

Solution ***Section 2.6 – Chain Rule***

Exercise

Find the derivative of $y = (3x^4 + 1)^4 (x^3 + 4)$

Solution

$$\begin{aligned} y' &= 4(12x^3)(3x^4 + 1)^3 (x^3 + 4) + 3x^2(3x^4 + 1)^4 \\ &= 48x^3(3x^4 + 1)^3 (x^3 + 4) + 3x^2(3x^4 + 1)^4 \\ &= 3x^2(3x^4 + 1)^3 [16x(x^3 + 4) + 3x^4 + 1] \\ &= 3x^2(3x^4 + 1)^3 [16x^4 + 64x + 3x^4 + 1] \\ &= \underline{3x^2(3x^4 + 1)^3 [19x^4 + 64x + 1]} \end{aligned}$$

Exercise

Find the derivative of $p(t) = \frac{(2t + 3)^3}{4t^2 - 1}$

Solution

$$\begin{aligned} P'(x) &= \frac{2(3)(2t + 3)^2(4t^2 - 1) - 8t(2t + 3)^3}{(4t^2 - 1)^2} \\ &= \frac{(2t + 3)^2 [6(4t^2 - 1) - 8t(2t + 3)]}{(4t^2 - 1)^2} \\ &= \frac{(2t + 3)^2 [24t^2 - 6 - 16t^2 - 24t]}{(4t^2 - 1)^2} \\ &= \frac{(2t + 3)^2 (8t^2 - 24t - 6)}{(4t^2 - 1)^2} \\ &= \underline{\underline{\frac{2(2t + 3)^2 (4t^2 - 12t - 3)}{(4t^2 - 1)^2}}} \end{aligned}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

Exercise

Find the derivative of $y = (x^3 + 1)^2$

Solution

$$u = x^3 + 1 \rightarrow y = u^2$$

$$\frac{d}{dx} y = \frac{dy}{du} \frac{du}{dx}$$

$$= 2u(3x^2)$$

$$y' = 2(x^3 + 1)(3x^2)$$
$$\underline{= 6x^2(x^3 + 1)}$$

Exercise

Find the derivative of $y = (x^2 + 3x)^4$

Solution

$$u = x^2 + 3x$$

$$y' = n(u)^{n-1} \frac{d}{dx}[u]$$

$$= 4(x^2 + 3x)^3 \frac{d}{dx}[x^2 + 3x]$$

$$\underline{= 4(2x + 3)(x^2 + 3x)^3}$$

Exercise

Find the derivative of $y = \frac{4}{2x+1}$

Solution

$$y = 4(2x+1)^{-1}$$

$$y' = -4(2x+1)^{-2}(2)$$

$$= -8(2x+1)^{-2}$$

$$\underline{= -\frac{8}{(2x+1)^2}}$$

Exercise

Find the derivative of $y = \frac{2}{(x-1)^3}$

Solution

$$y = 2(x-1)^{-3}$$

$$y' = 2(-3)(x-1)^{-4} (1)$$

$$= -\frac{6}{(x-1)^4} \Bigg|$$

Exercise

Find the derivative of $y = x^2 \sqrt{x^2 + 1}$

Solution

$$y = x^2 (x^2 + 1)^{1/2}$$

$$y' = x^2 \frac{d}{dx} [(x^2 + 1)^{1/2}] + (x^2 + 1)^{1/2} \frac{d}{dx} [x^2]$$

$$= x^2 \left[\frac{1}{2} (x^2 + 1)^{-1/2} (2x) \right] + (x^2 + 1)^{1/2} [2x]$$

$$= x^3 (x^2 + 1)^{-1/2} + 2x (x^2 + 1)^{1/2}$$

$$= \frac{(x^2 + 1)^{1/2}}{(x^2 + 1)^{1/2}} \left[x^3 (x^2 + 1)^{-1/2} + 2x (x^2 + 1)^{1/2} \right]$$

$$= \frac{x^3 (x^2 + 1)^{-1/2} (x^2 + 1)^{1/2} + 2x (x^2 + 1)^{1/2} (x^2 + 1)^{1/2}}{(x^2 + 1)^{1/2}}$$

$$= \frac{x^3 + 2x(x^2 + 1)}{(x^2 + 1)^{1/2}}$$

$$= \frac{x^3 + 2x^3 + 2x}{\sqrt{x^2 + 1}}$$

$$= \frac{3x^3 + 2x}{\sqrt{x^2 + 1}}$$

$$= \frac{x(3x^2 + 2)}{\sqrt{x^2 + 1}} \Bigg|$$

Exercise

Find the derivative of $y = \left(\frac{x+1}{x-5}\right)^2$

Solution

$$\begin{aligned} y' &= 2\left(\frac{x+1}{x-5}\right) \frac{-5-1}{(x-5)^2} & \left(\frac{ax+b}{cx+d}\right)' &= \frac{ad-bc}{(cx+d)^2} \\ &= -\frac{12(x+1)}{(x-5)^3} \end{aligned} \quad \left| \quad \begin{aligned} y' &= 2\left(\frac{x+1}{x-5}\right) \frac{d}{dx} \left[\frac{x+1}{x-5}\right] \\ &= 2\left(\frac{x+1}{x-5}\right) \left[\frac{(1)(x-5) - (1)(x+1)}{(x-5)^2} \right] \\ &= 2\left(\frac{x+1}{x-5}\right) \left(\frac{x-5-x-1}{(x-5)^2} \right) \\ &= 2\left(\frac{x+1}{x-5}\right) \left(\frac{-6}{(x-5)^2} \right) \\ &= -\frac{12(x+1)}{(x-5)^3} \end{aligned} \right|$$

Exercise

Find the derivative of $s(t) = \sqrt{2t^2 + 5t + 2}$

Solution

$$\begin{aligned} s(t) &= (2t^2 + 5t + 2)^{1/2} & U &= 2t^2 + 5t + 2 \rightarrow U' = 4t + 5 \\ s'(t) &= \frac{1}{2}(4t + 5)(2t^2 + 5t + 2)^{-1/2} & (U^n)' &= nU'U^{n-1} \\ &= \frac{1}{2} \frac{4t + 5}{\sqrt{2t^2 + 5t + 2}} \end{aligned}$$

