

Solution

Section 2.9 – Rank and the Fundamental Matrix Spaces

Exercise

Verify that $\text{rank}(A) = \text{rank}(A^T)$

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix} \quad \begin{array}{l} R_2 + 3R_1 \\ R_3 + 2R_1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 7 & 17 & 2 \\ 0 & 7 & 17 & 2 \end{bmatrix} \quad R_3 - R_2$$

$$\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 7 & 17 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad 7R_1 - 2R_2$$

$$\begin{bmatrix} 7 & 0 & -6 & -4 \\ 0 & 7 & 17 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \frac{1}{7}R_1 \\ \frac{1}{7}R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -\frac{6}{7} & -\frac{4}{7} \\ 0 & 1 & \frac{17}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 2$$

$$A^T = \begin{bmatrix} 1 & -3 & -2 \\ 2 & 1 & 3 \\ 4 & 5 & 9 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & -2 \\ 2 & 1 & 3 \\ 4 & 5 & 9 \\ 0 & 2 & 2 \end{bmatrix} \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 4R_1 \end{array}$$

$$\begin{bmatrix} 1 & -3 & -2 \\ 0 & 7 & 7 \\ 0 & 17 & 17 \\ 0 & 2 & 2 \end{bmatrix} \quad \begin{array}{l} \frac{1}{7}R_2 \\ \\ \\ \frac{1}{2}R_4 \end{array}$$

$$\begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & 1 \\ 0 & 17 & 17 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{array}{l} R_1 + 3R_2 \\ \\ R_3 - 17R_2 \\ R_4 - R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A^T) = 2$$

$$\underline{\text{rank}(A) = \text{rank}(A^T) = 2}$$

Exercise

Find the rank and nullity of the matrix; then verify that the values obtained satisfy

$$\text{rank}(A) + N(A) = n$$

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \quad \begin{array}{l} \\ R_2 - 5R_1 \\ R_3 - 7R_1 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix} \quad \begin{array}{l} R_1 + R_2 \\ \\ R_3 - R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 2$$

$$\text{nullity}(A) = 1$$

$$\text{rank}(A) + \text{nullity}(A) = 2 + 1 = 3 = \textcolor{red}{n} \leftarrow \text{number of columns}$$

Exercise

Find the rank and nullity of the matrix; then verify that the values obtained satisfy

$$\text{rank}(A) + N(A) = n$$

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{bmatrix} \begin{array}{l} 7R_1 + 4R_2 \\ \\ R_3 + R_2 \end{array}$$

$$\begin{bmatrix} 7 & 0 & 7 & -2 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \frac{1}{7}R_1 \\ -\frac{1}{7}R_2 \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 2$$

$$\text{nullity}(A) = 1$$

$$\text{rank}(A) + \text{nullity}(A) = 2 + 1 = 3 = \textcolor{red}{n}$$

Exercise

Find the rank and nullity of the matrix; then verify that the values obtained satisfy

$$\text{rank}(A) + N(A) = n$$

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix} \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 + R_1 \\ R_4 - 2R_1 \end{array}$$

$$\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & -14 & -14 & -14 & -28 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -5 & -5 & -5 & -10 \end{bmatrix} \begin{array}{l} \\ -\frac{1}{14}R_2 \\ \frac{1}{4}R_3 \\ -\frac{1}{5}R_4 \end{array}$$

$$\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix} \begin{array}{l} R_1 - 4R_2 \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 2$$

$$\text{nullity}(A) = 2$$

$$\text{rank}(A) + \text{nullity}(A) = 2 + 2 = 4 = n$$

Exercise

Find the rank and nullity of the matrix; then verify that the values obtained satisfy

$$\text{rank}(A) + N(A) = n$$

$$A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix} \begin{array}{l} \\ R_3 - 2R_1 \\ R_4 - 3R_1 \\ R_5 + 2R_1 \end{array}$$

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & 6 & 0 & -3 \end{bmatrix} \begin{array}{l} \\ R_4 - R_3 \\ R_5 - R_2 \end{array}$$

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 + R_2 \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 8 & 2 & -2 \\ 0 & 3 & 6 & 0 & -3 \\ 0 & 0 & -12 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \frac{1}{3}R_2 \\ -\frac{1}{12}R_3 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 8 & 2 & -2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 & -\frac{5}{12} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 - 8R_3 \\ R_2 - 2R_3 \\ \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & \frac{4}{3} \\ 0 & 1 & 0 & 0 & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & -\frac{5}{12} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 3$$

$$NS(A) = 2$$

$$\text{Number of columns} = 5$$

$$\text{rank}(A) + NS(A) = 3 + 2 = 5 = n$$

Exercise

If A is an $m \times n$ matrix, what is the largest possible value for its rank and the smallest possible value of the nullity of A .

Solution

The largest possible value for the rank of an $m \times n$ matrix:

- n if $m \geq n$ (when every column of the $rref(A)$ contains a leading 1)
- m if $m < n$ (when every row of the $rref(A)$ contains a leading 1)

The smallest possible value for the nullity of an $m \times n$ matrix:

- 0 if $m \geq n$ (when every column of the $rref(A)$ contains a leading 1)
- $n - m$ if $m < n$ (when every row of the $rref(A)$ contains a leading 1)

Exercise

Discuss how the rank of A varies with t .

$$a) A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix} \quad b) A = \begin{bmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{bmatrix}$$

Solution

$$\begin{aligned} a) \quad \begin{vmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{vmatrix} &= t + t + t - t^3 - 1 - 1 \\ &= -t^3 + 3t - 2 = 0 \end{aligned}$$

Solve for t : $\underline{t = 1, -2, -2}$

Therefore, $\text{rank}(A) = 3$ for $\forall t - \{1, -2, -2\}$

$$\text{If } t = 1, A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 1$$

$$\text{If } t = -2, A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \begin{array}{l} \\ \\ R_3 + R_2 \end{array}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ -\frac{1}{3}R_2 \\ \end{array}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 - R_2 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 2$$

$$\begin{aligned} b) \quad \begin{vmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{vmatrix} &= 6t^2 + 6 + 9 - 6 - 6t - 9t \\ &= 6t^2 - 15t + 9 = 0 \end{aligned}$$

Solve for t : $t=1, \frac{3}{2}$

Therefore, $\text{rank}(A)=3$ for $\forall t - \left\{1, \frac{3}{2}\right\}$

$$\text{If } t=1, A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & 1 \end{bmatrix} \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 + R_1 \end{array}$$

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 + R_2 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} -\frac{1}{3}R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A)=2$$

$$\text{If } t=\frac{3}{2}, A = \begin{bmatrix} \frac{3}{2} & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & \frac{3}{2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2} & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & \frac{3}{2} \end{bmatrix} \begin{array}{l} 2R_1 \\ \\ 2R_3 \end{array}$$

$$\begin{bmatrix} 3 & 6 & -2 \\ 3 & 6 & -2 \\ -2 & -6 & 3 \end{bmatrix} \begin{array}{l} \\ R_2 - R_1 \\ 3R_3 + 2R_1 \end{array}$$

$$\begin{bmatrix} 3 & 6 & -2 \\ 0 & 0 & 0 \\ 0 & -6 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & -2 \\ 0 & -6 & 5 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 + R_2$$

$$\begin{bmatrix} 3 & 0 & 3 \\ 0 & -6 & 5 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \frac{1}{3}R_1 \\ -\frac{1}{6}R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{5}{6} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(A) = 2$$

Exercise

Are there values of r and s for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

Has rank 1? Has rank 2? If so, find those values.

Solution

Since the third column will always have a nonzero entry, the *rank* will never be 1. (row 1 and row 4 never have a nonzero entry).

If $r = 2$ and $s = 1$, that implies to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\rightarrow \text{rank} = 2$$

Exercise

Find the row reduced form \mathbf{R} and the rank r of \mathbf{A} (those depend on c).

Which are the pivot columns of \mathbf{A} ? Which variables are free? What are the special solutions and the nullspace matrix \mathbf{N} (always depending on c)?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & c \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} c & c \\ c & c \end{bmatrix}$$

Solution

$$a) \quad c \neq 4 \quad \mathbf{R} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$\text{rank}(\mathbf{A}) = 2$, the pivot columns are 1 and 3, the second variable x_2 is free.

$$\text{The special solution: } x_2 = 1 \text{ which yields to } \mathbf{N} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$c = 4 \quad \mathbf{R} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$\text{rank}(\mathbf{A}) = 1$, the pivot column is column 1, the second and third variables x_2, x_3 are free.

$$\text{The special solution goes into } \mathbf{N} = \begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$b) \quad c \neq 0 \quad \mathbf{R} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$\text{rank}(\mathbf{A}) = 1$, the pivot column is the first column, the second variable x_2 is free.

$$\text{The special solution: } x_2 = 1 \text{ which yields to } \mathbf{N} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$c = 0 \quad \mathbf{R} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\text{rank}(\mathbf{A}) = 0$, the matrix has no pivot column, and both variables are free.

$$\text{The special solution is: } \mathbf{N} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Exercise

Find the row reduced form R and the rank r of A (those depend on c).

Which are the pivot columns of A ? Which variables are free? What are the special solutions and the nullspace matrix N (always depending on c)?

