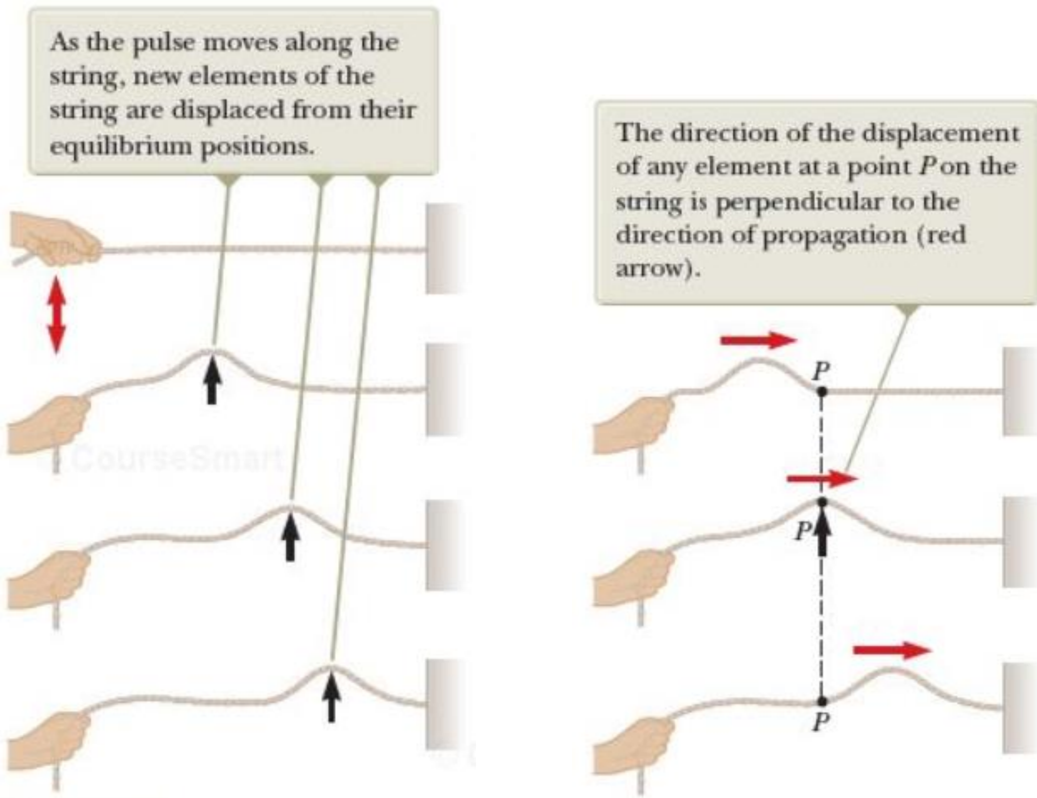
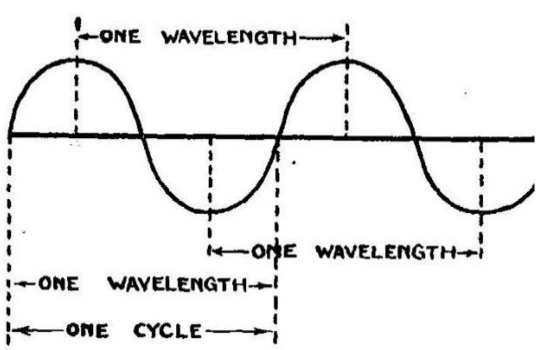
1.1 – Wave Motion

A wave is a phenomenon in which a certain physical quantity varies as a function position and the time resulting in the transfer of energy from one point to another through a medium. For example ocean waves the physical quantity that varies as a function of position and time is the up and down vertical displacement of the water molecules resulting in the transmission of energy from the source say to the beaches. If a snap shot of the ocean is taken, a variation of the vertical displacement (y) of the water molecules as a function of position can be observed. The distance between two consecutive peaks of this variation is called wavelength (λ) of the wave.





If we look at a particular point in the ocean, a molecule moving up and down as a function of time can be observed. The time taken for a certain molecule to make one complete up and down oscillation is called the period (T) of the wave.

A peak of a wave travels a distance of one wavelength in one period. Therefore the speed of a wave can be obtained as a ratio b/n the wavelength and period.

$$speed(v) = \frac{distance}{time} = \frac{wavelength\ (\lambda)}{period(T)}$$
$$\therefore v = \lambda T$$

or with $T = \frac{1}{f} \Rightarrow \boxed{v = \lambda f}$

v: speed of wave
λ: wavelength
f: frequency

Example

Consecutive peaks of an ocean are separated by a distance of 3m. If the peaks of the wave are approaching the beach with a speed of 5m/s, calculate the time taken for the water molecules to complete one up and down oscillation.

