Solution Section 2.8 – Derivatives of Logarithmic & Exponential Functions

Exercise

Find the derivative of $y = \ln\sqrt{x+5}$

Solution

$$y = \ln(x+5)^{1/2}$$

$$= \frac{1}{2}\ln(x+5)$$

$$\boxed{y' = \frac{1}{2(x+5)}}$$

Exercise

Find the Derivatives of $y = (3x+7)\ln(2x-1)$

Solution

$$f = 3x+7 \qquad f' = 3$$

$$g = \ln(2x-1) \qquad g' = \frac{2}{2x-1}$$

$$\boxed{y' = 3\ln(2x-1) + \frac{2(3x+7)}{2x-1}}$$

Exercise

Find the Derivatives of $f(x) = \ln\sqrt[3]{x+1}$

Solution

$$f(x) = \ln(x+1)^{1/3}$$

$$= \frac{1}{3}\ln(x+1)$$

$$u = x+1 \Rightarrow \frac{du}{dx} = 1$$

$$f'(x) = \frac{1}{3}\frac{1}{x+1}$$

$$\boxed{= \frac{1}{3(x+1)}}$$

Exercise

Find the Derivatives of $f(x)=\ln\left[x^2\sqrt{x^2+1}\right]$

Solution

$$f(x)=\ln(x^2)+\ln\sqrt{x^2+1} \qquad \textit{Product Property}$$

$$f(x)=\ln(x^2)+\ln(x^2+1)^{1/2}$$

$$f(x)=2\ln x+\tfrac{1}{2}\ln\left(x^2+1\right) \qquad \textit{Power Property}$$

$$f'(x)=2\frac{1}{x}+\frac{1}{2}\frac{2x}{x^2+1} \qquad \textit{Differentiate}$$

$$=\frac{2}{x}+\frac{x}{x^2+1}$$

Exercise

Find the Derivatives of $y=\ln\dfrac{x^2}{x^2+1}$

Solution

$$y=\ln x^2-\ln x^2+1$$

$$y'=\frac{2x}{x^2}-\frac{2x}{x^2+1}$$

$$=\frac{2}{x}-\frac{2x}{x^2+1}$$

Exercise

Find the Derivatives of $y=\ln\left[\dfrac{x^2(x+1)^3}{(x+3)^{1/2}}\right]$

Solution

$$y=\ln\left[x^2(x+1)^3\right]-\ln(x+3)^{1/2} \qquad \textit{Quotient Rule}$$

$$=\ln x^2+\ln(x+1)^3-\ln(x+3)^{1/2} \qquad \textit{Product Rule}$$

$$=2\ln x+3\ln(x+1)-\tfrac{1}{2}\ln(x+3) \qquad \textit{Power Rule}$$

$$y' = \boxed{\frac{2}{x} + \frac{3}{x+1} - \frac{1}{2(x+3)}}$$

Exercise

Find the Derivatives of $y = \ln(x^2 + 1)$

Solution

$$y' = \boxed{\frac{2x}{x^2 + 1}}$$

$$(\ln U)' = \frac{U'}{U}$$

Exercise

Find the Derivatives of $f(x) = \ln(x^2 - 4)$

Solution

Let u = $x^2 - 4 \Rightarrow \frac{du}{dx} = 2x$

$$f'(x) = \frac{1}{x^2 - 4}(2x)$$

$$= \boxed{\frac{2x}{x^2 - 4}}$$

Exercise

Find the derivative $f(x) = 2\ln\left(x^2 - 3x + 4\right)$

Solution

$$f'(x) = 2\frac{2x-3}{x^2 - 3x + 4}$$

$$= \boxed{\frac{4x-6}{x^2 - 3x + 4}}$$

Exercise

Find the derivative $f(x) = 3\ln\left(1 + x^2\right)$

Solution

$$f'(x) = 3\frac{2x}{1+x^2}$$
$$\boxed{= \frac{6x}{1+x^2}}$$

Exercise

Find the derivative $\quad f(x) = (1+\ln x)^3$

Solution

$$f'(x) = 3(1+\ln x)^2 (1+\ln x)'$$
$$= 3(1+\ln x)^2 \left(\frac{1}{x}\right)$$
$$\boxed{= \frac{3}{x}(1+\ln x)^2}$$

Exercise

Find the derivative $\quad f(x) = (x - 2\ln x)^4$

Solution

$$f'(x) = 4(x - 2\ln x)^3 (x - 2\ln x)'$$
$$= 4(x - 2\ln x)^3 \left(1 - \frac{2}{x}\right)$$
$$= 4(x - 2\ln x)^3 \left(\frac{x-2}{x}\right)$$
$$\boxed{= \frac{4x-8}{x}(x - 2\ln x)^3}$$

Exercise

Find the Derivatives of $f(x) = x^2 \ln x$

Solution

$$f' = x^2\left(\frac{1}{x}\right) + 2x\ln x \qquad\qquad (fg)' = f'g + fg'$$
$$= x + 2x\ln x$$
$$\boxed{= x(1+2\ln x)}$$

Exercise

Find the Derivatives of $f(x) = -\dfrac{\ln x}{x^2}$

Solution

$$f' = -\frac{x^2 \frac{d}{dx}[\ln x] - \ln x \frac{d}{dx}\left[x^2\right]}{\left(x^2\right)^2}$$

