Lecture Two – Second & Higher Order Equations

Section 2.1- Definitions and Examples

A second-order differential equation is an equation involving the independent variable t and unknown function y.

$$y'' = f(t, y, y')$$

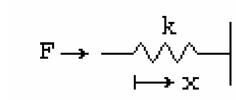
Linear equation: $y'' + p(t)y' + q(t)y = g(t)$

The coefficient $p, q, and\ g$ can be arbitrary functions.

The equation is said to be ***homogeneous*** when:

$$y'' + p(t)y' + q(t)y = 0$$

Newton's - Hooke's law



$$\boldsymbol{F = kx}$$

Hooke's law: Force is **p**roportional to the **d**isplacement that has been displaced from its real position.

$$F - T = K_s x$$

$$1\ kg.m/s^2 = 1\ N \qquad (Newton)$$

Example

A 4-*kg* weight is suspended from a spring. The displacement of the spring-mass equilibrium from the spring equilibrium is measured to be 49 *cm*. What is the spring constant?

Solution

$$mg = kx_0$$

$$k = \frac{mg}{x_0}$$

$$= \frac{4(9.8)}{0.49}$$

$$\underline{= 80\ N/m}$$

Proposition

$$y'' + p(t)y' + q(t)y = 0$$

Solutions: $y = C_1 y_1 + C_2 y_2$

C_1, C_2 are any constant.

$y_1(t) \;\&\; y_2(t)$ are linearly independent solutions forming a ***fundamental set of solutions***.

Definition

A linear combination of the two functions $u \,\&\, v$ is any function of the form

$$w = Au + Bv$$

Definition

Two functions $u \,\&\, v$ are said to be linearly independent on the interval (α, β), if neither is a constant multiple of the order on that interval. If one is a constant multiple of the other on (α, β), they said to be linearly dependent there.

Wronskian

The Wronskian is a function named after the Polish mathematician Józef Hoene-Wroński and it is used to determine whether a set of differentiable functions (solutions) is ***linearly independent*** on a given interval.

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_n(x) \\ f_1'(x) & f_2'(x) & f_n'(x) \\ & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & f_n^{(n-1)}(x) \end{vmatrix}$$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu'$$

If $W = 0 \;\Rightarrow\; u \;\&\; v$ are linearly dependent.

If $W \neq 0 \;\Rightarrow\; u \;\&\; v$ are linearly independent.

Theorem

Suppose that y_1 and y_2 are two solutions of the homogeneous second-order linear equation

$$y'' + p(x)y' + q(x)y = 0$$

On an open interval *I* on which *p* and *q* are continuous

1. If y_1 and y_2 are linearly dependent, then $W(y_1, y_2) \equiv 0$ on *I*.
2. If y_1 and y_2 are linearly independent, then $W(y_1, y_2) \neq 0$ at each point of *I*.

Example

Use the Wronskian to show that $\mathbf{f}_1 = x, \quad \mathbf{f}_2 = \sin x$ are linearly independence

Solution

The Wronskian is

$$W(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x \neq 0$$

This function is not identically zero. Thus the functions are linearly independent.

Exercises *Section* 2.1 - Definitions and Examples

(*Exercises* 1- 4) Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous.

1. $t^2 y'' = 4y' - \sin t$

2. $ty'' + (\sin t)y' = 4y - \cos 5t$

3. $t^2 y'' + 4yy' = 0$

4. $y'' + 4y' + 7y = 3e^{-t}\sin t$

(*Exercises*5- 6) Show by direct substitution that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation. Then verify by direct substitution, that any linear combination $C_1 y_1(t) + C_2 y_2(t)$ of the 2 given solutions is also a solution.

5. $y'' + 4y = 0;\quad y_1(t) = \cos 2t \quad y_2(t) = \sin 2t$

6. $y'' - 2y' + 2y = 0;\quad y_1(t) = e^t \cos t \quad y_2(t) = e^t \sin t$

7. Explain why $y_1(t)$ and $y_2(t)$ are linearly independent solutions. Calculate Wronskian and use it to explain the independence of the given solutions.

$$y'' + 9y = 0;\quad y_1(t) = \cos 3t \quad y_2(t) = \sin 3t$$

8. Show that $y_1(t) = e^t$ and $y_2(t) = e^{-3t}$ form a fundamental set of solutions for $y'' + 2y' - 3y = 0,$ then find a solution satisfying $y(0) = 1$ and $y'(0) = -2$.

9. Use the Wronskian to show that $\mathbf{f}_1 = 1,\ \mathbf{f}_2 = e^x,\ \mathbf{f}_3 = e^{2x}$ are linearly independence

10. Determine whether $\left\{e^x,\ xe^x,\ (x+1)e^x\right\}$ is a set of linearly independent.

11. Show that the functions $y_1(x) = e^{-3x},\quad y_2(x) = \cos 2x,\quad y_3(x) = \sin 2x$ are linearly independent.

Section 2.2 - Second-Order Equations and Systems

A Planar System of 1^{st}- order equations is a set of two first-order differential equations involving two unknown

$$x' = f(t,x,y)$$
$$y' = g(t,x,y)$$

where f *and* g are functions of the independent variable t and the unknown x and y.

Second-Order Equations and Planar Systems

$$y'' = f(t,\ y,\ y')$$

Let's re-write in first-order system:

$$y' = v$$
$$v' = F(t,y,v)$$

$$y'' + p(t)y' + q(t)y = F(t)$$
$$y'' = F(t) - p(t)y' - q(t)y$$
$$v' = F(t) - p(t)v - q(t)y$$

$$y' = v$$
$$v' = F(t) - p(t)v - q(t)y$$

Example

Consider a damped unforced spring: $y'' + 0.4y' + 3y = 0$

which satisfies the initial conditions $y(0) = 2$ and $v(0) = y'(0) = -1$

Solution

$$\begin{cases} y' = v \\ v' = -0.4v - 3y \end{cases}$$

$$v' + 0.4v = -3y$$

$$ve^{\int 0.4dy} = \int -3ye^{\int 0.4dy} + C$$

$$ve^{0.4y} = -3\int ye^{0.4y} + C \qquad\qquad \int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$= -3\frac{e^{0.4y}}{0.4^2}(0.4y-1) + C$$

$$= -18.75e^{0.4y}(0.4y-1) + C$$

$$v = -7.5y + 18.75 + Ce^{-0.4y}$$

$$v(0) = -7.5(0) + 18.75 + Ce^{-0.4(0)}$$
$$-1 = 18.75 + C$$
$$C = -19.75$$

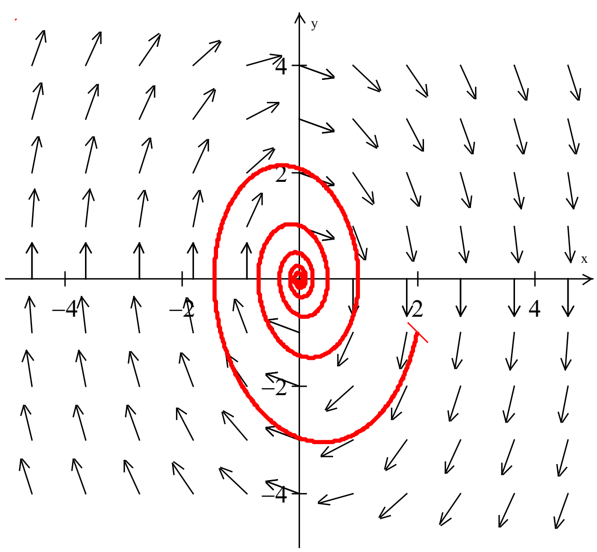
$$v(y) = -7.5y + 18.75 - 19.75e^{-0.4y}$$

$$y' = v = -7.5y + 18.75 - 19.75e^{-0.4y}$$

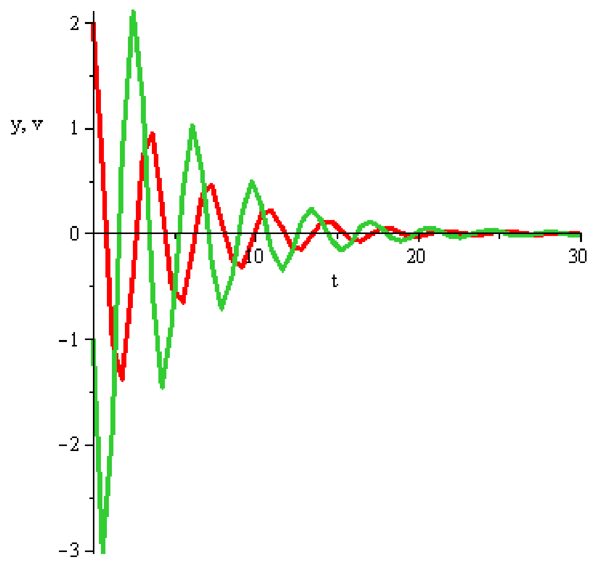
$$\frac{dy}{dt} = -7.5y + 18.75 - 19.75e^{-0.4y}$$

$$y(t) = -\frac{3\sqrt{74}}{74}e^{-t/5}\sin\left(\frac{\sqrt{74}}{5}t\right) + 2e^{-t/5}\cos\left(\frac{\sqrt{74}}{5}t\right)$$

The *yv*-plane is called the ***phase plane***.



Phase Plane Plot



Displacement *y* and the velocity *v*

Exercises *Section* 2.2 - Second-Order Equations and Systems

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation.

1. $y'' + 2y' - 3y = 0$

2. $y'' + 3y' + 4y = 2\cos 2t$

3. $y'' + 2y' + 2y = 2\sin 2\pi t$

4. $y'' + \mu\left(t^2 - 1\right)y' + y = 0$

5. $4y'' + 4y' + y = 0$

(*Exercises* 6-7) Given the mass, damping, and spring constants of an undriven spring-mass system

$$my'' + \mu y' + ky = 0$$

a) Provide separate plots of the position versus time (y vs. t) and the velocity versus time (v vs. t)
b) Provide a combined plot of both position and velocity versus time
c) Provide a plot of the velocity versus position (v vs. y) in the yv phase plane.

6. $m = 1\ kg, \quad \mu = 0\ kg/s, \quad k = 4kg/s^2, \quad y(0) = -2\ m, \quad y'(0) = -2\ m/s$

7. $m = 1\ kg, \quad \mu = 2\ kg/s, \quad k = 1kg/s^2, \quad y(0) = -3\ m, \quad y'(0) = -2\ m/s$

Section 2.3 - Linear, Homogeneous Equations with Constant Coefficients

The equations of the form:

$$y'' + py' + qy = 0$$

This is a class of equations that we can solve easily.

The analogous first-order, linear, homogeneous equation:

$$y' + py = 0$$

It is separable and easily solved, its general solution is

$$y(t) = Ce^{-pt}$$

Let look for a solution of the type

$$y(t) = e^{\lambda t}$$

$$y' = \lambda e^{\lambda t}$$

$$y'' = \lambda^2 e^{\lambda t}$$

$$\begin{aligned} y'' + py' + qy &= \lambda^2 e^{\lambda t} + p\lambda e^{\lambda t} + qe^{\lambda t} \\ &= \left(\lambda^2 + p\lambda + q\right)e^{\lambda t} \\ &= 0 \end{aligned}$$

$\lambda^2 + p\lambda + q = 0$ *This is called the **characteristic equation***

We can rewrite the differential equation and its characteristic equations

$$y'' + py' + qy = 0$$

$$\lambda^2 + p\lambda + q = 0$$

The roots are: $\lambda_{1,2} = \dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$

If $p^2 - 4q > 0 \Rightarrow$ *Two distinct real roots*

If $p^2 - 4q < 0 \Rightarrow$ *Two distinct complex roots*

If $p^2 - 4q = 0 \Rightarrow$ One *repeated real root*

Case 1: *Distinct Real Root*

$y_1 = C_1 e^{\lambda_1 t}$ *and* $y_2 = C_2 e^{\lambda_2 t}$ are both solutions.

