Solution — *Section* 1.3 – Volumes by Slicing

Exercise

Find the volume of the solid formed by revolving the region about the x-*axis* : $y=-x+1$

Solution

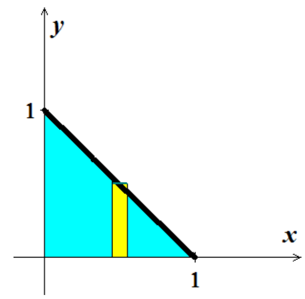
$$V=\pi\int_0^1(-x+1)^2\,dx \qquad\qquad V=\pi\int_a^b\left(f(x)^2-g(x)^2\right)dx$$

$$=\pi\int_0^1\left(x^2-2x+1\right)dx$$

$$=\pi\left(\frac{1}{3}x^3-x^2+x\right.\Big|_0^1$$

$$=\pi\left(\frac{1}{3}-1+1\right)$$

$$\underline{=\frac{\pi}{3}\quad unit^3}$$



Exercise

Find the volume of the solid formed by revolving the region about the x-*axis* : $y=4-x^2$

Solution

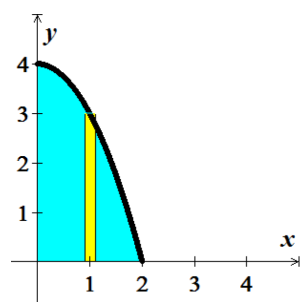
$$V=\pi\int_0^2\left(4-x^2\right)^2\,dx \qquad\qquad V=\pi\int_a^b\left(f(x)^2-g(x)^2\right)dx$$

$$=\pi\int_0^2\left(16-8x^2+x^4\right)dx$$

$$=\pi\left(16x-\frac{8}{3}x^3+\frac{1}{5}x^5\right.\Big|_0^2$$

$$=\pi\left(32-\frac{64}{3}+\frac{32}{5}\right)$$

$$\underline{=\frac{256\pi}{15}\quad unit^3}$$



Exercise

Find the volume of the solid formed by revolving the region about the x-*axis* : $y=\sqrt{x}$

Solution

$$V = \pi \int_1^4 \left(\sqrt{x}\right)^2 \, dx$$

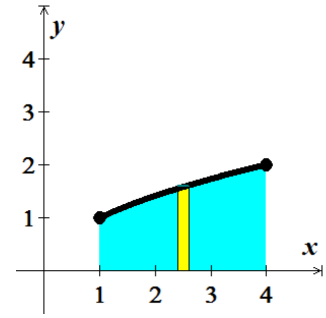
$$V = \pi \int_a^b \left(f(x)^2 - g(x)^2\right) dx$$

$$= \pi \int_1^4 x \, dx$$

$$= \pi \left(\frac{1}{2}x^2 \Big|_1^4\right.$$

$$= \pi\left(8 - \frac{1}{2}\right)$$

$$= \frac{15\pi}{2} \quad unit^3$$



Exercise

Find the volume of the solid formed by revolving the region about the *x-axis* : $y = \sqrt{9 - x^2}$

Solution

$$V = \pi \int_0^3 \left(\sqrt{9 - x^2}\right)^2 \, dx$$

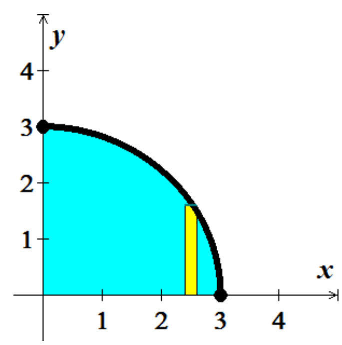
$$V = \pi \int_a^b \left(f(x)^2 - g(x)^2\right) dx$$

$$= \pi \int_0^3 \left(9 - x^2\right) dx$$

$$= \pi \left(9x - \frac{1}{3}x^3 \Big|_0^3\right.$$

$$= \pi(27 - 9)$$

$$= 18\pi \quad unit^3$$



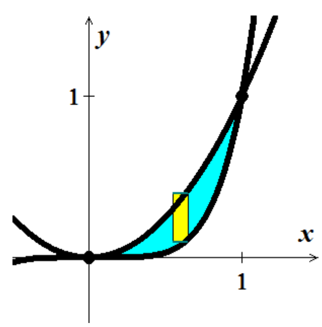
Exercise

Find the volume of the solid formed by revolving the region about the *x-axis* : $y = x^2, \quad y = x^5$

Solution

$$V = \pi \int_0^1 \left(x^4 - x^{10}\right) dx$$

$$V = \pi \int_a^b \left(f(x)^2 - g(x)^2\right) dx$$



$$= \pi \left(\frac{1}{5}x^5 - \frac{1}{11}x^{11} \Big|_0^1\right.$$

$$= \pi\left(\frac{1}{5} - \frac{1}{11}\right)$$

$$= \frac{6\pi}{55} \quad unit^3$$

Exercise

Find the volume of the solid formed by revolving the region about the *x-axis* : $y = 2, \quad y = 4 - \frac{x^2}{4}$

Solution

$$y = 4 - \frac{x^2}{4} = 2$$

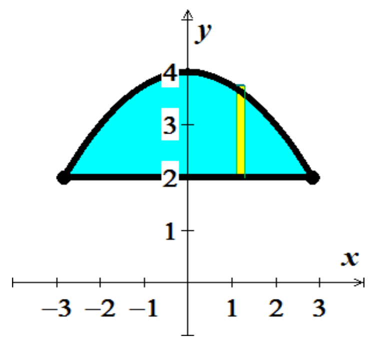
$$\frac{x^2}{4} = 2$$

$$x^2 = 8$$

$$\boxed{x = \pm 2\sqrt{2}}$$

$$V = \pi \int_{-2\sqrt{2}}^{2\sqrt{2}} \left(\left(4 - \frac{x^2}{4}\right)^2 - 4 \right) dx \qquad V = \pi \int_a^b \left(f(x)^2 - g(x)^2 \right) dx$$

$$= 2\pi \int_0^{2\sqrt{2}} \left(16 - 2x^2 + \frac{1}{16}x^4 - 4\right) dx$$



$$= 2\pi \int_0^{2\sqrt{2}} \left(12 - 2x^2 + \frac{1}{16}x^4\right) dx$$

$$= 2\pi \left(12x - \frac{2}{3}x^3 + \frac{1}{80}x^5 \right) \Big|_0^{2\sqrt{2}}$$

$$= 2\pi \left(24\sqrt{2} - \frac{32}{3}\sqrt{2} + \frac{128}{80}\sqrt{2}\right)$$

$$= 16\pi\sqrt{2}\left(3 - \frac{4}{3} + \frac{16}{80}\right)$$

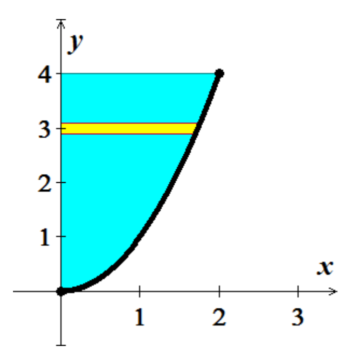
$$= 16\pi\sqrt{2}\left(\frac{448}{240}\right)$$

$$\boxed{= \frac{448\sqrt{2}}{15}\pi \quad unit^3}$$

Exercise

Find the volume of the solid formed by revolving the region about the *y-axis* : $y = x^2$

Solution



$$y = x^2 \rightarrow x = \sqrt{y}$$

$$V = \pi \int_0^4 \left(\sqrt{y}\right)^2 dy \qquad V = \pi \int_c^d \left(p(y)^2 - q(y)^2 \right) dy$$

$$= \pi \int_0^4 y\, dy$$

$$= \pi \left(\frac{1}{2} y^2 \right|_0^4$$

$$\boxed{= 8\pi \quad unit^3}$$

Exercise

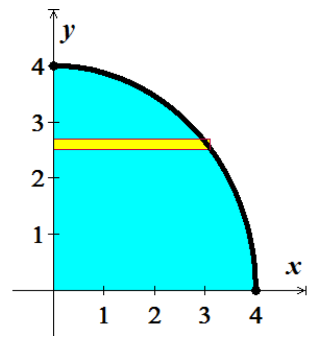
Find the volume of the solid formed by revolving the region about the *y-axis*: $\quad y = \sqrt{16 - x^2}$

Solution

$$y = \sqrt{16 - x^2}$$

$$x = \sqrt{16 - y^2}$$

$$V = \pi \int_0^4 \left(16 - y^2\right) dy \qquad\qquad V = \pi \int_c^d \left(p(y)^2 - q(y)^2\right) dy$$



$$= \pi \left(16y - \frac{1}{3} y^3 \right|_0^4$$

$$= \pi \left(64 - \frac{64}{3} \right)$$

$$\boxed{= \frac{128\pi}{3} \quad unit^3}$$

Exercise

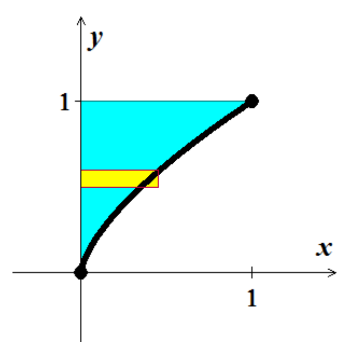
Find the volume of the solid formed by revolving the region about the *y-axis*: $\quad y = x^{2/3}$

Solution

$$y = x^{2/3}$$

$$x = y^{3/2}$$

$$V = \pi \int_0^1 \left(y^{3/2}\right)^2 dy \qquad\qquad V = \pi \int_c^d \left(p(y)^2 - q(y)^2\right) dy$$



$$= \pi \int_0^1 y^3 \, dy$$

$$= \pi \left(\frac{1}{4} y^4 \right|_0^1$$

$$\boxed{= \frac{\pi}{4} \quad unit^3}$$

Exercise

Find the volume of the solid formed by revolving the region about the y-*axis*: $x = -y^2 + 4y$

Solution

$$V = \pi \int_1^4 \left(-y^2 + 4y\right)^2 \, dy \qquad V = \pi \int_c^d \left(p(y)^2 - q(y)^2\right) dy$$

$$= \pi \int_1^4 \left(y^4 - 8y^3 + 16y^2\right) dy$$

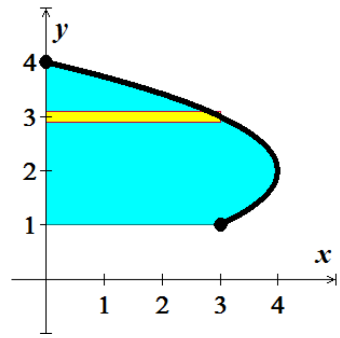
$$= \pi \left(\frac{1}{5}y^5 - 2y^4 + \frac{16}{3}y^3 \right|_1^4$$

$$= \pi \left(\frac{1024}{5} - 512 + \frac{1024}{3} - \frac{1}{5} + 2 - \frac{16}{3}\right)$$

$$= \pi \left(\frac{1023}{5} + \frac{1008}{3} - 510\right)$$

$$= \pi \left(\frac{459}{15}\right)$$

$$= \frac{153\pi}{5} \quad unit^3$$



Exercise

Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $y = \sqrt{x}, \quad y = 0, \quad x = 3$

a) the x-*axis* *b)* the y-*axis* *c)* the line $x = 3$ *d)* the line $x = 6$

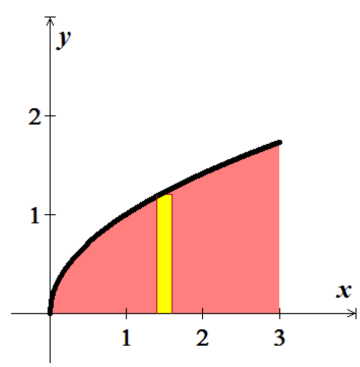
Solution

a) $\sqrt{x} = 0 \rightarrow x = 0$

$$V = \pi \int_0^3 x \, dx \qquad V = \pi \int_a^b f(x)^2 \, dx$$

$$= \pi \left(\frac{1}{2}x^2 \right|_0^3$$

$$= \frac{9\pi}{2} \quad unit^3$$



b) $y = \sqrt{x} = \sqrt{3}$

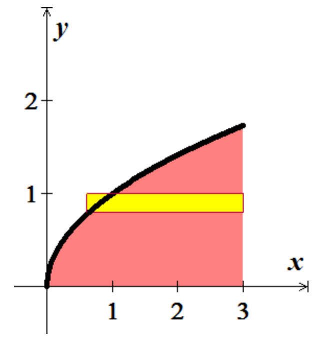
$x = y^2$

$$V = \pi \int_0^{\sqrt{3}} \left(9 - y^4\right) dy \qquad V = \pi \int_c^d \left(p(y)^2 - q(y)^2\right) dy$$

$$= \pi \left(9x - \frac{1}{5}x^5 \right|_0^{\sqrt{3}}$$

$$= \pi\left(9\sqrt{3} - \frac{9}{5}\sqrt{3}\right)$$

$$\boxed{= \frac{36\pi\sqrt{3}}{5} \quad unit^3}$$



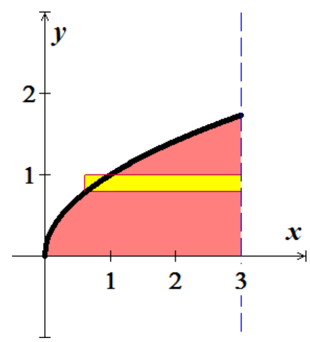
c) $V = \pi \int_0^{\sqrt{3}} \left(3 - y^2\right)^2 \, dy$

$$= \pi \int_0^{\sqrt{3}} \left(9 - 6y^2 + y^4\right) \, dy$$

$$= \pi \left(9y - 2y^3 + \frac{1}{5}y^5 \right|_0^{\sqrt{3}}$$

$$= \pi\left(9\sqrt{3} - 6\sqrt{3} + \frac{9}{5}\sqrt{3}\right)$$

$$\boxed{= \frac{24\sqrt{3}}{5}\pi \quad unit^3}$$



d) $R(y) = 3 + \left(3 - y^2\right) = 6 - y^2$

$r(y) = 3$

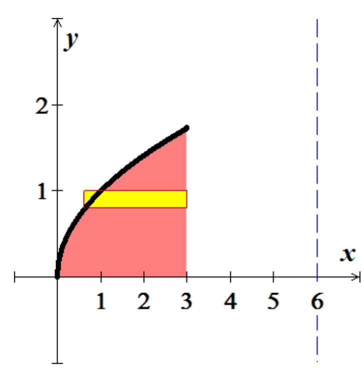
$$V = \pi \int_0^{\sqrt{3}} \left(\left(6 - y^2\right)^2 - 9\right) \, dy \qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right) dy$$

$$= \pi \int_0^{\sqrt{3}} \left(36 - 12y^2 + y^4 - 9\right) \, dy$$

$$= \pi \left(27y - 4y^3 + \frac{1}{5}y^5 \right|_0^{\sqrt{3}}$$

$$= \pi\left(27\sqrt{3} - 12\sqrt{3} + \frac{9}{5}\sqrt{3}\right)$$

$$\boxed{= \frac{84\sqrt{3}}{5}\pi \quad unit^3}$$



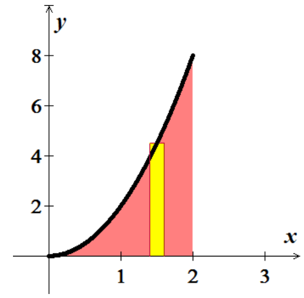
Exercise

Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $\quad y=2x^2, \quad y=0, \quad x=2$

a) the *x-axis* $\qquad$ *b)* the *y-axis* $\qquad$ *c)* the line $y=8$ $\qquad$ *d)* the line $x=2$

