Section 4.5 – Partial Fraction Decomposition

1- Decompose $\frac{P}{Q}$, where Q has Only Non-repeated Linear Factor

Under the assumption that Q has only non-repeated linear factors, the polynomial Q has the form

$$Q(x)=\left(x-a_1\right)\left(x-a_2\right)\ \cdots\cdots\ \left(x-a_n\right)$$

Where no 2 of the number $a_1, a_2, \ldots, a_n$ are equal. In this case, the partial fraction decomposition of $\frac{P}{Q}$ is of the form

$$\frac{P}{Q}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}\ \cdots\cdot+\frac{A_n}{x-a_n}$$

Where the numbers $A_1, A_2, \ldots, A_n$ are to be determined.

Example

Write the partial fraction decomposition of $\dfrac{x}{x^2-5x+6}$

Solution

First factor the denominator, $x^2-5x+6=(x-2)(x-3)$

$$\frac{x}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3}$$

$$\frac{x}{x^2-5x+6}=\frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$$x=A(x-3)+B(x-2)$$

$$x=Ax-3A+Bx-2B$$

$$x=(A+B)x-3A-2B \qquad\qquad 1x+0=(A+B)x-3A-2B$$

$$\to\begin{cases}1=A+B\\0=-3A-2B\end{cases}$$

$\begin{pmatrix}1&1&1\\-3&-2&0\end{pmatrix}\xrightarrow{rref}\begin{pmatrix}1&0&-2\\0&1&3\end{pmatrix}$	$\begin{array}{l}A+B=1\\A=-\frac{2}{3}B\end{array}\ \to\ \begin{array}{l}-\frac{2}{3}B+B=1\Rightarrow\frac{1}{3}B=1\Rightarrow B=3\\A=-\frac{2}{3}(3)=-2\end{array}$

Solve for A and B using any method, we get $A=-2\quad B=3$

Therefore; $\boxed{\dfrac{x}{x^2-5x+6}=\dfrac{-2}{x-2}+\dfrac{3}{x-3}}$

2- Decompose $\frac{P}{Q}$, where Q has Repeated Linear Factors

If a polynomial Q has a repeated linear factor, say $(x-a)^n$, $n \ge 2$ n is an integer, then in the partial fraction decomposition of $\frac{P}{Q}$, we allow for the terms

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} \cdots + \frac{A_n}{(x-a)^n}$$

Where the numbers $A_1, A_2, \ldots, A_n$ are to be determined.

Example

Write the partial fraction decomposition of $\frac{x+2}{x^3-2x^2+x}$

Solution

First factor the denominator, $x^3-2x^2+x = x(x-1)^2$

$$\frac{x+2}{x^3-2x^2+x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\begin{aligned} x+2 &= A(x-1)^2 + Bx(x-1) + Cx \\ &= A\left(x^2-2x+1\right) + Bx^2 - Bx + Cx \\ &= Ax^2 - 2Ax + A + Bx^2 - Bx + Cx \\ &= (A+B)x^2 + (-2A-B+C)x + A \end{aligned}$$

$$\begin{cases} A+B=0 \\ -2A-B+C=1 \\ A=2 \end{cases} \rightarrow \begin{cases} B=-A=-2 \\ C=1+2A+B=1+4-2=3 \end{cases}$$

$$\frac{x+2}{x^3-2x^2+x} = \frac{2}{x} + \frac{-2}{x-1} + \frac{3}{(x-1)^2}$$

$$\frac{x+2}{x^3-2x^2+x} = \frac{2}{x} - \frac{2}{x-1} + \frac{3}{(x-1)^2}$$

Example

Write the partial fraction decomposition of $\dfrac{x^3-8}{x^2(x-1)^3}$

Solution

$$\frac{x^3-8}{x^2(x-1)^3}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{(x-1)^2}+\frac{E}{(x-1)^3}$$

$$x^3-8=Ax(x-1)^3+B(x-1)^3+Cx^2(x-1)^2+Dx^2(x-1)+Ex^2$$

Let $x=0 \quad \rightarrow \quad -8=B(-1)^3 \quad \Rightarrow B=8$

$$x^3-8=Ax(x-1)^3+8(x-1)^3+Cx^2(x-1)^2+Dx^2(x-1)+Ex^2$$

Let $x=1 \quad \rightarrow \; 1 \; -8=E \quad \Rightarrow E=-7$

$$x^3-8=Ax\left(x^3-3x^2+3x-1\right)+8\left(x^3-3x^2+3x-1\right)+Cx^2\left(x^2-2x+1\right)+Dx^2(x-1)-7x^2$$

$$x^3-8-8\left(x^3-3x^2+3x-1\right)+7x^2$$
$$=Ax^4-3Ax^3+3Ax^2-Ax+Cx^4-2Cx^3+Cx^2+Dx^3-Dx^2$$

$$x^3-8-8x^3+24x^2-24x+8+7x^2$$
$$=(A+C)x^4+(-3A-2C+D)x^3+(3A+C-D)x^2-Ax$$

$$-7x^3+31x^2-24x=(A+C)x^4+(-3A-2C+D)x^3+(3A+C-D)x^2-Ax$$

$$\rightarrow\begin{cases} A+C=0 & C=-A=-24 \\ -3A-2C+D=-7 & \\ 3A+C-D=31 & \\ -A=-24 & \rightarrow A=24 \end{cases} \qquad D=-7+3A+2C=-7+72-48=17$$

$$\frac{x^3-8}{x^2(x-1)^3}\;\boxed{=\frac{24}{x}+\frac{8}{x^2}-\frac{24}{x-1}+\frac{17}{(x-1)^2}-\frac{7}{(x-1)^3}}$$

3- Decompose $\frac{P}{Q}$, where Q has a Non-repeated Irreducible Quadratic Factor

If Q contains a no-repeated irreducible quadratic factor of the form ax^2+bx+c, then in the partial fraction decomposition of $\frac{P}{Q}$, we allow for the term

$$\frac{Ax+B}{ax^2+bx+c}$$

Where the numbers A and B are to be determined.

