Solution Section 1.3 – Linear Differential Equations

Exercise

Find the general solution of $y' - y = 3e^t$

Solution

$$y_h = e^{\int -dt} = e^{-t}$$

$$\int 3e^t e^{-t} dt = \int 3dt = 3t$$

$$y(t) = \frac{1}{e^{-t}}(3t + C)$$

$$\boxed{y(t) = 3te^t + Ce^t}$$

Exercise

Find the general solution of $y' + y = \sin t$

Solution

$$e^{\int dt} = e^t$$

$$\int e^t \sin t\ dt = e^t \sin t - \int e^t \cos t\ dt$$

$$= e^t \sin t - e^t \cos t - \int e^t \sin t\ dt$$

$u = \sin t \qquad dv = e^t dt$

$du = \cos t\ dt \qquad v = e^t$

$u = \cos t \qquad dv = e^t dt$

$du = -\sin t\ dt \qquad v = e^t$

$$2\int e^t \sin t\ dt = e^t \sin t - e^t \cos t$$

$$\int e^t \sin t\ dt = \frac{1}{2}e^t(\sin t - \cos t)$$

$$y(t) = \frac{1}{e^t}\left(\frac{1}{2}e^t(\sin t - \cos t) + C\right)$$

$$\boxed{= \frac{1}{2}\sin t - \frac{1}{2}\cos t + Ce^{-t}}$$

Exercise

Find the general solution of $y' + y = \frac{1}{1+e^t}$

Solution

$$e^{\int dt} = e^t$$

$$\int \frac{e^t}{1+e^t}\, dt = \int \frac{1}{1+e^t} d\left(1+e^t\right) = \ln\left(1+e^t\right)$$

$$y(t) = \frac{1}{e^t}\left(\ln\left(1+e^t\right) + C\right)$$

$$= e^{-t}\ln\left(1+e^t\right) + Ce^{-t}$$

Exercise

Find the general solution of $y' - y = e^{2t} - 1$

Solution

$$e^{-\int dt} = e^{-t}$$

$$\int \left(e^{2t} - 1\right)e^{-t}\, dt = \int \left(e^t - e^{-t}\right)dt = e^t + e^{-t}$$

$$y(t) = \frac{1}{e^{-t}}\left(e^t + e^{-t} + C\right)$$

$$= e^{2t} + 1 + Ce^t$$

Exercise

Find the general solution of $y' + y = te^{-t} + 1$

Solution

$$e^{\int dt} = e^t$$

$$\int \left(te^{-t} + 1\right)e^t\, dt = \int \left(t + e^t\right)dt = t + e^t$$

$$y(t) = \frac{1}{e^t}\left(t + e^t + C\right)$$

$$= te^{-t} + 1 + Ce^{-t}$$

Exercise

Find the general solution of $y' + y = 1 + e^{-x}\cos 2x$

Solution

$$e^{\int dx} = e^x$$

$$\int \left(1+e^{-x}\cos 2x\right)e^x dx = \int \left(e^x + \cos 2x\right)dx = e^x + \frac{1}{2}\sin 2x$$

$$y(x) = e^{-x}\left(e^x + \frac{1}{2}\sin 2x + C\right)$$

$$= e^x + \frac{1}{2}e^{-x}\sin 2x + Ce^{-x}$$

Exercise

Solve the differential equation: $y' + y\cot x = \cos x$

Solution

$$e^{\int \cot x dx} = e^{\ln|\sin x|} = \sin x$$

$$\int \cos x \sin x \, dx = \int \sin x \; d(\sin x) = \frac{1}{2}\sin^2 x$$

$$y(x) = \frac{1}{\sin x}\left(\frac{1}{2}\sin^2 x + C\right)$$

$$= \frac{1}{2}\sin x + \frac{C}{\sin x}$$

Exercise

Solve the differential equation: $y' + y\sin t = \sin t$

Solution

$$e^{\int \sin t dt} = e^{-\cos t}$$

$$\int (\sin t)e^{-\cos t} dt = \int e^{-\cos t} d(-\cos t) = e^{-\cos t}$$

$$y(x) = \frac{1}{e^{-\cos t}}\left(e^{-\cos t} + C\right)$$

$$= 1 + Ce^{\cos t}$$

Exercise

Find the general solution of $y' = \cos x - y\sec x$

Solution

$$y' + (\sec x)\, y = \cos x$$

$$e^{\int \sec x dx} = e^{\ln|\sec x + \tan x|} = \sec x + \tan x$$

$$\int \cos x(\sec x + \tan x)\, dx = \int \cos x\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right) dx$$

$$= \int (1 + \sin x)\, dx$$

$$= x - \cos x$$

$$\boxed{y(x) = \frac{1}{\sec x + \tan x}(x - \cos x + C)}$$

Exercise

Solve the differential equation: $y' + (\tan x)\, y = \cos^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

Solution

$$y' + (\tan x)\, y = \cos^2 x, \quad P(x) = \tan x, \quad Q(x) = \cos^2 x$$

$$y_h = e^{\int \tan x dx} = e^{\ln(\cos x)^{-1}} = (\cos x)^{-1} \qquad \int \tan x dx = -\ln|\cos x| = \ln(\cos x)^{-1}$$

$$\int \cos^2 x (\cos x)^{-1} dx = \int \cos x dx = \sin x$$

$$y(x) = \frac{1}{(\cos x)^{-1}}(\sin x + C)$$

$$y(x) = \cos x(\sin x + C)$$

$$\boxed{= \cos x \sin x + C\cos x}$$

Exercise

Solve the differential equation: $y' + (\cot t)\, y = 2t\csc t$

Solution

$$e^{\int \cot t\ dt} = e^{\ln|\sin t|} = \sin t$$

$$\int 2t\csc t \sin t dt = \int 2t dt = t^2$$

$$y(t) = \frac{1}{\sin t}\left(t^2 + C\right)$$
$$\boxed{= \left(t^2 + C\right)\csc t}$$

Exercise

Solve the differential equation: $y' + (1 + \sin t)\, y = 0$

Solution

$$e^{\int (1+\sin t)dt} = e^{t-\cos t}$$
$$y(x) = \frac{C}{e^{t-\cos t}}$$
$$\boxed{= C\, e^{\cos t - t}}$$

Exercise

Find the general solution of $y' + \left(\frac{1}{2}\cos x\right) y = -\frac{3}{2}\cos x$

Solution

$$e^{\int \frac{1}{2}\cos x dx} = e^{\frac{1}{2}\sin x}$$
$$\int \left(-\frac{3}{2}\cos x\right) e^{\frac{1}{2}\sin x} dx = -3\int e^{\frac{1}{2}\sin x} d\left(\frac{1}{2}\sin x\right) = -3e^{\frac{1}{2}\sin x}$$
$$y(x) = e^{-\frac{1}{2}\sin x}\left(-3e^{\frac{1}{2}\sin x} + C\right)$$
$$\boxed{= -3 + Ce^{-\frac{1}{2}\sin x}}$$

Exercise

Solve the differential equation: $\frac{dy}{dx} + y = e^{3x}$

Solution

$$e^{\int dx} = e^x$$
$$\int e^x e^{3x} dx = \int e^{4x} dx = \frac{1}{4}e^{4x}$$
$$y(x) = \frac{1}{e^x}\left(\frac{1}{4}e^{4x} + C\right)$$
$$\boxed{= \frac{1}{4}e^{3x} + Ce^{-x}}$$

Exercise

Solve the differential equation: $y' - ty = t$

Solution

$$e^{\int -t\,dt} = e^{-\frac{1}{2}t^2}$$

$$\int te^{-\frac{1}{2}t^2}\,dt = -\int e^{-\frac{1}{2}t^2}\,d\left(-\frac{1}{2}t^2\right) = -e^{-\frac{1}{2}t^2}$$

$$y(t) = e^{\frac{1}{2}t^2}\left(e^{-\frac{1}{2}t^2} + C\right)$$

$$\boxed{= 1 + Ce^{\frac{1}{2}t^2}}$$

Exercise

Solve the differential equation: $y' = 2y + x^2 + 5$

Solution

$$y' - 2y = x^2 + 5$$

$$e^{\int -2dx} = e^{-2x}$$

$$\int \left(x^2 + 5\right)e^{-2x}\,dx = \left(-\frac{1}{2}x^2 - \frac{5}{2} - \frac{1}{2}x - \frac{1}{4}\right)e^{-2x}$$

$$= \left(-\frac{1}{2}x^2 - \frac{1}{2}x - \frac{11}{4}\right)e^{-2x}$$

$$= -\frac{1}{4}\left(2x^2 + 2x + 11\right)e^{-2x}$$

		$\int e^{-2x}$
+	x^2+5	$-\frac{1}{2}e^{-2x}$
−	$2x$	$\frac{1}{4}e^{-2x}$
+	2	$-\frac{1}{8}e^{-2x}$

$$y(x) = e^{2x}\left(-\frac{1}{4}\left(2x^2 + 2x + 11\right)e^{-2x} + C\right)$$

$$\boxed{= -\frac{1}{4}\left(2x^2 + 2x + 11\right) + Ce^{2x}}$$

Exercise

Solve the differential equation: $xy' + 2y = 3$

Solution

$$y' + \frac{2}{x}y = \frac{3}{x}$$

$$e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$\int x^2\frac{3}{x}\,dx = \int 3x\,dx = \frac{3}{2}x^2$$

$$y(x)=\frac{1}{x^2}\left(\frac{3}{2}x^2+C\right)$$

$$\boxed{=\frac{3}{2}+\frac{C}{x^2}}$$

Exercise

Find the general solution of $\frac{dy}{dt}-2y=4-t$

Solution

$$e^{\int -2dt}=e^{-2t}$$

$$\int (4-t)e^{-2t}\ dt=\int \left(4e^{-2t}-te^{-2t}\right)\ dt$$

$$=-2e^{-2t}+\frac{1}{2}te^{-2t}+\frac{1}{4}e^{-2t}$$

$$=-\frac{7}{4}e^{-2t}+\frac{1}{2}te^{-2t}$$

		$\int e^{-2t}$
+	t	$-\frac{1}{2}e^{-2t}$
−	1	$\frac{1}{4}e^{-2t}$

$$y(t)=\frac{1}{e^{-2t}}\left(-\frac{7}{4}e^{-2t}+\frac{1}{2}te^{-2t}+C\right)$$

$$\boxed{y(t)=\frac{1}{2}t-\frac{7}{4}+Ce^{2t}}$$

Exercise

Solve the differential equation: $y'+2y=1$

Solution

$$e^{\int 2dx}=e^{2x}$$

$$\int e^{2x}\ dx=\frac{1}{2}e^{2x}$$

$$y(x)=\frac{1}{e^{2x}}\left(\frac{1}{2}e^{2x}+C\right)$$

$$\boxed{=\frac{1}{2}+Ce^{-2x}}$$

Exercise

Solve the differential equation: $y'+2y=e^{-t}$

Solution

$$e^{\int 2dt} = e^{2t}$$

$$\int e^{-t}e^{2t}\, dt = \int e^{t}\, dt = e^{t}$$

$$y(x) = \frac{1}{e^{2t}}\left(e^{t} + C\right)$$

$$\underline{= e^{-t} + Ce^{-2t}}$$

Exercise

Solve the differential equation: $y' + 2y = e^{-2t}$

Solution

$$e^{\int 2dt} = e^{2t}$$

$$\int e^{-2t}e^{2t}\, dt = t$$

$$\underline{y(x) = (t + C)e^{-2t}}$$

Exercise

Find the general solution of $y' - 2y = e^{3t}$

Solution

$$e^{\int -2dt} = e^{-2t}$$

$$\int e^{3t}e^{-2t}dt = e^{t}$$

$$y(t) = e^{2t}\left(e^{t} + C\right)$$

$$\underline{= e^{3t} + Ce^{2t}}$$

Exercise

Find the general solution of $y' + 2y = e^{-x} + x + 1$

Solution

		$\int e^{2x}$
+	$x+1$	$\frac{1}{2}e^{2x}$
–	1	$\frac{1}{4}e^{2x}$

$$e^{\int 2dx} = e^{2x}$$

$$\int \left(e^{-x} + x + 1\right)e^{2x}dx = \int \left(e^{x} + (x+1)e^{2x}\right)dx$$

$$= e^{x} + \left(\frac{1}{2}x + \frac{1}{2} - \frac{1}{4}\right)e^{2x}$$

$$= e^{x} + \left(\frac{1}{2}x + \frac{1}{4}\right)e^{2x}$$

$$y(x) = e^{-2x}\left(e^{x} + \left(\frac{1}{2}x + \frac{1}{4}\right)e^{2x} + C\right)$$

$$\boxed{= e^{-x} + \frac{1}{2}x + \frac{1}{4} + Ce^{-2x}}$$

Exercise

Solve the differential equation: $y' + 2xy = x$

Solution

$$e^{\int 2x dx} = e^{x^2}$$

$$\int xe^{x^2}\, dx = \frac{1}{2}\int e^{x^2}\, d\left(x^2\right) = \frac{1}{2}e^{x^2}$$

$$y(x) = \frac{1}{e^{x^2}}\left(\frac{1}{2}e^{x^2} + C\right)$$

$$\boxed{= \frac{1}{2} + Ce^{-x^2}}$$

Exercise

Solve the differential equation: $y' - 2ty = t$

Solution

$$e^{\int -2t dt} = e^{-t^2}$$

$$\int te^{-t^2}\, dt = -\frac{1}{2}\int e^{-t^2}\, d\left(-t^2\right) = -\frac{1}{2}e^{-t^2}$$

$$y(t) = \frac{1}{e^{-t^2}}\left(-\frac{1}{2}e^{-t^2} + C\right)$$

$$\boxed{= Ce^{t^2} - \frac{1}{2}}$$

Exercise

Find the general solution of $y' + 2ty = 5t$

Solution

$$u(t) = e^{\int 2t\,dt} = e^{t^2}$$

$$e^{t^2} y' + 2te^{t^2} y = 5te^{t^2}$$

$$\left(e^{t^2} y\right)' = 5te^{t^2}$$

$$e^{t^2} y = \int 5te^{t^2}\, dt \qquad\qquad de^{t^2} = 2te^{t^2}\,dt$$

$$= 5\int \tfrac{1}{2} de^{t^2}$$

$$= \tfrac{5}{2} e^{t^2} + C$$

$$\boxed{y(t) = \tfrac{5}{2} + Ce^{-t^2}}$$

Exercise

Solve the differential equation: $\quad y' - 2xy = e^{x^2}$

Solution

$$e^{\int -2x\,dx} = e^{-x^2}$$

$$\int e^{x^2} e^{-x^2}\, dx = \int dx = x$$

$$\boxed{y(x) = e^{x^2}(x + C)}$$

Exercise

Solve the differential equation: $\quad y' + 2xy = x^3$

Solution

$$e^{\int 2x\,dx} = e^{x^2}$$

$$\int x^3 e^{x^2}\, dx = \frac{1}{2}\int x^2 e^{x^2}\, d\left(x^2\right) = \frac{1}{2}\int ue^u\, d(u)$$

$$= \frac{1}{2}\left(x^2 - 1\right)e^{x^2}$$

		$\int e^u$
+	u	e^u
−	1	e^u

$$y(x) = e^{-x^2}\left(\frac{1}{2}\left(x^2 - 1\right)e^{x^2} + C\right)$$

$$\boxed{= \frac{1}{2}\left(x^2 - 1\right) + Ce^{-x^2}}$$

Exercise

Solve the differential equation: $y' - 2y = t^2 e^{2t}$

Solution

$$e^{\int -2dt} = e^{-2t}$$

$$\int e^{-2t} t^2 e^{2t} dt = \int t^2 dt = \frac{1}{3}t^3$$

$$y(t) = \frac{1}{e^{-2t}}\left(\frac{1}{3}t^3 + C\right)$$

$$\boxed{= e^{2t}\left(\frac{1}{3}t^3 + C\right)}$$

Exercise

Find the general solution of $x' - 2\frac{x}{t+1} = (t+1)^2$

Solution

$$e^{\int -\frac{2}{t+1}dt} = e^{-2\ln(t+1)} = e^{\ln(t+1)^{-2}} = (t+1)^{-2}$$

$$\int (t+1)^2 (t+1)^{-2} dt = \int dt = t$$

$$x(t) = \frac{1}{(t+1)^{-2}}(t+C)$$

$$= (t+1)^2 (t+C)$$

$$\boxed{= t(t+1)^2 + C(t+1)^2}$$

Exercise

Find the general solution of $y' + \frac{2}{t}y = \frac{\cos t}{t^2}$

Solution

$$e^{\int \frac{2}{t}dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$\int \frac{\cos t}{t^2} t^2 dt = \int \cos t \; dt = \sin t$$

$$\boxed{y(t) = \frac{1}{t^2}(\sin t + C)}$$

Exercise

Solve the differential equation: $y' - 2(\cos 2t)y = 0$

Solution

$$e^{\int -2\cos 2t\, dt} = e^{-\sin 2t}$$

$$\boxed{y(x) = C\, e^{\sin 2t}}$$

Exercise

Find the general solution of $y' + 2y = \cos 3t$

Solution

$$e^{\int 2dt} = e^{2t}$$

$$\int (\cos 3t)e^{2t}dt = \left(\frac{1}{3}\sin 3t + \frac{1}{18}\cos 3t\right)e^{2t} - \frac{1}{36}\int (\cos 3t)e^{2t}dt$$

$$\frac{37}{36}\int (\cos 3t)e^{2t}dt = \frac{1}{18}(6\sin 3t + \cos 3t)e^{2t}$$

$$\int (\cos 3t)e^{2t}dt = \frac{2}{37}(6\sin 3t + \cos 3t)e^{2t}$$

$$y(t) = e^{-2t}\left(\frac{2}{37}(6\sin 3t + \cos 3t)e^{2t} + C\right)$$

$$\boxed{= \frac{2}{37}(6\sin 3t + \cos 3t) + Ce^{-2t}}$$

		$\int \cos 3t$
+	e^{2t}	$\frac{1}{3}\sin 3t$
−	$\frac{1}{2}e^{2t}$	$-\frac{1}{9}\cos 3t$
+	$\frac{1}{4}e^{2t}$	

