

$$\rightarrow\begin{cases}x+27=0 & \rightarrow x=-27\\ 1-\frac{1728}{x^3}=0 & \rightarrow x^3=1728\end{cases}\Rightarrow \boxed{x=\sqrt[3]{1728}=12}$$

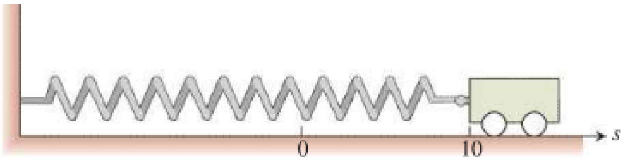
$$\boxed{L(12)}=\sqrt{\left(8+\frac{216}{12}\right)^2+(12+27)^2}\ \boxed{\approx 46.87\ ft}$$

Exercise

A small frictionless cart, attached to the wall by a spring, is pulled 10 cm from its rest position and released at time $t = 0$ to roll back and forth for 4 sec. Its position at time t is $s = 10\cos \pi t$

a) What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?

b) Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?



Solution

a) $s = 10\cos \pi t$

$v = s' = -10\pi \sin \pi t \quad \Rightarrow \quad speed = |v| = 10\pi |\sin \pi t|$

The car moving fast when $|\sin \pi t| = 1$

The maximum speed is: $= 10\pi \approx 31.42 \ cm/\sec$

The cart is moving fastest when $|\sin \pi t| = 1 \quad \Rightarrow \quad \pi t = \frac{\pi}{2} + k\pi$

$t = \frac{1}{2} + k; \quad 0 \le t \le 4$

$a = -10\pi^2 \cos \pi t$

$$\rightarrow \begin{cases} t = 0.5 & \rightarrow s = 10\cos \frac{\pi}{2} = 0 \\ t = 1.5 & \rightarrow s = 10\cos \frac{3\pi}{2} = 0 \\ t = 2.5 & \rightarrow s = 10\cos \frac{5\pi}{2} = 0 \\ t = 3.5 & \rightarrow s = 10\cos \frac{7\pi}{2} = 0 \end{cases} \quad \Rightarrow \quad \boxed{|a| = 0 \ \ cm/sec^2}$$

b) $|a| = 10\pi^2 |\cos \pi t|$

The magnitude of the acceleration is greatest when

$|\cos \pi t| = 1 \quad at \quad \boxed{t = 0.0, \ 1.0, \ 2.0, \ 3.0, \ 4.0 \ \sec}$

The position of the cart at these times is $|s| = 10 \ cm$

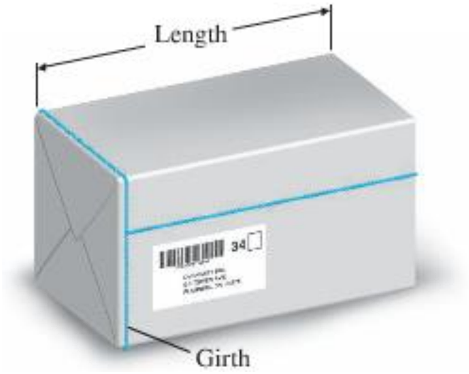
The speed is 0 *cm/sec*.

Exercise

A parcel delivery service will deliver a package only if the length plus girth (distance around) does not exceed 108 inches.

a) Find the dimensions of a rectangular box with square ends that satisfies the delivery service's restriction and has maximum volume. What is the maximum volume?

b) Find the dimensions (radius and height) of a cylinder container that meets the delivery service's requirement and has maximum volume. What is the maximum volume?



Solution

a) Let x: length of the side of the square.

$$L + 4x = 108 \quad \to \quad L = 108 - 4x$$

The volume of the box:

$$V = L \cdot x^2 = (108 - 4x) \cdot x^2$$

$$= 108x^2 - 4x^3$$

$$V' = 216x - 12x^2 = 0$$

$$x = 0 \qquad x = \frac{216}{12} \underline{= 18}$$

$$V(18) = 108(18)^2 - 4(18)^3 \ \underline{= 11,664}$$

$$L = 108 - 4(18) \ \underline{= 36}$$

Therefore, the volume is maximum at 11,664 in^3 for an 18″ by 18″ by 36″ container.

