Solution   *Section* 4.3 – Legendre's Equation

Exercise

Establish the recursion formula using the following two steps

a) Differentiate both sides of equation

$$g(t,\ x)=\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n \text{ with respect to } t \text{ to show that}$$

$$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\left(1-2xt+t^2\right)\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

b) Equate the coefficients of t^n in this equation to show that

$$P_1(x)=xP_0(x) \qquad \textit{and} \qquad (n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x) \quad \textit{for}\ \ n\ge 1$$

Solution

a) Let: $\left(1-2xt+t^2\right)^{-1/2}=\sum_{n=0}^{\infty}P_n(x)t^n$

Differentiate both sides with respect to t: $\left(\left(1-2xt+t^2\right)^{-1/2}\right)'=\left(\sum_{n=0}^{\infty}P_n(x)t^n\right)'$

$$-\frac{1}{2}(-2x+2t)\left(1-2xt+t^2\right)^{-3/2}=\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

$$(x-t)\left(1-2xt+t^2\right)^{-1/2}=\left(1-2xt+t^2\right)\sum_{n=1}^{\infty}nP_n(x)t^{n-1} \qquad \textit{Multiply both sides by: } 1-2xt+t^2$$

$$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\left(1-2xt+t^2\right)\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

b) $$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\sum_{n=0}^{\infty}xP_n(x)t^n-\underbrace{\sum_{n=0}^{\infty}P_n(x)t^{n+1}}_{n=n+1}$$

$$=\sum_{n=0}^{\infty}xP_n(x)t^n-\sum_{n=1}^{\infty}P_{n-1}(x)t^n$$

$$\left(1-2xt+t^2\right)\sum_{n=1}^{\infty} nP_n(x)t^{n-1} = \sum_{n=1}^{\infty} nP_n(x)t^{n-1} - \sum_{n=1}^{\infty} 2xnP_n(x)t^n + \sum_{n=1}^{\infty} nP_n(x)t^{n+1}$$

$$= \sum_{n=0}^{\infty}(n+1)P_{n+1}(x)t^n - \sum_{n=1}^{\infty} 2nxP_n(x)t^n + \sum_{n=2}^{\infty}(n-1)P_{n-1}(x)t^n$$

Thus,

$$\sum_{n=0}^{\infty} xP_n(x)t^n - \sum_{n=1}^{\infty} P_{n-1}(x)t^n = \sum_{n=0}^{\infty}(n+1)P_{n+1}(x)t^n - \sum_{n=1}^{\infty} 2nxP_n(x)t^n + \sum_{n=2}^{\infty}(n-1)P_{n-1}(x)t^n$$

Therefore;

$$0 = \left[xP_0(x) - P_1(x)\right]t^0 + \left[xP_1(x) - P_0(x) - 2P_2(x) + 2xP_1(x)\right]t^1$$

$$+\sum_{n=0}^{\infty}\left[xP_n(x) - P_{n-1}(x) - (n+1)P_{n+1}(x) + 2nxP_n(x) - (n-1)P_{n-1}(x)\right]t^n$$

$$0 = \left[xP_0(x) - P_1(x)\right]t^0 + \left[3xP_1(x) - P_0(x) - 2P_2(x)\right]t^1$$

$$+\sum_{n=0}^{\infty}\left[(2n+1)xP_n(x) - nP_{n-1}(x) - (n+1)P_{n+1}(x)\right]t^n$$

That implies:

$$xP_0(x) - P_1(x) = 0 \quad \Rightarrow \quad P_1(x) = xP_0(x)$$

$$3xP_1(x) - P_0(x) - 2P_2(x) = 0 \quad \Rightarrow \quad 2P_2(x) = P_0(x) - 3xP_1(x)$$

$$(2n+1)xP_n(x) - nP_{n-1}(x) - (n+1)P_{n+1}(x) = 0$$

$$\Rightarrow \quad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

If $n = 1$ then: $2P_2(x) = 3xP_1(x) - P_0(x)$ $\surd$

Exercise

Show that $P_{2n+1}(0)=0$ *and* $P_{2n}(0)=(-1)^n \dfrac{2n!}{2^{2n}(n!)^2}$

Solution

Given the formula:

$$nP_n(x)=(2n-1)xP_{n-1}(x)-(n-1)P_{n-2}(x) \qquad \textit{for } n\ge 2$$

By letting $x=0$, then the formula can be:

$$nP_n(0)=-(n-1)P_{n-2}(0)$$

Replacing n with $2n$, then

$$2nP_{2n}(0)=-(2n-1)P_{2n-2}(0)$$

$$P_{2n}(0)=\frac{1-2n}{2n}P_{2n-2}(0)$$

$$P_2(0)=\frac{1-2}{2}P_0(0)=-\frac{1}{2}P_0(0)$$

$$P_4(0)=\frac{1-4}{4}P_2(0)=\frac{1-2}{2}\cdot\frac{1-4}{4}P_0(0)=\frac{1\cdot 3}{2^2\cdot 1\cdot 2}P_0(0)$$

$$P_6(0)=\frac{1-6}{6}P_4(0)=\frac{1-2}{2}\cdot\frac{1-4}{4}\cdot\frac{1-6}{6}P_0(0)=-\frac{1\cdot 3\cdot 5}{2^3\cdot 1\cdot 2\cdot 3}P_0(0)$$

$$\vdots \qquad \vdots \quad \vdots$$

$$P_{2n}(0)=\frac{1-2}{2}\cdot\frac{1-4}{4}\cdot\frac{1-6}{6}\cdots\frac{1-2n}{2n}P_0(0)$$

$$=(-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\cdot 1\cdot 2\cdot 3\cdots n}P_0(0)$$

$$1\cdot 3\cdot 5\cdots(2n-1)=\frac{1\cdot 2\cdot 3\cdot 4\cdots(2n-1)(2n)}{2\cdot 4\cdot 6\cdots(2n)}$$

$$=\frac{(2n)!}{2^n n!}$$

$$=(-1)^n\frac{(2n)!}{2^n\cdot(n!)^2}P_0(0)$$

With $P_0(0)=1$

$$\boxed{P_{2n}(0)=(-1)^n\frac{(2n)!}{2^n\cdot(n!)^2}}$$

Exercise

Show that $P_n'(0) = (-1)^{n+1} P_n'(-1) = \frac{n(n+1)}{2}$

Hint: Use Legendre's equation $\left(1-x^2\right)y'' - 2xy' + n(n+1)y = 0$

Solution

Because $P_n(x)$ is a solution of Legendre's equation, then

$$\left(1-x^2\right)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$$

Let $x=1$, then

$$-2P_n'(1) + n(n+1)P_n(1) = 0$$

$$P_n'(1) = \frac{n(n+1)}{2}P_n(1)$$

Let $x=-1$, then

$$2P_n'(-1) + n(n+1)P_n(-1) = 0$$

$$P_n'(-1) = -\frac{n(n+1)}{2}P_n(-1)$$

However, $P_n(1) = P_n(-1) = 1$

$$\boxed{(-1)^{n+1}P_n'(-1) = \frac{n(n+1)}{2}}$$

Exercise

The differential equation $y'' + xy = 0$ is called ***Airy's equation***, and its solutions are called ***Airy functions***. Find the series for the solutions y_1 and y_2 where $y_1(0)=1$ and $y_1'(0)=0$, while $y_2(0)=0$ and $y_2'(0)=1$. What is the radius of convergence for these two series?

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y'' + xy = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}a_{n-1} x^{n} = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}a_{n-1} x^{n} = 0$$

$$2a_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_{n-1}\right]x^n = 0$$

$$2a_2 = 0 \quad or \quad (n+2)(n+1)a_{n+2} + a_{n-1} = 0$$

$$a_2 = 0 \quad or \quad a_{n+2} = \frac{-a_{n-1}}{(n+2)(n+1)} \qquad n \geq 1$$

$$a_3 = \frac{-a_0}{3\cdot 2} \qquad a_4 = -\frac{a_1}{4\cdot 3} \qquad a_5 = -\frac{a_2}{5\cdot 4} = 0$$

$$a_6 = -\frac{a_3}{6\cdot 5} = \frac{a_0}{6\cdot 5\cdot 3\cdot 2} \qquad a_7 = -\frac{a_4}{7\cdot 6} = \frac{a_1}{7\cdot 6\cdot 4\cdot 3} \qquad a_8 = -\frac{a_5}{8\cdot 7} = 0$$

$$a_9 = -\frac{a_6}{9\cdot 8} = -\frac{a_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2} \qquad a_{10} = -\frac{a_7}{10\cdot 9} = -\frac{a_1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3} \qquad a_{11} = 0$$

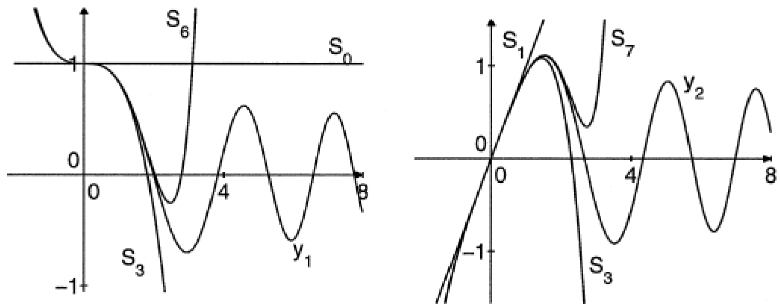
$$\vdots \quad \vdots \qquad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \qquad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots$$

$$a_{3n} = \frac{(-1)^n a_0}{(2\cdot 3)\cdot(5\cdot 6)\cdots(3n-1)(3n)} \qquad a_{3n+1} = \frac{(-1)^n a_1}{(3\cdot 4)\cdot(6\cdot 7)\cdots(3n)(3n+1)} \qquad a_{3n+2} = 0$$

$$y(x) = a_0\left[1 - \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 - \cdots\right] + a_1\left[x - \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}x^7 - \cdots\right]$$

$$y_1(x) = 1 - \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 - \cdots = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^{3n}}{(2\cdot 3)\cdot(5\cdot 6)\cdots(3n-1)(3n)}$$

$$y_2(x) = x - \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}x^7 - \cdots = x + \sum_{n=1}^{\infty}\frac{(-1)^n x^{3n+1}}{(3\cdot 4)\cdot(6\cdot 7)\cdots(3n)(3n+1)}$$



Exercise

The Hermite equation of order α is $\quad y'' - 2xy' + 2\alpha y = 0$

a) Find the general solution is $y(x) = a_0 y_1(x) + a_1 y_2(x)$

Show that $y_1(x)$ is a polynomial if α is an even integer, whereas $y_2(x)$ is a polynomial if α is an odd integer.

b) When $\alpha = n$, use $y_1(x)$ to find polynomial solutions for $n = 0$, $n = 2$, and $n = 4$, then use $y_2(x)$ to find polynomial solutions for $n = 1$, $n = 3$, and $n = 5$.

c) The Hermite polynomial of degree n is denoted by $H_n(x)$. It is the nth-degree polynomial solution of Hermite's equation, multiplied by a suitable constant so that the coefficient of x^n is 2^n. Use part (b) to show the first six Hermite polynomials are

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

d) A general formula for the Hermite polynomials is

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

Verify that this formula does in fact give an nth-degree polynomial.

Solution

a) $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - 2xy' + 2\alpha y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 2x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2\alpha \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} 2\alpha a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2\alpha a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} - 2(n-\alpha) a_n \right] x^n = 0$$

$$(n+1)(n+2) a_{n+2} - 2(n-\alpha) a_n = 0$$

$$a_{n+2} = \frac{2(n-\alpha)}{(n+1)(n+2)} a_n$$

a_0

$$n = 0 \ \rightarrow \ a_2 = -\frac{2\alpha}{2} a_0$$

$$n = 2 \ \rightarrow \ a_4 = \frac{2(2-\alpha)}{3 \cdot 4} a_2 = -\frac{2^2 \alpha (2-\alpha)}{4!} a_0$$

$$n = 4 \ \rightarrow \ a_6 = \frac{2(4-\alpha)}{5 \cdot 6} a_4 = -\frac{2^3 \alpha (2-\alpha)(4-\alpha)}{6!} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x) = 1 - \frac{2\alpha}{2!} x^2 - \frac{2^2 (2-\alpha)}{4!} x^4 - \frac{2^3 \alpha (2-\alpha)(4-\alpha)}{6!} x^6 - \cdots$$

$$= 1 - \frac{2\alpha}{2!} x^2 + \frac{2^2 (\alpha-2)}{4!} x^4 - \frac{2^3 \alpha (\alpha-2)(\alpha-4)}{6!} x^6 + \cdots$$

a_1

$$n = 1 \ \rightarrow \ a_3 = \frac{2(1-\alpha)}{6} a_1 = \frac{2(1-\alpha)}{3!} a_1$$

$$n=3 \to a_5 = \frac{2(3-\alpha)}{4\cdot 5}a_3 = \frac{2^2(1-\alpha)(3-\alpha)}{5!}a_1$$

$$n=5 \to a_7 = \frac{2(3-\alpha)}{6\cdot 7}a_5 = \frac{2^3(1-\alpha)(3-\alpha)(5-\alpha)}{7!}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = x + \frac{2(1-\alpha)}{3!}x^3 + \frac{2^2(1-\alpha)(3-\alpha)}{5!}x^5 + \frac{2^3(1-\alpha)(3-\alpha)(5-\alpha)}{7!}x^7 + \cdots$$

$$= x - \frac{2(\alpha-1)}{3!}x^3 + \frac{2^2(\alpha-1)(\alpha-3)}{5!}x^5 - \frac{2^3(\alpha-1)(\alpha-3)(\alpha-5)}{7!}x^7 + \cdots$$

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

$$= a_0\left(1 - \frac{2\alpha}{2!}x^2 + \frac{2^2\alpha(\alpha-2)}{4!}x^4 - \frac{2^3\alpha(\alpha-2)(\alpha-4)}{6!}x^6 + \cdots\right)$$

$$+ a_1\left(x - \frac{2(\alpha-1)}{3!}x^3 + \frac{2^2(\alpha-1)(\alpha-3)}{5!}x^5 - \frac{2^3(\alpha-1)(\alpha-3)(\alpha-5)}{7!}x^7 + \cdots\right)$$

$$= a_0 + a_0\sum_{m=1}^{\infty}(-1)^m \frac{2^m\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!}x^{2m}$$

$$+ a_1 x + a_1\sum_{m=1}^{\infty}(-1)^m \frac{2^m(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!}x^{2m+1}$$

$$\boxed{y_1(x) = 1 + \sum_{m=1}^{\infty}(-1)^m \frac{2^m\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!}x^{2m}}$$

$$\boxed{y_2(x) = x + \sum_{m=1}^{\infty}(-1)^m \frac{2^m(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!}x^{2m+1}}$$

b) $n=\alpha=0 \quad \to \quad y_1(x) \ \boxed{=1}$

$n=\alpha=2 \quad \to \quad y_1(x) = 1-\alpha x^2 \ \boxed{=1-2x^2}$

$n=\alpha=4 \quad \to \quad y_1(x) = 1-\alpha x^2 + \frac{\alpha(\alpha-2)}{6}x^4 \ \boxed{=1-4x^2+\frac{4}{3}x^4}$

$n=\alpha=1 \quad \to \quad y_2(x) \ \boxed{=x}$

$n=\alpha=3 \quad \to \quad y_2(x) = x - \frac{2(\alpha-1)}{3!}x^3 \ \boxed{=x-\frac{2}{3}x^3}$

$n=\alpha=5 \quad \rightarrow \quad y_2(x)=x-\frac{2(\alpha-1)}{3!}x^3+\frac{2^2(\alpha-1)(\alpha-3)}{5!}x^5$

$$\boxed{y_2(x)=x-\frac{4}{3}x^3+\frac{4}{15}x^5}$$

c) $H_0(x)=2^0\cdot 1=1$

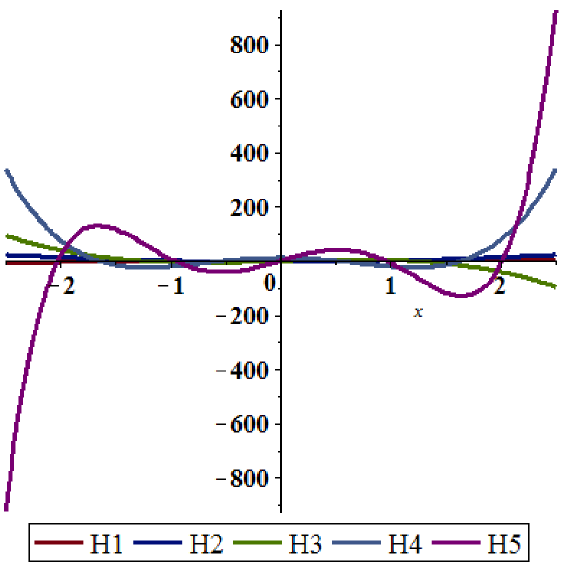
$H_1(x)=2^1\cdot x=2x$

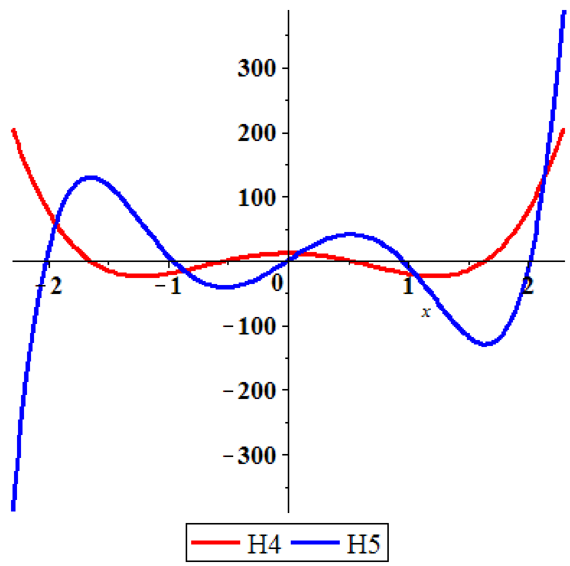
$H_2(x)=-2\left(1-2x^2\right)=4x^2-2$

$H_3(x)=-2^2\cdot 3\left(x-\frac{2}{3}x^3\right)=8x^3-12x$

$H_4(x)=2^2\cdot 3\left(1-4x^2+\frac{4}{3}x^4\right)=16x^4-48x^2+12$

$H_5(x)=2^3\cdot 3\cdot 5\left(\frac{4}{15}x^5-\frac{4}{3}x^3+x\right)=32x^5-160x^3+120x$





d) $\frac{d}{dx}\left(e^{-x^2}\right)=-2xe^{-x^2}$

$\frac{d^2}{dx^2}\left(e^{-x^2}\right)=-2\frac{d}{dx}\left(xe^{-x^2}\right)=-2\left(1-2x^2\right)e^{-x^2}$

$\frac{d^3}{dx^3}\left(e^{-x^2}\right)=2\frac{d}{dx}\left(\left(2x^2-1\right)e^{-x^2}\right)=2\left(4x-4x^3+2x\right)e^{-x^2}=\left(12x-8x^3\right)e^{-x^2}$

$\frac{d^4}{dx^4}\left(e^{-x^2}\right)=4\frac{d}{dx}\left(\left(3x-2x^3\right)e^{-x^2}\right)=4\left(3-6x^2-6x^2+4x^4\right)e^{-x^2}=\left(16x^4-48x^2+12\right)e^{-x^2}$

$H_1(x)=-e^{x^2}\frac{d}{dx}\left(e^{-x^2}\right)=2xe^{x^2}e^{-x^2}=2x$ √

$H_2(x)=e^{x^2}\frac{d^2}{dx^2}\left(e^{-x^2}\right)=-2e^{x^2}\left(1-2x^2\right)e^{-x^2}=4x^2-2$ √

$$H_3(x)=e^{x^2}\frac{d^3}{dx^3}\left(e^{-x^2}\right)=e^{x^2}\left(12x-8x^3\right)e^{-x^2}=12x-8x^3 \quad \checkmark$$

$$H_4(x)=e^{x^2}\frac{d^4}{dx^4}\left(e^{-x^2}\right)=e^{x^2}\left(16x^4-48x^2+12x\right)e^{-x^2}=16x^4-48x^2+12 \quad \checkmark$$

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

Exercise

Rodrigues's Formula is given by: $P_n(x)=\dfrac{1}{n!\,2^n}\dfrac{d^n}{dx^n}\left(x^2-1\right)^n$

For the nth-degree Legendre polynomial.

a) Show that $v=\left(x^2-1\right)^n$ satisfies the differential equation $\left(1-x^2\right)v'+2nxv=0$

Differentiate each side of this equation to obtain

$$\left(1-x^2\right)v''+2(n-1)xv'+2nv=0$$

b) Differentiate each side of the last equation n times in succession to obtain

$$\left(1-x^2\right)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}=0$$

which satisfies Legendre's equation of order n.

c) Show that the coefficient of x^n in u is $\dfrac{(2n)!}{n!}$; then state why this proves Rodrigues. Formula.

Note: That the coefficient of x^n in $P_n(x)$ is $\dfrac{(2n)!}{2^n(n!)^2}$

$$u=v^{(n)}=D^n\left(x^2-1\right)^n$$

Solution

a) $v=\left(x^2-1\right)^n \quad\rightarrow\quad v'=2nx\left(x^2-1\right)^{n-1}$

$$v'=2nx\left(x^2-1\right)^{n-1}$$

$$\begin{aligned}\left(1-x^2\right)v'+2nxv&=-2nx\left(x^2-1\right)\left(x^2-1\right)^{n-1}+2nx\left(x^2-1\right)^n\\&=-2nx\left(x^2-1\right)^n+2nx\left(x^2-1\right)^n\\&=0\end{aligned}$$

$$\frac{d}{dx}\left(\left(1-x^2\right)v'+2nxv\right)=0$$

$$\left(1-x^2\right)v'' - 2xv' + 2nxv' + 2nv = 0$$

$$\boxed{\left(1-x^2\right)v'' + 2(n-1)xv' + 2nv = 0}$$

b) $$\frac{d}{dx}\left(\left(1-x^2\right)v'' - 2xv' + 2nxv' + 2nv\right) = 0$$

$$\left(1-x^2\right)v^{(3)} - 2xv'' - 2v' - 2xv'' + 2nxv'' + 2nv' + 2nv'$$

$$\boxed{\left(1-x^2\right)v^{(3)} + 2(n-2)xv'' + 2(2n-1)v' = 0}$$

$n=1 \quad \rightarrow \quad \left(1-x^2\right)v^{(3)} - 2xv'' + 2v' = 0$

$$\boxed{\left(1-x^2\right)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0} \quad \checkmark$$

$$\frac{d}{dx}\left(\left(1-x^2\right)v^{(3)} + 2x(n-2)v'' + 2(2n-1)v'\right) = 0$$

$$\left(1-x^2\right)v^{(4)} - 2xv^{(3)} + 2x(n-2)v^{(3)} + 2(n-2)v'' + 2(2n-1)v'' = 0$$

$$\left(1-x^2\right)v^{(4)} + 2x(n-3)v^{(3)} + 6(n-1)v'' = 0$$

$$\boxed{\left(1-x^2\right)v^{(4)} + 2(n-3)xv^{(3)} + 3(2n-2)v'' = 0}$$

$n=2 \quad \rightarrow \quad \left(1-x^2\right)v^{(4)} - 2xv^{(3)} + 3\cdot 2v'' = 0$

$$\boxed{\left(1-x^2\right)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0} \quad \checkmark$$

$$\frac{d}{dx}\left(\left(1-x^2\right)v^{(4)} + 2(n-3)xv^{(3)} + 3(2n-2)v''\right) = 0$$

$$\left(1-x^2\right)v^{(5)} - 2xv^{(4)} + 2(n-3)xv^{(4)} + 2(n-3)v^{(3)} + 3(2n-2)v^{(3)} = 0$$

$$\left(1-x^2\right)v^{(5)} + (2n-6-2)xv^{(4)} + (2n-6+6n-6)v^{(3)} = 0$$

$$\left(1-x^2\right)v^{(5)} + (2n-8)xv^{(4)} + (8n-12)v^{(3)} = 0$$

$$\boxed{\left(1-x^2\right)v^{(5)} + 2(n-4)xv^{(4)} + 4(2n-3)v^{(3)} = 0}$$

$n=3 \quad \rightarrow \quad \left(1-x^2\right)v^{(5)} - 2xv^{(4)} + 4\cdot 3v^{(3)} = 0$

$$\boxed{\left(1-x^2\right)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0} \quad \checkmark$$

After m differentiations:

$$\left(1-x^2\right)v^{(m+2)} + 2(n-(m+1))xv^{(m+1)} + (m+1)(2n-m)v^{(m)} = 0$$

If we let $m = n$, then

$$\boxed{\left(1-x^2\right)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0}$$

Let assume that $\left(1-x^2\right)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0$ is true.

We need to prove that next derivative is also true.

$$\frac{d}{dx}\left(\left(1-x^2\right)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)}\right) = 0$$

$$\left(1-x^2\right)v^{(n+3)} - 2xv^{(n+2)} - 2v^{(n+1)} - 2xv^{(n+2)} + (2n-n)(n+1)v^{(n+1)} = 0$$

$$\left(1-x^2\right)v^{(n+3)} - 4xv^{(n+2)} + (2n-n-2)(n+1)v^{(n+1)} = 0$$

$$\left(1-x^2\right)v^{(n+3)} - 2(2)xv^{(n+2)} + (n-1)(n+2)v^{(n+1)} = 0$$

$$\left(1-x^2\right)v^{(n+3)} + 2(n-n-2)xv^{(n+2)} + (n-1)(n+2)v^{(n+1)} = 0$$

$$\left(1-x^2\right)v^{(n+3)} + 2(n-(n+2))xv^{(n+2)} + (2n-(n+1))((n+1)+1)v^{(n+1)} = 0$$

If we let $m = n+1$, then

$$\left(1-x^2\right)v^{(m+2)} + 2(n-(m+1))xv^{(m+1)} + (2n-m)(m+1)v^{(m)} = 0$$

If we let $m = n$, then

$$\left(1-x^2\right)v^{(n+2)} + 2(n-n-1)xv^{(m+1)} + (2n-n)(n+1)v^{(n)} = 0$$

$$\boxed{\left(1-x^2\right)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0} \quad \checkmark$$

c) $u = v^{(n)} = D^n\left(x^2-1\right)^n$

$$= \frac{d^n}{dx^n}\left(x^{2n} - nx^{2n-1} + \cdots - 1\right)$$

$$= 2n(2n-1)\cdots(2n-(n-1))x^n - \frac{d^n}{dx^n}\left(nx^{2n-1} + \cdots - 1\right)$$

$$= \frac{(2n)!}{n!}x^n - \frac{d^n}{dx^n}\left(nx^{2n-1} + \cdots - 1\right)$$

Since $u = v^{(n)}$ satisfies Legendre's equation of order n, $\dfrac{u}{2^n n!}$

From the notes:

The solution of Legendre polynomial of degree ***n*** is

$$P_n(x) = \sum_{k=0}^{K} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}$$

For the highest order, the result when:

$$k = 0 \quad \rightarrow \quad P_n(x) = \frac{(2n)!}{2^n (n!)^2} x^n$$

$$\frac{u}{2^n n!} = \frac{(2n)!}{2^n (n!)^2} x^n + \cdots$$

$$P_n(x) = \frac{1}{n!\ 2^n} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n$$

Solution *Section* 4.4 – Solution about Singular Points

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^2y'' + 3y' - xy = 0$$

Solution

$$y'' + \frac{3}{x^2}y' - \frac{x}{x^2}y = 0$$

$$P(x) = \frac{3}{x^2} \qquad Q(x) = -\frac{x}{x^2}$$

For $P(x) = \frac{3}{x^2} \quad \to \quad x = 0$

$\therefore\ p(x)$ is analytic except at $x = 0$

For $Q(x) = -\frac{x}{x^2} = -\frac{1}{x}$

$\therefore\ q(x)$ is not analytic at $x = 0$

The singular point is: $x = 0$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^2 + x\right)y'' + 3y' - 6xy = 0$$

Solution

$$y'' + \frac{3}{x(x+1)}y' - \frac{6}{x+1}y = 0$$

$$P(x) = \frac{3}{x(x+1)} \qquad Q(x) = -\frac{6x}{x(x+1)}$$

For $P(x) = \frac{3}{x(x+1)} \quad \to \quad x = 0, -1$

$\therefore\ p(x)$ is analytic except at $x = 0, -1$

For $q(x) = -\frac{6x}{x(x+1)} \quad \to \quad x = 0,\ -1$

$q(x) = -\frac{6x}{x(x+1)} = -\frac{6}{x+1}$; is actually analytic at $x = 0$

$\therefore\ q(x)$ is analytic except at $x = -1$

The singular points are: $x = 0,\ -1$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^2-1\right)y''+(1-x)y'+\left(x^2-2x+1\right)y=0$$

Solution

$$(x-1)(x+1)y''+(1-x)y'+(x-1)^2 y=0$$

$$y''-\frac{1}{x+1}y'+\frac{x-1}{x+1}y=0$$

$$P(x)=\frac{1-x}{x^2-1} \qquad Q(x)=\frac{(x-1)^2}{x^2-1}$$

For $p(x)=\frac{1-x}{(x-1)(x+1)} \quad \to \quad \underline{x=-1,\ 1}$

$p(x)=\frac{1-x}{(x-1)(x+1)}=-\frac{1}{x+1}$; is actually analytic at $x=1$

$\therefore\ p(x)$ is analytic except at $\underline{x=-1}$

For $q(x)=\frac{(x-1)^2}{(x-1)(x+1)} \quad \to \quad \underline{x=-1,\ 1}$

$q(x)=\frac{(x-1)^2}{(x-1)(x+1)}=\frac{x-1}{x+1}$; is actually analytic at $x=1$

$\therefore\ q(x)$ is analytic except at $\underline{x=-1}$

The singular point is: $\underline{x=-1}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$e^x y''-\left(x^2-1\right)y'+2xy=0$$

Solution

$$y''-\frac{x^2-1}{e^x}y'+\frac{2x}{e^x}y=0$$

$$P(x)=-\frac{x^2-1}{e^x} \qquad Q(x)=\frac{2x}{e^x}$$

Since $e^x\neq 0$, there are ***no*** singular points.

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\ln(x-1)y'' + (\sin 2x)y' - e^x y = 0$$

Solution

$$y'' + \frac{\sin 2x}{\ln(x-1)}y' - \frac{e^x}{\ln(x-1)}y = 0$$

$$P(x) = \frac{\sin 2x}{\ln(x-1)} \qquad Q(x) = -\frac{e^x}{\ln(x-1)}$$

$$\ln(x-1) = 0 \quad \to \quad x-1=1 \;\Rightarrow\; \underline{x=2}$$

The singular point is: $\underline{x \le 1,\ x = 2}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$xy'' + x(1-x)^{-1}y' + (\sin x)y = 0$$

Solution

$$y'' + \frac{x}{x(1-x)}y' + \frac{\sin x}{x}y = 0$$

$$P(x) = \frac{x}{x(1-x)} \qquad Q(x) = \frac{\sin x}{x}$$

For $p(x) = \dfrac{x}{x(1-x)} \quad \to \quad \underline{x = 0,\ 1}$

$p(x) = \dfrac{x}{x(1-x)} = \dfrac{1}{1-x}$; is actually analytic at $x=0$

$\therefore\ p(x)$ is analytic except at $\underline{x=1}$

For $q(x) = \dfrac{\sin x}{x} = \dfrac{x - \frac{1}{3!}x^3 + \cdots}{x} = 1 - \dfrac{1}{3!}x^2 + \cdots$ is analytic everywhere ($x=0$ is removable).

