# Solution    Section 4.8 – Divergence Theorem

### Exercise

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. $\vec{F} = \langle 2x,\ 3y,\ 4z \rangle \quad D = \left\{ (x, y, z):\ x^2 + y^2 + z^2 \le 4 \right\}$

**Solution**

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(4z)$$

$$= 9$$

$$\iiint_D \nabla \bullet \vec{F}\ dV = \iiint_D (9)\ dV$$

$$= 9 \cdot volume(D) \qquad x^2 + y^2 + z^2 = 4 = r^2$$

$$= 9 \cdot \frac{4\pi}{3} 2^3 \qquad volume(D) = \frac{4\pi r^3}{3}$$

$$= 96\pi$$

$$\vec{r} = \langle x,\ y,\ z \rangle \qquad R = \left\{ (\phi,\ \theta):\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi \right\}$$

$$= \langle 2\sin\phi\cos\theta,\ 2\sin\phi\sin\theta,\ 2\cos\phi \rangle$$

$$\vec{t}_\phi = \langle 2\cos\phi\cos\theta,\ 2\cos\phi\sin\theta,\ -2\sin\phi \rangle$$

$$\vec{t}_\theta = \langle -2\sin\phi\sin\theta,\ 2\sin\phi\cos\theta,\ 0 \rangle$$

$$\vec{t}_\phi \times \vec{t}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \left(4\sin^2\phi\cos\theta\right)\hat{i} + \left(4\sin^2\phi\sin\theta\right)\hat{\boldsymbol{j}} + \left(4\sin\phi\cos\phi\cos^2\theta + 4\sin\phi\cos\phi\sin^2\theta\right)\hat{\boldsymbol{k}}$$

$$= \left\langle 4\sin^2\phi\cos\theta,\ 4\sin^2\phi\sin\theta,\ 4\sin^2\phi\cos\phi \right\rangle$$

$$\vec{F} = \langle 2x,\ 3y,\ 4z \rangle$$

$$= \langle 2(2\sin\phi\cos\theta),\ 3(2\sin\phi\sin\theta),\ 4(2\cos\phi) \rangle$$

$$\vec{F} \bullet \left(\vec{t}_\phi \times \vec{t}_\theta\right) = \langle 4\sin\phi\cos\theta,\ 6\sin\phi\sin\theta,\ 8\cos\phi \rangle \bullet \left\langle 4\sin^2\phi\cos\theta,\ 4\sin^2\phi\sin\theta,\ 4\sin^2\phi\cos\phi \right\rangle$$

$$= 16\sin^3\phi\cos^2\theta + 24\sin^3\phi\sin^2\theta + 32\sin\phi\cos^2\phi$$

$$\iint_S \vec{F}\cdot\vec{n}\ dS = \iint_R \nabla\vec{F}\cdot\left(\vec{t}_\phi \times \vec{t}_\theta\right)\ dA$$

$$= 8\int_0^{2\pi}\int_0^{\pi}\left(2\cos^2\theta\sin^3\phi + 3\sin^2\theta\sin^3\phi + 4\sin\phi\cos^2\phi\right)\ d\phi d\theta$$

$$= 8\int_0^{2\pi}\int_0^{\pi}\left(\left(2\cos^2\theta + 3\sin^2\theta\right)\sin^2\phi + 4\cos^2\phi\right)\sin\phi\ d\phi d\theta$$

$$= -8\int_0^{2\pi}\int_0^{\pi}\left[\left(2\cos^2\theta + 3\sin^2\theta\right)\left(1-\cos^2\phi\right) + 4\cos^2\phi\right]\ d\left(\cos\phi\right)d\theta$$

$$= -8\int_0^{2\pi}\left[\left(2\cos^2\theta + 3\left(1-\cos^2\theta\right)\right)\left(\cos\phi - \frac{1}{3}\cos^3\phi\right) + \frac{4}{3}\cos^3\phi\right]_0^{\pi}\ d\theta$$

$$= -8(2)\int_0^{2\pi}\left[\left(3-\cos^2\theta\right)\left(-\frac{2}{3}\right) - \frac{4}{3}\right]\ d\theta$$

$$= -16\int_0^{2\pi}\left[\frac{2}{3}\left(\frac{1+\cos 2\theta}{2}\right) - \frac{10}{3}\right]\ d\theta$$

$$= -\frac{16}{3}\int_0^{2\pi}\left(1+\cos 2\theta - 10\right)\ d\theta$$

$$= -\frac{16}{3}\ \left(\frac{1}{2}\sin 2\theta - 9\theta\ \right|_0^{2\pi}$$

$$= -\frac{16}{3}\left(-18\pi\right)$$

$$\underline{= 96\pi}$$

### *Exercise*

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. $\vec{F} = \langle -x,\ -y,\ -z\rangle \quad D = \left\{\left(x,y,z\right):\ |x|\le 1,\ |y|\le 1,\ |z|\le 1\right\}$

***Solution***

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(-z)$$

$$\underline{= -3}$$

$$\iiint_D \nabla\cdot\vec{F}\ dV = \iiint_D (-3)\ dV$$

$$= -3 \cdot volume(D) \qquad volume(D) = 2 \cdot 2 \cdot 2 \quad (\textit{cube of distance between } -1 \ \& \ 1 = 2)$$

$$= -3 \cdot (2)^3$$

$$\underline{= -24}$$

Since the surface has a form of cube, therefore we have 6 surfaces

$$S_1 : x = -1 \quad \to \quad \vec{n}_1 = \langle -1, \ 0, \ 0 \rangle$$

$$\vec{F} \bullet \vec{n}_1 = \langle 1, \ -y, \ -z \rangle \bullet \langle -1, \ 0, \ 0 \rangle$$

$$\underline{= -1}$$

$$S_2 : x = 1 \quad \to \quad \vec{n}_2 = \langle 1, \ 0, \ 0 \rangle$$

$$\vec{F} \bullet \vec{n}_2 = \langle -1, \ -y, \ -z \rangle \bullet \langle 1, \ 0, \ 0 \rangle$$

$$\underline{= -1}$$

$$S_3 : y = -1 \quad \to \quad \vec{n}_3 = \langle 0, \ -1, \ 0 \rangle$$

$$\vec{F} \bullet \vec{n}_3 = \langle -x, \ 1, \ -z \rangle \bullet \langle 0, \ -1, \ 0 \rangle$$

$$\underline{= -1}$$

$$S_4 : y = 1 \quad \to \quad \vec{n}_4 = \langle 0, \ 1, \ 0 \rangle$$

$$\vec{F} \bullet \vec{n}_4 = \langle -x, \ -1, \ -z \rangle \bullet \langle 0, \ 1, \ 0 \rangle$$

$$\underline{= -1}$$

$$S_5 : z = -1 \quad \to \quad \vec{n}_5 = \langle 0, \ 0, \ -1 \rangle$$

$$\vec{F} \bullet \vec{n}_5 = \langle -x, \ -y, \ 1 \rangle \bullet \langle 0, \ 0, \ -1 \rangle$$

$$\underline{= -1}$$

$$S_6 : z = 1 \quad \to \quad \vec{n}_6 = \langle 0, \ 0, \ 1 \rangle$$

$$\vec{F} \bullet \vec{n}_6 = \langle -x, \ -y, \ -1 \rangle \bullet \langle 0, \ 0, \ 1 \rangle$$

$$\underline{= -1}$$

$$\iint_S \vec{F} \bullet \vec{n} \ dS = \sum_{k=1}^{6} \iint_{S_k} \vec{F} \bullet \vec{n}_k \ dS$$

$$= 6 \int_{-1}^{1} dz \int_{-1}^{1} dy \int_{-1}^{0} (-1) \ dx$$

$$= 6 \left( z \Big|_{-1}^{1} \right. \left( y \Big|_{-1}^{1} \right. \left( -x \Big|_{-1}^{0} \right.$$

$$= 6(2)(2)(-1)$$

$$\underline{= -24}$$

## ***Exercise***

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. $\vec{F} = \langle z-y,\ x,\ -x\rangle$ $\quad D = \left\{(x,y,z):\ \frac{x^2}{4} + \frac{y^2}{8} + \frac{z^2}{12} \le 1\right\}$

***Solution***

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(z-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(-x)$$

$$= 0$$

$$\iiint_D \nabla \cdot \vec{F}\ dV = \iiint_D (0)\ dV$$

$$= 0$$

$$\frac{x^2}{4} + \frac{y^2}{8} + \frac{z^2}{12} = 1 \quad \to \quad x^2 = 4,\ y^2 = 8,\ z^2 = 12$$

$$\vec{t} = \left\langle 2\sin u\cos v,\ 2\sqrt{2}\ \sin u\sin v,\ 2\sqrt{3}\ \cos u\right\rangle$$

$$\vec{t}_u = \left\langle 2\cos u\cos v,\ 2\sqrt{2}\ \cos u\sin v,\ -2\sqrt{3}\ \sin u\right\rangle$$

$$\vec{t}_v = \left\langle -2\sin u\sin v,\ 2\sqrt{2}\ \sin u\cos v,\ 0\right\rangle$$

$$\vec{n} = \vec{t}_u \times \vec{t}_v$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos u\cos v & 2\sqrt{2}\ \cos u\sin v & -2\sqrt{3}\ \sin u \\ -2\sin u\sin v & 2\sqrt{2}\ \sin u\cos v & 0 \end{vmatrix}$$

$$= \left\langle 4\sqrt{6}\ \sin^2 u\cos v,\ 4\sqrt{3}\ \sin^2 u\sin v,\ 4\sqrt{2}\ \sin u\cos u\cos^2 v + 4\sqrt{2}\ \sin u\cos u\sin^2 v\right\rangle$$

$$= \left\langle 4\sqrt{6}\ \sin^2 u\cos v,\ 4\sqrt{3}\ \sin^2 u\sin v,\ 4\sqrt{2}\ \sin u\cos u\right\rangle$$

$$\vec{F} = \langle z-y,\ x,\ -x\rangle$$

$$= \left\langle 2\sqrt{3}\cos u - 2\sqrt{2}\sin u\sin v,\ 2\sin u\cos v,\ -2\sin u\cos v\right\rangle$$

$$\vec{F} \cdot \vec{n} = \left\langle 2\sqrt{3}\cos u - 2\sqrt{2}\sin u\sin v,\ 2\sin u\cos v,\ -2\sin u\cos v\right\rangle$$

$$\cdot \left\langle 4\sqrt{6}\ \sin^2 u\cos v,\ 4\sqrt{3}\ \sin^2 u\sin v,\ 4\sqrt{2}\ \sin u\cos u\right\rangle$$

$$= 24\sqrt{2}\cos u\sin^2 u\cos v - 16\sqrt{3}\sin^3 u\cos v\sin v + 8\sqrt{3}\sin^3 u\sin v\cos v - 8\sqrt{2}\sin^2 u\cos u\cos v$$

$$= 16\sqrt{2}\cos u\sin^2 u\cos v - 8\sqrt{3}\sin^3 u\sin v\cos v$$

$$\iint_S \vec{F}\bullet\vec{n}\ dS = \int_0^{2\pi}\int_0^{\pi}\left(16\sqrt{2}\cos u\sin^2 u\cos v - 8\sqrt{3}\sin^3 u\sin v\cos v\right)\ dudv$$

$$= 8\left[\int_0^{2\pi}\int_0^{\pi}\left(2\sqrt{2}\sin^2 u\cos v\right)\ d(\sin u) + \int_0^{\pi}\sqrt{3}\left(1-\cos^2 u\right)\sin v\cos v\ d(\cos u)\right]\ dv$$

$$= 8\left[\int_0^{2\pi}\frac{2\sqrt{2}}{3}\cos v\left(\sin^2 u\ \Big|_0^{\pi}\ +\ \sqrt{3}\sin v\cos v\left(\cos u - \frac{1}{3}\cos^3 u\ \Big|_0^{\pi}\right]\ dv$$

$$= 8\int_0^{2\pi}\left(\frac{2\sqrt{2}}{3}\cos v(0) + \sqrt{3}\sin v\cos v\left(-2+\frac{2}{3}\right)\right)\ dv$$

$$= -\frac{64\sqrt{3}}{3}\int_0^{2\pi}\sin v\ d(\sin v)$$

$$= -\frac{64\sqrt{3}}{3}\ \left(\frac{1}{2}\sin v\ \Big|_0^{2\pi}\right.$$

$$\underline{= 0\ |}$$

### *Exercise*

Evaluate both integrals of the Divergence Theorem for the following vector field and region. Check for agreement. $\vec{F} = \left\langle x^2,\ y^2,\ z^2\right\rangle \quad D = \left\{(x,y,z):\ |x|\le 1,\ \ |y|\le 2,\ \ |z|\le 3\right\}$

***Solution***

$$\nabla\bullet\vec{F} = \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}\left(y^2\right) + \frac{\partial}{\partial z}\left(z^2\right)$$

$$\underline{= 2x + 2y + 2z\ |}$$

$$\iiint_D \nabla\bullet\vec{F}\ dV = 2\int_{-3}^{3}\int_{-2}^{2}\int_{-1}^{1}(x+y+z)\ dxdydz$$

$$= 2\int_{-3}^{3}\int_{-2}^{2}\int_{-1}^{1}(x+y+z)\ dxdydz$$

$$= 2\int_{-3}^{3}\int_{-2}^{2}\left(\frac{1}{2}x^2 + yx + zx\right)\Big|_{-1}^{1}\ dydz$$

$$= 2\int_{-3}^{3}\int_{-2}^{2}(2y+2z)\ dydz$$

$$= 2\int_{-3}^{3} \left( y^2 + 2zy \Big|_{-2}^{2} \right) dz$$

$$= 2\int_{-3}^{3} (8z)\, dz$$

$$= 2\left( 4z^2 \Big|_{-3}^{3} \right)$$

$$\underline{= 0}$$

$S_1 : x = -1 \quad \to \quad \vec{n}_1 = \langle -1,\ 0,\ 0 \rangle$

$$\vec{F} \cdot \vec{n}_1 = \left\langle x^2,\ y^2,\ z^2 \right\rangle \cdot \langle -1,\ 0,\ 0 \rangle$$

$$= -x^2 \Big|_{x=-1}$$

$$\underline{= -1}$$

$S_2 : x = 1 \quad \to \quad \vec{n}_2 = \langle 1,\ 0,\ 0 \rangle$

$$\vec{F} \cdot \vec{n}_2 = \left\langle x^2,\ y^2,\ z^2 \right\rangle \cdot \langle 1,\ 0,\ 0 \rangle$$

$$= x^2 \Big|_{x=1}$$

$$\underline{= 1}$$

$S_3 : y = -2 \quad \to \quad \vec{n}_3 = \langle 0,\ -1,\ 0 \rangle$

$$\vec{F} \cdot \vec{n}_3 = \left\langle x^2,\ y^2,\ z^2 \right\rangle \cdot \langle 0,\ -1,\ 0 \rangle$$

$$= -y^2 \Big|_{y=-2}$$

$$\underline{= -4}$$

$S_4 : y = 2 \quad \to \quad \vec{n}_4 = \langle 0,\ 1,\ 0 \rangle$

$$\vec{F} \cdot \vec{n}_4 = \left\langle x^2,\ y^2,\ z^2 \right\rangle \cdot \langle 0,\ 1,\ 0 \rangle$$

$$= y^2 \Big|_{y=2}$$

$$\underline{= 4}$$

$S_5 : z = -3 \quad \to \quad \vec{n}_5 = \langle 0,\ 0,\ -1 \rangle$

$$\vec{F} \cdot \vec{n}_5 = \left\langle x^2,\ y^2,\ z^2 \right\rangle \cdot \langle 0,\ 0,\ -1 \rangle$$

$$= -z^2 \Big|_{z=-3}$$

$$\underline{= -9}$$

$$S_6 : z = 3 \quad \rightarrow \quad \vec{n}_6 = \langle 0,\ 0,\ 1 \rangle$$

$$\vec{F} \bullet \vec{n}_6 = \left\langle x^2,\ y^2,\ z^2 \right\rangle \bullet \langle 0,\ 0,\ 1 \rangle$$

$$= z^2 \Big|_{z=3}$$

$$\underline{= 9}$$

$$\iint_S \vec{F} \bullet \vec{n}\ dS = \sum_{k=1}^{6} \iint_{S_k} \vec{F} \bullet \vec{n}_k\ dS$$

$$= \iint_S (-1+1-4+4-9+9)\,dS$$

$$= \iint_S (0)\,dS$$

$$\underline{= 0}$$

### *Exercise*

Use the Divergence Theorem to compute the outward flux of the vector field $\vec{F} = \langle -x,\ x-y,\ x-z \rangle$ across $S$ is the surface of the cube cut from the first octant by the planes $x = 1,\ y = 1,\ and\ z = 1$.

