Solution   *Section* 4.2 – Line Integrals

Exercise

Evaluate $\int_C (x+y)\,ds$ where C is the straight-line segment $x=t,\ y=(1-t),\ z=0$ from (0, 1, 0) to $(1,\ 0,\ 0)$.

Solution

$$\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= t\hat{i} + (1-t)\,\hat{j}$$

$$\frac{dr}{dt} = \boldsymbol{i} - \boldsymbol{j} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = \sqrt{1+1} = \sqrt{2}$$

$$\begin{matrix} x = t \\ y = 1-t \end{matrix} \quad \to \quad x + y = t + 1 - t = 1$$

$$\int_C f(x,y,z) = \int_0^1 f(t,1-t,0)\left|\frac{d\vec{r}}{dt}\right| dt$$
$$= \int_0^1 (1)\sqrt{2}dt$$
$$= \sqrt{2}\,t\,\Big|_0^1$$
$$= \sqrt{2}$$

Exercise

Evaluate $\int_C (x-y+z-2)\,ds$ where C is the straight-line segment $x=t,\ y=(1-t),\ z=1$ from $(0,\ 1,\ 1)$ to $(1,\ 0,\ 1)$.

Solution

$$\vec{r}(t) = t\hat{i} + (1-t)\,\hat{j} + \hat{k} \qquad 0 \le t \le 1$$

$$\frac{d\vec{r}}{dt} = \hat{i} - \hat{j} \quad \Rightarrow \quad \left|\frac{d\vec{r}}{dt}\right| = \sqrt{1+1} = \sqrt{2}$$

$$\begin{cases} x = t \\ y = 1 - t \\ z = 1 \end{cases}$$

$$x - y + z - 2 = t - 1 + t + 1 - 2$$

$$= 2t - 2$$

$$\int_C f(x, y, z) = \int_0^1 (2t - 2)\sqrt{2}\,dt$$

$$= \sqrt{2}\left(t^2 - 2t \right|_0^1$$

$$= \sqrt{2}(1 - 2)$$

$$= -\sqrt{2}$$

Exercise

Evaluate $\int_C (xy + y + z)\,ds$ along the curve $\vec{r}(t) = 2t\hat{i} + t\hat{j} + (2 - 2t)\hat{k}, \quad 0 \le t \le 1$

Solution

$$\vec{r}(t) = 2t\hat{i} + t\hat{j} + (2 - 2t)\hat{k}, \quad 0 \le t \le 1$$

$$\frac{d\vec{r}}{dt} = 2\hat{j} + \hat{j} - 2\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{4 + 1 + 4} = 3$$

$$\begin{aligned} x &= 2t \\ y &= t \quad \to \quad xy + y + z = 2t^2 + t + 2 - 2t = 2t^2 - t + 2 \\ z &= 2 - 2t \end{aligned}$$

$$\int_C (xy + y + z)\,ds = \int_0^1 \left(2t^2 - t + 2\right)(3)\,dt$$

$$= 3\left(\frac{2}{3}t^3 - \frac{1}{2}t^2 + 2t \right|_0^1$$

$$= 3\left(\frac{2}{3} - \frac{1}{2} + 2\right)$$

$$= 3\left(\frac{13}{6}\right)$$

$$= \frac{13}{2}$$

Exercise

Evaluate $\int_C \left(xz - y^2\right) ds$ C: is the line segment from $(0,\ 1,\ 2)$ to $(-3,\ 7,\ -1)$.

Solution

Equation of the line is:

$$\begin{cases} x = 0 + (-3-0)t \\ y = 1 + (7-1)t \\ z = 2 + (-1-2)t \end{cases} \quad \to \quad \langle -3t,\ 1+6t,\ 2-3t \rangle$$

$$\vec{r}(t) = \langle -3t,\ 1+6t,\ 2-3t \rangle \quad 0 \le t \le 1$$

$$\vec{r}'(t) = \langle -3,\ 6,\ -3 \rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{9+36+9}$$

$$= 3\sqrt{6}$$

$$\int_C \left(xz - y^2\right) ds = 3\sqrt{6} \int_0^1 \left((-3t)(2-3t) - (1+6t)^2\right) dt$$

$$= 3\sqrt{6} \int_0^1 \left(-6t + 9t^2 - 1 - 12t - 36t^2\right) dt$$

$$= 3\sqrt{6} \int_0^1 \left(-27t^2 - 18t - 1\right) dt$$

$$= 3\sqrt{6} \left(-9t^3 - 9t^2 - t \right)\Big|_0^1$$

$$= 3\sqrt{6}(-9-9-1)$$

$$= -57\sqrt{6}$$

Exercise

Evaluate $\int_C xy\ ds$; C: is the unit circle $\vec{r}(t) = \langle \cos t,\ \sin t \rangle$; $0 \le t \le 2\pi$

Solution

$$\vec{r}'(t) = \langle -\sin t,\ \cos t \rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{\sin^2 t + \cos^2 t}$$

$$= 1$$

$$\int_C xy\ ds = \int_0^{2\pi} \cos t \sin t\ dt$$

$$= \int_0^{2\pi} \sin t\ d(\sin t)$$

$$= \frac{1}{2} \sin^2 t \Big|_0^{2\pi}$$

$$\underline{= 0}$$

Exercise

Evaluate $\int_C (x+y)ds$ C: is the circle of radius 1 centered at $(0,\ 0)$

Solution

$$\vec{r}(t) = \langle \cos t,\ \sin t \rangle; \quad 0 \le t \le 2\pi$$

$$\vec{r}'(t) = \langle -\sin t,\ \cos t \rangle$$

$$|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t}$$

$$\underline{= 1}$$

$$\int_C (x+y)\ ds = \int_0^{2\pi} (\cos t + \sin t)\ dt$$

$$= \sin t - \cos t \Big|_0^{2\pi}$$

$$= -1 + 1$$

$$\underline{= 0}$$

Exercise

Evaluate $\int_C \left(x^2 - 2y^2\right) ds$ C: is the line $\vec{r}(t) = \left\langle \frac{t}{\sqrt{2}},\ \frac{t}{\sqrt{2}} \right\rangle; \quad 0 \le t \le 4$

Solution

$$\vec{r}'(t) = \left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}} \right\rangle$$

$$|\vec{r}'(t)| = \sqrt{\frac{1}{2} + \frac{1}{2}}$$

$$\boxed{=1}$$

$$\int_C \left(x^2 - 2y^2\right) ds = \int_0^4 \left(\frac{1}{2}t^2 - t^2\right) dt$$

$$= -\frac{1}{2}\int_0^4 t^2 dt$$

$$= -\frac{1}{6}t^3 \Big|_0^4$$

$$\boxed{= -\frac{32}{3}}$$

Exercise

Evaluate $\displaystyle\int_C x^2 y\ ds$ C: is the line $\vec{r}(t) = \left\langle \frac{t}{\sqrt{2}},\ 1 - \frac{t}{\sqrt{2}} \right\rangle; \quad 0 \le t \le 4$

Solution

$$\vec{r}'(t) = \left\langle \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}} \right\rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{\frac{1}{2} + \frac{1}{2}}$$

$$\boxed{=1}$$

$$\int_C x^2 y\ ds = \int_0^4 \frac{1}{2}t^2 \left(1 - \frac{1}{\sqrt{2}}t\right) dt$$

$$= \int_0^4 \left(\frac{1}{2}t^2 - \frac{1}{2\sqrt{2}}t^3\right) dt$$

$$= \frac{1}{6}t^3 - \frac{1}{8\sqrt{2}}t^4 \Big|_0^4$$

$$= \frac{32}{3} - \frac{32}{\sqrt{2}}$$

$$\boxed{= \frac{32 - 48\sqrt{2}}{3}}$$

Exercise

Evaluate $\int_C \left(x^2 + y^2\right) ds$ C: is the circle of radius 4 centered at $(0,\ 0)$

Solution

$$\vec{r}(t) = \langle 4\cos t,\ 4\sin t \rangle; \quad 0 \le t \le 2\pi$$

$$\vec{r}'(t) = \langle -4\sin t,\ 4\cos t \rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{16\sin^2 t + 16\cos^2 t}$$

$$= 4$$

$$\int_C \left(x^2 + y^2\right) ds = 4\int_0^{2\pi} \left(16\cos^2 t + 16\sin^2 t\right) dt$$

$$= 64\int_0^{2\pi} dt$$

$$= 128\pi$$

Exercise

Evaluate $\int_C \left(x^2 + y^2\right) ds$ C: is the line segment from $(0,\ 0)$ to $(5,\ 5)$

Solution

$$\vec{r}(t) = \langle 5t,\ 5t \rangle; \quad 0 \le t \le 1$$

$$\vec{r}'(t) = \langle 5,\ 5 \rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{25 + 25}$$

$$= 5\sqrt{2}$$

$$\int_C \left(x^2 + y^2\right) ds = 5\sqrt{2}\int_0^1 \left(25t^2 + 25t^2\right) dt$$

$$= 250\sqrt{2}\int_0^1 t^2 dt$$

$$= \frac{250}{3}\sqrt{2}\ t^3 \Big|_0^1$$

$$= \frac{250}{3}\sqrt{2}$$

Exercise

Evaluate $\displaystyle\int_C \frac{x}{x^2+y^2}\,ds$ C: is the line segment from $(1,\ 1)$ to $(10,\ 10)$

Solution

$$\vec{r}(t) = \langle t,\ t\rangle; \quad 1 \le t \le 10$$

$$\vec{r}'(t) = \langle 1,\ 1\rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{1+1}$$

$$= \sqrt{2}$$

$$\int_C \frac{x}{x^2+y^2}\,ds = \sqrt{2}\int_1^{10} \frac{t}{t^2+t^2}\,dt$$

$$= \frac{\sqrt{2}}{2}\int_1^{10} \frac{1}{t}\,dt$$

$$= \frac{\sqrt{2}}{2}\ \ln t\ \Big|_1^{10}$$

$$= \frac{\sqrt{2}}{2}\ln 10$$

Exercise

Evaluate $\displaystyle\int_C (xy)^{1/3}\,ds$ C: is the curve $y = x^2, \quad 0 \le x \le 1$

Solution

$$\vec{r}(t) = \left\langle t,\ t^2\right\rangle; \quad 0 \le t \le 1$$

$$\vec{r}'(t) = \langle 1,\ 2t\rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{1+4t^2}$$

$$\int_C (xy)^{1/3}\,ds = \int_0^1 \left(t^3\right)^{1/3}\ \sqrt{1+4t^2}\ dt$$

$$= \int_0^1 t\left(1+4t^2\right)^{1/2}\ dt$$

$$= \frac{1}{8}\int_0^1 \left(1+4t^2\right)^{1/2}\, d\left(1+4t^2\right)$$

$$= \frac{1}{12}\left(1+4t^2\right)^{3/2}\Big|_0^1$$

$$= \frac{1}{12}\left(5\sqrt{5}-1\right)$$

Exercise

Evaluate $\int_C xy\ ds$ C: is a portion of the ellipse $\frac{x^2}{4}+\frac{y^2}{16}=1$ in the first quadrant, oriented counterclockwise.

Solution

$$\vec{r}(t) = \langle 2\cos t,\ 4\sin t\rangle; \quad 0 \le t \le \pi$$

$$\vec{r}'(t) = \langle -2\sin t,\ 4\cos t\rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{4\sin^2 t + 16\cos^2 t}$$

$$= 2\sqrt{\sin^2 t + 4\cos^2 t}$$

$$\int_C xy\ ds = 2\int_0^\pi (8\cos t\sin t)\ \sqrt{1-\cos^2 t+4\cos^2 t}\ dt$$

$$= 16\int_0^\pi (\cos t\sin t)\left(1+3\cos^2 t\right)^{1/2}\ dt$$

$$= -\frac{8}{3}\int_0^\pi \left(1+3\cos^2 t\right)^{1/2}\ d\left(1+3\cos^2 t\right)$$

$$= -\frac{16}{9}\left(1+3\cos^2 t\right)^{3/2}\Big|_0^\pi$$

$$= -\frac{16}{9}(8-8)$$

$$= 0$$

Exercise

Evaluate $\int_C (2x-3y)\,ds$ C: is the line segment from $(-1,\ 0)$ to $(0,\ 1)$ followed by the line segment from $(0,\ 1)$ to $(1,\ 0)$

Solution

$(-1,\ 0)$ to $(0,\ 1)$

$\vec{r}_1(t) = \langle t-1,\ t\rangle \quad 0 \le t \le 1$

$\vec{r}_1'(t) = \langle 1,\ 1\rangle$

$\boxed{\left|\vec{r}_1'(t)\right| = \sqrt{2}}$

$(0,\ 1)$ to $(1,\ 0)$

$\vec{r}_2(t) = \langle t,\ 1-t\rangle$

$\vec{r}_2'(t) = \langle 1,\ -1\rangle$

$\boxed{\left|\vec{r}_2'(t)\right| = \sqrt{2}}$

$$\begin{aligned}
\int_C (2x-3y)\,ds &= \sqrt{2}\int_0^1 \big(2(t-1)-3t\big)\,dt + \sqrt{2}\int_0^1 (2t-3+3t)\,dt \\
&= \sqrt{2}\int_0^1 (-2-t)\,dt + \sqrt{2}\int_0^1 (5t-3)\,dt \\
&= \sqrt{2}\int_0^1 (-2-t+5t-3)\,dt \\
&= \sqrt{2}\int_0^1 (4t-5)\,dt \\
&= \sqrt{2}\left(2t^2-5t\ \right|_0^1 \\
&= \sqrt{2}(2-5) \\
&\boxed{= -3\sqrt{2}}
\end{aligned}$$

Exercise

Evaluate $\int_C (x+y+z)\,ds$; C is the circle $\vec{r}(t) = \langle 2\cos t,\ 0,\ 2\sin t\rangle \quad 0 \le t \le 2\pi$

Solution

$$\vec{r}'(t) = \langle -2\sin t,\ 0,\ 2\cos t\rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{4\sin^2 t + 4\cos^2 t}$$

$$= 2$$

$$\int_C (x+y+z)\,ds = \int_0^{2\pi} (2\cos t + 2\sin t)(2)\,dt$$

$$= 4\left(-\sin t + \cos t\right.\Big|_0^{2\pi}$$

$$= 0$$

Exercise

Evaluate $\int_C (x-y+2z)\ ds$; C is the circle $\vec{r}(t) = \langle 1,\ 3\cos t,\ 3\sin t\rangle \quad 0 \le t \le 2\pi$

Solution

$$\vec{r}'(t) = \langle 0,\ -3\sin t,\ 3\cos t\rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{9\sin^2 t + 9\cos^2 t}$$

$$= 3$$

$$\int_C (x-y+2z)\ ds = 3\int_0^{2\pi} (1 - 3\cos t + 6\sin t)\,dt$$

$$= 3\left(t - 3\sin t - 6\cos t\right.\Big|_0^{2\pi}$$

$$= 3(2\pi - 6 + 6)(t - 3\sin t - 6\cos t)$$

$$= 6\pi$$

Exercise

Evaluate $\displaystyle\int_C xyz\ ds$; C is the circle $\vec{r}(t) = \langle 1,\ 3\cos t,\ 3\sin t\rangle \quad 0 \le t \le 2\pi$

Solution

$$\vec{r}'(t) = \langle 0,\ -3\sin t,\ 3\cos t\rangle$$

$$|\vec{r}'(t)| = \sqrt{9\sin^2 t + 9\cos^2 t}$$

$$= 3$$

$$\int_C xyz\ ds = 3\int_0^{2\pi} (9\cos t \sin t)\,dt$$

$$= 27\int_0^{2\pi} \sin t\ d(\sin t)$$

$$= \frac{27}{2}\sin^2 t\,\Big|_0^{2\pi}$$

$$= 0$$

Exercise

Evaluate $\displaystyle\int_C xyz\ ds$; C is the line segment from $(0,\ 0,\ 0)$ to $(1,\ 2,\ 3)$

Solution

$$\vec{r}(t) = \langle t,\ 2t,\ 3t\rangle \qquad 0 \le t \le 1$$

$$\vec{r}'(t) = \langle 1,\ 2,\ 3\rangle$$

$$|\vec{r}'(t)| = \sqrt{1+4+9}$$

$$= \sqrt{14}$$

$$\int_C xyz\ ds = \sqrt{14}\int_0^1 6t^3\ dt$$

$$= \frac{3}{2}\sqrt{14}\ t^4\,\Big|_0^1$$

$$= \frac{3}{2}\sqrt{14}$$

Exercise

Evaluate $\int_C \frac{xy}{z}\,ds$; C is the line segment from $(1,\ 4,\ 1)$ to $(3,\ 6,\ 3)$

Solution

$$\vec{r}(t) = \langle 2t+1,\ 2t+4,\ 2t+1 \rangle \qquad 0 \le t \le 1$$

$$\vec{r}(t) = \langle 2,\ 2,\ 2 \rangle$$

$$\boxed{\left|\vec{r}'(t)\right| = 2\sqrt{3}}$$

$$\begin{aligned}
\int_C \frac{xy}{z}\,ds &= 2\sqrt{3}\int_0^1 \frac{(2t+1)(2t+4)}{2t+1}\,dt \\
&= 2\sqrt{3}\int_0^1 (2t+4)\,dt \\
&= 2\sqrt{3}\left(t^2+4t\right|_0^1 \\
&= \boxed{10\sqrt{3}}
\end{aligned}$$

Exercise

Evaluate $\int_C (y-z)\,ds$; C is the helix $\vec{r}(t) = \langle 3\cos t,\ 3\sin t,\ t \rangle \quad 0 \le t \le 2\pi$

Solution

$$\vec{r}'(t) = \langle -3\sin t,\ 3\cos t,\ 1 \rangle$$

$$\begin{aligned}
\left|\vec{r}'(t)\right| &= \sqrt{9\sin^2 t + 9\cos^2 t + 1} \\
&= \boxed{\sqrt{10}}
\end{aligned}$$

$$\begin{aligned}
\int_C (y-z)\,ds &= \sqrt{10}\int_0^{2\pi} (3\sin t - t)\,dt \\
&= \sqrt{10}\left(-3\cos t - \frac{1}{2}t^2\right|_0^{2\pi} \\
&= \sqrt{10}\left(-3 - 2\pi^2 + 3\right) \\
&= \boxed{-2\pi\sqrt{10}}
\end{aligned}$$

Exercise

Evaluate $\int_C xe^{yz}ds$; C is $\vec{r}(t)=\langle t,\ 2t,\ -4t\rangle \quad 1\le t\le 2$

Solution

$$\vec{r}'(t)=\langle 1,\ 2,\ -4\rangle$$

$$\boxed{|\vec{r}'(t)|=\sqrt{21}}$$

$$\int_C xe^{yz}ds=\sqrt{21}\int_1^2 te^{-8t^2}dt$$

$$=-\frac{\sqrt{21}}{16}\int_1^2 e^{-8t^2}d\left(-8t^2\right)$$

$$=-\frac{\sqrt{21}}{16}\ e^{-8t^2}\Big|_1^2$$

$$=-\frac{\sqrt{21}}{16}\left(e^{-32}-e^{-8}\right)$$

$$=-\frac{\sqrt{21}}{16e^8}\left(\frac{1}{e^{24}}-1\right)$$

$$\boxed{=\frac{\sqrt{21}}{16e^{32}}\left(e^{24}-1\right)}$$

Exercise

Find the integral of $f(x,y,z)=x+y+z$ over the straight–line segment from $(1,\ 2,\ 3)$ to $(0,\ -1,\ 1)$

Solution

$$\vec{r}(t)=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+t\left((0-1)\hat{i}+(-1-2)\hat{j}+(1-3)\hat{k}\right)$$

$$=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+t\left(-\hat{i}-3\hat{j}-2\hat{k}\right)$$

$$=(1-t)\hat{i}+(2-3t)\hat{j}+(3-2t)\hat{k},\quad 0\le t\le 1$$

$$\frac{d\vec{r}}{dt}=-\hat{i}-3\hat{j}-2\hat{k}$$

$$\left|\frac{dr}{dt}\right|=\sqrt{1+9+4}$$

$$\boxed{=\sqrt{14}}$$

$$x = 1 - t$$
$$y = 2 - 3t \quad \rightarrow \quad x + y + z = 1 - t + 2 - 3t + 3 - 2t$$
$$z = 3 - 2t$$

$$\underline{x + y + z = 6 - 6t}$$

$$\int_C (x + y + z)\,ds = \int_0^1 (6 - 6t)\left(\sqrt{14}\right)dt$$
$$= \sqrt{14}\,\left(6t - 3t^2\right|_0^1$$
$$\underline{= 3\sqrt{14}}$$

Exercise

Find the integral of $f(x, y, z) = \dfrac{\sqrt{3}}{x^2 + y^2 + z^2}$ over the curve $\vec{r}(t) = t\,\hat{i} + t\hat{j} + t\,\hat{k}, \quad 1 \le t \le \infty$

Solution

$$\vec{r}(t) = t\,\hat{i} + t\hat{j} + t\,\hat{k}, \quad 1 \le t \le \infty$$
$$\frac{d\vec{r}}{dt} = \hat{i} + \hat{j} + \hat{k}$$
$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{1 + 1 + 1}$$
$$\underline{= \sqrt{3}}$$

$$x^2 + y^2 + z^2 = t^2 + t^2 + t^2$$
$$\underline{= 3t^2}$$

$$\int_C \frac{\sqrt{3}}{x^2 + y^2 + z^2}\,ds = \int_1^\infty \frac{\sqrt{3}}{3t^2}\left(\sqrt{3}\right)dt$$
$$= -\frac{1}{t}\Big|_1^\infty$$
$$= -\left(\frac{1}{\infty} - 1\right)$$
$$\underline{= 1}$$

Exercise

Evaluate $\int_C x\, ds$ where C is

a) The straight-line segment $x = t,\ \ y = \frac{t}{2}$, from (0, 0) to (4, 2).

b) The parabolic curve $x = t,\ \ y = t^2$, from (0, 0) to (2, 4).

Solution

a) $x = t \quad \rightarrow \quad \begin{cases} x = 0 & t = 0 \\ x = 4 & t = 4 \end{cases}$

$t = 2y \quad \rightarrow \quad \begin{cases} y = 0 & t = 0 \\ y = 2 & t = 4 \end{cases}$

$\vec{r}(t) = t\,\hat{i} + \frac{t}{2}\hat{j}, \quad 0 \le t \le 4$

$\frac{d\vec{r}}{dt} = \hat{i} + \frac{1}{2}\hat{j}$

$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{1 + \frac{1}{4}}$

$= \frac{\sqrt{5}}{2}$

$$\int_C x\, ds = \int_0^4 t\frac{\sqrt{5}}{2}dt$$

$$= \frac{\sqrt{5}}{2}\left(\frac{1}{2}t^2 \right|_0^4$$

$$= 4\sqrt{5}$$

b) $x = t \quad \rightarrow \quad \begin{cases} x = 0 & t = 0 \\ x = 2 & t = 2 \end{cases}$

$t = \sqrt{y} \quad \rightarrow \quad \begin{cases} y = 0 & t = 0 \\ y = 4 & t = 2 \end{cases}$

$\vec{r}(t) = t\,\hat{i} + t^2\hat{j}, \quad 0 \le t \le 2$

$\frac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j}$

$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{1 + 4t^2}$

$$\int_C x\, ds = \int_0^2 t\sqrt{1+4t^2}\ dt \qquad\qquad d\left(1+4t^2\right) = 8tdt$$

$$=\frac{1}{8}\int_0^2 \left(1+4t^2\right)^{1/2} d\left(1+4t^2\right)$$

$$=\frac{1}{8}\left(\frac{2}{3}\left(1+4t^2\right)^{3/2}\right|_0^2$$

$$=\frac{1}{12}\left[(17)^{3/2}-1\right]$$

$$\underline{=\frac{1}{12}\left(17\sqrt{17}-1\right)}$$

Exercise

Evaluate $\int_C \sqrt{x+2y}\ ds$ where C is

a) The straight-line segment $x=t,\ \ y=4t$, from (0, 0) to (1, 4).

b) $C_1 \cup C_2 :\ C_1$ is the line segment (0, 0) to (1, 0) and C_2 is the line segment (1, 0) to (1, 2).

Solution

a) $x=t \quad \to \quad \begin{cases} x=0 & t=0 \\ x=1 & t=1 \end{cases}$

$t=\frac{y}{4} \quad \to \quad \begin{cases} y=0 & t=0 \\ y=4 & t=1 \end{cases}$

$\vec{r}(t)=t\,\hat{i}+4t\hat{j}, \quad 0\le t\le 1$

$\frac{d\vec{r}}{dt}=\hat{i}+4\hat{j}$

$\left|\frac{d\vec{r}}{dt}\right|=\sqrt{1+16}$

$\underline{=\sqrt{17}}$

$$\int_C \sqrt{x+2y}\ ds=\int_0^1 \sqrt{t+8t}\left(\sqrt{17}\right)dt$$

$$=\sqrt{17}\int_0^1 \sqrt{9t}\ dt$$

$$=3\sqrt{17}\left(\frac{2}{3}t^{3/2}\right|_0^1$$

$$\underline{=2\sqrt{17}}$$

b) $C_1:\ \ \vec{r}(t)=\left(0\,\hat{i}+0\hat{j}\right)+t\left(\hat{i}+0\hat{j}\right)\underline{=t\,\hat{i}} \quad 0\le t\le 1$

$\frac{d\vec{r}}{dt}=\hat{i} \quad \Rightarrow \quad \left|\frac{d\vec{r}}{dt}\right|=1$

$$C_2 : \quad \vec{r}(t) = \left(1\hat{i} + 0\hat{j}\right) + t\left((1-1)\,\hat{i} + (2-0)\,\hat{j}\right)$$

$$= \hat{i} + 2t\hat{j} \qquad\qquad 0 \le t \le 2$$

$$\frac{d\vec{r}}{dt} = 2\hat{j} \quad \Rightarrow \quad \left|\frac{d\vec{r}}{dt}\right| = 2$$

$$\int_C \sqrt{x+2y}\; ds = \int_0^1 \sqrt{t}\,(1)\,dt + \int_0^2 \sqrt{1+4t}\,(2)\,dt$$

$$= \frac{2}{3}t^{3/2}\Big|_0^1 + \frac{1}{2}\int_0^2 (1+4t)^{1/2}\; d(1+4t)$$

$$= \frac{2}{3} + \frac{1}{3}\left((1+4t)^{3/2}\right)\Big|_0^2$$

$$= \frac{2}{3} + \frac{1}{3}\left[(9)^{3/2} - 1\right]$$

$$= \frac{2}{3} + \frac{1}{3}(26)$$

$$\boxed{= \frac{28}{3}}$$

Exercise

Find the line integral of $f(x,y) = \dfrac{\sqrt{y}}{x}$ along the curve $\vec{r}(t) = t^3\,\hat{i} + t^4\hat{j}, \ \ \frac{1}{2} \le t \le 1$

Solution

$$\vec{r}(t) = t^3\,\hat{i} + t^4\hat{j}, \quad \frac{1}{2} \le t \le 1$$

$$\frac{d\vec{r}}{dt} = 3t^2\,\hat{i} + 4t^3\hat{j}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{9t^4 + 16t^6}$$

$$\boxed{= t^2\sqrt{9+16t^2}}$$

$$\int_C \frac{\sqrt{y}}{x}\, ds = \int_{1/2}^1 \frac{\sqrt{t^4}}{t^3}\left(t^2\sqrt{9+16t^2}\right)dt$$

$$= \int_{1/2}^1 t\left(9+16t^2\right)^{1/2} dt \qquad\qquad d\left(9+16t^2\right) = 32t\,dt$$

$$= \frac{1}{32}\int_{1/2}^1 \left(9+16t^2\right)^{1/2} d\left(9+16t^2\right)$$

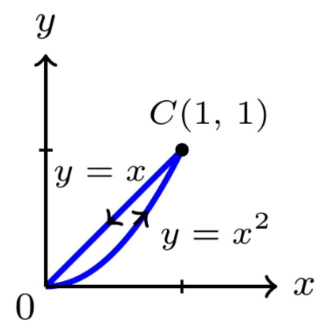
$$= \frac{1}{32}\left(\frac{2}{3}\right)\left(\left(9+16t^2\right)^{3/2}\right|_{1/2}^{1}$$

$$= \frac{1}{48}\left[(25)^{3/2} - (13)^{3/2}\right]$$

$$\boxed{= \frac{1}{48}\left(125 - 13\sqrt{13}\right)}$$

Exercise

Evaluate $\int_C \left(x+\sqrt{y}\right)\, ds$ where C is



Solution

$C_1: \ \vec{r}(t) = t\,\hat{i} + t^2\hat{j} \quad 0 \le t \le 1$

$$\frac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j} \quad \Rightarrow \quad \left|\frac{d\vec{r}}{dt}\right| = \sqrt{1+4t^2}$$

$C_2: \ \vec{r}(t) = \left(1\hat{i} + 1\hat{j}\right) + t\left(-\hat{i} - \hat{j}\right)$

$$= (1-t)\,\hat{i} + (1-t)\,\hat{j} \qquad 0 \le t \le 1$$

$$\frac{dr}{dt} = -\hat{i} - \hat{j} \quad \Rightarrow \quad \left|\frac{dr}{dt}\right| = \sqrt{2}$$

$$\int_C \left(x+\sqrt{y}\right)\, ds = \int_0^1 \left(t+\sqrt{t^2}\right)\left(\sqrt{1+4t^2}\right)dt + \int_0^1 \left(1-t+\sqrt{1-t}\right)\left(\sqrt{2}\right)dt$$

$$= \int_0^1 2t\left(\sqrt{1+4t^2}\right)dt - \sqrt{2}\int_0^1 \left((1-t)+\sqrt{1-t}\right)d(1-t)$$

$$= \frac{1}{4}\int_0^1 \left(1+4t^2\right)^{1/2} d\left(1+4t^2\right) - \sqrt{2}\left(\frac{1}{2}(1-t)^2 + \frac{2}{3}(1-t)^{3/2}\right|_0^1$$

$$= \frac{1}{6}\left(1+4t^2\right)^{3/2}\Big|_0^1 - \sqrt{2}\left(-\frac{1}{2} - \frac{2}{3}\right)$$

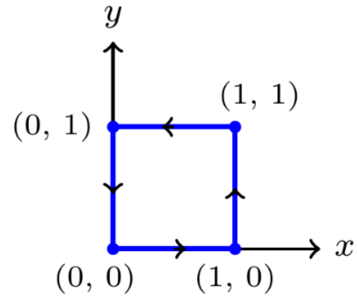
$$= \frac{1}{6}\left((5)^{3/2} - 1\right) + \frac{7\sqrt{2}}{6}$$

$$\boxed{= \frac{5\sqrt{5} - 1 + 7\sqrt{2}}{6}}$$

Exercise

Evaluate $\displaystyle\int_C \frac{1}{x^2+y^2+1}\,ds$ where C is

Solution



$C_1:\ \vec{r}(t)=t\,\hat{i} \qquad 0\le t\le 1$

$$\frac{d\vec{r}}{dt}=\hat{i} \quad\Rightarrow\quad \left|\frac{d\vec{r}}{dt}\right|=1$$

$C_2:\ \vec{r}(t)=\hat{i}+t\hat{j} \qquad 0\le t\le 1$

$$\frac{d\vec{r}}{dt}=\hat{j} \quad\Rightarrow\quad \left|\frac{d\vec{r}}{dt}\right|=1$$

$C_3:\ \vec{r}(t)=(1-t)\,\hat{i}+\hat{j} \qquad 0\le t\le 1$

$$\frac{d\vec{r}}{dt}=-\hat{i} \quad\Rightarrow\quad \left|\frac{d\vec{r}}{dt}\right|=1$$

$C_4:\ \vec{r}(t)=(1-t)\hat{j} \qquad 0\le t\le 1$

$$\frac{d\vec{r}}{dt}=-\hat{j} \quad\Rightarrow\quad \left|\frac{d\vec{r}}{dt}\right|=1$$

$$\int_C \frac{1}{x^2+y^2+1}\,ds=\int_0^1 \frac{1}{t^2+1}(1)\,dt+\int_0^1 \frac{1}{1+t^2+1}(1)\,dt$$

$$+\int_0^1 \frac{1}{(1-t)^2+1+1}(1)\,dt+\int_0^1 \frac{1}{(1-t)^2+1}(1)\,dt$$

$$=\int_0^1 \frac{1}{t^2+1}\,dt+\int_0^1 \frac{1}{t^2+2}\,dt-\int_0^1 \frac{1}{(1-t)^2+2}\,d(1-t)$$

$$-\int_0^1 \frac{1}{(1-t)^2+1}\,d(1-t)$$

$$=\tan^{-1}t+\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1-t}{\sqrt{2}}\right)-\tan^{-1}(1-t)\,\Bigg|_0^1$$

$$=\frac{\pi}{4}+\frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}+\frac{\pi}{4}$$

$$=\frac{\pi}{2}+\frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

Exercise

Find the line integral of $f(x,y)=\frac{x^3}{y}$ over the curve $C:\quad y=\frac{x^2}{2},\quad 0\le x\le 2$

Solution

$$\vec{r}(t)=x\,\hat{i}\;+y\hat{j}$$

$$=x\,\hat{i}\;+\tfrac{1}{2}x^2\hat{j} \qquad\qquad 0\le x\le 2$$

$$\frac{d\vec{r}}{dt}=\hat{i}\;+x\hat{j}$$

$$\left|\frac{d\vec{r}}{dt}\right|=\sqrt{1+x^2}$$

$$\int_C f(x,y)\,ds=\int_C \frac{x^3}{y^2}\,ds$$

$$=\int_0^2 2x\sqrt{1+x^2}\;dx \qquad\qquad d\left(1+x^2\right)=2xdx$$

$$=\int_0^2 \left(1+x^2\right)^{1/2}d\left(1+x^2\right)$$

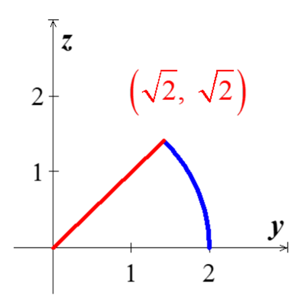
$$=\tfrac{2}{3}\left(1+x^2\right)^{3/2}\Big|_0^2$$

$$=\tfrac{2}{3}\left((5)^{3/2}-1\right)$$

$$=\frac{10\sqrt{5}-2}{3}$$

Exercise

Find the line integral of $f(x,y)=x^2-y$ over the curve $C:\quad x^2+y^2=4$ in the first quadrant from (0, 2) to $\left(\sqrt{2},\,\sqrt{2}\right)$



Solution

$$x=r\cos t \qquad y=r\sin t$$

$$\vec{r}(t)=(2\sin t)\,\hat{i}\;+(2\cos t)\,\hat{j} \qquad\qquad 0\le t\le \tfrac{\pi}{4}$$

$$\frac{d\vec{r}}{dt}=(2\cos t)\,\hat{i}\;-(2\sin t)\,\hat{j}$$

$$\left|\frac{d\vec{r}}{dt}\right|=\sqrt{4\cos^2 t+4\sin^2 t}$$

$$=2$$

$$f(x,y) = x^2 - y = 4\sin^2 t - 2\cos t$$

$$\int_C f(x,y)\,ds = \int_0^{\pi/4} \left(4\sin^2 t - 2\cos t\right)(2)\,dt \qquad \sin^2 t = \frac{1-\cos 2t}{2}$$

$$= 4\int_0^{\pi/4} (1 - \cos 2t - \cos t)\,dt$$

$$= 4\left(t - \frac{1}{2}\sin 2t - \sin t\right|_0^{\pi/4}$$

$$= 4\left(\frac{\pi}{4} - \frac{1}{2} - \frac{\sqrt{2}}{2}\right)$$

$$= 4\left(\frac{\pi}{4} - \frac{1+\sqrt{2}}{2}\right)$$

$$= \pi - 2\left(1+\sqrt{2}\right)$$

Exercise

Evaluate the line integral $\int_C \left(x^2 - 2xy + y^2\right)ds$; C is the upper half of a circle

$\vec{r}(t) = \langle 5\cos t,\ 5\sin t\rangle, \quad 0 \le t \le \pi \quad (ccw)$

Solution

$$\vec{r}\,' = \langle -5\sin t,\ 5\cos t\rangle$$

$$|\vec{r}\,'| = \sqrt{25\sin^2 t + 25\cos^2 t}$$

$$= 5$$

$$\int_C \left(x^2 - 2xy + y^2\right)ds = 5\int_0^{\pi} \left(25\cos^2 t - 50\cos t\sin t + 25\sin^2 t\right)dt$$

$$= 125\int_0^{\pi} (1 - 2\cos t\sin t)\,dt$$

$$= 125\int_0^{\pi} (1 - \sin 2t)\,dt$$

$$= 125\left(t + \frac{1}{2}\cos 2t\right|_0^{\pi}$$

$$= 125\left(\pi + \frac{1}{2} - \frac{1}{2}\right)$$

$$\underline{= 125\pi}$$

Exercise

Evaluate the line integral $\int_C y\,e^{-xz}\,ds$; C is the path $\vec{r}(t) = \langle t,\ 3t,\ -6t\rangle, \quad 0 \le t \le \ln 8$

Solution

$$\vec{r}\,' = \langle 1,\ 3,\ 6\rangle$$

$$|\vec{r}\,'| = \sqrt{1 + 9 + 36}$$

$$\underline{= \sqrt{46}}$$

$$\int_C ye^{-xz}ds = \sqrt{46}\int_0^{\ln 8} 3t\,e^{6t^2}\,dt$$

$$= \frac{\sqrt{46}}{4}\int_0^{\ln 8} e^{6t^2}\,d\left(6t^2\right)$$

$$= \frac{\sqrt{46}}{4}\left(e^{6t^2}\ \Big|_0^{3\ln 2}\right.$$

$$\underline{= \frac{\sqrt{46}}{4}\left(e^{54\ln 2} - 1\right)}$$

Exercise

Integrate $f(x, y, z) = \sqrt{x^2 + z^2}$ over the circle $\vec{r}(t) = (a\cos t)\,\hat{j} + (a\sin t)\,\hat{k}, \quad 0 \le t \le 2\pi$

Solution

$$\vec{r}\,' = \langle 0,\ -a\sin t,\ a\cos t\rangle$$

$$|\vec{r}\,'| = \sqrt{a^2\sin^2 t + a^2\cos^2 t}$$

$$\underline{= a}$$

$$f(t) = \sqrt{0 + a^2\sin^2 t}$$

$$\underline{= a|\sin t|}$$

$$\int_C f\ |r'|\,dt = a^2\int_0^{2\pi} |\sin t|\,dt$$

$$= a^2 \int_0^{\pi} \sin t \, dt \; + \; a^2 \int_{\pi}^{2\pi} \sin t \, dt$$

$$= -a^2 \left(\cos t \, \Big|_0^{\pi} \; - \; a^2 \left(\cos t \, \Big|_{\pi}^{2\pi}\right.\right.$$

$$= -a^2 (-1-1) - a^2 (1+1)$$

$$= 2a^2 + 2a^2$$

$$\underline{= 4a^2}$$

Exercise

Integrate $f(x,y,z) = \sqrt{x^2 + y^2}$ over the involute curve

$\vec{r}(t) = \langle \cos t + t\sin t, \quad \sin t - t\cos t \rangle, \quad 0 \le t \le \sqrt{3}$

Solution

$$\vec{r}' = \langle -\sin t + \sin t + t\cos t, \quad \cos t - \cos t + t\sin t \rangle$$

$$= \langle t\cos t, \quad t\sin t \rangle$$

$$|v| = \sqrt{t^2 \cos^2 t + t^2 \sin^2 t}$$

$$\underline{= t}$$

$$f(t) = \sqrt{(\cos t + t\sin t)^2 + (\sin t - t\cos t)^2}$$

$$= \sqrt{\cos^2 t + 2t\cos t \sin t + t^2 \sin^2 t + \sin^2 t - 2t\cos t \sin t + t^2 \cos^2 t}$$

$$= \sqrt{1 + t^2}$$

$$\int_C f \, |\vec{v}| \, dt = \int_0^{\sqrt{3}} t\sqrt{1+t^2} \; dt$$

$$= \frac{1}{2} \int_0^{\sqrt{3}} \left(1+t^2\right)^{1/2} \; dt$$

$$= \frac{1}{3}\left(1+t^2\right)^{3/2} \Big|_0^{\sqrt{3}}$$

$$= \frac{1}{3}(8-1)$$

$$\underline{= \frac{7}{3}}$$

Exercise

Find the average of the function on the given curves

$f(x, y) = x + 2y$ on the line segment from $(1, 1)$ to $(2, 5)$

Solution

$$\vec{r}(t) = \langle (2-1)t + 1,\ (5-1)t + 1 \rangle$$

$$= \langle t+1,\ 4t+1 \rangle$$

$$|\vec{r}'(t)| = \sqrt{1+16}$$

$$= \sqrt{17}$$

$$\int_C (x+2y)\,ds = \int_0^1 (t+1+2(4t+1)) \cdot \sqrt{17}\ dt$$

$$= \sqrt{17} \int_0^1 (9t+3)\ dt$$

$$= \sqrt{17} \left(\frac{9}{2}t^2 + 3t \right) \Big|_0^1$$

$$= \sqrt{17}\left(\frac{9}{2} + 3\right)$$

$$= \frac{15}{2}\sqrt{17}$$

The length of the line segment is $\sqrt{17}$

$\therefore$ The average value is $\frac{15}{2}$

Exercise

Find the average of the function on the given curves

$f(x, y) = x^2 + 4y^2$ on the circle of radius 9 centered at the origin.

Solution

$$\vec{r}(t) = \langle 9\cos t,\ 9\sin t \rangle \quad 0 \le t \le 2\pi$$

$$\vec{r}'(t) = \langle -9\sin t,\ 9\cos t \rangle$$

$$|\vec{r}'(t)| = \sqrt{81\sin^2 t + 81\cos^2 t}$$

$$= 9$$

$$\int_C \left(x^2 + 4y^2\right) ds = 9 \int_0^{2\pi} \left(81\cos^2 t + 324\sin^2 t\right)\ dt$$

$$= 729 \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}\cos 2t + 2 - 2\cos 2t \right) dt$$

$$= 729 \int_0^{2\pi} \left(\frac{5}{2} - \frac{3}{2}\cos 2t \right) dt$$

$$= 729 \left(\frac{5}{2}t - \frac{3}{4}\sin 2t \right. \Big|_0^{2\pi}$$

$$\underline{= 3,645\pi}$$

The circumference of the circle is $9(2\pi) = 18\pi$

$\therefore$ The average value is $\frac{3645\pi}{18\pi} = \frac{405}{2}$

Exercise

Find the average of the function on the given curves

$f(x, y) = xe^y$ on the circle of radius 1 centered at the origin.

Solution

$\vec{r}(t) = \langle \cos t, \ \sin t \rangle \quad 0 \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -\sin t, \ \cos t \rangle$

$\left| \vec{r}\,'(t) \right| = \sqrt{\sin^2 t + \cos^2 t}$

$\underline{= 1}$

$$\int_C xe^y ds = \int_0^{2\pi} \cos t \ e^{\sin t} \ dt$$

$$= \int_0^{2\pi} e^{\sin t} \ d(\sin t)$$

$$= e^{\sin t} \Big|_0^{2\pi}$$

$$= 1 - 1$$

$$\underline{= 0}$$

$\therefore$ The average value is 0

Exercise

Find the average of the function on the given curves

$$f(x, y) = \sqrt{4+9y^{2/3}} \text{ on the curve } y = x^{3/2}, \text{ for } 0 \le x \le 5$$

Solution

$$\vec{r}(t) = \left\langle t,\ t^{3/2} \right\rangle$$

$$\vec{r}'(t) = \left\langle 1,\ \frac{3}{2} t^{1/2} \right\rangle$$

$$\left|\vec{r}'(t)\right| = \sqrt{1 + \frac{9}{4} t}$$

$$= \frac{1}{2}\sqrt{4+9t}$$

$$\int_C \sqrt{4+9y^{2/3}}\, ds = \frac{1}{2}\int_0^5 \sqrt{4+9\left(t^{3/2}\right)^{2/3}}\ \sqrt{4+9t}\ dt$$

$$= \frac{1}{2}\int_0^5 \sqrt{4+9t}\ \sqrt{4+9t}\ dt$$

$$= \frac{1}{2}\int_0^5 (4+9t)\ dt$$

$$= \frac{1}{2}\left(4t + \frac{9}{2}t^2\right)\Big|_0^5$$

$$= \frac{1}{2}\left(20 + \frac{225}{2}\right)$$

$$= \frac{265}{4}$$

The length of the curve is

$$\int_0^5 \sqrt{4+9\left(x^{3/2}\right)^{2/3}}\ dx = \frac{1}{2}\int_0^5 \sqrt{4+9x}\ dx$$

$$= \frac{1}{18}\int_0^5 (4+9x)^{1/2}\ d(4+9x)$$

$$= \frac{1}{27}\ (4+9x)^{3/2}\Big|_0^5$$

$$= \frac{1}{27}(343-8)$$

$$=\boxed{\frac{335}{27}\quad unit}$$

$\therefore$ The average value is $=\frac{265}{4}\times\frac{27}{335}=\frac{1431}{268}$

Exercise

Find the length of the curve $\quad \vec{r}(t)=\left\langle 20\sin\frac{t}{4},\ 20\cos\frac{t}{4},\ \frac{t}{2}\right\rangle \quad 0\le t\le 2$

Solution

$$\vec{r}'(t)=\left\langle 5\cos\frac{t}{4},\ -5\sin\frac{t}{4},\ \frac{1}{2}\right\rangle$$

$$\begin{aligned}\left|\vec{r}'(t)\right| &= \sqrt{25\cos^2\frac{t}{4}+25\sin^2\frac{t}{4}+\frac{1}{4}} \\ &= \sqrt{25+\frac{1}{4}} \\ &= \boxed{\frac{1}{2}\sqrt{101}}\end{aligned}$$

$$\begin{aligned}L &= \int_0^2 \frac{1}{2}\sqrt{101}\ dt \\ &= \frac{1}{2}\sqrt{101}(2) \\ &= \boxed{\sqrt{101}\quad unit}\end{aligned}$$

Exercise

Find the length of the curve $\quad \vec{r}(t)=\langle 30\sin t,\ 40\sin t,\ 50\cos t\rangle \quad 0\le t\le 2\pi$

Solution

$$\begin{aligned}\left|\vec{r}'(t)\right| &= \sqrt{900\cos^2 t+1600\cos^2 t+2500\sin^2 t} \\ &= \sqrt{2500\cos^2 t+2500\sin^2 t} \\ &= \boxed{50}\end{aligned}$$

$$\begin{aligned}L &= \int_0^{2\pi} 50\ dt \\ &= \boxed{100\pi\quad unit}\end{aligned}$$

Exercise

Find the gradient field of the function $f(x,y,z)=\left(x^2+y^2+z^2\right)^{-1/2}$

Solution

$$\frac{\partial f}{\partial x}=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2x)$$
$$=-x\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial f}{\partial y}=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2y)$$
$$=-y\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial f}{\partial z}=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2z)$$
$$=-z\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\nabla f=-x\left(x^2+y^2+z^2\right)^{-3/2}\hat{i}-y\left(x^2+y^2+z^2\right)^{-3/2}\hat{j}-z\left(x^2+y^2+z^2\right)^{-3/2}\hat{k}$$
$$=\frac{-x\hat{i}-y\hat{j}-z\hat{k}}{\left(x^2+y^2+z^2\right)^{3/2}}$$

Exercise

Find the gradient field of the function $f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}$

Solution

$$f(x,\ y,\ z)=\ln\sqrt{x^2+y^2+z^2}$$
$$=\ln\left(x^2+y^2+z^2\right)^{1/2}$$
$$=\frac{1}{2}\ln\left(x^2+y^2+z^2\right)$$

$$\frac{\partial f}{\partial x}=\frac{1}{2}\frac{2x}{x^2+y^2+z^2}$$

$$= \frac{x}{x^2 + y^2 + z^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2} \frac{2y}{x^2 + y^2 + z^2}$$

$$= \frac{y}{x^2 + y^2 + z^2}$$

$$\frac{\partial f}{\partial z} = \frac{1}{2} \frac{2z}{x^2 + y^2 + z^2}$$

$$= \frac{z}{x^2 + y^2 + z^2}$$

$$\boxed{\nabla f = \frac{x\,\hat{i} + y\hat{j} + z\,\hat{k}}{x^2 + y^2 + z^2}}$$

Exercise

Find the gradient field of the function $f(x, y, z) = e^z - \ln\left(x^2 + y^2\right)$

Solution

$$\frac{\partial f}{\partial x} = -\frac{2x}{x^2 + y^2} \qquad \frac{\partial f}{\partial y} = -\frac{2y}{x^2 + y^2} \qquad \frac{\partial f}{\partial z} = e^z$$

$$\boxed{\nabla f = -\frac{2x}{x^2 + y^2}\,\hat{i} - \frac{2y}{x^2 + y^2}\,\hat{j} + e^z\,\hat{k}}$$

Exercise

Find the line integral of $\int_C (x - y)\,dx$ where $C:\ x = t, \quad y = 2t + 1, \quad for\ \ 0 \le t \le 3$

Solution

$x = t, \quad y = 2t + 1, \quad for\ \ 0 \le t \le 3$

$dx = dt$

$$\int_C (x - y)\,dx = \int_0^3 \left(t - (2t + 1)\right)dt$$

$$= \int_0^3 (-t - 1)\,dt$$

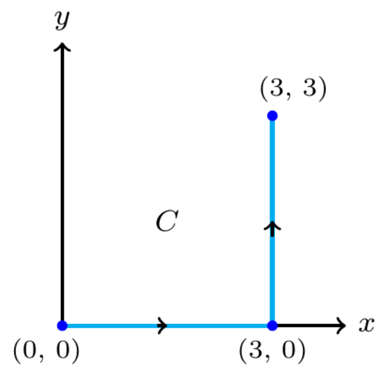
$$= -\left(\frac{1}{2}t^2 + t\right|_0^3$$

$$= -\left(\frac{9}{2} + 3\right)$$

$$\boxed{= -\frac{15}{2}}$$

Exercise

Find the line integral of $\int_C \left(x^2 + y^2\right) dy$ where C is



Solution

$C_1: \quad x = t, \quad y = 0, \quad 0 \le t \le 3 \quad \Rightarrow dy = 0$

$C_2: \quad x = 3, \quad y = t, \quad 0 \le t \le 3 \quad \Rightarrow dy = dt$

$$\int_C \left(x^2 + y^2\right) dy = \int_{C_1} \left(x^2 + y^2\right) dy + \int_{C_2} \left(x^2 + y^2\right) dy$$

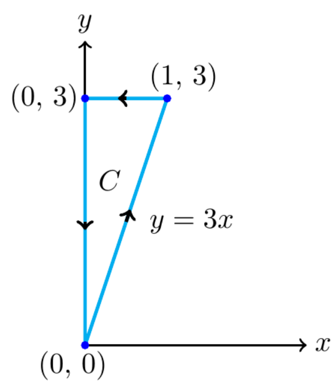
$$= \int_0^3 \left(t^2 + 0\right)(0) + \int_0^3 \left(9 + t^2\right) dt$$

$$= 9t + \frac{1}{3}t^3 \Big|_0^3$$

$$\boxed{= 36}$$

Exercise

Find the line integral of $\int_C \sqrt{x + y}\, dx$ where C is



Solution

$C_1: \quad x = t, \quad y = 3t, \quad 0 \le t \le 1$

$\Rightarrow dx = dt$

$C_2: \quad x = 1 - t, \quad y = 3, \quad 0 \le t \le 1$

$\Rightarrow dx = -dt$

$C_3: \quad x = 0, \quad y = 3 - t, \quad 0 \le t \le 3$

$\Rightarrow dx = 0$

$$\int_C \sqrt{x+y}\ dx = \int_{C_1} \sqrt{x+y}\ dx + \int_{C_2} \sqrt{x+y}\ dx + \int_{C_3} \sqrt{x+y}\ dx$$

$$= \int_0^1 \sqrt{t+3t}\ dt + \int_0^1 \sqrt{1-t+3}\ (-dt) + \int_0^3 \sqrt{3-t}\ (0)$$

$$= \int_0^1 2\sqrt{t}\ dt + \int_0^1 \sqrt{4-t}\ d(4-t)$$

$$= 2\left(\frac{2}{3}t^{3/2}\ \Big|_0^1 + \frac{2}{3}\left((4-t)^{3/2}\ \Big|_0^1\right.\right.$$

$$= \frac{4}{3} + \frac{2}{3}\left(3^{3/2} - 4^{3/2}\right)$$

$$= \frac{4}{3} + \frac{2}{3}\left(3\sqrt{3} - 8\right)$$

$$= \frac{4 + 6\sqrt{3} - 16}{3}$$

$$= \frac{6\sqrt{3} - 12}{3}$$

$$\boxed{= 2\sqrt{3} - 4}$$

Exercise

Find the work done by the force field $\vec{F} = xy\hat{i} + y\hat{j} - yz\hat{k}$ over the curve $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t\hat{k}, \quad 0 \le t \le 1$.

Solution

$$\frac{d\vec{r}}{dt} = \hat{i} + 2t\,\hat{j} + \hat{k}$$

$$\vec{F} = xy\hat{i} + y\hat{j} - yz\hat{k}$$

$$= t^3\hat{i} + t^2\hat{j} - t^3\hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left(t^3\hat{i} + t^2\hat{j} - t^3\hat{k}\right) \bullet \left(\hat{i} + 2t\hat{j} + \hat{k}\right)$$

$$= t^3 + 2t^3 - t^3$$

$$= 2t^3$$

$$Work = \int_0^1 \vec{F} \bullet \frac{d\vec{r}}{dt}\, dt$$

$$= \int_0^1 2t^3\ dt$$

$$=\frac{1}{2}t^4\Big|_0^1$$

$$=\frac{1}{2}$$

Exercise

Find the work done by the force field $\vec{F}=2y\hat{i}+3x\hat{j}+(x+y)\hat{k}$ over the curve $\vec{r}(t)=(\cos t)\hat{i}+(\sin t)\hat{j}+\frac{t}{6}\hat{k},\quad 0\le t\le 2\pi$.

Solution

$$\vec{F}=2y\hat{i}+3x\hat{j}+(x+y)\hat{k}$$

$$=(2\sin t)\hat{i}+(3\cos t)\hat{j}+(\cos t+\sin t)\hat{k}$$

$$\frac{d\vec{r}}{dt}=(-\sin t)\hat{i}+(\cos t)\hat{j}+\frac{1}{6}\hat{k}$$

$$\vec{F}\bullet\frac{d\vec{r}}{dt}=\left((2\sin t)\hat{i}+(3\cos t)\hat{j}+(\cos t+\sin t)\hat{k}\right)\bullet\left((-\sin t)\hat{i}+(\cos t)\hat{j}+\frac{1}{6}\hat{k}\right)$$

$$=-2\sin^2 t+3\cos^2 t+\frac{1}{6}\cos t+\frac{1}{6}\sin t$$

$$=-2\left(\frac{1-\cos 2t}{2}\right)+3\left(\frac{1+\cos 2t}{2}\right)+\frac{1}{6}\cos t+\frac{1}{6}\sin t$$

$$=\cos 2t-1+\frac{3}{2}+\frac{3}{2}\cos 2t+\frac{1}{6}\cos t+\frac{1}{6}\sin t$$

$$=\frac{1}{2}+\frac{5}{2}\cos 2t+\frac{1}{6}\cos t+\frac{1}{6}\sin t$$

$$Work=\int_0^{2\pi}\left(\frac{1}{2}+\frac{5}{2}\cos 2t+\frac{1}{6}\cos t+\frac{1}{6}\sin t\right)dt \qquad W=\int\vec{F}\bullet\frac{d\vec{r}}{dt}\,dt$$

$$=\frac{1}{2}t+\frac{5}{4}\sin 2t+\frac{1}{6}\sin t-\frac{1}{6}\cos t\Big|_0^{2\pi}$$

$$=\left(\pi-\frac{1}{6}\right)-\left(-\frac{1}{6}\right)$$

$$=\pi$$

Exercise

Find the work done by the force field $\vec{F}=z\hat{i}+x\hat{j}+y\hat{k}$ over the curve $\vec{r}(t)=(\sin t)\hat{i}+(\cos t)\hat{j}+t\hat{k},\quad 0\le t\le 2\pi$.

Solution

$$\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$$
$$= t\hat{i} + (\sin t)\,\hat{j} + (\cos t)\,\hat{k}$$

$$\frac{d\vec{r}}{dt} = (\cos t)\,\hat{i} + (-\sin t)\,\hat{j} + \hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left(t\hat{i} + (\sin t)\,\hat{j} + (\cos t)\,\hat{k}\right) \bullet \left((\cos t)\,\hat{i} + (-\sin t)\,\hat{j} + \hat{k}\right)$$
$$= t\cos t - \sin^2 t + \cos t$$
$$= t\cos t - \frac{1}{2} + \frac{1}{2}\cos 2t + \cos t$$

$$Work = \int_0^{2\pi} \vec{F} \bullet \frac{d\vec{r}}{dt}\,dt$$
$$= \int_0^{2\pi} \left(t\cos t - \frac{1}{2} + \frac{1}{2}\cos 2t + \cos t\right)dt$$

		$\int \cos t$
+	t	$\sin t$
−	1	$-\cos t$

$$= t\sin t + \cos t - \frac{1}{2}t + \frac{1}{4}\sin 2t + \sin t \,\Big|_0^{2\pi}$$
$$= (1-\pi) - (1)$$
$$\underline{= -\pi}$$

Exercise

Find the work required to move an object with given force field $\vec{F} = \langle -y,\ z,\ x\rangle$ on the path consisting of the line segments from $(0,\ 0,\ 0)$ to $(0,\ 1,\ 0)$ followed by the line segment from $(0,\ 1,\ 0)$ to $(0,\ 1,\ 4)$

Solution

$(0,\ 0,\ 0)$ to $(0,\ 1,\ 0)$ $\rightarrow \vec{r}_1(t) = \langle 0,\ t,\ 0\rangle$

$(0,\ 1,\ 0)$ to $(0,\ 1,\ 4)$ $\rightarrow \vec{r}_2(t) = \langle 0,\ 1,\ 4t\rangle$

$$\vec{r}_1'(t) = \langle 0,\ 1,\ 0\rangle$$
$$\vec{r}_2'(t) = \langle 0,\ 0,\ 4\rangle$$

$$\vec{F} \bullet \vec{r}_1'(t) = \langle -t,\ 0,\ 0\rangle \bullet \langle 0,\ 1,\ 0\rangle = 0$$
$$\vec{F} \bullet \vec{r}_2'(t) = \langle -1,\ 4t,\ 0\rangle \bullet \langle 0,\ 0,\ 4\rangle = 0$$

$$W = \int_0^1 (0+0)\,dt \qquad\qquad W = \int_C \vec{F}\cdot d\vec{r}$$

$$= 0$$

Exercise

Find the work required to move an object with given force field $\vec{F} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}}$ on the path $\vec{r}(t) = \left\langle t^2,\ 3t^2,\ -t^2\right\rangle$ for $1 \le t \le 2$

Solution

$$\frac{d\vec{r}}{dt} = \langle 2t,\ 6t,\ -2t\rangle$$

$$W = \int_1^2 \frac{\left\langle t^2,\ 3t^2,\ -t^2\right\rangle \bullet \langle 2t,\ 6t,\ -2t\rangle}{\left(t^4+9t^4+t^4\right)^{3/2}}\,dt \qquad\qquad W = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_1^2 \frac{2t^3+18t^3+2t^3}{\left(11t^4\right)^{3/2}}\,dt$$

$$= \frac{1}{11\sqrt{11}}\int_1^2 \frac{22t^3}{t^6}\,dt$$

$$= \frac{2}{\sqrt{11}}\int_1^2 t^{-3}\,dt$$

$$= -\frac{1}{\sqrt{11}}\,t^{-2}\Big|_1^2$$

$$= -\frac{1}{\sqrt{11}}\left(\frac{1}{4}-1\right)$$

$$= \frac{3}{4\sqrt{11}}$$

Exercise

Evaluate $\int_C \vec{F}\bullet\vec{T}\,ds$ for the vector field $\vec{F}=x^2\hat{i}-y\hat{j}$ along the curve $x=y^2$ from $(4,\ 2)$ to $(1,\ -1)$

Solution

$$\vec{r}=x\hat{i}+y\hat{j}$$
$$=y^2\hat{i}+y\hat{j} \qquad -1\le y\le 2$$
$$\vec{F}=x^2\hat{i}-y\hat{j}$$
$$=y^4\hat{i}-y\hat{j}$$
$$\frac{d\vec{r}}{dy}=2y\hat{i}+\hat{j}$$
$$\vec{F}\bullet\frac{d\vec{r}}{dy}=\left(y^4\hat{i}-y\hat{j}\right)\bullet\left(2y\hat{i}+\hat{j}\right)$$
$$=2y^5-y$$
$$\int_C \vec{F}\bullet\vec{T}\,ds=\int_2^{-1}\vec{F}\bullet\frac{d\vec{r}}{dy}\,dy$$
$$=\int_2^{-1}\left(2y^5-y\right)dy$$
$$=\frac{1}{3}y^6-\frac{1}{2}y^2\Big|_2^{-1}$$
$$=\left(\frac{1}{3}-\frac{1}{2}\right)-\left(\frac{64}{3}-2\right)$$
$$=-\frac{39}{2}$$

Exercise

Find the circulation and flux of the fields $\vec{F}_1=x\hat{i}+y\hat{j}$ *and* $\vec{F}_2=-y\hat{i}+x\hat{j}$ around and across each of the following curves.

a) The circle $\vec{r}(t)=(\cos t)\hat{i}+(\sin t)\hat{j},\quad 0\le t\le 2\pi$

b) The ellipse $\vec{r}(t)=(\cos t)\hat{i}+(4\sin t)\hat{j},\quad 0\le t\le 2\pi$

Solution

a) $\vec{r}(t)=(\cos t)\hat{i}+(\sin t)\hat{j},\quad 0\le t\le 2\pi$

$$\frac{d\vec{r}}{dt}=(-\sin t)\hat{i}+(\cos t)\hat{j}$$

$$\vec{F}_1 = x\hat{i} + y\hat{j}$$
$$= (\cos t)\hat{i} + (\sin t)\hat{j}$$

$$\vec{F}_1 \bullet \frac{d\vec{r}}{dt} = ((\cos t)\hat{i} + (\sin t)\hat{j}) \bullet ((-\sin t)\hat{i} + (\cos t)\hat{j})$$
$$= -\cos t \sin t + \sin t \cos t$$
$$= 0$$

$$\vec{F}_2 = -y\hat{i} + x\hat{j}$$
$$= -(\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\vec{F}_2 \bullet \frac{d\vec{r}}{dt} = (-(\sin t)\hat{i} + (\cos t)\hat{j}) \bullet ((-\sin t)\hat{i} + (\cos t)\hat{j})$$
$$= \sin^2 t + \cos^2 t$$
$$= 1$$

$$Cir_1 = \int_0^{2\pi} \left(\vec{F}_1 \bullet \frac{d\vec{r}}{dt}\right) dt$$
$$= \int_0^{2\pi} 0 dt$$
$$= 0$$

$$Cir_2 = \int_0^{2\pi} \left(\vec{F}_2 \bullet \frac{d\vec{r}}{dt}\right) dt$$
$$= \int_0^{2\pi} dt$$
$$= 2\pi$$

$$dx = -\sin t \; dt, \quad dy = \cos t \; dt$$
$$M_1 = x = \cos t, \quad N_1 = y = \sin t$$
$$M_2 = -y = -\sin t, \quad N_2 = x = \cos t$$

$$Flux_1 = \int_C M_1 dy - N_1 dx$$
$$= \int_0^{2\pi} \left(\cos^2 t + \sin^2 t\right) dt$$
$$= \int_0^{2\pi} dt$$
$$= 2\pi$$

$$Flux_2 = \int_C M_2 dy - N_2 dx$$

$$= \int_0^{2\pi} (-\sin t \cos t + \sin t \cos t) dt$$

$$= \int_0^{2\pi} (0) dt$$

$$\underline{= 0}$$

b) $\vec{r}(t) = (\cos t)\hat{i} + (4\sin t)\hat{j}, \quad 0 \le t \le 2\pi$

$$\frac{d\vec{r}}{dt} = (-\sin t)\hat{i} + (4\cos t)\hat{j}$$

$$\vec{F}_1 = x\hat{i} + y\hat{j}$$

$$= (\cos t)\hat{i} + (4\sin t)\hat{j}$$

$$\vec{F}_1 \bullet \frac{d\vec{r}}{dt} = ((\cos t)\hat{i} + (4\sin t)\hat{j}) \bullet ((-\sin t)\hat{i} + (4\cos t)\hat{j})$$

$$= -\cos t \sin t + 16 \sin t \cos t$$

$$= 15 \sin t \cos t$$

$$\vec{F}_2 = -y\hat{i} + x\hat{j}$$

$$= -(4\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\vec{F}_2 \bullet \frac{d\vec{r}}{dt} = ((-4\sin t)\hat{i} + (\cos t)\hat{j}) \bullet ((-\sin t)\hat{i} + (4\cos t)\hat{j})$$

$$= 4\sin^2 t + 4\cos^2 t$$

$$\underline{= 4}$$

$$Cir_1 = \int_0^{2\pi} \left(\vec{F}_1 \bullet \frac{d\vec{r}}{dt}\right) dt$$

$$= \int_0^{2\pi} 15 \sin t \cos t dt \qquad d(\sin t) = \cos t \ dt$$

$$= 15 \int_0^{2\pi} \sin t \ d(\sin t)$$

$$= \frac{15}{2} \left(\sin^2 t \ \right|_0^{2\pi}$$

$$= \frac{15}{2}(1 - 1)$$

$$\underline{= 0}$$

$$Cir_2 = \int_0^{2\pi} \left(\vec{F}_2 \cdot \frac{d\vec{r}}{dt}\right) dt$$

$$= \int_0^{2\pi} 4\, dt$$

$$= 4t \Big|_0^{2\pi}$$

$$\underline{= 8\pi}$$

$$dx = -\sin t\ dt, \quad dy = 4\cos t\ dt$$
$$M_1 = x = \cos t, \quad N_1 = y = 4\sin t$$
$$M_2 = -y = -4\sin t, \quad N_2 = x = \cos t$$

$$Flux_1 = \int_C M_1 dy - N_1 dx$$

$$= \int_0^{2\pi} \left(4\cos^2 t + 4\sin^2 t\right) dt$$

$$= 4\int_0^{2\pi} dt$$

$$\underline{= 8\pi}$$

$$Flux_2 = \int_C M_2 dy - N_2 dx$$

$$= -15 \int_0^{2\pi} (\sin t \cos t)\, dt$$

$$= -15 \int_0^{2\pi} \sin t\ d(\sin t)$$

$$= -15\left(\frac{1}{2}\sin^2 t \right|_0^{2\pi}$$

$$\underline{= 0}$$

Exercise

Find the circulation and flux of the fields $\vec{F}_1 = 2x\hat{i} - 3y\hat{j}$ *and* $\vec{F}_2 = 2x\hat{i} + (x-y)\hat{j}$ across the circle $\vec{r}(t) = (a\cos t)\hat{i} + (a\sin t)\hat{j}, \quad 0 \le t \le 2\pi$

Solution

$$\frac{d\vec{r}}{dt} = (-a\sin t)\hat{i} + (a\cos t)\hat{j}$$

$$\vec{F}_1 = 2x\hat{i} - 3y\hat{j}$$

$$= (2a\cos t)\hat{i} - (3a\sin t)\hat{j}$$

$$\vec{F}_1 \cdot \frac{d\vec{r}}{dt} = \big((2a\cos t)\hat{i} - (3a\sin t)\hat{j}\big) \cdot \big((-a\sin t)\hat{i} + (a\cos t)\hat{j}\big)$$

$$= -5a^2\cos t\sin t$$

$$\vec{F}_2 = 2x\hat{i} + (x-y)\hat{j}$$

$$= (2a\cos t)\hat{i} + a(\cos t - \sin t)\hat{j}$$

$$\vec{F}_2 \cdot \frac{d\vec{r}}{dt} = \big((2a\cos t)\hat{i} + a(\cos t - \sin t)\hat{j}\big) \cdot \big((-a\sin t)\hat{i} + (a\cos t)\hat{j}\big)$$

$$= -2a^2\cos t\sin t + a^2\cos^2 t - a^2\cos t\sin t$$

$$= a^2\left(\cos^2 t - 3\cos t\sin t\right)$$

$$Cir_1 = \int_0^{2\pi}\left(\vec{F}_1 \cdot \frac{d\vec{r}}{dt}\right)dt$$

$$= -5a^2\int_0^{2\pi} \sin t\cos t\,dt$$

$$= -5a^2\int_0^{2\pi} \sin t\; d(\sin t)$$

$$= -5a^2\left(\sin^2 t\right.\Big|_0^{2\pi}$$

$$= 0$$

$$Cir_2 = \int_0^{2\pi}\left(\vec{F}_2 \cdot \frac{d\vec{r}}{dt}\right)dt$$

$$= a^2\int_0^{2\pi}\left(\cos^2 t - 3\cos t\sin t\right)dt$$

$$= a^2\int_0^{2\pi}\left(\frac{1}{2} + \frac{1}{2}\cos 2t\right)dt \; - \; 3a^2\int_0^{2\pi}(\sin t)\,d(\sin t)$$

$$= a^2 \left(\frac{1}{2}t + \frac{1}{4}\sin 2t - 0 \right|_0^{2\pi}$$

$$\underline{= \pi a^2}$$

$dx = -a\sin t\ dt, \quad dy = a\cos t\ dt$

$M_1 = 2x = 2a\cos t, \quad N_1 = -3y = -3a\sin t$

$M_2 = 2a\cos t, \quad N_2 = a\cos t - a\sin t$

$$Flux_1 = \int_C M_1 dy - N_1 dx$$

$$= \int_0^{2\pi} \left(2a^2\cos^2 t - 3a^2\sin^2 t\right) dt \qquad \cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t, \quad \sin^2 t = \frac{1}{2} - \frac{1}{2}\cos 2t$$

$$= a^2 \int_0^{2\pi} \left(1 + \cos 2t - \frac{3}{2} + \frac{3}{2}\cos 2t\right) dt$$

$$= a^2 \int_0^{2\pi} \left(\frac{5}{2}\cos 2t - \frac{1}{2}\right) dt$$

$$= a^2 \left(\frac{5}{4}\sin 2t - \frac{1}{2}t \right|_0^{2\pi}$$

$$= a^2 \left[0 - \frac{1}{2}(2\pi)\right]$$

$$\underline{= -\pi a^2}$$

$$Flux_2 = \int_C M_2 dy - N_2 dx$$

$$= \int_0^{2\pi} \left(2a^2\cos^2 t - a^2\sin^2 t + a^2\cos t\sin t\right) dt$$

$$= a^2 \int_0^{2\pi} \left(1 + \cos 2t - \frac{1}{2} + \frac{1}{2}\cos 2t\right) dt \ + \ a^2 \int_0^{2\pi} (\sin t)\, d(\sin t)$$

$$= a^2 \left(\frac{1}{2}t + \frac{3}{4}\sin 2t + \frac{1}{2}\sin^2 t \right|_0^{2\pi}$$

$$= a^2 \frac{1}{2}(2\pi)$$

$$\underline{= \pi a^2}$$

Exercise

Find a field $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ in the *xy*-plane with the property that at each point $(x,y) \neq (0,0)$, $\vec{F}$ points toward the origin and $|\vec{F}|$ is

a) The distance from (*x, y*) to the origin

b) Inversely proportional to the distance from (*x, y*) to the origin. (The field is undefined at (0, 0).)

Solution

a) The slope of the line through the origin and a point (x,y) is: $m = \frac{y}{x}$

The vector parallel to the line is given by: $\vec{v} = x\hat{i} + y\hat{j}$

Pointing away from the origin: $\vec{F} = -\frac{\vec{v}}{|\vec{v}|} = -\frac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}}$ is the unit vector pointing toward the origin.

$$|\vec{F}| = \sqrt{x^2+y^2}$$

$$\vec{F} = \sqrt{x^2+y^2}\left(-\frac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}}\right)$$

$$= -x\hat{i} - y\hat{j}$$

b) $|\vec{F}| = \frac{C}{\sqrt{x^2+y^2}}, \quad C \neq 0$

$$\vec{F} = \frac{C}{\sqrt{x^2+y^2}}\left(-\frac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}}\right)$$

$$= -C\left(\frac{x\hat{i} + y\hat{j}}{x^2+y^2}\right)$$

Exercise

A fluid's velocity field is $\vec{F} = -4xy\hat{i} + 8y\hat{j} + 2\hat{k}$. Find the flow along the curve $\vec{r}(t) = t\hat{i} + t^2\hat{j} + \hat{k}, \quad 0 \leq t \leq 2$

Solution

$$\frac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j}$$

$$\vec{F} = -4xy\hat{i} + 8y\hat{j} + 2\hat{k}$$

$$= -4t^3\hat{i} + 8t^2\hat{j} + 2\hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left(-4t^3\hat{i} + 8t^2\hat{j} + 2\hat{k}\right) \bullet \left(\hat{i} + 2t\,\hat{j}\right)$$

$$= -4t^3 + 16t^3 = 12t^3$$

$$Flow = \int_R \vec{F} \bullet \frac{d\vec{r}}{dt}\,dt$$

$$= \int_0^2 12t^3\,dt$$

$$= 3t^4 \Big|_0^2$$

$$\underline{= 48}$$

Exercise

A fluid's velocity field is $\vec{F} = x^2\hat{i} + yz\hat{j} + y^2\hat{k}$. Find the flow along the curve $\vec{r}(t) = 3t\,\hat{j} + 4t\,\hat{k}, \quad 0 \le t \le 1$

Solution

$$\frac{d\vec{r}}{dt} = 3\,\hat{i} + 4\hat{j}$$

$$\vec{F} = x^2\,\hat{i} + yz\hat{j} + y^2\,\hat{k}$$

$$= 12t^2\,\hat{j} + 9t^2\,\hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left(12t^2\,\hat{j} + 9t^2\hat{k}\right) \bullet \left(3\hat{i} + 4\,\hat{j}\right)$$

$$= 36t^2 + 36t^2$$

$$\underline{= 72t^2}$$

$$Flow = \int_0^1 72t^2\,dt$$

$$Flow = \int_C \vec{F} \bullet \frac{d\vec{r}}{dt}\,dt$$

$$= 24t^3 \Big|_0^1$$

$$\underline{= 24}$$

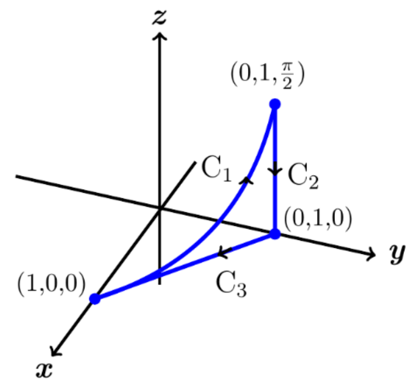
Exercise

Find the circulation of $\vec{F} = 2x\hat{i} + 2z\hat{j} + 2y\hat{k}$ around the closed path consisting of the following three curves traversed in the direction of increasing *t*.

