3.7. change of variables

$$x = g(u,v)$$
$$y = h(u,v)$$

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(g,h)\,|J(u,v)|\,du\,dv$$

$J(u,v) \rightarrow$ Jacobian determinant.

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\int_0^4 \int_{\frac{1}{2}y}^{\frac{1}{2}y+1} \frac{2x-y}{2}\,dx\,dy$$

$$\begin{cases} u = \dfrac{2x-y}{2} \rightarrow 2x - y = 2u \quad ① \\ v = \dfrac{y}{2} \rightarrow y = 2v \quad ② \end{cases} \qquad 2x = 2u + 2v$$

① $x = u + v$ ③

$x = \frac{1}{2}y$	$u+v = v$	$u = 0$
$x = \frac{1}{2}y + 1$	$u+v = v+1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$

$$J(u,v) = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2 \qquad \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\int_0^4 \int_{\frac{1}{2}y}^{\frac{1}{2}y+1} \frac{2x-y}{2}\,dx\,dy = \int_0^2 dv \int_0^1 u\,(2)\,du\,\cancel{dv}$$

$$= 2\ u^2\Big|_0^1$$

$$= 2$$

Ex $\displaystyle\int_0^1 \int_0^{1-x} \underbrace{\sqrt{x+y}}_{u}\ \underbrace{(y-2x)}_{v}{}^2\, dy\, dx$

Soln let $u = x+y$

$v = -2x+y$

$\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3 \quad \Delta_y = \begin{vmatrix} 1 & u \\ -2 & v \end{vmatrix}$

$\Delta_x = \begin{vmatrix} u & 1 \\ v & 1 \end{vmatrix}$

$$\begin{cases} x = \frac{1}{3}u - \frac{1}{3}v \\ y = \frac{2}{3}u + \frac{1}{3}v \end{cases}$$

$y = 0 \qquad \frac{2}{3}u + \frac{1}{3}v = 0 \to 2u = -v \longrightarrow v = -2u$

$y = 1-x \qquad \frac{1}{3}u - \frac{1}{3}v + \frac{2}{3}u + \frac{1}{3}v = 1 \qquad u = 1$

$x = 0 \qquad \frac{1}{3}u - \frac{1}{3}v = 0 \qquad u = v$

$x = 1 \qquad \frac{1}{3}u - \frac{1}{3}v = 1 \to u - v = 3 \qquad u = 0$

$$J = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$$

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\, dy\, dx = \frac{1}{3}\int_0^1 \int_{-2u}^{u} u^{1/2} v^2\, dv\, du$$

$$= \frac{1}{9}\int_0^1 u^{1/2} v^3 \Big|_{-2u}^{u} du \qquad u^3 - (-2u)^3$$

$$= \frac{1}{9}\int_0^1 u^{1/2}\big(\underbrace{u^3 + 8u^3}_{9u^3}\big)\, du$$

$$= \int_0^1 u^{7/2}\, du$$

$$= \frac{2}{9} u^{9/2} \Big|_0^1$$

$$= \frac{2}{9}$$

Ex $\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$ $\qquad e^{\sqrt{\ }}$

soln $\quad u = \sqrt{xy} \rightarrow u^2 = xy$ (1)

$v = \sqrt{\frac{y}{x}} \rightarrow v^2 = \frac{y}{x} \rightarrow y = xv^2$ (2)

(1) $\rightarrow u^2 = x^2 v^2 \Rightarrow \boxed{x = \frac{u}{v}}$

(2) $\rightarrow \underline{y = uv}$

$x = \frac{1}{y} \qquad \frac{u}{v} = \frac{1}{uv} \qquad u = 1$

$x = y \qquad \frac{u}{v} = uv \qquad v = 1$

$y = 1 \qquad uv = 1 \qquad u = 2$

$y = 2 \qquad uv = 2 \rightarrow v = \frac{2}{u}$

$$J = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix}$$

$$= \frac{u}{v} + \frac{u}{v}$$

$$= \boxed{\frac{2u}{v}}$$

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = 2\int_1^2 \int_1^{2/u} v \cdot e^{u} \frac{u}{v}\, dv\, du$$

$$= 2 \int_1^2 \int_1^{2/u} u e^{u}\, dv\, du$$

$$= 2 \int_1^2 u e^{u} \left(\frac{2}{u} - 1\right) du$$

$$= 2\int_1^2 (2-u)e^u\,du \qquad \begin{array}{cc|c} & & \int e^u \\ \hline + & 2-u & e^u \\ - & -1 & e^u \end{array}$$

$$= 2\,(\underbrace{\overbrace{2-u+1}^{3}}_{(3-u)})\,e^u\Big|_1^2$$

$$= 2\,(e^2 - 2e)$$

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Ex $\int_0^3\int_0^4\int_{\frac{1}{2}y}^{\frac{1}{2}y+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz$