$$= -\frac{x^2 \frac{1}{x} - 2x\ln x}{x^4}$$

$$= -\frac{x - 2x\ln x}{x^4}$$

$$= -\frac{x(1-2\ln x)}{x^4}$$

$$= -\frac{1-2\ln x}{x^3}$$

Exercise

Find the derivative of $y = \ln\left(t^2\right)$

Solution

$$y' = \frac{\left(t^2\right)'}{t^2}$$

$$= \frac{2t}{t^2}$$

$$= \frac{2}{t}$$

Exercise

Find the derivative of $y = \ln(2\theta + 2)$

Solution

$$y' = \frac{2}{2\theta + 2}$$

$$= \frac{1}{\theta + 1}$$

Exercise

Find the derivative of $y = (\ln x)^3$

Solution

$$y' = 3(\ln x)^2 (\ln x)' = 3(\ln x)^2 \frac{1}{x}$$

$$\boxed{= \frac{3(\ln x)^2}{x}}$$

Exercise

Find the derivative of $y = x(\ln x)^2$

Solution

$$y' = (\ln x)^2 + x\left(2(\ln x)\frac{1}{x}\right)$$

$$\boxed{= (\ln x)^2 + 2\ln x}$$

Exercise

Find the derivative of $y = \frac{x^4}{4}\ln x - \frac{x^4}{16}$

Solution

$$y' = \frac{4x^3}{4}\ln x + \frac{x^4}{4}\frac{1}{x} - \frac{4x^3}{16}$$

$$= x^3 \ln x + \frac{1}{4}x^3 - \frac{1}{4}x^3$$

$$\boxed{= x^3 \ln x}$$

Exercise

Find the derivative of $y = \frac{1+\ln t}{t}$

Solution

$$y' = \frac{\frac{1}{t}t - (1+\ln t)}{t^2}$$

$$= \frac{1-1-\ln t}{t^2}$$

$$\boxed{= -\frac{\ln t}{t^2}}$$

Exercise

Find the derivative $\quad f(x)=\dfrac{\ln x}{1+x}$

Solution

$$f'(x)=\dfrac{\left(\frac{1}{x}\right)(1+x)-\ln x}{(1+x)^2} \qquad u=\ln x \quad v=1+x \quad u'=\frac{1}{x} \quad v'=1$$

$$=\frac{1}{x}\dfrac{1+x-x\ln x}{(1+x)^2}$$

$$=\dfrac{1+x-x\ln x}{x(1+x)^2}$$

Exercise

Find the derivative $\quad f(x)=\dfrac{2x}{1+\ln x}$

Solution

$$f'(x)=\dfrac{2(1+\ln x)-(2x)\frac{1}{x}}{(1+\ln x)^2} \qquad u=2x \quad v=1+\ln x \quad u'=2 \quad v'=\frac{1}{x}$$

$$=\dfrac{2+2\ln x-2}{(1+\ln x)^2}$$

$$=\dfrac{2\ln x}{(1+\ln x)^2}$$

Exercise

Find the derivative $\quad f(x)=x^3\ln x$

Solution

$$u=x^3 \qquad v=\ln x$$

$$u'=3x^2 \qquad v'=\frac{1}{x}$$

$$f'(x)=3x^2\ln x+x^3\frac{1}{x}$$

$$=3x^2\ln x+x^2$$

$$=(3\ln x+1)x^2$$

Exercise

Find the derivative $f(x) = 6x^4 \ln x$

Solution

$$u = 6x^4 \qquad v = \ln x$$
$$u' = 24x^3 \qquad v' = \frac{1}{x}$$

$$f'(x) = 24x^3 \ln x + 6x^4 \frac{1}{x}$$
$$= 24x^3 \ln x + 6x^3$$
$$\boxed{= 6x^3 (4\ln x + 1)}$$

