Solution   Section 1.1 – Functions

Exercise

Find the domain: $f(x) = 7x + 4$

Solution

Domain: $(-\infty,\ \infty)$

Exercise

Find the domain: $f(x) = |3x - 2|$

Solution

Domain: $(-\infty,\ \infty)$

Exercise

Find the domain: $f(x) = x^2 - 2x - 15$

Solution

Domain: $(-\infty,\ \infty)$

Exercise

Find the domain: $g(x) = \dfrac{3}{x-4}$

Solution

Domain: $x - 4 \neq 0 \Rightarrow x \neq 4 \qquad (-\infty, 4) \cup (4, \infty)$

Exercise

Find the domain $y = \dfrac{2}{x-3}$

Solution

Domain: $x - 3 \neq 0 \Rightarrow x \neq 3 \qquad \Rightarrow (-\infty, 3) \cup (3, \infty)$

Exercise

Find the domain $y = \dfrac{-7}{x-5}$

Solution

$x - 5 \neq 0 \Rightarrow x \neq 5$ **Domain**: $(-\infty, 5) \cup (5, \infty)$

Exercise

Find the domain $f(x) = 4 - \frac{2}{x}$

Solution

$x \neq 0$ ***Domain***: $(-\infty, 0) \cup (0, \infty)$

Exercise

Find the domain $f(x) = \frac{1}{x^4}$

Solution

$x \neq 0$ ***Domain***: $(-\infty, 0) \cup (0, \infty)$

Exercise

Find the domain $f(x) = \frac{x+5}{2-x}$

Solution

$2 - x \neq 0 \Rightarrow x \neq 2$ ***Domain***: $(-\infty, 2) \cup (2, \infty)$

Exercise

Find the domain $f(x) = \frac{8}{x+4}$

Solution

$x + 4 \neq 0 \Rightarrow x \neq -4$ ***Domain***: $(-\infty, -4) \cup (-4, \infty)$

Exercise

Find the domain $f(x) = \frac{1}{x^2 - 4x - 5}$

Solution

$x^2 - 4x - 5 \neq 0$

$(x+1)(x-5) \neq 0$

$x \neq -1$ and $x \neq 5$

Domain: $(-\infty, \ -1) \cup (-1, \ 5) \cup (5, \ \infty)$

Exercise

Find the domain $g(x) = \dfrac{2}{x^2 + x - 12}$

Solution

$$x^2 + x - 12 \neq 0$$

$$(x+4)(x-3) \neq 0$$

$$x \neq -4 \qquad x \neq 3$$

Domain: $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$

Exercise

Find the domain $h(x) = \dfrac{5}{\frac{4}{x} - 1}$

Solution

$$x \neq 0 \qquad \frac{4}{x} - 1 \neq 0$$

$$\frac{4-x}{x} \neq 0$$

$$4 - x \neq 0$$

$$x \neq 4$$

Domain: $(-\infty, 0) \cup (0, 4) \cup (4, \infty)$

Exercise

Find the domain $y = \sqrt{x}$

Solution

$\boxed{x \geq 0}$

Domain: $[0, \infty)$

Exercise

Find the domain $y = \sqrt{4x+1}$

Solution

$4x + 1 \geq 0 \Rightarrow x \geq -\frac{1}{4}$

Domain: $\left[-\frac{1}{4}, \infty\right)$

Exercise

Find the domain $y = \sqrt{7 - 2x}$

Solution

$7 - 2x \ge 0$

$\Rightarrow -2x \ge -7 \Rightarrow \boxed{x \le \frac{7}{2}}$ ***Domain***: $\left(-\infty, \frac{7}{2}\right]$

Exercise

Find the domain $\quad f(x) = \sqrt{8 - x}$

Solution

$8 - x \ge 0 \Rightarrow -x \ge -8$

$\boxed{x \le 8}$ ***Domain***: $(-\infty,\ 8]$

Exercise

Find the domain $\quad f(x) = \dfrac{\sqrt{x+1}}{x}$

Solution

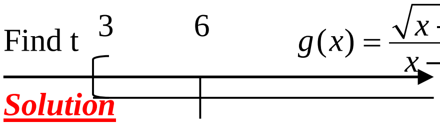
$x + 1 \ge 0 \qquad x \ne 0$

$x \ge -1$

Domain: $[-1,\ 0) \cup (0,\ \infty)$

Exercise

Find t $\quad g(x) = \dfrac{\sqrt{x-3}}{x-6}$



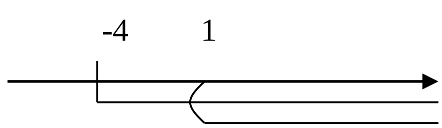
Solution

$\rightarrow \begin{cases} x \ge 3 \\ x \ne 6 \end{cases}$ ***Domain***: $[3,\ 6) \cup (6,\ \infty)$

Exercise

Find the domain $\quad f(x) = \dfrac{\sqrt{x+4}}{\sqrt{x-1}}$

Solution

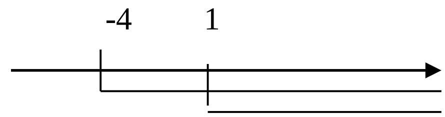


$\rightarrow \begin{cases} x \ge -4 \\ x > 1 \end{cases}$

Domain: $(1,\ \infty)$

Exercise

Find the domain $\sqrt{x+4}-\sqrt{x-1}$



Solution

$$\to \begin{cases} x \ge -4 \\ x \ge 1 \end{cases}$$

Domain: $[1,\ \infty)$

Exercise

Find the domain of $f(x)=\sqrt{2x+7}$

Solution

$$2x+7 \ge 0 \Rightarrow 2x \ge -7 \to x \ge -\frac{7}{2}$$

Domain: $x \ge -\frac{7}{2}$

Exercise

Find the domain of $f(x)=\sqrt{8-3x}$

Solution

$$8-3x \ge 0 \Rightarrow -3x \ge -8 \to x \le \frac{-8}{-3}$$

Domain: $\boxed{x \le \frac{8}{3}}$

Exercise

Find the domain of $f(x)=\sqrt{9-x^2}$

Solution

$$9-x^2 \ge 0 \Rightarrow -x^2 \ge -9$$

$$x^2 \le 9$$

$$-3 \le x \le 3$$

Domain: $-3 \le x \le 3$ *or* $[-3,\ 3]$

Exercise

Find the domain of $f(x)=\sqrt{x^2-25}$

Solution

$$x^2-25 \ge 0 \Rightarrow x^2 \ge 25$$

$$\Rightarrow x \le -5 \quad x \ge 5$$

Domain: $x \le -5,\ x \ge 5$ *or* $(-\infty,\ -5] \cup [5,\ \infty)$

Exercise

Find the domain of $f(x)=\dfrac{x+1}{x^3-4x}$

Solution

$$x^3-4x\neq 0 \Rightarrow x\left(x^2-4\right)\neq 0$$

$$x\neq 0 \quad x^2-4\neq 0$$

Domain: $x\neq 0;\quad x\neq 2;\quad x\neq -2$

$$(-\infty,-2)\cup(-2,0)\cup(0,2)\cup(2,\infty)$$

Exercise

Find the domain of $f(x)=\dfrac{4x}{6x^2+13x-5}$

Solution

$$6x^2+13x-5\neq 0 \Rightarrow \boxed{x\neq -\frac{5}{2},\ \frac{1}{3}}$$

Exercise

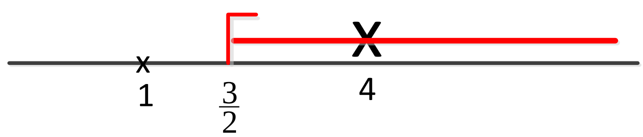
Find the domain of $f(x)=\dfrac{\sqrt{2x-3}}{x^2-5x+4}$

Solution

$2x-3\geq 0 \qquad x^2-5x+4\neq 0$

$2x\geq 3 \qquad x\neq 1,\ 4$

$x\geq \dfrac{3}{2}$



Domain: $\boxed{\left[\frac{3}{2},\ 4\right)\cup(4,\ \infty)}$

Exercise

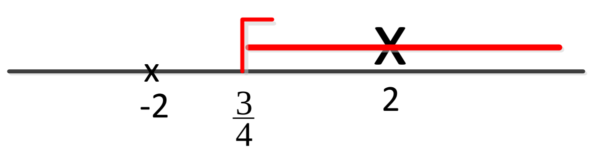
Find the domain of $f(x)=\dfrac{\sqrt{4x-3}}{x^2-4}$

Solution

$4x-3\geq 0 \qquad x^2-4\neq 0$

$4x\geq 3 \qquad x\neq \pm 2$

$x\geq \dfrac{3}{4}$



Domain: $\boxed{\left[\frac{3}{4},\ 2\right)\cup(2,\ \infty)}$

Exercise

Find the domain of $f(x)=\dfrac{x-4}{\sqrt{x-2}}$

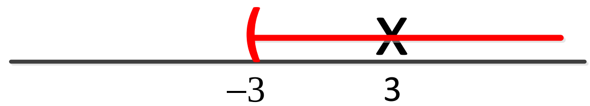
Solution

$x-2>0 \Rightarrow x>2$ **Domain**: $\boxed{x>2}$ $(2,\ \infty)$

Exercise

Find the domain of $f(x)=\dfrac{1}{(x-3)\sqrt{x+3}}$

Solution



$x-3\neq 0 \qquad x+3>0$

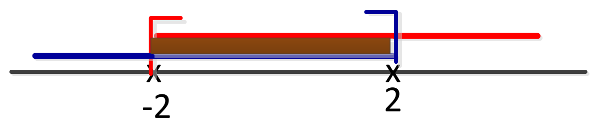
$x\neq 3 \qquad x>-3$

Domain: $\{x \mid x>-3 \text{ and } x\neq 3\}$ $(-3,\ 3)\cup(3,\ \infty)$

Exercise

Find the domain of $f(x)=\sqrt{x+2}+\sqrt{2-x}$

Solution



$x+2\geq 0 \qquad 2-x\geq 0$

$x\geq -2 \qquad -x\geq -2 \to x\leq 2$

Domain: $\{x \mid -2\leq x\leq 2\}$

Exercise

Find the domain of $f(x)=\sqrt{(x-2)(x-6)}$

Solution

$x-2\geq 0 \quad x-6\geq 0$

$x\geq 2 \quad x\geq 6$

Domain: $\{x \mid x\leq 2, x\geq 6\}$

2 6

−	+	+
−	−	+
+	−	+

Exercise

For the function *f* given by $f(x)=\sqrt{x-3}$, find the difference quotient $\dfrac{f(x)-f(a)}{x-a}$

Solution

$f(a)=\sqrt{a-3}$

$\dfrac{f(x)-f(a)}{x-a}=\dfrac{\sqrt{x-3}-\sqrt{a-3}}{x-a}$

Exercise

Given the function: $f(x) = 2x^2$. Find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned} f(x+h) &= 2(x+h)^2 \\ &= 2\left(x^2 + 2hx + h^2\right) \\ &= 2x^2 + 4hx + 2h^2 \end{aligned}$$

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{2x^2 + 4hx + 2h^2 - 2x^2}{h} \\ &= \frac{4hx + 2h^2}{h} \\ &= \frac{4hx}{h} + \frac{2h^2}{h} \\ &= 4x + 2h \end{aligned}$$

Exercise

For the function f given by $f(x) = 9x + 5$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$f(\mathbf{x} + \mathbf{h}) = 9(\mathbf{x} + \mathbf{h}) + 5 = 9x + 9h + 5$$

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\overbrace{9x + 9h + 5}^{f(x+h)} \; - \; \overbrace{(9x+5)}^{f(x)}}{h} \\ &= \frac{9x + 9h + 5 \; - \; 9x - 5}{h} \\ &= \frac{9h}{h} \\ &= 9 \end{aligned}$$

Exercise

For the function f given by $f(x) = 6x + 2$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{6(x+h) + 2 - (6x+2)}{h} \\ &= \frac{6x + 6h + 2 - 6x - 2}{h} \\ &= \frac{6h}{h} \\ &= 6 \end{aligned}$$

Exercise

For the function *f* given by $f(x) = 4x + 11$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{4(x+h) + 11 - (4x + 11)}{h} \\ &= \frac{4x + 4h + 11 - 4x - 11}{h} \\ &= \frac{4h}{h} \\ &= 4\end{aligned}$$

Exercise

For the function f given by $f(x) = 2x^2 - x - 3$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}f(x+h) &= 2(x+h)^2 - (x+h) - 3 \\ &= 2\left(x^2 + 2hx + h^2\right) - x - h - 3 \\ &= 2x^2 + 4hx + 2h^2 - x - h - 3\end{aligned}$$

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{2x^2 + 2h^2 + 4hx - x - h - 3 - \left(2x^2 - x - 3\right)}{h} \\ &= \frac{2x^2 + 2h^2 + 4hx - x - h - 3 - 2x^2 + x + 3}{h} \\ &= \frac{2h^2 + 4hx - h}{h} \\ &= \frac{2h^2}{h} + \frac{4hx}{h} - \frac{h}{h} \\ &= 2h + 4x - 1\end{aligned}$$

Exercise

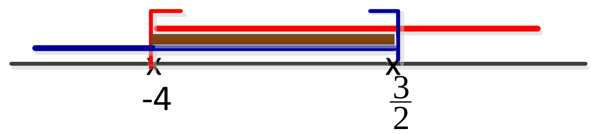
Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, *and* $(f/g)(x)$ and the domain of

$f(x) = \sqrt{3 - 2x}, \quad g(x) = \sqrt{x + 4}$

Solution

$$f(x) + g(x) = \sqrt{3 - 2x} + \sqrt{x + 4}$$

$$3-2x\ge 0 \qquad x+4\ge 0$$
$$-2x\ge -3 \qquad x\ge -4$$
$$x\le \frac{3}{2}$$



Domain: $\left\{x \mid -4\le x\le \frac{3}{2}\right\}$

$$f(x)-g(x)=\sqrt{3-2x}-\sqrt{x+4}$$
$$3-2x\ge 0 \qquad x+4\ge 0$$
$$-2x\ge -3 \qquad x\ge -4$$
$$x\le \frac{3}{2}$$

Domain: $\left\{x \mid -4\le x\le \frac{3}{2}\right\}$

$$(f\cdot g)(x)=\left(\sqrt{3-2x}\right)\left(\sqrt{x+4}\right)=\sqrt{(3-2x)(x+4)}=\sqrt{-2x^2-5x+12}$$
$$3-2x\ge 0 \qquad x+4\ge 0$$
$$-2x\ge -3 \qquad x\ge -4$$
$$x\le \frac{3}{2}$$

Domain: $\left\{x \mid -4\le x\le \frac{3}{2}\right\}$

$$(f/g)(x)=\frac{\sqrt{3-2x}}{\sqrt{x+4}}\frac{\sqrt{x+4}}{\sqrt{x+4}}=\frac{\sqrt{-2x^2-5x+12}}{x+4}$$
$$3-2x\ge 0 \qquad x+4> 0$$
$$-2x\ge -3 \qquad x> -4$$
$$x\le \frac{3}{2}$$

Domain: $\left\{x \mid -4< x\le \frac{3}{2}\right\}$ $\qquad \left(-4,\ \frac{3}{2}\right]$

Exercise

Find $(f+g)(x)$, $(f-g)(x)$, $(f\cdot g)(x)$, *and* $(f/g)(x)$ and the domain of

$$f(x)=\frac{2x}{x-4}, \quad g(x)=\frac{x}{x+5}$$

Solution

$$(f+g)(x)=\frac{2x}{x-4}+\frac{x}{x+5}$$
$$=\frac{2x(x+5)+x(x-4)}{(x-4)(x+5)}$$

$$= \frac{2x^2 + 10x + x^2 - 4x}{(x-4)(x+5)}$$

$$= \frac{3x^2 + 6x}{(x-4)(x+5)}$$

$x - 4 \neq 0 \qquad x + 5 \neq 0$

$x \neq 4 \qquad x \neq -5$

Domain: $\{x \mid x \neq -5, 4\}$ $\qquad (-\infty,\ -5) \cup (-5,\ 4) \cup (4,\ \infty)$

$$(f - g)(x) = \frac{2x}{x-4} - \frac{x}{x+5}$$

$$= \frac{2x(x+5) - x(x-4)}{(x-4)(x+5)}$$

$$= \frac{2x^2 + 10x - x^2 + 4x}{(x-4)(x+5)}$$

$$= \frac{x^2 + 14x}{(x-4)(x+5)}$$

$x \neq 4 \qquad x \neq -5$

Domain: $\{x \mid x \neq -5, 4\}$

$$(f \bullet g)(x) = \frac{2x}{x-4}\frac{x}{x+5} = \frac{2x^2}{(x-4)(x+5)}$$

$x \neq 4 \qquad x \neq -5$

Domain: $\{x \mid x \neq -5, 4\}$

$$(f / g)(x) = \frac{2x}{x-4} \div \frac{x}{x+5} == \frac{2x}{x-4}\frac{x+5}{x} = 2\frac{x+5}{x-4}$$

$x \neq 4 \qquad x \neq -5$

Domain: $\{x \mid x \neq -5, 4\}$

Exercise

Let $f(x) = \sqrt{4x-1}$ and $g(x) = \frac{1}{x}$. Find each of the following and give the domain

a) $(f+g)(x)$

b) $(f-g)(x)$

c) $(fg)(x)$

d) $\left(\frac{f}{g}\right)(x)$

Solution

a) $(f+g)(x)$

$$(f+g)(x)=\sqrt{4x-1}+\frac{1}{x}$$

$$4x-1\ge 0 \quad\Rightarrow\quad x\ge\frac{1}{4} \qquad x\ne 0$$

Domain: $\left[\frac{1}{4},\infty\right)$

b) $(f-g)(x)$

$$(f-g)(x)=\sqrt{4x-1}-\frac{1}{x}$$

$$4x-1\ge 0 \qquad\qquad x\ne 0$$

$$x\ge\frac{1}{4}$$

Domain: $\left[\frac{1}{4},\infty\right)$

c) $(fg)(x)$

$$(fg)(x)=\sqrt{4x-1}\left(\frac{1}{x}\right)=\frac{\sqrt{4x-1}}{x}$$

$$4x-1\ge 0\Rightarrow x\ge\frac{1}{4} \qquad x\ne 0$$

Domain: $\left[\frac{1}{4},\infty\right)$

d) $\left(\frac{f}{g}\right)(x)$

$$\left(\frac{f}{g}\right)(x)=\frac{\sqrt{4x-1}}{\frac{1}{x}}$$

Domain: $x\ne 0$

$$=x\sqrt{4x-1}$$

$$4x-1\ge 0 \qquad x\ge\frac{1}{4}$$

Domain: $\left[\frac{1}{4},\infty\right)$

Exercise

Given that $f(x)=x+1$ and $g(x)=\sqrt{x+3}$

a) Find $(f+g)(x)$

b) Find the domain of $(f+g)(x)$

c) Find: $(f+g)(6)$

Solution

a) $(f+g)(x)=f(x)+g(x)$

$$=x+1+\sqrt{x+3}$$

b) $x+3 \ge 0 \to x \ge -3$

Domain $= [-3,\ \infty)$

c) $(f+g)(6) = 6+1+\sqrt{6+3} = 10$

Exercise

Given that $f(x) = x^2 - 4$ and $g(x) = x+2$

a) Find $(f+g)(x)$ and its domain

b) Find $(f/g)(x)$ and its domain

Solution

a) $(f+g)(x) = x^2 - 4 + x + 2$

$$= x^2 + x - 2$$

Domain $= (-\infty,\ \infty)$

b) $\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x^2-4}{x+2}$

$x \ne -2$

Domain: $(-\infty,-2)\cup(-2,\infty)$

Exercise

Let $f(x) = x^2 + 1$ and $g(x) = 3x+5$. Find $(f+g)(1)$, $(f-g)(-3)$, $(fg)(5)$, and $\left(\frac{f}{g}\right)(0)$

Solution

a) $(f+g)(1) = f(1) + g(1)$

$$= 1^2 + 1 + 3(1) + 5$$
$$= 1+1+3+5$$
$$= 10$$

b) $(f-g)(-3) = f(-3) - g(-3)$

$$= (-3)^2 + 1 - (3(-3)+5)$$
$$= 10$$

c) $(fg)(5) = f(5)\cdot g(5)$

$$= \left(5^2+1\right)\cdot(3(5)+5)$$
$$= (26)\cdot(20)$$
$$= 520$$

d) $\left(\frac{f}{g}\right)(0)=\frac{f(0)}{g(0)}$

$$=\frac{0^2+1}{3(0)+5}$$

$$=\frac{1}{5}$$

Exercise

Find $(f\circ g)(x),\quad (g\circ f)(x),\quad f\left(g(-2)\right)$ *and* $g\left(f(3)\right)$: $f(x)=2x^2+3x-4,\quad g(x)=2x-1$

Solution

$$\begin{aligned}
f\left(g(x)\right) &= f\left(2x-1\right)\\
&= 2\left(2x-1\right)^2+3\left(2x-1\right)-4\\
&= 2\left(4x^2-4x+1\right)+6x-3-4\\
&= 8x^2-8x+2+6x-7\\
&= 8x^2-2x-5
\end{aligned}$$

$$\begin{aligned}
g\left(f(x)\right) &= g\left(2x^2+3x-4\right)\\
&= 2\left(2x^2+3x-4\right)-1\\
&= 4x^2+6x-8-1\\
&= 4x^2+6x-9
\end{aligned}$$

$$f\left(g(-2)\right)=8(-2)^2-2(-2)-5\ \underline{=31|}$$

$$g\left(f(3)\right)=4(3)^2+6(3)-9\ \underline{=45|}$$

Exercise

Find $(f\circ g)(x),\quad (g\circ f)(x),\quad f\left(g(-2)\right)$ *and* $g\left(f(3)\right)$: $f(x)=x^3+2x^2,\quad g(x)=3x$

Solution

$$\begin{aligned}
f\left(g(x)\right) &= f\left(3x\right)\\
&= \left(3x\right)^3+2\left(3x\right)^2\\
&= 27x^3+18x^2
\end{aligned}$$

$$\begin{aligned}
g\left(f(x)\right) &= g\left(x^3+2x^2\right)\\
&= 3\left(x^3+2x^2\right)
\end{aligned}$$

$$= 3x^3 + 6x^2$$

$$f\left(g(-2)\right) = 27(-2)^3 + 18(-2)^2 \ \underline{= -144|}$$

$$g\left(f(3)\right) = 3(3)^3 + 6(3)^2 \ \underline{= 135|}$$

Exercise

Find $(f \circ g)(x)$, $(g \circ f)(x)$, $f\left(g(-2)\right)$ *and* $g\left(f(3)\right)$: $f(x) = |x|$, $g(x) = -7$

Solution

$$f\left(g(x)\right) = f(-7)$$

$$= |-7|$$

$$= 7$$

$$g\left(f(x)\right) = g\left(|x|\right)$$

$$= -7$$

$$f\left(g(-2)\right) \ \underline{= 7|}$$

$$g\left(f(3)\right) \ \underline{= -7|}$$

Exercise

Let $f(x) = x^2 - 3x$ *and* $g(x) = \sqrt{x+2}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(\sqrt{x+2}\right)$ $\qquad x + 2 \ge 0 \Rightarrow x \ge -2$

$$= \left(\sqrt{x+2}\right)^2 - 3\sqrt{x+2}$$

$$= x + 2 - 3\sqrt{x+2} \qquad x + 2 \ge 0 \Rightarrow x \ge -2$$

Domain: $\{x \mid x \ge -2\}$

b) $g\left(f(x)\right) = g\left(x^2 - 3x\right)$ $\qquad \mathbb{R}$

$$= \sqrt{x^2 - 3x + 2} \qquad x^2 - 3x + 2 \ge 0 \Rightarrow (x = 1,\ 2) \leftrightarrow x \le 1,\ x \ge 2$$

Domain: $\{x \mid x \le 1,\ x \ge 2\}$

Exercise

Let $f(x) = \sqrt{x-2}$ *and* $g(x) = \sqrt{x+5}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\sqrt{x+5}\right)$ $\qquad x+5 \ge 0 \Rightarrow x \ge -5$

$= \sqrt{\sqrt{x+5}-2}$ $\qquad \sqrt{x+5}-2 \ge 0 \Rightarrow \sqrt{x+5} \ge 2$

$x+5 \ge 4 \ \to x \ge -1$

Domain: $\{x \mid x \ge -1\}$

b) $g(f(x)) = g\left(\sqrt{x-2}\right)$ $\qquad x-2 \ge 0 \Rightarrow x \ge 2$

$= \sqrt{\sqrt{x-2}+5}$ $\qquad \sqrt{x-2}+5 \ge 0 \Rightarrow \sqrt{x-2} \ge -5$ *Always true when* $x \ge 2$

Domain: $\{x \mid x \ge 2\}$

Exercise

Let $f(x) = \dfrac{3x+5}{2}$ *and* $g(x) = \dfrac{2x-5}{3}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\frac{2x-5}{3}\right)$ $\qquad \mathbb{R}$

$$= \frac{3\frac{2x-5}{3}+5}{2}$$

$$= \frac{2x-5+5}{2}$$

$= x$ $\qquad$ ***Domain***: $\mathbb{R}$

b) $g(f(x)) = g\left(\frac{3x+5}{2}\right)$ $\qquad \mathbb{R}$

$$= \frac{2\frac{3x+5}{2}-5}{3}$$

$$= \frac{3x+5-5}{3}$$

$= x$ $\qquad$ ***Domain***: $\mathbb{R}$

Exercise

Let $f(x)=\dfrac{x-1}{x-2}$ *and* $g(x)=\dfrac{x-3}{x-4}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f\left(g(x)\right)=f\left(\dfrac{x-3}{x-4}\right)$ $\qquad x-4\neq 0 \Rightarrow x\neq 4$

$$=\frac{\frac{x-3}{x-4}-1}{\frac{x-3}{x-4}-2}$$

$$=\frac{\frac{x-3-(x-4)}{x-4}}{\frac{x-3-2(x-4)}{x-4}}$$

$$=\frac{x-3+x+4}{x-3-2x+8}$$

$$=\frac{2x+1}{-x+5}$$ $\qquad -x+5\neq 0 \Rightarrow x\neq 5$

Domain: $\{x \mid x\neq 4,\ 5\}$

b) $g\left(f(x)\right)=g\left(\dfrac{x-1}{x-2}\right)$ $\qquad x-2\neq 0 \Rightarrow x\neq 2$

$$=\frac{\frac{x-1}{x-2}-3}{\frac{x-1}{x-2}-4}$$

$$=\frac{\frac{x-1-3(x-2)}{x-2}}{\frac{x-1-4(x-2)}{x-2}}$$

$$=\frac{x-1-3x+6}{x-1-4x+8}$$

$$=\frac{-2x+5}{-3x+7}$$ $\qquad -3x+7\neq 0 \Rightarrow -3x\neq -7 \to x\neq \dfrac{7}{3}$

Domain: $\left\{x \mid x\neq 2,\ \dfrac{7}{3}\right\}$

Exercise

Given $f(x)=\sqrt{x}$ and $g(x)=x+3$, find $(f\circ g)(x)$, $(g\circ f)(x)$ and their domain.

Solution

$$(f\circ g)(x)=f(g(x))$$

$$=f(x+3) \qquad \textit{Domain}:\ (-\infty,\infty)$$

$$=\sqrt{x+3}$$

$$x+3\ge 0 \quad\Rightarrow\quad x\ge -3$$

Domain: $[-3,\infty)$

$$(g\circ f)(x)=g(f(x))$$

$$=g\left(\sqrt{x}\right) \qquad \textit{Domain}:\ x\ge 0$$

$$=\sqrt{x}+3$$

$$x\ge 0$$

Domain: $[0,\infty)$

Exercise

Given that $f(x)=\sqrt{x}$ and $g(x)=2-3x$, find $(f\circ g)(x)$, $(g\circ f)(x)$ and their domain.

Solution

$$(f\circ g)(x)=f(g(x))$$

$$=f(2-3x) \qquad \textit{Domain}:\ (-\infty,\infty)$$

$$=\sqrt{2-3x}$$

$$2-3x\ge 0\to -3x\ge -2\Rightarrow \boxed{x\le \frac{2}{3}}$$

Domain: $\left(-\infty,\ \frac{2}{3}\right]$

$$g(f(x))=g\left(\sqrt{x}\right) \qquad \textit{Domain}:\ x\ge 0$$

$$=2-3\sqrt{x}$$

$$x\ge 0$$

Domain: $[0,\infty)$

Exercise

Given that $f(x)=\dfrac{1}{x-2}$ and $g(x)=\dfrac{x+2}{x}$, find $(f\circ g)(x)$, $(g\circ f)(x)$ and their domain.

