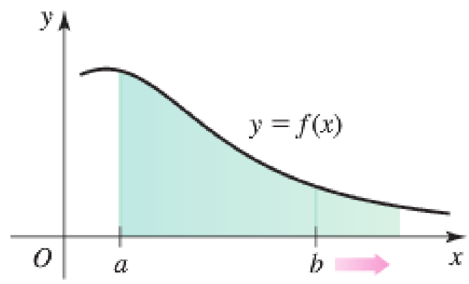
Section 2.6 – Improper Integrals

Definition

Integrals with infinite limits of integration are ***improper integrals***.

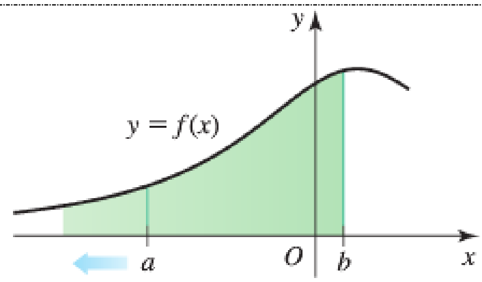
1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$



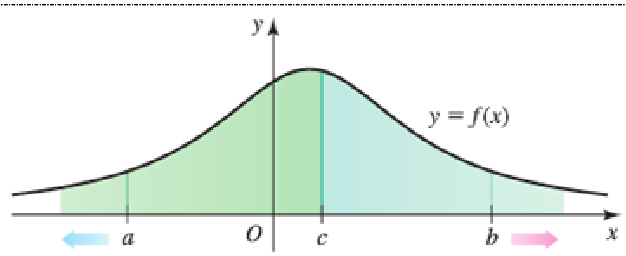
2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$



3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$



In each case, if the limit is finite we say that the improper integral ***converges*** and that the limit is the ***value*** of the improper integral. If the limit fails to exist, the improper integral ***diverges***.

Example

Is the area under the curve $y = \dfrac{\ln x}{x^2}$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?

Solution

$$\int_1^b \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x\Big|_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$$

$$u = \ln x \qquad dv = \frac{dx}{x^2}$$

$$du = \frac{1}{x}\,dx \qquad v = -\frac{1}{x}$$

$$= -\left(\frac{1}{b}\ln b - \ln 1\right) + \int_1^b \frac{1}{x^2}\,dx$$

$$= -\frac{1}{b}\ln b + \left[-\frac{1}{x}\right]_1^b$$

$$= -\frac{1}{b}\ln b - \left(\frac{1}{b} - 1\right)$$

$$= -\frac{1}{b}\ln b - \frac{1}{b} + 1$$

$$\int_1^\infty \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}dx$$

$$= \lim_{b\to\infty}\left(-\frac{1}{b}\ln b - \frac{1}{b} + 1\right)$$

$$= -\lim_{b\to\infty}\left(\frac{1/b}{1}\right) - 0 + 1 \qquad \textbf{\textit{L'Hôpital Rule}}$$

$$= -0 + 1$$

$$\boxed{=1}$$

Example

Evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$

Solution

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \int_{-\infty}^{0}\frac{dx}{1+x^2} + \int_0^\infty \frac{dx}{1+x^2}$$

$$\int_{-\infty}^{0}\frac{dx}{1+x^2} = \lim_{a\to-\infty}\int_a^0 \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty}\tan^{-1}x\Big|_a^0$$

$$= \lim_{a\to-\infty}\left(\tan^{-1}0 - \tan^{-1}a\right)$$

$$= 0 - \left(-\frac{\pi}{2}\right)$$

$$\boxed{= \frac{\pi}{2}}$$

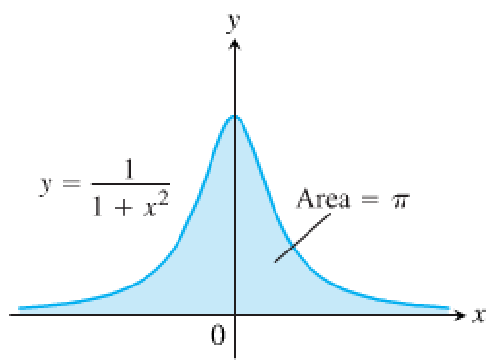
$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty}\tan^{-1}x\Big|_0^b$$

