5. The "delta function" and Green's functions

a) "delta functions"

(1) 1-dimension

(a) PHYSICAL EXAMPLE:

The voltage induced by a rapidly changing current through an inductor:

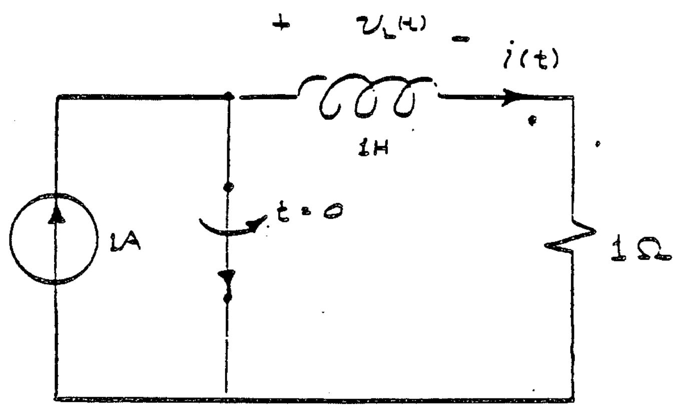
Consider the circuit in figure 1.

In this circuit our idealized model (by which we analyzed the network) yields us non-sense: namely, that $v_L(0)$ is infinite!

Does this mean that our physical model is "no good?"

No! It only means that it is <u>approximate</u>.

We got non-sense out because we put non-sense in by insisting that the current change instantaneously.



$$v_L(t) = 1H \cdot \frac{di(t)}{dt}$$

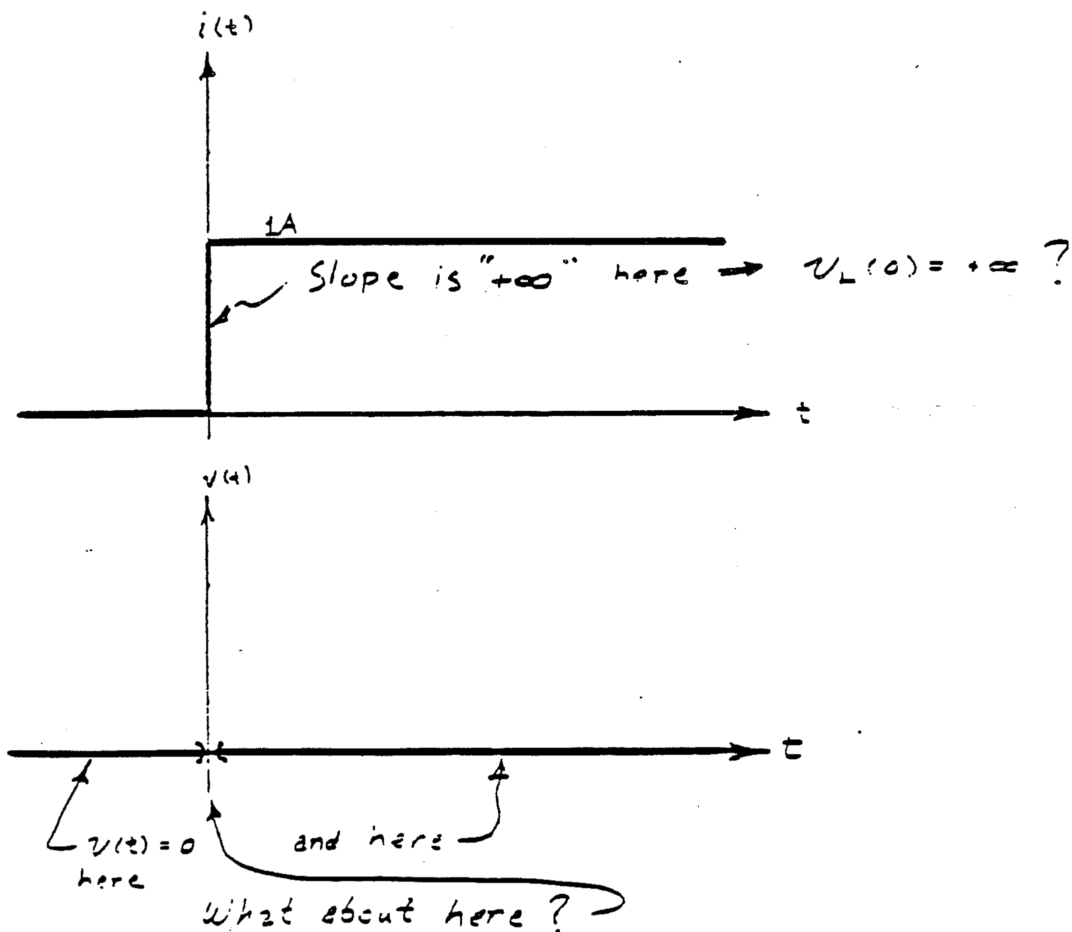


Fig. 1

A more realistic model is to assume ı (t) varies rapidly, but continuously as in figure 2.

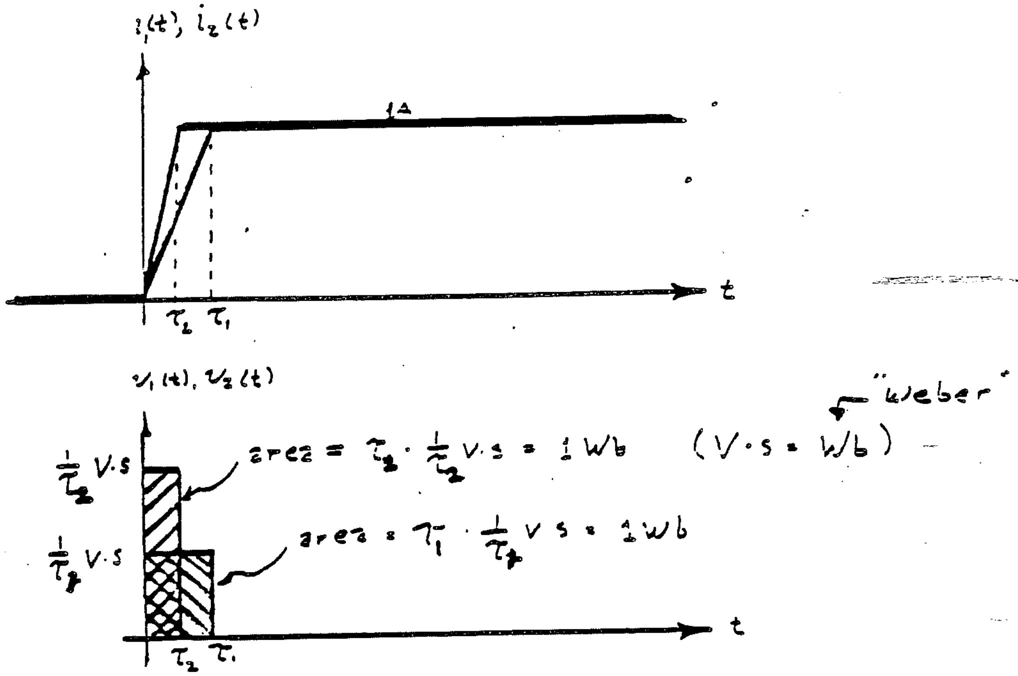

$i_1(t), i_2(t)$
1A
t
τ_2 τ_1
$v_1(t), v_2(t)$
"Weber"
$\frac{1}{\tau_2}$ V·s
area = $\tau_2 \cdot \frac{1}{\tau_2}$ V·s = 1 Wb
(V·s = Wb)
$\frac{1}{\tau_1}$ V·s
area = $\tau_1 \cdot \frac{1}{\tau_1}$ V s = 1 Wb
t
τ_2 τ_1

FIG. 2

The more rapidly i(t) varies, the more peaked the induced voltage becomes. But the area under each voltage pulse remains the same.

