Section 2.5 – Variation Parameters

In this section, we will introduce a technique called ***variation of parameters***.

The inhomogeneous equation is given by: $y'' + p(t)y' + q(t)y = g(t)$

A fundamental set of solutions y_1 and y_2 to associated homogeneous equation $y'' + py' + qy = 0.$

Then the general solution to the inhomogeneous equation is given by

$$y_k = C_1 y_1 + C_2 y_2$$

C_1 *and* C_2 are arbitrary constants.

General Case

A differential system can be written in a form:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

y_1 and y_2 fundamental set of solution to the homogenous equation, they are linearly independent Then the determinant will be recognized as the Wronskian:

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

Which we can obtain:

$$v_1' = -\frac{y_2 g(t)}{y_1 y_2' - y_1' y_2} = -\frac{y_2}{W} g(t) \qquad \Rightarrow v_1(t) = -\int \frac{y_2 g(t)}{W} dt$$

$$v_2' = \frac{y_1 g(t)}{y_1 y_2' - y_1' y_2} = \frac{y_1}{W} g(t) \qquad \Rightarrow v_2(t) = \int \frac{y_1 g(t)}{W} dt$$

$$\boxed{y_p = v_1 y_1 + v_2 y_2}$$

Example

$\left\{y_1(x)=x^4,\ y_2(x)=x^2\right\}$ is a fundamental set of solutions of $y''-\frac{5}{x}y'+\frac{8}{x^2}y=4x^3$.

Find a particular solution of the equation?

Solution

$$W=\begin{vmatrix} x^4 & x^2 \\ 4x^3 & 2x \end{vmatrix}=-2x^5\neq 0$$

$$v_1(x)=-\int\frac{x^2\left(4x^3\right)}{-2x^5}dx=\int 2dx\ \underline{=2x|}\qquad\qquad v_1(x)=-\int\frac{y_2 g(x)}{W}dx$$

$$v_2(x)=\int\frac{x^4\left(4x^3\right)}{-2x^5}dx=-2\int x^2dx\ \underline{=-\frac{2}{3}x^3|}\qquad\qquad v_2(x)=\int\frac{y_1 g(x)}{W}dx$$

The particular solution:

$$\begin{aligned}y_p&=v_1y_1+v_2y_2\\&=(2x)\left(x^4\right)-\frac{2}{3}x^3\left(x^2\right)\\&=\frac{4}{3}x^5\end{aligned}$$

The general solution: $\boxed{y(x)=C_1x^4+C_2x^2+\frac{4}{3}x^5}$

Example

$\left\{y_1(x)=e^{2x},\ y_2(x)=xe^{2x}\right\}$ is a fundamental set of solutions of $y''-4y'+4y=\frac{e^{2x}}{x}$.

Find a particular solution of the equation?

Solution

$$W=\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}+2xe^{2x} \end{vmatrix}=e^{4x}+2xe^{4x}-2xe^{4x}\ \underline{=e^{4x}\neq 0|}$$

$$v_1(x)=-\int\frac{xe^{2x}}{e^{4x}}\frac{e^{2x}}{x}dx=-\int dx\ \underline{=-x|}\qquad\qquad v_1(x)=-\int\frac{y_2 g(x)}{W}dx$$

$$v_2(x)=\int\frac{e^{2x}}{e^{4x}}\frac{e^{2x}}{x}dx=\int\frac{1}{x}dx\ \underline{=\ln|x||}\qquad\qquad v_2(x)=\int\frac{y_1 g(x)}{W}dx$$

The particular solution:

$$y_p = v_1 y_1 + v_2 y_2$$
$$= -xe^{2x} + \ln|x|\left(xe^{2x}\right)$$

The general solution:

$$y(x) = C_1 e^{2x} + C_2 xe^{2x} - xe^{2x} + xe^{2x}\ln|x|$$
$$= C_1 e^{2x} + \left(C_2 - 1\right)xe^{2x} + xe^{2x}\ln|x|$$
$$\underline{= C_1 e^{2x} + C_3 xe^{2x} + xe^{2x}\ln|x|}$$

