Linear Momentum and Collisions (Chapter 9 Lecture 1)

Momentum is a measure of the amount of motion an object has. It is defined to be the product of the mass and the velocity of the object.

$$\boxed{\vec{P} = m\vec{v}}$$

$\vec{P} \rightarrow$ momentum
$m \rightarrow$ mass
$\vec{v} \rightarrow$ velocity

Unit of measurement of momentum is kg*m/s

Example: Obtain the momentum of a 5 kg object moving with a velocity of $(2\hat{\imath} + 3\hat{\jmath})$ m/s

$m = 5\,kg$
$\vec{v} = (2\hat{\imath} + 3\hat{\jmath})$ m/s
$\vec{P} = ??$

$$\vec{P} = m\vec{v}$$
$$= 5kg(2\hat{\imath} + 3\hat{\jmath}) \text{ m/s}$$
$$= [10\hat{\imath} + 15\hat{\jmath}] \text{ kg*m/s}$$

9.1 Relationship between momentum & force

From Newton's 2nd Law

$$\vec{F} = m\vec{a} \text{ but } a = \frac{d\vec{v}}{dt}$$

$$\therefore \quad \vec{F} = \frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} \quad \text{but } m\vec{v} = \vec{P}$$

$$\boxed{\vec{F} = \frac{d\vec{P}}{dt}}$$

The relationship between force and momentum is that force acting on an object is equal to the rate of change of its momentum with time.

$$\vec{F} = \frac{d\vec{P}}{dt} \Rightarrow d\vec{P} = \vec{F}dt$$

$$\int_{\vec{P}_i}^{\vec{P}_f} d\vec{P} = \int_{t_i}^{t_f} \vec{F}dt$$

$$\Delta\vec{P} = \vec{P}_f - \vec{P}_i = \int_{t_i}^{t_f} \vec{P}\,dt$$

Change in momentum of an object is equal to the integral of the force acting on it in time.

The change in momentum of an object is also called the impulse acting on the object.

$$\boxed{\vec{I} = \Delta\vec{P} = \int_{t_i}^{t_f} \vec{F}\,dt}$$

$\vec{I} \rightarrow$ Impulse

Example: The force acting on an object varies with time according to the equation $F = -t^2 + 2t$.

a) Obtain the change in its momentum in the first 2 seconds

$t_i = 0$
$t_f = 2$
$F = -t^2 + 2t$
$\Delta P = ??$

$$\Delta\vec{P} = \int_{t_i}^{t_f} \vec{F}\,dt$$

$$\Delta\vec{P} = \int_0^2 (-t^2 + 2t)dt = \left[-\frac{t^3}{3} + t^2\right]\Big|_0^2$$

$$= \frac{4}{3} kg * \frac{m}{s}$$

b) Calculate the impulse acting on it in the first 2 seconds.

$I = ??$

$$I = \Delta P = \frac{4}{3} kg * \frac{m}{s}$$

c) If its initial velocity is 4 m/s, calculate its velocity after 2 seconds (mass= 2kg)

v_i =4 m/s

$\Delta P = \frac{4}{3} kg * \frac{m}{s}$

$m = 2\ kg$

$$\Delta P = mv_f - mv_i = \frac{4}{3}$$
$$2v_f - 2(4) = \frac{4}{3}$$
$$2v_f = \frac{4}{3} + 8 = \frac{28}{3}$$
$$v_f = \frac{14}{3} \text{ m/s}$$

<u>Example:</u> An object of mass 5 kg hits a wall with a velocity of $[8\hat{\imath} + 3\hat{\jmath}]$ m/s and bounces back with a velocity of $[-6\hat{\imath} + 2\hat{\jmath}]$ m/s

a) Calculate the change in its momentum

$\Delta \vec{P} = ??$

$$\Delta \vec{P} = \vec{P}_f - \vec{P}_i = m\vec{v}_f - m\vec{v}_i$$
$$= 5[-6\hat{\imath} + 2\hat{\jmath}] - 5[8\hat{\imath} + +3\hat{\jmath}]$$
$$= [-70\hat{\imath} - 5\hat{\jmath}] kg * \frac{m}{s}$$

b) Calculate the impulse impacted by the wall on the object.

$$\vec{I} = \Delta \vec{P} = [-70\hat{\imath} - 5\hat{\jmath}] kg * \frac{m}{s}$$

c) If the object was in contact with the wall for 0.2 seconds, calculate the average force exerted by the ball on the object.

