Section 3.3 – Algebra of Matrices

$$A = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Equality of Matrices

Definition of Equality of Matrices

Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal if and only if they have the same order (size) $m \text{ x } n$ and if each pair corresponding elements is equal

$a_{ij} = b_{ij}$ for $i = 1, 2, \ldots, m$ ***and*** $j = 1, 2, \ldots, n$

Example

Find the values of the variables for which each statement is true, if possible.

a) $\begin{bmatrix} 2 & 1 \\ p & q \end{bmatrix} = \begin{bmatrix} x & y \\ -1 & 0 \end{bmatrix}$

$x = 2, \ y = 1, \ \ p = -1, \ \ q = 0$

b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$

can't be true

c) $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

$\Rightarrow \begin{bmatrix} w = 9 & x = 17 \\ 8 = y & -12 = z \end{bmatrix}$

Addition and *Subtraction* of Matrices

Definition

If $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ are $m\times n$ matrices, their sum $A+B$, is the $m\times n$ matrix obtained by adding the corresponding entries; that is

$$\left[a_{ij}\right]+\left[b_{ij}\right]=\left[a_{ij}+b_{ij}\right]$$

Matrices can be added if their shapes are the same, meaning have the same ***order***.

$$\begin{bmatrix}1 & 2\\ 3 & 4\\ 0 & 0\end{bmatrix}+\begin{bmatrix}2 & 2\\ 4 & 4\\ 9 & 9\end{bmatrix}=\begin{bmatrix}1+2 & 2+2\\ 3+4 & 4+4\\ 0+9 & 0+9\end{bmatrix}$$

$$=\begin{bmatrix}3 & 4\\ 7 & 8\\ 9 & 9\end{bmatrix}$$

Scalar Multiplication Matrices

Definition

If k is a scalar and $A=\left[a_{ij}\right]$ is an $m\times n$ matrices, then scalar product $\boldsymbol{kA}$ is the $m\times n$ matrix obtained by multiplying each entry of A by k; that is

$$k\left[a_{ij}\right]=\left[ka_{ij}\right]$$

$$kA=k\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$$

$$=\begin{bmatrix}ka_{11} & ka_{12}\\ ka_{21} & ka_{22}\end{bmatrix}$$

Example

$$2\begin{bmatrix}1 & 2\\ 3 & 4\\ 0 & 0\end{bmatrix}=\begin{bmatrix}(2)1 & (2)2\\ (2)3 & (2)4\\ (2)0 & (2)0\end{bmatrix}$$

$$=\begin{bmatrix}2 & 4\\ 6 & 8\\ 0 & 0\end{bmatrix}$$

Definition

If $A_1, A_2, \ldots, A_n$ are matrices of the same size, and if $c_1, c_2, \ldots, c_n$ are scalars, then expression of the form

$$c_1 A_1 + c_2 A_2 + \ldots + c_n A_n$$

Is called a ***linear combination*** of $A_1, A_2, \ldots, A_n$ with *coefficients* $c_1, c_2, \ldots, c_n$.

Matrix Multiplication

Product of Two Matrices

Let ***A*** be an *m x n* matrix and let ***B*** be an *n x k* matrix. To find the element in the i^{th} row and j^{th} column of the product matrix ***AB***, multiply each element in the i^{th} row of ***A*** by the corresponding element in the j^{th} column of ***B***, and then add these products. The product matrix ***AB*** is an *m x k* matrix.

*Matrix **A*** *Matrix B*

m x n *n x k*

Inner *must be equal*

Outer: Order of *AB* is *m x k*

- ✓ To multiply ***AB*** or dot product, if ***A*** has ***n*** columns, ***B*** must have ***n*** rows.
- ✓ Squares matrices can be multiplied if and only if (***iff***) they have the same size.
- ✓ The entry in row *i* and column *j* of AB is $\boxed{(row\ i\ of\ A).(col\ j\ of\ B)}$

