

## ***Solution***      **Section 4.4 – Eigenvalues & Eigenvectors**

### ***Exercise***

Find the eigenvalues and eigenvectors of  $A$ ,  $A^2$ ,  $A^{-1}$ , and  $A + 4I$  :

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

Check the trace  $\lambda_1 + \lambda_2$  and the determinant  $\lambda_1 \lambda_2$  for  $A$  and also  $A^2$ .

### **Solution**

***For  $A$ :***

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2 - \lambda & -1 \\ -1 & 2 - \lambda \end{vmatrix} \\ &= (2 - \lambda)^2 - 1 \\ &= \lambda^2 - 4\lambda + 3 = \mathbf{0} \end{aligned}$$

The eigenvalues of  $A$  are  $\lambda_1 = 1$  and  $\lambda_2 = 3$ .

The trace of a square matrix  $A$  is the sum of the elements on the main diagonal:  $2 + 2$  agrees with  $1 + 3$ . The  $\det(A) = 3$  agrees with the product  $\lambda_1 \lambda_2$ .

The eigenvectors for  $A$  are:

$$\lambda_1 = 1: \quad (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 2-1 & -1 \\ -1 & 2-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x - y = 0 \end{cases}$$
$$\Rightarrow \underline{x = y}$$

$$\text{Therefore, the eigenvector } V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 3: \quad (A - \lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -x - y = 0 \end{cases}$$
$$\Rightarrow \underline{x = -y}$$

$$\text{Therefore, the eigenvector } V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For  $A^2$ :

$$\begin{aligned}\det(A^2 - \lambda I) &= \begin{vmatrix} 5-\lambda & -4 \\ -4 & 5-\lambda \end{vmatrix} \\ &= (5-\lambda)^2 - 16 \\ &= \lambda^2 - 10\lambda + 9 = 0\end{aligned}$$

The eigenvalues of  $A^2$  are  $\lambda_1 = 1$  and  $\lambda_2 = 9$ . Or  $\lambda_1 = 1^2 = 1$  and  $\lambda_2 = 3^2 = 9$

$$\begin{cases} \text{tr}(A) = 5 + 5 = 10 \\ \lambda_1 + \lambda_2 = 1 + 9 = 10 \end{cases}$$

$$\underline{\text{tr}(A) = \lambda_1 + \lambda_2}$$

$$\begin{cases} |A^2| = \begin{vmatrix} 5 & -4 \\ -4 & 5 \end{vmatrix} = 9 \\ \lambda_1 \lambda_2 = 1(9) = 9 \end{cases}$$

$$\Rightarrow \underline{|A^2| = \lambda_1 \lambda_2}$$

$$\lambda_1 = 1: (A^2 - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 4x - 4y = 0 \\ -4x + 4y = 0 \end{cases}$$
$$\Rightarrow \underline{x = y}$$

Therefore; the eigenvector  $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\lambda_2 = 9: (A^2 - \lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -4x - 4y = 0 \\ -4x - 4y = 0 \end{cases}$$
$$\Rightarrow \underline{x = -y}$$

Therefore; the eigenvector  $V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For  $A^{-1}$ :

$$\begin{aligned}A^{-1} &= \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\det(A^{-1} - \lambda I) &= \begin{vmatrix} \frac{2}{3} - \lambda & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} - \lambda \end{vmatrix} \\ &= \left(\frac{2}{3} - \lambda\right)^2 - \frac{1}{9} \\ &= \lambda^2 - \frac{4}{3}\lambda + \frac{1}{3} = 0\end{aligned}$$

The eigenvalues of  $A^{-1}$  are  $\lambda_1 = 1$  and  $\lambda_2 = \frac{1}{3}$ .

$$\lambda_1 = 1: \quad (A^{-1} - \lambda_1 I)V_1 = 0$$

$$\begin{aligned}\begin{pmatrix} \frac{2}{3} - 1 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -\frac{1}{3}x + \frac{1}{3}y = 0 \end{cases} \\ &\rightarrow \underline{x = y} \end{aligned}$$

Therefore; the eigenvector  $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\lambda_2 = \frac{1}{3}: \quad (A^{-1} - \lambda_2 I)V_2 = 0$$

$$\begin{aligned}\begin{pmatrix} \frac{2}{3} - \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} - \frac{1}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \frac{1}{3}x + \frac{1}{3}y = 0 \end{cases} \\ &\rightarrow \underline{x = -y} \end{aligned}$$

Therefore; the eigenvector  $V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

**For  $A + 4I$ :**

$$\begin{aligned}A + 4I &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}
 \det(A^{-1} - \lambda I) &= \begin{vmatrix} 6-\lambda & 1 \\ 1 & 6-\lambda \end{vmatrix} \\
 &= (6-\lambda)^2 - 1 \\
 &= \lambda^2 - 12\lambda + 35 = 0
 \end{aligned}$$

The eigenvalues of  $A^{-1}$  are  $\lambda_1 = 5$  and  $\lambda_2 = 7$ .

$$\lambda_1 = 5: (A + 4I - \lambda_1 I)V_1 = 0$$

$$\begin{aligned}
 \begin{pmatrix} 6-5 & 1 \\ 1 & 6-5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x+y=0 \\ x+y=0 \end{cases} \\
 &\rightarrow \underline{x = -y}
 \end{aligned}$$

$$\text{Therefore; the eigenvector } V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = 7: (A + 4I - \lambda_2 I)V_2 = 0$$

$$\begin{aligned}
 \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x-y=0 \\ x-y=0 \end{cases} \\
 &\rightarrow \underline{x = y}
 \end{aligned}$$

$$\text{Therefore; the eigenvector } V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The eigenvalues  $(A) = \lambda$

The eigenvalues  $(A^2) = \lambda^2$

The eigenvalues  $(A^{-1}) = \frac{1}{\lambda}$

### ***Exercise***

Show directly that the given vectors are eigenvectors of the given matrix. What are the corresponding eigenvalues?

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$$

### **Solution**

$$\begin{aligned} A\vec{v}_1 &= \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ -21 \end{bmatrix} \\ &= 7 \begin{bmatrix} 1 \\ -3 \end{bmatrix} \\ &= \underline{7\vec{v}_1} \end{aligned}$$

$\vec{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$  is an eigenvector corresponding to the eigenvalue 7

$$\begin{aligned} A\vec{v}_2 &= \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= 0 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \underline{0\vec{v}_2} \end{aligned}$$

$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  is an eigenvector corresponding to the eigenvalue 0

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & -2 \\ -3 & 6-\lambda \end{vmatrix} \\ &= (1-\lambda)(6-\lambda) - 6 \\ &= 6 - 7\lambda + \lambda^2 - 6 \\ &= \lambda^2 - 7\lambda = \mathbf{0} \end{aligned}$$

The eigenvalues are:  $\lambda_1 = 0$  and  $\lambda_2 = 7$

### ***Exercise***

For which real numbers  $c$  does this matrix  $A$  have

$$A = \begin{pmatrix} 2 & -c \\ -1 & 2 \end{pmatrix}$$

- a) Two real eigenvalues and eigenvectors.
- b) A repeated eigenvalue with only one eigenvector
- c) Two complex eigenvalues and eigenvectors.

### **Solution**

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 2 - \lambda & -c \\ -1 & 2 - \lambda \end{vmatrix} \\ &= (2 - \lambda)^2 - c \\ &= \lambda^2 - 4\lambda + 4 - c = 0 \end{aligned}$$

$$\lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-c)}}{2(1)}$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 + 4c}}{2}$$

- a) Two real eigenvalues and eigenvectors, when

$$16 + 4c > 0$$

$$4c > -16$$

$$\underline{c > -4}$$

- b) A repeated eigenvalue with only one eigenvector, when

$$16 + 4c = 0$$

$$\underline{c = -4}$$

- c) Two complex eigenvalues and eigenvectors, when

$$16 + 4c < 0$$

$$\underline{c < -4}$$

### Exercise

Find the eigenvalues of  $A$ ,  $B$ ,  $AB$ , and  $BA$ :

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

- a) The eigenvalues of  $AB$  (are equal to) (are not equal to) eigenvalues of  $A$  times eigenvalues of  $B$ .
- b) The eigenvalues of  $AB$  (are equal to) (are not equal to) eigenvalues of  $BA$ .

### Solution

Since  $A$  is a lower triangular, then  $\lambda_1 = \lambda_2 = 1$

Since  $B$  is an upper triangular, then  $\lambda_1 = \lambda_2 = 1$

$$\begin{aligned} \det(AB - I) &= \begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)(2-\lambda) - 1 \\ &= \lambda^2 - 3\lambda + 1 = 0 \end{aligned}$$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{5}}{2}$$

$$\begin{aligned} \det(BA - I) &= \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} \\ &= (2-\lambda)(1-\lambda) - 1 \\ &= \lambda^2 - 3\lambda + 1 = 0 \end{aligned}$$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{5}}{2}$$

- a) The eigenvalues of  $AB$  are **not** equal to eigenvalues of  $A$  times eigenvalues of  $B$ .
- b) The eigenvalues of  $AB$  are equal to the eigenvalues of  $BA$ .

### Exercise

When  $a + b = c + d$  show that  $(1, 1)$  is an eigenvector and find both eigenvalues of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

### Solution

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} a+b \\ c+d \end{pmatrix} \\ &= \begin{pmatrix} a+b \\ a+b \end{pmatrix} \\ &= (a+b) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

$$\text{If } a + b = c + d = \lambda_1$$

$$\text{tr}(A) = a + d = \lambda_1 + \lambda_2$$

$$\lambda_2 = (a + d) - \lambda_1$$

$$= a + d - (a + b)$$

$$= a + d - a - b$$

$$= d - b \quad \text{or} \quad = a - c$$

The eigenvalues for  $\lambda_2$  :

$$\begin{pmatrix} a - \lambda_2 & b \\ c & d - \lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a - (a - c) & b \\ c & d - (d - b) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c & b \\ c & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \{cx + by = 0$$

$$\underline{cx = -by} \mid$$

$$\text{The eigenvector: } \mathbf{v}_2 = \begin{pmatrix} b \\ -c \end{pmatrix}$$



### Exercise

The eigenvalues of  $A$  equal to the eigenvalues of  $A^T$ . This is because  $\det(A - \lambda I)$  equals  $\det(A^T - \lambda I)$ .

That is true because \_\_\_\_\_. Show by an example that the eigenvectors of  $A$  and  $A^T$  are not the same.

### Solution

$$\begin{aligned}\det(A - \lambda I) &= \det(A - \lambda I)^T \\ &= \det(A^T - (\lambda I)^T) \\ &= \det(A^T - \lambda I)\end{aligned}$$

Therefore,  $A$  and  $A^T$  have the same eigenvalues.

Let consider the matrix:

$$A = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$$

$$\begin{aligned}|A - \lambda I| &= \begin{vmatrix} -\lambda & 1 \\ 4 & -\lambda \end{vmatrix} \\ &= \lambda^2 - 4 = 0\end{aligned}$$

The eigenvalues of  $A$  are:  $\lambda_{1,2} = \pm 2$

$$\begin{aligned}\text{For } \lambda_1 = -2: \quad (A - \lambda_1 I)V_1 &= 0 \\ \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x + y = 0 \\ y = -2x \end{cases}\end{aligned}$$

$$V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\begin{aligned}\text{For } \lambda_2 = 2: \quad (A - \lambda_2 I)V_2 &= 0 \\ \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -2x + y = 0 \\ y = 2x \end{cases}\end{aligned}$$

$$V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For the transpose matrix  $A^T$

$$|A^T - \lambda I| = \begin{vmatrix} -\lambda & 4 \\ 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2 - 4 = 0$$

The eigenvalues of  $A^T$  are:  $\lambda_{1,2} = \pm 2$

For  $\lambda_1 = -2$ :  $(A^T - \lambda_1 I)V_3 = 0$

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x + 2y = 0 \end{cases}$$

$$\underline{x = -2y}$$

$$V_3 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

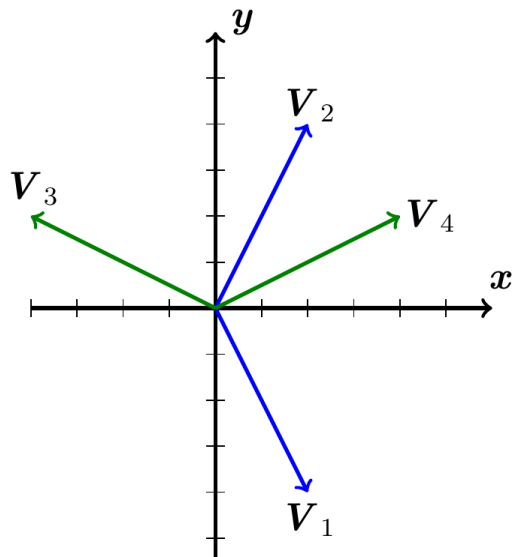
For  $\lambda_2 = 2$ :  $(A^T - \lambda_2 I)V_4 = 0$

$$\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x - 2y = 0 \end{cases}$$

$$\underline{x = 2y}$$

$$V_4 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad V_3 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad V_4 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$



The eigenvectors of  $A$  and  $A^T$  are not the same and from the graph they are not on same line.

### Exercise

Let  $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ . Compute the eigenvalues and eigenvectors of  $A$ .

### Solution

$$\det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & -1 \\ 1 & 2 - \lambda \end{vmatrix} \\ = (2 - \lambda)^2 + 1 = 0$$

$$(2 - \lambda)^2 = -1$$

$$2 - \lambda = \pm \sqrt{-1}$$

$$\underline{= \pm i}$$

The eigenvalues of  $A$  are:  $\lambda_{1,2} = 2 \pm i$

$$\text{For } \lambda_1 = 2 - i \Rightarrow (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 2 - (2 - i) & -1 \\ 1 & 2 - (2 - i) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x + iy = 0 \end{cases}$$

$$\underline{x = -iy}$$

$$\text{The eigenvector is: } V_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$\text{For } \lambda_2 = 2 + i \Rightarrow (A - \lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x - iy = 0 \end{cases}$$

$$\underline{x = iy}$$

$$\text{The eigenvector is: } V_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

### Exercise

$$\text{Let } A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

- What is the characteristic polynomial for  $A$  (i.e. compute  $\det(A - \lambda I)$ )?
- Verify that 1 is an eigenvalue of  $A$ . What is a corresponding eigenvector?
- What are the other eigenvalues of  $A$ ?