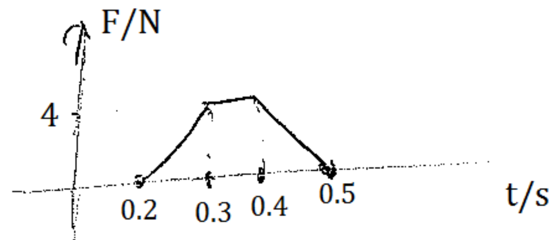
$\Delta \vec{P} = [-70\hat{\imath} - 5\hat{\jmath}] kg * \frac{m}{s}$

$\Delta t = 0.2$

$$\vec{F} = \frac{\Delta \vec{P}}{\Delta t}$$
$$\vec{F} = \frac{[-70\hat{\imath} - 5\hat{\jmath}]}{0.2} = [-350\hat{\imath} - 25\hat{\jmath}] N$$

Impulse or change in momentum can be obtained from a graph of force versus time as the area enclosed between the force versus time curve and the time axis.

<u>Example:</u> The following is a graph of force versus time for a certain particle of mass 3 kg.



a) Calculate the impulse acting on it.

I = area(⏢)

I = area (◺ 4 / 0.1) + area(□ 4 / 0.1) + area(4 ◺ / 0.1)

$$I = \frac{0.4}{2} + 0.4 + \frac{0.4}{2} = 0.8\ Ns$$

b) Calculate the change in its momentum

$$\Delta P = I = 0.8\ Ns$$

c) If initial speed is 2 m/s (i.e. at t = 0.2), calculate its final speed (i.e. at t = 0.5)

$m = 2kg$

$$\Delta P = mv_f - mv_i$$

$v_i = 2$ m/s
$\Delta P = 0.8\ Ns$
$v_f = ??$

$0.8 = 2v_f - 2(2)$
$2v_f = 4.8$
$v_f = \underline{2.4 \text{ m/s}}$

9.2 Conservation of Momentum

If the net force acting on an object is zero, then

$$\vec{F} = \frac{d\vec{P}}{dt} = 0 \quad \Rightarrow \quad d\vec{P} = 0$$

$$\int_{\vec{P}_i}^{\vec{P}_f} d\vec{P} = 0$$

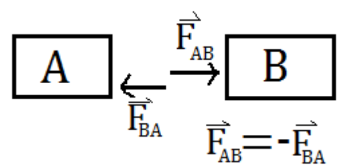
$$\Delta\vec{P} = \vec{P}_f - \vec{P}_i = 0 \ \text{ if } \ \vec{F} = 0$$

The principle of conservation of momentum states that if the net force acting on an object is zero, then the momentum of the object is conserved.

$$\boxed{\text{If } \vec{F} \text{ then } \vec{P}_f = \vec{P}_i}$$

The momentum of an isolated system of particles (an isolated system that does not interact with other systems) is conserved because all the forces involved are internal forces, and the sum of all internal forces is zero since for every action there is a reaction.

The momentum of two colliding objects is conserved if they are treated as a single system because the net force acting on the system is zero since the forces exerted on each other are action-reaction forces.



$$\vec{F}_{net} = \vec{F}_{AB} + \vec{F}_{BA}$$
$$= \vec{F}_{AB} + (-\vec{F}_{AB}) = 0$$

If an object of momentum of $\vec{P}_{1i}$ and an object of momentum $\vec{P}_{2i}$ collide and $\vec{P}_{1f}$ & $\vec{P}_{2f}$ are their momentums after collision, then

$$\vec{P}_{1i} + \vec{P}_{2i} = \vec{P}_{1f} + \vec{P}_{2f}$$

Or $\boxed{m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}}$ → conservation of momentum

For one dimensional collisions (collisions in a straight line becomes

$$\boxed{m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}}$$

Example: An object of mass 10 kg going to the right with a speed of 5 m/s collides with a 2 kg object moving with a speed of 3 m/s to the left. After the collision, the 2 kg object moves to the right with a speed of 20 m/s. Calculate the velocity of the 10 kg object after collision.

$m_1 = 10\ kg$
$v_{1i} = 5$ m/s
$v_{if} = ??$

$m_2 = 2\ kg$
$v_{2i} =$ -3 m/s (negative b/c its moving to the left)
$v_{2f} = 20$ m/s

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$
$$(10)(5) + (2)(-3) = (10)v_{if} + (2)(20)$$
$$44 = 10v_{if} + 40$$
$$10v_{if} = 4$$
$$v_{if} = 0.4 \text{ m/s}$$

9.3 Completely inelastic collisions

A completely inelastic collision is a collision where the two objects stick together after collision.