b) Let x: radius and L: height of the cylindrical container.

$$L + 2\pi x = 108 \quad \to \quad L = 108 - 2\pi x$$

$$V = \pi x^2 L$$

$$= \pi x^2 (108 - 2\pi x)$$

$$= 108\pi x^2 - 2\pi^2 x^3$$

$$V' = 216\pi x - 6\pi^2 x^2$$

$$= 6\pi x(36 - \pi x)$$

$$x = 0 \qquad \boxed{x = \frac{36}{\pi}}$$

$$V\left(\frac{36}{\pi}\right) = 108\pi\left(\frac{36}{\pi}\right)^2 - 2\pi^2\left(\frac{36}{\pi}\right)^3 \ \underline{= 14,851 \ in^3}$$

$$L = 108 - 2\pi\left(\frac{36}{\pi}\right) \underline{= 36}$$

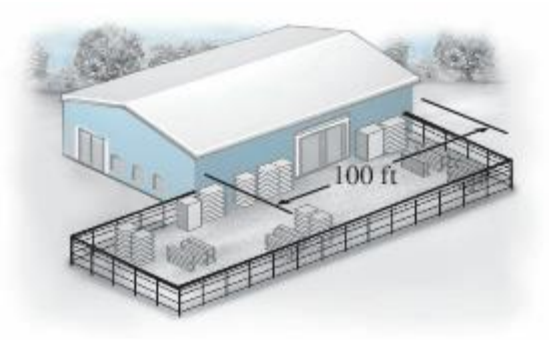
Therefore, the volume is maximum at 14,851 in^3 for a container of radius $\frac{36}{\pi}$ and height of 36″.

Exercise

The owner of a retail lumber store wants to construct a fence an outdoor storage are adjacent to the store, using all of the store as part of one side of the area. Find the dimensions that will enclose the largest area if

a) 240 feet fencing material are used.

b) 400 feet fencing material are used.



Solution

a) Let x and y be the width and the length of the rectangle respectively.

$$2x+2y-100=240$$

$$2x+2y=340$$

$$x+y=170 \quad \to \quad y=170-x$$

The area: $A=xy$

$$=x(170-x)$$

$$=170x-x^2 \qquad 100\le x\le 170$$

$$A'=170-2x=0$$

$\underline{x=\frac{170}{2}=85}$ which is not in the domain.

$$A(100)=170(100)-(100)^2=7,000$$

$$A(170)=170(170)-(170)^2=0$$

Thus, the maximum occurs when $x=100$ and $y=170-100=\underline{70}$

b) $2x+2y-100=400$

$$2x+2y=500$$

$$x+y=250 \quad \to \quad y=250-x$$

$$A=xy=x(250-x)$$

$$=250x-x^2 \qquad 100\le x\le 250$$

$$A'=250-2x=0 \quad \to \quad \boxed{x=125}$$

$$A(125)=250(125)-(125)^2=15,625$$

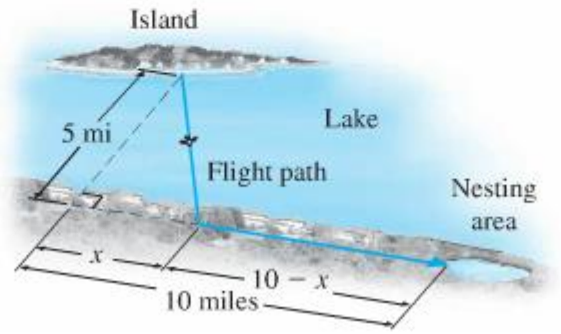
Thus, the maximum occurs when $x=125\ ft$ and $y=250-125=\underline{125\ \ ft}$

Exercise

Some birds tend to avoid flights over large bodies of water during daylight hours. Suppose that an adult bird with this tendency is taken from its nesting area on the edge of a large lake to an island 5 miles offshore and is then release.

a) If it takes only 1.4 times as much energy to fly over water as land, how far up the shore (x, in miles) should the bird head to minimize the total energy expended in returning to the nesting area?

b) If it takes only 1.1 times as much energy to fly over water as land, how far up the shore should the bird head to minimize the total energy expended in returning to the nesting area?



