Lecture Four

Section 4.1 – Inferences about Two Population Proportions

Objectives

Test a claim two population proportions or construct a confidence interval estimate of the difference between two population properties.

Distinguish between Independent and Dependent Sampling

A sampling method is ***independent*** when the individuals selected for one sample do not dictate which individuals are to be in a second sample. A sampling method is ***dependent*** when the individuals selected to be in one sample are used to determine the individuals to be in the second sample.

Dependent samples are often referred to as matched-pairs samples. It is possible for an individual to be matched against him- or herself.

Example

For each of the following, determine whether the sampling method is independent or dependent.

A researcher wants to know whether the price of a one night stay at a Holiday Inn Express is less than the price of a one night stay at a Red Roof Inn. She randomly selects 8 towns where the location of the hotels is close to each other and determines the price of a one night stay.

> ***The sampling method is dependent since the 8 Holiday Inn Express hotels can be matched with one of the 8 Red Roof Inn hotels by town***

A researcher wants to know whether the "state" quarters (introduced in 1999) have a mean weight that is different from "traditional" quarters. He randomly selects 18 "state" quarters and 16 "traditional" quarters and compares their weights.

> ***The sampling method is independent since the "state" quarters which were sampled had no bearing on which "traditional" quarters were sampled.***

Notation for Two Proportions

For population 1, we let:

p_1 = population proportion

n_1 = size of the sample

x_1 = number of successes in the sample (the sample proportion)

The corresponding notations apply to which come from population 2.

Pooled Sample Proportion

The pooled sample proportion is denoted by p and is given by:

$$\bar{p} = \frac{x_1 + n_2}{n_1 + n_2} \qquad \bar{q} = 1 - \bar{p}$$

- We denote the complement of p by q, **so $q = 1 - p$**

Requirements

We have proportions from two independent simple random samples.
For each of the two samples, the number of successes is at least 5 and the number of failures is at least 5.

Test Statistic for Two Proportions

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{\bar{p}\bar{q}}{n_1} + \frac{\bar{p}\bar{q}}{n_2}}} \qquad \text{where } p_1 - p_2 = 0 \text{ (assumed in the null hypothesis)}$$

$$\hat{p}_1 = \frac{x_1}{n_1} \quad \textit{and} \quad \hat{p}_2 = \frac{x_2}{n_2} \qquad \text{(sample proportions)}$$

$$\bar{p}_1 = \frac{x_1 + x_2}{n_1 + n_2} \qquad \text{(pooled sample proportion)}$$

$$\bar{q} = 1 - \bar{p}$$

P-value: Use Standard Normal Distribution Table. (Use the computed value of the test statistic z and find its P-value)

Critical values: Use Standard Normal Distribution Table. (Based on the significance level a, find critical values.)

Sampling Distribution of the Difference between Two Proportions

Use the positive and negative values of z (for two tails) and solve for $p_1 - p_2$. The results are the limits of the confidence interval given earlier.

The difference can be approximated by a normal distribution with mean $p_1 - p_2$ and variance

$$\sigma^2_{(\hat{p}_1 - \hat{p}_2)} = \sigma^2_{\hat{p}_1} + \sigma^2_{\hat{p}_2} = \frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}$$

The variance of the *differences* between two independent random variables is the *sum* of their individual variances.

The preceding variance leads to $\sigma_{\left(\hat{p}_1-\hat{p}_2\right)} = \sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}$

Provided that $n_1 \hat{p}_1 \hat{q}_1 \geq 10; \quad n_2 \hat{p}_2 \hat{q}_2 \geq 10$ and each sample size no more than 5% pf the population size.