Exercise

Find the derivative $f(x) = \ln x^8$

Solution

$$f(x) = \ln x^8 = 8\ln x$$

Power Rule

$$\boxed{f'(x) = \frac{8}{x}}$$

$$(\ln x)' = \frac{1}{x}$$

Exercise

Find the derivative $f(x) = 5x - \ln x^5$

Solution

$$f(x) = 5x - \ln x^5$$
$$= 5x - 5\ln x$$

Power Rule

$$\boxed{f'(x) = 5 - \frac{5}{x}}$$

$$(\ln x)' = \frac{1}{x}$$

Exercise

Find the derivative $f(x) = \ln x^{10} + 2\ln x$

Solution

$$f(x) = 10\ln x + 2\ln x$$

Power Rule

$$= 12\ln x$$

$$\boxed{f'(x) = \frac{12}{x}}$$

$$(\ln x)' = \frac{1}{x}$$

Exercise

Find the derivative $f(x) = \dfrac{\ln x}{2x+5}$

Solution

$$u = \ln x \quad v = 2x+5$$

$$u' = \frac{1}{x} \qquad v' = 2$$

$$f'(x) = \frac{\frac{1}{x}(2x+5) - (2)\ln x}{(2x+5)^2} \bullet \frac{x}{x}$$

$$\boxed{= \frac{2x+5-2x\ln x}{x(2x+5)^2}}$$

Exercise

Find the derivative $f(x) = -2\ln x + x^2 - 4$

Solution

$$\boxed{f'(x) = -\frac{2}{x} + 2x}$$

Exercise

Find the derivative of $y = \ln\left(\dfrac{1}{x\sqrt{x+1}}\right)$

Solution

$$y = \ln(1) - \ln\left(x\sqrt{x+1}\right)$$

$$= -\ln x - \ln(x+1)^{1/2}$$

$$= -\ln x - \frac{1}{2}\ln(x+1)$$

$$y' = -\frac{1}{x} - \frac{1}{2}\frac{1}{x+1}$$

$$= -\frac{2(x+1)+x}{2x(x+1)}$$

$$\boxed{= -\frac{3x+2}{2x(x+1)}}$$

Exercise

Find the derivative of $y = \ln(\ln(\ln x))$

Solution

$$y' = \frac{1}{\ln(\ln x)} \cdot (\ln(\ln x))'$$

$$= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot (\ln x)'$$

$$= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$$

$$\boxed{= \frac{1}{x(\ln x)(\ln(\ln x))}}$$

Exercise

Find the derivative of $y = \ln(\sec(\ln x))$

Solution

$$y' = \frac{1}{\sec(\ln x)} \cdot (\sec(\ln x))'$$

$$= \frac{1}{\sec(\ln x)} \cdot (\sec(\ln x)\tan(\ln x)) \cdot (\ln x)'$$

$$= \frac{\sec(\ln x)}{\sec(\ln x)} \tan(\ln x) \cdot \frac{1}{x}$$

$$\boxed{= \frac{\tan(\ln x)}{x}}$$

Exercise

Find the derivative of $y = \ln\left(\frac{(x^2+1)^5}{\sqrt{1-x}}\right)$

Solution

$$y = \ln(x^2+1)^5 - \ln(1-x)^{1/2}$$

$$= 5\ln(x^2+1) - \frac{1}{2}\ln(1-x)$$

$$y' = 5\frac{2x}{x^2+1} - \frac{1}{2}\frac{-1}{1-x}$$

$$\boxed{= \frac{10x}{x^2+1} + \frac{1}{2(1-x)}}$$

Exercise

Find the derivative of $y = \ln\sqrt{\dfrac{(x+1)^5}{(x+2)^{20}}}$

Solution

$$y = \frac{1}{2}\ln\left(\frac{(x+1)^5}{(x+2)^{20}}\right)$$

$$= \frac{1}{2}\left[\ln(x+1)^5 - \ln(x+2)^{20}\right]$$

$$= \frac{1}{2}\left[5\ln(x+1) - 20\ln(x+2)\right]$$

$$y' = \frac{1}{2}\left[5\frac{1}{x+1} - 20\frac{1}{x+2}\right]$$

$$= \frac{5}{2}\left[\frac{1}{x+1} - \frac{4}{x+2}\right]$$

$$= \frac{5}{2}\left(\frac{x+2-4x-4}{(x+1)(x+2)}\right)$$

$$= \frac{5}{2}\left[\frac{-3x-2}{(x+1)(x+2)}\right]$$

$$= -\frac{5}{2}\frac{3x+2}{(x+1)(x+2)}$$

Exercise

Find the derivative of $f(x) = e^{3x}$

Solution

$$f'(x) = 3e^{3x}$$

Exercise

Find the derivative of $f(x) = e^{-2x^3}$

Solution

$$f'(x) = e^{-2x^3}\left(-6x^2\right)$$

$$= -\frac{6x^2}{e^{2x^3}}$$

Exercise

Find the derivative of $f(x)=4e^{x^2}$

Solution

$$f'(x)=4e^{x^2}(2x)$$

$$\boxed{=8xe^{x^2}}$$

Exercise

Find the derivative of $f(x)=x^2e^x$

Solution

$$f'(x)=e^x\frac{d}{dx}[x^2]+x^2\frac{d}{dx}[e^x]$$

$$=e^x(2x)+x^2e^x$$

$$\boxed{=xe^x(2+x)}$$

Exercise

Find the derivative $\quad f(x)=2x^3e^x$

Solution

$$f'(x)=6x^2e^x+2x^3e^x$$

$$u=2x^3 \quad v=e^x$$

$$u'=6x^2 \quad v'=e^x$$

$$\boxed{=2x^2e^x(3+x)}$$

Exercise

Find the derivative $\quad f(x)=\dfrac{3e^x}{1+e^x}$

Solution

$$f'(x)=\frac{3e^x(1+e^x)-3e^xe^x}{(1+e^x)^2}$$

$$u=3e^x \quad v=1+e^x$$

$$u'=3e^x \quad v'=e^x$$

$$=\frac{3e^x+3e^{2x}-3e^{2x}}{(1+e^x)^2}$$

$$\boxed{=\frac{3e^x}{(1+e^x)^2}}$$

Exercise

Find the derivative $f(x) = 5e^x + 3x + 1$

Solution

$$\boxed{f'(x) = 5e^x + 3}$$

Exercise

Find the derivative of $f(x) = \frac{e^x + e^{-x}}{2}$

Solution

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f'(x) = \frac{1}{2}\left(\frac{d}{dx}\left[e^x\right] + \frac{d}{dx}\left[e^{-x}\right]\right)$$

$$\boxed{= \frac{1}{2}\left(e^x - e^{-x}\right)}$$

Exercise

Find the derivative of $f(x) = \frac{e^x}{x^2}$

Solution

$$f'(x) = \frac{x^2 e^x - e^x(2x)}{x^4}$$

$$= \frac{x^2 e^x - 2xe^x}{x^4}$$

$$= \frac{xe^x(x-2)}{x^4}$$

$$\boxed{= \frac{e^x(x-2)}{x^3}}$$

Exercise

Find the derivative of $f(x) = x^2 e^x - e^x$

Solution

$$f'(x) = e^x \frac{d}{dx}[x^2] + x^2 \frac{d}{dx}[e^x] - \frac{d}{dx}[e^x]$$

$$= e^{x}(2x) + x^{2}e^{x} - e^{x}$$

$$\boxed{= e^{x}\left(x^{2} + 2x - 1\right)}$$

Exercise

Find the derivative of $f(x) = (1+2x)e^{4x}$

Solution

$$f'(x) = (2)e^{4x} + (1+2x)(4e^{4x})$$

$$= 2e^{4x} + (1+2x)(4e^{4x})$$

$$= 2e^{4x}(1 + 2(1+2x))$$

$$= 2e^{4x}(1+2+4x)$$

$$\boxed{= 2e^{4x}(3+4x)}$$

Exercise

Find the derivative of $y = x^{2}e^{5x}$

Solution

$$y' = x^{2}\left(5e^{5x}\right) + 2x\left(e^{5x}\right)$$

$$\boxed{= xe^{5x}(5x+2)}$$

Exercise

Find the derivative of $y = x^{2}e^{-2x}$

Solution

$$y' = 2xe^{-2x} - 2x^{2}e^{-2x}$$

$$\boxed{= 2xe^{-2x}(1-x)}$$

Exercise

Find the derivative $f(x)=\dfrac{e^x}{x^2+1}$

Solution

$$f'(x)=\frac{e^x\left(x^2+1\right)-2xe^x}{\left(x^2+1\right)^2} \qquad u=e^x \quad v=x^2+1 \qquad u'=e^x \quad v'=2x$$

$$=\boxed{\frac{\left(x^2+1-2x\right)e^x}{\left(x^2+1\right)^2}}$$

Exercise

Find the derivative $f(x)=\dfrac{1-e^x}{1+e^x}$

Solution

$$f'(x)=\frac{-e^x\left(1+e^x\right)-e^x\left(1-e^x\right)}{\left(1+e^x\right)^2} \qquad u=1-e^x \quad v=1+e^x \qquad u'=-e^x \quad v'=e^x$$