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & c-1 & 0 & 0 \end{bmatrix} \quad \text{Interchange } R_2 \text{ \& } R_1$$

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) If $c = 1$, then

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This has only one pivot (first column) and 3 free variables x_2, x_3, x_4 .

$$\text{The nullspace matrix: } \begin{pmatrix} -1 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b) If $c \neq 1$, then

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \frac{1}{c-1} R_2$$

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_2 - R_1$$

$$\begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are two pivots (C_1, C_2) and 2 free variables x_3, x_4

The nullspace matrix: $\begin{pmatrix} -2 & -2 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$$

a) If $c = 1$

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad R_1 - 2R_2$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow 0a + b = 0 \Rightarrow b = 0$$

This has a single pivot in the second column and one free variable with the nullspace matrix

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

b) If $c = 2$

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \quad -R_1$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow a - 2b = 0$$

$$\text{if } b = 1 \quad a = 2$$

This has a single pivot in the first column with the nullspace matrix $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

c) Otherwise $c \neq 1, 2$

$$A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix} \quad \frac{1}{1-c} R_1$$

$$\begin{bmatrix} 1 & \frac{2}{1-c} \\ 0 & 2-c \end{bmatrix} \quad \frac{1}{2-c} R_2$$

$$\begin{bmatrix} 1 & \frac{2}{1-c} \\ 0 & 1 \end{bmatrix} \quad R_1 - \frac{2}{1-c} R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The result is the identity matrix with 2 pivots, which has $(2 - 2) = 0$ null space.

Exercise

If A has a rank r , then it has an r by r sub-matrix S that is invertible. Remove $m - r$ rows and $n - r$ columns to find an invertible sub-matrix S inside each A (you could keep the pivot rows and pivot columns of A).

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution

If a matrix A has rank r , then the

(dimension of the column space) = (dimension of the row space) = r

For the invertible sub-matrix S , we need to find r linearly independent rows and r linearly independent columns.

For matrix A :

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \quad R_2 - R_1$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} \quad R_1 - 3R_2$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The 1st and 3rd columns are linearly independent, and the 1st and 2nd rows are also linearly independent.

Rank (A) = 2.

The sub matrices are: $S_A = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$ $S_A = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$ $S_A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

For matrix **B**:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \quad R_2 - 2R_1$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Rank (**B**) = 1.

The submatrix is: $S_A = (1)$

For matrix **C**:

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rank (**C**) = 2.

The submatrix is by disregarding (deleting) 1st column and 2nd row: $S_A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Exercise

Suppose that column 3 of 4 x 6 matrix is all zero. Then x_3 must be a _____ variable. Give one special solution for this matrix.

Solution

The x_3 must be a ***free variable***.

A special solution for this variable can be taken to be.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Exercise

Fill in the missing numbers to make A rank 1, rank 2, rank 3. (your solution should be 3 matrices)

$$A = \begin{pmatrix} & -3 & \\ 1 & 3 & -1 \\ & 9 & -3 \end{pmatrix}$$

Solution

$$A = \begin{pmatrix} a & -3 & b \\ 1 & 3 & -1 \\ c & 9 & -3 \end{pmatrix}$$

If $\text{rank}(A) = 1$, then we need the 1st and 3rd to be multiple of the 2nd row to get zero in these rows.

$$A = \begin{pmatrix} a & -3 & b \\ 1 & 3 & -1 \\ c & 9 & -3 \end{pmatrix} \begin{array}{l} R_1 + R_2 \\ \\ R_3 - 3R_2 \end{array}$$

$$\begin{pmatrix} a+1 & 0 & b-1 \\ 1 & 3 & -1 \\ c-3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} a+1=0 \\ b-1=0 \\ c-3=0 \end{cases} \rightarrow \begin{cases} a=-1 \\ b=1 \\ c=3 \end{cases}$$

$$A = \begin{pmatrix} -1 & -3 & 1 \\ 1 & 3 & -1 \\ 3 & 9 & -3 \end{pmatrix}$$

If $\text{rank}(A) = 2$, then we need the 1st **or** 3rd to be multiple of the 2nd row to get zero row.

$$A = \begin{pmatrix} a & -3 & b \\ 1 & 3 & -1 \\ c & 9 & -3 \end{pmatrix} \begin{array}{l} R_1 + R_2 \\ \\ R_3 - 3R_2 \end{array}$$

$$\begin{pmatrix} a+1 & 0 & b-1 \\ 1 & 3 & -1 \\ c-3 & 0 & 0 \end{pmatrix} \quad \boxed{c \neq 3}$$

$$A = \begin{pmatrix} -1 & -3 & 1 \\ 1 & 3 & -1 \\ 2 & 9 & -3 \end{pmatrix}$$

If $\text{rank}(A) = 3$ (full rank), then the appropriate to start using 0's or 1's to fill the blank.

$$A = \begin{pmatrix} 0 & -3 & 0 \\ 1 & 3 & -1 \\ 1 & 9 & -3 \end{pmatrix} \quad \text{Interchange } R_1 \text{ \& } R_2$$

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & -3 & 0 \\ 1 & 9 & -3 \end{pmatrix} \quad R_3 - R_1$$

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & -3 & 0 \\ 0 & 6 & -2 \end{pmatrix} \quad -\frac{1}{3}R_2$$

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 6 & -2 \end{pmatrix} \quad \begin{matrix} R_1 - 3R_2 \\ R_1 - 6R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad -\frac{1}{2}R_3$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_1 + R_3$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, it has *rank* 3.

Exercise

Fill out these matrices so that they have rank 1:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{pmatrix} \quad B = \begin{pmatrix} 2 & & \\ 1 & & \\ 2 & 6 & -3 \end{pmatrix} \quad M = \begin{pmatrix} a & b \\ c & \end{pmatrix}$$

Solution

Rank = 1 means that all the rows are multiples of each other.

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & a & b \\ 4 & c & d \end{pmatrix} \xrightarrow[R_3=4R_1]{R_2=2R_1} \begin{matrix} a=2(2) & b=2(4) \\ c=4(2) & d=4(4) \end{matrix}$$

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & a & b \\ 1 & c & d \\ 2 & 6 & -3 \end{pmatrix} \xrightarrow[R_2=\frac{1}{2}R_3]{R_1=R_3} \begin{matrix} a=6 & b=-3 \\ c=3 & d=-\frac{3}{2} \end{matrix}$$

$$B = \begin{pmatrix} 2 & 6 & -3 \\ 1 & 3 & -\frac{3}{2} \\ 2 & 6 & -3 \end{pmatrix}$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{R_2=\frac{c}{a}R_1} d = \frac{c}{a}b$$

$$M = \begin{pmatrix} a & b \\ c & \frac{bc}{a} \end{pmatrix}$$

Exercise

Suppose A and B are n by n matrices, and $AB = I$. Prove from $\text{rank}(AB) \leq \text{rank}(A)$ that the $\text{rank}(A) = n$. So, A is invertible and B must be its two-sided inverse. Therefore $BA = I$ (which is not so obvious!).

Solution

Since A is n by $n \Rightarrow \text{rank}(A) \leq n$

$$n = \text{rank}(I_n) = \text{rank}(AB) \leq \text{rank}(A)$$

Exercise

Every m by n matrix of rank r reduces to $(m$ by $r)$ times $(r$ by $n)$:

$$A = (\text{pivot columns of } A) (\text{first } r \text{ rows of } R) = (COL)(ROW)^T$$

Write the 3 by 4 matrix $A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix}$ as the product of the 3 by 2 from the pivot columns and the 2 by 4 matrix from R .

Solution

$$A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix} \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \end{pmatrix} \begin{array}{l} \\ \\ R_3 - 2R_2 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} R_1 - R_2 \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The pivots columns are the 1st and 2nd column.

$$A = (\text{pivot columns of } A) (\text{first } r \text{ rows of } R) = (COL)(ROW)^T$$

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{aligned}$$

Exercise

Suppose R is m by n matrix of rank r , with pivot columns first: $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

- What are the shapes of those 4 blocks?
- Find the right-inverse B with $RB = I$ if $r = m$.
- Find the right-inverse C with $CR = I$ if $r = n$.
- What is the reduced row echelon form of R^T (with shapes)?
- What is the reduced row echelon form of $R^T R$ (with shapes)?

Prove that $R^T R$ has the same nullspace as R . Then show that $A^T A$ always has the same nullspace as A (a value fact).

- Suppose you allow elementary column operations on A as well as elementary row operations (which get to R). What is the “row-and-column reduced form” for an m by n matrix of rank r ?