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$
$$\vec{v}_{1f} = \vec{v}_{2f} = \vec{v}$$
$$\boxed{m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = (m_1 + m_2)\vec{v}} \rightarrow$$ Equation of completely inelastic collision.

If the collision is one dimensional, then this becomes
$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2)\vec{v}$$
For completely inelastic collision, kinetic energy is not conserved because some of the energy will be converted to hear or internal energy.

<u>Example:</u> A 6 kg object moving to the right with a speed of 5 m/s collides with a 4 kg object moving in the same direction with a speed of 3 m/s. If the collision is completely inelastic.

a) Calculate their velocity after collision

$\mathrm{m}_1 = 6$ kg $\mathrm{v}_{1\mathrm{i}} = 5$ m/s	$\mathrm{m}_2 = 4$ kg $\mathrm{v}_{2\mathrm{i}} = 3$ m/s $\vec{v} = ??$	$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2)\vec{v}$ $(6)(5) + (4)(3) = (6+4)\vec{v}$ $42 = 10\vec{v}$ $\vec{v} = \frac{42}{10} = \frac{21}{5} = 4.2$ m/s

b) Calculate the kinetic energy lost during the collision

$\Delta KE = KE_f - KE_i = ??$

$$\Delta KE = KE_f - KE_i$$
$$= \left[\frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2\right] - \left[\frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2\right]$$
$$\text{but } v_{if} = v_{2f} = v$$
$$\Delta KE = \frac{1}{2}(m_1 + m_2)v^2 - \left[\frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2\right]$$
$$\frac{1}{2}(6+4)\left(\frac{21}{5}\right)^2 - \left[\frac{1}{2}(6)(5)^2 + \frac{1}{2}(4)(3)^2\right]$$
$$\Delta KE = 88.2 - 93 = -4.8\,J$$

<u>The Ballistic Pendulum</u>: is a pendulum used to measure the speed of a bullet. Suppose a bullet of mass m_b & speed v_b is fired horizontally into a pendulum of mass m_p which was initially at rest. And suppose the bullet is embedded in the pendulum and the bullet-pendulum system moves with a speed of v after collision. Thus, since the collision is completely inelastic
$$m_b v_b = (m_b + m_p)v$$
$$v_b = \frac{(m_p + m_b)v}{m_b}$$
If the pendulum rises by a height of h, the kinetic energy at the bottom will be converted to potential energy
$$\frac{1}{2}(m_b + m_p)v^2 = (m_b + m_p)|g|h$$
$$v^2 = 2|g|h$$
$$v = \sqrt{2|g|h}$$
$$\therefore \quad \boxed{v_b = \frac{m_p + m_b}{m_b}\sqrt{2|g|h}}$$

9.4 Completely Elastic Collisions (Lecture 2)

A completely elastic collision is a collision where not only momentum but also kinetic energy is conserved. Therefore in this case there are two conservation equations. Assuming a one dimensional collision

Conservation of momentum

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \;\ldots\ldots(1)$$

Conservation of kinetic energy

$$\frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2 \;\ldots\ldots(2)$$

These two equations can be simplified to two linear equations.

Rearranging eq (2) with terms containing m_1 on one side and terms containing m_2 on the other side.

$$\frac{1}{2}m_1 v_{1i}^2 - \frac{1}{2}m_1 v_{1f}^2 = \frac{1}{2}m_2 v_{2f}^2 - \frac{1}{2}m_2 v_{2i}^2$$
$$m_1\left(v_{1i}^2 - v_{1f}^2\right) = m_2\left(v_{2f}^2 - v_{2i}^2\right)$$
$$m_1\left(v_{1i} + v_{1f}\right)\left(v_{1i} - v_{1f}\right) = m_2\left(v_{2f} + v_{2i}\right)\left(v_{2f} - v_{2i}\right) \;\ldots\ldots(2)$$

Rearranging eq (1) with terms containing m_1 on one side and those containing m_2 on the other side.

$$m_1 v_{1i} - m_1 v_{1f} = m_2 v_{2f} - m_2 v_{2i}$$
$$m_1\left(v_{1i} - v_{1f}\right) = m_2\left(v_{2f} - v_{2i}\right) \;\ldots\ldots(1)$$

Dividing eq.(2) by eq.(1) results in

$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$
$$v_{1i} - v_{2i} = v_{2f} - v_{1f}$$
$$v_{1i} - v_{2i} = -\left(v_{1f} - v_{2f}\right) \;\ldots\ldots(3)$$

Replacing eq.(2) by eq.(3), the equations of completely elastic collisions become

$$\boxed{\begin{gathered} m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \\ v_{1i} - v_{2i} = -\left(v_{1f} - v_{2f}\right) \end{gathered}}$$

Example: An object of mass 4 kg moving to the right with a speed of 10 m/s collides with a 2 kg object moving in the same direction with a speed of 5 m/s. Calculate their speed after collision.

