Lecture One

Section 1.1 – Propositional Logic

Introduction

The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments.

The logic rules are used in the design of computer circuits, the construction of computer programs, and the verification of the correctness of programs.

Propositions

A proposition is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

All the following declarative sentences are propositions

- ✓ Washington, D.C. is the capital of the United States of America. ***True***
- ✓ 1 + 1 = 2 ***True***
- ✓ 2 + 2 = 3 ***False***

Example

Consider the following sentences

1. What time is it?
2. Read this carefully
3. $x+1=2$
4. $x+y=z$

Solution

Sentences 1 and 2 are not propositions because they are not declarative sentences.
Sentences 3 and 4 are not propositions because they are not true (***T***) or false (***F***).

Definition

Let p be a proposition. The negation of p, denoted by $\neg p$ (also denoted by $\overline{p}$), is the statement

"It is not the case that p."

The proposition $\neg p$ is read "not p". The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

Example

Find the negation of the proposition: “Michael’s PC runs Linux” and express this in simple English.

Solution

The negation: Michael’s PC does not run Linux

Example

Find the negation of the proposition: “Vandana’s smartphone has at least 32GB of memory” and express this in simple English.

Solution

The negation: Vandana’s smartphone has less than 32GB of memory
Vandana’s smartphone does not have at least 32GB of memory

***Table*: Truth table of the Negation of a Proposition**

p	$\neg p$
T	***F***
F	***T***

Definition

Let p and q be propositions. The ***conjunction*** of p and q, denoted by $p \wedge q$, is the proposition “p and q.“
The *conjunction* $p \wedge q$ p and q is true for both are true and it’s false otherwise.

Example

Find the conjunction of the propositions p and q where p is the proposition “your PC has more than 16GB free hard disk space” and q is the proposition “your PC processor runs faster than 1 GHz.”

Solution

The conjunction is $p \wedge q$ and can be expressed as:

- ✓ Your PC has more than 16GB free hard disk space and its processor runs faster than 1 GHz.

For this conjunction to be true, both conditions given must be true.
It is false when one or both of these conditions are false.

Definition

Let p and q be propositions. The ***disjunction*** of p and q, denoted by $p \vee q$, is the proposition "p or q."
The disjunction $p \vee q$ is false when both p and q are false and it's true otherwise.

Truth Table for the Conjunction of Two Propositions.		
p	q	$p \wedge q$
T	***T***	***T***
T	***F***	***F***
F	***T***	***F***
F	***F***	***F***

Truth Table for the Disjunction of Two Propositions.		
p	q	$p \vee q$
T	***T***	***T***
T	***F***	***T***
F	***T***	***T***
F	***F***	***F***

Example

Find the disjunction of the propositions p and q where p is the proposition "your PC has more than 16GB free hard disk space" and q is the proposition "your PC processor runs faster than 1 GHz."

Solution

The disjunction is $p \vee q$ and can be expressed as:

✓ Your PC has more than 16GB free hard disk space, or the processor in your PC runs faster than 1 GHz.

Definition

Let p and q be propositions. The ***exclusive or*** of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

Truth Table for the Exclusive Or of Two Propositions.		
p	q	$p \oplus q$
T	***T***	***F***
T	***F***	***T***
F	***T***	***T***
F	***F***	***F***

Truth Table for the Conditional Statement $p \rightarrow q$.		
p	q	$p \rightarrow q$
T	***T***	***T***
T	***F***	***F***
F	***T***	***T***
F	***F***	***T***

Definition

Let p and q be propositions. The ***conditional statement*** $p \to q$, is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

If p, then q	***p implies q***
If p, q	***p only if q***
p is sufficient for q	***a sufficient condition for q is p***
q if p	***q whenever p***
q when p	***q necessary for p***
a necessary condition for p is q	***q follows from p***
q unless $\neg p$	

Example

Let p be the statement "Maria learns discrete mathematics" and q the statement "Maria will find a good job". Express the statement $p \to q$ as a statement in English.

Solution

$p \to q$ represents the statement:

- ✓ If Maria learns discrete mathematics, then she will find a good job.

There are many other way to express this conditional statement.

- ✓ Maria will find a good job when she learns discrete mathematics.
- ✓ For Maria to get a good job, it is sufficient for her to learn discrete mathematics.
- ✓ Maria will find a good job unless she does not learn discrete mathematics.

Example

What is the value of the variable x after the statement

if $2+2=4$ ***then*** $x := x+1$

If $x=0$ before the statement is encountered? (The symbol $:=$ stands for assignment. The statement $x := x+1$ means the assignment of the value of $x+1$ to x.)

Solution

Because 2 + 2 = 4 is true, the assignment statement $x := x+1$ is executed.

Hence, x has the value $0+1=1$ after this statement is encountered.

Converse, Contrapositive, and Inverse.

The ***converse*** of $p \to q$ is the proposition $q \to p$

The ***contrapositive*** of $p \to q$ is the proposition $\neg q \to \neg p$

The ***inverse*** of $p \to q$ is the proposition $\neg p \to \neg q$

Example

What are the contrapositive, the converse, and the inverse of the conditional statement

"The home team wins whenever it is raining?"

Solution

Because "*q* whenever *p*" is one of these ways to express the conditional statement $p \to q$, the original statement can be written as

✓ If it is raining, then the home team wins.

The contrapositive of this conditional statement is:

✓ If the home team does not win, then it is not raining.

The converse: If the home team wins, then it is raining.

The inverse: If it is not raining, then the home team does not win.

Only the contrapositive is equivalent to the original statement.

Definition

Let *p* and *q* be propositions. The ***biconditional statement*** $p \leftrightarrow q$ is the proposition "*p* if and only if *q*." The biconditional statement $p \leftrightarrow q$ is true when *p* and *q* have the same truth values, and is false otherwise. Biconditional statements are also called ***bi-implications***.

p is necessary and sufficient for *q*

If *p* then *q*, and conversely

p iff *q*

$p \leftrightarrow q$ has exactly the truth value as $(p \to q) \land (q \to p)$

Example

Let *p* be the statement "You can take the flight," and let *q* be the statement "You buy a ticket." Then $p \leftrightarrow q$ is the statement: "You can take the flight if and only if you buy a ticket."

Solution

This statement is true if *p* and *q* are either both true or false, that is, if you buy a ticket and can take the flight or if you do not buy a ticket and you cannot take the flight.

It is false when *p* and *q* have opposite truth values, that is, when you do not buy a ticket, but you can take the flight, and when you buy a ticket but you cannot take the flight.

Truth Tables of Compound Propositions

Example

Construct the truth table of the compound proposition $(p \vee \neg q) \to (p \wedge q)$

Solution

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \to (p \wedge q)$
T	***T***	***F***	***T***	***T***	***T***
T	***F***	***T***	***T***	***F***	***F***
F	***T***	***F***	***F***	***F***	***T***
F	***F***	***T***	***T***	***F***	***F***

Precedence of Logical Operators

Precedence of Logical Operators	
Operator	***Precedence***
$\neg$	***1***
$\wedge$	***2***
$\vee$	***3***
$\to$	***4***
$\leftrightarrow$	***5***

$p \wedge q \vee r$ ***means*** $(p \wedge q) \vee r$

Logic and Bit Opertions

True Value	***Bit***
T	***1***
F	***0***

Bit Operators OR, AND, and XOR				
x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	***0***	***0***	***0***	***0***
0	***1***	***1***	***0***	***1***
1	***0***	***1***	***0***	***1***
1	***1***	***1***	***1***	***0***

Definition

A bit string is a sequence of zero or more bits. The length of this string is the number of bits in the string.

Exercises *Section* 1.1 – Propositional Logic

1. Which of these sentences are propositions? What are truth values of those that are propositions?

a) Boston is the capital of Massachusetts.
b) Miami is the capital of Florida
c) $2+3=5$
d) $5+7=10$
e) $x+2=11$
f) Answer this question
g) Do not pass go
h) What time is it?
i) The moon is made of green cheese
j) $2^n \ge 100$

2. What is the negation if each of these propositions?

a) Mei has an MP3 player
b) There is no pollution in Texas
c) $2+1=3$
d) There are 13 items in a baker's dozen,
e) 121 is a perfect square

3. Suppose the Smartphone *A* has 256 MB RAM and 32 GB ROM, and the resolution of its camera is 8 MP; Smartphone *B* has 288 MB RAM and 64 GB ROM, and the resolution of its camera is 4 MP; Smartphone *C* has 128 MB RAM and 32 GB ROM, and the resolution of its camera is 5 MP. Determine the truth value of each of these propositions.

a) Smartphone *B* has the most RAM of these three smartphones
b) Smartphone *C* has more ROM or higher resolution camera than Smartphone *B*.
c) Smartphone *B* has more RAM, more ROM, and a higher resolution camera than Smartphone *A*.
d) If Smartphone *B* has more RAM and more ROM than Smartphone *C*, then it also has a higher resolution camera.
e) Smartphone *A* has more RAM than Smartphone *B* if and only if Smartphone *B* has more RAM than Smartphone *A*.

4. Let *p* and *q* be the proposition

p: I bought a lottery ticket this week
q: I won the million dollar jackpot

a) $\neg p$	*b)* $p \lor q$	*c)* $p \to q$	*d)* $p \land q$
e) $p \leftrightarrow q$	*f)* $\neg p \to \neg q$	*g)* $\neg p \land \neg q$	*h)* $\neg p \lor (p \land q)$

5. Let p and q be the proposition

p: Swimming at the New Jersey shore is allowed

q: Sharks have been spotted new the shore

$a)$ $\neg q$ $b)$ $p \wedge q$ $c)$ $\neg p \vee q$ $d)$ $p \to \neg q$

$e)$ $\neg q \to p$ $f)$ $\neg p \to \neg q$ $g)$ $p \leftrightarrow \neg q$ $h)$ $\neg p \wedge (p \vee \neg q)$

6. Let p, q and r be the proposition

p: You have the flu

q: You miss the final examination

r: You pass the course

Express each of these proposition as an English sentence

$a)$ $p \to q$ $b)$ $\neg q \leftrightarrow r$ $c)$ $q \to \neg r$ $d)$ $p \vee q \vee r$

$e)$ $(p \to \neg r) \vee (q \to \neg r)$ $f)$ $(p \wedge q) \vee (\neg q \wedge r)$

7. Determine whether each of these conditional statements is true or false.

a) If $1+1=2$, then 2 + 2 = 5
b) If 1 + 1 = 3, then 2 + 2 = 4
c) If 1 + 1 = 3, then 2 + 2 = 5
d) If monkeys can fly, then 1 + 1 = 3
e) If 1 + 1 = 3, then unicorns exist
f) If 1 + 1 = 3, then dogs can fly
g) If 1 + 1 = 2, the dogs can fly
h) If 2 + 2 = 4, then 1 + 2 = 3

8. Write each of these propositions in the form "*p* if and only if *q*" in English
a) If it is hot outside you buy an ice cream cone, and if you buy an ice cream cone it is hot outside.
b) For you to win the contest it is necessary and sufficient that you have only winning ticket.
c) You get promoted only if you have connections, and you have connections only if you get promoted.
d) If you watch television your mind will decay, and conversely.
e) The trains run late on exactly those days when I take it.
f) For you to get an ***A*** in this course, it is necessary and sufficient that you learn how to solve discrete mathematics problems.
g) If you read the newspaper every day, you will be informed, and conversely.
h) It rains if it is a weekend day, and it is a weekend day if it rains.
i) You can see the wizard only if the wizard is not in, and the wizard is not in only if you can see him

9. Construct a truth table for each of these compound propositions.

a) $p \wedge \neg p$

b) $p \vee \neg p$

c) $p \rightarrow \neg p$

d) $p \leftrightarrow \neg p$

e) $p \rightarrow \neg q$

f) $\neg p \leftrightarrow q$

g) $(p \vee \neg q) \rightarrow q$

h) $(p \vee q) \rightarrow (p \wedge q)$

i) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$

j) $(p \rightarrow q) \rightarrow (q \rightarrow p)$

k) $p \oplus (p \vee q)$

l) $(p \wedge q) \rightarrow (p \vee q)$

m) $(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q)$

n) $(p \rightarrow q) \vee (\neg p \rightarrow q)$

o) $(p \rightarrow q) \wedge (\neg p \rightarrow q)$

p) $(p \vee q) \vee r$

q) $(p \vee q) \wedge r$

r) $(p \wedge q) \vee r$

s) $(p \wedge q) \wedge r$

t) $(p \vee q) \wedge \neg r$

Section 1.2 – Propositional Equivalences

Introduction

An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value.

Definition

A compound proposition that is always true, no matter what the truth values of the proposition variables that occur in it, is called a ***tautology***. A compound proposition that is always false is called ***contradiction***. A compound proposition that is neither a tautology nor a contradiction is called a ***contingency***.

Example

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	***F***	***T***	***F***
F	***T***	***T***	***F***

$p \vee \neg p$ is always true, it is tautology

$p \wedge \neg p$ is always false, it is contradiction.

Logical Equivalences

Definition

Compound propositions p and q are called ***logically equivalent*** if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

De Morgan's Laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Example

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Solution

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	***T***	***T***	***F***	***F***	***F***	***F***
T	***F***	***T***	***F***	***F***	***T***	***F***
F	***T***	***T***	***F***	***T***	***F***	***F***
F	***F***	***F***	***T***	***T***	***T***	***T***

The truth table shows that $\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$ is a tautology and these compound propositions are logically equivalent.

Example

Show that $p \to q$ and $\neg p \vee q$ are logically equivalent.

Solution

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	***T***	***F***	***T***	***T***
T	***F***	***F***	***F***	***F***
F	***T***	***T***	***T***	***T***
F	***F***	***T***	***T***	***T***

The truth table shows that $p \to q$ and $\neg p \vee q$ are logically equivalent.

Example

Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. This is the *distributive law* of disjunction over conjunction.

Solution

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	***T***	***T***	***T***	***T***	***T***	***T***	***T***
T	***T***	***F***	***F***	***T***	***T***	***T***	***T***
T	***F***	***T***	***F***	***T***	***T***	***T***	***T***
T	***F***	***F***	***F***	***T***	***T***	***T***	***T***
F	***T***	***T***	***T***	***T***	***T***	***T***	***T***
F	***T***	***F***	***F***	***F***	***T***	***F***	***F***
F	***F***	***T***	***F***	***F***	***F***	***T***	***F***
F	***F***	***F***	***F***	***F***	***F***	***F***	***F***

The truth table Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

In these equivalences, ***T*** denotes the compound proposition that is always true and ***F*** denotes the compound proposition that is always false.

