

Solution **Section 2.4 – Integration of Rational Functions by Partial Fractions**

Exercise

Evaluate $\int \frac{dx}{x^2 + 2x}$

Solution

$$\frac{1}{x^2 + 2x} = \frac{A}{x} + \frac{B}{x+2}$$

$$1 = Ax + 2A + Bx$$

$$x \quad 2A = 1 \quad \rightarrow A = \frac{1}{2}$$

$$x^0 \quad A + B = 0 \quad \rightarrow B = -\frac{1}{2}$$

$$\begin{aligned} \int \frac{1}{x^2 + 2x} dx &= \frac{1}{2} \int \frac{1}{x} dx - \frac{1}{2} \int \frac{1}{x+2} dx \\ &= \frac{1}{2} \ln|x| - \frac{1}{2} \ln|x+2| + C \end{aligned}$$

Exercise

Evaluate $\int \frac{2x+1}{x^2 - 7x + 12} dx$

Solution

$$\frac{2x+1}{x^2 - 7x + 12} = \frac{A}{x-4} + \frac{B}{x-3}$$

$$2x+1 = Ax - 3A + Bx - 4B$$

$$x \quad A + B = 2$$

$$x^0 \quad -3A - 4B = 1$$

$$A = \frac{\begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -3 & -4 \end{vmatrix}} = \frac{-9}{-1} = 9 \quad B = \frac{\begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix}}{-1} = \frac{7}{-1} = -7$$

$$\int \frac{2x+1}{x^2 - 7x + 12} dx = 9 \int \frac{dx}{x-4} - 7 \int \frac{dx}{x-3}$$

$$= 9 \ln|x-4| - 7 \ln|x-3| + C$$

$$= \ln \left| \frac{(x-4)^9}{(x-3)^7} \right| + C$$

Exercise

Evaluate $\int \frac{x+3}{2x^3-8x} dx$

Solution

$$\begin{aligned} \frac{x+3}{2x^3-8x} &= \frac{1}{2} \frac{x+3}{x(x^2-4)} \\ &= \frac{1}{2} \left(\frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} \right) \\ &= \frac{1}{2} \frac{A(x+2)(x-2) + Bx(x-2) + Cx(x+2)}{x(x+2)(x-2)} \end{aligned}$$

$$Ax^2 - 4A + Bx^2 - 2Bx + Cx^2 + 2Cx = x + 3$$

$$x^2 \quad A + B + C = 0$$

$$x \quad -2B + 2C = 1$$

$$x^0 \quad -4A = 3 \quad \rightarrow A = -\frac{3}{4}$$

$$\begin{cases} B + C = \frac{3}{4} \\ -2B + 2C = 1 \end{cases} \Rightarrow \begin{cases} 2B + 2C = \frac{3}{2} \\ -2B + 2C = 1 \end{cases}$$

$$4C = \frac{5}{2} \rightarrow C = \frac{5}{8}$$

$$B = \frac{3}{4} - \frac{5}{8} \rightarrow B = \frac{1}{8}$$

$$\int \frac{x+3}{2x^3-8x} dx = \frac{1}{2} \int -\frac{3}{4} \frac{dx}{x} + \frac{1}{2} \int \frac{1}{8} \frac{dx}{x+2} + \frac{1}{2} \int \frac{5}{8} \frac{dx}{x-2}$$

$$= -\frac{3}{8} \ln|x| + \frac{1}{16} \ln|x+2| + \frac{5}{16} \ln|x-2| + K$$

$$= \frac{1}{16} (\ln|x+2| + 5 \ln|x-2| - 6 \ln|x|) + K$$

$$= \frac{1}{16} \ln \left| \frac{(x+2)(x-2)^5}{x^6} \right| + K$$

Exercise

Evaluate $\int \frac{x^2}{(x-1)(x^2+2x+1)} dx$

Solution

$$\begin{aligned} \frac{x^2}{(x-1)(x^2+2x+1)} &= \frac{x^2}{(x-1)(x+1)^2} \\ &= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \end{aligned}$$

$$\begin{aligned} x^2 &= A(x+1)^2 + B(x-1)(x+1) + C(x-1) \\ &= Ax^2 + 2Ax + A + Bx^2 - B + Cx - C \end{aligned}$$

$$x^2 \quad A + B = 1 \quad \rightarrow B = 1 - A$$

$$x \quad 2A + C = 0 \quad \rightarrow C = -2A$$

$$x^0 \quad A - B - C = 0 \quad \rightarrow A - 1 + A + 2A = 0$$

$$\underline{A = \frac{1}{4} \quad B = \frac{3}{4} \quad C = -\frac{1}{2}}$$

$$\begin{aligned} \int \frac{x^2}{(x-1)(x^2+2x+1)} dx &= \frac{1}{4} \int \frac{dx}{x-1} + \frac{3}{4} \int \frac{dx}{x+1} - \frac{1}{2} \int \frac{dx}{(x+1)^2} \\ &= \frac{1}{4} \ln|x-1| + \frac{3}{4} \ln|x+1| + \frac{1}{2} \frac{1}{(x+1)} + K \\ &= \frac{1}{4} \left(\ln|x-1| + \ln|x+1|^3 \right) + \frac{1}{2(x+1)} + K \\ &= \frac{1}{4} \ln|(x-1)(x+1)^3| + \frac{1}{2(x+1)} + K \end{aligned}$$

Exercise

Evaluate $\int \frac{8x^2+8x+2}{(4x^2+1)^2} dx$

Solution

$$\begin{aligned} \frac{8x^2+8x+2}{(4x^2+1)^2} &= \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{(4x^2+1)^2} \\ &= \frac{(Ax+B)(4x^2+1) + Cx+D}{(4x^2+1)^2} \end{aligned}$$

$$8x^2 + 8x + 2 = 4Ax^3 + 4Bx^2 + (A + C)x + B + D$$

$$\begin{cases} x^3 & A = 0 \\ x^2 & 4B = 8 \\ x & A + C = 8 \\ x^0 & B + D = 2 \end{cases} \rightarrow \boxed{A = 0} \quad \boxed{B = 2} \quad \boxed{C = 8} \quad \boxed{D = 0}$$

$$\int \frac{8x^2 + 8x + 2}{(4x^2 + 1)^2} dx = \int \frac{2}{4x^2 + 1} dx + \int \frac{8x}{(4x^2 + 1)^2} dx$$

$$d(4x^2 + 1) = 8x dx$$

$$= \int \frac{2}{4x^2 + 1} dx + \int \frac{d(4x^2 + 1)}{(4x^2 + 1)^2}$$

$$\int \frac{du}{u^2} = -\frac{1}{u} \quad \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$= \tan^{-1} 2x - \frac{1}{4x^2 + 1} + K$$

Exercise

Evaluate $\int \frac{x^2 + x}{x^4 - 3x^2 - 4} dx$

Solution

$$\frac{x^2 + x}{x^4 - 3x^2 - 4} = \frac{x^2 + x}{(x^2 - 4)(x^2 + 1)}$$

$$= \frac{A}{x - 2} + \frac{B}{x + 2} + \frac{Cx + D}{x^2 + 1}$$

$$x^2 + x = A(x + 2)(x^2 + 1) + B(x - 2)(x^2 + 1) + (Cx + D)(x^2 - 4)$$

$$= Ax^3 + Ax + 2Ax^2 + 2A + Bx^3 + Bx - 2Bx^2 - 2B + Cx^3 - 4Cx + Dx^2 - 4D$$

$$= (A + B + C)x^3 + (2A - 2B + D)x^2 + (A + B - 4C)x + 2A - 2B - 4D$$

$$\begin{cases} x^3 & A + B + C = 0 & (1) \\ x^2 & 2A - 2B + D = 1 & (2) \\ x & A + B - 4C = 1 & (3) \\ x^0 & 2A - 2B - 4D = 0 & (4) \end{cases}$$

$$(1) - (3) \rightarrow 5C = -1 \quad C = -\frac{1}{5}$$

$$(2) - (4) \rightarrow 5D = 1 \quad D = \frac{1}{5}$$

$$\begin{cases} A+B=\frac{1}{5} \\ 2A-2B=\frac{4}{5} \end{cases} \rightarrow \begin{cases} 2A+2B=\frac{2}{5} \\ 2A-2B=\frac{4}{5} \end{cases}$$

$$4A=\frac{6}{5} \rightarrow \underline{A=\frac{3}{10}}$$

$$B=\frac{1}{5}-\frac{3}{10} \rightarrow \underline{B=-\frac{1}{10}}$$

$$\begin{aligned} \int \frac{x^2+x}{x^4-3x^2-4} dx &= \frac{3}{10} \int \frac{1}{x-2} dx - \frac{1}{10} \int \frac{1}{x+2} dx + \frac{1}{5} \int \frac{-x+1}{x^2+1} dx \\ &= \frac{3}{10} \ln|x-2| - \frac{1}{10} \ln|x+2| - \frac{1}{5} \int \frac{x}{x^2+1} dx + \frac{1}{5} \int \frac{1}{x^2+1} dx \quad d(x^2+1)=2xdx \\ &= \frac{3}{10} \ln|x-2| - \frac{1}{10} \ln|x+2| - \frac{1}{10} \int \frac{d(x^2+1)}{x^2+1} + \frac{1}{5} \int \frac{1}{x^2+1} dx \quad \int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \\ &= \underline{\frac{3}{10} \ln|x-2| - \frac{1}{10} \ln|x+2| - \frac{1}{10} \ln(x^2+1) + \frac{1}{5} \tan^{-1} x + K} \end{aligned}$$

Exercise

Evaluate $\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2 + 1)^3} d\theta$

Solution

$$\frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2 + 1)^3} = \frac{A\theta + B}{\theta^2 + 1} + \frac{C\theta + D}{(\theta^2 + 1)^2} + \frac{E\theta + F}{(\theta^2 + 1)^3}$$

$$\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1 = (A\theta + B)(\theta^2 + 1)^2 + (C\theta + D)(\theta^2 + 1) + E\theta + F$$

$$= (A\theta + B)(\theta^4 + 2\theta^2 + 1) + C\theta^3 + C\theta + D\theta^2 + D + E\theta + F$$

$$= A\theta^5 + B\theta^4 + (2A + C)\theta^3 + (2B + D)\theta^2 + (A + C + E)\theta + B + D + F$$

$$\begin{cases} \boxed{A=0} \\ \boxed{B=1} \\ 2A+C=-4 \\ 2B+D=2 \\ A+C+E=-3 \\ B+D+F=1 \end{cases} \rightarrow \boxed{C=-4} \quad \boxed{D=0} \quad \boxed{E=1} \quad \boxed{F=0}$$

$$\begin{aligned}
\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2 + 1)^3} d\theta &= \int \frac{1}{\theta^2 + 1} d\theta - 4 \int \frac{\theta}{(\theta^2 + 1)^2} d\theta + \int \frac{\theta}{(\theta^2 + 1)^3} d\theta \\
&= \int \frac{1}{\theta^2 + 1} d\theta - 2 \int \frac{d(\theta^2 + 1)}{(\theta^2 + 1)^2} + \frac{1}{2} \int \frac{d(\theta^2 + 1)}{(\theta^2 + 1)^3} \quad d(\theta^2 + 1) = 2\theta d\theta \\
&= \tan^{-1} \theta + 2 \frac{1}{\theta^2 + 1} - \frac{1}{4} \frac{1}{(\theta^2 + 1)^2} + K
\end{aligned}$$

Exercise

Evaluate $\int \frac{x^4}{x^2 - 1} dx$

Solution

$$\frac{x^4}{x^2 - 1} = x^2 + 1 + \frac{1}{(x-1)(x+1)}$$

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$1 = Ax + A + Bx - B$$

$$\begin{cases} A + B = 0 \\ A - B = 1 \end{cases} \rightarrow \boxed{A = \frac{1}{2}} \quad \boxed{B = -\frac{1}{2}}$$

$$\begin{array}{r}
x^2 + 1 \\
x^2 - 1 \overline{) x^4} \\
\underline{x^4 - x^2} \\
x^2 \\
\underline{x^2 - 1} \\
1
\end{array}$$

$$\begin{aligned}
\int \frac{x^4}{x^2 - 1} dx &= \int (x^2 + 1) dx + \frac{1}{2} \int \frac{1}{x-1} dx - \frac{1}{2} \int \frac{1}{x+1} dx \\
&= \frac{1}{3} x^3 + x + \frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| + C \\
&= \frac{1}{3} x^3 + x + \frac{1}{2} (\ln|x-1| - \ln|x+1|) + C \\
&= \frac{1}{3} x^3 + x + \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{16x^3}{4x^2 - 4x + 1} dx$

Solution

$$\frac{16x^3}{4x^2 - 4x + 1} = 4x + 4 + \frac{12x - 4}{(2x - 1)^2}$$

$$= 4x + 4 + \frac{A}{2x - 1} + \frac{B}{(2x - 1)^2}$$

$$12x - 4 = 2Ax - A + B$$

$$\begin{cases} 2A = 12 \\ -A + B = -4 \end{cases} \rightarrow \boxed{A = 6} \quad \boxed{B = 2}$$

$$\begin{array}{r} 4x + 4 \\ 4x^2 - 4x + 1 \overline{) 16x^3} \\ \underline{16x^3 - 16x^2 + 4x} \\ 16x^2 - 4x \\ \underline{16x^2 - 16x + 4} \\ 12x - 4 \end{array}$$

$$\begin{aligned} \int \frac{16x^3}{4x^2 - 4x + 1} dx &= \int (4x + 4) dx + 6 \int \frac{dx}{2x - 1} + 2 \int \frac{dx}{(2x - 1)^2} \\ &= 2x^2 + 4x + 6 \left(\frac{1}{2} \right) \ln |2x - 1| + 2 \left(-\frac{1}{2} \right) \frac{1}{2x - 1} + C \\ &= \underline{2x^2 + 4x + 3 \ln |2x - 1| - \frac{1}{2x - 1} + C} \end{aligned}$$

Exercise

Evaluate $\int \frac{e^{4x} + 2e^{2x} - e^x}{e^{2x} + 1} dx$

Solution

$$\begin{aligned} \int \frac{e^{4x} + 2e^{2x} - e^x}{e^{2x} + 1} dx &= \int \frac{e^x (e^{3x} + 2e^x - 1)}{e^{2x} + 1} dx & y = e^x \Rightarrow dy = e^x dx \\ &= \int \frac{y^3 + 2y - 1}{y^2 + 1} dy \\ &= \int \left(y + \frac{y - 1}{y^2 + 1} \right) dy \\ &= \int y dy + \int \frac{y}{y^2 + 1} dy - \int \frac{1}{y^2 + 1} dy \\ &= \int y dy + \frac{1}{2} \int \frac{1}{y^2 + 1} d(y^2 + 1) - \int \frac{1}{y^2 + 1} dy & d(y^2 + 1) = 2y dy \\ &= \frac{1}{2} y^2 + \frac{1}{2} \ln(y^2 + 1) - \tan^{-1} y + C \\ &= \underline{\frac{1}{2} e^{2x} + \frac{1}{2} \ln(e^{2x} + 1) - \tan^{-1} e^x + C} \end{aligned}$$

Exercise

Evaluate $\int \frac{\sin \theta d\theta}{\cos^2 \theta + \cos \theta - 2}$

Solution

Let $y = \cos \theta \Rightarrow dy = -\sin \theta d\theta$

$$\int \frac{\sin \theta d\theta}{\cos^2 \theta + \cos \theta - 2} = -\int \frac{dy}{y^2 + y - 2}$$

$$\frac{1}{y^2 + y - 2} = \frac{1}{(y+2)(y-1)} = \frac{A}{y+2} + \frac{B}{y-1}$$

$$1 = (A+B)y - A + 2B$$

$$\begin{cases} A+B=0 \\ -A+2B=1 \end{cases} \rightarrow \boxed{A=-\frac{1}{3}} \quad \boxed{B=\frac{1}{3}}$$

$$\int \frac{\sin \theta d\theta}{\cos^2 \theta + \cos \theta - 2} = -\left(-\frac{1}{3} \int \frac{dy}{y+2} + \frac{1}{3} \int \frac{dy}{y-1} \right)$$

$$= \frac{1}{3} \ln|y+2| - \frac{1}{3} \ln|y-1| + C$$

$$= \frac{1}{3} (\ln|y+2| - \ln|y-1|) + C$$

$$= \frac{1}{3} \ln \left| \frac{y+2}{y-1} \right| + C$$

$$= \frac{1}{3} \ln \left| \frac{\cos \theta + 2}{\cos \theta - 1} \right| + C$$

Exercise

Evaluate $\int \frac{(x-2)^2 \tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2} dx$

Solution

$$\int \frac{(x-2)^2 \tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2} dx = \int \frac{(x-2)^2 \tan^{-1}(2x)}{(4x^2+1)(x-2)^2} dx - \int \frac{12x^3+3x}{(4x^2+1)(x-2)^2} dx$$

$$= \int \frac{\tan^{-1}(2x)}{4x^2+1} dx - \int \frac{3x(4x^2+1)}{(4x^2+1)(x-2)^2} dx$$

$$= \int \frac{\tan^{-1}(2x)}{4x^2+1} dx - \int \frac{3x}{(x-2)^2} dx$$

$$d\left(\tan^{-1} 2x\right)=\frac{dx}{(2x)^2+1}=\frac{dx}{4x^2+1}$$

$$\frac{3x}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}=\frac{Ax-2A+B}{(x-2)^2}$$

$$\begin{cases} \boxed{A=3} \\ -2A+B=0 \end{cases} \rightarrow \boxed{B=6}$$

$$\begin{aligned} \int \frac{(x-2)^2 \tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2} dx &= \frac{1}{2} \int \tan^{-1}(2x) d\left(\tan^{-1}(2x)\right) - 3 \int \frac{dx}{x-2} - 6 \int \frac{dx}{(x-2)^2} \\ &= \frac{1}{4} \left(\tan^{-1}(2x)\right)^2 - 3 \int \frac{d(x-2)}{x-2} - 6 \int \frac{d(x-2)}{(x-2)^2} \\ &= \frac{1}{4} \left(\tan^{-1}(2x)\right)^2 - 3 \ln|x-2| - \frac{6}{x-2} + C \end{aligned}$$

Exercise

Evaluate $\int \frac{\sqrt{x+1}}{x} dx$

Solution

Let $x+1=u^2 \Rightarrow dx=2udu$

$$u^2-1 \sqrt{\frac{1}{u^2 \frac{u^2-1}{1}}}$$

$$\int \frac{\sqrt{x+1}}{x} dx = \int \frac{u}{u^2-1} 2udu$$

$$= 2 \int \frac{u^2}{u^2-1} du$$

$$= 2 \int \left(1 + \frac{1}{u^2-1}\right) du$$

$$= 2 \int du + 2 \int \frac{1}{u^2-1} du$$

$$\frac{1}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1} = \frac{(A+B)u + A-B}{(u-1)(u+1)}$$

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \Rightarrow \boxed{A=\frac{1}{2}} \quad \boxed{B=-\frac{1}{2}}$$

$$= 2 \int du + 2 \int \left(\frac{1}{2} \frac{1}{u-1} - \frac{1}{2} \frac{1}{u+1}\right) du$$

$$\begin{aligned}
&= 2u + \int \frac{1}{u-1} du - \int \frac{1}{u+1} du \\
&= 2u + \ln|u-1| - \ln|u+1| + C \\
&= 2\sqrt{x+1} + \ln|\sqrt{x+1}-1| - \ln|\sqrt{x+1}+1| + C \\
&= \underline{2\sqrt{x+1} + \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C}
\end{aligned}$$