Exercise

Find the general solution of $y' - 3y = 5$

Solution

$$u(t) = e^{-\int 3dt} = e^{-3t}$$

$$e^{-3t}y' - 3e^{-3t}y = 5e^{-3t}$$

$$\left(e^{-3t}y\right)' = 5e^{-3t}$$

$$e^{-3t}y = \int 5e^{-3t}dt$$

$$e^{-3t}y = -\frac{5}{3}e^{-3t} + C$$

$$\boxed{y(t) = -\frac{5}{3} + Ce^{3t}}$$

Exercise

Solve the differential equation: $y' + 3y = 2xe^{-3x}$

Solution

$$e^{\int 3dx} = e^{3x}$$

$$\int 2xe^{-3x}e^{3x}\, dx = \int 2x\, dx = x^2$$

$$\boxed{y(x) = \frac{1}{e^{3x}}\left(x^2 + C\right)}$$

Exercise

Find the general solution of $y' + 3t^2 y = t^2$

Solution

$$e^{\int 3t^2 dt} = e^{t^3}$$

$$\int t^2 e^{t^3}\, dt = \frac{1}{3}\int e^{t^3}\, d\left(t^3\right) = \frac{1}{3}e^{t^3}$$

$$y(t) = \frac{1}{e^{t^3}}\left(\frac{1}{3}e^{t^3} + C\right)$$

$$\boxed{= \frac{1}{3} + Ce^{-t^3}}$$

Exercise

Solve the differential equation: $y' + 3x^2 y = x^2$

Solution

$$e^{\int 3x^2 dx} = e^{x^3}$$

$$\int x^2 e^{x^3} dx = \frac{1}{3}\int e^{x^3} d\left(e^{x^3}\right) = \frac{1}{3}e^{x^3}$$

$$y(x) = \frac{1}{e^{x^3}}\left(\frac{1}{3}e^{x^3} + C\right)$$

$$\boxed{= \frac{1}{3} + Ce^{-x^3}}$$

Exercise

Find the general solution of $y' + \frac{3}{t}y = \frac{\sin t}{t^3}, \quad (t \neq 0)$

Solution

$$e^{\int \frac{3}{t}dt} = e^{3\ln t} = e^{\ln t^3} = t^3$$

$$\int \frac{\sin t}{t^3}t^3\ dt = \int \sin t\ dt = -\cos t$$

$$y(t) = \frac{1}{t^3}(-\cos t + C)$$

$$\boxed{= \frac{C}{t^3} - \frac{\cos t}{t^3}}$$

Exercise

Find the general solution of $y' + \frac{3}{x}y = 1 + \frac{1}{x}$

Solution

$$e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

$$\int \left(1 + \frac{1}{x}\right)x^3 dx = \int \left(x^3 + x^2\right)dx = \frac{1}{4}x^4 + \frac{1}{3}x^3$$

$$y(x) = \frac{1}{x^3}\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + C\right)$$

$$\boxed{= \frac{1}{4}x + \frac{1}{3}x + \frac{C}{x^3}}$$

Exercise

Find the general solution of $y' + \frac{3}{2}y = \frac{1}{2}e^x$

Solution

$$e^{\int \frac{3}{2}dx} = e^{3x/2}$$

$$\int \left(\frac{1}{2}e^x\right)e^{3x/2}dx = \frac{1}{2}\int e^{5x/2}dx = \frac{1}{5}e^{5x/2}$$

$$y(x) = e^{-3x/2}\left(\frac{1}{5}e^{5x/2} + C\right)$$

$$\boxed{= \frac{1}{5}e^x + Ce^{-3x/2}}$$

Exercise

Find the general solution of $y' + 5y = t + 1$

Solution

		$\int e^{5t}$
+	$t+1$	$\frac{1}{5}e^{5t}$
−	1	$\frac{1}{25}e^{5t}$

$$e^{\int 5dt} = e^{5t}$$

$$\int (t+1)e^{5t}dt = \left(\frac{1}{5}t + \frac{1}{5} + \frac{1}{25}\right)e^{5t} = \frac{1}{5}\left(t + \frac{6}{5}\right)e^{5t}$$

$$y(t) = \frac{1}{e^{5t}}\left(\frac{1}{5}\left(t + \frac{6}{5}\right)e^{5t} + C\right)$$

$$\boxed{= \frac{1}{5}\left(t + \frac{6}{5}\right) + Ce^{-5t}}$$

Exercise

Solve the differential equation: $xy' - y = x^2 \sin x$

Solution

		$\int \sin x$
+	x^2	$-\cos x$
−	$2x$	$-\sin x$
+	2	$\cos x$

$$y' - \frac{1}{x}y = x\sin x$$

$$e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x\sin x + 2\cos x$$

$$y(x) = \frac{1}{x}\left(-x^2 \cos x + 2x\sin x + 2\cos x + C\right)$$

$$\boxed{= -x\cos x + 2\sin x + \frac{2}{x}\cos x + \frac{C}{x}}$$

Exercise

Solve the differential equation: $x\frac{dy}{dx} + y = e^x, \quad x > 0$

Solution

$$y' + \frac{1}{x}y = \frac{e^x}{x}$$

$$e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int \frac{e^x}{x}e^{\int \frac{1}{x}dx}dx = \int x\frac{e^x}{x}dx = \int e^x dx = e^x$$

$$\boxed{y(x) = \frac{1}{x}\left(e^x + C\right)}, \quad x > 0$$

Exercise

Solve the differential equation: $x\frac{dy}{dx}+2y=1-\frac{1}{x}, \quad x>0$

Solution

$$\frac{dy}{dx}+\frac{2}{x}y=\frac{1}{x}-\frac{1}{x^2}$$

$$e^{\int \frac{2}{x}dx}=e^{2\ln x}=e^{\ln x^2}=x^2$$

$$\int\left(\frac{1}{x}-\frac{1}{x^2}\right)x^2dx=\int(x-1)\,dx=\frac{1}{2}x^2-x$$

$$y(x)=\frac{1}{x^2}\left(\frac{1}{2}x^2-x+C\right)$$

$$\boxed{=\frac{1}{2}-\frac{1}{x}+\frac{C}{x^2}, \quad x>0}$$

Exercise

Find the general solution of $y\frac{dx}{dy}+2x=5y^3$

Solution

$$x'+\frac{2}{y}x=5y^2$$

$$e^{\int \frac{2}{y}dy}=e^{2\ln y}=y^2$$

$$\int 5y^2y^2dy=5\int y^4dx=y^5$$

$$x(y)=\frac{1}{y^2}\left(y^5+C\right)$$

$$\boxed{=y^3+\frac{C}{y^2}}$$

Exercise

Find the general solution of $ty'+y=\cos t$

Solution

$$y'+\frac{1}{t}y=\frac{\cos t}{t}$$

$$e^{\int \frac{1}{t}dt}=e^{\ln t}=t$$

$$\int t\frac{\cos t}{t}dt = \int \cos t\ dt = \sin t$$

$$\boxed{y(t) = \frac{1}{t}(\sin t + C)}$$

Exercise

Solve the differential equation: $xy' + 2y = x^2$

Solution

$$y' + \frac{2}{x}y = x$$

$$e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$\int x^3\ dx = \frac{1}{4}x^4$$

$$y(x) = \frac{1}{x^2}\left(\frac{1}{4}x^4 + C\right)$$

$$\boxed{= \frac{1}{4}x^2 + \frac{C}{x^2}}$$

Exercise

Solve the differential equation: $xy' = 2y + x^3\cos x$

Solution

$$y' - \frac{2}{x}y = x^2\cos x$$

$$e^{\int \frac{-2}{x}dx} = e^{-2\ln x} = x^{-2}$$

$$\int x^{-2}x^2\cos x\ dx = \int \cos x\ dx = \sin x$$

$$\boxed{y(x) = x^2(\sin x + C)}$$

Exercise

Find the general solution of $xy' + 2y = x^{-3}$

Solution

$$y' + \frac{2}{x}y = x^{-4}$$

$$e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$\int x^{-4}x^2 dx = \int x^{-2}dx = -\frac{1}{x}$$

$$y(x) = \frac{1}{x^2}\left(-\frac{1}{x} + C\right)$$

$$= \frac{C}{x^2} - \frac{1}{x^3}$$

Exercise

Find the general solution of $ty' + 2y = t^2$

Solution

$$y' + \frac{2}{t}y = t$$

$$e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$$

$$\int t^2(t)dt = \int t^3 dt = \frac{1}{4}t^4$$

$$y(t) = \frac{1}{t^2}\left(\frac{1}{4}t^4 + C\right)$$

$$= \frac{1}{4}t^2 + \frac{C}{t^2}$$

Exercise

Find the general solution of $xy' + 2\left(y + x^2\right) = \frac{\sin x}{x}$

Solution

$$y' + \frac{2}{x}y = \frac{\sin x}{x^2} - 2x$$

$$e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$\int \left(\frac{\sin x}{x^2} - 2x\right)x^2 dx = \int \left(\sin x - 2x^3\right)dx = -\cos x - \frac{1}{2}x^4$$

$$y(x) = \frac{1}{x^2}\left(-\cos x - \frac{1}{2}x^4 + C\right)$$

$$= -\frac{\cos x}{x^2} - \frac{1}{2}x^2 + \frac{C}{x^2}$$

Exercise

Solve the differential equation: $xy' + 4y = x^3 - x$

Solution

$$y' + \frac{4}{x}y = x^2 - 1$$

$$e^{\int \frac{4}{x}dx} = e^{4\ln x} = x^4$$

$$\int x^4\left(x^2 - 1\right)dx = \int\left(x^6 - x^4\right)dx = \frac{1}{7}x^7 - \frac{1}{5}x^5$$

$$y(x) = \frac{1}{x^4}\left(\frac{1}{7}x^7 - \frac{1}{5}x^5 + C\right)$$

$$= \frac{1}{7}x^3 - \frac{1}{5}x + Cx^{-4}$$

Exercise

Solve the differential equation: $xy' + (x+1)y = e^{-x}\sin 2x$

Solution

$$y' + \left(1 + \frac{1}{x}\right)y = \frac{1}{x}e^{-x}\sin 2x$$

$$e^{\int\left(1+\frac{1}{x}\right)dx} = e^{x+\ln x} = e^x e^{\ln x} = xe^x$$

$$\int xe^x \frac{1}{x}e^{-x}\sin 2x\, dx = \int \sin 2x\, dx = -\frac{1}{2}\cos 2x$$

$$y(x) = \frac{1}{xe^x}\left(\frac{1}{2}\cos 2x + C\right)$$

Exercise

Solve the differential equation: $xy' + (3x+1)y = e^{3x}$

Solution

$$y' + \left(3 + \frac{1}{x}\right)y = \frac{e^{3x}}{x}$$

$$e^{\int\left(3+\frac{1}{x}\right)dx} = e^{3x+\ln x} = xe^{3x}$$

$$\int xe^{3x}\frac{e^{3x}}{x}\, dx = \int e^{3x}\, dx = \frac{1}{3}e^{3x}$$

$$y(x) = \frac{1}{xe^{3x}}\left(e^{3x} + C\right)$$

$$=\boxed{\frac{1}{x}+Ce^{3x}} \quad x>0$$

Exercise

Solve the differential equation: $xy'+(2x-3)y=4x^4$

Solution

$$y'+\left(2-\frac{3}{x}\right)y=4x^3$$

$$e^{\int\left(2-\frac{3}{x}\right)dx}=e^{2x-3\ln x}=e^{2x}e^{-3\ln x}=x^{-3}e^{2x}$$

$$\int 4x^3x^{-3}e^{2x}\,dx=4\int e^{2x}dx=2e^{2x}$$

$$y(x)=\frac{1}{x^{-3}e^{2x}}\left(2e^{2x}+C\right)$$

$$=\boxed{2x^3+Cx^3e^{-2x}}$$

Exercise

Solve the differential equation: $2xy''-3y=9x^3$

Solution

$$y''-\frac{3}{2x}y=\frac{9}{2}x^2$$

$$e^{\int\frac{-3}{2x}dx}=e^{\frac{-3}{2}\ln x}=x^{-3/2}$$

$$\int\frac{9}{2}x^2x^{-3/2}\,dx=\frac{9}{2}\int x^{1/2}\,dx=3x^{3/2}$$

$$y(x)=x^{3/2}\left(3x^{3/2}+C\right)$$

$$=\boxed{3+Cx^{3/2}}$$

Exercise

Solve the differential equation: $2y'+3y=e^{-t}$

Solution

$$y'+\frac{3}{2}y=\frac{1}{2}e^{-t}$$

$$e^{\int\frac{3}{2}dt}=e^{3t/2}$$

$$\int e^{3t/2}e^{-t}dt = \int e^{t/2}dt = 2e^{t/2}$$

$$y(t) = \frac{1}{e^{3t/2}}\left(2e^{t/2} + C\right)$$

$$\boxed{= 2e^{-t} + Ce^{-3t/2}}$$

Exercise

Solve the differential equation: $2y' + 2ty = t$

Solution

$$y' + ty = \frac{1}{2}t$$

$$e^{\int tdt} = e^{t^2/2}$$

$$\int te^{t^2/2}dt = \int e^{t^2/2}d\left(\frac{1}{2}t^2\right) = e^{t^2/2}$$

$$y(t) = \frac{1}{e^{t^2/2}}\left(e^{t^2/2} + C\right)$$

$$\boxed{= 1 + Ce^{t^2/2}}$$

Exercise

Solve the differential equation: $3xy' + y = 10\sqrt{x}$

Solution

$$y' + \frac{1}{3x}y = \frac{10}{3}x^{-1/2}$$

$$e^{\int \frac{1}{3x}dx} = e^{\frac{1}{3}\ln x} = x^{1/3}$$

$$\int \frac{10}{3}x^{-1/2}x^{1/3}\, dx = \frac{10}{3}\int x^{-1/6}\, dx = 4x^{5/6}$$

$$y(x) = x^{-1/3}\left(4x^{5/6} + C\right)$$

$$\boxed{= 4x^{1/2} + Cx^{-1/3}}$$

Exercise

Solve the differential equation: $3xy' + y = 12x$

Solution

$$y' + \frac{1}{3x} y = 4$$

$$e^{\int \frac{1}{3x} dx} = e^{\frac{1}{3}\ln x} = x^{1/3}$$

$$\int 4x^{1/3}\, dx = 3x^{4/3}$$

$$y(x) = x^{-1/3}\left(3x^{4/3} + C\right)$$

$$\underline{= 3x + Cx^{-1/3}}$$

Exercise

Solve the differential equation: $x^2 y' + xy = 1$

Solution

$$y' + \frac{1}{x} y = \frac{1}{x^2}$$

$$e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$\int x \frac{1}{x^2} dx = \int \frac{1}{x} dx = \ln x$$

$$\underline{y(x) = \frac{1}{x}(\ln x + C)}$$

Exercise

Solve the differential equation: $x^2 y' + x(x+2)y = e^x$

Solution

$$y' + \left(1 + \frac{2}{x}\right) y = \frac{e^x}{x^2}$$

$$e^{\int \left(1+\frac{2}{x}\right)dx} = e^{x+2\ln x} = e^x e^{\ln x^2} = x^2 e^x$$

$$\int x^2 e^x \frac{e^x}{x^2} dx = \int e^{2x} dx = \frac{1}{2} e^{2x}$$

$$y(x) = \frac{1}{x^2 e^x}\left(\frac{1}{2} e^{2x} + C\right)$$

$$\underline{= \frac{1}{x^2}\left(\frac{1}{2} e^x + Ce^{-x}\right)}$$

Exercise

Find the general solution of $y^2 + (y')^2 = 1$

Solution

$$\left(\frac{dy}{dx}\right)^2 = 1 - y^2$$

$$\frac{dy}{dx} = \pm\sqrt{1-y^2}$$

$$\frac{dy}{\sqrt{1-y^2}} = \pm dx$$

$$\int \frac{dy}{\sqrt{1-y^2}} = \pm\int dx$$

$$\sin^{-1} y = \pm(x+c)$$

$$y = \sin\left(\pm(x+c)\right)$$

$$\underline{y(x) = \pm\sin(x+c)}$$

Exercise

Solve the differential equation: $(1+x)y' + y = \sqrt{x}$

Solution

$$\frac{dy}{dx} + \frac{1}{1+x}y = \frac{\sqrt{x}}{1+x}$$

$$e^{\int \frac{1}{1+x}dx} = e^{\ln(1+x)} = 1+x$$

$$\int \frac{\sqrt{x}}{1+x}(1+x)\,dx = \int x^{1/2}dx = \frac{2}{3}x^{3/2}$$

$$y(x) = \frac{1}{1+x}\left(\frac{2}{3}x^{3/2} + C\right)$$

$$\underline{= \frac{2x^{3/2}}{3(1+x)} + \frac{C}{1+x}}$$

Exercise

Find the general solution of $(1+x)y' + y = \cos x$

Solution

$$y' + \frac{1}{1+x}y = \frac{\cos x}{1+x}$$

$$y_h = e^{\int \frac{1}{1+x}dx} = e^{\ln(1+x)} = 1+x$$

$$\int \frac{\cos x}{1+x}(1+x)\,dx = \int \cos x dx = \sin x$$

$$y(x) = \frac{1}{x+1}(\sin x + C)$$

$$\boxed{y(x) = \frac{\sin x + C}{x+1}}$$

Exercise

Solve the differential equation: $(x+1)y' + (x+2)y = 2xe^{-x}$

Solution

$$y' + \frac{x+2}{x+1}y = \frac{2xe^{-x}}{x+1} \qquad \frac{x+2}{x+1} = 1 + \frac{1}{x+1}$$

$$e^{\int \left(1+\frac{1}{x+1}\right)dx} = e^{x+\ln(x+1)} = (x+1)e^{x}$$

$$\int (x+1)e^{x}\frac{2xe^{-x}}{x+1}\,dx = \int 2x\,dx = x^2$$

$$\boxed{y(x) = \frac{e^{-x}}{x+1}\left(x^2 + C\right)}$$

Exercise

Solve the differential equation: $(x+1)y' - xy = x + x^2$

Solution

$$y' - \frac{x}{x+1}y = \frac{x(x+1)}{x+1} = x$$

$$e^{\int \left(\frac{-x}{x+1}\right)dx} = e^{\int \left(\frac{1}{x+1}-1\right)dx} = e^{\ln|x+1|-x} = e^{\ln(x+1)}e^{-x} = (x+1)e^{-x}$$

$$\int x(x+1)e^{-x}\,dx = \left(-x^2 - x - 2x - 1 - 2\right)e^{-x}$$

$$= \left(-x^2 - 3x - 3\right)e^{-x}$$

		$\int e^{-x}$
+	$x^2 + x$	$-e^{-x}$
−	$2x+1$	e^{-x}
+	2	$-e^{-x}$

$$y(x) = \frac{e^{x}}{x+1}\left(-\left(x^2 + 3x + 3\right)e^{-x} + C\right)$$

$$\boxed{= \frac{x^2 + 3x + 3}{x+1} + \frac{Ce^{x}}{x+1}} \qquad x > -1$$

Exercise

Find the general solution of $\left(1+x^3\right)y' = 3x^2 y + x^2 + x^5$

Solution

$$y' - \frac{3x^2}{1+x^3}y = \frac{x^2\left(1+x^3\right)}{1+x^3} = x^2$$

$$e^{\int -\frac{3x^2}{1+x^3}dx} = e^{-\int \frac{d\left(1+x^3\right)}{1+x^3}} = e^{-\ln\left(1+x^3\right)} = e^{\ln\left(1+x^3\right)^{-1}} = \frac{1}{1+x^3}$$

$$\int \frac{x^2}{1+x^3}dx = \frac{1}{3}\int \frac{d\left(1+x^3\right)}{1+x^3} = \frac{1}{3}\ln\left(1+x^3\right)$$

$$y(x) = \left(1+x^3\right)\left(\frac{1}{3}\ln\left(1+x^3\right)+C\right)$$

$$= \frac{1}{3}\left(1+x^3\right)\ln\left(1+x^3\right) + C\left(1+x^3\right)$$

Exercise

Solve the differential equation: $\quad (t+1)\frac{ds}{dt} + 2s = 3(t+1) + \frac{1}{(t+1)^2}, \quad t > -1$

