Solution   *Section* **3.5 – Triple Integrals in Cylindrical and Spherical Coordinates**

Exercise

Evaluate the cylindrical coordinate integral $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\ rdr\ d\theta$

Solution

$$\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\ rdr\ d\theta = \int_0^{2\pi}\int_0^1\left(\sqrt{2-r^2}-r\right)rdr\ d\theta$$

$$= \int_0^{2\pi}\int_0^1\left(r\left(2-r^2\right)^{1/2}-r^2\right)drd\theta \qquad d\left(2-r^2\right)=-2rdr$$

$$= \int_0^{2\pi}\left(\int_0^1\left(-\frac{1}{2}\left(2-r^2\right)^{1/2}d\left(2-r^2\right)-r^2dr\right)\right)d\theta$$

$$= \int_0^{2\pi}\left[-\frac{1}{3}\left(2-r^2\right)^{3/2}-\frac{1}{3}r^3\right]_0^1 d\theta$$

$$= \int_0^{2\pi}\left[\left(-\frac{1}{3}-\frac{1}{3}\right)-\left(-\frac{1}{3}2^{3/2}\right)\right]d\theta$$

$$= \int_0^{2\pi}\left(-\frac{2}{3}+\frac{2^{3/2}}{3}\right)d\theta$$

$$= \frac{2\sqrt{2}-2}{3}[\theta]_0^{2\pi}$$

$$\underline{= 4\pi\frac{\sqrt{2}-1}{3}}$$

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Exercise

Evaluate the cylindrical coordinate integral $\displaystyle\int_0^{2\pi}\int_0^{\theta/(2\pi)}\int_0^{3+24r^2} dz\ rdr\ d\theta$

Solution

$$\int_0^{2\pi}\int_0^{\theta/(2\pi)}\int_0^{3+24r^2} dz\ rdr\ d\theta = \int_0^{2\pi}\int_0^{\theta/(2\pi)}\left(3+24r^2\right)rdr\ d\theta$$

$$= \int_0^{2\pi}\int_0^{\theta/(2\pi)}\left(3r+24r^3\right)dr\ d\theta$$

$$= \int_0^{2\pi}\left[\frac{3}{2}r^2+6r^4\right]_0^{\theta/(2\pi)} d\theta$$

$$= \int_0^{2\pi}\left[\frac{3}{8\pi^2}\theta^2+\frac{6}{16r^4}\theta^4\right] d\theta$$

$$= \left[\frac{1}{8\pi^2}\theta^3+\frac{3}{8r^4}\frac{1}{5}\theta^5\right]_0^{2\pi}$$

$$= \frac{1}{8\pi^2}8\pi^3+\frac{3}{8r^4}\frac{1}{5}32\pi^5$$

$$= \pi+\frac{12}{5}\pi$$

$$= \underline{\frac{17}{5}\pi}$$

Exercise

Evaluate the cylindrical coordinate integral $\displaystyle\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} zdz\ rdr\ d\theta$

Solution

$$\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} zdz\ rdr\ d\theta = \int_0^{\pi}\int_0^{\theta/\pi}\left[\frac{1}{2}z^2\right]_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} rdr\ d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\int_0^{\theta/\pi}\left[9\left(4-r^2\right)-\left(4-r^2\right)\right]rdr\ d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\int_0^{\theta/\pi}8\left(4-r^2\right)rdr\ d\theta$$

$$= 4 \int_0^\pi \int_0^{\theta/\pi} \left(4r - r^3\right) dr \, d\theta$$

$$= 4 \int_0^\pi \left[2r^2 - \frac{1}{4}r^4\right]_0^{\theta/\pi} d\theta$$

$$= 4 \int_0^\pi \left(2\frac{\theta^2}{\pi^2} - \frac{1}{4}\frac{\theta^4}{\pi^4}\right) d\theta$$

$$= 4\left[\frac{2}{3}\frac{\theta^3}{\pi^2} - \frac{1}{20}\frac{\theta^5}{\pi^4}\right]_0^\pi$$

$$= 4\left[\frac{2}{3}\frac{\pi^3}{\pi^2} - \frac{1}{20}\frac{\pi^5}{\pi^4}\right]$$

$$= 4\left(\frac{2}{3}\pi - \frac{1}{20}\pi\right)$$

$$= 4\left(\frac{37}{60}\pi\right)$$

$$= \frac{37}{15}\pi$$

Exercise

Evaluate the cylindrical coordinate integral $\int_0^{2\pi} \int_0^1 \int_{-1/2}^{1/2} \left(r^2 \sin^2\theta + z^2\right) dz \, rdr \, d\theta$

Solution

$$\int_0^{2\pi} \int_0^1 \int_{-1/2}^{1/2} \left(r^2 \sin^2\theta + z^2\right) dz \, rdr \, d\theta = \int_0^{2\pi} \int_0^1 \left[zr^2 \sin^2\theta + \frac{1}{3}z^3\right]_{-1/2}^{1/2} rdr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left[\frac{1}{2}r^2 \sin^2\theta + \frac{1}{24} - \left(-\frac{1}{2}r^2 \sin^2\theta - \frac{1}{24}\right)\right] rdrd\theta$$

$$= \int_0^{2\pi} \int_0^1 \left(r^2 \sin^2\theta + \frac{1}{12}\right) rdrd\theta$$

$$= \int_0^{2\pi} \int_0^1 \left(r^3 \sin^2\theta + \frac{1}{12}r\right) drd\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{4}r^4 \sin^2\theta + \frac{1}{24}r^2\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{4}\sin^2\theta + \frac{1}{24} \right) d\theta \qquad \int \sin^2 ax\, dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$= \left[\frac{1}{4}\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta \right) + \frac{1}{24}\theta \right]_0^{2\pi}$$

$$= \left[\frac{\theta}{8} - \frac{1}{16}\sin 2\theta + \frac{1}{24}\theta \right]_0^{2\pi}$$

$$= \frac{2\pi}{8} + \frac{1}{24}2\pi$$

$$= \underline{\frac{\pi}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{2\pi} \int_0^3 \int_0^{z/3} r^3 dr\, dz\, d\theta$

$$\int_0^{2\pi} \int_0^3 \int_0^{z/3} r^3 dr\, dz\, d\theta = \int_0^{2\pi} \int_0^3 \left[\frac{1}{4}r^4 \right]_0^{z/3} dz\, d\theta$$

$$= \frac{1}{324} \int_0^{2\pi} \int_0^3 z^4 \, dz\, d\theta$$

$$= \frac{1}{324} \int_0^{2\pi} \left[\frac{1}{5}z^5 \right]_0^3 d\theta$$

$$= \frac{243}{1620} \int_0^{2\pi} d\theta$$

$$= \frac{3}{20}[\theta]_0^{2\pi}$$

$$= \underline{\frac{3\pi}{10}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1\int_0^{\sqrt{z}}\int_0^{2\pi}\left(r^2\cos^2\theta+z^2\right)r\,d\theta\,dr\,dz$

$$\int_0^1\int_0^{\sqrt{z}}\int_0^{2\pi}\left(r^2\cos^2\theta+z^2\right)r\,d\theta\,dr\,dz=\int_0^1\int_0^{\sqrt{z}}\left[r^2\left(\frac{\theta}{2}+\frac{1}{4}\sin 2\theta\right)+z^2\theta\right]_0^{2\pi}r\,dr\,dz$$

$$=\int_0^1\int_0^{\sqrt{z}}\left(\pi r^2+2\pi z^2\right)r\,dr\,dz$$

$$=\int_0^1\int_0^{\sqrt{z}}\left(\pi r^3+2\pi z^2 r\right)dr\,dz$$

$$=\int_0^1\left[\frac{1}{4}\pi r^4+\pi z^2 r^2\right]_0^{\sqrt{z}}dz$$

$$=\int_0^1\left(\frac{1}{4}\pi z^2+\pi z^3\right)dz$$

$$=\left[\frac{1}{12}\pi z^3+\frac{1}{4}\pi z^4\right]_0^1$$

$$=\frac{1}{12}\pi+\frac{1}{4}\pi$$

$$=\frac{\pi}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi}\left(r\sin\theta+1\right)r\,d\theta\,dz\,dr$

$$\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi}\left(r\sin\theta+1\right)r\,d\theta dz dr=\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\left[-r\cos\theta+\theta\right]_0^{2\pi}r\,dz dr$$

$$=\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\left(-r+2\pi-(-r)\right)r\,dz dr$$

$$=\int_0^2\int_{r-2}^{\sqrt{4-r^2}}2\pi r\,dz dr$$

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$$= 2\pi \int_0^2 r[z]_{r-2}^{\sqrt{4-r^2}} \, dr$$

$$= 2\pi \int_0^2 r\left[\left(4-r^2\right)^{1/2} -(r-2)\right] dr$$

$$= 2\pi \int_0^2 \left[r\left(4-r^2\right)^{1/2} -r^2 +2r \right] dr \qquad d\left(4-r^2\right) = -2rdr$$

$$= 2\pi \left[-\frac{1}{3}\left(4-r^2\right)^{3/2} -\frac{1}{3}r^3 +r^2 \right]_0^2$$

$$= 2\pi \left[-\frac{8}{3} +4 -\left(-\frac{1}{3}(4)^{3/2}\right)\right]$$

$$= 2\pi \left(\frac{4}{3} +\frac{8}{3}\right)$$

$$\underline{= 8\pi}$$

Exercise

Evaluate the integral $\displaystyle \int_{-1}^{5}\int_{0}^{\frac{\pi}{2}}\int_{0}^{3} r\cos\theta \, drd\theta dz$

Solution

$$\int_{-1}^{5}\int_{0}^{\frac{\pi}{2}}\int_{0}^{3} r\cos\theta \, drd\theta dz = \int_{-1}^{5} dz \int_{0}^{\frac{\pi}{2}} \cos\theta d\theta \int_{0}^{3} r \, dr$$

$$= z\Big|_{-1}^{5} \ \ (\sin\theta)\Big|_{0}^{\frac{\pi}{2}} \ \ \left(\frac{1}{2}r^2\right)\Big|_{0}^{3}$$

$$= \frac{1}{2}(5+1)(1)(9)$$

$$\underline{= 27}$$

Exercise

Evaluate the integral $\displaystyle \int_{0}^{\frac{\pi}{4}}\int_{0}^{6}\int_{0}^{6-r} rz \, dzdrd\theta$

Solution

$$\int_0^{\frac{\pi}{4}} \int_0^6 \int_0^{6-r} rz \, dzdrd\theta = \frac{1}{2} \int_0^{\frac{\pi}{4}} d\theta \int_0^6 rz^2 \Big|_0^{6-r} \, dr$$

$$= \frac{\pi}{8} \int_0^6 \left(36r - 12r^2 + r^3 \right) dr$$

$$= \frac{\pi}{8} \left(18r^2 - 4r^3 + \frac{1}{4} r^4 \right) \Big|_0^6$$

$$= \frac{\pi}{8} \left(648 - 864 + 324 \right)$$

$$= \underline{\frac{27\pi}{2}}$$

Exercise

Evaluate the integral $\displaystyle \int_0^{\frac{\pi}{2}} \int_0^{2\cos^2 \theta} \int_0^{4-r^2} r\sin\theta \, dzdrd\theta$

Solution

$$\int_0^{\frac{\pi}{2}} \int_0^{2\cos^2 \theta} \int_0^{4-r^2} r\sin\theta \, dzdrd\theta = \int_0^{\frac{\pi}{2}} \int_0^{2\cos^2 \theta} r\sin\theta \, z \Big|_0^{4-r^2} drd\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\cos^2 \theta} \sin\theta \left(4r - r^3 \right) drd\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta \left(2r^2 - \frac{1}{4} r^4 \right) \Big|_0^{2\cos^2 \theta} d\theta$$

$$= -\int_0^{\frac{\pi}{2}} \left(8\cos^4 \theta - 4\cos^8 \theta \right) d(\cos\theta)$$

$$= \left(\frac{4}{9} \cos^9 \theta - \frac{8}{5} \cos^5 \theta \right) \Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{4}{9} + \frac{8}{5}$$

$$= \underline{\frac{52}{45}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^4 \int_0^z \int_0^{\frac{\pi}{2}} re^r \, d\theta dr dz$

Solution

$$\int_0^4 \int_0^z \int_0^{\frac{\pi}{2}} re^r \, d\theta dr dz = \int_0^4 \int_0^z re^r \theta \Big|_0^{\frac{\pi}{2}} \, dr dz$$

$$= \frac{\pi}{2} \int_0^4 \int_0^z re^r \, dr dz$$

$$= \frac{\pi}{2} \int_0^4 \left(re^r - e^r \right)\Big|_0^z \, dz$$

$$= \frac{\pi}{2} \int_0^4 \left(ze^z - e^z + 1 \right) dz$$

$$= \frac{\pi}{2} \left(ze^z - e^z - e^z + z \right)\Big|_0^4$$

$$= \frac{\pi}{2} \left(4e^4 - 2e^4 + 4 + 2 \right)$$

$$= \frac{\pi}{2} \left(2e^4 + 6 \right)$$

$$= \pi \left(e^4 + 3 \right)$$

		$\int e^r$
$+$	r	e^r
$-$	1	e^r

Exercise

Evaluate the integral $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^3 \int_0^{e^{-r^2}} r \, dz dr d\theta$

Solution

$$\int_0^{\frac{\pi}{2}} \int_0^3 \int_0^{e^{-r^2}} r \, dz dr d\theta = \int_0^{\frac{\pi}{2}} d\theta \int_0^3 rz \Big|_0^{e^{-r^2}} \, dr$$

$$= \frac{\pi}{2} \int_0^3 re^{-r^2} \, dr$$

$$= -\frac{\pi}{4} \int_0^3 e^{-r^2} d\left(-r^2 \right)$$

$$= -\frac{\pi}{4} e^{-r^2} \Big|_0^3$$

$$= \frac{\pi}{4}\left(1 - e^{-9}\right)$$

Exercise

Evaluate the integral $\displaystyle \int_0^{2\pi} \int_0^{\sqrt{5}} \int_0^{5-r^2} r\, dz\, dr\, d\theta$

Solution

$$\int_0^{2\pi} \int_0^{\sqrt{5}} \int_0^{5-r^2} r\, dz\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^{\sqrt{5}} r\, z\Big|_0^{5-r^2}\, dr$$

$$= 2\pi \int_0^{\sqrt{5}} \left(5r - r^3\right) dr$$

$$= 2\pi \left(\frac{5}{2} r^2 - \frac{1}{4} r^4\right)\Big|_0^{\sqrt{5}}$$

$$= 2\pi \left(\frac{25}{2} - \frac{25}{4}\right)$$

$$= \frac{25\pi}{2}$$

Exercise

Evaluate the integral $\displaystyle \int_0^{\pi} \int_0^{\cos\theta} \int_{2r^2}^{2r\cos\theta} r\, dz\, dr\, d\theta$

Solution

$$\int_0^{\pi} \int_0^{\cos\theta} \int_{2r^2}^{2r\cos\theta} r\, dz\, dr\, d\theta = \int_0^{\pi} \int_0^{\cos\theta} r\, z\Big|_{2r^2}^{2r\cos\theta}\, dr\, d\theta$$

$$= \int_0^{\pi} \int_0^{\cos\theta} \left(2r^2 \cos\theta - 2r^3\right) dr\, d\theta$$

$$= \int_0^{\pi} \left(\frac{2}{3} r^3 \cos\theta - \frac{1}{2} r^4\right)\Big|_0^{\cos\theta}\, d\theta$$

$$= \int_0^{\pi} \left(\frac{2}{3} \cos^4 \theta - \frac{1}{2} \cos^4 \theta\right) d\theta$$

$$= \frac{1}{6} \int_0^\pi \cos^4 \theta \, d\theta$$

$$= \frac{1}{24} \int_0^\pi \left(1 + \cos 2\theta\right)^2 \, d\theta$$

$$= \frac{1}{24} \int_0^\pi \left(1 + 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{1}{24} \int_0^\pi \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{1}{24} \left(\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right)\Bigg|_0^\pi$$

$$= \underline{\frac{\pi}{16}}$$

Exercise

Evaluate the integral $\displaystyle \int_0^\pi \int_0^{a\cos\theta} \int_0^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta$

Solution

$$\int_0^\pi \int_0^{a\cos\theta} \int_0^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta = \int_0^\pi \int_0^{a\cos\theta} r \, z \Bigg|_0^{\sqrt{a^2-r^2}} dr \, d\theta$$

$$= \int_0^\pi \int_0^{a\cos\theta} r \left(a^2 - r^2\right)^{1/2} dr \, d\theta$$

$$= -\frac{1}{2} \int_0^\pi \int_0^{a\cos\theta} \left(a^2 - r^2\right)^{1/2} d\left(a^2 - r^2\right) d\theta$$

$$= -\frac{1}{3} \int_0^\pi \left(a^2 - r^2\right)^{3/2} \Bigg|_0^{a\cos\theta} d\theta$$

$$= -\frac{1}{3} \int_0^\pi \left[\left(a^2 - a^2 \cos^2\theta\right)^{3/2} - a^3\right] d\theta$$

$$= -\frac{1}{3} \int_0^\pi \left[a^3 \left(1 - \cos^2\theta\right)^{3/2} - a^3\right] d\theta$$

$$= -\frac{a^3}{3} \int_0^\pi \left[\left(\sin^2\theta\right)^{3/2} - 1\right] d\theta$$

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$$= \frac{a^3}{3} \int_0^\pi \left(1 - \sin^3 \theta\right) d\theta$$

$$= \frac{a^3}{3} \int_0^\pi d\theta - \frac{a^3}{3} \int_0^\pi \sin^2 \theta \sin \theta \, d\theta$$

$$= \frac{a^3 \pi}{3} + \frac{a^3}{3} \int_0^\pi \left(1 - \cos^2 \theta\right) d(\cos \theta)$$

$$= \frac{a^3 \pi}{3} + \frac{a^3}{3} \left(\cos \theta - \frac{1}{3} \cos^3 \theta\right)\Big|_0^\pi$$

$$= \frac{a^3 \pi}{3} + \frac{a^3}{3} \left(-1 + \frac{1}{3} - 1 + \frac{1}{3}\right)$$

$$= \frac{a^3 \pi}{3} - \frac{4a^3}{9}$$

$$= \frac{a^3}{9} \left(3\pi - 4\right)$$

Exercise

Evaluate the integral $\displaystyle \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{a\cos\theta} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta$

Solution

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{a\cos\theta} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{a\cos\theta} rz \Big|_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} dr \, d\theta$$

$$= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{a\cos\theta} r \left(a^2 - r^2\right)^{1/2} dr \, d\theta$$

$$= - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{a\cos\theta} \left(a^2 - r^2\right)^{1/2} d\left(a^2 - r^2\right) d\theta$$

$$= -\frac{2}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(a^2 - r^2\right)^{3/2} \Big|_0^{a\cos\theta} d\theta$$

$$= -\frac{2}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\left(a^2 - a^2 \cos^2 \theta\right)^{3/2} - a^3\right) d\theta$$

$$= -\frac{2a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\left(\sin^2\theta\right)^{3/2} - 1 \right) d\theta$$

$$= -\frac{2a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\sin^3\theta - 1\right) d\theta$$

$$= \frac{2a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta - \frac{2a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \sin\theta \, d\theta$$

$$= \frac{2a^3\pi}{3} + \frac{2a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 - \cos^2\theta\right) d\left(\cos\theta\right)$$

$$= \frac{2a^3\pi}{3} + \frac{2a^3}{3} \left(\cos\theta - \frac{1}{3}\cos^3\theta\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{2a^3\pi}{3} + \frac{2a^3}{3}\left(-1 + \frac{1}{3} - 1 + \frac{1}{3}\right)$$

$$= \frac{2a^3\pi}{3} - \frac{8a^3}{9}$$

$$= \frac{2a^3}{9}\left(3\pi - 4\right)$$

Exercise

Convert $\displaystyle\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} 3dz \, rdrd\theta, \quad r \geq 0$

a) Rectangular coordinates with order of integration dzdxdy.

b) Spherical coordinates

c) Evaluate one of the integrals.

