Lecture One – First Order Equations

Section 1.1 – Differential Equations & Solutions

In the mathematical, engineering and sciences models are developed to assist in the understanding of physical phenomena. These models often yield an equation that contains some derivatives of an unknown function. Such an equation is called a ***differential equation***. We introduce few major methods ***geometrical***, ***analytical*** and ***numerical*** for getting the solutions of these problems.

Differential equations all about solving equations involving $y', y'', y^{(3)}, \dots$.

1.1-1 Ordinary Differential Equations

Ordinary differential equations involve an unknown function of a single variable with one or more of its derivatives.

$$\frac{dy}{dt} = y - t$$

y: $y(t)$ is unknown function

t: independent variable

Some other examples:

$$y' = y^2 - t$$
$$ty' = y$$
$$y' + 4y = e^{-3t}$$
$$yy'' + t^2 y = \cos t$$
$$y' = \cos(ty)$$

$\therefore$ The order of a differential equation is the order of the highest derivative that occurs in the equation.

y'' : *second order*

$$\frac{\partial^2 \omega}{\partial t^2} = c^2 \frac{\partial^2 \omega}{\partial x^2}$$ is not an ODE (ω is dependent on x and t)

This equation is called a ***partial differential equation***.

1.1-2 *Definition*

A first-order differential equation of the form $\dfrac{dy}{dt} = y' = f(t, y)$ is said to be in normal form.

$y^{(n)} = f\left(t, y, y', \ldots, y^{(n-1)}\right)$ is said to be in normal form.

$f:$ is a given function of two variables t & y (***rate function***).

1.1-3 Solutions

A solution of the first-order, ordinary differential equation $f(t, y, y') = 0$ is a differentiable function $y(t)$ such that $f(t, y(t), y'(t)) = 0$ for all t in the interval where $y(t)$ is defined.

1. Can be found in explicit and implicit form by applying manipulation (integration)
2. No real solution.

Example 1

Show that $y(t) = Ce^{-t^2}$ is a solution of the first-order equation $y' = -2ty$

Solution

$$y(t) = Ce^{-t^2} \Rightarrow y' = -2tCe^{-t^2}$$

$$y' = -2tCe^{-t^2}$$

$$y' = -2t\, y(t)$$ True; it is a solution

$y(t)$ is called the ***general solution***.
The solutions from the graph are called ***solution curves***.

Example 2

Is the function $y(t) = \cos t$ a solution to the differential equation $y' = 1 + y^2$

Solution

$$y' = -\sin t$$

$$y' = 1 + y^2 = -\sin t$$

$$1 + \cos^2 t \stackrel{?}{=} -\sin t$$

False; it is not a solution.

Exercises *Section* 1.1 – Differential Equations & Solutions

1. Show that $y(t) = Ce^{-(1/2)t^2}$ is a solution of the first-order equation $y' = -ty$ *for* $-3 \le C \le 3$

2. Show that $y(t) = \dfrac{4}{1+Ce^{-4t}}$ is a solution of the first- order equation $y' = y(4-y)$

3. Show that $y(x) = x^{-3/2}$ is a solution of $4x^2y'' + 12xy' + 3y = 0$ for $x > 0$

4. A general solution $y(t) = \dfrac{4}{1+Ce^{-4t}}$ may fail to produce all solutions of a differential equation $y' = y(4-y)$. Show that $y = 0$ is a solution of the differential equation, but no value of C in the given general solution will produce this solution.

5. Use the given general solution to find a solution of the differential equation having the given initial condition. $ty' + y = t^2, \quad y(t) = \frac{1}{3}t^2 + \frac{C}{t}, \qquad y(1) = 2$

6. Show that $y(t) = 2t - 2 + Ce^{-t}$ is a solution of the first-order equation $y' + y = 2t$ *for* $-3 \le C \le 3$

7. Use the given general solution to find a solution of the differential equation having the given initial condition. $y' + 4y = \cos t, \quad y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t + Ce^{-4t}, \qquad y(0) = -1$

8. Use the given general solution to find a solution of the differential equation having the given initial condition. $ty' + (t+1)y = 2te^{-t}, \quad y(t) = e^{-t}\left(t + \frac{C}{t}\right), \quad y(1) = \frac{1}{e}$

9. Use the given general solution to find a solution of the differential equation having the given initial condition. $y' = y(2+y), \quad y(t) = \dfrac{2}{-1+Ce^{-2t}}, \quad y(0) = -3$

10. Find the values of ***m*** so that the function $y = e^{mx}$ is a solution of the given differential equation

a) $y' + 2y = 0$

b) $5y' - 2y = 0$

c) $y'' - 5y' + 6y = 0$

d) $2y'' + 7y' - 4y = 0$

11. Let $x = c_1 \cos t + c_2 \sin t$ is 2-parameter family solutions of the second order differential equation of $x'' + x = 0$. Find a solution of the second-order consisting of this differential equation and the given initial conditions.

a) $x(0) = -1, \quad x'(0) = 8$

b) $x\left(\frac{\pi}{2}\right) = 0, \quad x'\left(\frac{\pi}{2}\right) = 1$

c) $x\left(\frac{\pi}{6}\right) = \frac{1}{2}, \quad x'\left(\frac{\pi}{6}\right) = 0$

d) $x\left(\frac{\pi}{4}\right) = \sqrt{2}, \quad x'\left(\frac{\pi}{4}\right) = 2\sqrt{2}$

12. Find values of *r* such that $y(x) = x^r$ is a solution of $x^2 y'' - 4xy' + 6y = 0$

(13 – 14) Solve the differential equation:

13. $y' = 3x^2 - 2x + 4$

14. $y'' = 2x + \sin 2x$

15. Given the differential equation $x^2 y'' - 2xy' + 2y = 4x^3$, is the given equation a solution?

a) $y = 2x^3 + x^2$

b) $y = 2x + x^2$

Section 1.2 – Separable Equations

1.2-1 *Separable* Equation

A simple class of first-order differential equations using a class of ***separable equations.*** These equations can be written with its variables separated and then easily solved.

If f is independent of $y \Rightarrow f(x,y) = g(x) = \dfrac{dy}{dx}$

$$dy = g(x)dx \qquad \textit{Integrate both sides}$$

$$y = \int g(x)dx$$

1.2-2 *Definition*

A first-order differential equation of the form $\dfrac{dy}{dx} = g(x)h(y)$ is said to be separable or to have separable variables.

$$\frac{dy}{h(y)} = g(x)dx$$

$$\frac{dy}{dx} = y^2xe^{3x+4y} = \left(xe^{3x}\right)\left(y^2e^{4y}\right)$$

$$\frac{dy}{dx} = y + \sin x \qquad \textit{not separable}$$

Example 1

At time t the sample contains $N(t)$ radioactive nuclei and is given by the differential equation:

$$N' = -\lambda N$$

This is called the ***exponential equation***.

$$N' = -\lambda N$$

$$\frac{dN}{dt} = -\lambda N \qquad \textit{Separable equation}$$

$$\frac{dN}{N} = -\lambda dt \qquad \textit{Integrate both sides}$$

$$\int \frac{dN}{N} = -\int \lambda dt$$

$$\ln|N| = -\lambda t + C \qquad \textit{Convert to exponential}$$

$$|N(t)| = e^{-\lambda t + C}$$

$$= e^{C} e^{-\lambda t}$$

$$N(t) = \begin{cases} e^{C} e^{-\lambda t} & \text{if } N > 0 \\ -e^{C} e^{-\lambda t} & \text{if } N < 0 \end{cases}$$

$$N(t) = Ae^{-\lambda t} \qquad A = \begin{cases} e^{C} & \text{if } N > 0 \\ -e^{C} & \text{if } N < 0 \end{cases}$$

Example 2

Solve the differential equation $y' = ty^2$

Solution

$$\frac{dy}{dt} = ty^2$$

$$\frac{dy}{y^2} = tdt \qquad \textit{Integrate both sides}$$

$$\int y^{-2} dy = \int tdt$$

$$-y^{-1} = \frac{1}{2}t^2 + C$$

$$-\frac{1}{y} = \frac{t^2 + 2C}{2} \qquad \textit{Cross multiplication}$$

$$-\frac{2}{t^2 + 2C} = y$$

$$\boxed{y(t) = -\frac{2}{t^2 + 2C}}$$

General Method

1. Separate the variables
2. Integrate both sides
3. Solve for the solution $y(t)$, if possible

1.2-3 Newton's Law of Cooling

Newton's Law of Cooling states that the rate of change of an object's temperature (***T***) is proportional to the difference between its temperature and the ambient temperature (***A***) (i.e. the temperature of its surroundings).

$$\frac{dT}{dt} = -k\left(T - A\right)$$

Example 3

A can of beer at 40° *F* is placed into a room when the temperature is 70° *F*. After 10 *minutes* the temperature of the beer is 50° *F*. What is the temperature of the beer as a function of time? What is the temperature of the beer 30 *minutes* after the beer was placed into the room?

Solution

By Newton's Law of cooling: The rate of change of an object's temperature (***T***) is proportional to the difference between its temperature and the ambient temperature (***A***).

$$\frac{dT}{dt} = -k\left(T - A\right)$$

$$\frac{dT}{T-A} = -kdt \qquad \textit{Integrate both sides}$$

$$\int_{T_0}^{T} \frac{dT}{T-A} = -\int_0^t kdt$$

$$\ln\left|T - A\right| \Big|_{T_0}^{T} = -kt \Big|_0^t$$

$$\ln\left|T - A\right| - \ln\left|T_0 - A\right| = -kt$$

$$\ln\frac{\left|T-A\right|}{\left|T_0-A\right|} = -kt \qquad \textit{Quotient Rule}$$

$$\frac{T-A}{T_0-A} = e^{-kt}$$

$$\Rightarrow\ T - A = \left(T_0 - A\right)e^{-kt}$$

Given: $T_0 = 40°F \quad \& \quad A = 70°F$

$$T(t) = 70 + \left(40 - 70\right)e^{-kt}$$

$$= \underline{70 - 30e^{-kt}}$$

$$T(t = 10) = 70 - 30e^{-10k} = 50$$

$$-30e^{-10k} = -20$$

$$e^{-10k} = \frac{2}{3}$$

$$-10k = \ln\frac{2}{3}$$

$$k = \frac{\ln\frac{2}{3}}{-10}$$

$$\approx 0.0405$$

$$T(t) = 70 - 30e^{-0.0405t}$$

$$T(t = 30) = 70 - 30e^{-0.0405(30)}$$

$$\approx 61.1\ \ ^\circ F$$

1.2-4 *Losing a solution*

When we use separate variables, the variable divisors could be zero at a point.

Example 4

Find a general solution to $\frac{dy}{dx} = y^2 - 4$

Solution

$$\frac{dy}{y^2 - 4} = dx$$

$$\left(\frac{1/4}{y-2} - \frac{1/4}{y+2}\right)dy = dx \qquad\qquad y = \pm 2 \text{ Critical points}$$

$$\frac{1}{4}\left(\int \frac{dy}{y-2} - \int \frac{dy}{y+2}\right) = \int dx$$

$$\frac{1}{4}\left(\ln|y-2| - \ln|y+2|\right) = x + c_1$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x + c_2$$

$$\left|\frac{y-2}{y+2}\right| = e^{4x+c_2}$$

$$\frac{y-2}{y+2} = \pm e^{c_2} e^{4x}$$

$$y - 2 = Ce^{4x}(y+2)$$

$$y - Ce^{4x}y = 2Ce^{4x} + 2$$

$$\left(1 - Ce^{4x}\right)y = 2\left(Ce^{4x} + 1\right)$$

$$\boxed{y(x) = 2\frac{1+Ce^{4x}}{1-Ce^{4x}}}$$

If $y = -2 \Rightarrow -2 = 2\frac{1+Ce^{4x}}{1-Ce^{4x}}$

$$-1 = \frac{1+Ce^{4x}}{1-Ce^{4x}}$$

$$-1 + Ce^{4x} = 1 + Ce^{4x}$$

$\Rightarrow \quad -1 = 1$ *impossible*

If $y = 2 \quad \Rightarrow 2 = 2\frac{1+Ce^{4x}}{1-Ce^{4x}}$

$$1 - Ce^{4x} = 1 + Ce^{4x}$$

$$-Ce^{4x} = Ce^{4x} \quad \Rightarrow \quad -C = C$$

$$y = 2 \;\Rightarrow\; \boxed{C = 0}$$

1.2-5 *Implicitly Defined Solutions*

Example 5

Find the solutions of the equation $y' = \frac{e^x}{1+y}$, having initial conditions $y(0) = 1$ and $y(0) = -4$

Solution

$$\frac{dy}{dx} = \frac{e^x}{1+y}$$

$$(1+y)dy = e^x dx$$

$$\int (1+y)dy = \int e^x dx$$

$$y + \frac{1}{2}y^2 = e^x + c$$

$$y^2 + 2y - 2\left(e^x + c\right) = 0$$

$$y(x) = \frac{1}{2}\left(-2 \pm \sqrt{4 + 8\left(e^x + c\right)}\right) \qquad \textit{Quadratic Formula}$$

$$= -1 \pm \sqrt{1 + 2\left(e^x + c\right)} \qquad \textit{Implicit}$$

$$y(0) = -1 + \sqrt{1 + 2\left(e^0 + c\right)} = 1$$

$$\sqrt{1+2(1+c)}=2$$
$$1+2+2c=4$$
$$2c=1$$
$$\underline{c=\frac{1}{2}}$$

$$y(0)=-1-\sqrt{1+2\left(e^0+c\right)}=-4$$
$$-\sqrt{1+2+2c}=-3$$
$$1+2+2c=9$$
$$2c=6$$
$$\underline{c=3}$$

$$\begin{cases} y(t)=-1+\sqrt{2+2e^x} \\ y(t)=-1-\sqrt{7+2e^x} \end{cases}$$

$\therefore\ y\neq -1$ *from* y', but it never it will be.

Explicit Solutions: $y=-1+\sqrt{\ }$

Implicit solutions: y^2+by+c

Example 6

Find the solutions to the differential equation $x'=\frac{2tx}{1+x}$, having $x(0)=1,\ -2,\ 0$

Solution

$$\frac{dx}{dt}=\frac{2tx}{1+x}$$
$$\frac{1+x}{x}dx=2tdt$$
$$\left(\frac{1}{x}+1\right)dx=2tdt$$
$$\int\left(\frac{1}{x}+1\right)dx=\int 2tdt$$
$$\ln|x|+x=t^2+c$$

For $x(0)=1$

$$1=0^2+c$$
$$c=1$$

$\ln|x| + x = t^2 + c \qquad x > 0$

We can't solve for $x(t)$

$\Rightarrow$ This solution is defined as implicit.

For $x(0) = -2$

$\ln|-2| + (-2) = 0^2 + c$

$c = -2 + \ln 2$

$\ln|x| + x = t^2 - 2 + \ln 2$

Since the initial condition < 0, then:

$x + \ln(-x) = t^2 - 2 + \ln 2$

For $x(0) = 0$

$0 = 0^2 + c \qquad$ *True statement*

$y' = 0 \ \Rightarrow \ x(t) = 0$ is a solution

Exercises *Section* 1.2 – Solutions to Separable Equations

(1 – 56) Find the general solution of the differential equation.

