

Solution

Section 2.6 – Tangent Planes and Linear Approximation

Exercise

Find the tangent plane and normal line of the surface $x^2 + y^2 + z^2 = 3$ at the point $P_0(1, 1, 1)$

Solution

$$f(x, y, z) = x^2 + y^2 + z^2$$

$$f_x = 2x, \quad f_y = 2y, \quad f_z = 2z$$

$$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\nabla f(1, 1, 1) = 2\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\textbf{Tangent Line: } f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

$$2(x - 1) + 2(y - 1) + 2(z - 1) = 0$$

$$2x + 2y + 2z = 6$$

$$\underline{x + y + z = 3}$$

$$\textbf{Normal Line: } x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$$

$$\underline{x = 1 + 2t, \quad y = 1 + 2t, \quad z = 1 + 2t}$$

Exercise

Find the tangent plane and normal line of the surface $x^2 + 2xy - y^2 + z^2 = 7$ at the point $P_0(1, -1, 3)$

Solution

$$f(x, y, z) = x^2 + 2xy - y^2 + z^2$$

$$\rightarrow f_x = 2x + 2y, \quad f_y = 2x - 2y, \quad f_z = 2z$$

$$\nabla f = (2x + 2y)\hat{i} + (2x - 2y)\hat{j} + 2z\hat{k}$$

$$\nabla f(1, -1, 3) = 4\hat{j} + 6\hat{k}$$

$$\textbf{Tangent Line: } f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

$$0(x - 1) + 4(y + 1) + 6(z - 3) = 0$$

$$4y + 6z = 14$$

$$\underline{2y + 3z = 7}$$

Normal Line: $x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$
 $\underline{x = 1, \quad y = -1 + 4t, \quad z = 3 + 6t}$

Exercise

Find the tangent plane and normal line of the surface $\cos \pi x - x^2 y + e^{xz} + yz = 4$ at the point $P_0(0, 1, 2)$

Solution

$$f(x, y, z) = \cos \pi x - x^2 y + e^{xz} + yz$$

$$\rightarrow f_x = -\pi \sin \pi x - 2xy + ze^{xz}, \quad f_y = -x^2 + z, \quad f_z = xe^{xz} + y$$

$$\nabla f = (-\pi \sin \pi x - 2xy + ze^{xz})\hat{i} + (z - x^2)\hat{j} + (xe^{xz} + y)\hat{k}$$

$$\nabla f(0, 1, 2) = 2\hat{i} + 2\hat{j} + \hat{k}$$

Tangent Line: $f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$
 $2(x - 0) + 2(y - 1) + (z - 2) = 0$
 $\underline{2x + 2y + z - 4 = 0}$

Normal Line: $x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$
 $\underline{x = 2t, \quad y = 1 + 2t, \quad z = 2 + t}$

Exercise

Find the tangent plane and normal line of the surface $x^2 - xy - y^2 - z = 0$ at the point $P_0(1, 1, -1)$

Solution

$$f(x, y, z) = x^2 - xy - y^2 - z$$

$$\rightarrow f_x = 2x - y, \quad f_y = -x - 2y, \quad f_z = -1$$

$$\nabla f = (2x - y)\hat{i} - (x + 2y)\hat{j} - \hat{k}$$

$$\nabla f(1, 1, -1) = \hat{i} - 3\hat{j} - \hat{k}$$

Tangent Line: $f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$
 $(x - 1) - 3(y - 1) - (z + 1) = 0$

$$\underline{x - 3y - z + 1 = 0}$$

Normal Line: $x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$

$$\underline{x = 1 + t, \quad y = 1 - 3t, \quad z = -1 - t}$$

Exercise

Find the tangent plane and normal line of the surface $x^2 + y^2 - 2xy - x + 3y - z = -4$ at the point $P_0(2, -3, 18)$

Solution

$$f(x, y, z) = x^2 + y^2 - 2xy - x + 3y - z$$

$$\rightarrow f_x = 2x - 2y - 1, \quad f_y = 2y - 2x + 3, \quad f_z = -1$$

$$\nabla f = (2x - 2y - 1)\hat{i} - (2y - 2x + 3)\hat{j} - \hat{k}$$

$$\nabla f(2, -3, 18) = 9\hat{i} - 7\hat{j} - \hat{k}$$

Tangent Line: $f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$

$$9(x - 2) - 7(y + 3) - (z - 18) = 0$$

$$\underline{9x - 7y - z = 21}$$

Normal Line: $x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$

$$\underline{x = 2 + 9t, \quad y = -3 - 7t, \quad z = 18 - t}$$

Exercise

Find an equation for the plane that is tangent to the surface $z = \ln(x^2 + y^2)$ at the point $(1, 0, 0)$

Solution

$$z = f(x, y) = \ln(x^2 + y^2)$$

$$f_x = \frac{2x}{x^2 + y^2} \rightarrow f_x(1, 0) = 2$$

$$f_y = \frac{2y}{x^2 + y^2} \rightarrow f_y(1, 0) = 0$$

Tangent Line: $2(x - 1) - (y - 0) - z = 0$

$$\underline{2x - z - 2 = 0}$$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) - (z - z_0) = 0$$

Exercise

Find an equation for the plane that is tangent to the surface $z = e^{-x^2-y^2}$ at the point $(0, 0, 1)$

Solution

$$z = f(x, y) = e^{-x^2-y^2}$$

$$f_x = -2xe^{-x^2-y^2} \rightarrow f_x(0, 0) = 0$$

$$f_y = -2ye^{-x^2-y^2} \rightarrow f_y(0, 0) = 0$$

$$\text{Tangent Line: } -(z-1) = 0$$

$$\underline{z = 1}$$

$$f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) - (z-z_0) = 0$$

Exercise

Find an equation for the plane that is tangent to the surface $z = \sqrt{y-x}$ at the point $(1, 2, 1)$

Solution

$$z = f(x, y) = \sqrt{y-x}$$

$$f_x = -\frac{1}{2}(y-x)^{-1/2} \rightarrow f_x(1, 2) = -\frac{1}{2}$$

$$f_y = \frac{1}{2}(y-x)^{-1/2} \rightarrow f_y(1, 2) = \frac{1}{2}$$

$$\text{Tangent Line: } -\frac{1}{2}(x-1) + \frac{1}{2}(y-2) - (z-1) = 0$$

$$-\frac{1}{2}x + \frac{1}{2}y - z + \frac{1}{2} = 0$$

$$\underline{x - y + 2z - 1 = 0}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$z = 2x^2 + y^2; \quad (1, 1, 3) \text{ and } (0, 2, 4)$$

Solution

$$f(x, y, z) = 2x^2 + y^2 - z$$

$$\nabla f = 4x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$\nabla f(1, 1, 3) = 4\hat{i} + 2\hat{j} - \hat{k}$$

The equation of the tangent plane:

$$4(x-1) + 2(y-1) - (z-3) = 0$$

$$\underline{4x + 2y - z = 3}$$

$$\nabla f(0, 2, 3) = 4\hat{j} - \hat{k}$$

The equation of the tangent plane:

$$4(y-2) - (z-4) = 0$$

$$\underline{4y - z = 4}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$x^2 + \frac{1}{4}y^2 - \frac{1}{9}z^2 = 1; \quad (0, 2, 0) \text{ and } \left(1, 1, \frac{3}{2}\right)$$

Solution

$$f(x, y, z) = x^2 + \frac{1}{4}y^2 - \frac{1}{9}z^2 - 1$$

$$\nabla f = 2x\hat{i} + \frac{1}{2}y\hat{j} - \frac{2}{9}z\hat{k}$$

$$\nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$\nabla f(0, 2, 0) = \hat{j}$$

The equation of the tangent plane:

$$y - 2 = 0$$

$$\underline{y = 2}$$

$$\nabla f\left(1, 1, \frac{3}{2}\right) = 2\hat{i} + \frac{1}{2}\hat{j} - \frac{1}{3}\hat{k}$$

The equation of the tangent plane:

$$2(x-1) + \frac{1}{2}(y-1) - \frac{1}{3}\left(z - \frac{3}{2}\right) = 0$$

$$2x - 2 + \frac{1}{2}y - \frac{1}{2} - \frac{1}{3}z + \frac{1}{2} = 0$$

$$2x + \frac{1}{2}y - \frac{1}{3}z = 2$$

$$\underline{12x + 3y - 2z = 12}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$xy \sin z - 1 = 0; \quad \left(1, 2, \frac{\pi}{6}\right) \text{ and } \left(-2, -1, \frac{5\pi}{6}\right)$$

Solution

$$f(x, y, z) = xy \sin z - 1$$

$$\nabla f = y \sin z \hat{i} + x \sin z \hat{j} + xy \cos z \hat{k}$$

$$\nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$

$$\begin{aligned} \nabla f\left(1, 2, \frac{\pi}{6}\right) &= 2 \sin \frac{\pi}{6} \hat{i} + \sin \frac{\pi}{6} \hat{j} + 2 \cos \frac{\pi}{6} \hat{k} \\ &= \hat{i} + \frac{1}{2} \hat{j} + \sqrt{3} \hat{k} \end{aligned}$$

The equation of the tangent plane:

$$(x-1) + \frac{1}{2}(y-2) + \sqrt{3}\left(z - \frac{\pi}{6}\right) = 0$$

$$x + \frac{1}{2}y + \sqrt{3}z = 2 + \frac{\pi}{6}\sqrt{3}$$

$$\underline{6x + 3y + 6\sqrt{3}z = 12 + \pi\sqrt{3} \quad |}$$

$$\begin{aligned} \nabla f\left(-2, -1, \frac{5\pi}{6}\right) &= -2 \sin \frac{5\pi}{6} \hat{i} - \sin \frac{5\pi}{6} \hat{j} + 2 \cos \frac{5\pi}{6} \hat{k} \\ &= -\frac{1}{2} \hat{i} - \hat{j} - \sqrt{3} \hat{k} \end{aligned}$$

The equation of the tangent plane:

$$-\frac{1}{2}(x+2) - (y+1) - \sqrt{3}\left(z - \frac{5\pi}{6}\right) = 0$$

$$-\frac{1}{2}x - y - \sqrt{3}z = 2 - \frac{5\pi}{6}\sqrt{3}$$

$$\underline{3x + 6y + 6\sqrt{3}z = 5\pi\sqrt{3} - 12 \quad |}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$yze^{xz} - 8 = 0; \quad (0, 2, 4) \text{ and } (0, -8, -1)$$

