

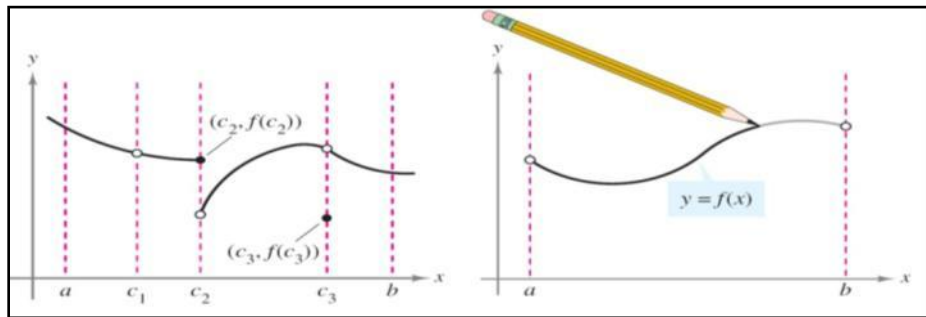
Section 1.6 – Continuity and Rates of Change

Definition of Continuity

Let c be a number in the interval (a, b) , and let f be a function whose domain contains the interval (a, b) . The function f is continuous at the point c if the following conditions are true.

1. $f(c)$ is defined
2. $\lim_{x \rightarrow c} f(x)$ exists
3. $\lim_{x \rightarrow c} f(x) = f(c)$

If f is continuous at every point in the interval (a, b) , then it is continuous on an open interval (a, b)



The Continuity of Polynomial & Rational functions:

- 1- A Polynomial function is continuous @ every real number
- 2- A rational function is continuous @ every point in its domain
 $x \neq c \Rightarrow \text{Continuous } (-\infty, c) \text{ and } (c, \infty)$

Example

Find all values of x where the following function is discontinuous

$$f(x) = \begin{cases} x+1 & \text{if } x < 1 \\ x^2 - 3x + 4 & \text{if } 1 \leq x \leq 3 \\ 5-x & \text{if } x > 3 \end{cases}$$

Solution

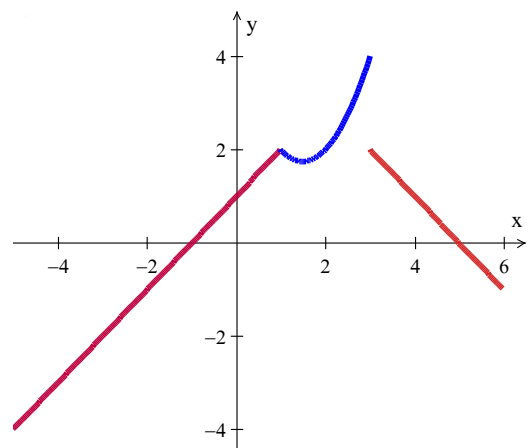
$$\lim_{x \rightarrow 1^-} (x+1) = 1+1 = 2$$

$$f(1) = (1)^2 - 3(1) + 4 = 2$$

$$f(3) = (3)^2 - 3(3) + 4 = 4$$

$$\lim_{x \rightarrow 3^+} (5-x) = 5-3 = 2$$

So f is discontinue at $x = 3$



Example

a) $f(x) = \frac{1}{x-1}$

Consists of all real number except $x = 1$.