Solution

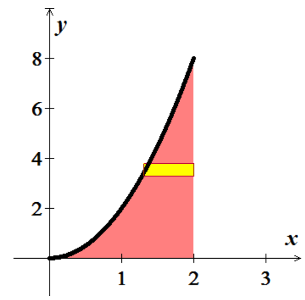
a) $V=\pi\int_0^2 4x^4\,dx \qquad V=\pi\int_a^b f(x)^2\,dx$



$$=\frac{4\pi}{5}x^5\Big|_0^2$$

$$=\frac{128\pi}{5}\quad unit^3$$

b) $R(y)=2$



$$x^2=\frac{y}{2}$$

$$r(y)=\sqrt{\frac{y}{2}}$$

$$x=2\ \to\ y=2(2)^2=8$$

$$V=\pi\int_0^8\left(4-\frac{1}{2}y\right)dy \qquad V=\pi\int_c^d\left(R(y)^2-r(y)^2\right)dy$$

$$=\pi\left(4y-\frac{1}{4}y^2\right)\Big|_0^8$$

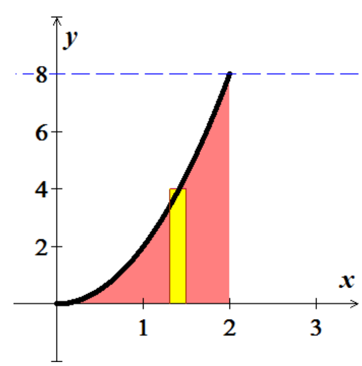
$$=\pi(32-16)$$

$$=16\pi\quad unit^3$$

c) $R(x)=8 \quad r(x)=8-2x^2$

$$V=\pi\int_0^2\left(64-\left(64-32x^2+4x^4\right)\right)dx \qquad V=\pi\int_a^b\left(R(x)^2-r(x)^2\right)dx$$

$$=4\pi\int_0^2\left(8x^2-x^4\right)dx$$



$$=4\pi\left(\frac{8}{3}x^3-\frac{1}{5}x^5\right)\Big|_0^2$$

$$=4\pi\left(\frac{64}{3}-\frac{32}{5}\right)$$

$$=\frac{896\pi}{15}\quad unit^3$$

d) $R(y)=2-\sqrt{\frac{1}{2}y} \quad r(y)=0$

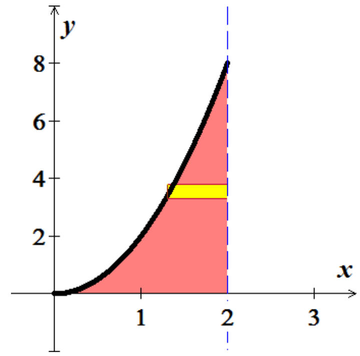
$$V=\pi\int_0^8\left(4-4\sqrt{\frac{1}{2}}y^{1/2}+\frac{1}{2}y\right)dy \qquad V=\pi\int_c^d\left(R(y)^2-r(y)^2\right)dy$$

$$=\pi\left(4y-\frac{4\sqrt{2}}{3}y^{3/2}+\frac{1}{4}y^2\right)\Big|_0^8$$

$$=\pi\left(32-\frac{4\sqrt{2}}{3}\left(2^{9/2}\right)+16\right)$$

$$=\pi\left(32-\frac{128}{3}+16\right)$$

$$=\frac{16\pi}{3} \quad unit^3$$



Exercise

Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $y=x^2, \quad y=4x-x^2$

a) the $x\text{-}axis$ *b)* the line $y=6$

Solution

$$x^2=4x-x^2$$

$$2x^2-4x=0$$

$$x=0,\ 2$$

a) $R(x)=4x-x^2 \quad r(x)=x^2$

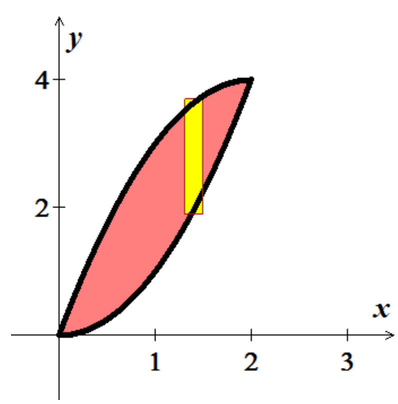
$$V=\pi\int_0^2\left(16x^2-8x^3+x^4-x^4\right)dx \qquad V=\pi\int_a^b\left(R(x)^2-r(x)^2\right)dx$$

$$=\pi\int_0^2\left(16x^2-8x^3\right)dx$$

$$=\pi\left(\frac{16}{3}x^3-2x^4\right)\Big|_0^2$$

$$=\pi\left(\frac{128}{3}-32\right)$$

$$=\frac{32\pi}{3} \quad unit^3$$



b) $R(x)=6-x^2 \quad r(x)=6-4x+x^2$

$$V = \pi \int_0^2 \left(36 - 12x^2 + x^4 - \left(36 - 48x + 28x^2 - 8x^3 + x^4\right)\right) dx$$

$$= \pi \int_0^2 \left(48x - 40x^2 + 8x^3\right) dx$$

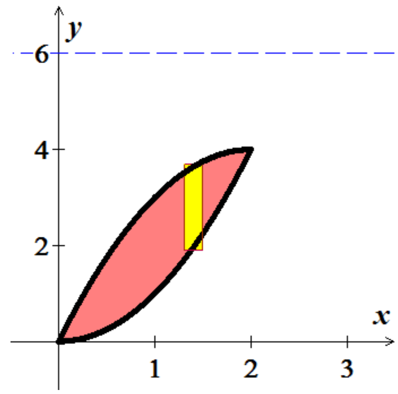
$$= 8\pi \int_0^2 \left(6x - 5x^2 + x^3\right) dx$$

$$= 8\pi \left(3x^2 - \frac{5}{3}x^3 + \frac{1}{4}x^4\right) \Big|_0^2$$

$$= 8\pi \left(12 - \frac{40}{3} + 4\right)$$

$$= 8\pi \left(\frac{8}{3}\right)$$

$$= \frac{64\pi}{3} \quad unit^3$$



Exercise

Find the volumes of the solids generated by revolving the region by the graphs of the equations about the given lines. $y = -x^2 + 2x + 4, \quad y = 4 - x$

a) the *x-axis* *b)* the line $y = 1$

Solution

$$y = -x^2 + 2x + 4 = 4 - x$$

$$-x^2 + 3x = 0$$

$$x = 0, \ 3$$

a) $R(x) = -x^2 + 2x + 4 \quad r(x) = 4 - x$

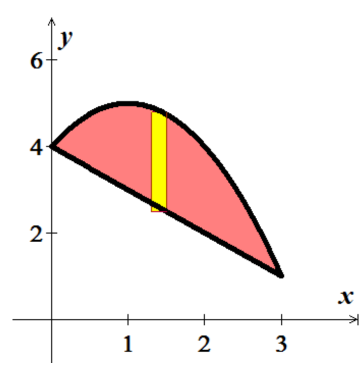
$$V = \pi \int_0^3 \left(x^4 - 4x^3 - 4x^2 + 16x + 16 - \left(16 - 8x + x^2\right)\right) dx$$

$$= \pi \int_0^3 \left(x^4 - 4x^3 - 5x^2 + 24x\right) dx$$

$$= \pi \left(\frac{1}{5}x^5 - x^4 - \frac{5}{3}x^3 + 12x^2\right) \Big|_0^3$$

$$= \pi \left(\frac{243}{5} - 81 - 45 + 108\right)$$

$$= \frac{153\pi}{5} \quad unit^3$$



b) $R(x)=\left(-x^2+2\text{x}+4\right)-1 \quad r(x)=(4-x)-1$

$$V=\pi\int_0^3\left(\left(-x^2+2x+3\right)^2-(3-x)^2\right)dx$$

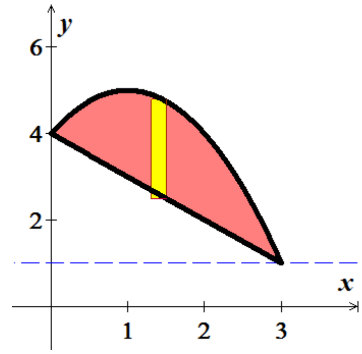
$$=\pi\int_0^3\left(x^4-4x^3-2x^2+12x+9-9+6x-x^2\right)dx$$

$$=\pi\int_0^3\left(x^4-4x^3-3x^2+18x\right)dx$$

$$=\pi\left(\frac{1}{5}x^5-x^4-x^3+9x^2\right|_0^3$$

$$=\pi\left(\frac{243}{5}-81-27+81\right)$$

$$=\frac{108\pi}{5}\quad unit^3$$



Exercise

The solid lies between planes perpendicular to the x-axis at $x=0$ and $x=4$. The cross-sections perpendicular to the axis on the interval $0\le x\le 4$ are squares whose diagonals run from the parabola $y=-\sqrt{x}$ to the parabola $y=\sqrt{x}$. Find the volume of the solid.

Solution

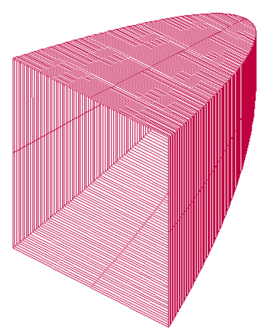
$$A(x)=\tfrac{1}{2}(diagonal)^2$$

$$=\frac{1}{2}\left(\sqrt{x}-\left(-\sqrt{x}\right)\right)^2$$

$$=\frac{1}{2}\left(2\sqrt{x}\right)^2$$

$$=\frac{1}{2}(4x)$$

$$=2x\quad unit^2 \qquad a=0,\quad b=4;$$



$$V=\int_a^b A(x)\,dx$$

$$=\int_0^4 2x\,dx$$

$$=x^2\Big|_0^4$$

$= 4^2 - 0$

$\boxed{= 16 \quad unit^3}$

Exercise

The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 1$. The cross-sections perpendicular to the x-axis between these planes are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = \sqrt{x}$. Find the volume of the solid.

Solution

Given: $y = x^2 \quad y = \sqrt{x} \quad 0 \le x \le 1$

$$Area = \frac{\pi}{4} D^2$$
$$= \frac{\pi}{4}\left(\sqrt{x} - x^2\right)^2$$
$$= \frac{\pi}{4}\left(x - 2x^{5/2} + x^4\right)$$

$$V = \frac{\pi}{4}\int_0^1 \left(x - 2x^{5/2} + x^4\right)dx$$
$$= \frac{\pi}{4}\left(\frac{1}{2}x^2 - \frac{4}{7}x^{7/2} + \frac{1}{5}x^5\right.\Big|_0^1$$
$$= \frac{\pi}{4}\left(\frac{1}{2} - \frac{4}{7} + \frac{1}{5}\right)$$
$$\boxed{= \frac{9\pi}{280} \quad unit^3}$$

Exercise

The base of the solid is the region in the first quadrant between the line $y = x$ and the parabola $y = 2\sqrt{x}$. The cross-sections of the solid perpendicular to the x-axis are equilateral triangles whose bases stretch from the line to the curve. Find the volume of the solid.

Solution

$$y = x = 2\sqrt{x}$$
$$x^2 = 4x \quad \to \quad \boxed{x = 0,\ 4}$$

$$Area(equilateral\ \Delta) = \frac{1}{2}(side)^2 \sin\frac{\pi}{3}$$
$$= \frac{1}{2}\left(2\sqrt{x} - x\right)^2\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\sqrt{3}}{4}\left(4x - 4x\sqrt{x} + x^2\right)$$

$$V=\frac{\sqrt{3}}{4}\int_0^4\left(4x-4x^{3/2}+x^2\right)dx$$

$$=\frac{\sqrt{3}}{4}\left(2x^2-\frac{8}{5}x^{5/2}+\frac{1}{3}x^3\right|_0^4$$

$$=\frac{\sqrt{3}}{4}\left(32-\frac{8}{5}(32)+\frac{64}{3}\right)$$

$$=2\sqrt{3}\left(4-\frac{32}{5}+\frac{8}{3}\right)$$

$$=2\sqrt{3}\left(\frac{60-96+40}{15}\right)$$

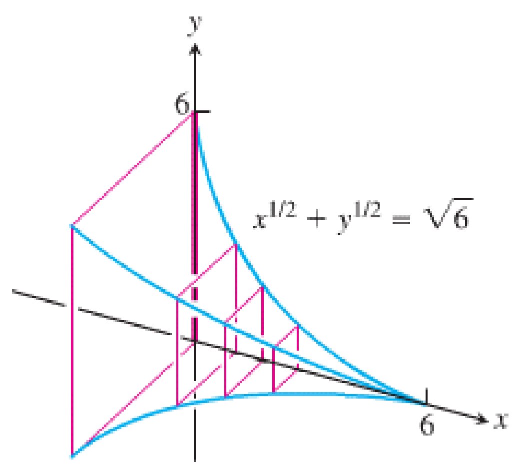
$$=\frac{8\sqrt{3}}{15}\quad unit^3$$

Exercise

The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 6$. The cross-sections between these planes are squares whose bases run from the x-axis up to the curve $x^{1/2}+y^{1/2}=\sqrt{6}$. Find the volume of the solid.

Solution

Given: $x^{1/2}+y^{1/2}=\sqrt{6}\quad 0\le x\le 6$



$$y^{1/2}=\sqrt{6}-\sqrt{x}$$

$$y=\left(\sqrt{6}-\sqrt{x}\right)^2$$

$$Area(square)=(side)^2$$

$$=\left(\left(\sqrt{6}-\sqrt{x}\right)^2-0\right)^2$$

$$=\left(6-2\sqrt{6}\sqrt{x}+x\right)^2$$

$$=36-24\sqrt{6}x^{1/2}+36x-4\sqrt{6}x^{3/2}+x^2$$

$$V=\int_0^6\left(36-24\sqrt{6}x^{1/2}+36x-4\sqrt{6}x^{3/2}+x^2\right)dx$$

$$=\left(36x-16\sqrt{6}\,x^{3/2}+18x^2-\frac{8}{5}\sqrt{6}x^{5/2}+\frac{1}{3}x^3\right|_0^6$$

$$=216-16(6)^2+18(36)-\frac{8}{5}(6)^3+72$$

$$=216-576+648-\frac{1,728}{5}+72$$

$$= 360 - \frac{1,728}{5}$$

$$\boxed{= \frac{72}{5} \quad unit^3}$$

Exercise

The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$. Find the volume of the solid.