***Solution***

$$\nabla \bullet \vec{F} = \nabla \bullet \langle -x,\ x-y,\ x-z \rangle$$

$$= \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(x-y) + \frac{\partial}{\partial z}(x-z)$$

$$= -1-1-1$$

$$\underline{= -3}$$

$$\iint_S \vec{F} \bullet \vec{n}\ dS = \iiint_D \nabla \bullet \vec{F}\ dV$$

$$= -3 \times (Volume\ of\ the\ cube)$$

$$= -3 \times (1)$$

$$\underline{= -3}$$

## Exercise

Use the Divergence Theorem to compute the outward flux of the vector field $\vec{F} = \frac{1}{3}\left\langle x^3,\ y^3,\ z^3 \right\rangle$ across $S$ is the sphere $\left\{(x,\ y,\ z):\ x^2 + y^2 + z^2 = 9\right\}$

**Solution**

$$\begin{aligned}\nabla \bullet \vec{F} &= \frac{1}{3}\nabla \bullet \left\langle x^3,\ y^3,\ z^3 \right\rangle \\ &= \frac{1}{3}\left(\frac{\partial}{\partial x}\left(x^3\right) + \frac{\partial}{\partial y}\left(y^3\right) + \frac{\partial}{\partial z}\left(z^3\right)\right) \\ &= \frac{1}{3}\left(3x^2 + 3y^2 + 3z^2\right) \\ &= x^2 + y^2 + z^2\end{aligned}$$

$$\begin{aligned}\iint_S \vec{F} \bullet \vec{n}\ dS &= \iiint_D \left(x^2 + y^2 + z^2\right)\ dV \\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{3} \left(r^2\right) r^2 \sin u\ drdudv \\ &= \int_0^{2\pi} dv \int_0^{\pi} \sin u\ du \int_0^{3} r^4\ dr \\ &= (2\pi)\ \left(-\cos u\ \Big|_0^{\pi}\right)\ \left(\frac{1}{5}r^5\ \Big|_0^{3}\right) \\ &= (2\pi)(2)\left(\frac{243}{5}\right) \\ &= \frac{972\pi}{5}\end{aligned}$$

## Exercise

Use the Divergence Theorem to compute the outward flux of the vector field $\vec{F} = \frac{\vec{r}}{|\vec{r}|^3} = \frac{\langle x,\ y,\ z\rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ across $D$ is the region between two spheres of radius 1 and 2 centered at $(5,\ 5,\ 5)$

**Solution**

$$\vec{F} = \frac{\langle x,\ y,\ z\rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$\frac{\partial}{\partial x}\vec{F} = \frac{x^2+y^2+z^2-3x^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$= \frac{-2x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial y}\vec{F} = \frac{x^2+y^2+z^2-3y^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$= \frac{x^2-2y^2+z^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\frac{\partial}{\partial z}\vec{F} = \frac{x^2+y^2+z^2-3z^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$= \frac{x^2+y^2-2z^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\nabla \bullet \vec{F} = \frac{1}{\left(x^2+y^2+z^2\right)^{5/2}}\left(-2x^2+y^2+z^2+x^2-2y^2+z^2+x^2+y^2-2z^2\right)$$

$$\underline{= 0}$$

So, the flux is zero across any surface that bounds a region where $\vec{F}$ is defined and differentiable; the given region does not include zero.

Thus, the net outward flux is zero.

### *Exercise*

Use the Divergence Theorem to compute the outward flux of the vector field $\vec{F} = \left\langle x^2,\ y^2,\ z^2\right\rangle$; $S$ is the cylinder $\left\{(x,y,z):\ x^2+y^2=4,\ 0\le z\le 8\right\}$

***Solution***

$$\nabla \bullet \vec{F} = \nabla \bullet \left\langle x^2,\ y^2,\ z^2\right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^2\right)+\frac{\partial}{\partial y}\left(y^2\right)+\frac{\partial}{\partial z}\left(z^2\right)$$

$$\underline{= 2x+2y+2z}$$

$$x^2+y^2=4 \to\ 0\le r\le 2$$

$$0 \le z \le 8, \quad 0 \le \theta \le 2\pi$$

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$\iint_S \vec{F} \bullet \vec{n}\, dS = \iiint_D (2x + 2y + 2z)\, dV$$

$$= 2\int_0^8 \int_0^{2\pi} \int_0^2 (r\cos\theta + r\sin\theta + z) r\, dr d\theta dz$$

$$= 2\int_0^8 \int_0^{2\pi} \int_0^2 \left((\cos\theta + \sin\theta) r^2 + zr\right) dr d\theta dz$$

$$= 2\int_0^8 \int_0^{2\pi} \left(\frac{1}{3}(\cos\theta + \sin\theta) r^3 + \frac{1}{2} zr^2 \Big|_0^2\right. d\theta dz$$

$$= 2\int_0^8 \int_0^{2\pi} \left(\frac{8}{3}(\cos\theta + \sin\theta) + 2z\right) d\theta dz$$

$$= 2\int_0^8 \left(\frac{8}{3}(\sin\theta - \cos\theta) + 2z\theta \Big|_0^{2\pi}\right. dz$$

$$= 2\int_0^8 \left(-\frac{8}{3} + 4\pi z + \frac{8}{3}\right) dz$$

$$= 4\pi \int_0^8 (2z)\, dz$$

$$= 4\pi\, z^2 \Big|_0^8$$

$$= 256\pi$$

### Exercise

Find the net outward flux of the field $\vec{F} = \langle 2z - y,\ x,\ -2x \rangle$ across the sphere of radius 1 centered at the origin.

***Solution***

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(2z - y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(-2x)$$

$$= 0$$

So, by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla \bullet \vec{F}$ is zero.

### Exercise

Find the net outward flux of the field $\vec{F} = \langle bz - cy,\ cx - az,\ ay - bx \rangle$ across any smooth closed surface $\mathbb{R}^3$, where $a$, $b$, and $c$ are constants.

**Solution**

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(bz - cy) + \frac{\partial}{\partial y}(cx - az) + \frac{\partial}{\partial z}(ay - bx)$$

$$= 0$$

So, by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla \cdot \vec{F}$ is zero.

### Exercise

Find the net outward flux of the field $\vec{F} = \langle z - y,\ x - z,\ y - x \rangle$ across the boundary of the cube $\{(x,\ y,\ z):\ |x| \le 1,\ \ |y| \le 1\ \ |z| \le 1\}$

**Solution**

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(z - y) + \frac{\partial}{\partial y}(x - z) + \frac{\partial}{\partial z}(y - x)$$

$$= 0$$

So, by the Divergence Theorem, the net outward flux is zero since the volume integral of $\nabla \cdot \vec{F}$ is zero.

### Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface $S$. $\quad \vec{F} = \langle x,\ -2y,\ 3z \rangle$; $S$ is the sphere $\{(x, y, z):\ x^2 + y^2 + z^2 = 6\}$

**Solution**

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(3z)$$

$$= 1 - 2 + 3$$

$$= 2$$

The sphere has a radius $\sqrt{6}$, therefore the volume of the sphere is $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi 6\sqrt{6} = 8\pi\sqrt{6}$

$$\iiint_D \nabla \cdot \vec{F}\ dV = 2 \cdot (\text{Volume of sphere})$$

$$= 2\left(8\pi\sqrt{6}\right)$$

$$= 16\pi\sqrt{6}$$

## *Exercise*

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

$\vec{F} = \left\langle x^2,\ 2xz,\ y^2 \right\rangle$; *S* is surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$

***Solution***

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}(2xz) + \frac{\partial}{\partial z}\left(y^2\right)$$

$$\underline{= 2x\,|}$$

$$\iiint_D \nabla \bullet \vec{F}\ dV = \iiint_D 2x\ dV$$

$$= \int_0^1 \int_0^1 \int_0^1 (2x)\,dxdydz$$

$$= \int_0^1 dz \int_0^1 dy \quad \left(x^2 \Big|_0^1\right.$$

$$= z\Big|_0^1 \ \ y\Big|_0^1 \ \ (1)$$

$$\underline{= 1\,|}$$

## *Exercise*

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

$\vec{F} = \langle x,\ 2y,\ z\rangle$; *S* is boundary of the tetrahedron in the first octant formed by the plane $x + y + z = 1$

***Solution***

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(z)$$

$$\underline{= 4|}$$

So, by the Divergence Theorem, the net outward flux is 4 times the volume of the tetrahedron.

Volume of the tetrahedron $= \frac{1}{3}(\textit{area of the base})(\textit{height})$

$$\textit{Area of the base} = \frac{1}{2}(x)(y)$$

$$= \frac{1}{2}(1)(1)$$

$$=\boxed{\frac{1}{2}}$$

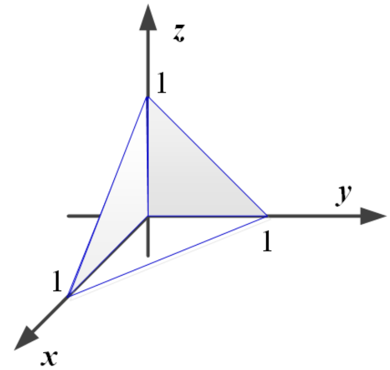
$$V=\frac{1}{3}(\text{area of the base})(\text{height})$$

$$=\frac{1}{3}\left(\frac{1}{2}\right)(1)$$

$$=\frac{1}{6}$$

$$\iiint\limits_D \nabla\cdot\vec{F}\,dV=4\left(\frac{1}{6}\right)$$

$$=\boxed{\frac{2}{3}}$$




## Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface $S$.

$\vec{F}=\left\langle x^2,\ y^2,\ z^2\right\rangle$; $S$ is the sphere $\left\{(x,y,z):\ x^2+y^2+z^2=25\right\}$

**Solution**

$$\nabla\cdot\vec{F}=\frac{\partial}{\partial x}\left(x^2\right)+\frac{\partial}{\partial y}\left(y^2\right)+\frac{\partial}{\partial z}\left(z^2\right)$$

$$=\boxed{2(x+y+z)}$$

So, by the Divergence Theorem, the net outward flux is

$$\iiint\limits_D \nabla\cdot\vec{F}\,dV=2\iiint\limits_D (x+y+z)\,dV$$

$$=2\int_0^{2\pi}\int_0^{\pi}\int_0^{5} 5r(\sin\varphi\cos\theta+\sin\varphi\sin\theta+\cos\varphi)\,dr\,d\varphi\,d\theta$$

$$=5\left(r^2\,\Big|_0^5\right.\int_0^{2\pi}\int_0^{\pi}(\sin\varphi\cos\theta+\sin\varphi\sin\theta+\cos\varphi)\,d\varphi\,d\theta$$

$$=125\int_0^{2\pi}\left(-\cos\varphi\cos\theta-\cos\varphi\sin\theta+\sin\varphi\,\Big|_0^{\pi}\right.d\theta$$

$$=125\int_0^{2\pi}2(\cos\theta+\sin\theta)\,d\theta$$

$$= 250\left(\sin\theta - \cos\theta\right)\Big|_0^{2\pi}$$

$$= 250\,(0)$$

$$\underline{= 0}$$

### *Exercise*

Use the Divergence Theorem to compute the net outward flux of the following fields across the given surface *S or D*.

$\vec{F} = \left\langle y - 2x,\ x^3 - \text{y},\ y^2 - z\right\rangle$; *S* is the sphere $\left\{(x, y, z):\ x^2 + y^2 + z^2 = 4\right\}$

***Solution***

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(y - 2x) + \frac{\partial}{\partial y}\left(x^3 - y\right) + \frac{\partial}{\partial z}\left(y^2 - z\right)$$

$$= -2 - 1 - 1$$

$$\underline{= -4}$$

The net outward flux:

$$\iiint\limits_D \nabla \bullet \vec{F}\ dV = -4\iiint\limits_D dV$$

$$= -4 \times (\textit{volume of the sphere})$$

$$= -4 \times \left(\frac{4\pi}{3}2^2\right)$$

$$\underline{= -\frac{128}{3}\pi}$$

### *Exercise*

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *S*.

$\vec{F} = \langle x,\ y,\ z\rangle$; *S* is the surface of the paraboloid $z = 4 - x^2 - y^2$, for $z \ge 0$, plus its base in the *xy*-plane

***Solution***

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$\underline{= 3}$$

So, by the Divergence Theorem, the net outward flux is

$$\iiint_D \nabla \cdot \vec{F}\, dV = \int_0^{2\pi} \int_0^{2} (3) rz\ dr d\theta$$

$$= 3\int_0^{2\pi} d\theta \int_0^{2} r\left(4-r^2\right) dr$$

$$= 3(2\pi)\left(2r^2 - \frac{1}{4}r^4\right)\Big|_0^2$$

$$= 6\pi(8-4)$$

$$= 24\pi$$

### Exercise

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface $S$.

$\vec{F} = \langle x,\ y,\ z\rangle$; $S$ is the surface of the cone $z^2 = x^2 + y^2$, for $0 \le z \le 4$, plus its top surface in the plane $z = 4$

**Solution**

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 3$$

Volume of a cone $= \frac{1}{3}(\textit{area of the base})(\textit{height})$

$$= \frac{1}{3}\left(\pi r^2\right)(4)$$

$$= \frac{4\pi}{3}(16)$$

$$= \frac{64\pi}{3}$$

So, by the Divergence Theorem, the net outward flux is

$$\iiint_D \nabla \cdot \vec{F}\, dV = (3)\left(\frac{64\pi}{3}\right)$$

$$= 64\pi$$

### *Exercise*

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *D*.

$\vec{F} = \langle z - x,\ x - y,\ 2y - z \rangle$; *D* is the region between the spheres of radius 2 and 4 centered at origin.

***Solution***

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(z - x) + \frac{\partial}{\partial y}(x - y) + \frac{\partial}{\partial z}(2y - z)$$

$$= -3$$

$$\text{Volume between 2 spheres } = \frac{4}{3}\pi\left(4^3 - 2^3\right)$$

$$= \frac{224}{3}\pi$$

$$\iiint_D \nabla \cdot \vec{F}\, dV = (-3)\left(\frac{224\pi}{3}\right)$$

$$= -224\pi$$

### *Exercise*

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *D*.

$\vec{F} = \vec{r}\,|\vec{r}| = \langle x,\ y,\ z \rangle \sqrt{x^2 + y^2 + z^2}$ ; *D* is the region between the spheres of radius 1 and 2 centered at origin.

***Solution***

$$\left(U^n V^m\right)' = U^{n-1} V^{m-1}\left(nU'V + mUV'\right)$$

$$\frac{\partial}{\partial x}\left(x\left(x^2 + y^2 + z^2\right)^{1/2}\right) = \left(x^2 + y^2 + z^2\right)^{-1/2}\left[x^2 + y^2 + z^2 + \frac{1}{2}(2x)(x)\right]$$

$$= \left(2x^2 + y^2 + z^2\right)\left(x^2 + y^2 + z^2\right)^{-1/2}$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}\left(x\left(x^2 + y^2 + z^2\right)^{1/2}\right) + \frac{\partial}{\partial y}\left(y\left(x^2 + y^2 + z^2\right)^{1/2}\right) + \frac{\partial}{\partial z}\left(z\left(x^2 + y^2 + z^2\right)^{1/2}\right)$$

$$= \left(x^2 + y^2 + z^2\right)^{-1/2}\left(2x^2 + y^2 + z^2 + x^2 + 2y^2 + z^2 + x^2 + y^2 + 2z^2\right)$$

$$= 4\left(x^2 + y^2 + z^2\right)^{-1/2}\left(x^2 + y^2 + z^2\right)$$

$$= 4\sqrt{x^2 + y^2 + z^2}$$

$$= 4\left|\vec{r}\right| \;\Big|$$

$$\iiint\limits_D \nabla \cdot \vec{F}\, dV = \iiint\limits_D 4\sqrt{x^2+y^2+z^2}\; dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{r} (4\rho)\rho^2 \sin\varphi \; d\rho\, d\varphi\, d\theta$$

$$= 4\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \int_0^{r} \rho^3 \; d\rho$$

$$= 4(2\pi)\ \left(-\cos\varphi \,\Big|_0^1 \quad \left(\tfrac{1}{4}\rho^4 \,\Big|_0^r\right.\right.$$

$$= 4(2\pi)(2)\left(\tfrac{1}{4}r^4\right)$$

$$= 4\pi r^4$$

$$= 4\pi\left(2^4 - 1^4\right)$$

$$= 60\pi \;\Big|$$

### *Exercise*

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface $D$.

$\vec{F} = \dfrac{\vec{r}}{|\vec{r}|} = \dfrac{\langle x,\ y,\ z\rangle}{\sqrt{x^2+y^2+z^2}}$; $D$ is the region between the spheres of radius 1 and 2 centered at origin.

***Solution***

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = \frac{x^2+y^2+z^2-x^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$= \frac{y^2+z^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) = \frac{x^2+y^2+z^2-y^2}{x^2+y^2+z^2}$$

$$= \frac{x^2+z^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)=\frac{x^2+y^2+z^2-z^2}{x^2+y^2+z^2}$$

$$=\frac{x^2+y^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\nabla \bullet \vec{F}=\frac{y^2+z^2+x^2+z^2+x^2+y^2}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$=\frac{2\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$=\frac{2}{\sqrt{x^2+y^2+z^2}}$$

$$\boxed{=\frac{2}{|\vec{r}|}}$$

$$\iiint_D \nabla \bullet \vec{F}\, dV = 2\iiint_D \frac{1}{\sqrt{x^2+y^2+z^2}}\, dV$$

$$=2\int_0^{2\pi}\int_0^{\pi}\int_0^{r}\left(\frac{1}{\rho}\right)\rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$=2\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\, d\varphi \int_0^{r} \rho\, d\rho$$

$$=2(2\pi)\ \left(-\cos\varphi \,\Big|_0^1\ \ \left(\frac{1}{2}\rho^2\ \Big|_0^r\right.\right.$$

$$=2(2\pi)(2)\left(\frac{1}{2}r^2\right)$$

$$\boxed{=4\pi r^2}$$

$D$ is the region between the spheres of radius 1 and 2 centered at origin

$$=4\pi\left(2^2-1^2\right)$$

$$\boxed{=12\pi}$$

### *Exercise*

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface $D$.