### Solution

$$\begin{aligned} a) \quad \det(A - \lambda I) &= \begin{vmatrix} 2 - \lambda & -2 & 3 \\ 1 & 1 - \lambda & 1 \\ 1 & 3 & -1 - \lambda \end{vmatrix} \\ &= (2 - \lambda)(1 - \lambda)(-1 - \lambda) - 2 + 9 - 3(1 - \lambda) - 3(2 - \lambda) + 2(-1 - \lambda) \\ &= (2 - 3\lambda + \lambda^2)(-1 - \lambda) + 7 - 3 + 3\lambda - 6 + 3\lambda - 2 - 2\lambda \\ &= -2 + 3\lambda - \lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3 + 4\lambda - 4 \\ &= \underline{-\lambda^3 + 2\lambda^2 + 5\lambda - 6} \end{aligned}$$

$$\begin{aligned}
 b) \quad \text{If } \lambda = 1 &\rightarrow -\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0 \\
 &-1^3 + 2(1)^2 + 5(1) - 6 = 0 \\
 &-1 + 2 + 5 - 6 = 0 \\
 &\boxed{0 = 0}
 \end{aligned}$$

1 is an eigenvalue of  $A$ .

$$\begin{pmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \rightarrow \begin{cases} x - 2y + 3z = 0 \\ x + z = 0 \\ x + 3y - 2z = 0 \end{cases}$$

$$\begin{cases} \underline{x = -z} \\ 3y = 2z - x = 2z + z = 3z \Rightarrow \underline{y = z} \end{cases}$$

The eigenvector for  $\lambda = 1$  is  $V = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$$\begin{aligned}
 c) \quad &-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0 \\
 &\underline{\lambda_1 = 1 \quad \lambda_2 = -2 \quad \lambda_3 = 3}
 \end{aligned}$$

## Exercise

For the matrix:  $\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

## Solution

$$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$$

$$i. \quad \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix}$$

$$= (3 - \lambda)(-1 - \lambda) - 0$$

$$= \lambda^2 - 2\lambda - 3$$

The characteristic equation:  $\lambda^2 - 2\lambda - 3$

**ii.**  $\lambda^2 - 2\lambda - 3 = 0$

The eigenvalues are  $\lambda_1 = -1$  and  $\lambda_2 = 3$

**iii.**  $\lambda_1 = -1 \rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 0 \\ 8 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 4x = 0 \\ 8x = 0 \end{cases}$$

$$\underline{x = 0}$$

Therefore, the eigenvector  $V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda_2 = 3 \rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 8x - 4y = 0 \end{cases}$$

$$\underline{2x = y}$$

Therefore; the eigenvector  $V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

The eigenvectors are given by:  $V = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

### **Exercise**

For the matrix:  $\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

**i.** Find the characteristic equation

**ii.** Find the eigenvalues

**iii.** Find the eigenvectors

### **Solution**

$$A = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$$

**i.**  $\det(A - \lambda I) = \begin{vmatrix} 10 - \lambda & -9 \\ 4 & -2 - \lambda \end{vmatrix}$

$$= (10 - \lambda)(-2 - \lambda) + 36$$

$$= \lambda^2 - 8\lambda + 16$$

The characteristic equation:  $\lambda^2 - 8\lambda + 16 = 0$

ii.  $\lambda^2 - 8\lambda + 16 = 0$

The eigenvalues are  $\lambda_{1,2} = 4$

iii.  $\lambda_1 = 4 \rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 4x - 6y = 0 \end{cases}$$

$$\underline{2x = 3y}$$

Therefore; the eigenvector  $V_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

For the second eigenvector  $V_2 \Rightarrow AV_2 = V_1$

$$\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \rightarrow \begin{cases} 4x - 2y = 2 \end{cases}$$

$$\underline{y = 2x - 1}$$

If  $x = 1 \Rightarrow y = 1$

$$\underline{V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

### Exercise

For the matrix:  $\begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}$$

i.  $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 3 \\ 4 & -\lambda \end{vmatrix}$   
 $= \lambda^2 - 12$

The characteristic equation:  $\lambda^2 - 12 = 0$

ii.  $\lambda^2 - 12 = 0$

The eigenvalues are  $\lambda_{1,2} = \pm\sqrt{12}$

iii. For  $\lambda_1 = -\sqrt{12} \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} \sqrt{12} & 3 \\ 4 & \sqrt{12} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \sqrt{12}x + 3y = 0 \\ 4x + \sqrt{12}y = 0 \end{cases}$$

$$\rightarrow \sqrt{12}x = -3y$$

Therefore; the eigenvector  $V_1 = \begin{pmatrix} -3 \\ \sqrt{12} \end{pmatrix}$

For  $\lambda_2 = \sqrt{12} \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -\sqrt{12} & 3 \\ 4 & -\sqrt{12} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -\sqrt{12}x + 3y = 0 \\ 4x - \sqrt{12}y = 0 \end{cases}$$

$$\rightarrow \sqrt{12}x = 3y$$

Therefore; the eigenvector  $V_2 = \begin{pmatrix} 3 \\ \sqrt{12} \end{pmatrix}$

The vectors are given by:  $V = \begin{pmatrix} -3 & 3 \\ \sqrt{12} & \sqrt{12} \end{pmatrix}$

### Exercise

For the matrix:  $\begin{pmatrix} -2 & -7 \\ 1 & 2 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} -2 & -7 \\ 1 & 2 \end{pmatrix}$$

i.  $\begin{vmatrix} -2-\lambda & -7 \\ 1 & 2-\lambda \end{vmatrix} = (-2-\lambda)(2-\lambda) + 7$

$$\begin{aligned}
 &= -4 + \lambda^2 + 7 \\
 &= \lambda^2 + 3
 \end{aligned}$$

The characteristic equation:  $\lambda^2 + 3 = 0$

**ii.**  $\lambda^2 = -3$

The eigenvalues are:  $\lambda_{1,2} = \pm i\sqrt{3}$

**iii.** For  $\lambda_1 = -i\sqrt{3} \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -2+i\sqrt{3} & -7 \\ 1 & 2+i\sqrt{3} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x_1 + (2+i\sqrt{3})y_1 = 0 \end{cases}$$

$$\underline{x_1 = -(2+i\sqrt{3})y_1}$$

Therefore; the eigenvector  $V_1 = \begin{pmatrix} 2+i\sqrt{3} \\ -1 \end{pmatrix}$

For  $\lambda_2 = i\sqrt{3} \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -2-i\sqrt{3} & -7 \\ 1 & 2-i\sqrt{3} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x_2 + (2-i\sqrt{3})y_2 = 0 \end{cases}$$

$$\underline{x_2 = -(2-i\sqrt{3})y_2}$$

Therefore; the eigenvector  $V_2 = \begin{pmatrix} 2-i\sqrt{3} \\ -1 \end{pmatrix}$

The vectors are given by:  $V = \begin{pmatrix} 2+i\sqrt{3} & 2-i\sqrt{3} \\ -1 & -1 \end{pmatrix}$

### Exercise

For the matrix:  $\begin{pmatrix} 12 & 14 \\ -7 & -9 \end{pmatrix}$

**i.** Find the characteristic equation

**ii.** Find the eigenvalues

**iii.** Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 12 & 14 \\ -7 & -9 \end{pmatrix}$$



$$\begin{aligned}
 i. \quad |A - \lambda I| &= \begin{vmatrix} 12 - \lambda & 14 \\ -7 & -9 - \lambda \end{vmatrix} \\
 &= (12 - \lambda)(-9 - \lambda) - (14)(-7) \\
 &= -108 - 12\lambda + 9\lambda + \lambda^2 + 98 \\
 &= \lambda^2 - 3\lambda - 10
 \end{aligned}$$

The characteristic equation:  $\lambda^2 - 3\lambda - 10 = 0$

ii. The eigenvalues are:  $\lambda_1 = -2$  and  $\lambda_2 = 5$

iii. For  $\lambda_1 = -2$ , we have:  $(A - \lambda_1 I)V_1 = 0$

$$\begin{aligned}
 \begin{pmatrix} 12 + 2 & 14 \\ -7 & -9 + 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
 \begin{pmatrix} 14 & 14 \\ -7 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 14x + 14y = 0 \\ x = -y \end{cases}
 \end{aligned}$$

$$\Rightarrow V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For  $\lambda_2 = 5$ , we have  $(A - \lambda_2 I)V_2 = 0$

$$\begin{aligned}
 \begin{pmatrix} 7 & 14 \\ -7 & -14 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 7x + 14y = 0 \\ x = -2y \end{cases}
 \end{aligned}$$

$$\Rightarrow V_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

The vectors are given by:  $V = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$

### Exercise

For the matrix:  $\begin{pmatrix} -4 & 1 \\ -2 & 1 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} -4 & 1 \\ -2 & 1 \end{pmatrix}$$

$$\begin{aligned} i. \quad \det(A - \lambda I) &= \begin{vmatrix} -4 - \lambda & 1 \\ -2 & 1 - \lambda \end{vmatrix} \\ &= (-4 - \lambda)(1 - \lambda) + 2 \\ &= \lambda^2 + 3\lambda - 2 \end{aligned}$$

$$\text{The characteristic equation: } \lambda^2 + 3\lambda - 2 = 0$$

$$ii. \quad \text{Thus, the eigenvalues are: } \lambda_{1,2} = \frac{-3 \pm \sqrt{17}}{2}$$

$$iii. \quad \text{For } \lambda_1 = \frac{-3 - \sqrt{17}}{2} : (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} -4 - \frac{-3 - \sqrt{17}}{2} & 1 \\ -2 & 1 - \frac{-3 - \sqrt{17}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-5 + \sqrt{17}}{2} & 1 \\ -2 & \frac{5 + \sqrt{17}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} \frac{-5 + \sqrt{17}}{2} x + y = 0 \\ -2x + \frac{5 + \sqrt{17}}{2} y = 0 \end{cases}$$

$$x = \left( \frac{5 + \sqrt{17}}{4} \right) y$$

$$\Rightarrow V_1 = \begin{pmatrix} \frac{5 + \sqrt{17}}{4} \\ 1 \end{pmatrix}$$

$$\text{For } \lambda_2 = \frac{-3+\sqrt{17}}{2} : \quad (A - \lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} -4 - \frac{-3+\sqrt{17}}{2} & 1 \\ -2 & 1 - \frac{-3+\sqrt{17}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-5-\sqrt{17}}{2} & 1 \\ -2 & \frac{5-\sqrt{17}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} \frac{-5-\sqrt{17}}{2}x + y = 0 \\ -2x + \frac{5-\sqrt{17}}{2}y = 0 \end{cases}$$

$$\underline{x = \left( \frac{5-\sqrt{17}}{4} \right) y}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} \frac{5-\sqrt{17}}{4} \\ 1 \end{pmatrix}}$$

$$\text{The eigenvectors can be written: } \begin{pmatrix} \frac{5+\sqrt{17}}{4} & \frac{5-\sqrt{17}}{4} \\ 1 & 1 \end{pmatrix}$$

### Exercise

For the matrix:  $\begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$$

$$\begin{aligned} i. \quad \det(A - \lambda I) &= \begin{vmatrix} 5 - \lambda & 3 \\ -6 & -4 - \lambda \end{vmatrix} \\ &= (-4 - \lambda)(5 - \lambda) + 18 \\ &= \lambda^2 - \lambda - 2 \end{aligned}$$

The characteristic equation:  $\lambda^2 - \lambda - 2 = 0$

ii. Thus, the eigenvalues are:  $\lambda_1 = -1$  and  $\lambda_2 = 2$

iii. For  $\lambda_1 = -1$ :  $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 6 & 3 \\ -6 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 6x + 3y = 0 \\ -6x - 3y = 0 \end{cases}$$

$$\underline{y = -2x}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

For  $\lambda_2 = 2$   $\Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 3 & 3 \\ -6 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 3x + 3y = 0 \\ -6x - 6y = 0 \end{cases}$$

$$\underline{y = -x}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

### Exercise

For the matrix:  $\begin{pmatrix} -2 & 3 \\ 0 & -5 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} -2 & 3 \\ 0 & -5 \end{pmatrix}$$

$$\begin{aligned} i. \quad \det(A - \lambda I) &= \begin{vmatrix} -2 - \lambda & 3 \\ 0 & -5 - \lambda \end{vmatrix} \\ &= (-2 - \lambda)(-5 - \lambda) - 0 \\ &= (2 + \lambda)(5 + \lambda) \end{aligned}$$

The characteristic equation:  $(2 + \lambda)(5 + \lambda) = 0$

ii. Thus, the eigenvalues are:  $\lambda_1 = -5$  and  $\lambda_2 = -2$

iii. For  $\lambda_1 = -5$ :  $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 3x + 3y = 0 \\ y = -x \end{cases}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

For  $\lambda_2 = -2$ :  $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 3 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 3y = 0 \\ -3y = 0 \end{cases}$$

$$\underline{y = 0}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

### Exercise

For the matrix:  $\begin{pmatrix} 2 & 0 \\ -4 & -2 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 2 & 0 \\ -4 & -2 \end{pmatrix}$$

$$\begin{aligned} i. \quad |A - \lambda I| &= \begin{vmatrix} 2 - \lambda & 0 \\ -4 & -2 - \lambda \end{vmatrix} \\ &= (2 - \lambda)(-2 - \lambda) - 0 \\ &= \lambda^2 - 4 \end{aligned}$$

The characteristic equation:  $\lambda^2 - 4 = 0$

ii. Thus, the eigenvalues are:  $\lambda_1 = -2$  and  $\lambda_2 = 2$

iii. For  $\lambda_1 = -2$  :  $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 0 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 4x = 0 \end{cases}$$

$$\underline{x = 0}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

For  $\lambda_2 = 2$  :  $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -4x - 4y = 0 \end{cases}$$

$$\underline{x = -y}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

### Exercise

For the matrix:  $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} \\ &= \lambda^2 - 4\lambda - 5 \end{aligned}$$

The characteristic equation:  $\lambda^2 - 4\lambda - 5 = 0$

ii. The eigenvalues are:  $\lambda_1 = -1$  and  $\lambda_2 = 5$

iii. For  $\lambda_1 = -1$  :  $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x + 2y = 0 \end{cases}$$

$$\underline{x = -y}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

For  $\lambda_2 = 5$  :  $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -4x + 2y = 0 \end{cases}$$

$$\underline{2x = y}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

### Exercise

For the matrix:  $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} 2 - \lambda & 2 \\ 1 & 3 - \lambda \end{vmatrix} \\ &= \lambda^2 - 5\lambda + 4 \end{aligned}$$

The characteristic equation:  $\lambda^2 - 5\lambda + 4 = 0$

ii. The eigenvalues are:  $\lambda_1 = 1$  and  $\lambda_2 = 4$

iii. For  $\lambda_1 = 1$  :  $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x + 2y = 0 \end{cases}$$

$$\underline{x = -2y}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$

For  $\lambda_2 = 4$   $\Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x - y = 0 \end{cases}$$

$$\underline{x = y}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$



### Exercise

For the matrix:  $\begin{pmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{pmatrix}$

- i.* Find the characteristic equation
- ii.* Find the eigenvalues
- iii.* Find the eigenvectors

### Solution

$$A = \begin{pmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{pmatrix}$$

$$\begin{aligned} i. \quad |A - \lambda I| &= \begin{vmatrix} -4 - \lambda & 2 \\ -\frac{5}{2} & 2 - \lambda \end{vmatrix} \\ &= \lambda^2 + 2\lambda - 3 \end{aligned}$$

The characteristic equation:  $\lambda^2 + 2\lambda - 3 = 0$

*ii.* The eigenvalues are:  $\lambda_1 = -3$  and  $\lambda_2 = 1$

*iii.* For  $\lambda_1 = -3$  :  $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -1 & 2 \\ -\frac{5}{2} & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -x + 2y = 0 \\ \end{cases}$$

$$\underline{x = 2y}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

For  $\lambda_2 = 1$  :  $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -5 & 2 \\ -\frac{5}{2} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -5x + 2y = 0 \\ \end{cases}$$

$$\underline{5x = 2y}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}}$$

### Exercise

For the matrix:  $\begin{pmatrix} -\frac{5}{2} & 2 \\ \frac{3}{4} & -2 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} -\frac{5}{2} & 2 \\ \frac{3}{4} & -2 \end{pmatrix}$$

$$\begin{aligned} i. \quad |A - \lambda I| &= \begin{vmatrix} -\frac{5}{2} - \lambda & 2 \\ \frac{3}{4} & -2 - \lambda \end{vmatrix} \\ &= \lambda^2 + \frac{9}{2}\lambda + \frac{7}{2} \end{aligned}$$