Logical Equivalences	
Equivalence	***Name***
$p \wedge \mathbf{T} = p$	***Identity laws***
$p \vee \mathbf{F} = p$	
$p \vee \mathbf{T} \equiv \mathbf{T}$	***Domination laws***
$p \wedge \mathbf{F} \equiv \mathbf{F}$	
$p \vee p \equiv p$	***Idempotent laws***
$p \wedge p \equiv p$	
$\neg(\neg p) \equiv p$	***Double negation law***
$p \vee q \equiv q \vee p$	***Commutative laws***
$p \wedge q \equiv q \wedge p$	
$(p \vee q) \vee r \equiv p \vee (q \vee r)$	***Associative laws***
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	***Distributive laws***
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	
$\neg(p \wedge q) \equiv \neg p \vee \neg q$	***De Morgan's laws***
$\neg(p \vee q) \equiv \neg p \wedge \neg q$	
$p \vee (p \wedge q) \equiv p$	***Absorption laws***
$p \wedge (p \vee q) \equiv p$	
$p \vee \neg p \equiv \mathbf{T}$	***Negation laws***
$p \wedge \neg p \equiv \mathbf{F}$	

Logical Equivalences Involving Conditional Statements
$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \vee \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Logical Equivalences Involving Biconditional Statements
$p \leftrightarrow q \equiv (p \to q) \wedge (q \to r)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

$$\neg\left(p_1 \vee p_2 \vee \cdots \vee p_n\right) \equiv \left(\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n\right)$$

$$\neg\left(p_1 \wedge p_2 \wedge \cdots \wedge p_n\right) \equiv \left(\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n\right)$$

Using De Morgan's Laws

The two logical equivalences known as De Morgan's laws are particularly important. The equivalence $\neg(p \vee q) \equiv \neg p \wedge \neg q$ and similarly $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Example

Use De Morgan's laws to express the negations of "Miguel has a cellphone and he has a laptop computer" and "Heather will go to the concert or Steve will go to the concert."

Solution

Let: p be "Miguel has a cellphone"
q be "Miguel has a laptop computer"

can be expressed as $p \wedge q$

By De Morgan's laws $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$ .We can express the negation of our original statement as "*Miguel does not have a cellphone* ***or*** *he does not have a laptop computer*"

Let: r be "Heather will go to the concert"
s be "Steve will go to the concert"

can be expressed as $r \vee s$

By De Morgan's laws $\neg(r \vee s) \equiv \neg r \wedge \neg s$. We can express the negation of our original statement as "*Heather will not go to the concert* ***and*** *Steve will not go to the concert*."

Example

Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

Solution

$$\begin{aligned} \neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) \\ &\equiv \neg(\neg p) \wedge \neg q \\ &\equiv p \wedge \neg q \end{aligned}$$

p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg q$	$p \wedge \neg q$
T	***T***	***T***	***F***	***F***	***F***
T	***F***	***F***	***T***	***T***	***T***
F	***T***	***T***	***F***	***F***	***F***
F	***F***	***T***	***F***	***T***	***F***

Example

Show that $\neg\left(p\vee\left(\neg p\wedge q\right)\right)$ and $\neg p\wedge\neg q$ are logically equivalent by developing a series of logical equivalences.

Solution

$\neg\left(p\vee\left(\neg p\wedge q\right)\right)\equiv\neg p\wedge\neg\left(\neg p\wedge q\right)$	*By De Morgan's law*
$\equiv\neg p\wedge\left(p\vee\neg q\right)$	*Double negation law*
$\equiv\left(\neg p\wedge p\right)\vee\left(\neg p\wedge\neg q\right)$	*Distribution law*
$\equiv \boldsymbol{F}\vee\left(\neg p\wedge\neg q\right)$	$\neg p\wedge p\equiv \boldsymbol{F}$
$\equiv\left(\neg p\wedge\neg q\right)\vee \boldsymbol{F}$	*Commutative law for disjunction*
$\equiv\neg p\wedge\neg q$	*Identity law*

Example

Show that $\left(p\wedge q\right)\rightarrow\left(p\vee q\right)$ is a tautology.

Solution

$\left(p\wedge q\right)\rightarrow\left(p\vee q\right)\equiv\neg\left(p\wedge q\right)\vee\left(p\vee q\right)$	
$\equiv\left(\neg p\vee\neg q\right)\vee\left(p\vee q\right)$	*By De Morgan's law*
$\equiv\left(\neg p\vee p\right)\vee\left(\neg q\vee q\right)$	*By Associative and commutative laws*
$\equiv \boldsymbol{T}\vee \boldsymbol{T}$	
$\equiv \boldsymbol{T}$	

Exercises *Section* 1.2 – Propositional Equivalences

1. Use the truth table to verify these equivalences

a) $p \wedge \boldsymbol{T} \equiv p$

b) $p \vee \boldsymbol{F} \equiv p$

c) $p \wedge \boldsymbol{F} \equiv \boldsymbol{F}$

d) $p \vee \boldsymbol{T} \equiv \boldsymbol{T}$

e) $p \vee p \equiv p$

f) $p \wedge p \equiv p$

2. Show that $\neg(\neg p)$ and p are logically equivalent

3. Use the truth table to verify the commutative laws

a) $p \vee q \equiv q \vee p$

b) $p \wedge q \equiv q \wedge p$

4. Use the truth table to verify the associative laws

a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

5. Show that each of these conditional statements is a tautology by using truth result tables.

a) $(p \wedge q) \rightarrow p$

b) $p \rightarrow (p \vee q)$

c) $\neg p \rightarrow (p \rightarrow q)$

d) $(p \wedge q) \rightarrow (p \rightarrow q)$

e) $\neg(p \rightarrow q) \rightarrow p$

f) $[\neg p \wedge (p \vee q)] \rightarrow q$

g) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

h) $[p \wedge (p \rightarrow q)] \rightarrow q$

6. Show that $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent

7. Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent

8. Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent

9. Show that $\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$ are logically equivalent

10. Show that $(p \rightarrow q) \vee (p \rightarrow r)$ and $p \rightarrow (q \vee r)$ are logically equivalent

11. Show that $(p \rightarrow r) \vee (q \rightarrow r)$ and $(p \wedge q) \rightarrow r$ are logically equivalent

12. Show that $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is a tautology

13. Show that $(p \vee q) \vee (\neg p \vee r) \rightarrow (q \vee r)$ is a tautology

14. Show that | (NAND) is functionally complete

Section 1.3 – Predicates and Quantifiers

Introduction

To express the meaning of a wide range of statements in mathematics and computer science in ways that permit us to reason and explore relationships between object, are called ***predicate logic***.

Predicates

Statements involving variables, such as

$$"x > 3," \quad "x = y + 3," \quad \textit{and} \quad "\textit{computer } x \textit{ is under attack by an intruder.}"$$

are often found in mathematical assertions, in computer programs, and in system specifications

Example

Let $P(x)$ denote the statement $x > 3$. What are the truth values of $P(4)$ and $P(2)$?

Solution

We obtain the statement $P(4)$ by setting $x = 4$ in the statement $x > 3$. Hence, $P(4)$, which is the statement $4 > 3$ is true.

However, $P(2)$, which is the statement $2 > 3$ is false.

Example

Let $Q(x, y)$ denote the statement $x = y + 3$. What are the truth values of propositions $Q(1,\ 2)$ and $Q(3,\ 0)$?

Solution

To obtain $Q(1,\ 2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$. Hence, $Q(1,\ 2)$ is the statement $1 = 2 + 3$ which is false.

The statement $Q(3,\ 0)$ is the proposition $3 = 0 + 3$ which is true.

Example

Let $A(c,n)$ denote the statement "Computer *c* is connected to netwrok *n*," where *c* is a variable representing a computer and *n* is a variable representing a network. Suppose that the computer MATH1 is connected to network CAMPUS2, but not to network CAMPUS1. What are the values of $A(MATH1, CAMPUS1)$ and $A(MATH1, CAMPUS2)$?

Solution

Because MATH1 is not connected to the CAMPUS1 network, we see that $A(MATH1, CAMPUS1)$ is false.
However, because MATH1 is connected to the CAMPUS2 network, we see that $A(MATH1, CAMPUS2)$ is true.

- Consider the statement ***if*** $x > 0$ **then** $x := x+1$
  When this statement is encountered in a program, the value of the variable *x* at the point in the execution of the program is inserted into $P(x)$, which is $x > 0$. If $P(x)$ is true for this value of *x*, the assignment statement $x := x+1$ is executed. So the value of *x* is increased by 1. If $P(x)$ is false for this value of *x*, the assignment statement is not executed, so the value of *x* is not changed.

Preconditions and Postconditions

Predicates are also used to establish the correctness of computer programs, that is, to show that computer programs always produce desired output given valid input.
The statements that describe valid input are known as ***preconditions*** and the conditions that the output should satisfy when the program has run are known as ***postconditions***.

Quantifiers

To create a proposition from a propositional function is called ***quantification***. Quantification expresses the extent to which a predicate is true over a range of elements. The words ***all, some, many, none*** and ***few*** are used in quantifications.
The area of logic that deals with predicates and quantifiers is called the ***predicate calculus***.

There are two quantifiers

- Existential Quantifier "$\exists$" reads "there exists"
- Universal Quantifier "$\forall$" reads "for all"

Each is placed in front of a propositional function and ***binds*** it to obtain a proposition with semantic value.

Definition

The ***universal quantification*** of $P(x)$ is the statement

“$P(x)$ for all values of *x* in the domain.”

The notation $\forall x\ P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the universal quantifier. We read $\forall x\ P(x)$ as “for all $x\ P(x)$” or “for every $x\ P(x)$”. An element for which $P(x)$ is false is called a counterexample of $\forall x\ P(x)$.

Statement	***When True?***	***When False?***
$\forall x\ P(x)$	$P(x)$ ***is true for every x.***	***There is an x for which*** $P(x)$ ***is false.***
$\forall x\ P(x)$	***There is an x for which*** $P(x)$ ***is true.***	$P(x)$ ***is false for every x.***

Example

Let $P(x)$ be the statement $"x+1>x"$. What is the truth value of the quantification $\forall x\ P(x)$, where the domain consists of all real numbers?

Solution

Because $P(x)$ is true for all real numbers *x*, the quantification $\forall x\ P(x)$ is true.

Example

Let $Q(x)$ be the statement $"x<2"$. What is the truth value of the quantification $\forall x\ Q(x)$, where the domain consists of all real numbers?

Solution

$Q(x)$ is not true for every real number *x*, because, $Q(3)$ is false.
That is, $x=3$ is a counterexample for the statement $\forall x\ Q(x)$. Thus $\forall x\ Q(x)$ is false.

Example

What is the truth value of $\forall x\ P(x)$, where $P(x)$ is the statement $"x^2<10"$ and the domain consists of the positive integers not exceeding 4?

Solution

The domain consists of the integers 1, 2, 3, and 4. Since $"4^2<10"$ is false, it follows that $\forall x\ P(x)$ is false.

Example

What is the truth value of $\forall x\left(x^2 \geq x\right)$, if the domain consists of real number? What is the truth value of this statement of the domain consists of all integers?

Solution

- $\left(\frac{1}{2}\right)^2 \not\geq \frac{1}{2}$, it follows that $\forall x\left(x^2 \geq x\right)$ is false.

  Note: $x^2 \geq x \quad iff \quad x^2 - x = x(x-1) \geq 0 \;\Rightarrow\; x \leq 0 \quad x \geq 1$

- If the domain consists of the integers, $\forall x\left(x^2 \geq x\right)$ is true, because there are no integers *x* with $0 < x < 1$

Definition

The ***existential quantification*** of $P(x)$ is the proposition

"There exists an element *x* in the domain such that $P(x)$"

We use the notation $\exists x\, P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the ***existential quantifier***.

Example

Let $P(x)$ denote the statement $"x > 3"$. What is the truth value of the quantification $\exists x\, P(x)$, where the domain consists of all real numbers?

Solution

Because $"x > 3"$ is sometimes true, for instance, when $x = 4$ – the existence quantification of $P(x)$, which is $\exists x\, P(x)$, is true.

Example

Let $Q(x)$ denote the statement $"x = x + 1"$. What is the truth value of the quantification $\exists x\, Q(x)$, where the domain consists of all real numbers?

Solution

$Q(x)$ is false for every real number *x*, the existential quantification of $Q(x)$, which is $\exists x\, Q(x)$, is false.

Example

What is the truth value of $\exists x\, P(x)$, where $P(x)$ is the statement "$x^2 > 10$" and the universe discourse of the positive integers not exceeding 4?

Solution

The domain consists of the integers 1, 2, 3, and 4. Since "$4^2 > 10$" is true, it follows that $\exists x\, P(x)$ is true.

Quantifiers with Restricted Domains

What do the statement $\forall x < 0\ \left(x^2 > 0\right)$, $\forall y \neq 0\ \left(y^3 \neq 0\right)$, and $\forall z > 0\ \left(z^2 = 0\right)$ mean, where the domain in each case consists of the real numbers?

- The statement $\forall x < 0\ \left(x^2 > 0\right)$ states that for every real numbers *x* with $x < 0$, $x^2 < 0$. That is, it states "the square of a negative real number is positive." This statement is the same as $\forall x\ \left(x < 0 \rightarrow x^2 > 0\right)$.
- The statement $\forall y \neq 0\ \left(y^3 \neq 0\right)$ states that for every real numbers *y* with $y \neq 0$, $y^3 \neq 0$. That is, it states "the cube of every nonzero real number is nonzero." This statement is the equivalent to $\forall y\ \left(y \neq 0 \rightarrow y^3 \neq 0\right)$.
- The statement $\forall z > 0\ \left(z^2 = 2\right)$ states that for every real numbers *z* with $z > 0$ such that $z^2 = 2$. That is, it states "There is a positive square root of 2." This statement is the equivalent to $\exists z\ \left(z > 0\ \wedge\ z^2 = 2\right)$.

Binding Variables

When a quantifier is used on the variable x, we say that this occurrence of the variable is bound. An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be ***free***.

Logical Equivalences Involving Quantifiers

Definition

Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ involving predicates and quantifiers are logically equivalent.

Negating Quantified Expressions

We will often want to consider the negation of a quantified expression. For instance, consider the negation of the statemnt

"Every student in your class has taken a course in calculus"

This statement is a universal quantification, namely, $\forall x\ P(x)$

where $P(x)$ is the statement "*x* has taken a course in calculus" and the domain consists of the students in your class.

The negation of this statement is " It is not the case that every student in your class who has not taken a course in calculus". This is simply the existential quantification of the negation of the original proposition function, namely, $\exists x\ \neg P(x)$.

