Solution   Section 3.8 – Dot product and Orthogonality

Exercise

If $\|\vec{v}\| = 5$ and $\|\vec{w}\| = 3$, what are the smallest and largest possible values of $\|\vec{v} - \vec{w}\|$ and $\vec{v}.\vec{w}$?

Solution

$$\|\vec{v} - \vec{w}\| \le \|\vec{v}\| + \|\vec{w}\| = 5 + 3 = 8$$

$$\|\vec{v} - \vec{w}\| \ge \|\vec{v}\| - \|\vec{w}\| = 5 - 3 = 2$$

$$|\vec{v}.\vec{w}| = \|\vec{v}\|.\|\vec{w}\|.\cos\theta \le \|\vec{v}\|.\|\vec{w}\|$$

$$-\|\vec{v}\|.\|\vec{w}\| \le |\vec{v}.\vec{w}| \le \|\vec{v}\|.\|\vec{w}\|$$

$$-(3)(5) \le |\vec{v}.\vec{w}| \le (3)(5)$$

$$-15 \le |\vec{v}.\vec{w}| \le 15$$

The minimum value occurs when the dot product is a small as possible, v and w are parallel, but point in opposite directions. Thus the smallest value is -15.

The maximum value occurs when the dot product is a large as possible, v and w are parallel and point in same direction. Thus the largest value is 15.

Exercise

If $\|\vec{v}\| = 7$ and $\|\vec{w}\| = 3$, what are the smallest and largest possible values of $\|\vec{v} + \vec{w}\|$ and $\vec{v}.\vec{w}$?

Solution

$$\|\vec{v} + \vec{w}\| \le \|\vec{v}\| + \|\vec{w}\| = 7 + 3 = 10$$

$$\|\vec{v} + \vec{w}\| \ge \|\vec{v}\| - \|\vec{w}\| = 7 - 3 = 4$$

$$|\vec{v}.\vec{w}| \le \|\vec{v}\|.\|\vec{w}\|$$

$$-\|\vec{v}\|.\|\vec{w}\| \le |\vec{v}.\vec{w}| \le \|\vec{v}\|.\|\vec{w}\|$$

$$-(7)(3) \le |\vec{v}.\vec{w}| \le (7)(3)$$

$$-21 \le |\vec{v}.\vec{w}| \le 21$$

The minimum value occurs when the dot product is a small as possible, v and w are parallel, but point in opposite directions. Thus the smallest value is -21. $\vec{v} = (7,\ 0,\ 0,\ \cdots)$ and $\vec{w} = (-3,\ 0,\ 0,\ \cdots)$

The maximum value occurs when the dot product is a large as possible, v and w are parallel and point in same direction. Thus the largest value is 21. $\vec{v} = (7,\ 0,\ 0,\ \cdots)$ and $\vec{w} = (3,\ 0,\ 0,\ \cdots)$

Exercise

Given that $cos(\alpha)=\frac{v_1}{\|v\|}$ and $sin(\alpha)=\frac{v_2}{\|v\|}$. Similarly, $cos(\beta)=$ ______ and $sin(\beta)=$ ______. The angle θ is $\beta-\alpha$. Substitute into the trigonometry formula $cos(\alpha)cos(\beta)+sin(\alpha)sin(\beta)$ for $cos(\beta-\alpha)$ to find $cos(\theta)=\frac{v.w}{\|v\|.\|w\|}$

Solution

$$cos(\beta)=\frac{w_1}{\|w\|}$$

$$sin(\beta)=\frac{w_2}{\|w\|}$$

$$\begin{aligned}cos(\beta-\alpha)&=cos(\alpha)cos(\beta)+sin(\alpha)sin(\beta)\\&=\frac{v_1}{\|v\|}\frac{w_1}{\|w\|}+\frac{v_2}{\|v\|}\frac{w_2}{\|w\|}\\&=\frac{v_1w_1+v_2w_2}{\|v\|.\|w\|}\\&=\frac{v.w}{\|v\|.\|w\|}\end{aligned}$$

Exercise

Can three vectors in the *xy* plane have $u.v<0$ and $v.w<0$ and $u.w<0$?

Solution

Let consider: $u=(1,\ 0)$, $v=\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, $w=\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

$$u.v=(1)\left(-\frac{1}{2}\right)+0=-\frac{1}{2}$$

$$\begin{aligned}v.w&=\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)+\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)\\&=\frac{1}{4}-\frac{3}{4}\\&=-\frac{1}{2}\end{aligned}$$

$$u.w=(1)\left(-\frac{1}{2}\right)+(0)\left(-\frac{\sqrt{3}}{2}\right)=-\frac{1}{2}$$

Yes, it is.