Solution

$$a) \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} r \times r & r \times (n-r) \\ (m-r) \times r & (m-r) \times (n-r) \end{bmatrix}$$

$$b) R = \begin{bmatrix} I & F \end{bmatrix}$$

$$RB = I \Rightarrow \begin{bmatrix} I & F \end{bmatrix} B = I$$

$$\begin{bmatrix} I & F \end{bmatrix} \begin{pmatrix} M \\ N \end{pmatrix} = I$$

$$IM + FN = I$$

$$\Rightarrow \begin{cases} M = I \\ N = 0 \end{cases} \rightarrow F : r \times (n-r)$$

$$B = \begin{bmatrix} I_{r \times r} \\ 0_{(n-r) \times r} \end{bmatrix}$$

$$c) R = \begin{bmatrix} I & 0 \end{bmatrix}$$

$$CR = I \Rightarrow C \begin{bmatrix} I & 0 \end{bmatrix} = I$$

$$C = \begin{bmatrix} I_{r \times r} & 0_{r \times (m-r)} \end{bmatrix}$$

$$d) R^T = \begin{bmatrix} I_{r \times r} & 0_{(m-r) \times r} \\ F_{r \times (n-r)} & 0_{(m-r) \times (n-r)} \end{bmatrix} \quad R_2 - F_{r \times (n-r)} R_1$$

$$rref(R^T) = \begin{bmatrix} I_{r \times r} & 0_{(m-r) \times r} \\ 0_{(n-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix}$$

$$\begin{aligned} e) \quad R^T R &= \begin{bmatrix} I & 0 \\ F & 0 \end{bmatrix} \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I & IF \\ FI & F^2 \end{bmatrix} \end{aligned}$$

$FI: r \times (n-r) \quad r \times r$, the inner is not equal but to make work, we can use the F transpose.

$$(n-r) \times r \quad r \times r \Rightarrow F^T I = F^T$$

$$\begin{aligned} R^T R &= \begin{bmatrix} I & 0 \\ F & 0 \end{bmatrix} \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I & F \\ F & F^2 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} I & F \\ F & F^2 \end{bmatrix} \quad R_2 - FR_1$$

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

$$rref(R^T R) = \begin{bmatrix} I_{r \times r} & F_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix}$$

$$= R$$

So, that $N(A) = N(rref(A))$ for any matrix A . So, $N(A) = N(R^T R)$

f) After getting to R we can use the column operations to get rid of F .

$$\begin{bmatrix} I_{r \times r} & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix}$$

Exercise

True or False (check addition or give a counterexample)

- a) The symmetric matrices in M (with $A^T = A$) form a subspace.
- b) The skew-symmetric matrices in M (with $A^T = -A$) form a subspace.
- c) The un-symmetric matrices in M (with $A^T \neq A$) form a subspace.
- d) Invertible matrices
- e) Singular matrices

Solution

a) True: $A^T = A$ and $B^T = B$ lead to $(A+B)^T = A^T + B^T = A+B$

b) True: $A^T = -A$ and $B^T = -B$ lead to $(A+B)^T = A^T + B^T = -A-B = -(A+B)$

c) False: $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

d) False: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ are invertible matrices but $A+B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ is not invertible.

\therefore The zero matrix is not invertible but any linear subspace should contain the zero matrix. So, invertible matrices do not form a linear subspace.

e) False: $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ are singular matrices

But $A+B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ is not singular.

Exercise

$$\text{Let } A = \begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 2 & 4 & -3 & 7 & 0 \\ 3 & 6 & -5 & 10 & -2 \\ 5 & 10 & -9 & 16 & 0 \end{pmatrix}$$

- a) Reduce A to row-reduced echelon form.
- b) What is the rank of A ?
- c) What are the pivots?
- d) What are the free variables?
- e) Find the special solutions. What is the nullspace $N(A)$?
- f) Exhibit an $r \times r$ submatrix of A which is invertible, where $r = \text{rank}(A)$. (An $r \times r$ submatrix of A is obtained by keeping r rows and r columns of A)

Solution

$$a) \quad A = \begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 2 & 4 & -3 & 7 & 0 \\ 3 & 6 & -5 & 10 & -2 \\ 5 & 10 & -9 & 16 & 0 \end{pmatrix} \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 5R_1 \end{array}$$

$$\begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \quad \begin{array}{l} R_3 - R_2 \\ R_4 - R_2 \end{array}$$

$$\begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- b) $\text{Rank}(A) = 3$
- c) The pivots are x_1, x_3, x_5
- d) The free variables are x_2, x_4

$$e) \quad \begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad -\frac{1}{2}R_3$$

$$\begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad R_1 + 2R_2$$

$$\begin{pmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $x = x_2 s_2 + x_4 s_4$

$$Rx = \begin{pmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} x_1 + 2x_2 + 5x_4 = 0 \\ x_3 + x_4 = 0 \\ x_5 = 0 \end{cases}$$

$$1. \text{ Set } x_2 = 1, \quad x_4 = 0 \rightarrow \begin{cases} x_1 + 2 = 0 \Rightarrow x_1 = -2 \\ x_3 = 0 \\ x_5 = 0 \end{cases}$$

The special solution: $\vec{s}_2 = (-2, 1, 0, 0, 0)$

$$2. \text{ Set } x_2 = 0, \quad x_4 = 1 \rightarrow \begin{cases} x_1 + 5 = 0 \Rightarrow x_1 = -5 \\ x_3 + 1 = 0 \Rightarrow x_3 = -1 \\ x_5 = 0 \end{cases}$$

The special solution: $\vec{s}_3 = (-5, 0, -1, 1, 0)$

The nullspace is the set $\left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

- f) The pivot rows and columns must be included in a submatrix. To do that, just take the rows and columns of A containing pivots, which are columns 1, 3, 5 and rows 1, 2, 3. That will give us a 3 by 3 submatrix. Therefore, this submatrix of A will be invertible.

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -3 & 0 \\ 3 & -5 & -2 \end{pmatrix}$$

Exercise

Let $A = \begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 2 & 1 & 0 & 0 & -15 \\ 6 & -1 & -8 & -1 & -47 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix}$

- Reduce A to (ordinary) echelon form.
- What the pivots?
- What are the free variables?
- Reduce A to row-reduced echelon form.
- Find the special solutions. What is the nullspace $N(A)$?
- What is the rank of A ?

g) Give the complete solution to $Ax = b$, where $b = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Solution

a) $A = \begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 2 & 1 & 0 & 0 & -15 \\ 6 & -1 & -8 & -1 & -47 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix} \quad \begin{matrix} R_2 + 2R_1 \\ R_3 + 6R_1 \end{matrix}$

$$\begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 0 & 5 & 10 & 0 & -5 \\ 0 & 11 & 22 & -1 & -17 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix} \quad \begin{matrix} 5R_3 - 11R_2 \\ 5R_4 - 2R_2 \end{matrix}$$

$$\begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 0 & 5 & 10 & 0 & -5 \\ 0 & 0 & 0 & -5 & -30 \\ 0 & 0 & 0 & 15 & 90 \end{pmatrix} \quad R_4 + 3R_3$$

$$\begin{pmatrix} \boxed{-1} & 2 & 5 & 0 & 5 \\ 0 & \boxed{5} & 10 & 0 & -5 \\ 0 & 0 & 0 & \boxed{-5} & -30 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

b) The pivots are -1 , 5 , and -5 (Columns 1, 2, 4)

c) The free variables are 3rd and 5th (x_3, x_5)

$$d) \begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 0 & 5 & 10 & 0 & -5 \\ 0 & 0 & 0 & -5 & -30 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} -R_1 \\ \frac{1}{5}R_2 \\ -\frac{1}{5}R_3 \end{array}$$

$$\begin{pmatrix} 1 & -2 & -5 & 0 & -5 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad R_1 + 2R_2$$

$$R = \begin{pmatrix} 1 & 0 & -1 & 0 & -7 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

e) Let $x = x_3 s_1 + x_5 s_2$

$$R\vec{x} = \begin{pmatrix} 1 & 0 & -1 & 0 & -7 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} x_1 - x_3 - 7x_5 = 0 \\ x_2 + 2x_3 - x_5 = 0 \\ x_4 + 6x_5 = 0 \end{cases}$$

1. Set $x_3 = 1, x_5 = 0$

$$\rightarrow \begin{cases} x_1 - 1 = 0 & \underline{x_1 = 1} \\ x_2 + 2 = 0 & \underline{x_2 = -2} \\ x_4 = 0 \end{cases}$$

The special solution: $\vec{s}_1 = (1, -2, 1, 0, 0)$

2. Set $x_3 = 0, x_5 = 1$

$$\rightarrow \begin{cases} x_1 - 7 = 0 & \underline{x_1 = 7} \\ x_2 - 1 = 0 & \underline{x_2 = 1} \\ x_4 + 6 = 0 & \underline{x_4 = -6} \end{cases}$$

The special solution: $\vec{s}_2 = (7, 1, 0, -6, 1)$

The nullspace is the set $\left\{ x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 7 \\ 1 \\ 0 \\ -6 \\ 1 \end{pmatrix} \right\}$

f) $\text{Rank}(A) = 3$

g) $A\vec{x} = \vec{b}$, where $\vec{b} = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

The complete solution = (the particular solution) + (special solution)

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 7 \\ 1 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

Exercise

$$\text{Let } A = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- a) Reduce A to row-reduced echelon form.
- b) What is the rank of A ?
- c) What the pivots variables?
- d) What are the free variables?
- e) Find the special solutions.
- f) What is the nullspace $N(A)$?

Solution

$$a) \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} \\ R_2 - R_1 \\ \\ R_3 - 2R_1 \\ \end{array}$$

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & -1 & -4 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} \\ -R_2 \\ R_3 - R_2 \\ R_4 - 2R_2 \\ \end{array}$$

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} \\ \\ R_5 - R_4 \\ \end{array}$$

$$R = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{array}{l} x_1 = -2x_2 - 3x_4 \\ x_3 = -4x_4 \\ x_5 = 0 \end{array}$$

- b) $\text{Rank}(A) = 3$
- c) The pivots variables are: x_1, x_3, x_5

d) The free variables are: x_2, x_4

e) Let $x = x_2 s_1 + x_4 s_2$

$$\begin{cases} x_1 = -2x_2 - 3x_4 \\ x_3 = -4x_4 \end{cases}$$

Set $x_2 = 1, x_4 = 0$

The special solution: $\vec{s}_1 = (-2, 1, 0, 0, 0)$

Set $x_2 = 0, x_4 = 1$;

The special solution: $\vec{s}_2 = (-3, 0, -4, 1, 0)$

f) The nullspace is the set $\left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix} \right\}$

$$N(A) = \begin{pmatrix} -2 & -3 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Exercise

$$\text{Let } A = \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}$$

a) Reduce A to row-reduced echelon form.

b) What is the rank of A ?

c) What the pivots?

d) What are the free variables?

e) Find the special solutions.

f) What is the nullspace $N(A)$?

