SOLUTION *Section* 4.1 – First-Order Systems

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $x'' + 3x' + 7x = t^2$

Solution

Let $x_1 = x, \quad x_2 = x' = x_1'$

Yield the system $\begin{cases} x_1' = x_2 \\ x_2' = -7x_1 - 3x_2 + t^2 \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $x^{(4)} + 6x'' - 3x' + x = \cos 3t$

Solution

Let $x_1 = x, \quad x_2 = x' = x_1', \quad x_3 = x'' = x_2', \quad x_4 = x''' = x_3'$

Yield the system $\begin{cases} x_1' = x_2, \quad x_2' = x_3, \quad x_3' = x_4 \\ x_4' = -x_1 + 3x_2 - 6x_3 + \cos 3t \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $t^2 x'' + tx' + \left(t^2 - 1\right)x = 0$

Solution

Let $x_1 = x, \quad x_2 = x' = x_1'$

Yield the system $\begin{cases} x_1' = x_2 \\ t^2 x_2' = \left(1 - t^2\right)x_1 - tx_2 \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $t^3 x^{(3)} - 2t^2 x'' + 3tx' + 5x = \ln t$

Solution

Let $x_1 = x, \quad x_2 = x' = x_1', \quad x_3 = x'' = x_2'$

Yield the system $\begin{cases} x_1' = x_2, \quad x_2' = x_3, \quad x_3' = x_4 \\ t^3 x_3' = -5x_1 - 3tx_2 + 2t^2 x_3 + \ln t \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $x'' - 5x + 4y = 0, \quad y'' + 4x - 5y = 0$

Solution

Let $\begin{array}{ll} x_1 = x & x_2 = x' = x_1' \\ y_1 = y & y_2 = y' = y_1' \end{array}$

$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = 5x_1 - 4y_1 \end{cases} \qquad \begin{cases} y_1' = y_2 \\ y_2' = -4x_1 + 5y_1 \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $x'' - 3x' + 4x - 2y = 0, \quad y'' + 2y' - 3x + y = \cos t$

Solution

Let $\begin{array}{ll} x_1 = x & x_2 = x' = x_1' \\ y_1 = y & y_2 = y' = y_1' \end{array}$

$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = -4x_1 + 2y_1 + 3x_2 \end{cases} \qquad \begin{cases} y_1' = y_2 \\ y_2' = 3x_1 - y_1 - 2y_2 + \cos t \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $x'' = 3x - y + 2z, \quad y'' = x + y - 4z, \quad z'' = 5x - y - z$

Solution

Let $\begin{array}{ll} x_1 = x & x_2 = x' = x_1' \\ y_1 = y & y_2 = y' = y_1' \\ z_1 = z & z_2 = z' = z_1' \end{array} \quad \Rightarrow \begin{cases} x_1' = x_2, \quad y_1' = y_2, \quad z_1' = z_2 \\ x_2' = 3x_1 - y_1 + 2z_1 \\ y_2' = x_1 + y_1 - 4z_1 \\ z_2' = 5x_1 - y_1 - z_1 \end{cases}$

Exercise

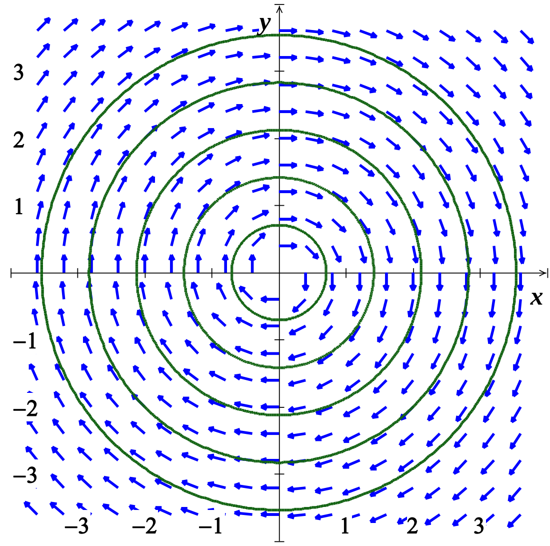
Transform the given differential equation or system into an equivalent system of 1st-order differential equation $\quad x'' = (1-y)x, \quad y'' = (1-x)y$