$C_1: \quad \vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}, \quad 0 \le t \le \frac{\pi}{2}$

$C_2: \quad \vec{r}(t) = \hat{j} + \frac{\pi}{2}(1-t)\hat{k}, \quad 0 \le t \le 1$

$C_3: \quad \vec{r}(t) = t\hat{i} + (1-t)\hat{j}, \quad 0 \le t \le 1$



Solution

$C_1: \quad \vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}, \quad 0 \le t \le \frac{\pi}{2}$

$$\frac{d\vec{r}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j} + \hat{k}$$

$$\vec{F} = 2x\hat{i} + 2z\hat{j} + 2y\hat{k}$$

$$= (2\cos t)\hat{i} + 2t\hat{j} + (2\sin t)\hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left((2\cos t)\hat{i} + 2t\hat{j} + (2\sin t)\hat{k}\right) \bullet \left((-\sin t)\hat{i} + (\cos t)\hat{j} + \hat{k}\right)$$

$$= -2\sin t\cos t + 2t\cos t + 2\sin t$$

$$= -\sin 2t + 2t\cos t + 2\sin t$$

$$Flow_1 = \int_0^{\pi/2} \left(-\sin 2t + 2t\cos t + 2\sin t\right)dt$$

		$\int \cos t$
+	t	$\sin t$
−	1	$-\cos t$

$$= \left(\frac{1}{2}\cos 2t + 2t\sin t + 2\cos t - 2\cos t\right|_0^{\pi/2}$$

$$= \left(\frac{1}{2}\cos 2t + 2t\sin t\right|_0^{\pi/2}$$

$$= \left(-\frac{1}{2} + 2\frac{\pi}{2}\right) - \left(\frac{1}{2}\right)$$

$$\underline{= \pi - 1}$$

$C_2: \quad \vec{r}(t) = \hat{j} + \frac{\pi}{2}(1-t)\hat{k}, \quad 0 \le t \le 1$

$$\frac{d\vec{r}}{dt} = -\frac{\pi}{2}\hat{k}$$

$$\vec{F} = 2x\hat{i} + 2z\hat{j} + 2y\hat{k}$$

$$= \pi(1-t)\hat{j} + 2\hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left(\pi(1-t)\hat{j} + 2\hat{k}\right) \bullet \left(-\frac{\pi}{2}\hat{k}\right)$$
$$= -\pi$$

$$Flow_2 = \int_0^1 (-\pi)\,dt$$
$$= -\pi\ t \Big|_0^1$$
$$= -\pi$$

$C_3: \ \vec{r}(t) = t\ \hat{i} + (1-t)\hat{j}, \quad 0 \le t \le 1$

$$\frac{d\vec{r}}{dt} = \hat{i} - \hat{j}$$
$$\vec{F} = 2x\hat{i} + 2z\hat{j} + 2y\hat{k}$$
$$= 2t\hat{i} + 2(1-t)\hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left(2t\hat{i} + 2(1-t)\hat{k}\right) \bullet \left(\hat{i} - \hat{j}\right)$$
$$= 2t$$

$$Flow_3 = \int_0^1 (2t)\,dt$$
$$= t^2 \Big|_0^1$$
$$= 1$$

$$Circulation = Flow_1 + Flow_2 + Flow_3$$
$$= \pi - 1 - \pi + 1$$
$$= 0$$

Exercise

The field $\vec{F} = xy\hat{i} + y\hat{j} - yz\hat{k}$ is the velocity field of a flow in space. Find the flow from $(0,\ 0,\ 0)$ to $(1,\ 1,\ 1)$ along the curve of intersection of the cylinder $y = x^2$ and the plane $z = x$. (***Hint***: Use $t = x$ as the parameter.)

Solution

Let $x = t \quad \Rightarrow \quad y = x^2 = t^2$

$z = x = t$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$= t\hat{i} + t^2\hat{j} + t\hat{k} \qquad 0 \le t \le 1$$

$$\frac{d\vec{r}}{dt} = \hat{i} + 2t\,\hat{j} + \hat{k}$$

$$\vec{F} = xy\hat{i} + y\hat{j} - yz\hat{k}$$

$$= t^3\hat{i} + t^2\hat{j} - t^3\hat{k}$$

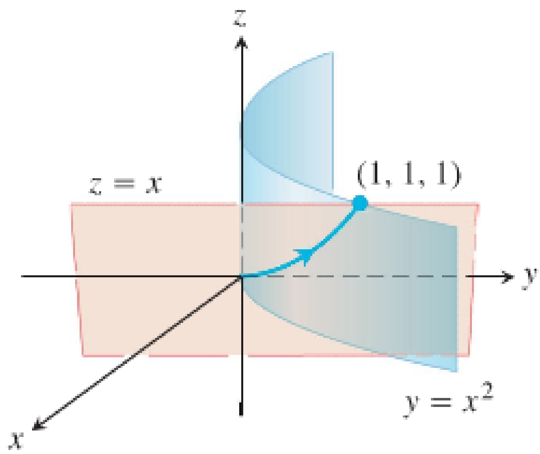
$$\vec{F} \cdot \frac{d\vec{r}}{dt} = \left(t^3\hat{i} + t^2\hat{j} - t^3\hat{k}\right) \cdot \left(\hat{i} + 2t\,\hat{j} + \hat{k}\right)$$

$$= t^3 + 2t^3 - t^3 = 2t^3$$

$$Flow = \int_0^1 \left(2t^3\right) dt$$

$$= \frac{1}{2}\, t^4 \Big|_0^1$$

$$= \frac{1}{2}$$



Exercise

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \nabla\left(x^2 y\right)$; $\quad C:\ \vec{r}(t) = \left\langle 9 - t^2,\ t\right\rangle, \quad for \ \ 0 \le t \le 3$

Solution

$$\vec{F} = \nabla\left(x^2 y\right)$$

$$= \left\langle 2xy,\ x^2\right\rangle$$

$$= \left\langle 18t - 2t^3,\ 81 - 18t^2 + t^4\right\rangle$$

$$\vec{r}'(t) = \langle -2t,\ 1\rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^3 \left\langle 18t - 2t^3,\ 81 - 18t^2 + t^4\right\rangle \cdot \langle -2t,\ 1\rangle\, dt$$

$$= \int_0^3 \left(-36t^2 + 4t^4 + 81 - 18t^2 + t^4\right) dt$$

$$= \int_0^3 \left(5t^4 - 54t^2 + 81\right) dt$$

$$= t^5 - 18t^3 + 81t \Big|_0^3$$

$$= 243 - 486 + 243$$

$$\underline{= 0}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \nabla(xyz); \quad C: \vec{r}(t) = \left\langle \cos t,\ \sin t,\ \frac{t}{\pi} \right\rangle, \quad for\ \ 0 \le t \le \pi$

Solution

$$\vec{F} = \nabla(xyz)$$

$$= \langle yz,\ xz,\ xy \rangle$$

$$= \left\langle \frac{t}{\pi}\sin t,\ \frac{t}{\pi}\cos t,\ \cos t \sin t \right\rangle$$

$$\vec{r}'(t) = \left\langle -\sin t,\ \cos t,\ \frac{1}{\pi} \right\rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^\pi \left\langle \frac{t}{\pi}\sin t,\ \frac{t}{\pi}\cos t,\ \cos t \sin t \right\rangle \cdot \left\langle -\sin t,\ \cos t,\ \frac{1}{\pi} \right\rangle dt$$

$$= \int_0^\pi \left(-\frac{t}{\pi}\sin^2 t + \frac{t}{\pi}\cos^2 t + \frac{1}{\pi}\cos t \sin t \right) dt$$

$$= \frac{1}{\pi}\int_0^\pi \left(t\cos 2t + \frac{1}{2}\sin 2t \right) dt$$

		$\int \cos 2t$
+	t	$\frac{1}{2}\sin 2t$
−	1	$-\frac{1}{4}\cos 2t$

$$= \frac{1}{\pi}\left(\frac{1}{2}t\sin 2t + \frac{1}{4}\cos 2t - \frac{1}{4}\cos 2t \right) \Big|_0^\pi$$

$$= \frac{1}{2\pi}\ \left(t\sin 2t \right. \Big|_0^\pi$$

$$\underline{= 0}$$

Or

$$\vec{F}=\nabla(xyz)=\nabla\varphi$$

$$\int_C \vec{F}\cdot d\vec{r}=\varphi(\pi)-\varphi(0)$$

$$=\varphi\left(\cos\pi\sin\pi\left(\frac{\pi}{\pi}\right)\right)-\varphi\left(\cos 0\sin 0\left(\frac{0}{\pi}\right)\right)$$

$$=0-0$$

$$\underline{=0}$$

Exercise

Evaluate the line integral $\int_C \vec{F}\bullet d\vec{r}$ .for the vector fields $\vec{F}$ and curves C.

$\vec{F}=\langle x,\ -y\rangle$; C is the square with vertices $(\pm 1,\ \pm 1)$ with counterclockwise orientation.

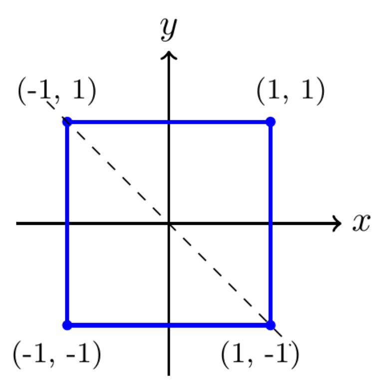
Solution

$(-1,\ -1)\ \to\ (1,\ -1)$

$$\begin{cases} x=-1+(1+1)t \\ y=-1+(-1+1)t \end{cases}$$

$$\vec{r}_1(t)=\langle -1+2t,\ -1\rangle$$

$$\vec{r}_1{}'(t)=\langle 2,\ 0\rangle$$



$(1,\ -1)\ \to\ (1,\ 1)$

$$\begin{cases} x=1+(1-1)t \\ y=-1+(1+1)t \end{cases}$$

$$\vec{r}_2(t)=\langle 1,\ -1+2t\rangle$$

$$\vec{r}_2{}'(t)=\langle 0,\ 2\rangle$$

$(1,\ 1)\ \to\ (-1,\ 1)$

$$\vec{r}_3(t)=\langle 1-2t,\ 1\rangle$$

$$\vec{r}_3{}'(t)=\langle -2,\ 0\rangle$$

$(-1,\ 1)\ \to\ (-1,\ -1)$

$$\vec{r}_4(t)=\langle -1,\ 1-2t\rangle$$

$$\vec{r}_4{}'(t)=\langle 0,\ -2\rangle$$

$$\vec{F}_1=\langle -1+2t,\ 1\rangle$$

$$\vec{F}_1 \cdot \vec{r}_1{}'(t) = \langle -1+2t,\ 1\rangle \cdot \langle 2,\ 0\rangle$$

$$= 4t - 2$$

$$\vec{F}_2 = \langle 1,\ 1-2t\rangle$$

$$\vec{F}_2 \cdot \vec{r}_2{}'(t) = \langle 1,\ 1-2t\rangle \cdot \langle 0,\ 2\rangle$$

$$= 2 - 4t$$

$$\vec{F}_3 = \langle 1-2t,\ -1\rangle$$

$$\vec{F}_3 \cdot \vec{r}_3{}'(t) = \langle 1-2t,\ -1\rangle \cdot \langle -2,\ 0\rangle$$

$$= 4t - 2$$

$$\vec{F}_4 = \langle -1,\ -1+2t\rangle$$

$$\vec{F}_4 \cdot \vec{r}_4{}'(t) = \langle -1,\ -1+2t\rangle \cdot \langle 0,\ -2\rangle$$

$$= 2 - 4t$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (4t-2+2-4t+4t-2+2-4t)\,dt$$

$$= 0$$

Or

$$\vec{F} = \nabla(xyz) = \nabla\varphi$$

$$= \nabla\left(\frac{1}{2}\left(x^2+y^2\right)\right)$$

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

Since the integral around any closed curve is 0.

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \langle y,\ z,\ -x\rangle; \quad C: \vec{r}(t) = \langle \cos t,\ \sin t,\ 4\rangle, \quad$ *for* $\ 0 \le t \le 2\pi$

Solution

$$\vec{F} = \langle \sin t,\ 4,\ -\cos t\rangle$$

$$\vec{r}'(t) = \langle -\sin t,\ \cos t,\ 0\rangle$$

$$\vec{F} \cdot \vec{r}'(t) = \langle \sin t,\ 4,\ -\cos t\rangle \cdot \langle -\sin t,\ \cos t,\ 0\rangle$$

$$= -\sin^2 t + 4\cos t$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(4\cos t - \sin^2 t\right) dt$$

$$= \int_0^{2\pi} \left(4\cos t - \frac{1}{2} - \frac{1}{2}\cos 2t\right) dt$$

$$= 4\sin t - \frac{1}{2}t - \frac{1}{2}\cos 2t \Big|_0^{2\pi}$$

$$\underline{= -\pi}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ for the vector fields $\vec{F}$ and curves C

$\vec{F} = \left\langle y^2,\ x \right\rangle$; where C is the arc of the parabola $x = 4 - y^2$ from $(-5,\ -3)$ to $(0,\ 2)$

Solution

Let $y = t \quad \rightarrow \quad -3 \le t \le 2$

$\vec{r}(t) = \left\langle 4 - t^2,\ t \right\rangle$

$\vec{r}'(t) = \langle -2t,\ 1 \rangle$

$\vec{F} = \left\langle t^2,\ 4 - t^2 \right\rangle$

$$\vec{F} \cdot \vec{r}' = \left\langle t^2,\ 4 - t^2 \right\rangle \cdot \langle -2t,\ 1 \rangle$$
$$= -2t^3 + 4 - t^2$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{-3}^{2} \left(-2t^3 + 4 - t^2\right) dt$$

$$= -\frac{1}{2}t^4 + 4t - \frac{1}{3}t^3 \Big|_{-3}^{2}$$

$$= -8 + 8 - \frac{8}{3} + \frac{81}{2} + 12 - 9$$

$$= \frac{-16 + 243 + 18}{6}$$

$$\underline{= \frac{245}{6}}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ for the vector fields $\vec{F}$ and curves C

$\vec{F} = \left\langle x^2 + y^2,\ 4x + y^2 \right\rangle$; where C is the straight line segment from $(6,\ 3)$ to $(6,\ 0)$

Solution

$$\vec{r}(t) = \langle 6,\ 3 - 3t \rangle$$

$$\vec{r}'(t) = \langle 0,\ -3 \rangle$$

$$\vec{F} = \left\langle 36 + 9 - 18t + 9t^2,\ 24 + 9 - 18t + 9t^2 \right\rangle$$

$$= \left\langle 45 - 18t + 9t^2,\ 33 - 18t + 9t^2 \right\rangle$$

$$\vec{F} \cdot \vec{r}' = \left\langle 45 - 18t + 9t^2,\ 33 - 18t + 9t^2 \right\rangle \cdot \langle 0,\ -3 \rangle$$

$$= -99 + 54t - 27t^2$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left(-99 + 54t - 27t^2 \right) dt$$

$$= -99t + 27t^2 - 9t^3 \Big|_0^1$$

$$= -99 + 27 - 9$$

$$\underline{= -81}$$

OR

$(6,\ 3)$ to $(6,\ 0)$ is just a straight parallel to the x-axis.

$x = 6 \quad \& \ dx = 0$

$$\int_C \vec{F} \cdot d\vec{r} = \oint_C \left(x^2 + y^2 \right) dx + \left(4x + y^2 \right) dy$$

$$= \oint_C 0 + \left(24 + y^2 \right) dy$$

$$= \int_3^0 \left(24 + y^2 \right) dy$$

$$= 24y + \frac{1}{3} y^3 \Big|_3^0$$

$$= -72 - 9$$

$$\underline{= -81}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot \vec{T}\, ds$ for the vector fields $\vec{F}$ and curves C.

$$\vec{F} = \langle x,\ y \rangle \text{ on the parabola } \vec{r}(t) = \left\langle 4t,\ t^2 \right\rangle \quad 0 \le t \le 1$$

Solution

$$\vec{F} = \left\langle 4t,\ t^2 \right\rangle$$

$$\vec{r}' = \langle 4,\ 2t \rangle$$

$$\vec{F} \cdot \vec{r}' = \left\langle 4t,\ t^2 \right\rangle \cdot \langle 4,\ 2t \rangle$$

$$= 16t + 2t^3$$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_0^1 \left(16t + 2t^3\right) dt$$

$$= 8t^2 + \frac{1}{2}t^4 \Big|_0^1$$

$$= 8 + \frac{1}{2}$$

$$= \frac{17}{2}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot \vec{T}\, ds$ for the vector fields $\vec{F}$ and curves C.

$$\vec{F} = \langle -y,\ x \rangle \text{ on the semicircle } \vec{r}(t) = \langle 4\cos t,\ 4\sin t \rangle \quad 0 \le t \le \pi$$

Solution

$$\vec{F} = \langle -4\sin t,\ 4\cos t \rangle$$

$$\vec{r}' = \langle -4\sin t,\ 4\cos t \rangle$$

$$\vec{F} \cdot \vec{r}' = \langle -4\sin t,\ 4\cos t \rangle \cdot \langle -4\sin t,\ 4\cos t \rangle$$

$$= 16\sin^2 t + 16\cos^2 t$$
$$= 16$$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_0^\pi 16\, dt$$

$$= 16t \Big|_0^\pi$$

$$= 16\pi$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot \vec{T}\, ds$ for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \langle y,\ x \rangle$ on the line segment from $(1,\ 1)$ to $(5,\ 10)$

Solution

$$\begin{aligned}\vec{r}(t) &= \langle (5-1)t+1,\ (10-1)t+1 \rangle \\ &= \langle 4t+1,\ 9t+1 \rangle\end{aligned}$$

$$\vec{F} = \langle 9t+1,\ 4t+1 \rangle$$

$$\vec{r}\,' = \langle 4,\ 9 \rangle$$

$$\begin{aligned}\vec{F} \cdot \vec{r}\,' &= \langle 9t+1,\ 4t+1 \rangle \cdot \langle 4,\ 9 \rangle \\ &= 36t+4+36t+9 \\ &= 72t+13\end{aligned}$$

$$\begin{aligned}\int_C \vec{F} \cdot \vec{T}\, ds &= \int_0^1 (72t+13)\, dt \\ &= 36t^2+13t \Big|_0^1 \\ &= 36+13 \\ &= 49\end{aligned}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot \vec{T}\, ds$ for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \langle -y,\ x \rangle$ on the parabola $y = x^2$ from $(0,\ 0)$ to $(1,\ 1)$

Solution

$$\vec{r}(t) = \left\langle t,\ t^2 \right\rangle \qquad\qquad \langle x = t,\ y \rangle$$

$$\vec{F} = \left\langle -t^2,\ t \right\rangle$$

$$\vec{r}\,' = \langle 1,\ 2t \rangle$$

$$\begin{aligned}\vec{F} \cdot \vec{r}\,' &= \left\langle -t^2,\ t \right\rangle \cdot \langle 1,\ 2t \rangle \\ &= t^2\end{aligned}$$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_0^1 t^2 dt$$

$$= \frac{1}{3}t^3 \Big|_0^1$$

$$\underline{= \frac{1}{3}}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot \vec{T}\, ds$ for the vector fields $\vec{F}$ and curves C.

$$\vec{F} = \frac{\langle x, y\rangle}{\left(x^2 + y^2\right)^{3/2}} \text{ on the curve } \vec{r}(t) = \left\langle t^2,\ 3t^2\right\rangle \quad 1 \le t \le 2$$

Solution

$$\vec{F} = \frac{\left\langle t^2,\ 3t^2\right\rangle}{\left(t^4 + 9t^4\right)^{3/2}}$$

$$= \frac{\left\langle t^2,\ 3t^2\right\rangle}{\left(10t^4\right)^{3/2}}$$

$$= \frac{1}{10\sqrt{10}} \frac{\left\langle t^2,\ 3t^2\right\rangle}{t^6}$$

$$= \frac{1}{10\sqrt{10}} \left\langle \frac{1}{t^4},\ \frac{3}{t^4}\right\rangle$$

$$\vec{r}' = \langle 2t,\ 6t\rangle$$

$$\vec{F} \cdot \vec{r}' = \frac{1}{10\sqrt{10}} \left\langle \frac{1}{t^4},\ \frac{3}{t^4}\right\rangle \cdot \langle 2t,\ 6t\rangle$$

$$= \frac{1}{10\sqrt{10}} \left(\frac{2}{t^3} + \frac{18}{t^3}\right)$$

$$= \frac{2}{\sqrt{10}} \frac{1}{t^3}$$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \frac{2}{\sqrt{10}} \int_1^2 t^{-3} dt$$

$$= -\frac{1}{\sqrt{10}}\ t^{-2}\Big|_1^2$$

$$= -\frac{\sqrt{10}}{10}\left(\frac{1}{4} - 1\right)$$

$$= \frac{3\sqrt{10}}{40}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot \vec{T}\ ds$ for the vector fields $\vec{F}$ and curves C.

$$\vec{F} = \frac{\langle x,\ y\rangle}{x^2 + y^2} \text{ on the line } \vec{r}(t) = \langle t,\ 4t\rangle \quad 1 \le t \le 10$$

Solution

$$\vec{F} = \frac{\langle t,\ 4t\rangle}{t^2 + 16t^2}$$

$$= \frac{1}{17}\left\langle \frac{1}{t},\ \frac{4}{t}\right\rangle$$

$$\vec{r}' = \langle 1,\ 4\rangle$$

$$\vec{F} \cdot \vec{r}' = \frac{1}{17}\left\langle \frac{1}{t},\ \frac{4}{t}\right\rangle \cdot \langle 1,\ 4\rangle$$

$$= \frac{1}{17}\left(\frac{1}{t} + \frac{16}{t}\right)$$

$$= \frac{1}{t}$$

$$\int_C \vec{F} \cdot \vec{T}\ ds = \int_1^{10} \frac{1}{t}\ dt$$

$$= \ln t\Big|_1^{10}$$

$$= \ln 10$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \langle y, \ -x \rangle$ on the path consisting of the line segment from $(1, \ 2)$ to $(0, \ 0)$ followed by the line segment from $(0, \ 0)$ to $(0, \ 4)$

Solution

$(1, \ 2)$ to $(0, \ 0)$

$$\vec{r}_1(t) = \langle 1-t, \ 2-2t \rangle$$

$$\vec{r}_1{}'(t) = \langle -1, \ -2 \rangle$$

$$\vec{F} = \langle 2-2t, \ t-1 \rangle$$

$$\begin{aligned} \vec{F} \bullet \vec{r}_1{}'(t) &= \langle 2-2t, \ t-1 \rangle \bullet \langle -1, \ -2 \rangle \\ &= -2+2t-2t+2 \\ &= 0 \end{aligned}$$

$(0, \ 0)$ to $(0, \ 4)$

$$\vec{r}_2(t) = \langle 0, \ 4t \rangle$$

$$\vec{r}_2{}'(t) = \langle 0, \ 4 \rangle$$

$$\vec{F} = \langle 4t, \ 0 \rangle$$

$$\begin{aligned} \vec{F} \bullet \vec{r}_2{}'(t) &= \langle 4t, \ 0 \rangle \bullet \langle 0, \ 4 \rangle \\ &= 0 \end{aligned}$$

$$\begin{aligned} \int_C \vec{F} \bullet d\vec{r} &= \int_0^1 \vec{F} \bullet \vec{r}_1{}' \ dt + \int_0^1 \vec{F} \bullet \vec{r}_2{}' \ dt \\ &= 0 \end{aligned}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \langle x, \ y \rangle$ on the path consisting of the line segment from $(-1, \ 0)$ to $(0, \ 8)$ followed by the line segment from $(0, \ 8)$ to $(2, \ 8)$

Solution

$(-1, \ 0)$ to $(0, \ 8)$

$$\vec{r}_1(t) = \langle t-1, \ 8t \rangle$$

$$\vec{r}_1{}'(t) = \langle 1, \ 8 \rangle$$

$$\vec{F} = \langle t-1,\ 8t\rangle$$

$$\vec{F} \bullet \vec{r}_1{}'(t) = \langle t-1,\ 8t\rangle \bullet \langle 1,\ 8\rangle$$
$$= t - 1 + 64t$$
$$= 65t - 1$$

$(0,\ 8)$ to $(2,\ 8)$

$$\vec{r}_2(t) = \langle 2t,\ 8\rangle$$

$$\vec{r}_2{}'(t) = \langle 2,\ 0\rangle$$

$$\vec{F} = \langle 2t,\ 8\rangle \text{o}$$

$$\vec{F} \bullet \vec{r}_2{}'(t) = \langle 2t,\ 8\rangle \bullet \langle 2,\ 0\rangle$$
$$= 4t$$

$$\int_C \vec{F} \bullet d\vec{r} = \int_0^1 \vec{F} \bullet \vec{r}_1{}'\ dt + \int_0^1 \vec{F} \bullet \vec{r}_2{}'\ dt$$

$$= \int_0^1 (65t - 1 + 4t)\, dt$$

$$= \frac{69}{2}t^2 - t \Big|_0^1$$

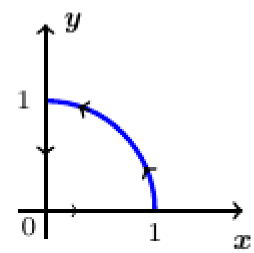
$$= \frac{69}{2} - 1$$

$$= \frac{67}{2}$$

Exercise

Find the work required to move an object on the given oriented curve $\vec{F} = \left\langle x^2,\ -xy\right\rangle$ on runs from $(1,\ 0)$ to $(0,\ 1)$ along the unit circle and then from $(0,\ 1)$ to $(0,\ 0)$ along the y-axis.

Solution



Along the unit circle: $\left(0 \le t \le \frac{\pi}{2}\right)$

$$\vec{r}_1(t) = \langle \cos t,\ \sin t\rangle$$

$$\vec{r}_1{}'(t) = \langle -\sin t,\ \cos t\rangle$$

$$\vec{F}_1 = \left\langle \cos^2 t,\ -\cos t \sin t\right\rangle$$

$$\overrightarrow{F_1} \cdot \vec{r}_1{}'(t) = \left\langle \cos^2 t,\ -\cos t \sin t \right\rangle \cdot \langle -\sin t,\ \cos t \rangle$$

$$= -\sin t \cos^2 t - \sin t \cos^2 t$$

$$\underline{= -2\sin t \cos^2 t\ |}$$

$(0,\ 1)$ to $(0,\ 0)$: $(0 \le t \le 1)$

$$\vec{r}_2(t) = \langle 0,\ t \rangle$$

$$\vec{r}_2{}'(t) = \langle 0,\ 1 \rangle$$

$$\vec{F}_1 = \langle 0,\ 0 \rangle$$

$$\overrightarrow{F_1} \cdot \vec{r}_1{}'(t)\ \underline{= 0\ |}$$

$$W = \int_0^{\frac{\pi}{2}} \left(-2\sin t \cos^2 t\right) dt + 0$$

$$W = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= 2\int_0^{\frac{\pi}{2}} \cos^2 t\ d(\cos t)$$

$$= \frac{2}{3} \cos^3 t \Big|_0^{\frac{\pi}{2}}$$

$$\underline{= -\frac{2}{3}\ |}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \langle y,\ x \rangle$ on the parabola $y = 2x^2$ from $(0,\ 0)$ to $(2,\ 8)$

Solution

$$\vec{r}(t) = \left\langle x,\ 2x^2 \right\rangle$$

$$= \left\langle 2t,\ 8t^2 \right\rangle \qquad 0 \le t \le 1$$

$$\vec{F} = \left\langle 8t^2,\ 2t \right\rangle$$

$$\vec{r}' = \langle 2,\ 16t \rangle$$

$$\vec{F} \cdot \vec{r}'(t) = \left\langle 8t^2,\ 2t \right\rangle \cdot \langle 2,\ 16t \rangle$$

$$= 16t^2 + 32t^2$$

$$= 48t^2$$

$$\int_C \vec{F} \bullet d\vec{r} = \int_0^1 48t^2\ dt$$

$$= 16\ t^3 \Big|_0^1$$

$$\underline{= 16\ }|$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \langle y,\ -x \rangle$ on the line $y = 10 - 2x$ from $(1,\ 8)$ to $(3,\ 4)$

Solution

$$\vec{r}(t) = \langle 2t + 1,\ -4t + 8 \rangle$$

$$\vec{F} = \langle 8 - 4t,\ -2t - 1 \rangle$$

$$\vec{r}' = \langle 2,\ -4 \rangle$$

$$\vec{F} \bullet \vec{r}'(t) = \langle 8 - 4t,\ -2t - 1 \rangle \bullet \langle 2,\ -4 \rangle$$

$$= 16 - 8t + 8t + 4$$

$$= 20$$

$$\int_C \vec{F} \bullet d\vec{r} = \int_0^1 20\ dt$$

$$\underline{= 20\ }|$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \langle x,\ y,\ z \rangle$ on the tilted ellipse $\vec{r}(t) = \langle 4\cos t,\ 4\sin t,\ 4\cos t \rangle \quad 0 \le t \le 2\pi$

Solution

$$\vec{F} = \langle 4\cos t,\ 4\sin t,\ 4\cos t \rangle$$

$$\vec{r}' = \langle -4\sin t,\ 4\cos t,\ -4\sin t \rangle$$

$$\vec{F} \bullet \vec{r}' = \langle 4\cos t,\ 4\sin t,\ 4\cos t \rangle \bullet \langle -4\sin t,\ 4\cos t,\ -4\sin t \rangle$$

$$= -16\cos t \sin t + 16\sin t \cos t - 16\cos t \sin t$$

$$= -16\cos t \sin t$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-16\cos t \sin t)\ dt$$

$$= \int_0^{2\pi} 16 \sin t\ d(\cos t)$$

$$= 8\sin^2 t \Big|_0^{2\pi}$$

$$\underline{= 0}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \langle -y,\ x,\ z \rangle$ on the helix $\vec{r}(t) = \left\langle 2\cos t,\ 2\sin t,\ \frac{t}{2\pi} \right\rangle \quad 0 \le t \le 2\pi$

Solution

$$\vec{F} = \left\langle -2\sin t,\ 2\cos t,\ \frac{t}{2\pi} \right\rangle$$

$$\vec{r}' = \left\langle -2\sin t,\ 2\cos t,\ \frac{1}{2\pi} \right\rangle$$

$$\vec{F} \cdot \vec{r}' = \left\langle -2\sin t,\ 2\cos t,\ \frac{t}{2\pi} \right\rangle \cdot \left\langle -2\sin t,\ 2\cos t,\ \frac{1}{2\pi} \right\rangle$$

$$= 4\sin^2 t + 4\cos^2 t + \frac{t}{4\pi^2}$$

$$= 4 + \frac{1}{4\pi^2} t$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(4 + \frac{1}{4\pi^2} t \right) dt$$

$$= 4t + \frac{1}{8\pi^2} t^2 \Big|_0^{2\pi}$$

$$\underline{= 8\pi + \frac{1}{2}}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ on the line segment from $(1,\ 1,\ 1)$ to $(10,\ 10,\ 10)$

Solution

$$\vec{r}(t) = \langle t+1,\ t+1,\ t+1 \rangle \quad 0 \le t \le 9$$

$$\vec{r}' = \langle 1,\ 1,\ 1 \rangle$$

$$\vec{F} = \frac{\langle t+1,\ t+1,\ t+1 \rangle}{\left(3(t+1)^2\right)^{3/2}}$$

$$= \frac{1}{3\sqrt{3}} \frac{\langle t+1,\ t+1,\ t+1 \rangle}{(t+1)^3}$$

$$= \frac{1}{3\sqrt{3}} \left\langle \frac{1}{(t+1)^2},\ \frac{1}{(t+1)^2},\ \frac{1}{(t+1)^2} \right\rangle$$

$$\vec{F} \bullet \vec{r}' = \frac{1}{3\sqrt{3}} \left\langle \frac{1}{(t+1)^2},\ \frac{1}{(t+1)^2},\ \frac{1}{(t+1)^2} \right\rangle \bullet \langle 1,\ 1,\ 1 \rangle$$

$$= \frac{1}{\sqrt{3}} \frac{1}{(t+1)^2}$$

$$\int_C \vec{F} \bullet d\vec{r} = \frac{1}{\sqrt{3}} \int_0^9 \frac{1}{(t+1)^2}\, dt$$

$$= \frac{1}{\sqrt{3}} \int_0^9 \frac{1}{(t+1)^2}\, d(t+1)$$

$$= -\frac{1}{\sqrt{3}} \left. \frac{1}{t+1} \right|_0^9$$

$$= -\frac{\sqrt{3}}{3}\left(\frac{1}{10} - 1\right)$$

$$= \frac{3\sqrt{3}}{10}$$

Exercise

Find the work required to move an object on the given oriented curve

$$\vec{F} = \frac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}} \text{ on the path } \vec{r}(t) = \left\langle t^2,\ 3t^2,\ -t^2 \right\rangle, \quad 1 \le t \le 2$$

Solution

$$\vec{r}' = \langle 2t,\ 6t,\ -2t \rangle$$

$$\vec{F} = \frac{\left\langle t^2,\ 3t^2,\ -t^2 \right\rangle}{\left(t^4 + 9t^4 + t^4\right)^{3/2}}$$

$$= \frac{1}{11\sqrt{11}}\ \frac{\left\langle t^2,\ 3t^2,\ -t^2 \right\rangle}{t^6}$$

$$= \frac{1}{11\sqrt{11}} \left\langle \frac{1}{t^4},\ \frac{3}{t^4},\ -\frac{1}{t^4} \right\rangle$$

$$\vec{F} \bullet \vec{r}\,' = \frac{1}{11\sqrt{11}} \left\langle \frac{1}{t^4},\ \frac{3}{t^4},\ -\frac{1}{t^4} \right\rangle \bullet \langle 2t,\ 6t,\ -2t \rangle$$

$$= \frac{1}{11\sqrt{11}} \left(\frac{2}{t^3} + \frac{18}{t^3} + \frac{2}{t^3} \right)$$

$$= \frac{2\sqrt{11}}{11}\ \frac{1}{t^3}$$

$$W = \frac{2\sqrt{11}}{11} \int_1^2 t^{-3}\ dt \qquad\qquad W = \int_C \vec{F} \bullet d\vec{r}$$

$$= -\frac{\sqrt{11}}{11}\ t^{-2} \Big|_1^2$$

$$= -\frac{\sqrt{11}}{11}\left(\frac{1}{4} - 1\right)$$

$$= \frac{3\sqrt{11}}{44}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \dfrac{\langle x,\ y \rangle}{\left(x^2 + y^2\right)^{3/2}}$ over the plane curve $\vec{r}(t) = \left\langle e^t \cos t,\ e^t \sin t \right\rangle$ from the point $(1,\ 0)$ to the point $\left(e^{2\pi},\ 0\right)$ by using the parametrization of the curve to evaluate the work integral

Solution

$$x = e^t \cos t \quad y = e^t \sin t$$

$$(1,\ 0) \ \Rightarrow \begin{cases} 1 = e^t \cos t \\ 0 = e^t \sin t \quad \rightarrow t = 0 \end{cases}$$

$$\left(e^{2\pi},\ 0\right) \Rightarrow \begin{cases} e^{2\pi} = e^t \cos t & \rightarrow\ t = 2\pi \\ 0 = e^t \sin t \end{cases}$$

$$\underline{0 \le t \le 2\pi}|$$

$$\vec{r}' = \left\langle e^t\left(\cos t - \sin t\right),\ e^t\left(\cos t + \sin t\right)\right\rangle$$

$$\vec{F} = \frac{\left\langle e^t \cos t,\ e^t \sin t\right\rangle}{\left(e^{2t}\cos^2 t + e^{2t}\sin^2 t\right)^{3/2}}$$

$$= \frac{\left\langle e^t \cos t,\ e^t \sin t\right\rangle}{e^{3t}}$$

$$= \left\langle \frac{\cos t}{e^{2t}},\ \frac{\sin t}{e^{2t}}\right\rangle$$

$$\vec{F} \cdot \vec{r}' = \left\langle \frac{\cos t}{e^{2t}},\ \frac{\sin t}{e^{2t}}\right\rangle \cdot \left\langle e^t\left(\cos t - \sin t\right),\ e^t\left(\cos t + \sin t\right)\right\rangle$$

$$= e^{-t}\left(\cos^2 t - \cos t \sin t + \sin^2 t + \cos t \sin t\right)$$

$$\underline{= e^{-t}}|$$

$$W = \int_0^{2\pi} e^{-t}\, dt \qquad\qquad W = \int_C \vec{F} \cdot d\vec{r}$$

$$= -e^{-t}\Big|_0^{2\pi}$$

$$\underline{= 1 - e^{-2\pi}}|$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \dfrac{\langle x,\ y,\ z\rangle}{x^2 + y^2 + z^2}$ on the line segment from $(1,\ 1,\ 1)$ to $(8,\ 4,\ 2)$

Solution

$$\vec{r}(t) = \langle 7t + 1,\ 3t + 1,\ t + 1\rangle \quad 0 \le t \le 1$$

$$\vec{r}' = \langle 7,\ 3,\ 1\rangle$$

$$\vec{F} = \frac{\langle 7t + 1,\ 3t + 1,\ t + 1\rangle}{(7t + 1)^2 + (3t + 1)^2 + (t + 1)^2}$$

$$= \frac{\langle 7t+1,\ 3t+1,\ t+1\rangle}{49t^2+14t+1+9t^2+6t+1+t^2+2t+1}$$

$$= \frac{\langle 7t+1,\ 3t+1,\ t+1\rangle}{59t^2+22t+3}$$

$$\vec{F}\cdot\vec{r}\,' = \frac{\langle 7t+1,\ 3t+1,\ t+1\rangle}{59t^2+22t+3}\cdot\langle 7,\ 3,\ 1\rangle$$

$$= \frac{49t+7+9t+3+t+1}{59t^2+22t+3}$$

$$= \frac{59t+11}{59t^2+22t+3}$$

$$W = \int_0^1 \frac{59t+11}{59t^2+22t+3}\,dt \qquad\qquad W = \int_C \vec{F}\cdot d\vec{r}$$

$$= \frac{1}{2}\int_0^1 \frac{1}{59t^2+22t+3}\,d\left(59t^2+22t+3\right)$$

$$= \frac{1}{2}\ln\left(59t^2+22t+3\right)\Big|_0^1$$

$$= \frac{1}{2}(\ln 84 - \ln 3)$$

$$= \frac{1}{2}\ln\frac{84}{3}$$

$$= \frac{1}{2}\ln 28$$

$$\boxed{= \ln\left(2\sqrt{7}\right)}$$

Exercise

Let C be the circle of radius 2 centered at the origin with counterclockwise orientation

a) Give the unit outward vector at any point $(x,\ y)$ on C.

b) Find the normal component of the vector field $\vec{F} = 2\langle y,\ -x\rangle$ at any point on C.

c) Find the normal component of the vector field $\vec{F} = \dfrac{\langle x,\ y\rangle}{x^2+y^2}$ at any point on C.

Solution

$r = 2$ @ origin, *ccw*.

a) $\langle x,\ y\rangle$ outward normal

$$\left|\langle x,\ y\rangle\right| = \sqrt{x^2+y^2}$$

$= r$

$= 2$

$\therefore$ unit outward normal: $\frac{1}{2}\langle x,\ y\rangle$

b) Normal component is:

$$\vec{F} \cdot \vec{n} = 2\langle y,\ -x\rangle \cdot \frac{1}{2}\langle x,\ y\rangle$$

$$= xy - xy$$

$$= 0$$

c) Normal component is:

$$\vec{F} \cdot \vec{n} = \frac{\langle x,\ y\rangle}{x^2 + y^2} \cdot \frac{1}{2}\langle x,\ y\rangle$$

$$= \frac{1}{2}\frac{x^2 + y^2}{x^2 + y^2}$$

$$= \frac{1}{2}$$

Exercise

Find the flow of the field $\vec{F} = \nabla\left(x^2 z e^y\right)$

a) Once around the ellipse C in which the plane $x + y + z = 1$ intersects the cylinder $x^2 + z^2 = 25$, clockwise as viewed from the positive y-axis.

b) Along the curved boundary of the helicoid $\vec{r}(r,\ \theta) = (r\cos\theta)\hat{i} + (r\sin\theta)\hat{j} + \theta\,\hat{k}$ from $(1,\ 0,\ 0)$ to $(1,\ 0,\ 2\pi)$

Solution

a) For any closed path C.

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

$\vec{F}$ is conservative.

b)

$$\int_C \vec{F} \cdot d\vec{r} = \int_{(1,\ 0,\ 0)}^{(1,\ 0,\ 2\pi)} \nabla\left(x^2 z e^y\right) dr$$

$$= \varphi(1,\ 0,\ 2\pi) - \varphi(1,\ 0,\ 0)$$

$$= x^2 z e^y \Big|_{(1,\ 0,\ 2\pi)} - x^2 z e^y \Big|_{(1,\ 0,\ 0)}$$

$$= 2\pi - 0$$

$$= 2\pi$$

Solution Section 4.4 – Green's Theorem

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = (x-y)\,\hat{i} + (y-x)\,\hat{j}$ and curve C is the square bounded by $x=0,\ x=1,\ y=0,\ y=1$

Solution

$$M = x - y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1$$

$$N = y - x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = -1, \quad \frac{\partial N}{\partial y} = 1$$

$$\begin{aligned}
Flux &= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy \\
&= \iint_R (1+1)\,dxdy \\
&= 2\int_0^1 \int_0^1 dxdy \\
&= 2\int_0^1 dy \\
&= 2
\end{aligned}$$

$$\begin{aligned}
Circulation &= \iint_R \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dxdy \\
&= \int_0^1 \int_0^1 \big(-1-(-1)\big)\,dxdy \\
&= 0
\end{aligned}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \left(x^2+4y\right)\hat{i} + \left(x+y^2\right)\hat{j}$ and curve C is the square bounded by $x=0,\ x=1,\ y=0,\ y=1$

Solution

$$M = x^2 + 4y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 2x, \quad \frac{\partial M}{\partial y} = 4$$

$$N = x + y^2 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = 2y$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_0^1 (2x + 2y)\, dxdy$$

$$= \int_0^1 \left(x^2 + 2yx \Big|_0^1 \right) dy$$

$$= \int_0^1 (1 + 2y)\, dy$$

$$= y + y^2 \Big|_0^1$$

$$\underline{= 2}$$

$$Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_0^1 (1 - 4)\, dxdy$$

$$= -3 \int_0^1 dy$$

$$\underline{= -3}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = (x + y)\,\hat{i} - \left(x^2 + y^2\right)\hat{j}$ and curve C is the triangle bounded by $y = 0,\ x = 1,\ y = x$

Solution

$$M = x + y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = 1$$

$$N = -\left(x^2 + y^2\right) \quad \Rightarrow \quad \frac{\partial N}{\partial x} = -2x, \quad \frac{\partial N}{\partial y} = -2y$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_0^x (1-2y)\,dydx$$

$$= \int_0^1 \left(y - y^2 \Big|_0^x \right. dx$$

$$= \int_0^1 \left(x - x^2\right) dx$$

$$= \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}$$

$$Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_0^x (-2x-1)\,dydx$$

$$= \int_0^1 \left(-2xy - y \Big|_0^x \right. dx$$

$$= \int_0^1 \left(-2x^2 - x\right) dx$$

$$= -\frac{2}{3}x^3 - \frac{1}{2}x^2 \Big|_0^1$$

$$= -\frac{2}{3} - \frac{1}{2}$$

$$= -\frac{7}{6}$$

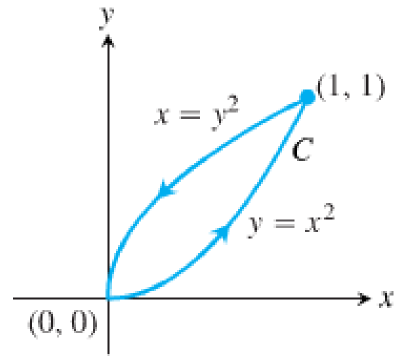
Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \left(xy + y^2\right)\hat{i} + (x-y)\hat{j}$ and curve C

Solution

$$M = xy + y^2 \quad \Rightarrow \quad \frac{\partial M}{\partial x} = y, \quad \frac{\partial M}{\partial y} = x + 2y$$

$$N = x - y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = -1$$



$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (y-1)\,dydx$$

$$= \int_0^1 \left(\frac{1}{2} y^2 - y \ \Big|_{x^2}^{\sqrt{x}} \ dx \right.$$

$$= \int_0^1 \left(\frac{1}{2} x - \sqrt{x} - \left(\frac{1}{2} x^4 - x^2 \right) \right) dx$$

$$= \int_0^1 \left(\frac{1}{2} x - x^{1/2} - \frac{1}{2} x^4 + x^2 \right) dx$$

$$= \frac{1}{4} x^2 - \frac{2}{3} x^{3/2} - \frac{1}{10} x^5 + \frac{1}{3} x^3 \Big|_0^1$$

$$= \frac{1}{4} - \frac{2}{3} - \frac{1}{10} + \frac{1}{3}$$

$$= -\frac{11}{60}$$

$$Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (1 - x - 2y)\,dydx$$

$$= \int_0^1 \left(y - xy - y^2 \ \Big|_{x^2}^{\sqrt{x}} \ dx \right.$$

$$= \int_0^1 \left(\sqrt{x} - x\sqrt{x} - x - x^2 + x^3 + x^4 \right) dx$$

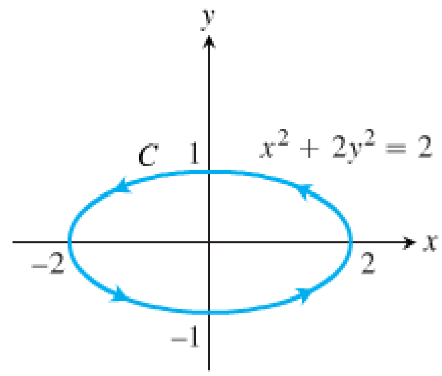
$$= \frac{2}{3} x^{3/2} - \frac{2}{5} x^{5/2} - \frac{1}{2} x^2 - \frac{1}{3} x^3 + \frac{1}{4} x^4 + \frac{1}{5} x^5 \Big|_0^1$$

$$= \frac{2}{3} - \frac{2}{5} - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

$$= -\frac{7}{60}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = (x+3y)\,\hat{i} + (2x-y)\,\hat{j}$ and curve C



Solution

$$M = x+3y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = 3$$

$$N = 2x-y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2, \quad \frac{\partial N}{\partial y} = -1$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{\left(2-x^2\right)/2}}^{\sqrt{\left(2-x^2\right)/2}} (1-1)\,dydx$$

$$= 0$$

$$Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{\left(2-x^2\right)/2}}^{\sqrt{\left(2-x^2\right)/2}} (2-3)\,dydx$$

$$= -\int_{-\sqrt{2}}^{\sqrt{2}} \left(\sqrt{\frac{2-x^2}{2}} + \sqrt{\frac{2-x^2}{2}}\right) dx$$

$$= -\frac{2}{\sqrt{2}} \int_{-\sqrt{2}}^{\sqrt{2}} \left(\sqrt{2-x^2}\right) dx$$

$$x = \sqrt{2}\,\sin\alpha \qquad \sqrt{2-x^2} = \sqrt{2}\,\cos\alpha$$

$$dx = \sqrt{2}\,\cos\alpha\; d\alpha$$

$$\int \sqrt{2-x^2}\; dx = \int \sqrt{2}\cos\alpha\left(\sqrt{2}\cos\alpha\right) d\alpha$$

$$= 2\int \cos^2\alpha\; d\alpha$$

$$= \int (1+\cos 2\alpha)\; d\alpha$$

$$= \alpha + \frac{1}{2}\sin 2\alpha$$

$$= \alpha + \sin\alpha\ \cos\alpha$$

$$= \sin^{-1}\frac{x}{\sqrt{2}} + \frac{x}{\sqrt{2}}\frac{\sqrt{2-x^2}}{\sqrt{2}}$$

$$= \sin^{-1}\frac{x}{\sqrt{2}} + \frac{1}{2}x\sqrt{2-x^2}$$

$$= -\frac{2}{\sqrt{2}}\left(\sin^{-1}\frac{x}{\sqrt{2}} + \frac{1}{2}x\sqrt{2-x^2}\right.\Bigg|_{-\sqrt{2}}^{\sqrt{2}}$$

$$= -\frac{2}{\sqrt{2}}\left[\sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) - \sin^{-1}\left(\frac{-\sqrt{2}}{\sqrt{2}}\right)\right]$$

$$= -\frac{2}{\sqrt{2}}\left(\sin^{-1}(1) + \sin^{-1}(1)\right)$$

$$= -\frac{2}{\sqrt{2}}\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= -\frac{2\pi}{\sqrt{2}}$$

$$\underline{= -\pi\sqrt{2}\ |}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \left(x + e^x \sin y\right)\hat{i} + \left(x + e^x \cos y\right)\hat{j}$ and curve C is the right-hand loop of the lemniscate $r^2 = \cos 2\theta$

Solution

$$M = x + e^x \sin y \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 1 + e^x \sin y, \quad \frac{\partial M}{\partial y} = e^x \cos y$$

$$N = x + e^x \cos y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1 + e^x \cos y, \quad \frac{\partial N}{\partial y} = -e^x \sin y$$

$$Flux = \iint_R \left(1 + e^x \sin y - e^x \sin y\right) dxdy \qquad\qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \iint_R dxdy$$

$$= \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r\ drd\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2}r^2 \Big|_0^{\sqrt{\cos 2\theta}}\right) d\theta$$

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos 2\theta \, d\theta$$

$$= \frac{1}{4} \sin 2\theta \Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{4}\big(1-(-1)\big)$$

$$= \frac{1}{2}$$

$$Circulation = \iint_R \left(1+e^x \cos y - e^x \cos y\right) dxdy \qquad Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \iint_R dxdy$$

$$= \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r \, dr d\theta$$

$$= \frac{1}{2}$$

Exercise

Use Green's Theorem to find the counterclockwise circulation and outward flux for the field and curves

Square: $\vec{F} = (2xy + x)\,\hat{i} + (xy - y)\,\hat{j}$ $\quad C:$ *The square bounded by* $x = 0,\ x = 1,\ y = 0,\ y = 1$

Solution

$$M = 2xy + x \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 2y + 1, \quad \frac{\partial M}{\partial y} = 2x$$

$$N = xy - y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = y, \quad \frac{\partial N}{\partial y} = x - 1$$

$$Flux = \iint_R (2y + 1 + x - 1)\, dxdy \qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \int_0^1 \int_0^1 (2y + x)\, dydx$$

$$= \int_0^1 \left(y^2 + xy \left. \right|_0^1 dx \right.$$

$$= \int_0^1 (1+x)\,dx$$

$$= x + \frac{1}{2}x^2 \Big|_0^1$$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2}$$

$$Cir = \int_0^1 \int_0^1 (y - 2x)\,dy\,dx \qquad\qquad Circulation = \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \int_0^1 \left(\frac{1}{2}y^2 - 2xy \left. \right|_0^1 dx \right.$$

$$= \int_0^1 \left(\frac{1}{2} - 2x \right) dx$$

$$= \frac{1}{2}x - x^2 \Big|_0^1$$

$$= \frac{1}{2} - 1$$

$$= -\frac{1}{2}$$

Exercise

Use Green's Theorem to find the counterclockwise circulation and outward flux for the field and curves

Triangle: $\vec{F} = \left(y - 6x^2 \right)\hat{i} + \left(x + y^2 \right)\hat{j}$

C: *The triangle made by the lines* $y = 0,\ y = x,\ and\ x = 1$

Solution

$M = y - 6x^2 \qquad N = x + y^2$

$\frac{\partial M}{\partial x} = -12x \qquad \frac{\partial N}{\partial x} = 1$

$\frac{\partial M}{\partial x} = -12x \qquad \frac{\partial N}{\partial y} = 2y$

$$Flux = \int_0^1 \int_y^1 (-12x + 2y)\, dxdy$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \int_0^1 \left(-6x^2 + 2yx \Big|_y^1 \right) dy$$

$$= \int_0^1 \left(-6 + 2y + 6y^2 - 2y^2 \Big|_y^1 \right) dy$$

$$= \int_0^1 \left(4y^2 + 2y - 6 \right) dy$$

$$= \frac{4}{3}y^3 + y^2 - 6y \Big|_0^1$$

$$= \frac{4}{3} + 1 - 6$$

$$= -\frac{11}{3}$$

$$Cir = \iint_R (1-1)\, dydx$$

$$Circulation = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= 0$$

Exercise

Find the circulation and the outward flux of the vector field $\vec{F} = \langle y - x,\ y \rangle$ for the curve $\vec{r}(t) = \langle 2\cos t,\ 2\sin t \rangle, \quad 0 \le t \le 2\pi$

Solution

$$\vec{F} = \langle 2\sin t - 2\cos t,\ 2\sin t \rangle$$

$$\vec{r}\,' = \langle -2\sin t,\ 2\cos t \rangle$$

$$\vec{F} \bullet \vec{r}\,' = \langle 2\sin t - 2\cos t,\ 2\sin t \rangle \bullet \langle -2\sin t,\ 2\cos t \rangle$$

$$= -4\sin^2 t + 4\cos t \sin t + 4\sin t \cos t$$

$$= -4\sin^2 t + 8\cos t \sin t$$

$$Cir = \int_0^{2\pi} (2\cos 2t - 2 + 4\sin 2t)\, dt$$

$$Circulation = \int_C \vec{F} \bullet \vec{T}\, ds$$

$$= \sin 2t - 2t - 2\cos 2t \Big|_0^{2\pi}$$

$$= -4\pi - 2 + 2$$

$$\underline{= -4\pi}\ \rfloor$$

$$dy = d(2\sin t) = 2\cos t\ dt$$

$$dx = d(2\cos t) = -2\sin 2t\ dt$$

$$Flux = \int_0^{2\pi} \big((2\sin t - 2\cos t)(2\cos t) - (2\sin t)(-2\sin t)\big)dt \qquad Flux = \int_C (Mdy - Ndx)\ dt$$

$$= \int_0^{2\pi} \left(4\sin t\cos t - 4\cos^2 t + 4\sin^2 t\right)dt$$

$$= \int_0^{2\pi} (2\sin 2t - 4\cos 2t)\,dt$$

$$= -\cos 2t - 2\sin 2t \Big|_0^{2\pi}$$

$$= -1 + 1$$

$$\underline{= 0}\ \rfloor$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \langle x,\ y\rangle$; where R is the half-annulus $\{(r,\ \theta):\ \ 1 \le r \le 2,\quad 0 \le \theta \le \pi\}$

Solution

$$M = y \quad \to \quad M_y = 1$$

$$N = x \quad \to \quad N_x = 1$$

$$Cir = \iint_R (1 - 1)\,dA \qquad Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

$$\underline{= 0}\ \rfloor$$

$$M = x \quad \to \quad M_x = 1$$

$$N = y \quad \to \quad N_y = 1$$

$$Flux = \iint_R (1 + 1)\,dA \qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dxdy$$

$$= 2\int_0^{\pi} d\theta \quad \int_1^2 r\ dr$$

$$= 2\pi \left(\frac{1}{2}r^2\right|_1^2$$

$$\underline{= 3\pi}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \langle -y,\ x\rangle$; where R is the annulus $\{(r,\ \theta):\ \ 1 \le r \le 3,\quad 0 \le \theta \le 2\pi\}$

Solution

$$Cir = \iint_R \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right) dA \qquad Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \iint_R (1+1)\, dA$$

$$= 2\int_0^{2\pi} d\theta \int_1^3 r\ dr$$

$$= 4\pi \left(\frac{1}{2}r^2\right|_1^3$$

$$\underline{= 16\pi}$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x)\right) dA \qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \iint_R (0-0)\, dA$$

$$\underline{= 0}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \langle 2x+y,\ x-4y\rangle$; where R is the quarter-annulus $\left\{(r,\ \theta):\ \ 1 \le r \le 4,\quad 0 \le \theta \le \frac{\pi}{2}\right\}$

Solution

$$Cir = \iint_R \left(\frac{\partial}{\partial x}(x-4y) - \frac{\partial}{\partial y}(2x+y)\right) dA \qquad Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \iint_R (1-1)\, dA$$

$$\underline{=0}$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}(2x+y) + \frac{\partial}{\partial y}(x-4y)\right) dA \qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \iint_R (2-4)\, dA$$

$$= -2\int_0^{\frac{\pi}{2}} d\theta \int_1^4 r\, dr$$

$$= -\pi \left(\frac{1}{2}r^2\right)\Big|_1^4$$

$$\underline{= -\frac{15}{2}\pi}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \langle x-y,\ 2y-x \rangle$; where R is the parallelogram $\{(x, y): \ 1-x \le y \le 3-x, \quad 0 \le x \le 1\}$

Solution

$$Cir = \iint_R \left(\frac{\partial}{\partial x}(2y-x) - \frac{\partial}{\partial y}(x-y)\right) dA \qquad Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \iint_R (-1+1)\, dA$$

$$\underline{=0}$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}(2y-x)\right) dA \qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \iint_R (1+2)\, dA$$

$$= 3\int_0^1 \int_{1-x}^{3-x} dydx$$

$$= 3\int_0^1 y \Big|_{1-x}^{3-x} dx$$

$$= 3\int_0^1 2\ dx$$

$$\underline{= 6}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \left\langle \ln\left(x^2 + y^2\right),\ \ \tan^{-1}\frac{y}{x}\right\rangle$; where R is the annulus $\{(r,\ \theta):\ \ 1 \le r \le 2,\quad 0 \le \theta \le 2\pi\}$

Solution

$$Cir = \iint_R \left(\frac{\partial}{\partial x}\left(\tan^{-1}\frac{y}{x}\right) - \frac{\partial}{\partial y}\left(\ln\left(x^2+y^2\right)\right)\right)dA \qquad Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

$$= \iint_R \left(-\frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} - \frac{2y}{x^2+y^2}\right)dA \qquad \left(\tan^{-1}u\right)' = \frac{u'}{1+u^2}$$

$$= \iint_R \left(-\frac{y}{x^2+y^2} - \frac{2y}{x^2+y^2}\right)dA$$

$$= -3\iint_R \left(\frac{y}{x^2+y^2}\right)dA$$

$$= -3\int_0^{2\pi}\int_1^2 \frac{r\sin\theta}{r^2} r\ drd\theta$$

$$= -3\int_0^{2\pi} \sin\theta\ d\theta \int_1^2 dr$$

$$= 3\ \left(\cos\theta\ \Big|_0^{2\pi}\right)\ \left(r\ \Big|_1^2\right)$$

$$= 3(1-1)(1)$$

$$\underline{= 0}$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}\left(\ln\left(x^2+y^2\right)\right) + \frac{\partial}{\partial y}\left(\tan^{-1}\frac{y}{x}\right)\right)dA \qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dxdy$$

$$= \iint_R \left(\frac{2x}{x^2+y^2} + \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} \right) dA \qquad \left(\tan^{-1} u\right)' = \frac{u'}{1+u^2}$$

$$= \iint_R \left(\frac{2x}{x^2+y^2} + \frac{x}{x^2+y^2} \right) dA$$

$$= 3 \iint_R \frac{x}{x^2+y^2} \, dA$$

$$= 3 \int_0^{2\pi} \int_1^2 \frac{r\cos\theta}{r^2} r \, dr d\theta$$

$$= 3 \int_0^{2\pi} \cos\theta \, d\theta \int_1^2 dr$$

$$= 3 \left(\sin\theta \Big|_0^{2\pi} \right) \left(r \Big|_1^2 \right)$$

$$= 3(0)(1)$$

$$= 0$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \nabla\left(\sqrt{x^2+y^2}\right)$; where R is the half-annulus $\{(r, \theta): \ 1 \le r \le 3, \ \ 0 \le \theta \le \pi\}$

Solution

$$\vec{F} = \nabla\left(\sqrt{x^2+y^2}\right)$$

$$= \left\langle \frac{x}{\sqrt{x^2+y^2}}, \ \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

$$Cir = \iint_R \left(\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}} \right) - \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}} \right) \right) dA \qquad Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \iint_R \left(-\frac{xy}{\left(x^2+y^2\right)^{3/2}} + \frac{xy}{\left(x^2+y^2\right)^{3/2}} \right) dA$$

$$= 0$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}} \right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}} \right) \right) dA \qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

$$= \iint_R \left(\frac{x^2+y^2-x^2}{\left(x^2+y^2\right)^{3/2}} + \frac{x^2+y^2-y^2}{\left(x^2+y^2\right)^{3/2}} \right) dA$$

$$= \iint_R \frac{x^2+y^2}{\left(x^2+y^2\right)^{3/2}} dA$$

$$= \iint_R \frac{1}{\left(x^2+y^2\right)^{1/2}} dA$$

$$= \int_0^{\pi} \int_1^3 \frac{1}{r} r \; dr d\theta$$

$$= \int_0^{\pi} d\theta \int_1^3 dr$$

$$\underline{= 2\pi}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \langle y\cos x, \; -\sin x \rangle$; where R is the square $\left\{ (x, y): \quad 0 \le x \le \frac{\pi}{2}, \quad 0 \le y \le \frac{\pi}{2} \right\}$

Solution

$$Cir = \iint_R \left(\frac{\partial}{\partial x}(-\sin x) - \frac{\partial}{\partial y}(y\cos x) \right) dA \qquad Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \iint_R (-\cos x - \cos x) \, dA$$

$$= -2\int_0^{\frac{\pi}{2}} dy \int_0^{\frac{\pi}{2}} \cos x \; dx$$

$$= -\pi \; \sin x \Big|_0^{\frac{\pi}{2}}$$

$$\underline{= -\pi}$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}(y\cos x) + \frac{\partial}{\partial y}(-\sin x)\right)dA$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dxdy$$

$$= \iint_R (-y\sin x + 0)\,dA$$

$$= -\int_0^{\frac{\pi}{2}} y\,dy \quad \int_0^{\frac{\pi}{2}} \sin x\,dx$$

$$= \frac{1}{2}y^2 \Big|_0^{\frac{\pi}{2}} \quad \cos x \Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{\pi^2}{8}$$

Exercise

Use Green's theorem to find the counterclockwise circulation and outward flux for the field $\vec{F} = \left\langle x + y^2,\ x^2 - y\right\rangle$; where $R = \left\{(x,\ y):\ \ 3y^2 \le x \le 36 - y^2\right\}$

Solution

$$x = 36 - y^2 = 3y^2$$

$$4y^2 = 36 \ \rightarrow\ y = \pm 3$$

$$Cir = \iint_R \left(\frac{\partial}{\partial x}\left(x^2 - y\right) - \frac{\partial}{\partial y}\left(x + y^2\right)\right)dA$$

$$Cir = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

$$= \iint_R (2x - 2y)\,dA$$

$$= 2\int_{-3}^{3}\int_{3y^2}^{36-y^2} (x - y)\,dxdy$$

$$= 2\int_{-3}^{3} \left(\frac{1}{2}x^2 - yx \Big|_{3y^2}^{36-y^2}\right. dy$$

$$= 2\int_{-3}^{3} \left(648 - 36y^2 + \frac{1}{2}y^4 - 36y + y^3 - \frac{9}{2}y^4 + 3y^3\right)dy$$

$$= 2\int_{-3}^{3} \left(648 - 36y - 36y^2 + 4y^3 - 4y^4\right)dy$$

$$= 8\left(162y - \frac{9}{2}y^2 - 3y^3 + \frac{1}{4}y^4 - \frac{1}{5}y^5\right|_{-3}^{3}$$

$$= 8\left(486 - \frac{81}{2} - 81 + \frac{81}{4} - \frac{243}{5} + 486 + \frac{81}{2} - 81 - \frac{81}{4} - \frac{243}{5}\right)$$

$$= 8\left(810 - \frac{486}{5}\right)$$

$$= \frac{28,512}{5}$$

$$Flux = \iint_R \left(\frac{\partial}{\partial x}\left(x + y^2\right) + \frac{\partial}{\partial y}\left(x^2 - y\right)\right) dA \qquad Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \iint_R (1-1)\, dA$$

$$= 0$$

Exercise

Find the outward flux for the field $\boldsymbol{F} = \left(3xy - \frac{x}{1+y^2}\right)\boldsymbol{i} + \left(e^x + \tan^{-1} y\right)\boldsymbol{j}$ across the cardioid $r = a(1+\cos\theta),\ \ a > 0$

Solution

$$M = 3xy - \frac{x}{1+y^2} \quad \Rightarrow \quad \frac{\partial M}{\partial x} = 3y - \frac{1}{1+y^2}$$

$$N = e^x + \tan^{-1} y \quad \Rightarrow \quad \frac{\partial N}{\partial y} = \frac{1}{1+y^2}$$

$$Flux = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \iint_R \left(3y - \frac{1}{1+y^2} + \frac{1}{1+y^2}\right) dxdy$$

$$= \iint_R 3y\ dxdy$$

$$= 3\int_0^{2\pi}\int_0^{a(1+\cos\theta)} (r\sin\theta)\ rdrd\theta$$

$$= 3\int_0^{2\pi} \frac{1}{3}\sin\theta\ \left(r^3\ \right|_0^{a(1+\cos\theta)}\ d\theta$$

$$= a^3 \int_0^{2\pi} \sin\theta (1+\cos\theta)^3 \; d\theta \qquad\qquad d(1+\cos\theta) = -\sin\theta d\theta$$

$$= -a^3 \int_0^{2\pi} (1+\cos\theta)^3 \; d(1+\cos\theta)$$

$$= -\frac{1}{4}a^3 (1+\cos\theta)^4 \Big|_0^{2\pi}$$

$$= -\frac{1}{4}a^3 \left(2^4 - 2^4\right)$$

$$\underline{= 0}$$

Exercise

Find the work done by $\boldsymbol{F} = 2xy^3\boldsymbol{i} + 4x^2y^2\boldsymbol{j}$ in moving a particle once counterclockwise around the curve C: The boundary of the triangular region in the first quadrant enclosed by the x-axis, the line $x = 1$ and the curve $y = x^3$

Solution

$$M = 2xy^3 \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 6xy^2$$

$$N = 4x^2y^2 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 8xy^2$$

$$Work = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy \qquad\qquad Work = \oint_C Mdx + Ndy$$

$$= \int_0^1 \int_0^{x^3} \left(8xy^2 - 6xy^2\right) dydx$$

$$= \int_0^1 \int_0^{x^3} \left(2xy^2\right) dydx$$

$$= \int_0^1 \left(\frac{2}{3}xy^3 \Big|_0^{x^3} \; dx\right.$$

$$= \frac{2}{3}\int_0^1 x^{10} \, dx$$

$$= \frac{2}{33} x^{11} \Big|_0^1$$

$$\underline{= \frac{2}{33}}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_C \left(y^2dx + x^2dy\right)$ C: The triangle bounded by

$x=0,\ x+y=1,\ y=0$

Solution

$$M = y^2 \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2y$$

$$N = x^2 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2x$$

$$\begin{aligned}
\oint_C \left(y^2dx + x^2dy\right) &= \int_0^1 \int_0^{1-x} (2x-2y)\, dydx \\
&= \int_0^1 \left(2xy - y^2 \Big|_0^{1-x}\right. dx \\
&= \int_0^1 \left(2x(1-x) - (1-x)^2\right) dx \\
&= \int_0^1 \left(2x - 2x^2 - 1 + 2x - x^2\right) dx \\
&= \int_0^1 \left(-3x^2 + 4x - 1\right) dx \\
&= -x^3 + 2x^2 - x \Big|_0^1 \\
&= -1 + 2 - 1 \\
&= 0
\end{aligned}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_C (3ydx + 2xdy)$ C: The boundary of

$0 \le x \le \pi, \quad 0 \le y \le \sin x$

Solution

$$M = 3y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 3$$

$$N = 2x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2$$

$$\oint_C (3ydx + 2xdy) = \int_0^\pi \int_0^{\sin x} (2-3)\, dydx$$

$$= -\int_0^\pi \left(y \Big|_0^{\sin x} \right. dx$$

$$= -\int_0^\pi \sin x \; dx$$

$$= \cos x \Big|_0^\pi$$

$$\underline{= -2}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_C \left(3y - e^{\sin x}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy$: where C is the circle $x^2 + y^2 = 9$

Solution

$$\oint_C \left(3y - e^{\sin x}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy = \iint_R \left(\frac{\partial}{\partial x}\left(7x + \sqrt{y^4+1}\right) - \frac{\partial}{\partial y}\left(3y - e^{\sin x}\right)\right)dA$$

$$= \iint_R (7-3)\, dA$$

$$= 4\iint_R \; dA$$

$$= 4\int_0^{2\pi} d\theta \quad \int_0^3 r\; dr$$

$$= 8\pi \left(\frac{1}{2}r^2 \Big|_0^3\right.$$

$$\underline{= 36\pi}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_C (3x-5y)\,dx + (x-6y)\,dy$: where C is the ellipse $\frac{x^2}{4}+y^2=1$

Solution

$$\oint_C (3x-5y)\,dx+(x-6y)\,dy = \iint_R \left(\frac{\partial}{\partial x}(x-6y)-\frac{\partial}{\partial y}(3x-5y)\right)dA$$

$$= \iint_R (1-(-5))\,dA$$

$$= 6\iint_R dA$$

$$= 6 \times \text{Area of ellipse}$$

$$\frac{x^2}{4}+y^2=1$$

$$x = 2\cos t \quad \to \quad dx = -2\sin t\ dt$$

$$y=\sin t \quad \to \quad dy = \cos t\ dt$$

$$A = \frac{1}{2}\int_0^{2\pi}\left(2\cos t(\cos t)-\sin t(-2\sin t)\right)dt \qquad\qquad A=\frac{1}{2}\oint_C xdy - ydx$$

$$= \int_0^{2\pi}\left(\cos^2 t+\sin^2 t\right)dt$$

$$= \int_0^{2\pi} dt$$

$$= 2\pi$$

$$\oint_C (3x-5y)\,dx+(x-6y)\,dy \ = 12\pi$$

Exercise

Use either form of Green's Theorem to evaluate the line integral $\oint_C \left(x^3+xy\right)dy+\left(2y^2-2x^2y\right)dx$; C is the square with vertices $(\pm 1,\ \pm 1)$ with *counterclockwise* orientation

Solution

$N=x^3+xy \quad\to\quad N_x=3x^2+y$

$M=2y^2-2x^2y \quad\to\quad M_y=4y-2x^2$

$$\begin{aligned}\oint_C \left(x^3+xy\right)dy+\left(2y^2-2x^2y\right)dx &= \int_{-1}^{1}\int_{-1}^{1}\left(3x^2+y-4y+2x^2\right)dydx\\ &= \int_{-1}^{1}\int_{-1}^{1}\left(5x^2-3y\right)dydx\\ &= \int_{-1}^{1}\left(5x^2y-\frac{3}{2}y^2\ \Big|_{-1}^{1}\right)dx\\ &= \int_{-1}^{1}\left(5x^2-\frac{3}{2}+5x^2+\frac{3}{2}\right)dx\\ &= \int_{-1}^{1}10x^2\ dx\\ &= \frac{10}{3}x^3\ \Big|_{-1}^{1}\\ &= \frac{20}{3}\end{aligned}$$

Exercise

Use either form of Green's Theorem to evaluate the line integral $\oint_C 3x^3dy-3y^3dx$; C is the circle of radius 4 centered at the origin with *clockwise* orientation.

Solution

$N=3x^3 \quad\to\quad N_x=9x^2$

$M=-3y^3 \quad\to\quad M_y=-9y^2$

$$\oint_C 3x^3dy-3y^3dx=\iint_R\left(9x^2+9y^2\right)dA$$

$$= 9\int_0^{2\pi}\int_0^4 r^2\ r\,drd\theta$$

$$= 9\int_0^{2\pi} d\theta\ \int_0^4 r^3\,dr$$

$$= 9(2\pi)\ \left(\frac{1}{4}r^4\ \Big|_0^4\right.$$

$$= 18\pi(64)$$

$$\underline{= 1152\pi\ |}$$

Since the orientation is *cw*: -1152π

Exercise

Evaluate $\int_C (x-y)dx + (x+y)dy$ counterclockwise around the triangle with vertices (0, 0), $(1,\ 0)$ and $(0,\ 1)$

Solution

Along $(0,0)\to(1,0)$: $\quad \vec{r}(t) = t\,\hat{i}\,, \quad 0 \le t \le 1$

$$\vec{F} = (x-y)\,\hat{i} + (x+y)\,\hat{j}$$

$$= t\,\hat{i} + t\,\hat{j}$$

$$\frac{d\vec{r}}{dt} = \hat{i}$$

$$\vec{F}\bullet\frac{d\vec{r}}{dt} = \left(t\,\hat{i} + t\,\hat{j}\right)\bullet\left(\hat{i}\right)$$

$$\underline{= t\ |}$$

Along $(1,0)\to(0,1)$: $\quad \vec{r}(t) = (1-t)\,\hat{i} + t\,\hat{j}\,, \quad 0 \le t \le 1$

$$\vec{F} = (x-y)\,\hat{i} + (x+y)\,\hat{j}$$

$$= (1-2t)\,\hat{i} + \hat{j}$$

$$\frac{d\vec{r}}{dt} = -\hat{i} + \hat{j}$$

$$\vec{F}\bullet\frac{d\vec{r}}{dt} = \left((1-2t)\,\hat{i} + \hat{j}\right)\bullet\left(-\hat{i} + \hat{j}\right)$$

$$= -1 + 2t + 1$$

$$\underline{= 2t\ |}$$

Along $(0,1)\to(0,0)$: $\quad \vec{r}(t) = (1-t)\,\hat{j}\,, \quad 0 \le t \le 1$

$$\vec{F} = (x-y)\,\hat{i} + (x+y)\,\hat{j}$$

$$= (t-1)\,\hat{i} + (1-t)\,\hat{j}$$

$$\frac{d\vec{r}}{dt} = -\hat{j}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left((t-1)\,\hat{i} + (1-t)\,\hat{j}\right) \bullet \left(-\hat{j}\right)$$

$$\underline{= t-1\,|}$$

$$\int_C (x-y)\,dx + (x+y)\,dy = \int_0^1 t\,dt + \int_0^1 2t\,dt + \int_0^1 (t-1)\,dt$$

$$= \int_0^1 (t+2t+t-1)\,dt$$

$$= \int_0^1 (4t-1)\,dt$$

$$= 2t^2 - t \Big|_0^1$$

$$= 2-1$$

$$\underline{= 1\,|}$$

Exercise

Use Green's theorem to evaluate the line integral $\oint xy^2dx + x^2ydy$; C is the triangle with vertices $(0,\ 0),\ (2,\ 0),\ (0,\ 2)$ with counterclockwise orientation.