Exercise

Evaluate $\int \frac{x^3 - 2x^2 + 3x - 4}{x^2 + 1} dx$

Solution

$$\begin{aligned}
\int \frac{x^3 - 2x^2 + 3x - 4}{x^2 + 1} dx &= \int \left(x - 2 + \frac{2x-2}{x^2+1} \right) dx \\
&= \int (x-2) dx + \int \frac{2x}{x^2+1} dx - 2 \int \frac{1}{x^2+1} dx \\
&= \int (x-2) dx + \int \frac{d(x^2+1)}{x^2+1} - 2 \int \frac{1}{x^2+1} dx \\
&= \underline{\frac{1}{2}x^2 - 2x + \ln(x^2+1) - 2 \tan^{-1}(x) + C}
\end{aligned}$$

$$\begin{array}{r}
x-2 \\
\hline
x^2+1 \overline{) x^3 - 2x^2 + 3x - 4} \\
\underline{x^3 + x} \\
-2x^2 + 2x - 4 \\
\underline{-2x^2 - 2} \\
2x - 2
\end{array}$$

Exercise

Evaluate $\int \frac{4x^2 + 2x + 4}{x+1} dx$

Solution

$$\begin{aligned}
\int \frac{4x^2 + 2x + 4}{x+1} dx &= \int \left(4x + 2 + \frac{6}{x+1} \right) dx \\
&= \int (4x+2) dx + \int \frac{6}{x+1} dx \\
&= \int (4x+2) dx + 6 \int \frac{d(x+1)}{x+1} \\
&= \underline{2x^2 - 2x + 6 \ln|x+1| + C}
\end{aligned}$$

$$\int \frac{d(U)}{U} = \ln|U|$$

Exercise

Evaluate $\int \frac{3x^2 + 7x - 2}{x^3 - x^2 - 2x} dx$

Solution

$$\frac{3x^2 + 7x - 2}{x^3 - x^2 - 2x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

$$3x^2 + 7x - 2 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1)$$

$$= Ax^2 - Ax - 2A$$

$$Bx^2 - 2Bx$$

$$Cx^2 + Cx$$

$$\begin{cases} A + B + C = 3 \\ -A - 2B + C = 7 \\ -2A = -2 \quad \rightarrow \underline{A=1} \end{cases}$$

$$\begin{cases} B + C = 2 \\ -2B + C = 8 \end{cases} \rightarrow \underline{B=-2} \quad \underline{C=4}$$

$$\begin{aligned} \int \frac{3x^2 + 7x - 2}{x^3 - x^2 - 2x} dx &= \int \left(\frac{1}{x} - \frac{2}{x+1} + \frac{4}{x-2} \right) dx \\ &= \ln|x| - 2\ln|x+1| + 4\ln|x-2| + K \\ &= \ln \frac{|x|(x-2)^4}{(x+1)^2} + K \end{aligned}$$

Exercise

Evaluate $\int \frac{3x^2 + 2x + 5}{(x-1)(x^2 - x - 20)} dx$

Solution

$$\frac{3x^2 + 2x + 5}{(x-1)(x^2 - x - 20)} = \frac{A}{x-1} + \frac{B}{x-5} + \frac{C}{x+4}$$

$$3x^2 + 2x + 5 = (A + B + C)x^2 + (-A + 3B - 6C)x - 20A - 4B + 5C$$

$$\begin{cases} x^2 & A + B + C = 3 \\ x & -A + 3B - 6C = 2 \\ x^0 & -20A - 4B + 5C = 5 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 3 & -6 \\ -20 & -4 & 5 \end{vmatrix} = 180$$

$$D_A = \begin{vmatrix} 3 & 1 & 1 \\ 2 & 3 & -6 \\ 5 & -4 & 5 \end{vmatrix} = -90$$

$$D_B = \begin{vmatrix} 1 & 3 & 1 \\ -1 & 2 & -6 \\ -20 & 5 & 5 \end{vmatrix} = 450$$

$$D_C = \begin{vmatrix} 1 & 1 & 3 \\ -1 & 3 & 2 \\ -20 & -4 & 5 \end{vmatrix} = 180$$

$$\underline{A = \frac{1}{2}, \quad B = \frac{5}{2}, \quad C = 1}$$

$$\begin{aligned} \int \frac{3x^2 + 2x + 5}{(x-1)(x^2 - x - 20)} dx &= \int \left(\frac{1}{2} \frac{1}{x-1} + \frac{5}{2} \frac{1}{x-5} + \frac{1}{x+4} \right) dx \\ &= \underline{\frac{1}{2} \ln|x-1| + \frac{5}{2} \ln|x-5| + \ln|x+4| + K} \end{aligned}$$

Exercise

Evaluate $\int \frac{5x^2 - 3x + 2}{x^3 - 2x^2} dx$

Solution

$$\frac{5x^2 - 3x + 2}{x^3 - 2x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$5x^2 - 3x + 2 = Ax^2 - 2Ax + Bx - 2B + Cx^2$$

$$\begin{cases} x^2 & A + C = 5 & C = 4 \\ x & -2A + B = -3 & A = 1 \\ x^0 & -2B = 2 & \rightarrow B = -1 \end{cases}$$

$$\begin{aligned} \int \frac{5x^2 - 3x + 2}{x^3 - 2x^2} dx &= \int \frac{dx}{x} - \int \frac{dx}{x^2} + 4 \int \frac{dx}{x-2} \\ &= \underline{\ln|x| + \frac{1}{x} + 4 \ln|x-2| + K} \end{aligned}$$

Exercise

Evaluate $\int \frac{7x^2 - 13x + 13}{(x-2)(x^2 - 2x + 3)} dx$

Solution

$$\frac{7x^2 - 13x + 13}{(x-2)(x^2 - 2x + 3)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 - 2x + 3}$$

$$7x^2 - 13x + 13 = Ax^2 - 2Ax + 3A + Bx^2 - 2Bx + Cx - 2C$$

$$\begin{cases} x^2 & A + B = 7 \\ x^1 & -2A - 2B + C = -13 \\ x^0 & 3A - 2C = 13 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 0 \\ -2 & -2 & 1 \\ 3 & 0 & -2 \end{vmatrix} = 3$$

$$D_A = \begin{vmatrix} 7 & 1 & 0 \\ -13 & -2 & 1 \\ 13 & 0 & -2 \end{vmatrix} = 15$$

$$D_B = \begin{vmatrix} 1 & 7 & 0 \\ -2 & -13 & 1 \\ 3 & 13 & -2 \end{vmatrix} = 6$$

$$D_C = \begin{vmatrix} 1 & 1 & 7 \\ -2 & -2 & -13 \\ 3 & 0 & 13 \end{vmatrix} = 3$$

$$\underline{A = 5; \quad B = 2; \quad C = 1}$$

$$\begin{aligned} \int \frac{7x^2 - 13x + 13}{(x-2)(x^2 - 2x + 3)} dx &= \int \frac{5dx}{x-2} + \int \frac{2x+1}{x^2 - 2x + 3} dx \\ &= 5 \ln|x-2| + \int \frac{2x - 2 + 3}{x^2 - 2x + 3} dx \\ &= 5 \ln|x-2| + \int \frac{2x-2}{x^2 - 2x + 3} dx + \int \frac{3}{(x-1)^2 + 3} dx \\ &= 5 \ln|x-2| + \ln(x^2 - 2x + 3) + \frac{3}{\sqrt{2}} \tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + K \end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{1 + \sin x}$

Solution

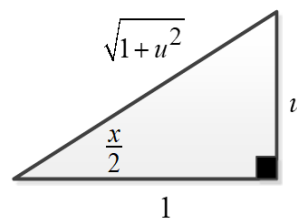
Let $u = \tan\left(\frac{x}{2}\right) \quad x = 2 \tan^{-1} u$

$$dx = \frac{2du}{1+u^2}$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

$$= 2 \frac{u}{\sqrt{1+u^2}} \frac{1}{\sqrt{1+u^2}}$$

$$= \frac{2u}{1+u^2}$$



$$\begin{aligned}
 \int \frac{dx}{1+\sin x} &= \int \frac{1}{1+\frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2} du \\
 &= \int \frac{2}{u^2+2u+1} du \\
 &= \int \frac{2}{(u+1)^2} d(u+1) \\
 &= -\frac{2}{u+1} + C \\
 &= -\frac{2}{\tan\left(\frac{x}{2}\right)+1} + C
 \end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{2+\cos x}$

Solution

Let $u = \tan\left(\frac{x}{2}\right)$ $x = 2 \tan^{-1} u$

$$dx = \frac{2du}{1+u^2}$$

$$\cos x = 2 \cos^2 \frac{x}{2} - 1$$

$$= 2 \frac{1}{1+u^2} - 1$$

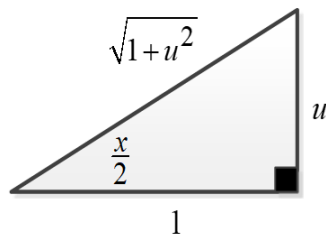
$$= \frac{1-u^2}{1+u^2}$$

$$\int \frac{dx}{2+\cos x} = \int \frac{1}{2+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} du$$

$$= 2 \int \frac{1}{u^2+3} du$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{u}{\sqrt{3}} \right) + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{1}{\sqrt{3}} \tan \frac{x}{2} \right) + C$$



$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$$

Exercise

Evaluate $\int \frac{dx}{1 - \cos x}$

Solution

$$\text{Let } u = \tan\left(\frac{x}{2}\right) \quad x = 2 \tan^{-1} u$$

$$dx = \frac{2du}{1+u^2}$$

$$\cos x = 2 \cos^2 \frac{x}{2} - 1$$

$$= 2 \frac{1}{1+u^2} - 1$$

$$= \frac{1-u^2}{1+u^2}$$

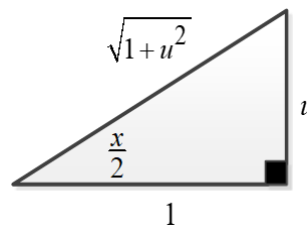
$$\int \frac{dx}{1 - \cos x} = \int \frac{1}{1 - \frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} du$$

$$= \int \frac{1}{u^2} du$$

$$= -\frac{1}{u} + C$$

$$= -\frac{1}{\tan \frac{x}{2}} + C$$

$$= -\cot \frac{x}{2} + C$$



Exercise

Evaluate $\int \frac{dx}{1 + \sin x + \cos x}$

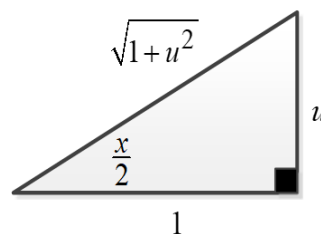
Solution

$$\text{Let } u = \tan\left(\frac{x}{2}\right) \quad x = 2 \tan^{-1} u$$

$$dx = \frac{2du}{1+u^2}$$

$$\cos x = 2 \cos^2 \frac{x}{2} - 1$$

$$= 2 \frac{1}{1+u^2} - 1$$



$$= \frac{1-u^2}{1+u^2}$$

$$\begin{aligned}\sin x &= 2 \sin \frac{x}{2} \cos \frac{x}{2} \\ &= 2 \frac{u}{\sqrt{1+u^2}} \frac{1}{\sqrt{1+u^2}} \\ &= \frac{2u}{1+u^2}\end{aligned}$$

$$\begin{aligned}\int \frac{dx}{1 + \sin x + \cos x} &= \int \frac{1}{1 + \frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} du \\ &= 2 \int \frac{1}{2+2u} du \\ &= \int \frac{1}{1+u} d(1+u) \\ &= \ln|1+u| + C \\ &= \ln\left|1 + \tan \frac{x}{2}\right| + C\end{aligned}$$

Exercise

Evaluate $\int \frac{1}{x^2 - 5x + 6} dx$

Solution

$$\frac{1}{x^2 - 5x + 6} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$Ax - 3A + Bx - 2B = 1$$

$$\rightarrow \begin{cases} A + B = 0 \\ -3A - 2B = 1 \end{cases} \rightarrow A = -1 \quad B = 1$$

$$\begin{aligned}\int \frac{1}{x^2 - 5x + 6} dx &= \int \left(\frac{-1}{x-2} + \frac{1}{x-3} \right) dx \\ &= \ln|x-3| - \ln|x-2| + C \\ &= \ln\left|\frac{x-3}{x-2}\right| + C\end{aligned}$$

Exercise

Evaluate $\int \frac{1}{x^2 - 5x + 5} dx$

Solution

$$\frac{1}{x^2 - 5x + 5} = \frac{A}{x - \frac{5 + \sqrt{5}}{2}} + \frac{B}{x - \frac{5 - \sqrt{5}}{2}}$$

$$Ax - \left(\frac{5 - \sqrt{5}}{2}\right)A + Bx - \left(\frac{5 + \sqrt{5}}{2}\right)B = 1$$

$$\begin{cases} A + B = 0 \\ -\frac{5 - \sqrt{5}}{2}A - \frac{5 + \sqrt{5}}{2}B = 1 \end{cases} \rightarrow \begin{cases} \frac{5 - \sqrt{5}}{2}A + \frac{5 - \sqrt{5}}{2}B = 0 \\ -\frac{5 - \sqrt{5}}{2}A - \frac{5 + \sqrt{5}}{2}B = 1 \end{cases}$$

$$-\sqrt{5}B = 1 \rightarrow B = -\frac{1}{\sqrt{5}} \Rightarrow A = \frac{1}{\sqrt{5}}$$

$$\begin{aligned} \int \frac{1}{x^2 - 5x + 5} dx &= \int \left(\frac{\frac{1}{\sqrt{5}}}{x - \frac{5 + \sqrt{5}}{2}} - \frac{\frac{1}{\sqrt{5}}}{x - \frac{5 - \sqrt{5}}{2}} \right) dx \\ &= \frac{\sqrt{5}}{5} \ln |2x - 5 - \sqrt{5}| - \frac{\sqrt{5}}{5} \ln |2x - 5 + \sqrt{5}| + C \end{aligned}$$

Exercise

Evaluate $\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} dx$

Solution

$$\begin{aligned} \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} &= \frac{5x^2 + 20x + 6}{x(x+1)^2} \\ &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \end{aligned}$$

$$Ax^2 + 2Ax + A + Bx^2 + Bx + Cx = 5x^2 + 20x + 6$$

$$\begin{cases} A + B = 5 \\ 2A + B + C = 20 \\ A = 6 \end{cases} \rightarrow \begin{cases} B = -1 \\ C = 9 \end{cases}$$

$$\begin{aligned} \int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} dx &= \int \left(\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2} \right) dx \\ &= 6 \ln |x| - \ln |x+1| - \frac{9}{x+1} + C \end{aligned}$$

$$= \ln \left| \frac{x^6}{x+1} - \frac{9}{x+1} + C \right|$$

Exercise

Evaluate $\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)} dx$

Solution

$$\frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)} = \frac{2x^3 - 4x - 8}{x(x-1)(x^2 + 4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx + D}{x^2 + 4}$$

$$Ax^3 - Ax^2 + 4Ax - 4A + Bx^3 + 4Bx + Cx^3 - Cx^2 + Dx^2 - Dx = 2x^3 - 4x - 8$$

$$\begin{cases} x^3 & A + B + C = 2 \\ x^2 & -A - C + D = 0 \\ x^1 & 4A + 4B - D = -4 \\ x^0 & -4A = -8 \end{cases} \rightarrow \begin{cases} B + C = 0 \\ -C + D = 2 \\ 4B - D = -12 \\ \underline{A = 2} \end{cases}$$

$$\Rightarrow \begin{cases} B + D = 2 \\ 4B - D = -12 \end{cases}$$

$$\underline{A = 2 \quad B = -2 \quad C = 2 \quad D = 4}$$

$$\begin{aligned} \int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)} dx &= \int \left(\frac{2}{x} - \frac{2}{x-1} + \frac{2x}{x^2 + 4} + \frac{4}{x^2 + 4} \right) dx & \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} \\ &= \underline{2 \ln |x| - 2 \ln |x-1| + \ln(x^2 + 4) + 2 \tan^{-1} \frac{x}{2} + C} \end{aligned}$$

Exercise

Evaluate $\int \frac{8x^3 + 13x}{(x^2 + 2)^2} dx$

Solution

$$\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 2)^2}$$

$$Ax^3 + 2Ax + Bx^2 + 2B + Cx + D = 8x^3 + 13x$$

$$\begin{cases} x^3 & A = 8 \\ x^2 & B = 0 \\ x^1 & 2A + C = 13 \\ x^0 & D = 0 \end{cases} \rightarrow \underline{C = -3}$$

$$\begin{aligned} \int \frac{8x^3 + 13x}{(x^2 + 2)^2} dx &= \int \frac{8x}{x^2 + 2} dx - \int \frac{3x}{(x^2 + 2)^2} dx \\ &= 2 \int \frac{1}{x^2 + 2} d(x^2 + 2) - \frac{3}{2} \int \frac{1}{(x^2 + 2)^2} d(x^2 + 2) \\ &= \underline{2 \ln(x^2 + 2) + \frac{3}{2} \frac{1}{x^2 + 2} + C} \end{aligned}$$

Exercise

Evaluate $\int \frac{\sin x}{\cos x + \cos^2 x} dx$

Solution

$$\frac{\sin x}{\cos x + \cos^2 x} = \frac{A}{\cos x} + \frac{B}{1 + \cos x}$$

$$A + A \cos x + B \cos x = \sin x$$

$$\begin{cases} A = \sin x \\ A + B = 0 \end{cases} \rightarrow \underline{B = -\sin x}$$

$$\begin{aligned} \int \frac{\sin x}{\cos x + \cos^2 x} dx &= \int \frac{\sin x}{\cos x} dx - \int \frac{\sin x}{1 + \cos x} dx \\ &= -\int \frac{1}{\cos x} d(\cos x) + \int \frac{1}{1 + \cos x} d(1 + \cos x) \\ &= -\ln|\cos x| + \ln|1 + \cos x| + C \\ &= \underline{\ln \left| \frac{1 + \cos x}{\cos x} \right| + C} \\ &= \underline{\ln |\sec x + 1| + C} \end{aligned}$$

Exercise

Evaluate $\int \frac{5 \cos x}{\sin^2 x + 3 \sin x - 4} dx$

Solution

$$\frac{5 \cos x}{\sin^2 x + 3 \sin x - 4} = \frac{A}{\sin x - 1} + \frac{B}{\sin x + 4}$$

$$A \sin x + 4A + B \sin x - B = 5 \cos x$$

$$\begin{cases} 4A - B = 5 \cos x \\ A + B = 0 \end{cases}$$

$$\underline{A = \cos x} \quad \underline{B = -\cos x}$$

$$\begin{aligned} \int \frac{5 \cos x}{\sin^2 x + 3 \sin x - 4} dx &= \int \frac{\cos x}{\sin x - 1} dx - \int \frac{\cos x}{\sin x + 4} dx \\ &= \int \frac{1}{\sin x - 1} d(\sin x - 1) - \int \frac{1}{\sin x + 4} d(\sin x + 4) \\ &= \ln |\sin x - 1| - \ln |\sin x + 4| + C \\ &= \ln \left| \frac{\sin x - 1}{\sin x + 4} \right| + C \end{aligned}$$

