Solution   *Section* **1.5 – Length of Curves**

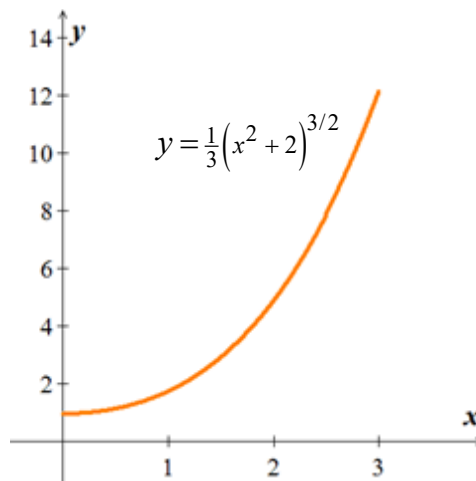## *Exercise*

Find the length of the curve $y = \frac{1}{3}\left(x^2 + 2\right)^{3/2}$ from $x = 0$ to $x = 3$.

Solution

$$\frac{dy}{dx} = \frac{1}{3}\frac{3}{2}\left(x^2 + 2\right)^{1/2}(2x)$$

$$= x\left(x^2 + 2\right)^{1/2}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + x^2\left(x^2 + 2\right)}$$

$$= \sqrt{1 + x^4 + 2x^2}$$

$$= \sqrt{\left(x^2 + 1\right)^2}$$

$$= x^2 + 1$$

$$L = \int_0^3 \left(x^2 + 1\right) dx$$

$$= \frac{1}{3}x^3 + x \Big|_0^3$$

$$= 9 + 3$$

$$= 12 \quad unit$$

$y = \frac{1}{3}\left(x^2 + 2\right)^{3/2}$

Exercise

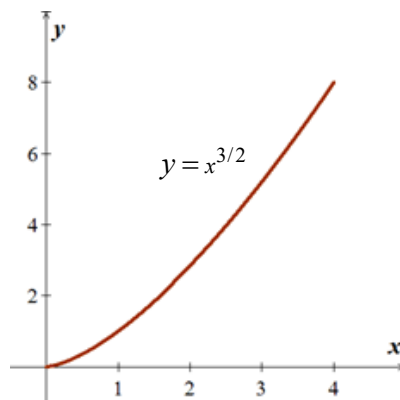Find the length of the curve $y = (x)^{3/2}$ from $x = 0$ to $x = 4$.

Solution

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{9}{4}x}$$

$$= \sqrt{\frac{4 + 9x}{4}}$$

$$= \frac{1}{2}\sqrt{4 + 9x}$$

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$$L = \int_0^4 \tfrac{1}{2}(4+9x)^{1/2}\,dx$$

$$= \tfrac{1}{18}\int_0^4 (4+9x)^{1/2}\,d(4+9x)$$

$$= \tfrac{1}{18}\,(4+9x)^{27}\,\Big|_0^4$$

$$= \tfrac{1}{27}\left(40^{3/2} - 4^{3/2}\right)$$

$$= \tfrac{1}{27}\left(80\sqrt{10} - 8\right)$$

$$= \tfrac{8}{27}\left(10\sqrt{10} - 1\right) \quad unit$$

$y = x^{3/2}$

Exercise

Find the length of the curve $x = \dfrac{y^{3/2}}{3} - y^{1/2}$ from $y = 1$ to $y = 9$.

Solution

$$a = \tfrac{1}{3}, \quad m = \tfrac{3}{2}, \quad b = -1, \quad n = \tfrac{1}{2}$$

1. $m + n = \dfrac{3}{2} + \dfrac{1}{2} = 2$ ✓

2. $abmn = \dfrac{1}{3}(-1)\left(\dfrac{3}{2}\right)\left(\dfrac{1}{2}\right) = -\dfrac{1}{4}$ ✓

$$L = \left(\tfrac{1}{3}y^{3/2} + y^{1/2}\right)\Big|_1^9$$

$$= 9 + 3 - \tfrac{4}{3}$$

$$= \tfrac{32}{3} \quad unit$$

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$$\frac{dx}{dy} = \tfrac{1}{2}y^{1/2} - \tfrac{1}{2}y^{-1/2}$$

$$= \tfrac{1}{2}\left(y^{1/2} - \frac{1}{y^{1/2}}\right)$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \tfrac{1}{4}\left(y^{1/2} - \frac{1}{y^{1/2}}\right)^2}$$

$$= \sqrt{1 + \tfrac{1}{4}\left(y - 2 + \frac{1}{y}\right)}$$

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$$= \sqrt{1 + \frac{1}{4}y - \frac{1}{2} + \frac{1}{4y}}$$

$$= \sqrt{\frac{1}{4}y + \frac{1}{2} + \frac{1}{4y}}$$

$$= \sqrt{\frac{1}{4}\left(y + 2 + \frac{1}{y}\right)}$$

$$= \frac{1}{2}\sqrt{\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2}$$

$$= \frac{1}{2}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)$$

$$L = \frac{1}{2}\int_1^9 \left(y^{1/2} + y^{-1/2}\right) dy$$

$$= \frac{1}{2}\left(\frac{2}{3}y^{3/2} + 2y^{1/2}\right)\Big|_1^9$$

$$= \frac{1}{3}y^{3/2} + y^{1/2}\Big|_1^9$$

$$= \frac{1}{3}9^{3/2} + 3 - \left(\frac{1}{3} + 1\right)$$

$$= 9 + 3 - \frac{4}{3}$$

$$= \frac{32}{3} \quad unit$$

Exercise

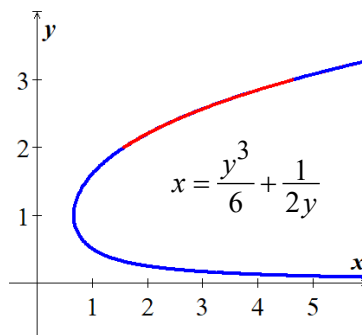Find the length of the curve $x = \frac{y^3}{6} + \frac{1}{2y}$ from $y = 2$ to $y = 3$.

