

Exercise

Find all positive integers n for which the given statement is not true

a) $3^n > 6n$ b) $3^n > 2n + 1$ c) $2^n > n^2$ d) $n! > 2n$

Solution

a) $n = 1$ $3 < 6$

$n = 2$ $3^2 < 18$

$n = 3$, $27 > 18$

The statement is true for all $n \geq 3$ $3^n > 6n$

The statement is not true for $\boxed{n = 1, 2}$

b) $n = 1$; $3 = 3$

$n = 2$; $9 > 5$

The statement is true for all $n \geq 2$ $3^n > 2n + 1$

The statement is not true for $\boxed{n = 1}$

c) $n = 1$; $2 < 4$

$n = 2$; $4 = 4$

$n = 3$; $8 < 9$

$n = 4$; $16 = 16$

$n = 5$; $32 > 25$

The statement is true for all $n \geq 5$; $2^n > n^2$

The statement is not true for $\boxed{n = 1, 2, 3, 4}$

d) $n = 1$; $1 < 2$

$n = 2$; $2 < 4$

$n = 3$; $6 = 6$

$n = 4$; $12 > 8$

The statement is true for all $n \geq 4$; $n! > 2n$

The statement is not true for $\boxed{n = 1, 2, 3}$

Exercise

Prove that the statement is true for every positive integer n . $2 + 4 + 6 + \dots + 2n = n(n + 1)$

Solution

(1) For $n = 1 \Rightarrow 2 = 1(1 + 1) = 2$; hence P_1 is true.

(2) Assume $2 + 4 + 6 + \dots + 2k = k(k + 1)$ is true

$$\Rightarrow 2 + 4 + 6 + \dots + 2k + 2(k + 1) = (k + 1)(k + 1 + 1) \text{ ?}$$

$$2 + 4 + 6 + \dots + 2k + 2(k + 1) = 2 + 4 + 6 + \dots + 2k + 2(k + 1)$$

$$= k(k + 1) + 2(k + 1)$$

Factor (k + 1)

$$= (k + 1)(k + 2)$$

$$= (k + 1)(k + 1 + 1) \quad \checkmark$$

Hence P_{k+1} is also true.

\therefore By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n . $1 + 3 + 5 + \dots + (2n - 1) = n^2$

Solution

(1) For $n = 1 \Rightarrow 1 = 1^2 = 1$; hence P_1 is true.

(2) Assume $1 + 3 + 5 + \dots + (2k - 1) = k^2$ is true

$$\Rightarrow 1 + 3 + 5 + \dots + (2(k + 1) - 1) = (k + 1)^2 \text{ ?}$$

$$1 + 3 + 5 + \dots + (2k - 1) + (2(k + 1) - 1) = 1 + 3 + 5 + \dots + (2k - 1) + (2k + 2 - 1)$$

$$= k^2 + (2k + 1)$$

$$= k^2 + 2k + 1$$

$$= (k + 1)^2 \quad \checkmark$$

Hence P_{k+1} is also true.

\therefore By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n . $2 + 7 + 12 + \dots + (5n - 3) = \frac{1}{2}n(5n - 1)$

Solution

(1) For $n = 1 \Rightarrow 2 = \frac{1}{2}(\overset{?}{1})(5(\overset{?}{1}) - 1) = \frac{1}{2}(4) = \overset{?}{2}$; hence P_1 is true.

(2) Assume $2 + 7 + 12 + \dots + (5k - 3) = \frac{1}{2}k(5k - 1)$ is true

$$2 + 7 + 12 + \dots + (5(k + 1) - 3) = \frac{1}{2}(k + 1)(5(k + 1) - 1) \quad ?$$

$$2 + 7 + 12 + \dots + (5k - 3) + (5(k + 1) - 3) = 2 + 7 + 12 + \dots + (5k - 3) + (5k + 5 - 3)$$

$$= \frac{1}{2}k(5k - 1) + (5k + 2) \quad \frac{2}{2}$$

$$= \frac{1}{2}[5k^2 - k + 10k + 4]$$

$$= \frac{1}{2}[5k^2 - k + 5k + 5k + 5 - 1]$$

$$= \frac{1}{2}[k(5k - 1 + 5) + 5k + 5 - 1]$$

$$= \frac{1}{2}[(k + 1)(5k + 5 - 1)]$$

$$= \frac{1}{2}[(k + 1)(5(k + 1) - 1)] \quad \checkmark \quad P_{k+1} \text{ is also true.}$$

\therefore By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n . $1 + 2.2 + 3.2^2 + \dots + n.2^{n-1} = 1 + (n - 1).2^n$

Solution

For $n = 1$

$$1 = 1 + (\overset{?}{1} - 1)2^{\overset{?}{1}} = 1 - 0 = \overset{?}{1}$$

Hence P_1 is true.