$\vec{F}=\langle z-y,\ x-z,\ 2y-x\rangle;\ \ D=\{(x,y,z):\ 1\le|x|\le 3,\ 1\le|y|\le 3,\ 1\le|z|\le 3\}$ is the region between two cubes

***Solution***

$$\nabla\cdot\vec{F}=\frac{\partial}{\partial x}(z-y)+\frac{\partial}{\partial y}(x-z)+\frac{\partial}{\partial z}(2y-x)$$
$$=0$$

Therefore, by the Divergence Theorem, the net outward flux is ***zero***.

### *Exercise*

Use the Divergence Theorem to compute the net outward flux of the following fields across the given surface $S$ *or* $D$.

$\vec{F}=\langle y+z,\ x+z,\ x+y\rangle$; $S$ consists of the faces of the cube $\{(x,\ y,\ z):\ \ |x|\le 1,\ \ |y|\le 1\ \ \ |z|\le 1\}$

***Solution***

$$\nabla\cdot\vec{F}=\frac{\partial}{\partial x}(y+z)+\frac{\partial}{\partial y}(x+z)+\frac{\partial}{\partial z}(x+y)$$
$$=0$$

The net outward flux:

$$\iiint_D \nabla\cdot\vec{F}\ dV=\iiint_D (0)\ dV$$
$$=0$$

### *Exercise*

Use the Divergence Theorem to compute the net outward flux of the following fields

$\vec{F}=\left\langle x^2,\ -y^2,\ z^2\right\rangle$; $D$ is the region in the first octant between the planes $z=4-x-y$ and $z=2-x-y$

***Solution***

$$\nabla\cdot\vec{F}=\frac{\partial}{\partial x}\left(x^2\right)+\frac{\partial}{\partial y}\left(-y^2\right)+\frac{\partial}{\partial z}\left(z^2\right)$$
$$=2x-2y+2z$$
$$=2(x-y+z)$$

| *Top plane* | *Bottom plane* |
|---|---|
| $0 \le z \le 4-x-y$ | $0 \le z \le 2-x-y$ |
| $z = 4-x-y = 0 \quad \rightarrow \quad 0 \le y \le 4-x$ | $z = 2-x-y = 0 \quad \rightarrow \quad 0 \le y \le 2-x$ |
| $y = 4-x = 0 \quad \rightarrow \quad 0 \le x \le 4$ | $y = 2-x = 0 \quad \rightarrow \quad 0 \le x \le 2$ |

The net outward flux:

$$\iiint_D \nabla \bullet \vec{F}\, dV = 2\iiint_D (x-y+z)\, dV$$

$$= 2\int_0^4 \int_0^{4-x} \int_0^{4-x-y} (x-y+z)\,dzdydx \;-\; 2\int_0^2 \int_0^{2-x} \int_0^{2-x-y} (x-y+z)\,dzdydx$$

$$= 2\int_0^4 \int_0^{4-x} \left((x-y)z + \frac{1}{2}z^2 \,\Big|_0^{4-x-y}\right) dydx$$

$$-2\int_0^2 \int_0^{2-x} \left((x-y)z + \frac{1}{2}z^2 \,\Big|_0^{2-x-y}\right) dydx$$

$$= 2\int_0^4 \int_0^{4-x} \left((x-y)(4-x-y) + \frac{1}{2}(4-x-y)^2\right) dydx$$

$$-2\int_0^2 \int_0^{2-x} \left((x-y)(2-x-y) + \frac{1}{2}(2-x-y)^2\right) dydx$$

$$= 2\int_0^4 \int_0^{4-x} \left(4x - x^2 - 4y + y^2 + 8 - 4x - 4y + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2\right) dydx$$

$$-2\int_0^2 \int_0^{2-x} \left(2x - x^2 - 2y + y^2 + 2 - 2x + \frac{1}{2}x^2 + xy - 2y + \frac{1}{2}y^2\right) dydx$$

$$= 2\int_0^4 \int_0^{4-x} \left(-\frac{1}{2}x^2 + 8 - 8y + xy + \frac{3}{2}y^2\right) dydx$$

$$-2\int_0^2 \int_0^{2-x} \left(-\frac{1}{2}x^2 + 2 + xy - 4y + \frac{3}{2}y^2\right) dydx$$

$$= 2\int_0^4 \left(-\frac{1}{2}x^2 y + 8y - 4y^2 + \frac{1}{2}xy^2 + \frac{1}{2}y^3 \,\Big|_0^{4-x}\; dx\right.$$

$$\left. -2\int_0^2 \left(-\frac{1}{2}x^2 y + 2y + \frac{1}{2}xy^2 - 2y^2 + \frac{1}{2}y^3 \,\Big|_0^{2-x}\; dx\right.$$

$$= 2\int_0^4 \left(-2x^2 + \frac{1}{2}x^3 + 32 - 8x + 32x - 4x^2 + 8x - 4x^2 + \frac{1}{2}x^3 + 32 - 24x + 6x^2 - \frac{1}{2}x^3\right)dx$$

$$-2\int_0^2 \left(-x^2 + \frac{1}{2}x^3 + 4 - 2x + 2x - 2x^2 + \frac{1}{2}x^3 - 8 + 8x + 2x^2 + 4 - 6x + 3x^2 - \frac{1}{2}x^3\right)dx$$

$$= 2\int_0^4 \left(8x - 4x^2 + \frac{1}{2}x^3\right)dx \;-\; 2\int_0^2 \left(\frac{1}{2}x^3 + 2x - 2x^2\right)dx$$

$$= 2\left(4x^2 - \frac{4}{3}x^3 + \frac{1}{8}x^4 \,\Big|_0^4 \;-\; 2\left(\frac{1}{8}x^4 + x^2 - \frac{2}{3}x^3 \,\Big|_0^2\right.\right.$$

$$= 2\left(64 - \frac{256}{3} + 32\right) - 2\left(2 + 4 - \frac{16}{3}\right)$$

$$= 2\left(96 - \frac{256}{3}\right) - \frac{4}{3}$$

$$= \frac{64}{3} - \frac{4}{3}$$

$$\underline{= 20}$$

## *Exercise*

Use the Divergence Theorem to compute the net outward flux of the following field across the given surface *D*.

$\vec{F} = \langle x,\ 2y,\ 3z\rangle$; *D* is the region between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ for $0 \le z \le 8$

***Solution***

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z)$$

$$\underline{= 6}$$

Volume of the sphere $x^2 + y^2 = 4$

$$V = \int_0^8 \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\, dx\, dz$$

$$= \int_0^8 dz \int_{-2}^{2} \left(y \Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\right) dx$$

$$= 16 \int_{-2}^{2} \sqrt{4-x^2}\, dx$$

$x = 2\sin\alpha \qquad \sqrt{4-x^2} = 2\cos\alpha$

$dx = 2\cos\alpha\, d\alpha$

$$\int \sqrt{4-x^2}\, dx = \int 2\cos\alpha\ (2\cos\alpha)\ d\alpha$$

$$= 4\int \cos^2\alpha\ d\alpha$$

$$= 2\int (1+\cos 2\alpha)\ d\alpha$$

$$= 2\left(\alpha + \frac{1}{2}\sin 2\alpha\right)$$

$$= 2\left(\alpha + \sin\alpha\cos\alpha\right)$$

$$= 2\left(\sin^{-1}\frac{x}{2} + \frac{x}{2}\frac{\sqrt{4-x^2}}{2}\right)$$

$$= 2\sin^{-1}\frac{x}{2} + \frac{1}{2}x\sqrt{4-x^2}$$

$$= 16\left(2\sin^{-1}\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2}\ \Big|_{-2}^{2}\right.$$

$$= 16\left(2\left(\frac{\pi}{2}\right) - 2\left(-\frac{\pi}{2}\right)\right)$$

$$\underline{= 32\pi \quad unit^3}$$

***Or***

$$V_1 = z\left(\pi r^2\right)$$

$$= 8(4\pi)$$

$$\underline{= 32\pi}$$

Volume of the sphere $x^2 + y^2 = 1$

$$V = \int_0^8 \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\, dx\, dz$$

$$= \int_0^8 dz \int_{-1}^{1} y \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= 8\int_{-1}^{1} 2\sqrt{1-x^2}\ dx$$

$x = \sin\alpha \qquad \sqrt{1-x^2} = \cos\alpha$

$dx = \cos\alpha\, d\alpha$

$$\int \sqrt{1-x^2}\ dx = \int \cos^2\alpha\ d\alpha$$

$$= \frac{1}{2}\int (1+\cos 2\alpha)\ d\alpha$$

$$= \frac{1}{2}\left(\alpha + \frac{1}{2}\sin 2\alpha\right)$$

$$= \frac{1}{2}\left(\alpha + \sin\alpha\cos\alpha\right)$$

$$= \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{4-x^2}$$

$$= 16\left(\frac{1}{2}\sin^{-1}x + \frac{x}{2}\sqrt{1-x^2}\right.\Bigg|_{-1}^{1}$$

$$= 16\left[\frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\left(-\frac{\pi}{2}\right)\right]$$

$$= 8\pi \quad unit^3$$

***Or***

$$V_2 = z\left(\pi r^2\right)$$

$$= 8(\pi)$$

$$= 8\pi$$

Therefore, the net outward flux is $\quad 6(32\pi - 8\pi)\ = 144\pi$

## *Exercise*

Compute the outward flux of the following vector field across the given surface

$\vec{F} = \left\langle x^2e^y\cos z,\ -4xe^y\cos z,\ 2xe^y\sin z\right\rangle$; $S$ is the boundary of the ellipsoid $\frac{x^2}{4} + y^2 + z^2 = 1$

***Solution***

$$\nabla\bullet\vec{F} = \frac{\partial}{\partial x}\left(x^2e^y\cos z\right) + \frac{\partial}{\partial y}\left(-4xe^y\cos z\right) + \frac{\partial}{\partial z}\left(2xe^y\sin z\right)$$

$$= 2xe^y\cos z - 4xe^y\cos z + 2xe^y\cos z$$

$$= 0$$

Therefore, by the Divergence Theorem, the net outward flux is ***zero***.

### Exercise

Compute the outward flux of the following vector field across the given surface $\vec{F} = \langle -yz,\ xz,\ 1\rangle$; $S$ is the boundary of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{4} + z^2 = 1$

**Solution**

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(-yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(1)$$

$$= 0$$

Therefore, by the Divergence Theorem, the net outward flux is ***zero***.

### Exercise

Compute the outward flux of the following vector field across the given surface
$\vec{F} = \langle x\sin y,\ -\cos y,\ z\sin y\rangle$; $S$ is the boundary of the region bounded by the planes $x = 1$, $y = 0$, $y = \frac{\pi}{2}$, $z = 0$, and $z = x$

**Solution**

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x\sin y) + \frac{\partial}{\partial y}(-\cos y) + \frac{\partial}{\partial z}(z\sin y)$$

$$= \sin y + \sin y + \sin y$$

$$= 3\sin y$$

$$\iiint_D \nabla \cdot \vec{F}\ dV = \int_0^{\pi/2} \int_0^1 \int_0^x 3\sin y\ dz dx dy$$

$$= \int_0^{\pi/2} \sin y\ dy \quad \int_0^1 3z \Big|_0^x dx$$

$$= -\cos y \Big|_0^{\pi/2} \int_0^1 3x\ dx$$

$$= \frac{3}{2}x^2 \Big|_0^1$$

$$= \frac{3}{2}$$

## *Exercise*

Compute the outward flux of the following vector field across the given surface

$\vec{F}=\left\langle x^2+x\sin y,\ y^2+2\cos y,\ z^2+z\sin y\right\rangle$ across the surface $S$ that is the boundary of the prism bounded by the planes $y=1-x,\ x=0,\ y=0,\ z=0,\ z=4$

***Solution***

$$\begin{aligned}
\nabla\bullet\vec{F} &= \nabla\bullet\left\langle x^2+x\sin y,\ y^2+2\cos y,\ z^2+z\sin y\right\rangle \\
&= \frac{\partial}{\partial x}\left(x^2+x\sin y\right)+\frac{\partial}{\partial y}\left(y^2+2\cos y\right)+\frac{\partial}{\partial z}\left(z^2+z\sin y\right) \\
&= 2x+\sin y+2y-2\sin y+2z+\sin y \\
&= \underline{2(x+y+z)}
\end{aligned}$$

$y=1-x,\ x=0,\ y=0,\ z=0,\ z=4$

$y=1-x=0\ \to\ x=1$

$0\le y\le 1-x,\quad 0\le x\le 1,\quad 0\le z\le 4$

$$\begin{aligned}
\iint_S \vec{F}\bullet\vec{n}\ dS &= 2\iiint_D (x+y+z)\ dV \\
&= 2\int_0^4\int_0^1\int_0^{1-x}(x+y+z)\ dydxdz \\
&= 2\int_0^4\int_0^1\left(xy+\frac{1}{2}y^2+zy\ \Big|_0^{1-x}\ dxdz\right. \\
&= 2\int_0^4\int_0^1\left(x-x^2+\frac{1}{2}-x+\frac{1}{2}x^2+z-xz\right)dxdz \\
&= 2\int_0^4\int_0^1\left(-\frac{1}{2}x^2+\frac{1}{2}+z-xz\right)dxdz \\
&= 2\int_0^4\left(-\frac{1}{6}x^3+\frac{1}{2}x+zx-\frac{1}{2}zx^2\ \Big|_0^1\ dz\right. \\
&= 2\int_0^4\left(-\frac{1}{6}+\frac{1}{2}+z-\frac{1}{2}z\right)dz \\
&= 2\int_0^4\left(\frac{1}{3}+\frac{1}{2}z\right)dz
\end{aligned}$$

$$= 2\left(\frac{1}{3}z + \frac{1}{4}z^2\right|_0^4$$

$$= 2\left(\frac{4}{3} + 4\right)$$

$$= \frac{32}{3}$$

### *Exercise*

Compute the outward flux of the following vector field across the given surface $\vec{F} = \langle x,\ -2y,\ 4z\rangle$ out of the sphere $S$ with $x^2 + y^2 + z^2 = a^2, \quad a > 0$

***Solution***

$$\nabla \cdot \vec{F} = \nabla \cdot \langle x,\ -2y,\ 4z\rangle$$

$$= 1 - 2 + 4$$

$$= 3$$

$$\iint_S \vec{F} \cdot \vec{n}\ dS = 3\iiint_D dV$$

$$= 3 \times (volume\ of\ a\ sphere)$$

$$= 3 \times \left(\frac{4\pi}{3}a^3\right)$$

$$= 4\pi a^3$$

### *Exercise*

Compute the outward flux of the following vector field across the given surface $\vec{F} = \left\langle ye^z,\ x^2e^z,\ xy\right\rangle$ out of the sphere $S$ with $x^2 + y^2 + z^2 = a^2, \quad a > 0$

***Solution***

$$\nabla \cdot \vec{F} = \nabla \cdot \left\langle ye^z,\ x^2e^z,\ xy\right\rangle$$

$$= \frac{\partial}{\partial x}\left(ye^z\right) + \frac{\partial}{\partial y}\left(x^2e^z\right) + \frac{\partial}{\partial z}(xy)$$

$$= 0$$

$$\iint_S \vec{F} \cdot \vec{n}\ dS = \iiint_D (0)\,dV$$

$$= 0$$

### *Exercise*

Compute the outward flux of the following vector field across the given surface

$\vec{F} = \left\langle x^2 + y^2,\ y^2 - z^2,\ z \right\rangle$ of the sphere $S$ with $x^2 + y^2 + z^2 = a^2, \quad a > 0$

