# *Solution*   *Section* **3.4 – Orthogonal Matrices**

## *Exercise*

Show that the matrix is orthogonal

a) $A = \begin{bmatrix} \dfrac{4}{5} & 0 & -\dfrac{3}{5} \\[2mm] -\dfrac{9}{25} & \dfrac{4}{5} & -\dfrac{12}{25} \\[2mm] \dfrac{12}{25} & \dfrac{3}{5} & \dfrac{16}{25} \end{bmatrix}$

b) $A = \begin{bmatrix} \dfrac{1}{3} & \dfrac{2}{3} & \dfrac{2}{3} \\[2mm] \dfrac{2}{3} & -\dfrac{2}{3} & \dfrac{1}{3} \\[2mm] -\dfrac{2}{3} & -\dfrac{1}{3} & \dfrac{2}{3} \end{bmatrix}$

## *Solution*

a) $AA^T = \begin{bmatrix} \dfrac{4}{5} & 0 & -\dfrac{3}{5} \\[2mm] -\dfrac{9}{25} & \dfrac{4}{5} & -\dfrac{12}{25} \\[2mm] \dfrac{12}{25} & \dfrac{3}{5} & \dfrac{16}{25} \end{bmatrix} \begin{bmatrix} \dfrac{4}{5} & -\dfrac{9}{25} & \dfrac{12}{25} \\[2mm] 0 & \dfrac{4}{5} & \dfrac{3}{5} \\[2mm] -\dfrac{3}{5} & -\dfrac{12}{25} & \dfrac{16}{25} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$A^T A = \begin{bmatrix} \dfrac{4}{5} & -\dfrac{9}{25} & \dfrac{12}{25} \\[2mm] 0 & \dfrac{4}{5} & \dfrac{3}{5} \\[2mm] -\dfrac{3}{5} & -\dfrac{12}{25} & \dfrac{16}{25} \end{bmatrix} \begin{bmatrix} \dfrac{4}{5} & 0 & -\dfrac{3}{5} \\[2mm] -\dfrac{9}{25} & \dfrac{4}{5} & -\dfrac{12}{25} \\[2mm] \dfrac{12}{25} & \dfrac{3}{5} & \dfrac{16}{25} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\therefore$ **A** is an orthogonal

b) $AA^T = \begin{bmatrix} \dfrac{1}{3} & \dfrac{2}{3} & \dfrac{2}{3} \\[2mm] \dfrac{2}{3} & -\dfrac{2}{3} & \dfrac{1}{3} \\[2mm] -\dfrac{2}{3} & -\dfrac{1}{3} & \dfrac{2}{3} \end{bmatrix} \begin{bmatrix} \dfrac{1}{3} & \dfrac{2}{3} & -\dfrac{2}{3} \\[2mm] \dfrac{2}{3} & -\dfrac{2}{3} & -\dfrac{1}{3} \\[2mm] \dfrac{2}{3} & \dfrac{1}{3} & \dfrac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$AA^T = I$

$\therefore$ **A** is an orthogonal

## *Exercise*

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b) $\begin{bmatrix} \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \\[2mm] \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{bmatrix}$

d) $\begin{bmatrix} -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} \\[2mm] 0 & -\dfrac{2}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} \\[2mm] \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} \end{bmatrix}$

f) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{1}{\sqrt{3}} & -\dfrac{1}{2} & 0 \\[2mm] 0 & \dfrac{1}{\sqrt{3}} & 0 & 1 \\[2mm] 0 & \dfrac{1}{\sqrt{3}} & \dfrac{1}{2} & 0 \end{bmatrix}$

$c)$ $\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

$e)$ $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$

### Solution

**$a)$** $AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

*A* is orthogonal with inverse $A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**$b)$** $AA^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ *A* is orthogonal with inverse $A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ (It is a standard matrix for a rotation of 45°)

**$c)$** $AA^T = \begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & & \\ & & \\ & & \end{bmatrix} \neq I$

Or $\|r_1\| = \sqrt{0 + 1^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{3}{2}} \neq 1|$ $\quad\quad \therefore A$ is **not** orthogonal

**$d)$** $AA^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\therefore$ *A* is orthogonal with inverse $A^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

e) $AA^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\therefore A$ is orthogonal with inverse $A^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$

f) $\|r_2\| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{3} + \frac{1}{4}} = \sqrt{\frac{7}{12}} \neq 1$

*Or*

$AA^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{5}{6} & & \\ & & & \\ & & & \end{pmatrix} \neq I$

$\therefore$ The matrix is **not** an orthogonal

## *Exercise*

Prove that if $A$ is orthogonal, then $A^T$ is orthogonal.