$1 + 2.2 + 3.2^2 + \dots + k.2^{k-1} = 1 + (k - 1).2^k$ is true

$$1 + 2.2 + 3.2^2 + \dots + k.2^{k-1} + (k + 1).2^{(k+1)-1} = 1 + ((k + 1) - 1).2^{k+1} \quad ?$$

$$1 + 2.2 + 3.2^2 + \dots + k.2^{k-1} + (k + 1).2^{(k+1)-1} = 1 + (k - 1).2^k + (k + 1).2^{k+1-1}$$

$$= 1 + k.2^k - 1.2^k + (k + 1).2^k$$

$$= 1 + k.2^k - 1.2^k + k.2^k + 1.2^k$$

$$= 1 + 2^1 k.2^k$$

$$\begin{aligned}
&= 1 + (k+0) \cdot 2^{k+1} \\
&= 1 + ((k+1)-1) \cdot 2^{k+1} \quad \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n . $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Solution

For $n = 1$

$$1^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = \frac{6}{6} = 1 \quad \checkmark$$

Hence P_1 is true.

$$1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6} \text{ is true}$$

$$1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \quad ?$$

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\
&= \frac{(k+1)[2k^2 + k + 6k + 6]}{6} \\
&= \frac{(k+1)[2k^2 + 7k + 6]}{6} \\
&= \frac{(k+1)((k+2)(2k+3))}{6} \\
&= \frac{(k+1)((k+1+1)(2k+2+1))}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \quad \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n . $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Solution

For $n = 1$

$$\frac{1}{1 \cdot 2} \stackrel{?}{=} \frac{1}{1+1} = \frac{1}{2} = \frac{1}{1 \cdot 2} \quad \checkmark$$

Hence P_1 is true.

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1} \text{ is true}$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{(k+1)+1} \quad ?$$

$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2)+1}{(k+1)(k+2)} \\ &= \frac{k^2+2k+1}{(k+1)(k+2)} \\ &= \frac{(k+1)(k+1)}{(k+1)(k+2)} \\ &= \frac{k+1}{(k+1)+1} \\ &= \frac{k+1}{(k+1)+1} \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

\therefore By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

Solution

For $n = 1$

$$\frac{1}{2} \stackrel{?}{=} 1 - \frac{1}{2} = \frac{1}{2} \quad \checkmark$$

Hence, P_1 is true.

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^k} = 1 - \frac{1}{2^k} \text{ is true}$$

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}} \text{ ?}$$

$$\begin{aligned} \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^k} + \frac{1}{2^{k+1}} &= 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \\ &= 1 - \frac{1}{2^k} + \frac{1}{2^k \cdot 2} \\ &= \frac{2^{k+1} - 2 + 1}{2^{k+1}} \\ &= \frac{2^{k+1} - 1}{2^{k+1}} \\ &= \frac{2^{k+1}}{2^{k+1}} - \frac{1}{2^{k+1}} \\ &= 1 - \frac{1}{2^{k+1}} \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + \dots + \frac{1}{(3n-2) \cdot (3n+1)} = \frac{n}{3n+1}$

Solution

For $n = 1$

$$\frac{1}{1 \cdot 4} \stackrel{?}{=} \frac{1}{3(1)+1} = \frac{1}{4} \quad \checkmark$$

Hence, P_1 is true.

$$\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \dots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1} \text{ is true}$$

$$\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \dots + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3(k+1)-2)(3(k+1)+1)} \stackrel{?}{=} \frac{k+1}{3(k+1)+1}$$

$$\begin{aligned} \frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \dots + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3(k+1)-2)(3(k+1)+1)} &= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)} \\ &= \frac{3k^2 + 4k + 1}{(3k+1)(3k+4)} \end{aligned}$$

$$\begin{aligned}
&= \frac{(3k+1)(k+1)}{(3k+1)(3k+3+1)} \\
&= \frac{k+1}{3(k+1)+1} \quad \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\frac{4}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \dots + \frac{4}{5^n} = 1 - \frac{1}{5^n}$

