Exercise

Sketch the following vectors with initial points located at the origin

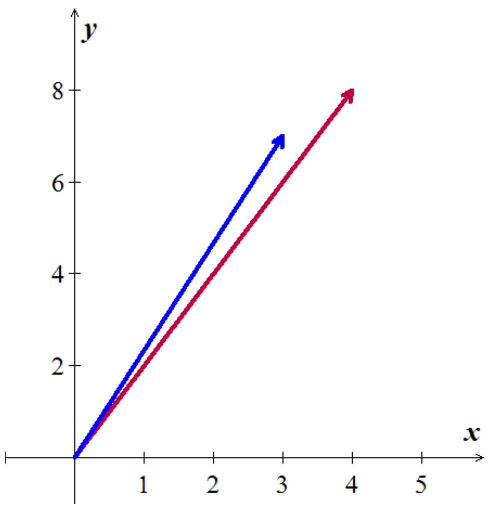
a) $P_1(4,\ 8)\quad P_2(3,\ 7)$

b) $P_1(-1,\ 0,\ 2)\quad P_2(0,\ -1,\ 0)$

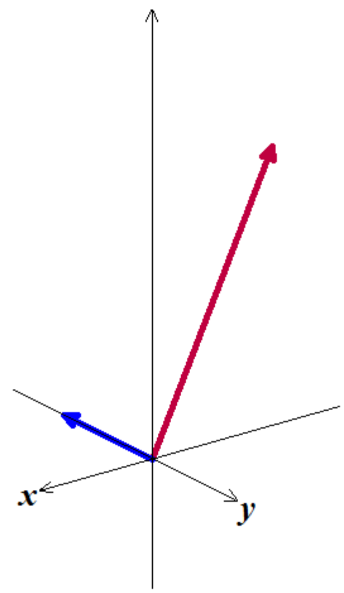
c) $P_1(3,\ -7,\ 2)\quad P_2(-2,\ 5,\ -4)$

Solution

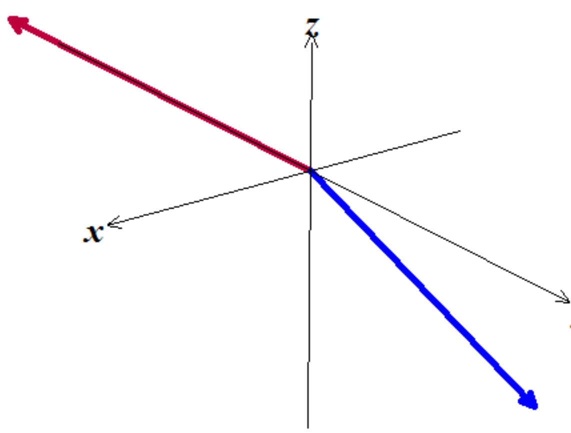
a)



b)



c)



Exercise

Find the components of the vector $\overrightarrow{P_1P_2}$

a) $P_1(3,\ 5)\quad P_2(2,\ 8)$

b) $P_1(5,\ -2,\ 1)\quad P_2(2,\ 4,\ 2)$

c) $P_1(0,\ 0,\ 0)\quad P_2(-1,\ 6,\ 1)$

Solution

a) $\overrightarrow{P_1P_2} = (2-3,\ 8-5)\ \underline{=(-1,\ 3)|}$

b) $\overrightarrow{P_1P_2} = (2-5,\ 4-(-2),\ 2-1)\ \underline{=(-3,\ 6,\ 1)|}$

c) $\overrightarrow{P_1P_2} = (-1-0,\ 6-0,\ 1-0)\ \underline{=(-1,\ 6,\ 1)|}$

Exercise

Find the terminal point of the vector that is equivalent to ***u*** = (1, 2) and whose initial point is $A(1,\ 1)$

Solution

The terminal point: $B(b_1,\ b_2)$

$(b_1-1,\ b_2-1) = (1,\ 2)$

$$\begin{cases} b_1 - 1 = 1 & \Rightarrow b_1 = 2 \\ b_2 - 1 = 2 & \Rightarrow b_2 = 3 \end{cases}$$

The terminal point: $\underline{B(2,\ 3)|}$

Exercise

Find the initial point of the vector that is equivalent to $\vec{u}$ = (1, 1, 3) and whose terminal point is $B(-1,\ -1,\ 2)$

