First Order Differential Equations

Section 2.7 – First-Order Linear Equations

General First-Order Differential Equations and Solutions

A ***first-order differential equation*** is an equation

$$\frac{dy}{dx} = f(x, y)$$

In which $f(x, y)$ is a function of two variables defined on a region in the *xy*-plane.

Example

Show that every member of the family of functions $y = \frac{C}{x} + 2$ is a solution of the first-order differential equation $\frac{dy}{dx} = \frac{1}{x}(2 - y)$ on the interval $(0, \infty)$, where C is any constant.

Solution

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{C}{x} + 2\right)$$
$$= -\frac{C}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{x}(2 - y)$$
$$-\frac{C}{x^2} = \frac{1}{x}\left(2 - \left(\frac{C}{x} + 2\right)\right)$$
$$-\frac{C}{x^2} = \frac{1}{x}\left(2 - \frac{C}{x} - 2\right)$$
$$-\frac{C}{x^2} = \frac{1}{x}\left(-\frac{C}{x}\right)$$
$$-\frac{C}{x^2} = -\frac{C}{x^2} \quad \checkmark$$

Therefore, for every value of C, the function $y = \frac{C}{x} + 2$ is a solution of the first-order differential equation $\frac{dy}{dx} = \frac{1}{x}(2 - y)$.

Example

Show that the function $y=(x+1)-\frac{1}{3}e^{x}$ is a solution of the first-order initial value problem

$\frac{dy}{dx}=y-x \quad y(0)=\frac{2}{3}$.

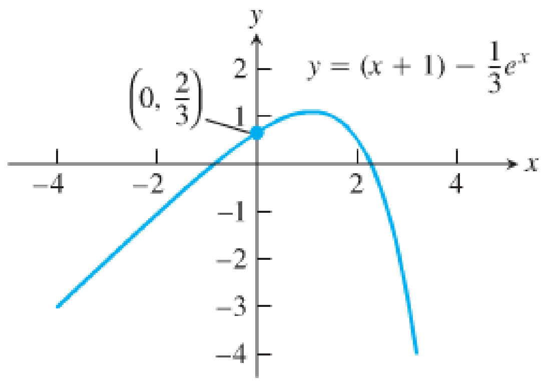
Solution

$$\frac{dy}{dx}=\frac{d}{dx}\left(x+1-\frac{1}{3}e^{x}\right)$$
$$=1-\frac{1}{3}e^{x}$$

$$y-x=1-\frac{1}{3}e^{x}$$

$$y=x+1-\frac{1}{3}e^{x}$$

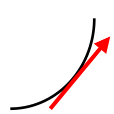
$$y(0)=(0+1)-\frac{1}{3}e^{0}$$
$$=1-\frac{1}{3}$$
$$\boxed{=\frac{2}{3}}$$

