

Solution **Section 1.7 – Length of Curves**

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = (2 \cos t) \hat{i} + (2 \sin t) \hat{j} + \sqrt{5}t \hat{k}; \quad 0 \leq t \leq \pi$$

Solution

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -(2 \sin t) \hat{i} + (2 \cos t) \hat{j} + \sqrt{5} \hat{k}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{4 \sin^2 t + 4 \cos^2 t + 5} \\ &= \sqrt{4 + 5} \\ &= 3 \end{aligned}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = -\frac{2 \sin t}{3} \hat{i} + \frac{2 \cos t}{3} \hat{j} + \frac{\sqrt{5}}{3} \hat{k}$$

$$\begin{aligned} \text{Length: } s &= \int_0^{\pi} |\vec{v}(t)| \, dt \\ &= \int_0^{\pi} 3 \, dt \\ &= 3t \Big|_0^{\pi} \\ &= 3\pi \end{aligned}$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = t \hat{i} + \frac{2}{3} t^{3/2} \hat{k}; \quad 0 \leq t \leq 8$$

Solution

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \hat{i} + t^{1/2} \hat{k} \quad \Rightarrow \quad |\vec{v}| = \sqrt{1+t}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{1+t}} \hat{i} + \frac{t^{1/2}}{\sqrt{1+t}} \hat{k}$$

$$\text{Length: } s = \int_0^8 |\vec{v}(t)| \, dt$$

$$\begin{aligned}
&= \int_0^8 (1+t)^{1/2} dt \\
&= \left[\frac{2}{3} (1+t)^{3/2} \right]_0^8 \\
&= \frac{2}{3} \left[(9)^{3/2} - 1 \right] \\
&= \frac{2}{3} (27 - 1) \\
&= \frac{52}{3}
\end{aligned}$$

$$= \int_0^8 (1+t)^{1/2} d(1+t)$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = (2+t)\hat{i} - (t+1)\hat{j} + t\hat{k}; \quad 0 \leq t \leq 3$$

Solution

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \hat{i} - \hat{j} + \hat{k}$$

$$|\vec{v}| = \sqrt{1+1+1} = \sqrt{3}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{3}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

$$\begin{aligned}
\text{Length: } s &= \int_0^3 |\vec{v}(t)| dt \\
&= \int_0^3 \sqrt{3} dt \\
&= \sqrt{3}t \Big|_0^3 \\
&= 3\sqrt{3}
\end{aligned}$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = (\cos^3 t)\hat{i} + (\sin^3 t)\hat{k}; \quad 0 \leq t \leq \frac{\pi}{2}$$

Solution

$$\vec{v}(t) = -\left(3\cos^2 t \sin t\right)\hat{i} + \left(3\sin^2 t \cos t\right)\hat{k}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned}
|\vec{v}| &= \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} \\
&= 3\sqrt{\cos^2 t \sin^2 t (\cos^2 t + \sin^2 t)} \\
&= 3\sqrt{\cos^2 t \sin^2 t} \\
&= 3|\cos t \sin t|
\end{aligned}$$

$$\begin{aligned}
\vec{T} &= -\left(\frac{3\cos^2 t \sin t}{3|\cos t \sin t|}\right)\hat{i} + \left(\frac{3\sin^2 t \cos t}{3|\cos t \sin t|}\right)\hat{k} & \vec{T} &= \frac{\vec{v}}{|\vec{v}|} \\
&= -(\cos t)\hat{i} + (\sin t)\hat{k}
\end{aligned}$$

$$\begin{aligned}
L &= \int_0^{\pi/2} 3\cos t \sin t \, dt & L &= \int_a^b |\vec{v}(t)| \, dt \\
&= \frac{3}{2} \int_0^{\pi/2} \sin 2t \, dt & \sin 2t &= 2\cos t \sin t \\
&= \frac{3}{2} \left[-\frac{1}{2} \cos 2t \right]_0^{\pi/2} \\
&= -\frac{3}{4} (\cos \pi - \cos 0) \\
&= -\frac{3}{4} (-2) \\
&= \frac{3}{2}
\end{aligned}$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = (t \cos t)\hat{i} + (t \sin t)\hat{j} + \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\hat{k}; \quad 0 \leq t \leq \pi$$

