Solution Section 3.1 – Inner Products

Exercise

Let $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u}=(1,\ 1)$, $\boldsymbol{v}=(3,\ 2)$, $\boldsymbol{w}=(0,\ -1)$, and $k=3$. Compute the following.

a) $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle$

b) $\langle k\boldsymbol{v},\ \boldsymbol{w}\rangle$

c) $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w}\rangle$

d) $\|\boldsymbol{v}\|$

e) $d(\boldsymbol{u},\ \boldsymbol{v})$

f) $\|\boldsymbol{u}-k\boldsymbol{v}\|$

Solution

a) $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 1(3)+1(2)=\underline{5}$

b) $\langle k\boldsymbol{v},\ \boldsymbol{w}\rangle = \langle 3v,\ w\rangle$

$=9\cdot 0+6\cdot(-1)$

$=\underline{-6}$

c) $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w}\rangle = \langle \boldsymbol{u},\ \boldsymbol{w}\rangle + \langle \boldsymbol{v},\ \boldsymbol{w}\rangle$

$=1\cdot 0+1\cdot(-1)+3\cdot 0+2\cdot(-1)$

$\underline{=-3}$

d) $\|\boldsymbol{v}\| = \sqrt{\langle \boldsymbol{v},\boldsymbol{v}\rangle} = \sqrt{3^2+2^2} = \underline{\sqrt{13}}$

e) $d(\boldsymbol{u},\ \boldsymbol{v}) = \|\boldsymbol{u}-\boldsymbol{v}\|$

$=\|(-2,\ -1)\|$

$=\sqrt{(-2)^2+(-1)^2}$

$=\underline{\sqrt{5}}$

f) $\|\boldsymbol{u}-k\boldsymbol{v}\| = \|(1,1)-3(3,2)\|$

$=\|(-8,\ -5)\|$

$=\sqrt{(-8)^2+(-5)^2}$

$=\underline{\sqrt{89}}$

Exercise

Let $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u}=(1,\ 1)$, $\boldsymbol{v}=(3,\ 2)$, $\boldsymbol{w}=(0,\ -1)$, and $k=3$. Compute the following for the weighted Euclidean inner product $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 2u_1v_1 + 3u_2v_2$.

a) $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle$ *c)* $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w}\rangle$ *e)* $d(\boldsymbol{u},\ \boldsymbol{v})$

b) $\langle k\boldsymbol{v},\ \boldsymbol{w}\rangle$ *d)* $\|\boldsymbol{v}\|$ *f)* $\|\boldsymbol{u}-k\boldsymbol{v}\|$

Solution

a) $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 2(1)(3)+3(1)(2) = \underline{12}$

b) $\langle k\boldsymbol{v},\ \boldsymbol{w}\rangle = 2(3\cdot 3)(0)+3(3\cdot 2)(-1) = \underline{-18}$

c) $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w}\rangle = \langle \boldsymbol{u},\ \boldsymbol{w}\rangle + \langle \boldsymbol{v},\ \boldsymbol{w}\rangle$

$= 1\cdot 0 + 1\cdot(-1) + 3\cdot 0 + 2\cdot(-1)$

$\underline{= -3}$

d) $\|\boldsymbol{v}\| = \sqrt{\langle \boldsymbol{v},\boldsymbol{v}\rangle} = \sqrt{2(3)(3)+3(2)(2)} = \underline{\sqrt{30}}$

e) $d(\boldsymbol{u},\ \boldsymbol{v}) = \|\boldsymbol{u}-\boldsymbol{v}\|$

$= \|\langle(-2,\ -1)\rangle\|$

$= \sqrt{2(-2)(-2)+3(-1)(-1)}$

$= \underline{\sqrt{11}}$

f) $\|\boldsymbol{u}-k\boldsymbol{v}\| = \|(1,1)-3(3,2)\|$

$= \|\langle(-8,\ -5)\rangle\|$

$= \sqrt{2(-8)^2+3(-5)^2}$

$= \underline{\sqrt{203}}$

Exercise

Let $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u}=(3,\ -2)$, $\boldsymbol{v}=(4,\ 5)$, $\boldsymbol{w}=(-1,\ 6)$, and $k=-4$. Verify the following.

a) $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = \langle \boldsymbol{v},\ \boldsymbol{u}\rangle$

b) $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w}\rangle = \langle \boldsymbol{u},\ \boldsymbol{w}\rangle + \langle \boldsymbol{v},\ \boldsymbol{w}\rangle$

c) $\langle \boldsymbol{u},\ \boldsymbol{v}+\boldsymbol{w}\rangle = \langle \boldsymbol{u},\ \boldsymbol{v}\rangle + \langle \boldsymbol{u},\ \boldsymbol{w}\rangle$

d) $\langle k\boldsymbol{u},\ \boldsymbol{v}\rangle = k\langle \boldsymbol{u},\ \boldsymbol{v}\rangle$

e) $\langle \mathbf{0},\ \mathbf{v}\rangle = \langle \mathbf{v},\ \mathbf{0}\rangle = 0$

Solution

a) $\langle \mathbf{u},\ \mathbf{v}\rangle = 3\cdot 4 + (-2)\cdot(5)\ \underline{=2}$

$\langle \mathbf{v},\ \mathbf{u}\rangle = 4\cdot 3 + (5)\cdot(-2)\ \underline{=2}$

b) $\langle \mathbf{u}+\mathbf{v},\ \mathbf{w}\rangle = \langle (7,3),\ (-1,6)\rangle = 7(-1)+3(6) = \underline{11}$

$\langle \mathbf{u},\ \mathbf{w}\rangle + \langle \mathbf{v},\ \mathbf{w}\rangle = (3)(-1)+(-2)(6)+(4)(-1)+(5)(6) = \underline{11}$

c) $\langle \mathbf{u},\ \mathbf{v}+\mathbf{w}\rangle = \langle (3,-2),\ (3,11)\rangle = 3(3)+(-2)(11) = \underline{-13}$

$\langle \mathbf{u},\ \mathbf{v}\rangle + \langle \mathbf{u},\ \mathbf{w}\rangle = (3)(4)+(-2)(5)+(3)(-1)+(-2)(6) = \underline{-13}$

d) $\langle k\mathbf{u},\ \mathbf{v}\rangle = (-4\cdot 3)\cdot 4 + ((-4)(-2))\cdot(5) = \underline{-8}$

$k\langle \mathbf{u},\ \mathbf{v}\rangle = (-4)(3\cdot 4 + (-2)\cdot(5)) = \underline{-8}$

e) $\langle \mathbf{0},\ \mathbf{v}\rangle = 0\cdot 4 + 0\cdot(5) = \underline{0}$

$\langle \mathbf{v},\ \mathbf{0}\rangle = 4\cdot 0 + (5)\cdot(0) = \underline{0}$

Exercise

Let $\langle \mathbf{u},\ \mathbf{v}\rangle$ be the Euclidean inner product on R^2, and let $\mathbf{u}=(3,\ -2)$, $\mathbf{v}=(4,\ 5)$, $\mathbf{w}=(-1,\ 6)$, and $k=-4$. Verify the following for the weighted Euclidean inner product $\langle \mathbf{u},\ \mathbf{v}\rangle = 4u_1v_1 + 5u_2v_2$.

a) $\langle \mathbf{u},\ \mathbf{v}\rangle = \langle \mathbf{v},\ \mathbf{u}\rangle$

b) $\langle \mathbf{u}+\mathbf{v},\ \mathbf{w}\rangle = \langle \mathbf{u},\ \mathbf{w}\rangle + \langle \mathbf{v},\ \mathbf{w}\rangle$

c) $\langle \mathbf{u},\ \mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{u},\ \mathbf{v}\rangle + \langle \mathbf{u},\ \mathbf{w}\rangle$

d) $\langle k\mathbf{u},\ \mathbf{v}\rangle = k\langle \mathbf{u},\ \mathbf{v}\rangle$

e) $\langle \mathbf{0},\ \mathbf{v}\rangle = \langle \mathbf{v},\ \mathbf{0}\rangle = 0$

Solution

a) $\langle \mathbf{u},\ \mathbf{v}\rangle = 4\cdot 3\cdot 4 + 5\cdot(-2)\cdot(5) = \underline{-2}$

$\langle \mathbf{v},\ \mathbf{u}\rangle = 4\cdot 4\cdot 3 + 5\cdot(5)\cdot(-2) = \underline{-2}$

b) $\langle \mathbf{u}+\mathbf{v},\ \mathbf{w}\rangle = \langle (7,3),\ (-1,6)\rangle = 4\cdot 7(-1)+5\cdot 3(6) = \underline{62}$

$\langle \mathbf{u},\ \mathbf{w}\rangle + \langle \mathbf{v},\ \mathbf{w}\rangle = 4\cdot(3)(-1)+5\cdot(-2)(6)+4\cdot(4)(-1)+5\cdot(5)(6) = \underline{62}$

c) $\langle \mathbf{u},\ \mathbf{v}+\mathbf{w}\rangle = \langle (3,-2),\ (3,11)\rangle = 4\cdot 3(3)+5\cdot(-2)(11) = \underline{-74}$

$\langle \mathbf{u},\ \mathbf{v}\rangle + \langle \mathbf{u},\ \mathbf{w}\rangle = 4\cdot(3)(4)+5\cdot(-2)(5)+4\cdot(3)(-1)+5\cdot(-2)(6) = \underline{-74}$

d) $\langle k\boldsymbol{u},\ \boldsymbol{v}\rangle = 4\cdot(-4\cdot 3)\cdot 4+5\cdot((-4)(-2))\cdot(5) = \underline{8}|$

$k\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = (-4)(4\cdot 3\cdot 4+5\cdot(-2)\cdot(5)) = \underline{8}|$

e) $\langle \boldsymbol{0},\ \boldsymbol{v}\rangle = 4\cdot 0\cdot 4+5\cdot 0\cdot(5) = \underline{0}|$

$\langle \boldsymbol{v},\ \boldsymbol{0}\rangle = 4\cdot 4\cdot 0+5\cdot(5)\cdot(0) = \underline{0}|$

Exercise

Let $\boldsymbol{u}=(u_1,\ u_2)$ and $\boldsymbol{v}=(v_1,\ v_2)$. Show that the following are inner product on R^3 by verifying that the inner product axioms hold. $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 3u_1v_1+5u_2v_2$

Solution

Axiom 1: $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 3u_1v_1+5u_2v_2 = 3v_1u_1+5v_2u_2 = \langle \boldsymbol{v},\ \boldsymbol{u}\rangle$

Axiom 2:
$$\begin{aligned}\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w}\rangle &= 3(u_1+v_1)w_1 + 5(u_2+v_2)w_2\\ &= 3(u_1w_1+v_1w_1) + 5(u_2w_2+v_2w_2)\\ &= 3u_1w_1+3v_1w_1 + 5u_2w_2+5v_2w_2\\ &= (3u_1w_1 + 5u_2w_2)+(3v_1w_1+5v_2w_2)\\ &= \langle \boldsymbol{u},\ \boldsymbol{w}\rangle+\langle \boldsymbol{v},\ \boldsymbol{w}\rangle\end{aligned}$$

Axiom 3:
$$\begin{aligned}\langle k\boldsymbol{u},\ \boldsymbol{v}\rangle &= 3(ku_1)v_1 + 5(ku_2)v_2\\ &= k(3u_1v_1 + 5u_2v_2)\\ &= k\langle \boldsymbol{u},\ \boldsymbol{v}\rangle\end{aligned}$$

Axiom 4:
$$\begin{aligned}\langle \boldsymbol{v},\ \boldsymbol{v}\rangle &= 3v_1v_1 + 5v_2v_2\\ &= 3v_1^2 + 5v_2^2 \ge 0\end{aligned}$$
$$v_1 = v_2 = 0 \quad iff \quad \boldsymbol{v}=\boldsymbol{0}$$

Exercise

Show that the following identity holds for the vectors in any inner product space

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2+\|\boldsymbol{u}-\boldsymbol{v}\|^2 = 2\|\boldsymbol{u}\|^2+2\|\boldsymbol{v}\|^2$$

Solution

$$\begin{aligned}\|\boldsymbol{u}+\boldsymbol{v}\|^2+\|\boldsymbol{u}-\boldsymbol{v}\|^2 &= \langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{u}+\boldsymbol{v}\rangle+\langle \boldsymbol{u}-\boldsymbol{v},\ \boldsymbol{u}-\boldsymbol{v}\rangle\\ &= \langle \boldsymbol{u},\boldsymbol{u}+\boldsymbol{v}\rangle+\langle \boldsymbol{v},\boldsymbol{u}+\boldsymbol{v}\rangle+\langle \boldsymbol{u},\boldsymbol{u}-\boldsymbol{v}\rangle-\langle \boldsymbol{v},\boldsymbol{u}-\boldsymbol{v}\rangle\end{aligned}$$

$$= \langle \boldsymbol{u},\boldsymbol{u}\rangle + \langle \boldsymbol{u},\boldsymbol{v}\rangle + \langle \boldsymbol{v},\boldsymbol{u}\rangle + \langle \boldsymbol{v},\boldsymbol{v}\rangle + \langle \boldsymbol{u},\boldsymbol{u}\rangle - \langle \boldsymbol{u},\boldsymbol{v}\rangle - \langle \boldsymbol{v},\boldsymbol{u}\rangle + \langle \boldsymbol{v},\boldsymbol{v}\rangle$$

$$= 2\langle \boldsymbol{u},\boldsymbol{u}\rangle + 2\langle \boldsymbol{v},\boldsymbol{v}\rangle$$

$$= 2\|\boldsymbol{u}\|^2 + 2\|\boldsymbol{v}\|^2 \quad \checkmark$$

Exercise

Show that the following identity holds for the vectors in any inner product space

$$\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = \frac{1}{4}\left(\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2\right)$$

Solution

$$\|\vec{u}+\vec{v}\|^2 = \langle \vec{u}+\vec{v},\ \vec{u}+\vec{v}\rangle = \|\vec{u}\|^2 + 2\langle \vec{u},\ \vec{v}\rangle + \|\vec{v}\|^2$$

$$\|\vec{u}-\vec{v}\|^2 = \langle \vec{u}-\vec{v},\ \vec{u}-\vec{v}\rangle = \|\vec{u}\|^2 - 2\langle \vec{u},\ \vec{v}\rangle + \|\vec{v}\|^2$$

$$\begin{array}{rl} & \|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\langle \vec{u},\ \vec{v}\rangle + \|\vec{v}\|^2 \\ - & \underline{\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 - 2\langle \vec{u},\ \vec{v}\rangle + \|\vec{v}\|^2} \\ & \|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2 = 4\langle \vec{u},\ \vec{v}\rangle \end{array}$$

$$\langle \vec{u},\ \vec{v}\rangle = \frac{1}{4}\left(\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2\right) \quad \checkmark$$

Exercise

Prove that $\|k\boldsymbol{v}\| = |k|\|\boldsymbol{v}\|$

Solution

$$\|k\vec{v}\|^2 = \langle k\vec{v},\ \vec{v}\rangle$$

$$= k^2\langle \vec{v},\ \vec{v}\rangle$$

$$= k^2\|\vec{v}\|^2$$

$$\|k\vec{v}\| = k\ \|\vec{v}\| \quad \checkmark$$

Solution Section 3.2 – Angle and Orthogonality in Inner Product Spaces

Exercise

Which of the following form orthonormal sets?

a) $(1, 0), (0, 2)$ in $\boldsymbol{R}^2$

b) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\boldsymbol{R}^2$

c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\boldsymbol{R}^2$

d) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ in $\boldsymbol{R}^3$

e) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ in $\boldsymbol{R}^3$

f) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ in $\boldsymbol{R}^3$

<u>***Solution***</u>

a) $(1,\ 0)\cdot(0,\ 2) = 1(0) + 0(2) = 0$, they are ***orthonormal*** sets

b) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\bullet\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} = \frac{1}{2} - \frac{1}{2} = 0$, they are orthonormal sets

c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\bullet\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} = -\frac{1}{2} - \frac{1}{2}\ \underline{= -1}$

They are ***not orthonormal***

d) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\bullet\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)\bullet\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{2}}\right) + 0 + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{2}}$$

$$= -\frac{1}{2}\frac{1}{\sqrt{3}} - \frac{1}{2}\frac{1}{\sqrt{3}}$$

$$= -\frac{1}{\sqrt{3}} \neq 0$$

They are ***not orthonormal*** sets

e) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\bullet\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)\bullet\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

$$=\frac{2}{3}\frac{2}{3}\frac{1}{3}+\left(-\frac{2}{3}\right)\frac{1}{3}\frac{2}{3}+\frac{1}{3}\left(-\frac{2}{3}\right)\frac{2}{3}$$

$$=\frac{4}{27}-\frac{4}{27}-\frac{4}{27}$$

$$=-\frac{4}{27}\neq 0$$

They are ***not orthonormal*** sets

f) $\left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right)\bullet\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)=\frac{1}{\sqrt{6}}\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{2}}\right)+0\ \underline{=0}$

They are ***orthonormal*** sets

Exercise

Find the cosine of the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

a) $\boldsymbol{u}=(1,\ -3),\quad \boldsymbol{v}=(2,\ 4)$

b) $\boldsymbol{u}=(-1,\ 0),\quad \boldsymbol{v}=(3,\ 8)$

c) $\boldsymbol{u}=(-1,\ 5,\ 2);\quad \boldsymbol{v}=(2,\ 4,\ -9)$

d) $\boldsymbol{u}=(4,\ 1,\ 8);\quad \boldsymbol{v}=(1,\ 0,\ -3)$

e) $\boldsymbol{u}=(1,\ 0,\ 1,\ 0);\quad \boldsymbol{v}=(-3,\ -3,\ -3,\ -3)$

f) $\boldsymbol{u}=(2,\ 1,\ 7,\ -1);\quad \boldsymbol{v}=(4,\ 0,\ 0,\ 0)$

g) $\boldsymbol{u}=(1,\ 3,\ -5,\ 4),\quad \boldsymbol{v}=(2,\ -43,\ 4,\ 1)$

h) $\boldsymbol{u}=(1,\ 2,\ 3,\ 4),\quad \boldsymbol{v}=(-1,\ -2,\ -3,\ -4)$

Solution

a) $\boldsymbol{u}=(1,\ -3),\quad \boldsymbol{v}=(2,\ 4)$

$$\|\boldsymbol{u}\|=\sqrt{1^2+(-3)^2}=\underline{\sqrt{10}}$$

$$\|\boldsymbol{v}\|=\sqrt{2^2+4^2}=\underline{\sqrt{20}}$$

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle=1(2)+(-3)(4)=\underline{-10}$$

$$\cos\theta=\frac{\langle \boldsymbol{u},\ \boldsymbol{v}\rangle}{\|\boldsymbol{u}\|.\|\boldsymbol{v}\|}$$

$$=\frac{-10}{\sqrt{10}\sqrt{20}}$$

$$=-\frac{10}{\sqrt{200}}$$

$$=-\frac{1}{\sqrt{2}}$$

b) $\boldsymbol{u}=(-1,\ 0);\quad \boldsymbol{v}=(3,\ 8)$

$$\|\boldsymbol{u}\|=\sqrt{(-1)^2+0^2}=\underline{1}$$

$$\|\boldsymbol{v}\|=\sqrt{3^2+8^2}=\underline{\sqrt{73}}$$

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle=(-1)(3)+(0)(8)=\underline{-3}$$

$$\cos\theta=\frac{-3}{1\sqrt{73}}$$

$$\cos\theta=\frac{\langle \boldsymbol{u},\ \boldsymbol{v}\rangle}{\|\boldsymbol{u}\|.\|\boldsymbol{v}\|}$$

$$=-\frac{3}{\sqrt{73}}$$

c) $\boldsymbol{u}=(-1,\ 5,\ 2);\quad \boldsymbol{v}=(2,\ 4,\ -9)$

$$\|\boldsymbol{u}\|=\sqrt{(-1)^2+5^2+2^2}=\underline{\sqrt{30}}$$

$$\|\boldsymbol{v}\|=\sqrt{2^2+4^2+(-9)^2}=\underline{\sqrt{101}}$$

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle=(-1)(2)+(5)(4)+(2)(-9)=\underline{0}$$

$$\cos\theta=\underline{0}$$

$$\cos\theta=\frac{\langle \boldsymbol{u},\ \boldsymbol{v}\rangle}{\|\boldsymbol{u}\|.\|\boldsymbol{v}\|}$$

d) $\boldsymbol{u}=(4,\ 1,\ 8);\quad \boldsymbol{v}=(1,\ 0,\ -3)$

$$\|\boldsymbol{u}\|=\sqrt{4^2+1^2+8^2}=\underline{9}$$

$$\|\boldsymbol{v}\|=\sqrt{1^2+0^2+(-3)^2}=\underline{\sqrt{10}}$$

$$\langle \boldsymbol{u},\ \boldsymbol{v}\rangle=(4)(1)+(1)(0)+(8)(-3)=\underline{-20}$$

$$\cos\theta=-\frac{20}{9\sqrt{10}}$$

$$\cos\theta=\frac{\langle \boldsymbol{u},\ \boldsymbol{v}\rangle}{\|\boldsymbol{u}\|.\|\boldsymbol{v}\|}$$

e) $\boldsymbol{u}=(1,\ 0,\ 1,\ 0);\quad \boldsymbol{v}=(-3,\ -3,\ -3,\ -3)$

$$\|\boldsymbol{u}\|=\underline{\sqrt{2}}$$

$$\|\boldsymbol{v}\|=\sqrt{9+9+9+9}=\underline{12}$$

$$\langle \boldsymbol{u},\ \boldsymbol{v}\rangle=(1)(-3)+(0)(-3)+(1)(-3)+(0)(-3)$$

$$=\underline{-6}$$

$$\cos\theta=\frac{-6}{12\sqrt{2}}$$

$$\cos\theta=\frac{\langle \boldsymbol{u},\ \boldsymbol{v}\rangle}{\|\boldsymbol{u}\|.\|\boldsymbol{v}\|}$$

$$=-\frac{1}{2\sqrt{2}}$$

f) $\boldsymbol{u}=(2,\ 1,\ 7,\ -1);\quad \boldsymbol{v}=(4,\ 0,\ 0,\ 0)$

$$\|\boldsymbol{u}\| = \sqrt{2^2 + 1^2 + 7^2 + (-1)^2} = \underline{\sqrt{55}}$$

$$\|\boldsymbol{v}\| = \sqrt{4^2 + 0} = \underline{4}$$

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = (2)(4) + (1)(0) + (7)(0) + (-1)(0)$$

$$= 8$$

$$\cos\theta = \frac{8}{4\sqrt{55}} \qquad \cos\theta = \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\|\boldsymbol{u}\| . \|\boldsymbol{v}\|}$$

$$= \frac{2}{\sqrt{55}}$$

g) $\boldsymbol{u} = (1, 3, -5, 4), \quad \boldsymbol{v} = (2, -4, 4, 1)$

$$\|\boldsymbol{u}\| = \sqrt{1 + 9 + 25 + 16} = \underline{\sqrt{51}}$$

$$\|\boldsymbol{v}\| = \sqrt{4 + 16 + 16 + 1} = \underline{\sqrt{37}}$$

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 2 - 12 - 20 + 4$$

$$= -26$$

$$\cos\theta = \frac{-26}{\sqrt{51}\sqrt{37}} \qquad \cos\theta = \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\|\boldsymbol{u}\| . \|\boldsymbol{v}\|}$$

h) $\boldsymbol{u} = (1, 2, 3, 4), \quad \boldsymbol{v} = (-1, -2, -3, -4)$

$$\|\boldsymbol{u}\| = \sqrt{1 + 4 + 9 + 16} = \underline{\sqrt{30}}$$

$$\|\boldsymbol{v}\| = \sqrt{1 + 4 + 9 + 16} = \underline{\sqrt{30}}$$

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = -1 - 4 - 9 - 16 = \underline{-30}$$

$$\cos\theta = \frac{-30}{\sqrt{30}\sqrt{30}} \qquad \cos\theta = \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\|\boldsymbol{u}\| . \|\boldsymbol{v}\|}$$

$$= -1$$

Exercise

Find the cosine of the angle between ***A*** and ***B***.