1. $y' = xy$
2. $xy' = 2y$
3. $y' = e^{x-y}$
4. $y' = \left(1+y^2\right)e^x$
5. $y' = xy + y$
6. $y' = ye^x - 2e^x + y - 2$
7. $y' = \dfrac{x}{y+2}$
8. $y' = \dfrac{xy}{x-1}$
9. $y' = \dfrac{y^2+ty+t^2}{t^2}$
10. $\dfrac{dy}{dx} = \dfrac{4x-x^3}{4+y^3}$
11. $y' = \dfrac{2xy+2x}{x^2-1}$
12. $\dfrac{dy}{dx} = \sin 5x$
13. $\dfrac{dy}{dx} = (x+1)^2$
14. $dx + e^{3x}dy = 0$
15. $dy - (y-1)^2\,dx = 0$
16. $x\dfrac{dy}{dx} = 4y$
17. $\dfrac{dx}{dy} = y^2 - 1$
18. $\dfrac{dy}{dx} = e^{2y}$
19. $\dfrac{dy}{dx} + 2xy^2 = 0$
20. $\dfrac{dy}{dx} = e^{3x+2y}$
21. $e^x y\dfrac{dy}{dx} = e^{-y} + e^{-2x-y}$
22. $y\ln x\dfrac{dx}{dy} = \left(\dfrac{y+1}{x}\right)^2$
23. $\dfrac{dy}{dx} = \left(\dfrac{2y+3}{4x+5}\right)^2$
24. $\csc y\,dx + \sec^2 x\,dy = 0$
25. $\sin 3x\,dx + 2y\cos^3 3x\,dy = 0$
26. $\left(e^y+1\right)^2 e^{-y}dx + \left(e^x+1\right)^3 e^{-x}dy = 0$
27. $x\left(1+y^2\right)^{1/2}dx = y\left(1+x^2\right)^{1/2}dy$
28. $\dfrac{dy}{dx} = y\sin x$
29. $(1+x)\dfrac{dy}{dx} = 4y$
30. $2\sqrt{x}\dfrac{dy}{dx} = \sqrt{1-y^2}$
31. $\dfrac{dy}{dx} = 3\sqrt{xy}$
32. $\dfrac{dy}{dx} = (64xy)^{1/3}$
33. $\dfrac{dy}{dx} = 2x\sec y$
34. $\left(1-x^2\right)\dfrac{dy}{dx} = 2y$
35. $(1+x)^2\dfrac{dy}{dx} = (1+y)^2$
36. $\dfrac{dy}{dx} = xy^3$
37. $y\dfrac{dy}{dx} = x\left(y^2+1\right)$
38. $y^3\dfrac{dy}{dx} = \left(y^4+1\right)\cos x$
39. $\dfrac{dy}{dx} = \dfrac{1+\sqrt{x}}{1+\sqrt{y}}$
40. $\dfrac{dy}{dx} = \dfrac{(x-1)y^5}{x^2\left(2y^3-y\right)}$

41. $\left(x^2+1\right)(\tan y)\,y'=x$

42. $x^2y'=1-x^2+y^2-x^2y^2$

43. $xy'+4y=0$

44. $\left(x^2+1\right)y'+2xy=0$

45. $\dfrac{y'}{\left(x^2+1\right)y}=3$

46. $y+e^xy'=0$

47. $\dfrac{dx}{dt}=3xt^2$

48. $x\dfrac{dy}{dx}=\dfrac{1}{y^3}$

49. $\dfrac{dy}{dx}=\dfrac{x}{y^2\sqrt{x+1}}$

50. $\dfrac{dx}{dt}-x^3=x$

51. $\dfrac{dy}{dx}=\dfrac{x}{ye^{x+2y}}$

52. $\dfrac{dy}{dx}=\dfrac{\sec^2 y}{1+x^2}$

53. $x\dfrac{dv}{dx}=\dfrac{1-4v^2}{3v}$

54. $\dfrac{dy}{dx}=3x^2\left(1+y^2\right)^{3/2}$

55. $\dfrac{1}{y}dy+ye^{\cos x}\sin x\,dx=0$

56. $\left(x+xy^2\right)dx+e^{x^2}y\,dy=0$

(57 – 116) Find the exact solution of the initial value problem. Indicate the interval of existence.

57. $y'=\dfrac{y}{x},\quad y(1)=-2$

58. $y'=-\dfrac{2t\left(1+y^2\right)}{y},\quad y(0)=1$

59. $y'=\dfrac{\sin x}{y},\quad y\left(\frac{\pi}{2}\right)=1$

60. $4t\,dy=\left(y^2+ty^2\right)dt,\quad y(1)=1$

61. $y'=\dfrac{1-2t}{y},\quad y(1)=-2$

62. $y'=y^2-4,\quad y(0)=0$

63. $\dfrac{dy}{dx}=\dfrac{3x^2+4x+2}{2(y-1)},\quad y(0)=-1$

64. $y'=\dfrac{x}{1+2y},\quad y(-1)=0$

65. $\left(e^{2y}-y\right)\cos x\dfrac{dy}{dx}=e^y\sin 2x,\quad y(0)=0$

66. $\dfrac{dy}{dx}=e^{-x^2},\quad y(3)=5$

67. $\dfrac{dy}{dx}+2y=1,\quad y(0)=\frac{5}{2}$

68. $\sqrt{1-y^2}\,dx-\sqrt{1-x^2}\,dy=0,\quad y(0)=\frac{\sqrt{3}}{2}$

69. $\left(1+x^4\right)dy+x\left(1+4y^2\right)dx=0,\quad y(1)=0$

70. $\dfrac{1}{t^2}\dfrac{dy}{dt}=y,\quad y(0)=1$

71. $\dfrac{dy}{dt}=-y^2e^{2t};\quad y(0)=1$

72. $\dfrac{dy}{dt}-(2t+1)y=0;\quad y(0)=2$

73. $\dfrac{dy}{dt}+4ty^2=0;\quad y(0)=1$

74. $\dfrac{dy}{dx}=ye^x;\quad y(0)=2e$

75. $\dfrac{dy}{dx}=3x^2\left(y^2+1\right);\quad y(0)=1$

76. $2y\dfrac{dy}{dx}=\dfrac{x}{\sqrt{x^2-16}};\quad y(5)=2$

77. $\dfrac{dy}{dx}=4x^3y-y;\quad y(1)=-3$

78. $\dfrac{dy}{dx}+1=2y;\quad y(1)=1$

79. $(\tan x)\frac{dy}{dx} = y;\quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

80. $e^{-2t}\frac{dy}{dt} = \frac{1+e^{-2t}}{y},\quad y(0) = 0$

81. $\frac{dy}{dt} = y\cos t + y,\quad y(0) = 2$

82. $\frac{dy}{dt} = \frac{t+2}{y},\quad y(0) = 2$

83. $x\frac{dy}{dx} - y = 2x^2 y;\quad y(1) = 1$

84. $\frac{dy}{dx} = 2xy^2 + 3x^2y^2;\quad y(1) = -1$

85. $\frac{dy}{dx} = 6e^{2x-y};\quad y(0) = 0$

86. $2\sqrt{x}\frac{dy}{dx} = \cos^2 y;\quad y(4) = \frac{\pi}{4}$

87. $y' + 3y = 0;\quad y(0) = -3$

88. $2y' - y = 0;\quad y(-1) = 2$

89. $2xy - y' = 0;\quad y(1) = 3$

90. $y\frac{dy}{dx} - \sin x = 0;\quad y\left(\frac{\pi}{2}\right) = -2$

91. $\frac{dy}{dt} = \frac{1}{y^2};\quad y(1) = 2$

92. $y' + \frac{1}{y+1} = 0;\quad y(1) = 0$

93. $y' + e^y t = e^y \sin t;\quad y(0) = 0$

94. $y' - 2ty^2 = 0;\quad y(0) = -1$

95. $\frac{dy}{dx} = 1 + y^2;\quad y\left(\frac{\pi}{4}\right) = -1$

96. $\frac{dy}{dt} = t - ty^2;\quad y(0) = \frac{1}{2}$

97. $3y^2\frac{dy}{dt} + 2t = 1;\quad y(-1) = -1$

98. $e^x y' + (\cos y)^2 = 0;\quad y(0) = \frac{\pi}{4}$

99. $(2y - \sin y)y' + x = \sin x;\quad y(0) = 0$

100. $e^y y' + \frac{x}{y+1} = \frac{2}{y+1};\quad y(1) = 2$

101. $(\ln y)y' + x = 1;\quad y(3) = e$

102. $y' = x^3(1-y);\quad y(0) = 3$

103. $y' = \left(1+y^2\right)\tan x;\quad y(0) = \sqrt{3}$

104. $\frac{1}{2}\frac{dy}{dx} = \sqrt{1+y}\ \cos x;\quad y(\pi) = 0$

105. $x^2\frac{dy}{dx} = \frac{4x^2 - x - 2}{(x+1)(y+1)};\quad y(1) = 1$

106. $\frac{1}{\theta}\frac{dy}{d\theta} = \frac{y\sin\theta}{y^2+1}\quad y(\pi) = 1$

107. $x^2 dx + 2y dy = 0;\quad y(0) = 2$

108. $\frac{1}{t}\frac{dy}{dt} = 2\cos^2 y;\quad y(0) = \frac{\pi}{4}$

109. $\frac{dy}{dx} = 8x^3 e^{-2y};\quad y(1) = 0$

110. $\frac{dy}{dx} = x^2(1+y);\quad y(0) = 3$

111. $\sqrt{y}dx + (1+x)dy = 0;\quad y(0) = 1$

112. $\frac{dy}{dx} = 6y^2 x,\quad y(1) = \frac{1}{25}$

113. $\frac{dy}{dx} = \frac{3x^2 + 4x - 4}{2y - 4},\quad y(1) = 3$

114. $y' = e^{-y}(2x - 4)\quad y(5) = 0$

115. $\frac{dr}{d\theta} = \frac{r^2}{\theta},\quad r(1) = 2$

116. $\frac{dy}{dt} = e^{y-t}\left(1+t^2\right)\sec y,\quad y(0) = 0$

117. A thermometer reading $100°F$ is placed in a medium having a constant temperature of $70°F$. After 6 *min*, the thermometer reads $80°F$. What is the reading after 20 *min*?

118. Blood plasma is stored at $40°F$. Before the plasma can be used, it must be at $90°F$. When the plasma is placed in an oven at $120°F$, it takes 45 *min* for the plasma to warm to $90°F$. How long will it take for the plasma to warm to $90°F$ if the oven temperature is set at:

a) $100°F$.
b) $140°F$.
c) $80°F$.

119. A pot of boiling water at $100°C$ is removed from a stove at time $t = 0$ and left to cool in the kitchen. After 5 *min*, the water temperature has decreased to $80°C$, and another 5 *min* later it has dropped to $65°C$. Assuming Newton's Law for cooling, determine the (constant) temperature of the kitchen.

120. A murder victim is discovered at midnight and the temperature of the body is recorded at $31°C$. One hour later, the temperature of the body is $29°C$. Assume that the surrounding air temperature remains constant at $21°C$. Use Newton's Law of cooling to calculate the victim's time of death. *Note*: The normal temperature of a living human being is approximately $37°C$.

121. Suppose a cold beer at $40°F$ is placed into a warn room at $70°F$. suppose 10 *minutes* later, the temperature of the beer is $48°F$. Use Newton's Law of cooling to find the temperature 25 *minutes* after the beer was placed into the room.

122. A thermometer is removed from a room where the temperature is $70°\ F$ and is taken outside, where the air temperature is $10°\ F$. After one-half minute the thermometer reads $50°\ F$.
a) What is the reading of the thermometer at $t = 1\ min$?
b) How long will it take for the thermometer to reach $15°\ F$?

123. A thermometer is taken from an inside room to the outside, where the air temperature is $5°\ F$. After 1 *minute* the thermometer reads $55°\ F$, and after 5 *minutes* the thermometer reads $30°\ F$. What is the initial temperature of the inside room?

124. The temperature inside a house is $70°\ F$. A thermometer is taken outside after being inside the house for enough time for it to read $70°\ F$. The outside air temperature is $10°\ F$. After three *minutes* the thermometer reading is found to be $25°\ F$. Find the reading on the thermometer as a function of time.

125. A metal bar at a temperature of $100°\ F$ is placed in a room at a constant temperature of $0°\ F$. If after 20 minutes the temperature of the bar is $50°\ F$.
a) Find the time it will take the bar to reach a temperature of $25°\ F$
b) Find the temperature of the bar after 10 *minutes*.

126. A small metal bar, whose initial temperature was $20°\ C$, is dropped into a large container of boiling water.
a) How long will it take the bar to reach $90°\ C$ if it is known that its temperature increases $2°$ in 1 *second*?
b) How long will it take the bar to reach $98°\ C$

127. Two large containers ***A*** and ***B*** of the same size are filled with different fluids. The fluids in containers ***A*** and ***B*** are maintained at $0°\ C$ and $100°\ C$, respectively. A small metal bar, whose initial temperature is $100°\ C$, is lowered into container ***A***. After 1 *minute* the temperature of the bar is $90°\ C$. After 2 *minutes* the bar is removed and instantly transferred to the other container. After 1 *minute* in container ***B*** the temperature of the bar rises $10°\ C$. How long, measured from the start of the entire process, will it take the bar to reach $99.9°\ C$?

128. A thermometer reading $70°\ F$ is placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer records that the thermometer reads $110°\ F$ after $\frac{1}{2}$ *minute* and $145°\ F$ after 1 *minute*. How hot is the oven?

129. At $t=0$ a sealed test tube containing a chemical is immersed in a liquid bath. The initial temperature of the chemical in the test tube is $80°\ F$. the liquid bath has a controlled temperature given by $T_m(t)=100-40e^{-0.1t}$, $t \ge 0$, where t is measured in *minutes*.

a) Assume that $k=-0.1$, describe in words what you expect the temperature $T(t)$ of the chemical to be like in the short term. In the long term.

b) Solve the initial-value problem.

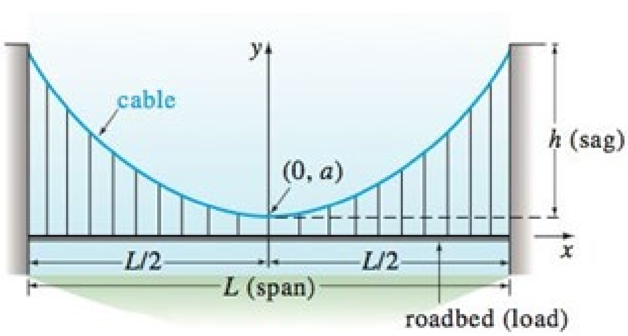
c) Graph $T(t)$.

130. The mathematical model for the shape of a flexible cable strung between two vertical supports is given by

$$\frac{dy}{dx}=\frac{W}{T_1}$$

Where W denotes the portion of the total vertical load between the points P_1 and P_2

The model is separable under the following conditions that describe a suspension bridge.



Let assume that the x-axis runs along the horizontal roadbed, and the y-axis passes through $(0,\ a)$, which is the lowest point on one cable over the span of the bridge, coinciding with the interval $\left[-\frac{L}{2},\ \frac{L}{2}\right]$.

In the case of a suspension bridge, the usual assumption is that the vertical load in the given equation is only a uniform roadbed, distributed along the horizontal axis. In other words, it is assumed that the weight of all cables is negligible in comparison to the weight of the roadbed and that the weight per unit length of the roadbed (lb/ft) is a constant ρ. Use this information to set up and solve an appropriate initial-value problem from which the shape (a curve with equation $y=\varphi(x)$) of each of the two cables in a suspension bridge is determined.

