Lecture Four – Integration

Section 4.1 – Antiderivatives, Substitution and General Power Rule

Antiderivatives

$$f(x) = x^3 \qquad \Rightarrow \qquad f'(x) = 3x^2$$

Definition of Antiderivative

A Function F is an Antiderivative of a function f if for every x in the domain of f, it follows that

$$F'(x) = f(x)$$

Notation for Antiderivatives and indefinite integrals

The notation

$$\int f(x)dx = F(x) + C$$

where C is an arbitrary constant, means that F is an Antiderivative of f.

That is $F'(x) = f(x)$ for all x in the domain of f.

$\int f(x)dx$ Indefinite integral

Integrand

Integral sign

$$\int f(x)dx = \overbrace{F(x)+C}^{\text{Antiderivative}}$$

Differential

Basic Integration Rules

$$\int k\,dx = kx + C$$

$$\int kf(x)\,dx = k\int f(x)\,dx$$

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int [f(x)-g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

Example

Find each indefinite integral.

a) $\int 5\,dx = 5x + C$

b) $\int -1\,dr = -r + C$

c) $\int 2\,dt = 2t + C$

Example

Find indefinite integral. $\int 5x\,dx$

<u>*Solution*</u>

$$\int 5x\,dx = \int 5x^{1}\,dx$$

$$= 5\frac{x^{1+1}}{1+1} + C$$

$$= \frac{5}{2}x^2 + C$$

Example

Find each indefinite integral.

a) $\int \frac{1}{x^2} dx = \int x^{-2} dx$

$= \frac{x^{-2+1}}{-2+1} + C$

$= \frac{x^{-1}}{-1} + C$

$= -\frac{1}{x} + C$

b) $\int \sqrt[3]{x} dx = \int x^{1/3} dx$

$= \frac{x^{1/3+1}}{1/3+1} + C$

$= \frac{x^{4/3}}{4/3} + C$

$= \frac{3}{4} x^{4/3} + C \qquad or \qquad = \frac{3}{4} x \sqrt[3]{x} + C$

Example

Find each indefinite integral.

a) $\int (x+4) dx = \int x dx + \int 4 dx$

$= \frac{1}{2} x^2 + 4x + C$

b) $\int (4x^3 - 5x + 2) dx = \int 4x^3 dx - \int 5x dx + \int 2 dx$

$= 4\frac{x^4}{4} - 5\frac{x^2}{2} + 2x + C$

$= x^4 - \frac{5}{2} x^2 + 2x + C$

Example

Find the integral $\int \frac{x^2+1}{\sqrt{x}}dx$

Solution

$$\int \frac{x^2+1}{\sqrt{x}}dx = \int \left(\frac{x^2}{x^{1/2}} + \frac{1}{x^{1/2}}\right)dx$$

$$= \int \left(x^{3/2} + x^{-1/2}\right)dx$$

$$= \frac{x^{5/2}}{5/2} + \frac{x^{1/2}}{1/2} + C$$

$$\boxed{= \frac{2}{5}x^{5/2} + 2x^{1/2} + C}$$

Particular Solutions

In many applications of integrations, we are given enough information to determine a particular solution. To do this, we need to know the value of *F(x)* for one value of *x*. This information is called an initial condition.

Example

Find the general solution of $F'(x) = 4x + 2$, and find the particular solution that satisfies the initial condition $F(1) = 8$.

Solution

$$F(x) = \int (4x+2)dx$$

$$= 4\frac{x^2}{2} + 2x + C$$

$$= 2x^2 + 2x + C$$

$$F(x) = 2(1)^2 + 2(1) + C = 8$$

$$2 + 2 + C = 8$$

$$4 + C = 8$$

$$C = 4$$

$$\boxed{F(x) = 2x^2 + 2x + 4}$$

Example

The marginal cost function for producing x units of a product is modeled by

$$\frac{dC}{dx} = 28 - 0.02x$$

It costs \$40 to produce one unit. Find the cost of producing 200 units.