$m_1 = 4kg$	$m_2 = 2kg$
$v_{1i} = 10$ m/s	$v_{2i} = 5$ m/s
$v_{1f} =??$	$v_{2f} =??$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$
$$(4)(10) + (2)(5) = (4)v_{1f} + (2)v_{2f}$$
$$4v_{1f} + 2v_{2f} = 50$$
$$2v_{1f} + v_{2f} = 25 \;\ldots\ldots(1)$$

$$v_{1i} - v_{2i} = -\left(v_{1f} - v_{2f}\right)$$
$$10 - 5 = -\left(v_{1f} - v_{2f}\right)$$
$$v_{1f} - v_{2f} = -5$$
$$v_{1f} = -5 + v_{2f} \;\ldots\ldots(2)$$

Substituting for v_{1f} in eq.(1) from eq.(2)

$$2v_{1f} + v_{2f} = 25$$
$$2\left(-5 + v_{2f}\right) + v_{2f} = 25$$
$$-10 + 2v_{2f} + v_{2f} = 25$$
$$3v_{2f} = 35$$
$$v_{2f} = \frac{35}{3}\text{ m/s}$$

And $v_{1f} = -5 + v_{2f}$

$$= -5 + \frac{35}{3}$$
$$= \frac{-15 + 35}{3}$$
$$v_{1f} = \frac{20}{3}\text{ m/s}$$

9.5 Two Dimensional (glancing) collisions

A two dimensional collision is a collision where different directions in a plane are involved for the velocities. Therefore in this case the vector form of the conservation of momentum should be used.

$$\vec{P}_{1i} + \vec{P}_{2i} = \vec{P}_{1f} + \vec{P}_{2f}$$

$$\Rightarrow \quad \boxed{m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}}$$

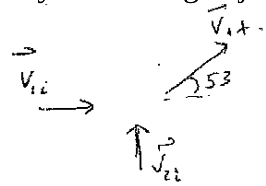
And in component form

$$\boxed{\begin{aligned} m_1v_{1ix} + m_2v_{2ix} &= m_1v_{1fx} + m_2v_{2fx} \\ m_1v_{1iy} + m_2v_{2iy} &= m_1v_{1fy} + m_2v_{2fy} \end{aligned}}$$

Example: An object of mass 5 kg going east with a speed of 4 m/s collided with a 3 kg object going north with a speed of 2 m/s. After collision the 4 kg object movies with a speed of 3 m/s making an angle of 53° with the horizontal.

a) Determine the x & y components of the velocity of the 3 kg object after the collision.

$m_1 = 4kg$
$v_{1i} = 4$ m/s
$\theta_{1i} = 0$ (east)



$$\vec{v}_{1i} = v_{1i}\cos\theta_{1i}\,\hat{\imath} + v_{1i}\sin\theta_{1i}\,\hat{\jmath}$$
$$= 4\cos 0\,\hat{\imath} + 4\sin 0\,\hat{\jmath}$$
$$= 4 \text{ m/s}\,\hat{\imath} + 0 \text{ m/s}\,\hat{\jmath} \Rightarrow v_{1ix} = 4\,;\, v_{1iy} = 0$$

$$\vec{v}_{1f} = 3 \text{ m/s}$$
$$\theta_{1f} = 53°$$
$$\vec{v}_{1f} = 3\cos 53\,\hat{\imath} + 3\sin 53\,\hat{\jmath}$$
$$= 2.4 \text{ m/s}\,\hat{\imath} + 1.8 \text{ m/s}\,\hat{\jmath}$$
$$\Rightarrow v_{1fx} = 2.4 \text{ m/s}\,;\, v_{1fy} = 1.8 \text{ m/s}$$

$m_2 = 3kg$
$v_{2i} = 2$ m/s
$\theta_{2i} = 90°$ (north)

$$\vec{v}_{2i} = v_{2i}\cos\theta_{1i}\,\hat{\imath} + v_{2i}\sin\theta_{1i}\,\hat{\jmath}$$
$$= 2\cos 90°\,\hat{\imath} + 2\sin 90°\,\hat{\jmath}$$
$$= 0\hat{\imath} + 2 \text{ m/s}\,\hat{\jmath}$$
$$\Rightarrow \quad v_{2ix} = 0\,;\, v_{2iy} = 2 \text{ m/s}$$