When constructing the confidence interval estimate of the difference between two proportions, we don't assume that the two proportions are equal, and we estimate the standard deviation as

$$\sigma = \sqrt{\frac{\hat{p}_1 \hat{q}_1}{n_1} + \frac{\hat{p}_2 \hat{q}_2}{n_2}}$$

In the test statistic $z = \dfrac{\left(\hat{p}_1 - \hat{p}_2\right) - \left(p_1 - p_2\right)}{\sqrt{\dfrac{p_1 q_1}{n_1} + \dfrac{p_2 q_2}{n_2}}}$

The best point estimate of p is called the ***pooled estimate*** of p, denoted $\hat{p}$, where

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

Test statistic for Comparing Two Population Proportions

$$z_0 = \frac{\hat{p}_1 - \hat{p}_2}{\sigma_{\hat{p}_1 - \hat{p}_2}} = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\hat{q}}\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$$

McNemar's Test

	Success	**Failure**
Success	f_{11}	f_{12}
Failure	f_{21}	f_{22}

$$z_0 = \frac{\left|f_{12} - f_{21}\right| - 1}{\sqrt{f_{12} + f_{21}}}$$

Hypothesis Test Regarding the Difference between Two Population Proportions

To test hypotheses regarding two population proportion:

- ✓ The samples are independently obtained using simple random sampling
- ✓ $n_1 \hat{p}_1 \hat{q}_1 \geq 10; \quad n_2 \hat{p}_2 \hat{q}_2 \geq 10$
- ✓ $n_1 \leq 0.05 N_1 \quad and \quad n_2 \leq 0.05 N_2$ (the sample size is more than 5% of the population size); this requirement ensures the independence necessary for a binomial experiment.

***Step* 1**: Determine the null and alternative hypotheses. The hypotheses can be structured in one of three ways:

Two–Tailed	***Left–Tailed***	***Right–Tailed***
$H_0: p_1 = p_2$	$H_0: p_1 = p_2$	$H_0: p_1 = p_2$
$H_1: p_1 \neq p_2$	$H_1: p_1 < p_2$	$H_1: p_1 > p_2$

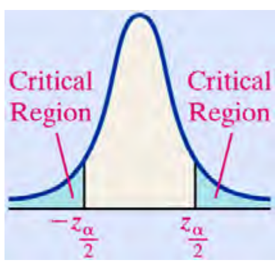
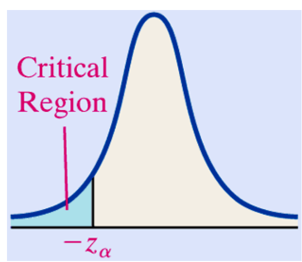
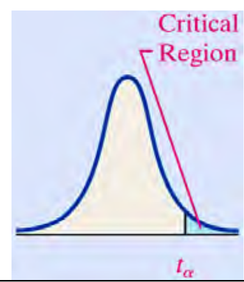
Note: p_1 is the population proportion for population 1, and p_2 is the population proportion for population 2.

***Step* 2:** Select a level of significance, α, based on the seriousness of making a Type I error.

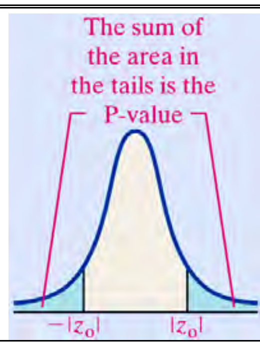
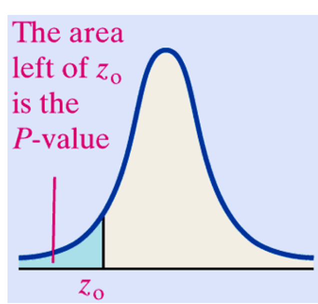
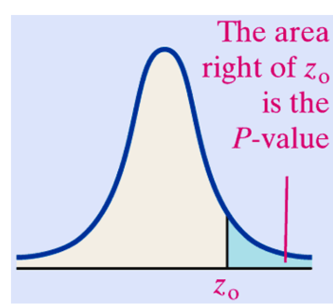
***Step* 3**: Compute the test statistic

$$z_0 = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\hat{q}}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \quad where \quad \hat{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