The only singular point is $\underline{x=1}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x(x+3)^2 y'' - y = 0$$

Solution

$$y'' - \frac{1}{x(x+3)^2} y = 0$$

$$P(x) = 0 \qquad Q(x) = -\frac{1}{x(x+3)^2}$$

For $q(x) = -\frac{1}{x(x+3)^2} \quad \rightarrow \quad \underline{x = 0, \ -3}$, is analytic elsewhere

The *Regular* singular points are $\underline{x = 0, \ -3}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^2 - 9\right)^2 y'' + (x+3) y' + 2y = 0$$

Solution

$$y'' + \frac{x+3}{\left(x^2-9\right)^2} y' + \frac{2}{\left(x^2-9\right)^2} y = 0$$

$$P(x) = \frac{x+3}{\left(x^2-9\right)^2} \qquad Q(x) = \frac{2}{\left(x^2-9\right)^2}$$

For $P(x) = \frac{x+3}{\left(x^2-9\right)^2} \quad \rightarrow \quad \underline{x = \pm 3}$

$$p(x) = \frac{x+3}{\left((x+3)(x-3)\right)^2} = \frac{(x+3)(x-3)}{(x+3)(x-3)^2}; \text{ is analytic at } x = -3$$

For $Q(x) = \frac{2\left(x^2-9\right)^2}{\left(x^2-9\right)^2} \quad \rightarrow \quad \underline{x = \pm 3}$

$\therefore \ q(x)$ is analytic at $\underline{x = \pm 3}$

The *Regular* singular point: $\underline{x = -3}$, and *Irregular* singular point: $\underline{x = 3}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$y'' - \frac{1}{x} y' + \frac{1}{(x-1)^3} y = 0$$

Solution

$$y'' - \frac{1}{x}y' + \frac{1}{(x-1)^3}y = 0$$

$$P(x) = -\frac{1}{x} \qquad Q(x) = \frac{1}{(x-1)^3}$$

For $P(x) = -\frac{1}{x} \quad \rightarrow \quad \underline{x=0}$

$$p(x) = \frac{x}{x} = 1 \text{ is analytic at } \underline{x=0}$$

For $Q(x) = \frac{1}{(x-1)^3} \quad \rightarrow \quad \underline{x=1}$

$$q(x) = \frac{(x-1)^2}{(x-1)^3} = \frac{1}{x-1} \text{ is } \textit{not} \text{ an analytic at } x=1$$

The *Regular* singular point: $\underline{x=0}$, and *Irregular* singular point: $\underline{x=1}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^3+4x\right)y'' - 2xy' + 6y = 0$$

Solution

$$y'' - \frac{2x}{x\left(x^2+4\right)}y' + \frac{6}{x\left(x^2+4\right)}y = 0$$

$$P(x) = -\frac{2x}{x\left(x^2+4\right)} \qquad Q(x) = \frac{6}{x\left(x^2+4\right)}$$

For $P(x) = -\frac{2x}{x\left(x^2+4\right)} \quad \rightarrow \quad x=0, \ \pm 2i$

$$p(x) = -\frac{2}{x^2+4} \text{ is analytic at } x = \pm 2i$$

For $Q(x) = \frac{6}{x\left(x^2+4\right)} \quad \rightarrow \quad x=0, \ \pm 2i$ is analytic

The *Regular* singular points: $\underline{x=0, \ \pm 2i}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^2(x-5)^2 y'' + 4xy' + \left(x^2-25\right)y = 0$$

Solution

$$y'' + \frac{4x}{x^2(x-5)^2}y' + \frac{x^2-25}{x^2(x-5)^2}y = 0$$

$$P(x) = \frac{4x}{x^2(x-5)^2} \qquad Q(x) = \frac{x^2-25}{x^2(x-5)^2}$$

For $P(x) = \dfrac{4x}{x^2(x-5)^2} \quad \to \quad x = 0,\ 5$

$p(x) = \dfrac{4x}{x^2(x-5)^2} = x(x-5)\dfrac{4}{x(x-5)^2}$ is not an analytic at $x = 5$

For $Q(x) = \dfrac{(x-5)(x+5)}{x^2(x-5)^2} \quad \to \quad x = 0,\ 5$

$q(x) = x^2(x-5)^2\dfrac{x+5}{x^2(x-5)}$ is an analytic at $x = 0,\ 5$

The *Regular* singular point: $\underline{x=0}|$, and *Irregular* singular point: $\underline{x=5}|$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^2+x-6\right)y'' + (x+3)y' + (x-2)y = 0$$

Solution

$$y'' + \frac{x+3}{x^2+x-6}y' + \frac{x-2}{x^2+x-6}y = 0$$

$$P(x) = \frac{x+3}{(x+3)(x-2)} \qquad Q(x) = \frac{x-2}{(x+3)(x-2)}$$

For $P(x) = \dfrac{x+3}{(x+3)(x-2)} \quad \to \quad x = -3,\ 2$

$p(x) = \dfrac{1}{x-2}$ is an analytic at $x = 2$

For $Q(x) = \dfrac{x-2}{(x+3)(x-2)} \quad \to \quad x = -3,\ 2$

$q(x) = \dfrac{1}{x+3}$ is an analytic at $x = -3$

The *Regular* singular points: $\underline{x=-3,\ 2}|$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x\left(x^2+1\right)^2 y'' + y = 0$$

Solution

$$y'' + \frac{1}{x\left(x^2+1\right)^2} y = 0$$

$$P(x) = 0 \qquad Q(x) = \frac{1}{x\left(x^2+1\right)^2}$$

For $Q(x) = \dfrac{1}{x\left(x^2+1\right)^2} \quad \rightarrow \quad x = 0, \ \pm i$

$$q(x) = x^2\left(x^2+1\right)^2 \frac{1}{x\left(x^2+1\right)^2} \text{ is an analytic at } x = 0, \ \pm i$$

The *Regular* singular points: $\underline{x = 0, \ \pm i}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^3\left(x^2-25\right)(x-2)^2 y'' + 3x(x-2)y' + 7(x+5)y = 0$$

Solution

$$y'' + \frac{3x(x-2)}{x^3\left(x^2-25\right)(x-2)^2} y' + \frac{7(x+5)}{x^3\left(x^2-25\right)(x-2)^2} y = 0$$

$$P(x) = \frac{3x(x-2)}{x^3(x-5)(x+5)(x-2)^2} \qquad Q(x) = \frac{7(x+5)}{x^3(x-5)(x+5)(x-2)^2}$$

For $P(x) = \dfrac{3x(x-2)}{x^3(x-5)(x+5)(x-2)^2} \quad \rightarrow \quad x = 0, \ \pm 5, \ 2$

$$p(x) = \frac{3x(x-5)(x+5)(x-2)}{x^2(x-5)(x+5)(x-2)} \text{ is not an analytic at } x = 0$$

For $Q(x) = \dfrac{7(x+5)}{x^3(x-5)(x+5)(x-2)^2} \quad \rightarrow \quad x = 0, \ \pm 5, \ 2$

$$q(x)=\frac{3x^2(x-5)^2(x+5)^2(x-2)^2}{x^3(x-5)(x-2)^2} \text{ is not an analytic at } x=0$$

The *Regular* singular point: $\boxed{x=2,\ \pm 5}$, and *Irregular* singular point: $\boxed{x=0}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(x^3-2x^2-3x\right)^2 y''+x(x-3)^2 y'-(x+1)y=0$$

Solution

$$y''+\frac{x(x-3)^2}{\left(x^3-2x^2-3x\right)^2}y'-\frac{x+1}{\left(x^3-2x^2-3x\right)^2}y=0$$

$$P(x)=\frac{x(x-3)^2}{x^2(x-3)^2(x+1)^2} \qquad Q(x)=-\frac{x+1}{x^2(x-3)^2(x+1)^2}$$

For $P(x)=\dfrac{x(x-3)^2}{x^2(x-3)^2(x+1)^2} \quad \rightarrow \quad x=0,\ -1,\ 3$

$$p(x)=\frac{1}{x(x+1)^2} \text{ is not an analytic at } x=-1$$

For $Q(x)=\dfrac{x+1}{x^2(x-3)^2(x+1)^2} \quad \rightarrow \quad x=0,\ -1,\ 3$

$$q(x)=\frac{x^2(x-3)^2(x+1)^2}{x^2(x-3)^2(x+1)} \text{ is an analytic at } x=0,\ -1,\ 3$$

The *Regular* singular point: $\boxed{x=0,\ 3}$, and *Irregular* singular point: $\boxed{x=-1}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$\left(1-x^2\right)y''+(\tan x)y'+x^{5/3}y=0$$

Solution

$$y''+\frac{\tan x}{1-x^2}y'+\frac{x^{5/3}}{1-x^2}y=0$$

$P(x)=\dfrac{\tan x}{1-x^2} \qquad Q(x)=\dfrac{x^{5/3}}{1-x^2}$

For $P(x)=\dfrac{\tan x}{1-x^2} \quad \to \quad x=\pm 1$

$\tan x=\pm\infty \quad \to \quad x=\pm\frac{\pi}{2}$ (Vertical Asymptotes).

For $Q(x)=\dfrac{x^{5/3}}{1-x^2} \quad \to \quad x=\pm 1$ is not analytic

The second derivatices doesn't exist at $x=0$

The *Regular* singular point: $\underline{x=0,\ \ \pm 1,\ \ \pm\frac{\pi}{2}}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x(x-1)^2(x+2)y''+x^2y'-\left(x^3+2x-1\right)y=0$$

Solution

$$y''+\frac{x^2}{x(x-1)^2(x+2)}y'-\frac{x^3+2x-1}{x(x-1)^2(x+2)}y=0$$

$$P(x)=\frac{x}{(x-1)^2(x+2)} \quad \& \quad Q(x)=-\frac{x^3+2x-1}{x(x-1)^2(x+2)}$$

For $P(x)=\dfrac{x}{(x-1)^2(x+2)} \quad \to \quad x=1,\ -2$

$$p_0=\lim_{x\to 1}(x-1)P(x)=\lim_{x\to 1}\frac{x}{(x-1)(x+2)}\underline{=\infty} \text{ is not analytic}$$

$$p_0=\lim_{x\to -2}(x+2)P(x)=\lim_{x\to -2}\frac{x}{(x-1)^2}\underline{=-\frac{2}{9}}$$

For $Q(x)=-\dfrac{x^3+2x-1}{x(x-1)^2(x+2)} \quad \to \quad x=0,\ 1,\ -2$

$$q_0=\lim_{x\to 0}x^2Q(x)=\lim_{x\to 0}\frac{x\left(x^3+2x-1\right)}{(x-1)^2(x+2)}\underline{=0}$$

$$q_0=\lim_{x\to 1}(x-1)^2Q(x)=\lim_{x\to 1}\frac{x^3+2x-1}{x(x+2)}\underline{=\frac{2}{3}}$$

$$q_0=\lim_{x\to -2}(x+2)^2Q(x)=-\lim_{x\to -2}\frac{\left(x^3+2x-1\right)(x+2)}{x(x-1)^2}\underline{=0}$$

The *Regular* singular point: $\underline{x=0,\ \ -2}$, and *Irregular* singular point: $\underline{x=1}$

Exercise

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

$$x^4\left(x^2+1\right)(x-1)^2 y'' + 4x^3(x-1)y' + (x+1)y = 0$$

Solution

$$y'' + \frac{4x^3(x-1)}{x^4\left(x^2+1\right)(x-1)^2}y' + \frac{x+1}{x^4\left(x^2+1\right)(x-1)^2}y = 0$$

$$P(x) = \frac{4}{x\left(x^2+1\right)(x-1)} \quad \& \quad Q(x) = \frac{x+1}{x^4\left(x^2+1\right)(x-1)^2}$$

For $P(x) = \frac{4}{x\left(x^2+1\right)(x-1)} \quad \rightarrow \quad \underline{x = 0,\ 1,\ \pm i}$

$$p_0 = \lim_{x\to 0} xP(x) = \lim_{x\to 0} \frac{4}{\left(x^2+1\right)(x-1)} \underline{= -4}$$

$$p_0 = \lim_{x\to 1} (x-1)P(x) = \lim_{x\to 1} \frac{4}{x\left(x^2+1\right)} \underline{= 2}$$

$$p_0 = \lim_{x\to i} (x-i)P(x) = \lim_{x\to i} \frac{4}{x(x-1)(x+i)} = -\frac{2}{i-1} = -\frac{2}{i-1}\frac{i+1}{i+1} \underline{= i+1}$$

$$p_0 = \lim_{x\to -i} (x+i)P(x) = \lim_{x\to -i} \frac{4}{x(x-1)(x-i)} = \frac{2}{i-1} = \frac{2}{i-1}\frac{i+1}{i+1} \underline{= -i-1}$$

For $Q(x) = \frac{x+1}{x^4\left(x^2+1\right)(x-1)^2} \quad \rightarrow \quad x = 0,\ 1,\ \pm i$

$$q_0 = \lim_{x\to 0} x^2 Q(x) = \lim_{x\to 0} \frac{x+1}{x^2\left(x^2+1\right)(x-1)^2} \underline{= \infty} \text{ is not analytic}$$

$$q_0 = \lim_{x\to 1} (x-1)^2 Q(x) = \lim_{x\to 0} \frac{x+1}{x^4\left(x^2+1\right)} \underline{= 1}$$

$$q_0 = \lim_{x\to \pm i} \left(x^2+1\right)^2 Q(x) = \lim_{x\to \pm i} \frac{(x+1)\left(x^2+1\right)}{x^2(x-1)^2} \underline{= 0}$$

The *Regular* singular point: $\underline{x = 0,\ \pm i}$, and *Irregular* singular point: $\underline{x = 0}$

Exercise

Determine whether $x = 0$ is an ordinary point, singular point, or irregular singular point of the given differential equation $\quad xy'' + (1-\cos x)y' + x^2 y = 0$

Solution

$$y'' + \frac{1-\cos x}{x}y' + xy = 0$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\frac{1-\cos x}{x} = \frac{1}{x}\left(1 - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right)$$

$$= \boxed{\frac{x}{2} - \frac{x^3}{24} + \frac{x^5}{720} - \cdots}, \quad \text{is analytic at } x = 0$$

$\therefore$ $x = 0$ is an ordinary point of the differential equation.

Exercise

Determine whether $x = 0$ is an ordinary point, singular point, or irregular singular point of the given differential equation $\left(e^x - 1 - x\right)y'' + xy = 0$

Solution

$$x^2 y'' + x^2 \frac{x}{e^x - 1 - x}y = 0$$

$$x^2 y'' + \frac{x^3}{e^x - 1 - x}y = 0$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x - 1 - x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots - 1 - x$$

$$= \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$\frac{x^3}{e^x - 1 - x} = \frac{1}{\frac{1}{x^3}\left(\frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)}$$

$$= \frac{1}{\frac{1}{2x} + \frac{1}{6} + \frac{x}{24} + \cdots}$$

$\therefore$ $x = 0$ is a regular singular point of the differential equation

Exercise

Find the Frobenius series solutions of $2x^2 y'' + 3xy' - \left(1 + x^2\right)y = 0$

Solution

$$y'' + \frac{3}{2}\frac{1}{x}y' - \frac{1}{2}\frac{1+x^2}{x^2}y = 0$$ ***Divide each term by*** $2x^2$

Therefore, $x = 0$ is a regular singular point, and that $p_0 = \frac{3}{2}, \quad q_0 = -\frac{1}{2}$

$p(x) \equiv \frac{3}{2}, \quad q(x) = -\frac{1}{2} - \frac{1}{2}x^2$ are polynomials.

The Frobenius series will converge for all $x > 0$. The indicial equation is

$$r(r-1) + \frac{3}{2}r - \frac{1}{2} = r^2 + \frac{1}{2}r - \frac{1}{2} = (r+1)\left(r - \frac{1}{2}\right) = 0$$

So the roots are $r_1 = \frac{1}{2}$ *and* $r_2 = -1$.

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2}\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{-1}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' + 3xy' - \left(1 + x^2\right)y = 0$$

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 3x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty}a_n x^{n+r} - x^2\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-2)a_n x^{n+r} + \sum_{n=0}^{\infty}3(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}a_n x^{n+r} - \sum_{n=0}^{\infty}a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2) + 3(n+r) - 1\right]a_n x^{n+r} - \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r+1) - 1\right]a_n x^{n+r} - \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0$$

$$\left(2r^2 + r - 1\right)a_0 + \left(2r^2 + 5r + 2\right)a_1 x +$$

$$\sum_{n=2}^{\infty}\left[(n+r)(2n+2r+1) - 1\right]a_n x^{n+r} - \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0$$

$$\left(2r^2+r-1\right)a_0+\left(2r^2+5r+2\right)a_1x+\sum_{n=2}^{\infty}\left(\left[(n+r)(2n+2r+1)-1\right]a_n-a_{n-2}\right)x^{n+r}=0$$

For $n=0 \quad\rightarrow\quad \left(2r^2+r-1\right)a_0=0$

$$r=-1 \quad or \quad r=\tfrac{1}{2} \quad \checkmark$$

For $n=1 \quad\rightarrow\quad \left(2r^2+5r+2\right)a_1=0$

$$\cancel{r=-2 \quad or \quad r=-\tfrac{1}{2}} \quad \text{Therefore } a_1=0$$

$$\left[(n+r)(2n+2r+1)-1\right]a_n-a_{n-2}=0$$

$$a_n=\frac{1}{(n+r)(2(n+r)+1)-1}a_{n-2}$$

$$=\frac{1}{2(n+r)^2+(n+r)-1}a_{n-2} \qquad for\ \ n\ge 2$$

$r=\frac{1}{2}$

$$a_n=\frac{1}{2\left(n+\frac{1}{2}\right)^2+\left(n+\frac{1}{2}\right)-1}a_{n-2}=\frac{1}{2n^2+3n}a_{n-2}$$

$r=-1$

$$b_n=\frac{1}{2(n-1)^2+(n-1)-1}b_{n-2}$$

$a_2=\frac{1}{14}a_0$

$a_3=\frac{1}{24}a_1=0$

$a_4=\frac{1}{44}a_2=\frac{1}{616}a_0$

$a_5=0$

$a_6=\frac{1}{90}a_4=\frac{1}{55440}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$b_2=\frac{1}{2}b_0$

$b_3=\frac{1}{9}b_1=0$

$b_4=\frac{1}{20}b_2=\frac{1}{40}b_0$

$b_5=0$

$b_6=\frac{1}{54}b_4=\frac{1}{2160}b_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_1(x)=a_0x^{1/2}\left(1+\frac{x^2}{14}+\frac{x^4}{616}+\frac{x^6}{55,440}+\cdots\right) \qquad y_2(x)=b_0x^{-1}\left(1+\frac{x^2}{2}+\frac{x^4}{40}+\frac{x^6}{2160}+\cdots\right)$$

$$y(x)=C_1\left(1+\sum_{n=0}^{\infty}\frac{4}{n!\,1\cdot3\cdot5\cdots(2n-1)}x^n\right)+C_2x^{1/2}\left(1+\sum_{n=0}^{\infty}\frac{4^n}{n!\,3\cdot5\cdots(2n+1)}x^n\right)$$

Exercise

Find the Frobenius series solutions of $2x^2y'' - xy' + \left(1+x^2\right)y = 0$

Solution

$$y'' - \frac{1}{2}\frac{1}{x}y' + \frac{1}{2}\frac{1+x^2}{x^2}y = 0 \qquad \textit{\textbf{Divide each term by }} 2x^2$$

Therefore, $x = 0$ is a *regular singular* point, and that $p_0 = -\frac{1}{2}, \quad q_0 = \frac{1}{2}$

$p(x) = -\frac{1}{2}, \quad q(x) = \frac{1}{2} + \frac{1}{2}x^2$ are polynomials.

The Frobenius series will converge for all $x > 0$. The indicial equation is

$$r(r-1) - \frac{1}{2}r + \frac{1}{2} = r^2 - \frac{3}{2}r + \frac{1}{2} = \frac{1}{2}\left(2r^2 - 3r + 1\right) = 0$$

So the roots are $r_1 = \frac{1}{2} \quad and \quad r_2 = 1$.

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2}\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^1\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$2x^2y'' - xy' + \left(1+x^2\right)y = 0$$

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r} + x^2\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2) - (n+r) + 1\right]a_n x^{n+r} + \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-3) + 1\right]a_n x^{n+r} + \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0$$

$$\left(2r^2-3r+1\right)a_0+\left((1+r)(2r-1)+1\right)a_1x+$$

$$\sum_{n=2}^{\infty}\left[(n+r)(2n+2r-3)+1\right]a_nx^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}=0$$

$$\left(2r^2-3r+1\right)a_0+\left(2r^2+r\right)a_1x+\sum_{n=2}^{\infty}\left[\left((n+r)(2n+2r-3)+1\right)a_n+a_{n-2}\right]x^{n+r}=0$$

For $n=0 \quad \rightarrow \quad \left(2r^2-3r+1\right)a_0=0 \quad \Rightarrow \quad \boxed{r=1 \;\; or \;\; r=\frac{1}{2}}$ $\surd$

For $n=1 \quad \rightarrow \quad \left(2r^2+r\right)a_1=0 \quad \Rightarrow r \;\; \boxed{\cancel{=0,-\frac{1}{2}}}$ $\qquad$ Therefore $\boxed{a_1=0}$

$$\left((n+r)(2n+2r-3)+1\right)a_n+a_{n-2}=0$$

$$\boxed{a_n=-\frac{1}{(n+r)(2n+2r-3)+1}a_{n-2}} \qquad for \;\; n\geq 2$$

$r=\frac{1}{2}$

$$a_n=-\frac{1}{\left(n+\frac{1}{2}\right)(2n-2)+1}a_{n-2}\;\;\boxed{=-\frac{1}{2n^2-n}a_{n-2}}$$

$n=2 \rightarrow a_2=-\frac{1}{6}a_0$ $\qquad$ $n=3 \rightarrow a_3=-\frac{1}{15}a_1=0$

$n=4 \rightarrow a_4=-\frac{1}{28}a_2=\frac{1}{168}a_0$ $\qquad$ $n=5 \rightarrow a_5=0$

$n=6 \rightarrow a_6=-\frac{1}{66}a_4=-\frac{1}{11,088}a_0$

$$y_1(x)=a_0x^{1/2}\left(1-\frac{x^2}{6}+\frac{x^4}{168}-\frac{x^6}{11,088}+\cdots\right)$$

$$\boxed{=a_0\left(x^{1/2}-\frac{1}{6}x^{5/2}+\frac{1}{168}x^{9/2}-\frac{1}{11,088}x^{13/2}+\cdots\right)}$$

$r=1$

$$b_n=-\frac{1}{(n+1)(2n-1)+1}b_{n-2}\;\;\boxed{=-\frac{1}{2n^2+n}b_{n-2}}$$

$n=2 \rightarrow b_2=-\frac{1}{10}b_0$ $\qquad$ $n=3 \rightarrow b_3=-\frac{1}{21}b_1=0$

$n=4 \rightarrow b_4=-\frac{1}{36}b_2=\frac{1}{360}b_0$ $\qquad$ $n=5 \rightarrow b_5=0$

$n=6 \rightarrow b_6=-\frac{1}{78}b_4=-\frac{1}{28,080}b_0$ $\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_2(x) = b_0 x\left(1 - \frac{x^2}{10} + \frac{x^4}{360} - \frac{x^6}{28,080} + \cdots\right)$$

$$= b_0\left(x - \frac{1}{10}x^3 + \frac{1}{360}x^5 - \frac{1}{28,080}x^7 + \cdots\right)$$

$$y(x) = a_0\left(x^{1/2} - \frac{1}{6}x^{5/2} + \frac{1}{168}x^{9/2} - \frac{1}{11,088}x^{13/2} + \cdots\right) + b_0\left(x - \frac{1}{10}x^3 + \frac{1}{360}x^5 - \frac{1}{28,080}x^7 + \cdots\right)$$

Exercise

Find the general solution to the equation $2xy'' + (1+x)y' + y = 0$

Solution

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$2xy'' + (1+x)y' + y = 0$$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + (1+x)\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)c_n x^{n-1}x^r + \sum_{n=0}^{\infty}(n+r)c_n x^{n-1}x^r + \sum_{n=0}^{\infty}(n+r)c_n x^n x^r + \sum_{n=0}^{\infty} c_n x^n x^r = 0$$

$$x^r\left(\sum_{n=0}^{\infty}(n+r)(2n+2r-2+1)c_n x^{n-1} + \sum_{n=0}^{\infty}(n+r+1)c_n x^n\right) = 0$$

$$x^r\left(\sum_{n=0}^{\infty}(n+r)(2n+2r-1)c_n x^{n-1} + \sum_{n=0}^{\infty}(n+r+1)c_n x^n\right) = 0$$

$$x^r\left(c_0 r(2r-1)x^{-1}+\underbrace{\sum_{n=1}^{\infty}c_n(n+r)(2n+2r-1)x^{n-1}}_{k=n-1}+\underbrace{\sum_{n=0}^{\infty}(n+r+1)c_n x^n}_{k=n}\right)=0$$

$$x^r\left(c_0 r(2r-1)x^{-1}+\sum_{k=0}^{\infty}c_{k+1}(r+k+1)(2k+2r+1)x^k+\sum_{k=0}^{\infty}c_k(r+k+1)x^k\right)=0$$

$$x^r\left(c_0 r(2r-1)x^{-1}+\sum_{k=0}^{\infty}\left[c_{k+1}(r+k+1)(2k+2r+1)+c_k(r+k+1)\right]x^k\right)=0$$

$$\begin{cases} c_0 r(2r-1)=0 & \Rightarrow \quad \boxed{r=0} \quad \boxed{r=\frac{1}{2}} \\ c_{k+1}(r+k+1)(2k+2r+1)+c_k(r+k)=0 & \Rightarrow \quad \boxed{c_{k+1}=-\dfrac{r+k+1}{(r+k+1)(2k+2r+1)}c_k} \end{cases}$$

$r=0$

$$c_{k+1}=-\frac{1}{2k+1}c_k$$

$$c_1=-\frac{1}{1}c_0$$

$$c_2=-\frac{1}{3}c_1=\frac{1}{3}c_0$$

$$c_3=-\frac{1}{5}c_2=-\frac{1}{1\cdot 3\cdot 5}c_0$$

$$c_4=-\frac{1}{7}c_3=\frac{1}{1\cdot 3\cdot 5\cdot 7}c_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$c_n=\frac{(-1)^n}{1\cdot 3\cdot 5\cdots(2n-1)}c_0$$

$r=\frac{1}{2}$

$$c_{k+1}=-\frac{k+\frac{3}{2}}{\left(k+\frac{3}{2}\right)(2k+2)}c_k=-\frac{1}{2(k+1)}c_k$$

$$c_1=-\frac{1}{2}c_0$$

$$c_2=-\frac{1}{2\cdot 2}c_1=\frac{1}{2\cdot 2\cdot 2}c_0$$

$$c_3=-\frac{1}{2\cdot 3}c_2=-\frac{1}{2^3(2\cdot 3)}c_0=-\frac{1}{2^3\cdot 3!}c_0$$

$$c_4=-\frac{1}{2\cdot 4}c_3=\frac{1}{2^4\cdot 4!}c_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$c_n=\frac{(-1)^n}{2^n n!}c_0$$

$$y_1(x)=c_0\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{1\cdot 3\cdot 5\cdots(2n-1)}x^n\right) \qquad y_2(x)=c_0 x^{1/2}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n!}x^n\right)$$

$$y(x)=C_1 y_1(x)+C_2 y_2(x)$$

$$=C_1\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{1\cdot 3\cdot 5\cdots(2n-1)}x^n\right)+C_2\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}x^{n+1/2}\right)$$

Exercise

Find the Frobenius series solutions of $xy'' + 2y' + xy = 0$

Solution

$$\frac{1}{x}\left(xy'' + 2y' + xy = 0\right)$$

$$y'' + 2\frac{1}{x}y' + \frac{x^2}{x^2}y = 0$$

$\therefore\ x = 0$ is a regular singular point with $p_0 = 2$ and $q_0 = 0$

The indicial equation is: $r(r-1) + 2r = r(r+1) = 0 \quad \rightarrow \quad \underline{r = 0,\ -1}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-1}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$xy'' + 2y' + xy = 0$$

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + x\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}2(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1) + 2(n+r)\right]a_n x^{n+r-1} + \sum_{n=2}^{\infty}a_{n-2}x^{n+r-1} = 0$$

$$r(r+1)a_0 x^{r-1} + (r+1)(r+2)a_1 x^r + \sum_{n=2}^{\infty}(n+r)(n+r+1)a_n x^{n+r-1} + \sum_{n=2}^{\infty}a_{n-2}x^{n+r-1} = 0$$

For $n = 0 \quad \rightarrow \quad r(r+1)a_0 = 0 \quad \Rightarrow \quad \underline{r = 0 \ \text{ or } \ r = -1}$ √

For $n = 1 \quad \rightarrow \quad (r+1)(r+2)a_1 = 0 \quad \Rightarrow \quad \cancel{r = -1,\ -2} \qquad \therefore\ \underline{a_1 = 0}$

$$(n+r)(n+r+1)a_n + a_{n-2} = 0$$

$$\boxed{a_n = -\frac{1}{(n+r)(n+r+1)} a_{n-2}}$$

$$r = 0 \quad \rightarrow \quad \boxed{a_n = -\frac{1}{n(n+1)} a_{n-2}}$$

$n=2 \rightarrow a_2 = -\frac{1}{2\cdot 3} a_0$ $\qquad$ $n=3 \rightarrow a_3 = -\frac{1}{12} a_1 = 0$

$n=4 \rightarrow a_4 = -\frac{1}{4\cdot 5} a_2 = \frac{1}{5!} a_0$ $\qquad$ $n=5 \rightarrow a_5 = 0$

$n=6 \rightarrow a_6 = -\frac{1}{6\cdot 7} a_4 = -\frac{1}{7!} a_0$ $\qquad$ $\vdots \quad \vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_1(x) = a_0 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)$$

$$\boxed{= \frac{a_0}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)}$$

$$r = -1 \quad \rightarrow \quad \boxed{b_n = -\frac{1}{n(n-1)} b_{n-2}}$$

$n=2 \rightarrow b_2 = -\frac{1}{2\cdot 1} b_0$ $\qquad$ $n=3 \rightarrow b_3 = -\frac{1}{6} b_1 = 0$

$n=4 \rightarrow b_4 = -\frac{1}{4\cdot 3} b_2 = \frac{1}{4!} b_0$ $\qquad$ $n=5 \rightarrow b_5 = 0$

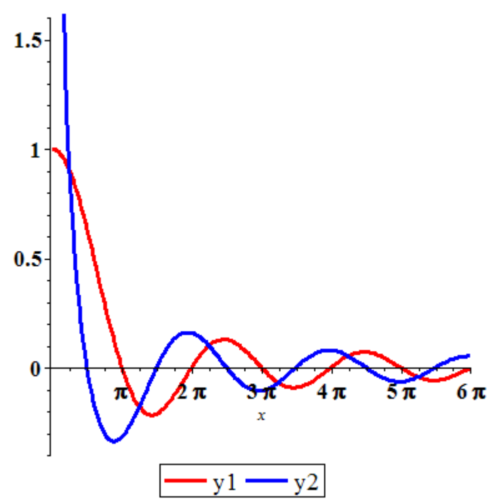
$n=6 \rightarrow b_6 = -\frac{1}{6\cdot 5} b_4 = -\frac{1}{6!} b_0$ $\qquad$ $\vdots \quad \vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = b_0 x^{-1}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)}$$

$$y(x) = \frac{a_0}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) + \frac{b_0}{x}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$\boxed{= a_0 \frac{\sin x}{x} + b_0 \frac{\cos x}{x}}$$



Exercise

Find the Frobenius series solutions of $2xy'' - y' + 2y = 0$

Solution

$$\left(\frac{x}{2}\right)2xy'' - \left(\frac{x}{2}\right)y' + \left(\frac{x}{2}\right)2y = 0$$

$$x^2y'' - \frac{1}{2}xy' + xy = 0$$

That implies to $p(x) = -\frac{1}{2}$ and $q(x) = x$, both are analytic.

