

Section 1.3 – Rational Functions

A function f is a **rational function** if $f(x) = \frac{g(x)}{h(x)}$,

Where $g(x)$ and $h(x)$ are polynomials. The domain of f consists of all real numbers **except** the zeros of the denominator $h(x)$.

Notation	Terminology
$x \rightarrow a^-$	x approaches a from the left (through values less than a)
$x \rightarrow a^+$	x approaches a from the right (through values greater than a)
$f(x) \rightarrow \infty$	$f(x)$ increases without bound (can be made as large positive as desired)
$f(x) \rightarrow -\infty$	$f(x)$ decreases without bound (can be made as large negative as desired)

The Domain of a Rational Function

Example

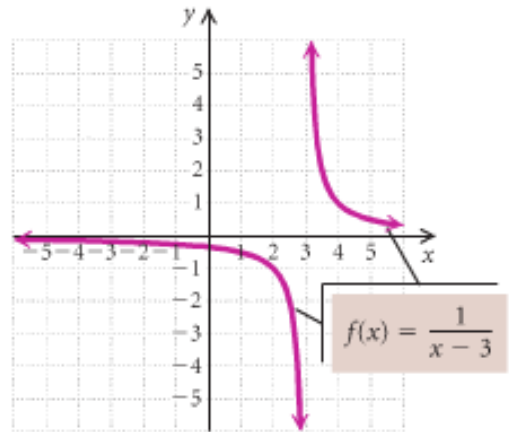
Consider: $f(x) = \frac{1}{x-3}$

Find the domain and graph f .

Solution

$$x - 3 = 0 \Rightarrow \boxed{x = 3}$$

Thus the domain is: $\{x | x \neq 3\}$ **or** $(-\infty, 3) \cup (3, \infty)$



Function	Domain	
$f(x) = \frac{1}{x}$	$\{x x \neq 0\}$	$(-\infty, 0) \cup (0, \infty)$
$f(x) = \frac{1}{x^2}$	$\{x x \neq 0\}$	$(-\infty, 0) \cup (0, \infty)$
$f(x) = \frac{x-3}{x^2+x-2}$	$\{x x \neq -2 \text{ and } x \neq 1\}$	$(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$
$f(x) = \frac{2x+5}{2x-6} = \frac{2x+5}{2(x-3)}$	$\{x x \neq 3\}$	$(-\infty, 3) \cup (3, \infty)$

Asymptotes

Vertical Asymptote (VA) - Think Domain

The line $x = a$ is a **vertical asymptote** for the graph of a function f if

$$f(x) \rightarrow \infty \quad \text{or} \quad f(x) \rightarrow -\infty$$

As x approaches a from either the left or the right

When the denominator and the numerator have both 0, then both the numerator and denominator can be factored by using $(x - a)$ and can be cancelled out. This means there is a **hole** in the function at this point.

Horizontal Asymptote (HA)

The line $y = c$ is a **horizontal asymptote** for the graph of a function f if

$$f(x) \rightarrow c \quad \text{as} \quad x \rightarrow -\infty \quad \text{or} \quad x \rightarrow \infty$$

Let $f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0} = \frac{a_n x^n}{b_m x^m}$ be a rational function.

1. If the degree of numerator is less than of denominator ($n < m$) $\Rightarrow y = 0$

$$y = \frac{2x+1}{4x^2+5} \Rightarrow \underline{y=0}$$

2. If the degree of numerator is equal of denominator ($n = m$) $\Rightarrow y = \frac{a_n}{b_m}$

$$y = \frac{2x^2+1}{4x^2+5} \Rightarrow y = \frac{2}{4} = \underline{\frac{1}{2}}$$

3. If the degree of numerator is greater than of denominator ($n > m$) \Rightarrow No horizontal asymptote

$$y = \frac{2x^3+1}{4x^2+5} \Rightarrow \text{No HA}$$

If **no** Horizontal Asymptote, then there is an **Oblique Asymptote**.

