Ex $\lim_{x \to c} (x^3+4x^2-3) = c^3+4c^2-3$

$f(x) = x^3+4x^2-3$

$f(c)$

Ex $\lim_{x \to c} \frac{x^4+x^2-1}{x^2+5} = \frac{c^4+c^2-1}{c^2+5}$

Ex $\lim_{x \to -2} \sqrt{4x^2-3} = \sqrt{4(-2)^2-3}$

$= \sqrt{13}$

Ex $\lim_{x \to -1} \frac{x^3+4x^2-3}{x^2+5} = \frac{-1+4-3}{1+5}$

$= \frac{0}{6}$

$= 0$

Ex $\lim_{x \to 1} \frac{x^2+x-2}{x^2-x} = \frac{1+1-2}{1-1} = \frac{0}{0}$ $= \frac{2}{1}$

$= \lim_{x \to 1} \frac{(x-1)(x+2)}{x(x-1)}$

$= \lim_{x \to 1} \frac{x+2}{x}$

$= 3$

Ex. $\lim\limits_{x \to 0} \dfrac{\sqrt{x^2+100}-10}{x^2} = \dfrac{10-10}{0} = \dfrac{0}{0}$

$$= \lim_{x \to 0} \frac{\sqrt{x^2+100}-10}{x^2} \cdot \frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$$

$$= \lim_{x \to 0} \frac{x^2+100-100}{x^2(\sqrt{x^2+100}+10)}$$

$$= \lim_{x \to 0} \frac{x^2}{x^2(\sqrt{x^2+100}+10)}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{x^2+100}+10}$$

$$= \frac{1}{20}$$

---

Sandwich Theorem

$-1 \le \begin{matrix}\text{cosine} \\ \text{sine}\end{matrix} \le 1$

$\left\{ \dfrac{1}{0} = \infty, \quad \dfrac{1}{\infty} = 0 \right\}$

$\lim\limits_{x \to 0} \sin\left(\dfrac{1}{x}\right) = \sin \infty$

$\lim\limits_{x \to 0^-} \sin\left(\dfrac{1}{x}\right) = -1$

$\lim\limits_{x \to 0^+} \sin\left(\dfrac{1}{x}\right) = 1$

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

$$\frac{\sin\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

$$\lim_{\theta\to 0} \cos\theta = 1$$

$$\lim_{\theta\to 0} \cos\theta < \lim_{\theta\to 0} \frac{\sin\theta}{\theta} < 1$$

$$1 \leq \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \leq 1$$

$$\boxed{\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1}$$

1) $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$

$\cos 2x = 1 - 2\sin^2 x$

$\cos x = 1 - 2\sin^2 \frac{x}{2}$

$\lim_{x \to 0} \frac{\cos x - 1}{x} = \frac{0}{0}$

$= \lim_{x \to 0} \frac{1 - 2\sin^2 \frac{x}{2} - 1}{x}$

$= \lim_{\frac{x}{2} \to 0} \frac{\sin \frac{x}{2}}{2 \cdot \frac{x}{2}} \sin \frac{x}{2}$

$\frac{1}{2} \cdot \frac{x}{2}$

$= \frac{1}{2}(1)\, 0$

$= 0$

$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$

$= \lim_{x \to 0} \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1}$

$= \lim_{x \to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)}$ $\quad c^2 + s^2 = 1$

$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{\cos x + 1}$

$= 1 \cdot \frac{0}{2}$

$= 0$

$$\lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$$

$$\lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{0}{0}$$

$$= \lim_{2x \to 0} \frac{2}{5} \frac{\sin 2x}{2x} \qquad \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

$$= \frac{2}{5} \lim_{2x \to 0} \frac{\sin 2x}{2x}$$

$$= \frac{2}{5} (1)$$

$$= \frac{2}{5}$$

$$\lim_{x \to 0} \frac{\sin ax}{bx} = \frac{a}{b}$$

$$= \lim_{x \to 0} \frac{a}{b} \frac{\sin ax}{ax}$$

---

$$\lim_{x \to 0} \frac{\tan x \sec 2x}{3x} = \frac{0}{0}$$

$$= \frac{1}{3} \lim_{x \to 0} \frac{1}{x} \frac{\sin x}{\cos x} \cdot \frac{1}{\cos 2x}$$

$$= \frac{1}{3} \lim_{x \to 0} \left( \frac{\sin x}{x} \cdot \frac{1}{\cos x} \cdot \frac{1}{\cos 2x} \right)$$