Example

Write the partial fraction decomposition of $\frac{3x-5}{x^3-1}$

Solution

$$\frac{3x-5}{x^3-1}=\frac{3x-5}{(x-1)\left(x^2+x+1\right)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$

$$3x-5=A\left(x^2+x+1\right)+(x-1)(Bx+C)$$

$$3x-5=Ax^2+Ax+A+Bx^2+Cx-Bx-C$$

$$\begin{cases} A+B=0 \\ A-B+C=3 \\ A-C=-5 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 3 \\ 1 & 0 & -1 & -5 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{2}{3} \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & \frac{13}{3} \end{array}\right)$$

$$A=-\frac{2}{3} \quad B=\frac{2}{3} \quad C=\frac{13}{3}$$

$$\boxed{\frac{3x-5}{x^3-1}=\frac{-\frac{2}{3}}{x-1}+\frac{\frac{2}{3}x+\frac{13}{3}}{x^2+x+1}}$$

$$=-\frac{2}{3}\frac{1}{x-1}+\frac{1}{3}\frac{2x+13}{x^2+x+1}$$

4- Decompose $\frac{P}{Q}$, where Q has a Repeated Irreducible Quadratic Factor

If Q contains a repeated irreducible quadratic factor of the form $\left(ax^2+bx+c\right)^n$, $n \geq 2$, n an integer, then in the partial fraction decomposition of $\frac{P}{Q}$, we allow for the terms

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+\cdots+\frac{A_nx+B_n}{\left(ax^2+bx+c\right)^n}$$

Where the numbers A_1, B_1, A_2, B_2, …, A_n, B_n are to be determined.

Example

Write the partial fraction decomposition of $\dfrac{x^3+x^2}{\left(x^2+4\right)^2}$

Solution

$$\frac{x^3+x^2}{\left(x^2+4\right)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{\left(x^2+4\right)^2}$$

$$\begin{aligned} x^3+x^2 &= (Ax+B)\left(x^2+4\right)+Cx+D \\ &= Ax^3+4Ax+Bx^2+4B+Cx+D \\ &= Ax^3+Bx^2+(4A+C)x+4B+D \end{aligned}$$

$$\begin{cases} A=1 \\ B=1 \\ 4A+C=0 \\ 4B+D=0 \end{cases} \rightarrow \quad \begin{matrix} \\ \\ C=-4A=-4 \\ D=-4B=-4 \end{matrix}$$

$$A=1, \quad B=1, \quad C=-4, \quad D=-4$$

$$\boxed{\frac{x^3+x^2}{\left(x^2+4\right)^2}=\frac{x+1}{x^2+4}+\frac{-4x-4}{\left(x^2+4\right)^2}}$$

Exercises *Section* 4.5 – Partial Fraction Decomposition

Write the partial fraction decomposition of each rational expression

1. $\dfrac{4}{x(x-1)}$

2. $\dfrac{3x}{(x+2)(x-1)}$

3. $\dfrac{1}{x\left(x^2+1\right)}$

4. $\dfrac{1}{(x+1)\left(x^2+4\right)}$

5. $\dfrac{x^2}{(x-1)^2(x+1)^2}$

6. $\dfrac{x+1}{x^2(x-2)^2}$

7. $\dfrac{x-3}{(x+2)(x+1)^2}$

8. $\dfrac{x^2+x}{(x+2)(x-1)^2}$

9. $\dfrac{10x^2+2x}{(x-1)^2\left(x^2+2\right)}$

10. $\dfrac{x^2+2x+3}{(x+1)\left(x^2+2x+4\right)}$

11. $\dfrac{x^2-11x-18}{x\left(x^2+3x+3\right)}$

12. $\dfrac{1}{(2x+3)(4x-1)}$

13. $\dfrac{x^2+2x+3}{\left(x^2+4\right)^2}$

14. $\dfrac{x^3+1}{\left(x^2+16\right)^2}$

15. $\dfrac{7x+3}{x^3-2x^2-3x}$

16. $\dfrac{x^2}{x^3-4x^2+5x-2}$

17. $\dfrac{x^3}{\left(x^2+16\right)^3}$

18. $\dfrac{4}{2x^2-5x-3}$

19. $\dfrac{2x+3}{x^4-9x^2}$

20. $\dfrac{x^2+9}{x^4-2x^2-8}$

21. $\dfrac{y}{y^2-2y-3}$

22. $\dfrac{x+3}{2x^3-8x}$

23. $\dfrac{x^2}{(x-1)\left(x^2+2x+1\right)}$

24. $\dfrac{3x^2+x+4}{x^3+x}$

25. $\dfrac{8x^2+8x+2}{\left(4x^2+1\right)^2}$