

## ***Solution***      ***Section 1.6 – Surface Area***

### ***Exercise***

Find the lateral (side) surface area of the cone generated by revolving the line segment  $y = \frac{x}{2}$ ,  $0 \leq x \leq 4$ , about the  $x$ -axis. Check your answer with the geometry formula

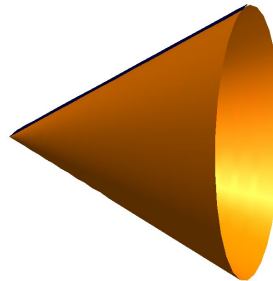
$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}$$

### **Solution**

$$\frac{dy}{dx} = \frac{1}{2}$$

$$\begin{aligned}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{1 + \frac{1}{4}} \\ &= \frac{\sqrt{5}}{2}\end{aligned}$$

$$\begin{aligned}S &= 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= 2\pi \int_0^4 \left(\frac{x}{2}\right) \frac{\sqrt{5}}{2} dx \\ &= \frac{\pi\sqrt{5}}{2} \int_0^4 x dx \\ &= \frac{\pi\sqrt{5}}{2} \left( \frac{1}{2}x^2 \right) \Big|_0^4 \\ &= \frac{\pi\sqrt{5}}{4} (4^2 - 0) \\ &= \underline{4\pi\sqrt{5} \text{ unit}^2}\end{aligned}$$



$$\text{base circumference} = 2\pi r = 2\pi(2) = 4\pi$$

$$\text{slant height} = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$$

$$\begin{aligned}\text{Lateral surface area} &= \frac{1}{2} \times \text{base circumference} \times \text{slant height} \\ &= \frac{1}{2} \times (4\pi) \times (2\sqrt{5}) \\ &= \underline{4\pi\sqrt{5} \text{ unit}^2}\end{aligned}$$

### Exercise

Find the lateral surface area of the cone generated by revolving the line segment  $y = \frac{x}{2}$ ,  $0 \leq x \leq 4$ , about the  $y$ -axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}$$

### Solution

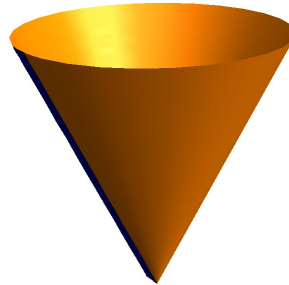
$$y = \frac{x}{2} \Rightarrow x = 2y$$

$$\rightarrow \begin{cases} x = 0 & \rightarrow y = 0 \\ x = 4 & \rightarrow y = 2 \end{cases}$$

$$\frac{dx}{dy} = 2$$

$$\begin{aligned} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{1 + 4} \\ &= \sqrt{5} \end{aligned}$$

$$\begin{aligned} S &= 2\pi \int_c^d x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \\ &= 2\pi \int_0^2 2\sqrt{5} y dy \\ &= 2\pi\sqrt{5} y^2 \Big|_0^2 \\ &= 2\pi\sqrt{5}(4 - 0) \\ &= \underline{8\pi\sqrt{5} \text{ unit}^2} \end{aligned}$$



$$\text{base circumference} = 2\pi(4) = \underline{8\pi}$$

$$\begin{aligned} \text{slant height} &= \sqrt{4^2 + 2^2} \\ &= \sqrt{20} \\ &= \underline{2\sqrt{5}} \end{aligned}$$

$$\begin{aligned} \text{Lateral surface area} &= \frac{1}{2} \times \text{base circumference} \times \text{slant height} \\ &= \frac{1}{2} \times (8\pi) \times (2\sqrt{5}) \\ &= \underline{8\pi\sqrt{5} \text{ unit}^2} \end{aligned}$$

### Exercise

Find the lateral surface area of the cone frustum generated by revolving the line segment

$y = \frac{x}{2} + \frac{1}{2}$ ,  $1 \leq x \leq 3$ , about the  $x$ -axis. Check your answer with the geometry formula

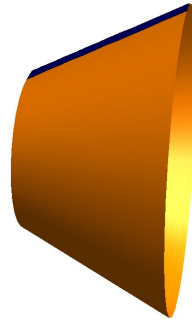
$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}$$

### Solution

$$\frac{dy}{dx} = \frac{1}{2}$$

$$\begin{aligned}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{1 + \frac{1}{4}} \\ &= \frac{\sqrt{5}}{2}\end{aligned}$$

$$\begin{aligned}S &= 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= 2\pi \int_1^3 \left(\frac{x}{2} + \frac{1}{2}\right) \left(\frac{\sqrt{5}}{2}\right) dx \\ &= \pi \frac{\sqrt{5}}{2} \int_1^3 (x+1) dx \\ &= \pi \frac{\sqrt{5}}{2} \left( \frac{1}{2}x^2 + x \right) \Big|_1^3 \\ &= \pi \frac{\sqrt{5}}{2} \left( \frac{9}{2} + 3 - \frac{3}{2} \right) \\ &= \pi \frac{\sqrt{5}}{2} (6) \\ &= \underline{3\pi\sqrt{5} \text{ unit}^2}\end{aligned}$$



$$r_1 = \frac{1}{2} + \frac{1}{2} = 1 \quad r_2 = \frac{3}{2} + \frac{1}{2} = 2$$

$$\begin{aligned}\text{slant height} &= \sqrt{(2-1)^2 + (3-1)^2} \\ &= \sqrt{5}\end{aligned}$$

$$\begin{aligned}\text{Frustum surface area} &= \pi(r_1 + r_2) \times \text{slant height} \\ &= \pi(1+2)\sqrt{5} \\ &= \underline{3\pi\sqrt{5} \text{ unit}^2}\end{aligned}$$

### Exercise

Find the lateral surface area of the cone frustum generated by revolving the line segment

$y = \frac{x}{2} + \frac{1}{2}$ ,  $1 \leq x \leq 3$ , about the  $y$ -axis. Check your answer with the geometry formula

$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}$$

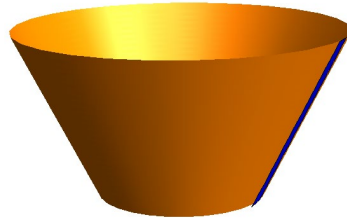
### Solution

$$y = \frac{x}{2} + \frac{1}{2}$$

$$x = 2y - 1$$

$$\frac{dx}{dy} = 2$$

$$\begin{aligned}\sqrt{1 + \left(\frac{dx}{dy}\right)^2} &= \sqrt{1 + 4} \\ &= \sqrt{5}\end{aligned}$$



$$S = 2\pi \int_1^2 (2y - 1)(\sqrt{5}) dy$$

$$= 2\pi\sqrt{5} \int_1^2 (2y - 1) dy$$

$$= 2\pi\sqrt{5} \left( y^2 - y \right) \Big|_1^2$$

$$= 2\pi\sqrt{5} [4 - 2 - (1 - 1)]$$

$$= 4\pi\sqrt{5} \text{ unit}^2$$

$$r_1 = 1 \quad r_2 = 3$$

$$\begin{aligned}\text{slant height} &= \sqrt{(2-1)^2 + (3-1)^2} \\ &= \sqrt{5}\end{aligned}$$

$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}$$

$$= \pi(1 + 3)\sqrt{5}$$

$$= 4\pi\sqrt{5} \text{ unit}^2$$

### Exercise

Set up and evaluate the definite integral for the area of the surface generated by revolving the curve  $y = \frac{1}{3}x^3$  about the  $x$ -axis

#### Solution

$$y = \frac{1}{3}x^3$$

$$y' = x^2$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + x^4}$$

$$S = 2\pi \int_0^3 \frac{1}{3}x^3 \sqrt{1 + x^4} \, dx$$

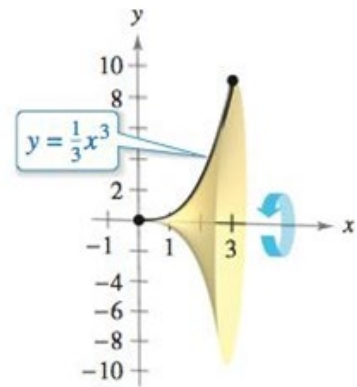
$$= \frac{\pi}{6} \int_0^3 (1 + x^4)^{1/2} d(1 + x^4)$$

$$= \frac{\pi}{9} (1 + x^4)^{3/2} \Big|_0^3$$

$$= \frac{\pi}{9} ((82)^{3/2} - 1)$$

$$= \frac{\pi}{9} (82\sqrt{82} - 1) \text{ unit}^2$$

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$



### Exercise

Set up and evaluate the definite integral for the area of the surface generated by revolving the curve  $y = 2\sqrt{x}$  about the  $x$ -axis

#### Solution

$$y = 2\sqrt{x}$$

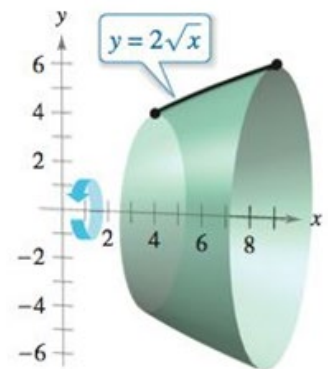
$$y' = \frac{1}{\sqrt{x}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{1}{x}}$$

$$S = 2\pi \int_4^9 2\sqrt{x} \sqrt{\frac{x+1}{x}} \, dx$$

$$= 4\pi \int_4^9 (1 + x)^{1/2} d(1 + x)$$

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$



$$= \frac{8}{3} \pi (1+x)^{3/2} \Big|_4^9$$

$$= \frac{8}{3} \pi (10^{3/2} - 5^{3/2}) \quad \text{unit}^2 \quad \approx 171.285$$

### Exercise

Find the area of the surface generated by  $y = \frac{x^3}{9}$ ,  $0 \leq x \leq 2$ ,  $x$ -axis

### Solution

$$\frac{dy}{dx} = \frac{1}{3} x^2$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{1}{9} x^4}$$

$$= \frac{1}{3} \sqrt{9 + x^4}$$

$$S = 2\pi \int_0^2 \frac{x^3}{9} \cdot \frac{1}{3} \sqrt{9 + x^4} dx$$

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \frac{2\pi}{27} \int_0^2 x^3 \sqrt{9 + x^4} dx$$

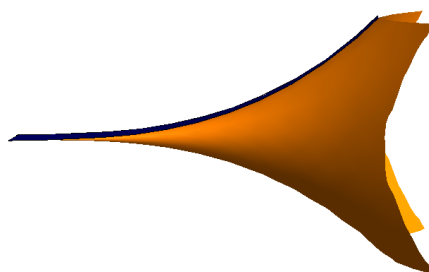
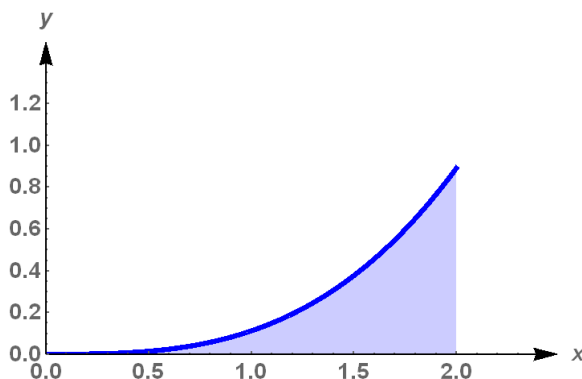
$$= \frac{\pi}{54} \int_0^2 (9 + x^4)^{1/2} d(x^4 + 9)$$

$$= \frac{\pi}{81} (x^4 + 9)^{3/2} \Big|_0^2$$

$$= \frac{\pi}{81} (25^{3/2} - 9^{3/2})$$

$$= \frac{\pi}{81} (125 - 27)$$

$$= \frac{98\pi}{81} \quad \text{unit}^2$$



### Exercise

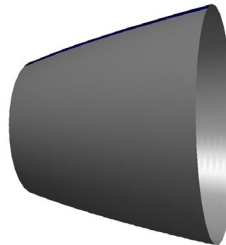
Find the area of the surface generated by  $y = \sqrt{x+1}$ ,  $1 \leq x \leq 5$ ,  $x$ -axis

### Solution

$$y = \sqrt{x+1}$$

$$\frac{dy}{dx} = \frac{1}{2}(x+1)^{-1/2}$$

$$\begin{aligned}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{1 + \frac{1}{4}(x+1)^{-1}} \\ &= \sqrt{1 + \frac{1}{4(x+1)}} \\ &= \sqrt{\frac{4x+4+1}{4(x+1)}} \\ &= \frac{1}{2}\sqrt{\frac{4x+5}{x+1}}\end{aligned}$$



$$\begin{aligned}S &= 2\pi \int_1^5 \sqrt{x+1} \left(\frac{1}{2}\right) \frac{\sqrt{4x+5}}{\sqrt{x+1}} dx \\ &= \pi \int_1^5 \sqrt{4x+5} dx \\ &= \frac{\pi}{4} \int_1^5 (4x+5)^{1/2} d(4x+5) \\ &= \frac{\pi}{6} (4x+5)^{3/2} \Big|_1^5 \\ &= \frac{\pi}{6} (25^{3/2} - 9^{3/2}) \\ &= \frac{\pi}{6} (98) \\ &= \frac{49\pi}{3} \text{ unit}^2\end{aligned}$$