Solution

$$f(g(x))=f\left(\tfrac{x+2}{x}\right)$$ ***Domain***: $\boxed{x\neq 0}$

$$=\frac{1}{\frac{x+2}{x}-2}$$

$$=\frac{1}{\frac{x+2-2x}{x}}$$

$$=\frac{x}{2-x}$$ ***Domain***: $\boxed{x\neq 2}$

Domain: $(-\infty,0)\cup(0,2)\cup(2,\infty)$

$$g(f(x))=g\left(\tfrac{1}{x-2}\right)$$ ***Domain***: $\boxed{x\neq 2}$

$$=\frac{\frac{1}{x-2}+2}{\frac{1}{x-2}}$$

$$=\frac{\frac{1+2x-4}{x-2}}{\frac{1}{x-2}}$$

$$=2x-3$$ ***Domain***: $\boxed{\mathbb{R}}$

Domain: $(-\infty,\ 2)\cup(2,\ \infty)$

Exercise

Given that $f(x)=2x-5$ and $g(x)=x^2-3x+8$, find $(f\circ g)(x)$, $(g\circ f)(x)$ and their domain then find $(f\circ g)(7)$

Solution

$$f(g(x))=f\left(x^2-3x+8\right)$$ *Domain:* $(-\infty,\infty)$

$$=2(\text{------------})-5$$

$$=2\left(2x^2-3x+8\right)-5$$

$$=2x^2-6x+16-5$$

$$=2x^2-6x+11$$ *Domain:* $(-\infty,\infty)$

Domain: $(-\infty,\infty)$

$g\left(f\left(x\right)\right) = g\left(2x-5\right)$ *Domain:* $\left(-\infty,\infty\right)$

$= \left(---\right)^2 - 3\left(---\right) + 8$

$= \left(2x-5\right)^2 - 3\left(2x-5\right) + 8$

$= 4x^2 - 20x + 25 - 6x + 15 + 8$

$= 4x^2 - 26x + 48$ *Domain:* $\left(-\infty,\infty\right)$

Domain*:* $\left(-\infty,\infty\right)$

$f\left(g\left(7\right)\right) = 2(7)^2 - 6(7) + 11 \underline{= 67}$

Exercise

Given that $f(x) = \sqrt{x}$ and $g(x) = x - 1$, find

a) $(f \circ g)(x) = f(g(x))$ *b)* $(g \circ f)(x) = g(f(x))$ *c)* $(f \circ g)(2) = f(g(2))$

Solution

a) $(f \circ g)(x) = f(g(x))$

$= f(x-1)$

$\underline{= \sqrt{x-1}}$

b) $(g \circ f)(x) = g(f(x))$

$= g(\sqrt{x})$

$\underline{= \sqrt{x} - 1}$

c) $(f \circ g)(2) = f(g(2))$ $= \sqrt{x-1}$

$= \sqrt{2-1}$

$\underline{= 1}$

Exercise

Given that $f(x) = \dfrac{x}{x+5}$ and $g(x) = \dfrac{6}{x}$, find

a) $(f \circ g)(x) = f(g(x))$ *b)* $(g \circ f)(x) = g(f(x))$ *c)* $(f \circ g)(2) = f(g(2))$

Solution

a) $(f \circ g)(x) = f(g(x))$

$= f\left(\dfrac{6}{x}\right)$

$$=\frac{\frac{6}{x}}{\frac{6}{x}+5}$$

$$=\frac{\frac{6}{x}}{\frac{6+5x}{x}}$$

$$\boxed{=\frac{6}{6+5x}}$$

b) $(g\circ f)(x)=g(f(x))$

$$=g\left(\frac{x}{x+5}\right)$$

$$=\frac{6}{\frac{x}{x+5}}$$

$$\boxed{=\frac{6(x+5)}{x}}$$

c) $(f\circ g)(2)=f(g(2))$

$$=\frac{6}{6+5(2)} \quad =\frac{6}{16}$$

$$\boxed{=\frac{3}{8}}$$

Exercise

Determine whether f is even, odd, or neither: $f(x)=3x^4+2x^2-5$

Solution

$$f(-x)=3(-x)^4+2(-x)^2-5$$

$$=3x^4+2x^2-5$$

$$=f(x)$$

$\therefore$ The function is ***even***.

Exercise

Determine whether f is even, odd, or neither: $f(x)=8x^3-3x^2$

Solution

$$f(-x)=8(-x)^3-3(-x)^2$$

$$=-8x^3-3x^2$$

$\therefore$ The function is ***neither***.

Exercise

Determine whether f is even, odd, or neither: $f(x)=\sqrt{x^2+4}$

Solution

$$f(-x)=\sqrt{(-x)^2+4}$$

$$=\sqrt{x^2+4}$$

$$=f(x) \qquad \therefore \text{ The function is } \textbf{\textit{even}}.$$

Exercise

Determine whether f is even, odd, or neither: $f(x)=3x^2-5x+1$

Solution

$$f(-x)=3(-x)^2-5(-x)+1$$

$$=3x^2+5x+1 \qquad \therefore \text{ The function is } \textbf{\textit{neither}}.$$

Exercise

Determine whether f is even, odd, or neither: $f(x)=\sqrt[3]{x^3-x}$

Solution

$$f(-x)=\sqrt[3]{(-x)^3-(-x)}$$

$$=\sqrt[3]{-x^3+x}$$

$$=\sqrt[3]{-\left(x^3-x\right)}$$

$$=-\sqrt[3]{x^3-x}$$

$$=-f(x) \qquad \therefore \text{ The function is } \textbf{\textit{odd}}.$$

Exercise

Determine whether f is even, odd, or neither: $f(x)=|x|-3$

Solution

$$f(-x)=|-x|-3$$

$$=|(-)x|-3$$

$$=|-1||x|-3$$

$$=|x|-3$$

$$=f(x) \qquad \therefore \text{ The function is } \textbf{\textit{even}}.$$

Exercise

Determine whether f is even, odd, or neither: $f(x) = x^3 - \frac{1}{x}$

Solution

$$f(-x) = (-x)^3 - \frac{1}{(-x)}$$

$$= -x^3 + \frac{1}{x}$$

$$= -\left(x^3 - \frac{1}{x}\right)$$

$$= -f(x)$$ $\therefore$ The function is ***odd***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = -x^3 + 2x$

Solution

$$f(-x) = -(-x)^3 + 2(-x)$$

$$= x^3 - 2x$$

$$= -f(x)$$ $\therefore$ The function is ***odd***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = x^5 - 2x^3$

Solution

$$f(-x) = (-x)^5 - 2(-x)^3$$

$$= -x^5 + 2x^3$$

$$= -f(x)$$ $\therefore$ The function is ***odd***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = .5x^4 - 2x^2 + 6$

Solution

$$f(-x) = .5(-x)^4 - 2(-x)^2 + 6$$

$$= .5x^4 - 2x^2 + 6$$

$$= f(x)$$ $\therefore$ The function is ***even***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = .75x^2 + |x| + 4$

Solution

$$f(-x) = .75(-x)^2 + |-x| + 4$$
$$= .75x^2 + |x| + 4$$
$$= f(x) \qquad \therefore \text{ The function is } \textbf{\textit{even}}.$$

Exercise

Decide whether each function is even, odd, or neither $f(x) = x^3 - x + 9$

Solution

$$f(-x) = (-x)^3 - (-x) + 9$$
$$= -x^3 + x + 9 \qquad \therefore \text{ The function is } \textbf{\textit{neither}}.$$

Exercise

Decide whether each function is even, odd, or neither $f(x) = x^4 - 5x + 8$

Solution

$$f(-x) = (-x)^4 - 5(-x) + 8$$
$$= x^4 + 5x + 8 \qquad \therefore \text{ The function is } \textbf{\textit{neither}}.$$

Exercise

Decide whether each function is even, odd, or neither $f(x) = x^3 + x$

Solution

$$f(-x) = (-x)^3 + (-x)$$
$$= -x^3 - x$$
$$= -f(x) \qquad \therefore \text{ The function is } \textbf{\textit{odd}}.$$

Exercise

Decide whether each function is even, odd, or neither $g(x) = x^2 - x$

Solution

$$g(-x) = (-x)^2 + (-x)$$

$= x^2 - x$ $\quad\therefore$ The function is ***neither***.

Exercise

Decide whether each function is even, odd, or neither $h(x) = 2x^2 + x^4$

Solution

$$h(-x) = 2(-x)^2 + (-x)^4$$
$$= 2x^2 + x^4$$
$$= h(x)$$
$\therefore$ The function is ***even***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = 2x^2 + x^4 + 1$

Solution

$$f(-x) = 2(-x)^2 + (-x)^4 + 1$$
$$= 2x^2 + x^4 + 1$$
$$= f(x)$$
$\therefore$ The function is ***even***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = \frac{1}{5}x^6 - 3x^2$

Solution

$$f(-x) = \frac{1}{5}(-x)^6 - 3(-x)^2$$
$$= \frac{1}{5}x^6 - 3x^2$$
$$= f(x)$$
$\therefore$ The function is ***even***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = x\sqrt{1-x^2}$

Solution

$$f(-x) = -x\sqrt{1-(-x)^2}$$
$$= -x\sqrt{1-x^2}$$
$$= -f(x)$$
$\therefore$ The function is ***odd***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = x^2\sqrt{1-x^2}$

Solution

$$\begin{aligned} f(-x) &= (-x)^2\sqrt{1-(-x)^2} \\ &= x^2\sqrt{1-x^2} \\ &= f(x) \end{aligned}$$

$\therefore$ The function is ***even***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = 5x^7 - 6x^3 - 2x$

Solution

$$\begin{aligned} f(-x) &= 5(-x)^7 - 6(-x)^3 - 2(-x) \\ &= -5x^7 + 6x^3 + 2x \\ &= -(5x^7 - 6x^3 - 2x) \\ &= -f(x) \end{aligned}$$

$\therefore$ The function is ***odd***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = 5x^6 - 3x^2 - 7$

Solution

$$\begin{aligned} f(-x) &= 5(-x)^6 - 3(-x)^2 - 7 \\ &= 5x^6 - 3x^2 - 7 \\ &= f(x) \end{aligned}$$

$\therefore$ The function is ***even***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = x^2 + 6$

Solution

$$\begin{aligned} f(-x) &= (-x)^2 + 6 \\ &= x^2 + 6 \\ &= f(x) \end{aligned}$$

$\therefore$ The function is ***even***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = 7x^3 - x$

Solution

$$f(-x) = 7(-x)^3 - (-x)$$

$$= -7x^3 + x$$
$$= -(7x^3 - x)$$
$$= -f(x) \qquad \therefore \text{ The function is } \textbf{\textit{odd}}.$$

Exercise

Decide whether each function is even, odd, or neither $h(x) = x^5 + 1$

Solution

$$h(-x) = (-x)^5 + 1$$
$$= -x^5 + 1 \begin{cases} \neq x^5 + 1 \\ \neq -(x^5 + 1) \end{cases} \quad \textit{Neither}$$

Exercise

$$f(x) = \begin{cases} 2 + x & \text{if } x < -4 \\ -x & \text{if } -4 \le x \le 2 \\ 3x & \text{if } x > 2 \end{cases}$$

Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

Solution

a) $f(-5) = 2 - 5 = -3$

b) $f(-1) = -(-1) = 1$

c) $f(0) = -0 = 0$

d) $f(3) = 3(3) = 9$

Exercise

$$f(x) = \begin{cases} -2x & \text{if } x < -3 \\ 3x - 1 & \text{if } -3 \le x \le 2 \\ -4x & \text{if } x > 2 \end{cases}$$

Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

Solution

a) $f(-5) = -2(-5) = 10$

b) $f(-1) = 3(-1) - 1 = -4$

c) $f(0) = 3(0) - 1 = -1$

d) $f(3) = -4(3) = -12$

Exercise

$$f(x)=\begin{cases} x^3+3 & if \quad -2\le x\le 0 \\ x+3 & if \quad 0<x<1 \\ 4+x-x^2 & if \quad 1\le x\le 3 \end{cases} \qquad \text{Find: } f(-5),\ f(-1),\ f(0),\ and\ f(3)$$

Solution

a) $f(-5)=\ doesn't\ exist$

b) $f(-1)=(-1)^3+3=2$

c) $f(0)=(0)^3+3=3$

d) $f(3)=4+(3)-(3)^2=-2$

Exercise

$$h(x)=\begin{cases} \dfrac{x^2-9}{x-3} & if \quad x\ne 3 \\ 6 & if \quad x=3 \end{cases} \qquad \text{Find: } h(5),\ h(0),\ and\ h(3)$$

Solution

a) $h(5)=\dfrac{5^2-9}{5-3}=8$

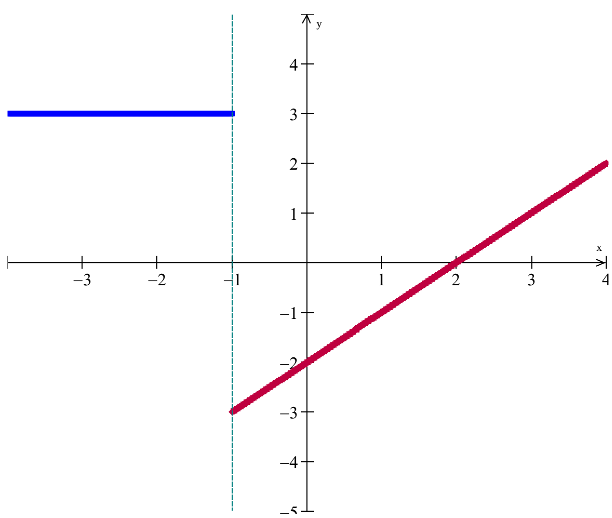
b) $h(0)=\dfrac{0^2-9}{0-3}=3$

c) $h(3)=6$

Exercise

Graph the piecewise function defined by $f(x)=\begin{cases} 3 & if \quad x\le -1 \\ x-2 & if \quad x>-1 \end{cases}$

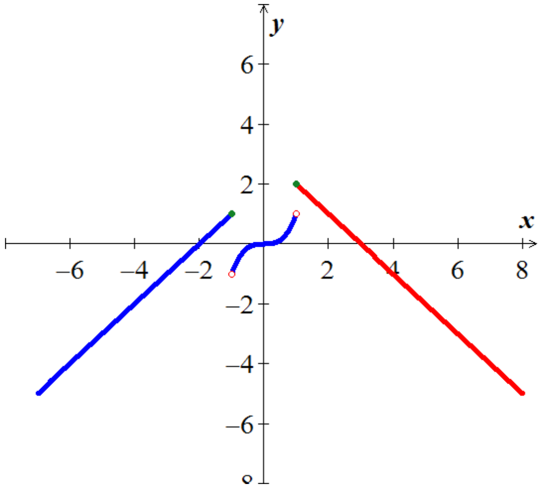
Solution



Exercise

Sketch the graph $f(x)=\begin{cases} x+2 & if \quad x\le -1 \\ x^3 & if \quad -1<x<1 \\ -x+3 & if \quad x\ge 1 \end{cases}$

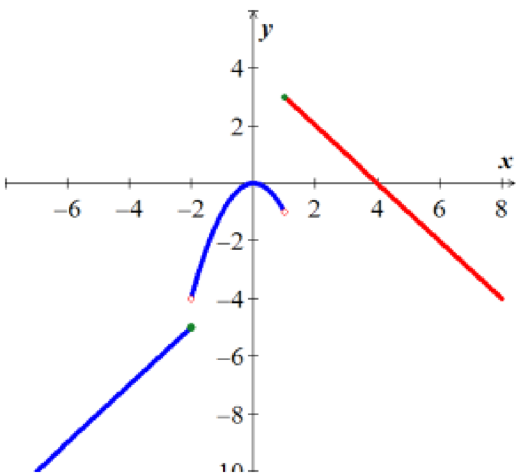
Solution



Exercise

Sketch the graph $f(x)=\begin{cases} x-3 & if \quad x\le -2 \\ -x^2 & if \quad -2<x<1 \\ -x+4 & if \quad x\ge 1 \end{cases}$

Solution



Solution *Section* 1.2 – Polynomial Functions & Graphs

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=2x^4-x^3+7x-12;\quad p(x)=x^2-3$

Solution

$$\begin{array}{r}
2x^2-x+6 \\
x^2-3\,\overline{)\,2x^4-x^3+0x^2+7x-12} \\
\underline{2x^4-6x^2} \\
-x^3+6x^2+7x \\
\underline{-x^3+3x} \\
6x^2+4x-12 \\
\underline{6x^2-18} \\
4x+6
\end{array}$$

$Q(x)=2x^2-x+6;\quad R(x)=4x+6$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=3x^3+2x-4;\quad p(x)=2x^2+1$

Solution

$$\begin{array}{r}
\frac{3}{2}x \\
2x^2+1\,\overline{)\,3x^3+0x^2+2x-4} \\
\underline{3x^3+\frac{3}{2}x} \\
\frac{1}{2}x-4
\end{array}$$

$Q(x)=\frac{3}{2}x;\quad R(x)=\frac{1}{2}x-4$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=7x+2;\quad p(x)=2x^2-x-4$

Solution

$Q(x)=0;\quad R(x)=7x+2$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=9x+4;\quad p(x)=2x-5$

Solution

$$\begin{array}{r|l} & \frac{9}{2} \\ 2x-5 & 9x+4 \\ & 9x-\frac{45}{2} \\ \hline & -\frac{37}{2} \end{array}$$

$$Q(x)=\frac{9}{2};\quad R(x)=-\frac{37}{2}$$

Exercise

Use the remainder theorem to find $f(c)$: $f(x)=x^4-6x^2+4x-8;\quad c=-3$

Solution

$$f(-3)=(-3)^4-6(-3)^2+4(-3)-8 \ \underline{=7}$$

Exercise

Use the remainder theorem to find $f(c)$: $f(x)=x^4+3x^2-12;\quad c=-2$

Solution

$$f(-2)=(-2)^4+3(-2)^2-12 \ \underline{=16}$$

Exercise

Use the factor theorem to show that $x-c$ is a factor of $f(x)$: $f(x)=x^3+x^2-2x+12;\quad c=-3$

Solution

$$f(-3)=(-3)^3+(-3)^2-2(-3)+12 \ \underline{=0}$$

From the factor theorem; $x+3$ is a factor of $f(x)$.

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $2x^3-3x^2+4x-5;\quad x-2$

Solution

$$\begin{array}{r|rrrr} 2 & 2 & -3 & 4 & -5 \\ & & 4 & 2 & 12 \\ \hline & 2 & 1 & 6 & \boxed{7} \end{array}$$

$$Q(x)=2x^2+x+6 \qquad R(x)=7$$

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $5x^3 - 6x^2 + 15; \quad x - 4$

Solution

$$\begin{array}{r|rrrr} 4 & 5 & -6 & 0 & 15 \\ & & 20 & 56 & 224 \\ \hline & 5 & 14 & 56 & \boxed{239} \end{array}$$

$Q(x) = 5x^2 + 14x + 56 \qquad R(x) = 239$

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $9x^3 - 6x^2 + 3x - 4; \quad x - \frac{1}{3}$

Solution

$$\begin{array}{r|rrrr} \frac{1}{3} & 9 & -6 & 3 & -4 \\ & & 3 & -1 & \frac{2}{3} \\ \hline & 9 & -3 & 2 & \boxed{-\frac{10}{3}} \end{array}$$

$Q(x) = 9x^2 - 3x + 2 \qquad R(x) = -\frac{10}{3}$

Exercise

Use the synthetic division to find $f(c)$: $f(x) = 2x^3 + 3x^2 - 4x + 4; \quad c = 3$

Solution

$$\begin{array}{r|rrrr} 3 & 2 & 3 & -4 & 4 \\ & & 9 & 36 & 93 \\ \hline & 3 & 12 & 32 & \boxed{97} \end{array}$$

$\boxed{f(3) = 97}$

Exercise

Use the synthetic division to find $f(c)$: $f(x) = 8x^5 - 3x^2 + 7; \quad c = \frac{1}{2}$

Solution

$$\begin{array}{r|rrrrrr} \frac{1}{2} & 8 & 0 & 0 & -3 & 0 & 7 \\ & & 4 & 2 & 1 & -1 & -\frac{1}{2} \\ \hline & 8 & 4 & 2 & -2 & -1 & \boxed{\frac{13}{2}} \end{array}$$

$\boxed{f\left(\frac{1}{2}\right) = \frac{13}{2}}$

Exercise

Use the synthetic division to find $f(c)$: $f(x)=x^3-3x^2-8; \quad c=1+\sqrt{2}$

Solution

$$\begin{array}{r|rrrr} 1+\sqrt{2} & 3 & -3 & 0 & -8 \\ & & 3+3\sqrt{2} & 6+3\sqrt{2} & 12+9\sqrt{2} \\ \hline & 3 & 3\sqrt{2} & 6+3\sqrt{2} & \boxed{4+9\sqrt{2}} \end{array}$$

$$\boxed{f\left(1+\sqrt{2}\right)=4+9\sqrt{2}}$$

Exercise

Use the synthetic division to show that c is a zero of $f(x)$: $f(x)=3x^4+8x^3-2x^2-10x+4; \quad c=-2$

Solution

$$\begin{array}{r|rrrrr} -2 & 3 & 8 & -2 & -10 & 4 \\ & & -6 & -4 & 12 & -4 \\ \hline & 3 & 2 & -6 & 2 & \boxed{0} \end{array}$$

$$\boxed{f(-2)=0}$$

Exercise

Use the synthetic division to show that c is a zero of $f(x)$: $f(x)=27x^4-9x^3+3x^2+6x+1; \quad c=-\frac{1}{3}$

Solution

$$\begin{array}{r|rrrrr} -\frac{1}{3} & 27 & -9 & 3 & 6 & 1 \\ & & -9 & 6 & -3 & -1 \\ \hline & 27 & -18 & 9 & 3 & \boxed{0} \end{array}$$

$$\boxed{f\left(-\frac{1}{3}\right)=0}$$

Exercise

Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

$f(x)=kx^3+x^2+k^2x+3k^2+11; \quad x+2$

Solution

$$\begin{array}{r|rrrr} -2 & k & 1 & k^2 & 3k^2+11 \\ & & -2k & 4k-2 & -2k^2-8k+4 \\ \hline & k & 1-2k & k^2+4k-2 & k^2-8k+15 \end{array}$$

$$k^2-8k+15=0 \Rightarrow \boxed{k=3,\ 5}$$

Exercise

Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

$f(x) = x^3 + k^3x^2 + +2kx - 2k^4; \quad x - 1.6$

Solution

$$\begin{array}{r|cccc} 1.6 & 1 & k^3 & 2k & -2k^4 \\ & & 1.6 & 1.6k^3 + 2.56 & 2.56k^3 + 3.2k + 4.096 \\ \hline & 1 & k^3 + 1.6 & 1.6k^3 + 2k + 2.56 & -2k^4 + 2.56k^3 + 3.2k + 4.096 \end{array}$$

$-2k^4 + 2.56k^3 + 3.2k + 4.096 = 0$

Using the calculator, the result will show that the solutions are: $x = -0.75,\ 1.96$ $\quad 0.032 \pm 1.18i$

Exercise

Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

$f(x) = k^2x^3 - 4kx + 3; \quad x - 1$

Solution

$$\begin{array}{r|cccc} 1 & k^2 & 0 & -4k & 3 \\ & & k^2 & k^2 - 4k & k^2 - 4k \\ \hline & k^2 & k^2 & k^2 - 4k & k^2 - 4k + 3 \end{array}$$

$k^2 - 4k + 3 = 0 \quad \Rightarrow \quad k = 1,\ 3$

Exercise

Find all solutions of the equation: $\quad x^3 - x^2 - 10x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{8}{1}\right\} = \pm\{1,\ 2,\ 4,\ 8\}$

$$\begin{array}{r|cccc} -1 & 1 & -1 & -10 & -8 \\ & & -1 & 2 & 8 \\ \hline & 1 & -2 & -8 & 0 \end{array} \to \quad x^2 - 2x - 8 = 0$$

The solutions are: $x = -1,\ -2,\ 4$

Exercise

Find all solutions of the equation: $x^3 + x^2 - 14x - 24 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{24}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24\}$

Using the calculator, the result will show that one solution is: $\boxed{x = -2}$

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -14 & -24 \\ & & -2 & 2 & 24 \\ \hline & 1 & -1 & -12 & 0 \end{array} \rightarrow \quad x^2 - x - 12 = 0$$

The solutions are: $x = -2,\ -3,\ 4$

Exercise

Find all solutions of the equation: $2x^3 - 3x^2 - 17x + 30 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{30}{2}\right\} = \pm\left\{1,\ 2,\ 3,\ 5,\ 6,\ 10,\ 15,\ 30,\ \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \frac{15}{2}\right\}$

Using the calculator, the result will show that one solution is: $\boxed{x = 2}$

$$\begin{array}{r|rrrr} 2 & 2 & -3 & -17 & 30 \\ & & 4 & 2 & -30 \\ \hline & 2 & 1 & -15 & \boxed{0} \end{array} \rightarrow \quad 2x^2 + x - 15 = 0$$

The solutions are: $x = 2,\ -3,\ \frac{5}{2}$

Exercise

Find all solutions of the equation: $12x^3 + 8x^2 - 3x - 2 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{2}{12}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{2},\ \frac{1}{6},\ \frac{1}{12}\right\}$

Using the calculator, the result will show that one solution is: $\boxed{x = \frac{1}{2}}$

$$\begin{array}{r|rrrr} \frac{1}{2} & 12 & 8 & -3 & -2 \\ & & 6 & 7 & 2 \\ \hline & 12 & 14 & 4 & \boxed{0} \end{array} \rightarrow \quad 12x^2 + 14x + 4 = 0$$

The solutions are: $x = \frac{1}{2},\ -\frac{1}{2},\ -\frac{2}{3}$

Exercise

Find all solutions of the equation: $x^4+3x^3-30x^2-6x+56=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{56}{1}\right\}=\pm\{1,\ 2,\ 4,\ 7,\ 8,\ 14,\ 28,\ 56\}$

Using the calculator, the result will show that one solution is: $\boxed{x=4}$

$$\begin{array}{r|rrrrr}
4 & 1 & 3 & -3 & -6 & 56 \\
 & & 4 & 28 & -8 & -56 \\ \hline
-7 & 1 & 7 & -2 & -14 & 0 \\
 & & -7 & 0 & 14 & \\ \hline
 & 1 & 0 & -2 & &
\end{array}$$

$\to x^3+7x^2-2x-14=0 \Rightarrow \frac{c}{d}=\pm\left\{\frac{14}{1}\right\}=\pm\{1,\ 2,\ 7,\ 14\}$

$\to x^2-2=0 \Rightarrow x=\pm\sqrt{2}$

The solutions are: $x=4,\ -7,\ \pm\sqrt{2}$

Exercise

Find all solutions of the equation: $3x^5-10x^4-6x^3+24x^2+11x-6=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{6}{3}\right\}=\pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$

Using the calculator, the result will show that one solution is: $\boxed{x=-1}$

$$\begin{array}{r|rrrrrr}
-1 & 3 & -10 & -6 & 24 & 11 & -6 \\
 & & -3 & 13 & -7 & -17 & 6 \\ \hline
-1 & 3 & -13 & 7 & 17 & -6 & 0 \\
 & & -3 & 16 & -23 & 6 & \\ \hline
2 & 3 & -16 & 23 & -6 & 0 & \\
 & & 6 & 20 & 6 & & \\ \hline
 & 3 & -10 & 3 & 0 & &
\end{array}$$

$x^4-13x^3+7x^2+17x-6=0 \to \pm\left\{\frac{6}{3}\right\}=\pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$3x^3-16x^2+26x-6=0 \to \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$3x^2-10x+3=0 \Rightarrow x=\frac{1}{3},\ 3$

The solutions are: $x=-1,\ -1,\ \frac{1}{3},\ 2,\ 3$

Exercise

Find all solutions of the equation: $6x^5+19x^4+x^3-6x^2=0$

Solution

$x^2\left(6x^3+19x^2+x-6\right)=0 \to \boxed{x=0,\ 0}$

$6x^3+19x^2+x-6=0$

possibilities for $\frac{c}{d}: \pm\left\{\frac{6}{6}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3}\right\}$

Using the calculator, the result will show that one solution is: $\boxed{x=-3}$

$$\begin{array}{r|rrrr} -3 & 6 & 19 & 1 & -6 \\ & & -18 & -3 & 6 \\ \hline & 6 & 1 & -2 & \boxed{0} \end{array} \quad 6x^2+x-2=0 \to x=\frac{1}{2},\ -\frac{2}{3}$$

The solutions are: $x=0,\ 0,\ -\frac{2}{3},\ -3,\ \frac{1}{2}$

Exercise

Find all solutions of the equation: $x^4-x^3-9x^2+3x+18=0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{18}{1}\right\} = \pm\{1,\ 2,\ 3,\ 6,\ 9,\ 18\}$

Using the calculator, the result will show that one solution is: $x=-2$

$$\begin{array}{r|rrrrr} -2 & 1 & -1 & -9 & 3 & 18 \\ & & -2 & 6 & 6 & -18 \\ \hline 3 & 1 & -3 & -3 & 9 & 0 \\ & & 3 & 0 & -9 & \\ \hline & 1 & 0 & -3 & 0 & \end{array}$$

$\to x^3-3x^2-3x+9=0 \to \pm\left\{\frac{9}{1}\right\} = \pm\{1,\ 3,\ 9\}$

$\to x^2-3=0 \Rightarrow x=\pm\sqrt{3}$

The solutions are: $x=-2,\ 3,\ \pm\sqrt{3}$

Exercise

Find all solutions of the equation: $2x^4-9x^3+9x^2+x-3=0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$

Using the calculator, the result will show that one solution is: $x=1$

$$\begin{array}{r|rrrrr} 1 & 2 & -9 & 9 & 1 & -3 \\ & & 2 & -7 & 2 & 3 \\ \hline 1 & 2 & -7 & 2 & 3 & 0 \\ & & 2 & -5 & -3 & \\ \hline & 2 & -5 & -3 & 0 & \end{array}$$

$\to 2x^3-7x^2+2x+3=0 \to \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$\to 2x^2-5x-3=0 \Rightarrow x=-\frac{1}{2},\ 3$

The solutions are: $x=1,\ 1,\ -\frac{1}{2},\ 3$

Exercise

Find all solutions of the equation: $8x^3+18x^2+45x+27=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{27}{8}\right\}=\pm\left\{\frac{1,\ 3,\ 9,\ 27}{1,\ 2,\ 4,\ 8}\right\}$

$$=\pm\left\{1,\ 3,\ 9,\ 27,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{3}{2},\ \frac{3}{4},\ \frac{3}{8},\ \frac{9}{2},\ \frac{9}{4},\ \frac{9}{8},\ \frac{27}{2},\ \frac{27}{4},\ \frac{27}{8}\right\}$$

Using the calculator, the result will show that one solution is: $x=-\frac{3}{4}$

$$\begin{array}{r|rrrr} -\frac{3}{4} & 8 & 18 & 45 & 27 \\ & & -6 & -9 & -27 \\ \hline & 8 & 12 & 36 & \boxed{0} \end{array} \quad \to \ 8x^2+12x+36=0$$

The solutions are: $x=-\frac{3}{4},\ -\frac{3}{4}\pm i\frac{3\sqrt{7}}{4}$

Exercise

Find all solutions of the equation: $3x^3-x^2+11x-20=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{20}{3}\right\}=\pm\left\{\frac{1,\ 2,\ 4,\ 5,\ 10,\ 20}{1,\ 3}\right\}=\pm\left\{1,\ 2,\ 4,\ 5,\ 10,\ 20,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{5}{3},\ \frac{10}{3},\ \frac{20}{3}\right\}$

Using the calculator, the result will show that one solution is: $x=\frac{4}{3}$

$$\begin{array}{r|rrrr} \frac{4}{3} & 3 & -1 & 11 & -20 \\ & & 4 & 4 & 20 \\ \hline & 3 & 3 & 15 & \boxed{0} \end{array} \quad \to \ 3x^2+3x+15=0$$

The solutions are: $x=\frac{4}{3},\ -\frac{1}{1}\pm i\frac{\sqrt{19}}{1}$

Exercise

Find all solutions of the equation: $6x^4+5x^3-17x^2-6x=0$

Solution

$x\left(6x^3+5x^2-17x-6\right)=0 \quad \to \quad x=0$

possibilities for $\frac{c}{d}:\pm\left\{\frac{6}{6}\right\}=\pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3}\right\}$

Using the calculator, the result will show that one solution is: $x=2$

$$\begin{array}{r|rrrr} -2 & 6 & 5 & -17 & -6 \\ & & -12 & 14 & 6 \\ \hline & 6 & -7 & -3 & \boxed{0} \end{array} \rightarrow 6x^2 - 7x - 3 = 0$$

The solutions are: $x = 0,\ -2,\ -\frac{1}{3},\ \frac{3}{2}$

Exercise

If $f(x) = 3x^3 - kx^2 + x - 5k$, find a number k such that the graph of f contains the point $(-1, 4)$.

