Solutions Section 3.5 – Introduction & Basic Theory of Linear Systems

Exercise

Determine if the system is linear, and if so determine which is homogeneous? or inhomogeneous?

$$\begin{cases} x_1' = -2x_1 + x_1 x_2 \\ x_2' = -3x_1 - x_2 \end{cases}$$

Solution

The system is nonlinear because of the term $x_1 x_2$

Exercise

Determine if the system is linear, and if so determine which is homogeneous? or inhomogeneous?

$$\begin{cases} x_1' = -x_2 \\ x_2' = \sin x_1 \end{cases}$$

Solution

The system is nonlinear because of the term $\sin x_1$

Exercise

Determine if the system is linear, and if so determine which is homogeneous? or inhomogeneous?

$$\begin{cases} x_1' = x_1 + (\sin t) x_2 \\ x_2' = 2t x_1 - x_2 \end{cases}$$

Solution

The system is linear and homogeneous, because $f_1(t) = f_2(t) = 0$

$$x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + f_1(t)$$
$$x_2' = a_{21}(t)x_1 + a_{2n}(t)x_2 + f_2(t)$$

Exercise

Write the given system of equations in matrix-form then show that the given vector is a solution to the system

$$\begin{cases} x_1' = -3x_1 + x_2 \\ x_2' = -2x_1 \end{cases} \qquad v = \left(-e^{-2t} + e^{-t}, -e^{-2t} + 2e^{-t}\right)^T$$

Solution

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_1 = -e^{-2t} + e^{-t} \qquad x_2 = -e^{-2t} + 2e^{-t}$$

$$x_1' = -3x_1 + x_2$$

$$2e^{-2t} - e^{-t} = -3\left(-e^{-2t} + e^{-t}\right) + \left(-e^{-2t} + 2e^{-t}\right)$$

$$2e^{-2t} - e^{-t} = 3e^{-2t} - 3e^{-t} - e^{-2t} + 2e^{-t}$$

$$2e^{-2t} - e^{-t} = 2e^{-2t} - e^{-t}$$

$$x_2' = \left(-e^{-2t} + 2e^{-t}\right)'$$

$$= 2e^{-2t} - 2e^{-t}$$

$$= -2\left(-e^{-2t} + e^{-t}\right)$$

$$= -2x_1$$

Exercise

Write the given system of equations in matrix-form then show that the given vector is a solution to the system

$$\begin{cases} x_1' = -x_1 + 4x_2 \\ x_2' = \quad 3x_2 \end{cases} \qquad v = \left(e^{3t} - e^{-t}, e^{3t}\right)^T$$

Solution

$$x_1 = e^{3t} - e^{-t} \qquad x_2 = e^{3t}$$

$$x_1' = 3e^{3t} + e^{-t}$$

$$= 4e^{3t} - e^{3t} + e^{-t}$$

$$= -\left(e^{3t} - e^{-t}\right) + 4e^{3t}$$

$$= -x_1 + 4x_2$$

$$x_2' = 3e^{3t} = 3x_2$$

Exercise

Verify by substitution that $x_1(t)$ and $x_2(t)$ are solutions of the given homogenous equation. Show also that the solutions $x_1(t)$ and $x_2(t)$ are linearly independent. Find the solution of the given homogeneous equation with the initial condition $x(0) = x_0$

$$x_1(t) = \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix}, \qquad x_2(t) = \begin{pmatrix} -e^{-2t} \\ e^{-2t} \end{pmatrix}$$

$$x' = \begin{pmatrix} -6 & -4 \\ 8 & 6 \end{pmatrix} x \qquad x(0) = \begin{pmatrix} -5 \\ 8 \end{pmatrix}$$

Solution

$$x_1'(t) = \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix}' = \begin{pmatrix} -2e^{2t} \\ 4e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} -6 & -4 \\ 8 & 6 \end{pmatrix} x_1 = \begin{pmatrix} -6 & -4 \\ 8 & 6 \end{pmatrix} \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} -2e^{2t} \\ 4e^{2t} \end{pmatrix} \qquad \Rightarrow \text{Therefore, } x_1 \text{ is a solution.}$$

$$x_2'(t) = \begin{pmatrix} -e^{-2t} \\ e^{-2t} \end{pmatrix}' = \begin{pmatrix} 2e^{2t} \\ -2e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} -6 & -4 \\ 8 & 6 \end{pmatrix} x_2 = \begin{pmatrix} -6 & -4 \\ 8 & 6 \end{pmatrix} \begin{pmatrix} -e^{2t} \\ e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} -2e^{2t} \\ 2e^{2t} \end{pmatrix} \qquad \Rightarrow x_2 \text{ is also a solution.}$$

$$x_1(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \qquad x_2(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$x(t) = C_1 \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} -e^{-2t} \\ e^{-2t} \end{pmatrix}$$

$$x(0) = C_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -5 \\ 8 \end{pmatrix} = \begin{pmatrix} -C_1 - C_2 \\ 2C_1 + C_2 \end{pmatrix} \qquad \Rightarrow \begin{cases} -C_1 - C_2 = -5 \\ 2C_1 + C_2 = 8 \end{cases} \rightarrow \qquad \boxed{C_1 = 3} \qquad \boxed{C_2 = 2}$$

$$x(t) = 3\begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix} + 2\begin{pmatrix} -e^{-2t} \\ e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} -3e^{2t} - 2e^{-2t} \\ 6e^{2t} + 2e^{-2t} \end{pmatrix}$$

Exercise

Verify by substitution that $x_1(t)$ and $x_2(t)$ are solutions of the given homogenous equation. Show also that the solutions $x_1(t)$ and $x_2(t)$ are linearly independent. Find the solution of the given homogeneous equation with the initial condition $x(0) = x_0$

$$x_1(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}, \qquad x_2(t) = \begin{pmatrix} e^{2t}(t+2) \\ e^{2t}(t+1) \end{pmatrix}$$

$$x' = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} x \qquad x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solution

$$x_1'(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}' = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} x_1(t) = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix} \underline{= x_1'}$$

$$x_2'(t) = \begin{pmatrix} e^{2t}(t+2) \\ e^{2t}(t+1) \end{pmatrix}'$$

$$= \begin{pmatrix} 2e^{2t}(t+2) + e^{2t} \\ 2e^{2t}(t+1) + e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2te^{2t} + 4e^{2t} + e^{2t} \\ 2te^{2t} + 2e^{2t} + e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2te^{2t} + 5e^{2t} \\ 2te^{2t} + 3e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} x_2(t) = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} te^{2t} + 2e^{2t} \\ te^{2t} + e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2te^{2t} + 5e^{2t} \\ 2te^{2t} + 3e^{2t} \end{pmatrix} \underline{= x_2'}$$

$$x_1(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad x_2(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The vectors are independent (x_1 is not a multiple of x_2), so the x_1 and x_2 are independent.