Solution

a) $\quad z = r = \sqrt{x^2 + y^2}$

$z = \sqrt{4 - r^2} = \sqrt{4 - x^2 - y^2}$

$r \leq \sqrt{2} \quad \rightarrow \quad r^2 \leq 2 \quad 0 \leq \theta \leq 2\pi$

$x^2 + y^2 \leq 2 \quad \rightarrow \quad -\sqrt{2 - y^2} \leq x \leq \sqrt{2 - y^2}$

$x = 0 \quad \rightarrow \quad -\sqrt{2} \leq y \leq \sqrt{2}$

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$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 3dz\ rdrd\theta = \int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}}\int_{\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} 3\sqrt{x^2+y^2}\ dzdxdy$$

b) Spherical coordinates

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \end{cases} \rightarrow \quad x^2+y^2 = \rho^2\sin^2\varphi$$

$$0 \le \theta \le 2\pi$$

$$z = \rho\cos\varphi = \sqrt{x^2+y^2}$$

$$\rho\cos\varphi = \rho\sin\varphi \quad \rightarrow \quad \underline{\varphi = \dfrac{\pi}{4}}$$

$$\rho = \frac{r}{\sin\varphi} = \frac{r}{\sin\frac{\pi}{4}} = \frac{\sqrt{2}}{\frac{\sqrt{2}}{2}} = 2$$

$$\underline{0 \le \rho \le 2}$$

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 3dz\ rdrd\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^2 3\rho^2\sin\varphi\ d\rho d\varphi d\theta$$

c) $$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 3r\ dzdrd\theta = 3\int_0^{2\pi} d\theta \int_0^{\sqrt{2}} rz\Big|_r^{\sqrt{4-r^2}}\ dr$$

$$= 3\theta\Big|_0^{2\pi}\int_0^{\sqrt{2}} r\left(\sqrt{4-r^2}-r\right) dr$$

$$= 6\pi\int_0^{\sqrt{2}} \left(r\sqrt{4-r^2}-r^2\right) dr$$

$$= -3\pi\int_0^{\sqrt{2}} \left(4-r^2\right)^{1/2} d\left(4-r^2\right) - 6\pi\int_0^{\sqrt{2}} r^2 dr$$

$$= -2\pi\left(4-r^2\right)^{3/2}\Big|_0^{\sqrt{2}} - 2\pi r^3\Big|_0^{\sqrt{2}}$$

$$= -2\pi\left(2\sqrt{2}-8\right) - 4\pi\sqrt{2}$$

$$= -2\pi\left(2\sqrt{2}-8+2\sqrt{2}\right)$$

$$= -8\pi\left(\sqrt{2}-2\right)$$

$$\underline{= 8\pi\left(2-\sqrt{2}\right)}$$

Exercise

Convert the integral $\int_{-1}^{1} \int_0^{\sqrt{1-y^2}} \int_0^x \left(x^2 + y^2\right) dz\,dx\,dy$ to an equivalent integral in cylindrical coordinates and evaluate the result.

$$\int_{-\pi/2}^{\pi/2} \int_0^1 \int_0^{r\cos\theta} r^3\,dz\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \int_0^1 r^3 [z]_0^{r\cos\theta}\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^1 r^3 r\cos\theta\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\tfrac{1}{5}r^5\cos\theta\right]_0^1 d\theta$$

$$= \tfrac{1}{5}\int_{-\pi/2}^{\pi/2} \cos\theta\,d\theta$$

$$= \tfrac{1}{5}[\sin\theta]_{-\pi/2}^{\pi/2}$$

$$= \tfrac{1}{5}(1+1)$$

$$= \underline{\tfrac{2}{5}}$$

Exercise

Set up an integral in rectangular coordinates equivalent to the integral

$$\int_0^{\pi/2} \int_1^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} r^3\left(\sin\theta\cos\theta\right)z^2\,dz\,dr\,d\theta$$

Arrange the order of integration to be z first, then y, then x.

$$\int_0^{\pi/2} \int_1^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} r^2\left(\sin\theta\cos\theta\right)z^2\,dz\,r\,dr\,d\theta$$

$$r^2\left(\sin\theta\cos\theta\right)z^2 = (r\sin\theta)(r\cos\theta)z^2$$

$$= xyz^2$$

$$1 \le z \le \sqrt{4-r^2} \quad \rightarrow \quad 1 \le z \le \sqrt{4-x^2-y^2}$$

$$1 \le r \le \sqrt{3}$$

$$1 \le r^2 \le 3$$

$$1 \le x^2 + y^2 \le 3$$

$$1 - x^2 \le y^2 \le 3 - x^2$$

$$\sqrt{1 - x^2} \le y \le \sqrt{3 - x^2}$$

$$0 \le \theta \le \frac{\pi}{2}$$

$$\theta = 0 \rightarrow \begin{cases} r = 1 \;\Rightarrow\; x = r\cos\theta = 1 \\ r = \sqrt{3} \;\Rightarrow\; x = r\cos\theta = \sqrt{3} \end{cases}$$

$$\theta = \frac{\pi}{2} \;\rightarrow\; x = r\cos\theta = 0$$

$$\int_0^{\pi/2} \int_1^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} r^3 (\sin\theta\cos\theta) z^2 \; dz\,dr\,d\theta$$

$$= \int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} z^2\, yx\; dz\,dy\,dx + \int_1^{\sqrt{3}} \int_0^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} z^2\, yx\; dz\,dy\,dx$$

Exercise

Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,dr\,d\theta$ over

the region D that is the right circular cylinder whose base is the circle
$r = 2\sin\theta$ in the xy-plane and whose top lies in the plane $z = 4 - y$

Solution

$$0 \le z \le 4 - y \Rightarrow 0 \le z \le 4 - r\sin\theta$$

$$\int_0^{\pi} \int_0^{2\sin\theta} \int_0^{4-r\sin\theta} f(r,\theta,z)\,dz \; r\,dr \; d\theta$$
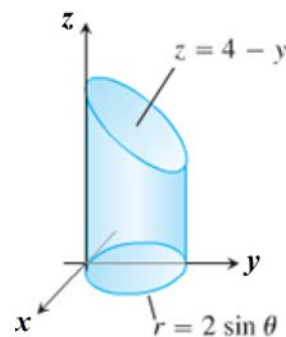
z = 4 − *y*

r = 2 sin *θ*

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Exercise

Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dzdrd\theta$ over the

region D which is the solid right cylinder whose base is the region in the xy-plane that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$
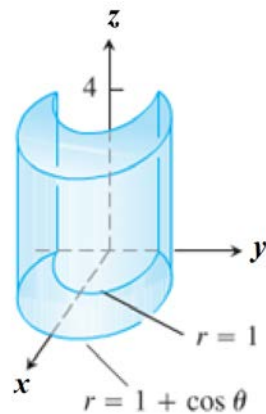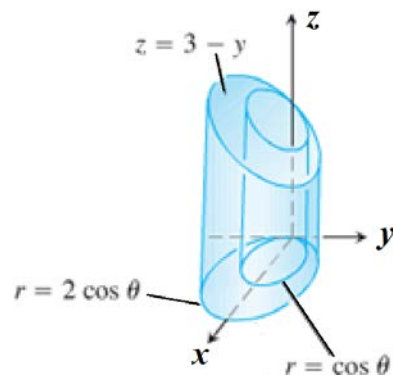
Solution

$$0 \le z \le 4 \quad 1 \le r \le 1 + \cos\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos\theta} \int_{0}^{4} f(r,\theta,z)\,dz\,rdr\,d\theta$$

Exercise

Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dzdrd\theta$

over the region D which is the solid right cylinder whose base is the region between the circles $r = \cos\theta$ and $r = 2\cos\theta$ and whose top lies in the plane $z = 3 - y$

Solution

$$\int_{-\pi/2}^{\pi/2} \int_{\cos\theta}^{2\cos\theta} \int_{0}^{3-r\sin\theta} f(r,\theta,z)\,dz\,rdr\,d\theta$$

Exercise

Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dzdrd\theta$ over the

region D which is the prism whose base is the triangle in the xy-plane bounded by the y-axis and the lines $y = x$ and $y = 1$ and whose top lies in the plane $z = 2 - x$
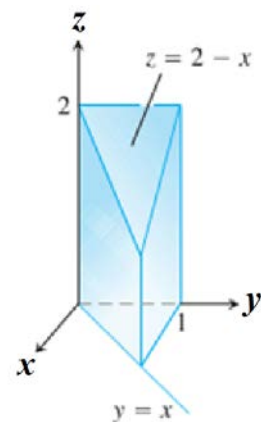
Solution

$$0 \le z \le 2 - x \quad \rightarrow 0 \le z \le 2 - r\cos\theta$$

$$y = 1 \quad \rightarrow \quad r\sin\theta = 1 \rightarrow r = \frac{1}{\sin\theta} = \csc\theta$$

$$\int_{\pi/4}^{\pi/2} \int_{0}^{\csc\theta} \int_{0}^{2-r\sin\theta} f(r,\theta,z)\,dz\,rdr\,d\theta$$

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Exercise

Evaluate the integrals in cylindrical coordinates. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^3 \left(x^2+y^2\right)^{3/2} dz\,dy\,dz$

Solution

$\begin{cases} 0 \le z \le 3 \\ 0 \le y \le \sqrt{9-x^2} \end{cases} \rightarrow 0 \le r \le 3$

$0 \le x \le 3 \quad (y \in QI) \rightarrow 0 \le \theta \le \dfrac{\pi}{2}$

$$\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^3 \left(x^2+y^2\right)^{3/2} dz\,dy\,dz = \int_0^3\int_0^{\frac{\pi}{2}}\int_0^3 \left(r^2\right)^{3/2} r\,dr\,d\theta\,dz$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^3 dz \int_0^3 r^4 dr$$

$$= \theta \Big|_0^{\frac{\pi}{2}} \;\; z \Big|_0^3 \;\; \frac{1}{5}r^5 \Big|_0^3$$

$$= \theta \Big|_0^{\frac{\pi}{2}} \;\; z \Big|_0^3 \;\; \frac{1}{5}r^5 \Big|_0^3$$

$$= \frac{\pi}{2}(3)\frac{243}{5}$$

$$= \frac{729\pi}{10}$$

Exercise

Evaluate the integrals in cylindrical coordinates. $\displaystyle\int_{-2}^2\int_{-1}^1\int_0^{\sqrt{1-z^2}} \frac{1}{\left(1+x^2+z^2\right)^2} dx\,dy\,dz$

Solution

$0 \le x \le \sqrt{1-z^2} \quad \rightarrow \quad 0 \le r \le 1$

$-1 \le y \le 1 \quad \rightarrow \quad -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

$$\int_{-2}^{2}\int_{-1}^{1}\int_{0}^{\sqrt{1-z^2}} \frac{1}{\left(1+x^2+z^2\right)^2} dxdydz = \int_{-2}^{2} dz \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_{0}^{1} \frac{1}{\left(1+r^2\right)^2} rdr$$

$$= z\Big|_{-2}^{2} \quad \theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \quad \frac{1}{2}\int_{0}^{1} \frac{1}{\left(1+r^2\right)^2} d\left(1+r^2\right)$$

$$= 4(\pi) \quad \left(-\frac{1}{2}\right)\frac{1}{1+r^2}\Big|_{0}^{1}$$

$$= -2\pi\left(\frac{1}{2}-1\right)$$

$$\underline{= \pi}$$

Exercise

Evaluate the spherical coordinate integral $\int_{0}^{\pi}\int_{0}^{\pi}\int_{0}^{2\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

Solution

$$\int_{0}^{\pi}\int_{0}^{\pi}\int_{0}^{2\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{1}{3}\int_{0}^{\pi}\int_{0}^{\pi} \sin\phi\left[\rho^3\right]_{0}^{2\sin\phi} d\phi \, d\theta$$

$$= \frac{8}{3}\int_{0}^{\pi}\int_{0}^{\pi} \sin^4\phi \, d\phi \, d\theta$$

$$\int \sin^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right)dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right)dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right)dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right)$$

$$= \frac{8}{3}\int_{0}^{\pi} \left[\frac{3}{8}\phi - \frac{1}{4}\sin 2\phi + \frac{1}{32}\sin 4\phi\right]_{0}^{\pi} d\theta$$

$$= \frac{8}{3}\int_{0}^{\pi} \left[\frac{3\pi}{8}\right]d\theta$$

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$$= \pi \left[\theta \right]_{0}^{\pi}$$

$$\underline{= \pi^2}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_{0}^{2\pi} \int_{0}^{\pi/4} \int_{0}^{2} (\rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

Solution

$$\int_{0}^{2\pi} \int_{0}^{\pi/4} \int_{0}^{2} (\rho\cos\phi)\rho^2 \sin\phi \, d\rho d\phi d\theta = \int_{0}^{2\pi} \int_{0}^{\pi/4} (\cos\phi\sin\phi)\left[\frac{1}{4}\rho^4\right]_{0}^{2} d\phi d\theta$$

$$= 4\int_{0}^{2\pi} \int_{0}^{\pi/4} (\cos\phi\sin\phi) \, d\phi d\theta$$

$$= 4\int_{0}^{2\pi} \int_{0}^{\pi/4} \sin\phi \, d(\sin\phi) d\theta$$

$$= 2\int_{0}^{2\pi} \left[\sin^2\phi\right]_{0}^{\pi/4} d\theta$$

$$= \int_{0}^{2\pi} d\theta$$

$$\underline{= 2\pi}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_{0}^{3\pi/2} \int_{0}^{\pi} \int_{0}^{1} 5\rho^3 \sin^3\phi \, d\rho \, d\phi \, d\theta$

Solution

$$\int_{0}^{3\pi/2} \int_{0}^{\pi} \int_{0}^{1} 5\rho^3 \sin^3\phi \, d\rho d\phi d\theta = \frac{5}{4}\int_{0}^{3\pi/2} \int_{0}^{\pi} \sin^3\phi \left[\rho^4\right]_{0}^{1} d\phi \, d\theta$$

$$= \frac{5}{4}\int_{0}^{3\pi/2} \int_{0}^{\pi} \sin^2\phi\sin\phi \, d\phi d\theta \qquad d(\cos\phi) = -\sin\phi$$

$$= -\frac{5}{4}\int_{0}^{3\pi/2} \int_{0}^{\pi} \left(1-\cos^2\phi\right) d(\cos\phi) d\theta$$

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$$= -\frac{5}{4} \int_0^{3\pi/2} \left[\cos\phi - \frac{1}{3}\cos^3\phi \right]_0^\pi d\theta$$

$$= -\frac{5}{4} \int_0^{3\pi/2} \left(-1 + \frac{1}{3} - \left(1 - \frac{1}{3} \right) \right) d\theta$$

$$= \frac{5}{3} \int_0^{3\pi/2} d\theta$$

$$= \frac{5}{3}\left(\frac{3\pi}{2} \right)$$

$$= \underline{\frac{5\pi}{2}}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{2\pi} \int_0^{\pi/2} \int_0^{2\cos\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$

Solution

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^{2\cos\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/2} \frac{1}{3}\sin\varphi \, \rho^3 \bigg|_0^{2\cos\varphi} d\varphi$$

$$= \frac{8}{3} \theta \bigg|_0^{2\pi} \int_0^{\pi/2} \sin\varphi \, \cos^3\varphi \, d\varphi$$

$$= -\frac{16\pi}{3} \int_0^{\pi/2} \cos^3\varphi \, d(\cos\varphi)$$

$$= -\frac{4\pi}{3}\cos^4\varphi \bigg|_0^{\pi/2}$$

$$= \underline{\frac{4\pi}{3}}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{\pi} \int_0^{\pi/4} \int_{2\sec\varphi}^{4\sec\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$

Solution

$$\int_0^\pi \int_0^{\pi/4} \int_{2\sec\varphi}^{4\sec\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \int_0^\pi d\theta \int_0^{\pi/4} \frac{1}{3}\sin\varphi \, \rho^3 \Big|_{2\sec\varphi}^{4\sec\varphi} d\varphi$$

$$= \frac{1}{3}\theta \Big|_0^\pi \int_0^{\pi/4} \sin\varphi \left(64\sec^3\varphi - 8\sec^3\varphi\right) d\varphi$$

$$= \frac{\pi}{3} \int_0^{\pi/4} \sin\varphi \left(56\sec^3\varphi\right) d\varphi$$

$$= -\frac{56\pi}{3} \int_0^{\pi/4} \cos^{-3}\varphi \, d\left(\cos\varphi\right)$$

$$= \frac{28\pi}{3} \frac{1}{\cos^2\varphi} \Big|_0^{\pi/4}$$

$$= \frac{28\pi}{3}\left(2-1\right)$$

$$= \underline{\frac{28\pi}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 \int_{-\pi}^0 \int_{\pi/4}^{\pi/2} \rho^3 \sin 2\phi \, d\phi \, d\theta \, d\rho$

Solution

$$\int_0^2 \int_{-\pi}^0 \int_{\pi/4}^{\pi/2} \rho^3 \sin 2\phi \, d\phi d\theta d\rho = -\frac{1}{2}\int_0^2 \int_{-\pi}^0 \rho^3 \left[\cos 2\phi\right]_{\pi/4}^{\pi/2} d\theta d\rho$$

$$= -\frac{1}{2}\int_0^2 \int_{-\pi}^0 \rho^3\left(-1-0\right) d\theta d\rho$$

$$= \frac{1}{2}\int_0^2 \int_{-\pi}^0 \rho^3 d\theta d\rho$$

$$= \frac{1}{2}\int_0^2 \rho^3\left(\pi\right) d\rho$$

$$= \frac{\pi}{8}\left[\rho^4\right]_0^2$$

$$= \underline{2\pi}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2} 5\rho^4\sin^3\phi\, d\rho\, d\theta\, d\phi$

Solution

$$\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2} 5\rho^4\sin^3\phi\, d\rho d\theta d\phi = \int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\sin^3\phi\left[\rho^5\right]_{\csc\phi}^{2} d\theta d\phi$$

$$= \int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\sin^3\phi\left(32-\csc^5\phi\right) d\theta d\phi$$

$$= \int_{\pi/6}^{\pi/2}\left(32\sin^3\phi-\csc^2\phi\right)\left[\theta\right]_{-\pi/2}^{\pi/2} d\phi$$

$$= \pi\left(\int_{\pi/6}^{\pi/2}32\sin^3\phi d\phi - \int_{\pi/6}^{\pi/2}\csc^2\phi d\phi\right)$$

$$= 32\pi\int_{\pi/6}^{\pi/2}\sin^2\phi\sin\phi d\phi - \pi\int_{\pi/6}^{\pi/2}\csc^2\phi d\phi$$

$$= 32\pi\int_{\pi/6}^{\pi/2}\left(1-\cos^2\phi\right) d\left(\cos\phi\right) + \pi\left[\cot\phi\right]_{\pi/6}^{\pi/2}$$

$$= 32\pi\left[\cos\phi-\frac{1}{3}\cos^3\phi\right]_{\pi/6}^{\pi/2} + \pi\left(-\sqrt{3}\right)$$

$$= 32\pi\left(\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{8}\right)-\pi\sqrt{3}$$

$$= 12\pi\sqrt{3}-\pi\sqrt{3}$$

$$\underline{= 11\pi\sqrt{3}}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_{0}^{2\pi}\int_{0}^{\frac{\pi}{4}}\int_{0}^{3}\rho^2\sin\phi\, d\rho d\phi d\theta$