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

### Exercise

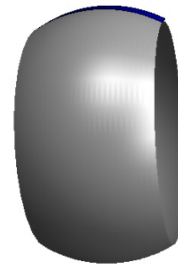
Find the area of the surface generated by  $y = \sqrt{2x-x^2}$ ,  $0.5 \leq x \leq 1.5$ ,  $x$ -axis

### Solution

$$\frac{dy}{dx} = \frac{1}{2}(2x-x^2)^{-1/2} (2-2x)$$

$$= (1-x)(2x-x^2)^{-1/2}$$

$$\begin{aligned}\sqrt{1+\left(\frac{dy}{dx}\right)^2} &= \sqrt{1+(1-x)^2(2x-x^2)^{-1}} \\ &= \sqrt{1+\frac{1-2x+x^2}{2x-x^2}} \\ &= \sqrt{\frac{2x-x^2+1-2x+x^2}{2x-x^2}} \\ &= \sqrt{\frac{1}{2x-x^2}} \\ &= \frac{1}{\sqrt{2x-x^2}}\end{aligned}$$



$$\begin{aligned}S &= 2\pi \int_{.5}^{1.5} \sqrt{2x-x^2} \frac{1}{\sqrt{2x-x^2}} dx \\ &= 2\pi \int_{.5}^{1.5} dx \\ &= 2\pi x \Big|_{.5}^{1.5} = 2\pi(1.5-.5) \\ &= \underline{2\pi \text{ unit}^2}\end{aligned}$$

$$S = 2\pi \int_a^b y \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx$$

### Exercise

Find the area of the surface generated by  $y = 3x + 4$ ,  $0 \leq x \leq 6$ , revolved about  $x$ -axis

### Solution

$$y' = 3$$

$$\begin{aligned}S &= 2\pi \int_0^6 (3x+4) \sqrt{1+9} dx \\ &= 2\pi\sqrt{10} \left( \frac{3}{2}x^2 + 4x \right) \Big|_0^6 \\ &= 2\pi\sqrt{10}(54+24) \\ &= \underline{156\pi\sqrt{10} \text{ unit}^2}\end{aligned}$$

$$S = 2\pi \int_a^b y \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx$$



### Exercise

Find the area of the surface generated by  $y = 12 - 3x$ ,  $1 \leq x \leq 3$ , revolved about  $x$ -axis

#### Solution

$$y' = -3$$

$$\begin{aligned} S &= 2\pi \int_1^3 (12 - 3x) \sqrt{1 + 9} \, dx \\ &= 2\pi\sqrt{10} \left( 12x - \frac{3}{2}x^2 \right) \Big|_1^3 \\ &= 2\pi\sqrt{10} \left( 36 - \frac{27}{2} - 12 + \frac{3}{2} \right) \\ &= \underline{24\pi\sqrt{10} \text{ unit}^2} \end{aligned}$$

$$S = 2\pi \int_a^b y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx$$

### Exercise

Find the area of the surface generated by  $y = 8\sqrt{x}$ ,  $9 \leq x \leq 20$ , revolved about  $x$ -axis

#### Solution

$$y' = \frac{4}{\sqrt{x}}$$

$$\begin{aligned} S &= 2\pi \int_9^{20} 8\sqrt{x} \sqrt{1 + \frac{16}{x}} \, dx \\ &= 16\pi \int_9^{20} \sqrt{x} \frac{\sqrt{x+16}}{\sqrt{x}} \, dx \\ &= 16\pi \int_9^{20} (x+16)^{1/2} \, d(x+16) \\ &= \frac{32\pi}{3} (x+16)^{3/2} \Big|_9^{20} \\ &= \frac{32\pi}{3} \left( (36)^{3/2} - (25)^{3/2} \right) \\ &= \frac{32\pi}{3} (216 - 125) \\ &= \underline{\frac{2912\pi}{3} \text{ unit}^2} \end{aligned}$$

$$S = 2\pi \int_a^b y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx$$

### Exercise

Find the area of the surface generated by  $y = x^3$ ,  $0 \leq x \leq 1$ , revolved about  $x$ -axis

### Solution

$$y' = 3x^2$$

$$\begin{aligned} S &= 2\pi \int_0^1 x^3 \sqrt{1+9x^4} \, dx \\ &= \frac{\pi}{18} \int_0^1 (1+9x^4)^{1/2} d(1+9x^4) \\ &= \frac{\pi}{27} (1+9x^4)^{3/2} \Big|_0^1 \\ &= \frac{\pi}{27} \left( (10)^{3/2} - 1 \right) \\ &= \frac{\pi}{27} (10\sqrt{10} - 1) \text{ unit}^2 \end{aligned}$$

$$S = 2\pi \int_a^b y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx$$

### Exercise

Find the area of the surface generated by  $y = x^{3/2} - \frac{1}{3}x^{1/2}$ ,  $1 \leq x \leq 2$ , revolved about  $x$ -axis

### Solution

$$\begin{aligned} y' &= \frac{3}{2}x^{1/2} - \frac{1}{6}x^{-1/2} \\ &= \frac{3}{2}\sqrt{x} - \frac{1}{6\sqrt{x}} \end{aligned}$$

$$a = 1, \quad m = \frac{3}{2}, \quad b = -\frac{1}{3}, \quad n = \frac{1}{2}$$

$$f(x) = ax^m + bx^n$$

$$1. \quad m + n = \frac{3}{2} + \frac{1}{2} = 2 \quad \checkmark$$

$$2. \quad abmn = (1)\left(\frac{3}{2}\right)\left(-\frac{1}{3}\right)\left(\frac{1}{2}\right) = -\frac{1}{4} \quad \checkmark$$

$$\begin{aligned} S &= 2\pi \int_1^2 \left( x^{3/2} - \frac{1}{3}x^{1/2} \right) \left( \frac{3}{2}\sqrt{x} + \frac{1}{6\sqrt{x}} \right) dx \\ &= 2\pi \int_1^2 \left( \frac{3}{2}x^2 - \frac{1}{3}x - \frac{1}{18} \right) dx \\ &= 2\pi \left( \frac{1}{2}x^3 - \frac{1}{6}x^2 - \frac{1}{18}x \right) \Big|_1^2 \end{aligned}$$

$$S = 2\pi \int_a^b f(x) \overline{f'(x)} \, dx$$

$$\begin{aligned}
&= 2\pi \left( 4 - \frac{2}{3} - \frac{1}{9} - \frac{1}{2} + \frac{1}{6} + \frac{1}{18} \right) \\
&= 2\pi \left( 3 - \frac{1}{18} \right) \\
&= \frac{53\pi}{9} \text{ unit}^2
\end{aligned}$$

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$$\begin{aligned}
S &= 2\pi \int_1^2 \left( x^{3/2} - \frac{1}{3}x^{1/2} \right) \sqrt{1 + \frac{(9x-1)^2}{36x}} \, dx & S &= 2\pi \int_a^b y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \\
&= \frac{2}{3}\pi \int_1^2 \left( 3x^{3/2} - x^{1/2} \right) \frac{\sqrt{36x + 81x^2 - 18x + 1}}{6\sqrt{x}} \, dx \\
&= \frac{\pi}{9} \int_1^2 (3x-1) \sqrt{81x^2 + 18x + 1} \, dx \\
&= \frac{\pi}{9} \int_1^2 (3x-1) \sqrt{(9x+1)^2} \, dx \\
&= \frac{\pi}{9} \int_1^2 (3x-1)(9x+1) \, dx \\
&= \frac{\pi}{9} \int_1^2 (27x^2 - 6x - 1) \, dx \\
&= \frac{\pi}{9} \left( 9x^3 - 3x^2 - x \right) \Big|_1^2 \\
&= \frac{\pi}{9} (72 - 12 - 2 - 9 + 3 + 1) \\
&= \frac{53\pi}{9} \text{ unit}^2
\end{aligned}$$

### Exercise

Find the area of the surface generated by  $y = \sqrt{4x+6}$ ,  $0 \leq x \leq 5$ , revolved about  $x$ -axis

### Solution

$$\begin{aligned}
y' &= \frac{2}{\sqrt{4x+6}} \\
S &= 2\pi \int_0^5 \sqrt{4x+6} \sqrt{1 + \frac{4}{4x+6}} \, dx & S &= 2\pi \int_a^b y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\end{aligned}$$

$$\begin{aligned}
&= 2\pi \int_0^5 \sqrt{4x+6} \sqrt{\frac{4x+6+4}{4x+6}} dx \\
&= 2\pi \int_0^5 (4x+10)^{1/2} dx \\
&= \frac{\pi}{2} \int_0^5 (4x+10)^{1/2} d(4x+10) \\
&= \frac{\pi}{3} (4x+10)^{3/2} \Big|_0^5 \\
&= \frac{\pi}{3} (30^{3/2} - 10^{3/2}) \\
&= \frac{\pi}{3} (30\sqrt{30} - 10\sqrt{10}) \\
&= \frac{10\pi\sqrt{10}}{3} (3\sqrt{3} - 1) \quad \text{unit}^2
\end{aligned}$$

### Exercise

Find the area of the surface generated by  $y = \frac{1}{4}(e^{2x} + e^{-2x})$ ,  $-2 \leq x \leq 2$ , revolved about  $x$ -axis

### Solution

$$y' = \frac{1}{2}(e^{2x} - e^{-2x})$$

$$a = \frac{1}{4}, \quad m = 2, \quad b = \frac{1}{4}, \quad n = -2$$

$$f(x) = ae^{mx} + be^{nx}$$

1.  $m = -n$  ✓

2.  $abmn = \frac{1}{4}(\frac{1}{4})(2)(-2) = -\frac{1}{4}$  ✓

$$S = 2\pi \int_{-2}^2 \frac{1}{4}(e^{2x} + e^{-2x}) \cdot \frac{1}{2}(e^{2x} + e^{-2x}) dx$$

$$S = 2\pi \int_a^b f(x) \overline{f'(x)} dx$$

$$= \frac{\pi}{4} \int_{-2}^2 (e^{4x} + 1 + e^{-4x}) dx$$

$$= \frac{\pi}{4} \left( \frac{1}{4}e^{4x} + 2x - \frac{1}{4}e^{-4x} \right) \Big|_{-2}^2$$

$$= \frac{\pi}{4} \left( \frac{1}{4}e^8 + 4 - \frac{1}{4}e^{-8} - \frac{1}{4}e^{-8} + 4 + \frac{1}{4}e^8 \right)$$

$$= \frac{\pi}{4} \left( \frac{1}{2}e^8 + 8 - \frac{1}{2}e^{-8} \right)$$

$$= \frac{\pi}{8} (e^8 + 16 - e^{-8}) \quad \text{unit}^2 \quad \Bigg|$$

$$S = 2\pi \int_{-2}^2 \frac{1}{4} (e^{2x} + e^{-2x}) \sqrt{1 + \frac{1}{4} (e^{2x} - e^{-2x})^2} \, dx$$

$$= \frac{\pi}{4} \int_{-2}^2 (e^{2x} + e^{-2x}) \sqrt{4 + e^{4x} - 2 + e^{-4x}} \, dx$$

$$= \frac{\pi}{4} \int_{-2}^2 (e^{2x} + e^{-2x}) \sqrt{e^{4x} + 2 + e^{-4x}} \, dx$$

$$= \frac{\pi}{4} \int_{-2}^2 (e^{2x} + e^{-2x}) \sqrt{(e^{2x} + e^{-2x})^2} \, dx$$

$$= \frac{\pi}{4} \int_{-2}^2 (e^{2x} + e^{-2x})^2 \, dx$$

$$= \frac{\pi}{4} \int_{-2}^2 (e^{4x} + 2 + e^{-4x}) \, dx$$

$$= \frac{\pi}{4} \left( \frac{1}{4} e^{4x} + 2x - \frac{1}{4} e^{-4x} \right) \Bigg|_{-2}^2$$

$$= \frac{\pi}{4} \left( \frac{1}{4} e^8 + 4 - \frac{1}{4} e^{-8} - \frac{1}{4} e^{-8} + 4 + \frac{1}{4} e^8 \right)$$

$$= \frac{\pi}{4} \left( \frac{1}{2} e^8 + 8 - \frac{1}{2} e^{-8} \right)$$

$$= \frac{\pi}{8} (e^8 + 16 - e^{-8}) \quad \text{unit}^2 \quad \Bigg|$$

$$S = 2\pi \int_a^b y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx$$

### Exercise

Find the area of the surface generated by  $y = \frac{1}{8}x^4 + \frac{1}{4x^2}$ ,  $1 \leq x \leq 2$ , revolved about  $x$ -axis