Solution

$$\begin{aligned}
 a) \quad & \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix} \begin{array}{l} 3R_2 - R_1 \\ 3R_3 - 2R_1 \\ R_4 - 2R_1 \end{array} \\
 & \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 0 & 0 & -3 & -15 & -3 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & -1 & -5 & 0 \end{pmatrix} \begin{array}{l} -R_2 \\ 3R_4 + R_2 \end{array} \\
 & \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 0 & 0 & 3 & 15 & 3 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \begin{array}{l} \frac{1}{3}R_1 \\ \frac{1}{3}R_2 \\ \frac{1}{3}R_3 \\ R_4 - R_3 \end{array} \\
 & \begin{pmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} R_2 - R_3 \end{array} \\
 & \begin{pmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} x_1 = -7x_2 - 3x_4 \\ x_3 = -5x_4 \\ x_5 = 0 \end{array}
 \end{aligned}$$

b) $\text{Rank}(A) = 3$

c) The pivots variables are: x_1, x_3, x_5

d) The free variables are: x_2, x_4

e) Let $x = x_2 s_1 + x_4 s_2$

$$\begin{cases} x_1 = -7x_2 - 3x_4 \\ x_3 = -5x_4 \end{cases}$$

Set $x_2 = 1, x_4 = 0$

The special solution: $\vec{s}_1 = (-7, 1, 0, 0, 0)$

Set $x_2 = 0, x_4 = 1$;

The special solution: $\vec{s}_2 = (-3, 0, -5, 1, 0)$

f) The nullspace is the set $\left\{ x_2 \begin{pmatrix} -7 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix} \right\}$

$$N(A) = \begin{pmatrix} -7 & -3 \\ 1 & 0 \\ 0 & -5 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Exercise

The 3 by 3 matrix A has rank 2.

$$A\vec{x} = \vec{b} \quad \text{is} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 + 5x_4 &= b_1 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 &= b_2 \\ 3x_1 + 6x_2 + 7x_3 + 13x_4 &= b_3 \end{aligned}$$

- Reduce $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ to $\begin{bmatrix} U & \vec{c} \end{bmatrix}$, so that $A\vec{x} = \vec{b}$ becomes triangular system $U\vec{x} = \vec{c}$.
- Find the condition on (b_1, b_2, b_3) for $A\vec{x} = \vec{b}$ to have a solution
- Describe the column space of A . Which plane in \mathbb{R}^3 ?
- Describe the nullspace of A . Which special solutions in \mathbb{R}^4 ?
- Find a particular solution to $A\vec{x} = (0, 6, -6)$ and then complete solution.

Solution

$$a) \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{array} \right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & -2 & -2 & b_3 - 3b_1 \end{array} \right] \quad R_3 + R_2$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{array} \right]$$

b) The last equation $b_3 + b_2 - 5b_1 = 0$ shows the solvability condition.

c) (i) The column space is the plane containing all combinations of the pivot columns: 1st (1, 2, 3) and 3rd (3, 8, 7).

(ii) The column space contains all vectors with $b_3 + b_2 - 5b_1 = 0$. That makes $A\vec{x} = \vec{b}$

solvable, so \vec{b} is in the column space. All columns of A pass this test $b_3 + b_2 - 5b_1 = 0$. This is the equation for the plane in **(i)**.

d) The special solutions have free variables:

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 5x_4 = 0 \\ 2x_3 + 2x_4 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = -2x_2 - 2x_4 \\ x_3 = -x_4 \end{cases}$$

Let $x_2 = 1, x_4 = 0$

$$\Rightarrow \begin{cases} x_1 = -2 \\ x_3 = 0 \end{cases}$$

$$\vec{s}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Let $x_2 = 0, x_4 = 1$

$$\Rightarrow \begin{cases} x_1 = -2 \\ x_3 = -1 \end{cases}$$

$$\vec{s}_2 = \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

The nullspace $N(A)$ in \mathbb{R}^4 contains all

$$\vec{x}_n = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

e) One particular solution x_p has free variables = zero.

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 5x_4 = 0 \\ 2x_3 + 2x_4 = 6 \end{cases}$$

$$\begin{cases} x_1 = -2x_2 - 3x_3 - 5x_4 \\ x_3 = 3 - x_4 \end{cases}$$

$$\begin{cases} x_1 = -2x_2 - 9 - 2x_4 \\ x_3 = 3 - x_4 \end{cases}$$

Let $x_2 = x_4 = 0$

$$\begin{cases} x_1 = -9 \\ x_3 = 3 \end{cases}$$

$$\vec{x}_p = \begin{pmatrix} -9 \\ 0 \\ -3 \\ 0 \end{pmatrix}$$

The complete solution to $A\vec{x} = (0, 6, -6)$ is $\vec{x} = \vec{x}_p + \text{all } \vec{x}_n$

$$\vec{x} = \begin{pmatrix} -9 \\ 0 \\ -3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Exercise

Find the special solutions and describe the complete solution to $A\vec{x} = \vec{0}$ for

$$A_1 = 3 \text{ by } 4 \text{ zero matrix} \quad A_2 = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \quad A_3 = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$$

Which are the pivot columns? Which are the free variables? What is the R (Reduced Row Echelon matrix) in each case?

Solution

$A_1 \vec{x} = \vec{0}$ has 4 solutions. They are the columns $\vec{s}_1, \vec{s}_2, \vec{s}_3, \vec{s}_4$ of the identity matrix (4 by 4).

The Nullspace is of \mathbb{R}^4 .

The complete solution: $\vec{x} = c_1 \vec{s}_1 + c_2 \vec{s}_2 + c_3 \vec{s}_3 + c_4 \vec{s}_4$ in \mathbb{R}^4 .

There are no pivot columns; all variables are free; the reduced R is the same zero matrix as A_1 .

$$A_2 \vec{x} = \vec{0}$$

$$A_2 \vec{x} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x_1 + 2x_2 = 0$$

The vector solution: $\vec{s} = (-2, 1)$, The first column of A_2 is its pivot column, and x_2 is the free variable.

$$\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \quad \frac{1}{3}R_1$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad R_2 - R_1$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

$$R_3 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

All variables are free. There are three special solutions to $A_3 \vec{x} = 0$

$$\vec{s}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{s}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{s}_3 = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The complete solution:

$$\vec{x} = c_1 \vec{s}_1 + c_2 \vec{s}_2 + c_3 \vec{s}_3$$

$$= c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Exercise

Create a 3 by 4 matrix whose special solutions to $A\vec{x} = \vec{0}$ are \vec{s}_1 and \vec{s}_2 :

$$\vec{s}_1 = \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{s}_2 = \begin{pmatrix} -2 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

You could create the matrix A in row reduced form R . Then describe all possible matrices A with the required Nullspace $N(A) = \text{all combinations of } \vec{s}_1 \text{ and } \vec{s}_2$.

Solution

We can write the solution:

$$\vec{x} = x_2 \vec{s}_1 + x_4 \vec{s}_2$$

$$x_2 \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -6 \\ 1 \end{pmatrix} = \begin{pmatrix} -3x_2 - 2x_4 \\ 2x_2 \\ -6x_4 \\ x_4 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3x_2 - 2x_4 \\ 2x_2 \\ -6x_4 \\ x_4 \end{pmatrix} \rightarrow \begin{cases} x_1 = -3x_2 - 2x_4 \\ x_3 = -6x_4 \end{cases}$$

$$\rightarrow \begin{cases} x_1 + 3x_2 + 2x_4 = 0 \\ x_3 + 6x_4 = 0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The entries 3, 2, 6 are the negatives of -3 , -2 , -6 in the special solutions.

Every 3 by 4 matrix has at least one special solution. These A 's have two.

Exercise

The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - z = 0$. One particular point on this plane is $(12, 0, 0)$. All points on the plane have the form (fill the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solution

$$x - 3y - z = 12$$

$$x = 3y + z + 12$$

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{pmatrix} 12 + 3y + z \\ y \\ z \end{pmatrix} \\ &= \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Exercise

Construct a matrix whose column space contains $(1, 1, 5)$ and $(0, 3, 1)$ and whose Nullspace contains $(1, 1, 2)$.

Solution

$$A = \begin{pmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 + 2a \\ 1 + 3 + 2b \\ 5 + 1 + 2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 1 + 2a = 0 & \rightarrow 2a = -1 & \Rightarrow a = -\frac{1}{2} \\ 4 + 2b = 0 & \rightarrow 2b = -4 & \Rightarrow b = -2 \\ 6 + 2c = 0 & \rightarrow 2c = -6 & \Rightarrow c = -3 \end{cases}$$

$$A = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{pmatrix}$$

Exercise

Construct a matrix whose column space contains $(1, 1, 0)$ and $(0, 1, 1)$ and whose Nullspace contains $(1, 0, 1)$ and $(0, 0, 1)$.