Slant or Oblique Asymptotes

When the degree of the numerator is one greater than the degree of the denominator, the graph has a slant or oblique asymptote and it is a line $y = ax + b$, $a \neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

Example

Find all the asymptotes and sketch the graph of f if $f(x) = \frac{x^2 - 9}{2x - 4}$

Solution

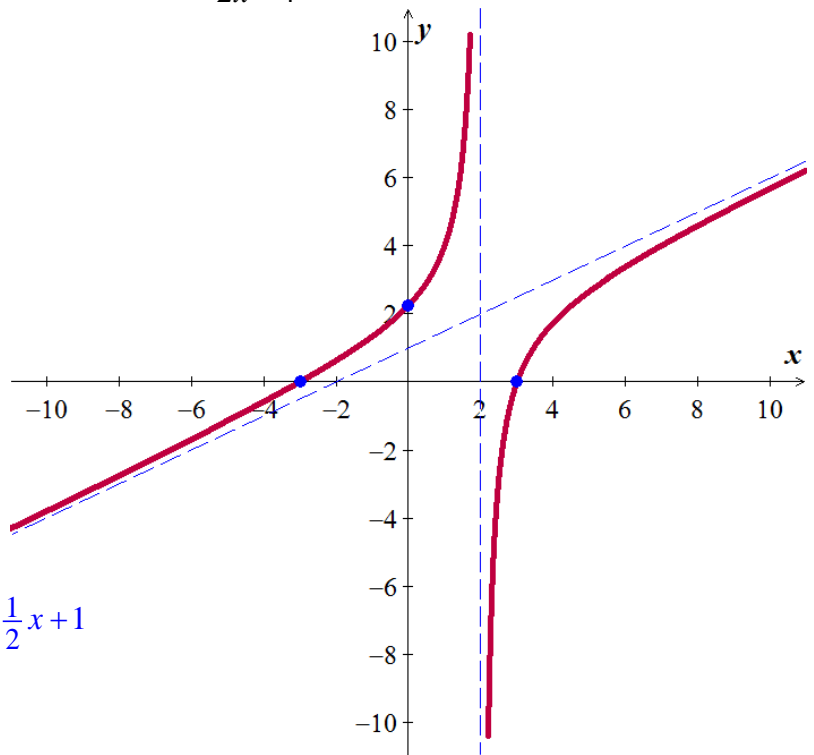
$$\begin{array}{r} \frac{1}{2}x + 1 \\ 2x - 4 \overline{) x^2 - 9} \\ \underline{x^2 - 2x} \\ 2x - 9 \\ \underline{2x - 4} \\ -5 \end{array}$$

$$f(x) = \left(\frac{1}{2}x + 1\right) - \frac{5}{2x - 4}$$

VA: $x = 2$ HA: n/a

Hole: n/a Oblique asymptote: $y = \frac{1}{2}x + 1$

x	y
0	$\frac{9}{4}$
± 3	0



Example

Find the vertical asymptote of $f(x) = \frac{1}{x - 2}$, and sketch the graph.

