

# Lecture R – Introduction to Differential Equation

## **Solution**      **Section R.1 – Derivative**

### **Exercise**

Find the derivative of  $f(t) = -3t^2 + 2t - 4$

### **Solution**

$$f'(t) = \underline{-6t + 2}$$

### **Exercise**

Find the derivative of  $g(x) = 4\sqrt[3]{x} + 2$

### **Solution**

$$g(x) = 4x^{1/3} + 2$$

$$g'(x) = \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3x^{2/3}}$$

$$= \underline{\frac{4}{3\sqrt[3]{x^2}}}$$

### **Exercise**

Find the derivative of  $f(x) = x(x^2 + 1)$

### **Solution**

$$f(x) = x^3 + x$$

$$f'(x) = \underline{3x^2 + 1}$$

### **Exercise**

Find the derivative of  $f(x) = \frac{2x^2 - 3x + 1}{x}$

### **Solution**

$$f(x) = \frac{2x^2}{x} - \frac{3x}{x} + \frac{1}{x}$$

$$= 2x - 3 + \frac{1}{x}$$

$$f'(x) = \underline{2 - \frac{1}{x^2}}$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

### Exercise

Find the derivative of  $f(x) = \frac{4x^3 - 3x^2 + 2x + 5}{x^2}$

### Solution

$$f(x) = 4x - 3 + \frac{2}{x} + 5x^{-2} \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$\begin{aligned} f'(x) &= 4 - \frac{2}{x^2} - 10x^{-3} \\ &= 4 - \frac{2}{x^2} - \frac{10}{x^3} \end{aligned}$$

### Exercise

Find the derivative of  $f(x) = \frac{-6x^3 + 3x^2 - 2x + 1}{x}$

### Solution

$$f(x) = -6x^2 + 3x - 2 + \frac{1}{x} \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$f'(x) = -12x + 3 - \frac{1}{x^2}$$

### Exercise

Find the derivative of  $f(x) = x\left(1 - \frac{2}{x+1}\right)$

### Solution

$$f(x) = x - \frac{2x}{x+1}$$

$$\left(\frac{2x}{x+1}\right)' \Rightarrow \begin{array}{ll} f = 2x & f' = 2 \\ g = x+1 & g' = 1 \end{array}$$

$$\begin{aligned} f'(x) &= 1 - \frac{2(x+1) - 2x}{(x+1)^2} \\ &= 1 - \frac{2x + 2 - 2x}{(x+1)^2} \\ &= 1 - \frac{2}{(x+1)^2} \end{aligned}$$

### Exercise

Find the derivative of  $g(s) = \frac{s^2 - 2s + 5}{\sqrt{s}}$

### Solution

$$\begin{aligned}g(s) &= \frac{s^2}{s^{1/2}} - 2\frac{s}{s^{1/2}} + \frac{5}{s^{1/2}} \\&= s^{3/2} - 2s^{1/2} + 5s^{-1/2} \\g'(s) &= \frac{3}{2}s^{1/2} - 2\frac{1}{2}s^{-1/2} + 5\left(-\frac{1}{2}\right)s^{-3/2} \\&= \frac{3}{2}s^{1/2} - s^{-1/2} - \frac{5}{2}s^{-3/2} \\&= \frac{3}{2}\sqrt{s} - \frac{1}{\sqrt{s}} - \frac{5}{2s^{3/2}} \\&= \frac{\frac{3}{2}\sqrt{s} - \frac{1}{\sqrt{s}} - \frac{5}{2s\sqrt{s}}}{\quad}\end{aligned}$$

### Exercise

Find the derivative of  $f(x) = \frac{x+1}{\sqrt{x}}$

### Solution

$$\begin{aligned}f(x) &= \frac{x}{x^{1/2}} + \frac{1}{x^{1/2}} \\&= x^{1/2} + x^{-1/2} \\f'(x) &= \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} \\&= \frac{\frac{1}{2x^{1/2}} - \frac{1}{2x^{3/2}}}{\quad}\end{aligned}$$

### Exercise

Find the derivative to the following functions  $y = 3x(2x^2 + 5x)$

### Solution

$$\begin{aligned}y &= 6x^3 + 15x^2 \\&\Rightarrow \underline{y' = 18x^2 + 30x}\end{aligned}$$

**Exercise**

Find the derivative to the following functions  $y = 3(2x^2 + 5x)$

**Solution**

$$y = 6x^2 + 15x$$

$$\Rightarrow \boxed{y' = 12x + 15}$$

**Exercise**

Find the derivative to the following functions  $y = \frac{x^2 + 4x}{5}$

**Solution**

$$y = \frac{1}{5} [x^2 + 4x]$$

$$\boxed{y' = \frac{1}{5} (2x + 4)}$$

**Exercise**

Find the derivative to the following functions  $y = \frac{3x^4}{5}$

**Solution**

$$y = \frac{3}{5} x^4$$

$$\boxed{y' = \frac{12}{5} x^3}$$

**Exercise**

Find the derivative to the following functions  $y = \frac{x^2 - 4}{2x + 5}$

**Solution**

$$y' = \frac{(2x + 5)(2x) - (x^2 - 4)(2)}{(2x + 5)^2}$$

$$= \frac{4x^2 + 10x - 2x^2 + 8}{(2x + 5)^2}$$

$$= \frac{2x^2 + 10x + 8}{(2x + 5)^2}$$

### Exercise

Find the derivative to the following functions  $y = \frac{(1+x)(2x-1)}{x-1}$

### Solution

$$\begin{aligned}y' &= \frac{(x-1) \frac{d}{dx}[(1+x)(2x-1)] - (1+x)(2x-1) \frac{d}{dx}[x-1]}{(x-1)^2} \\&= \frac{(x-1)[(1)(2x-1) + 2(1+x)] - (1+x)(2x-1)(1)}{(x-1)^2} \\&= \frac{(x-1)(2x-1+2+2x) - (2x-1+2x^2-x)}{(x-1)^2} \\&= \frac{(x-1)(4x+1) - 2x+1-2x^2+x}{(x-1)^2} \\&= \frac{4x^2+x-4x-1-2x+1-2x^2+x}{(x-1)^2} \\&= \frac{2x^2-4x}{(x-1)^2}\end{aligned}$$