$$= \lim_{b\to\infty}\left(\tan^{-1}b - \tan^{-1}0\right)$$

$$= \frac{\pi}{2} - 0$$

$$\boxed{= \frac{\pi}{2}}$$



$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$\underline{= \pi|}$$

Example

For what value of p does the integral $\int_1^{\infty} \frac{dx}{x^p}$ converge? When the integral does converge, what is its value?

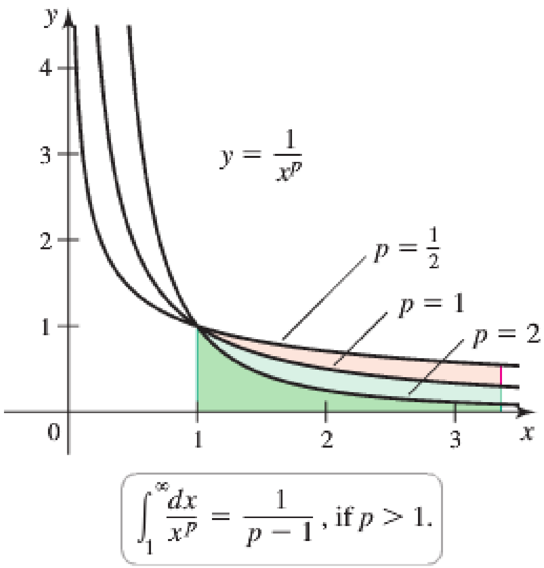
Solution

If $p \neq 1$ $\quad \int_1^b \frac{dx}{x^p} = \left.\frac{x^{-p+1}}{-p+1}\right|_1^b = \frac{1}{1-p}\left(b^{1-p}-1\right)$

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^p}$$

$$= \lim_{b\to\infty} \left[\frac{1}{1-p}\left(b^{1-p}-1\right)\right]$$

$$= \begin{cases} \frac{1}{p-1}, & p>1 \\ \infty, & p<1 \end{cases}$$



If $p = 1$ $\quad \int_1^{\infty} \frac{dx}{x^p} = \int_1^{\infty} \frac{dx}{x}$

$$= \lim_{b\to\infty} \int_1^b \frac{dx}{x}$$

$$= \lim_{b\to\infty} \left[\ln x\right]_1^b$$

$$= \lim_{b\to\infty} \left(\ln b - \ln 1\right)$$

$$\underline{= \infty|}$$

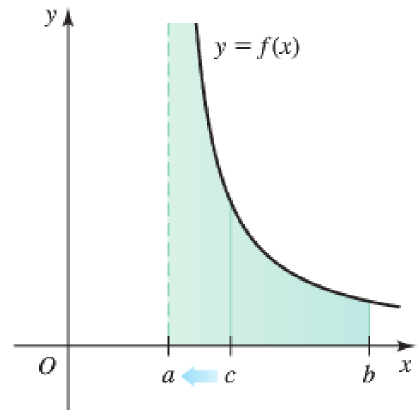
Integrands with Vertical Asymptotes

Definition

Integrals of functions that become infinite at a point within the interval of integration are ***improper integrals***.
If the limit is finite we say that the improper integral ***converges*** and that the limit is the ***value*** of the improper integral. If the limit does not exist, the integral ***diverges***.