<u>Solution</u>

$$C = \int (28 - 0.02x)dx$$

$$= 28x - 0.02\frac{x^2}{2} + K$$

Cost \$40 for one unit $\Rightarrow$ C(x = 1) = 40

$$C(x = 1) = 28(1) - 0.01(1)^2 + K = 40$$

K = 12.01

$$C(x) = -0.01x^2 + 28x + 12.01$$

$$C(200) = -0.01(200)^2 + 28(200) + 12.01 \underline{= \$5212.01}$$

General Power Rule

The Simple Power Rule is given by:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

$$\int \overbrace{\left(x^2+1\right)^3}^{u^3} \underbrace{2xdx}_{du} = \int u^3 du$$

$$= \frac{u^4}{4} + C$$

General Power Rule for Integration

If u is a differentiable function of x, then

$$\int u^n \frac{du}{dx} dx = \int u^n du = \frac{u^{n+1}}{n+1} + C \qquad n \neq -1$$

Example

Find the indefinite integral. $\int (3x^2+6)(x^3+6x)^2\,dx$

Solution

Let $u=x^3+6x \Rightarrow du=(3x^2+6)dx$

$$\int u^2 du = \frac{u^3}{3}+C$$

$$\boxed{=\frac{(x^3+6x)^3}{3}+C}$$

Example

Find the indefinite integral. $\int 2x\sqrt{x^2-2}\,dx$

Solution

$$u=x^2-2 \Rightarrow du=2xdx$$

$$\int 2x\sqrt{x^2-2}\,dx=\int\sqrt{u}\,du$$

$$=\int u^{1/2}\,du$$

$$=\frac{u^{3/2}}{3/2}+C$$

$$=\frac{2}{3}u^{3/2}+C$$

$$\boxed{=\frac{2}{3}(x^2-2)^{3/2}+C}$$

Example

Evaluate $\int x^3(3x^4+1)^2 dx$

Solution

$$u=3x^4+1 \Rightarrow du=12x^3dx \quad \Rightarrow \frac{1}{12}du=x^3dx$$

$$\int x^3(3x^4+1)^2 dx=\int \frac{1}{12}u^2du$$

$$=\frac{1}{12}\frac{(3x^4+1)^3}{3}+C$$

$$\boxed{=\frac{1}{36}(3x^4+1)^3+C}$$

Failure of the General Power Rule

Example

Find $\int 2(3x^4+1)^2\,dx$

Solution

$$\int 2(3x^4+1)^2\,dx = \int 2\left((3x^4)^2 + 2(3x^4)(1) + 1^2\right)dx \qquad (a+b)^2 = a^2+2ab+b^2$$

$$= \int 2\left(9x^8 + 6x^4 + 1\right)dx$$

$$= \int \left(18x^8 + 12x^4 + 2\right)dx$$

$$= 18\frac{x^9}{9} + 12\frac{x^5}{5} + 2x + C$$

$$= 2x^9 + \frac{12}{5}x^5 + 2x + C$$

Example

Find $\int 5x\sqrt{x^2-1}\,dx$

Solution

$$u = x^2 - 1 \quad \Rightarrow du = 2xdx$$

$$\Rightarrow \frac{1}{2}du = xdx$$

$$\int 5x\left(x^2-1\right)^{1/2}\,dx = 5\int u^{1/2}\ \frac{1}{2}du \qquad \textit{Substitute for x and dx}$$

$$= 5\int u^{1/2}\ \frac{1}{2}du$$

$$= \frac{5}{2}\int u^{1/2}du$$

$$= \frac{5}{2}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{5}{3}u^{3/2} + C$$

$$= \frac{5}{3}(x^2-1)^{3/2} + C$$

Example

The marginal propensity to consume income x can be modeled by $\dfrac{dQ}{dx}=\dfrac{0.98}{(x-19,999)^{0.02}};\ \ x\geq 20,000$

Where Q represents the income consumed. Estimate the amount by a family of four whose income was $30,000.00, with initial condition of 19,999.

<u>*Solution*</u>

$$Q=\int\frac{0.98}{(x-19,999)^{0.02}}dx$$

$$=\int 0.98(x-19,999)^{-0.02}\,dx$$

$$Q=(x-19,999)^{0.98}+19,999$$

$$30,000=(x-19,999)^{0.98}+19,999$$

$$30,000-19,999=(x-19,999)^{0.98}$$

$$10,001=(x-19,999)^{0.98}$$

$$x-19,999=10,001^{1/0.98}$$

$$x=10,001^{1/0.98}+19,999$$

$$=\$32,068.16$$

Exercises *Section* 4.1 – Antiderivatives, Substitution and General Power Rule

Find each indefinite integral.