The characteristic equation:  $2\lambda^2 + 9\lambda + 7 = 0$

ii. The eigenvalues are:  $\lambda_1 = -\frac{7}{2}$  and  $\lambda_2 = -1$

iii. For  $\lambda_1 = -\frac{7}{2}$  :  $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 2 \\ \frac{3}{4} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x + 2y = 0 \end{cases}$$

$$\underline{x = -2y}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$

For  $\lambda_2 = -1$  :  $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -\frac{3}{2} & 2 \\ \frac{3}{4} & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -\frac{3}{2}x + 2y = 0 \end{cases}$$

$$\underline{3x = 4y}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}}$$

### Exercise

For the matrix:  $\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} 3 - \lambda & -1 \\ 9 & -3 - \lambda \end{vmatrix} \\ &= (3 - \lambda)(-3 - \lambda) + 9 \\ &= \lambda^2 \end{aligned}$$

The characteristic equation:  $\lambda^2 = 0$

ii. The eigenvalues are:  $\lambda_{1,2} = 0$

iii. For  $\lambda_1 = 0$  :  $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 3x - y = 0 \\ 3x - y = 0 \end{cases}$$

$$\underline{3x = y}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}}$$

For the second eigenvector  $V_2 \Rightarrow AV_2 = V_1$

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \rightarrow \begin{cases} 3x - y = 1 \\ 3x - y = 1 \end{cases}$$

$$\rightarrow \text{if } x = 1 \Rightarrow y = 2$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

### Exercise

For the matrix:  $\begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} -6 - \lambda & 5 \\ -5 & 4 - \lambda \end{vmatrix} \\ &= -24 + 2\lambda + \lambda^2 + 25 \\ &= \lambda^2 + 2\lambda + 1 \end{aligned}$$

The characteristic equation:  $\lambda^2 + 2\lambda + 1 = 0$

$$\text{ii. The eigenvalues are: } \lambda_{1,2} = -1$$

$$\text{iii. For } \lambda_1 = 0 : (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -5x + 5y = 0 \end{cases}$$

$$\underline{x = y}$$

$$\Rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For the second eigenvector  $V_2 \Rightarrow AV_2 = V_1$

$$\begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{cases} -6x_2 + 5y_2 = 1 \end{cases}$$

$$\rightarrow \text{if } x_2 = 0 \rightarrow y_2 = \frac{1}{5}$$

$$\Rightarrow V_2 = \begin{pmatrix} 0 \\ \frac{1}{5} \end{pmatrix}$$

### Exercise

For the matrix:  $\begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix}$

- i.* Find the characteristic equation
- ii.* Find the eigenvalues
- iii.* Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix}$$

$$\begin{aligned} i. \quad |A - \lambda I| &= \begin{vmatrix} 6 - \lambda & -1 \\ 5 & 2 - \lambda \end{vmatrix} \\ &= \lambda^2 - 8\lambda + 17 \end{aligned}$$

The characteristic equation:  $\lambda^2 - 8\lambda + 17 = 0$

$$ii. \quad \lambda_{1,2} = \frac{8 \pm \sqrt{64 - 68}}{2}$$

The eigenvalues are:  $\lambda_{1,2} = 4 \pm i$

$$iii. \text{ For } \lambda_1 = 4 - i : (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 2 - i & -1 \\ 5 & -2 - i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} (2 - i)x - y = 0 \\ 5x - (2 + i)y = 0 \end{cases}$$

$$(2 - i)x = y$$

$$\Rightarrow V_1 = \begin{pmatrix} 1 \\ 2 - i \end{pmatrix}$$

$$\Rightarrow V_2 = \begin{pmatrix} 1 \\ 2 + i \end{pmatrix}$$

### Exercise

For the matrix:  $\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} 1 - \lambda & 1 \\ -2 & -1 - \lambda \end{vmatrix} \\ &= \lambda^2 + 1 \end{aligned}$$

The characteristic equation:  $\lambda^2 + 1 = 0$

$$\text{ii. } \lambda^2 = -1$$

The eigenvalues are:  $\lambda_{1,2} = \pm i$

$$\text{iii. For } \lambda_1 = -i : (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 1+i & 1 \\ -2 & -1+i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} (1+i)x + y = 0 \end{cases}$$

$$(1+i)x = -y$$

$$\Rightarrow V_1 = \begin{pmatrix} -1 \\ 1+i \end{pmatrix}$$

$$\Rightarrow V_2 = \begin{pmatrix} -1 \\ 1-i \end{pmatrix}$$

### Exercise

For the matrix:  $\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}$

- i.* Find the characteristic equation
- ii.* Find the eigenvalues
- iii.* Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}$$

$$\begin{aligned} i. \quad |A - \lambda I| &= \begin{vmatrix} 5 - \lambda & 1 \\ -2 & 3 - \lambda \end{vmatrix} \\ &= 15 - 8\lambda + \lambda^2 + 2 \\ &= \lambda^2 - 8\lambda + 17 \end{aligned}$$

The characteristic equation:  $\lambda^2 - 8\lambda + 17 = 0$

$$ii. \quad \lambda_{1,2} = \frac{8 \pm \sqrt{64 - 68}}{2}$$

The eigenvalues are:  $\lambda_{1,2} = 4 \pm i$

$$iii. \text{ For } \lambda_1 = 4 - i : (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 1+i & 1 \\ -2 & -1+i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} (1+i)x + y = 0 \end{cases}$$

$$(1+i)x = -y$$

$$\Rightarrow V_1 = \begin{pmatrix} -1 \\ 1+i \end{pmatrix}$$

$$\Rightarrow V_2 = \begin{pmatrix} -1 \\ 1-i \end{pmatrix}$$

### Exercise

For the matrix:  $\begin{pmatrix} 4 & 5 \\ -2 & 6 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 4 & 5 \\ -2 & 6 \end{pmatrix}$$

$$\begin{aligned} i. \quad |A - \lambda I| &= \begin{vmatrix} 4 - \lambda & 5 \\ -2 & 6 - \lambda \end{vmatrix} \\ &= 24 - 10\lambda + \lambda^2 + 10 \\ &= \lambda^2 - 10\lambda + 34 \end{aligned}$$

$$\text{The characteristic equation: } \lambda^2 - 10\lambda + 34 = 0$$

$$ii. \quad \lambda_{1,2} = \frac{10 \pm \sqrt{100 - 136}}{2}$$

$$\text{The eigenvalues are: } \lambda_{1,2} = 5 \pm 3i$$

$$iii. \text{ For } \lambda_1 = 5 - 3i : (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} -1 + 3i & 5 \\ -2 & 1 + 3i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} (-1 + 3i)x + 5y = 0 \end{cases}$$

$$\underline{(-1 + 3i)x = -5y}$$

$$\Rightarrow V_1 = \begin{pmatrix} -1 \\ -1 + 3i \end{pmatrix}$$

$$\Rightarrow V_2 = \begin{pmatrix} -1 \\ -1 - 3i \end{pmatrix}$$



### Exercise

For the matrix:  $\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} 5 - \lambda & -4 \\ 2 & -1 - \lambda \end{vmatrix} \\ &= -5 - 4\lambda + \lambda^2 + 8 \\ &= \lambda^2 - 4\lambda + 3 \end{aligned}$$

The characteristic equation:  $\lambda^2 - 4\lambda + 3 = 0$

ii. The eigenvalues are:  $\lambda_{1,2} = 1, 3$

iii. For  $\lambda_1 = 1$  :  $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 4x - 4y = 0 \end{cases}$$

$$\underline{x = y}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

For  $\lambda_2 = 3$  :  $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x - 4y = 0 \end{cases}$$

$$\underline{x = 2y}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

### Exercise

For the matrix:  $\begin{pmatrix} 6 & -6 \\ 4 & -4 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 6 & -6 \\ 4 & -4 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} 6 - \lambda & -6 \\ 4 & -4 - \lambda \end{vmatrix} \\ &= -24 - 2\lambda + \lambda^2 + 24 \\ &= \lambda^2 - 2\lambda \end{aligned}$$

The characteristic equation:  $\lambda^2 - 2\lambda = 0$

ii. The eigenvalues are:  $\lambda_{1,2} = 0, 2$

iii. For  $\lambda_1 = 0$  :  $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 6 & -6 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 6x - 6y = 0 \\ x = y \end{cases}$$

$$\Rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For  $\lambda_2 = 2$  :  $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 4 & -6 \\ 4 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 4x - 6y = 0 \\ 2x = 3y \end{cases}$$

$$\Rightarrow V_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

### Exercise

For the matrix:  $\begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix}$

- i.* Find the characteristic equation
- ii.* Find the eigenvalues
- iii.* Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix}$$

$$\begin{aligned} i. \quad |A - \lambda I| &= \begin{vmatrix} 5 - \lambda & -3 \\ 2 & -\lambda \end{vmatrix} \\ &= \lambda^2 - 5\lambda + 6 \end{aligned}$$

The characteristic equation:  $\lambda^2 - 5\lambda + 6 = 0$

$$ii. \quad \text{The eigenvalues are: } \lambda_{1,2} = 2, 3$$

$$iii. \quad \text{For } \lambda_1 = 2 : (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 3x - 3y = 0 \\ 2x - 2y = 0 \end{cases}$$
$$\underline{x = y}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$\text{For } \lambda_2 = 3 : (A - \lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x - 3y = 0 \\ 2x - 3y = 0 \end{cases}$$
$$\underline{2x = 3y}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}}$$

### Exercise

For the matrix:  $\begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} 5 - \lambda & -4 \\ 3 & -2 - \lambda \end{vmatrix} \\ &= -10 - 3\lambda + \lambda^2 + 12 \\ &= \lambda^2 - 3\lambda + 2 \end{aligned}$$

The characteristic equation:  $\lambda^2 - 3\lambda + 2 = 0$

ii. The eigenvalues are:  $\lambda_{1,2} = 1, 2$

iii. For  $\lambda_1 = 1$  :  $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & -4 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 4x - 4y = 0 \end{cases}$$

$$\underline{x = y}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

For  $\lambda_1 = 2$  :  $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 3 & -4 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 3x - 4y = 0 \end{cases}$$

$$\underline{3x = 4y}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}}$$

### Exercise

For the matrix:  $\begin{pmatrix} 6 & -10 \\ 2 & -3 \end{pmatrix}$

- i.* Find the characteristic equation
- ii.* Find the eigenvalues
- iii.* Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 6 & -10 \\ 2 & -3 \end{pmatrix}$$

$$\begin{aligned} i. \quad |A - \lambda I| &= \begin{vmatrix} 6 - \lambda & -10 \\ 2 & -3 - \lambda \end{vmatrix} \\ &= -18 - 3\lambda + \lambda^2 + 20 \\ &= \lambda^2 - 3\lambda + 2 \end{aligned}$$

The characteristic equation:  $\lambda^2 - 3\lambda + 2 = 0$

*ii.* The eigenvalues are:  $\lambda_{1,2} = 1, 2$

*iii.* For  $\lambda_1 = 1$  :  $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 5 & -10 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x - 4y = 0 \\ x = 2y \end{cases}$$

$$\Rightarrow V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For  $\lambda_1 = 2$  :  $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 4 & -10 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x - 5y = 0 \\ 2x = 5y \end{cases}$$

$$\Rightarrow V_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

### Exercise

For the matrix:  $\begin{pmatrix} 11 & -15 \\ 6 & -8 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 11 & -15 \\ 6 & -8 \end{pmatrix}$$

$$\begin{aligned} i. \quad |A - \lambda I| &= \begin{vmatrix} 11 - \lambda & -15 \\ 6 & -8 - \lambda \end{vmatrix} \\ &= -88 - 3\lambda + \lambda^2 + 90 \\ &= \lambda^2 - 3\lambda + 2 \end{aligned}$$

The characteristic equation:  $\lambda^2 - 3\lambda + 2 = 0$

$$ii. \quad \text{The eigenvalues are: } \lambda_{1,2} = 1, 2$$

$$iii. \quad \text{For } \lambda_1 = 1 : (A - \lambda_1 I)V_1 = 0$$

$$\begin{aligned} \begin{pmatrix} 10 & -15 \\ 6 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 6x - 9y = 0 \\ 2x = 3y \end{cases} \\ \Rightarrow V_1 &= \begin{pmatrix} 3 \\ 2 \end{pmatrix} \end{aligned}$$

$$\text{For } \lambda_1 = 2 : (A - \lambda_2 I)V_2 = 0$$

$$\begin{aligned} \begin{pmatrix} 9 & -15 \\ 6 & -10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 6x - 10y = 0 \\ 3x = 5y \end{cases} \\ \Rightarrow V_2 &= \begin{pmatrix} 5 \\ 3 \end{pmatrix} \end{aligned}$$

### Exercise

For the matrix:  $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{vmatrix} \\ &= 9 - 6\lambda + \lambda^2 - 1 \\ &= \lambda^2 - 6\lambda + 8 \end{aligned}$$

The characteristic equation:  $\lambda^2 - 6\lambda + 8 = 0$

$$\begin{aligned} \text{ii. } \lambda_{1,2} &= \frac{6 \pm \sqrt{36 - 32}}{2} \\ &= \frac{6 \pm 2}{2} \end{aligned}$$

The eigenvalues are:  $\lambda_1 = 2$  &  $\lambda_2 = 4$

$$\text{iii. For } \lambda_1 = 2 : (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x + y = 0 \\ x = -y \end{cases}$$

$$\Rightarrow V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\text{For } \lambda_2 = 4 : (A - \lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -x + y = 0 \\ x = y \end{cases}$$

$$\Rightarrow V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

### Exercise

For the matrix:  $\begin{pmatrix} 9 & 2 \\ 2 & 6 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 9 & 2 \\ 2 & 6 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} 9 - \lambda & 2 \\ 2 & 6 - \lambda \end{vmatrix} \\ &= 54 - 15\lambda + \lambda^2 - 4 \\ &= \lambda^2 - 15\lambda + 50 \end{aligned}$$

The characteristic equation:  $\lambda^2 - 15\lambda + 50 = 0$

$$\begin{aligned} \text{ii. } \lambda_{1,2} &= \frac{15 \pm \sqrt{225 - 200}}{2} \\ &= \frac{15 \pm 5}{2} \end{aligned}$$

The eigenvalues are:  $\lambda_1 = 5$  &  $\lambda_2 = 10$

$$\text{iii. For } \lambda_1 = 5 : (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x + y = 0 \\ 2x = -y \end{cases}$$

$$\Rightarrow V_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\text{For } \lambda_2 = 10 : (A - \lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -x + 2y = 0 \\ x = 2y \end{cases}$$

$$\Rightarrow V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$



### Exercise

For the matrix:  $\begin{pmatrix} 13 & 4 \\ 4 & 7 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 13 & 4 \\ 4 & 7 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} 13 - \lambda & 4 \\ 4 & 7 - \lambda \end{vmatrix} \\ &= 91 - 20\lambda + \lambda^2 - 16 \\ &= \lambda^2 - 20\lambda + 75 \end{aligned}$$

The characteristic equation:  $\lambda^2 - 20\lambda + 75 = 0$

$$\begin{aligned} \text{ii. } \lambda_{1,2} &= \frac{20 \pm \sqrt{400 - 300}}{2} \\ &= \frac{20 \pm 10}{2} \end{aligned}$$

The eigenvalues are:  $\lambda_1 = 5$  &  $\lambda_2 = 15$

$$\text{iii. For } \lambda_1 = 5 : (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 4x + 2y = 0 \\ 2x = -y \end{cases}$$

$$\Rightarrow V_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\text{For } \lambda_2 = 15 : (A - \lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -2x + 4y = 0 \\ x = 2y \end{cases}$$

$$\Rightarrow V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

### Exercise

For the matrix:  $\begin{pmatrix} 5 & -1 \\ 3 & -1 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 5 & -1 \\ 3 & -1 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} 5 - \lambda & -1 \\ 3 & -1 - \lambda \end{vmatrix} \\ &= -5 - 4\lambda + \lambda^2 + 3 \\ &= \lambda^2 - 4\lambda - 2 \end{aligned}$$

