Section 2.11 – Piecewise Continuous with Nonhomogeneous Terms

Mathematical models of mechanical or electrical systems often involve functions with discontinuities corresponding to external forces that are turned abruptly on or off. One such simple on-off function is the unit step function.

Definition

Let the function $f = f(x)$ be defined on an interval *I* and continuous except at a point $c \in I$.

If $\lim\limits_{x \to c^-} f(x)$ and $\lim\limits_{x \to c^+} f(x)$ exist, but $\lim\limits_{x \to c^-} f(x) \neq \lim\limits_{x \to c^+} f(x)$, then f is said to have a ***jump*** (or finite) ***discontinuity*** at *c*.

Definition

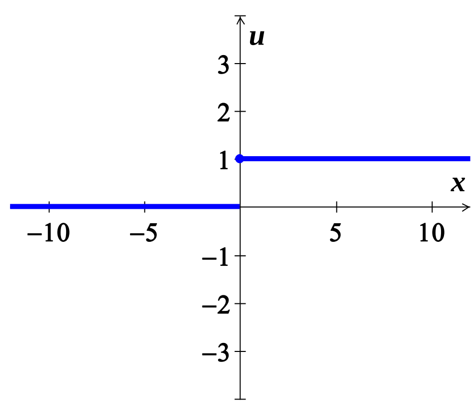
A function f defined on an interval *I* is ***piecewise continuous*** on *I* if it is continuous on *I* except for at most a finite number of points $c_1, c_2, \ldots, c_n$ of I at which it has jump discontinuities.

Theorem

If the function *f* is piecewise continuous on $[0, \infty)$, and of exponential order λ, then the Laplace transform $\mathcal{L}\{f(x)\}$ exists for $s > \lambda$.

Unit Step Functions

The function $u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$ is called the ***Heaviside function***.

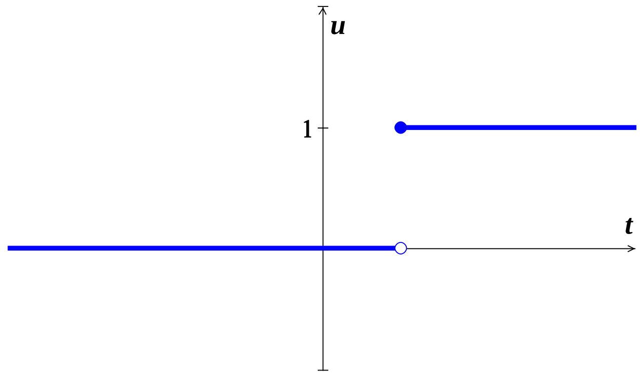


The ***unit step function*** at $t = a$ is defined by

$$u_a(t) = u(t-a) = \begin{cases} 0 & if\ \ t < a \\ 1 & if\ \ t \ge a \end{cases}$$

The notation $u_a(t)$ indicates succinctly where the unit upward step in value takes place whereas $u(t-a)$ connotes the sometimes-useful idea of a "time delay" a before the step is made.

a



If $a \ge 0$, then the Laplace transform of a unit step function is given by:

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

Theorem

If $\mathcal{L}\{f(t)\}$ exists for $s > c$, then

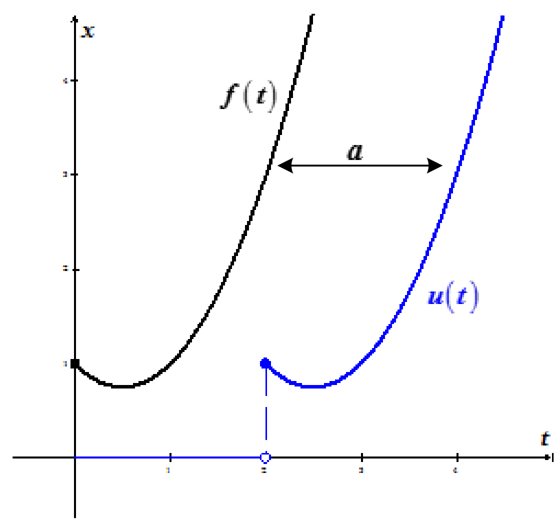
$$\mathcal{L}\{u(t-a)\,f(t-a)\} = e^{-as}F(s)$$

$$and \quad \mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = u(t-a)\,f(t-a)$$

For $s > c + a$

Then the translation of f is

$$f(t-a)u(t-a) = \begin{cases} 0 & if\ \ t < a \\ f(t-a) & if\ \ t \ge a \end{cases}$$