Exercise

Evaluate $\int \frac{e^x}{(e^x - 1)(e^x + 4)} dx$

Solution

$$\text{Let } u = e^x \rightarrow du = e^x dx$$

$$\int \frac{e^x}{(e^x - 1)(e^x + 4)} dx = \int \frac{du}{(u - 1)(u + 4)}$$

$$\frac{1}{(u - 1)(u + 4)} = \frac{A}{u - 1} + \frac{B}{u + 4}$$

$$Au + 4A + Bu - B = 1$$

$$\Rightarrow \begin{cases} A + B = 0 \\ 4A - B = 1 \end{cases} \rightarrow \underline{A = \frac{1}{5}, B = -\frac{1}{5}}$$

$$\begin{aligned} \int \frac{du}{(u - 1)(u + 4)} &= \frac{1}{5} \int \frac{1}{u - 1} du + \frac{4}{5} \int \frac{1}{u + 4} du \\ &= \frac{1}{5} \int \frac{1}{u - 1} d(u - 1) + \frac{4}{5} \int \frac{1}{u + 4} d(u + 4) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{5} \ln |e^x - 1| - \frac{1}{5} \ln(e^x + 4) + C \\
&= \frac{1}{5} \ln \left| \frac{e^x - 1}{e^x + 4} \right| + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{e^x}{(e^{2x} + 1)(e^x - 1)} dx$

Solution

Let $u = e^x \rightarrow du = e^x dx$

$$\int \frac{e^x}{(e^{2x} + 1)(e^x - 1)} dx = \int \frac{du}{(u^2 + 1)(u - 1)}$$

$$\frac{1}{(u^2 + 1)(u - 1)} = \frac{Au + B}{u^2 + 1} + \frac{C}{u - 1}$$

$$Au^2 - Au + Bu - B + Cu^2 + C = 1$$

$$\begin{cases} u^2 & A + C = 0 \\ u^1 & -A + B = 0 \\ u^0 & -B + C = 1 \end{cases} \rightarrow \begin{cases} B + C = 0 \\ -B + C = 1 \end{cases}$$

$$\underline{C = \frac{1}{2} \quad B = -\frac{1}{2} \quad A = -\frac{1}{2}}$$

$$\begin{aligned}
\int \frac{du}{(u^2 + 1)(u - 1)} &= -\frac{1}{2} \int \frac{u}{u^2 + 1} du - \frac{1}{2} \int \frac{du}{u^2 + 1} + \frac{1}{2} \int \frac{du}{u - 1} \\
&= -\frac{1}{4} \int \frac{1}{u^2 + 1} d(u^2 + 1) - \frac{1}{2} \arctan u + \frac{1}{2} \ln |u - 1| \\
&= -\frac{1}{4} \ln(e^{2x} + 1) - \frac{1}{2} \arctan e^x + \frac{1}{2} \ln |e^x - 1| + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{\sqrt{x}}{x - 4} dx$

Solution

Let $u = \sqrt{x} \rightarrow du = \frac{1}{2\sqrt{x}} dx \Rightarrow 2udu = dx$

$$\begin{aligned}
\int \frac{\sqrt{x}}{x-4} dx &= \int \frac{u}{u^2-4} 2u du \\
&= \int \frac{2u^2}{u^2-4} du \\
&= \int \left(2 + \frac{8}{u^2-4} \right) du \\
&\quad \frac{8}{u^2-4} = \frac{A}{u-2} + \frac{B}{u+2} \\
&\quad Au + 2A + Bu - 2B = 8 \\
&\quad \rightarrow \begin{cases} A+B=0 \\ 2A-2B=8 \end{cases} \Rightarrow \underline{A=2 \quad B=-2} \\
&= \int \left(2 + \frac{2}{u-2} - \frac{2}{u+2} \right) du \\
&= 2\sqrt{x} + 2\ln|\sqrt{x}-2| - 2\ln|\sqrt{x}+2| + C \\
&= \underline{2\sqrt{x} + 2\ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right| + C}
\end{aligned}$$

Exercise

Evaluate $\int \frac{1}{\sqrt{x}-\sqrt[3]{x}} dx$

Solution

Let $u = x^{1/6} \rightarrow u^6 = x \rightarrow 6u^5 du = dx$

$u^2 = x^{1/3} \quad u^3 = x^{1/2}$

$$\begin{aligned}
\int \frac{1}{\sqrt{x}-\sqrt[3]{x}} dx &= \int \frac{6u^5}{u^3-u^2} du \\
&= \int \frac{6u^3}{u-1} du \\
&= \int \left(6u^2 + 6u + 6 + \frac{6}{u-1} \right) du \\
&= 2u^3 + 3u^2 + 6u + 6\ln|u-1| + C \\
&= \underline{2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln|\sqrt[6]{x}-1| + C}
\end{aligned}$$

$$\begin{array}{r}
\begin{array}{r}
6u^2+6u+6 \\
\hline
6u^3 \\
-6u^3+6u^2 \\
\hline
6u^2 \\
-6u^2+6u \\
\hline
6u \\
-6u+6 \\
\hline
6
\end{array} \\
u-1 \overline{) }
\end{array}$$

Exercise

Evaluate $\int \frac{1}{x^2 - 9} dx$

Solution

$$\frac{1}{x^2 - 9} = \frac{A}{x - 3} + \frac{B}{x + 3}$$

$$Ax + 3A + Bx - 3B = 1$$

$$\Rightarrow \begin{cases} A + B = 0 \\ 3A - 3B = 1 \end{cases} \rightarrow \underline{A = \frac{1}{6} \quad B = -\frac{1}{6}}$$

$$\begin{aligned} \int \frac{1}{x^2 - 9} dx &= \frac{1}{6} \int \frac{1}{x - 3} dx - \frac{1}{6} \int \frac{1}{x + 3} dx \\ &= \frac{1}{6} \ln|x - 3| - \frac{1}{6} \ln|x + 3| + C \\ &= \underline{\frac{1}{6} \ln \left| \frac{x - 3}{x + 3} \right| + C} \end{aligned}$$

Exercise

Evaluate $\int \frac{2}{9x^2 - 1} dx$

Solution

$$\frac{2}{9x^2 - 1} = \frac{A}{3x - 1} + \frac{B}{3x + 1}$$

$$3Ax + A + 3Bx - B = 2$$

$$\Rightarrow \begin{cases} 3A + 3B = 0 \\ A - B = 2 \end{cases} \rightarrow \underline{A = 1 \quad B = -1}$$

$$\begin{aligned} \int \frac{2}{9x^2 - 1} dx &= \int \frac{1}{3x - 1} dx - \int \frac{1}{3x + 1} dx \\ &= \frac{1}{3} \ln|3x - 1| - \frac{1}{3} \ln|3x + 1| + C \\ &= \underline{\frac{1}{3} \ln \left| \frac{3x - 1}{3x + 1} \right| + C} \end{aligned}$$

Exercise

Evaluate $\int \frac{5}{x^2 + 3x - 4} dx$

Solution

$$\frac{5}{x^2 + 3x - 4} = \frac{A}{x-1} + \frac{B}{x+4}$$

$$Ax + 4A + Bx - B = 5$$

$$\Rightarrow \begin{cases} A + B = 0 \\ 4A - B = 5 \end{cases} \rightarrow \underline{A = 1 \quad B = -1}$$

$$\begin{aligned} \int \frac{5}{x^2 + 3x - 4} dx &= \int \frac{1}{x-1} dx - \int \frac{1}{x+4} dx \\ &= \ln|x-1| - \ln|x+4| + C \\ &= \ln \left| \frac{x-1}{x+4} \right| + C \end{aligned}$$

Exercise

Evaluate $\int \frac{3-x}{3x^2 - 2x - 1} dx$

Solution

$$\frac{3-x}{3x^2 - 2x - 1} = \frac{A}{x-1} + \frac{B}{3x+1}$$

$$3Ax + A + Bx - B = 3 - x$$

$$\Rightarrow \begin{cases} 3A + B = -1 \\ A - B = 3 \end{cases} \rightarrow \underline{A = \frac{1}{2} \quad B = -\frac{5}{2}}$$

$$\begin{aligned} \int \frac{3-x}{3x^2 - 2x - 1} dx &= \frac{1}{2} \int \frac{1}{x-1} dx - \frac{5}{2} \int \frac{1}{3x+1} dx \\ &= \frac{1}{2} \ln|x-1| - \frac{5}{6} \ln|3x+1| + C \end{aligned}$$

Exercise

Evaluate $\int \frac{x^2 + 12x + 12}{x^3 - 4x} dx$

Solution

$$\frac{x^2 + 12x + 12}{x^3 - 4x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

$$Ax^2 - 4A + Bx^2 + 2Bx + Cx^2 - 2Cx = x^2 + 12x + 12$$

$$\begin{cases} x^2 & A + B + C = 1 \\ x^1 & 2B - 2C = 12 \rightarrow A = -3 \quad B = 5 \quad C = -1 \\ x^0 & -4A = 12 \end{cases}$$

$$\int \frac{x^2 + 12x + 12}{x^3 - 4x} dx = -\frac{3}{x} + \frac{5}{x-2} - \frac{1}{x+2}$$

$$= \underline{-3 \ln|x| + 5 \ln|x-2| - \ln|x+2| + C}$$

Exercise

Evaluate $\int \frac{x^3 - x + 3}{x^2 + x - 2} dx$

Solution

$$\frac{x^3 - x + 3}{x^2 + x - 2} = x - 1 + \frac{2x + 1}{x^2 + x - 2}$$

$$\frac{2x + 1}{x^2 + x - 2} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$Ax + 2A + Bx - B = 2x + 1$$

$$\Rightarrow \begin{cases} A + B = 2 \\ 2A - B = 1 \end{cases} \rightarrow \underline{A = 1 \quad B = 1}$$

$$\int \frac{x^3 - x + 3}{x^2 + x - 2} dx = \int \left(x - 1 + \frac{1}{x-1} + \frac{1}{x+2} \right) dx$$

$$= \underline{\frac{1}{2}x^2 - x + \ln|x-1| + \ln|x+2| + C}$$

$$x^2 + x - 2 \left(\frac{x-1}{x^3 - x + 3} \right)$$

$$\frac{-x^3 - x^2 + 2x}{-x^2 + x + 3}$$

$$\frac{x^2 + x - 2}{2x - 1}$$

Exercise

Evaluate $\int \frac{5x - 2}{(x-2)^2} dx$

Solution

$$\frac{5x - 2}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$$

$$Ax - 2A + B = 5x - 2$$

$$\Rightarrow \begin{cases} A = 5 \\ -2A + B = -2 \end{cases} \rightarrow \underline{B = 8}$$

$$\int \frac{5x - 2}{(x-2)^2} dx = \frac{5}{x-2} + \frac{8}{(x-2)^2}$$

$$\underline{= 5 \ln|x-2| - \frac{8}{x-2} + C}$$

Exercise

Evaluate $\int \frac{2x^3 - 4x^2 - 15x + 4}{x^2 - 2x - 8} dx$

Solution

$$\int \frac{2x^3 - 4x^2 - 15x + 4}{x^2 - 2x - 8} dx = \int 2x dx + \int \frac{x+4}{x^2 - 2x - 8} dx$$

$$\frac{x+4}{x^2 - 2x - 8} = \frac{A}{x-4} + \frac{B}{x+2}$$

$$Ax + 2A + Bx - 4B = x + 4$$

$$\Rightarrow \begin{cases} A+B=1 \\ 2A-4B=4 \end{cases} \rightarrow \underline{A=\frac{4}{3} \quad B=-\frac{1}{3}}$$

$$\begin{aligned} \int \frac{2x^3 - 4x^2 - 15x + 4}{x^2 - 2x - 8} dx &= x^2 + \frac{4}{3} \int \frac{1}{x-4} dx - \frac{1}{3} \int \frac{1}{x+2} dx \\ &= \underline{x^2 + \frac{4}{3} \ln|x-4| - \frac{1}{3} \ln|x+2| + C} \end{aligned}$$

$$x^2 - 2x - 8 \overline{\left) \begin{array}{r} 2x^3 - 4x^2 - 15x + 4 \\ 2x^3 - 4x^2 - 16x \\ \hline 16x + 4 \end{array} \right.}$$

Exercise

Evaluate $\int \frac{x+2}{x^2 + 5x} dx$

Solution

$$\frac{x+2}{x^2 + 5x} = \frac{A}{x} + \frac{B}{x+5}$$

$$Ax + 5A + Bx = x + 2$$

$$\Rightarrow \begin{cases} A+B=1 \\ 5A=2 \end{cases} \rightarrow \underline{A=\frac{2}{5} \quad B=\frac{3}{5}}$$

$$\begin{aligned} \int \frac{x+2}{x^2 + 5x} dx &= \frac{2}{5} \int \frac{1}{x} dx + \frac{3}{5} \int \frac{1}{x+5} dx \\ &= \underline{\frac{2}{5} \ln|x| + \frac{3}{5} \ln|x+5| + C} \end{aligned}$$

Exercise

Evaluate $\int \frac{\sec^2 x}{\tan^2 x + 5 \tan x + 6} dx$

Solution

Let $u = \tan x \quad du = \sec^2 x \, dx$

$$\int \frac{\sec^2 x}{\tan^2 x + 5 \tan x + 6} dx = \int \frac{1}{u^2 + 5u + 6} du$$

$$\frac{1}{u^2 + 5u + 6} = \frac{1}{(u+2)(u+3)} = \frac{A}{u+2} + \frac{B}{u+3}$$

$$1 = Au + 3A + Bu + 2B$$

$$\begin{cases} A + B = 0 \\ 3A + 2B = 1 \end{cases} \rightarrow A = 1 \quad B = -1$$

$$\begin{aligned} \int \frac{\sec^2 x}{\tan^2 x + 5 \tan x + 6} dx &= \int \frac{1}{u+2} du - \int \frac{1}{u+3} du \\ &= \ln|\tan x + 2| - \ln|\tan x + 3| + C \\ &= \ln \left| \frac{\tan x + 2}{\tan x + 3} \right| + C \end{aligned}$$

Exercise

Evaluate $\int \frac{\sec^2 x}{\tan x(\tan x + 1)} dx$

Solution

Let $u = \tan x \quad du = \sec^2 x \, dx$

$$\int \frac{\sec^2 x}{\tan x(\tan x + 1)} dx = \int \frac{du}{u(u+1)}$$

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

$$1 = Au + A + Bu$$

$$\begin{cases} A + B = 0 \\ A = 1 \end{cases} \rightarrow B = -1$$

$$\begin{aligned} \int \frac{\sec^2 x}{\tan x(\tan x + 1)} dx &= \int \frac{du}{u} - \int \frac{du}{u+1} \\ &= \ln|\tan x| - \ln|\tan x + 1| + C \\ &= \ln \left| \frac{\tan x}{\tan x + 1} \right| + C \end{aligned}$$

Exercise

Evaluate $\int \frac{x \, dx}{x^2 + 4x + 3}$

Solution

$$\frac{x}{x^2 + 4x + 3} = \frac{A}{x+1} + \frac{B}{x+3}$$

$$x = Ax + 3A + Bx + B$$

$$\begin{cases} \textcolor{red}{x} & A + B = 1 \\ \textcolor{red}{x^0} & 3A + B = 0 \end{cases} \rightarrow A = -\frac{1}{2} \quad B = \frac{3}{2}$$

$$\begin{aligned} \int \frac{x \, dx}{x^2 + 4x + 3} &= -\frac{1}{2} \int \frac{dx}{x+1} + \frac{3}{2} \int \frac{dx}{x+3} \\ &= -\frac{1}{2} \ln|x+1| + \frac{3}{2} \ln|x+3| + C \\ &= \frac{1}{2} (3 \ln|x+3| - \ln|x+1|) + C \\ &= \frac{1}{2} \ln \left| \frac{(x+3)^3}{x+1} \right| + C \\ &= \ln \sqrt{\left| \frac{(x+3)^3}{x+1} \right|} + C \end{aligned}$$

Exercise

Evaluate $\int \frac{x+1}{x^2(x-1)} \, dx$

Solution

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$x+1 = Ax^2 - Ax + Bx - B + Cx^2$$

$$\textcolor{red}{x^2} \quad A + C = 0 \quad \underline{C = 2}$$

$$\textcolor{red}{x} \quad -A + B = 1 \quad \underline{A = -2}$$

$$\textcolor{red}{x^0} \quad -B = 1 \quad \rightarrow \underline{B = -1}$$

$$\int \frac{x+1}{x^2(x-1)} \, dx = -2 \int \frac{dx}{x} - \int \frac{dx}{x^2} + 2 \int \frac{dx}{x-1}$$

$$\underline{= -2 \ln|x| + \frac{1}{x} + 2 \ln|x-1| + C}$$

Exercise

Evaluate $\int \frac{2x^3 + x^2 - 21x + 24}{x^2 + 2x - 8} dx$

Solution

$$\begin{array}{r} x^2 - 2x - 8 \overline{) 2x^3 + x^2 - 21x + 24} \\ \underline{2x^3 - 4x^2 - 16x} \\ -3x^2 - 5x + 24 \\ \underline{-3x^2 - 6x + 24} \\ x \end{array}$$

$$\int \frac{2x^3 + x^2 - 21x + 24}{x^2 + 2x - 8} dx = \int \left(2x - 3 + \frac{x}{x^2 + 2x - 8} \right) dx$$

$$\frac{x}{x^2 + 2x - 8} = \frac{A}{x+4} + \frac{B}{x-2}$$

$$x = Ax - 2A + Bx + 4B$$

$$\begin{cases} \textcolor{red}{x} & A + B = 1 \\ \textcolor{red}{x}^0 & -2A + 4B = 0 \end{cases} \rightarrow B = \frac{1}{3} \quad A = \frac{2}{3}$$

$$\begin{aligned} \int \frac{2x^3 + x^2 - 21x + 24}{x^2 + 2x - 8} dx &= \int (2x - 3) dx + \frac{2}{3} \int \frac{dx}{x+4} + \frac{1}{3} \int \frac{dx}{x-2} \\ &= x^2 - 3x + \frac{2}{3} \ln|x+4| + \frac{1}{3} \ln|x-2| + C \end{aligned}$$

Exercise

Evaluate $\int \frac{8x+5}{2x^2+3x+1} dx$

Solution

$$\frac{8x+5}{2x^2+3x+1} = \frac{A}{2x+1} + \frac{B}{x+1}$$

$$8x+5 = Ax + A + 2Bx + B$$

$$\begin{cases} \textcolor{red}{x} & A + 2B = 8 \\ \textcolor{red}{x}^0 & A + B = 5 \end{cases} \rightarrow B = 3 \quad A = 2$$

$$\begin{aligned}
\int \frac{8x+5}{2x^2+3x+1} dx &= \int \frac{2}{2x+1} dx + \int \frac{3}{x+1} dx \\
&= \int \frac{1}{2x+1} d(2x+1) + 3 \int \frac{1}{x+1} d(x+1) \\
&= \ln|2x+1| + 3\ln|x+1| + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{3x^3+4x^2+6x}{(x+1)^2(x^2+4)} dx$