The characteristic equation:  $\lambda^2 - 4\lambda - 2 = 0$

$$\begin{aligned} \text{ii. } \lambda_{1,2} &= \frac{4 \pm \sqrt{16 + 8}}{2} \\ &= \frac{4 \pm 2\sqrt{6}}{2} \end{aligned}$$

The eigenvalues are:  $\lambda_{1,2} = 2 \pm \sqrt{6}$

$$\text{iii. For } \lambda_1 = 2 - \sqrt{6} : (A - \lambda_1 I)V_1 = 0$$

$$\begin{aligned} \begin{pmatrix} 3 + \sqrt{6} & -1 \\ 3 & -3 + \sqrt{6} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} (3 + \sqrt{6})x - y = 0 \\ (3 + \sqrt{6})x = y \end{cases} \\ \Rightarrow V_1 &= \begin{pmatrix} 1 \\ 3 + \sqrt{6} \end{pmatrix} \end{aligned}$$

$$\text{For } \lambda_2 = 2 + \sqrt{6} : (A - \lambda_2 I)V_2 = 0$$

$$\begin{aligned} \begin{pmatrix} 3 - \sqrt{6} & -1 \\ 3 & -3 - \sqrt{6} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} (3 - \sqrt{6})x - y = 0 \\ (3 - \sqrt{6})x = y \end{cases} \\ \Rightarrow V_2 &= \begin{pmatrix} 1 \\ 3 - \sqrt{6} \end{pmatrix} \end{aligned}$$

### Exercise

For the matrix:  $\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} 1 - \lambda & 1 \\ 4 & -2 - \lambda \end{vmatrix} \\ &= -2 + \lambda + \lambda^2 - 4 \\ &= \lambda^2 + \lambda - 6 \end{aligned}$$

The characteristic equation:  $\lambda^2 + \lambda - 6 = 0$

$$\text{ii. } \lambda_{1,2} = \frac{-1 \pm \sqrt{1+24}}{2}$$

The eigenvalues are:  $\lambda_1 = -3$  &  $\lambda_2 = 2$

$$\text{iii. For } \lambda_1 = -3 : (A - \lambda_1 I)V_1 = 0$$

$$\begin{aligned} \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 4x + y = 0 \\ 4x = -y \end{cases} \\ \Rightarrow V_1 &= \begin{pmatrix} -1 \\ 4 \end{pmatrix} \end{aligned}$$

$$\text{For } \lambda_2 = 2 \Rightarrow (A - \lambda_2 I)V_2 = 0$$

$$\begin{aligned} \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -x + y = 0 \\ x = y \end{cases} \\ \Rightarrow V_2 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

### Exercise

For the matrix:  $\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} -1 - \lambda & -4 \\ 1 & -1 - \lambda \end{vmatrix} \\ &= 1 + 2\lambda + \lambda^2 + 4 \\ &= \lambda^2 + 2\lambda + 5 \end{aligned}$$

The characteristic equation:  $\lambda^2 + 2\lambda + 5 = 0$

$$\text{ii. } \lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

The eigenvalues are:  $\lambda_{1,2} = -1 \pm 2i$

$$\text{iii. For } \lambda_1 = -1 - 2i : (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 2i & -4 \\ 1 & 2i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x + 2iy = 0 \end{cases}$$

$$\underline{x = -2iy}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} -2i \\ 1 \end{pmatrix}}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 2i \\ 1 \end{pmatrix}}$$

### Exercise

For the matrix:  $\begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -6 \\ -2 & 3 & 4 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -6 \\ -2 & 3 & 4 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} -1-\lambda & 0 & 0 \\ 2 & -5-\lambda & -6 \\ -2 & 3 & 4-\lambda \end{vmatrix} \\ &= (-1-\lambda)(-5-\lambda)(4-\lambda) + 18(-1-\lambda) \\ &= (5+6\lambda+\lambda^2)(4-\lambda) - 18 - 18\lambda \\ &= 20 - 5\lambda + 24\lambda - 6\lambda^2 + 4\lambda^2 - \lambda^3 - 18 - 18\lambda \\ &= -\lambda^3 - 2\lambda^2 + \lambda + 2 \end{aligned}$$

The characteristic equation:  $\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$

ii.  $\lambda = 1$

$$\begin{array}{c|ccc} 1 & 1 & 2 & -1 & -2 \\ & & 1 & 3 & 2 \\ \hline & 1 & 3 & 2 & 0 \end{array} \rightarrow \lambda^2 + 3\lambda + 2 = 0$$

The eigenvalues are:  $\lambda_1 = -2$   $\lambda_2 = -1$  and  $\lambda_3 = 1$

iii. For  $\lambda_1 = -2$  :  $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & -6 \\ -2 & 3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \underline{x=0} \\ 2x-3y-6z=0 \\ -2x+3y+6z=0 \end{cases} \quad (1)$$

$$(1) \rightarrow \underline{y = -2z}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}}$$

$$\text{For } \lambda_2 = -1 \Rightarrow (A - \lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & -4 & -6 \\ -2 & 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 0 = 0 \\ 2x - 4y - 6z = 0 & (1) \\ -2x + 3y + 5z = 0 & (2) \end{cases}$$

$$(1) + (2) \rightarrow -y - z = 0 \Rightarrow \underline{y = -z}$$

$$(1) \rightarrow 2x = -4z + 6z \Rightarrow \underline{x = z}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}$$

$$\text{For } \lambda_3 = 1 \Rightarrow (A - \lambda_3 I)V_3 = 0$$

$$\begin{pmatrix} -2 & 0 & 0 \\ 2 & -6 & -6 \\ -2 & 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -2x = 0 \rightarrow \underline{x = 0} \\ 2x - 6y - 6z = 0 & (1) \\ -2x + 3y + 3z = 0 & (2) \end{cases}$$

$$(1) + (2) \rightarrow -3y - 3z = 0 \Rightarrow \underline{y = -z}$$

$$\Rightarrow \underline{V_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}$$

$$\text{The vectors are given by: } V = \begin{pmatrix} 0 & 1 & 0 \\ -2 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

### ***Exercise***

$$\text{For the matrix: } \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

- i.** Find the characteristic equation
- ii.** Find the eigenvalues
- iii.** Find the eigenvectors

### **Solution**

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$i. \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & -1-\lambda \end{vmatrix}$$

$$= -(1-\lambda^2)(2-\lambda)$$

The characteristic equation:  $\underline{(1-\lambda^2)(2-\lambda) = 0}$

ii. The eigenvalues are:  $\underline{\lambda_1 = -1 \quad \lambda_2 = 1 \quad \text{and} \quad \lambda_3 = 2}$

iii. For  $\lambda_1 = -1$  :  $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x + y - z = 0 \\ \underline{y = 0} \end{cases} \quad (1)$$

$$(1) \rightarrow \underline{2x = z}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}$$

r  $\lambda_2 = 1$  :  $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} y - z = 0 \\ \underline{y = 0} \end{cases} \quad \underline{z = 0}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

For  $\lambda_3 = 2$  :  $(A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -x + y - z = 0 \\ \underline{y = 3z} \end{cases} \quad (1)$$

$$(1) \rightarrow \underline{x = 2z}$$

$$\Rightarrow \underline{V_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}$$

### Exercise

For the matrix:  $\begin{pmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} 2-\lambda & -7 & 0 \\ 5 & 10-\lambda & 4 \\ 0 & 5 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)^2(10-\lambda) - 20(2-\lambda) + 35(2-\lambda) \\ &= (2-\lambda)((10-\lambda)(2-\lambda) + 15) \\ &= (2-\lambda)(35 - 12\lambda + \lambda^2) \end{aligned}$$

$$\text{The characteristic equation: } \underline{(2-\lambda)(\lambda^2 - 12\lambda + 35) = 0}$$

$$\begin{aligned} \text{ii. } \lambda &= \frac{12 \pm \sqrt{144 - 140}}{2} \\ &= \frac{12 \pm 2}{2} \end{aligned}$$

$$\text{The eigenvalues are: } \underline{\lambda_1 = 2 \quad \lambda_2 = 5 \quad \text{and} \quad \lambda_3 = 7}$$

$$\text{iii. For } \lambda_1 = 2: (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 0 & -7 & 0 \\ 5 & 8 & 4 \\ 0 & 5 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \underline{y = 0} \\ 5x + 8y + 4z = 0 \\ \underline{5x = -4z} \end{cases}$$
$$\Rightarrow \underline{V_1 = \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}}$$

$$\text{For } \lambda_2 = 5: (A - \lambda_2 I)V_2 = 0$$



$$\begin{pmatrix} -5 & -7 & 0 \\ 5 & 3 & 4 \\ 0 & 5 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -5x - 7y = 0 & \rightarrow \underline{-5x = 7y} \\ 5y - 5z = 0 & \rightarrow \underline{z = y} \end{cases}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 7 \\ -5 \\ -5 \end{pmatrix}}$$

For  $\lambda_3 = 7$ :  $(A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} 0 & -7 & 0 \\ 5 & 8 & 4 \\ 0 & 5 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \underline{y = 0} \\ 5x + 8y + 4z = 0 & \rightarrow \underline{5x = -4z} \end{cases}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}}$$

### Exercise

For the matrix:  $\begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

- i.* Find the characteristic equation
- ii.* Find the eigenvalues
- iii.* Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\begin{aligned} i. \quad |A - \lambda I| &= \begin{vmatrix} 3 - \lambda & -1 & -1 \\ 1 & 1 - \lambda & -1 \\ 1 & -1 & 1 - \lambda \end{vmatrix} \\ &= (1 - 2\lambda + \lambda^2)(3 - \lambda) + 2 + 2 - 2\lambda - 3 + \lambda \\ &= 3 - 7\lambda + 5\lambda^2 - \lambda^3 + 1 - \lambda \\ &= -\lambda^3 + 5\lambda^2 - 8\lambda + 4 \end{aligned}$$

The characteristic equation:  $\underline{-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0}$

ii.  $\lambda = 1$

$$\begin{array}{c|cccc} \textcolor{red}{1} & -1 & 5 & -8 & 4 \\ & & -1 & 4 & -4 \\ \hline & -1 & 4 & -4 & 0 \end{array} \rightarrow \underline{-\lambda^2 + 4\lambda - 4 = 0}$$

The eigenvalues are:  $\lambda_1 = \textcolor{blue}{1}$  and  $\lambda_{2,3} = \textcolor{blue}{2}$

iii. For  $\lambda_1 = 1 \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x - y + z = 0 \\ x - z = 0 \\ x - y = 0 \end{cases} \rightarrow \begin{array}{l} \underline{x = z} \\ \underline{x = y} \end{array}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

For  $\lambda_2 = 2 \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x - y - z = 0 \end{cases} \quad \textcolor{red}{(1)}$$

$$\textcolor{red}{(1)} \rightarrow \underline{x = y + z} \quad \text{let } \underline{y = 0}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$$

$$\textcolor{red}{(1)} \rightarrow \text{let } \underline{\textcolor{red}{z} = 0} \quad \underline{x = y}$$

$$\Rightarrow \underline{V_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

### Exercise

For the matrix:  $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} \\ &= (-\lambda)(3-\lambda)^2 + 16 + 16 + 16\lambda - 4(3-\lambda) - 4(3-\lambda) \\ &= -9\lambda + 6\lambda^2 - \lambda^3 + 32 + 16\lambda - 12 + 4\lambda - 12 + 4\lambda \\ &= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 \end{aligned}$$

The characteristic equation:  $-\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0$

ii.  $\lambda = -1$

$$\begin{array}{c|ccc} -1 & -1 & 6 & 15 & 8 \\ & & 1 & -7 & -8 \\ \hline & -1 & 7 & 8 & 0 \end{array} \rightarrow -\lambda^2 + 7\lambda + 8 = 0$$

The eigenvalues are:  $\lambda_{1,2} = -1$  and  $\lambda_3 = 8$

iii. For  $\lambda_{1,2} = -1 \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x + y + 2z = 0 \end{cases} \quad (1)$$

Assume  $z = 0 \rightarrow 2x = -y$

$$\Rightarrow V_1 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

Assume  $y = 0 \rightarrow x = -z$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

$$\text{For } \lambda_3 = 8 \Rightarrow (A - \lambda_3 I)V_3 = 0$$

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 5x - 2y - 4z = 0 & (1) \\ x - 4y + z = 0 & (2) \\ 4x + 2y - 5z = 0 & (3) \end{cases}$$

$$(1) + (3) \rightarrow 9x - 9z = 0$$

$$\underline{x = z}$$

$$(2) \rightarrow 4y = 2z$$

$$\text{Assume } z = 2 = x \rightarrow \underline{y = 1}$$

$$\Rightarrow \underline{V_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}}$$

### Exercise

$$\text{For the matrix: } \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}$$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -8 & -5 & -3-\lambda \end{vmatrix} \\ &= (1-2\lambda+\lambda^2)(-3-\lambda) - 2 + 3 - 3\lambda + 6 + 2\lambda \\ &= -\lambda^3 - \lambda^2 + 4\lambda + 4 \end{aligned}$$

$$\text{The characteristic equation: } \underline{-\lambda^3 - \lambda^2 + 4\lambda + 4 = 0}$$

$$\text{ii. } \lambda = -1$$

$$\begin{array}{c|cccc} \textcolor{red}{-1} & -1 & -1 & 4 & 4 \\ & & 1 & 0 & -4 \\ \hline & -1 & 0 & 4 & \textcolor{red}{0} \end{array} \rightarrow \underline{-\lambda^2 + 4 = 0}$$

The eigenvalues are:  $\lambda_1 = -2$ ,  $\lambda_2 = -1$ , and  $\lambda_3 = 2$

iii. For  $\lambda_1 = -2 \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & -1 \\ -8 & -5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 3x + y + z = 0 \\ 2x + 3y - z = 0 \\ -8x - 5y - z = 0 \end{cases}$$

$$\text{Assume } \textcolor{red}{z} = 1 \rightarrow \begin{cases} 3x + y = -1 \\ 2x + 3y = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} = 7 \quad \Delta_x = \begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix} = -4 \quad \Delta_y = \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = 5$$

$$x = -\frac{4}{7} \quad y = \frac{5}{7} \quad z = 1$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} -4 \\ 5 \\ 7 \end{pmatrix}}$$

For  $\lambda_2 = -1 \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ -8 & -5 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x + y + z = 0 \\ 2x + 2y - z = 0 \\ -8x - 5y - 2z = 0 \end{cases}$$

$$\textcolor{red}{z} = 1 \rightarrow \begin{cases} 2x + y = -1 \\ 2x + 2y = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 2 \quad \Delta_x = \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} = -3 \quad \Delta_y = \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = 4$$

$$x = -\frac{3}{2} \quad y = 2 \quad z = 1$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}}$$

For  $\lambda_3 = 2 \Rightarrow (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -8 & -5 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -x + y + z = 0 \\ 2x - y - z = 0 \\ -8x - 5y - 5z = 0 \end{cases}$$

$$\textcolor{red}{x=0} \rightarrow \begin{cases} y + z = 0 \\ -y - z = 0 \end{cases}$$

$$\underline{y = -z}$$

$$\Rightarrow V_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

### Exercise

For the matrix:  $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$

- i.* Find the characteristic equation
- ii.* Find the eigenvalues
- iii.* Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

$$\begin{aligned} i. \quad |A - \lambda I| &= \begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} \\ &= -(1-\lambda^2)(2-\lambda) + 14 - 16 + 8\lambda + 1 - \lambda - 3 - 3\lambda \\ &= -\lambda^3 + 2\lambda^2 + 5\lambda - 6 \end{aligned}$$