1. $\int v^2 dv$

2. $\int x^{1/2} dx$

3. $\int e^{3t} dt$

4. $\int (6x^2 - 2e^x) dx$

5. $\int 4y^{-3} dy$

6. $\int (x^3 - 4x + 2) dx$

7. $\int \left(3z^2 - 4z + 5\right) dz$

8. $\int \left(x^2 - 1\right)^2 dx$

9. $\int \left(\sqrt[4]{x^3} + 1\right) dx$

10. $\int \sqrt{x}(x+1) dx$

11. $\int (1+3t)t^2 dt$

12. $\int \frac{x^2 - 5}{x^2} dx$

13. $\int (-40x + 250) dx$

14. $\int \frac{x+2}{\sqrt{x}} dx$

15. $\int \left(\frac{2}{\sqrt[3]{x}} - 6\sqrt{x} \right) dx$

16. $\int \left(x^2 - 1 \right)^3 (2x) dx$

17. $\int \frac{6x}{\left(1+x^2\right)^3} dx$

18. $\int u^3 \sqrt{u^4 + 2}\ du$

19. $\int \frac{t + 2t^2}{\sqrt{t}}\ dt$

20. $\int \left(1 + \frac{1}{t}\right)^3 \frac{1}{t^2}\ dt$

21. $\int \left(7 - 3x - 3x^2\right)(2x+1)\ dx$

22. $\int \sqrt{x} \left(4 - x^{3/2}\right)^2\ dx$

23. $\int \frac{1}{\sqrt{x} + \sqrt{x+1}} dx$

24. $\int \sqrt{1-x}\ dx$

25. $\int x\sqrt{x^2 + 4}\ dx$

26. Derive the position function if a ball is thrown upward with initial velocity of 32 ft per second from an initial height of 48 ft. When does the ball hit the ground? With what velocity does the ball hit the ground?

27. Suppose a publishing company has found that the marginal cost at a level of production of x thousand books is given by

$$\frac{dC}{dx} = \frac{50}{\sqrt{x}}$$

And that the fixed cost (the cost before the first book can be produced) is a \$25,000. Find the cost function $C(x)$.

28. If the marginal cost of producing x units of a commodity is given by

$$C'(x) = 0.3x^2 + 2x$$

And the fixed cost is \$2,000, find the cost function $C(x)$ and the cost of producing 20 units.

29. A satellite radio station is launching an aggressive advertising campaign in order to increase the number of daily listeners. The station currently has 27,000 daily listeners, and management expects the number of daily listeners, $S(t)$, to grow at the rate of

$$S'(t) = 60t^{1/2}$$

Listeners per day, where *t* is the number of days since the campaign began. How long should the campaign last if the station wants the number of daily listeners to grow to 41,000?

30. In 2007, U.S. consumption of renewable energy was 6.8 quadrillion Btu (or 6.8×10^{15} Btu). Since the 1960s, consumption has been growing at a rate (in quadrillion Btu's per year) given by

$$f'(t) = 0.004t + 0.062$$

Where *t* is years after 1960. Find $f(t)$ and estimate U.S. consumption of renewable energy in 2020.

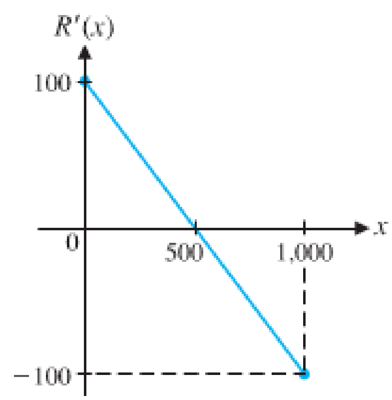
31. The graph of the marginal revenue function from the sale of *x* sports watches is given in the figure.

a) Using the graph shown, describe the shape of the graph of the revenue function $R(x)$ as *x* increases from 0 to 1.000.

b) Find the equation of the marginal revenue function. (linear function)

c) Find the equation of the revenue function that satisfies $R(0) = 0$. Graph the revenue function over the interval [0, 1,000]. Check the shape of the graph relative to the analysis in part (*a*).

d) Find the price-demand equation and determine the price when the demand is 700 units.



32. The rate of change of the monthly sales of a newly released football game is given by

$$S'(t) = 500t^{1/4} \qquad S(0) = 0$$

Where *t* is the number of months since the game was released and $S(t)$ is the number of games sold each month. Find $S(t)$. When will monthly sales reach 20,000 games?

33. If the rate of labor is given by: $g(x) = 2{,}000x^{-1/3}$

And if the first 8 control units require 12,000 labor-hours, how many labor-hours, $L(x)$, will be required for the first *x* control units? The first 27 control units?

34. The area A of a healing wound changes at a rate given approximately by

$$\frac{dA}{dt} = -4t^{-3} \qquad 1 \le t \le 10$$

Where t is time in days and $A(1) = 2\ cm^2$. What will the area of the wound be in 10 days?

35. The marginal revenue (in thousands of dollars) from the sale of x gadgets is given by the following function $R'(x) = 4x\left(x^2 + 27,000\right)^{-2/3}$

a) Find the total revenue function if the revenue from 115 gadgets is $55,581.

b) How many gadgets must be sold for a revenue of at least $50,000.