Proposition

If the characteristic equations $\lambda^2 + p\lambda + q = 0$ has two distinct real roots λ_1 and λ_2, then the ***general solution*** to $y'' + py' + qy = 0$ is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

Where C_1 and C_2 are arbitrary constants.

Example

Find the general solution to the equation $y'' - 3y' + 2y = 0$

Find the unique solution corresponding to the initial conditions $y(0) = 2$ and $y'(0) = 1$

Solution

The characteristic equation:

$$y'' - 3y' + 2y = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

The solution: $\lambda_{1,2} = 1,\ 2$

The general solution

$$y(t) = C_1 e^t + C_2 e^{2t}$$

$$y' = C_1 e^t + 2C_2 e^{2t}$$

$y(0) = 2 \qquad y(0) = C_1 e^0 + C_2 e^{2(0)}$

$$2 = C_1 + C_2$$

$y'(0) = 1 \qquad y'(0) = C_1 e^0 + 2C_2 e^{2(0)}$

$$1 = C_1 + 2C_2$$

$$\begin{matrix} C_1 + C_2 = 2 \\ C_1 + 2C_2 = 1 \end{matrix} \Rightarrow C_2 = -1 \quad C_1 = 3$$

The unique solution is: $\boxed{y(t) = 3e^t - e^{2t}}$

Case 2: Complex Roots

Proposition

If the characteristic equations $\lambda^2 + p\lambda + q = 0$ has two complex conjugate roots $\lambda = a + ib$ and $\bar{\lambda} = a - ib$.

1. The functions

$$z = e^{(a+ib)t} \text{ and } \bar{z} = e^{(a-ib)t}$$

So the general solution is

$$w(t) = C_1 e^{(a+ib)t} + C_2 e^{(a-ib)t}$$

Where C_1 and C_2 are arbitrary complex constants.

2. The functions

$$y_1(t) = e^{at}\cos(bt) \text{ and } y_2(t) = e^{at}\sin(bt)$$

So the general solution is

$$y(t) = e^{at}\left(A_1 \cos bt + A_2 \sin bt\right)$$

Where A_1 and A_2 are constants.

Example

Find the general solution to the equation $y'' + 2y' + 2y = 0$

Find the unique solution corresponding to the initial conditions $y(0) = 2$ and $y'(0) = 3$

Solution

The characteristic equation:

$$y'' + 2y' + 2y = 0$$

$$\lambda^2 + 2\lambda + 2 = 0$$

The solution: $\lambda_{1,2} = -1 \pm i = a \pm ib$

$a = -1; \quad b = 1$

The general solution

$$y(t) = e^{-t}\left(C_1 \cos t + C_2 \sin t\right)$$

$$y(0) = e^{-(0)}\left(C_1 \cos(0) + C_2 \sin(0)\right)$$

$$2 = 1\left(C_1 + C_2(0)\right)$$

$$\Rightarrow \boxed{C_1 = 2}$$

$$y' = -e^{-t}\left(C_1 \cos t + C_2 \sin t\right) + e^{-t}\left(-C_1 \sin t + C_2 \cos t\right)$$

$$y'(0) = -e^{-(0)}\left(C_1 \cos(0) + C_2 \sin(0)\right) + e^{-(0)}\left(-C_1 \sin(0) + C_2 \cos(0)\right)$$

$$3 = -\left(C_1\right) + \left(C_2\right)$$

$$C_2 - C_1 = 3$$

$$\underline{|C_2} = 3 + 2 \underline{= 5|}$$

$$\underline{y(t) = e^{-t}\left(2\cos t + 5\sin t\right)|}$$

Example

Find the general solution to the equation $y'' - 4y' + 13y = 0$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 13 = 0$

The solutions: $\lambda_{1,2} = \dfrac{4 \pm \sqrt{16-52}}{2} = \dfrac{4 \pm \sqrt{-36}}{2} = \dfrac{4 \pm 6i}{2} \underline{= 2 \pm 3i|}$

$a = 2;\ \ b = 3$

The general solution: $\underline{y(x) = e^{2x}\left(C_1 \cos 3x + C_2 \sin 3x\right)|}$

Case 3: Repeated Roots

If the roots of the characteristic equations are repeated

$$\lambda^2 + p\lambda + q = 0$$

$$(\lambda - \lambda_1)^2 = 0$$

$$\lambda_{1,2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

$$p^2 - 4q = 0 \;\Rightarrow\; q = \frac{p^2}{4}$$

$$\lambda_{1,2} = -\frac{p}{2}$$

$$y_1 = C_1 e^{\lambda_1 t}$$

$$= C_1 e^{-pt/2}$$

$$y_2 = v(t)y_1(t)$$

$$= v(t)e^{-pt/2}$$

$$y'' + py' + qy = 0$$

$$y'' + py' + \frac{p^2}{4}y = 0$$

$$y_2{}' = v'e^{-pt/2} - \frac{P}{2}ve^{-pt/2}$$

$$y_2{}'' = v''e^{-pt/2} - \frac{P}{2}v'e^{-pt/2} - \frac{P}{2}v'e^{-pt/2} + \frac{P^2}{4}ve^{-pt/2}$$

$$v''e^{-pt/2} - \frac{P}{2}v'e^{-pt/2} - \frac{P}{2}v'e^{-pt/2} + \frac{P^2}{4}ve^{-pt/2} + p\left(v'e^{-pt/2} - \frac{P}{2}ve^{-pt/2}\right) + \frac{P^2}{4}ve^{-pt/2} = 0$$

$$v''e^{-pt/2} = 0$$

$$v'' = 0$$

$$\Rightarrow v' = a$$

$$\Rightarrow v = at + b$$

$$v = t$$

$$y_2 = te^{-pt/2}$$

Proposition

If the characteristic equations $\lambda^2 + p\lambda + q = 0$ has one double root λ_1, then the ***general solution*** to $y'' + py' + qy = 0$ is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}$$
$$= \left(C_1 + C_2 t\right) e^{\lambda_1 t}$$

Where C_1 and C_2 are arbitrary constants.

Example

Find the general solution to the equation $y'' - 2y' + y = 0$

Find the unique solution corresponding to the initial conditions $y(0) = 2$ and $y'(0) = -1$

Solution

The characteristic equation:

$$\lambda^2 - 2\lambda + 1 = 0$$

The solution: $\lambda_{1,2} = 1$

$$y(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}$$
$$= C_1 e^t + C_2 t e^t$$

$$y(0) = C_1 e^{(0)} + C_2 (0) e^{(0)} \quad \Rightarrow \quad 2 = C_1$$

$$y' = C_1 e^t + C_2 e^t + C_2 t e^t$$
$$y'(0) = 2e^{(0)} + C_2 e^{(0)} + C_2 (0) e^{(0)}$$
$$-1 = 2 + C_2 \quad \Rightarrow \quad \boxed{C_2 = -3}$$

$$\boxed{y(t) = 2e^t - 3te^t}$$

Example

Find the general solution to the equation $y'' - 10y' + 25y = 0$

Solution

The characteristic equation: $\lambda^2 - 10\lambda + 25 = (\lambda - 5)^2 = 0$

The solutions are: $\lambda_{1,2} = 5$

The general solution: $\boxed{y(t) = C_1 e^{5t} + C_2 t e^{5t}}$

Higher-Order Equations

In general, to solve an nth-order differential equation, we must solve an nth degree characteristic polynomial equation

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

If all roots are real and distinct, then the general solution is

$$y(x) = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x} + \cdots + C_ne^{\lambda_n x}$$

If all roots are equal to λ, then the general solution is

$$y(x) = C_1e^{\lambda x} + C_2xe^{\lambda x} + C_3x^2e^{\lambda x} + \cdots + C_nx^{n-1}e^{\lambda x}$$

Example

Find the general solution of $\quad y''' + 3y'' - 4y = 0$

Solution

$\lambda^3 + 3\lambda^2 - 4 = 0 \qquad$ ***Solve for λ***

$\lambda_1 = 1, \quad \lambda_{2,\,3} = -2$

$$\boxed{y(x) = C_1e^x + (C_2 + C_3x)e^{-2x}}$$

Rational zero theorem: $\pm\left\{\frac{4}{1}\right\} = \pm\{1,\ 2,\ 4\}$

$\lambda_1 = 1, \quad \lambda_2 = -2$

$(\lambda - 1)(\lambda + 2)(\lambda - a) = 0$

$(-1)(2)(-a) = -4 \quad \Rightarrow \quad a = -2$

Example

Find the general solution of $\quad \lambda^4(\lambda + 1)(\lambda + 2)^2(\lambda^2 + 4) = 0$

Solution

$\lambda^2 + 4 = 0 \quad \Rightarrow \quad \lambda^2 = -4 \ \rightarrow \ \lambda = \pm 2i$

The solution: $\lambda = 0,\ 0,\ 0,\ 0,\ -1,\ -2,\ -2,\ \pm 2i$

$$\boxed{y(x) = C_1 + C_2x + C_3x^2 + C_4x^3 + C_5e^{-x} + (C_6 + C_7x)e^{-2x} + C_8\cos 2x + C_9\sin 2x}$$

Summary

The equation: $y'' + py' + qy = 0$

The characteristic equations $\lambda^2 + p\lambda + q = 0$

If $p^2 - 4q > 0$	$y_1(t) = C_1 e^{\lambda_1 t}$ *and* $y_1(t) = C_2 e^{\lambda_2 t}$	$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$
If $p^2 - 4q < 0$	$y_1(t) = e^{at} \cos bt$ *and* $y_1(t) = e^{at} \sin bt$	$y(t) = e^{at}\left(A_1 \cos bt + A_2 \sin bt\right)$
If $p^2 - 4q = 0$	$y_1 = e^{\lambda t}$ *and* $y_1 = te^{\lambda t}$	$y(t) = \left(C_1 + C_2 t\right) e^{\lambda_1 t}$

Exercises *Section* 2.3 - Linear, Homogeneous Equations with Constant Coefficients

Find the general solution.

1. $y'' - y' - 2y = 0$
2. $2y'' - 3y' - 2y = 0$
3. $y'' + 5y' + 6y = 0$
4. $y'' + 2y' + 17y = 0$
5. $y'' - 4y' + 4y = 0$
6. $y'' - 6y' + 9y = 0$
7. $6y'' + 5y' - 6y = 0$
8. $2y'' - 5y' - 3y = 0$
9. $y'' - 10y' + 25y = 0$
10. $y'' + y' + y = 0$
11. $3y'' - y' = 0$
12. $2y'' + 5y' = 0$
13. $y'' - 3y' + 2y = 0$
14. $y'' - y' - 6y = 0$
15. $3y'' + 2y' + y = 0$
16. $2y'' + 2y' + y = 0$
17. $y'' + 14y' + 49y = 0$

Find the general solution of the given higher-order differential equation

18. $y''' + 3y'' + 3y' + y = 0$
19. $3y''' - 19y'' + 36y' - 10y = 0$
20. $y''' - 6y'' + 12y' - 8y = 0$
21. $y''' + 5y'' + 7y' + 3y = 0$
22. $y^{(3)} + y' - 10y = 0$
23. $y^{(4)} + 2y'' + y = 0$
24. $y^{(4)} + y''' + y'' = 0$
25. $y^{(4)} + 4y = 0$
26. $y^{(4)} + 2y''' + 9y'' - 2y' - 10y = 0$
27. $y^{(5)} - 2y^{(4)} + 17y''' = 0$
28. $\left(D^2 + 6D + 13\right)^2 y = 0$
29. $\lambda^3(\lambda - 1)(\lambda - 2)^3\left(\lambda^2 + 9\right) = 0$

Find the solution of the given initial value problem.

