Section 1.2 – Propositional Equivalences

Introduction

An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value.

Definition

A compound proposition that is always true, no matter what the truth values of the proposition variables that occur in it, is called a ***tautology***. A compound proposition that is always false is called ***contradiction***. A compound proposition that is neither a tautology nor a contradiction is called a ***contingency***.

Example

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	***F***	***T***	***F***
F	***T***	***T***	***F***

$p \vee \neg p$ is always true, it is tautology

$p \wedge \neg p$ is always false, it is contradiction.

Logical Equivalences

Definition

Compound propositions *p* and *q* are called ***logically equivalent*** if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that *p* and *q* are logically equivalent.

De Morgan's Laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Example

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Solution

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	***T***	***T***	***F***	***F***	***F***	***F***
T	***F***	***T***	***F***	***F***	***T***	***F***
F	***T***	***T***	***F***	***T***	***F***	***F***
F	***F***	***F***	***T***	***T***	***T***	***T***

The truth table shows that $\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$ is a tautology and these compound propositions are logically equivalent.

Example

Show that $p \to q$ and $\neg p \vee q$ are logically equivalent.

Solution

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	***T***	***F***	***T***	***T***
T	***F***	***F***	***F***	***F***
F	***T***	***T***	***T***	***T***
F	***F***	***T***	***T***	***T***

The truth table shows that $p \to q$ and $\neg p \vee q$ are logically equivalent.

Example

Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. This is the *distributive law* of disjunction over conjunction.

Solution

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	***T***	***T***	***T***	***T***	***T***	***T***	***T***
T	***T***	***F***	***F***	***T***	***T***	***T***	***T***
T	***F***	***T***	***F***	***T***	***T***	***T***	***T***
T	***F***	***F***	***F***	***T***	***T***	***T***	***T***
F	***T***	***T***	***T***	***T***	***T***	***T***	***T***
F	***T***	***F***	***F***	***F***	***T***	***F***	***F***
F	***F***	***T***	***F***	***F***	***F***	***T***	***F***
F	***F***	***F***	***F***	***F***	***F***	***F***	***F***

The truth table Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

In these equivalences, $\boldsymbol{T}$ denotes the compound proposition that is always true and $\boldsymbol{F}$ denotes the compound proposition that is always false.

Logical Equivalences	
Equivalence	***Name***
$p \wedge \boldsymbol{T} = p$	***Identity laws***
$p \vee \boldsymbol{F} = p$	
$p \vee \boldsymbol{T} \equiv \boldsymbol{T}$	***Domination laws***
$p \wedge \boldsymbol{F} \equiv \boldsymbol{F}$	
$p \vee p \equiv p$	***Idempotent laws***
$p \wedge p \equiv p$	
$\neg(\neg p) \equiv p$	***Double negation law***
$p \vee q \equiv q \vee p$	***Commutative laws***
$p \wedge q \equiv q \wedge p$	
$(p \vee q) \vee r \equiv p \vee (q \vee r)$	***Associative laws***
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	***Distributive laws***
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	
$\neg(p \wedge q) \equiv \neg p \vee \neg q$	***De Morgan's laws***
$\neg(p \vee q) \equiv \neg p \wedge \neg q$	
$p \vee (p \wedge q) \equiv p$	***Absorption laws***
$p \wedge (p \vee q) \equiv p$	
$p \vee \neg p \equiv \boldsymbol{T}$	***Negation laws***
$p \wedge \neg p \equiv \boldsymbol{F}$	

Logical Equivalences Involving Conditional Statements
$p \rightarrow q \equiv \neg p \vee q$
$p \rightarrow q \equiv \neg q \vee \neg p$
$p \vee q \equiv \neg p \rightarrow q$
$p \wedge q \equiv \neg(p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv p \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical Equivalences Involving Biconditional Statements
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow r)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

$$\neg\left(p_1 \vee p_2 \vee \cdots \vee p_n\right) \equiv \left(\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n\right)$$

$$\neg\left(p_1 \wedge p_2 \wedge \cdots \wedge p_n\right) \equiv \left(\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n\right)$$

Using De Morgan's Laws

The two logical equivalences known as De Morgan's laws are particularly important. The equivalence $\neg(p \vee q) \equiv \neg p \wedge \neg q$ and similarly $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Example

Use De Morgan's laws to express the negations of "Miguel has a cellphone and he has a laptop computer" and "Heather will go to the concert or Steve will go to the concert."