Example

Find the particular solution for $y'' + y = \tan t$

Solution

The homogeneous equation for the differential equation $\lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm i$

Therefore; $y_1 = \cos t$ *and* $y_2 = \sin t$

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1 \neq 0$$

The system has a solution

$$v_1' = \sin t \ \tan t = \frac{\sin^2 t}{\cos t}$$

$$v_1 = \int \frac{\sin^2 t}{\cos t} dt$$
$$\underline{= -\ln|\sec t + \tan t| + \sin t}$$

$$v_2' = \cos t \ \tan t = \sin t$$

$$v_2 = \int \sin t \ dt$$
$$\underline{= -\cos t}$$

$$y_p = v_1 \cos t + v_2 \sin t$$
$$= \left(-\ln|\sec t + \tan t| + \sin t\right)\cos t + \left(-\cos t\right)\sin t$$
$$= -\cos t \ln|\sec t + \tan t| + \sin t \cos t - \cos t \sin t$$
$$\underline{= -\left(\cos t\right) \ \ln|\sec t + \tan t|}$$

***Higher*-Order Equations**

The nonhomogeneous higher-order equation is given by:

$$y^{(n)}+p_{n-1}(t)y^{(n-1)}+\dots+p_1(t)y'+p_0(t)y=g(t)$$

A fundamental set of solutions $y_1,\ y_2,\ \dots,\ y_n$ to associated homogeneous equation

$$y_h=c_1y_1+c_2y_2+\dots+\ c_ny_n$$

Then the particular solution is $y_p=u_1y_1+u_2y_2+\dots+\ u_ny_n$

Where u'_k, $k=1,2,\dots,n$ are determined by the n equations:

$$\begin{array}{cccc} u'_1y_1+ & u'_2y_2 & +\dots+ & u'_ny_n=0 \\ u'_1y'_1+ & u'_2y'_2 & +\dots+ & u'_ny'_n=0 \\ \vdots & \vdots & \vdots & \vdots \\ u'_1y_1^{(n-1)}+ & u'_2y_2^{(n-1)} & +\dots+ & u'_ny_n^{(n-1)}=g(t) \end{array}$$

Using Cramer's Rule give: $\quad u'_k=\dfrac{W_k}{W}$

For the 3rd-order differential equation:

$$W=\begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix} \qquad W_1=\begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y'_2 & y'_3 \\ g(t) & y''_2 & y''_3 \end{vmatrix} \qquad W_2=\begin{vmatrix} y_1 & 0 & y_3 \\ y'_1 & 0 & y'_3 \\ y''_1 & g(t) & y''_3 \end{vmatrix} \qquad W_3=\begin{vmatrix} y_1 & y_2 & 0 \\ y'_1 & y'_2 & 0 \\ y''_1 & y''_2 & g(t) \end{vmatrix}$$

$$u_1=\int\frac{W_1}{W} \quad u_2=\int\frac{W_2}{W} \quad u_3=\int\frac{W_3}{W}$$

Exercises *Section* 2.5 – Variation Parameters

1. $\left\{y_1(x)=e^{2x},\ y_2(x)=e^{-3x}\right\}$ is a fundamental set of solutions of $y''+y'-6y=3e^{2x}$.

Find a particular solution of the equation?

Find a particular solution to the given second-order differential equation (Use ***variation of parameters***):

2. $y''-y=t+3$

3. $y''-2y'+y=e^{t}$

4. $x''-4x'+4x=e^{2t}$

5. $x''+x=\tan^2 t$

6. $y''+25y=-2\tan(5x)$

7. $y''-6y'+9y=5e^{3x}$

8. $y''+4y=2\cos 2x$

9. $y''-5y'+6y=4e^{2x}+3$

10. Verify that $y_1(t)=t$ and $y_2(t)=t^{-3}$ are solution to the homogenous equation

$$t^2y''(t)+3ty'(t)-3y(t)=0$$

Use variation of parameters to find the general solution to

$$t^2y''(t)+3ty'(t)-3y(t)=\frac{1}{t}$$

Find the general solution to the given differential equation (Use ***variation of parameters***).