### Solution

$$y' = \frac{1}{2}x^3 - \frac{1}{2x^3}$$

$$a = \frac{1}{8}, \quad m = 4, \quad b = \frac{1}{4}, \quad n = -2$$

$$f(x) = ax^m + bx^n$$

$$1. \quad m + n = 4 - 2 = 2 \quad \checkmark$$

$$2. \quad abmn = \frac{1}{8} \left( \frac{1}{4} \right) (4) (-2) = -\frac{1}{4} \quad \checkmark$$

$$\begin{aligned}
S &= 2\pi \int_1^2 \left( \frac{1}{8}x^4 + \frac{1}{4x^2} \right) \left( \frac{1}{2}x^3 + \frac{1}{2x^3} \right) dx \\
&= \pi \int_1^2 \left( \frac{1}{4}x^7 + \frac{3}{4}x + \frac{1}{2}x^{-5} \right) dx \\
&= \frac{\pi}{8} \left( \frac{1}{8}x^8 + \frac{3}{2}x^2 - \frac{1}{2}x^{-4} \right) \Big|_1^2 \\
&= \frac{\pi}{8} \left( 32 + 6 - \frac{1}{32} - \frac{1}{8} - \frac{3}{2} + \frac{1}{2} \right) \\
&= \frac{\pi}{8} \left( 37 - \frac{5}{32} \right) \\
&= \frac{1179\pi}{256} \text{ unit}^2
\end{aligned}$$

$$S = 2\pi \int_a^b f(x) \overline{f'(x)} dx$$

$$\begin{aligned}
S &= 2\pi \int_1^2 \left( \frac{1}{8}x^4 + \frac{1}{4x^2} \right) \sqrt{1 + \left( \frac{x^6 - 1}{2x^3} \right)^2} dx \\
&= \frac{\pi}{4} \int_1^2 \left( \frac{x^6 + 2}{x^2} \right) \sqrt{1 + \frac{x^{12} - 2x^6 + 1}{4x^6}} dx \\
&= \frac{\pi}{4} \int_1^2 \left( \frac{x^6 + 2}{x^2} \right) \sqrt{\frac{x^{12} + 2x^6 + 1}{4x^6}} dx \\
&= \frac{\pi}{4} \int_1^2 \left( \frac{x^6 + 2}{x^2} \right) \frac{\sqrt{(x^6 + 1)^2}}{2x^3} dx \\
&= \frac{\pi}{4} \int_1^2 \frac{(x^6 + 2)(x^6 + 1)}{2x^5} dx \\
&= \frac{\pi}{4} \int_1^2 \frac{x^{12} + 3x^6 + 2}{2x^5} dx \\
&= \frac{\pi}{8} \int_1^2 \left( x^7 + 3x + 2x^{-5} \right) dx \\
&= \frac{\pi}{8} \left( \frac{1}{8}x^8 + \frac{3}{2}x^2 - \frac{1}{2}x^{-4} \right) \Big|_1^2 \\
&= \frac{\pi}{8} \left( 32 + 6 - \frac{1}{32} - \frac{1}{8} - \frac{3}{2} + \frac{1}{2} \right) \\
&= \frac{\pi}{8} \left( 37 - \frac{5}{32} \right) \\
&= \frac{1179\pi}{256} \text{ unit}^2
\end{aligned}$$

$$S = 2\pi \int_a^b y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx$$

### Exercise

Find the area of the surface generated by  $y = \frac{1}{3}x^3 + \frac{1}{4x}$ ,  $\frac{1}{2} \leq x \leq 2$ , revolved about  $x$ -axis

### Solution

$$y' = x^2 - \frac{1}{4x^2}$$

$$= \frac{4x^4 - 1}{4x^2}$$

$$a = \frac{1}{3}, \quad m = 3, \quad b = \frac{1}{4}, \quad n = -1$$

$$f(x) = ax^m + bx^n$$

$$1. \quad m + n = 3 - 1 = 2 \quad \checkmark$$

$$2. \quad abmn = \frac{1}{3} \left( \frac{1}{4} \right) (3)(-1) = -\frac{1}{4} \quad \checkmark$$

$$S = 2\pi \int_{1/2}^2 \left( \frac{1}{3}x^3 + \frac{1}{4x} \right) \left( x^2 + \frac{1}{4x^2} \right) dx$$

$$S = 2\pi \int_a^b f(x) \overline{f'(x)} dx$$

$$= 2\pi \int_{1/2}^2 \left( \frac{1}{3}x^5 + \frac{1}{3}x + \frac{1}{16}x^{-3} \right) dx$$

$$= 2\pi \left( \frac{1}{18}x^6 + \frac{1}{6}x^2 - \frac{1}{32}x^{-2} \right) \Big|_{1/2}^2$$

$$= 2\pi \left( \frac{32}{9} + \frac{2}{3} - \frac{1}{128} - \frac{1}{1,152} - \frac{1}{24} + \frac{1}{8} \right)$$

$$= 2\pi \left( \frac{4,096 + 768 - 9 - 1 - 48 + 144}{1,152} \right)$$

$$= 2\pi \left( \frac{4,950}{1,152} \right)$$

$$= \frac{275\pi}{32} \text{ unit}^2$$

$$S = 2\pi \int_{1/2}^2 \left( \frac{1}{3}x^3 + \frac{1}{4x} \right) \sqrt{1 + \left( \frac{4x^4 - 1}{4x^2} \right)^2} dx$$

$$S = 2\pi \int_a^b y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx$$

$$= 2\pi \int_{1/2}^2 \left( \frac{4x^4 + 3}{12x} \right) \sqrt{1 + \frac{16x^8 - 8x^4 + 1}{16x^4}} dx$$

$$= \frac{\pi}{6} \int_{1/2}^2 \left( \frac{4x^4 + 3}{x} \right) \sqrt{\frac{16x^8 + 8x^4 + 1}{16x^4}} dx$$

$$\begin{aligned}
&= \frac{\pi}{6} \int_{1/2}^2 \left( \frac{4x^4+3}{x} \right) \frac{\sqrt{(4x^4+1)^2}}{4x^2} dx \\
&= \frac{\pi}{24} \int_{1/2}^2 \left( \frac{4x^4+3}{x^3} \right) (4x^4+1) dx \\
&= \frac{\pi}{24} \int_{1/2}^2 (4x+3x^{-3})(4x^4+1) dx \\
&= \frac{\pi}{24} \int_{1/2}^2 (16x^5+16x+3x^{-3}) dx \\
&= \frac{\pi}{24} \left( \frac{8}{3}x^6+8x^2-\frac{3}{2}x^{-2} \right) \Big|_{1/2}^2 \\
&= \frac{\pi}{24} \left( \frac{512}{3}+32-\frac{3}{8}-\frac{1}{24}-2+6 \right) \\
&= \frac{\pi}{24} \left( \frac{4086}{24}+36 \right) \\
&= \frac{\pi}{24} \left( \frac{681}{4}+36 \right) \\
&= \frac{\pi}{24} \left( \frac{825}{4} \right) \\
&= \frac{275\pi}{32} \text{ unit}^2
\end{aligned}$$

### Exercise

Find the area of the surface generated by  $y = \sqrt{5x-x^2}$ ,  $1 \leq x \leq 4$ , revolved about  $x$ -axis

### Solution

$$\begin{aligned}
y' &= \frac{5-2x}{2\sqrt{5x-x^2}} \\
1 + \left( \frac{dy}{dx} \right)^2 &= 1 + \frac{(5-2x)^2}{4(5x-x^2)} \\
&= \frac{20x-4x^2+25-20x+4x^2}{4(5x-x^2)} \\
&= \frac{25}{4(5x-x^2)}
\end{aligned}$$

$$S = 2\pi \int_1^4 \sqrt{5x-x^2} \sqrt{\frac{25}{4(5x-x^2)}} dx$$

$$S = 2\pi \int_a^b y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx$$



$$\begin{aligned}
 &= 5\pi \int_1^4 dx \\
 &= 5\pi x \Big|_1^4 \\
 &= \underline{15\pi \text{ unit}^2}
 \end{aligned}$$

### Exercise

Set up and evaluate the definite integral for the area of the surface generated by revolving the curve about the

$$x\text{-axis} \quad y = \frac{1}{6}x^3 + \frac{1}{2x}, \quad 1 \leq x \leq 2$$

### Solution

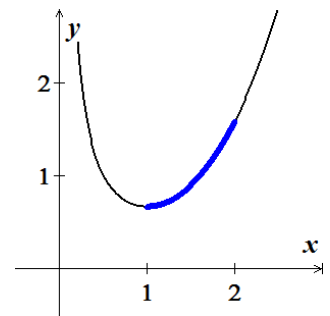
$$y' = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

$$a = \frac{1}{6}, \quad m = 3, \quad b = \frac{1}{2}, \quad n = -1$$

$$1. \quad m + n = 3 - 1 = 2 \quad \checkmark$$

$$2. \quad abmn = \frac{1}{6} \left( \frac{1}{2} \right) (3)(-1) = -\frac{1}{4} \quad \checkmark$$

$$f(x) = ax^m + bx^n$$



$$S = 2\pi \int_1^2 \left( \frac{1}{6}x^3 + \frac{1}{2x} \right) \left( \frac{1}{2}x^2 + \frac{1}{2x^2} \right) dx$$

$$S = 2\pi \int_a^b f(x) \overline{f'(x)} dx$$

$$= 2\pi \int_1^2 \left( \frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3} \right) dx$$

$$= 2\pi \left( \frac{1}{72}x^6 + \frac{1}{6}x^2 - \frac{1}{8x^2} \right) \Big|_1^2$$

$$= 2\pi \left( \frac{64}{72} + \frac{2}{3} - \frac{1}{32} - \frac{1}{72} - \frac{1}{6} + \frac{1}{8} \right)$$

$$= 2\pi \left( \frac{63}{72} + \frac{19}{32} \right)$$

$$= 2\pi \left( \frac{423}{288} \right)$$

$$= \underline{\frac{47\pi}{16} \text{ unit}^2}$$

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$$\begin{aligned}
 \sqrt{1 + (y')^2} &= \sqrt{1 + \frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4x^4}} \\
 &= \sqrt{\frac{1}{4}x^4 + \frac{1}{2} + \frac{1}{4x^4}}
 \end{aligned}$$

$$= \sqrt{\left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)^2}$$

$$= \frac{1}{2}x^2 + \frac{1}{2x^2}$$

$$S = 2\pi \int_1^2 \left(\frac{1}{6}x^3 + \frac{1}{2x}\right) \left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right) dx$$

$$= 2\pi \int_1^2 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3}\right) dx$$

$$= 2\pi \left( \frac{1}{72}x^6 + \frac{1}{6}x^2 - \frac{1}{8x^2} \right) \Big|_1^2$$

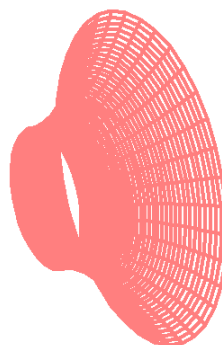
$$= 2\pi \left( \frac{64}{72} + \frac{2}{3} - \frac{1}{32} - \frac{1}{72} - \frac{1}{6} + \frac{1}{8} \right)$$

$$= 2\pi \left( \frac{63}{72} + \frac{19}{32} \right)$$

$$= 2\pi \left( \frac{423}{288} \right)$$

$$= \frac{47\pi}{16} \text{ unit}^2$$

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$



## Exercise

Set up and evaluate the definite integral for the area of the surface generated by revolving the curve about the

$$x\text{-axis} \quad y = \sqrt{4 - x^2}, \quad -1 \leq x \leq 1$$

## Solution

$$y' = \frac{-x}{\sqrt{4 - x^2}}$$

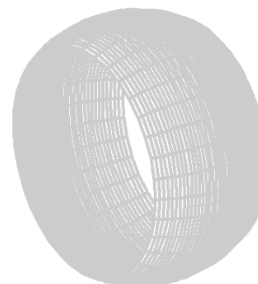
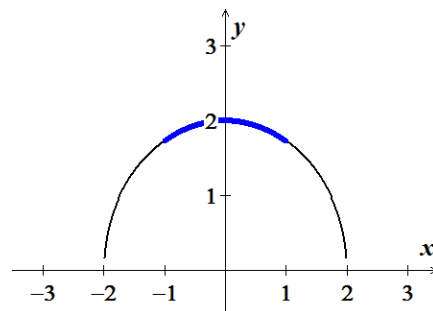
$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{4 - x^2}}$$

$$= \sqrt{\frac{4}{4 - x^2}}$$

$$S = 2\pi \int_{-1}^1 \sqrt{4 - x^2} \cdot \frac{2}{\sqrt{4 - x^2}} dx$$

$$= 4\pi \int_{-1}^1 dx$$

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$



$$= 4\pi x \Big|_{-1}^1$$

$$= 8\pi \text{ unit}^2$$

### Exercise

Set up and evaluate the definite integral for the area of the surface generated by revolving the curve about the

$x$ -axis  $y = \sqrt{9 - x^2}, \quad -2 \leq x \leq 2$

### Solution

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{9 - x^2}}$$

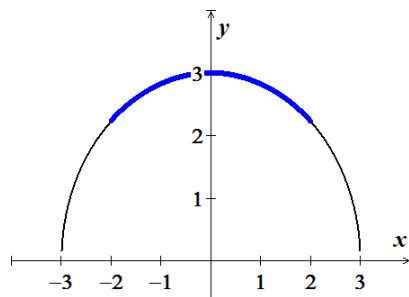
$$= \sqrt{\frac{9}{9 - x^2}}$$

$$S = 2\pi \int_{-2}^2 \sqrt{9 - x^2} \cdot \frac{3}{\sqrt{9 - x^2}} dx$$

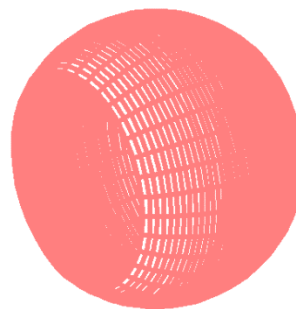
$$= 6\pi \int_{-2}^2 dx$$

$$= 6\pi x \Big|_{-2}^2$$

$$= 24\pi \text{ unit}^2$$



$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$



### Exercise

Set up and evaluate the definite integral for the area of the surface generated by revolving the curve about the  $y$ -axis