Solution

$$f(x, y, z) = yze^{xz} - 8$$

$$\nabla f = yz^2 e^{xz} \hat{i} + ze^{xz} \hat{j} + (y + xyz) e^{xz} \hat{k}$$

$$\nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$

$$\nabla f(0, 2, 4) = 32\hat{i} + 4\hat{j} + 2\hat{k}$$

The equation of the tangent plane:

$$32(x-0) + 4(y-2) + 2(z-4) = 0$$

$$\underline{32x + 4y + 2z = 16 \quad |}$$

$$\nabla f(0, -8, -1) = -8\hat{i} - \hat{j} - 8\hat{k}$$

The equation of the tangent plane:

$$-8(x-0)-(y+8)-8(z+1)=0$$

$$\underline{8x + y + 8z = 16}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$z = x^2 e^{x-y}; \quad (2, 2, 4) \text{ and } (-1, -1, 1)$$

Solution

$$f(x, y, z) = x^2 e^{x-y} - z$$

$$\nabla f = (2x + x^2) e^{x-y} \hat{i} - x^2 e^{x-y} \hat{j} - \hat{k} \qquad \nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$

$$\nabla f(2, 2, 4) = 8\hat{i} - 4\hat{j} - \hat{k}$$

The equation of the tangent plane:

$$8(x-2) - 4(y-2) - (z-4) = 0$$

$$\underline{8x - 4y - z = 4}$$

$$\nabla f(-1, -1, 1) = -\hat{i} - \hat{j} - \hat{k}$$

The equation of the tangent plane:

$$-(x+1) - (y+1) - (z-1) = 0$$

$$\underline{x + y + z = -1}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$z = \ln(1 + xy); \quad (1, 2, \ln 3) \text{ and } (-2, -1, \ln 3)$$

Solution

$$f(x, y) = \ln(1 + xy)$$

$$\nabla f = \frac{y}{1+xy} \hat{i} + \frac{x}{1+xy} \hat{j} \qquad \nabla f = f_x \hat{i} + f_y \hat{j}$$

$$\nabla f(1, 2) = \frac{2}{3} \hat{i} + \frac{1}{3} \hat{j}$$

The equation of the tangent plane:

$$z = f(1, 2) + f_x(1, 2)(x-1) + f_y(1, 2)(y-2)$$

$$= \ln 3 + \frac{2}{3}(x-1) + \frac{1}{3}(y-2)$$

$$= \ln 3 + \frac{2}{3}x - \frac{2}{3} + \frac{1}{3}y - \frac{2}{3}$$

$$\underline{= \frac{2}{3}x + \frac{1}{3}y - \frac{4}{3} + \ln 3}$$

$$\nabla f(-2, -1) = -\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j}$$

The equation of the tangent plane:

$$z = f(-2, -1) + f_x(-2, -1)(x+2) + f_y(-2, -1)(y+1)$$

$$= \ln 3 - \frac{1}{3}(x+2) - \frac{2}{3}(y+1)$$

$$= \ln 3 - \frac{1}{3}x - \frac{2}{3} - \frac{2}{3}y - \frac{2}{3}$$

$$\underline{= -\frac{2}{3}x - \frac{1}{3}y - \frac{4}{3} + \ln 3}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$z = f(x, y) = \frac{1}{x^2 + y^2} \text{ at the point } \left(1, 1, \frac{1}{2}\right)$$

Solution

$$f_x = -\frac{2x}{(x^2 + y^2)^2} \bigg|_{\left(1, 1, \frac{1}{2}\right)}$$

$$= -\frac{2}{(1+1)^2}$$

$$\underline{= -\frac{1}{2}}$$

$$f_y = -\frac{2y}{(x^2 + y^2)^2} \bigg|_{\left(1, 1, \frac{1}{2}\right)}$$

$$= -\frac{2}{4}$$

$$\underline{= -\frac{1}{2}}$$

$$f(x, y, z) = \frac{1}{x^2 + y^2} - 1$$

$$\underline{f_z = -1}$$

Tangent plane:

$$-\frac{1}{2}(x-1) - \frac{1}{2}(y-1) - \left(z - \frac{1}{2}\right) = 0$$

$$-x + 1 - y + 1 - 2z + 1 = 0$$

$$\underline{x + y + 2z = 3}$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$x^2 + y + z = 3; \quad (1, 1, 1) \text{ and } (2, 0, -1)$$

Solution

$$f(x, y, z) = x^2 + y + z - 3$$

$$\nabla f = \langle 2x, 1, 1 \rangle$$

At $(1, 1, 1)$:

$$\nabla f = \langle 2, 1, 1 \rangle$$

Tangent plane:

$$2(x-1) + (y-1) + (z-1) = 0$$

$$\underline{2x + y + z = 4}$$

At $(2, 0, -1)$:

$$\nabla f = \langle 4, 1, 1 \rangle$$

Tangent plane:

$$4(x-2) + y + (z+1) = 0$$

$$\underline{4x + y + z = 7}$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$x^2 + y^3 + z^4 = 2; \quad (1, 0, 1) \text{ and } (-1, 0, 1)$$

Solution

$$f(x, y, z) = x^2 + y^3 + z^4 - 2$$

$$\nabla f = \langle 2x, 3y^2, 4z^3 \rangle$$

At $(1, 0, 1)$:

$$\nabla f = \langle 2, 0, 4 \rangle$$

Tangent plane:

$$2(x-1) + 4(z-1) = 0$$

$$2x + 4z = 6$$

$$\underline{x + 2z = 3}$$

At $(-1, 0, 1)$:

$$\nabla f = \langle -2, 0, 4 \rangle$$

Tangent plane:

$$-2(x-1) + 4(z-1) = 0$$

$$\underline{x - 2z = -3}$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$xy + xz + yz = 12; \quad (2, 2, 2) \text{ and } (2, 0, 6)$$

Solution

$$f(x, y, z) = xy + xz + yz - 12$$

$$\nabla f = \langle y + z, x + z, x + y \rangle$$

At $(2, 2, 2)$:

$$\nabla f = \langle 4, 4, 4 \rangle$$

Tangent plane:

$$4(x-2) + 4(y-2) + 4(z-2) = 0$$

$$\underline{x + y + z = 6}$$

At $(2, 0, 6)$:

$$\nabla f = \langle 6, 8, 2 \rangle$$

Tangent plane:

$$6(x-2) + 8y + 2(z-6) = 0$$

$$\underline{3x + 4y + z = 12}$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$x^2 + y^2 - z^2 = 0; \quad (3, 4, 5) \text{ and } (-4, -3, 5)$$

Solution

$$f(x, y, z) = x^2 + y^2 - z^2$$

$$\nabla f = \langle 2x, 2y, -2z \rangle$$

At $(3, 4, 5)$:

$$\nabla f = \langle 6, 8, -10 \rangle$$

Tangent plane:

$$6(x-3) + 8(y-4) - 10(z-5) = 0$$

$$\underline{3x + 4y - 5z = 0}$$

At $(-4, -3, 5)$:

$$\nabla f = \langle -8, -6, -10 \rangle$$

Tangent plane:

$$-8(x+4) - 6(y+3) - 10(z-5) = 0$$

$$\underline{4x + 3y + 5z = 0}$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$xy \sin z = 1; \quad \left(1, 2, \frac{\pi}{6}\right) \text{ and } \left(-2, -1, \frac{5\pi}{6}\right)$$

Solution

$$f(x, y, z) = xy \sin z - 1$$

$$\nabla f = \langle y \sin z, x \sin z, xy \cos z \rangle$$

At $\left(1, 2, \frac{\pi}{6}\right)$:

$$\nabla f = \left\langle 1, \frac{1}{2}, \sqrt{3} \right\rangle$$

Tangent plane:

$$(x-1) + \frac{1}{2}(y-2) + \sqrt{3}\left(z - \frac{\pi}{6}\right) = 0$$

$$\underline{x + \frac{1}{2}y + \sqrt{3}z = 2 + \frac{\pi\sqrt{3}}{6}}$$

At $\left(-2, -1, \frac{5\pi}{6}\right)$:

$$\nabla f = \left\langle -\frac{1}{2}, -1, -\sqrt{3} \right\rangle$$

Tangent plane:

$$-\frac{1}{2}\left(x + \frac{1}{2}\right) - (y+1) - \sqrt{3}\left(z - \frac{5\pi}{6}\right) = 0$$

$$\underline{\frac{1}{2}x + y + \sqrt{3}z = \frac{5\pi\sqrt{3}}{6} - 2}$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$yze^{xz} = 8; \quad (0, 2, 4) \text{ and } (0, -8, -1)$$

Solution

$$f(x, y, z) = yze^{xz} - 8$$

$$\nabla f = \langle yz^2 e^{xz}, ze^{xz}, e^{xz}(y + xyz) \rangle$$

At $(0, 2, 4)$:

$$\nabla f = \langle 32, 4, 8 \rangle$$

Tangent plane:

$$32x + 4(y - 2) + 8(z - 4) = 0$$

$$\underline{8x + y + 2z = 10}$$

At $(0, -8, -1)$:

$$\nabla f = \langle -8, -1, 8 \rangle$$

Tangent plane:

$$-8x - (y + 8) + 8(z + 1) = 0$$

$$\underline{8x + y - 8z = 0}$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$z^2 - \frac{x^2}{16} - \frac{y^2}{9} = 1; \quad (4, 3, -\sqrt{3}) \text{ and } (-8, 9, \sqrt{14})$$

Solution

$$f(x, y, z) = z^2 - \frac{x^2}{16} - \frac{y^2}{9} - 1$$

$$\nabla f = \langle -\frac{1}{8}x, -\frac{2}{9}y, 2z \rangle$$

At $(4, 3, -\sqrt{3})$:

$$\nabla f = \langle -\frac{1}{2}, -\frac{2}{3}, -2\sqrt{3} \rangle$$

Tangent plane:

$$-\frac{1}{2}(x - 4) - \frac{2}{3}(y - 3) - 2\sqrt{3}(z + \sqrt{3}) = 0$$

$$\underline{\frac{1}{2}x + \frac{2}{3}y + 2\sqrt{3}z = -2}$$

At $(-8, 9, \sqrt{14})$:

$$\nabla f = \langle 1, -2, 2\sqrt{14} \rangle$$

Tangent plane:

$$(x+8) - 2(y-9) + 2\sqrt{14}(z-\sqrt{14}) = 0$$

$$\underline{x - 2y + 2\sqrt{14}z = 2}$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$2x + y^2 - z^2 = 0; \quad (0, 1, 1) \text{ and } (4, 1, -3)$$