Solution

$$f(-1) = 3(-1)^3 - k(-1)^2 + (-1) - 5k$$

$$4 = -3 - k - 1 - 5k$$

$$4 = -4 - 6k$$ *Add 4 on both side*

$$8 = -6k$$

$$k = -\frac{8}{6} = -\frac{4}{3}$$

Exercise

If $f(x) = kx^3 + x^2 - kx + 2$, find a number k such that the graph of f contains the point $(2, 12)$.

Solution

$$f(2) = k(2)^3 + (2)^2 - k(2) + 2$$

$$12 = 8k + 4 - 2k + 2$$

$$12 = 6k + 6$$

$$6k = 6$$

$$\boxed{k = 1}$$

Exercise

If one zero of $f(x) = x^3 - 2x^2 - 16x + 16k$ is 2, find two other zeros.

Solution

$$f(x) = x^2(x-2) - 16(x-k) \qquad k = 2$$

$$= (x-2)\left(x^2 - 16\right)$$

$$= (x-2)(x-4)(x+4)$$

The other zeros are: $4, -4$

Exercise

If one zero of $f(x) = x^3 - 3x^2 - kx + 12$ is -2, find two other zeros.

Solution

$$f(x) = x^2(x-3) - k\left(x - \frac{12}{k}\right) \qquad \frac{12}{k} \text{ has to be equal to } 3. \Rightarrow k = 4$$

$$f(x) = x^2(x-3) - 4(x-3)$$

$$= (x-3)\left(x^2-4\right)$$

$$= (x-3)(x-2)(x+2)$$

The zeros of $f(x)$ are: 3, −2, 2

Exercise

Find a polynomial $f(x)$ of degree 3 that has the zeros −1, 2, 3; and satisfies the given condition: $f(-2) = 80$

Solution

$$f(x) = k(x+1)(x-2)(x-3)$$

$$= k\left(x^2 - x - 2\right)(x-3)$$

$$= k\left(x^3 - 3x^2 - x^2 + 3x - 2x + 6\right)$$

$$= k\left(x^3 - 4x^2 + x + 6\right)$$

$$f(-2) = k\left((-2)^3 - 4(-2)^2 + (-2) + 6\right)$$

$$80 = k(-20)$$

$$k = \frac{80}{-20} = -4$$

$$f(x) = -4\left(x^3 - 4x^2 + x + 6\right)$$

$$\boxed{f(x) = -4x^3 + 16x^2 - 4x - 24}$$

Exercise

Find a polynomial $f(x)$ of degree 3 that has the zeros $-2i,\ 2i,\ 3$; and satisfies the given condition: $f(1)=20$

Solution

$$f(x) = k(x+2i)(x-2i)(x-3)$$
$$= k\left(x^2+4\right)(x-3)$$
$$= k\left(x^3-3x^2+4x-12\right)$$

$$f(1) = k\left((1)^3-3(1)^2+4(1)-12\right)$$

$$20 = k(-10) \Rightarrow k=-2$$

$$f(x) = -2\left(x^3-3x^2+4x-12\right)$$

$$\boxed{f(x) = -2x^3+6x^2-8x+24}$$

Exercise

Find a polynomial $f(x)$ of degree 4 with leading coefficient 1 such that both -4 and 3 are zeros of multiplicity 2, and sketch the graph of f.

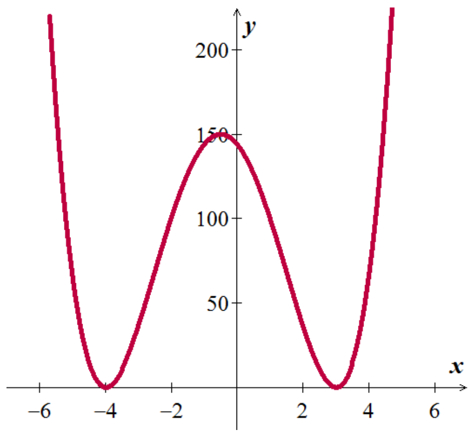
Solution

$$f(x) = a\left(x-x_1\right)\left(x-x_2\right)\left(x-x_3\right)\left(x-x_4\right)$$

$a=1 \quad x_1=x_2=-4 \quad x_3=x_4=3$

$$f(x) = (x+4)(x+4)(x-3)(x-3)$$
$$= \left(x^2+8x+16\right)\left(x^2-6x+9\right)$$
$$= x^4-6x^3+9x^2+8x^3-48x^2+72x+16x^2-96x+144$$
$$= x^4+2x^3-23x^2-24x+144$$

x	**y**
–5	64
–4	0
–2	100
0	144
2	36
3	0
4	64



Exercise

Find the zeros of $f(x) = x^2(3x+2)(2x-5)^3$, and state the multiplicity of each zero.

Solution

$$f(x) = x^2(3x+2)(2x-5)^3 = 0$$

The zeros are: $x = 0$ $(\textit{multiplicity of } 2)$

$$x = -\frac{2}{3}$$

$$x = \frac{5}{2} \quad (\textit{multiplicity of } 3)$$

Exercise

Find the zeros of $f(x) = 4x^5 + 12x^4 + 9x^3$, and state the multiplicity of each zero.

Solution

$$f(x) = x^3\left(4x^2 + 12x + 9\right) = 0$$

$$= x^3(2x+3)^2 = 0$$

The zeros are: $x = 0$ $(\textit{multiplicity of } 3)$

$$x = -\frac{3}{2} \quad (\textit{multiplicity of } 2)$$

Exercise

Find the zeros of $f(x) = \left(x^2 + x - 12\right)^3\left(x^2 - 9\right)^2$, and state the multiplicity of each zero.

Solution

$$f(x) = \left(x^2 + x - 12\right)^3\left(x^2 - 9\right)^2 = 0$$

$$x^2 + x - 12 = 0 \qquad\qquad x^2 - 9 = 0$$

$$x = -4,\ 3 \qquad\qquad x = \pm 3$$

The zeros are: $x = -4$ $(\textit{multiplicity of } 3)$

$$x = -3 \quad (\textit{multiplicity of } 2)$$

$$x = 3 \quad (\textit{multiplicity of } 5)$$

Exercise

Find the zeros of $f(x) = \left(6x^2 + 7x - 5\right)^4\left(4x^2 - 1\right)^2$, and state the multiplicity of each zero.

Solution

$$f(x) = \left(6x^2 + 7x - 5\right)^4\left(4x^2 - 1\right)^2 = 0$$

$$6x^2 + 7x - 5 = 0 \qquad\qquad 4x^2 - 1 = 0 \to x^2 = \frac{1}{4}$$

$$x = -\frac{5}{3},\ \frac{1}{2} \qquad\qquad x = \pm\frac{1}{2}$$

The zeros are: $x = -\frac{5}{3}$ $(\textit{multiplicity of } 4)$

$x = -\frac{1}{2}$ $(\textit{multiplicity of } 2)$

$x = \frac{1}{2}$ $(\textit{multiplicity of } 6)$

Exercise

Find the zeros of $f(x) = x^4 + 7x^2 - 144$, and state the multiplicity of each zero.

Solution

$$f(x) = x^4 + 7x^2 - 144$$

$$= \left(x^2 - 9\right)\left(x^2 + 16\right) = 0$$

$$x^2 - 9 = 0 \qquad\qquad x^2 + 16 = 0$$

$$x = \pm 3 \qquad\qquad x^2 = -16 \quad (\mathbb{C})$$

The zeros are: $\boxed{x = \pm 3}$

Exercise

Find the zeros of $f(x) = x^4 + 21x^2 - 100$, and state the multiplicity of each zero.

Solution

$$f(x) = x^4 + 21x^2 - 100$$

$$= \left(x^2 - 4\right)\left(x^2 + 25\right) = 0$$

$$x^2 - 4 = 0 \qquad\qquad x^2 + 25 = 0$$

$$x = \pm 2 \qquad\qquad x^2 = -25 \quad (\mathbb{C})$$

The zeros are: $\boxed{x = \pm 2}$

Exercise

Let $f(x) = x^4 - 4x^2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

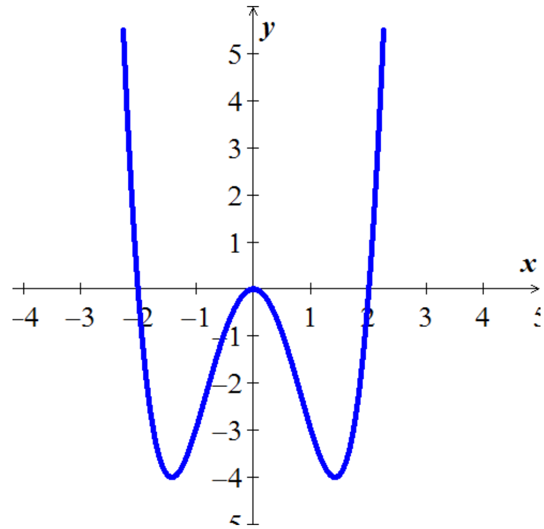
$$f(x) = x^2\left(x^2 - 4\right) = x^2(x-2)(x+2)$$

The zeros are: 0, 0, 2, –2.

$-\infty$	-2	$0,0$	2	∞
$+$		$-$		$+$

$f(x) < 0 \quad \underline{(-2,\ 0)\cup(0,\ 2)}$

$f(x) > 0 \quad \underline{(-\infty,\ -2)\cup(2,\ \infty)}$



Exercise

Let $f(x) = x^4 + 3x^3 - 4x^2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

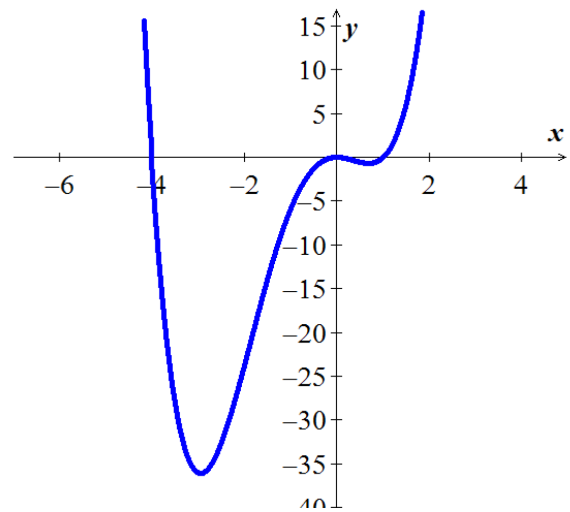
$$f(x) = x^2\left(x^2 + 3x - 4\right)$$

The zeros are: 0, 0, 1, –4.

$-\infty$	-4	$0,0$	1	∞
$+$		$-$		$+$

$f(x) > 0 \quad \underline{(-\infty,\ -4)\cup(1,\ \infty)}$

$f(x) < 0 \quad \underline{(-4,\ 0)\cup(0,\ 1)}$



Exercise

Let $f(x) = x^3 + 2x^2 - 4x - 8$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

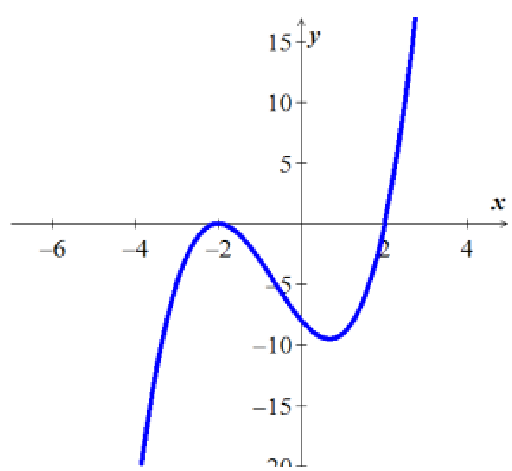
Solution

$$\begin{aligned} f(x) &= x^2(x+2) - 4(x+2) \\ &= (x+2)\left(x^2 - 4\right) \\ &= (x+2)(x+2)(x-2) = 0 \end{aligned}$$

The zeros are: 2, –2, –2

$-\infty$	-2	0	2	∞
$-$		$-$		$+$

$f(x) > 0 \quad \underline{(2,\ \infty)}$ $\qquad f(x) < 0 \quad \underline{(-\infty,\ -2)\cup(-2,\ 2)}$



Exercise

Let $f(x) = x^3 - 3x^2 - 9x + 27$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

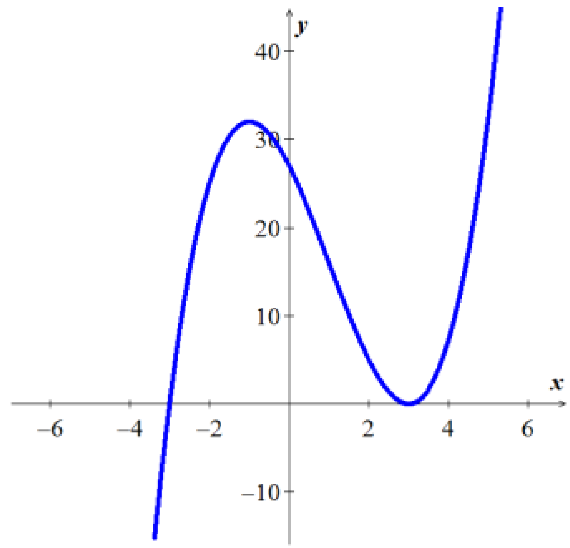
$$f(x) = x^2(x-3) - 9(x-3)$$
$$= (x-3)\left(x^2 - 9\right)$$
$$= (x-3)(x-3)(x+3)$$

The zeros are: –3, 3 (multiplicity)

$-\infty$	-3	0	3	∞
–		+		+

$f(x) > 0$ $\quad \underline{(-3,\ 3) \cup (3,\ \infty)}$

$f(x) < 0$ $\quad \underline{(-\infty,\ -3)}$



Exercise

Let $f(x) = -x^4 + 12x^2 - 27$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$x^2 = \frac{-12 \pm \sqrt{12^2 - 4(-1)(-27)}}{2(-1)} = \frac{-12 \pm \sqrt{36}}{-2} = \frac{-12 \pm 6}{-2}$$

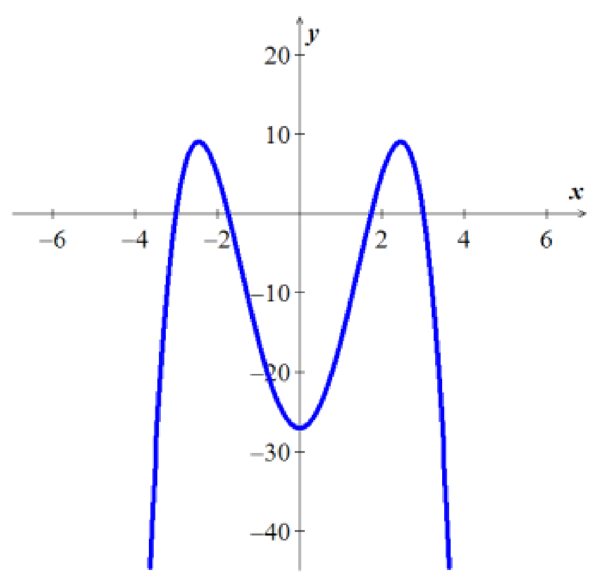
$$= \begin{cases} \frac{-12-6}{-2} = 9 \\ \frac{-12+6}{-2} = 3 \end{cases}$$

$$\to \begin{cases} x^2 = 9 \\ x^2 = 3 \end{cases} \Rightarrow \begin{matrix} x = \pm 3 \\ x = \pm\sqrt{3} \end{matrix}$$

	-3		$-\sqrt{3}$		$\sqrt{3}$		3	
–		+		–		+		–

$f(x) > 0$ $\quad \underline{\left(-3,\ -\sqrt{3}\right) \cup \left(\sqrt{3},\ 3\right)}$

$f(x) < 0$ $\quad \underline{(-\infty,\ -3) \cup \left(-\sqrt{3},\ \sqrt{3}\right) \cup (3,\ \infty)}$



Exercise

Let $f(x)=x^2(x+2)(x-1)^2(x-2)$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

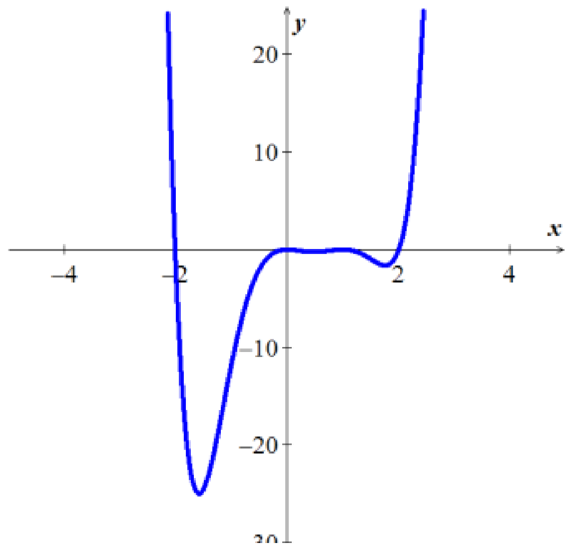
Solution

The zeros are: –2, 2, 0, 0, 1, 1

	–2	0,0 1,1	2	
	+	–	+	

$f(x)>0 \quad \boxed{(-\infty,\ -2)\cup(2,\ \infty)}$

$f(x)<0 \quad \boxed{(-2,\ 0)\cup(0,\ 1)\cup(1,\ 2)}$



Exercise

Let $f(x)=2x^3+11x^2-7x-6$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

Solution

$$possibilities: \pm\left\{\frac{6}{2}\right\}=\pm\left\{\frac{1,\ 2,\ 3,\ 6}{1,\ 2}\right\}=\pm\left\{1,\ 2,\ 3,\ 5,\ \frac{1}{2},\ \frac{3}{2}\right\}$$

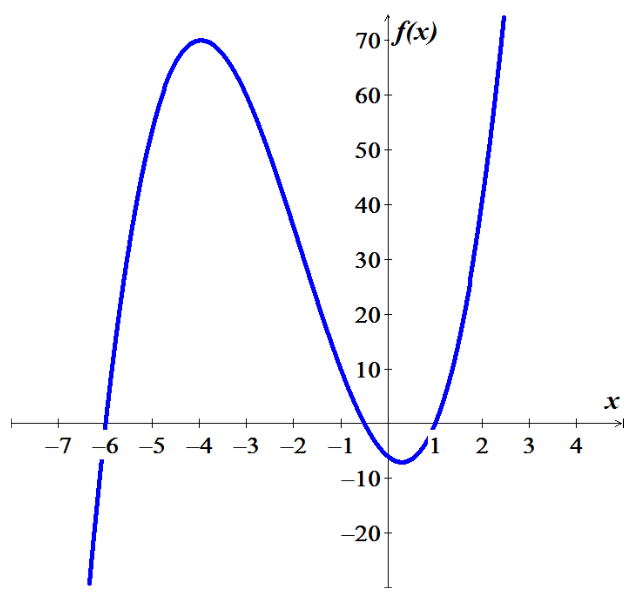
$$\begin{array}{r|rrrr} 1 & 2 & 11 & -7 & -6 \\ & & 2 & 13 & 6 \\ \hline & 2 & 13 & 6 & \boxed{0} \end{array} \quad \to \quad 2x^2+13x+6=0$$

The zeros are: $\boxed{x=1,\ -\frac{1}{2},\ -6}$

–6		$-\frac{1}{2}$		1	
–	+		–		+

$f(x)>0 \quad \boxed{\left(-6,\ -\frac{1}{2}\right)\cup(1,\ \infty)}$

$f(x)<0 \quad \boxed{(-\infty,\ -6)\cup\left(-\frac{1}{2},\ 1\right)}$



Exercise

Let $f(x) = x^3 + 2x^2 - 5x - 6$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$\textit{possibilities}\ : \pm\left\{\frac{6}{1}\right\} = \pm\{1,\ 2,\ 3,\ 6\}$$

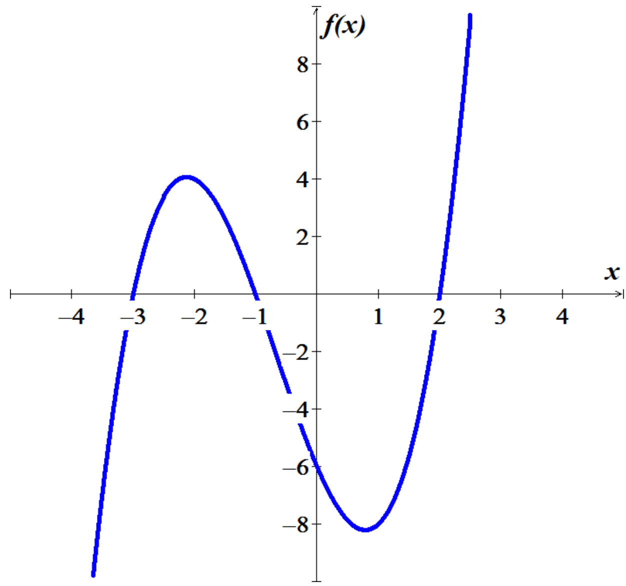
$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & & -1 & -1 & 6 \\ \hline & 1 & 1 & -6 & \boxed{0} \end{array} \quad \to x^2 + x - 6 = 0$$

The zeros are: $x = -1,\ -3,\ 2$

	-3	-1	2	
$-$		$+$	$-$	$+$

$f(x) > 0 \quad (-3,\ -1) \cup (2,\ \infty)$

$f(x) < 0 \quad (-\infty,\ -3) \cup (-1,\ 2)$



Exercise

Let $f(x) = x^3 + 8x^2 + 11x - 20$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$\textit{possibilities}\ : \pm\left\{\frac{20}{1}\right\} = \pm\{1,\ 2,\ 4,\ 5,\ 20,\ 20\}$$

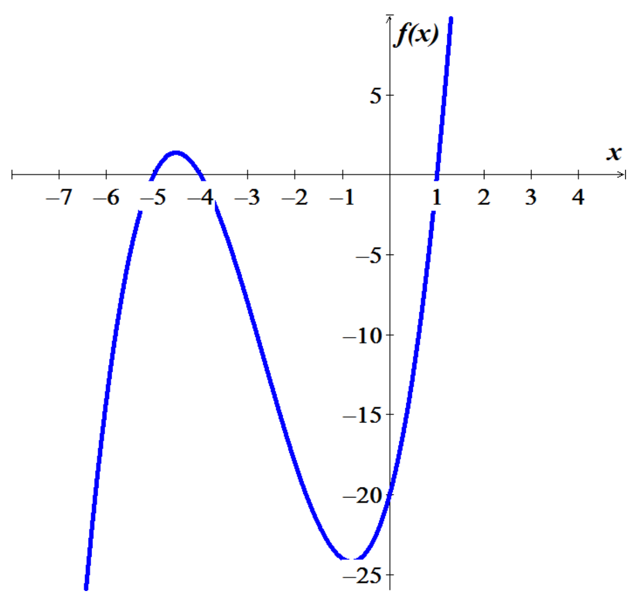
$$\begin{array}{r|rrrr} 1 & 1 & 8 & 11 & -20 \\ & & 1 & 9 & 20 \\ \hline & 1 & 9 & 20 & \boxed{0} \end{array} \quad \to x^2 + 9x + 20 = 0$$

The zeros are: $x = -5,\ -4,\ 1$

	-5	-4	1	
$-$		$+$	$-$	$+$

$f(x) > 0 \quad (-5,\ -1) \cup (1,\ \infty)$

$f(x) < 0 \quad (-\infty,\ -5) \cup (-4,\ 1)$



Exercise

Let $f(x)=x^4+x^2-2$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

Solution

possibilities : $\pm\{1,\ 2\}$

$$\begin{array}{r|rrrrr}
1 & 1 & 0 & 1 & 0 & -2 \\
 & & 1 & 1 & 2 & 1 \\ \hline
-1 & 1 & 1 & 2 & 2 & 0 \\
 & & -1 & 0 & -2 & \\ \hline
 & 1 & 0 & 2 & 0 &
\end{array}
\to x^3+x^2+2x+1=0 \ \to \pm\{1,\ 2\}$$

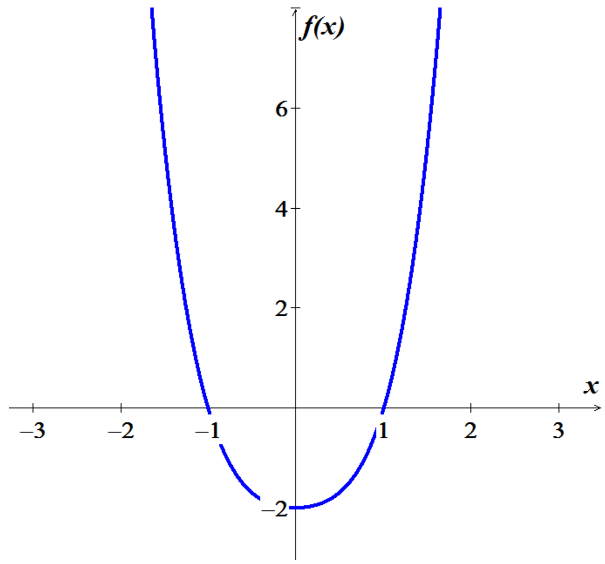
$\to x^2+2=0 \Rightarrow x=\pm i\sqrt{2}$