Hence, $x_0 = 0$ is a regular point

The indicial equation is: $r(r-1) - \frac{1}{2}r = r\left(r - \frac{3}{2}\right) = 0 \quad \rightarrow \quad \underline{r = 0,\ \frac{3}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{3/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' - y' + 2y = 0$$

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + 2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+r)(n+r-1) - (n+r)\right] a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-3) a_n + 2a_{n-1}\right] x^{n+r-1} = 0$$

$$(n+r)(2n+2r-3) a_n + 2a_{n-1} = 0$$

$$\boxed{a_n = -\frac{2}{(n+r)(2n+2r-3)}a_{n-1}}$$

$$r=0 \quad \to \quad \boxed{a_n = -\frac{2}{n(2n-3)}a_{n-1}}$$

$n=1 \ \to \ a_1 = 2a_0$

$n=2 \ \to \ a_2 = -a_1 = -2a_0$

$n=3 \ \to \ a_3 = -\frac{2}{9}a_2 = \frac{4}{9}a_0$

$n=4 \ \to \ a_4 = -\frac{1}{10}a_3 = -\frac{2}{45}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1+2x-2x^2+\frac{4}{9}x^3-\frac{2}{45}x^4+\cdots\right)}$$

$$r=\frac{3}{2} \quad \to \quad \boxed{b_n = -\frac{1}{n\left(n+\frac{3}{2}\right)}b_{n-1}}$$

$n=1 \ \to \ b_1 = -\frac{2}{5}b_0$

$n=2 \ \to \ b_2 = -\frac{1}{7}b_1 = \frac{2}{35}b_0$

$n=3 \ \to \ b_3 = -\frac{2}{27}b_2 = -\frac{4}{945}b_0$

$n=4 \ \to \ b_4 = -\frac{1}{22}b_3 = \frac{2}{20,790}b_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_2(x) = b_0x^{3/2}\left(1-\frac{2}{5}x+\frac{2}{35}x^2-\frac{4}{945}x^3+\frac{2}{20,790}x^4-\cdots\right)$$

$$= b_0\sqrt{x}\left(x-\frac{2}{5}x^2+\frac{2}{35}x^3-\frac{4}{945}x^4+\frac{2}{20,790}x^5-\cdots\right)$$

$$\boxed{y(x) = a_0\left(1+2x-2x^2+\frac{4}{9}x^3-\frac{2}{45}x^4+\cdots\right)+b_0\sqrt{x}\left(x-\frac{2}{5}x^2+\frac{2}{35}x^3-\frac{4}{945}x^4+\frac{2}{20,790}x^5-\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2xy''+5y'+xy=0$

Solution

$$\left(\frac{x}{2}\right)2xy''+5\left(\frac{x}{2}\right)y'+\left(\frac{x}{2}\right)xy=0$$

$$x^2y''+\frac{5}{2}xy'+\frac{1}{2}x^2y=0$$

That implies to $p(x)=\frac{5}{2}$ and $q(x)=\frac{1}{2}x^2$, both are analytic.

Hence, $x_0 = 0$ is a regular point

The indicial equation is: $r(r-1)+\frac{5}{2}r = r^2 + \frac{3}{2}r = 0 \quad \rightarrow \quad \boxed{r = 0, \ -\frac{3}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{-3/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$2xy'' + 5y' + xy = 0$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty} 5(n+r)a_n x^{n+r-1} + x\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} 5(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty}\left[2(n+r)(n+r-1)+5(n+r)\right]a_n x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0$$

$$r(2r+3)a_0 + (r+1)(2r+5)a_1 + \sum_{n=2}^{\infty}(n+r)(2n+2r+3)a_n x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r-1} = 0$$

$$r(2r+3)a_0 + (r+1)(2r+5)a_1 + \sum_{n=2}^{\infty}\left[(n+r)(2n+2r+3)a_n + a_{n-2}\right]x^{n+r-1} = 0$$

For $n = 0 \quad \rightarrow \quad r(2r+3)a_0 = 0 \quad \Rightarrow \quad \boxed{r = 0 \ \ or \ r = -\frac{3}{2}}$ √

For $n = 1 \quad \rightarrow \quad (r+1)(2r+5)a_1 = 0 \quad \Rightarrow \quad \boxed{\cancel{r = -1, \ -\frac{5}{2}}} \quad \rightarrow \quad \boxed{a_1 = 0}$

$$(n+r)(2n+2r+3)a_n + a_{n-2} = 0$$

$$\boxed{a_n = -\frac{1}{(n+r)(2n+2r+3)}a_{n-2}}$$

$$r=0 \quad \rightarrow \quad \boxed{a_n = -\frac{1}{n(2n+3)} a_{n-2}}$$

$$n=2 \ \rightarrow \ a_2 = -\frac{1}{14} a_0 \qquad\qquad n=3 \ \rightarrow \ a_3 = -\frac{1}{27} a_1 = 0$$

$$n=4 \ \rightarrow \ a_4 = -\frac{1}{88} a_2 = \frac{1}{616} a_0 \qquad\qquad n=5 \ \rightarrow \ a_5 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 \left(1 - \frac{1}{14} x^2 + \frac{1}{616} x^4 - \cdots\right)}$$

$$r = -\frac{3}{2} \quad \rightarrow \quad b_n = -\frac{1}{2n\left(n-\frac{3}{2}\right)} b_{n-2} \ \boxed{= -\frac{1}{n(2n-3)} b_{n-2}}$$

$$n=2 \ \rightarrow \ b_2 = -\frac{1}{2} b_0 \qquad\qquad n=3 \ \rightarrow \ b_3 = -\frac{1}{9} b_1 = 0$$

$$n=4 \ \rightarrow \ b_4 = -\frac{1}{20} b_2 = \frac{1}{40} b_0 \qquad\qquad n=5 \ \rightarrow \ b_5 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\begin{aligned} y_2(x) &= b_0 x^{-3/2} \left(1 - \frac{1}{2} x^2 + \frac{1}{40} x^3 - \cdots\right) \\ &\boxed{= b_0 \left(x^{-3/2} - \frac{1}{2} x^{1/2} + \frac{1}{40} x^{3/2} - \cdots\right)} \end{aligned}$$

$$\boxed{y(x) = a_0 \left(1 - \frac{1}{14} x^2 + \frac{1}{616} x^4 - \cdots\right) + b_0 \left(x^{-3/2} - \frac{1}{2} x^{1/2} + \frac{1}{40} x^{3/2} - \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $4xy'' + \frac{1}{2} y' + y = 0$

Solution

$$\left(\frac{x}{4}\right) 4xy'' + \frac{1}{2}\left(\frac{x}{4}\right) y' + \left(\frac{x}{4}\right) y = 0$$

$$x^2 y'' + \frac{1}{8} xy' + \frac{1}{4} x^2 y = 0$$

$$y'' + \frac{1}{8x} y' + \frac{1}{4} y = 0$$

That implies to $p(x) = \frac{1}{8x}$ and $q(x) = \frac{1}{4}$

$$p_0 = \lim_{x \to 0} xp(x) = \lim_{x \to 0} x \frac{1}{8x} \ \boxed{= \frac{1}{8}}$$

$$q_0 = \lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 \frac{1}{8} \ \boxed{= 0}$$

The indicial equation is: $r(r-1) + \frac{1}{8} r = r^2 - \frac{7}{8} r = 0 \quad \rightarrow \quad \boxed{r = 0,\ \frac{7}{8}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{7/8} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$8xy'' + y' + 2y = 0$$

$$8x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 8(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[8(n+r)(n+r-1) + (n+r)\right] a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$r(8r-7)a_0 + \sum_{n=0}^{\infty} (n+r)(8n+8r-7) a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$r(8r-7)a_0 + \sum_{n=0}^{\infty} \left[(n+r)(8n+8r-7) a_n + 2a_{n-1}\right] x^{n+r-1} = 0$$

For $n = 0 \;\rightarrow\; r(8r-7)a_0 = 0 \;\Rightarrow\; \boxed{r = 0,\ \frac{7}{8}}$ $\checkmark$

$$(n+r)(8n+8r-7)a_n + 2a_{n-1} = 0$$

$$\boxed{a_n = -\frac{2}{(n+r)(8n+8r-7)} a_{n-1}}$$

$$r = 0 \;\rightarrow\; \boxed{a_n = -\frac{2}{n(8n-7)} a_{n-1}}$$

$$n = 1 \;\rightarrow\; a_1 = -2a_0$$

$$n = 2 \;\rightarrow\; a_2 = -\tfrac{1}{9} a_1 = \tfrac{2}{9} a_0$$

$$n = 3 \;\rightarrow\; a_3 = -\tfrac{2}{51} a_2 = -\frac{4}{459} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0\left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \cdots\right)}$$

$$r = \frac{7}{8} \quad \to \quad b_n = -\frac{2}{\left(n + \frac{7}{8}\right)(8n)} b_{n-1} = \boxed{-\frac{2}{n(8n+7)} b_{n-1}}$$

$$n = 1 \;\to\; b_1 = -\frac{2}{15} b_0$$

$$n = 2 \;\to\; b_2 = -\frac{1}{23} b_1 = \frac{2}{345} b_0$$

$$n = 3 \;\to\; b_3 = -\frac{2}{93} b_2 = -\frac{4}{32{,}085} b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{7/8}\left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32{,}085}x^3 + \cdots\right)}$$

$$\boxed{y(x) = a_0\left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \cdots\right) + b_0 x^{7/8}\left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32{,}085}x^3 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2x^2y'' - xy' + \left(x^2 + 1\right)y = 0$

Solution

$$\frac{1}{2}2x^2y'' - \frac{1}{2}xy' + \frac{1}{2}\left(x^2 + 1\right)y = 0$$

$$x^2y'' - \frac{1}{2}xy' + \frac{1}{2}\left(x^2 + 1\right)y = 0$$

$$y'' - \frac{1}{2x}y' + \left(\frac{1}{2} + \frac{1}{2x^2}\right)y = 0$$

That implies to $p(x) = -\frac{1}{2x}$ and $q(x) = \frac{1}{2} + \frac{1}{2x^2}$.

$$p_0 = \lim_{x\to 0} xp(x) = -\lim_{x\to 0} x\frac{1}{2x} = \boxed{-\frac{1}{2}}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\left(\frac{1}{2} + \frac{1}{2x^2}\right) = \lim_{x\to 0}\left(\frac{1}{2}x^2 + \frac{1}{2}\right) = \boxed{\frac{1}{2}}$$

The indicial equation is: $r(r-1) - \frac{1}{2}r + \frac{1}{2} = r^2 - \frac{3}{2}r + \frac{1}{2} = 0 \quad \to \quad \boxed{r = 1,\ \frac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^1\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$2x^2y''-xy'+\left(x^2+1\right)y=0$$

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}-x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\left(x^2+1\right)\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r}-\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}+\sum_{n=0}^{\infty}a_n x^{n+r+2}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left(2(n+r)(n+r-1)-(n+r)+1\right)a_n x^{n+r}+\sum_{n=2}a_{n-2}x^{n+r}$$

$$\left(r(2r-3)+1\right)a_0+\left((r+1)(2r-1)+1\right)a_1+\sum_{n=2}\left((n+r)(2n+2r-3)+1\right)a_n x^{n+r}+\sum_{n=2}a_{n-2}x^{n+r}=0$$

$$\left(2r^2-3r+1\right)a_0+\left(2r^2+r\right)a_1+\sum_{n=2}^{\infty}\left[\left((n+r)(2n+2r-3)+1\right)a_n+a_{n-2}\right]x^{n+r}=0$$

For $n=0 \quad \rightarrow \quad \left(2r^2-3r+1\right)a_0=0 \quad \Rightarrow \quad \boxed{r=1,\ \frac{1}{2}}$ $\checkmark$

For $n=1 \quad \rightarrow \quad \left(2r^2+r\right)a_1=0 \quad \Rightarrow \quad \boxed{\cancel{r=0,\ -\frac{1}{2}}} \quad \rightarrow \quad \boxed{a_1=0}$

$$\left((n+r)(2n+2r-3)+1\right)a_n+a_{n-2}=0$$

$$\boxed{a_n=-\frac{1}{(n+r)(2n+2r-3)+1}a_{n-2}}$$

$$r=1 \quad \rightarrow \quad a_n=-\frac{1}{(n+1)(2n-1)+1}a_{n-2}=\boxed{-\frac{1}{2n^2+n}a_{n-2}}$$

$n=2 \rightarrow a_2=-\frac{1}{10}a_0$ $\qquad$ $n=3 \rightarrow a_3=-\frac{1}{21}a_1=0$

$n=4 \rightarrow a_4=-\frac{1}{36}a_2=\frac{1}{360}a_0$ $\qquad$ $n=5 \rightarrow a_5=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$ $\qquad$ $\vdots \quad \vdots \quad \vdots \quad \vdots$

$$y_1(x) = a_0 x\left(1 - \frac{1}{10}x^2 + \frac{1}{360}x^4 - \cdots\right)$$

$$= a_0\left(x - \frac{1}{10}x^3 + \frac{1}{360}x^5 - \cdots\right)$$

$$r = \frac{1}{2} \quad \rightarrow \quad b_n = -\frac{1}{\left(n+\frac{1}{2}\right)(2n-2)+1} b_{n-2} = -\frac{1}{2n^2 - n} b_{n-2}$$

$$n = 2 \rightarrow b_2 = -\frac{1}{6}b_0 \qquad\qquad n = 3 \rightarrow a_3 = -\frac{1}{15}a_1 = 0$$

$$n = 4 \rightarrow b_4 = -\frac{1}{28}b_2 = \frac{1}{168}b_0 \qquad\qquad n = 5 \rightarrow a_5 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{1/2}\left(1 - \frac{1}{6}x^2 + \frac{1}{168}x^4 - \cdots\right)$$

$$y(x) = a_0\left(x - \frac{1}{10}x^3 + \frac{1}{360}x^5 - \cdots\right) + b_0 x^{1/2}\left(1 - \frac{1}{6}x^2 + \frac{1}{168}x^4 - \cdots\right)$$

Exercise

Find the Frobenius series solutions of $3xy'' + (2-x)y' - y = 0$

Solution

$$\frac{x}{3}3xy'' + \frac{x}{3}(2-x)y' - \frac{x}{3}y = 0$$

$$x^2 y'' + \left(\frac{2}{3}x - \frac{1}{3}x^2\right)y' - \frac{x}{3}y = 0$$

$$y'' + \left(\frac{2}{3x} - \frac{1}{3}\right)y' - \frac{1}{3x}y = 0$$

That implies to $p(x) = \frac{2}{3x} - \frac{1}{3}$ and $q(x) = -\frac{1}{3x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(\frac{2}{3x} - \frac{1}{3}\right) = \lim_{x\to 0}\left(\frac{2}{3} - \frac{1}{3}x\right) = \frac{2}{3}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = -\lim_{x\to 0} x^2 \frac{1}{3x} = \lim_{x\to 0}\frac{x}{3} = 0$$

The indicial equation is: $r(r-1) + \frac{2}{3}r = r^2 - \frac{1}{3}r = 0 \quad \rightarrow \quad r = 0, \ \frac{1}{3}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$3xy'' + (2-x)y' - y = 0$$

$$3x\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + (2-x)\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(3n+3r-3) + 2(n+r)\right] a_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r+1) a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(3n+3r-1) a_n x^{n+r-1} - \sum_{n=1}^{\infty} (n+r) a_{n-1} x^{n+r-1} = 0$$

$$r(3r-1)a_0 + \sum_{n=1}^{\infty} (n+r)(3n+3r-1) a_n x^{n+r-1} - \sum_{n=1}^{\infty} (n+r) a_{n-1} x^{n+r-1} = 0$$

$$r(3r-1)a_0 + \sum_{n=1}^{\infty} \left[(n+r)(3n+3r-1) a_n - (n+r) a_{n-1}\right] x^{n+r-1} = 0$$

For $n=0 \quad \rightarrow \quad r(3r-1)a_0 = 0 \quad \Rightarrow \quad \boxed{r = 0,\ \frac{1}{3}} \quad \checkmark$

$$(n+r)(3n+3r-1)a_n - (n+r)a_{n-1} = 0$$

$$\boxed{a_n = \frac{1}{3n+3r-1} a_{n-1}}$$

$$r = 0 \quad \rightarrow \quad \boxed{a_n = \frac{1}{3n-1} a_{n-1}}$$

$$n = 1 \ \rightarrow \ a_1 = \frac{1}{2} a_0$$

$$n = 2 \ \rightarrow \ a_2 = \frac{1}{5} a_1 = \frac{1}{10} a_0$$

$$n = 3 \ \rightarrow \ a_3 = \frac{1}{8} a_2 = \frac{1}{80} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0\left(1+\frac{1}{2}x+\frac{1}{10}x^2+\frac{1}{80}x^3+\cdots\right)}$$

$$r=\frac{1}{3} \quad \to \quad \boxed{b_n = \frac{1}{3n}b_{n-1}}$$

$$n=1 \ \to \ b_1 = \frac{1}{3}b_0$$

$$n=2 \ \to \ b_2 = \frac{1}{6}b_1 = \frac{1}{18}b_0$$

$$n=3 \ \to \ b_3 = \frac{1}{9}b_2 = \frac{1}{162}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/3}\left(1+\frac{1}{3}x+\frac{1}{18}x^2+\frac{1}{162}x^3+\cdots\right)}$$

$$\boxed{y(x) = a_0\left(1+\frac{1}{2}x+\frac{1}{10}x^2+\frac{1}{80}x^3+\cdots\right)+b_0 x^{1/3}\left(1+\frac{1}{3}x+\frac{1}{18}x^2+\frac{1}{162}x^3+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2xy'' - (3+2x)y' + y = 0$

Solution

$$\frac{x}{2}2xy'' - \frac{x}{2}(3+2x)y' + \frac{x}{2}y = 0$$

$$x^2y'' - \left(\frac{3}{2}x + x^2\right)y' + \frac{1}{2}xy = 0$$

$$y'' - \left(\frac{3}{2x}+1\right)y' + \frac{1}{2x}y = 0$$

That implies to $p(x) = -\frac{3}{2x} - 1$ and $q(x) = \frac{1}{2x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(-\frac{3}{2x}-1\right) = \lim_{x\to 0}\left(-\frac{3}{2}-x\right) \boxed{= -\frac{3}{2}}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\frac{1}{2x} = \lim_{x\to 0}\frac{x}{2} \boxed{=0}$$

The indicial equation is: $r(r-1) - \frac{3}{2}r = r^2 - \frac{5}{2}r = 0 \quad \to \quad \boxed{r = 0, \ \frac{5}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{5/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$2xy'' - (3+2x)y' + y = 0$$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - (3+2x)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty}3(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty}2(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[2(n+r)(n+r-1) - 3(n+r)\right]a_n x^{n+r-1} - \sum_{n=0}^{\infty}(2n+2r-1)a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-5)a_n x^{n+r-1} - \sum_{n=1}^{\infty}(2n+2r-3)a_{n-1} x^{n+r-1} = 0$$

$$r(2r-5)a_0 + \sum_{n=1}^{\infty}(n+r)(2n+2r-5)a_n x^{n+r-1} - \sum_{n=1}^{\infty}(2n+2r-3)a_{n-1} x^{n+r-1} = 0$$

$$r(2r-5)a_0 + \sum_{n=1}^{\infty}\left[(n+r)(2n+2r-5)a_n - (2n+2r-3)a_{n-1}\right]x^{n+r-1} = 0$$

For $n=0 \quad \rightarrow \quad r(2r-5)a_0 = 0 \quad \Rightarrow \quad \underline{r = 0, \ \tfrac{5}{2}}$ √

$$(n+r)(2n+2r-5)a_n - (2n+2r-3)a_{n-1} = 0$$

$$\boxed{a_n = \frac{2n+2r-3}{(n+r)(2n+2r-5)}a_{n-1}}$$

$$\boxed{r = 0 \quad \rightarrow \quad a_n = \frac{2n-3}{n(2n-5)}a_{n-1}}$$

$n=1 \ \rightarrow \ a_1 = \frac{1}{3}a_0$

$n=2 \ \rightarrow \ a_2 = -\frac{1}{2}a_1 = -\frac{1}{6}a_0$

$n=3 \ \rightarrow \ a_3 = -a_2 = -\frac{1}{6}a_0$

$n=4 \ \rightarrow \ a_4 = \frac{5}{12}a_3 = -\frac{5}{72}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x)=a_0\left(1+\frac{1}{3}x-\frac{1}{6}x^2-\frac{1}{6}x^3-\frac{5}{72}x^4-\cdots\right)}$$

$$r=\frac{5}{2} \quad\rightarrow\quad b_n=\frac{2n+2}{2n\left(n+\frac{5}{2}\right)}b_{n-1} \quad \boxed{=\frac{2n+2}{n(2n+5)}b_{n-1}}$$

$n=1 \ \rightarrow\ b_1=\frac{4}{7}b_0$

$n=2 \ \rightarrow\ b_2=\frac{1}{3}b_1=\frac{4}{21}b_0$

$n=3 \ \rightarrow\ b_3=\frac{8}{33}b_2=\frac{32}{693}b_0$

$n=4 \ \rightarrow\ b_4=\frac{5}{26}b_3=\frac{80}{9,009}b_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x)=b_0x^{5/2}\left(1+\frac{4}{7}x+\frac{4}{21}x^2+\frac{32}{693}x^3+\frac{80}{9,009}x^4+\cdots\right)}$$

$$\boxed{y(x)=a_0\left(1+\frac{1}{3}x-\frac{1}{6}x^2-\frac{1}{6}x^3-\frac{5}{72}x^4-\cdots\right)+b_0x^{5/2}\left(1+\frac{4}{7}x+\frac{4}{21}x^2+\frac{32}{693}x^3+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $xy''+(x-6)y'-3y=0$

Solution

$xxy''+x(x-6)y'-3xy=0$

$x^2y''+\left(x^2-6x\right)y'-3xy=0$

$y''+\left(1-\frac{6}{x}\right)y'-\frac{3}{x}y=0$

That implies to $p(x)=1-\frac{6}{x}$ and $q(x)=-\frac{3}{x}$.

$p_0=\lim\limits_{x\to 0}xp(x)=\lim\limits_{x\to 0}x\left(1-\frac{6}{x}\right)=\lim\limits_{x\to 0}(x-6)\ \boxed{=-6}$

$q_0=\lim\limits_{x\to 0}x^2q(x)=-\lim\limits_{x\to 0}x^2\frac{3}{x}=-\lim\limits_{x\to 0}3x\ \boxed{=0}$

The indicial equation is: $r(r-1)-6r=r^2-7r=0 \quad\rightarrow\quad \boxed{r=0,\ 7}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=x^0\sum_{n=0}^{\infty}a_nx^n \quad \textit{and} \quad y_2(x)=x^7\sum_{n=0}^{\infty}b_nx^n$$

$$y=\sum_{n=0}^{\infty}a_nx^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$xy'' + (x-6)y' - 3y = 0$$

$$x\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + (x-6)\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - 3\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} 6(n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} 3a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) - 6(n+r)\right] a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r-3) a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-7) a_n x^{n+r-1} + \sum_{n=1}^{\infty} (n+r-4) a_{n-1} x^{n+r-1} = 0$$

$$r(r-7)a_0 + \sum_{n=1}^{\infty} (n+r)(n+r-7) a_n x^{n+r-1} + \sum_{n=1}^{\infty} (n+r-4) a_{n-1} x^{n+r-1} = 0$$

$$r(r-7)a_0 + \sum_{n=1}^{\infty} \left[(n+r)(n+r-7) a_n + (n+r-4) a_{n-1}\right] x^{n+r-1} = 0$$

For $n=0 \quad \to \quad r(r-7)a_0 = 0 \quad \Rightarrow \quad \boxed{r = 0,\ \ 7}$ $\checkmark$

$$(n+r)(n+r-7)a_n + (n+r-4)a_{n-1} = 0$$

$$\boxed{a_n = -\frac{n+r-4}{(n+r)(n+r-7)} a_{n-1}}$$

$$r = 0 \quad \to \quad \boxed{a_n = -\frac{n-4}{n(n-7)} a_{n-1}}$$

$n = 1 \ \to \ a_1 = -\frac{1}{2} a_0$

$n = 2 \ \to \ a_2 = -\frac{1}{5} a_1 = \frac{1}{10} a_0$

$n = 3 \ \to \ a_3 = -\frac{1}{12} a_2 = -\frac{1}{120} a_0$

$n = 4 \ \to \ a_4 = 0 a_3 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3\right)}$$

$$\boxed{r = 7 \quad \to \quad b_n = -\frac{n+3}{n(n+7)}b_{n-1}}$$

$$n = 1 \ \to \ b_1 = -\frac{1}{2}b_0$$

$$n = 2 \ \to \ b_2 = -\frac{5}{18}b_1 = \frac{5}{36}b_0$$

$$n = 3 \ \to \ b_3 = -\frac{1}{5}b_2 = -\frac{1}{36}b_0$$

$$n = 4 \ \to \ b_4 = -\frac{7}{44}b_3 = \frac{7}{1{,}584}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0x^7\left(1 - \frac{1}{2}x + \frac{5}{36}x^2 - \frac{1}{36}x^3 + \frac{7}{1{,}584}x^4 - \cdots\right)}$$

$$\boxed{y(x) = a_0\left(1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3\right) + b_0x^7\left(1 - \frac{1}{2}x + \frac{5}{36}x^2 - \frac{1}{36}x^3 + \frac{7}{1{,}584}x^4 - \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $x(x-1)y'' + 3y' - 2y = 0$

Solution

$$\frac{1}{x}x(x-1)y'' + 3\frac{1}{x}y' - 2\frac{1}{x}y = 0$$

$$(x-1)y'' + \frac{3}{x}y' - \frac{2}{x}y = 0$$

That implies to $p(x) = \frac{3}{x}$ and $q(x) = -\frac{2}{x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{3}{x} \ \underline{= 3|}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = -\lim_{x\to 0} x^2\frac{2}{x} = -\lim_{x\to 0} 2x \ \underline{= 0|}$$

The indicial equation is: $-r(r-1) + 3r = -r^2 + 4r = 0 \quad \to \quad \underline{r = 0,\ 4\,|}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_nx^n \quad \textit{and} \quad y_2(x) = x^4\sum_{n=0}^{\infty} b_nx^n$$

$$y = \sum_{n=0}^{\infty} a_nx^{n+r}$$

$$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$x(x-1)y'' + 3y' - 2y = 0$$

$$\left(x^2 - x\right)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 3\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - 2\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1}$$

$$+\sum_{n=0}^{\infty}3(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty}2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)-2\right]a_n x^{n+r} - \sum_{n=0}^{\infty}\left[(n+r)(n+r-1)-3(n+r)\right]a_n x^{n+r-1} = 0$$

$$\sum_{n=1}^{\infty}\left[(n-1+r)(n+r-2)-2\right]a_{n-1} x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)(n+r-4)a_n x^{n+r-1} = 0$$

$$\sum_{n=1}^{\infty}\left[(n-1+r)(n+r-2)-2\right]a_{n-1} x^{n+r-1} - r(r-4)a_0 - \sum_{n=1}^{\infty}(n+r)(n+r-4)a_n x^{n+r-1} = 0$$

$$-r(r-4)a_0 + \sum_{n=1}^{\infty}\left[\left((n+r-1)(n+r-2)-2\right)a_{n-1} - (n+r)(n+r-4)a_n\right]x^{n+r-1} = 0$$

For $n=0 \quad \rightarrow \quad -r(r-4)a_0 = 0 \quad \Rightarrow \quad \underline{r = 0,\ \ 4}$ √

$$\left((n+r-1)(n+r-2)-2\right)a_{n-1} - (n+r)(n+r-4)a_n = 0$$

$$\boxed{a_n = \frac{(n+r-1)(n+r-2)-2}{(n+r)(n+r-4)}a_{n-1}}$$

$$\boxed{r = 0 \quad \rightarrow \quad a_n = \frac{(n-1)(n-2)-2}{n(n-4)}a_{n-1}}$$

$$n = 1 \ \rightarrow \ a_1 = \frac{2}{3}a_0$$

$$n = 2 \ \rightarrow \ a_2 = \frac{1}{2}a_1 = \frac{1}{3}a_0$$

$$n=3 \;\to\; a_3 = \frac{0}{3}a_2 = 0$$

$$n=4 \;\to\; a_4 = 0a_3 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0\left(1+\frac{2}{3}x+\frac{1}{3}x^2\right)}$$

$$\boxed{r=4 \quad \to \quad b_n = \frac{(n+3)(n+2)-2}{n(n+4)}b_{n-1}}$$

$$n=1 \;\to\; b_1 = 2b_0$$

$$n=2 \;\to\; b_2 = \frac{3}{2}b_1 = 3b_0$$

$$n=3 \;\to\; b_3 = \frac{28}{21}b_2 = 4b_0$$

$$n=4 \;\to\; b_4 = \frac{5}{4}b_3 = 5b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0x^4\left(1+2x+3x^2+4x^3+5x^4+\cdots\right)}$$

$$\boxed{y(x) = a_0\left(1+\frac{2}{3}x+\frac{1}{3}x^2\right) + b_0\left(x^4+2x^5+3x^6+4x^7+5x^8+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $x^2y'' - \left(x-\frac{2}{9}\right)y = 0$

Solution

$$x^2y'' - \left(x-\frac{2}{9}\right)y = 0$$

$$y'' - \left(\frac{1}{x} - \frac{2}{9x^2}\right)y = 0$$

That implies to $p(x)=0$ and $q(x) = \frac{2}{9x^2} - \frac{1}{x}$.