Solution

For $n = 1$

$$\frac{4}{5} \stackrel{?}{=} 1 - \frac{1}{5} = \frac{4}{5} \quad \checkmark$$

Hence, P_1 is true.

$$\frac{4}{5} + \frac{4}{5^2} + \dots + \frac{4}{5^k} = 1 - \frac{1}{5^k} \text{ is true}$$

$$\frac{4}{5} + \frac{4}{5^2} + \dots + \frac{4}{5^k} + \frac{4}{5^{k+1}} \stackrel{?}{=} 1 - \frac{1}{5^{k+1}}$$

$$\begin{aligned}
\frac{4}{5} + \frac{4}{5^2} + \dots + \frac{4}{5^k} + \frac{4}{5^{k+1}} &= 1 - \frac{1}{5^k} + \frac{4}{5^{k+1}} \\
&= 1 - \left(\frac{1}{5^k} - \frac{4}{5^{k+1}} \right) \\
&= 1 - \frac{5-4}{5^{k+1}} \\
&= 1 - \frac{1}{5^{k+1}} \quad \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$

Solution

For $n = 1$

$$1^3 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1 \quad \checkmark$$

Hence, P_1 is true.

$$\frac{4}{5} + \frac{4}{5^2} + \dots + \frac{4}{5^k} = 1 - \frac{1}{5^k} \text{ is true}$$

$$1^3 + 2^3 + \dots + k^3 + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$$

$$\begin{aligned} 1^3 + 2^3 + \dots + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} \\ &= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} \\ &= \frac{(k+1)^2((k+1)+1)^2}{4} \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n . $3 + 3^2 + 3^3 + \dots + 3^n = \frac{3}{2}(3^n - 1)$

Solution

For $n = 1$

$$3 = \frac{3}{2}(3^1 - 1) = \frac{3}{2} \cdot 2 = 3 \quad \checkmark$$

Hence, P_1 is true.

$$3 + 3^2 + \dots + 3^k = \frac{3}{2}(3^k - 1) \text{ is true}$$

$$3 + 3^2 + \dots + 3^k + 3^{k+1} = \frac{3}{2}(3^{k+1} - 1)$$

$$\begin{aligned}
3 + 3^2 + \dots + 3^k + 3^{k+1} &= \frac{3}{2}(3^k - 1) + 3^{k+1} \\
&= \frac{1}{2}3^{k+1} - \frac{3}{2} + 3^{k+1} \\
&= \frac{3}{2}3^{k+1} - \frac{3}{2} \\
&= \frac{3}{2}(3^{k+1} - 1) \quad \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $x^{2n} + x^{2n-1}y + \dots + xy^{2n-1} + y^{2n} = \frac{x^{2n+1} - y^{2n+1}}{x - y}$

Solution

For $n = 1$

$$\begin{aligned}
x^2 + xy + y^2 &\stackrel{?}{=} \frac{x^3 - y^3}{x - y} \\
&= \frac{(x - y)(x^2 + xy + y^2)}{x - y} \\
&= x^2 + xy + y^2 \quad \checkmark
\end{aligned}$$

Hence, P_1 is true.

$$x^{2k} + x^{2k-1}y + \dots + xy^{2k-1} + y^{2k} = \frac{x^{2k+1} - y^{2k+1}}{x - y} \text{ is true}$$

$$x^{2(k+1)} + x^{2(k+1)-1}y + \dots + xy^{2(k+1)-1} + y^{2(k+1)} \stackrel{?}{=} \frac{x^{2(k+1)+1} - y^{2(k+1)+1}}{x - y}$$

$$\begin{aligned}
x^{2k+2} + x^{2k+1}y + \dots + xy^{2k+1} + y^{2k+2} &= x^2 \left(x^{2k} + x^{2k-1}y + \dots + y^{2k} \right) + xy^{2k+1} + y^{2k+2} \\
&= x^2 \left(\frac{x^{2k+1} - y^{2k+1}}{x - y} \right) + xy^{2k+1} + y^{2k+2} \\
&= \frac{x^{2k+3} - x^2y^{2k+1} + x^2y^{2k+1} + xy^{2k+2} - xy^{2k+2} - y^{2(k+1)+1}}{x - y} \\
&= \frac{x^{2(k+1)+1} - y^{2(k+1)+1}}{x - y} \quad \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $5 \cdot 6 + 5 \cdot 6^2 + 5 \cdot 6^3 + \dots + 5 \cdot 6^n = 6(6^n - 1)$