It is the area under the pulse that is usually of prime interest, not the details of the pulse.

(b) IMPULSE RESPONSE:

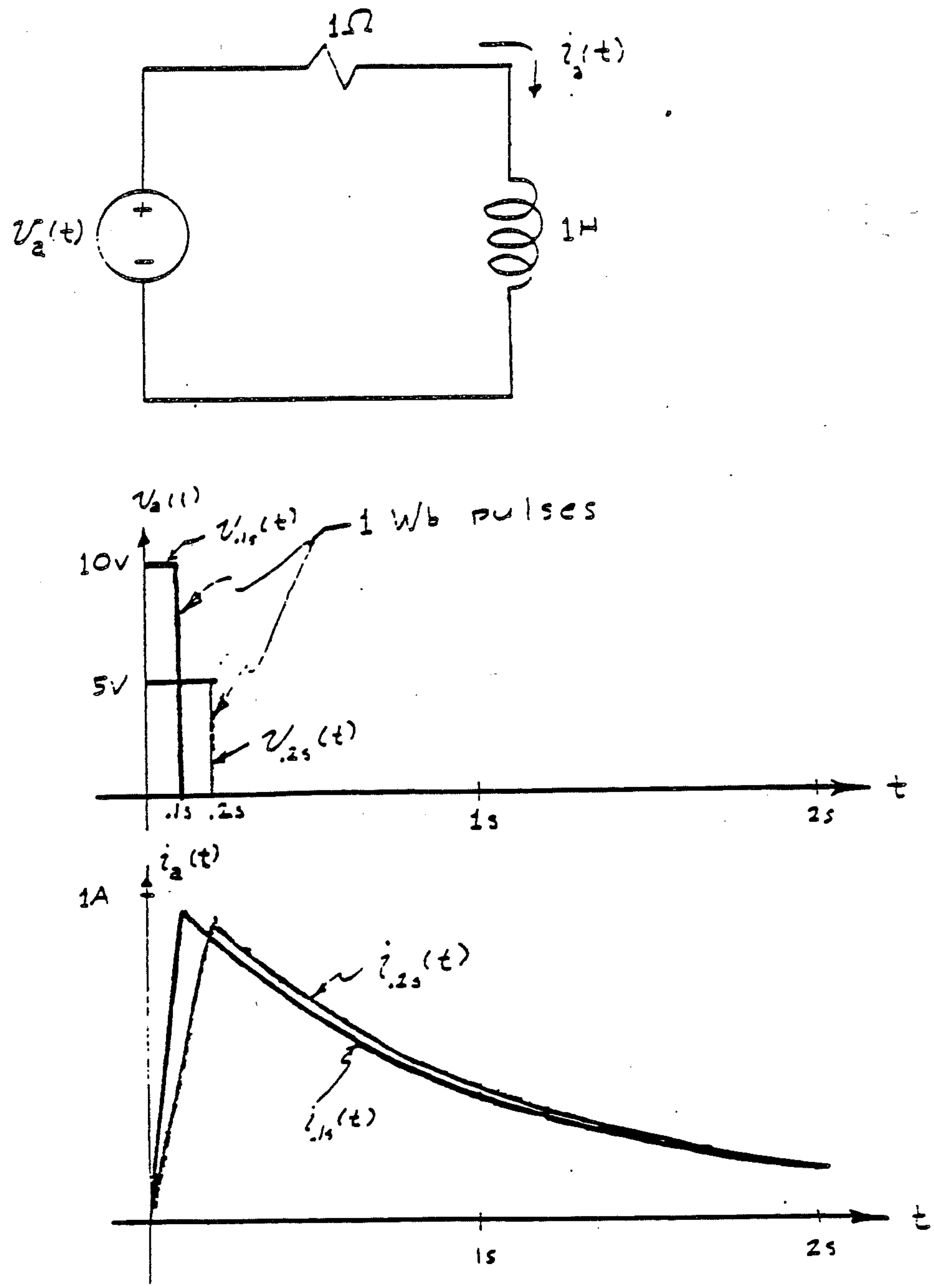
For example, consider the circuit in figure 3 which is driven by such sources.

Note that the response due to two substantially different sources are almost identical as long as the areas under the sources are the same.

Let a approach zero.

Then, $v_a(t)$ becomes increasingly taller and narrower in such a way its area remains constant.

Note that $v_a(t)$ has no limit in any ordinary sense.



1Ω
$i_a(t)$
$v_a(t)$
+
-
1H
$v_a(t)$
$v_{.1s}(t)$
1 Wb pulses
10V
5V
$v_{.2s}(t)$
.1s .2s
1s
2s
t
$i_a(t)$
1A
$i_{.2s}(t)$
$i_{.1s}(t)$
1s
2s
t

FIG. 3

However, the response, $i_a(t)$, does approach a limit, $h(t)$, which is called the impulse response.

This impulse response, though not actually physically produceble, well approximates the response which is physically producible by a highly peaked source.

Thus, from the standpoint of mathematical convenience, we introduce the concept of the "delta function" which is the limit in a new sense, the "sense of distributions," of $v_a(t)$ as a goes to zero.

The impulse response is then the response due to this "delta function" whose only important detail is its area.

(c) FORMAL INTRODUCTION TO THE "DELTA FUNCTION" BY THE THEORY OF DISTRIBUTIONS

Let's look at some properties of $v_a(t)$.

Let G be the set of complex valued, continuous functions of a real variable. Then for any f in G,

$$\int_{-\infty}^{\infty} v_a(t)\, f(t)\, dt = \frac{1}{a}\int_0^a f(t)\, dt.$$

Since f is continuous,

$$\lim_{a \to 0} \int_{-\infty}^{\infty} v_a(t) f(t)\, dt = f(0).$$

A set of functions, $\{\delta_a(t)\}$, which depends on the real parameter a is said to converge in the "sense of distributions" to $\delta(t)$ as $a \longrightarrow a_0$, if

$$\lim_{a \to a_0} \int_{-\infty}^{\infty} \delta_a(t) f(t)\, dt = f(0)$$

for any f in G.

Thus,

$$v_a(t) \longrightarrow \delta(t) \quad \text{as} \quad a \longrightarrow 0.$$

Some other examples are

$$\left.\begin{aligned} &\delta_k(t) = \frac{\sin(kt)}{\pi t} \longrightarrow \delta(t) \quad \text{as} \quad k \longrightarrow \infty, \\ &\text{or} \\ &\delta_a(t) = \frac{1}{\sqrt{\pi a}}\, e^{-t^2/a} \longrightarrow \delta(t) \quad \text{as} \quad a \longrightarrow 0. \end{aligned}\right\}$$

these are continously differentia

There are many other sets of functions, $\delta_a(t)$ such that $\delta_a(t) \to \delta(t)$ as $a \to 0$.