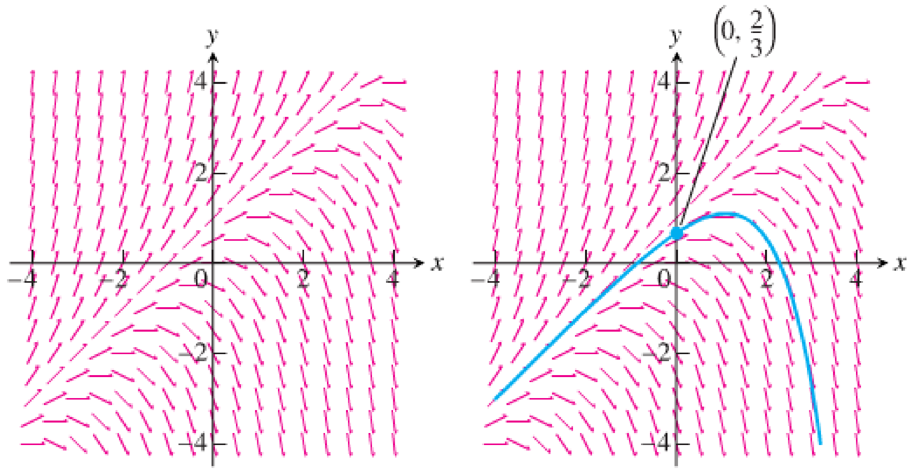


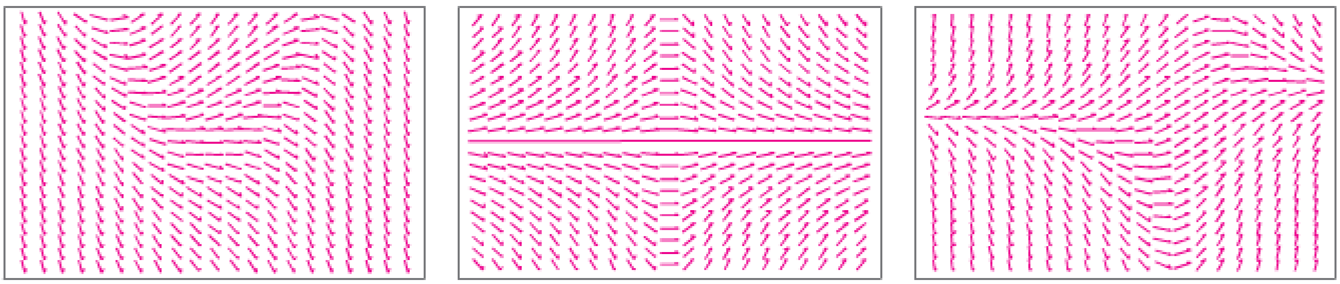
Slope Fields: Viewing Solution Curves

Each time we specify an initial condition $y(x_0)=y_0$ for the solution of a differential equation $y'=f(x,y)$, the solution curve is required to pass through the point (x_0,y_0) and to have a slope $f(x_0,y_0)$ there.

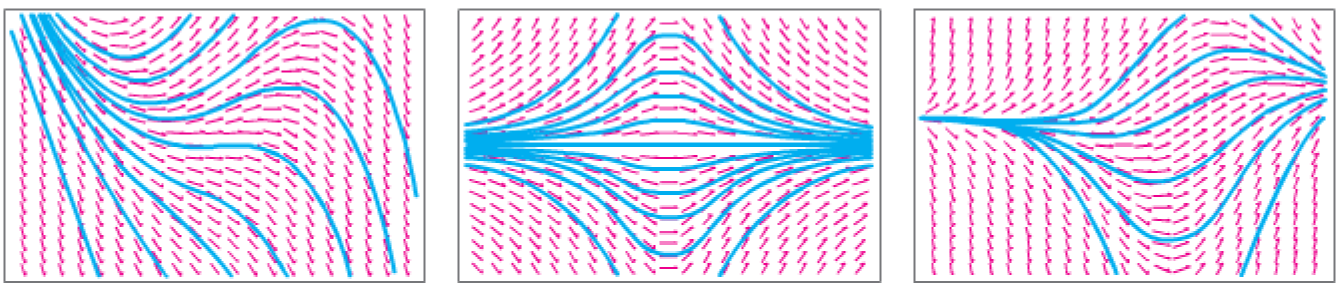
What we draw a lineal element at each point (x,y) with slope $f(x,y)$ then the collection of these lineal elements is called a ***direction field*** or a ***slope field*** of the differential equation $\frac{dy}{dx}=f(x,y)$.





