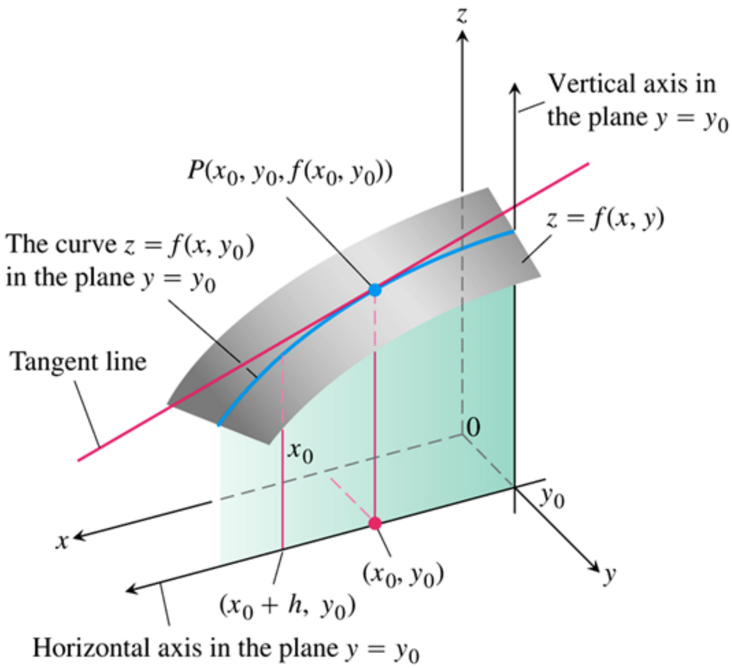
Section 2.3 – Partial Derivatives

Partial Derivatives of a Function of Two Variables

We define the partial derivative of *f* with respect to *x* at the point $\left(x_0, y_0\right)$ as the ordinary derivative of $f\left(x, y_0\right)$ with respect to *x* at the point $x = x_0$.

To distinguish partial derivatives from ordinary derivatives we use the symbol ∂ ***rather*** than the **d** symbol.



Definition

The ***partial derivative*** of $f(x, y)$ with ***respect to x*** at the point $\left(x_0, y_0\right)$ is

$$\left.\frac{\partial f}{\partial x}\right|_{\left(x_0, y_0\right)} = \lim_{h \to 0} \frac{f\left(x_0 + h, y_0\right) - f\left(x_0, y_0\right)}{h}$$

provided the limit exists.

Definition

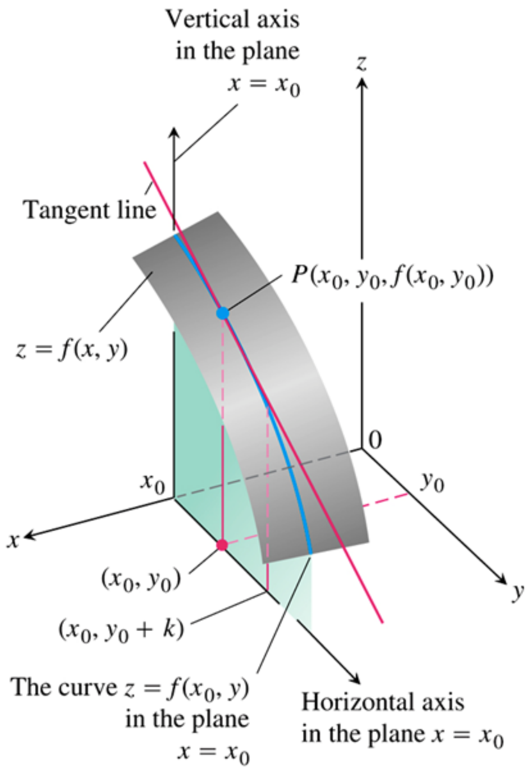
The ***partial derivative*** of $f(x, y)$ with ***respect to y*** at the point $\left(x_0, y_0\right)$ is