The characteristic equation:  $\underline{-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0}$

*ii.*  $\lambda = 1$

$$\begin{array}{c|cccc} \textcolor{red}{1} & -1 & 2 & 5 & -6 \\ & & -1 & 1 & 6 \\ \hline & -1 & 1 & 6 & \textcolor{red}{0} \end{array} \rightarrow \underline{-\lambda^2 + \lambda + 6 = 0}$$

The eigenvalues are:  $\underline{\lambda_1 = -2, \lambda_2 = 1, \text{ and } \lambda_3 = 3}$

$$\text{iii. For } \lambda_1 = -2 \Rightarrow (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 3x - y + 4z = 0 \\ 3x + 4y - z = 0 \\ 2x + y + z = 0 \end{cases}$$

$$\text{Assume } z = 1 \rightarrow \begin{cases} 3x - y = -4 \\ 3x + 4y = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} 3 & -1 \\ 3 & 4 \end{vmatrix} = 15 \quad \Delta_x = \begin{vmatrix} -4 & -1 \\ 1 & 4 \end{vmatrix} = -15 \quad \Delta_y = \begin{vmatrix} 3 & -4 \\ 3 & 1 \end{vmatrix} = 15$$

$$\underline{x = -1 \quad y = 1 \quad |}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad |}$$

$$\text{For } \lambda_2 = 1 \Rightarrow (A - \lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -y + 4z = 0 \\ 3x + y - z = 0 \\ 2x + y - 2z = 0 \end{cases}$$

$$\text{Assume } z = 1 \rightarrow \begin{cases} \underline{y = 4} \\ 3x + y = 1 \end{cases} \rightarrow \underline{x = -1 \quad |}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \quad |}$$

$$\text{For } \lambda_3 = 3 \Rightarrow (A - \lambda_3 I)V_3 = 0$$

$$\begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -2x - y + 4z = 0 \\ 3x - y - z = 0 \\ 2x + y - 4z = 0 \end{cases}$$

$$\text{Assume } z = 1 \rightarrow \begin{cases} -2x - y = -4 \\ 3x - y = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} -2 & -1 \\ 3 & -1 \end{vmatrix} = 5 \quad \Delta_x = \begin{vmatrix} -4 & -1 \\ 1 & -1 \end{vmatrix} = 5 \quad \Delta_y = \begin{vmatrix} -2 & -4 \\ 3 & 1 \end{vmatrix} = 10$$

$$\underline{x = -1 \quad y = 2 \quad |}$$

$$\Rightarrow \underline{V_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad |}$$

### Exercise

For the matrix: 
$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} \\ &= (1-2\lambda+\lambda^2)(3-\lambda) - (3-\lambda) \\ &= (3-\lambda)(\lambda^2-2\lambda) \end{aligned}$$

$$\text{The characteristic equation: } \underline{(3-\lambda)(\lambda^2-2\lambda)=0}$$

$$\text{ii. The eigenvalues are: } \underline{\lambda_1=0, \lambda_2=2, \text{ and } \lambda_3=3}$$

$$\text{iii. For } \lambda_1=0 \Rightarrow (A-\lambda_1 I)V_1=0$$

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x+y=0 \\ 3z=0 \end{cases} \\ \underline{x=-y} \quad \underline{z=0} \\ \Rightarrow V_1 &= \underline{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}} \end{aligned}$$

$$\text{For } \lambda_2=2 \Rightarrow (A-\lambda_2 I)V_2=0$$

$$\begin{aligned} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -x+y=0 \\ z=0 \end{cases} \rightarrow \underline{x=y} \\ \underline{z=0} \end{aligned}$$



$$\Rightarrow V_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{For } \lambda_3 = 3 \Rightarrow (A - \lambda_3 I)V_3 = 0$$

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x = y \\ x = 2y \end{cases}$$

$$\underline{x = y = 0}$$

$$\Rightarrow V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

### Exercise

$$\text{For the matrix: } \begin{pmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{pmatrix}$$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{pmatrix}$$

$$i. \quad |A - \lambda I| = \begin{vmatrix} -\frac{5}{2} - \lambda & 1 & 1 \\ 1 & -\frac{5}{2} - \lambda & 1 \\ 1 & 1 & -\frac{5}{2} - \lambda \end{vmatrix}$$

$$= -\left(\frac{5}{2} + \lambda\right)^3 + 2 + 3\left(\frac{5}{2} + \lambda\right)$$

$$= -\frac{49}{8} - \frac{63}{4}\lambda - \frac{15}{2}\lambda^2 - \lambda^3$$

$$-\frac{49}{8} - \frac{63}{4}\lambda - \frac{15}{2}\lambda^2 - \lambda^3 = 0$$

$$\text{The characteristic equation: } \underline{8\lambda^3 + 60\lambda^2 + 126\lambda + 49 = 0}$$

$$ii. \lambda = -\frac{1}{2}$$

$$\begin{array}{c|cccc} -\frac{1}{2} & 8 & 60 & 126 & 49 \\ & & -4 & -28 & -49 \\ \hline & 8 & 56 & 98 & 0 \end{array} \rightarrow \underline{8\lambda^2 + 56\lambda + 98 = 0}$$

$$4\lambda^2 + 28\lambda + 49 = 0$$

$$\lambda = \frac{-28 \pm \sqrt{784 - 784}}{8}$$

$$\text{The eigenvalues are: } \underline{\lambda_1 = -\frac{1}{2} \quad \& \quad \lambda_{2,3} = -\frac{7}{2}}$$

$$iii. \text{ For } \lambda_1 = -\frac{7}{2} \Rightarrow (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow x + y + z = 1$$

$$\text{Assume } z = 0 \rightarrow x + y = 1$$

$$y = 1 \Rightarrow \underline{x = -1}$$

$$\rightarrow \underline{V_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$$

$$\text{Assume } y = 0 \rightarrow x + z = 1$$

$$z = 1 \Rightarrow \underline{x = -1}$$

$$\rightarrow \underline{V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

$$\text{For } \lambda_3 = -\frac{1}{2} \Rightarrow (A - \lambda_3 I)V_3 = 0$$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{array}{l} -2x + y + z = 0 \\ x - 2y + z = 0 \\ x + y - 2z = 0 \end{array}$$

$$\text{Assume } z = 1 \rightarrow \begin{cases} -2x + y = -1 \\ x - 2y = -1 \end{cases}$$

$$\Delta = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 3 \quad \Delta_x = \begin{vmatrix} -2 & -1 \\ 1 & -1 \end{vmatrix} = 3$$

$$\underline{x=1} \quad y = -1 + 2 = \underline{1}$$

$$\rightarrow \underline{V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

### Exercise

For the matrix:  $\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} \text{i. } \det(A - \lambda I) &= \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} \\ &= (4-\lambda)(1-\lambda)(1-\lambda) + 2(1-\lambda) \\ &= (1-\lambda)[(4-\lambda)(1-\lambda) + 2] \\ &= (1-\lambda)(\lambda^2 - 5\lambda + 6) \end{aligned}$$

The characteristic equation:  $\underline{-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0}$

$$\text{ii. } -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0 \rightarrow \lambda_1 = 1$$

$$\begin{array}{c|ccc} 1 & -1 & 6 & -11 & 6 \\ & & -1 & 5 & -6 \\ \hline & -1 & 5 & -6 & \textcolor{red}{0} \end{array} \rightarrow \underline{-\lambda^2 + 5\lambda - 6 = 0}$$

Therefore; the eigenvalues are:  $\underline{\lambda_{1,2,3} = 1, 2, 3}$

$$\text{iii. } \text{For } \lambda_1 = 1 \Rightarrow (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 3x_1 + x_3 = 0 & (1) \\ -2x_1 = 0 & (2) \end{cases}$$

$$(2) \Rightarrow \underline{x_1 = 0}$$

$$(1) \Rightarrow \underline{x_3 = x_1 = 0}$$

Therefore; the eigenvector  $V_1 = \underline{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$

For  $\lambda_2 = 2 \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} 2x_1 + x_3 = 0 \Rightarrow x_3 = -2x_1 \\ -2x_1 - x_2 = 0 \Rightarrow x_2 = -2x_1 \\ -2x_1 - x_3 = 0 \end{cases}$$

Therefore; the eigenvector  $V_2 = \underline{\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}}$

For  $\lambda_3 = 3 \Rightarrow (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} x_1 + x_3 = 0 \Rightarrow x_3 = -x_1 \\ -2x_1 - 2x_2 = 0 \Rightarrow x_2 = -x_1 \\ -2x_1 - 2x_3 = 0 \end{cases}$$

Therefore; the eigenvector  $V_3 = \underline{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}}$

### Exercise

For the matrix: 
$$\begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$$

$$\begin{aligned} \text{i. } \det(A - \lambda I) &= \begin{vmatrix} 3-\lambda & 0 & -5 \\ \frac{1}{5} & -1-\lambda & 0 \\ 1 & 1 & -2-\lambda \end{vmatrix} \\ &= (3-\lambda)(-1-\lambda)(-2-\lambda) - 1 + 5(-1-\lambda) \\ &= (3-\lambda)(\lambda^2 + 3\lambda + 2) - 1 - 5 - 5\lambda \\ &= 3\lambda^2 + 9\lambda + 6 - \lambda^3 - 3\lambda^2 - 2\lambda - 6 - 5\lambda \\ &= -\lambda^3 + 2\lambda \end{aligned}$$

The characteristic equation:  $\underline{-\lambda^3 + 2\lambda = 0}$

$$\text{ii. } -\lambda(\lambda^2 - 2) = 0$$

Therefore; the eigenvalues are:  $\underline{\lambda_{1,2,3} = 0, \pm\sqrt{2}}$

$$\text{iii. } \text{For } \lambda_1 = -\sqrt{2} \Rightarrow (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 3+\sqrt{2} & 0 & -5 \\ \frac{1}{5} & -1+\sqrt{2} & 0 \\ 1 & 1 & -2+\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\begin{cases} (3+\sqrt{2})x_1 - 5x_3 = 0 & \Rightarrow x_3 = \frac{3+\sqrt{2}}{5}x_1 \\ \frac{1}{5}x_1 + (-1+\sqrt{2})x_2 = 0 & \Rightarrow x_2 = -\frac{1}{5(-1+\sqrt{2})}x_1 \\ x_1 + x_2 + (-2+\sqrt{2})x_3 = 0 \end{cases}$$

Therefore; the eigenvector  $V_1 = \begin{pmatrix} 1 \\ \frac{1}{5(1-\sqrt{2})} \\ \frac{3+\sqrt{2}}{5} \end{pmatrix}$

For  $\lambda_2 = 0 \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3x_1 - 5x_3 = 0 & \Rightarrow x_3 = \frac{3}{5}x_1 \\ \frac{1}{5}x_1 - x_2 = 0 & \Rightarrow x_2 = \frac{1}{5}x_1 \\ x_1 + x_2 - 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = \frac{3}{5}x_1 \\ x_2 = \frac{1}{5}x_1 \end{cases}$$

Therefore; the eigenvector  $V_2 = \begin{pmatrix} 5 \\ \frac{1}{5} \\ \frac{3}{5} \end{pmatrix}$

For  $\lambda_3 = \sqrt{2} \Rightarrow (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} 3-\sqrt{2} & 0 & -5 \\ \frac{1}{5} & -1-\sqrt{2} & 0 \\ 1 & 1 & -2-\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} (3-\sqrt{2})x_1 - 5x_3 = 0 & \Rightarrow x_3 = \frac{3-\sqrt{2}}{5}x_1 \\ \frac{1}{5}x_1 + (-1-\sqrt{2})x_2 = 0 & \Rightarrow x_2 = \frac{1}{5(1+\sqrt{2})}x_1 \\ x_1 + x_2 + (-2-\sqrt{2})x_3 = 0 \end{cases}$$

Therefore; the eigenvector  $V_3 = \begin{pmatrix} 1 \\ \frac{1}{5(1+\sqrt{2})} \\ \frac{3-\sqrt{2}}{5} \end{pmatrix}$

### Exercise

For the matrix:  $\begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$$

$$\begin{aligned} i. \quad \det(A - \lambda I) &= \begin{vmatrix} 3-\lambda & 0 & -5 \\ \frac{1}{5} & -1-\lambda & 0 \\ 1 & 1 & -2-\lambda \end{vmatrix} \\ &= (3-\lambda)(-1-\lambda)(-2-\lambda) - 1 + 5(-1-\lambda) \\ &= (3-\lambda)(\lambda^2 + 3\lambda + 2) - 1 - 5 - 5\lambda \\ &= 3\lambda^2 + 9\lambda + 6 - \lambda^3 - 3\lambda^2 - 2\lambda - 6 - 5\lambda \\ &= -\lambda^3 + 2\lambda \end{aligned}$$

The characteristic equation:  $-\lambda^3 + 2\lambda = 0$

$$ii. \quad -\lambda(\lambda^2 - 2) = 0$$

Therefore; the eigenvalues are:  $\lambda_{1,2,3} = 0, \pm\sqrt{2}$

$$iv. \quad \text{For } \lambda_1 = -\sqrt{2} \Rightarrow (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 3+\sqrt{2} & 0 & -5 \\ \frac{1}{5} & -1+\sqrt{2} & 0 \\ 1 & 1 & -2+\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} (3+\sqrt{2})x_1 - 5x_3 = 0 & \Rightarrow \underline{x_3 = \frac{3+\sqrt{2}}{5}x_1} \\ \frac{1}{5}x_1 + (-1+\sqrt{2})x_2 = 0 & \Rightarrow \underline{x_2 = -\frac{1}{5(-1+\sqrt{2})}x_1} \\ x_1 + x_2 + (-2+\sqrt{2})x_3 = 0 \end{cases}$$

Therefore; the eigenvector  $V_1 = \begin{pmatrix} 1 \\ \frac{1}{5(1-\sqrt{2})} \\ \frac{3+\sqrt{2}}{5} \end{pmatrix}$

For  $\lambda_2 = 0 \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3x_1 - 5x_3 = 0 & \Rightarrow x_3 = \frac{3}{5}x_1 \\ \frac{1}{5}x_1 - x_2 = 0 & \Rightarrow x_2 = \frac{1}{5}x_1 \\ x_1 + x_2 - 2x_3 = 0 \end{cases}$$

Therefore; the eigenvector  $V_2 = \begin{pmatrix} 5 \\ \frac{1}{5} \\ \frac{3}{5} \end{pmatrix}$

For  $\lambda_3 = \sqrt{2} \Rightarrow (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} 3-\sqrt{2} & 0 & -5 \\ \frac{1}{5} & -1-\sqrt{2} & 0 \\ 1 & 1 & -2-\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} (3-\sqrt{2})x_1 - 5x_3 = 0 & \Rightarrow x_3 = \frac{3-\sqrt{2}}{5}x_1 \\ \frac{1}{5}x_1 + (-1-\sqrt{2})x_2 = 0 & \Rightarrow x_2 = \frac{1}{5(1+\sqrt{2})}x_1 \\ x_1 + x_2 + (-2-\sqrt{2})x_3 = 0 \end{cases}$$

Therefore; the eigenvector  $V_3 = \begin{pmatrix} 1 \\ \frac{1}{5(1+\sqrt{2})} \\ \frac{3-\sqrt{2}}{5} \end{pmatrix}$