The zeros are: $x=\pm 1$

	-1		1	
$+$		$-$		$+$

$f(x)>0 \quad (-\infty,\ -1)\cup(1,\ \infty)$

$f(x)<0 \quad (-1,\ 1)$



Exercise

Let $f(x)=x^4-x^3-6x^2+4x+8$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

Solution

possibilities : $\pm\{1,\ 2,\ 4,\ 8\}$

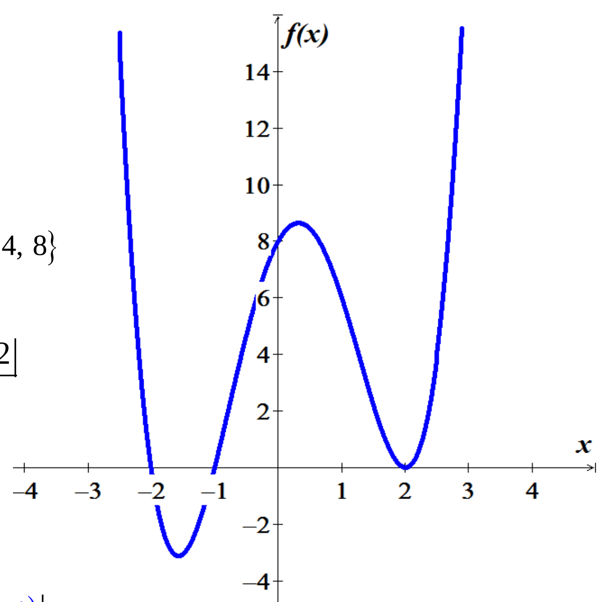
$$\begin{array}{r|rrrrr}
-1 & 1 & -1 & -6 & 4 & 8 \\
 & & -1 & 2 & 4 & -8 \\ \hline
-2 & 1 & -2 & -4 & 8 & 0 \\
 & & -2 & 8 & -8 & \\ \hline
 & 1 & -4 & 4 & 0 &
\end{array}
\to x^3-2x^2-4x+8=0 \to \pm\{1,\ 2,\ 4,\ 8\}$$

$\to x^2-4x+4=0 \Rightarrow x=2,\ 2$

The zeros are: $x=-2,\ -1,\ 2,\ 2$

	-2		-1		2	
$+$		$-$		$+$		$+$

$f(x)>0 \quad (-\infty,\ -1)\cup(1,\ \infty)$ $\qquad f(x)<0 \quad (-1,\ 1)$



Exercise

Let $f(x)=2x^4-x^3-5x^2+2x+2$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

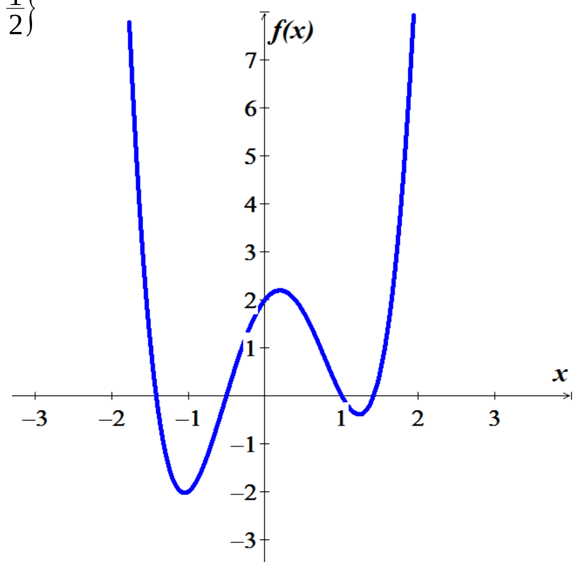
Solution

$possibilities:\ \pm\left\{\frac{2}{2}\right\}=\pm\left\{1,\ 2,\ \frac{1}{2}\right\}$

$$\begin{array}{r|rrrrr}
1 & 2 & -1 & -5 & 2 & 2 \\
 & & 2 & 1 & -4 & -2 \\
\hline
-\frac{1}{2} & 2 & 1 & -4 & -2 & \color{red}{0} \\
 & & -1 & 0 & 2 & \\
\hline
 & 2 & 0 & -4 & \color{red}{0} &
\end{array}$$

$\to\ 2x^3+x^2-4x-2=0\ \to \pm\left\{1,\ 2,\ \frac{1}{2}\right\}$

$\to\ 2x^2-4=0 \Rightarrow \underline{x=\pm\sqrt{2}}$



The zeros are: $\underline{x=-\frac{1}{2},\ 1,\ -\sqrt{2},\ \sqrt{2}}$

	$-\sqrt{2}$	$-\frac{1}{2}$	1	$\sqrt{2}$	
+	−	+	−	+	

$f(x)>0 \quad \underline{\left(-\infty,\ -\sqrt{2}\right)\cup\left(-\frac{1}{2},\ 1\right)\cup\left(\sqrt{2},\ \infty\right)}$

$f(x)<0 \quad \underline{\left(-\sqrt{2},\ -\frac{1}{2}\right)\cup\left(1,\ \sqrt{2}\right)}$

Exercise

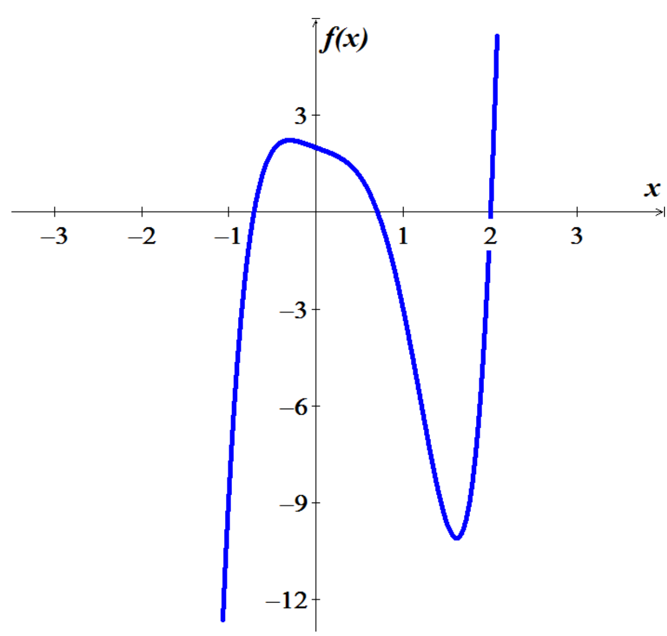
Let $f(x)=4x^5-8x^4-x+2$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

Solution

$$\begin{aligned} f(x) &= 4x^4(x-2)-(x-2) \\ &= (x-2)\left(4x^4-1\right)\color{red}{=0} \end{aligned}$$

$$4x^4-1=0 \quad \Rightarrow \begin{cases} x^2=-\frac{1}{2} & \mathbb{C} \\ x^2=\frac{1}{2} & x=\pm\frac{\sqrt{2}}{2} \end{cases}$$

The zeros are: $\underline{x=2,\ -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}}$



$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	2	
−	+	−	+

$f(x) > 0 \quad \left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right) \cup (2,\ \infty)$ $\qquad$ $f(x) < 0 \quad \left(-\infty,\ -\frac{\sqrt{2}}{2}\right) \cup \left(\frac{\sqrt{2}}{2},\ 2\right)$

Exercise

Let $f(x) = x^5 - 7x^4 + 19x^3 - 37x^2 + 60x - 36$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$possibilities:\ \pm\left\{\frac{36}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12,\ 18,\ 36\}$

$$\begin{array}{c|cccccc}
1 & 1 & -7 & 19 & -37 & 60 & -36 \\
 & & 1 & -6 & 13 & -24 & 36 \\
\hline
3 & 1 & -6 & 13 & -24 & 36 & 0 \\
 & & 3 & -9 & 12 & -36 & \\
\hline
3 & 1 & -3 & 4 & -12 & 0 & \\
 & & 3 & 0 & 12 & & \\
\hline
 & 1 & 0 & 4 & 0 & &
\end{array}$$

$x^4 - 6x^3 + 13x^2 - 24x + 36 = 0 \to \pm\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

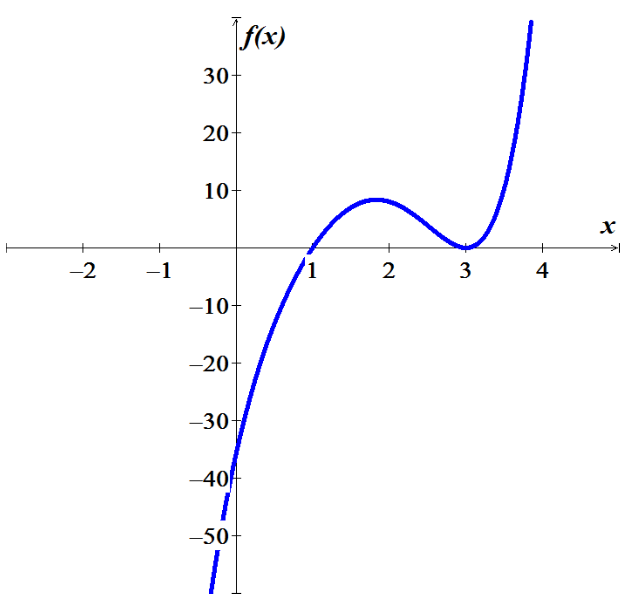
$x^3 - 3x^2 + 4x - 12 = 0 \to \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$x^2 + 4 = 0 \quad \Rightarrow \quad x = \pm 2i$

The zeros are: $x = 1,\ 3,\ 3$

	1		3	
$-$		$+$		$+$

$f(x) > 0 \quad (1,\ 3) \cup (3,\ \infty)$ $\qquad$ $f(x) < 0 \quad (-\infty,\ 1)$

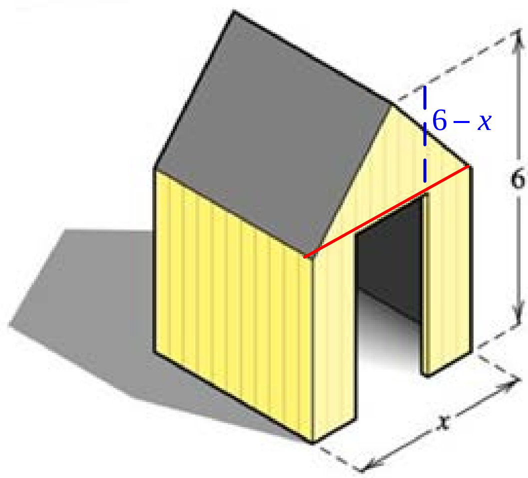


Exercise

A storage shelter is to be constructed in the shape of a cube with a triangular prism forming the roof. The length x of a side of the cube is yet to be determined.

a) If the total height of the structure is 6 *feet*, show that its volume V is given by $V = x^3 + \frac{1}{2}x^2(6-x)$

b) Determine x so that the volume is $80\ ft^3$



Solution

a) $V = V_{cube} + V_{triangle}$

$$= x^3 + \frac{1}{2}x(x)(6-x)$$

$$= \frac{1}{2}x^2(2x+6-x)$$

$$= \frac{1}{2}x^2(x+6)$$

b) $V = \frac{1}{2}x^2(x+6) = 80$

$$x^3 + 6x^2 - 160 = 0$$

possibilities: $\pm\left\{\frac{160}{1}\right\} = \pm\{1,\ 2,\ 4,\ 5,\ 8,\ 10,\ 16,\ 20,\ 32,\ 40,\ 80,\ 160\}$

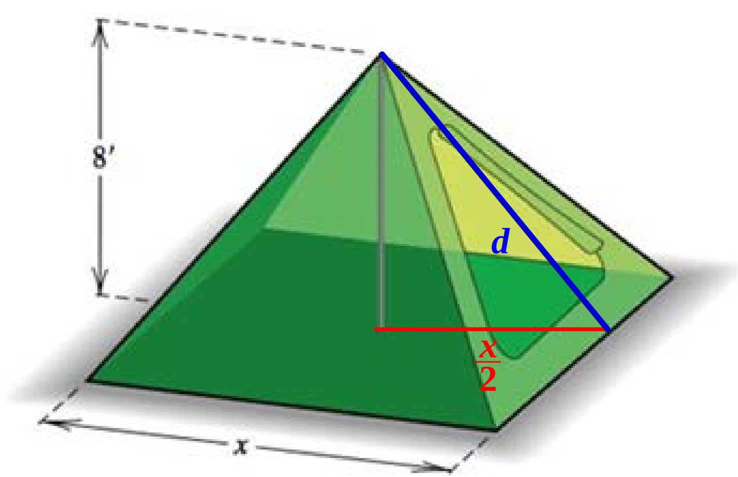
$$\begin{array}{r|rrrr} 4 & 1 & 6 & 0 & -160 \\ & & 4 & 40 & 160 \\ \hline & 1 & 10 & 40 & \boxed{0} \end{array} \quad \to x^2 + 10x + 40 = 0 \Rightarrow x = -5 \pm i\sqrt{15}$$

The solution is: $x = 4$

Exercise

A canvas camping tent is to be constructed in the shape of a pyramid with a square base. An 8–foot pole will form the center support. Find the length x of a side of the base so that the total amount of canvas needed for the sides and bottom is $384\ ft^2$.

Solution



$$d = \sqrt{64 + \frac{x^2}{4}} = \frac{1}{2}\sqrt{x^2 + 256}$$

$$A_{bottom} = x^2$$

$$A_{1-side} = \frac{1}{2}xd = \frac{1}{4}x\sqrt{x^2+256}$$

$$A_{total} = A_{bottom} + 4A_{1-side}$$

$$= x^2 + x\sqrt{x^2+256} = 384$$

$$x\sqrt{x^2+256}=384-x^2$$

$$\left(x\sqrt{x^2+256}\right)^2=\left(384-x^2\right)^2$$

$$x^2\left(x^2+256\right)=147,456-768x^2+x^4$$

$$-1,024x^2+147,456=0$$

$$x=\pm\sqrt{\frac{147,456}{1,024}}\ \underline{=12\ \ ft}$$

Solution   *Section* 1.3 – Rational Functions

Exercise

Determine all asymptotes of the function: $y = \dfrac{3x}{1-x}$

Solution

VA: $x = 1$ ***HA***: $y = -3$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{x^2}{x^2+9}$

Solution

VA: n/a $x^2 + 9 \ne 0$ ***HA***: $y = 1$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{x-2}{x^2-4x+3}$

Solution

$$x^2 - 4x + 3 = 0 \Rightarrow x = 1,\ 3$$

$$y = \frac{x}{x^2} \to 0$$

VA: $x = 1,\ x = 3$ ***HA***: $y = 0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{3}{x-5}$

Solution

VA: $x = 5$ ***HA***: $y = 0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{x^3-1}{x^2+1}$

Solution

VA: *none*

HA: *none*

Hole: n/a

Oblique asymptote: $y = x$

$$\begin{array}{r} x \\ x^2+1 \overline{\big) x^3 - 1} \\ \underline{-x^3 - x} \\ -x-1 \end{array}$$

$$y = x - \frac{x+1}{x^2+1}$$

Exercise

Determine all asymptotes of the function: $y = \dfrac{3x^2 - 27}{(x+3)(2x+1)}$

Solution

$$y = \frac{3x^2-27}{(x+3)(2x+1)} = \frac{3\left(x^2-9\right)}{(x+3)(2x+1)} = \frac{3(x+3)(x-3)}{(x+3)(2x+1)} = \frac{3(x-3)}{(2x+1)}$$

VA: $x = -3, \ -\frac{1}{2}$

HA: $y = \frac{3}{2}$

Hole: n/a

Oblique asymptote: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{x^3 + 3x^2 - 2}{x^2 - 4}$

Solution

VA: $x = \pm 2$

HA: n/a

Hole: n/a

Oblique asymptote: $y = x + 3$

$$\begin{array}{r} x+3 \\ x^2-4 \overline{\big) x^3 + 3x^2 - 2} \\ \underline{-x^3 + 4x} \\ 3x^2 + 4x - 2 \\ \underline{-3x^2 \qquad +12} \\ 4x + 10 \end{array}$$

$$y = x + 3 + \frac{4x+10}{x^2-4}$$

Exercise

Determine all asymptotes of the function: $y = \dfrac{x-3}{x^2-9}$

Solution

$$x^2 - 9 = 0 \rightarrow \boxed{x = \pm 3}$$

$$y = \frac{x-3}{(x-3)(x+3)} = \frac{1}{x+3}$$

VA: $x = 3$ ***HA***: $y = 0$

Hole: $x = 3 \to y = \frac{1}{6}$ ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \frac{6}{\sqrt{x^2 - 4x}}$

Solution

$$x^2 - 4x = 0 \Rightarrow x(x-4) = 0 \to \boxed{x = 0, 4}$$

VA: $x = 0,\ x = 4$ ***HA***: $y = 0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \frac{5x-1}{1-3x}$

Solution

VA: $x = \frac{1}{3}$ ***HA***: $y = -\frac{5}{3}$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $f(x) = \frac{2x-11}{x^2+2x-8}$

Solution

VA: $x = 2,\ x = -4$ ***HA***: $y = 0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $f(x) = \frac{x^2-4x}{x^3-x}$

Solution

$$f(x) = \frac{x(x-4)}{x\left(x^2-1\right)} = \frac{x-4}{x^2-1}$$

VA: $x = -1,\ x = 1$ ***HA***: $y = 0$

Hole: $x = 0 \to y = 4$ ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $f(x)=\dfrac{x-2}{x^3-5x}$

Solution

VA: $x=0,\ x=\pm\sqrt{5}$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{4x}{x^2+10x}$

Solution

$x^2+10x=0 \quad \rightarrow \quad x=0,\ -10$ ***Domain***: $(-\infty,\ -10)\cup(-10,\ 0)\cup(0,\ \infty)$

$$f(x)=\frac{4x}{x(x+10)}=\frac{4}{x+10}$$

VA: $x=-10$ ***HA***: $y=0$

Hole: $x=0 \quad \rightarrow \quad y=\frac{4}{10} \Rightarrow$ *hole* $\left(0,\ \frac{2}{5}\right)$ ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{3-x}{(x-4)(x+6)}$

Solution

VA: $x=-6$ *and* $x=4$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3}{2x^3-x^2-3x}$

Solution

$2x^3-x^2-3x=x\left(2x^2-x-3\right)=0 \quad \rightarrow \quad x=0,\ -1,\ \frac{3}{2}$

$$f(x)=\frac{x^3}{2x^3-x^2-3x}=\frac{x^3}{x\left(2x^2-x-3\right)}=\frac{x^2}{2x^2-x-3}$$

VA: $x=-1$ *and* $x=\frac{3}{2}$ ***HA***: $y=\frac{1}{2}$

Hole: $x=0 \quad \rightarrow \quad y=0 \Rightarrow$ *hole* $(0,\ 0)$ ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{3x^2+5}{4x^2-3}$

Solution

$4x^2-3=0 \quad \to \quad x=\pm\dfrac{\sqrt{3}}{2}$

Domain: $\left(-\infty,\ -\dfrac{\sqrt{3}}{2}\right)\cup\left(-\dfrac{\sqrt{3}}{2},\ \dfrac{\sqrt{3}}{2}\right)\cup\left(\dfrac{\sqrt{3}}{2},\ \infty\right)$

VA: $x=-\dfrac{\sqrt{3}}{2}$ *and* $x=\dfrac{\sqrt{3}}{2}$

HA: $y=\dfrac{3}{4}$

Hole: n/a

Oblique asymptote: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x+6}{x^3+2x^2}$

Solution

$x^3+2x^2=x^2(x+2)=0 \quad \to \quad x=0,-2$

Domain: $(-\infty,\ -2)\cup(-2,\ 0)\cup(0,\ \infty)$

VA: $x=0$ *and* $x=2$

HA: $y=0$

Hole: n/a

Oblique asymptote: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2+4x-1}{x+3}$

Solution

VA: $x=-3$

HA: n/a

Hole: n/a

Oblique asymptote: $y=x+1$

$$\begin{array}{r|l} & \quad x+1 \\ \hline x+3 & x^2+4x-1 \\ & \underline{-x^2-3x} \\ & \qquad x-1 \\ & \qquad \underline{-x-3} \\ & \qquad\ \ -4 \end{array}$$

$$f(x)=\frac{x^2+4x-1}{x+3}=x+1-\frac{4}{x+3}$$

Exercise

Determine all asymptotes of the function $f(x) = \dfrac{x^2 - 6x}{x - 5}$

Solution

$x - 5 = 0 \quad \to \quad x = 5$ **Domain**: $(-\infty,\ 5) \cup (5,\ \infty)$

VA: $x = 5$

HA: N/A

Hole: N/A

Oblique asymptote: $y = x - 1$

$$\begin{array}{r|l} & x - 1 \\ \hline x - 5 & x^2 - 6x \\ & \underline{-x^2 + 5x} \\ & \quad -x \\ & \quad \underline{x - 5} \\ & \quad\ -5 \end{array}$$

$$f(x) = \frac{x^2 - 6x}{x - 5} = x - 1 - \frac{5}{x - 5}$$

Exercise

Determine all asymptotes of the function $f(x) = \dfrac{x^3 - x^2 + x - 4}{x^2 + 2x - 1}$

Solution

$x^2 + 2x - 1 = 0 \quad \to \quad x = -1 \pm \sqrt{2}$

Domain: $\left(-\infty,\ -1 - \sqrt{2}\right) \cup \left(-1 - \sqrt{2},\ -1 + \sqrt{2}\right) \cup \left(-1 + \sqrt{2},\ \infty\right)$

VA: $x = -1 \pm \sqrt{2}$

HA: N/A

Hole: N/A

Oblique asymptote: $y = x - 3$

$$\begin{array}{r|l} & x - 3 \\ \hline x^2 + 2x - 1 & x^3 - x^2 + x - 4 \\ & \underline{-x^3 - 2x^2 + x} \\ & \quad -3x^2 + 2x - 4 \\ & \quad \underline{3x^2 + 6x - 3} \\ & \qquad 8x - 7 \end{array}$$

$$f(x) = \frac{x^3 - x^2 + x - 4}{x^2 + 2x - 1}$$

$$= x - 3 + \frac{8x - 7}{x^2 + 2x - 1}$$

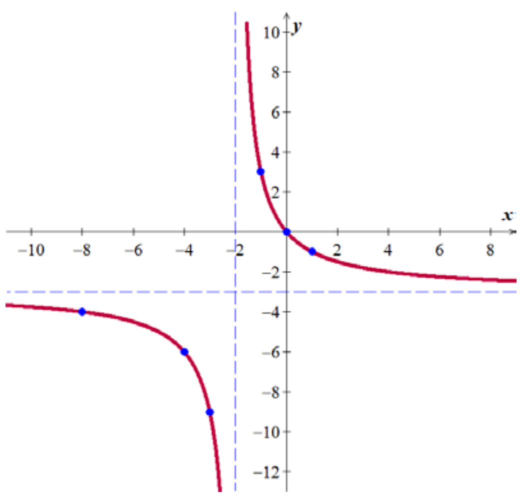
Exercise

Sketch the graph of $f(x) = \dfrac{-3x}{x + 2}$

Solution

VA: $x = -2$ ***HA***: $y = -3$

Hole: n/a ***OA***: n/a



Exercise

Sketch the graph of $f(x)=\dfrac{x+1}{x^2+2x-3}$

Solution

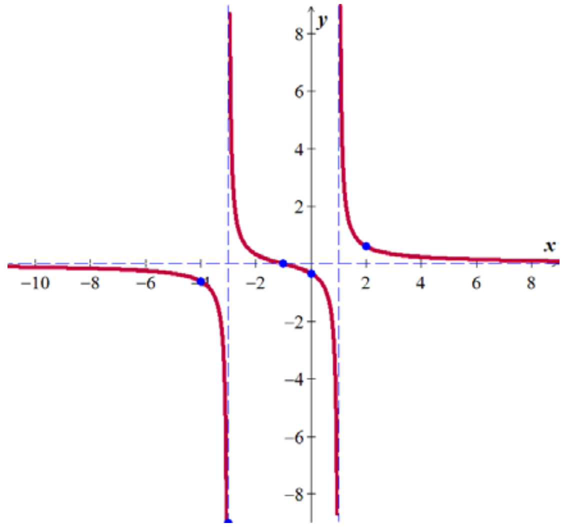
VA: $x=1,\ x=-3$

HA: $y=0$

Hole: n/a

Oblique asymptote: n/a

x	y
−5	−0.33
−2	0.33
0	−1/3
4	0.24



Exercise

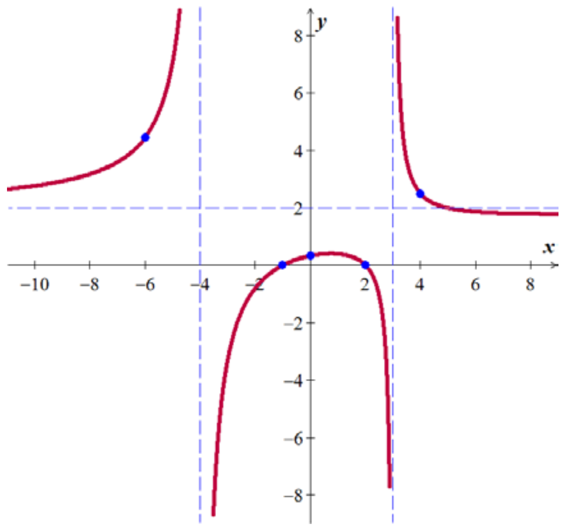
Sketch the graph of $f(x)=\dfrac{2x^2-2x-4}{x^2+x-12}$

Solution

VA: $x=-4,\ 3$ ***HA***: $y=2$

Hole: n/a ***OA***: n/a

x	y
−5	7
−2	−0.8
0	1/3
4	2.5
5	2



Exercise

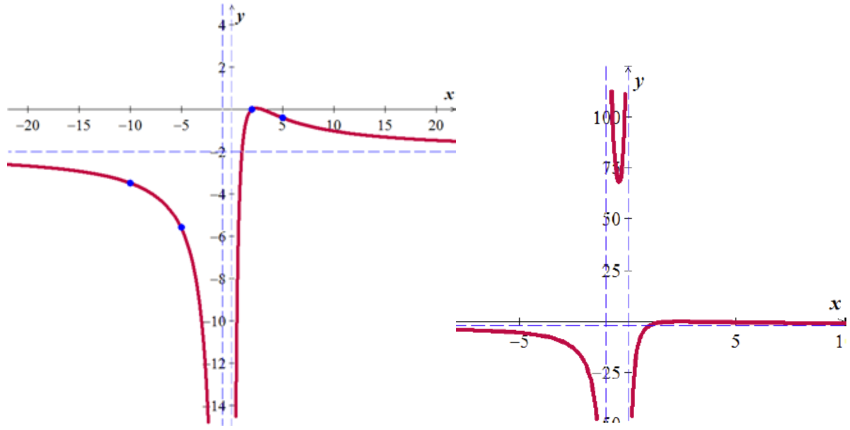
Sketch the graph of $f(x)=\dfrac{-2x^2+10x-12}{x^2+x}$

Solution

VA: $x=-1,\ 0$ ***HA***: $y=-2$

Hole: n/a ***OA***: n/a

x	y
−5	−5.6
−0.5	70
0	1/3
4	2.5
5	2



Exercise

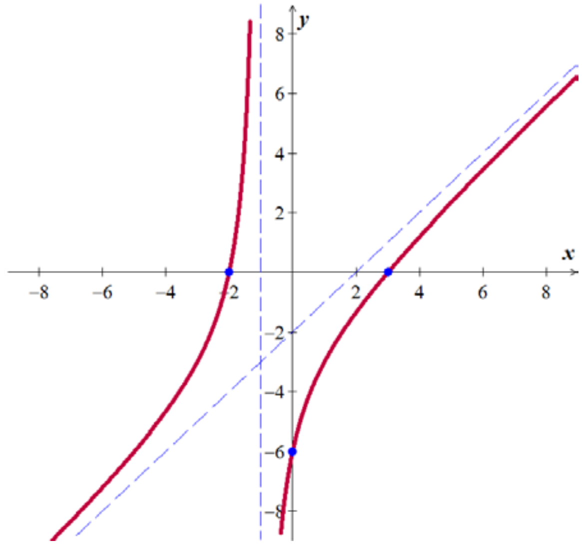
Find the oblique asymptote, and sketch the graph of $f(x) = \dfrac{x^2 - x - 6}{x+1}$