  The result: $\sum a_{ik} b_{kj}$

$$\begin{bmatrix} * & * & & & \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{i5} \\ * & & & & \\ * & & & & \end{bmatrix} \begin{bmatrix} * & * & b_{1j} & * & * & * \\ & & b_{2j} & & & \\ & & \vdots & & & \\ & & \vdots & & & \\ & & b_{5j} & & & \end{bmatrix} = \begin{bmatrix} & & * & & & \\ * & * & (AB)_{ij} & * & * & * \\ & & * & & & \\ & & * & & & \end{bmatrix}$$

$4 \ \ by \ \ 5$ $5 \ \ by \ \ 6$ $4 \ \ by \ \ 6$

$$\boldsymbol{AB} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$2x2 \qquad 2x2 \quad \to \quad 2x2$$

$$a_{11} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & \_ \\ \_ & \_ \end{bmatrix}$$

$$a_{12} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & af+bh \\ \_ & \_ \end{bmatrix}$$

$$a_{21} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ ce+dg & \_ \end{bmatrix}$$

$$a_{22} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ \_ & cf+dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Example

Find: $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}$

<u>Solution</u>

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1(5)+1(1) & 1(6)+1(0) \\ 2(5)-1(1) & 2(6)-1(0) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 \\ 9 & 12 \end{bmatrix}$$

Special Case

When A is a square matrix, then

$$A \quad times \quad A^2 = A^2 \quad times \quad A \ \underline{= A^3}$$

$$A^p = AA \cdots A \qquad (p \ factors)$$

$$\left(A^p\right)\left(A^q\right) = A^{p+q}$$

$$\left(A^p\right)^q = A^{pq}$$

Block Multiplication

If the cuts between columns of ***A*** match the cuts between rows of ***B***, then the block multiplication of ***AB*** allowed.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \left[\begin{array}{c|c} a_{11}b_{11} + a_{12}b_{21} & a_{12}b_{21} + a_{12}b_{22} \\ \hline a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{21} + a_{22}b_{22} \end{array}\right]$$

Important special case

$$\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix}$$

Matrix Form of the Equations

The coefficient matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}$

The equivalent matrix equation is in the form $AX = b$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Multiplication by ***rows*** $\quad AX = \begin{bmatrix} (row\ 1).X \\ (row\ 2).X \\ (row\ 3).X \end{bmatrix}$

Multiplication by ***columns*** $\quad AX = x\ (column\ 1) + y\ (column\ 2) + z\ (column\ 3)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Identity Matrix

The identity matrix is given by the form: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \boxed{Ix = x}$

Properties of Matrix

Addition and Scalar Multiplication

$A+B=B+A$	*Commutative Property of Addition*
$A+(B+C)=(A+B)+C$	*Associative Property of Addition*
$(kl)A=k(lA)$	*Associative Property of Scalar Multiplication*
$k(A+B)=kA+kB$	*Distributive Property*
$k(A-B)=kA-kB$	*Distributive Property*
$(k+l)A=kA+lA$	*Distributive Property*
$(k-l)A=kA-lA$	*Distributive Property*
$A+0=0+A=A$	*Additive Identity Property*
$A+(-A)=(-A)+A=0$	*Additive Inverse Property*
$k(AB)=kA(B)=A(kB)$	

Multiplication

$AB \neq BA$	*Commutative "**law**" is usually broken*
$A(BC)=(AB)C$	*Associative Property of Multiplication* (***Parentheses not needed***)
$A(B+C)=AB+AC$	*Distributive Property*
$(B+C)A=BA+CA$	*Distributive Property*
$A(B-C)=AB-AC$	*Distributive Property*
$(B-C)A=BA-CA$	*Distributive Property*

Exercises Section 3.3 – Algebra of Matrices

1. For the matrices: $A=\begin{bmatrix} p & 0 \\ q & r \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, when does $AB=BA$

2. A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

a) AB *b*) BA *c*) ABD *d*) DBA

e) ABC *f*) $ABCD$ *g*) $A(B+C)$

3. What rows or columns or matrices do you multiply to find.

a) The third column of AB?

b) The second column of AB?

c) The first row of AB?

d) The second row of AB?

e) The entry in row 3, column 4 of AB?

f) The entry in row 2, column 3 of AB?