$$=\frac{-e^x-e^{2x}-e^x+e^{2x}}{\left(1+e^x\right)^2}$$

$$=\boxed{-\frac{2e^x}{\left(1+e^x\right)^2}}$$

Exercise

Find the derivative of $y=\dfrac{e^x+e^{-x}}{x}$

Solution

$$y=\frac{\left(e^x-e^{-x}\right)x-\left(e^x+e^{-x}\right)}{x^2} \qquad f=e^x+e^{-x} \quad g=x \qquad f'=e^x-e^{-x} \quad g'=1$$

$$=\frac{xe^x-xe^{-x}-e^x-e^{-x}}{x^2}$$

$$=\boxed{\frac{(x-1)e^x-(x+1)e^{-x}}{x^2}}$$

Exercise

Find the derivative of $y=\sqrt{e^{2x^2}+e^{-2x^2}}$

Solution

$$y=\sqrt{e^{2x^2}+e^{-2x^2}}=\left(e^{2x^2}+e^{-2x^2}\right)^{1/2}=U^{1/2}$$

$$U=e^{2x^2}+e^{-2x^2} \qquad \left(e^{2x^2}\right)'=\left(2x^2\right)'e^{2x^2}=4xe^{2x^2}$$

$$U'=4xe^{2x^2}-4xe^{-2x^2}$$

$$y'=\frac{1}{2}\left(4xe^{2x^2}-4xe^{-2x^2}\right)\left(e^{2x^2}+e^{-2x^2}\right)^{-1/2}$$

$$=\frac{1}{2}\frac{4x\left(e^{2x^2}-e^{-2x^2}\right)}{\left(e^{2x^2}+e^{-2x^2}\right)^{1/2}}$$

$$=\frac{2x\left(e^{2x^2}-e^{-2x^2}\right)}{\sqrt{e^{2x^2}+e^{-2x^2}}}$$

Exercise

Find the derivative of $y=\dfrac{x}{e^{2x}}$

Solution

$$y'=\frac{1\left(e^{2x}\right)-x\left(2e^{2x}\right)}{\left(e^{2x}\right)^2} \qquad f=x \quad g=e^{2x} \qquad f'=1 \quad g'=2e^{2x}$$

$$=\frac{e^{2x}(1-2x)}{\left(e^{2x}\right)^2}$$

$$=\frac{1-2x}{e^{2x}}$$

Exercise

Find the derivative of $y = 3e^{5x^3+1}$

Solution

$$y' = 3(15x^2)e^{5x^3+1}$$

$$y' = 45x^2 e^{5x^3+1}$$

$$y'' = 45\left(2xe^{5x^3+1} + \left(x^2\right)15x^2 e^{5x^3+1}\right)$$

$$f = x^2 \qquad g = e^{5x^3+1}$$

$$f' = 2x \quad g' = 15x^2 e^{5x^3+1}$$

$$= 45e^{5x^3+1}\left(2x + 15x^4\right)$$

$$\boxed{= 45xe^{5x^3+1}\left(2 + 15x^3\right)}$$

Exercise

Find the derivative of $\left(x^2 - 2x + 2\right)e^x$

Solution

$$y = \left(x^2 - 2x + 2\right)e^x$$

$$y' = (2x - 2)e^x + \left(x^2 - 2x + 2\right)e^x$$

$$= \left(2x - 2 + x^2 - 2x + 2\right)e^x$$

$$\boxed{= x^2 e^x}$$

Exercise

Find the derivative of $e^\theta(\sin\theta + \cos\theta)$

Solution

$$\frac{d}{d\theta}e^\theta(\sin\theta + \cos\theta) = e^\theta(\sin\theta + \cos\theta) + e^\theta(\cos\theta - \sin\theta)$$

$$= e^\theta(\sin\theta + \cos\theta + \cos\theta - \sin\theta)$$

$$\boxed{= 2e^\theta \cos\theta}$$

Exercise

Find the derivative of $\ln\left(3\theta e^{-\theta}\right)$

Solution

$\frac{d}{d\theta}\ln\left(3\theta e^{-\theta}\right)=\frac{\left(3\theta e^{-\theta}\right)'}{3\theta e^{-\theta}}$ $=3\frac{e^{-\theta}-\theta e^{-\theta}}{3\theta e^{-\theta}}$ $=\frac{e^{-\theta}(1-\theta)}{\theta e^{-\theta}}$ $=\frac{1-\theta}{\theta}$	$\ln\left(3\theta e^{-\theta}\right)=\ln(3)+\ln(\theta)+\ln\left(e^{-\theta}\right)$ $=\ln 3+\ln\theta-\theta$ $\frac{d}{d\theta}\ln\left(3\theta e^{-\theta}\right)=\frac{1}{\theta}-1$

Exercise

Find the derivative of $\theta^3 e^{-2\theta}\cos 5\theta$

Solution

$$\frac{dy}{d\theta}=\left(\theta^3\right)' e^{-2\theta}\cos 5\theta+\theta^3\left(e^{-2\theta}\right)'\cos 5\theta+\theta^3 e^{-2\theta}(\cos 5\theta)'$$

$$=3\theta^2 e^{-2\theta}\cos 5\theta-2\theta^3 e^{-2\theta}\cos 5\theta-5\theta^3 e^{-2\theta}\sin 5\theta$$

$$=\theta^3 e^{-2\theta}(3\cos 5\theta-2\theta\cos 5\theta-5\theta\sin\theta)$$

Exercise

Find the derivative of $\ln\left(\frac{\sqrt{\theta}}{1+\sqrt{\theta}}\right)$

Solution

$$\frac{d}{d\theta}\ln\left(\frac{\sqrt{\theta}}{1+\sqrt{\theta}}\right)=\frac{d}{d\theta}\left[\ln\theta^{1/2}-\ln\left(1+\sqrt{\theta}\right)\right]$$

$$=\frac{d}{d\theta}\left[\frac{1}{2}\ln\theta-\ln\left(1+\sqrt{\theta}\right)\right]$$