(d) SURFACE SOURCES

$\delta(t)$ represents a physical quantity which is "concentrated" in time. We can also represent physical quantities which are concentrated in space.

An example is the <u>surface</u> <u>charge</u> in figure 4.

(2) 2-dimensions: line sources
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We could further concentrate charge (or any other quantity) in space into a <u>line charge</u> as in figure 5.

(3) 3-dimensions: point sources
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Finally, we could concentrate charge into a <u>single point</u> as in figure 6.

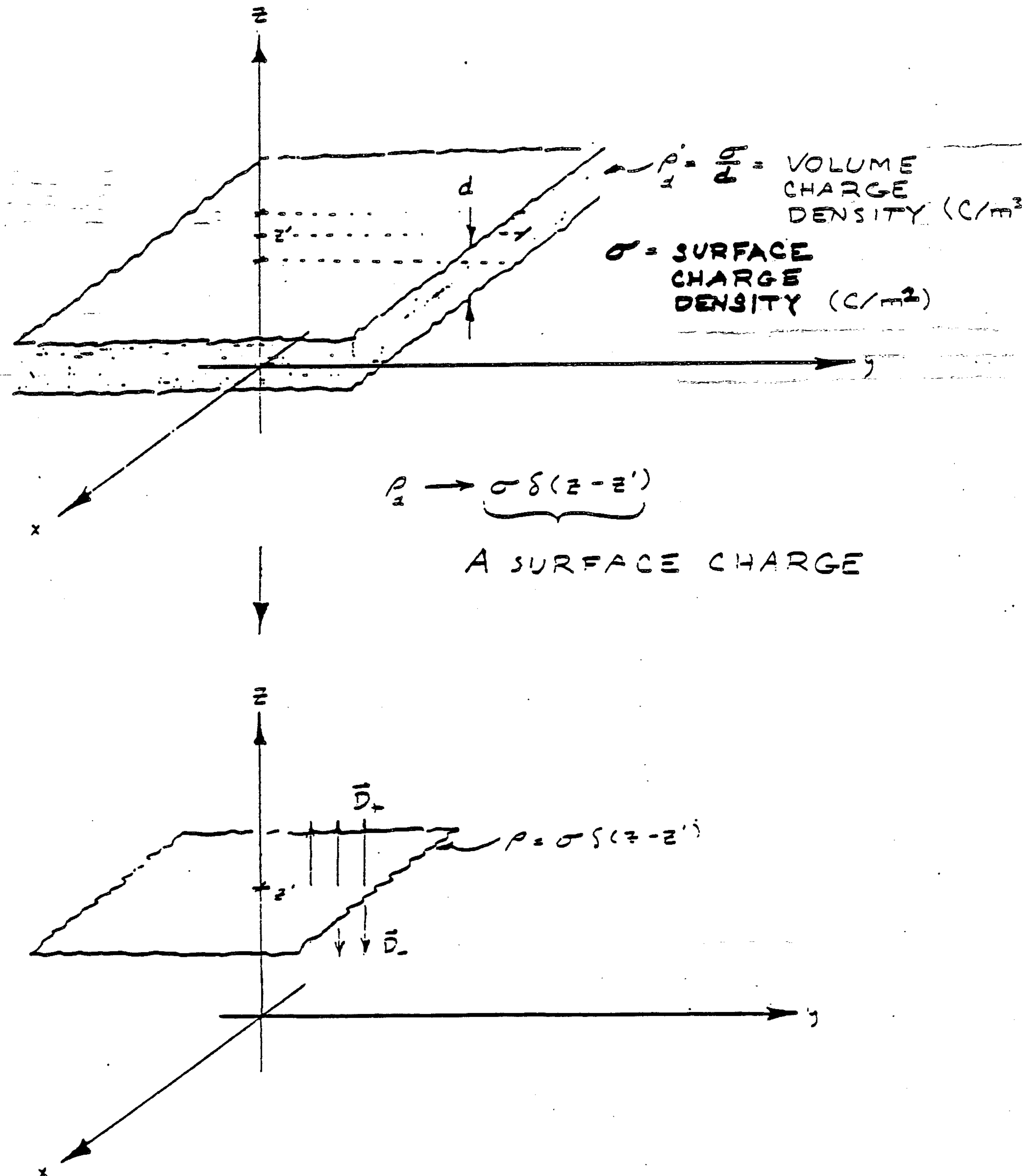

z
d
z'
$\rho_d' = \frac{\sigma}{d}$ = VOLUME CHARGE DENSITY (C/m³)
σ = SURFACE CHARGE DENSITY (C/m²)
y
x
$\rho_d \longrightarrow \underbrace{\sigma\,\delta(z-z')}$
A SURFACE CHARGE
z
$\vec{D}_+$
$\rho = \sigma\,\delta(z-z')$
z'
$\vec{D}_-$
y
x