Solution

$$a = \frac{1}{6}, \quad m = 3, \quad b = \frac{1}{2}, \quad n = -1$$

1. $m + n = 3 - 1 = 2$ ✓
2. $abmn = \frac{1}{6}\left(\frac{1}{2}\right)(3)(-1) = -\frac{1}{4}$ ✓

$$L = \left(\frac{y^3}{6} - \frac{1}{2y}\right)\Big|_2^3$$

$$= \frac{1}{2}\left[\frac{27}{2} - \frac{1}{3} - \left(\frac{8}{3} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left(\frac{26}{3} - \frac{13}{6}\right)$$

$$x = \frac{y^3}{6} + \frac{1}{2y}$$

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$$= \frac{13}{4} \quad unit$$

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$$\frac{dx}{dy} = \frac{1}{2}y^2 - \frac{1}{2y^2}$$

$$= \frac{1}{2}\left(y^2 - y^{-2}\right)$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \frac{1}{4}\left(y^2 - y^{-2}\right)^2}$$

$$= \frac{1}{2}\sqrt{4 + \left(y^4 - 2 + y^{-4}\right)}$$

$$= \frac{1}{2}\sqrt{y^4 + 2 + y^{-4}}$$

$$= \frac{1}{2}\sqrt{\left(y^2 + y^{-2}\right)^2}$$

$$= \frac{1}{2}\left(y^2 + y^{-2}\right)$$

$$L = \frac{1}{2}\int_2^3 \left(y^2 + y^{-2}\right)dy$$

$$= \left(\frac{y^3}{6} - \frac{1}{2y}\right)\Big|_2^3$$

$$= \frac{1}{2}\left[9 - \frac{1}{3} - \left(\frac{8}{3} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left(\frac{26}{3} - \frac{13}{6}\right)$$

$$= \frac{13}{4} \quad unit$$

Exercise

Find the length of the curve $f(x) = x^3 + \frac{1}{12x}$ *for* $\frac{1}{2} \le x \le 2$

Solution

$$a = 1, \quad m = 3, \quad b = \frac{1}{12}, \quad n = -1$$

1. $m + n = 2$ ✔   **2.** $abmn = -\frac{1}{4}$ ✔

$$L = \left(x^3 - \frac{1}{12x}\right)\Big|_{1/2}^2$$

$$= 8 - \frac{1}{24} - \frac{1}{8} + \frac{1}{6}$$

$$= 8 \quad unit$$

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Exercise

Find the length of the curve of $\qquad f(x) = \frac{1}{5}x^5 + \frac{1}{12x^3}\quad 1 \le x \le 2$

Solution

$a = \frac{1}{5}, \quad m = 5, \quad b = \frac{1}{12}, \quad n = -3$

   1. $\quad m + n = 5 - 3 = 2$ ✓   **2.** $\quad abmn = \frac{1}{5}\left(\frac{1}{12}\right)(5)(-3) = -\frac{1}{4}$ ✓

$L = \frac{1}{5}x^5 - \frac{1}{12x^3}\Big|_1^2$

$= \frac{32}{5} - \frac{1}{96} - \frac{1}{5} + \frac{1}{12}$

$= \frac{31}{5} + \frac{7}{96}$

$\underline{= \frac{3011}{480}\quad unit}$

Exercise

Find the length of the curve of $\qquad y = \frac{1}{3}x^{1/2} - x^{3/2}, \quad 0 \le x \le \frac{1}{3}$

Solution

$a = \frac{1}{3}, \quad m = \frac{1}{2}, \quad b = -1, \quad n = \frac{3}{2}$

   1. $\quad m + n = \frac{1}{2} + \frac{3}{2} = 2$ ✓   **2.** $\quad abmn = \frac{1}{3}(-1)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = -\frac{1}{4}$ ✓

$L = \frac{1}{3}x^{1/2} + x^{3/2}\Big|_0^{1/3}$

$= \frac{1}{3\sqrt{3}} + \frac{1}{3\sqrt{3}}$

$\underline{= \frac{2}{3\sqrt{3}}\quad unit} \qquad\qquad \underline{= \frac{2\sqrt{3}}{9}}$

Exercise

Find the length of the curve of $\qquad y = \frac{1}{3}x^3 + \frac{1}{4x}, \quad 1 \le x \le 2$

Solution

$a = \frac{1}{3}, \quad m = 3, \quad b = \frac{1}{4}, \quad n = -1$

   1. $\quad m + n = 3 - 1 = 2$ ✓   **2.** $\quad abmn = \frac{1}{3}\left(\frac{1}{4}\right)(3)(-1) = -\frac{1}{4}$ ✓

$L = \frac{1}{3}x^3 - \frac{1}{4x}\Big|_1^2$

$$= \frac{8}{3} - \frac{1}{8} - \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{3} + \frac{1}{8}$$

$$\underline{= \frac{59}{24} \quad unit}$$

Exercise

Find the length of the curve of $\qquad y = 2e^x + \frac{1}{8}e^{-x} \quad 0 \le x \le \ln 2$

Solution

$$a = 2, \quad m = 1, \quad b = \frac{1}{8}, \quad n = -1$$

1. $m = -n = 1$ ✓ $\qquad\qquad$ **2.** $abmn = 2\left(\frac{1}{8}\right)(1)(-1) = -\frac{1}{4}$ ✓

$$L = 2e^x - \frac{1}{8}e^{-x} \ \bigg|_0^{\ln 2}$$

$$= 2e^{\ln 2} - \frac{1}{8}e^{-\ln 2} - 2 + \frac{1}{8}$$

$$= 4 - \frac{1}{16} - \frac{15}{8}$$

$$\underline{= \frac{33}{16} \quad unit}$$

Exercise

Find the length of the curve of $\qquad y = e^{2x} + \frac{1}{16}e^{-2x}, \quad 0 \le x \le \ln 3$