Solution

a) Let the energy to fly over land be 1 unit; then the energy to fly over the water is 1.4 units.

$$(\,flight\ path)^2 = x^2 + 5^2$$

Total Energy:

$$E(x) = (1.4)\sqrt{x^2+25} + (1)(10-x)$$

$$= 1.4\sqrt{x^2+25} + 10 - x$$

$$E'(x) = \frac{1.4(2x)}{2\sqrt{x^2+25}} - 1$$

$$= \frac{1.4x}{\sqrt{x^2+25}} - 1 = 0$$

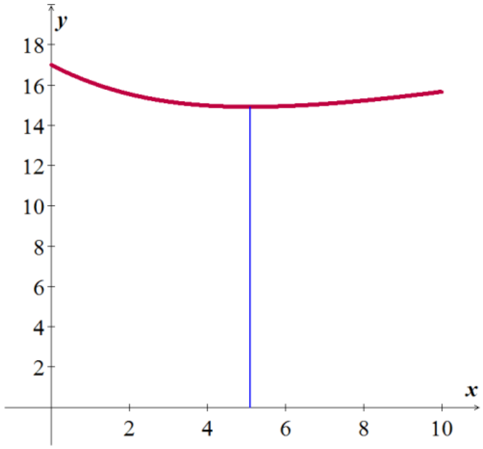
$$\frac{1.4x}{\sqrt{x^2+25}} = 1$$

$$1.4x = \sqrt{x^2+25} \qquad (1.4x)^2 = \left(\sqrt{x^2+25}\right)^2$$

$$1.96x^2 = x^2 + 25$$

$$0.96x^2 = 25$$

$$x^2 = 26.04 \quad \rightarrow \quad x = \pm 5.1$$



0	5.1
$E'(1) < 0$	$E'(6) > 0$
Decreasing	***Increasing***

Thus the critical value is $\underline{x = 5.1}$

Thus, the energy will be minimum when $x = 5.1$.

$$E(5.1) = 1.4\sqrt{(5.1)^2 + 25} + 10 - 5.1 = \underline{14.9}$$

$$E(10) = 1.4\sqrt{(10)^2 + 25} + 10 - 10 = 15.65$$

Thus, the absolute minimum occurs when $x = 5.1$ miles.

b) $E(x)=1.1\sqrt{x^2+25}+10-x \qquad 0\le x\le 10$

$$E'(x)=\frac{1.1x}{\sqrt{x^2+25}}-1=0$$

$$1.1x=\sqrt{x^2+25} \qquad (1.1x)^2=\left(\sqrt{x^2+25}\right)^2$$

$$1.21x^2=x^2+25$$

$$0.21x^2=25$$

$$x^2=119.05 \quad \rightarrow \quad x=\pm 10.91$$

The critical value $x=10.91>10$

$$E(0)=1.1\sqrt{(0)^2+25}+10-0=\underline{15.5}|$$

$$E(10)=1.1\sqrt{(10)^2+25}+10-10=\underline{12.30}|$$

Therefore, the absolute minimum occurs when $x=10$

Exercise

A pharmacy has a uniform annual demand for 200 bottles of a certain antibiotic. It costs \$10 to store one bottle for one year and \$40 to place an order. How many times during the year should the pharmacy order the antibiotic in order to minimize the total storage and reorder costs?

Solution

Let x be the number of times the pharmacy places order.
Let y be the number of demands.

The cost: $C=40x+10\left(\frac{y}{2}\right)=40x+5y$

We also have $xy=200 \quad \Rightarrow \quad y=\frac{200}{x}$

$$C(x)=40x+5\left(\frac{200}{x}\right)=40x+\frac{1{,}000}{x}$$

$$C'(x)=40-\frac{1{,}000}{x^2}=0$$

$$\frac{1{,}000}{x^2}=40 \quad \rightarrow \quad x^2=\frac{1{,}000}{40}=25 \quad \Rightarrow x=\pm 5 \quad \rightarrow \quad \boxed{x=5} \text{ since } x>0$$

$C''(x)=\frac{2{,}000}{x^3}>0$ the function has an absolute minimum at $x=5$

$C(5)=40(5)+\frac{1{,}000}{5}\ \underline{=\$400}|$ is the minimum cost.