Solution

$$\frac{ds}{dt} + \frac{2}{t+1}s = 3 + \frac{1}{(t+1)^3}$$

$$e^{\int \frac{2}{t+1}dt} = e^{2\ln(t+1)} = e^{\ln(t+1)^2} = (t+1)^2$$

$$\int \left(3 + \frac{1}{(t+1)^3}\right)(t+1)^2\,dt = \int \left(3(t+1)^2 + \frac{1}{t+1}\right)dt \qquad d(t+1) = dt$$

$$= 3\int (t+1)^2\,d(t+1) + \int \frac{1}{t+1}d(t+1)$$

$$= (t+1)^3 + \ln(t+1)$$

$$s(t) = \frac{1}{(t+1)^2}\left((t+1)^3 + \ln(t+1) + C\right)$$

$$= t+1 + \frac{\ln(t+1)}{(t+1)^2} + \frac{C}{(t+1)^2}, \quad t > -1$$

Exercise

Solve the differential equation: $(x+2)^2 y' = 5 - 8y - 4xy$

Solution

$$y' + \frac{4}{x+2} y = 5(x+2)^{-2}$$

$$e^{\int \left(\frac{4}{x+2}\right)dx} = e^{4\ln(x+2)} = (x+2)^4$$

$$\int 5(x+2)^{-2}(x+2)^4 \, dx = 5\int (x+2)^2 \, d(x+2) = \frac{5}{3}(x+2)^3$$

$$y(x) = (x+2)^{-4}\left(\frac{5}{3}(x+2)^3 + C\right)$$

$$\boxed{= \frac{5}{3}(x+2)^{-1} + C(x+2)^{-4}}$$

Exercise

Solve the differential equation: $\left(x^2 - 1\right)y' + 2y = (x+1)^2$

Solution

$$y' + \frac{2}{(x-1)(x+1)} y = \frac{(x+1)^2}{(x-1)(x+1)} = \frac{x+1}{x-1}$$

$$e^{\int \left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx} = e^{\ln|x-1| - \ln|x+1|} = e^{\ln|x-1|}e^{-\ln|x+1|} = \frac{x-1}{x+1} \qquad \frac{2}{(x-1)(x+1)} = \frac{1}{x-1} - \frac{1}{x+1}$$

$$\int \frac{x-1}{x+1}\frac{x+1}{x-1} \, dx = \int dx = x$$

$$\boxed{y(x) = \frac{x+1}{x-1}(x+C)} \quad -1 < x < 1$$

Exercise

Solve the differential equation: $\left(x^2 + 4\right)y' + 2xy = x^2\left(x^2 + 4\right)$

Solution

$$y' + \frac{2x}{x^2+4} y = x^2$$

$$e^{\int \frac{2x}{x^2+4}dx} = e^{\ln\left(x^2+4\right)} = x^2 + 4$$

$$\int x^2\left(x^2+4\right) dx = \int \left(x^4 + 4x^2\right) dx = \frac{1}{5}x^5 + \frac{4}{3}x^3$$

$$\boxed{y(x) = \frac{1}{x^2+4}\left(\frac{1}{5}x^5 + \frac{4}{3}x^3 + C\right)}$$

Exercise

Find the general solution of $\left(1+e^t\right)y' + e^t y = 0$

Solution

$$y' + \frac{e^t}{1+e^t}y = 0$$

$$P(t) = \frac{e^t}{1+e^t}, \quad Q(t) = 0$$

$$e^{\int \frac{e^t}{1+e^t}dt} = e^{\int \frac{1}{1+e^t}d\left(1+e^t\right)} = e^{\ln\left(1+e^t\right)} = 1+e^t$$

$$y(t) = \frac{1}{1+e^t}(0+c)$$

$$= \frac{c}{1+e^t}$$

$$\frac{dy}{dt} = -\frac{e^t}{1+e^t}y$$

$$\frac{dy}{y} = -\frac{e^t}{1+e^t}dt$$

$$\int \frac{dy}{y} = -\int \frac{1}{1+e^t}d\left(1+e^t\right)$$

$$\ln y = -\ln\left(1+e^t\right) + C$$

$$\ln y = \ln\left(\frac{1}{1+e^t}\right) + \ln c$$

$$\ln y = \ln\left(\frac{c}{1+e^t}\right)$$

$$y(t) = \frac{c}{1+e^t}$$

Exercise

Find the general solution of $\left(t^2+9\right)y' + ty = 0$

Solution

$$y' + \frac{t}{t^2+9}y = 0$$

$$e^{\int \frac{t}{t^2+9}dt} = e^{\frac{1}{2}\int \frac{1}{t^2+9}d\left(t^2+9\right)}$$

$$d\left(t^2+9\right) = 2tdt \Rightarrow \frac{1}{2}d\left(t^2+9\right) = tdt$$

$$= e^{\frac{1}{2}\ln\left(t^2+9\right)} = e^{\ln\sqrt{t^2+9}} = \sqrt{t^2+9}$$

$$y(t) = \frac{1}{\sqrt{t^2+9}}(0+c)$$

$$= \frac{c}{\sqrt{t^2+9}}$$

Exercise

Solve the differential equation: $e^{2x}y' + 2e^{2x}y = 2x$

Solution

$$y' + 2y = 2xe^{-2x}$$

$$e^{\int 2dx} = e^{2x}$$

$$\int 2xe^{-2x}\left(e^{2x}\right)dx = 2\int xdx = x^2$$

$$\underline{|y(x)} = \frac{1}{e^{2x}}\left(x^2 + C\right)$$

$$\underline{= x^2e^{-2x} + Ce^{-2x}|}$$

Exercise

Solve the differential equation: $\tan\theta \frac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \frac{\pi}{2}$

Solution

$$\frac{dr}{d\theta} + \frac{1}{\tan\theta}r = \frac{\sin^2\theta}{\tan\theta}$$

$$\frac{dr}{d\theta} + \frac{1}{\tan\theta}r = \sin^2\theta\frac{\cos\theta}{\sin\theta}$$

$$\frac{dr}{d\theta} + \frac{1}{\tan\theta}r = \sin\theta\cos\theta, \quad P(\theta) = \frac{1}{\tan\theta} = \cot\theta \quad Q(\theta) = \sin\theta\cos\theta$$

$$e^{\int \cot\theta d\theta} = e^{\ln|\sin\theta|} = \sin\theta, \quad 0 < \theta < \frac{\pi}{2}$$

$$\int(\sin\theta\cos\theta)(\sin\theta)d\theta = \int\left(\sin^2\theta\cos\theta\right)d\theta \qquad d(\sin\theta) = \cos\theta d\theta$$

$$= \int \sin^2\theta d(\sin\theta)$$

$$= \frac{1}{3}\sin^3\theta$$

$$\underline{|r(\theta)} = \frac{1}{\sin\theta}\left(\frac{1}{3}\sin^3\theta + C\right)$$

$$\underline{= \frac{1}{3}\sin^2\theta + \frac{C}{\sin\theta}|}$$

Exercise

Find the general solution of $(\cos t)y' + (\sin t)y = 1$

Solution

$$y' + (\tan t)y = \frac{1}{\cos t}$$

$$e^{\int \tan t\,dt} = e^{-\ln|\cos t|} = e^{\ln\frac{1}{|\cos t|}} = \frac{1}{|\cos t|} = \sec t$$

$$\int \sec^2 t\ dt = \tan t$$

$$y(t) = \frac{1}{\sec t}(\tan t + C)$$

$$= \cos t\left(\frac{\sin t}{\cos t} + C\right)$$

$$\underline{= \sin t + C\cos t}$$

Exercise

Solve the differential equation: $\cos x\frac{dy}{dx} + (\sin x)y = 1$

Solution

$$y' + (\tan x)y = \sec x$$

$$e^{\int(\tan x)dx} = e^{\ln|\sec x|} = \sec x$$

$$\int \sec^2 x\ dx = \tan x$$

$$y(x) = \cos x(\tan x + C)$$

$$\underline{= \sin x + C\cos x}$$

Exercise

Solve the differential equation: $\cos^2 x\sin x\frac{dy}{dx} + \left(\cos^3 x\right)y = 1$

Solution

$$y' + (\cot x)y = \frac{1}{\cos^2 x\sin x}$$

$$e^{\int(\cot x)dx} = e^{\ln|\sin x|} = \sin x$$

$$\int \frac{\sin x}{\cos^2 x\sin x}\ dx = \int \sec^2 x\ dx = \tan x$$

$$y(x)=\frac{1}{\sin x}(\tan x+C)$$
$$\boxed{=\sec x+\frac{C}{\sin x}}$$

Exercise

Solve the differential equation: $\frac{dr}{d\theta}+r\sec\theta=\cos\theta$

Solution

$$e^{\int(\sec\theta)d\theta}=e^{\ln|\sec\theta+\tan\theta|}=\sec\theta+\tan\theta$$
$$\int\cos\theta(\sec\theta+\tan\theta)\,d\theta=\int(1+\sin\theta)\,d\theta=\theta-\cos\theta$$
$$\boxed{r(\theta)=\frac{1}{\sec\theta+\tan\theta}(\theta-\cos\theta+C)}$$

Exercise

Find the general solution of $\frac{dr}{d\theta}+r\tan\theta=\sec\theta$

Solution

$$e^{\int\tan\theta d\theta}=e^{\ln|\sec\theta|}=\sec\theta$$
$$\int\sec^2\theta d\theta=\tan\theta$$
$$r(\theta)=\frac{1}{\sec\theta}(\tan\theta+C)$$
$$\boxed{=\sin\theta+C\cos\theta}$$

Exercise

Solve the differential equation: $\frac{dP}{dt}+2tP=P+4t-2$

Solution

$$P'+(2t-1)P=4t-2$$
$$e^{\int(2t-1)dt}=e^{t^2-t}$$
$$\int e^{t^2-t}(4t-2)\,dt=2\int e^{t^2-t}\,d\left(t^2-t\right)=2e^{t^2-t}$$
$$P(t)=\frac{1}{e^{t^2-t}}\left(2e^{t^2-t}+C\right)$$
$$\boxed{=2+Ce^{t-t^2}}$$

Exercise

Solve the differential equation: $ydx - 4\left(x + y^6\right)dy = 0$

Solution

$$y\frac{dx}{dy} - 4x - 4y^6 = 0$$

$$\frac{dx}{dy} - \frac{4}{y}x = 4y^5$$

$$e^{\int -\frac{4}{y}dy} = e^{-4\ln y} = e^{\ln y^{-4}} = y^{-4}$$

$$\int 4y^5 y^{-4}\, dx = 4\int y\, dy = 2y^2$$

$$x(y) = y^4\left(2y^2 + C\right)$$

$$\boxed{= 2y^6 + Cy^4}$$

Exercise

Solve the differential equation: $ydx = \left(ye^y - 2x\right)dy$

Solution

$$y\frac{dx}{dy} = ye^y - 2x \quad \to \quad \frac{dx}{dy} + \frac{2}{y}x = e^y$$

$$e^{\int \frac{2}{y}dy} = e^{2\ln y} = e^{\ln y^2} = y^2$$

$$\int y^2 e^y\, dx = \left(y^2 - 2y + 2\right)e^y$$

$$\boxed{x(y) = \frac{1}{y^2}\left(\left(y^2 - 2y + 2\right)e^y + C\right)}$$

		$\int e^y$
+	y^2	e^y
–	$2y$	e^y
+	2	e^y

Exercise

Find the general solution of $(x + y + 1)dx - dy = 0$

Solution

$$\frac{dy}{dx} = x + y + 1$$

$$y' - y = x + 1$$

$$e^{\int -dx} = e^{-x}$$

$$\int (x+1)e^{-x}dx = (-x - 2)e^{-x}$$

		$\int e^{-x}$
+	$x+1$	$-e^{-x}$
–	1	e^{-x}

$$y(x) = e^{x}\left((-x-2)e^{-x} + C\right)$$

$$\boxed{= -x - 2 + Ce^{x}}$$

Exercise

Find the general solution of $\frac{dy}{dx} = x^2 e^{-4x} - 4y$

Solution

$$y' + 4y = x^2 e^{-4x}$$

$$e^{\int 4dx} = e^{4x}$$

$$\int x^2 e^{-4x} e^{4x} dx = \int x^2 dx = \frac{1}{3}x^3$$

$$\boxed{y(x) = e^{-4x}\left(\frac{1}{3}x^3 + C\right)}$$

Exercise

Find the general solution of $\left(x^2+1\right)y' + xy - x = 0$

Solution

$$y' + \frac{x}{x^2+1}y = \frac{x}{x^2+1}$$

$$e^{\int \frac{x}{x^2+1}dx} = e^{\frac{1}{2}\ln\left(x^2+1\right)} = \left(x^2+1\right)^{1/2}$$

$$\int \frac{x}{x^2+1}\left(x^2+1\right)^{1/2} dx = \frac{1}{2}\int \left(x^2+1\right)^{-1/2} d\left(x^2+1\right) = \sqrt{x^2+1}$$

$$y(x) = \frac{1}{\sqrt{x^2+1}}\left(\sqrt{x^2+1} + C\right)$$

$$\boxed{= 1 + \frac{C}{\sqrt{x^2+1}}}$$

Exercise

Find the general solution of $\frac{dx}{dt} = 9.8 - 0.196x$

Solution

$$x' + 0.196x = 9.8$$

$$e^{\int .196 dx} = e^{0.196t}$$

$$\int 9.8e^{0.196t} dt = 50e^{0.196t}$$

$$x(t) = \frac{1}{e^{0.196t}}\left(50e^{0.196t} + C\right)$$

$$\boxed{= 50 + Ce^{-0.196t}}$$

Exercise

Find the general solution of $\frac{di}{dt} + 500i = 10\sin\omega t$

		$\int \sin\omega t$
+	e^{500t}	$-\frac{1}{\omega}\cos\omega t$
−	$500e^{500t}$	$-\frac{1}{\omega^2}\sin\omega t$
+	$25\times10^4 e^{500t}$	$-\int \frac{1}{\omega^2}\sin\omega t$

Solution

$$e^{\int 500 dt} = e^{500t}$$

$$\int 10(\sin\omega t)e^{500t} dt$$

$$\int (\sin\omega t)e^{500t} dt = \left(-\frac{1}{\omega}\cos\omega t + \frac{500}{\omega^2}\sin\omega t\right)e^{500t} - \frac{25\times10^4}{\omega^2}\int (\sin\omega t)e^{500t} dt$$

$$\left(\frac{\omega^2 + 25\times10^4}{\omega^2}\right)\int (\sin\omega t)e^{500t} dt = \frac{1}{\omega^2}(-\omega\cos\omega t + 500\sin\omega t)e^{500t}$$

$$\int 10(\sin\omega t)e^{500t} dt = \frac{10}{\omega^2 + 25\times10^4}(-\omega\cos\omega t + 500\sin\omega t)e^{500t}$$

$$i(t) = e^{-500t}\left(\frac{10}{\omega^2 + 25\times10^4}(-\omega\cos\omega t + 500\sin\omega t)e^{500t} + C\right)$$

$$\boxed{= \frac{10}{\omega^2 + 25\times10^4}(-\omega\cos\omega t + 500\sin\omega t) + Ce^{-500t}}$$

Exercise

Find the general solution of $2\frac{dQ}{dt}+100Q=10\sin 60t$

		$\int \sin 60t$
+	e^{50t}	$-\frac{1}{60}\cos 60t$
−	$50e^{50t}$	$-\frac{1}{3600}\sin 60t$
+	$2500e^{50t}$	$-\frac{1}{3600}\int \sin 60t$