4. Add AB to AC and compare with $A(B+C)$:

$$A=\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \quad and \quad B=\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad and \quad C=\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

5. True or False

a) If A^2 is defined then A is necessarily square.

b) If AB and BA are defined then A and B are square.

c) If AB and BA are defined then AB and BA are square.

d) If $AB=B$, then $A=I$

6. *a*) Find a nonzero matrix A such that $A^2=0$

b) Find a matrix that has $A^2 \neq 0$ but $A^3=0$

7. Suppose you solve $Ax=b$ for three special right sides b:

$$Ax_1=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad Ax_2=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad Ax_3=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If the three solutions x_1, x_2, x_3 are the columns of a matrix X, what is A times X?

8. Show that $(A+B)^2$ is different from $A^2+2AB+B^2$, when

$$A=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix} \quad and \quad B=\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}$$

Write down the correct rule for $(A+B)(A+B)=A^2+$ __________ $+B^2$

9. Find the product of the 2 matrices by rows or by columns:

a) $\begin{bmatrix}2 & 3\\ 5 & 1\end{bmatrix}\begin{bmatrix}4\\ 2\end{bmatrix}$

b) $\begin{bmatrix}1 & 2 & 4\\ 2 & 0 & 1\end{bmatrix}\begin{bmatrix}3\\ 1\\ 1\end{bmatrix}$

c) $\begin{bmatrix}3 & 6\\ 6 & 12\end{bmatrix}\begin{bmatrix}2\\ -1\end{bmatrix}$

d) $\begin{bmatrix}1 & 2 & 4\\ -2 & 3 & 1\\ -4 & 1 & 2\end{bmatrix}\begin{bmatrix}2\\ 2\\ 3\end{bmatrix}$

10. Given $A=\begin{bmatrix}4 & 1 & 3\\ 3 & -1 & -2\\ 0 & 0 & 4\end{bmatrix}$ $\quad B=\begin{bmatrix}-3 & -2 & -3\\ -1 & 0 & 0\\ 8 & -2 & -4\end{bmatrix}$ Find $A+B,\quad 2A,\quad and \quad -B$

11. Given $A=\begin{bmatrix}3 & 2 & -3\\ 0 & 1 & 0\end{bmatrix}$ $\quad B=\begin{bmatrix}3 & -4\\ 0 & 1\\ 1 & 0\end{bmatrix}$ Find $AB \quad and \quad BA \quad if\ possible$

12. Given $A=\begin{bmatrix}5 & 3\\ -1 & 0\end{bmatrix}$ $\quad B=\begin{bmatrix}4 & -2\\ -2 & 0\\ 9 & 1\end{bmatrix}$ Find $AB \quad and \quad BA \quad if\ possible$

13. Given $A=\begin{bmatrix}3 & 0\\ -1 & 2\\ 1 & 1\end{bmatrix}$ $\quad B=\begin{bmatrix}4 & -1\\ 0 & 2\end{bmatrix}$ Find $AB \quad and \quad BA \quad if\ possible$

14. Consider the matrices

$$A=\begin{bmatrix}3 & 0\\ -1 & 2\\ 1 & 1\end{bmatrix} \quad B=\begin{bmatrix}4 & -1\\ 0 & 2\end{bmatrix} \quad C=\begin{bmatrix}1 & 4 & 2\\ 3 & 1 & 5\end{bmatrix} \quad D=\begin{bmatrix}1 & 5 & 2\\ -1 & 0 & 1\\ 3 & 2 & 4\end{bmatrix} \quad E=\begin{bmatrix}6 & 1 & 3\\ -1 & 1 & 2\\ 4 & 1 & 3\end{bmatrix}$$

Compute the following (where possible):

a) $D+E$ ***b***) $D-E$ ***c***) $5A$ ***d***) $-7C$ ***e***) $2B-C$ ***g***) $-3(D+2E)$