Solution

$$f(x, y, z) = 2x + y^2 - z^2$$

$$\nabla f = \langle 2, 2y, -2z \rangle$$

At $(0, 1, 1)$:

$$\nabla f = \langle 2, 2, -2 \rangle$$

Tangent plane:

$$2x + 2(y-1) - 2(z-1) = 0$$

$$\underline{x + y - z = 0}$$

At $(4, 1, -3)$:

$$\nabla f = \langle 2, 2, 6 \rangle$$

Tangent plane:

$$2(x-4) + 2(y-1) + 6(z+3) = 0$$

$$\underline{x - y + 3z = -4}$$

Exercise

Find parametric equation for the line tangent to the curve of intersection of the surfaces

$$x + y^2 + 2z = 4, \quad x = 1 \quad \text{at the point } (1, 1, 1)$$

Solution

$$f_x = 1, \quad f_y = 2y, \quad f_z = 2$$

$$\nabla f = \hat{i} + 2y\hat{j} + 2\hat{k} \Big|_{(1, 1, 1)}$$

$$= \hat{i} + 2\hat{j} + 2\hat{k}$$

$$\nabla g = \hat{i}$$

$$\vec{v} = \nabla f \times \nabla g$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \underline{2\hat{j} - 2\hat{k}}$$

Tangent Line: $x = 1, \quad y = 1 + 2t, \quad z = 1 - 2t$

Exercise

Find parametric equation for the line tangent to the curve of intersection of the surfaces $xyz = 1, \quad x^2 + 2y^2 + 3z^2 = 6$ at the point $(1, 1, 1)$

Solution

$$f_x = yz, \quad f_y = xz, \quad f_z = xy$$

$$\nabla f = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$\nabla f(1, 1, 1) = \hat{i} + \hat{j} + \hat{k}$$

$$g_x = 2x, \quad g_y = 4y, \quad g_z = 6z$$

$$\nabla g = 2x\hat{i} + 4y\hat{j} + 6z\hat{k}$$

$$\nabla g(1, 1, 1) = 2\hat{i} + 4\hat{j} + 6\hat{k}$$

$$v = \nabla f \times \nabla g$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 4 & 6 \end{vmatrix}$$

$$= \underline{2\hat{i} - 4\hat{j} + 2\hat{k}}$$

Tangent Line: $x = 1 + 2t, \quad y = 1 - 4t, \quad z = 1 + 2t$

Exercise

Find parametric equation for the line tangent to the curve of intersection of the surfaces $x^3 + 3x^2y^2 + y^3 + 4xy - z^2 = 0, \quad x^2 + y^2 + z^2 = 11$ at the point $(1, 1, 3)$

Solution

$$f_x = 3x^2 + 6xy^2 + 4y \Big|_{(1, 1, 3)}$$

$$= \underline{13}$$

$$f_y = 6x^2y + 3y^2 + 4x \Big|_{(1, 1, 3)} \\ = 13$$

$$f_z = -2z \Big|_{(1, 1, 3)} \\ = -6$$

$$\nabla f(1, 1, 3) = 13\hat{i} + 13\hat{j} - 6\hat{k}$$

$$g_x = 2x, \quad g_y = 2y, \quad g_z = 2z$$

$$\nabla g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\nabla g(1, 1, 3) = 2\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 13 & 13 & -6 \\ 2 & 2 & 6 \end{vmatrix} \qquad \vec{v} = \nabla f \times \nabla g \\ = 90\hat{i} - 90\hat{j}$$

Tangent Line: $\underline{x = 1 + 90t, \quad y = 1 - 90t, \quad z = 3}$

Exercise

Find parametric equation for the line tangent to the curve of intersection of the surfaces

$$x^2 + y^2 = 4, \quad x^2 + y^2 - z = 0 \quad \text{at the point } (\sqrt{2}, \sqrt{2}, 4)$$

Solution

$$f_x = 2x, \quad f_y = 2y, \quad f_z = 0$$

$$\nabla f = 2x\hat{i} + 2y\hat{j}$$

$$\nabla f(\sqrt{2}, \sqrt{2}, 4) = 2\sqrt{2}\hat{i} + 2\sqrt{2}\hat{j}$$

$$g_x = 2x, \quad g_y = 2y, \quad g_z = -1$$

$$\nabla g = 2x\hat{i} + 2y\hat{j} - \hat{k} \Rightarrow \nabla g(\sqrt{2}, \sqrt{2}, 4) = 2\sqrt{2}\hat{i} + 2\sqrt{2}\hat{j} - \hat{k}$$

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\sqrt{2} & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 2\sqrt{2} & -1 \end{vmatrix} \qquad \vec{v} = \nabla f \times \nabla g \\ = -2\sqrt{2}\hat{i} + 2\sqrt{2}\hat{j}$$

Tangent Line: $\underline{x = \sqrt{2} - 2\sqrt{2}t, \quad y = \sqrt{2} + 2\sqrt{2}t, \quad z = 4}$

Exercise

Find an equation for the plane tangent to the level surface $f(x, y, z) = x^2 - y - 5z$ at the point $P_0(2, -1, 1)$. Also, find parametric equations for the line is normal to the surface at P_0 .

Solution

$$\begin{aligned}\nabla f &= 2x\hat{i} - \hat{j} - 5\hat{k} \Big|_{(2, -1, 1)} \\ &= 4\hat{i} - \hat{j} - 5\hat{k}\end{aligned}$$

Tangent Plane:

$$\begin{aligned}4(x-2) - (y+1) - 5(z-1) &= 0 \\ 4x - 8 - y - 1 - 5z + 5 &= 0 \\ \underline{4x - y - 5z = 4}\end{aligned}$$

Normal Line:

$$\begin{cases} x = 2 + 4t \\ y = -1 - t \\ z = 1 - 5t \end{cases}$$

Exercise

By about how much will $f(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2}$ change if the point $P(x, y, z)$ moves from $P_0(3, 4, 12)$ a distance of $ds = 0.1$ unit in the direction of $3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$?

Solution

$$\begin{aligned}f(x, y, z) &= \ln \sqrt{x^2 + y^2 + z^2} & (\ln u)' &= \frac{u'}{u} \\ &= \frac{1}{2} \ln(x^2 + y^2 + z^2)\end{aligned}$$

$$\begin{aligned}f_x &= \frac{x}{x^2 + y^2 + z^2} \Big|_{(3, 4, 12)} \\ &= \frac{3}{9 + 16 + 144} \\ &= \frac{3}{169}\end{aligned}$$

$$\begin{aligned}f_y &= \frac{y}{x^2 + y^2 + z^2} \Big|_{(3, 4, 12)} \\ &= \frac{4}{9 + 16 + 144} \\ &= \frac{4}{169}\end{aligned}$$

$$\begin{aligned}
 f_z &= \frac{z}{x^2 + y^2 + z^2} \Big|_{(3,4,12)} \\
 &= \frac{12}{9+16+144} \\
 &= \frac{12}{169}
 \end{aligned}$$

$$\nabla f = \frac{3}{169} \hat{i} + \frac{4}{169} \hat{j} + \frac{12}{169} \hat{k}$$

$$\begin{aligned}
 \vec{u} &= \frac{3\hat{i} + 4\hat{j} - 2\hat{k}}{\sqrt{9+16+4}} & \vec{u} &= \frac{\vec{v}}{|\vec{v}|} \\
 &= \frac{3}{7} \hat{i} + \frac{4}{7} \hat{j} - \frac{2}{7} \hat{k}
 \end{aligned}$$

$$\begin{aligned}
 \nabla f \cdot \vec{u} &= \left(\frac{3}{169} \hat{i} + \frac{4}{169} \hat{j} + \frac{12}{169} \hat{k} \right) \cdot \left(\frac{3}{7} \hat{i} + \frac{4}{7} \hat{j} - \frac{2}{7} \hat{k} \right) \\
 &= \frac{9}{1183}
 \end{aligned}$$

$$\begin{aligned}
 df &= (\nabla f \cdot \vec{u}) ds \\
 &= \frac{9}{1183} (0.1) \\
 &= \frac{9}{11830} \\
 &\approx 0.0008
 \end{aligned}$$

Exercise

By about how much will $f(x, y, z) = e^x \cos yz$ change if the point $P(x, y, z)$ moves from origin a distance of $ds = 0.1$ unit in the direction of $2\hat{i} + 2\hat{j} - 2\hat{k}$?

Solution

$$f_x = e^x \cos yz \Rightarrow f_x(0,0,0) = 1$$

$$f_y = -ze^x \sin yz \Rightarrow f_y(0,0,0) = 0$$

$$f_z = -ze^x \sin yz \Rightarrow f_z(0,0,0) = 0$$

$$\nabla f = \hat{i}$$

$$\begin{aligned}
 \vec{u} &= \frac{2\hat{i} + 2\hat{j} - 2\hat{k}}{\sqrt{4+4+4}} & \vec{u} &= \frac{\vec{v}}{|\vec{v}|} \\
 &= \frac{2}{2\sqrt{3}} \hat{i} + \frac{2}{2\sqrt{3}} \hat{j} - \frac{2}{2\sqrt{3}} \hat{k}
 \end{aligned}$$

$$= \frac{1}{\sqrt{3}} \hat{i} + \frac{1}{\sqrt{3}} \hat{j} - \frac{1}{\sqrt{3}} \hat{k}$$

$$\begin{aligned} \nabla f \cdot \vec{u} &= (\hat{i}) \cdot \left(\frac{1}{\sqrt{3}} \hat{i} + \frac{1}{\sqrt{3}} \hat{j} - \frac{1}{\sqrt{3}} \hat{k} \right) \\ &= \frac{1}{\sqrt{3}} \end{aligned}$$

$$df = (\nabla f \cdot \vec{u}) ds$$

$$= \frac{1}{\sqrt{3}} (0.1)$$

$$= \frac{1}{10\sqrt{3}} \quad \left| \quad \approx 0.0577 \right|$$

Exercise

Find the linearization $L(x, y)$ of $f(x, y) = x^2 + y^2 + 1$ at the point $(0, 0)$ and $(1, 1)$

