Section 1.4 – Inverse, Exponential & Logarithmic Functions

One-to-One Function

A function f is one-to-one (1 – 1) if different inputs have different outputs that is,

$$if\ \ a \neq b, \quad then \quad f(a) \neq f(b)$$

Or $$if\ \ f(a) = f(b), \quad then \ \ a = b$$

Definition of Inverse Function

Let f be one-to-one function with domain *D* and range *R*. A function g with domain *R* and range *D* is the ***inverse function*** of f, provided the following condition is true for every *x* in *D* and every *y* in *R*:

$$y = f(x) \quad iff \quad x = g(y)$$

If the inverse of a function f is also a function, it is named f^{-1} read " f – inverse"

The -1 in f^{-1} is not an exponent! And is not equal to $\cancel{\frac{1}{f(x)}}$

***Domain* and *Range* of f and f^{-1}**

$$domain\ of\ f^{-1} = range\ of\ f$$

$$range\ of\ f^{-1} = domain\ of\ f$$

Example

For the given function $f(x) = \dfrac{2x+3}{x+5}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{2a+3}{a+5} = \frac{2b+3}{b+5}$$

$$2ab + 10a + 3b + 15 = 2ab + 10b + 3a + 15$$

$$7a = 7b$$

$$a = b \quad \checkmark$$

$f(x)$ is $1-1$

b) $y = \dfrac{2x+3}{x+5}$

$xy + 5y = 2x + 3$

$x(y-2) = 3 - 5y$

$x = \dfrac{-5y+3}{y-2}$

$\boxed{f^{-1}(x) = \dfrac{-5x+3}{x-2}}$

c) Domain of $f(x)$ = Range of $f^{-1}(x)$: $\mathbb{R} - \{-5\}$

Range of $f(x)$ = Domain of $f^{-1}(x)$: $\mathbb{R} - \{2\}$

Definition **(Exponential Functions)**

The exponential function f with base ***b*** is defined by

$$f(x) = b^x \qquad \text{or} \qquad y = b^x$$

Base

where $b > 0$, $b \neq 1$ and $\boldsymbol{x}$ is any real number.

Graphing Exponential

1. Define the Horizontal Asymptote $f(x) = b^x \pm d$

$y = 0 \pm \text{d}$

The exponential function always equals to 0

$x \to \infty$ *or* $x \to -\infty \Rightarrow f(x) \to 0$

2. Define/Make a table

(Force your exponential to = 0, then solve for x)

x	***f(x)***
$x-1$	
$\boldsymbol{x}$	
$x+1$	

Domain: $(-\infty, \infty)$

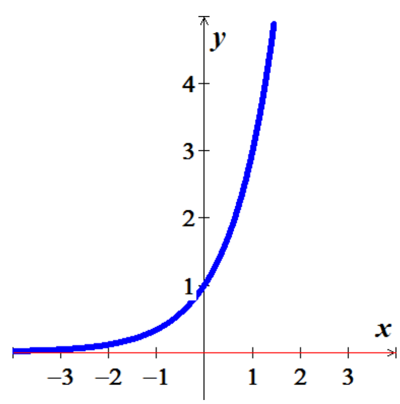
Range: (d, ∞)