Solution

It is impossible. Matrix A must be 3 by 3.

Since the nullspace is supposed to contain two independent vectors, A can have at most $3 - 2 = 1$ pivots.

Since the column space supposes to contain two independent vectors. A must have at least 2 pivots.

These conditions can't both be met.

Exercise

Construct a matrix whose column space contains $(1, 1, 1)$ and whose Nullspace contains $(1, 1, 1, 1)$.

Solution

The matrix needs to be 3 by 4 matrix.

$$\begin{pmatrix} 1 & a & b & c \\ 1 & d & e & f \\ 1 & g & h & i \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 1 + a + b + c = 0 \\ 1 + d + e + f = 0 \\ 1 + g + h + i = 0 \end{cases}$$

$$\begin{cases} a + b + c = -1 \\ d + e + f = -1 \\ g + h + i = -1 \end{cases}$$

$$\begin{cases} \text{if } b = c = 0 & a = -1 \\ \text{if } d = f = 0 & e = -1 \\ \text{if } g = h = 0 & i = -1 \end{cases}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

Exercise

How is the Nullspace $N(C)$ related to the spaces $N(A)$ and $N(B)$, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

Solution

$$Cx = \begin{bmatrix} Ax \\ Bx \end{bmatrix} = 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If and only if $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$

$$N(C) = N(A) \cap N(B)$$

Exercise

Why does no 3 by 3 matrix have a nullspace that equals its column space?

Solution

If nullspace = column space, then $n - r = r$ (there are r pivots).

For $n = 3 \Rightarrow 3 = 2r$ is impossible.

Exercise

If $AB = 0$ then the column space B is contained in the _____ of A . Give an example of A and B .

Solution

If $AB = 0$ then the column space B is contained in the **nullspace** of A .

$$\text{Example: } A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Exercise

True or false (with reason if true or example to show it is false)

- a) A square matrix has no free variables.
- b) An invertible matrix has no free variables.
- c) An m by n matrix has no more than n pivot variables.
- d) An m by n matrix has no more than m pivot variables.

Solution

- a) False. Any matrix with fewer than full number of pivots will. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- b) True. Since it is invertible, we will get the full number of pivots. The nullspace has dimension, so we have 0 free variables.
- c) True, the number of pivot variables is the dimension of the nullspace, which is at most the number of columns. The nullspace dimension + column space dimension = number of columns.
- d) True, in reduced echelon matrix the pivot columns are all 0 except for a single 1, and there are only up to m vectors of this type.

Exercise

Suppose an m by n matrix has r pivots. The number of special solutions is _____.

The Nullspace contains only $x = 0$ when $r =$ _____.

The column space is all of \mathbf{R}^m when $r =$ _____.

Solution

Suppose an m by n matrix has r pivots. The number of special solutions is $n - r$.

The Nullspace contains only $x = 0$ when $r =$ n .

The column space is all of \mathbf{R}^m when $r =$ m .

Exercise

Find the complete solution in the form $x_p + x_n$ to these full rank system:

- a) $x + y + z = 4$
- b) $\begin{cases} x + y + z = 4 \\ x - y + z = 4 \end{cases}$

Solution

- a) $x + y + z = 4$

The equivalent matrix is given by: $\begin{cases} Ax = 4 \\ A = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \end{cases}$

The complete solution in the form $\vec{x} = \vec{x}_p + \vec{x}_n$

\vec{x}_n is the homogeneous solution to $A\vec{x}_n = \vec{0}$

Size of A is $m = 1$ and $n = 3$, $\text{rank}(A) = r = 1$

$$A\vec{x}_n = \vec{0}$$

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + x_2 + x_3 = 0 \Rightarrow \underline{x_1 = -x_2 - x_3}$$

Set $x_2 = 1, \quad x_3 = 0$

The special solution: $\vec{s}_1 = (-1, 1, 0)$

Set $x_2 = 0, \quad x_3 = 1$

The special solution: $\vec{s}_2 = (-1, 0, 1)$

The nullspace is the set $\left\{ x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$x = 4 - y - z$$

$$\Rightarrow x_1 = 4 - x_2 - x_3$$

Set $x_2 = 0, \quad x_3 = 0$ that implies to the particular solution: $\vec{x}_p = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$

The complete solution in the form $\vec{x} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Note: that the null space of A is spanned by the two linearly independent vectors

$$(-1, 1, 0)^T \quad \text{and} \quad (-1, 0, 1)^T$$

b)
$$\begin{cases} x + y + z = 4 \\ x - y + z = 4 \end{cases}$$

The equivalent matrix is given by:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \text{ and } A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & 4 \end{array} \right] \quad R_2 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & 0 \end{array} \right] \quad -\frac{1}{2}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{array} \right] \quad R_1 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{array} \right]$$

The pivots are x_1, x_2 ; The free variable is x_3

Rank $r = 2$, $n = 2$, $m = 3$.

The nullspace has dimension $m - r = 1$.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 + x_3 = 0 \rightarrow x_1 = -x_3 \\ x_2 = 0 \end{cases}$$

$$\text{If } x_3 = 1 \Rightarrow x_1 = -1$$

The special solution: $\vec{s}_1 = (-1, 0, 1)$

The nullspace is the set $\left\{ x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Set $x_3 = 0$ that implies

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 = 4 \\ x_2 = 0 \end{cases}$$

Then the particular solution: $\vec{x}_p = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$

The complete solution in the form:

$$\vec{x} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Exercise

Find the complete solution in the form $\vec{x}_p + \vec{x}_n$ to the system:

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

Solution

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array} \right) \quad R_2 - 2R_1$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 2 & 4 & 1 \end{array} \right) \quad \begin{array}{l} 2R_1 - R_2 \\ R_3 - R_2 \end{array}$$

$$\left(\begin{array}{cccc|c} 2 & 6 & 0 & 0 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} \frac{1}{2}R_1 \\ \frac{1}{2}R_2 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

The pivots are x_1, x_3 ; The free variables are x_2, x_4

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 + 3x_2 = 0 & \underline{x_1 = -3x_2} \\ x_3 + 2x_4 = 0 & \underline{x_3 = -2x_4} \end{cases}$$

1. Set $x_2 = 1, x_4 = 0$

The special solution: $\vec{s}_1 = (-3, 1, 0, 0)$

2. Set $x_2 = 0, x_4 = 1$

The special solution: $\vec{s}_2 = (0, 0, -2, 1)$

The special solution: $\vec{x}_n = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 + 3(0) = \frac{1}{2} & \underline{x_1 = \frac{1}{2}} \\ x_3 + 2(0) = \frac{1}{2} & \underline{x_3 = \frac{1}{2}} \end{cases}$$

Then the particular solution:

$$\vec{x}_p = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

The complete solution in the form:

$$\vec{x} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

Exercise

If A is 3×7 matrix, its largest possible rank is _____. In this case, there is a pivot in every _____ of U and R , the solution to $Ax = b$ _____ (always exists or is unique), and the column space of A is _____. Construct an example of such a matrix A .

Solution

If A is 3×7 matrix, its largest possible rank is **3**. In this case, there is a pivot in every **row** of U and R , the solution to $Ax = b$ **always exists**, and the column space of A is \mathbb{R}^3 .

$$A = \begin{pmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{pmatrix}$$

$\text{rank}(A) \leq 3$, that implies that you have 3 pivots (1 each row)

$$\begin{aligned} A &= \begin{pmatrix} 1 & 0 & 0 & * & * & * & * \\ 0 & 1 & 0 & * & * & * & * \\ 0 & 0 & 1 & * & * & * & * \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 5 & 6 & 7 & 8 \\ 0 & 0 & 1 & 9 & 10 & 11 & 12 \end{pmatrix} \end{aligned}$$

Exercise

If A is 6×3 matrix, its largest possible rank is _____. In this case, there is a pivot in every _____ of U and R , the solution to $A\vec{x} = \vec{b}$ _____ (always exists or is unique), and the nullspace of A is _____. Construct an example of such a matrix A .

Solution

If A is 6×3 matrix, its largest possible rank is **3**. In this case, there is a pivot in every **column** of U and R , the solution to $A\vec{x} = \vec{b}$ **is unique**, and the column space of A is $\{\vec{0}\}$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Exercise

Find the rank of A , $A^T A$ and AA^T for $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix}$

Solution

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} \quad R_3 + R_1$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{pmatrix} \quad \frac{1}{2}R_2$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 3 \end{pmatrix} \quad R_3 - 3R_2$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\underline{\text{rank}(A) = 2}$$

$$\begin{aligned} A^T A &= \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ -1 & 9 \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 9 \end{pmatrix} \quad 2R_2 + R_1$$

$$\begin{pmatrix} 2 & -1 \\ 0 & 17 \end{pmatrix}$$

$$\underline{\text{rank}(A^T A) = 2}$$

$$AA^T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 4 \\ 1 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 4 \\ 1 & 4 & 5 \end{pmatrix} \quad \begin{array}{l} R_2 - R_1 \\ 2R_3 - R_1 \end{array}$$

$$\begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 6 & 9 \end{pmatrix} \quad R_3 - 3R_2$$

$$\begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\text{rank}(A^T A) = 2}$$

$\therefore \text{rank}(A) = \text{rank}(A^T A) = \text{rank}(AA^T)$ for any matrix, A .