Exercise

Find the norm of v, a unit vector that has the same direction as v, and a unit vector that is oppositely directed.

a) $v = (4, -3)$

b) $v = (1, -1, 2)$

c) $v = (-2, 3, 3, -1)$

Solution

a) $\|v\| = \sqrt{4^2 + (-3)^2} = \underline{5}$

Same direction unit vector: $u_1 = \dfrac{v}{\|v\|} = \frac{1}{5}(4, -3) = \underline{\left(\frac{4}{5}, -\frac{3}{5}\right)}$

Opposite direction unit vector: $u_2 = -\dfrac{v}{\|v\|} = -\frac{1}{5}(4, -3) = \underline{\left(-\frac{4}{5}, \frac{3}{5}\right)}$

b) $\|v\| = \sqrt{1^2 + (-1)^2 + 2^2} = \underline{\sqrt{6}}$

Same direction unit vector:

$$u_1 = \frac{v}{\|v\|} = \frac{1}{\sqrt{6}}(1, -1, 2) = \underline{\left(\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)}$$

Opposite direction unit vector:

$$u_2 = -\frac{v}{\|v\|} = -\frac{1}{\sqrt{6}}(1, -1, 2) = \underline{\left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right)}$$

c) $\|v\| = \sqrt{(-2)^2 + (3)^2 + (3)^2 + (-1)^2} = \underline{\sqrt{23}}$

Same direction unit vector:

$$u_1 = \frac{v}{\|v\|} = \frac{1}{\sqrt{23}}(-2, 3, 3, -1) = \underline{\left(\frac{-2}{\sqrt{23}},\ \frac{3}{\sqrt{23}},\ \frac{3}{\sqrt{23}},\ -\frac{1}{\sqrt{23}}\right)}$$

Opposite direction unit vector:

$$u_2 = -\frac{v}{\|v\|} = -\frac{1}{\sqrt{23}}(-2, 3, 3, -1) = \underline{\left(\frac{2}{\sqrt{23}},\ -\frac{3}{\sqrt{23}},\ -\frac{3}{\sqrt{23}},\ \frac{1}{\sqrt{23}}\right)}$$

Exercise

Evaluate the given expression with $\boldsymbol{u} = (2, -2, 3)$, $\boldsymbol{v} = (1, -3, 4)$, and $\boldsymbol{w} = (3, 6, -4)$

a) $\|u+v\|$

b) $\|-2u+2v\|$

c) $\|3u-5v+w\|$

d) $\|3v\|-3\|v\|$

e) $\|u\|+\|-2v\|+\|-3w\|$

Solution

a)
$$\begin{aligned}\|u+v\| &= \|(2,-2,3)+(1,-3,4)\| \\ &= \|(3,-5,7)\| \\ &= \sqrt{3^2+(-5)^2+7^2} \\ &= \sqrt{83}\end{aligned}$$

b)
$$\begin{aligned}\|-2u+2v\| &= \|(-4,4,-6)+(2,-6,8)\| \\ &= \|(-2,-2,2)\| \\ &= \sqrt{(-2)^2+(-2)^2+2^2} \\ &= \sqrt{12} \\ &= 2\sqrt{3}\end{aligned}$$

c)
$$\begin{aligned}\|3u-5v+w\| &= \|(6,-6,9)-(5,-15,20)+(3,6,-4)\| \\ &= \|(4,15,-15)\| \\ &= \sqrt{(4)^2+(15)^2+(-15)^2} \\ &= \sqrt{466}\end{aligned}$$

d)
$$\begin{aligned}\|3v\|-3\|v\| &= \|(3,-9,12)\|-3\|(1,-3,4)\| \\ &= \sqrt{3^2+(-9)^2+12^2}-3\sqrt{1^2+(-3)^2+4^2} \\ &= \sqrt{234}-3\sqrt{26} \\ &= 3\sqrt{26}-3\sqrt{26} \\ &= 0\end{aligned}$$

$$\|3v\|-3\|v\| = 3\|v\|-3\|v\| = 0$$

e)
$$\begin{aligned}\|u\|+\|-2v\|+\|-3w\| &= \|u\|-2\|v\|-3\|w\| \\ &= \sqrt{2^2+(-2)^2+3^2}-2\sqrt{1^2+(-3)^2+4^2}-3\sqrt{3^2+6^2+(-4)^2} \\ &= \sqrt{17}-2\sqrt{26}-3\sqrt{61}\end{aligned}$$

Exercise

Let $\boldsymbol{v}$ = (1, 1, 2, –3, 1). Find all scalars k such that $\|kv\| = 5$

Solution

$$\begin{aligned}\|kv\| &= |k|\|v\| \\ &= |k|\ \|(1,1,2,-3,1)\| \\ &= |k|\ \sqrt{1^2+1^2+2^2+(-3)^2+1^2} \\ &= |k|\sqrt{49} \\ &= 7|k|\end{aligned}$$

$$7|k| = 5 \quad \rightarrow \quad |k| = \frac{5}{7} \Rightarrow \boxed{k = \pm\frac{5}{7}}$$