Solution

λ = 3m; v = 5m/s; T = ?

$$T = \frac{\lambda}{v} = \frac{3m}{5\frac{m}{s}} = 0.6s$$

The Wave Equation

The wave equation is an equation that results from application of Newton's 2nd law ($F = ma$) to the particle through which the wave is propagating. If y is the physical quantity that varies as a function of position and time and the wave is travelling along the x-axis, the wave equation can be written as

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad (\textit{the wave equation})$$

As will be shown later, v is the speed of the wave. The solution of the wave equation is any function whose argument is $x \pm vt$ that is $y = f(x \pm vt)$

Example

For any function f, shown that $f(x - vt)$ is a solution of the wave equation by direct substitution.

Solution

Let $x - vt = u$

Then $f(x - vt) = f(u)$

$$\frac{\partial u}{\partial x} = \frac{\partial(x - vt)}{\partial x} = 1$$

$$\frac{\partial u}{\partial t} = \frac{\partial(x - vt)}{\partial x} = -v$$

With $y = f(u)$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 f(u)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f(u)}{\partial t^2}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f(u)}{\partial x}\right) = \frac{1}{v^2}\frac{\partial}{\partial t}\left(\frac{\partial f(u)}{\partial t^2}\right)$$

But $\dfrac{\partial f\left(u\right)}{\partial x} = \dfrac{\partial f\left(u\right)}{\partial u}\dfrac{\partial u}{\partial x} = \dfrac{\partial f\left(u\right)}{\partial u}$

Because $\dfrac{\partial u}{\partial x} = 1$

And $\dfrac{\partial f\left(u\right)}{\partial t} = \dfrac{\partial f\left(u\right)}{\partial u}\dfrac{\partial u}{\partial t} = -v\dfrac{\partial f\left(u\right)}{\partial u}$

Because $\dfrac{\partial u}{\partial t} = -v$

$$\therefore \frac{\partial}{\partial x}\left(\frac{\partial f(u)}{\partial x}\right) = \frac{1}{v^2}\frac{\partial}{\partial t}\left(\frac{\partial f(u)}{\partial u}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f(u)}{\partial x}\right) = \frac{1}{v^2}\frac{\partial}{\partial t}\left(-v\,\frac{\partial f(u)}{\partial u}\right)$$

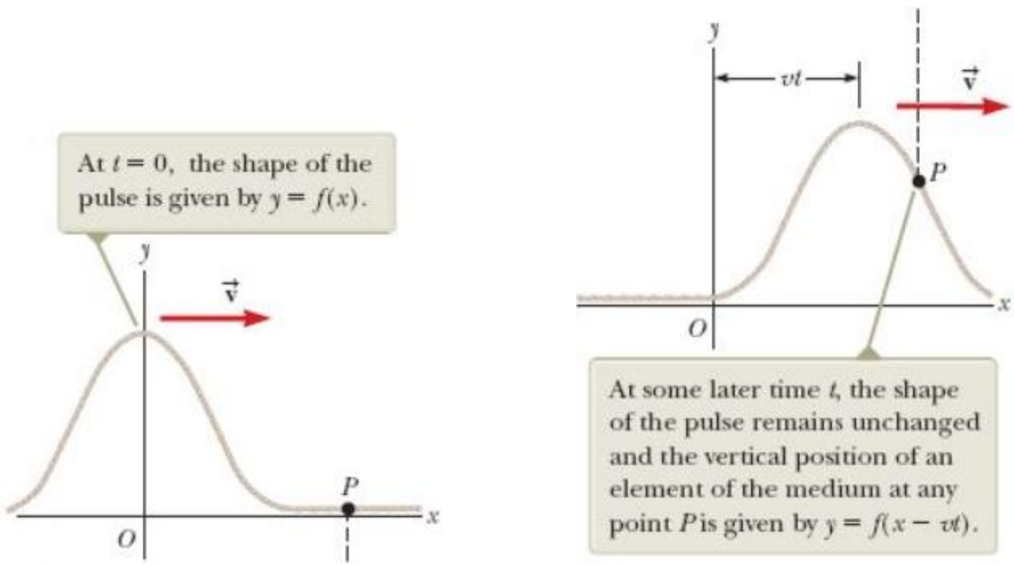
$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial f(u)}{\partial x}\right)\frac{\partial u}{\partial x} = \frac{1}{v^2}(-v)\frac{\partial}{\partial u}\frac{\partial f(u)}{\partial u}\frac{\partial u}{\partial t}$$