Solution

The initial point: $A(x, y, z)$

$(-1-x,\ -1-y,\ 2-z) = (1,\ 1,\ 3)$

$$\begin{cases} -1 - x = 1 & \Rightarrow x = -2 \\ -1 - y = 1 & \Rightarrow y = -2 \\ 2 - z = 3 & \Rightarrow z = -1 \end{cases}$$

The initial point: $\underline{A(-2,\ -2,\ -1)|}$

Exercise

Find a nonzero vector ***u*** with initial point $P(-1, 3, -5)$ such that

a) ***u*** has the same direction as ***v*** = (6, 7, –3)

b) ***u*** is oppositely directed as ***v*** = (6, 7, –3)

Solution

a) ***u*** has the same direction as ***v*** ⇒ ***u*** = ***v*** = (6, 7, –3)

The initial point $P(-1, 3, -5)$ then the terminal point :

$$(-1+6,\ 3+7,\ -5-3) = \underline{(5,\ 10,\ -8)}$$

b) ***u*** is oppositely as ***v*** ⇒ ***u*** = –***v*** = (–6, –7, 3)

The initial point $P(-1, 3, -5)$ then the terminal point :

$$(-1-6,\ 3-7,\ -5+3) = \underline{(-7,\ -4,\ -2)}$$

Exercise

Let ***u*** = (–3, 1, 2), ***v*** = (4, 0, –8), and ***w*** = (6, –1, –4). Find the components

a) $\vec{v}-\vec{w}$

b) $6\vec{u}+2\vec{v}$

c) $5(\vec{v}-4\vec{u})$

d) $-3(\vec{v}-8\vec{w})$

e) $(2\vec{u}-7\vec{w})-(8\vec{v}+\vec{u})$

f) $-\vec{u}+(\vec{v}-4\vec{w})$

Solution

a)
$$\begin{aligned}\vec{v}-\vec{w} &= (4-6,\ 0-(-1),\ -8-(-4))\\ &= \underline{(-2,\ 1,\ -4)}\end{aligned}$$

b)
$$\begin{aligned}6\vec{u}+2\vec{v} &= (-18,6,12)+(8,0,-16)\\ &= \underline{(-10,\ 6,\ -4)}\end{aligned}$$

c)
$$\begin{aligned}5(\vec{v}-4\vec{u}) &= 5(4-(-12),\ 0-4,\ -8-8)\\ &= 5(16,\ -4,\ -16)\\ &= \underline{(80,\ -20,\ -80)}\end{aligned}$$

d)
$$\begin{aligned}-3(\vec{v}-8\vec{w}) &= -3(4-48,\ 0-(-8),\ -8-(-32))\\ &= -3(-44,\ 8,\ 24)\\ &= \underline{(32,\ -24,\ -72)}\end{aligned}$$

e)
$$\begin{aligned}(2\vec{u}-7\vec{w})-(8\vec{v}+\vec{u}) &= [(-6,2,\ 4)-(42,\ -7,\ -28)]-[(32,\ 0,\ -64)+(-3,\ 1,\ 2)]\\ &= (-48,\ 9,\ 32)-(29,\ 1,\ -62)\\ &= \underline{(-77,\ 8,\ 94)}\end{aligned}$$

f) $$-u+(v-4w) = (3,-1,-2)+\left[(4,0,-8)-(24,-4,-16)\right]$$
$$= (3,\ -1,\ -2)+(-20,\ 4,\ 8)$$
$$\boxed{= (-17,\ 3,\ 6)}$$

Exercise

Let $\boldsymbol{u}$ = (2, 1, 0, 1, –1) and $\boldsymbol{v}$ = (–2, 3, 1, 0, 2). Find scalars a and b so that $a\boldsymbol{u} + b\boldsymbol{v}$ = (–8, 8, 3, –1, 7)

Solution

$$a\vec{u}+b\vec{v} = a(2,\ 1,\ 0,\ 1,\ -1)+b(-2,\ 3,\ 1,\ 0,\ 2)$$
$$= (a-2b,\ a+3b,\ b,\ a,\ -a+2b)$$
$$\boxed{= (-8,\ 8,\ 3,\ -1,\ 7)}$$

$$\begin{cases} a-2b=-8 \\ a+3b=8 \\ b=3 \\ a \quad =-1 \\ -a+2b=7 \end{cases} \qquad \rightarrow \quad a=-1 \quad b=3 \ \textit{Unique solution}$$