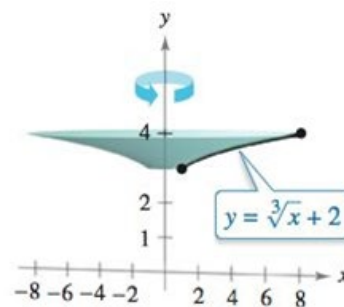
### Solution

$$y = \sqrt[3]{x} + 2$$

$$\sqrt[3]{x} = y - 2$$

$$x = (y - 2)^3$$

$$\frac{dx}{dy} = 3(y - 2)^2$$



$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+9(y-2)^4}$$

$$S = 2\pi \int_3^4 (y-2)^3 \sqrt{1+9(y-2)^4} dy$$

$$= \frac{\pi}{18} \int_3^4 \left(1+9(y-2)^4\right)^{1/2} d\left(1+9(y-2)^4\right)$$

$$= \frac{\pi}{27} \left(1+9(y-2)^4\right)^{3/2} \Big|_3^4$$

$$= \frac{\pi}{27} \left(145^{3/2} - 10^{3/2}\right)$$

$$= \frac{\pi}{27} \left(145\sqrt{145} - 10\sqrt{10}\right) \text{ unit}^2$$

$$S = 2\pi \int_a^b x(y) \sqrt{1+\left(\frac{dx}{dy}\right)^2} dy$$

$$y' = \frac{1}{3}x^{-2/3}$$

$$= \frac{1}{3x^{2/3}}$$

$$\begin{aligned} \sqrt{1+(y')^2} &= \sqrt{1+\frac{1}{9x^{4/3}}} \\ &= \frac{\sqrt{9x^{4/3}+1}}{3x^{2/3}} \end{aligned}$$

$$S = 2\pi \int_1^8 x \frac{\sqrt{9x^{4/3}+1}}{3x^{2/3}} dx$$

$$= \frac{2}{3}\pi \int_1^8 x^{1/3} \sqrt{9x^{4/3}+1} dx$$

$$= \frac{\pi}{18} \int_1^8 \left(9x^{4/3}+1\right)^{1/2} d\left(9x^{4/3}+1\right)$$

$$= \frac{\pi}{27} \left(9x^{4/3}+1\right)^{3/2} \Big|_1^8$$

$$= \frac{\pi}{27} \left( \left(72(8)^{1/3}+1\right)^{3/2} - 10^{3/2} \right)$$

$$= \frac{\pi}{27} \left(145\sqrt{145} - 10\sqrt{10}\right) \text{ unit}^2$$

$$S = 2\pi \int_a^b x \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx$$

### Exercise

Set up and evaluate the definite integral for the area of the surface generated by revolving the curve about the  $y$ -axis

#### Solution

$$y' = -2x$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + 4x^2}$$

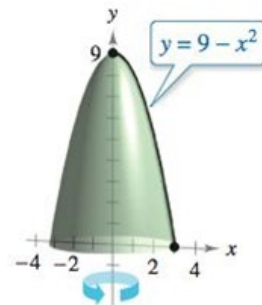
$$S = 2\pi \int_0^3 x \sqrt{1 + 4x^2} \, dx$$

$$= \frac{\pi}{4} \int_0^3 (1 + 4x^2)^{1/2} d(1 + 4x^2)$$

$$= \frac{\pi}{6} (1 + 4x^2)^{3/2} \Big|_0^3$$

$$= \frac{\pi}{6} (37\sqrt{37} - 1) \text{ unit}^2$$

$$S = 2\pi \int_a^b x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$



### Exercise

Find the area of the surface generated by  $y = (3x)^{1/3}$ ;  $0 \leq x \leq \frac{8}{3}$  about  $y$ -axis

#### Solution

$$3x = y^3 \rightarrow x = \frac{1}{3}y^3$$

$$x' = y^2$$

$$\begin{cases} x = 0 & \rightarrow y = 0 \\ x = \frac{8}{3} & \rightarrow y = \left(3 \cdot \frac{8}{3}\right)^{1/3} = 2 \end{cases}$$

$$S = 2\pi \int_0^2 \frac{1}{3}y^3 \sqrt{1 + y^4} \, dy$$

$$= \frac{\pi}{6} \int_0^2 (1 + y^4)^{1/2} d(1 + y^4)$$

$$= \frac{\pi}{9} (1 + y^4)^{3/2} \Big|_0^2$$

$$= \frac{\pi}{9} ((17)^{3/2} - 1)$$

$$= \frac{\pi}{9} (17\sqrt{17} - 1) \text{ unit}^2$$

$$S = 2\pi \int_c^d x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

### Exercise

Find the area of the surface generated of the curve  $y = 4x - 1$  between the points  $(1, 3)$  and  $(4, 15)$  about  $y$ -axis

### Solution

$$y = 4x - 1$$

$$x = \frac{1}{4}(y + 1)$$

$$x' = \frac{1}{4}$$

$$S = 2\pi \int_3^{15} \frac{1}{4}(y + 1) \sqrt{1 + \frac{1}{16}} dy$$

$$= \frac{\pi}{2} \sqrt{\frac{17}{16}} \int_3^{15} (y + 1) dy$$

$$= \frac{\pi\sqrt{17}}{8} \left( \frac{1}{2}y^2 + y \right) \Big|_3^{15}$$

$$= \frac{\pi\sqrt{17}}{8} \left( \frac{225}{2} + 15 - \frac{9}{2} - 3 \right)$$

$$= \frac{\pi\sqrt{17}}{8} (120)$$

$$= \underline{15\pi\sqrt{17} \text{ unit}^2}$$

$$S = 2\pi \int_c^d x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} dy$$

### Exercise

Find the area of the surface generated of the curve  $y = \frac{1}{2} \ln \left( 2x + \sqrt{4x^2 - 1} \right)$  between the points  $\left( \frac{1}{2}, 0 \right)$  and  $\left( \frac{17}{16}, \ln 2 \right)$  about  $y$ -axis

### Solution

$$2y = \ln \left( 2x + \sqrt{4x^2 - 1} \right)$$

$$\left( 2x + \sqrt{4x^2 - 1} \right)^2 = \left( e^{2y} \right)^2$$

$$4x^2 + 4x\sqrt{4x^2 - 1} + 4x^2 - 1 = e^{4y}$$

$$4x \left( 2x + \sqrt{4x^2 - 1} \right) = e^{4y} + 1$$

$$2x + \sqrt{4x^2 - 1} = e^{2y}$$

$$4x \left( e^{2y} \right) = e^{4y} + 1$$

$$x = \frac{e^{4y} + 1}{4e^{2y}}$$

$$= \frac{1}{4} (e^{2y} + e^{-2y})$$

$$x' = \frac{1}{2} (e^{2y} - e^{-2y})$$

$$a = \frac{1}{4}, \quad m = 2, \quad b = \frac{1}{4}, \quad n = -2$$

$$f(x) = ae^{mx} + be^{nx}$$

$$1. \quad m = -n \quad \checkmark$$

$$2. \quad abmn = \frac{1}{4} \left( \frac{1}{4} \right) (2)(-2) = -\frac{1}{4} \quad \checkmark$$

$$S = 2\pi \int_0^{\ln 2} \frac{1}{4} (e^{2y} + e^{-2y}) \cdot \frac{1}{2} (e^{2y} + e^{-2y}) dy$$

$$S = 2\pi \int_a^b f(x) \overline{f'(x)} dx$$

$$= \frac{\pi}{4} \int_0^{\ln 2} (e^{2y} + e^{-2y})^2 dy$$

$$= \frac{\pi}{4} \int_0^{\ln 2} (e^{4y} + 2 + e^{-4y}) dy$$

$$= \frac{\pi}{4} \left( \frac{1}{4} e^{4y} + 2y - \frac{1}{4} e^{-4y} \right) \Big|_0^{\ln 2}$$

$$= \frac{\pi}{4} \left( \frac{1}{4} e^{4 \ln 2} + 2 \ln 2 - \frac{1}{4} e^{-4 \ln 2} - \frac{1}{4} + \frac{1}{4} \right)$$

$$= \frac{\pi}{4} \left( \frac{1}{4} e^{\ln 2^4} + 2 \ln 2 - \frac{1}{4} e^{\ln 2^{-4}} \right)$$

$$= \frac{\pi}{4} \left( \frac{1}{4} 2^4 + 2 \ln 2 - \frac{1}{4} 2^{-4} \right)$$

$$= \frac{\pi}{4} \left( 4 + 2 \ln 2 - \frac{1}{64} \right)$$

$$= \frac{\pi}{4} \left( \frac{255}{64} + 2 \ln 2 \right) \quad \text{unit}^2$$

$$S = 2\pi \int_0^{\ln 2} \frac{1}{4} (e^{2y} + e^{-2y}) \sqrt{1 + \frac{1}{4} (e^{2y} - e^{-2y})^2} dy$$

$$S = 2\pi \int_c^d x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} dy$$

$$= \frac{\pi}{4} \int_0^{\ln 2} (e^{2y} + e^{-2y}) \sqrt{4 + e^{4y} - 2 + e^{-4y}} dy$$

$$= \frac{\pi}{4} \int_0^{\ln 2} (e^{2y} + e^{-2y}) \sqrt{(e^{2y} + e^{-2y})^2} dy$$

$$\begin{aligned}
&= \frac{\pi}{4} \int_0^{\ln 2} \left( e^{2y} + e^{-2y} \right)^2 dy \\
&= \frac{\pi}{4} \int_0^{\ln 2} \left( e^{4y} + 2 + e^{-4y} \right) dy \\
&= \frac{\pi}{4} \left( \frac{1}{4} e^{4y} + 2y - \frac{1}{4} e^{-4y} \right) \Big|_0^{\ln 2} \\
&= \frac{\pi}{4} \left( \frac{1}{4} e^{4 \ln 2} + 2 \ln 2 - \frac{1}{4} e^{-4 \ln 2} - \frac{1}{4} + \frac{1}{4} \right) \\
&= \frac{\pi}{4} \left( \frac{1}{4} e^{\ln 2^4} + 2 \ln 2 - \frac{1}{4} e^{\ln 2^{-4}} \right) \\
&= \frac{\pi}{4} \left( \frac{1}{4} 2^4 + 2 \ln 2 - \frac{1}{4} 2^{-4} \right) \\
&= \frac{\pi}{4} \left( 4 + 2 \ln 2 - \frac{1}{64} \right) \\
&= \frac{\pi}{4} \left( \frac{255}{64} + 2 \ln 2 \right) \text{ unit}^2
\end{aligned}$$

### Exercise

Find the area of the surface generated by  $x = \sqrt{12y - y^2}$ ;  $2 \leq y \leq 10$  about  $y$ -axis

### Solution

$$\begin{aligned}
x' &= \frac{6 - y}{\sqrt{12y - y^2}} \\
S &= 2\pi \int_2^{10} \sqrt{12y - y^2} \sqrt{1 + \frac{(6 - y)^2}{12y - y^2}} dy \\
&= 2\pi \int_2^{10} \sqrt{12y - y^2 + 36 - 12y + y^2} dy \\
&= 12\pi \int_2^{10} dy \\
&= 12\pi y \Big|_2^{10} \\
&= 96\pi \text{ unit}^2
\end{aligned}$$

$$S = 2\pi \int_c^d x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} dy$$



### Exercise

Find the area of the surface generated by  $x = 4y^{3/2} - \frac{1}{12}y^{1/2}$ ;  $1 \leq y \leq 4$  about  $y$ -axis

