

Section 4.3 – Multiplicative Inverses of Matrices

Identity Matrix

The $n \times n$ identity matrix with 1's on the main diagonal and 0's elsewhere and is denoted by

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3} \quad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

The Multiplicative Identity Matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then $AI = IA = A$

Example

$$A = \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution

$$\begin{aligned} AI &= \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4(1) - 7(0) & 4(0) - 7(1) \\ -3(1) + 2(0) & -3(0) + 2(1) \end{bmatrix} \\ &= \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix} = A \\ &= A \end{aligned}$$

Multiplicative inverse of a matrix

Multiplicative inverse of a matrix $A_{n \times n}$ and $A^{-1}_{n \times n}$ if exists, then:

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

Example

Show that B is Multiplicative inverse of A .

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Solution

$$\begin{aligned} A \cdot B &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2(1) - 1(1) & 2(-1) + 1(2) \\ 1(1) + 1(-1) & 1(-1) + 1(2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

$$\begin{aligned} BA &= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

$\therefore B$ is multiplicative inverse of a matrix A : $B = A^{-1}$

Finding Inverse matrix

To find inverse matrix using Gauss-Jordan method:

$$\left[A | I \right] \rightarrow \left[I | A^{-1} \right] \quad \text{where } A^{-1} \text{ read as "A inverse"}$$

For 2 by 2 matrices (*only*)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \begin{bmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix} \end{aligned}$$

If $ad - bc = 0$, then A^{-1} doesn't exist

Example

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix} \Rightarrow A^{-1} = ?$$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{(-1)(4) - (-2)(3)} \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} \end{aligned}$$

Example

$$A = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \Rightarrow A^{-1} = ?$$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{(3)(1) - (-2)(-1)} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \end{aligned}$$

To find inverse matrix using Gauss-Jordan method:

$$\left[A | I \right] \rightarrow \left[I | A^{-1} \right] \quad \text{where } A^{-1} \text{ read as "A inverse"}$$

Example

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \quad \text{Find } A^{-1}$$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{ccc|ccc} -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \end{array} \quad \begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & -1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array} \right] 2R_3 + R_2$$

$$\begin{array}{ccc|ccc} 0 & -2 & -4 & -2 & 0 & 2 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right] \begin{array}{l} R_1 - 2R_3 \\ R_2 - 5R_3 \\ \end{array}$$

$$\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 2 & -2 & -4 \\ 0 & 0 & 0 & 3 & -2 & -4 \end{array} \quad \begin{array}{ccc|ccc} 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & 0 & -5 & 5 & -5 & -10 \\ 0 & 2 & 0 & 6 & -4 & -10 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 2 & 0 & 6 & -4 & -10 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right] \frac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \quad \text{Find } A^{-1}$$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccccc|cccccc} -1 & 2 & 3 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 1 & 0 & 0 & -1 & 0 & -2 & -1 & 0 & 0 \\ \hline 0 & 2 & 5 & 1 & 1 & 0 & 0 & -1 & -2 & -1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array} \right] \frac{1}{2}R_2$$

$$0 \quad 1 \quad \frac{5}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array} \right] R_3 + R_2$$

$$\begin{array}{cccccc|cccccc} 0 & -1 & -2 & -1 & 0 & 1 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ \hline 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array} \right] 2R_3$$

$$0 \quad 0 \quad 1 \quad -1 \quad 1 \quad 2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right] \begin{array}{l} R_1 - 2R_3 \\ R_2 - \frac{5}{2}R_3 \end{array}$$

$$\begin{array}{cccccc|cccccc} 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -\frac{5}{2} & \frac{5}{2} & -\frac{5}{2} & -5 & 0 & 0 & -2 & 2 & -2 & -4 \\ \hline 0 & 1 & 0 & 3 & -2 & -5 & 1 & 0 & 0 & 3 & -2 & -4 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

Solving a System Using A^{-1}

To solve the matrix equation $AX = B$.

- X : matrix of the variables
- A : Coefficient matrix
- B : Constant matrix

$$AX = B$$

$$A^{-1}(AX) = A^{-1}B \quad \text{Multiply both side by } A^{-1}$$

$$(A^{-1}A)X = A^{-1}B \quad \text{Associate property}$$

$$IX = A^{-1}B \quad \text{Multiplicative inverse property}$$

$$X = A^{-1}B \quad \text{Identity property}$$

Example

Solve the system using A^{-1}

$$\begin{array}{rcrcrcrcrcrcl} x & & & + & 2z & = & 6 \\ -x & + & 2y & + & 3z & = & -5 \\ x & - & y & & & = & 6 \end{array}$$

$$\text{Given } A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix}$$

$A \quad X \quad = \quad B$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} 3(6)-2(-5)-4(6) \\ 3(6)-2(-5)-5(6) \\ -1(6)+1(-5)+2(6) \end{bmatrix} = \begin{bmatrix} 18+10-24 \\ 18+10-30 \\ -6-5+12 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

Solution: $\{(4, -2, 1)\}$

Example

Use the inverse of the coefficient matrix to solve the linear system

$$2x - 3y = 4$$

$$x + 5y = 2$$

Solution

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The solution of the system is $(2, 0)$