$u=\frac{2x-y}{2}$ $v=\frac{y}{2}$ $w=\frac{z}{3}$

$z=3w$

$y=2v$

$2u=2x-2v \to x=u+v$

$$J=\begin{vmatrix}1&1&0\\0&2&0\\0&0&3\end{vmatrix}=6$$

$x=\frac{1}{2}y$ $u+v=v$ $u=0$

$x=\frac{1}{2}y+1$ $u+v=v+1$ $u=1$

$y=0$ $2v=0$ $v=0$

$y=4$ $2v=4$ $v=2$

$z=0=3w$ $w=0$

$z=3=3w$ $w=1$

$$\int_0^2 dv\int_0^1\int_0^1(u+w)(6)\,du\,\cancel{dv}\,dw$$

$$=6(2)\int_0^1\left(\frac{1}{2}u^2+wu\right|_0^1 dw$$

$$=12\int_0^1\left(\frac{1}{2}+w\right)dw$$

$$=12\left(\frac{1}{2}w+\frac{1}{2}w^2\right|_0^1$$

$$=12$$

EX $\iiint_D xz\,dV$

$$y - x = 0 \qquad \to \quad u = y - x \quad ② \quad 0 \le u \le 2$$
$$y = 2 + x \quad y - x = 2$$

$$\left.\begin{matrix} z - y = 0 \\ z - y = 2 \end{matrix}\right\} \qquad v = z - y \quad ① \quad 0 \le v \le 2$$

$$\left.\begin{matrix} z = 0 \\ z = 3 \end{matrix}\right\} \qquad z = w \qquad 0 \le w \le 3$$

$$z = w$$
① $y = w - v$
$$u = w - v - x \to \quad x = -u - v + w$$

$$J = \begin{vmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \underline{1}$$

$$\iiint_D xz\,dV = \int_0^3 \int_0^2 \int_0^2 (w - u - v)w \, du\,dv\,dw$$
$$= \int_0^3 \int_0^2 \int_0^2 (w^2 - uw - vw)\,du\,dv\,dw$$
$$= \int_0^3 \int_0^2 \left(w^2 u - \tfrac{1}{2} w u^2 - vwu\right)\Big|_0^2 dv\,dw$$
$$= \int_0^3 \int_0^2 (2w^2 - 2w - 2vw)\,dv\,dw$$

$$= \int_0^3 \left(2w^2v - 2wv - wv^2\right|_0^2 dw$$

$$= \int_0^3 (4w^2 - 4w - 4w)\, dw$$

$$= \int_0^3 (4w^2 - 8w)\, dw$$

$$= \frac{4}{3}w^3 - 4w^2 \Big|_0^3$$

$$= 36 - 36$$

$$= 0$$

33 $\iiint_D dV$

$$\begin{cases} y-2x=0 \\ y-2x=2 \end{cases} \qquad \begin{cases} z-3y=0 \\ z-3y=1 \end{cases} \qquad \begin{cases} z-4x=0 \\ z-4x=3 \end{cases}$$

$$u = y-2x \qquad v = z-3y \qquad w = z-4x$$

$$0 \le u \le 2 \qquad 0 \le v \le 1 \qquad 0 \le w \le 3$$

$$\begin{aligned} -2x+y &= u \\ -3y+z &= v \quad +3y \\ -4x+z &= w \end{aligned} \qquad \Delta = \begin{vmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -4 & 0 & 1 \end{vmatrix} = 2$$

$$\Delta_x = \begin{vmatrix} u & 1 & 0 \\ v & -3 & 1 \\ w & 0 & 1 \end{vmatrix}$$

