Solution Section 4.3 – Multiplicative Inverses of Matrices

Exercise

Show that B is Multiplicative inverse of A

$$A = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

Solution

$$AB = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{pmatrix} -6 & \\ & \end{pmatrix}$$

$$\neq I$$

B is not multiplicative inverse of A

Exercise

Show that B is Multiplicative inverse of A

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

Solution

$$AB = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I$$

$$BA = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I$$

$\therefore B$ is Multiplicative inverse of A

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{-4+6}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ -\frac{1}{3} & 1 \end{bmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{10-10}\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

$$= \frac{1}{0}\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$

Solution

$$\left[\begin{array}{cc|cc} -2 & 3 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{array}\right] \quad -\frac{1}{2}R_1$$

$$\begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ -3 & 4 & 0 & 1 \end{array}\right] \quad R_2 + 3R_1$$

$$\begin{array}{cccc} -3 & 4 & 0 & 1 \\ 3 & -\frac{9}{2} & -\frac{3}{2} & 0 \\ \hline 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right] \quad -2R_2 \qquad\qquad \begin{array}{cccc} 0 & 1 & 3 & -2 \end{array}$$

$$\begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & \frac{9}{2} & -3 \\ \hline 1 & 0 & 4 & -3 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 3 & -2 \end{array}\right] \quad R_1 + \frac{3}{2}R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 4 & -3 \\ 0 & 1 & 3 & -2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} a & b \\ 3 & 3 \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{3a-3b}\begin{bmatrix} 3 & -b \\ -3 & a \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{3(a-b)} & \frac{-b}{3(a-b)} \\ \frac{-3}{3(a-b)} & \frac{a}{3(a-b)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{a-b} & \frac{-b}{3(a-b)} \\ \frac{-1}{a-b} & \frac{a}{3(a-b)} \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} -2 & a \\ 4 & a \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{-2a-4a}\begin{bmatrix} a & -a \\ -4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{-6a} & \frac{-a}{-6a} \\ \frac{-4}{-6a} & \frac{-2}{-6a} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{6} & \frac{1}{6} \\ \frac{2}{3a} & \frac{1}{3a} \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} 4 & 4 \\ b & a \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{4a-4b} \begin{bmatrix} a & -4 \\ -b & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{4(a-b)} & \frac{-4}{4(a-b)} \\ \frac{-b}{4(a-b)} & \frac{4}{4(a-b)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{4(a-b)} & \frac{-1}{a-b} \\ \frac{-b}{4(a-b)} & \frac{1}{a-b} \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{-1+4} \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

Exercise

Find the inverse of $A=\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}$

Solution

$$A^{-1}=\frac{1}{3}\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{1}{3} & \frac{2}{3}\\ -\frac{2}{3} & \frac{1}{3}\end{pmatrix}$$

Exercise

Find the inverse of $A=\begin{pmatrix}2 & 4\\ 3 & -1\end{pmatrix}$

Solution

$$A^{-1}=-\frac{1}{14}\begin{pmatrix}-1 & -4\\ -3 & 2\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{14} & \frac{2}{7}\\ \frac{3}{14} & -\frac{1}{7}\end{pmatrix}$$

Exercise

Find the inverse of $A=\begin{pmatrix}-1 & 3\\ 2 & -1\end{pmatrix}$

Solution

$$A^{-1}=-\frac{1}{5}\begin{pmatrix}-1 & -3\\ -2 & -1\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{5} & \frac{3}{5}\\ \frac{2}{5} & \frac{1}{5}\end{pmatrix}$$

Exercise

Find the inverse of $A=\begin{pmatrix}1 & 3\\ -2 & 5\end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{11}\begin{pmatrix} 5 & -3 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{11} & -\frac{3}{11} \\ \frac{2}{11} & \frac{1}{11} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{0}\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} -6 & 9 \\ 2 & -3 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{18-18}\begin{pmatrix} & \\ & \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} -2 & 7 \\ 0 & 2 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -7 \\ 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{7}{4} \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Exercise

Find the inverse of $A=\begin{pmatrix}4 & -16\\ 1 & -4\end{pmatrix}$

Solution

$$A=\frac{1}{-16+16}\begin{pmatrix}4 & -16\\ 1 & -4\end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A=\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$

Solution

$$A^{-1}=\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}$$

Exercise

Find the inverse of $A=\begin{pmatrix}2 & 1\\ a & a\end{pmatrix}$

Solution

$$A^{-1}=\frac{1}{a}\begin{pmatrix}a & -1\\ -a & 2\end{pmatrix}$$

$$=\begin{pmatrix}1 & -\frac{1}{a}\\ -1 & \frac{2}{a}\end{pmatrix}$$

Exercise

Find the inverse of $A=\begin{pmatrix}b & 3\\ b & 2\end{pmatrix}$

Solution

$$A^{-1}=-\frac{1}{b}\begin{pmatrix}2 & -3\\ -b & b\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{2}{b} & \frac{3}{b}\\ 1 & -1\end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & a \\ 3 & a \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{2a}\begin{pmatrix} a & -a \\ -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2a} & -\frac{1}{2a} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{a^2 - 4}\begin{pmatrix} a & -2 \\ -2 & a \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a}{a^2-4} & \frac{-2}{a^2-4} \\ \frac{-2}{a^2-4} & \frac{a}{a^2-4} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{0}\begin{pmatrix} & \\ & \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} -3 & \frac{1}{2} \\ 6 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{0}\begin{pmatrix} & \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find A^{-1} *if* $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccccc} 2 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -2 & 0 & 0 \\ \hline 0 & -2 & -3 & -2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 3 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & -3 & 0 & 0 \\ \hline 0 & 0 & -3 & -3 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \quad -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \quad -\tfrac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - \frac{3}{2}R_3 \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ \hline 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array} \qquad \begin{array}{cccccc} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ \hline 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