Solution

$$\oint xy^2dx + x^2ydy = \iint_R \left(\frac{\partial}{\partial x}\left(x^2y\right) - \frac{\partial}{\partial x}\left(x^2y\right)\right) dxdy$$

$$= \iint_R (2xy - 2xy)\,dxdy$$

$$\underline{= 0\,|}$$

Exercise

Use Green's theorem to evaluate the line integral $\oint \left(-3y + x^{3/2}\right)dx + \left(x - y^{2/3}\right)dy;$ C is the boundary of the half disk $\left\{(x, y):\ x^2 + y^2 \le 2,\ y \ge 0\right\}$ with counterclockwise orientation.

Solution

$$\oint \left(-3y + x^{3/2}\right)dx + \left(x - y^{2/3}\right)dy = \iint_C \left(\frac{\partial}{\partial x}\left(x - y^{2/3}\right) - \frac{\partial}{\partial y}\left(-3y + x^{3/2}\right)\right)dA$$

$$= \iint_C (1+3)\,dA$$

$$= \iint_C 4\ dA \qquad\qquad \textit{Semicircle } A = \pi$$

$$= 4\pi$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_{(0,\ 1)} \left(2x + e^{y^2}\right)dy - \left(4y^2 + e^{x^2}\right)dx:$ C is the boundary of the square with vertices $(0,\ 0),\ (1,\ 0),\ (1,\ 1)$ with counterclockwise orientation.

Solution

$$\oint_{(0,\ 1)} \left(2x + e^{y^2}\right)dy - \left(4y^2 + e^{x^2}\right)dx = \iint_C \left(\frac{\partial}{\partial x}\left(2x + e^{y^2}\right) + \frac{\partial}{\partial y}\left(4y^2 + e^{x^2}\right)\right)dA$$

$$= \iint_C (2+8y)\,dA$$

$$= \int_0^1 \int_0^1 (2+8y)\,dxdy$$

$$= \int_0^1 (2+8y)\,dy$$

$$= 2y + 4y^2 \Big|_0^1$$

$$= 6$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint_C (2x-3y)dy-(3x+4y)dx$: C is the unit circle

Solution

$$\oint_C (2x-3y)dy-(3x+4y)dx = \iint_C \left(\frac{\partial}{\partial x}(2x-3y)+\frac{\partial}{\partial y}(3x+4y)\right)dA$$

$$= \iint_C (2+4)dA$$

$$= 6\times(\text{Area of the unit circle})$$

$$\underline{= 6\pi}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint fdy - gdx$; where $\langle f,\ g\rangle = \langle 0,\ xy\rangle$ and C is the triangle with vertices $(0,\ 0)$, $(2,\ 0)$, $(0,\ 4)$ with counterclockwise orientation.

Solution

$(2,\ 0)-(0,\ 4): \to\ y = \frac{4}{-2}x+4\ \underline{=4-2x}$

$$\oint fdy - gdx = \iint_R \left(\frac{\partial}{\partial x}(f)+\frac{\partial}{\partial y}(g)\right)dA$$

$$= \iint_R \left(\frac{\partial}{\partial x}(0)+\frac{\partial}{\partial y}(xy)\right)dA$$

$$= \int_0^2 \int_0^{4-2x} x\ dydx$$

$$= \int_0^2 xy \Big|_0^{4-2x} dx$$

$$= \int_0^2 \left(4x-2x^2\right)dx$$

$$= 2x^2 - \frac{2}{3}x^3 \Big|_0^2$$

$$= 8 - \frac{16}{3}$$

$$\underline{= \frac{8}{3}}$$

Exercise

Apply Green's Theorem to evaluate the integral $\oint fdy - gdx$; where $\langle f,\ g\rangle = \langle x^2,\ 2y^2\rangle$ and C is the upper half of the unit circle and the line segment $-1 \le x \le 1$ with clockwise orientation.

Solution

$x^2 + y^2 = 1 \quad \to \quad y = \sqrt{1-x^2}$ *upper half of the unit circle*

$$\oint fdy - gdx = -\iint_R \left(\frac{\partial}{\partial x}(f) + \frac{\partial}{\partial y}(g)\right) dA \qquad \textit{clockwise orientation}$$

$$= -\iint_R \left(\frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}\left(2y^2\right)\right) dA$$

$$= -\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (2x+4y)\, dydx$$

$$= -\int_{-1}^{1} \left(2xy + 2y^2 \Big|_0^{\sqrt{1-x^2}}\right) dx$$

$$= -\int_{-1}^{1} \left(2x\sqrt{1-x^2} + 2\left(1-x^2\right)\right) dx$$

$$= \int_{-1}^{1} \left(1-x^2\right)^{1/2} d\left(1-x^2\right) - 2\int_{-1}^{1} \left(1-x^2\right) dx$$

$$= \frac{2}{3}\left(1-x^2\right)^{3/2} - 2x + \frac{2}{3}x^3 \Big|_{-1}^{1}$$

$$= -2 + \frac{2}{3} - 2 + \frac{2}{3}$$

$$\boxed{= -\frac{8}{3}}$$

Exercise

Apply Green's Theorem to evaluate the integral, the circulation line integral of $\vec{F} = \langle x^2 + y^2,\ 4x + y^3\rangle$, where C is the boundary of $\{(x,\ y):\ 0 \le y \le \sin x, \quad 0 \le x \le \pi\}$

Solution

Using Circulation form

$$\iint_C \left(\frac{\partial}{\partial x}\left(4x + y^3\right) - \frac{\partial}{\partial y}\left(x^2 + y^2\right)\right) dA = \iint_C (4-2y)\, dA$$

$$= \int_0^{\pi} \int_0^{\sin x} (4 - 2y)\, dydx$$

$$= \int_0^{\pi} \left(4y - y^2 \Big|_0^{\sin x} \, dx\right.$$

$$= \int_0^{\pi} \left(4\sin x - \sin^2 x\right) dx$$

$$= \int_0^{\pi} \left(4\sin x - \frac{1}{2} + \frac{1}{2}\cos 2x\right) dx$$

$$= -4\cos x - \frac{1}{2}x + \frac{1}{4}\sin 2x \Big|_0^{\pi}$$

$$= 4 - \frac{\pi}{2} + 4$$

$$\boxed{= 8 - \frac{\pi}{2}}$$

Exercise

Apply Green's Theorem to evaluate the integral, the circulation line integral of $\vec{F} = \left\langle 2xy^2 + x,\ 4x^3 + y \right\rangle$, where C is the boundary of $\left\{(x,\ y):\ 0 \le y \le \sin x, \quad 0 \le x \le \pi\right\}$

Solution

Using Circulation form

$$\iint_C \left(\frac{\partial}{\partial x}\left(4x^3 + y\right) - \frac{\partial}{\partial y}\left(2xy^2 + x\right)\right) dA = \iint_C \left(12x^2 - 4xy\right) dA$$

$$= \int_0^{\pi} \int_0^{\sin x} \left(12x^2 - 4xy\right) dydx$$

$$= \int_0^{\pi} \left(12x^2 y - 2xy^2 \Big|_0^{\sin x} \, dx\right.$$

$$= \int_0^{\pi} \left(12x^2 \sin x - 2x\sin^2 x\right) dx$$

$$= \int_0^{\pi} \left(12x^2 \sin x - 2x\left(\frac{1 - \cos 2x}{2}\right)\right) dx$$

$$= \int_0^{\pi} \left(12x^2 \sin x - x + x\cos 2x\right) dx$$

		$\int \sin x$			$\int \cos 2x$
+	$12x^2$	$-\cos x$	+	x	$\frac{1}{2}\sin 2x$
–	$24x$	$-\sin x$	–	1	$-\frac{1}{4}\cos 2x$
+	24	$\cos x$			

$$= \left(-12x^2\cos x + 24x\sin x + 24\cos x - \frac{1}{2}x^2 + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x \right|_0^{\pi}$$

$$= 12\pi^2 - 24 - \frac{\pi^2}{2} + \frac{1}{4} + 12\pi^2 - 24 - \frac{1}{4}$$

$$\boxed{= \frac{23\pi^2}{2} - 48}$$

Exercise

Apply Green's Theorem to evaluate the integral, the flus line integral of $\vec{F} = \left\langle e^{x-y},\ e^{y-x} \right\rangle$, where C is the boundary of $\{(x,\ y):\ 0 \le y \le x, \quad 0 \le x \le 1\}$

Solution

Using flux form

$$\iint_C \left(\frac{\partial}{\partial x}\left(e^{x-y}\right) + \frac{\partial}{\partial y}\left(e^{y-x}\right)\right) dA = \iint_C \left(e^{x-y} + e^{y-x}\right) dA$$

$$= \int_0^1 \int_0^x \left(e^{x-y} + e^{y-x}\right) dy\,dx$$

$$= \int_0^1 \left(-e^{x-y} + e^{y-x} \right|_0^x dx$$

$$= \int_0^1 \left(-1 + 1 + e^x - e^{-x}\right) dx$$

$$= \int_0^1 \left(e^x - e^{-x}\right) dx$$

$$= e^x + e^{-x} \Big|_0^1$$

$$\boxed{= e + e^{-1} - 2}$$

Exercise

Evaluate $\displaystyle\int_C y^2dx + x^2dy$ $\quad C$ is the circle $x^2 + y^2 = 4$

Solution

$$M = y^2 \quad \rightarrow \quad M_y = 2y$$

$$N = x^2 \quad \rightarrow \quad N_x = 2x$$

$$\int_C y^2dx + x^2dy = \int_C (2x - 2y)\, dxdy$$

$$= 2\int_0^{2\pi}\int_0^2 (r\cos\theta - r\sin\theta)\ rdrd\theta$$

$$= 2\int_0^{2\pi} (\cos\theta - \sin\theta)\, d\theta \int_0^2 r^2\, dr$$

$$= 2\ \left(\sin\theta + \cos\theta\right|_0^{2\pi} \quad \left(\frac{1}{3}r^3\right|_0^2$$

$$= 2(1-1)\left(\frac{8}{3}\right)$$

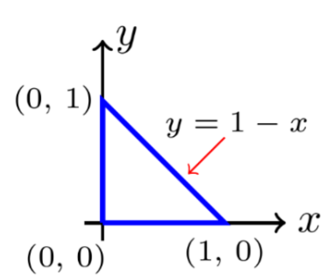
$$= 0$$

Exercise

Use the flux form to Green's Theorem to evaluate $\displaystyle\iint_R \left(2xy + 4y^3\right) dA$, where R is the triangle with vertices $(0,\ 0)$, $(1,\ 0)$, and $(0,\ 1)$.

Solution

$(1,\ 0) - (0,\ 1)$: $\quad y = -x + 1$



$$\iint_R \left(2xy + 4y^3\right) dA = \int_0^1\int_0^{1-x} \left(2xy + 4y^3\right) dydx$$

$$= \int_0^1 \left(xy^2 + y^4\right|_0^{1-x} dx$$

$$= \int_0^1 \left(x - 2x^2 + x^3 + 1 - 4x + 6x^2 - 4x^3 + x^4\right) dx$$

$$= \int_0^1 \left(1 - 3x + 4x^2 - 3x^3 + x^4\right) dx$$

$$= x - \frac{3}{2}x^2 + \frac{4}{3}x^3 - \frac{3}{4}x^4 + \frac{1}{5}x^5 \Big|_0^1$$

$$= 1 - \frac{3}{2} + \frac{4}{3} - \frac{3}{4} + \frac{1}{5}$$

$$= \frac{-30 + 80 - 45 + 12}{60}$$

$$= \frac{17}{60}$$

Exercise

Show that $\oint_C \ln x \sin y \, dy - \frac{\cos y}{x} dx = 0$ for any closed curve C to which Green's Theorem applies.

Solution

$$M = -\frac{\cos y}{x} \quad \to \quad M_y = \frac{\sin y}{x}$$

$$N = \ln x \sin y \quad \to \quad N_x = \frac{\ln y}{x}$$

$$\oint_C \ln x \sin y dy - \frac{\cos y}{x} dx = \iint_R \left(\frac{\sin y}{x} - \frac{\sin y}{x}\right) dx dy$$

$$= 0 \quad \checkmark$$

Exercise

Prove that the radial field $\vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$ where $\vec{r} = \langle x, y\rangle$ and p is a real number, is conservative on $\mathbb{R}^2$ with the origin removed. For what value of p is $\vec{F}$ conservative on $\mathbb{R}^2$ (including the origin)?

Solution

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$$

$$= \frac{\langle x, y\rangle}{\left(\sqrt{x^2 + y^2}\right)^p}$$

$$\varphi_x = \frac{x}{\left(x^2+y^2\right)^{p/2}}; \quad \varphi_y = \frac{y}{\left(x^2+y^2\right)^{p/2}}$$

$$\varphi = \int \frac{x}{\left(x^2+y^2\right)^{p/2}}\,dx$$

$$= \frac{1}{2}\int \left(x^2+y^2\right)^{-p/2} d\left(x^2+y^2\right)$$

$$= \frac{1}{2}\frac{1}{\frac{2-p}{2}}\left(x^2+y^2\right)^{1-p/2} + C$$

$$= \frac{1}{2-p}\left(x^2+y^2\right)^{1-p/2} + C(x,y) \qquad \textit{for } \; p \neq 2$$

For $p \neq 2$

$$\varphi = \frac{1}{2-p}\left(x^2+y^2\right)^{1-p/2} + C(x,y)$$

$$\varphi_y = \frac{1}{2-p}\frac{2-p}{2}(2y)\left(x^2+y^2\right)^{1-\frac{p}{2}-1} + C_y$$

$$= y\left(x^2+y^2\right)^{-\frac{p}{2}} + C_y = \frac{y}{\left(x^2+y^2\right)^{p/2}}$$

$$\Rightarrow \; \boxed{C_y = 0}$$

$$\therefore \; \varphi = \frac{1}{(2-p)\left(x^2+y^2\right)^{\frac{p-2}{2}}}$$

$$= \frac{-1}{(p-2)\left(r^2\right)^{\frac{p-2}{2}}}$$

$$= \frac{-1}{(p-2)\;|r|^{p-2}}$$

For $p = 2$

$$\vec{F} = \frac{\langle x, y\rangle}{\left(\sqrt{x^2+y^2}\right)^2}$$

$$= \frac{\langle x, y\rangle}{x^2+y^2}$$

$$\varphi_x = \frac{x}{x^2+y^2}; \quad \varphi_y = \frac{y}{x^2+y^2}$$

$$\varphi = \int \frac{x}{x^2+y^2}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x^2+y^2}\,d\left(x^2+y^2\right)$$

$$= \frac{1}{2}\ln\left(x^2+y^2\right) + C(x,y)$$

$$\varphi_y = \frac{y}{x^2+y^2} + C_y = \frac{y}{x^2+y^2}$$

$$\Rightarrow \ C_y = 0$$

$$\varphi = \frac{1}{2}\ln\left(|r|^2\right)$$

Thus $\vec{F}$ is conservative on all $\mathbb{R}^2$ for $p<0$

Exercise

Find the area of the elliptical region cut from the plane $x+y+z=1$ by the cylinder $x^2+y^2=1$

Solution

$$f(x,y,z) = x+y+z-1$$

$$\nabla f = \langle 1,\ 1,\ 1\rangle$$

$$|\nabla f| = \sqrt{3}$$

$$Area = \sqrt{3}\int_0^{2\pi}\int_0^1 r\,dr d\theta \qquad Area = \iint_R |\nabla f|\,dA$$

$$= \sqrt{3}\int_0^{2\pi} d\theta \quad \left(\frac{1}{2}r^2\right|_0^1$$

$$= \sqrt{3}(2\pi)\frac{1}{2}$$

$$= \pi\sqrt{3}\ \ unit^2$$

Exercise

Find the area of the cap cut from the paraboloid $x^2+y^2+z^2=1$ by the plane $z=\frac{\sqrt{2}}{2}$

Solution

$$f(x,y,z)=x^2+y^2+z^2-1$$

$$\nabla f=\langle 2x,\ 2y,\ 2z\rangle$$

$$\begin{aligned}|\nabla f|&=\sqrt{4x^2+4y^2+4z^2}\\&=2\sqrt{x^2+y^2+z^2}\\&=2\end{aligned}$$

$$\begin{aligned}|\nabla f\bullet\vec{p}|&=\langle 2x,\ 2y,\ 2z\rangle\bullet\hat{k}\\&=2|z|\\&=2z\end{aligned}$$

$$Area=\iint_R \frac{|\nabla f|}{|\nabla f\bullet\vec{p}|}\,dA$$

$$\begin{aligned}Area&=\iint_R \frac{2}{2z}\,dA\\&=\iint_R \frac{1}{\sqrt{1-x^2-y^2}}\,dxdy\\&=\int_0^{2\pi}d\theta\int_0^{\frac{1}{\sqrt{2}}}\frac{1}{\sqrt{1-r^2}}\,r\,dr\\&=-\pi\int_0^{\frac{1}{\sqrt{2}}}\left(1-r^2\right)^{-1/2}d\left(1-r^2\right)\\&=-2\pi\left(1-r^2\right)^{1/2}\Big|_0^{\frac{1}{\sqrt{2}}}\\&=-2\pi\left(\frac{1}{\sqrt{2}}-1\right)\\&=2\pi\left(1-\frac{\sqrt{2}}{2}\right)\\&=\pi\left(2-\sqrt{2}\right)\ unit^2\end{aligned}$$

Exercise

Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

$\vec{F} = \langle x,\ y\rangle;\quad R = \left\{(x,\ y):\ \ x^2 + y^2 \le 2\right\}$

Solution

$$M = x \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 0$$

$$N = y \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 0$$

$$Curl = 0 - 0 \qquad\qquad Curl = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$= 0$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (0 - 0)\, dA$$

$$= 0$$

$$\vec{r}(t) = \left\langle \sqrt{2}\cos t,\ \sqrt{2}\sin t\right\rangle$$

$$\vec{r}' = \left\langle -\sqrt{2}\sin t,\ \sqrt{2}\cos t\right\rangle$$

$$\vec{F} = \left\langle \sqrt{2}\cos t,\ \sqrt{2}\sin t\right\rangle$$

$$\begin{aligned}\vec{F} \bullet \frac{d\vec{r}}{dt} &= \left\langle \sqrt{2}\cos t,\ \sqrt{2}\sin t\right\rangle \bullet \left\langle -\sqrt{2}\sin t,\ \sqrt{2}\cos t\right\rangle \\ &= -2\cos t \sin t + 2\sin t \cos t \\ &= 0\end{aligned}$$

$$\int_C \vec{F} \bullet d\vec{r} = \int_0^{2\pi} 0\ dt$$

$$= 0$$

$\therefore$ The vector field is conservative since its curl is zero.

Exercise

Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

$\vec{F} = \langle y,\ x\rangle$; R is the square with vertices $(0,\ 0), (1,\ 0), (1,\ 1), (0,\ 1)$

Solution

$$M = y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 1$$

$$N = x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1$$

$Curl = 1 - 1$ $\qquad\qquad$ $Curl = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

$\underline{= 0}$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (1-1)\, dA$$

$\underline{= 0}$

$(0,\ 0) - (1,\ 0)$

$\vec{r}_1(t) = \langle t,\ 0\rangle$

$\vec{r}_1{}' = \langle 1,\ 0\rangle$

$\vec{F}_1 = \langle 0,\ t\rangle$

$\vec{F}_1 \cdot \vec{r}_1{}' = \langle 0,\ t\rangle \cdot \langle 1,\ 0\rangle$

$\underline{= 0}$

$(1,\ 0) - (1,\ 1)$

$\vec{r}_2(t) = \langle 1,\ t\rangle$

$\vec{r}_2{}' = \langle 0,\ 1\rangle$

$\vec{F}_2 = \langle t,\ 1\rangle$

$\vec{F}_2 \cdot \vec{r}_2{}' = \langle t,\ 1\rangle \cdot \langle 0,\ 1\rangle$

$\underline{= 1}$

$(1,\ 1) - (0,\ 1)$

$\vec{r}_3(t) = \langle 1-t,\ 1\rangle$

$\vec{r}_3{}' = \langle -1,\ 0\rangle$

$\vec{F}_3 = \langle 1,\ 1-t\rangle$

$\vec{F}_3 \cdot \vec{r}_3{}' = \langle 1,\ 1-t\rangle \cdot \langle -1,\ 0\rangle$

$\underline{= -1}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}_1 \cdot \vec{r}_1{}'\ dt + \int_0^1 \vec{F}_2 \cdot \vec{r}_2{}'\ dt + \int_0^1 \vec{F}_3 \cdot \vec{r}_3{}'\ dt$$

$= 0 + 1 - 1$

$\underline{= 0}$

$\therefore$ The vector field is conservative since its curl is zero.

Exercise

Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

$\vec{F} = \langle 2y,\ -2x \rangle$; R is the region bounded by $y = \sin x$ and $y = 0$ for $0 \le x \le \pi$

Solution

$$M = 2y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2$$

$$N = -2x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = -2$$

$$Curl = -2 - 2 \qquad\qquad Curl = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$\underline{= -4}$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \iint_R (-4)\, dA$$

$$= -4 \int_0^\pi \sin x\, dx$$

$$= 4 \cos x \Big|_0^\pi$$

$$= 4(-1-1)$$

$$\underline{= -8}$$

$y = 0$

$$\vec{r}_1(t) = \langle t,\ 0 \rangle$$

$$\vec{r}_1' = \langle 1,\ 0 \rangle$$

$$\vec{F} = \langle 0,\ -2t \rangle$$

$$\vec{F}_1 \cdot \vec{r}_1' = \langle 0,\ -2t \rangle \cdot \langle 1,\ 0 \rangle$$

$$\underline{= 0}$$

$y = \sin x$

$$\vec{r}_2(t) = \langle t,\ \sin t \rangle$$

$$\vec{r}_2' = \langle 1,\ \cos t \rangle$$

$$\vec{F}_2 = \langle 2\sin t,\ -2t \rangle$$

$$\vec{F}_2 \cdot \vec{r}_2' = \langle 2\sin t,\ -2t \rangle \cdot \langle 1,\ \cos t \rangle$$

$$\underline{= 2\sin t - 2t\cos t}$$

$$\int_C \vec{F}\bullet d\vec{r} = \int_0^{\pi} \vec{F}_1 \bullet \vec{r}_1{}'\ dt + \int_{\pi}^{0} \vec{F}_2 \bullet \vec{r}_2{}'\ dt$$

$$= 0 + \int_{\pi}^{0} (2\sin t - 2t\cos t)\,dt$$

$$= -2\cos t - 2t\sin t - 2\cos t \Big|_{\pi}^{0}$$

$$= -4\cos t - 2t\sin t \Big|_{\pi}^{0}$$

$$= -4 - 4$$

$$\underline{= -8}$$

∴ The vector field is ***not*** conservative since its curl is nonzero.

Exercise

Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

$\vec{F} = \langle -3y,\ 3x\rangle$; R is the triangle with vertices $(0,\ 0),\ (1,\ 0),\ (0,\ 2)$

Solution

$$M = -3y \quad\Rightarrow\quad \frac{\partial M}{\partial y} = -3$$

$$N = 3x \quad\Rightarrow\quad \frac{\partial N}{\partial x} = 3$$

$$Curl = 3 + 3 \qquad\qquad Curl = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$\underline{= 6}$$

$$y = \frac{2-0}{0-1}(x-1)$$

$$\underline{= -2x + 2}$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R 6\ dA$$

$$= 6\int_0^1 (2-2x)\,dx$$

$$= 6\left(2x - x^2\right)\Big|_0^1$$

$$\underline{= 6}$$

$(0,\ 0) - (1,\ 0)$

$$\vec{r}_1(t) = \langle t,\ 0\rangle$$

$$\vec{r}_1{}' = \langle 1,\ 0\rangle$$

$$\vec{F}_1 = \langle 0,\ 3t\rangle$$

$$\vec{F}_1 \bullet \vec{r}_1{}' = \langle 0,\ 3t\rangle \bullet \langle 1,\ 0\rangle$$

$$\underline{= 0\,|}$$

$(1,\ 0) - (0,\ 2)$

$$\vec{r}_2(t) = \langle 1-t,\ 2t\rangle$$

$$\vec{r}_2{}' = \langle -1,\ 2\rangle$$

$$\vec{F}_2 = \langle -6t,\ 3-3t\rangle$$

$$\vec{F}_2 \bullet \vec{r}_2{}' = \langle -6t,\ 3-3t\rangle \bullet \langle -1,\ 2\rangle$$

$$= 6t + 6 - 6t$$

$$\underline{= 6\,|}$$

$(0,\ 2) - (0,\ 0)$

$$\vec{r}_3(t) = \langle 0,\ 2-2t\rangle$$

$$\vec{r}_3{}' = \langle 0,\ -2\rangle$$

$$\vec{F}_3 = \langle 6t-6,\ 0\rangle$$

$$\vec{F}_3 \bullet \vec{r}_3{}' = \langle 6t-6,\ 0\rangle \bullet \langle 0,\ -2\rangle$$

$$\underline{= 0\,|}$$

$$\int_C \vec{F} \bullet d\vec{r} = \int_0^1 \vec{F}_1 \bullet \vec{r}_1{}'\ dt + \int_0^1 \vec{F}_2 \bullet \vec{r}_2{}'\ dt + \int_0^1 \vec{F}_3 \bullet \vec{r}_3{}'\ dt$$

$$= 0 + \int_0^1 6\ dt + 0$$

$$\underline{= 6\,|}$$

∴ The vector field is conservative since its curl is zero.

Exercise

Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

$\vec{F} = \left\langle 2xy,\ x^2 - y^2\right\rangle$; R is the region bounded by $y = x(2-x)$ and $y = 0$

Solution

$M = 2xy \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2x$

$N = x^2 - y^2 \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 2x$

$Curl = 2x - 2x \qquad\qquad Curl = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

$\quad \underline{= 0}$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (2x - 2x)\, dA$$

$$\underline{= 0}$$

$y = 0$

$\vec{r}_1(t) = \langle t,\ 0\rangle$

$\vec{r}_1{}' = \langle 1,\ 0\rangle$

$\vec{F}_1 = \langle 0,\ t^2\rangle$

$\vec{F}_1 \cdot \vec{r}_1{}' = \langle 0,\ t^2\rangle \cdot \langle 1,\ 0\rangle$

$\qquad \underline{= 0}$

$y = 2x - x^2$

$\vec{r}_2(t) = \langle t,\ 2t - t^2\rangle$

$\vec{r}_2{}' = \langle 1,\ 2 - 2t\rangle$

$\vec{F}_2 = \langle 4t^2 - 2t^3,\ \ -3t^2 + 4t^3 - t^4\rangle$

$$\begin{aligned}
\vec{F}_2 \cdot \vec{r}_2{}' &= \langle 4t^2 - 2t^3,\ \ -3t^2 + 4t^3 - t^4\rangle \cdot \langle 1,\ 2 - 2t\rangle \\
&= 4t^2 - 2t^3 + \left(-3t^2 + 4t^3 - t^4\right)(2 - 2t) \\
&= 4t^2 - 2t^3 - 6t^2 + 8t^3 - 2t^4 + 6t^3 - 8t^4 + 2t^5 \\
&= 2t^5 - 10t^4 + 12t^3 - 2t^2
\end{aligned}$$

$y = 2t - t^2 = 0 \quad \to \quad \underline{t = 0,\ 2}$

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^2 \vec{F}_1 \cdot \vec{r}_1{}'\ dt + \int_2^0 \vec{F}_2 \cdot \vec{r}_2{}'\ dt \\
&= 0 + \int_2^0 \left(2t^5 - 10t^4 + 12t^3 - 2t^2\right) dt
\end{aligned}$$

$$=\frac{1}{3}t^6-2t^5+3t^4-\frac{2}{3}t^3\Big|_2^0$$

$$=-\frac{64}{3}+64-48+\frac{16}{3}$$

$$=-\frac{48}{3}+16$$

$$\underline{=0}$$

$\therefore$ The vector field is conservative since its curl is zero.

Exercise

Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

$\vec{F}=\left\langle 0,\ x^2+y^2\right\rangle;\quad R=\left\{(x,\ y):\ \ x^2+y^2\le 1\right\}$

Solution

$$M=0\quad\Rightarrow\quad \frac{\partial M}{\partial y}=0$$

$$N=x^2+y^2\quad\Rightarrow\quad \frac{\partial N}{\partial x}=2x$$

$$Curl=2x-0 \qquad\qquad Curl=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}$$

$$\underline{=2x}$$

$$\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy=\iint_R 2x\ dA$$

$$=\int_0^{2\pi}\int_0^1 2r\cos\theta\ r\ drd\theta$$

$$=\int_0^{2\pi}\cos\theta\ d\theta\ \int_0^1 2r^2\ dr$$

$$=\sin\theta\Big|_0^{2\pi}\ \left(\frac{2}{3}r^3\Big|_0^1\right.$$

$$\underline{=0}$$

$$\vec{r}(t)=\langle\cos t,\ \sin t\rangle$$

$$\frac{d\vec{r}}{dt}=\langle-\sin t,\ \cos t\rangle$$

$$\vec{F}=\left\langle 0,\ \cos^2 t+\sin^2 t\right\rangle$$

$$=\langle 0,\ 1\rangle$$

$$\vec{F} \cdot \frac{d\vec{r}}{dt} = \langle 0,\ 1 \rangle \cdot \langle -\sin t,\ \cos t \rangle$$

$$= \cos t$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \cos t\ dt$$

$$= \sin t \Big|_0^{2\pi}$$

$$= 0$$

$\therefore$ The vector field is ***not*** conservative since its curl is nonzero.

Exercise

Find the area of the region using line integral of the region enclosed by the ellipse $x^2 + 4y^2 = 16$

Solution

$$x^2 + 4y^2 = 16$$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

$$\begin{cases} x = 4\cos t \\ y = 2\sin t \end{cases} \qquad 0 \le t \le 2\pi$$

$$A = \frac{1}{2} \oint_C \left(4\cos t \frac{d}{dt}(2\sin t) - 2\sin t \frac{d}{dt}(4\cos t) \right) dt$$

$$= 4 \int_0^{2\pi} \left(\cos^2 t + \sin^2 t \right) dt$$

$$= 4 \int_0^{2\pi} dt$$

$$= 8\pi \ \ unit^2$$

Exercise

Find the area of the region using line integral of the region bounded by the hypocycloid

$\vec{r}(t) = \left\langle \cos^3 t,\ \sin^3 t \right\rangle$ for $0 \le t \le 2\pi$.

Solution

$$A = \frac{1}{2}\int_0^{2\pi}\left(\cos^3 t\,\frac{d}{dt}\left(\sin^3 t\right) - \sin^3 t\,\frac{d}{dt}\left(\cos^3 t\right)\right)dt \qquad A = \frac{1}{2}\oint_C xdy - ydx$$

$$= \frac{1}{2}\int_0^{2\pi}\left(\cos^3 t\left(3\sin^2 t\cos t\right) - \sin^3 t\left(-3\cos^2 t\sin t\right)\right)dt$$

$$= \frac{3}{2}\int_0^{2\pi}\left(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t\right)dt$$

$$= \frac{3}{2}\int_0^{2\pi}\sin^2 t\cos^2 t\left(\cos^2 t + \sin^2 t\right)dt$$

$$= \frac{3}{2}\int_0^{2\pi}\sin^2 t\cos^2 t\ dt$$

$$= \frac{3}{2}\int_0^{2\pi}\left(\frac{1-\cos 2t}{2}\right)\left(\frac{1+\cos 2t}{2}\right)\ dt$$

$$= \frac{3}{8}\int_0^{2\pi}\left(1-\cos^2 2t\right)\ dt$$

$$= \frac{3}{8}\int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 4t\right)\ dt$$

$$= \frac{3}{8}\left(\frac{t}{2} - \frac{1}{8}\sin 4t\right.\Big|_0^{2\pi}$$

$$\underline{= \frac{3\pi}{8}\ \ unit^2}$$

Exercise

Find the area of the region using line integral of the region enclosed by a disk of radius 5

Solution

$$x = 5\cos t \quad \to \quad dx = -5\sin t\ dt$$

$$y = 5\sin t \quad \to \quad dy = 5\cos t\ dt$$

$$A = \frac{1}{2}\int_0^{2\pi}\left(5\cos t\left(5\cos t\right) - 5\sin t\left(-5\sin t\right)\right)dt \qquad A = \frac{1}{2}\oint_C xdy - ydx$$

$$= \frac{25}{2}\int_0^{2\pi}\left(\cos^2 t + \sin^2 t\right)dt$$

$$= \frac{25}{2}\int_0^{2\pi} dt$$

$$= 25\pi$$

Exercise

Find the area of the region using line integral of the region bounded by an ellipse with semi-major and semi-minor axes of length 12 and 8, respectively.

Solution

$$\frac{x^2}{6^2} + \frac{y^2}{4^2} = 1$$

$x = 6\cos t \quad \rightarrow \quad dx = -6\sin t\ dt$

$y = 4\sin t \quad \rightarrow \quad dy = 4\cos t\ dt$

$$A = \frac{1}{2}\int_0^{2\pi} \left(6\cos t\left(4\cos t\right) - 4\sin t\left(-6\sin t\right)\right)dt \qquad\qquad A = \frac{1}{2}\oint_C xdy - ydx$$

$$= 12\int_0^{2\pi} \left(\cos^2 t + \sin^2 t\right)dt$$

$$= 12\int_0^{2\pi} dt$$

$$= 24\pi$$

Exercise

Find the area of the region using line integral of the region bounded by an ellipse $9x^2 + 25y^2 = 225$.

Solution

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

$x = 5\cos t \quad \rightarrow \quad dx = -5\sin t\ dt$

$y = 3\sin t \quad \rightarrow \quad dy = 3\cos t\ dt$

$$A = \frac{1}{2}\int_0^{2\pi} \left(5\cos t\left(3\cos t\right) - 3\sin t\left(-5\sin t\right)\right)dt \qquad\qquad A = \frac{1}{2}\oint_C xdy - ydx$$

$$= \frac{15}{2} \int_0^{2\pi} \left(\cos^2 t + \sin^2 t \right) dt$$

$$= \frac{15}{2} \int_0^{2\pi} dt$$

$$= 15\pi$$

Exercise

Find the area of the region using line integral of the region $\left\{ (x,\ y) :\ \ x^2 + y^2 \le 16 \right\}$

Solution

$$x = 4\cos t \quad \rightarrow \quad dx = -4\sin t\ dt$$

$$y = 4\sin t \quad \rightarrow \quad dy = 4\cos t\ dt$$

$$A = \frac{1}{2} \int_0^{2\pi} \left(4\cos t \left(4\cos t \right) - 4\sin t \left(-4\sin t \right) \right) dt \qquad\qquad A = \frac{1}{2} \oint_C xdy - ydx$$

$$= 8 \int_0^{2\pi} \left(\cos^2 t + \sin^2 t \right) dt$$

$$= 8 \int_0^{2\pi} dt$$

$$= 16\pi$$

Exercise

Find the area of the region using line integral of the region bounded by the parabolas $\vec{r}(t) = \left\langle t,\ 2t^2 \right\rangle$ and $\vec{r}(t) = \left\langle t,\ 12 - t^2 \right\rangle$ for $-2 \le t \le 2$

Solution

$$A = \frac{1}{2} \oint_C xdy - ydx$$

$$A = \frac{1}{2} \int_{-2}^{2} \left(t \frac{d}{dt}\left(2t^2\right) - 2t^2 \frac{d}{dt}(t) \right) dt \ - \ \frac{1}{2} \int_{-2}^{2} \left(t \frac{d}{dt}\left(12 - t^2\right) - \left(12 - t^2\right) \frac{d}{dt}(t) \right) dt$$

$$= \frac{1}{2} \int_{-2}^{2} \left(t(4t) - 2t^2 \right) dt \ - \ \frac{1}{2} \int_{-2}^{2} \left(t(-2t) - 12 + t^2 \right) dt$$

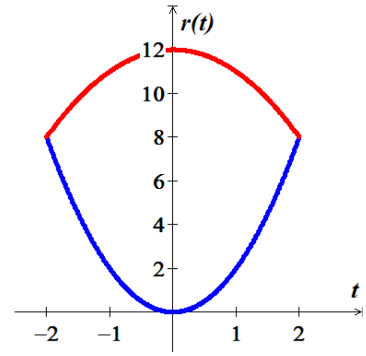
$$=\frac{1}{2}\int_{-2}^{2}\left(4t^2-2t^2+2t^2+12-t^2\right)dt$$

$$=\frac{1}{2}\int_{-2}^{2}\left(3t^2+12\right)dt$$

$$=\frac{1}{2}\left(t^3+12t\right.\Big|_{-2}^{2}$$

$$=\frac{1}{2}(8+24+8+24)$$

$$\underline{=32}$$



Exercise

Find the area of the region using line integral of the region bounded by the curve $\vec{r}(t)=\left\langle t\left(1-t^2\right),\ 1-t^2\right\rangle$ for $-1\le t\le 1$

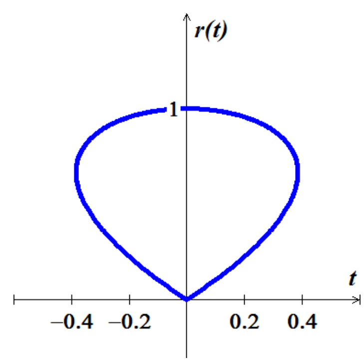
Solution

$\vec{r}(-1)=\langle 0,\ 0\rangle$

$\vec{r}\left(-\frac{1}{2}\right)=\left\langle \frac{1}{8},\ \frac{3}{4}\right\rangle$

$\vec{r}(0)=\langle 0,\ 1\rangle$

The curve travels in counterclockwise, therefore;



$$A=\frac{1}{2}\oint_C xdy-ydx$$

$$A=\frac{1}{2}\int_{1}^{-1}\left(\left(t-t^3\right)(-2t)-\left(1-t^2\right)\left(1-3t^2\right)\right)dt$$

$$=\frac{1}{2}\int_{1}^{-1}\left(-2t^2+2t^4-1+3t^2+t^2-3t^4\right)dt$$

$$=\frac{1}{2}\int_{1}^{-1}\left(2t^2-t^4-1\right)dt$$

$$=\frac{1}{2}\left(\frac{2}{3}t^3-\frac{1}{5}t^5-t\right.\Big|_{1}^{-1}$$

$$=-\frac{2}{3}+\frac{1}{5}+1$$

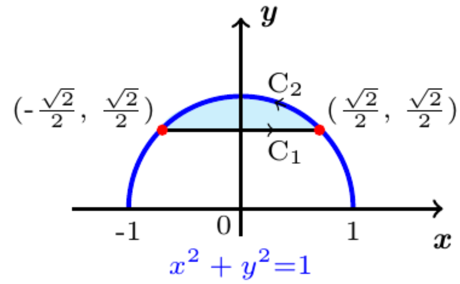
$$\underline{=\frac{8}{3}}$$

Exercise

Find the area of the region using line integral of the shaded region

Solution

For the path C_1 :



$$\begin{cases} t = 0 & \rightarrow x = -\frac{\sqrt{2}}{2} \\ t = 1 & \rightarrow x = \frac{\sqrt{2}}{2} \end{cases}$$

$$x = \frac{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}{1 - 0} t - \frac{\sqrt{2}}{2}$$

$$= \sqrt{2}t - \frac{\sqrt{2}}{2}$$

$$y = \frac{\sqrt{2}}{2}$$

$$C_1 : \ \vec{r}_1(t) = \left\langle \sqrt{2}t - \frac{\sqrt{2}}{2}, \ \ \frac{\sqrt{2}}{2} \right\rangle \quad 0 \le t \le 1$$

$$\frac{d\vec{r}_1}{dt} = \left\langle \sqrt{2}, \ 0 \right\rangle$$

For the path C_2 :

$$C_2 : \ \vec{r}_2(t) = \langle \cos t, \ \ \sin t \rangle \quad -\frac{\pi}{4} \le t \le \frac{\pi}{4}$$

$$\frac{d\vec{r}_2}{dt} = \langle -\sin t, \ \ \cos t \rangle$$

$$A = \frac{1}{2}\int_0^1 \left(\left(\sqrt{2}t - \frac{\sqrt{2}}{2} \right)(0) - \left(\frac{\sqrt{2}}{2} \right)\left(\sqrt{2} \right) \right) dt \ + \ \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\cos^2 t + \sin^2 t \right) dt \qquad A = \frac{1}{2}\oint_C xdy - ydx$$

$$= -\frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dt$$

$$= -\frac{1}{2} + \frac{1}{2}t \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$\boxed{= \frac{\pi}{4} - \frac{1}{2}}$$

Exercise

Prove the identity $\oint_C dx = \oint_C dy = 0$, where C is a simple closed smooth oriented curve

Solution

$$\oint_C dx = \oint_C dy$$

$$\oint_C dx - \oint_C dy = \oint_C (1dx - 1dy)$$

This is an outward flux of the constant vector field $\vec{F} = \langle 1,\ 1 \rangle$

$$\oint_C dx - \oint_C dy = \underbrace{\iint_R \left(\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(1) \right) dA}_{=0}$$

$$\oint_C dx = \oint_C dy = 0 \ \checkmark$$

Exercise

Prove the identity $\oint_C f(x)\,dx + g(y)\,dy = 0$, where f and g have continuous derivatives on the region enclosed by C (is a simple closed smooth oriented curve)

Solution

By Green's Theorem:

$$\oint_C f(x)\,dx + g(y)\,dy = \underbrace{\iint \left(\frac{\partial}{\partial x}(g(y)) - \frac{\partial}{\partial y}(f(x)) \right) dA}_{=0} \ \checkmark$$

Exercise

Show that the value of $\oint_C xy^2 dx + \left(x^2 y + 2x\right) dy$ depends only on the area of the region enclosed by C.

Solution

$$\oint_C xy^2 dx + \left(x^2 y + 2x\right) dy = \iint_R \left(\frac{\partial}{\partial x}\left(x^2 y + 2x\right) - \frac{\partial}{\partial y}\left(xy^2\right) \right) dA$$

$$= \iint_R (2xy + 2 - 2xy)\,dA$$

$$= 2\iint_R dA$$

$$\underline{= 2 \times \text{Area of } A}$$

$\therefore \oint_C xy^2dx + \left(x^2y + 2x\right)dy$ depends only on the area of the region

Exercise

In terms of the parameters a and b, how is the value of $\oint_C aydx + bxdy$ related to the area of the region enclosed by C, assuming counterclockwise orientation of C?

Solution

$$\oint_C aydx + bxdy = \iint_R \left(\frac{\partial}{\partial x}(bx) - \frac{\partial}{\partial y}(ay)\right)dA$$

$$= \iint_R (b-a)\,dA$$

$$\underline{= (b-a) \times \text{Area of } A}$$

Exercise

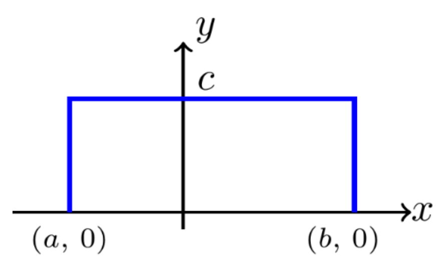
Show that if the circulation form of Green's Theorem is applied to the vector field $\left\langle 0,\ \frac{f(x)}{c}\right\rangle$ and $R = \{(x, y):\ \ a \le x \le b,\ \ 0 \le y \le c\}$, then the result is the Fundamental Theorem of Calculus,

$$\int_a^b \frac{df}{dx}dx = f(b) - f(a)$$

Solution

If $f(x)$ is continuous, then the circulation form of Green's Theorem is given

$$\oint_C \frac{f(x)}{c}dy = \frac{1}{c}\iint_R \frac{df}{dx}dA$$



$$\frac{1}{c}\iint_R \frac{df}{dx}\,dA = \frac{1}{c}\int_a^b \int_0^c \frac{df}{dx}\,dydx$$

$$= \frac{1}{c}\int_a^b \frac{df}{dx}\left(y \Big|_0^c\right) dx$$

$$= \int_a^b \frac{df}{dx}\,dx$$

$(a,\ 0)\ -\ (b,\ 0)$:

$$\vec{r}_1(t) = \langle (b-a)t + a,\ 0\rangle$$

$$\frac{d}{dt}\left(\vec{r}_1\right) = \langle b-a,\ 0\rangle$$

$$\vec{F}_1 = \left\langle 0,\ \frac{f\left((b-a)t+a\right)}{c}\right\rangle$$

$$\vec{F}_1 \bullet \vec{r}_1{}' = \left\langle 0,\ \frac{f\left((b-a)t+a\right)}{c}\right\rangle \bullet \langle b-a,\ 0\rangle$$

$$\underline{= 0\,|}$$

$(b,\ 0)\ -\ (b,\ c)$:

$$\vec{r}_2(t) = \langle b,\ ct\rangle$$

$$\vec{r}_2{}' = \langle 0,\ c\rangle$$

$$\vec{F}_2 = \left\langle 0,\ \frac{f(b)}{c}\right\rangle$$

$$\vec{F}_2 \bullet \vec{r}_2{}' = \left\langle 0,\ \frac{f(b)}{c}\right\rangle \bullet \langle 0,\ c\rangle$$

$$\underline{= f(b)\,|}$$

$(b,\ c)\ -\ (a,\ c)$:

$$\vec{r}_3(t) = \langle (a-b)t + b,\ c\rangle$$

$$\vec{r}_3{}' = \langle a-b,\ 0\rangle$$

$$\vec{F}_3 = \left\langle 0,\ \frac{f\left((a-b)t+b\right)}{c}\right\rangle$$

$$\vec{F}_3 \bullet \vec{r}_3{}' = \left\langle 0,\ \frac{f\left((a-b)t+b\right)}{c}\right\rangle \bullet \langle a-b,\ 0\rangle$$

$$\underline{= 0\,|}$$

$(a,\ c)\ -\ (a,\ 0)$:

$$\vec{r}_4(t) = \langle a,\ -ct + c \rangle$$

$$\vec{r}_4{}' = \langle 0,\ -c \rangle$$

$$\vec{F}_3 = \left\langle 0,\ \frac{f(a)}{c} \right\rangle$$

$$\vec{F}_3 \bullet \vec{r}_3{}' = \left\langle 0,\ \frac{f(a)}{c} \right\rangle \bullet \langle 0,\ -c \rangle$$

$$\boxed{= -f(a)}$$

$$\oint_C \frac{f(x)}{c} dy = \int_0^1 \big(0 + f(b) + 0 - f(a)\big) dt$$

$$= \int_0^1 \big(f(b) - f(a)\big) dt$$

$$= \big(f(b) - f(a)\big)\ t \Big|_0^1$$

$$\boxed{= f(b) - f(a)}$$

$$\boxed{\int_a^b \frac{df}{dx} dx = f(b) - f(a)}$$

Exercise

Show that if the flux form of Green's Theorem is applied to the vector field $\left\langle \frac{f(x)}{c},\ 0 \right\rangle$ and $R = \{(x,\ y):\ \ a \le x \le b,\ \ \ 0 \le y \le c\}$, then the result is the Fundamental Theorem of Calculus,

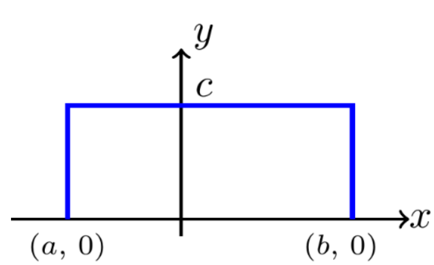
$$\int_a^b \frac{df}{dx} dx = f(b) - f(a)$$

Solution

If $f(x)$ is continuous, then the circulation form of Green's Theorem is given

$$\oint_C \frac{f(x)}{c} dy = \frac{1}{c} \iint_R \frac{df}{dx} dA$$

$$\frac{1}{c} \iint_R \frac{df}{dx} dA = \frac{1}{c} \int_a^b \int_0^c \frac{df}{dx} dydx$$



$$= \frac{1}{c} \int_a^b \frac{df}{dx} \left(y \,\Big|_0^c \right. dx$$

$$= \int_a^b \frac{df}{dx} dx$$

$(a,\ 0)\ -\ (b,\ 0)$:

$$\vec{r}_1(t) = \langle (b-a)t + a,\ 0 \rangle$$

$$\vec{r}_1{}' = \langle b-a,\ 0 \rangle$$

$$\vec{F}_1 = \left\langle \frac{f\left((b-a)t+a\right)}{c},\ 0 \right\rangle$$

$$\frac{f\left((b-a)t+a\right)}{c}(0) + 0(b-a) = 0 \quad (1)$$

$(b,\ 0)\ -\ (b,\ c)$:

$$\vec{r}_2(t) = \langle b,\ ct \rangle$$

$$\vec{r}_2{}' = \langle 0,\ c \rangle$$

$$\vec{F}_2 = \left\langle \frac{f(b)}{c},\ 0 \right\rangle$$

$$\frac{f(b)}{c}(c) + 0 = f(b) \quad (2)$$

$(b,\ c)\ -\ (a,\ c)$:

$$\vec{r}_3(t) = \langle (a-b)t + b,\ c \rangle$$

$$\vec{r}_3{}' = \langle a-b,\ 0 \rangle$$

$$\vec{F}_3 = \left\langle \frac{f\left((a-b)t+b\right)}{c},\ 0 \right\rangle$$

$$\frac{f\left((a-b)t+b\right)}{c}(0) + 0(a-b) = 0 \quad (3)$$

$(a,\ c)\ -\ (a,\ 0)$:

$$\vec{r}_4(t) = \langle a,\ -ct + c \rangle$$

$$\vec{r}_4{}' = \langle 0,\ -c \rangle$$

$$\vec{F}_3 = \left\langle \frac{f(a)}{c},\ 0 \right\rangle$$

$$\frac{f(a)}{c}(c) + 0 = f(a) \quad (2)$$

$$\oint_C \frac{f(x)}{c}dy = \int_0^1 \left(0 + f(b) + 0 - f(a)\right)dt$$

$$= \int_0^1 \left(f(b) - f(a)\right)dt$$

$$= \left(f(b) - f(a)\right)\, t \Big|_0^1$$

$$= f(b) - f(a)$$

$$\int_a^b \frac{df}{dx}dx = f(b) - f(a)$$

Solution Section 4.5 – Divergence and Curl

Exercise

Find the divergence of the following vector field $\vec{F} = \langle 2x,\ 4y,\ -3z \rangle$

Solution

$$div\vec{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(-3z) \qquad div\vec{F} = \nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= 2 + 4 - 3$$

$$= 3$$

Exercise

Find the divergence of the following vector field $\vec{F} = \langle -2y,\ 3x,\ z \rangle$

Solution

$$div\vec{F} = \frac{\partial}{\partial x}(-2y) + \frac{\partial}{\partial y}(3x) + \frac{\partial}{\partial z}(z) \qquad div\vec{F} = \nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= 1$$

Exercise

Find the divergence of the following vector field $\vec{F} = \left\langle x^2yz,\ -xy^2z,\ -xyz^2 \right\rangle$

Solution

$$div\vec{F} = \frac{\partial}{\partial x}\left(x^2yz\right) + \frac{\partial}{\partial y}\left(-xy^2z\right) + \frac{\partial}{\partial z}\left(-xyz^2\right) \qquad div\vec{F} = \nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= 2xyz - 2xyz - 2xyz$$

$$= -2xyz$$

Exercise

Find the divergence of the following vector field $\vec{F} = \left\langle x^2 - y^2,\ y^2 - z^2,\ z^2 - x^2 \right\rangle$

Solution

$$div\vec{F} = \frac{\partial}{\partial x}\left(x^2 - y^2\right) + \frac{\partial}{\partial y}\left(y^2 - z^2\right) + \frac{\partial}{\partial z}\left(z^2 - x^2\right) \qquad div\vec{F} = \nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= 2x + 2y + 2z$$

Exercise

Find the divergence of the following vector field $\vec{F} = \left\langle e^{-x+y},\ e^{-y+z},\ e^{-z+x} \right\rangle$

Solution

$$div\vec{F} = \frac{\partial}{\partial x}\left(e^{-x+y}\right) + \frac{\partial}{\partial y}\left(e^{-y+z}\right) + \frac{\partial}{\partial z}\left(e^{-z+x}\right) \qquad div\vec{F} = \nabla\cdot\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= -e^{-x+y} - e^{-y+z} - e^{-z+x}$$

Exercise

Find the divergence of the following vector field $\vec{F} = \langle yz\cos x,\ xz\cos y,\ xy\cos z \rangle$

Solution

$$div\vec{F} = \frac{\partial}{\partial x}(yz\cos x) + \frac{\partial}{\partial y}(xz\cos y) + \frac{\partial}{\partial z}(xy\cos z) \qquad div\vec{F} = \nabla\cdot\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= -yz\sin x - xz\sin y - xy\sin z$$

Exercise

Find the divergence of the following vector field $\vec{F} = \langle 12x,\ 4y,\ -3z \rangle$

Solution

$$div\vec{F} = \frac{\partial}{\partial x}(12x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(-3z) \qquad div\vec{F} = \nabla\cdot\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= 12 + 4 - 3$$

$$= 13$$

Exercise

Find the divergence of the following vector field $\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{1 + x^2 + y^2}$

Solution

$$div\vec{F} = \frac{\partial}{\partial x}\left(\frac{x}{1+x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{1+x^2+y^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{1+x^2+y^2}\right) \qquad div\vec{F} = \nabla\cdot\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= \frac{1+x^2+y^2-2x^2}{\left(1+x^2+y^2\right)^2} + \frac{1+x^2+y^2-2y^2}{\left(1+x^2+y^2\right)^2} + \frac{1}{\left(1+x^2+y^2\right)^2}$$

$$= \frac{3}{\left(1+x^2+y^2\right)^2}$$

Exercise

Calculate the divergence of the radial fields. $\vec{F} = \dfrac{\langle x,\ y,\ z\rangle}{x^2+y^2+z^2} = \dfrac{\vec{r}}{|\vec{r}|^2}$

Express the result in terms of the position vector $\vec{r}$ and its length $|\vec{r}|$.

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2+z^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{x^2+y^2+z^2}\right)$$

$$= \frac{-x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^2} + \frac{x^2-y^2+z^2}{\left(x^2+y^2+z^2\right)^2} + \frac{x^2+y^2-z^2}{\left(x^2+y^2+z^2\right)^2} \qquad \left(\frac{x}{x^2+y^2+z^2}\right)' = \frac{x^2+y^2+z^2-2x^2}{\left(x^2+y^2+z^2\right)^2}$$

$$= \frac{x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^2}$$

$$= \frac{\vec{r}}{|\vec{r}|^2}$$

Exercise

Calculate the divergence of the radial fields. $\vec{F} = \langle x,\ y,\ z\rangle\left(x^2+y^2+z^2\right) = 5|\vec{r}|^2$

Express the result in terms of the position vector $\vec{r}$ and its length $|\vec{r}|$.

Solution

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}\left(x\left(x^2+y^2+z^2\right)\right) + \frac{\partial}{\partial y}\left(y\left(x^2+y^2+z^2\right)\right) + \frac{\partial}{\partial z}\left(z\left(x^2+y^2+z^2\right)\right)$$

$$= \frac{\partial}{\partial x}\left(x^3+xy^2+xz^2\right) + \frac{\partial}{\partial y}\left(x^2y+y^3+yz^2\right) + \frac{\partial}{\partial z}\left(x^2z+y^2z+z^3\right)$$

$$= 3x^2+y^2+z^2+x^2+3y^2+z^2+x^2+y^2+3z^2$$

$$= 5\left(x^2+y^2+z^2\right)$$

$$= 5|\vec{r}|^2$$

Exercise

Calculate the divergence of the radial fields. $\vec{F} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}} = \dfrac{\vec{r}}{|\vec{r}|^3}$

Express the result in terms of the position vector $\vec{r}$ and its length $|\vec{r}|$.

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} + \frac{\partial}{\partial y}\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} + \frac{\partial}{\partial z}\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$= \frac{x^2+y^2+z^2-3x^2}{\left(x^2+y^2+z^2\right)^{5/2}} + \frac{x^2+y^2+z^2-3y^2}{\left(x^2+y^2+z^2\right)^{5/2}} + \frac{x^2+y^2+z^2-3z^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$= 0$$

Exercise

Calculate the divergence of the radial fields. $\vec{F} = \frac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^2} = \frac{\vec{r}}{|\vec{r}|^4}$

Express the result in terms of the position vector $\vec{r}$ and its length $|\vec{r}|$.

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}\frac{x}{\left(x^2+y^2+z^2\right)^2} + \frac{\partial}{\partial y}\frac{y}{\left(x^2+y^2+z^2\right)^2} + \frac{\partial}{\partial z}\frac{z}{\left(x^2+y^2+z^2\right)^2}$$

$$= \frac{x^2+y^2+z^2-4x^2}{\left(x^2+y^2+z^2\right)^3} + \frac{x^2+y^2+z^2-4y^2}{\left(x^2+y^2+z^2\right)^3} + \frac{x^2+y^2+z^2-4z^2}{\left(x^2+y^2+z^2\right)^3}$$

$$= -\frac{x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^3}$$

$$= -\frac{1}{|\vec{r}|^4}$$

Exercise

Consider the following vector fields $\vec{F} = \langle x,\ x+y\rangle$, the circle C, and two points P and Q.

a) Without computing the divergence, does the graph suggest that the divergence is positive or negative at P and Q?

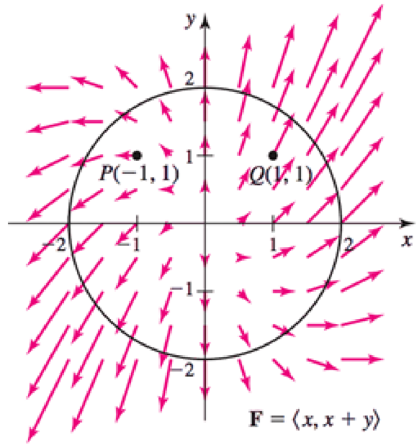
b) Compute the divergence and confirm your conjecture in part (*a*).

c) On what part of C is the flux outward? Inward?

d) Is the net outward flux across C positive or negative vector ?

Solution

a) At both P and Q, the arrows going away from the point are larger in both number and magnitude than those going in, so we would expect the divergence to be positive at both points.



b) $\nabla \cdot \vec{F} = \dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial y}(x+y)$

$= 1 + 1$

$\underline{= 2}\,| > 0$

It is positive everywhere.

c) The arrows all point roughly away from the origin, so we the flux is outward everywhere.

d) The net flux across C should be positive.

Exercise

Consider the following vector fields $\vec{F} = \left\langle x,\ y^2 \right\rangle$, the circle C, and two points P and Q.

a) Without computing the divergence, does the graph suggest that the divergence is positive or negative at P and Q?

b) Compute the divergence and confirm your conjecture in part (a).

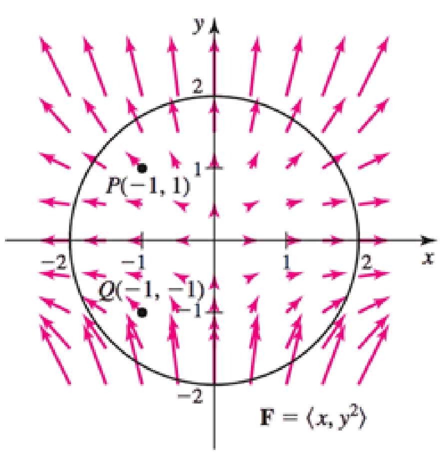
c) On what part of C is the flux outward? Inward?

d) Is the net outward flux across C positive or negative?

Solution

a) At P, the divergence should be positive.

At Q, the larger arrows point in towards Q, so the divergence should be negative.



b) $\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}\left(y^2\right)$

$= 1 + 2y$

At $P = (-1,\ 1) \quad \to \quad \nabla \bullet \vec{F} = 3$

At $Q = (-1,\ -1) \quad \to \quad \nabla \bullet \vec{F} = -1$

c) The flux is outward above the line $y = -1$; below this line, the flux is inward across *C*.

d) The size of the narrows pointing outward at the top of the circle seems to roughly equal those pointing inward at the bottom, so the remaining outward-pointing arrows result in a net positive flux across *C*.

Exercise

Consider the vector fields $\vec{F} = \langle 1,\ 0,\ 0\rangle \times \vec{r}$, where $\vec{r} = \langle x,\ y,\ z\rangle$

a) Compute the curl field and verify that it has the same direction as the axis of rotation

b) Compute the magnitude of the curl of the field

Solution

a) $\nabla \times \vec{F} = \nabla \times \left[\langle 1,\ 0,\ 0\rangle \times \langle x,\ y,\ z\rangle\right]$

$$= \nabla \times \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1 & 0 & 0 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left(-z\hat{j} + y\,\hat{k}\right)$$

$$= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -z & y \end{vmatrix}$$

$$= (1+1)\,\hat{i} + (0-0)\,\hat{j} + (0-0)\,\hat{k}$$

$$= 2\,\hat{i}$$

The curl is the same direction as the axis of rotation.

b) The magnitude of the curl is $\left|2\,\hat{i}\right| = 2$

Exercise

Consider the vector fields $\vec{F} = \langle 1,\ -1,\ 0 \rangle \times \vec{r}$, where $\vec{r} = \langle x,\ y,\ z \rangle$

a) Compute the curl field and verify that it has the same direction as the axis of rotation

b) Compute the magnitude of the curl of the field

Solution

a) $\nabla \times \vec{F} = \nabla \times \left[\langle 1,\ -1,\ 0 \rangle \times \langle x,\ y,\ z \rangle \right]$

$$= \nabla \times \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1 & -1 & 0 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left(-z\,\hat{i} - z\,\hat{j} + (x+y)\,\hat{k} \right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & -z & x+y \end{vmatrix}$$

$$= (1+1)\,\hat{i} + (-1-1)\,\hat{j} + (0-0)\,\hat{k}$$

$$= 2\,\hat{i} - 2\hat{j}$$

The curl is the same direction as the axis of rotation.

b) The magnitude of the curl is $\left| 2\,\hat{i} - 2\hat{j} \right| = 2\sqrt{2}$

Exercise

Consider the vector fields $\vec{F} = \langle 1,\ -1,\ 1 \rangle \times \vec{r}$, where $\vec{r} = \langle x,\ y,\ z \rangle$

a) Compute the curl field and verify that it has the same direction as the axis of rotation

b) Compute the magnitude of the curl of the field

Solution

a) $\nabla \times \vec{F} = \nabla \times \left[\langle 1,\ -1,\ 1 \rangle \times \langle x,\ y,\ z \rangle \right]$

$$= \nabla \times \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1 & -1 & 1 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left((-z-y)\,\hat{i} + (x-z)\,\hat{j} + (x+y)\,\hat{k} \right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z-y & x-z & x+y \end{vmatrix}$$

$$= (1+1)\,\hat{i} + (-1-1)\,\hat{j} + (1+1)\,\hat{k}$$

$$\boxed{= 2\,\hat{i} - 2\hat{j} + 2\,\hat{k}}$$

The curl is the same direction as the axis of rotation.

b) The magnitude of the curl is $\left|2\,\hat{i} - 2\hat{j} + 2\,\hat{k}\right| \boxed{= 2\sqrt{3}}$

Exercise

Consider the vector fields $\vec{F} = \langle 1,\ -2,\ -3\rangle \times \vec{r}$, where $\vec{r} = \langle x,\ y,\ z\rangle$

a) Compute the curl field and verify that it has the same direction as the axis of rotation

b) Compute the magnitude of the curl of the field

Solution

a) $\nabla \times \vec{F} = \nabla \times \left[\langle 1,\ -2,\ -3\rangle \times \langle x,\ y,\ z\rangle\right]$

$$= \nabla \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & -3 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left((-2z+3y)\,\hat{i} + (-3x - z)\,\hat{j} + (y+2x)\,\hat{k} \right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2z+3y & -3x-z & 2x+y \end{vmatrix}$$

$$= (1+1)\,\hat{i} + (-2-2)\,\hat{j} + (-3-3)\,\hat{k}$$

$$\boxed{= 2\,\hat{i} - 4\hat{j} - 6\hat{k}}$$

The curl is the same direction as the axis of rotation.

b) The magnitude of the curl is

$$\boxed{\left|2\,\hat{i} - 4\hat{j} - 6\hat{k}\right| = 2\sqrt{14}}$$

Exercise

Compute the curl of the vector field $\vec{F} = \left\langle x^2 - y^2,\ xy,\ z\right\rangle$

Solution

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & xy & z \end{vmatrix}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(xy)\right)\hat{i} + \left(\frac{\partial}{\partial z}\left(x^2 - y^2\right) - \frac{\partial}{\partial x}(z)\right)\hat{j} + \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}\left(x^2 - y^2\right)\right)\hat{k}$$

$$= (0-0)\hat{i} + (0-0)\hat{j} + (y+2y)\hat{k}$$

$$\underline{= 3y\,\hat{k}}$$

Exercise

Compute the curl of the vector field $\vec{F} = \left\langle 0,\ z^2 - y^2,\ -yz\right\rangle$

Solution

$$curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & z^2 - y^2 & -yz \end{vmatrix} \qquad curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(-yz) - \frac{\partial}{\partial z}\left(z^2 - y^2\right)\right)\hat{i} + \left(\frac{\partial}{\partial z}(0) - \frac{\partial}{\partial x}(-yz)\right)\hat{j} + \left(\frac{\partial}{\partial x}\left(z^2 - y^2\right) - \frac{\partial}{\partial y}(0)\right)\hat{k}$$

$$= (-z-2z)\hat{i} + (0-0)\hat{j} + (0-0)\hat{k}$$

$$\underline{= -3z\,\hat{i}}$$

Exercise

Compute the curl of the vector field $\vec{F} = \left\langle z^2 \sin y,\ xz^2 \cos y,\ 2xz \sin y\right\rangle$

Solution

$$curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 \sin y & xz^2 \cos y & 2xz \sin y \end{vmatrix} \qquad curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= (2xz \cos y - 2xz \cos y)\hat{i} + (2z \sin y - 2z \sin y)\hat{j} + \left(z^2 \cos y - z^2 \cos y\right)\hat{k}$$

$$\underline{= 0}$$

Exercise

Compute the curl of the vector field $\vec{F} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}} = \dfrac{\vec{r}}{|\vec{r}|^3}$

Solution

$$curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} & \frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} & \frac{z}{\left(x^2+y^2+z^2\right)^{3/2}} \end{vmatrix} \qquad curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$\left(U^n V^m\right)' = U^{n-1}V^{m-1}\left(nU'V + mUV'\right)$$

$$\left(z\left(x^2+y^2+z^2\right)^{-3/2}\right)'_y = (1)\left(x^2+y^2+z^2\right)^{-5/2}\left[(0)\left(x^2+y^2+z^2\right)-\frac{3}{2}z(2y)\right]$$

$$= \left(x^2+y^2+z^2\right)^{-5/2}(-3yz)$$

$$= \frac{1}{\left(x^2+y^2+z^2\right)^{5/2}}\left((-3yz+3yz)\,\hat{i} + (-3xz+3xz)\,\hat{j} + (-3xy+3xy)\,\hat{k}\right)$$

$$\underline{= 0}$$

Exercise

Compute the curl of the vector field $\vec{F} = \vec{r} = \langle x,\ y,\ z\rangle$

Solution

$$curl\ \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \qquad curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= (0-0)\,\hat{i} + (0-0)\,\hat{j} + (0-0)\,\hat{k}$$

$$\underline{= 0}$$

Exercise

Compute the curl of the vector field $\vec{F} = \left\langle 3xz^3e^{y^2},\ 2xz^3e^{y^2},\ 3xz^2e^{y^2}\right\rangle$

Solution

$$curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xz^3e^{y^2} & 2xz^3e^{y^2} & 3xz^2e^{y^2} \end{vmatrix} \qquad curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(6xyz^3e^{y^2} - 6xz^2e^{y^2}\right)\hat{i} + \left(9xz^2e^{y^2} - 3z^2e^{y^2}\right)\hat{j} + \left(2z^3e^{y^2} - 6xyz^3e^{y^2}\right)\hat{k}$$

$$= z^2e^{y^2}\left[(6xyz - 6x)\,\hat{i} + (9x-3)\,\hat{j} + (2z - 6xyz)\,\hat{k}\right]$$

Exercise

Compute the curl of the vector field $\vec{F} = \left\langle x^2 - z^2,\ 1,\ 2xz \right\rangle$

Solution

$$curl\ \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - z^2 & 1 & 2xz \end{vmatrix}$$

$$curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= (0-0)\hat{i} + (-2z - 2z)\,\hat{j} + (0-0)\,\hat{k}$$

$$= -4z\,\hat{j}$$

Exercise

Compute the curl of the vector field $\vec{F} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2 + y^2 + z^2\right)^{1/2}} = \dfrac{\vec{r}}{|\vec{r}|}$

Solution

$$curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\left(x^2+y^2+z^2\right)^{1/2}} & \frac{y}{\left(x^2+y^2+z^2\right)^{1/2}} & \frac{z}{\left(x^2+y^2+z^2\right)^{1/2}} \end{vmatrix}$$

$$= \frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}\left((-yz + yz)\hat{i} + (-xz + xz)\,\hat{j} + (-xy + xy)\,\hat{k}\right)$$

$$= 0$$

$$\left(U^nV^m\right)' = U^{n-1}V^{m-1}\left(nU'V + mUV'\right)$$

Exercise

Compute the divergence and curl of the following vector fields, state whether the field is *source-free* or *irrotational*. $\vec{F} = \langle yz,\ xz,\ xy\rangle$

Solution

$$div\vec{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) \qquad\qquad div\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$\underline{= 0}$$

$$curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} \qquad\qquad curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right)\hat{\imath} + \left(\frac{\partial}{\partial z}(yz) - \frac{\partial}{\partial x}(xy)\right)\hat{\jmath} + \left(\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz)\right)\hat{k}$$

$$= (x - x)\hat{\imath} + (y - y)\hat{\jmath} + (z - z)\hat{k}$$

$$\underline{= \vec{0}}$$

$\therefore$ The field is both source-free and irrotational.

Exercise

Compute the divergence and curl of the following vector fields, state whether the field is *source-free* or *irrotational*. $\vec{F} = \vec{r}\,|\vec{r}| = \langle x,\ y,\ z\rangle\sqrt{x^2 + y^2 + z^2}$

Solution

$$\frac{\partial}{\partial x}\left(x\sqrt{x^2 + y^2 + z^2}\right) = \frac{x^2 + y^2 + z^2 + x^2}{\sqrt{x^2 + y^2 + z^2}} \qquad\qquad \left(U^m V^n\right)' = U^{m-1}V^{n-1}\left(mU'V + nUV'\right)$$

$$= \frac{|\vec{r}|^2 + x^2}{|\vec{r}|}$$

$$\frac{\partial}{\partial y}\left(y\sqrt{x^2 + y^2 + z^2}\right) = \frac{x^2 + y^2 + z^2 + y^2}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{|\vec{r}|^2 + y^2}{|\vec{r}|}$$

$$\frac{\partial}{\partial z}\left(z\sqrt{x^2 + y^2 + z^2}\right) = \frac{x^2 + y^2 + z^2 + z^2}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{\left|\vec{r}\right|^2 + z^2}{\left|\vec{r}\right|}$$

$$div\vec{F} = \frac{\left|\vec{r}\right|^2 + x^2}{\left|\vec{r}\right|} + \frac{\left|\vec{r}\right|^2 + y^2}{\left|\vec{r}\right|} + \frac{\left|\vec{r}\right|^2 + z^2}{\left|\vec{r}\right|} \qquad div\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= \frac{3\left|\vec{r}\right|^2 + x^2 + y^2 + z^2}{\left|\vec{r}\right|}$$

$$= \frac{4\left|\vec{r}\right|^2}{\left|\vec{r}\right|}$$

$$\underline{= 4\left|\vec{r}\right|}$$

$$curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x\left|\vec{r}\right| & y\left|\vec{r}\right| & z\left|\vec{r}\right| \end{vmatrix} \qquad curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\left(z\left|\vec{r}\right|\right) - \frac{\partial}{\partial z}\left(y\left|\vec{r}\right|\right)\right)\hat{i} + \left(\frac{\partial}{\partial z}\left(x\left|\vec{r}\right|\right) - \frac{\partial}{\partial x}\left(z\left|\vec{r}\right|\right)\right)\hat{j} + \left(\frac{\partial}{\partial x}\left(y\left|\vec{r}\right|\right) - \frac{\partial}{\partial y}\left(x\left|\vec{r}\right|\right)\right)\hat{k}$$

$$= \left(\frac{yz}{\left|\vec{r}\right|} - \frac{yz}{\left|\vec{r}\right|}\right)\hat{i} + \left(\frac{xz}{\left|\vec{r}\right|} - \frac{zx}{\left|\vec{r}\right|}\right)\hat{j} + \left(\frac{yx}{\left|\vec{r}\right|} - \frac{xy}{\left|\vec{r}\right|}\right)\hat{k}$$

$$\underline{= \vec{0}}$$

∴ The field is irrotational but not a source-free.