### Exercise

For the matrix:  $\begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$$

$$\begin{aligned} i. \quad \det(A - \lambda I) &= \begin{vmatrix} -1-\lambda & 0 & 1 \\ -1 & 3-\lambda & 0 \\ -4 & 13 & -1-\lambda \end{vmatrix} \\ &= (-1-\lambda)^2(3-\lambda) - 13 + 4(3-\lambda) \\ &= (\lambda^2 + 2\lambda + 1)(3-\lambda) - 13 + 12 - 4\lambda \\ &= 3\lambda^2 + 6\lambda + 3 - \lambda^3 - 2\lambda^2 - \lambda - 1 - 4\lambda \\ &= -\lambda^3 + \lambda^2 + \lambda + 2 \end{aligned}$$

The characteristic equation:  $-\lambda^3 + \lambda^2 + \lambda + 2 = 0$

ii.  $\lambda = 2$

$$\begin{array}{c|cccc} 2 & -1 & 1 & 1 & 2 \\ & & -2 & -2 & 2 \\ \hline & -1 & -1 & -1 & 0 \end{array} \rightarrow -\lambda^2 - \lambda - 1 = 0$$

Therefore; the eigenvalues are:  $\lambda_{1,2,3} = 2, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

iii. For  $\lambda_1 = 2$ , we have:  $(A - \lambda_1 I)V_1 = 0$

$$\begin{aligned} \begin{pmatrix} -3 & 0 & 1 \\ -1 & 1 & 0 \\ -4 & 13 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{cases} -3x_1 + z_1 = 0 & \Rightarrow \underline{z_1 = 3x_1} \\ -x_1 + y_1 = 0 & \Rightarrow \underline{y_1 = x_1} \\ -4x_1 + 13y_1 - 3z_1 = 0 \end{cases} \end{aligned}$$

Therefore, the eigenvector  $V_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

For  $\lambda_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ , we have:  $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -\frac{1}{2} + i\frac{\sqrt{3}}{2} & 0 & 1 \\ -1 & \frac{7}{2} + i\frac{\sqrt{3}}{2} & 0 \\ -4 & 13 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow$$

$$\begin{cases} \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x_2 + z_2 = 0 & \Rightarrow z_2 = -\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x_2 \\ -x_2 + \left(\frac{7}{2} + i\frac{\sqrt{3}}{2}\right)y_2 = 0 & \Rightarrow y_2 = \left(\frac{2}{7 + i\sqrt{3}}\right)x_2 \\ -4x_2 + 13y_2 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)z_2 = 0 \end{cases}$$

Therefore; the eigenvector  $V_2 = \begin{pmatrix} 1 \\ \frac{1 - i\sqrt{3}}{2} \\ \frac{2}{7 + i\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1 - i\sqrt{3}}{2} \\ \frac{7 - i\sqrt{3}}{26} \end{pmatrix}$

$$V_3 = \begin{pmatrix} 1 \\ \frac{1 + i\sqrt{3}}{2} \\ \frac{2}{7 - i\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1 + i\sqrt{3}}{2} \\ \frac{7 + i\sqrt{3}}{26} \end{pmatrix}$$

### Exercise

For the matrix:  $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 0 & 0 \\ 4 & 3-\lambda & 2 \\ -8 & -4 & -3-\lambda \end{vmatrix} \\ &= (1-\lambda)(3-\lambda)(-3-\lambda) + 8(1-\lambda) \\ &= -9 + 9\lambda + \lambda^2 - \lambda^3 + 8 - 8\lambda \\ &= -\lambda^3 + \lambda^2 + \lambda - 1 \end{aligned}$$

The characteristic equation:  $-\lambda^3 + \lambda^2 + \lambda - 1 = 0$

ii.  $\lambda = 1$

$$\begin{array}{c|ccc} 1 & -1 & 1 & 1 & -1 \\ & & -1 & 0 & 1 \\ \hline & -1 & 0 & 1 & 0 \end{array} \rightarrow -x^2 + 1 = 0$$

Thus, the eigenvalues are:  $\lambda_{1,2,3} = 1, 1, -1$

iii. For  $\lambda_1 = 1 \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 4 & 2 & 2 \\ -8 & -4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 0 = 0 \\ 4x + 2y + 2z = 0 \\ -8x - 4y - 4z = 0 \end{cases} \Rightarrow \begin{cases} 2x = -y - z \\ 2x = -y - z \end{cases}$$

$$\text{If } x = 0 \Rightarrow y = -z$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}$$

$$\text{For } \lambda_2 = -1 \Rightarrow (A - \lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 4 & 4 & 2 \\ -8 & -4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x = 0 & \Rightarrow \underline{x = 0} \\ 4x + 4y + 2z = 0 & \Rightarrow \underline{z = -2y} \\ -8x - 4y - 2z = 0 \end{cases}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}}$$

$$\text{For } \lambda_3 = -1$$

$$AV_3 = V_2$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$$\begin{cases} \underline{x = 0} \\ 4x + 3y + 2z = 1 & \Rightarrow \underline{2z = 1 - 3y} \\ -8x - 4y - 3z = -2 & \Rightarrow \end{cases}$$

$$\Rightarrow \underline{V_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}$$

### Exercise

For the matrix: 
$$\begin{pmatrix} -1 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & -12 & 2 \end{pmatrix}$$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} -1 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & -12 & 2 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} -1-\lambda & -4 & -2 \\ 0 & 1-\lambda & 1 \\ -6 & -12 & 2-\lambda \end{vmatrix} \\ &= (-1-\lambda)(1-\lambda)(2-\lambda) + 24 - 12(1-\lambda) + 12(-1-\lambda) \\ &= -\lambda^3 + 2\lambda^2 + \lambda - 2 + 24 - 12 + 12\lambda - 12 - 12\lambda \\ &= -\lambda^3 + 2\lambda^2 + \lambda - 2 \end{aligned}$$

The characteristic equation:  $\underline{-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0}$

ii.  $\lambda = 1$

$$\begin{array}{c|cccc} 1 & -1 & 2 & 1 & -2 \\ & & -1 & 1 & 2 \\ \hline & -1 & 1 & 2 & \mathbf{0} \end{array} \rightarrow \underline{-x^2 + x + 2 = 0}$$

Thus, the eigenvalues are:  $\underline{\lambda_1 = -1 \quad \lambda_2 = 1 \quad \text{and} \quad \lambda_3 = 2}$

iii. For  $\lambda_1 = -1 \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & -4 & -2 \\ 0 & 2 & 1 \\ -6 & -12 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\begin{cases} -4y - 2z = 0 \\ 2y + z = 0 \\ -6x - 12y + 3z = 0 \end{cases} \rightarrow \begin{cases} 2y = -z \\ -6x = 12y - 3z \end{cases} \Rightarrow \begin{cases} y = -\frac{1}{2}z \\ x = \frac{3}{2}z \end{cases}$$

$$\Rightarrow V_1 = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

$$\text{For } \lambda_2 = 1 \Rightarrow (A - \lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} -2 & -4 & -2 \\ 0 & 0 & 1 \\ -6 & -12 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2x - 4y - 2z = 0 & \Rightarrow -2x - 4y = 0 \\ \quad \quad \quad \underline{z = 0} \\ -6x - 12y + 2z = 0 \\ \quad \quad \quad \underline{x = -2y} \end{cases}$$

$$\Rightarrow V_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{For } \lambda_3 = 2 \Rightarrow (A - \lambda_3 I)V_3 = 0$$

$$\begin{pmatrix} -3 & -4 & -2 \\ 0 & -1 & 1 \\ -6 & -12 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} -3x - 4y - 2z = 0 & \Rightarrow -3x = 6z \\ \quad -y + z = 0 & \Rightarrow \underline{y = z} \\ \quad -6x - 12y = 0 \end{cases}$$

$$\underline{x = -2z}$$

$$\Rightarrow V_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

### Exercise

For the matrix:  $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$$

$$\begin{aligned} \text{i. } |A - \lambda I| &= \begin{vmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{vmatrix} \\ &= (3-\lambda)(4-\lambda)(-1-\lambda) - 4 - 8 + 4(4-\lambda) + 4(3-\lambda) + 2\lambda + 2 \\ &= -\lambda^3 + 6\lambda^2 - 5\lambda - 12 - 12 + 16 - 4\lambda + 12 - 4\lambda + 2\lambda + 2 \\ &= -\lambda^3 + 6\lambda^2 - 11\lambda + 6 \end{aligned}$$

The characteristic equation:  $\underline{-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0}$

ii.  $\lambda = 1$

$$\begin{array}{c|ccc} 1 & -1 & 6 & -11 & 6 \\ & & -1 & 5 & -6 \\ \hline & -1 & 5 & -6 & \mathbf{0} \end{array} \rightarrow \underline{-\lambda^2 + 5\lambda - 6 = 0}$$

Thus, the eigenvalues are:  $\underline{\lambda_1 = 1 \quad \lambda_2 = 2 \quad \text{and} \quad \lambda_3 = 3}$

iii. For  $\lambda_1 = 1 \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x + 2y + 2z = 0 & (1) \\ x + 3y + z = 0 & (2) \\ -2x - 4y - 2z = 0 & (3) \end{cases}$$

$$(1) + (3) \rightarrow \underline{y = 0}$$

$$\begin{cases} (1) & 2x + 2z = 0 \\ (2) & \frac{x+z=0}{3x+3z=0} \end{cases} \Rightarrow \underline{x = -z}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

$$\text{For } \lambda_2 = 2 \Rightarrow (A - \lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x + 2y + 2z = 0 & (1) \\ x + 2y + z = 0 & (2) \\ -2x - 4y - 3z = 0 & (3) \end{cases}$$

$$\begin{cases} 2 \times (2) & 2x + 4y + 2z = 0 \\ (3) & -2x - 4y - 3z = 0 \end{cases} \Rightarrow \underline{z = 0}$$

$$(1) \rightarrow x + 2y = 0 \rightarrow \underline{x = -2y}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}$$

$$\text{For } \lambda_3 = 3 \Rightarrow (A - \lambda_3 I)V_3 = 0$$

$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ -2 & -4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2y + 2z = 0 \rightarrow \underline{y = -z} \\ x + y + z = 0 \rightarrow \underline{x = 0} \\ -2x - 4y - 4z = 0 \end{cases}$$

$$\Rightarrow \underline{V_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}$$



### Exercise

For the matrix:  $\begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}$

- i.* Find the characteristic equation
- ii.* Find the eigenvalues
- iii.* Find the eigenvectors

### Solution

$$A = \begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}$$

$$\begin{aligned} i. \quad |A - \lambda I| &= \begin{vmatrix} -6 - \lambda & 4 & 4 \\ -4 & 2 - \lambda & 4 \\ -10 & 8 & 4 - \lambda \end{vmatrix} \\ &= (-6 - \lambda)(2 - \lambda)(4 - \lambda) - 160 - 128 + 40(2 - \lambda) + 32(6 + \lambda) + 16(4 - \lambda) \\ &= -\lambda^3 + 4\lambda \end{aligned}$$

The characteristic equation:  $-\lambda^3 + 4\lambda = 0$

$$ii. \quad -\lambda(\lambda^2 - 4) = 0$$

Thus, the eigenvalues are:  $\lambda_1 = 0 \quad \lambda_2 = -2 \quad \text{and} \quad \lambda_3 = 2$

$$iii. \quad \text{For } \lambda_1 = 0 \Rightarrow (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -6x + 4y + 4z = 0 & (1) \\ -4x + 2y + 4z = 0 & (2) \\ -10x + 8y + 4z = 0 & (3) \end{cases}$$

$$(1) - (2) \rightarrow -2x + 2y = 0$$

$$\underline{x = y}$$

$$\begin{cases} (1) & -2x + 4z = 0 \\ (2) & -2x + 4z = 0 \end{cases} \rightarrow \underline{x = 2z}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}$$

$$\text{For } \lambda_2 = -2 \Rightarrow (A - \lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} -4 & 4 & 4 \\ -4 & 4 & 4 \\ -10 & 8 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -4x + 4y + 4z = 0 & (1) \\ -4x + 4y + 4z = 0 & (2) \\ -10x + 8y + 6z = 0 & (3) \end{cases}$$

$$\begin{cases} -x + y + z = 0 & (4) \\ -5x + 4y + 3z = 0 & (5) \end{cases}$$

$$5 \times (4) - (5) \rightarrow y + 2z = 0$$

$$\underline{y = -2z}$$

$$(5) \rightarrow -5x - 8z + 3z = 0$$

$$\underline{x = -z}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}}$$

$$\text{For } \lambda_3 = 2 \Rightarrow (A - \lambda_3 I)V_3 = 0$$

$$\begin{pmatrix} -8 & 4 & 4 \\ -4 & 0 & 4 \\ -10 & 8 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -8x + 4y + 4z = 0 \\ -4x + 4z = 0 \Rightarrow \underline{x = z} \\ -10x + 8y + 2z = 0 \end{cases}$$

$$\begin{cases} (1) & 4y - 4z = 0 \\ (3) & 8y - 8z = 0 \end{cases} \rightarrow \underline{y = z}$$

$$\Rightarrow \underline{V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

### Exercise

For the matrix:

$$\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- i.* Find the characteristic equation
- ii.* Find the eigenvalues
- iii.* Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} i. \quad \det(A - \lambda I) &= \begin{vmatrix} -\lambda & 0 & 2 & 0 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -2-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda) \begin{vmatrix} -\lambda & 0 & 2 \\ 1 & -\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix} \\ &= (1-\lambda) (\lambda^2(-2-\lambda) + 2 + \lambda) \\ &= (1-\lambda) (-\lambda^3 - 2\lambda^2 + \lambda + 2) \\ &= \lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2 \end{aligned}$$

The characteristic equation:  $\lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2 = 0$

*ii.*  $\lambda = 1$

$$\begin{array}{c|ccccc} 1 & 1 & 1 & -3 & -1 & 2 \\ & & 1 & 2 & -1 & -2 \\ \hline 1 & 1 & 2 & -1 & -2 & 0 \\ & & 1 & 3 & 2 & \\ \hline & 1 & 3 & 2 & 0 & \end{array} \rightarrow \begin{array}{l} \lambda^3 + 2\lambda^2 - \lambda + 2 = 0 \\ \lambda^2 + 3\lambda + 2 = 0 \end{array}$$

Thus, the eigenvalues are:  $\lambda_{1,2,3,4} = -2, -1, 1, 1$

iii. For  $\lambda_1 = -2 \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 0 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x_1 + 2x_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \\ x_2 = 0 \\ x_4 = 0 \end{cases}$$

$$\rightarrow \begin{cases} \underline{x_1 = -x_3} \\ \underline{x_2 = 0} \\ \underline{x_4 = 0} \end{cases}$$

Therefore; the eigenvector  $V_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

For  $\lambda_2 = -1 \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \\ x_2 - x_3 = 0 \\ \underline{x_4 = 0} \end{cases}$$

$$\rightarrow \begin{cases} \underline{x_1 = -2x_3} \\ \underline{x_2 = x_3} \end{cases}$$

Therefore; the eigenvector  $V_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

$$\text{For } \lambda_3 = 1 \Rightarrow (A - \lambda_3 I)V_3 = 0$$

$$\begin{pmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x_1 + 2x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_2 - 3x_3 = 0 \end{cases}$$

$$\rightarrow \begin{cases} x_1 = 2x_3 \\ x_2 = 3x_3 \\ \forall x_4 \end{cases}$$

$$\text{Therefore; the eigenvector } V_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x_1 + 2x_3 = 2 \\ x_1 - x_2 + x_3 = 3 \\ x_2 - 3x_3 = 1 \end{cases}$$

$$\begin{cases} x_1 = 2x_3 - 2 \\ x_2 = 1 + 3x_3 \end{cases}$$

$$\text{Therefore; the eigenvector } V_4 = \begin{pmatrix} 0 \\ 4 \\ 1 \\ 1 \end{pmatrix} \quad V_4 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