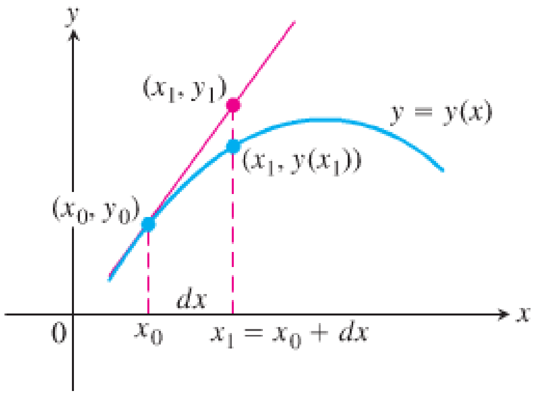


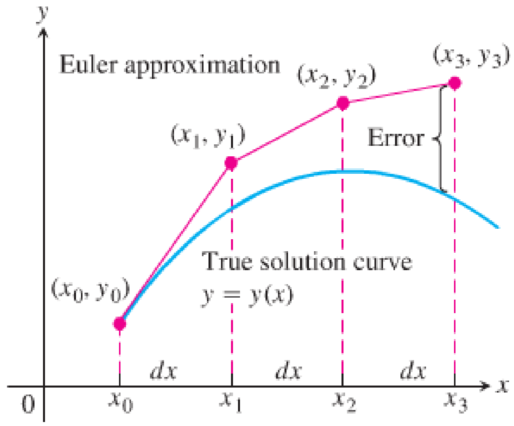
(a) $y' = y - x^2$ (b) $y' = -\dfrac{2xy}{1 + x^2}$ (c) $y' = (1 - x)y + \dfrac{x}{2}$



Euler's Method

Euler's method named after *Leonhard Euler* is an example of a ***fixed-step*** solver.
Euler's method is a first-order numerical procedure for solving ordinary differential equations (ODEs) with a given initial value. It is the most basic kind of explicit method for numerical integration of ordinary differential equations.





$$y' = f(x,y) \qquad y(x_0) = y_0$$

The setting size: $h = \frac{b-a}{k} > 0$; $\quad k = 1,\ 2,\ 3,\ \ldots$

Then, $\quad x_0 = a$

$$x_1 = x_0 + h = a + h$$

$$x_k = x_{k-1} + h = a + kh$$

Last point $\quad x_k = a + kh = b$

By the definition of the derivative:

$$y'\left(x_k\right) \approx \frac{y\left(x_{k+1}\right) - y\left(x_k\right)}{h}$$

$$y'\left(x_k\right) \approx \frac{y_{k+1} - y_k}{h} = f\left(x_k, y_k\right) : \textit{slope}$$

The tangent line at the point $\left(x_0, y(x_0)\right)$ is:

$$y_{k+1} = y_k + h.f\left(x_k, y_k\right)$$

$$y_{k+1} = y_k + \Delta x_{step}.f\left(x_k, y_k\right) \qquad \boxed{y_{k+1} = y_k + f\left(x_k, y_k\right)dx}$$

This method is known as *Euler's Method* with step size h.

Example

Find the first three approximations y_1, y_2, y_3 using Euler's method for the initial value problem

$$y' = 1 + y, \quad y(0) = 1$$

Starting at $x_0 = 0$ with $dx = 0.1$.

Solution

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)dx \\ &= y_0 + (1 + y_0)dx \\ &= 1 + (1+1)(0.1) \\ &= 1.2 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + f(x_1, y_1)dx \\ &= y_1 + (1 + y_1)dx \\ &= 1.2 + (1+1.2)(0.1) \\ &= 1.42 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + f(x_2, y_2)dx \\ &= y_2 + (1 + y_2)dx \\ &= 1.42 + (1+1.42)(0.1) \\ &= 1.662 \end{aligned}$$

