12. Static Equilibrium

12.1 Static equilibrium:

There are two kinds of equilibrium: translational equilibrium and rotational equilibrium.

Translational equilibrium: An object is said to be in translational equilibrium if it is either at rest or moving in a straight line with a constant speed.

Condition of translational equilibrium: An object will be in translational equilibrium if the net force acting on it is zero

$$\boxed{\sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots = 0}$$

In component form

$$\boxed{\begin{aligned} \sum \vec{F}_x &= \vec{F}_{1x} + \vec{F}_{2x} + \vec{F}_{3x} + \cdots = 0 \\ \sum \overrightarrow{F_y} &= \vec{F}_{1y} + \vec{F}_{2y} + \vec{F}_{3y} + \cdots = 0 \end{aligned}}$$

Rotational equilibrium: An object is said to be in rotational equilibrium if it is either at rest or rotating with a constant angular speed.

Conditions of rotational equilibrium: An object is said to be in rotational equilibrium if the net torque acting on it is zero.

$$\boxed{\sum \tau = \tau_1 + \tau_2 + \tau_3 + \cdots = 0}$$

Note: the torque due to the weight of an object can be obtained by assuming that the entire weight is concentrated at its center of mass

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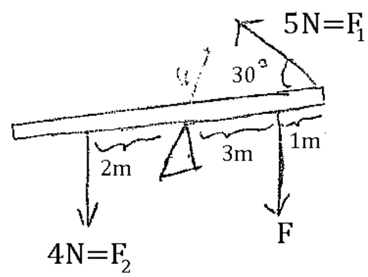


Example: Consider the uniform lever shown pivoted at its center. The lever is in equilibrium. Its weight is negligible.

a) Calculate the unknown force F.

Since it is in equilibrium the net torque acting on it must be zero.

$$\sum \tau = \tau_1 + \tau_2 + \tau_3 = 0$$

Using the pivot as the point of rotation

$F_1 = 5N$	$\theta_1 = 30°$	$r_1 = 4m$

$|\tau_1| = F_1 r_1 \sin\theta_1$
$= (5)(4)\sin 30$
$= 10Nm$
$\tau_1 = +10Nm$ (positive b/c its tendency is to produce counter clockwise rotation)

$F_2 = 4N$	$\theta_2 = 90°$	$r_2 = 2m$

$|\tau_2| = F_2 r_2 \sin\theta_2$
$= (4)(2)\sin 90°$
$= 8Nm$
$\tau_2 = +8Nm$ (positive b/c its tendency is to produce counter clockwise rotation)

$F = ??$	$\theta_F = 90°$	$r_F = 3m$

$|\tau_F| = F r_F \sin\theta_F$
$= F(3)\sin 90°$
$= 3F$
$\tau_F = -3F$ (negative b/c its tendency is to produce clockwise rotation)

$$\tau_1 + \tau_2 + \tau_3 = 0$$
$$10 + 8 - 3F = 0$$

$$-3F = 18$$
$$\underline{F = 6N}$$

b) Calculate the force exerted by the fulcrum
Since it is also in translational equilibrium the net force acting on it should be zero.
Let the force exerted by the fulcrum be represented by F_f

$$\vec{F}_{1x} + \vec{F}_{2x} + \vec{F}_{Fx} + F_{fx} = 0$$
$$\vec{F}_{1y} + \vec{F}_{2y} + \vec{F}_{Fy} + F_{fy} = 0$$

$$F_{1x} = F_1 \cos(180 - \theta_1)$$
$$= 5\cos 150° = -4.33$$
$$F_{2x} = 4\cos(-90°) = 0$$
$$F_{Fx} = 6\cos(-90°) = 0$$

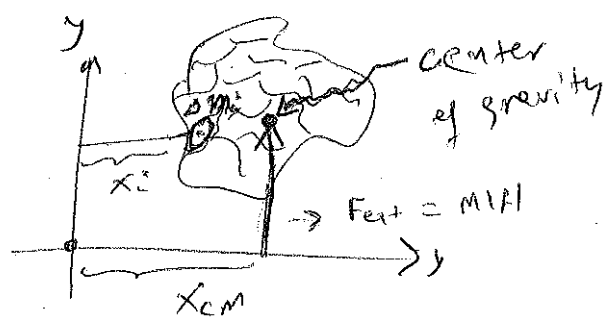
$$\vec{F}_{1x} + \vec{F}_{2x} + \vec{F}_{Fx} + F_{fx} = 0$$
$$-4.3 + 0 + 0 + F_{fx} = 0$$
$$F_{fx} = 4.3N$$