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\left(\frac{2}{9x^2}-\frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{2}{9}-x\right) = \boxed{\frac{2}{9}}$$

The indicial equation is: $r(r-1)+\frac{2}{9} = r^2 - r + \frac{2}{9} = 0$

$$9r^2 - 9r + 2 = 0 \quad \to \quad r = \frac{9\pm 3}{18} = \boxed{\frac{1}{3},\ \frac{2}{3}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/3}\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{2/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x^2 y'' - \left(x - \tfrac{2}{9}\right) y = 0$$

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - \left(x - \tfrac{2}{9}\right) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} + \sum_{n=0}^{\infty} \tfrac{2}{9} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + \tfrac{2}{9}\right] a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0$$

$$\left(r^2 - r + \tfrac{2}{9}\right) a_0 + \sum_{n=1}^{\infty} \left[(n+r)(n+r-1) + \tfrac{2}{9}\right] a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0$$

$$\left(r^2 - r + \tfrac{2}{9}\right) a_0 + \sum_{n=1}^{\infty} \left[\left((n+r)(n+r-1) + \tfrac{2}{9}\right) a_n - a_{n-1}\right] x^{n+r} = 0$$

For $n = 0 \quad \to \quad \left(r^2 - r + \tfrac{2}{9}\right) a_0 = 0 \quad \Rightarrow \quad \boxed{r = \tfrac{1}{3},\ \tfrac{2}{3}}$ √

$$\left((n+r)(n+r-1) + \tfrac{2}{9}\right) a_n - a_{n-1} = 0$$

$$\boxed{a_n = \frac{1}{(n+r)(n+r-1) + \frac{2}{9}} a_{n-1}}$$

$$r = \tfrac{1}{3} \quad \to \quad a_n = \frac{1}{\left(n + \frac{1}{3}\right)\left(n - \frac{2}{3}\right) + \frac{2}{9}} a_{n-1}$$

$$= \frac{1}{n^2 - \frac{1}{3} n} a_{n-1}$$

$$= \frac{3}{3n^2 - n} a_{n-1}$$

$$n = 1 \;\to\; a_1 = \frac{3}{2} a_0$$

$$n = 2 \;\to\; a_2 = \frac{3}{10} a_1 = \frac{9}{20} a_0$$

$$n = 3 \;\to\; a_3 = \frac{1}{8} a_2 = \frac{9}{160} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x) = a_0 x^{1/3}\left(1 + \frac{3}{2}x + \frac{9}{20}x^2 + \frac{9}{160}x^3 + \cdots\right)$$

$$r = \frac{2}{3} \quad \to \quad b_n = \frac{1}{\left(n + \frac{2}{3}\right)\left(n - \frac{1}{3}\right) + \frac{2}{9}} b_{n-1}$$

$$= \frac{3}{3n^2 + n} b_{n-1}$$

$$n = 1 \;\to\; b_1 = \frac{3}{4} b_0$$

$$n = 2 \;\to\; b_2 = \frac{3}{14} b_1 = \frac{9}{56} b_0$$

$$n = 3 \;\to\; b_3 = \frac{1}{10} b_2 = \frac{9}{560} b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{2/3}\left(1 + \frac{3}{4}x + \frac{9}{56}x^2 + \frac{9}{560}x^3 + \cdots\right)$$

$$y(x) = a_0 x^{1/3}\left(1 + \frac{3}{2}x + \frac{9}{20}x^2 + \frac{9}{160}x^3 + \cdots\right) + b_0 x^{2/3}\left(1 + \frac{3}{4}x + \frac{9}{56}x^2 + \frac{9}{560}x^3 + \cdots\right)$$

Exercise

Find the Frobenius series solutions of $x^2 y'' + x(3 + x)y' - 3y = 0$

Solution

$$\frac{1}{x^2} x^2 y'' + \frac{1}{x^2} x(3 + x)y' - 3\frac{1}{x^2} y = 0$$

$$y'' + \left(\frac{3}{x} + 1\right)y' - \frac{3}{x^2} y = 0$$

That implies to $p(x) = \frac{3}{x} + 1$ and $q(x) = -\frac{3}{x^2}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(\frac{3}{x} + 1\right) = \lim_{x\to 0}(3 + x) = 3$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = -\lim_{x\to 0} x^2 \frac{3}{x^2} = -3$$

The indicial equation is: $r(r-1)+3r-3=r^2+2r-3=0 \quad \rightarrow \quad \underline{r=1,\ -3|}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=x\sum_{n=0}^{\infty}a_n x^n \quad \text{and} \quad y_2(x)=x^{-3}\sum_{n=0}^{\infty}b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$x^2y''+x(3+x)y'-3y=0$$

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\left(3x+x^2\right)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-3\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}3(n+r)a_n x^{n+r}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1}-\sum_{n=0}^{\infty}3a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+3(n+r)-3\right]a_n x^{n+r}+\sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r}=0$$

$$\left(r^2+2r-3\right)a_0+\sum_{n=1}^{\infty}\left[(n+r)(n+r+2)-3\right]a_n x^{n+r}+\sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r}=0$$

$$\left(r^2+2r-3\right)a_0+\sum_{n=1}^{\infty}\left[\left((n+r)(n+r+2)-3\right)a_n+(n+r-1)a_{n-1}\right]x^{n+r}=0$$

For $n=0 \quad \rightarrow \quad \left(r^2+2r-3\right)a_0=0 \quad \Rightarrow \quad \underline{r=1,\ -3|}$ $\checkmark$

$$\left((n+r)(n+r+2)-3\right)a_n+(n+r-1)a_{n-1}=0$$

$$\underline{a_n=-\frac{n+r-1}{(n+r)(n+r+2)-3}a_{n-1}\Big|}$$

$$r=1 \quad \rightarrow \quad a_n=-\frac{n}{(n+1)(n+3)-3}a_{n-1}$$

$$\underline{=-\frac{n}{n^2+4n}a_{n-1}\Big|}$$

$$n=1 \;\to\; a_1 = -\frac{1}{5}a_0$$

$$n=2 \;\to\; a_2 = -\frac{2}{12}a_1 = \frac{1}{30}a_0$$

$$n=3 \;\to\; a_3 = -\frac{3}{21}a_2 = -\frac{1}{210}a_0$$

$$n=4 \;\to\; a_4 = -\frac{4}{32}a_3 = \frac{1}{1,680}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x\left(1-\frac{1}{5}x+\frac{1}{30}x^2-\frac{1}{210}x^3+\frac{1}{1,680}x^4-\cdots\right)}$$

$$r=-3 \;\to\; b_n = -\frac{n-4}{(n-3)(n-1)-3}b_{n-1}$$

$$\boxed{= -\frac{n-4}{n^2-4n}b_{n-1}}$$

$$n=1 \;\to\; b_1 = -b_0$$

$$n=2 \;\to\; b_2 = -\frac{2}{4}b_1 = \frac{1}{2}b_0$$

$$n=3 \;\to\; b_3 = -\frac{1}{3}b_2 = -\frac{1}{6}b_0$$

$$n=4 \;\to\; b_4 = 0b_3 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{-3}\left(1-x+\frac{1}{2}x^2-\frac{1}{6}x^3\right)}$$

$$\boxed{y(x) = a_0 x\left(1-\frac{1}{5}x+\frac{1}{30}x^2-\frac{1}{210}x^3+\frac{1}{1,680}x^4-\cdots\right)+b_0 x^{-3}\left(1-x+\frac{1}{2}x^2-\frac{1}{6}x^3\right)}$$

Exercise

Find the Frobenius series solutions of $x^2y''+\left(x^2-2x\right)y'+2y=0$

<u>Solution</u>

$$\frac{1}{x^2}x^2y''+\frac{1}{x^2}\left(x^2-2x\right)y'+2\frac{1}{x^2}y=0$$

$$y''+\left(1-\frac{2}{x}\right)y'+\frac{2}{x^2}y=0$$

That implies to $p(x)=1-\frac{2}{x}$ and $q(x)=\frac{2}{x^2}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(1-\frac{2}{x}\right) = \lim_{x\to 0}(x-2) \; \underline{=-2}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{2}{x^2} \underline{=2|}$$

The indicial equation is: $r(r-1)-2r+2 = r^2-3r+2=0 \quad\to\quad \underline{r=1,\ 2|}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^2\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$x^2y'' + \left(x^2-2x\right)y' + 2y = 0$$

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \left(x^2-2x\right)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} - \sum_{n=0}^{\infty}2(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)-2(n+r)+2\right]a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-3)+2\right]a_n x^{n+r} + \sum_{n=1}^{\infty}(n+r-1)a_{n-1} x^{n+r} = 0$$

$$\left(r^2-3r+2\right)a_0 + \sum_{n=1}^{\infty}\left[(n+r)(n+r-3)+2\right]a_n x^{n+r} + \sum_{n=1}^{\infty}(n+r-1)a_{n-1} x^{n+r} = 0$$

$$\left(r^2-3r+2\right)a_0 + \sum_{n=1}^{\infty}\left[\left((n+r)(n+r-3)+2\right)a_n + (n+r-1)a_{n-1}\right]x^{n+r} = 0$$

For $n=0 \quad\to\quad \left(r^2-3r+2\right)a_0 = 0 \quad\Rightarrow\quad \underline{r=1,\ 2|}$ $\checkmark$

$$\left((n+r)(n+r-3)+2\right)a_n + (n+r-1)a_{n-1} = 0$$

$$a_n = -\frac{n+r-1}{(n+r-1)(n+r-2)}a_{n-1} \ \boxed{= -\frac{1}{n+r-2}a_{n-1}}$$

$$r = 2 \quad \to \quad \boxed{a_n = = -\frac{1}{n}a_{n-1}}$$

$$n = 1 \ \to \ a_1 = -a_0$$

$$n = 2 \ \to \ a_2 = -\frac{1}{2}a_1 = \frac{1}{2}a_0$$

$$n = 3 \ \to \ a_3 = -\frac{1}{3}a_2 = -\frac{1}{3!}a_0$$

$$n = 4 \ \to \ a_4 = -\frac{1}{4}a_3 = \frac{1}{4!}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x\left(1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \cdots\right)}$$

$$r = 1 \quad \to \quad \boxed{b_n = -\frac{1}{n-1}b_{n-1}}$$

Since $n \neq 1$

$$\boxed{\begin{aligned} y(x) = a_0 x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots\right) + b_0 x\ln x\left(-x + x^2 - \frac{x^3}{2} + \frac{x^4}{6} - \cdots\right) \\ + x\ln x\left(1 - x + \frac{x^3}{4} - \frac{5}{36}x^4 + \cdots\right) \end{aligned}}$$

Exercise

Find the Frobenius series solutions of $x^2y'' + \left(x^2 + 2x\right)y' - 2y = 0$

Solution

$$\frac{1}{x^2}x^2y'' + \frac{1}{x^2}\left(x^2 + 2x\right)y' - 2\frac{1}{x^2}y = 0$$

$$y'' + \left(1 + \frac{2}{x}\right)y' - \frac{2}{x^2}y = 0$$

That implies to $p(x) = 1 + \frac{2}{x}$ and $q(x) = -\frac{2}{x^2}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(1 + \frac{2}{x}\right) = \lim_{x\to 0}(x+2) \ \boxed{= 2}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = -\lim_{x\to 0} x^2\frac{2}{x^2} \ \boxed{= -2}$$

The indicial equation is: $\quad r(r-1) + 2r - 2 = r^2 + r - 2 = 0 \quad \to \quad \boxed{r = 1, \ -2}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x^2 y'' + \left(x^2 + 2x\right) y' - 2y = 0$$

$$x^2\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \left(x^2 + 2x\right)\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r+1} + \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r} - 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1) + 2(n+r) - 2\right] a_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^{n+r} = 0$$

$$\left(r^2 + r - 2\right) a_0 + \sum_{n=1}^{\infty} \left[(n+r)(n+r+1) - 2\right] a_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^{n+r} = 0$$

$$\left(r^2 + r - 2\right) a_0 + \sum_{n=0}^{\infty} \left[\left((n+r)(n+r+1) - 2\right) a_n + (n+r-1) a_{n-1}\right] x^{n+r} = 0$$

For $n = 0 \;\rightarrow\; \left(r^2 + r - 2\right) a_0 = 0 \;\Rightarrow\; \underline{r = 1,\ 2}$ √

$$\left((n+r)^2 + (n+r) - 2\right) a_n + (n+r-1) a_{n-1} = 0$$

$$a_n = -\frac{n+r-1}{(n+r-1)(n+r+2)} a_{n-1} = -\frac{1}{n+r+2} a_{n-1}$$

$$r = 1 \;\rightarrow\; a_n == -\frac{1}{n+3} a_{n-1}$$

$$n = 1 \;\rightarrow\; a_1 = -\frac{1}{4} a_0$$

$$n = 2 \;\rightarrow\; a_2 = -\frac{1}{5} a_1 = \frac{1}{20} a_0$$

$$n = 3 \;\rightarrow\; a_3 = -\frac{1}{6} a_2 = -\frac{1}{120} a_0$$

$$n=4 \;\rightarrow\; a_4 = -\tfrac{1}{7}a_3 = \frac{1}{840}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x\left(1 - \tfrac{1}{4}x + \tfrac{1}{20}x^2 - \tfrac{1}{120}x^3 + \tfrac{1}{840}x^4 - \cdots\right)}$$

$$r=2 \;\rightarrow\; b_n == \boxed{-\frac{1}{n+4}b_{n-1}}$$

$$n=1 \;\rightarrow\; b_1 = -\tfrac{1}{5}b_0$$

$$n=2 \;\rightarrow\; b_2 = -\tfrac{1}{6}b_1 = \frac{1}{30}b_0$$

$$n=3 \;\rightarrow\; b_3 = -\tfrac{1}{7}b_2 = -\frac{1}{210}b_0$$

$$n=4 \;\rightarrow\; b_4 = -\tfrac{1}{8}b_3 = \frac{1}{1,680}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^2\left(1 - \frac{x}{5} + \frac{x^2}{30} - \frac{x^3}{210} + \frac{x^4}{1,680} - \cdots\right)}$$

$$\boxed{y(x) = a_0 x\left(1 - \frac{x}{4} + \frac{x^2}{20} - \frac{x^3}{120} + \frac{x^4}{840} - \cdots\right) + b_0 x^2\left(1 - \frac{x}{5} + \frac{x^2}{30} - \frac{x^3}{210} + \frac{x^4}{1,680} - \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2xy'' + 3y' - y = 0$

Solution

$$\frac{x}{2}2xy'' + 3\frac{x}{2}y' - \frac{x}{2}y = 0$$

$$x^2y'' + \tfrac{3}{2}xy' - \tfrac{1}{2}xy = 0$$

$$y'' + \frac{3}{2x}y' - \frac{1}{2x}y = 0$$

That implies to $p(x) = \dfrac{3}{2x}$ and $q(x) = -\dfrac{1}{2x}$

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{3}{2x} = \frac{3}{2}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\frac{1}{2x} = 0$$

The indicial equation is: $r(r-1) + \tfrac{3}{2}r = r^2 + \tfrac{1}{2}r = 0 \;\rightarrow\; \boxed{r = 0,\ -\tfrac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{-1/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$2xy''+3y'-y=0$$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\sum_{n=0}^{\infty}3(n+r)a_n x^{n+r-1}-\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}3(n+r)a_n x^{n+r-1}-\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[2(n+r)(n+r-1)a_n+3(n+r)\right]x^{n+r-1}-\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}=0$$

$$r(2r+1)a_0+\sum_{n=1}^{\infty}(n+r)(2n+2r+1)a_n x^{n+r-1}-\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}=0$$

$$r(2r+1)a_0+\sum_{n=1}^{\infty}\left[(n+r)(2n+2r+1)a_n-a_{n-1}\right]x^{n+r-1}=0$$

For $n=0 \rightarrow r(2r+1)a_0=0 \Rightarrow \boxed{r=0,\ -\tfrac{1}{2}}$ √

$$(n+r)(2n+2r+1)a_n-a_{n-1}=0$$

$$\boxed{a_n=\frac{1}{(n+r)(2n+2r+1)}a_{n-1}}$$

$$\boxed{r=0 \rightarrow a_n=\frac{1}{n(2n+1)}a_{n-1}}$$

$n=1 \rightarrow a_1=\frac{1}{3}a_0$

$n=2 \rightarrow a_2=\frac{1}{15}a_1=\frac{1}{30}a_0$

$n=3 \rightarrow a_3=\frac{1}{21}a_2=\frac{1}{630}a_0$

$n=4 \rightarrow a_4=\frac{1}{36}a_3=\frac{1}{22,680}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1+\frac{1}{3}x+\frac{1}{30}x^2+\frac{1}{630}x^3+\frac{1}{22,680}x^4+\cdots\right)}$$

$$\boxed{r=-\frac{1}{2} \quad \to \quad b_n = \frac{1}{n(2n-1)}b_{n-1}}$$

$$n=1 \ \to\ b_1 = b_0$$

$$n=2 \ \to\ b_2 = \frac{1}{6}b_1 = \frac{1}{6}b_0$$

$$n=3 \ \to\ b_3 = \frac{1}{15}b_2 = \frac{1}{90}b_0$$

$$n=4 \ \to\ b_4 = \frac{1}{28}b_3 = \frac{1}{2,520}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{-1/2}\left(1+x+\frac{1}{6}x^2+\frac{1}{90}x^3+\frac{1}{2,520}x^4+\cdots\right)}$$

$$\boxed{y(x) = a_0\left(1+\frac{1}{3}x+\frac{1}{30}x^2+\frac{1}{630}x^3+\frac{1}{22,680}x^4+\cdots\right)+b_0x^{-1/2}\left(1+x+\frac{1}{6}x^2+\frac{1}{90}x^3+\frac{1}{2,520}x^4+\cdots\right)}$$

$$\boxed{y(x) = a_0\sum_{n=0}^{\infty}\frac{x^n}{n!(2n+1)!!}+\frac{b_0}{\sqrt{x}}\left(1+\sum_{n=0}^{\infty}\frac{x^n}{n!(2n-1)!!}\right)}$$

Exercise

Find the Frobenius series solutions of $2xy''-y'-y=0$

Solution

$$\frac{1}{2x}2xy''-\frac{1}{2x}y'-\frac{1}{2x}y=0$$

$$y''-\frac{1}{2x}y'-\frac{1}{2x}y=0$$

That implies to $p(x)=-\frac{1}{2x}$ and $q(x)=\frac{1}{2x}$

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{-1}{2x} = -\frac{1}{2}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\frac{-1}{2x} = -\lim_{x\to 0}\frac{x}{2} = 0$$

The indicial equation is: $r(r-1)-\frac{1}{2}r = r^2-\frac{3}{2}r=0 \quad \to \quad \boxed{r=0,\ \frac{3}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty}a_nx^n \quad \textit{and} \quad y_2(x) = x^{3/2}\sum_{n=0}^{\infty}b_nx^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' - y' - y = 0$$

$$2x\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+r)(n+r-1) a_n - (n+r) \right] x^{n+r-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$r(2r-3)a_0 + \sum_{n=1}^{\infty} (n+r)(2n+2r-3) a_n x^{n+r-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$r(2r-3)a_0 + \sum_{n=1}^{\infty} \left[(n+r)(2n+2r-3) a_n - a_{n-1} \right] x^{n+r-1} = 0$$

For $n = 0 \quad \rightarrow \quad r(2r-3)a_0 = 0 \quad \Rightarrow \quad \boxed{r = 0, \ \frac{3}{2}}$ $\surd$

$$(n+r)(2n+2r-3)a_n - a_{n-1} = 0$$

$$\boxed{a_n = \frac{1}{(n+r)(2n+2r-3)} a_{n-1}}$$

$$\boxed{r = 0 \quad \rightarrow \quad a_n = \frac{1}{n(2n-3)} a_{n-1}}$$

$n = 1 \ \rightarrow \ a_1 = -a_0$

$n = 2 \ \rightarrow \ a_2 = \frac{1}{2} a_1 = -\frac{1}{2} a_0$

$n = 3 \ \rightarrow \ a_3 = \frac{1}{9} a_2 = -\frac{1}{18} a_0$

$n = 4 \ \rightarrow \ a_4 = \frac{1}{20} a_3 = -\frac{1}{360} a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1 - x - \frac{1}{2}x^2 - \frac{1}{18}x^3 - \frac{1}{360}x^4 - \cdots\right)}$$

$$\boxed{r = \frac{3}{2} \quad \to \quad b_n = \frac{1}{n(2n+3)}b_{n-1}}$$

$$n = 1 \;\to\; b_1 = \frac{1}{5}b_0$$

$$n = 2 \;\to\; b_2 = \frac{1}{14}b_1 = \frac{1}{70}b_0$$

$$n = 3 \;\to\; b_3 = \frac{1}{27}b_2 = \frac{1}{1890}b_0$$

$$n = 4 \;\to\; b_4 = \frac{1}{44}b_3 = \frac{1}{83,160}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{3/2}\left(1 + \frac{1}{5}x + \frac{1}{70}x^2 + \frac{1}{1890}x^3 + \frac{1}{83,160}x^4 + \cdots\right)}$$

$$\boxed{y(x) = a_0\left(1 - x - \frac{1}{2}x^2 - \frac{1}{18}x^3 - \frac{1}{360}x^4 + \cdots\right) + b_0 x^{3/2}\left(1 + \frac{1}{5}x + \frac{1}{70}x^2 + \frac{1}{1,890}x^3 + \frac{1}{83,160}x^4 - \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2xy'' + (1+x)y' + y = 0$

Solution

$$\frac{1}{2x}2xy'' + \frac{1}{2x}(1+x)y' + \frac{1}{2x}y = 0$$

$$y'' + \left(\frac{1}{2x} + \frac{1}{2}\right)y' + \frac{1}{2x}y = 0$$

That implies to $p(x) = \frac{1}{2x} + \frac{1}{2}$ and $q(x) = \frac{1}{2x}$

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(\frac{1}{2x} + \frac{1}{2}\right) = \lim_{x\to 0}\left(\frac{1}{2} + \frac{1}{2}x\right) = \frac{1}{2}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{1}{2x} = \lim_{x\to 0}\frac{x}{2} = 0$$

The indicial equation is: $r(r-1) + \frac{1}{2}r = r^2 - \frac{1}{2}r = 0 \quad \to \quad \boxed{r = 0,\ \frac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy'' + (1+x)y' + y = 0$$

$$2x\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + (1+x)\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+r)(n+r-1) + (n+r)\right] a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r+1) a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-1)\right] a_n x^{n+r-1} + \sum_{n=1}^{\infty} (n+r) a_{n-1} x^{n+r-1} = 0$$

$$r(2r-1)a_0 + \sum_{n=1}^{\infty} (n+r)(2n+2r-1) a_n x^{n+r-1} + \sum_{n=1}^{\infty} (n+r) a_{n-1} x^{n+r-1} = 0$$

$$r(2r-1)a_0 + \sum_{n=1}^{\infty} \left[(n+r)(2n+2r-1) a_n + (n+r) a_{n-1}\right] x^{n+r-1} = 0$$

For $n = 0 \quad \rightarrow \quad r(2r-1)a_0 = 0 \quad \Rightarrow \quad \boxed{r = 0, \ \frac{1}{2}}$ √

$$(n+r)(2n+2r-1)a_n + (n+r)a_{n-1} = 0$$

$$\boxed{a_n = -\frac{1}{2n+2r-1} a_{n-1}}$$

$$\boxed{r = 0 \quad \rightarrow \quad a_n = -\frac{1}{2n-1} a_{n-1}}$$

$n = 1 \rightarrow a_1 = -a_0$

$n = 2 \rightarrow a_2 = \frac{1}{3} a_1 = -\frac{1}{3} a_0$

$n = 3 \rightarrow a_3 = -\frac{1}{5} a_2 = \frac{1}{15} a_0$

$n = 4 \rightarrow a_4 = -\frac{1}{7} a_3 = -\frac{1}{105} a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1 - x + \frac{1}{3}x^2 - \frac{1}{15}x^3 + \frac{1}{105}x^4 - \cdots\right)}$$

$$\boxed{r = \frac{1}{2} \quad \to \quad b_n = -\frac{1}{2n}b_{n-1}}$$

$$n = 1 \ \to \ b_1 = -\frac{1}{2}b_0$$

$$n = 2 \ \to \ b_2 = -\frac{1}{4}b_1 = \frac{1}{8}b_0$$

$$n = 3 \ \to \ b_3 = -\frac{1}{6}b_2 = -\frac{1}{48}b_0$$

$$n = 4 \ \to \ b_4 = -\frac{1}{8}b_3 = -\frac{1}{384}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0x^{1/2}\left(1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4 + \cdots\right)}$$

$$\boxed{y(x) = a_0\left(1 - x + \frac{1}{3}x^2 - \frac{1}{15}x^3 + \frac{1}{105}x^4 - \cdots\right) + b_0x^{1/2}\left(1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2xy'' + \left(1 - 2x^2\right)y' - 4xy = 0$

Solution

$$\frac{x}{2}2xy'' + \frac{x}{2}\left(1 - 2x^2\right)y' - \frac{x}{2}4xy = 0$$

$$x^2y'' + \left(\frac{1}{2}x - x^3\right)y' + 2x^2y = 0$$

$$y'' + \left(\frac{1}{2x} - x\right)y' + 2y = 0$$

That implies to $p(x) = \frac{1}{2x} - x$ and $q(x) = 2$

$$p_0 = \lim_{x \to 0} xp(x) = \lim_{x \to 0} x\left(\frac{1}{2x} - x\right) = \lim_{x \to 0}\left(\frac{1}{2} - x^2\right) = \frac{1}{2}$$

$$q_0 = \lim_{x \to 0} x^2q(x) = \lim_{x \to 0} 2x^2 = 0$$

The indicial equation is: $r(r-1) + \frac{1}{2}r = r^2 - \frac{1}{2}r = 0 \quad \to \quad \boxed{r = 0, \ \frac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty}a_nx^n \quad \textit{and} \quad y_2(x) = x^{1/2}\sum_{n=0}^{\infty}b_nx^n$$

$$y=\sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2xy''+\left(1-2x^2\right)y'-4xy=0$$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\left(1-2x^2\right)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-4x\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-\sum_{n=0}^{\infty}2(n+r)a_n x^{n+r+1}-\sum_{n=0}^{\infty}4a_n x^{n+r+1}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2)+(n+r)\right]a_n x^{n+r-1}-\sum_{n=0}^{\infty}(2n+2r+4)a_n x^{n+r+1}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2)+(n+r)\right]a_n x^{n+r-1}-\sum_{n=2}^{\infty}(2n+2r)a_{n-2} x^{n+r-1}=0$$

$$r(2r-1)a_0+(r+2)(2r+1)a_1+\sum_{n=2}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1}-\sum_{n=2}^{\infty}(2n+2r)a_{n-2} x^{n+r-1}=0$$

$$r(2r-1)a_0+(r+2)(2r+1)a_1+\sum_{n=2}^{\infty}\left[(n+r)(2n+2r-1)a_n-2(n+r)a_{n-2}\right]x^{n+r-1}=0$$

For $n=0 \quad\rightarrow\quad r(2r+1)a_0=0 \quad\Rightarrow\quad \boxed{r=0,\ -\frac{1}{2}}$ √

For $n=1 \quad\rightarrow\quad (r+2)(2r+1)a_1=0 \quad\Rightarrow\quad \boxed{\cancel{r=-2,\ -\frac{1}{2}}} \qquad \boxed{a_1=0}$

$$(n+r)(2n+2r-1)a_n-2(n+r)a_{n-2}=0$$

$$\boxed{a_n=\frac{2}{2n+2r-1}a_{n-2}}$$

$$\boxed{r=0 \quad\rightarrow\quad a_n=\frac{2}{2n-1}a_{n-2}}$$

$n=2 \rightarrow a_2=\frac{2}{3}a_0$ $\qquad$ $n=3 \rightarrow a_3=\frac{2}{5}a_1=0$

$n=4 \rightarrow a_4=\frac{2}{7}a_2=\frac{4}{21}a_0$ $\qquad$ $n=5 \rightarrow a_5=\frac{2}{9}a_3=0$

$n=6 \rightarrow a_6 = \frac{2}{11}a_4 = \frac{8}{231}a_0$

$n=8 \rightarrow a_8 = \frac{2}{15}a_6 = \frac{16}{3,465}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$n=7 \rightarrow a_7 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0\left(1+\frac{2}{3}x^2+\frac{4}{21}x^4+\frac{8}{231}x^6+\frac{16}{3465}x^8+\cdots\right)}$$

$$\boxed{r=\frac{1}{2} \quad \rightarrow \quad b_n = \frac{1}{n}b_{n-2}}$$

$n=2 \rightarrow b_2 = \frac{1}{2}b_0$

$n=4 \rightarrow b_4 = \frac{1}{4}b_2 = \frac{1}{8}b_0$

$n=6 \rightarrow b_6 = \frac{1}{6}b_4 = \frac{1}{48}b_0$

$n=8 \rightarrow b_8 = \frac{1}{8}b_6 = \frac{1}{384}b_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$n=3 \rightarrow b_3 = \frac{1}{3}b_1 = 0$

$n=5 \rightarrow b_5 = \frac{1}{5}b_3 = 0$

$n=7 \rightarrow b_7 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = b_0 x^{1/2}\left(1+\frac{1}{2}x^2+\frac{1}{8}x^4+\frac{1}{48}x^6+\frac{1}{384}x^8+\cdots\right)}$$

$$\boxed{y(x) = a_0\left(1+\frac{2}{3}x^2+\frac{4}{21}x^4+\frac{8}{231}x^6+\frac{16}{3465}x^8+\cdots\right)+b_0 x^{1/2}\left(1+\frac{1}{2}x^2+\frac{1}{8}x^4+\frac{1}{48}x^6+\frac{1}{384}x^8+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2x^2y''+xy'-\left(1+2x^2\right)y=0$

Solution

$$\frac{1}{2}2x^2y''+\frac{1}{2}xy'-\frac{1}{2}\left(1+2x^2\right)y=0$$

$$x^2y''+\frac{1}{2}xy'-\frac{1}{2}\left(1+2x^2\right)y=0$$

$$y''+\frac{1}{2x}y'-\left(\frac{1}{2x^2}+1\right)y=0$$

That implies to $p(x)=\frac{1}{2x}$ and $q(x)=-\frac{1}{2x^2}-1$

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{1}{2x} = \lim_{x\to 0}\frac{1}{2} = \frac{1}{2}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\left(-\frac{1}{2x^2}-1\right) = \lim_{x\to 0} x^2\left(-\frac{1}{2}-x^2\right) = -\frac{1}{2}$$

The indicial equation is: $r(r-1)+\frac{1}{2}r-\frac{1}{2} = r^2-\frac{1}{2}r-\frac{1}{2}=0 \rightarrow \boxed{r=1,\ -\frac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^1 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' + xy' - \left(1+2x^2\right) y = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \left(1+2x^2\right) \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} 2a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-2) + (n+r) - 1\right] a_n x^{n+r} - \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r} = 0$$

$$\left(2r^2 - r - 1\right) a_0 + \left((r+1)(2r+1) - 1\right) a_1 + \sum_{n=2}^{\infty} \left[(n+r)(2n+2r-1) - 1\right] a_n x^{n+r} - \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r} = 0$$

$$\left(2r^2 - r - 1\right) a_0 + \left(2r^2 + 3r\right) a_1 + \sum_{n=2}^{\infty} \left[\left((n+r)(2n+2r-1) - 1\right) a_n - 2a_{n-2}\right] x^{n+r} = 0$$

For $n = 0 \quad \rightarrow \quad \left(2r^2 - r - 1\right) a_0 = 0 \quad \Rightarrow \quad r = 1, \; -\frac{1}{2}$ √

For $n = 1 \quad \rightarrow \quad r(2r+3) a_1 = 0 \quad \Rightarrow \quad \cancel{r = 0, -\frac{3}{2}} \qquad a_1 = 0$

$$\left((n+r)(2n+2r-1) - 1\right) a_n - 2a_{n-2} = 0$$

$$a_n = \frac{2}{(n+r)(2n+2r-1) - 1} a_{n-2}$$

$$r = 1 \quad \rightarrow \quad a_n = \frac{2}{(n+1)(2n+1) - 1} a_{n-2} = \frac{2}{2n^2 + 3n} a_{n-2}$$

$n=2 \;\rightarrow\; a_2 = \frac{2}{14}a_0 = \frac{1}{7}a_0$ $\qquad n=3 \;\rightarrow\; a_3 = \frac{2}{27}a_1 = 0$

$n=4 \;\rightarrow\; a_4 = \frac{2}{44}a_2 = \frac{1}{154}a_0$ $\qquad n=5 \;\rightarrow\; a_5 = \frac{2}{65}a_3 = 0$

$n=6 \;\rightarrow\; a_6 = \frac{2}{90}a_4 = \frac{1}{6,390}a_0$ $\qquad n=7 \;\rightarrow\; a_7 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0 x\left(1+\frac{1}{7}x^2+\frac{1}{154}x^4+\frac{1}{6,930}x^6+\cdots\right)}$$

$$\boxed{r=-\frac{1}{2} \;\rightarrow\; b_n = \frac{2b_{n-2}}{\left(n-\frac{1}{2}\right)(2n-2)-1} = \frac{2}{2n^2-3n}b_{n-2}}$$