Exercise

Find A^{-1}, where $A=\begin{bmatrix}1 & 2 & -1\\ 3 & 5 & 3\\ 2 & 4 & 3\end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 3 & 5 & 3 & 0 & 1 & 0\\ 2 & 4 & 3 & 0 & 0 & 1\end{array}\right] \begin{array}{l} \\ R_2-3R_1\\ R_3+2R_1\end{array}$$

$$\begin{array}{cccccc} 3 & 5 & 3 & 0 & 1 & 0\\ -3 & -6 & 3 & -3 & 0 & 0\\ \hline 0 & -1 & 6 & -3 & 1 & 0\end{array} \qquad \begin{array}{cccccc} 2 & 4 & 3 & 0 & 0 & 1\\ -2 & -4 & 2 & -2 & 0 & 0\\ \hline 0 & 0 & 5 & -2 & 0 & 1\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & -1 & 6 & -3 & 1 & 0\\ 0 & 0 & 5 & -2 & 0 & 1\end{array}\right] \ -R_2$$

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & 1 & -6 & 3 & -1 & 0\\ 0 & 0 & 5 & -2 & 0 & 1\end{array}\right] \ R_1-2R_2$$

$$\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0\\ 0 & -2 & 12 & -6 & 2 & 0\\ \hline 1 & 0 & 11 & -5 & 2 & 0\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 11 & -5 & 2 & 0\\ 0 & 1 & -6 & 3 & -1 & 0\\ 0 & 0 & 5 & -2 & 0 & 1\end{array}\right] \ \tfrac{1}{5}R_3$$

$$\begin{array}{cccccc} 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5}\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 11 & -5 & 2 & 0\\ 0 & 1 & -6 & 3 & -1 & 0\\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5}\end{array}\right] \begin{array}{l} R_1-11R_3\\ R_2+6R_3\end{array}$$

$$\begin{array}{cccccc} 0 & 1 & -6 & 3 & -1 & 0\\ 0 & 0 & 6 & -\frac{12}{5} & 0 & \frac{6}{5}\\ \hline 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5}\end{array} \qquad \begin{array}{cccccc} 1 & 0 & 11 & -5 & 2 & 0\\ 0 & 0 & -11 & \frac{22}{5} & 0 & -\frac{11}{5}\\ \hline 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5}\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5}\\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5}\\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5}\end{array}\right]$$

$$A^{-1}=\begin{bmatrix}-\frac{3}{5} & 2 & -\frac{11}{5}\\ \frac{3}{5} & -1 & \frac{6}{5}\\ -\frac{2}{5} & 0 & \frac{1}{5}\end{bmatrix}$$

Exercise

Find A^{-1}, where $A=\begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2+2R_1 \\ R_3-R_1 \end{array}$$

$$\begin{array}{cccccc} -2 & 0 & 1 & 0 & 1 & 0 \\ 2 & 4 & -2 & 2 & 0 & 0 \\ \hline 0 & 4 & -1 & 2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & -2 & 1 & -1 & 0 & 0 \\ \hline 0 & -3 & 1 & -1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 4 & -1 & 2 & 1 & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array}\right] \tfrac{1}{4}R_2 \qquad \begin{array}{cccccc} 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} R_1-2R_2 \\ \\ R_3+3R_2 \end{array}$$

$$\begin{array}{cccccc} 0 & -3 & 1 & -1 & 0 & 1 \\ 0 & 3 & -\frac{3}{4} & \frac{3}{2} & \frac{3}{4} & 0 \\ \hline 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{array} \qquad \begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & \frac{1}{2} & -1 & -\frac{1}{2} & 0 \\ \hline 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{array}\right] 4R_3 \qquad \begin{array}{cccccc} 0 & 0 & 1 & 2 & 3 & 4 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}\right] \begin{array}{l} R_1+\frac{1}{2}R_3 \\ R_2+\frac{1}{4}R_3 \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 \\ \hline 1 & 0 & 0 & 1 & 1 & 2 \end{array} \qquad \begin{array}{cccccc} 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ \hline 0 & 1 & 0 & 1 & 1 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}\right]$$

$$A^{-1}=\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

Exercise

Find A^{-1}, where $A=\begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \qquad \frac{1}{-2}R_1 \qquad 1 \quad -\frac{5}{2} \quad -\frac{3}{2} \quad -\frac{1}{2} \quad 0 \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \qquad R_2 - 4R_1 \qquad \begin{array}{cccccc} 4 & -1 & 3 & 0 & 1 & 0 \\ -4 & 10 & 6 & 2 & 0 & 0 \\ \hline 0 & 9 & 9 & 2 & 1 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \qquad R_3 - 7R_1 \qquad \begin{array}{cccccc} 7 & -2 & 5 & 0 & 0 & 1 \\ -7 & \frac{35}{2} & \frac{21}{2} & \frac{7}{2} & 0 & 0 \\ \hline 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}\right] \qquad \frac{1}{9}R_2 \qquad 0 \quad 1 \quad 1 \quad \frac{2}{9} \quad \frac{1}{9} \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}\right] \qquad R_3 - \frac{31}{2}R_2 \qquad \begin{array}{cccccc} 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \\ 0 & -\frac{31}{2} & -\frac{31}{2} & -\frac{31}{9} & -\frac{31}{18} & 0 \\ \hline 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array}\right]$$