$$F_{1y} = F_1 \sin(180 - \theta_1)$$
$$= 5\sin 150° = 2.5$$
$$F_{2y} = 4\sin(-90°) = -4$$
$$F_{Fy} = 6\sin(-90°) = -6$$

$$\vec{F}_{1y} + \vec{F}_{2y} + \vec{F}_{Fy} + F_{fy} = 0$$
$$2.5 - 4 - 6 + F_{fy} = 0$$
$$F_{fy} = 7.5N$$

$$F_f = \sqrt{F_{fx}^2 + F_{fy}^2} = \sqrt{(4.3)^2 + (7.5)^2} = 8.64\ N$$
$$\theta_f = \tan^{-1}\left(\frac{F_{fy}}{F_{fx}}\right) = \tan^{-1}\left(\frac{7.5}{4.3}\right) = 60°$$

Torque due to the weight of an object

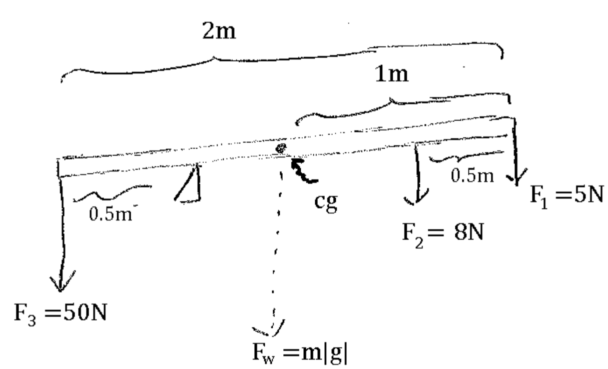


The point at which an object can be balanced completely is called the center of gravity of the object (which is the same with the center of mass of the object). Consider an object balanced by applying an external force on its center of gravity. Since it is in equilibrium, the net torque about any point should be zero. Let's take the origin shown to be as our point of rotation. Let's assume the object is divided into small masses $\Delta m_i's$. Then the torque due to each Δm_i is $-\Delta m_i|g|x_i$. The torque due to external balancing force is $M|g|x_{cm}$, where M is the mass of the object. Since it is in equilibrium the torque due to its weight and the torque due to the external force should be equal.

$$\sum \Delta m_i|g|x_i = M|g|x_{cm}$$

Therefore it follows that the torque due to the weight of an object can be calculated by assuming the whole mass of the object is concentrated at its center of gravity.

Example: The lever shown is uniform and is in equilibrium. Calculate the mass of the lever.



Solution

Since the lever is uniform, its center of gravity is located at its mid-point. The whole weight of the lever can be assumed to act at its midpoint.

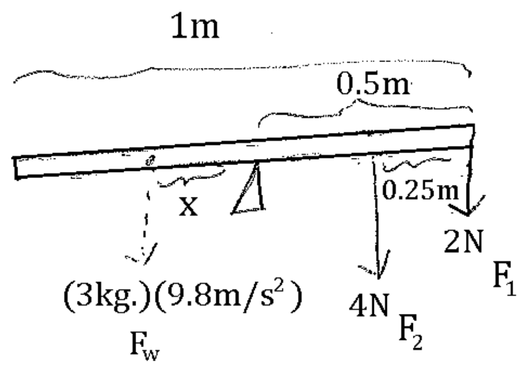
Taking torques about the pivot

$$\tau_1 + \tau_2 + \tau_3 + \tau_w = 0$$

$\tau_1 = -5(1.5)$ $= -7.5$	$\tau_2 = -8(1)$ $= -8$	$\tau_3 = 50(0.5)$ $= 25$	$\tau_w = -M\|g\|(0.5)$ Where M is the mass of the lever

$$-7.5 - 8 + 25 - M(9.8)(0.5) = 0$$
$$4.9M = 9.5$$
$$\underline{M = 1.94\ kg}$$

Example: The lever shown is non-uniform and is in equilibrium. It is pivoted at its midpoint. Calculate the location of its center of gravity if its mass is 3 kg.