Solution

$$\int_{0}^{2\pi}\int_{0}^{\frac{\pi}{4}}\int_{0}^{3}\rho^2\sin\phi\, d\rho d\phi d\theta = \int_{0}^{2\pi} d\theta\int_{0}^{\frac{\pi}{4}}\sin\phi\, d\phi\int_{0}^{3}\rho^2\, d\rho$$

$$= \theta \Big|_0^{2\pi} \quad (-\cos\phi)\Big|_0^{\frac{\pi}{4}} \quad \left(\tfrac{1}{3}\rho^3\right)\Big|_0^3$$

$$= (2\pi)\left(-\frac{1}{\sqrt{2}}+1\right)(9)$$

$$= 18\pi\left(1-\frac{1}{\sqrt{2}}\right)$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^3 \rho^3\cos\phi\sin\phi\,d\rho d\phi d\theta$

Solution

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^3 \rho^3\cos\phi\sin\phi\,d\rho d\phi d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/4}\sin\phi\, d(\sin\phi)\int_0^3 \rho^3 d\rho$$

$$= \theta\Big|_0^{2\pi}\quad\left(\tfrac{1}{2}\sin^2\phi\right)\Big|_0^{\pi/4}\quad\left(\tfrac{1}{4}\rho^4\right)\Big|_0^3$$

$$= (2\pi)\tfrac{1}{2}\left(\tfrac{1}{2}\right)\left(\tfrac{81}{4}\right)$$

$$= \frac{81\pi}{8}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^\pi\int_0^{\sin\theta} 2\cos\phi\,\rho^2\,d\rho d\theta d\phi$

Solution

$$\int_0^{\frac{\pi}{2}}\int_0^\pi\int_0^{\sin\theta} 2\cos\phi\,\rho^2\,d\rho d\theta d\phi = \frac{2}{3}\int_0^{\frac{\pi}{2}}\cos\phi d\phi\int_0^\pi \rho^3\Big|_0^{\sin\theta}\,d\theta$$

$$= \frac{2}{3}\sin\phi\Big|_0^{\frac{\pi}{2}}\int_0^\pi \sin^3\theta\,d\theta$$

$$= \frac{2}{3}\int_0^\pi \sin^2\theta\sin\theta\,d\theta$$

$$= -\frac{2}{3}\int_0^\pi\left(1-\cos^2\theta\right)d(\cos\theta)$$

$$= -\frac{2}{3}\left(\cos\theta - \frac{1}{3}\cos^3\theta\right)\Bigg|_0^\pi$$

$$= -\frac{2}{3}\left(-1 + \frac{1}{3} - 1 + \frac{1}{3}\right)$$

$$= -\frac{2}{3}\left(-\frac{4}{3}\right)$$

$$= \frac{8}{9}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^\pi\int_0^2 e^{-\rho^3}\rho^2\,d\rho d\theta d\varphi$

Solution

$$\int_0^{\frac{\pi}{2}}\int_0^\pi\int_0^2 e^{-\rho^3}\rho^2\,d\rho d\theta d\varphi = -\frac{1}{3}\int_0^{\frac{\pi}{2}}d\varphi\int_0^\pi d\theta\int_0^2 e^{-\rho^3}\,d\left(-\rho^3\right)$$

$$= -\frac{1}{3}\left(\frac{\pi}{2}\right)(\pi)e^{-\rho^3}\Bigg|_0^2$$

$$= -\frac{\pi^2}{6}\left(e^{-8}-1\right)$$

$$= \frac{\pi^2}{6}\left(1-e^{-8}\right)$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\varphi}\rho^2\sin\phi\,d\rho d\phi d\theta$

Solution

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\varphi}\rho^2\sin\phi\,d\rho d\phi d\theta = \int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}\sin\phi\left(\frac{1}{3}\rho^3\right)\Bigg|_0^{\cos\varphi}d\phi$$

$$= \frac{2\pi}{3}\int_0^{\frac{\pi}{4}}\sin\phi\left(\cos^3\varphi\right)d\phi$$

$$= -\frac{2\pi}{3}\int_0^{\frac{\pi}{4}}\left(\cos^3\varphi\right)d\left(\cos\phi\right)$$

$$= -\frac{\pi}{6}\left(\cos^4\varphi\right)\Big|_0^{\frac{\pi}{4}}$$

$$= -\frac{\pi}{6}\left(\frac{1}{4}-1\right)$$

$$= \frac{\pi}{8}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{\cos\theta}\rho^2\sin\varphi\cos\varphi\,d\rho d\theta d\varphi$

Solution

$$\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{\cos\theta}\rho^2\sin\varphi\cos\varphi\,d\rho d\theta d\varphi = \int_0^{\frac{\pi}{4}}\frac{1}{2}\sin 2\varphi\,d\varphi\int_0^{\frac{\pi}{4}}\frac{1}{3}\rho^3\Big|_0^{\cos\theta}\,d\theta$$

$$= -\frac{1}{12}\cos 2\varphi\Big|_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\cos^3\theta\,d\theta$$

$$= -\frac{1}{12}(-1)\int_0^{\frac{\pi}{4}}\left(1-\sin^2\theta\right)d(\sin\theta)$$

$$= \frac{1}{12}\left(\sin\theta-\frac{1}{3}\sin^3\theta\right)\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{12}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{12}\right)$$

$$= \frac{5\sqrt{2}}{144}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^4\rho^2\sin\varphi\,d\rho d\varphi d\theta$

Solution

$$\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^4\rho^2\sin\varphi\,d\rho d\varphi d\theta = \int_0^{2\pi}d\theta\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\sin\varphi d\varphi\left(\frac{1}{3}\rho^3\right)\Big|_0^4$$

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$$= \frac{64}{3}(2\pi)(-\cos\varphi)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{64\pi\sqrt{3}}{3}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{5} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$

Solution

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{5} \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \left(\frac{1}{3}\rho^3\right)\Big|_0^5$$

$$= \frac{125}{3}(2\pi)(-\cos\varphi)\Big|_0^{\pi}$$

$$= \frac{500\pi}{3}$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{\sin\theta} 2\cos\varphi \, \rho^2 \, d\rho d\theta d\varphi$

Solution

$$\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{\sin\theta} 2\cos\varphi \, \rho^2 \, d\rho d\theta d\varphi = \int_0^{\frac{\pi}{2}} \cos\varphi \, d\varphi \int_0^{\pi} \frac{2}{3}\rho^3 \Big|_0^{\sin\theta} \, d\theta$$

$$= \frac{2}{3}\sin\varphi\Big|_0^{\frac{\pi}{2}} \int_0^{\pi} \sin^3\theta \, d\theta$$

$$= \frac{2}{3} \int_0^{\pi} \sin^2\theta \sin\theta \, d\theta$$

$$= -\frac{2}{3} \int_0^{\pi} \left(1-\cos^2\theta\right) d\left(\cos\theta\right)$$

$$= -\frac{2}{3}\left(\cos\theta - \frac{1}{3}\cos^3\theta\right)\Big|_0^{\pi}$$

$$= -\frac{2}{3}\left(-1+\frac{1}{3}-1+\frac{1}{3}\right)$$

$$= \frac{8}{9}$$

Exercise

Evaluate the integral $\displaystyle\int_0^4 \int_0^{\frac{\sqrt{2}}{2}} \int_x^{\sqrt{1-x^2}} e^{-x^2-y^2}\, dy\,dx\,dz$

Solution

$y = \sqrt{1-x^2} \;\rightarrow\; x^2 + y^2 = 1 = r^2$

$\begin{cases} x = 0 = \cos\theta & \rightarrow \theta = \frac{\pi}{2} \\[2mm] x = \frac{\sqrt{2}}{2} = \cos\theta & \rightarrow \theta = \frac{\pi}{4} \end{cases} \quad\rightarrow\quad \frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$0 \le z \le 4$

$$\int_0^4 \int_0^{\frac{\sqrt{2}}{2}} \int_x^{\sqrt{1-x^2}} e^{-x^2-y^2}\, dy\,dx\,dz = \int_0^4 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 e^{-r^2} r\, dr\,d\theta\,dz$$

$$= -\frac{1}{2}\int_0^4 dz \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^1 e^{-r^2}\, d\!\left(-r^2\right)$$

$$= -\frac{1}{2}\, z\Big|_0^4 \;\; \theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \;\; e^{-r^2}\Big|_0^1$$

$$= -\frac{1}{2}(4)\left(\frac{\pi}{4}\right)\left(e^{-1} - 1\right)$$

$$= \frac{\pi}{2}\left(1 - e^{-1}\right)$$

Exercise

Evaluate the integral $\displaystyle\int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{\sqrt{x^2+y^2}}^4 dz\,dy\,dx$

Solution

$\sqrt{x^2+y^2} \le z \le 4 \;\rightarrow\; r \le z \le 4$

$y = \sqrt{16-x^2} \;\rightarrow\; x^2 + y^2 = 16 = r^2$

$0 \le r \le 4$

$-4 \le x \le 4 \;\rightarrow\; 0 \le \theta \le 2\pi$

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$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{\sqrt{x^2+y^2}}^{4} dz\,dy\,dx = \int_{0}^{2\pi}\int_{0}^{4}\int_{r}^{4} dz\ r\,dr\,d\theta$$

$$= \int_{0}^{2\pi} d\theta \int_{0}^{4} z\Big|_{r}^{4}\ r\,dr$$

$$= 2\pi \int_{0}^{4}\left(4r - r^2\right) dr$$

$$= 2\pi\left(2r^2 - \frac{1}{3}r^3\right)\Big|_{0}^{4}$$

$$= 2\pi\left(32 - \frac{64}{3}\right)$$

$$\underline{= \frac{64\pi}{3}}$$

Exercise

Evaluate the integral $\qquad \int_{0}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{x^2+y^2}}\left(x^2+y^2\right)^{-1/2} dz\,dy\,dx$

$$0 \le z \le \sqrt{x^2+y^2} \quad\rightarrow\quad 0 \le z \le r$$

$$y = \sqrt{9-x^2} \quad\rightarrow\quad x^2+y^2 = 9 = r^2 \qquad \underline{0 \le r \le 3|}$$

$$\begin{cases} x=0 = 3\cos\theta \quad\rightarrow\theta = \frac{\pi}{2} \\ x=3 = 3\cos\theta \quad\rightarrow\theta = 0 \end{cases} \quad\rightarrow\quad 0 \le \theta \le \frac{\pi}{2}$$

$$\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{x^2+y^2}}\left(x^2+y^2\right)^{-1/2} dz\,dy\,dx = \int_{0}^{\frac{\pi}{2}}\int_{0}^{3}\int_{0}^{r}\frac{1}{r} dz\ r\,dr\,d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} d\theta \int_{0}^{3} z\Big|_{0}^{r}\ dr$$

$$= \frac{\pi}{2}\int_{0}^{3} r\,dr$$

$$= \frac{\pi}{4}r^2\Big|_{0}^{3}$$

$$\underline{= \frac{9\pi}{4}}$$

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Exercise

Evaluate the integral $\displaystyle\int_{-1}^{1}\int_{0}^{\frac{1}{2}}\int_{\sqrt{3}y}^{\sqrt{1-y^2}}\sqrt{x^2+y^2}\;dxdydz$

Solution

$-1\le z\le 1$

$y=\sqrt{1-x^2}\;\rightarrow\;x^2+y^2=1=r^2 \qquad \underline{0\le r\le 1|}$

$\begin{cases} y=0=\sin\theta & \rightarrow\theta=0 \\ y=\dfrac{1}{2}=\sin\theta & \rightarrow\theta=\dfrac{\pi}{6}\end{cases}\;\rightarrow\;0\le\theta\le\dfrac{\pi}{6}$

$$\int_{-1}^{1}\int_{0}^{\frac{1}{2}}\int_{\sqrt{3}y}^{\sqrt{1-y^2}}\sqrt{x^2+y^2}\;dxdydz=\int_{-1}^{1}\int_{0}^{\frac{\pi}{6}}\int_{0}^{1} r\;rdrd\theta dz$$

$$=\int_{-1}^{1}dz\int_{0}^{\frac{\pi}{6}}d\theta\int_{0}^{1}r^2 dr$$

$$=z\Big|_{-1}^{1}\;\;\theta\Big|_{0}^{\frac{\pi}{6}}\;\;\frac{1}{3}r^3\Big|_{0}^{1}$$

$$=(2)\left(\frac{\pi}{6}\right)\left(\frac{1}{3}\right)$$

$$=\underline{\frac{\pi}{9}}$$

Exercise

Evaluate $\displaystyle\iiint_{D}\left(x^2+y^2+z^2\right)^{5/2}dV$; D is the unit ball.

Solution

$$\iiint_{D}\left(x^2+y^2+z^2\right)^{5/2}dV=\int_{0}^{2\pi}\int_{0}^{\pi}\int_{0}^{1}\left(\rho^2\right)^{5/2}\rho^2\sin\varphi d\rho d\varphi d\theta$$

$$=\int_{0}^{2\pi}d\theta\int_{0}^{\pi}\sin\varphi d\varphi\int_{0}^{1}\rho^7 d\rho$$

$$=2\pi(-\cos\varphi)\Big|_{0}^{\pi}\left(\frac{1}{8}\rho^8\right)\Big|_{0}^{1}$$

$$=2\pi(2)\left(\frac{1}{8}\right)$$

$$= \frac{\pi}{2}$$

Exercise

Evaluate $\displaystyle\iiint_D e^{-\left(x^2+y^2+z^2\right)^{3/2}} dV$; D is the unit ball.

Solution

$$\iiint_D e^{-\left(x^2+y^2+z^2\right)^{3/2}} dV = \int_0^{2\pi} \int_0^{\pi} \int_0^1 e^{-\rho^3} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= -\frac{1}{3} \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \int_0^1 e^{-\rho^3} d\left(-\rho^3\right)$$

$$= -\frac{2\pi}{3}\left(-\cos\varphi\right)\Big|_0^{\pi} \left(e^{-\rho^3}\right)\Big|_0^1$$

$$= -\frac{2\pi}{3}(2)\left(e^{-1}-1\right)$$

$$= \frac{4\pi}{3}\left(1-e^{-1}\right)$$

Exercise

Evaluate $\displaystyle\iiint_D \frac{1}{\left(x^2+y^2+z^2\right)^{3/2}} dV$; D is the solid between the spheres of radius 1 and 2 centered at the origin.

Solution

$$\iiint_D \left(x^2+y^2+z^2\right)^{-3/2} dV = \int_0^{2\pi} \int_0^{\pi} \int_1^2 \left(\rho^{-3}\right)\rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \int_1^2 \frac{1}{\rho} \, d\rho$$

$$= 2\pi\left(-\cos\varphi\right)\Big|_0^{\pi} (\ln\rho)\Big|_1^2$$

$$= 2\pi(2)(\ln 2)$$

$$= 4\pi \ln 2$$

Exercise

Evaluate $\iiint\limits_{D}\left(x^2 + y^2 + z^2\right)^{-3/2} dV$, where D is the region in the first octant between two spheres of radius 1 and 2 centered at the origin.

<u>*Solution*</u>

$$\iiint\limits_{D}\left(x^2 + y^2 + z^2\right)^{-3/2} dV = \int_{0}^{\pi/2}\int_{0}^{\pi/2}\int_{1}^{2}\left(\rho^{-3}\right)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_{0}^{\pi/2} d\theta \int_{0}^{\pi/2} \sin\varphi \, d\varphi \int_{1}^{2} \frac{1}{\rho} \, d\rho$$

$$= \frac{\pi}{2}\left(-\cos\varphi\right)\Big|_{0}^{\pi/2} \left(\ln\rho\right)\Big|_{1}^{2}$$

$$= \frac{\pi}{2}(1)(\ln 2)$$

$$= \underline{\frac{\pi}{2}\ln 2}$$
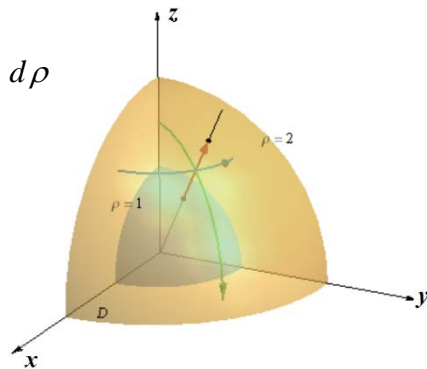
Exercise

Evaluate $\iiint\limits_{D} x^2 dV$; $D = \{(r, \theta, z): \ 0 \le r \le 1, \ 0 \le z \le 2r, \ 0 \le \theta \le 2\pi\}$

<u>*Solution*</u>

$$\iiint\limits_{D} x^2 dV = \int_{0}^{2\pi}\int_{0}^{1}\int_{0}^{2r} r^2 \cos^2\theta \, r \, dz \, dr \, d\theta$$

$$= \int_{0}^{2\pi} \frac{1}{2}(1 + \cos 2\theta) \, d\theta \int_{0}^{1} r^3 z\Big|_{0}^{2r} \, dr$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{0}^{2\pi} \int_{0}^{1} 2r^4 \, dr$$

$$= (2\pi)\left(\frac{1}{5}r^5\right)\Big|_{0}^{1}$$

$$= \underline{\frac{2\pi}{5}}$$

Exercise

Evaluate $\iiint\limits_{D} dV$; $D = \left\{ (r,\ \theta,\ z):\ \ 0 \le r \le 1,\ \ -\sqrt{4-r^2} \le z \le \sqrt{4-r^2},\ \ 0 \le \theta \le 2\pi \right\}$

Solution

$$\iiint\limits_{D} dV = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\ dz\ dr\ d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 rz \Big|_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}}\ dr$$

$$= 2\pi \int_0^1 2r\left(4-r^2\right)^{1/2} dr$$

$$= -2\pi \int_0^1 \left(4-r^2\right)^{1/2} d\left(4-r^2\right)$$

$$= -\frac{4}{3}\pi\left(4-r^2\right)^{3/2} \Big|_0^1$$

$$= -\frac{4\pi}{3}\left(3^{3/2} - 8\right)$$

$$= \frac{4\pi}{3}\left(8 - 3\sqrt{3}\right)$$

Exercise

Evaluate $\iiint\limits_{D} dV$; $D = \left\{ (r,\ \theta,\ z):\ \ 0 \le r \le 1,\ \ r \le z \le \sqrt{2-r^2},\ \ 0 \le \theta \le 2\pi \right\}$