Exercise

Compute the divergence and curl of the following vector fields, state whether the field is *source-free* or *irrotational*. $\vec{F} = \langle \sin xy,\ \cos yz,\ \sin xz \rangle$

Solution

$$div\vec{F} = \frac{\partial}{\partial x}(\sin xy) + \frac{\partial}{\partial y}(\cos yz) + \frac{\partial}{\partial z}(\sin xz) \qquad div\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$\underline{= y\cos xy - z\sin yz + x\cos xz}$$

$$curl\ \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin xy & \cos yz & \sin xz \end{vmatrix} \qquad curl\ \vec{F} = \begin{vmatrix} i & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(\sin xz) - \frac{\partial}{\partial z}(\cos yz) \right)\hat{i} + \left(\frac{\partial}{\partial z}(\sin xy) - \frac{\partial}{\partial x}(\sin xz) \right)\hat{j} + \left(\frac{\partial}{\partial x}(\cos yz) - \frac{\partial}{\partial y}(\sin xy) \right)\hat{k}$$

$$= (0 + y\,\text{sinyz})\hat{i} + (0 - z\cos xz)\hat{j} + (0 - x\cos xy)\hat{k}$$

$$= y\,\text{sinyz}\,\hat{i} - z\cos xz\hat{j} - x\cos xy\hat{k}$$

$\therefore$ The field is neither source-free nor irrotational.

Exercise

Compute the divergence and curl of the following vector fields, state whether the field is *source-free* or *irrotational*. $\vec{F} = \left\langle 2xy + z^4,\ x^2,\ 4xz^3 \right\rangle$

Solution

$$div\vec{F} = \frac{\partial}{\partial x}\left(2xy + z^4\right) + \frac{\partial}{\partial y}\left(x^2\right) + \frac{\partial}{\partial z}\left(4xz^3\right) \qquad div\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= 2y + 12xz^2$$

$$curl\ \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^4 & x^2 & 4xz^3 \end{vmatrix} \qquad curl\ \vec{F} = \begin{vmatrix} i & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\left(4xz^3\right) - \frac{\partial}{\partial z}\left(x^2\right) \right)\hat{i} + \left(\frac{\partial}{\partial z}\left(2xy + z^4\right) - \frac{\partial}{\partial x}\left(4xz^3\right) \right)\hat{j} + \left(\frac{\partial}{\partial x}\left(x^2\right) - \frac{\partial}{\partial y}\left(2xy + z^4\right) \right)\hat{k}$$

$$= (0)\hat{i} + \left(4z^3 - 4z^3\right)\hat{j} + (2x - 2x)\hat{k}$$

$$= \vec{0}$$

$\therefore$ The field is irrotational but not a source-free.

Exercise

Let $\vec{F} = \langle z,\ x,\ -y \rangle$

a) What are the components of curl $\vec{F}$ in the directions $\vec{n} = \langle 1,\ 0,\ 0 \rangle$ and $\vec{n} = \left\langle 0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}} \right\rangle$

b) In what direction is the scalar component of curl $\vec{F}$ a maximum?

Solution

a) $curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & -y \end{vmatrix}$ $\qquad$ $curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$

$= -\hat{i} + \hat{j} + \hat{k}$

Scalar component in the direction of $\vec{n} = \langle 1,\ 0,\ 0 \rangle$

$\langle -1,\ 1,\ 1 \rangle \bullet \langle 1,\ 0,\ 0 \rangle \underline{= -1}$

Scalar component in the direction of $\vec{n} = \left\langle 0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}} \right\rangle$

$\langle -1,\ 1,\ 1 \rangle \bullet \left\langle 0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}} \right\rangle \underline{= 0}$

b) The scalar component of the curl is a maximum in the direction of the curl, in the direction $\langle -1,\ 1,\ 1 \rangle$ whose unit direction vector is $\frac{1}{\sqrt{3}}\langle -1,\ 1,\ 1 \rangle$

Exercise

Let $\vec{F} = \langle z,\ 0,\ -y \rangle$

a) What are the components of curl $\vec{F}$ in the directions $\vec{n} = \langle 1,\ 0,\ 0 \rangle$ and $\vec{n} = \langle 1,\ -1,\ 1 \rangle$

b) In what direction $\vec{n}$ is $\left(curl\ \vec{F} \right) \bullet \vec{n}$ a maximum?

Solution

a) $curl\ \vec{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & 0 & -y \end{vmatrix}$ $\qquad$ $curl\ \vec{F} = \begin{vmatrix} \boldsymbol{i} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$

$= -\hat{i} + \hat{j}$

Scalar component in the direction of $\vec{n} = \langle 1,\ 0,\ 0 \rangle$

$\langle -1,\ 1,\ 0 \rangle \bullet \langle 1,\ 0,\ 0 \rangle \underline{= -1}$

Scalar component in the direction of $\vec{n} = \langle 1,\ -1,\ 1 \rangle$

$\langle -1,\ 1,\ 0 \rangle \bullet \langle 1,\ -1,\ 1 \rangle \underline{= -2}$

b) The component of the curl is a maximum in the direction of the curl, in the direction $\langle -1,\ 1,\ 0 \rangle$ whose unit direction vector is $\frac{1}{\sqrt{2}}\langle -1,\ 1,\ 0 \rangle$

Exercise

Within the cube $\{(x, y, z): \ -1 \le x \le 1, \ \ -1 \le y \le 1, \ \ -1 \le z \le 1\}$, where does $div\vec{F}$ have the greatest magnitude when $\vec{F} = \langle x^2 - y^2, \ xy^2 z, \ 2xz \rangle$

Solution

$$div\vec{F} = \frac{\partial}{\partial x}\left(x^2 - y^2\right) + \frac{\partial}{\partial y}\left(xy^2 z\right) + \frac{\partial}{\partial z}(2xz) \qquad\qquad div\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$= 2x + 2xyz + 2x$$

$$= 4x + 2xyz$$

For the greatest magnitude, x and y & z have the same sign

$(-1, \ -1, \ -1) \quad \left|div\vec{F}\right| = |-4-2|$

$$= 6$$

$(-1, \ 1, \ 1) \quad \left|div\vec{F}\right| = |-4-2|$

$$= 6$$

$(1, \ 1, \ 1) \quad \left|div\vec{F}\right| = |4+2|$

$$= 6$$

$(1, \ -1, \ -1) \quad \left|div\vec{F}\right| = |4+2|$

$$= 6$$

$div\vec{F}$ have the greatest magnitude of 6.

Exercise

Show that the general rotation field $\vec{F} = \vec{a} \times \vec{r}$, where $\vec{a}$ is a nonzero constant vector and $\vec{r} = \langle x, \ y, \ z \rangle$, has zero divergence.

Solution

Let $\vec{a} = \langle a_1, \ a_2, \ a_3 \rangle$

$$\vec{F} = \vec{a} \times \vec{r}$$

$$= \langle a_1, \ a_2, \ a_3 \rangle \times \langle x, \ y, \ z \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \left(a_2 z - a_3 y\right)\hat{i} + \left(a_3 x - a_1 z\right)\hat{j} + \left(a_1 y - a_2 x\right)\hat{k}$$

$$\nabla \times \vec{F} = \nabla \times \left\langle a_2 z - a_3 y,\ a_3 x - a_1 z,\ a_1 y - a_2 x \right\rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= \left(a_1 - a_1\right)\hat{i} + \left(a_2 - a_2\right)\hat{j} + \left(a_3 - a_3\right)\hat{k}$$

$$\underline{= 0}$$

Exercise

Let $\vec{a} = \langle 0,\ 1,\ 0\rangle$, $\vec{r} = \langle x,\ y,\ z\rangle$ and consider the rotation field $\vec{F} = \vec{a} \times \vec{r}$. Use the right-hand rule for cross product to find the direction of $\vec{F}$ at the points $(0,\ 1,\ 1)$, $(1,\ 1,\ 0)$, $(0,\ 1,\ -1)$, and $(-1,\ 1,\ 0)$

Solution

$$\langle 0,\ 1,\ 0\rangle \times \langle 0,\ 1,\ 1\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$\underline{= \langle 1,\ 0,\ 0\rangle}$; $\vec{F}$ points in the positive x-direction

$$\langle 0,\ 1,\ 0\rangle \times \langle 1,\ 1,\ 0\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix}$$

$\underline{= \langle 0,\ 0,\ -1\rangle}$; $\vec{F}$ points in the negative z-direction

$$\langle 0,\ 1,\ 0\rangle \times \langle 0,\ 1,\ -1\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix}$$

$\underline{= \langle -1,\ 0,\ 0\rangle}$; $\vec{F}$ points in the negative x-direction

$$\langle 0,\ 1,\ 0\rangle \times \langle -1,\ 1,\ 0\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{vmatrix}$$

$\underline{= \langle 0,\ 0,\ 1\rangle}$; $\vec{F}$ points in the positive z-direction

Exercise

Find the exact points on the circle $x^2 + y^2 = 2$ at which the field $\vec{F} = \langle f,\ g \rangle = \langle x^2,\ y \rangle$ switches from pointing inward to outward on the circle, or vice versa.

Solution

The field switches from inward-pointing to outward-pointing at points where it is tangent to the circle $x^2 + y^2 = 2$, where it is orthogonal to the normal to the circle.

The normal to the circle at (x, y) is a multiple of (x, y), so we want to find $x,\ y$ so that

$$\langle x,\ y \rangle \cdot \langle x^2,\ y \rangle = x^3 + y^2 = 0$$

$$x^2 + \left(-x^3\right) = 2$$

$$x^3 - x^2 + 2 = 0 \xrightarrow{\text{solutions}} x = -1,\ 1 \pm i$$

The solutions are: $\underline{x = -1 \quad \to y = \pm 1}$

Exercise

Suppose a solid object in $\mathbb{R}^3$ has a temperature distribution given by $T(x,\ y,\ z)$. The heat flow vector field in the object is $\vec{F} = -k\nabla T$, where the conductivity $k > 0$ is a property of the material. Note that the heat flow vector points in the direction opposite to that of the gradient, which is the direction of greatest temperature decrease. The divergence of the heat flow vector is $\vec{F} = -k\nabla \cdot \nabla T = -k\nabla^2 T$ (the Laplacian of *T*). Compute the heat flow vector field and its divergence for the following temperature distribution.

a) $T(x,\ y,\ z) = 100\, e^{-\sqrt{x^2+y^2+z^2}}$

b) $T(x,\ y,\ z) = 100\, e^{-x^2+y^2+z^2}$

c) $T(x,\ y,\ z) = 100\left(1 + \sqrt{x^2 + y^2 + z^2}\right)$

Solution

a) $T(x,\ y,\ z) = 100\, e^{-\sqrt{x^2+y^2+z^2}}$

$$\vec{F} = -k\nabla T \qquad\qquad \nabla T = \left(\frac{\partial}{\partial x}T\right)\hat{\boldsymbol{i}} + \left(\frac{\partial}{\partial y}T\right)\hat{\boldsymbol{j}} + \left(\frac{\partial}{\partial z}T\right)\hat{\boldsymbol{k}}$$

$$= -100\, k\ \nabla\ e^{-\sqrt{x^2+y^2+z^2}}$$

$$= \frac{100\, ke^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2 + y^2 + z^2}}\ \langle x,\ y,\ z \rangle$$

$$= 100\,k\left[\frac{\partial}{\partial x}\left(\frac{xe^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{ye^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{ze^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right)\right]$$

$$\frac{\partial}{\partial x}\left(\frac{xe^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right) = e^{-\sqrt{x^2+y^2+z^2}}\,\frac{\left(1-x^2\left(x^2+y^2+z^2\right)^{-1/2}\right)\sqrt{x^2+y^2+z^2}-x^2\left(x^2+y^2+z^2\right)^{-1/2}}{x^2+y^2+z^2}$$

$$= e^{-\sqrt{x^2+y^2+z^2}}\,\frac{x^2+y^2+z^2-x^2\left(x^2+y^2+z^2\right)^{1/2}-x^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$= \frac{y^2+z^2-x^2\left(x^2+y^2+z^2\right)^{1/2}}{\left(x^2+y^2+z^2\right)^{3/2}}\,e^{-\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial}{\partial y}\left(\frac{ye^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right) = e^{-\sqrt{x^2+y^2+z^2}}\,\frac{\left(1-y^2\left(x^2+y^2+z^2\right)^{-1/2}\right)\sqrt{x^2+y^2+z^2}-y^2\left(x^2+y^2+z^2\right)^{-1/2}}{x^2+y^2+z^2}$$

$$= \frac{x^2+z^2-y^2\left(x^2+y^2+z^2\right)^{1/2}}{\left(x^2+y^2+z^2\right)^{3/2}}\,e^{-\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial}{\partial z}\left(\frac{ye^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right) = e^{-\sqrt{x^2+y^2+z^2}}\,\frac{\left(1-z^2\left(x^2+y^2+z^2\right)^{-1/2}\right)\sqrt{x^2+y^2+z^2}-z^2\left(x^2+y^2+z^2\right)^{-1/2}}{x^2+y^2+z^2}$$

$$= \frac{x^2+y^2-z^2\left(x^2+y^2+z^2\right)^{1/2}}{\left(x^2+y^2+z^2\right)^{3/2}}\,e^{-\sqrt{x^2+y^2+z^2}}$$

$$\vec{F} = 100\,k\,\frac{e^{-\sqrt{x^2+y^2+z^2}}}{\left(x^2+y^2+z^2\right)^{3/2}}\left(2\left(x^2+y^2+z^2\right)-\left(x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{1/2}\right)$$

$$\boxed{= \frac{100\,k\,e^{-\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\left(2-\sqrt{x^2+y^2+z^2}\right)}$$

b) $T(x, y, z) = 100\,e^{-x^2+y^2+z^2}$

$$\vec{F} = -k\nabla T \qquad\qquad \nabla\,e^{-x^2+y^2+z^2} = e^{-x^2+y^2+z^2}\,\langle -2x,\ 2y,\ 2z\rangle$$

$$= -100\,k\,\nabla\,e^{-x^2+y^2+z^2}$$

$$= -200\,k\,e^{-x^2+y^2+z^2}\,\langle -x,\ y,\ z\rangle$$

$$= -200\, k\left[\frac{\partial}{\partial x}\left(-xe^{-x^2+y^2+z^2}\right)+\frac{\partial}{\partial y}\left(ye^{-x^2+y^2+z^2}\right)+\frac{\partial}{\partial z}\left(ze^{-x^2+y^2+z^2}\right)\right]$$

$$= -200\, ke^{-x^2+y^2+z^2}\left(-1+2x^2+1+2y^2+1+2z^2\right)$$

$$\boxed{= -200\, ke^{-x^2+y^2+z^2}\left(1+2x^2+2y^2+2z^2\right)}$$

c) $T(x, y, z) = 100\left(1+\sqrt{x^2+y^2+z^2}\right)$

$$\vec{F} = -k\nabla T$$

$$= -100\, k\ \nabla\left(1+\sqrt{x^2+y^2+z^2}\right)$$

$$= -100\, k\left[\left(x^2+y^2+z^2\right)^{-1/2}\langle x, y, z\rangle\right]$$

$$= -100\, k\left[\frac{\partial}{\partial x}\left(x\left(x^2+y^2+z^2\right)^{-1/2}\right)+\frac{\partial}{\partial y}\left(y\left(x^2+y^2+z^2\right)^{-1/2}\right)+\frac{\partial}{\partial z}\left(z\left(x^2+y^2+z^2\right)^{-1/2}\right)\right]$$

$$= -100\, k\left(x^2+y^2+z^2\right)^{-1/2}\left[3-\left(x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{-1}\right]$$

$$= -100\, k\left(x^2+y^2+z^2\right)^{-1/2}[3-1]$$

$$\boxed{= \frac{-200\, k}{\sqrt{x^2+y^2+z^2}}}$$

Exercise

Consider the rotational velocity field $\vec{v} = \langle -2y,\ 2z,\ 0\rangle$

a) If a paddle is placed in the *xy*-plane with its axis normal to this plane, what is its angular speed?

b) If a paddle is placed in the *xz*-plane with its axis normal to this plane, what is its angular speed?

c) If a paddle is placed in the *yz*-plane with its axis normal to this plane, what is its angular speed?

Solution

$$curl\ \vec{v} = \nabla\times\vec{v}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2y & 2z & 0 \end{vmatrix}$$

$$= \langle -2,\ 0,\ 2\rangle$$

a) In *xy*-plane with its axis normal to this plane, then the wheel is placed with its axis in the direction of *z*-axis $\langle 0,\ 0,\ 1\rangle$, so the component of velocity in that direction is

$$\langle -2,\ 0,\ 2\rangle \cdot \langle 0,\ 0,\ 1\rangle\ \underline{= 2}$$

its angular speed: $\omega = \frac{1}{2} \cdot 2 \ \boxed{= 1}$

b) In *xz*-plane with its axis normal to this plane, then the wheel is placed with its axis in the direction of *y*-axis $\langle 0,\ 1,\ 0 \rangle$, so the component of velocity in that direction is

$$\langle -2,\ 0,\ 2 \rangle \cdot \langle 0,\ 1,\ 0 \rangle \ \boxed{= 0}$$

The wheel ***does not*** turn.

c) In *yz*-plane with its axis normal to this plane, then the wheel is placed with its axis in the direction of *x*-axis $\langle 1,\ 0,\ 0 \rangle$, so the component of velocity in that direction is

$$\langle -2,\ 0,\ 2 \rangle \cdot \langle 1,\ 0,\ 0 \rangle \ \boxed{= -2}$$

its angular speed: $\omega = \frac{1}{2} \cdot |-2| \ \boxed{= 1}$

Exercise

Consider the rotational velocity field $\vec{v} = \langle 0,\ 10z,\ -10y \rangle$. If a paddle wheel is placed in the plane $x + y + z = 1$ with its axis normal to this plane, how fast does the paddle wheel spin (revolutions per unit time)?

Solution

$$curl\ \vec{v} = \nabla \times \vec{v}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 10z & -10y \end{vmatrix}$$

$$= \langle -20,\ 0,\ 0 \rangle$$

Since the wheel is placed in the plane $x + y + z = 1$ with its axis normal to this plane, then must point in the direction $\langle 1,\ 1,\ 1 \rangle$

The unit vector: $\frac{1}{\sqrt{3}} \langle 1,\ 1,\ 1 \rangle$

$$\frac{1}{\sqrt{3}} \langle 1,\ 1,\ 1 \rangle \cdot \langle -20,\ 0,\ 0 \rangle = -\frac{20}{\sqrt{3}}$$

The component of velocity along that direction is $-\frac{20}{\sqrt{3}}$

The angular velocity $\omega = \frac{1}{2} \left| -\frac{20}{\sqrt{3}} \right| \ \boxed{= \frac{10}{\sqrt{3}}}$

The paddle wheel spin: $\frac{10}{\sqrt{3}} \frac{1}{2\pi} \boxed{= \frac{5}{\pi\sqrt{3}} \quad rev/time}$

Exercise

The potential function for the gravitational force field due to a mass M at the origin acting on a mass m is $\phi = \dfrac{GMm}{|\vec{r}|}$, where $\vec{r} = \langle x,\ y,\ z \rangle$ is the position vector of the mass m and G is the gravitational constant.

a) Compute the gravitational force field $\vec{F} = -\nabla \phi$

b) Show that the field is irrotational; that is $\nabla \times \vec{F} = \vec{0}$

Solution

a) $\vec{F} = -\nabla \phi$

$$= -\nabla \left(\frac{GMm}{\sqrt{x^2 + y^2 + z^2}} \right)$$

$$= -GMm \left\langle \frac{\partial}{\partial x} \frac{1}{\sqrt{x^2 + y^2 + z^2}},\ \frac{\partial}{\partial y} \frac{1}{\sqrt{x^2 + y^2 + z^2}},\ \frac{\partial}{\partial z} \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right\rangle$$

$$= -\frac{GMm}{\left(x^2 + y^2 + z^2\right)^{3/2}} \langle -x,\ -y,\ -z \rangle$$

$$= \frac{GMm}{\left(x^2 + y^2 + z^2\right)^{3/2}} \langle x,\ y,\ z \rangle$$

$$\boxed{= GMm \frac{\vec{r}}{|\vec{r}|^3}}$$

b) $\nabla \times \vec{F} = GMm \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{|\vec{r}|^3} & \frac{y}{|\vec{r}|^3} & \frac{z}{|\vec{r}|^3} \end{vmatrix}$

$$\frac{\partial}{\partial y} \frac{x}{\left(x^2 + y^2 + z^2\right)^{3/2}} = \frac{-3xy}{\left(x^2 + y^2 + z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial z} \frac{x}{\left(x^2 + y^2 + z^2\right)^{3/2}} = \frac{-3xz}{\left(x^2 + y^2 + z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial x} \frac{y}{\left(x^2 + y^2 + z^2\right)^{3/2}} = \frac{-3xy}{\left(x^2 + y^2 + z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial z}\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}=\frac{-3yz}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial y}\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}=\frac{-3xy}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial z}\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}=\frac{-3yz}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\nabla\times\vec{F}=GMm\left((-3yz+3yz)\hat{i}+(-3xz+3xz)\hat{j}+(-3xy+3xy)\hat{k}\right)$$

$$\underline{=\vec{0}\,|}$$

Exercise

The potential function for the force field due to a charge q at the origin is $\phi=\frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}|}$, where $\vec{r}=\langle x,\ y,\ z\rangle$ is the position vector of the mass m and G is the gravitational constant.

a) Compute the force field $\vec{F}=-\nabla\phi$

b) Show that the field is irrotational; that is $\nabla\times\vec{F}=\vec{0}$

Solution

a) $\vec{F}=-\nabla\phi$

$$=-\frac{q}{4\pi\varepsilon_0}\nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$=-\frac{q}{4\pi\varepsilon_0}\left\langle\frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}},\ \frac{\partial}{\partial y}\frac{1}{\sqrt{x^2+y^2+z^2}},\ \frac{\partial}{\partial z}\frac{1}{\sqrt{x^2+y^2+z^2}}\right\rangle$$

$$=-\frac{q}{4\pi\varepsilon_0}\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}\langle -x,\ -y,\ -z\rangle$$

$$=\frac{q}{4\pi\varepsilon_0}\frac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\underline{=\frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{|\vec{r}|^3}\,|}$$

b) $\nabla \times \vec{F} = \dfrac{q}{4\pi\varepsilon_0} \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{|\vec{r}|^3} & \frac{y}{|\vec{r}|^3} & \frac{z}{|\vec{r}|^3} \end{vmatrix}$

$$\frac{\partial}{\partial y} \frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} = \frac{-3xy}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial z} \frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} = \frac{-3xz}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial x} \frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} = \frac{-3xy}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial z} \frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} = \frac{-3yz}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial y} \frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} = \frac{-3xy}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial z} \frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} = \frac{-3yz}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\nabla \times \vec{F} = \frac{q}{4\pi\varepsilon_0}\Big((-3yz+3yz)\,\hat{\imath} + (-3xz+3xz)\,\hat{\jmath} + (-3xy+3xy)\,\hat{k}\Big)$$

$$= \vec{0}$$

Exercise

The Navier-Stokes equation is the fundamental equation of fluid dynamics that models the motion of water in everything from bathtubs to oceans. In one of its many forms (incompressible, viscous flow), the equation is

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \bullet \nabla\right)\vec{V}\right) = -\nabla p + \mu(\nabla \bullet \nabla)\vec{V}$$

In this notation $\vec{V} = \langle u,\ v,\ w\rangle$ is the three-dimensional velocity field, p is the (scalar) pressure, ρ is the constant density of the fluid, and μ is the constant viscosity. Write out the three component equations of this vector equation.

Solution

$$\frac{\partial \vec{V}}{\partial t} = \left\langle \frac{\partial u}{\partial t},\ \frac{\partial v}{\partial t},\ \frac{\partial w}{\partial t} \right\rangle$$

$$\begin{aligned}
\left(\vec{V} \bullet \nabla\right)\vec{V} &= \left(\langle u,\ v,\ w\rangle \bullet \left\langle \frac{\partial}{\partial t},\ \frac{\partial}{\partial t},\ \frac{\partial}{\partial t} \right\rangle\right)\langle u,\ v,\ w\rangle \\
&= \left(u\frac{\partial}{\partial t} + v\frac{\partial}{\partial t} + w\frac{\partial}{\partial t}\right)\langle u,\ v,\ w\rangle \\
&= \left(u+v+w\right)\frac{\partial}{\partial t}\langle u,\ v,\ w\rangle \\
&= \left\langle \left(u+v+w\right)\frac{\partial u}{\partial t},\quad \left(u+v+w\right)\frac{\partial v}{\partial t},\quad \left(u+v+w\right)\frac{\partial w}{\partial t} \right\rangle
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \bullet \nabla\right)\vec{V} &= \left\langle \frac{\partial u}{\partial t},\ \frac{\partial v}{\partial t},\ \frac{\partial w}{\partial t} \right\rangle + \left\langle \left(u+v+w\right)\frac{\partial u}{\partial t},\quad \left(u+v+w\right)\frac{\partial v}{\partial t},\quad \left(u+v+w\right)\frac{\partial w}{\partial t} \right\rangle \\
&= \left\langle \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},\quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z},\quad \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right\rangle
\end{aligned}$$

$$\nabla p = \left\langle \frac{\partial p}{\partial t},\ \frac{\partial p}{\partial t},\ \frac{\partial p}{\partial t} \right\rangle$$

$$\begin{aligned}
\left(\nabla \bullet \nabla\right)\vec{V} &= \left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial t^2}\right)\langle u,\ v,\ w\rangle \\
&= \left\langle \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2},\quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2},\quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right\rangle
\end{aligned}$$

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \bullet \nabla\right)\vec{V}\right) = -\nabla p + \mu\left(\nabla \bullet \nabla\right)\vec{V}$$

$$\begin{aligned}
&\rho\left\langle \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},\quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z},\quad \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right\rangle = \\
&\qquad -\left\langle \frac{\partial p}{\partial t},\ \frac{\partial p}{\partial t},\ \frac{\partial p}{\partial t} \right\rangle + \mu\left\langle \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2},\quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2},\quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right\rangle
\end{aligned}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Exercise

One of Maxwell's equations for electromagnetic waves is $\nabla \times \vec{B} = C\dfrac{\partial \vec{E}}{\partial t}$, where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, and C is a constant.

a) Show that the fields $\vec{E}(z,\ t) = A\sin(kz - \omega t)\,\hat{i}$ $\qquad \vec{B}(z,\ t) = A\sin(kz - \omega t)\,\hat{j}$

Satisfy the equation for constants A, k, and ω, provided $\omega = \dfrac{k}{C}$

b) Make a rough sketch showing the directions of $\vec{E}$ and $\vec{B}$

Solution

a)
$$\nabla \times \vec{B} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & A\sin(kz-\omega t) & 0 \end{vmatrix}$$

$$= -\frac{\partial}{\partial z}\big(A\sin(kz-\omega t)\big)\hat{i} + 0\hat{j} + \frac{\partial}{\partial x}\big(A\sin(kz-\omega t)\big)\hat{k}$$

$$\boxed{= -Ak\sin(kz-\omega t)\,\hat{i}}$$

$$C\frac{\partial \vec{E}}{\partial t} = C\frac{\partial}{\partial t}\Big(A\sin(kz-\omega t)\,\hat{i}\Big)$$

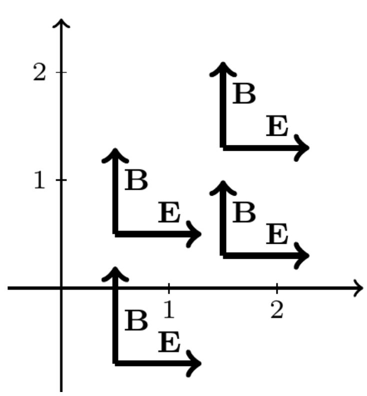
$$\boxed{= -AC\omega\sin(kz-\omega t)\,\hat{i}}$$

$$\nabla \times \vec{B} = C\frac{\partial \vec{E}}{\partial t}$$

$$-Ak\sin(kz-\omega t)\,\hat{i} = -AC\omega\sin(kz-\omega t)\,\hat{i}$$

$$\Rightarrow\ k = \omega C \ \rightarrow\ \boxed{\omega = \frac{k}{C}}$$

b)



Exercise

Prove that for a real number p, with $\vec{r} = \langle x,\ y,\ z \rangle$, $\nabla \bullet \dfrac{\langle x,\ y,\ z \rangle}{|\vec{r}|^p} = \dfrac{3-p}{|\vec{r}|^p}$

Solution

$$\nabla \bullet \frac{\langle x,\ y,\ z \rangle}{|\vec{r}|^p} = \left\langle \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z} \right\rangle \bullet \frac{\langle x,\ y,\ z \rangle}{\left(x^2+y^2+z^2\right)^{p/2}}$$

$$= \frac{\partial}{\partial x}\frac{x}{\left(x^2+y^2+z^2\right)^{p/2}} + \frac{\partial}{\partial y}\frac{y}{\left(x^2+y^2+z^2\right)^{p/2}} + \frac{\partial}{\partial z}\frac{z}{\left(x^2+y^2+z^2\right)^{p/2}}$$

$$= \frac{x^2+y^2+z^2-px^2}{\left(x^2+y^2+z^2\right)^{1+p/2}} + \frac{x^2+y^2+z^2-py^2}{\left(x^2+y^2+z^2\right)^{1+p/2}} + \frac{x^2+y^2+z^2-pz^2}{\left(x^2+y^2+z^2\right)^{1+p/2}}$$

$$= \frac{3\left(x^2+y^2+z^2\right)-p\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{1+p/2}}$$

$$= \frac{(3-p)\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{1+p/2}}$$

$$= \frac{3-p}{\left(x^2+y^2+z^2\right)^{p/2}}$$

$$= \frac{3-p}{|\vec{r}|^p} \quad \checkmark$$

Exercise

Prove that for a real number p, with $\vec{r} = \langle x,\ y,\ z \rangle$, $\nabla\left(\dfrac{1}{|\vec{r}|^p}\right) = \dfrac{-p\vec{r}}{|\vec{r}|^{p+2}}$

Solution

$$\nabla\left(\frac{1}{|\vec{r}|^p}\right) = \nabla\left(\frac{1}{\left(x^2+y^2+z^2\right)^{p/2}}\right)$$

$$= \left\langle \frac{\partial}{\partial x}\frac{1}{\left(x^2+y^2+z^2\right)^{p/2}},\ \frac{\partial}{\partial y}\frac{1}{\left(x^2+y^2+z^2\right)^{p/2}},\ \frac{\partial}{\partial z}\frac{1}{\left(x^2+y^2+z^2\right)^{p/2}} \right\rangle$$

$$= \frac{1}{\left(x^2+y^2+z^2\right)^{1+p/2}}\langle -px, \ -py, \ -pz\rangle$$

$$= \frac{-p}{\left(x^2+y^2+z^2\right)^{\frac{2+p}{2}}}\langle x, \ y, \ z\rangle$$

$$\boxed{= \frac{-p\vec{r}}{|\vec{r}|^{p+2}}} \quad \checkmark$$

Exercise

Prove that for a real number p, with $\vec{r} = \langle x, \ y, \ z\rangle$, $\nabla \bullet \nabla\left(\frac{1}{|\vec{r}|^p}\right) = \frac{p(p-1)}{|\vec{r}|^{p+2}}$

Solution

$$\nabla\left(\frac{1}{|\vec{r}|^p}\right) = \nabla\left(\frac{1}{\left(x^2+y^2+z^2\right)^{p/2}}\right)$$

$$= \left\langle \frac{\partial}{\partial x}\frac{1}{\left(x^2+y^2+z^2\right)^{p/2}}, \ \frac{\partial}{\partial y}\frac{1}{\left(x^2+y^2+z^2\right)^{p/2}}, \ \frac{\partial}{\partial z}\frac{1}{\left(x^2+y^2+z^2\right)^{p/2}}\right\rangle$$

$$= \frac{1}{\left(x^2+y^2+z^2\right)^{1+p/2}}\langle -px, \ -py, \ -pz\rangle$$

$$= \frac{-p}{\left(x^2+y^2+z^2\right)^{\frac{2+p}{2}}}\langle x, \ y, \ z\rangle$$

$$= -\frac{p\vec{r}}{|\vec{r}|^{p+2}}$$

$$\nabla \bullet \nabla\left(\frac{1}{|\vec{r}|^{p+2}}\right) = \left\langle \frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}, \ \frac{\partial}{\partial z}\right\rangle \bullet \frac{-p\langle x, \ y, \ z\rangle}{\left(x^2+y^2+z^2\right)^{\frac{p+2}{2}}}$$

$$= -p\frac{\partial}{\partial x}\frac{x}{\left(x^2+y^2+z^2\right)^{\frac{p+2}{2}}} - p\frac{\partial}{\partial y}\frac{y}{\left(x^2+y^2+z^2\right)^{\frac{p+2}{2}}} - p\frac{\partial}{\partial z}\frac{z}{\left(x^2+y^2+z^2\right)^{\frac{p+2}{2}}}$$

$$=-\frac{p}{\left(x^2+y^2+z^2\right)^{1+\frac{p+2}{2}}}\left(3\left(x^2+y^2+z^2\right)-(p+2)x^2-(p+2)y^2-(p+2)z^2\right)$$

$$=-\frac{p\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{1+\frac{p+2}{2}}}\left(3-(p+2)\right)$$

$$=\frac{p(p-1)}{\left(x^2+y^2+z^2\right)^{\frac{p+2}{2}}}$$

$$=\frac{p(p-1)}{|\vec{r}|^{p+2}}$$

√

Solution Section 4.6 – Surfaces and Area

Exercise

Find a parametrization of the surface: The paraboloid $z = x^2 + y^2, \; z \le 4$

Solution

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$z = x^2 + y^2 = r^2 \qquad z \le 4 \;\to\; r^2 \le 4 \Rightarrow 0 \le r \le 2$$

Then: $\vec{r}(r,\,\theta) = (r\cos\theta)\,\hat{i} + (r\sin\theta)\,\hat{j} + r^2\,\hat{k} \qquad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi$

Exercise

Find a parametrization of the surface: The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 4$

Solution

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$z = 2\sqrt{x^2 + y^2} = 2r \qquad \begin{aligned} z = 2 &\to r = 1 \\ z = 4 &\to r = 2 \end{aligned}$$

Then: $\vec{r}(r,\,\theta) = (r\cos\theta)\,\hat{i} + (r\sin\theta)\,\hat{j} + 2r\,\hat{k} \qquad 1 \le r \le 2, \quad 0 \le \theta \le 2\pi$

Exercise

Find a parametrization of the surface cut from the sphere $x^2 + y^2 + z^2 = 8$ by the plane $z = -2$

Solution

$$x^2 + y^2 + z^2 = 8 = \rho^2 \quad \to \quad \rho = 2\sqrt{2}$$

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

$$x = 2\sqrt{2}\sin\phi\cos\theta, \quad y = 2\sqrt{2}\sin\phi\sin\theta, \quad z = 2\sqrt{2}\cos\phi$$

For $z = -2$

$$2\sqrt{2}\cos\phi = -2$$

$$\cos\phi = -\frac{1}{\sqrt{2}}$$

$$\boxed{\phi = \frac{3\pi}{4}}$$

For $z = 2\sqrt{2}$

$$2\sqrt{2}\cos\phi = 2\sqrt{2}$$

$$\cos\phi = 1$$

$$\phi = 0$$

Then: $\vec{r}(\phi,\ \theta) = \left(2\sqrt{2}\sin\phi\cos\theta\right)\hat{i} + \left(2\sqrt{2}\sin\phi\sin\theta\right)\hat{j} + \left(2\sqrt{2}\cos\phi\right)\hat{k}$

$$0 \le \phi \le \frac{3\pi}{4}, \quad 0 \le \theta \le 2\pi$$

Exercise

Find a parametrization of the surface the plane $2x - 4y + 3z = 16$

Solution

$$z = \frac{1}{3}(16 - 2x + 4y)$$

Then: $\vec{r}(u,\ v) = \left\langle u,\ v,\ \frac{1}{3}(16 - 2u + 4v)\right\rangle \qquad u,\ v \in (-\infty,\ \infty)$

Exercise

Find a parametrization of the surface the cap of the sphere $x^2 + y^2 + z^2 = 16$ for $2\sqrt{2} \le z \le 4$

Solution

$$x^2 + y^2 + z^2 = 16 = \rho^2 \quad \rightarrow \quad \rho = 4$$

$$x = 4\sin\phi\cos\theta, \quad y = 4\sin\phi\sin\theta, \quad z = 4\cos\phi$$

For $z = 2\sqrt{2}$

$$4\cos\phi = 2\sqrt{2}$$

$$\cos\phi = \frac{\sqrt{2}}{2}$$

$$\phi = \frac{\pi}{4}$$

For $z = 4$

$$4\cos\phi = 4$$

$$\cos\phi = 1$$

$$\phi = 0$$

Then: $\vec{r}(\phi,\ \theta) = \left\langle 4\sin\phi\cos\theta,\ 4\sin\phi\sin\theta,\ 4\cos\phi\right\rangle \quad 0 \le \phi \le \frac{\pi}{4}, \quad 0 \le \theta \le 2\pi$

Exercise

Find a parametrization of the surface the frustum of the cone $z^2 = x^2 + y^2$ for $2 \le z \le 8$

Solution

$x = r\cos\theta, \quad y = r\sin\theta$

$z^2 = x^2 + y^2 = r^2 \quad \rightarrow \quad \underline{z = r}$

$z = 2 = r$

$z = 8 = r$

Then: $\vec{r}(r, \theta) = \langle r\cos\theta, \ r\sin\theta, \ r\rangle \qquad 2 \le r \le 8, \quad 0 \le \theta \le 2\pi$

Exercise

Find a parametrization of the surface the cone $z^2 = 4\left(x^2 + y^2\right)$ for $0 \le z \le 4$

Solution

$x = r\cos\theta, \quad y = r\sin\theta$

$z^2 = 4\left(x^2 + y^2\right) = 4r^2$

$\rightarrow \quad \underline{z = 2r}$

$z = 0 = 2r \quad \rightarrow \ r = 0$

$z = 4 = 2r \quad \rightarrow \ r = 2$

Then: $\vec{r}(r, \theta) = \left\langle \frac{1}{2}r\cos\theta, \ \frac{1}{2}r\sin\theta, \ r\right\rangle \qquad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi$

Exercise

Find a parametrization of the surface the portion of the cylinder $x^2 + y^2 = 9$ in the first octant, for $0 \le z \le 3$

Solution

$x = 3\cos\theta, \quad y = 3\sin\theta$

$z = r \quad \rightarrow \quad 0 \le r \le 3$

Then: $\vec{r}(r, \theta) = \langle 3\cos\theta, \ 3\sin\theta, \ r\rangle \qquad 0 \le r \le 3, \quad 0 \le \theta \le \frac{\pi}{2}$

Exercise

Find a parametrization of the surface the cylinder $y^2 + z^2 = 36$ for $0 \le x \le 9$

Solution

$$y = 6\cos\theta, \quad z = 6\sin\theta$$

$$x = r \quad \to \quad 0 \le r \le 9$$

Then: $\vec{r}(r,\ \theta) = \langle r,\ 6\cos\theta,\ 6\sin\theta \rangle \qquad\qquad 0 \le r \le 9, \quad 0 \le \theta \le 2\pi$

Exercise

Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of the plane $y + 2z = 2$ inside the cylinder $x^2 + y^2 = 1$

Solution

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$y + 2z = 2 \quad \to \quad z = \frac{2-y}{2} = \frac{2 - r\sin\theta}{2}$$

Then: $\vec{r}(r,\ \theta) = (r\cos\theta)\,\hat{i} + (r\sin\theta)\,\hat{j} + \left(\frac{2 - r\sin\theta}{2}\right)\hat{k} \qquad\qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$

$$\vec{r}_r = (\cos\theta)\,\hat{i} + (\sin\theta)\,\hat{j} - \left(\frac{\sin\theta}{2}\right)\hat{k}$$

$$\vec{r}_\theta = (-r\sin\theta)\,\hat{i} + (r\cos\theta)\,\hat{j} - \left(\frac{r\cos\theta}{2}\right)\hat{k}$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & -\frac{1}{2}\sin\theta \\ -r\sin\theta & r\cos\theta & -\frac{1}{2}r\cos\theta \end{vmatrix}$$

$$= \left(-\frac{1}{2}r\cos\theta\sin\theta + \frac{1}{2}r\cos\theta\sin\theta\right)\hat{i} - \left(-\frac{1}{2}r\cos^2\theta - \frac{1}{2}r\sin^2\theta\right)\hat{j}$$

$$+\left(r\cos^2\theta + r\sin^2\theta\right)\hat{k}$$

$$= \frac{1}{2}r\,\hat{j} + r\,\hat{k}$$

$$\left|\boldsymbol{r}_r \times \boldsymbol{r}_\theta\right| = \sqrt{\frac{r^2}{4} + r^2}$$

$$= \frac{\sqrt{5}}{2}r$$

$$A = \int_0^{2\pi} \int_0^1 \frac{\sqrt{5}}{2} r \, dr d\theta$$

$$= \frac{\sqrt{5}}{4} \int_0^{2\pi} \left(r^2 \, \Big|_0^1 \right. d\theta$$

$$= \frac{\sqrt{5}}{4} \int_0^{2\pi} d\theta$$

$$= \frac{\sqrt{5}}{4} (2\pi)$$

$$= \frac{\pi\sqrt{5}}{2} \quad unit^2$$

Exercise

Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of the cone $z = \frac{\sqrt{x^2 + y^2}}{3}$ between the planes $z = 1$ and $z = \frac{4}{3}$

Solution

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$z = \frac{\sqrt{x^2 + y^2}}{3} = \frac{r}{3} \qquad \begin{aligned} z = 1 &\rightarrow r = 3 \\ z = \frac{4}{3} &\rightarrow r = 4 \end{aligned}$$

Then: $\vec{r}(r, \theta) = (r\cos\theta)\,\hat{i} + (r\sin\theta)\,\hat{j} + \left(\frac{r}{3}\right)\hat{k} \qquad 3 \le r \le 4, \quad 0 \le \theta \le 2\pi$

$$\vec{r}_r = (\cos\theta)\,\hat{i} + (\sin\theta)\,\hat{j} + \frac{1}{3}\hat{k}$$

$$\vec{r}_\theta = (-r\sin\theta)\,\hat{i} + (r\cos\theta)\,\hat{j}$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & \frac{1}{3} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \left(0 - \frac{1}{3} r\cos\theta\right)\hat{i} - \left(0 + \frac{1}{3} r\sin\theta\right)\hat{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\hat{k}$$

$$= \left(-\frac{1}{3} r\cos\theta\right)\hat{i} - \left(\frac{1}{3} r\sin\theta\right)\hat{j} + r\,\hat{k}$$

$$\left|\vec{r}_r \times \vec{r}_\theta\right| = \sqrt{\frac{1}{9} r^2 \cos^2\theta + \frac{1}{9} r^2 \sin^2\theta + r^2}$$

$$= \sqrt{\frac{1}{9} r^2 + r^2}$$

$$=\boxed{\frac{\sqrt{10}}{3}r}$$

$$A=\int_0^{2\pi} d\theta \int_3^4 \frac{\sqrt{10}}{3} r\ dr$$

$$=\frac{\pi\sqrt{10}}{3}\left(r^2\ \Big|_3^4\right.$$

$$=\frac{\pi\sqrt{10}}{3}(16-9)$$

$$=\boxed{\frac{7\pi\sqrt{10}}{3}\quad unit^2}$$

Exercise

Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of the cylinder $x^2+z^2=10$ between the planes $y=-1$ and $y=1$

Solution

$$x=u\cos v,\quad z=u\sin v$$

$$x^2+z^2=10=u^2\cos^2 v+u^2\sin^2 v$$

$$u^2=10\to u=\sqrt{10}$$

Then: $\vec{r}(y,\ v)=(u\cos v)\ \hat{i}+y\hat{j}+(u\sin v)\ \hat{k}$

$$=\left(\sqrt{10}\cos v\right)\hat{i}+y\hat{j}\left(\sqrt{10}\sin v\right)\hat{k}$$

$$\vec{r}_y=\hat{j}$$

$$\vec{r}_v=\left(-\sqrt{10}\sin v\right)\hat{i}+\left(\sqrt{10}\cos v\right)\hat{k}$$

$$\vec{r}_y\times\vec{r}_v=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 1 & 0\\ -\sqrt{10}\sin v & 0 & \sqrt{10}\cos v\end{vmatrix}$$

$$=\left(\sqrt{10}\cos v\right)\hat{i}+\left(\sqrt{10}\sin v\right)\hat{k}$$

$$\left|\vec{r}_r\times\vec{r}_\theta\right|=\sqrt{10\cos^2 v+10\sin^2 v}$$

$$=\boxed{\sqrt{10}}$$

$$A=\int_0^{2\pi} dv\int_{-1}^{1}\sqrt{10}\ dy$$

$$= 2\pi\sqrt{10}\ \left(y \ \Big|_{-1}^{1}\right.$$

$$\boxed{= 4\pi\sqrt{10}\quad unit^2}$$

Exercise

Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of the cap cut from the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 4$

Solution

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$z = x^2 + y^2 = r^2$$

$$z = 1 \to r = 1$$

$$z = 4 \to r = 2$$

Then: $\vec{r}(r,\ \theta) = (r\cos\theta)\ \hat{i} + (r\sin\theta)\ \hat{j} + r^2\,\hat{k} \qquad\qquad 1 \le r \le 2, \quad 0 \le \theta \le 2\pi$

$$\vec{r}_r = (\cos\theta)\ \hat{i} + (\sin\theta)\ \hat{j} + 2r\,\hat{k}$$

$$\vec{r}_\theta = (-r\sin\theta)\ \hat{i} + (r\cos\theta)\ \hat{j}$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \left(0 - 2r^2\cos\theta\right)\hat{i} - \left(0 + 2r^2\sin\theta\right)\hat{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\hat{k}$$

$$= \left(-2r^2\cos\theta\right)\hat{i} - \left(2r^2\sin\theta\right)\hat{j} + r\,\hat{k}$$

$$\left|\vec{r}_r \times \vec{r}_\theta\right| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2}$$

$$= \sqrt{4r^4 + r^2}$$

$$\boxed{= r\sqrt{4r^2 + 1}}$$

$$A = \int_0^{2\pi}\int_1^2 r\sqrt{4r^2+1}\ dr d\theta \qquad\qquad d\left(4r^2+1\right) = 8rdr$$

$$= \frac{1}{8}\int_0^{2\pi} d\theta\ \int_1^2 \left(4r^2+1\right)^{1/2}\ d\left(4r^2+1\right)$$

$$= \frac{\pi}{6}\ \left(4r^2+1\right)^{3/2}\ \Big|_1^2$$

$$= \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)$$

$$= \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \ \ unit^2$$

Exercise

Find the area of the following surface using a parametric description of the surface:
The half cylinder $\{(r,\ \theta,\ z):\ \ r = 4,\ \ 0 \le \theta \le \pi.\ \ 0 \le z \le 7\}$

Solution

$x = 4\cos\theta, \quad y = 4\sin\theta$

$z = r$

Then: $\vec{r}(r,\ \theta) = \langle 4\cos\theta,\ 4\sin\theta,\ r\rangle$

$\vec{r}_r = \langle 0,\ 0,\ 1\rangle$

$\vec{r}_\theta = \langle -4\sin\theta,\ 4\cos\theta,\ 0\rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ -4\sin\theta & 4\cos\theta & 0 \end{vmatrix}$$

$$= \langle -4\cos\theta,\ -4\sin\theta,\ 0\rangle$$

$$\left|\vec{r}_r \times \vec{r}_\theta\right| = \sqrt{16\cos^2\theta + 16\sin^2\theta}$$

$$= 4$$

$$Area = \int_0^{\pi}\int_0^{7} 4\ dz\ d\theta$$

$$= 4(\pi)(7)$$

$$= 28\pi \ \ unit^2$$

Exercise

Find the area of the following surface using a parametric description of the surface:
The plane $z = 3 - x - 3y$ in the first octant

Solution

$\vec{r}(u,\ v) = \langle u,\ v,\ 3 - u - 3v\rangle$

$\vec{r}_u = \langle 1,\ 0,\ -1\rangle$

$\vec{r}_v = \langle 0,\ 1,\ -3\rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1 & 0 & -1 \\ 0 & 1 & -3 \end{vmatrix}$$

$$= \langle 1,\ 3,\ 1 \rangle$$

$$\left| \vec{r}_u \times \vec{r}_v \right| = \sqrt{1+9+1}$$

$$= \sqrt{11}$$

$$0 = 3 - u - 3v \quad \rightarrow \quad u = 3 - 3v$$

$$0 \le u \le 3 - 3v$$

$$u = 0 \quad \rightarrow \quad v = 3$$

$$0 \le v \le 3$$

$$Area = \int_0^1 \int_0^{3-3v} \sqrt{11}\ du\ dv$$

$$= \sqrt{11} \int_0^1 \left(u \Big|_0^{3-3v} \right) dv$$

$$= 3\sqrt{11} \int_0^1 (1 - v)\, dv$$

$$= 3\sqrt{11} \left(v - \frac{1}{2} v^2 \Big|_0^1 \right.$$

$$= 3\sqrt{11} \left(1 - \frac{1}{2} \right)$$

$$= \frac{3\sqrt{11}}{2} \quad unit^2$$

Exercise

Find the area of the following surface using a parametric description of the surface

The plane $z = 10 - x - y$ above the square $|x| \le 2, \quad |y| \le 2$

Solution

$$\vec{r}(u,\ v) = \langle u,\ v,\ 10 - u - v \rangle$$

$$\vec{r}_u = \langle 1,\ 0,\ -1 \rangle$$

$$\vec{r}_v = \langle 0,\ 1,\ -1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= \langle 1,\ 1,\ 1 \rangle$$

$$\left| \vec{r}_u \times \vec{r}_v \right| = \sqrt{1+1+1}$$

$$= \sqrt{3}$$

$$|x| \le 2 \quad \to -2 \le u \le 2$$

$$|y| \le 2 \quad \to -2 \le v \le 2$$

$$Area = \int_{-2}^{2} \int_{-2}^{2} \sqrt{3}\ du\ dv$$

$$= \sqrt{3} \int_{-2}^{2} dv \int_{-2}^{2} du$$

$$= \sqrt{3}\ v \Big|_{-2}^{2} \quad u \Big|_{-2}^{2}$$

$$= 16\sqrt{3}\ \ unit^2$$

Exercise

Find the area of the following surface using a parametric description of the surface

The hemisphere $x^2 + y^2 + z^2 = 100, \quad z \ge 0$

Solution

$$x^2 + y^2 + z^2 = 100 = \rho^2 \quad \to \quad \rho = 10$$

$$\vec{r} = \langle 10\sin u \cos v,\ 10\sin u \sin v,\ 10\cos u \rangle$$

$$\vec{r}_u = \langle 10\cos u \cos v,\ 10\cos u \sin v,\ -10\sin u \rangle$$

$$\vec{r}_v = \langle -10\sin u \sin v,\ 10\sin u \cos v,\ 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 10\cos u \cos v & 10\cos u \sin v & -10\sin u \\ -10\sin u \sin v & 10\sin u \cos v & 0 \end{vmatrix}$$

$$= \left\langle 100\sin^2 u \cos v,\ 100\sin^2 u \sin v,\ 100\sin u \cos u \cos^2 v + 100\sin u \cos u \sin^2 v \right\rangle$$

$$= \left\langle 100\sin^2 u \cos v,\ 100\sin^2 u \sin v,\ 100\sin u \cos u \right\rangle$$

$$\left|\vec{r}_u \times \vec{r}_v\right| = \sqrt{10^4 \sin^4 u \cos^2 v + 10^4 \sin^4 u \sin^2 v + 10^4 \sin^2 u \cos^2 u}$$

$$= 100\sqrt{\sin^4 u\left(\cos^2 v + \sin^2 v\right) + \sin^2 u \cos^2 u}$$

$$= 100\sqrt{\sin^4 u + \sin^2 u \cos^2 u}$$

$$= 100\ \sin u\ \sqrt{\sin^2 u + \cos^2 u}$$

$$\underline{= 100 \sin u\ |}$$

$$Area = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 100 \sin u\ du\ dv$$

$$= 100\int_0^{2\pi} dv\ \int_0^{\frac{\pi}{2}} \sin u\ du$$

$$= -200\pi\ \left(\cos u\ \right|_0^{\frac{\pi}{2}}$$

$$= -200\pi(-1)$$

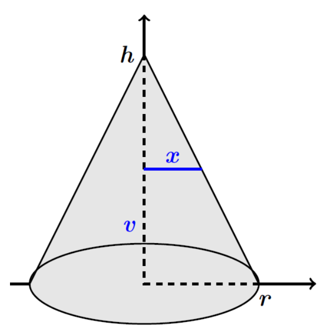
$$\underline{= 200\pi\ \ unit^2\ |}$$

Exercise

Find the area of the following surfaces using a parametric description of the surface
A cone with base radius r and height h, where r and h are positive constants.

Solution

Cone equation: $x^2 + y^2 - z = 0$ with $z \le h$



$x^2 + y^2 = r^2$

$\frac{x}{r} = \frac{v}{h} \quad \rightarrow \quad x = \frac{rv}{h}$

$0 \le v \le h, \quad 0 \le u \le 2\pi$

$$\vec{r}(u,\ v) = \left\langle \frac{r}{h} v \cos u,\ \frac{r}{h} v \sin u,\ v\right\rangle$$

$$\vec{r}_u = \left\langle -\frac{r}{h} v \sin u,\ \frac{r}{h} v \cos u,\ 0\right\rangle$$

$$\vec{r}_v = \left\langle \frac{r}{h} \cos u,\ \frac{r}{h} \sin u,\ 1\right\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ -\frac{r}{h} v \sin u & \frac{r}{h} v \cos u & 0 \\ \frac{r}{h} \cos u & \frac{r}{h} \sin u & 1 \end{vmatrix}$$

$$= \left\langle \frac{r}{h} v \cos u,\ \frac{r}{h} v \sin u,\ -\frac{r^2}{h^2} v \sin^2 u - \frac{r^2}{h^2} v \cos^2 u \right\rangle$$

$$= \left\langle \frac{r}{h} v \cos u,\ \frac{r}{h} v \sin u,\ -\frac{r^2}{h^2} v \right\rangle$$

$$\left| \vec{r}_u \times \vec{r}_v \right| = \sqrt{\frac{r^2}{h^2} v^2 \cos^2 u + \frac{r^2}{h^2} v^2 \sin^2 u + \frac{r^4}{h^4} v^2}$$

$$= \frac{rv}{h} \sqrt{\cos^2 u + \sin^2 u + \frac{r^2}{h^2}}$$

$$= \frac{rv}{h} \sqrt{1 + \frac{r^2}{h^2}}$$

$$= \frac{rv}{h^2} \sqrt{h^2 + r^2}$$

$$Area = \int_0^{2\pi} \int_0^h \frac{rv}{h^2} \sqrt{h^2 + r^2}\ dvdu$$

$$= \frac{r}{h^2} \sqrt{h^2 + r^2} \left(\frac{1}{2} v^2 \Big|_0^h \right) \int_0^{2\pi} du$$

$$= \frac{r}{h^2} \sqrt{h^2 + r^2} \left(\frac{1}{2} h^2 \right) (2\pi)$$

$$= \pi r \sqrt{h^2 + r^2}\ unit^2$$

Exercise

Find the area of the following surfaces using a parametric description of the surface

The cap of the sphere $x^2 + y^2 + z^2 = 4, \quad 1 \le z \le 2$

Solution

$$\vec{r} = \langle 2 \sin u \cos v,\ 2 \sin u \sin v,\ 2 \cos u \rangle$$

$$\vec{r}_u = \langle 2 \cos u \cos v,\ 2 \cos u \sin v,\ -2 \sin u \rangle$$

$$\vec{r}_v = \langle -2 \sin u \sin v,\ 2 \sin u \cos v,\ 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 2\cos u\cos v & 2\cos u\sin v & -2\sin u \\ -2\sin u\sin v & 2\sin u\cos v & 0 \end{vmatrix}$$

$$= \left\langle 4\sin^2 u\cos v,\ 4\sin^2 u\sin v,\ 4\sin u\cos u\cos^2 v + 4\sin u\cos u\sin^2 v \right\rangle$$

$$= \left\langle 4\sin^2 u\cos v,\ 4\sin^2 u\sin v,\ 4\sin u\cos u \right\rangle$$

$$\left|\vec{r}_u \times \vec{r}_v\right| = \sqrt{16\sin^4 u\cos^2 v + 16\sin^4 u\sin^2 v + 16\sin^2 u\cos^2 u}$$

$$= 4\sqrt{\sin^4 u\left(\cos^2 v + \sin^2 v\right) + \sin^2 u\cos^2 u}$$

$$= 4\sqrt{\sin^4 u + \sin^2 u\cos^2 u}$$

$$= 4\sin u\sqrt{\sin^2 u + \cos^2 u}$$

$$= 4\sin u$$

$$z = 1 = 2\cos u$$

$$\rightarrow \cos u = \frac{1}{2} \quad \Rightarrow \quad u = \frac{\pi}{3}$$

$$z = 2 = 2\cos u$$

$$\rightarrow \cos u = 1 \quad \Rightarrow \quad u = 0$$

$$Area = \int_0^{2\pi}\int_0^{\frac{\pi}{3}} 4\sin u\ du\ dv$$

$$= 4\int_0^{2\pi} dv \quad \int_0^{\frac{\pi}{3}} \sin u\ du$$

$$= -8\pi\ \left(\cos u\right|_0^{\frac{\pi}{3}}$$

$$= -8\pi\left(\frac{1}{2} - 1\right)$$

$$= 4\pi\ \ unit^2$$

Exercise

Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 2$.

Solution

$$\vec{p} = \hat{k}, \quad \nabla f = 2x\,\hat{i} + 2y\,\hat{j} - \hat{k}$$

$$|\nabla f| = \sqrt{(2x)^2 + (2y)^2 + 1}$$
$$= \sqrt{4x^2 + 4y^2 + 1}$$

$$|\nabla f \cdot \vec{p}| = 1$$

$$z = 2 \quad \Rightarrow \quad x^2 + y^2 = 2$$
$$x = r\cos\theta, \quad y = r\sin\theta$$
$$r^2 = x^2 + y^2 = 2 \Rightarrow r = \sqrt{2}$$

$$\text{Surface area} = \iint\limits_R \frac{|\nabla F|}{|\nabla F \cdot p|} dA$$
$$= \iint\limits_R \sqrt{4x^2 + 4y^2 + 1}\; dxdy$$
$$= \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + 1}\; r\,drd\theta$$
$$= \int_0^{2\pi} d\theta \quad \int_0^{\sqrt{2}} \sqrt{4r^2 + 1}\; r\,dr \qquad\qquad d\left(4r^2 + 1\right) = 8rdr$$
$$= \frac{1}{8}(2\pi) \int_0^{\sqrt{2}} \left(4r^2 + 1\right)^{1/2} d\left(4r^2 + 1\right)$$
$$= \frac{\pi}{6} \left(4r^2 + 1\right)^{3/2} \Big|_0^{\sqrt{2}}$$
$$= \frac{\pi}{6}(27 - 1)$$
$$= \frac{13\pi}{3} \quad unit^2$$

Exercise

Find the area of the portion of the surface $x^2 - 2z = 0$ that lies above the triangle bounded by the lines $x = \sqrt{3}$, $y = 0$, and $y = x$ in the *xy*-plane.

Solution

$$\vec{p} = \hat{k}$$
$$\nabla f = 2x\,\hat{i} - 2\hat{j}$$

$$\left|\nabla f\right| = \sqrt{4x^2+4}$$

$$= 2\sqrt{x^2+1}$$

$$\left|\nabla f \cdot \vec{p}\right| = \left|\left(2x\,\hat{i} - 2\,\hat{k}\right)\cdot\left(\hat{k}\right)\right|$$

$$\underline{= 2\,|}$$

$$\text{Surface area} = \iint_R \frac{\left|\nabla F\right|}{\left|\nabla F\cdot\vec{p}\right|}\,dA$$

$$= \int_0^{\sqrt{3}}\int_0^x \frac{2\sqrt{x^2+1}}{2}\,dydx$$

$$= \int_0^{\sqrt{3}} \sqrt{x^2+1}\,\left(y\,\Big|_0^x\right.\,dx$$

$$= \int_0^{\sqrt{3}} x\sqrt{x^2+1}\,dx \qquad\qquad d\left(x^2+1\right) = 2xdx$$

$$= \frac{1}{2}\int_0^{\sqrt{3}} \left(x^2+1\right)^{1/2} d\left(x^2+1\right)$$

$$= \frac{1}{2}\left(\frac{2}{3}\left(x^2+1\right)^{3/2}\right.\Big|_0^{\sqrt{3}}$$

$$= \frac{1}{3}\left(4^{3/2}-1\right)$$

$$= \frac{1}{3}(8-1)$$

$$\underline{= \frac{7}{3}\quad unit^2\,|}$$

Exercise

Find the area of the cap cut from the sphere $x^2+y^2+z^2=2$ by the cone $z=\sqrt{x^2+y^2}$.

Solution

$$\vec{p} = \hat{k}$$

$$\nabla f = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$$

$$\left|\nabla f\right| = \sqrt{4x^2+4y^2+4z^2}$$

$$= 2\sqrt{x^2+y^2+z^2}$$

$$= 2\sqrt{2}$$

$$\left|\nabla f \bullet \vec{p}\right| = \left|\left(2x\,\hat{i} + 2y\,\hat{i} + 2z\,\hat{k}\right) \bullet \left(\hat{k}\right)\right|$$

$$\underline{= 2z}$$

$$z = \sqrt{x^2 + y^2}$$

$$\to \quad x^2 + y^2 + z^2 = z^2 + z^2$$

$$= 2z^2 = 2$$

$$\Rightarrow \quad \underline{z = 1}$$

$$x^2 + y^2 + z^2 = 2 \quad \to \quad z = \sqrt{2 - \left(x^2 + y^2\right)}$$

$$Surface\ area = \iint_R \frac{2\sqrt{2}}{2z}\, dydx \qquad\qquad Surface\ area = \iint_R \frac{|\nabla F|}{|\nabla F \bullet \vec{p}|}\, dA$$

$$= \sqrt{2} \iint_R \frac{1}{\sqrt{2 - \left(x^2 + y^2\right)}}\, dydx$$

$$= \sqrt{2} \int_0^{2\pi} \int_0^1 \frac{1}{\sqrt{2 - r^2}}\, r\, drd\theta$$

$$= -\frac{\sqrt{2}}{2} \int_0^{2\pi} d\theta \int_0^1 \left(2 - r^2\right)^{-1/2} d\left(2 - r^2\right)$$

$$= -2\pi\sqrt{2} \left(2 - r^2\right)^{1/2} \Big|_0^1$$

$$= -2\pi\sqrt{2}\left(1 - \sqrt{2}\right)$$

$$\underline{= 2\pi\left(2 - \sqrt{2}\right) \quad unit^2}$$

Exercise

Find the area of the ellipse cut from the plane $z = cx$ (c a constant) by the cylinder $x^2 + y^2 = 1$.

Solution

$$cx - z = 0$$

$$\vec{p} = \hat{k}$$

$$\nabla f = c\,\hat{i} - \hat{k}$$

$$|\nabla f| = \sqrt{c^2+1}$$

$$|\nabla f \cdot \vec{p}| = \left|\left(c\,\hat{i} - \hat{k}\right)\cdot\left(\hat{k}\right)\right|$$

$$\underline{= 1 \,|}$$

$$Surface\ area = \iint\limits_R \sqrt{c^2+1}\ dxdy \qquad\qquad Surface\ area = \iint\limits_R \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|}\,dA$$

$$= \iint\limits_R \sqrt{c^2+1}\ dxdy$$

$$= \int_0^{2\pi} d\theta \int_0^1 \sqrt{c^2+1}\ r\,dr$$

$$= \pi\sqrt{c^2+1}\left(r^2 \right|_0^1$$

$$\underline{= \pi\sqrt{c^2+1} \quad unit^2 \,|}$$

Exercise

Find the area of the surface cut from the nose of the paraboloid $x = 1 - y^2 - z^2$ by yz-plane.

Solution

$$f_y(y,z) = -2y, \quad f_z(y,z) = -2z$$

$$\sqrt{f_y^2 + f_z^2 + 1} = \sqrt{4y^2 + 4z^2 + 1}$$

$$Area = \iint\limits_R \sqrt{4y^2+4z^2+1}\ dydz$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{4r^2+1}\ r\,drd\theta \qquad\qquad d\left(4r^2+1\right) = 8rdr$$

$$= \frac{1}{8}\int_0^{2\pi} d\theta \int_0^1 \left(4r^2+1\right)^{1/2} d\left(4r^2+1\right)$$

$$= \frac{\pi}{6}\left(4r^2+1\right)^{3/2}\Big|_0^1$$

$$\underline{= \frac{\pi}{6}\left(5\sqrt{5}-1\right) \quad unit^2 \,|}$$

Exercise

Find the area of the surface in the first octant cut from the cylinder $y=\frac{2}{3}z^{3/2}$ by the planes $x=1$ and $y=\frac{16}{3}$

Solution

$$y=\frac{2}{3}z^{3/2}$$

$$f_x(x,z)=0, \quad f_z(x,z)=z^{1/2}$$

$$\sqrt{f_x^2+f_z^2+1}=\sqrt{z+1}$$

$$y=\frac{2}{3}z^{3/2}$$

$$\underline{=\frac{16}{3}}$$

$$\Rightarrow z^{3/2}=8$$

$$z=8^{2/3}$$

$$\underline{=4}$$

$$Area=\int_0^4\int_0^1\sqrt{z+1}\ dxdz$$

$$=\int_0^4\left(x\sqrt{z+1}\ \Big|_0^1\ dz \qquad\qquad d(z+1)=dz\right.$$

$$=\int_0^4(z+1)^{1/2}\,d(z+1)$$

$$=(z+1)^{3/2}\ \Big|_0^4$$

$$=\frac{2}{3}\left(5^{3/2}-1\right)$$

$$\underline{=\frac{2}{3}\left(5\sqrt{5}-1\right)\quad unit^2}$$

Exercise

Use a surface integral to find the area of the helicoid

$\vec{r}(r,\ \theta) = (r\cos\theta)\,\hat{i} + (r\sin\theta)\,\hat{j} + \theta\hat{k}, \quad 0 \le \theta \le 2\pi, \quad 0 \le r \le 1$

Solution

$$\vec{r}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\vec{r}_\theta = -r\sin\theta\,\hat{i} + r\sin\theta\,\hat{j} + \hat{k}$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\sin\theta & 1 \end{vmatrix}$$

$$= \sin\theta\,\hat{i} - \cos\theta\,\hat{j} + r\,\hat{k}$$

$$\left|\vec{r}_r \times \vec{r}_\theta\right| = \sqrt{\sin^2\theta + \cos^2\theta + r^2}$$

$$= \sqrt{1+r^2}$$

$$Area = \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\ dr d\theta \qquad\qquad A = \int_c^d \int_a^b \left|\boldsymbol{r}_u \times \boldsymbol{r}_v\right| dudv$$

$$= \int_0^{2\pi} d\theta \ \int_0^1 \left(1+r^2\right)^{1/2}\ dr$$

Let $r = \tan x \quad \rightarrow dr = \sec^2 x\ dx$

$$\sqrt{1+r^2} = \sec x$$

$$\int \sqrt{1+r^2}\,dr = \int \sec^3 x\ dx$$

Let: $\quad u = \sec x \qquad dv = \sec^2 xdx$

$\quad du = \sec x \tan xdx \qquad v = \tan x$

$$\int \sec^3 xdx = \sec x \tan x - \int \tan x\left(\sec x \tan xdx\right)$$

$$= \sec x \tan x - \int \tan^2 x \sec xdx$$

$$= \sec x \tan x - \int \left(\sec^2 x - 1\right)\sec x\ dx$$

$$= \sec x \tan x - \int \left(\sec^3 x - \sec x\right)\ dx$$

$$= \sec x \tan x - \int \sec^3 x\ dx + \int \sec x\ dx$$

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

$$= \sec x \tan x + \ln\left|\sec x + \tan x\right| + C_1$$

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C$$

$$r = \tan x \quad \sec x = \sqrt{1+r^2}$$

$$= 2\pi \left(\frac{r}{2}\sqrt{1+r^2} + \frac{1}{2}\ln\left|r+\sqrt{1+r^2}\right|\right. \Bigg|_0^1$$

$$\boxed{= \pi\left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right) \quad unit^2}$$

Exercise

Use a surface integral to find the area of the surface $f(x, y) = \sqrt{2}\, xy$ above the origin $\{(r, \theta):\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

Solution

$$f_z(x, y) = \sqrt{2}\, y \quad f_y(x, y) = \sqrt{2}\, x$$

$$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{2y^2 + 2x^2 + 1}$$

$$= \sqrt{2\left(y^2 + x^2\right) + 1}$$

$$= \sqrt{2r^2 + 1}$$

$$Area = \int_0^{2\pi} \int_0^2 \sqrt{2r^2+1}\; r\, dr d\theta \qquad\qquad Area = \iint_S 1\, dS$$

$$= \frac{1}{4}\int_0^{2\pi} d\theta \int_0^2 \left(2r^2+1\right)^{1/2} d\left(2r^2+1\right)$$

$$= \frac{1}{4}(2\pi)\ \frac{2}{3}\left(2r^2+1\right)^{3/2} \Bigg|_0^2$$

$$= \frac{\pi}{3}(27-1)$$

$$\boxed{= \frac{26\pi}{3} \quad unit^2}$$

Exercise

Use a surface integral to find the area of the hemisphere $x^2+y^2+z^2=9$, for $z \ge 0$ (excluding the base).

Solution

$$\vec{r} = \langle 3\sin\varphi\cos\theta,\ 3\sin\varphi\sin\theta,\ 3\cos\varphi \rangle$$

$$\vec{r}_\varphi = \langle 3\cos\varphi\cos\theta,\ 3\cos\varphi\sin\theta,\ -3\sin\varphi \rangle$$

$$\vec{r}_\theta = \langle -3\sin\varphi\sin\theta,\ 3\sin\varphi\cos\theta,\ 0 \rangle$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\cos\varphi\cos\theta & 3\cos\varphi\sin\theta & -3\sin\varphi \\ -3\sin\varphi\sin\theta & 3\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= 9\sin^2\varphi\cos\theta\,\hat{i} + 9\sin^2\varphi\sin\theta\,\hat{j} + \left(9\sin\varphi\cos\varphi\cos^2\theta + 9\sin\varphi\cos\varphi\sin^2\theta\right)\hat{k}$$

$$= 9\sin^2\varphi\cos\theta\,\hat{i} + 9\sin^2\varphi\sin\theta\,\hat{j} + 9\sin\varphi\cos\varphi\,\hat{k}$$

$$\left|\vec{r}_\varphi \times \vec{r}_\theta\right| = \sqrt{81\sin^4\varphi\cos^2\theta + 81\sin^4\varphi\sin^2\theta + 81\sin^2\varphi\cos^2\varphi}$$

$$= 9\sqrt{\sin^4\varphi\left(\cos^2\theta + \sin^2\theta\right) + \sin^2\varphi\cos^2\varphi}$$

$$= 9\sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi}$$

$$= 9\sqrt{\sin^2\varphi\left(\sin^2\varphi + \cos^2\varphi\right)}$$

$$= 9\sqrt{\sin^2\varphi}$$

$$= 9\sin\varphi$$

$$S = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 9\sin\varphi\, d\varphi d\theta$$

$$= -9\ \left(\cos\varphi \right|_0^{\frac{\pi}{2}} \int_0^{2\pi} d\theta$$

$$= -9(-1)(2\pi)$$

$$= 18\pi\ \ unit^2$$

Exercise

Use a surface integral to find the area of the frustum of the cone $z^2 = x^2 + y^2$, for $2 \le z \le 4$ (excluding the bases).

Solution

$$\vec{r} = \langle v\cos u,\ v\sin u,\ v\rangle$$

$$\vec{r}_u = \langle -v\sin u,\ v\cos u,\ 0\rangle$$

$$\vec{r}_v = \langle \cos u,\ \sin u,\ 1\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix}$$

$$= \langle v\cos u,\ v\sin u,\ -v\rangle$$

$$\left|\vec{r}_u \times \vec{r}_v\right| = \sqrt{v^2\cos^2 u + v^2\sin^2 u + v^2}$$

$$= \sqrt{2v^2}$$

$$= v\sqrt{2}$$

$$S = \sqrt{2}\int_0^{2\pi} du \int_2^4 v\, dv$$

$$= \sqrt{2}(2\pi)\left(\frac{1}{2}v^2\right)\Big|_2^4$$

$$= \pi\sqrt{2}(16-4)$$

$$= 12\pi\sqrt{2}\ \ unit^2$$

Exercise

Use a surface integral to find the area of the plane $z = 6 - x - y$ above the square $|x| \le 1$, $|y| \le 1$.