Solution

Let $\quad \begin{array}{ll} x_1 = x & x_2 = x' = x_1' \\ y_1 = y & y_2 = y' = y_1' \end{array} \quad \Rightarrow \quad \begin{cases} x_1' = x_2, \quad y_1' = y_2 \\ x_2' = (1-y_1)x_1 \\ y_2' = (1-x_1)y_1 \end{cases}$

Exercise

Find the general solution $\quad x' = y, \quad y' = -x$



Solution

$x'' = y' = -x$

$x'' + x = 0 \quad \Rightarrow \quad \lambda^2 + 1 = 0$

The eigenvalues are: $\lambda_{1,2} = \pm i$

$x(t) = C_1 \cos t + C_2 \sin t$. Given $y = x'$

$\therefore$ General solution: $\begin{cases} x(t) = C_1 \cos t + C_2 \sin t \\ y(t) = -C_1 \sin t + C_2 \cos t \end{cases}$

Exercise

Find the general solution $\quad x' = y, \quad y' = -9x + 6y$

Solution

$x'' = y' = -9x + 6y$

$x'' = -9x + 6x'$

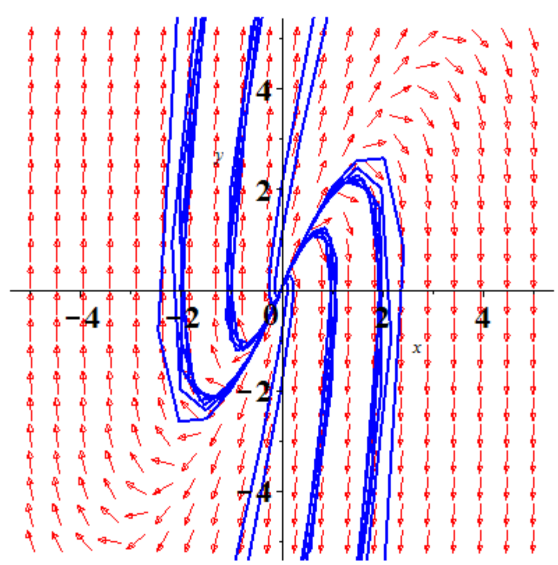
$x'' - 6x' + 9x = 0 \quad \Rightarrow \quad \lambda^2 - 6\lambda + 9 = 0$

The eigenvalues are: $\lambda_{1,2} = 3$

$x(t) = (C_1 + C_2 t)e^{3t}$

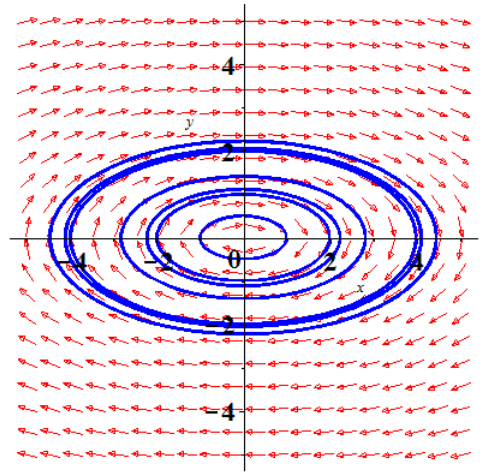
Given $y = x' = C_2 e^{3t} + 3(C_1 + C_2 t)e^{3t}$

$\therefore$ General solution: $\begin{cases} x(t) = (C_1 + C_2 t)e^{3t} \\ y(t) = (3C_1 + C_2 + 3C_2 t)e^{3t} \end{cases}$



Exercise

Find the general solution $\quad x' = 8y, \quad y' = -2x$



Solution

$$x'' = 8y' = -16x$$

$$x'' + 16x = 0 \quad \Rightarrow \quad \lambda^2 + 16 = 0$$

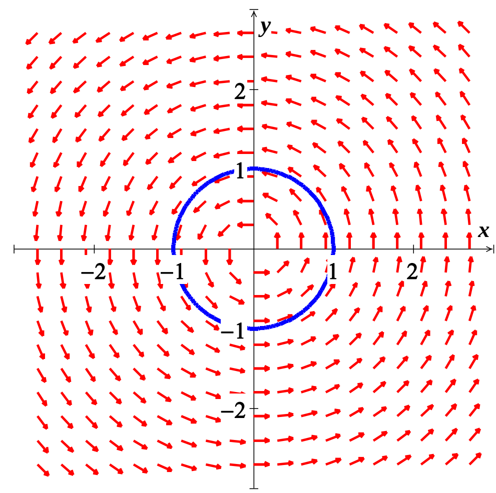
The eigenvalues are: $\lambda_{1,2} = \pm 4i$