Or

$$\begin{aligned}y &= \frac{(1+x)(2x-1)}{x-1} \\&= \frac{2x-1+2x^2-x}{x-1} \\&= \frac{2x^2+x-1}{x-1} \\y' &= \frac{(x-1)(4x+1) - (2x^2+x-1)(1)}{(x-1)^2} \\&= \frac{4x^2+x-4x-1-2x^2-x+1}{(x-1)^2} \\&= \frac{2x^2-4x}{(x-1)^2}\end{aligned}$$

**Exercise**

Find the derivative to the following functions  $y = \frac{4}{2x+1}$

**Solution**

$$y = 4(2x+1)^{-1}$$

$$y' = -4(2x+1)^{-2}(2)$$

$$= -8(2x+1)^{-2}$$

$$= -\frac{8}{(2x+1)^2}$$

**Exercise**

Find the derivative to the following functions  $y = \frac{2}{(x-1)^3} = 2(x-1)^{-3}$

**Solution**

$$y = 2(x-1)^{-3}$$

$$y' = 2(-3)(x-1)^{-4}(1)$$

$$= -\frac{6}{(x-1)^4}$$

**Exercise**

Find the derivative to the following functions  $y = \sqrt[3]{(x+4)^2}$

**Solution**

$$y = (x+4)^{2/3}$$

$$y' = \frac{2}{3}(x+4)^{-1/3}$$

$$= \frac{2}{3} \frac{1}{(x+4)^{1/3}}$$

$$= \frac{2}{3 \sqrt[3]{x+4}}$$

**Exercise**

Find the derivative of  $f(x) = \sqrt{2t^2 + 5t + 2}$

**Solution**

$$f(t) = (2t^2 + 5t + 2)^{1/2}$$

$$U = 2t^2 + 5t + 2 \rightarrow U' = 4t + 5$$

$$f'(t) = \frac{1}{2}(4t + 5)(2t^2 + 5t + 2)^{-1/2}$$

$$(U^n)' = nU'U^{n-1}$$

$$= \frac{1}{2} \frac{4t + 5}{\sqrt{2t^2 + 5t + 2}}$$

**Exercise**

Find the derivative of  $f(x) = \frac{1}{(x^2 - 3x)^2}$

**Solution**

$$f(x) = (x^2 - 3x)^{-2}$$

$$f'(x) = -2(2x - 3)(x^2 - 3x)^{-3}$$

$$= -\frac{2(2x - 3)}{(x^2 - 3x)^3}$$

**Exercise**

Find the derivative of  $y = t^2 \sqrt{t - 2}$

**Solution**

$$f = t^2$$

$$f' = 2t$$

$$g = (t - 2)^{1/2}$$

$$g' = \frac{1}{2}(t - 2)^{-1/2}$$

$$y' = 2t\sqrt{t - 2} + t^2 \frac{1}{2}(t - 2)^{-1/2}$$

$$= \left[ 2t(t - 2)^{1/2} + t^2 \frac{1}{2}(t - 2)^{-1/2} \right] \frac{2(t - 2)^{1/2}}{2(t - 2)^{1/2}}$$

$$\begin{aligned}
&= \frac{4t(t-2) + t^2}{2(t-2)^{1/2}} \\
&= \frac{4t^2 - 8t + t^2}{2\sqrt{t-2}} \\
&= \frac{5t^2 - 8t}{2\sqrt{t-2}}
\end{aligned}$$

### Exercise

Find the derivative of  $y = \left( \frac{6-5x}{x^2-1} \right)^2$

### Solution

$$f = 6 - 5x \quad f' = -5$$

$$g = x^2 - 1 \quad g' = 2x$$

$$\begin{aligned}
y' &= 2 \frac{-5(x^2-1) - 2x(6-5x)}{(x^2-1)^2} \left( \frac{6-5x}{x^2-1} \right) & (U^n)' = nU'U^{n-1} \\
&= 2 \frac{-5x^2 + 5 - 12x + 10x^2}{(x^2-1)^3} (6-5x) \\
&= \frac{2(5x^2 - 12x + 5)(6-5x)}{(x^2-1)^3}
\end{aligned}$$

### Exercise

Find the derivative to the following functions  $y = x^2 \sqrt{x^2 + 1}$

### Solution

$$\begin{aligned}
y &= x^2 (x^2 + 1)^{1/2} \\
y' &= x^2 \frac{d}{dx} \left[ (x^2 + 1)^{1/2} \right] + (x^2 + 1)^{1/2} \frac{d}{dx} [x^2] \\
&= x^2 \left[ \frac{1}{2} (x^2 + 1)^{-1/2} (2x) \right] + (x^2 + 1)^{1/2} [2x] \\
&= x^3 (x^2 + 1)^{-1/2} + 2x (x^2 + 1)^{1/2}
\end{aligned}$$



$$\begin{aligned}
&= \frac{(x^2+1)^{1/2}}{(x^2+1)^{1/2}} \left[ x^3(x^2+1)^{-1/2} + 2x(x^2+1)^{1/2} \right] \\
&= \frac{x^3(x^2+1)^{-1/2}(x^2+1)^{1/2} + 2x(x^2+1)^{1/2}(x^2+1)^{1/2}}{(x^2+1)^{1/2}} \\
&= \frac{x^3 + 2x(x^2+1)}{(x^2+1)^{1/2}} \\
&= \frac{x^3 + 2x^3 + 2x}{\sqrt{x^2+1}} \\
&= \frac{3x^3 + 2x}{\sqrt{x^2+1}} \\
&= \frac{x(3x^2+2)}{\sqrt{x^2+1}}
\end{aligned}$$