Example

$f(x) = 3^x$

Asymptote: $y = 0$

x	*f(x)*
–1	1/3
0	1
1	3



Example

Sketch $f(x) = 3^{x-2}$

Solution

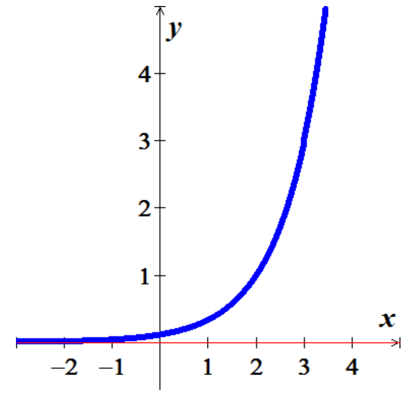
Shift right 2 unit

Asymptote: $y = 0$

Domain: $\mathbb{R}$

Range: $(0, \infty)$

x	$f(x)$
1	1/3
2	1
3	3



Example

Sketch the graph of $f(x) = 2^{-x^2}$

Solution

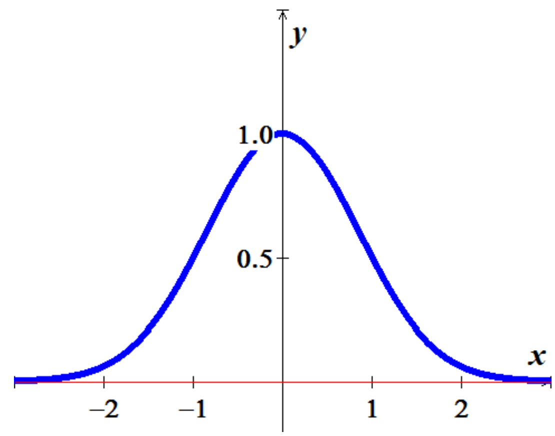
$$f(x) = \frac{1}{2^{x^2}}$$

Asymptote: $y = 0$

Domain: $\mathbb{R}$

Range: $(0, 1]$

x	$f(x)$
± 0	1
± 1	$\frac{1}{2}$
± 2	$\frac{1}{16}$



Natural Base *e*

The irrational number $e \approx 2.71828$ is called natural base

$f(x) = e^x$ is called natural exponential function

Example

Sketch $f(x) = e^{x+3} + 1$

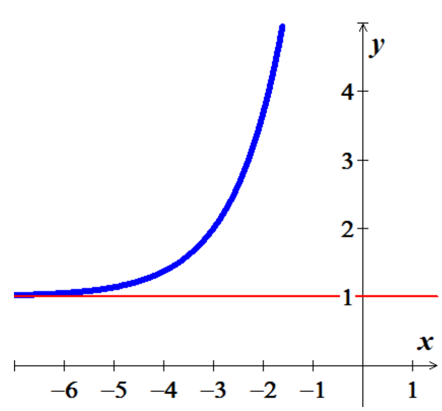
Solution

Asymptote: $y = 1$

Domain: $\mathbb{R}$

Range: $(1, \infty)$

x	$f(x)$
–4	1.4
–3	2
4	3.7



Logarithmic Function (*Definition*)

For $x > 0$ and $b > 0$, $b \neq 1$

$$y = \log_b x \quad \text{is equivalent to} \quad x = b^y$$

$$y = \log_b x \Leftrightarrow x = b^y$$

Base

The function $f(x) = \log_b x$ is the logarithmic function with base b.

$\log_b x$: <u>*read*</u> log base b of x

log x ***means*** $\log_{10} x$

ln x ***means*** $\log_e x$ $\qquad \ln x$ *read* "**el en of *x***"

Example

Write the equation in its equivalent exponential form:

$$3 = \log_7 x \qquad \Rightarrow x = 7^3$$

Write the equation in its equivalent logarithmic form:

$$2^5 = x \qquad \Rightarrow 5 = \log_2 x$$

Basic Logarithmic Properties

$$\log_b b = 1 \quad \to \quad b = b^1 \qquad\qquad \log_b 1 = 0 \quad \to \quad 1 = b^0$$

Inverse Properties

$$\log_b b^x = x \qquad\qquad b^{\log_b x} = x$$

Change-of-Base Logarithmic

$$\log_b M = \frac{\log_a M}{\log_a b} \qquad\qquad \log_b M = \frac{\log M}{\log b} \quad \textbf{or} \quad \log_b M = \frac{\ln M}{\ln b}$$

Domain

The domain of a logarithmic function of the form $f(x) = \log_b x$ is the set of all positive real numbers. (*Inside* the log has to be > 0)

Range: $\mathbb{R}$

Example

Find the domain of

a) $f(x) = \log_4(x-5)$ ***Domain***: $x > 5$

b) $f(x) = \ln(4-x)$ ***Domain***: $x < 4$

c) $h(x) = \ln(x^2)$ ***Domain***: $\mathbb{R} - \{0\}$ or $\{x \mid x \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$