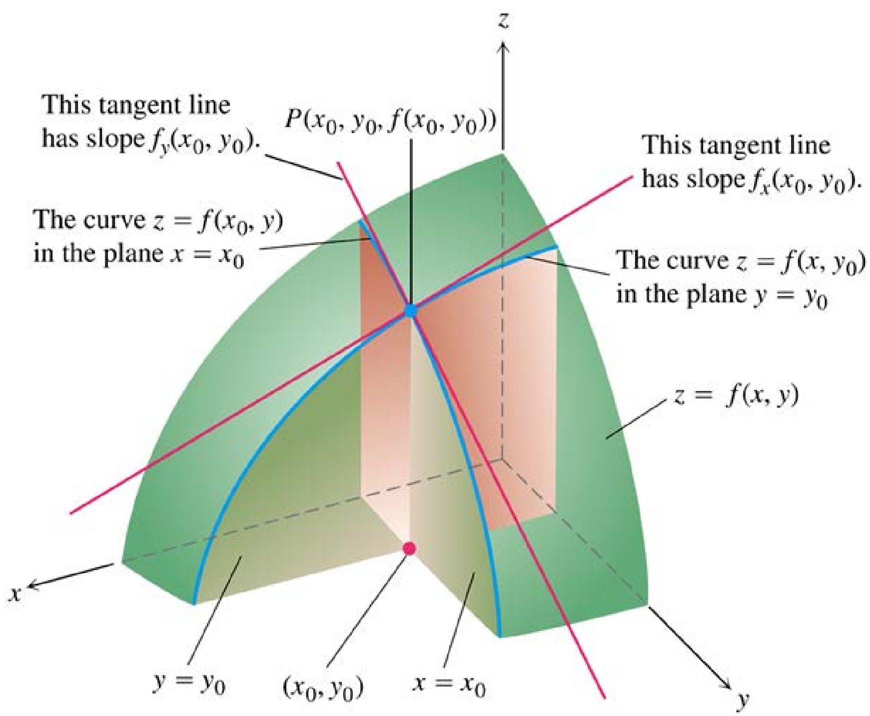
$$\left.\frac{\partial f}{\partial y}\right|_{\left(x_0, y_0\right)} = \lim_{h \to 0} \frac{f\left(x_0, y_0 + h\right) - f\left(x_0, y_0\right)}{h}$$

provided the limit exists.

The partial derivative with respect to y is denoted:

$$\frac{\partial f}{\partial y}\left(x_0, y_0\right), \quad f_y\left(x_0, y_0\right), \quad \frac{\partial f}{\partial y}, \quad f_y$$





Calculations

Example

Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4,\ -5)$ if $f(x,\ y) = x^2 + 3xy + y - 1$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + 3xy + y - 1\right)$$

$$= 2x + 3y$$

$$\left.\frac{\partial f}{\partial x}\right|_{(4,-5)} = 2(4) + 3(-5)$$

$$= -7$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + 3xy + y - 1\right)$$

$$= 3x + 1$$

$$\left.\frac{\partial f}{\partial y}\right|_{(4,-5)} = 3(4) + 1$$

$$= 13$$

Example

Find $\frac{\partial f}{\partial y}$ as a function if $f(x,\ y) = y \sin xy$

Solution

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy)$$

$$= \sin xy \frac{\partial}{\partial y}(y) + y \frac{\partial}{\partial y}(\sin xy)$$

$$= \sin xy + (y \cos xy)\frac{\partial}{\partial y}(xy)$$

$$= \sin xy + xy \cos xy$$

Example

Find f_x and f_y as a function if $f(x, y) = \dfrac{2y}{y + \cos x}$

Solution

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y+\cos x}\right) \qquad\qquad \left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$= \frac{(y+\cos x)\frac{\partial}{\partial x}(2y) - (2y)\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}$$

$$= \frac{(y+\cos x)(0) - 2y(-\sin x)}{(y+\cos x)^2}$$

$$= \frac{2y\sin x}{(y+\cos x)^2}$$

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y+\cos x}\right) \qquad\qquad \left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$= \frac{(y+\cos x)(2) - (2y)(1)}{(y+\cos x)^2}$$

$$= \frac{2y + 2\cos x - 2y}{(y+\cos x)^2}$$

$$= \frac{2\cos x}{(y+\cos x)^2}$$

OR

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{\cos x + y}\right) \qquad\qquad \left(\frac{ay+b}{cy+d}\right)' = \frac{ady' - bcy'}{(cy+d)^2}$$

$$= \frac{2y\sin x}{(y+\cos x)^2}$$

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y+\cos x}\right) \qquad\qquad \left(\frac{ay+b}{cy+d}\right)' = \frac{ad - bc}{(cy+d)^2}$$

$$= \frac{2\cos x}{(y+\cos x)^2}$$

Example

Find $\frac{\partial z}{\partial x}$ if the equation $yz - \ln z = x + y$ defines z as a function of the two independent variables x and y and the partial derivative exist.

Solution

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\left(\frac{yz-1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz-1}$$

Example

The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1,\ 2,\ 5)$.