$$\Rightarrow \frac{\partial^2 f(u)}{du^2} = \frac{(-v)(-v)}{v^2}\frac{\partial^2 f(u)}{du^2}$$

$$\Rightarrow \frac{\partial^2 f(u)}{du^2} = \frac{\partial^2 f(u)}{du^2}$$

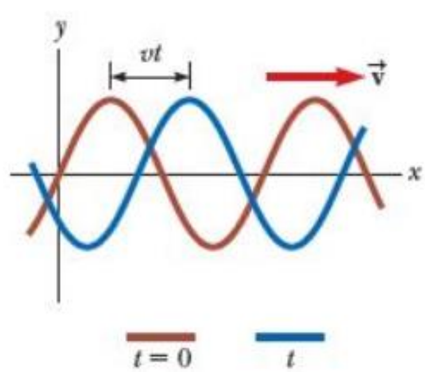
$f(x-vt)$ and $f(x+vt)$ as Travelling Wave

$f(x \pm vt)$ represents the value of the wave, *y*, at location and after a time inverted t. The value function corresponding to a constant argument will be constant. But for the argument to be a constant, $x \pm vt$ must be constant. For $x \pm vt$ to be constant *x* must be changing with time because time is always increasing. This means the position of the value of the function corresponding to a certain constant argument must be changing with time. In other words it must be travelling. For example the peak of an ocean wave is a constant corresponding to a constant argument of the function. But for the argument $x \pm vt$ to be constant, *x* must be changing with time because time is increasing. This implies the peak of the wave must be travelling with time which we know is the case.



The speed of a wave is defined to be the speed of a certain constant value of the function which can be taken to be the peak of the wave. But this necessitates that the argument $x \pm vt$ must be constant. $x \pm vt = C$. The speed of the wave is $\frac{dx}{dt}$. Taking the derivative of both sides.

$$\frac{dx}{dt} \pm v = 0 \quad \Rightarrow \quad \frac{dx}{dt} = \pm v$$



This shows that the variable *v* in the wave equation actually represents the speed of the wave. This also shows that argument $x \pm vt$ represents a wave moving to the left (*negative velocity*) and the argument $x - vt$ represents a wave moving to the right (*positive velocity*). That is, $f(x - vt)$ is a wave moving to the right and $f(x + vt)$ is a wave moving to the left.

Example

A stone is dropped in water initiating a disturbance that satisfies the wave equation $\frac{\partial^2 y}{\partial x^2} = 4\frac{\partial^2 y}{\partial t^2}$ where y is the vertical displacement (*disturbance*) of the water molecules. How far would the disturbance travel in 5 seconds?

<u>Solution</u>

$\Delta t = 5s$

Comparing the wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$ with this particular wave equation we see that

$\frac{1}{v^2} = 4 \;\Rightarrow\; v^2 = \frac{1}{4} \;\Rightarrow\; v = \frac{1}{2}$ $= 0.5\ s$

Since the disturbance is a constant value of the wave (*travels as it is*), its argument must be constant $\rightarrow$ $x - vt = constant$

Taking the example of both sides $\Delta x - v\Delta t = 0$

(*change of a constant is* $\rightarrow \Delta x - v\Delta t = (0.5)(5) = 2.5m$)

The disturbance will be 2.5*m* away after 5 seconds.

Harmonic (*Sinusoidal*) Wave

A typical wave in nature is sinusoidal wave which is a wave where the function f has a cosine or sine form. So far we have considered a function whose argument is distance $(x \pm vt)$. But the cosine and sine function take radians (*unit less*) as arguments. So the argument $x \pm vt$ needs to be modified to become unit less. Let's consider the argument $Kx - \omega t$ where K has units of 1/*m* and ω has units of 1/*s*.

Further since the periodicity of a cosine or sine function is 2π, we require that $Kx = 2\pi$ when $x = \lambda$ (*the wavelength*) and $\omega t = 2\pi$ when $t = T$ (*the period*)

$$Kx = 2\pi \Rightarrow K = \frac{2\pi}{x} \qquad \Rightarrow \omega T = 2\pi \quad \Rightarrow \quad \omega = \frac{2\pi}{T} = 2\pi f$$

K is called the wavenumber of the wave and ***ω*** is the angular frequency of the wave.