Exercise

Find the derivative of $f(x) = \frac{1}{(x^2 - 3x)^2}$

Solution

$$\begin{aligned} f(x) &= (x^2 - 3x)^{-2} \\ f'(x) &= -2(2x - 3)(x^2 - 3x)^{-3} \end{aligned}$$

$$= -\frac{2(2x-3)}{(x^2-3x)^3}$$

Exercise

Find the derivative of $y = t^2\sqrt{t-2}$

Solution

$$\begin{aligned} f &= t^2 & f' &= 2t \\ g &= (t-2)^{1/2} & g' &= \frac{1}{2}(t-2)^{-1/2} \\ y' &= 2t\sqrt{t-2} + t^2 \frac{1}{2}(t-2)^{-1/2} \\ &= \left[2t(t-2)^{1/2} + t^2 \frac{1}{2}(t-2)^{-1/2} \right] \frac{2(t-2)^{1/2}}{2(t-2)^{1/2}} \\ &= \frac{4t(t-2) + t^2}{2(t-2)^{1/2}} \\ &= \frac{4t^2 - 8t + t^2}{2\sqrt{t-2}} \\ &= \frac{5t^2 - 8t}{2\sqrt{t-2}} \end{aligned}$$

Exercise

Find the derivative of $y = \left(\frac{6-5x}{x^2-1} \right)^2$

Solution

$$\begin{aligned} y' &= 2 \frac{-5(x^2-1) - (2x)(6-5x)}{(x^2-1)^2} \left(\frac{6-5x}{x^2-1} \right) & (U^n)' &= nU'U^{n-1} & f &= 6-5x & f' &= -5 \\ & & & & g &= x^2-1 & g' &= 2x \\ &= 2 \frac{-5x^2 + 5 - 12x + 10x^2}{(x^2-1)^3} (6-5x) \\ &= \frac{2(5x^2 - 12x + 5)(6-5x)}{(x^2-1)^3} \end{aligned}$$

Exercise

Find the derivative of $y = 4x(3x + 5)^5$

Solution

$$\begin{aligned}y' &= 4(3x + 5)^5 + 5(3)(3x + 5)^4(4x) \\&= 4(3x + 5)^5 + 60x(3x + 5)^4 \\&= 4(3x + 5)^4(3x + 5 + 15x) \\&= \underline{4(3x + 5)^4(18x + 5)}\end{aligned}$$

Exercise

Find the derivative of $y = (3x^2 - 5x)^{1/2}$

Solution

$$\begin{aligned}u &= 3x^2 - 5x \quad \& \quad y = u^{1/2} \\ \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\&= \frac{1}{2}u^{-1/2}(6x - 5) \\&= \frac{1}{2}(6x - 5)(3x^2 - 5x)^{-1/2} \\&= \underline{\frac{6x - 5}{2(3x^2 - 5x)^{1/2}}}\end{aligned}$$

Exercise

Find the derivative of $D_x(x^2 + 5x)^8$

Solution

$$\begin{aligned}D_x(x^2 + 5x)^8 &= 8(x^2 + 5x)^7(x^2 + 5x)' \\&= 8(x^2 + 5x)^7(2x + 5) \\&= \underline{8(2x + 5)(x^2 + 5x)^7}\end{aligned}$$

Exercise

Find the derivative of $y = \frac{(3x+2)^7}{x-1}$

Solution

$$\begin{aligned} y' &= \frac{7(3)(3x+2)^6(x-1) - (1)(3x+2)^7}{(x-1)^2} \\ &= \frac{(3x+2)^6(21x-21-3x-2)}{(x-1)^2} \\ &= \frac{(3x+2)^6(18x-23)}{(x-1)^2} \end{aligned}$$

Exercise

Find the derivative of $y = \left(\frac{x^2}{8} + x - \frac{1}{x} \right)^4$

Solution

$$\begin{aligned} y' &= 4 \left(\frac{x^2}{8} + x - \frac{1}{x} \right)^3 \left(\frac{2x}{8} + 1 - \frac{-1}{x^2} \right) \\ &= 4 \left(\frac{x^2}{8} + x - \frac{1}{x} \right)^3 \left(\frac{x}{4} + 1 + \frac{1}{x^2} \right) \\ &= \left(x + 4 + \frac{4}{x^2} \right) \left(\frac{x^2}{8} + x - \frac{1}{x} \right)^3 \end{aligned}$$

Exercise

Find the derivative of $y = \sqrt{3x^2 - 4x + 6}$

Solution

$$y = (3x^2 - 4x + 6)^{1/2} = u^{1/2}$$

$$u = 3x^2 - 4x + 6 \Rightarrow u' = 6x - 4$$

$$y' = \frac{1}{2} u^{1/2} u'$$

$$= \frac{1}{2} (3x^2 - 4x + 6)^{-1/2} 2(3x - 4)$$

$$= \frac{3x-2}{\sqrt{3x^2-4x+6}} \Big|$$

Exercise

Find the derivative of $y = \cot\left(\pi - \frac{1}{x}\right)$

Solution

$$u = \pi - \frac{1}{x} \rightarrow u' = \frac{1}{x^2}$$

$$\begin{aligned} y' &= -\csc^2\left(\pi - \frac{1}{x}\right)\left(\frac{1}{x^2}\right) \\ &= -\frac{1}{x^2} \csc^2\left(\pi - \frac{1}{x}\right) \Big| \end{aligned}$$

Exercise

Find the derivative of $y = 5\cos^{-4}x$

Solution

$$y = 5\cos^{-4}x \quad u = \cos x \rightarrow u' = -\sin x$$

$$\begin{aligned} y' &= 5u^{-5}u' \\ &= 5(-4)\cos^{-5}x(-\sin x) \\ &= 20\sin x \cos^{-5}x \Big| \end{aligned}$$

Exercise

Find the derivative of $y = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right)$

Solution

$$\begin{aligned} y' &= \frac{3\pi}{2}\cos\left(\frac{3\pi t}{2}\right) + \frac{3\pi}{2}\left(-\cos\left(\frac{3\pi t}{2}\right)\right) \\ &= \frac{3\pi}{2}\cos\left(\frac{3\pi t}{2}\right) - \frac{3\pi}{2}\cos\left(\frac{3\pi t}{2}\right) \\ &= \frac{3\pi}{2}\left(\cos\left(\frac{3\pi t}{2}\right) - \cos\left(\frac{3\pi t}{2}\right)\right) \Big| \end{aligned}$$

Exercise

Find the derivative of $r = 6(\sec \theta - \tan \theta)^{3/2}$

Solution

$$r = 6(\sec \theta - \tan \theta)^{3/2} = 6u^{3/2} \Rightarrow u = \sec \theta - \tan \theta \rightarrow u' = \sec \theta \tan \theta - \sec^2 \theta$$

$$\Rightarrow u = \sec \theta - \tan \theta \rightarrow u' = \sec \theta \tan \theta - \sec^2 \theta$$

$$r' = 6\left(\frac{3}{2}\right)(\sec \theta - \tan \theta)^{3/2-1}(\sec \theta \tan \theta - \sec^2 \theta)$$

$$= 9(\sec \theta - \tan \theta)^{1/2}(\sec \theta \tan \theta - \sec^2 \theta)$$

$$= 9(\sec \theta \tan \theta - \sec^2 \theta) \sqrt{\sec \theta - \tan \theta}$$