$\therefore$ The inverse matrix ***doesn't exist***

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix}1&1&0\\-1&3&4\\0&4&3\end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc}1&1&0&1&0&0\\-1&3&4&0&1&0\\0&4&3&0&0&1\end{array}\right)\quad R_2+R_1 \qquad \begin{array}{cccccc}1&1&0&1&0&0\\-1&3&4&0&1&0\\\hline 0&4&4&1&1&0\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&4&4&1&1&0\\0&4&3&0&0&1\end{array}\right)\quad \tfrac{1}{4}R_2$$

$$\left(\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&1&1&\frac{1}{4}&\frac{1}{4}&0\\0&4&3&0&0&1\end{array}\right)\quad \begin{array}{l}R_1-R_2\\ \\ R_3-4R_2\end{array} \qquad \begin{array}{cccccc}0&4&3&0&0&1\\0&-4&-4&-1&-1&0\\\hline 0&0&-1&-1&-1&1\end{array} \qquad \begin{array}{cccccc}1&1&0&1&0&0\\0&-1&-1&-\frac{1}{4}&-\frac{1}{4}&0\\\hline 1&0&-1&\frac{3}{4}&-\frac{1}{4}&0\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&-1&\frac{3}{4}&-\frac{1}{4}&0\\0&1&1&\frac{1}{4}&\frac{1}{4}&0\\0&0&-1&-1&-1&1\end{array}\right)\quad -R_3$$

$$\left(\begin{array}{ccc|ccc}1&0&-1&\frac{3}{4}&-\frac{1}{4}&0\\0&1&1&\frac{1}{4}&\frac{1}{4}&0\\0&0&1&1&1&-1\end{array}\right)\quad \begin{array}{l}R_1+R_3\\R_2-R_3\end{array} \qquad \begin{array}{cccccc}1&0&-1&\frac{3}{4}&-\frac{1}{4}&0\\0&0&1&1&1&-1\\\hline 1&0&0&\frac{7}{4}&\frac{3}{4}&-1\end{array} \qquad \begin{array}{cccccc}0&1&1&\frac{1}{4}&\frac{1}{4}&0\\0&0&-1&-1&-1&1\\\hline 0&1&0&-\frac{3}{4}&-\frac{3}{4}&1\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&0&\frac{7}{4}&\frac{3}{4}&-1\\0&1&0&-\frac{3}{4}&-\frac{3}{4}&1\\0&0&1&1&1&-1\end{array}\right)$$

$$A^{-1}=\begin{pmatrix}\frac{7}{4}&\frac{3}{4}&-1\\-\frac{3}{4}&-\frac{3}{4}&1\\1&1&-1\end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ -2 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \quad R_3+2R_1 \qquad \begin{array}{cccccc} -2 & -3 & 0 & 0 & 0 & 1 \\ 2 & -2 & 2 & 2 & 0 & 0 \\ \hline 0 & -5 & 2 & 2 & 0 & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array}\right) \quad -\frac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1+R_2 \\ \\ R_3+5R_2 \end{array} \qquad \begin{array}{cccccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ \hline 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \end{array} \qquad \begin{array}{cccccc} 0 & -5 & 2 & 2 & 0 & 1 \\ 0 & 5 & -\frac{5}{2} & 0 & -\frac{5}{2} & 0 \\ \hline 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \end{array}\right) \quad -2R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -4 & 5 & -2 \end{array}\right) \quad \begin{array}{l} R_1-\frac{1}{2}R_3 \\ R_2+\frac{1}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & -4 & 5 & -2 \end{array}\right)$$

$$A^{-1}=\begin{pmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right) \quad \tfrac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right) \quad R_3 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right) \quad 2R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right) \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 - \frac{5}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & -2 & -2 & -5 \\ 0 & 0 & 1 & 3 & 1 & 2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3 & -2 & -4 \\ -2 & -2 & -5 \\ 3 & 1 & 2 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 1 & 2 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & -1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -4 & -3 & 1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array}\right) -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 2R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & 0 & 7 & 3 & -2 & 1 \end{array}\right) \frac{1}{7}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{array}\right) \begin{array}{l} R_1 + 3R_3 \\ R_2 - 4R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ 0 & 1 & 0 & \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 3 & 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \frac{1}{3}R_1 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & -3 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{7}{3} & -\frac{5}{3} & 3 & 1 \end{array}\right) \begin{array}{l} \\ \\ \frac{3}{7}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array}\right) \begin{array}{l} R_1 + \frac{1}{3}R_3 \\ R_2 - \frac{2}{3}R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ 0 & 1 & 0 & \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array}\right)$$

$$A^{-1}=\begin{pmatrix} \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $\qquad A=\begin{pmatrix}-3 & 1 & -1\\ 1 & -4 & -7\\ 1 & 2 & 5\end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc}-3 & 1 & -1 & 1 & 0 & 0\\ 1 & -4 & -7 & 0 & 1 & 0\\ 1 & 2 & 5 & 0 & 0 & 1\end{array}\right) \quad -\tfrac{1}{3}R_1$$

$$\left(\begin{array}{ccc|ccc}1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0\\ 1 & -4 & -7 & 0 & 1 & 0\\ 1 & 2 & 5 & 0 & 0 & 1\end{array}\right) \quad \begin{array}{l}R_2-R_1\\ R_3-R_1\end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0\\ 0 & -\frac{11}{3} & -\frac{22}{3} & \frac{1}{3} & 1 & 0\\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1\end{array}\right) \quad -\tfrac{3}{11}R_2$$

$$\left(\begin{array}{ccc|ccc}1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0\\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0\\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1\end{array}\right) \quad R_3-\tfrac{7}{3}R_2$$

$$\left(\begin{array}{ccc|ccc}1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0\\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0\\ 0 & 0 & 0 & & & \end{array}\right)$$

$\therefore$ Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $\qquad A=\begin{pmatrix}1 & 1 & -3\\ 2 & -4 & 1\\ -5 & 7 & 1\end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 2 & -4 & 1 & 0 & 1 & 0 \\ -5 & 7 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 5R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & -6 & 7 & -2 & 1 & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array}\right) \; -\frac{1}{6}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 12R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & & & \end{array}\right)$$