Solution

$$\iiint\limits_{D} dV = \int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} r\ dz\ dr\ d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 rz \Big|_r^{\sqrt{2-r^2}}\ dr$$

$$= 2\pi \int_0^1 \left(r\left(2-r^2\right)^{1/2} - r^2 \right) dr$$

$$= -\pi \int_0^1 \left(2 - r^2\right)^{1/2} d\left(2 - r^2\right) - 2\pi \int_0^1 r^2 dr$$

$$= -\frac{2\pi}{3}\left(2 - r^2\right)^{3/2}\bigg|_0^1 - \frac{2\pi}{3}r^3\bigg|_0^1$$

$$= -\frac{2\pi}{3}\left(1 - 2\sqrt{2}\right) - \frac{2\pi}{3}$$

$$= -\frac{2\pi}{3} + \frac{4\pi\sqrt{2}}{3} - \frac{2\pi}{3}$$

$$\underline{= \frac{2\pi}{3}\left(\sqrt{2} - 1\right)}$$

Exercise

Evaluate $\displaystyle\iiint_D dV$; $D = \left\{(r,\ \theta,\ z):\ \ 0 \le r \le 1,\ \ r^2 \le z \le \sqrt{2 - r^2},\ \ 0 \le \theta \le 2\pi\right\}$

Solution

$$\iiint_D dV = \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2 - r^2}} r\ dz\ dr\ d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 rz\bigg|_{r^2}^{\sqrt{2 - r^2}}\ dr$$

$$= 2\pi \int_0^1 \left(r\left(2 - r^2\right)^{1/2} - r^3\right) dr$$

$$= -\pi \int_0^1 \left(2 - r^2\right)^{1/2} d\left(2 - r^2\right) - 2\pi \int_0^1 r^3 dr$$

$$= -\frac{2\pi}{3}\left(2 - r^2\right)^{3/2}\bigg|_0^1 - \frac{\pi}{2}r^4\bigg|_0^1$$

$$= -\frac{2\pi}{3}\left(1 - 2\sqrt{2}\right) - \frac{\pi}{2}$$

$$= -\frac{2\pi}{3} + \frac{4\pi\sqrt{2}}{3} - \frac{\pi}{2}$$

$$\underline{= \left(\frac{4}{3}\sqrt{2} - \frac{7}{6}\right)\pi}$$

Exercise

Evaluate $\iiint\limits_{D} dV$; $D = \{(r, \theta, z): \ 0 \le r \le 4, \ \ 2r \le z \le 24 - r^2, \ \ 0 \le \theta \le 2\pi\}$

Solution

$$\iiint\limits_{D} dV = \int_{0}^{2\pi} \int_{0}^{4} \int_{2r}^{24-r^2} r \, dz \, dr \, d\theta$$

$$= \int_{0}^{2\pi} d\theta \int_{0}^{4} rz \Big|_{2r}^{24-r^2} dr$$

$$= 2\pi \int_{0}^{4} \left(24r - r^3 - 2r^2\right) dr$$

$$= 2\pi \left(12r^2 - \frac{1}{4}r^4 - \frac{2}{3}r^3\right)\Big|_{0}^{4}$$

$$= 2\pi \left(192 - 64 - \frac{128}{3}\right)$$

$$= \underline{\frac{512\pi}{3}}$$

Exercise

Evaluate $\iiint\limits_{D} y^2 z^2 dV$; $D = \{(\rho, \varphi, \theta): \ 0 \le \rho \le 1, \ \ 0 \le \varphi \le \frac{\pi}{3}, \ \ 0 \le \theta \le 2\pi\}$

Solution

$$x = \rho \sin\varphi \cos\theta \quad y = \rho \sin\varphi \sin\theta \quad z = \rho \cos\varphi$$

$$\iiint\limits_{D} y^2 z^2 dV = \int_{0}^{2\pi} \int_{0}^{\frac{\pi}{3}} \int_{0}^{1} \left(\rho^2 \sin^2\varphi \sin^2\theta\right)\left(\rho^2 \cos^2\varphi\right)\rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= \int_{0}^{2\pi} \sin^2\theta \, d\theta \int_{0}^{\frac{\pi}{3}} \sin^3\varphi \cos^2\varphi \, d\varphi \int_{0}^{1} \rho^6 \, d\rho$$

$$= \frac{1}{2} \int_{0}^{2\pi} (1 - \cos 2\theta) \, d\theta \int_{0}^{\frac{\pi}{3}} \sin^2\varphi \cos^2\varphi \sin\varphi \, d\varphi \left(\frac{1}{7}\rho^7\right)\Big|_{0}^{1}$$

$$= \frac{1}{14}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_{0}^{2\pi} \int_{0}^{\frac{\pi}{3}} -\left(1 - \cos^2\varphi\right)\cos^2\varphi \, d(\cos\varphi)$$

$$= \frac{\pi}{7} \int_0^{\frac{\pi}{3}} \left(\cos^4 \varphi - \cos^2 \varphi \right) d\left(\cos \varphi \right)$$

$$= \frac{\pi}{7} \left(\frac{1}{5} \cos^5 \varphi - \frac{1}{3} \cos^3 \varphi \right) \Big|_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{7} \left(\frac{1}{5} \left(\frac{1}{2} \right)^5 - \frac{1}{3} \frac{1}{8} - \frac{1}{5} + \frac{1}{3} \right)$$

$$= \frac{\pi}{7} \left(\frac{1}{160} - \frac{1}{24} + \frac{2}{15} \right)$$

$$= \underline{\frac{47\pi}{3360}}$$

Exercise

Evaluate $\iiint_D \left(x^2 + y^2 \right) dV$; $D = \{ (\rho, \varphi, \theta): \quad 2 \leq \rho \leq 3, \quad 0 \leq \varphi \leq \pi, \quad 0 \leq \theta \leq 2\pi \}$

Solution

$x = \rho \sin \varphi \cos \theta \quad y = \rho \sin \varphi \sin \theta \quad z = \rho \cos \varphi$

$$\iiint_D \left(x^2 + y^2 \right) dV = \int_0^{2\pi} \int_0^{\pi} \int_2^3 \left(\rho^2 \sin^2 \varphi \cos^2 \theta + \rho^2 \sin^2 \varphi \sin^2 \theta \right) \rho^2 \sin \varphi \, d\rho d\varphi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_2^3 \sin^2 \varphi \left(\cos^2 \theta + \sin^2 \theta \right) \rho^4 \sin \varphi \, d\rho d\varphi d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin^2 \varphi \sin \varphi \, d\varphi \int_2^3 \rho^4 \, d\rho$$

$$= 2\pi \int_0^{\pi} - \left(1 - \cos^2 \varphi \right) d\left(\cos \varphi \right) \left(\frac{1}{5} \rho^5 \right) \Big|_2^3$$

$$= \frac{2\pi}{5} \left(\frac{1}{3} \cos^3 \varphi - \cos \varphi \right) \Big|_0^{\pi} \left(243 - 32 \right)$$

$$= \frac{422\pi}{5} \left(-\frac{1}{3} + 1 - \frac{1}{3} + 1 \right)$$

$$= \underline{\frac{1688\pi}{15}}$$

Exercise

Evaluate $\iiint\limits_D y^2 dV$; $D = \{(\rho,\ \varphi,\ \theta):\ \ 0 \le \rho \le 3,\ \ 0 \le \varphi \le \pi,\ \ 0 \le \theta \le \pi\}$

Solution

$x = \rho \sin \varphi \cos \theta \quad y = \rho \sin \varphi \sin \theta \quad z = \rho \cos \varphi$

$$\iiint\limits_D y^2 dV = \int_0^\pi \int_0^\pi \int_0^3 \left(\rho^2 \sin^2 \varphi \sin^2 \theta\right) \rho^2 \sin \varphi \, d\rho d\varphi d\theta$$

$$= \int_0^\pi \sin^2 \theta \, d\theta \int_0^\pi \sin^2 \varphi \sin \varphi \, d\varphi \int_0^3 \rho^4 \, d\rho$$

$$= \frac{1}{2} \int_0^\pi (1 - \cos 2\theta) \, d\theta \int_0^\pi \left(\cos^2 \varphi - 1\right) d(\cos \varphi) \int_0^3 \rho^4 \, d\rho$$

$$= \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^\pi \left(\frac{1}{3}\cos^3 \varphi - \cos \varphi\right)\Big|_0^\pi \left(\frac{1}{5}\rho^5\right)\Big|_0^3$$

$$= \frac{1}{2}(\pi)\left(-\frac{1}{3} + 1 - \frac{1}{3} + 1\right)\left(\frac{243}{5}\right)$$

$$= \frac{243\pi}{10}\left(\frac{4}{3}\right)$$

$$= \underline{\frac{162\pi}{5}}$$

Exercise

Evaluate $\iiint\limits_D x e^{x^2 + y^2 + z^2} dV$; $D = \left\{(\rho,\ \varphi,\ \theta):\ \ 0 \le \rho \le 1,\ \ 0 \le \varphi \le \frac{\pi}{2},\ \ 0 \le \theta \le \frac{\pi}{2}\right\}$

Solution

$x = \rho \sin \varphi \cos \theta \quad y = \rho \sin \varphi \sin \theta \quad z = \rho \cos \varphi$

$x^2 + y^2 + z^2 = \rho^2$

$$\iiint\limits_D x e^{x^2 + y^2 + z^2} dV = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \left(\rho \sin \varphi \cos \theta \, e^{\rho^2}\right) \rho^2 \sin \varphi \, d\rho d\varphi d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos \theta d\theta \int_0^{\frac{\pi}{2}} \sin^2 \varphi d\varphi \int_0^1 \rho^3 e^{\rho^2} \, d\rho$$

$$u = \rho^2 \quad dv = \rho e^{\rho^2} d\rho = \frac{1}{2} e^{\rho^2} d\rho^2$$

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$$du = 2\rho d\rho \quad v = \frac{1}{2}e^{\rho^2}$$

$$\int \rho^3 e^{\rho^2}\, d\rho = \frac{1}{2}\rho^2 e^{\rho^2} - \int \rho e^{\rho^2}\, d\rho$$

$$= \frac{1}{2}\rho^2 e^{\rho^2} - \frac{1}{2}e^{\rho^2}$$

$$\iiint_D xe^{x^2+y^2+z^2}\, dV = \sin\theta\Big|_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2\varphi)\, d\varphi \ \left(\frac{1}{2}\rho^2 e^{\rho^2} - \frac{1}{2}e^{\rho^2}\right)\Big|_0^1$$

$$= \frac{1}{2}(\varphi - \sin 2\varphi)\Big|_0^{\frac{\pi}{2}} \ \left(\frac{1}{2}e - \frac{1}{2}e + \frac{1}{2}\right)$$

$$= \frac{1}{4}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{8}$$

Exercise

Evaluate $\displaystyle\iiint_D \sqrt{x^2+y^2+z^2}\, dV$; $D = \left\{(\rho,\ \varphi,\ \theta):\ 1\le\rho\le 2,\ \ 0\le\varphi\le\frac{\pi}{4},\ \ 0\le\theta\le 2\pi\right\}$

Solution

$$x = \rho\sin\varphi\cos\theta \quad y = \rho\sin\varphi\sin\theta \quad z = \rho\cos\varphi$$

$$x^2 + y^2 + z^2 = \rho^2$$

$$\iiint_D \sqrt{x^2+y^2+z^2}\, dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^2 \rho^3 \sin\varphi\, d\rho d\varphi d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\varphi\, d\varphi \int_1^2 \rho^3 d\rho$$

$$= 2\pi(-\cos\varphi)\Big|_0^{\frac{\pi}{4}} \ \left(\frac{1}{4}\rho^4\right)\Big|_1^2$$

$$= \frac{1}{2}\left(-\frac{\sqrt{2}}{2}+1\right)(16-1)$$

$$= \frac{15\pi}{2}\left(1 - \frac{\sqrt{2}}{2}\right)$$

Exercise

Find the volume of the solid whose height is 4 and whose base is the disk $\{(r,\ \theta):\ \ 0 \le r \le 2\cos\theta\}$

Solution

Base is the disk $\Rightarrow\ 0 \le \theta \le \pi$

$0 \le z \le 4$

$$V = \int_0^4 \int_0^\pi \int_0^{2\cos\theta} r\ drd\theta dz$$

$$= \frac{1}{2}\int_0^4 dz \int_0^\pi r^2 \Big|_0^{2\cos\theta}\ d\theta$$

$$= 8\int_0^\pi \cos^2\theta\ d\theta$$

$$= 4\int_0^\pi (1 + \cos 2\theta)\ d\theta$$

$$= 4\left(1 + \frac{1}{2}\sin 2\theta\right)\Big|_0^\pi$$

$$\underline{\underline{= 4\pi\ \ unit^3}}$$

Exercise

Find the volume of the solid in the first octant bounded by the cylinder $r = 1$ and the plane $z = x$

Solution

$0 \le z \le x = r\cos\theta \qquad 0 \le r \le 1$

first octant $0 \le \theta \le \dfrac{\pi}{2}$

$$V = \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{r\cos\theta} dz\ rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 z \Big|_0^{r\cos\theta}\ rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta d\theta \int_0^1 r^2 dr$$

$$= \sin\theta \Big|_0^{\frac{\pi}{2}} \left(\frac{1}{3}r^3\right)\Big|_0^1$$

$$= \frac{1}{3} \ unit^3$$

Exercise

Find the volume of the solid bounded by the cylinder $r = 1$ and $r = 2$ and the planes $z = 4 - x - y$ and $z = 0$

Solution

$r = 1$ and $r = 2$ $\rightarrow$ $1 \leq r \leq 2$

$z = 4 - x - y \rightarrow$ $0 \leq z \leq 4 - r\cos\theta - r\sin\theta$

$0 \leq \theta \leq 2\pi$

$$V = \int_0^{2\pi} \int_1^2 \int_0^{4-r\cos\theta-\sin\theta} dz \ rdrd\theta$$

$$= \int_0^{2\pi} \int_1^2 z \bigg|_0^{4-r\cos\theta-r\sin\theta} rdrd\theta$$

$$= \int_0^{2\pi} \int_1^2 \left(4r - r^2\cos\theta - r^2\sin\theta \right) drd\theta$$

$$= \int_0^{2\pi} \int_1^2 \left(4r - r^2 \left(\cos\theta + \sin\theta \right) \right) drd\theta$$

$$= \int_0^{2\pi} \left(2r^2 - \frac{1}{3}r^3 \left(\cos\theta + \sin\theta \right) \right) \bigg|_1^2 d\theta$$

$$= \int_0^{2\pi} \left(8 - \frac{8}{3} \left(\cos\theta + \sin\theta \right) - 2 + \frac{1}{3} \left(\cos\theta + \sin\theta \right) \right) d\theta$$

$$= \int_0^{2\pi} \left(6 - \frac{7}{3} \left(\cos\theta + \sin\theta \right) \right) d\theta$$

$$= \left(6\theta - \frac{7}{3} \left(\sin\theta - \cos\theta \right) \right) \bigg|_0^{2\pi}$$

$$= 12\pi + \frac{7}{3} - \frac{7}{3}$$

$$= 12\pi \ unit^3$$

Exercise

Find the volume of the solid D between the cone $z = \sqrt{x^2 + y^2}$ and the inverted paraboloid $z = 12 - x^2 - y^2$

Solution

$$\begin{cases} z = \sqrt{x^2 + y^2} = r \\ z = 12 - x^2 - y^2 = 12 - r^2 \end{cases} \rightarrow \underline{r \le z \le 12 - r^2}$$

$12 - r^2 = r \rightarrow r^2 + r - 12 = 0$

$\Rightarrow r = 3, \cancel{-4} \rightarrow 0 \le r \le 3$

$$V = \int_0^{2\pi} \int_0^3 \int_r^{12-r^2} dz\ rdrd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^3 z \Big|_r^{12-r^2} rdr$$

$$= 2\pi \int_0^3 \left(12r - r^3 - r^2\right) dr$$

$$= 2\pi \left(6r^2 - \frac{1}{4}r^4 - \frac{1}{3}r^3\right)\Big|_0^3$$

$$= 2\pi \left(54 - \frac{81}{4} - 9\right)$$

$$= \underline{\frac{99\pi}{2}\ unit^3}$$

Exercise

Find the volume of the solid region D that lies inside the cone $\phi = \frac{\pi}{6}$ and inside the sphere $\rho = 4$

Solution

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^4 \rho^2 \sin\varphi\ d\rho d\varphi d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{6}} \sin\varphi d\varphi \int_0^4 \rho^2\ d\rho$$

$$= 2\pi (-\cos\varphi)\Big|_0^{\frac{\pi}{6}} \left(\frac{1}{3}\rho^3\right)\Big|_0^4$$
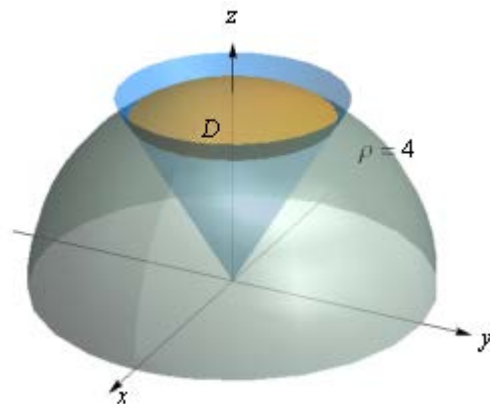
$$= \frac{2\pi}{3}\left(-\frac{\sqrt{3}}{2}+1\right)(64)$$

$$= \frac{64\pi}{3}\left(2-\sqrt{3}\right) \ unit^3$$

Exercise

Find the volume of the solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2, \ z \geq 0$

Solution

$$V = \int_0^{2\pi}\int_0^{\pi/2}\int_{\cos\phi}^2 \rho^2\sin\phi \ d\rho d\phi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi\left[\rho^3\right]_{\cos\phi}^2 d\phi d\theta \qquad\qquad d\left(\cos\phi\right) = -\sin\phi$$

$$= -\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2}\left(8-\cos^3\phi\right)d\left(\cos\phi\right)d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi}\left[8\cos\phi-\frac{1}{4}\cos^4\phi\right]_0^{\pi/2}d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi}\left(8-\frac{1}{4}\right)d\theta$$

$$= \frac{31}{12}\left[\theta\right]_0^{2\pi}$$

$$= \frac{31\pi}{6}\ unit^3$$
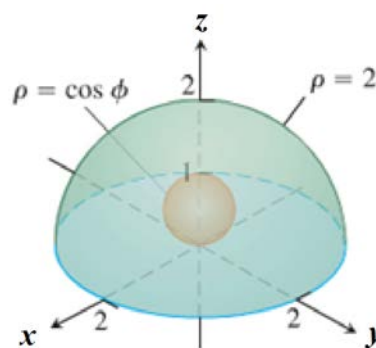
Exercise

Find the volume of the solid bounded below by the hemisphere $\rho = 1, \ z \geq 0$, and above the cardioid of revolution $\rho = 1 + \cos\phi$

Solution

$$V = \int_0^{2\pi}\int_0^{\pi/2}\int_1^{1+\cos\phi} \rho^2\sin\phi \ d\rho d\phi d\theta$$
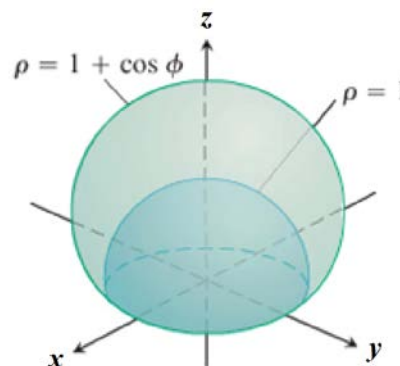
$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi\left[\rho^3\right]_1^{1+\cos\phi} d\phi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi\left[(1+\cos\phi)^3 - 1\right]d\phi d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2}\left[(1+\cos\phi)^3 - 1\right]d(1+\cos\phi)d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi}\left[\frac{1}{4}(1+\cos\phi)^4 - (1+\cos\phi)\right]_0^{\pi/2}d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi}\left[\frac{1}{4}-1-\left(\frac{1}{4}(2)^4 - (1+1)\right)\right]d\theta$$

$$= \frac{11}{12}\int_0^{2\pi}d\theta$$

$$= \frac{11}{12}[\theta]_0^{2\pi}$$

$$= \underline{\frac{11\pi}{6}\ unit^3}$$

Exercise

Find the volume of the solid

Solution

a) $\left(x^2+y^2\right)^2 - 1 \leq z \leq 4 - 4\left(x^2+y^2\right)$; $x^2+y^2 = r^2$