Therefore the solution of the wave equation in terms of a cosine function can be written as

$$y = A\sin(Kx \pm \omega t)$$

A: Amplitude (*maximum value*)
K: wavenumber
ω: angular frequency

A given value of the function (*say the peak of the wave*) occurs when the argument is a constant.

$$Kx \pm \omega t = C$$

Taking the derivative to obtain $\frac{dx}{dt}$ (*special*)

$$K\frac{dx}{dt} \pm \omega = 0$$

$$K\frac{dx}{dt} = I\omega \quad \Rightarrow \quad \frac{dx}{dt} = V = I\frac{\omega}{K}$$

That is the speed of the wave can also be expressed as the ratio between the angular frequency and the wavenumber. The negative sign (*corresponding to* $y = A\cos(Kx + \omega t)$) represents a wave to the left. And the positive sign (*corresponding to* $y = A\cos(Kx - \omega t)$) represents a wave travelling to the right. It can be shown by direct substitution that $y = A\cos(Kx \pm \omega t)$ is a solution to the wave equation with $v^2 = \frac{\omega^2}{K^2}$.

The displacement of the peak of the wave (*or other value*) can be obtained as speed times time

$$v = \frac{dx}{dt} = I\frac{\omega}{K}$$

$$\int_{x_i}^{x_f} dx = \pm\frac{\omega}{K}\int_0^t dt$$

$$\Delta x = I\frac{\omega}{K}t = Ivt$$

Example

A certain harmonic wave varies as a function of position and time according to the equation $y = 10\cos(10x - 20t)$. What is the maximum value of the wave?

a) Calculate the distance between two consecutive peaks of the wave.
b) Calculate the time taken for one complete up and down oscillation of the molecules.
c) Is the wave moving to the right or to the left?
d) Calculate the speed of the wave
e) Calculate the distance travelled by a peak of the wave in 10 seconds.

Solution

a) $K = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{K} = \frac{2\pi}{10} = 0.2\pi\ \frac{1}{m}$

b) $\omega = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{20} = 0.1\pi\ \frac{rad}{s}$

c) Since the equation is $10x - 20t$, it is going to the right

d) $v = \frac{\omega}{K} = \frac{20}{10} = 2\ \frac{m}{s}$

e) $\Delta x = \frac{\omega}{K}t = vt = (2)(10) = 20\ m$

Example

A disturbance at a certain point resulted in the following two waves: $y_1 = 4\cos(5x - 20t)$ and $y_2 = 4\cos(5x + 20t)$. Calculate the distance between the disturbances carried by the two waves after 4 seconds.

Solution

$y_1 = 4\cos(5x - 20t)$ represents a wave travelling to the right.

After 4 seconds the disturbance travels a distance of

$$\Delta x_1 = \frac{\omega}{K}t = \left(\frac{20}{4}\right)4 = 20m\ \text{to the right}$$

$y_2 = 4\cos(5x + 20t)$ represents a wave travelling to the left.

Therefore the disturbance would have travelled a distance of

$$\Delta x_2 = \frac{\omega}{K}t = \left(\frac{20}{4}\right)4 = 20m\ \text{to the left}$$

Therefore the distance between both disturbances after 4 seconds is

$$\Delta x_1 + \Delta x_2 = (20 + 20)m = 40\ m$$

Mechanical Energy of a Harmonic Wave

As shown in the previous chapter the displacement of a particle undergoing a harmonic oscillation is given by $y = A\cos(\omega t - \varphi)$ where φ is the phase angle and its mechanical energy is given by $E = \frac{1}{2}m\omega^2 A^2$, where m is the mass of the particle. In a harmonic wave, even though the disturbance is travelling, the particles of the medium carrying the wave are not travelling. Actually they are just oscillating back and forth like a harmonic oscillator. For example in water waves, even though the peak of the wave is moving away from the source, the wave molecules are just moving up and down like a harmonic oscillator.

Consider a particle at a certain location ($x = C$) in a medium where a harmonic wave is travelling. The particle oscillating is oscillating back and forth according to the equation

$$y = A\cos(Kx \pm \omega t) \quad or \quad y = A\cos(\omega t \pm Kx)$$

(*Remember* $\cos(-x) = \cos x$) with K*x* as its phase angle.

Therefore all the particles of the medium will have the same mechanical energy even though they might have different phase angles (*because of different locations or x*). The mechanical energy of a particle of mass d*m* at any location through which a harmonic wave is travelling is given by

$$dE = \frac{1}{2}dm\omega^2 A^2$$

And of course the total energy of all the particles undergoing harmonic oscillation can be obtained by interpreting this equation.

$$E_{total} = \frac{1}{2}M\omega^2 A^2$$

Where *M* is the total mass of all the particles.