Example

Use Euler's method to solve

$$y' = 1 + y, \quad y(0) = 1$$

On the interval $0 \le x \le 1$, starting at $x_0 = 0$ and taking

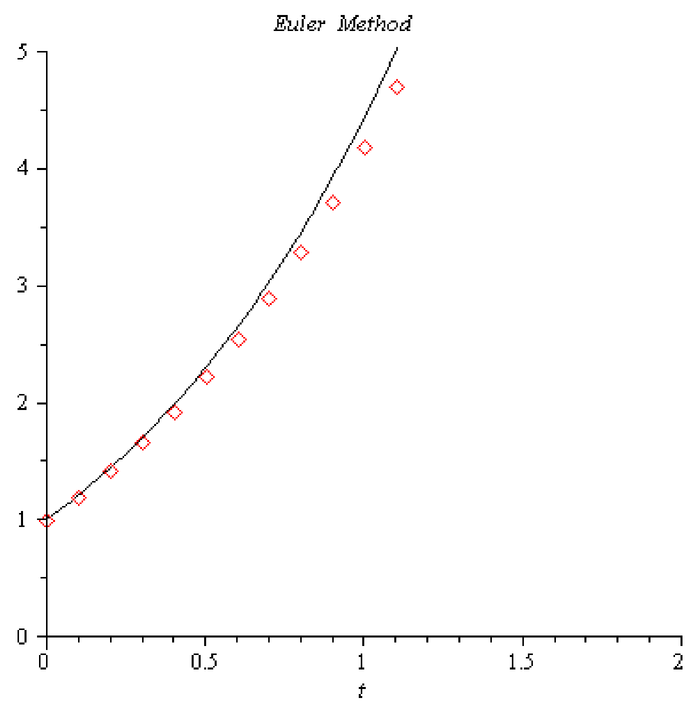
a) $dx = 0.1$.
b) $dx = 0.05$.

Compare the approximations with the values of the exact solution $y = 2e^x - 1$

Solution

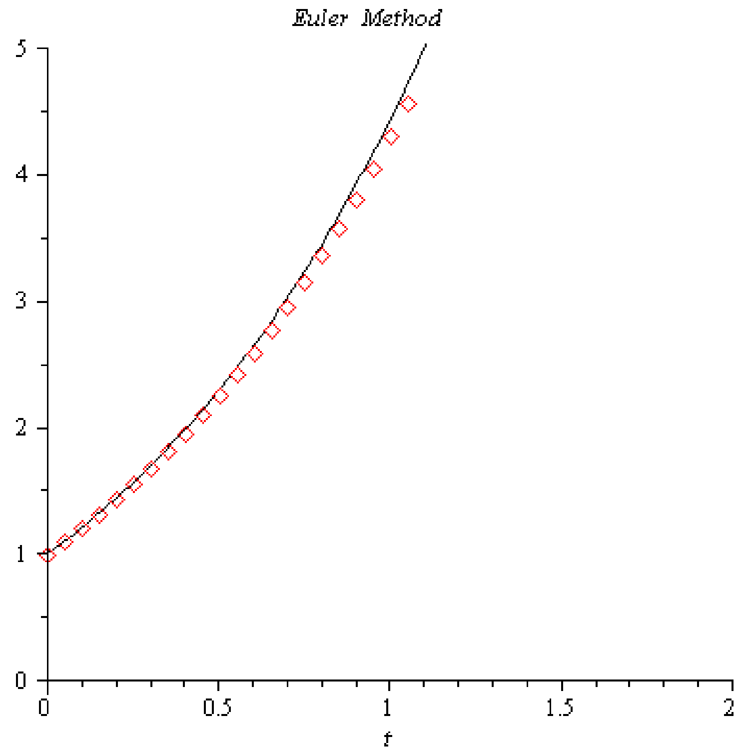
a) Euler Method ***dx* = 0.1**

t	***Approx.***	***Exact***	***Difference***
0.00	1.00000000	1.00000000	0.00000000
0.10	1.20000000	1.21034184	0.01034184
0.20	1.42000000	1.44280552	0.02280552
0.30	1.66200000	1.69971762	0.03771762
0.40	1.92820000	1.98364940	0.05544940
0.50	2.22102000	2.29744254	0.07642254
0.60	2.54312200	2.64423760	0.10111560
0.70	2.89743420	3.02750541	0.13007121
0.80	3.28717762	3.45108186	0.16390424
0.90	3.71589538	3.91920622	0.20331084
1.00	4.18748492	4.43656366	0.24907874