Therefore, the general solution is:

$$x(t) = C_1 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} e^{2t}(t+2) \\ e^{2t}(t+1) \end{pmatrix}$$

$$x(0) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 + 2C_2 \\ C_1 + C_2 \end{pmatrix} \quad \begin{cases} C_1 + 2C_2 = 0 \\ C_1 + C_2 = 1 \end{cases} \rightarrow \qquad \boxed{C_1 = 2} \qquad \boxed{C_2 = -1}$$

$$x(t) = 2\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} - \begin{pmatrix} te^{2t} + 2e^{2t} \\ te^{2t} + e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} -te^{2t} \\ -te^{2t} + e^{2t} \end{pmatrix}$$

Exercise

Verify by substitution that $x_1(t)$ and $x_2(t)$ are solutions of the given homogenous equation. Show also that the solutions $x_1(t)$ and $x_2(t)$ are linearly independent. Find the solution of the given homogeneous equation with the initial condition $x(0) = x_0$

$$x_1(t) = \begin{pmatrix} \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \cos t \end{pmatrix}, \qquad x_2(t) = \begin{pmatrix} \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \sin t \end{pmatrix}$$

$$x' = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} x \qquad\qquad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solution

$$x_1'(t) = \begin{pmatrix} \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \cos t \end{pmatrix}' = \begin{pmatrix} -\frac{1}{2}\sin t - \frac{1}{2}\cos t \\ -\sin t \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} x_1 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \cos t \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2}\cos t - \frac{1}{2}\sin t \\ -\sin t \end{pmatrix}$$

$$x_2'(t) = \begin{pmatrix} \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \sin t \end{pmatrix}' = \begin{pmatrix} -\frac{1}{2}\sin t + \frac{1}{2}\cos t \\ \cos t \end{pmatrix}'$$

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} x_2 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \sin t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \cos t \end{pmatrix}$$

$$x_1(0) = \begin{pmatrix} \frac{1}{2}\cos(0) - \frac{1}{2}\sin(0) \\ \cos(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \qquad x_2(0) = \begin{pmatrix} \frac{1}{2}\cos(0) + \frac{1}{2}\sin(0) \\ \sin(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

The vectors are independent (x_1 is not a multiple of x_2), so the x_1 and x_2 are independent.

Therefore, the general solution is:

$$x(t) = C_1 \begin{pmatrix} \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \cos t \end{pmatrix} + C_2 \begin{pmatrix} \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \sin t \end{pmatrix}$$

$$x(0) = C_1 \begin{pmatrix} \frac{1}{2}\cos(0) - \frac{1}{2}\sin(0) \\ \cos(0) \end{pmatrix} + C_2 \begin{pmatrix} \frac{1}{2}\cos(0) + \frac{1}{2}\sin(0) \\ \sin(0) \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}C_1 + \frac{1}{2}C_2 \\ C_1 \end{pmatrix} \Rightarrow \qquad \boxed{C_1 = 0} \qquad \boxed{C_2 = 2}$$

$$x(t) = 2\begin{pmatrix} \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \sin t \end{pmatrix} = \begin{pmatrix} \cos t + \sin t \\ 2\sin t \end{pmatrix}$$

Exercise

Rewrite the given equation into a system in normal form with initial value.

$y^{(4)} - y^{(3)} + 7y = \cos t\,; \qquad y(0) = y'(0) = 1, \quad y''(0) = 0, \quad y^{(3)}(0) = 2$

Solution

$$y_1 = y$$

$$y_2 = y_1' = y'$$

$$y_3 = y_2' = y''$$

$$y_4 = y_3' = y'''$$

$$y^{(4)} - y^{(3)} + 7y = \cos t \quad \rightarrow \quad \underline{y_4' = y_4 - 7y_1 + \cos t}$$

$$y_1(0) = y_2(0) = 1, \quad y_3(0) = 0, \quad y_4(0) = 2$$

Exercise

Rewrite the given equation into a system in normal form with initial value.

$$y^{(4)}+3y''-(\sin t)y'+8y=t^2\ , \quad y(0)=1, \quad y'(0)=2, \quad y''(0)=3, \quad y'''(0)=4$$

Solution

$$x_1=y$$

$$x_2=x_1'=y'$$

$$x_3=x_2'=y''$$

$$x_4=x_3'=y'''$$

$$x_4'=y^{(4)}$$

$$\boxed{=-3x_3+(\sin t)x_2-8x_1+t^2}$$

$$x_1(0)=1,\ \ x_2(0)=2,\ \ x_3(0)=3,\ \ x_4(0)=4$$

Exercise

Rewrite the given equation into a system in normal form with initial value.

$$y^{(6)}-(y')^3=e^{2t}-\sin y\,; \qquad y(0)=y'(0)=y''(0)=y^{(3)}(0)=y^{(4)}(0)=y^{(5)}(0)=0$$

Solution

$$x_1=y$$

$$x_2=x_1'=y'$$

$$x_3=x_2'=y''$$

$$x_4=x_3'=y'''$$

$$x_5=x_4'=y^{(4)}$$

$$x_6=x_5'=y^{(5)}$$

$$y^{(6)}-(y')^3=e^{2t}-\sin y \quad \rightarrow \quad \boxed{x_6'=x_2^3-\sin x_1+e^{2t}}$$

$$x_1(0)=x_2(0)=x_3(0)=x_4(0)=x_5(0)=x_6(0)=0$$

Exercise

Rewrite the given equation into a system in normal form with initial value.

$$\begin{cases} 3x'' = -5x + 2y \\ 4y'' = 6x - 2y \end{cases} \qquad \begin{cases} x(0) = -1, & x'(0) = 0 \\ y(0) = 1, & y'(0) = 2 \end{cases}$$

Solution

$x_1 = x$

$x_2 = x_1' = x'$

$x_3 = y$

$x_4 = x_3' = y'$

$$\begin{cases} x'' = -\frac{5}{3}x + \frac{3}{2}y \\ y'' = \frac{3}{2}x - \frac{1}{2}y \end{cases}$$

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{5}{3}x_1 + \frac{3}{2}x_3 \\ x_3' = x_4 \\ x_4' = \frac{3}{2}x_1 - \frac{1}{2}x_3 \end{cases} \qquad \begin{cases} x_1(0) = -1, & x_2(0) = 0 \\ x_3(0) = 1, & x_4(0) = 2 \end{cases}$$

Exercise

Rewrite the given equation into a system in normal form with initial value.

$$\begin{cases} x''' - y = t \\ 2x'' + 5y'' - 2y = 1 \end{cases} \qquad \begin{cases} x(0) = x'(0) = x''(0) = 4 \\ y(0) = y'(0) = 1 \end{cases}$$

Solution

$$\begin{cases} x''' = y + t \\ 5y'' = -2x'' + 2y + 1 \end{cases}$$

$x_1 = x \qquad x_2 = x_1' = x' \qquad x_3 = x_2' = x''$

$x_4 = y$

$x_5 = x_4' = y'$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 + t \\ x_4' = x_5 \\ x_5' = -\frac{2}{5}x_3 + \frac{2}{5}x_4 + \frac{1}{5} \end{cases} \qquad \begin{cases} x_1(0) = x_2(0) = x_3(0) = 4 \\ x_4(0) = x_5(0) = 1 \end{cases}$$

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $\quad x'' + 3x' + 7x = t^2$