Exercise

Find the derivative of $g(x) = \frac{\tan 3x}{(x+7)^4}$

Solution

$$g'(x) = \frac{(3\sec^2 3x)(x+7)^4 - 4(x+7)^3 \tan 3x}{(x+7)^8}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$u = \tan 3x \quad v = (x+7)^4$$

$$u' = 3\sec^2 3x \quad v' = 4(x+7)^3$$

$$= \frac{(x+7)^3 [3(x+7)\sec^2 3x - 4 \tan 3x]}{(x+7)^8}$$

$$= \frac{3(x+7)\sec^2 3x - 4 \tan 3x}{(x+7)^5}$$

Exercise

Find the derivative of $f(\theta) = \left(\frac{\sin \theta}{1 + \cos \theta}\right)^2$

Solution

$$f'(\theta) = 2\left(\frac{\sin \theta}{1 + \cos \theta}\right)\left(\frac{\sin \theta}{1 + \cos \theta}\right)'$$

$$= \frac{2 \sin \theta}{1 + \cos \theta} \left(\frac{\cos \theta (1 + \cos \theta) - (-\sin \theta) \sin \theta}{(1 + \cos \theta)^2} \right)$$

$$\begin{aligned}
&= \frac{2 \sin \theta}{1 + \cos \theta} \left(\frac{\cos \theta + \cos^2 \theta + \sin^2 \theta}{(1 + \cos \theta)^2} \right) \\
&= \frac{2 \sin \theta}{1 + \cos \theta} \left(\frac{\cos \theta + 1}{(1 + \cos \theta)^2} \right) \\
&= \frac{2 \sin \theta}{(1 + \cos \theta)^2} \quad |
\end{aligned}$$

Exercise

Find the derivative of $y = \sin^2(\pi t - 2)$

Solution

$$\begin{aligned}
y' &= 2 \sin(\pi t - 2) (\sin(\pi t - 2))' \\
&= 2 \sin(\pi t - 2) (\pi \cos(\pi t - 2)) \\
&= 2\pi \sin(\pi t - 2) \cos(\pi t - 2) \quad |
\end{aligned}$$

Exercise

Find the derivative of $y = (t \tan t)^{10}$

Solution

$$\begin{aligned}
y' &= 10(t \tan t)^9 (t \tan t)' \\
&= 10(t \tan t)^9 (\tan t + t \sec^2 t) \\
&= 10(t \tan t)^9 \tan t + 10t(t \tan t)^9 \sec^2 t \\
&= 10t^9 \tan^{10} t + 10t^{10} \tan^9 t \sec^2 t \quad |
\end{aligned}$$

Exercise

Find the derivative of $y = \cos\left(5 \sin\left(\frac{t}{3}\right)\right)$

Solution

$$\begin{aligned}
y' &= -\sin\left(5 \sin\left(\frac{t}{3}\right)\right) \left(5 \sin\left(\frac{t}{3}\right)\right)' \\
&= -\sin\left(5 \sin\left(\frac{t}{3}\right)\right) \left(5 \frac{1}{3} \cos\left(\frac{t}{3}\right)\right)
\end{aligned}$$

$$= -\frac{5}{3} \sin\left(5 \sin\left(\frac{t}{3}\right)\right) \cos\left(\frac{t}{3}\right) \Bigg|$$

Exercise

Find the derivative of $y = 4 \sin\left(\sqrt{1+\sqrt{t}}\right)$

Solution

$$\begin{aligned} y' &= 4 \cos\left(\sqrt{1+\sqrt{t}}\right) \left(\sqrt{1+\sqrt{t}}\right)' \\ &= \frac{1}{2} (1+\sqrt{t})^{-1/2} (t^{1/2})' \\ &= \frac{1}{2} (1+\sqrt{t})^{-1/2} \left(\frac{1}{2} t^{-1/2}\right) \\ &= \frac{1}{4} \frac{1}{\sqrt{t} \sqrt{1+\sqrt{t}}} \\ &= \frac{1}{4} \frac{1}{\sqrt{t(1+\sqrt{t})}} \end{aligned}$$

$$\begin{aligned} y' &= 4 \cos\left(\sqrt{1+\sqrt{t}}\right) \left(\frac{1}{4} \frac{1}{\sqrt{t+t\sqrt{t}}}\right) \\ &= \frac{\cos\left(\sqrt{1+\sqrt{t}}\right)}{\sqrt{t+t\sqrt{t}}} \Bigg| \end{aligned}$$

Exercise

Find the derivative of $y = \tan^2\left(\sin^3 x\right)$

Solution

$$\begin{aligned} u &= \sin^3 x \Rightarrow u' = 3 \sin^2 x (\sin x)' = 3 \sin^2 x (\cos x) \\ y' &= 2 \tan\left(\sin^3 x\right) \cdot \left(\tan\left(\sin^3 x\right)\right)' \\ &= 2 \tan\left(\sin^3 x\right) \cdot \sec^2\left(\sin^3 x\right) \cdot \left(\sin^3 x\right)' \\ &= 2 \tan\left(\sin^3 x\right) \cdot \sec^2\left(\sin^3 x\right) \cdot \left(3 \sin^2 x \cos x\right) \\ &= 6 \cos x \sin^2 x \cdot \tan\left(\sin^3 x\right) \cdot \sec^2\left(\sin^3 x\right) \Bigg| \end{aligned}$$

Exercise

Find the derivative of $f(x) = \left((x^2 + 3)^5 + x \right)^2$

Solution

$$\underline{f'(x) = 2 \left((x^2 + 3)^5 + x \right) \left(10x(x^2 + 3)^4 + 1 \right)}$$

Exercise

Find the derivative of $y = \left(\frac{3x-1}{x^2+3} \right)^2$

Solution

$$y = (3x-1)^2 (x^2+3)^{-2}$$

$$(U^m V^n)' = U^{m-1} V^{n-1} (mUV' + nUV')$$

$$\begin{aligned} y' &= (3x-1) (x^2+3)^{-3} (6x(x^2+3) - 4x(3x-1)) \\ &= \frac{3x-1}{(x^2+3)^3} (6x^3 + 18x - 12x^2 + 4x) \\ &= \frac{(3x-1)(6x^3 - 12x^2 + 22x)}{(x^2+3)^3} \end{aligned}$$