$x(t) = C_1 \cos 4t + C_2 \sin 4t$. Given $y = \frac{1}{8}x'$

$$\therefore \text{ General solution: } \begin{cases} x(t) = C_1 \cos 4t + C_2 \sin 4t \\ y(t) = -\frac{1}{2}C_1 \sin 4t + \frac{1}{2}C_2 \cos 4t \end{cases}$$

Exercise

Find the general solution $\quad x' = -2y, \quad y' = 2x; \quad x(0) = 1, \quad y(0) = 0$



Solution

$$x'' = -2y' = -4x$$

$$x'' + 4x = 0 \quad \Rightarrow \quad \lambda^2 + 4 = 0$$

The eigenvalues are: $\lambda_{1,2} = \pm 2i$

$x(t) = C_1 \cos 2t + C_2 \sin 2t$.

Given $y = -\frac{1}{2}x' \Rightarrow y(t) = C_1 \sin 2t - C_2 \cos 2t$

$x(0) = C_1 = 1 \quad and \quad y(0) = -C_2 = 0$

$$\therefore \text{ General solution: } \begin{cases} x(t) = \cos 2t \\ y(t) = \sin 2t \end{cases}$$

Exercise

Find the general solution $\quad x' = y, \quad y' = 6x - y; \quad x(0) = 1, \quad y(0) = 2$

Solution

$$x'' = y' = 6x - y$$

$$x'' = 6x - x'$$

$$x'' + x' - 6x = 0 \quad \Rightarrow \quad \lambda^2 + \lambda - 6 = 0$$

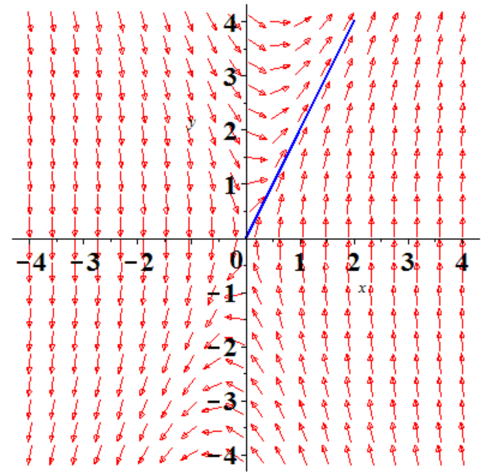
The eigenvalues are: $\lambda_1 = -3 \quad \lambda_2 = 2$

$$x(t) = C_1 e^{-3t} + C_2 e^{2t} \quad \Rightarrow \quad x(0) = C_1 + C_2 = 1$$

$$y(t) = x'(t) = -3C_1 e^{-3t} + 2C_2 e^{2t} \quad \Rightarrow \quad y(0) = -3C_1 + 2C_2 = 2$$

$$\begin{cases} C_1 + C_2 = 1 \\ -3C_1 + 2C_2 = 2 \end{cases} \quad \rightarrow \quad C_1 = 0, \ C_2 = 1$$

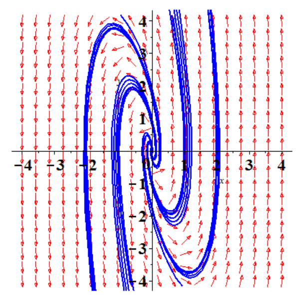
$\therefore$ General solution: $\begin{cases} x(t) = e^{2t} \\ y(t) = 2e^{2t} \end{cases}$



Exercise

Find the general solution $\quad x' = -y, \quad y' = 13x + 4y; \quad x(0) = 0, \quad y(0) = 3$