Solution

Let $\quad x_1 = x, \quad x_2 = x' = x_1'$

Yield the system $\begin{cases} x_1' = x_2 \\ x_2' = -7x_1 - 3x_2 + t^2 \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $\quad x^{(4)} + 6x'' - 3x' + x = \cos 3t$

Solution

Let $\quad x_1 = x, \quad x_2 = x' = x_1', \quad x_3 = x'' = x_2', \quad x_4 = x''' = x_3'$

Yield the system $\begin{cases} x_1' = x_2, \quad x_2' = x_3, \quad x_3' = x_4 \\ x_4' = -x_1 + 3x_2 - 6x_3 + \cos 3t \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $\quad t^2x'' + tx' + \left(t^2 - 1\right)x = 0$

Solution

Let $\quad x_1 = x, \quad x_2 = x' = x_1'$

Yield the system $\begin{cases} x_1' = x_2 \\ t^2x_2' = \left(1 - t^2\right)x_1 - tx_2 \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $\quad t^3x^{(3)} - 2t^2x'' + 3tx' + 5x = \ln t$

Solution

Let $\quad x_1 = x, \quad x_2 = x' = x_1', \quad x_3 = x'' = x_2'$

Yield the system $\begin{cases} x_1' = x_2, \quad x_2' = x_3, \quad x_3' = x_4 \\ t^3x_3' = -5x_1 - 3tx_2 + 2t^2x_3 + \ln t \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $\quad x'' - 5x + 4y = 0, \quad y'' + 4x - 5y = 0$

Solution

Let $\quad \begin{matrix} x_1 = x & x_2 = x' = x_1' \\ y_1 = y & y_2 = y' = y_1' \end{matrix} \quad \Rightarrow \quad \begin{cases} x_1' = x_2 \\ x_2' = 5x_1 - 4y_1 \end{cases}$

$\begin{cases} y_1' = y_2 \\ y_2' = -4x_1 + 5y_1 \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $\quad x'' - 3x' + 4x - 2y = 0, \quad y'' + 2y' - 3x + y = \cos t$

Solution

Let $\quad \begin{matrix} x_1 = x & x_2 = x' = x_1' \\ y_1 = y & y_2 = y' = y_1' \end{matrix} \Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = -4x_1 + 2y_1 + 3x_2 \end{cases} \quad \begin{cases} y_1' = y_2 \\ y_2' = 3x_1 - y_1 - 2y_2 + \cos t \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $\quad x'' = 3x - y + 2z, \quad y'' = x + y - 4z, \quad z'' = 5x - y - z$

Solution

Let $\quad \begin{matrix} x_1 = x & x_2 = x' = x_1' \\ y_1 = y & y_2 = y' = y_1' \\ z_1 = z & z_2 = z' = z_1' \end{matrix} \quad \Rightarrow \quad \begin{cases} x_1' = x_2, \quad y_1' = y_2, \quad z_1' = z_2 \\ x_2' = 3x_1 - y_1 + 2z_1 \\ y_2' = x_1 + y_1 - 4z_1 \\ z_2' = 5x_1 - y_1 - z_1 \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $\quad x'' = (1-y)x, \quad y'' = (1-x)y$

Solution

Let $\begin{matrix} x_1 = x & x_2 = x' = x_1' \\ y_1 = y & y_2 = y' = y_1' \end{matrix} \quad \Rightarrow \quad \begin{cases} x_1' = x_2, \quad y_1' = y_2 \\ x_2' = (1-y_1)x_1 \\ y_2' = (1-x_1)y_1 \end{cases}$

Exercise

Prove that the general solution of

$$X' = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} X$$

On the interval $(-\infty, \infty)$ is

$$X = C_1 \begin{pmatrix} 6 \\ -1 \\ -5 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} e^{-2t} + C_3 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} e^{3t}$$

Solution

Let $A = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$$X_1 = \begin{pmatrix} 6 \\ -1 \\ -5 \end{pmatrix} e^{-t} \quad \rightarrow \quad X_1' = \begin{pmatrix} -6 \\ 1 \\ 5 \end{pmatrix} e^{-t}$$

$$AX_1 = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ -1 \\ -5 \end{pmatrix} e^{-t} = \begin{pmatrix} -6 \\ 1 \\ 5 \end{pmatrix} e^{-t} \quad \rightarrow \quad X_1' = AX_1$$

$$X_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} e^{-2t} \quad \rightarrow \quad X_2' = \begin{pmatrix} 6 \\ -2 \\ -2 \end{pmatrix} e^{-2t}$$

$$AX_2 = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} e^{-2t} = \begin{pmatrix} 6 \\ -2 \\ -2 \end{pmatrix} e^{-2t} \quad \rightarrow \quad X_2' = AX_2$$

$$X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} e^{3t} \quad \rightarrow \quad X_3' = \begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix} e^{3t}$$

$$AX_3 = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} e^{3t} = \begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix} e^{3t} \quad \rightarrow \quad X_3' = AX_3$$

$$W = \begin{vmatrix} 6 & -3 & 2 \\ -1 & 1 & 1 \\ -5 & 1 & 1 \end{vmatrix} = 20 \neq 0$$

$\therefore\ X_1,\ X_2,\ and\ X_3$ form a fundamental set for $X' = AX$ on $(-\infty,\ \infty)$

Exercise

Prove that the general solution of

$$X' = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} X + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} 4 \\ -6 \end{pmatrix} t + \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

On the interval $(-\infty,\ \infty)$ is

$$X = C_1 \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix} e^{\sqrt{2}t} + C_2 \begin{pmatrix} 1 \\ -1+\sqrt{2} \end{pmatrix} e^{-\sqrt{2}t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} -2 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solution

Let $\quad A = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$

$$X_1 = \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix} e^{\sqrt{2}t} \quad \rightarrow \quad X_1' = \begin{pmatrix} \sqrt{2} \\ -\sqrt{2}-2 \end{pmatrix} e^{\sqrt{2}t}$$

$$AX_1 = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix} e^{\sqrt{2}t} = \begin{pmatrix} \sqrt{2} \\ -\sqrt{2}-2 \end{pmatrix} e^{\sqrt{2}t} \quad \rightarrow \quad X_1' = AX_1$$

$$X_2 = \begin{pmatrix} 1 \\ -1+\sqrt{2} \end{pmatrix} e^{-\sqrt{2}t} \quad \rightarrow \quad X_2' = \begin{pmatrix} -\sqrt{2} \\ \sqrt{2}-2 \end{pmatrix} e^{-\sqrt{2}t}$$

$$AX_1 = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix} e^{\sqrt{2}t} = \begin{pmatrix} \sqrt{2} \\ -\sqrt{2}-2 \end{pmatrix} e^{\sqrt{2}t} \quad \rightarrow \quad X_1' = AX_1$$

$$X_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} -2 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \rightarrow \quad X_p' = \begin{pmatrix} 2 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$AX_p = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} -2 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} 4 \\ -6 \end{pmatrix} t + \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} t^2 - 2t + 1 \\ 4t \end{pmatrix} + \begin{pmatrix} t^2 + 4t - 1 \\ t^2 - 6t + 5 \end{pmatrix}$$

$$= \begin{pmatrix} -t^2 - 2t - 1 \\ -t^2 + 6t - 1 \end{pmatrix} + \begin{pmatrix} t^2 + 4t - 1 \\ t^2 - 6t + 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2t - 2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$= X'_p$$

$$W = \begin{vmatrix} 1 & 1 \\ -1-\sqrt{2} & -1+\sqrt{2} \end{vmatrix} = 2\sqrt{2} \neq 0$$

$\therefore\ X_1$ *and* X_2 form a fundamental set for $X' = AX$ on $(-\infty,\ \infty)$

Exercise

$$\boldsymbol{x}' = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$$

a) Verify that the given vectors are solutions of the given system.
b) Use the Wronskian to show that they are linearly independent.
c) Write the general solution of the system.