a) $A = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$

b) $A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$

c) $A = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

d) $A = \begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$

Solution

a) $A = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$

$$\|A\| = \sqrt{\langle A, A\rangle}$$
$$= \sqrt{2^2 + 6^2 + 1^2 + (-3)^2}$$
$$= \sqrt{50}$$
$$= 5\sqrt{2}$$

$$\|B\| = \sqrt{\langle B, B\rangle}$$
$$= \sqrt{3^2 + 2^2 + 1^2 + 0^2}$$
$$= \sqrt{14}$$

$$\langle A, B\rangle = 2(3) + 6(2) + 1(1) + (-3)(0) \ \underline{=19|}$$

$$\cos\theta = \frac{19}{5\sqrt{2}\sqrt{14}} \qquad \cos\theta = \frac{\langle \boldsymbol{A}, \boldsymbol{B}\rangle}{\|\boldsymbol{A}\|.\|B\|}$$
$$\underline{= \frac{19}{10\sqrt{7}}|}$$

b) $A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$

$$\|A\| = \sqrt{\langle A, A\rangle}$$
$$= \sqrt{2^2 + 4^2 + (-1)^2 + 3^2}$$
$$\underline{= \sqrt{30}|}$$

$$\|B\| = \sqrt{\langle B, B\rangle}$$
$$= \sqrt{(-3)^2 + 1^2 + 4^2 + 2^2}$$
$$\underline{= \sqrt{30}|}$$

$$\langle A, B\rangle = 2(-3) + 4(1) + (-1)(4) + 3(2) \ \underline{=0|}$$

$$\cos\theta = \frac{0}{30} \ \underline{=0|} \qquad \cos\theta = \frac{\langle \boldsymbol{A}, \boldsymbol{B}\rangle}{\|\boldsymbol{A}\|.\|B\|}$$

c) $A = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

$$\|A\| = \sqrt{81 + 64 + 49 + 36 + 25 + 16} \qquad \|A\| = \sqrt{\langle A, A\rangle}$$
$$\underline{= \sqrt{271}|}$$

$$\|B\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2} \qquad \|B\| = \sqrt{\langle B, B\rangle}$$
$$\underline{= \sqrt{91}|}$$

$$\langle A,\ B\rangle = 9+16+21+24+25+24 \ \underline{=119|}$$

$$\cos\theta = \frac{119}{\sqrt{271}\sqrt{91}} \qquad\qquad \cos\theta = \frac{\langle \boldsymbol{A},\ \boldsymbol{B}\rangle}{\|\boldsymbol{A}\|.\|B\|}$$

d) $A = \begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$

$$\|A\| = \sqrt{1^2 + (-2)^2 + 7^2 + 6^2 + (-3)^2 + 4^2} \qquad\qquad \|A\| = \sqrt{\langle A,\ A\rangle}$$

$$\underline{= \sqrt{115}|}$$

$$\|B\| = \sqrt{1^2 + (-2)^2 + 3^2 + 6^2 + 5^2 + (-4)^2} \qquad\qquad \|B\| = \sqrt{\langle B,\ B\rangle}$$

$$\underline{= \sqrt{91}|}$$

$$\langle A,\ B\rangle = 1+4+21+36-15-16 \ \underline{=31|}$$

$$\cos\theta = \frac{31}{\sqrt{115}\sqrt{91}} \qquad\qquad \cos\theta = \frac{\langle \boldsymbol{A},\ \boldsymbol{B}\rangle}{\|\boldsymbol{A}\|.\|B\|}$$

Exercise

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

a) $\boldsymbol{u} = (-1,\ 3,\ 2);\quad \boldsymbol{v} = (4,\ 2,\ -1)$

b) $\boldsymbol{u} = (a,\ b);\quad \boldsymbol{v} = (-b,\ a)$

c) $\boldsymbol{u} = (-2,\ -2,\ -2);\quad \boldsymbol{v} = (1,\ 1,\ 1)$

d) $\boldsymbol{u} = (-4,\ 6,\ -10,\ 1);\quad \boldsymbol{v} = (2,\ 1,\ -2,\ 9)$

e) $\boldsymbol{u} = (-4,\ 6,\ -10,\ 1);\quad \boldsymbol{v} = (2,\ 1,\ -2,\ 9)$

Solution

a) $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = (-1)(4) + 3(2) + 2(-1) = \underline{0|}$ Therefore the given vectors are orthogonal.

b) $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = a(-b) + b(a) = \underline{0|}$ Therefore the given vectors are orthogonal.

c) $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = (-2)(1) + (-2)(1) + (-2)(1) = \underline{-6|}$ Therefore the given vectors are ***not*** orthogonal.

d) $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = (-4)(2) + (6)(1) + (-10)(-2) + (1)(9) = \underline{27|}$ Therefore the given vectors are ***not*** orthogonal.

e) $\|\boldsymbol{u}\| = \sqrt{(-4)^2 + 6^2 + (-10)^2 + 1^2} = \sqrt{153} = 3\sqrt{17}$

$$\|\boldsymbol{v}\| = \sqrt{2^2 + 1^2 + (-2)^2 + 9^2} = \sqrt{90} = 3\sqrt{10}$$

$$\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = (-4)(2) + 6(1) - 10(-2) + 1(9) \underline{= 27|}$$

$$\cos\theta = \frac{\langle \boldsymbol{u},\ \boldsymbol{v}\rangle}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$$

$$= \frac{27}{3\sqrt{17}\left(3\sqrt{10}\right)}$$

$$\boxed{= \frac{3}{\sqrt{170}}}$$

The vectors ***u*** and ***v*** are *NOT* orthogonal with respect to the Euclidean

Exercise

Do there exist scalars k and l such that the vectors $\boldsymbol{u} = (2,\ k,\ 6),\quad \boldsymbol{v} = (l,\ 5,\ 3),\quad and\quad \boldsymbol{w} = (1,\ 2,\ 3)$ are mutually orthogonal with respect to the Euclidean inner product?

Solution

$$\langle \boldsymbol{u},\boldsymbol{w}\rangle = (2)(1) + (k)(2) + (6)(3) = 20 + 2k = 0 \quad \Rightarrow \quad \boxed{k = -10}$$

$$\langle \boldsymbol{v},\boldsymbol{w}\rangle = (l)(1) + (5)(2) + (3)(3) = l + 19 = 0 \quad \Rightarrow \quad \boxed{l = -19}$$

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle = (2)(l) + (k)(5) + (6)(3) = 2l + 5k + 18 = 0$$

$$2(-19) + 5(-10) + 18 = -70 \neq 0$$

Thus, there are no scalars such that the vectors are mutually orthogonal

Exercise

Let $\boldsymbol{R}^3$ have the Euclidean inner product. For which values of k are ***u*** and ***v*** orthogonal?

a) $\boldsymbol{u} = (2,\ 1,\ 3),\quad \boldsymbol{v} = (1,\ 7,\ k)$

b) $\boldsymbol{u} = (k,\ k,\ 1),\quad \boldsymbol{v} = (k,\ 5,\ 6)$

Solution

a) $\langle \boldsymbol{u},\boldsymbol{v}\rangle = (2)(1) + (1)(7) + (3)(k)$

$= 9 + 3k = 0$

u and ***v*** are orthogonal for $\boxed{k = -3}$

b) $\langle \boldsymbol{u},\boldsymbol{v}\rangle = (k)(k) + (k)(5) + (1)(6)$

$= k^2 + 5k + 6 = 0$

u and ***v*** are orthogonal for $\boxed{k = -2,\ -3}$

Exercise

Let V be an inner product space. Show that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal unit vectors in V, then $\|\boldsymbol{u}-\boldsymbol{v}\|=\sqrt{2}$

Solution

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{v}\|^2 &= \langle \boldsymbol{u}-\boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}\rangle \\ &= \langle \boldsymbol{u}, \boldsymbol{u}-\boldsymbol{v}\rangle - \langle \boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}\rangle \\ &= \langle \boldsymbol{u}, \boldsymbol{u}\rangle - \langle \boldsymbol{u}, \boldsymbol{v}\rangle - \langle \boldsymbol{v}, \boldsymbol{u}\rangle + \langle \boldsymbol{v}, \boldsymbol{v}\rangle \\ &= \|\boldsymbol{u}\|^2 - 0 - 0 + \|\boldsymbol{v}\|^2 \\ &= 2\end{aligned}$$

Thus $\|\boldsymbol{u}-\boldsymbol{v}\|=\sqrt{2}$

Exercise

Let $\mathbf{S}$ be a subspace of $\mathbb{R}^n$. Explain what $\left(\mathbf{S}^{\perp}\right)^{\perp}=\mathbf{S}$ means and why it is true.

Solution

$\left(\mathbf{S}^{\perp}\right)^{\perp}$ is the orthogonal complement of , $\mathbf{S}^{\perp}$, which is itself the orthogonal complement of $\mathbf{S}$, so

$\left(\mathbf{S}^{\perp}\right)^{\perp}=\mathbf{S}$ means that $\mathbf{S}$ is the orthogonal of its orthogonal complement.

We need to show that $\mathbf{S}$ is contained in $\left(\mathbf{S}^{\perp}\right)^{\perp}$ and, conversely, that $\left(\mathbf{S}^{\perp}\right)^{\perp}$ is contained in $\mathbf{S}$ to be true.

i. Suppose $\vec{v}\in\mathbf{S}$ and $\vec{w}\in\mathbf{S}^{\perp}$. Then $\langle \vec{v}, \vec{w}\rangle = 0$ by definition of $\mathbf{S}^{\perp}$. Thus $\mathbf{S}$ is certainly contained is $\left(\mathbf{S}^{\perp}\right)^{\perp}$ (which consists of all vectors in $\mathbb{R}^n$ which are orthogonal to $\mathbf{S}^{\perp}$).

ii. Suppose $\vec{v}\in\left(\mathbf{S}^{\perp}\right)^{\perp}$ (means $\vec{v}$ is orthogonal to all vectors in $\mathbf{S}^{\perp}$); then we need to show that $\vec{v}\in\mathbf{S}$.

Let assume $\left\{\vec{u}_1,\vec{u}_2,\ldots,\vec{u}_p\right\}$ be a basis for $\mathbf{S}$ and let $\left\{\vec{w}_1,\vec{w}_2,\ldots,\vec{w}_q\right\}$ be a basis for $\mathbf{S}^{\perp}$. If $\vec{v}\notin\mathbf{S}$, then $\left\{\vec{u}_1,\vec{u}_2,\ldots,\vec{u}_p,\vec{v}\right\}$ is linearly independent set. Since each vector ifs that set is orthogonal to all of $\mathbf{S}^{\perp}$, the set $\left\{\vec{u}_1,\vec{u}_2,\ldots,\vec{u}_p,\ \vec{v},\ \vec{w}_1,\vec{w}_2,\ldots,\vec{w}_q\right\}$ is linearly independent. Since there are $p+q+1$ vectors in this set, this means that $p+q+1\le n \quad\Leftrightarrow\quad p+q\le n-1$. On the other hand, If A is the matrix whose i^{th} row is $\vec{u}_i^{\,T}$, then the row space of A is $\mathbf{S}$ and the nullspace of A is

$\mathbf{S}^{\perp}$. Since $\mathbf{S}$ is p-dimensional, the rank of A is p, meaning that the dimension of $\text{nul}(A) = \mathbf{S}^{\perp}$ is $q = n - p$. Therefore,

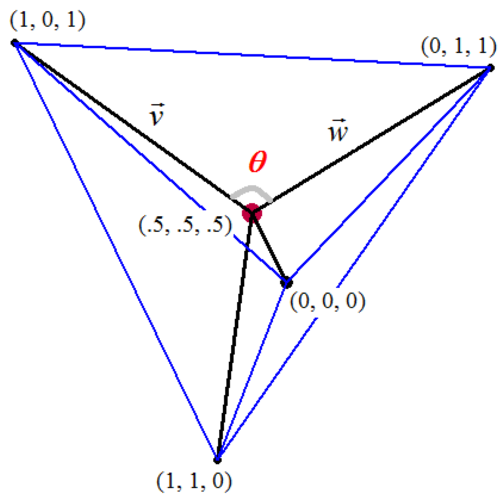
$$p + q = p + (n - p) = n$$

Which contradict the fact that $p + q \le n - 1$. From this, we see that, if $\vec{v} \in \left(\mathbf{S}^{\perp}\right)^{\perp}$, it must be the case that $\vec{v} \in \mathbf{S}$.

Exercise

The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0,\ 0,\ 0)$, $(1,\ 1,\ 0)$, $(1,\ 0,\ 1)$ and $(0,\ 1,\ 1)$ – (***note*** that all six edges have length $\sqrt{2}$, so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$ to the vertices?

Solution



Let $\vec{v}$ be the vector of the segment (1, 0, 1) and $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Let be the vector of the segment (0, 1, 1) and $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$

$$\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

We have:

$$\cos\theta = \frac{\langle \vec{v},\ \vec{w}\rangle}{\|\vec{v}\|\cdot\|\vec{w}\|}$$

$$= \frac{\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)\cdot\left(-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}\ \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}}$$

$$= \frac{-\frac{1}{4}}{\frac{3}{4}}$$

$$= -\frac{1}{3}$$

$$\theta \approx 109.47°$$

Exercise

Determine if the given vectors are orthogonal.

$$\boldsymbol{x}_1 = (1,\ 0,\ 1,\ 0),\quad \boldsymbol{x}_2 = (0,\ 1,\ 0,\ 1),\quad \boldsymbol{x}_3 = (1,\ 0,\ -1,\ 0),\quad \boldsymbol{x}_4 = (1,\ 1,\ -1,\ -1)$$

Solution

$$\boldsymbol{x}_1 \bullet \boldsymbol{x}_2 = (1,\ 0,\ 1,\ 0)\bullet(0,\ 1,\ 0,\ 1) = 0$$

$$\boldsymbol{x}_1 \bullet \boldsymbol{x}_3 = (1,\ 0,\ 1,\ 0)\bullet(1,\ 0,\ -1,\ 0) = 1-1 = 0$$

$$\boldsymbol{x}_1 \bullet \boldsymbol{x}_4 = (1,\ 0,\ 1,\ 0)\bullet(1,\ 1,\ -1,\ -1) = 1-1 = 0$$

$$\boldsymbol{x}_2 \bullet \boldsymbol{x}_3 = (0,\ 1,\ 0,\ 1)\bullet(1,\ 0,\ -1,\ 0) = 0$$

$$\boldsymbol{x}_2 \bullet \boldsymbol{x}_4 = (0,\ 1,\ 0,\ 1)\bullet(1,\ 1,\ -1,\ -1) = 1-1 = 0$$

$$\boldsymbol{x}_3 \bullet \boldsymbol{x}_4 = (1,\ 0,\ -1,\ 0)\bullet(1,\ 1,\ -1,\ -1) = 1-1 = 0$$

The given vectors are orthogonal

Exercise

Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

a) $\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\ \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)\ \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$

b) $\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right)\ \left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)\ \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)$

Solution

a) $\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\bullet\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}}\ \underline{= 0}$

$$\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\bullet\left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)=-\frac{1}{2}+0+\frac{1}{2}\ \underline{=0}|$$

$$\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)\bullet\left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)=-\frac{1}{\sqrt{6}}+0-\frac{1}{\sqrt{6}}=-\frac{2}{\sqrt{6}}\ \underline{\neq 0}|$$

Therefore the given vectors are ***not*** orthogonal.

b) $\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right)\bullet\left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)=\frac{4}{9}-\frac{2}{9}-\frac{2}{9}\ \underline{=0}|$

$$\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right)\bullet\left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)=\frac{2}{9}-\frac{4}{9}+\frac{2}{9}\ \underline{=0}|$$

$$\left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)\bullet\left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)=\frac{2}{9}+\frac{2}{9}-\frac{4}{9}\ \underline{=0}|$$

Therefore, the given vectors are orthogonal.

Exercise

Consider vectors $\vec{u}=(2,\ 3,\ 5)\quad \vec{v}=(1,\ -4,\ 3)\quad in\quad R^3$

a) $\langle\vec{u},\ \vec{v}\rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine between $\vec{u}$ *and* $\vec{v}$

Solution

a) $\langle\vec{u},\ \vec{v}\rangle=(2,\ 3,\ 5)\bullet(1,\ -4,\ 3)$

$$=2-12+15$$

$$\underline{=5}|$$

b) $\|\vec{u}\|=\sqrt{4+9+25}$

$$\underline{=\sqrt{38}}|$$

c) $\vec{v}=\sqrt{1+16+9}$

$$\underline{=\sqrt{26}}|$$

d) $\cos\theta=\frac{5}{\sqrt{38}\sqrt{26}}$ $\qquad\qquad$ $\cos\theta=\frac{\vec{u}\bullet\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$

Exercise

Consider vectors $\vec{u}=(1,\ 1,\ 1)\quad \vec{v}=(1,\ 2,\ -3)\quad in\quad R^3$

a) $\langle\vec{u},\ \vec{v}\rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine θ between $\vec{u}$ *and* $\vec{v}$

Solution

a) $\langle\vec{u},\ \vec{v}\rangle=(1,\ 1,\ 1)\bullet(1,\ 2,\ -3)$

$$=1+2-3$$

$$\underline{=0}|$$

b) $\|\vec{u}\| = \sqrt{1+1+1}$

$\boxed{= \sqrt{3}}$

c) $\|\vec{v}\| = \sqrt{1+4+9}$

$\boxed{= \sqrt{14}}$

d) $\boxed{\cos\theta = 0}$ $\qquad \cos\theta = \dfrac{\vec{u}\bullet\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$

$\vec{u}$ *and* $\vec{v}$ are orthogonal vectors.

Exercise

Consider vectors $\vec{u} = (1,\ 2,\ 5)$ $\quad \vec{v} = (2,\ -3,\ 5)$ $\quad \vec{w} = (4,\ 2,\ -3)$ $\quad in \quad R^3$

a) $\langle \vec{u},\ \vec{v} \rangle$

b) $\langle \vec{u},\ \vec{w} \rangle$

c) $\langle \vec{v},\ \vec{w} \rangle$

d) $\|\vec{u}\|$

e) $\|\vec{v}\|$

f) $\|\vec{w}\|$

g) Cosine α between $\vec{u}$ *and* $\vec{v}$

h) Cosine β between $\vec{u}$ *and* $\vec{w}$

i) Cosine θ between $\vec{v}$ *and* $\vec{w}$

j) $(\vec{u}+\vec{v})\bullet\vec{w}$

Solution

a) $\langle \vec{u},\ \vec{v} \rangle = (1,\ 2,\ 5)\bullet(2,\ -3,\ 5)$

$= 2 - 6 + 25$

$\boxed{= 21}$

b) $\langle \vec{u},\ \vec{w} \rangle = (1,\ 2,\ 5)\bullet(4,\ 2,\ -3)$

$= 4 + 4 - 15$

$\boxed{= -7}$

c) $\langle \vec{v},\ \vec{w} \rangle = (2,\ -3,\ 5)\bullet(4,\ 2,\ -3)$

$= 8 - 6 - 15$

$\boxed{= -13}$

d) $\|\vec{u}\| = \sqrt{1+4+25}$

$\boxed{= \sqrt{30}}$

e) $\|\vec{v}\| = \sqrt{4+9+25}$

$\boxed{= \sqrt{38}}$

f) $\|\vec{w}\| = \sqrt{16+4+9}$

$\boxed{= \sqrt{29}}$

g) $\boxed{\cos\alpha = \dfrac{21}{\sqrt{30}\sqrt{38}}}$ $\qquad \cos\alpha = \dfrac{\vec{u}\bullet\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$

h) $\boxed{\cos\beta = \dfrac{-7}{\sqrt{30}\sqrt{29}}}$ $\qquad \cos\beta = \dfrac{\vec{u}\bullet\vec{w}}{\|\vec{u}\|\|\vec{w}\|}$

i) $\boxed{\cos\theta = \dfrac{-13}{\sqrt{38}\sqrt{29}}}$ $\qquad \cos\theta = \dfrac{\vec{v}\bullet\vec{w}}{\|\vec{v}\|\|\vec{w}\|}$

j)
$$\begin{aligned}(\vec{u}+\vec{v})\bullet\vec{w} &= \left[(1,\ 2,\ 5)+(2,\ -3,\ 5)\right]\bullet(4,\ 2,\ -3)\\ &= (3,\ -1,\ 10)\bullet(4,\ 2,\ -3)\\ &= 12-2-30\\ &= -20\end{aligned}$$

Exercise

Consider polynomial $f(t)=3t-5;\quad g(t)=t^2 \quad in \quad \boldsymbol{P}(t)$

a) $\langle f,\ g\rangle$ *b)* $\|f\|$ *c)* $\|g\|$ *d)* Cosine between f *and* g

Solution

a)
$$\begin{aligned}\langle f,\ g\rangle &= \int_0^1 f(t)g(t)\,dt\\ &= \int_0^1 (3t-5)t^2 dt\\ &= \int_0^1 \left(3t^3-5t^2\right)dt\\ &= \frac{3}{4}t^4-\frac{5}{3}t^3 \Big|_0^1\\ &= \frac{3}{4}-\frac{5}{3}\\ &= -\frac{11}{12}\end{aligned}$$

b)
$$\begin{aligned}\langle f,\ f\rangle &= \int_0^1 f(t)f(t)\,dt\\ &= \int_0^1 (3t-5)^2\,dt\\ &= \frac{1}{3}\int_0^1 (3t-5)^2\,d(3t-5)\\ &= \frac{1}{9}(3t-5)^3\Big|_0^1\end{aligned}$$

$$=\frac{1}{9}(8-125)$$

$$\boxed{=13}$$

$$\|f\| = \sqrt{|\langle f, f\rangle|} \ \boxed{=\sqrt{13}}$$

c) $\langle g, g\rangle = \int_0^1 g(t)g(t)\,dt$

$$= \int_0^1 t^4 dt$$

$$= \frac{1}{5}t^5 \Big|_0^1$$

$$\boxed{=\frac{1}{5}}$$

$$\|g\| = \sqrt{|\langle g, g\rangle|} \ \boxed{=\frac{1}{\sqrt{5}}}$$

d) $\cos\theta = \dfrac{-\frac{11}{12}}{\sqrt{13}\,\frac{\sqrt{5}}{5}}$ $\qquad\qquad$ $\cos\theta = \dfrac{f \bullet g}{\|f\|\|g\|}$

$$\boxed{= \frac{-55}{12\sqrt{65}}}$$

Exercise

Consider polynomial $f(t) = t + 2$; $g(t) = 3t - 2$; $h(t) = t^2 - 2t - 3$ *in* $\boldsymbol{P}(t)$

a) $\langle f, g\rangle$ $\qquad$ *d)* $\|f\|$ $\qquad$ *g)* Cosine α between f *and* g

b) $\langle f, h\rangle$ $\qquad$ *e)* $\|g\|$ $\qquad$ *h)* Cosine β between f *and* h

c) $\langle g, h\rangle$ $\qquad$ *f)* $\|h\|$ $\qquad$ *i)* Cosine θ between g *and* h