19. $y'' - y' - 2y = 0; \quad y(0) = -1, \quad y'(0) = 2$

20. $y'' - 2y' + 17y = 0; \quad y(0) = -2, \quad y'(0) = 3$

21. $y'' + 25y = 0; \quad y(0) = 1, \quad y'(0) = -1$

22. $y'' + 10y' + 25y = 0; \quad y(0) = 2, \quad y'(0) = -1$

23. $y'' - 2y' - 3y = 0; \quad y(0) = 2, \quad y'(0) = -3$

24. $y'' - 4y' + 13y = 0; \quad y(0) = -1, \quad y'(0) = 2$

25. $y'' - 8y' + 17y = 0; \quad y(0) = 4, \quad y'(0) = -1$

26. $y'' - 4y' + 5y = 0; \quad y(0) = 1, \quad y'(0) = 5$

27. The roots of the characteristic equation of a certain differential equation are:

$$3, -5, 0, 0, 0, 0, -5, \ 2 \pm 3i \ \text{ and } \ 2 \pm 3i$$

Write a general solution of this homogeneous differential equation.

28. $y(x) = C_1 e^{2x} + C_2 e^{-2x} + C_3 \cos 2x + C_4 \sin 2x$ is the general solution of a homogeneous equation. What is the equation?

Section 2.4 - Harmonic Motion

Hooke's Law

The restoring force of a spring is proportional to the displacement

$$F = -ky, \quad k > 0 \quad (k : \textit{Spring constant})$$

Newton's Second Law

Force equals mass times acceleration

$$F = ma = m\frac{d^2y}{dt^2}$$

Mathematical model: $$m\frac{d^2y}{dt^2} = -ky$$

$$m\frac{d^2y}{dt^2} + ky = 0$$

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

$$\frac{d^2y}{dt^2} + \omega^2 y = 0; \qquad \omega = \sqrt{\frac{k}{m}}$$

$\frac{\omega}{2\pi}$ is called natural frequency of the system

Damped, Free Vibrations

A resistance force R (e.g. friction) proportional to the velocity $v = y'$ and acting in a direction opposite to the motion

$$R = -cy', \quad c > 0$$

Force equation: $$F = -ky(t) - cy'(t)$$

Mathematical model: $$my'' = -ky - cy$$

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0 \qquad (c,\ m,\ k \textit{ are constants})$$

The equation for the motion of a vibrating spring is given by

$$my'' + \mu y' + ky = F(t)$$

Where the constant coefficients are:

- m mass
- μ damping constant
- k spring constant
- F(t) external force

The differential equation that modeled simple *RLC* circuits is given by

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}I=\frac{dE}{dt}$$

Comparing the 2 systems are almost identical.

Combine the two systems:

$$y''+\frac{\mu}{m}y'+\frac{k}{m}y=\frac{1}{m}F(t)$$

$$\frac{d^2I}{dt^2}+\frac{R}{L}\frac{dI}{dt}+\frac{1}{LC}I=\frac{1}{L}\frac{dE}{dt}$$

If we let: $\begin{cases} c=\frac{\mu}{2m} & c=\frac{R}{2L} \\ \omega_0=\sqrt{k/m} & \omega_0=\sqrt{1/LC} \\ f(t)=\frac{1}{m}F(t) & f(t)=\frac{1}{L}\frac{dE}{dt} \\ x=y & x=I \end{cases}$

$$x''+2cx'+\omega_0^2x=f(t)$$

Where $c\ge 0$ *and* $\omega_0>0$ are constants.

This equation called ***harmonic motion***.

c ***damping*** constant

f ***forcing term***

Example

For a circuit without resistance $(R=0)$ and no source voltage, then the equation simplifies to

$$L\frac{d^2I}{dt^2}+\frac{1}{C}I=0 \qquad \textit{Divide by L}$$

$$\frac{d^2I}{dt^2}+\frac{1}{LC}I=0$$

$$\lambda^2+\frac{1}{LC}=0$$

$$\lambda^2=-\frac{1}{LC}$$

$$\lambda=\pm i\frac{1}{\sqrt{LC}} \qquad \lambda=a\pm ib\Rightarrow a=0;\ \ b=\frac{1}{\sqrt{LC}}$$

The general solution: $y(t)=e^{at}\left(A_1\cos bt+A_2\sin bt\right)$

$$\boxed{I(t)=C_1\cos\left(\frac{t}{\sqrt{LC}}\right)+C_2\sin\left(\frac{t}{\sqrt{LC}}\right)}$$

Simple Harmonic Motion

In the special case when there is no damping $(c=0)$ the motion is called ***simple harmonic motion***.

$$x'' + \omega_0^2 x = 0$$

The characteristic equation is:

$$\lambda^2 + \omega_0^2 = 0$$

The roots are $\lambda^2 = -\omega_0^2 \to \lambda = \pm i\omega_0$

$x(t) = a\cos\omega_0 t + b\sin\omega_0 t$

If we define $T = \dfrac{2\pi}{\omega_0} \Rightarrow T\omega_0 = 2\pi$

Then the periodic of the trigonometry functions implies that $x(t+T) = x(t)$ for all t.

Thus, the solution x is periodic with period T.

ω_0 is called the ***natural frequency***.

Amplitude and Phase Angle

$x(t) = a\cos\omega_0 t + b\sin\omega_0 t$

Consider the point (a,b), we can rewrite this in polar coordinates with a length of A.

$$a = A\cos\phi \quad b = A\sin\phi$$

$$\begin{aligned} x(t) &= a\cos\omega_0 t + b\sin\omega_0 t \\ &= A\cos\phi\cos\omega_0 t + A\sin\phi\sin\omega_0 t \\ &= A\cos\left(\omega_0 t - \phi\right) \end{aligned}$$

Where A ***amplitude*** of the oscillation $\qquad A = \sqrt{a^2 + b^2}$

ϕ ***Phase*** of the oscillation $\qquad \tan\phi = \dfrac{b}{a} \qquad \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

Period: $T = \dfrac{2\pi}{\omega_0}$

Frequency: $\nu = \dfrac{1}{T}$

Time lag of the motion is: $\delta = \dfrac{\varphi}{\omega_0}$

Example

A mass of 4 *kg* is attached to a spring with a spring constant of $k = 169\ kg/s^2$. It is then stretched 10 cm from the spring mass equilibrium and set to oscillating with an initial velocity is 130 *cm/s*. Assuming it oscillates without damping, find the frequency, amplitude, and phase of the vibration.

Solution

$$my'' + \mu y' + ky = F(t)$$

$$4y'' + 169y = 0 \qquad \textit{Divide by 4}$$

$$y'' + 42.25y = 0$$

The natural frequency: $\omega_0 = \sqrt{42.25} = 6.5$

$$y(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$$

$$= C_1 \cos 6.5t + C_2 \sin 6.5t$$

Stretched 10 cm $\rightarrow y(0) = 10\ cm = .1\ m$

$$y(0) = C_1 \cos 6.5(0) + C_2 \sin 6.5(0)$$

$$.1 = C_1$$

Initial velocity is 130 cm/s $\rightarrow y'(0) = 1.3\ m/s$

$$y'(t) = -6.5C_1 \sin 6.5t + 6.5C_2 \cos 6.5t$$

$$y'(0) = -6.5C_1 \sin 6.5(0) + 6.5C_2 \cos 6.5(0)$$

$$1.3 = 6.5C_2$$

$$C_2 = 0.2$$

$$y(t) = 0.1\cos 6.5t + 0.2\sin 6.5t$$

$$A = \sqrt{.1^2 + .2^2} \approx 0.2236\ m$$

$$\phi = \tan^{-1}\frac{b}{a}$$

$$= \tan^{-1}\frac{.2}{.1}$$

$$\approx 1.1071$$

$$\underline{y(t) = 0.2236\cos(6.5t - 1.1071)}$$

Damped Harmonic Motion

In this case, $c > 0$.

$$x'' + 2cx' + \omega_0^2 x = 0$$

The characteristic equation is:

$$\lambda^2 + 2c\lambda + \omega_0^2 = 0$$

The roots are $\lambda = -c \pm \sqrt{c^2 - \omega_0^2}$

There are 3 cases to consider damping and depend on the sign of the discriminant $c^2 - \omega_0^2$

1. $c^2 - \omega_0^2 < 0 \Rightarrow c < \omega_0$. This is the ***underdamped*** case. The roots are distinct complex numbers.

The general solution is

$$x(t) = e^{-ct}\left(C_1 \cos \omega t + C_2 \sin \omega t\right)$$

Where $\omega = \sqrt{\omega_0^2 - c^2}$

2. $c^2 - \omega_0^2 > 0 \Rightarrow c > \omega_0$. This is the ***overdamped*** case. The roots are distinct and real numbers.

The general solution is

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

Where $\sqrt{\omega_0^2 - c^2} < \sqrt{c^2} < c$

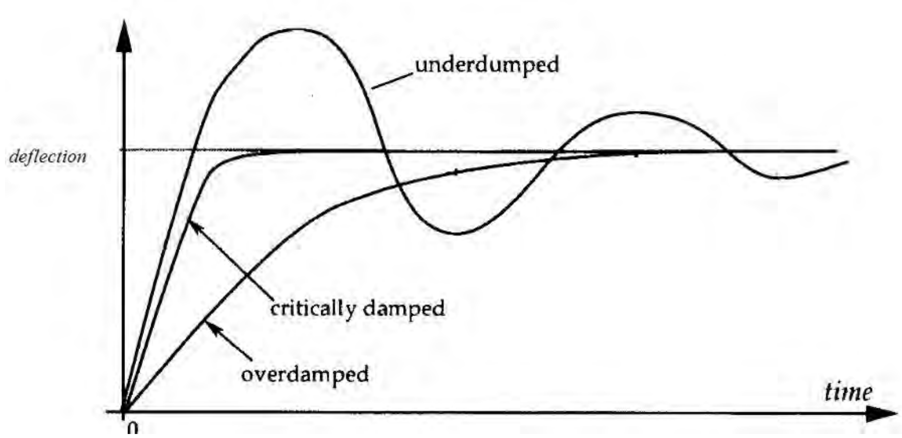
$$\lambda_1 < \lambda_2 < 0$$

3. $c^2 - \omega_0^2 = 0 \Rightarrow c = \omega_0$. This is the ***damped*** case. The root is a double root.

The general solution is

$$x(t) = C_1 e^{-ct} + C_2 t e^{-ct}$$

Where $\lambda = -c$



Example

A mass of 4 kg is attached to a spring with a spring constant of $k = 169\ kg/s^2$ and damping constant $\mu = 12.8\ kg/s$. With initial values of $y(0) = 0.1\ m$ and $y'(0) = 1.3 m/s$. Find the frequency, amplitude, and phase of the vibration.