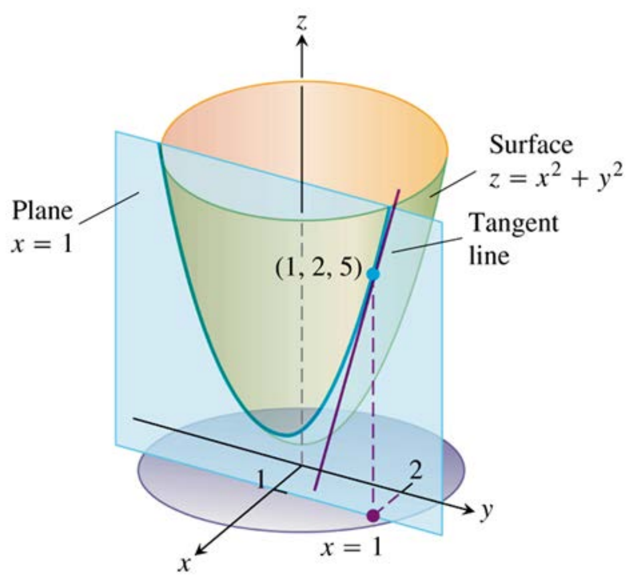
Solution

The slope is the value of the partial derivative $\frac{\partial z}{\partial y}$ at $(1,\ 2)$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}\left(x^2 + y^2\right)\right|_{(1,2)}$$

$$= \left.2y\right|_{(1,2)}$$

$$= 4$$



The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. $\Rightarrow$ $z = 1 + y^2$

$$\left.\frac{\partial z}{\partial y}\right|_{y=2} = \left.\frac{\partial}{\partial y}\left(1 + y^2\right)\right|_{y=2}$$

$$= \left.2y\right|_{y=2}$$

$$= 4$$

Functions of More than Two Variables

The partial derivatives of more than two variables are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

Example

If x, y, and z are independent variables and $f(x, y, z) = x\sin(y+3z)$.

Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x\sin(y+3z)\right)$$

$$= \sin(y+3z)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x\sin(y+3z)\right)$$

$$= x\cos(y+3z)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(x\sin(y+3z)\right)$$

$$= 3x\cos(y+3z)$$

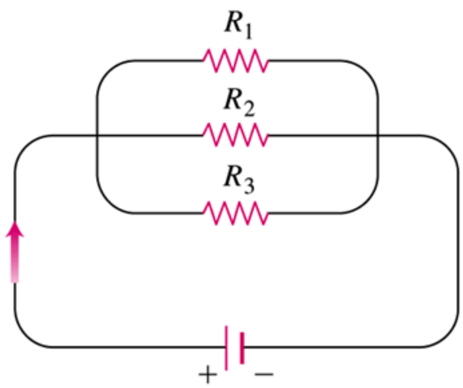
Example

If resistors of R_1, R_2, *and* R_3 ohms are connected in parallel to make an *R*-ohm resistor, the value of *R* can be found from the equation.

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}$$

Find the value of $\frac{\partial R}{\partial R_2}$ when $R_1=30\ \Omega,\ R_2=45\ \Omega,\ and\ R_3=90\ \Omega$

Solution



$$\frac{\partial}{\partial R_2}\left(\frac{1}{R}\right)=\frac{\partial}{\partial R_2}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)$$

$$-\frac{1}{R^2}\frac{\partial R}{\partial R_2}=\frac{\partial}{\partial R_2}\left(\frac{1}{R_2}\right)$$

$$-\frac{1}{R^2}\frac{\partial R}{\partial R_2}=-\frac{1}{R_2^2}$$

$$\frac{\partial R}{\partial R_2}=\frac{R^2}{R_2^2}=\left(\frac{R}{R_2}\right)^2$$

$$\frac{1}{R}=\frac{1}{30}+\frac{1}{45}+\frac{1}{90}=\frac{6}{90}=\frac{1}{15}$$

$$\Rightarrow\ R=15$$

$$\frac{\partial R}{\partial R_2}=\left(\frac{15}{45}\right)^2$$

$$\underline{=4}\,\rfloor$$

A small change in the resistance R_2 leads to a change in *R* about $\frac{1}{9}th$ as large.

