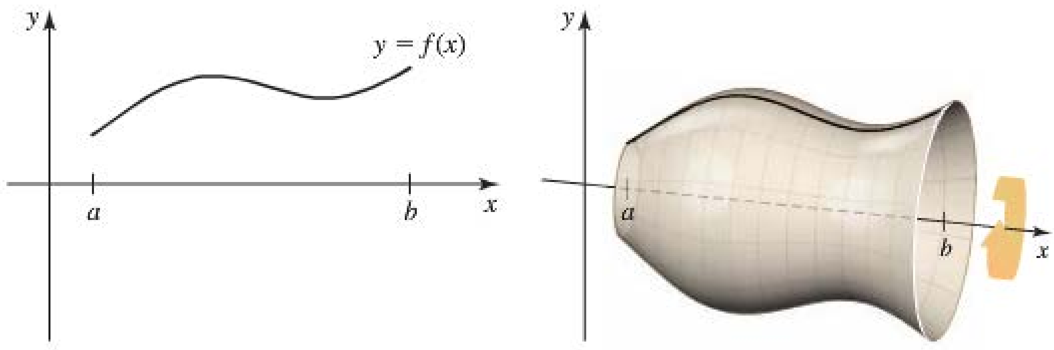
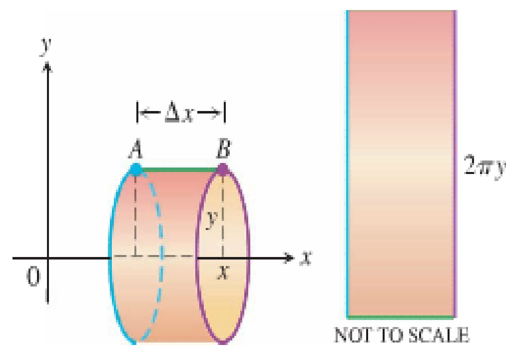
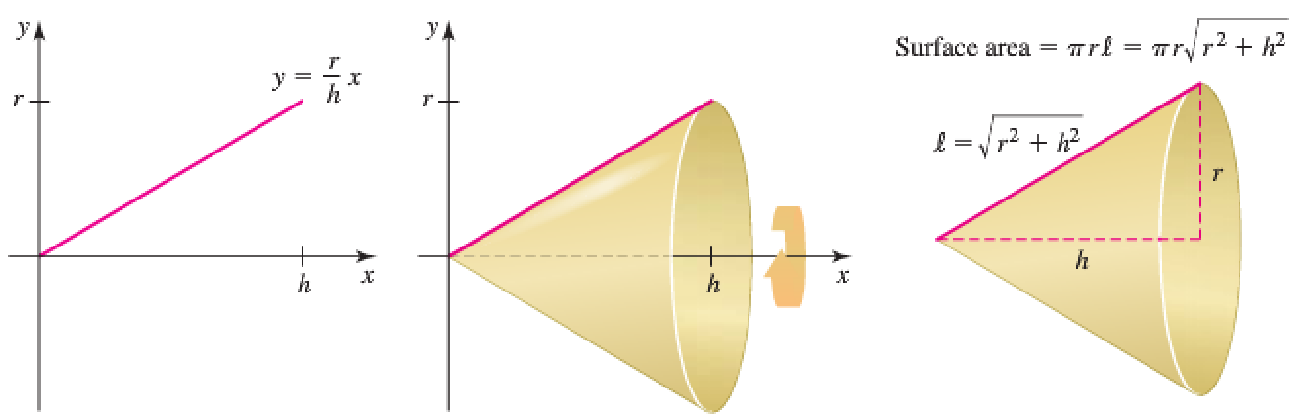
Section 1.6 – Surface Area

Consider a curve $y = f(x)$ on an interval $[a,\ b]$, where *f* is a nonnegative function with a continuous first derivative on $[a,\ b]$. Revolving the curve about the *x-axis* to generate a surface of revolution.