Exercise

Find the derivative of $y = \cos \sqrt{\sin(\tan \pi x)}$

Solution

$$\begin{aligned} y' &= -\left(\sin \sqrt{\sin(\tan \pi x)} \right) \left(\frac{1}{2} \frac{\pi \cos(\tan \pi x) \sec^2 \pi x}{\sqrt{\sin(\tan \pi x)}} \right) \\ &= -\frac{\pi \sec^2 \pi x \cos(\tan \pi x) \sin \sqrt{\sin(\tan \pi x)}}{2 \sqrt{\sin(\tan \pi x)}} \end{aligned}$$

Exercise

Find the derivative of $f(x) = \frac{x}{\sqrt{x^2+1}}$

Solution

$$f(x) = x(x^2 + 1)^{-1/2}$$

$$f'(x) = \frac{x^2 + 1 - \frac{1}{2}(2x^2)}{x^2 + 1}$$

$$= \frac{1}{x^2 + 1}$$

$$(U^m V^n)' = U^{m-1} V^{n-1} (mU'V + nUV')$$

Exercise

Find the derivative of $y = \cos(1 - 2x)^2$

Solution

$$y' = -(2(-2)(1 - 2x)) \sin(1 - 2x)^2$$

$$= 4(1 - 2x) \sin(1 - 2x)^2$$

Exercise

Find the derivative of $f(x) = (4x - 3)^2$

Solution

$$f'(x) = 8(4x - 3)$$

Exercise

Find the derivative of $f(x) = \frac{x}{\sqrt[3]{x^2 + 4}}$

Solution

$$f(x) = x(x^2 + 4)^{-1/3}$$

$$f'(x) = (x^2 + 4)^{-4/3} \left(x^2 + 4 - \frac{1}{3}(2x^2) \right)$$

$$= \frac{1}{3} \frac{x^2 + 12}{(x^2 + 4)^{4/3}}$$

$$(U^m V^n)' = U^{m-1} V^{n-1} (mU'V + nUV')$$

Exercise

Find the derivative of $f(x) = \left(\frac{x^2}{x^3 + 2}\right)^2$

Solution

$$f(x) = x^4 (x^3 + 2)^{-2}$$

$$f'(x) = x^3 (x^3 + 2)^{-3} (4x^3 + 8 - 2x^3)$$

$$= \frac{x^3 (2x^3 + 8)}{(x^3 + 2)^3}$$

$$(U^m V^n)' = U^{m-1} V^{n-1} (mUV' + nUV')$$

Exercise

Find the derivative of $y = \sin \sqrt[3]{x} + \sqrt[3]{\sin x}$

Solution

$$y' = \frac{1}{3} x^{-2/3} \cos \sqrt[3]{x} + \frac{1}{3} \cos x (\sin x)^{-2/3}$$

Exercise

Find the derivative of $f(\theta) = 4 \tan(\theta^2 + 3\theta + 2)$

Solution

$$f'(\theta) = 4(2\theta + 3) \sec^2(\theta^2 + 3\theta + 2)$$

Exercise

Find the derivative of $f(\theta) = \tan(\sin \theta)$

Solution

$$f'(\theta) = \cos \theta \sec^2(\sin \theta)$$

Exercise

Find the derivative of $y = 5x + \sin^3 x + \sin x^3$

Solution

$$y' = 5 + 3 \cos x \sin^2 x + 3x^2 \cos x^3$$

Exercise

Find the derivative of $y = \csc^5 3x$

Solution

$$y' = 15 \csc^4 3x (-\csc 3x \cot 3x) \\ = -15 \cot 3x \csc^5 3x$$

Exercise

Find the derivative of $y = 2x\sqrt{x^2 - 2x + 2}$

Solution

$$y' = 2\sqrt{x^2 - 2x + 2} + 2x(2x - 2)(x^2 - 2x + 2)^{-1/2} \\ = 2\sqrt{x^2 - 2x + 2} + \frac{4x^2 - 4x}{\sqrt{x^2 - 2x + 2}}$$

Exercise

Find the derivative of $\frac{d}{du} \left(\frac{4u^2 + u}{8u + 1} \right)^3$

Solution

$$(U^n)' = nU' U^{n-1} \quad \frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

$$\frac{d}{du} \left(\frac{4u^2 + u}{8u + 1} \right)^3 = 3 \left(\frac{4u^2 + u}{8u + 1} \right)^2 \frac{\begin{vmatrix} 4 & 1 \\ 0 & 8 \end{vmatrix} u^2 + \begin{vmatrix} 4 & 0 \\ 0 & 1 \end{vmatrix} u + \begin{vmatrix} 1 & 0 \\ 8 & 1 \end{vmatrix}}{(8u + 1)^2} \\ = 3(32u^2 + 4u + 1) \frac{(4u^2 + u)^2}{(8u + 1)^4}$$

Exercise

Find the derivative of $y = \frac{1}{2}x^2\sqrt{16-x^2}$

Solution

$$y = \frac{1}{2}x^2(16-x^2)^{1/2}$$

$$y' = \frac{1}{2}x(16-x^2)^{-1/2}(32-2x^2-x^2) \\ = \frac{1}{2} \frac{32x-3x^3}{\sqrt{16-x^2}}$$

$$(U^m V^n)' = U^{m-1} V^{n-1} (mU'V + nUV')$$

Exercise

Find the derivative of $y = \left(\frac{x-3}{2x+5}\right)^4$

Solution

$$y' = 4 \frac{5+6}{(2x+5)^2} \left(\frac{x-3}{2x+5}\right)^3 \\ = \frac{44(x-3)^3}{(2x+5)^5}$$

$$(U^n)' = nU' U^{n-1} \quad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \left(\frac{5x-3}{2x+5}\right)^5$

Solution

$$y' = 5 \frac{25+6}{(2x+5)^2} \left(\frac{5x-3}{2x+5}\right)^4 \\ = \frac{155(5x-3)^4}{(2x+5)^6}$$

$$(U^n)' = nU' U^{n-1} \quad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y = \left(\frac{6x-8}{2x-3}\right)^6$