Solution

For $n = 1$

$$5 \cdot 6 \stackrel{?}{=} 6(6^1 - 1) = 6(5) \quad \checkmark$$

Hence, P_1 is true.

$$5 \cdot 6 + 5 \cdot 6^2 + \dots + 5 \cdot 6^k = 6(6^k - 1) \text{ is true}$$

$$5 \cdot 6 + 5 \cdot 6^2 + \dots + 5 \cdot 6^k + 5 \cdot 6^{k+1} \stackrel{?}{=} 6(6^{k+1} - 1)$$

$$\begin{aligned} 5 \cdot 6 + 5 \cdot 6^2 + \dots + 5 \cdot 6^k + 5 \cdot 6^{k+1} &= 6(6^k - 1) + 5 \cdot 6^{k+1} \\ &= 6^{k+1} - 6 + 5 \cdot 6^{k+1} \\ &= 6^{k+1}(1 + 5) - 6 \\ &= 6 \cdot 6^{k+1} - 6 \\ &= 6(6^{k+1} - 1) \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $7 \cdot 8 + 7 \cdot 8^2 + 7 \cdot 8^3 + \dots + 7 \cdot 8^n = 8(8^n - 1)$

Solution

For $n = 1$

$$7 \cdot 8 \stackrel{?}{=} 8(8^1 - 1) = 8(7) \quad \checkmark$$

Hence, P_1 is true.

$$7 \cdot 8 + 7 \cdot 8^2 + \dots + 7 \cdot 8^k = 8(8^k - 1) \text{ is true}$$

$$7 \cdot 8 + 7 \cdot 8^2 + \dots + 7 \cdot 8^k + 7 \cdot 8^{k+1} \stackrel{?}{=} 8(8^{k+1} - 1)$$

$$\begin{aligned} 7 \cdot 8 + 7 \cdot 8^2 + \dots + 7 \cdot 8^k + 7 \cdot 8^{k+1} &= 8(8^k - 1) + 7 \cdot 8^{k+1} \\ &= 8^{k+1} - 8 + 7 \cdot 8^{k+1} \end{aligned}$$

$$\begin{aligned}
 &= 8^{k+1}(1+7) - 8 \\
 &= 8(8^{k+1} - 1) \quad \checkmark
 \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $3 + 6 + 9 + \dots + 3n = \frac{3n(n+1)}{2}$

Solution

For $n = 1$

$$3 = \frac{3(1)(1+1)}{2} = 3 \quad \checkmark$$

Hence, P_1 is true.

$$3 + 6 + 9 + \dots + 3k = \frac{3k(k+1)}{2} \text{ is true}$$

$$3 + 6 + 9 + \dots + 3k + 3(k+1) = \frac{3(k+1)(k+2)}{2}$$

$$\begin{aligned}
 3 + 6 + 9 + \dots + 3k + 3(k+1) &= \frac{3k(k+1)}{2} + 3(k+1) \\
 &= \frac{3k(k+1) + 6(k+1)}{2} \\
 &= \frac{(k+1)(3k+6)}{2} \\
 &= \frac{3(k+1)(k+2)}{2} \quad \checkmark
 \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $5 + 10 + 15 + \dots + 5n = \frac{5n(n+1)}{2}$

Solution

For $n = 1$

$$5 = \frac{? 5(1)(1+1)}{2} = 5 \quad \checkmark$$

Hence, P_1 is true.

$$5 + 10 + 15 + \dots + 5k = \frac{5k(k+1)}{2} \text{ is true}$$

$$5 + 10 + 15 + \dots + 5k + 5(k+1) = \frac{? 5(k+1)(k+2)}{2}$$

$$\begin{aligned} 5 + 10 + 15 + \dots + 5k + 5(k+1) &= \frac{5k(k+1)}{2} + 5(k+1) \\ &= \frac{5k(k+1) + 10(k+1)}{2} \\ &= \frac{(k+1)(5k+10)}{2} \\ &= \frac{5(k+1)(k+2)}{2} \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $1 + 3 + 5 + \dots + (2n-1) = n^2$