### *Solution*

If $A$ is orthogonal then $A^T = A^{-1}$ and $A = \left(A^T\right)^T$

Then $\left(A^T\right)^T A^T = AA^T = I \Rightarrow A^T$ is orthogonal

Another word, since $A$ is orthogonal, then both column and row vectors of $A$ form an orthonormal set. $A^T$ is just $A$ with its row and column vectors are swapped. The column vectors of $A^T$ (which are the row vectors of $A$) and row vectors of $A^T$ (which are the column vectors of $A$) form orthonormal sets, therefore $A^T$ is orthogonal

## Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{bmatrix} \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{6}} & \cdots \\[2mm] \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{6}} & \cdots \\[2mm] -\dfrac{1}{\sqrt{3}} & \dfrac{2}{\sqrt{6}} & \cdots \end{bmatrix}$$

### Solution

$$q_1 = \begin{bmatrix} \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{3}} \end{bmatrix}^T \quad \rightarrow \quad \|q_1\| = \sqrt{\dfrac{1}{3}+\dfrac{1}{3}+\dfrac{1}{3}} = 1$$

$$q_2 = \begin{bmatrix} \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{6}} & \dfrac{2}{\sqrt{6}} \end{bmatrix}^T \quad \rightarrow \quad \|q_2\| = \sqrt{\dfrac{1}{6}+\dfrac{1}{6}+\dfrac{4}{6}} = 1$$

Let $q_3 = \begin{bmatrix} x & y & z \end{bmatrix}^T$

$$q_1 \bullet q_3 = \dfrac{1}{\sqrt{3}}x + \dfrac{1}{\sqrt{3}}y - \dfrac{1}{\sqrt{3}}z = 0 \quad \rightarrow \quad x+y-z = 0$$

$$q_2 \bullet q_3 = \dfrac{1}{\sqrt{6}}x + \dfrac{1}{\sqrt{6}}y - \dfrac{2}{\sqrt{6}}z = 0 \quad \rightarrow \quad x+y-2z = 0$$

$$\begin{cases} x+y-z = 0 \\ x+y-2z = 0 \end{cases} \rightarrow \quad z = 0 \quad and \quad x+y=0 \Rightarrow x = -y$$

$$q_3 = \begin{bmatrix} -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$$

## Exercise

Determine if the given matrix is orthogonal. If it is, find its inverse

$$\begin{bmatrix} \dfrac{1}{9} & \dfrac{4}{5} & \dfrac{3}{7} \\[2mm] \dfrac{4}{9} & \dfrac{3}{5} & -\dfrac{2}{7} \\[2mm] \dfrac{8}{9} & -\dfrac{2}{5} & \dfrac{3}{7} \end{bmatrix}$$

### Solution

$$q_1 = \begin{bmatrix} \dfrac{1}{9} & \dfrac{4}{9} & \dfrac{8}{9} \end{bmatrix}^T \quad q_2 = \begin{bmatrix} \dfrac{4}{5} & \dfrac{3}{5} & -\dfrac{2}{5} \end{bmatrix}^T \quad q_3 = \begin{bmatrix} \dfrac{3}{7} & -\dfrac{2}{7} & \dfrac{3}{7} \end{bmatrix}^T$$

$$q_1 \bullet q_2 = \dfrac{4}{45} + \dfrac{12}{45} - \dfrac{16}{45} = 0$$

$$q_1 \bullet q_3 = \dfrac{3}{63} - \dfrac{8}{63} + \dfrac{24}{63} = \dfrac{19}{63} \neq 0$$

32

$$q_2 \cdot q_3 = \frac{12}{35} - \frac{6}{35} + \frac{6}{35} = \frac{12}{35} \neq 0$$

The given matrix is **not** orthogonal