Solution

$$z_x = -1 \quad z_y = -1$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{1+1+1}$$

$$= \sqrt{3}$$

$$Area = \int_{-1}^{1}\int_{-1}^{1} \sqrt{3}\ dxdy$$

$$= \sqrt{3} \int_{-1}^{1} dx \int_{-1}^{1} dy$$

$$= \sqrt{3}\ x \Big|_{-1}^{1}\ y \Big|_{-1}^{1}$$

$$\underline{= 4\sqrt{3}\ \ unit^2}$$

Exercise

Use a surface integral to find the area of: The cone $z^2 = 4\left(x^2 + y^2\right), \quad 0 \le z \le 4$

Solution

$$z^2 = 4x^2 + 4y^2$$

$$2zdz = 8xdx \quad \to \quad z_x = \frac{4x}{z}$$

$$2zdz = 8ydy \quad \to \quad z_y = \frac{4y}{z}$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{\frac{16x^2}{z^2} + \frac{16y^2}{z^2} + 1}$$

$$= \sqrt{\frac{16x^2 + 16y^2 + z^2}{z^2}}$$

$$= \sqrt{\frac{16x^2 + 16y^2 + 4x^2 + 4y^2}{4x^2 + 4y^2}}$$

$$= \sqrt{\frac{20\left(x^2 + y^2\right)}{4\left(x^2 + y^2\right)}}$$

$$\underline{= \sqrt{5}}$$

$$Area = \iint_R \sqrt{5}\ dA$$

$$\iint_R dA = area\ of\ the\ circle\ radius = 2$$

$$= \pi\sqrt{5}\left(\pi(2)^2\right)$$

$$\underline{= 4\pi\sqrt{5}\quad unit^2}$$

Exercise

Use a surface integral to find the area of: The paraboloid $z = 2\left(x^2 + y^2\right), \quad 0 \le z \le 8$

Solution

$$z = 2x^2 + 2y^2$$

$$z_x = 4x \quad z_y = 4y$$

$$\begin{aligned}\sqrt{z_x^2 + z_y^2 + 1} &= \sqrt{16x^2 + 16y^2 + 1}\\ &= \sqrt{16\left(x^2 + y^2\right) + 1}\\ &= \sqrt{16r^2 + 1}\end{aligned}$$

$$z = 2\left(x^2 + y^2\right) = 8 \quad \rightarrow \quad x^2 + y^2 = 4 = r^2$$

$$0 \le r \le 2 \quad 0 \le \theta \le 2\pi$$

$$\begin{aligned}Area &= \int_0^{2\pi} d\theta \int_0^2 \sqrt{16r^2 + 1}\; r\,dr\\ &= 2\pi \int_0^2 \frac{1}{32}\left(16r^2 + 1\right)^{1/2} d\left(16r^2 + 1\right)\\ &= \frac{\pi}{24}\left(16r^2 + 1\right)^{3/2} \Big|_0^2\\ &= \frac{\pi}{24}\left(65^{3/2} - 1\right)\\ &= \frac{\pi}{24}\left(65\sqrt{65} - 1\right) \quad unit^2\end{aligned}$$

Exercise

Use a surface integral to find the area of: The trough $z = x^2, \quad -2 \le x \le 2, \quad 0 \le y \le 4$

Solution

$$z_x = 2x \quad z_y = 0$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{4x^2 + 1}$$

$$Area = \int_0^4 \int_{-2}^2 \sqrt{4x^2 + 1}\; dxdy$$

$$= \int_0^4 dy \int_{-2}^{2} \sqrt{4x^2+1}\ dx \qquad\qquad \int \sqrt{a^2+x^2}\ dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|$$

$$2x = \tan\alpha \qquad \sqrt{4x^2+1} = \sec\alpha$$

$$2dx = \sec^2\alpha\, d\alpha$$

$$\int \sqrt{4x^2+1}\, dx = \int \sec\alpha\ \frac{1}{2}\sec^2\alpha\, d\alpha$$

$$= \frac{1}{2}\int \sec^3\alpha\, d\alpha$$

Let: $u = \sec\alpha \qquad dv = \sec^2\alpha d\alpha$

$du = \sec\alpha\tan\alpha d\alpha \qquad v = \tan\alpha$

$$\int \sec^3\alpha d\alpha = \sec x\tan\alpha - \int \tan\alpha\left(\sec x\tan x dx\right)$$

$$= \sec\alpha\tan\alpha - \int \tan^2\alpha\sec\alpha\, d\alpha$$

$$= \sec\alpha\tan\alpha - \int \left(\sec^2\alpha - 1\right)\sec\alpha\ d\alpha$$

$$= \sec\alpha\tan\alpha - \int \left(\sec^3\alpha - \sec\alpha\right)\ d\alpha$$

$$= \sec\alpha\tan\alpha - \int \sec^3\alpha\ d\alpha + \int \sec\alpha\ d\alpha$$

$$2\int \sec^3\alpha\ d\alpha = \sec\alpha\tan\alpha + \int \sec\alpha\ d\alpha$$

$$= \sec\alpha\tan\alpha + \ln\left|\sec\alpha + \tan\alpha\right|$$

$$\int \sec^3\alpha\ d\alpha = \frac{1}{2}\sec\alpha\tan\alpha + \frac{1}{2}\ln\left|\sec\alpha + \tan\alpha\right|$$

$$= \frac{1}{2}\sqrt{4x^2+1}\ (2x)\ +\ \frac{1}{2}\ln\left|\sqrt{4x^2+1}\ +\ 2x\right|$$

$$= x\sqrt{4x^2+1}\ +\ \frac{1}{2}\ln\left|2x+\sqrt{4x^2+1}\ \right|$$

$$= (4)\ \left(\frac{x}{2}\sqrt{4x^2+1}\ +\ \frac{1}{2}\ln\left|2x+\sqrt{4x^2+1}\right|\ \right|_{-2}^{2}$$

$$= 4\ \left(\sqrt{17} + \frac{1}{2}\ln\left|4+\sqrt{17}\right| + \sqrt{17} - \frac{1}{2}\ln\left|-4+\sqrt{17}\right|\right)$$

$$= 8\sqrt{17} + 2\ln\left|4+\sqrt{17}\right| - 2\ln\left|-4+\sqrt{17}\right|$$

$$= 8\sqrt{17} + \ln\left(4+\sqrt{17}\right)^2 - \ln\left(\sqrt{17}-4\right)^2$$

$$= 8\sqrt{17} + \ln\left(\frac{4+\sqrt{17}}{2}\right) - \ln\left(\frac{\sqrt{17}-4}{2}\right)$$

$$= 8\sqrt{17} + \ln\left(\frac{\sqrt{17}+4}{\sqrt{17}-4}\right)$$

$$= 8\sqrt{17} + \ln\left(\frac{\sqrt{17}+4}{\sqrt{17}-4}\cdot\frac{\sqrt{17}+4}{\sqrt{17}+4}\right)$$

$$= 8\sqrt{17} + \ln\left(4+\sqrt{17}\right)^2$$

$$= 8\sqrt{17} + 2\ln\left(4+\sqrt{17}\right) \quad unit^2$$

Exercise

Use a surface integral to find the area of: The part of the hyperbolic paraboloid $z = x^2 - y^2$ above the sector $R = \left\{(r,\ \theta):\quad 0 \le r \le 4,\quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}\right\}$

Solution

$$z_x = 2x \quad z_y = -2y$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1}$$

$$= \sqrt{4r^2 + 1}$$

$$Area = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \int_0^4 \sqrt{4r^2+1}\ r\,dr$$

$$= \frac{\pi}{2}\int_0^4 \frac{1}{8}\left(4r^2+1\right)^{1/2} d\left(4r^2+1\right)$$

$$= \frac{\pi}{24}\left(4r^2+1\right)^{3/2}\Big|_0^4$$

$$= \frac{\pi}{24}\left(65^{3/2}-1\right)$$

$$= \frac{\pi}{24}\left(65\sqrt{65}-1\right) \quad unit^2$$

Exercise

Use a surface integral to find the area of: $f(x, y, z) = xy$, where S is the plane $z = 2 - x - y$ in the first octant

Solution

$$z_x = -1 \quad z_y = -1$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{1+1+1}$$

$$= \sqrt{3}$$

$$z = 2 - x - y = 0 \quad \rightarrow \quad y = 2 - x$$

$$y = 2 - x = 0 \quad \rightarrow \quad x = 2$$

First octant: $0 \le y \le 2 - x \quad 0 \le x \le 2$

$$Area = \int_0^2 \int_0^{2-x} \sqrt{3}xy \, dydx$$

$$= \frac{\sqrt{3}}{2} \int_0^2 x\left(y^2 \Big|_0^{2-x}\right) dx$$

$$= \frac{\sqrt{3}}{2} \int_0^2 \left(4x - 4x^2 + x^3\right) dx$$

$$= \frac{\sqrt{3}}{2} \left(2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4 \Big|_0^2\right.$$

$$= \frac{\sqrt{3}}{2} \left(8 - \frac{32}{3} + 4\right)$$

$$= \frac{\sqrt{3}}{2} \left(\frac{4}{3}\right)$$

$$= \frac{2\sqrt{3}}{3} \quad unit^2$$

Exercise

Use a surface integral to find the area of: $f(x, y, z) = x^2 + y^2$, where S is the paraboloid $z = x^2 + y^2, \quad 0 \le z \le 4$

Solution

$$z_x = 2x \quad z_y = 2y$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1}$$

$$= \boxed{\sqrt{4r^2 + 1}}$$

$$z = x^2 + y^2 = r^2 = 0 \quad \to \quad r = 0$$

$$z = x^2 + y^2 = r^2 = 4 \quad \to \quad r = 2$$

$$0 \le r \le 2 \quad 0 \le \theta \le 2\pi$$

$$Area = \iint_R \sqrt{4r^2 + 1}\left(x^2 + y^2\right) dA$$

$$= \int_0^{2\pi} \int_0^2 r^2 \sqrt{4r^2 + 1}\ r\ dr d\theta$$

Let $u = 4r^2 + 1 \quad \to \quad du = 8rdr$

$$r^2 = \frac{1}{4}(u - 1)$$

$$\begin{cases} r = 2 & \to u = 17 \\ r = 0 & \to u = 1 \end{cases}$$

$$= \int_0^{2\pi} d\theta \int_1^{17} \frac{1}{4}(u-1)u^{1/2}\frac{1}{8}du$$

$$= \frac{1}{32}(2\pi)\int_1^{17}\left(u^{3/2} - u^{1/2}\right)du$$

$$= \frac{\pi}{16}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right|_1^{17}$$

$$= \frac{\pi}{16}\left(\frac{2}{5}17^2\sqrt{17} - \frac{2}{3}17\sqrt{17} - \frac{2}{5} + \frac{2}{3}\right)$$

$$= \frac{\pi}{16}\frac{1}{15}\left((1734 - 170)\sqrt{17} + 4\right)$$

$$= \frac{\pi}{240}\left(1564\sqrt{17} + 4\right)$$

$$= \boxed{\frac{\pi}{60}\left(391\sqrt{17} + 1\right) \quad unit^2}$$

Exercise

Use a surface integral to find the area of: $f(x, y, z) = 25 - x^2 - y^2$, where S is the hemisphere centered at the origin with radius 5, for $z \geq 0$

Solution

S is the hemisphere with radius 5: $x^2 + y^2 + z^2 = 25$

$$2xdx + 2zdz = 0 \quad z_x = -\frac{x}{z}$$

$$2ydy + 2zdz = 0 \quad z_y = -\frac{y}{z}$$

$$\begin{aligned}
\sqrt{z_x^2 + z_y^2 + 1} &= \sqrt{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1} \\
&= \sqrt{\frac{x^2 + y^2 + z^2}{z^2}} \\
&= \sqrt{\frac{25}{z^2}} \\
&= \frac{5}{z}
\end{aligned}$$

$$0 \leq r \leq 5 \quad 0 \leq \theta \leq 2\pi$$

$$\begin{aligned}
Area &= \iint_R \frac{5}{\sqrt{25 - x^2 - y^2}} \left(25 - x^2 - y^2\right) dA \\
&= 5 \iint_R \sqrt{25 - x^2 - y^2}\ dA \\
&= 5 \int_0^{2\pi} d\theta \int_0^5 \sqrt{25 - r^2}\ r\ dr \\
&= -5\pi \int_0^5 \left(25 - r^2\right)^{1/2} d\left(25 - r^2\right) \\
&= -5\pi \left(\frac{2}{3}\right) \left(25 - r^2\right)^{3/2} \Big|_0^5 \\
&= -\frac{10\pi}{3}(0 - 125) \\
&= \frac{1250\pi}{3} \quad unit^2
\end{aligned}$$

Exercise

Use a surface integral to find the area of: $f(x, y, z) = e^{x}$, where S is the plane $z = 8 - x - 2y$ in the first octant

Solution

$$z_x = -1 \quad z_y = -2$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{1+4+1}$$

$$= \sqrt{6}$$

$z = 8 - x - 2y = 0 \quad \to \quad x = 8 - 2y$

$x = 8 - 2y = 0 \quad \to \quad y = 4$

First octant: $0 \le y \le 4 \quad 0 \le x \le 8 - 2y$

$$Area = \int_0^4 \int_0^{8-2y} \sqrt{6}\, e^{x}\, dxdy$$

$$= \sqrt{6} \int_0^4 e^{x} \Big|_0^{8-2y} dy$$

$$= \sqrt{6} \int_0^4 \left(e^{8-2y} - 1\right) dy$$

$$= \sqrt{6} \left(-\frac{1}{2} e^{8-2y} - y \right|_0^4$$

$$= \sqrt{6} \left(-\frac{1}{2} - 4 + \frac{1}{2} e^{8}\right)$$

$$= \frac{\sqrt{6}}{2}\left(e^{8} - 9\right) \quad unit^2$$

Exercise

Use a surface integral to find the area of: $f(x, y, z) = e^{z}$, where S is the plane $z = 8 - x - 2y$ in the first octant

Solution

$$z_x = -1 \quad z_y = -2$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{1+4+1}$$

$$= \sqrt{6}$$

$z = 8 - x - 2y = 0 \quad \to \quad x = 8 - 2y$

$x = 8 - 2y = 0 \quad \to \quad y = 4$

First octant: $0 \le y \le 4 \quad 0 \le x \le 8 - 2y$

$$Area = \int_0^4 \int_0^{8-2y} \sqrt{6}\, e^z \; dxdy$$

$$= \sqrt{6} \int_0^4 \int_0^{8-2y} e^{8-x-2y} \; dxdy$$

$$= \sqrt{6}\, e^8 \int_0^4 \int_0^{8-2y} e^{-2y}\, e^{-x} \; dxdy$$

$$= -\sqrt{6}\, e^8 \int_0^4 e^{-2y}\, e^{-x} \Big|_0^{8-2y} \; dy$$

$$= -\sqrt{6}\, e^8 \int_0^4 e^{-2y} \left(e^{2y-8} - 1 \right) dy$$

$$= -\sqrt{6}\, e^8 \int_0^4 \left(e^{-8} - e^{-2y} \right) dy$$

$$= -\sqrt{6}\, e^8 \left(e^{-8} y + \frac{1}{2} e^{-2y} \right) \Big|_0^4$$

$$= -\sqrt{6}\, e^8 \left(4e^{-8} + \frac{1}{2} e^{-8} - \frac{1}{2} \right)$$

$$= -\sqrt{6} e^8 \left(\frac{9}{2} e^{-8} - \frac{1}{2} \right)$$

$$= \frac{\sqrt{6}}{2} \left(e^8 - 9 \right) \quad unit^2$$

Exercise

Evaluate the surface integral $\iint_S (1 + yz)\, dS$; S is the plane $x + y + z = 2$ in the first octant.

Solution

$z = 2 - x - y$

$z_x = -1 \quad z_y = -1$

$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{1 + 1 + 1}$

$= \sqrt{3}$

$z = 2 - x - y = 0$

$$\to \begin{cases} y = 2 - x \\ y = 0 \end{cases}$$

$y = 0 \quad \to \quad 0 \le x \le 2$

$$\iint_S (1 + yz)\, dS = \sqrt{3} \iint_R (1 + yz)\, dA$$

$$= \sqrt{3} \int_0^2 \int_0^{2-x} \left(1 + y(2 - x - y)\right) dydx$$

$$= \sqrt{3} \int_0^2 \int_0^{2-x} \left(1 + 2y - xy - y^2\right) dydx$$

$$= \sqrt{3} \int_0^2 \left(y + y^2 - \frac{1}{2}xy^2 - \frac{1}{3}y^3 \ \Big|_0^{2-x} \ dx\right.$$

$$= \sqrt{3} \int_0^2 \left(2 - x + 4 - 4x + x^2 - 2x + 2x^2 - \frac{1}{2}x^3 - \frac{8}{3} + 4x - 2x^2 + \frac{1}{3}x^3\right) dx$$

$$= \sqrt{3} \int_0^2 \left(\frac{10}{3} - 3x + x^2 - \frac{1}{6}x^3\right) dx$$

$$= \sqrt{3} \left(\frac{10}{3}x - \frac{3}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{24}x^4 \ \Big|_0^2\right.$$

$$= \sqrt{3}\left(\frac{20}{3} - 6 + \frac{8}{3} - \frac{2}{3}\right)$$

$$= \frac{8\sqrt{3}}{3}$$

Exercise

Evaluate the surface integral $\iint_S \langle 0, y, z \rangle \cdot \vec{n}\ dS$; S is the curve surface of the cylinder $y^2 + z^2 = a^2$, $|x| \le 8$ with outward normal vectors.

Solution

$\vec{n} = \langle 0,\ y,\ z \rangle$

$$\iint_S \langle 0,\ y,\ z \rangle \cdot \vec{n}\ dS = a \iint_R \langle 0,\ y,\ z \rangle \cdot \langle 0,\ y,\ z \rangle dA$$

$$= a \iint_R \left(y^2 + z^2\right) dA$$

$$= a^3 \iint_R dA$$

$$\iint_R dA = \textit{area of the circle radius } \frac{8}{2} = 4$$

$$= a^3 \left(2\pi 4^2\right)$$

$$= 32\pi a^3$$

Exercise

Evaluate the surface integral $\iint\limits_S (x - y + z)\,dS$; S is the entire surface including the base of the hemisphere $x^2 + y^2 + z^2 = 4$, for $z \geq 0$.

Solution

$$\vec{r} = \langle 2\sin\varphi\cos\theta,\ 2\sin\varphi\sin\theta,\ 2\cos\varphi \rangle$$

$$\vec{r}_\varphi = \langle 2\cos\varphi\cos\theta,\ 2\cos\varphi\sin\theta,\ -2\sin\varphi \rangle$$

$$\vec{r}_\theta = \langle -2\sin\varphi\sin\theta,\ 2\sin\varphi\cos\theta,\ 0 \rangle$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos\varphi\cos\theta & 2\cos\varphi\sin\theta & -2\sin\varphi \\ -2\sin\varphi\sin\theta & 2\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= 4\sin^2\varphi\cos\theta\ \hat{i} + 4\sin^2\varphi\sin\theta\ \hat{j} + \left(4\sin\varphi\cos\varphi\cos^2\theta + 4\sin\varphi\cos\varphi\sin^2\theta\right)\hat{k}$$

$$= 4\sin^2\varphi\cos\theta\,\hat{i} + 4\sin^2\varphi\sin\theta\,\hat{j} + 4\sin\varphi\cos\varphi\,\hat{k}$$

$$\left|\vec{r}_\varphi \times \vec{r}_\theta\right| = \sqrt{16\sin^4\varphi\cos^2\theta + 16\sin^4\varphi\sin^2\theta + 16\sin^2\varphi\cos^2\varphi}$$

$$= 4\sqrt{\sin^4\varphi\left(\cos^2\theta + \sin^2\theta\right) + \sin^2\varphi\cos^2\varphi}$$

$$= 4\sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi}$$

$$= 4\sqrt{\sin^2\varphi\left(\sin^2\varphi + \cos^2\varphi\right)}$$

$$= 4\sqrt{\sin^2\varphi}$$

$$= 4\sin\varphi$$

$$\iint_S (x-y+z)\,dS = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} (2\sin\varphi\cos\theta - 2\sin\varphi\sin\theta + 2\cos\varphi)(4\sin\varphi)\,d\varphi d\theta$$

$$= 8\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left(\sin^2\varphi\cos\theta - \sin^2\varphi\sin\theta + \sin\varphi\cos\varphi\right) d\varphi d\theta$$

$$= 8\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left((\cos\theta - \sin\theta)\left(\frac{1}{2} - \frac{1}{2}\cos 2\varphi\right) + \frac{1}{2}\sin 2\varphi\right) d\varphi d\theta$$

$$= 8\int_0^{2\pi} \left((\cos\theta - \sin\theta)\left(\frac{1}{2}\varphi - \frac{1}{4}\sin 2\varphi\right) - \frac{1}{4}\cos 2\varphi \,\Big|_0^{\frac{\pi}{2}}\right) d\theta$$

$$= 8\int_0^{2\pi} \left(\frac{\pi}{4}(\cos\theta - \sin\theta) + \frac{1}{4} + \frac{1}{4}\right) d\theta$$

$$= 8\left(\frac{\pi}{4}(\sin\theta + \cos\theta) + \frac{1}{2}\theta \,\Big|_0^{2\pi}\right.$$

$$= 8\left(\frac{\pi}{4} + \pi - \frac{\pi}{4}\right)$$

$$= 8\pi$$

Exercise

Evaluate $\iint_S \nabla \ln|\vec{r}| \cdot \vec{n}\, dS$, where S is the hemisphere $x^2 + y^2 + z^2 = a^2$, for $z \ge 0$, and where $\vec{r} = \langle x,\ y,\ z\rangle$. Assume normal vectors point either outward or in the positive z-direction.

Solution

$$\nabla \ln|\vec{r}| = \nabla \ln\sqrt{x^2 + y^2 + z^2}$$

$$= \frac{1}{x^2 + y^2 + z^2}\langle x,\ y,\ z\rangle \qquad x^2 + y^2 + z^2 = a^2$$

$$= \frac{1}{a^2}\langle x,\ y,\ z\rangle$$

$$2zdz + 2xdx = 0 \quad \to \quad z_x = -\frac{x}{z}$$

$$2zdz = 2ydy \quad \to \quad z_y = -\frac{y}{z}$$

Since the normal vector point either outward or in the positive z-direction

$$\vec{n} = \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle$$

$$\iint_S \nabla \ln|\vec{r}| \cdot \vec{n}\ dS = \iint_R \frac{1}{a^2}\langle x,\ y,\ z\rangle \cdot \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle dA$$

$$= \frac{1}{a^2} \iint_R \left(\frac{x^2}{z} + \frac{y^2}{z} + z \right) dA$$

$$= \frac{1}{a^2} \iint_R \left(\frac{x^2 + y^2 + z^2}{z} \right) dA$$

$$= \frac{1}{a^2} \iint_R \left(\frac{a^2}{z} \right) dA$$

$$= \iint_R \frac{1}{z}\ dA$$

$$= \iint_R \frac{1}{\sqrt{a^2 - x^2 - y^2}}\ dA$$

$$= \int_0^{2\pi} \int_0^{a} \frac{1}{\sqrt{a^2 - r^2}}\ r\ dr d\theta$$

$$= -\frac{1}{2} \int_0^{2\pi} d\theta \int_0^{a} \left(a^2 - r^2\right)^{-1/2} d\left(a^2 - r^2\right)$$

$$= -\pi(2) \left(a^2 - r^2\right)^{1/2} \Big|_0^a$$

$$= -2\pi(0 - a)$$

$$\underline{= 2\pi a}$$

Exercise

Evaluate $\iint_S |\vec{r}|\ dS$, where S is the cylinder $x^2 + y^2 = 4$, for $0 \le z \le 8$, and where $\vec{r} = \langle x,\ y,\ z\rangle$

Assume normal vectors point either outward or in the positive z-direction.

Solution

Parametrize the surface:

$$\vec{r}(u,\ v) = \langle 2\cos u,\ 2\sin u,\ v \rangle$$

$$\vec{r}_u = \langle -2\sin u,\ 2\cos u,\ 0 \rangle$$

$$\vec{r}_v = \langle 0,\ 0,\ 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle 2\cos u,\ 2\sin u,\ 0 \rangle$$

$$\left| \vec{r}_u \times \vec{r}_v \right| = \sqrt{4\cos^2 u + 4\sin^2 u}$$

$$\underline{= 2\ |}$$

$$0 \le z = v \le 8 \quad 0 \le u \le 2\pi$$

$$\iint_S |\vec{r}|\ dS = 2\iint_R \sqrt{x^2 + y^2 + z^2}\ dA$$

$$= 2\iint_R \sqrt{4 + z^2}\ dA$$

$$= 2\int_0^{2\pi} du \int_0^8 \sqrt{4 + v^2}\ dv$$

$$x = 2\tan\alpha \qquad \sqrt{v^2 + 4} = 2\sec\alpha$$

$$dx = 2\sec^2\alpha\, d\alpha$$

$$\int \sqrt{v^2 + 4}\, dx = \int 2\sec\alpha \left(2\sec^2\alpha\right) d\alpha$$

$$= 4\int \sec^3\alpha\ d\alpha$$

Let: $u = \sec\alpha \qquad dv = \sec^2\alpha d\alpha$

$du = \sec\alpha \tan\alpha d\alpha \qquad v = \tan\alpha$

$$\int \sec^3\alpha d\alpha = \sec x \tan\alpha - \int \tan\alpha \left(\sec x \tan x dx\right)$$

$$= \sec\alpha \tan\alpha - \int \tan^2\alpha \sec\alpha\ d\alpha$$

$$= \sec\alpha \tan\alpha - \int \left(\sec^2\alpha - 1\right) \sec\alpha\ d\alpha$$

$$= \sec\alpha \tan\alpha - \int \left(\sec^3\alpha - \sec\alpha\right)\ d\alpha$$

$$= \sec\alpha \tan\alpha - \int \sec^3 \alpha \, d\alpha + \int \sec\alpha \, d\alpha$$

$$2\int \sec^3 \alpha \, d\alpha = \sec\alpha \tan\alpha + \int \sec\alpha \, d\alpha$$

$$= \sec\alpha \tan\alpha + \ln\left|\sec\alpha + \tan\alpha\right|$$

$$\int \sec^3 \alpha \, d\alpha = \frac{1}{2}\sec\alpha \tan\alpha + \frac{1}{2}\ln\left|\sec\alpha + \tan\alpha\right|$$

$$= 4\pi \; (4)\left(\frac{1}{2}\frac{v}{2}\frac{\sqrt{4+v^2}}{2} + \frac{1}{2}\ln\left|\frac{1}{2}v + \frac{1}{2}\sqrt{4+v^2}\right| \; \right|_0^8$$

$$= 8\pi \; \left(\frac{1}{4}v\sqrt{4+v^2} + \; \ln\left|\frac{1}{2}v + \frac{1}{2}\sqrt{4+v^2}\right| \; \right|_0^8$$

$$= 8\pi \; \left(2\sqrt{68} + \ln\left(4 + \frac{1}{2}\sqrt{68}\right) - \ln 1\right)$$

$$= 8\pi\left(4\sqrt{17} + \ln\left(4 + \sqrt{17}\right)\right)$$

Exercise

Evaluate $\iint_S xyz \; dS$, where S is the part of the plane $z = 6 - y$ that lies on the cylinder $x^2 + y^2 = 4$

Assume normal vectors point either outward or in the positive z-direction.

Solution

$$z = 6 - y$$

$$z_x = 0 \quad z_y = -1$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{0+1+1}$$

$$= \sqrt{2}$$

$$\iint_S xyz \; dS = \sqrt{2} \iint_R xyz \; dA$$

$$= \sqrt{2} \int_0^{2\pi} \int_0^2 (r\cos\theta)(r\sin\theta)(6 - r\sin\theta) \; rdrd\theta$$

$$= \sqrt{2} \int_0^{2\pi} \int_0^2 \left(6r^3 \cos\theta \sin\theta - r^4 \cos\theta \sin^2\theta\right) \; drd\theta$$

$$= \sqrt{2} \int_0^{2\pi} \left(\frac{3}{2} r^4 \cos\theta \sin\theta - \frac{1}{5} r^5 \cos\theta \sin^2\theta \, \Big|_0^2 \, d\theta \right.$$

$$= \sqrt{2} \int_0^{2\pi} \left(12 \sin 2\theta - \frac{32}{5} \cos\theta \sin^2\theta \right) d\theta$$

$$= 12\sqrt{2} \int_0^{2\pi} \sin 2\theta \, d\theta \; - \; \frac{32\sqrt{2}}{5} \int_0^{2\pi} \sin^2\theta \, d(\sin\theta)$$

$$= -2\sqrt{2} \left(3\cos 2\theta + \frac{16}{15} \sin^3\theta \, \Big|_0^{2\pi} \right.$$

$$= -2\sqrt{2}\,(3-3)$$

$$\underline{= 0}$$

Exercise

Evaluate $\iint\limits_S \frac{\langle x,\ 0,\ z\rangle}{\sqrt{x^2+z^2}} \cdot \vec{n}\ dS$, where S is the cylinder $x^2+z^2=a^2,\quad |y| \le 2$. Assume normal vectors point either outward or in the positive z-direction.

Solution

$$\vec{n} = \langle x,\ 0,\ z\rangle$$

$$\iint\limits_S \frac{\langle x,\ 0,\ z\rangle}{\sqrt{x^2+z^2}} \cdot \vec{n}\ dS = \iint\limits_S \frac{\langle x,\ 0,\ z\rangle \cdot \langle x,\ 0,\ z\rangle}{\sqrt{x^2+z^2}}\ dS$$

$$= \iint\limits_R \frac{x^2+z^2}{\sqrt{x^2+z^2}}\ dA$$

$$= \iint\limits_R \sqrt{x^2+z^2}\ dA$$

$$= \iint\limits_R a\ dA$$

$$= a \int_0^{2\pi} du \int_{-2}^{2} dv$$

$$= a(2\pi)(4)$$

$$\underline{= 8\pi a}$$

Exercise

Evaluate the surface integral $\iint_S f(x, y, z)\,dS:\ f(x, y, z) = x^2 + y^2$, where S is the hemisphere $x^2 + y^2 + z^2 = 36, \quad z \ge 0$

Solution

$$\vec{r} = \langle 6\sin\varphi\cos\theta,\ 6\sin\varphi\sin\theta,\ 6\cos\varphi \rangle$$

$$\vec{r}_\varphi = \langle 6\cos\varphi\cos\theta,\ 6\cos\varphi\sin\theta,\ -6\sin\varphi \rangle$$

$$\vec{r}_\theta = \langle -6\sin\varphi\sin\theta,\ 6\sin\varphi\cos\theta,\ 0 \rangle$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6\cos\varphi\cos\theta & 6\cos\varphi\sin\theta & -6\sin\varphi \\ -6\sin\varphi\sin\theta & 6\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= 36\sin^2\varphi\cos\theta\ \hat{i} + 36\sin^2\varphi\sin\theta\ \hat{j} + \left(36\sin\varphi\cos\varphi\cos^2\theta + 36\sin\varphi\cos\varphi\sin^2\theta\right)\hat{k}$$

$$= 36\sin^2\varphi\cos\theta\ \hat{i} + 36\sin^2\varphi\sin\theta\ \hat{j} + 36\sin\varphi\cos\varphi\,\hat{k}$$

$$\left|\vec{r}_\varphi \times \vec{r}_\theta\right| = \sqrt{36^2\sin^4\varphi\cos^2\theta + 36^2\sin^4\varphi\sin^2\theta + 36^2\sin^2\varphi\cos^2\varphi}$$

$$= 36\sqrt{\sin^4\varphi\left(\cos^2\theta + \sin^2\theta\right) + \sin^2\varphi\cos^2\varphi}$$

$$= 36\sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi}$$

$$= 36\sqrt{\sin^2\varphi\left(\sin^2\varphi + \cos^2\varphi\right)}$$

$$= 36\sqrt{\sin^2\varphi}$$

$$\underline{= 36\sin\varphi\,|}$$

$$\iint_S f(x, y, z)\,dS = \iint_S \left(x^2 + y^2\right)dS$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left(36\sin^2\varphi\cos^2\theta + 36\sin^2\varphi\sin^2\theta\right)\left(36\sin\varphi\right)\,d\varphi\,d\theta$$

$$= 1{,}296\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin^2\varphi\left(\cos^2\theta + \sin^2\theta\right)\left(\sin\varphi\right)\,d\varphi\,d\theta$$

$$= 1{,}296\int_0^{2\pi} d\theta\ \int_0^{\frac{\pi}{2}} \sin^3\varphi\,d\varphi$$

$$= 1,296\pi \int_0^{\frac{\pi}{2}} -\left(1-\cos^2\varphi\right)\, d(\cos\varphi)$$

$$= 1,296\pi \left(\frac{1}{3}\cos^3\varphi - \cos\varphi\right|_0^{\frac{\pi}{2}}$$

$$= 1,296\pi \left(\frac{1}{3}-1\right)$$

$$= 1,728\pi$$

Exercise

Evaluate the surface integral $\iint_S f(x, y, z)\, dS$; $f(x, y, z) = y$, where S is the cylinder $x^2 + y^2 = 9, \quad 0 \le z \le 3$

Solution

Parametrize the surface:

$$\vec{r}(u, v) = \langle 3\cos u,\ 3\sin u,\ v\rangle$$

$$\vec{r}_u = \langle -3\sin u,\ 3\cos u,\ 0\rangle$$

$$\vec{r}_v = \langle 0,\ 0,\ 1\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\sin u & 3\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle 3\cos u,\ 3\sin u,\ 0\rangle$$

$$\left|\vec{r}_u \times \vec{r}_v\right| = \sqrt{9\cos^2 u + 9\sin^2 u}$$

$$= 3$$

$$0 \le z = v \le 3 \quad 0 \le u \le 2\pi$$

$$\iint_S f(x, y, z)\, dS = \iint_S y\, dS$$

$$= \int_0^3 dv \int_0^{2\pi} 3(3\sin u)\, du$$

$$= -9(3)\ \left(\cos u\right|_0^{2\pi}$$

$$= 0$$

Exercise

Evaluate the surface integral $\iint_S f(x, y, z)\,dS$; $f(x, y, z) = x$, where S is the cylinder $x^2 + z^2 = 1, \quad 0 \le y \le 3$

Solution

$$\vec{r}(u, v) = \langle \cos u,\ v,\ \sin u \rangle$$

$$\vec{r}_u = \langle -\sin u,\ 0,\ \cos u \rangle$$

$$\vec{r}_v = \langle 0,\ 1,\ 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin u & 0 & \cos u \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \langle -\cos u,\ 0,\ -\sin u \rangle$$

$$\left|\vec{r}_u \times \vec{r}_v\right| = \sqrt{\cos^2 u + \sin^2 u}$$

$$\underline{= 1|}$$

$$0 \le y = v \le 3 \quad 0 \le u \le 2\pi$$

$$\iint_S f(x, y, z)\,dS = \iint_S x\,dS$$

$$= \int_0^3 dv \int_0^{2\pi} \cos u\,du$$

$$= 3\left(\sin u \right|_0^{2\pi}$$

$$\underline{= 0\ |}$$

Exercise

Evaluate the surface integral $\iint_S f(x, y, z)\,dS$; $f(\rho, \varphi, \theta) = \cos\varphi$, where S is the part of the unit shpere in the first octant

Solution

$$x^2 + y^2 + z^2 = 1, \quad x, y, z \ge 0$$

$$\vec{r} = \langle \sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi \rangle$$

$$\vec{r}_\varphi = \langle \cos\varphi\cos\theta,\ \cos\varphi\sin\theta,\ -\sin\varphi \rangle$$

$$\vec{r}_\theta = \langle -\sin\varphi\sin\theta,\ \sin\varphi\cos\theta,\ 0 \rangle$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= \sin^2\varphi\cos\theta\ \hat{i} + \sin^2\varphi\sin\theta\ \hat{j} + \left(\sin\varphi\cos\varphi\cos^2\theta + \sin\varphi\cos\varphi\sin^2\theta\right)\hat{k}$$

$$= \sin^2\varphi\cos\theta\ \hat{i} + \sin^2\varphi\sin\theta\ \hat{j} + \sin\varphi\cos\varphi\,\hat{k}$$

$$\left|\vec{r}_\varphi \times \vec{r}_\theta\right| = \sqrt{\sin^4\varphi\cos^2\theta + \sin^4\varphi\sin^2\theta + \sin^2\varphi\cos^2\varphi}$$

$$= \sqrt{\sin^4\varphi\left(\cos^2\theta + \sin^2\theta\right) + \sin^2\varphi\cos^2\varphi}$$

$$= \sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi}$$

$$= \sqrt{\sin^2\varphi\left(\sin^2\varphi + \cos^2\varphi\right)}$$

$$= \sin\varphi$$

$$\iint_S f(x, y, z)\,dS = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} (\cos\varphi)(\sin\varphi)\ d\varphi d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} \sin 2\varphi\ d\varphi$$

$$= -\frac{\pi}{4}\ \cos 2\varphi \Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{\pi}{4}(-1-1)$$

$$= \frac{\pi}{4}$$

Exercise

Find the flux of $\vec{F} = \frac{\vec{r}}{|\vec{r}|}$ across the sphere of radius a centered at the origin, where $\vec{r} = \langle x, y, z \rangle$. Assume the normal vectors to the surface point outward.

Solution

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle x,\ y,\ z\rangle}{\sqrt{x^2+y^2+z^2}}$$

Using spherical to parametrize the sphere.

$$\sqrt{x^2+y^2+z^2} = a$$

$$\begin{aligned}\vec{F} &= \frac{1}{a}\langle a\sin u\cos v,\ \ a\sin u\sin v,\ a\cos u\rangle \\ &= \langle \sin u\cos v,\ \ \sin u\sin v,\ \ \cos u\rangle\end{aligned}$$

Using the table

$$\vec{n} = \left\langle a^2\sin^2 u\cos v,\ a^2\sin^2 u\sin v,\ a^2\sin u\cos u\right\rangle$$

$$\begin{aligned}\vec{F}\cdot\vec{n} &= \langle \sin u\cos v,\ \ \sin u\sin v,\ \ \cos u\rangle \cdot \left\langle a^2\sin^2 u\cos v,\ a^2\sin^2 u\sin v,\ a^2\sin u\cos u\right\rangle \\ &= a^2\sin^3 u\cos^2 v + a^2\sin^3 u\sin^2 v + a^2\sin u\cos^2 u \\ &= a^2\sin^3 u\left(\cos^2 v+\sin^2 v\right) + a^2\sin u\cos^2 u \\ &= a^2\sin^3 u + a^2\sin u\cos^2 u \\ &= a^2\sin u\left(\sin^2 u+\cos^2 u\right) \\ &= a^2\sin u\end{aligned}$$

$$\begin{aligned}\iint_S \vec{F}\cdot\vec{n}\ dS &= a^2\int_0^{2\pi}\int_0^{\pi}\sin u\ dudv \\ &= a^2\int_0^{2\pi} dv\quad \left(-\cos u\ \Big|_0^{\pi}\right. \\ &= a^2(2\pi)(1+1) \\ &= 4\pi a^2\end{aligned}$$

Exercise

Find the flux of the vector field $\vec{F} = \langle x,\ y,\ z\rangle$ across the curved surface of the cylinder $x^2+y^2=1$ for $|z|\le 8$

Solution

$$\vec{n} = \langle x,\ y,\ 0\rangle$$

$$\begin{aligned}|\vec{n}| &= \sqrt{x^2+y^2} \\ &= 1\end{aligned}$$

$$\iint_S \vec{F} \bullet \vec{n}\, dS = \iint_R \langle x,\ y,\ z \rangle \bullet \langle x,\ y,\ 0 \rangle\, dA$$

$$= \iint_R \left(x^2 + y^2\right) dA$$

$$= \iint_R dA$$

$$= \textit{area of the circle radius } \frac{8}{2} = 4$$

$$= 2\pi (4)^2$$

$$= 32\pi$$

Exercise

Find the flux of the vector fields across the given surface with the specified orientation $\vec{F} = \langle 0,\ 0,\ -1 \rangle$ across the slanted face of the tetrahedron $z = 4 - x - y$ in the first octant; normal vectors point upward

Solution

$z_x = -1, \quad z_y = -1$

Normal vectors point upward & first octant.

$\vec{n} = \langle 1,\ 1,\ 1 \rangle$

$z = 4 - x - y = 0 \quad \rightarrow \quad y = 4 - x$

$y = 4 - x = 0 \quad \rightarrow \quad x = 4$

$0 \le x \le 4 \quad 0 \le y \le 4 - x$

$$\iint_S \vec{F} \bullet \vec{n}\, dS = \iint_R \langle 0,\ 0,\ -1 \rangle \bullet \langle 1,\ 1,\ 1 \rangle\, dA$$

$$= \int_0^4 \int_0^{4-x} (0 + 0 - 1)\, dy\, dx$$

$$= -\int_0^4 y \Big|_0^{4-x} dx$$

$$= -\int_0^4 (4 - x)\, dx$$

$$= -\left(4x - \frac{1}{2}x^2 \right|_0^4$$

$$= -(16-8)$$

$$\underline{= -8}$$

Exercise

Find the flux of the vector fields across the given surface with the specified orientation $\vec{F} = \langle x,\ y,\ z\rangle$ across the slanted face of the tetrahedron $z = 10 - 2x - 5y$ in the first octant; normal vectors point upward

Solution

$$z_x = -2, \quad z_y = -5$$

Normal vectors point upward & first octant. $\vec{n} = \langle 2,\ 5,\ 1\rangle$

$$z = 10 - 2x - 5y = 0 \quad \rightarrow \quad y = \frac{1}{5}(10-2x)$$

$$y = \frac{1}{5}(10-2x) = 0 \quad \rightarrow \quad x = 5$$

$$0 \le x \le 5 \quad 0 \le y \le y = \frac{1}{5}(10-2x)$$

$$\iint_S \vec{F} \cdot \vec{n}\ dS = \iint_R \langle x,\ y,\ z\rangle \cdot \langle 2,\ 5,\ 1\rangle\, dA$$

$$= \int_0^5 \int_0^{\frac{1}{5}(10-2x)} (2x + 5y + z)\ dydx$$

$$= \int_0^5 \int_0^{\frac{1}{5}(10-2x)} (2x + 5y + 10 - 2x - 5y)\ dydx$$

$$= 10 \int_0^5 \int_0^{\frac{1}{5}(10-2x)} dydx$$

$$= 10 \int_0^5 y \Big|_0^{\frac{1}{5}(10-2x)} dx$$

$$= 10 \int_0^5 \frac{1}{5}(10-2x)\, dx$$

$$= 2\left(10x - x^2 \right|_0^5$$

$$= 2(50-25)$$

$$\underline{= 50}$$

Exercise

Find the flux of the vector fields across the given surface with the specified orientation

$\vec{F} = \langle x,\ y,\ z \rangle$ across the slanted face of the cone $z^2 = x^2 + y^2$ for $0 \le z \le 1$; normal vectors point upward

Solution

$$2zdz + 2xdx = 0 \quad \to \quad z_x = -\frac{x}{z}$$

$$2zdz = 2ydy \quad \to \quad z_y = -\frac{y}{z}$$

Normal vectors point upward: $\quad \vec{n} = \left\langle -\frac{x}{z},\ -\frac{y}{z},\ 1 \right\rangle$

$$\iint_S \vec{F} \bullet \vec{n}\ dS = \iint_R \langle x,\ y,\ z \rangle \bullet \left\langle -\frac{x}{z},\ -\frac{y}{z},\ 1 \right\rangle dA$$

$$= \iint_R \left(-\frac{x^2}{z} - \frac{y^2}{z} + z \right) dA$$

$$= \iint_R \left(\frac{-x^2 - y^2 + z^2}{z} \right) dA$$

$$= \iint_R \left(\frac{-x^2 - y^2 + x^2 + y^2}{z} \right) dA$$

$$= 0$$

Exercise

Find the flux of the vector fields across the given surface with the specified orientation

$\vec{F} = \left\langle e^{-y},\ 2z,\ xy \right\rangle$ across the curved sides of the surface

$S = \{(x,\ y,\ z):\ \ z = \cos y,\ \ -\pi \le y \le \pi,\ \ 0 \le x \le 4\}$; normal vectors point upward

Solution

$$z_x = 0 \quad z_y = -\sin y$$

Normal vectors point upward: $\quad \vec{n} = \langle 0,\ -\sin y,\ 1 \rangle$

$$\iint_S \vec{F} \bullet \vec{n}\ dS = \iint_R \left\langle e^{-y},\ 2z,\ xy \right\rangle \bullet \langle 0,\ -\sin y,\ 1 \rangle dA$$

$$= \iint_R \left(-2z\sin y + xy\right) dA$$

$$= \iint_R \left(-2\cos y \sin y + xy\right) dA$$

$$= \int_0^4 \int_{-\pi}^{\pi} \left(-\sin 2y + xy\right) dy dx$$

$$= \int_0^4 \left(\frac{1}{2}\cos 2y + \frac{1}{2}xy^2 \Big|_{-\pi}^{\pi} dx\right.$$

$$= \frac{1}{2}\int_0^4 \left(1 + \pi^2 x - 1 - \pi^2 x\right) dx$$

$$\underline{= 0}$$

Exercise

Find the flux of the vector fields across the given surface with the specified orientation

$\vec{F} = \frac{\vec{r}}{|\vec{r}|^3}$ across the sphere of radius a centered at the origin, where $\vec{r} = \langle x,\ y,\ z\rangle$; normal vectors point outward

Solution

$\vec{n} = \frac{\vec{r}}{|\vec{r}|}$ pointing outward

$$\iint_S \vec{F} \bullet \vec{n}\ dS = \iint_S \frac{\vec{r}}{|\vec{r}|^3} \bullet \frac{\vec{r}}{|\vec{r}|} dS$$

$$= \iint_S \frac{\vec{r}^{\,2}}{|\vec{r}|^4} dS$$

$$= \iint_S \frac{1}{|\vec{r}|^2} dS$$

$$= \iint_S \frac{1}{a^2} dS$$

$$= \frac{1}{a^2} \times \left(Area\ of\ a\ sphere\right)$$

$$= \frac{1}{a^2}\left(4\pi a^2\right)$$

$$= 4\pi$$

Exercise

Find the flux of the vector fields across the given surface with the specified orientation

$\vec{F} = \langle -y,\ x,\ 1\rangle$ across the cylinder $y = x^2$ for $0 \le x \le 1, \quad 0 \le z \le 4$; normal vectors point in the general direction of the positive y-axis

Solution

$$\vec{r}(u,\ v) = \left\langle u,\ u^2,\ v\right\rangle$$

$$\vec{r}_u = \langle 1,\ 2u,\ 0\rangle$$

$$\vec{r}_v = \langle 0,\ 0,\ 1\rangle$$

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle 2u,\ -1,\ 0\rangle$$

Normal vectors point in the general direction of the positive y-axis, then:

$$\vec{n} = \langle -2u,\ 1,\ 0\rangle$$

$$0 \le u \le 1, \quad 0 \le v \le 4$$

$$\iint_S \vec{F} \bullet \vec{n}\ dS = \iint_S \langle -y,\ x,\ 1\rangle \bullet \langle -2u,\ 1,\ 0\rangle\, dS$$

$$= \iint_R \left\langle -u^2,\ u,\ 1\right\rangle \bullet \langle -2u,\ 1,\ 0\rangle\, dA$$

$$= \int_0^4 dv \int_0^1 \left(2u^3 + u\right) du$$

$$= 4\ \left(\frac{1}{2}u^4 + \frac{1}{2}u^2\ \right|_0^1$$

$$= 4\left(\frac{1}{2} + \frac{1}{2}\right)$$

$$= 4$$

Exercise

Consider the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, where a ,b, and c are positive real numbers.

a) Show that the surface is described by the parametric equations

$$\vec{r}(u,v)=\langle a\cos u\sin v,\ \ b\sin u\sin v,\ \ c\cos v\rangle \quad \text{for } 0\le u\le 2\pi,\ \ 0\le v\le \pi$$

b) ***Write*** an integral for the surface area of the ellipsoid.

Solution

a) $\vec{r}(u,v)=\langle a\cos u\sin v,\ \ b\sin u\sin v,\ \ c\cos v\rangle$

$$\begin{aligned}\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}&=\frac{a^2\cos^2 u\sin^2 v}{a^2}+\frac{b^2\sin^2 u\sin^2 v}{b^2}+\frac{c^2\cos^2 v}{c^2}\\ &=\cos^2 u\sin^2 v+\sin^2 u\sin^2 v+\cos^2 v\\ &=\left(\cos^2 u+\sin^2 u\right)\sin^2 v+\cos^2 v\\ &=\sin^2 v+\cos^2 v\\ &=1\end{aligned}$$

b) $\vec{r}_u=\langle -a\sin u\sin v,\ \ b\cos u\sin v,\ \ 0\rangle$

$\vec{r}_v=\langle a\cos u\cos v,\ \ b\sin u\cos v,\ \ -c\sin v\rangle$

$$\vec{n}=\vec{r}_u\times\vec{r}_v=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -a\sin u\sin v & b\cos u\sin v & 0\\ b\cos u\sin v & b\sin u\cos v & -c\sin v\end{vmatrix}$$

$$=\left\langle -b\cos u\sin^2 v,\ \ ac\sin u\sin^2 v,\ \ -ab\sin v\cos v\right\rangle$$

$$|\vec{n}|=\sqrt{b^2\cos^2 u\sin^4 v+a^2c^2\sin^2 u\sin^4 v+a^2b^2\sin^2 v\cos^2 v}$$

$$=|\sin v|\sqrt{\left(b^2\cos^2 u+a^2c^2\sin^2 u\right)\sin^2 v+a^2b^2\cos^2 v}$$

$$\iint_S 1\,dS=\int_0^{2\pi}\int_0^{\pi}|\sin v|\ \sqrt{\left(b^2\cos^2 u+a^2c^2\sin^2 u\right)\sin^2 v+a^2b^2\cos^2 v}\ dudv$$

Exercise

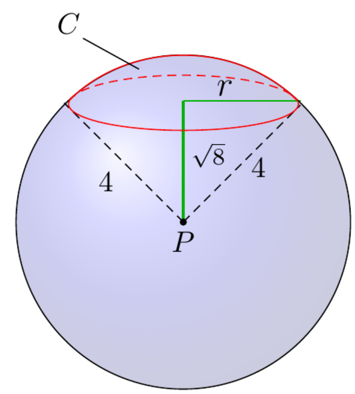
The cone $z^2 = x^2 + y^2, \quad z \ge 0$, cuts the sphere $x^2 + y^2 + z^2 = 16$ along a curve C.

a) Find the surface area of the sphere below C, for $z \ge 0$

b) Find the surface area of the sphere above C.

c) Find the surface area of the cone below C, for $z \ge 0$

Solution



$$\begin{cases} z^2 = x^2 + y^2 \\ x^2 + y^2 + z^2 = 16 \end{cases} \rightarrow \quad 2\left(x^2 + y^2\right) = 16$$

$x^2 + y^2 = 8$

$8 + z^2 = 16 \quad \rightarrow \quad \underline{z = 2\sqrt{2}|}$

$2zdz = 2xdx \quad \rightarrow \quad z_x = \frac{x}{z}$

$2zdz = 2ydy \quad \rightarrow \quad z_y = \frac{y}{z}$

Since the normal vector point outward & in the positive z-direction

$\vec{n} = \left\langle \frac{x}{z}, \ \frac{y}{z}, \ 1 \right\rangle$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1}$$

$$= \sqrt{\frac{x^2 + y^2 + z^2}{z^2}}$$

$$= \sqrt{\frac{16}{z^2}}$$

$$= \frac{4}{z}$$

$$= \frac{4}{\sqrt{16 - x^2 - y^2}}$$

a) $\text{Surface of } C = \int_0^{2\pi} \int_{\sqrt{8}}^{4} \frac{4}{\sqrt{16 - r^2}} \ r \ drd\theta$

$$= -2\int_0^{2\pi} d\theta \int_{\sqrt{8}}^{4} \left(16 - r^2\right)^{-1/2} \ dr\left(16 - r^2\right)$$

$$= -2(2\pi)(2) \left(16 - r^2\right)^{1/2} \Big|_{\sqrt{8}}^{4}$$

$$= -8\pi\left(0 - \sqrt{8}\right)$$

$$=16\pi\sqrt{2}$$

The total surface area of the sphere: $\pi r^3 = 64\pi$

Since the cone in the positive z-direction, then

Surface area of the sphere below $C = \frac{1}{2}64\pi + 16\pi\sqrt{2}$

$$= 16\pi\left(2+\sqrt{2}\right) \quad unit^2$$

b) $\displaystyle\iint_S 1\ dS = \iint_R \frac{4}{\sqrt{16-x^2-y^2}}\ dA$

$$= \int_0^{2\pi}\int_0^{\sqrt{8}} \frac{4}{\sqrt{16-r^2}}\ r\ drd\theta$$

$$= -2\int_0^{2\pi} d\theta \int_0^{\sqrt{8}} \left(16-r^2\right)^{-1/2}\ dr\left(16-r^2\right)$$

$$= -2(2\pi)\ (2)\ \left(16-r^2\right)^{1/2}\Big|_0^{\sqrt{8}}$$

$$= -8\pi\left(\sqrt{8}-4\right)$$

$$= 8\pi\left(4-2\sqrt{2}\right)$$

$$= 16\pi\left(2-\sqrt{2}\right)$$

c) $\displaystyle\iint_S 1\ dS = \iint_R \sqrt{2}\ dA$

$$= \sqrt{2}\pi\ \left(\sqrt{8}\right)^2$$

$$= 8\pi\sqrt{2}$$

Exercise

Consider the sphere $x^2+y^2+z^2=4$ and the cylinder $(x-1)^2+y^2=1$ for $z \ge 0$.

a) Find the surface area of the cylinder inside the sphere

b) Find the surface area of the sphere inside the cylinder.

Solution

a) $(x-1)^2+y^2=1 \quad \to \quad \begin{cases} x-1=\cos u & x=1+\cos u \\ y=\sin u \end{cases}$

$$\vec{r}(u,\ v) = \langle 1+\cos u,\ \ \sin u,\ \ v\rangle$$

$$\vec{r}_u = \langle -\sin u,\ \cos u,\ 0 \rangle$$

$$\vec{r}_v = \langle 0,\ 0,\ 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle \cos u,\ \sin u,\ 0 \rangle$$

$$\left|\vec{r}_u \times \vec{r}_v\right| = \sqrt{\cos^2 u + \sin^2 u}$$

$$\underline{= 1}$$

$$z^2 = 4 - x^2 - y^2$$

$$z = \sqrt{4 - \left(1 + 2\cos u + \cos^2 u\right) - \sin^2 u}$$

$$= \sqrt{3 - 2\cos u - \cos^2 u - \sin^2 u}$$

$$= \sqrt{2 - 2\cos u}$$

$$0 \le z = v \le \sqrt{2 - 2\cos u} \qquad 0 \le u \le 2\pi$$

$$\iint_S 1\, dS = \iint_R 1\, dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2-2\cos u}} dv\,du$$

$$= \int_0^{2\pi} v \Big|_0^{\sqrt{2-2\cos u}} du$$

$$= \sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos u}\ du \qquad \sin^2 \alpha = \frac{1 - \cos 2\alpha}{2} \ \rightarrow\ 1 - \cos u = 2\sin^2 \frac{u}{2}$$

$$= \sqrt{2} \int_0^{2\pi} \sqrt{2\sin^2 \frac{u}{2}}\ du$$

$$= 2\int_0^{2\pi} \sin\frac{u}{2}\ du$$

$$= -4\ \cos\frac{u}{2} \Big|_0^{2\pi}$$

$$= -4(-1 - 1)$$

$$\underline{= 8}$$

b) $\vec{r} = \langle 2\sin\varphi\cos\theta,\ 2\sin\varphi\sin\theta,\ 2\cos\varphi \rangle$

$$\vec{r}_\varphi = \langle 2\cos\varphi\cos\theta,\ 2\cos\varphi\sin\theta,\ -2\sin\varphi \rangle$$

$$\vec{r}_\theta = \langle -2\sin\varphi\sin\theta,\ 2\sin\varphi\cos\theta,\ 0 \rangle$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos\varphi\cos\theta & 2\cos\varphi\sin\theta & -2\sin\varphi \\ -2\sin\varphi\sin\theta & 2\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= 4\sin^2\varphi\cos\theta\ \hat{i} + 4\sin^2\varphi\sin\theta\ \hat{j} + \left(4\sin\varphi\cos\varphi\cos^2\theta + 4\sin\varphi\cos\varphi\sin^2\theta\right)\hat{k}$$

$$= 4\sin^2\varphi\cos\theta\ \hat{i} + 4\sin^2\varphi\sin\theta\ \hat{j} + 4\sin\varphi\cos\varphi\,\hat{k}$$

$$\left|\vec{r}_\varphi \times \vec{r}_\theta\right| = \sqrt{16\sin^4\varphi\cos^2\theta + 16\sin^4\varphi\sin^2\theta + 16\sin^2\varphi\cos^2\varphi}$$

$$= 4\sqrt{\sin^4\varphi\left(\cos^2\theta + \sin^2\theta\right) + \sin^2\varphi\cos^2\varphi}$$

$$= 4\sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi}$$

$$= 4\sqrt{\sin^2\varphi\left(\sin^2\varphi + \cos^2\varphi\right)}$$

$$\underline{= 4\sin\varphi}$$

$$\left(2\sin\varphi\cos\theta - 1\right)^2 + 4\sin^2\varphi\sin^2\theta = 1$$

$$4\sin^2\varphi\cos^2\theta - 4\sin\varphi\cos\theta + 1 + 4\sin^2\varphi\sin^2\theta = 1$$

$$4\sin^2\varphi\left(\cos^2\theta + \sin^2\theta\right) - 4\sin\varphi\cos\theta = 0$$

$$4\sin\varphi\left(\sin\varphi - \cos\theta\right) = 0$$

$$\begin{cases} \sin\varphi = 0 & \varphi = 0,\ \pi \quad \Rightarrow \quad \underline{0 \le u \le \pi} \\ \cos\theta = \sin\varphi & \theta = \cos^{-1}(\sin\varphi) \ \underline{= \frac{\pi}{2} - \varphi} \end{cases}$$

$$\iint\limits_S 1\, dS = \int_0^\pi \int_0^{\frac{\pi}{2}-\varphi} 4\sin\varphi\ d\theta d\varphi$$

$$= 4\int_0^\pi (\sin\varphi)\ \theta \Big|_0^{\frac{\pi}{2}-\varphi}\ d\varphi$$

$$= 4\int_0^\pi \left(\frac{\pi}{2}\sin\varphi - \varphi\sin\varphi\right) d\varphi$$

$$= 4\left(-\frac{\pi}{2}\cos\varphi + \varphi\cos\varphi - \sin\varphi\right|_0^{\pi}$$

$$= 4\left(\frac{\pi}{2} - \pi + \frac{\pi}{2}\right)$$

$$\underline{= 0}$$

Since it cannot be zero, we have to change $0 \le u \le \pi$ to half and multiply by 2.

$\therefore\ 0 \le u \le \frac{\pi}{2}$

$$\iint_S 1\, dS = 2 \times 4\left(-\frac{\pi}{2}\cos\varphi + \varphi\cos\varphi - \sin\varphi\right|_0^{\frac{\pi}{2}}$$

$$= 8\left(-1 + \frac{\pi}{2}\right)$$

$$\underline{= 4\pi - 8}$$

Exercise

Find the upward flux of the field $\vec{F} = \langle x, y, z\rangle$ across the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ in the first octant. Show that the flux equals c times the area if the base of the origin.

Solution

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \rightarrow \quad z = c - \frac{c}{a}x - \frac{c}{b}y$$

$$\frac{1}{a}dx + \frac{1}{c}dz = 0 \quad \rightarrow \quad z_x = -\frac{c}{a}$$

$$\frac{1}{b}dy + \frac{1}{c}dz = 0 \quad \rightarrow \quad z_y = -\frac{c}{b}$$

First octant $\vec{n} = \left\langle \frac{c}{a},\ \frac{c}{b},\ 1\right\rangle$

$$\iint_S \vec{F} \bullet \vec{n}\ dS = \iint_R \langle x, y, z\rangle \bullet \left\langle \frac{c}{a},\ \frac{c}{b},\ 1\right\rangle dA$$

$$= \iint_R \left(\frac{c}{a}x + \frac{c}{b}y + z\right) dA$$

$$= \iint_R \left(\frac{c}{a}x + \frac{c}{b}y + c - \frac{c}{a}x - \frac{c}{b}y\right) dA$$

$$= \iint_R c\ dA$$

$$\underline{= c \times (\text{Area of } A)}$$

As c increases, the slope of the plane gets closer to vertical, so that the x and y components of the vector field $\vec{F} = \langle x,\ y,\ z \rangle$ contribute more to the flux; also, the values of z get larger. This the flux increases as c does.

Exercise

Consider the field $\vec{F} = \langle x,\ y,\ z \rangle$ and the cone $z^2 = \dfrac{x^2 + y^2}{a^2}$, for $0 \le z \le 1$

a) Show that when $a = 1$, the outward flux across the cone is zero.

b) Find the outward flux (away from the z-axis); for any $a > 0$.

Solution

$$2z dz = 2\frac{x}{a^2}dx \quad \rightarrow \quad z_x = \frac{x}{a^2 z}$$

$$2z dz = 2\frac{y}{a^2}dy \quad \rightarrow \quad z_y = \frac{y}{a^2 z}$$

Since the normal is outward: $\quad \vec{n} = \left\langle -\dfrac{x}{a^2 z},\ -\dfrac{y}{a^2 z},\ 1 \right\rangle$

a) $a = 1 \ \rightarrow \ \vec{n} = \left\langle -\dfrac{x}{z},\ -\dfrac{y}{z},\ 1 \right\rangle$

$$\iint_S \vec{F} \bullet \vec{n}\ dS = \iint_R \langle x,\ y,\ z \rangle \bullet \left\langle -\frac{x}{z},\ -\frac{y}{z},\ 1 \right\rangle dA$$

$$= \iint_R \left(-\frac{x^2}{z} - \frac{y^2}{z} + z \right) dA$$

$$= \iint_R \left(-\frac{x^2 + y^2}{z} + z \right) dA \qquad\qquad z^2 = \frac{x^2 + y^2}{a^2}$$

$$= \iint_R \left(-\frac{z^2}{z} + z \right) dA$$

$$= \iint_R 0\ dA$$

$$\underline{= 0 \,|}$$

b)
$$\iint_S \vec{F} \bullet \vec{n}\ dS = \iint_R \langle x,\ y,\ z \rangle \bullet \left\langle -\frac{x}{a^2 z},\ -\frac{y}{a^2 z},\ 1 \right\rangle dA$$

$$= \iint_R \left(-\frac{x^2}{a^2 z} - \frac{y^2}{a^2 z} + z \right) dA$$

$$= \iint_R \left(-\frac{\left(x^2+y^2\right)}{a^2}\frac{1}{z} + z \right) dA$$

$$= \iint_R \left(-z^2\frac{1}{z} + z \right) dA$$

$$= \iint_R (-z+z)\, dA$$

$$= 0$$

The flow is a radial flow, so it is always tangent to the surface.

Exercise

A sphere of radius a is sliced parallel to the equatorial plane at a distance $a-h$ from the equatorial plane. Find the general formula for the surface area of the resulting spherical cap (excluding the base) with thickness h.

Solution

The sphere equation is: $x^2+y^2+z^2=a^2$

$2zdz = 2xdx \quad \rightarrow \quad z_x = \frac{x}{z}$

$2zdz = 2ydy \quad \rightarrow \quad z_y = \frac{y}{z}$

$\vec{n} = \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle$

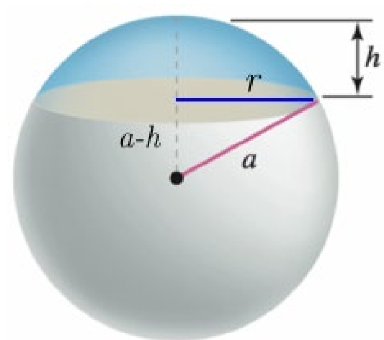
$$\sqrt{z_x^2+z_y^2+1} = \sqrt{\frac{x^2}{z^2}+\frac{y^2}{z^2}+1}$$

$$= \sqrt{\frac{x^2+y^2+z^2}{z^2}}$$

$$= \sqrt{\frac{a^2}{z^2}}$$

$$= \frac{a}{z}$$

$$= \frac{a}{\sqrt{a^2-x^2-y^2}}$$



$r^2 + (a-h)^2 = a^2$

$r^2 = a^2 - a^2 + 2ah - h^2$

$0 \le r \le \sqrt{2ah-h^2}$

$$\iint\limits_S 1\, dS = \iint\limits_R \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2ah-h^2}} \frac{a}{\sqrt{a^2 - r^2}}\, r\, dr d\theta$$

$$= -\frac{a}{2} \int_0^{2\pi} d\theta \int_0^{\sqrt{2ah-h^2}} \left(a^2 - r^2\right)^{-1/2} d\left(a^2 - r^2\right)$$

$$= -2a\pi \left(a^2 - r^2\right)^{1/2} \Big|_0^{\sqrt{2ah-h^2}}$$

$$= -2a\pi \left(\sqrt{a^2 - \left(2ah - h^2\right)} - a \right)$$

$$= -2a\pi \left(\sqrt{a^2 - 2ah + h^2} - a \right)$$

$$= -2a\pi \left(\sqrt{(a-h)^2} - a \right)$$

$$= -2a\pi (a - h - a)$$

$$\underline{= 2a\pi h}$$

Exercise

Consider the radial field $\vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$, where $\vec{r} = \langle x,\ y,\ z \rangle$ and *p* is a real number. Let *S* be he sphere of radius *a* centered at the origin. Show that the outward flux of $\vec{F}$ across the sphere is $\frac{4\pi}{a^{p-3}}$. It is instructive to do the calculation using both an explicit and parametric description of the sphere.

Solution

$$\vec{r} = \langle a\sin u \cos v,\ a\sin u \sin v,\ a\cos u \rangle$$

$$\vec{r}_u = \langle a\cos u \cos v,\ a\cos u \sin v,\ -a\sin u \rangle$$

$$\vec{r}_v = \langle -a\sin u \sin v,\ a\sin u \cos v,\ 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos u \cos v & a\cos u \sin v & -a\sin u \\ -a\sin u \sin v & a\sin u \cos v & 0 \end{vmatrix}$$

$$= \left\langle a^2 \sin^2 u \cos v,\ a^2 \sin^2 u \sin v,\ a^2 \sin u \cos u \cos^2 v + a^2 \sin u \cos u \sin^2 v \right\rangle$$

$$= \left\langle a^2 \sin^2 u \cos v,\ a^2 \sin^2 u \sin v,\ a^2 \sin u \cos u \right\rangle$$

$$\iint\limits_S \vec{F} \cdot \vec{n}\ dS = \iint\limits_R \frac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{p/2}} \cdot \left\langle a^2 \sin^2 u \cos v,\ a^2 \sin^2 u \sin v,\ a^2 \sin u \cos u \right\rangle\ dA$$

$$\left\langle a \sin u \cos v, a \sin u \sin v, a \cos u \right\rangle \cdot \left\langle a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \sin u \cos u \right\rangle$$

$$= \frac{1}{a^p} \iint\limits_R \left(a^3 \sin^3 u \cos^2 v + a^3 \sin^3 u \sin^2 v + a^3 \sin u \cos^2 u \right) dA$$

$$= \frac{1}{a^{p-3}} \iint\limits_R \sin u \left(\sin^2 u \left(\cos^2 v + \sin^2 v \right) + \cos^2 u \right) dA$$

$$= \frac{1}{a^{p-3}} \iint\limits_R \sin u \left(\sin^2 u + \cos^2 u \right) dA$$

$$= \frac{1}{a^{p-3}} \int_0^{2\pi} dv \int_0^{\pi} \sin u\ du$$

$$= \frac{2\pi}{a^{p-3}} \left(-\cos u \ \right|_0^{\pi}$$

$$\boxed{= \frac{4\pi}{a^{p-3}}}$$

Parametric

$$2zdz + 2xdx = 0 \quad \rightarrow \quad z_x = -\frac{x}{z}$$

$$2zdz + 2ydy = 0 \quad \rightarrow \quad z_y = -\frac{y}{z}$$

$$\vec{n} = \left\langle -\frac{x}{z},\ -\frac{y}{z},\ 1 \right\rangle$$

$$\iint\limits_S \vec{F} \cdot \vec{n}\ dS = \iint\limits_R \frac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{p/2}} \cdot \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle\ dA$$

$$= \frac{1}{\left(a^2\right)^{p/2}} \iint\limits_R \left(\frac{x^2}{z} + \frac{y^2}{z} + z \right) dA$$

$$= \frac{1}{a^p} \iint\limits_R \left(\frac{x^2+y^2+z^2}{z} \right) dA$$

$$= \frac{1}{a^p} \iint_R \left(\frac{a^2}{z}\right) dA$$

$$= a^{2-p} \int_0^{2\pi} \int_0^a \frac{r dr d\theta}{\sqrt{a^2 - r^2}}$$

$$= -\frac{1}{2} a^{2-p} \int_0^{2\pi} d\theta \int_0^a \left(a^2 - r^2\right)^{-1/2} d\left(a^2 - r^2\right)$$

$$= -\frac{2\pi}{a^{p-2}} \left(a^2 - r^2\right)^{1/2} \Big|_0^a$$

$$= -\frac{2\pi}{a^{p-2}} (-a)$$

$$\boxed{= \frac{2\pi}{a^{p-3}}}$$

$$\boxed{2 \times \frac{2\pi}{a^{p-3}} = \frac{4\pi}{a^{p-3}}}$$

Exercise

The heat flow vector field for conducting objects is $\vec{F} = -k\nabla T$, where $T(x, y, z)$ is the temperature in the object and $k > 0$ is a constant that depends on the material. Compute the outward flux of $\vec{F}$ across the following surface *S* for the given temperature distributions. Assume $k = 1$.

$T(x, y, z) = 100e^{-x-y}$; *S* consists of the faces of the cube $|x| \le 1, \quad |y| \le 1, \quad |z| \le 1$

Solution

$$\vec{F} = -\nabla T$$

$$= -\nabla\left(100e^{-x-y}\right)$$

$$= \left\langle 100e^{-x-y}, \ 100e^{-x-y}, \ 0 \right\rangle$$

Thus, the flow is parallel to the 2 sides where $z = \pm 1$, so the flus is zero.

For the side: $x = -1 \quad \to \langle -1, \ 0, \ 0 \rangle \quad S_1 : \ \langle -1, \ y, \ z \rangle$

$$\vec{t}_y = \langle 0, \ 1, \ 0 \rangle \quad \vec{t}_z = \langle 0, \ 0, \ 1 \rangle$$

$$\vec{t}_y \times \vec{t}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle 1, \ 0, \ 0 \rangle$$

$$\iint_S \vec{F} \cdot \vec{n}\ dS_1 = \iint_R \left\langle 100e^{-x-y},\ 100e^{-x-y},\ 0 \right\rangle \cdot \langle -1,\ 0,\ 0 \rangle\ dA$$

$$= -\iint_R 100\, e^{-x-y}\ dA$$

$$= -100 \int_{-1}^{1} \int_{-1}^{1} e^{1-y}\ dydz$$

$$= 100 \int_{-1}^{1} dz \int_{-1}^{1} e^{1-y}\ d(1-y)$$

$$= 100\ z \Big|_{-1}^{1} \quad e^{1-y} \Big|_{-1}^{1}$$

$$= 100(2)\left(1 - e^2\right)$$

$$\boxed{= 200\left(1 - e^2\right)}$$

For the side: $x = 1 \quad \rightarrow \langle 1,\ 0,\ 0 \rangle \quad S_2 : \ \langle 1,\ y,\ z \rangle$

$\vec{t}_y = \langle 0,\ 1,\ 0 \rangle \quad \vec{t}_z = \langle 0,\ 0,\ 1 \rangle$

$$\vec{t}_y \times \vec{t}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle 1,\ 0,\ 0 \rangle$$

$$\iint_S \vec{F} \cdot \vec{n}\ dS_2 = \iint_R \left\langle 100e^{-x-y},\ 100e^{-x-y},\ 0 \right\rangle \cdot \langle -1,\ 0,\ 0 \rangle\ dA$$

$$= -\iint_R 100e^{-x-y}\ dA$$

$$= 100 \int_{-1}^{1} \int_{-1}^{1} e^{-1-y}\ dydz$$

$$= -100 \int_{-1}^{1} dz \int_{-1}^{1} e^{-1-y}\ d(-1-y)$$

$$= -100\ \ z \Big|_{-1}^{1} \quad e^{-1-y} \Big|_{-1}^{1}$$

$$=-100(2)\left(e^{-2}-1\right)$$

$$\boxed{=200\left(1-e^{-2}\right)}$$

For the side: $y=-1$ $\rightarrow\langle 0,\ -1,\ 0\rangle$ $S_3:\ \langle x,\ -1,\ z\rangle$

$$\vec{t}_x=\langle 1,\ 0,\ 0\rangle \quad \vec{t}_z=\langle 0,\ 0,\ 1\rangle$$

$$\vec{t}_x\times\vec{t}_z=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 0 & 0\\ 0 & 0 & 1\end{vmatrix}$$

$$=\langle 0,\ -1,\ 0\rangle$$

$$\iint_S \vec{F}\bullet\vec{n}\ dS_3=\iint_R \left\langle 100e^{-x-y},\ 100e^{-x-y},\ 0\right\rangle\bullet\langle 0,\ -1,\ 0\rangle\ dA$$

$$=-\iint_R 100e^{-x-y}\ dA$$

$$=-100\int_{-1}^{1}\int_{-1}^{1} e^{1-x}\ dxdz$$

$$=100\int_{-1}^{1} dz\int_{-1}^{1} e^{1-x}\ d(1-x)$$

$$=100\ \ z\Big|_{-1}^{1}\ \ e^{1-x}\Big|_{-1}^{1}$$

$$=100(2)\left(1-e^{2}\right)$$

$$\boxed{=200\left(1-e^{2}\right)}$$

For the side: $y=1$ $\rightarrow\langle 0,\ 1,\ 0\rangle$ $S_4:\ \langle x,\ 1,\ z\rangle$

$$\vec{t}_x=\langle 1,\ 0,\ 0\rangle \quad \vec{t}_z=\langle 0,\ 0,\ 1\rangle$$

$$\vec{t}_x\times\vec{t}_z=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 0 & 0\\ 0 & 0 & 1\end{vmatrix}$$

$$=\langle 0,\ -1,\ 0\rangle$$

$$\iint_S \vec{F}\bullet\vec{n}\ dS_4=\iint_R \left\langle 100e^{-x-y},\ 100e^{-x-y},\ 0\right\rangle\bullet\langle 0,\ -1,\ 0\rangle\ dA$$

$$= -\iint_R 100e^{-x-y}\, dA$$

$$= 100\int_{-1}^{1}\int_{-1}^{1} e^{-1-x}\, dxdz$$

$$= -100\int_{-1}^{1} dz \quad \int_{-1}^{1} e^{-1-x}\, d(-1-x)$$

$$= -100 \quad z\Big|_{-1}^{1} \quad e^{-1-x}\Big|_{-1}^{1}$$

$$\boxed{= 200\left(1-e^{-2}\right)}$$

The total flux: $= 200 - 200e^2 + 200 - 200e^{-2} + 200 - 200e^2 + 200 - 200e^{-2}$

$$= 800 - 400e^2 - 400e^{-2}$$

$$= -100\left(e^2 + e^{-2} - 2\right)$$

$$= -100\left(e - e^{-1}\right)^2$$

$$\boxed{= -100\left(e - \frac{1}{e}\right)^2}$$

Exercise

The heat flow vector field for conducting objects is $\vec{F} = -k\nabla T$, where $T(x, y, z)$ is the temperature in the object and $k > 0$ is a constant that depends on the material. Compute the outward flux of $\vec{F}$ across the following surface S for the given temperature distributions. Assume $k = 1$.