Solution

$$y' = 6 \frac{-18+16}{(2x-3)^2} \left(\frac{6x-8}{2x-3}\right)^5$$

$$(U^n)' = nU' U^{n-1} \quad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

$$= -\frac{12(6x-8)^5}{(2x-3)^7}$$

Exercise

Find the derivative of $y = \left(\frac{3x^2-4}{2x^2-1} \right)^3$

Solution

$$y' = 3 \frac{2(-3+8)x}{(2x^2-1)^2} \left(\frac{3x^2-4}{2x^2-1} \right)^2$$

$$\left(U^n \right)' = nU' U^{n-1} \quad \left(\frac{ax^n+b}{cx^n+d} \right)' = \frac{n(ad-bc)x^{n-1}}{(cx^n+d)^2}$$

$$= \frac{30x(3x^2-4)^2}{(2x^2-1)^4}$$

Exercise

Find the derivative of $y = \left(\frac{3x^2+4}{2x^2+1} \right)^{-3}$

Solution

$$y' = (-3) \frac{2(3-8)x}{(2x^2+1)^2} \left(\frac{3x^2+4}{2x^2+1} \right)^{-4}$$

$$\left(U^n \right)' = nU' U^{n-1} \quad \left(\frac{ax^n+b}{cx^n+d} \right)' = \frac{n(ad-bc)x^{n-1}}{(cx^n+d)^2}$$

$$= \frac{15x}{(2x^2+1)^2} \left(\frac{2x^2+1}{3x^2+4} \right)^4$$

$$= \frac{15x(2x^2+1)^2}{(3x^2+4)^4}$$

Exercise

Find the derivative of $y = \left(\frac{2x^2-3}{x^2+1} \right)^{1/3}$

Solution

$$\begin{aligned}
 y' &= \frac{1}{3} \frac{2(2+3)x}{(x^2+1)^2} \left(\frac{2x^2-3}{x^2+1} \right)^{-2/3} \\
 &= \frac{10}{3} \frac{x}{(x^2+1)^2} \left(\frac{x^2+1}{2x^2-3} \right)^{2/3} \\
 &= \frac{10}{3} \frac{x}{(x^2+1)^{4/3} (2x^2-3)^{2/3}}
 \end{aligned}$$

$$(U^n)' = nU' U^{n-1} \quad \left(\frac{ax^n+b}{cx^n+d} \right)' = \frac{n(ad-bc)x^{n-1}}{(cx^n+d)^2}$$

Exercise

Find the derivative of $y = \sqrt{\frac{2x^3-3}{2x^3+1}}$

Solution

$$\begin{aligned}
 y' &= \frac{1}{2} \frac{3(2+6)x^2}{(x^3+1)^2} \left(\frac{2x^3-3}{x^3+1} \right)^{-1/2} \\
 &= \frac{12x^2}{(x^3+1)^2} \left(\frac{x^3+1}{2x^3-3} \right)^{1/2} \\
 &= \frac{12x^2}{(x^3+1)^{3/2} \sqrt{2x^3-3}}
 \end{aligned}$$

$$(U^n)' = nU' U^{n-1} \quad \left(\frac{ax^n+b}{cx^n+d} \right)' = \frac{n(ad-bc)x^{n-1}}{(cx^n+d)^2}$$

Exercise

Find the derivative of $y = \left(\frac{2x^4-3}{2x^4+1} \right)^5$

Solution

$$\begin{aligned}
 y' &= 5 \frac{4(2+6)x^3}{(2x^4+1)^2} \left(\frac{2x^4-3}{2x^4+1} \right)^4 \\
 &= \frac{160x^3(2x^4-3)^4}{(2x^4+1)^6}
 \end{aligned}$$

$$(U^n)' = nU' U^{n-1} \quad \left(\frac{ax^n+b}{cx^n+d} \right)' = \frac{n(ad-bc)x^{n-1}}{(cx^n+d)^2}$$

Exercise

Find the derivative of $y = \left(\frac{x^2 - 4x + 1}{5x^2 - 2x - 1} \right)^3$

Solution

$$(U^n)' = nU' U^{n-1} \quad \frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

$$y' = (3) \frac{\begin{vmatrix} 1 & -4 \\ 5 & -2 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} x + \begin{vmatrix} -4 & 1 \\ -2 & -1 \end{vmatrix}}{(5x^2 - 2x - 1)^2} \left(\frac{x^2 - 4x + 1}{5x^2 - 2x - 1} \right)^2$$

$$= \frac{(18x^2 - 12x + 6)(x^2 - 4x + 1)^2}{(5x^2 - 2x - 1)^4}$$

Exercise

Find the derivative of $y = \left(\frac{3x^2 - 4x + 2}{2x^2 + x - 1} \right)^{2/3}$

Solution

$$(U^n)' = nU' U^{n-1} \quad \frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2}$$

$$y' = \frac{2}{3} \frac{\begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} x^2 + 2 \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} x + \begin{vmatrix} -4 & 2 \\ 1 & -1 \end{vmatrix}}{(2x^2 + x - 1)^2} \left(\frac{3x^2 - 4x + 2}{2x^2 + x - 1} \right)^{-1/3}$$

$$= \frac{2}{3} \frac{11x^2 - 14x + 6}{(2x^2 + x - 1)^2} \left(\frac{2x^2 + x - 1}{3x^2 - 4x + 2} \right)^{1/3}$$

$$= \frac{2}{3} \frac{11x^2 - 14x + 6}{(2x^2 + x - 1)^{5/3} (3x^2 - 4x + 2)^{1/3}}$$

Exercise

Find the derivative of $f(x) = \left(\frac{3t^2-1}{3t^2+1}\right)^{-3}$

Solution

$$f(x) = \left(\frac{3t^2+1}{3t^2-1}\right)^3$$

$$f'(x) = 3 \frac{3(-3-3)t}{(3t^2-1)^2} \left(\frac{3t^2+1}{3t^2-1}\right)^2$$

$$= -\frac{6t(3t^2+1)^2}{(3t^2-1)^4}$$

$$(U^n)' = nU' U^{n-1} \quad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{(cx^n+d)^2}$$

Exercise

Find the derivative of $f(x) = \left(\frac{x}{3x^2+2x+1}\right)^{1/3}$

Solution

$$(U^n)' = nU' U^{n-1} \quad \frac{d}{dx} \left(\frac{ax^2+bx+c}{dx^2+ex+f} \right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2+ex+f)^2}$$

$$f'(x) = \frac{1}{3} \frac{\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} x^2 + 2 \begin{vmatrix} 0 & 0 \\ 3 & 1 \end{vmatrix} x + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}}{(3x^2+2x+1)^2} \left(\frac{x}{3x^2+2x+1}\right)^{-2/3}$$

$$= \frac{1}{3} \frac{-3x^2+1}{(3x^2+2x+1)^2} \left(\frac{3x^2+2x+1}{x}\right)^{2/3}$$

$$= \frac{-3x^2+1}{3x^{2/3}(3x^2+2x+1)^{4/3}}$$

Exercise

Find the derivative of $f(x) = (x^2+2x-3)^5 (2x+3)^6$

Solution

$$\left(U^m V^n\right)' = U^{m-1} V^{n-1} (mU'V + nUV')$$

$$\begin{aligned} f'(x) &= \left(x^2 + 2x - 3\right)^4 (2x + 3)^5 \left[5(2x + 2)(2x + 3) + 12(x^2 + 2x - 3) \right] \\ &= \left(x^2 + 2x - 3\right)^4 (2x + 3)^5 (20x^2 + 50x + 30 + 12x^2 + 24x - 36) \\ &= \left(x^2 + 2x - 3\right)^4 (2x + 3)^5 (32x^2 + 74x - 6) \end{aligned}$$