Or

Continuous on $(-\infty, 1)$ and $(1, \infty)$

b) $f(x) = \frac{x^2-4}{x-2}$

Continuous on $(-\infty, 2)$ and $(2, \infty)$

c) $f(x) = x^2 - 2x + 3$

Continuous @ every real number

d) $f(x) = x^3 + x$

Continuous @ every real number

If f is not continuous @ $x = c \Rightarrow$ function is said to have discontinuity @ c

\Rightarrow This type of discontinuity falls into 2 categories:

- | | | |
|------------------|-------------------------|--|
| 1. Removable | ex. $\frac{x^2-4}{x-2}$ | $\lim \frac{x^2-4}{x-2} = \text{exists} = \text{constant}$ |
| 2. Non-removable | ex. $\frac{1}{x-1}$ | $\lim = \text{doesn't exist} = \pm\infty$ |

Definition: Continuity on Close Interval

Let f be defined on a closed interval $[a, b]$, if f is continuous on the open interval

$$\Rightarrow \lim_{x \rightarrow a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \rightarrow b^-} f(x) = f(b) \quad \Rightarrow f \text{ is continuous } [a, b]$$

Example

Discuss the continuity of $f(x) = \sqrt{x-2}$

Solution

$$\text{Domain: } x - 2 \geq 0 \Rightarrow x \geq 2$$

$$\Rightarrow \lim_{x \rightarrow 2^+} \sqrt{x-2} = 0 = f(2)$$

Example

Discuss the continuity of

$$f(x) = \begin{cases} x+2 & -1 \leq x < 3 \\ 14-x^2 & 3 \leq x \leq 5 \end{cases}$$

Solution

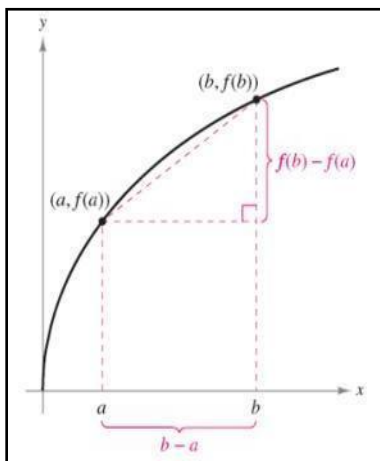
$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (x+2) = 5$$

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} (14-x^2) = 5$$

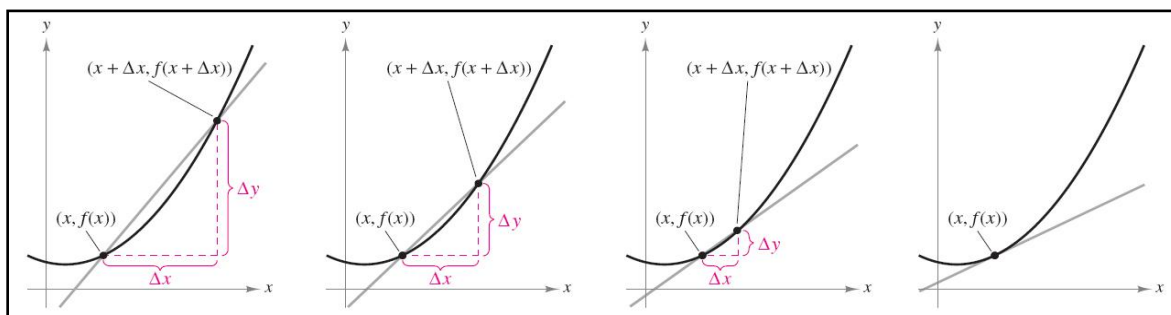
$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (14-x^2) = 5$$

Slope and Rate of change: Given $f(x)$ @ $(x, f(x))$

Definition of Average Rate of Change



$$\text{Average Rate of Change} = \frac{f(b) - f(a)}{b - a} = \frac{\Delta y}{\Delta x}$$



Slope and limit process

The secant line contains the points $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$. Using the slope formula, the slope of the secant line is

$$m_{\text{sec}} = \frac{f(x + \Delta x) - f(x)}{x + \Delta x - x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If you allow Δx to become smaller, the secant line will change slope and become closer to the slope of the tangent line. If you were to allow Δx to equal 0, the secant line becomes the tangent line. Of course you cannot allow Δx to equal 0, because the slope of the secant line would then be undefined. But you could let Δx approach 0. As Δx approaches 0, the secant line becomes the tangent line to the graph at the point $(x, f(x))$. Thus the slope of the secant line becomes the slope of the tangent line at the point $(x, f(x))$ is given by:

$$m = \lim_{\Delta x \rightarrow 0} m_{\text{sec}} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (\text{Difference Quotient})$$

As we know: $m = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

Example

Cigarette consumption in the United States has been declining since reaching a peak around 1960. Per capita cigarette consumption since 1980 can be closely approximated by the function

$$f(t) = 3870(0.970)^t$$

Where t is the number of years since 1980. Find the average rate of change of per capita consumption from 1985 to 2005.