$T(x, y, z) = 100e^{-x^2-y^2-z^2}$; S cis the sphere $x^2 + y^2 + z^2 = a^2$

Solution

$$\vec{F} = -\nabla T$$

$$= -\nabla\left(100e^{-x^2-y^2-z^2}\right)$$

$$= \left\langle 200xe^{-x^2-y^2-z^2},\ 200ye^{-x^2-y^2-z^2},\ 200ze^{-x^2-y^2-z^2}\right\rangle$$

$$x^2 + y^2 + z^2 = a^2 \quad \rightarrow \quad z = \sqrt{a^2 - x^2 - y^2}$$

$$2xdx + 2zdz = 0 \quad \rightarrow \quad z_x = -\frac{x}{z}$$

$$2ydy + 2zdz = 0 \quad \to \quad z_y = -\frac{y}{z}$$

$$\vec{n} = \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle$$

$$\iint_S \vec{F} \bullet \vec{n}\ dS = 200 \iint_R \left\langle xe^{-x^2-y^2-z^2},\ ye^{-x^2-y^2-z^2},\ ze^{-x^2-y^2-z^2} \right\rangle \bullet \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle dA$$

$$= 200 \iint_R \left(\frac{x^2}{z} + \frac{y^2}{z} + z \right) e^{-\left(x^2+y^2+z^2\right)}\ dA$$

$$= 200 \iint_R \left(\frac{x^2+y^2+z^2}{z} \right) e^{-a^2}\ dA$$

$$= 200a^2e^{-a^2} \iint_R \left(\frac{1}{z}\right) dA$$

$$= 200a^2e^{-a^2} \int_0^{2\pi} \int_0^a \frac{1}{\sqrt{a^2-r^2}}\ r\ drd\theta$$

$$= -100a^2e^{-a^2} \int_0^{2\pi} d\theta \int_0^a \left(a^2-r^2\right)^{-1/2} d\left(a^2-r^2\right)$$

$$= -400\pi a^2e^{-a^2} \left(a^2-r^2\right)^{1/2} \Big|_0^a$$

$$= 400\pi a^3e^{-a^2}$$

Because the vector field is symmetric, then the outward flux of $\vec{F}$ across is

$$2 \times 400\pi a^3e^{-a^2} = 800\pi a^3e^{-a^2}$$

Exercise

The heat flow vector field for conducting objects is $\vec{F} = -k\nabla T$, where $T(x,\ y,\ z)$ is the temperature in the object and $k > 0$ is a constant that depends on the material. Compute the outward flux of $\vec{F}$ across the following surface *S* for the given temperature distributions. Assume $k = 1$.

$T(x,\ y,\ z) = -\ln\left(x^2+y^2+z^2\right)$; *S* cis the sphere $x^2+y^2+z^2 = a^2$

Solution

$$\vec{F} = -\nabla T$$

$$= -\nabla\left(-\ln\left(x^2+y^2+z^2\right)\right)$$

$$= \left\langle \frac{2x}{x^2+y^2+z^2},\ \frac{2y}{x^2+y^2+z^2},\ \frac{2z}{x^2+y^2+z^2} \right\rangle$$

$$= \frac{2}{x^2+y^2+z^2}\langle x,\ y,\ z\rangle$$

$$x^2+y^2+z^2=a^2 \quad \to \quad z=\sqrt{a^2-x^2-y^2}$$

$$2xdx+2zdz=0 \quad \to \quad z_x=-\frac{x}{z}$$

$$2ydy+2zdz=0 \quad \to \quad z_y=-\frac{y}{z}$$

$$\vec{n}=\left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle$$

$$\iint_S \vec{F}\bullet\vec{n}\ dS = 2\iint_R \frac{\langle x,\ y,\ z\rangle}{x^2+y^2+z^2}\bullet\left\langle \frac{x}{z},\ \frac{y}{z},\ 1\right\rangle dA$$

$$= \frac{2}{a^2}\iint_R \left(\frac{x^2}{z}+\frac{y^2}{z}+z\right) dA$$

$$= \frac{2}{a^2}\iint_R \left(\frac{x^2+y^2+z^2}{z}\right) dA$$

$$= 2\iint_R \left(\frac{1}{z}\right) dA$$

$$= 2\int_0^{2\pi}\int_0^a \frac{1}{\sqrt{a^2-r^2}}\ r\ drd\theta$$

$$= -\int_0^{2\pi} d\theta \quad \int_0^a \left(a^2-r^2\right)^{-1/2} d\left(a^2-r^2\right)$$

$$= -4\pi\ \left(a^2-r^2\right)^{1/2}\Big|_0^a$$

$$\underline{= 4\pi a}$$

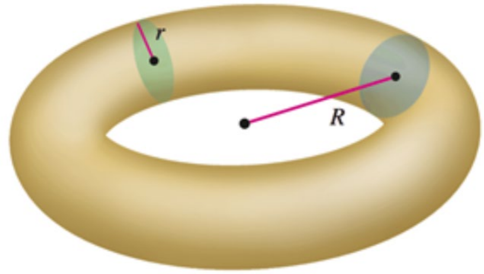
Because the vector field is symmetric, then the outward flux of $\vec{F}$ across is

$$2\times 4\pi a \ \underline{= 8\pi a}$$

Exercise

Given: $\vec{r}(u, v) = \langle (R + r\cos u)\cos v, \ (R + r\cos u)\sin v, \ r\sin u \rangle$

a) Show that a torus with radii $R > r$ may be described parametrically by $\vec{r}(u, v)$ for $0 \le u \le 2\pi, \quad 0 \le v \le 2\pi$



b) Show that the surface area of the torus is $4\pi^2 Rr$

Solution

a) If we let $\langle R\cos v, \ R\sin v, \ 0 \rangle$ the parametrized for the (outer) circle of radius R.

For the inner circle, that includes the z-axis, we can write the parametrization as: $\langle r\cos u\cos v, \ r\cos u\sin v, \ r\sin u \rangle$.

Therefore, the set of points on the torus can be parametrized by the sum of the se 2 vectors.

$$\langle R\cos v, \ R\sin v, \ 0 \rangle + \langle r\cos u\cos v, \ r\cos u\sin v, \ r\sin u \rangle$$
$$= \langle (R + r\cos u)\cos v, \ (R + r\cos u)\sin v, \ r\sin u \rangle$$

b) $\vec{r}(u, v) = \langle (R + r\cos u)\cos v, \ (R + r\cos u)\sin v, \ r\sin u \rangle$

$$\vec{t}_u = \langle -r\sin u\cos v, \ -r\sin s u\sin v, \ r\cos u \rangle$$

$$\vec{t}_v = \langle -(R + r\cos u)\sin v, \ (R + r\cos u)\cos v, \ 0 \rangle$$

$$\vec{t}_u \times \vec{t}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -r\sin u\cos v & -r\sin s u\sin v & r\cos u \\ -(R + r\cos u)\sin v & (R + r\cos u)\cos v & 0 \end{vmatrix}$$

$$= \left(-r(R + r\cos u)\cos u\cos v\right)\hat{i}$$
$$\left(-r(R + r\cos u)\cos u\sin v\right)\hat{j}$$
$$\left(-r(R + r\cos u)\sin u\cos^2 v - r(R + r\cos u)\sin u\sin^2\right)\hat{k}$$

$$= -r(R + r\cos u)\left\langle \cos u\cos v, \ \cos u\sin v, \ \sin u\left(\cos^2 v + \sin^2 v\right) \right\rangle$$

$$= -r(R + r\cos u)\langle \cos u\cos v, \ \cos u\sin v, \ \sin u \rangle$$

$$\left|\vec{t}_u \times \vec{t}_v\right| = r(R + r\cos u)\sqrt{\cos^2 u\cos^2 v + \cos^2 u\sin^2 v + \sin^2 u}$$

$$= r\left(R + r\cos u\right)\sqrt{\cos^2 u\left(\cos^2 v + \sin^2 v\right) + \sin^2 u}$$

$$= r\left(R + r\cos u\right)\sqrt{\cos^2 u + \sin^2 u}$$

$$= r\left(R + r\cos u\right)$$

$$Area\ of\ the\ torus = \int_0^{2\pi}\int_0^{2\pi} r\left(R + r\cos u\right)du\,dv$$

$$= r\int_0^{2\pi}\left(Ru + r\sin u\ \Big|_0^{2\pi}\ dv\right.$$

$$= 2\pi rR\int_0^{2\pi} dv$$

$$\underline{= 4\pi^2 rR \quad unit^2}$$

Solution Section 4.7 – Stokes' Theorem

Exercise

Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

$\vec{F} = \langle y,\ -x,\ 10\rangle$; *S* is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and *C* is the circle $x^2 + y^2 = 1$ in the *xy*-plane

Solution

$$\vec{F} = \langle y,\ -x,\ 10\rangle$$
$$= \langle \sin t,\ -\cos t,\ 10\rangle$$

$$x^2 + y^2 = 1 = r^2$$

$$\vec{r}(t) = \langle \cos t,\ \sin t,\ 0\rangle$$

$$\vec{r}'(t) = \langle -\sin t,\ \cos t,\ 0\rangle$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \langle \sin t,\ -\cos t,\ 10\rangle \cdot \langle -\sin t,\ \cos t,\ 0\rangle\, dA$$

$$= \int_0^{2\pi} \left(-\sin^2 t - \cos^2 t\right) dt \qquad \sin^2 t + \cos^2 t = 1$$

$$= -\int_0^{2\pi} dt$$

$$\underline{= -2\pi}$$

$$\nabla \times \vec{F} = \nabla \times \langle y,\ -x,\ 10\rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 10 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(10) + \frac{\partial}{\partial z}(x)\right)\hat{i} + \left(\frac{\partial}{\partial z}(y) - \frac{\partial}{\partial x}(10)\right)\hat{j} + \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right)\hat{k}$$

$$\underline{= \langle 0,\ 0,\ -2\rangle}$$

$$\iint_S \left(\nabla \times \vec{F}\right)\cdot \vec{n}\, dS = \iint_R \langle 0,\ 0,\ -2\rangle \cdot \left\langle \frac{x}{z},\ \frac{y}{z},\ 1\right\rangle dA$$

$$= \int_0^{2\pi} d\theta \int_0^1 -2r\,dr$$

$$= -(2\pi)\left(r^2 \Big|_0^1\right.$$

$$\underline{= -2\pi}$$

Or

Using the standard parametrization of the sphere

$$\to \vec{n} = \left\langle \sin^2\phi\cos\theta,\ \sin^2\phi\cos\theta,\ \cos\phi\sin\phi \right\rangle$$

$$\iint_S \left(\nabla\times\vec{F}\right)\cdot\vec{n}\,dS = \iint_R \langle 0,\ 0,\ -2\rangle \cdot \left\langle \sin^2\phi\cos\theta,\ \sin^2\phi\cos\theta,\ \cos\phi\sin\phi \right\rangle dA$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \left(-2\cos\phi\sin\phi\right) d\phi\, d\theta$$

$$= -\int_0^{2\pi} d\theta \int_0^{\pi/2} \sin 2\phi\, d\phi$$

$$= -(2\pi)\left(-\frac{1}{2}\cos 2\phi \Big|_0^{\pi/2}\right.$$

$$\underline{= -2\pi}$$

Exercise

Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces S, and closed curves C. Assume that C has counterclockwise orientation and S has a consistent orientation.

$\vec{F} = \langle 0,\ -x,\ y\rangle$; S is the upper half of the sphere $x^2+y^2+z^2=4$ and C is the circle $x^2+y^2=4$ in the xy-plane

Solution

$$x^2+y^2=4=r^2$$

$$\vec{r}(t) = \langle 2\cos t,\ 2\sin t,\ 0\rangle$$

$$\vec{r}\,'(t) = \langle -2\sin t,\ 2\cos t,\ 0\rangle$$

$$\vec{F} = \langle 0,\ -x,\ y\rangle$$

$$= \langle 0,\ -2\cos t,\ 2\sin t\rangle$$

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_R \langle 0,\ -2\cos t,\ 2\sin t \rangle \bullet \langle -2\sin t,\ 2\cos t,\ 0 \rangle dA$$

$$= \int_0^{2\pi} \left(-4\cos^2 t\right) dt$$

$$= -2\int_0^{2\pi} (1+\cos 2t)\, dt$$

$$= -2\left(t + \frac{1}{2}\sin 2t\right. \Big|_0^{2\pi}$$

$$\underline{= -4\pi}\,\big|$$

$$\nabla \times \vec{F} = \nabla \times \langle 0,\ -x,\ y \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -x & y \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x)\right)\hat{i} + \left(\frac{\partial}{\partial z}(0) - \frac{\partial}{\partial x}(y)\right)\hat{j} + \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(0)\right)\hat{k}$$

$$\underline{= \langle 1,\ 0,\ -1 \rangle}\,\big|$$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \iint_R \langle 1,\ 0,\ -1 \rangle \bullet \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle dA$$

$$= \iint_R \left(\frac{x}{z} - 1\right) dA$$

$$= \int_0^{2\pi} \int_0^2 \left(\frac{r\cos\theta}{\sqrt{4-r^2}} - 1\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^2 \left(\cos\theta\ \frac{r^2}{\sqrt{4-r^2}} - r\right) dr\,d\theta$$

$$r = 2\sin\alpha \qquad \sqrt{4-r^2} = 2\cos\alpha$$

$$dr = 2\cos\alpha\ d\alpha$$

$$\int \frac{r^2}{\sqrt{4-r^2}}\, dr = \int \frac{4\sin^2\alpha}{2\cos\alpha}\ (2\cos\alpha)\ d\alpha$$

$$= \int 4\sin^2\alpha\ d\alpha$$

$$= 2\int (1 - \cos 2\alpha)\ d\alpha$$
$$= 2\left(\alpha - \frac{1}{2}\sin 2\alpha\right)$$
$$= 2\alpha - 2\sin\alpha\cos\alpha$$
$$= 2\sin^{-1}\frac{r}{2} - 2\frac{r}{2}\frac{\sqrt{4-r^2}}{2}$$
$$= 2\sin^{-1}\frac{r}{2} - \frac{1}{2}r\sqrt{4-r^2}$$

$$= \int_0^{2\pi} \left(\left(2\sin^{-1}\left(\frac{r}{2}\right) - \frac{r}{2}\sqrt{4-r^2}\right)\cos\theta - \frac{1}{2}r^2 \ \Big|_0^2\ d\theta\right.$$
$$= \int_0^{2\pi} (\pi\cos\theta - 2)\,d\theta$$
$$= \pi\sin\theta - 2\theta \ \Big|_0^{2\pi}$$
$$\underline{= -4\pi}$$

Exercise

Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

$\vec{F} = \langle x,\ y,\ z\rangle$; *S* is the paraboloid $z = 8 - x^2 - y^2$ for $0 \le z \le 8$ and *C* is the circle $x^2 + y^2 = 8$ in the *xy*-plane

Solution

$$x^2 + y^2 = 8 = r^2$$
$$\vec{r}(t) = \langle 2\sqrt{2}\cos t,\ 2\sqrt{2}\sin t,\ 0\rangle$$
$$\frac{d\vec{r}}{dt} = \langle -2\sqrt{2}\sin t,\ 2\sqrt{2}\cos t,\ 0\rangle$$
$$\vec{F} = \langle x,\ y,\ z\rangle$$
$$= \langle 2\sqrt{2}\cos t,\ 2\sqrt{2}\sin t,\ 0\rangle$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \langle 2\sqrt{2}\cos t,\ 2\sqrt{2}\sin t,\ 0\rangle \cdot \langle -2\sqrt{2}\sin t,\ 2\sqrt{2}\cos t,\ 0\rangle dA$$
$$= \int_0^{2\pi} (-8\cos t\sin t + 8\cos t\sin t)\,dt$$
$$\underline{= 0}$$

Surface integral: $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS \underline{= 0}$

Exercise

Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces S, and closed curves C. Assume that C has counterclockwise orientation and S has a consistent orientation.

$\vec{F} = \langle 2z,\ -4x,\ 3y \rangle$; S is the cap of the sphere $x^2 + y^2 + z^2 = 169$ above the plane $z = 12$ and C is the boundary of S.

Solution

$$x^2 + y^2 + 12^2 = 169$$

$\rightarrow\quad x^2 + y^2 = 25$ is the intersection of the sphere with the plane $z = 12$.

$$\vec{r}(t) = \langle 5\cos t,\ 5\sin t,\ 0 \rangle$$

$$\frac{d\vec{r}}{dt} = \langle -5\sin t,\ 5\cos t,\ 0 \rangle$$

$$\begin{aligned}\vec{F} &= \langle 2z,\ -4x,\ 3y \rangle \\ &= \langle 2(12),\ -4 \times 5\cos t,\ 3 \times 5\sin t \rangle \\ &= \langle 24,\ -20\cos t,\ 15\sin t \rangle\end{aligned}$$

$$\begin{aligned}\oint_C \vec{F} \cdot d\vec{r} &= \iint_R \langle 24,\ -20\cos t,\ 15\sin t \rangle \cdot \langle -5\sin t,\ 5\cos t,\ 0 \rangle dA \\ &= \int_0^{2\pi} \left(-120\sin t - 100\cos^2 t \right) dt \\ &= 10\int_0^{2\pi} \left(-12\sin t - 5 - 5\cos 2t \right) dt \\ &= 10\left(12\cos t - 5t - \frac{5}{2}\sin 2t \right) \Big|_0^{2\pi} \\ &= 10(12 - 10\pi - 12) \\ &\underline{= -100\pi}\end{aligned}$$

$$\begin{aligned}\nabla \times \vec{F} &= \nabla \times \langle 2z,\ -4x,\ 3y \rangle \\ &= \begin{vmatrix} \hat{i} & \hat{i} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & -4x & 3y \end{vmatrix}\end{aligned}$$

$$= (3+0)\,\hat{i} + (2-0)\,\hat{j} + (-4-0)\,\hat{k}$$

$$= \langle 3,\ 2,\ -4 \rangle$$

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\, dS = \iint_R \langle 3,\ 2,\ -4 \rangle \cdot \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle dA$$

$$= \iint_R \left(\frac{3x}{z} + \frac{2y}{z} - 4 \right) dA$$

$$= \int_0^{2\pi} \int_0^5 \left(\frac{3r\cos\theta}{\sqrt{169-r^2}} + \frac{2r\sin\theta}{\sqrt{169-r^2}} - 4 \right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^5 \left(3\cos\theta\, \frac{r^2}{\sqrt{169-r^2}} + 2\sin\theta\, \frac{r^2}{\sqrt{169-r^2}} - 4r \right) dr\,d\theta$$

$$r = 13\sin\alpha \qquad \sqrt{169-r^2} = 13\cos\alpha$$

$$dr = 13\cos\alpha\, d\alpha$$

$$\int \frac{r^2}{\sqrt{169-r^2}}\, dr = \int \frac{169\sin^2\alpha}{13\cos\alpha} (13\cos\alpha)\, d\alpha$$

$$= \int 169\sin^2\alpha\, d\alpha$$

$$= \frac{169}{2} \int (1-\cos 2\alpha)\, d\alpha$$

$$= \frac{169}{2}\left(\alpha - \frac{1}{2}\sin 2\alpha\right)$$

$$= \frac{169}{2}(\alpha - \sin\alpha\cos\alpha)$$

$$= \frac{169}{2}\sin^{-1}\frac{r}{13} - \frac{169}{2}\frac{r}{13}\frac{\sqrt{169-r^2}}{13}$$

$$= \frac{169}{2}\sin^{-1}\frac{r}{13} - \frac{1}{2} r\sqrt{169-r^2}$$

$$= \int_0^{2\pi} \left((3\cos\theta + 2\sin\theta)\left(\frac{169}{2}\sin^{-1}\left(\frac{r}{13}\right) - \frac{r}{2}\sqrt{169-r^2} \right) - 2r^2 \right|_0^5 d\theta$$

$$= \int_0^{2\pi} \left((\cos\theta + \sin\theta)\left(\frac{507}{2}\sin^{-1}\left(\frac{5}{13}\right) - 90 \right) - 50 \right) d\theta$$

$$= \left(\frac{507}{2}\sin^{-1}\left(\frac{5}{13}\right) - 90 \right)(\sin\theta - \cos\theta) - 50\,\theta \Big|_0^{2\pi}$$

$$= -\left(169\sin^{-1}\left(\frac{5}{13}\right) - 60\right) - 100\pi + \left(169\sin^{-1}\left(\frac{5}{13}\right) - 60\right)$$

$$\underline{= -100\pi}$$

Exercise

Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

$\vec{F} = \langle y - z,\ z - x,\ x - y \rangle$; *S* is the cap of the sphere $x^2 + y^2 + z^2 = 16$ above the plane $z = \sqrt{7}$ and *C* is the boundary of *S*.

Solution

$$x^2 + y^2 + 7 = 16$$

$\rightarrow \quad x^2 + y^2 = 9$ is the intersection of the sphere with the plane $z = \sqrt{7}$.

$$\vec{r}(t) = \langle 3\cos t,\ 3\sin t,\ 0 \rangle$$

$$\frac{d\vec{r}}{dt} = \langle -3\sin t,\ 3\cos t,\ 0 \rangle$$

$$\vec{F} = \langle y - z,\ z - x,\ x - y \rangle$$

$$= \left\langle 3\sin t - \sqrt{7},\ \sqrt{7} - 3\cos t,\ 3\cos t - 3\sin t \right\rangle$$

$$\oint_C \vec{F} \bullet d\vec{r} = \iint_R \left\langle 3\sin t - \sqrt{7},\ \sqrt{7} - 3\cos t,\ 3\cos t - 3\sin t \right\rangle \bullet \langle -3\sin t,\ 3\cos t,\ 0 \rangle dA$$

$$= \int_0^{2\pi} \left(-9\sin^2 t + 3\sqrt{7}\sin t + 3\sqrt{7}\cos t - 9\cos^2 t\right) dt \qquad \sin^2 t + \cos^2 t = 1$$

$$= \int_0^{2\pi} \left(3\sqrt{7}\sin t + 3\sqrt{7}\cos t - 9\right) dt$$

$$= -3\sqrt{7}\cos t + 3\sqrt{7}\sin t - 9t \Big|_0^{2\pi}$$

$$= -3\sqrt{7} - 18\pi + 3\sqrt{7}$$

$$\underline{= -18\pi}$$

$$\nabla \times \vec{F} = \nabla \times \langle y - z,\ z - x,\ x - y \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z & z - x & x - y \end{vmatrix}$$

$$\underline{= \langle -2,\ -2,\ -2 \rangle}$$

$$\iint_S \left(\nabla\times\vec{F}\right)\bullet\vec{n}\,dS = \iint_R \langle -2,\ -2,\ -2\rangle \bullet \left\langle \frac{x}{z},\ \frac{y}{z},\ 1\right\rangle dA$$

$$= \iint_R \left(-2\frac{x}{z} - 2\frac{y}{z} - 2\right) dA$$

$$= -2\int_0^{2\pi}\int_0^3 \left(\frac{r\cos\theta}{\sqrt{16-r^2}} + \frac{r\sin\theta}{\sqrt{16-r^2}} + 1\right) r\,drd\theta$$

$$= -2\int_0^{2\pi}\int_0^3 \left((\cos\theta+\sin\theta)\frac{r^2}{\sqrt{16-r^2}} + r\right) drd\theta$$

$$r = 4\sin\alpha \qquad \sqrt{16-r^2} = 4\cos\alpha$$

$$dr = 4\cos\alpha\, d\alpha$$

$$\int \frac{r^2}{\sqrt{16-r^2}}\,dr = \int \frac{16\sin^2\alpha}{4\cos\alpha}\,(4\cos\alpha)\,d\alpha$$

$$= \int 16\sin^2\alpha\, d\alpha$$

$$= 8\int (1-\cos 2\alpha)\, d\alpha$$

$$= 8\left(\alpha - \frac{1}{2}\sin 2\alpha\right)$$

$$= 8(\alpha - \sin\alpha\cos\alpha)$$

$$= 8\sin^{-1}\frac{r}{4} - 8\frac{r}{4}\frac{\sqrt{16-r^2}}{4}$$

$$= 8\sin^{-1}\frac{r}{4} - \frac{1}{2}r\sqrt{16-r^2}$$

$$= -2\int_0^{2\pi} (\cos\theta+\sin\theta)\left(8\sin^{-1}\left(\frac{r}{4}\right) - \frac{r}{2}\sqrt{16-r^2}\right) + \frac{1}{2}r^2 \Big|_0^3\, d\theta$$

$$= -2\int_0^{2\pi} \left(\left(8\sin^{-1}\left(\frac{3}{4}\right) - \frac{3\sqrt{7}}{2}\right)(\cos\theta+\sin\theta) + \frac{9}{2}\right) d\theta$$

$$= -2\left(\left(8\sin^{-1}\left(\frac{3}{4}\right) - \frac{3\sqrt{7}}{2}\right)(-\sin\theta+\cos\theta) + \frac{9}{2}\theta \Big|_0^{2\pi}\right.$$

$$= -2\left(8\sin^{-1}\left(\frac{3}{4}\right) - \frac{3\sqrt{7}}{2} - 9\pi - 8\sin^{-1}\left(\frac{3}{4}\right) + \frac{3\sqrt{7}}{2}\right)$$

$$\underline{= -18\pi}$$

Exercise

Verify that the line integral and the surface integral of Stokes' Theorem are equal for the following vector fields, surfaces *S*, and closed curves *C*. Assume that *C* has counterclockwise orientation and *S* has a consistent orientation.

$\vec{F} = \langle -y,\ -x-z,\ y-x \rangle$; *S* is the part of the plane $z = 6 - y$ that lies in the cylinder $x^2 + y^2 = 16$ and *C* is the boundary of *S*.

Solution

$$\vec{r}(t) = \langle 4\cos t,\ 4\sin t,\ 6 - 4\sin t \rangle \qquad \vec{r}(t) = \langle x,\ y,\ z \rangle$$

$$\frac{d\vec{r}}{dt} = \langle -4\sin t,\ 4\cos t,\ -4\cos t \rangle$$

$$\vec{F} = \langle -y,\ -x-z,\ y-x \rangle$$

$$= \langle -4\sin t,\ -4\cos t - 6 + 4\sin t,\ 4\sin t - 4\cos t \rangle$$

$$\oint_C \vec{F} \bullet d\vec{r} = \int_0^{2\pi} \langle -4\sin t,\ -4\cos t - 6 + 4\sin t,\ 4\sin t - 4\cos t \rangle \bullet \langle -4\sin t,\ 4\cos t,\ -4\cos t \rangle\, dt$$

$$= \int_0^{2\pi} \left(16\sin^2 t - 16\cos^2 t - 24\cos t + 16\sin t\cos t - 16\sin t\cos t + 16\cos^2 t \right) dt$$

$$= \int_0^{2\pi} \left(16\sin^2 t - 24\cos t \right) dt$$

$$= \int_0^{2\pi} (8 - 8\cos 2t - 24\cos t)\, dt$$

$$= 8t - 4\sin 2t - 24\sin t \Big|_0^{2\pi}$$

$$= 16\pi$$

Exercise

Evaluate the line integral $\oint_C \vec{F} \bullet d\vec{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of *S*. Assume that *C* has a counterclockwise orientation,

$\vec{F} = \langle 2y,\ -z,\ x \rangle$; *C* is the circle $x^2 + y^2 = 12$ in the plane $z = 0$.

Solution

$$\nabla \times \vec{F} = \nabla \times \langle 2y,\ -z,\ x \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & -z & x \end{vmatrix}$$

$$\underline{= \langle 1, \ -1, \ -2 \rangle}$$

$$z = 0 \quad (0x + 0y) \quad \to \quad \vec{n} = \langle 0, \ 0, \ 1 \rangle$$

$$\iint_S \left(\nabla \times \vec{F} \right) \cdot \vec{n} \ dS = \iint_R \langle 1, \ -1, \ -2 \rangle \cdot \langle 0, \ 0, \ 1 \rangle \ dA$$

$$= \iint_R (-2) \, dA$$

$$= -2 \int_0^{2\pi} d\theta \int_0^{2\sqrt{3}} r \, dr$$

$$= -2(2\pi) \left(\frac{1}{2} r^2 \ \Big|_0^{2\sqrt{3}} \right.$$

$$\underline{= -24\pi}$$

Exercise

Evaluate the line integral $\oint_C \vec{F} \cdot d\vec{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of *S*. Assume that *C* has a counterclockwise orientation,

$\vec{F} = \langle y, \ xz, \ -y \rangle$; *C* is the ellipse $x^2 + \frac{y^2}{4} = 1$ in the plane $z = 1$.

Solution

$$\nabla \times \vec{F} = \nabla \times \langle y, \ xz, \ -y \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & -y \end{vmatrix}$$

$$\underline{= \langle -1 - x, \ 0, \ z - 1 \rangle}$$

$$z = 1 \quad (+0x + 0y) \quad \to \quad \vec{n} = \langle 0, \ 0, \ 1 \rangle$$

$$\iint_S \left(\nabla \times \vec{F} \right) \cdot \vec{n} \ dS = \iint_R \langle -1 - x, \ 0, \ z - 1 \rangle \cdot \langle 0, \ 0, \ 1 \rangle \, dA$$

$$= \iint_R (z-1)\, dA \qquad \text{Because } z = 1$$

$$= \iint_R (0)\, dA$$

$$\underline{= 0}$$

Exercise

Evaluate the line integral $\oint_C \vec{F} \bullet d\vec{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of *S*. Assume that *C* has a counterclockwise orientation,

$\vec{F} = \left\langle x^2 - z^2,\ y,\ 2xz \right\rangle$; *C* is the boundary of the plane $z = 4 - x - y$ in the plane first octant.

Solution

$$\nabla \times \vec{F} = \nabla \times \left\langle x^2 - z^2,\ y,\ 2xz \right\rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - z^2 & y & 2xz \end{vmatrix}$$

$$\underline{= \langle 0,\ -4z,\ 0 \rangle}$$

$$x + y + z = 4 \quad \to \quad \vec{n} = \langle 1,\ 1,\ 1 \rangle$$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \iint_R \langle 0,\ -4z,\ 0 \rangle \bullet \langle 1,\ 1,\ 1 \rangle\ dA$$

$$= \iint_R (-4z)\, dA$$

$$= -4 \int_0^4 \int_0^{4-x} (4 - x - y)\, dx dy$$

$$= -4 \int_0^4 \left(4y - xy - \tfrac{1}{2} y^2 \ \Big|_0^{4-x} \ dx\right.$$

$$= -4 \int_0^4 \left(16 - 4x - 4x + x^2 - \tfrac{1}{2}\left(16 - 8x + x^2\right)\right) dx$$

$$= -4\int_0^4 \left(\frac{1}{2}x^2 - 4x + 8\right)dx$$

$$= -4\left(\frac{1}{6}x^3 - 2x^2 + 8x \,\Big|_0^4\right.$$

$$= -4\left(\frac{32}{3} - 32 + 32\right)$$

$$\boxed{= -\frac{128}{3}}$$

Exercise

Evaluate the line integral $\oint_C \vec{F} \cdot d\vec{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of S. Assume that C has a counterclockwise orientation,

$\vec{F} = \left\langle y^2,\ -z^2,\ x\right\rangle$; C is the circle $\vec{r}(t) = \langle 3\cos t,\ 4\cos t,\ 5\sin t\rangle$ for $0 \le t \le 2\pi$.

Solution

$$\nabla \times \vec{F} = \nabla \times \left\langle y^2,\ -z^2,\ x\right\rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & -z^2 & x \end{vmatrix}$$

$$\boxed{= \langle -2z,\ -1,\ -2y\rangle}$$

S is the disk $\vec{t} = \langle 3r\cos t,\ 4r\cos t,\ 5r\sin t\rangle$

$\vec{t}_r = \langle 3\cos t,\ 4\cos t,\ 5\sin t\rangle$ & $\vec{t}_t = \langle -3r\sin t,\ -4r\sin t,\ 5r\cos t\rangle$

$$\vec{n} = \vec{t}_r \times \vec{t}_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\cos t & 4\cos t & 5\sin t \\ -3r\sin t & -4r\sin t & 5r\cos t \end{vmatrix}$$

$$\boxed{= \langle 20r,\ -15r,\ 0\rangle}$$

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS = \iint_R \langle -2z,\ -1,\ -2y\rangle \cdot \langle 20r,\ -15r,\ 0\rangle\ dA$$

$$= \int_0^{2\pi}\int_0^1 \left(-40rz + 15r\right)\ dr\ dt$$

$$= \int_0^{2\pi} \int_0^1 \left(-200r\sin t + 15r\right)\, dr\, dt$$

$$= \int_0^{2\pi} \left(-100r^2 \sin t + \frac{15}{2}r^2 \,\Big|_0^1\right. dt$$

$$= \int_0^{2\pi} \left(-100\sin t + \frac{15}{2}\right) dt$$

$$= 100\cos t + \frac{15}{2}t \,\Big|_0^{2\pi}$$

$$= 100 + 15\pi - 100$$

$$= 15\pi$$

Exercise

Evaluate the line integral $\oint_C \vec{F} \cdot d\vec{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of *S*. Assume that *C* has a counterclockwise orientation,

$\vec{F} = \left\langle 2xy\sin z,\ x^2 \sin z,\ x^2 y\cos z \right\rangle$; *C* is the boundary of the plane $z = 8 - 2x - 4y$ in the first octant.

Solution

$$\nabla \times \vec{F} = \nabla \times \left\langle 2xy\sin z,\ x^2 \sin z,\ x^2 y\cos z \right\rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy\sin z & x^2 \sin z & x^2 y\cos z \end{vmatrix}$$

$$= \left\langle x^2\cos z - x^2\cos z,\ 2xy\cos z - 2xy\cos z,\ 2x\sin z - 2x\sin z \right\rangle$$

$$= \langle 0,\ 0,\ 0\rangle$$

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS = 0$$

Exercise

Evaluate the line integral $\oint_C \vec{F} \cdot d\vec{r}$ using Stokes' Theorem, where $\vec{F} = \langle xz,\ yz,\ xy \rangle$; C: is the circle $x^2 + y^2 = 4$ in the xy-plane. Assume C has counterclockwise orientation.

Solution

$$\vec{r}(t) = \langle 2\cos t,\ 2\sin t,\ 0 \rangle$$

$$\frac{d\vec{r}}{dt} = \langle -2\sin t, 2\cos t,\ 0 \rangle$$

$$\begin{aligned} \vec{F} &= \langle xz,\ yz,\ xy \rangle \\ &= \langle 0,\ 0,\ 4\cos t \sin t \rangle \end{aligned}$$

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \langle 0,\ 0,\ 4\cos t \sin t \rangle \cdot \langle -2\sin t,\ 2\cos t,\ 0 \rangle\ dt \\ &= 0 \end{aligned}$$

Exercise

Evaluate the line integral $\oint_C \vec{F} \cdot d\vec{r}$ using the Stoke's Theorem $\vec{F} = \left\langle x^2 - y^2,\ x,\ 2yz \right\rangle$; C is the boundary of the plane $z = 6 - 2x - y$ in the first octant and has counterclockwise orientation.

Solution

$$2x + y + z = 6 \quad \to \quad \vec{n} = \langle 2,\ 1,\ 1 \rangle$$

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & x & 2yz \end{vmatrix} \\ &= \langle 2z,\ 0,\ 1 + 2y \rangle \end{aligned}$$

$$z = 6 - 2x - y = 0 \quad \to \quad 0 \le y \le 2x - 6$$

$$y = 2x - 6 = 0 \quad \to \quad 0 \le x \le 3$$

$$\begin{aligned} \iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS &= \iint_R \langle 2z,\ 0,\ 1 + 2y \rangle \cdot \langle 2,\ 1,\ 1 \rangle dA \\ &= \iint_R \langle 12 - 4x - 2y,\ 0,\ 1 + 2y \rangle \cdot \langle 2,\ 1,\ 1 \rangle dA \end{aligned}$$

$$= \iint_R (24 - 8x - 4y + 1 - 2y)\, dA$$

$$= \int_0^3 \int_0^{6-2x} (25 - 8x - 2y)\, dydx$$

$$= \int_0^3 \left(25y - 8xy - y^2 \, \Big|_0^{6-2x}\right) dx$$

$$= \int_0^3 \left(150 - 50x - 48x + 16x^2 - \left(36 - 24x + 4x^2\right)\right) dx$$

$$= \int_0^3 \left(114 - 74x + 12x^2\right) dx$$

$$= 114x - 37x^2 + 4x^3 \, \Big|_0^3$$

$$\underline{= 117}$$

Exercise

Evaluate the line integral $\oint_C \vec{F} \cdot d\vec{r}$ by evaluating the surface integral in Stokes' Theorem with an appropriate choice of S. Assume that C has a counterclockwise orientation

$\vec{F} = \left\langle x^2 - y^2,\ z^2 - x^2,\ y^2 - z^2 \right\rangle$; C is the boundary of the square $|x| \le 1, \quad |y| \le 1$ in the plane $z = 0$

Solution

Square bounded by $|x| \le 1, \quad |y| \le 1$, then $\vec{n} = \langle 0,\ 0,\ 1\rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & z^2 - x^2 & y^2 - z^2 \end{vmatrix}$$

$$= \langle 2y - 2z,\ 0,\ -2x + 2y\rangle$$

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\, dS = \iint_R \langle 2y - 2z,\ 0,\ -2x + 2y\rangle \cdot \langle 0,\ 0,\ 1\rangle\, dA$$

$$= \iint_R (2y - 2x)\ dA$$

$$= \int_{-1}^{1} \int_{-1}^{1} (2y - 2x)\,dydx$$

$$= \int_{-1}^{1} \left(y^2 - 2xy \ \Big|_{-1}^{1} \ dx \right.$$

$$= \int_{-1}^{1} (1 - 2x - 1 + 2x)\,dx$$

$$\underline{= 0}$$

Exercise

Evaluate the line integral in Stokes' Theorem to evaluate the surface integral $\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS$.

Assume that $\vec{n}$ points in an upward direction,

$\vec{F} = \langle x,\ y,\ z \rangle$; S is the upper half of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$

Solution

$$\nabla \times \vec{F} = \nabla \times \langle x,\ y,\ z \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$\underline{= \langle 0,\ 0,\ 0 \rangle}$$

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS \ \underline{= 0}$$

Let $z = 0 \quad \rightarrow \quad \frac{x^2}{4} + \frac{y^2}{9} = 1$

$$\vec{r}(t) = \langle 2\cos t,\ 3\sin t,\ 0 \rangle$$

$$\frac{d\vec{r}}{dt} = \langle -2\sin t,\ 3\cos t,\ 0 \rangle$$

$$\vec{F} = \langle x,\ y,\ z \rangle = \langle 2\cos t,\ 3\sin t,\ 0 \rangle$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \langle 2\cos t,\ 3\sin t,\ 0\rangle \cdot \langle -2\sin t,\ 3\cos t,\ 0\rangle\, dA$$

$$= \int_0^{2\pi} (-4\cos t \sin t + 9\sin t \cos t)\, dt$$

$$= \int_0^{2\pi} (5\sin t \cos t)\, dt$$

$$= \frac{5}{2}\int_0^{2\pi} \sin 2t\ dt$$

$$= \frac{5}{4}\left(-\cos 2t \right|_0^{2\pi}$$

$$= \frac{5}{2}(-1+1)$$

$$\underline{= 0\ |}$$

Exercise

Evaluate the line integral in Stokes' Theorem to evaluate the surface integral $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\ dS$.

Assume that $\vec{n}$ points in an upward direction,

$\vec{F} = \langle 2y,\ -z,\ x - y - z\rangle$; S is the cap of the sphere $x^2 + y^2 + z^2 = 25 \quad for\ 3 \le x \le 5$

Solution

The boundary of the surface is the intersection of the plane $x = 3$ and $x^2 + y^2 + z^2 = 25$

At $x = 3 \quad \to \quad y^2 + z^2 = 16$

$\vec{r}(t) = \langle 3,\ 4\cos t,\ 4\sin t\rangle$

$\frac{d\vec{r}}{dt} = \langle 0,\ -4\sin t,\ 4\cos t\rangle$

$\vec{F} = \langle 2y,\ -z,\ x - y - z\rangle$

$= \langle 8\cos t,\ -4\sin t,\ 3 - 4\cos t - 4\sin t\rangle$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \langle 8\cos t,\ -4\sin t,\ 3 - 4\cos t - 4\sin t\rangle \cdot \langle 0,\ -4\sin t,\ 4\cos t\rangle\, dA$$

$$= \int_0^{2\pi} \left(16\sin^2 t + 12\cos t - 16\cos^2 t - 16\sin t \cos t\right) dt \qquad \cos 2t = \cos^2 t - \sin^2 t$$

$$= \int_0^{2\pi} (12\cos t - 16\cos 2t - 8\sin 2t)\,dt$$

$$= 12\sin t - 8\sin 2t + 4\cos 2t \Big|_0^{2\pi}$$

$$= (0 - 8 + 4 - 0 + 8 - 4)$$

$$\underline{= 0}$$

$$\nabla \times \vec{F} = \nabla \times \langle x,\ y,\ z \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & -z & x - y - z \end{vmatrix}$$

$$\underline{= \langle 0,\ -1,\ -2 \rangle}$$

$$x = 3 \quad \rightarrow \quad \vec{n} = \langle 3,\ 0,\ 0 \rangle$$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS = \iint_R \langle 0,\ -1,\ -2 \rangle \cdot \langle 3,\ 0,\ 0 \rangle\, dA$$

$$\underline{= 0}$$

Exercise

Evaluate the line integral in Stokes' Theorem to evaluate the surface integral $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS$.

Assume that $\vec{n}$ points in an upward direction,

$\vec{F} = \langle x + y,\ y + z,\ x + z \rangle$; S is the tilted disk enclosed $\boldsymbol{r}(t) = \langle \cos t,\ 2\sin t,\ \sqrt{3}\cos t \rangle$

Solution

$$\vec{r}(t) = \langle \cos t,\ 2\sin t,\ \sqrt{3}\cos t \rangle$$

$$\frac{d\vec{r}}{dt} = \langle -\sin t,\ 2\cos t,\ -\sqrt{3}\sin t \rangle$$

$$\vec{F} = \langle x + y,\ y + z,\ x + z \rangle$$

$$= \langle \cos t + 2\sin t,\ 2\sin t + \sqrt{3}\cos t,\ \cos t + \sqrt{3}\cos t \rangle$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \langle \cos t + 2\sin t,\ 2\sin t + \sqrt{3}\cos t,\ \cos t + \sqrt{3}\cos t \rangle \cdot \langle -\sin t,\ 2\cos t,\ -\sqrt{3}\sin t \rangle\, dA$$

$$= \int_0^{2\pi} \left(-\cos t \sin t - 2\sin^2 t + 4\cos t \sin t + 2\sqrt{3}\cos^2 t - \sqrt{3}\sin t \cos t - 3\cos t \sin t \right) dt$$

$$= \int_0^{2\pi} \left(-2\sin^2 t + 2\sqrt{3}\cos^2 t - \sqrt{3}\sin t\cos t\right) dt$$

$$= \int_0^{2\pi} \left(-2\left(\frac{1-\cos 2t}{2}\right) + 2\sqrt{3}\left(\frac{1+\cos 2t}{2}\right) - \frac{\sqrt{3}}{2}\sin 2t\right) dt$$

$$= \int_0^{2\pi} \left(-1 + \cos 2t + \sqrt{3} + \sqrt{3}\cos 2t - \frac{\sqrt{3}}{2}\sin 2t\right) dt$$

$$= \left(\sqrt{3}-1\right)t + \frac{1}{2}\sin 2t + \frac{\sqrt{3}}{2}\sin 2t + \frac{\sqrt{3}}{4}\cos 2t \Big|_0^{2\pi}$$

$$= \left(\sqrt{3}-1\right)(2\pi) + \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}$$

$$= 2\pi\left(\sqrt{3}-1\right)$$

$$\nabla \times \vec{F} = \nabla \times \langle x,\ y,\ z\rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & y+z & x+z \end{vmatrix}$$

$$= \langle -1,\ -1,\ -1\rangle$$

S is the disk $\vec{t} = \left\langle r\cos t,\ 2r\sin t,\ \sqrt{3}r\cos t\right\rangle$

$$\vec{t}_r = \left\langle \cos t,\ 2\sin t,\ \sqrt{3}\cos t\right\rangle$$

$$\vec{t}_t = \left\langle -r\sin t,\ 2r\cos t,\ -r\sqrt{3}\sin t\right\rangle$$

$$\vec{n} = \vec{t}_r \times \vec{t}_t$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & 2\sin t & \sqrt{3}\cos t \\ -r\sin t & 2r\cos t & -r\sqrt{3}\sin t \end{vmatrix}$$

$$= \left\langle -2r\sqrt{3},\ 0,\ 2r\right\rangle$$

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS = \iint_R \langle -1,\ 0,\ -1\rangle \cdot \left\langle -2r\sqrt{3},\ -r\sqrt{3},\ 2r\right\rangle\ dA$$

$$= \int_0^{2\pi}\int_0^1 \left(2r\sqrt{3} - 2r\right)\ dr\ dt$$

$$= \int_0^{2\pi} dt \int_0^1 \left(2r\sqrt{3} - 2r\right) dr$$

$$= (2\pi) \left(\sqrt{3}\, r^2 - r^2 \right|_0^1$$

$$\underline{= 2\pi\left(\sqrt{3} - 1\right)}$$

Exercise

Evaluate the line integral in Stokes' Theorem to evaluate the surface integral $\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\, dS$.

Assume that $\vec{n}$ points in an upward direction

$\vec{F} = \frac{\vec{r}}{|\vec{r}|}$; S is the paraboloid $x = 9 - y^2 - z^2 \quad for \quad 0 \le x \le 9$ (excluding its base), and $\vec{r}(t) = \langle x,\ y,\ z\rangle$

Solution

$$x = 9 - y^2 - z^2 = 0 \quad \to \quad y^2 + z^2 = 9$$

$$\vec{r}(t) = \langle 0,\ 3\cos t,\ 3\sin t\rangle$$

$$\frac{d\vec{r}}{dt} = \langle 0,\ -3\sin t,\ 3\cos t\rangle$$

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|}$$

$$= \frac{\langle x,\ y,\ z\rangle}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{1}{3}\langle 0,\ 3\cos t,\ 3\sin t\rangle$$

$$\oint_C \vec{F} \cdot d\vec{r} = \frac{1}{3}\iint_R \langle 0,\ 3\cos t,\ 3\sin t\rangle \cdot \langle 0,\ -3\sin t,\ 3\cos t\rangle\, dA$$

$$= \frac{1}{3}\int_0^{2\pi} \left(-9\sin t\cos t + 9\sin t\cos t\right) dt$$

$$\underline{= 0}$$

Exercise

Use Stoke's Theorem to evaluate the surface integral $\iint_S \left(\nabla\times\vec{F}\right)\bullet\vec{n}\ dS$; $\vec{F}=\langle -z,\ x,\ y\rangle$, where S is the hyperboloid $z=10-\sqrt{1+x^2+y^2}$ *for* $z\ge 0$. Assume that $\vec{n}$ is the *outward normal*.

Solution

$$z=10-\sqrt{1+x^2+y^2}\ge 0$$

$$\sqrt{1+x^2+y^2}=10$$

$$1+x^2+y^2=100$$

$$x^2+y^2=99=r^2 \quad\rightarrow\quad r=\sqrt{99}$$

$$\vec{r}(t)=\left\langle \sqrt{99}\cos t,\ \sqrt{99}\sin t,\ 0\right\rangle$$

$$\frac{d\vec{r}}{dt}=\left\langle -\sqrt{99}\sin t,\ \sqrt{99}\cos t,\ 0\right\rangle$$

$$\vec{F}=\langle -z,\ x,\ y\rangle$$

$$=\left\langle 0,\ \sqrt{99}\cos t,\ \sqrt{99}\sin t\right\rangle$$

$$\iint_S \left(\nabla\times\vec{F}\right)\bullet\vec{n}\ dS=\oint_C \vec{F}\bullet d\vec{r}$$

$$=\oint_C \left\langle 0,\ \sqrt{99}\cos t,\ \sqrt{99}\sin t\right\rangle\bullet\left\langle -\sqrt{99}\sin t,\ \sqrt{99}\cos t,\ 0\right\rangle\ dt$$

$$=\int_0^{2\pi} 99\cos^2 t\ dt$$

$$=\frac{99}{2}\int_0^{2\pi}\left(1+\cos 2t\right)\ dt$$

$$=\frac{99}{2}\left(t+\frac{1}{2}\sin 2t\ \Big|_0^{2\pi}\right.$$

$$=99\pi$$

Exercise

Use Stokes' Theorem to evaluate the surface integral $\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS$, given $\vec{F} = \left\langle x^2 - z^2,\ y^2,\ xz \right\rangle$,

where S is the hemisphere $x^2 + y^2 + z^2 = 4$, for $y \ge 0$. Assume that $\vec{n}$ is the outward normal.

Solution

Let $y = 0 \quad \to \quad x^2 + z^2 = 4$

$\vec{r}(t) = \langle 2\cos t,\ 0,\ 2\sin t \rangle$

$\frac{d\vec{r}}{dt} = \langle -2\sin t,\ 0,\ 2\cos t \rangle$

$$\vec{F} = \left\langle x^2 - z^2,\ y^2,\ xz \right\rangle$$

$$= \left\langle 4\cos^2 t - 4\sin^2 t,\ 0,\ 4\cos t \sin t \right\rangle$$

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS = \oint_C \vec{F} \cdot d\vec{r}$$

$$= \oint_C \left\langle 4\cos^2 t - 4\sin^2 t,\ 0,\ 4\cos t \sin t \right\rangle \cdot \langle -2\sin t,\ 0,\ 2\cos t \rangle\ dt$$

$$= \int_0^{2\pi} \left(-8\cos^2 t \sin t + 8\sin^3 t + 8\cos^2 t \sin t\right) dt$$

$$= 8\int_0^{2\pi} \sin^2 t \sin t\ dt$$

$$= -8\int_0^{2\pi} \left(1 - \cos^2 t\right) d(\cos t)$$

$$= 8\left(\frac{1}{3}\cos^3 t - \cos t\right. \Big|_0^{2\pi}$$

$$= 8\left(\frac{1}{3} - 1 - \frac{1}{3} + 1\right)$$

$$\underline{= 0}$$

Exercise

Use Stokes' Theorem to find the circulation of the following vector fields around any simple closed smooth curve C. $\quad \vec{F} = \langle 2x,\ -2y,\ 2z \rangle$

Solution

This is a conservative vector field, and since it is around closed curve, then

$$\boxed{\oint_C \vec{F} \bullet d\vec{r} = 0} \qquad \text{(for any closed curve)}$$

Exercise

Use Stokes' Theorem to find the circulation of the following vector fields around any simple closed smooth curve *C*. $\vec{F} = \nabla\left(x \sin ye^z\right)$

Solution

This is a conservative vector field, and since it is around closed curve, then

$$\boxed{\oint_C \vec{F} \bullet d\vec{r} = 0}$$

Exercise

Use Stokes' Theorem to find the circulation of the following vector fields around any simple closed smooth curve *C*. $\vec{F} = \left\langle 3x^2 y,\ x^3 + 2yz^2,\ 2y^2 z \right\rangle$

Solution

This is a conservative vector field with $\varphi = x^3 y + y^2 z^2$, and since it is around closed curve, then

$$\boxed{\oint_C \vec{F} \bullet d\vec{r} = 0}$$

Exercise

Use Stokes' Theorem to find the circulation of the following vector fields around any simple closed smooth curve *C*. $\vec{F} = \left\langle y^2 z^3,\ 2xyz^3,\ 3xy^2 z^2 \right\rangle$

Solution

This is a conservative vector field with $\varphi = xy^2 z^3$, and since it is around closed curve, then

$$\boxed{\oint_C \vec{F} \bullet d\vec{r} = 0}$$

Exercise

Use Stokes' Theorem and a surface integral to find the circulation on C of the vector field $\vec{F} = \langle -y,\ x,\ 0\rangle$ as a function of φ. For what value of φ is the circulation a maximum?

Solution

$$\nabla \times \vec{F} = \nabla \times \langle x,\ y,\ z\rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$\boxed{= \langle 0,\ 0,\ 2\rangle}$$

$$\vec{t} = \langle r\cos\varphi\cos t,\ r\sin t,\ r\sin\varphi\cos t\rangle$$

$$\vec{t}_r = \langle \cos\varphi\cos t,\ \sin t,\ \sin\varphi\cos t\rangle$$

$$\vec{t}_t = \langle -r\cos\varphi\sin t,\ r\cos t,\ -r\sin\varphi\sin t\rangle$$

$$\vec{n} = \vec{t}_r \times \vec{t}_t$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\varphi\cos t & \sin t & \sin\varphi\cos t \\ -r\cos\varphi\sin t & r\cos t & -r\sin\varphi\sin t \end{vmatrix}$$

$$= \left\langle -r\sin\varphi\sin^2 t - r\sin\varphi\cos^2 t,\ 0,\ r\cos\varphi\cos^2 t + r\cos\varphi\sin^2 t\right\rangle$$

$$\boxed{= \langle -r\sin\varphi,\ 0,\ r\cos\varphi\rangle}$$

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS = \iint_R \langle 0,\ 0,\ 2\rangle \cdot \langle -r\sin\varphi,\ 0,\ r\cos\varphi\rangle\ dA$$

$$= \int_0^{2\pi}\int_0^1 (2r\cos\varphi)\ dr\ dt$$

$$= (2\pi)\left(r^2\cos\varphi\ \Big|_0^1\right.$$

$$\boxed{= 2\pi\cos\varphi}$$

The maximum value of the circulation when $\cos\varphi = 1 \Rightarrow \varphi = 0$ which is 2π

Exercise

A circle C in the plane $x+y+z=8$ has a radius of 4 and center (2, 3, 3). Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ for $\vec{F}=\langle 0,\ -z,\ 2y\rangle$ where C has a counterclockwise orientation when viewed from above. Does the circulation depend on the radius of the circle? Does it depend on the location of the center of the circle?

Solution

$$\nabla\times\vec{F}=\nabla\times\langle 0,\ -z,\ 2y\rangle$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & -z & 2y\end{vmatrix}$$

$$=\langle 3,\ 0,\ 0\rangle$$

$$x+y+z=8 \quad\rightarrow\quad \vec{n}=\langle 1,\ 1,\ 1\rangle$$

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\ dS=\iint_R \langle 3,\ 0,\ 0\rangle\cdot\langle 1,\ 1,\ 1\rangle\ dA$$

$$=\int_0^{2\pi}\int_0^4 (3)\ r dr\ dt$$

$$=(2\pi)\left(\frac{3}{2}r^2\ \Big|_0^4\right.$$

$$=48\pi$$

Exercise

Begin with the paraboloid $z=x^2+y^2$, for $0\le z\le 4$, and slice it with the plane $y=0$. Let S be the surface that remains for $y\ge 0$ (including the planar surface in the xz-plane). Let C be the semicircle and line segment that bound the cap of S in the plane $z=4$ with counterclockwise orientation. Let $\vec{F}=\langle 2z+y,\ 2x+z,\ 2y+x\rangle$

a) Describe the direction of the vectors normal to the surface that are consistent with the orientation of C.

b) Evaluate $\iint_S (\nabla\times\vec{F})\cdot\vec{n}\ dS$

c) Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ and check for argument with part (b).

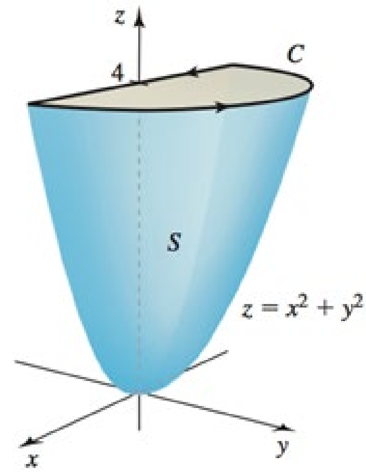
Solution

a) The normal vector point toward the z-axis on the curved surface of S and in the direction $\langle 0,\ 1,\ 0\rangle$ on the flat surface of S.

b) $\nabla\times\vec{F} = \nabla\times\langle 2z+y,\ 2x+z,\ 2y+x\rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z+y & 2x+z & 2y+x \end{vmatrix}$$

$$\underline{= \langle 1,\ 1,\ 1\rangle}$$



The planar surface in the xz-plane, then let S_1 be the surface parameterized by $\langle x,\ 0,\ z\rangle$.

Where, since $y=0$,

$z = x^2 + 0^2 \quad\Rightarrow\quad x^2 \le z \le 4$

and $z = 4 = x^2 \quad\Rightarrow\quad -2 \le x \le 0$

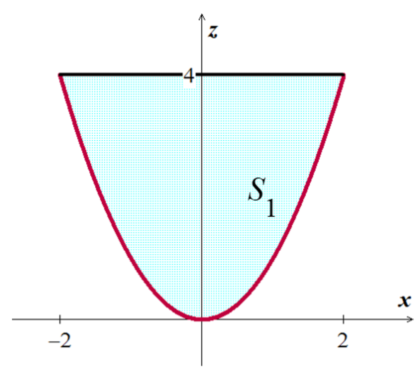
$\vec{t} = \langle x,\ 0,\ z\rangle$

$\vec{t}_x = \langle 1,\ 0,\ 0\rangle \quad \& \quad \vec{t}_z = \langle 0,\ 0,\ 1\rangle$

$\vec{n} = \vec{t}_x \times \vec{t}_z$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\underline{= \langle 0,\ -1,\ 0\rangle}$$



$$\iint_S \left(\nabla\times\vec{F}\right)\cdot\vec{n}\ dS = \iint_{S_1} \langle 1,\ 1,\ 1\rangle\cdot\langle 0,\ -1,\ 0\rangle\ dS$$

$$= \int_{-2}^{2}\int_{x^2}^{4} (-1)\ dz\ dx$$

$$= -\int_{-2}^{2} z\,\Big|_{x^2}^{4}\ dx$$

$$= -\int_{-2}^{2}\left(4-x^2\right)dx$$

$$= -\left(4x - \tfrac{1}{3}x^3\ \Big|_{-2}^{2}\right.$$

$$= -\left(8 - \frac{8}{3} + 8 - \frac{8}{3}\right)$$

$$\boxed{= -\frac{32}{3}}$$

Let S_2 be the surface of the half of the paraboloid for $y \geq 0$, parametrized as

$$\vec{t} = \left\langle r\cos\phi,\ r\sin\phi,\ r^2 \right\rangle; \quad 0 \leq r \leq 2; \quad -\pi \leq \phi \leq 0$$

$$\vec{t}_r = \langle \cos\phi,\ \sin\phi,\ 2r \rangle$$

$$\vec{t}_\phi = \langle -r\sin\phi,\ r\cos\phi,\ 0 \rangle$$

$$\vec{n} = \vec{t}_r \times \vec{t}_\phi$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\phi & \sin\phi & 2r \\ -r\sin\phi & r\cos\phi & 0 \end{vmatrix}$$

$$\boxed{= \left\langle -2r^2\cos\phi,\ -2r^2\sin\phi,\ r \right\rangle}$$

$$\iint_{S_2} \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS = \iint_{S_2} \langle 1,\ 1,\ 1 \rangle \cdot \left\langle -2r^2\cos\phi,\ -2r^2\sin\phi,\ r \right\rangle\ dS$$

$$= \int_{-\pi}^{0} \int_{0}^{2} \left(-2r^2\cos\phi - 2r^2\sin\phi + r\right) dr d\phi$$

$$= \int_{-\pi}^{0} \left(-\frac{2}{3}r^3\cos\phi - \frac{2}{3}r^3\sin\phi + \frac{1}{2}r^2 \Big|_0^2\ d\phi\right.$$

$$= \int_{-\pi}^{0} \left(-\frac{16}{3}\cos\phi - \frac{16}{3}\sin\phi + 2\right) d\phi$$

$$= -\frac{16}{3}\sin\phi + \frac{16}{3}\cos\phi + 2\phi \Big|_{-\pi}^{0}$$

$$= \frac{16}{3} + \frac{16}{3} + 2\pi$$

$$\boxed{= \frac{32}{3} + 2\pi}$$

$$\iint_{S} \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS = \iint_{S_1} \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS + \iint_{S_2} \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS$$

$$= -\frac{32}{3} + \frac{32}{3} + 2\pi$$

$$\boxed{= 2\pi}$$

c) $\oint_C \vec{F} \bullet d\vec{r} = \oint_{C_1} \vec{F} \bullet d\vec{r}_1 + \oint_{C_2} \vec{F} \bullet d\vec{r}_2$

$\vec{F} = \langle 2z + y,\ 2x + z,\ 2y + x \rangle$

$C_1 :\ \vec{r}_1 = \langle t,\ 0,\ 4 \rangle = \langle x,\ y,\ z \rangle \quad for \quad -2 \le t \le 2$

$$\frac{d\vec{r}_1}{dt} = \langle 1,\ 0,\ 0 \rangle$$

$C_2 :\ \vec{r}_2 = \langle 2\cos t,\ 2\sin t,\ 4 \rangle = \langle x,\ y,\ z \rangle \quad for \quad -\pi \le t \le 0$

$$\frac{d\vec{r}_2}{dt} = \langle -2\sin t,\ 2\cos t,\ 0 \rangle$$

$$\oint_{C_1} \vec{F} \bullet d\vec{r}_1 = -\int_{-2}^{2} \langle 2z + y,\ 2x + z,\ 2y + x \rangle \bullet \langle 1,\ 0,\ 0 \rangle\ dt$$

$$= -\int_{-2}^{2} (2z + y)\, dt$$

$$= -\int_{-2}^{2} (2(4) + 0)\, dt$$

$$= -\int_{-2}^{2} (8)\, dt$$

$$= -8t \Big|_{-2}^{2}$$

$$\underline{= -32\ |}$$

$$\oint_{C_2} \vec{F} \bullet d\vec{r}_2 = \int_{-\pi}^{0} \langle 2z + y,\ 2x + z,\ 2y + x \rangle \bullet \langle -2\sin t,\ 2\cos t,\ 0 \rangle\, dt$$

$$= \int_{-\pi}^{0} \langle 8 + 2\sin t,\ 4\cos t + 4,\ 4\sin t + 2\cos t \rangle \bullet \langle -2\sin t,\ 2\cos t,\ 0 \rangle\, dt$$

$$= \int_{-\pi}^{0} \left(-16\sin t - 4\sin^2 t + 8\cos^2 t + 8\cos t \right) dt \qquad \sin^2 t = 1 - \cos^2 t$$

$$= \int_{-\pi}^{0} \left(-16\sin t - 4\left(1 - \cos^2 t\right) + 8\cos^2 t + 8\cos t \right) dt$$

$$= \int_{-\pi}^{0} \left(-16\sin t - 4 + 12\cos^2 t + 8\cos t \right) dt \qquad \cos^2 t = \frac{1 + \cos 2t}{2}$$

$$= \int_{-\pi}^{0} (-16\sin t + 2 + 6\cos 2t + 8\cos t)\, dt$$

$$= 16\cos t + 2t + 3\sin 2t + 8\sin t \Big|_{-\pi}^{0}$$

$$= 32 + 2\pi$$

$$\oint_C \vec{F} \bullet d\vec{r} = \oint_{C_1} \vec{F} \bullet d\vec{r}_1 + \oint_{C_2} \vec{F} \bullet d\vec{r}_2$$

$$= -32 + 32 + 2\pi$$

$$= 2\pi$$

Exercise

The French Physicist André-Marie Ampère discovered that an electrical current I in a wire produces a magnetic field B. A special case of Ampère's Law relates the current to the magnetic field through the equation $\oint_C \boldsymbol{B} \cdot d\boldsymbol{r} = \mu I$, where C is any closed curve through which the wire passes and μ is a physical constant. Assume that the current I is given in terms of the current density $\boldsymbol{J}$ as $I = \iint_S \boldsymbol{J} \bullet \vec{n}\, dS$, where S is an oriented surface with C as a boundary. Use Stokes' Theorem to show that an equivalent form of Ampère's Law is $\nabla \times \boldsymbol{B} = \mu \boldsymbol{J}$.

Solution

$$\iint_S (\nabla \times B) \bullet \vec{n}\, dS = \oint_C \boldsymbol{B}\vec{r}_\varphi \times \vec{r}_\theta d\boldsymbol{r}$$

$$= \mu I$$

$$= \mu \iint_S \boldsymbol{J} \bullet \vec{n}\, dS$$

$$\iint_S (\nabla \times B) \bullet \vec{n}\, dS - \mu \iint_S \boldsymbol{J} \bullet \vec{n}\, dS = 0$$

Thus $$\iint_S \left[(\nabla \times B) - \mu \boldsymbol{J} \right] \bullet \vec{n}\, dS = 0$$

For all surfaces S bounded by any given closed curve C.

It is clear that given the freedom to choose C and S, that it follows that the integrand is identically zero, i.e. that for any surface S, $\left((\nabla \times B) - \mu \boldsymbol{J}\right) \bullet \vec{n} \ = 0$.

From this, it is easy to see that we must have $(\nabla \times B) = \mu \boldsymbol{J}$, since we are free to make normal vector point in any direction at any given point by choosing S appropriately.

Exercise

Let S be the paraboloid $z = a\left(1 - x^2 - y^2\right)$, for $z \ge 0$, where $a > 0$ is a real number. Let $\vec{F} = \langle x - y,\ \ y + z,\ \ z - x \rangle$. For what value(s) of a (if any) does $\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS$ have its maximum value?

Solution

$$For \quad z = a\left(1 - x^2 - y^2\right) = 0 \quad \Rightarrow \quad x^2 + y^2 = 1$$

$$\vec{r}(t) = \langle \cos t,\ \ \sin t,\ \ 0 \rangle$$

$$\frac{d\vec{r}}{dt} = \langle -\sin t,\ \ \cos t,\ \ 0 \rangle$$

$$\vec{F} = \langle x - y,\ y + z,\ z - x \rangle$$

$$= \langle \cos t - \sin t,\ \ \sin t,\ \ -\cos t \rangle$$

$$\oint_C \vec{F} \bullet d\vec{r} = \int_0^{2\pi} \langle \cos t - \sin t,\ \ \sin t,\ \ -\cos t \rangle \bullet \langle -\sin t,\ \ \cos t,\ \ 0 \rangle\ dt$$

$$= \int_0^{2\pi} \left(-\cos t \sin t + \sin^2 t + \cos t \sin t\right)\ dt$$

$$= \int_0^{2\pi} \sin^2 t\ dt$$

$$= \frac{1}{2}\int_0^{2\pi} (1 - \cos 2t)\ dt$$

$$= \frac{1}{2}\left(t - \frac{1}{2}\sin 2\mathrm{t}\ \right|_0^{2\pi}$$

$$= \pi$$

$\therefore$ The integral is independent of a.

Exercise

The goal is to evaluate $A = \iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS$, where $\vec{F} = \langle yz,\ -xz,\ xy \rangle$ and S ids the surface of the upper half of the ellipsoid $x^2 + y^2 + 8z^2 = 1 \ \ (z \ge 0)$

a) Evaluate a surface integral over a more convenient surface to find the value of A.

b) Evaluate A using a line integral.

Solution

a) The boundary of this surface is the circle $x^2 + y^2 = 0$ at $z = 0$

$$\nabla \times \vec{F} = \nabla \times \langle yz,\ -xz,\ xy \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{i} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz & xy \end{vmatrix}$$

$$= \langle 2x,\ 0,\ -2z \rangle$$

$$\nabla \times \vec{F}\Big|_{z=0} = \langle 2x,\ 0,\ 0 \rangle$$

At $z = 0 \quad \rightarrow \quad \vec{n} = \langle 0,\ 0,\ 1 \rangle$

$$\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS = \iint_S \langle 2x,\ 0,\ 0 \rangle \cdot \langle 0,\ 0,\ 1 \rangle\ dS$$

$$= \iint_S (0)\ dS$$

$$= 0$$

b) With the parameterization of the boundary circle and $z = 0$, we have

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle 0,\ 0,\ 1 \rangle \cdot \langle -\sin t,\ \cos t,\ 0 \rangle\ dt$$

$$= \int_0^{2\pi} 0\ dt$$

$$= 0$$

Exercise

Let $\vec{F} = \langle 2z,\ z,\ \ x+2y \rangle$ and let S be the hemisphere of radius a with its base in the xy-plane and center at the origin.

a) Evaluate $\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS$ by computing $\nabla \times \vec{F}$ and appealing to symmetry.

b) Evaluate the line integral using Stokes' Theorem to check part (a).

Solution

a) $\nabla \times \vec{F} = \nabla \times \langle 2z,\ z,\ \ x+2y \rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & z & x+2y \end{vmatrix}$$

$$= \langle 1,\ 1,\ \ 0 \rangle$$

S: $x^2 + y^2 + z^2 = a^2 \quad$ *with* $\quad z \ge 0$

$2xdx + 2zdz = 0 \quad \rightarrow \quad z_x = -\frac{x}{z}$

$2ydy + 2zdz = 0 \quad \rightarrow \quad z_y = -\frac{y}{z}$

$\vec{n} = \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle$

$$\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \iint_S \langle 1,\ 1,\ \ 0 \rangle \bullet \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle dS$$

$$= \iint_R \left(\frac{x}{z} + \frac{y}{z}\right) dA$$

$$= \iint_R \left(\frac{x+y}{z}\right) dA$$

By symmetry, the integral vanishes on each level curve, so it vanishes altogether.

b) Let $z = 0 \quad \rightarrow \quad x^2 + y^2 = a^2$

$\vec{r}(t) = \langle a\cos t,\ a\sin t,\ 0 \rangle$

$\frac{d\vec{r}}{dt} = \langle -a\sin t,\ a\cos t,\ 0 \rangle$

$\vec{F} = \langle 2z,\ z,\ \ x+2y \rangle$

$\quad = \langle 0,\ 0,\ a\cos t + 2a\sin t \rangle$

$$\iint_S \left(\nabla\times\vec{F}\right)\cdot\vec{n}\ dS = \oint_C \vec{F}\cdot d\vec{r}$$

$$= \oint_C \langle 0,\ 0,\ a\cos t + 2a\sin t\rangle \cdot \langle -a\sin t,\ a\cos t,\ 0\rangle\ dt$$

$$\underline{= 0}$$

Exercise

Let S be the disk enclosed by the curve $C:\ \vec{r}(t) = \langle \cos\varphi\cos t,\ \sin t,\ \sin\varphi\cos t\rangle$ fot $0 \le t \le 2\pi$, where $0 \le \varphi \le \frac{\pi}{2}$ is a fixed angle.

a) Find the a vector normal to *S*.

b) What is the areas of *S*?

c) Whant the length of *C*?

d) Use the Stokes' Theorem and a surface integral to find the ciurculation on *C* of the vector field $\vec{F} = \langle -y,\ x,\ 0\rangle$ as a function of φ. For what value of φ is the circulation a maximum?

e) What is the circulation on *C* of the vector field $\vec{F} = \langle -y,\ -z,\ x\rangle$ as a function of φ? For what value of φ is the circulation a maximum?

f) Consider the vector field $\vec{F} = \vec{a}\times\vec{r}$, where $\vec{a} = \langle a_1,\ a_2,\ a_3\rangle$ is a constant nonzero vector and $\vec{r} = \langle x,\ y,\ z\rangle$. Show that the circulation is a maximum when $\vec{a}$ points in the direection of the normal to *S*.