Proof

From the definition of $\mathcal{L}\{f(t)\}$, we get

$$e^{-as}F(s) = e^{-as}\int_0^\infty e^{-s\tau} f(\tau)\,d\tau = \int_0^\infty e^{-s(\tau+a)} f(\tau)\,d\tau$$

The substitution $t = \tau + a$

$$e^{-as}F(s) = \int_a^{\infty} e^{-st} f(t-a)\,dt$$

$$= \int_0^{\infty} e^{-st} u(t-a) f(t-a)\,dt$$

$$= \mathcal{L}\{u(t-a) f(t-a)\}$$

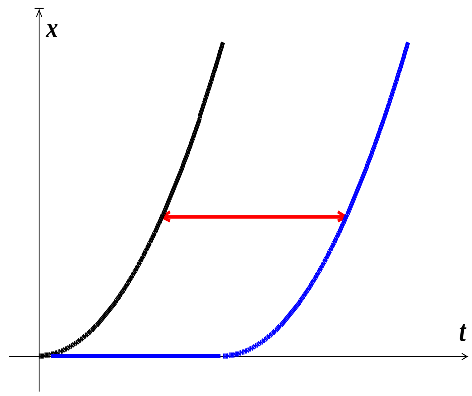
Because $u(t-a) f(t-a) = 0 \quad for \quad t < a$. This completes the proof.

Example

With $f(t) = \frac{1}{2}t^2$, then

$$\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s^3}\right\} = u(t-a)\frac{1}{2}(t-a)^2$$

$$= \begin{cases} 0 & if \ t < a \\ \frac{1}{2}(t-a)^2 & if \ t \ge a \end{cases}$$



Example

Find $\mathcal{L}\{g(t)\}$ if $\qquad g(t) = \begin{cases} 0 & if \ t < 3 \\ t^2 & if \ t \ge 3 \end{cases}$

Solution

When $t \ge 3 \ \rightarrow \ g(t) = t^2$ then $f(t) = (t+3)^2 \ \Rightarrow \ f(t-3) = t^2$

$$f(t) = (t+3)^2 = t^2 + 6t + 9$$

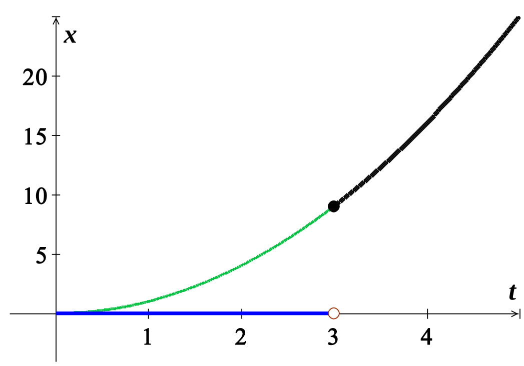
$$F(s) = \mathcal{L}\{t^2 + 6t + 9\}$$

$$= \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}$$

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{u(t-3) f(t-3)\}$$

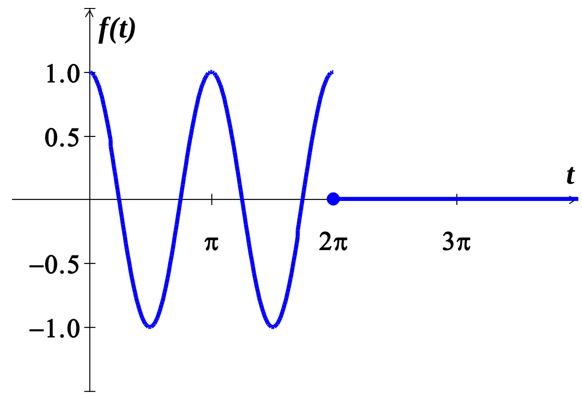
$$= e^{-3s}F(s)$$

$$\boxed{= e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)}$$



Example

Find $\mathcal{L}\{f(t)\}$ if $\quad f(t)=\begin{cases}\cos 2t & \text{if } 0\le t<2\pi\\ 0 & \text{if } t\ge 2\pi\end{cases}$



