Section 2.8 – Row and Column Spaces

Definition

For an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

The vectors

$$\begin{aligned} \vec{v}_1 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ \vec{v}_2 &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \\ \vdots & \qquad\qquad \vdots \\ \vec{v}_{\text{m}} &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{aligned}$$

In $\mathbb{R}^n$ that are formed from the rows of A are called the ***row vectors*** of A, and the vectors

$$\vec{v}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots \quad \vec{v}_3 = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

In $\mathbb{R}^m$ that are formed from the rows of A are called the ***column vectors*** of A.

Definition

If A is $m \times n$ matrix, then the subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the ***row space*** of A and is denoted by $RS(A)$ *or* $R(A)$, and the subspace $\mathbb{R}^m$ spanned by the row vectors of A is called the ***column space*** of A and is denoted by $CS(A)$ *or* $C(A)$. The solution space of the homogeneous system of equations $Ax = 0$, which is a subspace of $\mathbb{R}^n$, is called the null space of A.

The *Column Space* of A

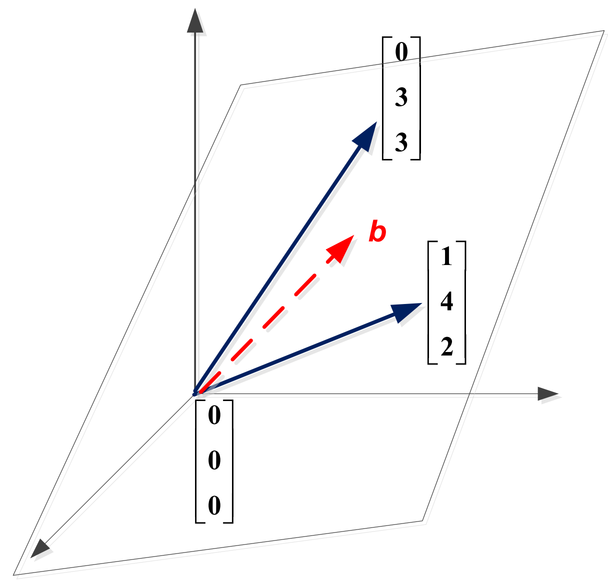
The most important subspaces are tied directly to a matrix A, to solve $A\vec{x} = b$.

Definition

The column space consists of all linear combinations of the columns. The combination are all possible vectors Ax. They fill the column space $C(A)$.

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$$

$$b = x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$



To solve $A\vec{x} = \vec{b}$ is to express b as a combination of the columns.

The column space $CS(A)$ is a plane that containing the two columns. $A\vec{x} = b$ is solvable when b in on that plane.

Theorem

The system $A\vec{x} = b$ is solvable if and only if b is in the column space of A.

Example

Let $A\vec{x} = b$ be the linear system

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Show that $\boldsymbol{b}$ is in the column space of A by expressing it as a linear combination of the column vectors of A.

Solution

$$\left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 1 & 2 & -3 & -9 \\ 2 & 1 & -2 & -3 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 0 & 5 & -1 & -8 \\ 0 & 7 & 2 & -1 \end{array}\right] \begin{array}{l} 5R_1 - 3R_2 \\ \\ 5R_3 - 7R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} -5 & 0 & 13 & 29 \\ 0 & 5 & -1 & -8 \\ 0 & 0 & 17 & 51 \end{array}\right] \begin{array}{l} 17R_1 - 13R_3 \\ 17R_2 + R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} -85 & 0 & 0 & -170 \\ 0 & 85 & 0 & -85 \\ 0 & 0 & 17 & 51 \end{array}\right] \begin{array}{l} -\frac{1}{85}R_1 \\ \frac{1}{85}R_2 \\ \frac{1}{17}R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

That implies to $x_1 = 2, \quad x_2 = -1, \quad x_3 = 3$

It follows that

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$`

Example

Describe the column spaces (they are subspaces of $\mathbb{R}^2$) for

$$I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \end{bmatrix}$$

Solution

The column space of I is the whole space $\mathbb{R}^2$. Every vector is a combination of the columns of I. In the space language $CS(I)$ *is* $\mathbb{R}^2$.