Solution

a) $\vec{x}_1{}' = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}' = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}$ $\qquad$ $\boldsymbol{x}' \vec{x}_1 = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} = \vec{x}_1{}'$ ✓

$\vec{x}_2{}' = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}' = \begin{bmatrix} 2e^{2t} \\ -2e^{2t} \end{bmatrix}$ $\qquad$ $\boldsymbol{x}' \vec{x}_2 = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ -2e^{2t} \end{bmatrix} = \vec{x}_2{}'$ ✓

b) $W = \begin{vmatrix} 2e^t & e^{2t} \\ -3e^t & -e^{2t} \end{vmatrix} = e^{3t} \neq 0$

The solutions x_1 *and* x_2 are linearly independent.

c) $\boldsymbol{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 = C_1 \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix} + C_2 \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} = \begin{pmatrix} 2C_1e^t + C_2e^{2t} \\ -3C_1e^t - C_2e^{2t} \end{pmatrix}$

Exercise

$$\mathbf{x}' = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\mathbf{x}; \quad \vec{x}_1 = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 5 \end{cases}$$

a) Verify that the given vectors are solutions of the given system.
b) Use the Wronskian to show that they are linearly independent.
c) Write the general solution of the system.
d) Find the particular solution that satisfies the given initial conditions

Solution

a) $\vec{x}_1{}' = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix}' = \begin{bmatrix} 3e^{3t} \\ 9e^{3t} \end{bmatrix}$ $\qquad \mathbf{x}' \cdot \vec{x}_1 = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ 9e^{3t} \end{bmatrix} = \vec{x}_1{}'$ ✓

$\vec{x}_2{}' = \begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix}' = \begin{bmatrix} -4e^{-2t} \\ -2e^{-2t} \end{bmatrix}$ $\qquad \mathbf{x}' \cdot \vec{x}_2 = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} -4e^{2t} \\ -2e^{2t} \end{bmatrix} = \vec{x}_2{}'$ ✓

b) $W = \begin{vmatrix} e^{3t} & 2e^{-2t} \\ 3e^{3t} & e^{-2t} \end{vmatrix} = -5e^{t} \neq 0$

The solutions x_1 *and* x_2 are linearly independent.

c) $\mathbf{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 = C_1\begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} + C_2\begin{pmatrix} 2e^{-2t} \\ e^{-2t} \end{pmatrix} = \begin{pmatrix} C_1e^{3t} + 2C_2e^{-2t} \\ 3C_1e^{3t} + C_2e^{-2t} \end{pmatrix}$

d) $x_1 = C_1e^{3t} + 2C_2e^{-2t} \qquad x_2 = 3C_1e^{3t} + C_2e^{-2t}$

$x_1(0) = C_1 + 2C_2 = 0 \qquad x_2(0) = 3C_1 + C_2 = 5$

$\Rightarrow \quad C_1 = 2 \quad C_2 = -1$

$$\begin{cases} x_1 = 2e^{3t} - 2e^{-2t} \\ x_2 = 6e^{3t} - e^{-2t} \end{cases}$$

Exercise

$$\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\mathbf{x}; \quad \vec{x}_1 = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}, \quad \begin{cases} x_1(0) = 5 \\ x_2(0) = -3 \end{cases}$$

a) Verify that the given vectors are solutions of the given system.
b) Use the Wronskian to show that they are linearly independent.
c) Write the general solution of the system.
d) Find the particular solution that satisfies the given initial conditions

Solution

a) $\vec{x}_1{}' = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}' = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} \end{bmatrix}$ $\qquad$ $\boldsymbol{x}' \cdot \vec{x}_1 = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} \end{bmatrix} = \vec{x}_1{}'$ ✓

$\vec{x}_2{}' = \begin{bmatrix} e^{-2t} \\ 5e^{-2t} \end{bmatrix}' = \begin{bmatrix} -2e^{-2t} \\ -10e^{-2t} \end{bmatrix}$ $\qquad$ $\boldsymbol{x}' \cdot \vec{x}_2 = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} e^{-2t} \\ 5e^{-2t} \end{bmatrix} = \begin{bmatrix} -2e^{2t} \\ -10e^{2t} \end{bmatrix} = \vec{x}_2{}'$ ✓

b) $W = \begin{vmatrix} e^{2t} & e^{-2t} \\ e^{2t} & 5e^{-2t} \end{vmatrix} = 4 \neq 0$

The solutions x_1 *and* x_2 are linearly independent.

c) $\boldsymbol{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 = C_1\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} + C_2\begin{pmatrix} e^{-2t} \\ 5e^{-2t} \end{pmatrix} = \begin{pmatrix} C_1e^{2t} + C_2e^{-2t} \\ C_1e^{2t} + 5C_2e^{-2t} \end{pmatrix}$

d) $x_1 = C_1e^{2t} + C_2e^{-2t} \qquad x_2 = C_1e^{2t} + 5C_2e^{-2t}$

$x_1(0) = C_1 + C_2 = 5 \qquad x_2(0) = C_1 + 5C_2 = -3$

$\Rightarrow \ \underline{C_1 = 7 \quad C_2 = -2}|$

$$\begin{cases} x_1 = 7e^{2t} - 2e^{-2t} \\ x_2 = 7e^{2t} - 10e^{-2t} \end{cases}$$

Exercise

$\boldsymbol{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 8 \\ x_2(0) = 0 \end{cases}$

a) Verify that the given vectors are solutions of the given system.
b) Use the Wronskian to show that they are linearly independent.
c) Write the general solution of the system.
d) Find the particular solution that satisfies the given initial conditions

Solution

a) $\vec{x}_1{}' = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}' = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix}$ $\qquad$ $\boldsymbol{x}' \cdot \vec{x}_1 = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} = \vec{x}_1{}'$ ✓

$\vec{x}_2{}' = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}' = \begin{bmatrix} -5e^{-2t} \\ -15e^{-2t} \end{bmatrix}$ $\qquad$ $\boldsymbol{x}' \cdot \vec{x}_2 = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} = \begin{bmatrix} -5e^{2t} \\ -15e^{2t} \end{bmatrix} = \vec{x}_2{}'$ ✓

b) $W = \begin{vmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{vmatrix} = 7e^{-3t} \neq 0$

The solutions x_1 *and* x_2 are linearly independent.