Solution

$$f(0, 0) = 1$$

$$f_x = 2x \Rightarrow f_x(0, 0) = 0$$

$$f_y = 2y \Rightarrow f_y(0, 0) = 0$$

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$L(x, y) = 1 + 0(x - 0) + 0(y - 0) \quad \underline{= 1}$$

$$f(1, 1) = 3$$

$$f_x = 2x \Rightarrow f_x(1, 1) = 2$$

$$f_y = 2y \Rightarrow f_y(1, 1) = 2$$

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$L(x, y) = 3 + 2(x - 1) + 2(y - 1)$$

$$\underline{= 2x + 2y - 1}$$

Exercise

Find the linearization $L(x, y)$ of $f(x, y) = (x + y + 2)^2$ at the point $(0, 0)$ and $(1, 2)$

Solution

$$f(0, 0) = 4$$

$$f_x = 2(x + y + 2) \Rightarrow f_x(0, 0) = 4$$

$$f_y = 2(x + y + 2) \Rightarrow f_y(0, 0) = 4$$

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$L(x, y) = 4 + 4(x - 0) + 4(y - 0)$$

$$\underline{= 4 + 4x + 4y}$$

$$f(1, 2) = (1 + 2 + 2)^2 = 25$$

$$f_x = 2(x + y + 2) \Rightarrow f_x(1, 2) = 10$$

$$f_y = 2(x + y + 2) \Rightarrow f_y(1, 2) = 10$$

$$L(x, y) = 25 + 10(x - 1) + 10(y - 2)$$

$$\underline{= 10x + 10y - 5}$$

Exercise

Find the linearization $L(x, y)$ of $f(x, y) = x^3y^4$ at the point $(1, 1)$ and $(0, 0)$

Solution

$$f(1, 1) = 1$$

$$f_x = 3x^2y^4 \Rightarrow f_x(1, 1) = 3$$

$$f_y = 4x^2y^3 \Rightarrow f_y(1, 1) = 4$$

$$L(x, y) = 1 + 3(x - 1) + 4(y - 1)$$

$$\underline{= 3x + 4y - 6}$$

$$f(0, 0) = 0$$

$$f_x = 3x^2y^4 \Rightarrow f_x(0, 0) = 0$$

$$f_y = 4x^2y^3 \Rightarrow f_y(0, 0) = 0$$

$$L(x, y) = 0 + 0(x - 0) + 0(y - 0)$$

$$\underline{= 0}$$

Exercise

Find the linearization $L(x, y)$ of $f(x, y) = e^{2y-x}$ at the point $(0, 0)$ and $(1, 2)$

Solution

$$f(0, 0) = e^0 = 1$$

$$f_x = -e^{2y-x} \Rightarrow f_x(0, 0) = -1$$

$$f_y = 2e^{2y-x} \Rightarrow f_y(0, 0) = 2$$

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$L(x, y) = 1 - 1(x - 0) + 2(y - 0) \\ = 1 - x + 2y$$

$$f(1, 2) = e^3$$

$$f_x = -e^{2y-x} \Rightarrow f_x(1, 2) = -e^3$$

$$f_y = 2e^{2y-x} \Rightarrow f_y(1, 2) = 2e^3$$

$$L(x, y) = e^3 - e^3(x - 1) + 2e^3(y - 2) \\ = -e^3x + 2e^3y - 2e^3$$

Exercise

Find the linearization $L(x, y, z)$ of $f(x, y, z) = x^2 + y^2 + z^2$ at the point $(1, 1, 1)$

Solution

$$f(1, 1, 1) = 3$$

$$f_x = 2x \Rightarrow f_x(1, 1, 1) = 2$$

$$f_y = 2y \Rightarrow f_y(1, 1, 1) = 2$$

$$f_z = 2z \Rightarrow f_z(1, 1, 1) = 2$$

$$L(x, y, z) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0)$$

$$L(x, y, z) = 3 + 2(x - 1) + 2(y - 1) + 2(z - 1) \\ = 2x + 2y + 2z - 3$$

Exercise

Find the linearization $L(x, y, z)$ of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point $(1, 2, 2)$

Solution

$$f(1, 2, 2) = \sqrt{1 + 4 + 4} = 3$$

$$\begin{aligned}
 f_x &= \frac{1}{2} \left(x^2 + y^2 + z^2 \right)^{-1/2} (2x) \\
 &= \frac{x}{\sqrt{x^2 + y^2 + z^2}} \Big|_{(1,2,2)} \\
 &= \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 f_y &= \frac{1}{2} \left(x^2 + y^2 + z^2 \right)^{-1/2} (2y) \\
 &= \frac{y}{\sqrt{x^2 + y^2 + z^2}} \Big|_{(1,2,2)} \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 f_z &= \frac{1}{2} \left(x^2 + y^2 + z^2 \right)^{-1/2} (2z) \\
 &= \frac{z}{\sqrt{x^2 + y^2 + z^2}} \Big|_{(1,2,2)} \\
 &= \frac{2}{3}
 \end{aligned}$$

$$L(x, y, z) = f(x_0, y_0, z_0) + f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0)$$

$$\begin{aligned}
 L(x, y, z) &= 3 + \frac{1}{3}(x - 1) + \frac{2}{3}(y - 2) + \frac{2}{3}(z - 2) \\
 &= \frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z
 \end{aligned}$$

Exercise

Find the linearization $L(x, y, z)$ of $f(x, y, z) = \frac{\sin xy}{z}$ at the point $(\frac{\pi}{2}, 1, 1)$

Solution

$$f\left(\frac{\pi}{2}, 1, 1\right) = \frac{\sin \frac{\pi}{2}}{1} = 1$$

$$\begin{aligned}
 f_x &= \frac{y \cos xy}{z} \Big|_{\left(\frac{\pi}{2}, 1, 1\right)} \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 f_y &= \frac{x \cos xy}{z} \Big|_{\left(\frac{\pi}{2}, 1, 1\right)} \\
 &= 0
 \end{aligned}$$

$$f_z = -\frac{\sin xy}{z^2} \bigg|_{\left(\frac{\pi}{2}, 1, 1\right)}$$

$$\underline{\underline{= -1}}$$

$$L(x, y, z) = f(x_0, y_0, z_0) + f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0)$$

$$L(x, y, z) = 1 + 0\left(x - \frac{\pi}{2}\right) + 0(y - 1) - (z - 1)$$

$$\underline{\underline{= 2 - z}}$$

Exercise

Find the linearization $L(x, y, z)$ of $f(x, y, z) = e^x + \cos(y + z)$ at the point $\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right)$

Solution

$$f\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right) = 1$$

$$f_x = e^x \bigg|_{\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right)}$$

$$\underline{\underline{= 1}}$$

$$f_y = -\sin(y + z) \bigg|_{\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right)}$$

$$\underline{\underline{= -1}}$$

$$f_z = -\sin(y + z) \bigg|_{\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right)}$$

$$\underline{\underline{= -1}}$$

$$L(x, y, z) = f(x_0, y_0, z_0) + f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0)$$

$$L(x, y, z) = 1 + x - \left(y - \frac{\pi}{4}\right) - \left(z - \frac{\pi}{4}\right)$$

$$\underline{\underline{= x - y - z + 1 + \frac{\pi}{2}}}$$

Exercise

Find the linear approximation to the function f at the point (a, b) and estimate the given function value

$$f(x, y) = 4 \cos(2x - y); \quad (a, b) = \left(\frac{\pi}{4}, \frac{\pi}{4}\right); \text{ estimate } f(0.8, 0.8)$$

Solution

$$\begin{aligned}
 f\left(\frac{\pi}{4}, \frac{\pi}{4}\right) &= 4 \cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) \\
 &= 4 \cos\left(\frac{\pi}{4}\right) \\
 &= \underline{2\sqrt{2}}
 \end{aligned}$$

$$f_x(x, y) = -8 \sin(2x - y)$$

$$\begin{aligned}
 f_x\left(\frac{\pi}{4}, \frac{\pi}{4}\right) &= -8 \sin\left(\frac{\pi}{4}\right) \\
 &= \underline{-4\sqrt{2}}
 \end{aligned}$$

$$f_y(x, y) = 4 \sin(2x - y)$$

$$\begin{aligned}
 f_y\left(\frac{\pi}{4}, \frac{\pi}{4}\right) &= 4 \sin\left(\frac{\pi}{4}\right) \\
 &= \underline{2\sqrt{2}}
 \end{aligned}$$

$$\begin{aligned}
 L(x, y) &= 2\sqrt{2} - 4\sqrt{2}\left(x - \frac{\pi}{4}\right) + 2\sqrt{2}\left(y - \frac{\pi}{4}\right) \\
 &= 2\sqrt{2} - 4\sqrt{2}x + \pi\sqrt{2} + 2\sqrt{2}y - \frac{\pi}{2}\sqrt{2} \\
 &= -4\sqrt{2}x + 2\sqrt{2}y + 2\sqrt{2} + \frac{\pi}{2}\sqrt{2}
 \end{aligned}$$

$$L(x, y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$$

$$\begin{aligned}
 f(0.8, 0.8) &= -4\sqrt{2}\frac{4}{5} + 2\sqrt{2}\frac{4}{5} + 2\sqrt{2} + \frac{\pi}{2}\sqrt{2} \\
 &= -\frac{8}{5}\sqrt{2} + 2\sqrt{2} + \frac{\pi}{2}\sqrt{2} \\
 &= \left(\frac{2}{5} + \frac{\pi}{2}\right)\sqrt{2} \\
 &= \underline{\underline{\left(4 + 5\pi\right)\frac{\sqrt{2}}{10}}} \\
 &\approx \underline{\underline{2.787}}
 \end{aligned}$$

Exercise

Find the linear approximation to the function f at the point (a, b) and estimate the given function value

$$f(x, y) = (x + y)e^{xy}; \quad (a, b) = (2, 0); \text{ estimate } f(1.95, 0.05)$$

Solution

$$\begin{aligned}
 f(2, 0) &= 2e^0 \\
 &= \underline{2}
 \end{aligned}$$

$$f_x(x, y) = (1 + xy + y^2)e^{xy}$$

$$\underline{f_x(2, 0) = 1}$$

$$f_y(x, y) = (1 + x^2 + xy)e^{xy}$$

$$\underline{f_y(2, 0) = 5}$$

$$\begin{aligned} L(x, y) &= 2 + (x - 2) + 5(y - 0) \\ &= x + 5y \end{aligned}$$

$$L(x, y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$$

$$\begin{aligned} f(1.95, 0.05) &= \frac{39}{20} + 5\left(\frac{1}{20}\right) \\ &= \underline{\frac{41}{20}} \\ &\approx \underline{2.205} \end{aligned}$$