Solution

$$2\frac{dQ}{dt}+\frac{1}{.01}Q=10\sin 60t \;\rightarrow\; \frac{dQ}{dt}+50Q=5\sin 60t$$

$$e^{\int 50dt}=e^{50t}$$

$$5\int e^{50t}(\sin 60t)dt=$$

$$\int e^{50t}(\sin 60t)dt=\left(-\frac{1}{60}\cos 60t+\frac{1}{72}\sin 60t\right)e^{50t}-\frac{25}{36}\int e^{50t}(\sin 60t)dt$$

$$\frac{61}{36}\int e^{50t}(\sin 60t)dt=\left(-\frac{1}{60}\cos 60t+\frac{1}{72}\sin 60t\right)e^{50t}$$

$$\int e^{50t}(\sin 60t)dt=\frac{36}{21,960}(-6\cos 60t+5\sin 60t)e^{50t}$$

$$5\int e^{50t}(\sin 60t)dt=\frac{1}{122}(-6\cos 60t+5\sin 60t)e^{50t}$$

$$Q(t)=\frac{1}{e^{50t}}\left(\frac{1}{122}(-6\cos 60t+5\sin 60t)e^{50t}+C\right)$$

$$=\frac{1}{122}(-6\cos 60t+5\sin 60t)+Ce^{-50t}$$

$$Q(0)=-\frac{6}{122}+C=0 \;\Rightarrow\; C=\frac{3}{61}$$

$$Q(t)=\frac{1}{122}\left(-5\cos 60t+6\sin 60t+6e^{-50t}\right)$$

Exercise

Find the general solution of $y'-3y=4$; $y(0)=2$

Solution

$$e^{-\int 3dt}=e^{-3t}$$

$$\int 4e^{-3t}dt=-\frac{4}{3}e^{-3t}$$

$$y(t)=e^{3t}\left(-\frac{4}{3}e^{-3t}+C\right)$$

$$=-\frac{4}{3}+Ce^{3t}$$

$$y(0)=-\frac{4}{3}+Ce^{3(0)}$$

$$2=-\frac{4}{3}+C \qquad C=\frac{4}{3}+2=\boxed{\frac{10}{3}}$$

$$\boxed{y(t)=-\frac{4}{3}+\frac{10}{3}e^{3t}}$$

Exercise

Find the general solution of $y'=y+2xe^{2x}$; $\quad y(0)=3$

Solution

$$y'-y=2xe^{2x}$$

$$e^{\int -1dx}=e^{-x}$$

$$\int 2xe^{2x}\left(e^{-x}\right)dx=2\int xe^{x}dx=2\left(xe^{x}-e^{x}\right)$$

$$y(x)=\frac{1}{e^{-x}}\left(2xe^{x}-2e^{x}+C\right)$$

$$=e^{x}\left(2xe^{x}-2e^{x}+C\right)$$

$$\boxed{=2xe^{2x}-2e^{2x}+Ce^{x}}$$

$$y(x=0)=2(0)e^{2(0)}-2e^{2(0)}+Ce^{(0)}$$

$$3=-2+C \qquad \rightarrow \boxed{C=5}$$

$$\boxed{y(x)=2xe^{2x}-2e^{2x}+5e^{x}}$$

Exercise

Find the general solution of $\left(x^2+1\right)y'+3xy=6x$; $\quad y(0)=-1$

Solution

$$y'+\frac{3x}{x^2+1}y=\frac{6x}{x^2+1}$$

$$e^{\int \frac{3x}{x^2+1}dx}=e^{\frac{3}{2}\ln\left(x^2+1\right)}=e^{\ln\left(x^2+1\right)^{\frac{3}{2}}}=\boxed{\left(x^2+1\right)^{\frac{3}{2}}}$$

$$\int\left(x^2+1\right)^{\frac{3}{2}}\frac{6x}{x^2+1}dx=3\int\left(x^2+1\right)^{\frac{1}{2}}d\left(x^2+1\right)=2\left(x^2+1\right)^{\frac{3}{2}}$$

$$y(x)=2+C\left(x^2+1\right)^{-\frac{3}{2}}$$

$$y(0)=2+C\left((0)^2+1\right)^{-\frac{3}{2}}$$

$$-1 = 2 + C(1)^{-\frac{3}{2}} \quad \rightarrow \underline{C = -3}$$

$$\boxed{y(x) = 2 - 3\left(x^2 + 1\right)^{-\frac{3}{2}}}$$

Exercise

Solve the initial value problem: $\quad t\dfrac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(2) = 1$

Solution

$$\frac{dy}{dt} + \frac{2}{t}y = t^2, \quad P(t) = \frac{2}{t}, \quad Q(t) = t^2$$

$$e^{\int \frac{2}{t}dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$\int t^2 t^2 dt = \int t^4 dt = \frac{1}{5}t^5$$

$$y(t) = \frac{1}{t^2}\left(\frac{1}{5}t^5 + C\right) = \frac{1}{5}t^3 + \frac{C}{t^2}$$

$$y(2) = \frac{1}{5}2^3 + \frac{C}{2^2}$$

$$1 = \frac{8}{5} + \frac{C}{4} \quad \rightarrow \frac{C}{4} = 1 - \frac{8}{5} = -\frac{3}{5} \Rightarrow \boxed{C = -\frac{12}{5}}$$

$$\boxed{y(t) = \frac{1}{5}t^3 - \frac{12}{5t^2}}$$

Exercise

Solve the initial value problem: $\quad \theta\dfrac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0, \quad y\left(\frac{\pi}{2}\right) = 1$

Solution

$$\frac{dy}{d\theta} + \frac{1}{\theta}y = \frac{\sin\theta}{\theta}, \quad P(\theta) = \frac{1}{\theta}, \quad Q(\theta) = \frac{\sin\theta}{\theta}$$

$$e^{\int \frac{1}{\theta}d\theta} = e^{\ln|\theta|} = \theta \quad (> 0)$$

$$\int \frac{\sin\theta}{\theta}\theta d\theta = \int \sin\theta d\theta = -\cos\theta$$

$$y(\theta) = \frac{1}{\theta}(-\cos\theta + C)$$

$$y\left(\frac{\pi}{2}\right) = \frac{2}{\pi}\left(-\cos\frac{\pi}{2} + C\right) 1 = \frac{2}{\pi}(0 + C) \qquad \Rightarrow 1 = \frac{2}{\pi}C \quad C = \frac{\pi}{2}$$

$$\boxed{y(\theta) = -\frac{\cos\theta}{\theta} + \frac{\pi}{2\theta}}$$

Exercise

Solve the initial value problem: $\frac{dy}{dx} + xy = x, \quad y(0) = -6$

Solution

$$\frac{dy}{dx} + xy = x$$

$$e^{\int x\,dx} = e^{x^2/2}$$

$$\int xe^{x^2/2}dx = \int e^{x^2/2}d\left(\frac{x^2}{2}\right) = e^{x^2/2} \qquad\qquad d\left(\frac{x^2}{2}\right) = x\,dx$$

$$y(x) = \frac{1}{e^{x^2/2}}\left(e^{x^2/2} + C\right)$$

$$y(0) = \frac{1}{e^{0^2/2}}\left(e^{0^2/2} + C\right)$$

$$-6 = 1 + C \to \underline{C = -7}$$

$$y(x) = \frac{1}{e^{x^2/2}}\left(e^{x^2/2} - 7\right)$$

$$\boxed{= 1 - \frac{7}{e^{x^2/2}}}$$

Exercise

Solve the initial value problem: $ty' + 2y = 4t^2, \quad y(1) = 2$

Solution

$$y' + \frac{2}{t}y = 4t$$

$$e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = e^{\ln t^2} = t^2$$

1st method

$$\int t^2(4t)\,dt = 4\int t^3\,dt = t^4$$

$$y(t) = \frac{1}{t^2}\left(t^4 + C\right) \qquad y = \frac{1}{e^{\int P\,dx}}\left(\int Q.e^{\int P\,dx}dx + C\right)$$

2nd method

$$t^2y' + t^2\frac{2}{t}y = 4t\left(t^2\right)$$

$$t^2y' + 2ty = 4t^3$$

$$\left(t^2y\right)' = 4t^3$$

$$t^2y = t^4 + C$$

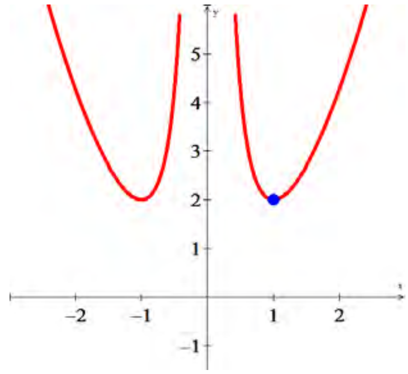
$$y(t) = \frac{1}{t^2}\left(t^4 + C\right)$$

$$y(1)=\frac{1}{1^2}\left(1^4+C\right)$$

$$2=1+C \rightarrow \underline{C=1}$$

$$y(t)=\frac{1}{t^2}\left(t^4+1\right)$$

$$\boxed{y(t)=t^2+\frac{1}{t^2}}$$



Exercise

Find the solution of the initial value problem $\left(1+t^2\right)y'+4ty=\left(1+t^2\right)^{-2}, \quad y(1)=0$

Solution

$$y'+\frac{4t}{1+t^2}y=\frac{\left(1+t^2\right)^{-2}}{1+t^2}$$

$$y'+\frac{4t}{1+t^2}y=\left(1+t^2\right)^{-3}$$

$$e^{\int\frac{4t}{1+t^2}dt}=e^{2\int\frac{d\left(1+t^2\right)}{1+t^2}}=e^{2\ln\left(1+t^2\right)}=e^{\ln\left(1+t^2\right)^2}=\left(1+t^2\right)^2$$

$$\int\left(1+t^2\right)^2\left(1+t^2\right)^{-3}dt=\int\frac{dt}{1+t^2}=\underline{\tan^{-1}|t|}$$

$$y(t)=\frac{1}{\left(1+t^2\right)^2}\left(\tan^{-1}|t|+C\right)$$

Given $y(1)=0$, then $\quad 0=\frac{1}{\left(1+1^2\right)^2}\left(\tan^{-1}|1|+C\right)$

$$0=\frac{\pi}{4}+C \quad\Rightarrow\quad C=-\frac{\pi}{4}$$

$$\boxed{y(t)=\frac{1}{\left(1+t^2\right)^2}\left(\tan^{-1}|t|-\frac{\pi}{4}\right)}$$

Exercise

Solve the initial value problem: $y' = x + 5y, \quad y(0) = 3$

Solution

$$y' - 5y = x$$

$$e^{\int -5dx} = \boxed{e^{-5x}}$$

$$\int xe^{-5x}dx = \left(-\frac{1}{5}x - \frac{1}{25}\right)e^{-5x}$$

		$\int e^{-5x}$
+	x	$-\frac{1}{5}e^{-5x}$
−	1	$\frac{1}{25}e^{-5x}$

$$y(x) = e^{5x}\left(\left(-\frac{1}{5}x - \frac{1}{25}\right)e^{-5x} + C\right)$$

$$= -\frac{1}{5}x - \frac{1}{25} + Ce^{5x} \qquad y(0) = 3$$

$$3 = -\frac{1}{25} + C \quad \rightarrow \quad C = \frac{76}{25}$$

$$\boxed{y(x) = -\frac{1}{5}x - \frac{1}{25} + \frac{76}{25}e^{5x}}$$

Exercise

Solve the initial value problem: $y' = 2x - 3y, \quad y(0) = \frac{1}{3}$

Solution

$$y' + 3y = 2x$$

$$e^{\int 3dx} = \boxed{e^{3x}}$$

$$\int 2xe^{3x}dx = \left(\frac{2}{3}x - \frac{2}{9}\right)e^{3x}$$

		$\int e^{3x}$
+	$2x$	$\frac{1}{3}e^{3x}$
−	2	$\frac{1}{9}e^{3x}$

$$y(x) = e^{-3x}\left(\left(\frac{2}{3}x - \frac{2}{9}\right)e^{3x} + C\right)$$

$$= \frac{2}{3}x - \frac{2}{9} + Ce^{-3x} \qquad y(0) = \frac{1}{3}$$

$$\frac{1}{3} = -\frac{2}{9} + C \quad \rightarrow \quad C = \frac{5}{9}$$

$$\boxed{y(x) = \frac{2}{3}x - \frac{2}{9} + \frac{5}{9}e^{-3x}}$$

Exercise

Solve the initial value problem: $xy' + y = e^x, \quad y(1) = 2$

Solution

$$y' + \frac{1}{x}y = \frac{e^x}{x}$$

$$e^{\int \frac{1}{x}dx} = e^{\ln x} = \boxed{x}$$

$$\int x\frac{e^x}{x}dx = \int e^x dx = \boxed{e^x}$$

$$y(x) = \frac{1}{x}\left(e^x + C\right)$$

$$y(1) = 2 \quad 2 = e + C \quad \rightarrow \quad \boxed{C = 2 - e}$$

$$\boxed{y(x) = \frac{1}{x}\left(e^x + 2 - e\right)}$$

Exercise

Solve the initial value problem: $\quad y\frac{dx}{dy} - x = 2y^2, \quad y(1) = 5$

Solution

$$\frac{dx}{dy} - \frac{1}{y}x = 2y$$

$$e^{\int -\frac{1}{y}dy} = e^{-\ln y} = e^{\ln y^{-1}} = \boxed{y^{-1}}$$

$$\int 2yy^{-1}\,dx = 2\int dy = 2y$$

$$x(y) = y(2y + C)$$

$$y(1) = 5 \quad \rightarrow 1 = 5(10 + C) \quad \Rightarrow \quad \boxed{C = -\frac{49}{5}}$$

$$\boxed{x(y) = 2y^2 - \frac{49}{5}y}$$

Exercise

Solve the initial value problem: $\quad xy' + y = 4x + 1, \quad y(1) = 8$

Solution

$$y' + \frac{1}{x}y = \frac{4x+1}{x}$$

$$e^{\int \frac{1}{x}dx} = e^{\ln x} = \boxed{x}$$

$$\int x\frac{4x+1}{x}dx = \int (4x+1)dx = \boxed{2x^2 + x}$$

$$y(x) = \frac{1}{x}\left(2x^2 + x + C\right) \qquad y(1) = 8$$

$$8 = 3 + C \quad \rightarrow \quad \boxed{C = 5}$$

$$\boxed{y(x) = 2x + 1 + \frac{5}{x}}$$

Exercise

Solve the initial value problem: $y' + 4xy = x^3 e^{x^2}, \quad y(0) = -1$

Solution

$$e^{\int 4x\,dx} = \boxed{e^{2x^2}}$$

$$\int x^3 e^{x^2} e^{2x^2}\,dx = \int x^3 e^{3x^2}\,dx = \frac{1}{6}\int x^2 e^{3x^2}\,d\left(3x^2\right)$$

$$= \frac{1}{18}\int u e^u\,d(u)$$

$$= \frac{1}{18}\left(3x^2 - 1\right)e^{3x^2}$$

	$u = 3x^2$	$\int e^u$
+	u	e^u
−	1	e^u

$$y(x) = \frac{1}{e^{2x^2}}\left(\frac{1}{18}\left(3x^2 - 1\right)e^{3x^2} + C\right)$$

$y(0) = -1 \qquad -1 = -\frac{1}{18} + C \quad \rightarrow \quad \boxed{C = -\frac{17}{18}}$

$$\boxed{y(x) = \frac{1}{18}\left(3x^2 - 1\right)e^{x^2} - \frac{17}{18}e^{2x^2}}$$

Exercise

Solve the initial value problem: $(x+1)y' + y = \ln x, \quad y(1) = 10$

Solution

$$y' + \frac{1}{x+1}y = \frac{\ln x}{x+1}$$

$$e^{\int \frac{dx}{x+1}} = e^{\ln(x+1)} = \boxed{x+1}$$

$$\int \frac{\ln x}{x+1}(x+1)\,dx = \int \ln x\,dx = x\ln x - x$$

$y(x) = \frac{1}{x+1}(x\ln x - x + C) \qquad y(1) = 10$

$10 = \frac{1}{2}(-1 + C) \quad \rightarrow \quad \boxed{C = 21}$

$$\boxed{y(x) = \frac{1}{x+1}(x\ln x - x + 21)}$$

Exercise

Solve the initial value problem: $y' - (\sin x)y = 2\sin x, \quad y\left(\frac{\pi}{2}\right) = 1$

Solution

$$e^{\int -\sin x\,dx} = \boxed{e^{\cos x}}$$

$$\int 2\sin x e^{\cos x} dx = -2\int e^{\cos x}\, d(\cos x) = -2e^{\cos x}$$

$$y(x) = \frac{1}{e^{\cos x}}\left(-2e^{\cos x} + C\right) \qquad y\left(\frac{\pi}{2}\right) = 1$$

$$1 = -2 + C \quad \rightarrow \quad \boxed{C = 3}$$

$$\boxed{y(x) = -2 + \frac{3}{e^{\cos x}}}$$

Exercise

Solve the initial value problem: $L\dfrac{di}{dt} + RI = E, \quad i(0) = i_0$

Solution

$$e^{\int Rdt} \ \boxed{= e^{Rt}}$$

$$\int Ee^{Rt} dt \ \boxed{= \frac{E}{R}e^{Rt}}$$

$$I(t) = \frac{1}{e^{Rt}}\left(\frac{E}{R}e^{Rt} + C\right) \qquad i(0) = i_0$$

$$i_0 = \frac{E}{R} + C \quad \rightarrow \quad \boxed{C = i_0 - \frac{E}{R}}$$

$$\boxed{I(t) = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right)e^{-Rt}}$$

Exercise

Solve the initial value problem: $\dfrac{dT}{dt} = k\left(T - T_m\right) \quad T(0) = T_0$