b) Euler Method ***dx* = 0.05**

t	*Approx.*	*Exact*	*Difference*
0.00	1.00000000	1.00000000	0.00000000
0.05	1.10000000	1.10254219	0.00254219
0.10	1.20500000	1.21034184	0.00534184
0.15	1.31525000	1.32366849	0.00841849
0.20	1.43101250	1.44280552	0.01179302
0.25	1.55256313	1.56805083	0.01548771
0.30	1.68019128	1.69971762	0.01952633
0.35	1.81420085	1.83813510	0.02393425
0.40	1.95491089	1.98364940	0.02873851
0.45	2.10265643	2.13662437	0.03396794
0.50	2.25778925	2.29744254	0.03965329
0.55	2.42067872	2.46650604	0.04582732
0.60	2.59171265	2.64423760	0.05252495
0.65	2.77129828	2.83108166	0.05978337
0.70	2.95986320	3.02750541	0.06764222
0.75	3.15785636	3.23400003	0.07614367
0.80	3.36574918	3.45108186	0.08533268
0.85	3.58403664	3.67929370	0.09525707
0.90	3.81323847	3.91920622	0.10596775
0.95	4.05390039	4.17141932	0.11751893
1.00	4.30659541	4.43656366	0.12996825



A ***first-order linear*** differential equation is one that can be written in the ***standard form***

$$\frac{dy}{dx} + P(x) \cdot y = Q(x)$$

Where P and Q are continuous functions of x

Solving Linear Equations

We solve the equation $\frac{dy}{dx} + P(x) \cdot y = Q(x)$

Separable Equation

Solution of the homogenous equation

$$\frac{dy}{dx} + P(x)y = 0$$

$$\frac{dy}{dx} = -P(x)y$$

$$\int \frac{dy}{y} = -\int P(x)dx$$

Integrate both sides

$$\ln|y| = -\int P(x)dx + C$$

Convert to exponential form

$$y(x) = e^{\int P(x)dx+C} = e^{\int P(x)dx} e^{C}$$

$$\boxed{y(x) = A.e^{\int P(x)dx}}$$

Example

Solve the differential equation $y' = ty^2$

Solution

$$\frac{dy}{dt} = ty^2$$

$$\frac{dy}{y^2} = tdt$$

$$\int y^{-2}dy = \int tdt$$

$$-y^{-1} = \frac{1}{2}t^2 + C$$

$$-\frac{1}{y} = \frac{t^2+2C}{2} \qquad \textit{Cross multiplication}$$

$$y(t) \ \boxed{= -\frac{2}{t^2 + 2C}}$$

General Method

1. Separate the variables
2. Integrate both sides
3. Solve for the solution $y(t)$, if possible

Example

Find the general solution of the differential equation. $y' = \dfrac{2xy + 2x}{x^2 - 1}$

Solution

$$\frac{dy}{dx} = \frac{2x(y+1)}{x^2 - 1}$$

$$\frac{dy}{y+1} = \frac{2x}{x^2 - 1}dx \qquad d\left(x^2 - 1\right) = 2xdx$$

$$\int \frac{d(y+1)}{y+1} = \int \frac{d\left(x^2 - 1\right)}{x^2 - 1}$$

$$\ln|y+1| = \ln\left|x^2 - 1\right| + C \quad \Rightarrow \quad y + 1 = e^{\ln\left|x^2-1\right| + C}$$

$$y = e^C e^{\ln\left|x^2-1\right|} - 1$$

$$\boxed{y(x) = Ae^{\ln\left|x^2-1\right|} - 1}$$

Solution of the Nonhomogeneous Equation $y' + p(x)y = f(x)$

Let assume: $y = y_h + y_p$ *where* $\begin{cases} y_h & \textit{Homogeneous Solution} \\ y_p & \textit{Paticular Solution} \end{cases}$

The homogeneous equation is given by $y'_h + p(x)y_h = 0$

$$y'_h = -p(x)y_h \quad \Rightarrow y_h = e^{-\int p\,dx}$$

$$y_p = u(x)y_h = u.e^{-\int p\,dx}$$

$$y'_p + p(x)y_p = f(x)$$

$$(uy_h)' + puy_h = f$$

$$u'y_h + uy'_h + puy_h = f$$

$$u'y_h + u(y'_h + py_h) = f \qquad \textit{Since } y'_h + py_h = 0 \qquad \textit{homogeneous}$$