Solution

$$a = 1, \quad m = 2, \quad b = \frac{1}{16}, \quad n = -2$$

1. $m = -n = 2$ ✓ $\qquad\qquad$ **2.** $abmn = 1\left(\frac{1}{16}\right)(2)(-2) = -\frac{1}{4}$ ✓

$$L = e^{2x} - \frac{1}{16}e^{-2x} \ \bigg|_0^{\ln 3}$$

$$= e^{2\ln 3} - \frac{1}{16}e^{-2\ln 3} - 1 + \frac{1}{16}$$

$$= 9 - \frac{1}{16}\left(\frac{1}{9}\right) - \frac{15}{16}$$

$$= \frac{1,160}{144}$$

$$\underline{= \frac{145}{18} \quad unit}$$

Exercise

Find the length of the curve $y = \ln(\cos x)$ $\quad 0 \le x \le \frac{\pi}{4}$

Solution

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$$

$$L = \int_0^{\pi/4} \sqrt{1 + \tan^2 x} \; dx$$

$$= \int_0^{\pi/4} \sqrt{\sec^2 x} \; dx$$

$$= \int_0^{\pi/4} \sec x \; dx$$

$$= \ln|\sec x + \tan x| \; \Big|_0^{\pi/4}$$

$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0|$$

$$= \ln\left|\sqrt{2} + 1\right| - \ln|1 + 0|$$

$$= \ln\left|\sqrt{2} + 1\right| - 0$$

$$= \ln\left(\sqrt{2} + 1\right) \quad unit$$

Exercise

Find the length of the curve $f(y) = 2e^{\sqrt{2}y} + \frac{1}{16}e^{-\sqrt{2}y}$ $\quad for \quad 0 \le y \le \frac{\ln 2}{\sqrt{2}}$

Solution

$$a = 2, \quad m = \sqrt{2}, \quad b = \frac{1}{16}, \quad n = -\sqrt{2}$$

1. $m = -n$ ✓

2. $abmn = 2\left(\sqrt{2}\right)\left(\frac{1}{16}\right)\left(-\sqrt{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(2e^{\sqrt{2}y} + \frac{1}{16}e^{-\sqrt{2}y}\right) \Big|_0^{\ln 2/\sqrt{2}}$$

$$= 2e^{\ln 2} + \frac{1}{16}e^{-\ln 2} - 2 - \frac{1}{16}$$

$$= 4 + \frac{1}{32} - \frac{33}{16}$$

$$= \frac{63}{32} \quad unit$$

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Exercise
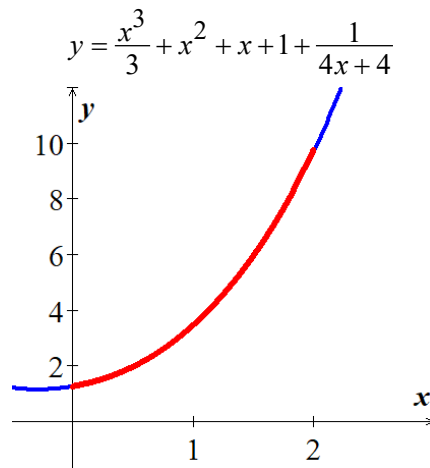
Find the length of the curve $y = \dfrac{x^3}{3} + x^2 + x + 1 + \dfrac{1}{4x+4}$ $0 \le x \le 2$

Solution

$\dfrac{dy}{dx} = x^2 + 2x + 1 - \dfrac{1}{4} \dfrac{1}{(x+1)^2}$

$= (x+1)^2 - \dfrac{1}{4} \dfrac{1}{(x+1)^2}$

$\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \sqrt{1 + \left((x+1)^2 - \dfrac{1}{4} \dfrac{1}{(x+1)^2}\right)^2}$

$= \sqrt{1 + (x+1)^4 - \dfrac{1}{2} - \dfrac{1}{16} \dfrac{1}{(x+1)^4}}$

$= \sqrt{(x+1)^4 + \dfrac{1}{2} - \dfrac{1}{16} \dfrac{1}{(x+1)^4}}$

$= \sqrt{\left((x+1)^2 + \dfrac{1}{4} \dfrac{1}{(x+1)^2}\right)^2}$

$= (x+1)^2 + \dfrac{1}{4}(x+1)^{-2}$

$L = \displaystyle\int_0^2 \left((x+1)^2 + \dfrac{1}{4}(x+1)^{-2}\right) dx$

$= \displaystyle\int_0^2 \left((x+1)^2 + \dfrac{1}{4}(x+1)^{-2}\right) d(x+1)$

$= \dfrac{1}{3}(x+1)^3 - \dfrac{1}{4}\dfrac{1}{x+1} \Big|_0^2$

$= 9 - \dfrac{1}{12} - \left(\dfrac{1}{3} - \dfrac{1}{4}\right)$

$= \underline{\dfrac{53}{6}}$ _unit_

$y = \dfrac{x^3}{3} + x^2 + x + 1 + \dfrac{1}{4x+4}$

Exercise

Find the length of the curve $y = \dfrac{x^3}{3} + x^2 + x + 1 + \dfrac{1}{4x+4}$ $0 \le x \le 4$

Solution

$$\frac{dy}{dx} = x^2 + 2x + 1 - \frac{1}{4}\frac{1}{(x+1)^2}$$

$$= (x+1)^2 - \frac{1}{4}\frac{1}{(x+1)^2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left((x+1)^2 - \frac{1}{4}\frac{1}{(x+1)^2}\right)^2}$$

$$= \sqrt{1 + (x+1)^4 - \frac{1}{2} - \frac{1}{16}\frac{1}{(x+1)^4}}$$

$$= \sqrt{(x+1)^4 + \frac{1}{2} - \frac{1}{16}\frac{1}{(x+1)^4}}$$

$$= \sqrt{\left((x+1)^2 + \frac{1}{4}\frac{1}{(x+1)^2}\right)^2}$$

$$= (x+1)^2 + \frac{1}{4}(x+1)^{-2}$$

$$L = \int_0^4 \left((x+1)^2 + \frac{1}{4}(x+1)^{-2}\right)dx$$

$$= \int_0^4 \left((x+1)^2 + \frac{1}{4}(x+1)^{-2}\right)d(x+1)$$

$$= \left(\frac{1}{3}(x+1)^3 - \frac{1}{4}(x+1)^{-1}\right)\Big|_0^4$$

$$= \frac{125}{3} - \frac{1}{20} - \frac{1}{3} + \frac{1}{4}$$

$$= \frac{124}{3} + \frac{1}{5}$$

$$= \frac{623}{15} \ unit$$

Exercise

Find the length of the curve $y = \ln\left(e^x - 1\right) - \ln\left(e^x + 1\right)$ $\ln 2 \le x \le \ln 3$