Express the solution of the IVP in terms of the sag h and span L.

131. The Brentano-Stevens Law in psychology models the way that a subject reacts to a stimulus. It states that if R represents the reaction to an amount S of stimulus, then the relative rates of increase are proportional:

$$\frac{1}{R}\frac{dR}{dt}=\frac{k}{S}\frac{dS}{dt}$$

Where k is a positive constant. Find R as a function of S.

132. Barbara weighs 60 *kg* and is on a diet of 1600 *calories* per day, of which 850 are used automatically by basal metabolism. She spends about 15 *cal/kg/day* times her weight doing exercises. If 1 *kg* of fat contains 10,000 *cal*. and we assume that the storage of calories in the form of fat is 100% efficient, formulate a differential equation and solve it to find her weight as a function of time. Does her weight ultimately approach an equilibrium weight?

133. When a chicken is removed from an oven, its temperature is measured at 300° *F*. Three minutes later its temperature is 200° *F*. How long will it take for the chicken to cool off to a room temperature of 70° *F*.

134. Suppose that a corpse was discovered in a motel room at midnight and its temperature was $80°\ F$. The temperature of the room is kept constant at $60°\ F$. Two hours later the temperature of the corpse dropped to $75°\ F$. Find the time of death.

135. Suppose that a corpse was discovered at 10 PM and its temperature was $85°\ F$. Two hours later, its temperature is $74°\ F$. If the ambient temperature is $68°\ F$. Estimate the time of death.

Section 1.3 – Models of Motions

In mathematics, the rate at which a quantity changes is the derivative of that quantity.
The 2nd way of computing the rate of change comes from the application itself and is different from on application to another.

Mechanics

1.3-1 Law of mechanics – Newton's Second Law (1665-1671)

The acceleration of an object as produced by a net force is directly proportional to the magnitude of the net force, in the same direction as the net force, and inversely proportional to the mass of the object.

The force acting on a mass is equal to the rate of change of momentum with respect to time. Momentum is defined as the product of mass and velocity ($m.v$).

The force is equal to the derivative of the momentum

$$\begin{aligned} F &= \frac{d}{dt}mv \\ &= m\frac{dv}{dt} \\ &= ma \end{aligned}$$

Position: $x(t) = -\frac{1}{2}gt^2 + v_0 t + x_0$

1.3-2 *Universal Law of gravitation*

Anybody with mass M attacks any other body with mass m directly toward the mass M, with a magnitude proportional to the product of the 2 masses and inversely proportional to the square of the distance separating them.

$$F = \frac{GMm}{r^2}$$

Acceleration: $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

Force: $F = -mg$

Gravity: $g = \frac{GM}{r^2}$

Motion ball: $a = -g = \frac{d^2x}{dt^2}$

$g = 32\ ft/\sec^2 = 9.8\ m/s^2$

1.3-3 *Air Resistance*

The application of force on an object causes an acceleration of that object. A force is not the only factor in the movement, or acceleration of an object. ***Air resistance*** involves movement through the air. Air resistance is a significant factor in how fast an object falls, according to this law.

$$R(x,v) = -r(x,v).v$$

R: resistance force (*has sign opposite of the velocity*)
r: is a function that is always nonnegative

- *when a ball is falling from a high altitude, the density of the air has to be taken into account.*

$$F = -mg + R(v)$$

$$m\frac{dv}{dt} = -mg - rv$$

$$\frac{dv}{dt} = -g - \frac{r}{m}v$$

$$dv = \left(-g - \frac{r}{m}v\right)dt$$

$$\frac{dv}{g+\frac{r}{m}v} = -dt$$ Integrate both sides

$$\int \frac{dv}{g+\frac{r}{m}v} = -\int dt$$

$$\frac{m}{r}\ln\left(g + \frac{r}{m}v\right) = -t + C_1$$

$$\ln\left(g + \frac{r}{m}v\right) = -\frac{r}{m}t + C_2$$

$$g + \frac{r}{m}v = e^{-\frac{r}{m}t+C_2}$$

$$v(t) = Ce^{-rt/m} - \frac{mg}{r}$$

When $t \to \infty \;\Rightarrow\; v = -\frac{mg}{r}$ (*Terminal Velocity*)

$$x(t) = -\frac{mC}{r}e^{-rt/m} - \frac{mg}{r}t + A$$ (*A*: is a constant)

Example 1

Suppose you drop a brick from the top of a building that is 250 *m* high. The brick has a mass of 2 *kg*, and the resistance force is given by $R=-4v$. How long will it take the brick to reach the ground? what will be its velocity at that time?

Solution

$$v(t)=Ce^{-rt/m}-\frac{mg}{r}$$

$$v(0)=0=C-\frac{mg}{r}$$

$$\begin{aligned}\Rightarrow C&=\frac{mg}{r}\\&=\frac{2(9.8)}{4}\\&=4.9\end{aligned}$$

$$\begin{aligned}\frac{dx}{dt}&=v(t)\\&=4.9\left(e^{-2t}-1\right)\end{aligned}$$

Integrate both sides

$$\int dx=\int 4.9\left(e^{-2t}-1\right)dt$$

$$x(t)=4.9\left(-\frac{1}{2}e^{-2t}-t\right)+A$$

$$x(0)=250=4.9\left(-\frac{1}{2}e^{-2(0)}-(0)\right)+A$$

$$250=4.9\left(-\frac{1}{2}\right)+A$$

$$250=-2.45+A$$

$$A=252.45$$

$$x(t)=4.9\left(-\frac{1}{2}e^{-2t}-t\right)+252.45$$

$$x(t)=0 \quad \Rightarrow \underline{t=51.52\ sec}$$

(*Using software to solve it*)

$$v(t)=4.9\left(e^{-2t}-1\right)$$

$$\underline{v(t=51.52)\approx -4.9\ m/s}$$

1.3-4 *Finding the displacement*

$$a = \frac{dv}{dt}$$
$$= \frac{dv}{dx}\frac{dx}{dt}$$
$$= \frac{dv}{dx}.v$$

$$F = -mg + R(v)$$
$$R = -k|v|.v$$
$$m\frac{dv}{dt} = -mg - k|v|v$$
$$\frac{dv}{dt} = -g - \frac{k}{m}v^2$$
$$= \frac{dv}{dx}.v$$

$$v\frac{dv}{dx} = -g - \frac{k}{m}v^2$$
$$= -\frac{mg + kv^2}{m}$$

Example 2

A ball of mass $m = 0.2\ kg$ is protected from the surface of the earth, with velocity $v_0 = 50\ m/s$. Assume that the force of air resistance is given by $R = -k|v|.v$, where $k = 0.02$. What is the maximum height reached by the ball?

Solution

$$v\frac{dv}{dx} = -\frac{mg + kv^2}{m}$$

$$\frac{vdv}{mg + kv^2} = -\frac{dx}{m}$$ ***Integrate both sides***

$$\int_{v_0}^{0} \frac{vdv}{mg + kv^2} = -\int_0^{x_{\max}} \frac{dx}{m}$$ $$d\left(mg + kv^2\right) = 2kvdv \quad \Rightarrow \quad \frac{d\left(mg+kv^2\right)}{2k} = vdv$$

$$\frac{1}{2k}\int_{v_0}^{0} \frac{d\left(mg + kv^2\right)}{mg + kv^2} = -\int_0^{x_{\max}} \frac{dx}{m}$$

$$\frac{1}{2k}\ln\left|mg+kv^2\right|_{50}^{0}=-\frac{x}{m}\Big|_{0}^{x_{max}}$$

$$\frac{1}{2k}\left[\ln(mg)-\ln\left(mg+k\left(50^2\right)\right)\right]=-\frac{x_{max}}{m}$$

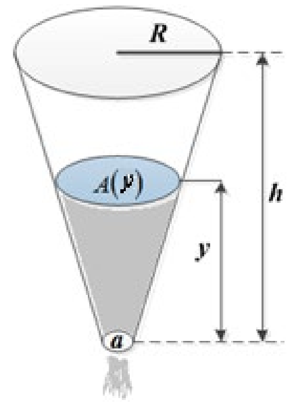
$$x_{max}=\frac{m}{2k}\left[\ln\left(mg+k\left(50^2\right)\right)-\ln(mg)\right]$$

$$=\frac{0.2}{2(.02)}\left[\ln\left(\frac{0.2(9.8)+(.02)(50)^2}{0.2(9.8)}\right)\right]$$

$$\underline{=16.4\ m}$$

1.3-5 *Torricelli's Law*

Suppose that a water tank has a ***hole with area a*** at its bottom, from which water is leaking.



Denote by $y(t)$ the depth of water in the tank at time *t*, and by $V(t)$ the volume of water in the tank then. It is plausible – and true, under ideal conditions – that the velocity of water exiting through the hole is

$$v = \sqrt{2gy}$$

Which is the velocity a drop of water would acquire in falling freely from the surface of the water to the hole. One can derive this formula beginning with the assumption that the sum of the kinetic and potential energy of the system remains constant. Under real conditions, taking into account the construction of a water jet from an orifice, $v = c\sqrt{2gy}$, where c is an empirical constant between 0 and 1 (usually about 0.6 for a small continuous stream of water). For simplicity, we take $c = 1$ in the following discussion.

$$\frac{dV}{dt} = -av = -a\sqrt{2gy}$$

$$\frac{dV}{dt} = -k\sqrt{y} \qquad \text{where} \quad k = a\sqrt{2g}$$

This is a statement of *Torricelli's* Law for a draining tank.

Let $A(y)$ denote the horizontal cross-sectional area of the tank at height *y*. Then, applied to a thin horizontal slice of water at height $\overline{y}$ with area $A(\overline{y})$ and thickness $d\,\overline{y}$, the integral method of cross sections gives

$$V(y) = \int_0^y A(\overline{y})\,d\overline{y}$$

The fundamental theorem of calculus therefore implies that $\frac{dV}{dy} = A(y)$ and hence that

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dy}\cdot\frac{dy}{dt}\\ &= A(y)\frac{dy}{dt}\end{aligned}$$

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy}$$

$$= -k\sqrt{y} \qquad \text{(An alternative form of } \textit{Torricelli's} \text{ Law)}$$

$$\frac{dh}{dt} = -c\frac{A_h}{A_w}\sqrt{2gh}$$

Where A_w and A_h are the cross-sectional areas of the water and the hole.

Example 3

A tank is shaped like a vertical cylinder; it initially contains water to a depth of 9 *ft.*, and a bottom plug is removed at time $t = 0$ (*hours*). After 1 *hr*. the depth of the water has dropped to 4 *ft*. how long does it take for all the water to drain from the tank?

Solution

$$\frac{dy}{dt} = k\sqrt{y}$$

$$\frac{dy}{y^{1/2}} = kdt \qquad \textit{Integrate both sides}$$

$$\int y^{-1/2}dy = \int kdt$$

$$2y^{1/2} = kt + C$$

With initial condition $y(0) = 9$

$$2\sqrt{9} = k(0) + C$$

$$\underline{C = 6}$$

$$2\sqrt{y} = kt + 6$$

$$y(1) = 4$$

$$2\sqrt{4} = k(1) + 6$$

$$k = 6 - 4 = -2$$

$$2\sqrt{y} = -2t + 6$$

$$\sqrt{y} = 3 - t$$

$$y(t) = (3 - t)^2$$

It will take 3 *hours* for the tank to empty.

Exercises *Section* 1.3 – Models of Motions

1. A body of mass m falls from rest subject to gravity in a medium offering resistance proportional to the square of the velocity. Determine the velocity and position of the body at t *seconds.*

2. A body of mass m, with initial velocity v_0, falls vertically. If the initial position is denoted s_0. Determine the velocity and position of the body at t seconds.
Assume the body acted upon by gravity alone and the air resistance proportional to the square of the velocity.

3. A body falls from a height of 300 *ft*. What distance has it traveled after 4 *sec*. if subject to g, the earth's acceleration?

4. A body falls from an initial velocity of $1,000\ ft/s$. What distance has it traveled after 3 *sec*. if subject to $g = 32\ ft/s^2$, the earth's acceleration?

5. A projectile is fired straight upwards with an initial velocity of $1,600\ ft/s$. What is its velocity at $40,000\ ft$. $\left(g = 32\ ft/s^2\right)$

6. A projectile is fired straight upwards with an initial velocity of $1,000\ ft/s$. What is its velocity at $8,000\ ft$. $\left(g = 32\ ft/s^2\right)$

7. An 8 *lbs*. weight falls from rest toward earth. Assuming that the weight is acted upon by air resistance, numerically equal to $2v$, but measured in pounds, find the velocity and distance fallen after t seconds. (The variable v represents the velocity measured in *ft./sec*.)

8. A stone is released from rest and dropped into a deep well. Eight seconds later, the sound of the stone splashing into the water at the bottom of the well returns to the ear of the person who released the stone. How long does it take the stone to drop to the bottom of the well? How deep is the well? Ignore air resistance. The speed of sound is 340 *m/s*.

9. A rocket is fired vertically and ascends with constant acceleration $a = 100\ m/s^2$ for 1.0 *min*. At that point, the rocket motor shuts off and the rocket continues upward under the influence of gravity. Find the maximum altitude acquired by the rocket and the total time elapsed from the take-off until the rocket returns to the earth. *Ignore air resistance*.

10. A ball having mass $m = 0.1\ kg$ falls from rest under the influence of gravity in a medium that provides a resistance that is proportional to its velocity. For a velocity of $0.2\ m/s$ the force due to the resistance of the medium is $-1\ N$. Find the terminal velocity of the ball.

1 N is the force required to accelerate a 1 kg mass at a rate of $1\ m/s^2$: $1N = 1\ \text{kg} \cdot m/s^2$

11. A ball is projected vertically upward with initial velocity v_0 from ground level. Ignore air resistance.

a) What is the maximum height acquired by the ball?

b) How long does it take the ball to reach its maximum height? How long does it take the ball to return to the ground? Are these times identical?

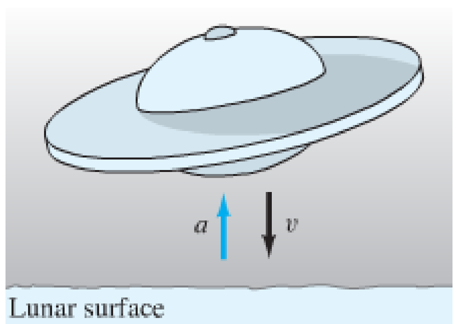
c) What is the speed of the ball when it impacts with the ground on its return?

12. An object having mass 70 *kg* falls from rest under the influence of gravity. The terminal velocity of the object is $-20\ m/s$. Assume that the air resistance is proportional to the velocity.

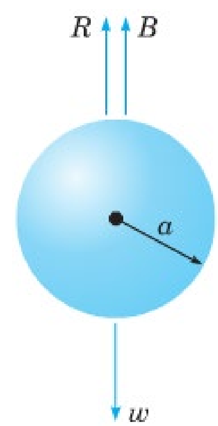
a) Find the velocity and distance traveled at the end of 2 seconds.

b) How long does it take the object to reach 80% of its terminal velocity?