Solution

For $n = 1$

$$1 = 1^2 = 1 \quad \checkmark$$

Hence, P_1 is true.

$$1 + 3 + 5 + \dots + (2k-1) = k^2 \text{ is true}$$

$$1 + 3 + 5 + \dots + (2k-1) + (2(k+1)-1) = \frac{?}{(k+1)^2}$$

$$\begin{aligned} 1 + 3 + 5 + \dots + (2k-1) + (2(k+1)-1) &= k^2 + 2k + 2 - 1 \\ &= k^2 + 2k + 1 \\ &= (k+1)^2 \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $4 + 7 + 10 + \dots + (3n + 1) = \frac{n(3n + 5)}{2}$

Solution

For $n = 1$

$$4 = \frac{1(3+5)}{2} = 4 \quad \checkmark$$

Hence, P_1 is true.

$$4 + 7 + 10 + \dots + (3k + 1) = \frac{k(3k + 5)}{2} \text{ is true}$$

$$4 + 7 + 10 + \dots + (3k + 1) + (3(k + 1) + 1) = \frac{(k + 1)(3(k + 1) + 5)}{2} = \frac{(k + 1)(3k + 8)}{2}$$

$$\begin{aligned} 4 + 7 + 10 + \dots + (3k + 1) + (3k + 4) &= \frac{k(3k + 5)}{2} + 3k + 4 \\ &= \frac{3k^2 + 5k + 6k + 8}{2} \\ &= \frac{3k^2 + 5k + 3k + 3k + 8}{2} \\ &= \frac{k(3k + 8) + (3k + 8)}{2} \\ &= \frac{(3k + 8)(k + 1)}{2} \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

\therefore By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction:

$$2 + 4 + 6 + \dots + 2(n - 1) + 2n = n(n + 1)$$

Solution

For $n = 1$

$$\begin{aligned} 2 &= 1(1 + 1) \\ 2 &= 2 \quad \checkmark \end{aligned}$$

Hence, P_1 is true.

$$\text{For } k: \quad 2 + 4 + 6 + \dots + 2(k - 1) + 2k = k(k + 1)$$

$$2 + 4 + \cdots + 2k + 2(k+1) = (k+1)(k+2)$$

$$\begin{aligned} 2 + 4 + \cdots + 2k + 2(k+1) &= k(k+1) + 2(k+1) \\ &= (k+1)(k+2) \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction:

$$1 + (1+2) + (1+2+3) + \cdots + (1+2+\cdots+n) = \frac{n(n+1)(n+2)}{6}$$

Solution

For $n = 1$

$$1 \stackrel{?}{=} \frac{1(1+1)(1+2)}{6}$$

$$1 \stackrel{?}{=} \frac{(2)(3)}{6}$$

$$1 = 1 \quad \checkmark$$

Hence, P_1 is true.

$$\text{For } k: \quad 1 + (1+2) + \cdots + (1+2+\cdots+k) = \frac{k(k+1)(k+2)}{6}$$

$$\text{Is } P_{k+1}: \quad 1 + (1+2) + \cdots + (1+2+\cdots+k) + (1+2+\cdots+k+(k+1)) \stackrel{?}{=} \frac{(k+1)(k+2)(k+3)}{6}$$

$$1 + (1+2) + \cdots + (1+2+\cdots+k) + (1+2+\cdots+k+(k+1)) = \frac{k(k+1)(k+2)}{6} + (1+2+\cdots+k+(k+1))$$

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$$

$$1 + 2 + \cdots + k + (k+1) = \frac{1}{2}k(k+1) + (k+1)$$

$$= (k+1)\left(\frac{1}{2}k+1\right)$$

$$= \frac{1}{2}(k+1)(k+2)$$

$$\begin{aligned} 1 + (1+2) + \cdots + (1+2+\cdots+k) + (1+2+\cdots+k+(k+1)) &= \frac{k(k+1)(k+2)}{6} + \frac{1}{2}(k+1)(k+2) \\ &= (k+1)(k+2)\left(\frac{k}{6} + \frac{1}{2}\right) \end{aligned}$$

$$= \frac{(k+1)(k+2)(k+3)}{6} \quad \checkmark$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction: $1+2+3+\dots+n < \frac{(2n+3)^2}{7}$