c) $\boldsymbol{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 = C_1\begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix} + C_2\begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix} = \begin{pmatrix} 3C_1e^{2t} + C_2e^{-5t} \\ 2C_1e^{2t} + 3C_2e^{-5t} \end{pmatrix}$

d) $x_1 = 3C_1e^{2t} + C_2e^{-5t} \qquad x_2 = 2C_1e^{2t} + 3C_2e^{-5t}$

$x_1(0) = 3C_1 + C_2 = 8 \qquad x_2(0) = 2C_1 + 3C_2 = 0 \;\Rightarrow\; \underline{C_1 = \frac{24}{7} \quad C_2 = -\frac{16}{7}}$

$$\begin{cases} x_1 = \frac{72}{7}e^{2t} - \frac{16}{7}e^{-2t} \\ x_2 = \frac{48}{7}e^{2t} - \frac{48}{7}e^{-2t} \end{cases}$$

Exercise

$$\boldsymbol{x}' = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 0 \\ x_3(0) = 4 \end{cases}$$

a) Verify that the given vectors are solutions of the given system.
b) Use the Wronskian to show that they are linearly independent.
c) Write the general solution of the system.
d) Find the particular solution that satisfies the given initial conditions

Solution

a) $\vec{x}_1{}' = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}' = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix} \qquad \boldsymbol{x}' \cdot \vec{x}_1 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix} = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix} = \vec{x}_1{}' \;\checkmark$

$\vec{x}_2{}' = \begin{bmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}' = \begin{bmatrix} -6e^{3t} \\ 0 \\ 3e^{3t} \end{bmatrix} \qquad \boldsymbol{x}' \cdot \vec{x}_2 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix} = \begin{bmatrix} -6e^{3t} \\ 0 \\ 3e^{3t} \end{bmatrix} = \vec{x}_2{}' \;\checkmark$

$\vec{x}_3{}' = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}' = \begin{bmatrix} 10e^{5t} \\ -10e^{5t} \\ 5e^{5t} \end{bmatrix} \qquad \boldsymbol{x}' \cdot \vec{x}_3 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix} = \begin{bmatrix} 10e^{5t} \\ -10e^{5t} \\ 5e^{5t} \end{bmatrix} = \vec{x}_3{}' \;\checkmark$

b) $W = \begin{vmatrix} 2e^{t} & -2e^{3t} & 2e^{5t} \\ 2e^{t} & 0 & -2e^{5t} \\ e^{t} & e^{3t} & e^{5t} \end{vmatrix} = 4e^{9t} + 4e^{9t} + 4e^{9t} + 4e^{9t} = 16e^{9t} \neq 0$

The solutions x_1, x_2 *and* x_3 are linearly independent.

c) $\boldsymbol{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 + C_3\vec{x}_3 = C_1\begin{pmatrix} 2e^{t} \\ 2e^{t} \\ e^{t} \end{pmatrix} + C_2\begin{pmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{pmatrix} + C_3\begin{pmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{pmatrix}$

$$= \begin{pmatrix} 2C_1e^{t} - 2C_2e^{3t} + 2C_3e^{5t} \\ 2C_1e^{t} \qquad\qquad -2C_3e^{5t} \\ C_1e^{t} + C_2e^{3t} + C_3e^{5t} \end{pmatrix}$$

d) $x_1 = 2C_1e^{t} - 2C_2e^{3t} + 2C_3e^{5t} \qquad x_2 = 2C_1e^{t} - 2C_3e^{5t} \qquad x_3 = C_1e^{t} + C_2e^{3t} + C_3e^{5t}$

$$\begin{cases} x_1(0) = 2C_1 - 2C_2 + 2C_3 = 0 \\ x_2(0) = 2C_1 - 2C_3 = 0 \\ x_3(0) = C_1 + C_2 + C_3 = 4 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & 1 & 1 & 4 \end{array}\right]$$

$$\Rightarrow \underline{C_1 = 1 \quad C_2 = 2 \quad C_3 = 1|}$$

$$\begin{cases} x_1(t) = 2e^{t} - 4e^{3t} + 2e^{5t} \\ x_2(t) = 2e^{t} - 2e^{5t} \\ x_3(t) = e^{t} + 2e^{3t} + e^{5t} \end{cases}$$

Exercise

$$\boldsymbol{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 10 \\ x_2(0) = 12 \\ x_3(0) = -1 \end{cases}$$

a) Verify that the given vectors are solutions of the given system.
b) Use the Wronskian to show that they are linearly independent.
c) Write the general solution of the system.
d) Find the particular solution that satisfies the given initial conditions

Solution

a) $\vec{x}_1{}' = \begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix}' = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{pmatrix}$ $\quad$ $\mathbf{x}' \cdot \vec{x}_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{pmatrix} = \vec{x}_1{}'$ $\checkmark$

$\vec{x}_2{}' = \begin{pmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix}' = \begin{pmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{pmatrix}$ $\quad$ $\mathbf{x}' \cdot \vec{x}_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{pmatrix} = \vec{x}_2{}'$ $\checkmark$

$\vec{x}_3{}' = \begin{pmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{pmatrix}' = \begin{pmatrix} 0 \\ -e^{-t} \\ e^{-t} \end{pmatrix}$ $\quad$ $\mathbf{x}' \cdot \vec{x}_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} 0 \\ -e^{-t} \\ e^{-t} \end{pmatrix} = \vec{x}_3{}'$ $\checkmark$

b) $W = \begin{vmatrix} e^{2t} & e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & -e^{-t} & -e^{-t} \end{vmatrix} = 3 \neq 0$ $\quad$ The solutions x_1, x_2 *and* x_3 are linearly independent.

c) $\mathbf{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 + C_3\vec{x}_3 = C_1\begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix} + C_2\begin{pmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix} + C_3\begin{pmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{pmatrix}$

$$= \begin{pmatrix} C_1e^{2t} + C_2e^{-t} \\ C_1e^{2t} \quad + C_3e^{-t} \\ C_1e^{2t} - C_2e^{-t} - C_3e^{-t} \end{pmatrix}$$

d) $x_1 = C_1e^{2t} + C_2e^{-t}$ $\quad$ $x_2 = C_1e^{2t} + C_3e^{-t}$ $\quad$ $x_3 = C_1e^{2t} - C_2e^{-t} - C_3e^{-t}$

$$\begin{cases} x_1(0) = C_1 + C_2 = 10 \\ x_2(0) = C_1 + C_3 = 12 \\ x_3(0) = C_1 - C_2 - C_3 = -1 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 1 & 0 & 1 & 12 \\ 1 & -1 & -1 & -1 \end{array}\right]$$

$$\Rightarrow \quad C_1 = 7 \quad C_2 = 3 \quad C_3 = 5$$

$$\begin{cases} x_1(t) = 7e^{2t} + 3e^{-t} \\ x_2(t) = 7e^{2t} + 5e^{-t} \\ x_3(t) = 7e^{2t} - 8e^{-t} \end{cases}$$

Exercise

$$\boldsymbol{x}' = \begin{bmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{bmatrix}, \quad \vec{x}_4 = \begin{bmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{bmatrix}, \quad \begin{cases} x_1(0) = 1 \\ x_2(0) = 3 \\ x_3(0) = 4 \\ x_4(0) = 7 \end{cases}$$

a) Verify that the given vectors are solutions of the given system.