$$= \frac{1}{3} (1 \cdot 1 \cdot 1)$$

$$= \frac{1}{3}$$

## Ex 1.2

$$\lim_{x \to a} c = c$$

1. $\lim_{x \to 3} (-1) = -1$

3. $\lim_{x \to 1000} (18\pi^2) = 18\pi^2$

4. $\lim_{x \to 1} \sqrt{5x+6} = \sqrt{11}$

10. $\lim_{x \to 2} (5x-6)^{3/2} = 4^{3/2} = (2^2)^{3/2}$

$= 8$

$4 = (2)^2$

20. $\lim_{x \to -2} (x^3 - 2x^2 + 4x + 8) = -8 - 8 - 8 + 8$

$= -16$

21. $\lim_{x \to 2} \dfrac{x^2 - 4}{x-2} = \dfrac{0}{0} = \dfrac{4-4}{2-2}$

$= \lim_{x \to 2} \dfrac{(x-2)(x+2)}{x-2}$

$= \lim_{x \to 2} (x+2)$

$= 4$

22. $\lim_{x \to 2} \dfrac{x^3 - 8}{x-2} = \dfrac{8-8}{2-2} = \dfrac{0}{0}$

$= \lim_{x \to 2} \dfrac{(x-2)(x^2+2x+4)}{x-2}$

$= \lim_{x \to 2} (x^2 + 2x + 4)$

$= 4 + 4 + 4$

$= 12$

**24** $\lim\limits_{x \to 0} \dfrac{\sqrt{x+4}-2}{x} = \dfrac{2-2}{0} = \dfrac{0}{0}$

$$= \lim_{x \to 0} \frac{\sqrt{x+4}-2}{x} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$$

$$= \lim_{x \to 0} \frac{x+4-4}{x\left(\sqrt{x+4}+2\right)}$$

$$= \lim_{x \to 0} \frac{x}{x\left(\sqrt{x+4}+2\right)}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{x+4}+2}$$

$$= \frac{1}{4}$$

**36** $\lim\limits_{x \to -1} \dfrac{\sqrt{x^2+8}-3}{x+1} = \dfrac{3-3}{-1+1} = \dfrac{0}{0}$

$$= \lim_{x \to -1} \frac{\sqrt{x^2+8}-3}{x+1} \cdot \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$$

$$= \lim_{x \to -1} \frac{x^2+8-9}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x \to -1} \frac{x^2-1}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x \to -1} \frac{(x+1)(x-1)}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x \to -1} \frac{x-1}{\sqrt{x^2+8}+3}$$

$$= \frac{-2}{6}$$

$$= -\frac{1}{3}$$

25) $\lim_{x \to -2} \frac{5}{x+2} = \frac{5}{0} = \infty$

41) $\lim_{x \to 0} \frac{1+x+\sin x}{3\cos x} = \frac{1+0+0}{3}$

$= \frac{1}{3}$

42) $\lim_{x \to -\pi} \sqrt{x+4}\,\cos(x+\pi) = \sqrt{-\pi+4}\,\cos 0$

$= \sqrt{4-\pi}$

51) $\lim_{x \to 0} \frac{\sin 3x}{4x} = \frac{3}{4}$

53) $\lim_{x \to 0} 6x^2(\cot x)(\csc 2x) =$

$= 6 \lim_{x \to 0} x^2 \frac{\overset{\to =1}{\cos x}}{\sin x} \cdot \frac{1}{\sin 2x}$

$= 6 \lim_{x \to 0} \frac{x}{\sin x} \frac{x}{2\sin x \cos x}$ $\cos 0 = 1$

$= 3 \lim_{x \to 0} \frac{1}{\frac{\sin x}{x}} \cdot \frac{1}{\frac{\sin x}{x}}$

$= 3$

## 1.3

$$\lim_{x \to \pm\infty} k = k$$

$$\lim_{x \to \pm\infty} \frac{1}{x} = 0$$

$$\boxed{\frac{1}{0} = \infty \qquad \frac{1}{\infty} = 0}$$

$$\lim_{x \to \pm\infty} \frac{ax^n \_\_\_}{bx^m \_\_\_}$$

$n = m \Rightarrow \lim_{x \to \pm\infty} : \frac{a}{b}$

$n < m \Rightarrow \lim_{x \to \pm\infty} = 0.$

$n > m \Rightarrow \qquad = \infty \to$ division

Ex $$\lim_{x \to \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \frac{5}{3}$$ ✓

$$\lim_{x \to \infty} \frac{5 + \frac{8x}{x^2} - \frac{3}{x^2}}{\frac{3x^2}{x^2} + \frac{2}{x^2}}$$

$$= \lim \frac{5 + \frac{8}{x} - \frac{3}{x^2}}{3 + \frac{2}{x^2}} \qquad \lim_{x \to \infty} \frac{1}{x} = 0$$

$$= \frac{5 + 0 - 0}{3 + 0}$$

$$= \frac{5}{3}$$

Ex $\lim_{x \to \infty} \frac{11x+2}{2x^2-1} = 0$ $\qquad$ $\lim \frac{11x}{2x^2} = \frac{11}{2x}$