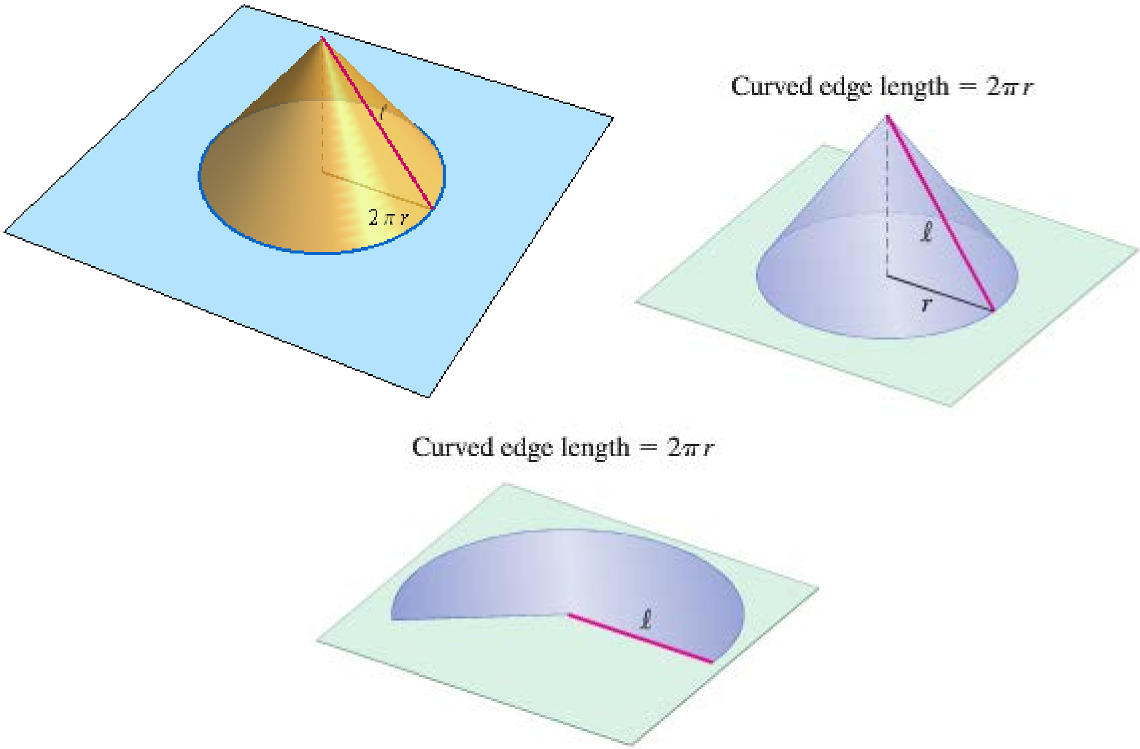


Consider the graph of $f(x) = \frac{r}{h}x$ on the interval $[0,\ h]$, where $h > 0$ and $r > 0$. When this line segment is revolved about the *x-axis*, it generates the surface of a cone of radius *r* and height *h*,



The surface area of a right circular cone, excluding the base, is $\pi r\sqrt{r^2 + h^2} = \pi r\ell$

One way to derive the formula for the surface area of a cone to cut the cone on a line from its base to its vertex. When the cone is unfolded it forms a sector of a circular disk of radius ℓ. So the area of the sector, which is also the surface area of the cone, is $\pi\ell^2 \frac{r}{\ell} = \pi r\ell$


2πr
ℓ
Curved edge length = 2πr
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r
Curved edge length = 2πr
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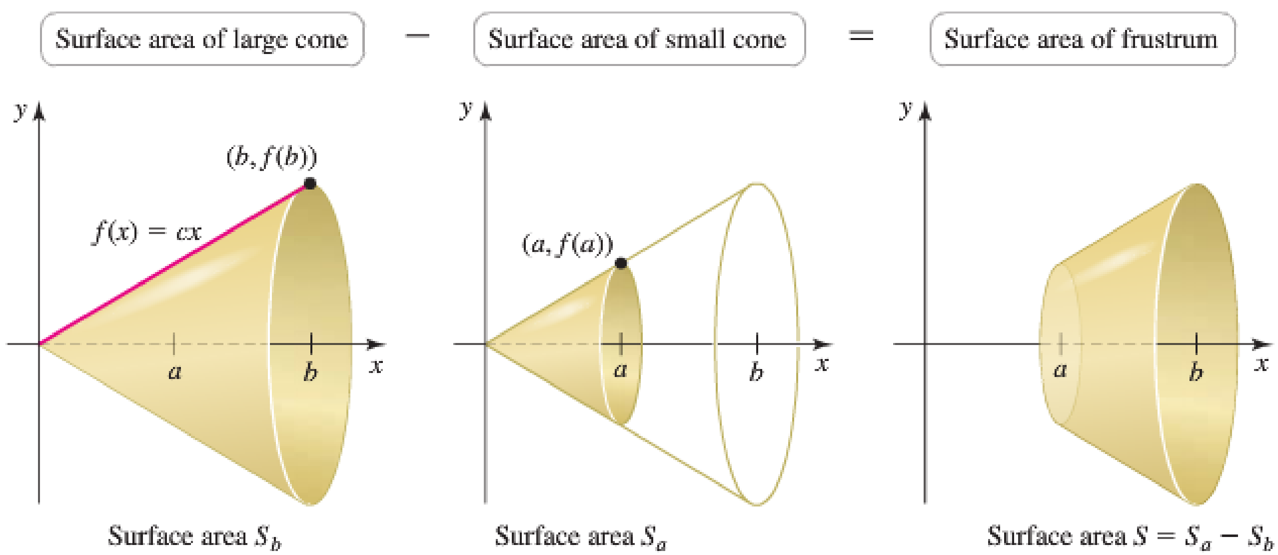