Exercise

Find $u \cdot v, \quad u \cdot u, \quad and \quad v \cdot v$

a) u = (3, 1, 4), v = (2, 2, –4)

b) u = (1, 1, 4, 6), v = (2, –2, 3, –2)

c) u = (2, –1, 1, 0, –2), v = (1, 2, 2, 2, 1)

Solution

a) $u \cdot v = (3,1,4)\cdot(2,2,-4) = 3(2)+1(2)+4(-4) \ \underline{=-8|}$

$u \cdot u = \|u\|^2 = 3^2+1^2+4^2 \ \underline{=26|}$

$v \cdot v = \|v\|^2 = 2^2+2^2+(-4)^2 \ \underline{=24|}$

b) $u \cdot v = (1,1,4,6)\cdot(2,-2,3,-2) = 1(2)+1(-2)+4(3)+6(-2) \ \underline{=0|}$

$u \cdot u = \|u\|^2 = 1^2+1^2+4^2+6^2 \ \underline{=54|}$

$v \cdot v = \|v\|^2 = 2^2+(-2)^2+3^2+(-2)^2 \ \underline{=21|}$

c) $u \cdot v = (2,-1,1,0,-2)\cdot(1,2,2,2,1) = 2(1)-1(2)+1(2)+0(2)-2(1) \ \underline{=0|}$

$u \cdot u = \|u\|^2 = 2^2+(-1)^2+1^2+0+(-2)^2 \ \underline{=10|}$

$v \cdot v = \|v\|^2 = 1^2+2^2+2^2+2^2+1^2 \ \underline{=14|}$

Exercise

Find the Euclidean distance between $\boldsymbol{u}$ and $\boldsymbol{v}$, then find the angle between them

a) $u = (3, 3, 3)$, $v = (1, 0, 4)$

b) $u = (1, 2, -3, 0)$, $v = (5, 1, 2, -2)$

c) $u = (0, 1, 1, 1, 2)$, $v = (2, 1, 0, -1, 3)$

Solution

a)
$$d = \|u - v\| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$$
$$= \sqrt{(-2)^2 + (-3)^2 + (1)^2}$$
$$\boxed{= \sqrt{14}}$$

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\ \|\boldsymbol{v}\|}$$
$$= \frac{3(1) + 3(0) + 3(4)}{\sqrt{3^2 + 3^2 + 3^2}\sqrt{1^2 + 0^2 + 4^2}}$$
$$= \frac{15}{\sqrt{27}\sqrt{17}}$$

$$\theta = \cos^{-1}\left(\frac{15}{\sqrt{27}\sqrt{17}}\right) = \boxed{45.56°}$$

b)
$$d = \|u - v\| = \sqrt{(1-5)^2 + (-2-1)^2 + (-3-2)^2 + (-2-0)^2}$$
$$= \sqrt{(-4)^2 + (-3)^2 + (-5)^2 + (-2)^2}$$
$$\boxed{= \sqrt{46}}$$

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\ \|\boldsymbol{v}\|}$$
$$= \frac{1(5) + 2(1) - 3(2) + 0(-2)}{\sqrt{1^2 + 2^2 + (-3)^2 + 0}\ \sqrt{5^2 + 1^2 + 2^2 + (-2)^2}}$$
$$= \frac{1}{\sqrt{14}\sqrt{34}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{14}\sqrt{34}}\right) = \boxed{87.37°}$$

c) $d=\|u-v\|=\sqrt{(0-2)^2+(1-1)^2+(1-0)^2+(1-(-1))^2+(2-3)^2}$

$$=\boxed{\sqrt{10}}$$

$$\cos\theta=\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\ \|\boldsymbol{v}\|}$$

$$=\frac{0(2)+1(1)+1(0)+1(-1)+2(3)}{\sqrt{0+1^2+1^2+1^2+2^2}\ \sqrt{2^2+1^2+0+(-1)^2+(3)^2}}$$

$$=\frac{6}{\sqrt{7}\sqrt{15}}$$

$$\theta=\cos^{-1}\left(\frac{6}{\sqrt{7}\sqrt{15}}\right)=\boxed{54.16^\circ}$$

Exercise

Find a unit vector that has the same direction as the given vector

a) $(-4,-3)$ *b)* $\left(-3,\ 2,\ \sqrt{3}\right)$ *c)* $(1, 2, 3, 4, 5)$

Solution

a) $u=\frac{u}{\|u\|}=\frac{(-4,-3)}{\sqrt{(-4)^2+(-3)^2}}$

$$=\frac{(-4,-3)}{\sqrt{25}}$$

$$=\boxed{\left(-\frac{4}{5},-\frac{3}{5}\right)}$$

b) $u=\frac{1}{\sqrt{(-3)^2+(2)^2+\left(\sqrt{3}\right)^2}}\left(-3,2,\sqrt{3}\right)$