$$u'y_h = f$$

$$\frac{du}{dx} = \frac{f}{y_h}$$

$$du = \frac{f}{e^{-\int p\,dx}}dx$$

$$= f.e^{\int p\,dx}dx$$

$$u = \int f.e^{\int p\,dx}dx$$

$$\boxed{y_p} = u.e^{-\int p\,dx} = \left(\int f.e^{\int p\,dx}dx\right)e^{-\int p\,dx} = \boxed{e^{-\int p\,dx}\int f.e^{\int p\,dx}dx}$$

$$y = y_h + y_p$$

$$= Ce^{-\int p\,dx} + e^{-\int p\,dx}\int f.e^{\int p\,dx}dx$$

$$= e^{-\int p\,dx}\left(C + \int f.e^{\int p\,dx}dx\right)$$

$$y' + p(x)y = f(x) \quad \Rightarrow \boxed{y = \frac{1}{e^{\int p\,dx}}\left(\int f.e^{\int p\,dx}dx + C\right)}$$

Example

Solve the equation $x\frac{dy}{dx} = x^2 + 3y, \quad x > 0$

Solution

$$y' - \frac{3}{x}y = x$$

$$e^{\int p\,dx} = e^{-3\int \frac{dx}{x}} = e^{-3\ln|x|} = e^{\ln x^{-3}} = \boxed{x^{-3}}$$

$$\int x.x^{-3}dx = \int x^{-2}dx = \frac{1}{x}$$

$$y(x) = \frac{1}{x^{-3}}\left(\frac{1}{x} + C\right)$$

$$= x^3\left(\frac{1}{x} + C\right)$$

$$\boxed{= x^2 + Cx^3} \quad x > 0$$

Example

Solve the equation $3xy' - y = \ln x + 1, \quad x > 0$, satisfying $y(1) = -2$

Solution

$$y' - \frac{1}{3x}y = \frac{\ln x + 1}{3x}$$

$$e^{\int p\,dx} = e^{\int\left(-\frac{1}{3x}\right)dx} = e^{-\frac{1}{3}\ln x} = e^{\ln x^{-1/3}} = \boxed{x^{-1/3}}$$

$$\int \left(x^{-1/3}\right)\frac{\ln x + 1}{3x}dx = \frac{1}{3}\int (\ln x + 1)x^{-4/3}\,dx$$

$$u = \ln x + 1 \quad dv = \int x^{-4/3}dx$$

$$du = \frac{1}{x}dx \qquad v = -3x^{-1/3}$$

$$= \frac{1}{3}\left(-3x^{-1/3}(\ln x + 1) + 3\int\left(x^{-4/3}\right)dx\right)$$

$$= \frac{1}{3}\left(-3x^{-1/3}(\ln x + 1) - 9x^{-1/3}\right)$$

$$= -x^{-1/3}(\ln x + 1) - 3x^{-1/3}$$

$$y(x) = x^{1/3}\left(-x^{-1/3}(\ln x + 1) - 3x^{-1/3} + C\right)$$

$$\boxed{= -\ln x - 4 + Cx^{1/3}}$$

$$y(1) = -\ln(1) - 4 + C(1)^{1/3}$$

$$-2 = -0 - 4 + C \qquad \boxed{2 = C}$$

$$\boxed{y = 2x^{1/3} - \ln x - 4}$$

Exercises Section 2.7 – First-Order Linear Equations

Write an equivalent first-order differential equation and initial condition for y.

1. $y = \int_1^x \frac{1}{t} dt$ **2.** $y = 2 - \int_0^x (1 + y(t)) \sin t dt$

Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

1. $y' = 1 - \frac{y}{x}, \quad y(2) = -1, \quad dx = 0.5$

2. $y' = x(1-y), \quad y(1) = 0, \quad dx = 0.2$

3. $y' = y^2(1+2x), \quad y(-1) = 1, \quad dx = 0.5$

4. $y' = ye^x, \quad y(0) = 2, \quad dx = 0.5$

5. Use the Euler method with $dx = 0.2$ to estimate $y(2)$ if $y' = \frac{y}{x}$ and $y(1) = 2$. What is the exact value of $y(2)$?