Solution

$$f(t)=\left[1-u(t-2\pi)\right]\cos 2t$$

$$=\cos 2t-u(t-2\pi)\cos 2(t-2\pi)$$

$$\mathcal{L}\{f(t)\}=\mathcal{L}\{\cos 2t\}-e^{-2\pi s}\mathcal{L}\{\cos 2t\}$$

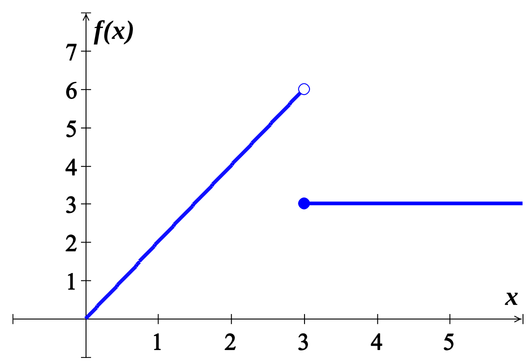
$$=\left(1-e^{-2\pi s}\right)\mathcal{L}\{\cos 2t\}$$

$$=\left(1-e^{-2\pi s}\right)\frac{s}{s^2+4} \qquad\qquad \mathcal{L}\{\cos 2t\}=\frac{s}{s^2+4}$$

$$\boxed{=\frac{s\left(1-e^{-2\pi s}\right)}{s^2+4}}$$

Example

Find $F(s)$ if $\quad f(x)=\begin{cases}2x & \text{if } 0\le x<3\\ 3 & \text{if } x\ge 3\end{cases}$



Solution

$$u(x-3)=\begin{cases}0 & \text{if } 0\le x<3\\ 1 & \text{if } x\ge 3\end{cases}$$

For $x\ge 3$, $\quad f(x)=3u(x-3)$

For $0\le x<3$, $\quad f(x)=2x-2x\cdot u(x-3)$

$$f(x)=2x-2x(x-3)u(x-3)+3u(x-3)$$

$$\mathcal{L}\{f(x)\}=2\mathcal{L}\{x\}-2\mathcal{L}\{(x-3)u(x-3)\}+3\mathcal{L}\{u(x-3)\}$$

$$F(s)=\frac{2}{s^2}-2e^{-3s}\frac{1}{s^2}-3e^{-3s}\frac{1}{s}$$

$$\boxed{=\frac{2}{s^2}-\frac{2e^{-3s}}{s^2}-\frac{3e^{-3s}}{s}}$$

Example

A mass that weighs 32 *lb* (mass *m* = 1 *slug*) is attached to the free end of a long light spring that is stretched 1 *ft* by a force of 4 *lb* (*k* = 4 *lb/ft*). The mass is initially at rest in its equilibrium position. Beginning at time $t = 0$ *sec*, an external force $F(t) = \cos 2t$ is applied to the mass, but at time $t = 2\pi$ this force is turned off (abruptly discontinued) and the mass is allowed to continue its motion unimpeded. Find the resulting position function $x(t)$ of the mass.

Solution

The initial value problem is

$$x'' + 4x = f(t); \quad x(0) = x'(0) = 0$$

$$\mathcal{L}\{x'' + 4x\} = \mathcal{L}\{\cos 2t\}$$

$$\left(s^2 X(s) - sx(0) - x'(0)\right) + 4X(x) = \frac{s\left(1 - e^{-2\pi s}\right)}{s^2 + 4}$$

$$\mathcal{L}\{\cos 2t\} = \frac{s\left(1 - e^{-2\pi s}\right)}{s^2 + 4}$$

$$\left(s^2 + 4\right) X(x) = \frac{s - se^{-2\pi s}}{s^2 + 4}$$

$$X(x) = \frac{s}{\left(s^2 + 4\right)^2} - e^{-2\pi s}\frac{s}{\left(s^2 + 4\right)^2}$$

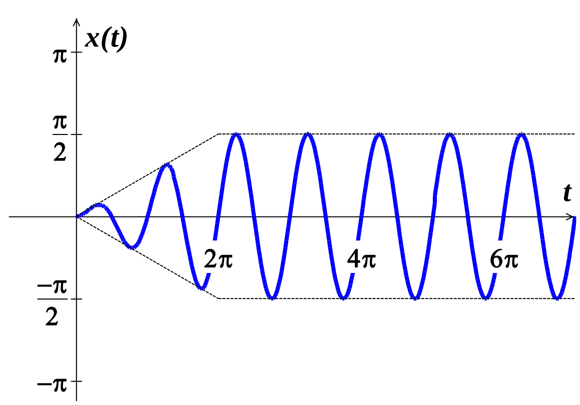
$$\mathcal{L}^{-1}\left\{\frac{s}{\left(s^2 + 4\right)^2}\right\} = \frac{1}{4}t\sin 2t$$