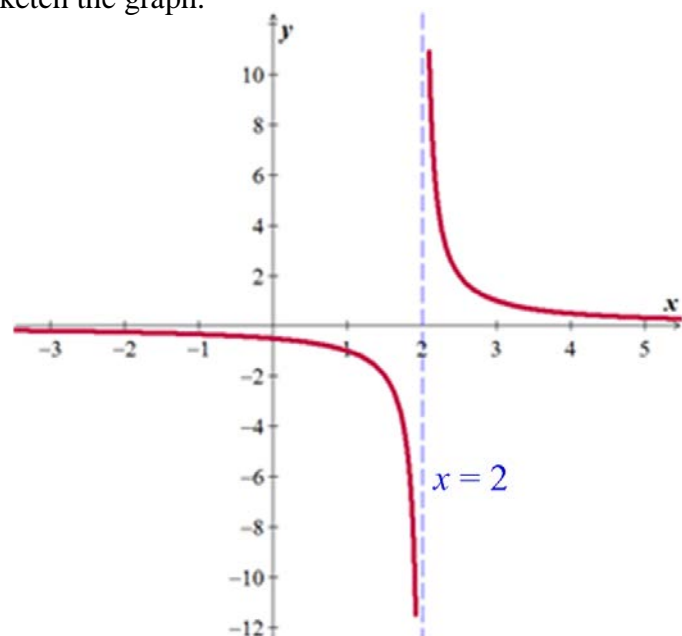
Solution

VA: $x = 2$ HA: $y = 0$

Hole: n/a Oblique asymptote: n/a

$$f(x) \rightarrow \infty \text{ as } x \rightarrow 2^+$$

$$f(x) \rightarrow -\infty \text{ as } x \rightarrow 2^-$$



Example

Sketch the graph of g if $g(x) = \frac{3x^2 + x - 4}{2x^2 - 7x + 5}$

Solution

$$g(x) = \frac{(3x+4)(x-1)}{(2x-5)(x-1)}$$

$$= \frac{3x+4}{2x-5}$$

$$f(x) = \frac{3x+4}{2x-5}$$

g has a hole at $x = 1$

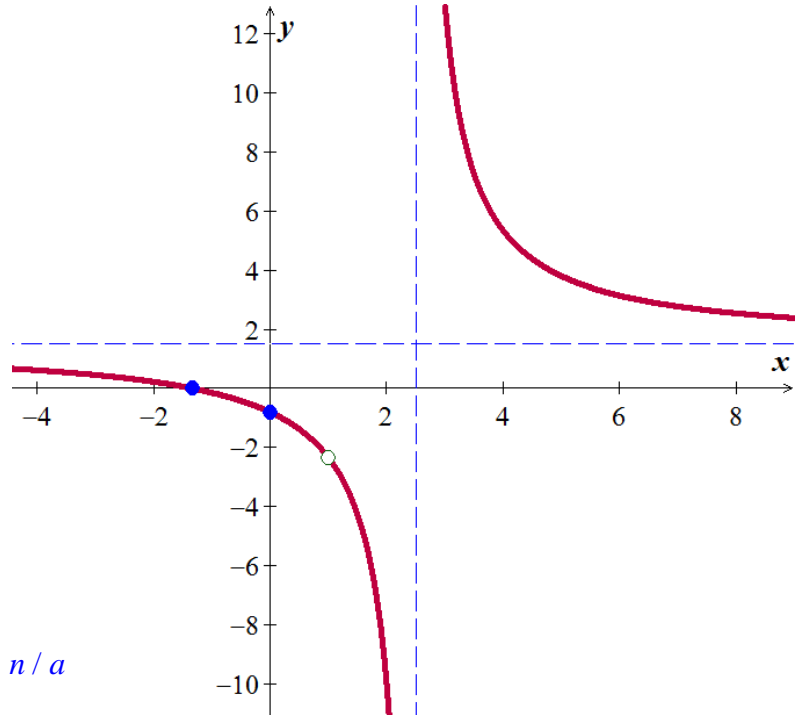
$$f(1) = -\frac{7}{3}$$

$$\text{VA: } x = \frac{5}{2}$$

$$\text{HA: } y = 0$$

$$\text{Hole: } \left(1, -\frac{7}{3}\right)$$

Oblique asymptote: n/a



Example

Find all asymptotes for the graph of f , if it exists

$$a) f(x) = \frac{3x-1}{x^2-x-6}$$

$$b) f(x) = \frac{5x^2+1}{3x^2-4}$$

$$c) f(x) = \frac{2x^4-3x^2+5}{x^2+1}$$

Solution

$$a) f(x) = \frac{3x-1}{x^2-x-6}$$

$$\text{VA: } x = -2, x = 3$$

$$\text{HA: } y = 0$$

Hole: n/a

Oblique asymptote: n/a

$$b) f(x) = \frac{5x^2+1}{3x^2-4}$$

$$3x^2-4=0 \rightarrow 3x^2=4 \rightarrow x^2=\frac{4}{3} \rightarrow x=\pm\frac{2}{\sqrt{3}}$$

$$\text{VA: } x = \pm\frac{2}{\sqrt{3}}$$

$$\text{HA: } y = \frac{5}{3}$$

Hole: n/a

Oblique asymptote: n/a

$$c) f(x) = \frac{2x^4 - 3x^2 + 5}{x^2 + 1}$$

VA: n/a

HA: n/a

Hole: n/a

Oblique asymptote: $y = 2x^2 - 5$

$$\begin{array}{r} 2x^2 - 5 \\ x^2 + 1 \overline{) 2x^4 - 3x^2 + 5} \\ \underline{-2x^4 - 2x^2} \\ -5x^2 + 5 \end{array}$$