b) Use the Wronskian to show that they are linearly independent.

c) Write the general solution of the system.

d) Find the particular solution that satisfies the given initial conditions

Solution

a)
$$\vec{x}_1{}' = \begin{pmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{pmatrix}' = \begin{pmatrix} -e^{-t} \\ 0 \\ 0 \\ -e^{-t} \end{pmatrix} \qquad \boldsymbol{x}' \cdot \vec{x}_1 = \begin{pmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ 0 \\ 0 \\ -e^{-t} \end{pmatrix} = \vec{x}_1{}' \quad \checkmark$$

$$\vec{x}_2{}' = \begin{pmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{pmatrix}' = \begin{pmatrix} 0 \\ 0 \\ -e^{-t} \\ 0 \end{pmatrix} \qquad \boldsymbol{x}' \cdot \vec{x}_2 = \begin{pmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -e^{-t} \\ 0 \end{pmatrix} = \vec{x}_2{}' \quad \checkmark$$

$$\vec{x}_3{}' = \begin{pmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{pmatrix}' = \begin{pmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{pmatrix} \qquad \boldsymbol{x}' \cdot \vec{x}_3 = \begin{pmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{pmatrix} = \begin{pmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{pmatrix} = \vec{x}_3{}' \quad \checkmark$$

$$\vec{x}_4{}' = \begin{pmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{pmatrix}' = \begin{pmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{pmatrix} \qquad \boldsymbol{x}' \cdot \vec{x}_3 = \begin{pmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{pmatrix} = \vec{x}_3{}' \quad \checkmark$$

b)
$$W = \begin{vmatrix} e^{-t} & 0 & 0 & e^{t} \\ 0 & 0 & e^{t} & 0 \\ 0 & e^{-t} & 0 & 3e^{t} \\ e^{-t} & 0 & -2e^{t} & 0 \end{vmatrix} = e^{-t}\begin{vmatrix} 0 & e^{t} & 0 \\ e^{-t} & 0 & 3e^{t} \\ 0 & -2e^{t} & 0 \end{vmatrix} - e^{t}\begin{vmatrix} 0 & 0 & e^{t} \\ 0 & e^{-t} & 0 \\ e^{-t} & 0 & -2e^{t} \end{vmatrix} = 0 - (-1) = 1 \neq 0$$

The solutions x_1, x_2 *and* x_3 are linearly independent.

c) $\boldsymbol{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 + C_3\vec{x}_3 + C_4\vec{x}_4 = C_1\begin{pmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{pmatrix} + C_2\begin{pmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{pmatrix} + C_3\begin{pmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{pmatrix} + C_4\begin{pmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} C_1e^{-t} + C_4e^{t} \\ C_3e^{t} \\ C_2e^{-t} + 3C_4e^{t} \\ C_1e^{-t} - 2C_3e^{t} \end{pmatrix}$$

d) $x_1(t) = C_1e^{-t} + C_4e^{t}, \quad x_2(t) = C_3e^{t}, \quad x_3(t) = C_2e^{-t} + 3C_4e^{t}, \quad x_4(t) = C_1e^{-t} - 2C_3e^{t}$

$$\begin{cases} x_1(0) = C_1 + C_4 = 1 \\ x_2(0) = C_3 = 3 \\ x_3(0) = C_2 + 3C_4 = 4 \\ x_4(0) = C_1 - 2C_3 = 7 \end{cases} \qquad \Rightarrow \quad \underline{C_1 = 13 \quad C_2 = 40 \quad C_3 = 3 \quad C_4 = -12|}$$

$$\begin{cases} x_1(t) = 13e^{-t} - 12e^{t} \\ x_2(t) = 3e^{t} \\ x_3(t) = 40e^{-t} - 36e^{t} \\ x_3(t) = 13e^{-t} - 6e^{t} \end{cases}$$

Exercise

Consider the *RLC* parallel circuit below. Let *V* represent the voltage drop across the capacitor and I represent the current across the inductor.

Show that: $\quad V' = -\frac{V}{RC} - \frac{1}{C} \qquad I' = \frac{V}{L}$

Solution

Using Kirchhoff's current law: $I_1 + I_2 + I_3 = 0$

In the RC loop: $V_1 - V_2 = 0$

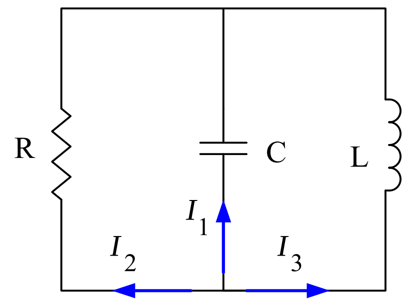
In the LC loop: $V_2 - V_3 = 0$

$V_2 = RI_2, \quad CV_1' = I_1, \quad LI_3' = V_3$

Since the circuit elements are in parallel, therefore $V_1 = V_2 = V_3 = V$

$$LI_3' = V_1 \Rightarrow \boxed{I_3' = \frac{V_1}{L}}$$

$$
\begin{aligned}
CV_1' &= I_1 \\
&= -I_2 - I_3 && V_2 = RI_2 \\
&= -\frac{V_2}{R} - I_3 && V_2 = V_1 \\
&= -\frac{V_1}{R} - I_3
\end{aligned}
$$



$$V_1' = -\frac{V_1}{CR} - \frac{I_3}{C}$$

Since $V_1 = V$ *and* $I_3 = I$

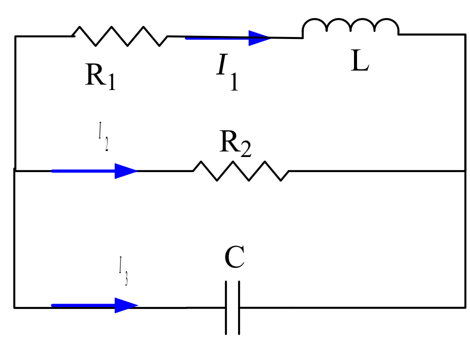
$$\Rightarrow \begin{cases} I' = \dfrac{V}{L} \\ V' = -\dfrac{V}{CR} - \dfrac{I}{C} \end{cases}$$

Exercise

Consider the *RLC* parallel circuit below. Let *V* represent the voltage drop across the capacitor and I represent the current across the inductor.