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\{X(x)\} \\ &= \frac{1}{4}t\sin 2t - u(t - 2\pi)\cdot\frac{1}{4}(t - 2\pi)\sin 2(t - 2\pi) \\ &= \frac{1}{4}t\sin 2t - \frac{1}{4}u(t - 2\pi)\cdot(t - 2\pi)\sin 2t \end{aligned}$$

$$\begin{aligned} \frac{1}{4}t\sin 2t - \frac{1}{4}u(t - 2\pi)\cdot(t - 2\pi)\sin 2t &= \frac{1}{4}t\sin 2t - \frac{1}{4}t\sin 2t + \frac{1}{4}2\pi\sin 2t \\ &= \frac{1}{2}\pi\sin 2t \end{aligned}$$

$$x(t) = \begin{cases} \frac{1}{4}t\sin 2t & \text{if } t < 2\pi \\ \frac{1}{2}\pi\sin 2t & \text{if } t \ge 2\pi \end{cases}$$

The mass oscillates with circular frequency $\omega = 2$ and with linearly increasing amplitude until the force is removed at time $t = 2\pi$. Thereafter, the mass continues to oscillate with the same frequency but with constant amplitude $\frac{\pi}{2}$.



The force $F(t) = \cos 2t$ would produce pure resonance if continued indefinitely.

Exercises *Section* 2.11 – Piecewise Continuous with Nonhomogeneous Terms

Find the inverse Laplace transform $f(t)$ of each function given:

1. $F(s)=\dfrac{e^{-3s}}{s^2}$

2. $F(s)=\dfrac{e^{-s}-e^{-3s}}{s^2}$

3. $F(s)=\dfrac{e^{-s}}{s+2}$

4. $F(s)=\dfrac{e^{-s}-e^{2-2s}}{s-1}$

5. $F(s)=\dfrac{e^{-\pi s}}{s^2+1}$

6. $F(s)=\dfrac{1-e^{-2\pi s}}{s^2+1}$

7. $F(s)=\dfrac{2s\left(e^{-\pi s}-e^{-2\pi s}\right)}{s^2+4}$

Find the Laplace transforms of the given functions

8. $f(t)=\begin{cases} 2 & \text{if } 0\le t<3 \\ 0 & \text{if } t\ge 3 \end{cases}$

9. $f(t)=\begin{cases} 1 & \text{if } 1\le t\le 4 \\ 0 & \text{if } t<1, \text{ or } t>4 \end{cases}$

10. $f(t)=\begin{cases} \sin t & \text{if } 0\le t\le 2\pi \\ 0 & \text{if } t>2\pi \end{cases}$

11. $f(t)=\begin{cases} \cos \pi t & \text{if } 0\le t\le 2 \\ 0 & \text{if } t>2 \end{cases}$

12. $f(t)=\begin{cases} \sin 2t & \text{if } \pi\le t\le 2\pi \\ 0 & \text{if } t<\pi \text{ or } t>2\pi \end{cases}$

Use the Laplace transform method to solve the initial-value problem

13. $y'+2y=f(x);\quad y(0)=1 \quad \text{where} \quad f(x)=\begin{cases} x & \text{if } 0\le x<3 \\ 1 & \text{if } x\ge 3 \end{cases}$

14. $y''+2y'+y=f(x);\quad y(0)=y'(0)=0 \quad \text{where} \quad f(x)=\begin{cases} 1 & \text{if } 0\le x<2 \\ x+1 & \text{if } x\ge 2 \end{cases}$

The values of mass *m*, spring constant *k*, dashpot resistance *c*, and force $f(t)$ are given for a mass-spring-dashpot system with external forcing function. Solve the initial value problem, and construct the graph position $x(t)$

$$mx''+cx'+kx=f(t);\quad x(0)=x'(0)=0$$

15. $m=1,\quad k=4,\quad c=0,\quad f(t)=\begin{cases} 1 & 0\le t<\pi \\ 0 & t\ge \pi \end{cases}$

16. $m=1,\quad k=4,\quad c=5,\quad f(t)=\begin{cases} 1 & 0\le t<2 \\ 0 & t\ge 2 \end{cases}$

17. $m = 1, \quad k = 9, \quad c = 0, \quad f(t) = \begin{cases} \sin t & 0 \le t \le 2\pi \\ 0 & t > 2\pi \end{cases}$

18. $m = 1, \quad k = 4, \quad c = 4, \quad f(t) = \begin{cases} t & 0 \le t \le 2 \\ 0 & t > 2 \end{cases}$