Solution

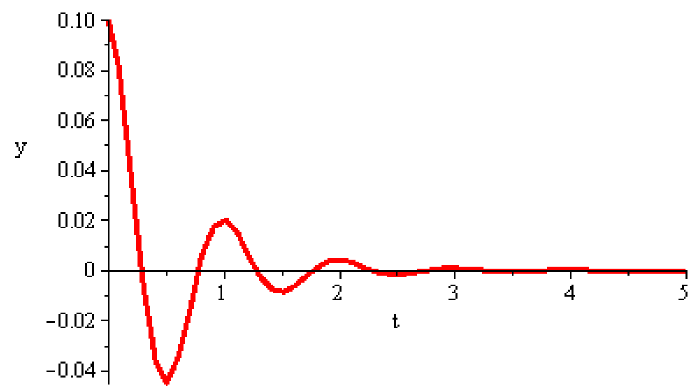
$$my'' + \mu y' + ky = F(t)$$
$$4y'' + 12.8y' + 169y = 0$$
$$y'' + 3.2y' + 42.25y = 0$$
$$\lambda^2 + 3.2\lambda + 42.25 = 0$$
$$\lambda = -1.6 \pm 6.3i$$

The general solution:

$$y(t) = e^{-1.6t}\left(C_1 \cos 6.3t + C_2 \sin 6.3t\right)$$

$$y(0) = e^{-1.6(0)}\left(C_1 \cos 6.3(0) + C_2 \sin 6.3(0)\right)$$
$$0.1 = C_1$$
$$y'(t) = -1.6e^{-1.6t}\left(C_1 \cos 6.3t + C_2 \sin 6.3t\right) + e^{-1.6t}\left(-6.3C_1 \sin 6.3t + 6.3C_2 \cos 6.3t\right)$$
$$y'(0) = -1.6e^{-1.6(0)}\left(C_1 \cos 6.3(0) + C_2 \sin 6.3(0)\right) + e^{-1.6(0)}\left(-6.3C_1 \sin 6.3(0) + 6.3C_2 \cos 6.3(0)\right)$$
$$1.3 = -1.6(0.1 + 0) + (1)\left(-0 + 6.3C_2\right)$$
$$1.3 = -0.16 + 6.3C_2$$
$$6.3C_2 = 1.46$$
$$C_2 \approx 0.2317$$

$$\boxed{y(t) = e^{-1.6t}(0.1\cos 6.3t + 0.2317\sin 6.3t)}$$



OR

$$A = \sqrt{.1^2 + .2317^2} \approx 0.2524\ m$$
$$\phi = \tan^{-1}\frac{b}{a}$$
$$= \tan^{-1}\frac{.2317}{.1}$$
$$\approx 1.1634$$

$$\boxed{y(t) = 0.2524\cos(6.3t - 1.1634)}$$

Example

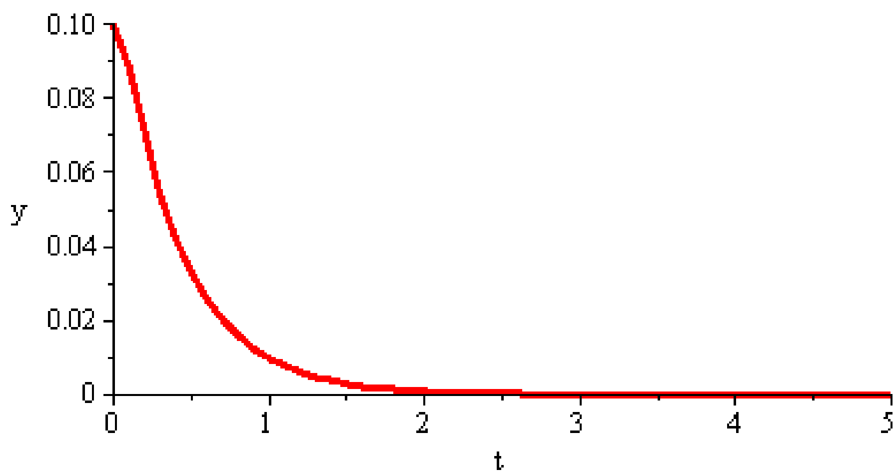
A mass of 4 kg is attached to a spring with a spring constant of $k = 169\ kg/s^2$ and damping constant $\mu = 77.6\ kg/s$. With initial values of $y(0) = 0.1\ m$ and $y'(0) = 1.3 m/s$. Find the general solution.

Solution

$$my'' + \mu y' + ky = F(t)$$
$$4y'' + 77.6y' + 169y = 0$$
$$y'' + 19.4y' + 42.25y = 0$$
$$\lambda^2 + 19.4\lambda + 42.25 = 0$$
$$\lambda_1 = -16.9,\ \lambda_2 = -2.5$$

The general solution:

$$y(t) = C_1 e^{-16.9t} + C_2 e^{-2.5t}$$
$$0.1 = C_1 e^{-16.9(0)} + C_2 e^{-2.5(0)}$$
$$0.1 = C_1 + C_2$$
$$y' = -16.9C_1 e^{-16.9t} - 2.5C_2 e^{-2.5t}$$
$$1.3 = -16.9C_1 e^{-16.9(0)} - 2.5C_2 e^{-2.5(0)}$$
$$1.3 = -16.9C_1 - 2.5C_2$$
$$\left.\begin{array}{r} 0.1 = C_1 + C_2 \\ 1.3 = -16.9C_1 - 2.5C_2 \end{array}\right\} \rightarrow C_1 = -\frac{31}{288},\ C_2 = \frac{299}{1440}$$
$$\boxed{y(t) = -\frac{31}{288} e^{-16.9t} + \frac{299}{1440} e^{-2.5t}}$$



Example

A mass of 4 kg is attached to a spring with a spring constant of $k = 169\ kg/s^2$; with initial values of $y(0) = 0.1\ m$ and $y'(0) = 1.3 m/s$. Find the damping constant μ for which there is critical damping

Solution

Critical damping occurs when $c = \omega_0$

Since $c = \frac{\mu}{2m} = \omega_0$

$$\mu = 2m\omega_0$$

$$= 2m\sqrt{\frac{k}{m}}$$

$$= 2(4)\sqrt{\frac{169}{4}}$$

$$= 52\ \ kg/s$$

$$4y'' + 52y' + 169y = 0$$

$$\lambda^2 + 13\lambda + 42.25 = 0$$

$$y(t) = C_1 e^{-6.5t} + C_2 te^{-6.5t}$$

$$0.1 = C_1$$

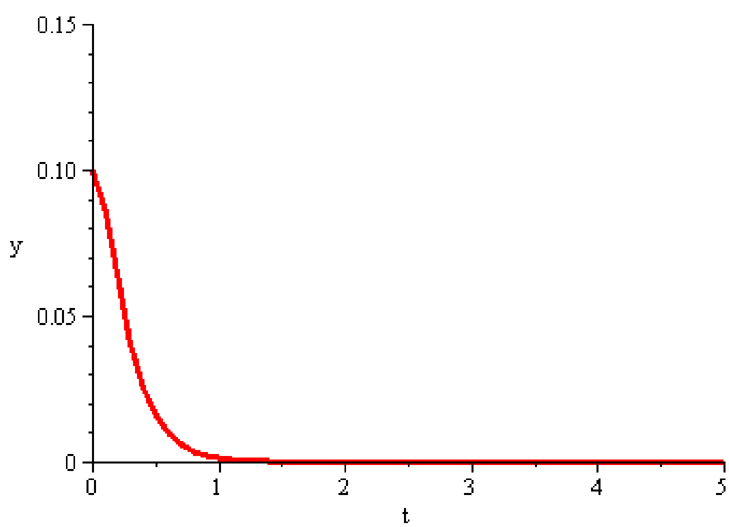
$$y' = -6.5C_1 e^{-6.5t} + C_2 e^{-6.5t} - 6.5C_2 te^{-6.5t}$$

$$1.3 = -6.5C_1 + C_2$$

$$1.3 = -6.5(0.1) + C_2$$

$$C_2 = 1.95$$

$$\boxed{y(t) = 0.1e^{-6.5t} + 1.95te^{-6.5t}}$$



Exercises *Section* 2.4 – Harmonic Motion

(***Exercises*** 1 - 2)

i. Plot the function

ii. Place the solution in the form $y = A\cos\left(\omega_0 t - \phi\right)$ and compare the graph with the plot in (***i***)

1. $y = \cos 2t + \sin 2t$

2. $y = \cos 4t + \sqrt{3}\sin 4t$

3. A 1-*kg* mass, when attached to a large spring, stretches the spring a distance of 4.9 *m*.

a) Calculate the spring constant.

b) The system is placed in a viscous medium that supplies a damping constant $\mu = 3\ kg/s$. The system is allowed to come to rest. Then the mass is displaced 1 *m* in the downward direction and given a sharp tap, imparting an instantaneous velocity of 1 *m/s* in the downward direction. Find the position of the mass as a function of time and plot the solution.

4. The undamped system

$$\frac{2}{5}x'' + kx = 0, \quad x(0) = 2 \quad x'(0) = v_0$$

is observed to have period $\frac{\pi}{2}$ and amplitude 2. Find k and v_0

5. A body with mass *m* = 0.5 *kg* is attached to the end of a spring that is stretched 2 *m* by a force of 100 *N*. It is set in motion with initial position $x_0 = 1m$ and initial velocity $v_0 = -5m/s$. (Note that these initial conditions indicate that he body is displaced to the right and is moving to the left at time *t* = 0.) Find the position function of the body as well as the amplitude, frequency, period of oscillation, and time lag of its motion.

Section 2.5 - Inhomogeneous Equations; the Method of Undetermined Coefficients

The second order ***nonhomogeneous*** equation is given by: $y'' + p(x)y' + q(x)y = f(x) \quad (N)$

The corresponding ***homogeneous*** equation: $y'' + p(x)y' + q(x)y = 0 \quad (H)$

Theorem

Suppose that y_p is a particular solution to the nonhomogeneous (or inhomogeneous) equation $y'' + py' + qy = f$ and that y_1 and y_2 form a fundamental set of solutions to the homogeneous equation $y'' + py' + qy = 0$. Then the general solution to the inhomogeneous equation is given by

$$y = y_p + C_1 y_1 + C_2 y_2$$

C_1 *and* C_2 are arbitrary constants.

Theorem

Let $y = y_1(x)$ *and* $y = y_2(x)$ be ***linearly independent*** $(W(x) \neq 0)$ solutions of the reduced equation (*H*) and let $y_p(x)$ be a ***particular solution*** of (*N*). Then the general solution of (*N*) consists of the general solution of the reduced equation (*H*) ***plus*** a particular solution of (*N*):

$$y(x) = \underbrace{y_p(x)}_{\text{a Particular Solution}} + \underbrace{C_1 y_1(x) + C_2 y_2(x)}_{\text{General Solution}}$$

Forcing Term

If the forcing term *f* has a form that is replicated under differentiation, then look for a solution with the same general form as the forcing term.