$n=2 \;\rightarrow\; b_2 = \frac{2}{2}b_0 = b_0$ $\qquad n=3 \;\rightarrow\; b_3 = \frac{2}{9}b_1 = 0$

$n=4 \;\rightarrow\; b_4 = \frac{2}{20}b_2 = \frac{1}{10}b_0$ $\qquad n=5 \;\rightarrow\; b_5 = \frac{2}{35}b_3 = 0$

$n=6 \;\rightarrow\; b_6 = \frac{2}{54}b_4 = \frac{1}{270}b_0$ $\qquad n=7 \;\rightarrow\; b_7 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = b_0 x^{-1/2}\left(1+x^2+\frac{1}{10}x^4+\frac{1}{270}x^6+\cdots\right)}$$

$$\boxed{y(x) = a_0 x\left(1+\frac{1}{7}x^2+\frac{1}{154}x^4+\frac{1}{6,930}x^6+\cdots\right)+b_0 x^{-1/2}\left(1+x^2+\frac{1}{10}x^4+\frac{1}{270}x^6+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $2x^2y''+xy'-\left(3-2x^2\right)y=0$

Solution

$$\frac{1}{2x^2}2x^2y''+\frac{1}{2x^2}xy'-\frac{1}{2x^2}\left(3-2x^2\right)y=0$$

$$y''+\frac{1}{2x}y'-\frac{1}{2x^2}\left(3-2x^2\right)y=0$$

That implies to $p(x)=\frac{1}{2x}$ and $q(x)=-\frac{1}{2x^2}\left(3-2x^2\right)$

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{1}{2x} = \boxed{\frac{1}{2}}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = -\lim_{x\to 0} x^2\left(\frac{1}{2x^2}\left(3-2x^2\right)\right) = -\lim_{x\to 0}\left(\frac{1}{2}\left(3-2x^2\right)\right) = \boxed{\frac{3}{2}}$$

The indicial equation is: $r(r-1)+\frac{1}{2}r-\frac{3}{2} = r^2-\frac{1}{2}r-\frac{3}{2}=0 \;\rightarrow\; \boxed{r=-1,\ \frac{3}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{-1}\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{3/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' + xy' - \left(3 - 2x^2\right) y = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \left(3-2x^2\right)\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} 3a_n x^{n+r} + \sum_{n=0}^{\infty} 2a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-2) + (n+r) - 3\right] a_n x^{n+r} + \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r} = 0$$

$$\left(2r^2 - r - 3\right) a_0 + \left((r+1)(2r+1) - 3\right) a_1 + \sum_{n=2}^{\infty} \left[(n+r)(2n+2r-1) - 3\right] a_n x^{n+r} + \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r} = 0$$

$$\left(2r^2 - r - 3\right) a_0 + \left(2r^2 + 3r - 2\right) a_1 + \sum_{n=2}^{\infty} \left[\left((n+r)(2n+2r-1) - 3\right) a_n + 2a_{n-2}\right] x^{n+r} = 0$$

For $n = 0 \quad \rightarrow \quad \left(2r^2 - r - 3\right) a_0 = 0 \quad \Rightarrow \quad \boxed{r = -1,\ \frac{3}{2}}$ √

For $n = 1 \quad \rightarrow \quad \left(2r^2 + 3r - 2\right) a_1 = 0 \quad \Rightarrow \quad \boxed{\cancel{r = -2,\ \frac{1}{2}}} \qquad \boxed{a_1 = 0}$

$$\left((n+r)(2n+2r-1) - 3\right) a_n + 2a_{n-2} = 0$$

$$\boxed{a_n = -\frac{2}{(n+r)(2n+2r-1) - 3} a_{n-2}}$$

$$r = -1 \quad \rightarrow \quad \boxed{a_n = -\frac{2}{(n-1)(2n-3) - 3} a_{n-2} = -\frac{2}{2n^2 - 5n} a_{n-2}}$$

$n = 2 \ \rightarrow \ a_2 = a_0 \qquad\qquad n = 3 \ \rightarrow \ a_3 = -\frac{2}{3} a_1 = 0$

$$n=4 \;\rightarrow\; a_4 = -\frac{2}{12}a_2 = -\frac{1}{6}a_0 \qquad n=5 \;\rightarrow\; a_5 = -\frac{2}{25}a_3 = 0$$

$$n=6 \;\rightarrow\; a_6 = -\frac{2}{252}a_4 = \frac{1}{126}a_0 \qquad n=7 \;\rightarrow\; a_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x^{-1}\left(1 + x^2 - \frac{1}{6}x^4 + \frac{1}{126}x^6 - \cdots\right)}$$

$$\boxed{r = \frac{3}{2} \;\rightarrow\; b_n = -\frac{2b_{n-2}}{\left(n+\frac{3}{2}\right)(2n+2)-3} = -\frac{2}{2n^2+5n}b_{n-2}}$$

$$n=2 \;\rightarrow\; b_2 = -\frac{2}{18}b_0 = -\frac{1}{9}b_0 \qquad n=3 \;\rightarrow\; b_3 = -\frac{2}{33}b_1 = 0$$

$$n=4 \;\rightarrow\; b_4 = -\frac{2}{568}b_2 = \frac{1}{234}b_0 \qquad n=5 \;\rightarrow\; b_5 = 0$$

$$n=6 \;\rightarrow\; b_6 = \frac{2}{102}b_4 = \frac{1}{11{,}934}b_0 \qquad n=7 \;\rightarrow\; b_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{3/2}\left(1 - \frac{1}{9}x^2 + \frac{1}{234}x^4 - \frac{1}{11{,}934}x^6 + \cdots\right)}$$

$$\boxed{y(x) = a_0 x^{-1}\left(1 + x^2 - \frac{1}{6}x^4 + \frac{1}{126}x^6 - \cdots\right) + b_0 x^{3/2}\left(1 - \frac{1}{9}x^2 + \frac{1}{234}x^4 - \frac{1}{11{,}934}x^6 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $3xy'' + 2y' + 2y = 0$

Solution

$$\frac{x}{3}3xy'' + 2\frac{x}{3}y' + 2\frac{x}{3}y = 0$$

$$x^2y'' + \frac{2}{3}xy' + \frac{2}{3}xy = 0$$

$$y'' + \frac{2}{3x}y' + \frac{2}{3x}y = 0$$

That implies to $p(x) = \frac{2}{3x}$ and $q(x) = \frac{2}{3x}$

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{2}{3x} = \boxed{\frac{2}{3}}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2\frac{2}{3x} = \lim_{x\to 0} \frac{2}{3}x = \boxed{0}$$

The indicial equation is: $r(r-1) + \frac{2}{3}r = r^2 - \frac{1}{3}r = 0 \;\rightarrow\; \boxed{r = 0,\ \frac{1}{3}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/3} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$3xy'' + 2y' + 2y = 0$$

$$3x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 2\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(3n+3r-3) + 2(n+r)\right] a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$r(3r-1)a_n + \sum_{n=1}^{\infty} (n+r)(3n+3r-1) a_n x^{n+r-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1} = 0$$

$$r(3r-1)a_n + \sum_{n=1}^{\infty} \left[(n+r)(3n+3r-1) a_n + 2a_{n-1}\right] x^{n+r-1} = 0$$

For $n = 0 \rightarrow r(3r-1)a_0 = 0 \Rightarrow \boxed{r = 0, \ \tfrac{1}{3}}$ ✓

$$(n+r)(3n+3r-1)a_n + 2a_{n-1} = 0$$

$$\boxed{a_n = -\frac{2}{(n+r)(3n+3r-1)} a_{n-1}}$$

$$\boxed{r = 0 \rightarrow a_n = -\frac{2}{3n^2 - n} a_{n-1}}$$

$n = 1 \rightarrow a_1 = -a_0$

$n = 2 \rightarrow a_2 = -\frac{1}{5} a_1 = \frac{1}{5} a_0$

$n = 3 \rightarrow a_3 = -\frac{2}{24} a_2 = -\frac{1}{60} a_0$

$$n=4 \;\rightarrow\; a_4 = -\frac{2}{44}a_3 = \frac{1}{1320}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x^0\left(1 - x + \frac{1}{5}x^2 - \frac{1}{60}x^3 + \frac{1}{1320}x^4 - \cdots\right)}$$

$$\boxed{r = \frac{1}{3} \;\rightarrow\; b_n = -\frac{2}{3n^2+n}b_{n-1}}$$

$$n=1 \;\rightarrow\; b_1 = -\frac{1}{2}b_0$$

$$n=2 \;\rightarrow\; b_2 = -\frac{2}{14}b_1 = \frac{1}{14}b_0$$

$$n=3 \;\rightarrow\; b_3 = -\frac{2}{30}b_2 = -\frac{1}{210}b_0$$

$$n=4 \;\rightarrow\; b_4 = -\frac{2}{52}b_3 = \frac{1}{5460}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/3}\left(1 - \frac{1}{2}x + \frac{1}{14}x^2 - \frac{1}{210}x^3 + \frac{1}{5460}x^4 - \cdots\right)}$$

$$\boxed{y(x) = a_0\left(1 - x + \frac{1}{5}x^2 - \frac{1}{60}x^3 + \frac{1}{1320}x^4 - \cdots\right) + b_0 x^{1/3}\left(1 - \frac{1}{2}x + \frac{1}{14}x^2 - \frac{1}{210}x^3 + \frac{1}{5460}x^4 - \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $3x^2y'' + 2xy' + x^2y = 0$

Solution

$$\frac{1}{3}3x^2y'' + 2\frac{1}{3}xy' + \frac{1}{3}x^2y = 0$$

$$x^2y'' + \frac{2}{3}xy' + \frac{1}{3}x^2y = 0$$

$$y'' + \frac{2}{3x}y' + \frac{1}{3}y = 0$$

That implies to $p(x) = \frac{2}{3x}$ and $q(x) = \frac{1}{3}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{2}{3x} \boxed{= \frac{2}{3}}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\frac{1}{3} \boxed{= 0}$$

The indicial equation is: $r(r-1) + \frac{2}{3}r = r^2 - \frac{1}{3}r = 0 \;\rightarrow\; \boxed{r = 0,\ \frac{1}{3}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$3x^2 y'' + 2xy' + x^2 y = 0$$

$$3x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 2x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + x^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(3n+3r-3) + 2(n+r)\right] a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$r(3r-1)a_0 + (r+1)(3r+2)a_1 + \sum_{n=2}^{\infty} (n+r)(3n+3r-1) a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

$$r(3r-1)a_0 + (r+1)(3r+2)a_1 + \sum_{n=2}^{\infty} \left[(n+r)(3n+3r-1) a_n + a_{n-2}\right] x^{n+r} = 0$$

For $n=0 \;\rightarrow\; r(3r-1)a_0 = 0 \;\Rightarrow\; \boxed{r = 0,\ \frac{1}{3}}$ √

For $n=1 \;\rightarrow\; (r+1)(3r+2)a_1 = 0 \;\Rightarrow\; \cancel{r = -1,\ -\frac{2}{3}}$ $\quad \boxed{a_1 = 0}$

$$(n+r)(3n+3r-1)a_n + a_{n-2} = 0$$

$$\boxed{a_n = -\frac{1}{(n+r)(3n+3r-1)} a_{n-2}}$$

$$\boxed{r = 0 \;\rightarrow\; a_n = -\frac{1}{n(3n-1)} a_{n-2}}$$

$n = 2 \;\rightarrow\; a_2 = -\frac{1}{10} a_0$

$n = 4 \;\rightarrow\; a_4 = -\frac{1}{44} a_2 = \frac{1}{440} a_0$

$n = 6 \;\rightarrow\; a_6 = -\frac{1}{102} a_4 = -\frac{1}{44{,}880} a_0$

$n = 3 \;\rightarrow\; a_3 = -\frac{1}{24} a_1 = 0$

$n = 5 \;\rightarrow\; a_5 = -\frac{1}{70} a_3 = 0$

$n = 7 \;\rightarrow\; a_7 = 0$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x^0 \left(1 - \frac{1}{10}x^2 + \frac{1}{440}x^4 - \frac{1}{44,880}x^6 + \cdots\right)}$$

$$\boxed{r = \frac{1}{3} \quad \rightarrow \quad b_n = -\frac{1}{\left(n+\frac{1}{3}\right)(3n)} b_{n-2} = -\frac{1}{n(3n+1)} b_{n-2}}$$

$n=2 \rightarrow b_2 = -\frac{1}{14}b_0$ $\qquad$ $n=3 \rightarrow b_3 = -\frac{1}{30}b_1 = 0$

$n=4 \rightarrow b_4 = -\frac{1}{72}b_2 = \frac{1}{728}b_0$ $\qquad$ $n=5 \rightarrow b_5 = 0$

$n=6 \rightarrow b_6 = -\frac{1}{114}b_4 = -\frac{1}{82,992}b_0$ $\qquad$ $n=7 \rightarrow b_7 = 0$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/3} \left(1 - \frac{1}{14}x^2 + \frac{1}{728}x^4 - \frac{1}{82,992}x^6 + \cdots\right)}$$

$$\boxed{y(x) = a_0 \left(1 - \frac{1}{10}x^2 + \frac{1}{440}x^4 - \frac{1}{44,880}x^6 + \cdots\right) + b_0 x^{1/3} \left(1 - \frac{1}{14}x^2 + \frac{1}{728}x^4 - \frac{1}{82,992}x^6 + \cdots\right)}$$

Exercise

Find the Frobenius series solutions of $3x^2y'' - xy' + y = 0$

Solution

$$\frac{1}{3}3x^2y'' - \frac{1}{3}xy' + \frac{1}{3}y = 0$$

$$x^2y'' - \frac{1}{3}xy' + \frac{1}{3}y = 0$$

$$y'' - \frac{1}{3x}y' + \frac{1}{3x^2}y = 0$$

That implies to $p(x) = -\frac{1}{3x}$ and $q(x) = \frac{1}{3x^2}$.

$$p_0 = \lim_{x \to 0} xp(x) = -\lim_{x \to 0} x\frac{1}{3x} \boxed{= -\frac{1}{3}}$$

$$q_0 = \lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 \frac{1}{3x^2} \boxed{= \frac{1}{3}}$$

The indicial equation is: $\quad r(r-1) - \frac{1}{3}r + \frac{1}{3} = r^2 - \frac{4}{3}r + \frac{1}{3} = 0 \quad \rightarrow \quad \boxed{r = 1, \ \frac{1}{3}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$3x^2 y'' - xy' + y = 0$$

$$3x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(3n+3r-3) - (n+r) + 1 \right] a_n x^{n+r} = 0$$

Since neither of λ, then let assume $a_n = 0, \quad n \ge 1$

$$y_1(x) = x \sum_{n=0}^{\infty} a_n x^n \ \underline{= a_0 x|}$$

$$y_2(x) = x^{1/3} \sum_{n=0}^{\infty} b_n x^n \ \underline{= b_0 x^{1/3}|}$$

$$\underline{y(x) = a_0 x + b_0 x^{1/3}\ |}$$

Exercise

Find the Frobenius series solutions of $4xy'' + 2y' + y = 0$

Solution

$$\frac{x}{4} 4xy'' + 2\frac{x}{4} y' + \frac{x}{4} y = 0$$

$$x^2 y'' + \frac{1}{2} xy' + \frac{x}{4} y = 0$$

$$y'' + \frac{1}{2x} y' + \frac{1}{4x} y = 0$$

That implies to $p(x) = \frac{1}{2x}$ and $q(x) = \frac{1}{4x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{1}{2x} \boxed{=\frac{1}{2}}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{1}{4x} = \lim_{x\to 0} \frac{x}{4} \boxed{=0}$$

The indicial equation is: $r(r-1)+\frac{1}{2}r = r^2 - \frac{1}{2}r = 0 \quad \to \quad \boxed{r = 0,\ \frac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$4xy'' + 2y' + y = 0$$

$$4x\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 2\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 4(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(4n+4r-4) + 2(n+r)\right] a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$r(4r-2)a_n + \sum_{n=1}^{\infty} (n+r)(4n+4r-2) a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

$$2r(2r-1)a_n + \sum_{n=1}^{\infty} \left[2(n+r)(2n+2r-1)a_n + a_{n-1}\right] x^{n+r-1} = 0$$

For $n = 0 \quad \to \quad 2r(2r-1)a_0 = 0 \quad \Rightarrow \quad \boxed{r = 0,\ \frac{1}{2}}$ $\surd$

$$2(n+r)(2n+2r-1)a_n + a_{n-1} = 0$$

$$\boxed{a_n = -\frac{1}{2(n+r)(2n+2r-1)} a_{n-1}}$$

$$r=0 \quad \rightarrow \quad a_n = -\frac{1}{2n(2n-1)}a_{n-1}$$

$$n=1 \rightarrow a_1 = -\frac{1}{2}a_0$$

$$n=2 \rightarrow a_2 = -\frac{1}{12}a_1 = \frac{1}{24}a_0$$

$$n=3 \rightarrow a_3 = -\frac{1}{30}a_2 = -\frac{1}{720}a_0$$

$$n=4 \rightarrow a_4 = -\frac{1}{42}a_3 = \frac{1}{30,240}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x) = a_0 x^0\left(1-\frac{1}{2}x+\frac{1}{24}x^2-\frac{1}{720}x^3+\frac{1}{30,240}x^4-\cdots\right)$$

$$r=\frac{1}{2} \quad \rightarrow \quad b_n = -\frac{1}{2\left(n+\frac{1}{2}\right)(2n)}b_{n-1} = -\frac{1}{4n^2+2n}b_{n-1}$$

$$n=1 \rightarrow b_1 = -\frac{1}{6}b_0$$

$$n=2 \rightarrow b_2 = -\frac{1}{20}b_1 = \frac{1}{120}b_0$$

$$n=3 \rightarrow b_3 = -\frac{1}{42}b_2 = -\frac{1}{5040}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x) = b_0 x^{1/2}\left(1-\frac{1}{6}x+\frac{1}{120}x^2-\frac{1}{5,040}x^3+\cdots\right)$$

$$y(x) = a_0\left(1-\frac{1}{2}x+\frac{1}{24}x^2-\frac{1}{720}x^3+\frac{1}{30,240}x^4-\cdots\right)+b_0x^{1/2}\left(1-\frac{1}{6}x+\frac{1}{120}x^2-\frac{1}{5,040}x^3+\cdots\right)$$

Exercise

Find the Frobenius series solutions of $6x^2y''+7xy'-\left(x^2+2\right)y=0$

Solution

$$\frac{1}{6}6x^2y''+\frac{1}{6}7xy'-\frac{1}{6}\left(x^2+2\right)y=0$$

$$x^2y''+\frac{7}{6}xy'-\frac{1}{6}\left(x^2+2\right)y=0$$

$$y''+\frac{7}{6x}y'-\frac{1}{6x^2}\left(x^2+2\right)y=0$$

That implies to $p(x)=\frac{7}{6x}$ and $q(x)=-\frac{1}{6x^2}\left(x^2+2\right)$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{7}{6x} = \boxed{\frac{7}{6}}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = -\lim_{x\to 0} x^2 \frac{1}{6x^2}\left(x^2+2\right) = -\lim_{x\to 0}\left(\frac{1}{6}x^2+\frac{1}{3}\right) = \boxed{-\frac{1}{3}}$$

The indicial equation is: $r(r-1)+\frac{7}{6}r-\frac{1}{3} = r^2+\frac{1}{6}r-\frac{1}{3} = 0$

$$6r^2+r-2=0 \quad \rightarrow \quad r = \frac{-1\pm 7}{12} \quad \boxed{r = \frac{1}{2},\ -\frac{2}{3}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2}\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{-2/3}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$6x^2 y'' + 7xy' - \left(x^2+2\right) y = 0$$

$$6x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 7x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - x^2\sum_{n=0}^{\infty}a_n x^{n+r} - 2\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}6(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}7(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}a_n x^{n+r+2} - \sum_{n=0}^{\infty}2a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[6(n+r)(n+r-1)+7(n+r)-2\right]a_n x^{n+r} - \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0$$

$$\left(6r^2+r-2\right)a_0 + \left((r+1)(6r+7)-2\right)a_1 + \sum_{n=2}^{\infty}\left[(n+r)(6n+6r+1)-2\right]a_n x^{n+r} - \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0$$

$$\left(6r^2+r-2\right)a_0 + \left(6r^2+13r+5\right)a_1 + \sum_{n=2}^{\infty}\left[\left((n+r)(6n+6r+1)-2\right)a_n - a_{n-2}\right]x^{n+r} = 0$$

For $n=0 \ \rightarrow \ \left(6r^2+r-2\right)a_0 = 0 \ \Rightarrow \ \boxed{r = \frac{1}{2},\ -\frac{2}{3}}$ √

For $n=1 \rightarrow \left(6r^2+13r+5\right)a_1=0 \Rightarrow r=\frac{-13\pm 7}{12}=\cancel{-\frac{1}{2}, -\frac{5}{3}}$ $\quad \boxed{a_1=0}$

$$\left((n+r)(6n+6r+1)-2\right)a_n - a_{n-2}=0$$

$$\boxed{a_n=\frac{1}{(n+r)(6n+6r+1)-2}a_{n-2}}$$

$$\boxed{r=\frac{1}{2} \rightarrow a_n=\frac{1}{n(6n+7)}a_{n-2}}$$

$n=2 \rightarrow a_2=\frac{1}{38}a_0 \qquad n=3 \rightarrow a_3=\frac{1}{75}a_1=0$

$n=4 \rightarrow a_4=\frac{1}{124}a_2=\frac{1}{4,712}a_0 \qquad n=5 \rightarrow a_5=\frac{1}{185}a_3=0$

$n=6 \rightarrow a_6=\frac{1}{258}a_4=\frac{1}{1,215,696}a_0 \qquad n=7 \rightarrow a_7=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x)=a_0x^{1/2}\left(1+\frac{1}{38}x^2+\frac{1}{4,712}x^4+\frac{1}{1,215,696}x^6+\cdots\right)}$$

$$\boxed{r=-\frac{2}{3} \rightarrow b_n=\frac{1}{n(6n-7)}b_{n-2}}$$

$n=2 \rightarrow b_2=\frac{1}{10}b_0 \qquad n=3 \rightarrow b_3=\frac{1}{33}b_1=0$

$n=4 \rightarrow b_4=\frac{1}{68}b_2=\frac{1}{680}b_0 \qquad n=5 \rightarrow b_5=0$

$n=6 \rightarrow b_6=\frac{1}{174}b_4=\frac{1}{118,320}b_0 \qquad n=7 \rightarrow b_7=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x)=b_0x^{-2/3}\left(1+\frac{1}{10}x^2+\frac{1}{680}x^4+\frac{1}{118,320}x^6+\cdots\right)}$$

$$\boxed{y(x)=a_0x^{1/2}\left(1+\frac{x^2}{38}+\frac{x^4}{4,712}+\frac{x^6}{1,215,696}+\cdots\right)+b_0x^{-2/3}\left(1+\frac{x^2}{10}+\frac{x^4}{680}+\frac{x^6}{118,320}+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $xy''+y'+2y=0$

Solution

$x\times \quad xy''+y'+2y=0$

$x^2y''+xy'+2xy=0$

$y''+\frac{1}{x}y'+\frac{2}{x}y=0$

That implies to $p(x)=\frac{1}{x}$ and $q(x)=\frac{2}{x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{1}{x} \underline{=1|}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{2}{x} = \lim_{x\to 0} 2x \underline{=0|}$$

The indicial equation is: $r^2 + (1-1)r = 0 \quad \rightarrow \quad \underline{r_{1,2} = 0|}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = y_1(x)\ln|x| + \sum_{n=0}^{\infty} c_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_n x^n \qquad (r = r_1 = 0)$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n+r-1} = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$xy'' + y' + 2y = 0$$

$$x\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n^2 a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n^2 a_n x^{n-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n-1} = 0$$

$$\sum_{n=1}^{\infty} n^2 a_n x^{n-1} + \sum_{n=1}^{\infty} 2a_{n-1} x^{n-1} = 0$$

$$\sum_{n=1}^{\infty} \left[n^2 a_n + 2a_{n-1} \right] x^{n-1} = 0$$

$$n^2 a_n + 2a_{n-1} = 0 \quad \Rightarrow \quad \underline{a_n = -\frac{2}{n^2} a_{n-1}|}$$

$$n = 1 \rightarrow a_1 = -2a_0$$

$$n=2 \to a_2 = -\frac{2}{2^2}a_1 = a_0 = \frac{2^2}{2^2}a_0$$

$$n=3 \to a_3 = -\frac{2}{9}a_2 = -\frac{2^3}{(2\cdot 3)^2}a_0 = -\frac{2}{9}a_0$$

$$n=4 \to a_4 = -\frac{2}{4^2}a_3 = \frac{2^4}{(4!)^2}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = \frac{(-1)^n 2^n}{(n!)^2}a_0}$$

$$\boxed{y_1(x) = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n 2^n}{(n!)^2}x^n}$$

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty}a_n x^n \quad (a_0 = 1)$$

$$y_2(x) = y_1(x)\ln x + \sum_{n=0}^{\infty}c_n x^n$$

$$y_2(x) = y_1(x)\ln x + x^{r_1}\sum_{n=0}^{\infty}c_n x^n$$

$$xy_2'' + y_2' + 2y_2 = 0$$

$$x\left(y_1(x)\ln x + \sum_{n=0}^{\infty}c_n x^n\right)'' + \left(y_1(x)\ln x + \sum_{n=0}^{\infty}c_n x^n\right)' + 2\left(y_1(x)\ln x + \sum_{n=0}^{\infty}c_n x^n\right) = 0$$

$$x\left(y_1'\ln x + \frac{1}{x}y_1\right)' + x\sum_{n=0}^{\infty}n(n-1)c_n x^{n-2} + y_1'\ln x + \frac{1}{x}y_1 + \sum_{n=0}^{\infty}nc_n x^{n-1} + 2y_1\ln x + \sum_{n=0}^{\infty}2c_n x^n = 0$$

$$x\left(y_1''\ln x + \frac{1}{x}y_1' - \frac{1}{x^2}y_1 + \frac{1}{x}y_1'\right) + y_1'\ln x + \frac{1}{x}y_1 + 2y_1\ln x$$

$$+\sum_{n=0}^{\infty}n(n-1)c_n x^{n-1} + \sum_{n=0}^{\infty}nc_n x^{n-1} + \sum_{n=0}^{\infty}2c_n x^n = 0$$

$$xy_1''\ln x + 2y_1' - \frac{1}{x}y_1 + \frac{1}{x}y_1 + \left(y_1' + 2y_1\right)\ln x + \sum_{n=0}^{\infty}n^2c_n x^{n-1} + \sum_{n=0}^{\infty}2c_n x^n = 0$$

$$\left(xy_1'' + y_1' + 2y_1\right)\ln x + 2y_1' + \sum_{n=0}^{\infty}n^2c_n x^{n-1} + \sum_{n=1}^{\infty}2c_{n-1}x^{n-1} = 0$$

Since: $xy_1'' + y_1' + 2y_1 = 0$

$$2y_1' + \sum_{n=1}^{\infty}n^2c_n x^{n-1} + \sum_{n=1}^{\infty}2c_{n-1}x^{n-1} = 0$$

$$2y_1' + \sum_{n=1}^{\infty}\left(n^2 c_n + 2c_{n-1}\right)x^{n-1} = 0$$

$$2\sum_{n=1}^{\infty}(-1)^n \frac{2^n}{(n!)^2} n x^{n-1} + \sum_{n=1}^{\infty}\left(n^2 c_n + 2c_{n-1}\right)x^{n-1} = 0$$

$$\sum_{n=1}^{\infty}\left[(-1)^n \frac{2^{n+1}}{(n-1)!n!} + n^2 c_n + 2c_{n-1}\right]x^{n-1} = 0$$

$$n^2 c_n + 2c_{n-1} + (-1)^n \frac{2^{n+1}}{(n-1)!n!} = 0$$

$$\boxed{c_n = -\frac{2}{n^2}c_{n-1} + (-1)^{n+1}\frac{2^{n+1}}{(n-1)!n!\,n^2}}$$

$n = 1 \ \to\ c_1 = -2c_0 + 4$ $\qquad c_0 = 0 \ \to\ c_1 = 4$

$n = 2 \ \to\ c_2 = -\frac{1}{2}c_1 - 1$ $\qquad c_2 = -3$

$n = 3 \ \to\ c_3 = -\frac{2}{9}c_2 + \frac{16}{12(9)}$ $\qquad c_3 = \frac{22}{27}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = y_1(x)\ln|x| + 4x - 3x^2 + \frac{22}{27}x^3 - \cdots}$$

$$y_2(x) = y_1(x)\ln x + x^{r_1}\sum_{n=0}^{\infty}c_n x^n$$

Exercise

Find the Frobenius series solutions of $xy'' - y = 0$

Solution

$x \times \quad xy'' - y = 0$

$x^2 y'' - xy = 0$

$y'' - \frac{1}{x}y = 0$

That implies to $p(x) = 0$ and $q(x) = -\frac{1}{x}$.

$$p_0 = \lim_{x\to 0} xp(x) \ \underline{= 0|}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2\left(-\frac{1}{x}\right) = -\lim_{x\to 0} x \ \underline{= 0|}$$

The indicial equation is: $r^2 - r = 0 \quad \to \quad \underline{r_1 = 1,\ r_2 = 0|}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1} \quad \textit{and} \quad y_2(x) = \alpha y_1(x)\ln|x| + \sum_{n=0}^{\infty} c_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_n x^{n+1} \qquad (r = r_1 = 1)$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n+r-1} = \sum_{n=0}^{\infty} n a_n x^n$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} = \sum_{n=0}^{\infty} n(n+1) a_n x^{n-1}$$

$$xy'' - y = 0$$

$$x\sum_{n=0}^{\infty} n(n+1) a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} n(n+1) a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} n(n+1) a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\sum_{n=1}^{\infty} n(n+1) a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\sum_{n=1}^{\infty} \left[n(n+1) a_n - a_{n-1} \right] x^n = 0$$

$$n(n+1) a_n - a_{n-1} = 0 \quad \Rightarrow \quad \boxed{a_n = \frac{1}{n(n+1)} a_{n-1}}$$

$$n = 1 \;\rightarrow\; a_1 = \frac{1}{2} a_0$$

$$n = 2 \;\rightarrow\; a_2 = \frac{1}{6} a_1 = a_0 = \frac{1}{(2)3!} a_0$$

$$n = 3 \;\rightarrow\; a_3 = \frac{1}{3 \cdot 4} a_2 = \frac{1}{(2 \cdot 3)4!} a_0$$

$$n = 4 \;\rightarrow\; a_4 = \frac{1}{4 \cdot 5} a_3 = \frac{1}{4!5!} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = \frac{1}{n!(n+1)!} a_0}$$

$$y_1(x)=x\sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^n=\sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^{n+1}$$

$$y_1(x)=x^{r_1}\sum_{n=0}^{\infty}a_n x^n \quad (a_0=1)$$

$$y_2(x)=\alpha y_1\ln x+\sum_{n=0}^{\infty}d_n x^n$$

$$y_2(x)=\alpha y_1(x)\ln x+\sum_{n=0}^{\infty}d_n x^n \quad (d_0=1)$$

$$y_2'=\alpha y_1'\ln x+\frac{\alpha}{x}y_1+\sum_{n=0}^{\infty}nd_n x^{n-1}$$

$$xy_2''-y_2=0$$

$$x\left(\alpha y_1'\ln x+\frac{\alpha}{x}y_1+\sum_{n=0}^{\infty}nd_n x^{n-1}\right)'-\left(\alpha y_1\ln x+\sum_{n=0}^{\infty}d_n x^n\right)=0$$

$$x\left(\alpha y_1'\ln x+\frac{\alpha}{x}y_1\right)'+x\sum_{n=0}^{\infty}n(n-1)d_n x^{n-2}-\alpha y_1\ln x-\sum_{n=0}^{\infty}d_n x^n=0$$

$$x\left(\alpha y_1''\ln x+\frac{\alpha}{x}y_1'-\frac{\alpha}{x^2}y_1+\frac{\alpha}{x}y_1'\right)-\alpha y_1(x)\ln x+\sum_{n=0}^{\infty}n(n-1)d_n x^{n-1}-\sum_{n=0}^{\infty}d_n x^n=0$$

$$\alpha\left(2y_1'-\frac{1}{x}y_1\right)+\alpha\left(xy_1''-y_1\right)\ln x+\sum_{n=0}^{\infty}n(n+1)d_{n+1}x^n-\sum_{n=0}^{\infty}d_n x^n=0$$