Solution

a) $\langle f, g\rangle = \int_0^1 (t+2)(3t-2)\,dt$ $\qquad\qquad$ $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$

$$= \int_0^1 \left(3t^2 + 4t - 4\right) dt$$

$$= t^3 + 2t^2 - 4t \Big|_0^1$$

$$= 1 + 2 - 4$$

$$\boxed{= -1}$$

b) $\langle f, h\rangle = \int_0^1 (t+2)\left(t^2 - 2t - 3\right) dt$ $\qquad$ $\langle f, h\rangle = \int_0^1 f(t)h(t)\,dt$

$$= \int_0^1 \left(t^3 - 7t - 6\right) dt$$

$$= \frac{1}{4}t^4 - \frac{7}{2}t^2 - 6t \Big|_0^1$$

$$= \frac{1}{4} - \frac{7}{2} - 6$$

$$\boxed{= -\frac{37}{4}}$$

c) $\langle g, h\rangle = \int_0^1 (3t-2)\left(t^2 - 2t - 3\right) dt$ $\qquad$ $\langle f, h\rangle = \int_0^1 g(t)h(t)\,dt$

$$= \int_0^1 \left(3t^3 - 8t^2 - 5t + 6\right) dt$$

$$= \frac{3}{4}t^4 - \frac{8}{3}t^3 - \frac{5}{2}t^2 + 6t \Big|_0^1$$

$$= \frac{3}{4} - \frac{8}{3} - \frac{5}{2} + 6$$

$$\boxed{= \frac{9}{4}}$$

d) $\langle f, f\rangle = \int_0^1 (t+2)^2\, dt$ $\qquad$ $\langle f, f\rangle = \int_0^1 f(t)f(t)\,dt$

$$= \frac{1}{3}(t+2)^3 \Big|_0^1$$

$$= \frac{1}{3}(27 - 8)$$

$$\boxed{= \frac{19}{3}}$$

$$\|f\| = \sqrt{|\langle f, f\rangle|} \ \boxed{= \sqrt{\frac{19}{3}}}$$

e) $\langle g, g\rangle = \int_0^1 (3t-2)^2\, dt$ $\qquad$ $\langle g, g\rangle = \int_0^1 g(t)g(t)\,dt$

$$= \frac{1}{3}\int_0^1 (3t-2)^2\, d(3t-2)$$

$$=\frac{1}{9}(3t-2)^3\Big|_0^1$$

$$=\frac{1}{9}(1+8)$$

$$\boxed{=1}$$

$$\|g\|=\sqrt{|\langle g,\ g\rangle|}\ \ \boxed{=1}$$

f) $\langle g,\ g\rangle=\int_0^1\left(t^2-2t-3\right)^2 dt$ $\qquad\qquad$ $\langle h,\ h\rangle=\int_0^1 h(t)h(t)dt$

$$=\int_0^1\left(t^4-4t^3-2t^2+12t+9\right)dt$$

$$=\left(\frac{1}{5}t^5-t^4-\frac{2}{3}t^3+6t^2+9t\right)\Big|_0^1$$

$$=\frac{1}{5}-1-\frac{2}{3}+6+9$$

$$\boxed{=\frac{203}{15}}$$

$$\|h\|=\sqrt{|\langle h,\ h\rangle|}\ \ \boxed{=\sqrt{\frac{203}{15}}}$$

g) $\cos\alpha=\frac{-1}{\sqrt{\frac{19}{3}}}$ $\qquad\qquad$ $\cos\alpha=\frac{f\bullet g}{\|f\|\|g\|}$

$$\boxed{=-\sqrt{\frac{3}{19}}}$$

h) $\cos\beta=-\frac{37}{4}\frac{1}{\sqrt{\frac{19}{3}}\sqrt{\frac{203}{15}}}$ $\qquad\qquad$ $\cos\beta=\frac{f\bullet h}{\|f\|\|h\|}$

$$\boxed{=-\frac{111}{4}\sqrt{\frac{5}{3{,}857}}}$$

i) $\cos\theta=\frac{9}{4}\frac{1}{\sqrt{\frac{203}{15}}}$ $\qquad\qquad$ $\cos\theta=\frac{g\bullet h}{\|g\|\|h\|}$

$$\boxed{=\frac{9}{4}\sqrt{\frac{15}{203}}}$$

Exercise

Suppose $\langle \vec{u},\ \vec{v} \rangle = 3 + 2\,i$ in a complex inner product space V. Find:

a) $\langle (2-4\,i)\vec{u},\ \vec{v} \rangle$ *b)* $\langle \vec{u},\ (4+3\,i)\vec{v} \rangle$ *c)* $\langle (3-6\,i)\vec{u},\ (5-2\,i)\vec{v} \rangle$ *d)* $\|\vec{u},\ \vec{v}\|$

Solution

a) $\langle (2-4\,i)\vec{u},\ \vec{v} \rangle = (2-4\,i)\langle \vec{u},\ \vec{v} \rangle$

$= (2-4\,i)(3+2\,i)$

$= 6 + 4i - 12i + 8 \qquad i^2 = -1$

$\boxed{= 14 - 8i}$

b) $\langle \vec{u},\ (4+3\,i)\vec{v} \rangle = (4+3\,i)\langle \vec{u},\ \vec{v} \rangle$

$= (4+3\,i)(3+2\,i)$

$= 12 + 8i + 9i - 6 \qquad i^2 = -1$

$\boxed{= 14 - 8i}$

c) $\langle (3-6\,i)\vec{u},\ (5-2\,i)\vec{v} \rangle = (3-6\,i)(5-2\,i)\langle \vec{u},\ \vec{v} \rangle$

$= (15 - 36i - 12)(3+2\,i)$

$= (3 - 36i)(3+2\,i)$

$= 9 - 102i + 72$

$\boxed{= 81 - 102i}$

d) $\|\vec{u},\ \vec{v}\| = \sqrt{\langle \vec{u},\ \vec{v} \rangle}$

$= \sqrt{9+4}$

$\boxed{= \sqrt{13}}$

Exercise

Find the Fourier coefficient c and the projection $c\vec{v}$ of $\vec{u} = (3+4\,i,\ 2-3\,i)$ along $\vec{v} = (5+i,\ 2i)$ in $\boldsymbol{C}^2$

Solution

$$c = \frac{(3+4\,i)\left(\overline{5+i}\right) + (2-3\,i)\left(\overline{2i}\right)}{5^2 + 1^2 + 2^2} \qquad c = \frac{\langle \vec{u},\ \vec{v} \rangle}{\langle \vec{v},\ \vec{v} \rangle}$$

$$= \frac{(3+4\,i)(5-i) + (2-3\,i)(-2i)}{25+1+4}$$

$$= \frac{15 + 17i + 4 - 4i - 6}{30}$$

$$= \frac{13 + 13i}{30}$$

$$=\boxed{\frac{13}{30}+\frac{13}{30}i}$$

$$proj(\vec{u},\ \vec{v}) = c\vec{v}$$

$$= \left(\frac{13}{30}+\frac{13}{30}i\right)(5+i,\ 2i)$$

$$= \left(\frac{13}{6}+\frac{13}{6}i+\frac{13}{30}i-\frac{13}{30},\quad \frac{13}{15}i-\frac{13}{15}\right)$$

$$= \left(\frac{52}{30}+\frac{78}{30}i,\ \ -\frac{13}{15}+\frac{13}{15}i\right)$$

$$=\boxed{\left(\frac{26}{15}+\frac{39}{30}i,\ \ -\frac{13}{15}+\frac{13}{15}i\right)}$$

Exercise

Suppose $\vec{v}=(1,\ 3,\ 5,\ 7)$. Find the projection of $\vec{v}$ onto W or find $\vec{w}\in W$ that minimizes $\|\vec{v}-\vec{w}\|$, where W is the subspace of R^4 spanned by:

a) $\vec{u}_1=(1,\ 1,\ 1,\ 1)$ *and* $\vec{u}_2=(1,\ -3,\ 4,\ -2)$

b) $\vec{v}_1=(1,\ 1,\ 1,\ 1)$ *and* $\vec{v}_2=(1,\ 2,\ 3,\ 2)$

Solution

a) $\vec{u}_1 \bullet \vec{u}_2 = (1,\ 1,\ 1,\ 1)\bullet(1,\ -3,\ 4,\ -2)$

$$= 1-3+4-2$$

$$=\boxed{0}$$

Therefore, $\vec{u}_1$ *and* $\vec{u}_2$ are orthogonal.

$$c_1 = \frac{\langle \vec{v},\ \vec{u}_1\rangle}{\langle \vec{u}_1,\ \vec{u}_1\rangle}$$

$$= \frac{(1,\ 3,\ 5,\ 7)\bullet(1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}$$

$$= \frac{1+3+5+7}{1+1+1+1}$$

$$= \frac{16}{4}$$

$$=\boxed{4}$$

$$c_2 = \frac{\langle \vec{v},\ \vec{u}_2\rangle}{\langle \vec{u}_2,\ \vec{u}_2\rangle}$$

$$= \frac{(1,\ 3,\ 5,\ 7) \bullet (1,\ -3,\ 4,\ -2)}{\|(1,\ -3,\ 4,\ -2)\|^2}$$

$$= \frac{1-9+20-14}{1+9+16+4}$$

$$= \frac{-2}{30}$$

$$= \boxed{\frac{1}{15}}$$

$$w = proj(\vec{v},\ W)$$

$$= c_1\vec{u}_1 + c_2\vec{u}_2$$

$$= 4(1,\ 1,\ 1,\ 1) + \frac{1}{15}(1,\ -3,\ 4,\ -2)$$

$$= \boxed{\left(\frac{59}{15},\ \frac{63}{5},\ \frac{56}{15},\ \frac{62}{15}\right)}$$

b) $\vec{v}_1 \bullet \vec{v}_2 = (1,\ 1,\ 1,\ 1) \bullet (1,\ 2,\ 3,\ 2)$

$$= 1+2+3+2$$

$$= \boxed{8 \ne 0}$$

Therefore, $\vec{v}_1$ *and* $\vec{v}_2$ are *not* orthogonal.

Applying Gram-Schmidt algorithm

$$\boxed{\vec{w}_1 = \vec{v}_1 = (1,\ 1,\ 1,\ 1)}$$

$$\vec{w}_2 = (1,\ 2,\ 3,\ 2) - \frac{(1,\ 2,\ 3,\ 2) \bullet (1,\ 1,\ 1,\ 1)}{4}(1,\ 1,\ 1,\ 1) \qquad \vec{w}_2 = \vec{v}_2 - \frac{\langle \vec{v},\ \vec{w}_1 \rangle}{\|\vec{w}_1\|^2}\vec{w}_1$$

$$= (1,\ 2,\ 3,\ 2) - 2(1,\ 1,\ 1,\ 1)$$

$$= \boxed{(-1,\ 0,\ 1,\ 0)}$$

$$c_1 = \frac{(1,\ 3,\ 5,\ 7) \bullet (1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2} \qquad c_1 = \frac{\langle \vec{v},\ \vec{w}_1 \rangle}{\langle \vec{w}_1,\ \vec{w}_1 \rangle}$$

$$= \frac{1+3+5+7}{1+1+1+1}$$

$$= \frac{16}{4}$$

$$= \boxed{4}$$

$$c_2 = \frac{(1,\ 3,\ 5,\ 7) \bullet (-1,\ 0,\ 1,\ 0)}{\|(-1,\ 0,\ 1,\ 0)\|^2} \qquad c_2 = \frac{\langle \vec{v},\ \vec{w}_2 \rangle}{\langle \vec{w}_2,\ \vec{w}_2 \rangle}$$

$$= \frac{-1+0+5+0}{2}$$

$$\boxed{=-3}$$

$$\begin{aligned} w &= proj\left(\vec{v},\ W\right) \\ &= c_1\vec{w}_1 + c_2\vec{w}_2 \\ &= 4(1,\ 1,\ 1,\ 1) - 3(-1,\ 0,\ 1,\ 0) \\ &= \boxed{(7,\ 4,\ 1,\ 4)} \end{aligned}$$

Exercise

Suppose $\left\{\vec{u}_1,\ \vec{u}_2,\ \dots,\ \vec{u}_n\right\}$ is an orthogonal set of vectors. Prove that (*Pythagoras*)

$$\left\|\vec{u}_1 + \vec{u}_2 + \ \dots + \vec{u}_n\right\|^2 = \left\|\vec{u}_1\right\|^2 + \left\|\vec{u}_2\right\|^2 + \ \dots + \left\|\vec{u}_n\right\|^2$$

Solution

$$\begin{aligned} \left\|\vec{u}_1 + \vec{u}_2 + \ \dots + \vec{u}_n\right\|^2 &= \left\langle \vec{u}_1 + \vec{u}_2 + \ \dots + \vec{u}_n,\ \vec{u}_1 + \vec{u}_2 + \ \dots + \vec{u}_n\right\rangle \\ &= \left\langle \vec{u}_1,\ \vec{u}_1\right\rangle + \left\langle \vec{u}_2,\ \vec{u}_2\right\rangle + \dots + \left\langle \vec{u}_n,\ \vec{u}_n\right\rangle \\ &= \left\|\vec{u}_1\right\|^2 + \left\|\vec{u}_2\right\|^2 + \ \dots + \left\|\vec{u}_n\right\|^2 \end{aligned}$$

Exercise

Suppose A is an orthogonal matrix. Show that $\langle \vec{u}A,\ \vec{v}A\rangle = \langle \vec{u},\ \vec{v}\rangle$ for any $\vec{u},\ \vec{v}\ \in V$

Solution

A is an orthogonal matrix $\Rightarrow AA^T = I$

And $\langle \vec{u},\ \vec{v}\rangle = \vec{u}^T\vec{v}$

$$\begin{aligned} \langle \vec{u}A,\ \vec{v}A\rangle &= (A\vec{u})^T (A\vec{v}) \\ &= \vec{u}^T\left(A^T A\right)\vec{v} \\ &= \vec{u}^T I\ \vec{v} \\ &= \vec{u}^T\vec{v} \\ &= \langle \vec{u},\ \vec{v}\rangle \quad \checkmark \end{aligned}$$

Exercise

Suppose A is an orthogonal matrix. Show that $\|\vec{u}A\| = \|\vec{u}\|$ for every $\vec{u} \in V$

Solution

A is an orthogonal matrix $\Rightarrow AA^T = I$ and $\langle \vec{u},\ \vec{u} \rangle = \vec{u}^T \vec{u}$

$$\begin{aligned}
\|\vec{u}A\|^2 &= \langle \vec{u}A,\ \vec{u}A \rangle \\
&= (A\vec{u})^T (A\vec{u}) \\
&= \vec{u}^T \left(A^T A \right) \vec{u} \\
&= \vec{u}^T I\ \vec{u} \\
&= \vec{u}^T \vec{u} \\
&= \langle \vec{u},\ \vec{u} \rangle \quad \checkmark
\end{aligned}$$

Solution Section 3.3 – Gram-Schmidt Process

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\boldsymbol{R}^m$.

$$\boldsymbol{u}_1 = (1,\ -3), \quad \boldsymbol{u}_2 = (2,\ 2)$$

Solution

$$v_1 = \frac{(1,\ -3)}{\sqrt{1^2 + (-3)^2}} = \frac{(1,\ -3)}{\sqrt{10}} \qquad\qquad v_1 = \frac{u_1}{\|u_1\|}$$

$$= \left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)$$

$$w_2 = u_2 - \left(u_2 . v_1\right) v_1$$

$$= (2,\ 2) - \left[(2,\ 2).\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)\right]\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)$$

$$= (2,\ 2) - \left[\frac{2}{\sqrt{10}} - \frac{6}{\sqrt{10}}\right]\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)$$

$$= (2,\ 2) - \left[-\frac{4}{\sqrt{10}}\right]\left(\frac{1}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\right)$$

$$= (2,\ 2) - \left(-\frac{4}{10},\ \frac{12}{10}\right)$$

$$= (2,\ 2) - \left(-\frac{2}{5},\ \frac{6}{5}\right)$$

$$= \left(\frac{12}{5},\ \frac{4}{5}\right)$$

$$\|w_2\| = \sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{144}{25} + \frac{16}{25}}$$

$$= \sqrt{\frac{160}{25}}$$

$$= \frac{4\sqrt{10}}{5}$$

$$v_2 = \frac{4\sqrt{10}}{5}\left(\frac{12}{5},\ \frac{4}{5}\right) \qquad\qquad v_2 = \frac{w_2}{\|w_2\|}$$

$$= \left(\frac{48\sqrt{10}}{25},\ \frac{16\sqrt{10}}{25}\right)$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\boldsymbol{R}^m$.

$$\boldsymbol{u}_1 = (1,\ 0), \quad \boldsymbol{u}_2 = (3,\ -5)$$

Solution

$$v_1 = \frac{(1,\ 0)}{\sqrt{1^2+0^2}} \qquad\qquad v_1 = \frac{u_1}{\|u_1\|}$$

$$\underline{= (1,\ 0)|}$$

$$w_2 = u_2 - (u_2.v_1)v_1 = (0,\ -5)$$

$$= (3,\ -5) - [(3,\ -5).(1,\ 0)](1,\ 0)$$

$$= (3,\ -5) - [3](1,\ 0)$$

$$= (3,\ -5) - (3,\ 0)$$

$$\underline{= (0,\ -5)|}$$

$$\|w_2\| = \sqrt{0^2 + (-5)^2} = 5$$

$$v_2 = \frac{1}{5}(0,\ -5) \qquad\qquad v_2 = \frac{w_2}{\|w_2\|}$$

$$\underline{= (0,\ -1)|}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ (1,\ 2,\ 1)\}$$

Solution

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,\ 1,\ 1)}{\sqrt{1^2+1^2+1^2}} = \frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$\underline{= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)|}$$

$$w_2 = v_2 - (v_2.u_1)u_1$$

$$= (-1,\ 1,\ 0) - \left[(-1,\ 1,\ 0)\bullet\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (-1,\ 1,\ 0) - \left[-\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 0\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (-1,\ 1,\ 0) - 0\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\underline{= (-1,\ 1,\ 0)}|$$

$$\|w_2\| = \sqrt{(-1)^2 + 1^2}\ \underline{= \sqrt{2}}|$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{(-1,\ 1,\ 0)}{\sqrt{2}}$$

$$\underline{= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)}|$$

$$v_3 \bullet u_1 = (1,\ 2,\ 1) \bullet \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}$$

$$= \frac{3}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}}$$

$$\underline{= \sqrt{3}}|$$

$$v_3 \bullet u_2 = (1,\ 2,\ 1) \bullet \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= -\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} + 0$$

$$\underline{= \frac{1}{\sqrt{2}}}|$$

$$w_3 = v_3 - (v_3 \bullet u_1)u_1 - (v_3 \bullet u_2)u_2$$

$$= (1,\ 2,\ 1) - \sqrt{3}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) - \sqrt{2}\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= (1,\ 2,\ 1) - (1,\ 1,\ 1) - (-1,\ 1,\ 0)$$

$$\underline{= (1,\ 0,\ 0)}|$$

$$u_3 = \frac{w_3}{\|w_3\|} = \frac{(1,\ 0,\ 0)}{\sqrt{1^2}}$$

$$\underline{= (1,\ 0,\ 0)}|$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(1,\ 1,\ 1),\ (0,\ 1,\ 1),\ (0,\ 0,\ 1)\}$$

Solution

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,\ 1,\ 1)}{\sqrt{1^2+1^2+1^2}} = \frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$w_2 = (0,\ 1,\ 1) - \left[(0,\ 1,\ 1)\bullet\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) \qquad \vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 . \vec{u}_1\right)\vec{u}_1$$

$$= (0,\ 1,\ 1) - \left[\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (0,\ 1,\ 1) - \left[\frac{2}{\sqrt{3}}\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (0,\ 1,\ 1) - \left(\frac{2}{3},\ \frac{2}{3},\ \frac{2}{3}\right)$$

$$= \left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$\|w_2\| = \sqrt{\left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \sqrt{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}} = \sqrt{\frac{6}{9}}$$

$$= \frac{\sqrt{6}}{3}$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}{\frac{\sqrt{6}}{3}}$$

$$= \frac{3}{\sqrt{6}}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$= \left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$v_3 \bullet u_1 = (0,\ 0,\ 1)\bullet\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}$$

$$v_3 \bullet u_2 = (0,\ 0,\ 1)\bullet\left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right) = \frac{1}{\sqrt{6}}$$

$$w_3 = (0,\ 0,\ 1) - \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) - \frac{1}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right) \quad \vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{u}_1\right)\vec{u}_1 - \left(\vec{v}_3 \bullet \vec{u}_2\right)\vec{u}_2$$

$$=(0,\ 0,\ 1)-\left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right)-\left(-\frac{1}{3},\ \frac{1}{6},\ \frac{1}{6}\right)$$

$$=\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)$$

$$u_3=\frac{w_3}{\|w_3\|}=\frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}}$$

$$=\frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\sqrt{\frac{1}{2}}}$$

$$=\frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\frac{1}{\sqrt{2}}}$$

$$=\sqrt{2}\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)$$

$$=\left(0,\ -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(1,\ 1,\ 1),\ (0,\ 2,\ 1),\ (1,\ 0,\ 3)\}$$

Solution

$$u_1=\frac{(1,1,1)}{\sqrt{1^2+1^2+1^2}}=\frac{(1,\ 1,\ 1)}{\sqrt{3}} \qquad \vec{u}_1=\frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$=\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$w_2=(0,\ 2,\ 1)-\left[(0,\ 2,\ 1)\cdot\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) \qquad \vec{w}_2=\vec{v}_2-\left(\vec{v}_2.\vec{u}_1\right)\vec{u}_1$$

$$=(0,\ 2,\ 1)-\left(\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$=(0,\ 2,\ 1)-\frac{3}{\sqrt{3}}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$=(0,\ 2,\ 1)-\sqrt{3}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$=(0,\ 2,\ 1)-(1,\ 1,\ 1)$$

$$=(-1,\ 1,\ 0)$$

$$\left\|w_2\right\| = \sqrt{(-1)^2 + (1)^2 + (0)^2}$$
$$= \sqrt{2}$$

$$u_2 = \frac{(-1,\ 1,\ 0)}{\sqrt{2}} \qquad\qquad \vec{u}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$$
$$= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$v_3 \cdot u_1 = (1,\ 0,\ 3) \cdot \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} + \frac{3}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

$$v_3 \cdot u_2 = (1,\ 0,\ 3) \cdot \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) = -\frac{1}{\sqrt{2}}$$

$$w_3 = (1,\ 0,\ 3) - \frac{4}{\sqrt{3}}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) \qquad \vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \cdot \vec{u}_1\right)\vec{u}_1 - \left(\vec{v}_3 \cdot \vec{u}_2\right)\vec{u}_2$$
$$= (1,\ 0,\ 3) - \left(\frac{4}{3},\ \frac{4}{3},\ \frac{4}{3}\right) + \left(-\frac{1}{2},\ \frac{1}{2},\ 0\right)$$
$$= \left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right)$$

$$u_3 = \frac{1}{\sqrt{\left(-\frac{5}{6}\right)^2 + \left(-\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2}}\left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right) \qquad \vec{u}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$
$$= \frac{1}{\sqrt{\frac{25}{36} + \frac{25}{36} + \frac{25}{9}}}\left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right)$$
$$= \frac{1}{\frac{5}{6}\sqrt{6}}\left(-\frac{5}{6},\ -\frac{5}{6},\ \frac{5}{3}\right)$$
$$= \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(2,\ 2,\ 2),\ (1,\ 0,\ -1),\ (0,\ 3,\ 1)\}$$

Solution

$$u_1 = \frac{(2,\ 2,\ 2)}{\sqrt{2^2+2^2+2^2}} = \frac{(2,\ 2,\ 2)}{\sqrt{12}} \qquad\qquad \vec{u}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$w_2 = (1,\ 0,\ -1) - \left[(1,\ 0,\ -1)\bullet\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) \qquad \vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 . \vec{u}_1\right)\vec{u}_1$$

$$= (1,\ 0,\ -1) - \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (0,\ 2,\ 1) - (0)\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$= (1,\ 0,\ -1)$$

$$u_2 = \frac{(1,\ 0,\ -1)}{\sqrt{2}} \qquad \vec{u}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$$

$$= \left(\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right)$$

$$v_3 \bullet u_1 = (0,\ 3,\ 1)\bullet\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) = \frac{3}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

$$v_3 \bullet u_2 = (0,\ 3,\ 1)\bullet\left(\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}$$

$$w_3 = (0,\ 3,\ 1) - \frac{4}{\sqrt{3}}\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right) \qquad \vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{u}_1\right)\vec{u}_1 - \left(\vec{v}_3 \bullet \vec{u}_2\right)\vec{u}_2$$

$$= (0,\ 3,\ 1) - \left(\frac{4}{3},\ \frac{4}{3},\ \frac{4}{3}\right) + \left(\frac{1}{2},\ 0,\ -\frac{1}{2}\right)$$

$$= \left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)$$

$$u_3 = \frac{1}{\sqrt{\left(-\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(-\frac{5}{6}\right)^2}}\left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right) \qquad \vec{u}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$

$$= \frac{1}{\sqrt{\frac{25}{36} + \frac{25}{36} + \frac{25}{9}}}\left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)$$

$$= \frac{1}{\frac{5}{6}\sqrt{6}}\left(-\frac{5}{6},\ \frac{5}{3},\ -\frac{5}{6}\right)$$

$$= \left(-\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}}\right)$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(1,\ -1,\ 0),\ (0,\ 1,\ 0),\ (2,\ 3,\ 1)\}$$

Solution

$$\vec{u}_1 = \frac{(1,\ -1,\ 0)}{\sqrt{2}} \qquad\qquad \vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)$$

$$\vec{w}_2 = (0,\ 1,\ 0) - \left[(0,\ 1,\ 0)\cdot\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)\right]\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right) \qquad\qquad \vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2.\vec{u}_1\right)\vec{u}_1$$

$$= (0,\ 1,\ 0) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)$$

$$= (0,\ 1,\ 0) + \left(\frac{1}{2},\ -\frac{1}{2},\ 0\right)$$

$$= \left(\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$\vec{u}_2 = \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}}}\left(\frac{1}{2},\ \frac{1}{2},\ 0\right) \qquad\qquad \vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\vec{v}_3\cdot\vec{u}_1 = (2,\ 3,\ 1)\cdot\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right) = \frac{2}{\sqrt{2}} - \frac{3}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

$$\vec{v}_3\cdot\vec{u}_2 = (2,\ 3,\ 1)\cdot\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$\vec{w}_3 = (2,\ 3,\ 1) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right) - 2\sqrt{2}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) \qquad\qquad \vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3\cdot\vec{u}_1\right)\vec{u}_1 - \left(\vec{v}_3\cdot\vec{u}_2\right)\vec{u}_2$$

$$= (2,\ 3,\ 1) + \left(\frac{1}{2},\ -\frac{1}{2},\ 0\right) - (2,\ 2,\ 0)$$

$$= \left(\frac{1}{2},\ \frac{1}{2},\ 1\right)$$

$$\vec{u}_3 = \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}\left(\frac{1}{2},\ \frac{1}{2},\ 1\right) \qquad\qquad \vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$= \frac{2}{\sqrt{6}}\left(\frac{1}{2},\ \frac{1}{2},\ 1\right)$$

$$=\boxed{\left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(3,\ 0,\ 4),\ (-1,\ 0,\ 7),\ (2,\ 9,\ 11)\}$$