Exercise

A 300-room hotel in Las Vegas is filled to capacity every night at \$80 a room. For each \$1 increase in rent, 3 fewer rooms are rented. If each rented room costs \$10 to service per day, how much should be the management charge for each room to maximize gross profit? What is the maximum gross profit?

Solution

Let x: number of dollar increases in the rate per night.
Total number of rooms rented: $300-3x$
Rate per night: $80+x$

Total income = (total number of rooms rented) (rate – 10)

$$y(x)=(300-3x)(80+x-10) \qquad 0\le x\le 100$$

$$=(300-3x)(70+x)$$

$$=21{,}000+90x-3x^2$$

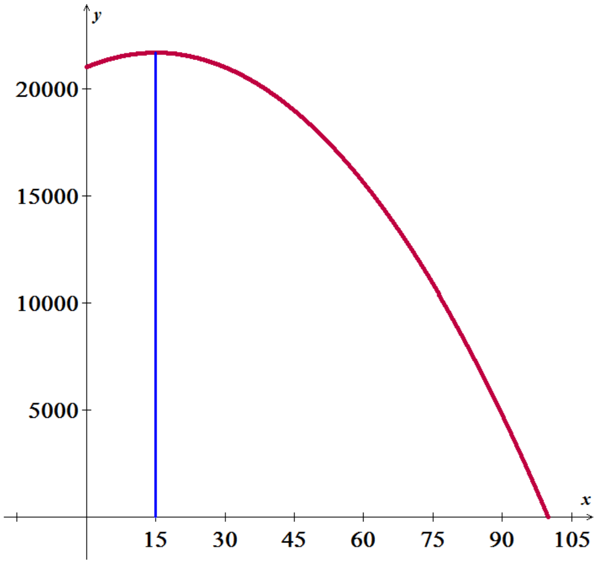
$$y'(x)=90-6x=0$$

$$6x=90 \quad\to\quad \boxed{x=15} \qquad (CN)$$

$$y''(x)=-6<0.$$

Maximum income: $y(15)=21{,}000+90(15)-3(15)^2 \ \underline{=\$21{,}675.00|}$

The rate per night: $80+15\ \underline{=\$95|}$



Exercise

A multimedia company anticipates that there will be a demand for 20,000 copies of a certain DVD during the next year. It costs the company \$0.50 to store a DVD for one year. Each time it must make additional DVDs, it costs \$200 to set up the equipment. How many DVDs should the company make during each production run to minimize its total storage and setup costs?

Solution

Let assume that there are 250 working days and then the daily demand is: $\frac{20,000}{250}=80\ \ DVDs$

Let: x: number of DVDs manufactured during each production run.
y: number of production runs.

The number of DVDs in storage between production runs will decrease from x to 0, and the average number in storage each day is $\frac{x}{2}$.

Since it costs \$0.50 to store a DVD for one year, the total storage cost is $(0.5)\frac{x}{2}=0.25x$.

The total cost is:

Total cost = setup cost + storage cost

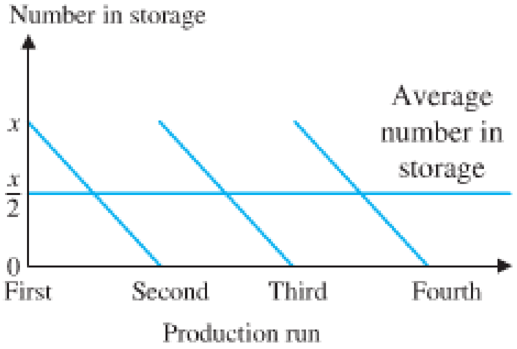
$$C=200y+0.25x$$

The total number of DVDs produced is xy. $xy=20,000$

$$y=\frac{20,000}{x}$$

$$C(x)=200\left(\frac{20,000}{x}\right)+0.25x \qquad 1\le x\le 20,000$$

$$=\frac{4,000,000}{x}+0.25x$$



$$C'(x)=-\frac{4,000,000}{x^2}+0.25=0$$

$$0.25=\frac{4,000,000}{x^2} \rightarrow x^2=\frac{4,000,000}{0.25}$$

$$\Rightarrow x=\sqrt{\frac{4,000,000}{0.25}}\ \underline{=4,000|}$$

0	4,000
$C'(1)<0$	$C'(10,000)>0$
Decreasing	***Increasing***

$$C(4,000)=\frac{4,000,000}{4,000}+0.25(4,000)$$

$$\underline{=\$2,000|}$$

$$y=\frac{20,000}{4,000}=\underline{5|}$$

The company will minimize its total cost by making 4,000 DVDs five times during the year.