Show that: $CV' = -I - \dfrac{V}{R_2}$ $\qquad LI' = -R_1 I + V$

Solution



Using Kirchhoff's current law: $I + I_2 + I_3 = 0$ **(1)**

In the R_1LR_2 loop: $R_1 I + LI' - R_2 I_2 = 0$ **(2)**

In the R_2C loop: $R_2 I_2 - V = 0$ **(3)**

From (3): $V = R_2 I_2 \Rightarrow I_2 = \dfrac{V}{R_2}$

From (2): $LI' = -R_1 I + R_2 I_2$ $\qquad V = R_2 I_2$

$$\boxed{= -R_1 I + V}$$

From (1): $I_2 = -I - I_3$

$$\frac{V}{R_2} = -I - I_3$$

However, the voltage drop across the capacitor is: $V = \dfrac{q}{C}$

$$\Rightarrow CV = q$$

$$CV' = q' \qquad I_3 = q'$$

$$CV' = I_3$$

$$\frac{V}{R_2} = -I - CV'$$

$$\boxed{CV' = -I - \frac{V}{R_2}}$$

Exercise

Let I_1 and I_2 represent the current flow across the indicators L_1 and L_2 respectively. Show that the circuit is modeled by the system

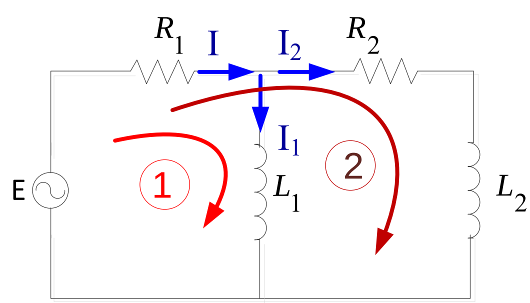
$$\begin{cases} L_1 I_1' = -R_1 I_1 - R_1 I_2 + E \\ L_2 I_2' = -R_1 I_1 - \left(R_1 + R_2\right) I_2 + E \end{cases}$$

Solution

By Kirchhoff's second law:

$$I = I_1 + I_2$$

From loop **1**:



$$-E + R_1 I + L_1 I_1' = 0$$

$$\begin{aligned} L_1 I_1' &= E - R_1 I \\ &= E - R_1\left(I_1 + I_2\right) \\ &\boxed{= -R_1 I_1 - R_1 I_2 + E} \end{aligned}$$

From loop **2**:

$$-E + R_1 I + R_2 I_2 + L_2 I_2' = 0$$

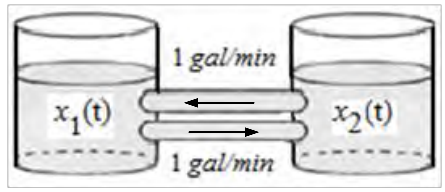
$$\begin{aligned} L_2 I_2' &= -R_1 I - R_2 I_2 + E \\ &= -R_1\left(I_1 + I_2\right) - R_2 I_2 + E \\ &\boxed{= -R_1 I_1 - R_1 I_2 - R_2 I_2 + E} \end{aligned}$$

Exercise

Two tanks are connected by two pipes. Each tank contains 500 *gallons* of a salt solution. Through on pipe solution is pumped from the first tank to the second at 1 *gal/min*. Through the other pipe, solution is pumped at the same rate from the second to the first tank. Show the salt content in each tank varies with time.

Solution

$x_1(t)$ and $x_2(t)$ represent the salt content.



$$\text{Rate out} = 1\ gal\ /\ min \times \frac{x_1}{500} lb\ /\ gal = \frac{x_1}{500} lb\ /\ min$$

$$\text{Rate in} = 1\ gal\ /\ min \times \frac{x_2}{500} lb\ /\ gal = \frac{x_2}{500} lb\ /\ min$$

$$\frac{dx_1}{dt} = Rate\ out - Rate\ in = \frac{x_2}{500} - \frac{x_1}{500}$$

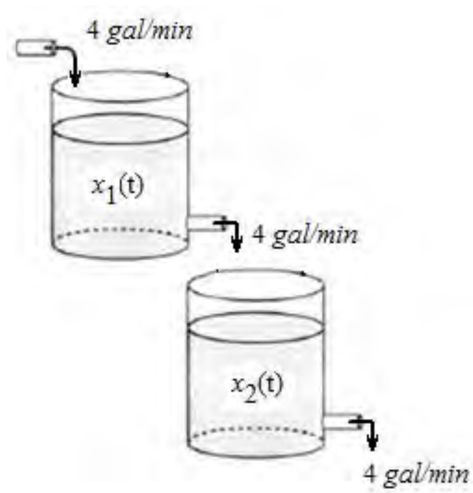
And $\frac{dx_2}{dt} = \frac{x_1}{500} - \frac{x_2}{500}$

$$x' = Ax \quad \rightarrow \quad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -\frac{1}{500} & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Exercise

Each tank contains 100 *gallons* of a salt solution. Pure water flows into the upper tank at a rate of 4 *gal/min*. Salt solution drains from the upper tank into the lower tank at a rate of 4 *gal/min*. Finally, salt solution drains from the lower tank at a rate of 4 *gal/min*, effectively keeping the volume of solution in each tank at a constant 100 *gal*. If the initial salt content of the upper and lower tanks is 10 and 20 *pounds*, respectively. Set up an initial value problem that models the amount of salt in each tank over time (do not solve).Write the model in matrix-vector form. Is the system homogeneous or inhomogeneous?

Solution



For the first tank: $\text{Rate out } = 4\frac{gal}{min} \times \frac{x_1}{100}\frac{lb}{gal} = \frac{x_1}{25} lb\ /\ min$

$$= \frac{x_1}{25} lb\ /\ min$$

$$\frac{dx_1}{dt} = Rate\ out\ - \ Rate\ in = -\frac{x_1}{25} \quad \text{Rate out - Rate in}$$

For the second tank: $\text{Rate out } = 4\ \frac{gal}{min} \times \frac{x_2}{100}\frac{lb}{gal} = \frac{x_2}{25}\frac{lb}{min}$

$$\frac{dx_2}{dt} = Rate\ out\ - \ Rate\ in = \frac{x_1}{25} - \frac{x_2}{25}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -\frac{1}{25} & 0 \\ \frac{1}{25} & -\frac{1}{25} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Exercise

Two masses on a frictionless tabletop are connected with a spring having spring constant k_2. The first mass is connected to a vertical support with a spring having spring constant k_1. The second mass is shaken harmonically via a force equaling $F = A\cos\omega t$. Let $x(t)$ and $y(t)$ measure the displacements of the masses m_1 and m_2, respectively, from their equilibrium positions as a function of time. If both masses start from rest at their equilibrium positions at time $t = 0$.

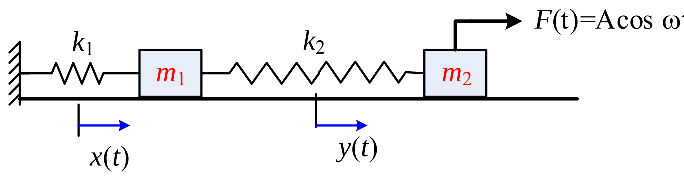
Set up an initial value problem that models the position of the masses over time (do not solve). Write the model in matrix-vector form. Is the system homogeneous or inhomogeneous?