$v_{2fx} =??$
$v_{2fy} =??$

$$m_1v_{1ix} + m_2v_{2ix} = m_1v_{1fx} + m_2v_{2fx}$$
$$(4)(4) + (3)(0) = (4)(2.4) + 3v_{2fx}$$
$$16 = 9.6 + 3v_{2fx}$$
$$3v_{2fx} = 6.4$$
$$v_{2fx} = 2.13 \text{ m/s}$$

$$m_1v_{1iy} + m_2v_{2iy} = m_1v_{1fy} + m_2v_{2fy}$$
$$(4)(0) + (3)(2) = (4)(1.8) + 3v_{2fy}$$
$$6 = 7.2 + 3v_{2fy}$$
$$-1.2 = 3v_{2fy} \Rightarrow v_{2fy} = -.4 \text{ m/s}$$

b) Determine the magnitude and direction of the velocity of the 3 kg object after collision

$v_{2f} =??\quad \theta_{2f} =??$

$$v_{2f} = \sqrt{v_{2fx}^2 + v_{2fy}^2} = \sqrt{(2.13)^2 + (-.4)^2}$$
$$= 2.17 \text{ m/s}$$

$$\theta_{2f} = \tan^{-1}\left(\frac{v_{2fy}}{v_{2fx}}\right)$$
$$= \tan^{-1}\left(\frac{2.14}{-.4}\right) + 180°$$
$$\theta_{2f} = 100.6°$$

9.6 Center of mass of a system of particles

The center of mass of a system of particles $\vec{r}_{cm}$ is an average of the position vectors of all the particles of the system weighted by their masses.

$$\boxed{\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}}$$

Where $\vec{r}_1, \vec{r}_2, \ldots$ are the position vectors of the particles and $m_1, m_2, \ldots$ are the masses of the particles

Since the x & y components of a position vector are the x & y coordinates respectively, in component form, this becomes

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$
$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

Where (x_{cm}, y_{cm}) is the coordinate of the center of mass and $(x_1, y_1), (x_2, y_2), \ldots$ are the coordinates of the particles

Example: A system is comprised of 3 particles. Particle 1 has a mass of 3 kg and is located at$(-2, 5)m$. Particle 2 has a mass of 4 kg and is located at$(3, 6)m$. Particle 3 has a mass of 2 kg and is located at $(4, -5)m$. Determine the location of the center of mass of the system.

$m_1 = 3kg$	$m_2 = 4\ kg$	$m_3 = 2kg$
$(x_1, y_1) = (-2, 5)m$	$(x_2, y_2) = (3,6)m$	$(x_3, y_3) = (4, -5)m$
$x_1 = -2m\ ; y_1 = 5\ m$	$x_2 = 3m\ ; y_2 = 6m$	$x_3 = 4m\ ; y_3 = -5m$

$$(x_{cm}\ , y_{cm}) = ??$$

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$
$$= \frac{(3)(-2) + (4)(3) + (2)(4)}{3 + 4 + 2}$$
$$x_{cm} = \frac{14}{9}m$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$
$$= \frac{(3)(5) + (4)(6) + (2)(-5)}{3 + 4 + 2}$$
$$= \frac{29}{9}m$$

$$\therefore\ (x_{cm}, y_{cm}) = \left(\frac{14}{9}m, \frac{29}{9}m\right)$$

9.6.1 Center of Mass of Solid Objects

A solid object may be divided into particles of small masses, Δm_i. Thus

$$\vec{r}_{cm} = \frac{\sum_i \Delta m_i \vec{r}_i}{\sum \Delta m_i} \qquad \sum \Delta m_i = M \text{ total mass of the system}$$
$$\vec{r}_{cm} = \frac{\sum_i \Delta m_i \vec{r}_i}{M}$$

If the limiting value of this expression as Δm_i approaches zero is taken, the summation becomes an integral.