13. A lunar lander is falling freely toward the surface of the moon at a speed of 450 *m/s*. Its retrorockets, when fired, provide a constant deceleration of 2.5 m/s^2 (the gravitational acceleration produced by the moon is assumed to be included in the given acceleration). At What height above the lunar surface should the retrorockets be activated to ensure a "soft touchdown? ($v = 0$ at impact)?



14. A body falling in a relatively dense fluid, oil for example, is acted on by three forces: a resistance force *R,* a buoyant force *B,* and its weight *w* due to gravity. The buoyant force is equal to the weight of the fluid displaced by the object. For a slowly moving spherical body of radius a, the resistive force is given by Stokes's Law $R = 6\pi\frac{\mu a}{v}$, where v is the velocity of the body, and μ is the coefficient of viscosity of the surrounding fluid?

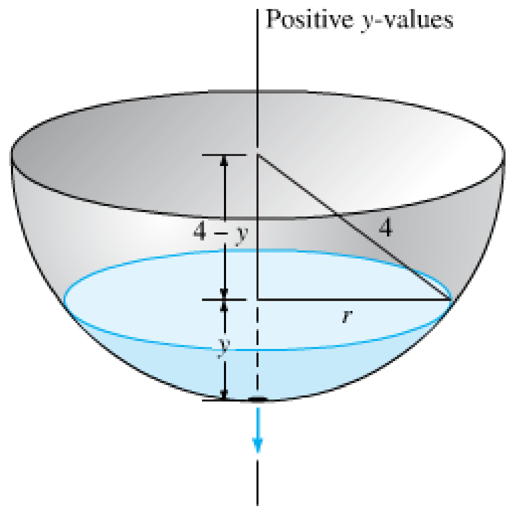


a) Find the limiting velocity of a solid sphere of radius *a* and density ρ falling freely in a medium of density ρ' and coefficient of viscosity μ.

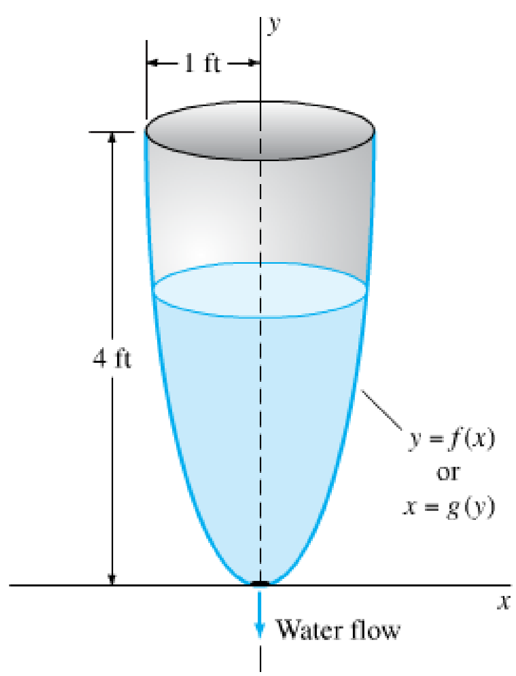
b) In 1910 R. A. Millikan studied the motion of tiny droplets of oil falling in an electric field. A field of strength *E* exerts a force E_e on a droplet with charge *e*. Assume that *E* has been adjusted so the droplet is held stationary $(v = 0)$ and that *w* and *B* are as given. Find an expression for *e*.

15. Suppose that the tank has a radius of 3 *ft*. and that its bottom hole is circular with radius 1 *in*. How long will it take the water (initially 9 *ft*. deep) to drain completely?

16. A hemispherical bowl has top radius of 4 *ft*. and at time $t = 0$ is full of water. At that moment a circular hole with diameter 1 *in*. is opened in the bottom of the tank. How long will it take for all the water to drain from the tank?



17. The clepsydra, or water clock – A 12-*hr* water clock is to be designed with the dimensions, shaped like the surface obtained by revolving the curve $y = f(x)$ around the *y*-axis. What should be this curve, and what should be the radius of the circular bottom hole, in order that the water level will fall at the constant rate of 4 *inches per hour*?



18. At time $t = 0$ the bottom plug (at the vertex) of a full conical water tank 16 *feet* high is removed. After 1 *hr*. the water in the tank is 9 *feet* deep. When will the tank be empty?

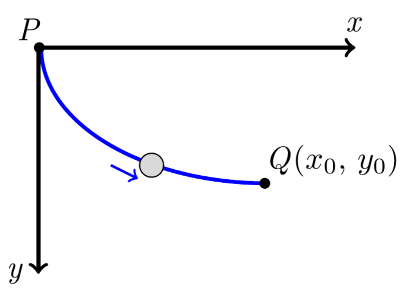
19. Suppose that a cylindrical tank initially containing V_0 gallons of water drains (through a bottom hole) in T minutes. Use Torricelli's Law to show that the volume of water in the tank after $t \le T$ minutes is $V = V_0\left(1-\frac{t}{T}\right)^2$

20. One of the famous problems in the history of mathematics is the brachistochrone problem: to find the curve along which a particle will slide without friction in the minimum time from one given point P to another point Q, the second point being lower than the first but not directly beneath it. This problem was posed by Johann Bernoulli in 1696 as a challenge problem to the mathematicians of his day. Correct solutions were found by Johann Bernoulli and his brother Jakob Bernoulli and by Isaac Newton, Gottfried Leibniz, and the Marquis de L'Hospital. The brachistochrone problem is important in the development of mathematics as one of the forerunners of the calculus of variations.

In solving this problem, it is convenient to take the origin as the upper point P and to orient the axes as shown. The lower point Q has coordinates $\left(x_0, y_0\right)$. It is then possible to show that the curve of minimum time is given by a function $y = \phi(x)$ that satisfies the differential equation

$$\left(1+y'^2\right)y = k^2 \qquad (eq.\ i)$$

Where k^2 is a certain positive constant to be determined later



a) Solve the equation $(eq.\ i)$ for y'. Why is it necessary to choose the positive square root?

b) Introduce the new variable t by the relation

$$y = k^2\sin^2 t \qquad (eq.\ ii)$$

Show that the equation found in part (a) then takes the form

$$k^2\sin^2 t\ dt = dx \qquad (eq.\ iii)$$

c) Letting $\theta = 2t$, show that the solution of $(eq.\ iii)$ for which $x = 0$ when $y = 0$ is given by

$$x = k^2\frac{\theta-\sin\theta}{2}, \quad y = k^2\frac{1-\cos\theta}{2} \qquad (eq.\ iv)$$

Equations (*iv*) are parametric equations of the solution of (eq. *i*) that passes through (0, 0). The graph of Eqs. (*iv*) is called a cycloid.

d) If we make a proper choice of the constant *k*, then the cycloid also passes through the point $\left(x_0, y_0\right)$ and is the solution of the brachistochrone problem. Find *k* if $x_0 = 1$ and $y_0 = 2$

21. Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the substance B at time $t = 0$, and if x is the amount of product at time t, then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt} = k(a-x)(b-x) \quad \textit{or} \quad \frac{1}{(a-x)(b-x)}\frac{dx}{dt} = k$$

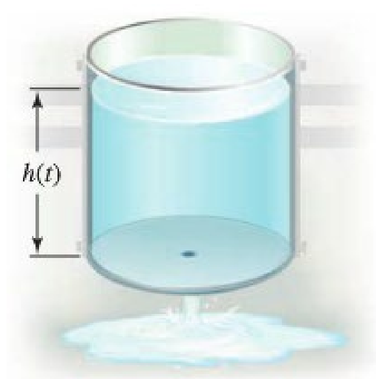
Where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t.

a) If $a = b$

b) If $a \neq b$

Assume in each case that $x = 0$ when $t = 0$

22. An open cylindrical tank initially filled with water drains through a hole in the bottom of the tank according to Torricelli's Law. If $h(t)$ is the depth of water in the tank for $t \geq 0$, then Torricelli's Law implies $h'(t) = -2k\sqrt{h}$, where k is a constant that includes the acceleration due to gravity, the radius of the tank, and the radius of the drain. Assume that the initial depth of the water is $h(0) = H$



a) Find the solution of the initial value problem.

b) Find the solution in the case that $k = 0.1$ and $H = 0.5\ m$.

c) In general, how long does it take the tank to drain in terms of k and H?

23. An object in free fall may be modeled by assuming that the only forces at work are the gravitational force and resistance (friction due to the medium in which the objects falls). By Newton's second Law (mass × acceleration = the sum of the external forces), the velocity of the object satisfies the differential equation

$$\underbrace{m}_{mass} \cdot \underbrace{v'(t)}_{acceleration} = \underbrace{mg + f(v)}_{external\ force}$$

Where f is a function that models the resistance and the positive direction is downward. One common assumption (often used for motion in air) is that $f(v) = -kv^2$, where $k > 0$ is a drag coefficient.

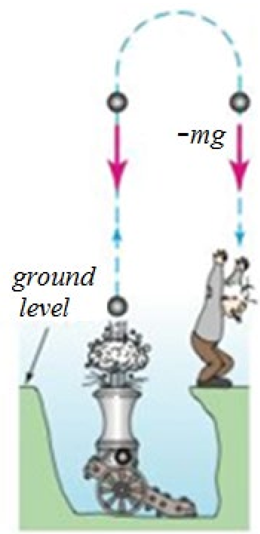
a) Show that the equation can be written in the form $v'(t) = g - av^2$ where $a = \frac{k}{m}$

b) For what (positive) value of v is $v'(t) = 0$? (This equilibrium solution is called the ***terminal velocity***.)

c) Find the solution of this separable equation assuming $v(0) = 0$ *and* $0 < v(t)^2 < \frac{g}{a}$ for $t \ge 0$

d) Graph the solution found in part (***c***) with $g = 9.8\ m/s^2,\ m = 1\ kg,\ and\ k = 0.1\ kg/m$, and verify the terminal velocity agrees with the value found in part (***b***).

24. Suppose a small cannonball weighing 16 *pounds* is shot vertically upward, with an initial velocity $v_0 = 300$ *ft/s*
The answer to the question "How high does the cannonball go?" depends on whether we take air resistance into account.



a) Suppose air resistance is ignored. If the positive direction is upward, then a model for the state of the cannonball is given by $\frac{d^2s}{dt^2} = -g$. Since $\frac{ds}{dt} = v(t)$ the last differential equation is the same as $\frac{dv}{dt} = -g$, where we take $g = 32\ ft/s^2$. Find the velocity $v(t)$ of the cannonball at time *t*.

b) Use the result in part (*a*) to determine the height $s(t)$ of the cannonball measured from ground level. Find the maximum height attained by the cannonball.

25. Two chemicals *A* and *B* are combined to form a chemical *C*. The resulting reaction between the two chemicals is such that for each *gram* of *A*, 4 *grams* of *B* is used. It is observed that 30 *grams* of the compound *C* is formed in 10 *minutes*.

a) Determine the amount of *C* at time *t* if the rate of the reaction is proportional to the amounts of *A* and *B* remaining and if initially there are 50 *grams* of *A* and 32 *grams* of *B*.

b) How much of the compound *C* is present at 15 *minutes*?

c) Interpret the solution as $t \to \infty$

26. Two chemicals A and B are combined to form a chemical C. The rate, or velocity, of the reaction is proportional to the product of the instantaneous amounts of A and B not converted to chemical C. Initially, there are 40 *grams* of A and 50 *grams* of B, and each gram of B, 2 *grams* of A is used. It is observed that 10 *grams* of C is formed in 5 *minutes*.

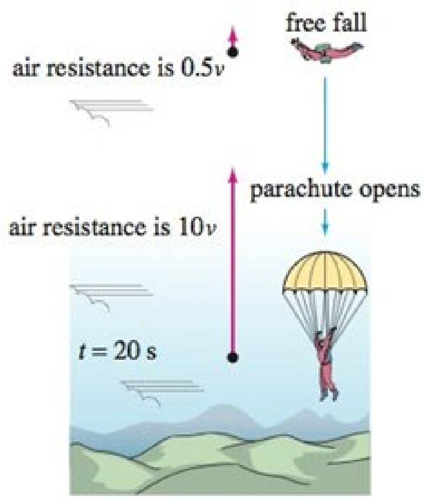
a) How much is formed in 20 *minutes*?

b) What is the limiting amount of C after a long time?

c) How much of chemicals A and B remains after a long time?

d) If 100 *grams* of chemical A is present initially, at what time is chemical C half-formed?

27. A skydiver weighs 125 *pounds*, and her parachute and equipment combined weigh another 35 *pounds*. After exiting from a plane at an altitude of 15,000 *feet*, she waits 15 *seconds* and opens her parachute. Assume that the constant of proportionality has the value $k = 0.5$ during free fall and $k = 10$ after the parachute is opened.



Assume that her initial velocity on leaving the plane is *zero*.

a) What is her velocity and how far has she traveled 20 *seconds* after leaving the plane?

b) How does her velocity at 20 *seconds* compare with her terminal velocity?

c) How long does it take her to reach the ground?

28. A tank in the form of a right-circular cylinder standing on end is leaking water through a circular hole in its bottom. When friction and contraction of water at the hole are ignored, the height h of water in the tank is described by

$$\frac{dh}{dt} = -\frac{A_h}{A_w}\sqrt{2gh}$$

Where A_w and A_h are the cross-sectional areas of the water and the hole, respectively.

a) Find $h(t)$ if the initial height of the water is H.

b) Sketch the graph $h(t)$ and give the interval I of definition in terms of the symbols A_w, A_h, and H. $\left(g = 32 \ ft/s^2\right)$

c) Suppose the tank is 10 *feet* high and has radius 2 *feet* and the circular hole has radius $\frac{1}{2}$ *inch*. If the tank is initially full, how long will it take to empty?

29. A tank in the form of a right-circular cylinder cone standing on end, vertex down, is leaking water through a circular hole in its bottom.

a) Suppose the tank is 20 *feet* high and has radius 8 *inches*. Show that the differential equation governing the height h of water leaking from a tank is

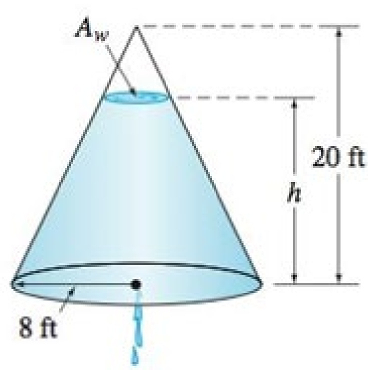
$$\frac{dh}{dt} = -\frac{5}{6h^{3/2}}$$

In this model, friction and contraction of the water at the hole were taken into account with $c = 0.6$ and $g = 32\ ft/s^2$. If the tank is initially full, how long will it take the tank to empty?

b) Suppose the tank has a vertex angle of 60° and the circular hole has radius 2 *inches*. Determine the differential equation governing the height h of water. Use $c = 0.6$ and $g = 32\ ft/s^2$.

c) If the height of the water is initially 9 *feet*, how long will it take the tank to empty?

30. Suppose that the conical tank is inverted and that water leaks out a circular hole of radius 2 *inches* in the center of its circular base. Is the time it takes to empty a full tank the same as for the tank with vertex down?



