Lecture Four

Section 4.1 – First-Order Systems

Consider a system of differential equations that can be solved for the highest-order derivatives of the dependent variables.

For instance, in the case of a system of two 2nd-order equations can be written in the form

$$\begin{cases} x_1' = f_1\left(t,\ x_1,\ x_2,\ x_1',\ x_2'\right) \\ x_2' = f_2\left(t,\ x_1,\ x_2,\ x_1',\ x_2'\right) \end{cases}$$

Any higher-order system can be transformed into an equivalent system of 1st-order equations. Consider a system consisting of the single nth-order equation.

$$x^{(n)} = f_2\left(t,\ x,\ x',\ \ldots,\ x^{(n-1)}\right)$$

We introduce the dependent variables $x_1,\ x_2,\ \ldots,\ x_n$ defined as follows:

$$x_1 = x, \quad x_2 = x', \quad x_3 = x'', \ \ldots \quad x_n = x^{(n-1)}$$

Note that $\quad x_1' = x', \quad x_2' = x'' = x_3, \quad \ldots$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \quad\vdots \\ x_{n-1}' = x_n \end{cases}$$

$$x_n' = f_2\left(t,\ x_1,\ x_2,\ \ldots,\ x_n\right)$$

Example

The 3rd-order equation $x''' + 3x'' + 2x' - 5x = \sin 3t$ can be written in the form

$$x''' = f\left(t,\ x,\ x',\ x''\right) = 5x - 2x' - 3x'' + \sin 3t$$

Let $\quad x_1 = x, \quad x_2 = x' = x_1', \quad x_3 = x'' = x_2'$

Yield the system

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = 5x_1 - 2x_2 - 3x_3 + \sin 3t \end{cases}$$

Example

Transform this system into an equivalent 1st-order system

$$\begin{cases} x'' = -3x + y \\ y'' = 2x - 2y + 20\sin 2t \end{cases}$$

Solution

Let $\begin{array}{ll} x_1 = x & x_2 = x' = x_1' \\ y_1 = y & y_2 = y' = y_1' \end{array}$

$$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = -3x_1 + y_1 \end{cases} \qquad \begin{cases} y_1' = y_2 \\ y_2' = 2x_1 - 2y_1 + 20\sin 2t \end{cases}$$

Of 4 1st-order equations in the dependent variables x_1, x_2, y_1, y_2

Simple 2–Dimensional Systems

The linear 2nd-order differential equation $x'' + px' + qx = 0$

Let $x' = y \quad \Rightarrow \quad x'' = y'$

$$\begin{cases} x' = y \\ y' = -qx - py \end{cases}$$

Example

Solve the 2-dimensional system

$$\begin{cases} x' = -2y \\ y' = \frac{1}{2}x \end{cases}$$

Then solve using the initial values $x(0) = 2, \quad y(0) = 0$

Solution

$$x'' = -2y' = -2\left(\frac{1}{2}x\right) = -x$$

$$x'' + x = 0 \quad \Rightarrow \quad \lambda^2 + 1 = 0 \quad \to \lambda_{1,2} = \pm i$$

$\therefore$ Have a general solution: $x(t) = A\cos t + B\sin t$

$$\begin{aligned} y(t) &= -\frac{1}{2}x'(t) \\ &= -\frac{1}{2}(-A\sin t + B\cos t) \end{aligned}$$

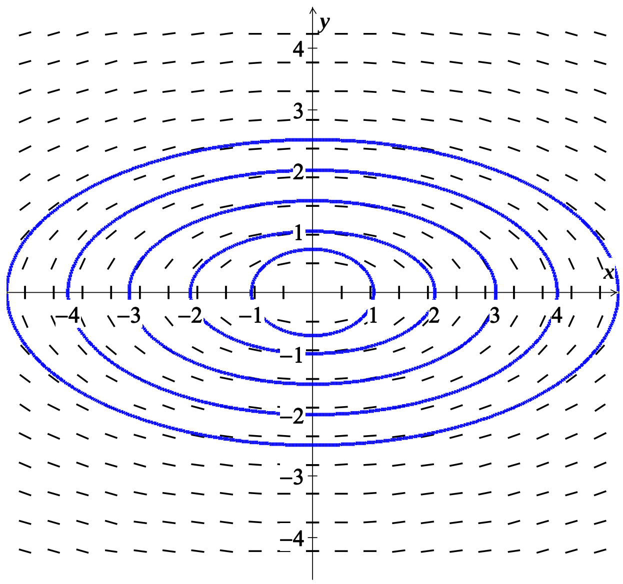
Let $A = C\cos\alpha \quad and \quad B = C\sin\alpha$

$$\begin{cases} x(t) = C\cos\alpha\cos t + C\sin\alpha\sin t = C\cos(t-\alpha) \\ y(t) = \frac{1}{2}(C\cos\alpha\sin t - C\sin\alpha\cos t) = \frac{1}{2}C\sin(t-\alpha) \end{cases}$$

$$\begin{cases} \cos(t-\alpha) = \dfrac{x(t)}{C} \\ \sin(t-\alpha) = \dfrac{2}{C}y(t) \end{cases}$$

$$\cos^2(t-\alpha) + \sin^2(t-\alpha) = 1$$

$$\frac{x^2}{C^2} + \frac{y^2}{(C/2)^2} = 1 \quad \therefore \quad \textit{Ellipse}$$