Graphs of *Logarithmic* Functions

Example

Graph $g(x) = \log(x-2) + 1$

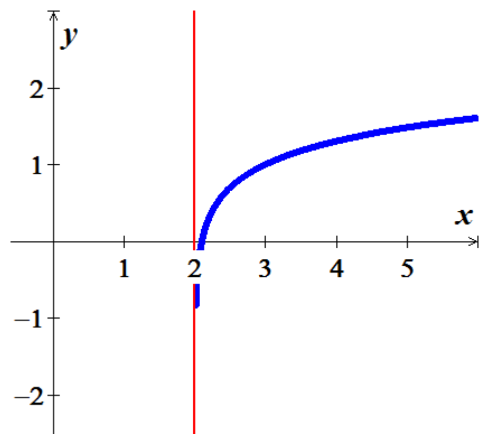
Solution

Asymptote: $x = 2$

Domain: $x > 2$

Range: $\mathbb{R}$

x	$g(x)$
2	
2.5	.7
3	1
4	1.3



Example

Graph $f(x) = \log_3 |x| \quad \text{for } x \neq 0$

Solution

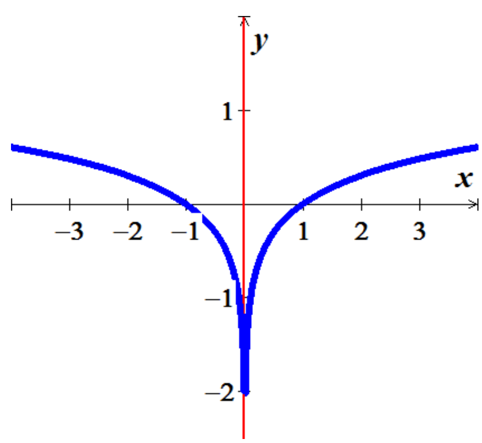
$f(-x) = \log_3 |-x| = \log_3 |x| = f(x)$

$\therefore$ The graph is symmetric with respect to the y-axis.