Take the friction/contraction coefficient to be $c = 0.6$ and $g = 32\ ft/s^2$

31. A differential equation for the velocity v of a falling mass m subjected to air resistance proportional to the square of the instantaneous velocity is

$$m\frac{dv}{dt} = mg - kv^2$$

Where $k > 0$ is a constant of proportionality. The positive direction is downward.

a) Solve the equation subject to the initial condition $v(0) = v_0$.

b) Use the solution in part (*a*) to determine the limiting, or terminal, velocity of the mass.

c) If the distance *s*, measured from the point where the mass was released above the ground, is related to velocity *v* by $\frac{ds}{dt} = v(t)$, find an explicit expression for $s(t)$ if $s(0) = 0$

32. An object is dropped from altitude y_0

a) Determine the impact velocity if the drag force is proportional to the square of velocity, with drag coefficient κ.

b) If the terminal velocity is known to −120 *mph* and the impact velocity was −90 *mph*, what was the initial altitude y_0 ?

33. An object is dropped from altitude y_0

a) Assume that the drag force is proportional to the velocity, with drag coefficient κ. Obtain an implicit solution relating velocity and altitude.

b) If the terminal velocity is known to −120 *mph* and the impact velocity was −90 *mph*, what was the initial altitude y_0 ?

34. An object of mass 3 *kg* is released from rest 500 *m* above the ground and allowed to fall under the influence of gravity. Assume the gravitational force constant, with $g = 9.81$ *m/s*^{2}, and the force due to air resistance is proportional to the velocity of the object with proportionality constant $\kappa = 3$ *N-sec/m*. Determine when the object will hit the ground.

35. A parachutist whose mass is 75 *kg* drops from helicopter hovering 4000 *m* above the ground and falls toward the earth under the influence of gravity. Assume the gravitational force is constant. Assume also that the force due to air resistance is proportional to the velocity of the parachutist, with the proportionality constant $\kappa_1 = 15$ *N-sec/m* when the chute is closed and with constant $\kappa_2 = 105$ *N-sec/m* when the chute is open. If the chute does not open until 1 *min* after the parachutist leaves the helicopter, after how many *seconds* will he reach the ground?

36. A parachutist whose mass is 75 *kg* drops from helicopter hovering 2000 *m* above the ground and falls toward the earth under the influence of gravity. Assume the gravitational force is constant. Assume also that the force due to air resistance is proportional to the velocity of the parachutist, with the proportionality constant $\kappa_1 = 30$ *N-sec/m* when the chute is closed and with constant $\kappa_2 = 90$ *N-sec/m* when the chute is open. If the chute does not open until the velocity of the parachutist reaches 20 *m/sec*, after how many seconds will he reach the ground?

37. An object of mass 5 *kg* is released from rest 1000 *m* above the ground and allowed to fall under the influence of gravity. Assume the gravitational force constant, with $g = 9.8\ m/s^2$, and the force due to air resistance is proportional to the velocity of the object with proportionality constant $\kappa = 50$ *N-sec/m*. Determine when the object will hit the ground.

38. An object of mass 500 *kg* is released from rest 1000 *m* above the ground and allowed to fall under the influence of gravity. Assume the gravitational force constant, with $g = 9.8\ m/s^2$, and the force due to air resistance is proportional to the velocity of the object with proportionality constant $\kappa = 50$ *N-sec/m*. Determine when the object will hit the ground.

39. A 400-*lbs* object is released from rest 500 *ft.* above the ground and allowed to fall under the influence of gravity. Assuming that the force in pounds due to air resistance is $-10v$, where v is the velocity of the object in *ft/s*, determine the equation of motion of the object. When will the object hit the ground?

40. An object of mass 8 *kg* is given an upward initial velocity of 20 *m/sec* and then allowed to fall under the influence of gravity. Assume that the force in Newton due to air resistance is $-16v$, where v is the velocity of the object in *m/sec*.

a) Determine the equation of motion of the object.

b) If the object is initially 100 *m* above the ground, determine when the object will hit the ground.

41. An object of mass 5 *kg* is given a downward initial velocity of 50 *m/sec* and then allowed to fall under the influence of gravity. Assume that the force in Newton due to air resistance is $-10v$, where v is the velocity of the object in *m/sec*.

a) Determine the equation of motion of the object.

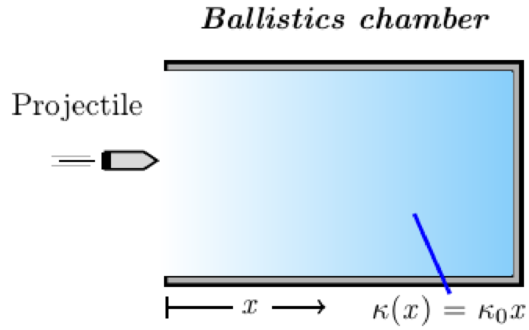
b) If the object is initially 100 *m* above the ground, determine when the object will hit the ground.

42. A shell of mass 2 *kg* is shot upward with an initial velocity of 200 *m/sec*. The magnitude of the force on the shell due to air resistance is $\frac{|v|}{20}$.

a) When will the shell reach its maximum height above the ground?

b) What is the maximum height?

43. We need to design a ballistics chamber to decelerate test projectiles fired into it. Assume the resistive force encountered by the projectile is proportional to the square of its velocity and neglect gravity.



The chamber is to be constructed so that the coefficient κ associated with this resistive force is not constant but is, in fact, a linearly increasing function of distance into the chamber:

Let $\kappa(x) = \kappa_0 x$, where κ_0 is a constant; the resistive force then has the form $\kappa(x)v^2 = \kappa_0 xv^2$.

If we use time t as the independent variable, Newton's Law of motion leads us to the differential equation

$$m\frac{dv}{dt} + \kappa_0 xv^2 = 0 \quad with \quad v = \frac{dx}{dt}$$

a) Adopt distance x into the chamber as the new independent variable and rewrite the given differential equation as a first order equation in terms of the new independent variable.

b) Determine the value κ_0 needed if the chamber is to reduce projectile velocity to 1% of its incoming value within d *units* of distance.

44. When the velocity v of an object is very large, the magnitude of the force due to air resistance is proportional to v^2 with the force acting in opposition to the motion of the object. A shell of mass 3 kg is hot upward from the ground with an initial velocity of 500 *m/sec*. If the magnitude of the force due to air resistance is $(0.1)v^2$.

a) When will the shell reach its maximum height above the ground?

b) What is the maximum height?

45. A sailboat has been running (on a straight course) under a light wind at 1 *m/sec*. Suddenly the wind picks up, blowing hard enough to apply a constant force of 600 N to the sailboat. The only other force acting on the boat is water resistance that is proportional to the velocity of the boat. If the proportionality constant for water resistance is $\kappa = 100$ *N-sec/m* and the mass of the sailboat is 50 *kg*.

a) Find the equation of motion of the sailboat.

b) What is the limiting velocity of the sailboat under this wind?

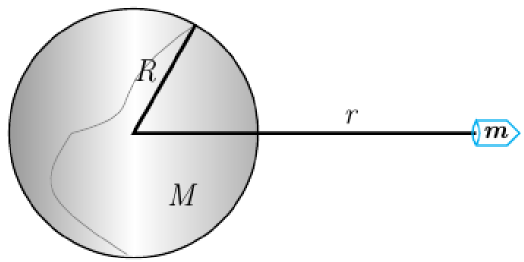
c) When the velocity of the sailboat reaches 5 *m/sec*, the boat begins to rise out of the water and plane. When this happens, the proportionality constant for the water resistance drop to $\kappa = 60$ *N-sec/m*. Find the equation of motion of the sailboat.

d) What is the limiting velocity of the sailboat under this wind as it is planning?

46. According to Newton's Law of gravitation, the attractive force between two objects varies inversely as the square of the distances between them. That is, $F_g = \frac{GM_1M_2}{r^2}$

Where M_1 and M_2 are the masses of the objects, r is the distance between them (center to center), F_g is the attractive force, and G is the constant of proportionality.

Consider ta projectile of constant mass m being fired vertically from Earth.
Let t represent time and v the velocity of the projectile.



Earth

a) Show that the motion of the projectile, under Earth's gravitational force, is governed by the equation

$$\frac{dv}{dt} = -\frac{gR^2}{r^2},$$

Where r is the distance between the projectile and the center of Earth, R is the radius of Earth, M is the mass of Earth, and $g = \frac{GM}{R^2}$.

b) Use the fact the $\frac{dr}{dt} = v$ to obtain $v\frac{dv}{dr} = -\frac{gR^2}{r^2}$

c) If the projectile leaves Earth's surface with velocity v_0, show that

$$v^2 = \frac{2gR^2}{r} + v_0^2 - 2gR$$

d) Use the result of part (c) to how that the velocity of the projectile remains positive if and only if $v_0^2 - 2gR > 0$. The velocity $v_e = \sqrt{2gR}$ is called the escape velocity?

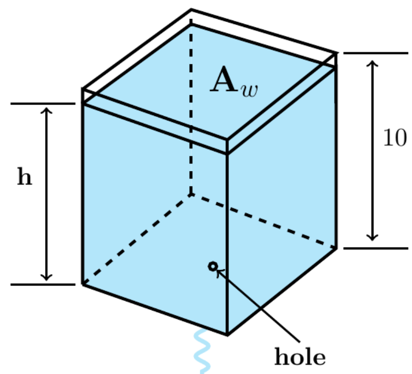
e) If $g = 9.81$ *m/sec*^{2} and $R = 6370$ *km* for Earth, what is Earth's escape velocity?

f) If the acceleration due to gravity for the Moon is $g_m = \frac{g}{6}$ and the radius of the Moon is $R_m = 1738$ *km*, what is the escape velocity of the Moon?

47. A 180-*lb* skydiver drops from a hot-air balloon. After 10 *sec* of free fall, a parachute is opened. The parachute immediately introduces a drag force proportional to velocity. After an additional 4 *sec*, the parachutist reaches the ground. Assume that air resistance is negligible during free fall and that the parachute is designed so that a 200-*lb* person will reach a terminal velocity of –10 *mph*.

a) What is the speed of the skydiver immediately before the parachute is opened?
b) What is the parachutist's impact velocity?
c) At what altitude was the parachute opened?
d) What is the balloon's altitude?

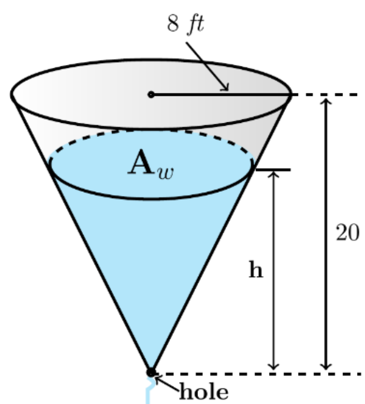
48. Suppose water is leaking from a tank through a circular hole of area A_h at its bottom. When water leaks through a hole, friction and contraction of the stream near the hole reduce the volume of water leaving the tank per second to $cA_h\sqrt{2gh}$, where c $(0 < c < 1)$ is an empirical constant.



Determine a differential equation for the height *h* of water at time *t* for the cubical tank. The radius of the hole is 2 *in*., $g = 32$ ft/s^2, and the friction/contraction factor is $c = 0.6$

49. The right-circular tank loses water out of a circular hole at its bottom.

The radius of the hole is 2 *in*., and $g = 32$ ft/s^2, and the friction/contraction factor is $c = 0.6$.



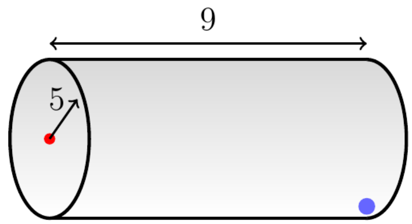
a) Determine a differential equation for the height *h* of water at time *t* for the cubical tank.
b) Find the height in function of time.

50. In meteorology, the term virga refers to falling rain drops or ice particles that evaporate before they reach the ground. Assume that a typical raindrop is spherical. Starting at some time, which we can designate as $t = 0$, the raindrop of radius r_0 falls from rest from a cloud and begins to evaporate.

a) If it is assumed that a raindrop evaporates in such a manner that its shape remains spherical, then it also makes sense to assume that the rate at which the raindrop evaporates – that is, the rate at which it loses mass – is proportional to its surface area, Show that this latter assumption implies that the rate at which the radius r of the raindrop decreases is a constant. Find $r(t)$.

b) If the positive direction is downward, construct a mathematical model for the velocity v of the falling raindrop at time $t > 0$. Ignore air resistance.

51. A horizontal cylindrical tank of length $9\ ft$, and radius $5\ ft$, is filled with oil. At $t = 0$ a plug at the lowest point of the tank is removed and a flow result.



Find y the depth of the oil in the tank at any time t while the tank is draining. The constriction coefficient is $k = \frac{1}{15}$

Section 1.4 – Linear Equations

Definition

A first order linear equation is given by the form:

$$y' + p(x)y = f(x)$$

Is said to be a linear equation in the variable y and where $p(x)$ and $f(x)$ are functions of x and called the coefficients

If $f(x) = 0 \rightarrow y' = p(x)y$.

This linear equation is said to be ***homogeneous***. (Otherwise it is ***nonhomogeneous or inhomogeneous***).

Linear	***Non-linear***
$x' = \sin(t)x$	$x' = t\sin x$
$y' = e^{2t}y + \cos t$	$y' = 1 - y^2$
$x' = (3t+2)x + t^2 - 1$	

It is possible for a differential equation to have no solutions.

1.4-1 *Solution of the homogeneous equation*

$$\frac{dx}{dt} = a(t)x$$

$$\frac{dx}{x} = a(t)dt$$

$$\int \frac{dx}{x} = \int a(t)dt$$

$$\ln|x| = \int a(t)dt + C \qquad \textit{Convert to exponential form}$$

$$|x| = e^{\int a(t)dt + C} \qquad \textit{Let } A = e^C$$

$$= e^C e^{\int a(t)dt}$$

$$\boxed{x(t) = A.e^{\int a(t)dt}}$$

Example 1

Solve: $x' = \sin(t)\, x$

Solution

$$\frac{dx}{dt} = \sin(t)\, x$$

$$x(t) = A.e^{\int \sin(t)dt}$$

$$\boxed{= A.e^{-\cos t}}$$

Second method

$$\frac{dx}{x} = \sin(t)\, dt$$

$$\int \frac{dx}{x} = \int \sin(t)\, dt$$

$$\ln|x| = \int \sin(t)dt + C$$

$$\ln|x| = -\cos(t) + C$$

$$\boxed{x(t) = e^{-\cos(t)+C}}$$

1.4-2 Solving a linear first-order Equation (*Properties*)

1. Put a linear equation into a standard form $y' + p(x)y = f(x)$
2. Identify $p(x)$ then find $y_h = e^{-\int pdx}$
3. Multiply the standard form by y_h
4. Integrate both sides

Solution of the Inhomogeneous Equation $u(t) = e^{-\int a(t)dt}$

$$x' = a(t)x + f(t)$$

$$x' - ax = f$$

$$(ux)' = u\left(x' - ax\right) = uf$$

$$u(t)x(t) = \int u(t)f(t)dt + C$$

Example 2

Find the general solution to: $x' = x + e^{-t}$

Solution

$$x' - x = e^{-t}$$

$$e^{-\int 1dt} = e^{-t}$$

$$e^{-t}\left(x' - x\right) = e^{-t}e^{-t}$$

$$\left(e^{-t}x\right)' = e^{-2t}$$

$$e^{-t}x(t) = \int e^{-2t}dt$$

$$e^{-t}x(t) = -\tfrac{1}{2}e^{-2t} + C$$

$$\boxed{x(t) = -\tfrac{1}{2}e^{-t} + Ce^{t}}$$

1.4-3 *Solution of the Nonhomogeneous Equation* $y' + p(x)y = f(x)$

Let assume: $y = y_h + y_p$ *where* $\begin{cases} y_h & \text{Homogeneous Solution} \\ y_p & \text{Paticular Solution} \end{cases}$

The homogeneous equation is given by $y_h' + p(x)y_h = 0$

$$y_h' = -p(x)y_h$$

$$\boxed{y_h = e^{-\int p\,dx}}$$

$$y_p = u(x)y_h = u.e^{-\int p\,dx}$$

$$y_p' + p(x)y_p = f(x)$$

$$\left(uy_h\right)' + puy_h = f$$

$$u'y_h + uy_h' + puy_h = f$$

$$u'y_h + u\left(y_h' + py_h\right) = f \qquad \text{Since } y_h' + py_h = 0 \qquad \text{homogeneous}$$

$$u'y_h = f$$

$$\frac{du}{dx} = \frac{f}{y_h}$$

$$du = \frac{f}{e^{-\int p\,dx}}dx$$

$$= f.e^{\int p\,dx}dx$$

$$u = \int f.e^{\int p\,dx}dx$$

$$y_p = u.e^{-\int p\,dx}$$

$$u = \left(\int f.e^{\int p\,dx}dx\right)e^{-\int p\,dx}$$

$$\boxed{y_p = e^{-\int p\,dx}\int f.e^{\int p\,dx}dx} \qquad y = y_h + y_p$$

$$\boxed{y = Ce^{-\int p\,dx} + e^{-\int p\,dx}\int f.e^{\int p\,dx}dx} \qquad \boxed{y = e^{-\int p\,dx}\left(C + \int f.e^{\int p\,dx}dx\right)}$$

Example 3

Find the general solution of $x' = x\sin t + 2te^{-\cos t}$ and the particular solution that satisfies $x(0) = 1$.