The column space of A is only a line, the second column (2, 4) is a multiple of the first column (1, 2) and (2, 4) and all other vectors $(c,\ 2c)$ along that line. The equation $A\vec{x} = \vec{b}$ is only solvable when $\vec{b}$ is on the line.

The column space $C(B)$ is all of $\mathbb{R}^2$. Every b is attainable. The vector $\vec{b} = (3,\ 4)$ is summation of column 1 and 2.

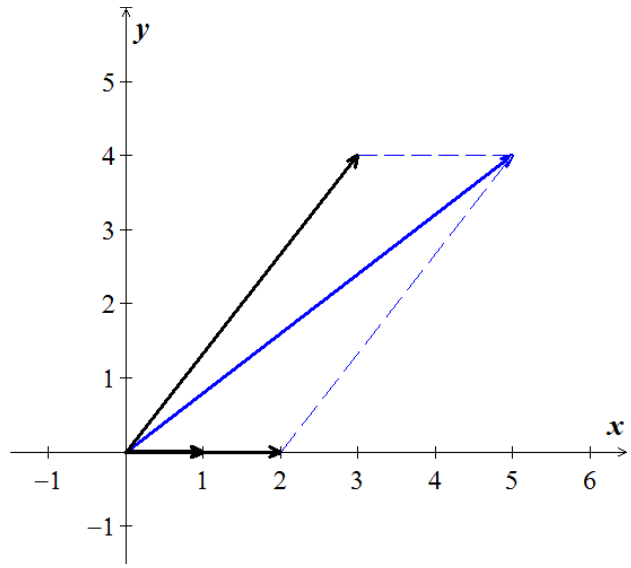
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ 4x_3 = 4 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 + 2x_2 = 2 \\ x_3 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 1 \end{cases} \quad or \quad \Rightarrow \begin{cases} x_1 = 2 \\ x_2 = 0 \end{cases}$$

$$x = (0,\ 1,\ 1) \quad also \quad x = (2,\ 0,\ 1)$$



This matrix has the same column as I and any $\vec{b}$ is allowed. $\vec{x}$ has an extra component (more solutions).

Pivot Columns

The pivot columns of R have 1's in the pivots and 0's everywhere else.

$$\text{Pivot columns: } A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix}$$

$$\text{Yields to: } R = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- ***The pivot columns are not combinations of earlier columns. The free columns are combinations of columns which are the special solutions!***

Complete **Solution** *to* $AX = B$

To solve $A\vec{x} = \vec{b}$, we need to put into an ***augmented*** form where $\vec{b}$ is not zero.

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \qquad B = \vec{b} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}$$

$$X = \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The augmented matrix is just $\begin{bmatrix} A & B \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} R & \boldsymbol{d} \end{bmatrix}$$

Special Solutions

Each special solution to $A\vec{x}=0$ and $R\vec{x}=0$ has one free variable equal to 1.

$$R\vec{x}=\begin{array}{c}\begin{bmatrix}1 & 3 & 0 & 2 & -1\\ 0 & 0 & 1 & 4 & -3\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\\ \begin{array}{ccccc} & F & & F & F\end{array}\end{array}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

The ***free variables*** are $x_2,\ x_4,\ x_5$

$$\rightarrow\begin{cases}x_1+3x_2+2x_4-x_5=0\\ x_3+4x_4-3x_5=0\end{cases}$$

1. Set $x_2=1,\ x_4=x_5=0\Rightarrow\begin{cases}x_1=-3\\ x_3=0\end{cases}$ **(*Column 2*)**

The special solution: $s_1=(-3,\ 1,\ 0,\ 0,\ 0)$

2. Set $x_4=1,\ x_2=x_5=0\Rightarrow\begin{cases}x_1=-2\\ x_3=-4\end{cases}$ **(*Column 4*)**

The special solution: $s_2=(-2,\ 0,\ -4,\ 1,\ 0)$

3. Set $x_5=1,\ x_2=x_4=0\Rightarrow\begin{cases}x_1=1\\ x_3=3\end{cases}$ **(*Column 5*)**

The special solution: $s_3=(1,\ 0,\ 3,\ 0,\ 1)$

The nullspace matrix N contains the 3 special solutions in its columns.

$$N=\begin{bmatrix}-3 & -2 & 1\\ 1 & 0 & 0\\ 0 & -4 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{array}{c}\textit{not free}\\ \textit{free}\\ \textit{not free}\\ \textit{free}\\ \textit{free}\end{array}$$

The linear combinations of these three columns give all vectors in the nullspace.