$$=\frac{1}{\sqrt{17}}\left(-3,\ 2,\ \sqrt{3}\right)$$

$$=\boxed{\left(-\frac{3}{\sqrt{17}},\ \frac{2}{\sqrt{17}},\ \frac{\sqrt{3}}{\sqrt{17}}\right)}$$

c) $u=\frac{1}{\sqrt{1^2+2^2+3^2+4^2+5^2}}(1,2,3,4,5)$

$$=\frac{1}{\sqrt{55}}(1,2,3,4,5)$$

$$=\boxed{\left(\frac{1}{\sqrt{55}},\ \frac{2}{\sqrt{55}},\ \frac{3}{\sqrt{55}},\ \frac{4}{\sqrt{55}},\ \frac{5}{\sqrt{55}}\right)}$$

Exercise

Find a unit vector that is oppositely to the given vector

a) $(-12, -5)$

b) $(3, -3, 3)$

c) $\left(-3,\ 1,\ \sqrt{6},\ 3\right)$

Solution

a) $$u = -\frac{1}{\sqrt{(-12)^2 + (-5)^2}}(-12,-5)$$

$$= -\frac{1}{\sqrt{169}}(-12,-5)$$

$$= \left(\frac{12}{13},\ \frac{5}{13}\right)$$

b) $$u = -\frac{1}{\sqrt{(3)^2 + (-3)^2 + (3)^2}}(3,-3,3)$$

$$= -\frac{1}{\sqrt{27}}(3,-3,3)$$

$$= -\frac{1}{3\sqrt{3}}(3,-3,3)$$

$$= \left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)$$

c) $$u = -\frac{1}{\sqrt{(-3)^2 + 1^2 + \left(\sqrt{6}\right)^2 + 3^2}}\left(-3,\ 1,\ \sqrt{6},\ 3\right)$$

$$= -\frac{1}{\sqrt{25}}\left(-3,\ 1,\ \sqrt{6},\ 3\right)$$

$$= \left(\frac{3}{5},\ -\frac{1}{5},\ -\frac{\sqrt{6}}{5},\ -\frac{3}{5}\right)$$

Exercise

Verify that the Cauchy-Schwarz inequality holds

a) $u = (-3, 1, 0)$, $v = (2, -1, 3)$

b) $u = (0, 2, 2, 1)$, $v = (1, 1, 1, 1)$

c) $u = (1, 3, 5, 2, 0, 1)$, $v = (0, 2, 4, 1, 3, 5)$

Solution

a)
$$\begin{aligned} |u \cdot v| &= |(-3,1,0)\bullet(2,-1,3)| \\ &= |-3(2)+1(-1)+0(3)| \\ &= |-7| \\ &= 7 \end{aligned}$$

$$\begin{aligned} \|u\|\|v\| &= \sqrt{(-3)^2+1^2+0}\ \sqrt{(2)^2+(-1)^2+3^2} \\ &= \sqrt{10}\sqrt{14} \\ &\approx 11.83 \end{aligned}$$

$|u \cdot v| \le \|u\|\|v\|$ Cauchy-Schwarz inequality holds

b)
$$\begin{aligned} |u \cdot v| &= |(0,2,2,1)\bullet(1,1,1,1)| \\ &= |0+2+2+1| \\ &= 5 \end{aligned}$$

$$\begin{aligned} \|u\|\|v\| &= \sqrt{0+2^2+2^2+1^2}\ \sqrt{1^2+1^2+1^2+1^2} \\ &= \sqrt{9}\sqrt{4} \\ &= 6 \end{aligned}$$

$|u \cdot v| \le \|u\|\|v\|$ Cauchy-Schwarz inequality holds

c)
$$\begin{aligned} |u \cdot v| &= |(1,3,5,2,0,1)\bullet(0,2,4,1,3,5)| \\ &= |0+6+20+2+0+5| \\ &= 23 \end{aligned}$$

$$\begin{aligned} \|u\|\|v\| &= \sqrt{1^2+3^2+5^2+2^2+0+1^2}\ \sqrt{0+2^2+4^2+1^2+3^2+5^2} \\ &= \sqrt{40}\sqrt{55} \\ &\approx 46 \end{aligned}$$

$|u \cdot v| \le \|u\|\|v\|$ Cauchy-Schwarz inequality holds

Exercise

Find $\boldsymbol{u} \cdot \boldsymbol{v}$ and then the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$ $\quad \boldsymbol{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$

