

Solution **Section 1.2 – Gaussian Elimination**

Exercise

When elimination is applied to the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix}$

- a) What are the first and second pivots?
- b) What is the multiplier l_{21} in the first step (l_{21} times row 1 is subtracted from row 2)?
- c) What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3?
- d) What is the multiplier $l_{31} = 0$, subtracting 0 times row 1 from row 3?

Solution

- a) The first pivot is 3 and when 2 times row 1 is subtracted from row 2, the second pivot is revealed as 7.

$$\begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix} \begin{array}{l} \text{subtract 2 times row.1} \\ \text{from row.2} \end{array} \begin{bmatrix} 3 & 1 & 0 \\ 0 & \boxed{7} & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

- b) The multiplier l_{21} in the first step is $\frac{6}{3} = 2$.
- c) If we reduce the entry 9 to 2, that drop of 7 in the a_{22} position would force a row exchange.

$$\begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix} \begin{array}{l} \text{subtract 7 times row.1} \\ \text{from row.2} \end{array} \begin{bmatrix} 3 & 1 & 0 \\ -15 & \boxed{2} & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

- d) The multiplier l_{31} is already zero because $a_{31} = 0$ and no needs row elimination.

Exercise

Use elimination to reach upper triangular matrices U . Solve by back substitution or explain why this is impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the $-x$ in equation (3).

$$\begin{cases} x + y + z = 7 \\ x + y - z = 5 \\ x - y + z = 3 \end{cases} \quad \begin{cases} x + y + z = 7 \\ x + y - z = 5 \\ -x - y + z = 3 \end{cases}$$

Solution

For the *first* system:

$$\begin{array}{lll} x + y + z = 7 & \text{subtract eqn.1} & x + y + z = 7 \\ x + y - z = 5 & \text{from eqn.2} & 0y - 2z = -2 \\ x - y + z = 3 & \text{from eqn.3} & -2y - 0z = -4 \\ \\ x + y + z = 7 & & 1x + y + z = 7 \\ x + y - z = 5 & \text{Exchange eqn.2} & -2y - 0z = -4 \\ x - y + z = 3 & \text{and eqn.3} & -2z = -2 \end{array}$$

The solutions are: $z = 1$ $y = 2$ $x = 4$ and the pivots are 1, -2, -2.

For the *second* system:

$$\begin{array}{lll} x + y + z = 7 & \text{Subtract eqn.1} & x + y + z = 7 \\ x + y - z = 5 & \text{from eqn.2} & 0y - 2z = -2 \\ -x - y + z = 3 & \text{Add eqn.1} & 0y + 2z = 10 \\ & \text{to eqn.3} & \\ \\ x + y + z = 7 & & x + y + z = 7 \\ 0y - 2z = -2 & \text{Add eqn.2} & 0y - 2z = -2 \\ 0y + 2z = 10 & \text{to eqn.3} & \boxed{0z = 8} \end{array}$$

The three planes don't meet. But if we change '3' in the last equation to '-5'

$$\begin{array}{lll} x + y + z = 7 & \text{Subtract eqn.1} & x + y + z = 7 \\ x + y - z = 5 & \text{from eqn.2} & 0y - 2z = -2 \\ -x - y + z = -5 & \text{Add eqn.1} & 0y + 2z = 2 \\ & \text{to eqn.3} & \\ \\ x + y + z = 7 & x + y = 6 & \\ 0y - 2z = -2 & & \text{There are unique infinite many solutions!} \\ 0y + 2z = 10 & z = 1 & \end{array}$$

The three planes now meet along a whole line.

Exercise

For which numbers a does the elimination break down (1) permanently (2) temporarily

$$ax + 3y = -3$$

$$4x + 6y = 6$$

Solve for x and y after fixing the second breakdown by a row change.