### ***Exercise***

Find the derivative to the following functions  $y = \left( \frac{x+1}{x-5} \right)^2$

### **Solution**

$$\begin{aligned}
y' &= 2 \left( \frac{x+1}{x-5} \right) \frac{d}{dx} \left[ \frac{x+1}{x-5} \right] \\
&= 2 \left( \frac{x+1}{x-5} \right) \left[ \frac{(1)(x-5) - (1)(x+1)}{(x-5)^2} \right] \\
&= 2 \left( \frac{x+1}{x-5} \right) \left( \frac{x-5-x-1}{(x-5)^2} \right) \\
&= 2 \left( \frac{x+1}{x-5} \right) \left( \frac{-6}{(x-5)^2} \right) \\
&= -\frac{12(x+1)}{(x-5)^3}
\end{aligned}$$

### Exercise

Find the derivative to the following functions  $y = x^2 \sin x$

#### Solution

$$u = x^2 \quad v = \sin x$$

$$u' = 2x \quad v' = \cos x$$

$$y' = \underline{2x \sin x + x^2 \cos x}$$

### Exercise

Find the derivative to the following functions  $y = \frac{\sin x}{x}$

#### Solution

$$u = \sin x \quad v = x$$

$$u' = \cos x \quad v' = 1$$

$$y' = \underline{\frac{x \cos x - \sin x}{x^2}}$$

### Exercise

Find the derivative to the following functions  $y = \frac{\cot x}{1 + \cot x}$

#### Solution

$$u = \cot x \quad v = 1 + \cot x$$

$$u' = -\csc^2 x \quad v' = -\csc^2 x$$

$$\begin{aligned} y' &= \frac{-\csc^2 x (1 + \cot x) + \csc^2 x \cot x}{(1 + \cot x)^2} \\ &= \frac{-\csc^2 x - \csc^2 x \cot x + \csc^2 x \cot x}{(1 + \cot x)^2} \\ &= \underline{\frac{-\csc^2 x}{(1 + \cot x)^2}} \end{aligned}$$

### Exercise

Find the derivative to the following functions  $y = x^2 \sin x + 2x \cos x - 2 \sin x$

#### Solution

$$\begin{aligned} y' &= 2x \sin x + x^2 \cos x + 2 \cos x - 2x \sin x - 2 \cos x \\ &= \underline{x^2 \cos x} \end{aligned}$$

### Exercise

Find the derivative to the following functions  $y = x^3 \sin x \cos x$

#### Solution

$$\begin{aligned} y' &= (x^3)' \sin x \cos x + x^3 (\sin x)' \cos x + x^3 \sin x (\cos x)' \\ &= \underline{3x^2 \sin x \cos x + x^3 \cos^2 x - x^3 \sin^2 x} \end{aligned}$$

### Exercise

Find the derivative to the following functions  $y = \frac{4}{\cos x} + \frac{1}{\tan x}$

#### Solution

$$\begin{aligned} y' &= \frac{-4 \sin x}{\cos^2 x} - \frac{\sec^2 x}{\tan^2 x} & \left(\frac{1}{u}\right)' &= -\frac{u'}{u^2} \\ &= -4 \frac{\sin x}{\cos x} \frac{1}{\cos x} - \frac{1}{\cos^2 x} \frac{\cos^2 x}{\sin^2 x} \\ &= \underline{-4 \tan x \sec x - \csc^2 x} \end{aligned}$$

### Exercise

Find the derivative to the following functions  $f(x) = x^2 e^x$

#### Solution

$$\begin{aligned} f'(x) &= e^x \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(e^x) \\ &= e^x (2x) + x^2 e^x \\ &= \underline{xe^x (2 + x)} \end{aligned}$$

### ***Exercise***

Find the derivative to the following functions  $f(x) = \frac{e^x + e^{-x}}{2}$

#### **Solution**

$$\begin{aligned} f(x) &= \frac{e^x + e^{-x}}{2} \\ &= \frac{1}{2}(e^x + e^{-x}) \end{aligned}$$

$$\begin{aligned} f'(x) &= \frac{1}{2} \left( \frac{d}{dx} [e^x] + \frac{d}{dx} [e^{-x}] \right) \\ &= \frac{1}{2} (e^x - e^{-x}) \end{aligned}$$

### ***Exercise***

Find the derivative to the following functions  $f(x) = \frac{e^x}{x^2}$

#### **Solution**

$$\begin{aligned} f'(x) &= \frac{x^2 e^x - e^x (2x)}{x^4} \\ &= \frac{x^2 e^x - 2x e^x}{x^4} \\ &= \frac{x e^x (x-2)}{x^4} \\ &= \frac{e^x (x-2)}{x^3} \end{aligned}$$

### ***Exercise***

Find the derivative to the following functions  $f(x) = x^2 e^x - e^x$

#### **Solution**

$$\begin{aligned} f'(x) &= e^x \frac{d}{dx} [x^2] + x^2 \frac{d}{dx} [e^x] - \frac{d}{dx} [e^x] \\ &= e^x (2x) + x^2 e^x - e^x \\ &= e^x (x^2 + 2x - 1) \end{aligned}$$