Solution

$$\frac{3x^3+4x^2+6x}{(x+1)^2(x^2+4)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+4}$$

$$\begin{aligned}
3x^3+4x^2+6x &= A(x+1)(x^2+4) + Bx^2+4B + (Cx+D)(x^2+2x+1) \\
&= Ax^3+4Ax+Ax^2+4A+Bx^2+4B+Cx^3+2Cx^2+Cx+Dx^2+2Dx+D
\end{aligned}$$

$$\begin{cases}
x^3 & A+C=3 \quad \rightarrow A=3-C \\
x^2 & A+B+2C+D=4 \\
x & 4A+C+2D=6 \\
x^0 & 4A+4B+D=0
\end{cases}$$

$$\begin{cases}
B+C+D=1 \\
-3C+2D=-6 \\
4B-4C+D=-12
\end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & 2 \\ 4 & -4 & 1 \end{vmatrix} = 25 \quad \Delta_B = \begin{vmatrix} 1 & 1 & 1 \\ -6 & -3 & 2 \\ -12 & -4 & 1 \end{vmatrix} = -25 \quad \Delta_C = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -6 & 2 \\ 4 & -12 & 1 \end{vmatrix} = 50$$

$$B = \frac{-25}{25} = -1 \quad C = \frac{50}{25} = 2 \quad \rightarrow \quad A = 3 - 2 = 1$$

$$2D = -6 + 6 \quad \rightarrow \quad D = 0$$

$$\begin{aligned}
\int \frac{3x^3+4x^2+6x}{(x+1)^2(x^2+4)} dx &= \int \frac{1}{x+1} dx - \int \frac{1}{(x+1)^2} dx + \int \frac{2x}{x^2+4} dx \\
&= \int \frac{1}{x+1} d(x+1) - \int \frac{1}{(x+1)^2} d(x+1) + \int \frac{1}{x^2+4} d(x^2+4)
\end{aligned}$$

$$\underline{= \ln|x+1| + \frac{1}{x+1} + \ln(x^2+4) + K}$$

Exercise

Evaluate $\int \frac{x^2-4}{x^2+4} dx$

Solution

$$x^2+4 \overline{\left. \begin{array}{l} 1 \\ x^2-4 \\ x^2+4 \\ 8 \end{array} \right\}}$$

$$\int \frac{x^2-4}{x^2+4} dx = \int \left(1 + \frac{8}{x^2+4} \right) dx$$

$$\underline{= x + 8 \arctan \frac{x}{2} + C}$$

Exercise

Evaluate $\int \frac{dx}{x^2-2x-15}$

Solution

$$\frac{1}{x^2-2x-15} = \frac{A}{x-5} + \frac{B}{x+3}$$

$$1 = Ax + 3A + Bx - 5B$$

$$\left\{ \begin{array}{ll} \textcolor{red}{x} & A+B=0 \quad \rightarrow B=-A \\ \textcolor{red}{x^0} & 3A-5B=1 \quad \rightarrow 8A=1 \Rightarrow \underline{A=\frac{1}{8}} \end{array} \right.$$

$$\underline{B=-\frac{1}{8}}$$

$$\int \frac{dx}{x^2-2x-15} = \frac{1}{8} \int \frac{dx}{x-5} - \frac{1}{8} \int \frac{dx}{x+3}$$

$$= \frac{1}{8} \ln|x-5| - \frac{1}{8} \ln|x+3| + C$$

$$\underline{= \frac{1}{8} \ln \left| \frac{x-5}{x+3} \right| + C}$$

Exercise

Evaluate $\int \frac{3x^2 + x - 3}{x^2 - 1} dx$

Solution

$$x^2 - 1 \overline{) \begin{array}{r} 3x^2 + x - 3 \\ \underline{3x^2 - 3} \\ x \end{array}}$$

$$\begin{aligned} \int \frac{3x^2 + x - 3}{x^2 - 1} dx &= \int \left(3 + \frac{x}{x^2 - 1} \right) dx \\ &= 3x + \frac{1}{2} \int \frac{1}{x^2 - 1} d(x^2 - 1) \\ &= \underline{3x + \frac{1}{2} \ln |x^2 - 1| + C} \end{aligned}$$

Exercise

Evaluate $\int \frac{2x^2 - 4x}{x^2 - 4} dx$

Solution

$$x^2 - 4 \overline{) \begin{array}{r} 2x^2 - 4x \\ \underline{2x^2 - 8} \\ -4x + 8 \end{array}}$$

$$\begin{aligned} \int \frac{2x^2 - 4x}{x^2 - 4} dx &= \int \left(2 - 4 \frac{x - 2}{x^2 - 4} \right) dx \\ &= \int \left(2 - 4 \frac{x - 2}{(x - 2)(x + 2)} \right) dx \\ &= \int \left(2 - \frac{4}{x + 2} \right) dx \\ &= \underline{2x + 4 \ln |x + 2| + C} \end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{x^3 - 2x^2}$

Solution

$$\frac{1}{x^3 - 2x^2} = \frac{1}{x^2(x-2)}$$
$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$1 = Ax(x-2) + Bx - 2B + Cx^2$$

$$x^2 \quad A + C = 0 \quad \underline{C = \frac{1}{4}}$$

$$x \quad -2A + B = 0 \quad \underline{A = -\frac{1}{4}}$$

$$x^0 \quad -2B = 1 \quad \rightarrow \underline{B = -\frac{1}{2}}$$

$$\int \frac{dx}{x^3 - 2x^2} = -\frac{1}{4} \int \frac{1}{x} dx - \frac{1}{2} \int \frac{1}{x^2} dx + \frac{1}{4} \int \frac{1}{x-2} dx$$
$$= -\frac{1}{4} \ln|x| + \frac{1}{2} \frac{1}{x} + \frac{1}{4} \ln|x-2| + K$$
$$= \underline{\frac{1}{2x} + \frac{1}{4} \ln \left| \frac{x-2}{x} \right| + K}$$

Exercise

Evaluate $\int \frac{dx}{x^2 - x - 2}$

Solution

$$\frac{1}{x^2 - x - 2} = \frac{A}{x+1} + \frac{B}{x-2}$$

$$1 = Ax - 2A + Bx + B$$

$$\begin{cases} x & A + B = 0 \\ x^0 & -2A + B = 1 \end{cases}$$

$$\underline{A = -\frac{1}{3} \quad B = \frac{1}{3}}$$

$$\int \frac{dx}{x^2 - x - 2} = -\frac{1}{3} \int \frac{dx}{x+1} + \frac{1}{3} \int \frac{dx}{x-2}$$

$$\begin{aligned}
&= -\frac{1}{3} \ln|x+1| + \frac{1}{3} \ln|x-2| + K \\
&= \frac{1}{3} \ln \left| \frac{x-2}{x+1} \right| + C
\end{aligned}$$

Exercise

Evaluate $\int \frac{4x^2 + 13x - 9}{x^3 + 2x^2 - 3x} dx$

Solution

$$\begin{aligned}
\frac{4x^2 + 13x - 9}{x^3 + 2x^2 - 3x} &= \frac{4x^2 + 13x - 9}{x(x^2 + 2x - 3)} \\
&= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+3}
\end{aligned}$$

$$4x^2 + 13x - 9 = A(x^2 + 2x - 3) + Bx(x+3) + Cx(x-1)$$

$$x^2 \quad A + B + C = 4$$

$$x \quad 2A + 3B - C = 13$$

$$x^0 \quad -3A = -9 \quad \rightarrow \underline{A = 3}$$

$$\begin{cases} B + C = 1 \\ 3B - C = 7 \end{cases} \rightarrow \underline{B = 2 \quad C = -1}$$

$$\begin{aligned}
\int \frac{4x^2 + 13x - 9}{x^3 + 2x^2 - 3x} dx &= \int \left(\frac{3}{x} + \frac{2}{x-1} - \frac{1}{x+3} \right) dx \\
&= \underline{3 \ln|x| + 2 \ln|x-1| - \ln|x+3| + K}
\end{aligned}$$

Exercise

Evaluate $\int \frac{3x^3 - 18x^2 + 29x - 4}{(x+1)(x-2)^3} dx$

Solution

$$\frac{3x^3 - 18x^2 + 29x - 4}{(x+1)(x-2)^3} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

$$3x^3 - 18x^2 + 29x - 4 = A(x-2)^3 + B(x+1)(x-2)^2 + C(x+1)(x-2) + Dx + D$$

$$= A(x^3 - 6x^2 + 12x - 8) + B(x+1)(x^2 - 4x + 4) + C(x^2 - x - 2) + Dx + D$$

$$x^3 \quad A + B = 3 \quad \rightarrow A = 3 - B$$

$$x^2 \quad -6A - 3B + C = -18$$

$$x \quad 12A - C + D = 29$$

$$x^0 \quad -8A + 4B - 2C + D = -4$$

$$\begin{cases} -18 + 6B - 3B + C = -18 \\ 36 - 12B - C + D = 29 \\ -24 + 8B + 4B - 2C + D = -4 \end{cases}$$

$$\begin{cases} 3B + C = 0 \\ -12B - C + D = -7 \\ 12B - 2C + D = 20 \end{cases}$$

$$\Delta = \begin{vmatrix} 3 & 1 & 0 \\ -12 & -1 & 1 \\ 12 & -2 & 1 \end{vmatrix} = 27 \quad \Delta_B = \begin{vmatrix} 0 & 1 & 0 \\ -7 & -1 & 1 \\ 20 & -2 & 1 \end{vmatrix} = 27$$

$$B = \frac{27}{27} = \underline{1}$$

$$C = -3B = \underline{-3}$$

$$D = -7 + 12 - 3 = \underline{2}$$

$$A = 3 - 1 = \underline{2}$$

$$\begin{aligned} \int \frac{3x^3 - 18x^2 + 29x - 4}{(x+1)(x-2)^3} dx &= \int \frac{2}{x+1} dx + \int \frac{1}{x-2} dx - 3 \int \frac{1}{(x-2)^2} dx + 2 \int \frac{1}{(x-2)^3} dx \\ &= 2 \int \frac{d(x+1)}{x+1} + \int \frac{d(x-2)}{x-2} - 3 \int \frac{d(x-2)}{(x-2)^2} + 2 \int (x-2)^{-3} d(x-2) \\ &= 2 \ln|x+1| + \ln|x-2| + \frac{3}{x-2} - \frac{1}{(x-2)^2} + K \end{aligned}$$

Exercise

Evaluate $\int \frac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4} dx$

Solution

$$\begin{aligned} 2x^3 - x^2 + 8x - 4 &= x^2(2x-1) + 4(2x-1) \\ &= (2x-1)(x^2+4) \end{aligned}$$

$$\frac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4} = \frac{A}{2x-1} + \frac{Bx+C}{x^2+4}$$

$$x^2 - x - 21 = Ax^2 + 4A + 2Bx^2 - Bx + 2Cx - C$$

$$x^2 \quad A + 2B = 1 \quad A = 1 - 2B$$

$$x \quad -B + 2C = -1 \quad C = \frac{1}{2}(B - 1)$$

$$x^0 \quad 4A - C = -21 \quad (1)$$

$$(1) \quad 4 - 8B - \frac{1}{2}B + \frac{1}{2} = -21$$

$$-\frac{17}{2}B = -21 - \frac{9}{2}$$

$$\frac{17}{2}B = \frac{51}{2} \rightarrow \underline{B=3}$$

$$C = \frac{1}{2}(3 - 1) = \underline{1}$$

$$A = 1 - 6 = \underline{-5}$$

$$\begin{aligned} \int \frac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4} dx &= \int \frac{5}{2x-1} dx + \int \frac{3x}{x^2+4} dx + \int \frac{1}{x^2+4} dx \\ &= \frac{5}{2} \int \frac{1}{2x-1} d(2x-1) + \frac{3}{2} \int \frac{1}{x^2+4} d(x^2+4) + \int \frac{1}{x^2+4} dx \\ &= \underline{\underline{\frac{5}{2} \ln|2x-1| + \frac{3}{2} \ln(x^2+4) + \frac{1}{2} \arctan\left(\frac{x}{2}\right) + K}} \end{aligned}$$

Exercise

Evaluate $\int \frac{5x^3 - 3x^2 + 7x - 3}{(x^2 + 1)^2} dx$

Solution

$$\frac{5x^3 - 3x^2 + 7x - 3}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}$$

$$5x^3 - 3x^2 + 7x - 3 = Ax^3 + Ax + Bx^2 + B + Cx + D$$

$$x^3 \quad \underline{A=5}$$

$$x^2 \quad \underline{B=-3}$$

$$x \quad A + C = 7 \rightarrow \underline{C=2}$$

$$x^0 \quad B + D = -3 \rightarrow \underline{D=0}$$

$$\begin{aligned}
\int \frac{5x^3 - 3x^2 + 7x - 3}{(x^2 + 1)^2} dx &= \int \frac{5x - 3}{x^2 + 1} dx + \int \frac{2x}{(x^2 + 1)^2} dx \\
&= \frac{5}{2} \int \frac{1}{x^2 + 1} d(x^2 + 1) - 3 \int \frac{1}{x^2 + 1} dx + \int \frac{1}{(x^2 + 1)^2} d(x^2 + 1) \\
&= \frac{5}{2} \ln(x^2 + 1) - 3 \arctan(x) - \frac{1}{x^2 + 1} + K
\end{aligned}$$

Exercise

Evaluate $\int \frac{2x^4 - 2x^3 + 6x^2 - 5x + 1}{x^3 - x^2 + x - 1} dx$

Solution

$$\begin{array}{r}
2x \\
\hline
x^3 - x^2 + x - 1 \overline{) 2x^4 - 2x^3 + 6x^2 - 5x + 1} \\
\underline{2x^4 - 2x^3 - 2x^2 + 2x} \\
8x^2 - 7x + 1
\end{array}$$

$$\frac{2x^4 - 2x^3 + 6x^2 - 5x + 1}{x^3 - x^2 + x - 1} = 2x + \frac{8x^2 - 7x + 1}{x^3 - x^2 + x - 1}$$

$$\begin{aligned}
x^3 - x^2 + x - 1 &= x^2(x - 1) + (x - 1) \\
&= (x - 1)(x^2 + 1)
\end{aligned}$$

$$\frac{8x^2 - 7x + 1}{x^3 - x^2 + x - 1} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 1}$$

$$8x^2 - 7x + 1 = Ax^2 + A + Bx^2 - Bx + Cx - C$$

$$x^2 \quad A + B = 8 \quad A = 8 - B$$

$$x \quad -B + C = -7 \quad C = B - 7$$

$$x^0 \quad A - C = 1 \quad (1)$$

$$(1) \rightarrow 8 - B - B + 7 = 1 \Rightarrow B = 7$$

$$A = 8 - 7 = 1$$

$$C = 7 - 7 = 0$$

$$\int \frac{2x^4 - 2x^3 + 6x^2 - 5x + 1}{x^3 - x^2 + x - 1} dx = \int 2x dx + \int \frac{1}{x - 1} dx + \int \frac{7x}{x^2 + 1} dx$$

$$\begin{aligned}
&= x^2 + \ln|x-1| + \frac{7}{2} \int \frac{1}{x^2+1} d(x^2+1) \\
&= x^2 + \ln|x-1| + \frac{7}{2} \ln(x^2+1) + K
\end{aligned}$$

Exercise

Evaluate $\int \frac{81}{x^3 - 9x^2} dx$

Solution

$$\begin{aligned}
\frac{81}{x^3 - 9x^2} &= \frac{81}{x^2(x-9)} \\
&= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-9}
\end{aligned}$$

$$81 = Ax^2 - 9Ax + Bx - 9B + Cx^2$$

$$x^2 \quad A + C = 0 \quad \rightarrow \quad \underline{C = 1}$$

$$x \quad -9A + B = 0 \quad \rightarrow \quad \underline{A = -1}$$

$$x^0 \quad -9B = 81 \quad \rightarrow \quad \underline{B = -9}$$

$$\begin{aligned}
\int \frac{81}{x^3 - 9x^2} dx &= \int \left(-\frac{1}{x} - \frac{9}{x^2} + \frac{1}{x-9} \right) dx \\
&= -\ln|x| + \frac{9}{x} + \ln|x-9| + K \\
&= \underline{\underline{\frac{9}{x} + \ln\left|\frac{x-9}{x}\right| + K}}
\end{aligned}$$

Exercise

Evaluate $\int \frac{10x}{x^2 - 2x - 24} dx$

Solution

$$\frac{10}{x^2 - 2x - 24} = \frac{A}{x-6} + \frac{B}{x+4}$$

$$10 = Ax + 4A + Bx - 6B$$

$$x \quad A + B = 10$$

$$x^0 \quad 4A - 6B = 0$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 4 & -6 \end{vmatrix} = -10 \quad \Delta_A = \begin{vmatrix} 10 & 1 \\ 0 & -6 \end{vmatrix} = -60$$

$$\underline{A = 6 \quad B = 4}$$

$$\int \frac{10x}{x^2 - 2x - 24} dx = \int \left(\frac{6}{x-6} + \frac{4}{x+4} \right) dx$$

$$\underline{= 6 \ln|x-6| + 4 \ln|x+4| + C}$$

Exercise

Evaluate $\int \frac{x+1}{x^2(x^2+4)} dx$

Solution

$$\frac{x+1}{x^2(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4}$$

$$x+1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Dx^2$$

$$x^3 \quad A + C = 0 \quad \rightarrow \quad \underline{C = -\frac{1}{4}}$$

$$x^2 \quad B + D = 0 \quad \rightarrow \quad \underline{D = -\frac{1}{4}}$$

$$x \quad 4A = 1 \quad \rightarrow \quad \underline{A = \frac{1}{4}}$$

$$x^0 \quad 4B = 1 \quad \rightarrow \quad \underline{B = \frac{1}{4}}$$

$$\int \frac{x+1}{x^2(x^2+4)} dx = \frac{1}{4} \int \frac{1}{x} dx + \frac{1}{4} \int \frac{1}{x^2} dx - \frac{1}{4} \int \frac{x}{x^2+4} dx - \frac{1}{4} \int \frac{1}{x^2+4} dx$$

$$= \frac{1}{4} \ln|x| - \frac{1}{4} \frac{1}{x} - \frac{1}{8} \int \frac{x}{x^2+4} d(x^2+4) - \frac{1}{8} \arctan\left(\frac{x}{2}\right) + K$$

$$\underline{= \frac{1}{4} \ln|x| - \frac{1}{4x} - \frac{1}{8} \ln(x^2+4) - \frac{1}{8} \arctan\left(\frac{x}{2}\right) + K}$$

Exercise

Evaluate $\int \frac{1+x^2}{(x+1)^3} dx$

Solution

$$\frac{1+x^2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$x^2+1 = A(x+1)^2 + Bx + B + C$$

$$\begin{aligned} x^2 \quad & \underline{A=1} \\ x \quad & 2A+B=0 \rightarrow \underline{B=-2} \\ x^0 \quad & A+B+C=1 \rightarrow \underline{C=2} \end{aligned}$$

$$\begin{aligned} \int \frac{1+x^2}{(x+1)^3} dx &= \int \frac{1}{x+1} dx - \int \frac{2}{(x+1)^2} dx + \int \frac{2}{(x+1)^3} dx \\ &= \int \frac{1}{x+1} d(x+1) - 2 \int \frac{1}{(x+1)^2} d(x+1) + 2 \int (x+1)^{-3} d(x+1) \\ &= \ln|x+1| + \frac{2}{x+1} - \frac{1}{(x+1)^2} + K \end{aligned}$$