Classical Approach

Two–Tailed	***Left–Tailed***	***Right–Tailed***
$z_0 < -z_{\alpha/2}$ *or* $z_0 > z_{\alpha/2}$ *Reject the null hypothesis* 	$z_0 < -z_\alpha$ *Reject the null hypothesis* 	$z_0 > z_\alpha$ *Reject the null hypothesis* 

***Step* 3**: Estimate the *P*–value

Two–Tailed	***Left–Tailed***	***Right–Tailed***
		

If* P-value < α*, reject the null hypothesis

Example

A recent General Social Survey asked the following two questions of a random sample of 1483 adult Americans under the hypothetical scenario that the government suspected that a terrorist act was about to happen:
Do you believe the authorities should have the right to tap people's telephone conversations?
Do you believe the authorities should have the right to detain people for as long as they want without putting them on trial?
The results of the survey are shown below

		Detain	
		Agree	**Disagree**
Tap	**Agree**	572	237
Phone	**Disagree**	224	450

Do the proportions who agree with each scenario differ significantly? Use the $\alpha = 0.05$ level of significance.

Solution

The sample proportion of individuals who believe that the authorities should be able to tap phones is

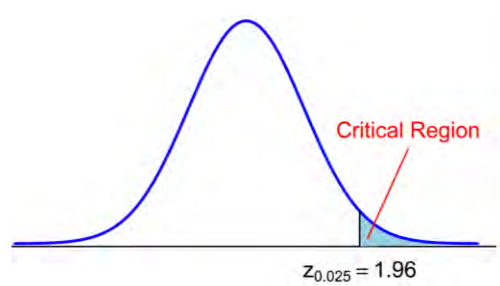
$$\hat{p}_1 = \frac{572 + 237}{1483} \underline{= 0.5455|}$$

The sample proportion of individuals who believe that the authorities should have the right to detain people is

$$\hat{p}_2 = \frac{572 + 224}{1483} \underline{= 0.5367|}$$

We want to determine whether the difference in sample proportions is due to sampling error or to the fact that the population proportions differ.

The samples are dependent and were obtained randomly. The total number of individuals who agree with one scenario, but disagree with the other is 237 + 224 = 461, which is greater than 10. We can proceed with McNemar's Test.



***Step* 1**: The hypotheses are as follows $\begin{cases} H_0: & \hat{p}_1 = \hat{p}_2 \\ H_1: & \hat{p}_1 \neq \hat{p}_2 \end{cases}$

***Step* 2:** The level of significance is $\alpha = 0.05$

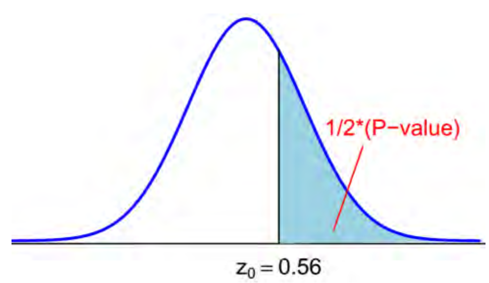
***Step* 3:** The test statistic is:

$$z_0 = \frac{|f_{12} - f_{21}| - 1}{\sqrt{f_{12} + f_{21}}} = \frac{|237 - 224| - 1}{\sqrt{237 + 224}}$$

$$z_{0.025} = 1.96 > 0.56 = z_0$$

$\Rightarrow$ We fail to reject the null hypothesis

$$P - value = 2 \bullet P(z > 0.56) \underline{\approx 0.5754|}$$



Since the P-value is greater than the level of significance $\alpha = 0.05$, we fail to reject the null hypothesis.

Conclusion

There is insufficient evidence at the $\alpha = 0.05$ level to conclude that there is a difference in the proportion of adult Americans who believe it is okay to phone tap versus detaining people for as long as they want without putting them on trial in the event that the government believed a terrorist plot was about to happen.