Solution

$$\begin{array}{r|l} & x-2 \\ \hline x+1 & x^2 - x - 6 \\ & \underline{x^2 + x} \\ & \quad -2x - 6 \\ & \quad \underline{-2x - 2} \\ & \qquad \boxed{-4} \end{array}$$

VA: $x = -1$ ***HA***: n/a

Hole: n/a ***OA***: $y = x - 2$



Exercise

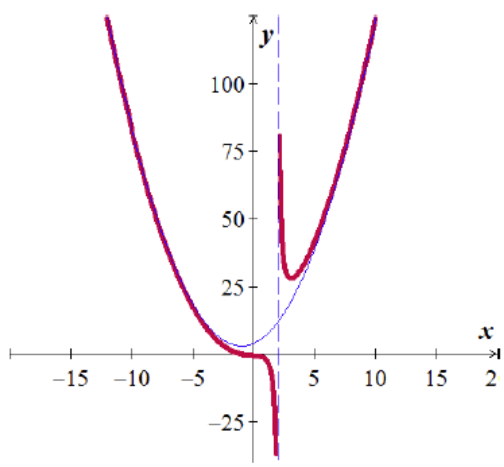
Find the oblique asymptote, and sketch the graph of $f(x) = \dfrac{x^3 + 1}{x-2}$

Solution

$$\begin{array}{r|l} & x^2 + 2x + 4 \\ \hline x-2 & x^3 - 1 \\ & \underline{x^3 - 2x^2} \\ & \quad 2x^2 \\ & \quad \underline{2x^2 - 4x} \\ & \qquad 4x - 1 \\ & \qquad \underline{4x - 8} \\ & \qquad\quad 7 \end{array}$$

VA: $x = 2$ ***HA***: n/a

Hole: n/a ***OA***: $y = x^2 + 2x + 4$



Exercise

Simplify and sketch the graph of $f(x) = \dfrac{2x^2 + x - 6}{x^2 + 3x + 2}$

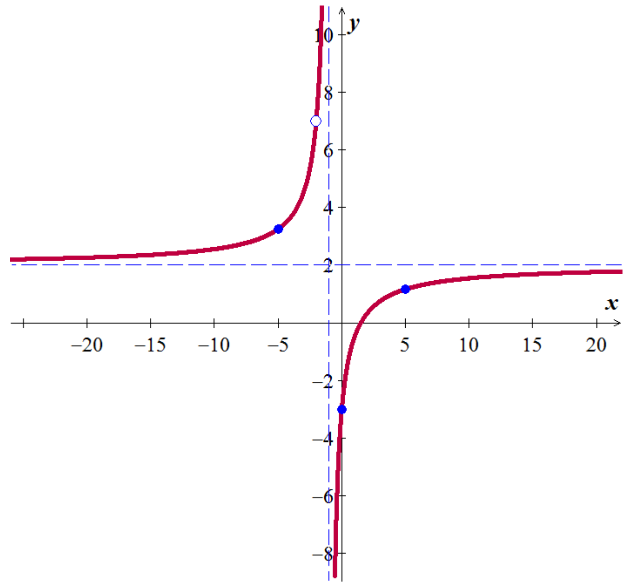
Solution

$$f(x) = \frac{(2x-3)(x+2)}{(x+1)(x+2)} = \frac{2x-3}{x+1}$$

VA: $x = -1$ ***HA***: $y = -2$

Hole: $(-2,\ 7)$ ***OA***: n/a

x	y
–4	3.6
0	–3
–3/2	0
4	1



Exercise

Simplify and sketch the graph of $f(x) = \dfrac{x-1}{1-x^2}$

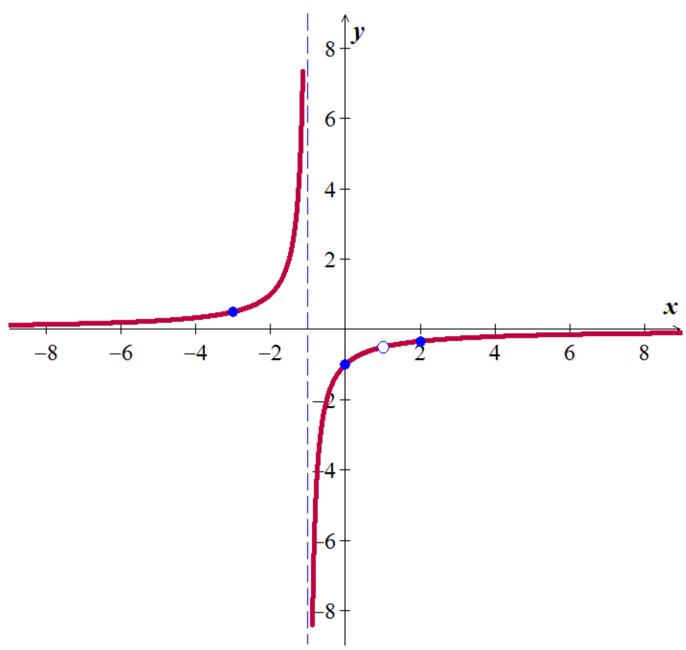
Solution

$$f(x) = \frac{x-1}{(x+1)(1-x)} = -\frac{1}{x+1}$$

VA: $x = -1$ ***HA***: $y = 0$

Hole: $\left(1,\ -\frac{1}{2}\right)$ ***OA***: n/a

x	y
–4	3.6
0	–3
–3/2	0
4	1



Exercise

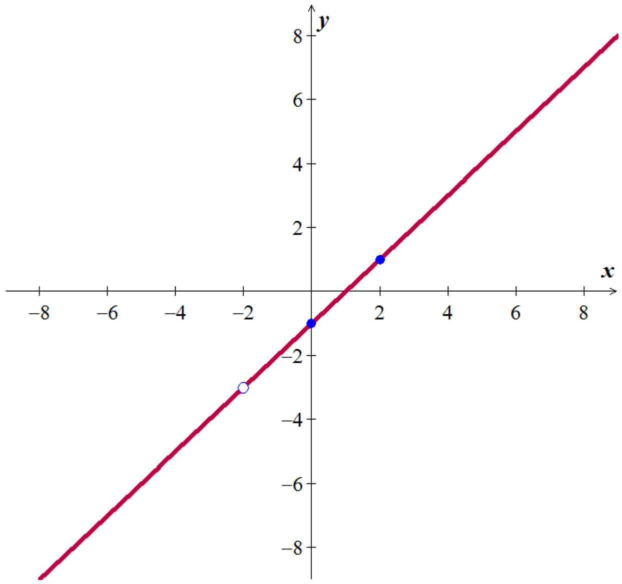
Simplify and sketch the graph of $f(x) = \dfrac{x^2 + x - 2}{x + 2}$

Solution

$$f(x) = \frac{(x+2)(x-1)}{x+2} = x - 1$$

VA: n/a ***HA***: n/a

Hole: $(-2,\ -3)$ ***OA***: n/a



Exercise

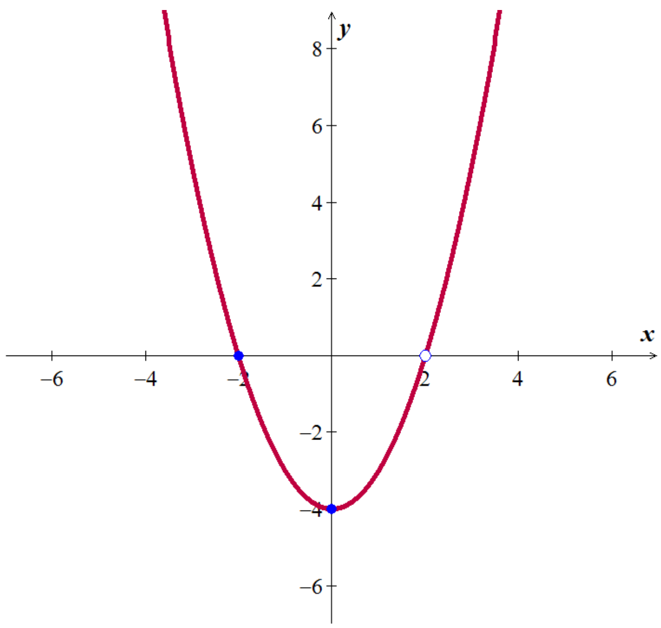
Simplify and sketch the graph of $f(x) = \dfrac{x^3 - 2x^2 - 4x + 8}{x - 2}$

Solution

$$f(x) = \frac{\left(x^2 - 4\right)(x-2)}{x-2} = x^2 - 4$$

VA: n/a ***HA***: n/a

Hole: $(2,\ 0)$ ***OA***: n/a

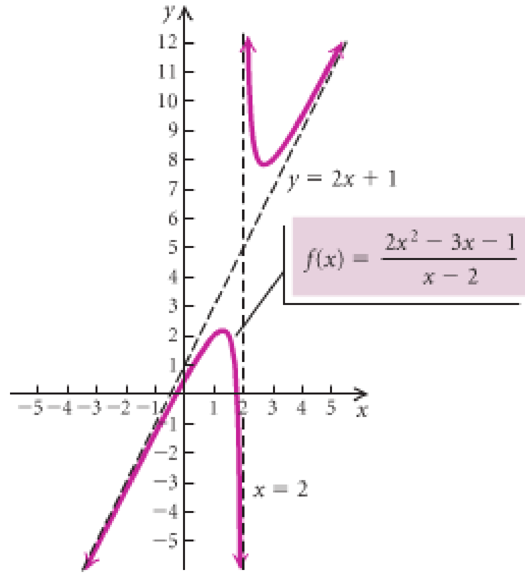


Exercise

Determine all asymptotes and sketch the graph of $f(x)=\dfrac{2x^2-3x-1}{x-2}$

Solution

$$\begin{array}{r}
2x+1 \\
x-2\,\overline{\smash{\big)}\,2x^2-3x-1} \\
\underline{-2x^2+4x} \\
x-1 \\
\underline{-x+2} \\
1
\end{array}$$



$$f(x)=\frac{2x^2-3x-1}{x-2}=(2x+1)+\frac{1}{x-2}$$

VA: $x=2$ ***HA***: $y=1$

Hole: $n/a\ (2,\ 0)$ ***OA***: $y=2x+1$

Exercise

Determine all asymptotes and sketch the graph of $f(x)=\dfrac{2x+3}{3x^2+7x-6}$

Solution

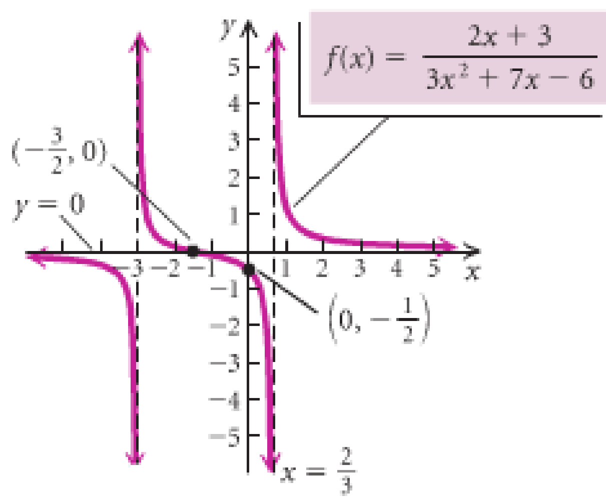
$3x^2+7x-6=0 \ \Rightarrow\ x=-3,\ \frac{2}{3}$

VA: $x=-3$ *and* $x=\frac{2}{3}$

HA: $y=0$

Hole: n/a

OA: n/a



Exercise

Determine all asymptotes and sketch the graph of $f(x) = \dfrac{x^2 - 1}{x^2 + x - 6}$

Solution

$x^2 + x - 6 = 0 \quad \Rightarrow \quad x = -3,\ 2$

VA: $x = -3$ *and* $x = 2$

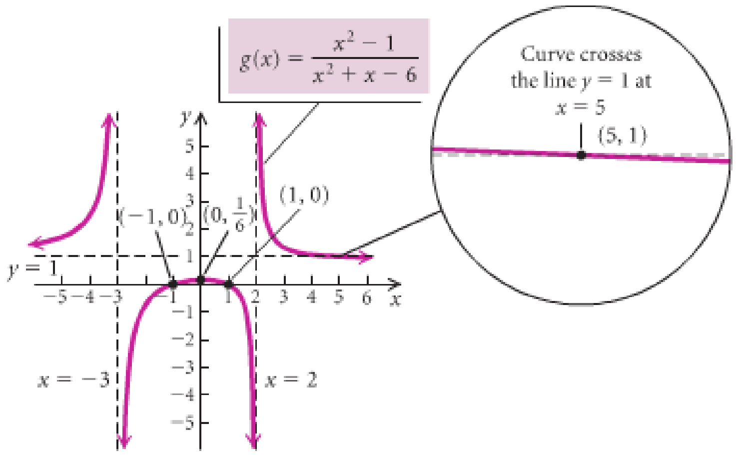
HA: $y = 1$

$1 = \dfrac{x^2 - 1}{x^2 + x - 6} \Rightarrow x^2 + x - 6 = x^2 - 1$

$\boxed{x = 5}$

Hole: n/a

OA: n/a



Exercise

Determine all asymptotes and sketch the graph of $f(x) = \dfrac{-2x^2 - x + 15}{x^2 - x - 12}$

Solution

$x^2 - x - 12 = 0 \quad \Rightarrow \quad x = -3,\ 4$

Domain: $(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$

$$f(x) = \frac{(-2x+5)(x+3)}{(x-4)(x+3)} = \frac{-2x+5}{x-4}$$

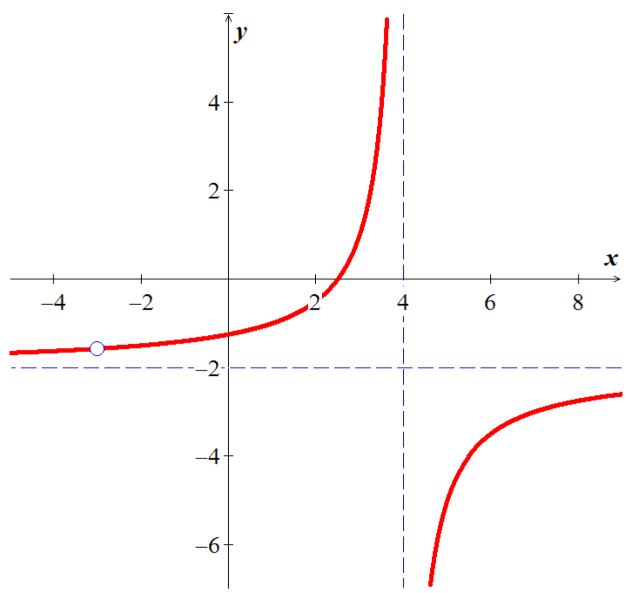
VA: $x = 4$

HA: $y = -2$

Hole: $x = -3 \quad \rightarrow \quad y = -\frac{11}{7}$

hole $\left(-3,\ -\frac{11}{7}\right)$

OA: n/a



Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \text{vertical asymptote}: \ x = 4 \\ \text{horizontal asymptote}: \ y = -1 \\ x - \text{intercept}: \ 3 \end{cases}$$

Solution

Vertical Asymptote: $\quad f(x) = \dfrac{}{x-4}$

Horizontal Asymptote: $\quad f(x) = \dfrac{-x+a}{x-4}$

x-intercept: $\quad f(x=3) = \dfrac{-3+a}{3-4} = 0 \quad \Rightarrow \quad a = 3$

$$\boxed{f(x) = \frac{-x+3}{x-4}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \text{vertical asymptote}: \ x = -3, x = 1 \\ \text{horizontal asymptote}: \ y = 0 \\ x - \text{intercept}: \ -1, \ \ f(0) = -2 \\ \text{hole at} \ \ x = 2 \end{cases}$$

Solution

Vertical Asymptote: $\quad f(x) = \dfrac{}{(x+3)(x-1)}$

Horizontal Asymptote: $\quad f(x) = \dfrac{ax+b}{(x+3)(x-1)}$

x-intercept: $\quad f(x=-1) = \dfrac{a(-1)+b}{(-1+3)(-1-1)} = \dfrac{-a+b}{-4} = 0$

$$-a + b = 0 \ \Rightarrow \ a = b$$

$$f(x=0) = \frac{a(0)+b}{(0+3)(0-1)} = \frac{b}{-3} = -2 \qquad \boxed{b = 6 = a}$$

$$f(x) = \frac{6x+6}{(x+3)(x-1)}$$

***Hole** at* $x = 2$: $\quad f(x) = \dfrac{6x+6}{(x+3)(x-1)} \dfrac{x-2}{x-2}$

$$= \frac{6(x+1)(x-2)}{\left(x^2+2x-3\right)(x-2)}$$

$$\boxed{= \frac{6\left(x^2-x-2\right)}{x^3-7x+6}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote: \ x=-4, x=5 \\ horizontal\ asymptote: \ y=\frac{3}{2} \\ x-intercept: \ -2 \end{cases}$$

Solution

Vertical Asymptote: $\qquad f(x)=\frac{}{(x+4)(x-5)}$

Horizontal Asymptote: $\qquad f(x)=\frac{3}{2}\frac{(x+a)(x+b)}{(x+4)(x-5)}$

x-intercept: $\qquad f(x=-2)=\frac{3}{2}\frac{(-2+a)(-2+b)}{}$

$$0=(-2+a)(-2+b)$$

$$\boxed{a=b=2}$$

$$f(x)=\frac{3}{2}\frac{(x-2)^2}{x^2-x-20}$$

$$=\frac{3x^2-12x+12}{2x^2-2x-40}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote: \ x=5 \\ horizontal\ asymptote: \ y=-1 \\ x-intercept: \ 2 \end{cases}$$

Solution

Vertical Asymptote: $\qquad f(x)=\frac{}{x-5}$

x-intercept : $\qquad f(x)=\frac{x-2}{x-5}$

Horizontal Asymptote: $\qquad f(x)=-\frac{x-2}{x-5}$

$$f(x)=-\frac{x-2}{x-5}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote: \ x=-2, \ x=0 \\ horizontal\ asymptote: \ y=0 \\ x-intercept: \ 2, \quad f(3)=1 \end{cases}$$

Solution

Vertical Asymptote: $\quad f(x)=\dfrac{\quad}{x(x+2)}$

x-intercept : $\quad f(x)=\dfrac{x-2}{x(x+2)}$

Horizontal Asymptote: $\quad f(x)=\dfrac{a(x-2)}{x(x+2)}$

$f(3)=1 \quad \rightarrow \dfrac{a(1)}{(3)(5)}=1 \quad \Rightarrow \underline{a=15}$

$$\boxed{f(x)=\frac{15x-30}{x^2+2x}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote: \ x=-3, \ x=1 \\ horizontal\ asymptote: \ y=0 \\ x-intercept: \ -1, \quad f(0)=-2 \\ \ hole: \ x=2 \end{cases}$$

Solution

Vertical Asymptote: $\quad f(x)=\dfrac{\quad}{(x+3)(x-1)}$

x-intercept : $\quad f(x)=\dfrac{(x+1)}{(x+3)(x-1)}$

Horizontal Asymptote: $\quad f(x)=\dfrac{a(x+1)}{(x+3)(x-1)}$

$f(0)=-2 \quad \rightarrow \dfrac{a}{-3}=-2 \quad \Rightarrow \underline{a=6}$

Hole at $x=2$**:** $\quad f(x)=\dfrac{6(x+1)(x-2)}{(x^2+2x-3)(x-2)}$

$$\boxed{f(x)=\frac{6x^2-6x-12}{x^3-7x+6}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote: \ x=-1, \ \ x=3 \\ horizontal\ asymptote: \ y=2 \\ x-intercept: \ \ -2, \ \ 1 \\ \ hole: \ \ x=0 \end{cases}$$

Solution

Vertical Asymptote: $\qquad f(x)=\dfrac{}{(x+1)(x-3)}$

Horizontal Asymptote: $\qquad f(x)=\dfrac{2}{(x+1)(x-3)}$

x-intercept : $\qquad f(x)=\dfrac{2(x+2)(x-1)}{(x+1)(x-3)}$

Hole at $x=0$**:** $\qquad f(x)=\dfrac{2x(x+2)(x-1)}{x(x+1)(x-3)}$

$$\boxed{f(x)=\dfrac{2x^3+2x^2-4x}{x^3-2x^2-3x}}$$

Solution Section 1.4 – Inverse Functions

Exercise

Determine whether the function is one-to-one: $f(x) = 3x - 7$

Solution

$$f(a) = f(b)$$
$$3a - 7 = 3b - 7$$
$$3a = 3b - 7 + 7$$
$$3a = 3b$$

Divide both sides by 3

$$a = b$$

$\therefore$ The function is ***one-to-one***

Exercise

Determine whether the function is one-to-one: $f(x) = x^2 - 9$

Solution

$1 \neq -1$ $1^2 - 9 \neq (-1)^2 - 9$ $-8 = -8 \rightarrow$ Contradict the definition	$f(a) = f(b)$ $a^2 - 9 = b^2 - 9$ $a^2 = b^2$ $a = \pm b$

The function is not ***one-to-one***

Exercise

Determine whether the function is one-to-one: $f(x) = \sqrt{x}$

Solution

$$f(a) = f(b)$$
$$\sqrt{a} = \sqrt{b}$$
$$\left(\sqrt{a}\right)^2 = \left(\sqrt{b}\right)^2$$

Square both sides

$$a = b$$

$\therefore$ The function is ***one-to-one***

Exercise

Determine whether the function is one-to-one: $f(x) = \sqrt[3]{x}$

Solution

$$f(a) = f(b)$$

$$\sqrt[3]{a} = \sqrt[3]{b}$$

$$\left(\sqrt[3]{a}\right)^3 = \left(\sqrt[3]{b}\right)^3 \qquad \textit{cube both sides}$$

$$a = b$$

$\therefore$ The function is ***one-to-one***

Exercise

Determine whether the function is one-to-one: $f(x) = |x|$

Solution

$$1 \ne -1$$

$$|1| \ne |-1|$$

$$1 \ne 1 \ (\textit{false})$$

$\therefore$ Function is ***not*** a ***one-to-one***

Exercise

Given the function f described by $f(x) = \dfrac{2}{x+3}$, prove that f is one-to-one.

Solution

$$f(a) = f(b)$$

$$\frac{2}{a+3} = \frac{2}{b+3}$$

$$(a+3)(b+3)\frac{2}{a+3} = \frac{2}{b+3}(a+3)(b+3)$$

$$2(b+3) = 2(a+3)$$

$$b+3 = a+3$$

$$a = b$$

$\therefore$ Function is ***one-to-one***

Exercise

Given the function f described by $f(x) = (x-2)^3$, prove that f is one-to-one.

Solution

$$f(a) = f(b)$$

$$(a-2)^3 = (b-2)^3$$

$$\left[(a-2)^3\right]^{1/3} = \left[(b-2)^3\right]^{1/3}$$

$a - 2 = b - 2$ *Add 2 on both sides*

$a = b$ $\therefore$ Function is ***one-to-one***

Exercise

Given the function *f* described by $y = x^2 + 2$, prove that *f* is one-to-one.

Solution

$f(a) = f(b)$

$a^2 + 2 = b^2 + 2$ *Subtract 2*

$a^2 = b^2$

$a = \pm\sqrt{b^2}$ Function is ***not*** a ***one-to-one***

The inverse function doesn't exist.

Exercise

Given the function *f* described by $f(x) = \frac{x+1}{x-3}$, prove that f is one-to-one.

Solution

$f(a) = f(b)$

$\frac{a+1}{a-3} = \frac{b+1}{b-3}$ ***Cross multiplication***

$(a+1)(b-3) = (b+1)(a-3)$

$ab - 3a + b - 3 = ab - 3b + a - 3$

$-3a - a = ab - 3b - 3 - b + 3 - ab$

$-4a = -4b$ ***Divide by −4***

$a = b$ Function is ***one-to-one***

Exercise

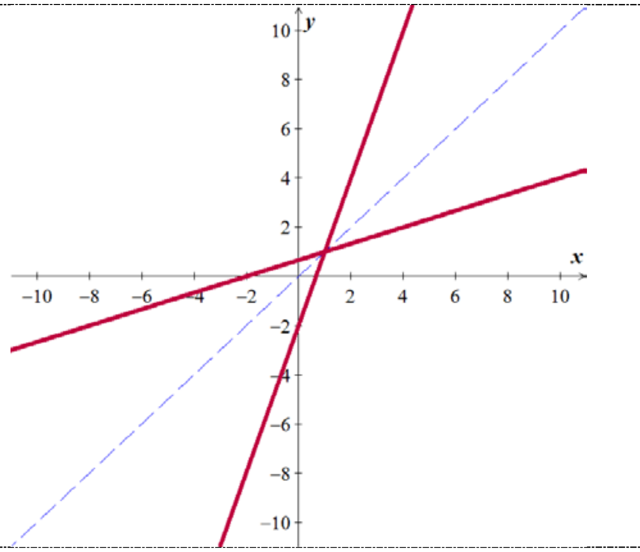
Prove the f *and* g are inverse functions of each other, and sketch the graphs of f *and* g:

$f(x) = 3x - 2 \quad g(x) = \frac{x+2}{3}$

Solution

$$f\left(g(x)\right) = f\left(\frac{x+2}{3}\right)$$
$$= 3\frac{x+2}{3} - 2$$
$$= x + 2 - 2$$
$$\underline{= x}$$

$$g\left(f(x)\right) = g\left(3x-2\right)$$
$$= \frac{3x-2+2}{3}$$
$$= \frac{3x}{3}$$
$$\underline{= x}$$



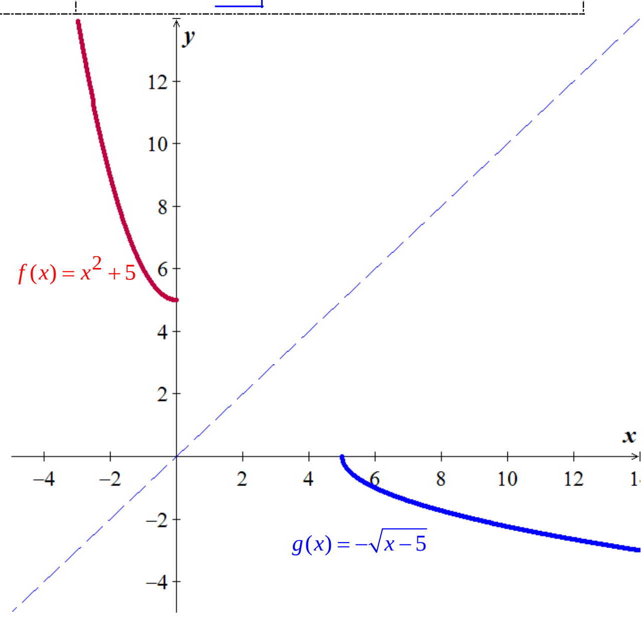
Exercise

Prove the f *and* g are inverse functions of each other, and sketch the graphs of f *and* g:

$f(x) = x^2 + 5,\ x \le 0 \quad g(x) = -\sqrt{x-5}, x \ge 5$

Solution

$f\left(g(x)\right) = f\left(-\sqrt{x-5}\right)$ $= \left(-\sqrt{x-5}\right)^2 + 5$ $= x - 5 + 5$ $\underline{= x}$	$g\left(f(x)\right) = g\left(x^2+5\right)$ $= -\sqrt{x^2+5-5}$ $= -\sqrt{x^2}$ $= -\lvert x\rvert$ $= -(-x)$ since $x < 0$ $\underline{= x}$



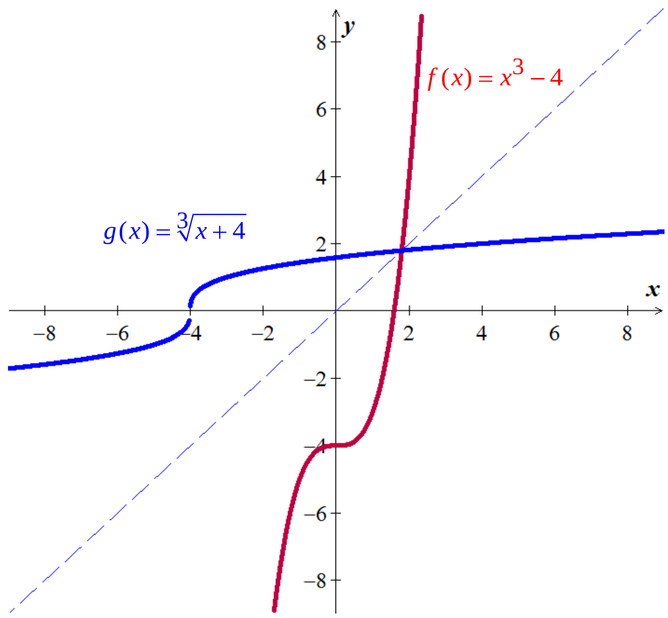
Exercise

Prove the f *and* g are inverse functions of each other, and sketch the graphs of f *and* g:

$f(x) = x^3 - 4; \quad g(x) = \sqrt[3]{x+4}$

Solution

$f\left(g(x)\right) = f\left(\sqrt[3]{x+4}\right)$ $= \left(\sqrt[3]{x+4}\right)^3 - 4$ $= x + 4 - 4$ $\underline{= x}$	$g\left(f(x)\right) = g\left(x^3 - 4\right)$ $= \sqrt[3]{x^3 - 4 + 4}$ $= \sqrt[3]{x^3}$ $\underline{= x}$



Exercise

Determine the domain and range of f^{-1}: $f(x) = -\dfrac{2}{x-1}$ (Hint: first find the domain and range of f)

Solution

$x - 1 \neq 0 \Rightarrow x \neq 1$

Range of f^{-1} = Domain of f: $\underline{\mathbb{R} - \{1\}}$ $\qquad (-\infty,\ 1) \cup (1,\ \infty)$

Domain of f^{-1} = Range of f: $\underline{\mathbb{R} - \{0\}}$ $\qquad (-\infty,\ 0) \cup (0,\ \infty)$

Exercise

Determine the domain and range of $f^{-1}:\ f(x)=\dfrac{5}{x+3}$ (Hint: first find the domain and range of *f*)

Solution

Domain of f^{-1} = Range of f: $\mathbb{R}-\{0\}$ $\quad (-\infty,\ 0)\cup(0,\ \infty)$

Range of f^{-1} = Domain of f: $\mathbb{R}-\{-3\}$ $\quad (-\infty,\ -3)\cup(-3,\ \infty)$

Exercise

Determine the domain and range of $f^{-1}:\ f(x)=\dfrac{4x+5}{3x-8}$ (Hint: first find the domain and range of *f*)

Solution

Domain of f^{-1} = Range of f: $\mathbb{R}-\left\{\frac{8}{3}\right\}$ $\quad \left(-\infty,\ \frac{8}{3}\right)\cup\left(\frac{8}{3},\ \infty\right)$

Range of f^{-1} = Domain of f: $\mathbb{R}-\left\{\frac{4}{3}\right\}$ $\quad \left(-\infty,\ \frac{4}{3}\right)\cup\left(\frac{4}{3},\ \infty\right)$

Exercise

For the given function $f(x)=3x+5$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$3a+5=3b+5$

$3a=3b$

$a=b$ $\qquad \therefore\ f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y=3x+5$

$x=3y+5$ ***Interchange x and y***

$x-5=3y$ ***Solve for y***

$\frac{x-5}{3}=y \quad \to f^{-1}(x)=\dfrac{x-5}{3}$

c) Domain of f^{-1} = Range of f: $\mathbb{R}$

Range of f^{-1} = Domain of f: $\mathbb{R}$

Exercise

For the given function $f(x)=\dfrac{1}{3x-2}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$\dfrac{1}{3a-2}=\dfrac{1}{3b-2}$

$3b-2=3a-2$

$3b=3a$

$a=b$ $\therefore$ $f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y=\dfrac{1}{3x-2}$

$x=\dfrac{1}{3y-2}$ *Interchange x and y*

$x(3y-2)=1$ *Solve for y*

$3xy-2x=1$

$3xy=1+2x$

$y=\boxed{\dfrac{1+2x}{3x}=f^{-1}(x)}$

c) Domain of f^{-1} = Range of f: $\mathbb{R}-\left\{\frac{2}{3}\right\}$

Range of f^{-1}= Domain of f: $\mathbb{R}-\{0\}$

Exercise

For the given function $f(x)=\dfrac{3x+2}{2x-5}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$\dfrac{3a+2}{2a-5}=\dfrac{3b+2}{2b-5}$

$6ab-15a+4b-10=6ab-15b+4a-10$

$19a=19b$

$a = b$ $\quad\therefore\ f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = \dfrac{3x+2}{2x-5}$

$x = \dfrac{3y+2}{2y-5}$ *Interchange x and y*

$2xy - 5x = 2y + 2$ *Solve for y*

$(2x-3)\,y = 5x + 2$

$y = \boxed{\dfrac{5x+2}{2x-3} = f^{-1}(x)}$

c) Domain of f^{-1} = Range of f : $\boxed{\mathbb{R} - \left\{\dfrac{5}{2}\right\}}$

Range of f^{-1} = Domain of f : $\boxed{\mathbb{R} - \left\{\dfrac{3}{2}\right\}}$

Exercise

For the given function $f(x) = \dfrac{4x}{x-2}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$\dfrac{4a}{a-2} = \dfrac{4b}{b-2}$

$4ab - 8a = 4ab - 8b$

$-8a = -8b$

$a = b$ $\quad\therefore\ f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = \dfrac{4x}{x-2}$

$x = \dfrac{4y}{y-2}$

$xy - 2x = 4y$

$(x-4)\,y = 2x$

$y = \boxed{\dfrac{2x}{x-4} = f^{-1}(x)}$

c) Domain of f^{-1} = Range of f : $\boxed{\mathbb{R} - \{2\}}$

Range of f^{-1} = Domain of f : $\boxed{\mathbb{R} - \{4\}}$

Exercise

For the given function $f(x) = 2 - 3x^2; \quad x \le 0$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$2 - 3a^2 = 2 - 3b^2$

$-3a^2 = -3b^2$

$a^2 = b^2$

$a = b$ since $x \le 0$ $\quad \therefore f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = 2 - 3x^2$

$x = 2 - 3y^2$

$3y^2 = 2 - x$

$y^2 = \dfrac{2-x}{3}$

$y = \boxed{-\sqrt{\dfrac{2-x}{3}} = f^{-1}(x)}$ $\qquad$ ***Since x < 0***

c) Domain of f^{-1} = Range of f: $\mathbb{R}$

Range of f^{-1} = Domain of f: $\mathbb{R}$

Exercise

For the given function $f(x) = 2x^3 - 5$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$2a^3 - 5 = 2b^3 - 5$

$a^3 = b^3$

$a = b$ $\qquad \therefore f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = 2x^3 - 5$

$$y+5=2x^3$$

$$\frac{y+5}{2}=x^3$$

$$x=\sqrt[3]{\frac{y+5}{2}}$$

$$\underline{f^{-1}(x)=\sqrt[3]{\frac{x+5}{2}}}$$

c) Domain of f^{-1} = Range of f: $\underline{\mathbb{R}}$

Range of f^{-1} = Domain of f: $\underline{\mathbb{R}}$

Exercise

For the given function $f(x)=\sqrt{3-x}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$$\left(\sqrt{3-a}\right)^2=\left(\sqrt{3-b}\right)^2$$

$$3-a=3-b$$

$a=b$ $\quad\therefore$ $f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y=\sqrt{3-x}$ $\qquad$ $\boldsymbol{y \geq 0}$

$$y=\sqrt{3-x}$$

$$y^2=3-x$$

$x=3-y^2$ $\qquad$ $\boldsymbol{x \geq 0}$

$$\underline{f^{-1}(x)=3-x^2}$$

c) Domain of f^{-1} = Range of f: $\underline{(-\infty,\ 3]}$

Range of f^{-1} = Domain of f: $\underline{[0,\ \infty)}$

Exercise

For the given function $f(x)=\sqrt[3]{x}+1$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$\sqrt[3]{a}+1 = \sqrt[3]{b}+1$

$\left(\sqrt[3]{a}\right)^3 = \left(\sqrt[3]{b}\right)^3$

$a = b$ $\qquad\therefore$ $f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = \sqrt[3]{x}+1$

$y = \sqrt[3]{x}+1$

$y-1 = \sqrt[3]{x}$

$(y-1)^3 = x$

$\boxed{f^{-1}(x) = (x-1)^3}$

c) Domain of f^{-1} = Range of f : $\mathbb{R}$

Range of f^{-1} = Domain of f : $\mathbb{R}$

Exercise

For the given function $f(x) = \left(x^3+1\right)^5$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$\left(a^3+1\right)^5 = \left(b^3+1\right)^5$

$\left(\left(a^3+1\right)^5\right)^{1/5} = \left(\left(b^3+1\right)^5\right)^{1/5}$

$a^3+1 = b^3+1$

$a^3 = b^3$

$a = b$ $\qquad\therefore$ $f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = \left(x^3+1\right)^5$

$$y = \left(x^3 + 1\right)^5$$

$$\sqrt[5]{y} = x^3 + 1$$

$$\sqrt[5]{y} - 1 = x^3$$

$$x = \sqrt[3]{\sqrt[5]{y} - 1}$$

$$\boxed{f^{-1}(x) = \sqrt[3]{\sqrt[5]{x} - 1}}$$

c) Domain of f^{-1} = Range of f: $\mathbb{R}$

Range of f^{-1} = Domain of f: $\mathbb{R}$

Exercise

For the given function $f(x) = x^2 - 6x; \quad x \geq 3$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$a^2 - 6a = b^2 - 6b$$

$$a^2 - b^2 = 6a - 6b$$

$$(a-b)(a+b) = 6(a-b)$$

$a = b$ $\quad\therefore$ $f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = x^2 - 6x$

$$x^2 - 6x - y = 0$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-y)}}{2(1)}$$

$$= \frac{6 \pm 4\sqrt{9+y}}{2}$$

$$= 3 \pm \sqrt{9+y}$$

Since $x \geq 3 \Rightarrow$ we can select $x = 3 + \sqrt{y+9}$

$$\therefore \boxed{f^{-1}(x) = 3 + \sqrt{x+9}}$$

c) Domain of f^{-1} = Range of f: $\mathbb{R}: \geq 3$

Range of f^{-1} = Domain of f: ≥ -9

Exercise

For the given function $f(x) = (x-2)^3$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$(a-2)^3 = (b-2)^3$

$a-2 = b-2$

$a = b$ $\quad\therefore$ $f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = (x-2)^3$

$x = (y-2)^3$

$x^{1/3} = \left[(y-2)^3\right]^{1/3}$

$x^{1/3} = y-2$

$\sqrt[3]{x} + 2 = y$

$\therefore$ $\underline{f^{-1}(x) = \sqrt[3]{x} + 2}$

c) Domain of f^{-1} = Range of f : $\underline{\mathbb{R}}$

Range of f^{-1} = Domain of f : $\underline{\mathbb{R}}$

Exercise

For the given function $f(x) = \dfrac{x+1}{x-3}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$\dfrac{a+1}{a-3} = \dfrac{b+1}{b-3}$

$ab - 3a + b - 3 = ab - 3b + a - 3$

$-4a = -4b$

$a = b$ $\quad\therefore$ $f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = \dfrac{x+1}{x-3}$

$x = \dfrac{y+1}{y-3}$

$x(y-3) = y+1$

$xy - 3x = y+1$

$xy - y = 3x+1$

$y(x-1) = 3x+1$

$y = \dfrac{3x+1}{x-1} = f^{-1}(x)$

c) Domain of f^{-1} = Range of f : $\mathbb{R} - \{3\}$

Range of f^{-1} = Domain of f : $\mathbb{R} - \{1\}$

Exercise

For the given function $f(x) = \dfrac{2x+1}{x-3}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$\dfrac{2a+1}{a-3} = \dfrac{2b+1}{b-3}$

$2ab - 6a + b - 3 = 2ab - 6b + a - 3$

$-7a = -7b$

$a = b$ $\qquad \therefore$ $f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = \dfrac{2x+1}{x-3}$

$x = \dfrac{2y+1}{y-3}$

$xy - 3x = 2y+1$

$y(x-2) = 3x+1$

$y = \dfrac{3x+1}{x-2} = f^{-1}(x)$

c) Domain of f^{-1} = Range of f : $\mathbb{R} - \{3\}$

Range of f^{-1} = Domain of f : $\mathbb{R} - \{2\}$

Exercise

Let $f(x)=x^3-1$ and $g(x)=\sqrt[3]{x+1}$, is g the inverse function of f?

Solution

$$\begin{aligned}(f\circ g)(x) &= f(g(x))\\ &= f\left(\sqrt[3]{x+1}\right)\\ &= \left(\sqrt[3]{x+1}\right)^3-1\\ &= x+1-1\\ &= x\end{aligned}$$

$$\begin{aligned}(g\circ f)(x) &= g(f(x))\\ &= g\left(x^3-1\right)\\ &= \sqrt[3]{x^3-1+1}\\ &= \sqrt[3]{x^3}\\ &= x\end{aligned}$$

g is the inverse function of f

Exercise

Given that $f(x)=5x+8$, use composition of functions to show that $f^{-1}(x)=\frac{x-8}{5}$

Solution

$$\begin{aligned}\left(f^{-1}\circ f\right)(x) &= f^{-1}(f(x))\\ &= f^{-1}(5x+8)\\ &= \frac{(5x+8)-8}{5}\\ &= \frac{5x}{5}\quad \underline{=x|}\end{aligned}$$

$$\left(f\circ f^{-1}\right)(x)=f(f^{-1}(x))=f^{-1}(\tfrac{x-8}{5})$$

$$=5(\tfrac{x-8}{5})+8=x-8+8\ \underline{=x|}$$

Exercise

Given the function $f(x)=(x+8)^3$

a) Find $f^{-1}(x)$

b) Graph f and f^{-1} in the same rectangular coordinate system

c) Find the domain and the range of f and f^{-1}

Solution

a) $y=(x+8)^3$ **Replace *f*(*x*) with *y***

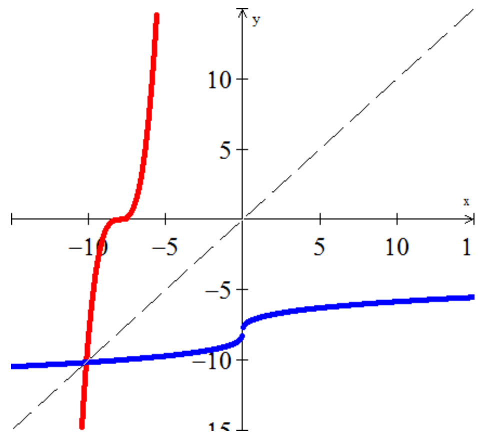
$x=(y+8)^3$ **Interchange *x* and *y***

$(x)^{1/3}=\left((y+8)^3\right)^{1/3}$

$x^{1/3}=y+8$ ***Subtract* 8 *from both sides*.**

$\boxed{x^{1/3}-8}=y=f^{-1}(x)$

b)



c) Domain of f = Range of f^{-1}: $(-\infty,\infty)$

Range of f = Domain of f^{-1}: $(-\infty,\infty)$

Solution *Section* 1.5 – Exponential Functions

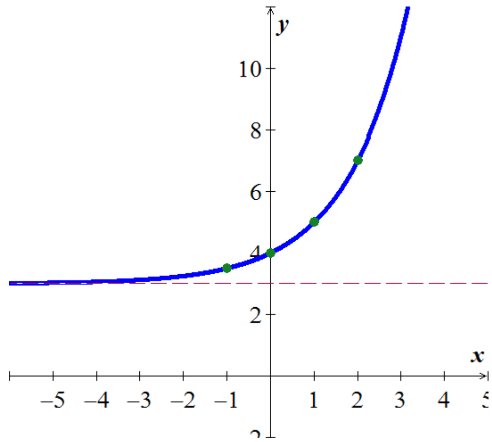
Exercise

Sketch the graph: $f(x)=2^{x}+3$

Solution

Asymptote: $y=3$

x	$f(x)$
–1	3.5
0	4
1	5
2	7



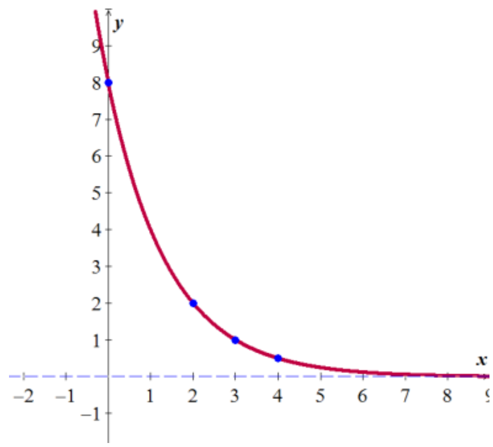
Exercise

Sketch the graph: $f(x)=2^{3-x}$

Solution

Asymptote: $y=0$

x	$f(x)$
1	4
2	2
3	1
4	.5



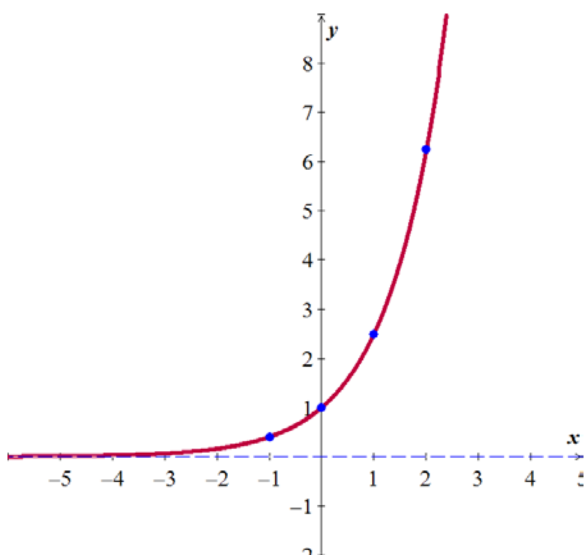
Exercise

Sketch the graph: $f(x)=\left(\frac{2}{5}\right)^{-x}$

Solution

Asymptote: $y=0$

x	$f(x)$
-1	0.4
0	1
1	2.5
2	6.25



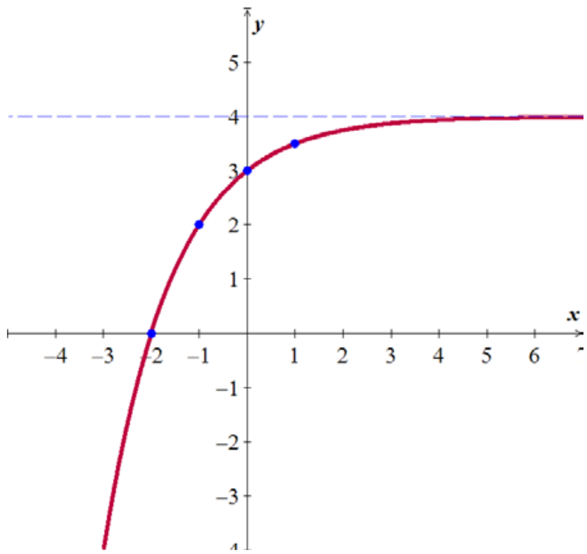
Exercise

Sketch the graph: $f(x) = -\left(\frac{1}{2}\right)^x + 4$

Solution

Asymptote: $y = 4$

x	$f(x)$
-2	0
-1	2
0	3
1	3.5



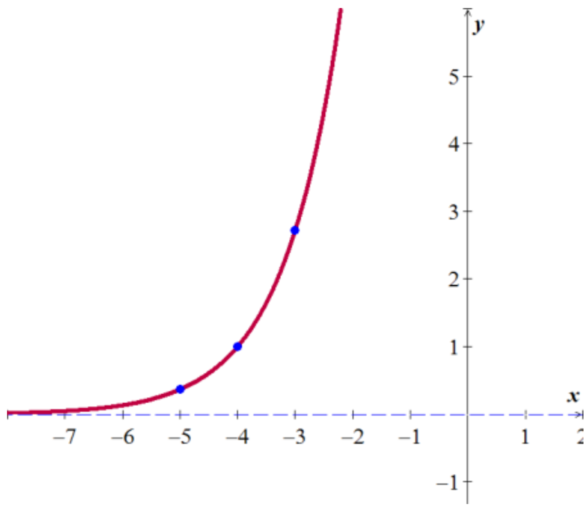
Exercise

Sketch the graph of $f(x) = e^{x+4}$

Solution

Asymptote: $y = 0$

x	$f(x)$
-5	0.4
-4	1
-3	2.7



Exercise

Simplify the expression $\dfrac{\left(e^x+e^{-x}\right)\left(e^x+e^{-x}\right)-\left(e^x-e^{-x}\right)\left(e^x-e^{-x}\right)}{\left(e^x+e^{-x}\right)^2}$

Solution

$$\frac{\left(e^x+e^{-x}\right)\left(e^x+e^{-x}\right)-\left(e^x-e^{-x}\right)\left(e^x-e^{-x}\right)}{\left(e^x+e^{-x}\right)^2} = \frac{\left(e^x+e^{-x}\right)^2-\left(e^x-e^{-x}\right)^2}{\left(e^x+e^{-x}\right)^2}$$

$$= \frac{\left[\left(e^x+e^{-x}\right)-\left(e^x-e^{-x}\right)\right]\left[\left(e^x+e^{-x}\right)+\left(e^x-e^{-x}\right)\right]}{\left(e^x+e^{-x}\right)^2}$$

$$= \frac{\left(e^x+e^{-x}-e^x+e^{-x}\right)\left(e^x+e^{-x}+e^x-e^{-x}\right)}{\left(e^x+e^{-x}\right)^2}$$

$$= \frac{\left(2e^{-x}\right)\left(2e^{x}\right)}{\left(e^{x}+e^{-x}\right)^2} \qquad e^{-x}e^{x} = e^{0} = 1$$

$$= \boxed{\frac{4}{\left(e^{x}+e^{-x}\right)^2}}$$

Exercise

Simplify the expression $\dfrac{\left(e^{x}-e^{-x}\right)^2-\left(e^{x}+e^{-x}\right)^2}{\left(e^{x}+e^{-x}\right)^2}$

Solution

$$\frac{\left(e^{x}-e^{-x}\right)^2-\left(e^{x}+e^{-x}\right)^2}{\left(e^{x}+e^{-x}\right)^2} = \frac{\left[\left(e^{x}-e^{-x}\right)-\left(e^{x}+e^{-x}\right)\right]\left[\left(e^{x}-e^{-x}\right)+\left(e^{x}+e^{-x}\right)\right]}{\left(e^{x}+e^{-x}\right)^2}$$

$$= \frac{\left(e^{x}-e^{-x}-e^{x}-e^{-x}\right)\left(e^{x}-e^{-x}+e^{x}+e^{-x}\right)}{\left(e^{x}+e^{-x}\right)^2}$$

$$= \frac{\left(-2e^{-x}\right)\left(2e^{x}\right)}{\left(e^{x}+e^{-x}\right)^2}$$

$$= \boxed{\frac{-4}{\left(e^{x}+e^{-x}\right)^2}}$$

Exercise

The exponential function $f(x) = 1066e^{0.042x}$ models the gray wolf population of the Western Great Lakes, $f(x)$, in billions, x years after 1978. Project the gray population in the recovery area in 2012.

Solution

$x = 2012 - 1978 = 34$

$f(x = 34) = 1066e^{0.042(34)}$ $\qquad$ 1066 e ^ (.042 * 34)

$= 4445.6$

$\boxed{\approx 4446}$

Exercise

The function $f(x) = 6.4e^{0.0123x}$ describes world population, $f(x)$, in billions, x years after 2004 subject to a growth rate of 1.23% annually. Use the function to predict world population in 2050.

Solution

$$x = 2050 - 2004 = 46$$

$$f(x = 46) = 6.4e^{0.0123(46)}$$

6.4 *e* ^ (0.0123 * 46)

$$\approx 11.27 \text{ billion}$$

Solution Section 1.6 – Logarithmic Functions and Properties

Exercise

Change to logarithm form $4^3 = 64$

Solution

$$4^3 = 64 \quad \Leftrightarrow \quad 3 = \log_4 64$$

Exercise

Change to logarithm form $4^{-3} = \frac{1}{64}$

Solution

$$4^{-3} = \frac{1}{64} \quad \Leftrightarrow \quad -3 = \log_4 \frac{1}{64}$$

Exercise

Change to logarithm form $3^x = 4 - t$

Solution

$$3^x = 4 - t \quad \Leftrightarrow \quad x = \log_3 (4 - t)$$

Exercise

Change to logarithm form $5^{7t} = \frac{a+b}{a}$

Solution

$$5^{7t} = \frac{a+b}{a} \quad \Leftrightarrow \quad 7t = \log_5 \frac{a+b}{a}$$

Exercise

Change to logarithm form $10^x = y + 1$

Solution

$$10^x = y + 1 \quad \Leftrightarrow \quad x = \log(y + 1)$$

Exercise

Change to logarithm form $e^7 = p$

Solution

$$e^7 = p \quad \Leftrightarrow \quad 7 = \ln p$$

Exercise

Change to logarithm form $e^{2t} = 3 - x$

Solution

$$e^{2t} = 3 - x \quad \Leftrightarrow \quad 2t = \ln(3 - x)$$

Exercise

Change to exponential form $\log_2 32 = 5$

Solution

$$\log_2 32 = 5 \quad \Leftrightarrow \quad 32 = 2^5$$

Exercise

Change to exponential form $\log_3 \frac{1}{243} = -5$

Solution

$$\log_3 \frac{1}{243} = -5 \quad \Leftrightarrow \quad \frac{1}{243} = 3^{-5}$$

Exercise

Change to exponential form $\log_3 (x + 2) = 5$

Solution

$$\log_3 (x + 2) = 5 \quad \Leftrightarrow \quad x + 2 = 3^5$$

Exercise

Change to exponential form $\log_2 m = 3x + 4$

Solution

$$\log_2 m = 3x + 4 \quad \Leftrightarrow \quad m = 2^{3x+4}$$

Exercise

Change to exponential form $\log x = 50$

Solution

$$\log x = 50 \quad \Leftrightarrow \quad x = 10^{50}$$

Exercise

Change to exponential form $\ln(z-2) = \frac{1}{6}$

Solution

$$\ln(z-2) = \frac{1}{6} \quad \Leftrightarrow \quad z - 2 = e^{1/6}$$

Exercise

Change to exponential form $\ln w = 4 + 3x$

Solution

$$\ln w = 4 + 3x \quad \Leftrightarrow \quad w = e^{4+3x}$$

Exercise

Find the number $\log_5 1$

Solution

$$\log_5 1 = 0$$

Exercise

Find the number $\log_7 7^2$

Solution

$$\log_7 7^2 = 2$$

Exercise

Find the number $3^{\log_3 8}$

Solution

$$3^{\log_3 8} = 8$$

Exercise

Find the number $10^{\log 3}$

Solution

$$10^{\log 3} = 3$$

Exercise

Find the number $e^{2+\ln 3}$

Solution

$$e^{2+\ln 3} = 22.1672$$

Exercise

Find the number $\ln e^{-3}$

Solution

$$\ln e^{-3} = -3$$

Exercise

Find $\log_5 8$ using common logarithms

Solution

$$\log_{\mathbf{5}} \mathbf{8} = \frac{\ln \mathbf{8}}{\ln \mathbf{5}} \approx 1.292$$

Exercise

Evaluate using the change of base formula (without a calculator) $\dfrac{\log_5 16}{\log_5 4}$

Solution

$$\frac{\log_5 16}{\log_5 4} = \frac{\log 4^2}{\log 4}$$

$$= \frac{2\log 4}{\log 4}$$

$$= 2$$

Exercise

Evaluate using the change of base formula (without a calculator) $\dfrac{\log_7 243}{\log_7 3}$

Solution

$$\frac{\log_7 243}{\log_7 3} = \frac{\frac{\log 3^5}{\log 7}}{\frac{\log 3}{\log 7}}$$

$$= \frac{5\log 3}{\log 3}$$

$$\underline{= 5}$$

Exercise

Sketch the graph of $f(x) = \log_4 (x-2)$

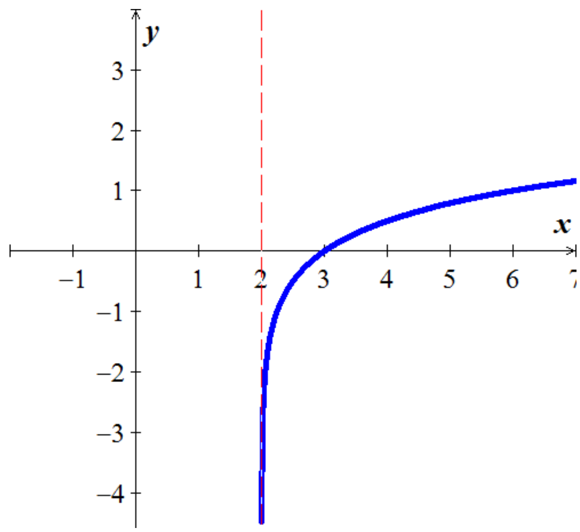
Solution

Asymptote: $x = 2$

Domain: $(2, \infty)$

Range: $(-\infty, \infty)$

x	***f(x)***
2	
2.5	−.5
3	0
4	.5



Exercise

Sketch the graph of $f(x) = \log_4 |x|$

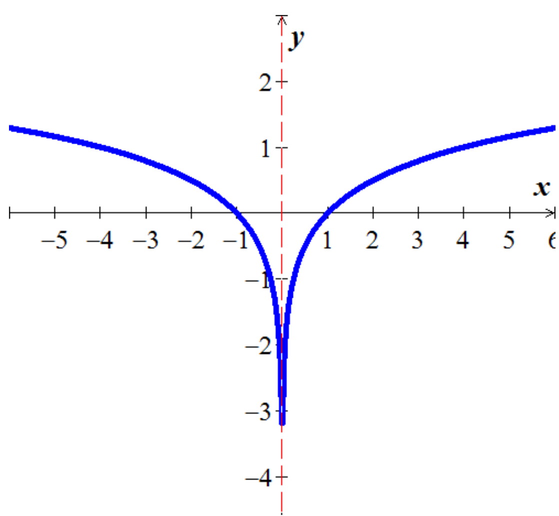
Solution

Asymptote: $x = 0$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, \infty)$

x	***f(x)***
0	
±.5	−.5
±1	0
±2	.5



Exercise

Sketch the graph of $f(x) = \left(\log_4 x\right) - 2$

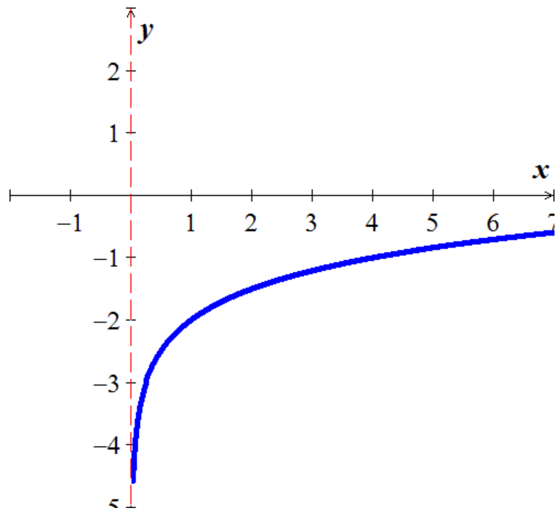
Solution

Asymptote: $x = 0$

Domain: $(0,\ \infty)$

Range: $(-\infty, \infty)$

x	$f(x)$
0	
0.5	−2.5
1	0
2	1.5



Exercise

Find the domain of $\log_5(x+4)$

Solution

$x > -4 \quad \rightarrow \quad \boxed{(-4, \infty)}$

Exercise

Find the domain of $\log_5(x+6)$

Solution

$x > -6 \quad \rightarrow \quad \boxed{(-6, \infty)}$

Exercise

Find the domain of $\log(2-x)$

Solution

$2 - x > 0$

$-x > -2 \qquad x < 2 \quad \rightarrow \quad \boxed{(-\infty, 2)}$

Exercise

Find the domain of $\log(7-x)$

Solution

$7 - x > 0 \qquad\qquad x < 7 \quad \rightarrow \quad \boxed{(-\infty, 7)}$

Exercise

Find the domain of $\ln(x-2)^2$

Solution

$$x-2\neq 0 \quad \Rightarrow \quad x\neq 2 \qquad\qquad \textbf{Domain}: (-\infty,2)\cup(2,\infty)$$

Exercise

Find the domain of $\ln(x-7)^2$

Solution

$$x-7\neq 0 \quad \Rightarrow \quad \boxed{x\neq 7}$$

Exercise

Find the domain of $\log\left(x^2-4x-12\right)$

Solution

$$x^2-4x-12\neq 0 \Rightarrow \boxed{x\neq -2, 6} \qquad (-\infty,-2)\cup(-2,6)\cup(6,\infty)$$

Exercise

Find the domain of $\log\left(\frac{x-2}{x+5}\right)$

Solution

$$\begin{cases} x\neq 2 \\ x\neq -5 \end{cases} \qquad (-\infty,-5)\cup(2,\infty)$$

-5	0	2
+	-	+

Exercise

Express $\log_a \frac{x^3 w}{y^2 z^4}$ in terms of logarithms of x, y, z, and w.