### Solution

$$x' = 6y^{1/2} - \frac{1}{24\sqrt{y}}$$

$$= \frac{144y-1}{24\sqrt{y}}$$

$$a = 4, \quad m = \frac{3}{2}, \quad b = -\frac{1}{12}, \quad n = \frac{1}{2}$$

$$f(x) = ax^m + bx^n$$

$$1. \quad m+n = \frac{3}{2} + \frac{1}{2} = 2 \quad \checkmark$$

$$2. \quad abmn = 4\left(-\frac{1}{12}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4} \quad \checkmark$$

$$S = 2\pi \int_1^4 \left(4y^{3/2} - \frac{1}{12}y^{1/2}\right) \left(6y^{1/2} - \frac{1}{24}y^{-1/2}\right) dy$$

$$S = 2\pi \int_a^b f(x) \overline{f'(x)} dx$$

$$= \frac{\pi}{144} \int_1^4 (48y-1)(144y+1) dy$$

$$= \frac{\pi}{144} \int_1^4 (6,912y^2 - 96y - 1) dy$$

$$= \frac{\pi}{144} \left( 2304y^3 - 48y^2 - y \right) \Big|_1^4$$

$$= \frac{\pi}{144} (147,456 - 768 - 4 - 2304 + 48 + 1)$$

$$= \frac{144,429\pi}{144}$$

$$= \frac{48,143\pi}{48} \quad \text{unit}^2$$

$$S = 2\pi \int_1^4 \left(4y^{3/2} - \frac{1}{12}y^{1/2}\right) \sqrt{1 + \frac{(144y-1)^2}{576y}} dy$$

$$S = 2\pi \int_c^d x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$= 2\pi \int_1^4 \left(4y^{3/2} - \frac{1}{12}y^{1/2}\right) \sqrt{\frac{576y + (144y)^2 - 288y + 1}{576y}} dy$$

$$= \frac{\pi}{12} \int_1^4 \left(4y^{3/2} - \frac{1}{12}y^{1/2}\right) \frac{1}{\sqrt{y}} \sqrt{(144y+1)^2} dy$$

$$\begin{aligned}
&= \frac{\pi}{144} \int_1^4 (48y-1)(144y+1) dy \\
&= \frac{\pi}{144} \int_1^4 (6,912y^2 - 96y - 1) dy \\
&= \frac{\pi}{144} \left( 2304y^3 - 48y^2 - y \right) \Big|_1^4 \\
&= \frac{\pi}{144} (147,456 - 768 - 4 - 2304 + 48 + 1) \\
&= \frac{144,429\pi}{144} \\
&= \frac{48,143 \pi}{48} \text{ unit}^2
\end{aligned}$$

### Exercise

Set up and evaluate the definite integral for the area of the surface generated by revolving the curve about the  $y$ -axis

$$y = 1 - \frac{1}{4}x^2, \quad 0 \leq x \leq 2$$

### Solution

$$y' = -\frac{1}{2}x$$

$$\begin{aligned}
\sqrt{1+(y')^2} &= \sqrt{1+\frac{x^2}{4}} \\
&= \frac{1}{2}\sqrt{4+x^2}
\end{aligned}$$

$$S = 2\pi \int_0^2 x \frac{\sqrt{4+x^2}}{2} dx$$

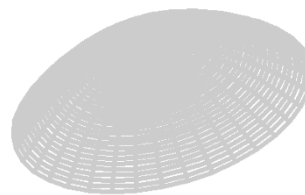
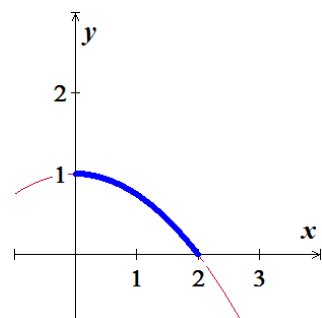
$$S = 2\pi \int_a^b x \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx$$

$$= \frac{\pi}{2} \int_0^2 (4+x^2)^{1/2} d(4+x^2)$$

$$= \frac{\pi}{3} (4+x^2)^{3/2} \Big|_0^2$$

$$= \frac{\pi}{3} (8^{3/2} - 4^{3/2})$$

$$= \frac{\pi}{3} (16\sqrt{2} - 8) \text{ unit}^2 \approx 15.318 \text{ unit}^2$$



### Exercise

Set up and evaluate the definite integral for the area of the surface generated by revolving the curve about the  $y$ -axis

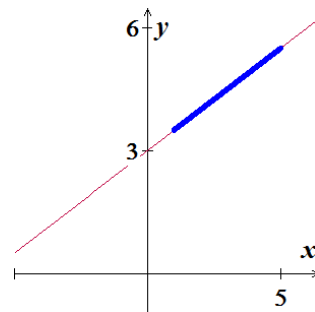
$$y = \frac{1}{2}x + 3, \quad 1 \leq x \leq 5$$

### Solution

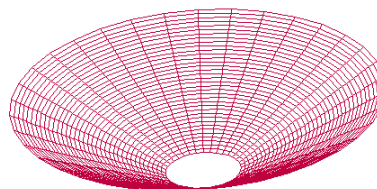
$$y' = \frac{1}{2}$$

$$\begin{aligned}\sqrt{1+(y')^2} &= \sqrt{1+\frac{1}{4}} \\ &= \frac{\sqrt{5}}{2}\end{aligned}$$

$$\begin{aligned}S &= \pi\sqrt{5} \int_1^5 x \, dx \\ &= \pi\sqrt{5} \left( \frac{1}{2}x^2 \right) \Big|_1^5 \\ &= \frac{\sqrt{5}}{2}\pi(25-1) \\ &= \underline{12\pi\sqrt{5} \text{ unit}^2}\end{aligned}$$



$$S = 2\pi \int_a^b x \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx$$



### Exercise

A right circular cone is generated by revolving the region bounded by  $y = \frac{3}{4}x$ ,  $y = 3$ , and  $x = 0$  about the  $y$ -axis. Find the lateral surface area of the cone.

### Solution

$$y' = \frac{3}{4}$$

$$\begin{aligned}\sqrt{1+(y')^2} &= \sqrt{1+\frac{9}{16}} \\ &= \frac{5}{4}\end{aligned}$$

$$y = 3 = \frac{3}{4}x \Rightarrow \underline{x = 4}$$

$$\begin{aligned}S &= \frac{5\pi}{2} \int_0^4 x \, dx \\ &= \frac{5\pi}{4} x^2 \Big|_0^4 \\ &= \underline{20\pi \text{ unit}^2}\end{aligned}$$

$$S = 2\pi \int_a^b x \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx$$

### Exercise

A right circular cone is generated by revolving the region bounded by  $y = \frac{h}{r}x$ ,  $y = h$ , and  $x = 0$  about the  $y$ -axis. Verify that the lateral surface area of the cone is  $S = \pi r \sqrt{r^2 + h^2}$

### Solution

$$y' = \frac{h}{r}$$

$$\begin{aligned}\sqrt{1+(y')^2} &= \sqrt{1+\frac{h^2}{r^2}} \\ &= \frac{\sqrt{r^2+h^2}}{r}\end{aligned}$$

$$y = h = \frac{h}{r}x \Rightarrow x = r$$

$$S = 2\pi \int_0^r x \frac{\sqrt{r^2+h^2}}{r} dx$$

$$= \frac{\pi \sqrt{r^2+h^2}}{r} \left( x^2 \right) \Big|_0^r$$

$$= \pi r \sqrt{r^2+h^2} \text{ unit}^2$$

$$S = 2\pi \int_a^b x \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx$$

### Exercise

Find the area of the zone of a sphere formed by revolving the graph of  $y = \sqrt{9-x^2}$ ,  $0 \leq x \leq 2$ , about the  $y$ -axis

### Solution

$$y' = \frac{-x}{\sqrt{9-x^2}}$$

$$\begin{aligned}\sqrt{1+(y')^2} &= \sqrt{1+\frac{x^2}{9-x^2}} \\ &= \frac{3}{\sqrt{9-x^2}}\end{aligned}$$

$$S = 2\pi \int_0^2 x \frac{3}{\sqrt{9-x^2}} dx$$

$$= -3\pi \int_0^2 (9-x^2)^{-1/2} d(9-x^2)$$

$$S = 2\pi \int_a^b x \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx$$

$$\begin{aligned}
&= -6\pi \left(9 - x^2\right)^{1/2} \Big|_0^2 \\
&= -6\pi(\sqrt{5} - 3) \\
&= \underline{6\pi(3 - \sqrt{5}) \text{ unit}^2}
\end{aligned}$$

### Exercise

Find the area of the zone of a sphere formed by revolving the graph of  $y = \sqrt{r^2 - x^2}$ ,  $0 \leq x \leq a$ , about the  $y$ -axis. Assume that  $a < r$ .

### Solution

$$y' = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$\begin{aligned}
\sqrt{1 + (y')^2} &= \sqrt{1 + \frac{x^2}{r^2 - x^2}} \\
&= \frac{r}{\sqrt{r^2 - x^2}}
\end{aligned}$$

$$S = 2\pi \int_0^a x \frac{r}{\sqrt{r^2 - x^2}} dx$$

$$= -\pi r \int_0^a (r^2 - x^2)^{-1/2} d(r^2 - x^2)$$

$$= -2\pi r \sqrt{r^2 - x^2} \Big|_0^a$$

$$= -2\pi r (\sqrt{r^2 - a^2} - r)$$

$$= \underline{2\pi r (r - \sqrt{r^2 - a^2}) \text{ unit}^2}$$

$$S = 2\pi \int_a^b x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

### Exercise

Find the area of the surface generated by the curve  $y = 1 + \sqrt{1 - x^2}$  between the points  $(1, 1)$  and  $\left(\frac{\sqrt{3}}{1}, \frac{3}{2}\right)$  about  $y$ -axis

### Solution

$$\left(\sqrt{1 - x^2}\right)^2 = (y - 1)^2$$

$$1 - x^2 = y^2 - 2y + 1$$

$$x = \sqrt{2y - y^2}$$

$$x' = \frac{1 - y}{\sqrt{2y - y^2}}$$

$$S = 2\pi \int_1^{3/2} \sqrt{2y - y^2} \sqrt{1 + \frac{(1 - y)^2}{2y - y^2}} dy$$

$$= 2\pi \int_1^{3/2} \sqrt{2y - y^2 + 1 - 2y + y^2} dy$$

$$= 2\pi \int_1^{3/2} dy$$

$$= 2\pi y \Big|_1^{3/2}$$

$$= \pi \text{ unit}^2$$

$$S = 2\pi \int_a^b x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

### Exercise

Find the area of the surface generated by  $y = \frac{1}{3}x^3$ ,  $0 \leq x \leq 1$ ,  $x$ -axis

### Solution

$$\sqrt{1 + (y')^2} = \sqrt{1 + x^4}$$

$$S = 2\pi \int_0^1 \frac{1}{3}x^3 \sqrt{1 + x^4} dx$$

$$= \frac{\pi}{6} \int_0^1 (1 + x^4)^{1/2} d(1 + x^4)$$

$$= \frac{\pi}{9} (1 + x^4)^{3/2} \Big|_0^1$$

$$= \frac{\pi}{9} ((2)^{3/2} - 1)$$

$$= \frac{\pi}{9} (2\sqrt{2} - 1) \text{ unit}^2$$

### Exercise

Find the area of the surface generated by  $x = \sqrt{4y - y^2}$ ,  $1 \leq y \leq 2$ ;  $y$ -axis

### Solution

$$x' = \frac{1}{2}(4 - 2y)(4y - y^2)^{-1/2}$$

$$= (2 - y)(4y - y^2)^{-1/2}$$

$$\sqrt{1 + (x')^2} = \sqrt{1 + (2 - y)^2 (4y - y^2)^{-1}}$$

$$= \sqrt{1 + \frac{4 - 4y + y^2}{4y - y^2}}$$

$$= \sqrt{\frac{4}{4y - y^2}}$$

$$= \frac{2}{\sqrt{4y - y^2}}$$

$$S = 2\pi \int_1^2 \sqrt{4y - y^2} \frac{2}{\sqrt{4y - y^2}} dy$$

$$= 4\pi \int_1^2 dy$$

$$= 2\pi(2 - 1)$$

$$= 4\pi \text{ unit}^2$$

### Exercise

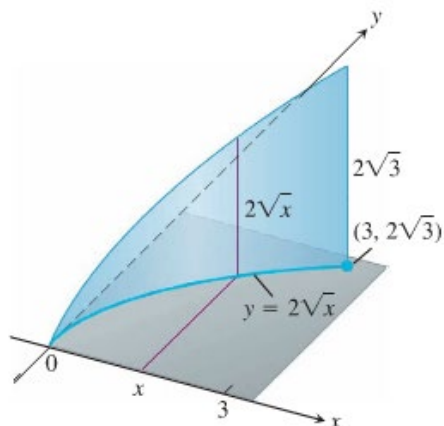
At points on the curve  $y = 2\sqrt{x}$ , line segments of length  $h = y$  are drawn perpendicular to the  $xy$ -plane. Find the area of the surface formed by these perpendiculars from  $(0, 0)$  to  $(3, 2\sqrt{3})$

### Solution

$$\sqrt{1 + (y')^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1 + \frac{1}{x}}$$

$$= \frac{\sqrt{x+1}}{\sqrt{x}}$$



$$\begin{aligned}
 S &= 2\pi \int_0^3 2\sqrt{x} \frac{\sqrt{x+1}}{\sqrt{x}} dx \\
 &= 4\pi \int_0^3 (1+x)^{1/2} d(1+x) \\
 &= \frac{8\pi}{3} (1+x)^{3/2} \Big|_0^3 \\
 &= \frac{8\pi}{3} \left( (4)^{3/2} - 1 \right) \\
 &= \frac{8\pi}{3} (8-1) \\
 &= \frac{56\pi}{3} \text{ unit}^2
 \end{aligned}$$

### Exercise

Find the area of the surface generated by  $x = 2\sqrt{4-y}$   $0 \leq y \leq \frac{15}{4}$ ,  $y$ -axis