Solution

$$\vec{u}_1 = \frac{(3,\ 0,\ 4)}{\sqrt{9+16}} \qquad \vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$=\boxed{\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)}$$

$$\vec{w}_2 = (-1,\ 0,\ 7) - \left[(-1,\ 0,\ 7)\bullet\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)\right]\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right) \qquad \vec{w}_2 = \vec{v}_2 - (\vec{v}_2 . \vec{u}_1)\vec{u}_1$$

$$= (-1,\ 0,\ 7) - \left(-\frac{3}{5} + \frac{28}{5}\right)\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)$$

$$= (-1,\ 0,\ 7) - 5\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)$$

$$= (-1,\ 0,\ 7) - (3,\ 0,\ 4)$$

$$=\boxed{(-4,\ 0,\ 3)}$$

$$\vec{u}_2 = \frac{1}{\sqrt{16+9}}(-4,\ 0,\ 3) \qquad \vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$=\boxed{\left(-\frac{4}{5},\ 0,\ \frac{3}{5}\right)}$$

$$\vec{v}_3 \bullet \vec{u}_1 = (2,\ 9,\ 11)\bullet\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right) = \frac{6}{5} + \frac{44}{5}\ \boxed{=10}$$

$$\vec{v}_3 \bullet \vec{u}_2 = (2,\ 9,\ 11)\bullet\left(-\frac{4}{5},\ 0,\ \frac{3}{5}\right) = -\frac{8}{5} + \frac{33}{5}\ \boxed{=5}$$

$$\vec{w}_3 = (2,\ 9,\ 11) - 10\left(\frac{3}{5},\ 0,\ \frac{4}{5}\right) - 5\left(-\frac{4}{5},\ 0,\ \frac{3}{5}\right) \qquad \vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \bullet \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \bullet \vec{u}_2)\vec{u}_2$$

$$=\boxed{(0,\ 9,\ 0)}$$

$$\vec{u}_3 = \frac{1}{9}(0,\ 9,\ 0) \qquad \vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$=\boxed{(0,\ 1,\ 0)}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(1,\ 1,\ 1,\ 1),\ (1,\ 2,\ 1,\ 0),\ (1,\ 3,\ 0,\ 0)\}$$

Solution

$$\boldsymbol{v}_1 = \boldsymbol{u}_1 \ \underline{=(1,\ 1,\ 1,\ 1)|}$$

$$\boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \frac{(1,\ 1,\ 1,\ 1)}{\sqrt{4}} = \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$\boldsymbol{v}_2 = (1,\ 2,\ 1,\ 0) - \frac{(1,\ 2,\ 1,\ 0)\bullet(1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}(1,\ 1,\ 1,\ 1) \qquad \boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1\rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

$$= (1,\ 2,\ 1,\ 0) - \frac{1+2+1}{4}(1,\ 1,\ 1,\ 1)$$

$$= (1,\ 2,\ 1,\ 0) - \frac{4}{4}(1,\ 1,\ 1,\ 1)$$

$$= (1,\ 2,\ 1,\ 0) - (1,\ 1,\ 1,\ 1)$$

$$\underline{= (0,\ 1,\ 0,\ -1)|}$$

$$\boldsymbol{q}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} = \frac{(0,\ 1,\ 0,\ -1)}{\sqrt{1+1}} = \left(0,\ \frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}\right)$$

$$\boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1\rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2\rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2$$

$$= (1,\ 3,\ 0,\ 0) - \frac{(1,\ 3,\ 0,\ 0)\bullet(1,\ 1,\ 1,\ 1)}{\|(1,\ 1,\ 1,\ 1)\|^2}(1,\ 1,\ 1,\ 1) - \frac{(1,\ 3,\ 0,\ 0)\bullet(0,\ 1,\ 0,\ -1)}{\|(0,\ 1,\ 0,\ -1)\|^2}(0,\ 1,\ 0,\ -1)$$

$$= (1,\ 3,\ 0,\ 0) - \frac{4}{4}(1,\ 1,\ 1,\ 1) - \frac{3}{2}(0,\ 1,\ 0,\ -1)$$

$$= (1,\ 3,\ 0,\ 0) - (1,\ 1,\ 1,\ 1) - \left(0,\ \frac{3}{2},\ 0,\ -\frac{3}{2}\right)$$

$$\underline{= \left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)|}$$

$$\boldsymbol{q}_3 = \frac{\boldsymbol{v}_3}{\|\boldsymbol{v}_3\|} = \frac{\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)}{\sqrt{\frac{1}{4}+1+\frac{1}{4}}} = \frac{\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)}{\frac{\sqrt{6}}{2}}$$

$$= \frac{2}{\sqrt{6}}\left(0,\ \frac{1}{2},\ -1,\ \frac{1}{2}\right)$$

$$\underline{= \left(0,\ \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)|}$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(0,\ 2,\ -1,\ 1),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 1,\ 1,\ -1)\}$$

Solution

$$\vec{u}_1=\frac{(0,\ 2,\ -1,\ 1)}{\sqrt{6}} \qquad \vec{u}_1=\frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$=\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$\vec{w}_2=(0,\ 0,\ 1,\ 1)-\left[(0,\ 0,\ 1,\ 1)\bullet\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)\right]\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right) \qquad \vec{w}_2=\vec{v}_2-\left(\vec{v}_2.\vec{u}_1\right)\vec{u}_1$$

$$=(0,\ 0,\ 1,\ 1)-\left[-\frac{1}{\sqrt{6}}+\frac{1}{\sqrt{6}}\right]\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$=(0,\ 0,\ 1,\ 1)-[0]\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$=(0,\ 0,\ 1,\ 1)$$

$$\vec{u}_2=\frac{(0,\ 0,\ 1,\ 1)}{\sqrt{2}} \qquad \vec{u}_2=\frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$=\left(0,\ 0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$\vec{v}_3\bullet\vec{u}_1=(-2,\ 1,\ 1,\ -1)\bullet\left(0,\ \frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)=\frac{2}{\sqrt{6}}-\frac{1}{\sqrt{6}}-\frac{1}{\sqrt{6}}\ \underline{=0}$$

$$\vec{v}_3\bullet\vec{u}_2=(-2,\ 1,\ 1,\ -1)\bullet\left(0,\ 0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\ \underline{=0}$$

$$\vec{w}_3=(-2,\ 1,\ 1,\ -1)-0-0 \qquad \vec{w}_3=\vec{v}_3-\left(\vec{v}_3\bullet\vec{u}_1\right)\vec{u}_1-\left(\vec{v}_3\bullet\vec{u}_2\right)\vec{u}_2$$

$$=(-2,\ 1,\ 1,\ -1)$$

$$\vec{u}_3=\frac{(-2,\ 1,\ 1,\ -1)}{\sqrt{(-2)^2+1^2+1^2+(-1)^2}} \qquad \vec{u}_3=\frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$=\frac{(-2,\ 1,\ 1,\ -1)}{\sqrt{7}}$$

$$=\left(-\frac{2}{\sqrt{7}},\ \frac{1}{\sqrt{7}},\ \frac{1}{\sqrt{7}},\ -\frac{1}{\sqrt{7}}\right)$$

Exercise

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of $\boldsymbol{R}^m$.

$$\boldsymbol{u}_1 = (1,\ 0,\ 0), \quad \boldsymbol{u}_2 = (3,\ 7,\ -2), \quad \boldsymbol{u}_3 = (0,\ 4,\ 1)$$

Solution

$$\vec{v}_1 = \frac{(1,\ 0,\ 0)}{\sqrt{1^2+0^2+0^2}} \qquad\qquad \vec{v}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|}$$

$$= (1,\ 0,\ 0)$$

$$\vec{w}_2 = (3,7,-2) - \left[(3,7,-2)\cdot(1,0,0)\right](1,0,0) \qquad\qquad \vec{w}_2 = \vec{u}_2 - \left(\vec{u}_2 . \vec{v}_1\right)\vec{v}_1$$

$$= (3,7,-2) - 3(1,0,0)$$

$$= (0,\ 7,\ -2)$$

$$\vec{v}_2 = \frac{1}{\sqrt{53}}(0,\ 7,\ -2) \qquad\qquad \vec{v}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{53}}\right)$$

$$\vec{u}_3 \cdot \vec{v}_1 = (0,\ 4,\ 1)\cdot(1,\ 0,\ 0) = 0$$

$$\vec{u}_3 \cdot \vec{v}_2 = (0,\ 4,\ 1)\cdot\left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{53}}\right) = \frac{26}{\sqrt{53}}$$

$$\vec{w}_3 = (0,\ 4,\ 1) - 0 - \frac{26}{\sqrt{53}}\left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{6}}\right) \qquad\qquad \vec{w}_3 = \vec{u}_3 - \left(\vec{u}_3 \cdot \vec{v}_1\right)\vec{v}_1 - \left(\vec{u}_3 \cdot \vec{v}_2\right)\vec{v}_2$$

$$= (0,\ 4,\ 1) - 0 - \frac{26}{\sqrt{53}}\left(0,\ \frac{7}{\sqrt{53}},\ -\frac{2}{\sqrt{6}}\right)$$

$$= (0,\ 4,\ 1) - \left(0,\ \frac{182}{53},\ -\frac{52}{53}\right)$$

$$= \left(0,\ \frac{30}{53},\ \frac{105}{53}\right)$$

$$\vec{v}_3 = \frac{\left(0,\ \frac{30}{53},\ \frac{105}{53}\right)}{\sqrt{\left(\frac{30}{53}\right)^2 + \left(\frac{105}{53}\right)^2}} \qquad\qquad \vec{v}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$$

$$= \frac{53}{\sqrt{11925}}\left(0,\ \frac{30}{53},\ \frac{105}{53}\right)$$

$$= \frac{53}{15\sqrt{53}}\left(0,\ \frac{30}{53},\ \frac{105}{53}\right)$$

$$= \left(0,\ \frac{2}{\sqrt{15}},\ \frac{7}{\sqrt{15}}\right)$$

Exercise

Use the Gram-Schmidt process to find an *orthonormal* basis for the subspaces of $\boldsymbol{R}^m$.

$$\vec{u}_1=(1,\ 1,\ 1,\ 1),\quad \vec{u}_2=(1,\ 2,\ 4,\ 5),\quad \vec{u}_3=(1,\ -3,\ -4,\ -2)$$

Solution

$$\vec{v}_1=\frac{(1,\ 1,\ 1,\ 1)}{\sqrt{4}} \qquad\qquad \vec{v}_1=\frac{\vec{u}_1}{\|\vec{u}_1\|}$$

$$=\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$\vec{w}_2=\vec{u}_2-\left(\vec{u}_2.\vec{v}_1\right)\vec{v}_1$$

$$=(1,\ 2,\ 4,\ 5)-\left[(1,\ 2,\ 4,\ 5)\bullet\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)\right]\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$=(1,\ 2,\ 4,\ 5)-6\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$=(1,\ 2,\ 4,\ 5)-(3,\ 3,\ 3,\ 3)$$

$$=(-2,\ -1,\ 1,\ 2)$$

$$\vec{v}_2=\frac{1}{\sqrt{10}}(-2,\ -1,\ 1,\ 2) \qquad\qquad \vec{v}_2=\frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$=\left(-\frac{2}{\sqrt{10}},\ -\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ \frac{2}{\sqrt{10}}\right)$$

$$\vec{u}_3\bullet\vec{v}_1=(1,\ -3,\ -4,\ -2)\bullet\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$=\frac{1-3-4-2}{2}=-\frac{8}{2}$$

$$=-4$$

$$\vec{u}_3\bullet\vec{v}_2=(1,\ -3,\ -4,\ -2)\bullet\left(-\frac{2}{\sqrt{10}},\ -\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ \frac{2}{\sqrt{10}}\right)$$

$$=\frac{-2+3-4-4}{\sqrt{10}}$$

$$=-\frac{7}{\sqrt{10}}$$

$$\vec{w}_3=\vec{u}_3-\left(\vec{u}_3\bullet\vec{v}_1\right)\vec{v}_1-\left(\vec{u}_3\bullet\vec{v}_2\right)\vec{v}_2$$

$$=(1,\ -3,\ -4,\ -2)+4\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)+\frac{7}{\sqrt{10}}\left(-\frac{2}{\sqrt{10}},\ -\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ \frac{2}{\sqrt{10}}\right)$$

$$=(1,\ -3,\ -4,\ -2)+(2,\ 2,\ 2,\ 2)+\left(-\frac{7}{5},\ -\frac{7}{10},\ \frac{7}{10},\ \frac{7}{5}\right)$$

$$=\left(\frac{8}{5},\ -\frac{17}{10},\ -\frac{27}{10},\ \frac{7}{5}\right)$$

$$\vec{v}_3=\frac{1}{\sqrt{\frac{64}{25}+\frac{289}{100}+\frac{289}{100}+\frac{49}{25}}}\left(\frac{8}{5},\ -\frac{17}{10},\ -\frac{27}{10},\ \frac{7}{5}\right) \qquad \vec{v}_3=\frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$

$$=\frac{1}{\sqrt{\frac{1030}{100}}}\left(\frac{8}{5},\ -\frac{17}{10},\ -\frac{27}{10},\ \frac{7}{5}\right)$$

$$=\left(\frac{16}{\sqrt{1030}},\ -\frac{17}{\sqrt{1030}},\ -\frac{27}{\sqrt{1030}},\ \frac{14}{\sqrt{1030}}\right)$$

Exercise

Use the Gram-Schmidt process to find an *orthonormal* basis for the subspaces of $\boldsymbol{R}^m$.

$$\vec{u}_1=(1,\ 1,\ 1,\ 1),\quad \vec{u}_2=(1,\ 1,\ 2,\ 4),\quad \vec{u}_3=(1,\ 2,\ -4,\ -3)$$

Solution

$$\vec{v}_1=\frac{(1,\ 1,\ 1,\ 1)}{\sqrt{4}} \qquad \vec{v}_1=\frac{\vec{u}_1}{\left\|\vec{u}_1\right\|}$$

$$=\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$\vec{w}_2=(1,\ 1,\ 2,\ 4)-\left[(1,\ 1,\ 2,\ 4)\bullet\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)\right]\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right) \qquad \vec{w}_2=\vec{u}_2-\left(\vec{u}_2.\vec{v}_1\right)\vec{v}_1$$

$$=(1,\ 1,\ 2,\ 4)-4\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$=(1,\ 1,\ 2,\ 4)-(2,\ 2,\ 2,\ 2)$$

$$=(-1,\ -1,\ 0,\ 2)$$

$$\vec{v}_2=\frac{1}{\sqrt{1+1+4}}(-1,\ -1,\ 0,\ 2) \qquad \vec{v}_2=\frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$$

$$=\left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0,\ \frac{2}{\sqrt{6}}\right)$$

$$\vec{u}_3\bullet\vec{v}_1=(1,\ 2,\ -4,\ -3)\bullet\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$=\frac{1+2-4-3}{2}$$

$$=-2$$

$$\vec{u}_3\bullet\vec{v}_2=(1,\ 2,\ -4,\ -3)\bullet\left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0,\ \frac{2}{\sqrt{6}}\right)$$

$$= \frac{-1-2-6}{\sqrt{6}}$$

$$\boxed{= -\frac{9}{\sqrt{6}}}$$

$$\vec{w}_3 = \vec{u}_3 - \left(\vec{u}_3 \cdot \vec{v}_1\right)\vec{v}_1 - \left(\vec{u}_3 \cdot \vec{v}_2\right)\vec{v}_2$$

$$= (1,\ 2,\ -4,\ -3) + 2\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right) + \frac{9}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0,\ \frac{2}{\sqrt{6}}\right)$$

$$= (1,\ 2,\ -4,\ -3) + (1,\ 1,\ 1,\ 1) + \left(-\frac{3}{2},\ -\frac{3}{2},\ 0,\ 3\right)$$

$$\boxed{= \left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)}$$

$$\vec{v}_3 = \frac{1}{\sqrt{\frac{1}{4}+\frac{9}{4}+9+1}}\left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right) \qquad \vec{v}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$

$$= \frac{2}{5\sqrt{2}}\left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)$$

$$\boxed{= \left(\frac{1}{5\sqrt{2}},\ \frac{3}{5\sqrt{2}},\ -\frac{6}{5\sqrt{2}},\ \frac{2}{5\sqrt{2}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an *orthonormal* basis for the subspaces of $\boldsymbol{R}^m$.

$$\boldsymbol{u}_1 = (0,\ 2,\ 1,\ 0); \quad \boldsymbol{u}_2 = (1,\ -1,\ 0,\ 0); \quad \boldsymbol{u}_3 = (1,\ 2,\ 0,\ -1); \quad \boldsymbol{u}_4 = (1,\ 0,\ 0,\ 1)$$

Solution

$$\vec{v}_1 = \frac{(0,\ 2,\ 1,\ 0)}{\sqrt{5}} \qquad \vec{v}_1 = \frac{\vec{u}_1}{\left\|\vec{u}_1\right\|}$$

$$\boxed{= \left(0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right)}$$

$$\vec{w}_2 = (1,-1,\ 0,\ 0) - \left[(1,-1,\ 0,\ 0)\cdot\left(0,\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}},0\right)\right]\left(0,\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}},0\right) \qquad \vec{w}_2 = \vec{u}_2 - \left(\vec{u}_2.\vec{v}_1\right)\vec{v}_1$$

$$= (1,-1,\ 0,\ 0) - \left(-\frac{2}{\sqrt{5}}\right)\left(0,\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}},0\right).$$

$$\boxed{= \left(1,-\frac{1}{5},\ \frac{2}{5},\ 0\right)}$$

$$\vec{v}_2 = \frac{1}{\sqrt{1+\frac{1}{25}+\frac{4}{25}+0}}\left(1,-\frac{1}{5},\ \frac{2}{5},\ 0\right) \qquad \vec{v}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$$

$$= \frac{5}{\sqrt{30}}\left(1, -\frac{1}{5},\ \frac{2}{5},\ 0\right)$$

$$= \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)$$

$$u_3 \bullet v_1 = (1,\ 2,\ 0,\ -1) \bullet \left(0, \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right) = \frac{4}{\sqrt{5}}$$

$$u_3 \bullet v_2 = (1,\ 2,\ 0,\ -1) \bullet \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right) = \frac{5}{\sqrt{30}} - \frac{2}{\sqrt{30}} = \frac{3}{\sqrt{30}}$$

$$w_3 = u_3 - (u_3 \bullet v_1)v_1 - (u_3 \bullet v_2)v_2$$

$$= (1,\ 2,\ 0,\ -1) - \left(\frac{4}{\sqrt{5}}\right)\left(0, \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right) - \left(\frac{3}{\sqrt{30}}\right)\left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right)$$

$$= (1,\ 2,\ 0,\ -1) - \left(0, \frac{8}{5},\ \frac{4}{5},\ 0\right) - \left(\frac{1}{2}, -\frac{1}{10},\ \frac{1}{5},\ 0\right)$$

$$= \left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$$

$$v_3 = \frac{w_3}{\|w_3\|} = \frac{\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (-1)^2 + (-1)^2}}$$

$$= \frac{1}{\sqrt{\frac{5}{2}}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$$

$$= \frac{\sqrt{2}}{\sqrt{5}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right) = \frac{2}{\sqrt{10}}\left(\frac{1}{2},\ \frac{1}{2},\ -1,\ -1\right)$$

$$= \left(\frac{1}{\sqrt{10}},\ \frac{1}{\sqrt{10}},\ -\frac{2}{\sqrt{10}},\ -\frac{2}{\sqrt{10}}\right)$$

$$u_4 \bullet v_1 = (1,0,0,1) \bullet \left(0, \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0\right) = 0$$

$$u_4 \bullet v_2 = (1,0,0,1) \bullet \left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}},\ \frac{2}{\sqrt{30}},\ 0\right) = \frac{5}{\sqrt{30}}$$

$$u_4 \bullet v_3 = (1,0,0,1) \bullet \left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) = -\frac{1}{\sqrt{10}}$$

$$w_4 = u_4 - (u_4 \bullet v_1)v_1 - (u_4 \bullet v_2)v_2 - (u_4 \bullet v_3)v_3$$

$$= (1,2,0,-1) - (0) - \left(\frac{5}{\sqrt{30}}\right)\left(\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0\right) + \left(\frac{1}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right)$$

$$= (1,2,0,-1) - \left(\frac{5}{6}, -\frac{1}{6}, \frac{1}{3}, 0\right) + \left(\frac{1}{10}, \frac{1}{10}, -\frac{1}{5}, -\frac{1}{5}\right)$$

$$=\left(\frac{4}{15},\frac{4}{15},-\frac{8}{15},\frac{4}{5}\right)$$

$$\boxed{v_4}=\frac{w_4}{\|w_4\|}=\frac{\left(\frac{4}{15},\frac{4}{15},-\frac{8}{15},\frac{4}{5}\right)}{\sqrt{\left(\frac{4}{15}\right)^2+\left(\frac{4}{15}\right)^2+\left(-\frac{8}{15}\right)^2+\left(\frac{4}{5}\right)^2}}$$

$$=\frac{1}{\sqrt{\frac{240}{225}}}\left(\frac{4}{15},\frac{4}{15},-\frac{8}{15},\frac{4}{5}\right)$$

$$=\frac{1}{\frac{4}{\sqrt{15}}}\left(\frac{4}{15},\frac{4}{15},-\frac{8}{15},\frac{4}{5}\right)$$

$$=\frac{15}{4\sqrt{15}}\left(\frac{4}{15},\frac{4}{15},-\frac{8}{15},\frac{4}{5}\right)$$

$$\boxed{=\left(\frac{1}{\sqrt{15}},\frac{1}{\sqrt{15}},-\frac{2}{\sqrt{15}},\frac{3}{\sqrt{15}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(1,\ 1,\ 0),\ (1,\ 2,\ 0),\ (0,\ 1,\ 2)\}$$

Solution

$$\boxed{\vec{v}_1=(1,\ 1,\ 0)}$$

$$\vec{v}_2=(0,\ 2,\ 1)-\frac{(1,\ 2,\ 0)\bullet(1,\ 1,\ 0)}{1+1+0}(1,\ 1,\ 0) \qquad \boldsymbol{v}_2=\boldsymbol{u}_2-\frac{\langle \boldsymbol{u}_2,\boldsymbol{v}_1\rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

$$=(0,\ 2,\ 1)-\frac{3}{2}(1,\ 1,\ 0)$$

$$\boxed{=\left(-\frac{3}{2},\ \frac{1}{2},\ 1\right)}$$

$$\frac{\langle \boldsymbol{u}_3,\vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1=\frac{(0,\ 1,\ 2)\bullet(1,\ 1,\ 0)}{2}(1,\ 1,\ 0)$$

$$=\left(\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$\frac{\langle \boldsymbol{u}_3,\vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2=\frac{1}{\frac{9}{4}+\frac{1}{4}+1}(0,\ 1,\ 2)\bullet\left(-\frac{3}{2},\ \frac{1}{2},\ 1\right)\left(-\frac{3}{2},\ \frac{1}{2},\ 1\right)$$

$$=\frac{2}{7}\frac{5}{2}\left(-\frac{3}{2},\ \frac{1}{2},\ 1\right)$$

$$=\left(-\frac{15}{14},\ \frac{5}{14},\ \frac{5}{7}\right)$$

$$\vec{v}_3=(0,\ 1,\ 2)-\left(\frac{1}{2},\ \frac{1}{2},\ 0\right)-\left(-\frac{15}{14},\ \frac{5}{14},\ \frac{5}{7}\right) \qquad \boldsymbol{v}_3=\boldsymbol{u}_3-\frac{\langle \boldsymbol{u}_3,\boldsymbol{v}_1\rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1-\frac{\langle \boldsymbol{u}_3,\boldsymbol{v}_2\rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2$$

$$=\left(\frac{4}{7},\ \frac{1}{7},\ \frac{9}{7}\right)$$

$$\boldsymbol{q}_1=\frac{1}{\sqrt{2}}(1,\ 1,\ 0) \qquad \boldsymbol{q}_1=\frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|}$$

$$=\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\boldsymbol{q}_2=\frac{1}{\sqrt{\frac{9}{4}+\frac{1}{4}+1}}\left(-\frac{3}{2},\ \frac{1}{2},\ 1\right) \qquad \boldsymbol{q}_2=\frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|}$$

$$=\frac{2}{\sqrt{14}}\left(-\frac{3}{2},\ \frac{1}{2},\ 1\right)$$

$$=\left(-\frac{2}{\sqrt{14}},\ \frac{1}{\sqrt{14}},\ \frac{2}{\sqrt{14}}\right)$$

$$\boldsymbol{q}_3=\frac{1}{\sqrt{\frac{16}{49}+\frac{1}{49}+\frac{81}{49}}}\left(\frac{4}{7},\ \frac{1}{7},\ \frac{9}{7}\right) \qquad \boldsymbol{q}_3=\frac{\boldsymbol{v}_3}{\|\boldsymbol{v}_3\|}$$

$$=\frac{7}{\sqrt{98}}\left(\frac{4}{7},\ \frac{1}{7},\ \frac{9}{7}\right)$$

$$=\frac{7}{7\sqrt{2}}\left(\frac{4}{7},\ \frac{1}{7},\ \frac{9}{7}\right)$$

$$=\left(\frac{4}{7\sqrt{2}},\ \frac{1}{7\sqrt{2}},\ \frac{9}{7\sqrt{2}}\right)$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(1,\ -2,\ 2),\ (2,\ 2,\ 1),\ (2,\ -1,\ -2)\}$$