Exercise

Find the linear approximation to the function f at the point (a, b) and estimate the given function value

$$f(x, y) = xy + x - y; \quad (a, b) = (2, 3); \text{ estimate } f(2.1, 2.99)$$

Solution

$$\begin{aligned} f(2, 3) &= 6 + 2 - 3 \\ &= \underline{5} \end{aligned}$$

$$\begin{aligned} f_x &= y + 1 \Big|_{(2, 3)} \\ &= \underline{4} \end{aligned}$$

$$\begin{aligned} f_y &= x - 1 \Big|_{(2, 3)} \\ &= \underline{1} \end{aligned}$$

$$\begin{aligned} L(x, y) &= 5 + 4(x - 2) + (y - 3) \\ &= 4x + y - 6 \end{aligned}$$

$$L(x, y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$$

$$\begin{aligned} f(2.1, 2.99) &= 4(2.1) + 2.99 - 6 \\ &= 4\frac{21}{10} + \frac{299}{100} - 6 \\ &= \frac{840 + 299 - 600}{100} \\ &= \frac{539}{100} \\ &= \underline{5.39} \end{aligned}$$

Exercise

Find the linear approximation to the function f at the point (a, b) and estimate the given function value

$$f(x, y) = 12 - 4x^2 - 8y^2; \quad (a, b) = (-1, 4); \text{ estimate } f(-1.05, 3.95)$$

Solution

$$\begin{aligned} f(-1, 4) &= 12 - 4 - 128 \\ &= -120 \end{aligned}$$

$$\begin{aligned} f_x &= -8x \Big|_{(-1, 4)} \\ &= 8 \end{aligned}$$

$$\begin{aligned} f_y &= -16y \Big|_{(-1, 4)} \\ &= -64 \end{aligned}$$

$$\begin{aligned} L(x, y) &= -120 + 8(x + 1) - 64(y - 4) \\ &= 8x - 64y + 144 \end{aligned}$$

$$L(x, y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$$

$$\begin{aligned} f(-1.05, 3.95) &= 8(-1.05) - 64(3.95) + 144 \\ &= -117.2 \end{aligned}$$

Exercise

Find the linear approximation to the function f at the point (a, b) and estimate the given function value

$$f(x, y) = -x^2 + 2y^2; \quad (a, b) = (3, -1); \text{ estimate } f(3.1, -1.04)$$

Solution

$$\begin{aligned} f(3, -1) &= -9 + 2 \\ &= -7 \end{aligned}$$

$$\begin{aligned} f_x &= -2x \Big|_{(3, -1)} \\ &= -6 \end{aligned}$$

$$\begin{aligned} f_y &= 4y \Big|_{(3, -1)} \\ &= -4 \end{aligned}$$

$$\begin{aligned} L(x, y) &= -7 - 6(x - 3) - 4(y + 1) \\ &= -6x - 4y + 7 \end{aligned}$$

$$L(x, y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$$

$$f(3.1, -1.04) = -6(3.1) - 4(-1.04) + 7$$

$$= -7.44$$

Exercise

Find the linear approximation to the function f at the point (a, b) and estimate the given function value

$$f(x, y) = \sqrt{x^2 + y^2}; \quad (a, b) = (3, -4); \text{ estimate } f(3.06, -3.92)$$

Solution

$$f(3, -4) = \sqrt{9 + 16}$$

$$= 5$$

$$f_x = \frac{x}{\sqrt{x^2 + y^2}} \Big|_{(3, -4)}$$

$$= \frac{3}{5}$$

$$f_y = \frac{y}{\sqrt{x^2 + y^2}} \Big|_{(3, -4)}$$

$$= -\frac{4}{5}$$

$$L(x, y) = 5 + \frac{3}{5}(x - 3) - \frac{4}{5}(y + 4)$$

$$= \frac{3}{5}x - \frac{4}{5}y$$

$$L(x, y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$$

$$f(3.06, -3.92) = \frac{3}{5}\left(\frac{306}{100}\right) - \frac{4}{5}\left(-\frac{392}{100}\right)$$

$$= \frac{918 + 1568}{500}$$

$$= \frac{1,243}{250}$$

$$= 4.972$$

Exercise

Find the linear approximation to the function f at the point (a, b) and estimate the given function value

$$f(x, y) = \ln(1 + x + y); \quad (a, b) = (0, 0); \text{ estimate } f(0.1, -0.2)$$

Solution

$$f(0, 0) = 0$$

$$f_x = \frac{1}{1+x+y} \Big|_{(0,0)} \\ = 1$$

$$f_y = \frac{1}{1+x+y} \Big|_{(0,0)} \\ = 1$$

$$L(x, y) = x + y$$

$$L(x, y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$$

$$f(0.1, -0.2) = 0.1 - 0.2 \\ = -0.1$$

Exercise

Find the linear approximation to the function f at the point (a, b) and estimate the given function value

$$f(x, y) = \frac{x+y}{x-y}; \quad (a, b) = (3, 2); \text{ estimate } f(2.95, 2.05)$$

Solution

$$f(3, 2) = 5$$

$$f_x = \frac{-2y}{(x-y)^2} \Big|_{(3,2)} \\ = -4$$

$$f_y = \frac{2x}{(x-y)^2} \Big|_{(3,2)} \\ = 6$$

$$L(x, y) = 5 - 4(x - 3) + 6(y - 2) \\ = -4x + 6y + 5$$

$$L(x, y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$$

$$f(2.95, 2.05) = -4(2.95) + 6(2.05) + 5 \\ = 5.5$$

Exercise

Estimate the change in the function $f(x, y) = -2y^2 + 3x^2 + xy$ when (x, y) changes from $(1, -2)$ to $(1.05, -1.9)$.

Solution

$$f_x = 6x + y$$

$$f_x(1, -2) = 4$$

$$f_y = -4y + x$$

$$f_y(1, -2) = 9$$

$$\begin{aligned}\Delta f &\approx f_x(1, -2)\Delta x + f_y(1, -2)\Delta y \\ &= 4(1.05 - 1) + 9(-1.9 + 2) \\ &= .2 + .9 \\ &= 1.1\end{aligned}$$

Exercise

What is the largest value that the directional derivative of $f(x, y, z) = xyz$ can have at the point $(1, 1, 1)$?

Solution

$$\begin{aligned}\nabla f &= yz\hat{i} + xz\hat{j} + xy\hat{k} \Big|_{(1, 1, 1)} \\ &= \hat{i} + \hat{j} + \hat{k}\end{aligned}$$

$$\begin{aligned}\text{The maximum value: } |\nabla f| &= \sqrt{1+1+1} \\ &= \sqrt{3}\end{aligned}$$

Exercise

You plan to calculate the volume inside a stretch of pipeline that is about 36 in. in diameter and 1 mile long. With which measurement should you be more careful, the length or the diameter? Why?

Solution

$$1 \text{ mile} = 5280 \text{ ft}$$

$$r = \frac{36 \text{ in}}{2} \frac{1 \text{ ft}}{12 \text{ in}} = \frac{3}{2} \text{ ft}$$

$$V = \pi r^2 h$$

$$\begin{aligned}dV &= 2\pi r h dr + \pi r^2 dh \\ &= 2\pi \left(\frac{3}{2}\right)(5280) dr + \pi \left(\frac{3}{2}\right)^2 dh \\ &= 15,840\pi dr + \frac{9\pi}{4} dh\end{aligned}$$

We have to be more careful with the diameter, since it has a greater effect on dV .

Exercise

The volume of a cylinder with radius r and height h is $V = \pi r^2 h$. Find the approximate percentage change in the volume when the radius decreases by 3% and the height increases by 2%.

Solution

$$\begin{aligned}\Delta V &= 2\pi r h \Delta r + \pi r^2 \Delta h \\ \frac{dV}{V} &= \frac{2\pi r h}{\pi r^2 h} dr + \frac{\pi r^2}{\pi r^2 h} dh \\ &= 2 \frac{dr}{r} + \frac{dh}{h} \\ &= 2(-3\%) + 2\% \\ &= \underline{4\%} \quad \text{Approximate change volume.}\end{aligned}$$

Exercise

The volume of an ellipsoid with axes of length $2a$, $2b$, and $2c$ is $V = \pi abc$. Find the percentage change in the volume when a increases by 2%, b increases by 1.5%, and c decreases by 2.5%.

Solution

$$\begin{aligned}dV &= \pi(bc\Delta a + ac\Delta b + ab\Delta c) \\ \frac{dV}{V} &= \frac{da}{a} + \frac{db}{b} + \frac{dc}{c} \\ &= 2\% + 1.5\% - 2.5\% \\ &= \underline{1\%} \quad \text{Approximate change volume.}\end{aligned}$$

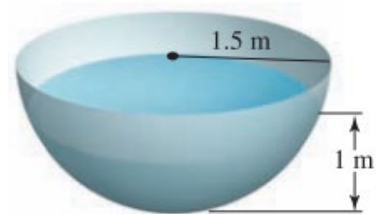
Exercise

A hemispherical tank with a radius of 1.50 m is filled with water to a depth of 1.00 m. Water level drops by 0.05 m (from 1.00 m to 0.95 m)

- a) Approximate the change in the volume of water in the tank. The volume of a spherical cap is $V = \frac{1}{3}\pi h^2(3r - h)$, where r is the radius of the sphere and h is the thickness of the cap (in this case, the depth of the water).
- b) Approximate the change in the surface area of the water in the tank.