Exercise

Explain why these are all false:

- The complete solution is any linear combination of \vec{x}_p and \vec{x}_n .
- A system $A\vec{x} = \vec{b}$ has at most one particular solution.
- The solution \vec{x}_p with all free variables zero is the shortest solution (minimum length $\|\vec{x}\|$). Find a 2 by 2 counterexample.
- If A is invertible there is no solution \vec{x}_n in the null space.

Solution

- The coefficient of \vec{x}_p must be one.
- If $\vec{x}_n \in N(A)$ is the nullspace of A and \vec{x}_p is one particular solution, then \vec{x}_p and \vec{x}_n is also a particular solution.
- If A is a 2 by 2 matrix of rank 1, then the solution to $A\vec{x} = \vec{b}$ form a line parallel to the line that the nullspace. The line $x + y = 1$ gives such an example.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{x}_p = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\begin{aligned}
 \text{Then } \|\vec{x}_p\| &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \\
 &= \sqrt{2 \cdot \frac{1}{4}} \\
 &= \frac{1}{\sqrt{2}} < 1
 \end{aligned}$$

while the particular solutions having some coordinate equal to zero are (1, 0) and (0, 1) and they both have $\|\vec{x}_p\| = 1$

d) There is always $\vec{x}_n = 0$

Exercise

Write down all known relation between r and m and n if $A\vec{x} = \vec{b}$ has

- a)* No solution for some \vec{b} .
- b)* Infinitely many solutions for every \vec{b} .
- c)* Exactly one solution for some \vec{b} , no solution for another \vec{b} .
- d)* Exactly one solution for every \vec{b} .

Solution

- a)* The system has less than full row rank: $r < m$.
- b)* The system has full row rank and less than full column rank: $m = r < n$.
- c)* The system has full column rank and less than full row rank: $n = r < m$.
- d)* The system has full row and column rank (it is invertible): $m = r = n$.

Exercise

Find a basis for its row space, find a basis for its column space, and determine its rank

$$\begin{array}{ll}
 a) \begin{bmatrix} 0 & 2 & -3 & 4 & 1 & 2 & 1 & 7 \\ 0 & 0 & 3 & -2 & 0 & 4 & -5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 6 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & b) \begin{bmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{bmatrix}
 \end{array}$$

Solution

a) Row Space: every row

$$\text{Column Space: } \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

$$\text{Rank} = 4$$

$$b) \begin{pmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{pmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ \end{array}$$

$$\begin{pmatrix} 3 & 2 & -1 \\ 0 & -1 & 7 \\ 0 & 1 & -7 \\ 0 & -1 & 7 \end{pmatrix} \begin{array}{l} R_1 + 2R_2 \\ \\ R_3 + R_2 \\ R_4 - R_2 \end{array}$$

$$\begin{pmatrix} 3 & 0 & 13 \\ 0 & -1 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \frac{1}{3}R_1 \\ -R_2 \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & \frac{13}{3} \\ 0 & 1 & -7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{Row Space: } [3 \ 2 \ -1], [6 \ 3 \ 5]$$

$$\text{Column Space: } \begin{bmatrix} 3 \\ 6 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \\ -1 \end{bmatrix}$$

$$\text{Rank} = 2$$

Exercise

Find a basis for the row space, find a basis for the null space, find $\dim RS$, find $\dim NS$, and verify $\dim RS + \dim NS = n$

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{bmatrix}$$

Solution

$$\begin{pmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{pmatrix} \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - 5R_1 \end{array}$$

$$\begin{pmatrix} 1 & -2 & 4 & 1 \\ 0 & 7 & -15 & -4 \\ 0 & 7 & -15 & -4 \end{pmatrix} \quad \begin{array}{l} 7R_1 + 2R_2 \\ R_3 - R_2 \end{array}$$

$$\begin{pmatrix} 7 & 0 & -2 & -1 \\ 0 & 7 & -15 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \frac{1}{7}R_1 \\ -\frac{1}{7}R_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & -\frac{2}{7} & -\frac{1}{7} \\ 0 & 1 & -\frac{15}{7} & -\frac{4}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Row Space: $\begin{bmatrix} 1 & -2 & 4 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 & -3 & -1 \end{bmatrix}$

Column Space: $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$

$$\dim RS = 2$$

$$\dim NS = 2$$

$$2 + 2 = 4 \Rightarrow \dim RS + \dim NS = n$$

Exercise

Determine if \vec{b} lies in the column space of the given matrix. If it does, express \vec{b} as linear combination of the column.

$$\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

Solution

$$\left[\begin{array}{cc|c} 2 & -3 & 4 \\ -4 & 6 & -6 \end{array} \right] \quad R_2 + 2R_1$$

$$\left[\begin{array}{cc|c} 2 & -3 & 4 \\ 0 & 0 & 2 \end{array} \right] \quad \begin{array}{l} \frac{1}{2}R_1 \\ \frac{1}{2}R_2 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 2 \\ 0 & 0 & 1 \end{array} \right] \quad R_1 - 2R_2$$

$$\left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 1 \end{array} \right]$$

\vec{b} does not lie in the column space

Exercise

Find the transition matrix from B to C and find $[\vec{x}]_C$

$$a) \quad B = \{(3, 1), (-1, -2)\}, \quad C = \{(1, -3), (5, 0)\}, \quad [\vec{x}]_B = [-1 \quad -2]^T$$

$$b) \quad B = \{(1, 1, 1), (-2, -1, 0), (2, 1, 2)\}, \quad C = \{(-6, -2, 1), (-1, 1, 5), (-1, -1, 1)\},$$

$$[\vec{x}]_B = [-3 \quad 2 \quad 4]^T$$

Solution

$$a) \quad \left[\begin{array}{cc|cc} 1 & 5 & 3 & -1 \\ -3 & 0 & 1 & -2 \end{array} \right] \quad R_2 + 3R_1$$

$$\left[\begin{array}{cc|cc} 1 & 5 & 3 & -1 \\ 0 & 15 & 10 & -5 \end{array} \right] \quad 3R_1 - R_2$$

$$\left[\begin{array}{cc|cc} 3 & 0 & -1 & 2 \\ 0 & 15 & 10 & -5 \end{array} \right] \quad \begin{array}{l} \frac{1}{3}R_1 \\ \frac{1}{15}R_2 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{array} \right]$$

$$[\vec{x}]_c = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$b) \left[\begin{array}{ccc|ccc} -6 & -1 & -1 & 1 & -2 & 2 \\ -2 & 1 & -1 & 1 & -1 & 1 \\ 1 & 5 & 1 & 1 & 0 & 2 \end{array} \right] \begin{array}{l} 3R_2 - R_1 \\ 6R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} -6 & -1 & -1 & 1 & -2 & 2 \\ 0 & 4 & -2 & 2 & -1 & 1 \\ 0 & 29 & 5 & 7 & -2 & 14 \end{array} \right] \begin{array}{l} 4R_1 + R_2 \\ 4R_3 - 29R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} -24 & 0 & -6 & 6 & -9 & 9 \\ 0 & 4 & -2 & 2 & -1 & 1 \\ 0 & 0 & 78 & -30 & 21 & 27 \end{array} \right] \begin{array}{l} 13R_1 + R_3 \\ 39R_2 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} -312 & 0 & 0 & 48 & -96 & 144 \\ 0 & 156 & 0 & 48 & -18 & 66 \\ 0 & 0 & 78 & -30 & 21 & 27 \end{array} \right] \begin{array}{l} -\frac{1}{312}R_1 \\ \frac{1}{156}R_2 \\ \frac{1}{78}R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{13} & \frac{4}{13} & -\frac{6}{13} \\ 0 & 1 & 0 & \frac{4}{13} & -\frac{3}{26} & \frac{11}{26} \\ 0 & 0 & 1 & -\frac{5}{13} & \frac{7}{26} & \frac{9}{26} \end{array} \right]$$

$$[\vec{x}]_c = \begin{bmatrix} -\frac{2}{13} & \frac{4}{13} & -\frac{6}{13} \\ \frac{4}{13} & -\frac{3}{26} & \frac{11}{26} \\ -\frac{5}{13} & \frac{7}{26} & \frac{9}{26} \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{10}{13} \\ \frac{17}{13} \\ \frac{35}{13} \end{bmatrix}$$

Exercise

Does A and A^T have the same number of pivots.

Solution

True

The number of pivots of A is its column rank, r .

We know that the column rank of A equals the row rank of A , which is the column rank of A^T .

Hence, A^T must have the same number of pivots as A .

Exercise

Let $A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ where $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

- a) What is the rank of A ?
- b) What is the dimension of A ?
- c) What are the pivots variables?
- d) What are the free variables?
- e) Find the special (homogeneous) solutions.
- f) What is the nullspace $N(A)$?
- g) Find the particular solution to $A\vec{x} = \vec{b}$
- h) Give the complete solution.