Exercise

Evaluate $\int \frac{6}{x^2-1} dx$

Solution

$$\frac{6}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$6 = Ax + A + Bx - B$$

$$x \quad A+B=0$$

$$x^0 \quad A-B=6$$

$$2A=6 \rightarrow \underline{A=3} \quad \underline{B=-3}$$

$$\begin{aligned} \int \frac{6}{x^2-1} dx &= \int \frac{3}{x-1} dx - \int \frac{3}{x+1} dx \\ &= 3 \ln|x-1| - 3 \ln|x+1| + C \\ &= 3 \ln \left| \frac{x-1}{x+1} \right| + C \end{aligned}$$

Exercise

Evaluate $\int \frac{21x^2}{x^3-x^2-12x} dx$

Solution

$$\int \frac{21x^2}{x^3-x^2-12x} dx = \int \frac{21x}{x^2-x-12} dx$$

$$\begin{aligned}\frac{21x^2}{x^3 - x^2 - 12x} &= \frac{21x^2}{x(x^2 - x - 12)} \\ &= \frac{21x^2}{x(x+3)(x-4)}\end{aligned}$$

$$\frac{21x}{x^2 - x - 12} = \frac{A}{x+3} + \frac{B}{x-4}$$

$$21x = Ax - 4A + Bx + 3B$$

$$x \quad A + B = 21$$

$$x^0 \quad -4A + 3B = 0$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ -4 & 3 \end{vmatrix} = 7 \quad \Delta_A = \begin{vmatrix} 21 & 1 \\ 0 & 3 \end{vmatrix} = 63$$

$$\underline{A = 9 \quad B = 12}$$

$$\begin{aligned}\int \frac{21x^2}{x^3 - x^2 - 12x} dx &= \int \frac{9}{x+3} dx + \int \frac{12}{x-4} dx \\ &= \underline{9 \ln|x+3| + 12 \ln|x-4| + C}\end{aligned}$$

Exercise

Evaluate $\int \frac{x+1}{x^3 + 3x^2 - 18x} dx$

Solution

$$\begin{aligned}\frac{x+1}{x^3 + 3x^2 - 18x} &= \frac{x+1}{x(x^2 + 3x - 18)} \\ &= \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+6}\end{aligned}$$

$$x+1 = Ax^2 + 3Ax - 18A + Bx^2 + 6Bx + Cx^2 - 3Cx$$

$$x^2 \quad A + B + C = 0$$

$$x \quad 3A + 6B - 3C = 1$$

$$x^0 \quad -18A = 1 \quad \rightarrow \underline{A = -\frac{1}{18}}$$

$$\begin{cases} B + C = \frac{1}{18} \\ 6B - 3C = \frac{7}{6} \end{cases} \rightarrow \begin{cases} 3B + 3C = \frac{1}{6} \\ 6B - 3C = \frac{7}{6} \end{cases}$$

$$9B = \frac{4}{3} \quad \rightarrow \underline{B = \frac{4}{27}}$$

$$C = \frac{1}{18} - \frac{4}{27} = -\frac{5}{54}$$

$$\int \frac{x+1}{x^3+3x^2-18x} dx = \int \left(-\frac{1}{18} \frac{1}{x} + \frac{4}{27} \frac{1}{x-3} - \frac{5}{54} \frac{1}{x+6} \right) dx$$

$$= -\frac{1}{18} \ln|x| + \frac{4}{27} \ln|x-3| - \frac{5}{54} \ln|x+6| + K$$

Exercise

Evaluate $\int \frac{x^2+12x-4}{x^3-4x} dx$

Solution

$$\frac{x^2+12x-4}{x^3-4x} = \frac{x^2+12x-4}{x(x^2-4)}$$

$$= \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

$$x^2+12x-4 = Ax^2-4A+Bx^2+2Bx+Cx^2-2Cx$$

$$x^2 \quad A+B+C=1$$

$$x \quad 2B-2C=12$$

$$x^0 \quad -4A=-4 \rightarrow \underline{A=1}$$

$$\rightarrow \begin{cases} B+C=0 \\ 2B-2C=12 \end{cases}$$

$$3B=12 \rightarrow \underline{B=4} \quad \underline{C=-4}$$

$$\int \frac{x^2+12x-4}{x^3-4x} dx = \int \left(\frac{1}{x} + \frac{4}{x-2} - \frac{4}{x+2} \right) dx$$

$$= \ln|x| + 4 \ln|x-2| - 4 \ln|x+2| + K$$

Exercise

Evaluate $\int \frac{6x^2}{x^4-5x^2+4} dx$

Solution

$$\frac{6x^2}{x^4-5x^2+4} = \frac{6x^2}{(x^2-1)(x^2-4)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2} + \frac{D}{x+2}$$

$$6x^2 = A(x+1)(x^2-4) + B(x-1)(x^2-4) + C(x+2)(x^2-1) + D(x-2)(x^2-1)$$

$$x^3 \quad A + B + C + D = 0$$

$$x^2 \quad A - B + 2C - 2D = 6$$

$$x^1 \quad -4A - 4B - C - D = 0$$

$$x^0 \quad -4A + 4B - 2C + 2D = 0$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -2 \\ -4 & -4 & -1 & -1 \\ -4 & 4 & -2 & 2 \end{vmatrix} = 72$$

$$\Delta_A = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 6 & -1 & 2 & -2 \\ 0 & -4 & -1 & -1 \\ 0 & 4 & -2 & 2 \end{vmatrix} = -72$$

$$\Delta_B = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 6 & 2 & -2 \\ -4 & 0 & -1 & -1 \\ -4 & 0 & -2 & 2 \end{vmatrix} = 72$$

$$\Delta_C = \begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 6 & -2 \\ -4 & -4 & 0 & -1 \\ -4 & 4 & 0 & 2 \end{vmatrix} = 144$$

$$\underline{A = -1 \quad B = 1 \quad C = 2 \quad D = -2}$$

$$\int \frac{6x^2}{x^4 - 5x^2 + 4} dx = \int \left(\frac{-1}{x-1} + \frac{1}{x+1} + \frac{2}{x-2} - \frac{2}{x+2} \right) dx$$

$$\underline{= -\ln|x-1| + \ln|x+1| + 2\ln|x-2| - 2\ln|x+2| + K}$$

Exercise

Evaluate $\int \frac{4x-2}{x^3-x} dx$

Solution

$$\frac{4x-2}{x^3-x} = \frac{4x-2}{x(x^2-1)}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$4x-2 = Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx$$

$$x^2 \quad A + B + C = 0$$

$$x^1 \quad B - C = 4$$

$$x^0 \quad -A = -2 \quad \rightarrow \underline{A = 2}$$

$$\begin{cases} B + C = -2 \\ B - C = 4 \end{cases} \rightarrow \underline{B = 1} \quad \underline{C = -3}$$

$$\int \frac{4x-2}{x^3-x} dx = \int \left(\frac{2}{x} + \frac{1}{x-1} - \frac{3}{x+1} \right) dx$$

$$\underline{= 2 \ln|x| + \ln|x-1| - 3 \ln|x+1| + K}$$

Exercise

Evaluate $\int \frac{16x^2}{(x-6)(x+2)^2} dx$

Solution

$$\frac{16x^2}{(x-6)(x+2)^2} = \frac{A}{x-6} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$16x^2 = A(x+2)^2 + Bx^2 - 4Bx - 12B + Cx - 6C$$

$$x^2 \quad A + B = 16$$

$$x^1 \quad 4A - 4B + C = 0$$

$$x^0 \quad 4A - 12B - 6C = 0$$

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ 4 & -4 & 1 \\ 4 & -12 & -6 \end{vmatrix} = 64$$

$$\Delta_A = \begin{vmatrix} 16 & 1 & 0 \\ 0 & -4 & 1 \\ 0 & -12 & -6 \end{vmatrix} = 576$$

$$A = \frac{576}{64} = 9$$

$$B = 16 - 9 = 7$$

$$C = -36 + 28 = -8$$

$$\int \frac{16x^2}{(x-6)(x+2)^2} dx = \int \frac{9}{x-6} dx + \int \frac{7}{x+2} dx - \int \frac{8}{(x+2)^2} dx$$

$$= \int \frac{9}{x-6} d(x-6) + \int \frac{7}{x+2} d(x+2) - \int \frac{8}{(x+2)^2} d(x+2)$$

$$\underline{= 9 \ln|x-6| + 7 \ln|x+2| + \frac{8}{x+2} + K}$$

Exercise

Evaluate $\int \frac{8(x^2 + 4)}{x(x^2 + 8)} dx$

Solution

$$\int \frac{8(x^2 + 4)}{x(x^2 + 8)} dx = 8 \int \frac{x^2 + 4}{x(x^2 + 8)} dx$$

$$\frac{x^2 + 4}{x(x^2 + 8)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 8}$$

$$x^2 + 4 = Ax^2 + 8A + Bx^2 + Cx$$

$$x^2 \quad A + B = 1 \quad \underline{B = \frac{1}{2}}$$

$$x^1 \quad \underline{C = 0}$$

$$x^0 \quad 8A = 4 \quad \underline{A = \frac{1}{2}}$$

$$\begin{aligned} 8 \int \frac{x^2 + 4}{x(x^2 + 8)} dx &= 8 \int \left(\frac{1}{2} \frac{1}{x} + \frac{1}{2} \frac{x}{x^2 + 8} \right) dx \\ &= 4 \int \frac{1}{x} dx + 2 \int \frac{x}{x^2 + 8} d(x^2 + 8) \\ &= \underline{4 \ln|x| + 2 \ln(x^2 + 8) + K} \end{aligned}$$

Exercise

Evaluate $\int \frac{x^2 + x + 2}{(x+1)(x^2 + 1)} dx$

Solution

$$\frac{x^2 + x + 2}{(x+1)(x^2 + 1)} = \frac{A}{x+1} + \frac{Bx + C}{x^2 + 1}$$

$$x^2 + x + 2 = Ax^2 + A + Bx^2 + Bx + Cx + C$$

$$x^2 \quad A + B = 1 \quad \rightarrow A = 1 - B$$

$$x^1 \quad B + C = 1 \quad \rightarrow C = 1 - B$$

$$x^0 \quad A + C = 2 \quad \rightarrow 1 - B + 1 - B = 2$$

$$\underline{B=0 \quad A=1 \quad C=1}$$

$$\int \frac{x^2+x+2}{(x+1)(x^2+1)} dx = \int \frac{1}{x+1} dx + \int \frac{1}{x^2+1} dx$$

$$\underline{= \ln|x+1| + \arctan(x) + K}$$

Exercise

Evaluate $\int \frac{2}{x(x^2+1)^2} dx$

Solution

$$\frac{2}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$2 = A(x^2+1)^2 + (Bx+C)(x^3+x) + Dx^2 + Ex$$

$$x^4 \quad A+B=0 \quad \rightarrow \underline{B=-2}$$

$$x^3 \quad \underline{C=0}$$

$$x^2 \quad 2A+B+D=0 \quad \rightarrow \underline{D=-2}$$

$$x^1 \quad C+E=0 \quad \rightarrow \underline{E=0}$$

$$x^0 \quad \underline{A=2}$$

$$\int \frac{2}{x(x^2+1)^2} dx = \int \frac{2}{x} dx - \int \frac{2x}{x^2+1} dx - \int \frac{2x}{(x^2+1)^2} dx$$

$$= 2 \ln|x| - \int \frac{1}{x^2+1} d(x^2+1) - \int \frac{1}{(x^2+1)^2} d(x^2+1)$$

$$\underline{= 2 \ln|x| - \ln(x^2+1) + \frac{1}{x^2+1} + K}$$

Exercise

Evaluate $\int \frac{1}{(x+1)(x^2+2x+2)} dx$

Solution

$$\frac{1}{(x+1)(x^2+2x+2)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+2} + \frac{Dx+E}{(x^2+2x+2)^2}$$

$$1 = A(x^2+2x+2)^2 + (Bx+C)(x+1)(x^2+2x+2) + (Dx+E)(x+1)$$

$$= Ax^4 + 4Ax^3 + 8Ax^2 + 8Ax + 4A + (Bx^2 + Bx + Cx + C)(x^2 + 2x + 2) + Dx^2 + Dx + Ex + E$$

$$x^4 \quad A + B = 0$$

$$x^3 \quad 4A + 3B + C = 0$$

$$x^2 \quad 8A + 4B + 3C + D = 0$$

$$x^1 \quad 8A + 2B + 4C + D + E = 0$$

$$x^0 \quad 4A + 2C + E = 1$$

$$\underline{A = 1, \quad B = -1, \quad C = -1, \quad D = -1, \quad E = -1}$$

$$\int \frac{1}{(x+1)(x^2+2x+2)^2} dx = \int \frac{dx}{x+1} - \int \frac{x+1}{x^2+2x+2} dx - \int \frac{x+1}{(x^2+2x+2)^2} dx$$

$$= \ln|x+1| - \frac{1}{2} \int \frac{d(x^2+2x+2)}{x^2+2x+2} - \frac{1}{2} \int \frac{d(x^2+2x+2)}{(x^2+2x+2)^2}$$

$$\underline{= \ln|x+1| - \frac{1}{2} \ln|x^2+2x+2| + \frac{1}{2} \frac{1}{x^2+2x+2} + K}$$

Exercise

Evaluate $\int \frac{2-x}{x^2+x} dx$

Solution

$$\frac{2-x}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}$$

$$2-x = Ax + A + Bx$$

$$x^1 \quad A + B = -1 \rightarrow \underline{B = -3}$$

$$x^0 \quad \underline{A = 2}$$

$$\int \frac{2-x}{x^2+x} dx = \int \left(\frac{2}{x} - \frac{3}{x+1} \right) dx$$

$$\underline{= 2 \ln|x| - 3 \ln|x+1| + C}$$

Exercise

Evaluate $\int \frac{3x+11}{(x+2)(x+3)} dx$

Solution

$$\frac{3x+11}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$3x+11 = Ax+3A+Bx+2B$$

$$x^1 \quad A+B=3$$

$$x^0 \quad 3A+2B=11$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -1 \quad \Delta_A = \begin{vmatrix} 3 & 1 \\ 11 & 2 \end{vmatrix} = -5 \quad \Delta_B = \begin{vmatrix} 1 & 3 \\ 3 & 11 \end{vmatrix} = 2$$

$$\underline{A=5 \quad B=-2}$$

$$\begin{aligned} \int \frac{3x+11}{(x+2)(x+3)} dx &= \int \left(\frac{5}{x+2} - \frac{1}{x+3} \right) dx \\ &= \underline{5 \ln|x+2| - 2 \ln|x+3| + C} \end{aligned}$$

Exercise

Evaluate $\int \frac{1}{x^2-a^2} dx$

Solution

$$\frac{1}{x^2-a^2} = \frac{A}{x-a} + \frac{B}{x+a}$$

$$1 = Ax + aA + Bx - aB$$

$$x^1 \quad A+B=0$$

$$x^0 \quad aA - aB = 1$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ a & -a \end{vmatrix} = -2a \quad \Delta_A = \begin{vmatrix} 0 & 1 \\ 1 & -a \end{vmatrix} = -1 \quad \Delta_B = \begin{vmatrix} 1 & 0 \\ a & 1 \end{vmatrix} = 1$$

$$\underline{A = \frac{1}{2a} \quad B = -\frac{1}{2a}}$$

$$\begin{aligned} \int \frac{1}{x^2-a^2} dx &= \int \left(\frac{1}{2a} \frac{1}{x-a} - \frac{1}{2a} \frac{1}{x+a} \right) dx \\ &= \underline{\frac{1}{2a} \ln|x-a| - \frac{1}{2a} \ln|x+a| + C} \end{aligned}$$

Exercise

Evaluate $\int \frac{1}{x^2 + 5x + 6} dx$

Solution

$$\frac{1}{x^2 + 5x + 6} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$1 = Ax + 3A + Bx + 2B$$

$$x^1 \quad A + B = 0$$

$$x^0 \quad 3A + 2B = 1$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -1 \quad \Delta_A = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1 \quad \Delta_B = \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1$$

$$\underline{A = 1 \quad B = -1}$$

$$\begin{aligned} \int \frac{1}{x^2 + 5x + 6} dx &= \int \left(\frac{1}{x+2} - \frac{1}{x+3} \right) dx \\ &= \ln|x+2| - \ln|x-3| + C \\ &= \ln \left| \frac{x+2}{x-3} \right| + C \end{aligned}$$