Surface area of large cone
−
Surface area of small cone
=
Surface area of frustrum
y
(b, f(b))
f(x) = cx
a
b
x
Surface area S_b
y
(a, f(a))
a
b
x
Surface area S_a
y
a
b
x
Surface area $S = S_a - S_b$

Definition

If the function $f(x) \ge 0$ is continuously differentiable on $[a, b]$, the area of the surface generated by revolving the graph of $y = f(x)$ about the *x-axis* is

$$S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\, dx$$

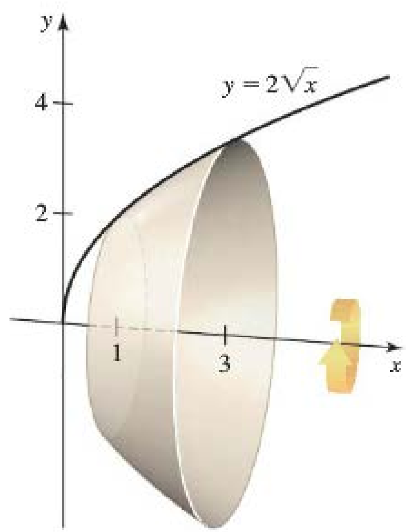
Example

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}, \quad 1 \le x \le 3$, about the *x-axis*.

Solution

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}, \quad a = 1, \quad b = 3$$

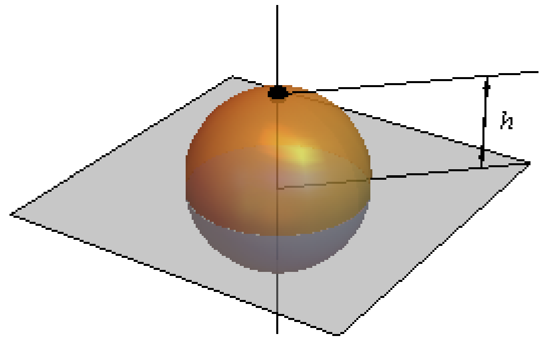
$$\begin{aligned}\sqrt{1+\left(\frac{dy}{dx}\right)^2} &= \sqrt{1+\left(\frac{1}{\sqrt{x}}\right)^2} \\ &= \sqrt{1+\frac{1}{x}} \\ &= \sqrt{\frac{x+1}{x}}\end{aligned}$$



$$\begin{aligned}S &= \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \\ &= 4\pi \int_1^3 \left(\sqrt{x}\right)\frac{\sqrt{x+1}}{\sqrt{x}}\, dx \\ &= 4\pi \int_1^3 (x+1)^{1/2}\, dx \\ &= \frac{8\pi}{3}(x+1)^{3/2}\Big|_1^3 \\ &= \frac{8\pi}{3}\left(4^{3/2} - 2^{3/2}\right) \\ &= \frac{8\pi}{3}\left(8 - 2\sqrt{2}\right) \\ &= \frac{16\pi}{3}\left(4 - \sqrt{2}\right)\ unit^2\end{aligned}$$

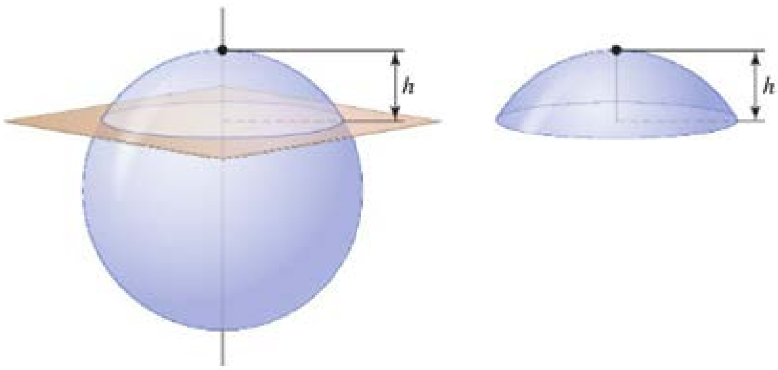
Example

A spherical cap is produced when a sphere of radius a is sliced by a horizontal plane that is a vertical distance h below the north pole of the sphere, where $0 \le h \le 2a$. We take the spherical cap to be that part of the sphere above the plane, so that h is the depth of the cap. Show that the area of a spherical cap of depth h cut from sphere of radius a is $2\pi ah$.



Solution

To generate the spherical surface, we revolved the curve $f(x) = \sqrt{a^2 - x^2}$ on the interval $[-a,\ a]$ about the x-$axis$.