Since: $xy_1''-y_1=0$

$$y_1(x)=\sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^{n+1} \rightarrow y_1'=\sum_{n=1}^{\infty}\frac{n+1}{n!(n+1)!}x^n$$

$$\alpha\left(2y_1'-\frac{1}{x}y_1\right)+\sum_{n=0}^{\infty}\left[n(n+1)d_{n+1}-d_n\right]x^n=0$$

$$\alpha\left(2\sum_{n=1}^{\infty}\frac{n+1}{n!(n+1)!}x^n-\frac{1}{x}\sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^{n+1}\right)+\sum_{n=0}^{\infty}\left[n(n+1)d_{n+1}-d_n\right]x^n=0$$

$$\alpha\left(\sum_{n=1}^{\infty}\frac{2n+2}{n!(n+1)!}x^n-\sum_{n=1}^{\infty}\frac{1}{n!(n+1)!}x^n\right)+\sum_{n=0}^{\infty}\left[n(n+1)d_{n+1}-d_n\right]x^n=0$$

$$\alpha\sum_{n=1}^{\infty}\frac{2n+1}{n!(n+1)!}x^n+\sum_{n=0}^{\infty}\left[n(n+1)d_{n+1}-d_n\right]x^n=0$$

$$\sum_{n=0}^{\infty}\left[n(n+1)d_{n+1}-d_n\right]x^n=-\alpha\sum_{n=1}^{\infty}\frac{2n+1}{n!(n+1)!}x^n$$

$$\boxed{n(n+1)d_{n+1}-d_n=-\alpha\frac{2n+1}{n!(n+1)!}} \qquad (d_0=1)$$

$$n=0 \ \rightarrow\ -d_0=-\alpha \qquad\qquad \Rightarrow\ \boxed{\alpha=d_0=1}$$

$$d_{n+1}=\frac{1}{n(n+1)}\left(d_n-\frac{2n+1}{n!(n+1)!}\right)$$

$$n=1 \ \rightarrow\ d_2=\frac{1}{2}\left(d_1-\frac{3}{2}\right)=\boxed{\frac{1}{2}d_1-\frac{3}{4}}$$

$$n=2 \ \rightarrow\ d_3=\frac{1}{6}\left(d_2-\frac{5}{12}\right)=\frac{1}{6}\left(\frac{1}{2}d_1-\frac{3}{4}-\frac{5}{12}\right)=\boxed{\frac{1}{12}d_1-\frac{7}{36}}$$

$$n=3 \ \rightarrow\ d_4=\frac{1}{12}\left(d_3-\frac{7}{144}\right)=\frac{1}{12}\left(\frac{1}{12}d_1-\frac{7}{36}-\frac{7}{144}\right)=\boxed{\frac{1}{144}d_1-\frac{35}{1,728}}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

If we let $d_1=0$

$$\boxed{y_2(x)=y_1(x)\ln x+1-\frac{3}{4}x^2-\frac{7}{36}x^3-\cdots} \qquad\qquad y_2(x)=y_1(x)\ln x+\sum_{n=0}^{\infty}d_nx^n$$

Exercise

Find the Frobenius series solutions of $2x(1-x)y''+(1+x)y'-y=0$

Solution

$$xy''+\frac{x+1}{2(1-x)}y'-\frac{1}{2(1-x)}y=0$$

$$x^2y''+\frac{1}{2}\frac{x(x+1)}{1-x}y'-\frac{x}{2(1-x)}y=0$$

$$y''+\frac{1}{2}\frac{x+1}{x(1-x)}y'-\frac{1}{2x(1-x)}y=0$$

That implies to $p(x)=\frac{1}{2}\frac{x+1}{x(1-x)}$ and $q(x)=-\frac{1}{2x(1-x)}$.

$$p_0=\lim_{x\to0}xp(x)=\frac{1}{2}\lim_{x\to0}x\frac{x+1}{x(1-x)}=\frac{1}{2}\lim_{x\to0}\frac{x+1}{1-x}=\boxed{\frac{1}{2}}$$

$$q_0=\lim_{x\to0}x^2q(x)=-\frac{1}{2}\lim_{x\to0}\frac{x}{1-x}=\boxed{0}$$

The indicial equation is: $r^2+\left(\frac{1}{2}-1\right)r=r^2-\frac{1}{2}r=0 \ \rightarrow\ \boxed{r=0,\ \frac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x)=x^{1/2}\sum_{n=0}^{\infty} b_n x^n$$

$$y=\sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$2x(1-x)y''+(1+x)y'-y=0$$

$$\left(2x-2x^2\right)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+(1+x)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r-1}-\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}-\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}\left[-2(n+r)(n+r-1)+n+r-1\right]a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}(n+r-1)(-2(n+r)+1)a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1}+\sum_{n=1}^{\infty}(n+r-2)(-2n-2r+3)a_{n-1} x^{n+r-1}=0$$

$$r(2r-1)a_0x^{r-1}+\sum_{n=1}^{\infty}(n+r)(2n+2r-1)a_n x^{n+r-1}-\sum_{n=1}^{\infty}(n+r-2)(2n+2r-3)a_{n-1} x^{n+r-1}=0$$

$$r(2r-1)a_0x^{r-1}+\sum_{n=1}^{\infty}\left[(n+r)(2n+2r-1)a_n-(n+r-2)(2n+2r-3)a_{n-1}\right]x^{n+r-1}=0$$

For $n=0 \quad \rightarrow \quad r(2r-1)a_0=0 \quad \Rightarrow \underline{r=0,\ \tfrac{1}{2}} \quad \surd$

$$(n+r)(2n+2r-1)a_n-(n+r-2)(2n+2r-3)a_{n-1}=0$$

$$\boxed{a_n = \frac{(n+r-2)(2n+2r-3)}{(n+r)(2n+2r-1)} a_{n-1}}$$

$$\boxed{r = 0 \quad \to \quad a_n = \frac{(n-2)(2n-3)}{n(2n-1)} a_{n-1}}$$

$n = 1 \ \to \ a_1 = a_0$

$n = 2 \ \to \ a_2 = 0a_1 = 0$

$n = 3 \ \to \ a_3 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_1(x) = a_0(1+x)}$$

$$\boxed{r = \frac{1}{2} \quad \to \quad b_n = \frac{\left(n-\frac{3}{2}\right)(2n-2)}{2n\left(n+\frac{1}{2}\right)} b_{n-1} = \frac{(2n-3)(n-1)}{n(2n+1)} b_{n-1}}$$

$n = 1 \ \to \ b_1 = 0b_0 = 0$

$n = 2 \ \to \ b_2 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y_2(x) = b_0 x^{1/2}}$$

$$\boxed{y(x) = a_0(1+x) + b_0\sqrt{x}}$$

Exercise

Find the Frobenius series solutions of $x^2y'' + \left(x^2 + \frac{1}{2}x\right)y' + xy = 0$

Solution

$$y'' + \left(1 + \frac{1}{2x}\right)y' + \frac{1}{x}y = 0$$

That implies to $p(x) = 1 + \frac{1}{2x}$ and $q(x) = \frac{1}{x}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(1 + \frac{1}{2x}\right) = \lim_{x\to 0}\left(x + \frac{1}{2}\right) \boxed{= \frac{1}{2}}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x \boxed{= 0}$$

The indicial equation is: $r^2 + \left(\frac{1}{2} - 1\right)r = r^2 - \frac{1}{2}r = 0 \quad \to \quad \boxed{r = 0, \ \frac{1}{2}}$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=\sum_{n=0}^{\infty}a_n x^n \quad \textit{and} \quad y_2(x)=x^{1/2}\sum_{n=0}^{\infty}b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$x^2y''+\left(x^2+\tfrac{1}{2}x\right)y'+xy=0$$

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\left(x^2+\tfrac{1}{2}x\right)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+x\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1}\sum_{n=0}^{\infty}\tfrac{1}{2}(n+r)a_n x^{n+r}+\sum_{n=0}^{\infty}a_n x^{n+r+1}=0$$

$$\sum_{n=0}^{\infty}(n+r+1)a_n x^{n+r+1}+\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+\tfrac{1}{2}(n+r)\right]a_n x^{n+r}=0$$

$$\sum_{n=1}^{\infty}(n+r)a_{n-1}x^{n+r}+\sum_{n=0}^{\infty}(n+r)\left(n+r-\tfrac{1}{2}\right)a_n x^{n+r}=0$$

$$\sum_{n=1}^{\infty}(n+r)a_{n-1}x^{n+r}+r\left(r-\tfrac{1}{2}\right)a_0+\sum_{n=1}^{\infty}(n+r)\left(n+r-\tfrac{1}{2}\right)a_n x^{n+r}=0$$

$$r\left(r-\tfrac{1}{2}\right)a_0+\sum_{n=1}^{\infty}\left[(n+r)\left(n+r-\tfrac{1}{2}\right)a_n+(n+r)a_{n-1}\right]x^{n+r}=0$$

For $n=0 \quad\rightarrow\quad r(2r-1)a_0=0 \quad\Rightarrow\quad \boxed{r=0,\ \tfrac{1}{2}}$ √

$$(n+r)\left(n+r-\tfrac{1}{2}\right)a_n+(n+r)a_{n-1}=0$$

$$\boxed{a_n=-2\frac{(n+r)}{(n+r)(2n+2r-1)}a_{n-1}}$$

$$\boxed{r=0 \quad\rightarrow\quad a_n=-\frac{2n}{n(2n-1)}a_{n-1}=-\frac{2}{2n-1}a_{n-1}}$$

$$n=1 \rightarrow a_1=-2a_0$$

$$n=2 \ \rightarrow\ a_2 = -\frac{2}{3}a_1 = \frac{4}{3}a_0$$

$$n=3 \ \rightarrow\ a_3 = -\frac{2}{5}a_2 = -\frac{8}{15}a_0$$

$$n=4 \ \rightarrow\ a_4 = -\frac{2}{7}a_3 = \frac{16}{105}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0\left(1-2x+\frac{4}{3}x^2-\frac{8}{15}x^3+\frac{16}{105}x^4-\cdots\right)}$$

$$\boxed{r=\frac{1}{2} \ \rightarrow\ b_n = -\frac{2\left(n+\frac{1}{2}\right)}{\left(n+\frac{1}{2}\right)(2n)}b_{n-1} = -\frac{1}{n}b_{n-1}}$$

$$n=1 \ \rightarrow\ b_1 = -b_0$$

$$n=2 \ \rightarrow\ b_2 = -\frac{1}{2}b_1 = \frac{1}{2}b_0$$

$$n=3 \ \rightarrow\ b_3 = -\frac{1}{3}b_2 = -\frac{1}{6}b_0$$

$$n=4 \ \rightarrow\ b_4 = -\frac{1}{4}b_3 = \frac{1}{24}b_0$$

$$n=5 \ \rightarrow\ b_5 = -\frac{1}{5}b_4 = -\frac{1}{120}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0x^{1/2}\left(1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5+\cdots\right)}$$

$$\boxed{y(x) = a_0\left(1-2x+\frac{4}{3}x^2-\frac{8}{15}x^3+\frac{16}{105}x^4-\cdots\right)+b_0\sqrt{x}\left(1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5+\cdots\right)}$$

Exercise

Find the Frobenius series solutions of $18x^2y'' + 3x(x+5)y' - (10x+1)y = 0$

Solution

$$y'' + \frac{x+5}{6x}y' - \frac{10x+1}{18x^2}y = 0$$

That implies to $p(x) = \frac{x+5}{6x}$ and $q(x) = -\frac{10x+1}{18x^2}$.

$$p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x\left(\frac{x+5}{6x}\right) = \lim_{x\to 0}\frac{x+5}{6} = \boxed{\frac{5}{6}}$$

$$q_0 = \lim_{x\to 0} x^2q(x) = -\lim_{x\to 0}\frac{10x+1}{18} = \boxed{-\frac{1}{18}}$$

The indicial equation is: $r^2 + \left(\frac{5}{6}-1\right)r - \frac{1}{18} = r^2 - \frac{1}{6}r - \frac{1}{18} = 0$

$$18r^2-3r-1=0 \quad \to \quad \boxed{r=-\frac{1}{6},\ \frac{1}{3}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x)=x^{-1/6}\sum_{n=0}^{\infty}a_n x^n \quad \text{and} \quad y_2(x)=x^{1/3}\sum_{n=0}^{\infty}b_n x^n$$

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$18x^2y''+3x(x+5)y'-(10x+1)y=0$$

$$18x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\left(3x^2+15x\right)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}-(10x+1)\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}18(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}3(n+r)a_n x^{n+r+1}+\sum_{n=0}^{\infty}15(n+r)a_n x^{n+r}$$
$$-\sum_{n=0}^{\infty}10a_n x^{n+r+1}-\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}\left[18(n+r)(n+r-1)+15(n+r)-1\right]a_n x^{n+r}+\sum_{n=0}^{\infty}(3n+3r-10)a_n x^{n+r+1}=0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(18n+18r-3)-1\right]a_n x^{n+r}+\sum_{n=1}^{\infty}(3n+3r-13)a_{n-1}x^{n+r}=0$$

$$\left(r(18r-3)-1\right)a_0x^r+\sum_{n=1}^{\infty}\left[(n+r)(18n+18r-3)-1\right]a_n x^{n+r}+\sum_{n=1}^{\infty}(3n+3r-13)a_{n-1}x^{n+r}=0$$

$$\left(r(18r-3)-1\right)a_0x^r+\sum_{n=1}^{\infty}\left[\left((n+r)(18n+18r-3)-1\right)a_n+(3n+3r-13)a_{n-1}\right]x^{n+r}=0$$

For $n=0 \quad \to \quad \left(18r^2-3r-1\right)a_0=0 \quad \Rightarrow \quad \boxed{r=-\frac{1}{6},\ \frac{1}{3}}$ √

$$\left((n+r)(18n+18r-3)-1\right)a_n+(3n+3r-13)a_{n-1}=0$$

$$\boxed{a_n = -\frac{3n+3r-13}{(n+r)(18n+18r-3)-1}a_{n-1}}$$

$$\boxed{r = -\frac{1}{6} \quad \to \quad a_n = -\frac{3n-\frac{1}{2}-13}{\left(n-\frac{1}{6}\right)(18n-6)-1}a_{n-1} = -\frac{1}{2}\frac{6n-27}{(6n-1)(3n-1)-1}a_{n-1}}$$

$$n=1 \;\to\; a_1 = -\frac{1}{2}\frac{-21}{9}a_0 = \frac{7}{6}a_0$$

$$n=2 \;\to\; a_2 = -\frac{1}{2}\frac{-15}{54}a_1 = \frac{5}{36}\frac{7}{6}a_0 = \frac{35}{216}a_0$$

$$n=3 \;\to\; a_3 = -\frac{1}{2}\frac{-9}{135}a_2 = \frac{1}{30}\frac{35}{216}a_0 = \frac{7}{1,296}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x^{-1/6}\left(1+\frac{7}{6}x+\frac{35}{216}x^2+\frac{7}{1,296}x^3+\cdots\right)}$$

$$\boxed{r = \frac{1}{3} \quad \to \quad b_n = -\frac{3n-12}{\left(n+\frac{1}{3}\right)(18n+3)-1}b_{n-1} = -\frac{3(n-4)}{(3n+1)(6n+1)-1}b_{n-1}}$$

$$n=1 \;\to\; b_1 = -\frac{-9}{27}b_0 = \frac{1}{3}b_0$$

$$n=2 \;\to\; b_2 = \frac{6}{90}b_1 = \frac{1}{15}\frac{1}{3}b_0 = \frac{1}{45}b_0$$

$$n=3 \;\to\; b_3 = \frac{3}{189}b_2 = \frac{1}{63}\frac{1}{45}b_0 = \frac{1}{2,835}b_0$$

$$n=4 \;\to\; b_4 = 0b_3 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/3}\left(1+\frac{1}{3}x+\frac{1}{45}x^2+\frac{1}{2835}x^3\right)}$$

$$\boxed{y(x) = a_0\frac{1}{x^{1/6}}\left(1+\frac{7}{6}x+\frac{35}{216}x^2+\frac{7}{1296}x^3+\cdots\right)+b_0 x^{1/3}\left(1+\frac{1}{3}x+\frac{1}{45}x^2+\frac{1}{2835}x^3\right)}$$

Exercise

Find the Frobenius series solutions of $2x^2y'' + 7x(x+1)y' - 3y = 0$

Solution

$$y'' + \frac{7}{2}\frac{x+1}{x}y' - \frac{3}{2x^2}y = 0$$

That implies to $p(x) = \frac{7}{2}\frac{x+1}{x}$ and $q(x) = -\frac{3}{2x^2}$.

$$p_0 = \lim_{x\to 0} xp(x) = \frac{7}{2}\lim_{x\to 0} x\left(\frac{x+1}{x}\right) = \frac{7}{2}\lim_{x\to 0}(x+1) = \boxed{\frac{7}{2}}$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = -\lim_{x\to 0} \frac{3}{2} = \boxed{-\frac{3}{2}}$$

The indicial equation is: $r^2 + \left(\frac{7}{2} - 1\right) r - \frac{3}{2} = r^2 + \frac{5}{2} r - \frac{3}{2} = 0$

$$2r^2 + 5r - 3 = 0 \quad \rightarrow \quad \boxed{r = -3,\ \frac{1}{2}}$$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{-3} \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1/2} \sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2 y'' + 7x(x+1) y' - 3y = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \left(7x^2 + 7x\right) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - 3 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} 7(n+r) a_n x^{n+r+1} + \sum_{n=0}^{\infty} 7(n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} 3a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-2) + 7(n+r) - 3\right] a_n x^{n+r} + \sum_{n=0}^{\infty} 7(n+r) a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r+5) - 3\right] a_n x^{n+r} + \sum_{n=1}^{\infty} 7(n+r-1) a_{n-1} x^{n+r} = 0$$

$$\left(2r^2 + 5r - 3\right) a_0 x^r + \sum_{n=1}^{\infty} \left[(n+r)(2n+2r+5) - 3\right] a_n x^{n+r} + \sum_{n=1}^{\infty} 7(n+r-1) a_{n-1} x^{n+r} = 0$$

$$\left(2r^2 + 5r - 3\right) a_0 x^r + \sum_{n=1}^{\infty} \left[\left((n+r)(2n+2r+5) - 3\right) a_n + 7(n+r-1) a_{n-1}\right] x^{n+r} = 0$$

For $n = 0 \quad \rightarrow \quad \left(2r^2 + 5r - 3\right) a_0 = 0 \quad \Rightarrow \quad \boxed{r = -3,\ \frac{1}{2}}$ $\surd$

$$\left((n+r)(2n+2r+5) - 3\right) a_n + 7(n+r-1) a_{n-1} = 0$$

$$\boxed{a_n = -\frac{7(n+r-1)}{(n+r)(2n+2r+5)-3}a_{n-1}}$$

$$\boxed{r=-3 \quad \to \quad a_n = -\frac{7(n-4)}{(n-3)(2n-1)-3}a_{n-1}}$$

$$n=1 \to a_1 = -\frac{21}{5}a_0$$

$$n=2 \to a_2 = -\frac{14}{6}a_1 = -\frac{7}{3}\left(-\frac{21}{5}\right)a_0 = \frac{49}{5}a_0$$

$$n=3 \to a_3 = -\frac{-7}{-3}a_2 = -\frac{7}{3}\frac{49}{5}a_0 = -\frac{343}{15}a_0$$

$$n=4 \to a_4 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_1(x) = a_0 x^{-3}\left(1-\frac{21}{5}x+\frac{49}{5}x^2-\frac{343}{15}x^3\right)}$$

$$\boxed{r=\frac{1}{2} \quad \to \quad b_n = -\frac{7\left(n-\frac{1}{2}\right)}{\left(n+\frac{1}{2}\right)(2n+6)-3}b_{n-1} = -\frac{7}{2}\frac{2n-1}{(2n+1)(n+3)-3}b_{n-1}}$$

$$n=1 \to b_1 = -\frac{7}{2}\frac{1}{9}b_0 = -\frac{7}{18}b_0$$

$$n=2 \to b_2 = -\frac{7}{2}\frac{3}{22}b_1 = -\frac{21}{44}\frac{-7}{18}b_0 = \frac{49}{264}b_0$$

$$n=3 \to b_3 = -\frac{7}{2}\frac{5}{39}b_2 = -\frac{35}{78}\frac{49}{264}b_0 = -\frac{1,715}{20,592}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y_2(x) = b_0 x^{1/2}\left(1-\frac{7}{18}x+\frac{49}{264}x^2-\frac{1,715}{20,592}x^3+\cdots\right)}$$

$$\boxed{y(x) = a_0\frac{1}{x^3}\left(1-\frac{21}{5}x+\frac{49}{5}x^2-\frac{343}{15}x^3\right)+b_0\sqrt{x}\left(1-\frac{7}{18}x+\frac{49}{264}x^2-\frac{1,715}{20,592}x^3+\cdots\right)}$$

Exercise

Find the Frobenius series solutions: $x(1-x)y''+\left[c-(a+b+1)x\right]y'-aby=0$ (*Gauss' Hypergeometric*)

Solution

$$y''+\frac{c-(a+b+1)x}{x(1-x)}y'-\frac{ab}{x(1-x)}y=0$$

That implies to $p(x)=\dfrac{c-(a+b+1)x}{x(1-x)}$ and $q(x)=-\dfrac{ab}{x(1-x)}$.

$$p_0 = \lim_{x\to 0} xp(x)$$

$$= \lim_{x\to 0} x\left(\frac{c-(a+b+1)x}{x(1-x)}\right)$$

$$= \lim_{x\to 0}\left(\frac{c-(a+b+1)x}{1-x}\right)$$

$$\underline{=c}\,\rfloor$$

$$q_0 = \lim_{x\to 0} x^2 q(x)$$

$$= -\lim_{x\to 0} x^2\frac{ab}{x(1-x)}$$

$$= -\lim_{x\to 0}\frac{abx}{1-x}$$

$$\underline{=0}\,\rfloor$$

$$p_1 = \lim_{x\to 1}(x-1)\,p(x)$$

$$= \lim_{x\to 1}(x-1)\left(\frac{c-(a+b+1)x}{x(1-x)}\right)$$

$$= \lim_{x\to 1}\left(-\frac{c-(a+b+1)x}{x}\right)$$

$$\underline{=a+b+1-c}\,\rfloor$$

$$q_1 = \lim_{x\to 1}(x-1)^2 q(x)$$

$$= -\lim_{x\to 1}(x-1)^2\frac{ab}{x(1-x)}$$

$$= \lim_{x\to 1}\frac{ab}{x}(x-1)$$

$$\underline{=0}\,\rfloor$$

The *Regular* singular points: $\underline{x=0,\ 1}\,\rfloor$

The indicial equation is: $r(r-1)-cr = r^2+(c-1)r = 0 \quad\to\quad \underline{r=0,\ 1-c}\,\rfloor$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^0\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{1-c}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$$

$$(x-x^2)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + [c-(a+b+1)x]\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - ab\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}c(n+r)a_n x^{n+r-1}$$

$$-\sum_{n=0}^{\infty}(a+b+1)(n+r)a_n x^{n+r} - ab\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+c(n+r)\right]a_n x^{n+r-1}$$

$$-\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)+(a+b+1)(n+r)+ab\right]a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1+c)a_n x^{n+r-1} - \sum_{n=1}^{\infty}\left[(n+r-1)(n+r-2+a+b+1)+ab\right]a_{n-1}x^{n+r-1} = 0$$

$$r(r+c-1)a_0 x^{r-1} + \sum_{n=1}^{\infty}(n+r)(n+r-1+c)a_n x^{n+r-1}$$

$$-\sum_{n=1}^{\infty}\left[(n+r-1)(n+r-1+a+b)+ab\right]a_{n-1}x^{n+r-1} = 0$$

$$r(r+c-1)a_0 x^{r-1} + \sum_{n=1}^{\infty}\left[(n+r)(n+r-1+c)a_n - \left((n+r-1)(n+r-1+a+b)+ab\right)a_{n-1}\right]x^{n+r-1} = 0$$

For $n=0 \quad \rightarrow \quad r(r+c-1)a_0 = 0 \quad \Rightarrow \quad \boxed{r = 0,\ 1-c}$ √

$$(n+r)(n+r-1+c)a_n - \left((n+r-1)(n+r-1+a+b)+ab\right)a_{n-1} = 0$$

$$(n+r)(n+r-1+c)a_n = \left((n+r-1)(n+r-1+a+b)+ab\right)a_{n-1}$$

$$\boxed{a_n = \frac{(n+r-1)(n+r-1+a+b)+ab}{(n+r)(n+r-1+c)}a_{n-1}}$$

$$\boxed{r=0 \quad \to \quad a_n = \frac{(n-1)(n-1+a+b)+ab}{n(n-1+c)}a_{n-1}}$$

$$n=1 \to a_1 = \frac{ab}{c}a_0$$

$$n=2 \to a_2 = \frac{1+a+b+ab}{2\cdot(c+1)}a_1 = \frac{(a+1)(b+1)}{2\cdot(c+1)}a_1 = \frac{ab(a+1)(b+1)}{2\cdot c\cdot(c+1)}a_0$$

$$n=3 \to a_3 = \frac{4+2a+2b+ab}{3\cdot(c+2)}a_2 = \frac{(a+2)(b+2)}{3\cdot(c+2)}a_2 = \frac{a(a+1)(a+2)\cdot b(b+1)(b+2)}{2\cdot 3\cdot c(c+1)(c+2)}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\to a_n = \frac{a(a+1)(a+2)\cdots(a+n-1)\bullet b(b+1)(b+2)\cdots(b+n-1)}{n!\cdot c(c+1)(c+2)\cdots(c+n-1)}a_0$$

$$y_1(x) = a_0\left(1+\frac{ab}{c}x+\frac{ab(a+1)(b+1)}{2\cdot c\cdot(c+1)}x^2+\frac{a(a+1)(a+2)\cdot b(b+1)(b+2)}{2\cdot 3\cdot c(c+1)(c+2)}x^3+\cdots\right)$$

$$\boxed{= a_0\left(1+\sum_{n=0}^{\infty}\frac{a(a+1)\cdots(a+n-1)\bullet b(b+1)\cdots(b+n-1)}{n!\cdot c(c+1)\cdots(c+n-1)}x^n\right)}$$

$$\boxed{r=1-c \quad \to \quad b_n = \frac{(n-c)(n-c+a+b)+ab}{n(n+1-c)}b_{n-1}}$$

$$n=1 \to b_1 = \frac{(1-c)(1-c+a+b)+ab}{2-c}b_0$$

$$n=2 \to b_2 = \frac{(2-c)(2-c+a+b)+ab}{2(3-c)}b_1 = \frac{(1-c)(1-c+a+b)+ab}{2-c}b_0$$

$$= \frac{((2-c)(2-c+a+b)+ab)((1-c)(1-c+a+b)+ab)}{2(2-c)(3-c)}b_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\to b_n = \frac{((n-c)(n-c+a+b)+ab)\cdots((1-c)(1-c+a+b)+ab)}{2(2-c)(3-c)\ \cdots(n+1-c)}b_0$$

$$\boxed{y_2(x) = b_0 x^{1-c}\left(\begin{array}{l} 1+\dfrac{(1-c)(1-c+a+b)+ab}{2-c}x \\ +\dfrac{((2-c)(2-c+a+b)+ab)((1-c)(1-c+a+b)+ab)}{2(2-c)(3-c)}x^2+\cdots \end{array}\right)}$$

$$
y(x) = a_0\left(1 + \sum_{n=0}^{\infty} \frac{a(a+1)\cdots(a+n-1)\bullet b(b+1)\cdots(b+n-1)}{n!\cdot c(c+1)\cdots(c+n-1)} x^n\right)
+ b_0 x^{1-c}\left(1 + \sum_{n=0}^{\infty} \frac{\big((n-c)(n-c+a+b)+ab\big)\cdots\big((1-c)(1-c+a+b)+ab\big)}{2(2-c)(3-c)\ \cdots(n+1-c)} x^n\right)
$$

Solution   *Section* 4.5 – Bessel's Equation and Bessel Functions

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + (x^2 - \upsilon^2)y = 0$: $\qquad x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$

Solution

$$\upsilon^2 = \frac{1}{9} \rightarrow \upsilon = \frac{1}{3}$$

The general solution is: $\boxed{y(x) = c_1 J_{1/3}(x) + c_2 J_{-1/3}(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + (x^2 - \upsilon^2)y = 0$: $\qquad x^2y'' + xy' + (x^2 - 1)y = 0$

Solution

$$\upsilon^2 = 1 \rightarrow \upsilon = 1$$

The general solution is: $\boxed{y(x) = c_1 J_1(x) + c_2 Y_1(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + (x^2 - \upsilon^2)y = 0$: $\qquad 4x^2y'' + 4xy' + (4x^2 - 25)y = 0$

Solution

$$\upsilon^2 = \frac{25}{4} \rightarrow \upsilon = \pm\frac{5}{2}$$

The general solution is: $\boxed{y(x) = c_1 J_{5/2}(x) + c_2 J_{-5/2}(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2y'' + xy' + (x^2 - \upsilon^2)y = 0$: $\qquad 16x^2y'' + 16xy' + (16x^2 - 1)y = 0$

Solution

$$\upsilon^2 = \frac{1}{15} \rightarrow \upsilon = \pm\frac{1}{4}$$

The general solution is: $\underline{y(x) = c_1 J_{1/4}(x) + c_2 J_{-1/4}(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2 y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0$: $\qquad xy'' + y' + xy = 0$

Solution

$\upsilon^2 = 0 \;\rightarrow\; \upsilon = 0$

The general solution is: $\underline{y(x) = c_1 J_0(x) + cY_0(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2 y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0$: $\qquad xy'' + y' + \left(x - \frac{4}{x}\right)y = 0$

Solution

$\upsilon^2 = 4 \;\rightarrow\; \upsilon = 2$

The general solution is: $\underline{y(x) = c_1 J_2(x) + c_2 Y_2(x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2 y'' + xy' + \left(\alpha^2 x^2 - \upsilon^2\right)y = 0$: $\qquad x^2 y'' + xy' + \left(9x^2 - 4\right)y = 0$

Solution

$$\begin{cases} \alpha^2 = 9 \;\rightarrow\; \alpha = 3 \\ \upsilon^2 = 4 \;\rightarrow\; \upsilon = 2 \end{cases}$$

The general solution is: $\underline{y(x) = c_1 J_2(3x) + c_2 Y_2(3x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2 y'' + xy' + \left(\alpha^2 x^2 - \upsilon^2\right)y = 0$: $\qquad x^2 y'' + xy' + \left(36x^2 - \frac{1}{4}\right)y = 0$

