2.6 Improper Integrals ∞

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

$\frac{1}{\infty} = 0$ $\frac{1}{0} = \infty$ $e^{\infty} = \infty$ $e^{-\infty} = 0$

$\int$ = constant converges

$\int \pm\infty$ diverges

Ex A? $y = \frac{\ln x}{x^2}$ x $1 \to \infty$

$$A = \int_1^\infty \frac{\ln x}{x^2}\,dx$$

$u = \ln x$ $du = \frac{dx}{x}$

$v = \int \frac{dx}{x^2} = -\frac{1}{x}$

$$= -\frac{\ln x}{x}\Big|_1^\infty + \int_1^\infty \frac{dx}{x^2}$$

$$= -\frac{\ln x}{x} - \frac{1}{x}\Big|_1^\infty$$

$$= -0 - 0 + 0 + 1$$

$$= \boxed{1 \text{ unit}^2}$$

Ex $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \tan^{-1}x\Big|_{-\infty}^{\infty}$

$$= \tan^{-1}\infty - \tan^{-1}(-\infty)$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \boxed{\pi}$$

$x = \tan\theta$

$1 + x^2 = \sec^2\theta$

$dx = \sec^2\theta\,d\theta$

$$\int_{-\infty}^{\infty} \frac{\sec^2\theta\,d\theta}{\sec^2\theta}$$

$$= \int_{-\infty}^{\infty} d\theta$$

$$= \theta\Big|_{-\infty}^{\infty}$$

Ex $\int_1^{\infty} \frac{dx}{x^p}$

$(-p+1)$

If $p \neq 1$: $\int_1^{\infty} x^{-p} dx = \frac{x^{1-p}}{1-p}\Big|_1^{\infty}$

$= \frac{1}{1-p}\left((\infty)^{1-p} - 1\right)$

$-\frac{1}{1-p}$

$= \begin{cases} \frac{1}{p-1} & p > 1 \\ \infty & p < 1 \end{cases}$

If $p = 1$: $\int_1^{\infty} \frac{dx}{x} = \ln x \Big|_1^{\infty}$

$= \underline{\infty}$

$\therefore\ p \leq 1 \Rightarrow$ diverges or $\int_1^{\infty} \frac{dx}{x^p} = \infty$

$p > 1 \Rightarrow \int_1^{\infty} \frac{dx}{x^p} = \frac{1}{p-1}$

$a \to \infty$, $-\infty \to b$, $-\infty, \infty \Big\}\begin{matrix} -\infty, 0 \\ 0, \infty \end{matrix}$

Ex $\int_0^1 \frac{dx}{1-x} = -\int_0^1 \frac{d(1-x)}{1-x}$

$= -\ln|1-x| \Big|_0^1$

$= \underline{\infty}$

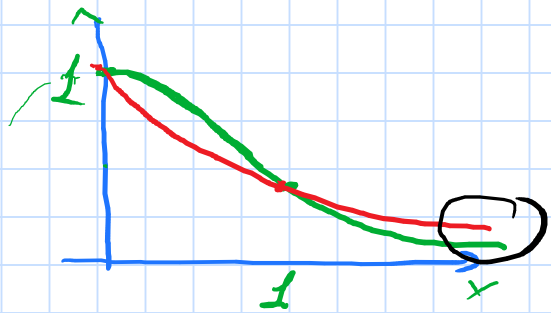
$\ln 0 = -\infty$

Ex $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^3 (x-1)^{-2/3}\, d(x-1)$ $\quad x \neq 1$

$$= 3\,(x-1)^{1/3}\Big|_0^3$$

$$= \underline{3\left(2^{1/3}+1\right)} \qquad 3\sqrt[3]{2}+3$$

Ex $\displaystyle\int_1^\infty e^{-x^2}\,dx$ $\qquad f(x) = e^{-x}$



$$\int_1^\infty e^{-x^2}\,dx \le \int_1^\infty e^{-x}\,dx$$

$$= -e^{-x}\Big|_1^\infty$$

$$= -\left(0-\frac{1}{e}\right)$$

$$= \underline{\frac{1}{e}}$$

Theorem: Direct Comparison Test

$0 \le f(x) \le g(x)$

(1) $\int_a^\infty f(x)$ converges if $\int_a^\infty g(x)$ converges

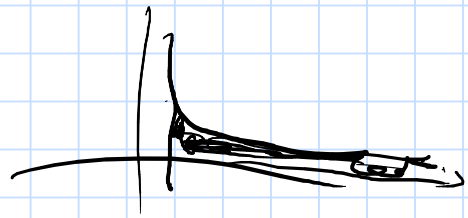
(2) $\int_a^\infty g(x)$ diverges if $\int_a^\infty f(x)$ diverges