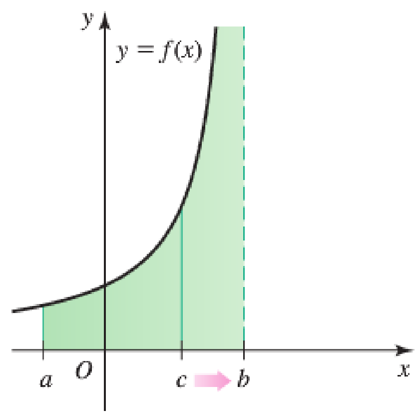
1. If $f(x)$ is continuous on $(a,\ b]$, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$$



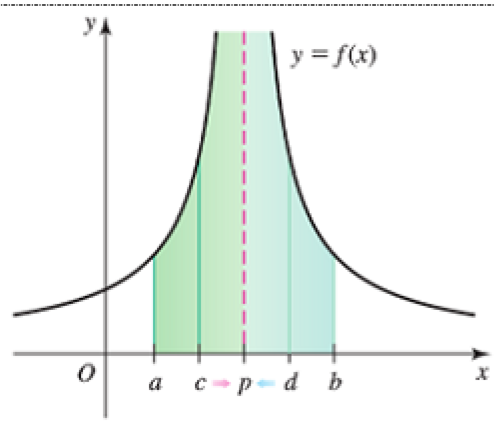
2. If $f(x)$ is continuous on $[a,\ b)$, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx$$



3. If $f(x)$ is continuous on $[a,\ b]$, then

$$\int_a^b f(x)\,dx = \lim_{c\to p^-}\int_a^c f(x)\,dx + \lim_{d\to p^+}\int_d^b f(x)\,dx$$



Example

Investigate the convergence of $\int_0^1 \frac{1}{1-x}dx$

Solution

$$\int_0^1 \frac{1}{1-x}dx = \lim_{b\to 1^-}\int_0^b \frac{1}{1-x}dx$$

$$= \lim_{b\to 1^-}\left[-\ln|1-x|\right]_0^b$$

$$= \lim_{b\to 1^-}\left[-\ln|1-b|+0\right]$$

$$\boxed{=\infty}$$

The limit is infinite, so the integral diverges.

Example

Evaluate $\int_0^3 \frac{dx}{(x-1)^{2/3}}$

Solution

The integrand has a vertical asymptote at $x = 1$ and is continuous on [0, 1) and (1, 3].

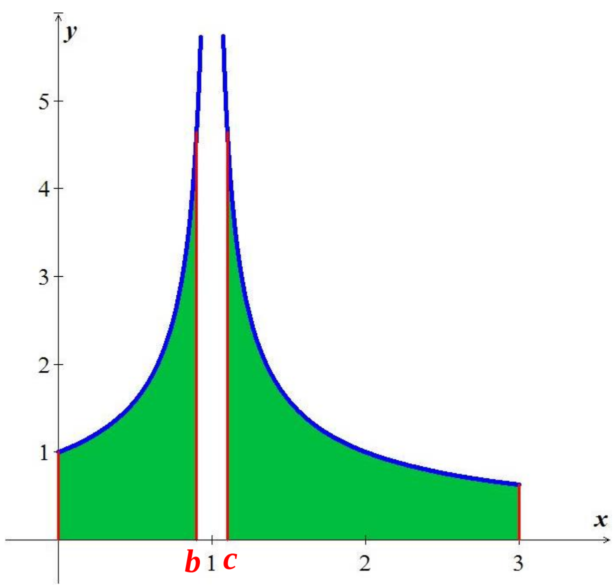
$$\int \frac{dx}{(x-1)^{2/3}} = \int (x-1)^{-2/3}d(x-1) = 3(x-1)^{1/3}$$

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \left[3(x-1)^{1/3}\right]_0^{1^-} + \left[3(x-1)^{1/3}\right]_{1^+}^3$$

$$= 3(0+1) + 3\left(\sqrt[3]{2}-0\right)$$

$$\boxed{= 3+3\sqrt[3]{2}}$$



Example

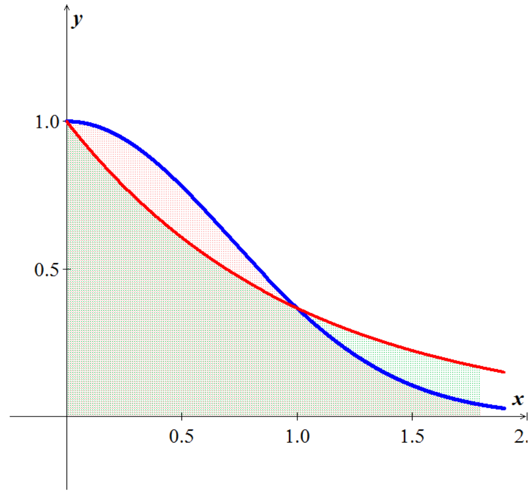
Does the integral $\int_1^{\infty} e^{-x^2}\,dx$ converge?