### Exercise

Find the derivative to the following functions  $f(x) = (1 + 2x)e^{4x}$

#### Solution

$$\begin{aligned} f'(x) &= (2)e^{4x} + (1 + 2x)(4e^{4x}) \\ &= 2e^{4x} + (1 + 2x)(4e^{4x}) \\ &= 2e^{4x}(1 + 2(1 + 2x)) \\ &= 2e^{4x}(1 + 2 + 4x) \\ &= \underline{2e^{4x}(3 + 4x)} \end{aligned}$$

### Exercise

Find the derivative to the following functions  $y = x^2e^{5x}$

#### Solution

$$\begin{aligned} y' &= x^2(5e^{5x}) + 2x(e^{5x}) \\ &= \underline{xe^{5x}(5x + 2)} \end{aligned}$$

### Exercise

Find the derivative to the following functions  $y = e^{x^2+1}\sqrt{5x+2}$

#### Solution

$$\begin{aligned} y &= (2x)e^{x^2+1}\sqrt{5x+2} + e^{x^2+1}\frac{5}{2\sqrt{5x+2}} \\ &= 2xe^{x^2+1}\sqrt{5x+2}\frac{2\sqrt{5x+2}}{2\sqrt{5x+2}} + \frac{5e^{x^2+1}}{2\sqrt{5x+2}} \\ &= \frac{4xe^{x^2+1}(5x+2)}{2\sqrt{5x+2}} + \frac{5e^{x^2+1}}{2\sqrt{5x+2}} \\ &= \frac{20x^2e^{x^2+1} + 8xe^{x^2+1} + 5e^{x^2+1}}{2\sqrt{5x+2}} \\ &= \underline{\frac{e^{x^2+1}(20x^2 + 8x + 5)}{2\sqrt{5x+2}}} \end{aligned}$$

### Exercise

Find the derivative to the following functions  $f(x) = \ln \sqrt[3]{x+1}$

#### Solution

$$\begin{aligned} f(x) &= \ln(x+1)^{1/3} \\ &= \frac{1}{3} \ln(x+1) \end{aligned}$$

$$u = x+1 \Rightarrow \frac{du}{dx} = 1$$

$$\begin{aligned} f'(x) &= \frac{1}{3} \frac{1}{x+1} \\ &= \frac{1}{3(x+1)} \end{aligned}$$

### Exercise

Find the derivative to the following functions  $f(x) = \ln \left[ x^2 \sqrt{x^2+1} \right]$

#### Solution

$$f(x) = \ln(x^2) + \ln \sqrt{x^2+1} \quad \text{Product Property}$$

$$f(x) = \ln(x^2) + \ln(x^2+1)^{1/2}$$

$$f(x) = 2 \ln x + \frac{1}{2} \ln(x^2+1) \quad \text{Power Property}$$

$$\begin{aligned} f'(x) &= 2 \frac{1}{x} + \frac{1}{2} \frac{2x}{x^2+1} \\ &= \frac{2}{x} + \frac{x}{x^2+1} \end{aligned} \quad \text{Differentiate}$$

### Exercise

Find the derivative to the following functions  $y = \ln \frac{x^2}{x^2+1}$

#### Solution

$$y = \ln x^2 - \ln x^2 + 1$$

$$\begin{aligned} y' &= \frac{2x}{x^2} - \frac{2x}{x^2+1} \\ &= \frac{2}{x} - \frac{2x}{x^2+1} \end{aligned}$$

### Exercise

Find the derivative to the following functions  $y = \ln \frac{1+e^x}{1-e^x}$

#### Solution

$$y = \ln(1+e^x) - \ln(1-e^x)$$

$$y' = \frac{e^x}{1+e^x} - \frac{-e^x}{1-e^x}$$

$$= \frac{e^x}{1+e^x} + \frac{e^x}{1-e^x}$$

$$= \frac{e^x - e^{2x} + e^x + e^{2x}}{(1+e^x)(1-e^x)}$$

$$= \frac{2e^x}{(1+e^x)(1-e^x)}$$

### Exercise

Find the derivative to the following functions  $y = x 3^{x+1}$

#### Solution

$$y' = 3^{x+1} + x 3^{x+1} \ln 3$$

$$= 3^{x+1} (1 + x \ln 3)$$

### Exercise

Find the derivative to the following functions  $f(t) = \frac{\log_8(t^{3/2}+1)}{t}$

#### Solution

$$f' = \frac{\frac{1}{\ln 8} \frac{\frac{3}{2} t^{1/2}}{t^{3/2}+1} \cdot t - \log_8(t^{3/2}+1)}{t^2}$$

$$= \frac{\frac{1}{\ln 8} \frac{\frac{3}{2} t^{3/2}}{t^{3/2}+1} - \log_8(t^{3/2}+1)}{t^2} \cdot \frac{2 \ln 8 (t^{3/2}+1)}{2 \ln 8 (t^{3/2}+1)}$$

$$= \frac{3t^{3/2} - 2(t^{3/2}+1)(\ln 8) \log_8(t^{3/2}+1)}{t^2(t^{3/2}+1) \ln 8}$$

## ***Solution*** Section R.2 – Integration

### ***Exercise***

Find each indefinite integral.  $\int \frac{x+2}{\sqrt{x}} dx$

### **Solution**

$$\begin{aligned}\int \frac{x+2}{\sqrt{x}} dx &= \int \left[ \frac{x}{x^{1/2}} + \frac{2}{x^{1/2}} \right] dx \\&= \int \frac{x}{x^{1/2}} dx + \int \frac{2}{x^{1/2}} dx \\&= \int x^{1/2} dx + 2 \int x^{-1/2} dx \\&= \frac{x^{3/2}}{3/2} + 2 \frac{x^{1/2}}{1/2} + C \\&= \underline{\frac{2}{3} x^{3/2} + 4x^{1/2} + C}\end{aligned}$$