Solution

$$y = \ln\left(e^x - 1\right) - \ln\left(e^x + 1\right) \Rightarrow \frac{dy}{dx} = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1}$$

$$= \frac{e^{2x} + e^x - e^{2x} + e^x}{e^{2x} - 1}$$

$$= \frac{2e^x}{e^{2x}-1}$$

$$L = \int_{\ln 2}^{\ln 3} \sqrt{1 + \left(\frac{2e^x}{e^{2x}-1}\right)^2}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{1 + \frac{4e^{2x}}{e^{4x} - 2e^{2x} + 1}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} - 2e^{2x} + 1 + 4e^{2x}}{\left(e^{2x}-1\right)^2}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} + 2e^{2x} + 1}{\left(e^{2x}-1\right)^2}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{\left(e^{2x}+1\right)^2}{\left(e^{2x}-1\right)^2}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{e^{2x}+1}{e^{2x}-1}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\frac{e^{2x}+1}{e^x}}{\frac{e^{2x}-1}{e^x}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\frac{e^{2x}}{e^x} + \frac{1}{e^x}}{\frac{e^{2x}}{e^x} - \frac{1}{e^x}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx \qquad \textit{or} \ \ \text{Let } u = e^x - e^{-x} \ \Rightarrow du = \left(e^x + e^{-x}\right)dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{1}{e^x - e^{-x}}\, d\!\left(e^x - e^{-x}\right) \qquad d\!\left(e^x - e^{-x}\right) = \left(e^x + e^{-x}\right)dx$$

$$= \ln\left|e^x - e^{-x}\right| \, \Big|_{\ln 2}^{\ln 3}$$

$$= \ln\left(3 - \tfrac{1}{3}\right) - \ln\left(2 - \tfrac{1}{2}\right)$$

$$= \ln\left(\frac{8}{3}\right) - \ln\left(\frac{3}{2}\right)$$

$$= \ln\left(\frac{16}{9}\right) \quad unit$$

Exercise

Find the length of the curve $\quad f(x) = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2} \quad 1 \le x \le 4$

Solution

$$a = \frac{2}{3}, \quad m = \frac{3}{2}, \quad b = -\frac{1}{2}, \quad n = \frac{1}{2}$$

1. $m + n = \frac{3}{2} + \frac{1}{2} = 2$ ✓

2. $abmn = \frac{2}{3}\left(\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(\frac{2}{3}x^{3/2} + \frac{1}{2}x^{1/2}\right)\Big|_1^4$$

$$= \frac{2}{3}4^{3/2} + 1 - \frac{2}{3} - \frac{1}{2}$$

$$= \frac{16}{3} - \frac{2}{3} + \frac{1}{2}$$

$$= \frac{31}{6} \quad unit$$

Exercise

Find the length of the curve $\quad f(x) = x^3 + \frac{1}{12x} \quad 1 \le x \le 4$

Solution

$$a = 1, \quad m = 3, \quad b = \frac{1}{12}, \quad n = -1$$

1. $m + n = 3 - 1 = 2$ ✓

2. $abmn = (1)\left(\frac{1}{12}\right)(3)(-1) = -\frac{1}{4}$ ✓

$$L = \left(x^3 - \frac{1}{12x}\right)\Big|_1^4$$

$$= 4^3 - \frac{1}{48} - 1 + \frac{1}{12}$$

$$= 63 + \frac{3}{48}$$

$$= \frac{3,027}{48} \quad unit$$

Exercise

Find the length of the curve $\quad f(x) = \frac{1}{8}x^4 + \frac{1}{4x^2} \quad 1 \le x \le 10$

Solution

$a = \frac{1}{8}, \quad m = 4, \quad b = \frac{1}{4}, \quad n = -2$

1. $\quad m + n = 4 - 2 = 2 \quad \checkmark$

2. $\quad abmn = \left(\frac{1}{8}\right)\left(\frac{1}{4}\right)(4)(-2) = -\frac{1}{4} \quad \checkmark$

$L = \left(\frac{1}{8}x^4 - \frac{1}{4x^2} \right) \Big|_1^{10}$

$= \frac{10^4}{8} - \frac{1}{400} - \frac{1}{8} + \frac{1}{4}$

$= \frac{9,999}{8} + \frac{99}{400}$

$= \frac{9}{8}\left(1111 + \frac{11}{50}\right)$

$= \frac{9}{8}\left(\frac{55,561}{50}\right)$

$= \frac{500,049}{400} \quad unit$

Exercise

Find the length of the curve $\quad f(x) = \frac{1}{4}x^4 + \frac{1}{8x^2} \quad 3 \le x \le 8$

Solution

$a = \frac{1}{4}, \quad m = 4, \quad b = \frac{1}{8}, \quad n = -2$

1. $\quad m + n = 4 - 2 = 2 \quad \checkmark$

2. $\quad abmn = \left(\frac{1}{4}\right)\left(\frac{1}{8}\right)(4)(-2) = -\frac{1}{4} \quad \checkmark$

$L = \left(\frac{1}{4}x^4 - \frac{1}{8x^2} \right) \Big|_3^{8}$

$= \frac{8^4}{4} - \frac{1}{8^3} - \frac{81}{4} + \frac{1}{72}$

$= \frac{4,015}{4} - \frac{1}{512} + \frac{1}{72}$

$= \frac{1}{4}\left(4,015 - \frac{1}{128} + \frac{1}{18}\right)$

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$$= \frac{1}{4}\left(4{,}015 + \frac{55}{1{,}152}\right)$$

$$= \frac{4{,}625{,}335}{4{,}608} \quad unit$$

Exercise

Find the length of the curve $\quad f(x) = \frac{1}{10}x^5 + \frac{1}{6x^3} \quad 1 \le x \le 7$

Solution

$$a = \frac{1}{10}, \quad m = 5, \quad b = \frac{1}{6}, \quad n = -3$$

1. $m + n = 5 - 3 = 2$ ✓

2. $abmn = \left(\frac{1}{10}\right)\left(\frac{1}{6}\right)(5)(-3) = -\frac{1}{4}$ ✓

$$L = \left(\frac{1}{10}x^5 - \frac{1}{6x^3}\right)\bigg|_1^7$$

$$= \frac{7^5}{10} - \frac{1}{2{,}058} - \frac{1}{10} + \frac{1}{6}$$

$$= \frac{8{,}403}{5} + \frac{57}{343}$$

$$= \frac{2{,}882{,}514}{1{,}715} \quad unit$$

Exercise

Find the length of the curve $\quad f(x) = \frac{3}{10}x^{1/3} - \frac{3}{2}x^{5/3} \quad 0 \le x \le 12\,\text{b}$