<u>***Solution***</u>

$$\int_1^{\infty} e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^{b} e^{-x^2}\,dx$$

$$\int_1^{b} e^{-x^2}\,dx \le \int_1^{b} e^{-x}\,dx = -e^{-b}+e^{-1} < e^{-1} \approx 0.36788$$

The integral converges



Theorem – Direct Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. $\int_a^{\infty} f(x)\,dx$ *converges if* $\int_a^{\infty} g(x)\,dx$ *converges*

2. $\int_a^{\infty} g(x)\,dx$ *diverges if* $\int_a^{\infty} f(x)\,dx$ *diverges*

Theorem – Limit Comparison Test

If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$$

Then

$$\int_a^{\infty} f(x)\,dx \quad \textit{and} \quad \int_a^{\infty} g(x)\,dx$$

Both converge or both diverge

Example

Show that $\int_1^\infty \frac{dx}{1+x^2}$ converges by comparison with $\int_1^\infty \frac{dx}{x^2}$. Find and compare the two integral values.

Solution

The functions $f(x)=\frac{1}{x^2}$ and $g(x)=\frac{1}{1+x^2}$ are positive and continuous on $[1, \infty)$. Also,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{1}{1+x^2}}$$

$$= \lim_{x\to\infty} \frac{1+x^2}{x^2}$$

$$\underline{=1|}$$

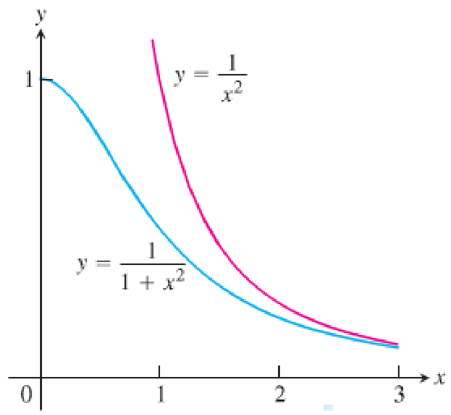
Therefore, $\int_1^\infty \frac{dx}{1+x^2}$ converges because $\int_1^\infty \frac{dx}{x^2}$ converges.

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 1\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$



Example

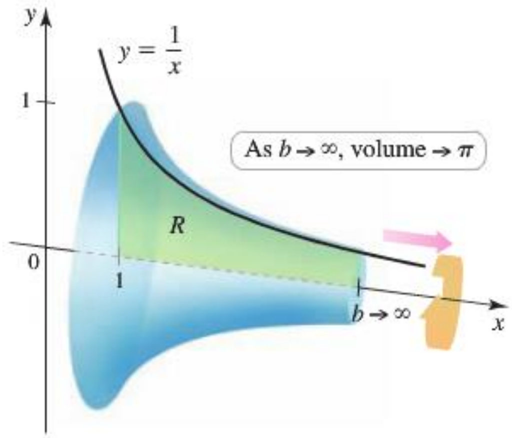
Let R be the region bounded by the graph of $y = x^{-1}$ and the x-*axis*, for $x \ge 1$.

a) What is the volume of the solid generated when R is revolved about the x-*axis*?

b) What is the surface area of the solid generated when R is revolved about the x-*axis*?

c) What is the volume of the solid generated when R is revolved about the y-*axis*?