FIG. 4

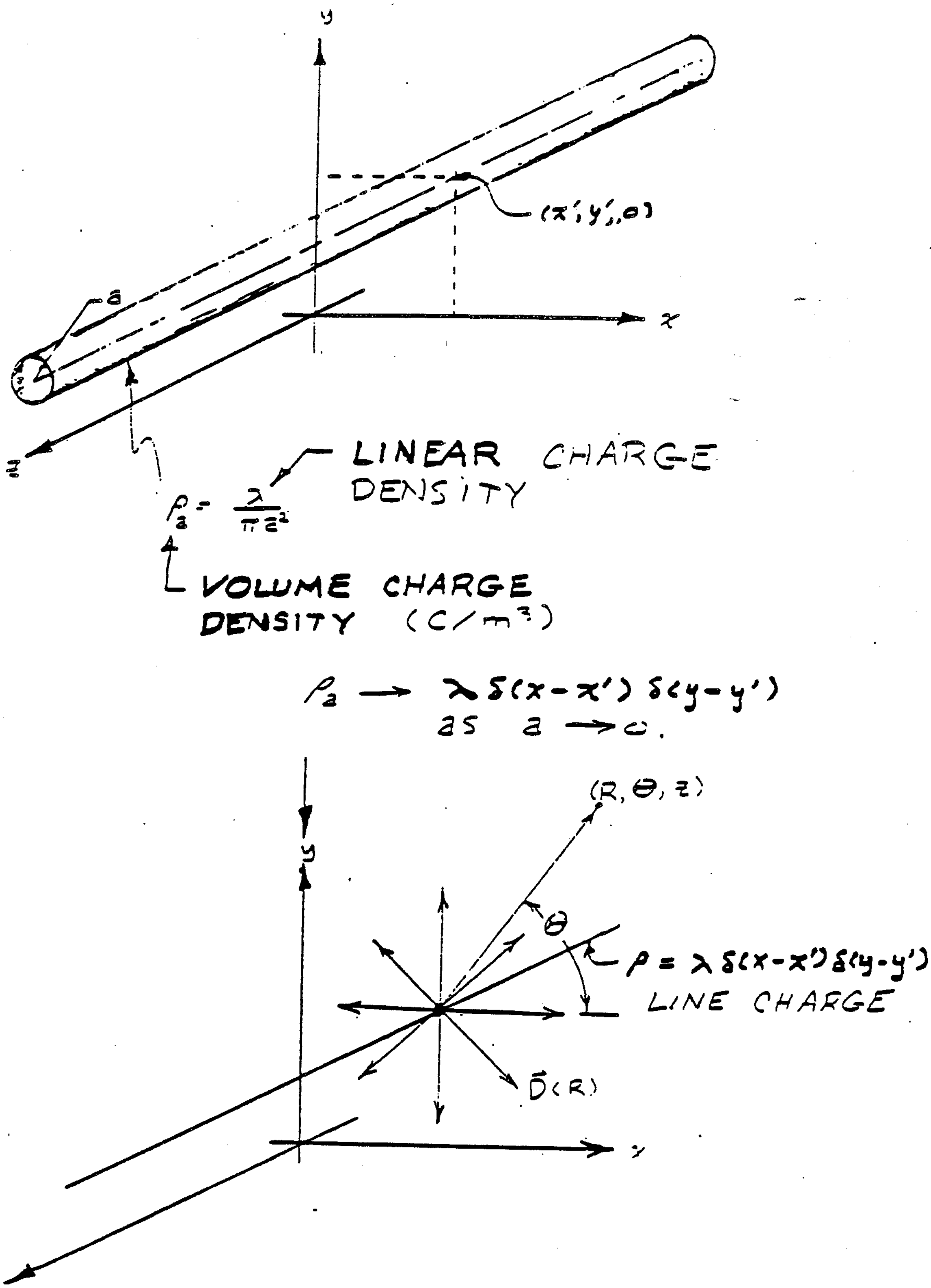

y
(x', y', 0)
a
x
z
LINEAR CHARGE DENSITY
$\rho_a = \frac{\lambda}{\pi a^2}$
VOLUME CHARGE DENSITY (C/m³)
$\rho_a \rightarrow \lambda\,\delta(x-x')\,\delta(y-y')$ as $a \rightarrow 0$.
y
(R, θ, z)
θ
$\rho = \lambda\,\delta(x-x')\,\delta(y-y')$
LINE CHARGE
$\vec{D}(R)$
x
z

FIG. 5

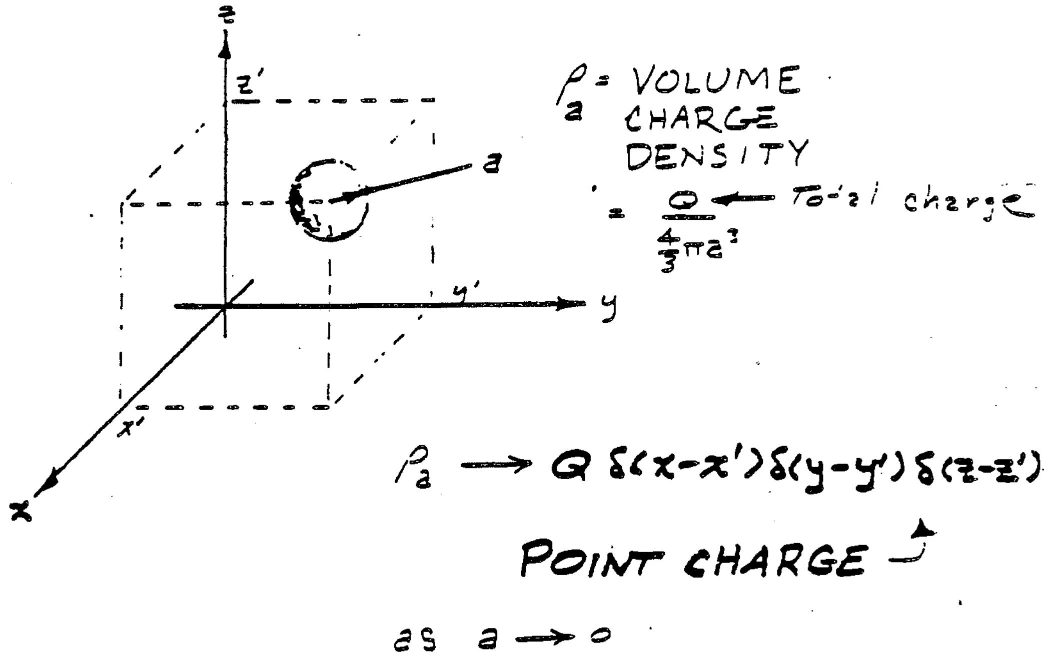

z
z'
a
y'
y
x'
x
ρ_a = VOLUME CHARGE DENSITY
$= \frac{Q}{\frac{4}{3}\pi a^3}$ ← Total charge
$\rho_a \longrightarrow Q\,\delta(x-x')\,\delta(y-y')\,\delta(z-z')$
POINT CHARGE
as $a \rightarrow 0$

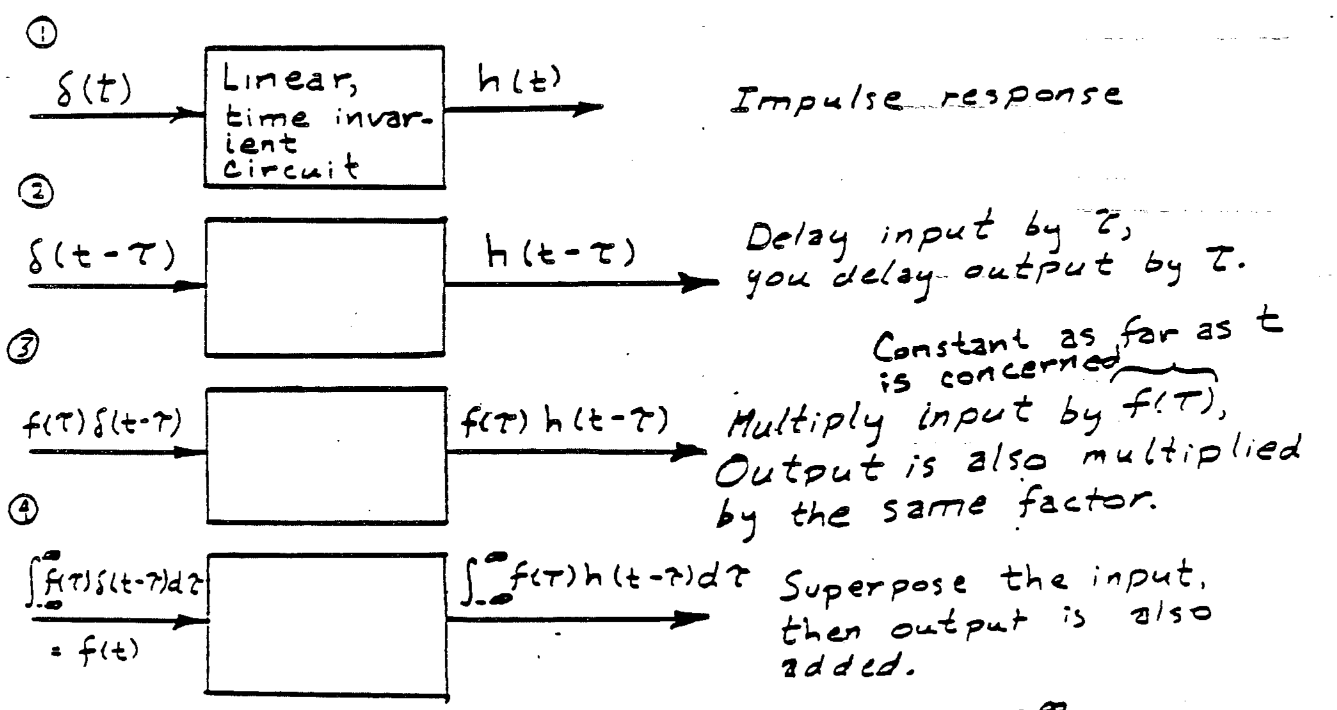
FIG. 6

b) Green's functions

(1) The impulse response and representation of the response due to arbitrary sources in terms of it.