***Solution***

$$\nabla \cdot \vec{F} = \nabla \cdot \left\langle x^2 + y^2,\ y^2 - z^2,\ z \right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^2 + y^2\right) + \frac{\partial}{\partial y}\left(y^2 - z^2\right) + \frac{\partial}{\partial z}(z)$$

$$= 2x + 2y + 1$$

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iiint_D (2x + 2y + 1)\, dV$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{a} (2a \sin\varphi \cos\theta + 2a \sin\varphi \sin\theta + 1)\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi} \left(2a \sin^2\varphi \cos\theta + 2a \sin^2\varphi \sin\theta + \sin\varphi\right) \rho^3 \Big|_0^a\, d\varphi\, d\theta$$

$$= \frac{a^3}{3} \int_0^{2\pi} \int_0^{\pi} \left(2a \sin^2\varphi \cos\theta + 2a \sin^2\varphi \sin\theta + \sin\varphi\right) d\varphi\, d\theta$$

$$= \frac{a^3}{3} \int_0^{2\pi} \int_0^{\pi} \left(a(1 - \cos 2\varphi)\cos\theta + a(1 - \cos 2\varphi)\sin\theta + \sin\varphi\right) d\varphi\, d\theta$$

$$= \frac{a^3}{3} \int_0^{2\pi} \int_0^{\pi} \left(a(\cos\theta + \sin\theta)(1 - \cos 2\varphi) + \sin\varphi\right) d\varphi\, d\theta$$

$$= \frac{a^3}{3} \int_0^{2\pi} \left(a(\cos\theta + \sin\theta)\left(\varphi - \frac{1}{2}\sin 2\varphi\right) - \cos\varphi \Big|_0^{\pi}\right) d\theta$$

$$= \frac{a^3}{3} \int_0^{2\pi} \left(a\pi(\cos\theta + \sin\theta) + 2\right) d\theta$$

$$= \frac{a^3}{3} \left(a\pi(\sin\theta - \cos\theta) + 2\theta\right) \Big|_0^{2\pi}$$

$$= \frac{a^3}{3}(-a\pi + 4\pi + a\pi)$$

$$= \frac{4}{3}\pi a^3$$

## *Exercise*

Compute the outward flux of the following vector field across the given surface

$\vec{F} = \left\langle x^3,\ 3yz^2,\ 3y^2z + x^2 \right\rangle$ out of the sphere $S$ with $x^2 + y^2 + z^2 = a^2, \quad a > 0$

***Solution***

$$\nabla \cdot \vec{F} = \nabla \cdot \left\langle x^3,\ 3yz^2,\ 3y^2z + x^2 \right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^3\right) + \frac{\partial}{\partial y}\left(3yz^2\right) + \frac{\partial}{\partial z}\left(3y^2z + x^2\right)$$

$$= 3x^2 + 3z^2 + 3y^2$$

$$\iiint\limits_D \nabla \cdot \vec{F}\ dV = 3\iiint\limits_D \left(x^2 + y^2 + z^2\right)\ dV$$

$$= 3\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\left(\rho^2\right)\rho^2 \sin\varphi\ d\rho\ d\varphi\ d\theta$$

$$= 3\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \int_0^{a} \rho^4\ d\rho$$

$$= 3(2\pi)\ \left(-\cos\varphi \Big|_0^1\ \ \left(\frac{1}{5}\rho^5 \Big|_0^a\right.\right.$$

$$= 3(2\pi)(2)\left(\frac{1}{5}a^5\right)$$

$$= \frac{12}{5}\pi a^5$$

## *Exercise*

$\vec{F} = \left\langle 2z,\ x,\ y^2 \right\rangle$; $S$ is the surface of the paraboloid $z = 4 - x^2 - y^2$, for $z \ge 0$, and the *xy*-plane.

***Solution***

$$\nabla \cdot \vec{F} = \nabla \cdot \left\langle 2z,\ x,\ y^2 \right\rangle$$

$$= \frac{\partial}{\partial x}(2z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}\left(y^2\right)$$

$$= 0$$

$$\iiint\limits_D \nabla \cdot \vec{F}\ dV = \iiint\limits_D (0)dV$$

$$= 0$$

### *Exercise*

$\vec{F} = \left\langle x,\ y^2,\ z \right\rangle$; $S$ is the solid region bounded by the coordinate planes and the plane $2x + 2y + z = 6$.

***Solution***

$$\nabla \bullet \vec{F} = \nabla \bullet \left\langle x,\ y^2,\ z \right\rangle$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}\left(y^2\right) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 2y + 1$$

$$= 2(1 + y)$$

$$2x + 2y + z = 6 \quad \rightarrow \quad 0 \le z \le 6 - 2x - 2y$$

$$z = 6 - 2x - 2y = 0 \quad \rightarrow \quad 0 \le x \le 3 - y$$

$$x = 3 - y = 0 \quad \rightarrow \quad 0 \le y \le 3$$

$$\iiint\limits_D \nabla \bullet \vec{F}\ dV = 2 \iiint\limits_D (1 + y)\,dV$$

$$= 2\int_0^3 \int_0^{3-y} \int_0^{6-2x-2y} (1 + y)\ dzdxdy$$

$$= 2\int_0^3 \int_0^{3-y} (1 + y)\ \left(z \Big|_0^{6-2x-2y}\right.\ dxdy$$

$$= 2\int_0^3 \int_0^{3-y} (1 + y)(6 - 2x - 2y)\,dxdy$$

$$= 2\int_0^3 \int_0^{3-y} \left(6 - 2x + 4y - 2xy - 2y^2\right)dxdy$$

$$= 2\int_0^3 \left(6x - x^2 + 4yx - yx^2 - 2y^2x \Big|_0^{3-y}\right.\ dy$$

$$= 2\int_0^3 \left(18 - 6y - 9 + 6y - y^2 + 12y - 4y^2 - 9y + 6y^2 - y^3 - 6y^2 + 2y^3\right)dy$$

$$= 2\int_0^3 \left(9 + 3y - 5y^2 + y^3\right)dy$$

$$= 2\ \left(9y + \frac{3}{2}y^2 - \frac{5}{3}y^3 + \frac{1}{4}y^4 \Big|_0^3\right.$$

$$= 2\left(27 + \frac{27}{2} - 45 + \frac{81}{4}\right)$$

$$= 2\left(\frac{135}{4} - 18\right)$$

$$\boxed{= \frac{63}{2}}$$

## Exercise

$\vec{F} = \left\langle x^2 + \sin z,\ xy + \cos z,\ e^y \right\rangle$; $S$ is the solid region bounded by the cylinder $x^2 + y^2 = 4$, the plane $x + z = 6$, and the $xy$-plane.

**Solution**

$$\nabla \bullet \vec{F} = \nabla \bullet \left\langle x^2 + \sin z,\ xy + \cos z,\ e^y \right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^2 + \sin z\right) + \frac{\partial}{\partial y}\left(xy + \cos z\right) + \frac{\partial}{\partial z}\left(e^y\right)$$

$$= 2x + x + 0$$

$$\boxed{= 3x}$$

$$x^2 + y^2 = 4 \quad \rightarrow \quad 0 \le r \le 2$$

$$x^2 + y^2 = 4 \quad \rightarrow \quad 0 \le \theta \le 2\pi$$

$$x + z = 6 \quad \rightarrow \quad 0 \le z \le 6 - r\cos\theta$$

$$\iiint_D \nabla \bullet \vec{F}\ dV = 3\iiint_D (x)\,dV$$

$$= 3\int_0^{2\pi}\int_0^2\int_0^{6-r\cos\theta} (r\cos\theta)r\ dzdrd\theta$$

$$= 3\int_0^{2\pi}\int_0^2 \left(r^2\cos\theta\right)\left(z\ \Big|_0^{6-r\cos\theta}\ drd\theta\right.$$

$$= 3\int_0^{2\pi}\int_0^2 \left(r^2\cos\theta\right)(6 - r\cos\theta)\,drd\theta$$

$$= 3\int_0^{2\pi}\int_0^2 \left(6r^2\cos\theta - r^3\cos^2\theta\right)drd\theta$$

$$= 3\int_0^{2\pi}\left(2r^3\cos\theta - \frac{1}{4}r^4\cos^2\theta\ \Big|_0^2\ d\theta\right.$$

$$= 3\int_0^{2\pi}\left(16\cos\theta - 4\cos^2\theta\right)d\theta$$

$$= 3\int_0^{2\pi} \left(16\cos\theta - 2 - 2\cos 2\theta\right) d\theta$$

$$= 3\left(16\sin\theta - 2\theta - \sin 2\theta\right. \Big|_0^{2\pi}$$

$$= 3(-4\pi)$$

$$\underline{= 12\pi}$$

## Exercise

Compute the outward flux of the following vector field $\vec{F} = \left\langle 2x^3,\ 2y^3,\ 2z^3 \right\rangle$ out of the sphere $S$ with $x^2 + y^2 + z^2 = 4$

**Solution**

$$\nabla \bullet \vec{F} = \nabla \bullet \left\langle 2x^3,\ 2y^3,\ 2z^3 \right\rangle$$

$$= \frac{\partial}{\partial x}\left(2x^3\right) + \frac{\partial}{\partial y}\left(2y^3\right) + \frac{\partial}{\partial z}\left(2z^3\right)$$

$$\underline{= 6x^2 + 6z^2 + 6y^2}$$

$$\iiint_D \nabla \bullet \vec{F}\, dV = 6\iiint_D \left(x^2 + y^2 + z^2\right) dV$$

$$= 6\int_0^{2\pi}\int_0^{\pi}\int_0^{2} \left(\rho^2\right)\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= 6\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \int_0^{2} \rho^4\, d\rho$$

$$= 6(2\pi)\left(-\cos\varphi\right. \Big|_0^{1} \left(\frac{1}{5}\rho^5\right. \Big|_0^{2}$$

$$= 6(2\pi)(2)\left(\frac{32}{5}\right)$$

$$\underline{= \frac{768\pi}{5}}$$

### Exercise

Compute the outward flux of the following vector field $\vec{F} = \langle x, \ y, \ z \rangle$ out of the sphere $S$ with $x^2 + y^2 + z^2 = 1$

***Solution***

$$\nabla \cdot \vec{F} = \nabla \cdot \langle x, \ y, \ z \rangle$$

$$= 3$$

$$\iiint_D \nabla \cdot \vec{F} \ dV = 3 \iiint_D dV$$

$$= 3 \times (\text{volume of a sphere})$$

$$= 3 \times \left( \frac{4\pi}{3} \right)$$

$$= 4\pi$$

### Exercise

Compute the outward flux of the following vector field $\vec{F} = \langle z, \ y, \ x \rangle$ out of the sphere $S$ with $x^2 + y^2 + z^2 = 1$

***Solution***

$$\nabla \cdot \vec{F} = \nabla \cdot \langle z, \ y, \ x \rangle$$

$$= 1$$

$$\iiint_D \nabla \cdot \vec{F} \ dV = \iiint_D dV$$

$$= \text{volume of a sphere } (r = 1)$$

$$= \frac{4\pi}{3}$$

### Exercise

Compute the outward flux of the following vector field $\vec{F} = \left\langle xy, \ y^2 + e^{xz^2}, \ \sin(xy) \right\rangle$; $S$ is the solid region bounded by the cylinder $z = 1 - x^2$, the planes $y + z = 2$, $z = 0$, and $y = 0$.

***Solution***

$$\nabla \cdot \vec{F} = \nabla \cdot \left\langle xy, \ y^2 + e^{xz^2}, \ \sin(xy) \right\rangle$$

$$= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(y^2 + e^{xz^2}\right) + \frac{\partial}{\partial z}(\sin(xy))$$

$$= y + 2y + 0$$

$$\boxed{= 3y}$$

$$y + z = 2 \quad \to \quad 0 \le y \le 2 - z$$

$$z = 1 - x^2 \quad \to \quad 0 \le z \le 1 - x^2$$

$$z = 1 - x^2 = 0 \quad \to \quad -1 \le x \le 1$$

$$\iiint_D \nabla \cdot \vec{F}\, dV = 3\iiint_D (y)\, dV$$

$$= 3\int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{2-z} y\, dydzdx$$

$$= \frac{3}{2}\int_{-1}^{1}\int_{0}^{1-x^2} y^2 \Big|_{0}^{2-z} dzdx$$

$$= \frac{3}{2}\int_{-1}^{1}\int_{0}^{1-x^2} (2-z)^2\, dzdx$$

$$= -\frac{3}{2}\int_{-1}^{1}\int_{0}^{1-x^2} (2-z)^2\, d(2-z)\,dx$$

$$= -\frac{1}{2}\int_{-1}^{1} (2-z)^3 \Big|_{0}^{1-x^2} dx$$

$$= -\frac{1}{2}\int_{-1}^{1}\left(\left(1+x^2\right)^3 - 8\right)dx$$

$$= -\frac{1}{2}\int_{-1}^{1}\left(-7 + 3x^2 + 3x^4 + x^6\right)dx$$

$$= -\left(-7x + x^3 + \frac{3}{5}x^5 + \frac{1}{7}x^7\right. \Big|_{0}^{1}$$

$$= 7 - 1 - \frac{3}{5} - \frac{1}{7}$$

$$= 6 - \frac{26}{35}$$

$$\boxed{= \frac{184}{35}}$$

## *Exercise*

Use the Divergence Theorem to compute the net outward flux of the following fields a $\vec{F} = \left\langle x^2,\ y^2,\ z^2 \right\rangle$; across the boundary of an ellipsoid $x^2 + y^2 + 4(z-1)^2 \le 4$

***Solution***

$$\nabla \cdot \vec{F} = \nabla \cdot \left\langle x^2,\ y^2,\ z^2 \right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}\left(y^2\right) + \frac{\partial}{\partial z}\left(z^2\right)$$

$$= 2x + 2y + 2z$$

$$x^2 + y^2 + 4(z-1)^2 = 4$$

$$r^2 = 4\left(1 - (z-1)^2\right)$$

$$0 \le r \le 2\sqrt{1-(z-1)^2}$$

$$4(z-1)^2 = 4$$

$$(z-1)^2 = 1$$

$$0 \le z \le 2$$

$$\iiint_D \nabla \cdot \vec{F}\ dV = 2\iiint_D (x+y+z)\,dV$$

$$= 2\int_0^{2\pi}\int_0^2\int_0^{2\sqrt{1-(z-1)^2}} (2\cos\theta + 2\sin\theta + z)\, r\ dr dz d\theta$$

$$= \int_0^{2\pi}\int_0^2 (2\cos\theta + 2\sin\theta + z)\ r^2 \Big|_0^{2\sqrt{1-(z-1)^2}}\ dz d\theta$$

$$= 4\int_0^{2\pi}\int_0^2 (2\cos\theta + 2\sin\theta + z)\left(-z^2 + 2z\right) dz d\theta$$

$$= 4\int_0^{2\pi}\int_0^2 \left((2\cos\theta + 2\sin\theta)\left(-z^2 + 2z\right) - z^3 + 2z^2\right)\ dz d\theta$$

$$= 4\int_0^{2\pi} \left((2\cos\theta + 2\sin\theta)\left(-\frac{1}{3}z^3 + z^2\right) - \frac{1}{4}z^4 + \frac{2}{3}z^3 \ \Big|_0^2\ d\theta$$

$$= 4\int_0^{2\pi} \left((2\cos\theta + 2\sin\theta)\left(-\frac{8}{3} + 4\right) - 4 + \frac{16}{3}\right) d\theta$$

$$= 4\int_0^{2\pi} \left(\frac{8}{6}(\cos\theta + \sin\theta) + \frac{4}{3}\right) d\theta$$

$$= 4 \left(\frac{8}{6}(\sin\theta - \cos\theta) + \frac{4}{3}\theta \right|_0^{2\pi}$$

$$= 4\left(-\frac{8}{6} + \frac{8\pi}{3} + \frac{8}{6}\right)$$

$$= \frac{32\pi}{3}$$

***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following fields a $\vec{F} = \left\langle x^2,\ y^2,\ z^2 \right\rangle$; across the boundary of the tetrahedron $x + y + z \le 3 \quad \& \quad x,\ y,\ z \ge 0$

***Solution***

$$\nabla \bullet \vec{F} = \nabla \bullet \left\langle x^2,\ y^2,\ z^2 \right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}\left(y^2\right) + \frac{\partial}{\partial z}\left(z^2\right)$$

$$= 2x + 2y + 2z$$

$$x + y + z = 3 \quad \to \quad 0 \le z \le 3 - y - x$$

$$z = 3 - y - x = 0 \quad \to \quad 0 \le y \le 3 - x$$

$$y = 3 - x = 0 \quad \to \quad 0 \le x \le 3$$

$$\iiint_D \nabla \bullet \vec{F}\ dV = 2\iiint_D (x + y + z)\, dV$$

$$= 2\int_0^3 \int_0^{3-x} \int_0^{3-y-x} (x + y + z)\, dz dy dx$$

$$= 2\int_0^3 \int_0^{3-x} \left(xz + yz + \frac{1}{2}z^2 \right|_0^{3-y-x} dy dx$$

$$= 2\int_0^3 \int_0^{3-x} \left(x(3 - y - x) + y(3 - y - x) + \frac{1}{2}(3 - y - x)^2\right) dy dx$$

$$= 2\int_0^3 \int_0^{3-x} \left(3x - xy - x^2 + 3y - y^2 - xy + \frac{1}{2}\left(9 - 6y - 6x + y^2 + 2xy + x^2\right)\right) dy dx$$

$$= 2\int_0^3 \int_0^{3-x} \left(-xy - \frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{9}{2}\right) dydx$$

$$= 2\int_0^3 \left(\frac{9}{2}y - \frac{1}{2}xy^2 - \frac{1}{2}x^2y - \frac{1}{6}y^3 \Big|_0^{3-x} dx\right.$$

$$= \int_0^3 \left(9(3-x) - x(3-x)^2 - x^2(3-x) - \frac{1}{3}(3-x)^3\right) dx$$

$$= \int_0^3 \left(27 - 9x - 9x + 6x^2 - x^3 - 3x^2 + x^3 - 9 + 9x - 3x^2 + \frac{1}{3}x^3\right) dx$$

$$= \int_0^3 \left(18 - 9x + \frac{1}{3}x^3\right) dx$$

$$= 18x - \frac{9}{2}x^2 + \frac{1}{12}x^4 \Big|_0^3$$

$$= 54 - \frac{81}{2} + \frac{27}{4}$$

$$= \frac{81}{4}$$

### *Exercise*

Use the Divergence Theorem to compute the net outward flux of the following fields a $\vec{F} = \left\langle x^2,\ y^2,\ z^2 \right\rangle$; across the boundary of the cylinder $x^2 + y^2 \le 2y \quad \& \quad 0 \le z \le 4$