Example

Sketch the graph of f if $f(x) = \frac{3x+4}{2x-5}$

Solution

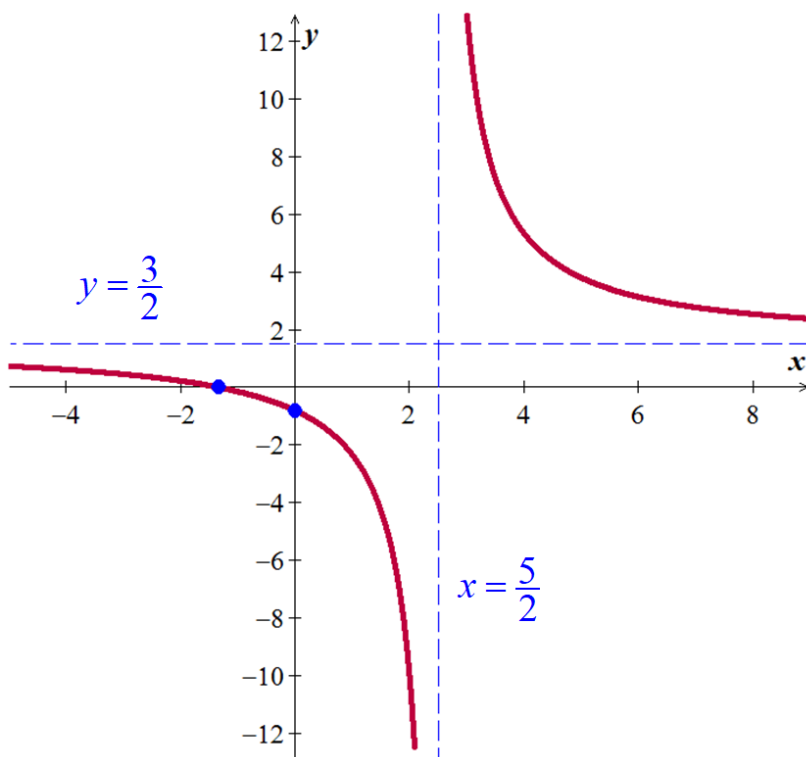
VA: $x = \frac{5}{2}$

HA: $y = -\frac{5}{3}$

Hole: n/a

Oblique asymptote: n/a

x	y
0	$-\frac{4}{5}$
$-\frac{4}{3}$	0
4	5.3



Example

Sketch the graph of f if $f(x) = \frac{x^2}{x^2 - x - 2}$

Solution

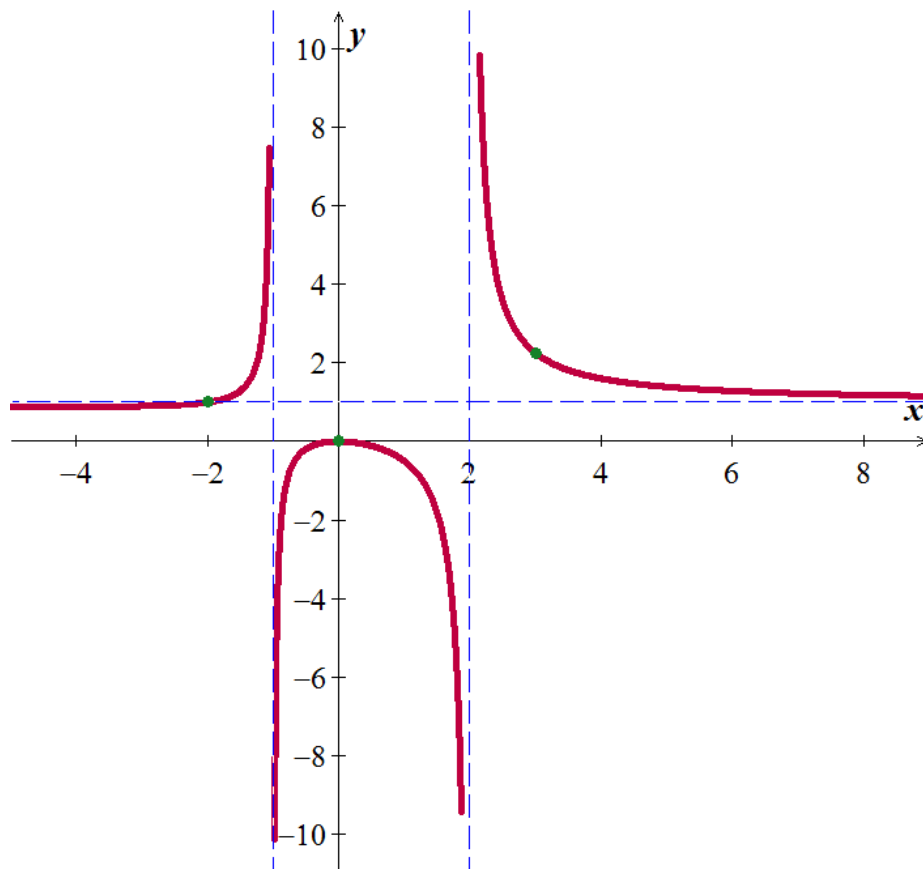
VA: $x = -1, 2$

HA: $y = 1$

Hole: n/a

Oblique asymptote: n/a

x	y
0	0
-4	0.88
-2	1
3	$\frac{9}{4}$



Example

Sketch the graph of f if $f(x) = \frac{x-1}{x^2-x-6}$

Solution

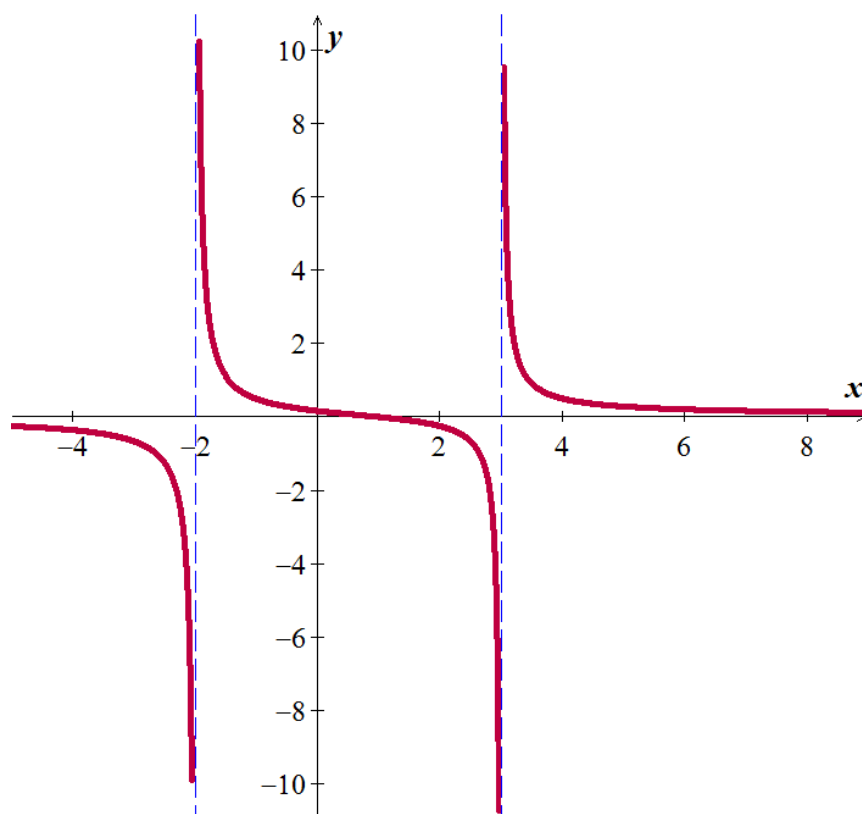
VA: $x = -2, 3$

HA: $y = 0$

Hole: n/a

Oblique asymptote: n/a

x	y
-4	-.36
-3	-.67
0	$\frac{1}{6}$
1	0
4	.5
5	$\frac{2}{7}$



Exercises Section 1.3 – Rational Functions

(1 – 21) Determine all asymptotes of the function