This example illustrates the following logical equivalence:

$$\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$$

$$\neg\exists x\ Q(x) \equiv \forall x\ \neg Q(x)$$

De Morgan's Laws for Quantifiers			
Negation	***Equivalent Statement***	***When Is Negation True?***	***When False?***
$\neg\exists x\ P(x)$	$\forall x\ \neg P(x)$	***For every x,*** $P(x)$ ***is false***	***There is an x for which*** $P(x)$ ***is true***
$\neg\forall x\ P(x)$	$\exists x\ \neg P(x)$	***There is an x for which*** $P(x)$ ***is false***	***For every x,*** $P(x)$ ***is true***

Example

What are the negations of the statements "There is an honest politician" and "All Americans eat cheeseburgers"?

Solution

Let $H(x)$ denote "x is honest."

Then the statement "there is an honest politician" is represented by $\exists x\ H(x)$

The negation statement is $\neg\exists x\ H(x)$, which is equivalent to $\forall x\ \neg H(x)$.

This negation can be expressed as "All politicians are not honest"

Let $G(x)$ denote "x eats cheeseburgers."

Then the statement "All Americans eat cheeseburgers" is represented by $\forall x\ G(x)$

The negation statement is $\neg\forall x\ G(x)$, which is equivalent to $\exists x\ \neg G(x)$.

This negation can be expressed as "There is an American who does not eat cheeseburgers."

Example

What are the negations of the statements $\forall x\ \left(x^2 > x\right)$ and $\exists x\ \left(x^2 = 2\right)$

Solution

The negation of $\forall x\ \left(x^2 > x\right)$ is the statement $\neg\forall x\ \left(x^2 > x\right) \equiv \exists x\ \neg\left(x^2 > x\right)$

Which can be written as $\exists x\ \left(x^2 \le x\right)$

The negation of $\exists x\ \left(x^2 = 2\right)$ is the statement $\neg\exists x\ \left(x^2 = 2\right) \equiv \forall x\ \neg\left(x^2 = 2\right)$

Which can be written as $\forall x\ \left(x^2 \ne 2\right)$

Example

Consider these statements, of which the first three are premises and the fourth is a valid conclusion

"All hummingbirds are richly colored"
"No large birds live on honey"
"Birds that do not live on honey are dull in color"
"Hummingbirds are small"

Let $P(x)$, $Q(x)$, $R(x)$ and $S(x)$ be the statements "x is a hummingbird, " "x is large," "x lives on honey," and " x is richly colored," respectively. Assuming that the domain consists of all birds, express the statements in the argument using quantifiers and $P(x)$, $Q(x)$, $R(x)$ and $S(x)$.

Solution

We can express the statements in the argument as

$$\forall x\left(P(x) \rightarrow S(x)\right)$$

$$\neg\exists x\left(Q(x) \wedge R(x)\right)$$

$$\forall x\left(\neg R(x) \rightarrow \neg S(x)\right)$$

$$\forall x\left(P(x) \rightarrow \neg Q(x)\right)$$

"***small***" is the same as "not large"

"***dull in color***" is the same as "not richly colored"

Exercises *Section* 1.3 – Predicates and Quantifiers

1. Let $P(x)$ denote the statement "$x \le 4$". What are these truth values?

a) $P(0)$ *b)* $P(4)$ *c)* $P(6)$

2. Let $P(x)$ be the statement "*the word* ***x*** *contains the letter* ***a***". What are these truth values?

a) $P(orange)$ *b)* $P(lemon)$ *c)* $P(true)$ *d)* $P(false)$

3. State the value of *x* after the statement **if** $P(x)$ **then** $x := 1$ is executed, where $P(x)$ is the statement "$x > 1$", if the value of *x* when the statement is reached is

a) $x = 0$ *b)* $x = 1$ *c)* $x = 2$

4. Let $P(x)$ be the statement "***x*** *spends more than five hours every weekday in class*," where the domain for *x* consists of all students. Express each of these quantifications in English.

a) $\exists x\, P(x)$ *b)* $\forall x\, P(x)$ *c)* $\exists x\, \neg P(x)$ *d)* $\forall x\, \neg P(x)$

5. Let $N(x)$ be the statement "***x*** *has visited North Dakota*," where the domain consists of the students in your class. Express each of these quantifications in English.

a) $\exists x\, N(x)$ *b)* $\forall x\, N(x)$ *c)* $\neg\exists x\, N(x)$ *d)* $\exists x\, \neg N(x)$

e) $\neg\forall x\, N(x)$ *f)* $\forall x\, \neg N(x)$

6. Let $C(x)$ be the statement "***x*** *has a cat*," let $D(x)$ be the statement "***x*** *has a dog*,", and let $F(x)$ be the statement "***x*** *has a ferret*." Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

a) A student in your class has a cat, a dog, and a ferret.
b) All students in your class have a cat, a dog, or a ferret.
c) Some student in your class has a cat and a ferret, but not a dog.
d) No student in your class has a cat, a dog, and a ferret.
e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

7. Let $Q(x)$ be the statement "$x+1 > 2x$". If the domain consists of all integers, what are these truth values?

a) $Q(0)$ *b)* $Q(-1)$ *c)* $Q(1)$ *d)* $\exists x\, Q(x)$

e) $\forall x\, Q(x)$ *f)* $\exists x\, \neg Q(x)$ *g)* $\forall x\, \neg Q(x)$

8. Determine the truth value of each of these statements if the domain consists of all integers

a) $\forall n(n+1 > n)$ *b)* $\exists n(2n = 3n)$ *c)* $\exists n(n = -n)$ *d)* $\forall n(3n \le 4n)$

9. Determine the truth value of each of these statements if the domain consists of all real numbers *a*) $\exists x\left(x^3=-1\right)$ *b*) $\exists x\left(x^4<x^2\right)$ *c*) $\forall x\left((-x)^2=x^2\right)$ *d*) $\forall x(2x>x)$

10. Suppose that the domain of the propositional function $P(x)$ consists of the integers 1, 2, 3, 4, and 5.

Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

a) $\exists x\ P(x)$ *b*) $\forall x\ P(x)$ *c*) $\neg\exists x\ P(x)$ *d*) $\neg\forall x\ P(x)$

e) $\forall x\ \left((x\neq 3)\rightarrow P(x)\right)\ \vee\ \exists x\ \neg P(x)$

11. For each of these statements find a domain for which the statement is true and a domain for which the statement is false.

a) Everyone is studying discrete mathematics.
b) Everyone is older than 21 years.
c) Every two people have the same mother.
d) No Two different people have the same grandmother.

12. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.
b) Not everyone is perfect.
c) All your friends are perfect.
d) At least one of your friends is perfect.
e) Everyone is your friend and is perfect.
f) Not everybody is your friend or someone is not perfect.

13. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) Something is not in the correct place.
b) All tools are in the correct place and are in excellent condition.
c) Everything is in the correct place and in excellent condition.
d) Nothing is in the correct place and is in excellent condition.
e) One of your tools is not in the correct place, but it is in excellent condition.

Section 1.4 – Nested Quantifiers

Introduction

Nested quantifiers commonly occur in mathematics and computer science. Nested quantifiers can sometimes be difficult to understand.
We will see how to use nested quantifiers to express mathematical statements such as “The sum of two positive integers is always positive.” We will show how nested quantifiers can be used to translate sentences such as “Everyone has exactly one best friend” into logical statements.

Example

Translate the statement $\forall x \forall y \left((x > 0) \land (y < 0) \rightarrow (xy < 0) \right)$
Where the domain for both variables consists of all real numbers.

Solution

This statement says that for every real number *x* and every real number *y*, if $x > 0$ and $y < 0$, then $xy < 0$. That is, this statement says that for all real numbers *x* and *y*, if *x* is positive and *y* is negative, then *xy* is negative.
This can be stated more succinctly as
“The product of a postitive real number and a negative real numer is always a negative real number.”

The Order of Quantifiers

Example

Let $P(x, y)$ be the statement "$x + y = y + x$". What are the truth values of the quantifications $\forall x \forall y\ P(x, y)$ and $\forall y \forall x\ P(x, y)$ where the domain for all variables consists of all real numbers?

Solution

The quantification $\forall x \forall y\ P(x, y)$ denotes the proposition

For all real numbers *x*, for all real numbers *y*, $x + y = y + x$

The quantification $\forall y \forall x\ P(x, y)$ denotes the proposition

For all real numbers *y*, for all real numbers *x*, $x + y = y + x$

Which they have the same meaning.
Therefore; $\forall x \forall y\ P(x, y)$ and $\forall y \forall x\ P(x, y)$ have the same meaning, and both are true. This illustrates the principle that the order of nested universal quantifiers in a statement without other quantifiers can be changed without changing the meaning of the quantified statement.

Example

Let $P(x, y)$ be the statement "$x+y=0$". What are the truth values of the quantifications $\exists y \forall x\ P(x, y)$ and $\forall x \exists y\ P(x, y)$ where the domain for all variables consists of all real numbers?

Solution

The quantification $\exists y \forall x\ P(x, y)$ denotes the proposition

There is a real number *y*, such that for every real number *x*, $P(x, y)$

No matter what value of *y* is chosen, there is only one value of *x* for which $x+y=0$. Because there is no real number *y* such that $x+y=0$ for all real numbers *x*, the statement $\exists y \forall x\ P(x, y)$ is false.

$x+1=0$

The quantification $\forall x \exists y\ P(x, y)$ denotes the proposition

For every real number *x*, there is a real number *y* such that $P(x, y)$

✓ *"For all x, there exists a y such that* $P(x, y)$ *"*

$x+y=0 \quad \Rightarrow \quad y=-x$

Hence, the statement $\forall x \exists y\ P(x, y)$ is true.

✓ ***There exists an x such that for all y*** $P(x, y)$ ***is true"***

Quantifications of Two variables

Statement	***When True?***	***When False?***
$\forall x \forall y\ P(x, y)$ $\forall x \forall y\ P(x, y)$	$P(x, y)$ ***is true for every pair*** x, y	***There is a pair*** x, y ***for which*** $P(x, y)$ ***is false.***
$\forall x \exists y\ P(x, y)$	***For every x there is a y for which*** $P(x, y)$ ***is true.***	***There is an x such that*** $P(x, y)$ ***is false for every y.***
$\exists x \forall y\ P(x, y)$	***There is an x for which*** $P(x, y)$ ***is true for every y.***	***For every x there is a y for which*** $P(x, y)$ ***is false.***
$\exists x \exists y\ P(x, y)$ $\exists y \exists x\ P(x, y)$	***There is a pair*** x, y ***for which*** $P(x, y)$ ***is true.***	$P(x, y)$ ***is false for every pair*** x, y

Example

Let $P(x, y, z)$ be the statement "$x+y=z$". What are the truth values of the statements $\forall x \forall y \exists z\ P(x, y, z)$ and $\exists z \forall x \forall y\ P(x, y, z)$ where the domain for all variables consists of all real numbers?

Solution

The statement $\forall x \forall y \exists z\ P(x, y, z)$ denotes the proposition

For all real numbers *x* and for all real numbers *y* there is a real number *z* such that $x+y=z$

This statement is true.

The statement $\exists z \forall x \forall y\ P(x, y, z)$ denotes the proposition

There is a real number *z* such that for all real numbers *x* and for all real numbers *y* it is true that $x+y=z$

This statement is false, because there is no value of z that satisfies the equation $x+y=z$ for all values of *x* and *y*.

Translating Mathematical Statements into Statements Involving Nested Quantifiers

Example

Translate the statement "The sum of two positive integers is always positive" into a logical expression.

Solution

Let *x* and *y* be the positive integers variables which: "For all positive integers *x* and *y*, $x+y$ is positive."
We can express as:

$$\forall x \forall y\big((x>0)\wedge(y>0)\rightarrow(x+y>0)\big)$$

We also can translate this using the positive integers as the domain.

$$\forall x \forall y\ (x+y>0)$$

Where the domain for both variables consists of all positive integers

Example

Translate the statement $\forall x\left(C(x)\vee\exists y\left(C(y)\wedge F(x,y)\right)\right)$ into English, Where $C(x)$ is "*x* has a computer," $F(x,\ y)$ is "*x* and *y* are friends," and the domain for both *x* and *y* consists of all students in your school.

Solution

The statement says

For every student *x* in your school, *x* has a computer or there is a student *y* such that *y* has a computer and *x* and *y* are friends.

In other words

Every student in your school has a computer or has a friend who has a computer.

Example

Translate the statement $\exists x\forall y\forall z\left(F(x,y)\wedge F(x,z)\wedge(y\neq z)\right)\to\neg F(y,z)$ into English, where $F(a,b)$ means *a* and *b* are friends and the domain for both *x*, *y* and *z* consists of all students in your school.

Solution

The original statement says:

There is a student *x* such that for all students *y* and all students *z* other than *y*, if *x* and *y* are friends and *x* and *z* are friends, then *y* and *z* are not friends.

In other words

There is a student none of whose friends are also friends with each other.

Example

Express the statement "Everyone has exactly one best friend" as a logical expression involving predicates, quantifiers with domain consisting of all people, and logical connectives.

Solution

For every person *x*, *x* has exactly one best friend *y*. "$\forall x$ (person *x* has exactly one best friend)" with domain consisting of all people.

For every person *z*, if person *z* is not person *y*, then *z* is not the best friend of *x*.

Let $B(x,y)$ be the statement "*y* is the best friend of *x*".