Solution

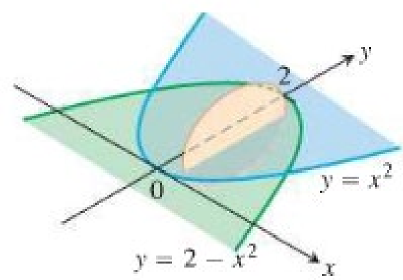
$$y = 2 - x^2 = x^2 \quad \Rightarrow \quad 2x^2 = 2 \to x^2 = 1 \quad \Rightarrow \boxed{x = \pm 1}$$

$$A(x) = \frac{\pi}{4}(diameter)^2 = \frac{\pi}{4}\left(2 - x^2 - x^2\right)^2$$

$$= \frac{\pi}{4}\left(2\left(1 - x^2\right)\right)^2$$

$$= \frac{\pi}{4}4\left(1 - 2x^2 + x^4\right)$$

$$\boxed{= \pi\left(1 - 2x^2 + x^4\right) \quad unit^2} \qquad a = -1, \quad b = 1;$$



$$V = \int_a^b A(x)\,dx$$

$$= \int_{-1}^{1} \pi\left(1 - 2x^2 + x^4\right) dx$$

$$= \pi\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_{-1}^{1}$$

$$= \pi\left[\left(1 - \frac{2}{3} + \frac{1}{5}\right) - \left(-1 + \frac{2}{3} - \frac{1}{5}\right)\right]$$

$$= 2\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$\boxed{= \frac{16\pi}{15} \quad unit^3}$$

Exercise

The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$. Find the volume of the solid.

Solution

$$A(x) = \frac{1}{2}(diagonal)^2$$

$$= \frac{1}{2}\left(\sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right)\right)^2$$

$$= \frac{1}{2}\left(2\sqrt{1-x^2}\right)^2$$

$$= 2\left(1-x^2\right) \quad unit^2 \qquad a=-1, \quad b=1;$$

$$V = \int_a^b A(x)\,dx$$

$$= \int_{-1}^{1} 2\left(1-x^2\right)dx$$

$$= 2\left(x - \frac{1}{3}x^3\right.\Big|_{-1}^{1}$$

$$= 2\left[\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)\right]$$

$$= 4\left(1-\frac{1}{3}\right)$$

$$= \frac{8}{3} \quad unit^3$$

Exercise

The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis. The cross-sections perpendicular to the x-axis are

Find the volume of the solid.

a) Equilateral triangles with bases running from the x-axis to the curve as shown

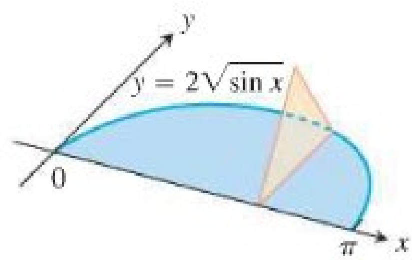
b) Squares with bases running from the x-axis to the curve.

Solution

a) $A(x) = \frac{1}{2}(side)(side)\cdot\sin\frac{\pi}{3}$ *Equilateral triangle* $\theta = \frac{\pi}{3}$

$$= \frac{1}{2}\left(2\sqrt{\sin x}\right)\left(2\sqrt{\sin x}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \sqrt{3}\sin x \quad unit^2 \qquad a=0, \quad b=\pi;$$



$$V = \sqrt{3}\int_0^{\pi}\sin x\,dx$$

$$= \sqrt{3}\left(-\cos x\right.\Big|_0^{\pi}$$

$$= -\sqrt{3}\left[\cos\pi - \cos 0\right]$$

$$\underline{= 2\sqrt{3} \quad unit^3}$$

b) $A(x) = (side)^2$ $\qquad a = 0, \quad b = \pi$

$$= \left(2\sqrt{\sin x}\right)^2$$

$$\underline{= 4\sin x \quad unit^2}$$

$$V = 4\int_0^{\pi} \sin x\, dx$$

$$= 4\left(-\cos x\right|_0^{\pi}$$

$$= -4(\cos\pi - \cos 0)$$

$$\underline{= 8 \quad unit^3}$$

Exercise

The base of the solid is the disk $x^2 + y^2 \le 1$. The cross-sections by planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.

Solution

$$x^2 + y^2 = 1$$

$$x^2 = 1 - y^2$$

$$\underline{x = \pm\sqrt{1 - y^2}}$$

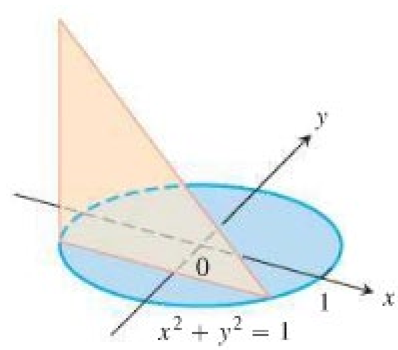
$$A(y) = \frac{1}{2}(leg)(leg)$$

$$= \frac{1}{2}\left(\sqrt{1-y^2} - \left(-\sqrt{1-y^2}\right)\right)^2$$

$$= \frac{1}{2}\left[2\sqrt{1-y^2}\right]^2$$

$$\underline{= 2\left(1 - y^2\right) \quad unit^2} \qquad c = -1, \quad d = 1;$$

$$V = \int_{-1}^{1} 2\left(1 - y^2\right) dy \qquad V = \int_c^d A(y)\, dy$$



$$= 2\left(y - \frac{1}{3}y^3\right|_{-1}^{1}$$

$$= 2\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right]$$

$$= 4\left(1-\frac{1}{3}\right)$$

$$= \frac{8}{3} \quad unit^3$$

Exercise

Find the volume of the given tetrahedron. (***Hint***: Consider slices perpendicular to one of the labeled edges)

Solution

Let consider the slices perpendicular to edge labeled 5 are triangles.

By similar triangles, we have: $\frac{height}{base} = \frac{h}{b} = \frac{3}{4}$

$$h = \frac{3}{4}b$$

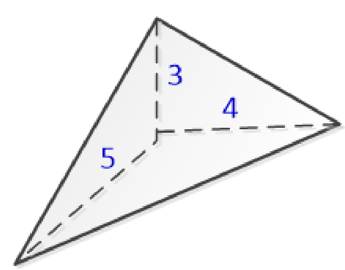
The equation of the line through (5, 0) and (0, 4) is:

$$y = \frac{4-0}{0-5}(x-0)+4$$

$$y = -\frac{4}{5}x+4$$

Therefore, the length of the base: $b = -\frac{4}{5}x+4$

$$h = \frac{3}{4}b = \frac{3}{4}\left(-\frac{4}{5}x+4\right) = -\frac{3}{5}x+3$$



$$A(x) = \frac{1}{2}(base)\cdot(height) = \frac{1}{2}\left(-\frac{4}{5}x+4\right)\left(-\frac{3}{5}x+3\right)$$

$$= \frac{1}{2}\left(\frac{12}{25}x^2 - \frac{24}{5}x+12\right)$$

$$= \frac{6}{25}x^2 - \frac{12}{5}x+5 \quad unit^2$$

$$V = \int_0^5 \left(\frac{6}{25}x^2 - \frac{12}{5}x+5\right)dx \qquad\qquad V = \int_a^b A(x)\,dx$$

$$= \frac{2}{25}x^3 - \frac{6}{5}x^2 + 5x \Big|_0^5$$

$$= 10 - 30 + 25$$

$$= 5 \quad unit^3$$

Exercise

Find the volume of the solid of revolution

Solution

$$V=\pi\int_{-\sqrt{3}/3}^{\sqrt{3}}\left(\frac{1}{\sqrt{1+x^2}}\right)^2 dx$$

$$=\pi\int_{-\sqrt{3}/3}^{\sqrt{3}}\frac{1}{1+x^2}dx$$

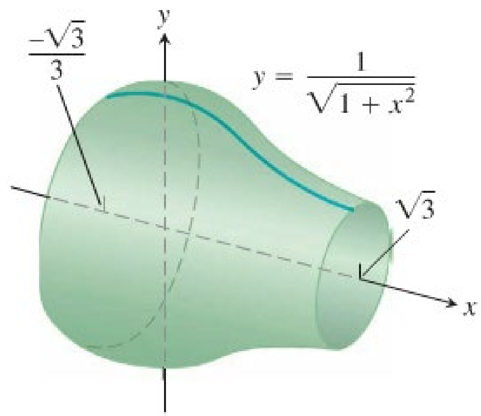
$$=\pi\left(\tan^{-1}x\right|_{-\sqrt{3}/3}^{\sqrt{3}}$$

$$=\pi\left(\tan^{-1}\sqrt{3}-\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right)$$

$$=\pi\left(\frac{\pi}{3}-\left(-\frac{\pi}{6}\right)\right)$$

$$=\pi\left(\frac{\pi}{3}+\frac{\pi}{6}\right)$$

$$\boxed{=\frac{\pi^2}{2}\quad unit^3}$$



Exercise

Find the volume of the solid generated by revolving the region bounded by $y=x^2$ and the lines $y=0$, $x=2$ about the x-axis.

Solution

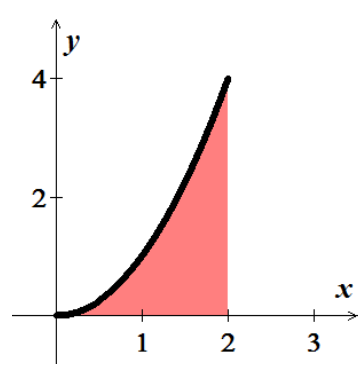
$$R(x)=x^2$$

$$V=\pi\int_0^2\left(x^2\right)^2 dx$$

$$=\pi\int_0^2 x^4 dx$$

$$=\pi\left(\frac{1}{5}x^5\right|_0^2$$

$$\boxed{=\frac{32\pi}{5}\quad unit^3}$$



Exercise

Find the volume of the solid generated by revolving the region bounded by $y = x - x^2$ and the line $y = 0$ about the x-axis.

Solution

$$R(x) = x - x^2$$

$$x - x^2 = 0 \rightarrow \boxed{x = 0,\ 1}$$

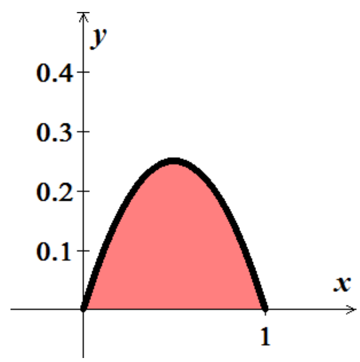
$$V = \int_0^1 \pi \left(x - x^2\right)^2 dx$$

$$= \pi \int_0^1 \left(x^2 - 2x^3 + x^4\right) dx$$

$$= \pi \left(\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right|_0^1$$

$$= \pi\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right)$$

$$\boxed{= \frac{\pi}{30} \quad unit^3}$$



Exercise

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{\cos x}$ and the lines $0 \le x \le \frac{\pi}{2}, \quad y = 0, \quad x = 0$ about the x-axis.

Solution

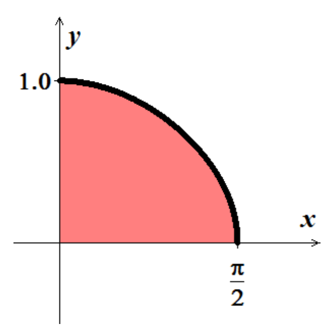
$$R(x) = \sqrt{\cos x}$$

$$V = \int_0^{\pi/2} \pi \left(\sqrt{\cos x}\right)^2 dx$$

$$= \pi \int_0^{\pi/2} \cos x dx$$

$$= \pi \left(\sin x\right|_0^{\pi/2}$$

$$\boxed{= \pi \quad unit^3}$$



Exercise

Find the volume of the solid generated by revolving the region bounded by $y = \sec x$ and the lines $y = 0, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$ about the x-axis.

Solution

$R(x) = \sec x$

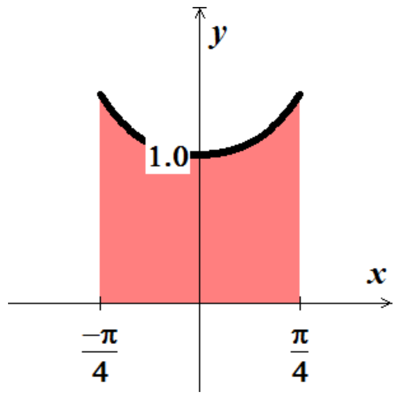
$$V = \pi \int_{-\pi/4}^{\pi/4} \sec^2 x \, dx$$

$$= \pi \left(\tan x \Big|_{-\pi/4}^{\pi/4} \right.$$

$$= \pi \left(\tan\left(\frac{\pi}{4}\right) - \tan\left(-\frac{\pi}{4}\right) \right)$$

$$= \pi \left(1 - (-1) \right)$$

$$= 2\pi \quad unit^3$$



Exercise

Find the volume of the solid generated by revolving the region bounded by $x = \sqrt{5}\, y^2$ and the lines $x = 0, \quad y = -1, \quad y = 1$ about the y-axis.

Solution

$R(y) = \sqrt{5} y^2$

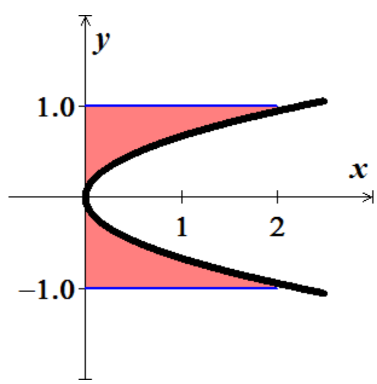
$$V = \pi \int_{-1}^{1} \left(\sqrt{5} y^2 \right)^2 dy$$

$$= \pi \int_{-1}^{1} 5y^4 dy$$

$$= \pi \left(y^5 \Big|_{-1}^{1} \right.$$

$$= \pi \left(1 - (-1) \right)$$

$$= 2\pi \quad unit^3$$



Exercise

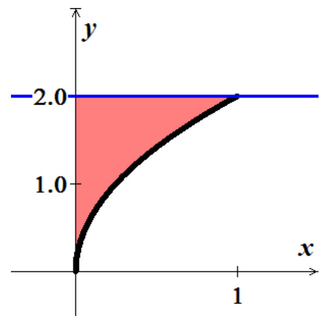
Find the volume of the solid generated by revolving the region bounded by $y = 2\sqrt{x}$ and the lines $y = 2, \quad x = 0$ about the *x*-axis.

Solution

$$r(x) = 2\sqrt{x} \quad \textit{and} \quad R(x) = 2$$

$$V = \pi \int_0^1 \left((2)^2 - \left(2\sqrt{x}\right)^2 \right) dx \qquad\qquad V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2 \right) dx$$

$$= \pi \int_0^1 (4 - 4x)\, dx$$



$$= 4\pi \left(x - \frac{1}{2}x^2 \right) \Big|_0^1$$

$$= 4\pi \left[\left(1 - \frac{1}{2}\right) - 0 \right]$$

$$= 2\pi \quad unit^3$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y = \sec x, \quad y = \tan x$ and the lines $x = 0, \quad x = 1$ about the *x*-axis.

Solution

$$r(x) = \tan x \quad \textit{and} \quad R(x) = \sec x$$

$$V = \pi \int_0^1 \left(\sec^2 x - \tan^2 x \right) dx \qquad\qquad V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2 \right) dx$$

$$= \pi \int_0^1 (1)\, dx$$

$$= \pi\, x \Big|_0^1$$

$$= \pi \quad unit^3$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $x = \sqrt{2\sin 2y}$ and the lines $0 \le y \le \frac{\pi}{2}, \quad x = 0$ about the y-axis.