Solution

$$\vec{v}(t) = (\cos t - t \sin t)\hat{i} + (\sin t + t \cos t)\hat{j} + \left(\sqrt{2} t^{1/2}\right)\hat{k} \quad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned}
|\vec{v}(t)| &= \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + 2t} \\
&= \sqrt{\cos^2 t - 2t \cos t + \sin^2 t + \sin^2 t + 2t \cos t + \cos^2 t + 2t} \\
&= \sqrt{2 + 2t}
\end{aligned}$$

$$\vec{T} = \frac{1}{\sqrt{2}} \left(\frac{\cos t - t \sin t}{\sqrt{1+t}} \right) \hat{i} + \frac{1}{\sqrt{2}} \left(\frac{\sin t + t \cos t}{\sqrt{1+t}} \right) \hat{j} + \left(\sqrt{\frac{t}{1+t}} t^{1/2} \right) \hat{k} \quad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\begin{aligned}
 L &= \int_0^{\pi} \sqrt{2} \sqrt{1+t} \, dt & L &= \int_a^b |\vec{v}(t)| \, dt \\
 &= \sqrt{2} \int_0^{\pi} \sqrt{1+t} \, d(1+t) \\
 &= \sqrt{2} \int_0^{\pi} \sqrt{1+t} \, d(1+t) \\
 &= \sqrt{2} (\sqrt{1+\pi} - 1) \quad |
 \end{aligned}$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\mathbf{r}(t) = (t \sin t + \cos t) \mathbf{i} + (t \cos t - \sin t) \mathbf{j}; \quad \sqrt{2} \leq t \leq 2$$

Solution

$$\begin{aligned}
 \vec{v}(t) &= (\sin t + t \cos t - \sin t) \hat{i} + (\cos t - t \sin t - \cos t) \hat{j} & \vec{v}(t) &= \frac{d\vec{r}}{dt} \\
 &= (t \cos t) \hat{i} - (t \sin t) \hat{j}
 \end{aligned}$$

$$\begin{aligned}
 |\vec{v}| &= \sqrt{t^2 \cos^2 t + t^2 \sin^2 t} \\
 &= \sqrt{t^2 (\cos^2 t + \sin^2 t)} \\
 &= \sqrt{t^2} \\
 &= |t| \\
 &= t \quad \text{because } \sqrt{2} \leq t \leq 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{T} &= \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{t \cos t}{t} \right) \mathbf{i} - \left(\frac{t \sin t}{t} \right) \mathbf{j} \\
 &= (\cos t) \mathbf{i} - (\sin t) \mathbf{j} \quad |
 \end{aligned}$$

$$\begin{aligned}
 L &= \int_{\sqrt{2}}^2 t \, dt & L &= \int_a^b |\vec{v}(t)| \, dt \\
 &= \frac{1}{2} t^2 \Big|_{\sqrt{2}}^2 \\
 &= \frac{1}{2} (4 - 2) \\
 &= 1 \quad |
 \end{aligned}$$

Exercise

Find the point on the curve $\vec{r}(t) = (5 \sin t)\hat{i} + (5 \cos t)\hat{j} + 12t\hat{k}$ at a distance 26π units along the curve from the point $(0, 5, 0)$ in the direction of increasing arc length.

Solution

$$\vec{v} = (5 \cos t)\hat{i} - (5 \sin t)\hat{j} + 12\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{25 \cos^2 t + 25 \sin^2 t + 144} \\ &= \sqrt{25(\cos^2 t + \sin^2 t) + 144} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

$$s = \int_0^{t_0} 13 \, dt \qquad s = \int_0^{t_0} |\vec{v}(t)| \, dt$$

$$= 13t_0$$

$$s = 26\pi = 13t_0$$

$$\underline{t_0 = 2\pi}$$

$$\begin{aligned} \vec{r}(t = 2\pi) &= (5 \sin 2\pi)\hat{i} + (5 \cos 2\pi)\hat{j} + 12(2\pi)\hat{k} \\ &= 0\hat{i} + 5\hat{j} + 24\pi\hat{k} \end{aligned}$$