$r^4 - 1 \leq z \leq 4 - 4r$

$4 - 4r = 0 \rightarrow r = 1 \qquad 0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi \rightarrow (4) \quad 0 \leq \theta \leq \frac{\pi}{2}$

$$V = 4\int_0^{\pi/2}\int_0^1\int_{r^4-1}^{4-4r^2} dz\ rdrd\theta$$

$$= 4\int_0^{\pi/2}\int_0^1\left[\left(4-4r^2\right)-\left(r^4-1\right)\right]rdrd\theta$$

$$= 4\int_0^{\pi/2}\int_0^1\left(5-4r^2-r^4\right)rdrd\theta$$

$$= 4\int_0^{\pi/2}\int_0^1\left(5r-4r^3-r^5\right)drd\theta$$

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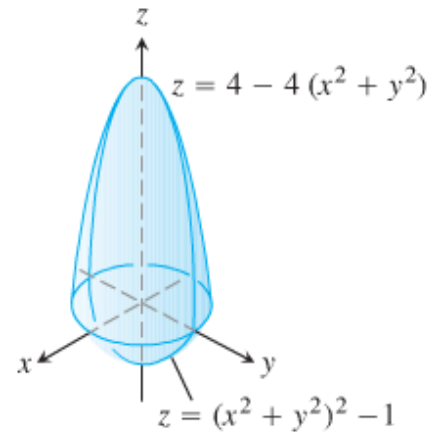
$$= 4 \int_0^{\pi/2} \left[\frac{5}{2} r^2 - r^4 - \frac{1}{6} r^6 \right]_0^1 d\theta$$

$$= 4 \left(\frac{5}{2} - 1 - \frac{1}{6} \right) \int_0^{\pi/2} d\theta$$

$$= \frac{16}{3} [\theta]_0^{\pi/2}$$

$$= \frac{8\pi}{3} \ unit^3$$

b) $$V = 4 \int_0^{\pi/2} \int_0^1 \int_{-\sqrt{1-r^2}}^{1-r} dz \ r dr d\theta$$

$z = 1 - r$

$z = -\sqrt{1-r^2}$

$$= 4 \int_0^{\pi/2} \int_0^1 \left[(1-r) + \sqrt{1-r^2} \right] r dr d\theta$$

$$= 4 \int_0^{\pi/2} \int_0^1 \left[r - r^2 + r \left(1 - r^2 \right)^{1/2} \right] dr d\theta$$

$$= 4 \int_0^{\pi/2} \left(\left[\frac{1}{2} r^2 - \frac{1}{3} r^3 \right]_0^1 - \frac{1}{2} \int_0^1 \left(1 - r^2 \right)^{1/2} d \left(1 - r^2 \right) \right) d\theta$$

$$= 4 \int_0^{\pi/2} \left(\left(\frac{1}{2} - \frac{1}{3} \right) - \frac{1}{3} \left[\left(1 - r^2 \right)^{3/2} \right]_0^1 \right) d\theta$$

$$= 4 \left(\frac{1}{6} + \frac{1}{3} \right) \int_0^{\pi/2} d\theta$$

$$= 2 [\theta]_0^{\pi/2}$$

$$= \pi \ unit^3$$
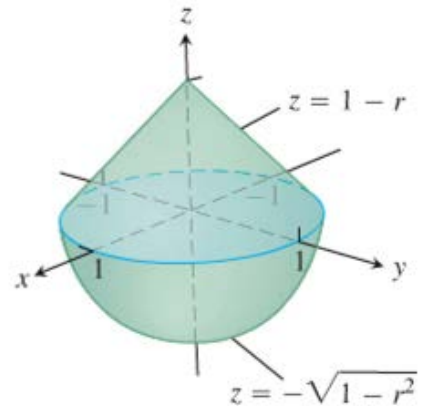
c) $$0 \le z \le \sqrt{1 - x^2 - y^2} = \sqrt{1 - r^2} \quad ; \quad x^2 + y^2 = r^2$$

$z = \sqrt{1 - x^2 - y^2}$

$r = \sin \theta$

$$V = \int_0^{\pi/2} \int_0^{\sin\theta} \int_0^{\sqrt{1-r^2}} dz \ r dr d\theta$$

$$= \int_0^{\pi/2} \int_0^{\sin\theta} \sqrt{1 - r^2} \ r dr d\theta \qquad\qquad d \left(1 - r^2 \right) = -2r dr$$
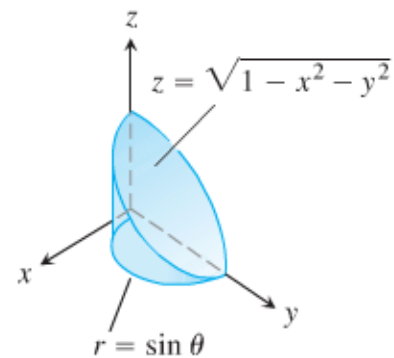
$$= -\frac{1}{2} \int_0^{\pi/2} \int_0^{\sin\theta} \left(1 - r^2 \right)^{1/2} d \left(1 - r^2 \right) d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/2} \left[\frac{2}{3}\left(1-r^2\right)^{3/2} \right]_0^{\sin\theta} d\theta$$

$$= -\frac{1}{3} \int_0^{\pi/2} \left[\left(1-\sin^2\theta\right)^{3/2} - 1 \right] d\theta$$

$$= -\frac{1}{3} \int_0^{\pi/2} \left[\left(\cos^2\theta\right)^{3/2} - 1 \right] d\theta$$

$$= -\frac{1}{3} \int_0^{\pi/2} \left(\cos^3\theta - 1 \right) d\theta$$

$$= -\frac{1}{3} \int_0^{\pi/2} \cos^2\theta\cos\theta\, d\theta + \frac{1}{3}\int_0^{\pi/2} d\theta \qquad\qquad d\left(\sin\theta\right) = \cos\theta\, d\theta$$

$$= -\frac{1}{3} \int_0^{\pi/2} \left(1-\sin^2\theta\right) d\left(\sin\theta\right) + \frac{\pi}{6}$$

$$= -\frac{1}{3} \left[\sin\theta - \frac{1}{3}\sin^3\theta \right]_0^{\pi/2} + \frac{\pi}{6}$$

$$= -\frac{1}{3}\left(1 - \frac{1}{3}\right) + \frac{\pi}{6}$$

$$= -\frac{2}{9} + \frac{\pi}{6}$$

$$= \underline{\frac{3\pi - 4}{18}\ unit^3}$$

d)
$$V = \int_0^{\pi/2} \int_0^{\cos\theta} \int_0^{3\sqrt{1-r^2}} dz\ rdr d\theta$$

$$= \int_0^{\pi/2} \int_0^{\cos\theta} 3r\sqrt{1-r^2}\, dr d\theta \qquad\qquad d\left(1-r^2\right) = -2r dr$$

$$= -\frac{3}{2} \int_0^{\pi/2} \int_0^{\cos\theta} \left(1-r^2\right)^{1/2} dr d\theta$$

$$= -\int_0^{\pi/2} \left[\left(1-r^2\right)^{3/2} \right]_0^{\cos\theta} d\theta$$

$$= -\int_0^{\pi/2} \left[\left(1-\cos^2\theta\right)^{3/2} - 1 \right] d\theta$$

$$= -\int_0^{\pi/2} \left(\sin^3\theta - 1 \right) d\theta$$

$z = 3\sqrt{1-x^2-y^2}$

$r = \cos\theta$
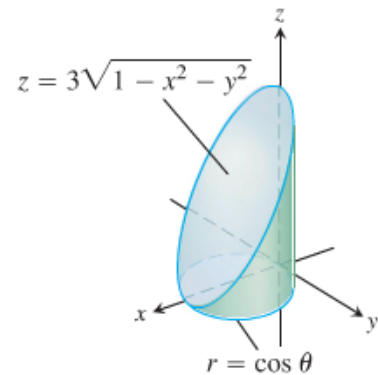
$$= -\int_0^{\pi/2} \sin^2 \theta \sin \theta d\theta + \int_0^{\pi/2} d\theta \qquad\qquad d(\cos\theta) = -\sin\theta d\theta$$

$$= \int_0^{\pi/2} \left(1 - \cos^2\theta\right) d(\cos\theta) + [\theta]_0^{\pi/2}$$

$$= \left[\cos\theta - \frac{1}{3}\cos^3\theta\right]_0^{\pi/2} + \frac{\pi}{2}$$

$$= -1 + \frac{1}{3} + \frac{\pi}{2}$$

$$= \frac{3\pi - 4}{6} \ unit^3$$

Exercise

Find the volume of the smaller region cut from the solid sphere $\rho \le 2$ by the plane $z = 1$

Solution

$$\cos\phi = \frac{z}{\rho} \Rightarrow \rho = \frac{1}{\cos\phi} = \sec\phi$$

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec\phi}^2 \rho^2 \sin\phi \, d\rho d\phi d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/3} \sin\phi \left[\rho^3\right]_{\sec\phi}^2 d\phi d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/3} \sin\phi \left[8 - \sec^3\phi\right] d\phi d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/3} \left(8\sin\phi - \tan\phi\sec^2\phi\right) d\phi d\theta \qquad d(\tan\phi) = \sec^2\phi d\phi$$

$$= \frac{1}{3} \int_0^{2\pi} \left(\int_0^{\pi/3} 8\sin\phi \, d\phi - \int_0^{\pi/3} \tan\phi \, d(\tan\phi)\right) d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \left[-8\cos\phi - \frac{1}{2}\tan^2\phi\right]_0^{\pi/3} d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \left[-4 - \frac{1}{2}(3) - (-8 - 0)\right] d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \frac{5}{2} d\theta$$

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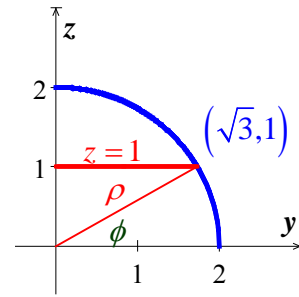
$$= \frac{5}{6}[\theta]_0^{2\pi}$$

$$= \frac{5\pi}{3} \ unit^3$$

Exercise

Find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2 + 1$

Solution

$$x^2 + y^2 \le z \le x^2 + y^2 + 1 \quad \rightarrow \quad r^2 \le z \le r^2 + 1$$

$$x^2 + y^2 = 1 = r^2 \quad \rightarrow \quad 0 \le r \le 1$$

$$0 \le \theta \le 2\pi$$

$$V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{r^2+1} dz \ rdrd\theta$$

$$= 4 \int_0^{\pi/2} \int_0^1 \left[r^2 + 1 - r^2 \right] rdrd\theta$$

$$= 4 \int_0^{\pi/2} \int_0^1 rdrd\theta$$

$$= 2 \int_0^{\pi/2} \left[r^2 \right]_0^1 d\theta$$

$$= 2 \int_0^{\pi/2} d\theta$$

$$= 2[\theta]_0^{\pi/2}$$

$$= 2\left(\frac{\pi}{2} \right)$$

$$= \pi \ unit^3$$

Exercise

Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$

Solution

$$V = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} \int_0^{\sqrt{2-r^2}} dz \; r dr d\theta$$

$$= 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} r[z]_0^{\sqrt{2-r^2}} \; dr d\theta$$

$$= 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} r\sqrt{2-r^2} \; dr d\theta \qquad\qquad\qquad d\left(1-r^2\right) = -2rdr$$

$$= -4 \int_0^{\pi/2} \int_1^{\sqrt{2}} \left(2-r^2\right)^{1/2} d\left(2-r^2\right) d\theta$$

$$= -\frac{8}{3} \int_0^{\pi/2} \left[\left(2-r^2\right)^{3/2}\right]_1^{\sqrt{2}} d\theta$$

$$= -\frac{8}{3} \int_0^{\pi/2} (-1) d\theta$$

$$= \frac{8}{3}[\theta]_0^{\pi/2}$$

$$= \frac{8}{3}\left(\frac{\pi}{2}\right)$$

$$= \underline{\frac{4\pi}{3} \; unit^3}$$

Exercise

Find the volume of the solid between the sphere $x^2 + y^2 + z^2 = 19$ and the hyperboloid $z^2 - x^2 - y^2 = 1$ for $z > 0$

Solution

$$z = \sqrt{19 - x^2 - y^2} \qquad z = \sqrt{1 + x^2 + y^2}$$

$$19 - x^2 - y^2 = 1 + x^2 + y^2 \;\Rightarrow\; 2y^2 = 18 - 2x^2 \Rightarrow \quad y = \pm\sqrt{9 - x^2}$$

$$9 - x^2 = 0 \;\rightarrow\; -3 \le x \le 3$$

$$V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{1+x^2+y^2}}^{\sqrt{19-x^2-y^2}} 1 \; dz dy dx$$

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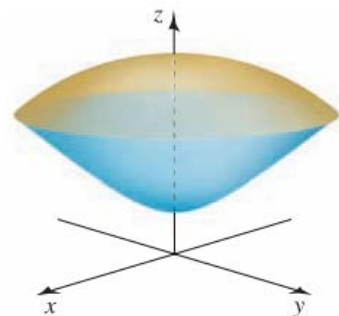
$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left(\sqrt{19 - x^2 - y^2} - \sqrt{1 + x^2 + y^2} \right) dy\, dx \qquad \textit{Convert to \textbf{Polar} coordinates}$$

$$= \int_{0}^{2\pi} \int_{0}^{3} \left(\sqrt{19 - r^2} - \sqrt{1 + r^2} \right) r\, dr\, d\theta$$

$$= \int_{0}^{2\pi} d\theta \left(-\frac{1}{2} \int_{0}^{3} \left(19 - r^2 \right)^{1/2} d\left(19 - r^2 \right) - \frac{1}{2} \int_{0}^{3} \left(1 + r^2 \right)^{1/2} d\left(1 + r^2 \right) \right)$$

$$= 2\pi \left(-\frac{1}{3} \left(19 - r^2 \right)^{3/2} - \frac{1}{3} \left(1 + r^2 \right)^{3/2} \right) \Big|_{0}^{3}$$

$$= -\frac{2}{3} \pi \left(10\sqrt{10} + 10\sqrt{10} - 19\sqrt{19} - 1 \right)$$

$$= \underline{\underline{\frac{2\pi}{3} \left(1 + 19\sqrt{19} - 20\sqrt{10} \right)}} \; unit^3$$

Exercise

Evaluate the integral in cylindrical coordinates $\qquad \int_{0}^{2\pi} \int_{0}^{1} \int_{-1}^{1} r\, dz\, dr\, d\theta$

Solution

$$\int_{0}^{2\pi} \int_{0}^{1} \int_{-1}^{1} r\, dz\, dr\, d\theta = \int_{0}^{2\pi} d\theta \int_{0}^{1} r\, dr \int_{-1}^{1} dz$$

$$= (2\pi)\left(\frac{1}{2} r^2 \right) \Big|_{0}^{1} \; z \Big|_{-1}^{1}$$

$$= (2\pi)\left(\frac{1}{2} \right)(2)$$

$$= \underline{2\pi}$$
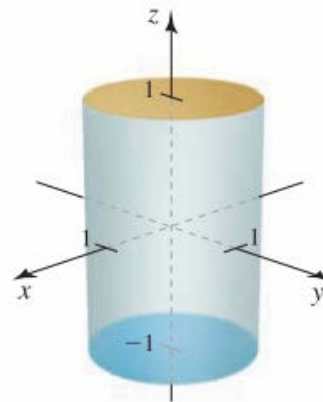
Exercise

Evaluate the integral in cylindrical coordinates $\qquad \int_{0}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_{0}^{9-3\sqrt{x^2+y^2}} dz\, dx\, dy$

Solution

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^{9-3\sqrt{x^2+y^2}} dzdxdy = \int_0^\pi \int_0^3 \int_0^{9-3r} r\, dzdrd\theta$$

$$= \int_0^\pi d\theta \int_0^3 rz \Big|_0^{9-3r} dr$$

$$= \pi \int_0^3 \left(9r - 3r^2\right) dr$$

$$= \pi \left(\frac{9}{2}r^2 - r^3\right)\Big|_0^3$$

$$= \pi \left(\frac{81}{2} - 27\right)$$

$$= \underline{\frac{27}{2}\pi}$$
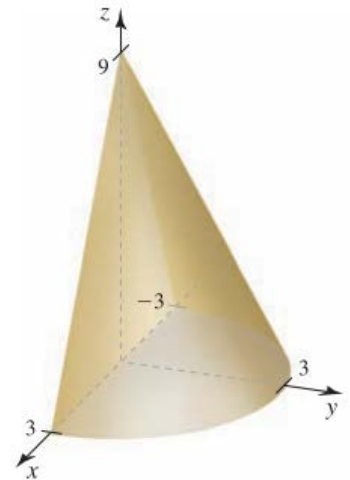
Exercise

Evaluate the integral in cylindrical coordinates $\quad \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{-1}^1 \left(x^2+y^2\right)^{3/2} dzdxdy$

Solution

$$\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{-1}^1 \left(x^2+y^2\right)^{3/2} dzdxdy = \int_0^{2\pi} \int_0^1 \int_{-1}^1 r^3\, dz\, rdrd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 r^4 dr\, (z)\Big|_{-1}^1$$

$$= 2\pi \left(\frac{1}{5}r^5\right)\Big|_0^1 (2)$$

$$= \underline{\frac{4\pi}{5}}$$

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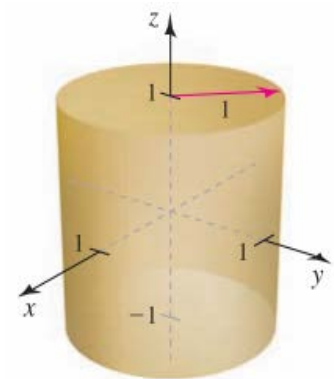
Exercise

Evaluate the integral in cylindrical coordinates $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{2}\frac{1}{1+x^2+y^2}\,dz\,dy\,dx$
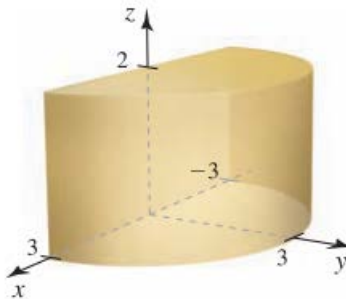
Solution

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{2}\frac{1}{1+x^2+y^2}\,dz\,dy\,dx = \int_{0}^{\pi}\int_{0}^{3}\int_{0}^{2}\frac{1}{1+r^2}\,dz\,r\,dr\,d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi}d\theta\int_{0}^{3}\frac{1}{1+r^2}\,d\!\left(1+r^2\right)\,[z]\Big|_{0}^{2}$$

$$= \pi\ln\!\left(1+r^2\right)\Big|_{0}^{3}$$

$$= \underline{\pi\ln(10)}$$

Exercise

Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the plane $z = 0$ and the hyperboloid $z = \sqrt{17} - \sqrt{1+x^2+y^2}$