Therefore; the statement can be expressed as:

$$\exists y\left(B(x,y)\wedge\forall z\left((z\neq y)\to\neg B(x,z)\right)\right)$$

The original statement can be expressed as:

$$\forall x\exists y\left(B(x,y)\wedge\forall z\left((z\neq y)\to\neg B(x,z)\right)\right)$$

Negating Multiple Quantifiers

The negation rules for single quantifiers:

- $\neg\forall xP(x) = \exists x\neg P(x)$
- $\neg\exists xP(x) = \forall x\neg P(x)$
- Essentially, you change the quantifier(s), and negate what it's quantifying

Example

Express the negation of the statement $\forall x\exists y(xy = 1)$ so that no negation precedes a quantifier

Solution

The negation is: $\neg\forall x\exists y(xy = 1)$

Which is equivalent: $\equiv \exists x\neg\exists y(xy = 1)$

$$\equiv \exists x\forall y\neg(xy = 1)$$

$$\equiv \exists x\forall y(xy \neq 1)$$

$$\neg(\forall x\exists y\forall zP(x, y, z)) = \exists x\neg(\exists y\forall zP(x, y, z))$$

$$= \exists x\forall y\neg(\forall zP(x, y, z))$$

$$= \exists x\forall y\exists z\neg P(x, y, z)$$

Consider $\neg(\forall x\exists y\ P(x,y)) = \exists x\forall y\ \neg P(x,y)$

- The left side is saying "for all *x*, there exists a *y* such that P is true"
- To disprove it (negate it), you need to show that "there exists an *x* such that for all *y*, P is false"

Consider $\neg(\exists x\forall y\ P(x,y)) = \forall x\exists y\ \neg P(x,y)$

- The left side is saying "there exists an *x* such that for all *y*, P is true"
- To disprove it (negate it), you need to show that "for all *x*, there exists a *y* such that P is false"

Exercises *Section* 1.4 – Nested Quantifiers

1. Translate these statements into English, where the domain for each variable consists of all real numbers

a) $\forall x \exists y (x < y)$

b) $\exists x \forall y (xy = y)$

c) $\forall x \forall y \left(\left((x \geq 0) \wedge (y < 0) \right) \rightarrow (x - y > 0) \right)$

d) $\forall x \forall y \left(\left((x \geq 0) \wedge (y \geq 0) \right) \rightarrow (xy \geq 0) \right)$

e) $\forall x \forall y \exists z (xy = z)$

f) $\forall x \forall y \exists z (x = y + z)$

2. Let $Q(x,\ y)$ be the statement "*x* has sent an e-mail message to *y*," where the domain for both *x* and *y* consists of all students in your class. Express each of these quantifications in English

a) $\exists x \exists y Q(x,\ y)$

b) $\exists x \forall y Q(x,\ y)$

c) $\forall x \exists y Q(x,\ y)$

d) $\exists y \forall x Q(x,\ y)$

e) $\forall y \exists x Q(x,\ y)$

f) $\forall x \forall y Q(x,\ y)$

3. Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators where the domain consists of all integers.

a) The product of two negative integers is positive.

b) The average of two positive integers is positive.

c) The difference of two negative integers is not necessarily negative.

d) The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers.

4. Rewrite these statements so that the negations only appear within the predicates

a) $\neg \exists y \forall x P(x, y)$

b) $\neg \forall x \exists y P(x, y)$

c) $\neg \exists y \left(Q(y) \wedge \forall x \neg R(x, y) \right)$

5. Express the negations of each of these statements so that all negation symbols immediately precede predicates.

a) $\forall x \exists y \forall z T(x, y, z)$

b) $\forall x \exists y P(x, y) \vee \forall x \exists y Q(x, y)$

6. Let $T(x, y)$ mean that student *x* likes cuisine *y*, where the domain for *x* consists of all students at your school and the domain for *y* consists of all cuisines. Express each of these statements by a simple English sentence.

a) $\neg T(A, J)$

b) $\exists x T(x, \ Korean) \wedge \forall x Tx(x, \ Mexican)$

c) $\exists y (T(Monique, \ y) \vee T(Jay, \ y))$

d) $\forall x \forall z \exists y ((x \neq z) \rightarrow \neg (T(x, y) \wedge T(z, y)))$

e) $\exists x \exists z \forall y (T(x, y) \leftrightarrow T(z, y))$

f) $\forall x \forall z \exists y (T(x, y) \leftrightarrow T(z, y))$

7. Let $L(x, y)$ be the statement " *x* loves *y*", where the domain for both *x* and *y* consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody loves Jerry.
b) Everybody loves somebody.
c) There is somebody whom everybody loves.
d) Nobody loves everybody.
e) There is somebody whom Lois does not love.
f) There is somebody whom no one loves.
g) There is exactly one person whom everybody loves.
h) There are exactly two people whom L loves.
i) Everyone loves himself or herself.
j) There is someone who loves no one besides himself or herself.

8. Let $S(x)$ be the predicate "*x* is a student," $F(x)$ the predicate "*x* is a faculty member," $A(x, y)$ the predicate "*x* has asked *y* a question," where the domain consists of all people associated with your school. Use quantifiers to express each of these statements.

a) Lois asked Professor Fred a question.
b) Every student has asked Professor Fred a question.
c) Every faculty member has either asked Professor Fred a question or been asked a question by Professor Miller.
d) Some student has not asked any faculty member a question.
e) There is a faculty member who has never been asked a question by a student.
f) Some student has asked every faculty member a question.
g) There is a faculty member who has asked every other faculty member a question.
h) Some student has never been asked a question by a faculty member.

9. Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.

a) Every user has access to exactly one mailbox.
b) There is a process that continues to run during all error conditions only if the kernel is working correctly.
c) All users on the campus network can access all websites whose url has a .edu extension.

10. Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers

a) $\exists x \forall y \left(x + y = y \right)$

b) $\forall x \forall y \left(\left(\left(x \geq 0 \right) \wedge \left(y < 0 \right) \right) \rightarrow \left(x - y > 0 \right) \right)$

c) $\exists x \exists y \left(\left(\left(x \leq 0 \right) \wedge \left(y \leq 0 \right) \right) \wedge \left(x - y > 0 \right) \right)$

d) $\forall x \forall y \left(\left(\left(x \neq 0 \right) \wedge \left(y \neq 0 \right) \right) \leftrightarrow \left(xy \neq 0 \right) \right)$

11. Determine the truth value of each of these statements if the domain for all variables consists of all integers

a) $\forall n \exists m \ \left(n^2 < m \right)$

b) $\exists n \forall m \ \left(n < m^2 \right)$

c) $\forall n \exists m \ \left(n + m = 0 \right)$

d) $\exists n \forall m \ \left(nm = m \right)$

e) $\exists n \exists m \ \left(n^2 + m^2 = 5 \right)$

f) $\exists n \exists m \ \left(n^2 + m^2 = 6 \right)$

g) $\exists n \exists m \ \left(n + m = 4 \ \wedge \ n - m = 1 \right)$

h) $\exists n \exists m \ \left(n + m = 4 \ \wedge \ n - m = 2 \right)$

i) $\forall n \forall m \exists p \ \left(p = \dfrac{m + n}{2} \right)$

Section 1.5 – Introduction to Proofs

Some Terminology

A ***theorem*** is a statement that can be shown to be true. Theorems can also be referred to as facts or results. We demonstrate that a theorem is true with a ***proof***. A proof is a valid argument that establishes the truth of a theorem. The statements used in a proof can include ***axioms*** (or ***postulates***), which are statements we assume to be true.

Less important theorems sometimes are called ***propositions***. A less important theorem that is helpful in the proof of other results is called a ***lemma*** (***plural lemmas or lemmata***).

A ***corollary*** is a theorem that can be established directly from a theorem that has been proved.
A ***conjecture*** is a statement that is being proposed to be true statement, usually on the basis of some partial evidence, a heuristic argument, or the intuition of an expert.

Direct Proofs

A direct proof is a conditional statement $p \to q$ is constructed when the first step is the assumption that *p* is true; subsequence steps are constructed using rules of inference, with the final step showing that *q* must be true.
Consider an implication: $p \to q$

- If *p* is false, then the implication is always true.
- Show that if *p* is true then *q* is true.

Definition

The integer *n* is ***even*** if there exists an integer *k* such that $n = 2k$, and *n* is ***odd*** if there exists an integer *k* such that $n = 2k + 1$. Two integers have the ***same parity*** when both are even or both are odd; they have ***opposite parity*** when one is even and the other is odd.

Example

Give a direct proof of the theorem "If *n* is an odd integer, then n^2 is odd"

Solution

This states: $\forall n\ P(n) \to Q(n)$, where

$P(n)$ is "*n* is an odd integer"

$Q(n)$ is "n^2 is an odd"

Using direct proof, we assume that n is odd, is a true statement. By the definition of an odd integer, it follows that $n = 2k+1$, where k is some integer. We need to show that n^2 is odd.

$$n^2 = (2k+1)^2 \qquad \textit{Square both sides}$$

$$= 4k^2 + 4k + 1$$

$$= 2\left(2k^2 + 2k\right) + 1 \qquad 2k^2 + 2k = K$$

$$= 2K + 1$$

By the definition of an odd integer, we can conclude that n^2 is also an odd integer.

Example

Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square.
(An integer a is a perfect square if there is an integer b such that $a = b^2$.)

Solution

Using direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, namely, we assume that m and n are both perfect squares.
By the definition of a perfect square:

$\exists s \ni m = s^2$ There is an integer s such that $m = s^2$

$\exists t \ni n = t^2$

The goal is to show that nm is also a perfect square.

$$mn = s^2t^2 = (ss)(tt) = (st)(st) = (st)^2$$

By the definition of a perfect square, it follows that nm is also a perfect square.

Proof by Contraposition

In logic, ***proof by contrapositive*** is a form of proof that establishes the truth or validity of a proposition by demonstrating the truth or validity of the converse of its negated parts.

To prove by contraposition, consider an implication $p \to q$, prove that $\neg q \to \neg p$,

- If the antecedent $\neg q$ is false, then the contrapositive is always true.
- Show that if $\neg q$ is true, then $\neg p$ is true

To perform an indirect proof, do a direct proof on the contrapositive.

Example

Prove that if *n* is an integer and 3*n* + 2 is odd, then *n* is odd.

Solution

The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "3*n* + 2 is odd, then *n* is odd" is false. Assume that *n* is even, then by the definition of an even integer, $n = 2k$ for some integer *k*.

$$3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$$

This show $3n + 2$ is even, because it is a multiple of 2, therefore not odd. This is the negation of the theorem of the conditional statement implies that the hypothesis is false; the original conditional statement is true.

Our proof by contraposition succeeded; we have proved the theorem "If 3*n* + 2 is odd, then *n* is odd."

Example

Prove that if $n = ab$ where *a* and *b* are positive integers, then $a \le \sqrt{n}$ or $b \le \sqrt{n}$.

Solution

By using proof by contraposition, let assume that the conclusion of the conditional statement "if $n = ab$ where *a* and *b* are positive integers, then $a \le \sqrt{n}$ or $b \le \sqrt{n}$" is false.

$\left(a \le \sqrt{n}\right) \vee \left(b \le \sqrt{n}\right)$ is false.

Using the meaning of disjunction together with De Morgan's law, that implies that both $a \le \sqrt{n}$ and $b \le \sqrt{n}$ are false $\Rightarrow \quad a > \sqrt{n}$ *and* $b > \sqrt{n}$

Then $\qquad ab > \sqrt{n}\sqrt{n} = n \quad \Rightarrow \quad ab \ne n$, which contradicts the statement $n = ab$.

This is the negation of the theorem of the conditional statement implies that the hypothesis is false; the original conditional statement is true. Our proof by contraposition succeeded; we have proved the theorem "If $n = ab$ where *a* and *b* are positive integers, then $a \le \sqrt{n}$ or $b \le \sqrt{n}$."

Vacuous and Trivial Proofs

If *p* is a conjunction of other hypotheses and we know one or more of these hypotheses is false, then *p* is false and so $p \to q$ is ***vacuously*** true regardless of the truth value of *q*.

If we know *q* is true then $p \to q$ is true regardless of the truth value of *p*, this called ***Trivial Proofs***.

Example

Show that the proposition $P(0)$ is true, where $P(n)$ is "If $n > 1$, then $n^2 > n$" and the domain consists of all integers.

Solution

Using a vacuous proof; $P(0)$ is "If $0 > 1$, then $0^2 > 0$". Indeed, the hypothesis $0 > 1$ is false. This tells us that $P(0)$ is automatically true.

Example

Let $P(n)$ be "If *a* and *b* are positive integers with $a \ge b$, then $a^n \ge b^n$," where the domain consists of all nonnegative integers. Show that $P(0)$ is true.

Solution

The proposition $P(0)$ is "If $a \ge b$, then $a^0 \ge b^0$." Because $a^0 = b^0 = 1$, the conclusion of the conditional statement is true. Hence, this conditional statement, which is $P(0)$, is true.

Definition

The real number *r* is rational if there exist integers *p* and *q* with $q \ne 0$ such that $r = \frac{p}{q}$. A real number that is not rational is called irrational.

Example

Prove that the sum of two rational numbers is rational. (Note that if we include the implicit quantifiers here, the theorem we want to prove is "For every real number *r* and every real number *s*, if *r* and *s* are rational numbers, the *r* + *s* is rational.)

Solution

From the definition of a rational number, that there exist integers *p* and *q* with $q \ne 0$ such that $r = \frac{p}{q}$, and integers *t* and *u* with $u \ne 0$ such that $s = \frac{t}{u}$.

$$r+s=\frac{p}{q}+\frac{t}{u}=\frac{pu+qt}{qu}$$

Because $q \neq 0$ and $u \neq 0$, it follows that $qu \neq 0$. Therefore; we have $r + s$ is rational.

Example

Prove that n is an integer and n^2 is odd, then n is odd

Solution

Suppose that n is an integer and n^2 is odd. Then, $\exists k \in \mathbb{Z} \ni n^2 = 2k+1$.
$\Rightarrow n = \pm\sqrt{2k+1}$ (which is not useful).

By using proof by contraposition, the statement n is not odd, that means n is even.
That implies that $\exists k \in \mathbb{Z} \ni n = 2k$.

To prove the theorem, we need to show that this hypothesis implies the conclusion that n^2 is not odd, that means n^2 is even.

$n^2 = (2k)^2 = 4k^2 = 2\left(2k^2\right)$, which implies that n^2 is even.

We have proved that n is an integer and n^2 is odd, then n is odd by a proof of contraposition.

Proofs by Contradiction

The basic ides of a proof of contradiction is to assume that the statement we want to prove is false. That is, the supposition that p is false followed necessarily by the conclusion q from not $\neg p$, where q is false, which implies that p is true.

Given a statement p, assume it is false, assume $\neg p$

- Prove that $\neg p$ cannot occur
    - A contradiction exists
    - Given a statement of the form $p \rightarrow q$
    - To assume it's false, you only have to consider the case where p is true and q is false

Example

Show that at least four of any 22 days must fall on the same day of the week.

Solution

Let p be the proposition "at least four of any 22 days must fall on the same day of the week"
Suppose that $\neg p$ is true $\Rightarrow$ "at most three of the 22 days fall on the same day of the week".
There are 7 days per week $\Rightarrow$ at most 3 of the chosen days could fall on that day. That contradicts the premise that we have 22 days under consideration.
If r is the statement that 22 days are chosen, that we have shown that $\neg p \rightarrow (r \wedge \neg r)$.

We know that p is true. We have proved that at least four of any 22 days must fall on the same day of the week.