Solution

The matrix form is:
$$\begin{pmatrix} a & 3 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$$

If $a = 0$, the elimination brakes down temporarily.

$$\begin{pmatrix} 4 & 6 \\ 0 & \boxed{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

The system is in upper triangular form and entry row 2 column 2 is not equal to zero, therefore the system has a solution.

If $a \neq 0$,

$$\begin{pmatrix} a & 3 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} \quad R_2 - \frac{4}{a}R_1$$

$$\begin{pmatrix} a & 3 \\ 0 & 6 - \frac{12}{a} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 + \frac{12}{a} \end{pmatrix}$$

$$6 - \frac{12}{a} = 0 \Rightarrow \frac{12}{a} = 6$$

$$\rightarrow \boxed{a = \frac{12}{6} = 2}$$

If $a = 2$,

$$\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \end{pmatrix}, \text{ the system will fail and has no solution.}$$

If $a \neq 2$;

$$\begin{pmatrix} a & 3 \\ 0 & 6 - \frac{12}{a} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 + \frac{12}{a} \end{pmatrix}, \text{ the system has a unique solution.}$$

Exercise

Find the pivots and the solution for these four equations:

$$2x + y = 0$$

$$x + 2y + z = 0$$

$$y + 2z + t = 0$$

$$z + 2t = 5$$

Solution

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} R_2 - \frac{1}{2}R_1$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 1.5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} R_3 - \frac{2}{3}R_2$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} R_4 - \frac{3}{4}R_3$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 5 \end{pmatrix} \begin{array}{l} 2x = -y \Rightarrow \boxed{x = -2\frac{1}{2} = -1} \\ \frac{3}{2}y + z = 0 \Rightarrow y = -z\frac{2}{3} = -(-3)\frac{2}{3} \rightarrow \boxed{y = 2} \\ \frac{4}{3}z + t = 0 \rightarrow \frac{4}{3}z = -t \rightarrow \boxed{z = -4\frac{3}{4} = -3} \\ \frac{5}{4}t = 5 \rightarrow \boxed{t = 4} \end{array}$$

The pivots are diagonal entries and the solution is: $(-1, 2, -3, 4)$

Exercise

Look for a matrix that has row sums 4 and 8, and column sums 2 and s .

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{array}{ll} a + b = 4 & a + c = 2 \\ c + d = 8 & b + d = s \end{array}$$

The four equations are solvable only if $s = \underline{\hspace{1cm}}$. Then find two different matrices that have the correct row and column sums.

Solution

$$\begin{array}{r} a + b = 4 \\ + \quad c + d = 8 \\ \hline a + c + b + d = 12 \end{array}$$

$$2 + s = 12$$

$$\boxed{s = 10}$$

Exercise

Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of A is a linear combination of the first two rows. Find a third equation that can't be solved together with $x + y + z = 0$ and $x - 2y - z = 1$

Solution

The system is singular if row 3 of A is a **linear combination** of the first two rows.

There are many possible of a third equation that can't be solved together with $x + y + z = 0$ and $x - 2y - z = 1$.

$$\begin{array}{r} 3 \text{ times } 1^{\text{st}} \text{ equation} \quad 3x + 3y + 3z \\ \text{minus } 2^{\text{nd}} \quad \quad \quad -x + 2y + z \\ \hline 2x + 5y + 4z = 1 \end{array}$$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array} \right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array} \quad \begin{array}{cccc} -1 & -2 & 3 & 1 \\ 1 & 1 & 2 & 8 \\ \hline 0 & -1 & 5 & 9 \end{array} \quad \begin{array}{cccc} 3 & -7 & 4 & 10 \\ -3 & -3 & -6 & -24 \\ \hline 0 & -10 & -2 & -14 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array} \right] -R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array} \right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 10R_2 \end{array} \quad \begin{array}{cccc} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ \hline 1 & 0 & 7 & 17 \end{array} \quad \begin{array}{cccc} 0 & -10 & -2 & -14 \\ 0 & 10 & -50 & -90 \\ \hline 0 & 0 & -52 & -104 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array} \right] -\frac{1}{52}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array} \right] \begin{array}{l} R_1 - 7R_3 \\ R_2 + 5R_3 \\ \end{array} \quad \begin{array}{cccc} 1 & 0 & 7 & 17 \\ 0 & 0 & -7 & -14 \\ \hline 1 & 0 & 0 & 3 \end{array} \quad \begin{array}{cccc} 0 & 1 & -5 & -9 \\ 0 & 0 & 5 & 10 \\ \hline 0 & 1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