Solution

- a) $\text{Rank}(A) = 2$
- b) Dimension of $A = 2$
- c) The pivots variables are: x_1, x_3
- d) The free variables are: x_2, x_4
- e) Let $x = x_2 s_1 + x_4 s_2$

$$A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = -3x_2 - 2x_4 \\ x_3 = -4x_4 \end{cases}$$

Set $x_2 = 1, x_4 = 0$

The special solution: $\vec{s}_1 = (-3, 1, 0, 0)$

Set $x_2 = 0, x_4 = 1$;

The special solution: $\vec{s}_2 = (-2, 0, -4, 1)$

f) The nullspace is the set $\left\{ x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix} \right\}$

$$N(A) = \begin{pmatrix} -3 & -2 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \end{pmatrix}$$

g) $\vec{x}_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

h) $\vec{x} = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Exercise

Let $A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ where $\vec{b} = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

- What is the rank of A ?
- What is the dimension of A ?
- What are the pivots variables?
- What are the free variables?
- Find the special (homogeneous) solutions.
- What is the nullspace $N(A)$?
- Find the particular solution to $A\vec{x} = \vec{b}$
- Give the complete solution.

Solution

- $\text{Rank}(A) = 3$
- Dimension of $A = 1$
- The pivots variables are: x_1, x_2, x_4

d) The free variables are: x_3

e) Let $x = x_3 s_1$

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{cases} x_1 = -2x_3 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

The special solution: $\vec{s}_1 = (-2, 0, 1, 0)$

$$f) N(A) = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$g) \vec{x}_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$h) \vec{x} = x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Exercise

Let $A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ where $\vec{b} = A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

- a) What is the rank of A ?
- b) What is the dimension of A ?
- c) What are the pivots variables?
- d) What are the free variables?
- e) Find the special (homogeneous) solutions.
- f) What is the nullspace $N(A)$?
- g) Find the particular solution to $A\vec{x} = \vec{b}$
- h) Give the complete solution.

Solution

$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{3}R_2}$$

$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -\frac{1}{3} & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 - 2R_2}$$

$$\begin{pmatrix} 1 & 0 & \frac{2}{3} & -4 \\ 0 & 1 & -\frac{1}{3} & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = -\frac{2}{3}x_3 + 4x_4 \\ x_2 = \frac{1}{3}x_3 - 4x_4 \end{cases}$$

a) $\text{Rank}(A) = 2$

b) Dimension of $A = 2$

c) The pivots variables are: x_1, x_2

d) The free variables are: x_3, x_4

e) Let $x = x_3 s_1 + x_4 s_2$

Set $x_3 = 1, x_4 = 0$

The special solution: $\vec{s}_1 = \left(-\frac{2}{3}, \frac{1}{3}, 1, 0\right)$

Set $x_3 = 0, x_4 = 1$;

The special solution: $\vec{s}_2 = (4, -4, 0, 1)$

f) The nullspace is the set $\left\{ x_3 \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ -4 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$N(A) = \begin{pmatrix} -\frac{2}{3} & 4 \\ \frac{1}{3} & -4 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$g) \quad \vec{x}_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$h) \quad \vec{x} = x_3 \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ -4 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Exercise

Let $A = \begin{pmatrix} 1 & 0 & 0 & \frac{13}{11} \\ 0 & 1 & 0 & -\frac{17}{11} \\ 0 & 0 & 1 & \frac{6}{11} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ where $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

- What is the rank of A ?
- What is the dimension of A ?
- What are the pivots variables?
- What are the free variables?
- Find the special (homogeneous) solutions.
- What is the nullspace $N(A)$?
- Find the particular solution to $A\vec{x} = \vec{b}$
- Give the complete solution.

Solution

- $\text{Rank}(A) = 3$
- Dimension of $A = 1$
- The pivots variables are: x_1, x_2, x_3
- The free variables are: x_4
- Let $x = x_4 \begin{pmatrix} s \\ 1 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 0 & 0 & \frac{13}{11} \\ 0 & 1 & 0 & -\frac{17}{11} \\ 0 & 0 & 1 & \frac{6}{11} \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = -\frac{13}{11} \\ x_2 = \frac{17}{11} \\ x_3 = -\frac{6}{11} \end{cases}$$

Set $x_4 = 1$

The special solution: $\vec{s}_1 = \left(-\frac{13}{11}, \frac{17}{11}, -\frac{6}{11}, 1\right)$

$$f) \quad N(A) = \begin{pmatrix} -\frac{13}{11} \\ \frac{17}{11} \\ -\frac{6}{11} \\ 1 \end{pmatrix}$$

$$g) \quad \vec{x}_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$h) \quad \vec{x} = x_4 \begin{pmatrix} -\frac{13}{11} \\ \frac{17}{11} \\ -\frac{6}{11} \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Exercise

Let $A = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ where $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

- a) What is the rank of A ?
- b) What is the dimension of A ?
- c) What are the pivots variables?
- d) What are the free variables?
- e) Find the special (homogeneous) solutions.
- f) What is the nullspace $N(A)$?
- g) Find the particular solution to $A\vec{x} = \vec{b}$
- h) Give the complete solution.

Solution

- a) $\text{Rank}(A) = 3$
- b) Dimension of $A = 1$
- c) The pivots variables are: x_1, x_2, x_3
- d) The free variables are: x_4
- e) Let $x = x_4 s_1$

$$A = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = \frac{1}{3} \\ x_2 = 1 \\ x_3 = \frac{1}{3} \end{cases}$$

Set $x_4 = 1$

The special solution: $\vec{s}_1 = \left(\frac{1}{3}, 1, \frac{1}{3}, 1\right)$

f) $N(A) = \begin{pmatrix} \frac{1}{3} \\ 1 \\ \frac{1}{3} \\ 1 \end{pmatrix}$

$$g) \quad \vec{x}_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$h) \quad \vec{x} = x_4 \begin{pmatrix} \frac{1}{3} \\ 1 \\ \frac{1}{3} \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Exercise

Let $A = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ where $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

- What is the rank of A ?
- What is the dimension of A ?
- What are the pivots variables?
- What are the free variables?
- Find the special (homogeneous) solutions.
- What is the nullspace $N(A)$?
- Find the particular solution to $A\vec{x} = \vec{b}$
- Give the complete solution.

Solution

- Rank(A) = 3
- Dimension of $A = 2$
- The pivots variables are: x_1, x_3, x_5
- The free variables are: x_2, x_4
- Let $x = x_2 s_1 + x_4 s_2$

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 = -2x_2 + x_4 \\ x_3 = -x_4 \\ x_5 = 0 \end{cases}$$

Set $x_2 = 1 \quad x_4 = 0$

The special solution: $\vec{s}_1 = (-2, 1, 0, 0, 0)$

Set $x_2 = 0$ $x_4 = 1$

The special solution: $\vec{s}_2 = (1, 0, -1, 1, 0)$

$$f) \quad N(A) = \begin{pmatrix} -2 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$g) \quad \vec{x}_p = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$h) \quad \vec{x} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Exercise

Find a basis for each of the four subspaces associated with the given matrix

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{pmatrix} \quad R_2 - 2R_1$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{pmatrix} \quad R_1 - 2R_2$$

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \end{pmatrix} \quad x_1 = -4x_3 \quad \leftarrow \text{Row space}$$

$$\text{Rank}(A) = 1$$

$$\text{Dimension of } A = 1$$

1. Basis for **row space**: $\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$

The pivots variables are: x_1, x_2

2. Basis of the **column spaces**: $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

The free variable is: x_3

Set $x_3 = 1 \Rightarrow s_1 = (-4, 0, 1)$

3. Basis of the **Nullspace**: $\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$

$$A^T = \begin{pmatrix} 1 & 2 \\ 2 & 5 \\ 4 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 5 \\ 4 & 8 \end{pmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{array}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{array}{l} R_1 - 2R_2 \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

4. Basis of the **Left Nullspace**: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Exercise

Find a basis for each of the four subspaces associated with the given matrix

$$B = \begin{pmatrix} 1 & 3 & 0 & 5 \\ 2 & 6 & 1 & 16 \\ 5 & 15 & 0 & 25 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 3 & 0 & 5 \\ 2 & 6 & 1 & 16 \\ 5 & 15 & 0 & 25 \end{pmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 5R_1 \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} x_1 = -3x_2 - 5x_4 \\ x_3 = -6x_4 \\ \\ \end{array} \begin{array}{l} \leftarrow \text{Row space} \\ \leftarrow \text{Row space} \\ \\ \end{array}$$

$$\text{Rank}(A) = 2$$

$$\text{Dimension of } A = 2$$

1. Basis for **row space**:

$$\begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 6 \end{pmatrix}$$

$$\text{The pivots variables are: } x_1, x_3$$

2. Basis of the **column spaces**:

$$\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{The free variable is: } x_2, x_4$$

$$\begin{cases} x_1 = -3x_2 - 5x_4 \\ x_3 = -6x_4 \end{cases}$$

$$\text{Set } x_2 = 1 \quad x_4 = 0$$

$$\text{The special solution: } s_1 = (-3, 1, 0, 0)$$

$$\text{Set } x_2 = 0 \quad x_4 = 1$$

$$\text{The special solution: } s_2 = (-5, 0, -6, 1)$$

3. Basis of the **Nullspace**:

$$\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -5 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 6 & 15 \\ 0 & 1 & 0 \\ 5 & 16 & 25 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 5 \\ 3 & 6 & 15 \\ 0 & 1 & 0 \\ 5 & 16 & 25 \end{pmatrix} \quad \begin{array}{l} R_2 - 3R_1 \\ R_4 - 5R_1 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 0 \end{pmatrix} \quad \begin{array}{l} R_4 - 2R_3 \\ R_4 - 6R_3 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad y_1 = -5y_3$$