Exercise

Find all scalars $c_1,\ c_2,\ and\ c_3$ such that $c_1(1,\ 2,\ 0)+c_2(2,\ 1,\ 1)+c_3(0,\ 3,\ 1)=(0,\ 0,\ 0)$

Solution

$$c_1(1,\ 2,\ 0)+c_2(2,\ 1,\ 1)+c_3(0,\ 3,\ 1) = (c_1+2c_2,\ 2c_1+c_2+3c_3,\ c_2+c_3)$$
$$= (0,\ 0,\ 0)$$

$$\begin{cases} c_1+2c_2 \quad = 0 \\ 2c_1+c_2+3c_3 = 0 \\ c_2+c_3 = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\boxed{c_1=c_2=c_3=0}$$

Exercise

Find the distance between the given points $[5 \quad 1 \quad 8 \quad -1 \quad 2 \quad 9]$, $[4 \quad 1 \quad 4 \quad 3 \quad 2 \quad 8]$

Solution

$$\begin{aligned} d &= \sqrt{(4-5)^2 + (1-1)^2 + (4-8)^2 + (3+1)^2 + (2-2)^2 + (8-9)^2} \\ &= \sqrt{1+0+16+16+0+1} \\ &= \boxed{\sqrt{34}} \end{aligned}$$

Exercise

Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operation on $\boldsymbol{u} = (u_1, u_2)$ $\quad \boldsymbol{v} = (v_1, v_2)$

$$\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1 + 1,\ u_2 + v_2 + 1) \qquad k\boldsymbol{u} = (ku_1,\ ku_2)$$

a) Compute $\boldsymbol{u} + \boldsymbol{v}$ and $k\boldsymbol{u}$ for $\boldsymbol{u} = (0, 4)$, $\boldsymbol{v} = (1, -3)$, and $k = 2$.

b) Show that $(0, 0) \neq \mathbf{0}$.

c) Show that $(-1, -1) = 0$.

d) Show that $\boldsymbol{u} + (-\boldsymbol{u}) = 0$ for $\boldsymbol{u} = (u_1, u_2)$

e) Find two vector space axioms that fail to hold.

Solution

a) $\boldsymbol{u} + \boldsymbol{v} = (0+1+1,\ 4-3+1) = \boxed{(2,\ 2)}$

$k\boldsymbol{u} = (ku_1,\ ku_2) = (2(0),\ 2(4)) = \boxed{(0,\ 8)}$

b)

$$\begin{aligned} (0,\ 0) + (u_1, u_2) &= (0 + u_1 + 1,\ 0 + u_2 + 1) \\ &= (u_1 + 1,\ u_2 + 1) \\ &\neq (u_1, u_2) \end{aligned}$$

Therefore (0, 0) is not the zero vector **0** required (by Axiom).

c)

$$\begin{aligned} (-1,\ -1) + (u_1, u_2) &= (-1 + u_1 + 1,\ -1 + u_2 + 1) \\ &= (u_1, u_2) \end{aligned}$$

$$\begin{aligned} (u_1, u_2) + (-1,\ -1) &= (u_1 - 1 + 1,\ u_2 - 1 + 1) \\ &= (u_1, u_2) \end{aligned}$$

Therefore $(-1, -1) = \mathbf{0}$ holds.

d) Let $\boldsymbol{u}=\left(u_1,\ u_2\right)$ $-\boldsymbol{u}=\left(-2-u_1,\ -2-u_2\right)$

$$\boldsymbol{u}+(-\boldsymbol{u})=\left(u_1+\left(-2-u_1\right)+1,\ u_2+\left(-2-u_2\right)+1\right)$$

$$=(-1,\ -1)$$

$$\underline{=\mathbf{0}}$$

$\boldsymbol{u}+(-\boldsymbol{u})=0$ holds

e) Axiom 7: $k(\boldsymbol{u}+\boldsymbol{v})\overset{?}{=}k\boldsymbol{u}+k\boldsymbol{v}$

$$k(\boldsymbol{u}+\boldsymbol{v})=k\left(u_1+v_1+1,\ u_2+v_2+1\right)=\left(ku_1+kv_1+k,\ ku_2+kv_2+k\right)$$

$$k\boldsymbol{u}+k\boldsymbol{v}=\left(ku_1,\ ku_2\right)+\left(kv_1,\ kv_2\right)=\left(ku_1+kv_1+1,\ ku_2+kv_2+1\right)$$