Solution

$$\boldsymbol{u} \cdot \boldsymbol{v} = 3 + 0 - 2 - 1 \,\underline{= 0|}$$

$$\theta = \cos^{-1} \frac{0}{\sqrt{15}\sqrt{3}} = \cos^{-1}(0) \,\underline{= 90^\circ|}$$

Exercise

Find the norm: $\|\boldsymbol{u}\| + \|\boldsymbol{v}\|$, $\|\boldsymbol{u} + \boldsymbol{v}\|$ for $\boldsymbol{u} = (3,\ -1,\ -2,\ 1,\ 4)$ $\quad \boldsymbol{v} = (1,\ 1,\ 1,\ 1,\ 1)$

Solution

$$\|\boldsymbol{u}\| + \|\boldsymbol{v}\| = \sqrt{3^2 + (-1)^2 + (-2)^2 + 1^2 + 4^2} + \sqrt{1+1+1+1+1} \,\underline{= \sqrt{31} + \sqrt{5}|}$$

$$\|\boldsymbol{u} + \boldsymbol{v}\| = \|(4,\ 0,\ -1,\ 2,\ 5)\| = \sqrt{16 + 0 + 1 + 4 + 25} \,\underline{= \sqrt{46}|}$$

Exercise

Find all numbers r such that: $\|r(1,\ 0,\ -3,\ -1,\ 4,\ 1)\| = 1$

Solution

$$r\sqrt{1+9+1+16+1} = \pm 1$$

$$r\sqrt{28} = \pm 1$$

$$r = \pm \frac{1}{2\sqrt{7}} \,\underline{= \pm \frac{\sqrt{7}}{14}|}$$

Exercise

Find the distance between $P_1(7, -5, 1)$ and $P_2(-7, -2, -1)$

Solution

$$\begin{aligned} \|P_1P_2\| &= \sqrt{(-7-7)^2 + (-2+5)^2 + (-1-1)^2} \\ &= \sqrt{14^2 + 3^2 + (-2)^2} \\ &= \sqrt{196 + 9 + 4} \\ &\underline{= \sqrt{209}|} \end{aligned}$$

Exercise

Given $\boldsymbol{u} = (1, -5, 4)$, $\boldsymbol{v} = (3, 3, 3)$

a) Find $\boldsymbol{u}\cdot\boldsymbol{v}$

b) Find the cosine of the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$.

Solution

a) $\boldsymbol{u}\cdot\boldsymbol{v} = 3-15+12 = \underline{0}$

b) $\cos\theta = \dfrac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\ \|\boldsymbol{v}\|}\ \underline{=0}$

Exercise

Determine whether $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

a) $\boldsymbol{u}=(-6,\ -2),\quad \boldsymbol{v}=(5,\ -7)$

b) $\boldsymbol{u}=(6,\ 1,\ 4),\quad \boldsymbol{v}=(2,\ 0,\ -3)$

c) $\boldsymbol{u}=(1,\ -5,\ 4),\quad \boldsymbol{v}=(3,\ 3,\ 3)$

d) $\boldsymbol{u}=(-2,\ 2,\ 3),\quad \boldsymbol{v}=(1,\ 7,\ -4)$

Solution

a) $\boldsymbol{u}\cdot\boldsymbol{v}=(-6)(5)+(-2)(-7)$

$=-30+14$

$\underline{=-16\neq 0}$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are not orthogonal

b) $\boldsymbol{u}\cdot\boldsymbol{v}=6(2)+1(0)+4(-3)\ \underline{=0}$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

c) $\boldsymbol{u}\cdot\boldsymbol{v}=1(3)-5(3)+4(3)\ \underline{=0}$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

d) $\boldsymbol{u}\cdot\boldsymbol{v}=-2(1)+2(7)+3(-4)\ \underline{=0}$

$\therefore$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

Exercise

Determine whether the vectors form an orthogonal set

a) $\boldsymbol{v}_1=(2,\ 3),\quad \boldsymbol{v}_2=(3,\ 2)$

b) $\boldsymbol{v}_1=(1,\ -2),\quad \boldsymbol{v}_2=(-2,\ 1)$

c) $\boldsymbol{u}=(-4,\ 6,\ -10,\ 1)\quad \boldsymbol{v}=(2,\ 1,\ -2,\ 9)$

d) $\boldsymbol{u}=(a,\ b)\quad \boldsymbol{v}=(-b,\ a)$

e) $\boldsymbol{v}_1=(-2,\ 1,\ 1),\quad \boldsymbol{v}_2=(1,\ 0,\ 2),\quad \boldsymbol{v}_3=(-2,\ -5,\ 1)$

f) $\boldsymbol{v}_1 = (1,\ 0,\ 1), \quad \boldsymbol{v}_2 = (1,\ 1,\ 1), \quad \boldsymbol{v}_3 = (-1,\ 0,\ 1)$

g) $\boldsymbol{v}_1 = (2,-2,1), \quad \boldsymbol{v}_2 = (2,1,-2), \quad \boldsymbol{v}_3 = (1,2,2)$