### ***Exercise***

Find each indefinite integral  $\int 4y^{-3} dy$

### **Solution**

$$\begin{aligned}\int 4y^{-3} dy &= 4 \frac{y^{-2}}{-2} + C \\&= \underline{-\frac{2}{y^2} + C}\end{aligned}$$

### ***Exercise***

Find each indefinite integral  $\int (x^3 - 4x + 2) dx$

### **Solution**

$$\int (x^3 - 4x + 2) dx = \underline{\frac{1}{4} x^4 - 2x^2 + 2x + C}$$



**Exercise**

Find each indefinite integral  $\int \left( \sqrt[4]{x^3} + 1 \right) dx$

**Solution**

$$\int \left( x^{3/4} + 1 \right) dx = \underline{\frac{4}{7} x^{7/4} + x + C}$$

**Exercise**

Find each indefinite integral  $\int \sqrt{x}(x+1) dx$

**Solution**

$$\begin{aligned} \int x^{1/2}(x+1) dx &= \int \left( x^{3/2} + x^{1/2} \right) dx \\ &= \underline{\frac{2}{5} x^{5/2} + \frac{2}{3} x^{3/2} + C} = x + 5x^{-1} + C \end{aligned}$$

**Exercise**

Find each indefinite integral  $\int (1+3t)t^2 dt$

**Solution**

$$\int \left( t^2 + 3t^3 \right) dt = \underline{\frac{1}{3} t^3 + \frac{3}{4} t^4 + C}$$

**Exercise**

Find each indefinite integral  $\int \frac{x^2-5}{x^2} dx$

**Solution**

$$\begin{aligned} \int \frac{x^2-5}{x^2} dx &= \int \left( 1 - \frac{5}{x^2} \right) dx \\ &= \int \left( 1 - 5x^{-2} \right) dx \\ &= \underline{x + \frac{5}{x} + C} \end{aligned}$$

### Exercise

Find each indefinite integral  $\int (-40x + 250) dx$

### Solution

$$\int (-40x + 250) dx = \underline{-20x^2 + 250x + C}$$

### Exercise

Find each indefinite integral  $\int (7 - 3x - 3x^2)(2x + 1) dx$

### Solution

$$\begin{aligned}\int (7 - 3x - 3x^2)(2x + 1) dx &= \int (14x + 7 - 6x^2 - 3x - 6x^3 - 3x^2) dx \\ &= \int (-6x^3 - 9x^2 + 11x + 7) dx \\ &= \underline{-\frac{3}{2}x^4 - 3x^3 + \frac{11}{2}x^2 + 7x + C}\end{aligned}$$

### Exercise

Evaluate the integral  $\int xe^{2x} dx$

### Solution

Let:  $u = x \Rightarrow du = dx$

$$dv = e^{2x} dx \Rightarrow v = \int dv = \int e^{2x} dx = \frac{1}{2}e^{2x}$$

$$\int u dv = uv - \int v du$$

$$\begin{aligned}\int xe^{2x} dx &= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x} dx \\ &= \frac{1}{2}xe^{2x} - \frac{1}{2} \frac{1}{2}e^{2x} + C \\ &= \underline{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}\end{aligned}$$

### Exercise

Evaluate the integral  $\int x \ln x dx$

### Solution

$$\text{Let: } u = \ln x \Rightarrow du = \frac{1}{x} dx$$

$$dv = x dx \Rightarrow v = \int dv = \int x dx = \frac{1}{2} x^2$$

$$\begin{aligned} \int x \ln x dx &= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x^2 \frac{dx}{x} \\ &= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x dx \\ &= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C \end{aligned}$$

### Exercise

Evaluate the integral  $\int x^2 \sin x dx$

### Solution

$g(x) = \sin x$ and its integrals			
$x^2$	(+)	$\rightarrow$	$-\cos x$
$2x$	(-)	$\rightarrow$	$-\sin x$
$2$	(+)	$\rightarrow$	$\cos x$
$0$			

$$\int x^2 \sin x dx = -x^2 \cos x - 2x \sin x + 2 \cos x + C$$

### Exercise

Evaluate the integral  $\int (x^2 - 2x + 1)e^{2x} dx$

### Solution

$\int e^{2x}$		
$x^2 - 2x + 1$	$(+)$	$\frac{1}{2}e^{2x}$
$2x - 2$	$(-)$	$\frac{1}{4}e^{2x}$
$2$	$(+)$	$\frac{1}{8}e^{2x}$
$0$		

$$\begin{aligned}\int (x^2 - 2x + 1)e^{2x} dx &= \frac{1}{2}(x^2 - 2x + 1)e^{2x} - \frac{1}{4}(2x - 2)e^{2x} + \frac{1}{8}(2)e^{2x} + C \\&= \left(\frac{1}{2}x^2 - x + \frac{1}{2} - \frac{1}{2}x + \frac{1}{2} + \frac{1}{4}\right)e^{2x} + C \\&= \left(\frac{1}{2}x^2 - \frac{3}{2}x + \frac{5}{4}\right)e^{2x} + C\end{aligned}$$