Solution

$$\log_a \frac{x^3 w}{y^2 z^4} = \log_a x^3 w - \log_a y^2 z^4 \qquad \textit{Quotient rule}$$

$$= \log_a x^3 + \log_a w - \left(\log_a y^2 + \log_a z^4\right) \qquad \textit{Product rule}$$

$$= \log_a x^3 + \log_a w - \log_a y^2 - \log_a z^4 \qquad \textit{Distribute minus}$$

$$= 3\log_a x + \log_a w - 2\log_a y - 4\log_a z \qquad \textit{Power rule}$$

Exercise

Express $\log_a \dfrac{\sqrt{y}}{x^4 \sqrt[3]{z}}$ in terms of logarithms of x, y, and z.

Solution

$$\log_a \frac{\sqrt{y}}{x^4 \sqrt[3]{z}} = \log_a y^{1/2} - \log_a x^4 z^{1/3} \qquad \textit{Quotient rule}$$

$$= \log_a y^{1/2} - \left(\log_a x^4 + \log_a z^{1/3}\right) \qquad \textit{Product rule}$$

$$= \log_a y^{1/2} - \log_a x^4 - \log_a z^{1/3} \qquad \textit{Distribute minus}$$

$$= \frac{1}{2}\log_a y - 4\log_a x - \frac{1}{3}\log_a z \qquad \textit{Power rule}$$

Exercise

Express $\ln \sqrt[4]{\dfrac{x^7}{y^5 z}}$ in terms of logarithms of x, y, and z.

Solution

$$\ln \sqrt[4]{\frac{x^7}{y^5 z}} = \ln\left(\frac{x^7}{y^5 z}\right)^{1/4}$$

$$= \frac{1}{4}\ln\left(\frac{x^7}{y^5 z}\right) \qquad \textit{Power rule}$$

$$= \frac{1}{4}\left(\ln x^7 - \ln y^5 z\right) \qquad \textit{Quotient rule}$$

$$= \frac{1}{4}\left(\ln x^7 - \left(\ln y^5 + \ln z\right)\right) \qquad \textit{Product rule}$$

$$= \frac{1}{4}\left(\ln x^7 - \ln y^5 - \ln z\right)$$

$$= \frac{1}{4}\left(7\ln x - 5\ln y - \ln z\right) \qquad \textit{Power rule}$$

$$= \frac{7}{4}\ln x - \frac{5}{4}\ln y - \ln z$$

Exercise

Express $\ln x \sqrt[3]{\dfrac{y^4}{z^5}}$ in terms of logarithms of x, y, and z.

Solution

$$\ln x \sqrt[3]{\frac{y^4}{z^5}} = \ln x + \ln\left(\frac{y^4}{z^5}\right)^{1/3} \qquad \textit{Product rule}$$

$$= \ln x + \ln\left(\frac{y^{4/3}}{z^{5/3}}\right)$$

$$= \ln x + \ln y^{4/3} - \ln z^{5/3} \qquad \textit{Quotient rule}$$

$$\underline{= \ln x + \frac{4}{3}\ln y - \frac{5}{3}\ln z} \qquad \textit{Power rule}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b\left(\frac{x^3 y}{z^2}\right)$

Solution

$$\log_b\left(\frac{x^3 y}{z^2}\right) = \log_b\left(x^3 y\right) - \log_b z^2$$

$$= \log_b x^3 + \log_b y - \log_b z^2$$

$$\underline{= 3\log_b x + \log_b y - 2\log_b z}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b\left(\frac{\sqrt[3]{x}\, y^4}{z^5}\right)$

Solution

$$\log_b\left(\frac{\sqrt[3]{x}\, y^4}{z^5}\right) = \log_b\left(\sqrt[3]{x}\, y^4\right) - \log_b\left(z^5\right)$$

$$= \log_b\left(x^{1/3}\right) + \log_b\left(y^4\right) - \log_b\left(z^5\right)$$

$$\underline{= \frac{1}{3}\log_b x + 4\log_b y - 5\log_b z}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log\left(\frac{100x^3\, \sqrt[3]{5-x}}{3(x+7)^2}\right)$

Solution

$$\log\left(\frac{100x^3\, \sqrt[3]{5-x}}{3(x+7)^2}\right) = \log\left(100x^3\, \sqrt[3]{5-x}\right) - \log\left(3(x+7)^2\right)$$

$$= \log 10^2 + \log x^3 + \log(5-x)^{1/3} - \left[\log 3 + \log\left((x+7)^2\right)\right]$$

$$= 2\log 10 + 3\log x + \frac{1}{3}\log(5-x) - \log 3 - 2\log(x+7)$$

$$= 2 + 3\log x + \frac{1}{3}\log(5-x) - \log 3 - 2\log(x+7)$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \sqrt[4]{\frac{m^8\ n^{12}}{a^3\ b^5}}$

Solution

$$\log_a \sqrt[4]{\frac{m^8\ n^{12}}{a^3\ b^5}} = \log_a \left(\frac{m^8 n^{12}}{a^3 b^5}\right)^{1/4} \qquad \textit{Power Rule}$$

$$= \frac{1}{4}\log_a \left(\frac{m^8 n^{12}}{a^3 b^5}\right) \qquad \textit{Quotient Rule}$$

$$= \frac{1}{4}\left[\log_a m^8 n^{12} - \log_a a^3 b^5\right] \qquad \textit{Product Rule}$$

$$= \frac{1}{4}\left[\log_a m^8 + \log_a n^{12} - \left(\log_a a^3 + \log_a b^5\right)\right] \qquad \textit{Power Rule}$$

$$= \frac{1}{4}\left[8\log_a m + 12\log_a n - 3 - 5\log_a b\right]$$

$$= 2\log_a m + 3\log_a n - \frac{3}{4} - \frac{5}{4}\log_a b$$

Exercise

Use the properties of logarithms to rewrite: $\log_p \sqrt[3]{\frac{m^5 n^4}{t^2}}$

Solution

$$\log_p \sqrt[3]{\frac{m^5 n^4}{t^2}} = \log_p \left(\frac{m^5 n^4}{t^2}\right)^{1/3} \qquad \textit{Power Rule}$$

$$= \frac{1}{3}\log_p \left(\frac{m^5 n^4}{t^2}\right) \qquad \textit{Quotient Rule}$$

$$= \frac{1}{3}\left(\log_p m^5 n^4 - \log_p t^2\right) \qquad \textit{Product Rule}$$

$$= \frac{1}{3}\left(\log_p m^5 + \log_p n^4 - \log_p t^2\right) \qquad \textit{Power Rule}$$

$$= \frac{1}{3}\left(5\log_p m + 4\log_p n - 2\log_p t\right)$$

$$= \frac{5}{3}\log_p m + \frac{4}{3}\log_p n - \frac{2}{3}\log_p t$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \sqrt[n]{\frac{x^3 y^5}{z^m}}$

Solution

$$\log_b \sqrt[n]{\frac{x^3 y^5}{z^m}} = \log_b \left(\frac{x^3 y^5}{z^m}\right)^{1/n}$$

$$= \frac{1}{n}\log_b \left(\frac{x^3 y^5}{z^m}\right) \qquad \textit{Power Rule}$$

$$= \frac{1}{n}\left(\log_b x^3 y^5 - \log_b z^m\right) \qquad \textit{Quotient Rule}$$

$$= \frac{1}{n}\left(\log_b x^3 + \log_b y^5 - \log_b z^m\right) \qquad \textit{Product Rule}$$

$$= \frac{1}{n}\left(3\log_b x + 5\log_b y - m\log_b z\right) \qquad \textit{Power Rule}$$

$$= \frac{3}{n}\log_b x + \frac{5}{n}\log_b y - \frac{m}{n}\log_b z$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \sqrt[3]{\frac{a^2 b}{c^5}}$

Solution

$$\log_a \sqrt[3]{\frac{a^2 b}{c^5}} = \log_a \left(\frac{a^2 b}{c^5}\right)^{1/3} \qquad \textit{Convert the radical to power}$$

$$= \frac{1}{3}\log_a \left(\frac{a^2 b}{c^5}\right) \qquad \textit{Power Rule}$$

$$= \frac{1}{3}\left[\log_a a^2 b - \log_a c^5\right] \qquad \textit{Quotient Rule}$$

$$= \frac{1}{3}\left[\log_a a^2 + \log_a b - \log_a c^5\right] \qquad \textit{Product Rule}$$

$$= \frac{1}{3}\left[2\log_a a + \log_a b - 5\log_a c\right] \qquad \textit{Power Rule}$$

$$= \frac{2}{3}\log_a a + \frac{1}{3}\log_a b - \frac{5}{3}\log_a c$$

$$= \frac{2}{3} + \frac{1}{3}\log_a b - \frac{5}{3}\log_a c$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b\left(x^4\sqrt[3]{y}\right)$

Solution

$$\begin{aligned}\log_b\left(x^4\sqrt[3]{y}\right) &= \log_b\left(x^4\right)+\log_b\left(\sqrt[3]{y}\right)\\ &= \log_b\left(x^4\right) + \log_b\left(y^{1/3}\right)\\ &= 4\log_b(x) + \frac{1}{3}\log_b(y)\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_5\left(\frac{\sqrt{x}}{25y^3}\right)$

Solution

$$\begin{aligned}\log_5\left(\frac{\sqrt{x}}{25y^3}\right) &= \log_5\left(x^{1/2}\right)-\log_5\left(25y^3\right)\\ &= \log_5\left(x^{1/2}\right) - \left[\log_5\left(5^2\right)+\log_5\left(y^3\right)\right]\\ &= \log_5\left(x^{1/2}\right) - \log_5\left(5^2\right) - \log_5\left(y^3\right)\\ &= \frac{1}{2}\log_5(x) - 2\log_5(5) - 3\log_5(y)\\ &= \frac{1}{2}\log_5(x)-2-3\log_5(y)\end{aligned}$$

Exercise

Write as a single logarithmic $\quad 4\ln x+7\ln y-3\ln z$

Solution

$$\begin{aligned}4\ln x+7\ln y-3\ln z &= \ln x^4+\ln y^7-\ln z^3\\ &= \ln\left(x^4y^7\right)-\ln z^3\\ &= \ln\left(\frac{x^4y^7}{z^3}\right)\end{aligned}$$

Exercise

Write as a single logarithmic $\frac{1}{3}\left[5\ln(x+6)-\ln x-\ln(x^2-25)\right]$

Solution

$$\begin{aligned}\frac{1}{3}\left[5\ln(x+6)-\ln x-\ln(x^2-25)\right] &= \frac{1}{3}\left[5\ln(x+6)-\left(\ln x+\ln(x^2-25)\right)\right]\\ &= \frac{1}{3}\left[\ln(x+6)^5-\ln x(x^2-25)\right]\\ &= \frac{1}{3}\left[\ln\frac{(x+6)^5}{x(x^2-25)}\right]\\ &= \ln\left(\frac{(x+6)^5}{x(x^2-25)}\right)^{1/3}\end{aligned}$$

Exercise

Write as a single logarithmic $\frac{2}{3}\left[\ln\left(x^2-4\right)-\ln\left(x+2\right)\right]+\ln(x+y)$

Solution

$$\begin{aligned}\frac{2}{3}\left[\ln\left(x^2-4\right)-\ln\left(x+2\right)\right]+\ln(x+y) &= \frac{2}{3}\left[\ln\frac{x^2-4}{x+2}\right]+\ln(x+y)\\ &= \frac{2}{3}\left[\ln\frac{(x+2)(x-2)}{x+2}\right]+\ln(x+y)\\ &= \frac{2}{3}\ln(x-2)+\ln(x+y)\\ &= \ln(x-2)^{2/3}+\ln(x+y)\\ &= \ln(x-2)^{2/3}(x+y)\\ &= \ln(x+y)\sqrt[3]{(x-2)^2}\end{aligned}$$

Exercise

Write as a single logarithmic $\frac{1}{2}\log_b m+\frac{3}{2}\log_b 2n-\log_b m^2n$

Solution

$$\begin{aligned}\frac{1}{2}\log_b m+\frac{3}{2}\log_b 2n-\log_b m^2n &= \log_b m^{1/2}+\log_b (2n)^{3/2}-\log_b m^2n\\ &= \log_b\left(m^{1/2}(2n)^{3/2}\right)-\log_b m^2n\\ &= \log_b\frac{m^{1/2}2^{3/2}n^{3/2}}{m^2n}\end{aligned}$$

$$= \log_b \frac{2^{3/2} n^{1/2}}{m^{3/2}}$$

$$= \log_b \left(\frac{2^3 n}{m^3}\right)^{1/2}$$

$$= \log_b \sqrt{\frac{8n}{m^3}}$$

Exercise

Write the expression as a single logarithm. $\frac{1}{2}\log_y p^3q^4 - \frac{2}{3}\log_y p^4q^3$

Solution

$$\frac{1}{2}\log_y p^3q^4 - \frac{2}{3}\log_y p^4q^3 = \log_y \left(p^3q^4\right)^{1/2} - \log_y \left(p^4q^3\right)^{2/3}$$

$$= \log_y \frac{\left(p^3q^4\right)^{1/2}}{\left(p^4q^3\right)^{2/3}}$$

$$= \log_y \frac{\left(p^3\right)^{1/2}\left(q^4\right)^{1/2}}{\left(p^4\right)^{2/3}\left(q^3\right)^{2/3}}$$

$$= \log_y \frac{p^{3/2}q^2}{p^{8/3}q^2}$$

$$= \log_y \frac{p^{3/2}}{p^{8/3}}$$

$$= \log_y \frac{1}{p^{8/3-3/2}}$$

$$= \log_y \frac{1}{p^{7/6}}$$

Exercise

Write the expression as a single logarithm. $\frac{1}{2}\log_a x + 4\log_a y - 3\log_a x$

Solution

$$\frac{1}{2}\log_a x + 4\log_a y - 3\log_a x = 4\log_a y - \frac{5}{2}\log_a x$$

$$= \log_a y^4 - \log_a x^{5/2}$$

$$= \log_a y^4 - \log_a \sqrt{x^5}$$

Exercise

Write the expression as a single logarithm. $\frac{2}{3}\left[\ln\left(x^2-9\right)-\ln(x+3)\right]+\ln(x+y)$

Solution

$$\begin{aligned}
\frac{2}{3}\left[\ln\left(x^2-9\right)-\ln(x+3)\right]+\ln(x+y) &= \frac{2}{3}\ln\frac{x^2-9}{x+3}+\ln(x+y)\\
&= \frac{2}{3}\ln\frac{(x+3)(x-3)}{x+3}+\ln(x+y)\\
&= \frac{2}{3}\ln(x-3)+\ln(x+y)\\
&= \ln(x-3)^{2/3}+\ln(x+y)\\
&= \ln\left((x-3)^{2/3}(x+y)\right)\\
&= \ln\left((x+y)\sqrt[3]{(x-3)^2}\right)
\end{aligned}$$

Exercise

Write the expression as a single logarithm. $\frac{1}{4}\log_b x-2\log_b 5-10\log_b y$

Solution

$$\begin{aligned}
\frac{1}{4}\log_b x-2\log_b 5-10\log_b y &= \log_b x^{1/4}-\log_b 5^2-\log_b y^{10}\\
&= \log_b x^{1/4}-\left[\log_b 5^2+\log_b y^{10}\right]\\
&= \log_b x^{1/4}-\left[\log_b\left(5^2y^{10}\right)\right]\\
&= \log_b\frac{\sqrt[4]{x}}{5^2y^{10}}
\end{aligned}$$

Exercise

Express as one logarithm: $2\log_a x+\frac{1}{3}\log_a(x-2)-5\log_a(2x+3)$

Solution

$$\begin{aligned}
2\log_a x+\tfrac{1}{3}\log_a(x-2)-5\log_a(2x+3) &= \log_a x^2+\log_a(x-2)^{1/3}-\log_a(2x+3)^5\\
&= \log_a x^2(x-2)^{1/3}-\log_a(2x+3)^5\\
&= \log_a\frac{x^2(x-2)^{1/3}}{(2x+3)^5}
\end{aligned}$$

Exercise

Express as one logarithm: $5\log_a x - \frac{1}{2}\log_a (3x-4) - 3\log_a (5x+1)$

Solution

$$\begin{aligned}
5\log_a x - \tfrac{1}{2}\log_a (3x-4) - 3\log_a (5x+1) &= \log_a x^5 - \log_a (3x-4)^{1/2} - \log_a (5x+1)^3 \\
&= \log_a x^5 - \left[\log_a (3x-4)^{1/2} + \log_a (5x+1)^3\right] \\
&= \log_a x^5 - \left[\log_a (3x-4)^{1/2}(5x+1)^3\right] \\
&= \log_a \frac{x^5}{(3x-4)^{1/2}(5x+1)^3}
\end{aligned}$$

Exercise

Express as one logarithm: $\log\left(x^3y^2\right) - 2\log\left(x\sqrt[3]{y}\right) - 3\log\left(\frac{x}{y}\right)$

Solution

$$\begin{aligned}
\log\left(x^3y^2\right) - 2\log\left(x\sqrt[3]{y}\right) - 3\log\left(\frac{x}{y}\right) &= \log\left(x^3y^2\right) - \log\left(xy^{1/3}\right)^2 - \log\left(xy^{-1}\right)^3 \\
&= \log\left(x^3y^2\right) - \left[\log\left(x^2y^{2/3}\right) + \log\left(x^3y^{-3}\right)\right] \\
&= \log\left(x^3y^2\right) - \log\left(x^2y^{2/3}x^3y^{-3}\right) \\
&= \log\left(x^3y^2\right) - \log\left(x^5y^{-7/3}\right) \\
&= \log\left(\frac{x^3y^2}{x^5y^{-7/3}}\right) \\
&= \log\left(\frac{y^2y^{7/3}}{x^2}\right) \\
&= \log\left(\frac{y^{13/3}}{x^2}\right) \\
&= \log\left(\frac{\sqrt[3]{y^{13}}}{x^2}\right) \\
&= \log\left(\frac{y^4\sqrt[3]{y}}{x^2}\right)
\end{aligned}$$

Exercise

Express as one logarithm: $\ln y^3 + \frac{1}{3}\ln\left(x^3y^6\right) - 5\ln y$

Solution

$$\begin{aligned}
\ln y^3 + \tfrac{1}{3}\ln\left(x^3y^6\right) - 5\ln y &= \ln y^3 + \ln\left(x^3y^6\right)^{1/3} - \ln y^5 \\
&= \ln y^3 + \ln\left(x^{3/3}y^{6/3}\right) - \ln y^5 \\
&= \ln y^3 + \ln\left(xy^2\right) - \ln y^5 \\
&= \ln\left(y^3xy^2\right) - \ln y^5 \\
&= \ln\left(\frac{y^5x}{y^5}\right) \\
&= \ln x
\end{aligned}$$

Exercise

Express as one logarithm: $2\ln x - 4\ln\left(\frac{1}{y}\right) - 3\ln(xy)$

Solution

$$\begin{aligned}
2\ln x - 4\ln\left(\frac{1}{y}\right) - 3\ln(xy) &= \ln x^2 - \ln\left(\frac{1}{y}\right)^4 - \ln(xy)^3 \\
&= \ln x^2 - \left[\ln\left(y^{-4}\right) + \ln\left(x^3y^3\right)\right] \\
&= \ln x^2 - \ln\left(y^{-4}x^3y^3\right) \\
&= \ln x^2 - \ln\left(y^{-1}x^3\right) \\
&= \ln\frac{x^2}{y^{-1}x^3} \\
&= \ln\frac{y}{x}
\end{aligned}$$

Exercise

On a study by psychologists Bornstein and Bornstein, it was found that the average walking speed w, in feet per second, of a person living in a city of population P, in ***thousands***, is given by the function

$$w(P) = 0.37\ln P + 0.05$$

a) The population is 124,848. Find the average walking speed of people living in Hartford.

b) The population is 1,236,249. Find the average walking speed of people living in San Antonio.

Solution

124,848 = 124.848 ***thousand***

a) $w(P=124.848) = 0.37 \ln(124.848) + 0.05 \approx 1.8$ ft/sec

b) $w(P=1,236.249) = 0.37 \ln(1,236.249) + 0.05 \approx 2.7$ ft/sec

Exercise

The loudness of sounds is measured in a unit called a decibel. To measure with this unit, we first assign an intensity of I_0 to a very faint sound, called the threshold sound. If a particular sound has intensity *I*, then the decibel rating of this louder sound is

$$d = 10 \log \frac{I}{I_0}$$

Find the exact decibel rating of a sound with intensity $10,000 I_0$

Solution

$$d = 10 \log \frac{10000 I_0}{I_0}$$
$$= 10 \log 10000$$
$$= 40$$

Exercise

A model for advertising response is given by the function

$$N(a) = 1000 + 200 \ln a, \qquad a \ge 1$$

Where *N(a)* is the number of units sold when a is the amount spent on advertising, in thousands of dollars.

a) $N(a = 1)$

b) $N(a = 5)$

Solution

a) $N(a=1) = 1000 + 200 \ln 1 = 1000$ *units*

b) $N(a=5) = 1000 + 200 \ln 5 = 1322$ *units*

Exercise

Students in an accounting class took a final exam and then took equivalent forms of the exam at monthly intervals thereafter. The average score *S(t)*, as a percent, after *t* months was found to be given by the function

$$S(t) = 78 - 15 \log(t + 1), \quad t \ge 0$$

a) What was the average score when the students initially took the test, t = 0?

b) What was the average score after 4 months? 24 months?

Solution

a) What was the average score when the students initially took the test, t = 0?

$t = 0 \rightarrow S(t) = 78 - 15\log(0 + 1) = 78\%$

b) What was the average score after 4 months? 24 months?