Solution

$$\frac{dT}{dt} - kT = -kT_m$$

$$e^{\int -kdt} \ \boxed{= e^{-kt}}$$

$$\int -kT_m e^{-kt} dt \ \boxed{= T_m e^{-kt}}$$

$$T(t) = \frac{1}{e^{-kt}}\left(T_m e^{-kt} + C\right) \qquad T(0) = T_0$$

$$T_0 = T_m + C \quad \rightarrow \quad \boxed{C = T_0 - T_m}$$

$$\boxed{T(t) = T_m + \left(T_0 - T_m\right)e^{kt}}$$

Exercise

Solve the initial value problem: $y' + y = 2, \quad y(0) = 0$

Solution

$$e^{\int dx} = e^{x}$$

$$\int 2e^{x} dx = 2e^{x}$$

$$y(x) = \frac{1}{e^{x}}\left(2e^{x} + C\right)$$

$$= 2 + Ce^{-x}$$

$$y(0) = 0 \quad \to 0 = 2 + C \quad \Rightarrow \underline{C = -2}$$

$$\underline{y(x) = 2 - 2e^{-x}}$$

Exercise

Solve the initial value problem: $y' - 2y = 3e^{2x}, \quad y(0) = 0$

Solution

$$e^{\int -2dx} = e^{-2x}$$

$$\int 3e^{2x}e^{-2x} dx = 3x$$

$$y(x) = \underline{e^{2x}(3x + C)}$$

$$y(0) = 0 \quad \to \underline{0 = C}$$

$$\underline{y(x) = 3xe^{2x}}$$

Exercise

Solve the initial value problem: $xy' + 2y = 3x, \quad y(1) = 5$

Solution

$$y' + \frac{2}{x} y = 3$$

$$e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^{2}$$

$$\int 3x^{2}\ dx = x^{3}$$

$$y(x) = \frac{1}{x^{2}}\left(x^{3} + C\right)$$

$$= x + \frac{C}{x^2}$$

$$y(1) = 5 \quad \rightarrow 5 = 1 + C \quad \Rightarrow \quad \underline{C = 4}$$

$$y(x) = x + \frac{4}{x^2}$$

Exercise

Solve the initial value problem: $\quad xy' + 5y = 7x^2, \quad y(2) = 5$

Solution

$$y' + \frac{5}{x}y = 7x$$

$$e^{\int \frac{5}{x}dx} = e^{5\ln x} = x^5$$

$$\int 7x^2x^5 \, dx = \frac{7}{8}x^8$$

$$y(x) = \frac{1}{x^5}\left(\frac{7}{8}x^8 + C\right)$$

$$= \frac{7}{8}x^3 + \frac{C}{x^5}$$

$$y(2) = 5 \quad \rightarrow 5 = 7 + \frac{1}{32}C \quad \Rightarrow \quad \underline{C = -64}$$

$$y(x) = \frac{7}{8}x^3 - \frac{64}{x^5}$$

Exercise

Solve the initial value problem: $\quad xy' - y = x, \quad y(1) = 7$

Solution

$$y' - \frac{1}{x}y = 1$$

$$e^{\int \frac{-1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\int \frac{1}{x} \, dx = \ln x$$

$$y(x) = x(\ln x + C)$$

$$= x\ln x + Cx$$

$$y(1) = 7 \quad \rightarrow \underline{7 = C}$$

$$y(x) = x\ln x + 7x$$

Exercise

Solve the initial value problem: $xy' + y = 3xy, \quad y(1) = 0$

Solution

$$xy' + (1-3x)y = 0$$

$$y' + \left(\frac{1}{x} - 3\right)y = 0$$

$$e^{\int\left(\frac{1}{x}-3\right)dx} = e^{\ln x - 3x} = e^{\ln x}e^{-3x} = xe^{-3x}$$

$$y(x) = \frac{1}{xe^{-3x}}C$$

$$= \boxed{\frac{Ce^{3x}}{x}}$$

$$y(1) = 0 \quad \rightarrow \boxed{0 = C}$$

$$\boxed{y(x) = 0}$$

Exercise

Solve the initial value problem: $xy' + 3y = 2x^5, \quad y(2) = 1$

Solution

$$y' + \frac{3}{x}y = 2x^4$$

$$e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

$$\int 2x^4 x^3\, dx = \frac{1}{4}x^8$$

$$y(x) = \frac{1}{x^3}\left(\frac{1}{4}x^8 + C\right)$$

$$= \boxed{\frac{1}{4}x^5 + Cx^{-3}}$$

$$y(2) = 1 \quad \rightarrow 1 = 8 + \frac{C}{8} \quad \Rightarrow \boxed{C = -56}$$

$$\boxed{y(x) = \frac{1}{4}x^5 - 56x^{-3}}$$

Exercise

Solve the initial value problem: $y' + y = e^x, \quad y(0) = 1$

Solution

$$e^{\int dx} = e^x$$

$$\int e^x e^x\,dx = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$$

$$y(x) = \frac{1}{e^x}\left(\frac{1}{2}e^{2x} + C\right)$$

$$= \frac{1}{2}e^x + Ce^{-x}$$

$$y(0) = 1 \quad \to 1 = \frac{1}{2} + C \quad \Rightarrow C = \frac{1}{2}$$

$$y(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$$

Exercise

Solve the initial value problem: $\quad xy' - 3y = x^3, \quad y(1) = 10$

Solution

$$y' - \frac{3}{x}y = x^2$$

$$e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = x^{-3}$$

$$\int x^{-3}x^3\,dx = \int dx = x$$

$$y(x) = x^3(x + C)$$

$$= x^4 + Cx^3$$

$$y(1) = 10 \quad \to 10 = 1 + C \quad \Rightarrow C = 9$$

$$y(x) = x^4 + 9x^3$$

Exercise

Solve the initial value problem: $\quad y' + 2xy = x, \quad y(0) = -2$

Solution

$$e^{\int 2x dx} = e^{x^2}$$

$$\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2}$$

$$y(x) = e^{-x^2}\left(\frac{1}{2}e^{x^2} + C\right)$$

$$= \frac{1}{2} + Ce^{-x^2}$$

$$y(0) = -2 \quad \to -2 = \frac{1}{2} + C \quad \Rightarrow C = -\frac{5}{2}$$

$$\boxed{y(x)=\frac{1}{2}-\frac{5}{2}e^{-x^2}}$$

Exercise

Solve the initial value problem: $\quad y'=(1-y)\cos x, \quad y(\pi)=2$

Solution

$$y'+(\cos x)y=\cos x$$

$$e^{\int \cos x dx}=e^{\sin x}$$

$$\int \cos x e^{\sin x}\, dx=e^{\sin x}$$

$$y(x)=\frac{1}{e^{\sin x}}\left(e^{\sin x}+C\right)$$

$$\underline{=1+Ce^{-\sin x}}$$

$$y(\pi)=2 \quad \rightarrow \quad 2=1+C \quad \Rightarrow \underline{C=1}$$

$$\boxed{y(x)=1+e^{-\sin x}}$$

Exercise

Solve the initial value problem: $\quad (1+x)y'+y=\cos x, \quad y(0)=1$

Solution

$$y'+\frac{1}{x+1}y=\frac{\cos x}{1+x}$$

$$e^{\int \frac{1}{1+x}dx}=e^{\ln(1+x)}=1+x$$

$$\int \frac{\cos x}{1+x}(1+x)\, dx=\sin x$$

$$y(x)=\frac{1}{1+x}(\sin x+C)$$

$$y(0)=1 \quad \Rightarrow \underline{C=1}$$

$$\boxed{y(x)=\frac{1}{1+x}(\sin x+1)}$$

Exercise

Solve the initial value problem: $\quad y'=1+x+y+xy, \quad y(0)=0$

Solution

$$y'-(1+x)y=1+x$$

$$e^{-\int(1+x)dx} = e^{-x-\frac{1}{2}x^2}$$

$$\int(1+x)e^{-\left(x+x^2/2\right)}\,dx = -e^{-\left(x+x^2/2\right)}$$

$$y(x) = e^{x+\frac{1}{2}x^2}\left(-e^{-\left(x+\frac{1}{2}x^2\right)}+C\right)$$

$$= -1+Ce^{x+\frac{1}{2}x^2}$$

$$y(0)=0 \quad \rightarrow \quad 0=-1+C \quad \Rightarrow \underline{C=1}$$

$$\boxed{y(x) = -1+e^{x+\frac{1}{2}x^2}}$$

Exercise

Solve the initial value problem: $\quad xy' = 3y + x^4\cos x, \quad y(2\pi)=0$

Solution

$$y' - \frac{3}{x}y = x^3\cos x$$

$$e^{-\int\frac{3}{x}dx} = e^{-3\ln x} = x^{-3}$$

$$\int x^{-3}x^3\cos x\,dx = \int\cos x\,dx = \sin x$$

$$y(x) = x^3(\sin x + C)$$

$$y(2\pi)=0 \quad \rightarrow \quad \underline{0=C}$$

$$\boxed{y(x) = x^3\sin x}$$

Exercise

Solve the initial value problem: $\quad y' = 2xy + 3x^2e^{x^2}, \quad y(0)=5$

Solution

$$y' - 2xy = 3x^2e^{x^2}$$

$$e^{-\int 2xdx} = e^{-x^2}$$

$$\int 3x^2e^{x^2}e^{-x^2}\,dx = \int 3x^2\,dx = x^3$$

$$y(x) = e^{x^2}\left(x^3+C\right)$$

$$y(0)=5 \quad \to \quad \boxed{5=C}$$

$$\boxed{y(x)=e^{x^2}\left(x^3+5\right)}$$

Exercise

Solve the initial value problem: $\left(x^2+4\right)y'+3xy=x, \quad y(0)=1$

Solution

$$y'+\frac{3x}{x^2+4}y=\frac{x}{x^2+4}$$

$$e^{\int\frac{3x}{x^2+4}dx}=e^{\frac{3}{2}\int\frac{1}{x^2+4}d\left(x^2+4\right)}=e^{\frac{3}{2}\ln\left(x^2+4\right)}=\left(x^2+4\right)^{3/2}$$

$$\int\frac{x}{x^2+4}\left(x^2+4\right)^{3/2}dx=\frac{1}{2}\int\left(x^2+4\right)^{1/2}d\left(x^2+4\right)=\frac{1}{3}\left(x^2+4\right)^{3/2}$$

$$y(x)=\left(x^2+4\right)^{-3/2}\left(\frac{1}{3}\left(x^2+4\right)^{3/2}+C\right)$$

$$=\frac{1}{3}+C\left(x^2+4\right)^{-3/2}$$

$$y(0)=1 \quad \to \quad 1=\frac{1}{3}+\frac{1}{8}C \quad \Rightarrow \boxed{C=\frac{16}{3}}$$

$$\boxed{y(x)=\frac{1}{3}+\frac{16}{3}\left(x^2+4\right)^{-3/2}}$$

Exercise

Solve the initial value problem: $\left(x^2+1\right)y'+3x^3y=6xe^{-3x^2/2}, \quad y(0)=1$

Solution

$$y'+\frac{3x^3}{x^2+1}y=\frac{6xe^{-3x^2/2}}{x^2+1}$$

$$y'+\left(3x-\frac{3x}{x^2+1}\right)y=\frac{6xe^{-3x^2/2}}{x^2+1}$$

$$e^{\int\left(3x-\frac{3x}{x^2+1}\right)dx}=e^{\frac{3}{2}x^2-\frac{3}{2}\ln\left(x^2+1\right)}$$

$$=e^{\frac{3}{2}x^2}e^{\ln\left(x^2+1\right)^{-3/2}}$$

$$=e^{\frac{3}{2}x^2}\left(x^2+1\right)^{-3/2}$$

$$\int \frac{6xe^{-3x^2/2}}{x^2+1} e^{\frac{3}{2}x^2} \left(x^2+1\right)^{-3/2} dx = 3\int \left(x^2+1\right)^{-5/2} d\left(x^2+1\right)$$

$$= -2\left(x^2+1\right)^{-3/2}$$

$$y(x) = e^{-3x^2/2}\left(x^2+1\right)^{3/2}\left(-2\left(x^2+1\right)^{-3/2} + C\right)$$

$$= e^{-3x^2/2}\left(-2 + C\left(x^2+1\right)^{3/2}\right)$$

$$y(0)=1 \quad \rightarrow \quad 1 = -2 + C \quad \Rightarrow \underline{C=3}$$

$$\underline{y(x) = e^{-3x^2/2}\left(-2 + 3\left(x^2+1\right)^{3/2}\right)}$$

Exercise

Solve the initial value problem: $\quad y' - 2y = e^{3x}; \quad y(0) = 3$

Solution

$$e^{\int -2dx} = e^{-2x}$$

$$\int e^{3x}e^{-2x}dx = \int e^{x}dx = e^{x}$$

$$y(x) = e^{2x}\left(e^{x} + C\right)$$

$$\underline{= e^{3x} + Ce^{2x}}$$

$$y(0)=1 \quad \rightarrow 1 = 1 + C \quad \Rightarrow \underline{C=0}$$

$$\underline{y(x) = e^{3x}}$$

Exercise

Solve the initial value problem: $\quad y' - 3y = 6; \quad y(0) = 1$

Solution

$$e^{\int -3dx} = e^{-3x}$$

$$\int 6e^{-3x}dx = -2e^{-3x}$$

$$y(x) = e^{3x}\left(-2e^{-3x} + C\right)$$

$$\underline{= -2 + Ce^{3x}}$$

$$y(0)=1 \quad \to 1=-2+C \quad \Rightarrow \boxed{C=3}$$

$$\boxed{y(x)=-2+3e^{3x}}$$

Exercise

Solve the initial value problem: $\quad 2y'+3y=e^{x}; \quad y(0)=0$

Solution

$$y'+\frac{3}{2}y=\frac{1}{2}e^{x}$$

$$e^{\int \frac{3}{2}dx}=e^{3x/2}$$

$$\int e^{x}e^{3x/2}dx=\int e^{5x/2}dx=\frac{2}{5}e^{5x/2}$$

$$y(x)=e^{-3x/2}\left(\frac{2}{5}e^{5x/2}+C\right)$$

$$=\boxed{\frac{2}{5}e^{x}+Ce^{-3x/2}}$$

$$y(0)=0 \quad \to \quad 0=\frac{2}{5}+C \quad \Rightarrow \boxed{C=-\frac{2}{5}}$$

$$\boxed{y(x)=\frac{2}{5}e^{x}-\frac{2}{5}e^{-3x/2}}$$

Exercise

Solve the initial value problem: $\quad y'+y=1+e^{-x}\cos 2x; \quad y\left(\frac{\pi}{2}\right)=0$

Solution

$$e^{\int dx}=e^{x}$$

$$\int e^{x}\left(1+e^{-x}\cos 2x\right)dx=\int \left(e^{x}+\cos 2x\right)dx=e^{x}+\frac{1}{2}\sin 2x$$

$$y(x)=e^{-x}\left(e^{x}+\frac{1}{2}\sin 2x+C\right)$$

$$=\boxed{1+\frac{1}{2}e^{-x}\sin 2x+Ce^{-x}}$$

$$y\left(\frac{\pi}{2}\right)=0 \quad \to \quad 0=1+Ce^{-\pi/2} \quad \Rightarrow \boxed{C=-e^{\pi/2}}$$

$$\boxed{y(x)=1+\frac{1}{2}e^{-x}\sin 2x-e^{-x+\pi/2}}$$

Exercise

Solve the initial value problem: $2y' + (\cos x)y = -3\cos x; \quad y(0) = -4$

Solution

$$y' + \left(\frac{1}{2}\cos x\right)y = -\frac{3}{2}\cos x$$

$$e^{\frac{1}{2}\int \cos x\, dx} = e^{\frac{1}{2}\sin x}$$

$$\int e^{\frac{1}{2}\sin x}(-3\cos x)dx = -6\int e^{\frac{1}{2}\sin x} d\left(\frac{1}{2}\sin x\right) = -6e^{\frac{1}{2}\sin x}$$

$$y(x) = e^{-\frac{1}{2}\sin x}\left(-6e^{\frac{1}{2}\sin x} + C\right)$$

$$= -6 + Ce^{-\frac{1}{2}\sin x}$$

$$y(0) = -4 \quad \rightarrow \quad -4 = -6 + C \quad \Rightarrow C = 2$$

$$y(x) = -6 + 2e^{-\frac{1}{2}\sin x}$$

Exercise

Solve the initial value problem: $y' + 2y = e^{-x} + x + 1; \quad y(-1) = e$

Solution

$$e^{\int 2dx} = e^{2x}$$

$$\int \left(e^{-x} + x + 1\right)e^{2x}dx = \int \left(e^{x} + (x+1)e^{2x}\right)dx$$

$$= e^{x} + \left(\frac{1}{2}x + \frac{1}{4}\right)e^{2x}$$

		$\int e^{2x}$
$+$	$x+1$	$\frac{1}{2}e^{2x}$
$-$	1	$\frac{1}{4}e^{2x}$

$$y(x) = e^{-2x}\left(e^{x} + \left(\frac{1}{2}x + \frac{1}{4}\right)e^{2x} + C\right)$$

$$= e^{-x} + \frac{1}{2}x + \frac{1}{4} + Ce^{-2x}$$

$$y(-1) = e \quad \rightarrow \quad e = e - \frac{1}{2} + \frac{1}{4} + Ce^{2} \quad \Rightarrow C = \frac{1}{4}e^{-2}$$

$$y(x) = e^{-x} + \frac{1}{2}x + \frac{1}{4} + \frac{1}{4}e^{-2x-2}$$

Exercise

Solve the initial value problem: $y' + \frac{y}{x} = xe^{-x}; \quad y(1) = e - 1$

Solution

$$e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int x^2 e^{-x} dx = \left(-x^2 - 2x - 2\right)e^{-x}$$