Therefore, $k(\boldsymbol{u}+\boldsymbol{v})\neq k\boldsymbol{u}+k\boldsymbol{v}$; Axiom 7 fails to hold

Axiom 8: $(k+m)\boldsymbol{u}\overset{?}{=}k\boldsymbol{u}+m\boldsymbol{u}$

$$(k+m)\boldsymbol{u}=\left((k+m)u_1,\ (k+m)u_2\right)=\left(ku_1+mu_1,\ ku_2+mu_2\right)$$

$$k\boldsymbol{u}+m\boldsymbol{u}=\left(ku_1,\ ku_2\right)+\left(mu_1,\ mu_2\right)=\left(ku_1+mu_1+1,\ ku_2+mu_2+1\right)$$

Therefore, $(k+m)\boldsymbol{u}\neq k\boldsymbol{u}+m\boldsymbol{u}$; Axiom 8 fails to hold

Exercise

Find $\vec{w}$ given that $10\vec{u}+3\vec{w}=4\vec{v}-2\vec{w}$, $\vec{u}=\begin{pmatrix}1\\-6\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-20\\5\end{pmatrix}$.

Solution

$$-10\vec{u}+10\vec{u}+3\vec{w}+2\vec{w}=-10\vec{u}+4\vec{v}-2\vec{w}+2\vec{w}$$

$$5\vec{w}=-10\vec{u}+4\vec{v}$$

$$\vec{w}=-2\vec{u}+\frac{4}{5}\vec{v}$$

$$=-2\begin{pmatrix}1\\-6\end{pmatrix}+\frac{4}{5}\begin{pmatrix}-20\\5\end{pmatrix}$$

$$=\begin{pmatrix}-2\\12\end{pmatrix}+\begin{pmatrix}-16\\4\end{pmatrix}$$

$$=\begin{pmatrix}-18\\16\end{pmatrix}$$

Exercise

Find $\vec{w}$ given that $\vec{u}+3\vec{v}-2\vec{w}=5\vec{u}+\vec{v}-4\vec{w}$, $\vec{u}=\begin{pmatrix}1\\-1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-2\\3\end{pmatrix}$

Solution

$$\vec{u}-\vec{u}+3\vec{v}-3\vec{v}-2\vec{w}+4\vec{w}=5\vec{u}-\vec{u}+\vec{v}-3\vec{v}-4\vec{w}+4\vec{w}$$

$$2\vec{w}=4\vec{u}-2\vec{v}$$

$$\vec{w}=2\vec{u}-\vec{v}$$

$$=2\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}-2\\3\end{pmatrix}$$

$$=\begin{pmatrix}2\\-2\end{pmatrix}+\begin{pmatrix}-2\\3\end{pmatrix}$$

$$\boxed{=\begin{pmatrix}0\\1\end{pmatrix}}$$

Exercise

Find $\vec{w}$ given that $2\vec{u}+\vec{v}-3\vec{w}=5\vec{u}+7\vec{v}+3\vec{w}$, $\vec{u}=\begin{pmatrix}1\\-1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-2\\3\end{pmatrix}$

Solution

$$2\vec{u}-2\vec{u}+\vec{v}-\vec{v}-3\vec{w}-3\vec{w}=5\vec{u}-2\vec{u}+7\vec{v}-\vec{v}+3\vec{w}-3\vec{w}$$

$$-6\vec{w}=3\vec{u}+6\vec{v}$$

$$\vec{w}=-\frac{1}{2}\vec{u}-\vec{v}$$

$$=-\frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}-\begin{pmatrix}-2\\3\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\end{pmatrix}-\begin{pmatrix}-2\\3\end{pmatrix}$$

$$\boxed{=\begin{pmatrix}\frac{3}{2}\\-\frac{5}{2}\end{pmatrix}}$$

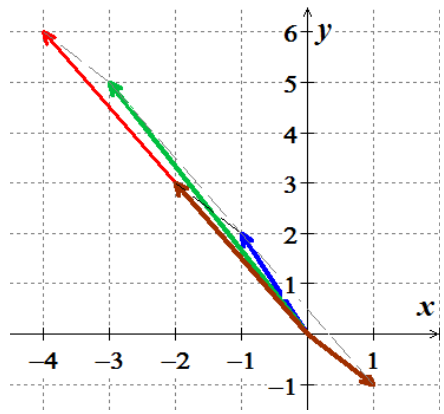
Exercise

Draw $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, and $\vec{u}+2\vec{v}$ $\quad\quad$ $\vec{u}=\begin{pmatrix}1\\-1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-2\\3\end{pmatrix}$