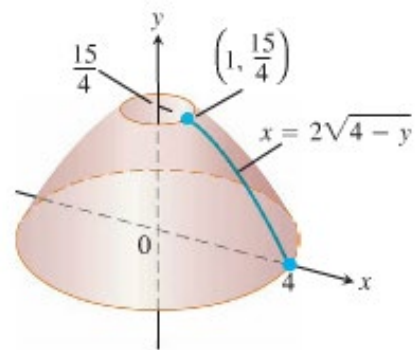
### Solution

$$\frac{dy}{dx} = 2 \frac{1}{2} (4-y)^{-1/2} (-1) = \frac{-1}{\sqrt{4-y}}$$

$$\begin{aligned}
 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{1 + \frac{1}{4-y}} \\
 &= \sqrt{\frac{4-y+1}{4-y}} \\
 &= \sqrt{\frac{5-y}{4-y}}
 \end{aligned}$$

$$\begin{aligned}
 S &= 2\pi \int_0^{15/4} 2\sqrt{4-y} \frac{\sqrt{5-y}}{\sqrt{4-y}} dy \\
 &= 4\pi \int_0^{15/4} \sqrt{5-y} dy \\
 &= -4\pi \int_0^{15/4} (5-y)^{1/2} d(5-y) \\
 &= -\frac{8\pi}{3} (5-y)^{3/2} \Big|_0^{15/4}
 \end{aligned}$$

$$d(5-y) = -dy$$





$$\begin{aligned}
&= -\frac{8\pi}{3} \left[ \left( 5 - \frac{15}{4} \right)^{3/2} - (5-0)^{3/2} \right] \\
&= -\frac{8\pi}{3} \left( \left( \frac{5}{4} \right)^{3/2} - 5^{3/2} \right) \\
&= -\frac{8\pi}{3} \left( \frac{5\sqrt{5}}{8} - 5\sqrt{5} \right) \\
&= -\frac{8\pi}{3} 5\sqrt{5} \left( \frac{1}{8} - 1 \right) \\
&= -\frac{8\pi}{3} 5\sqrt{5} \left( -\frac{7}{8} \right) \\
&= \underline{\underline{\frac{35\pi\sqrt{5}}{3} \text{ unit}^2}}
\end{aligned}$$

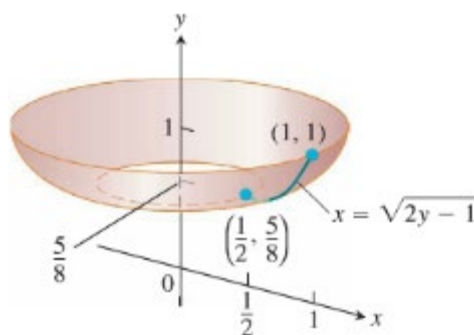
### Exercise

Find the area of the surface generated by  $x = \sqrt{2y-1}$   $\frac{5}{8} \leq y \leq 1$ ,  $y$ -axis

### Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2}(2y-1)^{-1/2} (2) \\
&= \frac{1}{\sqrt{2y-1}}
\end{aligned}$$

$$\begin{aligned}
\sqrt{1 + \left( \frac{dy}{dx} \right)^2} &= \sqrt{1 + \frac{1}{2y-1}} \\
&= \sqrt{\frac{2y}{2y-1}}
\end{aligned}$$



$$\begin{aligned}
S &= 2\pi \int_{5/8}^1 \sqrt{2y-1} \frac{\sqrt{2y}}{\sqrt{2y-1}} dy \\
&= 2\pi \int_{5/8}^1 \sqrt{2y} dy \\
&= 2\pi\sqrt{2} \int_{5/8}^1 y^{1/2} dy \\
&= \frac{4\pi\sqrt{2}}{3} \left( y^{3/2} \right) \Big|_{5/8}^1 \\
&= \frac{4\pi\sqrt{2}}{3} \left( 1 - \frac{5\sqrt{5}}{16\sqrt{2}} \right)
\end{aligned}$$

$$= \frac{4\pi\sqrt{2}}{3} \left( \frac{16\sqrt{2}-5\sqrt{5}}{16\sqrt{2}} \right)$$

$$= \frac{\pi}{12} (16\sqrt{2}-5\sqrt{5}) \text{ unit}^2 \Big|$$

### Exercise

$y = \frac{1}{3}(x^2 + 2)^{3/2}$ ,  $0 \leq x \leq \sqrt{2}$ ;  $y$ -axis (Hint: Express  $ds = \sqrt{dx^2 + dy^2}$  in terms of  $dx$ , and evaluate the integral  $S = \int 2\pi x \, ds$  with appropriate limits.)

### Solution

$$dy = \frac{1}{3} \frac{3}{2} (x^2 + 2)^{1/2} (2x) dx$$

$$= x\sqrt{x^2 + 2} \, dx$$

$$ds = \sqrt{dx^2 + \left(x\sqrt{x^2 + 2} \, dx\right)^2}$$

$$= \sqrt{dx^2 + x^2(x^2 + 2)dx^2}$$

$$= \sqrt{1 + x^4 + 2x^2} \, dx$$

$$= \sqrt{(1 + x^2)^2} \, dx$$

$$= (1 + x^2) \, dx$$

$$S = \int 2\pi x \, ds$$

$$= 2\pi \int_0^{\sqrt{2}} x(1 + x^2) \, dx$$

$$= \pi \int_0^{\sqrt{2}} (1 + x^2) \, d(1 + x^2)$$

$$= \frac{\pi}{2} (1 + x^2)^2 \Big|_0^{\sqrt{2}}$$

$$= \frac{\pi}{2} (9 - 1)$$

$$= 4\pi \text{ unit}^2 \Big|$$

$$d(1 + x^2) = 2x dx$$

### Exercise

Find the area of the surface generated by revolving the curve  $x = \frac{1}{2}(e^y + e^{-y})$ ,  $0 \leq y \leq \ln 2$ , about  $y$ -axis

### Solution

$$x' = \frac{1}{2}(e^y - e^{-y})$$

$$a = \frac{1}{2}, \quad m = 1, \quad b = \frac{1}{2}, \quad n = -1$$

$$f(x) = ae^{mx} + be^{nx}$$

$$1. \quad m = -n \quad \checkmark$$

$$2. \quad abmn = \frac{1}{2} \left( \frac{1}{2} \right) (1)(-1) = -\frac{1}{4} \quad \checkmark$$

$$S = 2\pi \int_0^{\ln 2} \frac{1}{2}(e^y + e^{-y}) \frac{1}{2}(e^y + e^{-y}) dy$$

$$S = 2\pi \int_a^b f(x) \overline{f'(x)} dx$$

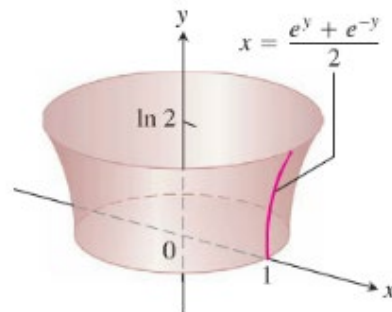
$$= \frac{\pi}{2} \int_0^{\ln 2} (e^{2y} + e^{-2y} + 2) dy$$

$$= \frac{\pi}{2} \left( \frac{1}{2}e^{2y} - \frac{1}{2}e^{-2y} + 2y \right) \Big|_0^{\ln 2}$$

$$= \frac{\pi}{2} \left( \frac{1}{2}e^{2\ln 2} - \frac{1}{2}e^{-2\ln 2} + 2\ln 2 - \frac{1}{2}e + \frac{1}{2} \right)$$

$$= \frac{\pi}{2} \left( \frac{1}{2} \cdot 4 - \frac{1}{2} \cdot \frac{1}{4} + 2\ln 2 \right)$$

$$= \frac{\pi}{2} \left( \frac{15}{8} + 2\ln 2 \right) \quad \text{unit}^2$$



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$$S = 2\pi \int_0^{\ln 2} \frac{1}{2}(e^y + e^{-y}) \sqrt{1 + \left( \frac{e^y - e^{-y}}{2} \right)^2} dy$$

$$= \pi \int_0^{\ln 2} (e^y + e^{-y}) \sqrt{1 + \frac{e^{2y} + e^{-2y} - 2}{4}} dy$$

$$= \pi \int_0^{\ln 2} (e^y + e^{-y}) \sqrt{\frac{4 + e^{2y} + e^{-2y} - 2}{4}} dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} (e^y + e^{-y}) \sqrt{e^{2y} + e^{-2y} + 2} dy$$

$$\begin{aligned}
&= \frac{\pi}{2} \int_0^{\ln 2} (e^y + e^{-y}) \sqrt{(e^y + e^{-y})^2} dy \\
&= \frac{\pi}{2} \int_0^{\ln 2} (e^y + e^{-y})^2 dy \\
&= \frac{\pi}{2} \int_0^{\ln 2} (e^{2y} + e^{-2y} + 2) dy \\
&= \frac{\pi}{2} \left( \frac{1}{2} e^{2y} - \frac{1}{2} e^{-2y} + 2y \right) \Big|_0^{\ln 2} \\
&= \frac{\pi}{2} \left( \frac{1}{2} e^{2 \ln 2} - \frac{1}{2} e^{-2 \ln 2} + 2 \ln 2 - \frac{1}{2} e + \frac{1}{2} \right) \\
&= \frac{\pi}{2} \left( \frac{1}{2} \cdot 4 - \frac{1}{2} \cdot \frac{1}{4} + 2 \ln 2 \right) \\
&= \frac{\pi}{2} \left( \frac{15}{8} + 2 \ln 2 \right) \quad \text{unit}^2
\end{aligned}$$

### Exercise

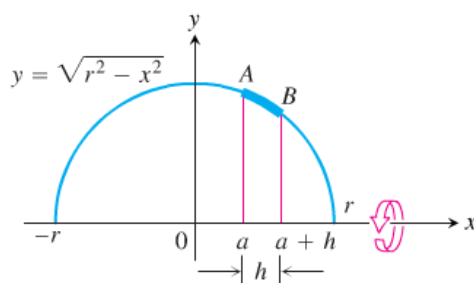
Did you know that if you can cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle  $y = \sqrt{r^2 - x^2}$  shown here is revolved about the  $x$ -axis to generate a sphere. Let  $AB$  be an arc of the semicircle that lies above an interval of length  $h$  on the  $x$ -axis. Show that the area swept out by  $AB$  does not depend on the location of the interval. (It does depend on the length of the interval.)

### Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-2x}{2\sqrt{r^2 - x^2}} \\
&= \frac{-x}{\sqrt{r^2 - x^2}}
\end{aligned}$$

$$\begin{aligned}
\sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{1 + \frac{x^2}{r^2 - x^2}} \\
&= \sqrt{\frac{r^2}{r^2 - x^2}}
\end{aligned}$$

$$S = 2\pi \int_a^{a+h} \sqrt{r^2 - x^2} \cdot \frac{r}{\sqrt{r^2 - x^2}} dx$$



$$\begin{aligned}
&= 2\pi r \int_a^{a+h} dx \\
&= 2\pi r x \Big|_a^{a+h} \\
&= 2\pi r(a+h-a) \\
&= \underline{2\pi r h \text{ unit}^2}
\end{aligned}$$

### Example

The curved surface of a funnel is generated by revolving the graph of  $y = f(x) = x^3 + \frac{1}{12x}$  on the interval  $[1, 2]$  about the  $x$ -axis. Approximately what volume of paint is needed to cover the outside of the funnel with a layer of paint 0.05 cm thick? Assume that  $x$  and  $y$  measured in centimeters.