Solution

$$t = 1985 - 1980 = 5$$

$$f(t = 5) = 3870(0.970)^5 = 3323.3$$

$$f(t = 2005 - 1980 = 25) = 3870(0.970)^{25} = 1807.2$$

$$\begin{aligned} \frac{f(25) - f(5)}{25 - 5} &= \frac{1807.2 - 3323.3}{20} \\ &= -76 \end{aligned}$$

The average rate decreased at a rate of 76 cigarettes per year.

Definition of Instantaneous rate of Change

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{b \rightarrow a} \frac{f(b) - f(a)}{b - a}$$

Example

The distance in feet of an object from a starting point is given by $s(t) = 2t^2 - 5t + 40$, where t is time in seconds.

a) Find the average velocity of the object from 2 sec to 4 sec.

$$\begin{aligned} \text{The average velocity: } &= \frac{s(4) - s(2)}{4 - 2} \\ &= \frac{(2(4)^2 - 5(4) + 40) - (2(2)^2 - 5(2) + 40)}{2} \\ &= 7 \text{ ft / sec} \end{aligned}$$

c) Find the instantaneous velocity at 4 sec.

$$\begin{aligned}
 \lim_{b \rightarrow 4} \frac{s(b) - s(4)}{b - 4} &= \lim_{b \rightarrow 4} \frac{\left(2(b)^2 - 5(b) + 40\right) - \left(2(4)^2 - 5(4) + 40\right)}{b - 4} \\
 &= \lim_{b \rightarrow 4} \frac{2b^2 - 5b + 40 - 52}{b - 4} \\
 &= \lim_{b \rightarrow 4} \frac{2b^2 - 5b - 12}{b - 4} \\
 &= \lim_{b \rightarrow 4} \frac{(2b + 3)(b - 4)}{b - 4} \\
 &= \lim_{b \rightarrow 4} (2b + 3) \\
 &= 11 \text{ ft / sec}
 \end{aligned}$$

Example

Find the slope of the graph of $f(x) = x^2$ at the point (2, 4)

Solution

$$\begin{aligned}
 m_{\text{sec}} &= \frac{f(2 + \Delta x) - f(2)}{\Delta x} \\
 &= \frac{(2 + \Delta x)^2 - 2^2}{\Delta x} \\
 &= \frac{4 + \Delta x^2 + 4\Delta x - 4}{\Delta x} \\
 &= \frac{\Delta x^2 + 4\Delta x}{\Delta x} \\
 &= \frac{\Delta x^2}{\Delta x} + \frac{4\Delta x}{\Delta x} \\
 &= \Delta x + 4
 \end{aligned}$$

$$\begin{aligned}
 m &= \lim_{\Delta x \rightarrow 0} m_{\text{sec}} \\
 &= \lim_{\Delta x \rightarrow 0} (\Delta x + 4) \\
 &= 4
 \end{aligned}$$

Exercises **Section 1.6 – Continuity and Rates of Change**

Determine whether $f(x)$ is continuous on the entire number line. Explain your reasoning.

1. $f(x) = \frac{x}{x^2 - 1}$

2. $f(x) = \frac{x - 5}{x^2 - 9x + 20}$

3. Find the slope of the graph of $f(x) = 2x + 5$

4. Find the slope of the graph of $f(x) = \sqrt{x}$