Solution: $\boxed{(3, 1, 2)}$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \quad \quad - 3w = -3 \end{cases}$$

Solution

Solution: $(w-1, 2z, z, w)$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \quad \quad - 3w = -3 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ -1 & 3 & -2 & 1 \\ 3 & 4 & -7 & 10 \end{array} \right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & -2 & -10 & -14 \end{array} \right] 5R_3 + 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & 0 & -52 & -52 \end{array} \right] \begin{array}{l} x + 2y + z = 8 \quad (3) \\ 5y - z = 9 \quad (2) \\ -52z = -52 \quad (1) \end{array}$$

$$(1) \Rightarrow z = 1$$

$$(2) \Rightarrow 5y = 9 + 1 = 10 \rightarrow y = 2$$

$$(3) \Rightarrow x = 8 - 4 - 1 = 3$$

\therefore Solution: $(3, 2, 1)$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} 2u - 3v + w - x + y = 0 \\ 4u - 6v + 2w - 3x - y = -5 \\ -2u + 3v - 2w + 2x - y = 3 \end{cases}$$

Solution

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 4 & -6 & 2 & -3 & -1 & -5 \\ -2 & 3 & -2 & 2 & -1 & 3 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -3 & -5 \\ 0 & 0 & -1 & 1 & 0 & 3 \end{array} \right] \begin{array}{l} 2u - 3v + w - x + y = 0 \quad (3) \\ -x - 3y = -5 \quad (2) \\ -w + x = 3 \quad (1) \end{array}$$

$$(2) \Rightarrow x = 5 - 3y$$

$$(1) \Rightarrow w = x - 3 = 2 - 3y$$

$$(3) \Rightarrow 2u = 3v - 2 + 3y + 5 - 3y - y = 3v - y + 3$$

$$u = \frac{3}{2}v - \frac{1}{2}y + \frac{3}{2}$$

$$\therefore \text{Solution: } \left(\frac{3}{2}v - \frac{1}{2}y + \frac{3}{2}, v, 2 - 3y, 5 - 3y, y \right)$$

Exercise

Solve the given linear system by any method

$$\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

Solution

$$\text{Solution: } (0, 0, 0)$$

Exercise

Solve the given linear system by any method

$$\begin{cases} 2x + 2y + 4z = 0 \\ -y - 3z + w = 0 \\ 3x + y + z + 2w = 0 \\ x + 3y - 2z - 2w = 0 \end{cases}$$

Solution

Solution: $\boxed{(-w, w, 0, w)}$

Exercise

Add 3 times the second row to the first of

$$\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

Solution

$$E = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$EA = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 27 & 8 & -1 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix} \xrightarrow{R_1 + 3R_2} \begin{bmatrix} 27 & 8 & -1 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

Exercise

Solve the system using Gaussian elimination

$$\begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array} \right] \begin{array}{l} \\ 3R_2 - 5R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & -1 & 4 & 26 \\ 0 & 0 & 0 & 0 \end{array} \right] R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} 3x_1 + 2x_2 - x_3 = -15 \quad (3) \\ -x_2 + 11x_3 = 75 \quad (2) \\ -7x_3 = -49 \quad (1) \end{array}$$

$$(1) \rightarrow x_3 = 7$$

$$(2) \rightarrow x_2 = 77 - 75 = 2$$

$$(1) \rightarrow 3x_1 = -15 - 4 + 7 = 12 \Rightarrow x_1 = 4$$

\therefore Solution: $(4, 2, 7)$