Example

Find a particular solution to the equation $y'' - y' - 2y = 2e^{-2t}$

Solution

The forcing term $f(t) = 2e^{-2t} \Rightarrow$ the particular solution $y = ae^{-2t}$

$$y' = -2ae^{-2t}$$

$$y'' = 4ae^{-2t}$$

$$4ae^{-2t} + 2ae^{-2t} - 2ae^{-2t} = 2e^{-2t}$$

$$4ae^{-2t} = 2e^{-2t}$$

$4a = 2$

$a = \frac{1}{2}$

$\boxed{y_p(t) = \frac{1}{2}e^{-2t}}$

Trigonometric Forcing Term

$f(t) = A\cos\omega t + B\sin\omega t$

The general solution: $y(t) = a\cos\omega t + b\sin\omega t$

Example

Find a particular solution to the equation $y'' + 2y' - 3y = 5\sin 3t$

Solution

The particular solution: $y(t) = a\cos 3t + b\sin 3t$

$$y' = -3a\sin 3t + 3b\cos 3t$$

$$y'' = -9a\cos 3t - 9b\sin 3t$$

$$\begin{aligned} y'' + 2y' - 3y &= -9a\cos 3t - 9b\sin 3t + 2(-3a\sin 3t + 3b\cos 3t) - 3(a\cos 3t + b\sin 3t) \\ &= -9a\cos 3t - 9b\sin 3t - 6a\sin 3t + 6b\cos 3t - 3a\cos 3t - 3b\sin 3t \\ &= (-12a + 6b)\cos 3t - (6a + 12b)\sin 3t \\ &= 5\sin 3t \end{aligned}$$

$$\begin{cases} -12a + 6b = 0 \\ -(6a + 12b) = 5 \end{cases} \Rightarrow a = -\frac{1}{6}, \ b = -\frac{1}{3}$$

$\boxed{y_p(t) = -\frac{1}{6}\cos 3t - \frac{1}{3}\sin 3t}$

The Complex Method

Example

Find a particular solution to the equation $y'' + 2y' - 3y = 5\sin 3t$

Solution

$$5e^{3it} = 5\cos 3t + 5i\sin 3t = 5cis3t$$

$$z'' + 2z' - 3z = 5e^{3it}$$

The particular solution: $z(t) = x(t) + i\, y(t)$

$$\begin{aligned} z'' + 2z' - 3z &= (x+iy)'' + 2(x+iy)' - 3(x+iy) \\ &= (x'' + 2x' - 3x) + i(y'' + 2y' - 3y) \\ &= 5\cos 3t + i\ 5\sin 3t \end{aligned}$$

$$x'' + 2x' - 3x = 5\cos 3t$$

$$z(t) = ae^{3it}$$

$$z' = 3iae^{3it}$$

$$z'' = 9i^2ae^{3it} = -9ae^{3it}$$

$$\begin{aligned} z'' + 2z' - 3z &= -9ae^{3it} + 2(3i)ae^{3it} - 3ae^{3it} \\ &= -12ae^{3it} + 6iae^{3it} \\ &= -6(2-i)ae^{3it} \\ &= 5e^{3it} \end{aligned}$$

$$-6(2-i)a = 5$$

$$\begin{aligned} a &= -\frac{5}{6(2-i)}\frac{2+i}{2+i} \\ &= -\frac{5(2+i)}{6(4+1)} \\ &= -\frac{2+i}{6} \end{aligned}$$

$$\begin{aligned} z(t) &= -\tfrac{1}{6}(2+i)e^{3it} \\ &= -\tfrac{1}{6}(2+i)(\cos 3t + i\sin 3t) \\ &= -\tfrac{1}{6}\left[(2\cos 3t - \sin 3t) + i(\cos 3t + 2\sin 3t)\right] \end{aligned}$$

$$\boxed{y_p(t) = -\tfrac{1}{6}(\cos 3t + 2\sin 3t)}$$

Polynomial Forcing Term

$$f(t) = a_0 t^n + a_1 t^{n-1} + \ldots + a_{n-1} t + a_n$$

Example

Find a particular solution to the equation $y'' + 2y' - 3y = 3t + 4$

Solution

The right-hand side is a polynomial of degree 1.

The particular solution: $y(t) = at + b$

$$y' = a$$
$$y'' = 0$$

$$\begin{aligned} y'' + 2y' - 3y &= 0 + 2a - 3(at + b) \\ &= 2a - 3b - 3at \\ &= 3t + 4 \end{aligned}$$

$$\rightarrow \begin{cases} -3a = 3 \\ 2a - 3b = 4 \end{cases} \quad \Rightarrow a = -1; \quad b = -2$$

$$\boxed{y_p(t) = -t - 2}$$

Exceptional Cases

Example

Find a particular solution to the equation $y'' - y' - 2y = 3e^{-t}$

Solution

The particular solution $y = ae^{-t}$

$$y'' - y' - 2y = ae^{-t} + ae^{-t} - 2ae^{-t}$$
$$= 0$$

The particular solution $y = ate^{-t}$ or $y = at^2e^{-t}$

$$y' = ae^{-t} - ate^{-t} = ae^{-t}(1-t)$$
$$y'' = -ae^{-t} - ae^{-t} + ate^{-t}$$
$$= ate^{-t} - 2ae^{-t}$$

$$y'' - y' - 2y = ate^{-t} - 2ae^{-t} - ae^{-t} + ate^{-t} - 2ate^{-t}$$
$$= -3ae^{-t}$$

$$-3ae^{-t} = 3e^{-t}$$
$$a = -1$$

The particular solution $\boxed{y_p = -te^{-t}}$

Summary

$f(t)$	y_p
Any Constant	A
$at + b$	$At + B$
$at^2 + c$	$At^2 + Bt + C$
$at^3 + \cdots + b$	$At^3 + Bt^2 + Ct + E$
$\sin at \quad or \quad \cos at$	$A\cos at + B\sin at$
e^{at}	Ae^{at}
$(at + b)e^{at}$	$(At + B)e^{at}$
t^2e^{at}	$\left(At^2 + Bt + C\right)e^{at}$
$e^{at}\sin bt$	$e^{at}(A\cos bt + B\sin bt)$
$t^2\sin bt$	$\left(At^2 + Bt + C\right)\cos bt + \left(Et^2 + Ft + G\right)\sin bt$
$te^{at}\cos bt$	$(At + B)\cos bt + (Ct + E)\sin bt$

Exercises *Section* 2.5 - Inhomogeneous Equations; the Method of Undetermined Coefficients

1. Show that the 3 solutions $y_1 = x, \quad y_2 = x\ln x, \quad y_3 = x^2$ of the 3rd order equation $x^3y''' - x^2y'' + 2xy' - 2y = 0$ are linearly independent on an open interval $x > 0$. Then find a particular solution that satisfies the initial conditions $y(1) = 3, \quad y'(1) = 2, \quad y''(1) = 1$

Find the particular solution for the given differential equation

2. $y'' + 3y' + 2y = 4e^{-3t}$

3. $y'' + 6y' + 8y = -3e^{-t}$

4. $y'' + 2y' + 5y = 12e^{-t}$

5. $y'' + 3y' - 18y = 18e^{2t}$

6. $y'' + 4y = \cos 3t$

7. $y'' + 7y' + 6y = 3\sin 2t$

8. $y'' + 5y' + 4y = 2 + 3t$

9. $y'' + 6y' + 8y = 2t - 3$

10. $y'' + 3y' + 4y = t^3$

11. $y'' + 2y' + 2y = 2 + \cos 2t$

12. $y'' - y = t - e^{-t}$

13. $y'' - 2y' + y = 10e^{-2t}\cos t$

14. $y''' - 4y'' + 4y' = 5t^2 - 6t + 4t^2e^t + 3e^{5t}$

Use the ***complex method*** to find the particular solution for

15. $y'' + 4y' + 3y = \cos 2t + 3\sin 2t$

16. $y'' + 4y = \cos 3t$

Find the general solution for the given differential equation

17. $y'' + 3y' + 2y = 4x^2$

18. $y'' - 3y' = 8e^{3x} + 4\sin x$

19. $y'' + 8y = 5x + 2e^{-x}$

20. $y'' + y = x\cos x - \cos x$

21. $y'' + 25y = 20\sin 5x$

22. $y''' + 8y'' = -6x^2 + 9x + 2$

23. $y''' - 3y'' + 3y' - y = e^x - x + 16$

Find a solution to the homogeneous equation; then find a particular solution to form a general solution. Then find the solution satisfying the given initial conditions

24. $y'' - 4y' - 5y = 4e^{-2t}; \quad y(0) = 0, \quad y'(0) = -1$

25. $y'' + 2y' + 2y = 2\cos 2t; \quad y(0) = -2, \quad y'(0) = 0$

26. $y'' - 2y' + y = t^3; \quad y(0) = 1, \quad y'(0) = 0$

27. $y'' + 4y' + 4y = 4 - t; \quad y(0) = -1, \quad y'(0) = 0$

28. $y'' - 64y = 16; \quad y(0) = 1, \quad y'(0) = 0$

29. $y'' - 5y' = t - 2 \quad y(0) = 0, \quad y'(0) = 2$

30. $y'' + y = 8\cos 2t - 4\sin t \quad y\left(\frac{\pi}{2}\right) = -1, \quad y'\left(\frac{\pi}{2}\right) = 0$

Section 2.6 - Variation of Parameters

In this section, we will introduce a technique called ***variation of parameters***.

The inhomogeneous equation is given by: $y'' + p(t)y' + q(t)y = g(t)$

A fundamental set of solutions y_1 and y_2 to associated homogeneous equation $y'' + py' + qy = 0$.
Then the general solution to the inhomogeneous equation is given by

$$y_k = C_1 y_1 + C_2 y_2$$

C_1 *and* C_2 are arbitrary constants.

General Case

A differential system can be written in a form:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

y_1 and y_2 fundamental set of solution to the homogenous equation, they are linearly independent Then the determinant will be recognized as the Wronskian:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

Which we can obtain:

$$v_1' = -\frac{y_2 g(t)}{y_1 y_2' - y_1' y_2} = -\frac{y_2}{W} g(t) \qquad \Rightarrow v_1(t) = -\int \frac{y_2 g(t)}{W} dt$$

$$v_2' = \frac{y_1 g(t)}{y_1 y_2' - y_1' y_2} = \frac{y_1}{W} g(t) \qquad \Rightarrow v_2(t) = \int \frac{y_1 g(t)}{W} dt$$

$$\boxed{y_p = v_1 y_1 + v_2 y_2}$$

Example

$\left\{y_1(x)=x^4,\ y_2(x)=x^2\right\}$ is a fundamental set of solutions of $y''-\frac{5}{x}y'+\frac{8}{x^2}y=4x^3$.

Find a particular solution of the equation?

Solution

$$W=\begin{vmatrix} x^4 & x^2 \\ 4x^3 & 2x \end{vmatrix}=-2x^5\neq 0$$

$$v_1(x)=-\int\frac{x^2\left(4x^3\right)}{-2x^5}dx=\int 2dx\ \underline{=2x|} \qquad v_1(x)=-\int\frac{y_2g(x)}{W}dx$$

$$v_2(x)=\int\frac{x^4\left(4x^3\right)}{-2x^5}dx=-2\int x^2dx\ \underline{=-\frac{2}{3}x^3|} \qquad v_2(x)=\int\frac{y_1g(x)}{W}dx$$

The particular solution:

$$\begin{aligned} y_p &= v_1y_1+v_2y_2 \\ &= (2x)\left(x^4\right)-\frac{2}{3}x^3\left(x^2\right) \\ &= \frac{4}{3}x^5 \end{aligned}$$

The general solution: $\quad y(x)=C_1x^4+C_2x^2+\frac{4}{3}x^5$

Example

$\left\{y_1(x)=e^{2x},\ y_2(x)=xe^{2x}\right\}$ is a fundamental set of solutions of $y''-4y'+4y=\frac{e^{2x}}{x}$.

Find a particular solution of the equation?

Solution

$$W=\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}+2xe^{2x} \end{vmatrix}=e^{4x}+2xe^{4x}-2xe^{4x}\ \underline{=e^{4x}\neq 0|}$$

$$v_1(x)=-\int\frac{xe^{2x}}{e^{4x}}\frac{e^{2x}}{x}dx=-\int dx\ \underline{=-x|} \qquad v_1(x)=-\int\frac{y_2g(x)}{W}dx$$

$$v_2(x)=\int\frac{e^{2x}}{e^{4x}}\frac{e^{2x}}{x}dx=\int\frac{1}{x}dx\ \underline{=\ln|x||} \qquad v_2(x)=\int\frac{y_1g(x)}{W}dx$$

The particular solution:

$$y_p=v_1y_1+v_2y_2$$

$$= -xe^{2x} + \ln|x|\left(xe^{2x}\right)$$

The general solution:

$$y(x) = C_1 e^{2x} + C_2 xe^{2x} - xe^{2x} + xe^{2x}\ln|x|$$

$$= C_1 e^{2x} + \left(C_2 - 1\right)xe^{2x} + xe^{2x}\ln|x|$$

$$\underline{= C_1 e^{2x} + C_3 xe^{2x} + xe^{2x}\ln|x|}$$

Example

Find the particular solution for $y'' + y = \tan t$

Solution

The homogeneous equation for the differential equation $\lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm i$

Therefore; $y_1 = \cos t$ *and* $y_2 = \sin t$

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1 \neq 0$$

The system has a solution

$$v_1{}' = \sin t \ \tan t = \frac{\sin^2 t}{\cos t}$$

$$v_1 = \int \frac{\sin^2 t}{\cos t}dt$$

$$\underline{= -\ln|\sec t + \tan t| + \sin t}$$

$$v_2{}' = \cos t \ \tan t = \sin t$$

$$v_2 = \int \sin t \ dt$$

$$\underline{= -\cos t}$$

$$y_p = v_1 \cos t + v_2 \sin t$$

$$= \left(-\ln|\sec t + \tan t| + \sin t\right)\cos t + \left(-\cos t\right)\sin t$$

$$= -\cos t\ln|\sec t + \tan t| + \sin t\cos t - \cos t\sin t$$

$$\underline{= -(\cos t)\ \ln|\sec t + \tan t|}$$

Exercises *Section* 2.6 - Variation of Parameters

1. $\left\{y_1(x)=e^{2x},\, y_2(x)=e^{-3x}\right\}$ is a fundamental set of solutions of $y''+y'-6y=3e^{2x}$.