$$x = -\frac{3}{2}u - \frac{1}{2}v + \frac{1}{2}w \qquad = -3u + w - v$$

$$\begin{aligned} y &= u - 3u - v + w \\ &= -2u - v + w \end{aligned}$$

$$\begin{aligned} z &= v - 6u - 3v + 3w \\ &= -6u - 2v + 3w \end{aligned}$$

$$J = \begin{vmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ -2 & -1 & 1 \\ -6 & -2 & 3 \end{vmatrix} = \frac{9}{2} + 3 + 2 - 3 - 3 - 3$$

$$= \frac{1}{2}$$

$$\iiint_D dV = \frac{1}{2}\int_0^2 du \int_0^1 dv \int_0^3 dw$$

$$= \frac{1}{2}(2)(1)(3)$$

$$= \underline{3}$$

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$$\iint_R e^{xy}\,dA \qquad \begin{cases} xy = 1 \\ xy = 4 \end{cases} \qquad \begin{matrix} \frac{y}{x} = 1 \\ \frac{y}{x} = 3 \end{matrix}$$

$$u = xy \rightarrow 1 \le u \le 4$$

$$v = \frac{y}{x} \rightarrow 1 \le v \le 3$$

$$y = vx$$

$$u = x^2 v \rightarrow x = \sqrt{\frac{u}{v}} = \frac{\sqrt{u}}{\sqrt{v}}$$

$$y = v\frac{u^{1/2}}{v^{1/2}} = \sqrt{uv}$$

$$J = \begin{vmatrix} \frac{1}{2\sqrt{uv}} & \frac{-\sqrt{u}}{2v^{3/2}} \\ \frac{v}{2\sqrt{uv}} & \frac{u}{2\sqrt{uv}} \end{vmatrix}$$

$$= \frac{u}{4uv} + \frac{\cancel{v}\sqrt{\cancel{u}}}{4\sqrt{\cancel{u}v}\,v^{3/2}} \quad \tfrac{1}{2}$$

$$= \frac{1}{4v} + \frac{1}{4v}$$

$$\boxed{= \frac{1}{2v}}$$

$$\iint_R e^{xy}\,dA = \frac{1}{2}\int_1^3 \frac{1}{v}\,dv \int_1^4 e^u\,du$$

$$= \frac{1}{2}\ln|v|\Big|_1^3 \; e^u\Big|_1^4$$

$$\boxed{= \frac{1}{2}(\ln 3)(e^4 - e)}$$

#32 $\iiint\limits_D xy\,dv$

$$\begin{cases} y-x=0 \\ y-x=2 \end{cases} \qquad \begin{cases} z-y=0 \\ z-y=2 \end{cases} \qquad \begin{cases} z=0 \\ z=3 \end{cases}$$

$$u=y-x \qquad v=z-y \qquad z=w$$

$$0\le u\le 2 \qquad 0\le v\le 2 \qquad 0\le w\le 3$$

$z=w$

$y=w-v$

$x=w-v-u$

$u=$

$$J=\begin{vmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \boxed{1}$$

$$\iiint\limits_D xy\,dv = \int_0^3\int_0^2\int_0^2 (w-v-u)(w-v)\,du\,dv\,dw$$

$$=\int_0^3\int_0^2\int_0^2 (w^2-2vw+v^2-uw+uv)\,du\,dv\,dw$$

$$=\int_0^3\int_0^2 \left(w^2u-2vwu+v^2u-\frac{1}{2}wu^2+\frac{1}{2}vu^2\right)\Big|_0^2\,dv\,dw$$