The spherical cap of height h corresponds to that part of the sphere on the interval $[-a+h,\ a]$ for $0 \le h \le 2a$

$$f'(x) = -x\left(a^2 - x^2\right)^{-1/2}$$

$$1 + f'(x)^2 = 1 + \frac{x^2}{a^2 - x^2} = \frac{a^2}{a^2 - x^2}$$

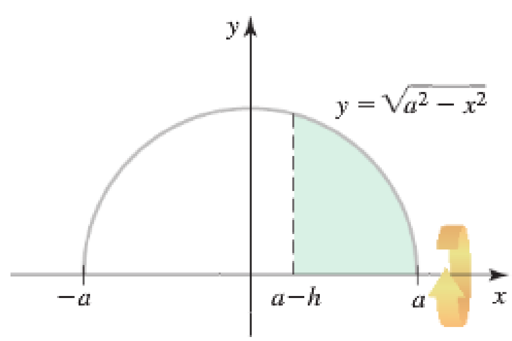
$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\,dx$$

$$= 2\pi \int_{a-h}^{a} \sqrt{a^2 - x^2}\,\frac{a}{\sqrt{a^2 - x^2}}\,dx$$

$$= 2\pi \int_{a-h}^{a} a\,dx$$

$$= 2\pi a x \Big|_{a-h}^{a}$$

$$\boxed{= 2\pi ah \quad unit^2}$$



Surface Area for revolution about the y-axis

If $x = g(y) \ge 0$ is continuously differentiable on [*c*, *d*], the area of the surface generated by revolving the graph of $x = g(y)$ about the *y*-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + \left(g'(y)\right)^2}\, dy$$

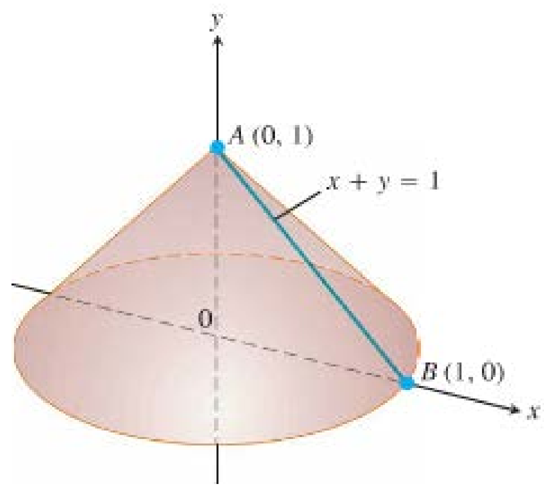
Example

The line segment $x = 1 - y$, $0 \le y \le 1$, is revolved about the *y*-axis to generate the cone. Find its lateral surface area (which excludes the base area)

Solution

$$Lateral\ Surface\ Area = \frac{base\ circumference}{2} \times slant\ height\ = \pi\sqrt{2}$$

$$x = 1 - y \quad \frac{dx}{dy} = -1, \quad c = 0, \quad d = 1$$



$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_0^1 2\pi(1 - y)\sqrt{1 + (-1)^2}\, dy$$

$$= 2\pi \int_0^1 (1 - y)\sqrt{2}\, dy$$

$$= 2\pi\sqrt{2}\left[y - \frac{y^2}{2}\right]_0^1$$

$$= 2\pi\sqrt{2}\left(1 - \frac{1}{2}\right)$$

$$= \pi\sqrt{2}\ \ unit^2$$

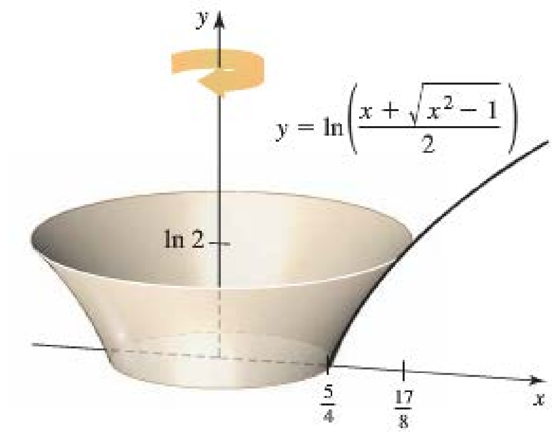
Example

Consider the function $y = \ln\left(\dfrac{x+\sqrt{x^2-1}}{2}\right)$

Find the area of the surface generated when the part of the curve between the points $\left(\frac{5}{4},\ 0\right)$ and $\left(\frac{17}{8},\ \ln 2\right)$ is revolved about *y-axis*.