Example

Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Solution

Let p be the proposition "$\sqrt{2}$ is irrational". Suppose that $\neg p$ is true $\Rightarrow$ "$\sqrt{2}$ is rational".

If $\sqrt{2}$ is rational, $\exists a \text{ and } b \ni \sqrt{2} = \frac{a}{b}$

$$\left(\sqrt{2}\right)^2 = \left(\frac{a}{b}\right)^2 \quad \Rightarrow \quad 2 = \frac{a^2}{b^2} \rightarrow \underline{2b^2 = a^2}$$

It follows that a^2 is even, that implies a must also be even. Therefore, by the definition of an even integer then we can let $a = 2c$ for some integer c. Thus, $2b^2 = 4c^2 \quad \Rightarrow \quad b^2 = 2c^2$

By the definition of even, this means that b^2 is even, that implies b must also be even as well.

The assumption of $\neg p$ leads to the equation $\sqrt{2} = \frac{a}{b}$, where a and b have no common factors, but both a and b are even, that is, 2 divides both a and b. However, our assumption $\neg p$ leads to the contradiction that 2 divides both a and b and 2 doesn't divide both a and b, $\neg p$ must be false.
That is, the statement p "$\sqrt{2}$ is irrational" is true.

Proofs of Equivalence

To prove a theorem that is a biconditional statement, that is, a statement of the form $p \leftrightarrow q$, we must show that $p \to q$ and $q \to p$ are both true. The validity of this approach is based on the tautology

$$(p \leftrightarrow q) \leftrightarrow (p \to q) \wedge (q \to p)$$

Example

Prove the theorem "If *n* is an integer, then *n* is odd if and only if n^2 is odd"

Solution

Let: *p* is "*n* is odd" and *q* is "n^2 is odd".
The theorem has the form: "p *iff* q". To prove this theorem, we need to show that $p \to q$ and $q \to p$ are both true.

Using direct proof, we assume that *n* is odd, is a true statement. By the definition of an odd integer, it follows that $n = 2k+1$, where *k* is some integer. We need to show that n^2 is odd.

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2\left(2k^2 + 2k\right) + 1 = 2K + 1 \qquad 2k^2 + 2k = K$$

By the definition of an odd integer, we can conclude that n^2 is also an odd integer. Therefore, $p \to q$ is true.

Suppose that *n* is an integer and n^2 is odd. Then, $\exists k \in \mathbb{Z} \ni n^2 = 2k+1 \Rightarrow n = \pm\sqrt{2k+1}$ (which is not useful). By using proof by contraposition, the statement *n* is not odd, that means *n* is even. That implies that $\exists k \in \mathbb{Z} \ni n = 2k$.

To prove the theorem, we need to show that this hypothesis implies the conclusion that n^2 is not odd, that means n^2 is even. $n^2 = (2k)^2 = 4k^2 = 2\left(2k^2\right)$, which implies that n^2 is even.

We have proved that *n* is an integer and n^2 is odd, then *n* is odd by a proof of contraposition. Therefore, $q \to p$ is true.

Because $p \to q$ and $q \to p$ are both true, we have shown that the theorem is true.

Example

Show that these statements about the integer *n* are equivalent:

p_1 : *n is even*

p_2 : $n-1$ *is odd*

p_3 : n^2 *is even*

Solution

We will show that these 3 statements are equivalent by showing that the condition statements $p_1 \to p_2$, $p_2 \to p_3$, $p_3 \to p_1$ are true.

Using a direct proof to show that $p_1 \to p_2$.

Suppose that *n* is even, then $n = 2k$ for some $k \in \mathbb{Z}$.

$$\begin{aligned} n-1 &= 2k-1 \\ &= 2(k-1)+1 \end{aligned}$$

This means that $n-1$ is odd because it is of the form $2m+1$, where m is the integer $k-1$. Therefore the statement $p_1 \to p_2$ is true.

Also using a direct proof to show that $p_2 \to p_3$.

Suppose that $n-1$ is even, then $n-1 = 2k+1$ for some $k \in \mathbb{Z}$.

$$\begin{aligned} n &= 2k+2 \\ n^2 &= (2k+2)^2 \\ &= 4k^2+8k+4 \\ &= 2\left(2k^2+4k+2\right) \end{aligned}$$

Hence, $n-1$ is even. Therefore the statement $p_2 \to p_3$ is true.

Using a proof by contraposition to prove $p_3 \to p_1$. That is, we have to prove that if n is not even, then n^2 is not even.

To prove the theorem, we need to show that this hypothesis implies the conclusion that n^2 is not odd, that means n^2 is even. $n^2 = (2k)^2 = 4k^2 = 2\left(2k^2\right)$, which implies that n^2 is even.

We have proved that *n* is an integer and n^2 is odd, then *n* is odd by a proof of contraposition. Therefore, $p_3 \to p_1$ is true.

This completes the proof.

Counterexamples

To show that a statement of the form $\forall x\, P(x)$ is false, we need only find a ***counterexample***, that is, an example of *x* for which $P(x)$ is false.

Example

Show that the statement "Every positive integer is the sum of the squares of two integers" is false.

Solution

To show that this statement is false, we look for a counterexample, which is a particular integer that is not the sum of the squares of two integers.

To choose a counterexample, we can select 3 because it cannot be written as the sum of the squares of two integers. Let use 0 and 1 which implies $0^2+1^2=0+1=1\neq 3$. Therefore, we can't get 3 as the sum of two terms of which is 0 or 1.

Consequently, we have shown that "Every positive integer is the sum of the squares of two integers" is false.

Mistakes in Proofs

Each step of a mathematical proof needs to be correct and the conclusion needs to follow logically from the steps that precede it. Many mistakes result from the introduction of steps that do not logically follow from those that precede it.

Example

What is wrong with this "proof?": If n^2 is positive, then *n* is positive.

Proof: Suppose that n^2 is positive. Because the conditional statement "If *n* is positive, then n^2 is positive" is true, we can conclude that *n* is positive.

Solution

Let $P(n)$ be "*n* is positive" and $Q(n)$ be "n^2 is positive."

The statement can be written: $\forall n\big(P(n)\to Q(n)\big)$.

A counterexample is supplied by $n=-1 \Rightarrow n^2=1$ is positive, but *n* is negative.

Exercises *Section* 1.5 – Introduction to Proofs

1. Show that the square of an even number is an even number

2. Prove that if n is an integer and n^3+5 is odd, then n is even

3. Show that $m^2=n^2$ if and only if $m=n$ *or* $m=-n$

4. Use a direct proof to show that the sum of two odd integers is even.

5. Use a direct proof to show that the sum of two even integers is even.

6. Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

7. Prove or disprove that the product of two irrational numbers is irrational.

8. Prove that if x is irrational, then $\frac{1}{x}$ is irrational.

9. Prove that if x is rational and $x \neq 0$, then $\frac{1}{x}$ is rational.

10. Prove the proposition $P(0)$, where $P(n)$ is the proposition "If n is a positive integer greater than 1, then $n^2 > n$." What kind of proof did you use?

11. Let $P(n)$ be the proposition "If a and b are positive real numbers, then $(a+b)^n \geq a^n + b^n$." Prove that $P(1)$ is true. What kind of proof did you use?

12. Show that these statements about the integer x are equivalent:

i) $3x+2$ *is even* ***ii***) $x+5$ *is odd* ***iii***) x^2 *is even*

13. Show that these statements about the real number x are equivalent:

i) x *is irrational* ***ii***) $3x+2$ *is irrational* ***iii***) $\frac{x}{2}$ *is irrational*

14. Prove that at least one of the real numbers $a_1, a_2, \ldots, a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?

Section 1.6 – Proof Methods and Strategy

Introduction

The strategy behind constructing proofs includes selecting a proof method and then successfully constructing an argument step by step, based on this method.

Exhaustive Proof and Proof by Cases

To prove a conditional statement of the form $\left(p_1 \vee p_2 \vee \cdots \vee p_n\right) \to q$

The tautology $\left[\left(p_1 \vee p_2 \vee \cdots \vee p_n\right) \to q\right] \leftrightarrow \left[\left(p_1 \to q\right) \wedge \left(p_2 \to q\right) \wedge \cdots \wedge \left(p_n \to q\right)\right]$

can be used as a rule of inference

Such an argument is called a ***proof by cases***. Sometimes to prove that a conditional statement $p \to q$ is true, it is convenient to use a disjunction $p_1 \vee p_2 \vee \cdots \vee p_n$ instead of p as the hypothesis of the conditional statement, where p and $p_1 \vee p_2 \vee \cdots \vee p_n$ are equivalent.

Exhaustive Proof

Also known as ***proof by cases***, ***perfect induction***, or the ***brute force method***, is a method of mathematical proof in which the statement to be proved is split into a finite number of cases and each case is checked to see if the proposition in question holds.

Theorem

A proposition that has been proved to be true

- Two special kinds of theorems: Lemma and Corollary.
- Lemma: A theorem that is usually not too interesting in its own right but is useful in proving another theorem.
- Corollary: A theorem that follows quickly from another theorem.

Example

Prove that $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$

Solution

Using a proof by exhaustion:

For $n = 1$: $(n+1)^3 = 2^3 = 8 \geq 3^1 = 3$

For $n = 2$: $(n+1)^3 = 3^3 = 27 \geq 3^2 = 9$

For $n = 3$: $(n+1)^3 = 4^3 = 64 \geq 3^3 = 27$

For $n = 4$: $(n+1)^3 = 5^3 = 125 \geq 3^4 = 81$

We have used the method of exhaustion to prove that $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$

Example

Prove that if n is an integer, then $n^2 \geq n$

Solution

Case 1: When $n = 0$, that implies to $0^2 \geq 0$. It follows that $n^2 \geq n$ is true.

Case 2: When $n \geq 1$, $\Rightarrow n \cdot n \geq 1 \cdot n$, we obtain $n^2 \geq n$ It follows that $n^2 \geq n$ is true.

Case 3: When $n \leq -1$, but $n^2 \geq 0$. It follows that $n^2 \geq n$ is true.

Because the inequality $n^2 \geq n$ holds in all three cases, we can conclude that if n is an integer, then $n^2 \geq n$.

Example

Show that if x and y are integers and both xy and $x + y$ are even, then both x and y are even.

Solution

Using the proof by contraposition:
Suppose that x and y are not both even. That is, x is odd or y is odd (or both).
Assume that x is odd, so that $x = 2k+1$ for some integer k.
Case 1: y even $\Rightarrow y = 2n$

$$x + y = 2k + 1 + 2n = 2(k+n) + 1 \text{ is } \textit{odd}$$

Case 2: y odd $\Rightarrow y = 2n+1$

$$xy = (2k+1)(2n+1) = 4kn + 2k + 2n + 1 = 2(2kn + k + n) + 1 \text{ is } \textit{odd}$$

This completes the proof by contraposition.

Existence Proofs

A statement $\exists x\, P(x)$ is called an ***existence proof***. There are several ways to proof a theorem of this type.

- ***Constructive***: Find a specific value of c for which $P(c)$ exists
- ***Nonconstructive***: Show that such a c exists, but don't actually find it. Assume it does not exist, and show a contradiction

Example

Show that a square exists that is the sum of two other squares

Solution

Proof: $3^2 + 4^2 = 5^2$

Because we have displayed a positive integer that can be written as the sum of two squares, we are done.

Example

Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

Solution

Proof: $1729 = 10^3 + 9^3 = 12^3 + 1^3$

We proved that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

Example

Show that a cube exists that is the sum of three other cubes

Solution

Proof: $3^3 + 4^3 + 5^3 = 6^3$

We proved that a cube exists that is the sum of three other cubes.

Uniqueness Proofs

A theorem may state that only one such value exists. Theorem statements that involve the word ``unique" are known as ***uniqueness theorems***. Typically the proof of such a statement follows the idea that we assume there are two elements that satisfy the conclusion of the statement and then show that these elements are identical.

Existence: We show that an element x with the desired property exists.
Uniqueness: We show that if $y \neq x$, then y does not have the desired property.

Equivalently, we can show that if x and y both have the desired property, then $x = y$.

Example

Show that if x and y are real numbers and $x \neq 0$, then there is a unique real number r such that $xr + y = 0$

Solution

The solution of $xr + y = 0$ is $r = -\frac{y}{x}$ because $x\left(-\frac{y}{x}\right) + y = -y + y = 0$. Consequently, a real number r exists for which $xr + y = 0$. This is the existence part of the proof.

Suppose that s is a real number such that $xs + y = 0$, then $xr + y = xs + y$, where $r = -\frac{y}{x}$.

$xr + y = xs + y \quad \Rightarrow \quad xr = xs \quad (x \neq 0) \quad \rightarrow \ r = s$

This means that if $s \neq r$, then $xs + y \neq 0$. This establishes the uniqueness part of the proof.

Proof Strategies

Usually, when you are working on a proof, you should use the logical forms of the givens and goals to guide you in choosing what proof strategies to use. Generally, if the statement is a conditional statement, we should try a direct proof; if this fails, we can try an indirect proof. If neither of these approaches works, you might try a proof by contradiction.

Example

Given two positive numbers x and y, their ***arithmetic mean*** is $\frac{x+y}{2}$ and their ***geometric mean*** is $\sqrt{xy}$.

When we compare the arithmetic and geometric means of pairs of distinct positive real numbers, we find that the arithmetic mean is always greater than the geometric mean. For example, when $x = 4$ and $y = 6$, we have $\frac{4+6}{2} = 5 > \sqrt{4 \cdot 6} = \sqrt{24}$. Can we prove that this inequality is always true?

Solution

To prove $\frac{x+y}{2} > \sqrt{xy}$

$$\left(\frac{x+y}{2}\right)^2 > \left(\sqrt{xy}\right)^2$$

$$\frac{x^2+2xy+y^2}{4} > xy$$

$$x^2+2xy+y^2 > 4xy$$

$$x^2-2xy+y^2 > 0$$

$$(x-y)^2 > 0$$

It is true inequality, since $(x-y)^2 > 0$ when $x \neq y$, it follows that $\frac{x+y}{2} > \sqrt{xy}$.

Suppose that x and y are distinct positive real numbers. Then $(x-y)^2 > 0$ because the square of a nonzero real number is positive.

$$x^2-2xy+y^2 > 0$$

$$x^2-2xy+y^2+4xy > 4xy$$

$$x^2+2xy+y^2 > 4xy$$

$$(x+y)^2 > 4xy$$

divide both sides by 4

$$\frac{(x+y)^2}{4} > xy$$

Square roots both sides

$$\frac{x+y}{2} > \sqrt{xy}$$

We conclude that if x and y are distinct positive real numbers, then their arithmetic mean $\frac{x+y}{2}$ is greater than the geometric mean $\sqrt{xy}$

Fermat's Last Theorem

The eqaution $\quad x^n + y^n = z^n$

Has no solutions in integers x, y, and z with $xyz \neq 0$ whenever n is an integer with $n > 2$.