		$\int e^{-x}$
+	x^2	$-e^{-x}$
–	$2x$	e^{-x}
+	2	$-e^{-x}$

$$y(x) = \frac{1}{x}\left(-\left(x^2 + 2x + 2\right)e^{-x} + C\right)$$

$$y(1) = e - 1 \quad \to e - 1 = -5e^{-1} + C \quad \Rightarrow \boxed{C = 5e^{-1} + e - 1}$$

$$\boxed{y(x) = \frac{1}{x}\left(-\left(x^2 + 2x + 2\right)e^{-x} + 5e^{-1} + e - 1\right)}$$

Exercise

Solve the initial value problem: $y' + 4y = e^{-x}; \quad y(1) = \frac{4}{3}$

Solution

$$e^{\int 4dx} = e^{4x}$$

$$\int e^{-x}e^{4x}dx = \int e^{3x}dx = \frac{1}{3}e^{3x}$$

$$y(x) = e^{-4x}\left(\frac{1}{3}e^{3x} + C\right)$$

$$= \frac{1}{3}e^{-x} + Ce^{-4x}$$

$$y(1) = \frac{4}{3} \quad \to \quad \frac{4}{3} = \frac{1}{3}e^{-1} + Ce^{-4} \quad \Rightarrow \boxed{C = \frac{1}{3}\left(4e^4 - e^3\right)}$$

$$\boxed{y(x) = \frac{1}{3}e^{-x} + \frac{1}{3}\left(4e^4 - e^3\right)e^{-4x}}$$

Exercise

Solve the initial value problem: $x^2 y' + 3xy = x^4 \ln x + 1; \quad y(1) = 0$

Solution

$$y' + \frac{3}{x}y = x^2 \ln x + \frac{1}{x^2}$$

$$e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

$$\int\left(x^2\ln x+\frac{1}{x^2}\right)x^3\,dx=\int\left(x^5\ln x+x\right)dx \qquad \begin{matrix} u=\ln x & dv=x^5 \\ du=\frac{1}{x} & v=\frac{1}{6}x^6 \end{matrix}$$

$$=\frac{1}{6}x^6\ln x-\frac{1}{6}\int x^5dx+\frac{1}{2}x^2$$

$$=\frac{1}{6}x^6\ln x-\frac{1}{36}x^6+\frac{1}{2}x^2$$

$$y(x)=\frac{1}{x^3}\left(\frac{1}{6}x^6\ln x-\frac{1}{36}x^6+\frac{1}{2}x^2+C\right)$$

$$=\frac{1}{6}x^3\ln x-\frac{1}{36}x^3+\frac{1}{2x}+\frac{C}{x^3}$$

$$y(1)=0 \quad \rightarrow 0=-\frac{1}{36}+\frac{1}{2}+C\Rightarrow C=-\frac{17}{36}$$

$$\boxed{y(x)=\frac{1}{6}x^3\ln x-\frac{1}{36}x^3+\frac{1}{2x}-\frac{17}{36x^3}}$$

Exercise

Find the solution of the initial value problem $\quad y'+\frac{3}{x}y=3x-2 \quad y(1)=1$

Solution

$$e^{\int\frac{3}{x}dx}=e^{3\ln x}=x^3$$

$$\int(3x-2)x^3dx=\int\left(3x^4-2x^3\right)dx=\frac{3}{5}x^5-\frac{1}{2}x^4$$

$$y(x)=\frac{1}{x^3}\left(\frac{3}{5}x^5-\frac{1}{2}x^4+C\right)$$

$$\boxed{=\frac{3}{5}x^2-\frac{1}{2}x+\frac{C}{x^3}}$$

$$y(1)=1 \quad \rightarrow 1=\frac{3}{5}-\frac{1}{2}+C \quad \Rightarrow \boxed{C=\frac{9}{10}}$$

$$\boxed{y(x)=\frac{3}{5}x^2-\frac{1}{2}x+\frac{9}{10}x^{-3}}$$

Exercise

Find the solution of the initial value problem $\quad (\cos x)y'+y\sin x=2x\cos^2 x \quad y\left(\frac{\pi}{4}\right)=\frac{-15\sqrt{2}\pi^2}{32}$

Solution

$$y'+(\tan x)y=2x\cos x$$

$$e^{\int\tan x dx}=e^{\ln\sec x}=\sec x$$

$$\int 2x\cos x(\sec x)dx = \int 2x\ dx = x^2$$

$$y(x) = \frac{1}{\sec x}\left(x^2 + C\right)$$

$$= \cos x\left(x^2 + C\right)$$

$$y\left(\frac{\pi}{4}\right) = \frac{-15\sqrt{2}\pi^2}{32} \quad \rightarrow \quad \frac{-15\sqrt{2}\pi^2}{32} = \frac{\sqrt{2}}{2}\left(\frac{\pi^2}{16} + C\right)$$

$$\Rightarrow\ C = \frac{-15\sqrt{2}\pi^2}{32} - \frac{\sqrt{2}\pi^2}{32} = -\frac{\sqrt{2}\pi^2}{2}$$

$$y(x) = \cos x\left(x^2 - \frac{\sqrt{2}\pi^2}{2}\right)$$

Exercise

Find the solution of the initial value problem $(\cos x)y' + (\sin x)y = 2\cos^3 x\sin x - 1 \quad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

Solution

$$y' + (\tan x)y = 2\cos^2 x\sin x - \sec x$$

$$e^{\int \tan x dx} = e^{\ln|\sec x|} = \sec x$$

$$\int \left(2\cos^2 x\sin x - \sec x\right)\sec x\ dx = \int \left(2\cos x\sin x - \sec^2 x\right)dx$$

$$= \int \left(\sin 2x - \sec^2 x\right)dx$$

$$= -\frac{1}{2}\cos 2x - \tan x$$

$$y(x) = \frac{1}{\sec x}\left(-\frac{1}{2}\cos 2x - \tan x + C\right)$$

$$y(x) = -\frac{1}{2}\cos 2x\cos x - \sin x + C\cos x \qquad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$$

$$3\sqrt{2} = -\frac{\sqrt{2}}{2} + C\frac{\sqrt{2}}{2} \quad \rightarrow \quad C = 7$$

$$y(x) = -\frac{1}{2}\cos 2x\cos x - \sin x + 7\cos x$$

Exercise

Find the solution of the initial value problem $t\,y' + 2y = t^2 - t + 1 \quad y(1) = \frac{1}{2}$

Solution

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

$$e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = t^2$$

$$\int \left(t-1+\frac{1}{t}\right)t^2 dt = \int \left(t^3-t^2+t\right)dt = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2$$

$$y(t) = \frac{1}{t^2}\left(\frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + C\right)$$

$$= \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{C}{t^2} \qquad y(1) = \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C \quad \to \quad C = \frac{1}{12}$$

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{2t^2}$$

Exercise

Find the solution of the initial value problem $\quad t\,y' - 2y = t^5 \sin 2t - t^3 + 4t^4 \quad y(\pi) = \frac{3}{2}\pi^4$

Solution

$$y' - \frac{2}{t}y = t^4 \sin 2t - t^2 + 4t^3$$

$$e^{\int -\frac{2}{t}dt} = e^{-2\ln|t|} = t^{-2}$$

$$\int \left(t^4 \sin 2t - t^2 + 4t^3\right)\frac{1}{t^2}dt = \int \left(t^2 \sin 2t - 1 + 4t\right)dt$$

		$\int \sin 2t$
+	t^2	$-\frac{1}{2}\cos 2t$
−	$2t$	$-\frac{1}{4}\sin 2t$
+	2	$\frac{1}{8}\cos 2t$

$$= -\frac{1}{2}t^2 \cos 2t + \frac{1}{2}t \sin 2t + \frac{1}{4}\cos 2t - t + 2t^2$$

$$y(t) = t^2\left(-\frac{1}{2}t^2 \cos 2t + \frac{1}{2}t \sin 2t + \frac{1}{4}\cos 2t - t + 2t^2 + C\right)$$

$$= -\frac{1}{2}t^4 \cos 2t + \frac{1}{2}t^3 \sin 2t + \frac{1}{4}t^2 \cos 2t - t^3 + 2t^4 + Ct^2 \qquad y(\pi) = \frac{3}{2}\pi^4$$

$$\frac{3}{2}\pi^4 = -\frac{1}{2}\pi^4 + \frac{1}{4}\pi^2 - \pi^3 + 2\pi^4 + \pi^2 C$$

$$C = \pi - \frac{1}{4}$$

$$y(t) = -\frac{1}{2}t^4 \cos 2t + \frac{1}{2}t^3 \sin 2t + \frac{1}{4}t^2 \cos 2t - t^3 + 2t^4 + \left(\pi - \frac{1}{4}\right)t^2$$

Exercise

Find the solution of the initial value problem $\quad 2y' - y = 4\sin 3t \quad y(0) = y_0$

Solution

$$y' - \frac{1}{2}y = 2\sin 3t$$

$$e^{\int -\frac{1}{2}dt} = e^{-\frac{t}{2}}$$

$$\int 2e^{-t/2}\sin 3t \ dt = e^{-t/2}\left(-\frac{2}{3}\cos 3t - \frac{1}{9}\sin 3t\right) - \frac{1}{18}\int e^{-t/2}\sin 3t \ dt$$

$$\frac{37}{18}\frac{1}{2}\int 2e^{-t/2}\sin 3t \ dt = e^{-t/2}\left(-\frac{2}{3}\cos 3t - \frac{1}{9}\sin 3t\right)$$

$$\int 2e^{-t/2}\sin 3t \ dt = \frac{36}{37}e^{-t/2}\left(-\frac{2}{3}\cos 3t - \frac{1}{9}\sin 3t\right)$$

$$= \left(-\frac{24}{37}\cos 3t - \frac{4}{37}\sin 3t\right)e^{-t/2}$$

		$\int \sin 3t$
$+$	$2e^{-t/2}$	$-\frac{1}{3}\cos 3t$
$-$	$-e^{-t/2}$	$-\frac{1}{9}\sin 3t$
$+$	$\frac{1}{2}e^{-t/2}$	

$$y(t) = e^{t/2}\left(\left(-\frac{24}{37}\cos 3t - \frac{4}{37}\sin 3t\right)e^{-t/2} + C\right)$$

$$\boxed{= -\frac{24}{37}\cos 3t - \frac{4}{37}\sin 3t + Ce^{t/2}} \qquad y(0) = y_0$$

$$y_0 = -\frac{24}{37} + C \quad \rightarrow \quad \boxed{C = y_0 + \frac{24}{37}}$$

$$\boxed{y(t) = -\frac{24}{37}\cos 3t - \frac{4}{37}\sin 3t + \left(y_0 + \frac{27}{37}\right)e^{t/2}}$$

Exercise

Find the solution of the initial value problem $\quad y' + 2y = 2 - e^{-4t} \quad y(0) = 1$

Solution

$$e^{\int 2dt} = e^{2t}$$

$$\int \left(2 - e^{-4t}\right)e^{2t}dt = \int \left(2e^{2t} - e^{-2t}\right)dt = e^{2t} + \frac{1}{2}e^{-2t}$$

$$y(t) = \frac{1}{e^{2t}}\left(e^{2t} + \frac{1}{2}e^{-2t} + C\right)$$

$$= 1 + \frac{1}{2}e^{-4t} + Ce^{-2t}$$

$$y(0) = 1 \quad \rightarrow \ 1 = 1 + \frac{1}{2} + C \quad \Rightarrow \ \boxed{C = -\frac{1}{2}}$$

$$\boxed{y(t) = 1 + \frac{1}{2}e^{-4t} - \frac{1}{2}e^{-2t}}$$

Exercise

Find the solution of the initial value problem $y' - y = -\frac{1}{2}e^{t/2}\sin 5t + 5e^{t/2}\cos 5t \quad y(0) = 0$

Solution

$$e^{\int -dt} = e^{-t}$$

$$\int \left(-\frac{1}{2}e^{t/2}\sin 5t + 5e^{t/2}\cos 5t\right)e^{-t}dt = -\frac{1}{2}\int \left(e^{-t/2}\sin 5t\right)dt + 5\int \left(e^{-t/2}\cos 5t\right)dt$$

		$\int \sin 5t$
+	$e^{-t/2}$	$-\frac{1}{5}\cos 5t$
−	$-\frac{1}{2}e^{-t/2}$	$-\frac{1}{25}\sin 5t$
+	$\frac{1}{4}e^{-t/2}$	

		$\int \cos 5t$
+	$e^{-t/2}$	$\frac{1}{5}\sin 5t$
−	$-\frac{1}{2}e^{-t/2}$	$-\frac{1}{25}\cos 5t$
+	$\frac{1}{4}e^{-t/2}$	

$$\int \left(e^{-t/2}\sin 5t\right)dt = \left(-\frac{1}{5}\cos 5t - \frac{1}{50}\sin 5t\right)e^{-t/2} - \frac{1}{100}\int \left(e^{-t/2}\sin 5t\right)dt$$

$$\frac{101}{100}\int \left(e^{-t/2}\sin 5t\right)dt = -\frac{1}{50}(10\cos 5t + \sin 5t)e^{-t/2}$$

$$\int \left(e^{-t/2}\sin 5t\right)dt = -\frac{2}{101}(10\cos 5t + \sin 5t)e^{-t/2}$$

$$\int \left(e^{-t/2}\cos 5t\right)dt = e^{-t/2}\left(\frac{1}{5}\sin 5t - \frac{1}{50}\cos 5t\right) - \frac{1}{100}\int \left(e^{-t/2}\cos 5t\right)dt$$

$$\frac{101}{100}\int \left(e^{-t/2}\cos 5t\right)dt = \frac{1}{50}e^{-t/2}(10\sin 5t - \cos 5t)$$

$$\int \left(e^{-t/2}\cos 5t\right)dt = \frac{2}{101}e^{-t/2}(10\sin 5t - \cos 5t)$$

$$\int \left(-\frac{1}{2}e^{t/2}\sin 5t + 5e^{t/2}\cos 5t\right)e^{-t}dt = \left(\frac{10}{101}\cos 5t + \frac{1}{101}\sin 5t + \frac{100}{101}\sin 5t - \frac{10}{101}\cos 5t\right)e^{-t/2}$$

$$= e^{-t/2}\sin 5t$$

$$y(t) = e^{t}\left(e^{-t/2}\sin 5t + C\right)$$

$$= e^{t/2}\sin 5t + Ce^{t}$$

$$y(0) = 0 \quad \rightarrow \quad \underline{C = 0}$$

$$\boxed{y(t) = e^{t/2}\sin 5t}$$

Exercise

Find the solution of the initial value problem $\quad y' + 2y = 3\ ; \quad y(0) = -1$

Solution

$$e^{\int 2dt} = e^{2t}$$

$$\int 3e^{2t}dt = \frac{3}{2}e^{2t}$$

$$y(t) = \frac{1}{e^{2t}}\left(\frac{3}{2}e^{2t} + C\right)$$

$$= \frac{3}{2} + Ce^{-2t}$$

$$y(0) = -1 \quad \rightarrow \quad \frac{3}{2} + C = -1 \quad \Rightarrow C = -\frac{5}{2}$$

$$y(t) = \frac{3}{2} - \frac{5}{2}e^{-2t}$$

Exercise

Find the solution of the initial value problem $\quad y' + (\cos t)y = \cos t\ ; \quad y(\pi) = 2$

Solution

$$e^{\int \cos t\, dt} = e^{\sin t}$$

$$\int (\cos t)e^{\sin t}dt = \int e^{\sin t}d(\sin t) = e^{\sin t}$$

$$y(t) = \frac{1}{e^{\sin t}}\left(e^{\sin t} + C\right)$$

$$= 1 + Ce^{-\sin t}$$

$$y(\pi) = 2 \quad \rightarrow \quad 1 + C = 2 \quad \Rightarrow C = 1$$

$$y(t) = 1 + e^{-\sin t}$$

Exercise

Find the solution of the initial value problem $\quad y' + 2ty = 2t\ ; \quad y(0) = 1$

Solution

$$e^{\int 2tdt} = e^{t^2}$$

$$\int (2t)e^{t^2}dt = \int e^{t^2}d\left(t^2\right) = e^{t^2}$$

$$y(t) = \frac{1}{e^{t^2}}\left(e^{t^2} + C\right)$$

$$= 1 + Ce^{-t^2}$$

$$y(0) = 1 \quad \rightarrow \; 1 + C = 1 \quad \Rightarrow C = 0$$

$$y(t) = 1$$

Exercise

Find the solution of the initial value problem $\quad y' + y = \frac{e^{-t}}{t^2}$; $\; y(1) = 0$

Solution

$$e^{\int dt} = e^t$$

$$\int \left(e^t\right)\frac{e^{-t}}{t^2}dt = \int t^{-2}dt = -\frac{1}{t}$$

$$y(t) = \frac{1}{e^t}\left(-\frac{1}{t} + C\right)$$

$$y(1) = 0 \quad \rightarrow \; \frac{1}{e}(-1 + C) = 0 \quad \Rightarrow C = 1$$

$$y(t) = \frac{1}{e^t}\left(-\frac{1}{t} + 1\right)$$

Exercise

Find the solution of the initial value problem $\quad ty' + 2y = \sin t$; $\; y(\pi) = \frac{1}{\pi}$

Solution

$$y' + \frac{2}{t}y = \frac{\sin t}{t}$$

$$e^{\int \frac{2}{t}dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$\int \left(t^2\right)\frac{\sin t}{t}dt = \int (t\sin t)dt = -t\cos t + \sin t$$

		$\int \sin t$
+	t	$-\cos t$
−	1	$-\sin t$

$$y(t) = \frac{1}{t^2}(\sin t - t\cos t + C)$$

$y(\pi)=\frac{1}{\pi} \quad \to \quad \frac{1}{\pi^2}(\pi+C)=\frac{1}{\pi} \quad \Rightarrow \underline{C=0}$

$$\boxed{y(t)=\frac{1}{t^2}(\sin t-t\cos t)}$$

Exercise

Find the solution of the initial value problem $\quad y'+\frac{2}{t}y=\frac{\cos t}{t^2}\ ;\quad y(\pi)=0$