Ex R: $y = \frac{1}{x}$ x-axis $(y=0)$ $x \geq 1$

a) V? rev. x-axis

$$V = \pi \int_1^\infty \frac{1}{x^2}\,dx$$

$$V = \pi \int_a^b (f(x))^2\,dx$$



$$= -\pi \frac{1}{x}\Big|_1^\infty$$

$$= -\pi(0-1)$$

$$= \pi \text{ unit}^3$$

b) S? rev. x-axis

$$\sqrt{1+(f')^2} = \sqrt{1+\left(-\frac{1}{x^2}\right)^2}$$

$$= \sqrt{\frac{x^4+1}{x^4}}$$

$$S = 2\pi \int_1^\infty \frac{1}{x} \frac{\sqrt{x^4+1}}{x^2}\,dx$$

$$= 2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\,dx$$

$$x^{-3}\sqrt{x^4+1} \quad \cancel{S<S}$$

$$\sqrt{x^4+1} > \sqrt{x^4}$$

$$= x^2$$

$$> 2\pi \int_1^\infty \frac{x^2}{x^3}\,dx$$

$$= 2\pi \int_1^\infty \frac{dx}{x}$$

$$= 2\pi \ln x\Big|_1^\infty$$

$$\ln \infty = \infty$$

$$= \infty \quad \text{unit}^2$$

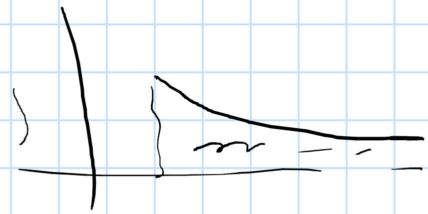
c) V? y-axis is

$$V = 2\pi\int_1^\infty x\,\frac{1}{x}\,dx$$

$$= 2\pi\int_1^\infty dx$$

$$= 2\pi\, x\Big|_1^\infty$$

$$= \infty \text{ unit}^3$$



#2 $$\int_0^4 \frac{dx}{\sqrt{4-x}} = -\int_0^4 (4-x)^{-1/2}\, d(4-x)$$

$$= -2(4-x)^{1/2}\Big|_0^4$$

$$= -2(0-2)$$

$$= 4$$

#25 $$\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}} = \frac{1}{2}\int_{-\infty}^{\infty}(x^2+4)^{-3/2}\, d(x^2+4)$$

$$= -(x^2+4)^{-1/2}\Big|_{-\infty}^{\infty}$$

$\infty^{-1} = \frac{1}{\infty}$

$$= -[0-0]$$

$$= 0$$

#6 $$\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx = -\int_{-\infty}^{\infty} e^{-x^2}\, d(-x^2)$$

$$= -e^{-x^2}\Big|_{-\infty}^{\infty}$$

$$= 0$$

#24 $\int_0^1 x \ln x \, dx = \int_0^1 e^y (y) e^y \, dy$

$y = \ln x \Rightarrow x = e^y$

$dy = \frac{dx}{x} \Rightarrow dx = e^y dy$

$= \int_0^1 y e^{2y} dy$

		$\int e^{2y}$
+	y	$\frac{1}{2}e^{2y}$
−	1	$\frac{1}{4}e^{2y}$

$= \left(\frac{1}{2}y - \frac{1}{4}\right)e^{2y} \Big|_0^1$

$= \left(\frac{1}{2}\ln x - \frac{1}{4}\right)x^2 \Big|_0^1$

$= -\frac{1}{4}$

#26 $\int_1^\infty (1-x)e^x dx$

		$\int e^x$
+	$1-x$	e^x
−	-1	e^x

$= (1 - x + 1)e^x \Big|_1^\infty$

$= (2-x)e^x \Big|_1^\infty$

$= \infty$

#27 $\int_{-\infty}^\infty \frac{e^x dx}{1+e^{2x}} = \int_{-\infty}^\infty \frac{d(e^x)}{1+(e^x)^2}$