***Solution***

$$\nabla \cdot \vec{F} = \nabla \cdot \left\langle x^2,\ y^2,\ z^2 \right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}\left(y^2\right) + \frac{\partial}{\partial z}\left(z^2\right)$$

$$= 2x + 2y + 2z$$

$$x^2 + y^2 = 2y$$

$$x^2 = 2y - y^2$$

$$-\sqrt{2y - y^2} \le x \le \sqrt{2y - y^2}$$

$$(y-1)^2 = 1 \quad \to \quad 0 \le y \le 2$$

$$\iiint_D \nabla \cdot \vec{F}\ dV = 2\iiint_D (x + y + z)\, dV$$

$$= 2\int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_0^4 (x+y+z)\ dzdxdy$$

$$= 2\int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \left(xz + yz + \frac{1}{2}z^2 \right|_0^4 dxdy$$

$$= 2\int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} (4x+4y+8)dxdy$$

$$= 2\int_0^2 \left(2x^2 + 4(y+2)x \right|_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} dy$$

$$= 4\int_0^2 \left(2y - y^2 + 2(y+2)\sqrt{2y-y^2} - 2y + y^2 + 2(y+2)\sqrt{2y-y^2}\right)dy$$

$$= 16\int_0^2 (y+2)\sqrt{2y-y^2}\ dy$$

$$= 16\int_0^2 (y+2)\sqrt{1-(y-1)^2}\ dy$$

Let $u = y-1 \quad \rightarrow \quad du = dy$

$$y = u+1 \quad \rightarrow \quad \begin{cases} y = 2 & \rightarrow u = 1 \\ y = 0 & \rightarrow u = -1 \end{cases}$$

$$= 16\int_{-1}^1 (u+3)\sqrt{1-u^2}\ dy$$

Let $u = \sin\theta \quad \rightarrow \quad du = \cos\theta d\theta$

$$\rightarrow \quad \begin{cases} u = 1 & \rightarrow \theta = \frac{\pi}{2} \\ u = -1 & \rightarrow \theta = -\frac{\pi}{2} \end{cases}$$

$$= 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3+\sin\theta)\sqrt{1-\sin^2\theta}\ \cos\theta\, d\theta$$

$$= 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3+\sin\theta)\cos\theta\ \cos\theta\, d\theta$$

$$= 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(3\cos^2\theta + \sin\theta\cos\theta\right)\, d\theta$$

$$= 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3}{2} + \frac{3}{2}\cos 2\theta + \frac{1}{2}\sin 2\theta\right)\, d\theta$$

$$= 16\left(\frac{3}{2}\theta + \frac{3}{4}\sin 2\theta - \frac{1}{4}\cos 2\theta \right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 16\left(\frac{3\pi}{4} + \frac{3\pi}{4}\right)$$

$$= 24\pi$$

### *Exercise*

Use the Divergence Theorem to compute the net outward flux of the following fields a

$\vec{F} = \left\langle x^2,\ y^2,\ z^2\right\rangle$; across the boundary of a ball $(x-2)^2 + y^2 + (z-3)^2 \le 9$

***Solution***

$$\nabla \bullet \vec{F} = \nabla \bullet \left\langle x^2,\ y^2,\ z^2\right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}\left(y^2\right) + \frac{\partial}{\partial z}\left(z^2\right)$$

$$= 2x + 2y + 2z$$

$$(x-2)^2 + y^2 + (z-3)^2 = 9$$

$$y^2 = 9 - (x-2)^2 - (z-3)^2$$

$$-\sqrt{9-(x-2)^2-(z-3)^2} \le y \le \sqrt{9-(x-2)^2-(z-3)^2}$$

$$y = 9-(x-2)^2-(z-3)^2 = 0 \quad \rightarrow \quad (x-2)^2 = 9-(z-3)^2$$

$$2-\sqrt{9-(z-3)^2} \le x \le 2+\sqrt{9-(z-3)^2}$$

$$9-(z-3)^2 = 0 \quad \rightarrow z-3 = \pm 3 \quad 0 \le z \le 6$$

$$Flux = \iiint_D \nabla \bullet \vec{F}\ dV$$

$$= 2\iiint_D (x+y+z)\, dV$$

$$= 2\int_0^6 \int_{2-\sqrt{9-(z-3)^2}}^{2+\sqrt{9-(z-3)^2}} \int_{-\sqrt{9-(x-2)^2-(z-3)^2}}^{\sqrt{9-(x-2)^2-(z-3)^2}} (x+y+z)\ dydxdz$$

$$= 2\int_0^6 \int_{2-\sqrt{9-(z-3)^2}}^{2+\sqrt{9-(z-3)^2}} \left(xy+\frac{1}{2}y^2+yz\ \Big|_{-\sqrt{9-(x-2)^2-(z-3)^2}}^{\sqrt{9-(x-2)^2-(z-3)^2}} dxdz$$

$$= 4\int_0^6 \int_{2-\sqrt{9-(z-3)^2}}^{2+\sqrt{9-(z-3)^2}} (x+z)\sqrt{9-(x-2)^2-(z-3)^2}\ dxdz$$

Let $u = x-2, \quad x = u+2, \quad du = dx$

$v = z-3, \quad z = v+3, \quad dv = dz$

$$\begin{cases} z = 6 \rightarrow & v = 3 \\ z = 0 \rightarrow & v = -3 \end{cases}$$

$$= 4\int_{-3}^{3} \int_{2-\sqrt{9-v^2}}^{2+\sqrt{9-v^2}} (u+v+5)\sqrt{9-\left(u^2+v^2\right)}\ dxdz$$

Let $u = r\cos\theta = 3\cos\theta$

$v = r\sin\theta = 3\sin\theta$

$0 \le r \le 3, \quad 0 \le \theta \le 2\pi$

$$= 4\int_0^{2\pi} \int_0^3 (3\cos\theta+3\sin\theta+5)\sqrt{9-r^2}\ rdrd\theta$$

$$= -2\int_0^{2\pi} (3\cos\theta+3\sin\theta+5)d\theta \int_0^3 \left(9-r^2\right)^{1/2} d\left(9-r^2\right)$$

$$= -\frac{4}{3}\ (3\sin\theta-3\cos\theta+5\theta\ \Big|_0^{2\pi} \quad \left(9-r^2\right)^{3/2}\ \Big|_0^3$$

$$= -\frac{4}{3}\ (10\pi)\ (-27)$$

$$\underline{= 360\pi}$$

### ***Exercise***

Use the Divergence Theorem to compute the net outward flux of the following fields a $\vec{F} = \left\langle x^4,\ -x^3z^2,\ 4xy^2z \right\rangle$; across the boundary of the cylinder $x^2 + y^2 = 1$ and the planes $z = x + 2 \quad \& \quad z = 0$

***Solution***

$$\nabla \cdot \vec{F} = \nabla \cdot \left\langle x^4,\ -x^3z^2,\ 4xy^2z \right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^4\right) + \frac{\partial}{\partial y}\left(-x^3z^2\right) + \frac{\partial}{\partial z}\left(4xy^2z\right)$$

$$= 4x^3 + 4xy^2$$

$$= 4x\left(x^2 + y^2\right)$$

$$x^2 + y^2 = 1 = r^2 \quad \to \quad 0 \le r \le 1$$

$$0 \le \theta \le 2\pi$$

$$z = x + 2 \quad \to \quad 0 \le z \le r\cos\theta + 2$$

$$Flux = \iiint_D \nabla \cdot \vec{F}\, dV$$

$$= 4\iiint_D x\left(x^2 + y^2\right) dV$$

$$= 4\int_0^{2\pi}\int_0^1\int_0^{2+r\cos\theta} r\cos\theta\left(r^2\right)\, dz\,dr\,d\theta$$

$$= 4\int_0^{2\pi}\int_0^1\int_0^{2+r\cos\theta} r^3\cos\theta\, dz\,dr\,d\theta$$

$$= 4\int_0^{2\pi}\int_0^1 r^3\cos\theta\ \left(z \,\Big|_0^{2+r\cos\theta}\right. dr\,d\theta$$

$$= 4\int_0^{2\pi}\int_0^1 r^3\cos\theta\left(2 + r\cos\theta\right) dr\,d\theta$$

$$= 4\int_0^{2\pi}\int_0^1 \left(2r^3\cos\theta + r^4\cos^2\theta\right) dr\,d\theta$$

$$= 4\int_0^{2\pi} \left(\frac{1}{2}r^4\cos\theta + \frac{1}{5}r^5\cos^2\theta \,\Big|_0^1\right. d\theta$$

$$= 4\int_0^{2\pi} \left(\frac{1}{2}\cos\theta + \frac{1}{5}\cos^2\theta\right) d\theta$$

$$= 4\int_0^{2\pi} \left(\frac{1}{2}\cos\theta + \frac{1}{10} + \frac{1}{10}\cos 2\theta\right) d\theta$$

$$= 4\left(\frac{1}{2}\sin\theta + \frac{1}{10}\theta + \frac{1}{20}\sin 2\theta \right|_0^{2\pi}$$

$$\boxed{= \frac{4\pi}{5}}$$

### *Exercise*

Use the Divergence Theorem to compute the net outward flux of the following fields a

$\vec{F} = \left\langle x^2z^3,\ 2xyz^3,\ xz^4 \right\rangle$; $S$ is the surface of the box with vertices $(\pm 1,\ \pm 2,\ \pm 3)$.

***Solution***

$$\nabla \cdot \vec{F} = \nabla \cdot \left\langle x^2z^3,\ 2xyz^3,\ xz^4 \right\rangle$$

$$= \frac{\partial}{\partial x}\left(x^2z^3\right) + \frac{\partial}{\partial y}\left(2xyz^3\right) + \frac{\partial}{\partial z}\left(xz^4\right)$$

$$= 2xz^3 + 2xz^3 + 4xz^3$$

$$\boxed{= 8xz^3}$$

$$Flux = \iiint_D \nabla \cdot \vec{F}\ dV$$

$$= \iiint_D \left(8xz^3\right) dV$$

$$= 8\int_{-1}^{1} x\,dx \int_{-2}^{2} dy \int_{-3}^{3} z^3\,dz$$

$$= 8\left(\frac{1}{2}x^2 \right|_{-1}^{1} \quad \left(y \right|_{-2}^{2} \quad \left(\frac{1}{4}z^4 \right|_{-3}^{3}$$

$$\boxed{= 0}$$

### *Exercise*

Use the Divergence Theorem to compute the net outward flux of the following fields a $\vec{F} = \left\langle z\tan^{-1}\left(y^2\right),\ z^3\ln\left(x^2+1\right),\ z \right\rangle$; across the part of the paraboloid $x^2+y^2+z=2$ that lies above the plane $z=1$ and is oriented upward.

***Solution***

$$\nabla \bullet \vec{F} = \nabla \bullet \left\langle z\tan^{-1}\left(y^2\right),\ z^3\ln\left(x^2+1\right),\ z \right\rangle$$

$$= \frac{\partial}{\partial x}\left(z\tan^{-1}\left(y^2\right)\right) + \frac{\partial}{\partial y}\left(z^3\ln\left(x^2+1\right)\right) + \frac{\partial}{\partial z}(z)$$

$$= 1$$

$$z = 2-\left(x^2+y^2\right) \quad \rightarrow \quad 1 \le z \le 2-r^2$$

$$z = 2-r^2 = 1 \quad \rightarrow \quad 0 \le r \le 1$$

$$0 \le \theta \le 2\pi$$

$$Flux = \iiint\limits_D \nabla \bullet \vec{F}\ dV$$

$$= \iiint\limits_D (1)\,dV$$

$$= \int_0^{2\pi}\int_0^1\int_1^{2-r^2} r\ dz\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \quad \int_0^1 r\ \left(z \Big|_1^{2-r^2}\right.\ dr$$

$$= 2\pi \int_0^1 r\left(1-r^2\right)dr$$

$$= 2\pi \int_0^1 \left(r-r^3\right)dr$$

$$= 2\pi \left(\frac{1}{2}r^2 - \frac{1}{4}r^4\right.\Big|_0^1$$

$$= 2\pi\left(\frac{1}{2}-\frac{1}{4}\right)$$

$$= \frac{\pi}{2}$$

## *Exercise*

Prove that $\nabla\left(\dfrac{1}{|\vec{r}|^4}\right) = -\dfrac{4\vec{r}}{|\vec{r}|^6}$ and use the result to prove that $\nabla \cdot \nabla\left(\dfrac{1}{|\vec{r}|^4}\right) = \dfrac{12}{|\vec{r}|^6}$

***Solution***

$$|\vec{r}|^4 = \left(x^2 + y^2 + z^2\right)^2$$

$$\nabla\left(\frac{1}{|\vec{r}|^4}\right) = \nabla\frac{1}{\left(x^2 + y^2 + z^2\right)^2}$$

$$= -\frac{1}{\left(x^2 + y^2 + z^2\right)^3}\langle 4x,\ 4y,\ 4z\rangle$$

$$= -\frac{4}{\left(|\vec{r}|^2\right)^3}\langle x,\ y,\ z\rangle$$

$$= -\frac{4\vec{r}}{|\vec{r}|^6} \quad \checkmark$$

$$\nabla \cdot \nabla\left(\frac{1}{|\vec{r}|^4}\right) = \nabla \cdot \left(\frac{-4\vec{r}}{|\vec{r}|^4}\right)$$

$$= -4\nabla \cdot \left(\frac{\langle x,\ y,\ z\rangle}{\left(x^2 + y^2 + z^2\right)^3}\right)$$

$$= -\frac{4}{\left(x^2 + y^2 + z^2\right)^4}\left(x^2 + y^2 + z^2 - 6x^2 + x^2 + y^2 + z^2 - 6y^2 + x^2 + y^2 + z^2 - 6z^2\right)$$

$$= -\frac{4}{\left(x^2 + y^2 + z^2\right)^4}\left(-3x^2 - 3y^2 - 3z^2\right)$$

$$= \frac{12}{\left(x^2 + y^2 + z^2\right)^4}\left(x^2 + y^2 + z^2\right)$$

$$= \frac{12}{\left(x^2 + y^2 + z^2\right)^3}$$

$$= \frac{12}{\left(\vec{r}^2\right)^3}$$

$$= \frac{12}{|\vec{r}|^6} \quad \checkmark$$

### ***Exercise***

Consider the radial vector field $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2\right)^{p/2}}$. Let $S$ be the sphere of radius $a$ at the origin.

*a)* Use the surface integral to show that the outward flux of $\vec{F}$ across $S$ is $4\pi a^{3-p}$. Recall that the unit normal to sphere is $\dfrac{\vec{r}}{|\vec{r}|}$.

*b)* For what values of $p$ does $\vec{F}$ satisfy the conditions of the Divergence Theorem? For these values of $p$, use the fact the $\nabla \cdot \vec{F} = \dfrac{3-p}{|\vec{r}|^p}$ to compute the flux around $S$ using the Divergence Theorem.

***Solution***

***a)*** $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p} \quad \& \quad \vec{n} = \dfrac{\vec{r}}{|\vec{r}|}$

$$\begin{aligned}
\vec{F} \cdot \vec{n} &= \frac{\vec{r}}{|\vec{r}|^p} \cdot \frac{\vec{r}}{|\vec{r}|} \\
&= \frac{|\vec{r}|^2}{|\vec{r}|^{p+1}} \\
&= |\vec{r}|^{1-p} \\
&= \left(\sqrt{x^2 + y^2 + z^2}\right)^{1-p} \\
&= \left(\sqrt{a^2}\right)^{1-p} \\
&= a^{1-p}
\end{aligned}$$

$$\begin{aligned}
\iint_S \vec{F} \cdot \vec{n}\ dS &= a^{1-p} \iint_S dS \\
&= a^{1-p} \times (area\ of\ sphere) \\
&= a^{1-p} \times 4\pi a^2 \\
&= 4\pi a^{3-p} \quad \checkmark
\end{aligned}$$

***b)*** The conditions of the Divergence Theorem require that $\vec{F}$ be defined and have continuous partials everywhere inside the sphere; in particular, this must hold at the origin. Thus, we must have $p \le -2$. Then the volume integral is:

$$\iiint_D \nabla \bullet \vec{F}\, dV = \iiint_D \frac{3-p}{|\vec{r}|^p}\, dV$$

$$= (3-p) \iiint_D r^{-p}\, dV$$

$$= (3-p) \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^{-p}\, \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= (3-p) \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, d\varphi \int_0^{a} \rho^{2-p}\, d\rho$$

$$= (3-p)(2\pi)\left(-\cos\varphi \Big|_0^{\pi}\right. \left(\frac{1}{3-p}\rho^{3-p}\Big|_0^{a}\right.$$

$$= (3-p)(2\pi)(2)\left(\frac{1}{3-p}a^{3-p}\right)$$

$$= 4\pi a^{3-p}$$

### *Exercise*

Consider the radial vector field $\vec{F} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle x,\ y,\ z\rangle}{\sqrt{x^2+y^2+z^2}}$.