Solution

$$V = \pi \int_0^{\pi/2} \left(\sqrt{2\sin 2y}\right)^2 dy$$

$$= 2\pi \int_0^{\pi/2} (\sin 2y)\, dy$$

$$= -\pi \cos 2y \Big|_0^{\pi/2}$$

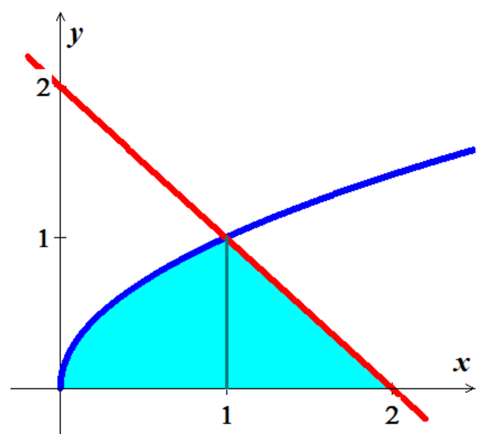
$$= -\pi(-1-1)$$

$$\boxed{= 2\pi \quad unit^3}$$

Exercise

What is the volume of the solid whose base is the region in the first quadrant bounded by $y = \sqrt{x}$, $y = 2 - x$, and the x-axis, and whose cross sections perpendicular to the base and parallel to the y-axis are squares?

Solution



$$y = 2 - x = \sqrt{x}$$

$$(2-x)^2 = x$$

$$x^2 - 5x + 4 = 0 \quad \rightarrow \quad \boxed{x = 1, \cancel{4}}$$

$$y = 2 - x = 0 \quad \rightarrow \quad \boxed{x = 2}$$

$$V = \int_0^1 \left(\sqrt{x}\right)^2 dx + \int_1^2 (2-x)^2\, dx$$

$$= \int_0^1 x\, dx + \int_1^2 \left(4 - 4x + x^2\right) dx$$

$$= \frac{1}{2}x^2 \Big|_0^1 + \left(4x - 2x^2 + \frac{1}{3}x^3\right|_1^2$$

$$= \frac{1}{2} + 8 - 8 + \frac{8}{3} - 4 + 2 - \frac{1}{3}$$

$$= \frac{3 + 14 - 12}{6}$$

$$\boxed{= \frac{5}{6} \quad unit^3}$$

Exercise

What is the volume of the solid whose base is the region in the first quadrant bounded by $y=\sqrt{x}$, $y=2-x$, and the x-axis, and whose cross sections perpendicular to the base and parallel to the y-axis are semicircles?

Solution

$0<x<1$ slices are semicircles with diameter $\sqrt{x}$

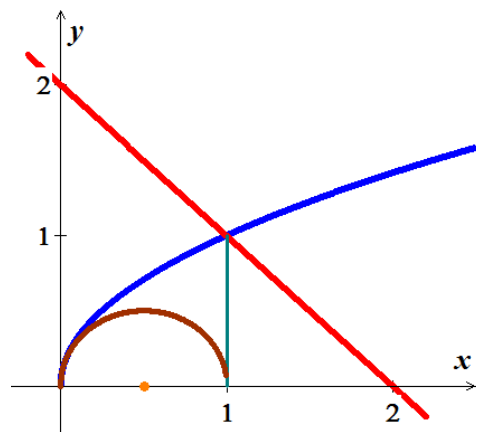
$$Area=\frac{1}{2}\pi\left(\frac{\sqrt{x}}{2}\right)^2$$

$$\boxed{=\frac{\pi}{8}x}$$

$1<x<2$ with diameter $2-x$

$$Area=\frac{1}{2}\pi\left(\frac{2-x}{2}\right)^2$$

$$\boxed{=\frac{\pi}{8}\left(4-4x+x^2\right)}$$



$$V=\frac{\pi}{8}\int_0^1 x\,dx+\frac{\pi}{8}\int_1^2\left(4-4x+x^2\right)dx$$

$$=\frac{\pi}{16}x^2\Big|_0^1+\frac{\pi}{8}\left(4x-2x^2+\frac{1}{3}x^3\right)\Big|_1^2$$

$$=\frac{\pi}{16}+\frac{\pi}{8}\left(8-8+\frac{8}{3}-4+2-\frac{1}{3}\right)$$

$$=\frac{\pi}{16}+\frac{\pi}{8}\left(\frac{7}{3}-2\right)$$

$$=\frac{\pi}{16}+\frac{\pi}{24}$$

$$\boxed{=\frac{5\pi}{48}\ unit^3}$$

Exercise

What is the volume of the solid whose base is the region in the first quadrant bounded by $y=\sqrt{x}$, $y=2-x$, and the y-axis, and whose cross sections perpendicular to the base and parallel to the x-axis are square?

Solution

$$\begin{cases} y=\sqrt{x} & \rightarrow \quad x=y^2 \\ y=2-x & \rightarrow \quad x=2-y \end{cases}$$

$$x=y^2=2-y$$

$$y^2+y-2=0 \quad\rightarrow\quad \boxed{y=1,\ \cancel{-2}}$$

$$V=\int_0^1\left(y^2\right)^2 dy+\int_1^2(2-y)^2 dy$$

$$=\int_0^1 y^4\, dy+\int_1^2\left(4-4y+y^2\right)dy$$

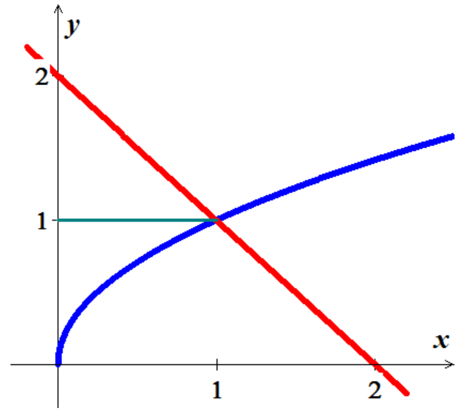
$$=\frac{1}{5}y^5\Big|_0^1+\left(4y-2y^2+\frac{1}{3}y^3\right)\Big|_1^2$$

$$=\frac{1}{5}+8-8+\frac{8}{3}-4+2-\frac{1}{3}$$

$$=\frac{1}{5}+\frac{7}{3}-2$$

$$=\frac{3+35-30}{15}$$

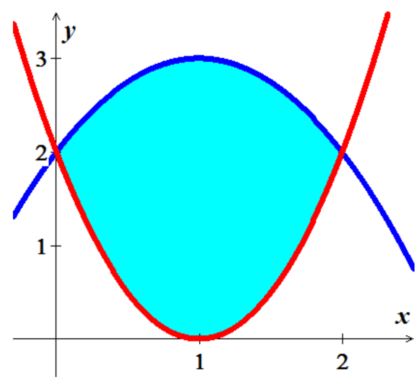
$$=\frac{8}{15}\ unit^3$$



Exercise

The region bounded by the curves $y=-x^2+2x+2$ and $y=2x^2-4x+2$ is revolved about the x-axis. What is the volume of the solid that is generated?

Solution



$$y=2x^2-4x+2=-x^2+2x+2$$

$$3x^2-6x=0 \quad\rightarrow\quad \underline{x=0,\ 2}$$

$$V=\pi\int_0^2\left[\left(-x^2+2x+2\right)^2-\left(2x^2-4x+2\right)^2\right]dx$$

$$=\pi\int_0^2\left(\begin{array}{l}x^4-2x^3-2x^2-2x^3+4x^2+4x-2x^2+4x+4\\ -4x^4+8x^3-4x^2+8x^3-16x^2+8x-4x^2+8x-4\end{array}\right)dx$$

$$=\pi\int_0^2\left(-3x^4+12x^3-24x^2+24x\right)dx$$

$$=\pi\left(-\frac{3}{5}x^5+3x^4-8x^3+12x^2\right)\Big|_0^2$$

$$=\pi\left(-\frac{96}{5}+48-64+48\right)$$

$$=\pi\left(32-\frac{96}{5}\right)$$

$$=\frac{64\pi}{5}\ unit^3$$

Exercise

The region bounded by the curves $y = 2e^{-x}$, $y = e^{x}$, and the y-axis is revolved about the x-axis. What is the volume of the solid that is generated?

Solution

$$y = 2e^{-x} = e^{x}$$

$$\left(e^{x}\right)^2 = 2$$

$$e^{x} = \sqrt{2} \quad \rightarrow \quad \underline{x = \ln\sqrt{2}}$$

$$V = \pi\int_0^{\ln\sqrt{2}} \left(\left(2e^{-x}\right)^2 - \left(e^{x}\right)^2\right)dx \qquad\qquad V = \pi\int_a^b \left(f(x)^2 - g(x)^2\right)dx$$

$$= \pi\int_0^{\ln\sqrt{2}} \left(4e^{-2x} - e^{2x}\right)dx$$

$$= \pi\left(-2e^{-2x} - \frac{1}{2}e^{2x}\right. \Bigg|_0^{\ln\sqrt{2}}$$

$$= \pi\left(-2e^{-2\ln\sqrt{2}} - \frac{1}{2}e^{2\ln\sqrt{2}} + 2 + \frac{1}{2}\right)$$

$$= \pi\left(-2e^{-\ln 2} - \frac{1}{2}e^{\ln 2} + \frac{5}{2}\right)$$

$$= \pi\left(-2\left(\frac{1}{2}\right) - \frac{1}{2}(2) + \frac{5}{2}\right)$$

$$= \pi\left(-2 + \frac{5}{2}\right)$$

$$= \frac{\pi}{2}\ unit^3$$

Exercise

The region bounded by the curves $y = \sec x$, $y = 2$, for $0 \le x \le \frac{\pi}{3}$ is revolved around the x-axis. What is the volume of the solid that is generated?

Solution

$$V = \pi\int_0^{\pi/3} \left((2)^2 - (\sec x)^2\right)dx \qquad\qquad V = \pi\int_a^b \left(f(x)^2 - g(x)^2\right)dx$$

$$= \pi\int_0^{\pi/3} \left(4 - \sec^2 x\right)dx$$

$$= \pi\left(4x - \tan x\right|_0^{\pi/3}$$

$$= \pi\left(\frac{4\pi}{3} - \sqrt{3}\right)$$

$$\boxed{= \frac{4}{3}\pi^2 - \pi\sqrt{3} \quad unit^3}$$

Exercise

The region bounded by the graph $y = (x-2)^2$ and $y = 4$ is revolved about the line $y = 4$. What is the volume of the resulting solid?

Solution

Disk *Method*:

$radius: \; 4 - (x-2)^2 = -x^2 + 4x$

$$V = \pi\int_0^4 \left(-x^2 + 4x\right)\left(4 - (x-2)^2\right) dx$$

$$= \pi\int_0^4 \left(-x^2 + 4x\right)^2 dx$$

$$= \pi\int_0^4 \left(x^4 - 8x^3 + 16x^2\right) dx$$

$$= \pi\left(\frac{1}{5}x^5 - 2x^4 + \frac{16}{3}x^3\right|_0^4$$

$$= \pi\left(\frac{4}{5}4^4 - 2\left(4^4\right) + \frac{4}{3}4^4\right)$$

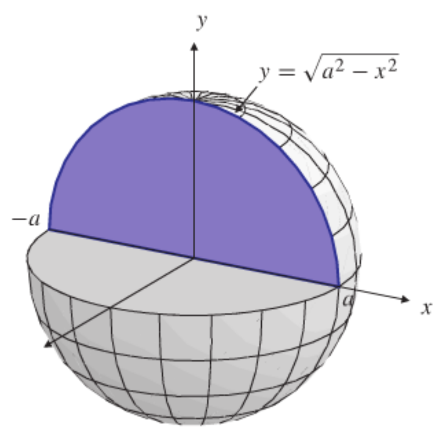
$$= 4^4\pi\left(\frac{4}{5} - 2 + \frac{4}{3}\right)$$

$$= 256\pi\left(\frac{12 - 30 + 20}{15}\right)$$

$$\boxed{= \frac{512\pi}{15} \quad unit^3}$$

Exercise

Find the volume of a solid ball having radius a.



Solution

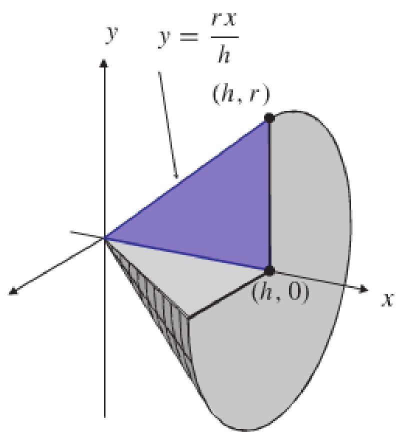
$$V = \pi\int_{-a}^{a} \left(\sqrt{a^2 - x^2}\right)^2 dx$$

$$= 2\pi \int_0^a \left(a^2 - x^2\right) dx$$

$$= 2\pi \left(a^2 x - \frac{x^3}{3}\right|_0^a$$

$$= 2\pi \left(a^3 - \frac{a^3}{3}\right)$$

$$= \frac{4}{3}\pi a^3 \quad unit^3$$



Exercise

You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentation at home persuades you that you can get one that hold about 3 *L* if you make it 9 *cm* deep and give the sphere a radius of 16 *cm*. To be sure, you picture the wok as a solid of revolution, and calculate its volume with an integral. To the nearest cubic centimeter, what volume do you really get?

$\left(1\ L = 1{,}000\ cm^3\right)$

Solution

$$x^2 + y^2 = 256$$

$$x^2 = 256 - y^2$$

$$R(y) = \sqrt{256 - y^2}$$

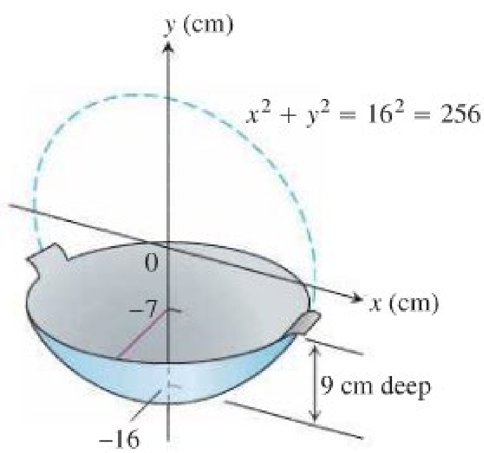
$$V = \pi \int_{-16}^{-7} \left(\sqrt{256 - y^2}\right)^2 dy$$

$$= \pi \int_{-16}^{-7} \left(256 - y^2\right) dy$$

$$= \pi \left(256y - \frac{1}{3}y^3\right|_{-16}^{-7}$$

$$= \pi \left[\left(256(-7) - \frac{1}{3}(-7)^3\right) - \left(256(-16) - \frac{1}{3}(-16)^3\right)\right]$$

$$= 1053\pi \ \ cm^3 \quad \approx 3308 \ \ cm^3$$



Exercise

A curved wedge is cut from a circular cylinder of radius 3 by two planes. On plane is perpendicular to the axis of the cylinder. The second place crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

Solution

The base of the cylinder is a circle $x^2 + y^2 = 9$.