But a more interesting quantity is dealing with waves is the rate of transmission of energy-which is the amount of energy that crosses perpendicular cross-sectional area per unit time. This is referred as transmission power (*P*). The average transmission rate can be obtained as the ratio between the amount of energy that crosses a perpendicular cross-sectional area in one cycle (or wavelength) to the time taken for one cycle (or one period).

$$\bar{P} = \frac{E_\lambda}{T}$$

P: average transmission power

E_λ : Energy that crosses a ⊥ area in one cycle.

T: period.

$$E_\lambda = \int_x^{x+\lambda} dE = \int_x^{x+d\lambda} \frac{1}{2}dm\omega^2 A^2$$

For simplicity, let's consider a harmonic wave travelling in a uniform string. If μ is the mass per unit length of the string, then $dm = \mu dx$

$$\therefore E_\lambda = \int_x^{x+\lambda} \frac{1}{2}\mu\omega^2 A^2 dx = \frac{1}{2}\mu\omega^2 A^2 \int_x^{x+\lambda} dx$$

$$\int_x^{x+\lambda} dx = \lambda$$

$$\Rightarrow E_\lambda = \frac{1}{2}\mu\omega^2 A^2 \lambda$$

$$\therefore \bar{P} = \frac{E_\lambda}{T} = \frac{\frac{1}{2}\mu\omega^2 A^2 \lambda}{T} = \frac{1}{2}\mu\omega^2 A^2 \left(\frac{\lambda}{T}\right)$$

But $\frac{\lambda}{T} = v$ (*speed of the wave*)

$$\therefore \bar{P} = \frac{1}{2}\mu\omega^2 A^2 v$$

(*transmission power for a harmonic wave travelling in a string of linear density μ, unit power is J/s or Watt (W)*)

The rate of transmission of energy is proportional to the square of the frequency (ω), to the square of the amplitude (A) and to the speed of the wave.

Example

A harmonic wave of the form $y = 0.01\cos(10x - 500t)$ is travelling in a string whose mass per unit length is 0.004 *kg/m*. Calculate the amount of energy that crosses a certain point of the string per a unit time.

Solution

Given: $\mu = 0.004 kg/m$; $A = 0.01m$; $\omega = 500 rad/s$; $K = 10$

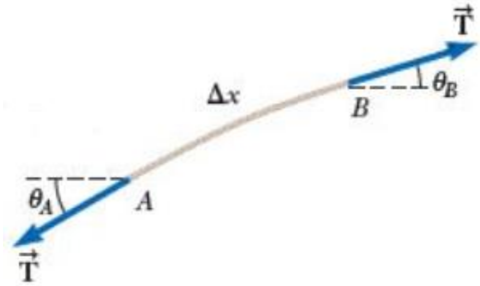
$$v = \frac{\omega}{K} = \frac{500}{10} = 50 m/s$$

$$\bar{P} = \frac{1}{2}\mu\omega^2 A^2 v$$

$$= \frac{1}{2}(0.004)(500)^2(0.01)^2(50)$$

$$= 2.5 \ Watt$$

Speed of a Wave in a String



The speed of a wave in a string can be obtained by comparing the general form of the wave equation, with the wave equation obtained by applying Newton's 2nd law to a small mass element of the string.

T = tension

A small mass element of a string is subjected to tension forces from both of its ends. The net vertical force is equal to $T\sin\theta_A - \sin\theta_B$. The tangents of θ_A and θ_B should be equal to the slopes of the string at the respective locations.

Therefore

$$\tan\theta_B = \left.\frac{dy}{dx}\right|_x \quad and \quad \tan\theta_A = \left.\frac{dy}{dx}\right|_{x+\Delta x}$$

$$\therefore F_{net}^{y} = T(\sin\theta_A - \sin\theta_B) = T\left(\sin\left[tan^{-1}\left(\left(\frac{dy}{dx}\right)_{x+\Delta x}\right)\right] - \sin\left[tan^{-1}\left(\frac{dy}{dx}\right)_x\right]\right)$$

But $\quad tan^{-1}\left(\left(\frac{dy}{dx}\right)_{x+\Delta x}\right) \approx \left(\frac{dy}{dx}\right)_{x+\Delta x}$