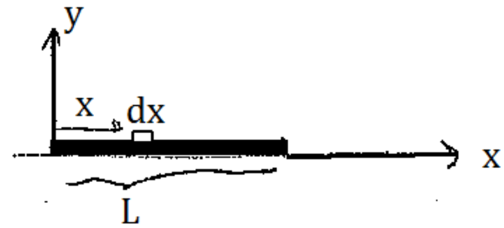
$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\, dm$$

And in component form

$$\boxed{\begin{aligned} x_{cm} &= \frac{1}{M}\int x\, dm \\ y_{cm} &= \frac{1}{M}\int y\, dm \end{aligned}}$$

Where M is the mass of the object

<u>Example:</u> Find the location of the center of mass of uniform rod of length L.



Uniform ⇒ constant linear density (ρ)

$$\rho = \frac{dm}{dx} = \frac{M}{L}$$
$$\therefore \quad dm = \rho\, dx$$
$$= \frac{M}{L} dx$$

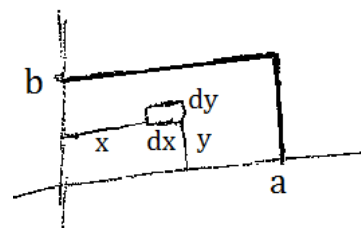
Where dm is mass of length element dx

$$x_{cm} = \frac{1}{M}\int_0^L x\, dm = \frac{1}{M}\int_0^L x\left(\frac{M}{L}dx\right)$$
$$= \frac{1}{L}\int_0^L x\, dx = \frac{1}{L}\left[\frac{x^2}{2}\Big|_0^L\right] = \frac{1}{L}\left[\frac{L^2}{2}\right] = \frac{L}{2}$$

$$\therefore \quad \boxed{x_{cm} = \frac{L}{2}}$$

The center of mass is located at the midpoint of the rod as expected.

<u>Example:</u> Find the location of the center of mass of a uniform rectangular object of sides <u>a & b.</u>



uniform ⇒ the area density (ρ) is a constant and is equal to the ratio between the total mass (m) & the total area ($A = ab$)

$$\rho = \frac{dm}{dA} = \frac{M}{ab}$$

dm is the mass of a small area element $dA = dxdy$

$$\therefore \quad dm = \rho dA = \left(\frac{M}{ab}\right) dx\, dy$$

&

$$= \frac{1}{ab}\int_0^b dy \int_0^a x\, dx = \frac{1}{ab}\int_0^b dy\left[\frac{x^2}{2}\Big|_0^a\right] = \frac{1}{ab}\int_0^b dy\left(\frac{a^2}{2}\right)$$
$$= \frac{a}{2b} y\big|_o^b = \frac{a}{2b} b = \frac{a}{2}$$
$$\therefore \quad x_{cm} = \frac{a}{2}$$

Similarly

$$y_{cm} = \frac{1}{M}\int y\, dm = \frac{1}{M}\iint y\left(\frac{M}{ab}\right) dx\, dy$$
$$= \frac{1}{ab}\int_0^b dx \int_0^a y\, dy = \frac{1}{ab}\int_0^a dx\left[\frac{y\text{^}2}{2}\Big|_0^b\right] = \frac{1}{ab}\int_0^a dx\left(\frac{b^2}{2}\right)$$

$$= \frac{b}{2a} x|_o^b = \frac{b}{2a} a = \frac{b}{2}$$

$$\therefore \quad y_{cm} = \frac{b}{2}$$

$$\boxed{(x_{cm}\,, y_{cm}) = \left(\frac{a}{2}\,, \frac{b}{2}\right)}$$

The center of mass is located at the geometric center of the rectangle as expected

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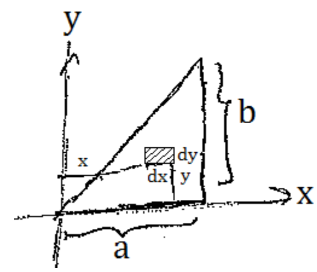
Example: Find the location of the center of mass of a uniform right angles triangular place of sides a & b

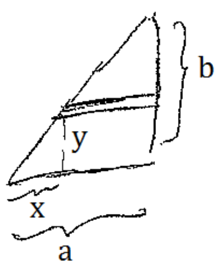
uniform ⇒ area density (ρ) is a constant and equal to mass divided by total area.