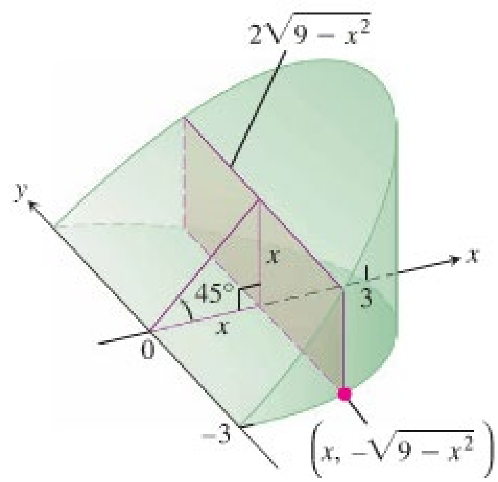
Since the second plane cut the base at the cylinder at the center, therefore, the base of the wedge is semi-circle. $y = \pm\sqrt{9 - x^2} = radius$.

When we slice the wedge by a plane perpendicular to the axis of the cylinder, we obtained a cross-section at x which is a rectangle of height x.

The area of this cross-section is: $A(x) = height \times width$

$$= x\left(2\sqrt{9 - x^2}\right)$$

$$= 2x\sqrt{9 - x^2}$$



The rectangles run from $x = 0$ to $x = 3$, so

$$V = \int_0^3 2x\sqrt{9 - x^2}\, dx \qquad or\ \ u = 9 - x^2 \quad \rightarrow \quad du = -2xdx$$

$$= -\int_0^3 \left(9 - x^2\right)^{1/2} d\left(9 - x^2\right)$$

$$= -\frac{2}{3}\left(9 - x^2\right)^{3/2} \Big|_0^3$$

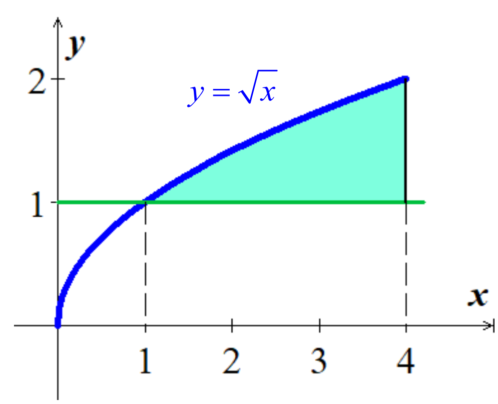
$$= -\frac{2}{3}\left(0 - 9^{3/2}\right)$$

$$\underline{= 18 \quad unit^3}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

Solution



$$V = \pi\int_1^4 \left(\sqrt{x} - 1\right)^2 dx \qquad\qquad V = \int_1^4 \pi\left[R(x)\right]^2 dx$$

$$= \pi\int_1^4 \left(x - 2\sqrt{x} + 1\right) dx$$

$$= \pi\left(\frac{x^2}{2} - 2\frac{2}{3}x^{3/2} + x\right|_1^4$$

$$= \pi\left[\left(8 - \frac{4}{3}4^{3/2} + 4\right) - \left(\frac{1}{2} - \frac{4}{3} + 1\right)\right]$$

$$= \frac{7\pi}{6} \quad unit^3$$

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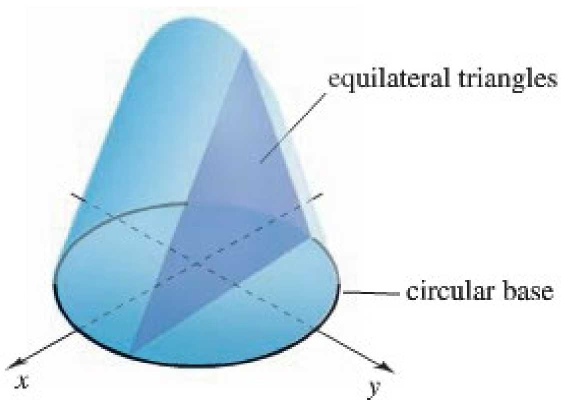
Exercise

The solid with a circular base of radius 5 whose cross sections perpendicular to the base and parallel to the *x*-axis are equilateral triangles. Use the general slicing method to find the volume of the solid.

Solution



$$x^2 + y^2 = 5^2$$

$$y = \pm\sqrt{25 - x^2}$$

$$A(x) = \left(2\sqrt{25 - x^2}\right)^2$$

$$= 100 - 4x^2$$

$$V = \int_0^5 \left(100 - 4x^2\right)dx$$

$$= 100x - \frac{4}{3}x^3 \Big|_0^5$$

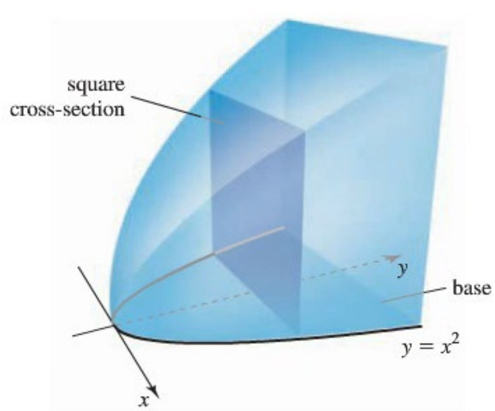
$$= 500 - \frac{500}{3}$$

$$= \frac{1000}{3} \quad unit^3$$

Exercise

The solid whose base is the region bounded by $y = x^2$ and the line $y = 1$ and whose cross sections perpendicular to the base and parallel to the *x*-axis squares.
Use the general slicing method to find the volume of the solid.

Solution



$$y = x^2$$

$$x = \sqrt{y}$$

$$V = \int_0^1 A(y)\,dy$$

$$= \int_0^1 \left(2\sqrt{y}\right)^2 dy$$

$$= \int_0^1 (4y)\,dy$$

$$= 2y^2 \Big|_0^1$$

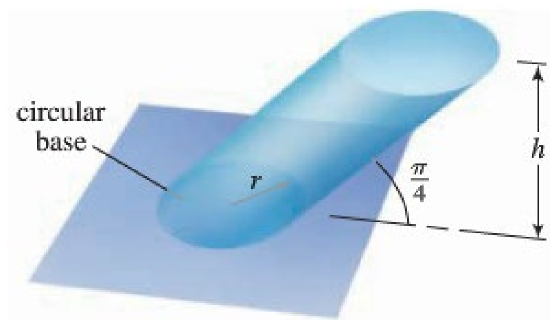
$$\boxed{= 2 \quad unit^3}$$

Exercise

A circular cylinder of radius r and height h whose curved surface is at an angle of $\frac{\pi}{4}$ rad . Use the general slicing method to find the volume of the solid

Solution

The cross sections are all circles with area πr^2



$$V = \int_0^h \left(\pi r^2\right) dz$$

$$= \pi r^2 z \Big|_0^h$$

$$\boxed{= \pi r^2 h \quad unit^3}$$

∴ The 45° angle does not affect the volume.

Exercise

Let R be the region bounded by $y = e^{-x}, \quad y = 0, \quad x = 0, \quad x = \ln 4$. Use the disk method to find the volume of the solid generated when R is revolved about x-axis.

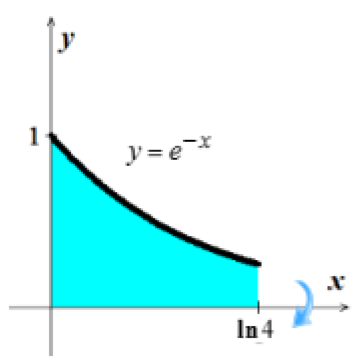
Solution

$$V = \pi \int_0^{\ln 4} e^{-2x}\,dx$$

$$= -\frac{\pi}{2} e^{-2x} \Big|_0^{\ln 4}$$

$$= -\frac{\pi}{2}\left(e^{-2\ln 4} - 1\right)$$

$$= -\frac{\pi}{2}\left(\frac{1}{16} - 1\right)$$



$$= \boxed{\frac{15\pi}{32} \quad unit^3}$$

Exercise

Let R be the region bounded by $y = \cos x, \quad y = 0, \quad x = 0$. Use the disk method to find the volume of the solid generated when R is revolved about x-axis,

Solution

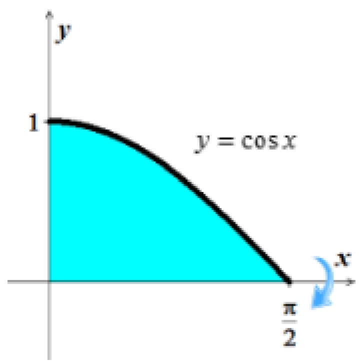
$$V = \pi \int_0^{\pi/2} \cos^2 x \, dx$$

$$= \frac{\pi}{2} \int_0^{\pi/2} (1 + \cos 2x) \, dx$$

$$= \frac{\pi}{2} \left(x + \frac{1}{2} \sin 2x \right. \Big|_0^{\pi/2}$$

$$= \frac{\pi}{2} \left(\frac{\pi}{2} \right)$$

$$= \boxed{\frac{\pi^2}{4} \quad unit^3}$$



Exercise

Let R be the region bounded by $y = e^{x/2}, \quad y = e^{-x}, \quad x = \ln 2, \quad x = \ln 8$. Use the disk method to find the volume of the solid generated when R is revolved about x-axis.

Solution

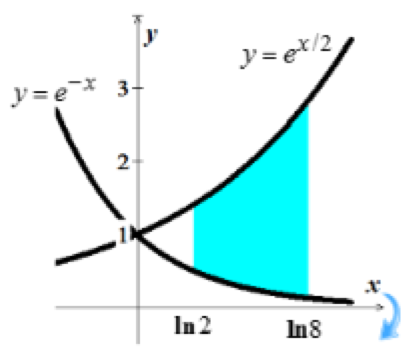
$$V = \pi \int_{\ln 2}^{\ln 8} \left(\left(e^{x/2}\right)^2 - \left(e^{-x}\right)^2 \right) dx$$

$$= \pi \int_{\ln 2}^{\ln 8} \left(e^{x} - e^{-2x} \right) dx$$

$$= \pi \left(e^{x} + \frac{1}{2} e^{-2x} \right. \Big|_{\ln 2}^{\ln 8}$$

$$= \pi \left(8 + \frac{1}{2} \frac{1}{64} - 2 - \frac{1}{8} \right)$$

$$= \boxed{\frac{753\pi}{128} \quad unit^3}$$



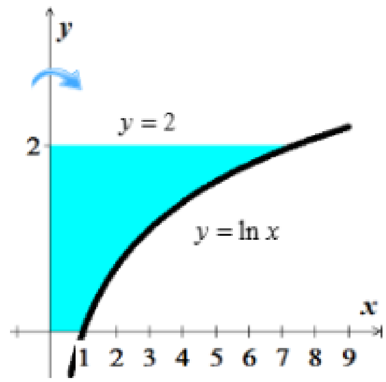
Exercise

Let R be the region bounded by $y=0, \quad y=\ln x, \quad y=2, \quad x=0$. Use the disk method to find the volume of the solid generated when R is revolved about y-axis.

Solution

$$y=\ln x$$

$$x=e^{y}$$



$$V=\pi\int_0^2 e^{2y}\,dy$$

$$=\frac{\pi}{2}e^{2y}\Big|_0^2$$

$$\boxed{=\frac{\pi}{2}\left(e^4-1\right)\quad unit^3}$$

Exercise

Let R be the region bounded by $y=\sin^{-1}x, \quad x=0, \quad y=\frac{\pi}{4}$. Use the disk method to find the volume of the solid generated when R is revolved about y-axis.

Solution

$$y=\sin^{-1}x$$

$$x=\sin y$$

$$V=\pi\int_0^{\pi/4}\sin^2 y\,dy$$

$$=\frac{\pi}{2}\int_0^{\pi/4}(1-\cos 2y)\,dy$$

$$=\frac{\pi}{2}\left(y-\frac{1}{2}\sin 2y\right)\Big|_0^{\pi/4}$$

$$=\frac{\pi}{2}\left(\frac{\pi}{4}-\frac{1}{2}\right)$$

$$\boxed{=\frac{\pi(\pi-2)}{8}\quad unit^3}$$

Exercise

Find the volume of the solid of revolution bounded by $y = \frac{\ln x}{\sqrt{x}}, \quad y = 0, \quad and \quad x = 2$ revolved about the x-axis. Sketch the region

Solution

$$y = \frac{\ln x}{\sqrt{x}} = 0 \rightarrow \underline{x = 1}|$$

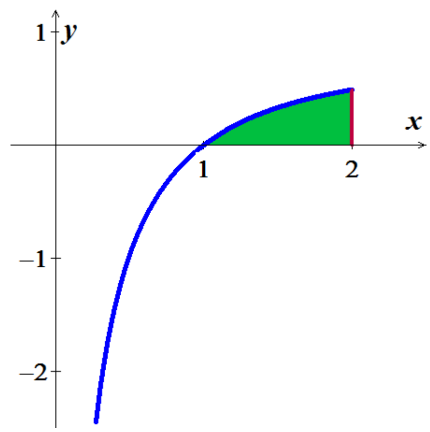
$$V = \pi \int_1^2 \left(\frac{\ln x}{\sqrt{x}}\right)^2 dx$$

$$= \pi \int_1^2 \frac{\ln^2 x}{x} dx$$

$$= \pi \int_1^2 \ln^2 x \, d(\ln x)$$

$$= \frac{\pi}{3} \ln^3 x \Big|_1^2$$

$$= \frac{\pi \ln^3 2}{3} \quad unit^3$$



Exercise

Find the volume of the solid of revolution bounded by $y = e^{-x}, \quad y = e^x, \quad x = 0, \quad x = \ln 4$ revolved about the x-axis. Sketch the region