The point is: $\underline{(0, 5, 24\pi)}$

Exercise

Find the arc length parameter along the curve from the point where $t = 0$. Also, find the length of the indicated portion of the curve. $\vec{r}(t) = (4 \cos t)\hat{i} + (4 \sin t)\hat{j} + 3t\hat{k}; \quad 0 \leq t \leq \frac{\pi}{2}$

Solution

$$\vec{v} = -(4 \sin t)\hat{i} + (4 \cos t)\hat{j} + 3\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{16 \sin^2 t + 16 \cos^2 t + 9} \\ &= \sqrt{16 + 9} \\ &= 5 \end{aligned}$$

$$s = \int_0^t 5 \, dt$$

$$= 5t \Big|$$

$$s\left(\frac{\pi}{2}\right) = \frac{5\pi}{2}$$

$$s = \int_0^t |\vec{v}(\tau)| \, d\tau$$

Exercise

Find the arc length parameter along the curve from the point where $t = 0$. Also, find the length of the indicated portion of the curve. $\vec{r}(t) = (e^t \cos t)\hat{i} + (e^t \sin t)\hat{j} + e^t \hat{k}; \quad -\ln 4 \leq t \leq 0$

Solution

$$\vec{v}(t) = (e^t \cos t - e^t \sin t)\hat{i} + (e^t \sin t + e^t \cos t)\hat{j} + e^t \hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + e^{2t}}$$

$$= \sqrt{e^{2t} \cos^2 t - e^{2t} \cos t \sin t + e^{2t} \sin^2 t + e^{2t} \sin^2 t + e^{2t} \cos^2 t + e^{2t} \cos t \sin t + e^{2t} \cos^2 t + e^{2t}}$$

$$= \sqrt{2e^{2t} (\cos^2 t + \sin^2 t) + e^{2t}}$$

$$= \sqrt{3e^{2t}}$$

$$= \sqrt{3} e^t$$

$$s(t) = \int_0^t \sqrt{3} e^\tau \, d\tau \qquad s = \int_0^t |\vec{v}(\tau)| \, d\tau$$

$$= \sqrt{3} e^\tau \Big|_0^t$$

$$= \sqrt{3} (e^t - 1)$$

$$s(0) = \sqrt{3} (e^0 - 1) = 0$$

$$s(-\ln 4) = \sqrt{3} (e^{-\ln 4} - 1)$$

$$= \sqrt{3} \left(e^{\ln \frac{1}{4}} - 1 \right)$$

$$= \sqrt{3} \left(\frac{1}{4} - 1 \right)$$

$$\begin{aligned}
&= \sqrt{3} \left(-\frac{3}{4} \right) \\
&= \frac{3\sqrt{3}}{4} \\
s(-\ln 4) - s(0) &= \underline{\underline{\frac{3\sqrt{3}}{4}}}
\end{aligned}$$

Exercise

Find the arc length parameter along the curve from the point where $t = 0$. Also, find the length of the indicated portion of the curve. $\vec{r}(t) = (1 + 2t)\hat{i} + (1 + 3t)\hat{j} + (6 - 6t)\hat{k}; \quad -1 \leq t \leq 0$

Solution

$$\vec{v} = 2\hat{i} + 3\hat{j} - 6\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{4 + 9 + 36} = 7$$

$$\begin{aligned}
s(t) &= \int_0^t 7 \, d\tau \\
&= 7t
\end{aligned}
\qquad
s = \int_0^t |\vec{v}(\tau)| \, d\tau$$

$$\text{Length: } s(0) - s(-1) = 0 - (-7) = \underline{\underline{7}}$$

Exercise

Find the arc length of the curve $\vec{r}(t) = \langle 2t^{9/2}, t^3 \rangle$ for $0 \leq t \leq 2$