Solution



a) $V = \pi \int_1^\infty \frac{1}{x^2}\, dx$ $\qquad V = \pi \int_a^b \left(f(x)\right)^2 dx$

$$= -\pi \frac{1}{x} \Big|_1^\infty$$

$$= -\pi(0-1)$$

$$\boxed{= \pi \ \ unit^3}$$

b) $S = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\, dx$ $\qquad S = 2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\, dx$

$$= 2\pi \int_1^\infty \frac{1}{x^3}\sqrt{x^4+1}\, dx$$

$$> 2\pi \int_1^\infty \frac{x^2}{x^3}\, dx \qquad \sqrt{x^4+1} > \sqrt{x^4} = x^2$$

$$= 2\pi \int_1^\infty \frac{1}{x}\, dx$$

$$= 2\pi(\ln x)\Big|_1^\infty$$

$$\boxed{= \infty \ \ unit^2}$$

c) $V = 2\pi \int_1^\infty x\frac{1}{x}\, dx$ $\qquad V = 2\pi \int_a^b x \cdot f(x)\, dx \quad (Shell\ method)$

$$= 2\pi x \Big|_1^\infty$$

$$\boxed{= \infty \ \ unit^3}$$

Exercises *Section* 2.6 – Improper Integrals

Evaluate the integrals

1. $\displaystyle\int_0^{\infty} \frac{dx}{x^2+1}$

2. $\displaystyle\int_0^{4} \frac{dx}{\sqrt{4-x}}$

3. $\displaystyle\int_{-\infty}^{2} \frac{2dx}{x^2+4}$

4. $\displaystyle\int_{-\infty}^{\infty} \frac{xdx}{\left(x^2+4\right)^{3/2}}$

5. $\displaystyle\int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}}$

6. $\displaystyle\int_{-\infty}^{\infty} 2xe^{-x^2}dx$

7. $\displaystyle\int_0^{1} (-\ln x)dx$

8. $\displaystyle\int_{-1}^{4} \frac{dx}{\sqrt{|x|}}$

9. $\displaystyle\int_0^{\infty} e^{-3x}\, dx$

10. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

11. $\displaystyle\int_1^{10} \frac{dx}{(x-2)^{1/3}}$

12. $\displaystyle\int_1^{\infty} \frac{dx}{x^2}$

13. $\displaystyle\int_0^{\infty} \frac{dx}{(x+1)^3}$

14. $\displaystyle\int_{-\infty}^{0} e^{x}\, dx$

15. $\displaystyle\int_1^{\infty} 2^{-x}\, dx$

16. $\displaystyle\int_{-\infty}^{0} \frac{dx}{\sqrt[3]{2-x}}$

17. $\displaystyle\int_{4/\pi}^{\infty} \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)dx$