$\therefore$ Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_3 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right) \; -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - 2R_2 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ \frac{1}{5}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right) \begin{array}{l} R_1 - 11R_3 \\ R_2 + 6R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $\qquad A = \begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -6 & 1 \\ -2 & -2 & 5 & 1 \end{bmatrix}$

<u>Solution</u>

$$\left[\begin{array}{cccc|cccc} -2 & -3 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -\frac{1}{2}R_1 \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 + 2R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\therefore$ Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 1 & -14 & 7 & 38 \\ -1 & 2 & 1 & -2 \\ 1 & 2 & -1 & -6 \\ 1 & -2 & 3 & 6 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & -6 & 0 & 0 & 1 & 0 \\ 1 & -2 & 3 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & -12 & 8 & 36 & 1 & 1 & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array}\right] \quad -\frac{1}{12}R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + 14R_2 \\ \\ R_3 - 16R_2 \\ R_4 - 12R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & \frac{8}{3} & 4 & \frac{1}{3} & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array}\right] \quad \frac{3}{8}R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + \frac{7}{3}R_3 \\ R_2 + \frac{2}{3}R_3 \\ \\ R_4 - 4R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & -2 & -\frac{1}{2} & -1 & -\frac{3}{2} & 1 \end{array}\right] -\frac{1}{2}R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array}\right] \begin{array}{l} R_1 + \frac{1}{2}R_4 \\ R_2 + 2R_4 \\ R_3 - \frac{3}{2}R_4 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ 0 & 0 & 1 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{bmatrix}$$

Exercise

Find the inverse, if exists, of $A=\begin{bmatrix}10 & 20 & -30 & 15\\ 3 & -7 & 14 & -8\\ -7 & -2 & -1 & 2\\ 4 & 4 & -3 & 1\end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc}10 & 20 & -30 & 15 & 1 & 0 & 0 & 0\\ 3 & -7 & 14 & -8 & 0 & 1 & 0 & 0\\ -7 & -2 & -1 & 2 & 0 & 0 & 1 & 0\\ 4 & 4 & -3 & 1 & 0 & 0 & 0 & 1\end{array}\right] \quad \frac{1}{10}R_1$$

$$\left[\begin{array}{cccc|cccc}1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0\\ 3 & -7 & 14 & -8 & 0 & 1 & 0 & 0\\ -7 & -2 & -1 & 2 & 0 & 0 & 1 & 0\\ 4 & 4 & -3 & 1 & 0 & 0 & 0 & 1\end{array}\right] \quad \begin{array}{l} R_2-3R_1\\ R_3+7R_1\\ R_4-4R_1\end{array}$$

$$\left[\begin{array}{cccc|cccc}1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0\\ 0 & -13 & 23 & -\frac{25}{2} & -\frac{3}{10} & 1 & 0 & 0\\ 0 & 12 & -22 & \frac{25}{2} & \frac{7}{10} & 0 & 1 & 0\\ 0 & -4 & 9 & -5 & -\frac{2}{5} & 0 & 0 & 1\end{array}\right] \quad -\frac{1}{13}R_2$$

$$\left[\begin{array}{cccc|cccc}1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0\\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0\\ 0 & 12 & -22 & \frac{25}{2} & \frac{7}{10} & 0 & 1 & 0\\ 0 & -4 & 9 & -5 & -\frac{2}{5} & 0 & 0 & 1\end{array}\right] \quad \begin{array}{l} R_1-2R_2\\ \\ R_3-12R_2\\ R_4+4R_2\end{array}$$

$$\left[\begin{array}{cccc|cccc}1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0\\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0\\ 0 & 0 & -\frac{10}{13} & \frac{25}{26} & \frac{11}{26} & \frac{12}{13} & 1 & 0\\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1\end{array}\right] \quad -\frac{13}{10}R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 - \frac{7}{13}R_3 \\ R_2 + \frac{23}{13}R_3 \\ \\ R_4 - \frac{25}{13}R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{19}{20} & -\frac{11}{5} & -\frac{23}{10} & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array}\right] \quad \begin{array}{l} R_2 + R_4 \\ R_3 + R_4 \end{array}.$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array}\right] \quad \begin{array}{l} \\ \\ \\ \frac{4}{5}R_4 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array}\right] \quad \begin{array}{l} R_1 - \frac{1}{4}R_4 \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{bmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $\quad A=[x]$

Solution

For A^{-1} exists, $x \neq 0$

$$AA^{-1}=I$$

$$[x][a]=[1]$$

$$xa=1$$

$$\boxed{a=\frac{1}{x}}$$

$$A^{-1}=\left[\frac{1}{x}\right]$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $\quad A=\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$

Solution

For A^{-1} exists, $\boxed{x, y \neq 0}$

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad AA^{-1}=I$$

$$\begin{cases} ax=1 & bx=0 \\ cy=0 & dy=1 \end{cases}$$

$$\begin{cases} a=\frac{1}{x} & b=0 \\ c=0 & d=\frac{1}{y} \end{cases}$$

$$A^{-1}=\begin{bmatrix} \frac{1}{x} & 0 \\ 0 & \frac{1}{y} \end{bmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $\quad A=\begin{bmatrix} 0 & 0 & x \\ 0 & y & 0 \\ z & 0 & 0 \end{bmatrix}$

Solution

For A^{-1} exists, $\underline{x, y, z \neq 0}|$

$$\begin{pmatrix} 0 & 0 & x \\ 0 & y & 0 \\ z & 0 & 0 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad AA^{-1}=I$$

$$\begin{pmatrix} xg & xh & xi \\ yd & ye & yf \\ za & zb & zc \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{cases} xg=1 & xh=0 & xi=0 \\ yd=0 & ye=1 & yf=0 \\ za=0 & zb=0 & zc=1 \end{cases}$$

$$\begin{cases} g=\frac{1}{x} & h=0 & i=0 \\ d=0 & e=\frac{1}{y} & f=0 \\ a=0 & b=0 & c=\frac{1}{z} \end{cases}$$

$$A^{-1}=\begin{pmatrix} 0 & 0 & \frac{1}{z} \\ 0 & \frac{1}{y} & 0 \\ \frac{1}{z} & 0 & 0 \end{pmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $\quad A=\begin{bmatrix} x & 1 & 1 & 1 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{bmatrix}$