Exercise

A university student center sells 1,600 cups of coffee per day at a price of \$2.40.

a) A market survey shows that for every \$0.05 reduction in price, 50 more cups of coffee will be sold. How much should be the student center charge for a cup of coffee in order to maximize revenue?

b) A different market survey shows that for every \$0.10 reduction in the original \$2.40 price, 60 more cups of coffee will be sold. Now how much should the student center charge for a cup of coffee in order to maximize revenue?

Solution

a) Let x: number of price reductions

The price of a cup of coffee will be: $p = 2.40 - 0.05x$

The number of cups sold will be: $1,600 + 50x$

Revenue: $R(x) = x \cdot p(x)$

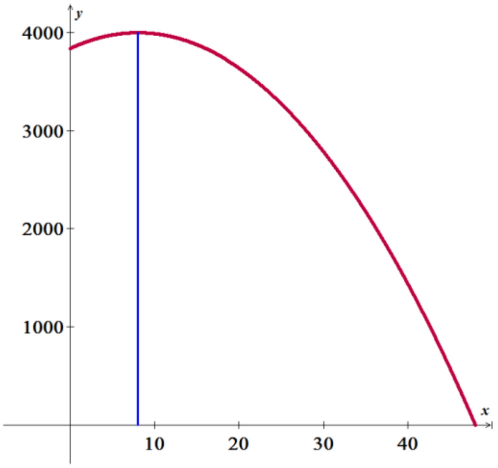
$$= (1,600 + 50x)(2.40 + 0.05x)$$

$$= 3,840 + 40x - 2.5x^2$$

$R'(x) = 40 - 5x = 0$

$$5x = 40 \quad \rightarrow \quad \boxed{x = 8} \quad (CN)$$

$R''(x) = -5 < 0$ that implies R has an absolute maximum.



Maximum revenue: $R(8) = 3,840 + 40(8) - 2.5(8)^2 \ \boxed{= \$4,000.00}$ when 1,600 cups of coffee sold at the price $p = 2.40 - 0.05(8) \ \boxed{= \$2.00}$ per cup.

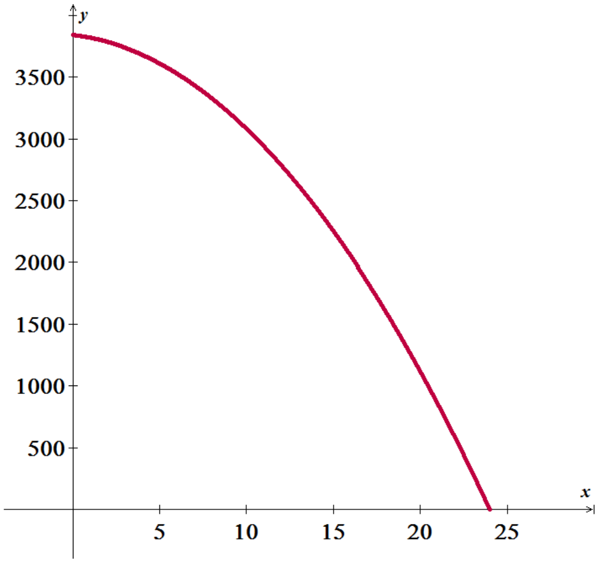
b) *Revenue*: $R(x) = (1,600 + 60x)(2.40 - .10x)$

$$= 3,840 - 16x - 6x^2$$

$R'(x) = -16 - 12x = 0$

$$12x = -16 \quad \rightarrow \quad \boxed{x = -\frac{4}{3} < 0} \quad (CN)$$

Thus, $R(x)$ is decreasing and its maximum occurs at $x = 0$ they should charge \$2.40 per cup.