Solution

For $n = 1$

$$1 < \frac{? (2+3)^2}{7}$$

$$1 < \frac{25}{7} > 1 \quad \checkmark$$

Hence, P_1 is true.

$$\text{For } k: \quad 1+2+\dots+k < \frac{(2k+3)^2}{7}$$

$$\text{Is } P_{k+1}: \quad 1+2+\dots+k+(k+1) < \frac{(2(k+1)+3)^2}{7}$$

$$< \frac{(2k+5)^2}{7} \quad ? \quad \frac{4k^2+20k+25}{7}$$

$$\begin{aligned} 1+2+\dots+k+(k+1) &< \frac{(2k+3)^2}{7} + (k+1) \\ &= \frac{4k^2+12k+9+7k+7}{7} \\ &= \frac{1}{7} (4k^2+19k+16+k+9-k-9) \\ &= \frac{1}{7} (4k^2+20k+25-(k+9)) \\ &= \frac{(2k+5)^2}{7} - \frac{k+9}{7} \\ &< \frac{(2k+5)^2}{7} \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction:

$$\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n-3) \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n-2) \cdot (2n)}$$

Solution

For $n = 1$

$$\frac{1}{2} \leq \frac{1}{2} \quad \checkmark$$

Hence, P_1 is true.

$$\text{For } k: \quad \frac{1}{2k} \leq \frac{1 \cdots (2k-3) \cdot (2k-1)}{2 \cdots (2k-2) \cdot (2k)}$$

$$\begin{aligned} \text{Is } P_{k+1}: \quad \frac{1}{2(k+1)} &\leq \frac{1 \cdots (2k-1) \cdot (2k+1)}{2 \cdots (2k) \cdot (2k+2)} \\ \frac{1 \cdots (2k-1) \cdot (2k+1)}{2 \cdots (2k) \cdot (2k+2)} &\geq \frac{1}{2(k+1)} \quad ? \end{aligned}$$

$$\begin{aligned} \frac{1 \cdots (2k-1) \cdot (2k+1)}{2 \cdots (2k) \cdot (2k+2)} &= \frac{1 \cdots (2k-1)}{2 \cdots (2k)} \cdot \frac{2k+1}{2k+2} \\ &\geq \frac{1}{2k} \cdot \frac{2k+1}{2k+2} \\ &= \frac{2k+1}{2k} \cdot \frac{1}{2(k+1)} \\ &= \left(1 + \frac{1}{2k}\right) \cdot \frac{1}{2(k+1)} \\ &\geq \frac{1}{2(k+1)} \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction:

$$\frac{2n+1}{2n+2} \leq \frac{\sqrt{n+1}}{\sqrt{n+2}}$$

Solution

For $n = 1$

$$\frac{2+1}{2+2} \stackrel{?}{\leq} \frac{\sqrt{1+1}}{\sqrt{1+2}}$$

$$\frac{3}{4} \stackrel{?}{\leq} \frac{\sqrt{2}}{\sqrt{3}}$$

$$3\sqrt{3} \stackrel{?}{\leq} 4\sqrt{2} \quad \text{Square both sides}$$

$$27 \leq 32 \quad \checkmark$$

Hence, P_1 is true.

$$\begin{aligned} \text{For } k: \quad \frac{2k+1}{2k+2} &\leq \frac{\sqrt{k+1}}{\sqrt{k+2}} \\ (2k+1)\sqrt{k+2} &\leq (2k+2)\sqrt{k+1} \end{aligned}$$

$$\text{Is } P_{k+1}: \quad \frac{2k+3}{2k+4} \stackrel{?}{\leq} \frac{\sqrt{k+2}}{\sqrt{k+3}}$$

$$(2k+3)\sqrt{k+3} \stackrel{?}{\leq} (2k+4)\sqrt{k+2}$$

$$(2k+3) \leq (2k+4)$$

$$\sqrt{k+3} \leq \sqrt{k+2}$$

$$(2k+3)\sqrt{k+3} \leq (2k+4)\sqrt{k+2} \quad \checkmark$$

Hence P_{k+1} is true.

\therefore By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction: $n! < n^n$ for $n > 1$

Solution

For $n = 2$

$$2! \stackrel{?}{<} 2^2$$

$$2 < 4 \quad \checkmark$$

Hence, P_1 is true.