The impulse reponse can be used to construct the response due to an arbitrary input.

The steps are illustrated in figure 7 below:



Therefore, the response due to $f(t)$ is $\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$.

Fig. 7

The impulse response is an example of a "<u>Green's function</u>."

<u>Most</u> simply stated, a <u>Green's function</u> is the <u>response due to a "point source</u>." (Here "point source" can represent a "delta function" of any dimensionality).

(2) The electrostatic potential due to a surface charge and stratified distributions of charge.

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We will now solve Poisson's equation,

$$\nabla^2 \varphi = -\rho/\epsilon$$

for a charge of density

$$\rho(x, y, z) = f(z),$$

distributed in an infinite empty space.

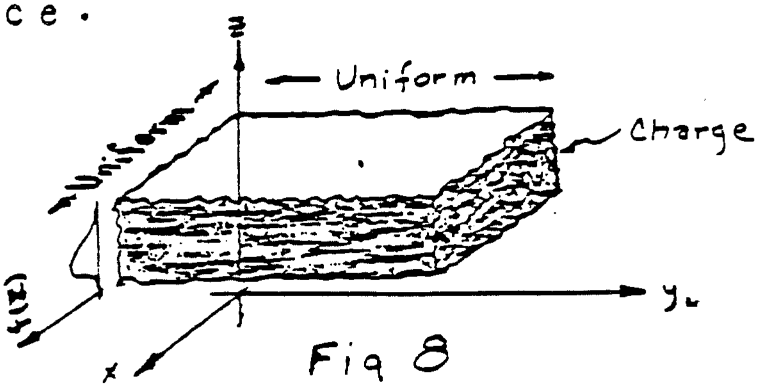


Fig 8

We will solve for φ by superposing the potentials generated by surface charges of

$$\delta(z-z'); \qquad f(z) = \int_{-\infty}^{\infty} f(z')\,\delta(z-z')\,dz'$$

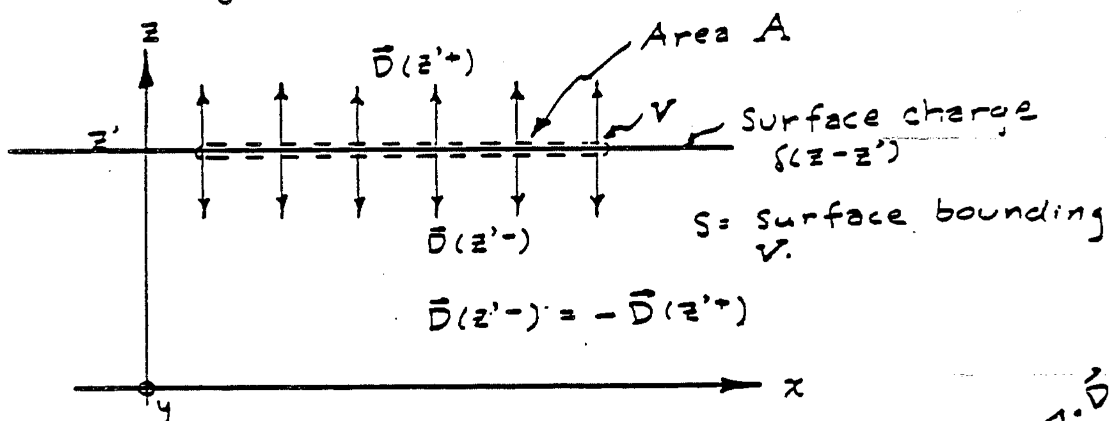
$$\nabla^2 G(z|z') = -\delta(z-z')/\epsilon;$$

$$\varphi(z) = \int_{-\infty}^{\infty} G(z|z')\,f(z')\,dz'$$

We can find the Green's function, G, by use of <u>symmetry</u> and <u>Gauss's law</u>.

Because of the symmetry, the electric flux density, $\vec{D}$, must be uniform and z directed.

Consider figure 9 below:



$$\oint\!\!\oint_S \vec{D}\cdot\hat{n}\,ds = (\vec{D}(z'+)\cdot\hat{z} - \vec{D}(z'-)\cdot\hat{z})A = 2D(z'+)A$$
$$= \iiint_V \delta(z-z')\,dv = A.$$

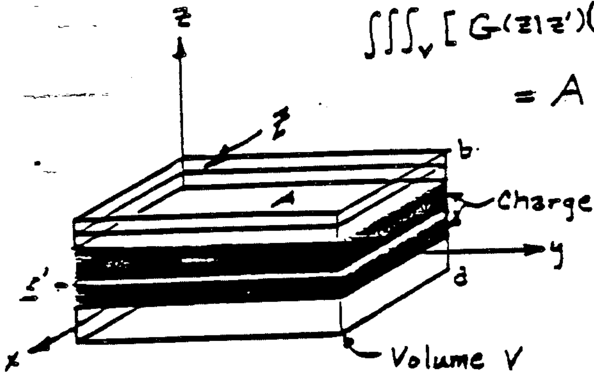
$\therefore\ D(z'+) = \frac{1}{2}$, and $D(z'-) = -\frac{1}{2}$. Or $\overbrace{\vec{D}(z) = \pm\hat{z}\tfrac{1}{2} \quad z \gtrless z'}^{\text{since } \nabla\cdot\vec{D} = 0 \text{ for } z \gtrless z'}$.

$$\therefore\ G(z|z') = -\frac{1}{\epsilon}\int_{z'}^{z} \vec{D}(\zeta)\cdot\hat{z}\,d\zeta = -\frac{1}{2\epsilon}|z-z'|.$$

Fig. 9

We could also have applied Green's theorem rather than our intuitive idea of superposition to obtain this expression for φ.

$$\left[\nabla' = \hat{x}\frac{\partial}{\partial x'} + \hat{y}\frac{\partial}{\partial y'} + \hat{z}\frac{\partial}{\partial z'}\right]$$



$$\iiint_V \left[G(z|z')(\nabla')^2\varphi(z') - \varphi(z')(\nabla')^2 G(z|z')\right] dv'$$

$$= A\int_a^b \left[\frac{f(z')}{\epsilon} G(z|z') - \varphi(z')\frac{\delta(z-z')}{\epsilon}\right] dz'$$

$$= \oiint_S \left[G(z|z')\,\nabla'\varphi(z') - \varphi(z')\nabla' G(z|z')\right] \cdot \hat{n}\, ds'.$$

$$\therefore\ \varphi(z) = \int_a^b f(z')G(z|z')\,dz' - \frac{\epsilon}{A}\oiint_S \left[G\nabla'\varphi - \varphi\nabla' G\right]\cdot \hat{n}\, ds'.$$

The surface integral in this expression is a constant. This is because both $G(z|z')$ and φ vary as $-z$ $(+z)$ + some constant for $z > b$ $(z < a)$, respectively. Thus, the z variation cancels and we are left with a constant.