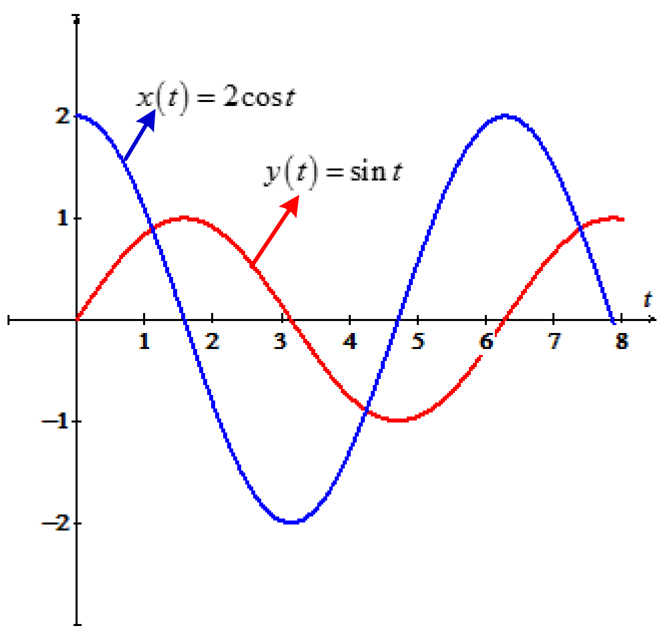


$$x(0) = 2, \quad y(0) = 0$$

$$x(0) = A = 2$$

$$y(0) = -\frac{1}{2}B = 0$$

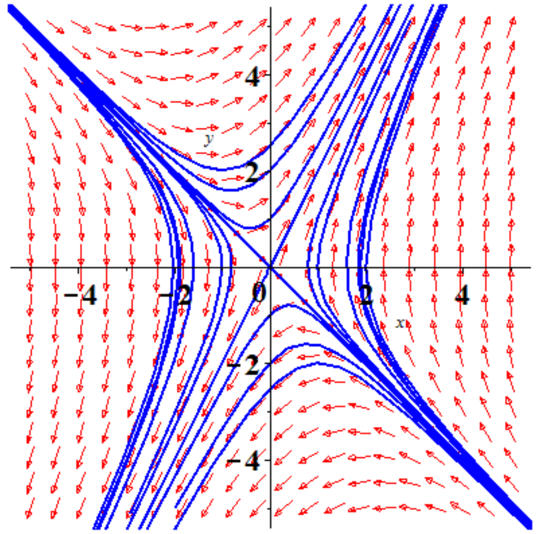
$$\begin{cases} x(t) = 2\cos t \\ y(t) = \sin t \end{cases}$$



Example

Find the general solution of the system

$$\begin{cases} x' = y \\ y' = 2x + y \end{cases}$$



Solution

$$x'' = y' = 2x + y$$
$$x'' = 2x + x'$$
$$x'' - x' - 2x = 0 \quad \Rightarrow \quad \lambda^2 - \lambda - 2 = 0$$

The eigenvalues are: $\lambda_1 = -1, \quad \lambda_2 = 2$

$\therefore$ General solution: $\begin{cases} x(t) = Ae^{-t} + Be^{2t} \\ y(t) = -Ae^{-t} + 2Be^{2t} \end{cases}$

Example

Solve the initial value problem

$$\begin{cases} x' = -y \\ y' = (1.01)x - (0.2)y \\ x(0) = 0, \quad y(0) = 1 \end{cases}$$

Solution

$$x'' = -y' = -1.01x + 0.2y$$
$$x'' = -y' = -1.01x - 0.2x'$$
$$x'' + 0.2x' + 1.01x = 0$$
$$\lambda^2 + 0.2\lambda + 1.01 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{-0.2 \pm \sqrt{0.04 - 4.04}}{2} = -0.1 \pm i$$
$$\underline{x(t) = e^{-0.1t}(A\cos t + B\sin t)}$$
$$x(0) = 0 \quad \rightarrow A = 0 \quad \Rightarrow \quad \underline{x(t) = Be^{-0.1t}\sin t}$$
$$y(t) = -x' = -0.1Be^{-0.1t}\sin t - Be^{-0.1t}\cos t$$
$$y(0) = 1 \quad \rightarrow -B = 1 \quad y(t) = -\tfrac{1}{10}e^{-t/10}\sin t + e^{-t/10}\cos t$$