After 4 months $\rightarrow S(t = 4) = 78 - 15\log(4 + 1) = 67.5\%$

24 months $\rightarrow S(t = 24) = 78 - 15\log(24 + 1) = 57\%$

Solution Section 1.7 – Exponential and Logarithmic Equations

Exercise

Solve $3^{5x-8}=9^{x+2}$

Solution

$$3^{5x-8}=\left(3^2\right)^{x+2}$$

$$3^{5x-8}=3^{2x+4}$$

$$5x-8=2x+4$$

$$5x-2x=8+4$$

$$3x=12 \qquad \textit{Divide by 3 both sides}$$

$$\boxed{x=4}$$

Exercise

Solve the equation: $7^{x+6}=7^{3x-4}$

Solution

$$x+6=3x-4$$

$$4+6=3x-x$$

$$10=2x$$

$$\boxed{x=5}$$

Exercise

Solve the equation: $2^{-100x}=(0.5)^{x-4}$

Solution

$$2^{-100x}=\left(\frac{1}{2}\right)^{x-4}$$

$$2^{-100x}=\left(2^{-1}\right)^{x-4}$$

$$2^{-100x}=2^{-x+4}$$

$$-100x=-x+4$$

$$-100x+x=4$$

$$-99x=4 \qquad \boxed{x=-\frac{4}{99}}$$

Exercise

Solve the equation: $4^{x}\left(\frac{1}{2}\right)^{3-2x}=8.\left(2^{x}\right)^{2}$

Solution

$$\left(2^{2}\right)^{x}\left(2^{-1}\right)^{3-2x}=2^{3}.2^{2x}$$

$$2^{2x}2^{2x-3}=2^{3+2x}$$

$$2^{2x+2x-3}=2^{3+2x}$$

$$2^{4x-3}=2^{3+2x}$$

$$4x-3=3+2x$$

$$4x-2x=3+3$$

$$2x=6$$

$$\boxed{x=3}$$

Exercise

Solve the equation: $5^{3x-6}=125$

Solution

$$5^{3x-6}=5^{3}$$

$$3x-6=3$$

$$3x=9$$

$$\Rightarrow \boxed{x=3}$$

Exercise

Solve the equation $e^{x^{2}}=e^{7x-12}$

Solution

$$e^{x^{2}}=e^{7x-12}$$

$$x^{2}=7x-12$$

$$x^{2}-7x+12=0$$

$$\boxed{x=3,\ 4}$$

Exercise

Solve the equation $f(x) = xe^{x} + e^{x}$

Solution

$$xe^{x} + e^{x} = 0$$

$$e^{x}(x+1) = 0$$

$$e^{x} = 0 \qquad x+1 = 0$$

$$\boxed{x = -1} \text{ (Only solution)}$$

Exercise

Solve the equation $f(x) = x^{3}\left(4e^{4x}\right) + 3x^{2}e^{4x}$

Solution

$$x^{3}\left(4e^{4x}\right) + 3x^{2}e^{4x} = 0$$

$$x^{2}e^{4x}(4x+3) = 0$$

$$x^{2} = 0 \qquad 4x+3 = 0$$

$$x = 0,\ 0 \qquad x = -\frac{3}{4}$$

The solutions are: $\boxed{x = 0,\ 0,\ -\frac{3}{4}}$

Exercise

Find the exact solution (2-decimal place approximation): $3^{x+4} = 2^{1-3x}$

Solution

$\ln 3^{x+4} = \ln 2^{1-3x}$ ***'ln' both sides***

$(x+4)\ln 3 = (1-3x)\ln 2$ ***Power Rule***

$x\ln 3 + 4\ln 3 = \ln 2 - 3x\ln 2$ ***Distribute***

$$x\ln 3 + 3x\ln 2 = \ln 2 - 4\ln 3$$

$$x(\ln 3 + 3\ln 2) = \ln 2 - 4\ln 3$$

$$\underline{x = \frac{\ln 2 - 4\ln 3}{\ln 3 + 3\ln 2} \approx -1.16}$$

Exercise

Find the exact solution (2-decimal place approximation): $3^{2-3x} = 4^{2x+1}$

Solution

$$\ln 3^{2-3x} = \ln 4^{2x+1} \qquad \textbf{'ln' both sides}$$

$$(2-3x)\ln 3 = (2x+1)\ln 4 \qquad \textbf{Power Rule}$$

$$2\ln 3 - 3x\ln 3 = 2x\ln 4 + \ln 4$$

$$-3x\ln 3 - 2x\ln 4 = \ln 4 - 2\ln 3$$

$$-x(3\ln 3 + 2\ln 4) = \ln 4 - 2\ln 3$$

$$x = -\frac{\ln 4 - 2\ln 3}{3\ln 3 + 2\ln 4}$$

$$= -\frac{\ln 4 - \ln 3^2}{\ln 3^3 + \ln 4^2}$$

$$= \frac{\ln 9 - \ln 4}{\ln 27 + \ln 16}$$

$$= \frac{\ln \frac{9}{4}}{\ln 432}$$

$$\approx 0.13$$

Exercise

Solve $\quad 7^{2x+1} = 3^{x+2}$

Solution

$$ln7^{2x+1} = ln3^{x+2}$$

$$(2x+1)ln7 = (x+2)ln3$$

$$2xln7 + ln7 = xln3 + 2ln3$$

$$2xln7 - xln3 = 2ln3 - ln7$$

$$x(2ln7 - ln3) = 2ln3 - ln7$$

$$\boxed{x = \frac{2ln3 - ln7}{2ln7 - ln3}}$$

Exercise

Solve: $4^{x+3} = 3^{-x}$

Solution

$$\ln 4^{x+3} = \ln 3^{-x}$$

$$(x+3)\ln 4 = -x\ln 3$$

$$x\ln 4 + 3\ln 4 = -x\ln 3$$

$$x\ln 4 + x\ln 3 = -3\ln 4$$

$$x(\ln 4 + \ln 3) = -3\ln 4$$

$$x = \frac{-3\ln 4}{(\ln 4 + \ln 3)}$$

$$\boxed{x \approx -1.6737}$$

Exercise

Find the exact solution (2-decimal place approximation): $2^{-x^2} = 5$

Solution

$$\ln 2^{-x^2} = \ln 5$$

$$-x^2 \ln 2 = \ln 5$$

$$x^2 = -\frac{\ln 5}{\ln 2} \quad \Rightarrow \quad \textit{No Solution}$$

Exercise

Find the exact solution (2-decimal place approximation): $2^{-x} = 8$

Solution

$$2^{-x} = 2^3$$

$$-x = 3 \qquad \boxed{x = -3}$$

Exercise

Find the exact solution (2-decimal place approximation): $\log\left(x^2+4\right) - \log(x+2) = 2 + \log(x-2)$

Solution

$$\log\left(x^2+4\right) - \log(x+2) - \log(x-2) = 2$$

$$\log\left(x^2+4\right) - \left[\log(x+2) + \log(x-2)\right] = 2$$

$$\log\left(x^2+4\right) - \log(x+2)(x-2) = 2$$

$$\log\left(\frac{x^2+4}{x^2-4}\right) = 2$$

$$\frac{x^2+4}{x^2-4} = 10^2$$

$$x^2 + 4 = 100x^2 - 400$$

$$400 + 4 = 100x^2 - x^2$$

$$99x^2 = 404$$

$$x^2 = \frac{404}{99}$$

$\boxed{x = 2.02}$ is the only solution

Exercise $x = \pm\sqrt{\frac{404}{99}} \approx \pm 2.02$

Find the exact solution (2-decimal place approximation): $5^x + 125\left(5^{-x}\right) = 30$

Solution

$$5^x 5^x + 125\left(5^{-x}\right)5^x = 30\left(5^x\right)$$

$$5^{2x} + 125 = 30\left(5^x\right)$$

$$5^{2x} - 30\left(5^x\right) + 125 = 0 \qquad \textit{Solve for } 5^x$$

$$5^x = 5 \qquad 5^x = 25 = 5^2$$

$$x = 1 \qquad x = 2$$

$$\boxed{x = 1,\ 2}$$

Exercise

Find the exact solution (2-decimal place approximation): $4^x - 3\left(4^{-x}\right) = 8$

Solution

$$4^x 4^x - 3\left(4^{-x}\right)4^x = 8\left(4^x\right)$$

$$4^{2x} - 3 = 8\left(4^x\right)$$

$$4^{2x} - 8\left(4^x\right) - 3 = 0 \qquad \textit{Solve for } 4^x$$

$$4^x = 4 + \sqrt{19} \qquad 4^x = 4 - \sqrt{19} < 0$$

$$x \ln 4 = \ln\left(4 + \sqrt{19}\right)$$

$$\underline{\left| x = \frac{\ln\left(4+\sqrt{19}\right)}{\ln 4} \approx 1.53 \right|}$$

Exercise

Solve the equation without using the calculator: $\log x^2 = (\log x)^2$

Solution

$$2\log x = (\log x)^2$$

$$(\log x)^2 - 2\log x = 0$$

$$\log x(\log x - 2) = 0$$

$$\log x = 0 \qquad\qquad \log x - 2 = 0$$

$$x = 1 \qquad\qquad \log x = 2$$

$$x = 10^2 = 100$$

$$\boxed{x = 1,\ 100}$$

Exercise

Solve the equation without using the calculator: $\log(\log x) = 2$

Solution

$\log x = 10^2$ ***Convert to exponential***

$$\boxed{x = 10^{100}}$$

Exercise

Solve the equation without using the calculator: $\log\sqrt{x^3 - 9} = 2$

Solution

$\sqrt{x^3 - 9} = 10^2$ ***Convert to exponential***

$$\left(\sqrt{x^3 - 9}\right)^2 = (100)^2$$

$$x^3 - 9 = 10000$$

$$x^3 = 10009$$

$$\boxed{x = \sqrt[3]{10,009}}$$

Exercise

Solve the equation without using the calculator: $e^{2x} + 2e^{x} - 15 = 0$

Solution

$$\left(e^{x}\right)^{2} + 2e^{x} - 15 = 0 \qquad \textit{Solve for } e^{x}$$

$$e^{x} = 3 \qquad e^{x} \cancel{=} -5 < 0$$

$$\boxed{x = \ln 3}$$

Exercise

Solve the equation: $\log_{3} x - \log_{9}(x+42) = 0$

Solution

$$\frac{\ln x}{\ln 3} - \frac{\ln(x+42)}{\ln 9} = 0$$

$$\frac{\ln x}{\ln 3} - \frac{\ln(x+42)}{\ln 3^{2}} = 0$$

$$\frac{\ln x}{\ln 3} - \frac{\ln(x+42)}{2\ln 3} = 0$$

$$\frac{\ln x}{\ln 3} - \frac{1}{2}\frac{\ln(x+42)}{\ln 3} = 0$$

$$\frac{\ln x - \ln(x+42)^{1/2}}{\ln 3} = 0$$

$$\ln x - \ln(x+42)^{1/2} = 0$$

$$\ln x = \ln(x+42)^{1/2}$$

$$x = (x+42)^{1/2}$$

$$(x)^{2} = \left((x+42)^{1/2}\right)^{2}$$

$$x^{2} = x + 42 \qquad \to x^{2} - x - 42 = 0 \Rightarrow x = -6,\ 7$$

The solution: $\boxed{x = 7}$

Exercise

Use common logarithms to solve for ***x*** in terms of ***y***: $y = \dfrac{10^x + 10^{-x}}{2}$

Solution

$$2y = 10^x + 10^{-x}$$

$$10^x\left(10^x\right) + 10^{-x}\left(10^x\right) - 2y\left(10^x\right) = 0$$

$$\left(10^x\right)^2 - 2y\left(10^x\right) + 1 = 0$$

Using the quadratic formula:

$$10^x = \frac{2y \pm \sqrt{(2y)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2y \pm \sqrt{4y^2 - 4}}{2}$$

$$= \frac{2y \pm 2\sqrt{y^2 - 1}}{2}$$

$$= y \pm \sqrt{y^2 - 1}$$

$y - \sqrt{y^2 - 1} > 0 \Rightarrow y > \sqrt{y^2 - 1} \Rightarrow y^2 > y^2 - 1$ (***True for any y*** **> 1**)

$y^2 - 1 \ge 0 \;\Rightarrow\; \cancel{y \le -1} \;\; or \;\; y \ge 1$

$$10^x = y - \sqrt{y^2 - 1} \qquad\qquad 10^x = y + \sqrt{y^2 - 1}$$

$$x = \log\left(y - \sqrt{y^2 - 1}\right) \qquad\qquad x = \log\left(y + \sqrt{y^2 - 1}\right)$$

Exercise

Use common logarithms to solve for ***x*** in terms of ***y***: $y = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$

Solution

$$y\left(10^x + 10^{-x}\right) = 10^x - 10^{-x}$$

$$y10^x + y10^{-x} = 10^x - 10^{-x}$$

$$y10^x - 10^x = -10^{-x} - y10^{-x}$$

$$10^x(y - 1) = -10^{-x}(1 + y)$$

$$10^x 10^x(y - 1) = -10^x 10^{-x}(1 + y)$$

$$\left(10^x\right)^2(y - 1) = -(1 + y)$$

$$\left(10^x\right)^2 = -\frac{y+1}{y-1}$$

$$\left(10^x\right)^2 = \frac{y+1}{1-y}$$

$$10^x = \left(\frac{y+1}{1-y}\right)^{1/2}$$

$$x = \log\left(\frac{y+1}{1-y}\right)^{1/2}$$

Exercise

Use natural logarithms to solve for ***x*** in terms of ***y***: $y = \frac{e^x - e^{-x}}{2}$

Solution

$$2y = e^x - e^{-x}$$

$$2ye^x = e^x e^x - e^{-x}e^x$$

$$2ye^x = \left(e^x\right)^2 - 1$$

$$\left(e^x\right)^2 - 2ye^x - 1 = 0$$

Using the quadratic formula: $e^x = \frac{2y \pm \sqrt{(2y)^2 - 4(1)(-1)}}{2(1)} = \frac{2y \pm \sqrt{4y^2+4}}{2} = \frac{2y \pm 2\sqrt{y^2+1}}{2}$

$$e^x = y \pm \sqrt{y^2+1}$$

$e^x = y - \sqrt{y^2+1} < 0$ (***not a solution***)

$$e^x = y + \sqrt{y^2+1} \Rightarrow \boxed{x = \ln\left(y + \sqrt{y^2+1}\right)}$$

Exercise

Use natural logarithms to solve for ***x*** in terms of ***y***: $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Solution

$$ye^x + ye^{-x} = e^x - e^{-x}$$

$$ye^{-x} + e^{-x} = e^x - ye^x$$

$$(y+1)e^{-x} = (1-y)e^x$$

$$(y+1)e^{-x}e^x = (1-y)e^x e^x$$

$$y+1=(1-y)\left(e^x\right)^2$$

$$\left(e^x\right)^2=\frac{y+1}{1-y}$$

$$e^x=\sqrt{\frac{y+1}{1-y}} \qquad \Rightarrow \quad \boxed{x=\ln\sqrt{\frac{y+1}{1-y}}}$$

Exercise

Solve: $ln\sqrt[4]{x}=\sqrt{lnx}$

Solution

$$\ln x^{1/4}=\sqrt{\ln x}$$

$$\frac{1}{4}\ln x=\sqrt{\ln x}$$

$$\left(\frac{1}{4}\ln x\right)^2=\left(\sqrt{\ln x}\right)^2$$

$$\frac{1}{6}\ln^2 x=\ln x$$

$$\ln^2 x=6\ln x$$

$$\ln^2 x-6\ln x=0$$

$$\ln x(\ln x-6)=0 \qquad \begin{cases} \ln x=0 \rightarrow \boxed{x=1} \\ \ln x-6=0 \Rightarrow \ln x=6 \rightarrow \boxed{x=e^6} \end{cases}$$

Exercise

Solve: $\sqrt{lnx}=ln\sqrt{x}$

Solution

$$\sqrt{\ln x}=\ln x^{1/2}$$

$$\sqrt{\ln x}=\frac{1}{2}\ln x$$

$$\left(\sqrt{\ln x}\right)^2=\left(\frac{1}{2}\ln x\right)^2$$

$$\ln x=\frac{1}{4}ln^2 x$$

$$4\ln x=ln^2 x$$

$$ln^2 x-4\ln x=0$$

$$\ln x(\ln x-4)=0 \qquad \begin{cases} \ln x=0 \rightarrow \boxed{x=1} \\ \ln x-4=0 \Rightarrow \ln x=4 \rightarrow \boxed{x=e^4} \end{cases}$$

Exercise

Solve for t using logarithms with base a: $2a^{t/3} = 5$

Solution

$$a^{t/3} = \frac{5}{2}$$

$$\log a^{t/3} = \log \frac{5}{2}$$

$$\frac{t}{3} \log a = \log \frac{5}{2}$$

$$\frac{t}{3} = \frac{\log \frac{5}{2}}{\log a}$$

$$\frac{t}{3} = \log_a \frac{5}{2}$$

$$\boxed{t = 3\log_a \frac{5}{2}}$$

Exercise

Solve for t using logarithms with base a: $K = H - Ca^t$

Solution

$$Ca^t = H - K$$

$$a^t = \frac{H-K}{C}$$

$$\log a^t = \log \frac{H-K}{C}$$

$$t \log a = \log \frac{H-K}{C}$$

$$t = \frac{\log \frac{H-K}{C}}{\log a} = \log_a \frac{H-K}{C}$$

Exercise

Solve the equation: $\log_4 x = \log_4 (8-x)$

Solution

$$x = 8 - x$$

$$x + x = 8$$

$$2x = 8 \rightarrow x = 4$$

Check: $\boxed{x = 4}$

Exercise

Solve the equation: $\log_7(x-5)=\log_7(6x)$

Solution

$$x-5=6x$$
$$x-6x=5$$
$$-5x=5 \ \Rightarrow x=-1$$

Check: $\log_7(-1-5)=\log_7(6(-1))$

No solution (no negative inside the log)

Exercise

Solve the equation: $\ln x^2=\ln(12-x)$

Solution

$$\ln x^2=\ln(12-x)$$
$$x^2=12-x$$
$$x^2+x-12=0 \quad \rightarrow \quad x=-4,\ 3$$

Check: $x=-4 \quad \Rightarrow \quad \ln(-4)^2=\ln(12+4)$

$x=3 \quad \Rightarrow \quad \ln(3)^2=\ln(12-3)$

The solutions are: $\boxed{x=-4,\ 3}$

Exercise

Solve the equation: $e^{x\ln 3}=27$

Solution

$$\ln e^{x\ln 3}=\ln 27$$
$$x\ln 3=\ln 3^3$$
$$x=\frac{3\ln 3}{\ln 3}=3$$

Exercise

Solve the equation $\log_6(2x-3)=\log_6 12-\log_6 3$

Solution

$$\log_6(2x-3)=\log_6\frac{12}{3}$$

$$\log_6(2x-3)=\log_6 4$$

$$2x-3=4$$

$$2x=7$$

$$\boxed{x=\frac{7}{2}}\ \ \textbf{Check}$$

Exercise

Solve the equation $\ln(-4-x)+\ln 3=\ln(2-x)$

Solution

$$\ln 3(-4-x)=\ln(2-x)$$

$$-12-3x=2-x$$

$$-12-2=3x-x$$

$$-14=2x$$

$$x=-7$$

Check: $\ln(-4-(-7))+\ln 3=\ln(2-(-7))$

$$\ln(3)+\ln 3=\ln(9)$$

$$\ln 3(3)=\ln(9)$$

The solution is $\boxed{x=-7}$

Exercise

Solve the equation $\log_2(x+7)+\log_2 x=3$

Solution

$$\log_2 x(x+7)=3$$

$$x(x+7)=2^3$$

Convert to Exponential Form

$$x^2+7x=8$$

$$x^2+7x-8=0 \quad \Rightarrow \quad x=1,\ \ -8$$

Check: $x=1 \Rightarrow \quad \log_2(1+7)+\log_2 1=3 \rightarrow \log_2 8=3$

$$x=-8 \Rightarrow \quad \log_2(-8+7)+\cancel{\log_2(-8)}=3$$

The solution is $\boxed{x=1}$

Exercise

Solve the equation $\log_3(x+3)+\log_3(x+5)=1$

Solution

$$\log_3(x+3)(x+5)=1$$

$$x^2+3x+5x+15=3^1 \qquad \textit{Convert to Exponential Form}$$

$$x^2+8x+15-3=0$$

$$x^2+8x+12=0 \qquad \to \underline{x=-2,\ -6|}$$

Check: $x=-2 \Rightarrow \log_3(1)+\log_3(3)=1$

$x=-6 \Rightarrow \log_3(-6+3)+\log_3(-6+5)=1$ $x=-2 \Rightarrow \log_3(-2+3)+\log_3(-2+5)=1$

$$\cancel{\log_3(-3)}+\log_3(-1)=1$$

The solution is $\boxed{x=-2}$

Exercise

Solve the equation $\ln x=1-\ln(x+2)$

Solution

$$\ln x+\ln(x+2)=1$$

$$\ln x(x+2)=1$$

$$x^2+2x=e^1 \qquad \textit{Convert to Exponential Form}$$

$$x^2+2x-e=0$$

$$x=\frac{-2\pm\sqrt{4+4e}}{2}=\frac{-2\pm2\sqrt{1+e}}{2}=\begin{cases}-1-\sqrt{1+e}<0\\ -1+\sqrt{1+e}=0.923\end{cases}$$

The solution is $\underline{x=-1+\sqrt{1+e}|}$

Exercise

Solve the equation $\ln x=1+\ln(x+1)$

Solution

$$\ln x-\ln(x+1)=1$$

$$\ln\frac{x}{x+1}=1$$

$$\frac{x}{x+1}=e^1$$

$$x=(x+1)e$$

$$x=ex+e$$

$$x - ex = e$$
$$x(1-e) = e$$
$$x = \frac{e}{1-e} < 0 \qquad \therefore \textbf{\textit{No solution}}$$

Exercise

Solve the equation $\log_3(x-2) = \log_3 27 - \log_3(x-4) - 5^{\log_5 1}$

Solution

$$\log_3(x-2) + \log_3(x-4) = \log_3 3^3 - 1$$
$$\log_3(x-2)(x-4) = 3 - 1$$
$$\log_3\left(x^2 - 6x + 8\right) = 2$$
$$x^2 - 6x + 8 = 3^2$$
$$x^2 - 6x + 8 = 9$$
$$x^2 - 6x - 1 = 0 \qquad \rightarrow \quad x = 3 \pm \sqrt{10}$$

Check: $x = 3 + \sqrt{10} \quad \Rightarrow \log_3\left(3 + \sqrt{10} - 2\right) = \log_3 27 - \log_3\left(3 + \sqrt{10} - 4\right) - 5^{\log_5 1}$

$x = 3 + \sqrt{10} \quad \Rightarrow \cancel{\log_3\left(3 - \sqrt{10} - 2\right)} = \log_3 27 - \log_3\left(3 - \sqrt{10} - 4\right) - 5^{\log_5 1}$

The solution is $\boxed{x = 3 + \sqrt{10}}$

Exercise

Solve the equation $\log_2(x+3) = \log_2(x-3) + \log_3 9 + 4^{\log_4 3}$

Solution

$$\log_2(x+3) - \log_2(x-3) = 2 + 3$$
$$\log_2 \frac{x+3}{x-3} = 5$$
$$\frac{x+3}{x-3} = 2^5$$
$$x + 3 = 32(x-3)$$
$$x + 3 = 32x - 96$$
$$96 + 3 = 32x - x$$
$$31x = 99$$
$$x = \frac{99}{31} > 3 \qquad \textbf{\textit{Domain}}: \ x > 3$$

The solution is: $\boxed{x = \frac{99}{31}}$

Exercise

Solve $\log_5(x-7)=2$

Solution

$$x-7=5^2$$
$$x=25+7$$
$$\boxed{x=32}$$

Exercise

Solve $\log_5 x+\log_5(4x-1)=1$

Solution

$$\log_5 x(4x-1)=1$$
$$x(4x-1)=5^1$$
$$4x^2-x=5$$
$$4x^2-x-5=0 \quad \to \begin{cases} x=-1 \\ x=\frac{5}{4} \end{cases} \to \textit{Check} \quad \boxed{x=\frac{5}{4}} \quad \textit{only solution}$$

Exercise

Solve: $\log x+\log(x-3)=1$

Solution

$$\log[x(x-3)]=1$$
$$x(x-3)=10^1=10$$
$$x^2-3x=10$$
$$x^2-3x-10=0$$
$$(x+2)(x-5)=0 \qquad \Rightarrow x=-2,\ 5$$

Check: $x=-2 \Rightarrow \cancel{\log(-2)+\log(x-3)=1}$

$x=5 \quad \Rightarrow \log(5)+\log(5-3)=1$

Exercise

Solve: $\log x-\log(x+3)=1$

Solution

$$\log\frac{x}{x+3}=1$$

$$\frac{x}{x+3} = 10^{1} = 10$$

$$x = 10x + 30$$

$$9x = -30$$

$$x = -\frac{10}{3}$$ ***No Solution***

Exercise

Solve: $\log_3 x = -2$

Solution

$$x = 3^{-2}$$ ***Convert to exponential***

$$x = \frac{1}{3^2}$$

$$\boxed{x = \frac{1}{9}}$$

Exercise

Solve: $\log(3x+2) + \log(x-1) = 1$

Solution

$$\log(3x+2) + \log(x-1) = 1$$ ***Product Rule***

$$\log\left[(3x+2)(x-1)\right] = 1$$ ***Convert to exponential form***

$$(3x+2)(x-1) = 10^1$$

$$3x^2 - x - 2 = 10$$

$$3x^2 - x - 12 = 0$$ ***Solve for x***

$$x = \frac{1-\sqrt{145}}{6} < 0 \qquad x = \frac{1+\sqrt{145}}{6} > 1$$

Solution: $\boxed{x = \frac{1+\sqrt{145}}{6}}$

Exercise

Solve: $\log_5(x+2) + \log_5(x-2) = 1$

Solution

$$\log_5\left[(x+2)(x-2)\right] = 1$$

$$(x+2)(x-2) = 5^1$$

$$x^2 - 4 = 5$$

$$x^2 = 5 + 4$$

$x^2 = 9$

$x = \pm 3$

$\log_5 [(-3)+2] + \log_5 [(-3)-2] = 1$

$\log_5 [(3)+2] + \log_5 [(3)-2] = 1$

Solution: $\boxed{x = 3}$

Exercise

Solve: $\log x + \log(x-9) = 1$

Solution

$\log x(x-9) = 1$

$x(x-9) = 10^1$

$x^2 - 9x - 10 = 0$

$\Rightarrow x = -1$ (Check; it is not a solution)

$\Rightarrow x = 10$ (only solution)

Exercise

Solve: $\log_2 (x+1) + \log_2 (x-1) = 3$

Solution

$\log_2 (x+1)(x-1) = 3$

$x^2 - 1 = 2^3$

$x^2 = 8+1 = 9 \Rightarrow x = \pm 3$

Check: $x = -3 \rightarrow \log_2 (-3+1) + \log_2 (-3-1) = 3 \quad \Rightarrow$ *It is* not a *Solution*

$x = 3 \rightarrow \log_2 (3+1) + \log_2 (3-1) = 3 \quad \Rightarrow$ *Solution*

Exercise

Solve: $\log_8 (x+1) - \log_8 x = 2$

Solution

$\log_8 \left(\frac{x+1}{x}\right) = 2$

$\frac{x+1}{x} = 8^2 = 64$

$x + 1 = 64x$

$1 = 63x$

Solution: $\boxed{x = \frac{1}{63}}$

Exercise

Solve: $\log(x+6)-\log(x+2)=\log x$

Solution

$\log(x+6)-\log(x+2)=\log x$ *Quotient Rule*

$\log\frac{x+6}{x+2}=\log x$

$\frac{x+6}{x+2}=x$ *Multiply by x + 2*

$x+6=x(x+2)$

$x+6=x^2+2x$

$0=x^2+2x-x-6$

$x^2+x-6=0$ ***Solve for x*** $\rightarrow x=-3,\ 2$

Check: $x=-3\rightarrow\log(-3+6)-\log(-3+2)=\log(-3)$ *Or* **Domain**

$x=2\rightarrow\log(2+6)-\log(2+2)=\log(2)$

Solution: $\boxed{x=2}$

Exercise

Solve: $\ln(x+8)+\ln(x-1)=2\ln x$

Solution

$\ln[(x+8)(x-1)]=\ln x^2$

$(x+8)(x-1)=x^2$

$x^2-x+8x-8=x^2$

$x^2-x+8x-8-x^2=0$

$7x-8=0$

$7x=8$ $\therefore\ x=\frac{8}{7}$ ***Check***: $\ln(\frac{8}{7}+8)+\ln(\frac{8}{7}-1)=2\ln\frac{8}{7}$

Exercise

Solve: $\ln(4x+6)-\ln(x+5)=\ln x$

Solution

$\ln\left(\frac{4x+6}{x+5}\right)=\ln x$

$\frac{4x+6}{x+5}=x$

$4x+6=x(x+5)$

$4x+6=x^2+5x$

$0 = x^2 + 5x - 4x - 6$

$0 = x^2 + x - 6$

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$\Rightarrow x = -3, \quad 2$

Check: $x = -3$ *no solution* $\quad \ln(4x+6) - \ln(x+5) = \ln(-3)$

$x = 2$ *(only solution)*

Exercise

Solve: $\ln(5+4x) - \ln(x+3) = \ln 3$

Solution

$\ln \frac{5+4x}{x+3} = \ln 3$

$\frac{5+4x}{x+3} = 3$

$5 + 4x = 3(x+3)$

$5 + 4x = 3x + 9$

$4x - 3x = 9 - 5$

$x = 4$ ***Check***: $\ln(5+4(4)) - \ln((4)+3) = \ln 3$

Solution: $\boxed{x = 4}$

Exercise

Solve $\ln(x-5) - \ln(x+4) = \ln(x-1) - \ln(x+2)$

Solution

$\ln \frac{x-5}{x+4} = \ln \frac{x-1}{x+2}$

$\frac{x-5}{x+4} = \frac{x-1}{x+2}$

$(x-5)(x+2) = (x-1)(x+4)$

$x^2 + 2x - 5x - 10 = x^2 + 4x - x - 4$

$x^2 - 3x - 10 = x^2 + 3x - 4$

$x^2 - 3x - 10 - x^2 - 3x + 4 = 0$

$-6x - 6 = 0$

$-6x = 6 \Rightarrow x = -1$ *No solution*

Exercise

Solve $\ln(x-3)=\ln(7x-23)-\ln(x+1)$

Solution

$$\ln(x-3)=\ln\left(\frac{7x-23}{x+1}\right)$$

$$x-3=\frac{7x-23}{x+1}$$

$$(x-3)(x+1)=7x-23$$

$$x^2-2x-3=7x-23$$

$$x^2-9x+20=0 \qquad \Rightarrow x=4,\ 5$$

Check: $x=4 \quad \Rightarrow \ln(4-3)=\ln(7(4)-23)-\ln(4+1)$

$x=5 \quad \Rightarrow \ln(5-3)=\ln(7(5)-23)-\ln(5+1)$

Solution: $\boxed{x=4,\ 5}$

Exercise

Solve the equation $\log_4(5+x)=3$

Solution

$\log_4(5+x)=3$ *Convert to exponential.* $\boxed{x=59}$

$5+x=4^3$

$x=64-5$

$x=59$ ***Check***: $\log_4(5+59)=3$ ***True statement***

Solution: $\boxed{x=59}$

Exercise

Solve the equation $\log_5(2x+3)=\log_5 11+\log_5 3$

Solution

$\log_5(2x+3)=\log_5(11\bullet 3)$

$\log_5(2x+3)=\log_5(33)$

$2x+3=33$

$2x=30$

$x=15$ ***Check***: $\log_5(2(15)+3)=\log_5 11+\log_5 3$

Solution: $\boxed{x=15}$