Asymptote: $x = 0$

Domain: $\mathbb{R} - \{0\}$

Range: $\mathbb{R}$



Exercises *Section* 1.4 – Inverse, Exponential & Logarithmic Functions

(**1 – 9**) Determine whether the function is *one*-to-*one*

1. $f(x) = 3x - 7$

2. $f(x) = x^2 - 9$

3. $f(x) = \sqrt{x}$

4. $f(x) = \sqrt[3]{x}$

5. $f(x) = |x|$

6. $f(x) = \dfrac{2}{x+3}$

7. $f(x) = (x-2)^3$

8. $y = x^2 + 2$

9. $f(x) = \dfrac{x+1}{x-3}$

10. Given the function $f(x) = (x+8)^3$

a) Find $f^{-1}(x)$

b) Graph f and f^{-1} in the same rectangular coordinate system

c) Find the domain and the range of f and f^{-1}

(**11 – 38**) For the given functions

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

11. $f(x) = \dfrac{2x}{x-1}$

12. $f(x) = \dfrac{x}{x-2}$

13. $f(x) = \dfrac{x+1}{x-1}$

14. $f(x) = \dfrac{2x+1}{x+3}$

15. $f(x) = \dfrac{3x-1}{x-2}$

16. $f(x) = \dfrac{2x}{x-1}$

17. $f(x) = \dfrac{x}{x-2}$

18. $f(x) = \dfrac{x+1}{x-1}$

19. $f(x) = \dfrac{2x+1}{x+3}$

20. $f(x) = \dfrac{3x-1}{x-2}$

21. $f(x) = \dfrac{3x-2}{x+4}$

22. $f(x) = \dfrac{-3x-2}{x+4}$

23. $f(x) = \sqrt{x-1} \quad x \ge 1$

24. $f(x) = \sqrt{2-x} \quad x \le 2$

25. $f(x) = x^2 + 4x \quad x \ge -2$

26. $f(x) = 3x + 5$

27. $f(x) = \dfrac{1}{3x-2}$

28. $f(x) = \dfrac{3x+2}{2x-5}$

29. $f(x) = \dfrac{4x}{x-2}$

30. $f(x) = 2 - 3x^2; \quad x \le 0$

31. $f(x) = 2x^3 - 5$

32. $f(x) = \sqrt{3-x}$

33. $f(x) = \sqrt[3]{x} + 1$

34. $f(x) = \left(x^3 + 1\right)^5$

35. $f(x) = x^2 - 6x; \quad x \ge 3$

36. $f(x) = (x-2)^3$

37. $f(x) = \dfrac{x+1}{x-3}$

38. $f(x) = \dfrac{2x+1}{x-3}$

39. Simplify the expression $\dfrac{\left(e^{x}+e^{-x}\right)\left(e^{x}+e^{-x}\right)-\left(e^{x}-e^{-x}\right)\left(e^{x}-e^{-x}\right)}{\left(e^{x}+e^{-x}\right)^{2}}$

40. Simplify the expression $\dfrac{\left(e^{x}-e^{-x}\right)^{2}-\left(e^{x}+e^{-x}\right)^{2}}{\left(e^{x}+e^{-x}\right)^{2}}$

(**41 – 52**)Write the equation in its equivalent logarithmic form

41. $2^6 = 64$

42. $5^4 = 625$

43. $5^{-3} = \frac{1}{125}$

44. $\sqrt[3]{64} = 4$

45. $b^3 = 343$

46. $8^y = 300$

47. $\sqrt[n]{x} = y$

48. $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$

49. $\left(\frac{1}{2}\right)^{-5} = 32$

50. $e^{x-2} = 2y$

51. $e = 3x$

52. $\sqrt[3]{e^{2x}} = y$

(**53 – 64**) Write the equation in its equivalent exponential form

53. $\log_5 125 = y$

54. $\log_4 16 = x$

55. $\log_5 \frac{1}{5} = x$

56. $\log_2 \frac{1}{8} = x$

57. $\log_6 \sqrt{6} = x$

58. $\log_3 \frac{1}{\sqrt{3}} = x$

59. $6 = \log_2 64$

60. $2 = \log_9 x$

61. $\log_{\sqrt{3}} 81 = 8$

62. $\log_4 \frac{1}{64} = -3$

63. $\log_4 26 = y$

64. $\ln M = c$

(**65 – 71**) Evaluate the expression without using a calculator

65. $\log_4 16$

66. $\log_2 \frac{1}{8}$

67. $\log_6 \sqrt{6}$

68. $\log_3 \frac{1}{\sqrt{3}}$

69. $\log_3 \sqrt[7]{3}$

70. $\log_3 \sqrt{9}$

71. $\log_{\frac{1}{2}} \sqrt{\frac{1}{2}}$

(**72 – 80**) Simplify

72. $\log_5 1$

73. $\log_7 7^2$

74. $3^{\log_3 8}$

75. $10^{\log 3}$

76. $e^{2+\ln 3}$

77. $\ln e^{-3}$

78. $\ln e^{x-5}$

79. $\log_b b^n$

80. $\ln e^{x^2+3x}$

(**81 – 108**) Find the domain of

81. $f(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

82. $f(x)=\dfrac{e^{|x|}}{1+e^x}$

83. $f(x)=\sqrt{1-e^x}$

84. $f(x)=\sqrt{e^x-e^{-x}}$

85. $f(x)=\log_5(x+4)$

86. $f(x)=\log_5(x+6)$

87. $f(x)=\log(2-x)$

88. $f(x)=\log(7-x)$

89. $f(x)=\ln(x-2)^2$

90. $f(x)=\ln(x-7)^2$

91. $f(x)=\log\left(x^2-4x-12\right)$

92. $f(x)=\log\left(\frac{x-2}{x+5}\right)$

93. $f(x)=\log\left(\frac{3-x}{x-2}\right)$

94. $f(x)=\ln\left(\dfrac{x^2}{x-4}\right)$

95. $f(x)=\log_3\left(x^3-x\right)$

96. $f(x)=\log\sqrt{2x-5}$

97. $f(x)=3\ln(5x-6)$

98. $f(x)=\log\left(\frac{x}{x-2}\right)$

99. $f(x)=\ln\left(x^2+4\right)$

100. $f(x)=\ln|4x-8|$

101. $f(x)=\ln\left(x^2-9\right)$

102. $f(x)=\ln|5-x|$

103. $f(x)=\ln(x-4)^2$

104. $f(x)=\ln\left(x^2-4\right)$

105. $f(x)=\ln\left(x^2-4x+3\right)$

106. $f(x)=\ln\left(2x^2-5x+3\right)$

107. $f(x)=\log\left(x^2+4x+3\right)$

108. $f(x)=\ln\left(x^4-x^2\right)$

(**109 – 129**) Find the ***asymptote***, ***domain***, and ***range*** of the given functions. Then, sketch the graph