For k : $k! < k^k$

Is P_{k+1} : $(k+1)! < (k+1)^{k+1}$

$$(k+1)! = k! (k+1)$$

$$< k^k (k+1)$$

$$k < k+1$$

$$\begin{aligned}
 & k^k < (k+1)^k \\
 & < (k+1)^k (k+1) \\
 & = (k+1)^{k+1} \quad \checkmark
 \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction: $(a^m)^n = a^{mn}$ (a and m are constant)

Solution

For $n = 1$

$$(a^m)^1 \stackrel{?}{=} a^{m(1)}$$

$$a^m = a^m \quad \checkmark$$

Hence, P_1 is true.

$$(a^m)^k = a^{mk} \text{ is true}$$

$$(a^m)^{(k+1)} \stackrel{?}{=} a^{m(k+1)}$$

$$\begin{aligned}
 (a^m)^{(k+1)} &= (a^m)^k a^m \\
 &= a^{km} a^m \\
 &= a^{km+m} \\
 &= a^{m(k+1)} \quad \checkmark
 \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n . $n < 2^n$

Solution

For $n = 1$

$$1 < 2^1 \quad \checkmark$$

Hence, P_1 is true.

Assume that P_k is true $k < 2^k$

We need to prove that P_{k+1} is true, that is $k+1 < 2^{k+1}$

$$\begin{aligned}k+1 &< k+k = 2k \\&< 2 \cdot 2^k \\&= 2^{k+1} \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n . 3 is a factor of $n^3 - n + 3$

Solution

For $n = 1$

$$1^3 - 1 + 3 = 3 = 3(1) \quad \checkmark$$

Hence, P_1 is true.

Assume that P_k is true 3 is a factor of $k^3 - k + 3$

We need to prove that P_{k+1} is true, that is $(k+1)^3 - (k+1) + 3$

$$\begin{aligned}(k+1)^3 - (k+1) + 3 &= k^3 + 3k^2 + 3k + 1 - k - 1 + 3 \\&= (k^3 - k + 3) + 3k^2 + 3k && k^3 - k + 3 = 3K \\&= 3K + 3k^2 + 3k \\&= 3(K + k^2 + k) \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n . 4 is a factor of $5^n - 1$

Solution

For $n = 1$

$$5^1 - 1 = 4 = 4(1) \quad \checkmark$$

Hence, P_1 is true.

Assume that P_k is true 4 is a factor of $5^k - 1$

We need to prove that P_{k+1} is true, that is $5^{k+1} - 1$

$$\begin{aligned} 5^{k+1} - 1 &= 5^k 5^1 - 5 + 4 \\ &= 5(5^k - 1) + 4 \\ &= 5(5^k - 1) + 4 \end{aligned}$$

By the induction hypothesis, 4 is a factor of $5^k - 1$ and 4 is a factor of 4, so 4 is a factor of the $(k+1)$ term. \checkmark

Hence P_{k+1} is true.

\therefore By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $2^n > 2n$ if $n \geq 3$

Solution

For $n = 3$

$$2^3 \geq 2(3)$$

$$8 \geq 6 \quad \checkmark$$

Hence, P_3 is true.

Assume that P_k is true: $2^k > 2k$

We need to prove that $P_{k+1} : 2^{k+1} > 2(k+1)$ is true

$$\begin{aligned} 2^k &> 2k \\ 2^k \cdot 2 &> 2k \cdot 2 \\ 2^{k+1} &> 4k = 2k + 2k \quad k \geq 3 \\ &> 2k + 2 \\ &= 2(k+1) \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

\therefore By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: If $0 < a < 1$, then $a^n < a^{n-1}$

Solution

For $n = 1$

$$a^1 < a^{1-1}$$

$$a < 1 \quad \checkmark$$

since $0 < a < 1 \Rightarrow P_1$ is true.

Assume that P_k is true: $a^k < a^{k-1}$

We need to prove that $P_{k+1} : a^{k+1} < a^k$ is true

$$a^k < a^{k-1} \rightarrow a^k \cdot a < a^{k-1} \cdot a$$

$$a^{k+1} < a^k \quad \checkmark$$

Hence P_{k+1} is true.

\therefore By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: If $n \geq 4$, then $n! > 2^n$

Solution

For $n = 4$

$$4! > 2^4$$

$$24 > 16 \quad \checkmark$$

Hence, P_4 is true.