$$=\frac{1}{2}\frac{1}{\theta}-\frac{\frac{1}{2}\theta^{-1/2}}{1+\sqrt{\theta}}$$

$$=\frac{1}{2\theta}-\frac{1}{2}\frac{1}{\sqrt{\theta}\left(1+\sqrt{\theta}\right)}$$

$$= \frac{1}{2}\left(\frac{1}{\theta} - \frac{1}{\sqrt{\theta}\left(1+\sqrt{\theta}\right)}\right)$$

$$= \frac{1}{2}\frac{\sqrt{\theta}\left(1+\sqrt{\theta}\right) - \theta}{\theta\sqrt{\theta}\left(1+\sqrt{\theta}\right)}$$

$$= \frac{1}{2}\frac{\sqrt{\theta} + \theta - \theta}{\theta\sqrt{\theta}\left(1+\sqrt{\theta}\right)}$$

$$= \frac{1}{2}\frac{\sqrt{\theta}}{\theta\sqrt{\theta}\left(1+\sqrt{\theta}\right)}$$

$$= \boxed{\frac{1}{2\theta\left(1+\sqrt{\theta}\right)}}$$

Exercise

Find the derivative of $e^{(\cos t + \ln t)}$

Solution

$$e^{(\cos t+\ln t)} = e^{\cos t}e^{\ln t}$$

$$= t\ e^{\cos t}$$

$$\frac{d}{dt}e^{(\cos t+\ln t)} = \frac{d}{dt}\left(t\ e^{\cos t}\right)$$

$$= e^{\cos t} + te^{\cos t}(-\sin t)$$

$$= \boxed{(1 - t\sin t)e^{\cos t}}$$

Exercise

Find the derivative of $e^{\sin t}\left(\ln t^2 + 1\right)$

Solution

$$\frac{d}{dt}\overset{u}{e^{\sin t}}\overset{v}{\left(\ln t^2+1\right)} = \overset{u'}{e^{\sin t}\cos t}\overset{v}{\left(\ln t^2+1\right)} + \overset{v'}{\frac{2}{t}}\overset{u}{e^{\sin t}}$$

$$= \boxed{e^{\sin t}\left[\left(\ln t^2+1\right)\cos t + \frac{2}{t}\right]}$$

Exercise

Find the Derivatives of $y = e^{x^2}\ln x$

Solution

$$\boxed{y' = 2xe^{x^2}\ln x + \frac{e^{x^2}}{x}}$$

$f = e^{x^2}$ $\quad g = \ln x$

$f' = 2xe^{x^2}$ $\quad g' = \frac{1}{x}$

Exercise

Find the derivative $\quad f(x) = e^x + x - \ln x$

Solution

$$\boxed{f'(x) = e^x + 1 - \frac{1}{x}}$$

Exercise

Find the derivative $\quad f(x) = \ln x + 2e^x - 3x^2$

Solution

$$\boxed{f'(x) = \frac{1}{x} + 2e^x - 6x}$$

Exercise

Find the derivative $\quad f(x) = \ln x^2 + 4e^x$

Solution

$$f(x) = 2\ln x + 4e^x$$

Power Rule

$$\boxed{f'(x) = \frac{2}{x} + 4e^x}$$

$$(\ln x)' = \frac{1}{x}$$

Exercise

Find the Derivatives of $y = \ln\frac{1+e^x}{1-e^x}$

Solution

$$y = \ln\left(1+e^x\right) - \ln\left(1-e^x\right)$$

$$y' = \frac{e^x}{1+e^x} - \frac{-e^x}{1-e^x}$$

$$= \frac{e^x}{1+e^x} + \frac{e^x}{1-e^x}$$

$$= \frac{e^x - e^{2x} + e^x + e^{2x}}{\left(1+e^x\right)\left(1-e^x\right)}$$

$$= \boxed{\frac{2e^x}{\left(1+e^x\right)\left(1-e^x\right)}}$$

Exercise

Find the Derivatives of $y = \dfrac{\ln x}{e^{2x}}$

Solution

$$y' = \frac{e^{2x}(1/x) - \ln x(2e^{2x})}{e^{4x}}$$

$$= \frac{e^{2x} - 2x\ln x(e^{2x})}{e^{4x}}$$

$$= \boxed{\frac{e^{2x}(1-2x\ln x)}{e^{4x}}}$$

Exercise

Find the Derivatives of $f(x) = e^{2x}\ln\left(xe^x + 1\right)$

Solution

$$f = e^{2x} \qquad U = 2x \to U' = 2 \qquad f' = 2e^{2x}$$

$$g = \ln\left(xe^x + 1\right) \quad U = xe^x + 1 \to U' = e^x + xe^x \quad g' = \frac{e^x + xe^x}{xe^x + 1}$$

$$f'(x) = 2e^{2x}\ln\left(xe^x + 1\right) + e^{2x}\frac{e^x + xe^x}{xe^x + 1}$$

$$= \boxed{e^{2x}\left[2\ln\left(xe^x + 1\right) + \frac{e^x(1+x)}{xe^x + 1}\right]}$$

Exercise

Find the Derivatives of $f(x)=\dfrac{xe^x}{\ln\left(x^2+1\right)}$

Solution

$$f'(x)=\frac{e^x(1+x)\ln\left(x^2+1\right)-\frac{2x}{x^2+1}xe^x}{\left[\ln\left(x^2+1\right)\right]^2}$$

$u=xe^x \qquad u'=e^x+xe^x$

$v=\ln\left(x^2+1\right) \qquad v'=\frac{2x}{x^2+1}$

$$=\frac{e^x\left[(1+x)\ln\left(x^2+1\right)-\frac{2x^2}{x^2+1}\right]}{\left[\ln\left(x^2+1\right)\right]^2}$$

$$=\frac{e^x\left[\frac{\left(x^2+1\right)(1+x)\ln\left(x^2+1\right)-2x^2}{x^2+1}\right]}{\left[\ln\left(x^2+1\right)\right]^2}$$

$$\boxed{=\frac{e^x\left[\left(x^2+1\right)(1+x)\ln\left(x^2+1\right)-2x^2\right]}{\left(x^2+1\right)\left[\ln\left(x^2+1\right)\right]^2}}$$