Partial Derivatives and Continuity

Example

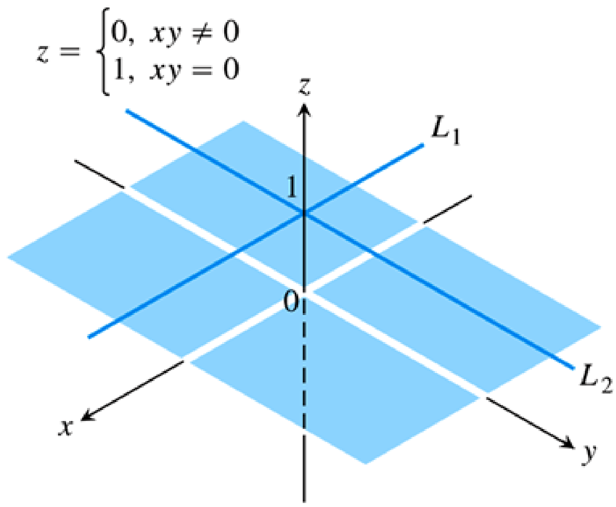
Let $f(x,y)=\begin{cases}0, & xy\neq 0\\ 1, & xy=0\end{cases}$

a) Find the limit of f as (x, y) approaches $(0, 0)$ along the line $y = x$.

b) Prove that f is not continuous at the origin.

c) Show that both partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at the origin.

Solution



a) Since $f(x, y)$ is constantly zero along the line $y = x$ (except at the origin)

$$\lim_{(x,y)\to(0,0)} f(x, y)\Big|_{y=x} = \lim_{(x,y)\to(0,0)} 0$$

$$\underline{=0}$$

b) Since $f(0, 0) = 1$, and the limit proves that f is ***not*** continuous at $(0, 0)$.

c) $\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \left.\frac{\partial}{\partial x}1\right|_{(0,0)}\ \underline{=0}$ is the slope of the line at any x.

The slope of the line at any y, $\left.\frac{\partial f}{\partial y}\right|_{(0,0)}\ \underline{=0}$

Second-Order Partial Derivatives

The second-order derivatives are denoted by

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \quad \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}, \quad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$\frac{\partial^2 f}{\partial y \partial x}$ or $f_{xy} = \left(f_x\right)_y$ Differentiate first with respect to x, then with respect to y.

Example

If $f(x, y) = x\cos y + ye^x$. Find the second derivatives $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}, \text{ and } \frac{\partial^2 f}{\partial x \partial y}$

Solution

$$\frac{\partial f}{\partial x} = \cos y + ye^x \qquad \frac{\partial f}{\partial y} = -x\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\cos y + ye^x\right)$$

$$= ye^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\cos y + ye^x\right)$$

$$= -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(-x\sin y + e^x\right)$$

$$= -x\cos y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(-x\sin y + e^x\right)$$

$$= -\sin y + e^x$$

Theorem – The Mixed Derivative Theorem

If $f(x, y)$ and its partial derivatives $f_x, f_y, f_{xy},$ *and* f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Example

Find $\dfrac{\partial^2 w}{\partial x \partial y}$ if $w = xy + \dfrac{e^y}{y^2+1}$

Solution

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(xy + \frac{e^y}{y^2+1}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(x + \frac{e^y\left(y^2+1\right) - 2ye^y}{\left(y^2+1\right)^2}\right)$$

$$\underline{= 1 \rfloor}$$

Partial Derivatives of Still Higher Order

Example

Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$

Solution

$$f_y = -4xyz + x^2$$

$$f_{yx} = -4yz + 2x$$

$$f_{yxy} = -4z$$

$$\underline{f_{yxyz} = -4 \rfloor}$$

Differentiability

Theorem – The Increment Theorem for Functions of Two Variables

Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region *R* containing a point (x_0, y_0) and that f_x *and* f_y are continuous at (x_0, y_0). Then the change

$$\Delta z = f\left(x_0 + \Delta x,\ y_0 + \Delta y\right) - f\left(x_0, y_0\right)$$

In the value of *f* that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x,\ y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x\left(x_0, y_0\right)\Delta x + f_y\left(x_0, y_0\right)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

In which each of $\varepsilon_1,\ \varepsilon_2 \to 0$ as both $\Delta x,\ \Delta y \to 0$

Definition

A function $z = f(x, y)$ is ***differentiable at*** (x_0, y_0) If $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and Δz satisfies an equation of the form

$$\Delta z = f_x\left(x_0, y_0\right)\Delta x + f_y\left(x_0, y_0\right)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

In which each of $\varepsilon_1,\ \varepsilon_2 \to 0$ as both $\Delta x,\ \Delta y \to 0$. We call *f* ***differentiable*** if it is differentiable at every point in its domain, and say that its graph is a ***smooth surface***.