Let $y_3 = 1$

4. Basis of the **Left Nullspace**: $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$

Exercise

Find a basis for each of the four subspaces associated with the given matrix

$$C = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad R_2 - R_1$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad \begin{array}{l} R_1 - 3R_2 \\ R_3 - R_2 \end{array}$$

$$\begin{pmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{ll} x_2 = -2x_3 + 2x_5 & \leftarrow \text{Row space} \\ x_4 = -2x_5 & \leftarrow \text{Row space} \end{array}$$

Rank (A) = 2

Dimension of $A = 2$

1. Basis for **row space**:

$$\begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ -2 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

The pivots variables are: x_2, x_4

2. Basis of the **column spaces**:

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

The free variable is: x_3, x_5

$$\begin{cases} x_2 = -2x_3 + 2x_5 \\ x_4 = -2x_5 \end{cases}$$

Set $x_3 = 1 \quad x_5 = 0$

The special solution: $s_1 = (0, -2, 1, 0, 2)$

Set $x_3 = 0 \quad x_5 = 1$

The special solution: $s_2 = (0, 2, 0, -2, 1)$

3. Basis of the **Nullspace**:

$$\begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

$$C^T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \end{pmatrix} \quad \begin{matrix} \\ R_3 - 2R_2 \\ R_4 - 3R_2 \\ R_5 - 4R_2 \end{matrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 2 & 2 \end{pmatrix} \begin{array}{l} \\ 3R_2 - R_4 \\ \\ 3R_5 - 2R_4 \\ \end{array}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{array}{l} \\ \frac{1}{3}R_4 \\ \frac{1}{4}R_5 \\ \\ \end{array}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{l} \\ R_2 + R_5 \\ \\ R_4 - \frac{1}{3}R_5 \\ \end{array}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Basis of the **Left Nullspace**:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Exercise

Find a basis for each of the four subspaces associated with the given matrix

$$D = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix} \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} R_2 + 2R_1 \\ -R_2 \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{ll} x_1 = -x_3 - x_4 & \leftarrow \text{Row space} \\ x_2 = -x_3 & \leftarrow \text{Row space} \end{array}$$

$$\text{Rank}(A) = 2$$

$$\text{Dimension of } A = 2$$

1. Basis for **row space**: $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

The pivots variables are: x_1, x_2

2. Basis of the **column spaces**: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

The free variable is: x_3, x_4

$$\begin{cases} x_1 = -x_3 - x_4 \\ x_2 = -x_3 \end{cases}$$

$$\text{Set } x_3 = 1 \quad x_4 = 0$$

$$\text{The special solution: } s_1 = (-1, 0, 1, 0)$$

$$\text{Set } x_3 = 0 \quad x_4 = 1$$

$$\text{The special solution: } s_2 = (-1, -1, 0, 1)$$

3. Basis of the **Nullspace**:

$$\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$D^T = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - R_1 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} R_1 + R_2 \\ \\ R_3 - 2R_2 \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \\ -R_2 \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad y_1 = -y_3$$

Let $y_3 = 1$

4. Basis of the **Left Nullspace**: $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Exercise

Find a basis for each of the four subspaces associated with the given matrix

$$M = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{pmatrix} \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 5R_1 \end{array}$$

$$\begin{pmatrix} 1 & -2 & 4 & 1 \\ 0 & 7 & -15 & -4 \\ 0 & 7 & -15 & -4 \end{pmatrix} \begin{array}{l} 7R_1 + 2R_2 \\ \\ R_3 - R_2 \end{array}$$

$$\begin{pmatrix} 7 & 0 & -2 & -1 \\ 0 & 7 & -15 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \frac{1}{7}R_1 \\ \frac{1}{7}R_2 \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & -\frac{2}{7} & -\frac{1}{7} \\ 0 & 1 & -\frac{15}{7} & -\frac{4}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} x_1 = \frac{2}{7}x_3 + \frac{1}{7}x_4 \\ x_2 = \frac{15}{7}x_3 + \frac{4}{7}x_4 \\ \end{array} \begin{array}{l} \leftarrow \text{Row space} \\ \leftarrow \text{Row space} \end{array}$$

$$\text{Rank}(A) = 2$$

$$\text{Dimension of } A = 2$$

1. Basis for **row space**:

$$\begin{pmatrix} 1 \\ 0 \\ -\frac{2}{7} \\ -\frac{1}{7} \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ -\frac{15}{7} \\ -\frac{4}{7} \end{pmatrix}$$

The pivots variables are: x_1, x_2

2. Basis of the **column spaces**:

$$\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \quad \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$$

The free variable is: x_3, x_4

$$\begin{cases} x_1 = \frac{2}{7}x_3 + \frac{1}{7}x_4 \\ x_2 = \frac{15}{7}x_3 + \frac{4}{7}x_4 \end{cases}$$

$$\text{Set } x_3 = 1 \quad x_4 = 0$$

$$\text{The special solution: } s_1 = \left(\frac{2}{7}, \frac{15}{7}, 1, 0 \right)$$

$$\text{Set } x_3 = 0 \quad x_4 = 1$$

$$\text{The special solution: } s_2 = \left(\frac{1}{7}, \frac{4}{7}, 0, 1 \right)$$

3. Basis of the **Nullspace**:

$$\begin{pmatrix} \frac{2}{7} \\ \frac{15}{7} \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} \frac{1}{7} \\ \frac{4}{7} \\ 0 \\ 1 \end{pmatrix}$$

$$M^T = \begin{pmatrix} 1 & 3 & 5 \\ -2 & 1 & -3 \\ 4 & -3 & 5 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 5 \\ -2 & 1 & -3 \\ 4 & -3 & 5 \\ 1 & -1 & 1 \end{pmatrix} \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 - 4R_1 \\ R_4 - R_1 \end{array}$$

$$\begin{pmatrix} 1 & 3 & 5 \\ 0 & 7 & 7 \\ 0 & -15 & -15 \\ 0 & -4 & -4 \end{pmatrix} \begin{array}{l} \\ \frac{1}{7}R_2 \\ \frac{1}{15}R_3 \\ \frac{1}{4}R_4 \end{array}$$

$$\begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{array}{l} R_1 - 3R_2 \\ \\ R_3 + R_2 \\ R_4 + R_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} y_1 = -2y_3 \\ y_1 = -y_3 \\ \\ \end{array}$$

Let $y_3 = 1$

4. Basis of the **Left Nullspace**: $\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

Exercise

Find a basis for each of the four subspaces associated with the given matrix

$$N = \begin{pmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{pmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ \end{array}$$

$$\begin{pmatrix} 3 & 2 & -1 \\ 0 & -1 & 7 \\ 0 & 1 & -7 \\ 0 & -1 & 7 \end{pmatrix} \begin{array}{l} R_1 + 2R_2 \\ \\ R_3 + R_2 \\ R_4 - R_2 \end{array}$$

$$\begin{pmatrix} 3 & 0 & 13 \\ 0 & -1 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \frac{1}{3}R_1 \\ -R_2 \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & \frac{13}{3} \\ 0 & 1 & -7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} x_1 = -\frac{13}{3}x_3 \\ x_2 = 7x_3 \\ \\ \end{array} \begin{array}{l} \leftarrow \text{Row space} \\ \leftarrow \text{Row space} \\ \\ \end{array}$$

$$\text{Rank}(A) = 2$$

$$\text{Dimension of } A = 2$$

1. Basis for **row space**: $\begin{pmatrix} 1 \\ 0 \\ \frac{13}{3} \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -7 \end{pmatrix}$

The pivots variables are: x_1, x_2

2. Basis of the **column spaces**: $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$

The free variable is: x_3, x_4

$$\begin{cases} x_1 = -\frac{13}{7}x_3 \\ x_2 = 7x_3 \end{cases}$$

Set $x_3 = 1$. The special solution: $s_1 = \left(-\frac{13}{3}, 7, 1\right)$

3. Basis of the **Nullspace**: $\begin{pmatrix} -\frac{13}{3} \\ 7 \\ 1 \end{pmatrix}$

$$N^T = \begin{pmatrix} 3 & 6 & -3 & 0 \\ 2 & 3 & -1 & -1 \\ -1 & 5 & -6 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 6 & -3 & 0 \\ 2 & 3 & -1 & -1 \\ -1 & 5 & -6 & 7 \end{pmatrix} \begin{array}{l} \\ 3R_2 - 3R_1 \\ 3R_3 + R_1 \end{array}$$

$$\begin{pmatrix} 3 & 6 & -3 & 0 \\ 0 & -9 & 6 & -3 \\ 0 & 21 & -21 & 21 \end{pmatrix} \begin{array}{l} \frac{1}{3}R_1 \\ -\frac{1}{9}R_2 \\ \frac{1}{21}R_3 \end{array}$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 1 & -1 & 1 \end{pmatrix} \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 - R_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{array}{l} R_1 + R_3 \\ R_1 - 2R_3 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{4}{3} \\ 0 & 1 & 0 & \frac{5}{3} \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \end{pmatrix} -3R_3$$

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{4}{3} \\ 0 & 1 & 0 & \frac{5}{3} \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{array}{l} y_1 = \frac{4}{3}y_4 \\ y_2 = -\frac{5}{3}y_4 \\ y_3 = -2y_4 \end{array}$$

Let $y_4 = 1$

4. Basis of the **Left Nullspace**: $\begin{pmatrix} \frac{4}{3} \\ -\frac{5}{3} \\ -2 \\ 1 \end{pmatrix}$