*a)* Evaluate a surface integral to show that $\iint_S \vec{F} \bullet \vec{n}\, dS = 4\pi a^2$, where $S$ is the surface of a sphere of radius $a$ centered at the origin.

*b)* Note that the first partial derivatives of the components of $\vec{F}$ are undefined at the origin, so the Divergence Theorem does not apply directly. Nevertheless, the flux across the sphere as computed in part (*a*) is finite. Evaluate the triple integral of the Divergence Theorem as an improper integral as follows. Integrate $div\vec{F}$ over the region between two spheres of radius $a$ and $0 < \varepsilon < a$. Then let $\varepsilon \to 0^+$ to obtain the flux computed in part (*a*).

***Solution***

***a)*** $\vec{F} = \frac{\vec{r}}{|\vec{r}|}$ & $\vec{n} = \frac{\vec{r}}{|\vec{r}|}$

$$\vec{F} \bullet \vec{n} = \frac{\vec{r}}{|\vec{r}|} \bullet \frac{\vec{r}}{|\vec{r}|}$$

$$= \frac{|\vec{r}|^2}{|\vec{r}|^2}$$

$$\underline{= 1}$$

$$\iint\limits_S \vec{F} \bullet \vec{n}\ dS = \iint\limits_S dS$$

$$= area\ of\ sphere$$

$$\underline{= 4\pi a^2}$$

***b)*** $\nabla \bullet \vec{F} = \nabla \bullet \left( \frac{\langle x,\ y,\ z \rangle}{\sqrt{x^2 + y^2 + z^2}} \right)$

$$\frac{\partial}{\partial x}\left( \frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) = \frac{x^2 + y^2 + z^2 - x^2}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$= \frac{y^2 + z^2}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$\frac{\partial}{\partial x}\left( \frac{y}{\sqrt{x^2 + y^2 + z^2}} \right) = \frac{x^2 + y^2 + z^2 - y^2}{x^2 + y^2 + z^2}$$

$$= \frac{x^2 + z^2}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$\frac{\partial}{\partial x}\left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) = \frac{x^2 + y^2 + z^2 - z^2}{x^2 + y^2 + z^2}$$

$$= \frac{x^2 + y^2}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$\nabla \bullet \vec{F} = \frac{y^2 + z^2 + x^2 + z^2 + x^2 + y^2}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$= \frac{2\left(x^2 + y^2 + z^2\right)}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$= \frac{2}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \boxed{\frac{2}{|\vec{r}|}}$$

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iiint_D \frac{2}{|\vec{r}|}\, dV$$

$$= 2\int_0^{2\pi}\int_0^{\pi}\int_{\varepsilon}^{a} \frac{1}{\rho}\,\rho^2 \sin\varphi\; d\rho d\varphi d\theta$$

$$= 2\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\, d\varphi \int_{\varepsilon}^{a} \rho\, d\rho$$

$$= 4\pi\, \left(-\cos\varphi \Big|_0^{\pi} \quad \left(\tfrac{1}{2}\rho^2 \Big|_{\varepsilon}^{a}\right.\right.$$

$$\boxed{= 4\pi\left(a^2 - \varepsilon^2\right)}$$

$$\lim_{\varepsilon \to 0} 4\pi\left(a^2 - \varepsilon^2\right) \boxed{= 4\pi a^2}$$

## *Exercise*

The electric field due to a point charge $Q$ is $\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}}{|\vec{r}|^3}$, where $\vec{r} = \langle x,\ y,\ z\rangle$ and $\varepsilon_0$ is a constant

*a)* Show that the flux of the field across a sphere of radius $a$ centered at the origin is

$$\iint_S \vec{E} \cdot \vec{n}\, dS = \frac{Q}{\varepsilon_0}$$

*b)* Let $S$ be the boundary of the origin between two spheres centered of radius $a$ and $b$ with $a < b$. Use the Divergence Theorem to show that the net outward flux across $S$ is zero.

*c)* Suppose there is a distribution of charge within a region $D$. Let $q(x,\ y,\ z)$ be the charge density (charge per unit volume). Interpret the statement that

$$\iint_S \vec{E} \cdot \vec{n}\, dS = \frac{1}{\varepsilon_0}\iiint_D q(x,\ y,\ z)\; dV$$

*d)* Assuming $\vec{E}$ satisfies the conditions of the Divergence Theorem, conclude from part ($c$) that

$$\nabla \cdot \vec{E} = \frac{q}{\varepsilon_0}$$

*e)* Because the electric force is conservative, it has a potential function $\phi$. From part ($d$) conclude that $\nabla^2\phi = \nabla \cdot \nabla\phi = \frac{q}{\varepsilon_0}$

***Solution***

***a)*** $\vec{F} \bullet \vec{n} = \dfrac{\vec{r}}{|\vec{r}|^3} \bullet \dfrac{\vec{r}}{|\vec{r}|}$

$$= \frac{|\vec{r}|^2}{|\vec{r}|^4}$$

$$= |\vec{r}|^{-2}$$

$$\iint_S \vec{F} \bullet \vec{n}\, dS = |\vec{r}|^{-2}$$

***Area*** of sphere $= |\vec{r}|^{-2}\left(4\pi r^2\right)$

$$= |a|^{-2}\left(4\pi a^2\right)$$

$$\underline{= 4\pi}$$

$$\iint_S \vec{E} \bullet \vec{n}\, dS = \frac{Q}{4\pi\varepsilon_0}(4\pi)$$

$$= \frac{Q}{\varepsilon_0}$$

***b)*** The net outward flux across $S$ is the difference of the fluxes across the inner and outer sphere; by part ($a$), these are equal (independent of the radius), so the net flux across $S$ is zero.

***c)*** The left-hand side is the flux across the boundary of $D$, while the right-hand side is the sum of the charge densities at each point of $D$.

$$\iint_S \vec{E} \bullet \vec{n}\, dS = \frac{1}{\varepsilon_0}\iiint_D q(x, y, z)\, dV$$

$$= \frac{Q}{\varepsilon_0}$$

$$\rightarrow \quad \iiint_D q(x, y, z)\, dV = Q$$

The statement says that the flux across the boundary, up to multiplication by a constant, is the sum of the charge densities in the region.

***d)*** $\dfrac{1}{\varepsilon_0}\displaystyle\iiint_D q(x, y, z)\, dV = \iint_S \vec{E} \bullet \vec{n}\, dS$

$$= \iiint_D \nabla \bullet \vec{E}\, dV$$

This holds for all regions $D$.

Therefore; that implies that $\nabla \bullet \vec{E} = \dfrac{q(x, y, z)}{\varepsilon_0}$

***e)*** $\nabla^2 \phi = \nabla \bullet \nabla \phi$

$$= \nabla \bullet \vec{E}$$

$$\boxed{= \dfrac{q}{\varepsilon_0}}$$

## *Exercise*

***Fourier's Law*** of heat transfer (or heat conduction) states that the heat flow vector $\vec{F}$ at a point is proportional to the negative gradient of the temperature that is, $\vec{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k > 0$ is called the *conductivity*, which has metric units of $J / m\text{-}s\text{-}K$. A temperature function for a region $D$ is given. Find the net outward heat flux $\iint\limits_S \vec{F} \bullet \vec{n}\ dS = -k \iint\limits_S \nabla T \bullet \vec{n}\ dS$ across the boundary $S$ of $D$. In some cases it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume $k = 1$.

$$T(x, y, z) = 100 + x + 2y + z; \quad D = \{(x, y, z):\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$$

***Solution***

$$\nabla T = T_x\ \hat{i} + T_y\ \hat{j} + T_z\ \hat{k}$$

$$= \hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{F} = -k\nabla T$$

$$= k\langle -1,\ -2,\ -1\rangle$$

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(-k) + \frac{\partial}{\partial y}(-2k) + \frac{\partial}{\partial z}(-k)$$

$$\boxed{= 0}$$

Therefore, the heat flux is ***zero***.

### Exercise

***Fourier's Law*** of heat transfer (or heat conduction) states that the heat flow vector $\vec{F}$ at a point is proportional to the negative gradient of the temperature that is, $\vec{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k > 0$ is called the *conductivity*, which has metric units of $J/m\text{-}s\text{-}K$. A temperature function for a region $D$ is given.

Find the net outward heat flux $\iint\limits_S \vec{F} \bullet \vec{n}\ dS = -k \iint\limits_S \nabla T \bullet \vec{n}\ dS$ across the boundary $S$ of $D$. In some cases, it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume $k = 1$.

$$T(x, y, z) = 100 + x^2 + y^2 + z^2; \quad D = \{(x, y, z):\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$$

**Solution**

$$\nabla T = T_x\ \hat{i} + T_y\ \hat{j} + T_z\ \hat{k}$$

$$= 2x\ \hat{i} + 2y\ \hat{j} + 2z\ \hat{k}$$

$$\vec{F} = -\nabla T$$

$$= \langle -2x,\ -2y,\ -2z \rangle$$

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(-2x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(-2z)$$

$$= -6$$

Therefore, the heat flux is **–6** times the volume of the region (or **–6**).

### Exercise

***Fourier's Law*** of heat transfer (or heat conduction) states that the heat flow vector $\vec{F}$ at a point is proportional to the negative gradient of the temperature that is, $\vec{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k > 0$ is called the *conductivity*, which has metric units of $J/m\text{-}s\text{-}K$. A temperature function for a region $D$ is given.

Find the net outward heat flux $\iint\limits_S \vec{F} \bullet \vec{n}\ dS = -k \iint\limits_S \nabla T \bullet \vec{n}\ dS$ across the boundary $S$ of $D$. In some cases, it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume $k = 1$.

$$T(x, y, z) = 100 + e^{-z}; \quad D = \{(x, y, z):\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$$

**Solution**

$$\nabla T = T_x\ \hat{i} + T_y\ \hat{j} + T_z\ \hat{k}$$

$$= -e^{-z}\ \hat{k}$$

$$\vec{F} = -\nabla T$$

$$=\left\langle 0,\ 0,\ e^{-z}\right\rangle$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}\left(e^{-z}\right)$$

$$\underline{= -e^{-z}}\,\Big|$$

Therefore, the heat flux is

$$\int_0^1\int_0^1\int_0^1 \left(-e^{-z}\right)dxdydz = \int_0^1 \left(-e^{-z}\right)dz \quad \int_0^1 dy \quad \int_0^1 dx$$

$$= e^{-z}\Big|_0^1 \ (1)(1)$$

$$\underline{= e^{-1} - 1}\,\Big|$$

## *Exercise*

***Fourier's Law*** of heat transfer (or heat conduction) states that the heat flow vector $\vec{F}$ at a point is proportional to the negative gradient of the temperature that is, $\vec{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k > 0$ is called the *conductivity*, which has metric units of $J/m\text{-}s\text{-}K$. A temperature function for a region *D* is given.

Find the net outward heat flux $\iint\limits_S \vec{F} \cdot \vec{n}\ dS = -k \iint\limits_S \nabla T \cdot \vec{n}\ dS$ across the boundary *S* of *D*. In some cases, it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume $k = 1$

$$T(x,\ y,\ z) = 100e^{-x^2-y^2-z^2};\quad D \text{ is the sphere of radius } a \text{ centered at the origin.}$$

***Solution***

$$\nabla T = T_x\ \hat{i} + T_y\ \hat{j} + T_z\ \hat{k}$$

$$= \left\langle -200xe^{-x^2-y^2-z^2},\ -200ye^{-x^2-y^2-z^2},\ -200ze^{-x^2-y^2-z^2}\right\rangle$$

$$\vec{F} = -\nabla T$$

$$= 200\left\langle xe^{-x^2-y^2-z^2},\ ye^{-x^2-y^2-z^2},\ ze^{-x^2-y^2-z^2}\right\rangle$$

$$\nabla \cdot \vec{F} = 200\frac{\partial}{\partial x}\left(xe^{-x^2-y^2-z^2}\right) + 200\frac{\partial}{\partial y}\left(ye^{-x^2-y^2-z^2}\right) + 200\frac{\partial}{\partial z}\left(ze^{-x^2-y^2-z^2}\right)$$

$$= 200e^{-x^2-y^2-z^2}\left(1 - 2x^2 + 1 - 2y^2 + 1 - 2z^2\right)$$

$$= 200e^{-x^2-y^2-z^2}\left(3 - 2x^2 - 2y^2 - 2z^2\right)$$

$$\iiint_D \nabla \cdot \vec{F}\, dV = \iiint_D 200e^{-x^2-y^2-z^2}\left(3-2x^2-2y^2-2z^2\right) dV$$

$$= 200\int_0^{2\pi}\int_0^{\pi}\int_0^{a} e^{-\rho^2}\left(3-2\rho^2\right)\rho^2 \ \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= 200\int_0^{2\pi} d\theta \ \int_0^{\pi} \sin\varphi \, d\varphi \ \int_0^{a}\left(3\rho^2-2\rho^4\right)e^{-\rho^2} \ d\rho$$

$$= 200(2\pi) \ \left(-\cos\varphi \ \Big|_0^{\pi} \ \left(\rho^3 e^{-\rho^2} \ \Big|_0^{a}\right.\right. \qquad \left(3\rho^2-2\rho^4\right)e^{-\rho^2} = \left(\rho^3 e^{-\rho^2}\right)'$$

$$\underline{= 800\pi a^3 \ e^{-a^2}}$$

### *Exercise*

Consider the surface *S* consisting of the quarter-sphere $x^2+y^2+z^2=a^2$, for $z \ge 0$ *and* $x \ge 0$, and the half disk in the *yz*-plane $y^2+z^2 \le a^2$, for $z \ge 0$. The boundary of *S* in the *xy*-plane is *C*, which consists of the semicircle $x^2+y^2=a^2$, for $x \ge 0$, and the line segment $[-a, \ a]$ on the *y*-axis, with a counterclockwise orientation. Let $\vec{F} = \langle 2z-y, \ x-z, \ y-2x \rangle$

*a)* Describe the direction in which the normal vectors point on *S*.

*b)* Evaluate $\oint_C \vec{F} \cdot d\vec{r}$

*c)* Evaluate $\iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n} \ dS$ and check for segment with part (*b*).

**Solution**

***a)*** Normal vectors point outwards everywhere on *S*; that is, on the curved surface. They point outwards and on the flat surface they point in the direction of negative *x*.