Solution

$$x' - x\sin t = 2te^{-\cos t} \qquad P(t) = \sin t, \quad Q(t) = 2te^{-\cos t}$$

$$x_h = e^{-\int \sin t\,dt} = e^{\cos t}$$

$$\int Q(t)x_h\,dt = \int 2te^{-\cos t}e^{\cos t}\,dt = \int 2t\,dt = t^2$$

$$x(t) = e^{-\cos t}\left(t^2 + C\right) \qquad x = \frac{1}{e^{\int P dt}}\left(\int Q.e^{\int P dt}dt + C\right)$$

$$x(0) = \left((0)^2 + C\right)e^{-\cos 0} = 1$$

$$Ce^{-1} = 1$$

$$C = e$$

$$\boxed{x(t) = \left(t^2 + e\right)e^{-\cos t}}$$

Example 4

Find the general solution of $x' = x\tan t + \sin t$ and the particular solution that satisfies $x(0) = 2$..

Solution

$$x' - (\tan t)x = \sin t \qquad P(t) = -\tan t, \quad Q(t) = \sin t$$

$$e^{-\int \tan t\,dt} = e^{\ln(\cos t)} \boxed{= \cos t}$$

$$\int (\sin t)(\cos t)\,dt = -\int \cos t\,d(\cos t) = -\frac{1}{2}\cos^2 t$$

$$x(t) = \frac{1}{\cos t}\left(-\frac{1}{2}\cos^2 t + C\right) = -\frac{1}{2}\cos t + \frac{1}{\cos t}C$$

$$\boxed{= -\frac{1}{2}\cos t + \frac{1}{\cos t}C}$$

$$x(0) = -\frac{1}{2}\cos(0) + \frac{C}{\cos(0)} = 2$$

$$-\frac{1}{2} + C = 2 \quad\Rightarrow\quad C = \frac{5}{2}$$

$$\boxed{x(t) = -\frac{1}{2}\cos t + \frac{5}{2\cos t}}$$

Notes

1. Integrating an expression that is not the differential of any elementary function is called non-elementary.

$$\int e^{x^2}dx \qquad \int x\tan x dx \qquad \int \frac{e^{-x}}{x}dx$$

$$\int \sin x^2 dx \qquad \int \cos x^2 dx \qquad \int \frac{\sin x}{x}dx \qquad \int \frac{\cos x}{x}dx$$

2. In math some important functions are defined in terms of non-elementary integrals. Two such functions are the error function and the complementary error function.

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \qquad erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$$

Exercises *Section* 1.4 – Linear Equations

(1 – 90) Find the general solution of the first-order, linear equation.

1. $y' - y = 3e^t$
2. $y' + y = \sin t$
3. $y' + y = \dfrac{1}{1+e^t}$
4. $y' - y = e^{2t} - 1$
5. $y' + y = te^{-t} + 1$
6. $y' + y = 1 + e^{-x}\cos 2x$
7. $y' + y\cot x = \cos x$
8. $y' + y\sin t = \sin t$
9. $y' = \cos x - y\sec x$
10. $y' + (\tan x)y = \cos^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$
11. $y' + (\cot t)y = 2t\csc t$
12. $y' + (1+\sin t)y = 0$
13. $y' + \left(\frac{1}{2}\cos x\right)y = -\frac{3}{2}\cos x$
14. $\dfrac{dy}{dx} + y = e^{3x}$
15. $y' - ty = t$
16. $y' = 2y + x^2 + 5$
17. $xy' + 2y = 3$
18. $\dfrac{dy}{dt} - 2y = 4 - t$
19. $y' + 2y = 1$
20. $y' + 2y = e^{-t}$
21. $y' + 2y = e^{-2t}$
22. $y' - 2y = e^{3t}$
23. $y' + 2y = e^{-x} + x + 1$
24. $y' + 2xy = x$
25. $y' - 2ty = t$
26. $y' + 2ty = 5t$
27. $y' - 2xy = e^{x^2}$
28. $y' + 2xy = x^3$
29. $y' - 2y = t^2e^{2t}$
30. $x' - 2\dfrac{x}{t+1} = (t+1)^2$
31. $y' + \dfrac{2}{t}y = \dfrac{\cos t}{t^2}$
32. $y' - 2(\cos 2t)y = 0$
33. $y' + 2y = \cos 3t$
34. $y' - 3y = 5$
35. $y' + 3y = 2xe^{-3x}$
36. $y' + 3t^2y = t^2$
37. $y' + 3x^2y = x^2$
38. $y' + \dfrac{3}{t}y = \dfrac{\sin t}{t^3}, \quad (t \ne 0)$
39. $y' + \dfrac{3}{x}y = 1 + \dfrac{1}{x}$
40. $y' + \dfrac{3}{2}y = \dfrac{1}{2}e^x$
41. $y' + 5y = t + 1$
42. $xy' - y = x^2\sin x$
43. $x\dfrac{dy}{dx} + y = e^x, \quad x > 0$
44. $x\dfrac{dy}{dx} + 2y = 1 - \dfrac{1}{x}, \quad x > 0$
45. $y\dfrac{dx}{dy} + 2x = 5y^3$
46. $ty' + y = \cos t$
47. $xy' + 2y = x^2$
48. $xy' = 2y + x^3\cos x$
49. $xy' + 2y = x^{-3}$
50. $ty' + 2y = t^2$
51. $xy' + 2\left(y + x^2\right) = \dfrac{\sin x}{x}$
52. $xy' + 4y = x^3 - x$

53. $xy' + (x+1)y = e^{-x}\sin 2x$

54. $xy' + (3x+1)y = e^{3x}$

55. $xy' + (2x-3)y = 4x^4$

56. $2xy' - 3y = 9x^3$

57. $2y' + 3y = e^{-t}$

58. $2y' + 2ty = t$

59. $3xy' + y = 10\sqrt{x}$

60. $3xy' + y = 12x$

61. $x^2y' + xy = 1$

62. $x^2y' + x(x+2)y = e^x$

63. $y^2 + (y')^2 = 1$

64. $(1+x)y' + y = \sqrt{x}$

65. $(1+x)y' + y = \cos x$

66. $(x+1)y' + (x+2)y = 2xe^{-x}$

67. $(x+1)y' - xy = x + x^2$

68. $\left(1+x^3\right)y' = 3x^2y + x^2 + x^5$

69. $(t+1)\dfrac{ds}{dt} + 2s = 3(t+1) + \dfrac{1}{(t+1)^2}, \quad t > -1$

70. $(x+2)^2 y' = 5 - 8y - 4xy$

71. $\left(x^2-1\right)y' + 2y = (x+1)^2$

72. $\left(x^2+4\right)y' + 2xy = x^2\left(x^2+4\right)$

73. $\left(1+e^t\right)y' + e^t y = 0$

74. $\left(t^2+9\right)y' + ty = 0$

75. $e^{2x}y' + 2e^{2x}y = 2x$

76. $\tan\theta\dfrac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \frac{\pi}{2}$

77. $(\cos t)y' + (\sin t)y = 1$

78. $\cos x\dfrac{dy}{dx} + (\sin x)y = 1$

79. $\cos^2 x\sin x\dfrac{dy}{dx} + \left(\cos^3 x\right)y = 1$

80. $\dfrac{dr}{d\theta} + r\sec\theta = \cos\theta$

81. $\dfrac{dr}{d\theta} + r\tan\theta = \sec\theta$

82. $\dfrac{dP}{dt} + 2tP = P + 4t - 2$

83. $ydx - 4\left(x + y^6\right)dy = 0$

84. $ydx = \left(ye^y - 2x\right)dy$

85. $(x+y+1)dx - dy = 0$

86. $\dfrac{dy}{dx} = x^2e^{-4x} - 4y$

87. $\left(x^2+1\right)y' + xy - x = 0$

88. $\dfrac{dx}{dt} = 9.8 - 0.196x$

89. $\dfrac{di}{dt} + 500i = 10\sin\omega t$

90. $2\dfrac{dQ}{dt} + 100Q = 10\sin 60t$

(91 – 168) Find the solution of the initial value problem

91. $y' - 3y = 4\,; \quad y(0) = 2$

92. $y' = y + 2xe^{2x}; \quad y(0) = 3$

93. $\left(x^2+1\right)y' + 3xy = 6x; \quad y(0) = -1$

94. $t\dfrac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(2) = 1$

95. $\theta\dfrac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0, \quad y\left(\frac{\pi}{2}\right) = 1$

96. $\dfrac{dy}{dx} + xy = x, \quad y(0) = -6$

97. $ty' + 2y = 4t^2, \quad y(1) = 2$

98. $\left(1+t^2\right)y' + 4ty = \left(1+t^2\right)^{-2}, \quad y(1) = 0$

99. $y' + y = e^t, \quad y(0) = 1$

100. $y' + \frac{1}{2}y = t, \quad y(0) = 1$

101. $y' = x + 5y, \quad y(0) = 3$

102. $y' = 2x - 3y, \quad y(0) = \frac{1}{3}$

103. $xy' + y = e^x, \quad y(1) = 2$

104. $y\frac{dx}{dy} - x = 2y^2, \quad y(1) = 5$

105. $xy' + y = 4x + 1, \quad y(1) = 8$

106. $y' + 4xy = x^3 e^{x^2}, \quad y(0) = -1$

107. $(x+1)y' + y = \ln x, \quad y(1) = 10$

108. $x(x+1)y' + xy = 1, \quad y(e) = 1$

109. $L\frac{di}{dt} + RI = E, \quad i(0) = i_0$

110. $\frac{dT}{dt} = k(T - T_m) \quad T(0) = T_0$

111. $y' + y = 2, \quad y(0) = 0$

112. $xy' + 2y = 3x, \quad y(1) = 5$

113. $y' - 2y = 3e^{2x}, \quad y(0) = 0$

114. $xy' + 5y = 7x^2, \quad y(2) = 5$

115. $xy' - y = x, \quad y(1) = 7$

116. $xy' + y = 3xy, \quad y(1) = 0$

117. $xy' + 3y = 2x^5, \quad y(2) = 1$

118. $y' + y = e^x, \quad y(0) = 1$

119. $xy' - 3y = x^3, \quad y(1) = 10$

120. $y' + 2xy = x, \quad y(0) = -2$

121. $y' = (1 - y)\cos x, \quad y(\pi) = 2$

122. $(1+x)y' + y = \cos x, \quad y(0) = 1$

123. $y' = 1 + x + y + xy, \quad y(0) = 0$

124. $xy' = 3y + x^4 \cos x, \quad y(2\pi) = 0$

125. $y' = 2xy + 3x^2 e^{x^2}, \quad y(0) = 5$

126. $\left(x^2 + 4\right)y' + 3xy = x, \quad y(0) = 1$

127. $y' - 2y = e^{3x}; \quad y(0) = 3$

128. $y' - 3y = 6; \quad y(0) = 1$

129. $2y' + 3y = e^x; \quad y(0) = 0$

130. $\left(x^2 + 1\right)y' + 3x^3 y = 6xe^{-3x^2/2}, \quad y(0) = 1$

131. $y' + y = 1 + e^{-x}\cos 2x; \quad y\left(\frac{\pi}{2}\right) = 0$

132. $2y' + (\cos x)y = -3\cos x; \quad y(0) = -4$

133. $y' + 2y = e^{-x} + x + 1; \quad y(-1) = e$

134. $y' + \frac{y}{x} = xe^{-x}; \quad y(1) = e - 1$

135. $y' + 4y = e^{-x}; \quad y(1) = \frac{4}{3}$

136. $x^2 y' + 3xy = x^4 \ln x + 1; \quad y(1) = 0$

137. $y' + \frac{3}{x}y = 3x - 2 \quad y(1) = 1$

138. $y' - (\sin x)y = 2\sin x, \quad y\left(\frac{\pi}{2}\right) = 1$

139. $y' + (\tan x)y = \cos^2 x, \quad y(0) = -1$

140. $(\cos x)y' + y\sin x = 2x\cos^2 x \quad y\left(\frac{\pi}{4}\right) = \frac{-15\sqrt{2}\pi^2}{32}$

141. $(\cos x)y' + (\sin x)y = 2\cos^3 x\sin x - 1 \quad y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

142. $ty' + 2y = t^2 - t + 1 \quad y(1) = \frac{1}{2}$

143. $ty' - 2y = t^5 \sin 2t - t^3 + 4t^4 \quad y(\pi) = \frac{3}{2}\pi^4$

144. $2y' - y = 4\sin 3t \quad y(0) = y_0$

145. $y' + 2y = 2 - e^{-4t} \quad y(0) = 1$

146. $y' - y = -\frac{1}{2}e^{t/2}\sin 5t + 5e^{t/2}\cos 5t \quad y(0) = 0$

147. $y' + 2y = 3; \quad y(0) = -1$

148. $y' + (\cos t)y = \cos t; \quad y(\pi) = 2$

149. $y' + 2ty = 2t; \quad y(0) = 1$

150. $y' + y = \frac{e^{-t}}{t^2}; \quad y(1) = 0$

151. $ty' + 2y = \sin t$; $y(\pi) = \frac{1}{\pi}$

152. $y' + \frac{2}{t}y = \frac{\cos t}{t^2}$; $y(\pi) = 0$

153. $(\sin t)y' + (\cos t)y = 0$; $y\left(\frac{3\pi}{4}\right) = 2$

154. $y' + 3t^2 y = t^2$; $y(0) = 2$

155. $ty' + y = t\sin t$; $y(\pi) = -1$

156. $y' + y = \sin t$; $y(\pi) = 1$

157. $y' + y = \cos 2t$; $y(0) = 5$

158. $y' + 3y = \cos 2t$; $y(0) = -1$

159. $y' - 2y = 7e^{2t}$; $y(0) = 3$

160. $y' - 2y = 3e^{-2t}$; $y(0) = 10$

161. $y' + 2y = t^2 + 2t + 1 + e^{4t}$; $y(0) = 0$

162. $y' - 3y = 2t - e^{4t}$; $y(0) = 0$

163. $y' + y = t^3 + \sin 3t$; $y(0) = 0$

164. $y' + 2y = \cos 2t + 3\sin 2t + e^{-t}$; $y(0) = 0$

165. $y' + y = e^{3t}$; $y(0) = y_0$

166. $t^2 y' - ty = 1$; $y(1) = y_0$

167. $y' + ay = e^{at}$; $y(0) = y_0$, $a \neq 0$

168. $3y' + 12y = 4$; $y(0) = y_0$

(169 – 172) Find a solution to the initial value problem that is continuous on the given interval $[a, b]$