$$\rho = \frac{dm}{dA} = \frac{M}{\frac{1}{2}ab} = \frac{2m}{ab}$$

Where dm is the mass of a small area element $dA = dx\, dy$

$$\therefore dm = \rho dA = \left(\frac{2M}{ab}\right) dx\, dy$$



From proportionality of similar triangles

$$\frac{x}{y} = \frac{a}{b} \; or \; x = \frac{a}{b} y$$

$$x_{cm} = \frac{1}{M}\int x\, dm = \frac{1}{M}\iint x \left(\frac{2M}{ab}\right) dx\, dy$$

$$= \frac{2}{ab}\int_0^b dy \int_{x=\frac{a}{b}y}^{a} x\, dx$$

$$\therefore \quad x_{cm} = \frac{2}{ab}\int_0^b \left[\frac{x^2}{2}\Big|_{\frac{a}{b}y}^{a}\right]$$

$$= \frac{2}{ab}\int_0^b dy \left[\frac{a^2}{2} - \frac{\left(\frac{ay}{b}\right)^2}{2}\right]$$

$$= \frac{2}{ab}\int_0^b dy \left[\frac{a^2}{2} - \frac{a^2 y^2}{2b^2}\right]$$

$$= \frac{2}{ab}\frac{a^2}{2}\int_0^b dy - \frac{2}{ab}\frac{a^2}{2b^2}\int_0^b y^2\, dy$$

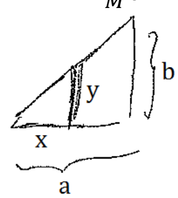
$$= \frac{2}{ab}\cdot\frac{a^2}{2}\cdot b - \frac{2}{ab}\cdot\frac{a^2}{2b^2}\left[\frac{y^3}{3}\Big|_0^b\right]$$

$$= a - \frac{a}{b^3}\cdot\frac{b^3}{3} = a - \frac{a}{3} = \frac{2}{3}a$$

$$x_{cm} = \frac{2}{3}a$$

Similarly

$$y_{cm} = \frac{1}{M}\int y\, dm = \frac{1}{M}\iint y \left(\frac{2M}{ab}\right) dx\, dy$$



$$\frac{y}{x} = \frac{b}{a} \quad or \quad y = \frac{b}{a}x$$

$$y_{cm} = \frac{2}{ab}\int_0^{\frac{b}{a}x} y\, dy = \frac{2}{ab}\int_0^a dx \left[\frac{y^2}{2}\Big|_0^{\frac{b}{a}x}\right]$$

$$= \frac{2}{ab}\int_0^a dx\, \frac{b^2 x^2}{2a^2}$$

$$= \frac{2b^2}{2a^3 b}\int_0^a x^2 dx = \frac{2b^2}{2a^3 b}\left[\frac{x^3}{3}\Big|_0^a\right]$$

$$= \frac{2b^2}{2a^3 b}\cdot\frac{a^3}{3} = \frac{b}{3}$$

$$y_{cm} = \frac{b}{3}$$

$$\therefore \quad \boxed{(x_{cm}\,, y_{cm}) = (\frac{2}{3}a\,, \frac{b}{3})}$$

Example: A rod extends from $x = 0$ to $x = 2\ m$ along the x-axis. The linear density of the rod depends on x according to the equation $\lambda = 2x + 1$. Find the location of its center of mass.
Solution

$$x_{cm} = \frac{1}{M}\int_{x=0}^{x=2} x\,dm \quad M = \int dm$$

$$\frac{dm}{dx} = \lambda = 2x + 1 \Rightarrow\ dm = (2x+1)dx$$

$$\therefore\quad m = \int dm = \int_{x=0}^{x=2} (2x+1)\,dx = (x^2 + x)|_{x=0}^{x=2} = 6\ kg.$$

$$x_{cm} = \frac{1}{6}\int_{x=0}^{x=2} x\,dm \quad \text{but} \quad dm = (2x+1)dx$$

$$\Rightarrow x_{cm} = \frac{1}{6}\int_{x=0}^{x=2} x(2x+1)\,dx = \frac{1}{6}\int_{x=0}^{x=2} (2x^2 + x)\,dx$$

$$= \frac{1}{6}\left[\frac{2x^3}{3} + \frac{x^2}{2}\right]|_{x=0}^{x=2} = \frac{1}{6}\left[\left(\frac{2(2)^3}{3} + \frac{2^2}{2}\right) - 0\right]$$

$$\underline{= 1.22\ m}$$

9.6.2 Velocity of the Center of Mass

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots}{M}$$

Where $M = m_1 + m_2 + \cdots$

$$\frac{d\vec{r}_{cm}}{dt} = \vec{v}_{cm} = \frac{m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \cdots}{M}$$

But $\frac{d\vec{r}_i}{dt} = \vec{v}_i \rightarrow$ velocity of the i^{th} particle

$$\therefore \boxed{\begin{aligned}\vec{v}_{cm} &= \frac{1}{M}(m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots) \\ &= \frac{1}{M}\sum_i m_i\vec{v}_i\end{aligned}}$$

That is, the velocity of the center of mass is equal to the average of the velocities of the particles weighed by their masses.