1. $y = \frac{3x}{1-x}$

2. $y = \frac{x^2}{x^2+9}$

3. $y = \frac{x-2}{x^2-4x+3}$

4. $y = \frac{3}{x-5}$

5. $y = \frac{x^3-1}{x^2+1}$

6. $y = \frac{3x^2-27}{(x+3)(2x+1)}$

7. $y = \frac{x^3+3x^2-2}{x^2-4}$

8. $y = \frac{x-3}{x^2-9}$

9. $y = \frac{6}{\sqrt{x^2-4x}}$

10. $y = \frac{5x-1}{1-3x}$

11. $f(x) = \frac{2x-11}{x^2+2x-8}$

12. $f(x) = \frac{x^2-4x}{x^3-x}$

13. $f(x) = \frac{x-2}{x^3-5x}$

14. $f(x) = \frac{4x}{x^2+10x}$

15. $f(x) = \frac{3-x}{(x-4)(x+6)}$

16. $f(x) = \frac{x^3}{2x^3-x^2-3x}$

17. $f(x) = \frac{3x^2+5}{4x^2-3}$

18. $f(x) = \frac{x+6}{x^3+2x^2}$

19. $f(x) = \frac{x^2+4x-1}{x+3}$

20. $f(x) = \frac{x^2-6x}{x-5}$

21. $f(x) = \frac{x^3-x^2+x-4}{x^2+2x-1}$

(22 – 53) Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote, Hole, Oblique Asymptote*) and sketch the graph of