Solution

$$z = \sqrt{17} - \sqrt{1+x^2+y^2} = 0 \quad \to \quad 17 = 1+x^2+y^2$$

$$x^2 + y^2 = 16 = r^2$$

$$V = \int_{0}^{2\pi}\int_{0}^{4}\int_{0}^{\sqrt{17}-\sqrt{1+r^2}}1\,dz\,r\,dr\,d\theta$$

$$= \int_{0}^{2\pi}d\theta\int_{0}^{4}z\Big|_{0}^{\sqrt{17}-\sqrt{1+r^2}}\,r\,dr$$

$z = \sqrt{17} - \sqrt{1+x^2+y^2}$
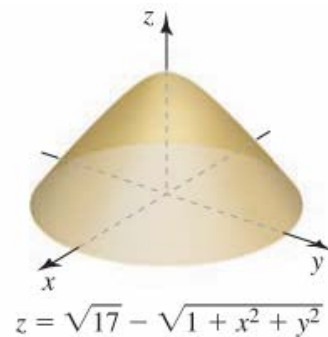
$$= 2\pi \int_0^4 \left(\sqrt{17} - \sqrt{1+r^2} \right) r\, dr$$

$$= 2\pi \int_0^4 \left(\sqrt{17}\, r - r\sqrt{1+r^2} \right) dr$$

$$= 2\pi \left(\frac{1}{2}\sqrt{17}\, r^2 \Big|_0^4 - \frac{1}{2}\int_0^4 \sqrt{1+r^2}\; d\left(1+r^2\right) \right)$$

$$= \pi \left(16\sqrt{17} - \frac{2}{3}\left(1+r^2\right)^{3/2} \Big|_0^4 \right)$$

$$= \pi \left(16\sqrt{17} - \frac{2}{3}17\sqrt{17} + \frac{2}{3} \right)$$

$$\underline{= \pi \left(\frac{14\sqrt{17}+2}{3} \right) unit^3}$$

Exercise

Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the plane $z = 25$ and the paraboloid $z = x^2 + y^2$

Solution

$$z = x^2 + y^2 = r^2 = 25 \;\rightarrow\; r = 5$$

$$V = \int_0^{2\pi} \int_0^5 \int_{r^2}^{25} 1\, dz\; r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^5 z \Big|_{r^2}^{25} r\, dr$$

$$= 2\pi \int_0^5 \left(25 - r^2 \right) r\, dr$$

$$= 2\pi \int_0^5 \left(25r - r^3 \right) dr$$

$$= 2\pi \left(\frac{25}{2} r^2 - \frac{1}{4} r^4 \right) \Big|_0^5$$

$$= 2\pi \left(\frac{1}{2} 5^4 - \frac{1}{4} 5^4 \right)$$

$$\underline{= \frac{625\pi}{2}\; unit^3}$$
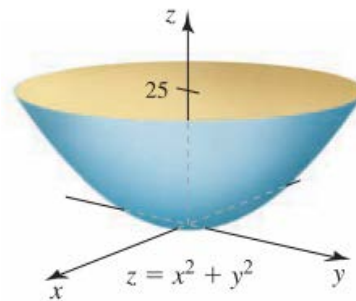
Exercise

Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the parabolic cylinders $z = y^2 + 1$ and $z = 2 - x^2$

Solution

$$2 - x^2 - \left(y^2 + 1\right) = 1 - \left(x^2 + y^2\right)$$

$$z = y^2 + 1 = 2 - x^2 \;\rightarrow\; x^2 + y^2 = 1 = r^2$$

$$V = \int_0^{2\pi} \int_0^1 \left(1 - r^2\right) r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 \left(r - r^3\right) dr$$

$$= 2\pi \left(\frac{1}{2} r^2 - \frac{1}{4} r^4\right)\Big|_0^1$$

$$= 2\pi \left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \underline{\frac{\pi}{2} \quad unit^3}$$
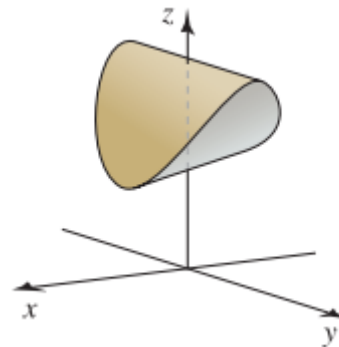
Exercise

Evaluate the integral $\displaystyle \int_0^{2\pi} \int_0^{\pi/3} \int_0^{4\sec\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

Solution

$$\int_0^{2\pi} \int_0^{\pi/3} \int_0^{4\sec\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/3} \frac{1}{3}\sin\varphi\, \rho^3 \Big|_0^{4\sec\varphi} d\varphi$$

$$= \frac{128\pi}{3} \int_0^{\pi/3} \sin\varphi\, \sec^3\varphi\, d\varphi$$

$$= -\frac{128\pi}{3} \int_0^{\pi/3} \cos^{-3}\varphi\, d(\cos\varphi)$$

$$= \frac{64\pi}{3} \frac{1}{\cos^2\varphi}\Big|_0^{\pi/3}$$

$$= \frac{64\pi}{3}(4 - 1)$$

$$= \underline{64\pi}$$
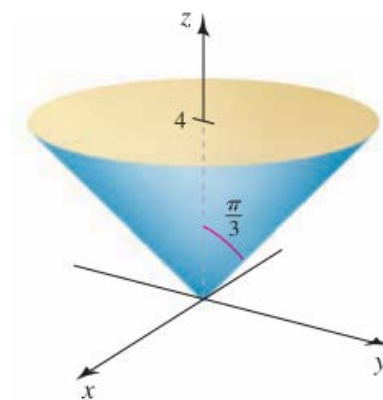
Exercise

Evaluate the integral $\displaystyle\int_0^{\pi}\int_0^{\pi/6}\int_{2\sec\varphi}^{4} \rho^2 \sin\varphi\, d\rho d\varphi d\theta$

Solution

$$\int_0^{\pi}\int_0^{\pi/6}\int_{2\sec\varphi}^{4} \rho^2 \sin\varphi\, d\rho d\varphi d\theta = \frac{1}{3}\int_0^{\pi} d\theta \int_0^{\pi/6} \sin\varphi\, \rho^3 \Big|_{2\sec\varphi}^{4}\, d\varphi$$

$$= \frac{\pi}{3}\int_0^{\pi/6} \sin\varphi\left(64 - 8\sec^3\varphi\right) d\varphi$$

$$= \frac{8\pi}{3}\int_0^{\pi/6} \left(\cos^{-3}\varphi - 8\right) d(\cos\varphi)$$

$$= \frac{8\pi}{3}\left(\frac{-1}{2\cos^2\varphi} - 8\cos\varphi\right)\Big|_0^{\pi/6}$$

$$= \frac{8\pi}{3}\left(-\frac{2}{3} - 4\sqrt{3} + \frac{1}{2} + 8\right)$$

$$= \frac{8\pi}{3}\left(\frac{47}{3} - 4\sqrt{3}\right)$$

$$= \left(\frac{188}{9} - \frac{32}{3}\sqrt{3}\right)\pi$$
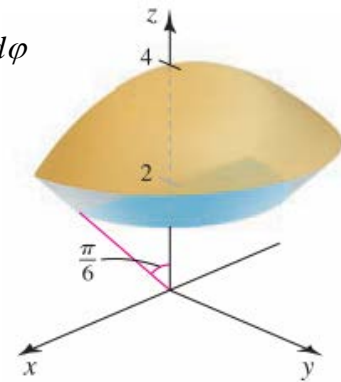
Exercise

Evaluate the integral $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_1^{2\sec\varphi} \left(\rho^{-3}\right)\rho^2 \sin\varphi\, d\rho d\varphi d\theta$
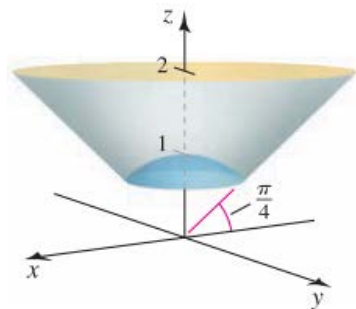
Solution

$$\int_0^{2\pi}\int_0^{\pi/4}\int_1^{2\sec\varphi} \left(\rho^{-3}\right)\rho^2 \sin\varphi\, d\rho d\varphi d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/4}\int_1^{2\sec\varphi} \sin\varphi\left(\frac{1}{\rho}\, d\rho\right) d\varphi$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\varphi\ln(\rho)\Big|_1^{2\sec\varphi}\, d\varphi$$

$$= 2\pi \int_0^{\pi/4} \sin\varphi \, \ln(2\sec\varphi) d\varphi$$

$$u = \ln(2\sec\varphi) \qquad\qquad dv = \sin\varphi d\varphi$$

$$du = \frac{2\sec\varphi\tan\varphi}{2\sec\varphi} = \tan\varphi \qquad v = -\cos\varphi$$

$$= 2\pi\left[-\cos\varphi \, \ln(2\sec\varphi)\Big|_0^{\pi/4} + \int_0^{\pi/4} \sin\varphi d\varphi \right]$$

$$= 2\pi\left(-\cos\varphi \, \ln(2\sec\varphi) - \cos\varphi\right)\Big|_0^{\pi/4}$$

$$= 2\pi\left(-\frac{\sqrt{2}}{2}\ln(2\sqrt{2}) - \frac{\sqrt{2}}{2} + \ln 2 + 1 \right)$$

$$= \underline{2\pi\left[\ln 2 - \frac{\sqrt{2}}{2}\ln(2\sqrt{2}) + 1 - \frac{\sqrt{2}}{2} \right]}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^{2\csc\varphi} \rho^2\sin\varphi \, d\rho d\varphi d\theta$

<u>Solution</u>

$$\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^{2\csc\varphi} \rho^2\sin\varphi \, d\rho d\varphi d\theta = \frac{1}{3}\int_0^{2\pi} d\theta \int_{\pi/6}^{\pi/3}\sin\varphi\left(\rho^3\right)\Big|_0^{2\csc\varphi} d\varphi$$

$$= \frac{16\pi}{3}\int_{\pi/6}^{\pi/3}\sin\varphi\csc^3\varphi \, d\varphi$$

$$= -\frac{16\pi}{3}\int_{\pi/6}^{\pi/3}\sin\varphi\csc\varphi \, d(\cot\varphi)$$

$$= -\frac{16\pi}{3}\int_{\pi/6}^{\pi/3} d(\cot\varphi)$$

$$= -\frac{16\pi}{3}(\cot\varphi)\Big|_{\pi/6}^{\pi/3}$$

$$= -\frac{16\pi}{3}\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right)$$

$$= \frac{32\pi}{3\sqrt{3}}$$

$$= \underline{\frac{32}{9}\pi\sqrt{3}}$$
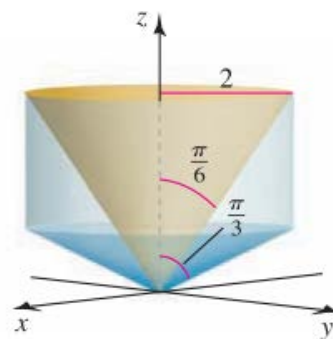
Exercise

Use the spherical coordinates to find the volume of a ball of radius $a > 0$

Solution

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \left(\rho^3\right)\Big|_0^{a}$$

$$= \frac{2\pi}{3} a^3 \left(-\cos\varphi\right)\Big|_0^{\pi}$$

$$= \underline{\frac{4}{3}\pi a^3} \quad unit^3$$

Exercise

Use the spherical coordinates to find the volume of the solid bounded by the sphere $\rho = 2\cos\varphi$ and the hemisphere $\rho = 1$, $z \geq 0$

Solution

$$\rho = 2\cos\varphi = 1 \quad \rightarrow \quad \varphi = \frac{\pi}{3}$$

$$z = \frac{1}{2} \rightarrow \cos\varphi = \frac{1}{2}\frac{1}{\rho} \quad \rightarrow \quad \rho = \frac{1}{2}\sec\varphi$$

$$V = 2\int_0^{2\pi} \int_0^{\pi/3} \int_{\frac{1}{2}\sec\varphi}^{1} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= \frac{2}{3} \int_0^{2\pi} d\theta \int_0^{\pi/3} \sin\varphi \left(\rho^3\right)\Big|_{\frac{1}{2}\sec\varphi}^{1} d\varphi$$

$$= \frac{4\pi}{3} \int_0^{\pi/3} \sin\varphi \left(1 - \frac{1}{8}\sec^3\varphi\right) d\varphi$$

$$= \frac{4\pi}{3} \left(\int_0^{\pi/3} \sin\varphi \, d\varphi + \frac{1}{8}\int_0^{\pi/3} \cos^{-3}\varphi \, d(\cos\varphi)\right)$$

$$= \frac{4\pi}{3} \left(-\cos\varphi - \frac{1}{16}\frac{1}{\cos^2\varphi}\right)\Big|_0^{\pi/3}$$

$$= \frac{4\pi}{3}\left(-\frac{1}{2} - \frac{1}{4} + 1 + \frac{1}{16}\right)$$
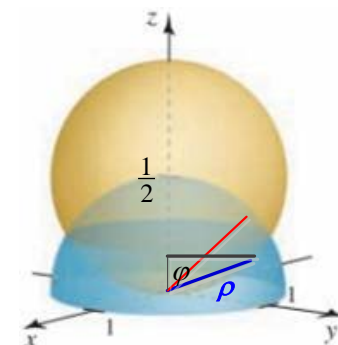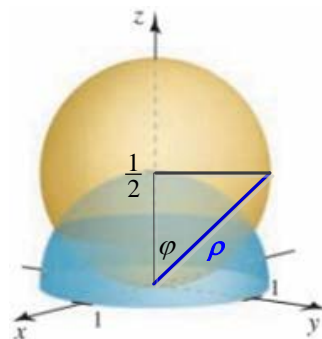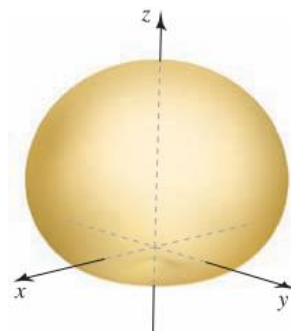
$$= \frac{5\pi}{12}$$

Exercise

Use the spherical coordinates to find the volume of the solid of revolution

$$D = \{(\rho, \varphi, \theta): \ 0 \le \rho \le 1 + \cos\varphi, \ 0 \le \varphi \le \pi, \ 0 \le \theta \le 2\pi\}$$

Solution

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{1+\cos\varphi} \rho^2 \sin\varphi \ d\rho d\varphi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \rho^3 \Big|_0^{1+\cos\varphi} d\varphi$$

$$= \frac{2\pi}{3}\int_0^{\pi} \sin\varphi(1+\cos\varphi)^3 \ d\varphi$$

$$= -\frac{2\pi}{3}\int_0^{\pi} (1+\cos\varphi)^3 \ d(1+\cos\varphi)$$

$$= -\frac{\pi}{6}(1+\cos\varphi)^4 \Big|_0^{\pi}$$

$$= \frac{\pi}{6}2^4$$

$$= \frac{8}{3}\pi \ unit^3$$

Exercise

Use the spherical coordinates to find the volume of the solid outside the cone $\varphi = \frac{\pi}{4}$ and inside the sphere $\rho = 4\cos\varphi$

Solution

$$V = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{4\cos\varphi} \rho^2 \sin\varphi \ d\rho d\varphi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_{\pi/4}^{\pi/2} \sin\varphi\left(\rho^3\right)\Big|_0^{4\cos\varphi} d\varphi$$
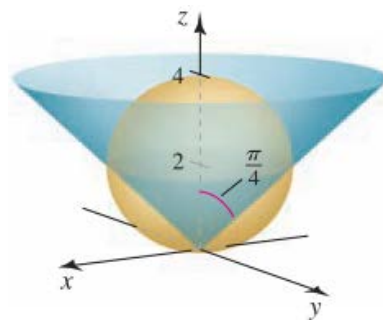
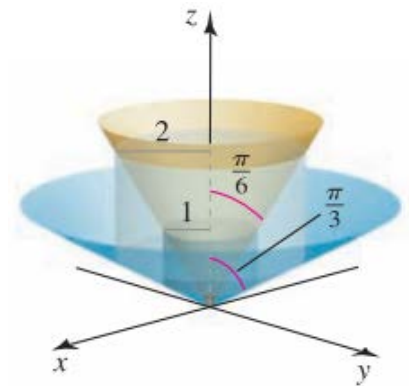$$= \frac{128}{3}\pi \int_{\pi/4}^{\pi/2} \sin\varphi\left(\cos^3\varphi\right) d\varphi$$

$$= \frac{128}{3}\pi \int_{\pi/4}^{\pi/2} \left(-\cos^3\varphi\right) d(\cos\varphi)$$

$$= \frac{32}{3}\pi \left(-\cos^4\varphi\right)\Big|_{\pi/4}^{\pi/2}$$

$$= \frac{32}{3}\pi \left(\frac{1}{4}\right)$$

$$= \frac{8}{3}\pi \quad unit^3$$

Exercise

Use the spherical coordinates to find the volume of the solid bounded by the cylinders $r = 1$ and $r = 2$, and the cone $\varphi = \frac{\pi}{6}$ and $\varphi = \frac{\pi}{3}$
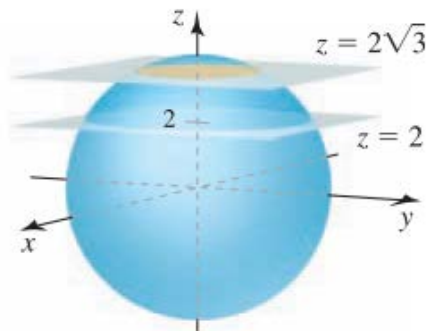
Solution

$$V = \int_0^{2\pi} \int_{\pi/6}^{\pi/3} \int_{\csc\varphi}^{2\csc\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_{\pi/6}^{\pi/3} \sin\varphi \left(\rho^3\right)\Big|_{\csc\varphi}^{2\csc\varphi} d\varphi$$

$$= \frac{14\pi}{3}\int_{\pi/6}^{\pi/3} \sin\varphi \left(\csc^3\varphi\right) d\varphi$$

$$= \frac{14\pi}{3}\int_{\pi/6}^{\pi/3} \csc^2\varphi \, d\varphi$$

$$= \frac{14\pi}{3}\left(-\cot\varphi\right)\Big|_{\pi/6}^{\pi/3}$$

$$= \frac{14\pi}{3}\left(-\frac{1}{\sqrt{3}} + \sqrt{3}\right)$$

$$= \frac{14\pi}{3}\left(\frac{2}{\sqrt{3}}\right)$$

$$= \frac{28}{9}\pi\sqrt{3} \quad unit^3$$

Exercise

Use the spherical coordinates to find the volume of the ball $\rho \le 4$ that lies between the planes $z = 2$ and $z = 2\sqrt{3}$

Solution

$z = 2\sqrt{3} \quad \to \quad \cos\varphi = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2} \quad \Rightarrow \quad \varphi = \dfrac{\pi}{6}$

$z = 2 \quad \to \quad \cos\varphi = \dfrac{2}{4} = \dfrac{1}{2} \quad \Rightarrow \quad \varphi = \dfrac{\pi}{3}$