Solution

$$\vec{v}_1=\vec{u}_1=(1,\ -2,\ 2)$$

$$\vec{v}_2=(2,\ 2,\ 1)-\frac{(2,\ 2,\ 1)\bullet(1,\ -2,\ 2)}{9}(1,\ -2,\ 2) \qquad \boldsymbol{v}_2=\boldsymbol{u}_2-\frac{\langle \boldsymbol{u}_2,\boldsymbol{v}_1\rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

$$=(2,\ 2,\ 1)-\frac{0}{9}(1,\ -2,\ 2)$$

$$\underline{=(2,\ 2,\ 1)}$$

$$\frac{\langle \boldsymbol{u}_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1=\frac{(2,\ -1,\ -2)\bullet(1,\ -2,\ 2)}{9}(1,\ -2,\ 2)$$

$$=\frac{0}{9}(1,\ -2,\ 2)$$

$$=(0,\ 0,\ 0)$$

$$\frac{\langle \boldsymbol{u}_3, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2=\frac{1}{9}\left[(2,\ -1,\ -2)\bullet(2,\ 2,\ 1)\right](2,\ 2,\ 1)$$

$$=(0,\ 0,\ 0)$$

$$\vec{v}_3=(2,\ -1,\ -2)-(0,\ 0,\ 0)-(0,\ 0,\ 0) \qquad \boldsymbol{v}_3=\boldsymbol{u}_3-\frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1\rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1-\frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2\rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2$$

$$\underline{=(2,\ -1,\ -2)}$$

$$\boldsymbol{q}_1=\frac{1}{3}(1,\ -2,\ 2) \qquad \boldsymbol{q}_1=\frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|}$$

$$\underline{=\left(\frac{1}{3},\ -\frac{2}{3},\ \frac{2}{3}\right)}$$

$$\boldsymbol{q}_2=\frac{1}{3}(2,\ 2,\ 1) \qquad \boldsymbol{q}_2=\frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|}$$

$$\underline{=\left(\frac{2}{3},\ \frac{2}{3},\ \frac{1}{3}\right)}$$

$$\boldsymbol{q}_3=\frac{1}{3}(2,\ -1,\ -2) \qquad \boldsymbol{q}_3=\frac{\boldsymbol{v}_3}{\|\boldsymbol{v}_3\|}$$

$$\underline{=\left(\frac{2}{3},\ -\frac{1}{3},\ -\frac{2}{3}\right)}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(1,\ 0,\ 0),\ (1,\ 1,\ 1),\ (1,\ 1,\ -1)\}$$

Solution

$$\underline{\vec{v}_1=\vec{u}_1=(1,\ 0,\ 0)}$$

$$\vec{v}_2 = (1,\ 1,\ 1) - \frac{(1,\ 1,\ 1)\bullet(1,\ 0,\ 0)}{1}(1,\ 0,\ 0) \qquad\qquad \boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

$$= (1,\ 1,\ 1) - (1,\ 0,\ 0)$$

$$\boxed{= (0,\ 1,\ 1)}$$

$$\frac{\langle \boldsymbol{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 1,\ -1)\bullet(1,\ 0,\ 0)}{1}(1,\ 0,\ 0)$$

$$= (1,\ 0,\ 0)$$

$$\frac{\langle \boldsymbol{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{1}\left[(1,\ 1,\ -1)\bullet(0,\ 1,\ 1)\right](0,\ 1,\ 1)$$

$$= 0(0,\ 1,\ 1)$$

$$= (0,\ 0,\ 0)$$

$$\vec{v}_3 = (1,\ 1,\ -1) - (1,\ 0,\ 0) - (0,\ 0,\ 0) \qquad\qquad \boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2$$

$$\boxed{= (0,\ 1,\ -1)}$$

$$\boldsymbol{q}_1 = \frac{1}{1}(1,\ 0,\ 0) \qquad\qquad \boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|}$$

$$\boxed{= (1,\ 0,\ 0)}$$

$$\boldsymbol{q}_2 = \frac{1}{\sqrt{2}}(0,\ 1,\ 1) \qquad\qquad \boldsymbol{q}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|}$$

$$\boxed{= \left(0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)}$$

$$\boldsymbol{q}_3 = \frac{1}{\sqrt{2}}(0,\ 1,\ -1) \qquad\qquad \boldsymbol{q}_3 = \frac{\boldsymbol{v}_3}{\|\boldsymbol{v}_3\|}$$

$$\boxed{= \left(0,\ \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(4,\ -3,\ 0),\ (1,\ 2,\ 0),\ (0,\ 0,\ 4)\}$$

Solution

$$\boxed{\vec{v}_1 = \vec{u}_1 = (4,\ -3,\ 0)}$$

$$\vec{v}_2 = (1,\ 2,\ 0) - \frac{(1,\ 2,\ 0)\bullet(4,\ -3,\ 0)}{25}(4,\ -3,\ 0) \qquad \boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

$$= (1,\ 2,\ 0) + \frac{2}{25}(4,\ -3,\ 0)$$

$$\boxed{= \left(\frac{33}{25},\ \frac{44}{25},\ 0\right)}$$

$$\frac{\langle \boldsymbol{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(0,\ 0,\ 4)\bullet(4,\ -3,\ 0)}{25}(4,\ -3,\ 0)$$

$$= (0,\ 0,\ 0)$$

$$\frac{\langle \boldsymbol{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{225}{3{,}025}\left[(0,\ 0,\ 4)\bullet\left(\frac{33}{25},\ \frac{44}{25},\ 0\right)\right]\left(\frac{33}{25},\ \frac{44}{25},\ 0\right)$$

$$= (0,\ 0,\ 0)$$

$$\vec{v}_3 = (0,\ 0,\ 4) - (0,\ 0,\ 0) - (0,\ 0,\ 0) \qquad \boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2$$

$$\boxed{= (0,\ 0,\ 4)}$$

$$\boldsymbol{q}_1 = \frac{1}{\sqrt{16+9}}(4,\ -3,\ 0) \qquad \boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|}$$

$$\boxed{= \left(\frac{4}{5},\ -\frac{3}{5},\ 0\right)}$$

$$\boldsymbol{q}_2 = \frac{25}{\sqrt{3{,}025}}\left(\frac{33}{25},\ \frac{44}{25},\ 0\right) \qquad \boldsymbol{q}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|}$$

$$= \frac{25}{55}\left(\frac{33}{25},\ \frac{44}{25},\ 0\right)$$

$$\boxed{= \left(\frac{3}{5},\ \frac{4}{5},\ 0\right)}$$

$$\boldsymbol{q}_3 = \frac{1}{4}(0,\ 0,\ 4) \qquad \boldsymbol{q}_3 = \frac{\boldsymbol{v}_3}{\|\boldsymbol{v}_3\|}$$

$$\boxed{= (0,\ 0,\ 1)}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(0,\ 1,\ 2),\ (2,\ 0,\ 0),\ (1,\ 1,\ 1)\}$$

Solution

$$\boxed{\vec{v}_1 = \vec{u}_1 = (0,\ 1,\ 2)}$$

$$\vec{v}_2 = (2,\ 0,\ 0) - \frac{(2,\ 0,\ 0)\bullet(0,\ 1,\ 2)}{5}(0,\ 1,\ 2) \qquad \boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1\rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

$$= (2,\ 0,\ 0) + \frac{0}{5}(0,\ 1,\ 2)$$

$$\boxed{= (2,\ 0,\ 0)}$$

$$\frac{\langle \boldsymbol{u}_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 1,\ 1)\bullet(0,\ 1,\ 2)}{5}(0,\ 1,\ 2)$$

$$= \frac{3}{5}(0,\ 1,\ 2)$$

$$= \left(0,\ \frac{3}{5},\ \frac{6}{5}\right)$$

$$\frac{\langle \boldsymbol{u}_3, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{4}\left[(1,\ 1,\ 1)\bullet(2,\ 0,\ 0)\right](2,\ 0,\ 0)$$

$$= \frac{1}{2}(2,\ 0,\ 0)$$

$$= (1,\ 0,\ 0)$$

$$\vec{v}_3 = (1,\ 1,\ 1) - (1,\ 0,\ 0) - \left(0,\ \frac{3}{5},\ \frac{6}{5}\right) \qquad \boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1\rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2\rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2$$

$$\boxed{= \left(0,\ \frac{2}{5},\ -\frac{1}{5}\right)}$$

$$\boldsymbol{q}_1 = \frac{1}{\sqrt{5}}(0,\ 1,\ 2) \qquad \boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|}$$

$$\boxed{= \left(0,\ \frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)}$$

$$\boldsymbol{q}_2 = \frac{1}{2}(2,\ 0,\ 0) \qquad \boldsymbol{q}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|}$$

$$\boxed{= (1,\ 0,\ 0)}$$

$$q_3 = \frac{5}{\sqrt{5}}\left(0,\ \frac{2}{5},\ -\frac{1}{5}\right) \qquad\qquad q_3 = \frac{v_3}{\|v_3\|}$$

$$= \boxed{\left(0,\ \frac{2}{\sqrt{5}},\ -\frac{1}{\sqrt{5}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(0,\ 1,\ 1),\ (1,\ 1,\ 0),\ (1,\ 0,\ 1)\}$$

Solution

$$\boxed{\vec{v}_1 = \vec{u}_1 = (0,\ 1,\ 1)}$$

$$\vec{v}_2 = (1,\ 1,\ 0) - \frac{(1,\ 1,\ 0)\cdot(0,\ 1,\ 1)}{2}(0,\ 1,\ 1) \qquad\qquad v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2}v_1$$

$$= (1,\ 1,\ 0) - \frac{1}{2}(0,\ 1,\ 1)$$

$$= \boxed{\left(1,\ \frac{1}{2},\ -\frac{1}{2}\right)}$$

$$\frac{\langle u_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 0,\ 1)\cdot(0,\ 1,\ 1)}{2}(0,\ 1,\ 1)$$

$$= \left(0,\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$\frac{\langle u_3, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{4}{6}\left[(1,\ 0,\ 1)\cdot\left(1,\ \frac{1}{2},\ -\frac{1}{2}\right)\right]\left(1,\ \frac{1}{2},\ -\frac{1}{2}\right)$$

$$= \frac{1}{3}\left(1,\ \frac{1}{2},\ -\frac{1}{2}\right)$$

$$= \left(\frac{1}{3},\ \frac{1}{6},\ -\frac{1}{6}\right)$$

$$\vec{v}_3 = (1,\ 0,\ 1) - \left(0,\ \frac{1}{2},\ \frac{1}{2}\right) - \left(\frac{1}{3},\ \frac{1}{6},\ -\frac{1}{6}\right) \qquad\qquad v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2}v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2}v_2$$

$$= \boxed{\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3}\right)}$$

$$q_1 = \frac{1}{\sqrt{2}}(0,\ 1,\ 1) \qquad\qquad q_1 = \frac{v_1}{\|v_1\|}$$

$$= \boxed{\left(0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)}$$

$$q_2 = \frac{2}{\sqrt{6}}\left(1,\ \frac{1}{2},\ -\frac{1}{2}\right) \qquad q_2 = \frac{v_2}{\|v_2\|}$$

$$= \boxed{\left(\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}}\right)}$$

$$q_3 = \frac{3}{\sqrt{12}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3}\right) \qquad q_3 = \frac{v_3}{\|v_3\|}$$

$$= \boxed{\left(\frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(1,\ 2,\ -2),\ (1,\ 0,\ -4),\ (5,\ 2,\ 0),\ (1,\ 1,\ -1)\}$$

Solution

$$\boxed{\vec{v}_1 = \vec{u}_1 = (1,\ 2,\ -2)}$$

$$\vec{v}_2 = (1,\ 0,\ -4) - \frac{(1,\ 0,\ -4)\bullet(1,\ 2,\ -2)}{9}(1,\ 2,\ -2) \qquad v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2}v_1$$

$$= (1,\ 0,\ -4) - (1,\ 2,\ -2)$$

$$= \boxed{(0,\ -2,\ -2)}$$

$$\frac{\langle u_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(5,\ 2,\ 0)\bullet(1,\ 2,\ -2)}{9}(1,\ 2,\ -2)$$

$$= (1,\ 2,\ -2)$$

$$\frac{\langle u_3, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{8}\left[(5,\ 2,\ 0)\bullet(0,\ -2,\ -2)\right](0,\ -2,\ -2)$$

$$= -\frac{1}{2}(0,\ -2,\ -2)$$

$$= (0,\ 1,\ 1)$$

$$\vec{v}_3 = (5,\ 2,\ 0) - (1,\ 2,\ -2) - (0,\ 1,\ 1) \qquad v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2}v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2}v_2$$

$$\boxed{=(4,\ -1,\ 1)}$$

$$\frac{\langle \boldsymbol{u}_4, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 1,\ -1)\bullet(1,\ 2,\ -2)}{9}(1,\ 2,\ -2)$$

$$=\frac{5}{9}(1,\ 2,\ -2)$$

$$=\left(\frac{5}{9},\ \frac{10}{9},\ -\frac{10}{9}\right)$$

$$\frac{\langle \boldsymbol{u}_4, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{8}\left[(1,\ 1,\ -1)\bullet(0,\ -2,\ -2)\right](0,\ -2,\ -2)$$

$$=(0,\ 0,\ 0)$$

$$\frac{\langle \boldsymbol{u}_4, \vec{v}_3\rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \frac{1}{18}\left[(1,\ 1,\ -1)\bullet(4,\ -1,\ 1)\right](4,\ -1,\ 1)$$

$$=\frac{1}{9}(4,\ -1,\ 1)$$

$$=\left(\frac{4}{9},\ -\frac{1}{9},\ \frac{1}{9}\right)$$

$$\vec{v}_4 = (1,\ 1,\ -1)-\left(\frac{5}{9},\ \frac{10}{9},\ -\frac{10}{9}\right)-\left(\frac{4}{9},\ -\frac{1}{9},\ \frac{1}{9}\right) \quad \boldsymbol{v}_4 = \boldsymbol{u}_4 - \frac{\langle \boldsymbol{u}_4, \boldsymbol{v}_1\rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_4, \boldsymbol{v}_2\rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2 - \frac{\langle \boldsymbol{u}_4, \boldsymbol{v}_3\rangle}{\|\boldsymbol{v}_3\|^2}\boldsymbol{v}_3$$

$$\boxed{=(0,\ 0,\ 0)}$$

$$\boldsymbol{q}_1 = \frac{1}{3}(1,\ 2,\ -2) \qquad \boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|}$$

$$\boxed{=\left(\frac{1}{3},\ \frac{2}{3},\ -\frac{2}{3}\right)}$$

$$\boldsymbol{q}_2 = \frac{1}{2\sqrt{2}}(0,\ -2,\ -2) \qquad \boldsymbol{q}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|}$$

$$\boxed{=\left(0,\ -\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right)}$$

$$\boldsymbol{q}_3 = \frac{1}{3\sqrt{2}}(4,\ -1,\ 1) \qquad \boldsymbol{q}_3 = \frac{\boldsymbol{v}_3}{\|\boldsymbol{v}_3\|}$$

$$\boxed{=\left(\frac{4}{3\sqrt{2}},\ -\frac{1}{3\sqrt{2}},\ \frac{1}{3\sqrt{2}}\right)}$$

$$\boxed{\boldsymbol{q}_4 = (0,\ 0,\ 0)} \qquad \boldsymbol{q}_4 = \frac{\boldsymbol{v}_4}{\|\boldsymbol{v}_4\|}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(-3,\ 1,\ 2),\ (1,\ 1,\ 1),\ (2,\ 0,\ -1),\ (1,\ -3,\ 2)\}$$

Solution

$$\boxed{\vec{v}_1 = \vec{u}_1 = (-3,\ 1,\ 2)}$$

$$\vec{v}_2 = (1,\ 1,\ 1) - \frac{(1,\ 1,\ 1)\bullet(-3,\ 1,\ 2)}{14}(-3,\ 1,\ 2)$$

$$\boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

$$= (1,\ 1,\ 1) - \frac{0}{14}(1,\ 2,\ -2)$$

$$\boxed{= (1,\ 1,\ 1)}$$

$$\frac{\langle \boldsymbol{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(2,\ 0,\ -1)\bullet(-3,\ 1,\ 2)}{14}(-3,\ 1,\ 2)$$

$$= -\frac{4}{7}(-3,\ 1,\ 2)$$

$$= \left(\frac{12}{7},\ -\frac{4}{7},\ -\frac{8}{7}\right)$$

$$\frac{\langle \boldsymbol{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{1}{3}\left[(2,\ 0,\ -1)\bullet(1,\ 1,\ 1)\right](1,\ 1,\ 1)$$

$$= \frac{1}{3}(1,\ 1,\ 1)$$

$$= \left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$\vec{v}_3 = (2,\ 0,\ -1) - \left(\frac{12}{7},\ -\frac{4}{7},\ -\frac{8}{7}\right) - \left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$\boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2$$

$$\boxed{= \left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)}$$

$$\frac{\langle \boldsymbol{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ -3,\ 2)\bullet(-3,\ 1,\ 2)}{14}(-3,\ 1,\ 2)$$

$$= -\frac{1}{7}(-3,\ 1,\ 2)$$

$$= \left(\frac{3}{7},\ -\frac{1}{7},\ -\frac{2}{7}\right)$$

$$\frac{\langle \boldsymbol{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{(1,\ -3,\ 2)\bullet(1,\ 1,\ 1)}{3}(1,\ 1,\ 1)$$

$$= (0,\ 0,\ 0)$$

$$\frac{\langle \boldsymbol{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \frac{441}{42}\left[(1,\ -3,\ 2)\cdot\left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)\right]\left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)$$

$$= \frac{441}{42}\left(-\frac{24}{21}\right)\left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right)$$

$$= \left(\frac{4}{7},\ -\frac{20}{7},\ \frac{16}{7}\right)$$

$$\boldsymbol{v}_4 = \boldsymbol{u}_4 - \frac{\langle \boldsymbol{u}_4, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_4, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2 - \frac{\langle \boldsymbol{u}_4, \boldsymbol{v}_3 \rangle}{\|\boldsymbol{v}_3\|^2}\boldsymbol{v}_3$$

$$\vec{v}_4 = (1,\ -3,\ 2) - \left(\frac{3}{7},\ -\frac{1}{7},\ -\frac{2}{7}\right) - (0,\ 0,\ 0) - \left(\frac{4}{7},\ -\frac{20}{7},\ \frac{16}{7}\right)$$

$$= (0,\ 0,\ 0)$$

$$\boldsymbol{q}_1 = \frac{1}{\sqrt{14}}(-3,\ 1,\ 2) \qquad \boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|}$$

$$= \left(-\frac{3}{\sqrt{14}},\ \frac{1}{\sqrt{14}},\ \frac{2}{\sqrt{14}}\right)$$

$$\boldsymbol{q}_2 = \frac{1}{\sqrt{3}}(1,\ 1,\ 1) \qquad \boldsymbol{q}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\boldsymbol{q}_3 = \frac{21}{\sqrt{42}}\left(-\frac{1}{21},\ \frac{5}{21},\ -\frac{4}{21}\right) \qquad \boldsymbol{q}_3 = \frac{\boldsymbol{v}_3}{\|\boldsymbol{v}_3\|}$$

$$= \left(-\frac{1}{\sqrt{42}},\ \frac{5}{\sqrt{42}},\ -\frac{4}{\sqrt{42}}\right)$$

$$\boldsymbol{q}_4 = (0,\ 0,\ 0) \qquad \boldsymbol{q}_4 = \frac{\boldsymbol{v}_4}{\|\boldsymbol{v}_4\|}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(2,\ 1,\ 1),\ (0,\ 3,\ -1),\ (3,\ -4,\ -2),\ (-1,\ -1,\ 3)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (2,\ 1,\ 1)$$

$$\vec{v}_2 = (0,\ 3,\ -1) - \frac{(0,\ 3,\ -1)\bullet(2,\ 1,\ 1)}{6}(2,\ 1,\ 1) \qquad \boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

$$= (0,\ 3,\ -1) - \frac{1}{3}(2,\ 1,\ 1)$$

$$\boxed{= \left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)}$$

$$\frac{\langle \boldsymbol{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(3,\ -4,\ -2)\bullet(2,\ 1,\ 1)}{6}(2,\ 1,\ 1)$$

$$= (0,\ 0,\ 0)$$

$$\frac{\langle \boldsymbol{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{9}{84}\left[(3,\ -4,\ -2)\bullet\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)\right]\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)$$

$$= \frac{3}{28}(-10)\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)$$

$$= \left(\frac{5}{7},\ -\frac{20}{7},\ \frac{10}{7}\right)$$

$$\vec{v}_3 = (3,\ -4,\ -2) - (0,\ 0,\ 0) - \left(\frac{5}{7},\ -\frac{20}{7},\ \frac{10}{7}\right) \qquad \boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2$$

$$\boxed{= \left(\frac{16}{7},\ -\frac{8}{7},\ -\frac{24}{7}\right)}$$

$$\frac{\langle \boldsymbol{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(-1,\ -1,\ 3)\bullet(2,\ 1,\ 1)}{6}(2,\ 1,\ 1)$$

$$= (0,\ 0,\ 0)$$

$$\frac{\langle \boldsymbol{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{9}{84}\left[(-1,\ -1,\ 3)\bullet\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)\right]\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)$$

$$= \frac{3}{28}\left(-\frac{18}{3}\right)\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right)$$

$$= \left(\frac{3}{7},\ -\frac{12}{7},\ \frac{6}{7}\right)$$

$$\frac{\langle \boldsymbol{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \frac{49}{896}\left[(-1,\ -1,\ 3)\bullet\left(\frac{16}{7},\ -\frac{8}{7},\ -\frac{24}{7}\right)\right]\left(\frac{16}{7},\ -\frac{8}{7},\ -\frac{24}{7}\right)$$

$$= \frac{7}{128}\left(-\frac{80}{7}\right)\left(\frac{16}{7},\ -\frac{8}{7},\ -\frac{24}{7}\right)$$

$$= \left(-\frac{10}{7},\ \frac{5}{7},\ \frac{15}{7}\right)$$

$$v_4 = u_4 - \frac{\langle u_4, v_1\rangle}{\|v_1\|^2}v_1 - \frac{\langle u_4, v_2\rangle}{\|v_2\|^2}v_2 - \frac{\langle u_4, v_3\rangle}{\|v_3\|^2}v_3$$

$$\vec{v}_4 = (-1,\ -1,\ 3) - (0,\ 0,\ 0) - \left(\frac{3}{7},\ -\frac{12}{7},\ \frac{6}{7}\right) - \left(-\frac{10}{7},\ \frac{5}{7},\ \frac{15}{7}\right)$$

$$= (0,\ 0,\ 0)$$

$$q_1 = \frac{1}{\sqrt{6}}(2,\ 1,\ 1) \qquad q_1 = \frac{v_1}{\|v_1\|}$$

$$= \left(\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)$$

$$q_2 = \frac{3}{2\sqrt{21}}\left(-\frac{2}{3},\ \frac{8}{3},\ -\frac{4}{3}\right) \qquad q_2 = \frac{v_2}{\|v_2\|}$$

$$= \left(-\frac{1}{\sqrt{21}},\ \frac{4}{\sqrt{21}},\ -\frac{2}{\sqrt{21}}\right)$$

$$q_3 = \frac{7}{8\sqrt{14}}\left(\frac{16}{7},\ -\frac{8}{7},\ -\frac{24}{7}\right) \qquad q_3 = \frac{v_3}{\|v_3\|}$$

$$= \left(\frac{2}{\sqrt{14}},\ -\frac{1}{\sqrt{14}},\ -\frac{3}{\sqrt{14}}\right)$$

$$q_4 = (0,\ 0,\ 0) \qquad q_4 = \frac{v_4}{\|v_4\|}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of R^m.