Exercises *Section* 2.3 – Partial Derivatives

(1 – 17) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

1. $f(x,y) = 2x^2 - 3y - 4$
2. $f(x,y) = x^2 - xy + y^2$
3. $f(x,y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$
4. $f(x,y) = (xy - 1)^2$
5. $f(x,y) = \left(x^3 + \frac{y}{2}\right)^{2/3}$
6. $f(x,y) = \frac{1}{x+y}$
7. $f(x,y) = \frac{x}{x^2 + y^2}$
8. $f(x,y) = \tan^{-1}\frac{y}{x}$
9. $f(x,y) = e^{-x}\sin(x+y)$
10. $f(x,y) = e^{xy}\ln y$
11. $f(x,y) = \sin^2(x - 3y)$
12. $f(x,y) = \cos^2\left(3x - y^2\right)$
13. $f(x,y) = x^y$
14. $f(x,\ y) = 3x^2y^5$
15. $f(x,\ y) = x\cos y - y\sin x$
16. $f(x,\ y) = \frac{x^2}{x^2 + y^2}$
17. $f(x,\ y) = xye^{xy}$

(18 – 30) Find $f_x,\ f_y,\ and\ f_z$

18. $f(x,y,z) = 1 + xy^2 - 2z^2$
19. $f(x,y,z) = xy + yz + xz$
20. $f(x,y,z) = x - \sqrt{y^2 + z^2}$
21. $f(x,y,z) = \left(x^2 + y^2 + z^2\right)^{-1/2}$
22. $f(x,y,z) = \sec^{-1}(x + yz)$
23. $f(x,y,z) = \ln(x + 2y + 3z)$
24. $f(x,y,z) = e^{-\left(x^2+y^2+z^2\right)}$
25. $f(x,y,z) = \tanh(x + 2y + 3z)$
26. $f(x,y,z) = \sinh\left(xy - z^2\right)$
27. $f(x,\ y,\ z) = \frac{xyz}{x+y}$
28. $f(x,\ y,\ z) = 4xyz^2 - \frac{3x}{y}$
29. $f(x,\ y,\ z) = e^{x+2y+3z}$
30. $f(x,\ y,\ z) = x^2\ \sqrt{y+z}$

(31 – 34) Find partial derivatives of the function with respect to each variable

31. $g(r,\theta) = r\cos\theta + r\sin\theta$
32. $f(x,y) = \frac{1}{2}\ln\left(x^2 + y^2\right) + \tan^{-1}\frac{y}{x}$
33. $h(x,y,z) = \sin(2\pi x + y - 3z)$
34. $f(r,l,T,w) = \frac{1}{2rl}\sqrt{\frac{T}{\pi w}}$

(**35 – 43**) Find all the second-order partial derivatives of

35. $f(x,y)=x+y+xy$

36. $f(x,y)=\sin xy$

37. $g(x,y)=x^2y+\cos y+y\sin x$

38. $r(x,y)=\ln(x+y)$

39. $w=x^2\tan(xy)$

40. $w=ye^{x^2-y}$

41. $g(x,y)=y+\frac{x}{y}$

42. $g(x,y)=e^x+y\sin x$

43. $f(x,y)=y^2-3xy+\cos y+7e^y$

(**44 – 45**) Verify that the function satisfies Laplace's equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

44. $u(x,\ y)=y\left(3x^2-y^2\right)$

45. $u(x,\ y)=\ln\left(x^2+y^2\right)$

46. Let $f(x,y)=2x+3y-4$. Find the slope of the line tangent to this surface at the point $(2,\ -1)$ and lying in the

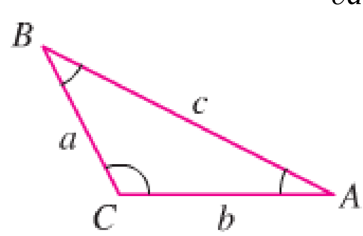
a) Plane $x=2$

b) Plane $y=-1$.

47. Let $w=f(x,y,z)$ be a function of three independent variables and writs the formal definition of the partial derivative $\frac{\partial f}{\partial y}$ at $\left(x_0,y_0,z_0\right)$. Use this definition to find $\frac{\partial f}{\partial y}$ at $(-1,\ 0,\ 3)$ for $f(x,y,z)=-2xy^2+yz^2$.

48. Find the value of $\frac{\partial x}{\partial z}$ at the point $(1,-1,-3)$ if the equation $xz+y\ln x-x^2+4=0$ defines *x* as a function of the two independent variables *y* and *z* and the partial derivative exists.

49. Express *A* implicitly as a function of *a*, *b*, and *c* and calculate $\frac{\partial A}{\partial a}$ and $\frac{\partial A}{\partial b}$.



50. An important partial differential equation that describes the distribution of heat in a region at time *t* can be represented by the *one-dimensional heat equation*

$$\frac{\partial f}{\partial t}=\frac{\partial^2 f}{\partial x^2}$$

Show that $u(x,t) = \sin(\alpha x) \cdot e^{-\beta t}$ satisfies the heat equation for constants α and β. What is the relationship between α and β for this function to be a solution?