Solution

$$y = \ln\left(\frac{x+\sqrt{x^2-1}}{2}\right) \quad \rightarrow \quad e^y = \frac{x+\sqrt{x^2-1}}{2}$$



$$\left(2e^y - x\right)^2 = \left(\sqrt{x^2-1}\right)^2$$

$$4e^{2y} - 4xe^y + x^2 = x^2 - 1$$

$$4xe^y = 4e^{2y} + 1 \quad \Rightarrow \quad x = e^y + \frac{1}{4}e^{-y} = g(y)$$

$$g'(y) = e^y - \frac{1}{4}e^{-y}$$

$$\sqrt{1+g'(y)^2} = \sqrt{1+\left(e^y - \frac{1}{4}e^{-y}\right)^2}$$

$$= \sqrt{1+e^{2y} - \frac{1}{2} + \frac{1}{16}e^{-2y}}$$

$$= \sqrt{e^{2y} + \frac{1}{2} + \frac{1}{16}e^{-2y}}$$

$$= \sqrt{\left(e^y + \frac{1}{4}e^{-y}\right)^2}$$

$$= e^y + \frac{1}{4}e^{-y}$$

$$S = 2\pi \int_0^{\ln 2} \left(e^y + \frac{1}{4}e^{-y}\right)^2 dy$$

$$= 2\pi \int_0^{\ln 2} \left(e^{2y} + \frac{1}{2} + \frac{1}{16}e^{-2y}\right) dy$$

$$= 2\pi \left(\frac{1}{2}e^{2y} + \frac{y}{2} - \frac{1}{32}e^{-2y}\right)\Bigg|_0^{\ln 2}$$

$$= 2\pi \left(2 + \frac{\ln 2}{2} - \frac{1}{128} - \frac{1}{2} + \frac{1}{32}\right)$$

$$= \pi\left(\frac{195}{64} + \ln 2\right) \ \ \textit{unit}^2$$

Exercises *Section* 1.6 – Surface Area

1. Find the lateral (side) surface area of the cone generated by revolving the line segment $y=\frac{x}{2}, \quad 0\le x\le 4$, about the *x*-axis. Check your answer with the geometry formula

$$Lateral\ surface\ area=\frac{1}{2}\times base\ circumference\times slant\ height$$

2. Find the lateral surface area of the cone generated by revolving the line segment $y=\frac{x}{2}, \quad 0\le x\le 4$, about the *y*-axis. Check your answer with the geometry formula

$$Lateral\ surface\ area=\frac{1}{2}\times base\ circumference\times slant\ height$$

3. Find the lateral surface area of the cone frustum generated by revolving the line segment $y=\frac{x}{2}+\frac{1}{2}, \quad 1\le x\le 3$, about the *x*-axis. Check your answer with the geometry formula

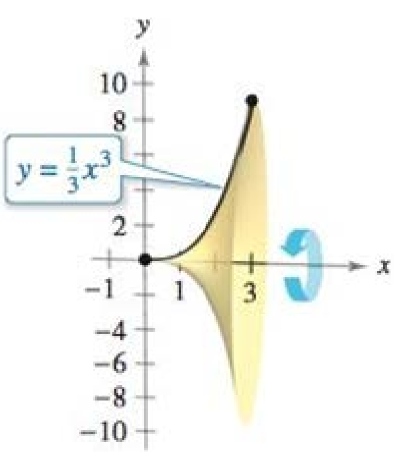
$$Frustum\ surface\ area=\pi\left(r_1+r_2\right)\times slant\ height$$

4. Find the lateral surface area of the cone frustum generated by revolving the line segment $y=\frac{x}{2}+\frac{1}{2}, \quad 1\le x\le 3$, about the *y*-axis. Check your answer with the geometry formula

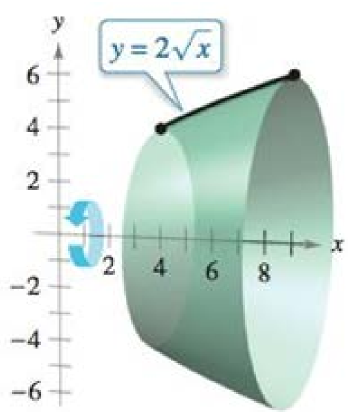
$$Frustum\ surface\ area=\pi\left(r_1+r_2\right)\times slant\ height$$

Find the area of the surface generated by revolving the curve about the *x-axis*

5.



6.