### Exercise

Evaluate the integral  $\int e^{2x} \cos 3x dx$

### Solution

Let:  $u = \cos 3x$        $dv = e^{2x} dx$   
 $du = -3\sin 3x dx$        $v = \frac{1}{2}e^{2x}$

$$\begin{aligned}\int e^{2x} \cos 3x dx &= \cos 3x \left(\frac{1}{2}e^{2x}\right) - \int \frac{1}{2}e^{2x} (-3\sin 3x dx) \\&= \frac{1}{2}e^{2x} \cos 3x + \frac{3}{2} \int e^{2x} \sin 3x dx\end{aligned}$$

$$\begin{aligned}u &= \sin 3x & dv &= e^{2x} dx \\du &= 3\cos 3x dx & v &= \frac{1}{2}e^{2x}\end{aligned}$$

$$\int e^{2x} \cos 3x dx = \frac{1}{2}e^{2x} \cos 3x + \frac{3}{2} \left[ \sin 3x \left(\frac{1}{2}e^{2x}\right) - \int \frac{1}{2}e^{2x} (3\cos 3x dx) \right]$$

$$\int e^{2x} \cos 3x dx = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x - \frac{9}{4} \int e^{2x} \cos 3x dx$$

$$\int e^{2x} \cos 3x dx + \frac{9}{4} \int e^{2x} \cos 3x dx = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x + C_1$$

$$\frac{13}{4} \int e^{2x} \cos 3x dx = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x + C_1$$

$$\frac{4}{13} \frac{13}{4} \int e^{2x} \cos 3x dx = \frac{4}{13} \frac{1}{2} e^{2x} \cos 3x + \frac{4}{13} \frac{3}{4} e^{2x} \sin 3x + \frac{4}{13} C_1$$

$$\int e^{2x} \cos 3x dx = \underline{\frac{e^{2x}}{13} (2 \cos 3x + 3 \sin 3x) + C}$$

### ***Exercise***

Find the general solution of the differential equation  $y' = 2t + 3$

### **Solution**

$$dy = (2t + 3) dt$$

$$\int dy = \int (2t + 3) dt$$

$$\boxed{y = t^2 + 3t + C}$$

### ***Exercise***

Find the general solution of the differential equation  $y' = 3t^2 + 2t + 3$

### **Solution**

$$\int dy = \int (3t^2 + 2t + 3) dt$$

$$\boxed{y = t^3 + t^2 + 3t + C}$$

### ***Exercise***

Find the general solution of the differential equation  $y' = \sin 2t + 2 \cos 3t$

### **Solution**

$$\int dy = \int (\sin 2t + 2 \cos 3t) dt$$

$$y(t) = -\frac{1}{2} \cos 2t + \frac{2}{3} \sin 3t + C$$

### ***Exercise***

Find the general solution of the differential equation:  $y' = x^3(3x^4 + 1)^2$

### **Solution**

$$\int x^3 (3x^4 + 1)^2 dx$$

$$u = 3x^4 + 1 \Rightarrow du = 12x^3 dx$$

$$\Rightarrow \frac{1}{12} du = x^3 dx$$

$$\begin{aligned} \int x^3 (3x^4 + 1)^2 dx &= \int \frac{1}{12} u^2 du \\ &= \frac{1}{12} \frac{(3x^4 + 1)^3}{3} + C \\ &= \frac{1}{36} (3x^4 + 1)^3 + C \end{aligned}$$

$$y = \frac{1}{36} (3x^4 + 1)^3 + C$$

### Exercise

Find the general solution of the differential equation:  $y' = 5x\sqrt{x^2 - 1}$

#### Solution

$$\int 5x\sqrt{x^2 - 1} \, dx$$

$$u = x^2 - 1 \Rightarrow du = 2x \, dx$$

$$\Rightarrow \frac{1}{2} du = x \, dx$$

$$\int 5x(x^2 - 1)^{1/2} \, dx$$

$$= 5 \int u^{1/2} \frac{1}{2} du$$

Substitute for  $x$  and  $dx$

$$= \frac{5}{2} \int u^{1/2} du$$

$$= \frac{5}{2} \frac{u^{3/2}}{3/2} + C$$

$$= \frac{5}{3} u^{3/2} + C$$

$$= \frac{5}{3} (x^2 - 1)^{3/2} + C$$

### Exercise

Find the general solution of the differential equation:  $y' = x\sqrt{x^2 + 4}$

#### Solution

$$u = x^2 + 4 \Rightarrow du = 2x \, dx$$

$$x \, dx = \frac{1}{2} du$$

$$\int \sqrt{x^2 + 4} \, x \, dx = \int u^{1/2} \frac{1}{2} du$$

$$= \frac{1}{2} \frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3} u^{3/2} + C$$

$$= \frac{1}{3} (x^2 + 4)^{3/2} + C$$

$$\boxed{y(x) = \frac{1}{3} (x^2 + 4)^{3/2} + C}$$

### Exercise

Find the general solution of the differential equation:  $y' = (2x+1)e^{x^2+x}$

### Solution

$$\int dy = \int (2x+1)e^{x^2+x} dx$$

$$u = x^2 + x \Rightarrow du = (2x+1)dx$$

$$\int dy = \int e^u du$$

$$y = e^u + C$$

$$y(x) = e^{x^2+x} + C$$

### Exercise

Find the general solution of the differential equation:  $y' = \frac{1}{6x-5}$

### Solution

$$\int dy = \int \frac{1}{6x-5} dx$$

$$\int dy = \frac{1}{6} \int \frac{1}{6x-5} d(6x-5)$$

$$y(x) = \frac{1}{6} \ln|6x-5| + C$$

### Exercise

Find the general solution of the differential equation:  $y' = \frac{x^2+2x+3}{x^3+3x^2+9x+1}$