Note that in one dimension, Green's theorem is simply the rule for "integration by parts."

As a specific example, consider a uniform slab of charge, 1cm thick, with a density of $1C/cm^3$, centered about the x-y plane.

Then using the Green's function derived above, the electrostatic potential is

$$-\varphi(z) = \frac{1}{2\epsilon}\int_{-\infty}^{\infty} f(z')\,|z-z'|\,dz'$$

$$= \frac{1C/cm^3}{2\epsilon}\int_{-\frac{1}{2}cm}^{\frac{1}{2}cm} (z'-z)\,dz' = \frac{1}{2\epsilon}\left(\frac{1}{2}\left(\tfrac{1}{2}cm - z\right)^2 - \frac{1}{2}\left(-\tfrac{1}{2}cm - z\right)^2\right)\frac{C}{c}$$

$$= -\frac{1C\,z/cm^2}{2\epsilon} = \frac{1C|z|/cm^2}{2\epsilon}; \quad \text{for } z < -\tfrac{1}{2}cm.$$

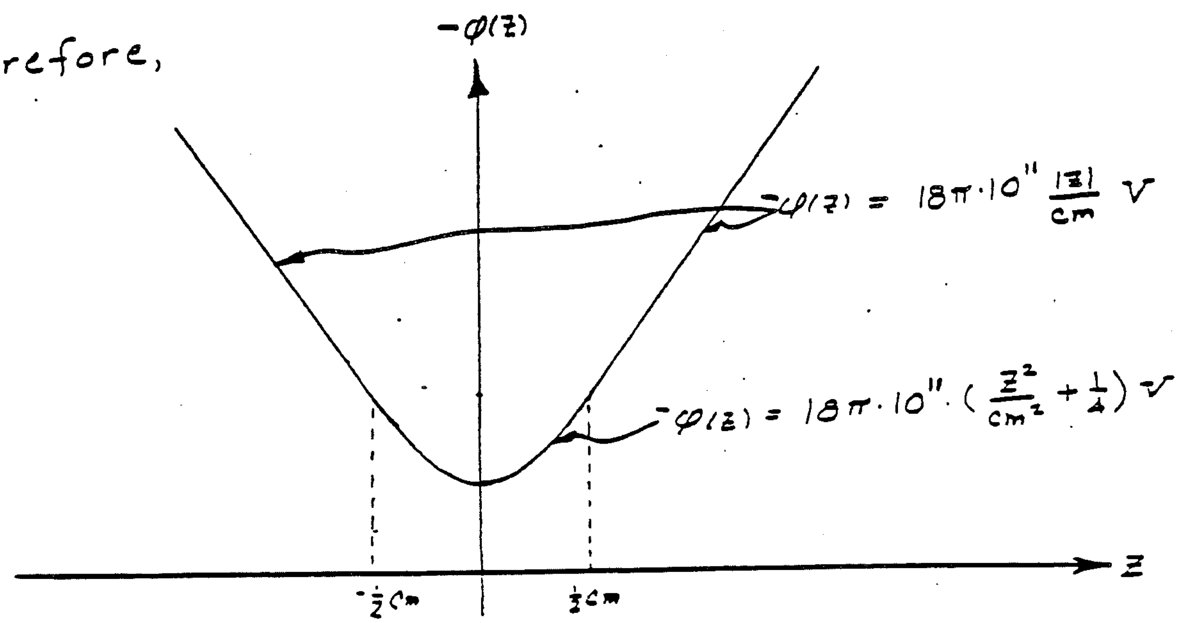
$$-\varphi(z) = \frac{1C/cm^3}{2\epsilon}\int_{-\frac{1}{2}cm}^{z} (z-z')\,dz' + \frac{1}{2\epsilon}\int_{z}^{\frac{1}{2}cm} (z'-z)\,dz'$$

$$= \frac{1C/cm^3}{2\epsilon}\left\{+\frac{1}{2}\left(\tfrac{1}{2}cm + z\right)^2 + \frac{1}{2}\left(z - \tfrac{1}{2}cm\right)^2\right\}$$

$$= \frac{1C/cm^3}{2\epsilon}\left\{z^2 + \frac{1}{4}cm^2\right\} \quad \text{for } -\tfrac{1}{2}cm < z < \tfrac{1}{2}cm.$$

$$-\varphi(z) = \frac{1C/cm^3}{2\epsilon}\int_{-\frac{1}{2}cm}^{\frac{1}{2}cm} (z-z')\,dz' = \frac{1C|z|/cm^2}{2\epsilon} \text{ for } z > \tfrac{1}{2}cm.$$

Therefore,



(5). The electrostatic potential due to a line charge and cylindrical distributions of charge.

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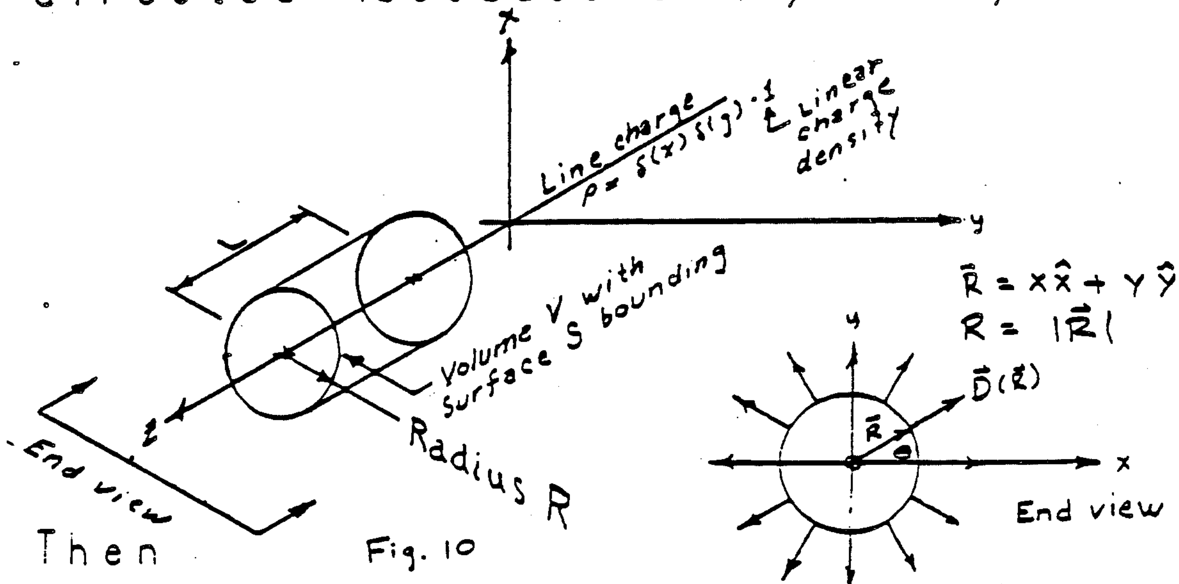
Consider figure 10 below. The $\vec{D}$ field at a radius, $\vec{R}$, must be constant in magnitude and radially directed (because of symmetry).