$$\vec{u}_1 = (1,\ 1,\ 0,\ -1), \quad \vec{u}_2 = (1,\ 3,\ 0,\ 1), \quad \vec{u}_3 = (4,\ 2,\ 2,\ 0)$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (1,\ 1,\ 0,\ -1)$$

$$\vec{v}_2 = (1,\ 3,\ 0,\ 1) - \frac{(1,\ 3,\ 0,\ 1)\cdot(1,\ 1,\ 0,\ -1)}{3}(1,\ 1,\ 0,\ -1) \qquad v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2}v_1$$

$$= (1,\ 3,\ 0,\ 1) - (1,\ 1,\ 0,\ -1)$$

$$= (0,\ 2,\ 0,\ 2)$$

$$\frac{\langle u_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(4,\ 2,\ 2,\ 0)\cdot(1,\ 1,\ 0,\ -1)}{3}(1,\ 1,\ 0,\ -1)$$

$$= (2,\ 2,\ 0,\ -2)$$

$$\frac{\langle \boldsymbol{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{(4,\ 2,\ 2,\ 0)\bullet(0,\ 2,\ 0,\ 2)}{8}(0,\ 2,\ 0,\ 2)$$

$$= \frac{1}{2}(0,\ 2,\ 0,\ 2)$$

$$= (0,\ 1,\ 0,\ 1)$$

$$\vec{v}_3 = (4,\ 2,\ 2,\ 0) - (2,\ 2,\ 0,\ -2) - (0,\ 1,\ 0,\ 1) \qquad \boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2$$

$$\boxed{= (2,\ -1,\ 2,\ 1)}$$

$$\boldsymbol{q}_1 = \frac{1}{\sqrt{3}}(1,\ 1,\ 0,\ -1) \qquad \boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|}$$

$$\boxed{= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ 0,\ -\frac{1}{\sqrt{3}}\right)}$$

$$\boldsymbol{q}_2 = \frac{1}{2\sqrt{2}}(0,\ 2,\ 0,\ 2) \qquad \boldsymbol{q}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|}$$

$$\boxed{= \left(0,\ \frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)}$$

$$\boldsymbol{q}_3 = \frac{1}{\sqrt{10}}(2,\ -1,\ 2,\ 1) \qquad \boldsymbol{q}_3 = \frac{\boldsymbol{v}_3}{\|\boldsymbol{v}_3\|}$$

$$\boxed{= \left(\frac{2}{\sqrt{10}},\ -\frac{1}{\sqrt{10}},\ \frac{2}{\sqrt{10}},\ -\frac{1}{\sqrt{10}}\right)}$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\boldsymbol{R}^m$.

$$\vec{u}_1 = (1,\ 1,\ 1,\ 1), \quad \vec{u}_2 = (1,\ 1,\ 2,\ 4), \quad \vec{u}_3 = (1,\ 2,\ -4,\ -3)$$

Solution

$$\boxed{\vec{v}_1 = \vec{u}_1 = (1,\ 1,\ 1,\ 1)}$$

$$\vec{v}_2 = (1,\ 1,\ 2,\ 4) - \frac{(1,\ 1,\ 2,\ 4)\bullet(1,\ 1,\ 1,\ 1)}{4}(1,\ 1,\ 1,\ 1) \qquad \boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

$$= (1,\ 1,\ 2,\ 4) - 2(1,\ 1,\ 1,\ 1)$$

$$\boxed{= (-1,\ -1,\ 0,\ 2)}$$

$$\frac{\langle \boldsymbol{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(1,\ 2,\ -4,\ -3)\bullet(1,\ 1,\ 1,\ 1)}{4}(1,\ 1,\ 1,\ 1)$$

$$= (-1,\ -1,\ -1,\ -1)$$

$$\frac{\langle \boldsymbol{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{(1,\ 2,\ -4,\ -3)\bullet(-1,\ -1,\ 0,\ 2)}{6}(-1,\ -1,\ 0,\ 2)$$

$$= -\frac{3}{2}(-1,\ -1,\ 0,\ 2)$$

$$= \left(\frac{3}{2},\ \frac{3}{2},\ 0,\ -3\right)$$

$$\vec{v}_3 = (1,\ 2,\ -4,\ -3) - (-1,\ -1,\ -1,\ -1) - \left(\frac{3}{2},\ \frac{3}{2},\ 0,\ -3\right) \qquad \boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2$$

$$= \left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)$$

$$\boldsymbol{q}_1 = \frac{1}{2}(1,\ 1,\ 1,\ 1) \qquad \boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|}$$

$$= \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$$

$$\boldsymbol{q}_2 = \frac{1}{\sqrt{6}}(-1,\ -1,\ 0,\ 2) \qquad \boldsymbol{q}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|}$$

$$= \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0,\ \frac{2}{\sqrt{6}}\right)$$

$$\boldsymbol{q}_3 = \frac{2}{\sqrt{50}}\left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right) \qquad \boldsymbol{q}_3 = \frac{\boldsymbol{v}_3}{\|\boldsymbol{v}_3\|}$$

$$= \frac{2}{5\sqrt{2}}\left(\frac{1}{2},\ \frac{3}{2},\ -3,\ 1\right)$$

$$= \left(\frac{1}{5\sqrt{2}},\ \frac{3}{5\sqrt{2}},\ -\frac{6}{5\sqrt{2}},\ \frac{2}{5\sqrt{2}}\right)$$

Exercise

Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\boldsymbol{R}^m$.

$$\{(3,\ 4,\ 0,\ 0),\ (-1,\ 1,\ 0,\ 0),\ (2,\ 1,\ 0,\ -1),\ (0,\ 1,\ 1,\ 0)\}$$

Solution

$$\vec{v}_1 = \vec{u}_1 = (3,\ 4,\ 0,\ 0)$$

$$\vec{v}_2 = (-1,\ 1,\ 0,\ 0) - \frac{(-1,\ 1,\ 0,\ 0)\bullet(3,\ 4,\ 0,\ 0)}{25}(3,\ 4,\ 0,\ 0) \qquad \boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

$$= (-1,\ 1,\ 0,\ 0) - \frac{1}{25}(3,\ 4,\ 0,\ 0)$$

$$= \boxed{\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)}$$

$$\frac{\langle \boldsymbol{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(2,\ 1,\ 0,\ -1)\bullet(3,\ 4,\ 0,\ 0)}{25}(3,\ 4,\ 0,\ 0)$$

$$= \frac{10}{25}(3,\ 4,\ 0,\ 0)$$

$$= \left(\frac{6}{5},\ \frac{8}{5},\ 0,\ 0\right)$$

$$\frac{\langle \boldsymbol{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{625}{1{,}225}\left[(2,\ 1,\ 0,\ -1)\bullet\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)\right]\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$= \frac{25}{49}\left(-\frac{35}{25}\right)\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$= \left(\frac{4}{5},\ -\frac{3}{5},\ 0,\ 0\right)$$

$$\vec{v}_3 = (2,\ 1,\ 0,\ -1) - \left(\frac{6}{5},\ \frac{8}{5},\ 0,\ 0\right) - \left(\frac{4}{5},\ -\frac{3}{5},\ 0,\ 0\right) \qquad \boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2$$

$$= \boxed{(0,\ 0,\ 0,\ -1)}$$

$$\frac{\langle \boldsymbol{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \frac{(0,\ 1,\ 1,\ 0)\bullet(3,\ 4,\ 0,\ 0)}{25}(3,\ 4,\ 0,\ 0)$$

$$= \frac{4}{25}(3,\ 4,\ 0,\ 0)$$

$$= \left(\frac{12}{25},\ \frac{16}{25},\ 0,\ 0\right)$$

$$\frac{\langle \boldsymbol{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{25}{49}\left[(0,\ 1,\ 1,\ 0)\bullet\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)\right]\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$= \frac{25}{49}\left(\frac{21}{25}\right)\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right)$$

$$= \left(-\frac{12}{25},\ \frac{9}{25},\ 0,\ 0\right)$$

$$\frac{\langle \boldsymbol{u}_4, \vec{v}_3\rangle}{\|\vec{v}_3\|^2}\vec{v}_3 = \left[(0,\ 1,\ 1,\ 0)\bullet(0,\ 0,\ 0,\ -1)\right](0,\ 0,\ 0,\ -1)$$

$$= (0,\ 0,\ 0,\ 0)$$

$$= \left(-\frac{10}{7},\ \frac{5}{7},\ \frac{15}{7}\right)$$

$$\boldsymbol{v}_4 = \boldsymbol{u}_4 - \frac{\langle \boldsymbol{u}_4, \boldsymbol{v}_1\rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_4, \boldsymbol{v}_2\rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2 - \frac{\langle \boldsymbol{u}_4, \boldsymbol{v}_3\rangle}{\|\boldsymbol{v}_3\|^2}\boldsymbol{v}_3$$

$$\vec{v}_4 = (0,\ 1,\ 1,\ 0) - \left(\frac{12}{25},\ \frac{16}{25},\ 0,\ 0\right) - \left(-\frac{12}{25},\ \frac{9}{25},\ 0,\ 0\right) - (0,\ 0,\ 0,\ 0)$$

$$\underline{= (0,\ 0,\ 1,\ 0)}$$

$$\boldsymbol{q}_1 = \frac{1}{5}(3,\ 4,\ 0,\ 0) \qquad \boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|}$$

$$\underline{= \left(\frac{3}{5},\ \frac{4}{5},\ 0,\ 0\right)}$$

$$\boldsymbol{q}_2 = \frac{25}{35}\left(-\frac{28}{25},\ \frac{21}{25},\ 0,\ 0\right) \qquad \boldsymbol{q}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|}$$

$$\underline{= \left(-\frac{4}{5},\ \frac{3}{5},\ 0,\ 0\right)}$$

$$\underline{\boldsymbol{q}_3 = (0,\ 0,\ 0,\ -1)} \qquad \boldsymbol{q}_3 = \frac{\boldsymbol{v}_3}{\|\boldsymbol{v}_3\|}$$

$$\underline{= \left(\frac{2}{\sqrt{14}},\ -\frac{1}{\sqrt{14}},\ -\frac{3}{\sqrt{14}}\right)}$$

$$\underline{\boldsymbol{q}_4 = (0,\ 0,\ 1,\ 0)} \qquad \boldsymbol{q}_4 = \frac{\boldsymbol{v}_4}{\|\boldsymbol{v}_4\|}$$

Exercise

Find the QR–decomposition of

a) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

Solution

a) Since $\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5 \neq 0$, The matrix is invertible

$$u_1(1,\ 2),\quad u_2 = (-1,\ 3)$$

$$v_1 = u_1 = (1,\ 2)$$

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{(1,\ 2)}{\sqrt{1^2+2^2}} = \frac{(1,\ 2)}{\sqrt{5}} = \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)$$

$$\begin{aligned} v_2 &= u_2 - (u_2.v_1)v_1 \\ &= (-1,\ 3) - \left[(-1,\ 3).\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)\right]\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \\ &= (-1,\ 3) - \left(\frac{5}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \\ &= (-1,\ 3) - (1,\ 2) \\ &= (-2,\ 1) \end{aligned}$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{(-2,1)}{\sqrt{(-2)^2+1^2}} = \left(-\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right)$$

$$\langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle = (1,2)\bullet\left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right) = \frac{5}{\sqrt{5}} = \sqrt{5}$$

$$\langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle = (-1,\ 3)\bullet\left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right) = \frac{5}{\sqrt{5}} = \sqrt{5}$$

$$\langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle = (-1,\ 3)\bullet\left(-\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right) = \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{5}} = \sqrt{5}$$

$$\begin{aligned} R &= \begin{bmatrix} \langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle & \langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle \\ 0 & \langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix} \end{aligned}$$

The *QR*-decomposition of the matrix is

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

b) The column vectors of are: $\boldsymbol{u}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$$\boldsymbol{v}_1 = \boldsymbol{u}_1 = (3,\ -4)$$

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{(3,\ -4)}{\sqrt{9+16}} = \boxed{\left(\frac{3}{5},\ -\frac{4}{5}\right)}$$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1$$

$$= (5,\ 0) - \frac{(5,\ 0)\bullet(3,\ -4)}{25}(3,\ -4)$$

$$= (5,\ 0) - \frac{15}{25}(3,\ -4)$$

$$= (5,\ 0) - \frac{3}{5}(3,\ -4)$$

$$= (5,\ 0) - \left(\frac{9}{5},\ -\frac{12}{5}\right)$$

$$= \left(\frac{16}{5},\ \frac{12}{5}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{\left(\frac{16}{5},\ \frac{12}{5}\right)}{\sqrt{\frac{256}{25}+\frac{144}{25}}} = \frac{1}{\sqrt{\frac{400}{25}}}\left(\frac{16}{5},\ \frac{12}{5}\right) = \frac{1}{\sqrt{16}}\left(\frac{16}{5},\ \frac{12}{5}\right) = \frac{1}{4}\left(\frac{16}{5},\ \frac{12}{5}\right) = \boxed{\left(\frac{4}{5},\ \frac{3}{5}\right)}$$

$$R = \begin{bmatrix} \langle u_1, q_1\rangle & \langle u_2, q_1\rangle \\ 0 & \langle u_2, q_2\rangle \end{bmatrix}$$

$$= \begin{bmatrix} 3\left(\frac{3}{5}\right) - 4\left(-\frac{4}{5}\right) & 5\left(\frac{3}{5}\right) + 0\left(-\frac{4}{5}\right) \\ 0 & 5\left(\frac{4}{5}\right) - 0\left(\frac{3}{5}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}}_{Q} \cdot \underbrace{\begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix}}_{R}$$

c) Since the column vectors $u_1\ (1,\ 0,\ 1)$, $u_2 = (2,\ 1,\ 4)$ are linearly independent, so has a QR–decomposition.

$$v_1 = u_1 = (1,\ 0,\ 1)$$

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{(1,\ 0,\ 1)}{\sqrt{1^2+0+1^2}} = \frac{(1,\ 0,\ 1)}{\sqrt{2}} \boxed{= \left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)}$$

$$v_2 = u_2 - (u_2.v_1)v_1$$

$$= (2,1,4) - \left[(2,1,4).\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\right]\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$= (2,1,4) - \left(\frac{6}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$$

$$= (2,1,4) - (3,\ 0,\ 3)$$

$$= (-1,\ 1,\ 1)$$

$$\boldsymbol{q}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} = \frac{(-1,\ 1,\ 1)}{\sqrt{(-1)^2 + 1^2 + 1^2}} = \left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$\langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle = (1,\ 0,\ 1)\bullet\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle = (2,\ 1,\ 4)\bullet\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}} = 3\sqrt{2}$$

$$\langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle = (2,\ 1,\ 4)\bullet\left(-\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) = -\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{4}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$R = \begin{bmatrix} \langle \boldsymbol{u}_1, \boldsymbol{q}_1 \rangle & \langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle \\ 0 & \langle \boldsymbol{u}_2, \boldsymbol{q}_2 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

The QR–decomposition of the matrix is $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$

d) Since $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{vmatrix} = -4 \neq 0$, The matrix is invertible, so it has a QR-decomposition.

$$u_1 (1,\ 1,\ 0), \quad u_2 = (2,\ 1,\ 3), \quad u_3 = (1,\ 1,\ 1)$$

$$v_1 = u_1 = (1,\ 1,\ 0)$$

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{(1,\ 1,\ 0)}{\sqrt{1^2 + 1^2 + 0}} = \frac{(1,\ 1,\ 0)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\vec{v}_2 = \vec{u}_2 - (\vec{u}_2 \bullet \vec{v}_1)\vec{v}_1$$

$$= (2,\ 1,\ 3) - \left[(2,\ 1,\ 3).\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)\right]\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= (2,\ 1,\ 3) - \frac{3}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= (2,\ 1,\ 3) - \left(\frac{3}{2},\ \frac{3}{2},\ 0\right)$$

$$= \left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{\left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + 3^2}}$$

$$= \frac{\left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)}{\sqrt{\frac{19}{2}}}$$

$$= \frac{\sqrt{2}}{\sqrt{19}}\left(\frac{1}{2},\ -\frac{1}{2},\ 3\right)$$

$$= \left(\frac{1}{\sqrt{38}},\ -\frac{1}{\sqrt{38}},\ \frac{6}{\sqrt{38}}\right) \qquad = \left(\frac{\sqrt{2}}{2\sqrt{19}},\ -\frac{\sqrt{2}}{2\sqrt{19}},\ \frac{3\sqrt{2}}{\sqrt{19}}\right)$$

$$\vec{v}_3 = \vec{u}_3 - \left(\vec{u}_3 \bullet \vec{v}_1\right)\vec{v}_1 - \left(\vec{u}_3 \bullet \vec{v}_2\right)\vec{v}_2$$

$$= (1,1,1) - \left[(1,1,1).\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)\right]\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$$

$$- \left[(1,1,1).\left(\frac{1}{\sqrt{38}},-\frac{1}{\sqrt{38}},\frac{6}{\sqrt{38}}\right)\right]\left(\frac{1}{\sqrt{38}},-\frac{1}{\sqrt{38}},\frac{6}{\sqrt{38}}\right)$$

$$= (1,1,1) - \frac{2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right) - \frac{6}{\sqrt{38}}\left(\frac{1}{\sqrt{38}},-\frac{1}{\sqrt{38}},\frac{6}{\sqrt{38}}\right)$$

$$= (1,1,1) - (1,1,0) - \left(\frac{3}{19},-\frac{3}{19},\frac{18}{19}\right)$$

$$= \left(-\frac{3}{19},\ \frac{3}{19},\ \frac{1}{19}\right)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \frac{\left(-\frac{3}{19},\ \frac{3}{19},\ \frac{1}{19}\right)}{\sqrt{\left(-\frac{3}{19}\right)^2 + \left(\frac{3}{19}\right)^2 + \left(\frac{1}{19}\right)^2}}$$

$$= \frac{19}{\sqrt{19}}\left(-\frac{3}{19},\ \frac{3}{19},\ \frac{1}{19}\right)$$

$$= \left(-\frac{3}{\sqrt{19}},\ \frac{3}{\sqrt{19}},\ \frac{1}{\sqrt{19}}\right)$$

$$\left\langle \boldsymbol{u}_1, \boldsymbol{q}_1 \right\rangle = (1,\ 1,\ 0) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\left\langle \boldsymbol{u}_2, \boldsymbol{q}_1 \right\rangle = (2,\ 1,\ 3) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) = \frac{3}{\sqrt{2}}$$

$$\langle u_2, q_2\rangle = (2,\ 1,\ 3)\cdot\left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}}\right) = \frac{2-1+18}{\sqrt{38}} = \frac{19}{\sqrt{38}} = \frac{19}{\sqrt{2}\sqrt{19}} = \frac{\sqrt{19}}{\sqrt{2}}$$

$$\langle u_3, q_1\rangle = (1,\ 1,\ 1)\cdot\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\langle u_3, q_2\rangle = (1,\ 1,\ 1)\cdot\left(\frac{1}{\sqrt{38}}, -\frac{1}{\sqrt{38}}, \frac{6}{\sqrt{38}}\right) = \frac{1-1+6}{\sqrt{38}} = \frac{6}{\sqrt{2}\sqrt{19}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{\sqrt{19}}$$

$$\langle u_3, q_3\rangle = (1,\ 1,\ 1)\cdot\left(-\frac{3}{\sqrt{19}},\ \frac{3}{\sqrt{19}},\ \frac{1}{\sqrt{19}}\right) = \frac{-3+3+1}{\sqrt{19}} = \frac{1}{\sqrt{19}}$$

$$R = \begin{bmatrix} \langle u_1, q_1\rangle & \langle u_2, q_1\rangle & \langle u_3, q_1\rangle \\ 0 & \langle u_2, q_2\rangle & \langle u_3, q_2\rangle \\ 0 & 0 & \langle u_3, q_3\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}$$

The QR–decomposition of the matrix is

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2\sqrt{19}} & -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{2\sqrt{19}} & \frac{3}{\sqrt{19}} \\ 0 & \frac{3\sqrt{2}}{\sqrt{19}} & \frac{1}{\sqrt{19}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \frac{\sqrt{19}}{\sqrt{2}} & \frac{3\sqrt{2}}{\sqrt{19}} \\ 0 & 0 & \frac{1}{\sqrt{19}} \end{bmatrix}$$

e) $$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is linearly dependent, so doesn't have a QR–decomposition.

Exercise

Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product

$$\boldsymbol{u} = (0,\ -2,\ 2,\ 1), \quad \boldsymbol{v} = (-1,\ -1,\ 1,\ 1)$$

Solution

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = 0 - 2(-1) + 2(1) + 1(1) = 5$$

$$\left\|\langle \boldsymbol{u},\boldsymbol{v}\rangle\right\| = \sqrt{5}$$

$$\|\boldsymbol{u}\|.\|\boldsymbol{v}\| = \sqrt{0+4+4+1}\sqrt{1+1+1+1}$$

$$= \sqrt{9}\sqrt{4}$$

$$\boxed{= 6}$$

$$\sqrt{5} < 6 \quad \Rightarrow \quad \left\|\langle \boldsymbol{u},\boldsymbol{v}\rangle\right\| \le \|\boldsymbol{u}\|.\|\boldsymbol{v}\|$$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $\quad f_1(x) = x+2, \quad f_2(x) = x^2 - 3x + 4$

Solution

Let $\quad \vec{u}_1 = f_1 = x+2, \quad \vec{u}_2 = f_2 = x^2 - 3x + 4$

$$\vec{v}_1 = \vec{u}_1 \ \boxed{= x+2}$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^{1} (x+2)^2\, dx$$

$$= \frac{1}{3}(x+2)^3 \Big|_{-1}^{1}$$

$$= \frac{1}{3}(27-1)$$

$$\boxed{= \frac{26}{3}}$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^{1} \left(x^2 - 3x + 4\right)(x+2)\ dx$$

$$= \int_{-1}^{1} \left(x^3 - x^2 - 2x + 8\right)\ dx$$

$$= \left(\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2 + 8x\right)\Big|_{-1}^{1}$$

$$= \frac{1}{4} - \frac{1}{3} - 1 + 8 - \frac{1}{4} - \frac{1}{3} + 1 + 8$$

$$\boxed{= \frac{46}{3}}$$

$$\vec{v}_2 = x^2 - 3x + 4 - \frac{46}{3}\left(\frac{3}{26}\right)(x+2) \qquad\qquad \boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

$= x^2 - 3x + 4 - \frac{23}{13}x - \frac{46}{13}$

$\boxed{= x^2 - \frac{62}{13}x + \frac{6}{13}}$

The orthogonal basis is $\left\{ x+2, \quad x^2 - \frac{62}{13}x + \frac{6}{13} \right\}$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} \left(x^2 - \frac{62}{13}x + \frac{6}{13} \right)^2 dx$$

$$= \frac{1}{169} \int_{-1}^{1} \left(13x^2 - 62x + 6 \right)^2 dx$$

$$= \frac{1}{169} \int_{-1}^{1} \left(169x^4 + 3{,}844x^2 + 36 - 1{,}612x^3 + 156x^2 - 744x \right) dx$$

$$= \frac{1}{169} \left(\frac{169}{5}x^5 + \frac{4{,}000}{3}x^3 + 36x - 403x^4 - 372x^2 \right) \Big|_{-1}^{1}$$

$$= \frac{1}{169} \left(\frac{169}{5} + \frac{4{,}000}{3} + 36 - 403 - 372 + \frac{169}{5} + \frac{4{,}000}{3} + 36 + 403 + 372 \right)$$

$$= \frac{1}{169} \left(\frac{338}{5} + \frac{8{,}000}{3} + 72 \right)$$

$$\boxed{= \frac{3{,}238}{195}}$$

$\vec{q}_1 \boxed{= \frac{\sqrt{3}}{\sqrt{26}}(x+2)}$ $\qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$

$\boxed{\vec{q}_2 = \sqrt{\frac{195}{3238}} \left(x^2 - \frac{62}{13}x + \frac{6}{13} \right)}$ $\qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$

The orthonormal basis is $\left\{ \frac{\sqrt{3}}{\sqrt{26}}(x+2), \quad \sqrt{\frac{195}{3238}} \left(x^2 - \frac{62}{13}x + \frac{6}{13} \right) \right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1, 1]$ spanned by the functions, using the inner product $f_1(x) = x, \quad f_2(x) = x^3, \quad f_3(x) = x^5$