Solution

$$\begin{cases} \alpha^2 = 36 \rightarrow \alpha = 6 \\ \upsilon^2 = \frac{1}{4} \rightarrow \upsilon = \frac{1}{2} \end{cases}$$

The general solution is: $\boxed{y(x) = c_1 J_{1/2}(6x) + c_2 J_{-1/2}(6x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2 y'' + xy' + \left(\alpha^2 x^2 - \upsilon^2\right) y = 0$: $\qquad x^2 y'' + xy' + \left(25x^2 - \frac{4}{9}\right) y = 0$

Solution

$$\begin{cases} \alpha^2 = 25 \rightarrow \alpha = 5 \\ \upsilon^2 = \frac{4}{9} \rightarrow \upsilon = \frac{2}{3} \end{cases}$$

The general solution is: $\boxed{y(x) = c_1 J_{2/3}(5x) + c_2 J_{-2/3}(5x)}$

Exercise

Find the general solution of the given differential equation on $(0, \infty)$ using Bessel equation

$x^2 y'' + xy' + \left(\alpha^2 x^2 - \upsilon^2\right) y = 0$: $\qquad x^2 y'' + xy' + \left(2x^2 - 64\right) y = 0$

Solution

$$\begin{cases} \alpha^2 = 2 \rightarrow \alpha = \sqrt{2} \\ \upsilon^2 = 64 \rightarrow \upsilon = 8 \end{cases}$$

The general solution is: $\boxed{y(x) = c_1 J_8\left(\sqrt{2}x\right) + c_2 Y_8\left(\sqrt{2}x\right)}$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$4x^2 y'' + 8xy' + \left(x^4 - 3\right) y = 0$$

Solution

$$\frac{1}{4} \times \quad 4x^2 y'' + 8xy' + \left(x^4 - 3\right) y = 0$$

$$x^2 y'' + 2xy' + \left(-\frac{3}{4} + \frac{1}{4}x^4\right) y = 0 \qquad\qquad x^2 \frac{d^2 y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right) y = 0$$

$A = 2, \quad B = -\frac{3}{4}, \quad C = \frac{1}{4}, \quad p = 4$

$$\alpha = \frac{1-2}{2} = -\frac{1}{2}, \quad \beta = \frac{4}{2} = 2, \quad k = \frac{2\sqrt{\frac{1}{4}}}{4} = \frac{1}{4}, \quad \upsilon = \frac{\sqrt{1+3}}{4} = \frac{1}{2}$$

$$y(x) = x^{-1/2}\left[c_1 J_{1/2}\left(\frac{1}{4}x^2\right) + c_2 J_{-1/2}\left(\frac{1}{4}x^2\right)\right]$$

$$y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

$$= x^{-1/2}\left(c_1\sqrt{\frac{2}{\pi z}}\sin z + c_2\sqrt{\frac{2}{\pi z}}\cos z\right)$$

$$= c_1\left(\frac{2}{\pi x}\right)^{1/2}\sin x + c_2\left(\frac{2}{\pi x}\right)^{1/2}\cos x$$

$$= x^{-1/2}\left(c_1\frac{2}{x}\sqrt{\frac{2}{\pi}}\sin\frac{x^2}{4} + c_2\frac{2}{x}\sqrt{\frac{2}{\pi}}\cos\frac{x^2}{4}\right)$$

$$z = kx^{\beta} = \frac{x^2}{4}$$

$$= x^{-3/2}\left(C_1\sqrt{\frac{2}{\pi}}\sin\frac{x^2}{4} + C_2\sqrt{\frac{2}{\pi}}\cos\frac{x^2}{4}\right)$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y'' + 9xy = 0$$

Solution

$$x^2 \times \quad y'' + 9xy = 0$$

$$x^2 y'' + 9x^3 y = 0$$

$$x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$A = 0, \quad B = 0, \quad C = 9, \quad p = 3$

$$\alpha = \frac{1}{2}, \quad \beta = \frac{3}{2}, \quad k = 2, \quad \upsilon = \frac{1}{3}$$

$$\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\boxed{y(x) = x^{1/2}\left[c_1 J_{1/3}\left(2x^{3/2}\right) + c_2 J_{-1/3}\left(2x^{3/2}\right)\right]}$$

$$y(x) = x^{\alpha}\left[c_1 J_{\upsilon}\left(kx^{\beta}\right) + c_2 J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' + (x-3)y = 0$$

Solution

$$x \times \quad xy'' - 3y + xy = 0$$

$$x^2 y'' - 3xy + x^2 y = 0$$

$$x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$A = -3, \quad B = 0, \quad C = 1, \quad p = 2$

$$\alpha = 2, \quad \beta = 1, \quad k = 1, \quad \upsilon = \frac{\sqrt{16}}{2} = 2 \qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2-4B}}{p}$$

$$\boxed{y(x) = x^2\left[c_1 Y_2(x) + c_2 J_2(x)\right]} \qquad y(x) = x^\alpha\left[c_1 J_\upsilon\left(kx^\beta\right) + c_2 J_{-\upsilon}\left(kx^\beta\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' + \left(4x^3 - 1\right)y = 0$$

Solution

$$x\times \quad xy'' - y + 4x^3 y = 0$$

$$x^2y'' - xy + 4x^4 y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = -1, \quad B = 0, \quad C = 4, \quad p = 4$$

$$\alpha = 1, \quad \beta = 2, \quad k = 1, \quad \upsilon = \frac{1}{2} \qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2-4B}}{p}$$

$$y(x) = x\left[c_1 J_{1/2}\left(x^2\right) + c_2 J_{-1/2}\left(x^2\right)\right] \qquad y(x) = x^\alpha\left[c_1 J_\upsilon\left(kx^\beta\right) + c_2 J_{-\upsilon}\left(kx^\beta\right)\right]$$

$$= x\left(c_1 \frac{1}{x}\sqrt{\frac{2}{\pi}}\sin x^2 + c_2 \frac{1}{x}\sqrt{\frac{2}{\pi}}\cos x^2\right) \qquad y(z) = x^\alpha\left(c_1\left(\frac{2}{\pi z}\right)^{1/2}\sin z + c_2\left(\frac{2}{\pi z}\right)^{1/2}\cos z\right)$$

$$= c_1\sqrt{\frac{2}{\pi}}\sin x^2 + c_2\sqrt{\frac{2}{\pi}}\cos x^2 \qquad z = kx^\beta = x^2$$

$$\boxed{= C_1 \sin x^2 + C_2 \cos x^2}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2y'' + xy' - \left(\frac{1}{4} + x^2\right)y = 0$$

Solution

$$x^2y'' + xy' + \left(-\frac{1}{4} - x^2\right)y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = 1, \quad B = -\frac{1}{4}, \quad C = -1, \quad p = 2$$

$$\alpha = 0, \quad \beta = 1, \quad k = i, \quad \upsilon = \frac{1}{2} \qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2-4B}}{p}$$

$$y(x) = c_1 I_{1/2}(x) + c_2 I_{-1/2}(x)$$

$$y(x) = x^{\alpha}\left[c_1 I_{\upsilon}\left(kx^{\beta}\right) + c_2 I_{-\upsilon}\left(kx^{\beta}\right)\right]$$

$$\boxed{y(x) = c_1\sqrt{\frac{2}{\pi x}}\sinh x + c_2\sqrt{\frac{2}{\pi x}}\cosh x}$$

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sinh x$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' + (2x+1)y' + (2x+1)y = 0$$

Solution

$$x\times \quad xy'' + (2x+1)y' + (2x+1)y = 0$$

$$x^2y'' + x(2x+1)y' + \left(2x^2 + x\right)y = 0$$

Let $Y = ye^{x} \quad \to \ y = Ye^{-x}$

$$x^2(Y'' - 2Y' + Y)e^{-x} + x(2x+1)(Y' - Y)e^{-x} + \left(2x^2 + x\right)Ye^{-x} = 0$$

$$x^2Y'' - 2x^2Y' + x^2Y + \left(2x^2 + x\right)Y' - \left(2x^2 + x\right)Y + \left(2x^2 + x\right)Y = 0$$

$$x^2Y'' + xY' + x^2Y = 0$$

$$x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^{p}\right)y = 0$$

$A = 1, \quad B = 0, \quad C = 1, \quad p = 2$

$$\alpha = 0, \quad \beta = 1, \quad k = 1, \quad \upsilon = 0$$

$$\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$Y(x) = c_1 J_0(x) + c_2 Y_0(x)$$

$$y(x) = x^{\alpha}\left[c_1 J_0\left(kx^{\beta}\right) + c_2 Y_0\left(kx^{\beta}\right)\right]$$

$$\boxed{y(x) = \left(c_1 J_0(x) + c_2 Y_0(x)\right)e^{-x}}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' - y' - xy = 0$$

Solution

$$x\times \quad xy'' - y' - xy = 0$$

$$x^2y'' - xy' - x^2y = 0$$

$$x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^{p}\right)y = 0$$

$A = -1, \quad B = 0, \quad C = -1 = i, \quad p = 2$

Let $Y = \dfrac{y}{x}$ & $X = ix$

$$y = xY \quad \& \quad x = -iX$$

$$x^2\left(2Y' + xY''\right) - x\left(Y + xY'\right) - x^3Y = 0$$

$$x^3Y'' + x^2Y' - x\left(x^2 + 1\right)Y = 0$$

$$x^2Y'' + xY' - \left(x^2 + 1\right)Y = 0$$

$$-X^2Y'' - iXY' - \left(-X^2 + 1\right)Y = 0$$

$$X^2Y'' + XY' + \left(X^2 - 1\right)Y = 0$$

$$x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$A = 1, \quad B = -1, \quad C = 1, \quad p = 2$

$$\alpha = 0, \quad \beta = 1, \quad k = 1, \quad \upsilon = 1$$

$$\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$Y = Z_1(X)$$

$$y(x) = xZ_1(ix)$$

$$y(x) = x^\alpha\left[c_1J_0\left(kx^\beta\right) + c_2Y_0\left(kx^\beta\right)\right]$$

$$\boxed{= x\left(c_1I_1(x) + c_2K_1(x)\right)}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^4y'' + a^2y = 0$$

Solution

$$\frac{1}{x^2}\times \quad x^4y'' + a^2y = 0$$

$$x^2y'' + \frac{a^2}{x^2}y = 0$$

Let $Y = \frac{y}{\sqrt{x}} \quad \to y = \sqrt{x}\,Y$

$$X = \frac{a}{x} \quad \to \quad x = \frac{a}{X}$$

$$X^2Y'' + XY' + \left(X^2 - K^2\right)Y = 0$$

$$Y = x^{-1/2}y$$

$$Y' = -\frac{1}{2}x^{-3/2}y + x^{-1/2}y'$$

$$Y'' = \frac{3}{4}x^{-5/2}y - x^{-3/2}y' + x^{-1/2}y''$$

$$x^2\left(x^{-1/2}y'' - x^{-3/2}y' + \frac{3}{4}x^{-5/2}y\right) + x\left(-\frac{1}{2}x^{-3/2}y + x^{-1/2}y'\right) + \left(x^2 - K^2\right)x^{-1/2}y = 0$$

$$x^{3/2}y'' - x^{1/2}y' + \frac{3}{4}x^{-1/2}y - \frac{1}{2}x^{-1/2}y + x^{1/2}y' + \left(x^2 - K^2\right)x^{-1/2}y = 0$$

$$x^{3/2}y'' + \left(x^2 - K^2 + \frac{1}{4}\right)x^{-1/2}y = 0 \quad \times x^{1/2}$$

$$x^2y'' + \left(x^2 - K^2 + \frac{1}{4}\right)y = 0$$

$$x^2 - K^2 + \frac{1}{4} = x^2$$

$$-K^2 + \frac{1}{4} = 0 \quad \rightarrow \quad K^2 = \frac{1}{4}$$

$$X^2Y'' + XY' + \left(X^2 - \frac{1}{4}\right)Y = 0$$

$$x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = 1, \quad B = -\frac{1}{4}, \quad C = 1, \quad p = 2$$

$$\alpha = 0, \quad \beta = 1, \quad k = 1, \quad \upsilon = \frac{1}{2}$$

$$\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$Y = Z_{1/2}(X)$$

$$y(x) = \sqrt{x}Z_{1/2}\left(\frac{a}{x}\right)$$

$$y(x) = x^\alpha\left[c_1J_\upsilon\left(kx^\beta\right) + c_2J_{-\upsilon}\left(kx^\beta\right)\right]$$

$$= \sqrt{x}\left(c_1J_{1/2}\left(\frac{a}{x}\right) + c_2J_{-1/2}\left(\frac{a}{x}\right)\right)$$

$$y(z) = x^\alpha\left(c_1\left(\frac{2}{\pi z}\right)^{1/2}\sin z + c_2\left(\frac{2}{\pi z}\right)^{1/2}\cos z\right)$$

$$= \sqrt{x}\left(c_1\sqrt{\frac{2x}{\pi a}}\sin\frac{a}{x} + c_2\sqrt{\frac{2x}{\pi a}}\cos\frac{a}{x}\right)$$

$$z = kX^\beta = \frac{a}{x}$$

$$= x\left(c_1\sqrt{\frac{2}{\pi a}}\sin\frac{a}{x} + c_2\sqrt{\frac{2}{\pi a}}\cos\frac{a}{x}\right)$$

$$\boxed{= x\left(C_1\sin\frac{a}{x} + C_2\cos\frac{a}{x}\right)}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y'' - x^2y = 0$$

Solution

$$x^2 \times \quad y'' - x^2y = 0$$

$$x^2y'' - x^4y = 0$$

$$A = 0, \quad B = 0, \quad C = -1, \quad p = 4$$

$$\alpha = \frac{1}{2}, \quad \beta = 1, \quad k = \frac{i}{2}, \quad \upsilon = 0$$

$$\alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

Let $Y = \frac{y}{\sqrt{x}} \quad \rightarrow y = \sqrt{x}\,Y$

$$X=\frac{1}{2}ix^2 \quad \to \quad x^2=-2iX$$

$$X^2Y''+XY'+\left(X^2-K^2\right)Y=0$$

$$Y=x^{-1/2}y$$

$$Y'=-\frac{1}{2}x^{-3/2}y+x^{-1/2}y'$$

$$Y''=\frac{3}{4}x^{-5/2}y-x^{-3/2}y'+x^{-1/2}y''$$

$$x^2\left(x^{-1/2}y''-x^{-3/2}y'+\frac{3}{4}x^{-5/2}y\right)+x\left(-\frac{1}{2}x^{-3/2}y+x^{-1/2}y'\right)+\left(x^2-K^2\right)x^{-1/2}y=0$$

$$x^{3/2}y''-x^{1/2}y'+\frac{3}{4}x^{-1/2}y-\frac{1}{2}x^{-1/2}y+x^{1/2}y'+\left(x^2-K^2\right)x^{-1/2}y=0$$

$$x^{3/2}y''+\left(x^2-K^2+\frac{1}{4}\right)x^{-1/2}y=0 \qquad \times x^{1/2}$$

$$x^2y''+\left(x^2-K^2+\frac{1}{4}\right)y=0$$

$$x^2-K^2+\frac{1}{4}=-x^4$$

$$K=\frac{1}{4} \quad \to \quad K^2=\frac{1}{16}$$

$$X^2Y''+XY'+\left(X^2-\frac{1}{16}\right)Y=0$$

$A=1, \quad B=-\frac{1}{16}, \quad C=1, \quad p=2$

$\alpha=0, \quad \beta=1, \quad k=1, \quad \upsilon=\frac{1}{4}$ $\qquad \alpha=\frac{1-A}{2},\ \beta=\frac{p}{2},\ k=\frac{2\sqrt{C}}{p},\ \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}$

$$Y=Z_{1/4}(X)$$

$$y(x)=\sqrt{x}\,Z_{\frac{1}{4}}\left(\frac{i}{2}x^2\right)$$

$$=\sqrt{x}\left(c_1 I_{\frac{1}{4}}\left(\frac{x^2}{2}\right)+c_2 I_{-\frac{1}{4}}\left(\frac{x^2}{2}\right)\right)$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2y''-xy'+\left(1+x^2\right)y=0$$

Solution

$x^2y''-xy'+\left(1+x^2\right)y=0$ $\qquad x^2\frac{d^2y}{dx^2}+Ax\frac{dy}{dx}+\left(B+Cx^p\right)y=0$

$A=-1, \quad B=1, \quad C=1, \quad p=2$

$\alpha = 1, \quad \beta = 1, \quad k = 1, \quad \upsilon = 0$ $\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$

$$\boxed{y(x) = x\left[c_1 J_0(x) + c_2 Y_0(x)\right]}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' + 3y' + xy = 0$$

Solution

$x \times \quad xy'' + 3y' + xy = 0$

$x^2y'' + 3xy' + x^2y = 0$ $\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$

$A = 3, \quad B = 0, \quad C = 1, \quad p = 2$

$\alpha = -1, \quad \beta = 1, \quad k = 1, \quad \upsilon = 1$ $\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$

$$\boxed{y(x) = x^{-1}\left[c_1 J_1(x) + c_2 Y_1(x)\right]}$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$xy'' - y' + 36x^3y = 0$$

Solution

$x \times \quad xy'' - y' + 36x^3y = 0$

$x^2y'' - xy' + 36x^4y = 0$ $\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$

$A = -1, \quad B = 0, \quad C = 36, \quad p = 4$

$\alpha = 1, \quad \beta = 2, \quad k = 3, \quad \upsilon = \frac{1}{2}$ $\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$

$$\boxed{y(x) = x\left[c_1 J_{1/2}\left(3x^2\right) + c_2 J_{-1/2}\left(3x^2\right)\right]}$$ $\qquad y(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2y''-5xy'+(8+x)y=0$$

Solution

$$x^2y''-5xy'+(8+x)y=0 \qquad x^2\frac{d^2y}{dx^2}+Ax\frac{dy}{dx}+\left(B+Cx^p\right)y=0$$

$$A=-5,\quad B=8,\quad C=1,\quad p=1$$

$$\alpha=3,\quad \beta=\frac{1}{2},\quad k=2,\quad \upsilon=2 \qquad \alpha=\frac{1-A}{2},\ \beta=\frac{p}{2},\ k=\frac{2\sqrt{C}}{p},\ \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}$$

$$\boxed{y(x)=x^3\left[c_1J_2\left(2x^{1/2}\right)+c_2Y_2\left(2x^{1/2}\right)\right]} \qquad y(x)=x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right)+c_2Y_{\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$36x^2y''+60xy'+\left(9x^3-5\right)y=0$$

Solution

$$x^2y''+\frac{5}{3}xy'+\left(\frac{1}{4}x^3-\frac{5}{36}\right)y=0 \qquad x^2\frac{d^2y}{dx^2}+Ax\frac{dy}{dx}+\left(B+Cx^p\right)y=0$$

$$A=\frac{5}{3},\quad B=-\frac{5}{36},\quad C=\frac{1}{4},\quad p=3$$

$$\alpha=-\frac{1}{3},\quad \beta=\frac{3}{2},\quad k=\frac{1}{3},\quad \upsilon=\frac{\sqrt{\left(-\frac{2}{3}\right)^2+\frac{5}{9}}}{3}=\frac{1}{3} \qquad \alpha=\frac{1-A}{2},\ \beta=\frac{p}{2},\ k=\frac{2\sqrt{C}}{p},\ \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}$$

$$\boxed{y(x)=x^{-1/3}\left[c_1J_{1/3}\left(\frac{1}{3}x^{3/2}\right)+c_2J_{-1/3}\left(\frac{1}{3}x^{3/2}\right)\right]} \qquad y(x)=x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right)+c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$16x^2y''+24xy'+\left(1+144x^3\right)y=0$$

Solution

$$x^2y''+\frac{3}{2}xy'+\left(\frac{1}{16}+9x^3\right)y=0 \qquad x^2\frac{d^2y}{dx^2}+Ax\frac{dy}{dx}+\left(B+Cx^p\right)y=0$$

$$A=\frac{3}{2},\quad B=\frac{1}{16},\quad C=9,\quad p=3$$

$\alpha = -\frac{1}{4}, \quad \beta = \frac{3}{2}, \quad k = 2, \quad \upsilon = \frac{\sqrt{\left(-\frac{1}{2}\right)^2 - \frac{1}{4}}}{3} = 0$

$\alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$

$\underline{y(x) = x^{-1/4}\left[c_1 J_0\left(2x^{3/2}\right) + c_2 Y_0\left(2x^{3/2}\right)\right]}$

$y(x) = x^{\alpha}\left[c_1 J_\upsilon\left(kx^{\beta}\right) + c_2 Y_\upsilon\left(kx^{\beta}\right)\right]$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$x^2 y'' + 3xy' + \left(1 + x^2\right)y = 0$$

Solution

$x^2 y'' + 3xy' + \left(1 + x^2\right)y = 0$

$x^2 \frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$

$A = 3, \quad B = 1, \quad C = 1, \quad p = 2$

$\alpha = -1, \quad \beta = 1, \quad k = 1, \quad \upsilon = \frac{\sqrt{(-2)^2 - 4}}{3} = 0$

$\alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$

$\underline{y(x) = x^{-1}\left[c_1 J_0(x) + c_2 Y_0(x)\right]}$

$y(x) = x^{\alpha}\left[c_1 J_\upsilon\left(kx^{\beta}\right) + c_2 Y_\upsilon\left(kx^{\beta}\right)\right]$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$4x^2 y'' - 12xy' + (15 + 16x)y = 0$$

Solution

$x^2 y'' - 3xy' + \left(\frac{15}{4} + 4x\right)y = 0$

$x^2 \frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$

$A = -3, \quad B = \frac{15}{4}, \quad C = 4, \quad p = 1$

$\alpha = 2, \quad \beta = \frac{1}{2}, \quad k = 4, \quad \upsilon = \frac{\sqrt{(4)^2 - 15}}{1} = 1$

$\alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$

$\underline{y(x) = x^{2}\left[c_1 J_1\left(4x^{1/2}\right) + c_2 Y_1\left(4x^{1/2}\right)\right]}$

$y(x) = x^{\alpha}\left[c_1 J_\upsilon\left(kx^{\beta}\right) + c_2 Y_\upsilon\left(kx^{\beta}\right)\right]$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$16x^2y'' - \left(5 - 144x^3\right)y = 0$$

Solution

$$x^2y'' + \left(9x^3 - \frac{5}{16}\right)y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = 0, \quad B = -\frac{5}{16}, \quad C = 9, \quad p = 3$$

$$\alpha = \frac{1}{2}, \quad \beta = \frac{3}{2}, \quad k = 2, \quad \upsilon = \frac{\sqrt{1 + \frac{5}{4}}}{3} = \frac{1}{2} \qquad \alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\boxed{y(x) = x^{1/2}\left[c_1J_{1/2}\left(2x^{3/2}\right) + c_2J_{-1/2}\left(2x^{3/2}\right)\right]} \qquad y(x) = x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right) + c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$2x^2y'' + 3xy' - \left(28 - 2x^5\right)y = 0$$

Solution

$$x^2y'' + \frac{3}{2}xy' + \left(x^5 - 14\right)y = 0 \qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$$A = \frac{3}{2}, \quad B = -14, \quad C = 1, \quad p = 5$$

$$\alpha = -\frac{1}{4}, \quad \beta = \frac{5}{2}, \quad k = \frac{2}{5}, \quad \upsilon = \frac{\sqrt{\left(-\frac{1}{2}\right)^2 + 56}}{5} = \frac{\frac{15}{2}}{5} = \frac{3}{2} \qquad \alpha = \frac{1-A}{2}, \ \beta = \frac{p}{2}, \ k = \frac{2\sqrt{C}}{p}, \ \upsilon = \frac{\sqrt{(1-A)^2 - 4B}}{p}$$

$$\boxed{y(x) = x^{-1/4}\left[c_1J_{3/2}\left(\frac{2}{5}x^{5/2}\right) + c_2J_{-3/2}\left(\frac{2}{5}x^{5/2}\right)\right]} \qquad y(x) = x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right) + c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y'' + x^4y = 0$$

Solution

$$x^2 \times \quad y'' + x^4y = 0$$

$$x^2y'' + x^6y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$A = 0, \quad B = 0, \quad C = 1, \quad p = 6$

$$\alpha = \frac{1}{2}, \quad \beta = 3, \quad k = \frac{1}{3}, \quad \upsilon = \frac{1}{6} \qquad\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2-4B}}{p}$$

$$\boxed{y(x) = x^{1/2}\left[c_1J_{1/6}\left(\tfrac{1}{3}x^3\right) + c_2J_{-1/6}\left(\tfrac{1}{3}x^3\right)\right]} \qquad\qquad y(x) = x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right) + c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Determine the general solution in terms of Bessel functions or, if possible, in terms of elementary functions.

$$y'' + 4x^3y = 0$$

Solution

$x^2 \times \quad y'' + 4x^3y = 0$

$$x^2y'' + 4x^5y = 0 \qquad\qquad x^2\frac{d^2y}{dx^2} + Ax\frac{dy}{dx} + \left(B + Cx^p\right)y = 0$$

$A = 0, \quad B = 0, \quad C = 4, \quad p = 5$

$$\alpha = \frac{1}{2}, \quad \beta = \frac{5}{2}, \quad k = \frac{4}{5}, \quad \upsilon = \frac{1}{5} \qquad\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2-4B}}{p}$$

$$\boxed{y(x) = x^{1/2}\left[c_1J_{1/5}\left(\tfrac{4}{5}x^{5/2}\right) + c_2J_{-1/5}\left(\tfrac{4}{5}x^{5/2}\right)\right]} \qquad\qquad y(x) = x^{\alpha}\left[c_1J_{\upsilon}\left(kx^{\beta}\right) + c_2J_{-\upsilon}\left(kx^{\beta}\right)\right]$$

Exercise

Find a Frobenius solution of Bessel's equation of order zero $x^2y'' + xy' + x^2y = 0$

Solution

$$y'' + \frac{1}{x}y' + y = 0$$

Therefore, $x = 0$ is a regular singular point, and that $p_0 = 1, \quad q_0 = 0$ and $p(x) \equiv 1, \quad q(x) = x^2$.

The indicial equation is: $r(r-1) + r = r^2 = 0 \quad \rightarrow \quad \boxed{r = 0}$

There is only one Frobenius series solution: $y(x) = \sum_{n=0}^{\infty} a_nx^n$

$$y = \sum_{n=0}^{\infty} a_nx^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$x^2 y'' + x y' + x^2 y = 0$$

$$x^2 \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} n a_n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} [n(n-1)+n] a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\sum_{n=0}^{\infty} n^2 a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0 \quad a_4 = -\frac{a_2}{4^2} = \frac{a_0}{2^2 \cdot 4^2}$$

$$0 + a_1 x + \sum_{n=2}^{\infty} n^2 a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$a_1 x + \sum_{n=2}^{\infty} \left(n^2 a_n + a_{n-2}\right) x^n = 0$$

$$a_1 = 0 \quad \rightarrow \quad a_{n(odd)} = 0$$

$$n^2 a_n + a_{n-2} = 0 \quad \Rightarrow \quad a_n = -\frac{a_{n-2}}{n^2} \quad (n \geq 2)$$

$$a_2 = -\frac{a_0}{4}$$

$$a_6 = -\frac{a_4}{6^2} = -\frac{a_0}{2^2 \cdot 4^2 \cdot 6^2}$$

$$a_{2n} = \frac{(-1)^n}{2^2 \cdot 4^2 \cdots (2n)^2} a_0 = \frac{(-1)^n}{2^{2n} \cdot (n!)^2} a_0$$

The choice $a_0 = 1$ gives us the Bessel function of order zero of the first kind.

$$J_0(x) = \frac{(-1)^n x^{2n}}{2^{2n} \cdot (n!)^2} \quad = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots$$

Exercise

Derive the formula $\quad x J_\upsilon'(x) = \upsilon J_\upsilon(x) - x J_{\upsilon+1}(x)$

Solution

$$x J_\upsilon(x) = x\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$x J_\upsilon'(x) = x\sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= 2\sum_{n=0}^{\infty}\frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon J_v(x)$$

$$= \upsilon J_v(x) + x\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-1)!\,\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \upsilon J_v(x) + x\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n!\,\Gamma(2+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon+1}$$

$$= \upsilon J_\upsilon(x) - x J_{\upsilon+1}(x) \quad \checkmark$$

Exercise

Derive the formula $\quad x J_\upsilon'(x) = -\upsilon J_\upsilon(x) + x J_{\upsilon-1}(x)$

Solution

$$x J_\upsilon'(x) = x\sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= 2\sum_{n=0}^{\infty}\frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$-\upsilon J_{\upsilon}(x)+xJ_{\upsilon-1}(x)=-\upsilon\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}+x\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^n\upsilon}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^n\upsilon}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}+\sum_{n=0}^{\infty}\frac{(-1)^n(\upsilon+n)}{n!\Gamma(1+\upsilon+n)}2\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(-\upsilon+2n+2\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$=x\sum_{n=0}^{\infty}\frac{(-1)^n(2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$\boxed{=xJ'_{\upsilon}(x)}\ \checkmark$$

Exercise

Derive the formula $\quad 2\upsilon J'_{\upsilon}(x)=xJ_{\upsilon+1}(x)+xJ_{\upsilon-1}(x)$

Solution

From previous proofs:

$$\begin{array}{rl} & xJ'_{\upsilon}(x)=\upsilon J_{\upsilon}(x)-xJ_{\upsilon+1}(x)\\ - & xJ'_{\upsilon}(x)=-\upsilon J_{\upsilon}(x)+xJ_{\upsilon-1}(x)\\ \hline & 0=2\upsilon J_{\upsilon}(x)-xJ_{\upsilon+1}(x)-xJ_{\upsilon-1}(x)\end{array}$$

$$\boxed{2\upsilon J'_{\upsilon}(x)=xJ_{\upsilon+1}(x)+xJ_{\upsilon-1}(x)}\ \checkmark$$

Exercise

Prove that $\quad \frac{d}{dx}\left[x^{\upsilon+1}J_{\upsilon+1}(x)\right]=x^{\upsilon+1}J_{\upsilon}(x)$

Solution

$$\frac{d}{dx}\left[x^{\upsilon+1}J_{\upsilon+1}(x)\right]=\frac{d}{dx}\left[x^{\upsilon+1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+1+n)}\left(\frac{x}{2}\right)^{2n+\upsilon+1}\right]$$

$$=\frac{d}{dx}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\upsilon+n+2)}\left(\frac{x}{2}\right)^{2n+2\upsilon+2}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(2n+2\upsilon+2)}{n!\Gamma(\upsilon+n+2)}\left(\frac{x}{2}\right)^{2n+2\upsilon+1}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n 2(n+\upsilon+1)}{n!2\Gamma(\upsilon+n+2)}\left(\frac{x}{2}\right)^{2n+2\upsilon+1} \qquad 2\Gamma(\upsilon+n+2)=2(\upsilon+n+1)\Gamma(\upsilon+n+1)$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n 2(n+\upsilon+1)}{n!\,2(\upsilon+n+1)\Gamma(\upsilon+n+1)}\left(\frac{x}{2}\right)^{2n+2\upsilon+1}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(\upsilon+n+1)}\left(\frac{x}{2}\right)^{2n+2\upsilon+1}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(\upsilon+n+1)}\left(\frac{x}{2}\right)^{2n+\upsilon}\left(\frac{x}{2}\right)^{\upsilon+1}$$

$$=x^{\upsilon+1}J_{\upsilon}(x) \quad \checkmark$$

Exercise

Show that $y=\sqrt{x}\,J_{3/2}(x)$ is a solution of $x^2y''+\left(x^2-2\right)y=0$