Exercise

Find the derivative of $f(x) = (2x^2 - 4x + 3)^4 (3x - 5)^5$

Solution

$$\left(U^m V^n\right)' = U^{m-1} V^{n-1} (mU'V + nUV')$$

$$\begin{aligned} f'(x) &= \left(2x^2 - 4x + 3\right)^3 (3x - 5)^4 \left[4(4x - 4)(3x - 5) + 15(2x^2 - 4x + 3) \right] \\ &= \left(2x^2 - 4x + 3\right)^3 (3x - 5)^4 (48x^2 - 128x + 80 + 30x^2 - 60x + 45) \\ &= \left(2x^2 - 4x + 3\right)^3 (3x - 5)^4 (88x^2 - 188x + 135) \end{aligned}$$

Exercise

Find the derivative of $f(x) = (x^2 + 2x - 3)^4 (x^2 + 3x + 5)^6$

Solution

$$\left(U^m V^n\right)' = U^{m-1} V^{n-1} (mU'V + nUV')$$

$$\begin{aligned} f'(x) &= \left(x^2 + 2x - 3\right)^3 (x^2 + 3x + 5)^5 \left[4(2x + 2)(x^2 + 3x + 5) + 6(2x + 3)(x^2 + 2x - 3) \right] \\ &= \left(x^2 + 2x - 3\right)^3 (x^2 + 3x + 5)^5 (8x^3 + 32x^2 + 64x + 40 + 12x^3 + 42x^2 - 54) \\ &= \left(x^2 + 2x - 3\right)^3 (x^2 + 3x + 5)^5 (20x^3 + 74x^2 + 64x - 14) \end{aligned}$$

Exercise

Find the derivative of $f(x) = (2x^3 - 5x)^3 (x^2 + 2x + 1)^4 (2x - 3)^5$

Solution

$$(U^m V^n W^p)' = U^{m-1} V^{n-1} W^{p-1} (mU'VW + nUV'W + pUVW')$$

$$\begin{aligned} f'(x) &= (2x^3 - 5x)^2 (x^2 + 2x + 1)^3 (2x - 3)^4 \left[3(6x^2 - 5)(x^2 + 2x + 1)(2x - 3) \right. \\ &\quad \left. + 4(2x + 2)(2x^3 - 5x)(2x - 3) + 5(2)(2x^3 - 5x)(x^2 + 2x + 1) \right] \\ &= (2x^3 - 5x)^2 (x^2 + 2x + 1)^3 (2x - 3)^4 \left[(18x^2 - 15)(2x^3 + x^2 - 4x - 3) \right. \\ &\quad \left. + (8x + 8)(4x^4 - 6x^3 - 10x^2 + 15x) + (20x^5 + 40x^4 - 20x^3 - 100x^2 - 50x) \right] \\ &= (2x^3 - 5x)^2 (x^2 + 2x + 1)^3 (2x - 3)^4 \end{aligned}$$

x^5	x^4	x^3	x^2	x	x^0
36	18	-72	-54	-60	45
32	-48	-30	-15	120	
20	32	-80	120	50	
	40	-48	-80		
		-20	-100		

$$= \underline{(2x^3 - 5x)^2 (x^2 + 2x + 1)^3 (2x - 3)^4 (88x^5 + 42x^4 - 250x^3 - 129x^2 + 110x + 45)}$$

Exercise

Find the derivative of $f(x) = (x^4 + 3x)^4 (x^3 + 2x)^5 (2x - 3)^6$

Solution

$$(U^m V^n W^p)' = U^{m-1} V^{n-1} W^{p-1} (mU'VW + nUV'W + pUVW')$$

$$\begin{aligned} f'(x) &= (x^4 + 3x)^3 (x^3 + 2x)^4 (2x - 3)^5 \\ &\quad \left[4(4x^3 + 3)(x^3 + 2x)(2x - 3) + 5(x^4 + 3x)(3x^2 + 2)(2x - 3) + 12(x^4 + 3x)(x^3 + 2x) \right] \end{aligned}$$

$$f'(x) = (x^4 + 3x)^3 (x^3 + 2x)^4 (2x - 3)^5 \left[(16x^3 + 9)(8x^4 - 9x^3 + 16x^2 - 18x) \right. \\ \left. + (5x^4 + 15x)(6x^3 - 9x^2 + 4x - 6) + (12x^4 + 36x)(x^3 + 2x) \right]$$

$$\begin{array}{ll} x^7 & 128 + 30 + 12 \\ x^6 & -144 - 45 \\ x^5 & 256 + 20 + 24 \\ x^4 & -288 + 72 - 30 + 90 + 36 \\ x^3 & -81 - 135 \\ x^2 & 144 + 60 + 72 \\ x^1 & -162 - 90 \end{array}$$

$$f'(x) = (x^4 + 3x)^3 (x^3 + 2x)^4 (2x - 3)^5 (170x^7 - 189x^6 + 300x^5 - 120x^4 - 216x^3 + 206x^2 - 252x)$$

Exercise

Find the derivative of $f(x) = \frac{(x^2 - 6x)^5}{(3x^2 + 5x - 2)^4}$

Solution

$$f(x) = (x^2 - 6x)^5 (3x^2 + 5x - 2)^{-4} \quad \left(U^m V^n \right)' = U^{m-1} V^{n-1} (mU'V + nUV')$$

$$\begin{aligned} f'(x) &= (x^2 - 6x)^4 (3x^2 + 5x - 2)^{-5} \left[5(2x - 6)(3x^2 + 5x - 2) - 4(x^2 - 6x)(6x + 5) \right] \\ &= (x^2 - 6x)^4 (3x^2 + 5x - 2)^{-5} \left[(10x - 30)(3x^2 + 5x - 2) - 4(6x^3 - 31x^2 - 30x) \right] \\ &= (x^2 - 6x)^4 (3x^2 + 5x - 2)^{-5} \end{aligned}$$

$$\begin{array}{ll} x^3 & 30 - 24 \\ x^2 & 50 - 90 + 124 \\ x & -20 - 150 + 120 \\ x^0 & 60 \end{array}$$

$$= \frac{(x^2 - 6x)^4 (6x^3 + 84x^2 - 50x + 60)}{(3x^2 + 5x - 2)^5}$$

Exercise

Find the derivative of $f(x) = \frac{(2x^2 + 3x + 1)^4}{(x^2 + 5x - 6)^5}$

Solution

$$f(x) = (2x^2 + 3x + 1)^4 (x^2 + 5x - 6)^{-5} \quad (U^m V^n)' = U^{m-1} V^{n-1} (mU'V + nUV')$$

$$f'(x) = (2x^2 + 3x + 1)^3 (x^2 + 5x - 6)^{-6} \left[4(4x + 3)(x^2 + 5x - 6) - 5(2x^2 + 3x + 1)(2x + 5) \right]$$

$$= \frac{(2x^2 + 3x + 1)^3}{(x^2 + 5x - 6)^6} \left[(16x + 12)(x^2 + 5x - 6) - (2x^2 + 3x + 1)(10x + 25) \right]$$

$$x^3 \quad 16x - 20$$

$$x^2 \quad 80x + 12 - 50x - 30$$

$$x \quad -96x + 60 - 75$$

$$x^0 \quad -7 - 25$$

$$f'(x) = \frac{(2x^2 + 3x + 1)^3}{(x^2 + 5x - 6)^6} (-4x^3 + 12x^2 - 111x - 32x)$$