Solution

$$a = \frac{1}{10}, \quad m = 5, \quad b = \frac{1}{6}, \quad n = -3$$

1. $m + n = 5 - 3 = 2$ ✓

2. $abmn = \left(\frac{1}{10}\right)\left(\frac{1}{6}\right)(5)(-3) = -\frac{1}{4}$ ✓

$$L = \left(\frac{3}{10}x^{1/3} + \frac{3}{2}x^{5/3}\right)\bigg|_0^{12}$$

$$= \frac{3}{10}\sqrt[3]{12} + \frac{3}{2}12\sqrt[3]{144}$$

$$= \frac{3}{10}\sqrt[3]{12} + 18\sqrt[3]{144} \quad unit \qquad = \frac{3}{10}\sqrt[3]{12}\left(1 + 600\sqrt[3]{12}\right)$$

Exercise

Find the length of the curve $f(x) = x^{1/2} - \frac{1}{3}x^{3/2}$ $2 \le x \le 9$

Solution

$a = 1, \quad m = \frac{1}{2}, \quad b = -\frac{1}{3}, \quad n = \frac{3}{2}$

1. $m + n = \frac{1}{2} + \frac{3}{2} = 2$ ✓

2. $abmn = (1)\left(-\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = -\frac{1}{4}$ ✓

$L = \left(x^{1/2} + \frac{1}{3}x^{3/2} \ \Big|_2^9\right.$

$= 3 + 9 - \sqrt{2} - \frac{2\sqrt{2}}{3}$

$= \frac{1}{3}\left(36 - 5\sqrt{2}\right)$ *unit*

Exercise

Find the length of the curve $y = x^{1/2} - \frac{1}{3}x^{3/2}$ $1 \le x \le 4$

Solution

$a = 1, \quad m = \frac{1}{2}, \quad b = -\frac{1}{3}, \quad n = \frac{3}{2}$

1. $m + n = \frac{1}{2} + \frac{3}{2} = 2$ ✓

2. $abmn = (1)\left(-\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = -\frac{1}{4}$ ✓

$L = \left(x^{1/2} + \frac{1}{3}x^{3/2} \ \Big|_1^4\right.$

$= 2 + \frac{8}{3} - 1 - \frac{1}{3}$

$= 1 + \frac{7}{3}$

$= \frac{10}{3}$ *unit*

Exercise

Find the length of the curve $x = y^{2/3}$, $1 \le y \le 8$

Solution

$x' = \frac{2}{3}y^{-1/3}$

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$$(x')^2 = \frac{4}{9}y^{-2/3}$$

$$L = \int_1^8 \sqrt{1 + \frac{4}{9y^{2/3}}}\ dy$$

$$= \int_1^8 \frac{1}{3y^{1/3}}\sqrt{9y^{2/3}+4}\ dy$$

$$= \frac{1}{3}\int_1^8 y^{-1/3}\sqrt{9y^{2/3}+4}\ dy$$

$$= \frac{1}{18}\int_1^8 \left(9y^{2/3}+4\right)^{1/2}\ d\left(9y^{2/3}+4\right)$$

$$= \frac{1}{27}\left(9y^{2/3}+4\right)^{3/2}\ \Big|_1^8$$

$$= \frac{1}{27}\left(\left(9\left(2^3\right)^{2/3}+4\right)^{3/2}-13^{3/2}\right)$$

$$\underline{= \frac{1}{27}\left(40^{3/2}-13^{3/2}\right)\quad unit}$$

Exercise

Find the length of the curve $y = 2x + 4 \quad -2 \le x \le 2$

Solution

$$\sqrt{1+(y')^2} = \sqrt{1+2^2}$$

$$= \sqrt{5}$$

$$L = \int_{-2}^2 \sqrt{5}\ dx$$

$$= \sqrt{5}\ x\ \Big|_{-2}^2$$

$$\underline{= 4\sqrt{5}\quad unit}$$

Exercise

Find the length of the curve $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ $x \in [1,\ 2]$

Solution

$a = \dfrac{1}{6}, \quad m = 3, \quad b = \dfrac{1}{2}, \quad n = -1$

   1. $m + n = 3 - 1 = 2$ ✓

   2. $abmn = \left(\dfrac{1}{6}\right)(3)\left(\dfrac{1}{2}\right)(-1) = -\dfrac{1}{4}$ ✓

$L = \dfrac{x^3}{6} - \dfrac{1}{2x} \Big|_1^2$

$= \dfrac{4}{3} - \dfrac{1}{4} - \dfrac{1}{6} + \dfrac{1}{2}$

$\underline{= \dfrac{7}{12} \quad unit}$

Exercise

Find the length of the curve $f(x) = x^{1/2} - \dfrac{1}{3}x^{3/2}$ $1 \le x \le 3$

Solution

$a = 1, \quad m = \dfrac{1}{2}, \quad b = -\dfrac{1}{3}, \quad n = \dfrac{3}{2}$

   1. $m + n = \dfrac{1}{2} + \dfrac{3}{2} = 2$ ✓

   2. $abmn = (1)\left(-\dfrac{1}{3}\right)\left(\dfrac{1}{2}\right)\left(\dfrac{3}{2}\right) = -\dfrac{1}{4}$ ✓

$L = \left(x^{1/2} + \dfrac{1}{3}x^{3/2}\right) \Big|_1^3$

$= \sqrt{3} + \sqrt{3} - 1 - \dfrac{1}{3}$

$\underline{= 2\sqrt{3} - \dfrac{4}{3} \quad unit}$

Exercise

Find the length of the curve $y = \dfrac{3}{4}x^{4/3} - \dfrac{3}{8}x^{2/3} + 5, \quad 1 \le x \le 8$

Solution

$a = \dfrac{3}{4}, \quad m = \dfrac{4}{3}, \quad b = -\dfrac{3}{8}, \quad n = \dfrac{2}{3}$

1. $m+n = \dfrac{4}{3} + \dfrac{2}{3} = 2$ ✓

2. $abmn = \left(\dfrac{3}{4}\right)\left(\dfrac{4}{3}\right)\left(-\dfrac{3}{8}\right)\left(\dfrac{2}{3}\right) = -\dfrac{1}{4}$ ✓

$$L = \left(\dfrac{3}{4}x^{4/3} + \dfrac{3}{8}x^{2/3}\right)\Bigg|_1^8$$

$$= \dfrac{3}{4}\left(2^3\right)^{4/3} + \dfrac{3}{8}\left(2^3\right)^{2/3} - \dfrac{3}{4} - \dfrac{3}{8}$$

$$= 12 + \dfrac{3}{2} - \dfrac{3}{4} - \dfrac{3}{8}$$

$$= \dfrac{96 + 12 - 6 - 3}{8}$$

$$= \dfrac{99}{8} \quad unit$$

Exercise

Find the length of the curve $\quad y = \ln x - \dfrac{1}{8}x^2; \quad 1 \le x \le 2$