### Solution

$$f'(x) = 3x^2 - \frac{1}{12x^2}$$

$$a = 1, \quad m = 3, \quad b = \frac{1}{12}, \quad n = -1$$

$$f(x) = ax^m + bx^n$$

$$1. \quad m + n = 3 - 1 = 2 \quad \checkmark$$

$$2. \quad abmn = (1)\left(\frac{1}{12}\right)(3)(-1) = -\frac{1}{4} \quad \checkmark$$

$$S = 2\pi \int_1^2 \left(x^3 + \frac{1}{12x}\right) \left(3x^2 - \frac{1}{12x^2}\right) dx$$

$$S = 2\pi \int_a^b x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

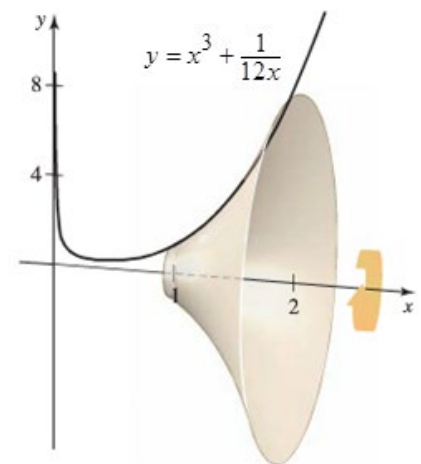
$$= 2\pi \int_1^2 \left(3x^5 + \frac{x}{3} + \frac{1}{144}x^{-3}\right) dx$$

$$= 2\pi \left( \frac{1}{2}x^6 + \frac{1}{6}x^2 - \frac{1}{288}x^{-2} \right) \Big|_1^2$$

$$= 2\pi \left( 32 + \frac{2}{3} - \frac{1}{1152} - \frac{1}{2} - \frac{1}{6} + \frac{1}{288} \right)$$

$$= 2\pi \left( \frac{36864 + 768 - 1 - 576 - 192 + 4}{1152} \right)$$

$$= \underline{\frac{12,289}{192} \pi \text{ cm}^2}$$



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$$1 + f'(x)^2 = 1 + \left(3x^2 - \frac{1}{12x^2}\right)^2$$

$$\begin{aligned}
&= 1 + 9x^4 - \frac{1}{2} + \frac{1}{144x^4} \\
&= 9x^4 + \frac{1}{2} + \frac{1}{144x^4} \\
&= \left( 3x^2 + \frac{1}{12x^2} \right)^2
\end{aligned}$$

$$\begin{aligned}
S &= 2\pi \int_1^2 \left( x^3 + \frac{1}{12x} \right) \sqrt{\left( 3x^2 + \frac{1}{12x^2} \right)^2} dx \\
&= 2\pi \int_1^2 \left( x^3 + \frac{1}{12x} \right) \left( 3x^2 + \frac{1}{12x^2} \right) dx \\
&= 2\pi \int_1^2 \left( 3x^5 + \frac{x}{3} + \frac{1}{144} x^{-3} \right) dx \\
&= 2\pi \left( \frac{1}{2} x^6 + \frac{1}{6} x^2 - \frac{1}{288} x^{-2} \right) \Big|_1^2 \\
&= 2\pi \left( 32 + \frac{2}{3} - \frac{1}{1152} - \frac{1}{2} - \frac{1}{6} + \frac{1}{288} \right) \\
&= 2\pi \left( \frac{36,864 + 768 - 1 - 576 - 192 + 4}{11,52} \right) \\
&= \frac{12,289}{192} \pi \text{ cm}^2
\end{aligned}$$

Because the paint layer is 0.05 cm thick, the approximate volume of paint needed is

$$\begin{aligned}
&= \left( \frac{12,289}{192} \pi \text{ cm}^2 \right) (0.05 \text{ cm}) \\
&\approx 10.1 \text{ cm}^3
\end{aligned}$$

### Exercise

When the circle  $x^2 + (y - a)^2 = r^2$  on the interval  $[-r, r]$  is revolved about the  $x$ -axis, the result is the surface of a torus, where  $0 < r < a$ . Show that the surface area of the torus is  $S = 4\pi^2 ar$ .

### Solution

$$\begin{aligned}
x^2 + (y - a)^2 &= r^2 \\
(y - a)^2 &= r^2 - x^2 \\
y &= a \pm \sqrt{r^2 - x^2}
\end{aligned}$$

$$f(x) = a + \sqrt{r^2 - x^2}$$

$$\begin{aligned} 1 + f'(x)^2 &= 1 + \left( \frac{-x}{\sqrt{r^2 - x^2}} \right)^2 \\ &= 1 + \frac{x^2}{r^2 - x^2} \\ &= \frac{r^2}{r^2 - x^2} \end{aligned}$$

$$\begin{aligned} S_1 &= 2\pi \int_{-r}^r \left( a + \sqrt{r^2 - x^2} \right) \frac{r}{\sqrt{r^2 - x^2}} dx \\ &= 4\pi \int_0^r \left( \frac{ar}{\sqrt{r^2 - x^2}} + r \right) dx \\ &= 4\pi \left( ar \sin^{-1}\left(\frac{x}{r}\right) + rx \right) \Big|_0^r \\ &= 4\pi \left( ar \frac{\pi}{2} + r^2 \right) \\ &= \underline{2\pi^2 ar + 4\pi r^2 \text{ unit}^2} \end{aligned}$$

$$\begin{aligned} S_2 &= 2\pi \int_{-r}^r \left( a - \sqrt{r^2 - x^2} \right) \frac{r}{\sqrt{r^2 - x^2}} dx \\ &= 4\pi \int_0^r \left( \frac{ar}{\sqrt{r^2 - x^2}} - r \right) dx \\ &= 4\pi \left( ar \sin^{-1}\left(\frac{x}{r}\right) - rx \right) \Big|_0^r \\ &= 4\pi \left( ar \frac{\pi}{2} - r^2 \right) \\ &= \underline{2\pi^2 ar - 4\pi r^2 \text{ unit}^2} \end{aligned}$$

$$\begin{aligned} S &= 2\pi^2 ar + 4\pi r^2 + 2\pi^2 ar - 4\pi r^2 \\ &= \underline{4\pi^2 ar \text{ unit}^2} \end{aligned}$$

### Exercise

A 1.5-mm layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the curve  $y = \sqrt{8x - x^2}$  on the interval  $[1, 7]$  is revolved about the  $x$ -axis. Assume  $x$  and  $y$  are in *meters*.

### Solution

$$y' = \frac{4-x}{\sqrt{8x-x^2}}$$

$$\begin{aligned} S &= 2\pi \int_1^7 \sqrt{8x-x^2} \sqrt{1 + \frac{(4-x)^2}{8x-x^2}} dx \\ &= 2\pi \int_1^7 \sqrt{8x-x^2} \frac{\sqrt{8x-x^2+16-8x+x^2}}{\sqrt{8x-x^2}} dx \\ &= 2\pi \int_1^7 \sqrt{16} dx \\ &= 8\pi x \Big|_1^7 \\ &= 48\pi \text{ m}^2 \end{aligned}$$

$$S = 2\pi \int_a^b f(x) \sqrt{1 + \left(\frac{df}{dx}\right)^2} dx$$

The volume of paint required to cover the surface to a thickness 0.0015 *m* is

$$V = 48\pi (0.0015)$$

$$\approx 0.226195 \text{ m}^3$$

$$1 \text{ m}^3 = 264.172052 \text{ gal}$$

$$= 0.226195 \times 264.172052$$

$$\approx 59.75 \text{ gal}$$

### Exercise

A 1.5-mm layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the upper portion of the circle  $x^2 + y^2 = 100$  on the interval  $[-8, 8]$  is revolved about the  $x$ -axis. Assume  $x$  and  $y$  are in *meters*.

### Solution

$$y = \sqrt{100 - x^2}$$

$$y' = \frac{-x}{\sqrt{100 - x^2}}$$



$$\begin{aligned}
 S &= 2\pi \int_{-8}^8 \sqrt{100-x^2} \sqrt{1+\frac{x^2}{100-x^2}} dx \\
 &= 2\pi \int_{-8}^8 \sqrt{100-x^2} \frac{\sqrt{100-x^2+x^2}}{\sqrt{100-x^2}} dx \\
 &= 20\pi \int_{-8}^8 dx \\
 &= 20\pi x \Big|_{-8}^8 \\
 &= \underline{320\pi \text{ m}^2}
 \end{aligned}$$

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

The volume of paint required to cover the surface to a thickness 0.0015 m is

$$\begin{aligned}
 V &= 320\pi(0.0015) \\
 &\approx \underline{1.507965 \text{ m}^3} \qquad 1 \text{ m}^3 = 264.172052 \text{ gal} \\
 &= 1.507965 \times 264.172052 \\
 &\approx \underline{398.36 \text{ gal}}
 \end{aligned}$$

### Exercise

Find the surface area of a cone (excluding the base) with radius 4 and height 8 using integration and a surface area integral.

#### Solution

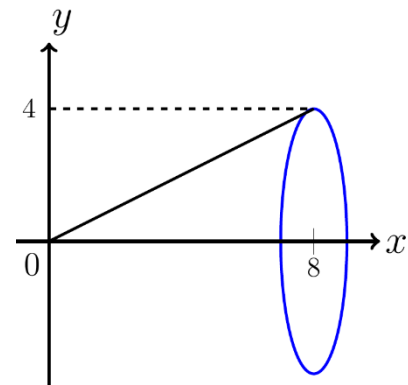
$$(0, 0) \rightarrow (8, 4)$$

$$y = \frac{4}{8}x$$

$$= \frac{1}{2}x$$

$$\begin{aligned}
 \sqrt{1+(y')^2} &= \sqrt{1+\left(\frac{1}{2}\right)^2} \\
 &= \frac{\sqrt{5}}{2}
 \end{aligned}$$

$$\begin{aligned}
 S &= 2\pi \int_0^8 \frac{x}{2} \frac{\sqrt{5}}{2} dx \\
 &= \frac{\pi\sqrt{5}}{2} x^2 \Big|_0^8 \\
 &= \underline{32\pi\sqrt{5} \text{ unit}^2}
 \end{aligned}$$



$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

### Exercise

Let  $f(x) = \frac{1}{3}x^3$  and let  $R$  be the region bounded by the graph of  $f$  and the  $x$ -axis on the interval  $[0, 2]$

- Find the area of the surface generated when the graph of  $f$  on  $[0, 2]$  is revolved about the  $x$ -axis.
- Find the volume of the solid generated when  $R$  is revolved about the  $y$ -axis.
- Find the volume of the solid generated when  $R$  is revolved about the  $x$ -axis.

### Solution

- a) Surface revolved about the  $x$ -axis

$$\sqrt{1+(f')^2} = \sqrt{1+(x^2)^2}$$

$$S = 2\pi \int_0^2 \frac{1}{3}x^3 \sqrt{1+x^4} \, dx$$

$$= \frac{\pi}{6} \int_0^2 (1+x^4)^{1/2} d(1+x^4)$$

$$= \frac{\pi}{9} (1+x^4)^{3/2} \Big|_0^2$$

$$= \frac{\pi}{9} (17^{3/2} - 1)$$

$$= \frac{\pi}{9} (17\sqrt{17} - 1) \text{ unit}^2$$

$$S = 2\pi \int_a^b y \sqrt{1+\left(\frac{dy}{dx}\right)^2} \, dx$$

- b) Using Shell Method about the  $y$ -axis

$$V = 2\pi \int_0^2 x \left( \frac{x^3}{3} \right) dx$$

$$= \frac{2\pi}{3} \int_0^2 x^4 \, dx$$

$$= \frac{2\pi}{15} x^5 \Big|_0^2$$

$$= \frac{64\pi}{15} \text{ unit}^3$$

- c) Using Disk Method about the  $x$ -axis

$$V = \pi \int_0^2 \left( \frac{x^3}{3} \right)^2 dx$$

$$= \frac{\pi}{9} \int_0^2 x^6 \, dx$$

$$= \frac{\pi}{63} x^7 \bigg|_0^2$$

$$= \frac{128\pi}{63} \text{ unit}^3$$

### Exercise

Let  $f(x) = \sqrt{3x - x^2}$  and let  $R$  be the region bounded by the graph of  $f$  and the  $x$ -axis on the interval  $[0, 3]$

- Find the area of the surface generated when the graph of  $f$  on  $[0, 3]$  is revolved about the  $x$ -axis.
- Find the volume of the solid generated when  $R$  is revolved about the  $x$ -axis.

### Solution

- Surface revolved about the  $x$ -axis

$$f' = \frac{3-2x}{2\sqrt{3x-x^2}}$$

$$\begin{aligned} \sqrt{1+(f')^2} &= \sqrt{1 + \left( \frac{3-2x}{2\sqrt{3x-x^2}} \right)^2} \\ &= \sqrt{1 + \frac{9-12x+4x^2}{4(3x-x^2)}} \\ &= \sqrt{\frac{12x-4x^2+9-12x+4x^2}{4(3x-x^2)}} \\ &= \frac{1}{2} \sqrt{\frac{9}{3x-x^2}} \\ &= \frac{3}{2} \frac{1}{\sqrt{3x-x^2}} \end{aligned}$$

$$\begin{aligned} S &= 2\pi \int_0^3 \sqrt{3x-x^2} \left( \frac{3}{2} \frac{1}{\sqrt{3x-x^2}} \right) dx \\ &= 3\pi \int_0^3 dx \\ &= 3\pi x \bigg|_0^3 \\ &= 9\pi \text{ unit}^2 \end{aligned}$$

$$S = 2\pi \int_a^b y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} dx$$

- Using Disk Method about the  $x$ -axis

$$\begin{aligned}
 V &= \pi \int_0^3 \left( \sqrt{3x - x^2} \right)^2 dx \\
 &= \pi \int_0^3 (3x - x^2) dx \\
 &= \pi \left( \frac{3}{2}x^2 - \frac{1}{3}x^3 \right) \Big|_0^3 \\
 &= \pi \left( \frac{27}{2} - 9 \right) \\
 &= \frac{9\pi}{2} \text{ unit}^3
 \end{aligned}$$

### Exercise

Let  $f(x) = \frac{1}{2}x^4 + \frac{1}{16x^2}$  and let  $R$  be the region bounded by the graph of  $f$  and the  $x$ -axis on the interval  $[1, 2]$

- Find the area of the surface generated when the graph of  $f$  on  $[1, 2]$  is revolved about the  $x$ -axis.
- Find the length of the curve  $y = f(x)$  on  $[1, 2]$
- Find the volume of the solid generated when  $R$  is revolved about the  $y$ -axis.
- Find the volume of the solid generated when  $R$  is revolved about the  $x$ -axis.