Exercise

Find the derivative of $f(x) = \frac{(x^3 - 3x)^3 (x^2 + 4x)^4}{(x^2 + 4x + 1)^2}$

Solution

$$(U^m V^n W^p)' = U^{m-1} V^{n-1} W^{p-1} (mU'VW + nUV'W + pUVW')$$

$$f'(x) = \frac{(x^3 - 3x)^2 (x^2 + 4x)^3}{(x^2 + 4x + 1)^3} \left[3(3x^2 - 3)(x^2 + 4x)(x^2 + 4x + 1) + 3(x^3 - 3x)(2x + 4)(x^2 + 4x + 1) - 2(2x + 4)(x^3 - 3x)(x^2 + 4x) \right]$$

$$= \frac{(x^3 - 3x)^2 (x^2 + 4x)^3}{(x^2 + 4x + 1)^3} \left[(9x^2 - 9)(x^4 + 8x^3 + 17x^2 + 4x) + (3x^3 - 9x)(2x^3 + 12x^2 + 18x + 4) - (4x + 8)(x^5 + 4x^4 - 3x^3 - 12x^2) \right]$$

$$\begin{array}{rcl}
 x^6 & & 9 + 6 - 4 \\
 x^5 & & 72 + 36 - 16 - 16 - 8 \\
 x^4 & & 153 - 9 + 54 - 18 + 12 - 32 \\
 x^3 & & 36 - 72 + 12 - 108 + 48 + 24 \\
 x^2 & & -153 - 162 + 96 \\
 x^1 & & -36 - 36
 \end{array}$$

$$f'(x) = \frac{(x^3 - 3x)^2 (x^2 + 4x)^3}{(x^2 + 4x + 1)^3} (11x^6 + 68x^5 + 160x^4 - 60x^3 - 219x^2 - 72x)$$

Exercise

Find the derivative of $f(x) = \frac{x^2 + 3}{(2x - 1)^3 (3x + 1)^4}$

Solution

$$f(x) = (x^2 + 3)(2x - 1)^{-3}(3x + 1)^{-4} \quad (U^m V^n W^p)' = U^{m-1} V^{n-1} W^{p-1} (mU'VW + nUV'W + pUVW')$$

$$\begin{aligned}
 f'(x) &= (2x - 1)^{-4} (3x + 1)^{-5} \\
 &\quad \left[2x(2x - 1)(3x + 1) - 6(x^2 + 3)(3x + 1) - 12(x^2 + 3)(2x - 1) \right] \\
 &= \frac{1}{(2x - 1)^4 (3x + 1)^5} \left((4x^2 - 2x)(3x + 1) - 6(3x^3 + x^2 + 9x + 3) - 12(2x^3 - x^2 + 6x - 3) \right)
 \end{aligned}$$

$$\begin{array}{rcl}
 x^3 & & 12 - 18 - 24 \\
 x^2 & & 4 - 6 - 6 + 12 \\
 x & & -2 - 54 - 72 \\
 x^0 & & -18 + 36
 \end{array}$$

$$f'(x) = \frac{-30x^3 + 4x^2 - 128x + 18}{(2x - 1)^4 (3x + 1)^5}$$

Exercise

Find the derivative of $f(x) = \frac{(x^3 - 3x)^3 (x^2 + 4x)^4}{(x^2 + 4x + 1)^2}$

Solution

$$f(x) = (x^3 - 3x)^3 (x^2 + 4x)^4 (x^2 + 4x + 1)^{-2}$$

$$(U^m V^n W^p)' = U^{m-1} V^{n-1} W^{p-1} (mU'VW + nUV'W + pUVW')$$

$$f'(x) = (x^3 - 3x)^2 (x^2 + 4x)^3 (x^2 + 4x + 1)^{-3} \left[3(3x^2 - 3)(x^2 + 4x)(x^2 + 4x + 1) \right. \\ \left. + 4(x^3 - 3x)(2x + 4)(x^2 + 4x + 1) - 2(x^3 - 3x)(x^2 + 4x)(2x + 4) \right]$$

$$f'(x) = (x^3 - 3x)^2 (x^2 + 4x)^3 (x^2 + 4x + 1)^{-3} \left[(9x^2 - 9)(x^4 + 8x^3 + 9x^2 + 4x) \right. \\ \left. + (4x^3 - 12x)(2x^3 + 12x^2 + 18x + 4) + (-2x^3 + 6x)(2x^3 + 12x^2 + 16x) \right]$$

$$\begin{array}{rcl} x^6 & & 9+8-4 \\ x^5 & & 72+48-24 \\ x^4 & & 81-9+72-24-32+12 \\ x^3 & & 36-72+16-144+72 \\ x^2 & & -81-216+96 \\ x^1 & & -36-48 \end{array}$$

$$f'(x) = \frac{(13x^6 + 96x^5 + 100x^4 - 92x^3 - 201x^2 - 84x)(x^3 - 3x)^2 (x^2 + 4x)^3}{(x^2 + 4x + 1)^3}$$