### Solution

$$\int dy = \int \frac{x^2+2x+3}{x^3+3x^2+9x+1} dx$$

$$u = x^3 + 3x^2 + 9x + 1 \quad du = 3(x^2 + 2x + 3)dx$$

$$\int dy = \frac{1}{3} \int \frac{du}{u}$$

$$y(x) = \frac{1}{3} \ln|u| + C$$

$$y(x) = \frac{1}{3} \ln|x^3 + 3x^2 + 9x + 1| + C$$



### Exercise

Find the general solution of the differential equation:  $y' = \frac{1}{x(\ln x)^2}$

### Solution

$$\int dy = \int \frac{1}{x(\ln x)^2} dx \qquad u = \ln x \quad du = \frac{dx}{x}$$

$$\int dy = \int \frac{1}{u^2} du$$

$$y = -\frac{1}{u} + C$$

$$\boxed{y(x) = -\frac{1}{\ln x} + C}$$

### Exercise

Evaluate the integrals  $\int_{-2}^2 (x^3 - 2x + 3) dx$

### Solution

$$\begin{aligned} \int_{-2}^2 (x^3 - 2x + 3) dx &= \left[ \frac{x^4}{4} - x^2 + 3x \right]_{-2}^2 \\ &= \left( \frac{(2)^4}{4} - (2)^2 + 3(2) \right) - \left( \frac{(-2)^4}{4} - (-2)^2 + 3(-2) \right) \\ &= 12 \end{aligned}$$

### Exercise

Evaluate the integrals  $\int_0^1 (x^2 + \sqrt{x}) dx$

### Solution

$$\begin{aligned} \int_0^1 (x^2 + \sqrt{x}) dx &= \left[ \frac{x^3}{3} + \frac{2}{3} x^{3/2} \right]_0^1 \\ &= \left( \frac{(1)^3}{3} + \frac{2}{3} (1)^{3/2} \right) - 0 \\ &= 1 \end{aligned}$$

### Exercise

Evaluate the integrals  $\int_0^{\pi/3} 4 \sec u \tan u \, du$

### Solution

$$\begin{aligned}\int_0^{\pi/3} 4 \sec u \tan u \, du &= 4 \sec u \Big|_0^{\pi/3} \\&= 4 \left( \sec \frac{\pi}{3} - \sec 0 \right) \\&= 4(2 - 1) \\&= \underline{4}\end{aligned}$$

### Exercise

Evaluate the integrals  $\int_{\pi/4}^{3\pi/4} \csc \theta \cot \theta \, d\theta$

### Solution

$$\begin{aligned}\int_{\pi/4}^{3\pi/4} \csc \theta \cot \theta \, d\theta &= -\csc \theta \Big|_{\pi/4}^{3\pi/4} \\&= -\left( \csc \frac{3\pi}{4} - \csc \frac{\pi}{4} \right) \\&= -(\sqrt{2} - \sqrt{2}) \\&= \underline{0}\end{aligned}$$

### Exercise

Evaluate the integrals  $\int_{-\pi/3}^{-\pi/4} \left( 4 \sec^2 t + \frac{\pi}{t^2} \right) dt$

### Solution

$$\begin{aligned}\int_{-\pi/3}^{-\pi/4} \left( 4 \sec^2 t + \frac{\pi}{t^2} \right) dt &= \int_{-\pi/3}^{-\pi/4} \left( 4 \sec^2 t + \pi t^{-2} \right) dt \\&= \left[ 4 \tan t - \pi t^{-1} \right]_{-\pi/3}^{-\pi/4} \\&= \left( 4 \tan \left( -\frac{\pi}{4} \right) - \pi \left( -\frac{4}{\pi} \right) \right) - \left( 4 \tan \left( -\frac{\pi}{3} \right) - \pi \left( -\frac{3}{\pi} \right) \right) \\&= (4(-1) + 4) - (4(-\sqrt{3}) + 3)\end{aligned}$$

$$= -(-4\sqrt{3} + 3)$$

$$= \underline{4\sqrt{3} - 3}$$

### Exercise

Evaluate the integrals  $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy$

### Solution

$$\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy = \int_{-3}^{-1} \left( \frac{y^5}{y^3} - \frac{2y}{y^3} \right) dy$$

$$= \int_{-3}^{-1} (y^2 - 2y^{-2}) dy$$

$$= \left[ \frac{1}{3}y^3 + 2y^{-1} \right]_{-3}^{-1}$$

$$= \left( \frac{1}{3}(-1)^3 + \frac{2}{-1} \right) - \left( \frac{1}{3}(-3)^3 + \frac{2}{-3} \right)$$

$$= \underline{\underline{\frac{22}{3}}}$$

### Exercise

Evaluate the integrals  $\int_1^8 \frac{(x^{1/3} + 1)(2 - x^{2/3})}{x^{1/3}} dx$

### Solution

$$\int_1^8 \frac{(x^{1/3} + 1)(2 - x^{2/3})}{x^{1/3}} dx = \int_1^8 \frac{2x^{1/3} - x + 2 - x^{2/3}}{x^{1/3}} dx$$

$$= \int_1^8 (2 - x^{2/3} + 2x^{-1/3} - x^{1/3}) dx$$

$$= \left[ 2x - \frac{3}{5}x^{5/3} + 3x^{2/3} - \frac{3}{4}x^{4/3} \right]_1^8$$

$$= \left( 2(8) - \frac{3}{5}(8)^{5/3} + 3(8)^{2/3} - \frac{3}{4}(8)^{4/3} \right) - \left( 2(1) - \frac{3}{5}(1)^{5/3} + 3(1)^{2/3} - \frac{3}{4}(1)^{4/3} \right)$$