18. $\displaystyle\int_{e^2}^{\infty} \frac{dx}{x\ln^p x} \quad p>1$

19. $\displaystyle\int_0^{\infty} \frac{p}{\sqrt[5]{p^2+1}}\, dp$

20. $\displaystyle\int_{-1}^{1} \ln y^2\, dy$

21. $\displaystyle\int_{-2}^{6} \frac{dx}{\sqrt{|x-2|}}$

22. $\displaystyle\int_0^{\infty} xe^{-x}dx$

23. $\displaystyle\int_0^{1} x\ln x\, dx$

24. $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}dx$

25. $\displaystyle\int_1^{\infty} (1-x)e^{x}\, dx$

26. $\displaystyle\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\, dx$

27. $\displaystyle\int_0^{1} \frac{dx}{\sqrt[3]{x}}$

28. $\displaystyle\int_1^{\infty} \frac{3}{\sqrt[3]{x}}\, dx$

29. $\displaystyle\int_1^{\infty} \frac{4}{\sqrt[4]{x}}\, dx$

30. $\displaystyle\int_0^{2} \frac{dx}{x^3}$

31. $\displaystyle\int_1^{\infty} \frac{dx}{x^3}$

32. $\displaystyle\int_1^{\infty} \frac{6}{x^4}\, dx$

33. $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$

34. $\displaystyle\int_{-\infty}^{0} xe^{-4x}dx$

35. $\displaystyle\int_0^{\infty} xe^{-x/3}dx$

36. $\displaystyle\int_0^{\infty} x^2e^{-x}dx$

37. $\displaystyle\int_0^{\infty} e^{-x}\cos x\, dx$

38. $\displaystyle\int_4^{\infty} \frac{1}{x(\ln x)^3}dx$

39. $\displaystyle\int_1^{\infty} \frac{\ln x}{x}dx$

40. $\displaystyle\int_{-\infty}^{\infty} \frac{4}{16+x^2}dx$

41. $\displaystyle\int_0^{\infty} \frac{x^3}{\left(x^2+1\right)^2}\,dx$

42. $\displaystyle\int_0^{\infty} \frac{1}{e^x+e^{-x}}\,dx$

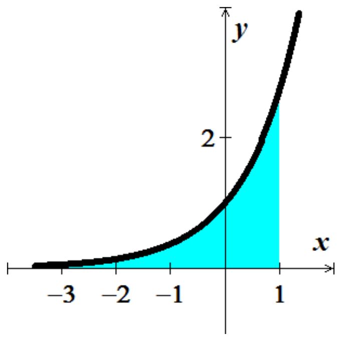
43. $\displaystyle\int_0^{\infty} \frac{e^x}{1+e^x}\,dx$

44. $\displaystyle\int_0^{\infty} \cos \pi x\,dx$

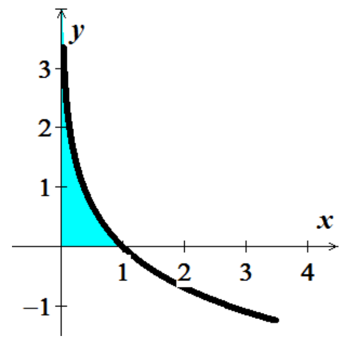
45. $\displaystyle\int_0^{\infty} \sin \frac{x}{2}\,dx$

Find the area of the unbounded shaded region

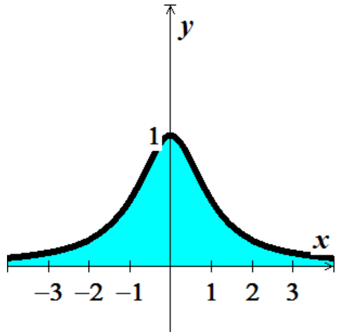
46. $y=e^x, \quad -\infty < x \le 1$



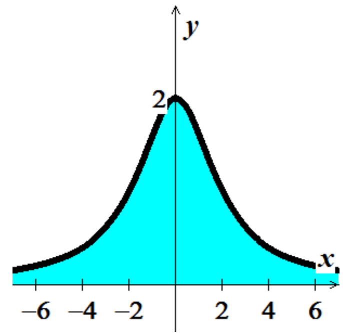
47. $y=-\ln x$



48. $y=\dfrac{1}{x^2+1}$



49. $y=\dfrac{8}{x^2+4}$



50. Find the area of the region R between the graph of $f(x)=\dfrac{1}{\sqrt{9-x^2}}$ and the x-*axis* on the interval $(-3,\ 3)$ (if it exists)

51. Find the volume of the region bounded by $f(x)=\left(x^2+1\right)^{-1/2}$ and the x-*axis* on the interval $[2,\ \infty)$ is revolved about the x-*axis*.

52. Find the volume of the region bounded by $f(x)=\sqrt{\dfrac{x+1}{x^3}}$ and the x-*axis* on the interval $[1,\ \infty)$ is revolved about the x-*axis*.

53. Find the volume of the region bounded by $f(x)=(x+1)^{-3}$ and the x-*axis* on the interval $[0,\ \infty)$ is revolved about the y-*axis*.

54. Find the volume of the region bounded by $f(x) = \frac{1}{\sqrt{x}\ln x}$ and the *x-axis* on the interval $[2, \infty)$ is revolved about the *x-axis*.

55. Find the volume of the region bounded by $f(x) = \frac{\sqrt{x}}{\sqrt[3]{x^2+1}}$ and the *x-axis* on the interval $[0, \infty)$ is revolved about the *x-axis*.