Confidence Interval Estimate of $p_1 - p_2$

The confidence interval estimate of the difference $p_1 - p_2$ is:

$$\left(\hat{p}_1 - \hat{p}_2\right) - E < \left(p_1 - p_2\right) < \left(\hat{p}_1 - \hat{p}_2\right) + E$$

Where the margin of error E is given by $E = z_{\alpha/2}\sqrt{\frac{\hat{P}_1\hat{q}_1}{n_1} + \frac{\hat{P}_2\hat{q}_2}{n_2}}$

Rounding: Round the confidence interval limits to three significant digits,

Example

The table below lists results from a simple random sample of front-seat occupants involved in car crashes. Use a 0.05 significance level to test the claim that the fatality rate of occupants is lower for those in cars equipped with airbags.

	Airbag Available	No Airbag Available
Occupant Fatalities	41	52
Total number of occupants	11,541	9,853

Solution

Requirements are satisfied: two simple random samples, two samples are independent; Each has at least 5 successes and 5 failures (11,500, 41; 9801, 52).
Use the P-value method.

Step 1: Express the claim as $p_1 < p_2$.

Step 2: If $p_1 < p_2$ is false, then $p_1 \geq p_2$.

Step 3: $p_1 < p_2$ does not contain equality so it is the alternative hypothesis. The null hypothesis is the statement of equality.

$H_0: \quad p_1 = p_2$

$H_a: \quad p_1 < p_2$ (original claim)

Step 4: Significance level is 0.05

Step 5: Use normal distribution as an approximation to the binomial distribution. Estimate the common values of p_1 and p_2

Step 6: Find the value of the test statistic.

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{\overline{p}\overline{q}}{n_1} + \frac{\overline{p}\overline{q}}{n_2}}}$$

$$= \frac{\left(\frac{41}{11{,}541} - \frac{52}{9{,}853}\right) - 0}{\sqrt{\frac{(0.004347)(0.995653)}{11{,}541} + \frac{(0.004347)(0.995653)}{9{,}853}}}$$

$$= -1.91$$

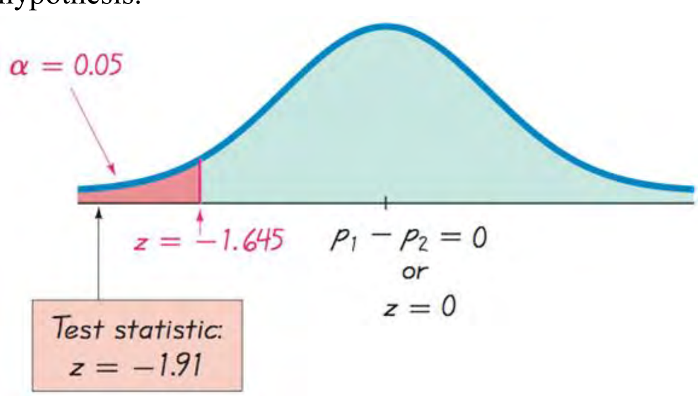
Left-tailed test. Area to left of $z = -1.91$ is 0.0281 (Standard Normal Distribution Table), so the *P*-value is 0.0281.

Step 7: Because the *P*-value of 0.0281 is less than the significance level of $\alpha = 0.05$, we reject the null hypothesis of $p_1 = p_2$.

Because we reject the null hypothesis, we conclude that there is sufficient evidence to support the claim that the proportion of accident fatalities for occupants in cars with airbags is less than the proportion of fatalities for occupants in cars without airbags. Based on these results, it appears that airbags are effective in saving lives.

Example: **Using the Traditional Method**

With a significance level of $\alpha = 0.05$ in a left- tailed test based on the normal distribution, we refer to Standard Normal Distribution Table and find that an area of $\alpha = 0.05$ in the left tail corresponds to the critical value of $z = -1.645$. The test statistic of does fall in the critical region bounded by the critical value of $z = -1.645$.
We again reject the null hypothesis.