7. $y=\frac{x^3}{9}, \quad 0\le x\le 2$

8. $y=\sqrt{x+1}, \quad 1\le x\le 5$

9. $y=\sqrt{2x-x^2}, \quad 0.5\le x\le 1.5$

10. $y=3x+4, \quad 0\le x\le 6$

11. $y=12-3x, \quad 1\le x\le 3$

12. $y=x^{3/2}-\frac{1}{3}x^{1/2}, \quad 1\le x\le 2$

13. $y=\sqrt{4x+6}, \quad 0\le x\le 5$

14. $y=\frac{1}{4}\left(e^{2x}+e^{-2x}\right), \quad -2\le x\le 2$

15. $y=\frac{1}{8}x^4+\frac{1}{4x^2}, \quad 1\le x\le 2$

16. $y = 8\sqrt{x}, \quad 9 \le x \le 20$

17. $y = x^3, \quad 0 \le x \le 1$

18. $y = \frac{1}{3}x^3 + \frac{1}{4x}, \quad \frac{1}{2} \le x \le 2$

19. $y = \sqrt{5x - x^2}, \quad 1 \le x \le 4$

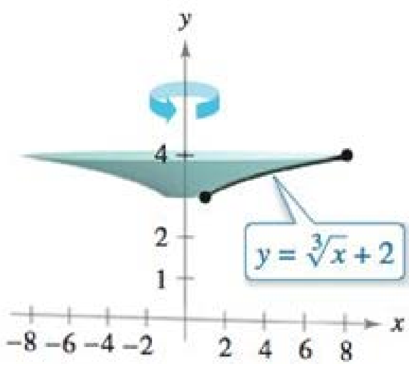
20. $y = \frac{1}{6}x^3 + \frac{1}{2x}, \quad 1 \le x \le 2$

21. $y = \sqrt{4 - x^2}, \quad -1 \le x \le 1$

22. $y = \sqrt{9 - x^2}, \quad -2 \le x \le 2$

Find the area of the surface generated by revolving the curve about the *y-axis*

23.



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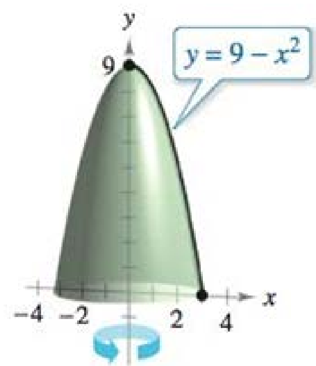


25. $y = (3x)^{1/3}; \quad 0 \le x \le \frac{8}{3}$

26. $x = \sqrt{12y - y^2}; \quad 2 \le y \le 10$

27. $x = 4y^{3/2} - \frac{1}{12}y^{1/2}; \quad 1 \le y \le 4$

28. $y = 1 - \frac{1}{4}x^2, \quad 0 \le x \le 2$

29. $y = \frac{1}{2}x + 3, \quad 1 \le x \le 5$

30. A right circular cone is generated by revolving the region bounded by $y = \frac{3}{4}x$, $y = 3$, and $x = 0$ about the *y-axis*. Find the lateral surface area of the cone.

31. A right circular cone is generated by revolving the region bounded by $y = \frac{h}{r}x$, $y = h$, and $x = 0$ about the *y-axis*. Verify that the lateral surface area of the cone is $S = \pi r\sqrt{r^2 + h^2}$

32. Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{9 - x^2}, \quad 0 \le x \le 2$, about the *y-axis*

33. Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{r^2 - x^2}, \quad 0 \le x \le a$, about the *y-axis*. Assume that $a < r$.

34. Find the area of the surface generated by part of the curve $y = 4x - 1$ between the points $(1, 3)$ and $(4, 15)$ *about y-axis*

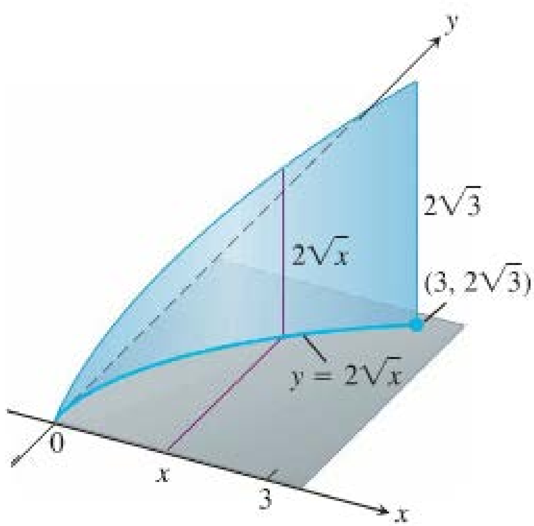
35. Find the area of the surface generated by part of the curve $y=\frac{1}{2}\ln\left(2x+\sqrt{4x^2-1}\right)$ between the points $\left(\frac{1}{2},\ 0\right)$ and $\left(\frac{17}{16},\ \ln 2\right)$ *about y-axis*

36. Find the area of the surface generated by $y=1+\sqrt{1-x^2}$ between the points $(1,\ 1)$ and $\left(\frac{\sqrt{3}}{1},\ \frac{3}{2}\right)$ *about y-axis*