Solution

$$\begin{aligned}a) \quad V &= \frac{1}{3}\pi h^2(3r - h) \\ &= \frac{1}{3}\pi(3rh^2 - h^3) \\ dV &= \frac{1}{3}\pi(6rh - 3h^2)dh\end{aligned}$$



$$\begin{aligned}
&= \pi(2rh - h^2)dh \\
&= \pi(2(1.5)(1) - 1^2)(-0.05) \\
&= \underline{-0.1\pi \text{ m}^3}
\end{aligned}$$

$$\begin{aligned}
b) \quad S &= \pi(2rh - h^2) \\
dS &= \pi(2r - 2h)dh \\
&= 2\pi(1.5 - 1)(-0.05) \\
&= \underline{-0.05\pi \text{ m}^2}
\end{aligned}$$

Exercise

Find the linearization $L(x, y, z)$ of $f(x, y, z) = e^x + \cos(y + z)$ at the point $(0, \frac{\pi}{4}, \frac{\pi}{4})$

Consider a closed rectangular box with a square base. If x is measured with error at most 2% and y is measured with error at most 3% use a differential to estimate the corresponding percentage error in computing the box's

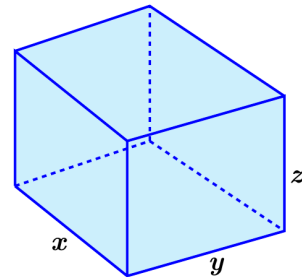
- a) Surface area
- b) Volume

Solution

Given: $\frac{dx}{x} \leq 0.02, \quad \frac{dy}{y} \leq 0.03$

a) $S = 2(xx + xy + xy) = 2x^2 + 4xy$

$$\begin{aligned}
dS &= (4x + 4y)dx + 4xdy \\
&= (4x + 4y)\left(x \frac{dx}{x}\right) + 4xy \frac{dy}{y} \\
&= (4x^2 + 4xy) \frac{dx}{x} + 4xy \frac{dy}{y} \\
&\leq (4x^2 + 4xy)(0.02) + 4xy(0.03) \\
&= 0.02(4x^2) + 0.02(4xy) + 0.03(4xy) \\
&= 0.04(2x^2) + 0.05(4xy) \\
&\leq 0.05(2x^2) + 0.05(4xy) \\
&= 0.05(2x^2 + 4xy) \\
&= \underline{0.05 S}
\end{aligned}$$



$$b) V = x^2 y$$

$$dV = 2xydx + x^2 dy$$

$$= 2x^2 y \frac{dx}{x} + x^2 y \frac{dy}{y}$$

$$\leq 2x^2 y (0.02) + x^2 y (.03)$$

$$= .07 (x^2 y)$$

$$= .07 V$$

Exercise

Consider a closed container in the shape of a cylinder of radius 10 cm and height 15 cm with a hemisphere on each end.

The container is coated with a layer of ice $\frac{1}{2}$ cm thick. Use a differential to estimate the total volume of ice.

(Hint: assume r is radius with $dr = \frac{1}{2}$ and h is height with $dh = 0$)

Solution

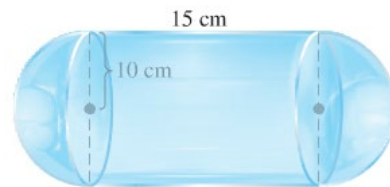
$$V = \frac{4\pi}{3} r^3 + \pi r^2 h$$

$$dV = 4\pi r^2 dr + 2\pi r h dr + \pi r^2 dh$$

$$= (4\pi r^2 + 2\pi r h) dr + \pi r^2 dh$$

$$= (4\pi (10)^2 + 2\pi (10)(15)) \left(\frac{1}{2} \right) + \pi (10)^2 (0)$$

$$= 350\pi \text{ cm}^3$$



Exercise

A standard 12-fl-oz can of soda is essentially a cylinder of radius $r = 1$ in and height $h = 5$ in.

- At these dimensions, how sensitive is the can's volume to a small change in radius versus a small change in height?
- Could you design a soda can that appears to hold more soda but in fact holds the same 12-fl-oz? What might its dimensions be? (There is more than one correct answer.)

Solution

Given: $r = 1$ in $h = 5$ in.

$$a) V = \pi r^2 h \Rightarrow dV = 2\pi r h dr + \pi r^2 dh$$

$$dV = 10\pi dr + \pi dh$$

$$= \pi(10dr + dh)$$

The volume is about 10 times more sensitive to a change in r .

$$b) \quad dV = 0 \Rightarrow 2\pi rhdr + \pi r^2 dh = 0$$

$$2hdr + r dh = 0$$

$$10dr + dh = 0 \Rightarrow dr = -\frac{1}{10} dh$$

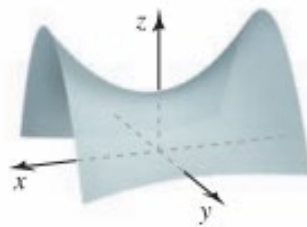
Assume $dh = 1.5$, then $dr = -.15$

$$2h(-0.15) + r(1.5) = 0$$

$$r = 0.85 \text{ in} \quad h = 6.5 \text{ in. is one solution for } \Delta V \approx dV = 0$$

Exercise

Consider the function $f(x, y) = 2x^2 - 4y^2 + 10$, whose graph is shown



- a) Fill in the table showing the value of the directional derivative at points (a, b) in the direction given by the unit vectors \mathbf{u} , \mathbf{v} , and \mathbf{w}
- b) Interpret each of the directional derivatives computed in part(a) at the point $(2, 0)$

Solution

$$a) \quad f_x = 4x \quad f_y = -8y$$

$$\begin{aligned} \nabla f \cdot \vec{u} &= (4a\hat{i} - 8b\hat{j}) \cdot \left(u_x \hat{i} + u_y \hat{j} \right) \\ &= 4au_x - 8bu_y \end{aligned}$$

	$(0, 0)$	$(a, b) = (2, 0)$	$(a, b) = (1, 1)$
$\vec{u} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$	0	$4(2)\frac{\sqrt{2}}{2} - 0 = 4\sqrt{2}$	$4(1)\left(\frac{\sqrt{2}}{2}\right) - 8(1)\left(\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$
$\vec{v} = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$	0	$4(2)\left(-\frac{\sqrt{2}}{2}\right) - 0 = -4\sqrt{2}$	$4(1)\left(-\frac{\sqrt{2}}{2}\right) - 8(1)\left(\frac{\sqrt{2}}{2}\right) = -6\sqrt{2}$
$\vec{w} = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right)$	0	$4(2)\left(-\frac{\sqrt{2}}{2}\right) - 0 = -4\sqrt{2}$	$4(1)\left(-\frac{\sqrt{2}}{2}\right) - 8(1)\left(-\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$

- b) The function is **increasing** @ $(2, 0)$ in the direction of \vec{u}

The function is **decreasing** @ $(2, 0)$ in the direction of $\vec{v} + \vec{w}$

Exercise

Two spheres have the same center and radii r and R , where $0 < r < R$. The volume of the region between the sphere is $V(r, R) = \frac{4\pi}{3}(R^3 - r^3)$.

- First use your intuition. If r is held fixed, how does V change as R increases? What is the sign of V_R ? If R is held fixed, how does V change as r increases (up to the value of R)? What is the sign of V_r ?
- Compute V_r and V_R . Are the results consistent with part (a)?
- Consider spheres with $R = 3$ and $r = 1$. Does the volume change more if R is increased by $\Delta R = 0.1$ (with r fixed) or if r is decreased by $\Delta r = 0.1$ (with R fixed)?

Solution

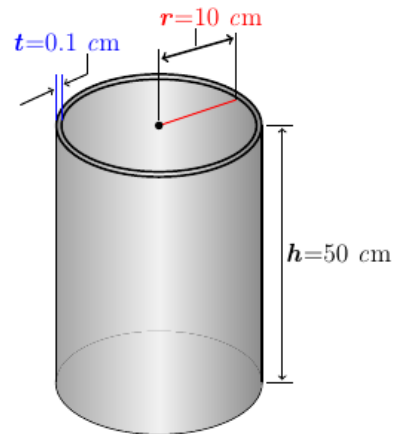
- r is fixed, then $V_R = 4\pi R^2 > 0$
 \therefore If R increases then V increases.
 R is fixed, then $V_r = -4\pi r^2 < 0$
 \therefore If r increases then V decreases.
- Yes, $V_r = -4\pi r^2 < 0$ $V_R = 4\pi R^2 > 0$
- If $R = 3$, $r = 1$, $\Delta r = 0.1$, and $\Delta R = 0.1$
 $\Delta R = 0.1 \Rightarrow \Delta V = 4\pi(3)^2(0.1) = 3.6\pi$
 If r is decreased by 0.1
 $\Delta V = -4\pi(1)^2(-0.1) = 0.4\pi$
 \therefore Volume changes more if R is increased.

Exercise

A company manufactures cylindrical aluminum tubes to rigid specifications. The tubes are designed to have an outside radius of $r = 10 \text{ cm}$, a height of $h = 50 \text{ cm}$, and a thickness of $t = 0.1 \text{ cm}$. The manufacturing process produces tubes with a maximum error of $\pm 0.05 \text{ cm}$ in the radius and height and a maximum error of $\pm 0.0005 \text{ cm}$ in the thickness. The volume of the material used to construct a cylindrical tube is $V(r, h, t) = \pi h t (2r - t)$. Estimate maximum error in the volume of the tube.

Solution

$$V(r, h, t) = 2\pi r h t - \pi h t^2$$



$$\begin{aligned}
dV &= 2\pi h t dr + \left(2\pi r t - \pi t^2\right) dh + 2\pi h(r-t) dt \\
&= 2\pi(50)(0.1)(.05) + \left(2\pi(10)(0.1) - \pi(0.1)^2\right)(.05) + 2\pi(50)(10-0.1)(.0005) \\
&= \pi(0.5 + 1.99(.05) + 990(.0005)) \\
&\approx \underline{3.4385}
\end{aligned}$$

The maximum error in the volume is approximately 3.4385 cm^3 .

The volume is far more sensitive to errors in the thickness, since for the thickness 990π is more than for the radius (10π) and height (1.99π)

Exercise

The volume of a right circular cone with radius r and height h is $V = \frac{1}{3}\pi hr^2$

- Approximate the change in the volume of the cone when the radius changes from $r = 6.5$ to $r = 6.6$ and the height changes from $h = 4.20$ to $h = 4.15$
- Approximate the change in the volume of the cone when the radius changes from $r = 5.4$ to $r = 5.37$ and the height changes from $h = 12.0$ to $h = 11.96$

Solution

$$\begin{aligned}
V &= \frac{1}{3}\pi hr^2 \\
dV &= \frac{1}{3}\pi(2rhdr + r^2dh) \\
a) \quad dV &= \frac{\pi}{3}\left(2(6.5)(4.2)(6.6-6.5) + (6.5)^2(4.15-4.2)\right) \\
&= \frac{\pi}{3}(54.6(0.5) + 42.25(-.05)) \\
&\approx \underline{3.505} \\
b) \quad dV &= \frac{\pi}{3}\left(2(5.4)(12)(-0.03) + (5.4)^2(-0.04)\right) \\
&\approx \underline{-5.293}
\end{aligned}$$

Exercise

The area of an ellipse with axes of length $2a$ and $2b$ is $A = \pi ab$. Approximate the percent change in the area when a increases by 2% and b increases by 1.5%.