$$= \left( -\frac{16}{5} \right) - \left( \frac{73}{20} \right)$$

$$= \underline{\underline{-\frac{137}{20}}}$$

**Exercise**

Evaluate:  $\int_0^1 (2t+3)^3 dt$

**Solution**

$$\begin{aligned}
 \int_0^1 (2t+3)^3 dt &= \int_0^1 u^3 \frac{1}{2} du & u = 2t+3 \Rightarrow du = 2dt \rightarrow \frac{du}{2} = dt \\
 &= \frac{1}{2} \int_0^1 u^3 du \\
 &= \frac{1}{2} \frac{u^4}{4} \Big|_0^1 \\
 &= \frac{1}{8} (2t+3)^4 \Big|_0^1 \\
 &= \frac{1}{8} \left[ (2(1)+3)^4 - (2(0)+3)^4 \right] \\
 &= \frac{1}{8} \left[ 5^4 - 3^4 \right] \\
 &= \underline{68}
 \end{aligned}$$

**Exercise**

Evaluate the integral  $\int_{-1}^1 r\sqrt{1-r^2} dr$

**Solution**

$$\begin{aligned}
 \text{Let } u = 1-r^2 &\Rightarrow du = -2rdr \rightarrow -\frac{1}{2}du = rdr \\
 \int_{-1}^1 r\sqrt{1-r^2} dr &= \int_{-1}^1 u^{1/2} \left(-\frac{1}{2}du\right) \\
 &= -\frac{1}{2} \frac{2}{3} u^{3/2} \Big|_{-1}^1 \\
 &= -\frac{1}{3} \left[ (1-r^2)^{3/2} \right]_{-1}^1 \\
 &= -\frac{1}{3} \left[ (1-(1)^2)^{3/2} - (1-(-1)^2)^{3/2} \right] \\
 &= -\frac{1}{3} [0-0] \\
 &= \underline{0}
 \end{aligned}$$

### ***Exercise***

Find the general solution of  $F'(x) = 4x + 2$ , and find the particular solution that satisfies the initial condition  $F(1) = 8$ .

### **Solution**

$$\begin{aligned} F(x) &= \int (4x + 2)dx \\ &= 4 \frac{x^2}{2} + 2x + C \\ &= 2x^2 + 2x + C \end{aligned}$$

$$F(x) = 2(1)^2 + 2(1) + C = 8$$

$$2 + 2 + C = 8$$

$$4 + C = 8$$

$$C = 4$$

$$\Rightarrow \boxed{F(x) = 2x^2 + 2x + 4}$$

### ***Exercise***

Find the general solution of the differential equation:  $y' = t \cos 3t$

### **Solution**

$$u = t \rightarrow du = dt$$

$$dv = \cos 3t \rightarrow v = \frac{1}{3} \sin 3t$$

$$\begin{aligned} y &= \frac{1}{3}t \sin 3t - \frac{1}{3} \int \sin 3t dt \\ &= \frac{1}{3}t \sin 3t - \frac{1}{3} \frac{1}{3} \cos 3t + C \end{aligned}$$

$$\boxed{y(t) = \frac{1}{3}t \sin 3t - \frac{1}{9} \cos 3t + C}$$

### ***Exercise***

A ball is thrown into the air from an initial height of 6 m with an initial velocity of 120 m/s. What will be the maximum height of the ball and at what time will this event occur?

### **Solution**

$$\frac{dv}{dt} = -g \Rightarrow dv = -g dt$$

$$v(t) = -gt + C_1$$

$$v(t=0) = -g(0) + C_1 = 120$$

$$C_1 = 120$$

$$v(t) = -9.8t + 120$$

$$\frac{dx}{dt} = v \Rightarrow dx = v dt$$

$$x(t) = \int (-9.8t + 120) dt$$

$$= -4.9t^2 + 120t + C_2$$

$$x(0) = -4.9(0)^2 + 120(0) + C_2 = 6$$

$$C_2 = 6$$

$$x(t) = -4.9t^2 + 120t + 6$$

$$v(t) = -9.8t + 120 = 0 \rightarrow t = \frac{120}{9.8} = 12.24 \text{ sec}$$

$$x(t=12.24) = -4.9(12.24)^2 + 120(12.24) + 6$$

$$\boxed{x(t) = 740.69 \text{ m}}$$

### ***Exercise***

Derive the position function if a ball is thrown upward with initial velocity of 32 ft per second from an initial height of 48 ft. When does the ball hit the ground? With what velocity does the ball hit the ground?

### **Solution**

$$s(t) = -16t^2 + 32t + 48$$

$$s(0) = 48$$

$$s''(t) = -32$$

$$s'(t) = \int -32 dt \quad s'(0) = 32$$

$$= -32t + C_1$$

$$s'(0) = -32(0) + C_1 = 32$$

$$\Rightarrow C_1 = 32$$

$$s'(t) = -32t + 32$$

$$s(t) = \int (-32t + 32) dt$$

$$= -32\frac{t^2}{2} + 32t + C_2$$

$$s(0) = -32\frac{0^2}{2} + 32(0) + C_2 = 48 \quad \Rightarrow C_2 = 48$$

$$s(t) = -16t^2 + 32t + 48$$

$$s(t) = -16t^2 + 32t + 48 = 0$$

$$-t^2 + 2t + 3 = 0 \quad \Rightarrow t = -1, t = 3$$

The ball hits the ground in **3** seconds

The velocity:  $v(t) = s'(t) = -32t + 32$

$$v(\textcolor{red}{t} = \textcolor{red}{3}) = -32(\textcolor{red}{3}) + 32 = \underline{\textcolor{blue}{-64} \text{ ft} / \text{sec}^2}$$