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants.

6. $y = Ce^{-5t}; \quad y'(t) + 5y = 0$

7. $y = Ct^{-3}; \quad ty'(t) + 3y = 0$

8. $y = C_1 \sin 4t + C_2 \cos 4t; \quad y''(t) + 16y = 0$

9. $y = C_1 e^{-x} + C_2 e^x; \quad y''(x) - y = 0$

10. $y' + 4y = \cos t, \quad y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t + Ce^{-4t}, \qquad y(0) = -1$

11. $ty' + (t+1)y = 2te^{-t}, \quad y(t) = e^{-t}\left(t + \frac{C}{t}\right), \quad y(1) = \frac{1}{e}$

12. $y' = y(2+y), \quad y(t) = \frac{2}{-1 + Ce^{-2t}}, \quad y(0) = -3$

Verify that the given function y is a solution of the initial value problem that follows it.

13. $y = 16e^{2t} - 10; \quad y' - 2y = 20, \quad y(0) = 6$

14. $y = 8t^6 - 3; \quad ty' - 6y = 18, \quad y(1) = 5$

15. $y = -3\cos 3t; \quad y'' + 9y = 0, \quad y(0) = -3, \ y'(0) = 0$

16. $y = \frac{1}{4}\left(e^{2x} - e^{-2x}\right); \quad y'' - 4y = 0, \quad y(0) = 0, \ y'(0) = 1$

Solve the differential equations

17. $y' = xy$

18. $xy' = 2y$

19. $y' = e^{x-y}$

20. $y' = \left(1+y^2\right)e^x$

21. $y' = xy + y$

22. $y' = ye^x - 2e^x + y - 2$

23. $y' = \dfrac{x}{y+2}$

24. $y' = \dfrac{xy}{x-1}$

25. $x\dfrac{dy}{dx} + y = e^x, \quad x > 0$

26. $y' + (\tan x)y = \cos^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

27. $(1+x)y' + y = \sqrt{x}$

28. $e^{2x}y' + 2e^{2x}y = 2x$

29. $x\dfrac{dy}{dx} + 2y = 1 - \dfrac{1}{x}, \quad x > 0$

30. $(t+1)\dfrac{ds}{dt} + 2s = 3(t+1) + \dfrac{1}{(t+1)^2}, \quad t > -1$

31. $\tan\theta\dfrac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \frac{\pi}{2}$

32. $y' = \cos x - y\sec x$

33. $\left(1+x^3\right)y' = 3x^2y + x^2 + x^5$

34. $\dfrac{dy}{dt} - 2y = 4 - t$

35. $y' + y = \dfrac{1}{1+e^t}$

36. $y' = 3y - 4$

37. $y' = -2y - 4$

38. $y' = -y + 2$

39. $y' = 2y + 6$

Solve the initial value problem

40. $t\dfrac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(2) = 1$

41. $\theta\dfrac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0, \quad y\left(\frac{\pi}{2}\right) = 1$

42. $\dfrac{dy}{dx} + xy = x, \quad y(0) = -6$

43. $y' = \dfrac{y}{x}, \qquad y(1) = -2$

44. $y' = \dfrac{\sin x}{y}, \qquad y\left(\frac{\pi}{2}\right) = 1$

45. $y' = y + 2xe^{2x}; \qquad y(0) = 3$

46. $\left(x^2+1\right)y' + 3xy = 6x; \qquad y(0) = -1$

47. $y' = \left(4t^3+1\right)y, \quad y(0) = 4$

48. $y' = \dfrac{e^t}{2y}, \quad y(\ln 2) = 1$

49. $(\sec x)y' = y^3, \quad y(0) = 3$

50. $\dfrac{dy}{dx} = e^{x-y}, \quad y(0) = \ln 3$

51. $y' = 2e^{3y-t}, \quad y(0) = 0$

52. $y' = 3y - 6, \quad y(0) = 9$

53. $y' = -y + 2, \quad y(0) = -2$

54. $y' = -2y - 4, \quad y(0) = 0$