Solution

By Newton's Law; the first mass:

$$m_1 x'' = -k_1 x + k_2 (y - x)$$

$$x'' = -\frac{k_1}{m_1} x + \frac{k_2}{m_1}(y - x)$$



The second mass:

$$m_2 y'' = -k_2 (y - x) + A\cos\omega t$$

$$y'' = -\frac{k_2}{m_2}(y - x) + \frac{A}{m_2}\cos\omega t$$

Let assume: $x_1 = x, \quad x_2 = x', \quad x_3 = y, \quad x_4 = y'$

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{k_1}{m_1} x_1 + \frac{k_2}{m_1}\left(x_3 - x_1\right) \\ x_3' = x_4 \\ x_4' = -\frac{k_2}{m_2}\left(x_3 - x_1\right) + \frac{A}{m_2}\cos\omega t \end{cases} \Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = -\left(\frac{k_1}{m_1} + \frac{k_2}{m_1}\right) x_1 + \frac{k_2}{m_1} x_3 \\ x_3' = x_4 \\ x_4' = \frac{k_2}{m_2} x_1 - \frac{k_2}{m_2} x_3 + \frac{A}{m_2}\cos\omega t \end{cases}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1 + k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{A}{m_2}\cos\omega t \end{pmatrix}$$

Exercise

Derive the equations $\begin{cases} m_1 x_1'' = -\left(k_1 + k_2\right)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - \left(k_2 + k_3\right)x_2 \end{cases}$

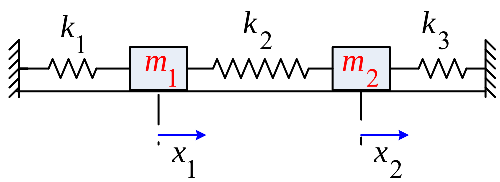
For the displacements (from equilibrium) of the 2 masses.

Solution

First spring is stretched by x_1

Second spring is stretched by $x_2 - x_1$

Third spring is stretched by x_2



Newton's second law gives:

For m_1 $\Rightarrow$ $m_1 x_1'' = -k_1 x_1 + k_2\left(x_2 - x_1\right)$

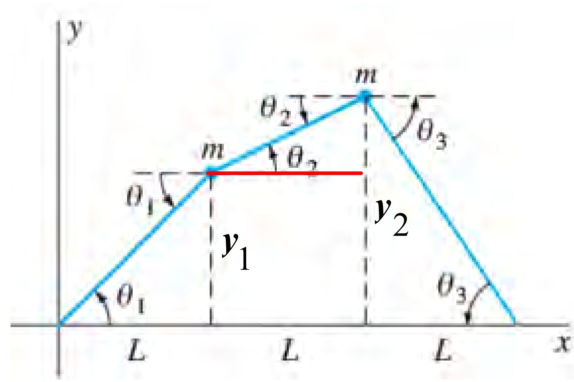
For m_2 $\Rightarrow$ $m_2 x_2'' = -k_2\left(x_2 - x_1\right) - k_3 x_2$

That implies to: $\begin{cases} m_1 x_1'' = -\left(k_1 + k_2\right)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - \left(k_2 + k_3\right)x_2 \end{cases}$

Exercise

Two particles each of mass *m* are attached to a string under (constant) tension *T*. Assume that the particles oscillate vertically (that is, parallel to the *y*-axis) with amplitudes so small that the sines of the angles shown are accurately approximated by their tangents. Show that the displacement y_1 and y_2 satisfy the equations

$$\begin{cases} k y_1'' = -2y_1 + y_2 \\ k y_2'' = y_1 - 2y_2 \end{cases} \qquad \text{where } k = \frac{mL}{T}$$



Solution

For the first mass:

$$m y_1'' = -T\sin\theta_1 + T\sin\theta_2$$

$$\approx -T\tan\theta_1 + T\tan\theta_2$$

$$m y_1'' = -T\frac{y_1}{L} + T\frac{y_2 - y_1}{L}$$

$$\frac{L}{T} m y_1'' = -\frac{L}{T}T\frac{y_1}{L} + \frac{L}{T}T\frac{y_2 - y_1}{L} \qquad \text{where } k = \frac{mL}{T}$$

$$\underline{\left|k y_1''\right.} = -y_1 + y_2 - y_1 \ \underline{= -2y_1 + y_2\Big|}$$

For the second mass:

$$my_2'' = -T\sin\theta_2 + T\sin\theta_3$$
$$\approx -T\tan\theta_2 + T\tan\theta_3$$

$$my_2'' = -T\frac{y_2 - y_1}{L} + T\frac{y_2}{L}$$

$$\frac{L}{T}my_2'' = -\frac{L}{T}T\frac{y_2 - y_1}{L} + \frac{L}{T}T\frac{y_2}{L} \qquad \text{where } k = \frac{mL}{T}$$

$$\boxed{ky_2'' = -y_2 + y_1 - y_2 = y_1 - 2y_2}$$

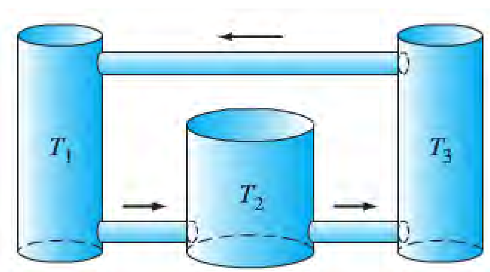
$$\Rightarrow \begin{cases} ky_1'' = -2y_1 + y_2 \\ ky_2'' = y_1 - 2y_2 \end{cases} \qquad \text{where } k = \frac{mL}{T}$$

Exercise

There 100-gal fermentation vats are connected, and the mixtures in each tank are kept uniform by stirring. Denote by $x_i(t)$ the amount (in pounds) of alcohol in tank T_i at time *t* (*i* = 1, 2, 3). Suppose that the mixture circulates between the tanks at the rate of 10 *gal/min*. Derive the equations

$$\begin{cases} 10x_1' = -x_1 \qquad + x_3 \\ 10x_2' = x_1 - x_2 \\ 10x_3' = \qquad x_2 - x_3 \end{cases}$$

Solution



Concentration of salt in each tank is: $c_i = \frac{X}{V} = \frac{x_i}{100}$

Rate in/out = Volume Rate x Concentration

Rate of change = Rate in – rate out

For T_1: $\quad x_1' = r\frac{x_3}{100} - r\frac{x_1}{100} = \frac{1}{10}(x_3 - x_1)$

For T_2: $\quad x_2' = r\frac{x_1}{100} - r\frac{x_2}{100} = \frac{1}{10}(x_1 - x_2)$

For T_3: $\quad x_3' = r\frac{x_2}{100} - r\frac{x_3}{100} = \frac{1}{10}(x_2 - x_3)$

That implies:

$$\begin{cases} 10x_1' = -x_1 \qquad + x_3 \\ 10x_2' = x_1 - x_2 \\ 10x_3' = \qquad x_2 - x_3 \end{cases}$$

Exercise

Suppose that a particle with mass m and electrical charge q moves in the xy-plane under the influence of the magnetic field $\vec{B} = B\hat{k}$ (thus a uniform field parallel to the z-axis), so the force on the particle is $\vec{F} = q\vec{v} \times \vec{B}$ if its velocity is $\vec{v}$. Show that the equations of motion of the particle are

$$mx'' = +qBy', \quad my'' = -qBx'$$

Solution

Let $\vec{r} = (x, y, z)$ be the position vector, then Newton's law

$$\vec{F} = mx''$$

$$\begin{aligned} \vec{F} = mx'' &= q\left(\vec{v} \times \vec{B}\right) \\ &= q\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x' & y' & z' \\ 0 & 0 & B \end{vmatrix} \\ &= qBy'\hat{i} - qBx'\hat{j} \end{aligned}$$

$$\Rightarrow mx'' = +qBy', \quad my'' = -qBx'$$