Section 3.6 – Alternating Series, Absolute and Conditional Convergence

A series in which the terms are alternately positive and negative is an alternating series.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots+\frac{(-1)^{n+1}}{n}+\cdots$$

$$-2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots+\frac{(-1)^{n}\,4}{2^{n}}+\cdots$$

Theorem – The Alternating Series Test (Leibniz's Test)

The series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$

Converges if all three of the following conditions are satisfied:

1. The u_n '*s* are all positive.
2. The positive u_n '*s* are (eventually) non-increasing: $u_n \ge u_{n+1}$ for all $n \ge N$, for some integer N.
3. $u_n \to 0$

Example

The alternating harmonic series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$

Solution

1. $\frac{1}{n} > 0$
2. $n < n+1 \;\to\; \frac{1}{n} > \frac{1}{n+1}$
3. $\frac{1}{n} \to 0$

Therefore, the series converges.

Theorem – The Alternating Series Estimation Theorem

IF the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions, then for $n \ge N$

$$s_n = u_1 - u_2 + u_3 - \cdots + (-1)^{n+1}u_n$$

Approximates the sum L of the series with an error whose absolute values is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1} and the remainder, $L - s_n$, has the same sign as the first unused term.

Absolute and Conditional Convergence

Definition

A series $\sum a_n$ ***converges absolutely*** (is ***absolutely convergent***) if the corresponding series of absolute values, $\sum |a_n|$, converges.

Definition

A series converges but does not converge absolutely ***converges conditionally***.

Theorem

If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.

Example

For $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$ the corresponding series of absolute values is the convergent series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

The original series converges because it converges absolutely.

Example

For $\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$, which contains both positive and negative terms, the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \frac{|\sin 3|}{9} + \cdots$$

Which converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ because $|\sin n| \le 1$ for every n.

The original series converges absolutely; therefore, it converges.

Rearranging Series

Theorem

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1,\ b_2,\ \ldots,\ b_n,\ \ldots$ is any arrangement of the sequence $\{a_n\}$, then

$\sum b_n$ converges absolutely and $\qquad \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$

Exercises Section 3.6 – Alternating Series, Absolute and Conditional Convergence

Determine if the alternating series converges or diverges

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$

2. $\sum_{n=2}^{\infty}(-1)^{n}\frac{4}{(\ln n)^2}$

3. $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n^2+1}$

4. $\sum_{n=1}^{\infty}(-1)^{n}\frac{n^2+5}{n^2+4}$

5. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^n$

6. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$

7. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}$

8. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{3n+2}$

9. $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{3^n}$

10. $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{e^n}$

11. $\sum_{n=1}^{\infty}(-1)^{n}\frac{5n-1}{4n+1}$

12. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+5}$

13. $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{\ln(n+1)}$

14. $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{\ln(n+1)}$

15. $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{\sqrt{n}}$

16. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^2+4}$

17. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+1}{\ln(n+1)}$

18. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln(n+1)}{n+1}$

19. $\sum_{n=1}^{\infty}\sin\frac{(2n-1)\pi}{2}$

20. $\sum_{n=1}^{\infty}\frac{1}{n}\cos n\pi$

21. $\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!}$

22. $\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(2n+1)!}$

23. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}}{n+2}$

24. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}}{\sqrt[3]{n}}$

Determine if the series converge absolutely or conditionally, or diverges

25. $\sum_{n=1}^{\infty}(-1)^{n+1}(0.1)^n$

26. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.1)^n}{n}$

27. $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{1+\sqrt{n}}$

28. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\sin n}{n^2}$

29. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3+n}{5+n}$

30. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}$

31. $\sum_{n=1}^{\infty}\frac{n\cos(n\pi)}{2^n}$

32. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$

33. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2+2n+1}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n^2+\ln n}$

36. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n}\left(n^2-1\right)}{n^2+1}$

37. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{(n+1)\ln(n+1)}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{n}}{n!}$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n\pi^n}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{100\cos(n\pi)}{2n+3}$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(-100)^n}$

42. $\displaystyle\sum_{n=10}^{\infty} \frac{\sin\left(n+\frac{1}{2}\right)\pi}{\ln\ln n}$

43. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{2^n}$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n!}$

46. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3}$

47. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

48. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

49. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{n^2}{(n+1)^2}$

50. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{2n+3}{n+10}$

51. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\ln n}$

52. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n} e^{-n^2}$

53. $\displaystyle\sum_{n=2}^{\infty} (-1)^{n}\frac{n}{n^3-5}$

54. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{4/3}}$

55. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+4}}$

56. $\displaystyle\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1}$

57. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\arctan n$

58. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$

59. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left[(n-1)\frac{\pi}{2}\right]}{n}$

For what values of *x* does the series converge absolutely? Converge conditionally? Diverge?

60. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^{n}}{n\ 2^n}$

61. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^{n}}{n^2\ 2^{2n}}$

62. $\displaystyle\sum_{n=1}^{\infty} (n+1)^2\left(\frac{x}{x+2}\right)^n$

63. $\displaystyle\sum_{n=0}^{\infty} (-1)^n\frac{(x-1)^{n}}{2n+3}$

64. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}\left(\frac{3x+2}{-5}\right)^n$

65. $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{2^n\ln n}$

66. $\displaystyle\sum_{n=1}^{\infty} \frac{(4x+1)^{n}}{n^3}$

67. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+3)^{n}}{n^{1/3}4^n}$

68. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(1+\frac{1}{x}\right)^n$

Use any method to determine if the series converges or diverges.

69. $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+1}$

70. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$

71. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{5}{n}$

72. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{3^{n-1}}{n!}$

73. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{3^n}{n2^n}$

74. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{1}{n}$

75. $\sum_{n=1}^{\infty} \frac{n}{(-2)^{n-1}}$

76. $\sum_{n=1}^{\infty} \frac{10}{n^{3/2}}$

77. $\sum_{n=1}^{\infty} \frac{2}{n^2+5}$

78. $\sum_{n=1}^{\infty} \frac{3^n}{n^2}$

79. $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

80. $\sum_{n=1}^{\infty} 5\left(\frac{7}{8}\right)^n$

81. $\sum_{n=1}^{\infty} \frac{3n^2}{2n^2+1}$

82. $\sum_{n=1}^{\infty} 100e^{-n/2}$

83. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+4}$

84. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{3n^2-1}$

85. $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

86. Use a Riemann sum argument to show that $ln\, n! \geq \int_1^n ln\, t \; dt = nln\, n - n + 1$

Then for what values of x does the series $\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$ converge absolutely? Converge conditionally? Diverge? (*Use the ratio test first*)

87. It can be proved that if a series converges absolutely, then its terms may be summed in any order without changing the value of the series. However, if a series converges conditionally, then the value of the series depends on the order of summation. For example, the (conditionally convergent) alternating harmonic series has the value

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\ln 2$$

Show that by rearranging the terms (so the sign pattern is ++–),

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\ldots=\frac{3}{2}\ln 2$$

88. A crew of workers is constructing a tunnel through a mountain. Understandably, the rate of construction decreases because rocks and earth must be removed a greater distance as the tunnel gets longer. Suppose that each week the crew digs 0.95 of the distance it dug the previous week. In the first week, the crew constructed 100 *m* of tunnel.

a) How far does the crew dig in 10 *weeks*? 20 *weeks*? *N* weeks?

b) What is the longest tunnel the crew can build at this rate?

c) The time required to dig 100 *m* increases by 10% each *week*, starting with 1 *week* to dig the first 100 *m*. Can the crew complete a 1.5 *km* tunnel in 10 *weeks*? Explain.

89. Consider the alternating series

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k \quad \textit{where} \quad a_k = \begin{cases} \dfrac{4}{k+1} & \textit{if } k \textit{ is odd} \\ \dfrac{2}{k} & \textit{if } k \textit{ is even} \end{cases}$$

a) Write out the first ten terms of the series, group them in pairs, and show that the even partial sums of the series form the (divergent) harmonic series.

b) Show that $\lim_{k\to\infty} a_k = 0$

c) Explain why the series diverges even though the terms of the series approach zero.