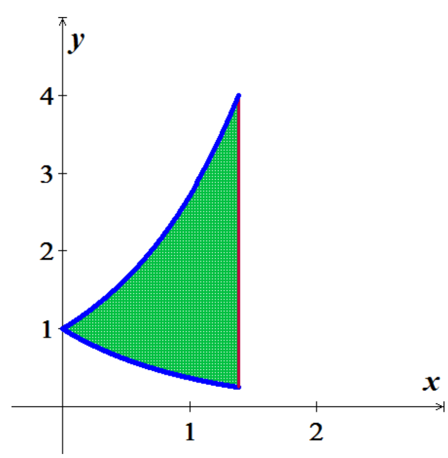
Solution

$$V = \pi \int_0^{\ln 4} \left(e^{2x} - e^{-2x}\right) dx$$

$$= \frac{\pi}{2} \left(e^{2x} + e^{-2x} \Big|_0^{\ln 4}\right.$$

$$= \frac{\pi}{2}\left(16 + \frac{1}{16}\right) \qquad e^{2\ln 4} = e^{\ln 4^2} = 16$$

$$= \frac{225\pi}{32} \quad unit^3$$



Exercise

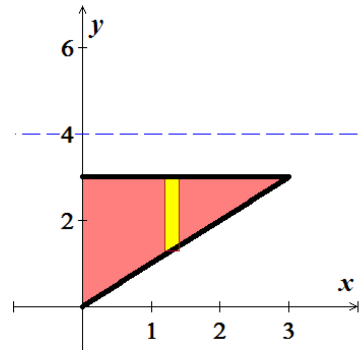
Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $y=4$

$$y=x, \quad y=3, \quad x=0$$

Solution

$\underline{y=x=3}|$

$R(x)=4-x \quad \& \quad r(x)=4-3=1$



$$V=\pi\int_0^3\left((4-x)^2-1\right)dx \qquad V=\pi\int_a^b\left(R(x)^2-r(x)^2\right)dx$$

$$=\pi\int_0^3\left(16-8x+x^2-1\right)dx$$

$$=\pi\left(15x-4x^2+\frac{1}{3}x^3\right|_0^3$$

$$=\pi(45-36+9)$$

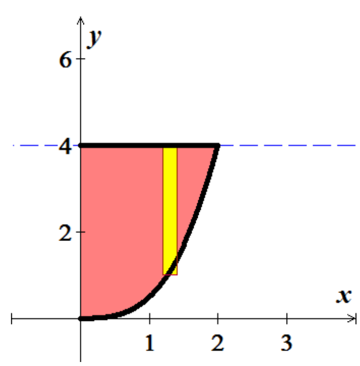
$$\underline{=18\pi \quad unit^3}|$$

Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $y=4$

$$y=\frac{1}{2}x^3, \quad y=4, \quad x=0$$

Solution



$y=\frac{1}{2}x^3=4$

$x^3=8$

$\underline{x=2}|$

$R(x)=4-\frac{1}{2}x^3 \quad \& \quad r(x)=0$

$$V=\pi\int_0^2\left(4-\frac{1}{2}x^3\right)^2dx \qquad V=\pi\int_a^b\left(R(x)^2-r(x)^2\right)dx$$

$$=\pi\int_0^2\left(16-4x^3+\frac{1}{4}x^6\right)dx$$

$$=\pi\left(16x-x^4+\frac{1}{28}x^7\right|_0^2$$

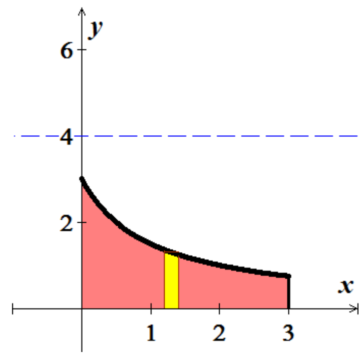
$$= \pi\left(32 - 16 + \frac{128}{28}\right)$$

$$= \frac{144\pi}{7} \quad unit^3$$

Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $y = 4$

$$y = \frac{3}{1+x}, \quad y = 0, \quad x = 0, \quad x = 3$$



Solution

$$R(x) = 4 \quad \& \quad r(x) = 4 - \frac{3}{1+x}$$

$$V = \pi\int_0^3 \left(16 - \left(4 - \frac{3}{1+x}\right)^2\right) dx \qquad V = \pi\int_a^b \left(R(x)^2 - r(x)^2\right) dx$$

$$= \pi\int_0^3 \left(\frac{24}{1+x} - \frac{9}{(1+x)^2}\right) dx \qquad \int \frac{d(x+1)}{(x+1)^2} = \frac{-1}{x+1}$$

$$= \pi\left(24\ln(1+x) + \frac{9}{1+x}\right)\Big|_0^3$$

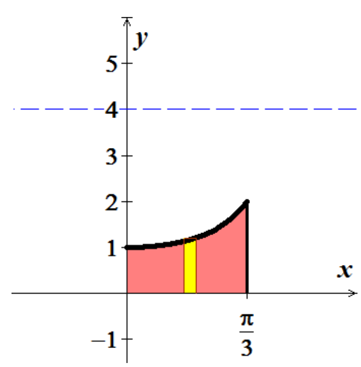
$$= \pi\left(24\ln 4 + \frac{9}{4} - 9\right)$$

$$= \left(48\ln 2 - \frac{27}{4}\right)\pi \quad unit^3$$

Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $y = 4$

$$y = \sec x, \quad y = 0, \quad 0 \le x \le \frac{\pi}{3}$$



Solution

$$R(x) = 4 \quad \& \quad r(x) = 4 - \sec x$$

$$V = \pi\int_0^{\pi/3} \left(16 - (4 - \sec x)^2\right) dx$$

$$= \pi\int_0^{\pi/3} \left(16 - 16 + 8\sec x - \sec^2 x\right) dx$$

$$\int \sec x dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{\sec^2 x + \sec \tan x}{\sec x + \tan x} dx \qquad d(\sec x + \tan x) = (\sec x \tan x + \sec^2 x) dx$$

$$= \int \frac{d(\sec x + \tan x)}{\sec x + \tan x}$$

$$= \ln|\sec x + \tan x|$$

$$= \pi \left(8 \ln|\sec x + \tan x| - \tan x \right|_0^{\pi/3}$$

$$= \pi\left(8 \ln\left(2+\sqrt{3}\right) - \sqrt{3}\right) \quad unit^3$$

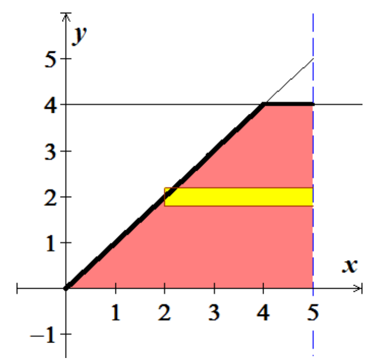
Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $x = 5$

$$y = x, \quad y = 0, \quad y = 4, \quad x = 5$$

Solution

$R(y) = 5 - y \quad \& \quad r(y) = 0$



$$V = \pi \int_0^4 (5-y)^2 dy \qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right) dy$$

$$= -\pi \frac{1}{3} (5-y)^3 \Big|_0^4$$

$$= -\pi \frac{1}{3}\left(1 - 5^3\right)$$

$$= \frac{124\pi}{3} \quad unit^3$$

Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $x = 5$

$$y = 3 - x, \quad y = 0, \quad y = 2, \quad x = 0$$

Solution

$R(y) = 5 \quad \& \quad r(y) = 5 - (3 - y)$

$\qquad\qquad\qquad\qquad = 2 + y$

$$V = \pi \int_0^2 \left(25 - (2+y)^2\right) dy \qquad\qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right) dy$$

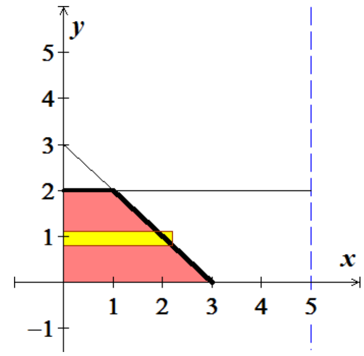
$$= \pi \int_0^2 \left(25 - 4 - 4y - y^2\right) dy$$

$$= \pi \int_0^2 \left(21 - 4y - y^2\right) dy$$

$$= \pi \left(21y - 2y^2 - \frac{1}{3}y^3 \ \Big|_0^2\right.$$

$$= \pi \left(42 - 8 - \frac{8}{3}\right)$$

$$= \frac{94\pi}{3} \quad unit^3$$



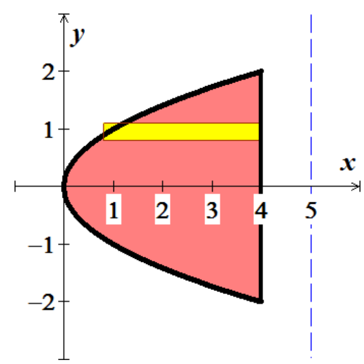
Exercise

Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $x = 5$ $\qquad x = y^2, \quad x = 4$

Solution

$$x = y^2 = 4 \ \rightarrow \ y = \pm 2$$

$$R(y) = 5 - y^2 \quad \& \quad r(y) = 5 - 4 = 1$$



$$V = \pi \int_{-2}^2 \left(\left(5 - y^2\right)^2 - 1\right) dy \qquad\qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right) dy$$

$$= 2\pi \int_0^2 \left(24 - 10y^2 + y^4\right) dy$$

$$= 2\pi \left(24y - \frac{10}{3}y^3 + \frac{1}{5}y^5 \ \Big|_0^2\right.$$

$$= 2\pi \left(48 - \frac{80}{3} + \frac{32}{5}\right)$$

$$= \frac{832\pi}{15} \quad unit^3$$

Exercise

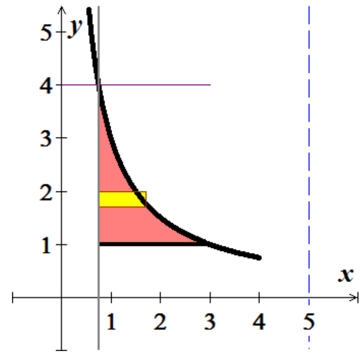
Find the volume of the solid generated by revolving the region boubbded by the graphs of the equations about the line $x = 5$ $\quad xy = 3, \quad y = 1, \quad y = 4, \quad x = 5$

Solution

$$xy = 3 \quad \to x = \frac{3}{y}$$

$$R(y) = 5 - \frac{3}{y} \quad \& \quad r(y) = 0$$

$$V = \pi \int_1^4 \left(5 - \frac{3}{y}\right)^2 dy \qquad\qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right) dy$$



$$= \pi \int_1^4 \left(25 - \frac{30}{y} + \frac{9}{y^2}\right) dy$$

$$= \pi \left(25y - 30 \ln y - \frac{9}{y}\right|_1^4$$

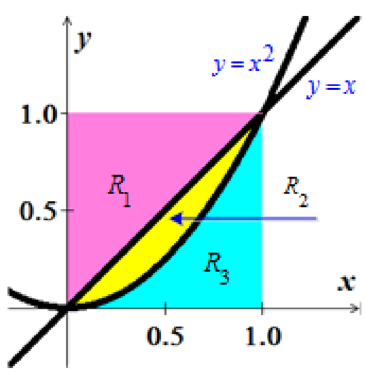
$$= \pi \left(100 - 30 \ln 4 - \frac{9}{4} - 25 + 9\right)$$

$$= \pi \left(84 - \frac{9}{4} - 30 \ln 2^2\right)$$

$$= \pi \left(\frac{327}{4} - 60 \ln 2\right) \quad unit^3$$

Exercise

Find the volume generated by rotating the given region $y = x^2$ *and* $y = x$ about the specified line.



a) R_1 *about* $x = 0$	*d)* R_2 *about* $y = 1$	*g)* R_2 *about* $x = 0$
b) R_1 *about* $x = 1$	*e)* R_3 *about* $x = 0$	*h)* R_2 *about* $x = 1$
c) R_2 *about* $y = 0$	*f)* R_3 *about* $x = 1$	

Solution

a) $V = \pi \int_0^1 y^2\, dy$ $\qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2 \right) dy$

$= \frac{\pi}{3} y^3 \Big|_0^1$

$\boxed{= \frac{\pi}{3} \quad unit^3}$

b) $R(y) = 1 \quad \& \quad r(y) = 1 - y$

$V = \pi \int_0^1 \left(1 - (1-y)^2 \right) dy$ $\qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2 \right) dy$

$= \pi \int_0^1 \left(2y - y^2 \right) dy$

$= \pi \left(y^2 - \frac{1}{3} y^3 \right. \Big|_0^1$

$= \pi \left(1 - \frac{1}{3} \right)$

$\boxed{= \frac{2\pi}{3} \quad unit^3}$

c) $R_2 \quad about \quad y = 0$

$R(x) = x \quad \& \quad r(x) = x^2$

$V = \pi \int_0^1 \left(x^2 - x^4 \right) dx$ $\qquad V = \pi \int_a^b \left(R(x)^2 - r(x)^2 \right) dx$

$= \pi \left(\frac{1}{3} x^3 - \frac{1}{5} x^5 \right. \Big|_0^1$

$= \pi \left(\frac{1}{3} - \frac{1}{5} \right)$

$\boxed{= \frac{2\pi}{15} \quad unit^3}$

d) $R_2 \quad about \quad y = 1$

$R(x) = 1 - x^2 \quad \& \quad r(x) = 1 - x$

$V = \pi \int_0^1 \left(\left(1 - x^2 \right)^2 - (1-x)^2 \right) dx$ $\qquad V = \pi \int_a^b \left(R(x)^2 - r(x)^2 \right) dx$

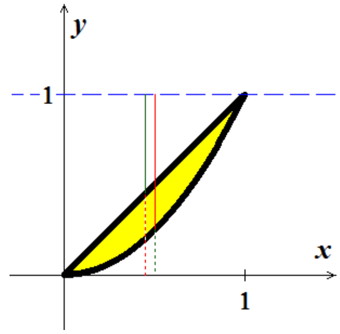
$= \pi \int_0^1 \left(1 - 2x^2 + x^4 - 1 + 2x - x^2 \right) dx$

$$= \pi \int_0^1 \left(x^4 - 3x^2 + 2x\right) dx$$

$$= \pi \left(\frac{1}{5}x^4 - x^3 + x^2 \right|_0^1$$

$$= \pi \left(\frac{1}{5} - 1 + 1\right)$$

$$\boxed{= \frac{\pi}{5} \quad unit^3}$$



e) R_3 *about* $x = 0$

$$y = x^2 \quad \to \quad x = \sqrt{y}$$

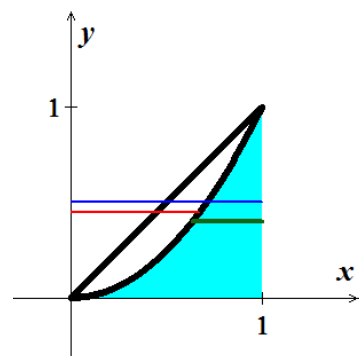
$$R(y) = 1 \quad \& \quad r(y) = \sqrt{y}$$

$$V = \pi \int_0^1 (1 - y)\, dy \qquad\qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right) dy$$

$$= \pi \left(y - \frac{1}{2}y^2 \right|_0^1$$

$$= \pi \left(1 - \frac{1}{2}\right)$$

$$\boxed{= \frac{\pi}{2} \quad unit^3}$$



f) R_3 *about* $x = 1$

$$R(y) = 1 - \sqrt{y} \quad \& \quad r(y) = 1 - 1 = 0$$

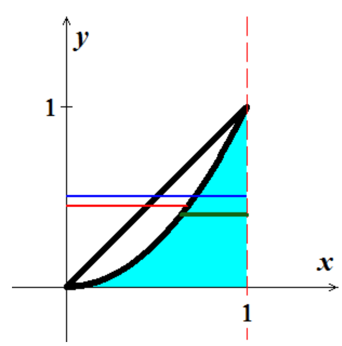
$$V = \pi \int_0^1 \left(1 - \sqrt{y}\right)^2 dy \qquad\qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right) dy$$

$$= \pi \int_0^1 \left(1 - 2\sqrt{y} + y\right) dy$$

$$= \pi \left(y - \frac{4}{3}y^{3/2} + \frac{1}{2}y^2 \right|_0^1$$

$$= \pi \left(1 - \frac{4}{3} + \frac{1}{2}\right)$$

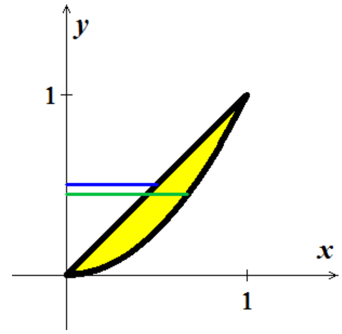
$$\boxed{= \frac{\pi}{6} \quad unit^3}$$



g) R_2 *about* $x = 0$

$$R(y) = \sqrt{y} \quad \& \quad r(y) = y$$

$$V = \pi \int_0^1 \left(y - y^2\right) dy \qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right) dy$$



$$= \pi \left(\frac{1}{2}y^2 - \frac{1}{3}y^3 \right|_0^1$$

$$= \pi\left(\frac{1}{2} - \frac{1}{3}\right)$$

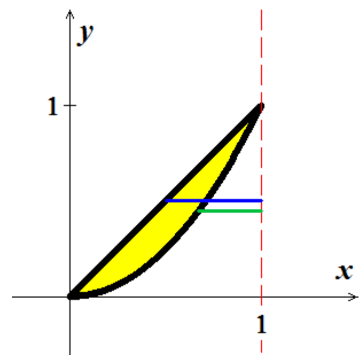
$$= \frac{\pi}{6} \quad unit^3$$

h) $R_2 \quad about \quad x = 1$

$R(y) = 1 - y$ & $r(y) = 1 - \sqrt{y}$

$$V = \pi \int_0^1 \left((1-y)^2 - \left(1-\sqrt{y}\right)^2\right) dy \qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right) dy$$

$$= \pi \int_0^1 \left(1 - 2y + y^2 - 1 + 2\sqrt{y} - y\right) dy$$



$$= \pi \int_0^1 \left(-3y + y^2 + 2y^{1/2}\right) dy$$

$$= \pi \left(-\frac{3}{2}y^2 + \frac{1}{3}y^3 + \frac{4}{3}y^{3/2} \right|_0^1$$

$$= \pi\left(-\frac{3}{2} + \frac{1}{3} + \frac{4}{3}\right)$$

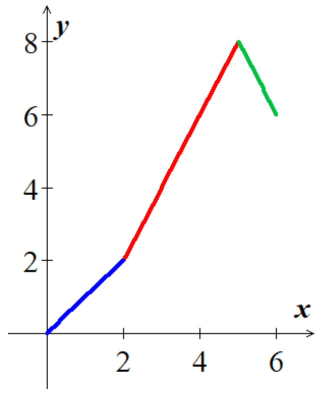
$$= \pi\left(\frac{5}{3} - \frac{3}{2}\right)$$