Solution

For A^{-1} exists, $\underline{x, y, z, w \neq 0}|$

$$\begin{pmatrix} x & 1 & 1 & 1 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad AA^{-1} = I$$

$$\begin{cases} xa_{11} + xa_{21} + xa_{31} + xa_{41} = 1 \\ xa_{12} + xa_{22} + xa_{32} + xa_{42} = 0 \\ xa_{13} + xa_{23} + xa_{33} + xa_{43} = 0 \\ xa_{14} + xa_{24} + xa_{34} + xa_{44} = 0 \end{cases}$$

$$\begin{cases} ya_{21} = 0 & a_{21} = 0 \\ ya_{22} = 1 & a_{22} = \dfrac{1}{y} \\ ya_{23} = 0 & a_{23} = 0 \\ ya_{24} = 0 & a_{24} = 0 \end{cases}$$

$$\begin{cases} za_{31} = 0 & a_{31} = 0 \\ za_{32} = 0 & a_{32} = 0 \\ za_{33} = 1 & a_{33} = \dfrac{1}{z} \\ za_{34} = 0 & a_{34} = 0 \end{cases}$$

$$\begin{cases} wa_{41} = 0 & a_{41} = 0 \\ wa_{42} = 0 & a_{42} = 0 \\ wa_{43} = 0 & a_{43} = 0 \\ wa_{44} = 1 & a_{44} = \dfrac{1}{w} \end{cases}$$

$$\rightarrow \begin{cases} xa_{11} = 1 & a_{11} = \dfrac{1}{x} \\ xa_{12} + \dfrac{x}{y} = 0 & a_{12} = -\dfrac{1}{y} \\ xa_{13} + \dfrac{x}{z} = 0 & a_{13} = -\dfrac{1}{z} \\ xa_{14} + \dfrac{x}{w} = 0 & a_{14} = -\dfrac{1}{w} \end{cases}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{x} & -\frac{1}{y} & -\frac{1}{z} & -\frac{1}{w} \\ 0 & \frac{1}{y} & 0 & 0 \\ 0 & 0 & \frac{1}{z} & 0 \\ 0 & 0 & 0 & \frac{1}{w} \end{pmatrix}$$

Exercise

Solve the system using A^{-1} $\begin{cases} x \qquad\quad +2z = 6 \\ -x + 2y + 3z = -5 \\ x - y \qquad\quad = 6 \end{cases}$ Given $A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

Solution

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3(6) - 2(-5) - 4(6) \\ 3(6) - 2(-5) - 5(6) \\ -1(6) + 1(-5) + 2(6) \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

Solution: $(4, \ -2, \ 1)$

Exercise

Solve the system using A^{-1} $\begin{cases} x + 2y + 5z = 2 \\ 2x + 3y + 8z = 3 \\ -x + \ y + 2z = 3 \end{cases}$

a) Write the linear system as a matrix equation in the form $AX = B$

b) Solve the system using the inverse that is given for the coefficient matrix

the inverse of $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix}$ *is* $\begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}$

Solution

a) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$

b) $\begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$

Exercise

Solve the system using A^{-1} $\begin{cases} x - y + z = 8 \\ \quad 2y - z = -7 \\ 2x + 3y \quad = 1 \end{cases}$

a) Write the linear system as a matrix equation in the form $AX = B$

b) Solve the system using the inverse that is given for the coefficient matrix

the inverse is $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

Solution

a) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -7 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$

Solution

$A = \begin{pmatrix} 3 & 2 \\ 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

$A^{-1} = -\frac{1}{7}\begin{pmatrix} -1 & -2 \\ -2 & 3 \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{pmatrix} \begin{pmatrix} -4 \\ 5 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{6}{7} \\ -\frac{23}{7} \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(\frac{6}{7},\ -\frac{23}{7}\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$

Solution

$$A = \begin{pmatrix} 2 & 5 \\ 5 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{29}\begin{pmatrix} -2 & -5 \\ -5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{29} & \frac{5}{29} \\ \frac{5}{29} & -\frac{2}{29} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{2}{29} & \frac{5}{29} \\ \frac{5}{29} & -\frac{2}{29} \end{pmatrix} \begin{pmatrix} 7 \\ -3 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{29} \\ -\frac{41}{7} \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(-\frac{1}{29},\ -\frac{41}{29}\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$

Solution

$$A = \begin{pmatrix} 4 & -7 \\ 2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} -16 \\ 9 \end{pmatrix}$$

$$A^{-1} = \frac{1}{34}\begin{pmatrix} 5 & 7 \\ -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{34} & \frac{7}{34} \\ -\frac{1}{17} & \frac{2}{17} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{5}{34} & \frac{7}{34} \\ -\frac{1}{17} & \frac{2}{17} \end{pmatrix}\begin{pmatrix} -16 \\ 9 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(-\frac{1}{2},\ 2\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$

Solution

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$A^{-1} = -1\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$$

$$X = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$\therefore$ ***Solution***: $(-2,\ 5)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 3x+4y=2 \\ 2x+5y=-1 \end{cases}$

Solution

$$A=\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} \quad B=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$A^{-1}=\frac{1}{7}\begin{pmatrix} 5 & -4 \\ -2 & 3 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{5}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{3}{7} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{5}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{3}{7} \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$\therefore$ ***Solution***: $(2,\ -1)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 5x-2y=4 \\ -10x+4y=7 \end{cases}$