Caution

When testing a claim about two population proportions, the *P*-value method and the traditional method are equivalent, but they are *not* equivalent to the confidence interval method. If you want to test a claim about two population proportions, use the *P*-value method or traditional method; if you want to estimate the difference between two population proportions, use a confidence interval.

Example

Use the sample data given in the preceding Example to construct a 90% confidence interval estimate of the difference between the two population proportions. (As shown in Table 8-2 on page 406, the confidence level of 90% is comparable to the significance level of $\alpha = 0.05$ used in the preceding left-tailed hypothesis test.) What does the result suggest about the effectiveness of airbags in an accident?

Solution

Requirements are satisfied as we saw in the preceding example.

90% confidence interval: $z_{\alpha/2} = 1.645$ Calculate the margin of error, E

$$E = z_{\alpha/2}\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}}$$

$$= 1.645\sqrt{\frac{\frac{41}{11{,}541}\cdot\frac{11{,}500}{11{,}541}}{11{,}541} + \frac{\frac{52}{9{,}853}\cdot\frac{9801}{9{,}853}}{9{,}853}}$$

$$\underline{= 0.001507}$$

TI-83 / 84 Calculator – For Hypothesis and confidence intervals
Press **STAT** Select **TESTS** Choose the option of **2-PropZTest** (for hypothesis test) Or **2-PropZInt** (for confidence test) ***Result***: Calculator will display a P-value instead of critical values, so the P-value method of hypothesis is used.

Exercises *Section* 4.1 – Inferences about Two Population Proportions

1. A Student surveyed her friends and found that among 20 males, 4 smoke and among 30 female, 6 smoke. Give two reasons why these results should not be used for a hypothesis test of the claim that the proportions of male smokers and female smokers are equal.

2. In clinical trials of the drug Zocor, some subjects were treated with Zocor and other were given a placebo. The 95% confidence interval estimate of the difference between the proportions of subjects who experienced headaches is $-0.0518 < p_1 - p_2 < 0.0194$. Write a statement interpreting that confidence interval.

3. Among 8834 malfunctioning pacemakers, in 15.8% the malfunctions were due to batteries. Find the number of successes *x*.

4. Among 129 subjects who took Chantix as an aid to stop smoking, 12.4% experienced nausea. Find the number of successes *x*.

5. Among 610 adults selected randomly from among the residents of one town, 26.1% said that they have favor stronger gun-control laws. Find the number of successes *x*.

6. A computer manufacturer randomly selects 2,410 of its computers for quality assurance and finds that 3.13% of these computer are found defective. Find the number of successes *x*.

7. Assume that you plan to use a significance level of $\alpha = 0.05$ to test the claim that $p_1 = p_2$. Use the given sample sizes and number of successes to find the pooled estimate $\overline{p}$

a) $n_1 = 288, \quad n_2 = 252, \quad x_1 = 75, \quad x_2 = 70$

b) $n_1 = 100, \quad n_2 = 100, \quad \hat{p}_1 = 0.2, \quad \hat{p}_2 = 0.18$

8. The numbers of online applications from simple random samples of college applications for 2003 and for the current year are given below.

	2003	Current Year
Number of application in sample	36	27
Number of online applications in sample	13	14

Assume that you plan to use a significance level of $\alpha = 0.05$ to test the claim that $p_1 = p_2$.

Find

a) The pooled estimate $\overline{p}$

b) The *x* test statistic

c) The critical *z* values

d) The *P*–value

Assume 95% confidence interval

e) The margin of error E
f) The 95% confidence interval.

9. Chantix is a drug used as an aid to stop smoking. The numbers of subjects experiencing insomnia for each of two treatment groups in a clinical trial of the drug Chantix are given below:

	Chantix Treatment	Placebo
Number in group	129	805
Number experiencing insomnia	19	13

Assume that you plan to use a significance level of $\alpha = 0.05$ to test the claim that $p_1 = p_2$.