Solution

$$\begin{aligned}
dA &= \pi(b da + a db) \\
\frac{dA}{A} &= \frac{\pi}{\pi ab}(b da + a db)
\end{aligned}$$

$$\begin{aligned}\frac{dA}{A} &= \frac{da}{a} + \frac{db}{b} \\ &= 2\% + 1.5\% \\ &= \underline{3.5\%}\end{aligned}$$

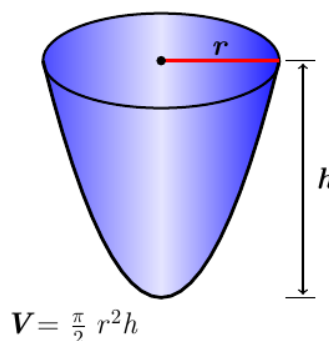
Exercise

The Volume of a segment of a circular paraboloid with radius r and height h is $V = \frac{1}{2}\pi hr^2$.

Approximate the percent change in the volume when the radius decreases by 1.5% and the height increases by 2.2%

Solution

$$\begin{aligned}dV &= \frac{\pi}{2}(2rhdr + r^2dh) \\ \frac{1}{V}dV &= \frac{1}{\frac{1}{2}\pi hr^2} \frac{\pi}{2}(2rhdr + r^2dh) \\ \frac{dV}{V} &= 2\frac{dr}{r} + \frac{dh}{h} \\ &= 2(-1.5\%) + 2.2\% \\ &= \underline{-0.8\%}\end{aligned}$$



Exercise

Batting averages in baseball are defined by $A = \frac{x}{y}$, where $x \geq 0$ is the total number of hits and $y > 0$ is

the total number of at-bats. Treat x and y as positive real numbers and note that $0 \leq A \leq 1$.

- Estimate the change in the batting average if the number of hits increases from 60 to 62 and the number of at-bats increases from 175 to 180.
- If a batter currently has a batting average of $A = 0.35$, does the average decrease if the batter fails to get a hit more than it increases if the batter gets a hit?
- Does the answer in part (b) depend on the current batting average? Explain.

Solution

$$\begin{aligned}a) \quad dA &= \frac{1}{y}dx - \frac{x}{y^2}dy \\ &= \frac{1}{175}(62 - 60) - \frac{60}{175^2}(180 - 175) \\ &= \frac{2}{175} - \frac{300}{175^2} \\ &= \frac{50}{30,625}\end{aligned}$$

$$= \frac{2}{1,225}$$

$$\approx 0.001633$$

b) If the batter fails to get a hit, the average decreases by

$$\frac{x}{y} - \frac{x}{y+1} = \frac{x}{y(y+1)}$$

$$= \frac{A}{y+1}$$

If the batter gets a hit, the average increases by

$$\frac{x+1}{y+1} - \frac{x}{y} = \frac{y-x}{y(y+1)}$$

$$= \frac{1-\frac{x}{y}}{y+1}$$

$$= \frac{1-A}{y+1}$$

If $A = 0.35$, the second of these quantities is larger, therefore the answer is no; the batting average changes more if the batter gets a hit than if he fails to get a hit.

c) The answer depends on whether A is less than or greater than 0.50.

Exercise

A conical tank with radius 0.50 m and height 2.0 m is filled with water. Water released from the tank, and the water level drops by 0.05 m (from 2.0 m to 1.95 m).

Approximate the change in volume of water in the tank.

(Hint: When the water level drops, both the radius and height of the cone of water change).

Solution

$$\frac{x}{r} = \frac{y}{h}$$

$$\frac{x}{.5} = \frac{1.95}{2}$$

$$x = .4875$$

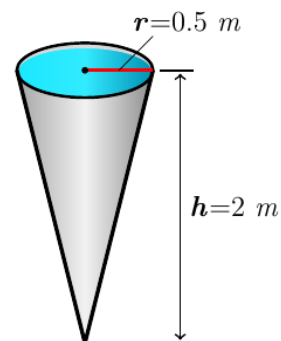
$$dr = 0.4875 - 0.5$$

$$= -0.0125$$

$$V = \frac{1}{3} \pi h r^2$$

$$dV = \frac{1}{3} \pi (2r h dr + r^2 dh)$$

$$\frac{1}{V} dV = \frac{1}{\frac{1}{3} \pi h r^2} \frac{\pi}{3} (2r h dr + r^2 dh)$$



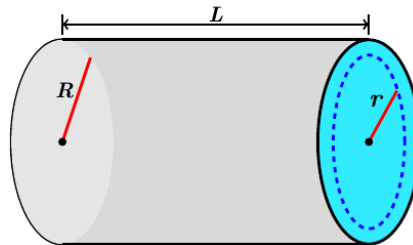
$$\begin{aligned}
\frac{dV}{V} &= 2 \frac{dr}{r} + \frac{dh}{h} \\
&= -2 \frac{0.0125}{0.5} - \frac{.05}{2} \\
&= -0.05 - \frac{.05}{2} \\
&= -0.075
\end{aligned}$$

$$\begin{aligned}
V &= \frac{1}{3} \pi (2) (0.5)^2 \\
&\approx 0.5236
\end{aligned}$$

$$\begin{aligned}
dV &= (-0.075)(.5236) \\
&\approx 0.03927 \text{ m}^3
\end{aligned}$$

Exercise

Poiseuille's law is a fundamental law of fluid dynamics that describes the flow velocity of a viscous incompressible fluid in a cylinder (it is used to model blood flow through veins and arteries). It says that in a cylinder of radius R and length L , the velocity of the fluid $r \leq R$ units from the centerline of the cylinder is $V = \frac{P}{4L\upsilon} (R^2 - r^2)$, where P is the difference in the pressure between the ends of the cylinder and υ is the viscosity of the fluid. Assuming that P and υ are constant, the velocity V along the centerline of the cylinder ($r = 0$) is $V = \frac{kR^2}{L}$, where k is a constant that we will take to be $k = 1$.



- Estimate the change in the centerline velocity ($r = 0$) if the radius of the flow cylinder increases from $R = 3 \text{ cm}$ to $R = 3.05 \text{ cm}$ and the length increases from $L = 50 \text{ cm}$ to $L = 50.5 \text{ cm}$.
- Estimate the percent change in the centerline velocity if the radius of the flow cylinder R decreases by 1% and the length increases by 2%.

Solution

$$k = 1 \rightarrow V = \frac{R^2}{L}$$

$$\begin{aligned}
a) \quad dV &= \frac{2R}{L} dR - \frac{R^2}{L^2} dL \\
&= \frac{2(3)}{50} (3.05 - 3) - \frac{3^2}{50^2} (50.5 - 50)
\end{aligned}$$

$$\begin{aligned}
&= \frac{3}{25}(0.05) - \frac{9}{2500}(0.5) \\
&= \frac{3}{500} - \frac{9}{5000} \\
&= \frac{21}{5,000} \\
&= 0.0042 \text{ cm}^3
\end{aligned}$$

$$b) \frac{1}{V} dV = \frac{L}{R^2} \left(\frac{2R}{L} dR - \frac{R^2}{L^2} dL \right)$$

$$\frac{dV}{V} = 2 \frac{dR}{R} - \frac{dL}{L}$$

$$= 2(-1\%) - (2\%)$$

$$= -4\%$$

R decreases by 1% and the length increases by 2%.

V will decrease by approximately 4%.

Exercise

Suppose that in a large group of people a fraction $0 \leq r \leq 1$ of the people have flu. The probability that in n random encounters, you will meet at least one person with flu is $P = f(n, r) = 1 - (1 - r)^n$. Although n is a positive integer, regard it as a positive real number.

a) Compute f_r and f_n .

b) How sensitive is the probability P to the flu rate r ? Suppose you meet $n = 20$ people.

Approximately how much does the probability P increase if the flu rate increases from $r = 0.1$ to $r = 0.11$ (with n fixed)?

c) Approximately how much does the probability P increase the flu rate increases from $r = 0.9$ to $r = 0.91$

d) Interpret the results of parts (b) and (c).

Solution

$$a) \underline{f_r = n(1-r)^{n-1}}$$

$$f = 1 - (1 - r)^n$$

$$f_n = -\frac{\partial}{\partial n} (1 - r)^n$$

$$\ln y = \ln (1 - r)^n$$

$$\ln y = n \ln (1 - r)$$

$$\frac{y_n}{y} = \ln (1 - r)$$

$$y_n = (1 - r)^n \ln (1 - r)$$

$$\underline{f_n = -(1-r)^n \ln(1-r)}$$

b) $n = 20$ $r = 0.1$

$$\Delta P \approx f_r(20, 0.1)(0.11 - 0.1)$$

$$= 20(1 - 0.1)^{19}(0.01)$$

$$\approx 0.027$$

$$f_r = n(1-r)^{n-1}$$

c) $n = 20$ $r = 0.9$

$$\Delta P \approx f_r(20, 0.9)(.01)$$

$$= 20(1 - 0.9)^{19}(0.01)$$

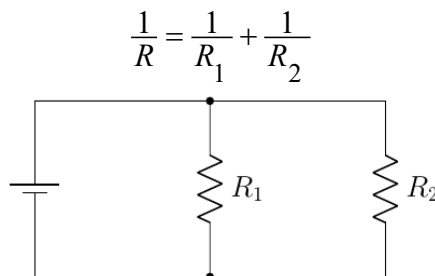
$$\approx 2 \times 10^{-20}$$

$$f_r = n(1-r)^{n-1}$$

- d)** Small changes in the flu rate have a greater effect on the probability of catching the flu when the flu rate is small compared to when the flu rate is large.

Exercise

When two electrical resistors with resistance $R_1 > 0$ and $R_2 > 0$ are wired in parallel in a circuit, the combined resistance R is given by



- a) Estimate the change in R if R_1 increases from 2Ω to 2.05Ω and R_2 decreases from 3Ω to 2.95Ω .
- b) Is it true that if $R_1 = R_2$ and R_1 increases by the same small amount as R_2 decreases, then R is approximately unchanged? Explain.
- c) Is it true that if R_1 and R_2 increase, then R increases? Explain.
- d) Suppose $R_1 > R_2$ and R_1 increases by the same small amount as R_2 decreases. Does R increase or decrease?