### Solution

- a) Surface revolved about the  $x$ -axis

$$f' = 2x^3 - \frac{1}{8x^3}$$

$$a = \frac{1}{2}, \quad m = 4, \quad b = \frac{1}{16}, \quad n = -2$$

$$f(x) = ax^m + bx^n$$

$$1. \quad m + n = 4 - 2 = 2 \quad \checkmark$$

$$2. \quad abmn = \frac{1}{2} \left( \frac{1}{16} \right) (4)(-2) = -\frac{1}{4} \quad \checkmark$$

$$\begin{aligned}
 S &= 2\pi \int_1^2 \left( \frac{1}{2}x^4 + \frac{1}{16x^2} \right) \left( 2x^3 + \frac{1}{8x^3} \right) dx \\
 &= 2\pi \int_1^2 \left( x^7 + \frac{1}{16}x + \frac{1}{8}x + \frac{1}{128x^5} \right) dx \\
 &= 2\pi \int_1^2 \left( x^7 + \frac{3}{16}x + \frac{1}{128}x^{-5} \right) dx
 \end{aligned}$$

$$\begin{aligned}
&= 2\pi \left( \frac{1}{8}x^8 + \frac{3}{32}x^2 - \frac{1}{512x^4} \right) \Big|_1^2 \\
&= 2\pi \left( 32 + \frac{3}{8} - \frac{1}{8192} - \frac{1}{8} - \frac{3}{32} + \frac{1}{512} \right) \\
&= \frac{263,439 \pi}{4,096} \quad \text{unit}^2
\end{aligned}$$

$$\begin{aligned}
\sqrt{1+(f')^2} &= \sqrt{1+\left(\frac{16x^6-1}{8x^3}\right)^2} \\
&= \sqrt{\frac{64x^6+256x^{12}-32x^6+1}{64x^6}} \\
&= \frac{\sqrt{256x^{12}+32x^6+1}}{8x^3} \\
&= \frac{\sqrt{(16x^6+1)^2}}{8x^3} \\
&= \frac{16x^6+1}{8x^3}
\end{aligned}$$

$$\begin{aligned}
S &= 2\pi \int_1^2 \left( \frac{1}{2}x^4 + \frac{1}{16x^2} \right) \left( 2x^3 + \frac{1}{8x^3} \right) dx \\
&= 2\pi \int_1^2 \left( x^7 + \frac{1}{16}x + \frac{1}{8}x + \frac{1}{128x^5} \right) dx \\
&= 2\pi \int_1^2 \left( x^7 + \frac{3}{16}x + \frac{1}{128}x^{-5} \right) dx \\
&= 2\pi \left( \frac{1}{8}x^8 + \frac{3}{32}x^2 - \frac{1}{512x^4} \right) \Big|_1^2 \\
&= 2\pi \left( 32 + \frac{3}{8} - \frac{1}{8192} - \frac{1}{8} - \frac{3}{32} + \frac{1}{512} \right) \\
&= \frac{263,439 \pi}{4,096} \quad \text{unit}^2
\end{aligned}$$

$$S = 2\pi \int_a^b y \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx$$

$$b) \quad a = \frac{1}{2}, \quad m = 4, \quad b = \frac{1}{16}, \quad n = -2$$

$$f(x) = ax^m + bx^n$$

$$1. \quad m+n = 4-2 = 2 \quad \checkmark$$

$$2. \quad abmn = \frac{1}{2} \left( \frac{1}{16} \right) (4)(-2) = -\frac{1}{4} \quad \checkmark$$

$$\begin{aligned} L &= \left( \frac{1}{2}x^4 - \frac{1}{16x^2} \right) \Big|_1^2 \\ &= 8 - \frac{1}{64} - \frac{1}{2} + \frac{1}{16} \\ &= \frac{483}{64} \quad \text{unit} \end{aligned}$$

c) Using Shell Method about the  $y$ -axis

$$\begin{aligned} V &= 2\pi \int_1^2 x \left( \frac{1}{2}x^4 + \frac{1}{16x^2} \right) dx \\ &= \pi \int_1^2 \left( x^5 + \frac{1}{8x} \right) dx \\ &= \pi \left( \frac{1}{6}x^6 + \frac{1}{8} \ln x \right) \Big|_1^2 \\ &= \pi \left( \frac{32}{3} + \frac{1}{8} \ln 2 - \frac{1}{6} \right) \\ &= \frac{21\pi}{2} + \frac{\pi}{8} \ln 2 \quad \text{unit}^3 \end{aligned}$$

d) Using Disk Method about the  $x$ -axis

$$\begin{aligned} V &= \pi \int_1^2 \left( \frac{1}{2}x^4 + \frac{1}{16x^2} \right)^2 dx \\ &= \pi \int_1^2 \left( \frac{1}{4}x^8 + \frac{1}{16}x^2 + \frac{1}{256}x^{-4} \right) dx \\ &= \frac{\pi}{4} \left( \frac{1}{9}x^9 + \frac{1}{12}x^3 - \frac{1}{192} \frac{1}{x^3} \right) \Big|_1^2 \\ &= \frac{\pi}{4} \left( \frac{512}{9} + \frac{2}{3} - \frac{1}{1536} - \frac{1}{9} - \frac{1}{12} + \frac{1}{192} \right) \\ &= \frac{264,341}{18,432} \pi \quad \text{unit}^3 \end{aligned}$$

### Exercise

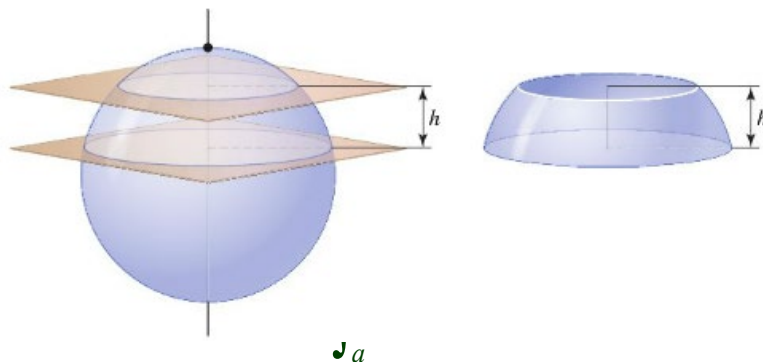
Suppose a sphere of radius  $r$  is sliced by two horizontal planes  $h$  units apart. Show that the surface area of the resulting zone on the sphere is  $2\pi h$ , independent of the location of the cutting planes.

### Solution

$$f(x) = \sqrt{r^2 - x^2}$$

$$\begin{aligned} 1 + f'(x)^2 &= 1 + \left( \frac{x}{\sqrt{r^2 - x^2}} \right)^2 \\ &= 1 + \frac{x^2}{r^2 - x^2} \\ &= \frac{r^2}{r^2 - x^2} \end{aligned}$$

$$\begin{aligned} S &= 2\pi \int_a^{a+h} \sqrt{r^2 - x^2} \cdot \frac{r}{\sqrt{r^2 - x^2}} dx \\ &= 2\pi r \left| x \right|_a^{a+h} \\ &= 2\pi r(a+h-a) \\ &= \underline{2\pi rh \text{ unit}^2} \end{aligned}$$



### Exercise

An ornamental light bulb is designed by revolving the graph of  $y = \frac{1}{3}x^{1/2} - x^{3/2}$ ,  $0 \leq x \leq \frac{1}{3}$  about the  $x$ -axis, where  $x$  and  $y$  are measured in feet. Find the surface area of the bulb and use the result to approximate the amount of glass needed to make the bulb. (Assume that the glass is 0.015 inch thick)

### Solution

$$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2}$$

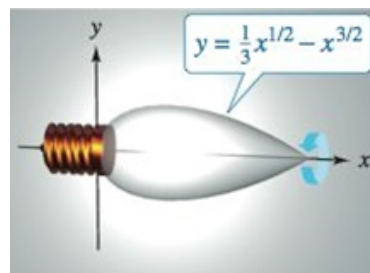
$$a = \frac{1}{3}, \quad m = \frac{1}{2}, \quad b = -1, \quad n = \frac{3}{2}$$

$$f(x) = ax^m + bx^n$$

$$1. \quad m + n = \frac{1}{2} + \frac{3}{2} = 2 \quad \checkmark$$

$$2. \quad abmn = \frac{1}{3}(-1)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = -\frac{1}{4} \quad \checkmark$$

$$S = 2\pi \frac{1}{6} \int_0^{1/3} \left( \frac{1}{3}x^{1/2} - x^{3/2} \right) \left( x^{-1/2} + 9x^{1/2} \right) dx$$



$$\begin{aligned}
&= \frac{\pi}{3} \int_0^{1/3} \left( \frac{1}{3} + 2x - 9x^2 \right) dx \\
&= \frac{\pi}{3} \left( \frac{1}{3}x + x^2 - 3x^3 \right) \Big|_0^{1/3} \\
&= \frac{\pi}{3} \left( \frac{1}{9} + \frac{1}{9} - \frac{1}{9} \right) \\
&= \frac{\pi}{27} \text{ ft}^2 \Big|
\end{aligned}$$


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$$\begin{aligned}
\sqrt{1+(y')^2} &= \sqrt{1 + \frac{1}{36}x^{-1} - \frac{1}{2} + \frac{9}{4}x} \\
&= \frac{1}{6} \sqrt{x^{-1} + 18 + 81x} \\
&= \frac{1}{6} \sqrt{\left( x^{-1/2} + 9x^{1/2} \right)^2} \\
&= \frac{1}{6} \left( x^{-1/2} + 9x^{1/2} \right) \\
S &= 2\pi \frac{1}{6} \int_0^{1/3} \left( \frac{1}{3}x^{1/2} - x^{3/2} \right) \left( x^{-1/2} + 9x^{1/2} \right) dx \\
&= \frac{\pi}{3} \int_0^{1/3} \left( \frac{1}{3} + 2x - 9x^2 \right) dx \\
&= \frac{\pi}{3} \left( \frac{1}{3}x + x^2 - 3x^3 \right) \Big|_0^{1/3} \\
&= \frac{\pi}{3} \left( \frac{1}{9} + \frac{1}{9} - \frac{1}{9} \right) \\
&= \frac{\pi}{27} \text{ ft}^2 \Big| \approx 0.1164 \text{ ft}^2 \approx 16.8 \text{ in}^2
\end{aligned}$$

Amount of glass needed:

$$\begin{aligned}
V &= \frac{\pi}{2} \left( \frac{0.015}{12} \right) \\
&\approx 0.00015 \text{ ft}^3 \\
&\approx 0.25 \text{ in}^3 \Big|
\end{aligned}$$



### Exercise

The shaded band is cut from a sphere of radius  $R$  by parallel planes  $h$  units apart. Show that the surface area of the band is  $2\pi Rh$

### Solution

$$y = \sqrt{R^2 - x^2}$$

$$\frac{dy}{dx} = \frac{1}{2} \frac{-2x}{\sqrt{R^2 - x^2}}$$

$$= \frac{-x}{\sqrt{R^2 - x^2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{-x}{\sqrt{R^2 - x^2}}\right)^2$$

$$= \frac{x^2}{R^2 - x^2}$$

$$S = 2\pi \int_a^{a+h} \sqrt{R^2 - x^2} \cdot \sqrt{1 + \frac{x^2}{R^2 - x^2}} dx$$

$$= 2\pi \int_a^{a+h} \sqrt{R^2 - x^2 + x^2} dx$$

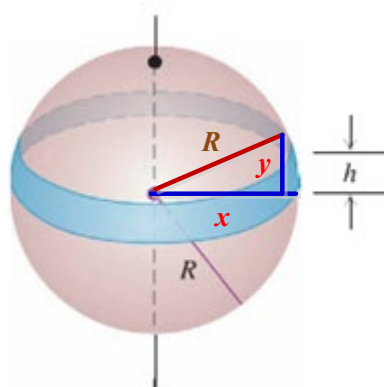
$$= 2\pi \int_a^{a+h} \sqrt{R^2} dx$$

$$= 2\pi R \int_a^{a+h} dx$$

$$= 2\pi R x \Big|_a^{a+h}$$

$$= 2\pi R((a+h) - a)$$

$$= \underline{2\pi Rh \text{ unit}^2}$$



$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

### Exercise

A drawing of a 90-ft dome is used by the National Weather Service. How much outside surface is there to paint (not counting the bottom)?

### Solution

$$x = \sqrt{R^2 - y^2} = \sqrt{45^2 - y^2}$$

$$\frac{dx}{dy} = \frac{1}{2} \frac{-2y}{\sqrt{45^2 - y^2}}$$

$$= \frac{-y}{\sqrt{45^2 - y^2}}$$

$$\left(\frac{dx}{dy}\right)^2 = \left(\frac{-y}{\sqrt{45^2 - y^2}}\right)^2$$

$$= \frac{y^2}{45^2 - y^2}$$

$$S = 2\pi \int_{-22.5}^{45} \sqrt{45^2 - y^2} \cdot \sqrt{1 + \frac{y^2}{45^2 - y^2}} dy$$

$$= 2\pi \int_{-22.5}^{45} \sqrt{45^2 - y^2 + y^2} dy$$

$$= 90\pi \int_{-22.5}^{45} dy$$

$$= 90\pi y \Big|_{-22.5}^{45}$$

$$= 90\pi(45 + 22.5)$$

$$= 6075\pi \text{ ft}^2 \Big| 19,085 \text{ ft}^2$$