Solution

$$\sqrt{1+(y')^2} = \sqrt{1 + \left(\dfrac{1}{x} - \dfrac{1}{4}x\right)^2}$$

$$= \sqrt{1 + \dfrac{1}{x^2} - \dfrac{1}{2} + \dfrac{1}{16}x^2}$$

$$= \sqrt{\dfrac{1}{x^2} + \dfrac{1}{2} + \dfrac{1}{16}x^2}$$

$$= \sqrt{\left(\dfrac{1}{x} + \dfrac{1}{4}x\right)^2}$$

$$= \dfrac{1}{x} + \dfrac{1}{4}x$$

$$L = \int_1^2 \left(\dfrac{1}{x} + \dfrac{1}{4}x\right) dx$$

$$= \ln x + \dfrac{1}{8}x^2 \Bigg|_1^2$$

$$= \ln 2 + \dfrac{1}{2} - \dfrac{1}{8}$$

$$= \ln 2 + \dfrac{3}{8} \quad unit$$

Exercise

Find the length of the curve $\quad y = \frac{1}{2}x^2 - \frac{1}{4}\ln x; \quad 1 \le x \le 3$

Solution

$$\sqrt{1+(y')^2} = \sqrt{1+\left(x-\frac{1}{4x}\right)^2}$$

$$= \sqrt{1+x^2 - \frac{1}{2} + \frac{1}{16x^2}}$$

$$= \sqrt{x^2 + \frac{1}{2} + \frac{1}{16x^2}}$$

$$= \sqrt{\left(x+\frac{1}{4x}\right)^2}$$

$$= \left. x + \frac{1}{4x} \right|$$

$$L = \int_1^3 \left(x+\frac{1}{4x}\right) dx$$

$$= \frac{1}{2}x^2 + \frac{1}{4}\ln x \Big|_1^3$$

$$= \frac{9}{2} + \frac{1}{4}\ln 3 - \frac{1}{2}$$

$$= 4 + \frac{1}{4}\ln 2 \quad unit$$

Exercise

Find the length of the curve $\quad y = \int_{-2}^x \sqrt{2t^4 - 1}\, dt \quad -2 \le x \le -1$

Solution

$$\frac{dy}{dt} = \sqrt{2t^4 - 1}$$

$$\sqrt{1+\left(\frac{dy}{dt}\right)^2} = \sqrt{1 + 2t^4 - 1}$$

$$= \sqrt{2t^4}$$

$$= \left. \sqrt{2}\, t^2 \right|$$

$$L = \sqrt{2} \int_{-2}^{-1} t^2\, dt$$

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$$= \frac{\sqrt{2}}{3} t^3 \Big|_{-2}^{-1}$$

$$= \frac{\sqrt{2}}{3}(-1+8)$$

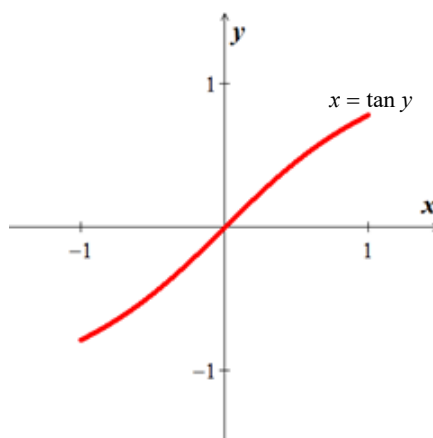$$= \frac{7\sqrt{2}}{3} \quad unit$$

Exercise

Find the length of the curve $x = \displaystyle\int_0^y \sqrt{\sec^4 t - 1} \; dt \quad -\frac{\pi}{4} \le y \le \frac{\pi}{4}$

Solution

$$\frac{dx}{dy} = \sqrt{\sec^4 y - 1}$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \sec^4 y - 1}$$

$$= \sqrt{\sec^4 y}$$

$$= \sec^2 y$$

$$L = \int_{-\pi/4}^{\pi/4} \sec^2 y \; dy$$

$$= \tan y \Big|_{-\pi/4}^{\pi/4}$$

$$= 1 - (-1)$$

$$= 2 \quad unit$$

$x = \tan y$

Exercise

Find the length of the curve $y = 3 - 2x \quad 0 \le x \le 2$. Check your answer by finding the length of the segment as the hypotenuse of a right triangle.

Solution

$$\frac{dy}{dx} = -2$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + 4}$$

$$= \sqrt{5}$$

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$$L = \int_0^2 \sqrt{5}\ dx$$

$$= \sqrt{5}\ x\ \Big|_0^2$$

$$\underline{= 2\sqrt{5}\quad unit}$$

$$\begin{cases} x = 0 & \to y = 3 \\ x = 2 & \to y = -1 \end{cases}$$

$$d = \sqrt{(2-0)^2 + (3+1)^2}$$

$$= \sqrt{4+16}$$

$$= \sqrt{20}$$

$$\underline{= 2\sqrt{5}}$$
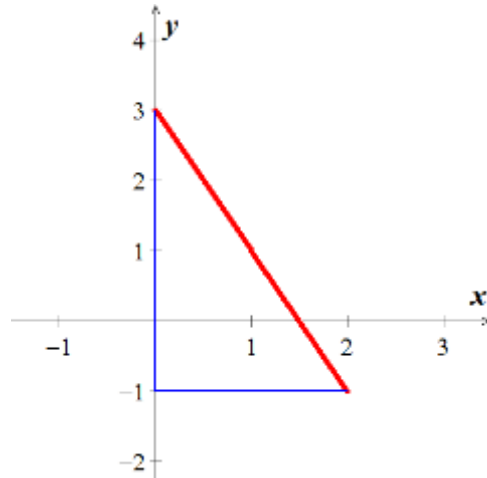
Exercise

Find a curve through the origin in the xy-plane whose length from $x = 0$ to $x = 1$ is $L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x}\ dx$

Solution

$$L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x}\ dx$$

$$\frac{dy}{dx} = \frac{e^{x/2}}{2} \qquad \to dy = \frac{e^{x/2}}{2}dx$$

$$y = \int \frac{e^{x/2}}{2}\ dx$$

$$\underline{= e^{x/2} + C}$$

$$0 = e^{0/2} + C$$

$$0 = 1 + C \qquad \Rightarrow \underline{C = -1}$$

$$\underline{y = e^{x/2} - 1}$$
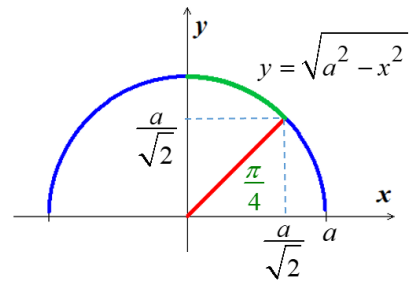
Exercise

Confirm that the circumference of a circle of radius a is $2\pi a$.