$$=\int_0^3\int_0^2 (2w^2-4vw+2v^2-2w+2v)\,dv\,dw$$

$$=\int_0^3 \left(2w^2v-2wv^2+\frac{2}{3}v^3-2wv+v^2\right)\Big|_0^2\,dw$$

$$= \int_0^3 \left(4w^2 - 8w + \frac{16}{3} - 4w + 4\right) dw$$

$$= \int_0^3 \left(4w^2 - 12w + \frac{28}{3}\right) dw$$

$$= \frac{4}{3}w^3 - 6w^2 + \frac{28}{3}w \Big|_0^3$$

$$= 36 - 54 + 28$$

$$= 10$$

#1 $\int_0^2 \int_0^{4-y^2} y \, dx\,dy = \int_0^2 y x \Big|_0^{4-y^2} dy$

$= \int_0^2 y(4-y^2)\,dy$

$= \int_0^2 (4y - y^3)\,dy$

$= 2y^2 - \frac{1}{4}y^4 \Big|_0^2$

$= 8 - 4$

$= \underline{4}$

#2 $$\int_0^1 \int_0^a \frac{1}{(1+x^2+y^2)^2}\,dy\,dx$$

$x^2+y^2=r^2$

$0 \le y \le a \qquad\qquad 0 \le x \le 1$

$y = r\sin\theta = 0 \to r=0$

$\qquad\qquad = a \to r = \frac{a}{\sin\theta} = a\csc\theta$

$x = r\cos\theta = 0 \to 0$

$\qquad\qquad = 1 \to r = \frac{1}{\cos\theta} = \sec\theta$

$\cancel{(0,0)}$

$(0,a) \to \theta = \tan^{-1}\frac{a}{0} = \frac{\pi}{2} \qquad\qquad 0 \quad \tan^{-1}a \quad \frac{\pi}{2}$

$(1,0) \to \theta = \tan^{-1}0 = 0$

$(1,a) \to \theta = \tan^{-1}a$

$$\int_0^{\tan^{-1}a}\int_0^{\sec\theta} \frac{r\,dr\,d\theta}{(1+r^2)^2} + \int_{\tan^{-1}a}^{\frac{\pi}{2}}\int_0^{a\csc\theta}\frac{r\,dr\,d\theta}{(1+r^2)^2}$$

$$= \frac{1}{2}\int_0^{\tan^{-1}a}\int_0^{\sec\theta}\frac{d(1+r^2)}{(1+r^2)^2}\,d\theta + \frac{1}{2}\int_{\tan^{-1}a}^{\frac{\pi}{2}}\int_0^{a\csc\theta}\frac{d(1+r^2)}{(1+r^2)^2}\,d\theta$$

$$= \frac{-1}{2}\int_0^{\tan^{-1}a}\frac{1}{1+r^2}\Big|_0^{\sec\theta}\,d\theta - \frac{1}{2}\int_{\tan^{-1}a}^{\pi/2}\frac{1}{1+r^2}\Big|_0^{a\csc\theta}\,d\theta$$

$$= -\frac{1}{2}\int_0^{\tan^{-1}a}\left(\frac{1}{1+\sec^2\theta}-1\right)d\theta - \frac{1}{2}\int_{\tan^{-1}a}^{\pi/2}\left(\frac{1}{1+a^2\csc^2\theta}-1\right)d\theta$$

$$= +\frac{1}{2}\int_0^{\tan^{-1}a}\frac{\sec^2\theta}{1+\sec^2\theta}\,d\theta + \frac{1}{2}\int_{\tan^{-1}a}^{\pi/2}\frac{a^2\csc^2\theta}{1+a^2\csc^2\theta}\,d\theta$$

$\sec^2\theta \to 1+\tan^2\theta$ $\qquad$ $\csc^2\theta \to 1+\cot^2\theta$ $\qquad$ $-\csc^2\theta\,d\theta$

$$= \frac{1}{2}\int_0^{\tan^{-1}a}\frac{d(\tan\theta)}{2+\tan^2\theta} - \frac{a^2}{2}\int_{\tan^{-1}a}^{\pi/2}\frac{d(\cot\theta)}{(1+a^2)+a^2\cot^2\theta}$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\tan\theta}{\sqrt{2}}\right)\Bigg|_0^{\tan^{-1}a} - \frac{1}{2}\int_{\tan^{-1}a}^{\pi/2}\frac{d(\cot\theta)}{\frac{1+a^2}{a^2}+\cot^2\theta}$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\frac{a}{\sqrt{2}} - \frac{1}{2}\frac{a}{\sqrt{1+a^2}}\tan^{-1}\left(\frac{a\cot\theta}{\sqrt{1+a^2}}\right)\Bigg|_{\tan^{-1}a}^{\pi/2}$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\frac{a}{\sqrt{2}} + \frac{1}{2}\frac{a}{\sqrt{1+a^2}}\tan^{-1}\left(\frac{1}{\sqrt{1+a^2}}\right)$$

$$\cot(\tan^{-1}a) = \frac{1}{a}$$

$$\tan^{-1}a = \alpha$$

$$\tan\alpha = \frac{a}{1}$$