Exercise

Find the Derivatives of $f(x)=xe^{-10x}$

Solution

$$\boxed{f'(x)=e^{-10x}-10xe^{-10x}}$$

Exercise

Find the Derivatives of $f(x)=x\ln^2 x$

Solution

$$f'(x)=\ln^2 x+x\left(2\frac{1}{x}\ln x\right)$$

$$\boxed{=\ln^2 x+2\ln x}$$

Exercise

Find the Derivatives of $f(x) = e^{-x}\ln x$

Solution

$$\boxed{f'(x) = -e^{-x}\ln x + \frac{e^{-x}}{x}}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt{x(x+1)}$

Solution

$$\ln y = \ln\left(x(x+1)\right)^{1/2} \quad = -\ln x - \tfrac{1}{2}\ln(x+1)$$

$$\frac{y'}{y} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x+1}\right)$$

$$\frac{y'}{y} = \frac{1}{2}\left(\frac{2x+1}{x(x+1)}\right)$$

$$y' = \frac{1}{2}\left(\frac{2x+1}{x(x+1)}\right)\bullet y$$

$$\boxed{= \frac{1}{2}\left(\frac{2x+1}{x(x+1)}\right)\sqrt{x(x+1)}}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt{\left(x^2+1\right)(x-1)^2}$

Solution

$$\ln y = \ln\left(\left(x^2+1\right)(x-1)^2\right)^{1/2}$$

$$= \frac{1}{2}\ln\left(\left(x^2+1\right)(x-1)^2\right)$$

$$= \frac{1}{2}\left[\ln\left(x^2+1\right) + \ln(x-1)^2\right]$$

$$= \frac{1}{2}\left[\ln\left(x^2+1\right) + 2\ln(x-1)\right]$$

$$= \frac{1}{2}\ln\left(x^2+1\right) + \ln(x-1)$$

$$\frac{y'}{y} = \frac{1}{2}\frac{2x}{x^2+1} + \frac{1}{x-1}$$

$$= \frac{x}{x^2+1} + \frac{1}{x-1}$$

$$= \frac{x(x-1) + \left(x^2+1\right)}{\left(x^2+1\right)(x-1)}$$

$$= \frac{x^2 - x + x^2 + 1}{\left(x^2+1\right)(x-1)}$$

$$= \frac{2x^2 - x + 1}{\left(x^2+1\right)(x-1)}$$

$$y' = \frac{2x^2 - x + 1}{\left(x^2+1\right)(x-1)} \cdot y$$

$$= \frac{2x^2 - x + 1}{\left(x^2+1\right)(x-1)} \sqrt{\left(x^2+1\right)(x-1)^2}$$

$$= \frac{2x^2 - x + 1}{\left(x^2+1\right)(x-1)} |x-1| \sqrt{\left(x^2+1\right)}$$

$$= \frac{\left(2x^2 - x + 1\right)|x-1|}{\left(x^2+1\right)(x-1)} \left(x^2+1\right)^{1/2}$$

$$= \frac{\left(2x^2 - x + 1\right)|x-1|}{\sqrt{x^2+1}\,(x-1)}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt{\frac{1}{t(t+1)}}$

Solution

$$y = \left(\frac{1}{t(t+1)}\right)^{1/2}$$

$$\ln y = \ln\left(\frac{1}{t(t+1)}\right)^{1/2}$$

$$\ln y = \frac{1}{2}\ln\left(\frac{1}{t(t+1)}\right)$$

$$= -\frac{1}{2}\ln\left(t(t+1)\right)$$

$$= -\frac{1}{2}\left[\ln t + \ln(t+1)\right]$$

$$\frac{y'}{y} = -\frac{1}{2}\left(\frac{1}{t} + \frac{1}{t+1}\right)$$

$$y' = -\frac{1}{2}\left(\frac{1}{t} + \frac{1}{t+1}\right)y$$

$$= -\frac{1}{2}\left(\frac{t+1+t}{t(t+1)}\right)\frac{1}{\left(t(t+1)\right)^{1/2}}$$

$$= -\frac{1}{2}\frac{2t+1}{\left(t(t+1)\right)^{3/2}}$$

$$\boxed{= -\frac{2t+1}{2\left(t^2+t\right)^{3/2}}}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \dfrac{\theta+5}{\theta\cos\theta}$

Solution

$$\ln y = \ln\left(\frac{\theta+5}{\theta\cos\theta}\right)$$

$$\ln y = \ln(\theta+5) - \ln(\theta\cos\theta)$$

$$\ln y = \ln(\theta+5) - \ln\theta - \ln(\cos\theta)$$

$$\frac{y'}{y} = \frac{1}{\theta+5} - \frac{1}{\theta} + \frac{\sin\theta}{\cos\theta}$$

$$y' = \left(\frac{1}{\theta+5} - \frac{1}{\theta} + \frac{\sin\theta}{\cos\theta}\right)y$$

$$\boxed{y' = \left(\frac{\theta+5}{\theta\cos\theta}\right)\left(\frac{1}{\theta+5} - \frac{1}{\theta} + \tan\theta\right)}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt[3]{\dfrac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}}$

Solution

$$\ln y = \ln\left(\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}\right)^{1/3}$$

$$= \frac{1}{3}\left[\ln x + \ln(x+1) + \ln(x-2) - \ln\left(x^2+1\right) - \ln(2x+3)\right]$$

$$\frac{y'}{y} = \frac{1}{3}\left[\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right]$$

$$y' = \frac{1}{3} \cdot \sqrt[3]{\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}} \cdot \left(\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right)$$