Then

Fig. 10

$$\oint\!\!\oint_S \vec{D}(\vec{R})\cdot\hat{n}\,ds = L\int_0^{2\pi} D(R)\,\hat{R}\cdot\hat{R}\cdot R\,d\theta$$

$$= L2\pi D\cdot R = L\cdot 1 \implies D(R) = \frac{1}{2\pi R}$$

Therefore, $\vec{D}(\vec{R}) = \frac{1}{2\pi R}\hat{R}$.

The potential due to this line charge is

$$\varphi(R) = -\frac{1}{\epsilon}\int_{R_0}^{R} \frac{1}{2\pi r}\hat{r}\cdot d\vec{r} = -\frac{1}{\epsilon}\int_{R_0}^{R}\frac{dr}{2\pi r} = -\frac{1}{2\pi\epsilon}\ln(R/R_0)$$

$$= -\frac{1}{2\pi\epsilon}\ln R + C,$$ where C is a constant and the path from R_0 to R is along a radial line.

Thus, the Green's function for a cylindrical charge distribution is

$$G(\vec{R}|\vec{R}') = -\frac{1}{2\pi\epsilon}\ln|\vec{R}-\vec{R}'| + C$$

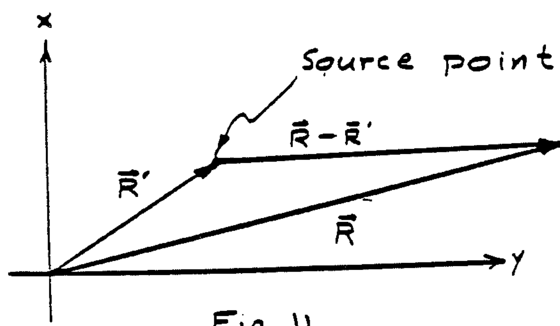


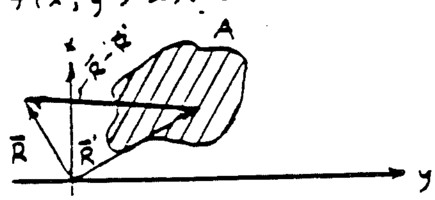
Fig 11

It is convenient (but not necessary) to take $C = 0$.

Then the potential due to a cylinder of charge of cross section A and density $\rho(\vec{R}) = f(x, y)$ is

$$\varphi(\vec{R}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y')\, G(\vec{R}|\vec{R}')\, dx'dy'$$

$$= -\frac{1}{2\pi\epsilon}\iint_A f(x', y') \ln\left[\sqrt{(x-x')^2+(y-y')^2}\right] dx'dy'$$



(We could have obtained this same result by applying Green's theorem as in the previous section).

EXAMPLE: Find the potential due to the strip of charge as shown in figure 12 below.

$$\rho = [u(y+a) - u(y-a)]\,\delta(x)\cdot 1\,C/m$$

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

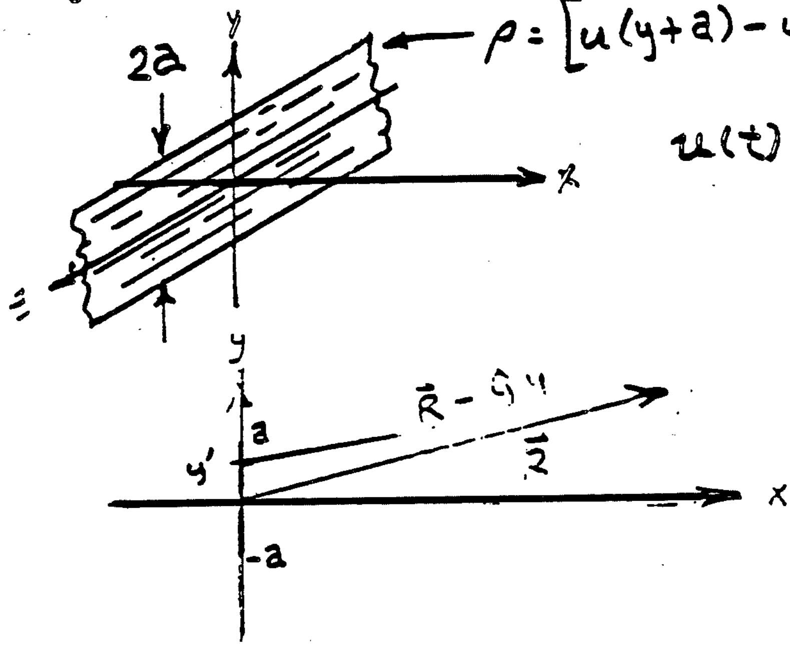


Fig 12

$$\varphi(\vec{R}) = -\frac{1}{2\pi\epsilon}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[u(y'+a) - u(y'-a)]\cdot \delta(x')\,1C/m\;G(\vec{R}\,|\,\vec{R}')\,dx'dy'$$

$$= -\frac{1}{2\pi\epsilon}\,1C/m\int_{-\infty}^{\infty}\ln[x^2 + (y-y')^2]^{1/2}\,dy'.$$

we use $\ln\psi^{1/2} = \frac{1}{2}\ln\psi$; $\ln(uv) = \ln u + \ln$

$x^2 + (y'-y)^2 = [(y'-y) + jx]\cdot[(y'-y) - jx]$,

where $j = \sqrt{-1}$

Therefore,

$$\ln[(y-y')^2+x^2]^{1/2}$$
$$= \tfrac{1}{2}\ln[(y'-y)+jx] + \tfrac{1}{2}\ln[(y'-y)-jx]$$
$$= Re\{\ln[(y'-y)+jx]\}.$$

We must integrate

$$\int_{-a}^{a} \ln[(y'-y)+jx]\,dy'.$$

Let $z = y'-y+jx$. Then
at $y'=-a$, $z=-a-y+jx$ and
at $y'=a$, $z=a-y+jx$.

NOTE: $d(z\ln z) = \ln z\,dz + z\frac{1}{z}dz$
$= \ln z\,dz + dz \longrightarrow$

$$\ln z\,dz = d[z\ln z - z].$$

Therefore,

$$\int_{-a}^{a} \ln[(y'-y)+jx]\,dy'$$

$$= \int_{-a-y+jx}^{a-y+jx} \ln z\,dz$$

$$= (a-y+jx)\ln(a-y+jx) - (a-y+jx)$$
$$- (-a-y+jx)\ln(-a-y+jx) + (-a-y+jx).$$

But $\ln(\pm a-y+jx) =$

$$\ln\left[\sqrt{(a\mp y)^2+x^2}\right] + j\arg[\pm a-y+jx].$$

($\arg(z)$ = the angle of the complex number z).