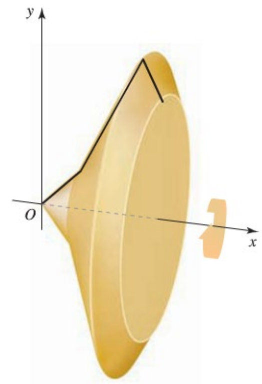
$$= \frac{\pi}{6} \quad unit^3$$

Exercise

Let $f(x) = \begin{cases} x & if \quad 0 \le x \le 2 \\ 2x - 2 & if \quad 2 < x \le 5 \\ -2x + 18 & if \quad 5 < x \le 6 \end{cases}$

Find the volume of the solid formed when the region bounded by the graph of *f*, the *x*-axis, and the line $x = 6$ is revolved about the *x*-axis.





Solution

$$V = \pi\int_0^2 x^2 dx \;+\; \pi\int_2^5 (2x-2)^2\, dx \;+\; \pi\int_5^6 (18-2x)^2\, dx$$

$$= \pi\int_0^2 x^2 dx + \frac{\pi}{2}\int_2^5 (2x-2)^2\, d(2x-2) - \frac{\pi}{2}\int_5^6 (18-2x)^2\, d(18-2x)$$

$$= \frac{\pi}{3}x^3\Big|_0^2 \;+\; \frac{\pi}{6}\left((2x-2)^3\right.\Big|_2^5 \;-\; \frac{\pi}{6}\left((18-2x)^3\right.\Big|_5^6$$

$$= \pi\left[\frac{8}{3}+\frac{1}{6}(512-8)-\frac{1}{6}(216-512)\right]$$

$$= \pi\left(\frac{8}{3}+\frac{252}{3}+\frac{148}{3}\right)$$

$$= 136\pi \quad unit^3$$

Exercise

Consider the solid formed by revolving the region bounded by $y=\sqrt{x}$, $y=0$, and $x=4$ about the *x-axis*

a) Find the value of x in the interval $[0,\ 4]$ that divides the solids into two parts of equal volume.

b) Find the values of x in the interval $[0,\ 4]$ that divide the solids into three parts of equal volume.

Solution

$$V = \pi\int_0^4 \left(\sqrt{x}\right)^2 dx$$

$$= \pi\int_0^4 x\; dx$$

$$= \frac{\pi}{2}x^2\Big|_0^4$$

$$= 8\pi \quad unit^3$$

a) Let $0 < c < 4$

$$V = \pi \int_0^c x\, dx$$

$$= \frac{\pi}{2} x^2 \Big|_0^c$$

$$= \frac{\pi}{2} c^2 = \frac{1}{2}(8\pi)$$

$$c^2 = 8 \Rightarrow \underline{c = 2\sqrt{2}}$$

$\therefore\ x = 2\sqrt{2}$, the solid is divided into 2 equal volume parts.

b) The first one third part:

$$\frac{\pi}{2} c^2 = \frac{1}{3}(8\pi)$$

$$c^2 = \frac{16}{3}$$

$$\underline{c = \frac{4\sqrt{3}}{3}}$$

The second part $\left(\frac{2}{3}\right)$:

$$\frac{\pi}{2} d^2 = \frac{2}{3}(8\pi)$$

$$d^2 = \frac{32}{3}$$

$$\underline{d = \frac{4\sqrt{2}}{\sqrt{3}} = \frac{4\sqrt{6}}{3}}$$

$\therefore\ x = \frac{4\sqrt{3}}{3},\ \frac{4\sqrt{6}}{3}$, the solid is divided into 3 equal volume parts.

Exercise

The arc of $y = 4 - \frac{1}{4}x^2$ on the interval $[0,\ 4]$ is revolved about the line $y = b$

a) Find the volume of the resulting solid as a function of *b*.

b) Graph the function in part (*a*), and approximate the value of *b* that minimizes the volume of the solid.

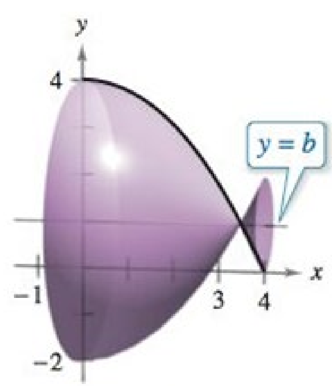
c) Find the value of *b* that minimizes the volume of the solid, and compare the result with the answer in part (*b*).

Solution

a) $y = 4 - \frac{1}{4}x^2 = b$

$$x^2 = 4(4-b)$$

$$\underline{x = 2\sqrt{4-b}}$$



$$V=\pi\int_0^{2\sqrt{4-b}}\left(4-\frac{1}{4}x^2-b\right)^2dx\ +\ \pi\int_{2\sqrt{4-b}}^{4}\left(b-4+\frac{1}{4}x^2\right)^2dx$$

$$=\pi\int_0^4\left(4-\frac{1}{4}x^2-b\right)^2dx$$

$$=\pi\int_0^4\left(16-2x^2+\frac{1}{2}bx^2+\frac{1}{16}x^4-8b+b^2\right)dx$$

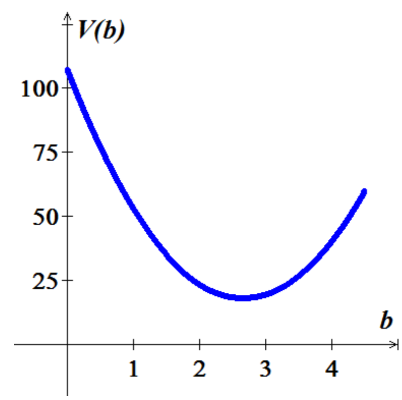
$$=\pi\left(16x-\frac{2}{3}x^3+\frac{1}{6}bx^3+\frac{1}{80}x^5-8bx+b^2x\right.\Big|_0^4$$

$$=\pi\left(64-\frac{16}{3}+\frac{64}{6}b+\frac{64}{5}-32b+4b^2\right)$$

$$=\pi\left(4b^2-\frac{64}{3}b+\frac{512}{15}\right)$$

$$\underline{V(b)=4\pi\left(b^2-\frac{16}{3}b+\frac{128}{15}\right)\quad unit^3}$$

b) From the graph (*using software*)



2.660	17.873
2.670	17.872
2.680	17.874

The minumium volume is 17.872 for $b=2.67$

c) $V'(b)=4\pi\left(2b-\frac{16}{3}\right)=0$

$CN:\quad \underline{b=\frac{8}{3}}$

$$V\left(b=\frac{8}{3}\right)=4\pi\left(\frac{64}{9}-\frac{128}{9}+\frac{128}{15}\right)$$

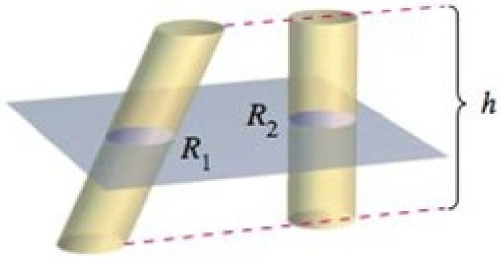
$$=4\pi\left(\frac{128}{15}-\frac{64}{9}\right)$$

$$=4\pi\left(\frac{64}{45}\right)$$

$$\underline{=\frac{256\pi}{45}\quad unit^3}\qquad \underline{\approx 17.872}$$

Exercise

Prove that if two solids have equal altitudes and all plane sections parallel to their bases and at equal distances from their bases have equal areas, then the solids hace the same volume.



$$\text{Area of } R_1 = \text{Area of } R_2$$

Solution

Since $A_1(x) = A_2(x)$ when $a \le x \le b$, then

$$V_1 = \int_a^b A_1(x)\,dx$$

$$= \int_a^b A_2(x)\,dx$$

$$= V_2$$

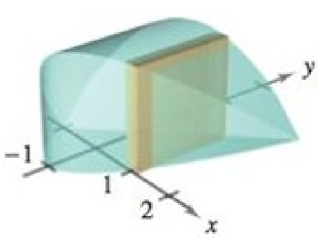
∴ The volume are the same.

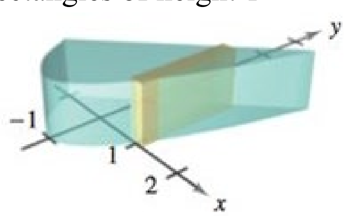
Exercise

Find the volumes of the solids whose bases are bounded by the graph of $y = x + 1$ and $y = x^2 - 1$, with the indicated cross sections taken perpendicular to the *x-axis*

a) Squares

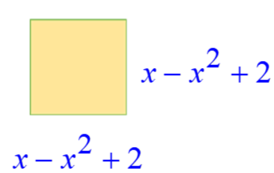
b) Rectangles of height 1





Solution

a) Base of cross section $= (x+1) - (x^2 - 1) = x - x^2 + 2$



$$A = b^2 = \left(2 + x - x^2\right)^2$$

$$= 4 + 4x - 3x^2 - 2x^3 + x^4$$

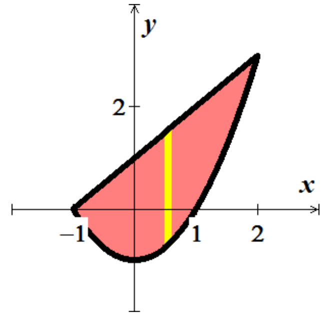
$$V = \int_{-1}^{2} \left(4 + 4x - 3x^2 - 2x^3 + x^4\right) dx$$

$$= 4x + 2x^2 - x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5 \Big|_{-1}^{2}$$

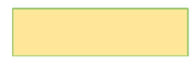
$$= 8 + 8 - 8 - 8 + \frac{32}{5} + 4 - 2 - 1 + \frac{1}{2} + \frac{1}{5}$$

$$= \frac{32}{5} + 1 + \frac{7}{10}$$

$$= \frac{81}{10} \quad unit^3$$



b) $A = bh = \left(2 + x - x^2\right)(1)$



$$V = \int_{-1}^{2} \left(2 + x - x^2\right) dx$$

$$= 2x + \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_{-1}^{2}$$

$$= 4 + 2 - \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3}$$

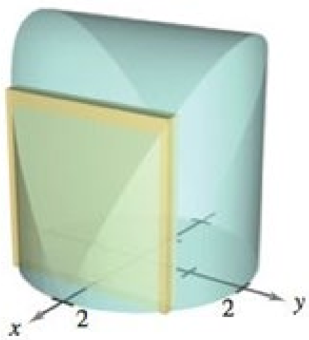
$$= 8 - 3 - \frac{1}{2}$$

$$= \frac{9}{2} \quad unit^3$$

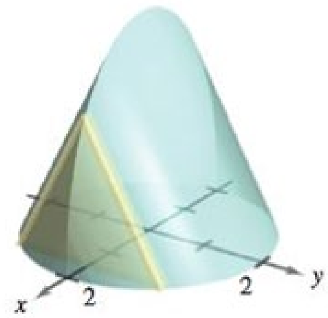
Exercise

Find the volumes of the solids whose bases are bounded by the circle $x^2 + y^2 = 4$, with the indicated cross sections taken perpendicular to the x-*axis*

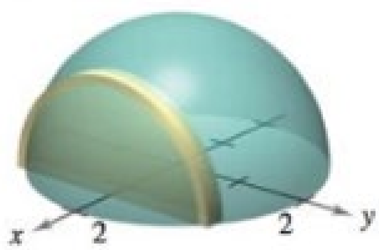
a) Squares



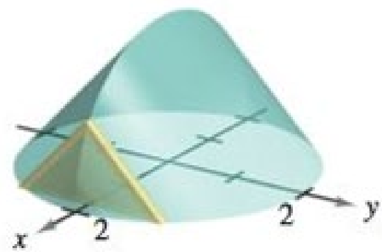
b) Equilateral triangles



c) Semicircles



d) Iscosceles right triangles

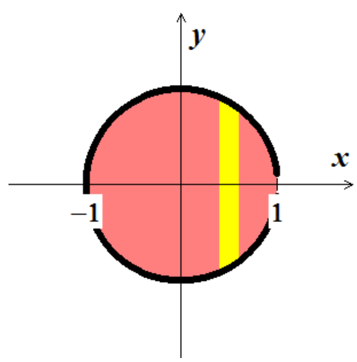


Solution

$$y = \pm\sqrt{4 - x^2}$$

Base cross section $y = 2\sqrt{4 - x^2}$

a) $A = b^2 = 4\left(4 - x^2\right)$



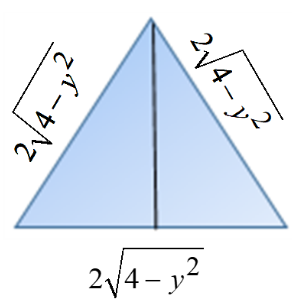
$$V = 4\int_{-2}^{2}\left(4 - x^2\right)dx$$

$$= 8\left(4x - \frac{1}{3}x^3\right.\Big|_0^2$$

$$= 8\left(8 - \frac{8}{3}\right)$$

$$= \frac{128}{3}$$

b) $\sin 60° = \dfrac{h}{2\sqrt{4 - x^2}}$



$$h = \sqrt{3}\sqrt{4 - x^2}$$

$$A = \frac{1}{2}bh$$

$$= \frac{1}{2}\left(2\sqrt{4-x^2}\right)\sqrt{3}\ \sqrt{4-x^2}$$

$$\boxed{= \sqrt{3}\left(4-x^2\right) \quad unit^2}$$

$$V = \sqrt{3}\int_{-2}^{2}\left(4-x^2\right)dx$$

$$= 2\sqrt{3}\ \left(4x - \frac{1}{3}x^3\ \right|_0^2$$

$$= 2\sqrt{3}\left(8-\frac{8}{3}\right)$$

$$\boxed{= \frac{32\sqrt{3}}{3} \quad unit^3}$$

c) $A = \frac{1}{2}\pi r^2$

$$= \frac{1}{2}\pi\left(\sqrt{4-x^2}\right)^2$$

$$\boxed{= \frac{\pi}{2}\left(4-x^2\right) \quad unit^2}$$

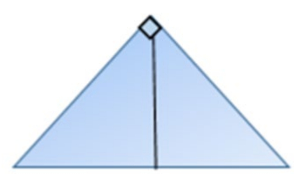
$$V = \frac{\pi}{2}\int_{-2}^{2}\left(4-x^2\right)dx$$

$$= \pi\ \left(4x - \frac{1}{3}x^3\ \right|_0^2$$

$$= \pi\left(8-\frac{8}{3}\right)$$

$$\boxed{= \frac{16\pi}{3} \quad unit^3}$$

d) $\tan 45° = \frac{h}{\sqrt{4-x^2}}$



$$h = \sqrt{4-x^2}$$

$$A = \frac{1}{2}bh$$

$$= \frac{1}{2}\left(2\sqrt{4-x^2}\right)\sqrt{4-x^2}$$

$$\boxed{= 4-x^2 \quad unit^2}$$

$$V = \int_{-2}^{2}\left(4-x^2\right)dx$$

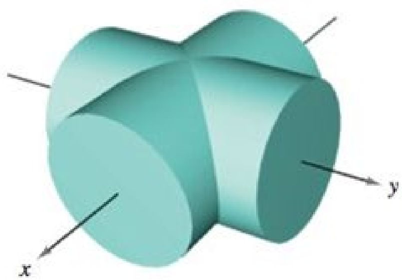
$$= 2\ \left(4x - \frac{1}{3}x^3\ \right|_0^2$$

$$= 2\left(8 - \frac{8}{3}\right)$$

$$= \frac{32}{3} \quad unit^3$$

Exercise

Find the volume of the solid of intersection (the solid common to both) of the two right circular cylinders of radius r whose axes meet at right angles.