Solution

a) $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

$$\frac{1}{R} = \frac{R_2 + R_1}{R_1 R_2}$$

$$\begin{aligned}
R &= \frac{R_1 R_2}{R_2 + R_1} \\
dR &= \frac{R_2^2}{(R_2 + R_1)^2} dR_1 + \frac{R_1^2}{(R_2 + R_1)^2} dR_2 \quad \left(\frac{ax+b}{cx+d} \right)' = \frac{ad-bc}{(cx+d)^2} \\
&= \frac{R_2^2}{(R_2 + R_1)^2} \frac{R_1^2}{R_1^2} dR_1 + \frac{R_1^2}{(R_2 + R_1)^2} \frac{R_2^2}{R_2^2} dR_2 \\
&= \left(\frac{R_1 R_2}{R_2 + R_1} \right)^2 \frac{dR_1}{R_1^2} + \left(\frac{R_1 R_2}{R_2 + R_1} \right)^2 \frac{dR_2}{R_2^2} \\
&= R^2 \left(\frac{dR_1}{R_1^2} + \frac{dR_2}{R_2^2} \right) \\
&= \left(\frac{6}{5} \right)^2 \left(\frac{2.05-2}{4} + \frac{2.95-3}{9} \right) \\
&= \frac{36}{25} \left(\frac{.05}{4} - \frac{.05}{9} \right) \\
&= \frac{36}{500} \left(\frac{5}{36} \right) \\
&= \frac{1}{100} \\
&= \underline{0.01 \, \Omega}
\end{aligned}$$

b) If $R_1 = R_2$

R_1 increases by the same small amount as R_2 decreases.

$$dR_1 = -dR_2$$

$$\begin{aligned}
dR &= R^2 \left(\frac{dR_1}{R_1^2} + \frac{dR_2}{R_2^2} \right) \\
&= R^2 \left(-\frac{dR_2}{R_2^2} + \frac{dR_2}{R_2^2} \right) \\
&= \underline{0}
\end{aligned}$$

c) If R_1 and R_2 increase

$$dR = R^2 \left(\frac{dR_1}{R_1^2} + \frac{dR_2}{R_2^2} \right) > 0$$

Therefore, R increases

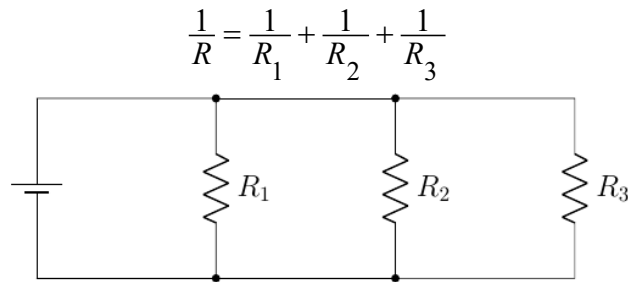
d) *Given:* $R_1 > R_2$

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

R is more sensitive to changes in R_2 , so if R_1 increases by the same small amount as R_2 decreases, then R will decrease.

Exercise

When three electrical resistors with resistance $R_1 > 0$, $R_2 > 0$ and $R_3 > 0$ are wired in parallel in a circuit, the combined resistance R is given by



Estimate the change in R if R_1 increases from 2Ω to 2.05Ω , R_2 decreases from 3Ω to 2.95Ω , and R_3 increases from 1.5Ω to 1.55Ω

Solution

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

$$\frac{1}{R} = \frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_1 R_2 R_3}$$

$$R = \frac{R_1 R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

$$-\frac{1}{R^2} dR = -\frac{1}{R_1^2} dR_1 - \frac{1}{R_2^2} dR_2 - \frac{1}{R_3^2} dR_3$$

$$dR = R^2 \left(\frac{1}{R_1^2} dR_1 + \frac{1}{R_2^2} dR_2 + \frac{1}{R_3^2} dR_3 \right)$$

$$= \left(\frac{2(3)(1.5)}{6 + 3 + 4.5} \right)^2 \left(\frac{1}{4}(2.05 - 2) + \frac{1}{9}(2.95 - 3) + \frac{100}{225}(1.55 - 1.5) \right)$$

$$\begin{aligned}
&= \left(\frac{9}{13.5}\right)^2 \left(\frac{1}{4}(.05) - \frac{1}{9}(.05) + \frac{100}{225}(.05)\right) \\
&= \left(\frac{90}{135}\right)^2 \left(\frac{5}{400} - \frac{5}{900} + \frac{1}{45}\right) \\
&= \left(\frac{2}{3}\right)^2 \left(\frac{1}{80} - \frac{1}{180} + \frac{1}{45}\right) \\
&= \frac{4}{9} \left(\frac{9-4+16}{720}\right) \\
&= \frac{1}{9} \left(\frac{21}{180}\right) \\
&= \frac{7}{540} \Omega \\
&\approx 0.013 \Omega
\end{aligned}$$

Exercise

Consider the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and the plane P given by $Ax + By + Cz + 1 = 0$. Let

$$h = \frac{1}{\sqrt{A^2 + B^2 + C^2}} \quad \text{and} \quad m = \sqrt{a^2 A^2 + b^2 B^2 + c^2 C^2}$$

- Find the equation of the plane tangent to the ellipsoid at the point (p, q, r) .
- Find the two points on the ellipsoid at which the tangent plane parallel to P and find equations of the tangent planes.
- Show that the distance between the origin and the plane P is h .
- Show that the distance between the origin and the tangent planes is hm .
- Find a condition that guarantees the plane P does not intersect the ellipsoid.

Solution

$$a) \quad f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

$$\begin{aligned}
\nabla f &= \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right\rangle \Big|_{(p, q, r)} \\
&= \left\langle \frac{2p}{a^2}, \frac{2q}{b^2}, \frac{2r}{c^2} \right\rangle
\end{aligned}$$

Equation of the **plane tangent** to the ellipsoid at the point (p, q, r) is:

$$\frac{2p}{a^2}(x-p) + \frac{2q}{b^2}(y-q) + \frac{2r}{c^2}(z-r) = 0$$

$$\frac{p}{a^2}x + \frac{q}{b^2}y + \frac{r}{c^2}z = \frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2}$$

$$\left. \begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= 1 \\ \frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} &= 1 \\ \therefore \frac{p}{a^2}x + \frac{q}{b^2}y + \frac{r}{c^2}z &= 1 \end{aligned} \right| (p, q, r)$$

b) Given: $Ax + By + Cz + 1 = 0 \rightarrow$ The vector will be $\langle A, B, C \rangle$ and $\left\langle \frac{p}{a^2}, \frac{q}{b^2}, \frac{r}{c^2} \right\rangle$ must be proportional.

$$\left\langle \frac{p}{a^2}, \frac{q}{b^2}, \frac{r}{c^2} \right\rangle = \lambda \langle A, B, C \rangle$$

$$\begin{cases} \frac{p}{a^2} = \lambda A \rightarrow p = \lambda A a^2 \\ \frac{q}{b^2} = \lambda B \rightarrow q = \lambda B b^2 \\ \frac{r}{c^2} = \lambda C \rightarrow r = \lambda C c^2 \end{cases}$$

$$\langle p, q, r \rangle = \lambda \langle A a^2, B b^2, C c^2 \rangle$$

$$\frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} = 1$$

$$\frac{\lambda^2 A^2 a^4}{a^2} + \frac{\lambda^2 B^2 b^4}{b^2} + \frac{\lambda^2 C^2 c^4}{c^2} = 1$$

$$\lambda^2 (A^2 a^2 + B^2 b^2 + C^2 c^2) = 1$$

$$m = \sqrt{a^2 A^2 + b^2 B^2 + c^2 C^2}$$

$$\lambda^2 m^2 = 1$$

$$\lambda = \pm \frac{1}{m}$$

$$\langle p, q, r \rangle = \pm \frac{1}{m} \langle A a^2, B b^2, C c^2 \rangle$$

Equations of the tangent planes: $\underline{(p, q, r) = \pm (A a^2, B b^2, C c^2)}$

c) The distance between the plane $Ax + By + Cz + 1 = 0$ to the origin:

Let $S = (x, y, z)$ be the point on the plane, then

$$\overrightarrow{OS} = \langle x, y, z \rangle$$

$$\vec{n} = \langle A, B, C \rangle$$

$$|\vec{n}| = \sqrt{A^2 + B^2 + C^2}$$

$$\begin{aligned}
d &= \left| \frac{\langle x, y, z \rangle \cdot \langle A, B, C \rangle}{\sqrt{A^2 + B^2 + C^2}} \right| \\
&= \left| \frac{Ax + By + Cz}{\sqrt{A^2 + B^2 + C^2}} \right| \\
&= \left| \frac{-1}{\sqrt{A^2 + B^2 + C^2}} \right| \\
&= \frac{1}{\sqrt{A^2 + B^2 + C^2}} \\
&= h \quad \checkmark
\end{aligned}$$

Distance from a Point to a Plane: $d = \left| \frac{\overrightarrow{OS} \cdot \vec{n}}{|\vec{n}|} \right|$

$$Ax + By + Cz + 1 = 0$$

d) The tangent plane at $Q(p, q, r) = \pm(Aa^2, Bb^2, Cc^2)$ has an equation $Ax + By + Cz = \pm m$

$$\overrightarrow{OQ} = \langle Aa^2, Bb^2, Cc^2 \rangle$$

$$\vec{n} = \langle A, B, C \rangle$$

$$|\vec{n}| = \sqrt{A^2 + B^2 + C^2}$$

$$\begin{aligned}
d &= \left| \frac{\langle Aa^2, Bb^2, Cc^2 \rangle \cdot \langle A, B, C \rangle}{\sqrt{A^2 + B^2 + C^2}} \right| \\
&= \frac{a^2 A^2 + b^2 B^2 + c^2 C^2}{\sqrt{A^2 + B^2 + C^2}} \\
&= hm \quad \checkmark
\end{aligned}$$

$$d = \left| \frac{\overrightarrow{OQ} \cdot \vec{n}}{|\vec{n}|} \right|$$

$$m = \sqrt{a^2 A^2 + b^2 B^2 + c^2 C^2} \quad h = \frac{1}{\sqrt{A^2 + B^2 + C^2}}$$

e) For the plane P does not intersect the ellipsoid if and only if the 2 tangent planes parallel to P are closer to the origin than P ; this is equivalent to the condition $m < 1$.