Solution

$$A=\begin{pmatrix} 5 & -2 \\ -10 & 4 \end{pmatrix} \quad B=\begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$A^{-1}=\frac{1}{0}\begin{pmatrix} 4 & 2 \\ 10 & 5 \end{pmatrix}$$

Inverse matrix ***doesn't exist***.

$$-\frac{1}{2}\begin{cases} 5x-2y=4 \\ 5x-2y=-\frac{7}{2} \end{cases}$$

$$4\neq-\frac{7}{2}$$

$\therefore$ ***No Solution***

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$

Solution

$$A=\begin{pmatrix} 1 & -4 \\ 5 & -20 \end{pmatrix} \quad B=\begin{pmatrix} -8 \\ -40 \end{pmatrix}$$

$$A^{-1}=\frac{1}{0}\begin{pmatrix} & \\ & \end{pmatrix}$$

Inverse matrix ***doesn't exist***.

$$\frac{1}{5}\begin{cases} x-4y=-8 \\ x-4y=-8 \end{cases}$$

∴ ***Solution***: $(4y-8,\ y)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

Solution

$$A=\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \quad B=\begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$A^{-1}=-\frac{1}{3}\begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}\begin{pmatrix} 3 \\ 3 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

∴ ***Solution***: $(2,\ -1)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x+10y=-14 \\ 7x-2y=-16 \end{cases}$

Solution

$$A=\begin{pmatrix} 2 & 10 \\ 7 & -2 \end{pmatrix} \quad B=\begin{pmatrix} -14 \\ -16 \end{pmatrix}$$

$$A^{-1}=-\frac{1}{74}\begin{pmatrix} -2 & -10 \\ -7 & 2 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{37} & \frac{5}{37} \\ \frac{7}{74} & -\frac{1}{37} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{1}{37} & \frac{5}{37} \\ \frac{7}{74} & -\frac{1}{37} \end{pmatrix}\begin{pmatrix} -14 \\ -16 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -\frac{94}{37} \\ -\frac{33}{37} \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(-\frac{94}{37},\ -\frac{33}{37}\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 4x-3y=24 \\ -3x+9y=-1 \end{cases}$

Solution

$$A=\begin{pmatrix} 4 & -3 \\ -3 & 9 \end{pmatrix} \quad B=\begin{pmatrix} 24 \\ -1 \end{pmatrix}$$

$$A^{-1}=\frac{1}{27}\begin{pmatrix} 9 & 3 \\ 3 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{27} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{27} \end{pmatrix}\begin{pmatrix} 24 \\ -1 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{71}{9} \\ \frac{68}{27} \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(\frac{71}{9},\ \frac{68}{27}\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 4x+2y=12 \\ 3x-2y=16 \end{cases}$

Solution

$$A = \begin{pmatrix} 4 & 2 \\ 3 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 12 \\ 16 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{14}\begin{pmatrix} -2 & -2 \\ -3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{3}{14} & -\frac{2}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{3}{14} & -\frac{2}{7} \end{pmatrix}\begin{pmatrix} 12 \\ 16 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$\therefore$ ***Solution***: $(4,\ -2)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x+2y=-1 \\ 4x-2y=6 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 2 \\ 4 & -2 \end{pmatrix} \quad B = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{10}\begin{pmatrix} -2 & -2 \\ -4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{10} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} -1 \\ 6 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\therefore$ ***Solution***: $\underline{(1, \ -1)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$

Solution

$$\frac{1}{2} \begin{cases} x - 2y = 5 \\ -5x + y = 2 \end{cases}$$

$$A = \begin{pmatrix} 1 & -2 \\ -5 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{9} \begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{9} & -\frac{2}{9} \\ -\frac{5}{9} & -\frac{1}{9} \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{1}{9} & -\frac{2}{9} \\ -\frac{5}{9} & -\frac{1}{9} \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$\therefore$ ***Solution***: $\underline{(-1, \ -3)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$

Solution

$$\begin{matrix} \frac{1}{3} \\ \frac{1}{15} \end{matrix} \to \begin{cases} 4x+5y=-9 \\ 2x-y=-1 \end{cases}$$

$$A=\begin{pmatrix} 4 & 5 \\ 2 & -1 \end{pmatrix} \quad B=\begin{pmatrix} -9 \\ -1 \end{pmatrix}$$

$$A^{-1}=-\frac{1}{14}\begin{pmatrix} -1 & -5 \\ -2 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{14} & \frac{5}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{1}{14} & \frac{5}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{pmatrix}\begin{pmatrix} -9 \\ -1 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$\therefore$ ***Solution***: $(-1,\ -1)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 4x-4y=-12 \\ 4x+4y=-20 \end{cases}$

Solution

$$\frac{1}{4} \to \begin{cases} x-y=-3 \\ x+y=-5 \end{cases}$$

$$A=\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad B=\begin{pmatrix} -3 \\ -5 \end{pmatrix}$$

$$A^{-1}=\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$$

$\therefore$ ***Solution***: $(-4, \ -1)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} -2x+3y=4 \\ -3x+4y=5 \end{cases}$

Solution

$$A = \begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$$

$$X = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\therefore$ ***Solution***: $(1, \ 2)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x-2y=6 \\ 4x+3y=2 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{11} \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{11} & \frac{2}{11} \\ -\frac{4}{11} & \frac{1}{11} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{3}{11} & \frac{2}{11} \\ -\frac{4}{11} & \frac{1}{11} \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

∴ ***Solution***: $\underline{(2, \ -2)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x - 3y = 7 \\ 4x + y = -7 \end{cases}$