***b)*** $\vec{F} = \langle 2z-y, \ x-z, \ y-2x \rangle$ & $C: \ x^2+y^2=a^2 \qquad [-a, \ a]$

Parameterized:

$\vec{r}_1 = \langle a\cos t, \ a\sin t, \ 0 \rangle$

$$\frac{d\vec{r}_1}{dt} = \langle -a\sin t, \ a\cos t, \ 0 \rangle$$

$$\vec{F} = \langle 2z-y, \ x-z, \ y-2x \rangle$$

$$= \langle -a\sin t, \ a\cos t, \ a\sin t - 2a\cos t \rangle$$

$$\vec{F}\cdot\frac{d\vec{r}_1}{dt}=\langle -a\sin t,\ a\cos t,\ a\sin t-2a\cos t\rangle\cdot\langle -a\sin t,\ a\cos t,\ 0\rangle$$

$$=a^2\sin^2 t+a^2\cos^2 t$$

$$\boxed{=a^2}$$

For $0\le t\le a$

$\vec{r}_2=\langle 0,\ a-2t,\ 0\rangle$

$$\frac{d\vec{r}_2}{dt}=\langle 0,\ -2,\ 0\rangle$$

$$\vec{F}=\langle 2z-y,\ x-z,\ y-2x\rangle$$

$$=\langle 2t-a,\ 0,\ a-2t\rangle$$

$$\vec{F}\cdot\frac{d\vec{r}_2}{dt}=\langle 2t-a,\ 0,\ a-2t\rangle\cdot\langle 0,\ -2,\ 0\rangle$$

$$\boxed{=0}$$

$$\oint_C \vec{F}\cdot d\vec{r}=\oint_C \vec{F}\cdot d\vec{r}_1+\oint_C \vec{F}\cdot d\vec{r}_2$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2\,dt+0$$

$$=a^2\,t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$\boxed{=\pi a^2}$$

***c)*** $\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2z-y & x-z & y-2x\end{vmatrix}$

$$=\langle 2,\ 4,\ 2\rangle$$

Spherical: $\langle a\sin\varphi\cos\theta,\ a\sin\varphi\sin\theta,\ a\cos\varphi\rangle$

$\vec{t}_\varphi=\langle a\cos\varphi\cos\theta,\ a\cos\varphi\sin\theta,\ -a\sin\varphi\rangle$

$\vec{t}_\theta=\langle -a\sin\varphi\sin\theta,\ a\sin\varphi\cos\theta,\ 0\rangle$

$\vec{n}_1=\vec{t}_\varphi\times\vec{t}_\theta$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= \left\langle a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\cos\varphi\sin\varphi\cos^2\theta + a^2\cos\varphi\sin\varphi\sin^2\theta \right\rangle$$

$$= \left\langle a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\cos\varphi\sin\varphi \right\rangle$$

$$\left(\nabla\times\vec{F}\right)\cdot\vec{n}_1 = \langle 2,\ 4,\ 2\rangle \cdot \left\langle a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\cos\varphi\sin\varphi \right\rangle$$

$$\underline{= 2a^2\sin^2\varphi\cos\theta + 4a^2\sin^2\varphi\sin\theta + 2a^2\cos\varphi\sin\varphi}$$

From part ($a$), the flat surface is pointing in $x$-negative direction

$$\vec{n}_2 = \langle -1,\ 0,\ 0\rangle$$

$$\left(\nabla\times\vec{F}\right)\cdot\vec{n}_2 = \langle 2,\ 4,\ 2\rangle \cdot \langle -1,\ 0,\ 0\rangle$$

$$\underline{= -2}$$

$$\iint\limits_S \left(\nabla\times\vec{F}\right)\cdot\vec{n}\ dS = \iint\limits_{S_1} \left(2a^2\sin^2\varphi\cos\theta + 4a^2\sin^2\varphi\sin\theta + 2a^2\cos\varphi\sin\varphi\right)dS - 2\iint\limits_{S_2} dS$$

$$= 2a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \left(\tfrac{1}{2}(1-\cos 2\varphi)\cos\theta + \tfrac{1}{2}(1-\cos 2\varphi)\sin\theta + \cos\varphi\sin\varphi\right)\, d\varphi d\theta$$
$$-2\times(semi-circle)$$

$$= a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \left(\cos\theta - \cos 2\varphi\cos\theta + \sin\theta - \cos 2\varphi\sin\theta + \sin 2\varphi\right)\, d\varphi d\theta$$
$$-2\left(\tfrac{1}{2}\pi a^2\right)$$

$$= a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left((\cos\theta+\sin\theta)\varphi - \tfrac{1}{2}(\cos\theta+\sin\theta)\sin 2\varphi - \tfrac{1}{2}\cos 2\varphi \,\Big|_0^{\frac{\pi}{2}}\right. d\theta - \pi a^2$$

$$= a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left((\cos\theta+\sin\theta)\frac{\pi}{2} + \frac{1}{2} + \frac{1}{2}\right)d\theta\ -\ \pi a^2$$

$$= a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi}{2}(\cos\theta+\sin\theta) + 1\right)d\theta\ -\ \pi a^2$$

$$= a^2 \left( \frac{\pi}{2}(\sin\theta - \cos\theta) + \theta \right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \pi a^2$$

$$= a^2\left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}\right) - \pi a^2$$

$$= 2\pi a^2 - \pi a^2$$

$$\underline{= \pi a^2}$$

## *Exercise*

Let $S$ be the hemisphere $x^2 + y^2 + z^2 = a^2$, for $z \ge 0$, and let $T$ be the paraboloid $z = a - \frac{1}{a}\left(x^2 + y^2\right)$, for $z \ge 0$, where $a > 0$. Assume the surfaces have outward normal vectors.

*a)* Verify that $S$ and $T$ have the same base $\left(x^2 + y^2 \le a^2\right)$ and the same high point $(0,\ 0,\ a)$.

*b)* Which surface has the greater area?

*c)* Show that the flux of the radial field $\vec{F} = \langle x,\ y,\ z\rangle$ across $S$ is $2\pi a^3$.

*d)* Show that the flux of the radial field $\vec{F} = \langle x,\ y,\ z\rangle$ across $T$ is $\frac{3\pi a^3}{2}$.

***Solution***

***a)*** $S$: base is the surface where $z = 0$

$$\Rightarrow x^2 + y^2 = a^2 \quad (circle)$$

$T$: $z = 0 = a - \frac{1}{a}\left(x^2 + y^2\right) \quad \Rightarrow x^2 + y^2 = a^2$

The high point of the hemisphere (maximum $z$-coordinate) occurs when

$x = y = 0 \quad \to z = a$

$\therefore\ x^2 + y^2 \le a^2$ ✓

***b)*** $S$: is the surface of hemisphere $\frac{1}{2}4\pi a^2 = 2\pi a^2$

For $T$: paraboloid

$z_x = -\frac{2x}{a} \quad z_y = -\frac{2y}{a}$

$$\vec{n} = \sqrt{z_x^2 + z_y^2 + 1}$$

$$= \sqrt{\frac{4x^2}{a^2} + \frac{4y^2}{a^2} + 1}$$

$$= \frac{1}{a}\sqrt{4\left(x^2 + y^2\right) + a^2}$$

$$\iint_S 1\,dS = \frac{1}{a}\iint_S \sqrt{4\left(x^2+y^2\right)+a^2}\;dS$$

$$= \frac{1}{a}\int_0^{2\pi} d\theta \quad \int_0^a \sqrt{4r^2+a^2}\; r\,dr$$

$$= \frac{\pi}{4a}\int_0^a \left(4r^2+a^2\right)^{1/2} d\left(4r^2+a^2\right)$$

$$= \frac{\pi}{6a}\left(4r^2+a^2\right)^{3/2}\Big|_0^a$$

$$= \frac{\pi}{6a}\left(\left(5a^2\right)^{3/2} - a^3\right)$$

$$= \frac{\pi}{6a}\left(5^{3/2}a^3 - a^3\right)$$

$$= \frac{\pi}{6}\left(5\sqrt{5}-1\right)a^2$$

$$\boxed{= \frac{5\sqrt{5}-1}{6}\pi a^2}$$

$$< 2\pi a^2$$

∴ Area of the paraboloid is smaller than the area of the hemisphere.

***c)*** $\vec{F} = \langle x,\ y,\ z\rangle$

$$= \langle a\sin\varphi\cos\theta,\ a\sin\varphi\sin\theta,\ a\cos\varphi\rangle$$

$$\vec{t}_\varphi = \langle a\cos\varphi\cos\theta,\ a\cos\varphi\sin\theta,\ -a\sin\varphi\rangle$$

$$\vec{t}_\theta = \langle -a\sin\varphi\sin\theta,\ a\sin\varphi\cos\theta,\ 0\rangle$$

$$\vec{n}_1 = \vec{t}_\varphi \times \vec{t}_\theta$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= \left\langle a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\cos\varphi\sin\varphi\cos^2\theta + a^2\cos\varphi\sin\varphi\sin^2\theta\right\rangle$$

$$= \left\langle a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\cos\varphi\sin\varphi\right\rangle$$

$$\vec{F}\bullet\vec{n} = \langle a\sin\varphi\cos\theta,\ a\sin\varphi\sin\theta,\ a\cos\varphi\rangle \bullet \left\langle a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\cos\varphi\sin\varphi\right\rangle$$

$$= a^3\sin^3\varphi\cos^2\theta + a^3\sin^3\varphi\sin^2\theta + a^3\cos^2\varphi\sin\varphi$$

$$= a^3\sin^3\varphi\left(\cos^2\theta+\sin^2\theta\right) + a^3\cos^2\varphi\sin\varphi$$

$$= a^3 \sin\varphi\left(\sin^2\varphi + \cos^2\varphi\right)$$

$$= a^3 \sin\varphi$$

$$\iint_S \vec{F}\bullet\vec{n}\; dS = a^3 \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi\; d\varphi$$

$$= -2\pi a^3 \left(\cos\varphi \Big|_0^{\frac{\pi}{2}}\right.$$

$$\underline{= 2\pi a^3}\,\Big| \quad \checkmark$$

***d)*** Across *T*: $\quad z = a - \frac{1}{a}\left(x^2 + y^2\right)$

Parameterization:

$$\left\langle r\cos\theta,\ r\sin\theta,\ a - \frac{r^2}{a}\right\rangle$$

$0 \le r \le a \quad 0 \le \theta \le 2\pi$

$$\vec{t}_r = \left\langle \cos\theta,\ \sin\theta,\ -\frac{2}{a}r\right\rangle$$

$$\vec{t}_\theta = \left\langle -r\sin\theta,\ r\cos\theta,\ 0\right\rangle$$

$$\vec{n} = \vec{t}_\varphi \times \vec{t}_\theta$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & -\frac{2}{a}r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \left\langle \frac{2r^2}{a}\cos\theta,\ \frac{2r^2}{a}\sin\theta,\ r\cos^2\theta + r\sin^2\theta\right\rangle$$

$$= \left\langle \frac{2r^2}{a}\cos\theta,\ \frac{2r^2}{a}\sin\theta,\ r\right\rangle$$

$$\vec{F}\bullet\vec{n} = \left\langle r\cos\theta,\ r\sin\theta,\ a - \frac{r^2}{a}\right\rangle \bullet \left\langle \frac{2r^2}{a}\cos\theta,\ \frac{2r^2}{a}\sin\theta,\ r\right\rangle$$

$$= \frac{2}{a}r^3\cos^2\theta + \frac{2r^2}{a}\sin^2\theta + ar - \frac{1}{a}r^3$$

$$= \frac{2}{a}r^3\left(\cos^2\theta + \sin^2\theta\right) + ar - \frac{1}{a}r^3$$

$$= \frac{2}{a}r^3 + ar - \frac{1}{a}r^3$$

$$= \frac{1}{a}\left(r^3 + a^2 r\right)$$

$$\iint_S \vec{F} \bullet \vec{n}\ dS = \frac{1}{a}\int_0^{2\pi} d\theta \quad \int_0^a \left(r^3 + a^2 r\right)\ dr$$

$$= \frac{2\pi}{a}\ \left(\frac{1}{4}r^4 + \frac{1}{2}a^2 r^2\ \right|_0^a$$

$$= \frac{2\pi}{a}\left(\frac{1}{4}a^4 + \frac{1}{2}a^4\right)$$

$$= \frac{2\pi}{a}\left(\frac{3}{4}a^4\right)$$

$$\boxed{= \frac{3\pi a^3}{2}} \quad \checkmark$$

## Exercise

The gravitational force due to a point mass $M$ is proportional to $\vec{F} = \dfrac{GM\vec{r}}{|\vec{r}|^3}$, where $\vec{r} = \langle x,\ y,\ z\rangle$ and $G$ is the gravitational constant.

*a)* Show that the flux force field across a sphere of radius $a$ centered at the origin is $\iint_S \vec{F} \bullet \vec{n}\ dS = 4\pi GM$

*b)* Let $S$ be the boundary of the region between two spheres centered at the origin of radius $a$ and $b$ with $a < b$. Use the Divergence Theorem to show that the net outward flux across $S$ is zero.

*c)* Suppose there is a distribution of mass within a region $D$ containing the origin. Let $\rho = (x,\ y,\ z)$ be the mass density (mass per unit volume). Interpret the statement that

$$\iint_S \vec{F} \bullet \vec{n}\ dS = 4\pi G \iiint_D \rho(x,\ y,\ z)\ dV$$

*d)* Assuming $\vec{F}$ satisfies the conditions of the Divergence Theorem, conclude from part (*c*) that $\nabla \bullet \vec{F} = 4\pi G\rho$

*e)* Because the gravitational force is conservative, it has a potential function $\phi$. From part (*d*) conclude that $\nabla^2\phi = 4\pi G\rho$

## Solution

***a)*** The unit normal to sphere is $\vec{n} = \dfrac{\vec{r}}{|\vec{r}|}$

$$\vec{F} \bullet \vec{n} = GM\frac{\vec{r}}{|\vec{r}|^3} \bullet \frac{\vec{r}}{|\vec{r}|}$$

$$= GM\frac{|\vec{r}|^2}{|\vec{r}|^4}$$

$$= GM\frac{1}{|\vec{r}|^2}$$

$$= GM\frac{1}{x^2+y^2+z^2}$$

$$\boxed{= \frac{GM}{a^2}}$$

$$\iint_S \vec{F}\bullet\vec{n}\ dS = \frac{GM}{a^2}\iint_S dS$$

$$= \frac{GM}{a^2}\times(area\ of\ sphere)$$

$$= \frac{GM}{a^2}\times 4\pi a^2$$

$$\boxed{= 4\pi GM} \quad \checkmark$$

**b)** $$\iint_S \vec{F}\bullet\vec{n}\ dS = 4\pi G\iiint_D \rho = (x,\ y,\ z)\ dV$$

Since the outward flux across a sphere, from part (a), is independent of the radius of the sphere, the outward flux across the spheres of radii $a$ and $b$ are equal, so their difference, which is the net flux across the spherical shell bounded by them, is zero.

**c)** $$\iint_S \vec{F}\bullet\vec{n}\ dS = 4\pi G\iiint_D \rho(x,\ y,\ z)\ dV \qquad \rho = (x,\ y,\ z)$$

The left-hand side is the flux across the boundary of $D$, while the right-hand side is the sum of the mass density inside $D$.

The statement says that the flux across the boundary is determined by (is a constant multiple of) the sum of the mass density inside $D$.

**d)** $$4\pi G\iiint_D \rho(x,\ y,\ z)\ dV = \iint_S \vec{F}\bullet\vec{n}\ dS$$

$$= \iiint_D \nabla\bullet\vec{F}\ dV$$

$$\iiint_D \nabla \bullet \vec{F}\, dV = \iiint_D 4\pi G\rho(x, y, z)\, dV$$

$$\nabla \bullet \vec{F} = 4\pi G\rho(x, y, z)$$

***e)*** $\nabla^2\phi = \nabla \bullet \nabla\phi$

$$= \nabla \bullet \vec{F}$$

$$\underline{= 4\pi G\rho(x, y, z)}$$

## *Exercise*

Let $\vec{F}$ be a radial field $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p}$, where *p* is a real number and $\vec{r} = \langle x, y, z\rangle$. With $p = 3$, $\vec{F}$ is an inverse square field.

*a)* Show that the net flux across a sphere centered at the origin is independent of the radius of the sphere only for $p = 3$

*b)* Explain the observation in part (*a*) by finding the flux of $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p}$ across the boundaries of a spherical box $\{(\rho, \varphi, \theta):\ a \le \rho \le b,\ \ \varphi_1 \le \varphi \le \varphi_2,\ \ \theta_1 \le \theta \le \theta_2\}$ for various values of *p*.

***Solution***

***c)*** $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p}$ & $\vec{n} = \dfrac{\vec{r}}{|\vec{r}|}$

$$\vec{F} \bullet \vec{n} = \frac{\vec{r}}{|\vec{r}|^p} \bullet \frac{\vec{r}}{|\vec{r}|}$$

$$= \frac{|\vec{r}|^2}{|\vec{r}|^{p+1}}$$

$$= |\vec{r}|^{1-p}$$

$$= \left(\sqrt{x^2 + y^2 + z^2}\right)^{1-p}$$

$$= \left(\sqrt{a^2}\right)^{1-p}$$

$$\underline{= a^{1-p}}$$

$$\iint_S \vec{F} \bullet \vec{n}\, dS = a^{1-p} \iint_S dS$$

$$= a^{1-p} \times (area\ of\ sphere)$$

$$= a^{1-p} \times 4\pi a^2$$

$$\boxed{= 4\pi a^{3-p}}$$

If $p = 3$, then

$$\boxed{\iint_S \vec{F} \bullet \vec{n}\ dS = 4\pi}$$

Which is independent of the radius of the sphere.

***d)*** $$\nabla \bullet \frac{\langle x,\ y,\ z\rangle}{|\vec{r}|^p} = \left\langle \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z} \right\rangle \bullet \frac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{p/2}}$$

$$= \frac{\partial}{\partial x}\frac{x}{\left(x^2+y^2+z^2\right)^{p/2}} + \frac{\partial}{\partial y}\frac{y}{\left(x^2+y^2+z^2\right)^{p/2}} + \frac{\partial}{\partial z}\frac{z}{\left(x^2+y^2+z^2\right)^{p/2}}$$

$$= \frac{x^2+y^2+z^2-px^2}{\left(x^2+y^2+z^2\right)^{1+p/2}} + \frac{x^2+y^2+z^2-py^2}{\left(x^2+y^2+z^2\right)^{1+p/2}} + \frac{x^2+y^2+z^2-pz^2}{\left(x^2+y^2+z^2\right)^{1+p/2}}$$

$$= \frac{3\left(x^2+y^2+z^2\right) - p\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{1+p/2}}$$

$$= \frac{(3-p)\left(x^2+y^2+z^2\right)}{\left(x^2+y^2+z^2\right)^{1+p/2}}$$

$$= \frac{3-p}{\left(x^2+y^2+z^2\right)^{p/2}}$$

$$\boxed{= \frac{3-p}{|\vec{r}|^p}}$$

$$\iiint_D \nabla \bullet \vec{F}\ dV = \iiint_D \frac{3-p}{|\vec{r}|^p}\ dV$$

$$= (3-p)\iiint_D r^{-p}\ dV$$

$$=(3-p)\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\int_a^b \rho^{-p}\,\rho^2\sin\varphi\,d\rho d\varphi d\theta$$

$$=(3-p)\int_{\theta_1}^{\theta_2} d\theta\int_{\varphi_1}^{\varphi_2}\sin\varphi\,d\varphi\int_a^b \rho^{2-p}\,d\rho$$

$$=(3-p)\left(\theta\Big|_{\theta_1}^{\theta_2}\right)\left(-\cos\varphi\Big|_{\varphi_1}^{\varphi_2}\right)\left(\frac{1}{3-p}\rho^{3-p}\Big|_a^b\right)$$

$$=\left(\theta_2-\theta_1\right)\left(\cos\varphi_1-\cos\varphi_2\right)\left(b^{3-p}-a^{3-p}\right)$$

In general, for $\iiint_D \nabla\cdot\vec{F}\,dV=0$, only if $3-p=0\ \rightarrow\ \boxed{p=3}$

## *Exercise*

Consider the potential function $\phi(x, y, z)=G(\rho)$, where $G$ is any twice differentiable function and $\rho=\sqrt{x^2+y^2+z^2}$; therefore, $G$ depends only on the distance from the origin.