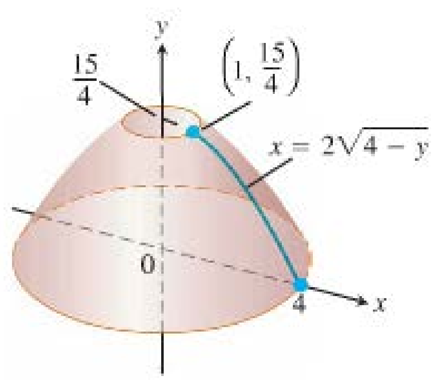
37. Find the area of the surface generated by $y=\frac{1}{3}x^3,\quad 0\le x\le 1,\quad x-axis$

38. Find the area of the surface generated by $x=\sqrt{4y-y^2},\quad 1\le y\le 2;\quad y-axis$

39. At points on the curve $y=2\sqrt{x}$, line segments of length $h=y$ are drawn perpendicular to the *xy*-plane. Find the area of the surface formed by these perpendiculars from $(0,\ 0)$ *to* $\left(3,\ 2\sqrt{3}\right)$

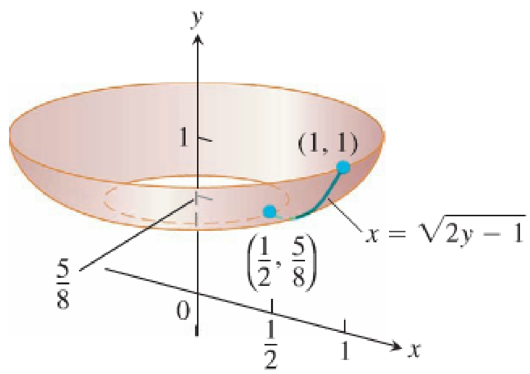


40. Find the area of the surface generated by $x=2\sqrt{4-y}\quad 0\le y\le \frac{15}{4},\quad y-axis$

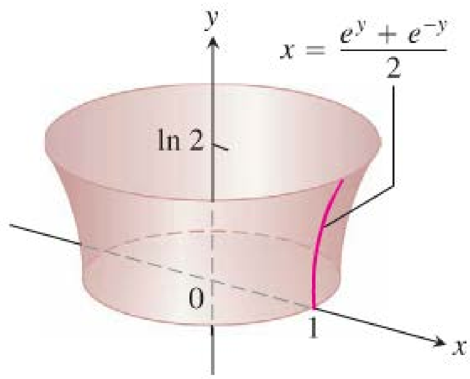


41. $y=\frac{1}{3}\left(x^2+2\right)^{3/2}$, $0\le x\le\sqrt{2}$; $y-axis$ (Hint: Express $ds=\sqrt{dx^2+dy^2}$ in terms of dy, and evaluate the integral $S=\int 2\pi y\,ds$ with appropriate limits.)

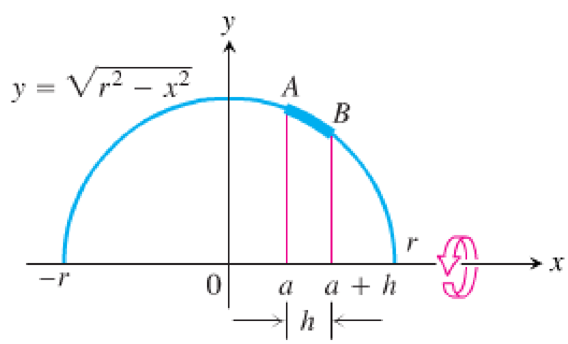
42. Find the area of the surface generated by $x=\sqrt{2y-1}$ $\frac{5}{8}\le y\le 1$, $y-axis$



43. Find the area of the surface generated by revolving the curve $x=\frac{1}{2}\left(e^y+e^{-y}\right)$, $0\le y\le\ln 2$, about y-axis



44. Did you know that if you can cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle $y=\sqrt{r^2-x^2}$ shown here is revolved about the x-axis to generate a sphere. Let AB be an arc of the semicircle that lies above an interval of length h on the x-axis. Show that the area swept out by AB does not depend on the location of the interval. (It does depend on the length of the interval.)



45. The curved surface of a funnel is generated by revolving the graph of $y = f(x) = x^3 + \frac{1}{12x}$ on the interval $[1,\ 2]$ about the x-*axis*. Approximately what volume of paint is needed to cover the outside of the funnel with a layer of paint 0.05 *cm* thick? Assume that *x* and *y* measured in centimeters.

46. When the circle $x^2 + (y-a)^2 = r^2$ on the interval $[-r,\ r]$ is revolved about the x-*axis*, the result is the surface of a torus, where $0 < r < a$. Show that the surface area of the torus is $S = 4\pi^2 ar$.

47. A 1.5–*mm* layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the curve $y = \sqrt{8x - x^2}$ on the interval $[1,\ 7]$ is revolved about the x-*axis*. Assume *x* and *y* are in *meters*.