Solution

Let $\vec{u}_1 = f_1 = x, \quad \vec{u}_2 = f_2 = x^3, \quad \vec{u}_3 = f_3 = x^5$

$\vec{v}_1 = \vec{u}_1 \boxed{= x}$

$$\left\langle \vec{v}_1, \vec{v}_1 \right\rangle = \int_{-1}^{1} x^2 dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$\boxed{= \frac{2}{3}}$$

$$\left\langle \vec{u}_2, \vec{v}_1 \right\rangle = \int_{-1}^{1} x^4 \, dx$$

$$= \frac{1}{5}x^5 \Big|_{-1}^{1}$$

$$\boxed{= \frac{2}{5}}$$

$$\vec{v}_2 = x^3 - \frac{2}{5}\left(\frac{3}{2}\right)(x) \qquad\qquad \boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\left\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \right\rangle}{\left\| \boldsymbol{v}_1 \right\|^2} \boldsymbol{v}_1$$

$$\boxed{= x^3 - \frac{3}{5}x}$$

$$\left\langle \vec{v}_2, \vec{v}_2 \right\rangle = \int_{-1}^{1} \left(x^3 - \frac{3}{5}x\right)^2 \, dx$$

$$= \int_{-1}^{1} \left(x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2\right) dx$$

$$= \left(\frac{1}{7}x^7 - \frac{6}{25}x^5 + \frac{3}{25}x^3\right) \Big|_{-1}^{1}$$

$$= 2\left(\frac{1}{7} - \frac{6}{25} + \frac{3}{25}\right)$$

$$\boxed{= \frac{8}{175}}$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \int_{-1}^{1} x^6 \, dx$$

$$= \frac{1}{7}x^7 \Big|_{-1}^{1}$$

$$\boxed{= \frac{2}{7}}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \int_{-1}^{1} x^5 \left(x^3 - \frac{3}{5}x\right) \, dx$$

$$= \int_{-1}^{1} \left(x^8 - \frac{3}{5}x^6\right) dx$$

$$= \left(\frac{1}{9}x^9 - \frac{3}{35}x^7\right)\Big|_{-1}^{1}$$

$$= 2\left(\frac{1}{9} - \frac{3}{35}\right)$$

$$= \boxed{\frac{16}{315}}$$

$$\vec{v}_3 = x^5 - \frac{16}{315}\left(\frac{175}{8}\right)\left(x^3 - \frac{3}{5}x\right) - \frac{2}{7}\left(\frac{3}{2}\right)x \qquad \boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

$$= x^5 - \frac{70}{63}\left(x^3 - \frac{3}{5}x\right) - \frac{3}{7}x$$

$$= x^5 - \frac{70}{63}x^3 + \frac{14}{21}x - \frac{3}{7}x$$

$$= \boxed{x^5 - \frac{70}{63}x^3 + \frac{5}{21}x}$$

The orthogonal basis is $\left\{x,\ \ x^3 - \frac{3}{5}x\ ,\ \ x^5 - \frac{70}{63}x^3 + \frac{5}{21}x\ \right\}$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^{1} \left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x\right)^2 dx$$

$$= \int_{-1}^{1} \frac{1}{3,969}\left(63x^5 - 70x^3 + 15x\right)^2 dx$$

$$= \frac{1}{3,969}\int_{-1}^{1} \left(3,969x^{10} - 8,820x^8 + 1,890x^6 - 2,100x^4 + 4,900x^6 + 225x^2\right) dx$$

$$= \frac{1}{3,969}\left(\frac{3,969}{11}x^{11} - 980x^9 + 970x^7 - 420x^5 + 75x^3\right)\Big|_{-1}^{1}$$

$$= \frac{2}{3,969}\left(\frac{3,969}{11} - 980 + 970 - 420 + 75\right)$$

$$= \frac{2}{3,969}\left(\frac{3,969}{11} - 355\right)$$

$$= \frac{2}{3,969}\left(\frac{64}{11}\right)$$

$$= \boxed{\frac{128}{43,659}}$$

$$\vec{q}_1 = \frac{x}{\sqrt{2/3}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{\sqrt{3}}{\sqrt{2}}x$$

$$\vec{q}_2 = \sqrt{\frac{175}{8}}\left(x^3 - \frac{3}{5}x\right) \qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{5\sqrt{7}}{2\sqrt{2}}\left(x^3 - \frac{3}{5}x\right)$$

$$\vec{q}_3 = \sqrt{\frac{43,659}{128}}\left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x\right) \qquad \vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \frac{63\sqrt{11}}{8\sqrt{2}}\left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x\right)$$

The orientation basis is $\left\{\frac{\sqrt{3}}{\sqrt{2}}x,\ \frac{5}{2}\sqrt{\frac{7}{2}}\left(x^3 - \frac{3}{5}x\right),\ \frac{63}{8}\sqrt{\frac{11}{2}}\left(x^5 - \frac{70}{63}x^3 + \frac{5}{21}x\right)\right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $\quad f_1(x) = 1,\quad f_2(x) = x,\quad f_3(x) = \frac{1}{2}\left(3x^2 - 1\right)$

Solution

Let $\vec{u}_1 = f_1 = 1,\quad \vec{u}_2 = f_2 = x,\quad \vec{u}_3 = f_3 = \frac{3}{2}x^2 - \frac{1}{2}$

$$\vec{v}_1 = \vec{u}_1 = 1$$

$$\left\langle \vec{v}_1, \vec{v}_1 \right\rangle = \int_{-1}^{1} 1dx$$

$$= x\Big|_{-1}^{1}$$

$$= 2$$

$$\left\langle \vec{u}_2, \vec{v}_1 \right\rangle = \int_{-1}^{1} x\, dx$$

$$= \frac{1}{2}x^2\Big|_{-1}^{1}$$

$$= 0$$

$$\vec{v}_2 = x \qquad \boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\left\langle \boldsymbol{u}_2, \boldsymbol{v}_1 \right\rangle}{\left\|\boldsymbol{v}_1\right\|^2}\boldsymbol{v}_1$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} x^2 \, dx$$

$$= \frac{1}{3} x^3 \Big|_{-1}^{1}$$

$$= \boxed{\frac{2}{3}}$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \frac{1}{2} \int_{-1}^{1} \left(3x^2 - 1\right) dx$$

$$= \frac{1}{2}\left(x^3 - x\right)\Big|_{-1}^{1}$$

$$= \boxed{0}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \frac{1}{2} \int_{-1}^{1} x\left(3x^2 - 1\right) dx$$

$$= \frac{1}{2} \int_{-1}^{1} \left(3x^3 - x\right) dx$$

$$= \frac{1}{2}\left(\frac{3}{4}x^4 - \frac{1}{2}x^2\right)\Big|_{-1}^{1}$$

$$= \boxed{0}$$

$$\boxed{\vec{v}_3 = \frac{1}{2}\left(3x^2 - 1\right)}$$

$$\boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

The orthogonal basis is $\left\{1, \ x, \ \frac{1}{2}\left(3x^2 - 1\right)\right\}$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \frac{1}{4} \int_{-1}^{1} \left(3x^2 - 1\right)^2 dx$$

$$= \frac{1}{4} \int_{-1}^{1} \left(9x^4 - 6x^2 + 1\right) dx$$

$$= \frac{1}{4}\left(\frac{9}{5}x^5 - 2x^3 + x\right)\Big|_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{9}{5} - 2 + 1\right)$$

$$= \boxed{\frac{2}{5}}$$

$$\boxed{\vec{q}_1 = \frac{1}{\sqrt{2}}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\boxed{\vec{q}_2 = \sqrt{\frac{3}{2}}\, x} \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\boxed{\vec{q}_3 = \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right)} \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

The orthonormal basis is $\left\{\frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}x\ ,\ \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right)\right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $\quad f_1(x) = 1, \quad f_2(x) = \sin \pi x, \quad f_3(x) = \cos \pi x$

Solution

Let $\vec{u}_1 = f_1 = 1, \quad \vec{u}_2 = f_2 = \sin \pi x, \quad \vec{u}_3 = f_3 = \cos \pi x$

$$\vec{v}_1 = \vec{u}_1 \ \boxed{= 1}$$

$$\begin{aligned}
\left\langle \vec{v}_1,\ \vec{v}_1 \right\rangle &= \int_{-1}^{1} 1dx \\
&= x\Big|_{-1}^{1} \\
&= 2
\end{aligned}$$

$$\begin{aligned}
\left\langle \vec{u}_2,\ \vec{v}_1 \right\rangle &= \int_{-1}^{1} \sin \pi x\ dx \\
&= -\frac{1}{\pi}\cos \pi x\Big|_{-1}^{1} \\
&= 0
\end{aligned}$$

$$\boxed{\vec{v}_2 = \sin \pi x} \qquad \boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\left\langle \boldsymbol{u}_2,\ \boldsymbol{v}_1 \right\rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

$$\begin{aligned}
\left\langle \vec{v}_2,\ \vec{v}_2 \right\rangle &= \int_{-1}^{1} \sin^2 \pi x\ dx \\
&= \frac{1}{2}\int_{-1}^{1} (1 - \cos 2\pi x)\ dx \\
&= \frac{1}{2}\left(x - \frac{1}{2\pi}\sin 2\pi x\right)\Big|_{-1}^{1}
\end{aligned}$$

$$\underline{=1}$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \int_{-1}^{1} \cos \pi x \, dx$$

$$= \frac{1}{\pi} \sin \pi x \Big|_{-1}^{1}$$

$$\underline{=0}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \int_{-1}^{1} \cos \pi x \sin \pi x \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} \sin 2\pi x \, dx$$

$$= -\frac{1}{4\pi} \cos 2\pi x \Big|_{-1}^{1}$$

$$\underline{=0}$$

$$\vec{v}_3 = \cos \pi x \qquad \boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\left\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \right\rangle}{\left\| \boldsymbol{v}_2 \right\|^2} \boldsymbol{v}_2 - \frac{\left\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \right\rangle}{\left\| \boldsymbol{v}_1 \right\|^2} \boldsymbol{v}_1$$

The orthogonal basis is $\left\{ 1, \ \sin \pi x - \frac{1}{\pi}, \ \cos \pi x \right\}$

$$\left\langle \vec{v}_3, \vec{v}_3 \right\rangle = \int_{-1}^{1} \cos^2 \pi x \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} (1 + \cos 2\pi x) \, dx$$

$$= \frac{1}{2} \left(x + \frac{1}{2\pi} \sin 2\pi x \right) \Big|_{-1}^{1}$$

$$\underline{=1}$$

$$\vec{q}_1 = \frac{1}{\sqrt{2}} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$\vec{q}_2 = \sin \pi x \qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\| \vec{v}_2 \right\|}$$

$$\vec{q}_3 = \cos \pi x \qquad \vec{q}_3 = \frac{\vec{v}_3}{\left\| \vec{v}_3 \right\|}$$

The orthonormal basis is $\left\{ \frac{1}{\sqrt{2}}, \ \sin \pi x, \ \cos \pi x \right\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $\quad f_1(x)=\sin\pi x,\quad f_2(x)=\sin 2\pi x,\quad f_3(x)=\sin 3\pi x$

Solution

Let $\vec{u}_1 = f_1 = \sin\pi x,\quad \vec{u}_2 = f_2 = \sin 2\pi x,\quad \vec{u}_3 = f_3 = \sin 3\pi x$

$\vec{v}_1 = \vec{u}_1 \ \underline{= \sin\pi x}$

$$\left\langle \vec{v}_1,\ \vec{v}_1\right\rangle = \int_{-1}^{1}\sin^2\pi x\ dx$$

$$= \frac{1}{2}\int_{-1}^{1}(1-\cos 2\pi x)\ dx$$

$$= \frac{1}{2}\left(x-\frac{1}{2\pi}\sin 2\pi x\right)\Big|_{-1}^{1}$$

$$\underline{=1}$$

$$\left\langle \vec{u}_2,\ \vec{v}_1\right\rangle = \int_{-1}^{1}\sin\pi x\sin 2\pi x\ dx \qquad \sin a\sin b=\tfrac{1}{2}\left[\cos(a+b)-\cos(a-b)\right]$$

$$= \frac{1}{2}\int_{-1}^{1}\left(\cos 3\pi x-\cos(-\pi x)\right)\ dx$$

$$= \frac{1}{2}\int_{-1}^{1}\left(\cos 3\pi x-\cos\pi x\right)\ dx$$

$$= \frac{1}{2}\left(\frac{1}{3\pi}\sin 3\pi x-\frac{1}{\pi}\sin\pi x\right)\Big|_{-1}^{1}$$

$$\underline{=0}$$

$\underline{\vec{v}_2 = \sin 2\pi x}$ $\qquad \boldsymbol{v}_2 = \boldsymbol{u}_2 - \dfrac{\left\langle \boldsymbol{u}_2,\ \boldsymbol{v}_1\right\rangle}{\left\|\boldsymbol{v}_1\right\|^2}\boldsymbol{v}_1$

$$\left\langle \vec{v}_2,\ \vec{v}_2\right\rangle = \int_{-1}^{1}\sin^2 2\pi x\ dx$$

$$= \frac{1}{2}\int_{-1}^{1}(1-\cos 4\pi x)\ dx$$

$$= \frac{1}{2}\left(x-\frac{1}{4\pi}\sin 4\pi x\right)\Big|_{-1}^{1}$$

$$\underline{=1}$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \int_{-1}^{1} \sin \pi x \sin 3\pi x \, dx \qquad \sin a \sin b = \frac{1}{2}\left[\cos(a+b) - \cos(a-b)\right]$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 4\pi x - \cos(-2\pi x)\right) dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 4\pi x - \cos 2\pi x\right) dx$$

$$= \frac{1}{2}\left(\frac{1}{4\pi}\sin 4\pi x - \frac{1}{2\pi}\sin 2\pi x\right)\Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \int_{-1}^{1} \sin 3\pi x \sin 2\pi x \, dx \qquad \sin a \sin b = \frac{1}{2}\left[\cos(a+b) - \cos(a-b)\right]$$

$$= \frac{1}{2}\int_{-1}^{1} \left(\cos 5\pi x - \cos \pi x\right) dx$$

$$= \frac{1}{2}\left(\frac{1}{5\pi}\sin 5\pi x - \frac{1}{\pi}\sin \pi x\right)\Big|_{-1}^{1}$$

$$\underline{= 0}$$

$$\boxed{\vec{v}_3 = \sin 3\pi x} \qquad \boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\left\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \right\rangle}{\left\|\boldsymbol{v}_2\right\|^2}\boldsymbol{v}_2 - \frac{\left\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \right\rangle}{\left\|\boldsymbol{v}_1\right\|^2}\boldsymbol{v}_1$$

The orthogonal basis is $\{\sin \pi x, \quad \sin 2\pi x, \quad \sin 3\pi x\}$

$$\left\langle \vec{v}_3, \vec{v}_3 \right\rangle = \int_{-1}^{1} \sin^2 3\pi x \, dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(1 - \cos 6\pi x\right) dx$$

$$= \frac{1}{2}\left(x - \frac{1}{6\pi}\sin 6\pi x\right)\Big|_{-1}^{1}$$

$$\underline{= 1}$$

$$\boxed{\vec{q}_1 = \sin \pi x} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$\boxed{\vec{q}_2 = \sin 2\pi x} \qquad \vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$\boxed{\vec{q}_3 = \sin 3\pi x} \qquad \vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

The orthonormal basis is $\{\sin \pi x, \quad \sin 2\pi x, \quad \sin 3\pi x\}$

Exercise

Apply the Gram-Schmidt ***orthonormalization*** process in $C^0[-1,\ 1]$ spanned by the functions, using the inner product $f_1(x)=\cos\pi x,\quad f_2(x)=\cos 2\pi x,\quad f_3(x)=\cos 3\pi x$

Solution

Let $\vec{u}_1 = f_1 = \cos\pi x,\quad \vec{u}_2 = f_2 = \cos 2\pi x,\quad \vec{u}_3 = f_3 = \cos 3\pi x$

$\vec{v}_1 = \vec{u}_1 = \cos\pi x$

$$\left\langle \vec{v}_1,\ \vec{v}_1\right\rangle = \int_{-1}^{1}\cos^2\pi x\ dx$$

$$=\frac{1}{2}\int_{-1}^{1}(1+\cos 2\pi x)\ dx$$

$$=\frac{1}{2}\left(x+\frac{1}{2\pi}\sin 2\pi x\right)\Big|_{-1}^{1}$$

$$=1$$

$$\left\langle \vec{u}_2,\ \vec{v}_1\right\rangle = \int_{-1}^{1}\cos 2\pi x\cos\pi x\ dx \qquad \cos a\cos b=\tfrac{1}{2}\left[\cos(a+b)+\cos(a-b)\right]$$

$$=\frac{1}{2}\int_{-1}^{1}(\cos 3\pi x+\cos\pi x)\ dx$$

$$=\frac{1}{2}\left(\frac{1}{3\pi}\sin 3\pi x+\frac{1}{\pi}\sin\pi x\right)\Big|_{-1}^{1}$$

$$=0$$

$$\vec{v}_2 = \cos 2\pi x \qquad \boldsymbol{v}_2=\boldsymbol{u}_2-\frac{\left\langle \boldsymbol{u}_2,\ \boldsymbol{v}_1\right\rangle}{\left\|\boldsymbol{v}_1\right\|^2}\boldsymbol{v}_1$$

$$\left\langle \vec{v}_2,\ \vec{v}_2\right\rangle = \int_{-1}^{1}\cos^2 2\pi x\ dx$$

$$=\frac{1}{2}\int_{-1}^{1}(1+\cos 4\pi x)\ dx$$

$$=\frac{1}{2}\left(x+\frac{1}{4\pi}\sin 4\pi x\right)\Big|_{-1}^{1}$$

$$=1$$

$$\left\langle \vec{u}_3,\ \vec{v}_1\right\rangle = \int_{-1}^{1}\cos 3\pi x\cos\pi x\ dx \qquad \cos a\cos b=\tfrac{1}{2}\left[\cos(a+b)+\cos(a-b)\right]$$

$$= \frac{1}{2}\int_{-1}^{1} (\cos 4\pi x + \cos 2\pi x)\ dx$$

$$= \frac{1}{2}\left(\frac{1}{4\pi}\sin 4\pi x + \frac{1}{2\pi}\sin 2\pi x\right)\Bigg|_{-1}^{1}$$

$$\boxed{= 0}$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^{1} \cos 3\pi x \cos 2\pi x\ dx$$

$$\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$$

$$= \frac{1}{2}\int_{-1}^{1} (\cos 5\pi x + \cos \pi x)\ dx$$

$$= \frac{1}{2}\left(\frac{1}{5\pi}\sin 5\pi x + \frac{1}{\pi}\sin \pi x\right)\Bigg|_{-1}^{1}$$

$$\boxed{= 0}$$

$$\boxed{\vec{v}_3 = \cos 3\pi x}$$

$$\boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1$$

The orthogonal basis is $\{\cos \pi x,\ \ \cos 2\pi x\ ,\ \ \cos 3\pi x\}$

$$\langle \vec{v}_3, \vec{v}_3 \rangle = \int_{-1}^{1} \cos^2 3\pi x\ dx$$

$$= \frac{1}{2}\int_{-1}^{1} (1 + \cos 6\pi x)\ dx$$

$$= \frac{1}{2}\left(x + \frac{1}{6\pi}\sin 6\pi x\right)\Bigg|_{-1}^{1}$$

$$\boxed{= 1}$$

$$\boxed{\vec{q}_1 = \cos \pi x} \qquad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\boxed{\vec{q}_2 = \cos 2\pi x} \qquad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$\boxed{\vec{q}_3 = \cos 3\pi x} \qquad \vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

The orthonormal basis is $\{\cos \pi x,\ \ \cos 2\pi x\ ,\ \ \cos 3\pi x\}$

Solution Section 3.4 – Orthogonal Matrices

Exercise

Show that the matrix is orthogonal

$$a)\quad A=\begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix} \qquad b)\quad A=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Solution

a) $$AA^T=\begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}\begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$A^TA=\begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix}\begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore$ ***A*** is an orthogonal

b) $$AA^T=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$AA^T=I$$

$\therefore$ ***A*** is an orthogonal

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is orthogonal with inverse $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Solution

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is orthogonal with inverse $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ (It is a standard matrix for a rotation of 45°)

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Solution

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad = I$$

$\therefore \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is orthogonal with inverse $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Solution

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}^T = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad = I$$

$\therefore \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ is orthogonal with an inverse $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 7 \\ 1 & 3 & -5 \\ -1 & 4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & & \\ & & \\ & & \end{pmatrix} \quad \neq I$$

$\therefore \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$ is not an orthogonal

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Solution

$$\begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2} & & \\ & & \\ & & \end{pmatrix} \neq I$$

Or $\left\| r_1 \right\| = \sqrt{0 + 1^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{3}{2}} \underline{\neq 1}$ $\quad \therefore$ ***A*** is ***not*** orthogonal

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Solution

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \text{is orthogonal with inverse } \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

Solution

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \text{is orthogonal with inverse } \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

Solution

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} \text{ is orthogonal with inverse } \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$$

Solution

$$\left\|r_2\right\| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{3}+\frac{1}{4}} = \sqrt{\frac{7}{12}} \neq 1$$

Or

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{5}{6} & & \\ & & & \\ & & & \end{pmatrix} \neq I$$

$\therefore$ The matrix is ***not*** an orthogonal

Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & \cdots \end{bmatrix}$$

<u>Solution</u>

$$\boldsymbol{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}^T \rightarrow \left\|\boldsymbol{q}_1\right\| = \sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 1$$

$$\boldsymbol{q}_2 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}^T \rightarrow \left\|\boldsymbol{q}_2\right\| = \sqrt{\frac{1}{6}+\frac{1}{6}+\frac{4}{6}} = 1$$

Let $\boldsymbol{q}_3 = [x \quad y \quad z]^T$

$$\boldsymbol{q}_1 \cdot \boldsymbol{q}_3 = \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y - \frac{1}{\sqrt{3}}z = 0 \quad \rightarrow \quad x+y-z=0$$

$$\boldsymbol{q}_2 \cdot \boldsymbol{q}_3 = \frac{1}{\sqrt{6}}x + \frac{1}{\sqrt{6}}y - \frac{2}{\sqrt{6}}z = 0 \quad \rightarrow \quad x+y-2z=0$$

$$\begin{cases} x+y-z=0 \\ x+y-2z=0 \end{cases} \rightarrow \quad z=0 \quad \text{and} \quad x+y=0 \Rightarrow x=-y$$

$$q_3 = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$$

Exercise

Determine if the given matrix is orthogonal. If it is, find its inverse

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{5} & \frac{3}{7} \\ \frac{4}{9} & \frac{3}{5} & -\frac{2}{7} \\ \frac{8}{9} & -\frac{2}{5} & \frac{3}{7} \end{bmatrix}$$

Solution

$$q_1 = \begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{8}{9} \end{bmatrix}^T \quad q_2 = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & -\frac{2}{5} \end{bmatrix}^T \quad q_3 = \begin{bmatrix} \frac{3}{7} & -\frac{2}{7} & \frac{3}{7} \end{bmatrix}^T$$

$$q_1 \bullet q_2 = \frac{4}{45} + \frac{12}{45} - \frac{16}{45} = 0$$

$$q_1 \bullet q_3 = \frac{3}{63} - \frac{8}{63} + \frac{24}{63} = \frac{19}{63} \neq 0$$

$$q_2 \bullet q_3 = \frac{12}{35} - \frac{6}{35} + \frac{6}{35} = \frac{12}{35} \neq 0$$

The given matrix is ***not*** orthogonal

Exercise

Prove that if A is orthogonal, then A^T is orthogonal.

Solution

Since A is orthogonal then $A^T = A^{-1}$ and $A = \left(A^T\right)^T$

Then $\left(A^T\right)^T A^T = AA^T = I \Rightarrow A^T$ is orthogonal

Another word, since A is orthogonal, then both column and row vectors of A form an orthonormal set. A^T is just A with its row and column vectors are swapped. The column vectors of A^T (which are the row vectors of A) and row vectors of A^T (which are the column vectors of A) form orthonormal sets, therefore A^T is orthogonal

Exercise

Prove that if A is orthogonal, then A^{-1} is orthogonal

Solution

Since A is orthogonal then $A^T = A^{-1}$ and $A = \left(A^{-1}\right)^{-1}$

$$\left(A^{-1}\right)^{-1} = \left(A^{T}\right)^{-1} \qquad A^T = A^{-1}$$

$$= \left(A^{-1}\right)^{T}$$

$\therefore\ A^{-1}$ is orthogonal

Exercise

Prove that if *A* and *B* are orthogonal, then AB is orthogonal

Solution

Since *A* is orthogonal then $A^T = A^{-1}$

and *B* is orthogonal then $B^T = B^{-1}$

$$(AB)^T = B^T A^T$$

$$= B^{-1}A^{-1}$$

$$= (AB)^{-1}$$

$\therefore\ AB$ is orthogonal

Exercise

Let *Q* be an $n \times n$ orthogonal matrix, and let *A* be an $n \times n$ matrix.