Solution

$$A = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 7 \\ -7 \end{pmatrix}$$

$$A^{-1} = \frac{1}{14}\begin{pmatrix} 1 & 3 \\ -4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{14} & \frac{3}{14} \\ -\frac{2}{7} & \frac{1}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{14} & \frac{3}{14} \\ -\frac{2}{7} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} 7 \\ -7 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

∴ ***Solution***: $\underline{(-1, \ -3)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$

Solution

$$A=\begin{pmatrix}1&1&1\\2&1&-1\\1&-1&1\end{pmatrix}\quad B=\begin{pmatrix}2\\5\\-2\end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\2&1&-1&0&1&0\\1&-1&1&0&0&1\end{array}\right)\quad \begin{array}{l}R_2-2R_1\\R_3-R_1\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&-1&-3&-2&1&0\\0&-2&0&-1&0&1\end{array}\right)\quad -R_2$$

$$\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&3&2&-1&0\\0&-2&0&-1&0&1\end{array}\right)\quad \begin{array}{l}R_1-R_2\\ \\R_3+2R_2\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&-2&-1&1&0\\0&1&3&2&-1&0\\0&0&6&3&-2&1\end{array}\right)\quad \frac{1}{6}R_3$$

$$\left(\begin{array}{ccc|ccc}1&0&-2&-1&1&0\\0&1&3&2&-1&0\\0&0&1&\frac{1}{2}&-\frac{1}{3}&\frac{1}{6}\end{array}\right)\quad \begin{array}{l}R_1+2R_3\\R_2-3R_3\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&0&0&\frac{1}{3}&\frac{1}{3}\\0&1&0&\frac{1}{2}&0&-\frac{1}{2}\\0&0&1&\frac{1}{2}&-\frac{1}{3}&\frac{1}{6}\end{array}\right)$$

$$A^{-1}=\begin{pmatrix}0&\frac{1}{3}&\frac{1}{3}\\\frac{1}{2}&0&-\frac{1}{2}\\\frac{1}{2}&-\frac{1}{3}&\frac{1}{6}\end{pmatrix}$$

$$X=\begin{pmatrix}0&\frac{1}{3}&\frac{1}{3}\\\frac{1}{2}&0&-\frac{1}{2}\\\frac{1}{2}&-\frac{1}{3}&\frac{1}{6}\end{pmatrix}\begin{pmatrix}2\\5\\-2\end{pmatrix}\qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$\therefore$ ***Solution***: $(1,\ 2,\ -1)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -1 \\ 9 \\ 9 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 - 2R_1 \\ R_2 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & -4 & 4 & -3 & 0 & 1 \end{array}\right) \quad -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & -4 & 4 & -3 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 4R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & -8 & 5 & -4 & 1 \end{array}\right) \quad -\tfrac{1}{8}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{array}\right) \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 + 3R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ 0 & 0 & 1 & -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{pmatrix} \begin{pmatrix} -1 \\ 9 \\ 9 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

∴ ***Solution***: $(2,\ 1,\ 4)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 5 & -1 \\ 0 & 3 & -1 \\ -3 & 6 & 2 \end{pmatrix} \quad B = \begin{pmatrix} -4 \\ -1 \\ 11 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ -3 & 6 & 2 & 0 & 0 & 1 \end{array}\right) \quad R_3 + 3R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & 21 & -1 & 3 & 0 & 1 \end{array}\right) \quad \frac{1}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 21 & -1 & 3 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 - 5R_2 \\ \\ R_3 - 21R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & 1 & -\frac{5}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 6 & 3 & -7 & 1 \end{array}\right) \quad \frac{1}{6}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & 1 & -\frac{5}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{array}\right) \quad \begin{array}{l} R_1 - \frac{2}{3}R_3 \\ R_2 + \frac{1}{3}R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & \frac{8}{9} & -\frac{1}{9} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{18} & \frac{1}{18} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{8}{9} & -\frac{1}{9} \\ \frac{1}{6} & -\frac{1}{18} & \frac{1}{18} \\ \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{2}{3} & \frac{8}{9} & -\frac{1}{9} \\ \frac{1}{6} & -\frac{1}{18} & \frac{1}{18} \\ \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} -4 \\ -1 \\ 11 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ ***Solution***: $\underline{(-3,\ 0,\ 1)}\,|$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x+3y+4z=14 \\ 2x-3y+2z=10 \\ 3x-y+z=9 \end{cases}$

Solution

$$A=\begin{pmatrix} 1 & 3 & 4 \\ 2 & -3 & 2 \\ 3 & -1 & 1 \end{pmatrix} \quad B=\begin{pmatrix} 14 \\ 10 \\ 9 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 2 & -3 & 2 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_2-2R_1 \\ R_3-3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -9 & -6 & -2 & 1 & 0 \\ 0 & -10 & -11 & -3 & 0 & 1 \end{array}\right) -\frac{1}{9}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & -\frac{1}{9} & 0 \\ 0 & -10 & -11 & -3 & 0 & 1 \end{array}\right) \begin{array}{l} R_1-3R_2 \\ \\ R_3+10R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & -\frac{1}{9} & 0 \\ 0 & 0 & -\frac{13}{3} & -\frac{7}{9} & -\frac{10}{9} & 1 \end{array}\right) -\frac{3}{13}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & -\frac{1}{9} & 0 \\ 0 & 0 & 1 & \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{array}\right) \begin{array}{l} R_2-2R_3 \\ R_2-\frac{2}{3}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{39} & -\frac{7}{39} & \frac{6}{13} \\ 0 & 1 & 0 & \frac{4}{39} & -\frac{11}{39} & \frac{2}{13} \\ 0 & 0 & 1 & \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{39} & -\frac{7}{39} & \frac{6}{13} \\ \frac{4}{39} & -\frac{11}{39} & \frac{2}{13} \\ \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{1}{39} & -\frac{7}{39} & \frac{6}{13} \\ \frac{4}{39} & -\frac{11}{39} & \frac{2}{13} \\ \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{pmatrix} \begin{pmatrix} 14 \\ 10 \\ 9 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