### Exercise

For the matrix:

$$\begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

- i. Find the characteristic equation
- ii. Find the eigenvalues
- iii. Find the eigenvectors

### Solution

$$A = \begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\begin{aligned} i. \quad \det(A - \lambda I) &= \begin{vmatrix} 10 - \lambda & -9 & 0 & 0 \\ 4 & -2 - \lambda & 0 & 0 \\ 0 & 0 & -2 - \lambda & -7 \\ 0 & 0 & 1 & 2 - \lambda \end{vmatrix} \\ &= (10 - \lambda) \begin{vmatrix} -2 - \lambda & 0 & 0 \\ 0 & -2 - \lambda & -7 \\ 0 & 1 & 2 - \lambda \end{vmatrix} + 9 \begin{vmatrix} 4 & 0 & 0 \\ 0 & -2 - \lambda & -7 \\ 0 & 1 & 2 - \lambda \end{vmatrix} \\ &= (10 - \lambda) \left[ (-2 - \lambda)^2 (2 - \lambda) + 7(-2 - \lambda) \right] + 9 \left[ (4)(-2 - \lambda)(2 - \lambda) + 28 \right] \\ &= (10 - \lambda)(-2 - \lambda)(3 + \lambda^2) + 9(4\lambda^2 + 12) \\ &= (3 + \lambda^2)(-8\lambda + \lambda^2 + 16) \\ &= (3 + \lambda^2)(\lambda - 4)^2 \end{aligned}$$

$$\Rightarrow \text{The characteristic equation: } \underline{(3 + \lambda^2)(\lambda - 4)^2 = 0}$$

$$ii. \quad \text{The eigenvalues are } \underline{\lambda_{1,2,3,4} = 4, 4, \pm i\sqrt{3}}$$

$$iii. \quad \text{For } \lambda_1 = 4 \Rightarrow (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 6 & -9 & 0 & 0 \\ 4 & -6 & 0 & 0 \\ 0 & 0 & -6 & -7 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 6x_1 - 9x_2 = 0 & \rightarrow \underline{2x_1 = 3x_2} \\ 4x_1 - 6x_2 = 0 \\ -6x_3 - 7x_4 = 0 & (1) \\ x_3 - 2x_4 = 0 & (2) \end{cases}$$

$$(1) \& (2) \rightarrow \underline{x_3 = x_4 = 0}$$

Therefore; the eigenvector  $V_1 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 6 & -9 & 0 & 0 \\ 4 & -6 & 0 & 0 \\ 0 & 0 & -6 & -7 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 6x_1 - 9x_2 = 3 & (3) \\ 4x_1 - 6x_2 = 2 & (4) \\ -6x_3 - 7x_4 = 0 & (5) \\ x_3 - 2x_4 = 0 & (6) \end{cases}$$

$$\begin{cases} 6x_1 - 9x_2 = 3 \\ 4x_1 - 6x_2 = 2 \end{cases} \quad \Delta = \begin{vmatrix} 6 & -9 \\ 4 & -6 \end{vmatrix} = 0 \quad \Delta_4 = \begin{vmatrix} 6 & 3 \\ 4 & 2 \end{vmatrix} = 0$$

$$(5) \& (6) \rightarrow \underline{x_3 = x_4 = 0}$$

Therefore; the eigenvector  $V_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

For  $\lambda_3 = -i\sqrt{3} \Rightarrow (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} 10+i\sqrt{3} & -9 & 0 & 0 \\ 4 & -2+i\sqrt{3} & 0 & 0 \\ 0 & 0 & -2+i\sqrt{3} & -7 \\ 0 & 0 & 1 & 2+i\sqrt{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} (10+i\sqrt{3})x_1 - 9x_2 = 0 \rightarrow x_1 = \frac{9}{10+i\sqrt{3}}x_2 \\ 4x_1 + (-2+i\sqrt{3})x_2 = 0 \rightarrow x_1 = \frac{-2+i\sqrt{3}}{4}x_2 \\ (-2+i\sqrt{3})x_3 - 7x_4 = 0 \rightarrow (6) \\ x_3 + (2+i\sqrt{3})x_4 = 0 \rightarrow x_3 = -(2+i\sqrt{3})x_4 \end{cases}$$

$$(6) \rightarrow \frac{7}{-2+i\sqrt{3}} \left( \frac{-2-i\sqrt{3}}{-2-i\sqrt{3}} \right) = -(2+i\sqrt{3})$$

$$\frac{9}{10+i\sqrt{3}} \stackrel{?}{=} \frac{-2+i\sqrt{3}}{4}$$

$$36 \neq (-2+i\sqrt{3})(10+i\sqrt{3})$$

$$\Rightarrow \underline{x_1 = x_2 = 0}$$

Therefore; the eigenvector  $V_3 = \begin{pmatrix} 0 \\ 0 \\ -(2+i\sqrt{3}) \\ 1 \end{pmatrix}$

Therefore; the eigenvector  $V_4 = \begin{pmatrix} 0 \\ 0 \\ -2+i\sqrt{3} \\ 1 \end{pmatrix}$

### Exercise

Find the eigenvalues of  $A^9$  for  $A = \begin{pmatrix} 1 & 3 & 7 & 11 \\ 0 & \frac{1}{2} & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

### Solution

Since the matrix is an upper triangular, then the eigenvalues are:  $\lambda = 1, \frac{1}{2}, 0, 2$

The eigenvalues of  $A^9$  are:  $1^9 = \underline{1} \quad \left( \frac{1}{2} \right)^9 = \underline{\frac{1}{512}}$

$0^9 = \underline{0} \quad 2^9 = \underline{512}$



### Exercise

Given:  $A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$ . Compute  $A^{11}$

### Solution

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 7 & -1 \\ 0 & 1-\lambda & 0 \\ 0 & 15 & -2-\lambda \end{vmatrix}$$
$$= (-1-\lambda)(1-\lambda)(-2-\lambda)$$

The eigenvalues are:  $\lambda_{1,2,3} = -1, 1, -2$

For  $\lambda_1 = -1$ , we have:  $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 7 & -1 \\ 0 & 2 & 0 \\ 0 & 15 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\begin{cases} 7y_1 - z_1 = 0 & \rightarrow \underline{z_1 = 7y_1} \\ 2y_1 = 0 & \rightarrow \underline{y_1 = 0} \\ 15y_1 - z_1 = 0 \end{cases}$$

The eigenvector  $V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

For  $\lambda_2 = 1$ , we have:  $(A - I)V_2 = 0$

$$\begin{pmatrix} -2 & 7 & -1 \\ 0 & 0 & 0 \\ 0 & 15 & -3 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\begin{cases} -2x_2 + 7y_2 - z_2 = 0 & \rightarrow 2x_2 = 7y_2 - z_2 \\ 15y_2 - 3z_2 = 0 & \rightarrow \underline{5y_2 = z_2} \end{cases}$$

$$2x_2 = 7y_2 - 5y_2$$

$$\underline{x_2 = y_2}$$

The eigenvector  $V_2 = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

For  $\lambda_3 = -2$ , we have:  $(A + 2I)V_3 = 0$

$$\begin{pmatrix} 1 & 7 & -1 \\ 0 & 3 & 0 \\ 0 & 15 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_3 + 7y_3 - z_3 = 0 & \rightarrow \underline{x_3 = z_3} \\ 3y_3 = 0 & \rightarrow \underline{y_3 = 0} \\ 15y_3 = 0 \end{cases}$$

The eigenvector  $V_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$D^{11} = \begin{pmatrix} (-1)^{11} & 0 & 0 \\ 0 & 1^{11} & 0 \\ 0 & 0 & (-2)^{11} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2048 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}$$

$$\left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 5 & 1 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 5R_2 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -5 & 1 \end{array} \right) \begin{array}{l} R_1 - R_3 \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & | & 1 & 4 & -1 \\ 0 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & -5 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$A^{11} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2048 \end{pmatrix} \begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & -2048 \\ 0 & 1 & 0 \\ 0 & 5 & -2048 \end{pmatrix} \begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 10237 & 2047 \\ 0 & 1 & 0 \\ 0 & 10245 & -2048 \end{pmatrix}$$

### ***Exercise***

Find the eigenvalues of the matrices

$$A = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0.61 & 0.52 \\ 0.39 & 0.48 \end{pmatrix}, \quad A^\infty = \begin{pmatrix} 0.57143 & 0.57143 \\ 0.42857 & 0.42857 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{pmatrix}$$

### **Solution**

The eigenvalues for  $A$  :

$$\begin{vmatrix} 0.7 - \lambda & 0.4 \\ 0.3 & 0.6 - \lambda \end{vmatrix} = (0.7 - \lambda)(0.6 - \lambda) - .12$$

$$= \lambda^2 - 1.3\lambda + .3 = 0$$

$$\lambda_{1,2} = \frac{1.3 \pm \sqrt{1.69 - 1.2}}{2}$$

$$= .65 \pm \frac{\sqrt{.49}}{2}$$

$$= 0.65 \pm 0.35$$

The eigenvalues are:  $\lambda_1 = 1 \quad \lambda_2 = 0.3$

The eigenvalues for  $A^2$  :

$$\lambda_1 = 1^2 = \underline{1}$$

$$\lambda_2 = 0.3^2 = \underline{0.09}$$

The eigenvalues for  $A^\infty$  :

$$\lambda^2 - \lambda = 0$$

$$\lambda_1 = 1^2 = \underline{1}$$

$$\lambda_2 = 0.3^\infty = \underline{0}$$

The eigenvalues for  $B$  :

$$\begin{vmatrix} 0.3 - \lambda & 0.6 \\ 0.7 & 0.4 - \lambda \end{vmatrix} = \lambda^2 - .7\lambda - .3 = 0$$

$$\lambda_{1,2} = 0.35 \pm 0.65$$

$$\text{The eigenvalues are: } \underline{\lambda_1 = 1 \quad \lambda_2 = -0.3}$$

### ***Exercise***

Given the matrix  $\begin{bmatrix} -1 & -3 \\ -3 & 7 \end{bmatrix}$

- Find the characteristic polynomial.
- Find the eigenvalues
- Find the bases for its eigenspaces
- Graph the eigenspaces
- Verify directly that  $A\vec{v} = \lambda\vec{v}$ , for all associated eigenvectors and eigenvalues.

### **Solution**

$$\begin{aligned} a) \quad \begin{vmatrix} -1 - \lambda & -3 \\ -3 & 7 - \lambda \end{vmatrix} &= (-1 - \lambda)(7 - \lambda) - 9 \\ &= -7 - 6\lambda + \lambda^2 - 9 \\ &= \lambda^2 - 6\lambda - 16 \end{aligned}$$

The characteristic polynomial is  $\underline{\lambda^2 - 6\lambda - 16 = 0}$

$$b) \quad \lambda^2 - 6\lambda - 16 = 0 \Rightarrow \lambda_1 = -2 \quad \text{and} \quad \lambda_2 = 8$$

$$c) \quad \text{For } \lambda_1 = -2, \text{ we have: } (A + 2I)V_1 = 0$$

$$\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 - 3y_1 = 0 \\ -3x_1 + 9y_1 = 0 \end{cases} \Rightarrow \underline{x_1 = 3y_1}$$

Therefore, the eigenvector  $\underline{V_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}}$

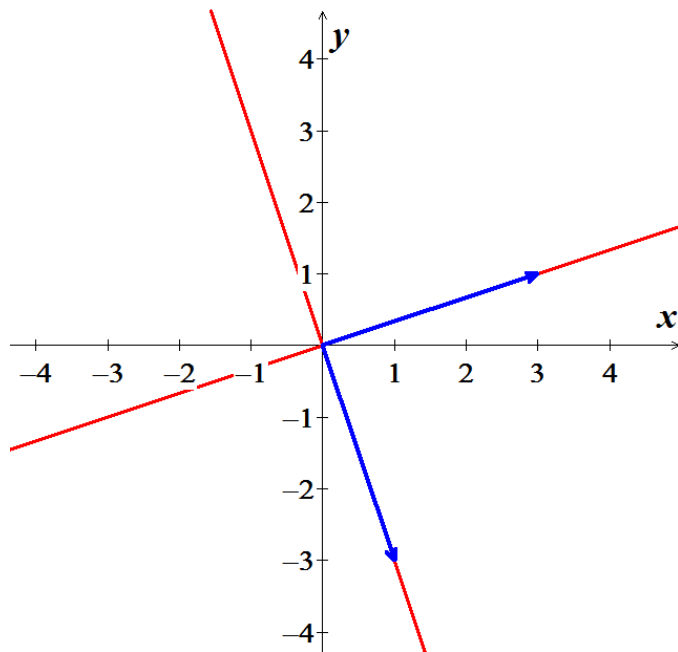
For  $\lambda_2 = 8$ , we have:  $(A + 8I)V_2 = 0$

$$\begin{pmatrix} -9 & -3 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -9x_2 - 3y_2 = 0 \\ -3x_2 - y_2 = 0 \end{cases} \Rightarrow \underline{y_2 = -3x_2}$$

Therefore, the eigenvector  $\underline{V_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}}$

d)



e)  $AV_1 = \lambda_1 V_1$

$$\begin{pmatrix} -1 & -3 \\ -3 & 7 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = -2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -6 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \end{pmatrix} \checkmark$$

$$AV_2 = \lambda_2 V_2$$

$$\begin{pmatrix} -1 & -3 \\ -3 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = 8 \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 8 \\ -24 \end{pmatrix} = \begin{pmatrix} -6 \\ -24 \end{pmatrix} \checkmark$$

### Exercise

Given the matrix  $\begin{bmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{bmatrix}$

- Find the characteristic polynomial.
- Find the eigenvalues
- Find the bases for its eigenspaces
- Graph the eigenspaces
- Verify directly that  $A\vec{v} = \lambda\vec{v}$ , for all associated eigenvectors and eigenvalues.