Solution

Let: p be "Miguel has a cellphone"
q be "Miguel has a laptop computer"
can be expressed as $p \wedge q$

By De Morgan's laws $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$. We can express the negation of our original statement as "*Miguel does not have a cellphone* ***or*** *he does not have a laptop computer*"

Let: r be "Heather will go to the concert"
s be "Steve will go to the concert"
can be expressed as $r \vee s$

By De Morgan's laws $\neg(r \vee s) \equiv \neg r \wedge \neg s$. We can express the negation of our original statement as "*Heather will not go to the concert* ***and*** *Steve will not go to the concert*."

Example

Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.

Solution

$$\begin{aligned}\neg(p \to q) &\equiv \neg(\neg p \vee q)\\ &\equiv \neg(\neg p) \wedge \neg q\\ &\equiv p \wedge \neg q\end{aligned}$$

p	q	$p \to q$	$\neg(p \to q)$	$\neg q$	$p \wedge \neg q$
T	***T***	***T***	***F***	***F***	***F***
T	***F***	***F***	***T***	***T***	***T***
F	***T***	***T***	***F***	***F***	***F***
F	***F***	***T***	***F***	***T***	***F***

Example

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution

$$\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \textit{By De Morgan's law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \textit{Double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \textit{Distribution law} \\
&\equiv \boldsymbol{F} \vee (\neg p \wedge \neg q) && \neg p \wedge p \equiv \boldsymbol{F} \\
&\equiv (\neg p \wedge \neg q) \vee \boldsymbol{F} && \textit{Commutative law for disjunction} \\
&\equiv \neg p \wedge \neg q && \textit{Identity law}
\end{aligned}$$

Example

Show that $(p \wedge q) \to (p \vee q)$ is a tautology.

Solution

$$\begin{aligned}
(p \wedge q) \to (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \textit{By De Morgan's law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \textit{By Associative and commutative laws} \\
&\equiv \boldsymbol{T} \vee \boldsymbol{T} \\
&\equiv \boldsymbol{T}
\end{aligned}$$

Exercises *Section* 1.2 – Propositional Equivalences

1. Use the truth table to verify these equivalences

a) $p \wedge \boldsymbol{T} \equiv p$

b) $p \vee \boldsymbol{F} \equiv p$

c) $p \wedge \boldsymbol{F} \equiv \boldsymbol{F}$

d) $p \vee \boldsymbol{T} \equiv \boldsymbol{T}$

e) $p \vee p \equiv p$

f) $p \wedge p \equiv p$

2. Show that $\neg(\neg p)$ and p are logically equivalent

3. Use the truth table to verify the commutative laws

a) $p \vee q \equiv q \vee p$

b) $p \wedge q \equiv q \wedge p$

4. Use the truth table to verify the associative laws

a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

5. Show that each of these conditional statements is a tautology by using truth result tables.

a) $(p \wedge q) \to p$

b) $p \to (p \vee q)$

c) $\neg p \to (p \to q)$

d) $(p \wedge q) \to (p \to q)$

e) $\neg(p \to q) \to p$

f) $\lfloor \neg p \wedge (p \vee q) \rfloor \to q$

g) $\lfloor (p \to q) \wedge (q \to r) \rfloor \to (p \to r)$

h) $\lfloor p \wedge (p \to q) \rfloor \to q$

6. Show that $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent

7. Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent

8. Show that $p \to q$ and $\neg q \to \neg p$ are logically equivalent

9. Show that $\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$ are logically equivalent

10. Show that $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$ are logically equivalent

11. Show that $(p \to r) \vee (q \to r)$ and $(p \wedge q) \to r$ are logically equivalent

12. Show that $\left(p \rightarrow q\right) \wedge \left(q \rightarrow r\right) \rightarrow \left(p \rightarrow r\right)$ is a tautology

13. Show that $\left(p \vee q\right) \vee \left(\neg p \vee r\right) \rightarrow \left(q \vee r\right)$ is a tautology

14. Show that | (NAND) is functionally complete