Solution

$$\vec{v}(t) = \langle 9t^{7/2}, 3t^2 \rangle \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$L = \int_0^2 \sqrt{(9t^{7/2})^2 + (3t^2)^2} \, dt \qquad L = \int_0^2 \left| \frac{d\vec{r}}{dt} \right| \, dt$$

$$= \int_0^2 \sqrt{81t^7 + 9t^4} \, dt$$

$$= \int_0^2 3t^2 \sqrt{9t^3 + 1} \, dt$$

$$= \frac{1}{9} \int_0^2 (9t^3 + 1)^{1/2} d(9t^3 + 1)$$

$$\begin{aligned}
&= \frac{2}{27} \left(9t^3 + 1 \right)^{3/2} \Big|_0^2 \\
&= \frac{2}{27} (73\sqrt{73} - 1) \quad \text{unit}
\end{aligned}$$

Exercise

Find the arc length of the curve $\vec{r}(t) = \left\langle t^2, \frac{4\sqrt{2}}{3}t^{3/2}, 2t \right\rangle$ for $1 \leq t \leq 3$

Solution

$$\begin{aligned}
\frac{d\vec{r}}{dt} &= \left\langle 2t, 2\sqrt{2} t^{1/2}, 2 \right\rangle \\
L &= \int_1^3 \sqrt{4t^2 + 8t + 4} \, dt & L &= \int_a^b |\vec{r}'(t)| \, dt \\
&= 2 \int_1^3 \sqrt{(t+1)^2} \, dt \\
&= 2 \int_1^3 (t+1) \, dt \\
&= 2 \left(\frac{1}{2}t^2 + t \right) \Big|_1^3 \\
&= 2 \left(\frac{9}{2} + 1 - \frac{1}{2} - 1 \right) \\
&= 12 \quad \text{unit}
\end{aligned}$$

Exercise

Find the arc length of the curve $\vec{r}(t) = \left\langle t, \ln \sec t, \ln(\sec t + \tan t) \right\rangle$ for $0 \leq t \leq \frac{\pi}{4}$

Solution

$$\begin{aligned}
\frac{d\vec{r}}{dt} &= \left\langle 1, \frac{\tan t \sec t}{\sec t}, \frac{\tan t \sec t + \sec^2 t}{\sec t + \tan t} \right\rangle \\
&= \left\langle 1, \tan t, \sec t \right\rangle \\
L &= \int_0^{\pi/4} \sqrt{1 + \tan^2 t + \sec^2 t} \, dt & L &= \int_a^b |\vec{r}'(t)| \, dt
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{\pi/4} \sqrt{2 \sec^2 t} \, dt \\
&= \sqrt{2} \int_0^{\pi/4} \sec t \, dt \\
&= \sqrt{2} \ln(\sec t + \tan t) \Big|_0^{\pi/4} \\
&= \sqrt{2} \ln(\sqrt{2} + 1)
\end{aligned}$$

Exercise

Find the lengths of the curves $\vec{r}(t) = (2 \cos t) \hat{i} + (2 \sin t) \hat{j} + t^2 \hat{k}; \quad 0 \leq t \leq \frac{\pi}{4}$

Solution

$$\frac{d\vec{r}}{dt} = (-2 \sin t) \hat{i} + (2 \cos t) \hat{j} + 2t \hat{k}$$

$$\left| \frac{d\vec{r}}{dt} \right| = \sqrt{4 \sin^2 t + 4 \cos^2 t + 4t^2}$$

$$= \sqrt{4 + 4t^2}$$

$$= 2\sqrt{1 + t^2}$$

$$L = 2 \int_0^{\pi/4} \sqrt{1 + t^2} \, dt$$

$$L = \int_a^b |\vec{r}'(t)| \, dt$$

$$= t\sqrt{1 + t^2} + \ln\left(t + \sqrt{1 + t^2}\right) \Big|_0^{\pi/4}$$

$$= \frac{\pi}{4} \sqrt{1 + \frac{\pi^2}{16}} + \ln\left(\frac{\pi}{4} + \sqrt{1 + \frac{\pi^2}{16}}\right) - 0 - \ln 1$$

$$= \frac{\pi}{4} \sqrt{1 + \frac{\pi^2}{16}} + \ln\left(\frac{\pi}{4} + \sqrt{1 + \frac{\pi^2}{16}}\right)$$