169. $y' + \frac{1}{x}y = f(x)$, $y(1) = 1$ $\qquad f(x) = \begin{cases} 3x, & 1 \le x \le 2 \\ 0, & 2 < x \le 3 \end{cases}$ $\qquad [a, b] = [1, 3]$

170. $y' + (\sin x)y = f(x)$, $y(0) = 3$ $\qquad f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ -\sin x, & \pi < x \le 2\pi \end{cases}$ $\qquad [a, b] = [0, 2\pi]$

171. $y' + p(t)y = 2$, $y(0) = 1$ $\qquad p(t) = \begin{cases} 0, & 0 \le t \le 1 \\ \frac{1}{t}, & 1 < t \le 2 \end{cases}$ $\qquad [a, b] = [0, 2]$

172. $y' + p(t)y = 0$, $y(0) = 3$ $\qquad p(t) = \begin{cases} 2t - 1, & 0 \le t \le 1 \\ 0, & 1 < t \le 3 \\ -\frac{1}{t}, & 3 < t \le 4 \end{cases}$ $\qquad [a, b] = [0, 4]$

Find the solution of the initial value problem. Discuss the interval of existence and provide a sketch of your solution

173. $xy' + 2y = \sin x$; $y\left(\frac{\pi}{2}\right) = 0$

174. $(2x + 3)y' = y + (2x + 3)^{1/2}$; $y(-1) = 0$

175. The following system of differential equations is encountered in the study of the decay of a special type of radioactive series of elements

$$\frac{dx}{dt} = -\lambda_1 x \qquad \frac{dy}{dt} = \lambda_1 x - \lambda_2 y$$

Where λ_1 *and* λ_2 are constants.

Discuss how to solve this system subject to $x(0) = x_0$, $y(0) = y_0$

176. Let $P(t)$ be the performance level of someone learning a skill as a function of the training time *t*. The graph of *P* is called a ***learning curve***. We proposed the differential equation

$$\frac{dP}{dt} = k\left(M - P(t)\right)$$

As a reasonable model for learning, where *k* is a positive constant. Solve it as a linear differential equation and use your solution to graph the learning curve.

177. A differential equation describing the velocity *v* of a falling mass subject to air resistance proportional to the instantaneous velocity is

$$m\frac{dv}{dt} = mg - kv$$

Where $k > 0$ is a constant of proportionality. The positive direction is downward.

a) Solve the equation subject to the initial condition $v(0) = v_0$

b) Use the solution in part (*a*) to determine the limiting, or terminal, velocity of the mass.

c) If the distance *s*, measured from the point where the mass was released above ground, is related to velocity *v* by $\frac{ds}{dt} = v(t)$, find an explicit expression for $s(t)$ if $s(0) = 0$

178. As a raindrop falls, it evaporates while retaining its spherical shape. If we make the further assumptions that the rate at which the raindrop evaporates is proportional to its surface and that air resistance is negligible, then a model for the velocity $v(t)$ of the raindrop is

$$\frac{dv}{dt} + \frac{3(k/\rho)}{\frac{k}{\rho}t + r_0}v = g$$

Here ρ is the density of water, r_0 is the radius of the raindrop at $t = 0$, $k < 0$ is the constant of proportionality, and downward direction is taken to be the positive direction.

a) Solve for $v(t)$ if the raindrop falls from rest.

b) Show that the radius of the raindrop at time *t* is $r(t) = \frac{k}{\rho}t + r_0$.

c) If $r_0 = 0.01\,ft$ and $r = 0.007\,ft$ 10 *seconds* after the raindrop falls from a cloud, determine the time at which the raindrop has evaporated completely.

179. A model that describes the population of a fishery in which harvesting takes place at a constant rate is given by

$$\frac{dP}{dt} = kP - h$$

Where *k* and *h* are positive constants.

a) Solve $P(t)$ given the initial value $P(0) = P_0$

b) Describe the behavior of the population $P(t)$ for increasing time in three cases $P_0 > \frac{h}{k}$, $P_0 = \frac{h}{k}$, and $P_0 < \frac{h}{k}$

c) Use the results from part (*b*) to determine whether the fish population will ever go extinct in finite time, that is, whether there exists a time $T > 0$ such that $P(T) = 0$. If the population goes extinct then find *T*.

180. A certain body weighing $45\ lb$, is heated to a temperature of $300°$. Then at $t = 0$ it is plunged into $100\ lb$ of water at a temperature of $50°$. Given that the specific heat of the body is $\frac{1}{9}$, find the formula for the temperature *T* of the body during its cooling.

Section 1.5 – Mixing Problems

A typical mixing problem investigates the behavior of a mixed solution of some substance. Typically, the solution is being mixed in a tank. A solution of a given concentration enters the mixture at some fixed rate and is thoroughly mixed in the tank. The tank is also being drained at some fixed rate.

The physical representation of the rate of change:

$$\begin{aligned}\frac{dx}{dt} &= \text{rate of change}\\ &= \text{rate in} - \text{rate out}\end{aligned}$$

This is referred to as a ***balance law***.

The rate in and out is given by:

Rate = Volume Rate (*gal/min*) *x* Concentration (*lb./gal*)

Example 1

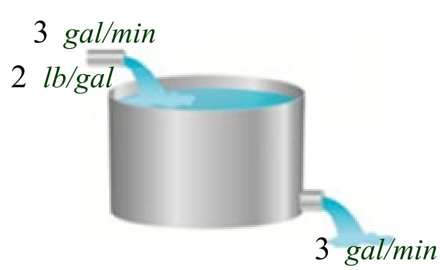
The tank initially holds 100 *gal* of pure water. At time $t=0$, a solution containing 2 *lb.* of salt per *gallon* begins to enter the tank at the rate of 3 *gallons* per *minute*. At the same time a drain is opened at the bottom of the tank so that the volume of solution in the tank remains constant.
How much salt is in the tank after 60 *min*?
What will be the eventual salt content in the tank?

Solution

$x(t)$: number of pounds of salt in the tank after *t* min.

Volume: $V(t) = 100 + (3-3)t = 100$

Concentration at time *t*: $c(t) = \dfrac{x(t)}{V(t)} = \dfrac{x(t)}{100}\ \ lb/gal$



Rate in = Volume Rate *x* Concentration

$$\begin{aligned}&= 3\frac{gal}{min} \times 2\frac{lb}{gal}\\ &= 6\ \ lb/\min\end{aligned}$$

Rate out = Volume Rate *x* Concentration

$$\begin{aligned}&= 3\frac{gal}{\min} \times \frac{x(t)}{100}\frac{lb}{gal}\\ &= \frac{3x(t)}{100}\ \ lb/\min\end{aligned}$$

$$\begin{aligned}\frac{dx}{dt} &= \text{rate of change}\\ &= \text{rate in} - \text{rate out}\end{aligned}$$

$$= 6 - \frac{3x}{100}$$

$$\frac{dx}{dt} + \frac{3}{100}x = 6$$

$$u(t) = e^{\int \left(\frac{3}{100}\right)dt} = e^{0.03t}$$

$$\int 6e^{0.03t}dt = \frac{6}{0.03}e^{0.03t} = 200e^{0.03t}$$

$$x(t) = e^{-0.03t}\left(200e^{0.03t} + C\right)$$

$$\boxed{x(t) = 200 + Ce^{-0.03t}}$$

Since there was no salt present in the tank initially, the initial condition is $x(0) = 0$

$$x(t = 0) = 200 + Ce^{-0.03(0)} = 0$$

$$200 + C = 0$$

$$C = -200$$

$$\boxed{x(t) = 200 - 200e^{-0.03t}}$$

After 60 *min*:

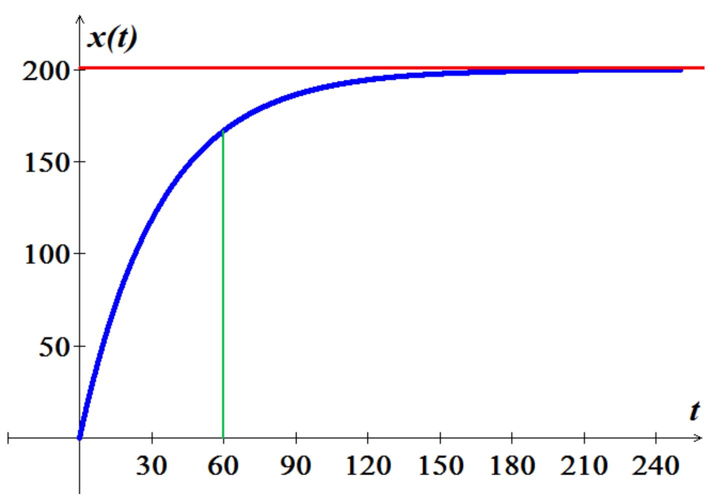
$$x(60) = 200 - 200e^{-0.03(60)}$$

$$\boxed{\approx 167\ lb}$$

As $t \to \infty$ then $x(t) = \lim\limits_{t\to\infty}\left(200 - 200e^{-0.03t}\right)$

$$= 200 - 200\lim_{t\to\infty}\left(e^{-0.03t}\right) \qquad \lim_{t\to\infty}\left(e^{-0.03t}\right) = e^{-\infty} = 0$$

$$\boxed{= 200\ \ lb}$$



Example 2

The 600-*gal* tank is filled with 300 *gal* of pure water. A spigot is opened above the tank and a salt solution containing 1.5 *lb*. of salt per gallon of solution begins flowing into the tank at the rate of 3 *gal/min*. simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave tank at a rate of 1 *gal/min*. What will be the salt content in the tank at the precise moment that the volume of solution in the tank is equal to the tank's capacity (600 *gal*)?

Solution

$$\begin{aligned} V(t) &= 300 + (3-1)t \\ &= 300 + 2t \end{aligned}$$

$$c(t) = \frac{x(t)}{300+2t}$$

$$\begin{aligned} \text{Rate in } &= 3\frac{gal}{min} \times 1.5\frac{lb}{gal} \\ &= \boxed{4.5 \ lb/min} \end{aligned}$$

$$\begin{aligned} \text{Rate out } &= 1 \times \frac{x}{300+2t} \\ &= \boxed{\frac{x}{300+2t} \ lb/\min} \end{aligned}$$

$$\frac{dx}{dt} = 4.5 - \frac{x}{300+2t}$$

$$\frac{dx}{dt} + \frac{1}{300+2t}x = 4.5$$

$$\begin{aligned} u(t) &= e^{\int \frac{1}{300+2t}dt} \qquad\qquad d(300+2t) = 2dt \\ &= e^{\frac{1}{2}\int \frac{1}{300+2t}d(300+2t)} \\ &= e^{\frac{1}{2}\ln(300+2t)} \\ &= e^{\ln(300+2t)^{1/2}} \\ &= \sqrt{300+2t} \end{aligned}$$

$$\int 4.5\sqrt{300+2t}\ dt = 4.5\frac{1}{2}\frac{2}{3}(300+2t)^{2/3}$$

$$\begin{aligned} x(t) &= \frac{1}{\sqrt{300+2t}}\left(1.5(300+2t)^{3/2} + C\right) \\ &= 1.5(300+2t) + \frac{C}{\sqrt{300+2t}} \\ &= 450 + 3t + \frac{C}{\sqrt{300+2t}} \end{aligned}$$

$$x(0) = 450 + 3(0) + \frac{C}{\sqrt{300+2(0)}} = 0$$

$$450 + \frac{C}{\sqrt{300}} = 0$$

$$C = -450\sqrt{300}$$

$$\underline{= -4500\sqrt{3}|}$$

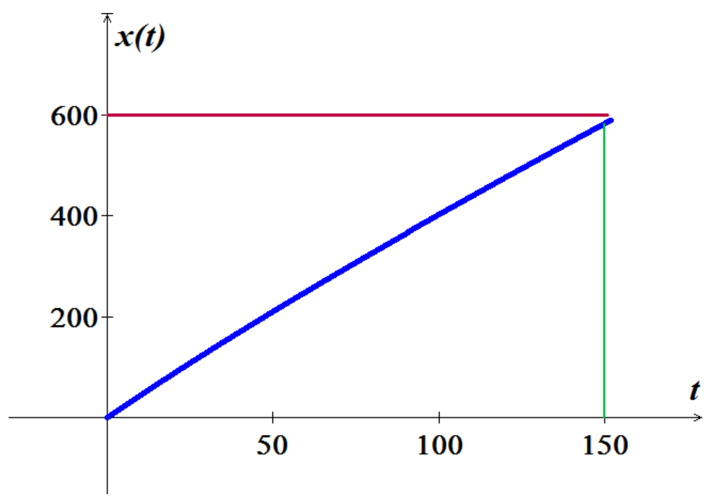
$$\underline{x(t) = 450 + 3t - \frac{4500\sqrt{3}}{\sqrt{300+2t}}}$$

$$V = 300 + 2t = 600$$

$$t = 150 \ min$$

$$x(t = 150) = 450 + 3(150) - \frac{4500\sqrt{3}}{\sqrt{300+2(150)}}$$

$$\underline{\approx 582 \ lb}$$



Exercises *Section* 1.5 – Mixing Problems

1. Consider two tanks, label tank ***A*** and tank ***B*** for reference. Tank ***A*** contains 100 *gal* of solution in which is dissolved 20 *lb.* of salt. Tank ***B*** contains 200 *gal* of solution which is dissolved 40 *lbs.* of salt. Pure water flows into the tank ***A*** at rate of 5 *gal/s*. There is a drain at the bottom of tank ***A***. The solution leaves tank ***A*** via the drain at a rate of 5 *gal/s* and flows immediately into tank ***B*** at the same rate. A drain at the bottom of tank ***B*** allows the solution to leave tank ***B*** at a rate of 2.5 *gal/s*. What is the salt content in tank ***B*** at the precise moment that tank ***B*** contains 250 *gal* of solution?

2. A tank contains 100 *gal* of pure water. At time zero, a sugar-water solution containing 0.2 *lb.* of sugar per gal enters the tank at a rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the sugar solution to leave the tank at 3 *gal/min*. Assume that the solution in the tank is kept perfectly mixed at all times.

a) What will be the sugar content in the tank after 20 *minutes*?
b) How long will it take the sugar content in the tank to reach 15 *lbs.*?
c) What will be the eventual sugar content in the tank?