Exercise

Find the **second** derivative $y = \frac{x^2 + 3}{(x-1)^3 + (x+1)^3}$

Solution

$$(x-1)^3 + (x+1)^3 = x^3 - 3x^2 + 3x - 1 + x^3 + 3x^2 + 3x + 1 \\ = 2x^3 + 6x$$

$$y = \frac{x^2 + 3}{2x^3 + 6x}$$

$$u = x^2 + 3 \quad v = 2x^3 + 6x \\ u' = 2x \quad v' = 6x^2 + 6$$

$$y' = \frac{4x^4 + 12x^2 - 6x^4 - 18x^2 - 6x^2 - 18}{(2x^3 + 6x)^2}$$

$$= \frac{-2x^4 - 12x^2 - 18}{(2x^3 + 6x)^2}$$

$$= -2 \frac{x^4 + 6x^2 + 9}{(2x^3 + 6x)^2} \quad \Bigg|$$

$$u = x^4 + 6x^2 + 9 \quad v = (2x^3 + 6x)^2$$

$$\begin{aligned} u' &= 4x^3 + 12x & v' &= 2(2x^3 + 6x)(6x^2 + 6) \\ &= 4x(x^2 + 3) \end{aligned}$$

$$y'' = -2 \frac{4x(x^2 + 3)(2x^3 + 6x)^2 - 2(2x^3 + 6x)(6x^2 + 6)(x^4 + 6x^2 + 9)}{(2x^3 + 6x)^4}$$

$$= -4(2x^3 + 6x) \frac{2x(2x^5 + 6x^3 + 6x^3 + 18x) - (6x^6 + 36x^4 + 54x^2 + x^4 + 36x^2 + 54)}{(2x^3 + 6x)^4}$$

$$= -4 \frac{4x^5 + 24x^3 + 36x^2 - 6x^6 - 37x^4 - 90x^2 - 54}{(2x^3 + 6x)^3}$$

$$= -4 \frac{-6x^6 + 4x^5 - 37x^4 + 24x^3 - 54x^2 - 54}{(2x^3 + 6x)^3} \quad \Bigg|$$

Exercise

Find the **second** derivative of $y = \left(1 + \frac{1}{x}\right)^3$

Solution

$$y' = 3\left(1 + \frac{1}{x}\right)^2 \left(1 + \frac{1}{x}\right)'$$

$$= 3\left(1 + \frac{1}{x}\right)^2 \left(-\frac{1}{x^2}\right)$$

$$= -\frac{3}{x^2} \left(1 + \frac{1}{x}\right)^2 \quad \Bigg|$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$\begin{aligned}
y'' &= \left(-\frac{3}{x^2} \right)' \left(1 + \frac{1}{x} \right)^2 + \left(-\frac{3}{x^2} \right) \left(\left(1 + \frac{1}{x} \right)^2 \right)' \\
&= \left(-\frac{-3(2x)}{x^4} \right) \left(1 + \frac{1}{x} \right)^2 + \left(-\frac{3}{x^2} \right) \left(2 \left(1 + \frac{1}{x} \right) \left(-\frac{1}{x^2} \right) \right) \\
&= \frac{6}{x^3} \left(1 + \frac{1}{x} \right)^2 + \frac{6}{x^4} \left(1 + \frac{1}{x} \right) \\
&= \frac{6}{x^3} \left(1 + \frac{1}{x} \right) \left(1 + \frac{1}{x} + \frac{1}{x} \right) \\
&= \frac{6}{x^3} \left(1 + \frac{1}{x} \right) \left(1 + \frac{2}{x} \right) \quad \boxed{}
\end{aligned}$$

Exercise

Find the **second** derivative of $y = 9 \tan\left(\frac{x}{3}\right)$

Solution

$$\begin{aligned}
y' &= 9 \sec^2\left(\frac{x}{3}\right) \cdot \left(\frac{x}{3}\right)' \\
&= 9 \sec^2\left(\frac{x}{3}\right) \cdot \left(\frac{1}{3}\right) \\
&= 3 \sec^2\left(\frac{x}{3}\right) \\
y'' &= 6 \sec\left(\frac{x}{3}\right) \cdot \left(\sec\left(\frac{x}{3}\right)\right)' \\
&= 6 \sec\left(\frac{x}{3}\right) \cdot \frac{1}{3} \sec\left(\frac{x}{3}\right) \cdot \tan\left(\frac{x}{3}\right) \\
&= 2 \sec^2\left(\frac{x}{3}\right) \cdot \tan\left(\frac{x}{3}\right) \quad \boxed{}
\end{aligned}$$

Exercise

Find the tangent line to the graph of $y = \sqrt[3]{(x+4)^2}$ when $x = 4$.

Solution

$$\begin{aligned}
y &= (x+4)^{2/3} \\
y' &= \frac{2}{3} (x+4)^{-1/3} \\
&= \frac{2}{3} \frac{1}{(x+4)^{1/3}}
\end{aligned}$$

$$= \frac{2}{3\sqrt[3]{x+4}}$$

$$x = 4 \rightarrow \underline{m = y' = \frac{2}{3\sqrt[3]{4+4}} = \frac{2}{3\sqrt[3]{2^3}} = \frac{2}{3(2)} = \frac{1}{3}} \quad \underline{\underline{\frac{1}{3}}}$$

$$x = 4 \rightarrow y = \sqrt[3]{(4+4)^2} = 4$$

$$y = \frac{1}{3}(x-4) + 4$$

$$y = \frac{1}{3}x - \frac{4}{3} + 4$$

$$\underline{y = \frac{1}{3}x + \frac{8}{3}} \quad \underline{\underline{\frac{1}{3}}}$$

Exercise

Evaluate the limit $\lim_{h \rightarrow 0} \frac{\sin^2\left(\frac{\pi}{4} + h\right) - \frac{1}{2}}{h}$

Solution

$$\lim_{h \rightarrow 0} \frac{\sin^2\left(\frac{\pi}{4} + h\right) - \frac{1}{2}}{h} = \frac{\frac{1}{2} - \frac{1}{2}}{0} = \underline{\underline{\frac{0}{0}}}$$

$$f\left(\frac{\pi}{4}\right) = \sin^2 \frac{\pi}{4} = \frac{1}{2}$$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\sin^2\left(\frac{\pi}{4} + h\right) - \frac{1}{2}}{h} &= f'\left(\frac{\pi}{4}\right) \\ &= 2 \sin \frac{\pi}{4} \cos \frac{\pi}{4} \\ &= 2 \frac{\sqrt{2}}{2} \frac{\sqrt{2}}{2} \\ &= \underline{\underline{1}} \end{aligned}$$

Exercise

Evaluate the limit $\lim_{x \rightarrow 5} \frac{\tan(\pi\sqrt{3x-11})}{x-5}$

Solution

$$\lim_{x \rightarrow 5} \frac{\tan(\pi\sqrt{3x-11})}{x-5} = \frac{\tan 2\pi}{0} = \underline{\underline{\frac{0}{0}}}$$

$$f(x) = \tan(\pi\sqrt{3x-11})$$

$$\begin{aligned}
 \lim_{x \rightarrow 5} \frac{f(x) - f(5)}{x - 5} &= f'(5) \\
 &= \frac{3\pi}{2\sqrt{3x-11}} \sec^2(\pi\sqrt{3x-11}) \Big|_{x=5} \\
 &= \frac{3\pi}{4} \sec^2(2\pi) \\
 &= \frac{3\pi}{4}
 \end{aligned}$$