Solution: Let the location of the center of gravity be a distance of x to the left of the pivot. Since the lever is in equilibrium, the net torque acting on the lever should be zero. Taking the pivot to be the point of rotation

$$\tau_1 + \tau_2 + \tau_w = 0$$

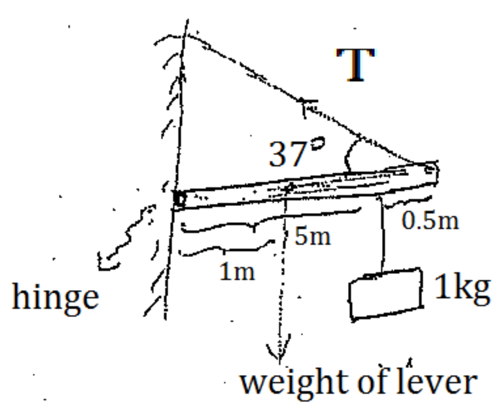
$\tau_1 = -2(0.5)$ $= -1Nm$	$\tau_2 = -4(.25)$ $= -2Nm$	$\tau_w = 3(9.8)x$ $= 29.4x$

$$29.4x - 1 - 2 = 0$$
$$x = \frac{3}{29.4} = 0.1m$$

Therefore the center of gravity is located <u>0.1m</u> to the left of the pivot.

Chapter 12 Section 2

Example: A uniform lever of mass 2 kg and length 2 m is hinged to a wall supported by a string as shown. There is a weight of mass 1 kg hanging at a distance of 1.5 m from the hinge.



a) Calculate the tension *(T)* in the string

The weight of the lever acts at the center of gravity of the lever which is 1m away from the hinge.

Taking the hinge as the point of rotation the torque due to the 1 kg weight and the weight of the lever are negative because they have clockwise tendencies.

$$\sum \tau = 0$$
$$T(4)\sin 37° - 2(9.9)(1) - 1(9.8)(1.5) = 0$$
$$2.4T - 29.4 = 0$$
$$T = \frac{29.4}{2.4} = 12.25\ N$$

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b) Calculate the force exerted by the hinge on the lever.
Let the force exerted by the hinge be represented by *F*.

$$\sum F = 0$$

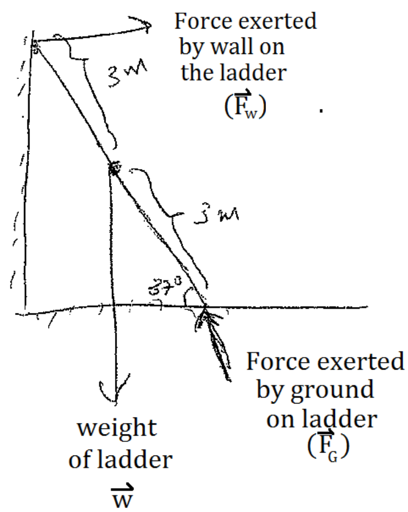
$$F_x - T\cos 37° = 0 \qquad T = 12.25$$
$$F_x = (12.25)(.8) = 9.8N$$

$$F_y + T\sin 37° - 2(9.8) + 1(9.8) = 0$$
$$F_y = -(12.25)(0.6) + 2(9.8) + 1(9.8)$$
$$= 22N$$

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{9.8^2 + 22^2} = 24.1N$$
$$\theta_F = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{22}{9.8}\right)$$
$$= 66°$$

Example: Consider the uniform ladder shown. Its length is 6m/ Its mass is 4 kg. The wall is frictionless.

a) Calculate the force exerted by the wall on the ladder.



Let the force exerted be represented by $\vec{F}_w$, the force exerted by the ground by $\vec{F}_G$ & the weight of the ladder by $\vec{w}$.
Since the wall is frictionless, the force exerted by the van on the ladder is only a normal force and is perpendicular to the wall.
Taking the point of rotation to be the point of contact with the ground.

$$\tau_{F_w} + \tau_w = 0$$
$$|\tau_{F_w}| = F_w \cdot r_{F_w}\sin(\theta_{F_w}) \qquad r_{F_w} = 6m$$
$$= F_w(6)\sin 37° \qquad \theta_{F_w} = 37°$$
$$= 3.6F_w$$

$\tau_{F_w} = -3..6F_w$(because its tendency is to produce clockwise rotation)

$$|\tau_w| = wr_w\sin(\theta_w) \qquad w = m|g| \qquad m = 4kg$$
$$= (4)(10)$$
$$= 40N$$
$r_w = 3m$ (center of gravity is at midpoint b/c it is uniform)
$$\theta_w = 90 - 37$$
$$= 53°$$

$$\therefore \quad |\tau_w| = (40)(3)\sin 53°$$
$$= 96\ N\cdot m$$
$$\tau_w = +96\ N\cdot m$$
(+ because its tendency is to produce counter clockwise rotation)
$$\tau_{F_w} + \tau_w = 0$$

$$-3.6F_w + 96 = 0$$

$$F_w = \frac{96}{3.6} = 26.7N \text{ perpedicular to the wall}$$

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b) Calculate the force exerted by the ground on the ladder.