Exercise

Evaluate $\int \frac{x^3 + 6x^2 + 3x + 6}{x^3 + 2x^2} dx$

Solution

$$\begin{aligned} x^3 + 2x^2 \overline{\int \frac{1}{x^3 + 6x^2 + 3x + 6}} \\ \underline{x^3 + 2x^2} \\ 4x^2 + 3x + 6 \end{aligned}$$

$$\frac{x^3 + 6x^2 + 3x + 6}{x^3 + 2x^2} = 1 + \frac{4x^2 + 3x + 6}{x^3 + 2x^2}$$

$$\begin{aligned} \frac{4x^2 + 3x + 6}{x^3 + 2x^2} &= \frac{4x^2 + 3x + 6}{x^2(x+2)} \\ &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} \end{aligned}$$

$$4x^2 + 3x + 6 = Ax(x+2) + Bx + 2B + Cx^2$$

$$x^2 \quad A + C = 4 \quad \rightarrow \underline{C = 4}$$

$$x^1 \quad 2A + B = 3 \quad \rightarrow \underline{A = 0}$$

$$x^0 \quad 2B = 6 \quad \rightarrow \underline{B = 3}$$

$$\int \frac{x^3 + 6x^2 + 3x + 6}{x^3 + 2x^2} dx = \int \left(1 + \frac{0}{x} + \frac{3}{x^2} + \frac{4}{x+2} \right) dx$$

$$\underline{= x - \frac{3}{x} + 4 \ln|x+2| + K}$$

Exercise

Evaluate $\int \frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x} dx$

Solution

$$\frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x} = \frac{2x^2 + 5x - 1}{x(x^2 + x - 2)}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$2x^2 + 5x - 1 = Ax^2 + Ax - 2A + Bx^2 + 2Bx + Cx^2 - Cx$$

$$x^2 \quad A + B + C = 2 \quad B + C = \frac{3}{2}$$

$$x^1 \quad A + 2B - C = 5 \quad 2B - C = \frac{9}{2}$$

$$x^0 \quad -2A = -1 \quad \rightarrow \underline{A = \frac{1}{2}}$$

$$3B = 6 \quad \rightarrow \underline{B = 2}$$

$$C = \frac{3}{2} - 2 \quad \rightarrow \underline{C = -\frac{1}{2}}$$

$$\int \frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x} dx = \int \left(\frac{1}{2} \frac{1}{x} + \frac{2}{x-1} - \frac{1}{2} \frac{1}{x+2} \right) dx$$

$$\underline{= \frac{1}{2} \ln|x| + 2 \ln|x-1| - \frac{1}{2} \ln|x+2| + K}$$

Exercise

Evaluate $\int \frac{3x + 6}{x^3 + 2x^2 - 3x} dx$

Solution

$$\frac{3x+6}{x^3+2x^2-3x} = \frac{3x+6}{x(x^2+2x-3)}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+3}$$

$$3x+6 = Ax^2 + 2Ax - 3A + Bx^2 + 3Bx + Cx^2 - Cx$$

$$x^2 \quad A+B+C=0 \quad B+C=2$$

$$x^1 \quad 2A+3B-C=3 \quad 3B-C=7$$

$$x^0 \quad -3A=6 \quad \rightarrow \underline{A=-2}$$

$$4B=9 \quad \rightarrow \underline{B=\frac{9}{4}}$$

$$C=2-\frac{9}{4} \quad \rightarrow \underline{C=-\frac{1}{4}}$$

$$\int \frac{3x+6}{x^3+2x^2-3x} dx = \int \left(-\frac{2}{x} + \frac{9}{4} \frac{1}{x-1} - \frac{1}{4} \frac{1}{x+3} \right) dx$$

$$= -2 \ln|x| + \frac{9}{4} \ln|x-1| - \frac{1}{4} \ln|x+3| + K$$

Exercise

Evaluate $\int \frac{3x^2+2x-2}{x^3-1} dx$

Solution

$$\frac{3x^2+2x-2}{x^3-1} = \frac{3x^2+2x-2}{(x-1)(x^2+x+1)}$$

$$= \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$3x^2+2x-2 = Ax^2 + Ax + A + Bx^2 - Bx + Cx - C$$

$$x^2 \quad A+B=3 \quad \rightarrow B=3-A$$

$$x^1 \quad A-B+C=2 \quad \rightarrow A-3+A+A+2=2$$

$$x^0 \quad A-C=-2 \quad \rightarrow C=A+2$$

$$3A=3 \quad \rightarrow \underline{A=1}$$

$$B=3-1=\underline{2}$$

$$C=1+2=\underline{3}$$

$$\int \frac{3x^2+2x-2}{x^3-1} dx = \int \left(\frac{1}{x-1} + \frac{2x+3}{x^2+x+1} \right) dx$$

$$\begin{aligned}
&= \ln|x-1| + \int \frac{2x+1+2}{x^2+x+1} dx \\
&= \ln|x-1| + \int \frac{2x+1}{x^2+x+1} dx + \int \frac{2}{x^2+x+1} dx \\
&= \ln|x-1| + \int \frac{d(x^2+x+1)}{x^2+x+1} + \int \frac{2}{\left(x+\frac{1}{2}\right)^2+1-\frac{1}{4}} dx \\
&= \ln|x-1| + \ln(x^2+x+1) + \int \frac{2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} d\left(x+\frac{1}{2}\right) \\
&= \ln|x-1| + \ln(x^2+x+1) + \int \frac{2}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} d\left(x+\frac{1}{2}\right) \\
&= \ln|x-1| + \ln(x^2+x+1) + 2 \frac{2}{\sqrt{3}} \arctan \frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + K \\
&= \ln|x-1| + \ln(x^2+x+1) + \frac{4\sqrt{3}}{3} \arctan \frac{2x+1}{\sqrt{3}} + K
\end{aligned}$$

Exercise

Evaluate $\int \frac{x^3+5x^2+2x-4}{x^4-1} dx$

Solution

$$\begin{aligned}
\frac{x^3+5x^2+2x-4}{x^4-1} &= \frac{x^3+5x^2+2x-4}{(x-1)(x+1)(x^2+1)} \\
&= \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}
\end{aligned}$$

$$x^3+5x^2+2x-4 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)$$

$$x^3 \quad A+B+C=1 \quad (1)$$

$$x^2 \quad A-B+D=5 \quad (2)$$

$$x^1 \quad A+B-C=2 \quad (3)$$

$$x^0 \quad A-B-D=-4 \quad (4)$$

$$(1) + (3) \rightarrow 2A + 2B = 3$$

$$(2) + (4) \rightarrow 2A - 2B = 1$$

$$\Delta = \begin{vmatrix} 2 & 2 \\ 2 & -2 \end{vmatrix} = -8 \quad \Delta_A = \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} = -8 \quad \Delta_B = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = -4$$

$$\underline{A = 1 \quad B = \frac{1}{2}}$$

$$(1) \rightarrow C = 1 - 1 - \frac{1}{2} = -\frac{1}{2}$$

$$(2) \rightarrow D = 5 - 1 + \frac{1}{2} = \frac{9}{2}$$

$$\begin{aligned} \int \frac{x^3 + 5x^2 + 2x - 4}{x^4 - 1} dx &= \int \frac{1}{x-1} dx + \frac{1}{2} \int \frac{1}{x+1} dx - \frac{1}{2} \int \frac{x}{x^2+1} dx + \frac{9}{2} \int \frac{1}{x^2+1} dx \\ &= \ln|x-1| + \frac{1}{2} \ln|x+1| - \frac{1}{4} \int \frac{1}{x^2+1} d(x^2+1) + \frac{9}{2} \arctan x + K \\ &= \ln|x-1| + \frac{1}{2} \ln|x+1| - \frac{1}{4} \ln(x^2+1) + \frac{9}{2} \arctan x + K \end{aligned}$$

Exercise

Evaluate $\int \frac{x^2 + 4x}{(x^2 + 4)(x-2)^2} dx$

Solution

$$\frac{x^2 + 4x}{(x^2 + 4)(x-2)^2} = \frac{Ax+B}{x^2+4} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

$$x^2 + 4x = (Ax+B)(x^2 - 4x + 4) + C(x^2 + 4) + Dx^2 + 4D$$

$$x^3 \quad A + C = 0 \quad \rightarrow A = -C$$

$$x^2 \quad -4A + B - 2C + D = 1$$

$$x^1 \quad 4A - 4B + 4C = 4$$

$$x^0 \quad 4B - 8C + 4D = 0$$

$$\begin{cases} B + 2C + D = 1 & 2C + D = 2 \\ -4B = 4 & \underline{B = -1} \\ 4B - 8C + 4D = 0 & -8C + 4D = 4 \end{cases}$$

$$\begin{cases} 2C + D = 2 \\ -2C + D = 1 \end{cases}$$

$$2D = 3 \rightarrow \underline{D = \frac{3}{2}}$$

$$2C = 2 - \frac{3}{2} = \frac{1}{2} \rightarrow \underline{C = \frac{1}{4}}$$

$$\underline{A = -\frac{1}{4}}$$

$$\begin{aligned} \int \frac{x^2 + 4x}{(x^2 + 4)(x-2)^2} dx &= -\frac{1}{4} \int \frac{x}{x^2 + 4} dx - \int \frac{1}{x^2 + 4} dx + \frac{1}{4} \int \frac{1}{x-2} dx + \frac{3}{2} \int \frac{1}{(x-2)^2} dx \\ &= -\frac{1}{8} \int \frac{1}{x^2 + 4} d(x^2 + 4) - \frac{1}{2} \arctan \frac{x}{2} + \frac{1}{4} \ln|x-2| + \frac{3}{2} \int \frac{1}{(x-2)^2} d(x-2) \\ &= \underline{-\frac{1}{8} \ln(x^2 + 4) - \frac{1}{2} \arctan \frac{x}{2} + \frac{1}{4} \ln|x-2| - \frac{3}{2} \frac{1}{x-2} + K} \end{aligned}$$

Exercise

Evaluate $\int \frac{x^2 + 2x + 3}{(x-1)(x+1)^2} dx$

Solution

$$\frac{x^2 + 2x + 3}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x^2 + 2x + 3 = Ax^2 + 2Ax + A + B(x+1)(x-1) + Cx - C$$

$$x^2 \quad A + B = 1 \quad B = 1 - A$$

$$x^1 \quad 2A + C = 2 \quad C = 2 - 2A$$

$$x^0 \quad A - B - C = 3 \quad (1)$$

$$(1) \rightarrow A - 1 + A - 2 + 2A = 3 \Rightarrow \underline{A = \frac{3}{2}}$$

$$B = 1 - \frac{3}{2} = \underline{-\frac{1}{2}}$$

$$C = 2 - 3 = \underline{-1}$$

$$\begin{aligned} \int \frac{x^2 + 2x + 3}{(x-1)(x+1)^2} dx &= \frac{3}{2} \int \frac{1}{x-1} dx - \frac{1}{2} \int \frac{1}{x+1} dx - \int \frac{1}{(x+1)^2} d(x+1) \\ &= \underline{\frac{3}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| + \frac{1}{x+1} + K} \end{aligned}$$

Exercise

Evaluate $\int \frac{x^4 - x^3 + 3x^2 - x + 2}{(x-1)(x^2+2)^2} dx$

Solution

$$\frac{x^4 - x^3 + 3x^2 - x + 2}{(x-1)(x^2+2)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}$$

$$\begin{aligned} x^4 - x^3 + 3x^2 - x + 2 &= A(x^4 + 4x^2 + 4) + (Bx+C)(x-1)(x^2+2) + (Dx+E)(x-1) \\ &= Ax^4 + 4Ax^2 + 4A + (Bx+C)(x^3 + 2x - x^2 - 2) + Dx^2 - Dx + Ex - E \end{aligned}$$

$$x^4 \quad A + B = 1 \quad \rightarrow A = 1 - B$$

$$x^3 \quad -B + C = -1 \quad \rightarrow C = B - 1$$

$$x^2 \quad 4A + 2B - C + D = 3$$

$$x^1 \quad -2B + 2C - D + E = 1$$

$$x^0 \quad 4A - 2C - E = 2$$

$$\begin{cases} 4 - 4B + 2B - B + 1 + D = 3 \\ -2B + 2B - 2 - D + E = 1 \\ 4 - 4B - 2B + 2 - E = 2 \end{cases}$$

$$\begin{cases} -3B + D = -2 \\ -D + E = 3 \\ -6B - E = -4 \end{cases} \rightarrow \begin{cases} -3B + E = 1 \\ -6B - E = -4 \end{cases}$$

$$-9B = -3 \rightarrow \underline{B = \frac{1}{3}}$$

$$E = 1 + 1 = \underline{2}$$

$$A = 1 - \frac{1}{3} = \underline{\frac{2}{3}}$$

$$C = \frac{1}{3} - 1 = \underline{-\frac{2}{3}}$$

$$D = -2 + 1 = \underline{-1}$$

$$\begin{aligned} \int \frac{x^4 - x^3 + 3x^2 - x + 2}{(x-1)(x^2+2)^2} dx &= \frac{2}{3} \int \frac{1}{x-1} dx + \frac{1}{3} \int \frac{x}{x^2+2} dx - \frac{2}{3} \int \frac{1}{x^2+2} dx - \int \frac{x}{(x^2+2)^2} dx \\ &\quad + \int \frac{2}{(x^2+2)^2} dx \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{3} \ln|x-1| + \frac{1}{6} \int \frac{d(x^2+2)}{x^2+2} - \frac{2}{3\sqrt{2}} \arctan \frac{x}{\sqrt{2}} - \frac{1}{2} \int \frac{d(x^2+2)}{(x^2+2)^2} \\
&\quad + \int \frac{2}{(x^2+2)^2} dx \\
&= \frac{2}{3} \ln|x-1| + \frac{1}{6} \ln(x^2+2) - \frac{2}{3\sqrt{2}} \arctan \frac{x}{\sqrt{2}} + \frac{1}{2(x^2+2)} + \int \frac{2}{(x^2+2)^2} dx
\end{aligned}$$

$$x = \sqrt{2} \tan \theta \rightarrow dx = \sqrt{2} \sec^2 \theta d\theta$$

$$x^2 + 2 = 2 \sec^2 \theta$$

$$\begin{aligned}
\int \frac{2}{(x^2+2)^2} dx &= \int \frac{2}{4 \sec^4 \theta} \sqrt{2} \sec^2 \theta d\theta \\
&= \frac{\sqrt{2}}{2} \int \frac{d\theta}{\sec^2 \theta} \\
&= \frac{\sqrt{2}}{2} \int \cos^2 \theta d\theta \\
&= \frac{\sqrt{2}}{4} \int (1 + \cos 2\theta) d\theta \\
&= \frac{\sqrt{2}}{4} \left(\theta + \frac{1}{2} \sin 2\theta \right) \\
&= \frac{\sqrt{2}}{4} (\theta + \sin \theta \cos \theta) \\
&= \frac{\sqrt{2}}{4} \left(\theta + \frac{\tan \theta}{\sec^2 \theta} \right) \\
&= \frac{\sqrt{2}}{4} \left(\arctan \frac{x}{\sqrt{2}} + \frac{x}{\sqrt{2}} \frac{2}{x^2+2} \right)
\end{aligned}$$

$$\begin{aligned}
\int \frac{x^4 - x^3 + 3x^2 - x + 2}{(x-1)(x^2+2)^2} dx &= \frac{2}{3} \ln|x-1| + \frac{1}{6} \ln(x^2+2) - \frac{2}{3\sqrt{2}} \arctan \frac{x}{\sqrt{2}} + \frac{1}{2(x^2+2)} \\
&\quad + \frac{\sqrt{2}}{4} \arctan \frac{x}{\sqrt{2}} + \frac{1}{2} \frac{x}{x^2+2} + K
\end{aligned}$$

Exercise

Evaluate $\int \frac{-x^2 + 11x + 18}{(x-1)(x+1)(x^2 + 3x + 3)} dx$

Solution

$$\frac{-x^2 + 11x + 18}{(x-1)(x+1)(x^2 + 3x + 3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx + D}{x^2 + 3x + 3}$$

$$-x^2 + 11x + 18 = A(x+1)(x^2 + 3x + 3) + B(x-1)(x^2 + 3x + 3) + (Cx + D)(x^2 - 1)$$

$$x^3 \quad A + B + C = 0 \quad (1)$$

$$x^2 \quad 4A + 2B + D = -1 \quad (2)$$

$$x^1 \quad 6A - C = 11 \quad \rightarrow C = 6A - 11$$

$$x^0 \quad 3A - 3B - D = 18 \quad (3)$$

$$\begin{cases} (1) \rightarrow 7A + B = 11 \\ (2) + (3) \rightarrow 7A - B = 17 \end{cases}$$

$$14A = 28 \rightarrow \underline{A = 2}$$

$$B = 11 - 14 = \underline{-3}$$

$$C = 12 - 11 = \underline{1}$$

$$(2) \rightarrow D = -1 - 8 + 6 = \underline{-3}$$

$$\begin{aligned} \int \frac{-x^2 + 11x + 18}{(x-1)(x+1)(x^2 + 3x + 3)} dx &= \int \frac{2}{x-1} dx - \int \frac{3}{x+1} dx + \int \frac{x-3}{x^2 + 3x + 3} dx \\ &= 2 \ln|x-1| - 3 \ln|x+1| + \int \frac{x-3}{x^2 + 3x + 3} dx \end{aligned}$$

$$\begin{aligned} \int \frac{x-3}{x^2 + 3x + 3} dx &= \frac{1}{2} \int \frac{2x - 6 + \color{red}{3} - \color{red}{3}}{x^2 + 3x + 3} dx \\ &= \frac{1}{2} \int \frac{2x+3}{x^2 + 3x + 3} dx - \frac{9}{2} \int \frac{1}{\left(x + \frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx \\ &= \frac{1}{2} \int \frac{1}{x^2 + 3x + 3} d\left(x^2 + 3x + 3\right) - \frac{9}{2} \frac{2}{\sqrt{3}} \arctan \frac{x + \frac{3}{2}}{\frac{\sqrt{3}}{2}} \\ &= \frac{1}{2} \ln|x^2 + 3x + 3| - 3\sqrt{3} \arctan \frac{2x+3}{\sqrt{3}} \end{aligned}$$

$$\int \frac{-x^2 + 11x + 18}{(x-1)(x+1)(x^2 + 3x + 3)} dx = 2 \ln|x-1| - 3 \ln|x+1| + \frac{1}{2} \ln|x^2 + 3x + 3| - 3\sqrt{3} \arctan \frac{2x+3}{\sqrt{3}} + K$$

Exercise

Evaluate $\int \frac{x^3 + 5x^2 + 2x - 4}{x(x^2 + 4)^2} dx$

Solution

$$\frac{x^3 + 5x^2 + 2x - 4}{x(x^2 + 4)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4} + \frac{Dx + E}{(x^2 + 4)^2}$$

$$x^3 + 5x^2 + 2x - 4 = A(x^4 + 8x^2 + 16) + (Bx^2 + Cx)(x^2 + 4) + Dx^2 + Ex$$

$$x^4 \quad A + B = 0 \quad \rightarrow B = -\frac{1}{4}$$

$$x^3 \quad C = 1$$

$$x^2 \quad 8A + 4B + D = 5 \quad \rightarrow D = 6$$

$$x^1 \quad 4C + E = 2 \quad \rightarrow E = -2$$

$$x^0 \quad 16A = -4 \quad \rightarrow A = -\frac{1}{4}$$

$$\begin{aligned} \int \frac{x^3 + 5x^2 + 2x - 4}{x(x^2 + 4)^2} dx &= -\frac{1}{4} \int \frac{dx}{x} + \frac{1}{4} \int \frac{x}{x^2 + 4} dx + \int \frac{dx}{x^2 + 4} + \int \frac{6x}{(x^2 + 4)^2} dx - \int \frac{2}{(x^2 + 4)^2} dx \\ &= -\frac{1}{4} \ln|x| + \frac{1}{8} \int \frac{d(x^2 + 4)}{x^2 + 4} + \frac{1}{2} \arctan \frac{x}{2} + 3 \int \frac{d(x^2 + 4)}{(x^2 + 4)^2} - \int \frac{2}{(x^2 + 4)^2} dx \\ &= -\frac{1}{4} \ln|x| + \frac{1}{8} \ln(x^2 + 4) + \frac{1}{2} \arctan \frac{x}{2} - \frac{3}{x^2 + 4} - \int \frac{2}{(x^2 + 4)^2} dx \end{aligned}$$

$$x = 2 \tan \theta \quad \rightarrow dx = 2 \sec^2 \theta d\theta$$

$$x^2 + 4 = 4 \sec^2 \theta$$

$$\begin{aligned} \int \frac{2}{(x^2 + 4)^2} dx &= \int \frac{2}{16 \sec^4 \theta} 2 \sec^2 \theta d\theta \\ &= \frac{1}{4} \int \frac{d\theta}{\sec^2 \theta} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \int \cos^2 \theta \, d\theta \\
&= \frac{1}{8} \int (1 + \cos 2\theta) \, d\theta \\
&= \frac{1}{8} \left(\theta + \frac{1}{2} \sin 2\theta \right) \\
&= \frac{1}{8} (\theta + \sin \theta \cos \theta) \\
&= \frac{1}{8} \left(\theta + \frac{\tan \theta}{\sec^2 \theta} \right) \\
&= \frac{1}{8} \left(\arctan \frac{x}{2} + \frac{x}{2} \frac{4}{x^2 + 4} \right) \\
&= \frac{1}{8} \arctan \frac{x}{2} + \frac{1}{4} \frac{x}{x^2 + 4} \Big|
\end{aligned}$$

$$\begin{aligned}
\int \frac{x^3 + 5x^2 + 2x - 4}{x(x^2 + 4)^2} \, dx &= -\frac{1}{4} \ln|x| + \frac{1}{8} \ln(x^2 + 4) + \frac{1}{2} \arctan \frac{x}{2} - \frac{3}{x^2 + 4} - \frac{1}{8} \arctan \frac{x}{2} - \frac{1}{4} \frac{x}{x^2 + 4} + K \\
&= -\frac{1}{4} \ln|x| + \frac{1}{8} \ln(x^2 + 4) + \frac{3}{8} \arctan \frac{x}{2} - \frac{1}{4} \frac{x+12}{x^2 + 4} + K \Big|
\end{aligned}$$