One *Particular* Solution

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} R & \boldsymbol{d} \end{bmatrix}$$

The ***free variables*** for R to be $x_2 = x_4$.

Then the equations give the ***pivot variables*** $x_1 = 1 \quad x_3 = 6$

The ***particular solution*** is: $\boxed{(1,\ 0,\ 6,\ 0)}$

The two special (nullspace) solutions to $Rx = 0$:

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{r} x_1 + 3x_2 + x_4 = 0 \\ x_3 + 4x_4 = 0 \end{array}$$

$$\Rightarrow \begin{cases} x_1 = -3x_2 - x_4 \\ x_3 = -4x_4 \end{cases}$$

$x_2 = 1,\ x_4 = 0$

$$\Rightarrow\ x_1 = -3,\ x_3 = 0 \quad \to \boxed{(-3,\ 1,\ 0,\ 0)}$$

$x_2 = 0,\ x_4 = 1$

$$\Rightarrow\ x_1 = -2,\ x_3 = -4 \ \to \boxed{(-2,\ 0,\ -4,\ 1)}$$

The ***complete solution***:

$$x = x_p + x_n$$

$$= \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix}$$

Example

Find the condition on (b_1, b_2, b_3) for $A\vec{x} = \vec{b}$ to be solvable, if

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Solution

The augmented form:

$$\begin{bmatrix} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{bmatrix} \qquad \begin{array}{l} \\ R_2 - R_1 \\ R_3 + 2R_1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & -1 & b_3 + 2b_1 \end{bmatrix} \qquad \begin{array}{l} R_1 - R_2 \\ \\ R_3 + R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 + b_1 + b_2 \end{bmatrix} \quad \to \quad \underline{b_1 + b_2 + b_3 = 0}$$

The last equation is 0 = 0 provided $b_1 + b_2 + b_3 = 0$.

There are ***no*** *free variables* and ***no*** *special solutions*.

The nullspace solution: $x_n = 0$

The complete solution:

$$x = x_p + x_n$$

$$= \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $b_1 + b_2 + b_3 \neq 0$, there is no solution to $A\vec{x} = \vec{b}$ and $\vec{x}_p$ doesn't exist.

Example

a) Find a subset of the vectors

$$\vec{v}_1 = (1,-2,0,3) \quad \vec{v}_2 = (2,-5,-3,6), \quad \vec{v}_3 = (0,1,3,0), \quad \vec{v}_4 = (2,-1,4,-7), \quad \vec{v}_5 = (5,-8,1,2)$$

That forms a basis for the space spanned by these vectors

b) Express each vector not in the basis as a linear combination of the basis vectors

Solution

a) Construct the vectors as its column vectors

$$\begin{array}{cl} \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} & \begin{array}{l} \\ R_2 + 2R_1 \\ \\ R_4 - 3R_1 \end{array} \end{array}$$

$$\begin{array}{cl} \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & -3 & 3 & 4 & 1 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix} & \begin{array}{l} R_1 + 2R_2 \\ \\ R_3 - 3R_2 \\ \\ \end{array} \end{array}$$

$$\begin{array}{cl} \begin{bmatrix} 5 & 0 & 10 & 0 & 5 \\ 0 & -5 & 5 & 0 & -5 \\ 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{l} \frac{1}{5}R_1 \\ -\frac{1}{5}R_2 \\ -\frac{1}{5}R_3 \\ \\ \end{array} \end{array}$$

$$\begin{array}{ccccc} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 & \vec{w}_4 & \vec{w}_5 \end{array} \end{array}$$

The leading 1's occurs in columns 1, 2, and 4.