### Solution

$$a) \begin{vmatrix} 5-\lambda & 0 & -4 \\ 0 & -3-\lambda & 0 \\ -4 & 0 & -1-\lambda \end{vmatrix} = (5-\lambda)(-3-\lambda)(-1-\lambda) - 16(-3-\lambda)$$

$$= (5-\lambda)(3+4\lambda+\lambda^2) + 48 + 16\lambda$$

$$= 15 + 20\lambda + 5\lambda^2 - 3\lambda - 4\lambda^2 - \lambda^3 + 48 + 16\lambda$$

$$= -\lambda^3 + \lambda^2 + 33\lambda + 63$$

The characteristic polynomial is  $-\lambda^3 + \lambda^2 + 33\lambda + 63 = 0$

b)  $\lambda = -3$

$$\begin{array}{c|cccc} -3 & -1 & 1 & 33 & 63 \\ & & 3 & -12 & -63 \\ \hline & -1 & 4 & 21 & 0 \end{array} \rightarrow -\lambda^2 + 4\lambda + 21 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16 + 84}}{-2}$$

The eigenvalues are:  $\lambda_{1,2,3} = -3, -3, 7$

c) For  $\lambda_{1,2} = -3$ , we have:  $(A + 3I)V_1 = 0$

$$\begin{pmatrix} 8 & 0 & -4 \\ 0 & 0 & 0 \\ -4 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 8x_1 - 4z_1 = 0 & \rightarrow \underline{z_1 = 2x_1} \\ -4x_1 + 2z_1 = 0 \end{cases}$$

Therefore, the eigenvector  $V_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$  and  $V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

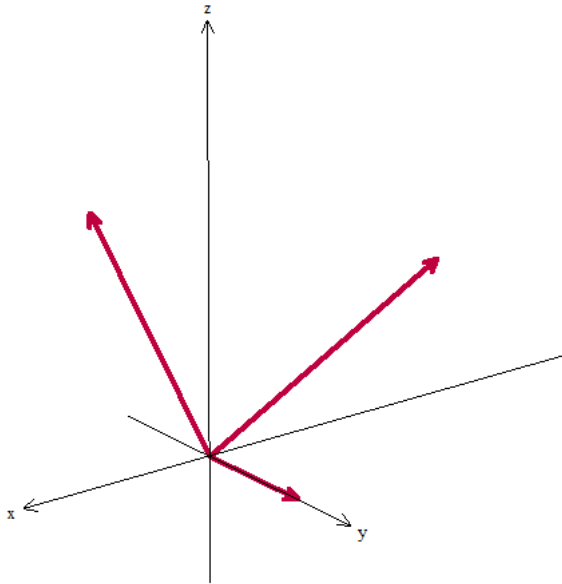
For  $\lambda_3 = 7$ , we have:  $(A - 7I)V_3 = 0$

$$\begin{pmatrix} -2 & 0 & -4 \\ 0 & -10 & 0 \\ -4 & 0 & -8 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2x_3 - 4z_3 = 0 & \rightarrow \underline{x_3 = -2z_3} \\ -10y_3 = 0 & \rightarrow \underline{y_3 = 0} \\ -4x_3 - 8z_3 = 0 \end{cases}$$

Therefore, the eigenvector  $V_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

d)



e)  $AV_1 = \lambda_1 V_1$

$$\begin{pmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -3 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -6 \end{pmatrix} \checkmark$$

$$AV_2 = \lambda_2 V_2$$

$$\begin{pmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} \checkmark$$

$$AV_3 = \lambda_3 V_3$$

$$\begin{pmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 7 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -14 \\ 0 \\ 7 \end{pmatrix} = \begin{pmatrix} -14 \\ 0 \\ 7 \end{pmatrix} \checkmark$$

### ***Exercise***

Explain why a  $2 \times 2$  matrix can have at most two distinct eigenvalues. Explain why an  $n \times n$  matrix can have at most  $n$  distinct eigenvalues

### **Solution**

A  $2 \times 2$  matrix has only 2 entries in the main diagonal. Then, Lambda exists only twice in those entries. By using the determinant, the product will produce a power two characteristics equation. A second-degree equation will produce 2 real distinct eigenvalues, or 2 repeated eigenvalues, or 2 complex eigenvalues. Therefore, a  $2 \times 2$  matrix can have at most two distinct eigenvalues.

The same for an  $n \times n$  matrix, the matrix has  $n$  entries in the main diagonal with lambda. Then the product of  $n^{th}$  lambda will produce a characteristic equation with  $n$  power. That means that will have  $n$  real distinct eigenvalues, or  $n$  repeated eigenvalues, or  $n$  complex eigenvalues.

Therefore, a  $n \times n$  matrix can have at most  $n$  distinct eigenvalues.



### Exercise

Construct an example of a  $2 \times 2$  matrix with only one distinct eigenvalue.

### Solution

A  $2 \times 2$  matrix with only one distinct eigenvalue, which means that we have repeated lambda. To do so, the other diagonal has a zero and the main diagonal has the same value.

Example for one zero in the diagonal.

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$
$$\begin{vmatrix} a-\lambda & b \\ 0 & a-\lambda \end{vmatrix} = (a-\lambda)^2 = 0$$
$$\lambda_{1,2} = a$$

Example:  $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

### Exercise

Let  $\lambda$  be an eigenvalue of an invertible matrix  $A$ . Show that  $\lambda^{-1}$  is an eigenvalue of  $A^{-1}$ .

### Solution

Since matrix  $A$  is invertible, then  $AA^{-1} = A^{-1}A = I$

Let  $\lambda$  be an eigenvalue of an invertible matrix  $A$ , then there is a nonzero eigenvector  $\vec{v}$  such that  $A\vec{v} = \lambda\vec{v}$

$$A^{-1}A\vec{v} = A^{-1}\lambda\vec{v}$$

$$I\vec{v} = \lambda(A^{-1}\vec{v})$$

$$\vec{v} = \lambda(A^{-1}\vec{v})$$

Since  $\vec{v} \neq \vec{0}$  and  $\lambda$  cannot be zero. Then

$$\frac{1}{\lambda} \vec{v} = A^{-1} \vec{v}$$

$$\lambda^{-1} \vec{v} = A^{-1} \vec{v}$$

That will prove that  $\lambda^{-1}$  is an eigenvalue of  $A^{-1}$

### Exercise

Show that if  $A^2$  is the zero matrix, then the only eigenvalue of  $A$  is 0

#### Solution

Assume that  $A^2$  is the zero matrix.

$$\text{If } A\vec{v} = \lambda\vec{v}$$

$$AA\vec{v} = A\lambda\vec{v}$$

$$A^2\vec{v} = \lambda(A\vec{v})$$

$$A^2\vec{v} = \lambda(\lambda\vec{v})$$

$$A^2\vec{v} = \lambda^2\vec{v}$$

Since  $\vec{v} \neq \vec{0}$  and  $A^2$  is the zero matrix. Then  $\lambda$  must be zero.

Therefore, each eigenvalue of  $A$  is zero.

### Exercise

Show that  $\lambda$  is an eigenvalue of  $A$  if and only if  $\lambda$  is an eigenvalue of  $A^T$ .

#### Solution

Suppose that  $\lambda$  is an eigenvalue of  $A$ , then  $A - \lambda I = 0$

$$\begin{aligned}(A - \lambda I)^T &= A^T - \lambda I^T \\ &= A^T - \lambda I = 0\end{aligned}$$

This will result that matrix and its transpose have the same characteristic equation.

Thus,  $\lambda$  is an eigenvalue of  $A$  if and only if  $\lambda$  is an eigenvalue of  $A^T$

### Exercise

For  $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ , find one eigenvalue, without calculation. Justify your answer.

#### Solution

Since the matrix  $A$  has the row then matrix  $A$  is not invertible (Columns are linearly dependent). Therefore, the eigenvalue is zero of the matrix.

### Exercise

For  $A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ , find one eigenvalue, and two linearly independent eigenvectors, without calculation.

Justify your answer.

### Solution

Since the matrix  $A$  has the row then matrix  $A$  is not invertible (Columns are linearly dependent). Therefore, the eigenvalue is zero of the matrix.

For  $\lambda = 0$ , then the eigenvector is given by  $(A - \lambda I)V = 0$

Since  $\lambda = 0$ , that implies to  $AV = 0$

Since matrix  $A$  is nonzero matrix that it will imply to  $2x + 2y + 2z = 0$  all rows are the same.

Which it will result to:  $x + y + z = 0$

The two linearly independent eigenvectors:

$$V_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad V_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

### Exercise

Consider an  $n \times n$  matrix  $A$  with the property that the row sums all equal the same number  $s$ . Show that  $s$  is an eigenvalue of  $A$ .

### Solution

Let consider a  $2 \times 2$  matrix with all ones as entries

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$s = 1 + 2 = 3$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 2 \\ = \lambda^2 - 3\lambda = 0$$

One of the eigenvalues is:  $\lambda = 3 = s$

$$\text{Let } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\text{With } \begin{cases} a + b = s \\ c + d = s \end{cases}$$

$$\begin{pmatrix} a \\ c \end{pmatrix} + \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} s \\ s \end{pmatrix}$$

$$1 \cdot \begin{pmatrix} a \\ c \end{pmatrix} + 1 \cdot \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} s \\ s \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A\vec{v} = s\vec{v} \quad \checkmark$$

For  $n \times n$  matrix  $A$ :

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$$\text{Where } s = \sum_{i=1}^n a_{1i} = \dots = \sum_{i=1}^n a_{ni}$$

$$\begin{pmatrix} a_{11} + \cdots + a_{1n} \\ \vdots \\ a_{n1} + \cdots + a_{nn} \end{pmatrix} = \begin{pmatrix} s \\ \vdots \\ s \end{pmatrix}$$

$$1 \cdot \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} + \cdots + 1 \cdot \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} = s \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = s \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$A\vec{v} = s\vec{v} \quad \checkmark$$

That prove that  $s$  is an eigenvalue of  $A$ .

### Exercise

Consider an  $n \times n$  matrix  $A$  with the property that the column sums all equal the same number  $s$ . Show that  $s$  is an eigenvalue of  $A$ .

### Solution

Given that the column sums of an  $n \times n$  matrix  $A$  all equal the same number  $s$ .

Then the transpose of the matrix  $A$  will imply that  $A^T$  has the row sums all equal the same number  $s$ .

In addition, the matrix  $A$  and  $A^T$  have the same eigenvalues.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \quad \text{Where } s = \sum_{i=1}^n a_{i1} = \cdots = \sum_{i=1}^n a_{in}$$

$$A^T = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{pmatrix} \quad \text{Where } s = \sum_{i=1}^n a_{i1} = \cdots = \sum_{i=1}^n a_{in}$$

$$\begin{pmatrix} a_{11} + \cdots + a_{n1} \\ \vdots \\ a_{1n} + \cdots + a_{nn} \end{pmatrix} = \begin{pmatrix} s \\ \vdots \\ s \end{pmatrix}$$

$$1 \cdot \begin{pmatrix} a_{11} \\ \vdots \\ a_{1n} \end{pmatrix} + \cdots + 1 \cdot \begin{pmatrix} a_{n1} \\ \vdots \\ a_{nn} \end{pmatrix} = s \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} + \cdots + a_{n1} \\ \vdots \\ a_{1n} + \cdots + a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = s \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$A^T \vec{v} = s \vec{v} \quad \checkmark$$

That show that  $s$  is an eigenvalue of  $A^T$  and since  $A$  and  $A^T$  have the same eigenvalues.

The prove is completed that  $s$  is an eigenvalue of  $A$ .

### Exercise

Let  $A$  be the matrix of the linear transformation  $T$  on  $\mathbb{R}^2$

$T$ : reflects points across some line through the origin.

Without writing  $A$ , find an eigenvalue of  $A$  and describe the eigenspace.

### Solution

Given  $T$  reflects points across some line through the origin in  $\mathbb{R}^2$ , which implies that the coordinates are equal ( $x = y$ ).

The linear transformation can be written in the form:  $T(\vec{v}) = \vec{v}$

This line more likely is the scalar nonzero product of the eigenvectors  $\vec{v}$ .

$$A\vec{v} = \lambda\vec{v}$$

Since  $A$  be the matrix of the linear transformation  $T$  on  $\mathbb{R}^2$ , then  $A\vec{v} = \vec{v}$ .

Thus, the eigenvalue  $\lambda = 1$  of the matrix  $A$  which will result to the corresponding eigenvector  $\vec{v}$ .

The other eigenvector  $\vec{u}$  can be generated by applying the orthogonal to the line and which leads to the eigenvalue  $\lambda = -1$ . The result form that each vector on the line through  $\vec{u}$  can be transformed into the opposite sign of that vector.

### Exercise

Let  $A$  be the matrix of the linear transformation  $T$  on  $\mathbb{R}^2$

$T$ : reflects points about some line through the origin.

Without writing  $A$ , find an eigenvalue of  $A$  and describe the eigenspace.

### Solution

Given  $T$  reflects points *about* some line through the origin.

If  $\vec{v} \in \mathbb{R}^2$  lines on the line, then the linear transformation can be written in the form:

$$T(\vec{v}) = A\vec{v} = \vec{v}$$

That implies to  $T$  rotates points around a given line, the points on the line are not moved at all.

Thus, the eigenvalue  $\lambda = 1$  of the matrix  $A$  which will result to the corresponding eigenvector  $\vec{v}$ .

The corresponding eigenspace is either just the line if  $T$  doesn't rotate full rotation ( $2\pi k$ ).

Therefore, the corresponding eigenspace is the line the points are being rotated around.

### Exercise

Show that if  $\vec{v}$  is an eigenvector of the matrix product  $AB$  and  $B\vec{v} \neq \vec{0}$ , then  $B\vec{v}$  is an eigenvector of  $BA$

### Solution

Since  $\vec{v}$  is an eigenvector of the matrix product  $AB$ , that must be some eigenvalue  $\lambda$  to satisfy.

Such that  $AB\vec{v} = \lambda\vec{v}$  and  $\vec{v} \neq \vec{0}$ .

Since  $B\vec{v} \neq \vec{0}$ , then we can rewrite

$$AB\vec{v} = \lambda\vec{v}$$

$$A(B\vec{v}) = \lambda\vec{v} \quad \text{Multiply both sides by matrix } B.$$

$$BA(B\vec{v}) = B\lambda\vec{v}$$

$$BA(B\vec{v}) = \lambda(B\vec{v})$$

Therefore, since  $B\vec{v} \neq \vec{0}$ , that is clearly that  $B\vec{v}$  is an eigenvector of  $BA$ .

### Exercise

Explain and demonstrate that the eigenspace of a matrix  $A$  corresponding to some eigenvalue  $\lambda$  is a subspace.

### Solution

$\lambda$  is an eigenvalue of a square matrix  $(n \times n)$ , then  $A\vec{v} = \lambda\vec{v}$  and  $\vec{v}$  is a non-zero vector.

That implies to:  $(A - \lambda I)\vec{v} = \vec{0}$ .

The eigenspace consists of the zero vector and all the eigenvectors  $\vec{v}$  corresponding to the eigenvalue  $\lambda$ .

This is equivalent to the null space of  $A - \lambda I$  which includes the trivial (zero vector) solution of  $(A - \lambda I)\vec{v} = \vec{0}$  as well as the non-trivial (non-zero) solutions. As the null space is definitely a subspace, and the eigenspace is essentially the same, then the eigenspace is a subspace too.

Is the eigenspace is closed under addition?

Suppose that  $\vec{v}_1$  and  $\vec{v}_2$  are eigenvectors corresponding to  $\lambda$ .

Let assume that  $A\vec{v}_1 = \lambda\vec{v}_1$  and  $A\vec{v}_2 = \lambda\vec{v}_2$

$$\begin{aligned} A(\vec{v}_1 + \vec{v}_2) &= A\vec{v}_1 + A\vec{v}_2 \\ &= \lambda\vec{v}_1 + \lambda\vec{v}_2 \\ &= \lambda(\vec{v}_1 + \vec{v}_2) \end{aligned}$$

Therefore,  $(\vec{v}_1 + \vec{v}_2)$  is in the eigenspace of  $\lambda$  under addition.

Is the eigenspace is closed under scalar multiplication?

Let  $\vec{v}_1$  be an eigenvector corresponding to  $\lambda$  and  $c$  be any real scalar.

$$\begin{aligned}cA(\vec{v}_1) &= A(c\vec{v}_1) \\&= c(\lambda\vec{v}_1) \\&= \lambda(c\vec{v}_1)\end{aligned}$$

Therefore,  $c\vec{v}_1$  is in the eigenspace of  $\lambda$  under scalar multiplication.

Therefore, the eigenspace of a matrix  $A$  corresponding to some eigenvalue  $\lambda$  is a subspace.

### ***Exercise***

If  $\lambda$  is an eigenvalue of the matrix  $A$ , prove that  $\lambda^2$  is an eigenvalue of  $A^2$ .

### **Solution**

Since  $\lambda$  is an eigenvalue of the matrix  $A$ , then  $A\vec{v} = \lambda\vec{v}$  where  $\vec{v} \neq \vec{0}$ .

$$A\vec{v} = \lambda\vec{v}$$

$$A(A\vec{v}) = A(\lambda\vec{v})$$

$$\begin{aligned}A^2\vec{v} &= \lambda(A\vec{v}) & A\vec{v} &= \lambda\vec{v} \\&= \lambda(\lambda\vec{v}) \\&= \lambda^2\vec{v}\end{aligned}$$

Therefore,  $\lambda^2$  is an eigenvalue of  $A^2$