$V = \displaystyle\int_0^{2\pi} \int_0^{\pi/6} \int_{2\sqrt{3}\sec\varphi}^{4} \rho^2 \sin\varphi \, d\rho d\varphi d\theta - \int_0^{2\pi} \int_0^{\pi/3} \int_{2\sec\varphi}^{4} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$

$= \dfrac{1}{3} \displaystyle\int_0^{2\pi} d\theta \int_0^{\pi/6} \sin\varphi \left(\rho^3\right)\Big|_{2\sqrt{3}\sec\varphi}^{4} d\varphi - \dfrac{1}{3} \int_0^{2\pi} d\theta \int_0^{\pi/3} \sin\varphi \left(\rho^3\right)\Big|_{2\sec\varphi}^{4} d\varphi$

$= \dfrac{2\pi}{3} \displaystyle\int_0^{\pi/6} \sin\varphi \left(64 - 24\sqrt{3}\sec^3\varphi\right) d\varphi - \dfrac{2\pi}{3} \int_0^{\pi/3} \sin\varphi \left(64 - 8\sec^3\varphi\right) d\varphi$

$= \dfrac{16\pi}{3} \displaystyle\int_0^{\pi/6} \left(3\sqrt{3}\cos^{-3}\varphi - 8\right) d(\cos\varphi) + \dfrac{16\pi}{3} \int_0^{\pi/3} \left(8 - \cos^{-3}\varphi\right) d(\cos\varphi)$

$= \dfrac{16\pi}{3} \left(-\dfrac{3\sqrt{3}}{2}\sec^2\varphi - 8\cos\varphi\right)\Big|_0^{\pi/6} + \dfrac{16\pi}{3}\left(8\cos\varphi + \dfrac{1}{2}\sec^2\varphi\right)\Big|_0^{\pi/3}$

$= \dfrac{16\pi}{3} \left(-2\sqrt{3} - 4\sqrt{3} + \dfrac{3\sqrt{3}}{2} + 8\right) + \dfrac{16\pi}{3}\left(4 + 2 - 8 - \dfrac{1}{2}\right)$

$= \dfrac{16\pi}{3} \left(-\dfrac{9\sqrt{3}}{2} + 8 - \dfrac{5}{2}\right)$

$= \dfrac{8\pi}{3}\left(9\sqrt{3} - 11\right) \ unit^3$
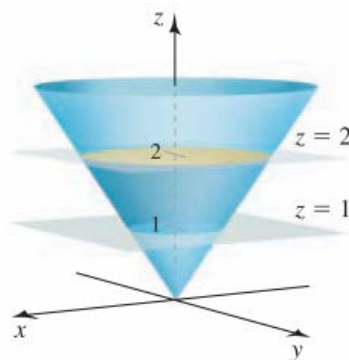
Exercise

Use the spherical coordinates to find the volume of the solid inside the cone $z = \left(x^2 + y^2\right)^{1/2}$ that lies between the planes $z = 1$ and $z = 2$

Solution

$$z = 2 \;\rightarrow\; x^2 + y^2 = 4 = r^2 \;\Rightarrow\; \varphi = \tan^{-1}\frac{2}{2} = \frac{\pi}{4}$$

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_{\sec\varphi}^{2\sec\varphi} \rho^2 \sin\varphi \; d\rho d\varphi d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\varphi \left(\rho^3\right)\Big|_{2\sec\varphi}^{\sec\varphi} \; d\varphi$$

$$= \frac{2\pi}{3} \int_0^{\pi/4} \left(-7\sec^3\varphi\right) d(\cos\varphi)$$

$$= \frac{7\pi}{3} \left(\frac{1}{\cos^2\varphi}\right)\Bigg|_0^{\pi/4}$$

$$= \frac{7\pi}{3} \; unit^3$$

Or: $Volume = \frac{1}{3} Ah = \frac{1}{3}\left(2^2 \pi \times 2 - 1^2 \pi \times 1\right) = \frac{7\pi}{3}$

Exercise

The *x*- and *y*-axes from the axes of two right circular cylinders with radius 1.
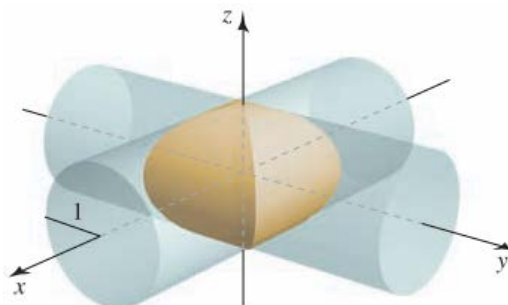Find the volume of the solid that is common to the two cylinders.

Solution

Due to symmetry, this region is made up of *eight* identical pieces, one in each octant.

$$y = 0 \;\rightarrow\; x^2 + z^2 = 1 \;\Rightarrow\; x = \sqrt{1 - z^2}$$

$$/\!/ \; x\text{-axis} \;\rightarrow\; y^2 + z^2 = 1 \;\Rightarrow\; y = \sqrt{1 - z^2}$$
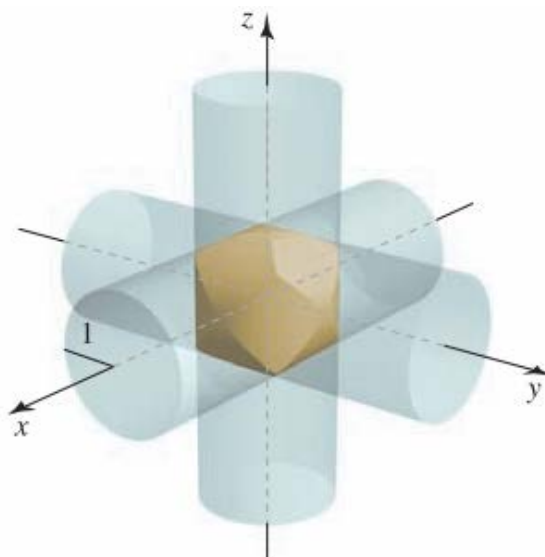
$$V = 8 \int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2}} 1 \; dydxdz$$

$$= 8 \int_0^1 \int_0^{\sqrt{1-z^2}} \sqrt{1 - z^2} \; dxdz$$

$$= 8 \int_0^1 \sqrt{1-z^2} \; x \Big|_0^{\sqrt{1-z^2}} \; dz$$

$$= 8 \int_0^1 \left(1 - z^2 \right) dz$$

$$= 8 \left(z - \frac{1}{3} z^3 \right) \Big|_0^1$$
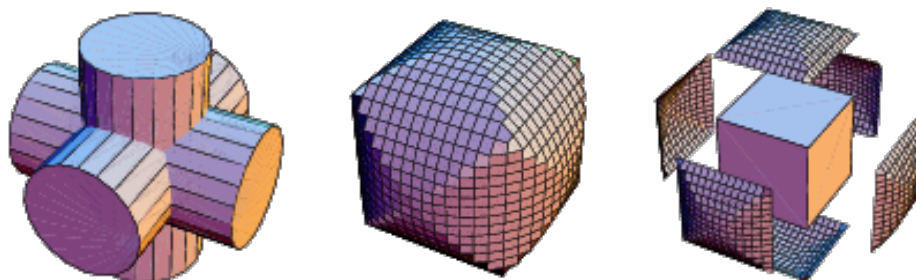
$$= \frac{16}{3} \; unit^3$$

Exercise

The coordinate axes from the axes of three right circular cylinders with radius 1.

Find the volume of the solid that is common to the three cylinders.

Solution

Due to symmetry, this region is made up of *eight* identical pieces, one in each octant.

$$y = 0 \;\; \rightarrow \; x^2 + z^2 = 1 \;\Rightarrow\; x = \sqrt{1-z^2}$$

$$\text{//} \; z\text{-}axis \;\; \rightarrow \; x^2 + y^2 = 1 \;\Rightarrow\; y = \sqrt{1-x^2}$$

If the particle starts at a point on the xz-plane for which $x < z$, then $\sqrt{1-z^2} < \sqrt{1-x^2}$

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$$V = 8 \left(\int_0^{\frac{\sqrt{2}}{2}} \int_x^{\sqrt{1-x^2}} \int_0^{\sqrt{1-z^2}} 1 \, dydzdx + \int_0^{\frac{\sqrt{2}}{2}} \int_z^{\sqrt{1-z^2}} \int_0^{\sqrt{1-x^2}} 1 \, dydxdz \right)$$

$$= 8 \left(\int_0^{\frac{\sqrt{2}}{2}} \int_x^{\sqrt{1-x^2}} \sqrt{1-z^2} \, dzdx + \int_0^{\frac{\sqrt{2}}{2}} \int_z^{\sqrt{1-z^2}} \sqrt{1-x^2} \, dxdz \right)$$

$$= 16 \int_0^{\frac{\sqrt{2}}{2}} \int_x^{\sqrt{1-x^2}} \sqrt{1-z^2} \, dzdx$$

$$= 16 \int_0^{\frac{\pi}{4}} \int_0^1 r\sqrt{1-r^2 \cos^2\theta} \, drd\theta \qquad\qquad w = r\cos\theta \quad \Rightarrow \quad dw = \cos\theta dr$$

$$= 16 \int_0^{\frac{\pi}{4}} \int_0^1 \frac{w}{\cos\theta} \sqrt{1-w^2} \, \frac{dw}{\cos\theta} d\theta$$

$$= -8 \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2\theta} \int_0^1 \sqrt{1-w^2} \, d\left(1-w^2\right) d\theta$$

$$= -\frac{16}{3} \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2\theta} \left(1-w^2\right)^{3/2} \Big|_0^1 \, d\theta$$

$$= -\frac{16}{3} \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2\theta} \left(1-r^2\cos^2\theta\right)^{3/2} \Big|_0^1 \, d\theta$$

$$= -\frac{16}{3} \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2\theta} \left(\left(1-\cos^2\theta\right)^{3/2} - 1 \right) d\theta$$

$$= -\frac{16}{3} \int_0^{\frac{\pi}{4}} \frac{\sin^3\theta - 1}{\cos^2\theta} d\theta$$

$$= -\frac{16}{3} \int_0^{\frac{\pi}{4}} \left(\tan^2\theta \sin\theta - \sec^2\theta \right) d\theta$$

$$= -\frac{16}{3} \int_0^{\frac{\pi}{4}} \frac{\cos^2\theta - 1}{\cos^2\theta} d\left(\cos\theta\right) + \frac{16}{3}\tan\theta \Big|_0^{\frac{\pi}{4}}$$

$$= -\frac{16}{3} \int_0^{\frac{\pi}{4}} \left(1 - \frac{1}{\cos^2\theta} \right) d\left(\cos\theta\right) + \frac{16}{3}$$

$$= -\frac{16}{3}\left(\cos\theta + \frac{1}{\cos\theta}\right)\Big|_{0}^{\frac{\pi}{4}} + \frac{16}{3}$$

$$= -\frac{16}{3}\left(\frac{\sqrt{2}}{2} + \sqrt{2} - 2\right) + \frac{16}{3}$$

$$= -8\sqrt{2} + \frac{32}{3} + \frac{16}{3}$$

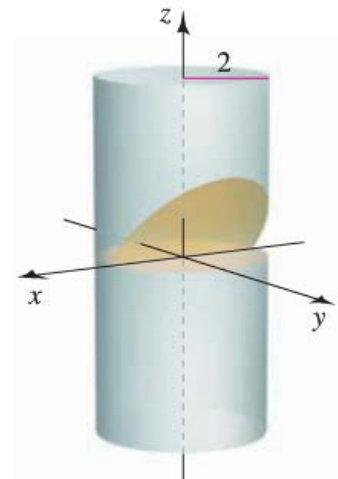$$= 16 - 8\sqrt{2}$$

$$= 8\left(2 - \sqrt{2}\right) \ unit^3$$

Exercise

Find the volume of one of the wedges formed when the cylinder $x^2 + y^2 = 4$ is cut by the planes $z = 0$ and $y = z$

Solution

$$x^2 + y^2 = 4 \ \rightarrow \ 0 \le r \le 2$$

$$z = 0 \quad y = z \ \rightarrow 0 \le \theta \le \pi$$

$$z = y = r\sin\theta \ \rightarrow \ 0 \le z \le r\sin\theta$$

$$V = \int_{0}^{2}\int_{0}^{\pi}\int_{0}^{r\sin\theta} r \, dz d\theta dr$$

$$= \int_{0}^{2}\int_{0}^{\pi} rz \Big|_{0}^{r\sin\theta} d\theta dr$$

$$= \int_{0}^{2} r^2 dr \int_{0}^{\pi} \sin\theta \, d\theta$$

$$= \frac{1}{3}r^3 \Big|_{0}^{2} \ (-\cos\theta) \Big|_{0}^{\pi}$$

$$= \frac{8}{3}(1+1)$$
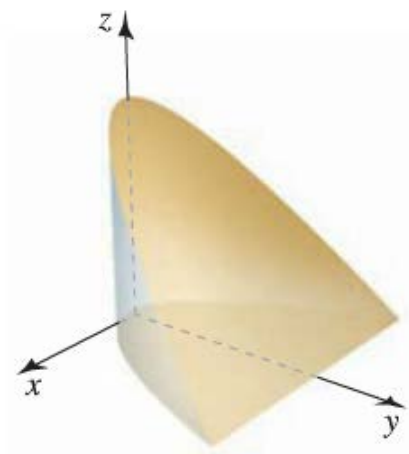
$$= \frac{16}{3} \ unit^3$$

Exercise

Find the volume of the region inside the parabolic cylinder $y = x^2$ between the planes $z = 3 - y$ and $z = 0$

Solution

$z = 3 - y = 0 \rightarrow y = 3 \quad x^2 \le y \le 3$

$y = x^2 = 3 \quad \rightarrow x = \pm\sqrt{3}$

$V = \int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^{3} \int_{0}^{3-y} dz\,dy\,dx$

$= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^{3} z \Big|_{0}^{3-y} dy\,dx$

$= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^{3} (3-y)\,dy\,dx$

$= \int_{-\sqrt{3}}^{\sqrt{3}} \left(3y - \frac{1}{2}y^2\right)\Big|_{x^2}^{3} dx$

$= \int_{-\sqrt{3}}^{\sqrt{3}} \left(9 - \frac{9}{2} - 3x^2 + \frac{1}{2}x^4\right) dx$

$= \left(\frac{9}{2}x - x^3 + \frac{1}{10}x^5\right)\Big|_{-\sqrt{3}}^{\sqrt{3}}$

$= 2\left(\frac{9}{2}\sqrt{3} - 3\sqrt{3} + \frac{3}{10}\sqrt{3}\right)$

$= 2\left(\frac{18}{10}\sqrt{3}\right)$

$= \underline{\frac{18\sqrt{3}}{5} \; unit^3}$

Exercise

Find the volume of the tetrahedron with vertices $(0,\ 0,\ 0)$, $(1,\ 0,\ 0)$, $(1,\ 1,\ 0)$, and $(1,\ 1,\ 1)$

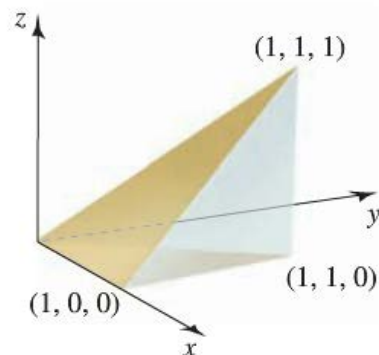### Solution

$0 \le x \le 1$
$0 \le y \le x$
$0 \le z \le y$

$$V = \int_0^1 \int_0^x \int_0^y dz\,dy\,dx$$

$$= \int_0^1 \int_0^x z \Big|_0^y dy\,dx$$

$$= \int_0^1 \int_0^x y\,dy\,dx$$

$$= \int_0^1 \frac{1}{2} y^2 \Big|_0^x dx$$

$$= \int_0^1 \frac{1}{2} x^2 dx$$

$$= \frac{1}{6} x^3 \Big|_0^1$$

$$= \frac{1}{6}\ unit^3$$
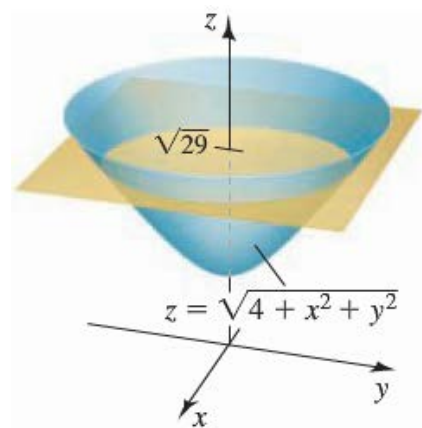

(1, 1, 1)

(1, 1, 0)

(1, 0, 0)

Exercise

Find the volume of the region bounded by the plane $z = \sqrt{29}$ and the hyperboloid $z = \sqrt{4 + x^2 + y^2}$.
Use integration in cylindrical coordinates.

Solution

$$z = \sqrt{4 + x^2 + y^2} = \sqrt{29}$$

$$4 + x^2 + y^2 = 29$$

$$x^2 + y^2 = 25 \quad \rightarrow \quad 0 \le r \le 5$$

$$0 \le \theta \le 2\pi$$

$$V = \int_0^{2\pi} \int_0^5 \int_{\sqrt{4+r^2}}^{\sqrt{29}} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^5 r\,z \Big|_{\sqrt{4+r^2}}^{\sqrt{29}} dr$$

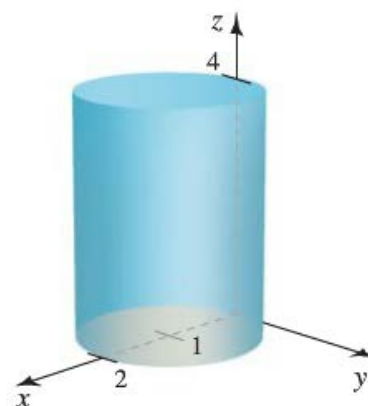$$= \theta \Big|_0^{2\pi} \int_0^5 r \left(\sqrt{29} - \sqrt{4+r^2} \right) dr$$

$\sqrt{29}$

$z = \sqrt{4 + x^2 + y^2}$

$$= 2\pi\sqrt{29} \int_0^5 r \, dr - 2\pi \int_0^5 r \left(4 + r^2\right)^{1/2} dr$$

$$= \pi\sqrt{29} \; r^2 \Big|_0^5 - \pi \int_0^5 \left(4 + r^2\right)^{1/2} d\left(4 + r^2\right)$$

$$= 25\pi\sqrt{29} - \frac{2\pi}{3}\left(4 + r^2\right)^{3/2} \Big|_0^5$$

$$= 25\pi\sqrt{29} - \frac{2\pi}{3}\left((29)^{3/2} - 8\right)$$

$$= 25\pi\sqrt{29} - \frac{58\pi}{3}\sqrt{29} + \frac{16\pi}{3}$$

$$= \frac{17\pi}{3}\sqrt{29} + \frac{16\pi}{3} \; unit^3$$

Exercise

Find the volume of the solid cylinder whose height is 4 and whose base is the disk $\{(r, \theta) : \ 0 \le r \le 2\cos\theta\}$. Use integration in cylindrical coordinates

Solution

$$V = \int_0^4 \int_0^\pi \int_0^{2\cos\theta} r \, dr \, d\theta \, dz$$

$$= \int_0^4 dz \int_0^\pi \frac{1}{2} r^2 \Big|_0^{2\cos\theta} d\theta$$

$$= \frac{1}{2} z \Big|_0^4 \int_0^\pi 4\cos^2\theta \, d\theta$$

$$= 4 \int_0^\pi \left(1 + \cos 2\theta\right) d\theta$$

$$= 4\left(\theta + \frac{1}{2}\sin 2\theta\right) \Big|_0^\pi$$
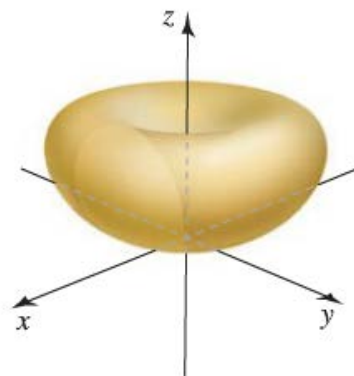
$$= 4\pi \; unit^3$$

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Exercise

Use integration in spherical coordinates to find the volume of the rose petal of revolution

$$D = \left\{ (\rho, \varphi, \theta): \ 0 \le \rho \le 4 \sin 2\varphi, \ 0 \le \varphi \le \frac{\pi}{2}, \ 0 \le \theta \le 2\pi \right\}$$

Solution

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{4\sin 2\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi \ \rho^3 \Big|_0^{4\sin 2\varphi} \, d\varphi$$

$$= \frac{1}{3} \theta \Big|_0^{2\pi} \int_0^{\frac{\pi}{2}} 64 \sin\varphi \ \sin^3 2\varphi \, d\varphi$$

$$= \frac{128\pi}{3} \int_0^{\frac{\pi}{2}} 8 \sin\varphi \ \sin^3 \varphi \cos^3 \varphi \, d\varphi$$

$$= \frac{1024\pi}{3} \int_0^{\frac{\pi}{2}} \sin^4 \varphi \cos^2 \varphi \cos\varphi \, d\varphi$$

$$= \frac{1024\pi}{3} \int_0^{\frac{\pi}{2}} \sin^4 \varphi \left(1 - \sin^2 \varphi\right) d\left(\sin\varphi\right)$$

$$= \frac{1024\pi}{3} \int_0^{\frac{\pi}{2}} \left(\sin^4 \varphi - \sin^6 \varphi\right) d\left(\sin\varphi\right)$$

$$= \frac{1024\pi}{3} \left(\frac{1}{5} \sin^5 \varphi - \frac{1}{7} \sin^7 \varphi\right) \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1024\pi}{3} \left(\frac{1}{5} - \frac{1}{7}\right)$$

$$= \frac{1024\pi}{3} \left(\frac{2}{35}\right)$$

$$= \underline{\frac{2048\pi}{105} \ unit^3}$$

Exercise

Use integration in spherical coordinates to find the volume of the region above the cone $\varphi = \frac{\pi}{4}$ and inside the sphere $\rho = 4\cos\varphi$.