For small angles and

$$tan^{-1}\left(\left(\frac{dy}{dx}\right)_x\right) \approx \left(\frac{dy}{dx}\right)_x$$

for small angles

$$\therefore F_{net}^{y} \approx T\left(\sin\left[\left(\frac{dy}{dx}\right)_{x+\Delta x}\right] - \sin\left[\left(\frac{dy}{dx}\right)_x\right]\right)$$

Again since $\sin x \approx x$ for small angles it follows that

$$F_{net}^{y} \approx T\left(\left(\frac{dy}{dx}\right)_{x+\Delta x} - \left(\frac{dy}{dx}\right)_x\right)$$

for small angles

Applying Newton's 2nd law to the vertical motion

$$F_{net}^{y} \approx T\left(\left(\frac{dy}{dx}\right)_{x+\Delta x} - \left(\frac{dy}{dx}\right)_x\right) = dm\frac{\partial^2 y}{\partial t^2}$$

If μ is the mass per unit length of the string then $dm = \mu dx$

$$\left(\left(\frac{dy}{dx}\right)_{x+\Delta x} - \left(\frac{dy}{dx}\right)_{x}\right) = \Delta x = \frac{1}{\frac{T}{\mu}}\frac{\partial^2 y}{\partial t^2}$$

But as Δx approaches zero, the left side becomes $\frac{\partial^2 y}{\partial x^2}$

$$\therefore \frac{\partial^2 y}{\partial x^2} = \frac{1}{\frac{T}{\mu}}\frac{\partial^2 y}{\partial t^2}$$

Comparing this with the general wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$ it follows that

$$v^2 = \frac{T}{\mu} \quad or \quad v = \sqrt{\frac{T}{\mu}}$$

V: speed of a wave in a string,
T: tension in the string,
μ: mass per unit length of the string

Example

It is found that a harmonic wave of the form $y = 0.001\cos(5x - 400t)$ travels in a string when an object of mass 10kg hangs from it. Calculate the mass per unit length of the string.

Solution

Given: $m = 10\ kg$; $A = 0.01m$; $\omega = 400\ rad/s$; $K = 5$

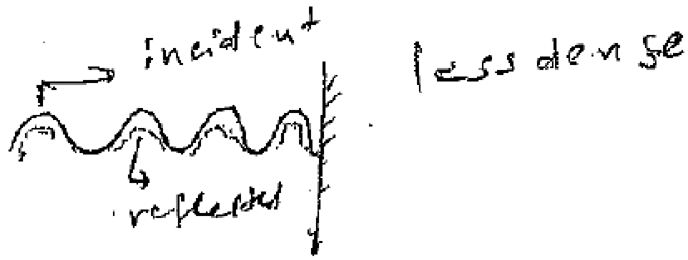
T = weight of hanging object
$= mg$
$= (10)(9.8)$
$= 98$ N

$$v = \frac{\omega}{K} = \frac{400}{5} = 80\ m/\text{s}$$

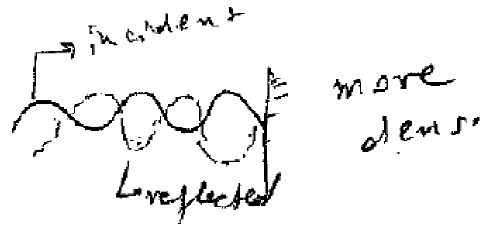
$$\mu = \frac{T}{v^2} = \frac{98}{80^2} \boxed{= 0.15\ kg/m}$$

Reflection and Transmission of Wave

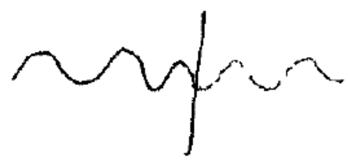
When the medium across which a wave is travelling changes some of the wave may be reflected and some of it may be transmitted. When a wave is reflected from a less dense medium, it is reflected without phase change.



When a wave is reflected from a denser medium, its phase angle changes by π radian or 180 degrees. In other words it is inverted



A transmitted wave does not undergo a phase change



Types of Waves

There are two kinds of waves. They are called transverse and longitudinal waves. A transverse wave is a wave where the direction of movement of the particles carrying the wave is perpendicular to the direction of propagation of energy. An example is an ocean wave. While the wave travels parallel to the surface of the ocean, the water molecules vibrate up and down perpendicular to the direction of propagation of energy.

Longitudinal waves are waves where the direction of vibration of the particles carrying the wave is parallel to the direction of propagation of energy. An example is sound wave. As sound travels through a medium the molecules of the medium vibrate back and forth in the direction of propagations of sound.