Find a particular solution of the equation?

Find a particular solution to the given second-order differential equation (Use ***variation of parameters***):

2. $y''-y=t+3$

3. $y''-2y'+y=e^{t}$

4. $x''-4x'+4x=e^{2t}$

5. $x''+x=\tan^2 t$

6. $y''+25y=-2\tan(5x)$

7. $y''-6y'+9y=5e^{3x}$

8. $y''+4y=2\cos 2x$

9. $y''-5y'+6y=4e^{2x}+3$

10. Verify that $y_1(t)=t$ and $y_2(t)=t^{-3}$ are solution to the homogenous equation

$$t^2y''(t)+3ty'(t)-3y(t)=0$$

Use variation of parameters to find the general solution to

$$t^2y''(t)+3ty'(t)-3y(t)=\frac{1}{t}$$

Find the general solution to the given differential equation.

11. $y''-4y'+4y=(x+1)e^{2x}$

12. $y''+9y=\csc 3x$

13. $y''-y=\frac{1}{x}$

14. $y''+y=\sin x$

15. $y''+y=\cos^2 x$

16. $y''-y=\cosh x$

17. $y''-4y=\frac{e^{x}}{x}$

18. $y''+3y'+2y=\sin e^{x}$

19. $y''-2y'+y=\frac{e^{x}}{1+x^2}$

20. $y''+2y'+y=e^{-x}\ln x$

Section 2.7 - Forced Harmonic Motion

A sinusoidal forcing is giving by the model:

$$x'' + 2cx' + \omega_0^2 x = A\cos\omega t$$

A: Amplitude if the driving force (constant)
ω: driving frequency.
c: damping constant.
ω_0 : natural frequency.

Forced undamped harmonic motion

The undamped equation has $c = 0$ or

$$x'' + \omega_0^2 x = A\cos\omega t$$

The homogeneous equation is: $x'' + \omega_0^2 x = 0$

With general solution: $x_h = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$

Case 1 $\omega \neq \omega_0$

The particular solution is given by the form: $x_p = a\cos\omega t + b\sin\omega t$

$$x_p' = -a\omega\sin\omega t + b\omega\cos\omega t$$

$$x_p'' = -a\omega^2\cos\omega t - b\omega^2\sin\omega t$$

$$\begin{aligned}
x_p'' + \omega_0^2 x_p &= -a\omega^2\cos\omega t - b\omega^2\sin\omega t + \omega_0^2\left(a\cos\omega t + b\sin\omega t\right) \\
&= -a\omega^2\cos\omega t - b\omega^2\sin\omega t + \omega_0^2 a\cos\omega t + \omega_0^2 b\sin\omega t \\
&= a\left(\omega_0^2 - \omega^2\right)\cos\omega t + b\left(\omega_0^2 - \omega^2\right)\sin\omega t \\
&= A\cos\omega t
\end{aligned}$$

$$A = a\left(\omega_0^2 - \omega^2\right) \qquad b\left(\omega_0^2 - \omega^2\right) = 0$$

$$a = \frac{A}{\left(\omega_0^2 - \omega^2\right)} \qquad b = 0 \qquad \textit{since } \omega_0^2 - \omega^2 \neq 0$$

$$x_p = \frac{A}{\left(\omega_0^2 - \omega^2\right)}\cos\omega t$$

$$x(t) = x_h(t) + x_p(t)$$
$$= C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{A}{\left(\omega_0^2 - \omega^2\right)} \cos \omega t$$

When the *motion starts at equilibrium*; this means $x(0) = x'(0) = 0$

$$C_1 + \frac{A}{\left(\omega_0^2 - \omega^2\right)} = 0 \quad \Rightarrow \quad C_1 = -\frac{A}{\left(\omega_0^2 - \omega^2\right)}$$

$$x' = -C_1 \omega_0 \sin \omega_0 t + C_2 \omega_0 \cos \omega_0 t - \frac{A}{\left(\omega_0^2 - \omega^2\right)} \omega_0 \sin \omega t$$

$$x'(0) = C_2 \omega_0 = 0 \quad \Rightarrow C_2 = 0$$

$$x(t) = -\frac{A}{\left(\omega_0^2 - \omega^2\right)} \cos \omega_0 t + \frac{A}{\left(\omega_0^2 - \omega^2\right)} \cos \omega t$$

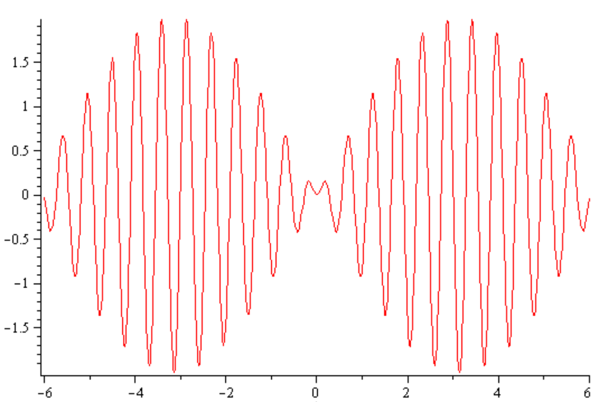
$$x(t) = \frac{A}{\left(\omega_0^2 - \omega^2\right)} \left(\cos \omega t - \cos \omega_0 t\right)$$

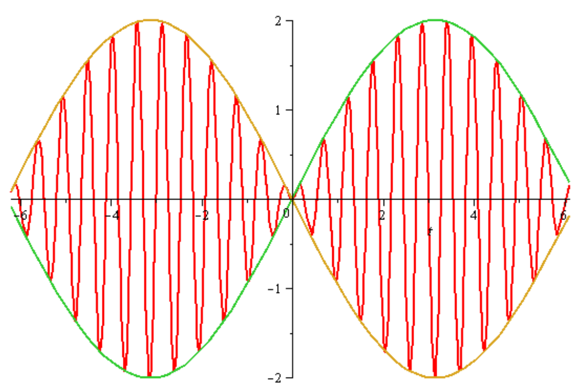
Example

Suppose $A = 23,\ \omega_0 = 11,\ \omega = 12$ with these values of the parameters the solution becomes

Solution

$$x(t) = \frac{A}{\left(\omega_0^2 - \omega^2\right)} \left(\cos \omega t - \cos \omega_0 t\right)$$
$$= \frac{23}{\left(11^2 - 12^2\right)} \left(\cos 12t - \cos 11t\right)$$
$$= -\left(\cos 12t - \cos 11t\right)$$
$$= \cos 11t - \cos 12t$$





Mean frequency: $\bar{\omega} = \dfrac{\omega_0 + \omega}{2}$ ***Half difference***: $\delta = \dfrac{\omega_0 - \omega}{2}$

Case 2 $\omega = \omega_0$

The particular solution is given by the form: $x_p = t\left(a\cos\omega_0 t + b\sin\omega_0 t\right)$

$$x'_p = a\cos\omega_0 t + b\sin\omega_0 t - at\omega_0 \sin\omega_0 t + bt\omega_0 \cos\omega_0 t$$

$$x''_p = -a\omega_0 \sin\omega_0 t + b\omega_0 \cos\omega_0 t - a\omega_0 \sin\omega_0 t + b\omega_0 \cos\omega_0 t - at\omega_0^2 \cos\omega_0 t - bt\omega_0^2 \sin\omega_0 t$$

$$= -2\omega_0\left(a\sin\omega_0 t - b\cos\omega_0 t\right) - t\omega_0^2\left(a\cos\omega_0 t + b\sin\omega_0 t\right)$$

$$x''_p + \omega_0^2 x_p = -2\omega_0\left(a\sin\omega_0 t - b\cos\omega_0 t\right) - t\omega_0^2\left(a\cos\omega_0 t + b\sin\omega_0 t\right) + t\omega_0^2\left(a\cos\omega_0 t + b\sin\omega_0 t\right)$$

$$= -2\omega_0\left(a\sin\omega_0 t - b\cos\omega_0 t\right)$$

$$= A\cos\omega t$$

$A = 2b\omega_0$ $\quad a = 0$

$$b = \frac{A}{2\omega_0}$$

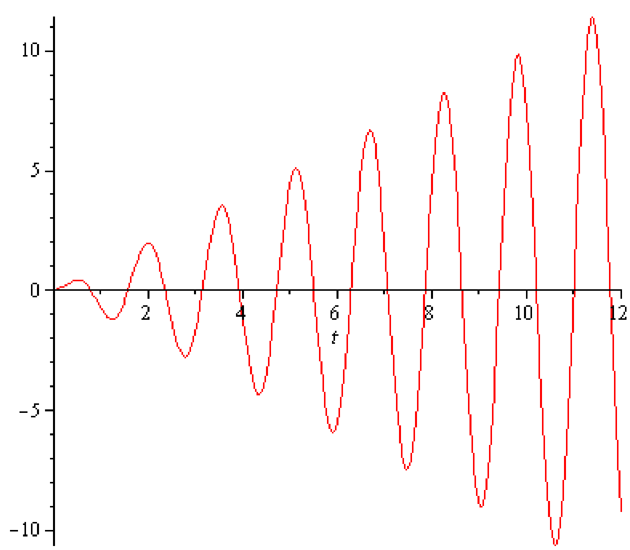
$$x_p = \frac{A}{2\omega_0} t\sin\omega_0 t$$

$$x(t) = x_h(t) + x_p(t)$$

$$= C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{A}{2\omega_0} t\sin\omega_0 t$$

$x(0) = C_1 = 0,\ x'(0) = C_2 = 0,\ A = 8,\ w_0 = 4$

$x(t) = t\sin 4t$



Forced Damped Harmonic Motion

Let's add damping to the system

$$x'' + 2cx' + \omega_0^2 x = A\cos\omega t$$

The homogeneous equation is: $x'' + 2cx' + \omega_0^2 x = 0$

$$\lambda^2 + 2c\lambda + \omega_0^2 = 0$$

$$\lambda_{1,2} = -c \pm \sqrt{c^2 - \omega_0^2}$$

Underdamped Case: $c < \omega_0$

$$x_h = e^{-ct}\left(C_1 \cos\eta t + C_2 \sin\eta t\right)$$

Where $\eta = \sqrt{\omega_0^2 - c^2}$

To determine the inhomogeneous equation, it is better to use complex method.