Solution

a) $\vec{r}(t) = \langle r\cos\varphi\cos t,\ r\sin t,\ r\sin\varphi\cos t\rangle$

$$\vec{t}_r = \langle \cos\varphi\cos t,\ \sin t,\ \sin\varphi\cos t\rangle$$

$$\vec{t}_t = \langle -r\cos\varphi\sin t,\ r\cos t,\ -r\sin\varphi\sin t\rangle$$

$$\vec{t}_\varphi\times\vec{t}_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\varphi\cos t & \sin t & \sin\varphi\cos t \\ -r\cos\varphi\sin t & r\cos t & -r\sin\varphi\sin t \end{vmatrix}$$

$$= \Big\langle -r\sin\varphi\sin^2 t - r\sin\varphi\cos^2 t,$$
$$-r\sin\varphi\cos\varphi\cos t\sin t + r\sin\varphi\cos\varphi\cos t\sin t,$$
$$r\cos\varphi\cos^2 t + r\cos\varphi\sin^2 t\Big\rangle$$

$$= \left\langle -r\sin\varphi\left(\sin^2 t + \cos^2 t\right),\ 0,\ r\cos\varphi\left(\cos^2 t\sin^2 t\right)\right\rangle$$

$$= \langle -r\sin\varphi,\ 0,\ r\cos\varphi\rangle$$

$$\vec{n} = \vec{t}_\varphi \times \vec{t}_t$$
$$\underline{= \langle -r\sin\varphi,\ 0,\ r\cos\varphi \rangle}$$

b) $$\left|\vec{t}_r \times \vec{t}_t\right| = \sqrt{r^2\sin^2\varphi + r^2\cos^2\varphi}$$
$$\underline{= r}$$

$$Area = \int_0^{2\pi}\int_0^1 \left|\vec{t}_r \times \vec{t}_t\right| drdt \qquad\qquad Surface\ Area = \iint_S 1\, dS$$

$$= \int_0^{2\pi} dt \int_0^1 r\, dr$$
$$= (2\pi)\left(\frac{1}{2}r^2\right|_0^1$$
$$\underline{= \pi \quad unit^2}$$

(this surface is simply the unit circle inclined at the angle φ to the xy-plane)

c) $$\vec{r}(t) = \langle \cos\varphi\cos t,\ \sin t,\ \sin\varphi\cos t \rangle$$
$$\frac{d\vec{r}}{dt} = \langle -\cos\varphi\sin t,\ \cos t,\ -\sin\varphi\sin t \rangle$$
$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\cos^2\varphi\sin^2 t + \cos^2 t + \sin^2\varphi\sin^2 t}$$
$$= \sqrt{\left(\cos^2\varphi + \sin^2\varphi\right)\sin^2 t + \cos^2 t}$$
$$= \sqrt{\sin^2 t + \cos^2 t}$$
$$\underline{= 1}$$

$$L = \int_0^{2\pi} 1\, dt$$
$$\underline{= 2\pi \quad unit}$$

(Because it just the circumference of the unit circle)

d) $$\vec{F} = \langle -y,\ x,\ 0 \rangle$$
$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$
$$= \langle 0,\ 0,\ 2 \rangle$$

$$\iint_S \left(\nabla\times\vec{F}\right)\cdot\vec{n}\ dS = \iint_R \langle 0,\ 0,\ 2\rangle \cdot \langle -r\sin\varphi,\ 0,\ r\cos\varphi\rangle\ dA$$

$$= \int_0^{2\pi}\int_0^1 2r\cos\varphi\ drdt$$

$$= \cos\varphi \int_0^{2\pi} dt \int_0^1 2r\ dr$$

$$= 2\pi\cos\varphi\ \left(r^2\ \Big|_0^1\right.$$

$$\underline{= 2\pi\cos\varphi\ |}$$

The maximum when $\cos\varphi = 1 \quad\to\quad \underline{\varphi = 0|}$

The circulation has a maximum of 2π at $\varphi = 0$.

e) $\vec{r}(t) = \langle \cos\varphi\cos t,\ \sin t,\ \sin\varphi\cos t\rangle$

$$\frac{d\vec{r}}{dt} = \langle -\cos\varphi\sin t,\ \cos t,\ -\sin\varphi\sin t\rangle$$

$$\vec{F} = \langle -y,\ -z,\ x\rangle$$

$$= \langle -\sin t,\ -\sin\varphi\cos t,\ \cos\varphi\cos t\rangle$$

$$\vec{F}\cdot\frac{d\vec{r}}{dt} = \langle -\sin t,\ -\sin\varphi\cos t,\ \cos\varphi\cos t\rangle \cdot \langle -\cos\varphi\sin t,\ \cos t,\ -\sin\varphi\sin t\rangle$$

$$= \cos\varphi\sin^2 t - \sin\varphi\cos^2 t - \cos\varphi\cos t\sin\varphi\sin t$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left(\cos\varphi\sin^2 t - \sin\varphi\cos^2 t - \cos\varphi\cos t\sin\varphi\sin t\right)dt$$

$$= \frac{1}{2}\cos\varphi\int_0^{2\pi}(1-\cos 2t)\,dt - \frac{1}{2}\sin\varphi\int_0^{2\pi}(1+\cos 2t)\,dt$$

$$+\cos\varphi\sin\varphi\int_0^{2\pi}\cos t\ d(\cos t)$$

$$= \frac{1}{2}\cos\varphi\left(t - \frac{1}{2}\sin 2t\ \Big|_0^{2\pi} - \frac{1}{2}\sin\varphi\left(t + \frac{1}{2}\sin 2t\ \Big|_0^{2\pi} + \frac{1}{2}\cos\varphi\sin\varphi\cos^2 t\ \Big|_0^{2\pi}\right.\right.$$

$$= \pi\cos\varphi - \pi\sin\varphi + \frac{1}{2}\cos\varphi\sin\varphi(1-1)$$

$$\underline{= \pi(\cos\varphi - \sin\varphi)\ |}$$

The maximum when $\cos\varphi - \sin\varphi = 1 \ \to \ \varphi = 0, \ \frac{3\pi}{2}$

The maximum circulation is π at $\varphi = 0$.

f) $\vec{F} = \vec{a} \times \vec{r} \qquad \vec{a} = \left\langle a_1,\, a_2,\, a_3 \right\rangle$

$$= \left\langle a_1,\, a_2,\, a_3 \right\rangle \times \langle x,\, y,\, z \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \left\langle a_2 z - a_3 y,\ a_3 x - a_1 z,\ a_1 y - a_2 x \right\rangle$$

$$\nabla \times \left(\vec{a} \times \vec{r}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= \left\langle 2a_1,\ 2a_2,\ 2a_3 \right\rangle$$

$$\vec{r}(t) = \left\langle r\cos\varphi\cos t,\ r\sin t,\ r\sin\varphi\cos t \right\rangle$$

$$\vec{n} = \left\langle -r\sin\varphi,\ 0,\ r\cos\varphi \right\rangle$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \left\langle 2a_1,\ 2a_2,\ 2a_3 \right\rangle \cdot \left\langle -r\sin\varphi,\ 0,\ r\cos\varphi \right\rangle\, dS$$

$$= \int_0^{2\pi} \int_0^1 \left(-2a_1 r\sin\varphi + 2a_3 r\cos\varphi\right) drdt$$

$$= 2\int_0^{2\pi} dt \int_0^1 \left(a_3 \cos\varphi - a_1 \sin\varphi\right) r\, dr$$

$$= (2\pi)\left(a_3 \cos\varphi - a_1 \sin\varphi\right) \left(r^2 \Big|_0^1\right.$$

$$\boxed{= 2\pi\left(a_3 \cos\varphi - a_1 \sin\varphi\right)}$$

When $\vec{a}$ points in the direection of the normal to S their cross-product is zero.

$$\left\langle a_1,\, a_2,\, a_3 \right\rangle \times \left\langle -r\sin\varphi,\ 0,\ r\cos\varphi \right\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ -r\sin\varphi & 0 & r\cos\varphi \end{vmatrix}$$

$$= \left\langle ra_2 \cos\varphi,\ -r\left(a_3 \sin\varphi + a_1 \cos\varphi\right),\ ra_2 \sin\varphi \right\rangle = 0$$

$$\left\langle a_2 \cos\varphi,\ \left(a_3 \sin\varphi + a_1 \cos\varphi\right),\ a_2 \sin\varphi \right\rangle = 0$$

$\underline{a_2 = 0|}$ & $\underline{a_3 \cos\varphi - a_1 \sin\varphi = 0|}$

Exercise

Let R be a region in a plane that has a unit normal vector $\vec{n} = \langle a,\ b,\ c\rangle$ and boundary C. Let $\vec{F} = \langle bz,\ cx,\ ay\rangle$

a) Show that $\nabla \times \vec{F} = \vec{n}$

b) Use Stokes' Theorem to show that

$$\text{Area of } R = \oint_C \vec{F} \bullet d\vec{r}$$

c) Consider the curve C given by $\vec{r}(t) = \langle 5\sin t,\ 13\cos t,\ 12\sin t\rangle$, for $0 \le t \le 2\pi$. Prove that C lies in a plane by showing that $\vec{r} \times \vec{r}'$ is constant for all t.

d) Use part (*b*) to find the area of the region enclosed by C in part (*c*). (*Hint*: Find the unit normal vector that is consistent with the orientation of C.)

Solution

a) $\nabla \times \vec{F} = \nabla \times \langle bz,\ cx,\ ay\rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz & cx & ay \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(ay) - \frac{\partial}{\partial z}(cx),\ \frac{\partial}{\partial z}(bz) - \frac{\partial}{\partial x}(ay),\ \frac{\partial}{\partial x}(cx) - \frac{\partial}{\partial y}(bz) \right\rangle$$

$$= \langle a,\ b,\ c\rangle$$

$$\underline{= \vec{n}\,|} \qquad \checkmark$$

b) $\text{Area of } R = \iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS$

$$= \iint_S \vec{n} \bullet \vec{n}\ dS$$

$$= \iint_R |\vec{n}|^2\ dA \qquad \text{Since } |\vec{n}| = 1$$

$$= \iint_R dA$$

$$= Area\ of\ R$$

$$= \oint_C \vec{F} \bullet d\vec{r} \qquad \checkmark$$

c) $\vec{r}(t) = \langle 5\sin t,\ 13\cos t,\ 12\sin t \rangle$

$$\frac{d\vec{r}}{dt} = \langle 5\cos t,\ -13\sin t,\ 12\cos t \rangle$$

$$\vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5\sin t & 13\cos t & 12\sin t \\ 5\cos t & -13\sin t & 12\cos t \end{vmatrix}$$

$$= \left\langle 156\cos^2 t + 156\sin^2 t,\ 70\cos t\sin t - 70\cos t\sin t,\ -65\sin^2 t - 65\cos^2 t \right\rangle$$

$$= \left\langle 156\left(\cos^2 t + \sin^2 t\right),\ 0,\ -65\left(\sin^2 t + \cos^2 t\right) \right\rangle$$

$$\underline{= \langle 156,\ 0,\ -65 \rangle}$$

$\therefore\ \vec{r} \times \frac{d\vec{r}}{dt}$ is constant for all t, so that $\vec{r}$ must lie in a plane.

d) $\vec{r} \times \frac{d\vec{r}}{dt} = \langle 156,\ 0,\ -65 \rangle$

$$\left| \vec{r} \times \frac{d\vec{r}}{dt} \right| = \sqrt{156^2 + 65^2}$$

$$= \sqrt{28,561}$$

$$\underline{= 169}$$

$$\vec{n} = \frac{\vec{r} \times \vec{r}'}{\left|\vec{r} \times \vec{r}'\right|}$$

$$= \frac{1}{169}\langle 156,\ 0,\ -65 \rangle$$

$$\underline{= \left\langle \frac{12}{13}\ 0,\ -\frac{5}{13} \right\rangle}$$

$$\underline{a = \frac{12}{13},\quad b = 0,\quad c = -\frac{5}{13}}$$

$$\vec{F} = \langle bz,\ cx,\ ay \rangle$$

$$= \left\langle 12(0)\sin t,\ 5\left(-\frac{5}{13}\right)\sin t,\ 13\left(\frac{12}{13}\right)\cos t \right\rangle$$

$$= \left\langle 0,\ \frac{25}{13}\sin t,\ 12\cos t \right\rangle$$

$$\frac{d\vec{r}}{dt} = \langle 5\cos t,\ -13\sin t,\ 12\cos t \rangle$$

$$\oint_C \vec{F} \bullet d\vec{r} = \int_0^{2\pi} \left\langle 0,\ \frac{25}{13}\sin t,\ 12\cos t \right\rangle \bullet \langle 5\cos t,\ -13\sin t,\ 12\cos t \rangle\ dt$$

$$= \int_0^{2\pi} \left(25\sin^2 t + 144\cos^2 t\right)\ dt$$

$$= \int_0^{2\pi} \left(\frac{25}{2} - \frac{1}{2}\cos 2t + 72 + \frac{1}{2}\cos 2t\right)\ dt$$

$$= \int_0^{2\pi} \frac{169}{2}\ dt$$

$$= \frac{169}{2}\ t \Big|_0^{2\pi}$$

$$\underline{= 169\pi}$$

Exercise

Consider the radial vector fields $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p}$, where p is a real number and $\vec{r} = \langle x,\ y,\ z \rangle$. Let C be any circle in the xy-plane centered at the origin.

a) Evaluate a line integral to show that the field has zero circulation on C.

b) For what values of p does Stokes' Theorem apply? For those values of p, use the surface integral in Stokes' Theorem to show that the field has zero circulation on C.

Solution

a) Let $C:\quad x^2 + y^2 = a^2$

$$\vec{r}(t) = \langle a\cos t,\ a\sin t,\ 0 \rangle$$

$$\frac{d\vec{r}}{dt} = \langle -a\sin t,\ a\cos t,\ 0 \rangle$$

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$$

$$= \frac{\langle a\cos t,\ a\sin t,\ 0 \rangle}{\left|a^2\cos^2 t + a^2\sin^2 t\right|^{p/2}}$$

$$= \frac{a\langle \cos t,\ \sin t,\ 0 \rangle}{\left|a^2\right|^{p/2}}$$

$$= \frac{a\langle \cos t,\ \sin t,\ 0\rangle}{a^p}$$

$$= a^{1-p}\langle \cos t,\ \sin t,\ 0\rangle$$

$$\oint_C \vec{F}\cdot d\vec{r} = a^{1-p}\int_0^{2\pi}\langle \cos t,\ \sin t,\ 0\rangle \cdot \langle -a\sin t,\ a\cos t,\ 0\rangle\ dt$$

$$= a^{2-p}\int_0^{2\pi}(-\cos t\sin t + \sin t\cos t)\ dt$$

$$\underline{= 0}$$

b) Stokes' Theorem will apply when the vector field is defined throughout the disk of radius a, which happens only $p \le 0$.

In this case, $\nabla\times\vec{F} = a^{-p}\langle 0,\ 0,\ 0\rangle$, so that the surface integral is zero.

Exercise

Consider the vector fierld $\vec{F} = \frac{-y}{x^2+y^2}\,\hat{i} + \frac{x}{x^2+y^2}\,\hat{j} + z\,\hat{k}$

a) Show that $\nabla\times\vec{F} = \vec{0}$

b) Show that $\oint_C \vec{F}\cdot d\vec{r}$ is not zero on circle C in the xy-plane enclosing the origin.

c) Explain why Stokes' Theorem does not apply in this case.

Solution

a) $$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & z \end{vmatrix}$$

$$= \left\langle 0,\ 0,\ \frac{y^2-x^2}{\left(x^2+y^2\right)^2} + \frac{x^2-y^2}{\left(x^2+y^2\right)^2}\right\rangle$$

$$\underline{= \langle 0,\ 0,\ 0\rangle}\quad \checkmark$$

b) Let $C:\ x^2+y^2=1$

$\vec{r}(t) = \langle \cos t,\ \sin t,\ 0\rangle$

$$\frac{d\vec{r}}{dt} = \langle -\sin t,\ \cos t,\ 0 \rangle$$

$$\vec{F} = \left\langle \frac{-y}{x^2+y^2},\ \frac{x}{x^2+y^2},\ z \right\rangle$$

$$= \langle -\sin t,\ \cos t,\ 0 \rangle$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle -\sin t,\ \cos t,\ 0 \rangle \cdot \langle -\sin t,\ \cos t,\ 0 \rangle\ dt$$

$$= \int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right)\ dt$$

$$= \int_0^{2\pi} dt$$

$$\underline{= 2\pi}$$

c) The Theroem does not apply because the vector field is not defined at the origin,, which is inside the curve *C*.
The limit of the *y*-coordinate is different depending on the direction.

Exercise

Let *S* be a small circular disk of radius *R* centered at the point *P* with a unit normal vector $\vec{n}$. Let *C* be the boundary of *S*.

a) Express the average circulation of the vector field $\vec{F}$ on *S* as a surface integral of $\nabla \times \vec{F}$

b) Argue for that small *R*, the average circulation approaches $\left(\nabla \times \vec{F}\right)\Big|_P \cdot \vec{n}$ (the component of $\nabla \times \vec{F}$ in the direction of $\vec{n}$ evaluated at *P*) with the approximation improving as $R \to 0$.

Solution

a) The circumference of the disk is $2\pi R$, so the average circulation is

$$\frac{1}{2\pi R} \iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS$$

b) As *R* becomes small, because the vector field $\vec{F}$ and thus $\nabla \times \vec{F}$ are continuous.
$\nabla \times \vec{F}$ can be made arbitrarily close to $\left(\nabla \times \vec{F}\right)\Big|_P$ everywhere on *S* by taking *R* small enough.

Approximately, then

$$\left(\nabla \times \vec{F}\right) \cdot \vec{n} \approx \left(\nabla \times \vec{F}\right)\Big|_P \cdot \vec{n}$$

So that

$$\frac{1}{2\pi R}\iint\limits_S \left(\nabla\times\vec{F}\right)\bullet\vec{n}\ dS \approx \frac{1}{2\pi R}\iint\limits_S \left(\nabla\times\vec{F}\right)_P \bullet\vec{n}\ dS$$

$$= \frac{1}{2\pi R}\left(\nabla\times\vec{F}\right)_P \bullet\vec{n}\iint\limits_S 1\ dS$$

$$= \left(\nabla\times\vec{F}\right)_P \bullet\vec{n}$$

As $R \to 0$, the approximation $\nabla\times\vec{F}$ becomes better, so the value of the integral does as well.

Solution   *Section* 4.8 – Divergence Theorem

Exercise

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. $\vec{F} = \langle 2x,\ 3y,\ 4z \rangle \quad D = \left\{ (x, y, z):\ x^2 + y^2 + z^2 \le 4 \right\}$

Solution

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(4z)$$

$$= 9$$

$$\iiint_D \nabla \bullet \vec{F}\ dV = \iiint_D (9)\ dV$$

$$= 9 \cdot volume(D) \qquad x^2 + y^2 + z^2 = 4 = r^2$$

$$= 9 \cdot \frac{4\pi}{3} 2^3 \qquad volume(D) = \frac{4\pi r^3}{3}$$

$$= 96\pi$$

$$\vec{r} = \langle x,\ y,\ z \rangle \qquad R = \left\{ (\phi,\ \theta):\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi \right\}$$

$$= \langle 2\sin\phi\cos\theta,\ 2\sin\phi\sin\theta,\ 2\cos\phi \rangle$$

$$\vec{t}_\phi = \langle 2\cos\phi\cos\theta,\ 2\cos\phi\sin\theta,\ -2\sin\phi \rangle$$

$$\vec{t}_\theta = \langle -2\sin\phi\sin\theta,\ 2\sin\phi\cos\theta,\ 0 \rangle$$

$$\vec{t}_\phi \times \vec{t}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \left(4\sin^2\phi\cos\theta\right)\hat{i} + \left(4\sin^2\phi\sin\theta\right)\hat{\boldsymbol{j}} + \left(4\sin\phi\cos\phi\cos^2\theta + 4\sin\phi\cos\phi\sin^2\theta\right)\hat{\boldsymbol{k}}$$

$$= \left\langle 4\sin^2\phi\cos\theta,\ 4\sin^2\phi\sin\theta,\ 4\sin^2\phi\cos\phi \right\rangle$$

$$\vec{F} = \langle 2x,\ 3y,\ 4z \rangle$$

$$= \langle 2(2\sin\phi\cos\theta),\ 3(2\sin\phi\sin\theta),\ 4(2\cos\phi) \rangle$$

$$\vec{F} \bullet \left(\vec{t}_\phi \times \vec{t}_\theta\right) = \langle 4\sin\phi\cos\theta,\ 6\sin\phi\sin\theta,\ 8\cos\phi \rangle \bullet \left\langle 4\sin^2\phi\cos\theta,\ 4\sin^2\phi\sin\theta,\ 4\sin^2\phi\cos\phi \right\rangle$$

$$= 16\sin^3\phi\cos^2\theta + 24\sin^3\phi\sin^2\theta + 32\sin\phi\cos^2\phi$$

$$\iint_S \vec{F}\bullet\vec{n}\ dS = \iint_R \nabla\vec{F}\bullet\left(\vec{t}_\phi\times\vec{t}_\theta\right)\ dA$$

$$= 8\int_0^{2\pi}\int_0^{\pi}\left(2\cos^2\theta\sin^3\phi + 3\sin^2\theta\sin^3\phi + 4\sin\phi\cos^2\phi\right)\ d\phi d\theta$$

$$= 8\int_0^{2\pi}\int_0^{\pi}\left(\left(2\cos^2\theta + 3\sin^2\theta\right)\sin^2\phi + 4\cos^2\phi\right)\sin\phi\ d\phi d\theta$$

$$= -8\int_0^{2\pi}\int_0^{\pi}\left[\left(2\cos^2\theta + 3\sin^2\theta\right)\left(1-\cos^2\phi\right) + 4\cos^2\phi\right]\ d\left(\cos\phi\right) d\theta$$

$$= -8\int_0^{2\pi}\left[\left(2\cos^2\theta + 3\left(1-\cos^2\theta\right)\right)\left(\cos\phi - \tfrac{1}{3}\cos^3\phi\right) + \tfrac{4}{3}\cos^3\phi\right]_0^{\pi}\ d\theta$$

$$= -8(2)\int_0^{2\pi}\left[\left(3-\cos^2\theta\right)\left(-\tfrac{2}{3}\right) - \tfrac{4}{3}\right]\ d\theta$$

$$= -16\int_0^{2\pi}\left[\tfrac{2}{3}\left(\frac{1+\cos 2\theta}{2}\right) - \tfrac{10}{3}\right]\ d\theta$$

$$= -\tfrac{16}{3}\int_0^{2\pi}\left(1+\cos 2\theta - 10\right)\ d\theta$$

$$= -\tfrac{16}{3}\left(\tfrac{1}{2}\sin 2\theta - 9\theta\right.\Big|_0^{2\pi}$$

$$= -\tfrac{16}{3}\left(-18\pi\right)$$

$$\underline{= 96\pi}\ \Big|$$

Exercise

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. $\vec{F} = \langle -x,\ -y,\ -z\rangle \quad D = \left\{(x,y,z):\ |x|\le 1,\ |y|\le 1,\ |z|\le 1\right\}$

Solution

$$\nabla\bullet\vec{F} = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(-z)$$

$$\underline{= -3}\ \Big|$$

$$\iiint_D \nabla\bullet\vec{F}\ dV = \iiint_D (-3)\ dV$$

$$= -3 \cdot volume(D) \qquad volume(D) = 2 \cdot 2 \cdot 2 \quad (cube\ of\ distance\ between\ -1\ \&\ 1 = 2)$$

$$= -3 \cdot (2)^3$$

$$\underline{= -24}$$

Since the surface has a form of cube, therefore we have 6 surfaces

$$S_1 : x = -1 \quad \rightarrow \quad \vec{n}_1 = \langle -1,\ 0,\ 0 \rangle$$

$$\vec{F} \bullet \vec{n}_1 = \langle 1,\ -y,\ -z \rangle \bullet \langle -1,\ 0,\ 0 \rangle$$

$$\underline{= -1}$$

$$S_2 : x = 1 \quad \rightarrow \quad \vec{n}_2 = \langle 1,\ 0,\ 0 \rangle$$

$$\vec{F} \bullet \vec{n}_2 = \langle -1,\ -y,\ -z \rangle \bullet \langle 1,\ 0,\ 0 \rangle$$

$$\underline{= -1}$$

$$S_3 : y = -1 \quad \rightarrow \quad \vec{n}_3 = \langle 0,\ -1,\ 0 \rangle$$

$$\vec{F} \bullet \vec{n}_3 = \langle -x,\ 1,\ -z \rangle \bullet \langle 0,\ -1,\ 0 \rangle$$

$$\underline{= -1}$$

$$S_4 : y = 1 \quad \rightarrow \quad \vec{n}_4 = \langle 0,\ 1,\ 0 \rangle$$

$$\vec{F} \bullet \vec{n}_4 = \langle -x,\ -1,\ -z \rangle \bullet \langle 0,\ 1,\ 0 \rangle$$

$$\underline{= -1}$$

$$S_5 : z = -1 \quad \rightarrow \quad \vec{n}_5 = \langle 0,\ 0,\ -1 \rangle$$

$$\vec{F} \bullet \vec{n}_5 = \langle -x,\ -y,\ 1 \rangle \bullet \langle 0,\ 0,\ -1 \rangle$$

$$\underline{= -1}$$

$$S_6 : z = 1 \quad \rightarrow \quad \vec{n}_6 = \langle 0,\ 0,\ 1 \rangle$$

$$\vec{F} \bullet \vec{n}_6 = \langle -x,\ -y,\ -1 \rangle \bullet \langle 0,\ 0,\ 1 \rangle$$

$$\underline{= -1}$$

$$\iint_S \vec{F} \bullet \vec{n}\ dS = \sum_{k=1}^{6} \iint_{S_k} \vec{F} \bullet \vec{n}_k\ dS$$

$$= 6 \int_{-1}^{1} dz \int_{-1}^{1} dy \int_{-1}^{0} (-1)\ dx$$

$$= 6\ \left(z \Big|_{-1}^{1}\right) \left(y \Big|_{-1}^{1}\right) \left(-x \Big|_{-1}^{0}\right)$$

$$= 6(2)(2)(-1)$$

$$\underline{= -24}$$

Exercise

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. $\vec{F} = \langle z - y,\ x,\ -x \rangle \quad D = \left\{ (x, y, z):\ \frac{x^2}{4} + \frac{y^2}{8} + \frac{z^2}{12} \le 1 \right\}$

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(z - y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(-x)$$

$$= 0$$

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iiint_D (0)\, dV$$

$$= 0$$

$$\frac{x^2}{4} + \frac{y^2}{8} + \frac{z^2}{12} = 1 \quad \to \quad x^2 = 4,\ y^2 = 8,\ z^2 = 12$$

$$\vec{t} = \left\langle 2\sin u \cos v,\ 2\sqrt{2}\, \sin u \sin v,\ 2\sqrt{3}\, \cos u \right\rangle$$

$$\vec{t}_u = \left\langle 2\cos u \cos v,\ 2\sqrt{2}\, \cos u \sin v,\ -2\sqrt{3}\, \sin u \right\rangle$$

$$\vec{t}_v = \left\langle -2\sin u \sin v,\ 2\sqrt{2}\, \sin u \cos v,\ 0 \right\rangle$$

$$\vec{n} = \vec{t}_u \times \vec{t}_v$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos u \cos v & 2\sqrt{2}\, \cos u \sin v & -2\sqrt{3}\, \sin u \\ -2\sin u \sin v & 2\sqrt{2}\, \sin u \cos v & 0 \end{vmatrix}$$

$$= \left\langle 4\sqrt{6}\, \sin^2 u \cos v,\ 4\sqrt{3}\, \sin^2 u \sin v,\ 4\sqrt{2}\, \sin u \cos u \cos^2 v + 4\sqrt{2}\, \sin u \cos u \sin^2 v \right\rangle$$

$$= \left\langle 4\sqrt{6}\, \sin^2 u \cos v,\ 4\sqrt{3}\, \sin^2 u \sin v,\ 4\sqrt{2}\, \sin u \cos u \right\rangle$$

$$\vec{F} = \langle z - y,\ x,\ -x \rangle$$

$$= \left\langle 2\sqrt{3} \cos u - 2\sqrt{2} \sin u \sin v,\ 2\sin u \cos v,\ -2\sin u \cos v \right\rangle$$

$$\vec{F} \cdot \vec{n} = \left\langle 2\sqrt{3} \cos u - 2\sqrt{2} \sin u \sin v,\ 2\sin u \cos v,\ -2\sin u \cos v \right\rangle$$

$$\cdot \left\langle 4\sqrt{6}\, \sin^2 u \cos v,\ 4\sqrt{3}\, \sin^2 u \sin v,\ 4\sqrt{2}\, \sin u \cos u \right\rangle$$

$$= 24\sqrt{2} \cos u \sin^2 u \cos v - 16\sqrt{3} \sin^3 u \cos v \sin v + 8\sqrt{3} \sin^3 u \sin v \cos v - 8\sqrt{2} \sin^2 u \cos u \cos v$$

$$= 16\sqrt{2} \cos u \sin^2 u \cos v - 8\sqrt{3} \sin^3 u \sin v \cos v$$

$$\iint_S \vec{F}\cdot\vec{n}\ dS = \int_0^{2\pi}\int_0^{\pi}\left(16\sqrt{2}\cos u\sin^2 u\cos v - 8\sqrt{3}\sin^3 u\sin v\cos v\right)\ dudv$$

$$= 8\left[\int_0^{2\pi}\int_0^{\pi}\left(2\sqrt{2}\sin^2 u\cos v\right)\ d(\sin u) + \int_0^{\pi}\sqrt{3}\left(1-\cos^2 u\right)\sin v\cos v\ d(\cos u)\right]\ dv$$

$$= 8\left[\int_0^{2\pi}\frac{2\sqrt{2}}{3}\cos v\left(\sin^2 u\ \Big|_0^{\pi}\ +\ \sqrt{3}\sin v\cos v\left(\cos u - \frac{1}{3}\cos^3 u\ \Big|_0^{\pi}\right]\ dv\right.\right.$$

$$= 8\int_0^{2\pi}\left(\frac{2\sqrt{2}}{3}\cos v(0) + \sqrt{3}\sin v\cos v\left(-2+\frac{2}{3}\right)\right)\ dv$$

$$= -\frac{64\sqrt{3}}{3}\int_0^{2\pi}\sin v\ d(\sin v)$$

$$= -\frac{64\sqrt{3}}{3}\ \left(\frac{1}{2}\sin v\ \Big|_0^{2\pi}\right.$$

$$\underline{= 0\ |}$$

Exercise

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. $\vec{F} = \left\langle x^2,\ y^2,\ z^2\right\rangle \quad D = \left\{(x, y, z):\ |x| \le 1,\ |y| \le 2,\ |z| \le 3\right\}$

Solution

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}\left(y^2\right) + \frac{\partial}{\partial z}\left(z^2\right)$$

$$\underline{= 2x + 2y + 2z\ |}$$

$$\iiint_D \nabla\cdot\vec{F}\ dV = 2\int_{-3}^{3}\int_{-2}^{2}\int_{-1}^{1}(x+y+z)\ dxdydz$$

$$= 2\int_{-3}^{3}\int_{-2}^{2}\int_{-1}^{1}(x+y+z)\ dxdydz$$

$$= 2\int_{-3}^{3}\int_{-2}^{2}\left(\frac{1}{2}x^2 + yx + zx\right)_{-1}^{1}\ dydz$$

$$= 2\int_{-3}^{3}\int_{-2}^{2}(2y+2z)\ dydz$$

$$= 2 \int_{-3}^{3} \left(y^2 + 2zy \,\Big|_{-2}^{2} \right) dz$$

$$= 2 \int_{-3}^{3} (8z)\, dz$$

$$= 2 \left(4z^2 \,\Big|_{-3}^{3} \right)$$

$$\underline{= 0}$$

$S_1 : x = -1 \quad \to \quad \vec{n}_1 = \langle -1,\ 0,\ 0 \rangle$

$$\vec{F} \cdot \vec{n}_1 = \left\langle x^2,\ y^2,\ z^2 \right\rangle \cdot \langle -1,\ 0,\ 0 \rangle$$

$$= -x^2 \,\Big|_{x=-1}$$

$$\underline{= -1}$$

$S_2 : x = 1 \quad \to \quad \vec{n}_2 = \langle 1,\ 0,\ 0 \rangle$

$$\vec{F} \cdot \vec{n}_2 = \left\langle x^2,\ y^2,\ z^2 \right\rangle \cdot \langle 1,\ 0,\ 0 \rangle$$

$$= x^2 \,\Big|_{x=1}$$

$$\underline{= 1}$$

$S_3 : y = -2 \quad \to \quad \vec{n}_3 = \langle 0,\ -1,\ 0 \rangle$

$$\vec{F} \cdot \vec{n}_3 = \left\langle x^2,\ y^2,\ z^2 \right\rangle \cdot \langle 0,\ -1,\ 0 \rangle$$

$$= -y^2 \,\Big|_{y=-2}$$

$$\underline{= -4}$$

$S_4 : y = 2 \quad \to \quad \vec{n}_4 = \langle 0,\ 1,\ 0 \rangle$

$$\vec{F} \cdot \vec{n}_4 = \left\langle x^2,\ y^2,\ z^2 \right\rangle \cdot \langle 0,\ 1,\ 0 \rangle$$

$$= y^2 \,\Big|_{y=2}$$

$$\underline{= 4}$$

$S_5 : z = -3 \quad \to \quad \vec{n}_5 = \langle 0,\ 0,\ -1 \rangle$

$$\vec{F} \cdot \vec{n}_5 = \left\langle x^2,\ y^2,\ z^2 \right\rangle \cdot \langle 0,\ 0,\ -1 \rangle$$

$$= -z^2 \,\Big|_{z=-3}$$

$$\underline{= -9}$$

$$S_6 : z = 3 \quad \rightarrow \quad \vec{n}_6 = \langle 0,\ 0,\ 1 \rangle$$

$$\begin{aligned}
\vec{F} \bullet \vec{n}_6 &= \left\langle x^2,\ y^2,\ z^2 \right\rangle \bullet \langle 0,\ 0,\ 1 \rangle \\
&= z^2 \Big|_{z=3} \\
&= 9
\end{aligned}$$

$$\begin{aligned}
\iint_S \vec{F} \bullet \vec{n}\ dS &= \sum_{k=1}^{6} \iint_{S_k} \vec{F} \bullet \vec{n}_k\ dS \\
&= \iint_S (-1+1-4+4-9+9)\,dS \\
&= \iint_S (0)\,dS \\
&= 0
\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the outward flux of the vector field $\vec{F} = \langle -x,\ x-y,\ x-z \rangle$ across S is the surface of the cube cut from the first octant by the planes $x=1,\ y=1,\ and\ z=1$.

Solution

$$\begin{aligned}
\nabla \bullet \vec{F} &= \nabla \bullet \langle -x,\ x-y,\ x-z \rangle \\
&= \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(x-y) + \frac{\partial}{\partial z}(x-z) \\
&= -1-1-1 \\
&= -3
\end{aligned}$$

$$\begin{aligned}
\iint_S \vec{F} \bullet \vec{n}\ dS &= \iiint_D \nabla \bullet \vec{F}\ dV \\
&= -3 \times (Volume\ of\ the\ cube) \\
&= -3 \times (1) \\
&= -3
\end{aligned}$$

Exercise

Use the Divergence Theorem to compute the outward flux of the vector field $\vec{F} = \frac{1}{3}\left\langle x^3,\ y^3,\ z^3 \right\rangle$ across S is the sphere $\left\{(x,\ y,\ z):\ x^2 + y^2 + z^2 = 9\right\}$

Solution

$$\nabla \cdot \vec{F} = \frac{1}{3}\nabla \cdot \left\langle x^3,\ y^3,\ z^3 \right\rangle$$

$$= \frac{1}{3}\left(\frac{\partial}{\partial x}\left(x^3\right) + \frac{\partial}{\partial y}\left(y^3\right) + \frac{\partial}{\partial z}\left(z^3\right) \right)$$

$$= \frac{1}{3}\left(3x^2 + 3y^2 + 3z^2\right)$$

$$= x^2 + y^2 + z^2$$

$$\iint_S \vec{F} \cdot \vec{n}\ dS = \iiint_D \left(x^2 + y^2 + z^2\right)\ dV$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{3} \left(r^2\right) r^2 \sin u\ drdudv$$

$$= \int_0^{2\pi} dv \int_0^{\pi} \sin u\ du \int_0^{3} r^4\ dr$$

$$= (2\pi)\ \left(-\cos u \Big|_0^{\pi}\right.\ \left.\left(\frac{1}{5}r^5\right.\right|_0^3$$

$$= (2\pi)(2)\left(\frac{243}{5}\right)$$

$$= \frac{972\pi}{5}$$

Exercise

Use the Divergence Theorem to compute the outward flux of the vector field $\vec{F} = \frac{\vec{r}}{|\vec{r}|^3} = \frac{\langle x,\ y,\ z\rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ across D is the region between two spheres of radius 1 and 2 centered at $(5,\ 5,\ 5)$

Solution

$$\vec{F} = \frac{\langle x,\ y,\ z\rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$\frac{\partial}{\partial x}\vec{F} = \frac{x^2+y^2+z^2-3x^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$= \frac{-2x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial y}\vec{F} = \frac{x^2+y^2+z^2-3y^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$= \frac{x^2-2y^2+z^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial z}\vec{F} = \frac{x^2+y^2+z^2-3z^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$= \frac{x^2+y^2-2z^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\nabla \bullet \vec{F} = \frac{1}{\left(x^2+y^2+z^2\right)^{5/2}}\left(-2x^2+y^2+z^2+x^2-2y^2+z^2+x^2+y^2-2z^2\right)$$

$$= 0$$

So, the flux is zero across any surface that bounds a region where $\vec{F}$ is defined and differentiable; the given region does not include zero.

Thus, the net outward flux is zero.

Exercise

Use the Divergence Theorem to compute the outward flux of the vector field $\vec{F} = \left\langle x^2,\ y^2,\ z^2\right\rangle$; S is the cylinder $\left\{(x,y,z):\ x^2+y^2=4,\ 0\le z\le 8\right\}$

Solution

$$\nabla \bullet \vec{F} = \nabla \bullet \left\langle x^2,\ y^2,\ z^2\right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^2\right)+\frac{\partial}{\partial y}\left(y^2\right)+\frac{\partial}{\partial z}\left(z^2\right)$$

$$= 2x+2y+2z$$

$$x^2+y^2=4 \to\ 0\le r\le 2$$

$0 \le z \le 8, \quad 0 \le \theta \le 2\pi$

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

$$\iint_S \vec{F} \cdot \vec{n}\ dS = \iiint_D (2x + 2y + 2z)\ dV$$

$$= 2\int_0^8 \int_0^{2\pi} \int_0^2 (r\cos\theta + r\sin\theta + z) r\ dr d\theta dz$$

$$= 2\int_0^8 \int_0^{2\pi} \int_0^2 \left((\cos\theta + \sin\theta) r^2 + zr\right)\ dr d\theta dz$$

$$= 2\int_0^8 \int_0^{2\pi} \left(\tfrac{1}{3}(\cos\theta + \sin\theta) r^3 + \tfrac{1}{2} zr^2 \ \Big|_0^2\right) d\theta dz$$

$$= 2\int_0^8 \int_0^{2\pi} \left(\tfrac{8}{3}(\cos\theta + \sin\theta) + 2z\right) d\theta dz$$

$$= 2\int_0^8 \left(\tfrac{8}{3}(\sin\theta - \cos\theta) + 2z\theta \ \Big|_0^{2\pi}\right) dz$$

$$= 2\int_0^8 \left(-\tfrac{8}{3} + 4\pi z + \tfrac{8}{3}\right) dz$$

$$= 4\pi \int_0^8 (2z)\, dz$$

$$= 4\pi\ z^2 \Big|_0^8$$

$$\underline{= 256\pi}$$

Exercise

Find the net outward flux of the field $\vec{F} = \langle 2z - y,\ x,\ -2x \rangle$ across the sphere of radius 1 centered at the origin.

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2z - y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(-2x)$$

$$\underline{= 0}$$

So, by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla \cdot \vec{F}$ is zero.

Exercise

Find the net outward flux of the field $\vec{F} = \langle bz - cy,\ cx - az,\ ay - bx \rangle$ across any smooth closed surface $\mathbb{R}^3$, where *a, b*, and *c* are constants.

Solution

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(bz - cy) + \frac{\partial}{\partial y}(cx - az) + \frac{\partial}{\partial z}(ay - bx)$$
$$= 0$$

So, by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla \bullet \vec{F}$ is zero.

Exercise

Find the net outward flux of the field $\vec{F} = \langle z - y,\ x - z,\ y - x \rangle$ across the boundary of the cube $\{(x,\ y,\ z):\ |x| \le 1,\ \ |y| \le 1\ \ |z| \le 1\}$

Solution

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(z - y) + \frac{\partial}{\partial y}(x - z) + \frac{\partial}{\partial z}(y - x)$$
$$= 0$$

So, by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla \bullet \vec{F}$ is zero.

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*. $\quad \vec{F} = \langle x,\ -2y,\ 3z \rangle$; *S* is the sphere $\{(x, y, z):\ x^2 + y^2 + z^2 = 6\}$

Solution

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(3z)$$
$$= 1 - 2 + 3$$
$$= 2$$

The sphere has a radius $\sqrt{6}$, therefore the volume of the sphere is $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi 6\sqrt{6} = 8\pi\sqrt{6}$

$$\iiint\limits_D \nabla \bullet \vec{F}\ dV = 2 \cdot (\textit{Volume of sphere})$$
$$= 2\left(8\pi\sqrt{6}\right)$$
$$= 16\pi\sqrt{6}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

$\vec{F} = \left\langle x^2,\ 2xz,\ y^2 \right\rangle$; *S* is surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}(2xz) + \frac{\partial}{\partial z}\left(y^2\right)$$

$$\underline{= 2x}$$

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iiint_D 2x\, dV$$

$$= \int_0^1 \int_0^1 \int_0^1 (2x)\, dxdydz$$

$$= \int_0^1 dz \int_0^1 dy \quad \left(x^2 \Big|_0^1\right.$$

$$= z\Big|_0^1 \ y\Big|_0^1 \ (1)$$

$$\underline{= 1}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

$\vec{F} = \langle x,\ 2y,\ z \rangle$; *S* is boundary of the tetrahedron in the first octant formed by the plane $x + y + z = 1$

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(z)$$

$$\underline{= 4}$$

So, by the Divergence Theorem, the net outward flux is 4 times the volume of the tetrahedron.

Volume of the tetrahedron $= \frac{1}{3}(\textit{area of the base})(\textit{height})$

$$\textit{Area of the base} = \frac{1}{2}(x)(y)$$

$$= \frac{1}{2}(1)(1)$$

$$\boxed{=\frac{1}{2}}$$

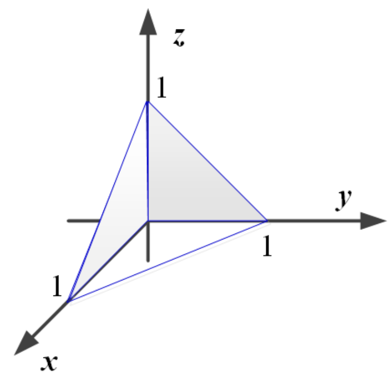
$$V=\frac{1}{3}(\text{area of the base})(\text{height})$$

$$=\frac{1}{3}\left(\frac{1}{2}\right)(1)$$

$$=\frac{1}{6}$$

$$\iiint\limits_D \nabla\cdot\vec{F}\,dV=4\left(\frac{1}{6}\right)$$

$$\boxed{=\frac{2}{3}}$$



Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

$\vec{F}=\left\langle x^2,\ y^2,\ z^2\right\rangle$; *S* is the sphere $\left\{(x,y,z):\ x^2+y^2+z^2=25\right\}$

Solution

$$\nabla\cdot\vec{F}=\frac{\partial}{\partial x}\left(x^2\right)+\frac{\partial}{\partial y}\left(y^2\right)+\frac{\partial}{\partial z}\left(z^2\right)$$

$$\boxed{=2(x+y+z)}$$

So, by the Divergence Theorem, the net outward flux is

$$\iiint\limits_D \nabla\cdot\vec{F}\,dV=2\iiint\limits_D (x+y+z)\,dV$$

$$=2\int_0^{2\pi}\int_0^{\pi}\int_0^{5} 5r(\sin\varphi\cos\theta+\sin\varphi\sin\theta+\cos\varphi)\,drd\varphi d\theta$$

$$=5\left(r^2\,\Big|_0^5\right.\int_0^{2\pi}\int_0^{\pi}(\sin\varphi\cos\theta+\sin\varphi\sin\theta+\cos\varphi)\,d\varphi d\theta$$

$$=125\int_0^{2\pi}\left(-\cos\varphi\cos\theta-\cos\varphi\sin\theta+\sin\varphi\,\Big|_0^{\pi}\right.d\theta$$

$$=125\int_0^{2\pi}2(\cos\theta+\sin\theta)\,d\theta$$

$$= 250\ \left(\sin\theta - \cos\theta\ \Big|_0^{2\pi}\right.$$

$$= 250\ (0)$$

$$\underline{= 0}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields across the given surface *S or D*.

$\vec{F} = \left\langle y - 2x,\ x^3 - \text{y},\ y^2 - z \right\rangle$; *S* is the sphere $\left\{(x, y, z):\ x^2 + y^2 + z^2 = 4\right\}$

Solution

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(y - 2x) + \frac{\partial}{\partial y}\left(x^3 - y\right) + \frac{\partial}{\partial z}\left(y^2 - z\right)$$

$$= -2 - 1 - 1$$

$$\underline{= -4}$$

The net outward flux:

$$\iiint\limits_D \nabla \bullet \vec{F}\ dV = -4\iiint\limits_D dV$$

$$= -4 \times (\textit{volume of the sphere})$$

$$= -4 \times \left(\frac{4\pi}{3}2^2\right)$$

$$\underline{= -\frac{128}{3}\pi}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

$\vec{F} = \langle x,\ y,\ z\rangle$; *S* is the surface of the paraboloid $z = 4 - x^2 - y^2$, for $z \ge 0$, plus its base in the *xy*-plane

Solution

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$\underline{= 3}$$

So, by the Divergence Theorem, the net outward flux is

$$\iiint\limits_D \nabla \cdot \vec{F}\, dV = \int_0^{2\pi} \int_0^2 (3) rz\; dr d\theta$$

$$= 3\int_0^{2\pi} d\theta \int_0^2 r\left(4-r^2\right) dr$$

$$= 3(2\pi)\left(2r^2 - \frac{1}{4}r^4\right)\Big|_0^2$$

$$= 6\pi(8-4)$$

$$= 24\pi$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

$\vec{F} = \langle x,\ y,\ z\rangle$; *S* is the surface of the cone $z^2 = x^2 + y^2$, for $0 \le z \le 4$, plus its top surface in the plane $z = 4$

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 3$$

Volume of a cone $= \frac{1}{3}(\textit{area of the base})(\textit{height})$

$$= \frac{1}{3}\left(\pi r^2\right)(4)$$

$$= \frac{4\pi}{3}(16)$$

$$= \frac{64\pi}{3}$$

So, by the Divergence Theorem, the net outward flux is

$$\iiint\limits_D \nabla \cdot \vec{F}\, dV = (3)\left(\frac{64\pi}{3}\right)$$

$$= 64\pi$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface D.

$\vec{F} = \langle z-x,\ x-y,\ 2y-z \rangle$; D is the region between the spheres of radius 2 and 4 centered at origin.

Solution

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(z-x) + \frac{\partial}{\partial y}(x-y) + \frac{\partial}{\partial z}(2y-z)$$

$$= -3$$

$$\text{Volume between 2 spheres } = \frac{4}{3}\pi\left(4^3 - 2^3\right)$$

$$= \frac{224}{3}\pi$$

$$\iiint_D \nabla \bullet \vec{F}\ dV = (-3)\left(\frac{224\pi}{3}\right)$$

$$= -224\pi$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface D.

$\vec{F} = \vec{r}\,|\vec{r}| = \langle x,\ y,\ z \rangle \sqrt{x^2+y^2+z^2}$; D is the region between the spheres of radius 1 and 2 centered at origin.

Solution

$$\left(U^n V^m\right)' = U^{n-1}V^{m-1}\left(nU'V + mUV'\right)$$

$$\frac{\partial}{\partial x}\left(x\left(x^2+y^2+z^2\right)^{1/2}\right) = \left(x^2+y^2+z^2\right)^{-1/2}\left[x^2+y^2+z^2+\frac{1}{2}(2x)(x)\right]$$

$$= \left(2x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{-1/2}$$

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}\left(x\left(x^2+y^2+z^2\right)^{1/2}\right) + \frac{\partial}{\partial y}\left(y\left(x^2+y^2+z^2\right)^{1/2}\right) + \frac{\partial}{\partial z}\left(z\left(x^2+y^2+z^2\right)^{1/2}\right)$$

$$= \left(x^2+y^2+z^2\right)^{-1/2}\left(2x^2+y^2+z^2+x^2+2y^2+z^2+x^2+y^2+2z^2\right)$$

$$= 4\left(x^2+y^2+z^2\right)^{-1/2}\left(x^2+y^2+z^2\right)$$

$$= 4\sqrt{x^2+y^2+z^2}$$

$$= 4|\vec{r}|$$

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iiint_D 4\sqrt{x^2+y^2+z^2}\; dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{r} (4\rho)\rho^2 \sin\varphi\; d\rho\, d\varphi\, d\theta$$

$$= 4\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \int_0^{r} \rho^3\, d\rho$$

$$= 4(2\pi)\;\left(-\cos\varphi\;\Big|_0^1 \quad \left(\tfrac{1}{4}\rho^4\;\Big|_0^r\right.\right.$$

$$= 4(2\pi)(2)\left(\tfrac{1}{4}r^4\right)$$

$$= 4\pi r^4$$

$$= 4\pi\left(2^4 - 1^4\right)$$

$$= 60\pi$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface D.

$\vec{F} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle x,\, y,\, z\rangle}{\sqrt{x^2+y^2+z^2}}$; D is the region between the spheres of radius 1 and 2 centered at origin.

Solution

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = \frac{x^2+y^2+z^2-x^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$= \frac{y^2+z^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) = \frac{x^2+y^2+z^2-y^2}{x^2+y^2+z^2}$$

$$= \frac{x^2+z^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)=\frac{x^2+y^2+z^2-z^2}{x^2+y^2+z^2}$$

$$=\frac{x^2+y^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\nabla\bullet\vec{F}=\frac{y^2+z^2+x^2+z^2+x^2+y^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$=\frac{2\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$=\frac{2}{\sqrt{x^2+y^2+z^2}}$$

$$\boxed{=\frac{2}{|\vec{r}|}}$$

$$\iiint_D \nabla\bullet\vec{F}\,dV=2\iiint_D \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV$$

$$=2\int_0^{2\pi}\int_0^{\pi}\int_0^{r}\left(\frac{1}{\rho}\right)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$=2\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi\,d\varphi\int_0^{r}\rho\,d\rho$$

$$=2(2\pi)\left(-\cos\varphi\Big|_0^1\right)\left(\frac{1}{2}\rho^2\Big|_0^r\right)$$

$$=2(2\pi)(2)\left(\frac{1}{2}r^2\right)$$

$$\boxed{=4\pi r^2}$$

D is the region between the spheres of radius 1 and 2 centered at origin

$$=4\pi\left(2^2-1^2\right)$$

$$\boxed{=12\pi}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *D*.

$\vec{F} = \langle z - y,\ x - z,\ 2y - x \rangle;\ \ D = \{(x, y, z):\ 1 \le |x| \le 3,\ 1 \le |y| \le 3,\ 1 \le |z| \le 3\}$ is the region between two cubes

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(z - y) + \frac{\partial}{\partial y}(x - z) + \frac{\partial}{\partial z}(2y - x)$$

$$= 0$$

Therefore, by the Divergence Theorem, the net outward flux is ***zero***.

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields across the given surface *S or D*.

$\vec{F} = \langle y + z,\ x + z,\ x + y \rangle$; *S* consists of the faces of the cube $\{(x,\ y,\ z):\ \ |x| \le 1,\ \ |y| \le 1\ \ |z| \le 1\}$

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(y + z) + \frac{\partial}{\partial y}(x + z) + \frac{\partial}{\partial z}(x + y)$$

$$= 0$$

The net outward flux:

$$\iiint_D \nabla \cdot \vec{F}\ dV = \iiint_D (0)\ dV$$

$$= 0$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields

$\vec{F} = \left\langle x^2,\ -y^2,\ z^2 \right\rangle$; *D* is the region in the first octant between the planes $z = 4 - x - y$ and $z = 2 - x - y$

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}\left(-y^2\right) + \frac{\partial}{\partial z}\left(z^2\right)$$

$$= 2x - 2y + 2z$$

$$= 2(x - y + z)$$

Top plane	*Bottom plane*
$0 \le z \le 4-x-y$	$0 \le z \le 2-x-y$
$z = 4-x-y = 0 \quad \to \quad 0 \le y \le 4-x$	$z = 2-x-y = 0 \quad \to \quad 0 \le y \le 2-x$
$y = 4-x = 0 \quad \to \quad 0 \le x \le 4$	$y = 2-x = 0 \quad \to \quad 0 \le x \le 2$

The net outward flux:

$$\iiint\limits_D \nabla \cdot \vec{F}\, dV = 2\iiint\limits_D (x-y+z)\, dV$$

$$= 2\int_0^4 \int_0^{4-x} \int_0^{4-x-y} (x-y+z)\,dzdydx \;-\; 2\int_0^2 \int_0^{2-x} \int_0^{2-x-y} (x-y+z)\,dzdydx$$

$$= 2\int_0^4 \int_0^{4-x} \left((x-y)z + \frac{1}{2}z^2 \,\Big|_0^{4-x-y} \right) dydx$$

$$-2\int_0^2 \int_0^{2-x} \left((x-y)z + \frac{1}{2}z^2 \,\Big|_0^{2-x-y} \right) dydx$$

$$= 2\int_0^4 \int_0^{4-x} \left((x-y)(4-x-y) + \frac{1}{2}(4-x-y)^2 \right) dydx$$

$$-2\int_0^2 \int_0^{2-x} \left((x-y)(2-x-y) + \frac{1}{2}(2-x-y)^2 \right) dydx$$

$$= 2\int_0^4 \int_0^{4-x} \left(4x - x^2 - 4y + y^2 + 8 - 4x - 4y + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 \right) dydx$$

$$-2\int_0^2 \int_0^{2-x} \left(2x - x^2 - 2y + y^2 + 2 - 2x + \frac{1}{2}x^2 + xy - 2y + \frac{1}{2}y^2 \right) dydx$$

$$= 2\int_0^4 \int_0^{4-x} \left(-\frac{1}{2}x^2 + 8 - 8y + xy + \frac{3}{2}y^2 \right) dydx$$

$$-2\int_0^2 \int_0^{2-x} \left(-\frac{1}{2}x^2 + 2 + xy - 4y + \frac{3}{2}y^2 \right) dydx$$

$$= 2\int_0^4 \left(-\frac{1}{2}x^2y + 8y - 4y^2 + \frac{1}{2}xy^2 + \frac{1}{2}y^3\ \Big|_0^{4-x}\ dx\right.$$

$$\left. - 2\int_0^2 \left(-\frac{1}{2}x^2y + 2y + \frac{1}{2}xy^2 - 2y^2 + \frac{1}{2}y^3\ \Big|_0^{2-x}\ dx\right.$$

$$= 2\int_0^4 \left(-2x^2 + \frac{1}{2}x^3 + 32 - 8x + 32x - 4x^2 + 8x - 4x^2 + \frac{1}{2}x^3 + 32 - 24x + 6x^2 - \frac{1}{2}x^3\right)dx$$

$$- 2\int_0^2 \left(-x^2 + \frac{1}{2}x^3 + 4 - 2x + 2x - 2x^2 + \frac{1}{2}x^3 - 8 + 8x + 2x^2 + 4 - 6x + 3x^2 - \frac{1}{2}x^3\right)dx$$

$$= 2\int_0^4 \left(8x - 4x^2 + \frac{1}{2}x^3\right)dx\ -\ 2\int_0^2 \left(\frac{1}{2}x^3 + 2x - 2x^2\right)dx$$

$$= 2\left(4x^2 - \frac{4}{3}x^3 + \frac{1}{8}x^4\ \Big|_0^4\ -\ 2\left(\frac{1}{8}x^4 + x^2 - \frac{2}{3}x^3\ \Big|_0^2\right.\right.$$

$$= 2\left(64 - \frac{256}{3} + 32\right) - 2\left(2 + 4 - \frac{16}{3}\right)$$

$$= 2\left(96 - \frac{256}{3}\right) - \frac{4}{3}$$

$$= \frac{64}{3} - \frac{4}{3}$$

$$= 20$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface D.

$\vec{F} = \langle x,\ 2y,\ 3z\rangle$; D is the region between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ for $0 \le z \le 8$

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z)$$

$$= 6$$

Volume of the sphere $x^2 + y^2 = 4$

$$V = \int_0^8 \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\ dx\ dz$$

$$= \int_0^8 dz \int_{-2}^{2} \left(y \Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \right. dx$$

$$= 16 \int_{-2}^{2} \sqrt{4-x^2}\ dx$$

$$x = 2\sin\alpha \qquad \sqrt{4-x^2} = 2\cos\alpha$$

$$dx = 2\cos\alpha\, d\alpha$$

$$\int \sqrt{4-x^2}\ dx = \int 2\cos\alpha\ (2\cos\alpha)\ d\alpha$$

$$= 4\int \cos^2\alpha\ d\alpha$$

$$= 2\int (1+\cos 2\alpha)\ d\alpha$$

$$= 2\left(\alpha + \frac{1}{2}\sin 2\alpha\right)$$

$$= 2\ (\alpha + \sin\alpha\cos\alpha)$$

$$= 2\left(\sin^{-1}\frac{x}{2} + \frac{x}{2}\frac{\sqrt{4-x^2}}{2}\right)$$

$$= 2\sin^{-1}\frac{x}{2} + \frac{1}{2}x\sqrt{4-x^2}$$

$$= 16\left(2\sin^{-1}\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2}\ \Big|_{-2}^{2}\right.$$

$$= 16\left(2\left(\frac{\pi}{2}\right) - 2\left(-\frac{\pi}{2}\right)\right)$$

$$= 32\pi \quad unit^3$$

Or

$$V_1 = z\left(\pi r^2\right)$$

$$= 8(4\pi)$$

$$= 32\pi$$

Volume of the sphere $x^2 + y^2 = 1$

$$V = \int_0^8 \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\ dx\ dz$$

$$= \int_0^8 dz \int_{-1}^{1} y \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= 8\int_{-1}^{1} 2\sqrt{1-x^2}\ dx$$

$x = \sin\alpha \qquad \sqrt{1-x^2} = \cos\alpha$

$dx = \cos\alpha\, d\alpha$

$$\int \sqrt{1-x^2}\ dx = \int \cos^2\alpha\ d\alpha$$

$$= \frac{1}{2}\int (1+\cos 2\alpha)\ d\alpha$$

$$= \frac{1}{2}\left(\alpha + \frac{1}{2}\sin 2\alpha\right)$$

$$= \frac{1}{2}(\alpha + \sin\alpha\cos\alpha)$$

$$= \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{4-x^2}$$

$$= 16\left(\frac{1}{2}\sin^{-1}x + \frac{x}{2}\sqrt{1-x^2}\right)\Big|_{-1}^{1}$$

$$= 16\left[\frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\left(-\frac{\pi}{2}\right)\right]$$

$$\underline{= 8\pi \quad unit^3}$$

Or

$$V_2 = z\left(\pi r^2\right)$$

$$= 8(\pi)$$

$$\underline{= 8\pi}$$

Therefore, the net outward flux is $6(32\pi - 8\pi)\ \underline{= 144\pi}$

Exercise

Compute the outward flux of the following vector field across the given surface

$\vec{F} = \left\langle x^2e^y\cos z,\ -4xe^y\cos z,\ 2xe^y\sin z\right\rangle$; S is the boundary of the ellipsoid $\frac{x^2}{4} + y^2 + z^2 = 1$

Solution

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}\left(x^2e^y\cos z\right) + \frac{\partial}{\partial y}\left(-4xe^y\cos z\right) + \frac{\partial}{\partial z}\left(2xe^y\sin z\right)$$

$$= 2xe^y\cos z - 4xe^y\cos z + 2xe^y\cos z$$

$$\underline{= 0}$$

Therefore, by the Divergence Theorem, the net outward flux is ***zero***.

Exercise

Compute the outward flux of the following vector field across the given surface $\vec{F} = \langle -yz,\ xz,\ 1\rangle$; S is the boundary of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{4} + z^2 = 1$

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(-yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(1)$$

$$= 0$$

Therefore, by the Divergence Theorem, the net outward flux is ***zero***.

Exercise

Compute the outward flux of the following vector field across the given surface
$\vec{F} = \langle x\sin y,\ -\cos y,\ z\sin y\rangle$; S is the boundary of the region bounded by the planes $x = 1$, $y = 0$, $y = \frac{\pi}{2}$, $z = 0$, and $z = x$

Solution

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x\sin y) + \frac{\partial}{\partial y}(-\cos y) + \frac{\partial}{\partial z}(z\sin y)$$

$$= \sin y + \sin y + \sin y$$

$$= 3\sin y$$

$$\iiint\limits_D \nabla \cdot \vec{F}\ dV = \int_0^{\pi/2}\int_0^1\int_0^x 3\sin y\ dzdxdy$$

$$= \int_0^{\pi/2} \sin y\ dy \quad \int_0^1 3z\Big|_0^x\ dx$$

$$= -\cos y\Big|_0^{\pi/2} \int_0^1 3x\ dx$$

$$= \frac{3}{2}x^2\Big|_0^1$$

$$= \frac{3}{2}$$

Exercise

Compute the outward flux of the following vector field across the given surface

$\vec{F} = \left\langle x^2 + x\sin y,\ y^2 + 2\cos y,\ z^2 + z\sin y \right\rangle$ across the surface S that is the boundary of the prism bounded by the planes $y = 1 - x,\ x = 0,\ y = 0,\ z = 0,\ z = 4$

Solution

$$\nabla \bullet \vec{F} = \nabla \bullet \left\langle x^2 + x\sin y,\ y^2 + 2\cos y,\ z^2 + z\sin y \right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^2 + x\sin y\right) + \frac{\partial}{\partial y}\left(y^2 + 2\cos y\right) + \frac{\partial}{\partial z}\left(z^2 + z\sin y\right)$$

$$= 2x + \sin y + 2y - 2\sin y + 2z + \sin y$$

$$= 2(x + y + z)$$

$$y = 1 - x,\ x = 0,\ y = 0,\ z = 0,\ z = 4$$

$$y = 1 - x = 0 \ \rightarrow\ x = 1$$

$$0 \le y \le 1 - x, \quad 0 \le x \le 1, \quad 0 \le z \le 4$$

$$\iint_S \vec{F} \bullet \vec{n}\ dS = 2\iiint_D (x + y + z)\ dV$$

$$= 2\int_0^4 \int_0^1 \int_0^{1-x} (x + y + z)\ dydxdz$$

$$= 2\int_0^4 \int_0^1 \left(xy + \frac{1}{2}y^2 + zy\ \Big|_0^{1-x}\ dxdz\right.$$

$$= 2\int_0^4 \int_0^1 \left(x - x^2 + \frac{1}{2} - x + \frac{1}{2}x^2 + z - xz\right)dxdz$$

$$= 2\int_0^4 \int_0^1 \left(-\frac{1}{2}x^2 + \frac{1}{2} + z - xz\right)dxdz$$

$$= 2\int_0^4 \left(-\frac{1}{6}x^3 + \frac{1}{2}x + zx - \frac{1}{2}zx^2\ \Big|_0^1\ dz\right.$$

$$= 2\int_0^4 \left(-\frac{1}{6} + \frac{1}{2} + z - \frac{1}{2}z\right)dz$$

$$= 2\int_0^4 \left(\frac{1}{3} + \frac{1}{2}z\right)dz$$

$$= 2\left(\frac{1}{3}z + \frac{1}{4}z^2 \,\right|_0^4$$

$$= 2\left(\frac{4}{3} + 4\right)$$

$$= \frac{32}{3}$$

Exercise

Compute the outward flux of the following vector field across the given surface $\vec{F} = \langle x,\ -2y,\ 4z\rangle$ out of the sphere S with $x^2 + y^2 + z^2 = a^2, \quad a > 0$

Solution

$$\nabla \bullet \vec{F} = \nabla \bullet \langle x,\ -2y,\ 4z\rangle$$

$$= 1 - 2 + 4$$

$$= 3$$

$$\iint_S \vec{F} \bullet \vec{n}\ dS = 3\iiint_D dV$$

$$= 3 \times (volume\ of\ a\ sphere)$$

$$= 3 \times \left(\frac{4\pi}{3}a^3\right)$$

$$= 4\pi a^3$$

Exercise

Compute the outward flux of the following vector field across the given surface $\vec{F} = \left\langle ye^z,\ x^2e^z,\ xy\right\rangle$ out of the sphere S with $x^2 + y^2 + z^2 = a^2, \quad a > 0$

Solution

$$\nabla \bullet \vec{F} = \nabla \bullet \left\langle ye^z,\ x^2e^z,\ xy\right\rangle$$

$$= \frac{\partial}{\partial x}\left(ye^z\right) + \frac{\partial}{\partial y}\left(x^2e^z\right) + \frac{\partial}{\partial z}(xy)$$

$$= 0$$

$$\iint_S \vec{F} \bullet \vec{n}\ dS = \iiint_D (0)\,dV$$

$$= 0$$

Exercise

Compute the outward flux of the following vector field across the given surface

$\vec{F} = \left\langle x^2 + y^2,\ y^2 - z^2,\ z \right\rangle$ of the sphere S with $x^2 + y^2 + z^2 = a^2, \quad a > 0$

Solution

$$\nabla \cdot \vec{F} = \nabla \cdot \left\langle x^2 + y^2,\ y^2 - z^2,\ z \right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^2 + y^2\right) + \frac{\partial}{\partial y}\left(y^2 - z^2\right) + \frac{\partial}{\partial z}(z)$$

$$= 2x + 2y + 1$$

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iiint_D (2x + 2y + 1)\, dV$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{a} (2a\sin\varphi\cos\theta + 2a\sin\varphi\sin\theta + 1)\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \int_0^{\pi} \left(2a\sin^2\varphi\cos\theta + 2a\sin^2\varphi\sin\theta + \sin\varphi\right) \rho^3 \Big|_0^a\, d\varphi\, d\theta$$

$$= \frac{a^3}{3}\int_0^{2\pi} \int_0^{\pi} \left(2a\sin^2\varphi\cos\theta + 2a\sin^2\varphi\sin\theta + \sin\varphi\right) d\varphi\, d\theta$$

$$= \frac{a^3}{3}\int_0^{2\pi} \int_0^{\pi} \left(a(1 - \cos 2\varphi)\cos\theta + a(1 - \cos 2\varphi)\sin\theta + \sin\varphi\right) d\varphi\, d\theta$$

$$= \frac{a^3}{3}\int_0^{2\pi} \int_0^{\pi} \left(a(\cos\theta + \sin\theta)(1 - \cos 2\varphi) + \sin\varphi\right) d\varphi\, d\theta$$

$$= \frac{a^3}{3}\int_0^{2\pi} \left(a(\cos\theta + \sin\theta)\left(\varphi - \frac{1}{2}\sin 2\varphi\right) - \cos\varphi \Big|_0^{\pi}\right) d\theta$$

$$= \frac{a^3}{3}\int_0^{2\pi} \left(a\pi(\cos\theta + \sin\theta) + 2\right) d\theta$$

$$= \frac{a^3}{3}\left(a\pi(\sin\theta - \cos\theta) + 2\theta\right) \Big|_0^{2\pi}$$

$$= \frac{a^3}{3}(-a\pi + 4\pi + a\pi)$$

$$= \frac{4}{3}\pi a^3$$

Exercise

Compute the outward flux of the following vector field across the given surface

$\vec{F} = \left\langle x^3,\ \ 3yz^2,\ \ 3y^2z + x^2 \right\rangle$ out of the sphere S with $x^2 + y^2 + z^2 = a^2, \quad a > 0$

Solution

$$\nabla \cdot \vec{F} = \nabla \cdot \left\langle x^3,\ \ 3yz^2,\ \ 3y^2z + x^2 \right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^3\right) + \frac{\partial}{\partial y}\left(3yz^2\right) + \frac{\partial}{\partial z}\left(3y^2z + x^2\right)$$

$$= 3x^2 + 3z^2 + 3y^2$$

$$\iiint_D \nabla \cdot \vec{F}\ dV = 3\iiint_D \left(x^2 + y^2 + z^2\right)\ dV$$

$$= 3\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\left(\rho^2\right)\rho^2 \sin\varphi\ d\rho\ d\varphi\ d\theta$$

$$= 3\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \int_0^{a} \rho^4\ d\rho$$

$$= 3(2\pi)\ \left(-\cos\varphi \Big|_0^1\right. \left(\frac{1}{5}\rho^5 \Big|_0^a\right.$$

$$= 3(2\pi)(2)\left(\frac{1}{5}a^5\right)$$

$$= \frac{12}{5}\pi a^5$$

Exercise

$\vec{F} = \left\langle 2z,\ x,\ y^2 \right\rangle$; S is the surface of the paraboloid $z = 4 - x^2 - y^2$, for $z \geq 0$, and the xy-plane.

Solution

$$\nabla \cdot \vec{F} = \nabla \cdot \left\langle 2z,\ x,\ y^2 \right\rangle$$

$$= \frac{\partial}{\partial x}(2z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}\left(y^2\right)$$

$$= 0$$

$$\iiint_D \nabla \cdot \vec{F}\ dV = \iiint_D (0)dV$$

$$= 0$$

Exercise

$\vec{F} = \left\langle x,\ y^2,\ z \right\rangle$; S is the solid region bounded by the coordinate planes and the plane $2x + 2y + z = 6$.

Solution

$$\nabla \bullet \vec{F} = \nabla \bullet \left\langle x,\ y^2,\ z \right\rangle$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}\left(y^2\right) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 2y + 1$$

$$= 2(1+y)$$

$$2x + 2y + z = 6 \quad \to \quad 0 \le z \le 6 - 2x - 2y$$

$$z = 6 - 2x - 2y = 0 \quad \to \quad 0 \le x \le 3 - y$$

$$x = 3 - y = 0 \quad \to \quad 0 \le y \le 3$$

$$\iiint\limits_D \nabla \bullet \vec{F}\ dV = 2\iiint\limits_D (1+y)\,dV$$

$$= 2\int_0^3 \int_0^{3-y} \int_0^{6-2x-2y} (1+y)\ dzdxdy$$

$$= 2\int_0^3 \int_0^{3-y} (1+y)\ \left(z \Big|_0^{6-2x-2y}\right.\ dxdy$$

$$= 2\int_0^3 \int_0^{3-y} (1+y)(6-2x-2y)\,dxdy$$

$$= 2\int_0^3 \int_0^{3-y} \left(6 - 2x + 4y - 2xy - 2y^2\right)dxdy$$

$$= 2\int_0^3 \left(6x - x^2 + 4yx - yx^2 - 2y^2x \Big|_0^{3-y}\right.\ dy$$

$$= 2\int_0^3 \left(18 - 6y - 9 + 6y - y^2 + 12y - 4y^2 - 9y + 6y^2 - y^3 - 6y^2 + 2y^3\right)dy$$

$$= 2\int_0^3 \left(9 + 3y - 5y^2 + y^3\right)dy$$

$$= 2\ \left(9y + \frac{3}{2}y^2 - \frac{5}{3}y^3 + \frac{1}{4}y^4 \Big|_0^3\right.$$

$$= 2\left(27 + \frac{27}{2} - 45 + \frac{81}{4}\right)$$

$$= 2\left(\frac{135}{4} - 18\right)$$

$$\boxed{= \frac{63}{2}}$$

Exercise

$\vec{F} = \left\langle x^2 + \sin z,\ xy + \cos z,\ e^y \right\rangle$; S is the solid region bounded by the cylinder $x^2 + y^2 = 4$, the plane $x + z = 6$, and the xy-plane.

Solution

$$\nabla \cdot \vec{F} = \nabla \cdot \left\langle x^2 + \sin z,\ xy + \cos z,\ e^y \right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^2 + \sin z\right) + \frac{\partial}{\partial y}(xy + \cos z) + \frac{\partial}{\partial z}\left(e^y\right)$$

$$= 2x + x + 0$$

$$\boxed{= 3x}$$

$x^2 + y^2 = 4 \quad \to \quad 0 \le r \le 2$

$x^2 + y^2 = 4 \quad \to \quad 0 \le \theta \le 2\pi$

$x + z = 6 \quad \to \quad 0 \le z \le 6 - r\cos\theta$

$$\iiint_D \nabla \cdot \vec{F}\ dV = 3\iiint_D (x)\,dV$$

$$= 3\int_0^{2\pi}\int_0^2\int_0^{6-r\cos\theta} (r\cos\theta)r\ dz\,dr\,d\theta$$

$$= 3\int_0^{2\pi}\int_0^2 \left(r^2\cos\theta\right)\left(z\ \Big|_0^{6-r\cos\theta}\right) dr\,d\theta$$

$$= 3\int_0^{2\pi}\int_0^2 \left(r^2\cos\theta\right)(6 - r\cos\theta)\,dr\,d\theta$$

$$= 3\int_0^{2\pi}\int_0^2 \left(6r^2\cos\theta - r^3\cos^2\theta\right)dr\,d\theta$$

$$= 3\int_0^{2\pi} \left(2r^3\cos\theta - \frac{1}{4}r^4\cos^2\theta\ \Big|_0^2\right) d\theta$$

$$= 3\int_0^{2\pi} \left(16\cos\theta - 4\cos^2\theta\right)d\theta$$

$$= 3\int_0^{2\pi} \left(16\cos\theta - 2 - 2\cos 2\theta\right) d\theta$$

$$= 3\ \left(16\sin\theta - 2\theta - \sin 2\theta\ \right|_0^{2\pi}$$

$$= 3\left(-4\pi\right)$$

$$\underline{= 12\pi\ }|$$

Exercise

Compute the outward flux of the following vector field $\vec{F} = \left\langle 2x^3,\ \ 2y^3,\ \ 2z^3 \right\rangle$ out of the sphere S with $x^2 + y^2 + z^2 = 4$

Solution

$$\nabla \bullet \vec{F} = \nabla \bullet \left\langle 2x^3,\ \ 2y^3,\ \ 2z^3 \right\rangle$$

$$= \frac{\partial}{\partial x}\left(2x^3\right) + \frac{\partial}{\partial y}\left(2y^3\right) + \frac{\partial}{\partial z}\left(2z^3\right)$$

$$\underline{= 6x^2 + 6z^2 + 6y^2\ }|$$

$$\iiint_D \nabla \bullet \vec{F}\ dV = 6\iiint_D \left(x^2 + y^2 + z^2\right)\ dV$$

$$= 6\int_0^{2\pi}\int_0^{\pi}\int_0^{2} \left(\rho^2\right)\rho^2 \sin\varphi\ d\rho\ d\varphi\ d\theta$$

$$= 6\int_0^{2\pi} d\theta \quad \int_0^{\pi} \sin\varphi d\varphi \quad \int_0^{2} \rho^4\ d\rho$$

$$= 6\left(2\pi\right)\ \left(-\cos\varphi\ \right|_0^1 \quad \left(\tfrac{1}{5}\rho^5\ \right|_0^2$$

$$= 6\left(2\pi\right)\left(2\right)\left(\frac{32}{5}\right)$$

$$\underline{= \frac{768\pi}{5}\ }|$$

Exercise

Compute the outward flux of the following vector field $\vec{F} = \langle x, \ y, \ z \rangle$ out of the sphere S with $x^2 + y^2 + z^2 = 1$

Solution

$$\nabla \cdot \vec{F} = \nabla \cdot \langle x, \ y, \ z \rangle$$

$$= 3$$

$$\iiint_D \nabla \cdot \vec{F} \, dV = 3 \iiint_D dV$$

$$= 3 \times (\text{volume of a sphere})$$

$$= 3 \times \left(\frac{4\pi}{3} \right)$$

$$= 4\pi$$

Exercise

Compute the outward flux of the following vector field $\vec{F} = \langle z, \ y, \ x \rangle$ out of the sphere S with $x^2 + y^2 + z^2 = 1$

Solution

$$\nabla \cdot \vec{F} = \nabla \cdot \langle z, \ y, \ x \rangle$$

$$= 1$$

$$\iiint_D \nabla \cdot \vec{F} \, dV = \iiint_D dV$$

$$= \text{volume of a sphere } (r = 1)$$

$$= \frac{4\pi}{3}$$

Exercise

Compute the outward flux of the following vector field $\vec{F} = \left\langle xy, \ y^2 + e^{xz^2}, \ \sin(xy) \right\rangle$; S is the solid region bounded by the cylinder $z = 1 - x^2$, the planes $y + z = 2$, $z = 0$, and $y = 0$.