Solution

$$e^{\int \frac{2}{t}dt}=e^{2\ln t}=e^{\ln t^2}=t^2$$

$$\int \left(t^2\right)\frac{\cos t}{t^2}dt=\int (\cos t)dt=\sin t$$

$$\boxed{y(t)=\frac{1}{t^2}(\sin t+C)}$$

$y(\pi)=0 \quad \to \quad \frac{1}{\pi^2}(C)=0 \quad \Rightarrow \underline{C=0}$

$$\boxed{y(t)=\frac{\sin t}{t^2}}$$

Exercise

Find the solution of the initial value problem $\quad (\sin t)y'+(\cos t)y=0\ ;\quad y\left(\frac{3\pi}{4}\right)=2$

Solution

$$y'+(\cot t)y=0$$

$$e^{\int (\cot t)dt}=e^{\ln(\sin t)}=\sin t$$

$$\boxed{y(t)=\frac{C}{\sin t}}$$

$y\left(\frac{3\pi}{4}\right)=2 \quad \to \quad C\left(\sqrt{2}\right)=2 \quad \Rightarrow \underline{C=\sqrt{2}}$

$$\boxed{y(t)=\sqrt{2}\csc t}$$

Exercise

Find the solution of the initial value problem $\quad y'+3t^2y=t^2\ ;\quad y(0)=2$

Solution

$$e^{\int 3t^2 dt} = e^{t^3}$$

$$\int \left(t^2\right) e^{t^3} dt = \frac{1}{3}\int e^{t^3} d\left(t^3\right) = \frac{1}{3}e^{t^3}$$

$$y(t) = \frac{1}{e^{t^3}}\left(\frac{1}{3}e^{t^3} + C\right)$$

$$= \frac{1}{3} + Ce^{-t^3}$$

$$y(0) = 2 \quad \to \quad \frac{1}{3} + C = 2 \quad \Rightarrow C = \frac{5}{3}$$

$$y(t) = \frac{1}{3} + \frac{5}{3}e^{-t^3}$$

Exercise

Find the solution of the initial value problem $\quad ty' + y = t\sin t\ ;\quad y(\pi) = -1$

Solution

$$y' + \frac{1}{t}y = \sin t$$

$$e^{\int \frac{1}{t}dt} = e^{\ln t} = t$$

$$\int t\sin t\ dt = \sin t - t\cos t$$

		$\int \sin t$
+	t	$-\cos t$
–	1	$-\sin t$

$$y(t) = \frac{1}{t}(\sin t - t\cos t + C)$$

$$y(\pi) = -1 \quad \to \quad \frac{1}{\pi}(\pi + C) = -1 \quad \Rightarrow C = -2\pi$$

$$y(t) = \frac{1}{t}(\sin t - t\cos t - 2\pi)$$

Exercise

Find the solution of the initial value problem $\quad y' + y = \sin t\ ;\quad y(\pi) = 1$

Solution

$$e^{\int dt} = e^t$$

$$\int e^t \sin t\ dt = e^t(-\cos t + \sin t) - \int e^t \sin t\ dt$$

$$2\int e^t \sin t\ dt = e^t(-\cos t + \sin t)$$

		$\int \sin t$
+	e^t	$-\cos t$
–	e^t	$-\sin t$
+	e^t	

$$\int e^t \sin t\ dt = \frac{1}{2}e^t\left(-\cos t + \sin t\right)$$

$$y(t) = \frac{1}{e^t}\left(\frac{1}{2}e^t\left(\sin t - \cos t\right) + C\right)$$

$$= \frac{1}{2}\left(\sin t - \cos t\right) + Ce^{-t}$$

$$y(\pi) = 1 \quad \rightarrow \quad \frac{1}{2} + Ce^{-\pi} = 1 \quad \Rightarrow C = \frac{1}{2}e^{\pi}$$

$$y(t) = \frac{1}{2}\left(\sin t - \cos t\right) + \frac{1}{2}e^{\pi}e^{-t}$$

Exercise

Find the solution of the initial value problem $y' + y = \cos 2t\ ;\quad y(0) = 5$

Solution

$$e^{\int dt} = e^t$$

		$\int \cos 2t$
+	e^t	$\frac{1}{2}\sin 2t$
−	e^t	$-\frac{1}{4}\cos 2t$
+	e^t	

$$\int e^t \cos 2t\ dt = e^t\left(\frac{1}{2}\sin 2t + \frac{1}{4}\cos 2t\right) - \frac{1}{4}\int e^t \cos 2t\ dt$$

$$\left(1 + \frac{1}{4}\right)\int e^t \cos 2t\ dt = e^t\left(\frac{1}{2}\sin 2t + \frac{1}{4}\cos 2t\right)$$

$$\frac{5}{4}\int e^t \cos 2t\ dt = \frac{1}{4}e^t\left(2\sin 2t + \cos 2t\right)$$

$$\int e^t \cos 2t\ dt = \frac{1}{5}e^t\left(2\sin 2t + \cos 2t\right)$$

$$y(t) = \frac{1}{e^t}\left(\frac{1}{5}e^t\left(2\sin 2t + \cos 2t\right) + C\right)$$

$$= \frac{1}{5}\left(2\sin 2t + \cos 2t\right) + Ce^{-t}$$

$$y(0) = 5 \quad \rightarrow \quad \frac{1}{5} + C = 5 \quad \Rightarrow C = \frac{24}{5}$$

$$y(t) = \frac{1}{5}\left(2\sin 2t + \cos 2t\right) + \frac{25}{5}e^{-t}$$

Exercise

Find the solution of the initial value problem $y' + 3y = \cos 2t\ ;\quad y(0) = -1$

Solution

$$e^{\int 3dt} = e^{3t}$$

$$\int e^{3t}\cos 2t\ dt = e^{3t}\left(\tfrac{1}{2}\sin 2t + \tfrac{3}{4}\cos 2t\right) - \tfrac{9}{4}\int e^{3t}\cos 2t\ dt$$

$$\left(1+\tfrac{9}{4}\right)\int e^{t}\cos 2t\ dt = e^{t}\left(\tfrac{1}{2}\sin 2t + \tfrac{3}{4}\cos 2t\right)$$

		$\int \cos 2t$
$+$	e^{3t}	$\frac{1}{2}\sin 2t$
$-$	$3e^{3t}$	$-\frac{1}{4}\cos 2t$
$+$	$9e^{3t}$	

$$\tfrac{13}{4}\int e^{t}\cos 2t\ dt = \tfrac{1}{4}e^{t}\left(2\sin 2t + 3\cos 2t\right)$$

$$\int e^{t}\cos 2t\ dt = \tfrac{1}{13}e^{t}\left(2\sin 2t + 3\cos 2t\right)$$

$$y(t) = \frac{1}{e^{3t}}\left(\frac{1}{13}e^{3t}\left(2\sin 2t + 3\cos 2t\right) + C\right)$$

$$\boxed{= \frac{1}{13}\left(2\sin 2t + 3\cos 2t\right) + Ce^{-3t}}$$

$$y(0) = -1 \quad \rightarrow \quad \frac{3}{13} + C = -1 \quad \Rightarrow \boxed{C = -\frac{16}{13}}$$

$$\boxed{y(t) = \frac{1}{13}\left(2\sin 2t + 3\cos 2t\right) - \frac{16}{13}e^{-3t}}$$

Exercise

Find the solution of the initial value problem $\quad y' - 2y = 7e^{2t}\ ;\quad y(0) = 3$

Solution

$$e^{\int -2dt} = e^{-2t}$$

$$\int 7e^{2t}e^{-2t}\ dt = 7\int dt = 7t$$

$$y(t) = \frac{1}{e^{-2t}}(7t + C)$$

$$\boxed{= e^{2t}(7t + C)}$$

$$y(0) = 3 \quad \rightarrow \quad \boxed{C = 3}$$

$$\boxed{y(t) = e^{2t}(7t + 3)}$$

Exercise

Find the solution of the initial value problem $\quad y' - 2y = 3e^{-2t}\ ;\quad y(0) = 10$

Solution

$$e^{\int -2dt} = e^{-2t}$$

$$\int 3e^{-2t}e^{-2t}\,dt = 3\int e^{-4t}\,dt = -\frac{3}{4}e^{-4t}$$

$$y(t) = \frac{1}{e^{-2t}}(7t + C)$$

$$= e^{2t}(7t + C)$$

$$y(0) = 3 \quad \to \quad C = 3$$

$$y(t) = e^{2t}(7t + 3)$$

Exercise

Find the solution of the initial value problem $\quad y' + 2y = t^2 + 2t + 1 + e^{4t}\ ; \quad y(0) = 0$

Solution

$$e^{\int 2dt} = e^{2t}$$

$$\int e^{2t}\left(t^2 + 2t + 1 + e^{4t}\right)dt = \int \left(t^2 + 2t + 1\right)e^{2t}\,dt + \int e^{6t}\,dt$$

$$= \left(\frac{1}{2}t^2 + t + \frac{1}{2} - \frac{1}{2}t - \frac{1}{2} + \frac{1}{4}\right)e^{2t} + \frac{1}{6}e^{6t}$$

$$y(t) = \frac{1}{e^{2t}}\left(\left(\frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{4}\right)e^{2t} + \frac{1}{6}e^{6t} + C\right)$$

$$= \left(\frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{4}\right)e^{2t} + \frac{1}{6}e^{4t} + Ce^{-2t}$$

$$y(0) = 0 \quad \to \quad \frac{1}{4} + \frac{1}{6} + C = 0 \quad \Rightarrow C = -\frac{5}{12}$$

$$y(t) = \left(\frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{4}\right)e^{2t} + \frac{1}{6}e^{4t} - \frac{5}{12}e^{-2t}$$

		$\int e^{2t}$
+	$t^2 + 2t + 1$	$\frac{1}{2}e^{2t}$
–	$2t + 2$	$\frac{1}{4}e^{2t}$
+	2	$\frac{1}{8}e^{2t}$

Exercise

Find the solution of the initial value problem $\quad y' - 3y = 2t - e^{4t}\ ; \quad y(0) = 0$

Solution

$$e^{\int -3dt} = e^{-3t}$$

$$\int e^{-3t}\left(2t - e^{4t}\right)dt = \int 2te^{-3t}\,dt - \int e^{t}\,dt$$

$$= \left(-\frac{2}{3}t - \frac{2}{9}\right)e^{-3t} - e^{t}$$

		$\int e^{-3t}$
+	$2t$	$-\frac{1}{3}e^{-3t}$
–	2	$\frac{1}{9}e^{-3t}$

$$y(t)=\frac{1}{e^{-3t}}\left(\left(-\frac{2}{3}t-\frac{2}{9}\right)e^{-3t}-e^{t}+C\right)$$

$$=-\frac{2}{3}t-\frac{2}{9}-e^{4t}+Ce^{3t}$$

$$y(0)=0 \quad \to \quad -\frac{2}{9}-1+C=0 \quad \Rightarrow C=\frac{11}{9}$$

$$y(t)=-\frac{2}{3}t-\frac{2}{9}-e^{4t}+\frac{11}{9}e^{3t}$$

Exercise

Find the solution of the initial value problem $y'+y=t^3+\sin 3t\ ;\quad y(0)=0$

Solution

$$e^{\int dt}=e^{t}$$

$$\int e^{t}\left(t^3+\sin 3t\right)\,dt=\int t^3e^{t}\,dt+\int e^{t}\sin 3t\,dt$$

		$\int \sin 3t$
+	e^t	$-\frac{1}{3}\cos 3t$
−	e^t	$-\frac{1}{9}\sin 3t$
+	e^t	

$$\int e^{t}\sin 3t\,dt=e^{t}\left(-\frac{1}{3}\cos 3t+\frac{1}{9}\sin 3t\right)-\frac{1}{9}\int e^{t}\sin 3t\,dt$$

$$\left(1+\frac{1}{9}\right)\int e^{t}\sin 3t\,dt=\frac{1}{9}e^{t}\left(\sin 3t-3\cos 3t\right)$$

$$\frac{10}{9}\int e^{t}\sin 3t\,dt=\frac{1}{9}e^{t}\left(\sin 3t-3\cos 3t\right)$$

$$\int e^{t}\sin 3t\,dt=\frac{1}{10}e^{t}\left(\sin 3t-3\cos 3t\right)$$

		$\int e^t$
+	t^3	e^t
−	$3t^2$	e^t
+	$6t$	e^t
−	6	e^t

$$\int e^{t}\left(t^3+\sin 3t\right)\,dt=\int t^3e^{t}\,dt+\int e^{t}\sin 3t\,dt$$

$$=\left(t^3-3t^2+6t-6\right)e^{t}+\frac{1}{10}e^{t}\left(\sin 3t-3\cos 3t\right)$$

$$=\left(t^3-3t^2+6t-6+\frac{1}{10}\sin 3t-\frac{3}{10}\cos 3t\right)e^{t}$$

$$y(t)=\frac{1}{e^{t}}\left(\left(t^3-3t^2+6t-6+\frac{1}{10}\sin 3t-\frac{3}{10}\cos 3t\right)e^{t}+C\right)$$

$$=t^3-3t^2+6t-6+\frac{1}{10}\sin 3t-\frac{3}{10}\cos 3t+Ce^{-t}$$

$$y(0)=0 \quad \to \quad -6-\frac{3}{10}+C=0 \quad \Rightarrow C=\frac{63}{10}$$

$$y(t)=t^3-3t^2+6t-6+\frac{1}{10}\sin 3t-\frac{3}{10}\cos 3t+\frac{63}{10}e^{-t}$$

Exercise

Find the solution of the initial value problem $y' + 2y = \cos 2t + 3\sin 2t + e^{-t}$; $y(0) = 0$

Solution

$$e^{\int 2dt} = e^{2t}$$

$$\int e^{2t}\left(\cos 2t + 3\sin 2t + e^{-t}\right) dt = \int e^{2t}\cos 2t\ dt + 3\int e^{2t}\sin 2t\ dt + \int e^{t} dt$$

$$\int e^{2t}\cos 2t\ dt = e^{2t}\left(\frac{1}{2}\sin 2t + \frac{1}{2}\cos 2t\right) - \int e^{2t}\cos 2t\ dt$$

		$\int \cos 2t$
+	e^{2t}	$\frac{1}{2}\sin 2t$
−	$2e^{2t}$	$-\frac{1}{4}\cos 2t$
+	$4e^{2t}$	

$$2\int e^{2t}\cos 2t\ dt = \frac{1}{2}e^{2t}\left(\sin 2t + \cos 2t\right)$$

$$\int e^{2t}\cos 2t\ dt = \boxed{\frac{1}{4}e^{2t}\left(\sin 2t + \cos 2t\right)}$$

$$\int e^{2t}\sin 2t\ dt = e^{2t}\left(-\frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t\right) - \int e^{2t}\sin 2t\ dt$$

		$\int \sin 2t$
+	e^{2t}	$-\frac{1}{2}\cos 2t$
−	$2e^{2t}$	$-\frac{1}{4}\sin 2t$
+	$4e^{2t}$	

$$2\int e^{2t}\sin 2t\ dt = \frac{1}{2}e^{2t}\left(\sin 2t - \cos 2t\right)$$

$$\int e^{2t}\sin 2t\ dt = \boxed{\frac{1}{4}e^{2t}\left(\sin 2t - \cos 2t\right)}$$

$$\begin{aligned}\int e^{2t}\left(\cos 2t + 3\sin 2t + e^{-t}\right) dt &= \int e^{2t}\cos 2t\ dt + 3\int e^{2t}\sin 2t\ dt + \int e^{t} dt \\ &= \frac{1}{4}e^{2t}\left(\sin 2t + \cos 2t\right) + \frac{3}{4}e^{2t}\left(\sin 2t - \cos 2t\right) + e^{t} \\ &= \frac{1}{4}e^{2t}\left(\sin 2t + \cos 2t + 3\sin 2t - 3\cos 2t\right) + e^{t} \\ &= \frac{1}{4}e^{2t}\left(4\sin 2t - 2\cos 2t\right) + e^{t} \\ &= e^{2t}\left(\sin 2t - \frac{1}{2}\cos 2t\right) + e^{t}\end{aligned}$$

$$\begin{aligned}y(t) &= \frac{1}{e^{2t}}\left(e^{2t}\left(\sin 2t - \frac{1}{2}\cos 2t\right) + e^{t} + C\right) \\ &= \boxed{\sin 2t - \frac{1}{2}\cos 2t + e^{-t} + Ce^{-2t}}\end{aligned}$$

$$y(0) = 0 \quad \to \quad -\frac{1}{2} + 1 + C = 0 \quad \Rightarrow \boxed{C = -\frac{1}{2}}$$

$$\boxed{y(t) = \sin 2t - \frac{1}{2}\cos 2t + e^{-t} - \frac{1}{2}e^{-2t}}$$

Exercise

Find the solution of the initial value problem $y' + y = e^{3t}$; $y(0) = y_0$

Solution

$$e^{\int dt} = e^t$$

$$\int e^t e^{3t}\, dt = \int e^{4t} dt = \frac{1}{4}e^{4t}$$

$$y(t) = \frac{1}{e^t}\left(\frac{1}{4}e^{4t} + C\right)$$

$$= \frac{1}{4}e^{3t} + Ce^{-t}$$

$$y(0) = y_0 \quad \to \quad \frac{1}{4} + C = y_0 \quad \Rightarrow C = y_0 - \frac{1}{4}$$

$$\boxed{y(t) = \frac{1}{4}e^{3t} + \left(y_0 - \frac{1}{4}\right)e^{-t}}$$

Exercise

Find the solution of the initial value problem $t^2 y' - ty = 1$; $y(1) = y_0$

Solution

$$y' - \frac{1}{t}y = \frac{1}{t^2}$$

$$e^{\int -\frac{1}{t}dt} = e^{-\ln t} = \frac{1}{t}$$

$$\int \frac{1}{t}\frac{1}{t^2}\, dt = \int t^{-3} dt = -\frac{1}{2}t^{-2}$$

$$y(t) = t\left(-\frac{1}{2t^2} + C\right)$$

$$= -\frac{1}{2t} + Ct$$

$$y(1) = y_0 \quad \to \quad -\frac{1}{2} + C = y_0 \quad \Rightarrow C = y_0 + \frac{1}{2}$$

$$\boxed{y(t) = -\frac{1}{2t} + \left(y_0 + \frac{1}{2}\right)t}$$

Exercise

Find the solution of the initial value problem $y' + ay = e^{at}$; $y(0) = y_0$, $a \neq 0$