Exercises *Section* 1.6 – Proof Methods and Strategy

1. Prove that $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$

2. Prove that there are no positive perfect cubes less than 1000 that are the sum of the cubes of two positive integers.

3. Prove that if x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$. (*Hint*: Use a proof by cases, with the two cases corresponding to $x \geq y$ and $x < y$, respectively.)

4. Prove the triangle inequality, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$

5. Prove that either $2 \cdot 10^{500} + 15$ or $2 \cdot 10^{500} + 16$ is not a perfect square

6. Prove that there exists a pair of consecutive integers such that one of these integers is a perfect square and the other is a perfect cube.

7. Suppose that a and b are odd integers with $a \neq b$. Show there is a unique integer c such that $|a - c| = |b - c|$

Section 1.7 – Sets

Introduction

A set is an unordered collection of objects, called elements or members of the set. A set is said to contain its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

Example

Colors of a rainbow: {red, orange, yellow, green, blue, purple}

Example

States of matter {solid, liquid, gas, plasma}

Example

The set V of all vowels in the English alphabet can be written as: $V = \{a,\ e,\ i,\ o,\ u\}$

Example

The set O of odd positive integers less than 10 can be expressed by $O = \{1, 3, 5, 7, 9\}$

Example

The set of positive integers less than 100 can be denoted by $\{1, 2, 3, \ldots, 99\}$

- Another way to describe a set is to use ***set builder*** notation.

For instance, the set O of odd positive integers less than 10 can be written as

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$$

Or, specifying the universe as the set of positive integers, as

$$O = \left\{x \in \mathbb{Z}^{+} \mid x \text{ is an odd and } x < 10\right\}$$

The set of ***Natural numbers***: $\mathbb{N} = \{0,\ 1,\ 2,\ 3,\ \ldots\}$

The set of ***Integers***: $\mathbb{Z} = \{\ldots,\ -2,\ -1,\ 0,\ 1,\ 2,\ \ldots\}$

The set of ***positive integers***: $\mathbb{Z}^{+} = \{1,\ 2,\ 3,\ \ldots\}$

The set of ***Rational numbers***: $\mathbb{Q} = \left\{\frac{p}{q} \middle| p \in \mathbb{Z},\ q \in \mathbb{Z},\ and\ q \neq 0\right\}$

The set of ***Real numbers***: $\mathbb{R}$

The set of ***positive Real numbers***: $\mathbb{R}^{+}$

The set of ***Complex numbers***: $\mathbb{C}$

Intervals

The notations for intervals of real numbers. When *a* and *b* are real numbers with *a* < *b*, we write

$$[a,\ b] = \{x | a \leq x \leq b\}$$

$$[a,\ b) = \{x | a \leq x < b\}$$

$$(a,\ b] = \{x | a < x \leq b\}$$

$$(a,\ b) = \{x | a < x < b\}$$

$[a,\ b]$ is called ***closed interval*** from *a* to *b*.

$(a,\ b)$ is called ***open interval*** from *a* to *b*.

Definition

Two sets are equal *iff* they have the same elements. Therefore, if *A* and *B* are sets, then *A* and *B* are equal *iff* $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if *A* and *B* are equal sets

Example

The set {1, 3, 5} and {3, 5, 1} are equal, because they have the same elements.

- Order of the elements of a set are listed does not matter.
  {1, 2, 3, 4, 5} = {5, 4, 3, 2, 1}

The Empty Set

There is a special set that has no elements. This set is called the ***empty set***, or ***null set***, and is denoted by $\varnothing$. The empty set can also denoted by { }.

A set with one element is called a ***singleton set***.

Venn Diagrams

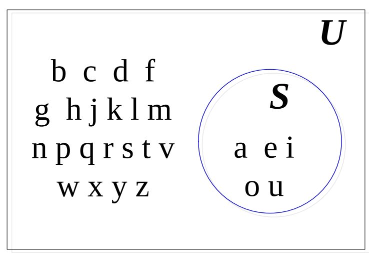
In Venn diagrams the ***universal set*** U, which contains all the objects under consideration, is represented by a rectangle.

Represents sets graphically

- ✓ The box represents the universal set
- ✓ Circles represent the set(s)

Consider set S, which is the set of all vowels in the alphabet

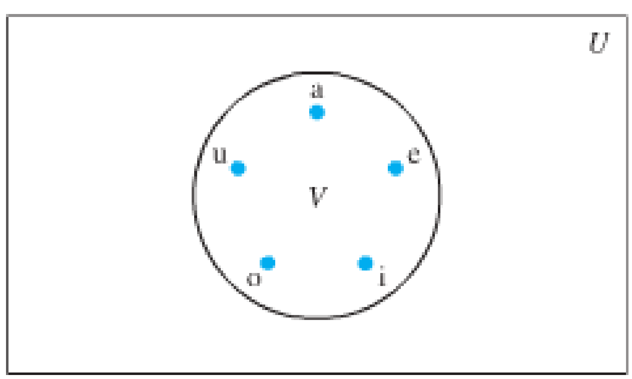
The individual elements are usually not written in a Venn diagram



Example

Draw a Venn diagram that represents *V*, the set of vowels in the English alphabet.

Solution



Subset

Set A is a subset of set B (written $A \subseteq B$) if and only if every element of A is also an element of B. Set A is a proper subset (written $A \subset B$) if $A \subseteq B$ and $A \neq B$

We see that $A \subseteq B$ if and only if the quantification:

$\forall x \ (x \in A \to x \in B)$ is true

Note that to show that A is not a subset of B we need only find one element $x \in A$ with $x \notin B$. Such an x is counterexample to the claim that $x \in A$ implies $x \in B$.

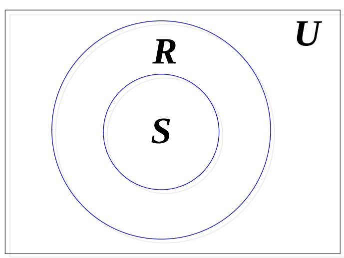
Showing that A is a Subset of B – To show that $A \subseteq B$, show that if x belong to A then a also belong to B.

Showing that A is Not a Subset of B – To show that $A \not\subseteq B$, find a single $x \in A$ such that $x \notin B$.

Example

{1, 2, 8} $\subseteq$ {1, 2, 3, 4, 5, 6, 7}

Proper subsets: Venn diagram $\mathrm{S} \subset \mathrm{R}$



Example

The set of people who have taken discrete mathematics at the school is not a subset of all computer science majors at the school if there is at least one student who has taken discrete mathematics who is not a computer science major.

Theorem

For every set S

i. $\varnothing \subseteq S$ and

ii. $S \subseteq S$

Proof (i)

Let S be a set. To show $\varnothing \subseteq S$, we must show that $\forall x\left(x \in \varnothing \to x \in S\right)$ is true.

Because the empty set contains no elements, it follows that $x \in \varnothing$ is always false. It follows that the conditional statement $x \in \varnothing \to x \in S$ is always true, because its hypothesis is always false and a conditional statement with a false hypothesis is true. Therefore, $\forall x\left(x \in \varnothing \to x \in S\right)$ is true.

This complete the proof of (i) using a vacuous proof.

Showing Two Sets are Equal – To show that two sets A and B are equals, show that $A \subseteq B$ and $B \subseteq A$.

Example

We have the sets $A = \{\varnothing,\ \{a\},\ \{b\},\ \{a,b\}\}$ *and* $B = \{x \mid x \text{ is a subset of the set } \{a,b\}\ \}$

Solution

These two sets are equal, that is, $A = B$.

Note: $\{a\} \in A$ *but* $a \notin A$

The Size of a Set

Definition

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the ***cardinality*** of S. The cardinality of S is denoted by $|S|$.

- Let A be the set of odd positive integers less than 10. $|A| = 5$
- Let S be the set of of letter in English alphabet. $|S| = 26$
- The null set has no elements. $|\varnothing| = 0$

Definition

A set is said to be infinite if it is not finite.

Example: The set of positive integers is infinite.

Power Sets

Definition

Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$

Note that the empty set and the set itself are memebers of the set of subsets.

Example

What is the power set of the set {0, 1, 2}?

Solution

$$\mathcal{P}(\{0,1,2\}) = \{\varnothing,\ \{0\},\ \{1\},\ \{2\},\ \{0,1\},\ \{0,2\},\ \{1,2\},\ \{0,1,2\}\}$$

Example

What is the power set of the empty set? What is the power set of the set $\{\varnothing\}$?

Solution

$$\mathcal{P}(\varnothing) = \{\varnothing\}$$

$$\mathcal{P}(\{\varnothing\}) = \{\varnothing,\ \{\varnothing\}\}$$

Cartesian Products

Definition

The ***order n-tuple*** $\left(a_1,\ a_2, \ldots,\ a_n\right)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, …, and a_n as its *n*th element.

Let *A* and *B* be sets. The Cartesian product of *A* and *B*, denoted by $A \times B$, is the set of all ordered pairs $(a,\ b)$, where $a \in A$ and $b \in B$. Hence

$$A \times B = \left\{(a,\ b) \middle| a \in A\ \wedge\ b \in B\right\}$$

Example

Let *A* represent the set of all students at a university, and let *B* represent the set of all courses offered at the university. What is the Cartesian product $A \times B$ and how can it be used?

Solution

The Cartesian product $A \times B$ consists of all the ordered pairs of the form $(a,\ b)$, where *a* is a student at the university and *b* is a course offered at the university. One way to use the set $A \times B$ is to represent all possible enrollments of students in courses at the university.

Example

What is the Cartesian product *A* = {1, 2} and *B* = {*a*, *b*, *c*}?

Solution

$$A \times B = \{(1,\ a),\ (1,\ b),\ (1,\ c),\ (2,\ a),\ (2,\ b),\ (2,\ c)\}$$

Example

Show that the Cartesian product $B \times A$ is not equal to $A \times B$, where *A* = {1, 2} and *B* = {*a*, *b*, *c*}?

Solution

$$A \times B = \{(1,\ a),\ (1,\ b),\ (1,\ c),\ (2,\ a),\ (2,\ b),\ (2,\ c)\}$$

$$B \times A = \{(a,\ 1),\ (a,\ 2),\ (b,\ 1),\ (b\ 2),\ (c,\ 1),\ (c,\ 2)\}$$

$$\Rightarrow\ A \times B \neq B \times A$$

Definition

The ***Cartesian product*** of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of ordered *n*-tuples $\left(a_1, a_2, \ldots, a_n\right)$, where a_i belongs to A_i for *i* = 1, 2, …, *n*. In other words,

$$A_1 \times A_2 \times \ldots \times A_n = \left\{\left(a_1, a_2, \ldots, a_n\right) \middle| a_i \in A_i \ \ for\ i = 1, 2, \ldots, n\right\}$$

Example

What is the Cartesian product $A \times B \times C$, where *A* = {0, 1}, *B* = {1, 2}, and *C* = {0, 1, 2}

Solution

$$\begin{aligned} A \times B \times C = \{&(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), \\ &(1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\} \end{aligned}$$

Example

Suppose that *A* = {1, 2}, find A^2 and A^3

Solution

$$A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

$$A^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$$

Example

What are the ordered pairs in the less than or equal relation, which contains (a, b) if $a \le b$, on the set $\{0, 1, 2, 3\}$?

Solution

The ordered pairs in ***R*** are:

(0, 0), (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)

Using Set Notation with Quantifiers

For example $\forall x \in S\left(P(x)\right)$ denotes

Universal quantification of $P(x)$ over all elements in the S

Shorthand for $\forall x\left(x \in S \to P(x)\right)$

$\exists x \in S\left(P(x)\right)$ denotes

Existential quantification of $P(x)$ over all elements in the S

Shorthand for $\exists x\left(x \in S \wedge P(x)\right)$

Example

What do the statements $\forall x \in \boldsymbol{R}\left(x^2 \ge 0\right)$ and $\exists x \in \boldsymbol{Z}\left(x^2 = 1\right)$ mean?

Solution

The statement $\forall x \in \boldsymbol{R}\left(x^2 \ge 0\right)$ states that for every real numbers *x*, $x^2 \ge 0$.

This statement can be expressed as "The square of every real number is nonnegative." This is a true statement.

The statement $\exists x \in \boldsymbol{Z}\left(x^2 = 1\right)$ states that there exists an integer *x*, $x^2 = 1$.

This statement can be expressed as "The is an integer whose square is 1." This is also a true statement because $x = 1$ *or* -1 such an integer.

Exercises Section 1.7 – Sets

1. List the members of these sets

a) $\left\{x \middle| x \text{ is a real number such that } x^2 = 1\right\}$

b) $\left\{x \middle| x \text{ is a positive integer less than } 12\right\}$

c) $\left\{x \middle| x \text{ is the square of an integer and } x < 100\right\}$

d) $\left\{x \middle| x \text{ is an integer such that } x^2 = 2\right\}$

2. Determine whether each these pairs of sets are equal.

a) $\{1,3,3,3,5,5,5,5,5\}, \quad \{5,3,1\}$

b) $\{\{1\}\}, \quad \{1, \{1\}\}$

c) $\varnothing, \quad \{\varnothing\}$

3. For each of the following sets, determine whether 2 is an element of that set.

a) $\left\{x \in \mathbb{R} \middle| x \text{ is an integer greater than } 1\right\}$

b) $\left\{x \in \mathbb{R} \middle| x \text{ is the square of an integer}\right\}$

c) $\{2, \{2\}\}$

d) $\{\{2\}, \{\{2\}\}\}$

e) $\{\{2\}, \{2, \{2\}\}\}$

f) $\{\{\{2\}\}\}$

4. Determine whether each of these statements is true or false

a) $0 \in \varnothing$

b) $\varnothing \in \{0\}$

c) $\{0\} \subset \varnothing$

d) $\varnothing \subset \{0\}$

e) $\{0\} \in \{0\}$

f) $\{0\} \subset \{0\}$

g) $\{\varnothing\} \subseteq \{\varnothing\}$

h) $x \in \{x\}$

i) $\{x\} \subseteq \{x\}$

j) $\{x\} \in \{x\}$

k) $\{x\} \in \{\{x\}\}$

l) $\varnothing \subseteq \{x\}$

m) $\varnothing \in \{x\}$

5. Use a Venn Diagram to illustrate the relationships $A \subset B$ and $B \subset C$.

6. Use a Venn Diagram to illustrate the relationships $A \subset B$ and $A \subset C$.

7. Suppose that *A*, *B*, and *C* are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$

8. What is the cardinality of each of these sets?

a) $\{a\}$

b) $\{\{a\}\}$

c) $\{a, \{a\}\}$

d) $\{a, \{a\}, \{a, \{a\}\}\}$

9. How many elements does each of these sets have where *a* and *b* are distinct elements?

a) $\mathcal{P}(\{a, b, \{a, b\}\})$

b) $\mathcal{P}(\{\varnothing, a, \{a\}, \{\{a\}\}\})$

c) $\mathcal{P}(\mathcal{P}(\varnothing))$

10. What is the Cartesian product $A \times B \times C$, where *A* is the set of all airlines and *B* and *C* are both the set of all cities in the United States? Give an example of how this Cartesian product can be used.