*a)* Show that the gradient vector field associated with $\phi$ is $\vec{F}=\nabla\phi=G'(\rho)\frac{\vec{r}}{\rho}$, where $\vec{r}=\langle x, y, z\rangle$ and $\rho=|\vec{r}|$.

*b)* Let $S$ be the sphere of radius $a$ centered at the origin and let $D$ be the region enclosed by $S$. show that the flux of $\vec{F}$ across $S$ is $\iint_S \vec{F}\cdot\vec{n}\,dS=4\pi a^2G'(a)$.

*c)* Show that $\nabla\cdot\vec{F}=\nabla\cdot\nabla\phi=\frac{2}{\rho}G'(\rho)+G''(\rho)$

*d)* Use part ($c$) to show that the flux across $S$ (as given in part ($b$)) is also obtained by the volume integral $\iiint_D \nabla\cdot\vec{F}\,dV$. (Hint: use spherical coordinates and integrate by parts.)

***Solution***

***a)*** $\phi(x, y, z)=G(\rho)$

$$\nabla\phi(x, y, z)=\langle\phi_x, \phi_y, \phi_z\rangle$$

$$=\left\langle G'(\rho)\rho_x,\ G'(\rho)\rho_y,\ G'(\rho)\rho_z\right\rangle$$

$$= G'(\rho)\left\langle \rho_x,\ \rho_y,\ \rho_z \right\rangle$$

$$= G'(\rho)\left\langle \frac{x}{\sqrt{x^2+y^2+z^2}},\ \frac{y}{\sqrt{x^2+y^2+z^2}},\ \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

$$= \frac{G'(\rho)}{\sqrt{x^2+y^2+z^2}}\langle x,\ y,\ y\rangle$$

$$= G'(\rho)\frac{\vec{r}}{\rho}$$

$\therefore \qquad \vec{F} = \nabla\phi = G'(\rho)\dfrac{\vec{r}}{\rho}$

***b)*** The sphere of radius $a$ centered at the origin $\rightarrow\ x^2+y^2+z^2=a^2$

$$2xdx + 2zdz = 0 \quad \rightarrow \quad z_x = -\frac{x}{z}$$

$$2ydy + 2zdz = 0 \quad \rightarrow \quad z_y = -\frac{y}{z}$$

$$\vec{n} = \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle$$

$$\vec{F}\bullet\vec{n} = \frac{G'(\rho)}{\rho}\langle x,\ y,\ z\rangle \bullet \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle$$

$$= \frac{G'(a)}{a}\left(\frac{x^2}{z} + \frac{y^2}{z} + z\right)$$

$$= \frac{G'(a)}{a}\left(\frac{x^2+y^2+z^2}{z}\right)$$

$$= \frac{G'(a)}{a}\left(\frac{a^2}{z}\right)$$

$$= G'(a)\frac{a}{z}$$

$$\iint_S \vec{F}\bullet\vec{n}\ dS = aG'(a)\iint_S \frac{1}{z}\ dS$$

$$= aG'(a)\int_0^{2\pi}\int_0^a \frac{1}{\sqrt{a^2-r^2}}\ r\ drd\theta$$

$$= -\frac{1}{2}aG'(a)\int_0^{2\pi} d\theta \quad \int_0^a \left(a^2-r^2\right)^{-1/2} d\left(a^2-r^2\right)$$

$$= -2\pi aG'(a)\ \left(a^2-r^2\right)^{1/2}\ \Big|_0^a$$

$$= 2\pi a^2 G'(a)$$

Since the surface area is *twice* the value

Total surface $\boxed{= 4\pi a^2 G'(a)}$

***c)*** $\vec{F} = \nabla\phi = G'(\rho)\dfrac{\vec{r}}{\rho} \qquad\qquad \rho = \sqrt{x^2 + y^2 + z^2}$

$$\nabla \bullet \vec{F} = \nabla \bullet \left( G'(\rho)\frac{\vec{r}}{\rho} \right)$$

$$= \nabla \bullet \frac{G'(\rho)}{\rho}\langle x,\ y,\ z\rangle$$

$$= \frac{\partial}{\partial x}\left( G'(\rho)\frac{x}{\rho} \right) + \frac{\partial}{\partial y}\left( G'(\rho)\frac{y}{\rho} \right) + \frac{\partial}{\partial z}\left( G'(\rho)\frac{z}{\rho} \right)$$

$$\frac{\partial}{\partial x}\left( G'(\rho)\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) = G''(\rho)\rho_x \frac{x}{\rho} + G'(\rho)\frac{x^2 + y^2 + z^2 - x^2}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$= G''(\rho)\frac{x}{\rho}\frac{x}{\rho} + G'(\rho)\frac{y^2 + z^2}{\rho^3}$$

$$= G''(\rho)\frac{x^2}{\rho^2} + G'(\rho)\frac{y^2 + z^2}{\rho^3}$$

$$\frac{\partial}{\partial y}\left( G'(\rho)\frac{y}{\sqrt{x^2 + y^2 + z^2}} \right) = G''(\rho)\rho_y \frac{y}{\rho} + G'(\rho)\frac{x^2 + y^2 + z^2 - y^2}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$= G''(\rho)\frac{y}{\rho}\frac{y}{\rho} + G'(\rho)\frac{y^2 + z^2}{\rho^3}$$

$$= G''(\rho)\frac{y^2}{\rho^2} + G'(\rho)\frac{y^2 + z^2}{\rho^3}$$

$$\frac{\partial}{\partial z}\left( G'(\rho)\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) = G''(\rho)\rho_z \frac{z}{\rho} + G'(\rho)\frac{x^2 + y^2 + z^2 - z^2}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$= G''(\rho)\frac{z}{\rho}\frac{z}{\rho} + G'(\rho)\frac{y^2 + z^2}{\rho^3}$$

$$= G''(\rho)\frac{z^2}{\rho^2} + G'(\rho)\frac{y^2 + z^2}{\rho^3}$$

$$\nabla \cdot \vec{F} = G''(\rho)\frac{z^2}{\rho^2} + G'(\rho)\frac{y^2+z^2}{\rho^3} + G''(\rho)\frac{y^2}{\rho^2} + G'(\rho)\frac{x^2+z^2}{\rho^3} + G''(\rho)\frac{z^2}{\rho^2} + G'(\rho)\frac{x^2+y^2}{\rho^3}$$

$$= G''(\rho)\left(\frac{x^2+y^2+z^2}{\rho^2}\right) + G'(\rho)\frac{2\left(x^2+y^2+z^2\right)}{\rho^3}$$

$$= \frac{\rho^2}{\rho^2}G''(\rho) + G'(\rho)\frac{2\rho^2}{\rho^3}$$

$$\boxed{= G''(\rho) + \frac{2}{\rho}G'(\rho)}$$

***d)*** By the divergence theorem, the flux:

$$\iiint_D \nabla \cdot \vec{F}\ dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2 \sin\varphi\left(G''(\rho) + \frac{2}{\rho}G'(\rho)\right)\ d\rho d\varphi d\theta$$

$$= \int_0^{2\pi} d\theta \quad \int_0^{\pi} \sin\varphi d\varphi \quad \int_0^{a} \rho^2\left(G''(\rho) + \frac{2}{\rho}G'(\rho)\right)\ d\rho$$

$$= 2\pi\ \left(-\cos\varphi\ \Big|_0^{\pi}\right. \quad \int_0^{a} \rho^2\left(G''(\rho) + \frac{2}{\rho}G'(\rho)\right)\ d\rho$$

$$= 4\pi\int_0^{a}\left(\rho^2 G''(\rho) + 2\rho G'(\rho)\right)\ d\rho \qquad\qquad \left(\rho^2 G'(\rho)\right)' = \rho^2 G''(\rho) + 2\rho G'(\rho)$$

$$= 4\pi\int_0^{a}\left(\rho^2 G'(\rho)\right)'\ d\rho$$

$$= 4\pi G'(\rho)\ \rho^2\ \Big|_0^{a}$$

$$\boxed{= 4\pi a^2 G'(\rho)}$$

## *Exercise*

Prove Green's Identity for scalar-valued functions $u$ and $v$ defined on a region $D$:

$$\iiint_D \left(u\nabla^2 v - v\nabla^2 u\right)\ dV = \iint_S\ \left(u\nabla v - v\nabla u\right) \cdot \vec{n}\ dS$$

***Solution***

$$\nabla \cdot \left(u\nabla v\right) = \nabla u \cdot \nabla v + u\nabla^2 v$$

$$\iiint_D \left(u\nabla^2 v + \nabla u \bullet \nabla v\right)\, dV = \iiint_D \left(\nabla \bullet (u\nabla v)\right)\, dV$$

$$= \iint_S u\nabla v \bullet \vec{n}\, dS$$

$$\iiint_D \left(v\nabla^2 u + \nabla u \bullet \nabla v\right)\, dV = \iiint_D \left(\nabla \bullet (v\nabla u)\right)\, dV$$

$$= \iint_S v\nabla u \bullet \vec{n}\, dS$$

$$\iiint_D \left(u\nabla^2 v + \nabla u \bullet \nabla v\right)\, dV - \iiint_D \left(v\nabla^2 u + \nabla u \bullet \nabla v\right)\, dV = \iint_S u\nabla v \bullet \vec{n}\, dS - \iint_S v\nabla u \bullet \vec{n}\, dS$$

$$\iiint_D \left(u\nabla^2 v + \nabla u \bullet \nabla v - v\nabla^2 u + \nabla u \bullet \nabla v\right)\, dV = \iint_S \left(u\nabla v \bullet \vec{n} - v\nabla u \bullet \vec{n}\right)\, dS$$

$$\iiint_D \left(u\nabla^2 v - v\nabla^2 u\right)\, dV = \iint_S \left(u\nabla v - v\nabla u\right) \bullet \vec{n}\, dS$$

## *Exercise*

Prove the identity: $\displaystyle\iiint_D \nabla \times \vec{F}\, dV = \iint_S \left(\vec{n} \times \vec{F}\right) dS$

***Solution***

Let $\vec{F} = \langle f, g, h\rangle$ and $\vec{n} = \left\langle n_1, n_2, n_3\right\rangle$, then:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z},\ \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x},\ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$

$$\vec{F}\times\vec{n}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ f & g & h\\ n_1 & n_2 & n_3\end{vmatrix}$$

$$=\left\langle n_2h-n_3g,\ n_3f-n_1h,\ n_1g-n_2f\right\rangle$$

$\hat{i}\quad\to\quad n_2h-n_3g=\langle 0,\ h,\ -g\rangle\cdot\left\langle n_1,\ n_2,\ n_3\right\rangle$

$\vec{F}_1=\langle 0,\ h,\ -g\rangle$

$$\iint_S\left(n_2h-n_3g\right)dS=\iint_S\vec{F}_1\cdot\vec{n}\ dS$$

$$=\iiint_D\left(\nabla\cdot\vec{F}_1\right)dR$$

$$=\iint_D\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)dA$$

$\hat{j}\quad\to\quad n_3f-n_1h=\langle -h,\ 0,\ f\rangle\cdot\left\langle n_1,\ n_2,\ n_3\right\rangle$

$\vec{F}_2=\langle -h,\ 0,\ f\rangle$

$$\iint_S\left(n_3f-n_1h\right)dS=\iint_S\vec{F}_2\cdot\vec{n}\ dS$$

$$=\iiint_D\left(\nabla\cdot\vec{F}_2\right)dR$$

$$=\iint_D\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)dA$$

$\hat{k}\quad\to\quad n_1g-n_2f=\langle g,\ -f,\ 0\rangle\cdot\left\langle n_1,\ n_2,\ n_3\right\rangle$

$\vec{F}_3=\langle g,\ -f,\ 0\rangle$

$$\iint_S\left(n_1g-n_2f\right)dS=\iint_S\vec{F}_3\cdot\vec{n}\ dS$$

$$=\iiint_D\left(\nabla\cdot\vec{F}_3\right)dR$$

$$= \iint_D \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

$$\iiint_D \left\langle \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z},\ \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x},\ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle dV = \iint_S \left\langle n_2 h - n_3 g,\ n_3 f - n_1 h,\ n_1 g - n_2 f \right\rangle dS$$

$$\iiint_D \nabla \times \vec{F}\ dV = \iint_S \left( \vec{n} \times \vec{F} \right) dS \quad \checkmark$$

### *Exercise*

Prove the identity: $\displaystyle\iint_S \left( \vec{n} \times \nabla\varphi \right) dS = \oint_C \varphi\, d\vec{r}$

***Solution***

Let $\varphi = \left\langle \varphi_x, \varphi_y, \varphi_z \right\rangle$ and $\vec{n} = \left\langle n_1, n_2, n_3 \right\rangle$, then:

$$\vec{n} \times \nabla\varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ n_1 & n_2 & n_3 \\ \varphi_x & \varphi_y & \varphi_z \end{vmatrix}$$

$$= \left\langle n_2\varphi_z - n_3\varphi_y,\ n_3\varphi_x - n_1\varphi_z,\ n_1\varphi_y - n_2\varphi_x \right\rangle$$

$$\hat{i} \quad \to \quad n_2\varphi_z - n_3\varphi_y = \left\langle 0,\ \varphi_z,\ -\varphi_y \right\rangle \cdot \left\langle n_1, n_2, n_3 \right\rangle$$

$$= \left( \nabla \times \langle \varphi,\ 0,\ 0 \rangle \right) \cdot \left\langle n_1, n_2, n_3 \right\rangle$$

$$\vec{F}_1 = \langle \varphi,\ 0,\ 0 \rangle$$

$$\iint_S \left( \left( \nabla \times \langle \varphi,\ 0,\ 0 \rangle \right) \cdot \left\langle n_1, n_2, n_3 \right\rangle \right) dS = \iint_S \left( \nabla \times \vec{F}_1 \right) \cdot \vec{n}\ dS$$

$$= \oint_C \vec{F}_1 \cdot d\vec{r}$$

$$= \oint_C \langle \varphi,\ 0,\ 0 \rangle \cdot d\vec{r}$$

$$= \oint_C \varphi \bullet d\vec{r}$$

$$\hat{j} \rightarrow \quad n_3\varphi_x - n_1\varphi_z = \left\langle -\varphi_z,\ 0,\ \varphi_x \right\rangle \bullet \left\langle n_1,\ n_2,\ n_3 \right\rangle$$

$$= \left(\nabla \times \langle 0,\ \varphi,\ 0\rangle\right) \bullet \left\langle n_1,\ n_2,\ n_3 \right\rangle$$

$$\vec{F}_2 = \langle 0,\ \varphi,\ 0\rangle$$

$$\iint_S \left(\left(\nabla \times \langle 0,\ \varphi,\ 0\rangle\right) \bullet \left\langle n_1,\ n_2,\ n_3 \right\rangle\right) dS = \iint_S \left(\nabla \times \vec{F}_2\right) \bullet \vec{n}\ dS$$

$$= \oint_C \vec{F}_2 \bullet d\vec{r}$$

$$= \oint_C \langle 0,\ \varphi,\ 0\rangle \bullet d\vec{r}$$

$$= \oint_C \varphi \bullet d\vec{r}$$

$$\hat{k} \rightarrow \quad n_1\varphi_y - n_2\varphi_x = \left\langle \varphi_y,\ -\varphi_x,\ 0 \right\rangle \bullet \left\langle n_1,\ n_2,\ n_3 \right\rangle$$

$$= \left(\nabla \times \langle 0,\ 0,\ \varphi\rangle\right) \bullet \left\langle n_1,\ n_2,\ n_3 \right\rangle$$

$$\vec{F}_3 = \langle 0,\ 0,\ \varphi\rangle$$

$$\iint_S \left(\left(\nabla \times \langle 0,\ 0,\ \varphi\rangle\right) \bullet \left\langle n_1,\ n_2,\ n_3 \right\rangle\right) dS = \iint_S \left(\nabla \times \vec{F}_3\right) \bullet \vec{n}\ dS$$

$$= \oint_C \vec{F}_3 \bullet d\vec{r}$$

$$= \oint_C \langle 0,\ 0,\ \varphi\rangle \bullet d\vec{r}$$

$$= \oint_C \varphi \bullet d\vec{r}$$

$$\iint_S \left(\vec{n} \times \nabla\varphi\right) dS = \oint_C \varphi\, d\vec{r} \quad \checkmark$$