Exercise

Find the derivative of $y = t^{1-e}$

Solution

$$y' = (1-e)t^{1-e-1}$$

$$= (1-e)t^{-e}$$

Exercise

Find the derivative of $y = 2^{\sin 3t}$

Solution

$$y = a^u \quad \Rightarrow y' = a^u \ln a \cdot (u')$$

$$y' = \left(2^{\sin 3t} \ln 2\right)(\cos 3t)(3)$$

$$= 3(\ln 2)\cos 3t\left(2^{\sin 3t}\right)$$

Exercise

Find the derivative of $y = \log_3 (1+\theta \ln 3)$

Solution

$$y = \frac{\ln(1+\theta \ln 3)}{\ln 3}$$

$$y' = \frac{1}{\ln 3} \cdot \frac{\ln 3}{1+\theta \ln 3} \qquad y = \ln u \quad \Rightarrow y' = \frac{u'}{u}$$

$$= \frac{1}{1+\theta \ln 3}$$

Exercise

Find the derivative of $y = \log_{25} e^x - \log_5 \sqrt{x}$

Solution

$$y=\frac{\ln e^x}{\ln 25}-\frac{\ln x^{1/2}}{\ln 5}$$
$$=\frac{x}{2\ln 5}-\frac{1}{2}\frac{\ln x}{\ln 5}$$
$$=\frac{1}{2\ln 5}(x-\ln x)$$
$$y'=\frac{1}{2\ln 5}\left(1-\frac{1}{x}\right)$$
$$\boxed{=\frac{x-1}{2x\ln 5}}$$

Exercise

Find the derivative of $y=\log_3 r\cdot\log_9 r$

Solution

$$y=\frac{\ln r}{\ln 3}\cdot\frac{\ln r}{\ln 9}$$
$$=\frac{1}{\ln 3\cdot\ln 9}\cdot\ln^2 r$$
$$y'=\frac{1}{\ln 3\cdot\ln 9}\cdot(2\ln r)\left(\frac{1}{r}\right)$$
$$\boxed{=\frac{2\ln r}{r\cdot\ln 3\cdot\ln 9}}$$

Exercise

Find the derivative of $y=\log_7\left(\frac{\sin\theta\cos\theta}{e^\theta 2^\theta}\right)$

Solution

$$y=\frac{\ln(\sin\theta)+\ln(\cos\theta)-\ln\left(e^\theta\right)-\ln\left(2^\theta\right)}{\ln 7}$$
$$=\frac{1}{\ln 7}\left[\ln(\sin\theta)+\ln(\cos\theta)-\theta-\theta\ln(2)\right]$$
$$y'=\frac{1}{\ln 7}\left[\frac{\cos\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta}-1-\ln(2)\right]$$
$$\boxed{=\frac{1}{\ln 7}(\cot\theta-\tan\theta-1-\ln 2)}$$

Exercise

Find the derivative of $y = 3\log_8 \left(\log_2 t\right)$

Solution

$$y = 3\frac{\ln\left(\log_2 t\right)}{\ln 8} = \frac{3}{\ln 8}\ln\left(\frac{\ln t}{\ln 2}\right)$$

$$y' = \frac{3}{\ln 2^3}\left(\frac{1}{\frac{\ln t}{\ln 2}}\right)\left(\frac{1}{\ln 2}\cdot\frac{1}{t}\right)$$

$$= \frac{3}{2\ln 2}\left(\frac{\ln 2}{\ln t}\right)\left(\frac{1}{t\ln 2}\right)$$

$$\boxed{= \frac{1}{t(\ln t)(\ln 2)}}$$

Exercise

Find the derivative of $y = t\ \log_3 \left(e^{(\sin t)(\ln 3)}\right)$

Solution

$$y = t\,\frac{\ln e^{(\sin t)(\ln 3)}}{\ln 3}$$

$$= \frac{1}{\ln 3}t(\sin t)(\ln 3)$$

$$= t\sin t$$

$$\boxed{y' = \sin t + t\cos t}$$

Exercise

Find the derivative of $\qquad f(x) = \log_3 (x+8)$

Solution

$$\boxed{f'(x) = \frac{1}{\ln 3}\left(\frac{1}{x+8}\right)} \qquad\qquad \frac{d}{dx}\left[\log_a u\right] = \left(\frac{1}{\ln a}\right)\left(\frac{1}{u}\right)\frac{du}{dx}$$

Exercise

Find the derivative of $\qquad f(x) = 2^{x^2-x}$

Solution

$$\boxed{f'(x) = (2x-1)(\ln 2)2^{x^2-x}} \qquad\qquad \frac{d}{dx}[a^u] = a^u\ln(a)\frac{du}{dx}$$

Exercise

Use logarithmic differentiation to find the derivative of $y=(x+1)^x$

Solution

$$\ln y=\ln(x+1)^x=x\cdot\ln(x+1)$$

$$\frac{y'}{y}=\ln(x+1)+x\cdot\frac{1}{x+1}$$

$$\boxed{y'=(x+1)^x\left(\ln(x+1)+\frac{x}{x+1}\right)}$$

Exercise

Use logarithmic differentiation to find the derivative of $y=x^2+x^{2x}$

Solution

$$y-x^2=x^{2x}$$

$$\ln\left(y-x^2\right)=\ln x^{2x}=\underline{2x\ln x}$$

$$\frac{1}{y-x^2}(y'-2x)=2\ln x+2x\frac{1}{x}$$

$$y'-2x=\left(y-x^2\right)(2\ln x+2)$$

$$y'-2x=\left(x^2+x^{2x}-x^2\right)(2\ln x+2)$$

$$y'=2x^{2x}(\ln x+1)+2x$$

$$\boxed{=2\left(x^{2x}\ln x+x^{2x}+x\right)}$$

Exercise

Use logarithmic differentiation to find the derivative of $y=(\sin x)^x$

Solution

$$\ln y=\ln(\sin x)^x$$

$$u=x\qquad v=\ln(\sin x)$$

$$\ln y=x\ln(\sin x)$$

$$u'=1\qquad v'=\frac{\cos x}{\sin x}$$

$$\frac{y'}{y}=\ln(\sin x)+x\frac{\cos x}{\sin x}$$

$$y'=y(\ln(\sin x)+x\cot x)$$

$$\boxed{=(\sin x)^x\left[\ln(\sin x)+x\cot x\right]}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = x^{\sin x}$