Exercise

Evaluate $\int_{-1}^2 \frac{5x}{x^2 - x - 6} \, dx$

Solution

$$\frac{5x}{x^2 - x - 6} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$5x = Ax + 2A + Bx - 3B$$

$$x^1 \quad A + B = 5$$

$$x^0 \quad 2A - 3B = 0$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5 \quad \Delta_A = \begin{vmatrix} 5 & 1 \\ 0 & -3 \end{vmatrix} = -15 \quad \Delta_B = \begin{vmatrix} 1 & 5 \\ 2 & 0 \end{vmatrix} = -10$$

$$A = \frac{-15}{-5} = 3 \quad B = \frac{-10}{-5} = 2$$

$$\int_{-1}^2 \frac{5x}{x^2 - x - 6} \, dx = \int_{-1}^2 \left(\frac{3}{x-3} + \frac{2}{x+2} \right) \, dx$$

$$\begin{aligned}
&= 3 \ln |x-3| + 2 \ln |x+2| \Big|_{-1}^2 \\
&= 3 \ln |-1| + 2 \ln 4 - 3 \ln |-4| - 2 \ln 1 \\
&= 2 \ln 4 - 3 \ln 4 \\
&= \underline{-\ln 4}
\end{aligned}$$

Exercise

Evaluate $\int_0^5 \frac{2}{x^2 - 4x - 32} dx$

Solution

$$\frac{2}{x^2 - 4x - 32} = \frac{A}{x-8} + \frac{B}{x+4}$$

$$2 = Ax + 4A + Bx - 8B$$

$$x^1 \quad A + B = 0$$

$$x^0 \quad 4A - 8B = 2$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 4 & -8 \end{vmatrix} = -12 \quad \Delta_A = \begin{vmatrix} 0 & 1 \\ 2 & -8 \end{vmatrix} = -2 \quad \Delta_B = \begin{vmatrix} 1 & 0 \\ 4 & 2 \end{vmatrix} = 2$$

$$A = \frac{-2}{-12} = \underline{\frac{1}{6}} \quad B = \frac{2}{-12} = \underline{-\frac{1}{6}}$$

$$\begin{aligned}
\int_0^5 \frac{2}{x^2 - 4x - 32} dx &= \frac{1}{6} \int_0^5 \left(\frac{1}{x-8} - \frac{1}{x+4} \right) dx \\
&= \frac{1}{6} \left(\ln |x-8| - \ln |x+4| \right) \Big|_0^5 \\
&= \frac{1}{6} \left(\ln |-3| - \ln 9 - \ln |-8| + \ln 4 \right) \\
&= \frac{1}{6} \left(\ln 3 - \ln 3^2 - \ln 8 + \ln 4 \right) \\
&= \frac{1}{6} \left(\ln 3 - 2 \ln 3 - 3 \ln 2 + 2 \ln 2 \right) \\
&= \frac{1}{6} \left(-\ln 3 - \ln 2 \right) \\
&= -\frac{1}{6} \left(\ln 3 + \ln 2 \right) \\
&= \underline{-\frac{\ln 6}{6}}
\end{aligned}$$

Exercise

Evaluate $\int_0^1 \frac{dx}{(x+1)(x^2+1)}$

Solution

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$1 = Ax^2 + A + Bx^2 + Bx + Cx + C$$

$$x^2 \quad A+B=0 \quad A=-B$$

$$x \quad B+C=0 \quad C=-B$$

$$x^0 \quad A+C=1 \quad (1)$$

$$(1) \rightarrow -B - B = 1 \quad \underline{B = -\frac{1}{2}}$$

$$\underline{A = C = \frac{1}{2}}$$

$$\begin{aligned} \int_0^1 \frac{dx}{(x+1)(x^2+1)} &= \frac{1}{2} \int_0^1 \frac{1}{x+1} dx - \frac{1}{2} \int_0^1 \frac{x}{x^2+1} dx + \frac{1}{2} \int_0^1 \frac{1}{x^2+1} dx \\ &= \frac{1}{2} \ln(x+1) - \frac{1}{4} \int_0^1 \frac{1}{x^2+1} d(x^2+1) + \frac{1}{2} \arctan x \\ &= \left(\frac{1}{2} \ln(x+1) - \frac{1}{4} \ln(x^2+1) + \frac{1}{2} \arctan x \right) \Big|_0^1 \\ &= \frac{1}{2} \ln 2 - \frac{1}{4} \ln 2 + \frac{1}{2} \arctan 1 - \frac{1}{2} \ln 1 - \frac{1}{4} \ln 1 + \frac{1}{2} \arctan 0 \\ &= \underline{\underline{\frac{1}{4} \ln 2 + \frac{\pi}{8}}} \end{aligned}$$

Exercise

Evaluate $\int_{-1/2}^{1/2} \frac{x^2+1}{x^2-1} dx$

Solution

$$\begin{aligned} \int_{-1/2}^{1/2} \frac{x^2+1}{x^2-1} dx &= \int_{-1/2}^{1/2} \left(1 + \frac{2}{x^2-1} \right) dx \\ \frac{2}{x^2-1} &= \frac{A}{x-1} + \frac{B}{x+1} \end{aligned}$$

$$2 = Ax + A + Bx - B$$

$$\begin{array}{l} x \quad A + B = 0 \\ x^0 \quad A - B = 2 \end{array} \rightarrow A = 1 \quad B = -1$$

$$\begin{aligned} \int_{-1/2}^{1/2} \frac{x^2+1}{x^2-1} dx &= \int_{-1/2}^{1/2} \left(1 + \frac{1}{x-1} - \frac{1}{x+1} \right) dx \\ &= x + \ln|x-1| - \ln|x+1| \Big|_{-1/2}^{1/2} \\ &= x + \ln \left| \frac{x-1}{x+1} \right| \Big|_{-1/2}^{1/2} \\ &= \frac{1}{2} + \ln \left| \frac{-\frac{1}{2}}{\frac{3}{2}} \right| + \frac{1}{2} - \ln \left| \frac{-\frac{3}{2}}{\frac{1}{2}} \right| \\ &= 1 + \ln \left| \frac{1}{3} \right| - \ln|-3| \\ &= 1 - \ln 3 - \ln 3 \\ &= \underline{1 - 2\ln 3} \end{aligned}$$

Exercise

Evaluate $\int_0^2 \frac{3}{4x^2+5x+1} dx$

Solution

$$\frac{3}{4x^2+5x+1} = \frac{A}{x+1} + \frac{B}{4x+1}$$

$$4Ax + A + Bx + B = 3$$

$$\Rightarrow \begin{cases} 4A + B = 0 \\ A + B = 3 \end{cases} \rightarrow \underline{A = -1 \quad B = 4}$$

$$\begin{aligned} \int_0^2 \frac{3}{4x^2+5x+1} dx &= -\int_0^2 \frac{1}{x+1} dx + \int_0^2 \frac{4}{4x+1} dx \\ &= -\ln(x+1) + \ln(4x+1) \Big|_0^2 \\ &= \ln \frac{4x+1}{x+1} \Big|_0^2 \\ &= \underline{\ln 3} \end{aligned}$$

Exercise

Evaluate $\int_1^5 \frac{x-1}{x^2(x+1)} dx$

Solution

$$\frac{x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$Ax^2 + Ax + Bx + B + Cx^2 = x - 1$$

$$\begin{cases} x^2 & A + C = 0 \\ x^1 & A + B = 1 \rightarrow A = 2 \quad C = -2 \\ x^0 & \underline{B = -1} \end{cases}$$

$$\begin{aligned} \int_1^5 \frac{x-1}{x^2(x+1)} dx &= \int_1^5 \left(\frac{2}{x} - \frac{1}{x^2} - \frac{2}{x+1} \right) dx \\ &= 2 \ln x + \frac{1}{x} - 2 \ln(x+1) \Big|_1^5 \\ &= 2 \ln 5 + \frac{1}{5} - 2 \ln 6 - 1 + 2 \ln 2 \\ &= \underline{2 \ln \frac{5}{3} - \frac{4}{5}} \end{aligned}$$

Exercise

Evaluate $\int_1^2 \frac{x+1}{x(x^2+1)} dx$

Solution

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$Ax^2 + A + Bx^2 + Cx = x + 1$$

$$\begin{cases} x^2 & A + B = 0 \\ x^1 & \underline{C = 1} \rightarrow \underline{B = -1} \\ x^0 & \underline{A = 1} \end{cases}$$

$$\begin{aligned} \int_1^2 \frac{x+1}{x(x^2+1)} dx &= \int_1^2 \frac{1}{x} dx - \int_1^2 \frac{x}{x^2+1} dx + \int_1^2 \frac{1}{x^2+1} dx \\ &= \int_1^2 \frac{1}{x} dx - \frac{1}{2} \int_1^2 \frac{1}{x^2+1} d(x^2+1) + \int_1^2 \frac{1}{x^2+1} dx \end{aligned}$$

$$\begin{aligned}
&= \ln x - \frac{1}{2} \ln(x^2 + 1) + \arctan x \Big|_1^2 \\
&= \ln 2 - \frac{1}{2} \ln 5 + \arctan 2 + \frac{1}{2} \ln 2 - \frac{\pi}{4} \\
&= \frac{1}{2} (3 \ln 2 - \ln 5) - \frac{\pi}{4} + \arctan 2 \\
&= \frac{1}{2} \ln \frac{8}{5} - \frac{\pi}{4} + \arctan 2
\end{aligned}$$

Exercise

Evaluate $\int_0^1 \frac{x^2 - x}{x^2 + x + 1} dx$

Solution

$$\begin{aligned}
\int_0^1 \frac{x^2 - x}{x^2 + x + 1} dx &= \int_0^1 \left(1 - \frac{2x + 1}{x^2 + x + 1} \right) dx \\
&= \int_0^1 dx - \int_0^1 \frac{1}{x^2 + x + 1} d(x^2 + x + 1) \\
&= x - \ln(x^2 + x + 1) \Big|_0^1 \\
&= 1 - \ln 3
\end{aligned}$$

Exercise

Evaluate $\int_4^8 \frac{y dy}{y^2 - 2y - 3}$

Solution

$$\frac{y}{y^2 - 2y - 3} = \frac{A}{y - 3} + \frac{B}{y + 1}$$

$$y = Ay + A + By - 3B$$

$$\rightarrow \begin{cases} A + B = 1 \\ A - 3B = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -4$$

$$\Delta_A = \begin{vmatrix} 1 & 1 \\ 0 & -3 \end{vmatrix} = -3$$

$$\Delta_B = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$\Rightarrow A = \frac{3}{4} \quad B = \frac{1}{4}$$

$$\begin{aligned}
\int_4^8 \frac{ydy}{y^2-2y-3} &= \frac{3}{4} \int_4^8 \frac{dy}{y-3} + \frac{1}{4} \int_4^8 \frac{dy}{y+1} \\
&= \left[\frac{3}{4} \ln|y-3| + \frac{1}{4} \ln|y+1| \right]_4^8 \\
&= \frac{3}{4} \ln|5| + \frac{1}{4} \ln|9| - \left(\frac{3}{4} \ln|1| + \frac{1}{4} \ln|5| \right) \\
&= \frac{3}{4} \ln 5 + \frac{1}{4} \ln 9 - \frac{1}{4} \ln 5 \\
&= \frac{1}{2} \ln 5 + \frac{1}{4} \ln 3^2 && \text{Power Rule} \\
&= \frac{1}{2} \ln 5 + \frac{1}{2} \ln 3 \\
&= \frac{1}{2} (\ln 5 + \ln 3) && \text{Product Rule} \\
&= \frac{1}{2} \ln 15
\end{aligned}$$

Exercise

Evaluate $\int_1^{\sqrt{3}} \frac{3x^2+x+4}{x^3+x} dx$

Solution

$$\frac{3x^2+x+4}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$3x^2+x+4 = Ax^2 + A + Bx^2 + Cx$$

$$\begin{cases} A+B=3 \rightarrow B=3-4 = \underline{-1} \\ \underline{C=1} \\ \underline{A=4} \end{cases}$$

$$\int_1^{\sqrt{3}} \frac{3x^2+x+4}{x^3+x} dx = \int_1^{\sqrt{3}} \frac{4}{x} dx + \int_1^{\sqrt{3}} \frac{-x+1}{x^2+1} dx$$

$$= 4 \int_1^{\sqrt{3}} \frac{1}{x} dx - \int_1^{\sqrt{3}} \frac{x}{x^2+1} dx + \int_1^{\sqrt{3}} \frac{1}{x^2+1} dx$$

$$d(x^2+1) = 2x dx$$

$$= 4 \int_1^{\sqrt{3}} \frac{1}{x} dx - \frac{1}{2} \int_1^{\sqrt{3}} \frac{d(x^2+1)}{x^2+1} + \int_1^{\sqrt{3}} \frac{1}{x^2+1} dx$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$= \left[4 \ln|x| - \frac{1}{2} \ln(x^2+1) + \tan^{-1} x \right]_1^{\sqrt{3}}$$

$$\begin{aligned}
&= 4 \ln \sqrt{3} - \frac{1}{2} \ln 4 + \tan^{-1} \sqrt{3} - \left(4 \ln 1 - \frac{1}{2} \ln 2 + \tan^{-1} 1 \right) \\
&= 4 \ln 3^{1/2} - \frac{1}{2} \ln 2^2 + \frac{\pi}{3} + \frac{1}{2} \ln 2 - \frac{\pi}{4} \\
&= 2 \ln 3 - \ln 2 + \frac{\pi}{12} + \frac{1}{2} \ln 2 \\
&= 2 \ln 3 - \frac{1}{2} \ln 2 + \frac{\pi}{12} \\
&= \ln \left(\frac{9}{\sqrt{2}} \right) + \frac{\pi}{12}
\end{aligned}$$

Exercise

Evaluate $\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$

Solution

$$u = \tan\left(\frac{x}{2}\right) \rightarrow x = 2 \tan^{-1} u$$

$$dx = \frac{2du}{1+u^2}$$

$$\cos x = 2 \cos^2 \frac{x}{2} - 1$$

$$= 2 \frac{1}{1+u^2} - 1$$

$$= \frac{1-u^2}{1+u^2}$$

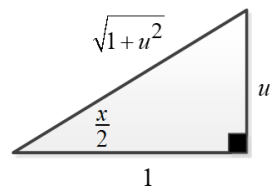
$$\sin x = 2 \frac{u}{\sqrt{1+u^2}} \frac{1}{\sqrt{1+u^2}}$$

$$= \frac{2u}{1+u^2}$$

$$\int_0^{\pi/2} \frac{dx}{\sin x + \cos x} = \int_0^{\pi/2} \frac{1}{\frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} du$$

$$= 2 \int_0^{\pi/2} \frac{du}{2u+1-u^2}$$

$$= -2 \int_0^{\pi/2} \frac{du}{u^2 - 2u - 1}$$



$$\begin{aligned}
&= -\frac{1}{\sqrt{2}} \int_0^{\pi/2} \left(\frac{1}{u-1-\sqrt{2}} - \frac{1}{u-1+\sqrt{2}} \right) du \\
&\quad \frac{2}{u^2 - 2u - 1} = \frac{A}{u-1-\sqrt{2}} + \frac{B}{u-1+\sqrt{2}} \\
&\quad 2 = Au + (-1+\sqrt{2})A + Bu + (-1-\sqrt{2})B \\
&\quad \begin{cases} \textcolor{red}{x} & A+B=0 \\ \textcolor{red}{x^0} & (-1+\sqrt{2})A - (1+\sqrt{2})B = 2 \end{cases} \\
&\quad \rightarrow \begin{cases} B = -A = -\frac{1}{\sqrt{2}} \\ 2\sqrt{2}A = 2 \end{cases} \\
&= -\frac{1}{\sqrt{2}} \left(\ln \left| \frac{1}{u-1-\sqrt{2}} \right| - \ln \left| \frac{1}{u-1+\sqrt{2}} \right| \right) \Big|_0^{\pi/2} \\
&= \frac{1}{\sqrt{2}} \left(\ln \left| \frac{u-1+\sqrt{2}}{u-1-\sqrt{2}} \right| \right) \Big|_0^{\pi/2} \\
&= \frac{1}{\sqrt{2}} \left(\ln \left| \frac{\tan \frac{x}{2} - 1 + \sqrt{2}}{\tan \frac{x}{2} - 1 - \sqrt{2}} \right| \right) \Big|_0^{\pi/2} \\
&= \frac{1}{\sqrt{2}} \left(\ln |-1| - \ln \left| \frac{-1+\sqrt{2}}{-1-\sqrt{2}} \right| \right) \\
&= \frac{1}{\sqrt{2}} \ln \left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right) \Big|
\end{aligned}$$

Exercise

Evaluate $\int_0^{\pi/3} \frac{\sin \theta}{1 - \sin \theta} d\theta$

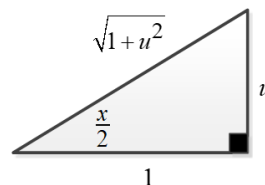
Solution

$$u = \tan\left(\frac{x}{2}\right) \rightarrow x = 2 \tan^{-1} u$$

$$dx = \frac{2du}{1+u^2}$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

$$= 2 \frac{u}{\sqrt{1+u^2}} \frac{1}{\sqrt{1+u^2}}$$



$$= \frac{2u}{1+u^2}$$

$$\begin{aligned} \int_0^{\pi/3} \frac{\sin \theta}{1 - \sin \theta} d\theta &= \int_0^{\pi/3} \frac{1}{\csc \theta - 1} d\theta \\ &= \int_0^{\pi/3} \frac{1}{\frac{1+u^2}{2u} - 1} \cdot \frac{2}{1+u^2} du \\ &= \int_0^{\pi/3} \frac{4u}{(1+u^2-2u)(1+u^2)} du \\ &= \int_0^{\pi/3} \frac{4u}{(u-1)^2(1+u^2)} du \end{aligned}$$

$$\frac{4u}{(u-1)^2(1+u^2)} = \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{Cu+D}{1+u^2}$$

$$4u = Au + Au^3 - A - Au^2 + B + Bu^2 + Cu^3 - 2Cu^2 + Cu + Du^2 - 2Du + D$$

$$\begin{cases} A + C = 0 & \rightarrow A = -C \\ -A + B - 2C + D = 0 \\ C - 2D = 4 & \rightarrow D = \frac{1}{2}C - 2 \\ -A + B + D = 0 \end{cases}$$

$$\begin{cases} C + B - 2C + \frac{1}{2}C - 2 = 0 \\ C + B + \frac{1}{2}C - 2 = 0 \end{cases}$$

$$\begin{cases} B - \frac{1}{2}C = 2 \\ B + \frac{3}{2}C = 2 \end{cases} \rightarrow \underline{C=0} \quad \underline{B=2}$$

$$\underline{A=0; \quad D=-2}$$

$$\begin{aligned} &= \int_0^{\pi/3} \left(\frac{2}{(u-1)^2} - \frac{2}{1+u^2} \right) du \\ &= \frac{-2}{u-1} - 2 \tan^{-1} u \Big|_0^{\pi/3} \\ &= \frac{-2}{\tan \frac{x}{2} - 1} - 2 \tan^{-1} \left(\tan \frac{x}{2} \right) \Big|_0^{\pi/3} \end{aligned}$$

$$\begin{aligned}
&= \frac{-2}{\tan \frac{x}{2} - 1} - x \Big|_0^{\pi/3} \\
&= \frac{-2}{\frac{1}{\sqrt{3}} - 1} - \frac{\pi}{3} - 2 \\
&= \frac{-2\sqrt{3}}{1 - \sqrt{3}} - \frac{\pi}{3} - 2 \\
&= \frac{-2}{1 - \sqrt{3}} - \frac{\pi}{3} \\
&= \underline{1 + \sqrt{3} - \frac{\pi}{3}}
\end{aligned}
\qquad
= \frac{-2}{1 - \sqrt{3}} \frac{1 + \sqrt{3}}{1 + \sqrt{3}} - \frac{\pi}{3}$$