Exercise

Find the lengths of the curves $\vec{r}(t) = (3 \cos t) \hat{i} + (3 \sin t) \hat{j} + 2t^{3/2} \hat{k}; \quad 0 \leq t \leq 3$

Solution

$$\frac{d\vec{r}}{dt} = (-3 \sin t) \hat{i} + (3 \cos t) \hat{j} + 3t^{1/2} \hat{k}$$

$$\begin{aligned} \left| \frac{d\vec{r}}{dt} \right| &= \sqrt{9 \sin^2 t + 9 \cos^2 t + 9t} \\ &= 3\sqrt{1+t^2} \end{aligned}$$

$$L = 3 \int_0^3 \sqrt{1+t} \, dt$$

$$= 3 \int_0^3 (1+t)^{1/2} \, d(1+t)$$

$$= 2(1+t)^{3/2} \Big|_0^3$$

$$= 2(4^{3/2} - 1)$$

$$= 2(8 - 1)$$

$$= 14$$

$$L = \int_a^b |\vec{r}'(t)| \, dt$$

Exercise

The acceleration of a wayward firework is given by $\vec{a}(t) = \sqrt{2}\hat{j} + 2t\hat{k}$ for $0 \leq t \leq 3$. Suppose the initial velocity of the firework is $\vec{v}(0) = 1$.

a) Find the velocity of the firework, for $0 \leq t \leq 3$.

b) Find the length of the trajectory of the firework over the interval $0 \leq t \leq 3$

Solution

$$a) \quad \vec{v} = \int \langle 0, \sqrt{2}, 2t \rangle \, dt$$

$$= \langle 0, \sqrt{2}t, t^2 \rangle + \vec{C}$$

$$\vec{v}(0) = 1 = \langle 1, 0, 0 \rangle$$

$$\langle 1, 0, 0 \rangle = \langle 0, 0, 0 \rangle + \vec{C}$$

$$\vec{C} = \langle 1, 0, 0 \rangle$$

$$\vec{v}(t) = \langle 0, \sqrt{2}t, t^2 \rangle + \langle 1, 0, 0 \rangle$$

$$= \langle 1, \sqrt{2}t, t^2 \rangle$$

$$\begin{aligned}
 b) \quad L &= \int_0^3 \sqrt{1+2t^2+t^4} \, dt \\
 &= \int_0^3 \sqrt{(1+t^2)^2} \, dt \\
 &= \int_0^3 (1+t^2) \, dt \\
 &= t + \frac{1}{3}t^3 \Big|_0^3 \\
 &= 3+9 \\
 &= 12 \quad \text{unit}
 \end{aligned}$$

$$L = \int_a^b |\vec{r}'(t)| \, dt$$

Exercise

If a string wound around a fixed circle in unwound while held taut in the plane of the circle, its end P traces an involute of the circle. The circle in question is the circle $x^2 + y^2 = 1$ and the tracing point starts at $(1, 0)$. The unwound portion of the string is tangent to the circle at Q , and t is the radian measure of the angle from the position x -axis to segment OQ . Derive the parametric equations

$$x = \cos t + t \sin t, \quad y = \sin t - t \cos t, \quad t > 0 \quad \text{of the point } P(x, y) \text{ for the involute.}$$

Solution

$$\angle PQB = \angle QOB = t$$

$$PQ = \text{arc}(AQ) = t$$

PQ = Length of the unwound string

$$\text{arc}(AQ)$$

$$\Delta PDQ : \begin{cases} \sin t = \frac{DP}{PQ} = \frac{DP}{t} \rightarrow \underline{DP = t \sin t} \\ \cos t = \frac{QD}{PQ} = \frac{QD}{t} \rightarrow \underline{QD = t \cos t} \end{cases}$$

$$\begin{aligned}
 x &= OB + BC \\
 &= OB + DP \\
 &= \cos t + t \sin t
 \end{aligned}$$

$$\begin{aligned}
 y &= PC \\
 &= QB - QD \\
 &= \sin t - t \cos t
 \end{aligned}$$