Solution

$$f(x) = \sqrt{a^2 - x^2} \quad for \quad -a \le x \le a$$

$$f'(x) = -\frac{x}{\sqrt{a^2 - x^2}} \qquad but \quad \underline{x \ne \pm a}$$

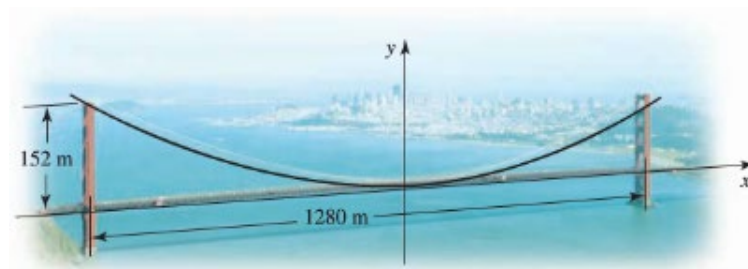$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2}}$$

$$= \frac{a}{\sqrt{a^2 - x^2}}$$

Let's compute the length of $\frac{1}{8}$ of the circle on $\left[0, \frac{a}{\sqrt{2}}\right]$

$$L = 8a \int_0^{a/\sqrt{2}} \frac{dx}{\sqrt{a^2 - x^2}}$$

$$= 8a \; \sin^{-1}\left(\frac{x}{a}\right) \Big|_0^{a/\sqrt{2}}$$

$$= 8a \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= 8a\left(\frac{\pi}{4}\right)$$

$$\underline{= 2\pi a \quad unit}$$

Exercise

The profile of the cables on a suspension bridge may be modeled by a parabola. The central span of the Golden Gate Bridge is 1280 m long and 152 m high. The parabola $y = 0.00037x^2$ gives a good fit to the shape of the cables, where $|x| \le 640$, and x and y are measured in *meters*. Approximate the length of the cables that stretch between the tops of the two towers.

Solution

$$y' = 0.00074x$$

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$$L = \int_{-640}^{640} \sqrt{1+(.00074x)^2}\ dx \qquad\qquad \int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|$$

$$= \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\left|x+\sqrt{1+x^2}\right|\ \Big|_{-640}^{640}$$

$$= 320\sqrt{1+640^2} + \frac{1}{2}\ln\left|640+\sqrt{1+640^2}\right| + 320\sqrt{1+x^2} - \frac{1}{2}\ln\left|-640+\sqrt{1+640^2}\right|$$

$$\approx 1326.4 \quad m$$

Exercise

Electrical wires suspended between two towers form a caternary modeled by the equation

$$y = 20\cosh\frac{x}{20}, \quad -20 \le x \le 20$$

Where x and y are measured in *meters*. The towers are 40 *meters* apart. Find the length of the suspended cable.
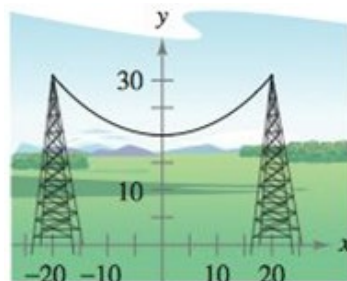
Solution

$$y = 20\cosh\frac{x}{20}$$

$$y' = \sinh\frac{x}{20}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\sinh^2\frac{x}{20}}$$

$$= \sqrt{\cosh^2\frac{x}{20}}$$

$$= \cosh\frac{x}{20}$$

$$L = \int_{-20}^{20} \cosh\frac{x}{20}\ dx$$

$$= 2(20)\ \sinh\frac{x}{20}\ \Big|_0^{20}$$

$$= 40(\sinh 1 - \sinh 0)$$

$$= 40\sinh 1 \quad unit$$

$$= 20\left(e-e^{-1}\right) \quad unit$$

Exercise

A barn is 100 *feet* long and 40 *feet* wide. A cross section of the roof is the inverted caternary $y = 31 - 10\left(e^{x/20} + e^{-x/20}\right)$. Find the number of **square feet** of roofing on the barn.

Solution

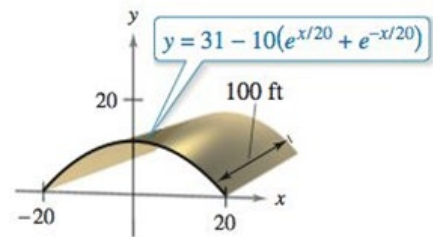$$a = 10, \quad m = \frac{1}{20}, \quad b = 10, \quad n = -\frac{1}{20}$$

1. $m = -n$ ✓
2. $abmn = 10(10)\left(\frac{1}{20}\right)\left(-\frac{1}{20}\right) = -\frac{1}{4}$ ✓

$$L = 10 \left(e^{x/20} - e^{-x/20} \; \Big|_{-20}^{20}\right)$$

$$= 10\left(e - \frac{1}{e} - \frac{1}{e} + e\right)$$

$$= 20\left(e - \frac{1}{e}\right) \; ft \quad \approx 47 \, ft$$

$y = 31 - 10(e^{x/20} + e^{-x/20})$

100 ft

∴ There are $100(47) = 4,700 \, ft^2$ of roofing on the barn

Exercise

A cable for a suspension bridge has the shape of a parabola with equation $y = kx^2$. Let h represent the height of the cable from it lowest point to its highest point and let $2w$ represent the total span of the bridge.