Solution

$$x^2y''+\left(x^2-2\right)y=0$$

$J_{3/2}(x)$ is the solution of Bessel's equation of order $\frac{3}{2}$:

$$x^2J''_{3/2}(x)+xJ'_{3/2}(x)+\left(x^2-\frac{9}{4}\right)J_{3/2}(x)=0$$

$$x^2\left(\sqrt{x}\,J_{3/2}(x)\right)''+\left(x^2-2\right)\sqrt{x}\,J_{3/2}(x)=$$

$$=x^2\left[-\frac{1}{4}x^{-3/2}J_{3/2}(x)+x^{-1/2}J'_{3/2}(x)+x^{1/2}J''_{3/2}(x)\right]+\left(x^2-2\right)\sqrt{x}\,J_{3/2}(x)$$

$$=-\frac{1}{4}x^{1/2}J_{3/2}(x)+x^{3/2}J'_{3/2}(x)+x^{5/2}J''_{3/2}(x)+x^{5/2}J_{3/2}(x)-2\sqrt{x}\,J_{3/2}(x)$$

$$= \sqrt{x}\left[x^2 J''_{3/2}(x) + x J'_{3/2}(x) + \left(x^2 - \frac{9}{4}\right) J_{3/2}(x)\right]$$

$$\underline{= 0}$$

Exercise

Show that $\quad 4J''_{\upsilon}(x) = J_{\upsilon-2}(x) - 2J_{\upsilon}(x) + J_{\upsilon+2}(x)$

Solution

$$J_{\upsilon}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$x J'_{\upsilon}(x) = x\sum_{n=0}^{\infty} \frac{(-1)^n (2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= 2\sum_{n=0}^{\infty} \frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$-\upsilon J_{\upsilon}(x) + x J_{\upsilon-1}(x) = -\upsilon\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + x\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n \upsilon}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n \upsilon}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon} + \sum_{n=0}^{\infty} \frac{(-1)^n (\upsilon+n)}{n!\Gamma(1+\upsilon+n)} 2\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (-\upsilon+2n+2\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= x\sum_{n=0}^{\infty} \frac{(-1)^n (2n+\upsilon)}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$\underline{= x J'_{\upsilon}(x)}$$

$$x J'_{\upsilon}(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\upsilon)}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\upsilon)}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= 2\sum_{n=0}^{\infty} \frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n n}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon} + \upsilon J_v(x)$$

$$= \upsilon J_v(x) + x \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!\,\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon-1}$$

$$= \upsilon J_v(x) + x \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n!\,\Gamma(2+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon+1}$$

$$\boxed{= \upsilon J_{\upsilon}(x) - x J_{\upsilon+1}(x)}$$

$$x J'_{\upsilon}(x) = -\upsilon J_{\upsilon}(x) + x J_{\upsilon-1}(x)$$

$$\begin{array}{rl} & x J'_{\upsilon}(x) = \upsilon J_{\upsilon}(x) - x J_{\upsilon+1}(x) \\ + & x J'_{\upsilon}(x) = -\upsilon J_{\upsilon}(x) + x J_{\upsilon-1}(x) \\ \hline & 2x J'_{\upsilon}(x) = -x J_{\upsilon+1}(x) + x J_{\upsilon-1}(x) \end{array}$$

$$J'_{\upsilon}(x) = \frac{1}{2}\left(J_{\upsilon-1}(x) - J_{\upsilon+1}(x)\right)$$

$$J''_{\upsilon}(x) = \frac{1}{2}\left(J'_{\upsilon-1}(x) - J'_{\upsilon+1}(x)\right)$$

$$J'_{\upsilon}(x) = \frac{1}{2}\left(J_{\upsilon-1}(x) - J_{\upsilon+1}(x)\right) \quad \rightarrow (\upsilon = \upsilon - 1) \quad J'_{\upsilon-1}(x) = \frac{1}{2}\left(J_{\upsilon-2}(x) - J_{\upsilon}(x)\right)$$

$$J'_{\upsilon}(x) = \frac{1}{2}\left(J_{\upsilon-1}(x) - J_{\upsilon+1}(x)\right) \quad \rightarrow (\upsilon = \upsilon + 1) \quad J'_{\upsilon+1}(x) = \frac{1}{2}\left(J_{\upsilon}(x) - J_{\upsilon+2}(x)\right)$$

$$J''_{\upsilon}(x) = \frac{1}{2}\left(J'_{\upsilon-1}(x) - J'_{\upsilon+1}(x)\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}J_{\upsilon-2}(x) - \frac{1}{2}J_{\upsilon}(x) - \frac{1}{2}J_{\upsilon}(x) + \frac{1}{2}J_{\upsilon+2}(x)\right)$$

$$= \frac{1}{4}\left(J_{\upsilon-2}(x) - 2J_{\upsilon}(x) + J_{\upsilon+2}(x)\right)$$

$$\boxed{4J''_{\upsilon}(x) = J_{\upsilon-2}(x) - 2J_{\upsilon}(x) + J_{\upsilon+2}(x)} \quad \checkmark$$

Exercise

Show that $y = x^{1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$ is a solution of Airy's differential equation $y'' + \alpha^2 xy = 0$, $x > 0$, whenever w is a solution of Bessel's equation of order $\frac{2}{3}$, that is, $t^2w'' + tw' + \left(t^2 - \frac{1}{9}\right)w = 0$, $t > 0$.

[*Hint*: After differentiating, substituting, and simplifying, then let $t = \frac{2}{3}\alpha x^{3/2}$].

Solution

$$y = x^{1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$$

$$\begin{aligned} y' &= \frac{1}{2}x^{-1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right) + x^{1/2}\left(\alpha x^{1/2}\right)w'\left(\frac{2}{3}\alpha x^{3/2}\right) \\ &= \alpha x w'\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{1}{2}x^{-1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right) \end{aligned}$$

$$\begin{aligned} y'' &= \alpha x\left(\alpha x^{1/2}\right)w''\left(\frac{2}{3}\alpha x^{3/2}\right) + \alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{1}{2}x^{-1/2}\left(\alpha x^{1/2}\right)w'\left(\frac{2}{3}\alpha x^{3/2}\right) - \frac{1}{4}x^{-3/2}w\left(\frac{2}{3}\alpha x^{3/2}\right) \\ &= \alpha^2 x^{3/2}w''\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{3}{2}\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right) - \frac{1}{4}x^{-3/2}w\left(\frac{2}{3}\alpha x^{3/2}\right) \end{aligned}$$

$$y'' + \alpha^2 xy = 0$$

$$\alpha^2 x^{3/2}w''\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{3}{2}\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right) - \frac{1}{4}x^{-3/2}w\left(\frac{2}{3}\alpha x^{3/2}\right) + \alpha^2 x^{3/2}w\left(\frac{2}{3}\alpha x^{3/2}\right) = 0$$

$$\alpha^2 x^{3/2}w''\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{3}{2}\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right) + \left(\alpha^2 x^{3/2} - \frac{1}{4x^{3/2}}\right)w\left(\frac{2}{3}\alpha x^{3/2}\right) = 0$$

$$t = \frac{2}{3}\alpha x^{3/2} \quad \rightarrow \quad \alpha x^{3/2} = \frac{3}{2}t$$

$$\frac{3}{2}\frac{\alpha}{t}\left[t^2 w''(t) + tw'(t) + \left(t^2 - \frac{1}{9}\right)w(t)\right] = 0$$

$$\boxed{t^2 w'' + tw' + \left(t^2 - \frac{1}{9}\right)w = 0} \quad \checkmark$$

Exercise

Use the relation $\Gamma(x+1) = x\Gamma(x)$ and if p is nonnegative integer, then show that

$$J_\upsilon(x) = \frac{1}{\Gamma(\upsilon+1)}\left(\frac{x}{2}\right)^\upsilon \left[1 + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)}\left(\frac{x}{2}\right)^{2n}\right]$$

Solution

$$J_\upsilon(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(\upsilon+n+1)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

Given: $\Gamma(x+1) = x\Gamma(x)$

$$\Gamma(\upsilon+n+1)=(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)\Gamma(\upsilon+n)$$

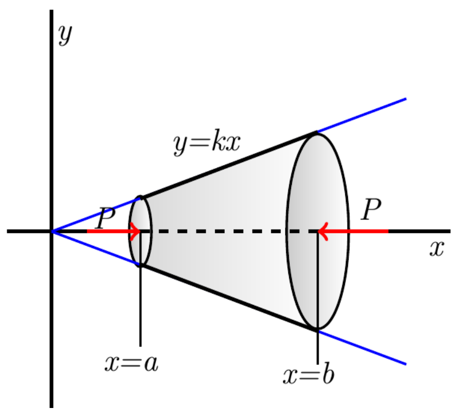
$$J_\upsilon(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)\Gamma(\upsilon+n)}\left(\frac{x}{2}\right)^{2n}\left(\frac{x}{2}\right)^{\upsilon}$$

$$=\frac{1}{\Gamma(\upsilon+1)}\left(\frac{x}{2}\right)^{\upsilon}\left[1+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)}\left(\frac{x}{2}\right)^{2n}\right] \quad \checkmark$$

Exercise

A linearly tapered rod with circular cross section, subject to an axial force P of compression. Its deflection curve $y=y(x)$ satisfies the endpoint value problem

$$EIy''+Py=0\ ; \quad y(a)=y(b)=0 \quad (1)$$



Here, however, the moment of inertia $I=I(x)$ of the cross section at x is given by

$$I(x)=\frac{1}{4}\pi(kx)^4=I_0\left(\frac{x}{b}\right)^4 \quad (2)$$

Where $I_0=I(b)$, the value of I at $x=b$. Substitution of $I(x)$ in the differential equation (1) yields to the eigenvalue problem

$$x^4y''+\lambda y=0\ ; \quad y(a)=y(b)=0 \quad (3)$$

Where $\lambda=\mu^2=\frac{Pb^4}{EI_0}$

a) Show that the general solution of $x^4y''+\mu^2y=0$ is $\quad y(x)=x\left(A\cos\frac{\mu}{x}+B\sin\frac{\mu}{x}\right)$

b) Conclude that the nth eigenvalue is given by $\mu_n=n\pi\frac{ab}{L}$, where $L=b-a$ is the length of the rod, and hence that the nth buckling force is

$$P_n=\frac{n^2\pi^2}{L^2}\left(\frac{a}{b}\right)^2 EI_0$$

Solution

a) $x^{-2}\times \quad x^4y''+\mu^2y=0$

$$x^2y''+\mu^2x^{-2}y=0$$

$$A=0,\quad B=0,\quad C=\mu^2,\quad p=-2$$

$$\alpha=\frac{1}{2},\quad \beta=-1,\quad k=\mu,\quad \upsilon=\frac{1}{2} \qquad\qquad \alpha=\frac{1-A}{2},\ \beta=\frac{p}{2},\ k=\frac{2\sqrt{C}}{p},\ \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}$$

$$y(x)=x^{1/2}\left[c_1J_{1/2}\left(\mu x^{-1}\right)+c_2J_{-1/2}\left(\mu x^{-1}\right)\right] \qquad y(x)=x^{\alpha}\left(c_1J_{1/2}\left(kx^{\beta}\right)+c_2J_{-1/2}\left(kx^{\beta}\right)\right)$$

$$=\sqrt{x}\left(c_1\sqrt{\frac{2x}{\pi\mu}}\cos\left(\frac{\mu}{x}\right)+c_2\sqrt{\frac{2x}{\pi\mu}}\sin\left(\frac{\mu}{x}\right)\right)$$

$$=x^{\alpha}\left(c_1\left(\frac{2}{\pi kx^{\beta}}\right)^{1/2}\sin\left(kx^{\beta}\right)+c_2\left(\frac{2}{\pi kx^{\beta}}\right)^{1/2}\cos\left(kx^{\beta}\right)\right)$$

$$=x\left(c_1\sqrt{\frac{2}{\pi\mu}}\cos\left(\frac{\mu}{x}\right)+c_2\sqrt{\frac{2}{\pi\mu}}\sin\left(\frac{\mu}{x}\right)\right) \qquad A=c_1\sqrt{\frac{2}{\pi\mu}},\quad B=c_2\sqrt{\frac{2}{\pi\mu}}$$

$$=x\left(A\cos\left(\frac{\mu}{x}\right)+B\sin\left(\frac{\mu}{x}\right)\right)$$

b) ***Given***: $\mu_n=n\pi\frac{ab}{L};\quad y(a)=y(b)=0,\quad L=b-a$

$$\begin{cases} y(a)=a\left(A\cos\left(\frac{\mu}{a}\right)+B\sin\left(\frac{\mu}{a}\right)\right)=0\\ y(b)=b\left(A\cos\left(\frac{\mu}{b}\right)+B\sin\left(\frac{\mu}{b}\right)\right)=0\end{cases}$$

$$\begin{cases} A\cos\left(\frac{\mu}{a}\right)+B\sin\left(\frac{\mu}{a}\right)=0\\ A\cos\left(\frac{\mu}{b}\right)+B\sin\left(\frac{\mu}{b}\right)=0\end{cases} \qquad (a,\ b\neq 0)$$

$$\Delta=\begin{vmatrix}\cos\frac{\mu}{a} & \sin\frac{\mu}{a}\\ \cos\frac{\mu}{b} & \sin\frac{\mu}{b}\end{vmatrix}$$

$$=\cos\frac{\mu}{a}\sin\frac{\mu}{b}-\sin\frac{\mu}{a}\cos\frac{\mu}{b}$$

$$=\sin\left(\frac{\mu}{b}-\frac{\mu}{a}\right)$$

$$=\sin\left(\frac{b-a}{ab}\mu\right)$$

$$=\sin\left(\frac{L}{ab}\mu\right)$$

$$\lambda=\mu^2=\frac{Pb^4}{EI_0}$$

$$P=\frac{EI_0}{b^4}\mu^2$$

$$=\frac{EI_0}{b^4}\left(n\pi\frac{ab}{L}\right)^2$$

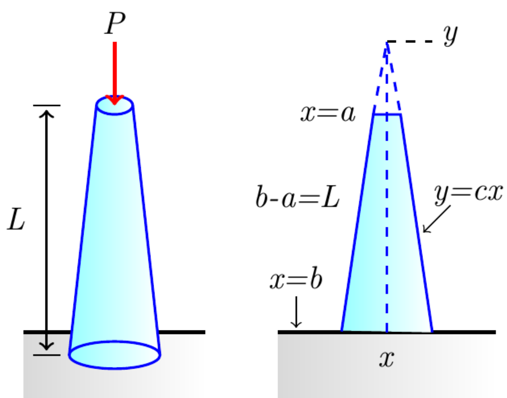
$$\boxed{=\frac{n^2\pi^2}{L^2}\left(EI_0\right)\left(\frac{a}{b}\right)^2}$$

Exercise

When a constant vertical compressive force or load *P* was applied to a thin column of uniform cross section, the deflection $y(x)$ was a solution of the boundary-value problem

$$EI\frac{d^2y}{dy^2}+Py=0\ ; \quad y(0)=0, \quad y(L)=0$$

The assumption here is that the column is hinged at both ends. The column will buckle or deflect only when the compression force is a critical load P_n



a) Let assume that the column is of length *L*, is hinged at both ends, has circular cross sections, and is tapered. If the column, a truncated cone, has a linear taper $y=cx$ in cross section, the moment of inertia of a cross section with respect to an axis perpendicular to the $xy-plane$ is $I=\frac{1}{4}\pi r^4$, where $r=y$ and $y=cx$. Hence we can write $I(x)=I_0\left(x\ b\right)^4$, where $I_0=I(b)=\frac{1}{4}\pi(cb)^4$. Substituting $I(x)$ into the differential equation, we see that the deflection in this case is determine from the *BVP*?

$$x^4\frac{d^2y}{dx^2}+\lambda y=0\ ; \quad y(a)=0, \quad y(b)=0$$

Where $\lambda=Pb^4EI_0$

Find the critical loads P_n for the tapered column. Use an appropriate identity to express the buckling modes $y_n(x)$ as a single function.

b) Plot the graph of the first buckling mode $y_1(x)$ corresponding to the Euler load P_1 when $b=11$ and $a=1$

Solution

c) $x^{-2}\times \quad x^4y''+\lambda y=0$

$$x^2y''+\lambda x^{-2}y=0$$

$A=0, \quad B=0, \quad C=\lambda, \quad p=-2$

$$\alpha=\frac{1}{2}, \quad \beta=-1, \quad k=\sqrt{\lambda}, \quad \upsilon=\frac{1}{2} \qquad\qquad \alpha=\frac{1-A}{2},\ \beta=\frac{p}{2},\ k=\frac{2\sqrt{C}}{p},\ \upsilon=\frac{\sqrt{(1-A)^2-4B}}{p}$$

$$y(x)=x^{1/2}\left[c_1J_{1/2}\left(\sqrt{\lambda}x^{-1}\right)+c_2J_{-1/2}\left(\sqrt{\lambda}x^{-1}\right)\right] \qquad y(x)=x^{\alpha}\left(c_1J_{1/2}\left(kx^{\beta}\right)+c_2J_{-1/2}\left(kx^{\beta}\right)\right)$$

$$=\sqrt{x}\left(c_1\sqrt{\frac{2x}{\pi\sqrt{\lambda}}}\cos\left(\frac{\sqrt{\lambda}}{x}\right)+c_2\sqrt{\frac{2x}{\pi\sqrt{\lambda}}}\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right)$$

$$=x^{\alpha}\left(c_1\sqrt{\frac{2}{\pi kx^{\beta}}}\sin\left(kx^{\beta}\right)+c_2\sqrt{\frac{2}{\pi kx^{\beta}}}\cos\left(kx^{\beta}\right)\right)$$

$$=x\left(c_1\sqrt{\frac{2x}{\pi\sqrt{\lambda}}}\cos\left(\frac{\sqrt{\lambda}}{x}\right)+c_2\sqrt{\frac{2x}{\pi\sqrt{\lambda}}}\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right) \qquad A=c_1\sqrt{\frac{2}{\pi\sqrt{\lambda}}},\quad B=c_2\sqrt{\frac{2}{\pi\sqrt{\lambda}}}$$

$$\boxed{=x\left(A\cos\left(\frac{\sqrt{\lambda}}{x}\right)+B\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right)}$$

Given: $\quad \lambda=Pb^4EI_0\ ; \quad y(a)=y(b)=0, \quad L=b-a$

$$\begin{cases} y(a)=a\left(A\cos\left(\frac{\sqrt{\lambda}}{a}\right)+B\sin\left(\frac{\sqrt{\lambda}}{a}\right)\right)=0 \\ y(b)=b\left(A\cos\left(\frac{\sqrt{\lambda}}{b}\right)+B\sin\left(\frac{\sqrt{\lambda}}{b}\right)\right)=0 \end{cases}$$

$$\begin{cases} A\cos\left(\frac{\sqrt{\lambda}}{a}\right)+B\sin\left(\frac{\sqrt{\lambda}}{a}\right)=0 \\ A\cos\left(\frac{\sqrt{\lambda}}{b}\right)+B\sin\left(\frac{\sqrt{\lambda}}{b}\right)=0 \end{cases} \qquad (a,\ b\neq 0)$$

$$\Delta=\begin{vmatrix} \cos\frac{\sqrt{\lambda}}{a} & \sin\frac{\sqrt{\lambda}}{a} \\ \cos\frac{\sqrt{\lambda}}{b} & \sin\frac{\sqrt{\lambda}}{b} \end{vmatrix}$$

$$=\cos\frac{\sqrt{\lambda}}{a}\sin\frac{\sqrt{\lambda}}{b}-\sin\frac{\sqrt{\lambda}}{a}\cos\frac{\sqrt{\lambda}}{b}$$

$$=\sin\left(\frac{\sqrt{\lambda}}{b}-\frac{\sqrt{\lambda}}{a}\right)$$

$$= \sin\left(\frac{b-a}{ab}\sqrt{\lambda}\right)$$

$$= \sin\left(\frac{L}{ab}\sqrt{\lambda}\right) = 0$$

$$\frac{L}{ab}\sqrt{\lambda} = n\pi \quad \rightarrow \quad \sqrt{\lambda} = \frac{n\pi ab}{L} \qquad (n \in \mathbb{N})$$

$$\lambda = \frac{n^2\pi^2a^2b^2}{L^2} = Pb^4 EI_0$$

$$\boxed{P_n = \frac{n^2\pi^2}{L^2}\left(EI_0\right)\left(\frac{a}{b}\right)^2}$$

If we let $B = -A\dfrac{\sin\frac{\sqrt{\lambda}}{a}}{\cos\frac{\sqrt{\lambda}}{a}}$

$$y(x) = x\left(A\cos\left(\frac{\sqrt{\lambda}}{x}\right) + B\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right)$$

$$= x\left(A\cos\left(\frac{\sqrt{\lambda}}{x}\right) - A\frac{\sin\frac{\sqrt{\lambda}}{a}}{\cos\frac{\sqrt{\lambda}}{a}}\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right)$$

$$= \frac{A}{\cos\frac{\sqrt{\lambda}}{a}}x\left(\cos\frac{\sqrt{\lambda}}{a}\cos\left(\frac{\sqrt{\lambda}}{x}\right) - \sin\frac{\sqrt{\lambda}}{a}\sin\left(\frac{\sqrt{\lambda}}{x}\right)\right) \qquad \frac{A}{\cos\frac{\sqrt{\lambda}}{a}} = C$$

$$= Cx\sin\left(\frac{\sqrt{\lambda}}{x} - \frac{\sqrt{\lambda}}{a}\right)$$

$$\boxed{= Cx\sin\sqrt{\lambda}\left(\frac{1}{x} - \frac{1}{a}\right)}$$

$$y_n(x) = Cx\sin\sqrt{\lambda}\left(\frac{1}{x} - \frac{1}{a}\right) \qquad \left(\sqrt{\lambda} = \frac{n\pi ab}{L}\right)$$

$$= Cx\sin\frac{n\pi ab}{L}\left(\frac{1}{x} - \frac{1}{a}\right)$$

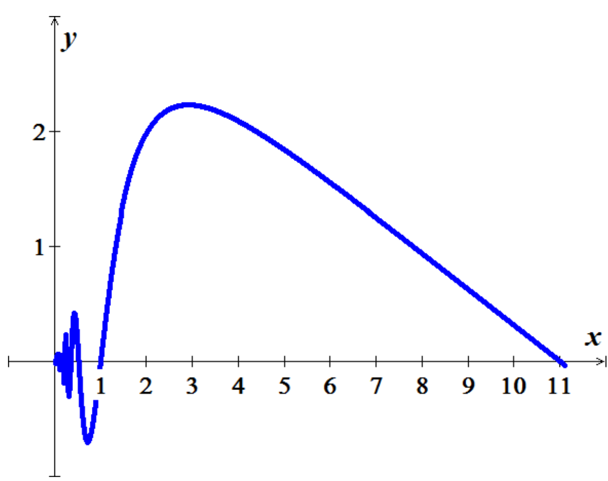
$$= Cx\sin\frac{n\pi b}{L}\left(\frac{a}{x} - 1\right)$$

$$\boxed{= C_1 x\sin\frac{n\pi b}{L}\left(1 - \frac{a}{x}\right)}$$

d) ***Given***: $n = 1, \quad a = 1, \quad b = 11$

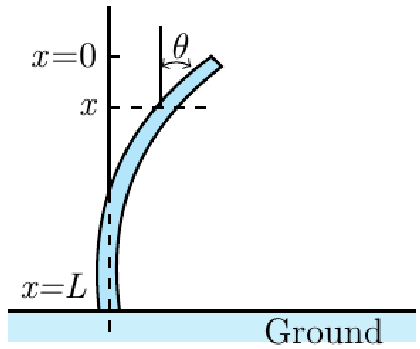
Let $C_1 = 1$

$$y_1(x) = x\sin\frac{11\pi}{10}\left(1 - \frac{1}{x}\right)$$



Exercise

For a practical application, we now consider the problem of determining when a uniform vertical column will buckle under its own weight (after, perhaps, being nudged laterally just a bit by a passing breeze). We take $x = 0$ at the free top end of the column and $x = L > 0$ at its bottom; we assume that the bottom is rigidly imbedded in the ground, perhaps in concrete.



Denote the angular deflection of the column at the point *x* by $\theta(x)$. From the theory of elasticity it follows that

$$EI\frac{d^2\theta}{dx^2} + g\rho x\theta = 0$$

Where *E* is the Young's modulus of the material of the column,

I is its cross-sectional moment of inertia

ρ is the linear density of the column

g is gravitational acceleration.

For physical reasons – no bending at the free top of the column and no deflection at its imbedded bottom – the boundary conditions are $\theta'(0) = 0, \quad \theta(L) = 0$

Determine the general equation of the length *L*.

Solution

$EI\theta'' + g\rho x\theta = 0$

$\theta'' + \frac{g\rho}{EI}x\theta = 0$

Let $\lambda = \frac{g\rho}{EI} = \gamma^2$

$x^2 \times \quad \theta'' + \gamma^2 x\theta = 0$

$x^2\theta'' + \gamma^2 x^3\theta = 0$; $\theta'(0) = 0, \quad \theta(L) = 0$

$A = 0, \quad B = 0, \quad C = \gamma^2, \quad p = 3$

$\alpha = \frac{1}{2}, \quad \beta = \frac{3}{2}, \quad k = \frac{2\gamma}{3}, \quad \upsilon = \frac{1}{3}$ $\qquad \alpha = \frac{1-A}{2},\ \beta = \frac{p}{2},\ k = \frac{2\sqrt{C}}{p},\ \upsilon = \frac{\sqrt{(1-A)^2-4B}}{p}$

$\theta(x) = x^{1/2}\left[c_1 J_{1/3}\left(\frac{2}{3}\gamma x^{3/2}\right) + c_2 J_{-1/3}\left(\frac{2}{3}\gamma x^{3/2}\right)\right]$ $\qquad y(x) = x^{\alpha}\left(c_1 J_{\nu}\left(kx^{\beta}\right) + c_2 J_{-\nu}\left(kx^{\beta}\right)\right)$

$$J_{1/3}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma\left(1+\frac{1}{3}+n\right)}\left(\frac{x}{2}\right)^{2n+\frac{1}{3}}$$

$$J_{\upsilon}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\upsilon+n)}\left(\frac{x}{2}\right)^{2n+\upsilon}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma\left(n+\frac{4}{3}\right)}\left(\frac{x}{2}\right)^{2n+\frac{1}{3}}$$

$$J_{\upsilon}(x)=\frac{x^{\upsilon}}{2^{\upsilon}\Gamma(\upsilon+1)}\left\{1-\frac{x^2}{2(2\upsilon+2)}+\frac{x^4}{2\cdot 4\cdot(2\upsilon+2)(2\upsilon+4)}-\cdots\right\}$$

$$=\frac{x^{1/3}}{2^{1/3}\Gamma\left(\frac{4}{3}\right)}\left\{1-\frac{x^2}{2\left(\frac{2}{3}+2\right)}+\frac{x^4}{2\cdot 4\cdot\left(\frac{2}{3}+2\right)\left(\frac{2}{3}+4\right)}-\cdots\right\}$$

$$=\frac{x^{1/3}}{2^{1/3}\Gamma\left(\frac{4}{3}\right)}\left\{1-\frac{3x^2}{2^3}+\frac{3^2x^4}{112\times 2^3}-\cdots\right\}$$

$$J_{1/3}\left(\frac{2}{3}\gamma x^{3/2}\right)=\frac{1}{2^{1/3}\Gamma\left(\frac{4}{3}\right)}\left(\frac{2}{3}\gamma x^{3/2}\right)^{1/3}\left\{1-\frac{3}{2^3}\left(\frac{2}{3}\gamma x^{3/2}\right)^2+\frac{3^2}{896}\left(\frac{2}{3}\gamma x^{3/2}\right)^4-\cdots\right\}$$

$$=\frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}x^{1/2}\left\{1-\frac{1}{12}\gamma^2x^3+\frac{1}{504}\gamma^4x^6-\cdots\right\}$$

$$J_{-1/3}\left(\frac{2}{3}\gamma x^{3/2}\right)=\frac{1}{2^{-1/3}\Gamma\left(\frac{4}{3}\right)}\left(\frac{2}{3}\gamma x^{3/2}\right)^{-1/3}\left\{1-\frac{1}{2\left(2-\frac{1}{3}\right)}\left(\frac{2}{3}\gamma x^{3/2}\right)^2+\frac{1}{8\left(2-\frac{1}{3}\right)\left(4-\frac{1}{3}\right)}\left(\frac{2}{3}\gamma x^{3/2}\right)^4-\cdots\right\}$$

$$=\frac{3^{1/3}}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)}x^{-1/2}\left\{1-\frac{1}{6}\gamma^2x^3+\frac{1}{180}\gamma^4x^6-\cdots\right\}$$

$$\theta(x)=x^{1/2}\left[c_1J_{1/3}\left(\frac{2}{3}\gamma x^{3/2}\right)+c_2J_{-1/3}\left(\frac{2}{3}\gamma x^{3/2}\right)\right]$$

$$=x^{1/2}\left[c_1\frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}x^{1/2}\left\{1-\frac{1}{12}\gamma^2x^3+\frac{1}{504}\gamma^4x^6-\cdots\right\}+c_2\frac{3^{1/3}}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)}x^{-1/2}\left\{1-\frac{1}{6}\gamma^2x^3+\frac{1}{180}\gamma^4x^6-\cdots\right\}\right]$$

$$\boxed{=c_1\frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}\left\{x-\frac{1}{12}\gamma^2x^4+\frac{1}{504}\gamma^4x^7-\cdots\right\}+c_2\frac{3^{1/3}}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)}\left\{1-\frac{1}{6}\gamma^2x^3+\frac{1}{180}\gamma^4x^6-\cdots\right\}}$$

Given: $\theta(L)=0, \quad \theta'(0)=0$

$$\theta'(x)=c_1\frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}\left\{1-\frac{1}{3}\gamma^2x^3+\frac{1}{72}\gamma^4x^6-\cdots\right\}+\frac{3^{1/3}}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)}\left\{\frac{1}{2}\gamma^2x^2+\frac{1}{30}\gamma^4x^5-\cdots\right\}$$

$$\theta'(0)=c_1\frac{\gamma^{1/3}}{3^{1/3}\Gamma\left(\frac{4}{3}\right)}=0\quad\to\quad\boxed{c_1=0}$$

$$\frac{3^{1/3}c_2}{\gamma^{1/3}\Gamma\left(\frac{2}{3}\right)}\left\{1-\frac{1}{6}\gamma^2L^3+\frac{1}{180}\gamma^4L^6-\cdots\right\}=0$$

$$c_2J_{-1/3}\left(\frac{2}{3}\gamma L^{3/2}\right)=0 \quad\rightarrow\quad J_{-1/3}\left(\frac{2}{3}\gamma L^{3/2}\right)=0$$

$$J_{-1/3}\left(z=\frac{2}{3}\gamma L^{3/2}\right)=0$$

Using MatLab:

```
fzero('besselj(-1/3,z)',2)
syms z y
fplot(besselj(-1/3, z))
axis([0 10 -0.5 1.1])
grid on
ylabel('J_-1/3(z)')
legend('J_-1/3')
```

$$\boxed{z=1.8664}$$

$$z=\frac{2}{3}\gamma L^{3/2} \quad\rightarrow\quad L=\left(\frac{3z}{2\gamma}\right)^{2/3}$$

$$L=\left(\frac{3(1.86635)}{2\sqrt{\frac{g\rho}{EI}}}\right)^{2/3}$$

$$\boxed{\approx 1.986352\left(\frac{EI}{g\rho}\right)^{1/3}}$$