Find
a) The pooled estimate $\overline{p}$
b) The *x* test statistic
c) The critical *z* values
d) The *P*–value
Assume 95% confidence interval
e) The margin of error E
f) The 95% confidence interval.

10. In a 1993 survey of 560 college students, 171 said that they used illegal drugs during the previous year. In a recent survey of 720 college students, 263 said that they used illegal drugs during the previous year. Use a 0.05 significance level to test the claim that the proportion of college students using illegal drugs in 1993 was less than it is now.

11. In a 1993 survey of 560 college students, 171 said that they used illegal drugs during the previous year. In a recent survey of 720 college students, 263 said that they used illegal drugs during the previous year. Construct the confidence interval corresponding to the hypothesis test conducted with a 0.05 significance level. What conclusion does the confidence interval suggest?

12. A simple random sample of front-seat occupants involved in car crashes is obtained. Among 2823 occupants not wearing seat belts, 31 were killed. Among 7765 occupants wearing seat belts, 16 were killed. Construct a 90% confidence interval estimate of the difference between the fatality rates for those not wearing seat belts and those wearing seat belts. What does the result suggest about the effectiveness of seat belts?

13. A Pew Research Center poll asked randomly selected subjects if they agreed with the statement that "It is morally wrong fir married people to have an affair/" Among the 386 women surveyed, 347 agrees with the statement. Among the 359 men surveyed, 305 agreed with the statement.
a) Use a 0.05 significance level to test the claim that the percentage of women who agree is difference from the percentage of men who agree. Does there appear to be a difference in the way women and men feel about this issue?
b) Construct the confidence interval corresponding to the hypothesis test conducted with a 0.05 significance level. What conclusion does the confidence interval suggest?

14. Tax returns include an option of designating \$3 for presidential election campaigns, and it does not cost the taxpayer anything to make that designation. In a simple random sample of 250 tax returns from 1976, 27.6% of the returns designated the \$3 for the campaign. In a simple random sample of 300 recent tax returns, 7.3% of the returns designated the \$3 for the campaign. Use a 0.05 significance level to test the claim that the percentage of returns designating the \$3 for the campaign was greater in 1973 than it is now.

15. In an experiment, 16% of 734 subjects treated with Viagra experienced headaches. In the same experiment, 4% of 725 subjects given a placebo experienced headaches.

a) Use a 0.01 significance level to test the claim that the proportion of headaches is greater for those treated with Viagra. Do headaches appear to be a concern for those who take Viagra?

b) Construct the confidence interval corresponding to the hypothesis test conducted with a 0.01 significance level. What conclusion does the confidence interval suggest?

16. Two different simple random samples are drawn from two different populations. The first sample consists of 20 people with 10 having a common attribute. The second sample consists of 2000 people with 1404 of them having the same common attribute. Compare the results from a hypothesis test of $p_1 = p_2$ (with a 0.05 significance level) and a 95% confidence interval estimate of $p_1 - p_2$.

17. A report on the nightly news broadcast stated that 11 out of 142 households with pet dogs were burglarized and 21 out of 217 without pet dogs were burglarized. Find the *z* test statistic for the hypothesis test. Assume that you plan to use a significance level of $\alpha = 0.05$ to test the claim that $p_1 = p_2$.

18. Assume that the samples are independent and that they have been randomly selected. Construct a 90% confidence interval for the difference between population proportions $p_1 = p_2$

19. The sample size needed to estimate the difference between two population proportions ti within a margin of error *E* with a confidence level of $1 - \alpha$ can be found as follows:

$$E = z_{\alpha/2}\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}.$$

In this expression, replace n_1 and n_2 by n (assuming both samples have the same size) and replace each of p_1, q_1, p_2 *and* q_2 by 0.5 (because their values are not known). Then solve for n. Use this approach to find the size pf each sample of you want to estimate the difference between the proportions of men and women who plan to vote in the next presidential election. Assume that you want 99% confidence that your error is no more than 0.05.