56. Find the volume of the region bounded by $f(x) = \left(x^2 - 1\right)^{-1/4}$ and the *x-axis* on the interval $(1, 2]$ is revolved about the *y-axis*.

57. Find the volume of the region bounded by $f(x) = \tan x$ and the *x-axis* on the interval $\left[0, \frac{\pi}{2}\right)$ is revolved about the *x-axis*.

58. Find the volume of the region bounded by $f(x) = -\ln x$ and the *x-axis* on the interval $(0, 1]$ is revolved about the *x-axis*.

59. Find the volume of the solid generated by revolving the region bounded by the graphs of $y = xe^{-x}$, $y = 0$, and $x = 0$ about the *x-axis*.

Consider the region satisfying the inequalities

a) Find the area of the region
b) Find the volume of the solid generated by revolving the region about the *x-axis*.
c) Find the volume of the solid generated by revolving the region about the *y-axis*.

60. $y \le e^{-x}, \quad y \ge 0, \quad x \ge 0$

61. $y \le \frac{1}{x^2}, \quad y \ge 0, \quad x \ge 1$

62. Find the perimeter of the hypocycloid of four cusps $x^{2/3} + y^{2/3} = 4$

63. Find the arc length of the graph $y = \sqrt{16 - x^2}$ over the interval $[0, 4]$

64. The region bounded by $(x-2)^2 + y^2 = 1$ is revolved about the *y-axis* to form a torus. Find the surface area of the torus.

65. Find the surface area formed by revolving the graph $y = 2e^{-x}$ on the interval $[0, \infty)$ about the *x-axis*

66. The magnetic potential *P* at a point on the axis of a circular coil is given by

$$P = \frac{2\pi NIr}{k} \int_c^{\infty} \frac{1}{\left(r^2 + x^2\right)^{3/2}}\, dx$$

Where *N, I, r, k,* and *c* are constants. Find *P*.

67. A "semi-infinite" uniform rod occupies the nonnegative x-*axis*. The rod has a linear density δ, which means that a segment of length dx has a mass of δdx. A particle of mass M is located at the point $(-a,\ 0)$. The gravitational force F that the rod exerts on the mass is given by

$$F = \int_0^\infty \frac{GM\delta}{(a+x)^2}\, dx$$

Where G is the gravitational constant. Find F.

68. Let R be the region bounded by the graph of $f(x) = x^{-p}$ and the x-*axis*

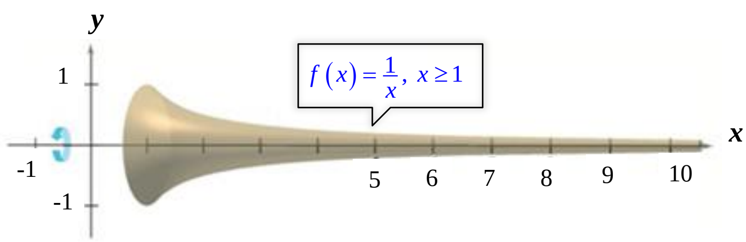
a) Let S be the solid generated when R is revolved about the x-*axis*. For what values of p is the volume of S finite for $0 < x \le 1$?

b) Let S be the solid generated when R is revolved about the y-*axis*. For what values of p is the volume of S finite for $0 < x \le 1$?

c) Let S be the solid generated when R is revolved about the x-*axis*. For what values of p is the volume of S finite for $x \ge 1$?

d) Let S be the solid generated when R is revolved about the y-*axis*. For what values of p is the volume of S finite for $x \ge 1$?

69. The solid formed by revolving (about the x-*axis*) the unbounded region lying between the graph of $f(x) = \frac{1}{x}$ and the x-*axis* $(x \ge 1)$ is called ***Gabriel's Horn***.



Show that this solid has a finite volume and an infinite surface area

70. Water is drained from a 3000-*gal* tank at a rate that starts at 100 *gal/hr*. and decreases continuously by 5% /*hr*. If the drain left open indefinitely, how much water drains from the tank? Can a full tank be emptied at this rate?