Solution

$$\ln y = \ln x^{\sin x}$$

$$\ln y = \sin x \ln x$$

$$\frac{y'}{y} = \cos x \ln x + \frac{\sin x}{x}$$

$$\frac{y'}{y} = \frac{x \cos x \ln x + \sin x}{x}$$

$$y' = y \frac{x \cos x \ln x + \sin x}{x}$$

$$\boxed{= x^{\sin x}\left[\frac{\sin x + x(\ln x)(\cos x)}{x}\right]}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = (\ln x)^{\ln x}$

Solution

$$\ln y = \ln (\ln x)^{\ln x}$$

$$\ln y = (\ln x) \ln (\ln x)$$

$$\frac{y'}{y} = \frac{1}{x} \ln (\ln x) + \ln x \frac{\frac{1}{x}}{\ln x}$$

$$y' = \left(\frac{1}{x} \ln (\ln x) + \frac{1}{x}\right) y$$

$$\boxed{= \left(\frac{\ln (\ln x) + 1}{x}\right)(\ln x)^{\ln x}}$$

Exercise

Find the second derivative of $y = 3e^{5x^3+1}$

Solution

$$y' = 45x^2 e^{5x^3+1}$$

$$\boxed{y'' = \left(90x + 675x^5\right) e^{5x^3+1}}$$

Exercise

Find the equations of the tangent lines to $f(x) = e^x$ at the points (0, 1)

Solution

$$f'(x) = e^x$$

$$(0,\ 1) \Rightarrow m = f'(x=0)$$

$$= e^0$$

$$= 1$$

$$y - 1 = 1(x - 0) + 1 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$\boxed{y = x + 1}$$

Exercise

Find the equations of the tangent lines to $f(x) = e^x$ at the points $(1,\ e)$

Solution

$$f'(x) = e^x$$

$$(1,\ e) \Rightarrow m = f'(x=1) = e^1 = e$$

$$y = e(x-1) + e \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$\boxed{y = ex}$$

Exercise

Find the equations of the tangent lines to $y = 4xe^{-x} + 5$ at $x = 1$

Solution

$$y' = 4e^{-x} - 4xe^{-x} = 4e^{-x}(1-x)$$

$$= 4e^{-x}(1-x)$$

$$m = y'(x=1)$$

$$= 4e^{-1}(1-1) = 0$$

$$\Rightarrow x = 1 \rightarrow y = 4e^{-1} + 5 \qquad \left(1,\ 4e^{-1} + 5\right)$$

$$y = 0\left(x-1\right) + 4e^{-1} + 5 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$\boxed{y = 4e^{-1} + 5}$$

Exercise

Find the equation of the tangent lines to $f(x) = 4e^{-8x}$ at the points (0, 4)

Solution

$$f'(x) = -32e^{-8x}$$

$$m = f'(0) = -32e^{-8(0)} = -32$$

$$y = -32(x-0)+4 \qquad\qquad y = m(x - x_1) + y_1$$

$$y = -32x + 4$$

Exercise

The following formula accurately models the relationship between the size of a certain type of tumor and the amount of time that it has been growing:

$$V(t) = 450(1 - e - 0.0022t)^3$$

where t is in months and $V(t)$ is measured in cubic centimeters. Calculate the rate of change of tumor volume at 80 months.

Solution

$$U = 1 - e - 0.0022t \qquad V = 450U^3$$

$$U' = -.0022 \qquad V' = 450(3)U^2U'$$

$$V'(t) = 450(3)(1 - e - 0.0022t)^2(-.0022)$$

$$= 2.97(1 - e - 0.0022t)^2$$

$$V'(t = 80) = 2.97(1 - e - 0.0022(80))^2$$

$$\approx 10.66$$

Exercise

A yeast culture at room temperature (68° F) is placed in a refrigerator set at a constant temperature of 38° F. After t hours, the temperature T of the culture is given approximately by

$$T = 30e^{-0.58t} + 38 \quad t \ge 0$$

What is the rate of change of temperature of the culture at the end of 1 *hour*? At the end of 4 *hours*?

Solution

$$T' = 30(-0.58)e^{-0.58t} = -17.4e^{-0.58t}$$

$$T'(1) = -17.4e^{-0.58(1)} \underline{\approx -9.74° \; F / hr}$$

$$T'(4) = -17.4e^{-0.58(4)} \underline{\approx -1.71° \; F / hr}$$

Exercise

A mathematical model for the average age of a group of people learning to type is given by

$$N(t) = 10 + 6\ln t \quad t \geq 1$$

Where $N(t)$ is the number of words per minute typed after *t hours* of instruction and practice (2 hours per day, 5 days per week). What is the rate of learning after 10 *hours* of instruction and practice? After 100 *hours*?

Solution

$$N'(t) = \frac{6}{t}$$

$$N'(10) = \frac{6}{10} \underline{= 0.6}$$

After 10 *hours* of instruction and practice, the rate of learning is 0.6 words/minute per hour of instruction and practice.

$$N'(100) = \frac{6}{100} \underline{= 0.06}$$

After 100 *hours* of instruction and practice, the rate of learning is 0.06 words/minute per hour of instruction and practice.

Exercise

The population of coyotes in the northwestern portion of Alabama is given by the formula $P(t) = (t^2 + 100)\ln(t+2)$, where *t* represents the time in years since 2000 (the year 2000 corresponds to $(t = 0)$ Find the rate of change of the coyote population in $2013 \; (t = 13)$.

Solution

$$P'(t) = 2t\ln(t+2) + \frac{1}{t+2}(t^2 + 100) \qquad P' = f'g + g'f \qquad f = t^2 + 100 \quad g = \ln(t+2)$$

$$f' = 2t \qquad g' = \frac{1}{t+2}$$

$$= 2t\ln(t+2) + \frac{t^2 + 100}{t+2}$$

$$P'(t = 13) = 2(13)\ln(13+2) + \frac{13^2 + 100}{13+2} \qquad 2 * 13\ln(13+2) + (13 \wedge 2 + 100) / (13+2)$$

$$\underline{\approx 88.34}$$