$$z'' + 2cz' + \omega_0^2 z = Ae^{i\omega t}$$

However, $x(t) = \mathrm{Re}(z)$

The particular solution: $z(t) = ae^{i\omega t}$

$$z'' + 2cz' + \omega_0^2 z = (i\omega)^2 ae^{i\omega t} + 2c(i\omega)ae^{i\omega t} + \omega_0^2 ae^{i\omega t}$$

$$= \left((i\omega)^2 + 2c(i\omega) + \omega_0^2\right)ae^{i\omega t}$$

$$P(i\omega) = (i\omega)^2 + 2c(i\omega) + \omega_0^2$$

$$P(i\omega)ae^{i\omega t} = Ae^{i\omega t}$$

$$\Rightarrow a = \frac{A}{P(i\omega)}$$

$$z(t) = \frac{A}{P(i\omega)}e^{i\omega t}$$

$$= H(i\omega)e^{i\omega t}$$

$H(i\omega)$ is called the transfer function

$$P(i\omega) = -\omega^2 + 2ic\omega + \omega_0^2$$

$$= \omega_0^2 - \omega^2 + 2ic\omega$$

$$P(i\omega) = Re^{i\phi}$$

$$= R\left(\cos\phi + i\sin\phi\right)$$

Polar Coordinates:

$$R\cos\phi = \omega_0^2 - \omega^2 \qquad\qquad R\sin\phi = 2c\omega$$

$$R = \sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4c^2\omega^2}$$

$$\cos\phi = \frac{\omega_0^2 - \omega^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4c^2\omega^2}} \qquad\qquad \sin\phi = \frac{2c\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4c^2\omega^2}}$$

$$\cot\phi = \frac{\omega_0^2 - \omega^2}{2c\omega} \quad\Rightarrow\quad \phi(\omega) = arc\cot\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right) \qquad 0 < \phi < \pi$$

$$H(i\omega) = \frac{1}{P(i\omega)} = \frac{1}{R}e^{-i\phi}$$

We will define the ***gain*** G by:

$$G(\omega) = \frac{1}{R} = \frac{1}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4c^2\omega^2}} \qquad\qquad H(i\omega) = G(\omega)e^{-i\phi}$$

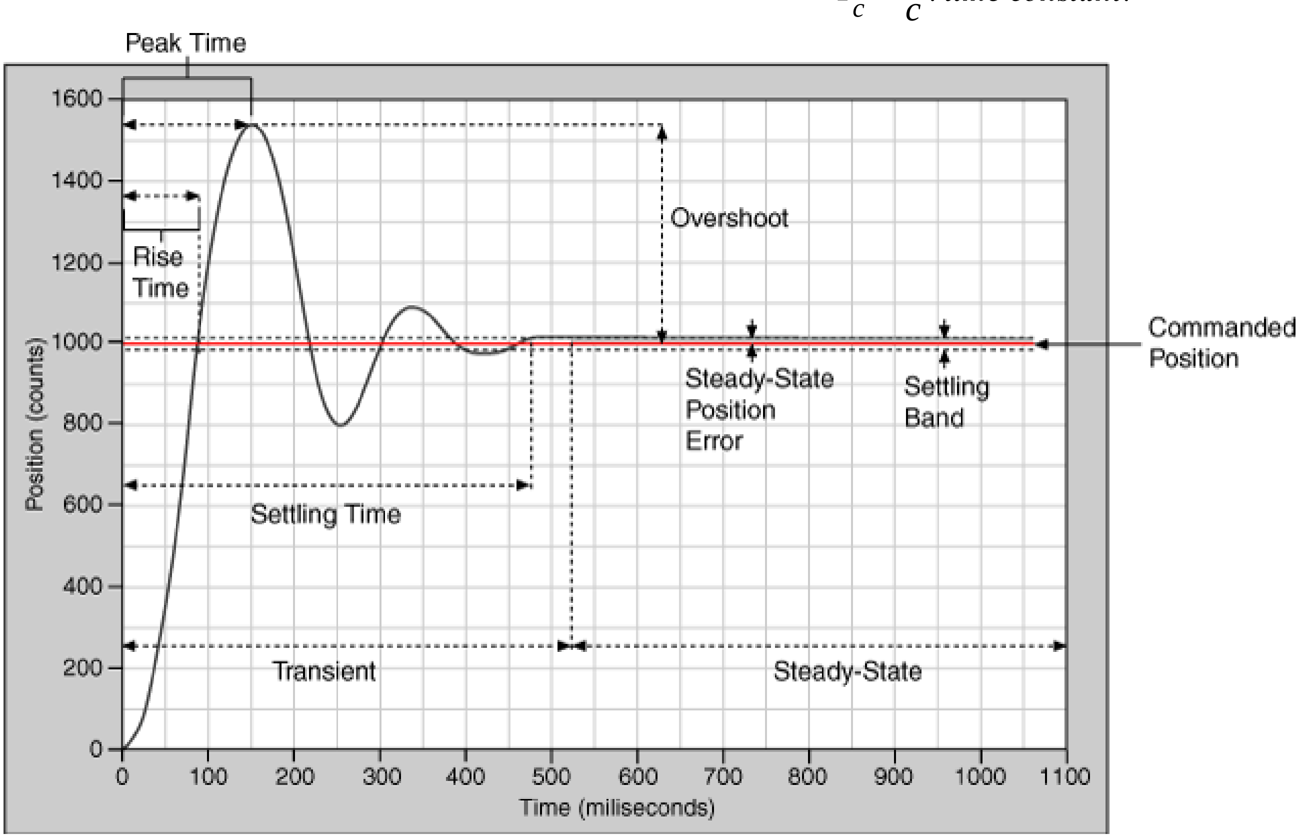
The solution: $\quad z(t) = H(i\omega)Ae^{i\omega t} = G(\omega)Ae^{i(\omega t - \phi)}$

$\quad x_p(t) = \operatorname{Re} z(t) == G(\omega)A\cos(\omega t - \phi)$

$$\boxed{x = e^{-ct}\left(C_1\cos\eta t + C_2\sin\eta t\right) + G(\omega)A\cos(\omega t - \phi)}$$

e^{-ct} : ***transient*** term.

$T_c = \frac{1}{c}$: ***time constant***.



Exercises *Section* 2.7- Forced Harmonic Motion

1. A 1-*kg* mass is attached to a spring $k = 4kg / s^2$ and the system is allowed to come to rest. The spring-mass system is attached to a machine that supplies external driving force $f(t) = 4\cos\omega t$ *Newtons*. The system is started from equilibrium; the mass is having no initial displacement or velocity. Ignore any damping forces.

a) Find the position of the mass as a function of time

b) Place your answer in the form $s(t) = A\sin\delta t\sin\overline{\omega}t$. Select an ω near the natural frequency of the system to demonstrate the "beating" of the system. Sketch a plot shows the "beats:" and include the envelope of the beating motion in your plot.

Find a particular solution to the differential equation using undetermined coefficients. Find and plot the solution of the initial value problem. Superimpose the plots of the transient response and the steady state solution.

2. $x'' + 7x' + 10x = 3\cos 3t \quad x(0) = -1, \ \ x'(0) = 0$

3. $x'' + 4x' + 5x = 3\sin t \quad x(0) = 0, \ \ x'(0) = -3$

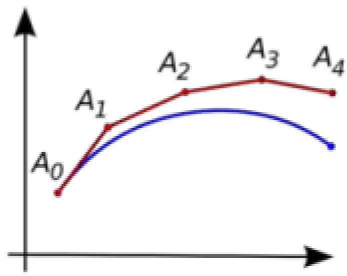
4. Find a particular solution of $y'' - 2y' + 5y = 2\cos 3x - 4\sin 3x + e^{2x}$ given the set

$y_p = A\cos 3x + B\sin 3x + Ce^{2x}$ where *A, B, C* are to be determined

Section 2.8 – Euler's & Runge-Kutta Methods

Euler's method named after ***Leonhard Euler*** is an example of a ***fixed-step*** solver.

Euler's method is a first-order numerical procedure for solving ordinary differential equations (***ODE***s) with a given initial value. It is the most basic kind of explicit method for numerical integration of ordinary differential equations.



$y' = f(x, y) \qquad y(x_0) = y_0$

The setting size: $h = \dfrac{b-a}{k} > 0$; $\quad k \in \mathbb{N}$

$$\begin{aligned}
\text{Then,} \qquad & x_0 = a \\
& x_1 = x_0 + h = a + h \\
& x_k = x_{k-1} + h = a + kh \\
\textit{Last point} \qquad & x_k = a + kh = b
\end{aligned}$$

By the definition of the derivative:

$$y'\left(x_k\right) \approx \frac{y\left(x_{k+1}\right) - y\left(x_k\right)}{h}$$

$$y'\left(x_k\right) \approx \frac{y_{k+1} - y_k}{h} = f\left(x_k, y_k\right) : \textit{slope}$$

The tangent line at the point $\left(x_0, y(x_0)\right)$ is:

$$y_{k+1} = y_k + h.f\left(x_k, y_k\right)$$

$$y_{k+1} = y_k + \Delta x_{step} \cdot f\left(x_k, y_k\right)$$

This method is known as *Euler's Method* with step size h.

Example

Compute the first four step in the Euler's method approximation to the solution of $y' = y - x$ with $y(1) = 1$, using the step size $h = 0.1$. Compare the result with the actual solution to the initial value problem.

Solution

$y(1) = 1 \Rightarrow x_0 = 1 \;\; and \;\; y_0 = 1$

The *first* step:

$$\begin{aligned} y_1 &= y_0 + h\left(y_0 - x_0\right) \\ &= 1 + 0.1(1-1) \\ &= 1 \end{aligned}$$

$$x_1 = x_0 + h = 1 + 0.1 = 1.1$$

The *second* step:

$$\begin{aligned} y_2 &= y_1 + h\left(y_1 - x_1\right) \\ &= 1 + 0.1(1-1.1) \\ &= 0.99 \end{aligned}$$

$$x_2 = x_1 + h = 1.1 + 0.1 = 1.2$$

The *third* step:

$$\begin{aligned} y_3 &= y_2 + h\left(y_2 - x_2\right) \\ &= 0.99 + 0.1(0.99 - 1.2) \\ &= 0.969 \end{aligned}$$

$$x_3 = x_2 + h = 1.2 + 0.1 = 1.3$$

The *fourth* step:

$$\begin{aligned} y_4 &= y_3 + h\left(y_3 - x_3\right) \\ &= 0.969 + 0.1(0.969 - 1.3) \\ &= 0.9359 \end{aligned}$$

$$x_3 = x_2 + h = 1.3 + 0.1 = 1.4$$

The exact solution to $y' = y - x$ is $y(x) = 1 + x - e^{x-1}$

x_k	y_k : Euler's	y_k - exact	*Error*
1.0	1.0	1.0	0
1.1	1.0	0.9948	-0.0052
1.2	0.990	0.9786	-0.0114
1.3	0.969	0.9501	-0.0189
1.4	0.9359	0.9082	-0.0277

Runge-Kutta Methods

Like Euler's method, the Runge-Kutta methods are fixed-step solvers.

The second-Order Runge-Kutta Method

The second-Order Runge-Kutta method is also known as the improved Euler's method.

Starting from the initial value point $\left(x_0, y_0\right)$, we compute two slopes:

$$s_1 = f\left(t_0, y_0\right)$$

$$s_2 = f\left(t_0 + h, y_0 + hs_1\right)$$

$$y_1 = y_0 + h\frac{s_1 + s_2}{2}$$

But an analysis using Taylor's theorem reveals that there is an improvement in the estimate for the truncation error.
For the second-Order Runge-Kutta method, we have

$$\left|y(t_1) - y_1\right| \le Mh^3$$

The constant M depends on the function $f(t, y)$.
The second-Order Runge-Kutta method is controlled by the cube of the step size instead of the square.

Input t_0 *and* y_0
For k = 1 to N

$$s_1 = f\left(t_{k-1}, y_{k-1}\right)$$

$$s_2 = f\left(t_{k-1} + h, y_{k-1} + hs_1\right)$$

$$y_k = y_{k-1} + h\frac{s_1 + s_2}{2}$$

$$t_k = t_{k-1} + h$$

Example

Compute the first four step in the second-Order Runge-Kutta method approximation to the solution of $y' = y - t$ with $y(1) = 1$, using the step size $h = 0.1$. Compare the result with the actual solution to the initial value problem.