Solution

a) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 2(3) + 3(2) \ \underline{= 12 \neq 0|}$

$\therefore$ Vectors don't form an orthogonal set

b) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 1(-2) - 2(1) \ \underline{= -4 \neq 0|}$

$\therefore$ Vectors don't form an orthogonal set

c) $\boldsymbol{u}\bullet\boldsymbol{v} = -8 + 6 + 20 + 9 \ \underline{= 27 \neq 0|}$; These vectors are not orthogonal

d) $\boldsymbol{u}\bullet\boldsymbol{v} = -ab + ab \ \underline{= 0|}$; These vectors are orthogonal

e) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = -2(1) + 1(0) + 1(2) \underline{= 0|}$

$\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = -2(-2) + 1(-5) + 1(1) \underline{= 0|}$

$\boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 1(-2) + 0(-5) + 2(1) \underline{= 0|}$

$\therefore$ Vectors form an orthogonal set

f) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 1(1) + 0(1) + 1(1) \ \underline{= 2 \neq 0|}$

$\therefore$ Vectors don't form an orthogonal set

g) $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 2(2) - 2(1) + 1(-2) \underline{= 0|}$

$\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = 2(1) - 2(2) + 1(2) \underline{= 0|}$

$\boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 2(1) + 1(2) - 2(2) \underline{= 0|}$

$\therefore$ Vectors form an orthogonal set

Exercise

Find a unit vector that is orthogonal to both $\boldsymbol{u} = (1, 0, 1)$ and $\boldsymbol{v} = (0, 1, 1)$

Solution

Let $\boldsymbol{w} = (w_1, w_2, w_3)$ be the unit vector that is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$.

$$\boldsymbol{u} \cdot \boldsymbol{w} = 1(w_1) + 0(w_2) + 1(w_3) = \underline{w_1 + w_3 = 0|}$$

$$\boxed{w_3 = -w_1}$$

$$\boldsymbol{v} \cdot \boldsymbol{w} = 0(w_1) + 1(w_2) + 1(w_3) = \underline{w_2 + w_3 = 0|}$$

$$\boxed{w_3 = -w_2}$$

$w_1 = w_2 = -w_3$

The orthogonal vector to both $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{w} = (1,\ 1,\ -1)$, therefore the unit vector is

$$\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = \frac{1}{\sqrt{1^2+1^2+(-1)^2}}(1,\ 1,\ -1)$$

$$= \frac{1}{\sqrt{3}}(1,\ 1,\ -1)$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)$$

The possible vectors are: $\boxed{\pm\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)}$

Exercise

a) Show that $\boldsymbol{v} = (a, b)$ and $\boldsymbol{w} = (-b, a)$ are orthogonal vectors.

b) Use the result to find two vectors that are orthogonal to $\boldsymbol{v} = (2, -3)$.

c) Find two unit vectors that are orthogonal to $(-3, 4)$

Solution

a) $\boldsymbol{v}\cdot\boldsymbol{w} = a(-b)+b(a) = -ab+ab = 0$ are orthogonal vectors.

b) $(2, 3)$ and $(-2, 3)$.

c) $u_1 = \frac{1}{\sqrt{4^2+3^2}}(4,3) = \underline{\left(\frac{4}{5},\ \frac{3}{5}\right)|}$

$u_2 = -\frac{1}{\sqrt{4^2+3^2}}(4,3) = \underline{\left(-\frac{4}{5},\ -\frac{3}{5}\right)|}$

Exercise

Show that if $\boldsymbol{v}$ is orthogonal to both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, then $\boldsymbol{v}$ is orthogonal to $k_1\boldsymbol{w}_1 + k_2\boldsymbol{w}_2$ for all scalars k_1 and k_2.

Solution

$$\boldsymbol{v}\cdot\left(k_1\boldsymbol{w}_1 + k_2\boldsymbol{w}_2\right) = \boldsymbol{v}\cdot\left(k_1\boldsymbol{w}_1\right) + \boldsymbol{v}\cdot\left(k_2\boldsymbol{w}_2\right)$$

$= k_1\left(\boldsymbol{v}\cdot\boldsymbol{w}_1\right) + k_2\left(\boldsymbol{v}\cdot\boldsymbol{w}_2\right)$ ***If v is orthogonal to*** $\boldsymbol{w}_1$ & $\boldsymbol{w}_2 \to \boldsymbol{v}\cdot\boldsymbol{w}_1 = \boldsymbol{v}\cdot\boldsymbol{w}_2 = 0$

$$= k_1(0) + k_2(0)$$

$$\underline{= 0|}$$

Exercise

Show that $\vec{u}-\vec{v}$ is orthogonal to $\vec{u}+\vec{v}$ if and only if $\|\vec{u}\|=\|\vec{v}\|$