$\lim_{x \to 0^+} \frac{1}{x} = \infty \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty$

Vertical Asymptote 

Ex $\lim_{x \to 3^+} \frac{2-5x}{x-3} = \frac{-13}{0^+}$ ← sign

$= -\infty$

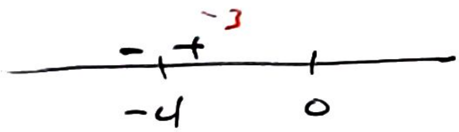
$\lim_{x \to 3^-} \frac{2-5x}{x-3} = \frac{-13}{0^-}$

$= \infty$

Ex $\lim_{x \to -4^+} \frac{-x^3+5x^2-6x}{-x^3-4x^2} = \frac{64+80+24}{64-64}$

$-\cancel{x^2}(x+4)$ $\qquad$ $= \frac{168}{0^+}$ ← sign

$= \infty$



Ex $f(x) = \dfrac{x^2-4x+3}{x^2-1}$

a) $\lim\limits_{x\to 1} f(x) = \dfrac{1-4+3}{1-1} = \dfrac{0}{0}$

$= \lim\limits_{x\to 1} \dfrac{(x-1)(x-3)}{(x-1)(x+1)}$

$= \lim\limits_{x\to 1} \dfrac{x-3}{x+1}$

$= \dfrac{-2}{2}$

$= \boxed{-1}$

b) $\lim\limits_{x\to -1^-} f(x) = \dfrac{1+4+3}{1-1}$

$= \dfrac{8}{0^+}$

$= \boxed{\infty}$

c) $\lim\limits_{x\to -1^+} f(x) = \dfrac{8}{0^-} = \boxed{-\infty}$

---

$\lim\limits_{\theta\to 0^+} \cot\theta = \lim\limits_{\theta\to 0^+} \dfrac{\cos\theta}{\sin\theta}$

$= \dfrac{1}{0^+}$

$= \boxed{\infty}$

$\lim\limits_{\theta\to 0^-} \cot\theta = \dfrac{1}{0^-}$

$= \boxed{-\infty}$

1.4 $\quad f(x) = \dfrac{x^3 - 2}{|x|^3 + 1} \qquad \lim_{x \to 0} f(x) = -2$

$x \geq 0 \quad \lim_{x \to +\infty} f(x) = \lim_{x \to \infty} \dfrac{x^3}{|x|^3}$

$= \lim_{x \to \infty} \dfrac{x^3}{x^3}$

$= 1$

$x \leq 0 \quad \lim_{x \to -\infty} \dfrac{x^3}{|x|^3} = \lim_{x \to -\infty} \dfrac{x^3}{(-x)^3}$

$= -1$

$-x$

$(-\cancel{x}) = +$

$|x| = -x \to x < 0$

---

$\lim_{x \to \infty} \sin\left(\dfrac{1}{x}\right) = \sin \dfrac{1}{\infty}$

$= \sin 0$

$= 0$

$\lim_{x \to 0} \dfrac{\sin x}{x} = 1$

$\lim_{x \to \pm\infty} x \sin \dfrac{1}{x} = \lim_{x \to \pm\infty} \dfrac{\sin \frac{1}{x}}{\frac{1}{x}} \qquad \dfrac{1}{x} \xrightarrow[x \to \infty]{} 0$

$= \lim_{\frac{1}{x} \to 0} \dfrac{\sin \frac{1}{x}}{\frac{1}{x}} \qquad \left.\dfrac{1}{x}\right|_{x=0}$

$= 1$

$$\lim_{x\to\infty}\left(2+\frac{\sin x}{x}\right) = 2 + \lim_{x\to\pm\infty}\frac{\sin x}{x}$$

$$\frac{[-1,1]}{\infty} = 0$$

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

$$\frac{1}{x} \to 0$$

$$0 \le \lim_{x\to\infty}\frac{\sin x}{x} \le 0$$

$$= 2+0$$

$$= \underline{2}$$

---

$$\lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right) = \infty - \infty$$

$$= \lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right)\frac{x+\sqrt{x^2+16}}{x+\sqrt{x^2+16}}$$

$$= \lim_{x\to\infty}\frac{x^2-x^2-16}{x+\sqrt{x^2+16}}$$

$$= \lim_{x\to\infty}\frac{-16}{x+\underline{\sqrt{x^2+16}}}$$

$$= -\lim_{x\to\infty}\frac{16}{x+x}$$

$$= -\lim_{x\to\infty}\frac{8}{x} \qquad \frac{16}{2x}$$

$$= \underline{0}$$

---

$\frac{0}{0}$ { factor, conjugate

$\frac{\neq 0}{0} = \infty$ $\qquad$ $\frac{0}{\neq 0} = 0$