Solution

$\Rightarrow t_0 = 1$ *and* $y_0 = 1$

The *first* step:

$$\begin{aligned} s_1 &= f\left(t_0, y_0\right) \\ &= y_0 - t_0 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} s_2 &= f\left(t_0 + h, y_0 + hs_1\right) \\ &= \left(y_0 + hs_1\right) - \left(t_0 + h\right) \\ &= (1 + .1(0)) - (1 + .1) \\ &= 1 - 1.1 \\ &= -0.1 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + h\frac{s_1 + s_2}{2} \\ &= 1 + 0.1\left(\frac{0 - 0.1}{2}\right) \\ &= 0.995 \end{aligned}$$

$$\begin{aligned} t_1 &= t_0 + h \\ &= 1 + 0.1 \\ &= 1.1 \end{aligned}$$

The *second* step:

$$s_1 = y_1 - t_1 = 0.995 - 1.1 = -0.105$$

$$s_2 = \left(y_1 + hs_1\right) - \left(t_1 + h\right) = (0.995 + .1(-0.105)) - (1.1 + .1) = -.2155$$

$$y_2 = y_1 + h\frac{s_1 + s_2}{2} = .995 + 0.1\left(\frac{-.105 - .2155}{2}\right) = .978975$$

$$t_2 = t_1 + h = 1.1 + .1 = 1.2$$

The *third* step:

$$s_1 = y_2 - t_2 = 0.978975 - 1.2 = -0.221025$$

$$s_2 = \left(y_2 + hs_1\right) - \left(t_2 + h\right) = \left(0.978975 + .1(-0.221025)\right) - \left(1.2 + .1\right) = -.3431275$$

$$y_3 = y_2 + h\frac{s_1 + s_2}{2} = .978975 + 0.1\left(\frac{-.221025 - .3431275}{2}\right) = 0.9507673$$

$$t_3 = t_2 + h = 1.3$$

The *fourth* step:

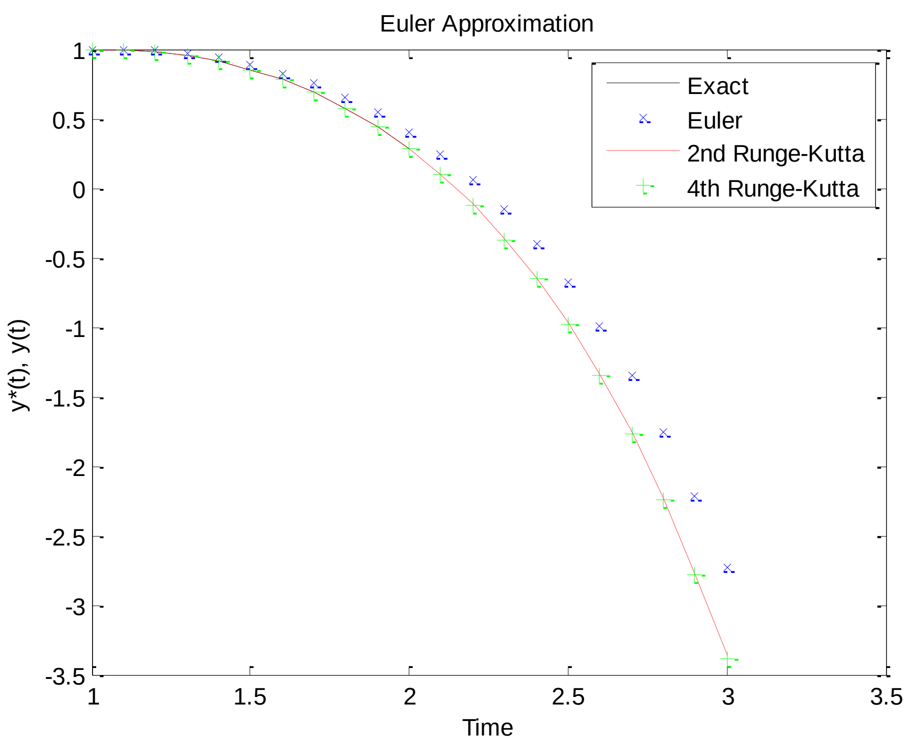
$$s_1 = y_3 - t_3 = 0.9507673 - 1.3 = -0.3492327$$

$$s_2 = \left(y_3 + hs_1\right) - \left(t_3 + h\right) = \left(0.9507673 + .1(-0.3492327)\right) - \left(1.3 + .1\right) = -.48415597$$

$$y_4 = y_3 + h\frac{s_1 + s_2}{2} = .9507673 + 0.1\left(\frac{-.3492327 - .48415597}{2}\right) = 0.9090979$$

$$t_4 = t_3 + h = 1.4$$

t_k	y_k : Runge-Kutta	y_k - Exact	***Runge-Kutta Error***	***Euler's Error***
1.0	1.0	1.0	**0**	**0**
1.1	0.9950000	0.994829081	**–0.000170918**	**–0.0052**
1.2	0.9789750	0.978597241	**–0.000377758**	**–0.0114**
1.3	0.9507673	0.950141192	**–0.000626182**	**–0.0189**
1.4	0.9090979	0.908175302	**–0.000922647**	**–0.0277**



Fourth-Order **Runge-Kutta Method**

This method is the most commonly used solution algorithm. For most equations and systems it is suitably fast and accurate.

Starting from the initial value point $\left(t_0\ ,y_0\right)$, we compute two slopes:

$$s_1 = f\left(t_0, y_0\right)$$

$$s_2 = f\left(t_0 + \frac{h}{2},\ y_0 + \frac{h}{2}s_1\right)$$

$$s_3 = f\left(t_0 + \frac{h}{2},\ y_0 + \frac{h}{2}s_2\right)$$

$$s_4 = f\left(t_0 + h,\ y_0 + hs_3\right)$$

$$y_1 = y_0 + h\frac{s_1 + 2s_2 + 2s_3 + s_4}{6}$$

Example

Compute the first four step in the second-Order Runge-Kutta method approximation to the solution of $y' = y - t$ with $y(1) = 1$, using the step size $h = 0.1$. Compare the result with the actual solution to the initial value problem.

Solution

$$\Rightarrow t_0 = 1 \ \ and \ \ y_0 = 1$$

The *first* step:

$$s_1 = f\left(t_0, y_0\right)$$
$$= 1 - 1$$
$$= 0$$

$$s_2 = f\left(t_0 + \frac{h}{2},\ y_0 + \frac{h}{2}s_1\right)$$
$$= f\left(1.05\ ,1\right)$$
$$= 1 - 1.05$$
$$= -0.05$$

$$s_3 = f\left(t_0 + \frac{h}{2},\ y_0 + \frac{h}{2}s_2\right)$$
$$= f\left(1.05\ ,.9975\right)$$
$$= .9975 - 1.05$$
$$= -0.0525$$

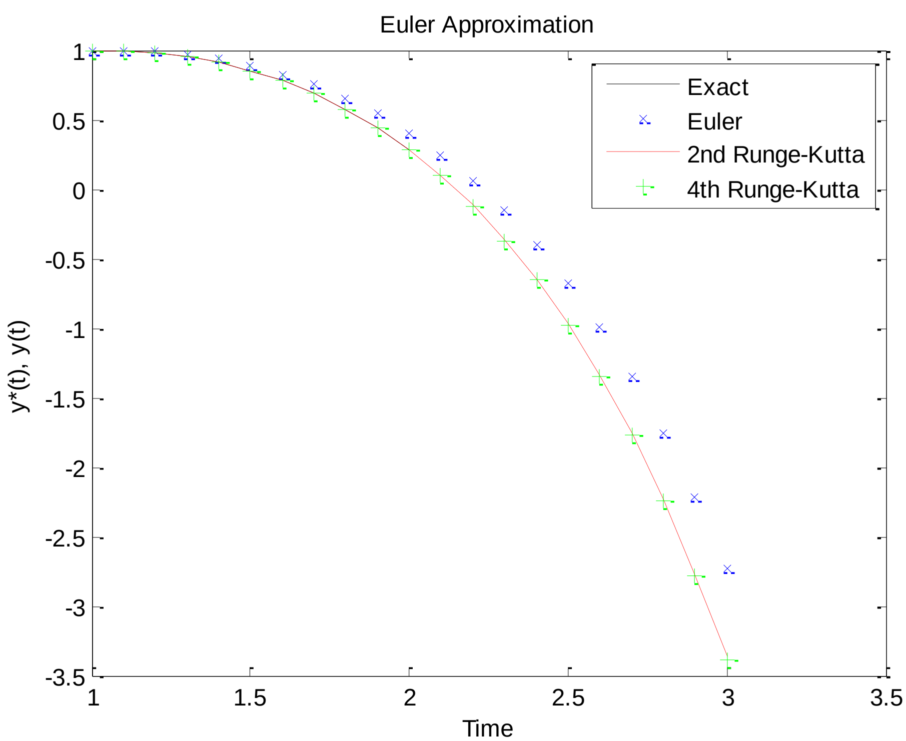
$$s_4 = f\left(t_0 + h,\ y_0 + hs_3\right)$$

$$= f(1.1, .99475)$$
$$= .99475 - 1.1$$
$$= -0.10525$$

$$y_1 = y_0 + h\frac{s_1 + 2s_2 + 2s_3 + s_4}{6}$$
$$= 1 + 0.1\left(\frac{0 + 2(-.05) + 2(-.0525) + (-.10525)}{6}\right)$$
$$= 0.99482916667$$

$$t_1 = t_0 + h$$
$$= 1.1$$

t_k	y_k : Runge-Kutta	y_k - Exact	***Runge-Kutta Error***
1.0	1.0	1.0	**0**
1.1	0.994829167	0.994829081	**–0.000000086**
1.2	0.978597429	0.978597241	**0.000000295**
1.3	0.950141502	0.950141192	**–0.000000310**
1.4	0.908175759	0.908175302	**–0.000000457**



Exercises *Section* 2.8 - Euler's & Runge-Kutta Methods

Calculate the first five iterations of Euler's method with step $h = 0.1$ of

1. $y' = ty \quad y(0) = 1$

2. $z' = x - 2z \quad z(0) = 1$

3. $z' = 5 - z \quad z(0) = 0$

4. Given: $y' + 2xy = x \quad y(0) = 8$

a) Use a computer and Euler's method to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

5. Given: $z' - 2z = xe^{2x} \quad z(0) = 1$

a) Use a computer and Euler's method to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a third with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

6. Consider the initial value problem $y' = 12y(4 - y) \quad y(0) = 1$
Use Euler's method with step size $h = 0.04$ to sketch solution on the interval $[0,\ 2]$

7. You've seen that the error in Euler's method varies directly as the first power of the step size $\left(i.e\ E_h \approx \lambda h\right)$. This makes Euler's method an order to halve the error? How does this affect the number of required iterations?

8. Use Euler's method to provide an approximate solution over the given time interval using the given steps sizes. Provide a plot of ***v*** versus ***y*** for each step size

$$y'' + 4y = 0, \quad y(0) = 4, \quad y'(0) = 0, \quad [0,\ 2\pi]; \quad h = 0.1,\ 0.01,\ 0.001$$

9. Given $z' + z = \cos x \quad z(0) = 1$

a) Use a computer and Runge-Kutta method to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

10. Given $x' = \frac{t}{x} \quad x(0) = 1$

a) Use a computer and Runge-Kutta method to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

11. Consider the initial value problem $y' = \frac{t}{y^2} \quad y(0) = 1$

Use Runge-Kutta method with step size $h = 0.04$ to sketch solution on the interval $[0,\ 2]$