Solution

Suppose that $\vec{u}-\vec{v}$ is orthogonal to $\vec{u}+\vec{v}$. Then

$$\begin{aligned}0&=\langle\vec{u}-\vec{v},\ \vec{u}+\vec{v}\rangle\\ &=(\vec{u}-\vec{v})^T(\vec{u}+\vec{v})\\ &=\left(\vec{u}^T-\vec{v}^T\right)(\vec{u}+\vec{v})\\ &=\vec{u}^T\vec{u}+\vec{u}^T\vec{v}-\vec{v}^T\vec{u}-\vec{v}^T\vec{v}\\ &=\langle\vec{u},\ \vec{u}\rangle+\langle\vec{u},\ \vec{v}\rangle-\langle\vec{v},\ \vec{u}\rangle-\langle\vec{v},\ \vec{v}\rangle \qquad \langle\vec{u},\ \vec{v}\rangle=\langle\vec{v},\ \vec{u}\rangle\\ &=\langle\vec{u},\ \vec{u}\rangle-\langle\vec{v},\ \vec{v}\rangle\end{aligned}$$

So $\langle\vec{u},\ \vec{u}\rangle=\langle\vec{v},\ \vec{v}\rangle$. Therefore, $\|\vec{u}\|^2=\|\vec{v}\|^2 \quad\Rightarrow\quad \|\vec{u}\|=\|\vec{v}\|$.

Suppose $\|\vec{u}\|=\|\vec{v}\|$. Then

$$\begin{aligned}\langle\vec{u}-\vec{v},\ \vec{u}+\vec{v}\rangle&=(\vec{u}-\vec{v})^T(\vec{u}+\vec{v})\\ &=\left(\vec{u}^T-\vec{v}^T\right)(\vec{u}+\vec{v})\\ &=\vec{u}^T\vec{u}+\vec{u}^T\vec{v}-\vec{v}^T\vec{u}-\vec{v}^T\vec{v}\\ &=\langle\vec{u},\ \vec{u}\rangle+\langle\vec{u},\ \vec{v}\rangle-\langle\vec{v},\ \vec{u}\rangle-\langle\vec{v},\ \vec{v}\rangle \qquad \langle\vec{u},\ \vec{v}\rangle=\langle\vec{v},\ \vec{u}\rangle\\ &=\langle\vec{u},\ \vec{u}\rangle-\langle\vec{v},\ \vec{v}\rangle\\ &=\|\vec{u}\|^2-\|\vec{v}\|^2\\ &=0\end{aligned}$$

So we can see that $\vec{u}-\vec{v}$ is orthogonal to $\vec{u}+\vec{v}$

We conclude that $\vec{u}-\vec{v}$ is orthogonal to $\vec{u}+\vec{v}$ if and only if $\|\vec{u}\|=\|\vec{v}\|$, as desired.

Exercise

Given $\boldsymbol{u}=(3,\ -1,\ 2)$ $\boldsymbol{v}=(4,\ -1,\ 5)$ *and* $\boldsymbol{w}=(8,\ -7,\ -6)$

a) Find $3\boldsymbol{v}-4(5\boldsymbol{u}-6\boldsymbol{w})$

b) Find $\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{v}$ and then the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$.

Solution

a) $$\begin{aligned}3\boldsymbol{v}-4(5\boldsymbol{u}-6\boldsymbol{w})&=3(4,\ -1,\ 5)-4\left(5(3,\ -1,\ 2)-6(8,\ -7,\ -6)\right)\\ &=(12,\ -3,\ 15)-4\left((15,\ -5,\ 10)-(48,\ -42,\ -36)\right)\\ &=(12,\ -3,\ 15)-4(-33,\ 37,\ 46)\\ &=(12,\ -3,\ 15)-(-132,\ 148,\ 184)\end{aligned}$$

$$\boxed{=(144,\ \ -151,\ \ -169)}$$

b) $\boldsymbol{u}\bullet\boldsymbol{v}=(3,\ \ -1,\ \ 2)\bullet(1,\ 1,\ \ -1)$

$$=3-1-2$$

$$\boxed{=0}$$

$$\boxed{\theta=90°}$$

Exercise

a) Show that $\boldsymbol{v}=(a,b)$ and $\boldsymbol{w}=(-b,a)$ are orthogonal vectors

b) Use the result in part (*a*) to find two vectors that are orthogonal to $\boldsymbol{v}=(2,-3)$

c) Find two unit vectors that are orthogonal to $(-3, 4)$

Solution

a) $\boldsymbol{u}\cdot\boldsymbol{v}=-ab+ba=0$; 2 vectors are orthogonal vectors.