11. What is the Cartesian product $A \times B$, where *A* is the set of all courses offered by the mathematics department and *B* is the set of mathematics professors at this university? Give an example of how this Cartesian product can be used.

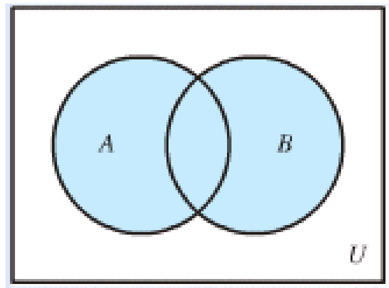
12. Let *A* be a set. Show that $\varnothing \times A = A \times \varnothing = \varnothing$

Section 1.8 – Set Operations

Union of Two Sets

Let *A* and *B* be sets, the ***union*** of the sets *A* and *B*, denoted by $A \cup B$, is the set that contains those elements that are either in *A* or in *B*, or in both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$



Example

Let $A = \{1,3,5,7,9,11\}$, $B = \{3,6,9,12\}$. Find each set $A \cup B$

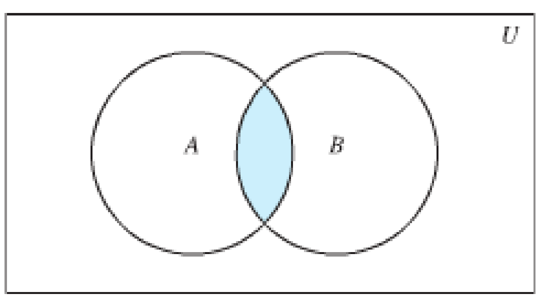
Solution

$$A \cup B = \{1,3,5,6,7,9,11,12\}$$

Intersection of Two Sets

Let *A* and *B* be sets, the ***intersection*** of the sets *A* and *B*, denoted by $A \cap B$, is the set containing those elements in both *A* or in *B*.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$



Example

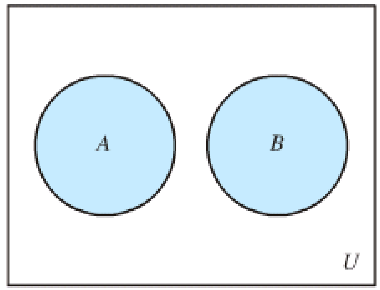
Let $A = \{3,6,9\}$, $B = \{2,4,6,8\}$, find $A \cap B$

Solution

$$A \cap B = \{6\}$$

Disjoint Sets

For any sets *A* and *B*, if *A* and *B* are ***disjoint*** sets, then their intersection is the empty set $A \cap B = \phi$



Example

Let $A = \{1,3,5,7,9\}$, $B = \{2,4,6,8,10\}$, find $A \cap B$

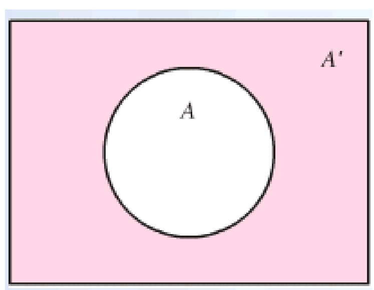
Solution

$A \cap B = \varnothing$. Therefore, *A* and *B* are disjoint.

Complement of a Set

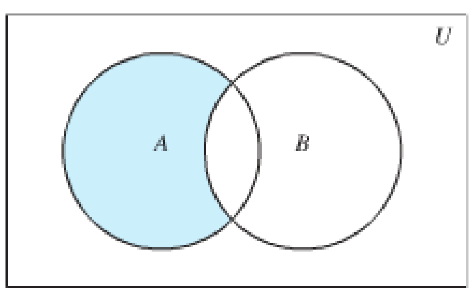
Let *A* be any set, with *U* representing the universal set, then the complement of *A*.

$$A' \text{ or } \bar{A} = \{x \mid x \notin A \text{ and } x \in U\}$$



Difference of two Sets

Let *A* and *B* be sets, the ***difference*** of *A* and *B*, denoted by $A - B$, is the set containing those elements that are *A* but not in *B*. The difference of *A* and *B* is also called the complement of *B* with respect to *A*.



Example

Find {1, 3, 5} – {1, 2, 3}

Solution

{1, 3, 5} – {1, 2, 3} = {5}

Example

What is the difference of the set of computer science majors at the school and the set of mathematics majors at the school?

Solution

The difference is the set of all computer science majors at your school are not also mathematics majors.

Example

Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Find $\overline{A}$

Solution

$$\overline{A} = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10\}$$

Set Identities

Identity	Name
$A\bigcap U = A$ $A\bigcup \varnothing = A$	Identity laws
$A\bigcup U = U$ $A\bigcap \varnothing = \varnothing$	Domination laws
$A\bigcup A = A$ $A\bigcap A = A$	Idempotent laws
$\overline{\left(\overline{A}\right)} = A$	Complementation laws
$A\bigcup B = B\bigcup A$ $A\bigcap B = B\bigcap A$	Commutative laws
$A\bigcup(B\bigcup C)=(A\bigcup B)\bigcup C$ $A\bigcap(B\bigcap C)=(A\bigcap B)\bigcap(C)$	Associative laws
$A\bigcup(B\bigcap C)=(A\bigcup B)\bigcap(A\bigcup C)$ $A\bigcap(B\bigcup C)=(A\bigcap B)\bigcup(A\bigcap C)$	Distributive laws
$\overline{A\bigcap B} = \overline{A}\bigcup\overline{B}$ $\overline{A\bigcup B} = \overline{A}\bigcap\overline{B}$	De Morgan's laws
$A\bigcup(A\bigcap B) = A$ $A\bigcap(A\bigcup B) = A$	Absorption laws
$A\bigcup\overline{A} = U$ $A\bigcap\overline{A} = \varnothing$	Complement laws

Example

Prove that $\overline{A\bigcap B} = \overline{A}\bigcup\overline{B}$

Solution

1. We need to show that $\overline{A\bigcap B} \subseteq \overline{A}\bigcup\overline{B}$

   Suppose that $x \in \overline{A\bigcap B} \quad \Rightarrow \quad x \notin A\bigcap B$ (by the definition of complement)

   Using the definition of the intersection, we see that the proposition $\neg\left((x\in A)\wedge(x\in B)\right)$ is true.

   $\neg(x\in A)$ *or* $\neg(x\in B)$ — *By applying De Morgan's law of the proposition*

   $x\notin A$ *or* $x\notin B$ — *Using the definition of the negation of proposition*

   $x\in\overline{A}$ *or* $x\in\overline{B}$ — *Using the complement of a set*

   $x\in\overline{A}\bigcup\overline{B}$ — *Using the definition of union*

   $\overline{A\bigcap B}\subseteq\overline{A}\bigcup\overline{B}$

2. We need to show that $\overline{A}\bigcup\overline{B}\subseteq\overline{A\bigcap B}$

   Suppose that $x\in\overline{A}\bigcup\overline{B} \quad \Rightarrow \quad x\in\overline{A}$ *or* $x\in\overline{B}$ (by the definition of union)

   $x\notin A$ *or* $x\notin B$ — *Using the definition of the complement*

$\neg(x \in A) \ \vee \ \neg(x \in B)$ *True*

$\neg(x \in A) \ \wedge \ \neg(x \in B)$ *By applying De Morgan's law of the proposition*

$\neg(x \in A \cap B)$ *Using the definition of the intersection*

$x \in \overline{A \cap B}$ *Using the definition of complement*

That shows that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Therefore; $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Example

Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution

$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$ *By definition of complement*

$= \{x \mid \neg(x \in (A \cap B))\}$

$= \{x \mid \neg(x \in A \ \wedge \ x \in B)\}$ *By definition of complement*

$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$ *By the first De Morgan law for logical equivalences*

$= \{x \mid x \notin A \ \vee \ x \notin B\}$ *By definition of does not belong symbol*

$= \{x \mid x \in \overline{A} \ \vee \ x \in \overline{B}\}$ *By definition of complement*

$= \{x \mid x \in \overline{A} \cup \overline{B}\}$ *By definition of union*

$= \overline{A} \cup \overline{B}$

Example

Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution

A Membership Table for the Distributive Property

A	**B**	**C**	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	**1**	**1**	**1**	**1**	**1**	**1**	**1**
1	**1**	**0**	**1**	**1**	**1**	**0**	**1**
1	**0**	**1**	**1**	**1**	**0**	**1**	**1**
1	**0**	**0**	**0**	**0**	**0**	**0**	**0**
0	**1**	**1**	**1**	**0**	**0**	**0**	**0**
0	**1**	**0**	**1**	**0**	**0**	**0**	**0**
0	**0**	**1**	**1**	**0**	**0**	**0**	**0**
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**

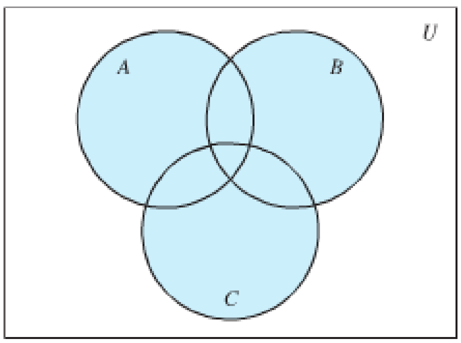
Example

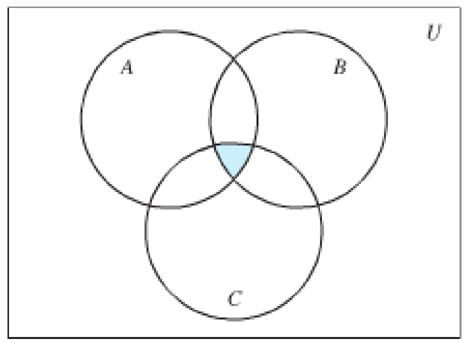
Let A, B, and C be sets. Show that $\overline{A\cup(B\cap C)}=(\bar{C}\cup\bar{B})\cap\bar{A}$

Solution

$$\begin{aligned}
\overline{A\cup(B\cap C)} &= \bar{A}\cap\left(\overline{B\cap C}\right) && \textit{By the first De Morgan law}\\
&= \bar{A}\cap\left(\bar{B}\cup\bar{C}\right) && \textit{By the second De Morgan law}\\
&= \left(\bar{B}\cup\bar{C}\right)\cap\bar{A} && \textit{By the commutative law for intersection}\\
&= \left(\bar{C}\cup\bar{B}\right)\cap\bar{A} && \textit{By the commutative law for union}
\end{aligned}$$

Generalized Unions and Intersections





$$A\cup(B\cup C)=(A\cup B)\cup C$$
$$A\cap(B\cap C)=(A\cap B)\cap C$$

Example

Let A = {0, 2, 4, 6, 8}, B = {0, 1, 2, 3, 4}, and C = {0, 3, 6, 9}. What are $A\cup B\cup C$ and $A\cap B\cap C$

Solution

$$A\cup B\cup C=\{0,\ 1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 9\}$$
$$A\cap B\cap C=\{0\}$$

Definition

The ***union*** of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

$$A_1\cup A_2\cup\cdots\cup A_n=\bigcup_{i=1}^{n}A_i$$

Definition

The ***intersection*** of a collection of sets is the set that contains those elements that are members of at all the sets in the collection.

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

For i = 1, 2, …, let $A_i = \{i,\ i+1,\ i+2,\ \ldots\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i,\ i+1,\ i+2,\ \ldots\} = \{1,\ 2,\ 3,\ \ldots\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i,\ i+1,\ i+2,\ \ldots\} = \{n,\ n+1,\ n+2,\ \ldots\} = A_n$$

Exercises *Section* 1.8 – Set Operations

1. Let A be the set of students who live within one mile of school and let B be the set of students who walk to classes. Describe the students in each of these sets.

a) $A \cap B$

b) $A \cup B$

c) $A - B$

d) $B - A$

2. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$

a) $A \cup B$

b) $A \cap B$

c) $A - B$

d) $B - A$

3. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$

a) $A \cup B$

b) $A \cap B$

c) $A - B$

d) $B - A$

4. Prove the domination laws by showing that

a) $A \cup U = U$

b) $A \cap U = A$

c) $A \cup \varnothing = A$

d) $A \cap \varnothing = \varnothing$

5. Prove the complement laws by showing that

a) $A \cup \overline{A} = U$

b) $A \cap \overline{A} = \varnothing$

6. Show that

a) $A - \varnothing = A$

b) $\varnothing - A = \varnothing$

7. Prove the absorption law by showing that if A and B are sets, then

a) $A \cap (A \cup B) = A$

b) $A \cup (A \cap B) = A$

8. Show that if *A*, *B*, and *C* are sets, then $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

9. Let *A* and *B* be sets. Show that

a) $(A \cap B) \subseteq A$

b) $A \subseteq (A \cup B)$

c) $(A - B) \subseteq A$

d) $A \cap (B - A) = \varnothing$

e) $A \cup (B - A) = A \cup B$

10. Draw the Venn diagrams for each of these combinations of the sets *A*, *B*, and *C*.

a) $A \cap (B - C)$

b) $(A \cap B) \cup (A \cap C)$

c) $(A \cap \overline{B}) \cup (A \cap \overline{C})$

d) $\overline{A} \cap \overline{B} \cap \overline{C}$

e) $(A - B) \cup (A - C) \cup (B - C)$

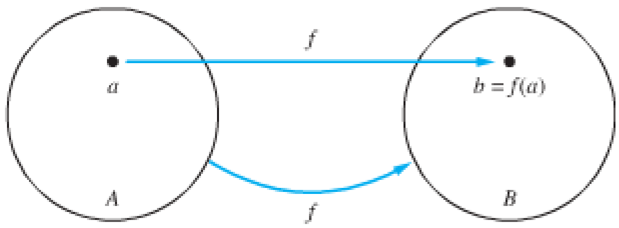
11. Show that $A \oplus B = (A \cup B) - (A \cap B)$

12. Show that $A \oplus B = (A - B) \cup (B - A)$

Section 1.9 – Functions

Definition

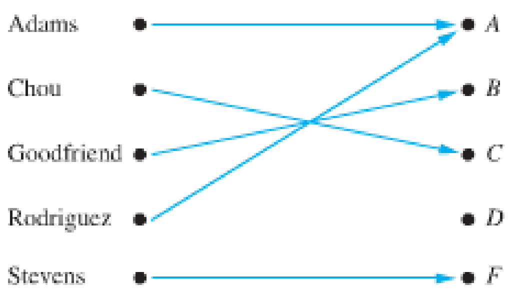
Let *A* and *B* be nonempty sets. A function f from *A* and *B* is an assignment of exactly one element of *B* to each of *A*. We write $f(a) = b$ if b is the unique element of *B* assigned by the function f to the element *a* of *A*. If f is a function from *A* to *B*, we write $f : A \to B$



When we define a function we specify its domain, its codomain, and the mapping of elements of the domain to elements in the codomain. Two functions are ***equal*** when they have the same domain, have the same codomain, and map each element of their common domain to the same element in their common codomain.