Example: Three particles of masses 1kg, 2kg, & 3kg are moving with velocities $(-2\hat{\imath} + 3\hat{\jmath})$m/s, $(5\hat{\imath} - 7\hat{\jmath})$m/s, and $-4\hat{\imath}$ m/s, respectively. Determine the velocity of the center of mass.

$m_1 = 1kg$	$m_2 = 2kg$	$m_3 = 3\ kg$
$\vec{v}_1 = (-2\hat{\imath} + 3\hat{\jmath})$ m/s	$\vec{v}_2 = (5\hat{\imath} - 7\hat{\jmath})$ m/s	$\vec{v}_3 = (-4\hat{\imath})$ m/s

$$\vec{v}_{cm} = ??$$

$$M = m_1 + m_2 + m_3 = (1 + 2 + 3)kg = 6kg$$

$$\therefore\ \vec{v}_{cm} = \tfrac{1}{6}[1(-2\hat{\imath} + 3\hat{\jmath}) + 2(5\hat{\imath} - 7\hat{\jmath}) + 3(-4\hat{\imath})]\ \text{m/s}$$

$$= \tfrac{1}{6}[-2\hat{\imath} + 3\hat{\jmath} + 10\hat{\imath} - 14\hat{\jmath} - 12\hat{\imath}]\ \text{m/s}$$

$$= \frac{1}{6}[(-2 + 10 - 12)\hat{\imath} + (3 - 14)\hat{\jmath}]$$
$$= \frac{1}{6}[-4\hat{\imath} - 11\hat{\jmath}] \text{ m/s} = [-\frac{2}{3}\hat{\imath} - \frac{11}{6}] \text{ m/s}$$

9.6.3 Acceleration of the center of mass of a system of particles

$$\vec{v}_{cm} = \frac{1}{M}(m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots)$$
$$\vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{1}{M}\left(m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + m_3\frac{d\vec{v}_3}{dt} + \cdots\right)$$

But $\frac{d\vec{v}_i}{dt} = \vec{a}_i \rightarrow$ acceleration of the i^{th} particle

$$\boxed{\vec{a}_{cm} = \frac{1}{M}(m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots)}$$
$$\vec{a}_{cm} = \frac{1}{M}\sum_i \vec{a}_i\, m_i$$

That is, the acceleration of the center of mass is equal to the average of the acceleration of the particles weighted by their masses.

Example: The center of mass of the system of two particles of masses 10 kg & 20 kg is moving with an acceleration of $(4\hat{\imath} - 3\hat{\jmath})$ m/s^2. If the 10 kg object is moving with an acceleration of $(6\hat{\jmath})$ m/s^2, find the acceleration of the 20 kg object.

$m_1 = 10kg \quad \vec{a}_{cm} = (4\hat{\imath} - 3\hat{\jmath})$ m/s^2

$\vec{a}_1 = (6\hat{\jmath})$ m/$s^2 \quad m_2 = 20kg \quad \vec{a}_2 = ??$

$$m\vec{a}_{cm} = m\vec{a}_1 + m\vec{a}_2 \qquad M = (10 + 20)kg = 30kg$$
$$30(4\hat{\imath} - 3\hat{\jmath}) = 10(6\hat{\jmath}) + 20\vec{a}_2$$
$$120\hat{\imath} - 90\hat{\jmath} = 60\hat{\jmath} + 20\vec{a}_2$$
$$20\vec{a}_2 = 120\hat{\imath} - 150\hat{\jmath}$$
$$\vec{a}_2 = \frac{120\hat{\imath} - 150\hat{\jmath}}{20} = \underline{[6\hat{\imath} - 7.5\hat{\jmath}] \text{ m/}s^2}$$

Net force acting on system of particles

The internal forces exerted among the particles themselves will add up to zero because they are action reaction pairs. That is, for every force exerted by one particle on another there will be a reaction force equal but opposite in direction. Therefore the net force acting on a system of particles is equal to the net external force (forces exerted by particles outside the system). The net external force is equal to the sum of all the external forces exerted on the particles in the system.

$$F_{ext} = m_a\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots$$

But $m_a\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots = m\vec{a}_{cm}$

$$\therefore \quad \boxed{F_{ext} = m\vec{a}_{cm}}$$

The net external force on a system of particles is equal to the product of the total mass of the particles and the acceleration of the center of mass of the particles.