$$\sum F_x = 0 \quad \Longrightarrow \quad F_{wx} + F_{Gx} + w_x = 0$$

$$\sum F_y = 0 \quad \Longrightarrow \quad F_{wy} + F_{Gy} + w_y = 0$$

Since $\vec{F}_w$ is a horizontal force to the right

$$F_{wx} = F_w = 26.7N$$

$$F_{wy} = 0$$

Since the weight is a downward vertical force

$$w_x = 0$$

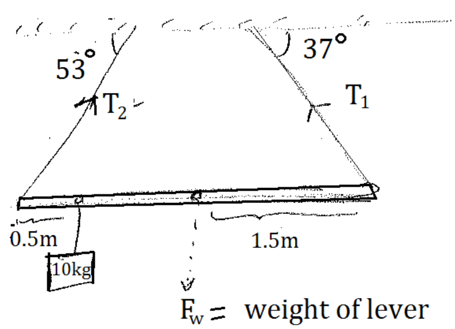
$$w_y = -m|g| = -4(10) = -40N$$

$F_{wx} + F_{Gx} + w_x = 0$	$F_{wy} + F_{Gy} + w_y = 0$
$26.7 + F_{Gx} + 0 = 0$	$0 + F_{Gy} - 40 = 0$
$F_{GX} = -26.7\ N$	$F_{Gy} = 40N$

$$F_G = \sqrt{F_{Gx}^2 + F_{Gy}^2} = \sqrt{(-26.7)^2 + (40)^2} = 48N$$

$$\theta_G = \tan^{-1}\left(\frac{F_{Gy}}{F_{Gx}}\right) = \tan^{-1}\left(\frac{40}{-26.7}\right) + 180 = 123.7°$$

Example: A uniform rod of length 3 m and mass 5 kg is suspended by 2
strings as shown. An object of mass 12 kg is hanging 0.5 m away from the left end.



a) Calculate the tension in the string attached to the right end.
   Solution
   Taking the pivot to be the left end (so that we don't have to worry about T_2)

$$\sum \tau = 0 \quad \Rightarrow T_1 \sin 37°\,(3m) - (10kg)\left(9.8\frac{m}{s^2}\right)(0.5) - (12kg)\left(9.8\frac{m}{s^2}\right)(1.5) = 0$$

$$T_1 = 125N$$

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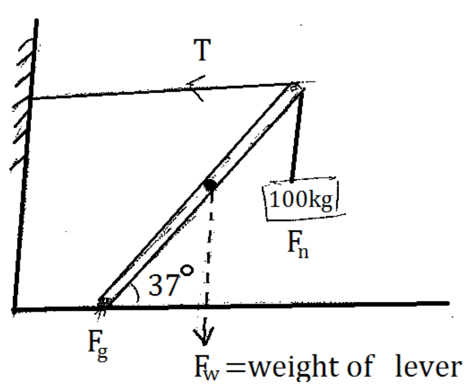
b) Calculate the tension in the string attached to the left end.
   Solution
   Taking the pivot to be at the right end

$$-T_2 \sin 53\,(3) + 10(9.8)(2.5) + (12)(9.8)(1.5) = 0$$

$$T_2 = 175N$$

Example: A uniform rod of length 2 m and mass 4 kg is pivoted at the ground as shown. It is supported by a string



attached to the wall as shown. An object of mass 100kg is hanging from its top end.

a) Calculate the tension in the string.
   Solution
   Taking the pivot at the ground contact.

$$\sum \tau = 0 \quad \Rightarrow \quad T \sin 37\,(2) - 100(9.8) \sin 53\,(2) - (4)(9.8) \sin 53\,(1) = 0$$
$$T = 1332.8\, N$$

b) Calculate the force exerted by the ground on the lever.
   Solution
   Let the forces exerted by the ground and hanging object be $\vec{F_g}$ & $\vec{F_n}$, respectively, and let the force due to its own weight be $\vec{F_w}$. Since it is in equilibrium $\vec{F_g} + \vec{F_w} + \vec{F_n} + T = 0$

$\vec{T} = -1332.8\,\hat{\imath}$	$\vec{F_n} = -100(9.8)\hat{\jmath}$ $= -980\hat{\jmath}$	$\vec{F_w} = -4(9.8)\hat{\jmath} = -39.2\hat{\jmath}$

$$\therefore \quad \vec{F_g} = -\vec{F_w} - \vec{F_n} - \vec{T} = 39.2\hat{\jmath} + 980\hat{\jmath} + 1332.8\hat{\imath}$$
$$\vec{F_g} = (1332.8\hat{\imath} + 1019.2\hat{\jmath})N$$