Show that the length C of the cable is given by

$$C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2} \; dx$$

Solution

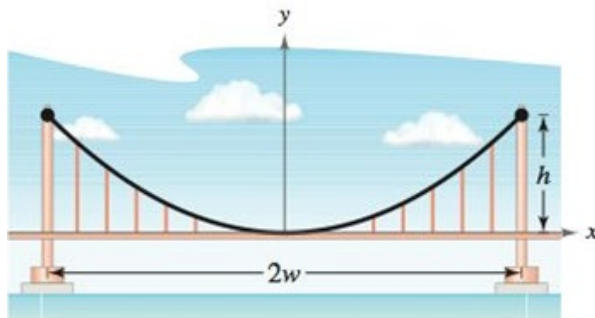$$y' = 2kx$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + 4k^2x^2}$$

At $(w, h) \rightarrow h = kw^2$

$$\Rightarrow k = \frac{h}{w^2}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{4h^2}{w^4}x^2}$$

∴ By symmetry:

$$C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2} \; dx$$

2w

h

Exercise

Find the total length of the graph of the astroid $x^{2/3} + y^{2/3} = 4$

Solution

$x^{2/3} + y^{2/3} = 4$

$y = \left(4 - x^{2/3}\right)^{3/2}$

$y' = \dfrac{3}{2}\left(-\dfrac{2}{3}x^{-1/3}\right)\left(4 - x^{2/3}\right)^{1/2}$

$\quad = -\dfrac{1}{x^{1/3}}\left(4 - x^{2/3}\right)^{1/2}$

$1 + \left(y'\right)^2 = 1 + \dfrac{1}{x^{2/3}}\left(4 - x^{2/3}\right)$

$\quad\quad\quad = \dfrac{4}{x^{2/3}}$

$y = 0 \;\rightarrow\; x^{2/3} = 4$

$\underline{x = 4^{3/2} = 8}$

$L = 4\displaystyle\int_0^8 \sqrt{\dfrac{4}{x^{2/3}}}\; dx$

$\quad = 8\displaystyle\int_0^8 x^{-1/3}\; dx$

$\quad = 12\, x^{2/3} \Big|_0^8$

$\quad = 12(4 - 0)$

$\underline{= 48 \quad unit}$
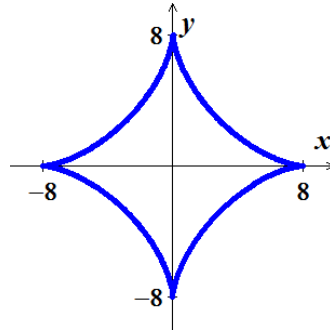
Exercise

Find the arc length from $(0,\,3)$ clockwise to $\left(2,\,\sqrt{5}\right)$ along the circle $x^2 + y^2 = 9$

Solution

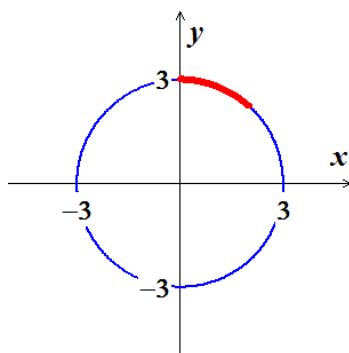$y = \sqrt{9 - x^2}$

$y' = -\dfrac{x}{\sqrt{9 - x^2}}$

200

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{x^2}{9-x^2}}$$

$$= \sqrt{\frac{9}{9-x^2}}$$

$$= \frac{3}{\sqrt{9-x^2}}$$

$$L = \int_0^2 \frac{3}{\sqrt{9-x^2}}\,dx$$

$$= 3\arcsin\frac{x}{3}\Big|_0^2$$

$$= 3\arcsin\frac{2}{3} \quad unit \quad\Big|\quad \approx 2.1892\Big|$$

Exercise

Find the arc length from $(-3,\ 4)$ clockwise to $(4,\ 3)$ along the circle $x^2 + y^2 = 25$. Show that the result is one-fourth the circumference of the circle.
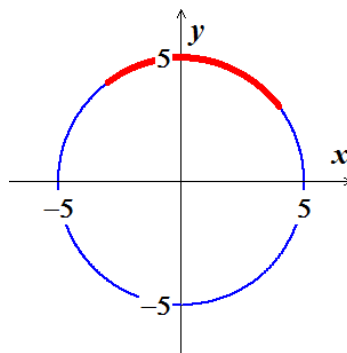
Solution

$$y = \sqrt{25-x^2}$$

$$y' = -\frac{x}{\sqrt{25-x^2}}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{x^2}{25-x^2}}$$

$$= \sqrt{\frac{25}{25-x^2}}$$

$$= \frac{5}{\sqrt{25-x^2}}$$

$$L = \int_{-3}^4 \frac{5}{\sqrt{25-x^2}}\,dx$$

$$= 5\arcsin\frac{x}{5}\Big|_{-3}^4$$

$$= 5\left(\arcsin\frac{4}{5}+\arcsin\frac{3}{5}\right) \quad unit \quad\Big|\quad \approx 7.854\Big|$$

Exercise

$y = \ln x$ between $x = 1$ and $x = b > 1$ that

$$\int \frac{\sqrt{x^2 + a^2}}{x} dx = \sqrt{x^2 + a^2} - a \ln\left(\frac{a + \sqrt{x^2 + a^2}}{x}\right) + C$$

Use any means to approximate the value of b for which the curve has length 2.

Solution

Given: $L = 2$

$$y = \ln x \quad \rightarrow \quad y' = \frac{1}{x}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{1}{x^2}}$$

$$= \frac{\sqrt{x^2 + 1}}{x}$$

$$L = \int_1^b \frac{\sqrt{x^2 + 1}}{x} dx$$

$$= \sqrt{x^2 + 1} - \ln\left(\frac{1 + \sqrt{x^2 + 1}}{x}\right) \Bigg|_1^b$$

$$= \sqrt{b^2 + 1} - \ln\left(\frac{1 + \sqrt{b^2 + 1}}{b}\right) - \sqrt{2} + \ln\left(1 + \sqrt{2}\right) = 2$$

Using Mapple:

$$fsolve\left(\sqrt{b^2 + 1} - \ln\left(\frac{1 + \sqrt{b^2 + 1}}{b}\right) - \sqrt{2} + \ln\left(1 + \sqrt{2}\right) = 2, \ b\right)$$

$$b = 2.714999998$$

$b \approx 2.715$