22. $f(x) = \frac{-3x}{x+2}$

23. $f(x) = \frac{x+1}{x^2+2x-3}$

24. $f(x) = \frac{2x^2-2x-4}{x^2+x-12}$

25. $f(x) = \frac{-2x^2+10x-12}{x^2+x}$

26. $f(x) = \frac{x^2-x-6}{x+1}$

27. $f(x) = \frac{x^3+1}{x-2}$

28. $f(x) = \frac{2x^2+x-6}{x^2+3x+2}$

29. $f(x) = \frac{x-1}{1-x^2}$

30. $f(x) = \frac{x^2+x-2}{x+2}$

31. $f(x) = \frac{x^3-2x^2-4x+8}{x-2}$

32. $f(x) = \frac{2x^2-3x-1}{x-2}$

33. $f(x) = \frac{2x+3}{3x^2+7x-6}$

34. $f(x) = \frac{x^2-1}{x^2+x-6}$

35. $f(x) = \frac{-2x^2-x+15}{x^2-x-12}$

36. $f(x) = \frac{1}{x-3}$

37. $f(x) = \frac{-2}{x+3}$

38. $f(x) = \frac{x}{x+2}$

39. $f(x) = \frac{x-5}{x+4}$

40. $f(x) = \frac{2x^2-2}{x^2-9}$

41. $f(x) = \frac{x^2-3}{x^2+4}$

42. $f(x) = \frac{x^2+4}{x^2-3}$

$$43. \quad f(x) = \frac{x^2}{x^2 - 6x + 9}$$

$$44. \quad f(x) = \frac{x^2 + x + 4}{x^2 + 2x - 1}$$

$$45. \quad f(x) = \frac{2x^2 + 14}{x^2 - 6x + 5}$$

$$46. \quad f(x) = \frac{x^2 - 4x - 5}{2x + 5}$$

$$47. \quad f(x) = \frac{x - 3}{x^2 - 3x + 2}$$

$$48. \quad f(x) = \frac{x^2 + 2}{x^2 + 3x + 2}$$

$$49. \quad f(x) = \frac{x - 2}{x^2 - 3x + 2}$$

$$50. \quad f(x) = \frac{x^2 + x}{x + 1}$$

$$51. \quad f(x) = \frac{x^2 - 2x}{x - 2}$$

$$52. \quad f(x) = \frac{x^2 - 3x}{x + 3}$$

$$53. \quad f(x) = \frac{x^3 + 3x^2 - 4x + 6}{x + 2}$$

(54 – 59) Find an equation of a rational function f that satisfies the given conditions

$$54. \quad \begin{cases} \text{vertical asymptote: } x = 4 \\ \text{horizontal asymptote: } y = -1 \\ x\text{-intercept: } 3 \end{cases}$$

$$55. \quad \begin{cases} \text{vertical asymptote: } x = -4, x = 5 \\ \text{horizontal asymptote: } y = \frac{3}{2} \\ x\text{-intercept: } -2 \end{cases}$$

$$56. \quad \begin{cases} \text{vertical asymptote: } x = 5 \\ \text{horizontal asymptote: } y = -1 \\ x\text{-intercept: } 2 \end{cases}$$

$$57. \quad \begin{cases} \text{vertical asymptote: } x = -2, x = 0 \\ \text{horizontal asymptote: } y = 0 \\ x\text{-intercept: } 2, \quad f(3) = 1 \end{cases}$$

$$58. \quad \begin{cases} \text{vertical asymptote: } x = -3, x = 1 \\ \text{horizontal asymptote: } y = 0 \\ x\text{-intercept: } -1, \quad f(0) = -2 \\ \text{hole at } x = 2 \end{cases}$$

$$59. \quad \begin{cases} \text{vertical asymptote: } x = -1, x = 3 \\ \text{horizontal asymptote: } y = 2 \\ x\text{-intercept: } -2, 1 \\ \text{hole: } x = 0 \end{cases}$$