48. A 1.5–*mm* layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the upper portion of the circle $x^2 + y^2 = 100$ on the interval $[-8,\ 8]$ is revolved about the x-*axis*. Assume *x* and *y* are in *meters*.

49. Find the surface area of a cone (excluding the base) with radius 4 and height 8 using integration and a surface area integral.

50. Let $f(x) = \frac{1}{3}x^3$ and let *R* be the region bounded by the graph of *f* and the x-*axis* on the interval $[0,\ 2]$

a) Find the area of the surface generated when the graph of *f* on $[0,\ 2]$ is revolved about the x-*axis*.

b) Find the volume of the solid generated when *R* is revolved about the y-*axis*.

c) Find the volume of the solid generated when *R* is revolved about the x-*axis*.

51. Let $f(x) = \sqrt{3x - x^2}$ and let *R* be the region bounded by the graph of *f* and the x-*axis* on the interval $[0,\ 3]$

a) Find the area of the surface generated when the graph of *f* on $[0,\ 3]$ is revolved about the x-*axis*.

b) Find the volume of the solid generated when *R* is revolved about the x-*axis*.

52. Let $f(x) = \frac{1}{2}x^4 + \frac{1}{16x^2}$ and let *R* be the region bounded by the graph of *f* and the x-*axis* on the interval $[1,\ 2]$

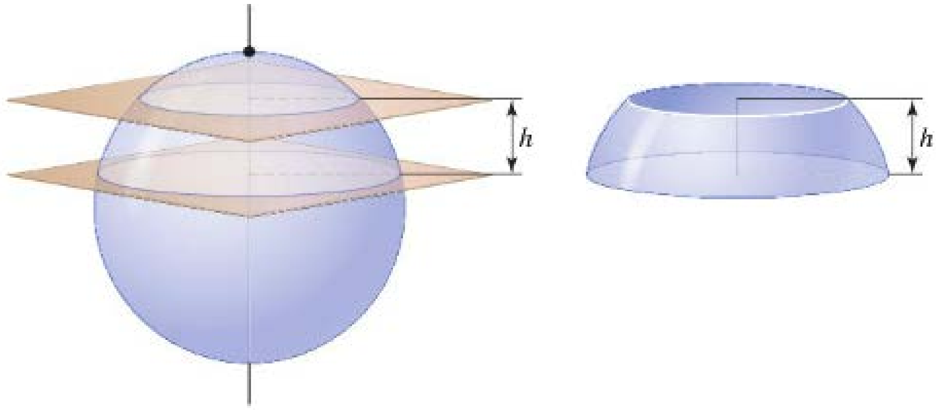
a) Find the area of the surface generated when the graph of *f* on $[1,\ 2]$ is revolved about the x-*axis*.

b) Find the length of the curve $y = f(x)$ on $[1,\ 2]$

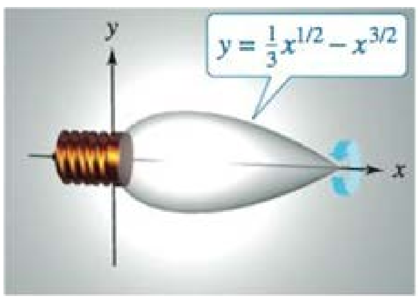
c) Find the volume of the solid generated when *R* is revolved about the y-*axis*.

d) Find the volume of the solid generated when *R* is revolved about the x-*axis*.

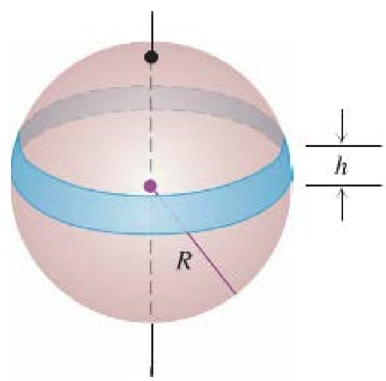
53. Suppose a sphere of radius r is sliced by two horizontal planes h units apart. Show that the surface area of the resulting zone on the sphere is $2\pi h$, independent of the location of the cutting planes.



54. An ornamental light bulb is designed by revolving the graph of $y = \frac{1}{3}x^{1/2} - x^{3/2}, \quad 0 \le x \le \frac{1}{3}$ about the x-*axis*, where x and y are mesured in *feet*. Find the surface area of the bulb and use the result to approximate the amount of glass needed to make the bulb.
(Assume that the glass is 0.015 *inch* thick)



55. The shaded band is cut from a sphere of radius R by parallel planes h units apart. Show that the surface area of the band is $2\pi Rh$



56. A drawing of a 90-*ft* dome is used by the National Weather Service. How much outside surface is there to paint (not counting the bottom)?