11. $y''-y=\frac{1}{x}$

12. $y''-y=\sinh 2x$

13. $y''-y=x$

14. $y''-y=\cosh x$

15. $y''-y=\sin x$

16. $y''-y=e^{x}$

17. $y''+y=\sec x$

18. $y''+y=\tan x$

19. $y''+y=\sin x$

20. $y''+y=\csc x$

21. $y''+y=\cos^2 x$

22. $y''+y=\csc^2 x$

23. $y''+y=\sec^2 x$

24. $y''+y=\sec x\tan x$

25. $y''+y'=x$

26. $y''-y'=e^{x}\cos x$

28. $y''+y'-2y=e^{3x}$

29. $y''+2y'+y=e^{-x}\ln x$

30. $y''-2y'+y=\dfrac{e^{x}}{1+x^2}$

31. $y''+2y'+y=e^{-x}$

32. $y''-2y'-8y=3e^{-2x}$

33. $y''+3y'+2y=\sin e^{x}$

34. $y''+3y'+2y=4e^{x}$

35. $y''+3y'+2y=\dfrac{1}{1+e^{x}}$

36. $y''-4y=\sinh 2x$

37. $y''+4y=\sec 2x$

38. $y''+4y=\cos 3x$

39. $y''+4y=\sin^2 x$

40. $y''+4y=\sin^2 2t$

27. $y'' + y' - 2y = xe^{x}$

42. $y'' - 4y = xe^{x}$

43. $y'' - 4y' + 4y = 2e^{2x}$

44. $y'' - 4y' + 4y = (x+1)e^{2x}$

45. $y'' + 4y' + 5y = 10$

46. $y'' - 9y = \dfrac{9x}{e^{3x}}$

47. $y'' + 9y = \csc 3x$

48. $y'' + 9y = 3\tan 3t$

49. $y'' + 9y = \sin 3x$

50. $y'' + 9y = 2\sec 3x$

41. $y'' - 4y = \dfrac{e^{x}}{x}$

51. $4y'' + 36y = \csc 3x$

52. $\left(D^2 + 5D + 6\right)y = x^2 + 2x$

53. $\left(D^2 - 3D + 2\right)y = \dfrac{1}{1+e^{-x}}$

54. $y''' + y' = \sec x$

55. $y''' - 3y'' + 2y' = \dfrac{e^{x}}{1+e^{-x}}$

56. $y''' - 6y'' + 11y' - 6y = e^{x}$

57. $x^3 y^{(3)} - 4x^2 y'' + 8xy' - 8y = 4\ln x$

Find the general solution by to ***variation of parameters*** with the given initial conditions.

58. $y'' + y = \sec t$; $y(0) = 1,\quad y'(0) = 2$

59. $y'' + y = \sec^3 t$; $y(0) = 1,\quad y'(0) = \frac{1}{2}$

60. $y'' - y = t + \sin t$; $y(0) = 2,\quad y'(0) = 3$

61. $y'' - 2y' + y = \dfrac{e^{x}}{x}$; $y(0) = 1,\quad y'(0) = 0$

62. $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$; $y(0) = 1,\quad y'(0) = 0$

63. $y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x$; $y(0) = 1,\quad y'(0) = 2$

64. $y'' + 4y = \sin^2 2t$; $y\left(\frac{\pi}{8}\right) = 0,\quad y'\left(\frac{\pi}{8}\right) = 0$

65. $y'' + 4y = \sin^2 2t$; $y(0) = 0,\quad y'(0) = 0$

66. $y'' - 4y' + 4y = \left(12x^2 - 6x\right)e^{2x}$; $y(0) = 1,\quad y'(0) = 0$

67. $2y'' + y' - y = x + 1$; $y(0) = 1,\quad y'(0) = 0$

68. $4y'' - y = xe^{x/2}$; $y(0) = 1,\quad y'(0) = 0$

69. $t^2 y'' - ty' + y = t$; $y(1) = 1,\quad y'(1) = 4$