Solution

$$\vec{u}+\vec{v}=\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}-2\\3\end{pmatrix}$$
$$=\begin{pmatrix}-1\\2\end{pmatrix}$$

$$\vec{u}+2\vec{v}=\begin{pmatrix}1\\-1\end{pmatrix}+2\begin{pmatrix}-2\\3\end{pmatrix}$$
$$=\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}-4\\6\end{pmatrix}$$
$$=\begin{pmatrix}-3\\5\end{pmatrix}$$



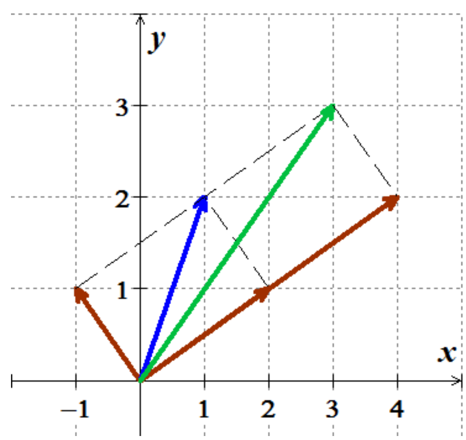
Exercise

Draw $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, and $\vec{u}+2\vec{v}$ $\quad\quad$ $\vec{u}=\begin{pmatrix}-1\\1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}2\\1\end{pmatrix}$

Solution

$$\vec{u}+\vec{v}=\begin{pmatrix}-1\\1\end{pmatrix}+\begin{pmatrix}2\\1\end{pmatrix}$$
$$=\begin{pmatrix}1\\2\end{pmatrix}$$

$$\vec{u}+2\vec{v}=\begin{pmatrix}-1\\1\end{pmatrix}+2\begin{pmatrix}2\\1\end{pmatrix}$$
$$=\begin{pmatrix}-1\\1\end{pmatrix}+\begin{pmatrix}4\\2\end{pmatrix}$$
$$=\begin{pmatrix}3\\3\end{pmatrix}$$



Exercise

Draw $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, and $\vec{u}+2\vec{v}$ $\qquad \vec{u}=\begin{pmatrix}1\\0\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}1\\2\end{pmatrix}$

Solution

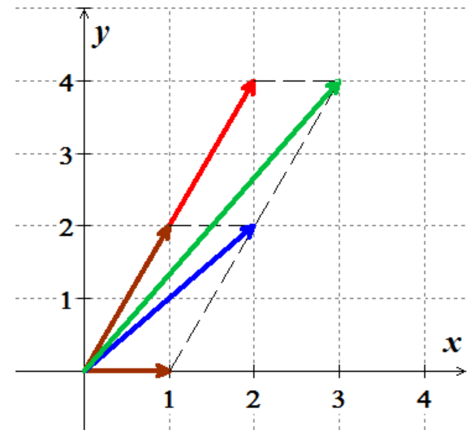
$$\vec{u}+\vec{v}=\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}1\\2\end{pmatrix}$$

$$=\begin{pmatrix}2\\2\end{pmatrix}$$

$$\vec{u}+2\vec{v}=\begin{pmatrix}1\\0\end{pmatrix}+2\begin{pmatrix}1\\2\end{pmatrix}$$

$$=\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}2\\4\end{pmatrix}$$

$$=\begin{pmatrix}3\\4\end{pmatrix}$$



Exercise

Draw $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, and $\vec{u}+2\vec{v}$ $\qquad \vec{u}=\begin{pmatrix}-1\\0\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}1\\-1\end{pmatrix}$

Solution

$$\vec{u}+\vec{v}=\begin{pmatrix}-1\\0\end{pmatrix}+\begin{pmatrix}1\\-1\end{pmatrix}$$

$$=\begin{pmatrix}0\\-1\end{pmatrix}$$

$$\vec{u}+2\vec{v}=\begin{pmatrix}-1\\0\end{pmatrix}+2\begin{pmatrix}1\\-1\end{pmatrix}$$

$$=\begin{pmatrix}-1\\0\end{pmatrix}+\begin{pmatrix}2\\-2\end{pmatrix}$$

$$=\begin{pmatrix}1\\-2\end{pmatrix}$$