Solution

$$\nabla \cdot \vec{F} = \nabla \cdot \left\langle xy, \ y^2 + e^{xz^2}, \ \sin(xy) \right\rangle$$

$$= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(y^2 + e^{xz^2}\right) + \frac{\partial}{\partial z}(\sin(xy))$$

$$= y + 2y + 0$$

$$\boxed{= 3y}$$

$y + z = 2 \quad \rightarrow \quad 0 \le y \le 2 - z$

$z = 1 - x^2 \quad \rightarrow \quad 0 \le z \le 1 - x^2$

$z = 1 - x^2 = 0 \quad \rightarrow \quad -1 \le x \le 1$

$$\iiint_D \nabla \cdot \vec{F}\, dV = 3 \iiint_D (y)\, dV$$

$$= 3 \int_{-1}^{1} \int_{0}^{1-x^2} \int_{0}^{2-z} y\, dydzdx$$

$$= \frac{3}{2} \int_{-1}^{1} \int_{0}^{1-x^2} y^2 \Big|_{0}^{2-z} dzdx$$

$$= \frac{3}{2} \int_{-1}^{1} \int_{0}^{1-x^2} (2-z)^2\, dzdx$$

$$= -\frac{3}{2} \int_{-1}^{1} \int_{0}^{1-x^2} (2-z)^2\, d(2-z)\, dx$$

$$= -\frac{1}{2} \int_{-1}^{1} (2-z)^3 \Big|_{0}^{1-x^2} dx$$

$$= -\frac{1}{2} \int_{-1}^{1} \left(\left(1+x^2\right)^3 - 8\right) dx$$

$$= -\frac{1}{2} \int_{-1}^{1} \left(-7 + 3x^2 + 3x^4 + x^6\right) dx$$

$$= -\left(-7x + x^3 + \frac{3}{5}x^5 + \frac{1}{7}x^7 \Big|_{0}^{1}\right.$$

$$= 7 - 1 - \frac{3}{5} - \frac{1}{7}$$

$$= 6 - \frac{26}{35}$$

$$\boxed{= \frac{184}{35}}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields a $\vec{F}=\left\langle x^2,\ y^2,\ z^2\right\rangle$; across the boundary of an ellipsoid $x^2+y^2+4(z-1)^2\le 4$

Solution

$$\nabla\cdot\vec{F}=\nabla\cdot\left\langle x^2,\ y^2,\ z^2\right\rangle$$

$$=\frac{\partial}{\partial x}\left(x^2\right)+\frac{\partial}{\partial y}\left(y^2\right)+\frac{\partial}{\partial z}\left(z^2\right)$$

$$=2x+2y+2z$$

$$x^2+y^2+4(z-1)^2=4$$

$$r^2=4\left(1-(z-1)^2\right)$$

$$0\le r\le 2\sqrt{1-(z-1)^2}$$

$$4(z-1)^2=4$$

$$(z-1)^2=1$$

$$0\le z\le 2$$

$$\iiint\limits_D \nabla\cdot\vec{F}\ dV=2\iiint\limits_D (x+y+z)\,dV$$

$$=2\int_0^{2\pi}\int_0^2\int_0^{2\sqrt{1-(z-1)^2}}(2\cos\theta+2\sin\theta+z)\,r\ drdzd\theta$$

$$=\int_0^{2\pi}\int_0^2(2\cos\theta+2\sin\theta+z)\ r^2\Big|_0^{2\sqrt{1-(z-1)^2}}\ dzd\theta$$

$$=4\int_0^{2\pi}\int_0^2(2\cos\theta+2\sin\theta+z)\left(-z^2+2z\right)dzd\theta$$

$$=4\int_0^{2\pi}\int_0^2\left((2\cos\theta+2\sin\theta)\left(-z^2+2z\right)-z^3+2z^2\right)\ dzd\theta$$

$$=4\int_0^{2\pi}\left((2\cos\theta+2\sin\theta)\left(-\frac{1}{3}z^3+z^2\right)-\frac{1}{4}z^4+\frac{2}{3}z^3\ \Big|_0^2\ d\theta\right.$$

$$=4\int_0^{2\pi}\left((2\cos\theta+2\sin\theta)\left(-\frac{8}{3}+4\right)-4+\frac{16}{3}\right)d\theta$$

$$= 4\int_0^{2\pi} \left(\frac{8}{6}(\cos\theta + \sin\theta) + \frac{4}{3}\right) d\theta$$

$$= 4\left(\frac{8}{6}(\sin\theta - \cos\theta) + \frac{4}{3}\theta \right|_0^{2\pi}$$

$$= 4\left(-\frac{8}{6} + \frac{8\pi}{3} + \frac{8}{6}\right)$$

$$= \frac{32\pi}{3}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields a $\vec{F} = \left\langle x^2,\ y^2,\ z^2 \right\rangle$; across the boundary of the tetrahedron $x + y + z \le 3 \quad \& \quad x,\ y,\ z \ge 0$

Solution

$$\nabla \bullet \vec{F} = \nabla \bullet \left\langle x^2,\ y^2,\ z^2 \right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}\left(y^2\right) + \frac{\partial}{\partial z}\left(z^2\right)$$

$$= 2x + 2y + 2z$$

$x + y + z = 3 \quad \to \quad 0 \le z \le 3 - y - x$

$z = 3 - y - x = 0 \quad \to \quad 0 \le y \le 3 - x$

$y = 3 - x = 0 \quad \to \quad 0 \le x \le 3$

$$\iiint_D \nabla \bullet \vec{F}\ dV = 2\iiint_D (x + y + z)\, dV$$

$$= 2\int_0^3 \int_0^{3-x} \int_0^{3-y-x} (x + y + z)\, dz\,dy\,dx$$

$$= 2\int_0^3 \int_0^{3-x} \left(xz + yz + \frac{1}{2}z^2 \right|_0^{3-y-x} dy\,dx$$

$$= 2\int_0^3 \int_0^{3-x} \left(x(3 - y - x) + y(3 - y - x) + \frac{1}{2}(3 - y - x)^2\right) dy\,dx$$

$$= 2\int_0^3 \int_0^{3-x} \left(3x - xy - x^2 + 3y - y^2 - xy + \frac{1}{2}\left(9 - 6y - 6x + y^2 + 2xy + x^2\right)\right) dy\,dx$$

$$= 2\int_0^3 \int_0^{3-x} \left(-xy - \frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{9}{2}\right) dydx$$

$$= 2\int_0^3 \left(\frac{9}{2}y - \frac{1}{2}xy^2 - \frac{1}{2}x^2 y - \frac{1}{6}y^3 \,\Big|_0^{3-x}\right) dx$$

$$= \int_0^3 \left(9(3-x) - x(3-x)^2 - x^2(3-x) - \frac{1}{3}(3-x)^3\right) dx$$

$$= \int_0^3 \left(27 - 9x - 9x + 6x^2 - x^3 - 3x^2 + x^3 - 9 + 9x - 3x^2 + \frac{1}{3}x^3\right) dx$$

$$= \int_0^3 \left(18 - 9x + \frac{1}{3}x^3\right) dx$$

$$= 18x - \frac{9}{2}x^2 + \frac{1}{12}x^4 \,\Big|_0^3$$

$$= 54 - \frac{81}{2} + \frac{27}{4}$$

$$= \frac{81}{4}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields a $\vec{F} = \left\langle x^2,\, y^2,\, z^2 \right\rangle$; across the boundary of the cylinder $x^2 + y^2 \le 2y \quad \& \quad 0 \le z \le 4$

Solution

$$\nabla \cdot \vec{F} = \nabla \cdot \left\langle x^2,\, y^2,\, z^2 \right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}\left(y^2\right) + \frac{\partial}{\partial z}\left(z^2\right)$$

$$= 2x + 2y + 2z$$

$$x^2 + y^2 = 2y$$

$$x^2 = 2y - y^2$$

$$-\sqrt{2y - y^2} \le x \le \sqrt{2y - y^2}$$

$$(y-1)^2 = 1 \quad \rightarrow \quad 0 \le y \le 2$$

$$\iiint_D \nabla \cdot \vec{F}\; dV = 2\iiint_D (x + y + z)\, dV$$

$$= 2\int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_0^4 (x+y+z)\ dzdxdy$$

$$= 2\int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \left(xz+yz+\frac{1}{2}z^2 \right|_0^4 dxdy$$

$$= 2\int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} (4x+4y+8)dxdy$$

$$= 2\int_0^2 \left(2x^2+4(y+2)x \right|_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} dy$$

$$= 4\int_0^2 \left(2y-y^2+2(y+2)\sqrt{2y-y^2}-2y+y^2+2(y+2)\sqrt{2y-y^2}\right)dy$$

$$= 16\int_0^2 (y+2)\sqrt{2y-y^2}\ dy$$

$$= 16\int_0^2 (y+2)\sqrt{1-(y-1)^2}\ dy$$

Let $u = y-1 \quad \rightarrow \quad du = dy$

$$y = u+1 \quad \rightarrow \quad \begin{cases} y=2 & \rightarrow u=1 \\ y=0 & \rightarrow u=-1 \end{cases}$$

$$= 16\int_{-1}^1 (u+3)\sqrt{1-u^2}\ dy$$

Let $u = \sin\theta \quad \rightarrow \quad du = \cos\theta d\theta$

$$\rightarrow \quad \begin{cases} u=1 & \rightarrow \theta = \frac{\pi}{2} \\ u=-1 & \rightarrow \theta = -\frac{\pi}{2} \end{cases}$$

$$= 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3+\sin\theta)\sqrt{1-\sin^2\theta}\ \cos\theta\, d\theta$$

$$= 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3+\sin\theta)\cos\theta\ \cos\theta\, d\theta$$

$$= 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(3\cos^2\theta + \sin\theta\cos\theta\right)\, d\theta$$

$$= 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3}{2} + \frac{3}{2}\cos 2\theta + \frac{1}{2}\sin 2\theta\right)\, d\theta$$

$$= 16\left(\frac{3}{2}\theta + \frac{3}{4}\sin 2\theta - \frac{1}{4}\cos 2\theta\right.\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 16\left(\frac{3\pi}{4} + \frac{3\pi}{4}\right)$$

$$\underline{= 24\pi}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields a $\vec{F} = \left\langle x^2,\, y^2,\, z^2\right\rangle$; across the boundary of a ball $(x-2)^2 + y^2 + (z-3)^2 \le 9$

Solution

$$\nabla \cdot \vec{F} = \nabla \cdot \left\langle x^2,\, y^2,\, z^2\right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}\left(y^2\right) + \frac{\partial}{\partial z}\left(z^2\right)$$

$$\underline{= 2x + 2y + 2z}$$

$$(x-2)^2 + y^2 + (z-3)^2 = 9$$

$$y^2 = 9 - (x-2)^2 - (z-3)^2$$

$$-\sqrt{9 - (x-2)^2 - (z-3)^2} \le y \le \sqrt{9 - (x-2)^2 - (z-3)^2}$$

$$y = 9 - (x-2)^2 - (z-3)^2 = 0 \quad \rightarrow \quad (x-2)^2 = 9 - (z-3)^2$$

$$2 - \sqrt{9 - (z-3)^2} \le x \le 2 + \sqrt{9 - (z-3)^2}$$

$$9 - (z-3)^2 = 0 \quad \rightarrow z - 3 = \pm 3 \quad 0 \le z \le 6$$

$$Flux = \iiint_D \nabla \cdot \vec{F}\, dV$$

$$= 2\iiint_D (x + y + z)\, dV$$

$$= 2\int_0^6 \int_{2-\sqrt{9-(z-3)^2}}^{2+\sqrt{9-(z-3)^2}} \int_{-\sqrt{9-(x-2)^2-(z-3)^2}}^{\sqrt{9-(x-2)^2-(z-3)^2}} (x+y+z)\ dydxdz$$

$$= 2\int_0^6 \int_{2-\sqrt{9-(z-3)^2}}^{2+\sqrt{9-(z-3)^2}} \left(xy+\frac{1}{2}y^2+yz\ \Big|_{-\sqrt{9-(x-2)^2-(z-3)^2}}^{\sqrt{9-(x-2)^2-(z-3)^2}} dxdz\right.$$

$$= 4\int_0^6 \int_{2-\sqrt{9-(z-3)^2}}^{2+\sqrt{9-(z-3)^2}} (x+z)\sqrt{9-(x-2)^2-(z-3)^2}\ dxdz$$

Let $u = x-2, \quad x = u+2, \quad du = dx$

$v = z-3, \quad z = v+3, \quad dv = dz$

$$\begin{cases} z = 6 \rightarrow & v = 3 \\ z = 0 \rightarrow & v = -3 \end{cases}$$

$$= 4\int_{-3}^3 \int_{2-\sqrt{9-v^2}}^{2+\sqrt{9-v^2}} (u+v+5)\sqrt{9-\left(u^2+v^2\right)}\ dxdz$$

Let $u = r\cos\theta = 3\cos\theta$

$v = r\sin\theta = 3\sin\theta$

$0 \le r \le 3, \quad 0 \le \theta \le 2\pi$

$$= 4\int_0^{2\pi} \int_0^3 (3\cos\theta+3\sin\theta+5)\sqrt{9-r^2}\ rdrd\theta$$

$$= -2\int_0^{2\pi} (3\cos\theta+3\sin\theta+5)d\theta \int_0^3 \left(9-r^2\right)^{1/2} d\left(9-r^2\right)$$

$$= -\frac{4}{3}\left(3\sin\theta-3\cos\theta+5\theta\ \Big|_0^{2\pi} \quad \left(9-r^2\right)^{3/2}\ \Big|_0^3\right.$$

$$= -\frac{4}{3}(10\pi)(-27)$$

$$\underline{= 360\pi}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields a $\vec{F} = \left\langle x^4,\ -x^3z^2,\ 4xy^2z \right\rangle$; across the boundary of the cylinder $x^2 + y^2 = 1$ and the planes $z = x + 2 \quad \& \quad z = 0$

Solution

$$\nabla \cdot \vec{F} = \nabla \cdot \left\langle x^4,\ -x^3z^2,\ 4xy^2z \right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^4\right) + \frac{\partial}{\partial y}\left(-x^3z^2\right) + \frac{\partial}{\partial z}\left(4xy^2z\right)$$

$$= 4x^3 + 4xy^2$$

$$= 4x\left(x^2 + y^2\right)$$

$x^2 + y^2 = 1 = r^2 \quad \to \quad 0 \le r \le 1$

$0 \le \theta \le 2\pi$

$z = x + 2 \quad \to \quad 0 \le z \le r\cos\theta + 2$

$$Flux = \iiint\limits_D \nabla \cdot \vec{F}\ dV$$

$$= 4\iiint\limits_D x\left(x^2 + y^2\right) dV$$

$$= 4\int_0^{2\pi}\int_0^1\int_0^{2+r\cos\theta} r\cos\theta\left(r^2\right)\ dzdrd\theta$$

$$= 4\int_0^{2\pi}\int_0^1\int_0^{2+r\cos\theta} r^3\cos\theta\ dzdrd\theta$$

$$= 4\int_0^{2\pi}\int_0^1 r^3\cos\theta\ \left(z\ \Big|_0^{2+r\cos\theta}\right.\ drd\theta$$

$$= 4\int_0^{2\pi}\int_0^1 r^3\cos\theta\left(2 + r\cos\theta\right)drd\theta$$

$$= 4\int_0^{2\pi}\int_0^1 \left(2r^3\cos\theta + r^4\cos^2\theta\right)drd\theta$$

$$= 4\int_0^{2\pi}\left(\frac{1}{2}r^4\cos\theta + \frac{1}{5}r^5\cos^2\theta\ \Big|_0^1\right.\ d\theta$$

$$= 4\int_0^{2\pi} \left(\frac{1}{2}\cos\theta + \frac{1}{5}\cos^2\theta\right) d\theta$$

$$= 4\int_0^{2\pi} \left(\frac{1}{2}\cos\theta + \frac{1}{10} + \frac{1}{10}\cos 2\theta\right) d\theta$$

$$= 4\left(\frac{1}{2}\sin\theta + \frac{1}{10}\theta + \frac{1}{20}\sin 2\theta \,\Big|_0^{2\pi}\right.$$

$$\underline{= \frac{4\pi}{5}}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields a $\vec{F} = \left\langle x^2z^3,\ 2xyz^3,\ xz^4 \right\rangle$; S is the surface of the box with vertices $(\pm 1,\ \pm 2,\ \pm 3)$.

Solution

$$\nabla \cdot \vec{F} = \nabla \cdot \left\langle x^2z^3,\ 2xyz^3,\ xz^4 \right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^2z^3\right) + \frac{\partial}{\partial y}\left(2xyz^3\right) + \frac{\partial}{\partial z}\left(xz^4\right)$$

$$= 2xz^3 + 2xz^3 + 4xz^3$$

$$\underline{= 8xz^3}$$

$$Flux = \iiint\limits_D \nabla \cdot \vec{F}\ dV$$

$$= \iiint\limits_D \left(8xz^3\right) dV$$

$$= 8\int_{-1}^{1} x\,dx \int_{-2}^{2} dy \int_{-3}^{3} z^3\,dz$$

$$= 8\left(\frac{1}{2}x^2 \,\Big|_{-1}^{1}\right.\quad \left(y \,\Big|_{-2}^{2}\right.\quad \left(\frac{1}{4}z^4 \,\Big|_{-3}^{3}\right.$$

$$\underline{= 0}$$

Exercise

Use the Divergence Theorem to compute the net outward flux of the following fields a $\vec{F} = \left\langle z\tan^{-1}\left(y^2\right),\ z^3\ln\left(x^2+1\right),\ z\right\rangle$; across the part of the paraboloid $x^2+y^2+z=2$ that lies above the plane $z=1$ and is oriented upward.

Solution

$$\nabla\bullet\vec{F} = \nabla\bullet\left\langle z\tan^{-1}\left(y^2\right),\ z^3\ln\left(x^2+1\right),\ z\right\rangle$$

$$= \frac{\partial}{\partial x}\left(z\tan^{-1}\left(y^2\right)\right)+\frac{\partial}{\partial y}\left(z^3\ln\left(x^2+1\right)\right)+\frac{\partial}{\partial z}(z)$$

$$\underline{=1}$$

$$z=2-\left(x^2+y^2\right) \quad\rightarrow\quad 1\le z\le 2-r^2$$

$$z=2-r^2=1 \quad\rightarrow\quad 0\le r\le 1$$

$$0\le\theta\le 2\pi$$

$$Flux = \iiint\limits_D \nabla\bullet\vec{F}\ dV$$

$$= \iiint\limits_D (1)\,dV$$

$$= \int_0^{2\pi}\int_0^1\int_1^{2-r^2} r\ dzdrd\theta$$

$$= \int_0^{2\pi} d\theta \quad \int_0^1 r\ \left(z\ \Big|_1^{2-r^2}\right.\ dr$$

$$= 2\pi\int_0^1 r\left(1-r^2\right)dr$$

$$= 2\pi\int_0^1 \left(r-r^3\right)dr$$

$$= 2\pi\left(\frac{1}{2}r^2-\frac{1}{4}r^4\ \Big|_0^1\right.$$

$$= 2\pi\left(\frac{1}{2}-\frac{1}{4}\right)$$

$$\underline{=\frac{\pi}{2}}$$

Exercise

Prove that $\nabla\left(\frac{1}{|\vec{r}|^4}\right) = -\frac{4\vec{r}}{|\vec{r}|^6}$ and use the result to prove that $\nabla \bullet \nabla\left(\frac{1}{|\vec{r}|^4}\right) = \frac{12}{|\vec{r}|^6}$

Solution

$$|\vec{r}|^4 = \left(x^2 + y^2 + z^2\right)^2$$

$$\begin{aligned}
\nabla\left(\frac{1}{|\vec{r}|^4}\right) &= \nabla \frac{1}{\left(x^2 + y^2 + z^2\right)^2} \\
&= -\frac{1}{\left(x^2 + y^2 + z^2\right)^3}\langle 4x,\ 4y,\ 4z\rangle \\
&= -\frac{4}{\left(|\vec{r}|^2\right)^3}\langle x,\ y,\ z\rangle \\
&= \boxed{-\frac{4\vec{r}}{|\vec{r}|^6}} \quad \checkmark
\end{aligned}$$

$$\begin{aligned}
\nabla \bullet \nabla\left(\frac{1}{|\vec{r}|^4}\right) &= \nabla \bullet \left(\frac{-4\vec{r}}{|\vec{r}|^4}\right) \\
&= -4\nabla \bullet \left(\frac{\langle x,\ y,\ z\rangle}{\left(x^2 + y^2 + z^2\right)^3}\right) \\
&= -\frac{4}{\left(x^2 + y^2 + z^2\right)^4}\left(x^2 + y^2 + z^2 - 6x^2 + x^2 + y^2 + z^2 - 6y^2 + x^2 + y^2 + z^2 - 6z^2\right) \\
&= -\frac{4}{\left(x^2 + y^2 + z^2\right)^4}\left(-3x^2 - 3y^2 - 3z^2\right) \\
&= \frac{12}{\left(x^2 + y^2 + z^2\right)^4}\left(x^2 + y^2 + z^2\right) \\
&= \frac{12}{\left(x^2 + y^2 + z^2\right)^3} \\
&= \frac{12}{\left(\vec{r}^2\right)^3} \\
&= \boxed{\frac{12}{|\vec{r}|^6}} \quad \checkmark
\end{aligned}$$

Exercise

Consider the radial vector field $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{p/2}}$. Let S be the sphere of radius a at the origin.

a) Use the surface integral to show that the outward flux of $\vec{F}$ across S is $4\pi a^{3-p}$. Recall that the unit normal to sphere is $\dfrac{\vec{r}}{|\vec{r}|}$.

b) For what values of p does $\vec{F}$ satisfy the conditions of the Divergence Theorem? For these values of p, use the fact the $\nabla \cdot \vec{F} = \dfrac{3-p}{|\vec{r}|^p}$ to compute the flux around S using the Divergence Theorem.

Solution

a) $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p} \quad \& \quad \vec{n} = \dfrac{\vec{r}}{|\vec{r}|}$

$$\begin{aligned}
\vec{F}\cdot\vec{n} &= \frac{\vec{r}}{|\vec{r}|^p}\cdot\frac{\vec{r}}{|\vec{r}|}\\
&= \frac{|\vec{r}|^2}{|\vec{r}|^{p+1}}\\
&= |\vec{r}|^{1-p}\\
&= \left(\sqrt{x^2+y^2+z^2}\right)^{1-p}\\
&= \left(\sqrt{a^2}\right)^{1-p}\\
&= \boxed{a^{1-p}}
\end{aligned}$$

$$\begin{aligned}
\iint_S \vec{F}\cdot\vec{n}\ dS &= a^{1-p}\iint_S dS\\
&= a^{1-p}\times(\mathit{area\ of\ sphere})\\
&= a^{1-p}\times 4\pi a^2\\
&= \boxed{4\pi a^{3-p}}
\end{aligned}$$ ✓

b) The conditions of the Divergence Theorem require that $\vec{F}$ be defined and have continuous partials everywhere inside the sphere; in particular, this must hold at the origin. Thus, we must have $p \le -2$. Then the volume integral is:

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iiint_D \frac{3-p}{|\vec{r}|^p}\, dV$$

$$= (3-p) \iiint_D r^{-p}\, dV$$

$$= (3-p) \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^{-p}\, \rho^2 \sin\varphi\, d\rho d\varphi d\theta$$

$$= (3-p) \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\, d\varphi \int_0^{a} \rho^{2-p}\, d\rho$$

$$= (3-p)(2\pi)\left(-\cos\varphi \Big|_0^{\pi}\right.\left(\frac{1}{3-p}\rho^{3-p}\Big|_0^a\right.$$

$$= (3-p)(2\pi)(2)\left(\frac{1}{3-p}a^{3-p}\right)$$

$$= 4\pi a^{3-p}$$

Exercise

Consider the radial vector field $\vec{F} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}}$.

a) Evaluate a surface integral to show that $\iint_S \vec{F} \cdot \vec{n}\, dS = 4\pi a^2$, where S is the surface of a sphere of radius a centered at the origin.

b) Note that the first partial derivatives of the components of $\vec{F}$ are undefined at the origin, so the Divergence Theorem does not apply directly. Nevertheless, the flux across the sphere as computed in part (*a*) is finite. Evaluate the triple integral of the Divergence Theorem as an improper integral as follows. Integrate $div\vec{F}$ over the region between two spheres of radius a and $0 < \varepsilon < a$. Then let $\varepsilon \to 0^+$ to obtain the flux computed in part (*a*).

Solution

a) $\vec{F} = \frac{\vec{r}}{|\vec{r}|}$ & $\vec{n} = \frac{\vec{r}}{|\vec{r}|}$

$$\vec{F} \cdot \vec{n} = \frac{\vec{r}}{|\vec{r}|} \cdot \frac{\vec{r}}{|\vec{r}|}$$

$$= \frac{|\vec{r}|^2}{|\vec{r}|^2}$$

$$\underline{= 1}$$

$$\iint_S \vec{F} \cdot \vec{n}\ dS = \iint_S dS$$

$$= \textit{area of sphere}$$

$$\underline{= 4\pi a^2}$$

b) $\nabla \cdot \vec{F} = \nabla \cdot \left(\frac{\langle x,\ y,\ z \rangle}{\sqrt{x^2+y^2+z^2}} \right)$

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}} \right) = \frac{x^2+y^2+z^2-x^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$= \frac{y^2+z^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}} \right) = \frac{x^2+y^2+z^2-y^2}{x^2+y^2+z^2}$$

$$= \frac{x^2+z^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}} \right) = \frac{x^2+y^2+z^2-z^2}{x^2+y^2+z^2}$$

$$= \frac{x^2+y^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\nabla \cdot \vec{F} = \frac{y^2+z^2+x^2+z^2+x^2+y^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$= \frac{2\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$= \frac{2}{\sqrt{x^2+y^2+z^2}}$$

$$= \boxed{\frac{2}{|\vec{r}|}}$$

$$\iiint\limits_D \nabla \cdot \vec{F}\, dV = \iiint\limits_D \frac{2}{|\vec{r}|}\, dV$$

$$= 2\int_0^{2\pi}\int_0^{\pi}\int_\varepsilon^a \frac{1}{\rho}\, \rho^2 \sin\varphi\, d\rho d\varphi d\theta$$

$$= 2\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\, d\varphi \int_\varepsilon^a \rho\, d\rho$$

$$= 4\pi \left(-\cos\varphi \,\Big|_0^{\pi} \quad \left(\tfrac{1}{2}\rho^2 \,\Big|_\varepsilon^a\right.\right.$$

$$\boxed{= 4\pi\left(a^2 - \varepsilon^2\right)}$$

$$\lim_{\varepsilon\to 0} \; 4\pi\left(a^2 - \varepsilon^2\right) \boxed{= 4\pi a^2}$$

Exercise

The electric field due to a point charge Q is $\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}}{|\vec{r}|^3}$, where $\vec{r} = \langle x,\ y,\ z\rangle$ and ε_0 is a constant

a) Show that the flux of the field across a sphere of radius a centered at the origin is

$$\iint_S \vec{E} \cdot \vec{n}\, dS = \frac{Q}{\varepsilon_0}$$

b) Let S be the boundary of the origin between two spheres centered of radius a and b with $a < b$. Use the Divergence Theorem to show that the net outward flux across S is zero.

c) Suppose there is a distribution of charge within a region D. Let $q(x,\ y,\ z)$ be the charge density (charge per unit volume). Interpret the statement that

$$\iint_S \vec{E} \cdot \vec{n}\, dS = \frac{1}{\varepsilon_0}\iiint_D q(x,\ y,\ z)\, dV$$

d) Assuming $\vec{E}$ satisfies the conditions of the Divergence Theorem, conclude from part (c) that

$$\nabla \cdot \vec{E} = \frac{q}{\varepsilon_0}$$

e) Because the electric force is conservative, it has a potential function ϕ. From part (d) conclude that $\nabla^2\phi = \nabla \cdot \nabla\phi = \frac{q}{\varepsilon_0}$

Solution

a) $\vec{F} \bullet \vec{n} = \dfrac{\vec{r}}{|\vec{r}|^3} \bullet \dfrac{\vec{r}}{|\vec{r}|}$

$$= \frac{|\vec{r}|^2}{|\vec{r}|^4}$$

$$= |\vec{r}|^{-2}$$

$$\iint_S \vec{F} \bullet \vec{n}\ dS = |\vec{r}|^{-2}$$

Area of sphere $= |\vec{r}|^{-2}\left(4\pi r^2\right)$

$$= |a|^{-2}\left(4\pi a^2\right)$$

$$\underline{= 4\pi}$$

$$\iint_S \vec{E} \bullet \vec{n}\ dS = \frac{Q}{4\pi\varepsilon_0}(4\pi)$$

$$\boxed{= \frac{Q}{\varepsilon_0}}$$

b) The net outward flux across S is the difference of the fluxes across the inner and outer sphere; by part (a), these are equal (independent of the radius), so the net flux across S is zero.

c) The left-hand side is the flux across the boundary of D, while the right-hand side is the sum of the charge densities at each point of D.

$$\iint_S \vec{E} \bullet \vec{n}\ dS = \frac{1}{\varepsilon_0}\iiint_D q(x, y, z)\ dV$$

$$= \frac{Q}{\varepsilon_0}$$

$$\rightarrow \iiint_D q(x, y, z)\ dV = Q$$

The statement says that the flux across the boundary, up to multiplication by a constant, is the sum of the charge densities in the region.

d) $\dfrac{1}{\varepsilon_0}\displaystyle\iiint_D q(x, y, z)\ dV = \iint_S \vec{E} \bullet \vec{n}\ dS$

$$= \iiint_D \nabla \bullet \vec{E}\ dV$$

This holds for all regions *D*.

Therefore; that implies that $\nabla \bullet \vec{E} = \dfrac{q(x, y, z)}{\varepsilon_0}$

e) $\nabla^2 \phi = \nabla \bullet \nabla \phi$

$$= \nabla \bullet \vec{E}$$

$$\boxed{= \dfrac{q}{\varepsilon_0}}$$

Exercise

Fourier's Law of heat transfer (or heat conduction) states that the heat flow vector $\vec{F}$ at a point is proportional to the negative gradient of the temperature that is, $\vec{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k > 0$ is called the *conductivity*, which has metric units of $J\,/\,m\text{-}s\text{-}K$. A temperature function for a region *D* is given. Find the net outward heat flux $\iint\limits_S \vec{F} \bullet \vec{n}\ dS = -k \iint\limits_S \nabla T \bullet \vec{n}\ dS$ across the boundary *S* of *D*. In some cases it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume $k = 1$.

$$T(x, y, z) = 100 + x + 2y + z; \quad D = \{(x, y, z):\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$$

Solution

$$\nabla T = T_x\ \hat{i} + T_y\ \hat{j} + T_z\ \hat{k}$$

$$= \hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{F} = -k\nabla T$$

$$= k\langle -1,\ -2,\ -1\rangle$$

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(-k) + \frac{\partial}{\partial y}(-2k) + \frac{\partial}{\partial z}(-k)$$

$$\boxed{= 0}$$

Therefore, the heat flux is ***zero***.

Exercise

Fourier's Law of heat transfer (or heat conduction) states that the heat flow vector $\vec{F}$ at a point is proportional to the negative gradient of the temperature that is, $\vec{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k > 0$ is called the *conductivity*, which has metric units of $J / m\text{-}s\text{-}K$. A temperature function for a region *D* is given.

Find the net outward heat flux $\iint\limits_S \vec{F} \bullet \vec{n}\ dS = -k \iint\limits_S \nabla T \bullet \vec{n}\ dS$ across the boundary *S* of *D*. In some cases, it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume $k = 1$.

$$T(x, y, z) = 100 + x^2 + y^2 + z^2; \quad D = \{(x, y, z):\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$$

Solution

$$\nabla T = T_x\ \hat{i} + T_y\ \hat{j} + T_z\ \hat{k}$$

$$= 2x\ \hat{i} + 2y\ \hat{j} + 2z\ \hat{k}$$

$$\vec{F} = -\nabla T$$

$$= \langle -2x,\ -2y,\ -2z \rangle$$

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(-2x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(-2z)$$

$$= -6$$

Therefore, the heat flux is **–6** times the volume of the region (or **–6**).

Exercise

Fourier's Law of heat transfer (or heat conduction) states that the heat flow vector $\vec{F}$ at a point is proportional to the negative gradient of the temperature that is, $\vec{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k > 0$ is called the *conductivity*, which has metric units of $J / m\text{-}s\text{-}K$. A temperature function for a region *D* is given.

Find the net outward heat flux $\iint\limits_S \vec{F} \bullet \vec{n}\ dS = -k \iint\limits_S \nabla T \bullet \vec{n}\ dS$ across the boundary *S* of *D*. In some cases, it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume $k = 1$.

$$T(x, y, z) = 100 + e^{-z}; \quad D = \{(x, y, z):\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$$

Solution

$$\nabla T = T_x\ \hat{i} + T_y\ \hat{j} + T_z\ \hat{k}$$

$$= -e^{-z}\ \hat{k}$$

$$\vec{F} = -\nabla T$$

$$= \left\langle 0,\ 0,\ e^{-z} \right\rangle$$

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}\left(e^{-z}\right)$$

$$\underline{= -e^{-z}}$$

Therefore, the heat flux is

$$\int_0^1 \int_0^1 \int_0^1 \left(-e^{-z}\right) dxdydz = \int_0^1 \left(-e^{-z}\right) dz \quad \int_0^1 dy \quad \int_0^1 dx$$

$$= e^{-z} \Big|_0^1 \ (1)(1)$$

$$\underline{= e^{-1} - 1}$$

Exercise

Fourier's Law of heat transfer (or heat conduction) states that the heat flow vector $\vec{F}$ at a point is proportional to the negative gradient of the temperature that is, $\vec{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k > 0$ is called the *conductivity*, which has metric units of $J / m\text{-}s\text{-}K$. A temperature function for a region *D* is given.

Find the net outward heat flux $\iint\limits_S \vec{F} \bullet \vec{n}\ dS = -k \iint\limits_S \nabla T \bullet \vec{n}\ dS$ across the boundary *S* of *D*. In some cases, it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume $k = 1$

$T(x, y, z) = 100e^{-x^2-y^2-z^2}$; *D* is the sphere of radius *a* centered at the origin.

Solution

$$\nabla T = T_x\ \hat{i} + T_y\ \hat{j} + T_z\ \hat{k}$$

$$= \left\langle -200xe^{-x^2-y^2-z^2},\ -200ye^{-x^2-y^2-z^2},\ -200ze^{-x^2-y^2-z^2} \right\rangle$$

$$\vec{F} = -\nabla T$$

$$= 200\left\langle xe^{-x^2-y^2-z^2},\ ye^{-x^2-y^2-z^2},\ ze^{-x^2-y^2-z^2} \right\rangle$$

$$\nabla \bullet \vec{F} = 200\frac{\partial}{\partial x}\left(xe^{-x^2-y^2-z^2}\right) + 200\frac{\partial}{\partial y}\left(ye^{-x^2-y^2-z^2}\right) + 200\frac{\partial}{\partial z}\left(ze^{-x^2-y^2-z^2}\right)$$

$$= 200e^{-x^2-y^2-z^2}\left(1 - 2x^2 + 1 - 2y^2 + 1 - 2z^2\right)$$

$$= 200e^{-x^2-y^2-z^2}\left(3 - 2x^2 - 2y^2 - 2z^2\right)$$

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iiint_D 200e^{-x^2-y^2-z^2}\left(3-2x^2-2y^2-2z^2\right)\, dV$$

$$= 200\int_0^{2\pi}\int_0^{\pi}\int_0^{a} e^{-\rho^2}\left(3-2\rho^2\right)\rho^2\ \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= 200\int_0^{2\pi} d\theta\ \int_0^{\pi}\sin\varphi\, d\varphi\ \int_0^{a}\left(3\rho^2-2\rho^4\right)e^{-\rho^2}\ d\rho$$

$$= 200(2\pi)\ \left(-\cos\varphi\ \Big|_0^{\pi}\ \left(\rho^3 e^{-\rho^2}\ \Big|_0^{a}\right.\right. \qquad \left(3\rho^2-2\rho^4\right)e^{-\rho^2} = \left(\rho^3 e^{-\rho^2}\right)'$$

$$\boxed{= 800\pi a^3\ e^{-a^2}}$$

Exercise

Consider the surface *S* consisting of the quarter-sphere $x^2+y^2+z^2=a^2$, for $z \ge 0$ *and* $x \ge 0$, and the half disk in the *yz*-plane $y^2+z^2 \le a^2$, for $z \ge 0$. The boundary of *S* in the *xy*-plane is *C*, which consists of the semicircle $x^2+y^2=a^2$, for $x \ge 0$, and the line segment $[-a,\ a]$ on the *y*-axis, with a counterclockwise orientation. Let $\vec{F} = \langle 2z-y,\ x-z,\ y-2x\rangle$

a) Describe the direction in which the normal vectors point on *S*.

b) Evaluate $\oint_C \vec{F} \cdot d\vec{r}$

c) Evaluate $\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\ dS$ and check for segment with part (*b*).

Solution

a) Normal vectors point outwards everywhere on *S*; that is, on the curved surface. They point outwards and on the flat surface they point in the direction of negative *x*.

b) $\vec{F} = \langle 2z-y,\ x-z,\ y-2x\rangle\ \&\ C:\ x^2+y^2=a^2 \qquad [-a,\ a]$

Parameterized:

$\vec{r}_1 = \langle a\cos t,\ a\sin t,\ 0\rangle$

$$\frac{d\vec{r}_1}{dt} = \langle -a\sin t,\ a\cos t,\ 0\rangle$$

$$\vec{F} = \langle 2z-y,\ x-z,\ y-2x\rangle$$

$$= \langle -a\sin t,\ a\cos t,\ a\sin t - 2a\cos t\rangle$$

$$\vec{F} \cdot \frac{d\vec{r}_1}{dt} = \langle -a\sin t,\ a\cos t,\ a\sin t - 2a\cos t \rangle \cdot \langle -a\sin t,\ a\cos t,\ 0 \rangle$$

$$= a^2 \sin^2 t + a^2 \cos^2 t$$

$$\underline{= a^2}$$

For $0 \le t \le a$

$$\vec{r}_2 = \langle 0,\ a - 2t,\ 0 \rangle$$

$$\frac{d\vec{r}_2}{dt} = \langle 0,\ -2,\ 0 \rangle$$

$$\vec{F} = \langle 2z - y,\ x - z,\ y - 2x \rangle$$

$$= \langle 2t - a,\ 0,\ a - 2t \rangle$$

$$\vec{F} \cdot \frac{d\vec{r}_2}{dt} = \langle 2t - a,\ 0,\ a - 2t \rangle \cdot \langle 0,\ -2,\ 0 \rangle$$

$$\underline{= 0}$$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot d\vec{r}_1 + \oint_C \vec{F} \cdot d\vec{r}_2$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2\, dt + 0$$

$$= a^2\ t \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$\underline{= \pi a^2}$$

c) $$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z - y & x - z & y - 2x \end{vmatrix}$$

$$= \langle 2,\ 4,\ 2 \rangle$$

Spherical: $\langle a\sin\varphi\cos\theta,\ a\sin\varphi\sin\theta,\ a\cos\varphi \rangle$

$$\vec{t}_\varphi = \langle a\cos\varphi\cos\theta,\ a\cos\varphi\sin\theta,\ -a\sin\varphi \rangle$$

$$\vec{t}_\theta = \langle -a\sin\varphi\sin\theta,\ a\sin\varphi\cos\theta,\ 0 \rangle$$

$$\vec{n}_1 = \vec{t}_\varphi \times \vec{t}_\theta$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= \left\langle a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\cos\varphi\sin\varphi\cos^2\theta + a^2\cos\varphi\sin\varphi\sin^2\theta \right\rangle$$

$$= \left\langle a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\cos\varphi\sin\varphi \right\rangle$$

$$\left(\nabla\times\vec{F}\right)\cdot\vec{n}_1 = \langle 2,\ 4,\ 2\rangle \cdot \left\langle a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\cos\varphi\sin\varphi \right\rangle$$

$$= 2a^2\sin^2\varphi\cos\theta + 4a^2\sin^2\varphi\sin\theta + 2a^2\cos\varphi\sin\varphi$$

From part (a), the flat surface is pointing in x-negative direction

$\vec{n}_2 = \langle -1,\ 0,\ 0\rangle$

$$\left(\nabla\times\vec{F}\right)\cdot\vec{n}_2 = \langle 2,\ 4,\ 2\rangle \cdot \langle -1,\ 0,\ 0\rangle$$

$$= -2$$

$$\iint_S \left(\nabla\times\vec{F}\right)\cdot\vec{n}\ dS = \iint_{S_1}\left(2a^2\sin^2\varphi\cos\theta + 4a^2\sin^2\varphi\sin\theta + 2a^2\cos\varphi\sin\varphi\right)dS - 2\iint_{S_2} dS$$

$$= 2a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}(1-\cos 2\varphi)\cos\theta + \frac{1}{2}(1-\cos 2\varphi)\sin\theta + \cos\varphi\sin\varphi\right)\ d\varphi d\theta$$
$$-2\times(semi-circle)$$

$$= a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left(\cos\theta - \cos 2\varphi\cos\theta + \sin\theta - \cos 2\varphi\sin\theta + \sin 2\varphi\right)\ d\varphi d\theta$$
$$-2\left(\frac{1}{2}\pi a^2\right)$$

$$= a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left((\cos\theta+\sin\theta)\varphi - \frac{1}{2}(\cos\theta+\sin\theta)\sin 2\varphi - \frac{1}{2}\cos 2\varphi\ \Big|_0^{\frac{\pi}{2}}\ d\theta - \pi a^2\right.$$

$$= a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left((\cos\theta+\sin\theta)\frac{\pi}{2} + \frac{1}{2} + \frac{1}{2}\right)d\theta\ -\ \pi a^2$$

$$= a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\pi}{2}(\cos\theta+\sin\theta) + 1\right)d\theta\ -\ \pi a^2$$

$$= a^2 \left(\frac{\pi}{2}(\sin\theta - \cos\theta) + \theta \right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \pi a^2$$

$$= a^2\left(\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}\right) - \pi a^2$$

$$= 2\pi a^2 - \pi a^2$$

$$\underline{= \pi a^2}$$

Exercise

Let S be the hemisphere $x^2 + y^2 + z^2 = a^2$, for $z \ge 0$, and let T be the paraboloid $z = a - \frac{1}{a}\left(x^2 + y^2\right)$, for $z \ge 0$, where $a > 0$. Assume the surfaces have outward normal vectors.

a) Verify that S and T have the same base $\left(x^2 + y^2 \le a^2\right)$ and the same high point $(0,\ 0,\ a)$.

b) Which surface has the greater area?

c) Show that the flux of the radial field $\vec{F} = \langle x,\ y,\ z\rangle$ across S is $2\pi a^3$.

d) Show that the flux of the radial field $\vec{F} = \langle x,\ y,\ z\rangle$ across T is $\frac{3\pi a^3}{2}$.

Solution

a) S: base is the surface where $z = 0$

$$\Rightarrow x^2 + y^2 = a^2 \quad (circle)$$

T: $z = 0 = a - \frac{1}{a}\left(x^2 + y^2\right) \quad \Rightarrow x^2 + y^2 = a^2$

The high point of the hemisphere (maximum z-coordinate) occurs when

$x = y = 0 \quad \to z = a$

$\therefore\ x^2 + y^2 \le a^2$ ✓

b) S: is the surface of hemisphere $\frac{1}{2}4\pi a^2 = 2\pi a^2$

For T: paraboloid

$z_x = -\frac{2x}{a} \quad z_y = -\frac{2y}{a}$

$$\vec{n} = \sqrt{z_x^2 + z_y^2 + 1}$$

$$= \sqrt{\frac{4x^2}{a^2} + \frac{4y^2}{a^2} + 1}$$

$$= \frac{1}{a}\sqrt{4\left(x^2 + y^2\right) + a^2}$$

$$\iint_S 1\, dS = \frac{1}{a}\iint_S \sqrt{4\left(x^2+y^2\right)+a^2}\, dS$$

$$= \frac{1}{a}\int_0^{2\pi} d\theta \int_0^a \sqrt{4r^2+a^2}\; r\, dr$$

$$= \frac{\pi}{4a}\int_0^a \left(4r^2+a^2\right)^{1/2} d\left(4r^2+a^2\right)$$

$$= \frac{\pi}{6a}\left(4r^2+a^2\right)^{3/2}\Big|_0^a$$

$$= \frac{\pi}{6a}\left(\left(5a^2\right)^{3/2} - a^3\right)$$

$$= \frac{\pi}{6a}\left(5^{3/2}a^3 - a^3\right)$$

$$= \frac{\pi}{6}\left(5\sqrt{5}-1\right)a^2$$

$$\boxed{= \frac{5\sqrt{5}-1}{6}\pi a^2}$$

$$< 2\pi a^2$$

∴ Area of the paraboloid is smaller than the area of the hemisphere.

c) $\vec{F} = \langle x,\ y,\ z\rangle$

$$= \langle a\sin\varphi\cos\theta,\ a\sin\varphi\sin\theta,\ a\cos\varphi\rangle$$

$$\vec{t}_\varphi = \langle a\cos\varphi\cos\theta,\ a\cos\varphi\sin\theta,\ -a\sin\varphi\rangle$$

$$\vec{t}_\theta = \langle -a\sin\varphi\sin\theta,\ a\sin\varphi\cos\theta,\ 0\rangle$$

$$\vec{n}_1 = \vec{t}_\varphi \times \vec{t}_\theta$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= \left\langle a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\cos\varphi\sin\varphi\cos^2\theta + a^2\cos\varphi\sin\varphi\sin^2\theta\right\rangle$$

$$= \left\langle a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\cos\varphi\sin\varphi\right\rangle$$

$$\vec{F}\bullet\vec{n} = \langle a\sin\varphi\cos\theta,\ a\sin\varphi\sin\theta,\ a\cos\varphi\rangle \bullet \left\langle a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\cos\varphi\sin\varphi\right\rangle$$

$$= a^3\sin^3\varphi\cos^2\theta + a^3\sin^3\varphi\sin^2\theta + a^3\cos^2\varphi\sin\varphi$$

$$= a^3\sin^3\varphi\left(\cos^2\theta+\sin^2\theta\right) + a^3\cos^2\varphi\sin\varphi$$

$$= a^3 \sin\varphi\left(\sin^2\varphi + \cos^2\varphi\right)$$

$$= a^3 \sin\varphi$$

$$\iint_S \vec{F} \bullet \vec{n}\ dS = a^3 \int_0^{2\pi} d\theta\ \int_0^{\frac{\pi}{2}} \sin\varphi\ d\varphi$$

$$= -2\pi a^3\ \left(\cos\varphi\ \Big|_0^{\frac{\pi}{2}}\right.$$

$$\underline{= 2\pi a^3} \quad \checkmark$$

d) Across T: $\quad z = a - \frac{1}{a}\left(x^2 + y^2\right)$

Parameterization:

$$\left\langle r\cos\theta,\ r\sin\theta,\ a - \frac{r^2}{a}\right\rangle$$

$0 \le r \le a \quad 0 \le \theta \le 2\pi$

$$\vec{t}_r = \left\langle \cos\theta,\ \sin\theta,\ -\frac{2}{a}r\right\rangle$$

$$\vec{t}_\theta = \left\langle -r\sin\theta,\ r\cos\theta,\ 0\right\rangle$$

$$\vec{n} = \vec{t}_\varphi \times \vec{t}_\theta$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & -\frac{2}{a}r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \left\langle \frac{2r^2}{a}\cos\theta,\ \frac{2r^2}{a}\sin\theta,\ r\cos^2\theta + r\sin^2\theta\right\rangle$$

$$= \left\langle \frac{2r^2}{a}\cos\theta,\ \frac{2r^2}{a}\sin\theta,\ r\right\rangle$$

$$\vec{F} \bullet \vec{n} = \left\langle r\cos\theta,\ r\sin\theta,\ a - \frac{r^2}{a}\right\rangle \bullet \left\langle \frac{2r^2}{a}\cos\theta,\ \frac{2r^2}{a}\sin\theta,\ r\right\rangle$$

$$= \frac{2}{a}r^3\cos^2\theta + \frac{2r^2}{a}\sin^2\theta + ar - \frac{1}{a}r^3$$

$$= \frac{2}{a}r^3\left(\cos^2\theta + \sin^2\theta\right) + ar - \frac{1}{a}r^3$$

$$= \frac{2}{a}r^3 + ar - \frac{1}{a}r^3$$

$$= \frac{1}{a}\left(r^3 + a^2 r\right)$$

$$\iint_S \vec{F} \cdot \vec{n}\ dS = \frac{1}{a}\int_0^{2\pi} d\theta \int_0^a \left(r^3 + a^2 r\right) dr$$

$$= \frac{2\pi}{a}\left(\frac{1}{4}r^4 + \frac{1}{2}a^2 r^2\right)\Big|_0^a$$

$$= \frac{2\pi}{a}\left(\frac{1}{4}a^4 + \frac{1}{2}a^4\right)$$

$$= \frac{2\pi}{a}\left(\frac{3}{4}a^4\right)$$

$$= \boxed{\frac{3\pi a^3}{2}} \quad \checkmark$$

Exercise

The gravitational force due to a point mass M is proportional to $\vec{F} = \dfrac{GM\vec{r}}{|\vec{r}|^3}$, where $\vec{r} = \langle x,\ y,\ z\rangle$ and G is the gravitational constant.

a) Show that the flux force field across a sphere of radius a centered at the origin is $\iint_S \vec{F} \cdot \vec{n}\ dS = 4\pi GM$

b) Let S be the boundary of the region between two spheres centered at the origin of radius a and b with $a < b$. Use the Divergence Theorem to show that the net outward flux across S is zero.

c) Suppose there is a distribution of mass within a region D containing the origin. Let $\rho = (x,\ y,\ z)$ be the mass density (mass per unit volume). Interpret the statement that

$$\iint_S \vec{F} \cdot \vec{n}\ dS = 4\pi G \iiint_D \rho(x,\ y,\ z)\ dV$$

d) Assuming $\vec{F}$ satisfies the conditions of the Divergence Theorem, conclude from part (*c*) that $\nabla \cdot \vec{F} = 4\pi G\rho$

e) Because the gravitational force is conservative, it has a potential function ϕ. From part (*d*) conclude that $\nabla^2\phi = 4\pi G\rho$

Solution

a) The unit normal to sphere is $\vec{n} = \dfrac{\vec{r}}{|\vec{r}|}$

$$\vec{F} \cdot \vec{n} = GM\frac{\vec{r}}{|\vec{r}|^3} \cdot \frac{\vec{r}}{|\vec{r}|}$$

$$= GM\frac{|\vec{r}|^2}{|\vec{r}|^4}$$

$$= GM\frac{1}{|\vec{r}|^2}$$

$$= GM\frac{1}{x^2+y^2+z^2}$$

$$= \boxed{\frac{GM}{a^2}}$$

$$\iint_S \vec{F}\bullet\vec{n}\ dS = \frac{GM}{a^2}\iint_S dS$$

$$= \frac{GM}{a^2}\times(area\ of\ sphere)$$

$$= \frac{GM}{a^2}\times 4\pi a^2$$

$$= \boxed{4\pi GM}$$ ✓

b) $$\iint_S \vec{F}\bullet\vec{n}\ dS = 4\pi G\iiint_D \rho = (x,\ y,\ z)\ dV$$

Since the outward flux across a sphere, from part (a), is independent of the radius of the sphere, the outward flux across the spheres of radii a and b are equal, so their difference, which is the net flux across the spherical shell bounded by them, is zero.

c) $$\iint_S \vec{F}\bullet\vec{n}\ dS = 4\pi G\iiint_D \rho(x,\ y,\ z)\ dV \qquad\qquad \rho = (x,\ y,\ z)$$

The left-hand side is the flux across the boundary of D, while the right-hand side is the sum of the mass density inside D.

The statement says that the flux across the boundary is determined by (is a constant multiple of) the sum of the mass density inside D.

d) $$4\pi G\iiint_D \rho(x,\ y,\ z)\ dV = \iint_S \vec{F}\bullet\vec{n}\ dS$$

$$= \iiint_D \nabla\bullet\vec{F}\ dV$$

$$\iiint\limits_D \nabla \cdot \vec{F}\, dV = \iiint\limits_D 4\pi G\rho(x,\ y,\ z)\ dV$$

$$\nabla \cdot \vec{F} = 4\pi G\rho(x,\ y,\ z)$$

e) $\nabla^2\phi = \nabla \cdot \nabla\phi$

$$= \nabla \cdot \vec{F}$$

$$\underline{= 4\pi G\rho(x,\ y,\ z)}$$

Exercise

Let $\vec{F}$ be a radial field $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p}$, where p is a real number and $\vec{r} = \langle x,\ y,\ z\rangle$. With $p = 3$, $\vec{F}$ is an inverse square field.

a) Show that the net flux across a sphere centered at the origin is independent of the radius of the sphere only for $p = 3$

b) Explain the observation in part (*a*) by finding the flux of $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p}$ across the boundaries of a spherical box $\left\{(\rho,\ \varphi,\ \theta):\ \ a \le \rho \le b,\ \ \varphi_1 \le \varphi \le \varphi_2,\ \ \theta_1 \le \theta \le \theta_2\right\}$ for various values of p.

Solution

c) $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p}$ & $\vec{n} = \dfrac{\vec{r}}{|\vec{r}|}$

$$\vec{F} \cdot \vec{n} = \frac{\vec{r}}{|\vec{r}|^p} \cdot \frac{\vec{r}}{|\vec{r}|}$$

$$= \frac{|\vec{r}|^2}{|\vec{r}|^{p+1}}$$

$$= |\vec{r}|^{1-p}$$

$$= \left(\sqrt{x^2 + y^2 + z^2}\right)^{1-p}$$

$$= \left(\sqrt{a^2}\right)^{1-p}$$

$$\underline{= a^{1-p}}$$

$$\iint\limits_S \vec{F} \cdot \vec{n}\ dS = a^{1-p} \iint\limits_S\ dS$$

$$= a^{1-p} \times (area\ of\ sphere)$$

$$= a^{1-p} \times 4\pi a^2$$

$$\boxed{= 4\pi a^{3-p}}$$

If $p = 3$, then

$$\boxed{\iint_S \vec{F} \cdot \vec{n}\ dS = 4\pi}$$

Which is independent of the radius of the sphere.

d) $$\nabla \cdot \frac{\langle x,\ y,\ z\rangle}{|\vec{r}|^p} = \left\langle \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right\rangle \cdot \frac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{p/2}}$$

$$= \frac{\partial}{\partial x}\frac{x}{\left(x^2+y^2+z^2\right)^{p/2}} + \frac{\partial}{\partial y}\frac{y}{\left(x^2+y^2+z^2\right)^{p/2}} + \frac{\partial}{\partial z}\frac{z}{\left(x^2+y^2+z^2\right)^{p/2}}$$

$$= \frac{x^2+y^2+z^2-px^2}{\left(x^2+y^2+z^2\right)^{1+p/2}} + \frac{x^2+y^2+z^2-py^2}{\left(x^2+y^2+z^2\right)^{1+p/2}} + \frac{x^2+y^2+z^2-pz^2}{\left(x^2+y^2+z^2\right)^{1+p/2}}$$

$$= \frac{3\left(x^2+y^2+z^2\right) - p\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{1+p/2}}$$

$$= \frac{(3-p)\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{1+p/2}}$$

$$= \frac{3-p}{\left(x^2+y^2+z^2\right)^{p/2}}$$

$$\boxed{= \frac{3-p}{|\vec{r}|^p}}$$

$$\iiint_D \nabla \cdot \vec{F}\ dV = \iiint_D \frac{3-p}{|\vec{r}|^p}\ dV$$

$$= (3-p)\iiint_D r^{-p}\ dV$$

$$=(3-p)\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\int_a^b \rho^{-p}\,\rho^2\sin\varphi\; d\rho d\varphi d\theta$$

$$=(3-p)\int_{\theta_1}^{\theta_2} d\theta \int_{\varphi_1}^{\varphi_2}\sin\varphi\; d\varphi \int_a^b \rho^{2-p}\; d\rho$$

$$=(3-p)\left(\theta\Big|_{\theta_1}^{\theta_2}\right)\left(-\cos\varphi\Big|_{\varphi_1}^{\varphi_2}\right)\left(\frac{1}{3-p}\rho^{3-p}\Big|_a^b\right)$$

$$=\left(\theta_2-\theta_1\right)\left(\cos\varphi_1-\cos\varphi_2\right)\left(b^{3-p}-a^{3-p}\right)$$

In general, for $\iiint\limits_D \nabla\cdot\vec{F}\; dV = 0$, only if $3-p=0 \;\rightarrow\; \underline{p=3}$

Exercise

Consider the potential function $\phi(x, y, z)=G(\rho)$, where G is any twice differentiable function and $\rho=\sqrt{x^2+y^2+z^2}$; therefore, G depends only on the distance from the origin.

a) Show that the gradient vector field associated with ϕ is $\vec{F}=\nabla\phi=G'(\rho)\frac{\vec{r}}{\rho}$, where $\vec{r}=\langle x, y, z\rangle$ and $\rho=|\vec{r}|$.

b) Let S be the sphere of radius a centered at the origin and let D be the region enclosed by S. show that the flux of $\vec{F}$ across S is $\iint\limits_S \vec{F}\cdot\vec{n}\; dS = 4\pi a^2 G'(a)$.

c) Show that $\nabla\cdot\vec{F}=\nabla\cdot\nabla\phi=\frac{2}{\rho}G'(\rho)+G''(\rho)$

d) Use part (*c*) to show that the flux across S (as given in part (*b*)) is also obtained by the volume integral $\iiint\limits_D \nabla\cdot\vec{F}\; dV$. (Hint: use spherical coordinates and integrate by parts.)

Solution

a) $\phi(x, y, z)=G(\rho)$

$$\nabla\phi(x, y, z)=\left\langle \phi_x,\ \phi_y,\ \phi_z\right\rangle$$

$$=\left\langle G'(\rho)\rho_x,\ G'(\rho)\rho_y,\ G'(\rho)\rho_z\right\rangle$$

$$= G'(\rho)\left\langle \rho_x,\ \ \rho_y,\ \ \rho_z \right\rangle$$

$$= G'(\rho)\left\langle \frac{x}{\sqrt{x^2+y^2+z^2}},\ \frac{y}{\sqrt{x^2+y^2+z^2}},\ \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

$$= \frac{G'(\rho)}{\sqrt{x^2+y^2+z^2}}\langle x,\ y,\ y\rangle$$

$$= G'(\rho)\frac{\vec{r}}{\rho}$$

$$\therefore \qquad \vec{F} = \nabla\phi = G'(\rho)\frac{\vec{r}}{\rho}$$

b) The sphere of radius a centered at the origin $\rightarrow$ $x^2+y^2+z^2=a^2$

$$2xdx + 2zdz = 0 \quad \rightarrow \quad z_x = -\frac{x}{z}$$

$$2ydy + 2zdz = 0 \quad \rightarrow \quad z_y = -\frac{y}{z}$$

$$\vec{n} = \left\langle \frac{x}{z},\ \frac{y}{z},\ 1\right\rangle$$

$$\vec{F}\bullet\vec{n} = \frac{G'(\rho)}{\rho}\langle x,\ y,\ z\rangle \bullet \left\langle \frac{x}{z},\ \frac{y}{z},\ 1\right\rangle$$

$$= \frac{G'(a)}{a}\left(\frac{x^2}{z} + \frac{y^2}{z} + z\right)$$

$$= \frac{G'(a)}{a}\left(\frac{x^2+y^2+z^2}{z}\right)$$

$$= \frac{G'(a)}{a}\left(\frac{a^2}{z}\right)$$

$$= G'(a)\frac{a}{z}$$

$$\iint_S \vec{F}\bullet\vec{n}\ dS = aG'(a)\iint_S \frac{1}{z}\ dS$$

$$= aG'(a)\int_0^{2\pi}\int_0^a \frac{1}{\sqrt{a^2-r^2}}\ r\ drd\theta$$

$$= -\frac{1}{2}aG'(a)\int_0^{2\pi} d\theta \quad \int_0^a \left(a^2-r^2\right)^{-1/2} d\left(a^2-r^2\right)$$

$$= -2\pi aG'(a)\ \left(a^2-r^2\right)^{1/2}\ \Big|_0^a$$

$$= 2\pi a^2 G'(a)$$

Since the surface area is *twice* the value

Total surface $\boxed{= 4\pi a^2 G'(a)}$

c) $\vec{F} = \nabla\phi = G'(\rho)\dfrac{\vec{r}}{\rho}$ $\qquad\qquad \rho = \sqrt{x^2 + y^2 + z^2}$

$$\nabla \bullet \vec{F} = \nabla \bullet \left(G'(\rho)\frac{\vec{r}}{\rho} \right)$$

$$= \nabla \bullet \frac{G'(\rho)}{\rho}\langle x,\, y,\, z\rangle$$

$$= \frac{\partial}{\partial x}\left(G'(\rho)\frac{x}{\rho} \right) + \frac{\partial}{\partial y}\left(G'(\rho)\frac{y}{\rho} \right) + \frac{\partial}{\partial z}\left(G'(\rho)\frac{z}{\rho} \right)$$

$$\frac{\partial}{\partial x}\left(G'(\rho)\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) = G''(\rho)\rho_x\frac{x}{\rho} + G'(\rho)\frac{x^2 + y^2 + z^2 - x^2}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$= G''(\rho)\frac{x}{\rho}\frac{x}{\rho} + G'(\rho)\frac{y^2 + z^2}{\rho^3}$$

$$= G''(\rho)\frac{x^2}{\rho^2} + G'(\rho)\frac{y^2 + z^2}{\rho^3}$$

$$\frac{\partial}{\partial y}\left(G'(\rho)\frac{y}{\sqrt{x^2 + y^2 + z^2}} \right) = G''(\rho)\rho_y\frac{y}{\rho} + G'(\rho)\frac{x^2 + y^2 + z^2 - y^2}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$= G''(\rho)\frac{y}{\rho}\frac{y}{\rho} + G'(\rho)\frac{y^2 + z^2}{\rho^3}$$

$$= G''(\rho)\frac{y^2}{\rho^2} + G'(\rho)\frac{y^2 + z^2}{\rho^3}$$

$$\frac{\partial}{\partial z}\left(G'(\rho)\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) = G''(\rho)\rho_z\frac{z}{\rho} + G'(\rho)\frac{x^2 + y^2 + z^2 - z^2}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$= G''(\rho)\frac{z}{\rho}\frac{z}{\rho} + G'(\rho)\frac{y^2 + z^2}{\rho^3}$$

$$= G''(\rho)\frac{z^2}{\rho^2} + G'(\rho)\frac{y^2 + z^2}{\rho^3}$$

$$\nabla \bullet \vec{F} = G''(\rho)\frac{z^2}{\rho^2} + G'(\rho)\frac{y^2+z^2}{\rho^3} + G''(\rho)\frac{y^2}{\rho^2} + G'(\rho)\frac{x^2+z^2}{\rho^3} + G''(\rho)\frac{z^2}{\rho^2} + G'(\rho)\frac{x^2+y^2}{\rho^3}$$

$$= G''(\rho)\left(\frac{x^2+y^2+z^2}{\rho^2}\right) + G'(\rho)\frac{2\left(x^2+y^2+z^2\right)}{\rho^3}$$

$$= \frac{\rho^2}{\rho^2}G''(\rho) + G'(\rho)\frac{2\rho^2}{\rho^3}$$

$$= G''(\rho) + \frac{2}{\rho}G'(\rho)$$

d) By the divergence theorem, the flux:

$$\iiint_D \nabla \bullet \vec{F}\; dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2 \sin\varphi\left(G''(\rho) + \frac{2}{\rho}G'(\rho)\right)\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} d\theta \quad \int_0^{\pi} \sin\varphi\, d\varphi \quad \int_0^{a} \rho^2\left(G''(\rho) + \frac{2}{\rho}G'(\rho)\right)\, d\rho$$

$$= 2\pi\;\left(-\cos\varphi\;\Big|_0^{\pi}\right. \quad \int_0^{a} \rho^2\left(G''(\rho) + \frac{2}{\rho}G'(\rho)\right)\, d\rho$$

$$= 4\pi\int_0^{a}\left(\rho^2 G''(\rho) + 2\rho G'(\rho)\right)\, d\rho \qquad\qquad \left(\rho^2 G'(\rho)\right)' = \rho^2 G''(\rho) + 2\rho G'(\rho)$$

$$= 4\pi\int_0^{a}\left(\rho^2 G'(\rho)\right)'\, d\rho$$

$$= 4\pi G'(\rho)\;\rho^2\;\Big|_0^{a}$$

$$= 4\pi a^2 G'(\rho)$$

Exercise

Prove Green's Identity for scalar-valued functions u and v defined on a region D:

$$\iiint_D \left(u\nabla^2 v - v\nabla^2 u\right)\, dV = \iint_S \left(u\nabla v - v\nabla u\right) \bullet \vec{n}\; dS$$

Solution

$$\nabla \bullet \left(u\nabla v\right) = \nabla u \bullet \nabla v + u\nabla^2 v$$

$$\iiint_D \left(u\nabla^2 v + \nabla u \bullet \nabla v\right)\, dV = \iiint_D \left(\nabla \bullet \left(u\nabla v\right)\right)\, dV$$

$$= \iint_S u\nabla v \bullet \vec{n}\ dS$$

$$\iiint_D \left(v\nabla^2 u + \nabla u \bullet \nabla v\right)\, dV = \iiint_D \left(\nabla \bullet \left(v\nabla u\right)\right)\, dV$$

$$= \iint_S v\nabla u \bullet \vec{n}\ dS$$

$$\iiint_D \left(u\nabla^2 v + \nabla u \bullet \nabla v\right)\, dV - \iiint_D \left(v\nabla^2 u + \nabla u \bullet \nabla v\right)\, dV = \iint_S u\nabla v \bullet \vec{n}\ dS - \iint_S v\nabla u \bullet \vec{n}\ dS$$

$$\iiint_D \left(u\nabla^2 v + \nabla u \bullet \nabla v - v\nabla^2 u + \nabla u \bullet \nabla v\right)\, dV = \iint_S \left(u\nabla v \bullet \vec{n} - v\nabla u \bullet \vec{n}\right)\, dS$$

$$\iiint_D \left(u\nabla^2 v - v\nabla^2 u\right)\, dV = \iint_S \left(u\nabla v - v\nabla u\right) \bullet \vec{n}\ dS$$

Exercise

Prove the identity: $\displaystyle \iiint_D \nabla \times \vec{F}\ dV = \iint_S \left(\vec{n} \times \vec{F}\right) dS$

Solution

Let $\vec{F} = \langle f,\ g,\ h\rangle$ and $\vec{n} = \left\langle n_1,\ n_2,\ n_3\right\rangle$, then:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z},\ \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x},\ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$

$$\vec{F}\times\vec{n}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ f & g & h\\ n_1 & n_2 & n_3\end{vmatrix}$$

$$=\left\langle n_2h-n_3g,\ n_3f-n_1h,\ n_1g-n_2f\right\rangle$$

$\hat{i}\quad\to\quad n_2h-n_3g=\langle 0,\ h,\ -g\rangle\bullet\left\langle n_1,\ n_2,\ n_3\right\rangle$

$\vec{F}_1=\langle 0,\ h,\ -g\rangle$

$$\iint\limits_S\left(n_2h-n_3g\right)dS=\iint\limits_S\vec{F}_1\bullet\vec{n}\ dS$$

$$=\iiint\limits_D\left(\nabla\bullet\vec{F}_1\right)dR$$

$$=\iint\limits_D\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)dA$$

$\hat{j}\quad\to\quad n_3f-n_1h=\langle -h,\ 0,\ f\rangle\bullet\left\langle n_1,\ n_2,\ n_3\right\rangle$

$\vec{F}_2=\langle -h,\ 0,\ f\rangle$

$$\iint\limits_S\left(n_3f-n_1h\right)dS=\iint\limits_S\vec{F}_2\bullet\vec{n}\ dS$$

$$=\iiint\limits_D\left(\nabla\bullet\vec{F}_2\right)dR$$

$$=\iint\limits_D\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)dA$$

$\hat{k}\quad\to\quad n_1g-n_2f=\langle g,\ -f,\ 0\rangle\bullet\left\langle n_1,\ n_2,\ n_3\right\rangle$

$\vec{F}_3=\langle g,\ -f,\ 0\rangle$

$$\iint\limits_S\left(n_1g-n_2f\right)dS=\iint\limits_S\vec{F}_3\bullet\vec{n}\ dS$$

$$=\iiint\limits_D\left(\nabla\bullet\vec{F}_3\right)dR$$

$$= \iint\limits_{D} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

$$\iiint\limits_{D} \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z},\ \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x},\ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle dV = \iint\limits_{S} \left\langle n_2 h - n_3 g,\ n_3 f - n_1 h,\ n_1 g - n_2 f \right\rangle dS$$

$$\iiint\limits_{D} \nabla \times \vec{F}\ dV = \iint\limits_{S} \left(\vec{n} \times \vec{F} \right) dS \quad \checkmark$$

Exercise

Prove the identity: $\displaystyle \iint\limits_{S} \left(\vec{n} \times \nabla \varphi \right) dS = \oint_{C} \varphi\, d\vec{r}$

Solution

Let $\varphi = \left\langle \varphi_x,\ \varphi_y,\ \varphi_z \right\rangle$ and $\vec{n} = \left\langle n_1,\ n_2,\ n_3 \right\rangle$, then:

$$\vec{n} \times \nabla \varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ n_1 & n_2 & n_3 \\ \varphi_x & \varphi_y & \varphi_z \end{vmatrix}$$

$$= \left\langle n_2 \varphi_z - n_3 \varphi_y,\ n_3 \varphi_x - n_1 \varphi_z,\ n_1 \varphi_y - n_2 \varphi_x \right\rangle$$

$$\hat{i} \ \rightarrow \ n_2 \varphi_z - n_3 \varphi_y = \left\langle 0,\ \varphi_z,\ -\varphi_y \right\rangle \cdot \left\langle n_1,\ n_2,\ n_3 \right\rangle$$

$$= \left(\nabla \times \langle \varphi,\ 0,\ 0 \rangle \right) \cdot \left\langle n_1,\ n_2,\ n_3 \right\rangle$$

$$\vec{F}_1 = \langle \varphi,\ 0,\ 0 \rangle$$

$$\iint\limits_{S} \left(\left(\nabla \times \langle \varphi,\ 0,\ 0 \rangle \right) \cdot \left\langle n_1,\ n_2,\ n_3 \right\rangle \right) dS = \iint\limits_{S} \left(\nabla \times \vec{F}_1 \right) \cdot \vec{n}\ dS$$

$$= \oint_{C} \vec{F}_1 \cdot d\vec{r}$$

$$= \oint_{C} \langle \varphi,\ 0,\ 0 \rangle \cdot d\vec{r}$$

$$= \oint_C \varphi \bullet d\vec{r}$$

$$\hat{j} \quad \rightarrow \quad n_3 \varphi_x - n_1 \varphi_z = \left\langle -\varphi_z,\ 0,\ \varphi_x \right\rangle \bullet \left\langle n_1,\ n_2,\ n_3 \right\rangle$$

$$= \left(\nabla \times \langle 0,\ \varphi,\ 0 \rangle\right) \bullet \left\langle n_1,\ n_2,\ n_3 \right\rangle$$

$$\vec{F}_2 = \langle 0,\ \varphi,\ 0 \rangle$$

$$\iint_S \left(\left(\nabla \times \langle 0,\ \varphi,\ 0 \rangle\right) \bullet \left\langle n_1,\ n_2,\ n_3 \right\rangle\right) dS = \iint_S \left(\nabla \times \vec{F}_2\right) \bullet \vec{n}\ dS$$

$$= \oint_C \vec{F}_2 \bullet d\vec{r}$$

$$= \oint_C \langle 0,\ \varphi,\ 0 \rangle \bullet d\vec{r}$$

$$= \oint_C \varphi \bullet d\vec{r}$$

$$\hat{k} \quad \rightarrow \quad n_1 \varphi_y - n_2 \varphi_x = \left\langle \varphi_y,\ -\varphi_x,\ 0 \right\rangle \bullet \left\langle n_1,\ n_2,\ n_3 \right\rangle$$

$$= \left(\nabla \times \langle 0,\ 0,\ \varphi \rangle\right) \bullet \left\langle n_1,\ n_2,\ n_3 \right\rangle$$

$$\vec{F}_3 = \langle 0,\ 0,\ \varphi \rangle$$

$$\iint_S \left(\left(\nabla \times \langle 0,\ 0,\ \varphi \rangle\right) \bullet \left\langle n_1,\ n_2,\ n_3 \right\rangle\right) dS = \iint_S \left(\nabla \times \vec{F}_3\right) \bullet \vec{n}\ dS$$

$$= \oint_C \vec{F}_3 \bullet d\vec{r}$$

$$= \oint_C \langle 0,\ 0,\ \varphi \rangle \bullet d\vec{r}$$

$$= \oint_C \varphi \bullet d\vec{r}$$

$$\iint_S \left(\vec{n} \times \nabla \varphi\right) dS = \oint_C \varphi\, d\vec{r} \quad \checkmark$$