Exercises *Section* 3.6 – Newton's Method

Exercise

Use Newton's method to estimate the on real solution of $x^3+3x+1=0$. Start with $x_0=0$ and then find x_2

Solution

$$y=x^3+3x+1 \quad\rightarrow\quad y'=3x^2+3$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{x_n^3+3x_n+1}{3x_n^2+3}$$

$$x_0=0$$

$$x_1=x_0-\frac{x_0^3+3x_0+1}{3x_0^2+3}=0-\frac{0+3(0)+1}{3(0)+3}=-\frac{1}{3}$$

$$x_2=x_1-\frac{x_1^3+3x_1+1}{3x_1^2+3}=-\frac{1}{3}-\frac{\left(-\frac{1}{3}\right)^3+3\left(-\frac{1}{3}\right)+1}{3\left(-\frac{1}{3}\right)+3}=-0.3222$$

Exercise

Use Newton's method to estimate the on real solution of $x^4+x-3=0$. Start with $x_0=-1$ for the left-hand zero and with $x_0=1$ for the zero on the right. Then, in each case, find x_2

Solution

$$y=x^4+x-3 \quad\rightarrow\quad y'=4x^3+1$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{x_n^4+x_n-3}{4x_n^3+1}$$

$$\boxed{x_0=-1}$$

$$x_1=x_0-\frac{x_0^4+x_0-3}{4x_0^3+1}=-1-\frac{(-1)^4+(-1)-3}{4(-1)^3+1}=-2$$

$$x_2=x_1-\frac{x_1^4+x_1-3}{4x_1^3+1}=-2-\frac{(-2)^4+(-2)-3}{4(-2)^3+1}=-1.64516$$

$$\boxed{x_0 = 1}$$

$$x_1 = x_0 - \frac{x_0^4 + x_0 - 3}{4x_0^3 + 1} = 1 - \frac{(1)^4 + (1) - 3}{4(1)^3 + 1} = \frac{6}{5}$$

$$x_2 = x_1 - \frac{x_1^4 + x_1 - 3}{4x_1^3 + 1} = \frac{6}{5} - \frac{\left(\frac{6}{5}\right)^4 + \left(\frac{6}{5}\right) - 3}{4\left(\frac{6}{5}\right)^3 + 1} = 1.16542$$

Exercise

Use Newton's method to estimate the on real solution of $2x - x^2 + 1 = 0$. Start with $x_0 = 0$ for the left-hand zero and with $x_0 = 2$ for the zero on the right. Then, in each case, find x_2

Solution

$$y = 2x - x^2 + 1 \quad \to \quad y' = 2 - 2x$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2x_n - x_n^2 + 1}{2 - 2x_n}$$

$$\boxed{x_0 = 0}$$

$$x_1 = x_0 - \frac{2x_0 - x_0^2 + 1}{2 - 2x_0} = 0 - \frac{2(0) - (0)^2 + 1}{2 - 2(0)} = -0.5$$

$$x_2 = x_1 - \frac{2x_1 - x_1^2 + 1}{2 - 2x_1} = -0.5 - \frac{2(-0.5) - (-0.5)^2 + 1}{2 - 2(-0.5)} = -0.41667$$

$$\boxed{x_0 = 2}$$

$$x_1 = x_0 - \frac{2x_0 - x_0^2 + 1}{2 - 2x_0} = 2 - \frac{2(2) - (2)^2 + 1}{2 - 2(2)} = 2.5$$

$$x_2 = x_1 - \frac{2x_1 - x_1^2 + 1}{2 - 2x_1} = 2.5 - \frac{2(2.5) - (2.5)^2 + 1}{2 - 2(2.5)} = 2.41667$$

Exercise

Use Newton's method to estimate the on real solution of $x^4 - 2 = 0$. Start with $x_0 = 1$ and then find x_2

Solution

$$y = x^4 - 2 \quad \to \quad y' = 4x^3$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 2}{4x_n^3}$$

$$\boxed{x_0 = 1}$$

$$x_1 = x_0 - \frac{x_0^4 - 2}{4x_0^3} = 1 - \frac{(1)^4 - 2}{4(1)^3} = 1.25$$

$$x_2 = x_1 - \frac{x_1^4 - 2}{4x_1^3} = 1.25 - \frac{(1.25)^4 - 2}{4(1.25)^3} \approx 1.1935$$