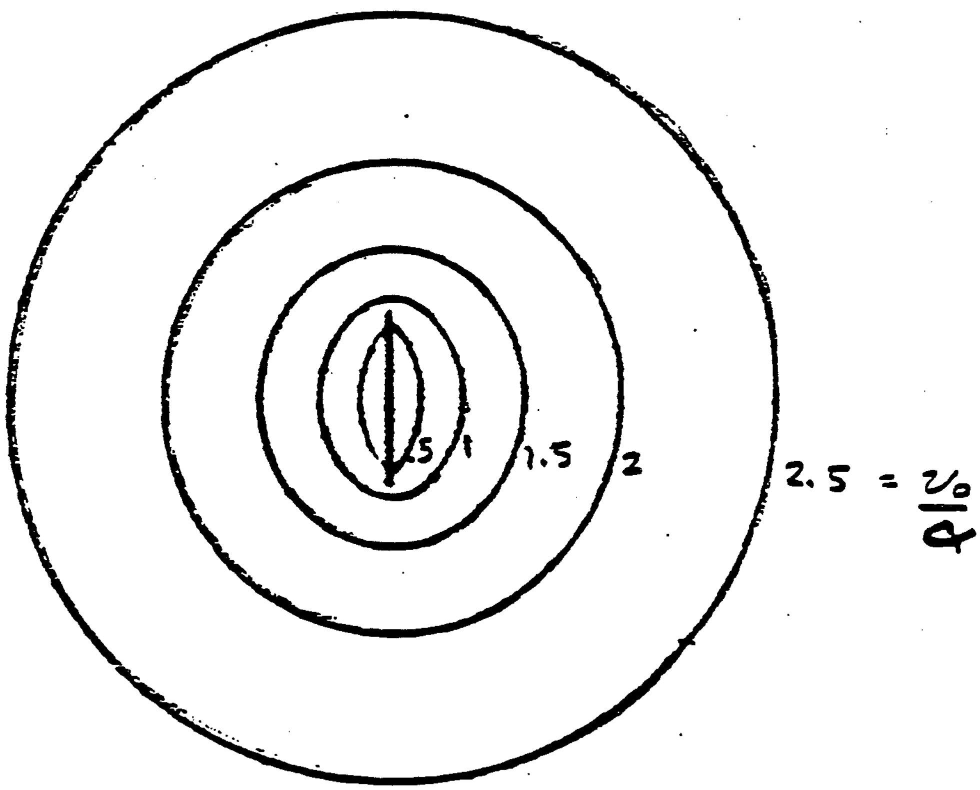
Putting it all together,

$$\varphi(\vec{R}) = -\mathrm{Re}\left\{\int_{-a}^{a} \ln[(y'-y)+jx]\,dy'\right\}\cdot\frac{1\,C/m}{2\pi\epsilon}$$

$$= -\Big\{\tfrac{1}{2}(a-y)\ln[(y-a)^2+x^2]$$
$$+\tfrac{1}{2}(a+y)\ln[(y+a)^2+x^2]$$
$$-x\arg[a-y+jx] + x\arg[-a-y+jx]$$
$$-2a\Big\}\cdot 1\,C/m\,\frac{1}{2\pi\epsilon}$$

The equipotential surfaces (ie; $\phi(\vec{R}) = v_0$ = constant) are plotted below.

$\underline{\underline{a = 1\,m.}}$



α is a normalization constant,

$$\alpha = -1\ C/m \cdot \frac{1}{2\pi\epsilon}$$

(4) The electrostatic potential due to a point source and a volume distribution of charge.

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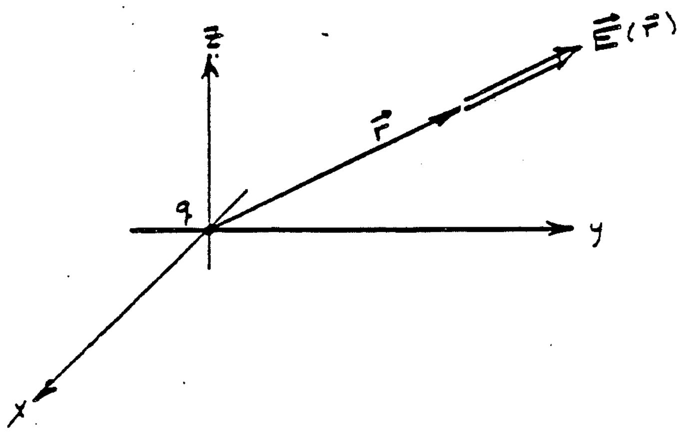
Coulomb's law requires

$$\vec{E} = \frac{1}{4\pi\epsilon} \cdot q \cdot \frac{\hat{r}}{r^2}$$



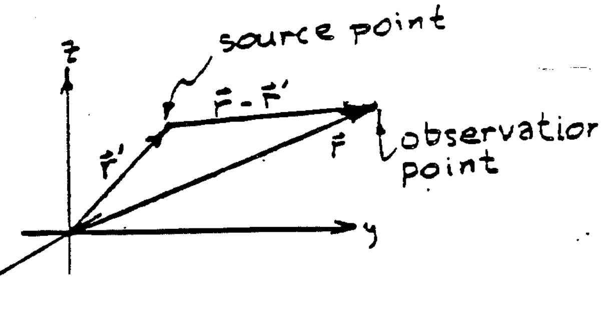
The potential due to a point charge of q at the origin is

$$-\frac{q}{4\pi\epsilon}\int_{\infty}^{r}\frac{\hat{r}'}{(r')^2}\cdot d\vec{r}' = \frac{q}{4\pi\epsilon}\int_{r}^{\infty}\frac{dr'}{(r')^2} = \frac{q}{4\pi\epsilon}\ \frac{1}{r}.$$

Here we have (arbitrarily but conveniently) taken the potential "at ∞" to be zero.

Thus the potential at $\vec{r}$ due to a unit point charge at $\vec{r}\,'$ is

$$G(\vec{r}\,|\,\vec{r}\,') = \frac{1}{4\pi\epsilon}\ \frac{1}{|\vec{r}-\vec{r}\,'|}$$



Thus, for a distribution of charges in V with density ρ, the potential is

$$\varphi(\vec{r}) = \iiint_V \rho(\vec{r}\,')\,\frac{1}{4\pi\epsilon}\ \frac{1}{|\vec{r}-\vec{r}\,'|}\,dx'dy'dz'$$

(5) Summary:

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Thus, we have seen that a solution to

$$\nabla^2 \varphi = -\rho/\epsilon$$

can be constructed by first finding the solution due to a point, line, or surface charge depending on whether ρ is constant in none, one, or two of the variables (x, y, z):

$\rho(x,y,z) = f(z)$: $G(z|z') = -\frac{1}{2\epsilon}|z - z'|$ 1-Dimension

$\rho(x,y,z) = f(x,y)$: $G(\vec{R}|\vec{R}') = -\frac{1}{2\pi\epsilon}\ln|\vec{R} - \vec{R}'|$ 2-Dimensions
where $\vec{R} = \hat{x}x + \hat{y}y$,

$\rho(x,y,z) = f(x,y,z)$: $G(\vec{r}|\vec{r}') = \frac{1}{4\pi\epsilon}\frac{1}{|\vec{r} - \vec{r}'|}$ 3-Dimensions
$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$.

These Green's functions are <u>only</u> valid for <u>static</u> <u>charges</u> <u>in</u> an otherwise <u>empty</u> <u>space</u>.

For other cases such as charges within a rectangular metal box; other Green's functions apply. (We will see examples of these later).

Also, for different differential equations, different Green's functions also apply. For example,

$$G(\bar{r}|\bar{r}') = \frac{1}{4\pi}\frac{e^{-jk|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}$$

is the appropriate Green's function for the Helmholtz equation over an infinite space,

$$\nabla^2\varphi + k^2\varphi = \psi$$

$$\varphi(\bar{r}) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-jk|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}\psi(\bar{r}')\,dx'dy'dz'.$$