$\{\vec{w}_1, \vec{w}_2, \vec{w}_4\}$ is a basis for the column space, and consequently $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$

b) $\vec{w}_1 = (1,\ 0,\ 0,\ 0) \quad \vec{w}_2 = (0,\ 1,\ 0,\ 0), \quad \vec{w}_3 = (2,\ -1,\ 0,\ 0)$

$\vec{w}_4 = (0,\ 0,\ 1,\ 0), \quad \vec{w}_5 = (1,\ 1,\ 1,\ 0)$

$$\vec{w}_3 = 2\vec{w}_1 - \vec{w}_2$$

$$\vec{w}_3 = \vec{w}_1 + \vec{w}_2 + \vec{w}_4$$

We call these ***dependency equations***

The corresponding relationships are:

$$\vec{v}_3 = 2\vec{v}_1 - \vec{v}_2$$

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 + \vec{v}_4$$

Solving $Ax = 0$ by ***elimination***

Matrix A is rectangular and we still use the elimination.

1. Forward elimnation from A to a triangular U.
2. Back substitution in $Ax = 0$ to find x.

Consider the matrix $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix} \quad \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix} \quad \begin{matrix} \\ \\ R_3 - 4R_2 \end{matrix}$$

Triangular U: $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

P: The ***pivot*** variables are x_1 *and* x_3, since columns 1 and 3 contains pivots.

F: The ***free*** variables are x_2 *and* x_4, since columns 2 and 4 have no pivots.

Special solutions to:

$$\begin{cases} x_1 + x_2 + 2x_3 + 3x_4 = 0 \\ \quad 4x_3 + 4x_4 = 0 \end{cases} \rightarrow \begin{cases} x_1 = -x_2 - x_4 \\ x_3 = -x_4 \end{cases}$$

Complete solution: $$x = x_2 \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{Special} + x_4 \underbrace{\begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{Special} = \underbrace{\begin{pmatrix} -x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix}}_{Complete}$$

The special solution are in the nullspace $NS(A)$, and their combinations fill out the whole Nullspace.

Exercises Section 2.8 – Row and Column Spaces

1. List the row vectors and column vectors of the matrix

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ 3 & 5 & 7 & -1 \\ 1 & 4 & 2 & 7 \end{bmatrix}$$

(2 – 4) Express the product $A\vec{x}$ as a linear combination of the column vectors of A.

2. $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$

4. $\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$

(5 – 8) Determine whether $\vec{b}$ is in the column space of A, and if so, express $\vec{b}$ as a linear combination of the column vectors of A.

5. $A = \begin{bmatrix} 1 & 3 \\ 4 & -6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$

7. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 7 \end{bmatrix}$

9. Suppose that $x_1 = -1, \quad x_2 = 2, \quad x_3 = 4, \quad x_4 = -3$ is a solution of a nonhomogeneous linear system $A\vec{x} = \vec{b}$ and that the solution set of the homogeneous system $A\vec{x} = \vec{0}$ is given by the formulas $\quad x_1 = -3r + 4s, \quad x_2 = r - s, \quad x_3 = r, \quad x_4 = s$

a) Find a vector form of the general solution of $A\vec{x} = \vec{0}$

b) Find a vector form of the general solution of $A\vec{x} = \vec{b}$

(10 – 13) Find the vector form of the general solution of the given linear system $A\vec{x} = \vec{b}$; then use that result to find the vector form of the general solution of $A\vec{x} = \vec{0}$.

10. $\begin{cases} x_1 - 3x_2 = 1 \\ 2x_1 - 6x_2 = 2 \end{cases}$

11. $\begin{cases} x_1 + x_2 + 2x_3 = 5 \\ x_1 \qquad + x_3 = -2 \\ 2x_1 + x_2 + 3x_3 = 3 \end{cases}$

12. $\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 = 4 \\ -2x_1 + x_2 + 2x_3 + x_4 = -1 \\ -x_1 + 3x_2 - x_3 + 2x_4 = 3 \\ 4x_1 - 7x_2 \qquad - 5x_4 = -5 \end{cases}$

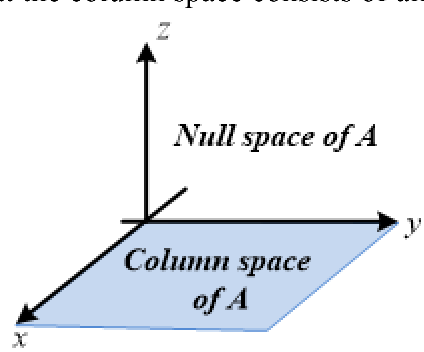
13. $\begin{cases} x_1 - 2x_2 + x_3 + 2x_4 = -1 \\ 2x_1 - 4x_2 + 2x_3 + 4x_4 = -2 \\ -x_1 + 2x_2 - x_3 - 2x_4 = 1 \\ 3x_1 - 6x_2 + 3x_3 + 6x_4 = -3 \end{cases}$