Solution

$$e^{\int a\,dt} = e^{at}$$

$$\int e^{at}e^{at}\,dt = \int e^{2at}dt = \frac{1}{2a}e^{2at}$$

$$y(t) = \frac{1}{e^{at}}\left(\frac{1}{2a}e^{2at} + C\right)$$

$$= \frac{1}{2a}e^{at} + Ce^{-at}$$

$$y(0) = y_0 \quad \to \quad \frac{1}{2a} + C = y_0 \quad \Rightarrow C = y_0 - \frac{1}{2a}$$

$$y(t) = \frac{1}{2a}e^{at} + \left(y_0 - \frac{1}{2a}\right)e^{-at}$$

Exercise

Find the solution of the initial value problem $3y' + 12y = 4$; $y(0) = y_0$

Solution

$$y' + 4y = \frac{4}{3}$$

$$e^{\int 4dt} = e^{4t}$$

$$\int \frac{4}{3}e^{4t}\,dt = \frac{1}{3}e^{4t}$$

$$y(t) = \frac{1}{e^{4t}}\left(\frac{1}{3}e^{4t} + C\right)$$

$$= \frac{1}{3} + Ce^{-4t}$$

$$y(0) = y_0 \quad \to \quad \frac{1}{3} + C = y_0 \quad \Rightarrow C = y_0 - \frac{1}{3}$$

$$y(t) = \frac{1}{3} + \left(y_0 - \frac{1}{3}\right)e^{-4t}$$

Exercise

Find a solution to the initial value problem that is continuous on the given interval $[a,\ b]$

$$y' + \frac{1}{x}y = f(x), \quad y(1) = 1 \qquad\qquad f(x) = \begin{cases} 3x, & 1 \le x \le 2 \\ 0, & 2 < x \le 3 \end{cases} \qquad [a,\ b] = [1,\ 3]$$

Solution

For $1 \le x \le 2$:

$$y' + \frac{1}{x}y = 3x, \quad y(1) = 1$$

$$e^{\int \frac{1}{x}dx} = e^{\ln x} = \underline{x}$$

$$\int 3x^2 dx = x^3$$

$$y(x) = \frac{1}{x}\left(x^3 + C\right)$$

$$= x^2 + \frac{C}{x}$$

$$y(1) = 1 \quad \rightarrow 1 = 1 + C \ \Rightarrow \underline{C = 0}$$

$$\boxed{y(x) = x^2}$$

For $2 \le x \le 3$:

$$y' + \frac{1}{x}y = 0 \qquad x = 2 \ \Rightarrow \ y = 4$$

$$y(x) = \frac{C}{x}$$

$$y(2) = 4 \quad \rightarrow 4 = \frac{C}{2} \ \Rightarrow \ \underline{C = 8}$$

$$\boxed{y(x) = \frac{8}{x}}$$

Exercise

Find a solution to the initial value problem that is continuous on the given interval $[a,\ b]$

$$y' + (\sin x)y = f(x), \quad y(0) = 3 \qquad\qquad f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ -\sin x, & \pi < x \le 2\pi \end{cases} \qquad [a,\ b] = [0,\ 2\pi]$$

Solution

For $0 \le x \le \pi$:

$$y' + (\sin x)y = \sin x, \quad y(0) = 3$$

$$e^{\int \sin x dx} = e^{-\cos x}$$

$$\int \sin x\, e^{-\cos x} dx = \int e^{-\cos x} d(-\cos x) = e^{-\cos x}$$

$$y(x)=e^{\cos x}\left(e^{-\cos x}+C\right)$$

$$=1+Ce^{\cos x}$$

$$y(0)=3 \quad \rightarrow\ 3=1+Ce \ \Rightarrow \boxed{C=2e^{-1}}$$

$$\boxed{y(x)=1+2e^{\cos x-1}} \qquad y(\pi)=1+2e^{-2}$$

For $\pi \le x \le 2\pi$:

$$y'+(\sin x)y=-\sin x \qquad y(\pi)=1+2e^{-2}$$

$$e^{\int \sin x dx}=e^{-\cos x}$$

$$\int -\sin x\, e^{-\cos x}dx=\int -e^{-\cos x}d(-\cos x)=-e^{-\cos x}$$

$$y(x)=e^{\cos x}\left(-e^{-\cos x}+C\right)$$

$$=-1+Ce^{\cos x}$$

$$y(\pi)=1+2e^{-2} \quad \rightarrow\ 1+2e^{-2}=-1+Ce^{-1}$$

$$Ce^{-1}=2+2e^{-2} \ \Rightarrow\ C=2e+2e^{-1}$$

$$\boxed{y(x)=-1+\left(2e+2e^{-1}\right)e^{\cos x-1}}$$

Exercise

Find a solution to the initial value problem that is continuous on the given interval $[a,\ b]$

$$y'+p(t)y=2, \quad y(0)=1 \qquad p(t)=\begin{cases}0, & 0\le t\le 1\\ \frac{1}{t}, & 1<t\le 2\end{cases} \qquad [a,\ b]=[0,\ 2]$$

Solution

For $0\le t\le 1$:

$$y'=2, \quad y(0)=1$$

$$\int dy=\int 2dt$$

$$y(t)=2t+C$$

$$y(0)=1 \quad \Rightarrow \boxed{C=1}$$

$$\boxed{y(t)=2t+1} \qquad t=1 \ \Rightarrow\ y=3$$

For $1\le t\le 2$:

$$y'+\frac{1}{t}y=2, \quad y(1)=3$$

$$e^{\int \frac{1}{t}dt} = e^{\ln t} = t$$

$$\int 2t\ dt = t^2$$

$$y(t) = \frac{1}{t}\left(t^2 + C\right)$$

$$y(1) = 3 \quad \to \quad 3 = 1 + C \quad \Rightarrow \quad \underline{C = 2}$$

$$\underline{y(t) = t + \frac{2}{t}}$$

Exercise

Find a solution to the initial value problem that is continuous on the given interval $[a,\ b]$

$$y' + p(t)y = 0, \quad y(0) = 3 \qquad\qquad p(t) = \begin{cases} 2t - 1, & 0 \le t \le 1 \\ 0, & 1 < t \le 3 \\ -\frac{1}{t}, & 3 < t \le 4 \end{cases} \qquad\qquad [a,\ b] = [0,\ 4]$$

Solution

For $0 \le t \le 1$:

$$y' + (2t - 1)y = 0, \quad y(0) = 3$$

$$e^{\int (2t-1)dt} = e^{t^2 - t}$$

$$y(t) = Ce^{-t^2 + t}$$

$$y(0) = 3 \quad \Rightarrow \underline{C = 3}$$

$$\underline{y(t) = 3e^{t - t^2}} \qquad\qquad t = 1 \ \Rightarrow \ y = 3$$

For $1 \le t \le 3$:

$$y' = 0, \quad y(1) = 3$$

$$y(t) = C$$

$$y(1) = 3 \quad \Rightarrow \quad \underline{C = 3}$$

$$\underline{y(t) = 3} \qquad\qquad t = 3 \ \Rightarrow \ y = 3$$

For $3 \le t \le 4$:

$$y' - \frac{1}{t}y = 0, \quad y(3) = 3$$

$$e^{\int \frac{-1}{t}dt} = e^{-\ln t} = \frac{1}{t}$$

$$y(t) = Ct$$

$$y(3)=3 \quad \Rightarrow \underline{C=1}$$

$$\boxed{y(t)=t}$$

Exercise

Solve $xy'+2y=\sin x$ *for* y' $\qquad y\left(\frac{\pi}{2}\right)=0$

Solution

$$xy'+2y=\sin x$$

$$y'+\frac{2}{x}y=\frac{\sin x}{x} \qquad x\neq 0$$

$$e^{\int \frac{2}{x}dx}=e^{2\ln|x|}=e^{\ln x^2}=x^2$$

$$\int x^2\frac{\sin x}{x}dx=\int x\sin x\,dx$$

$$=-x\cos x+\sin x$$

		$\int \sin x$
+	x	$-\cos x$
−	1	$-\sin x$

$$y(x)=\frac{1}{x^2}\left(-x\cos x+\sin x+C\right)$$

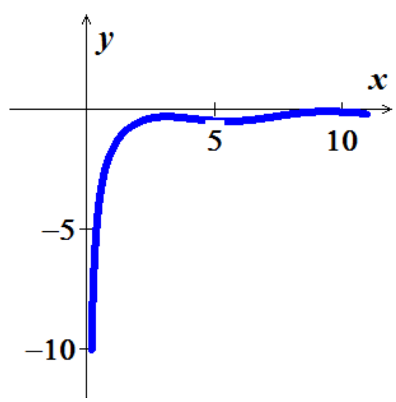
$$\boxed{-\frac{1}{x}\cos x+\frac{1}{x^2}\sin x+\frac{C}{x^2}}$$

$$y\left(\frac{\pi}{2}\right)=-\frac{1}{\left(\frac{\pi}{2}\right)}\cos\left(\frac{\pi}{2}\right)+\frac{1}{\left(\frac{\pi}{2}\right)^2}\sin\left(\frac{\pi}{2}\right)+\frac{C}{\left(\frac{\pi}{2}\right)^2}$$

$$0=\frac{4}{\pi^2}+\frac{4}{\pi^2}C$$

$$\frac{4}{\pi^2}C=-\frac{4}{\pi^2} \quad \to \underline{C=-1}$$

$$\boxed{y(x)=-\frac{1}{x}\cos x+\frac{1}{x^2}\sin x-\frac{1}{x^2}} \qquad x\neq 0$$



Exercise

Find the solution of the initial value problem. Discuss the interval of existence and provide a sketch of your solution $(2x+3)y'=y+(2x+3)^{1/2}$; $\quad y(-1)=0$

Solution

$$(2x+3)y'-y=(2x+3)^{1/2}$$

$$y'-\frac{1}{2x+3}y=(2x+3)^{-1/2}$$

$$e^{\int \frac{-1}{2x+3}dx}=e^{-\frac{1}{2}\int \frac{1}{2x+3}d(2x+3)}=e^{-\frac{1}{2}\ln(2x+3)}=e^{\ln(2x+3)^{-1/2}}=|2x+3|^{-1/2}$$

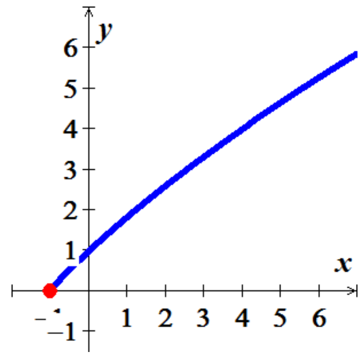
$$\int (2x+3)^{-1/2}(2x+3)^{-1/2}\,dx = \int (2x+3)^{-1}\,dx$$

$$= \frac{1}{2}\int \frac{d(2x+3)}{2x+3} \qquad\qquad d(2x+3) = 2dx$$

$$= \frac{1}{2}\ln|2x+3|$$

$$y(x) = \frac{1}{(2x+3)^{-1/2}}\left(\frac{1}{2}\ln(2x+3) + C\right)$$

$$= \frac{1}{2}(2x+3)^{1/2}\ln(2x+3) + C(2x+3)^{1/2}$$



$$0 = \frac{1}{2}(2(-1)+3)^{1/2}\ln(2(-1)+3) + C(2(-1)+3)^{1/2}$$

$$0 = \frac{1}{2}(1)^{1/2}\ln(1) + C(1)^{1/2} \quad \rightarrow \quad \boxed{C = 0}$$

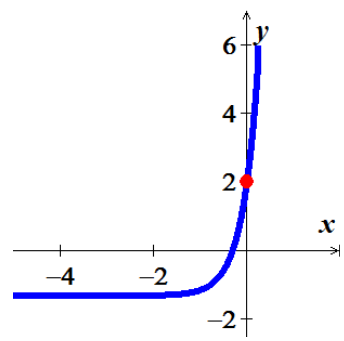
$$\boxed{y(x) = \frac{1}{2}(2x+3)^{1/2}\ln(2x+3)}$$

Exercise

Find the general solution of the given differential equation. Then find the particular solution satisfying the given initial condition of $y' - 3y = 4, \quad y(0) = 2$

Solution

$$e^{\int -3dx} = e^{-3x}$$



$$\int 4e^{-3x}dx = -\frac{4}{3}e^{-3x}$$

$$y(x) = \frac{1}{e^{-3x}}\left(-\frac{4}{3}e^{-3x} + C\right)$$

$$= -\frac{4}{3} + Ce^{3x}$$

$$y(0) = 2 \quad \rightarrow 2 = -\frac{4}{3} + Ce^{0} \quad \Rightarrow \quad \boxed{C = \frac{10}{3}}$$

$$y(x) = -\frac{4}{3} + \frac{10}{3}e^{3x}$$

$$= \boxed{\frac{10e^{3x} - 4}{3}}$$

Exercise

Find the general solution of the given differential equation. Then find the particular solution satisfying the given initial condition of

$$y' + \frac{1}{2}y = t, \quad y(0) = 1$$

Solution

$$e^{\int \frac{1}{2}dt} = e^{t/2}$$

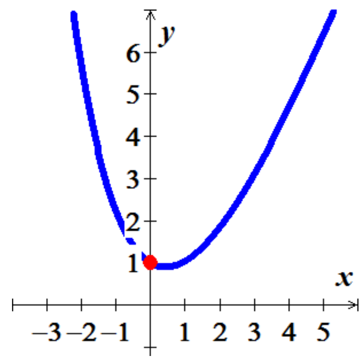
$$\left(e^{t/2}y\right)' = te^{t/2}$$

$$e^{t/2}y = \int te^{t/2}dt \qquad\qquad \int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$= \frac{e^{t/2}}{\left(\frac{1}{2}\right)^2}\left(\frac{t}{2}-1\right) + C$$

$$= 4e^{t/2}\left(\frac{t}{2}-1\right) + C$$

$$= (2t-4)e^{t/2} + C$$



$$y(t) = (2t-4) + Ce^{-t/2}$$

$$y(0) = 1 \quad \rightarrow \ 1 = -4 + C \quad \Rightarrow \ \boxed{C = 5}$$

$$\boxed{y(t) = 2t - 4 + 5e^{-t/2}}$$

Exercise

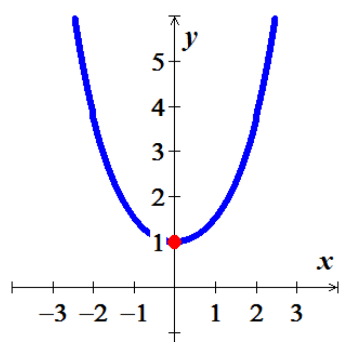
Find the general solution of the given differential equation. Then find the particular solution satisfying the given initial condition of

$$y' + y = e^t, \quad y(0) = 1$$

Solution

$$e^{\int dt} = e^t$$

$$\int e^t e^t dt = \int e^{2t}dt = \frac{1}{2}e^{2t}$$



$$y(t) = \frac{1}{e^t}\left(\frac{1}{2}e^{2t} + C\right)$$

$$\boxed{= \frac{1}{2}e^t + Ce^{-t}}$$

$$y(0) = 1 \quad \rightarrow 1 = \frac{1}{2} + C \quad \Rightarrow \ \boxed{C = \frac{1}{2}}$$

$$\boxed{y(t) = \frac{1}{2}\left(e^t + e^{-t}\right)}$$

Exercise

The following system of differential equations is encountered in the study of the decay of a special type of radioactive series of elements

$$\frac{dx}{dt} = -\lambda_1 x$$

$$\frac{dy}{dt} = \lambda_1 x - \lambda_2 y$$

Where λ_1 *and* λ_2 are constants.

Discuss how to solve this system subject to $x(0) = x_0,\ y(0) = y_0$

Solution

$$\int \frac{dx}{x} = -\lambda_1 \int dt$$

$$\ln x = -\lambda_1 t + C$$

$$x(t) = e^{-\lambda_1 t + C} = Ae^{-\lambda_1 t}$$

$$x(0) = A = x_0$$

$$\boxed{x(t) = x_0 e^{-\lambda_1 t}}$$

$$\frac{dy}{dt} = x_0 \lambda_1 e^{-\lambda_1 t} - \lambda_2 y$$

$$y' + \lambda_2 y = x_0 \lambda_1 e^{-\lambda_1 t}$$

$$e^{\int \lambda_2 dt} = e^{\lambda_2 t}$$

$$x_0 \lambda_1 \int e^{-\lambda_1 t} e^{\lambda_2 t} dt = x_0 \lambda_1 \int e^{(\lambda_2 - \lambda_1)t} dt$$

$$= \frac{x_0 \lambda_1}{\lambda_2 - \lambda_1} e^{(\lambda_2 - \lambda_1)t}$$

$$y(t) = \frac{1}{e^{\lambda_2 t}} \left(\frac{x_0 \lambda_1}{\lambda_2 - \lambda_1} e^{(\lambda_2 - \lambda_1)t} + C \right)$$

$$\boxed{= \frac{x_0 \lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + Ce^{-\lambda_2 t}}$$

$$y(0) = \frac{x_0 \lambda_1}{\lambda_2 - \lambda_1} + C = y_0$$

$$C = y_0 - \frac{x_0 \lambda_1}{\lambda_2 - \lambda_1} = \frac{\lambda_2 y_0 - \lambda_1 y_0 - x_0 \lambda_1}{\lambda_2 - \lambda_1}$$

$$y(t) = \frac{x_0 \lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \frac{\lambda_2 y_0 - \lambda_1 y_0 - x_0 \lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}$$