Two intersecting cylinders

Solid of intersection

Solution

The cross sections are squares.

By symmetry, we can divide the volume to 8 equal sections.

$$x^2 + y^2 = r^2$$

$$x = \sqrt{r^2 - y^2}$$

$$A(y) = b^2$$

$$= \left(\sqrt{r^2 - y^2}\right)^2$$

$$= r^2 - y^2 \quad unit^2$$

$$V = 8\int_0^r \left(r^2 - y^2\right)dy$$

$$= 8\left(r^2 y - \frac{1}{3}y^3\right)\Big|_0^r$$

$$= 8\left(r^3 - \frac{1}{3}r^3\right)$$

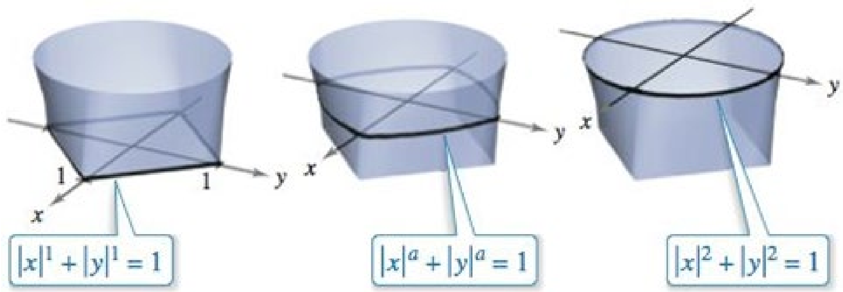
$$= \frac{16}{3}r^3 \quad unit^3$$

Exercise

The solid shown in the figure has cross sections bounded by the graph $|x|^a + |y|^a = 1$ where $1 \le a \le 2$.

a) Describe the cross section when $a = 1$ and $a = 2$.

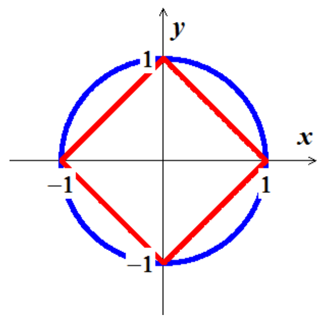
b) Describe a procedure for approximating the volume of the solid.



Solution

a) When $a = 1 \quad \Rightarrow \quad |x| + |y| = 1$ represents a square.

When $a = 2 \quad \Rightarrow \quad |x|^2 + |y|^2 = 1$ represents a circle.



b) $|x|^a + |y|^a = 1$

$$|y| = \left(1 - |x|^a\right)^{1/a}$$

$$A = 2\int_{-1}^{1} \left(1 - |x|^a\right)^{1/a} dx$$

$$= 4\int_{0}^{1} \left(1 - |x|^a\right)^{1/a} dx$$

To approximate the volume of the solid, from *n* slices, each of whose area is approximated by the integral above,
Then sum the volumes of these *n* slices.

Exercise

Two planes cut a right circular cylinder to form a wedge. One plane is perpendicular to the axis of the cylinder and the second makes an angle of θ degrees with the first.

a) Find the volume of the wedge if $\theta = 45°$.

b) Find the volume of the wedge for an arbitrary angle θ. Assuming that the cylinder has sufficient length, how does the volume of the wedge change as θ increases from 0° to 90°?

Solution

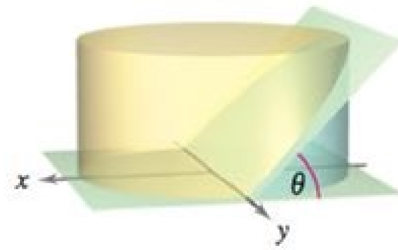
a) Since $\theta = 45°$, then the cross sections are isosceles right triangles.

$$x^2 + y^2 = r^2$$

$$x = \pm\sqrt{r^2 - y^2}$$

$$\tan 45° = \frac{h}{\sqrt{r^2 - y^2}} = 1$$

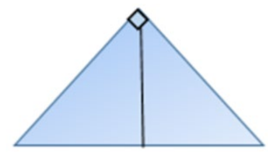
$$\boxed{h = \sqrt{r^2 - y^2}}$$



$$A(y) = \frac{1}{2} bh$$

$$= \frac{1}{2} \sqrt{r^2 - y^2} \sqrt{r^2 - y^2}$$

$$\boxed{= \frac{1}{2}\left(r^2 - y^2\right) \quad unit^2}$$



$$V = \frac{1}{2} \int_{-r}^{r} \left(r^2 - y^2\right) dy$$

$$= r^2 y - \frac{1}{3} y^3 \Big|_0^r$$

$$\boxed{= \frac{2}{3} r^3 \quad unit^3}$$

b) $\tan\theta = \dfrac{h}{\sqrt{r^2 - y^2}}$

$$h = (\tan\theta)\sqrt{r^2 - y^2}$$

$$A(y) = \frac{1}{2} bh$$

$$\boxed{= \frac{1}{2}\left(r^2 - y^2\right)\tan\theta \quad unit^2}$$

$$V = \frac{1}{2}\tan\theta \int_{-r}^{r} \left(r^2 - y^2\right) dy$$

$$= \tan\theta \left(r^2 y - \frac{1}{3} y^3\right) \Big|_0^r$$

$$\boxed{= \frac{2}{3} r^3 \tan\theta \quad unit^3}$$

As $\theta \to 90° \quad \Rightarrow \quad V \to \infty$

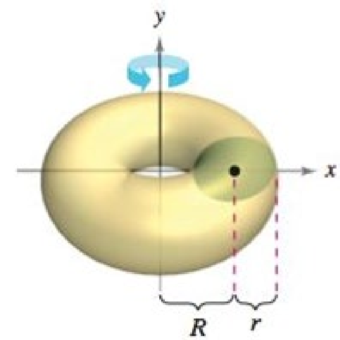
Exercise

For the given torus (donut).

a) Show that the volume of the torus is given by the integral

$$8\pi R\int_0^r \sqrt{r^2-y^2}\,dy \quad where \quad R>r>0$$

b) Find the volume of the torus



Solution

a) $(x-R)^2+y^2=r^2$

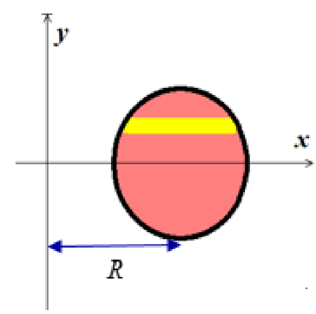
$x=R\pm\sqrt{r^2-y^2}$

$$V=\pi\int_{-r}^{r}\left[\left(R+\sqrt{r^2-y^2}\right)^2-\left(R-\sqrt{r^2-y^2}\right)^2\right]dy$$

$$=2\pi\int_0^r\left(R^2+2R\sqrt{r^2-y^2}+r^2-y^2-R^2+2R\sqrt{r^2-y^2}-r^2+y^2\right)dy$$

$$=8\pi R\int_0^r\sqrt{r^2-y^2}\,dy \quad \checkmark$$

b) $\int_0^r\sqrt{r^2-y^2}\,dy$ is $\frac{1}{4}$ of the area of a circle of radius r.



$A=\frac{1}{4}\pi r^2$

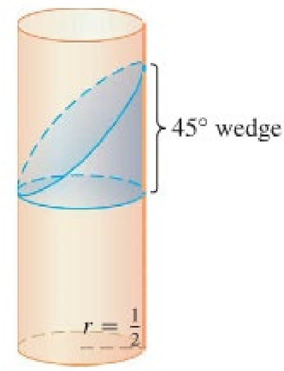
$$V=8\pi R\int_0^r\sqrt{r^2-y^2}\,dy$$

$$=8\pi R\left(\frac{1}{4}\pi r^2\right)$$

$$\underline{=2\pi^2r^2R \quad unit^3}$$

Exercise

Consider a right-circular cylinder of diameter 1. Form a wedge by making one slice parallel to the base of the cylinder completely through the cylinder, and another slice at an angle of 45° to the first slice and intersecting the first slice at the opposite edge of the cylinder.
Find the volume of the wedge.



Solution

Since the slice at an angle of 45°, the volume of the wedge is half of the cylinder of radius 0.5 and height 1.

$$V = \frac{1}{2}\pi\left(\frac{1}{2}\right)^2(1)$$

$$= \frac{\pi}{8} \quad unit^3$$

Exercise

A martini glass in the shape of a right-circular cone of height h and semi-vertical angle α is filled with liquid. Slowly a ball is lowered into the glass, displacing liquid and causing it to overflow. Find the radius R of the ball that causes the greatest volume of liquid to overflow out the glass.

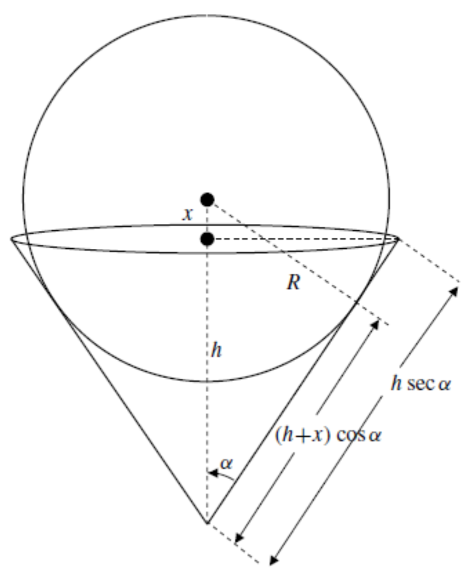
Solution

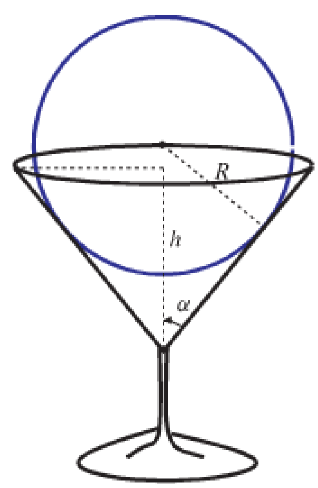
Let $\boldsymbol{x}$: the distance of the center above the glass top.

$$\sin\alpha = \frac{R}{x+h}$$

$$R = (x+h)\sin\alpha$$

$$y^2 = R^2 - x^2$$





$$V = \pi\int_x^R \left(R^2 - x^2\right)dx$$

$$= \pi\left(R^2x - \frac{1}{3}x^3\right)\Big|_x^R$$

$$= \pi\left(R^3 - \frac{1}{3}R^3 - \left(R^2x - \frac{1}{3}x^3\right)\right)$$

$$= \pi\left(\frac{2}{3}R^3 - R^2x + \frac{1}{3}x^3\right)$$

$$= \frac{\pi}{3}\left(2R^3 - 3R^2x + x^3\right) \quad unit^3$$

$$\frac{dV}{dx} = \frac{\pi}{3}\left(6R^2\frac{dR}{dx} - 6Rx\frac{dR}{dx} - 3R^2 + 3x^2\right) = 0$$

$$\left(6R^2 - 6Rx\right)\frac{dR}{dx} - 3R^2 + 3x^2 = 0$$

$$6R(R-x)\frac{dR}{dx}=3\left(R^2-x^2\right)$$

$$2R(R-x)\frac{dR}{dx}=(R-x)(R+x)$$

$$2R\frac{dR}{dx}=R+x$$

$$R=x\sin\alpha+h\sin\alpha$$

$$\frac{dR}{dx}=\sin\alpha$$

$$2R\sin\alpha=R+\frac{R}{\sin\alpha}-h$$

$$R+\frac{R}{\sin\alpha}-2R\sin\alpha=h$$

$$R\left(\sin\alpha+1-2\sin^2\alpha\right)=h\sin\alpha$$

$$R=\frac{h\sin\alpha}{\sin\alpha+1-2\sin^2\alpha}$$

$$=\frac{h\sin\alpha}{\sin\alpha+\cos 2\alpha}$$

$$\le h\frac{\sin^2\alpha}{\cos^2\alpha}$$

$$x=\frac{1}{\sin\alpha}R-h$$

$$=\frac{1}{\sin\alpha}\frac{h\sin\alpha}{\sin\alpha+1-2\sin^2\alpha}-h$$

$$=h\left(\frac{1}{\sin\alpha+1-2\sin^2\alpha}-1\right)$$

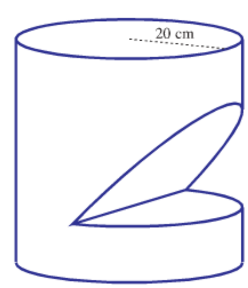
$$=h\left(\frac{2\sin^2\alpha-\sin\alpha}{\sin\alpha+1-2\sin^2\alpha}\right)\quad \textit{unit}$$

$$-R\le x=h\left(\frac{2\sin^2\alpha-\sin\alpha}{\sin\alpha+1-2\sin^2\alpha}\right)\le R\le h\tan^2\alpha$$

Exercise

A 45° notch is cut to the center of a cylindrical log having radius 20 *cm*. One plane face the notch is perpendicular to the axis of the log.

What volume of wood was removed from the log by cutting the notch?



Solution

$$V=2\int_0^{20}\frac{1}{2}\left(\sqrt{400-x^2}\right)^2 dx$$

$$= 400x - \frac{1}{3}x^3 \Big|_0^{20}$$

$$= 400(20) - \frac{1}{3}20^3$$

$$\underline{= 5333.33 \ cm^3}$$