Solution

$$0 \le \varphi \le \frac{\pi}{4} \qquad 0 \le \rho \le 4\cos\varphi \qquad 0 \le \theta \le 2\pi$$

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{4\cos\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

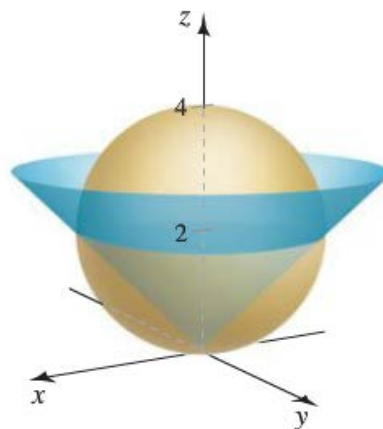$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\varphi\rho^3 \Big|_0^{4\cos\varphi} d\varphi$$

$$= \frac{128\pi}{3}\int_0^{\frac{\pi}{4}} \sin\varphi\cos^3\varphi \, d\varphi$$

$$= -\frac{128\pi}{3}\int_0^{\frac{\pi}{4}} \cos^3\varphi \, d(\cos\varphi)$$

$$= -\frac{32\pi}{3}\cos^4\varphi \Big|_0^{\frac{\pi}{4}}$$

$$= -\frac{32\pi}{3}\left(\frac{1}{4}-1\right)$$

$$= 8\pi \ unit^3$$

Exercise
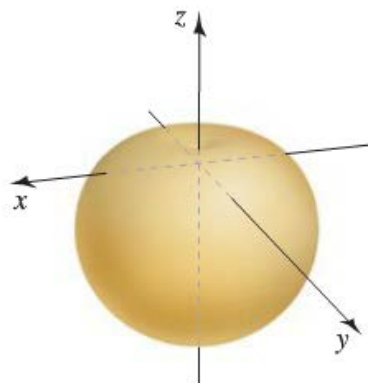
Find the volume of the cardioid of revolution $D = \left\{(\rho,\varphi,\theta): \ 0 \le \rho \le \frac{1-\cos\varphi}{2}, \ 0 \le \varphi \le \pi, \ 0 \le \theta \le 2\pi\right\}$

Solution

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{\frac{1-\cos\varphi}{2}} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, \rho^3 \Big|_0^{\frac{1-\cos\varphi}{2}} d\varphi$$

$$= \frac{\pi}{12}\int_0^{\pi} \sin\varphi \, (1-\cos\varphi)^3 \, d\varphi$$

$$= \frac{\pi}{12} \int_0^{\pi} (1 - \cos \varphi)^3 \, d(1 - \cos \varphi)$$

$$= \frac{\pi}{48} (1 - \cos \varphi)^4 \Big|_0^{\pi}$$

$$= \frac{\pi}{48} (16)$$

$$\underline{= \frac{\pi}{3} \ unit^3}$$

Exercise

A cake is shaped like a solid cone with radius 4 and height 2, with its base on the *xy*-plane. A wedge of the cake is removed by making two slices from the axis of the cone outward, perpendicular to the *xy*-plane separated by an angle of Q radians, where $0 < Q < 2\pi$

   a) Find the volume of the slice for $Q = \frac{\pi}{4}$. Use geometry to check your answer.

   b) Find the volume of the slice for $0 < Q < 2\pi$. Use geometry to check your answer.

Solution

Volume of a cone $= \frac{\pi}{3}(4)^2 (2)$ $\qquad\qquad\qquad V = \frac{\pi}{3} r^2 h$

$$\underline{= \frac{32\pi}{3}}$$

Equation of the cone in cylindrical coordinates is:

$$\begin{cases} r = 4 & \to z = 0 \\ r = 0 & \to z = 2 = h \end{cases}$$

$$m = \frac{2 - 0}{0 - 4} = -\frac{1}{2}$$

$$\underline{z = -\frac{1}{2} r + 2}$$

a) $V = \displaystyle\int_0^{\frac{\pi}{4}} \int_0^4 \int_0^{2 - \frac{1}{2}r} r \, dz \, dr \, d\theta$

$$= \int_0^{\frac{\pi}{4}} d\theta \int_0^4 rz \Big|_0^{2 - \frac{1}{2}r} dr$$

$$= \frac{\pi}{4} \int_0^4 \left(2r - \frac{1}{2} r^2 \right) dr$$

$$= \frac{\pi}{4} \left(r^2 - \frac{1}{6} r^3 \right) \Big|_0^4$$

$$= \frac{\pi}{4} \left(16 - \frac{32}{3} \right)$$

$$\underline{= \frac{4\pi}{3} \ unit^3}$$

Since $Q = \frac{\pi}{4}$, then the volume of the slice is equal to $\frac{1}{8}$ of the cone volume

$$V = \frac{1}{8} \frac{32\pi}{3} = \underline{\frac{4\pi}{3}}$$

b) $V = \displaystyle\int_0^Q \int_0^4 \int_0^{2-\frac{1}{2}r} r \ dz dr d\theta$

$$= \int_0^Q d\theta \int_0^4 rz \bigg|_0^{2-\frac{1}{2}r} dr$$

$$= Q \int_0^4 \left(2r - \frac{1}{2}r^2 \right) dr$$

$$= Q \left(r^2 - \frac{1}{6}r^3 \right) \bigg|_0^4$$

$$= Q \left(16 - \frac{32}{3} \right)$$

$$\underline{= \frac{16}{3} Q}$$

Geometrically, since Q in radians, then $\dfrac{Q}{2\pi}$ of a circle.

$\therefore$ Volume of the slice is $\dfrac{Q}{2\pi}$ times of the curve.

Exercise

A spherical fish tank with a radius of 1 *ft* is filled with water to a level 6 *in*. below the top of the tank.

a) Determine the volume and weight of the water in the fish tank. (The weight density of water is about 62.5 *lb* / *ft*^{3}.)

b) How much additional water must be added to completely fill the tank?
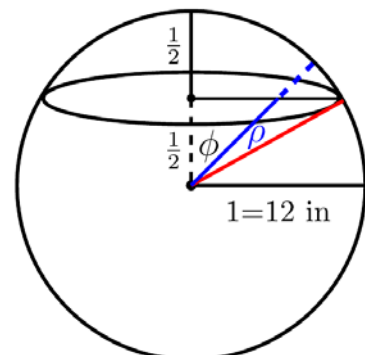
Solution

$$\varphi = \cos^{-1} \frac{6}{12} = \frac{\pi}{3}$$

$$0 \le \theta \le 2\pi$$

$$\cos\varphi = \frac{\frac{1}{2}}{\rho} \quad \rightarrow \quad \rho = \frac{1}{2}\sec\varphi$$

$$\frac{1}{2}\sec\varphi \le \rho \le 1$$

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a) Volume of empty spherical cap:

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_{\frac{1}{2}\sec\varphi}^{1} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{3}} \sin\varphi \, \rho^3 \Big|_{\frac{1}{2}\sec\varphi}^{1} \, d\varphi$$

$$= \frac{2\pi}{3} \int_0^{\frac{\pi}{3}} \sin\varphi \left(1 - \frac{1}{8}\sec^3\varphi\right) d\varphi$$

$$= \frac{2\pi}{3} \int_0^{\frac{\pi}{3}} \sin\varphi \, d\varphi - \frac{\pi}{12} \int_0^{\frac{\pi}{3}} \sin\varphi \cos^{-3}\varphi \, d\varphi$$

$$= -\frac{2\pi}{3}\cos\varphi \Big|_0^{\frac{\pi}{3}} + \frac{\pi}{12} \int_0^{\frac{\pi}{3}} \cos^{-3}\varphi \, d(\cos\varphi)$$

$$= -\frac{2\pi}{3}\left(\frac{1}{2} - 1\right) - \frac{\pi}{24}\cos^{-2}\varphi \Big|_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{3} - \frac{\pi}{24}(4 - 1)$$

$$= \frac{\pi}{3} - \frac{\pi}{8}$$

$$= \underline{\frac{5\pi}{24} \ ft^3}$$

Volume of a sphere is $\frac{4\pi}{3}$

$\therefore$ Volume of water $\frac{4\pi}{3} - \frac{5\pi}{24} = \underline{\frac{9\pi}{8} \ ft^3}$

Weights $= (6.25)\frac{9\pi}{8} \approx \underline{220.893 \ lbs}$

b) The addition water to fill the tank is $= \underline{\frac{5\pi}{24} \ ft^3}$

Exercise

A spherical cloud of electric charge has known charge density $Q(\rho)$, where ρ is the spherical coordinate. Find the total charge in the cloud in the following cases.

a) $Q(\rho) = \dfrac{2 \times 10^{-4}}{\rho^4}$, $\quad 1 \le \rho < \infty$

b) $Q(\rho) = \dfrac{2 \times 10^{-4}}{1 + \rho^3}$, $\quad 1 \le \rho < \infty$

c) $Q(\rho) = 2 \times 10^{-4} e^{-0.01\rho^3}$, $\quad 0 \le \rho < \infty$

Solution

a) $\displaystyle \int_0^{2\pi} \int_0^{\pi} \int_1^{\infty} \frac{2 \times 10^{-4}}{\rho^4} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = 2 \times 10^{-4} \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, d\varphi \int_1^{\infty} \frac{1}{\rho^2} d\rho$

$$= 2 \times 10^{-4} (2\pi)(-\cos\varphi)\Big|_0^{\pi} \left(-\frac{1}{\rho}\right)\Big|_1^{\infty}$$

$$= 4\pi \times 10^{-4} (2) \ (1) \qquad \frac{1}{\rho} \underset{\rho \to \infty}{\longrightarrow} 0$$

$$\underline{= 8\pi \times 10^{-4}}$$

b) $\displaystyle \int_0^{2\pi} \int_0^{\pi} \int_1^{\infty} \frac{2 \times 10^{-4}}{1 + \rho^3} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \frac{2}{3} \times 10^{-4} \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, d\varphi \int_1^{\infty} \frac{1}{1 + \rho^3} d\left(1 + \rho^3\right)$

$$= \frac{2}{3} \times 10^{-4} (2\pi)(-\cos\varphi)\Big|_0^{\pi} \left(\ln\left(1 + \rho^3\right)\right)\Big|_1^{\infty}$$

$$\underline{= \infty} \qquad\qquad \textcolor{red}{\ln\left(1 + \rho^3\right) \to \infty}$$

c) $\displaystyle 2 \times 10^{-4} \int_0^{2\pi} \int_0^{\pi} \int_1^{\infty} e^{-0.01\rho^3} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta =$

$$= -\frac{2}{.003} \times 10^{-4} \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, d\varphi \int_1^{\infty} e^{-0.01\rho^3} d\left(-0.01\rho^3\right)$$

$$= -\frac{2}{3} \times 10^{-2} (2\pi)(-\cos\varphi)\Big|_0^{\pi} \left(e^{-0.01\rho^3}\right)\Big|_1^{\infty}$$

$$= -\frac{4}{3} \pi \times 10^{-4} (2) \ (-1) \qquad \frac{1}{\rho} \underset{\rho \to \infty}{\longrightarrow} 0$$

$$\underline{= \frac{8\pi}{3} \times 10^{-4}}$$

Exercise

A point mass m is a distance d from the center of a thin spherical shell of mass M and radius R. The magnitude of the gravitational force on the point mass is given by the integral

$$F(d) = \frac{GMm}{4\pi} \int_0^{2\pi} \int_0^{\pi} \frac{(d - R\cos\phi)\sin\phi}{\left(R^2 + d^2 - 2Rd\cos\phi\right)^{3/2}} d\phi d\theta$$

Where G is the gravitational constant.

a) Use the change of variable $x = \cos\phi$ to evaluate the integral and show that if $d > R$, then

$F(d) = \dfrac{GMm}{d^2}$, which means the force is the same as if the mass of the shell were concentrated

at its center.

b) Show that is $d < r$ (the point mass is inside the shell), then $F = 0$.

Solution

a) $\quad x = \cos\phi \quad \rightarrow \quad \sin\varphi = \sqrt{1 - x^2}$

$dx = -\sin\varphi d\varphi \quad \rightarrow \quad d\varphi = -\dfrac{dx}{\sqrt{1 - x^2}}$

$\begin{cases} \varphi = 0 & \rightarrow x = 1 \\ \varphi = \pi & \rightarrow x = -1 \end{cases}$

$$F(d) = -\frac{GMm}{4\pi} \int_0^{2\pi} d\theta \int_{-1}^{1} \frac{(d - Rx)\sqrt{1 - x^2}}{\left(R^2 + d^2 - 2xRd\right)^{3/2}} \frac{-dx}{\sqrt{1 - x^2}}$$

$$= \frac{1}{2} GMm \int_{-1}^{1} \left(\frac{d}{\left(R^2 + d^2 - 2xRd\right)^{3/2}} - \frac{Rx}{\left(R^2 + d^2 - 2xRd\right)^{3/2}} \right) dx$$

$$= \frac{GMm}{2} \left(-\frac{1}{2R} \int_{-1}^{1} \frac{d\left(R^2 + d^2 - 2dRx\right)}{\left(R^2 + d^2 - 2dRx\right)^{3/2}} - \int_{-1}^{1} \frac{Rx}{\left(R^2 + d^2 - 2dRx\right)^{3/2}} dx \right)$$

$u = R^2 + d^2 - 2dRx \quad \rightarrow \quad du = -2dRdx$

$Rx = \dfrac{1}{2d}\left(R^2 + d^2 - u\right)$

$$\int \frac{Rx}{\left(R^2 + d^2 - 2dRx\right)^{3/2}} dx = \frac{1}{2d} \int \left(R^2 + d^2 - u\right)\left(u^{-3/2}\right)\left(-\frac{1}{2Rd}\right) du$$

$$= -\frac{1}{3Rd^2} \int \left(\left(R^2 + d^2\right)u^{-3/2} - u^{-1/2}\right) du$$

$$= -\frac{1}{4Rd^2}\left(-2\left(R^2 + d^2\right)u^{-1/2} - 2u^{1/2}\right)$$

$$= \frac{1}{2Rd^2\sqrt{R^2 + d^2 - 2dRx}}\left(R^2 + d^2 + R^2 + d^2 - 2dRx\right)$$

$$= \frac{R^2 + d^2 - dRx}{Rd^2\sqrt{R^2 + d^2 - 2dRx}}$$

$$F(d) = \frac{GMm}{2}\left(\frac{1}{R\sqrt{R^2 + d^2 - 2dRx}} - \frac{R^2 + d^2 - dRx}{Rd^2\sqrt{R^2 + d^2 - 2dRx}}\right)\Bigg|_{-1}^{1}$$

$$= \frac{GMm}{2}\left(\frac{dRx - R^2}{Rd^2\sqrt{R^2 + d^2 - 2dRx}}\right)\Bigg|_{-1}^{1}$$

$$= \frac{GMm}{2}\left(\frac{Rd - R^2}{Rd^2\sqrt{R^2 + d^2 - 2Rd}} - \frac{-Rd - R^2}{Rd^2\sqrt{R^2 + d^2 + 2Rd}}\right)$$

$$= \frac{GMm}{2}\left(\frac{Rd - R^2}{Rd^2\sqrt{R^2 + d^2 - 2Rd}} + \frac{Rd + R^2}{Rd^2\sqrt{R^2 + d^2 + 2Rd}}\right)$$

$$= \frac{GMm}{2}\left(\frac{R(d - R)}{Rd^2\sqrt{(R - d)^2}} + \frac{R(d + R)}{Rd^2(R + d)}\right)$$

If $d > R$, then

$$F(d) = \frac{GMm}{2}\left(\frac{1}{d^2} + \frac{1}{d^2}\right)$$

$$\underline{= \frac{GMm}{d^2}}$$

b) If $d < R$, then

$$F(d) = \frac{GMm}{2}\left(-\frac{1}{d^2} + \frac{1}{d^2}\right)$$

$$\underline{= 0}$$

Exercise

Before a gasoline-powered engine is started, water must be drained from the bottom of the fuel tank. Suppose the tank is a right circular cylinder on its side with a length of 2 *feet* and a radius of 1 *foot*. If the water level is 6 *inches*. above the lowest part of the tank, determine how much water must be drained from the tank.

Solution

$$\cos\theta = \frac{\frac{1}{2}}{r} \quad \to \quad r = \frac{1}{2}\sec\theta$$

$$\theta = \cos^{-1}\frac{1}{2} = \pm\frac{\pi}{3}$$

$$V = \int_0^2 \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{\frac{1}{2}\sec\theta}^1 r\, dr d\theta dz$$

$$= \frac{1}{2}\int_0^2 dz \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} r^2 \Big|_{\frac{1}{2}\sec\theta}^1 \, d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(1 - \frac{1}{4}\sec^2\theta\right) d\theta$$

$$= \left(\theta - \frac{1}{4}\tan\theta\right)\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{4} + \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

$$= \underline{\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\ ft^3}$$