$\therefore$ General solution: $\begin{cases} x(t) = -e^{-t/10}\sin t \\ y(t) = e^{-t/10}\left(\cos t - \frac{1}{10}\sin t\right) \end{cases}$

Exercises Section 4.1 – First-Order Systems

Transform the given differential equation or system into an equivalent system of 1st-order differential equation

1. $x'' + 3x' + 7x = t^2$

2. $x^{(4)} + 6x'' - 3x' + x = \cos 3t$

3. $t^2 x'' + tx' + \left(t^2 - 1\right)x = 0$

4. $t^3 x^{(3)} - 2t^2 x'' + 3tx' + 5x = \ln t$

5. $x'' - 5x + 4y = 0, \quad y'' + 4x - 5y = 0$

6. $x'' - 3x' + 4x - 2y = 0, \quad y'' + 2y' - 3x + y = \cos t$

7. $x'' = 3x - y + 2z, \quad y'' = x + y - 4z, \quad z'' = 5x - y - z$

8. $x'' = (1 - y)x, \quad y'' = (1 - x)y$

Find the general solution

9. $x' = y, \quad y' = -x$

10. $x' = y, \quad y' = -9x + 6y$

11. $x' = 8y, \quad y' = -2x$

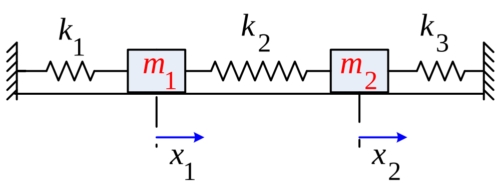
12. $x' = -2y, \quad y' = 2x; \quad x(0) = 1, \quad y(0) = 0$

13. $x' = y, \quad y' = 6x - y; \quad x(0) = 1, \quad y(0) = 2$

14. $x' = -y, \quad y' = 13x + 4y; \quad x(0) = 0, \quad y(0) = 3$

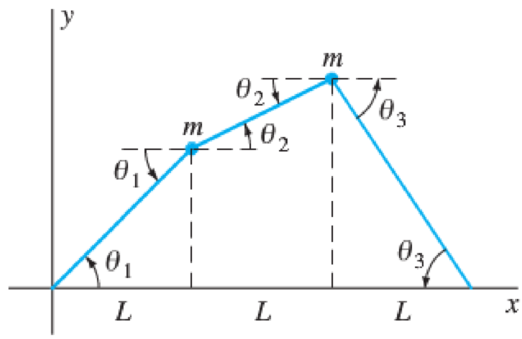
Derive the equations $\begin{cases} m_1 x_1'' = -\left(k_1 + k_2\right)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - \left(k_2 + k_3\right)x_2 \end{cases}$

For the displacements (from equilibrium) of the 2 masses.



15. Two particles each of mass m are attached to a string under (constant) tension T. Assume that the particles oscillate vertically (that is, parallel to the y-axis) with amplitudes so small that the sines of the angles shown are accurately approximated by their tangents. Show that the displacement y_1 and y_2 satisfy the equations

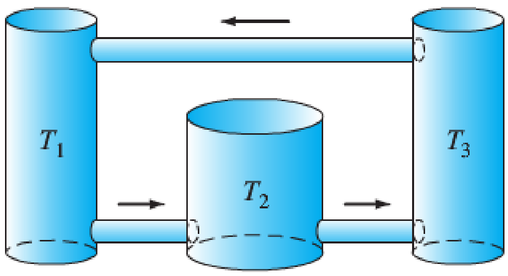
$$\begin{cases} ky_1'' = -2y_1 + y_2 \\ ky_2'' = y_1 - 2y_2 \end{cases} \qquad \text{where } k = \frac{mL}{T}$$



16. There 100-gal fermentation vats are connected, and the mixtures in each tank are kept uniform by stirring. Denote by $x_i(t)$ the amount (in pounds) of alcohol in tank T_i at time t (i = 1, 2, 3).

Suppose that the mixture circulates between the tanks at the rate of 10 gal/min. Derive the equations

$$\begin{cases} 10x_1' = -x_1 \qquad + x_3 \\ 10x_2' = x_1 - x_2 \\ 10x_3' = \qquad x_2 - x_3 \end{cases}$$



17. Suppose that a particle with mass m and electrical charge q moves in the xy-plane under the influence of the magnetic field $\vec{B} = B\hat{k}$ (thus a uniform field parallel to the z-axis), so the force on the particle is $\vec{F} = q\vec{v} \times \vec{B}$ if its velocity is $\vec{v}$. Show that the equations of motion of the particle are

$$mx'' = +qBy', \quad my'' = -qBx'$$