$\int \frac{du}{1+u^2}$

$= \tan^{-1} e^x \Big|_{-\infty}^\infty$

$= \tan^{-1}\infty - \tan^{-1} 0$

$= \frac{\pi}{2} - 0$

$= \frac{\pi}{2}$

1^{st} O.D.E (Ordinary Differential Eqns)

$$\frac{dy}{dx} = f(x, y) = y'$$

$$f(x, y, y') = y''$$

Ex $y = \frac{c}{x} + 2$ ← soln $\quad \frac{dy}{dx} = \frac{1}{x}(2-y)$ ←

$$\frac{dy}{dx} = \frac{1}{x}\left(2 - \frac{c}{x} + 2\right)$$

$$= -\frac{c}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{x}(2-y)$$

$$-\frac{c}{x^2} = \frac{1}{x}\left(2 - \frac{c}{x} + 2\right)$$

$$-\frac{c}{x^2} = -\frac{c}{x^2}$$

$$y = \frac{c}{x} + 2$$

$$\frac{dy}{dx} = -\frac{c}{x^2}$$

Ex $y = (x+1) - \frac{1}{3}e^x \rightarrow y' = y - x, \; y(0) = \frac{2}{3}$

$$y' = 1 - \frac{1}{3}e^x \qquad\qquad y' = x + 1 - \frac{1}{3}e^x - x$$

$$= 1 - \frac{1}{3}e^x$$

$$y(0) = 0 + 1 - \frac{1}{3}e^0$$

$$\frac{2}{3} \overset{?}{=} 1 - \frac{1}{3}$$

$$\frac{2}{3} = \frac{2}{3} \checkmark$$

Uniqueness & Existence

direction field

slope field

$$y' + Py = Q$$

Separable Eqn

$$y' + Py = 0$$

homogeneous eqn

$$\frac{dy}{dx} = -P(x)\, y$$

$$\int \frac{dy}{y} = -\int P(x)\, dx$$

$$\ln|y| = -\int P(x)\, dx + C$$

$$y = e^{-\int P(x)\, dx + C}$$

$$= e^{-\int P(x)\, dx}\, \underbrace{e^{C}}$$

$$= A e^{-\int P(x)\, dx}$$

Ex $\quad y' = ty^2$

$$\frac{dy}{dt} = ty^2$$

$$\int \frac{dy}{y^2} = \int t\,dt$$

$$-\frac{1}{y} = \frac{1}{2}t^2 + C$$

$$= \frac{t^2 + 2C}{2} \qquad C_1 = 2C$$

$$y(t) = \frac{2}{t^2 + C_1}$$

Ex $\quad y' = \dfrac{2xy + 2x}{x^2 - 1}$

$$\frac{dy}{dx} = \frac{2x(y+1)}{x^2-1}$$

$$\frac{dy}{y+1} = \frac{2x}{x^2-1}\,dx$$

$$\int \frac{d(y+1)}{y+1} = \int \frac{d(x^2-1)}{x^2-1}$$

$$\ln|y+1| = \ln|x^2-1| + \ln C \qquad C>0$$

$$= \ln(C|x^2-1|)$$

$$y+1 = C|x^2-1|$$

$$y(x) = C|x^2-1| - 1$$

Ex $\quad x\dfrac{dy}{dx} = x^2 + 3y \qquad x > 0$

$y' + py = q$

$e^{\int p\,dx}$

$xy' - 3y = x^2$

$y' - \dfrac{3}{x}y = x$

$e^{-\int \frac{3}{x}dx} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3}$

$= \dfrac{1}{x^3}$

$\int \dfrac{1}{x^3} x\,dx = \int \dfrac{dx}{x^2} = -\dfrac{1}{x}$

$y(x) = \dfrac{1}{\frac{1}{x^3}}\left(-\dfrac{1}{x} + c\right)$

$= x^3\left(-\dfrac{1}{x} + c\right)$

$= -x^2 + cx^3$

Ex $3xy' - y = \ln x + 1$ $\boxed{y(1) = -2}$

$$y' - \frac{1}{3x} y = \frac{1}{3x}(\ln x + 1)$$

$$e^{-\int \frac{dx}{3x}} = e^{-\frac{1}{3}\ln x} = x^{-1/3}$$

$e^{a \ln u} = e^{\ln u^a} = u^a$

$$\int x^{-1/3} \frac{1}{3x}(\ln x + 1)\, dx$$

$$= \frac{1}{3}\int x^{-4/3}(\ln x + 1)\, dx$$

$u = 1 + \ln x$

$du = \frac{dx}{x}$

$v = \int x^{-4/3} dx$

$= -3x^{-1/3}$

$$= \frac{1}{3}\left[-3x^{-1/3}(1 + \ln x) + 3\int x^{-4/3} dx\right]$$

$$= -x^{-1/3}(1 + \ln x) - 3x^{-1/3}$$

$$y(x) = x^{1/3}\left(-x^{-1/3}(1 + \ln x) - 3x^{-1/3} + C\right)$$

$$= -1 - \ln x - 3 + Cx^{1/3}$$

$$= -4 - \ln x + Cx^{1/3}$$

$y(1) = -4 + C$

$-2 = -4 + C$

$C = 2$

$$y(x) = -4 - \ln x + 2x^{1/3}$$