Show that $\det\left(QAQ^T\right) = \det(A)$

Solution

$$\det\left(QAQ^T\right) = \det(Q)\det(A)\det\left(Q^T\right)$$

$$= \det(A)\det\left(QQ^T\right)$$ Since *Q* is an orthogonal matrix $\det\left(QQ^T\right) = \det(I)$

$$= \det(A)\det(I)$$

$$= \det(A) \quad \checkmark$$

Exercise

Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$

a) Is matrix *A* an orthogonal matrix?

b) Let *B* be the matrix obtained by normalizing each row of *A*, find *B*.

c) Is *B* an orthogonal matrix?

d) Are the columns of B orthogonal?

Solution

a) $AA^T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 7 \\ 1 & 3 & -5 \\ -1 & 4 & 2 \end{pmatrix}$

$= \begin{pmatrix} 3 & & \\ & & \\ & & \end{pmatrix} \neq I$

$\therefore \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$ is ***not*** an orthogonal

b) $\|(1,\ 1,\ -1)\| = \sqrt{1+1+1} \ \underline{=\sqrt{3}}$

$\|(1,\ 3,\ 4)\| = \sqrt{1+9+16} \ \underline{=\sqrt{26}}$

$\|(7,\ -5,\ 2)\| = \sqrt{49+25+4} \ \underline{=\sqrt{78}}$

$$B = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{26}} & \frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \\ \frac{7}{\sqrt{78}} & -\frac{5}{\sqrt{78}} & \frac{2}{\sqrt{78}} \end{pmatrix}$$

c) Yes, since the rows are orthogonal with unit vectors.

$$BB^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{26}} & \frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \\ \frac{7}{\sqrt{78}} & -\frac{5}{\sqrt{78}} & \frac{2}{\sqrt{78}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{26}} & \frac{7}{\sqrt{78}} \\ \frac{1}{\sqrt{3}} & \frac{3}{\sqrt{26}} & -\frac{5}{\sqrt{78}} \\ -\frac{1}{\sqrt{3}} & \frac{4}{\sqrt{26}} & \frac{2}{\sqrt{78}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

d) Yes, since the rows of B form an orthonormal set of vectors. Then, the column of B must form an orthonormal set.

$$\left\|\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{26}},\ \frac{7}{\sqrt{78}}\right)\right\| = \sqrt{\frac{1}{3}+\frac{1}{26}+\frac{49}{78}}$$

$$= \sqrt{\frac{26+3+49}{78}}$$

$$= \sqrt{\frac{78}{78}}$$

$$\underline{= 1}$$

$$\left\| \left(\frac{1}{\sqrt{3}},\ \frac{3}{\sqrt{26}},\ -\frac{5}{\sqrt{78}} \right) \right\| = \sqrt{\frac{1}{3} + \frac{9}{26} + \frac{25}{78}}$$

$$= \sqrt{\frac{26 + 27 + 25}{78}}$$

$$= \sqrt{\frac{78}{78}}$$

$$\underline{= 1}$$

$$\left\| \left(-\frac{1}{\sqrt{3}},\ \frac{4}{\sqrt{26}},\ \frac{2}{\sqrt{78}} \right) \right\| = \sqrt{\frac{1}{3} + \frac{16}{26} + \frac{4}{78}}$$

$$= \sqrt{\frac{78}{78}}$$

$$\underline{= 1}$$

Solution Section 3.5 – Least Squares Analysis

Exercise

Find the equation of the line that best fits the given points in the least-squares sense.

a) $\{(0,\ 2),\ (1,\ 2),\ (2,\ 0)\}$

b) $\{(1,\ 5),\ (2,\ 4),\ (3,\ 1),\ (4,\ 1),\ (5,-1)\}$

c) $\{(0,\ 1),\ (1,\ 3),\ (2,\ 4),\ (3,\ 4)\}$

d) $\{(-2,\ 0),\ (-1,\ 0),\ (0,\ 1),\ (1,\ 3),\ (2,\ 5)\}$

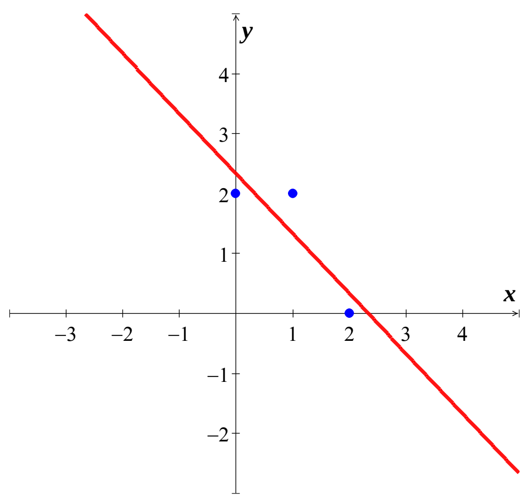
Solution

a) $\{(0,\ 2),\ (1,\ 2),\ (2,\ 0)\}$

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \quad x = \begin{bmatrix} m \\ b \end{bmatrix} \quad y = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$



The normal equation formula: $A^T A\mathbf{x} = A^T \mathbf{y}$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

We have: $m = -1$ *and* $b = \frac{7}{3}$.

Thus, $\boxed{y = -x + \frac{7}{3}}$

b) $\{(1,\ 5),\ (2,\ 4),\ (3,\ 1),\ (4,\ 1),\ (5,-1)\}$

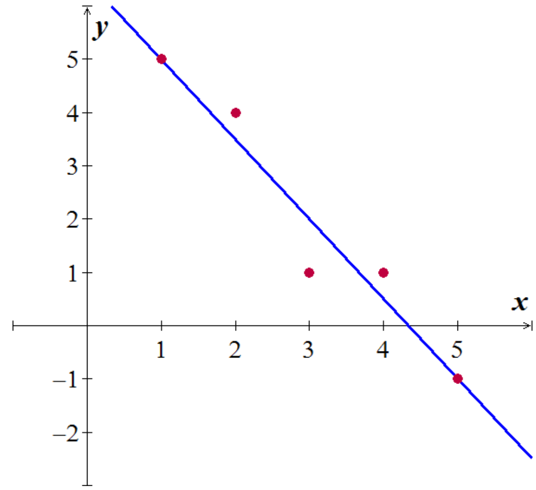
Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

where $A=\begin{bmatrix}1&1\\2&1\\3&1\\4&1\\5&1\end{bmatrix}$ $x=\begin{bmatrix}m\\b\end{bmatrix}$ $y=\begin{bmatrix}5\\4\\1\\1\\-1\end{bmatrix}$

The normal equation: $A^T A\mathbf{x}=A^T\mathbf{y}$

$$\begin{bmatrix}1&2&3&4&5\\1&1&1&1&1\end{bmatrix}\begin{bmatrix}1&1\\2&1\\3&1\\4&1\\5&1\end{bmatrix}\begin{bmatrix}m\\b\end{bmatrix}=\begin{bmatrix}1&2&3&4&5\\1&1&1&1&1\end{bmatrix}\begin{bmatrix}5\\4\\1\\1\\-1\end{bmatrix}$$



$$\begin{bmatrix}55&15\\15&5\end{bmatrix}\begin{bmatrix}m\\b\end{bmatrix}=\begin{bmatrix}15\\10\end{bmatrix}$$

We have: $m=-\frac{3}{2}$ *and* $b=\frac{13}{2}$.

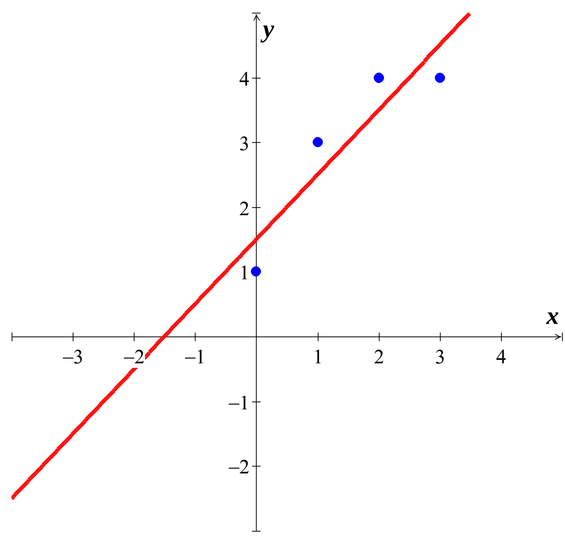
Thus, $\boxed{y=-1.5x+6.5}$

c) $\{(0,1),\ (1,3),\ (2,4),\ (3,4)\}$

Let $y=mx+b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix}0&1\\1&1\\2&1\\3&1\end{pmatrix}\begin{pmatrix}m\\b\end{pmatrix}=\begin{pmatrix}1\\3\\4\\4\end{pmatrix}$$

where $A=\begin{pmatrix}0&1\\1&1\\2&1\\3&1\end{pmatrix}$ $x=\begin{pmatrix}m\\b\end{pmatrix}$ $y=\begin{pmatrix}1\\3\\4\\4\end{pmatrix}$



The normal equation: $A^T A\mathbf{x}=A^T\mathbf{y}$

$$\begin{pmatrix}0&1&2&3\\1&1&1&1\end{pmatrix}\begin{pmatrix}0&1\\1&1\\2&1\\3&1\end{pmatrix}\begin{pmatrix}m\\b\end{pmatrix}=\begin{pmatrix}0&1&2&3\\1&1&1&1\end{pmatrix}\begin{pmatrix}1\\3\\4\\4\end{pmatrix}$$

$$\begin{pmatrix}14&6\\6&4\end{pmatrix}\begin{pmatrix}m\\b\end{pmatrix}=\begin{pmatrix}23\\12\end{pmatrix}$$

$$\begin{pmatrix}m\\b\end{pmatrix}=\frac{1}{20}\begin{pmatrix}4&-6\\-6&14\end{pmatrix}\begin{pmatrix}23\\12\end{pmatrix}=\begin{pmatrix}1\\\frac{3}{2}\end{pmatrix}$$

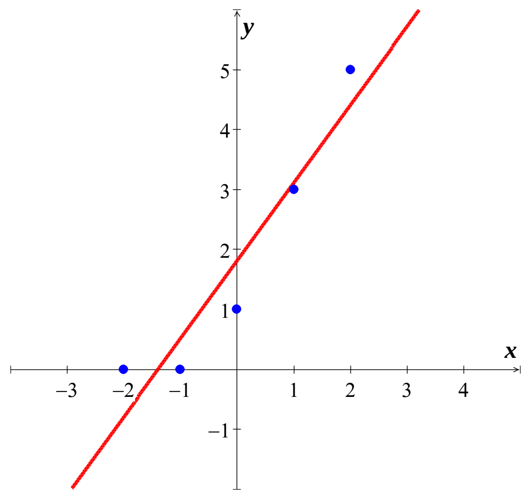
We have: $m = 1$ *and* $b = \frac{3}{2}$.

Thus, $\boxed{y = x + 1.5}$

d) $\{(-2,\ 0),\ (-1,\ 0),\ (0,\ 1),\ (1,\ 3),\ (2,\ 5)\}$

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$



where $A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$ $x = \begin{pmatrix} m \\ b \end{pmatrix}$ $y = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$

The normal equation: $A^T A\mathbf{x} = A^T \mathbf{y}$

$$\begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{50}\begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}\begin{pmatrix} 13 \\ 9 \end{pmatrix} = \begin{pmatrix} \frac{13}{10} \\ \frac{9}{5} \end{pmatrix}$$

We have: $m = 1.3$ *and* $b = 1.8$.

Thus, $\boxed{y = 1.3x + 1.8}$

Exercise

Find the orthogonal projection of the vector $\boldsymbol{u}$ on the subspace of $\boldsymbol{R}^4$ spanned by the vectors

a) $\boldsymbol{u} = (-3,\ -3,\ 8,\ 9)$; $\boldsymbol{v}_1 = (3,\ 1,\ 0,\ 1)$, $\boldsymbol{v}_2 = (1,\ 2,\ 1,\ 1)$, $\boldsymbol{v}_3 = (-1,\ 0,\ 2,\ -1)$

b) $\boldsymbol{u} = (6,\ 3,\ 9,\ 6)$; $\boldsymbol{v}_1 = (2,\ 1,\ 1,\ 1)$, $\boldsymbol{v}_2 = (1,\ 0,\ 1,\ 1)$, $\boldsymbol{v}_3 = (-2,\ -1,\ 0,\ -1)$

c) $\boldsymbol{u} = (-2,\ 0,\ 2,\ 4)$; $\boldsymbol{v}_1 = (1,\ 1,\ 3,\ 0)$, $\boldsymbol{v}_2 = (-2,\ -1,\ -2,\ 1)$, $\boldsymbol{v}_3 = (-3,\ -1,\ 1,\ 3)$

Solution

a) Let $A=\begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$

$$A^T A=\begin{pmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{pmatrix}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

$$=\begin{pmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{pmatrix}$$

$$A^T \boldsymbol{u}=\begin{pmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{pmatrix}\begin{pmatrix} -3 \\ -3 \\ 8 \\ 9 \end{pmatrix}$$

$$=\begin{pmatrix} -3 \\ 8 \\ 10 \end{pmatrix}$$

The normal solution is $A^T A\boldsymbol{x} = A^T\boldsymbol{u}$

$$\begin{pmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} -3 \\ 8 \\ 10 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{x}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

So $proj_W\,\boldsymbol{u} = A\boldsymbol{x} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

$$=\begin{pmatrix} -2 \\ 3 \\ 4 \\ 0 \end{pmatrix}$$

$\boxed{proj_W\,\boldsymbol{u} = (-2,\ 3,\ 4,\ 0)}$

b) $\boldsymbol{u}=(6,\ 3,\ 9,\ 6);\quad \boldsymbol{v}_1=(2,\ 1,\ 1,\ 1),\quad \boldsymbol{v}_2=(1,\ 0,\ 1,\ 1),\quad \boldsymbol{v}_3=(-2,\ -1,\ 0,\ -1)$

Let $A=\begin{pmatrix}2&1&-2\\1&0&-1\\1&1&0\\1&1&-1\end{pmatrix}$

$$A^TA=\begin{pmatrix}2&1&1&1\\1&0&1&1\\-2&-1&0&-1\end{pmatrix}\begin{pmatrix}2&1&-2\\1&0&-1\\1&1&0\\1&1&-1\end{pmatrix}$$

$$=\begin{pmatrix}7&4&-6\\4&3&-3\\-6&-3&6\end{pmatrix}$$

$$A^T\boldsymbol{u}=\begin{pmatrix}2&1&1&1\\1&0&1&1\\-2&-1&0&-1\end{pmatrix}\begin{pmatrix}6\\3\\9\\6\end{pmatrix}$$

$$=\begin{pmatrix}30\\21\\-21\end{pmatrix}$$

The normal solution is $A^TA\boldsymbol{x}=A^T\boldsymbol{u}$

$$\begin{pmatrix}7&4&-6\\4&3&-3\\-6&-3&6\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}30\\21\\-21\end{pmatrix}\quad\Rightarrow\quad \boldsymbol{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}6\\3\\4\end{pmatrix}$$

So $proj_W\,\boldsymbol{u}=A\boldsymbol{x}=\begin{pmatrix}2&1&-2\\1&0&-1\\1&1&0\\1&1&-1\end{pmatrix}\begin{pmatrix}6\\3\\4\end{pmatrix}=\begin{pmatrix}7\\2\\9\\5\end{pmatrix}$

$proj_W\,\boldsymbol{u}=(7,\ 2,\ 9,\ 5)$

c) $\boldsymbol{u}=(-2,\ 0,\ 2,\ 4);\quad \boldsymbol{v}_1=(1,\ 1,\ 3,\ 0),\quad \boldsymbol{v}_2=(-2,\ -1,\ -2,\ 1),\quad \boldsymbol{v}_3=(-3,\ -1,\ 1,\ 3)$

Let $A=\begin{pmatrix}1&-2&-3\\1&-1&-1\\3&-2&1\\0&1&3\end{pmatrix}$

$$A^T A = \begin{pmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{pmatrix}$$

$$A^T \boldsymbol{u} = \begin{pmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix}$$

The normal solution is $A^T A\boldsymbol{x} = A^T \boldsymbol{u}$

$$\begin{pmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} \\ -\frac{8}{5} \\ \frac{8}{5} \end{pmatrix}$$

So $proj_W \boldsymbol{u} = A\boldsymbol{x} = \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} \\ -\frac{8}{5} \\ \frac{8}{5} \end{pmatrix}$

$$= \begin{pmatrix} -\frac{12}{5} \\ -\frac{4}{5} \\ \frac{12}{5} \\ \frac{16}{5} \end{pmatrix}$$

$$proj_W \boldsymbol{u} = \left(-\frac{12}{5},\ -\frac{4}{5},\ \frac{12}{5},\ \frac{16}{5}\right)$$

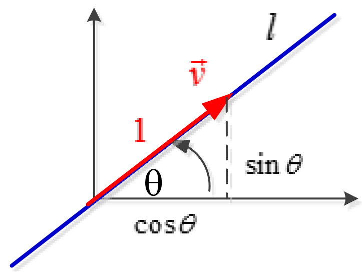
Exercise

Find the standard matrix for the orthogonal projection P of $\boldsymbol{R}^2$ on the line passes through the origin and makes an angle θ with the positive x-axis.

Solution

Since the line l in 2-dimensional, than we can take $\vec{v} = (\cos\theta,\ \sin\theta)$ as a basis for this subspace

$$A = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$



$$[P] = A^T A = [\cos\theta \quad \sin\theta]\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

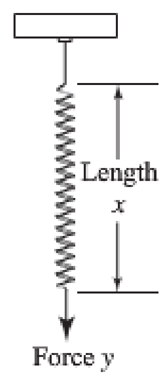
$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

Exercise

Hooke's law in physics states that the length x of a uniform spring is a linear function of the force y applied to it. If we express the relationship as $y = mx + b$, then the coefficient m is called the spring constant. Suppose a particular unstretched spring has a measured length of 6.1 *inches*.(i.e., x = 6.1 when y = 0). Forces of 2 pounds, 4 pounds, and 6 pounds are then applied to the spring, and the corresponding lengths are found to be 7.6 inches, 8.7 inches, and 10.4 inches. Find the spring constant.

Solution

$$M = \begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$



The normal equation: $A^T A\boldsymbol{x} = A^T \boldsymbol{y}$

$$\begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 278.82 & 32.8 \\ 32.8 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 112.4 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{39.44}\begin{pmatrix} 4 & -32.8 \\ -32.8 & 278.2 \end{pmatrix}\begin{pmatrix} 112.4 \\ 12 \end{pmatrix} = \begin{pmatrix} 1.4 \\ -8.8 \end{pmatrix}$$

Thus, the estimated value of the spring constant is ≈ 1.4 *pounds*.

Exercise

Prove: If A has a linearly independent column vectors, and if $\boldsymbol{b}$ is orthogonal to the column space of A, then the least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x} = 0$.

Solution

If A has linearly independent column vectors, then $A^T A$ is invertible and the least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$ is the solution of $A^T A\boldsymbol{x} = A^T \boldsymbol{b}$, but since $\boldsymbol{b}$ is orthogonal to the column space of A. $A^T \boldsymbol{b} = 0$, so $\boldsymbol{x}$ is a solution of $A^T A\boldsymbol{x} = 0$. Thus $\boldsymbol{x} = 0$ since $A^T A$ is invertible.

Exercise

Let A be an $m \times n$ matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of R^n onto the row space of A.

Solution

A^T will have linearly independent column vectors, and the column space A^T is the row space of A. Thus, the standard matrix for the orthogonal projection of R^n onto the row space of A is

$$[P] = A^T\left[\left(A^T\right)^T A^T\right]^{-1}\left(A^T\right)^T = A^T\left(AA^T\right)^{-1} A$$

Exercise

Let W be the line with parametric equations $x = 2t, \quad t = -t, \quad z = 4t$

a) Find a basis for W.

b) Find the standard matrix for the orthogonal projection on W.

c) Use the matrix in part (b) to find the orthogonal projection of a point $P_0\left(x_0, y_0, z_0\right)$ on W.

d) Find the distance between the point $P_0(2, 1, -3)$ and the line W.

Solution

a) $W = span\{(2, -1, 4)\}$ so that the vector $(2, -1, 4)$ forms a basis for W (linear independence)

b) Let $A = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$

$$[P] = A\left(A^T A\right)^{-1} A^T$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \left(\begin{bmatrix} 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} [21]^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \frac{1}{21} \begin{bmatrix} 4 & -2 & 8 \\ -2 & 1 & -4 \\ 8 & -4 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix}$$

c) $$\begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \frac{4}{21} x_0 - \frac{2}{21} y_0 + \frac{8}{21} z_0 \\ -\frac{2}{21} x_0 + y_0 - \frac{4}{21} z_0 \\ \frac{8}{21} x_0 - \frac{4}{21} y_0 + \frac{16}{21} z_0 \end{bmatrix}$$

d) $$\begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{6}{7} \\ \frac{3}{7} \\ -\frac{12}{7} \end{bmatrix}$$

The distance between P_0 and W equals to the distance between P_0 and its projection on W.

The distance between $(2, 1, -3)$ and $\left(-\frac{6}{7}, \frac{3}{7}, -\frac{12}{7}\right)$ is

$$d = \sqrt{\left(2+\frac{6}{7}\right)^2 + \left(1-\frac{3}{7}\right)^2 + \left(-3+\frac{12}{7}\right)^2}$$

$$= \sqrt{\frac{400}{49} + \frac{16}{49} + \frac{81}{49}}$$

$$= \frac{\sqrt{497}}{7}$$

Exercise

In R^3, consider the line l given by the equations $\quad x=t, \quad t=t, \quad z=t$

And the line m given by the equations $\quad x=s, \quad t=2s-1, \quad z=1$

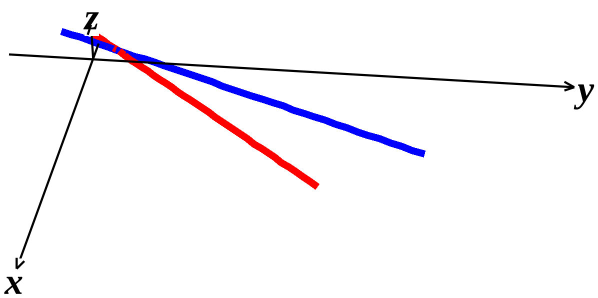
Let P be the point on l, and let Q be a point on m. Find the values of t and s that minimize the distance between the lines by minimizing the squared distance $\|P-Q\|^2$

Solution

When $t=1 \quad \Rightarrow \quad Let\ P=(1,\ 1,\ 1)$ is on line l

When $s=1 \quad \Rightarrow \quad Let\ Q=(1,\ 1,\ 1)$ is on line m

$$\|P-Q\| = \sqrt{(1-1)^2+(1-1)^2+(1-1)^2} = 0 \geq 0$$



Thus, these are the values $P=(1,\ 1,\ 1)$ and $Q=(1,\ 1,\ 1)$ are the values for $s=t=1$ that minimize the distance between the lines.

Exercise

Determine whether the statement is true or false,

a) If A is an $m\times n$ matrix, then $A^T A$ is a square matrix.

b) If $A^T A$ is invertible, then A is invertible.

c) If A is invertible, then $A^T A$ is invertible.

d) If $A\boldsymbol{x}=\boldsymbol{b}$ is a consistent linear system, then $A^T A\boldsymbol{x}=A^T\boldsymbol{b}$ is also consistent.

e) If $A\boldsymbol{x}=\boldsymbol{b}$ is an inconsistent linear system, then $A^T A\boldsymbol{x}=A^T\boldsymbol{b}$ is also inconsistent.

f) Every linear system has a least squares solution.

g) Every linear system has a unique least squares solution.

h) If A is an $m\times n$ matrix with linearly independent columns and $\boldsymbol{b}$ is in R^m, then $A\boldsymbol{x}=\boldsymbol{b}$ has a unique least squares solution.

Solution

a) ***True***; $A^T A$ is an $n\times n$ matrix

b) ***False***; only square matrix has inverses, but $A^T A$ can be invertible when A is not square matrix.

c) ***True***; if A is invertible, so is A^T, so the product $A^T A$ is also invertible

d) ***True***

e) ***False***; the system $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$ may be consistent

f) ***True***

g) ***False***; the least squares solution may involve a parameter

h) ***True***; if A has linearly independent column vectors; then $A^T A$ is invertible, so $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$ has a unique solution