14. Given the vectors $\vec{v}_1 = (1,\ 2,\ 0)$ and $\vec{v}_2 = (2,\ 3,\ 0)$

a) Are they linearly independent?
b) Are they a basis for any space?
c) What space **V** do they span?
d) What is the dimension of that space?
e) What matrices $\boldsymbol{A}$ have **V** as their column space?
f) Which matrices have **V** as their nullspace?
g) Describe all vectors $\vec{v}_3$ that complete a basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$ for $\mathbb{R}^3$.

15. *a*) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Show that relative to an *xyz*-coordinate system in 3-space the null space of A consists of all points on the z-axis and that the column space consists of all points in the *xy*-plane.



b) Find a 3 x 3 matrix whose null space is the x-axis and whose column space is the *yz*-plane.

16. If we add an extra column $\vec{b}$ to a matrix A, then the column space gets larger unless _____. Give an example where the column space gets larger and an example where it doesn't. Why is $A\vec{x} = \vec{b}$ solvable exactly when the column space doesn't get larger – it is the same for A and $\begin{bmatrix} A & \vec{b} \end{bmatrix}$?

17. For which right sides (find a condition on b_1, b_2, b_3) are these solvable. (Use the column space $C(A)$ and the equation $A\vec{x} = \vec{b}$)

a) $$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

b) $$\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

18. Show that the matrices A and $\begin{bmatrix} A & AB \end{bmatrix}$ (with extra columns) have the same column space. But find a square matrix with $C\left(A^2\right)$ smaller than $C(A)$. Important point: An n by n matrix has $C(A) = \mathbb{R}^n$ exactly when A is an ______ matrix.

19. The column of AB are combinations of the columns of A. This means: The column space of AB is contained in (possibly equal to) to the column space of A. Give an example where the column spaces A and AB are not equal.

20. Find a square matrix A where $C\left(A^2\right)$ (the column space of A^2 is smaller than $C(A)$.

21. Suppose $A\vec{x} = \vec{b}$ and $C\vec{x} = \vec{b}$ have the same (complete) solutions for every $\vec{b}$. Is true that $A = C$?

22. Apply Gauss-Jordan elimination to $U\vec{x} = 0$ and $U\vec{x} = c$. Reach $R\vec{x} = 0$ and $R\vec{x} = d$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \qquad \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

Solve $R\vec{x} = 0$ to find x_n (its free variable is $x_2 = 1$).

Solve $R\vec{x} = d$ to find x_p (its free variable is $x_2 = 0$).

The subspace requirements: $x+y$ ***and*** cx ***(and then all linear combinations*** $cx+dy$***) must stay in the subspace.***

23. Which of the following subsets of $\mathbb{R}^3$ are actually subspaces?

a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$

b) The plane of vectors with $b_1 = 1$.

c) The vectors with $b_1 b_2 b_3 = 0$.

d) All linear combinations of $v = (1, 4, 0)$ and $w = (2, 2, 2)$.

e) All vectors that satisfies $b_1 + b_2 + b_3 = 0$

f) All vectors with $b_1 \le b_2 \le b_3$.

24. We are given three different vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$. Construct a matrix so that the equations $A\vec{x} = \vec{b}_1$ and $A\vec{x} = \vec{b}_2$ are solvable, but $A\vec{x} = \vec{b}_3$ is not solvable.

a) How can you decide if this possible?

b) How could you construct A?

25. For which vectors (b_1, b_2, b_3) do these systems have a solution?

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

26. Find a basis for the null space of A. $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

27. Is it true that is $m = n$ then the row space of A equals the column space.

28. If the row space equals the column space the $A^T = A$

29. If $A^T = -A$, then the row space of A equals the column space.

30. Does the matrices A and $-A$ share the same 4 subspaces?

31. Is A and B share the same 4 subspaces then A is multiple of B.

32. Suppose $A\vec{x} = \vec{b}$ & $C\vec{x} = \vec{b}$ have the same (complete) solutions for every $\vec{b}$. Is it true that $A = C$

33. A and A^T have the same left nullspace?