Solution



$$x'' = -y' = -13x - 4y$$

$$x'' + 4x' + 13x = 0 \quad \Rightarrow \quad \lambda^2 + 4\lambda + 13 = 0$$

The eigenvalues are: $\lambda = \dfrac{-4 \pm \sqrt{-36}}{2} \quad \lambda_{1,2} = -2 \pm 3i$

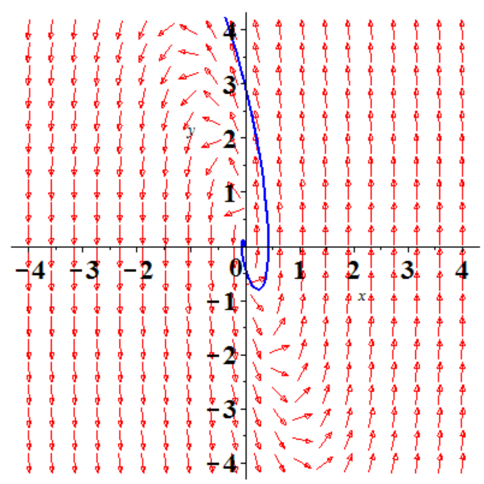
$$x(t) = e^{-2t}\left(C_1 \cos 3t + C_2 \sin 3t\right) \quad \Rightarrow \quad x(0) = \boxed{C_1 = 0}$$

Given $y = -x'$

$$\Rightarrow y(t) = \left(-3C_1 \sin 3t + C_2 \cos 3t\right)e^{-2t} - 2\left(C_1 \cos 3t + C_2 \sin 3t\right)e^{-2t}$$

$$y(t) = \left(-\left(3C_1 + 2C_2\right)\sin 3t + \left(C_2 - 2C_1\right)\cos 3t\right)e^{-2t}$$

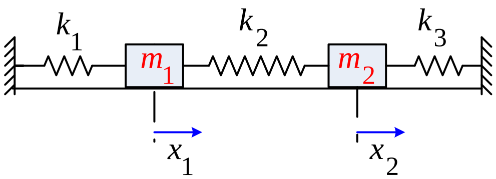
$$y(0) = C_2 - 2C_1 = 3 \quad \Rightarrow \quad \boxed{C_2 = 3}$$



$\therefore$ General solution: $\begin{cases} x(t) = (3\sin 3t)e^{-2t} \\ y(t) = (-6\sin 3t + 3\cos 3t)e^{-2t} \end{cases}$

Exercise

Derive the equations $\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 \end{cases}$



For the displacements (from equilibrium) of the 2 masses.

Solution

First spring is stretched by x_1

Second spring is stretched by $x_2 - x_1$

Third spring is stretched by x_2

Newton's second law gives:

For $m_1 \quad \Rightarrow \quad m_1 x_1'' = -k_1 x_1 + k_2 (x_2 - x_1)$

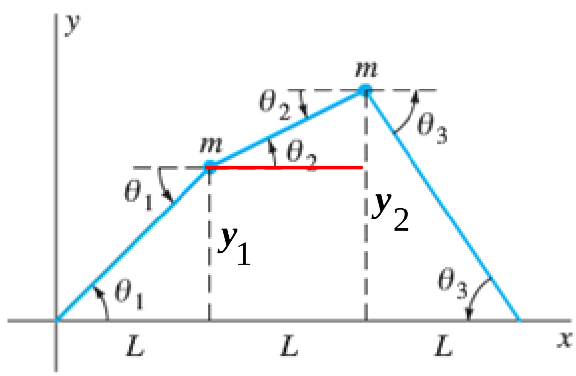
For $m_2 \quad \Rightarrow \quad m_2 x_2'' = -k_2 (x_2 - x_1) - k_3 x_2$

That implies to: $\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 \end{cases}$

Exercise

Two particles each of mass *m* are attached to a string under (constant) tension *T*. Assume that the particles oscillate vertically (that is, parallel to the *y*-axis) with amplitudes so small that the sines of the angles shown are accurately approximated by their tangents. Show that the displacement y_1 and y_2 satisfy the equations

$$\begin{cases} ky_1'' = -2y_1 + y_2 \\ ky_2'' = y_1 - 2y_2 \end{cases} \qquad \textit{where } k = \frac{mL}{T}$$