Assume that P_k is true: $k! > 2^k$

We need to prove that $P_{k+1} : (k+1)! > 2^{k+1}$ is true

$$(k+1)! = k!(k+1)$$

$$> 2^k (k+1) \quad \text{Since } k \geq 4 \Rightarrow k+1 > 2$$

$$> 2^k \cdot 2$$

$$= 2^{k+1} \quad \checkmark$$

Hence P_{k+1} is true.

\therefore By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $3^n > 2n + 1$ if $n \geq 2$

Solution

For $n = 2$

$$3^2 > 2(2) + 1$$

$$9 > 5 \quad \checkmark$$

Hence, P_2 is true.

Assume that P_k is true: $3^k > 2k + 1$;

We need to prove that P_{k+1} : $3^{k+1} > 2(k+1) + 1$ is true

$$\begin{aligned} 3^k > 2k + 1 &\Rightarrow 3^k \cdot 3 > (2k + 1) \cdot 3 \\ 3^{k+1} &> 6k + 3 \\ &> 2k + 2 + 1 \\ &= 2(k+1) + 1 \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

\therefore By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $2^n > n^2$ for $n > 4$

Solution

For $n = 5$

$$2^5 > 5^2$$

$$32 > 25 \quad \checkmark$$

Hence, P_5 is true.

Assume that P_k is true: $2^k > k^2$

We need to prove that P_{k+1} : $2^{k+1} > (k+1)^2$ is true

$$\begin{aligned} 2^k &> k^2 \\ 2^k \cdot 2 &> k^2 \cdot 2 \\ 2^{k+1} &> 2k^2 \\ &= k^2 + k^2 \\ &> k^2 + 2k + 1 \end{aligned} \quad \begin{aligned} k < k+1 &\Rightarrow k \cdot k > k + k + 1 \Rightarrow k^2 > 2k + 1 \end{aligned}$$

$$= (k+1)^2 \quad \checkmark$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $4^n > n^4$ for $n \geq 5$

Solution

For $n = 5$

$$4^5 > 5^4$$

$$1024 > 625 \quad \checkmark$$

Hence, P_5 is true.

Assume that P_k is true: $4^k > k^4$

We need to prove that $P_{k+1} : 4^{k+1} > (k+1)^4$ is true

$$4^k > k^4$$

$$4^k \cdot 4 > k^4 \cdot 4$$

$$4^{k+1} > 4k^4$$

$$k < k+1$$

$$4k > k+1$$

$$4k^4 > (k+1)^4$$

$$> (k+1)^4 \quad \checkmark$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

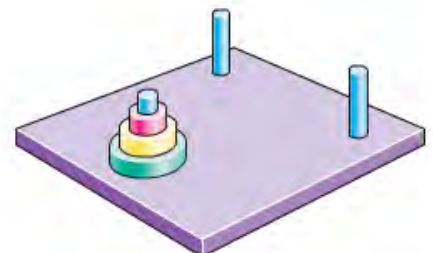
Exercise

A pile of n rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must be moved, one at a time, to a different peg with no ring ever placed on top of a smaller ring.

Find the least number of moves that would be required.

Prove your result by mathematical induction.

Solution



With 1 ring, 1 move is required.

With 2 rings, 3 moves are required $\Rightarrow 3 = 2 + 1$

With 3 rings, 7 moves are required $\Rightarrow 7 = 2^2 + 2 + 1$

With n rings, $2^{n-1} + \dots + 2^2 + 2^1 + 2^0 = 2^n - 1$ moves are required

For $n = 1$

$$2^0 = 2^1 - 1 = 1 \quad \checkmark$$

Hence, P_1 is true.

Assume that P_k is true: $2^{k-1} + \dots + 2^2 + 2^1 + 2^0 = 2^k - 1$

$$2^k + 2^{k-1} + \dots + 2^2 + 2^1 + 1 \stackrel{?}{=} 2^{k+1} - 1$$

$$\begin{aligned} 2^k + 2^{k-1} + \dots + 2^2 + 2^1 + 1 &= 2^k + 2^k - 1 \\ &= 2 \cdot 2^k - 1 \\ &= 2^{k+1} - 1 \quad \checkmark \end{aligned}$$

\therefore By the mathematical induction, the given statement is true.