109. $f(x)=2^x+3$

110. $f(x)=2^{3-x}$

111. $f(x)=\left(\frac{2}{5}\right)^{-x}$

112. $f(x)=-\left(\frac{1}{2}\right)^x+4$

113. $f(x)=4^x$

114. $f(x)=2-4^x$

115. $f(x)=-3+4^{x-1}$

116. $f(x)=1+\left(\frac{1}{4}\right)^{x+1}$

117. $f(x)=e^{x-2}$

118. $f(x)=3-e^{x-2}$

119. $f(x)=e^{x+4}$

120. $f(x)=2+e^{x-1}$

121. $f(x)=\log_4(x-2)$

122. $f(x)=\log_4|x|$

123. $f(x)=\left(\log_4 x\right)-2$

124. $f(x)=\log(3-x)$

125. $f(x)=2-\log(x+2)$

126. $f(x)=\ln(x-2)$

127. $f(x)=\ln(3-x)$

128. $f(x)=2+\ln(x+1)$

129. $f(x)=1-\ln(x-2)$

130. On a study by psychologists Bornstein and Bornstein, it was found that the average walking speed w, in feet per second, of a person living in a city of population P, in ***thousands***, is given by the function:

$$w(P) = 0.37 \ln P + 0.05$$

a) The population is 124,848. Find the average walking speed of people living in Hartford.
b) The population is 1,236,249. Find the average walking speed of people living in San Antonio.

131. The loudness of sounds is measured in a unit called a decibel. To measure with this unit, we first assign an intensity of I_0 to a very faint sound, called the threshold sound. If a particular sound has intensity I, then the decibel rating of this louder sound is

$$d = 10 \log \frac{I}{I_0}$$

Find the exact decibel rating of a sound with intensity $10{,}000 I_0$

132. Students in an accounting class took a final exam and then took equivalent forms of the exam at monthly intervals thereafter. The average score $S(t)$, as a percent, after t months was found to be given by the function

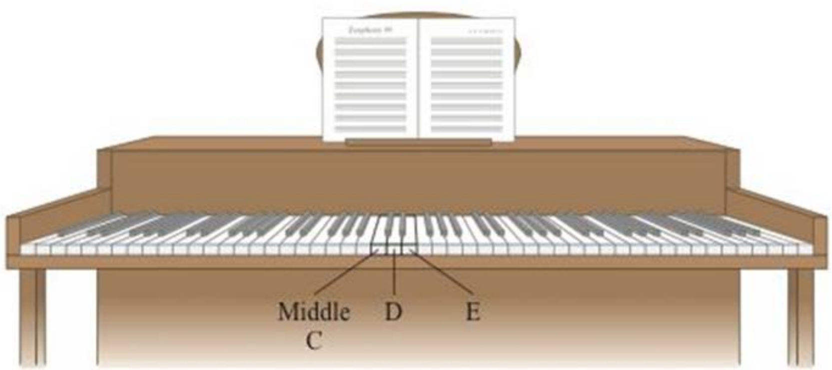
$$S(t) = 78 - 15 \log(t+1); \quad t \geq 0$$

a) What was the average score when the students initially took the test, $t = 0$?

b) What was the average score after 4 *months*? 24 *months*?

133. Starting on the left side of a standard 88–*key* piano, the frequency, in *vibrations* per *second*, of the nth note is given by

$$f(n) = (27.5)\, 2^{\frac{n-1}{12}}$$



a) Determine the frequency of middle C, key number 40 on an 88–*key* piano.
b) Is the difference in frequency between middle C (key number 40) and D (key number 42) the same as the difference in frequency between D (key number 42) and E (key number 44)?