Solution

For the first mass:

$$my_1'' = -T\sin\theta_1 + T\sin\theta_2$$

$$\approx -T\tan\theta_1 + T\tan\theta_2$$

$$my_1'' = -T\frac{y_1}{L} + T\frac{y_2 - y_1}{L}$$

$$\frac{L}{T}my_1'' = -\frac{L}{T}T\frac{y_1}{L} + \frac{L}{T}T\frac{y_2 - y_1}{L} \qquad \textit{where } k = \frac{mL}{T}$$

$$\boxed{ky_1'' = -y_1 + y_2 - y_1 = -2y_1 + y_2}$$

For the second mass:

$$my_2'' = -T\sin\theta_2 + T\sin\theta_3$$

$$\approx -T\tan\theta_2 + T\tan\theta_3$$

$$my_2'' = -T\frac{y_2 - y_1}{L} + T\frac{y_2}{L}$$

$$\frac{L}{T}my_2'' = -\frac{L}{T}T\frac{y_2 - y_1}{L} + \frac{L}{T}T\frac{y_2}{L} \qquad \text{where } k = \frac{mL}{T}$$

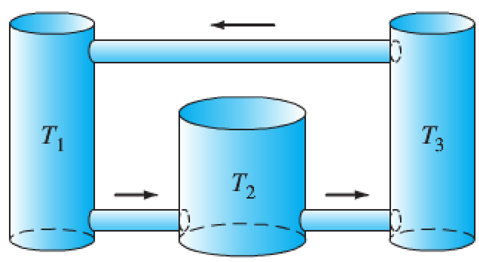
$$\boxed{ky_2'' = -y_2 + y_1 - y_2 = y_1 - 2y_2}$$

$$\Rightarrow \begin{cases} ky_1'' = -2y_1 + y_2 \\ ky_2'' = y_1 - 2y_2 \end{cases} \qquad \text{where } k = \frac{mL}{T}$$

Exercise

There 100-gal fermentation vats are connected, and the mixtures in each tank are kept uniform by stirring. Denote by $x_i(t)$ the amount (in pounds) of alcohol in tank T_i at time *t* (*i* = 1, 2, 3). Suppose that the mixture circulates between the tanks at the rate of 10 *gal/min*. Derive the equations

$$\begin{cases} 10x_1' = -x_1 \qquad + x_3 \\ 10x_2' = x_1 - x_2 \\ 10x_3' = \qquad x_2 - x_3 \end{cases}$$



Solution

Concentration of salt in each tank is: $c_i = \frac{X}{V} = \frac{x_i}{100}$

Rate in/out = Volume Rate x Concentration

Rate of change** = **Rate in – rate out

For T_1: $\quad x_1' = r\frac{x_3}{100} - r\frac{x_1}{100} = \frac{1}{10}\left(x_3 - x_1\right)$

For T_2: $\quad x_2' = r\frac{x_1}{100} - r\frac{x_2}{100} = \frac{1}{10}\left(x_1 - x_2\right)$

For T_3: $\quad x_3' = r\frac{x_2}{100} - r\frac{x_3}{100} = \frac{1}{10}\left(x_2 - x_3\right)$

That implies:

$$\begin{cases} 10x_1' = -x_1 \qquad + x_3 \\ 10x_2' = x_1 - x_2 \\ 10x_3' = \qquad x_2 - x_3 \end{cases}$$

Exercise

Suppose that a particle with mass m and electrical charge q moves in the xy-plane under the influence of the magnetic field $\vec{B} = B\hat{k}$ (thus a uniform field parallel to the z-axis), so the force on the particle is $\vec{F} = q\vec{v} \times \vec{B}$ if its velocity is $\vec{v}$. Show that the equations of motion of the particle are

$$mx'' = +qBy', \quad my'' = -qBx'$$

Solution

Let $\vec{r} = (x,\ y,\ z)$ be the position vector, then Newton's law

$$\vec{F} = mx''$$

$$\begin{aligned} \vec{F} = mx'' &= q\left(\vec{v} \times \vec{B}\right) \\ &= q\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x' & y' & z' \\ 0 & 0 & B \end{vmatrix} \\ &= qBy'\hat{i} - qBx'\hat{j} \end{aligned}$$

$$\Rightarrow\ mx'' = +qBy', \quad my'' = -qBx'$$