Exercise

Find the volume of the solid generated by the revolving the shaded region about x -axis

Solution

$$V = \pi \int_{0.5}^{2.5} y^2 dx$$

$$= \pi \int_{0.5}^{2.5} \frac{9}{3x - x^2} dx$$

$$= 9\pi \int_{0.5}^{2.5} \frac{1}{3x - x^2} dx$$

$$\frac{1}{3x - x^2} = \frac{1}{x(3 - x)}$$

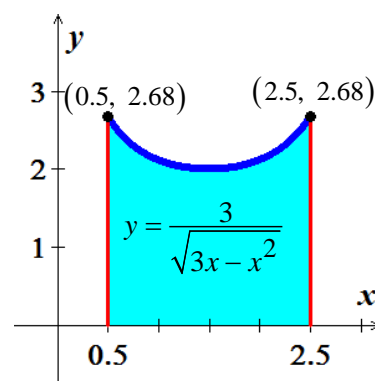
$$= \frac{A}{x} + \frac{B}{3 - x}$$

$$1 = 3A - Ax + Bx$$

$$\begin{cases} B - A = 0 \\ 3A = 1 \end{cases} \Rightarrow \underline{A = \frac{1}{3}} \quad \underline{B = \frac{1}{3}}$$

$$= 9\pi \int_{0.5}^{2.5} \frac{1}{3} \left(\frac{1}{x} + \frac{1}{3 - x} \right) dx$$

$$= 3\pi \int_{0.5}^{2.5} \left(\frac{1}{x} - \frac{1}{x - 3} \right) dx$$



$$\begin{aligned}
&= 3\pi \left[\int_{0.5}^{2.5} \frac{1}{x} dx - \int_{0.5}^{2.5} \frac{1}{x-3} dx \right] \\
&= 3\pi \left[\ln|x| - \ln|x-3| \right]_{0.5}^{2.5} \\
&= 3\pi \left[\ln \left| \frac{x}{x-3} \right| \right]_{0.5}^{2.5} \\
&= 3\pi \left[\ln \left| \frac{2.5}{-0.5} \right| - \ln \left| \frac{0.5}{-2.5} \right| \right] \\
&= 3\pi \left[\ln 5 - \ln \frac{1}{5} \right] \\
&= 3\pi [\ln 5 + \ln 5] \\
&= 3\pi [2 \ln 5] \\
&= \underline{3\pi \ln 25}
\end{aligned}$$

Exercise

Find the area of the region bounded by the graphs of $y = \frac{12}{x^2 + 5x + 6}$, $y = 0$, $x = 0$, and $x = 1$

Solution

$$A = \int_0^1 \frac{12}{x^2 + 5x + 6} dx$$

$$\frac{12}{x^2 + 5x + 6} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$12 = Ax + 3A + Bx + 2B$$

$$\begin{cases} A + B = 0 \\ 3A + 2B = 12 \end{cases} \rightarrow A = 12 \quad B = -12$$

$$A = \int_0^1 \frac{12}{x+2} dx - \int_0^1 \frac{12}{x+3} dx$$

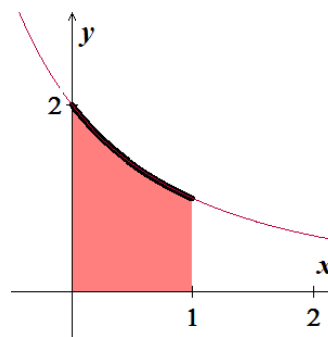
$$= 12 \left(\ln|x+2| - \ln|x+3| \right) \Big|_0^1$$

$$= 12 (\ln 3 - \ln 4 - \ln 2 + \ln 3)$$

$$= 12 (2 \ln 3 - 3 \ln 2)$$

$$= 12 (\ln 9 - \ln 8)$$

$$= \underline{12 \ln \frac{9}{8}}$$



Exercise

Find the area of the region bounded by the graphs of $y = \frac{7}{16-x^2}$ and $y = 1$

Solution

$$\begin{aligned} A &= 2 \int_0^3 \left(1 - \frac{7}{16-x^2} \right) dx \\ &= 2 \int_0^3 dx - 2 \int_0^3 \frac{7}{16-x^2} dx \end{aligned}$$

$$\begin{aligned} x &= 4 \sin \theta & 16 - x^2 &= 16 \cos^2 \theta \\ dx &= 4 \cos \theta d\theta \end{aligned}$$

$$= 2x \Big|_0^3 - 14 \int_0^3 \frac{1}{4 \cos \theta} d\theta$$

$$= 6 - \frac{7}{2} \int_0^3 \sec \theta d\theta$$

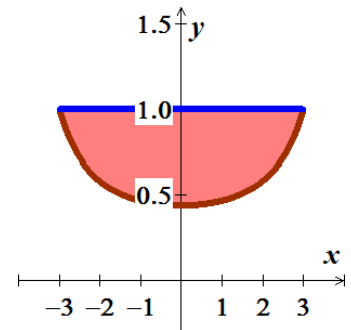
$$= 6 - \frac{7}{2} \ln |\sec \theta + \tan \theta| \Big|_0^3$$

$$= 6 - \frac{7}{2} \ln \left| \frac{4+x}{\sqrt{16-x^2}} \right| \Big|_0^3$$

$$= 6 - \frac{7}{2} \ln \left| \frac{7}{\sqrt{7}} \right|$$

$$= 6 - \frac{7}{2} \ln \sqrt{7}$$

$$= \underline{6 - \frac{7}{4} \ln 7} \approx 2.595$$



Exercise

Consider the region bounded by the graphs $y = \frac{2x}{x^2+1}$, $y = 0$, $x = 0$, and $x = 3$.

- Find the volume of the solid generated by revolving the region about the x -axis
- Find the centroid of the region.

Solution

$$\begin{aligned} a) \quad V &= \pi \int_0^3 \left(\frac{2x}{x^2+1} \right)^2 dx \\ &= 4\pi \int_0^3 \frac{x^2}{(x^2+1)^2} dx \end{aligned}$$

$$\frac{x^2}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$x^2 = Ax^3 + Ax + Bx^2 + B + Cx + D$$

$$\begin{cases} x^3 & A=0 \\ x^2 & B=1 \\ x & A+C=0 \rightarrow C=0 \\ x^0 & B+D=0 \rightarrow D=-1 \end{cases}$$

$$= 4\pi \int_0^3 \frac{1}{x^2+1} dx - 4\pi \int_0^3 \frac{1}{(x^2+1)^2} dx$$

$$= 4\pi \arctan x \Big|_0^3 - 4\pi \int_0^3 \frac{1}{\sec^2 \theta} d\theta$$

$$= 4\pi \arctan 3 - 4\pi \int_0^3 \cos^2 \theta d\theta$$

$$= 4\pi \arctan 3 - 2\pi \int_0^3 (1 + \cos 2\theta) d\theta$$

$$= 4\pi \arctan 3 - 2\pi (\theta + \sin \theta \cos \theta) \Big|_0^3$$

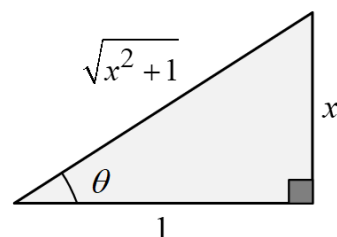
$$= 4\pi \arctan 3 - 2\pi \left(\arctan x + \frac{x}{x^2+1} \right) \Big|_0^3$$

$$= 4\pi \arctan 3 - 2\pi \left(\arctan 3 + \frac{3}{10} \right)$$

$$= \underline{2\pi \arctan 3 - \frac{3\pi}{5}} \quad \approx 5.963$$

$$x = \tan \theta \quad x^2 + 1 = \sec^2 \theta$$

$$dx = \sec^2 \theta d\theta$$

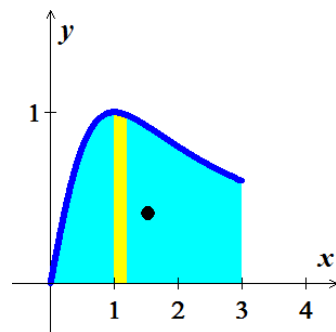


b) $A = \int_0^3 \frac{2x}{x^2+1} dx$

$$= \int_0^3 \frac{1}{x^2+1} d(x^2+1)$$

$$= \ln(x^2+1) \Big|_0^3$$

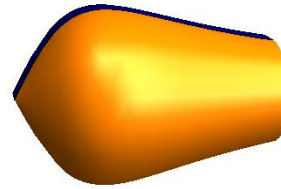
$$= \underline{\ln 10}$$



$$\bar{x} = \frac{1}{\ln 10} \int_0^3 \frac{2x^2}{x^2+1} dx$$

$$\bar{x} = \frac{1}{A} \int_a^b x \cdot f(x) dx$$

$$\begin{aligned}
&= \frac{1}{\ln 10} \int_0^3 \left(2 - \frac{2}{x^2 + 1} \right) dx \\
&= \frac{1}{\ln 10} (2x - 2 \arctan x) \Big|_0^3 \\
&= \frac{2}{\ln 10} (3 - \arctan 3) \Big| \approx 1.521
\end{aligned}$$



$$\begin{aligned}
\bar{y} &= \frac{1}{2} \frac{1}{\ln 10} \int_0^3 \left(\frac{2x}{x^2 + 1} \right)^2 dx & \bar{x} &= \frac{1}{A} \int_a^b x \cdot f(x) dx \\
&= \frac{2}{\ln 10} \int_0^3 \frac{x^2}{(x^2 + 1)^2} dx \\
&= \frac{2}{\ln 10} \int_0^3 \frac{1}{x^2 + 1} dx - \frac{2}{\ln 10} \int_0^3 \frac{1}{(x^2 + 1)^2} dx \\
&= \frac{2}{\ln 10} \left(\arctan x - \frac{1}{2} \arctan x - \frac{1}{2} \frac{x}{x^2 + 1} \right) \Big|_0^3 \\
&= \frac{2}{\ln 10} \left(\frac{1}{2} \arctan 3 - \frac{3}{20} \right) \\
&= \frac{1}{\ln 10} \left(\arctan 3 - \frac{3}{10} \right) \Big| \approx 0.412
\end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(\frac{2}{\ln 10} (3 - \arctan 3), \frac{1}{\ln 10} \left(\arctan 3 - \frac{3}{10} \right) \right) \Big| \approx (1.521, 0.412)$$

Exercise

Consider the region bounded by the graph $y^2 = \frac{(2-x)^2}{(1+x)^2}$ $0 \leq x \leq 1$.

Find the volume of the solid generated by revolving this region about the x -axis.

Solution

$$\begin{aligned}
V &= \pi \int_0^1 \frac{(2-x)^2}{(1+x)^2} dx \\
&= 4\pi \int_0^1 \frac{1}{(1+x)^2} dx - 4\pi \int_0^1 \frac{x}{(1+x)^2} dx + \pi \int_0^1 \frac{x^2}{(1+x)^2} dx
\end{aligned}$$

$$\frac{x}{(1+x)^2} = \frac{A}{x+1} + \frac{B}{(1+x)^2}$$

$$Ax + A + B = x$$

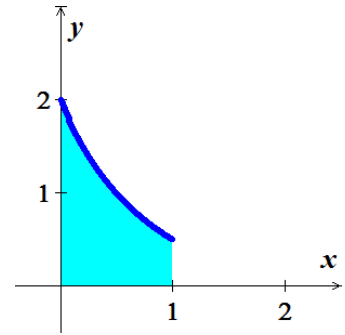
$$\underline{A=1, \quad B=-1}$$

$$\frac{x^2}{x^2 + 2x + 1} = 1 - \frac{2x+1}{(1+x)^2}$$

$$= 1 - \left(\frac{C}{x+1} + \frac{D}{(1+x)^2} \right)$$

$$Cx + C + D = 2x + 1$$

$$\underline{C=2, \quad D=-1}$$



$$= -4\pi \frac{1}{1+x} \Big|_0^1 - 4\pi \int_0^1 \frac{1}{x+1} dx + 4\pi \int_0^1 \frac{1}{(1+x)^2} dx + \pi \int_0^1 dx - 2\pi \int_0^1 \frac{1}{x+1} dx + \pi \int_0^1 \frac{1}{(1+x)^2} dx$$

$$= 2\pi + \left(-4\pi \ln(x+1) - 4\pi \frac{1}{x+1} + \pi x - 2\pi \ln(x+1) - \pi \frac{1}{x+1} \right) \Big|_0^1$$

$$= 2\pi - \left(6\pi \ln(x+1) + 5\pi \frac{1}{x+1} - \pi x \right) \Big|_0^1$$

$$= 2\pi - \left(6\pi \ln(2) + \frac{5}{2}\pi - \pi - 5\pi \right)$$

$$= 2\pi - 6\pi \ln 2 + \frac{7}{2}\pi$$

$$\underline{= \frac{\pi}{2}(11 - 12\ln 2)}$$



Exercise

A single infected individual enters a community of n susceptible individuals. Let x be the number of newly infected individuals at time t . The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. So,

$$\frac{dx}{dt} = k(x+1)(n-x) \text{ and you obtain}$$

$$\int \frac{1}{(x+1)(n-x)} dx = \int k dt$$

Solve for x as a function of t .

Solution

$$\frac{1}{(x+1)(n-x)} = \frac{A}{x+1} + \frac{B}{n-x}$$

$$1 = An - Ax + Bx + B$$

$$\begin{cases} -A + B = 0 \\ nA + B = 1 \end{cases}$$

$$(n+1)A = 1 \Rightarrow \underline{A = \frac{1}{n+1} = B}$$

$$\begin{aligned} \int \frac{1}{(x+1)(n-x)} dx &= \frac{1}{n+1} \int \frac{1}{x+1} dx + \frac{1}{n+1} \int \frac{1}{n-x} dx \\ &= \frac{1}{n+1} (\ln|x+1| - \ln|n-x|) \\ &= \frac{1}{n+1} \ln \left| \frac{x+1}{n-x} \right| \end{aligned}$$

$$\int k dt = kt + C$$

$$\frac{1}{n+1} \ln \left| \frac{x+1}{n-x} \right| = kt + C$$

$$x(t=0) = 0 \quad \rightarrow \quad \underline{\frac{1}{n+1} \ln \left| \frac{1}{n} \right| = C}$$

$$\frac{1}{n+1} \ln \left| \frac{x+1}{n-x} \right| = kt + \frac{1}{n+1} \ln \left| \frac{1}{n} \right|$$

$$\ln \left| \frac{x+1}{n-x} \right| - \ln \left| \frac{1}{n} \right| = (n+1)kt$$

$$\ln \left| \frac{nx+n}{n-x} \right| = (n+1)kt$$

$$\frac{nx+n}{n-x} = e^{(n+1)kt}$$

$$nx+n = ne^{(n+1)kt} - xe^{(n+1)kt}$$

$$\left(n + e^{(n+1)kt} \right) x = ne^{(n+1)kt} - n$$

$$\underline{x = \frac{ne^{(n+1)kt} - n}{n + e^{(n+1)kt}}} \quad \lim_{t \rightarrow \infty} x = n$$

Exercise

Evaluate $\int_0^1 \frac{x}{1+x^4} dx$ in **two** different ways.

Solution

1- Partial method

$$\frac{x}{1+x^4} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}$$

$$x = Ax^3 - \sqrt{2}Ax^2 + Ax + Bx^2 - \sqrt{2}Bx + B + Cx^3 + \sqrt{2}Cx^2 + Cx + Dx^2 + \sqrt{2}Dx + D$$

$$x^3 \quad A + C = 0 \rightarrow C = -A$$

$$x^2 \quad -\sqrt{2}A + B + \sqrt{2}C + D = 0$$

$$x \quad A - \sqrt{2}B + C + \sqrt{2}D = 1$$

$$x^0 \quad B + D = 0 \rightarrow D = -B$$

$$\begin{cases} -2\sqrt{2}A = 0 \rightarrow \underline{A = 0 = C} \\ -2\sqrt{2}B = 1 \Rightarrow \underline{B = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}} \end{cases}$$

$$\rightarrow \underline{D = \frac{\sqrt{2}}{4}}$$

$$\begin{aligned} \int_0^1 \frac{x}{1+x^4} dx &= -\frac{\sqrt{2}}{4} \int_0^1 \frac{1}{x^2 + \sqrt{2}x + 1} dx + \frac{\sqrt{2}}{4} \int_0^1 \frac{1}{x^2 - \sqrt{2}x + 1} dx \\ &= -\frac{\sqrt{2}}{4} \int_0^1 \frac{1}{\left(x + \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} dx + \frac{\sqrt{2}}{4} \int_0^1 \frac{1}{\left(x - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} dx \\ &= \frac{\sqrt{2}}{4} \left(-\sqrt{2} \arctan \sqrt{2} \left(x + \frac{\sqrt{2}}{2} \right) + \sqrt{2} \arctan \sqrt{2} \left(x - \frac{\sqrt{2}}{2} \right) \right) \Big|_0^1 \\ &= \frac{1}{2} \left(-\arctan(\sqrt{2}x + 1) + \arctan(\sqrt{2}x - 1) \right) \Big|_0^1 \\ &= \frac{1}{2} \left(-\arctan(\sqrt{2} + 1) + \arctan(\sqrt{2} - 1) + \arctan 1 - \arctan(-1) \right) \\ &= \underline{\underline{\frac{1}{2} \left(-\arctan(\sqrt{2} + 1) + \arctan(\sqrt{2} - 1) + \frac{\pi}{2} \right)}} \end{aligned}$$

2- Let $u = x^2 \rightarrow du = 2xdx$

$$\begin{aligned} \int_0^1 \frac{x}{1+x^4} dx &= \frac{1}{2} \int_0^1 \frac{1}{1+u^2} du \\ &= \frac{1}{2} \arctan x^2 \Big|_0^1 \\ &= \underline{\underline{\frac{\pi}{8}}} \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \left(\arctan(\sqrt{2} - 1) - \arctan(\sqrt{2} + 1) + \frac{\pi}{2} \right) &= \frac{1}{2} \left(\arctan \left(\frac{\sqrt{2} - 1 - \sqrt{2} - 1}{1 + (\sqrt{2} - 1)(\sqrt{2} + 1)} \right) + \frac{\pi}{2} \right) \\ &= \frac{1}{2} \left(\arctan(-1) + \frac{\pi}{2} \right) \end{aligned}$$

$$= \frac{1}{2} \left(-\frac{\pi}{4} + \frac{\pi}{2} \right)$$

$$= \frac{\pi}{8}$$