∴ ***Solution***: $\underline{(2,\ 0,\ 3)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{array}\right) \quad R_2 - R_1$$

∴ ***Solution***: $\underline{(1,\ 4,\ -3)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 2 & -3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 17 \\ 7 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array}\right) \quad R_3 - 2R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & -7 & 0 & -2 & 0 & 1 \end{array}\right) \quad \frac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -7 & 0 & -2 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 7R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{7}{2} & -2 & \frac{7}{2} & 1 \end{array}\right) \quad -\frac{2}{7}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{4}{7} & -1 & -\frac{2}{7} \end{array}\right) \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 + \frac{1}{2}R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{7} & 1 & \frac{4}{7} \\ 0 & 1 & 0 & \frac{2}{7} & 0 & -\frac{1}{7} \\ 0 & 0 & 1 & \frac{4}{7} & -1 & -\frac{2}{7} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{7} & 1 & \frac{4}{7} \\ \frac{2}{7} & 0 & -\frac{1}{7} \\ \frac{4}{7} & -1 & -\frac{2}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{1}{7} & 1 & \frac{4}{7} \\ \frac{2}{7} & 0 & -\frac{1}{7} \\ \frac{4}{7} & -1 & -\frac{2}{7} \end{pmatrix} \begin{pmatrix} 17 \\ 7 \\ -1 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$$

$\therefore$ ***Solution***: $\underline{(4,\ 5,\ 3)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} -2x+6y+7z=3 \\ -4x+5y+3z=7 \\ -6x+3y+5z=-4 \end{cases}$

Solution

$$A = \begin{pmatrix} -2 & 6 & 7 \\ -4 & 5 & 3 \\ -6 & 3 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} -2 & 6 & 7 & 1 & 0 & 0 \\ -4 & 5 & 3 & 0 & 1 & 0 \\ -6 & 3 & 5 & 0 & 0 & 1 \end{array}\right) \quad -\tfrac{1}{2}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -\frac{7}{2} & -\frac{1}{2} & 0 & 0 \\ -4 & 5 & 3 & 0 & 1 & 0 \\ -6 & 3 & 5 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2+4R_1 \\ R_3+6R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -\frac{7}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & -7 & -11 & -2 & 1 & 0 \\ 0 & -15 & -16 & -3 & 0 & 1 \end{array}\right) \quad -\tfrac{1}{7}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -\frac{7}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & -15 & -16 & -3 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1+3R_2 \\ \\ R_3+15R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{17}{14} & \frac{5}{14} & -\frac{3}{7} & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & \frac{53}{7} & \frac{9}{7} & -\frac{15}{7} & 1 \end{array}\right) \quad \tfrac{7}{53}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{17}{14} & \frac{5}{14} & -\frac{3}{7} & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & 1 & \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{array}\right) \quad \begin{array}{l} R_1 - \frac{17}{14}R_3 \\ R_2 - \frac{11}{7}R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{8}{53} & -\frac{9}{106} & -\frac{17}{106} \\ 0 & 1 & 0 & \frac{1}{53} & \frac{16}{53} & -\frac{11}{53} \\ 0 & 0 & 1 & \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{8}{53} & -\frac{9}{106} & -\frac{7}{106} \\ \frac{1}{53} & \frac{16}{53} & -\frac{11}{53} \\ \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{8}{53} & -\frac{9}{106} & -\frac{7}{106} \\ \frac{1}{53} & \frac{16}{53} & -\frac{11}{53} \\ \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{pmatrix}\begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 3 \\ -2 \end{pmatrix}$$

$\therefore$ ***Solution***: $\boxed{\left(\frac{1}{2},\ 3,\ -2\right)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

Solution

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -3 & 4 \\ 4 & -2 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 3 & -3 & 4 & 0 & 1 & 0 \\ 4 & -2 & 3 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \\ 2R_2 - 3R_1 \\ 2R_3 - 4R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 5 & -3 & 2 & 0 \\ 0 & 0 & 2 & -4 & 0 & 2 \end{array}\right) \quad \begin{array}{l} 3R_1 - R_2 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 6 & 0 & -2 & 6 & -2 & 0 \\ 0 & -3 & 5 & -3 & 2 & 0 \\ 0 & 0 & 2 & -4 & 0 & 2 \end{array}\right) \quad \begin{array}{l} R_1 + R_3 \\ 2R_2 - 5R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 6 & 0 & 0 & 2 & -2 & 2 \\ 0 & -6 & 0 & 14 & 4 & -10 \\ 0 & 0 & 2 & -4 & 0 & 2 \end{array}\right) \quad \begin{array}{l} \frac{1}{6}R_1 \\ -\frac{1}{6}R_2 \\ \frac{1}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{7}{6} & -\frac{2}{3} & \frac{5}{3} \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{7}{6} & -\frac{2}{3} & \frac{5}{3} \\ -2 & 0 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{7}{6} & -\frac{2}{3} & \frac{5}{3} \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$\therefore$ ***Solution***: $(0,\ 1,\ 2)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

Solution

$$A=\begin{pmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad B=\begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2-2R_1 \\ R_3+R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 3 & -2 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{array}\right) \quad \frac{1}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1+2R_2 \\ \\ R_3+R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & 1 \end{array}\right) \quad \begin{array}{l} R_1-R_3 \\ R_2-R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & -1 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & 1 \end{array}\right)$$

$$A^{-1}=\begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & -1 \\ -1 & 0 & -1 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}$$

$$X=\begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & -1 \\ -1 & 0 & -1 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ 6 \end{pmatrix}$$

$\therefore$ ***Solution***: $(-4,\ -6,\ 6)$