Exercise

Section 4.3 – Multiplicative Inverses of Matrices

Show that B is Multiplicative inverse of A

$$1. \quad A = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$2. \quad A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

Find the inverse, if exists, of

$$3. \quad A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$$

$$14. \quad A = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$$

$$25. \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$$

$$4. \quad A = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

$$15. \quad A = \begin{pmatrix} -6 & 9 \\ 2 & -3 \end{pmatrix}$$

$$26. \quad A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$$

$$5. \quad A = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$$

$$16. \quad A = \begin{pmatrix} -2 & 7 \\ 0 & 2 \end{pmatrix}$$

$$27. \quad A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$6. \quad A = \begin{bmatrix} a & b \\ 3 & 3 \end{bmatrix}$$

$$17. \quad A = \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$$

$$28. \quad A = \begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$$

$$7. \quad A = \begin{bmatrix} -2 & a \\ 4 & a \end{bmatrix}$$

$$18. \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$29. \quad A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}$$

$$8. \quad A = \begin{bmatrix} 4 & 4 \\ b & a \end{bmatrix}$$

$$19. \quad A = \begin{pmatrix} 2 & 1 \\ a & a \end{pmatrix}$$

$$30. \quad A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$$

$$9. \quad A = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$20. \quad A = \begin{pmatrix} b & 3 \\ b & 2 \end{pmatrix}$$

$$31. \quad A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$$

$$10. \quad A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$21. \quad A = \begin{pmatrix} 1 & a \\ 3 & a \end{pmatrix}$$

$$32. \quad A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 1 & 2 \end{pmatrix}$$

$$11. \quad A = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$$

$$22. \quad A = \begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$$

$$12. \quad A = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$$

$$23. \quad A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$$

$$13. \quad A = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$$

$$24. \quad A = \begin{pmatrix} -3 & \frac{1}{2} \\ 6 & -1 \end{pmatrix}$$

$$33. \quad A = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

$$34. \quad A = \begin{pmatrix} -3 & 1 & -1 \\ 1 & -4 & -7 \\ 1 & 2 & 5 \end{pmatrix}$$

$$35. \quad A = \begin{pmatrix} 1 & 1 & -3 \\ 2 & -4 & 1 \\ -5 & 7 & 1 \end{pmatrix}$$

$$36. \quad A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{pmatrix}$$

$$37. \quad A = \begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -6 & 1 \\ -2 & -2 & 5 & 1 \end{bmatrix}$$

$$38. \quad A = \begin{bmatrix} 1 & -14 & 7 & 38 \\ -1 & 2 & 1 & -2 \\ 1 & 2 & -1 & -6 \\ 1 & -2 & 3 & 6 \end{bmatrix}$$

$$39. \quad A = \begin{bmatrix} 10 & 20 & -30 & 15 \\ 3 & -7 & 14 & -8 \\ -7 & -2 & -1 & 2 \\ 4 & 4 & -3 & 1 \end{bmatrix}$$

State the conditions under which A^{-1} exists. Then find a formula for A^{-1}

$$40. \quad A = [x]$$

$$41. \quad A = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

$$42. \quad A = \begin{bmatrix} 0 & 0 & x \\ 0 & y & 0 \\ z & 0 & 0 \end{bmatrix}$$

$$43. \quad A = \begin{bmatrix} x & 1 & 1 & 1 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{bmatrix}$$

$$44. \quad \text{Solve the system using } A^{-1} \quad \begin{cases} x & + 2z = 6 \\ -x + 2y + 3z = -5 \\ x - y & = 6 \end{cases} \quad \text{Given } A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

45. Solve the system using A^{-1}

$$\begin{cases} x + 2y + 5z = 2 \\ 2x + 3y + 8z = 3 \\ -x + y + 2z = 3 \end{cases}$$

a) Write the linear system as a matrix equation in the form $AX = B$

b) Solve the system using the inverse that is given for the coefficient matrix

$$\text{the inverse of } \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix} \text{ is } \begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}$$

46. Solve the system using A^{-1}

$$\begin{cases} x - y + z = 8 \\ 2y - z = -7 \\ 2x + 3y = 1 \end{cases}$$

a) Write the linear system as a matrix equation in the form $AX = B$

b) Solve the system using the inverse that is given for the coefficient matrix

$$\text{the inverse is } \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

(47–75) Use the *inverse* of the coefficient matrix to solve the linear system

47. $\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$

58. $\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$

67. $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

48. $\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$

59. $\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$

68. $\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$

49. $\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$

60. $\begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$

50. $\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$

61. $\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$

69. $\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$

51. $\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$

62. $\begin{cases} -2x + 3y = 4 \\ -3x + 4y = 5 \end{cases}$

70. $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

52. $\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$

63. $\begin{cases} x - 2y = 6 \\ 4x + 3y = 2 \end{cases}$

53. $\begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$

64. $\begin{cases} 2x - 3y = 7 \\ 4x + y = -7 \end{cases}$

71. $\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$

54. $\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$

65. $\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$

72. $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

55. $\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$

66. $\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$

73. $\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$

56. $\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$

57. $\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$