b) $\boldsymbol{v}=(2,-3)\ \Rightarrow \boldsymbol{w}=(-3,-2)$ and $\boldsymbol{w}=(3,2)$

c) $(-3,4)\ \Rightarrow\ \boldsymbol{u}=\dfrac{(-3,\ 4)}{\sqrt{9+16}}=\left(-\dfrac{3}{5},\ \dfrac{4}{5}\right)$

$$\boldsymbol{u}_1=\left(\frac{4}{5},\frac{3}{5}\right)\quad \text{and}\quad \boldsymbol{u}_2=\left(-\frac{4}{5},-\frac{3}{5}\right)$$

Exercise

Show that $A(3, 0, 2)$, .$B(4, 3, 0)$, and $C(8, 1, -1)$ are vertices of a right triangle. At which vertex is the right angle?

Solution

$AB=(4-3,3-0,0-2)=(1,3,-2)\qquad AC=(5,1,-3)\qquad BC=(4,-2,-1)$

$AB\bullet AC=5+3+6=14$

$AB\bullet BC=4-6+2=0$

$AC\bullet BC=20-2+3=21$

The right triangle at point B

Exercise

Establish the identity: $\quad \boldsymbol{u}\bullet\boldsymbol{v}=\frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2-\frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2$

Solution

Let $\boldsymbol{u}(u_1,\ u_2,\ \dots,\ u_n)$ *and* $\boldsymbol{v}=(v_1,\ v_2,\ \dots,\ v_n)$

$$\boxed{\boldsymbol{u}\bullet\boldsymbol{v}=u_1v_1+u_2v_2+\dots+u_nv_n}$$

$$\boldsymbol{u}+\boldsymbol{v}=\left(u_1+v_1,\ u_2+v_2,\ \ldots, u_n+v_n\right)$$

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2=\left(u_1+v_1\right)^2+\left(u_2+v_2\right)^2+\cdots+\left(u_n+v_n\right)^2$$

$$=u_1^2+v_1^2+2u_1v_1+u_2^2+v_2^2+2u_2v_2+\cdots+u_2^2+v_n^2+2u_nv_n$$

$$\boldsymbol{u}-\boldsymbol{v}=\left(u_1-v_1,\ u_2-v_2,\ \ldots, u_n-v_n\right)$$

$$\|\boldsymbol{u}-\boldsymbol{v}\|^2=\left(u_1-v_1\right)^2+\left(u_2-v_2\right)^2+\cdots+\left(u_n-v_n\right)^2$$

$$=u_1^2+v_1^2-2u_1v_1+u_2^2+v_2^2-2u_2v_2+\cdots+u_2^2+v_n^2-2u_nv_n$$

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2-\|\boldsymbol{u}-\boldsymbol{v}\|^2=u_1^2+v_1^2+2u_1v_1+u_2^2+v_2^2+2u_2v_2+\cdots+u_2^2+v_n^2+2u_nv_n$$

$$-\left(u_1^2+v_1^2-2u_1v_1+u_2^2+v_2^2-2u_2v_2+\cdots+u_2^2+v_n^2-2u_nv_n\right)$$

$$=u_1^2+v_1^2+2u_1v_1+u_2^2+v_2^2+2u_2v_2+\cdots+u_2^2+v_n^2+2u_nv_n$$

$$-u_1^2-v_1^2+2u_1v_1-u_2^2-v_2^2+2u_2v_2-\cdots-u_2^2-v_n^2+2u_nv_n$$

$$=4u_1v_1+4u_2v_2+\cdots+4u_nv_n$$

$$\frac{1}{4}\left(\|\boldsymbol{u}+\boldsymbol{v}\|^2-\|\boldsymbol{u}-\boldsymbol{v}\|^2\right)=u_1v_1+u_2v_2+\cdots+u_nv_n$$

Therefore; $\boldsymbol{u}\bullet\boldsymbol{v}=\frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2-\frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2$ is true.

2nd *method*:

$$\frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2-\frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2=\frac{1}{4}\left[(\boldsymbol{u}+\boldsymbol{v})(\boldsymbol{u}+\boldsymbol{v})-(\boldsymbol{u}-\boldsymbol{v})(\boldsymbol{u}-\boldsymbol{v})\right]$$

$$=\frac{1}{4}\left[\boldsymbol{uu}+2\boldsymbol{uv}+\boldsymbol{vv}-(\boldsymbol{uu}-2\boldsymbol{uv}+\boldsymbol{vv})\right]$$

$$=\frac{1}{4}\left[\boldsymbol{uu}+2\boldsymbol{uv}+\boldsymbol{vv}-\boldsymbol{uu}+2\boldsymbol{uv}-\boldsymbol{vv}\right]$$

$$=\frac{1}{4}(4\boldsymbol{uv})$$

$$=\boldsymbol{u}\bullet\boldsymbol{v}$$