Example

What are the domain, codomain, and range of the function that assigns grades to students shown below



Solution

The domain is the set ***G*** = {Adams, Chou, Goodfriend, Rodriguez, Stevens}

The codomain is the set {*A*, *B*, *C*, *D*, *F*}

The range of ***G*** is the set {*A*, *B*, *C*, *F*}

Example

Let $f : \mathbb{Z} \to \mathbb{Z}$ assign the square of an integer to this integer. Then $f(x) = x^2$, where the domain of f is the set of all integers, the codomain of f is the set of all integers, and the range of f is the set of all integers that are perfect squares, namely, {0, 1, 4, 9, . . .}

Definition

Let f_1 and f_2 be functions from A to **R**. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to **R** defined for all $x \in A$ by

$$\left(f_1 + f_2\right)(x) = f_1(x) + f_2(x)$$

$$\left(f_1 f_2\right)(x) = f_1(x) f_2(x)$$

Example

Let f_1 and f_2 be functions from **R** to **R** such that $f_1(x) = x^2$ and $f_2 = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

Solution

$$\left(f_1 + f_2\right)(x) = x^2 + \left(x - x^2\right) = x$$

$$\left(f_1 f_2\right)(x) = x^2\left(x - x^2\right) = x^3 - x^4$$

Definition

Let f be function from A to B and Let S be a subset of A. The ***image*** of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so

$$f(S) = \left\{ t \,\middle|\, \exists s \in S \;\; \left(t = f(s)\right) \right\}$$

We also use the shorthand $\left\{ f(S) \,\middle|\, s \in S \right\}$ to denote this set.

One-to-One and Onto Functions

Definition

A function f is said to be *one-to-one*, or an ***injunction***, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be ***injective*** if it is one-to-one.

Note:

A function *f* is one-to-one (1 – 1) if different inputs have different outputs that is,

$$if \;\; a \neq b, \qquad then \quad f(a) \neq f(b)$$

A function *f* is one-to-one (1 – 1) if different outputs the same, the inputs are the same – that is,

$$if \;\; f(a) = f(b), \qquad then \;\; a = b$$

Remark

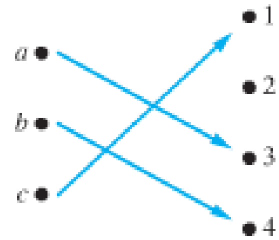
We can express that f is one-to-one using the qualifier as $\forall a \forall b \left(f(a) = f(b) \to a = b \right)$ or equivalently $\forall a \forall b \left(a \neq b \to f(a) \neq f(b) \right)$, where the universe of disclosure is the domain of the function.

Example

Determine whether the function f from {*a*, *b*, *c*, *d*} to {1, 2, 3, 4, 5} with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.

Solution

The function is one-to-one because *f* takes on different values of the four elements of its domain.



Example

Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Solution

The function is ***not*** one-to-one because $f(-1) = f(1) = 1$ but $1 \neq -1$

Example

Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one.

Solution

The function is one-to-one because $x + 1 \neq y + 1$ when $x \neq y$

Definition

A function *f* whose domain and codomain are subsets of the set of real numbers is called ***increasing*** if $f(x) \leq f(y)$, and ***strictly increasing*** if $f(x) < f(y)$, whenever $x < y$ and *x* and *y* are in the domain of *f*. Similarly, *f* is called ***decreasing*** if $f(x) \geq f(y)$, and ***strictly increasing*** if $f(x) > f(y)$, whenever $x < y$ and *x* and *y* are in the domain of *f*. (The word ***strictly*** in this definition indicates a strict inequality.)

Definition

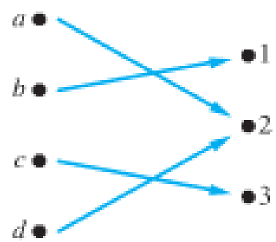
A function f from A to B is called ***onto***, or a ***surjection***, *iff* for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called ***surjective*** if it is onto.

Example

Let f be function from {*a*, *b*, *c*, *d*} to {1, 2, 3} defined by $f(a)=3$, $f(b)=2$, $f(c)=1$, and $f(d)=3$. Is f an onto function?

Solution

Because all three elements of the codomain are images of elements in the domain, we see that f is onto.



Example

Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution

The function is ***not*** onto because there is no integer *x* with $x^2 = -1$.

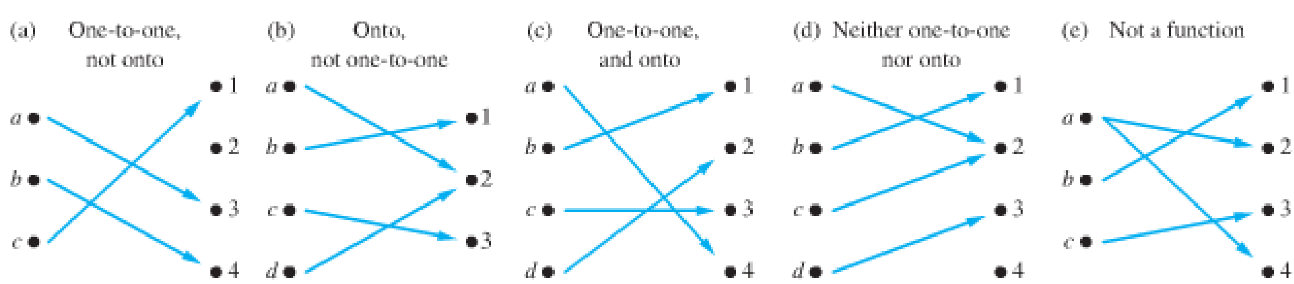
Example

Is the function $f(x) = x+1$ from the set of integers to the set of integers onto?

Solution

The function is onto because for every integer *y* there is an integer *x* such that $f(x) = y$.

$f(x) = y$ *iff* $x+1 = y$, which holds if and only if $x = y-1$



Definition

The function f is *one-to-one correspondence*, or a ***bijection***, if it is both one-to-one and onto. We say also that such a function is ***bijective***.

Example

Let f be function from {*a*, *b*, *c*, *d*} to {1, 2, 3, 4} defined by $f(a)=4$, $f(b)=2$, $f(c)=1$, and $f(d)=3$. Is f a bijection?

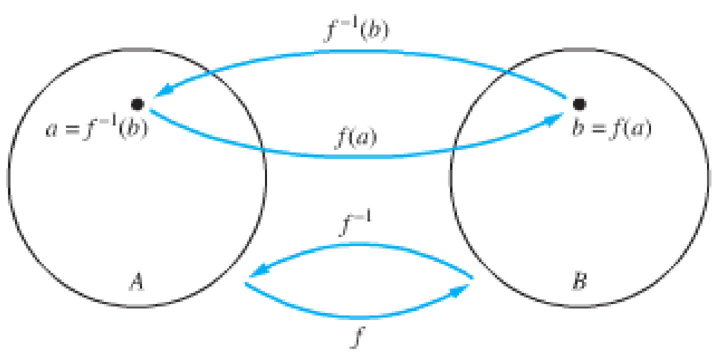
Solution

The function *f* is one-to-one and onto.
It is one-to-one because no two values in the domain are assigned the same function value.
It is onto because all four elements of the codomain are images of elements in the domain.
Hence, *f* is a bijection.

Inverse Functions and Compositions of Functions

Definition

Let f be a one-to-one correspondence from the set *A* to the set *B*. The inverse function of *f* is the function that assigns to an element *b* belonging to *B* the unique element *a* in *A* such that $f(a)=b$. The inverse function of *f* is denoted by f^{-1}. Hence, $f^{-1}(b)=a$ when $f(a)=b$



Example

Let f be function from {*a*, *b*, *c*} to {1, 2, 3} defined by $f(a)=2$, $f(b)=3$, and $f(c)=1$. Is f invertible, and if it is, what is its inverse?

Solution

The function f is invertible since it is a one-to-one.

The inverse function: $f^{-1}(1)=c$, and $f^{-1}(2)=a$, $f^{-1}(3)=b$

Example

Let $f:\mathbb{Z}\to\mathbb{Z}$ be such that $f(x)=x+1$. Is f invertible, and if it is, what is its inverse?

Solution

The function f is invertible since it is a one-to-one.

$y=x+1 \quad \Rightarrow \quad x=y-1$

$f^{-1}(y)=y-1$

Example

Let $f:\mathbb{R}\to\mathbb{R}$ be such that $f(x)=x^2$ Is f invertible?

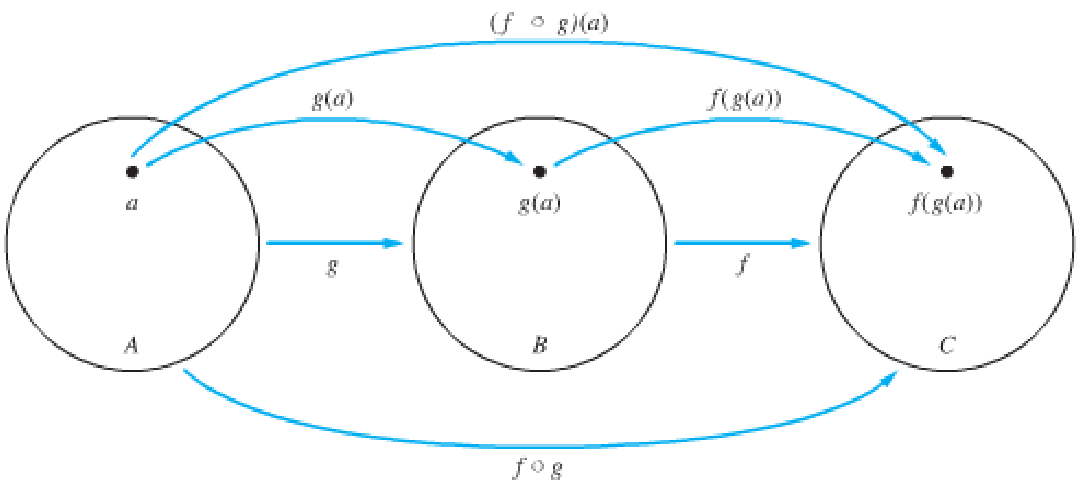
Solution

The function is ***not*** one-to-one. Hence, f is ***not*** invertible.

Definition

Let *g* be a function from the set *A* to the set *B* and let *f* be a function from the set *B* to the set *C*. The composition of the function *f* and *g*, denoted for all $a\in A$ by $f\circ g$, is defined by

$$(f\circ g)(a)=f(g(a))$$



Example

Let *g* be the function from the set {*a*, *b*, *c*} to itself such that $g(a)=b$, $g(b)=c$, and $g(c)=a$.

Let *f* be the function from the set {*a*, *b*, *c*} to the set {1, 2, 3} such that $f(a)=3$, $f(b)=2$, and $f(c)=1$. What is the composition of *f* and *g*, and what is the composition of *g* and *f*?

Solution

$(f\circ g)(a)=f(g(a))=f(b)=2$

$$(f \circ g)(b) = f(g(b)) = f(c) = 1$$
$$(f \circ g)(c) = f(g(c)) = f(a) = 3$$

$$(g \circ f)(a) = g(f(a)) = g(3) \quad \nexists.$$

Therefore; $g \circ f$ is not defined, because the range of *f* is not a subset of the domain of *g*.

Example

Let *f* and *g* be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of *f* and *g*, and what is the composition of *g* and *f*?

Solution

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(3x+2) \\ &= 2(3x+2)+3 \\ &= \underline{6x+7}|\end{aligned}$$

$$(g \circ f)(x) = g(f(x)) = g(2x+3) = 3(2x+3) + 2 = \underline{6x+11}|.$$

Exercises *Section* 1.9 – Functions

1. Why is *f* not a function from $\mathbb{R}$ to $\mathbb{R}$ if

a) $f(x)=\frac{1}{x}$?

b) $f(x)=\sqrt{x}$?

c) $f(x)=\pm\sqrt{x^2+1}$?

2. Determine whether *f* is a function from $\mathbb{Z}$ to $\mathbb{R}$ if

a) $f(x)=\pm x$?

b) $f(x)=\sqrt{x^2+1}$?

c) $f(x)=\frac{1}{x^2-4}$?

3. Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.

a) The function that assigns to each bit string the number of ones in the string minus the number of zeros in the string.

b) The function that assigns to each bit string twice the number of zeros in that string.

c) The function that assigns the number of bits over when a bit string is split into bytes (which are blocks of 8 bits).

4. Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one and onto.

a) $f(a)=b,\ f(b)=a,\ f(c)=c,\ f(d)=d$

b) $f(a)=b,\ f(b)=b,\ f(c)=d,\ f(d)=c$

c) $f(a)=d,\ f(b)=b,\ f(c)=c,\ f(d)=d$

5. Determine whether the function $f:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ is onto if

a) $f(m,\ n)=m+n$

b) $f(m,\ n)=m^2+n^2$

c) $f(m,\ n)=m$

d) $f(m,\ n)=|n|$

e) $f(m,\ n)=m-n$

6. Determine whether each of these functions is a bijection from $\mathbb{R}\to\mathbb{R}$

a) $f(x)=2x+1$

b) $f(x)=x^2+1$

c) $f(x)=x^3$

d) $f\left(x\right)=\dfrac{x^2+1}{x^2+2}$

e) $f\left(x\right)=x^5+1$

7. Suppose that g is a function from A to B and f is a function from B to C.

a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.

b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.