3. A tank initially contains 50 *gal* of sugar water having a concentration of 2 *lb*. of sugar for each gal of water. At time zero, pure water begins pouring into the tank at a rate of 2 *gal* per *minute*. Simultaneously, a drain is opened at the bottom of the tank so that the volume of sugar-water solution in the tank remains constant.

a) How much sugar is in the tank after 10 *minutes*?
b) How long will it take the sugar content in the tank to dip below 20 *lbs*.?
c) What will be the eventual sugar content in the tank?

4. A 50-*gal* tank initially contains 20 *gal* of pure water. Salt-water solution containing 0.5 *lb.* of salt for each gallon of water begins entering the tank at a rate of 4 *gal/min.* simultaneously; a drain is opened at the bottom of the tank, allowing the salt-water solution to leave the tank at a rate of 2 *gal/min*. What is the salt content (*lb.*) in the tank at the precise moment that the tank is full of salt-water solution?

5. A tank contains 500 *gal* of a salt-water solution containing 0.05 *lb.* of salt per gallon of water. Pure water is poured into the tank and a drain at the bottom of the tank is adjusted so as to keep the volume of solution in the tank constant. At what rate (*gal/min*) should the water be poured into the tank to lower the salt concentration to 0.01 *lb./gal* of water in less than one hour?

6. Suppose that a large tank initially holds 300 *gallons* of water in which 50 *pounds* of salt have been dissolved. Pure water is pumped into the tank at a rate of 3 *gal/min*, and when the solution is well stirred, it is then pumped out at the same rate. Determine a differential equation for the amount of salt $x(t)$ in the tank at time $t > 0$.

7. Suppose that a large mixing tank initially holds 300 *gallons* of water is which 50 *pounds* of salt have been dissolved. Another brine solution is pumped into the tank at a rate of 3 *gal/min*, and when the solution is well stirred, it is then pumped out at a slower rate if 2 *gal/min*. If the concentration of the solution entering is 2 *lb./gal*, determine a differential equation for the amount of salt $x(t)$ in the tank at time $t>0$

8. Suppose that a large mixing tank initially holds 300 *gallons* of water is which 50 *pounds* of salt have been dissolved. Another brine solution is pumped into the tank at a rate of 3 *gal/min*, and when the solution is well stirred, it is then pumped out at a faster rate if 3.5 *gal/min*. If the concentration of the solution entering is 2 *lbs./gal*, determine a differential equation for the amount of salt $x(t)$ in the tank at time $t>0$.

9. A tank contains 100 *gal* of fresh water. A solution containing 1 *lb./gal* of soluble lawn fertilizer runs into the tank at the rate of 1 *gal/min*, and the mixture is pumped out of the tank at a rate of 3 *gal/min*. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.

10. A 200-*gal* tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 *lb./gal* of concentrate enters the tank at the rate of 5 *gal/min*, and the well-stirred mixture is withdrawn at the rate of 3 *gal/min*.

a) At what time will the tank be full?
b) At the time the tank is full, how many pounds of concentrate will it contain?

11. A 1500-gallon tank initially contains 600 *gallons* of water with 5 *lbs*. of salt dissolved in it. Water enters the tank at a rate of 9 *gal/hr.* and the water entering the tank at a rate has a salt concentration of $\frac{1}{5}(1+\cos t)$ *lbs./gal*. If a well-mixed solution leaves the tank at a rate of 6 *gal/hr.*, how much salt is in the tank when it overflows?

12. Suppose that an Iowa class battleship has mass 51,000 metric tons (51,000,000 *kg*) and $k \approx 59{,}000\ kg/\sec$. Assume that the ship loses power when it is moving at a speed of 9 *m/sec*.

a) About how far will the ship coast before it is dead in the water?
b) About how long will it take the ship's speed to drop to 1 m/sec?

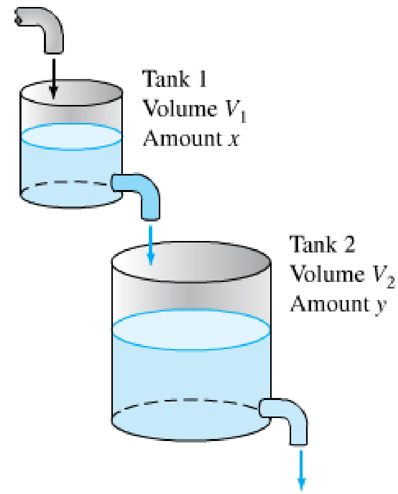
13. A 66-*kg* cyclist on a 7-*kg* bicycle starts coasting on level ground at 9 *m/sec*. The $k \approx 3.9\ kg/\sec$

a) About how far will the cyclist coast before reaching a complete stop?
b) How long will it take the cyclist's speed to drop to 1 *m/sec*?

14. An Executive conference room of a corporation contains 4500 ft^3 of air initially free of carbon monoxide. Starting at time $t = 0$, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 $ft^3/\min$. A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 $ft^3/\min$.

Find the time when the concentration of carbon monoxide in the room reaches 0.01%.

15. Consider the cascade of 2 tanks with $V_1 = 100\ gal$ and $V_2 = 200\ gal$ the volumes of brine in the 2 tanks. Each tank also initially contains 50 *lb*s. of salt. The three flow rates indicated in the figure are each 5 *gal/min*, with pure water flowing into tank.



a) Find the amount $x(t)$ of salt in tank 1 at time *t*.

b) Suppose that $y(t)$ is the amount of salt in tank 2 at time *t*.

Show first that $\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200}$

And then solve for $y(t)$, using the function $x(t)$ found in part (*a*).

c) Finally, find the maximum amount of salt ever in tank 2.

16. Suppose that in the cascade tank 1 initially 100 *gal* of pure ethanol and tank 2 initially contains 100 *gal* of pure water. Pure water flows into tank 1 at 10 *gal/min*, and the other two flow rates are also 10 *gal/min*.

a) Find the amounts $x(t)$ and $y(t)$ of ethanol in the two tanks at time $t \geq 0$.

b) Find the maximum amount of ethanol ever in tank 2.

17. A multiple cascade is shown in the figure. At time $t = 0$, tank 0 contains 1 *gal* of ethanol and 1 *gal* of water; all the remaining tanks contain 2 *gal* of pure water each. Pure water is pumped into tank 0 at 1 *gal/min*, and the varying mixture in each tank is pumped into the one below it at the same rate. Assume, as usual, that the mixtures are kept perfectly uniform by stirring. Let $x_n(t)$ denote the amount of ethanol in tank *n* at time *t*.

a) Show that $x_0(t) = e^{-t/2}$

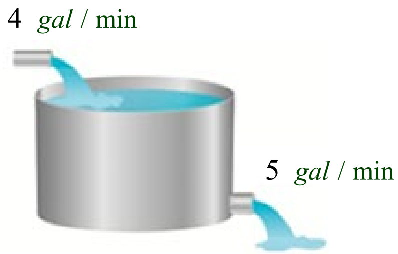
b) Show that the maximum value of $x_n(t)$ for $n > 0$ is $M_n = x_n(2n) = \frac{n^n e^{-n}}{n!}$

18. Assume that Lake Erie has a volume of 480 km^3 and that its rate of inflow (from Lake Huron) and outflow (to Lake Ontario) are both 350 km^3 per year. Suppose that at the time $t = 0$ (*years*), the pollutant concentration of Lake Erie – caused by past industrial pollution that has now been ordered to cease – is 5 times that of Lake Huron. If the outflow henceforth is perfectly mixed lake water, how long will it take to reduce the pollution concentration in Lake Erie to twice that of Lake Huron?

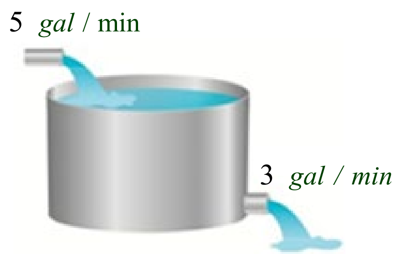
19. A 120-gal tank initially contains 90 *lbs*. of salt dissolved in 90 *gal* of water. Brine containing 2 *lb./gal* of salt flows into the tank at rate of 4 *gal/min*, and the well-stirred mixture flows out the tank at the rate of 3 *gal/min*. How much salt does the tank contain when it is full?

20. A 1000 *gallon* holding tank that catches runoff from some chemical process initially has 800 *gallons* of water with 2 *ounces* of pollution dissolved in it. Polluted water flows into the tank at a rate of 3 *gal/hr*. and contains 5 *ounces/gal* of pollution in it. A well-mixed solution leaves the tank at 3 *gal/hr*. as well. When the amount of pollution in the holding tank reaches 500 *ounces* the inflow of polluted water is cut off and fresh water will enter the tank at a decreased rate of 2 *gallons* while the outflow is increased to 4 *gal/hr*. Determine the amount of pollution in the tank at time *t*.

21. A tank contains 50 *gallons* of a solution composed of 90% water and 10% alcohol. A second solution containing 50% water and 50% alcohol is added to the tank at the rate of 4 $gal/\min$. As the second solution is being added, the tank is being drained at a rate of 5 $gal/\min$. The solution in the tank is stirred constantly. How much alcohol is in the tank after 10 *minutes*?

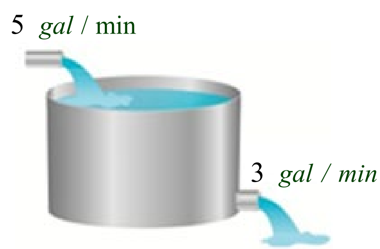


22. A 200-*gallon* tank is half full of distilled water. At time $t=0$, a concentrate solution containing 0.5 *lb/gal* enters the tank at the rate of 5 $gal/\min$, and well-stirred mixture is withdrawn at the rate of 3 $gal/\min$.



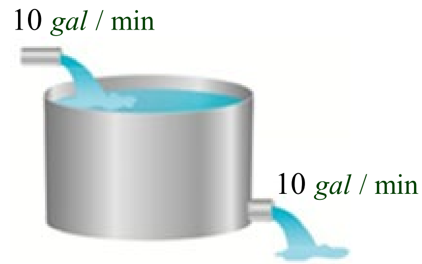
a) At what time will the tank be full?
b) At the time the tank is full, how many pounds of concentrate will it contain?

23. A 200-*gallon* tank is half full of distilled water. At time $t=0$, a concentrate solution containing 1 *lb/gal* enters the tank at the rate of 5 $gal/\min$, and well-stirred mixture is withdrawn at the rate of 3 $gal/\min$.



a) At what time will the tank be full?
b) At the time the tank is full, how many pounds of concentrate will it contain?

24. A 200-*gallon* tank is full of a concentrate solution containing 25 *lb*. Starting at time $t = 0$, distilled water is admitted to the tank at the rate of 10 *gal* / min, and well-stirred mixture is withdrawn at the same rate.



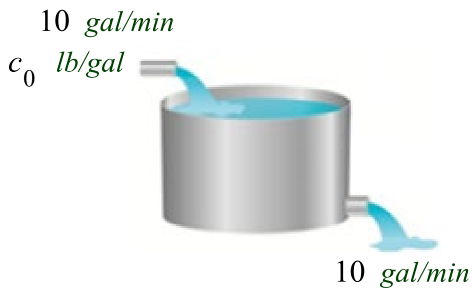
a) Find the amount of concentrate in the solution as a function of *t*.
b) Find the time at which the amount of concentrate in the tank reaches 15 *pounds*.
c) Find the quantity of the concentrate in the solution as $t \to \infty$.

25. A 500-*gallon* tank is full of a concentrate solution containing 50 *lb*. Starting at time $t = 0$, distilled water is admitted to the tank at the rate of 10 *gal* / min, and well-stirred mixture is withdrawn at the rate 15 *gal* / min.

a) At what time will the tank be empty?
b) Find the amount of concentrate in the solution as a function of *t*.

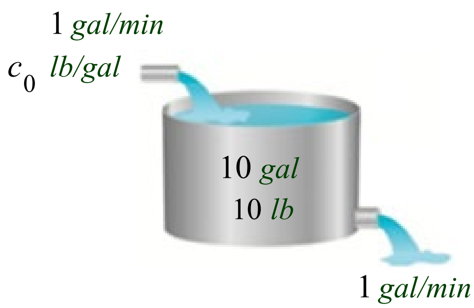
26. A tank contains 300 *liters* of fluid in which 20 *grams* of salt is dissolved. Brine containing 1 *g* of salt per *liter* is then pumped into the tank at a rate of 4 *L/min*; the well-mixed solution is pumped out at the same rate. Find the number $x(t)$ of grams of salt in the tank at time *t*.

27. A 100-*gallon* tank is full of a concentrate solution containing y_0 *lb*. Starting at time $t = 0$, Brine containing c_0 *lb/gal* is then pumped into the tank at a rate of 10 *gal/min*, and well-stirred mixture is withdrawn at the rate 10 *gal/min*.



Find the amount of concentrate in the solution as a function of *t*.

28. A 10-*gallon* tank is full of a concentrate solution containing 10 *lb*. Starting at time $t=0$, Brine containing c_0 *lb/gal* is then pumped into the tank at a rate of 1 *gal/min*, and well-stirred mixture is withdrawn at the rate 1 *gal/min*.



a) Find the amount of concentrate in the solution as a function of *t*.
b) Find the quantity of the concentrate in the solution as $t \to \infty$.

29. A tank contains 200 *liters* of fluid in which 30 *grams* of salt is dissolved. Brine containing 1 *gram* of salt per liter is then pumped into the tank at a rate of 4 *L/min*; the well-mixed solution is pumped out at the same rate.

a) Find the number $A(t)$ of grams of salt in the tank at time *t*.
b) Solve by assuming that pure water is pumped into the tank.

30. A large tank is filled to capacity with 500 *gallons* of pure water. Brine containing 2 *pounds* of salt per gallon is pumped into the tank at a rate of 5 *gal/min*. The well-mixed solution is pumped out at the same rate.

a) Find the number $A(t)$ of grams of salt in the tank at time *t*.
b) What is the concentration $c(t)$ of the salt in the tank at time *t*? At time $t=5$ *min*?
c) What is the concentration of the salt in the tank after a long time, that is, as $t \to \infty$?
d) What is the concentration of the salt in the tank equal to one-half this limiting value?
e) Solve under assumption that the solution is pumped out at a faster rate of 10 *gal/min*. when tis the tank empty?

31. A large tank is filled to capacity with 100 *gallons* of fluid in which 10 *pounds* of salt is dissolved. Brine containing $\frac{1}{2}$ *pound* of salt per gallon is pumped into the tank at a rate of 6 *gal/min*. The well-mixed solution is pumped out at the slower rate of 4 *gal/min*. Find the number of pounds of salt in the tank after 30 *minutes*.

32. A 5000-*gal* tank is maintained with a pumping system that passes 100 *gal* of water per minute through the tank. To treat a certain fish malady, a soluble antibiotic is introduced into the inflow system. Assume that the inflow concentration of medicine is $10te^{-t/50}$ *mg/gal*, where *t* is measured in *minutes*. The well-stirred mixture flows out of the tank at the same rate.

a) Solve for the amount of medicine in the tank as function of time.

b) What is the maximum concentration of medicine achieved by this dosing and when does it occur?

c) For the antibiotic to be effective, its concentration must exceed 100 *mg/gal* for a minimum of 60 *min*. was the dosing effective?

33. A tank initially contains 400 *gal* of fresh water. At time $t = 0$, a brine solution with a concentration of 0.1 *lb*. of salt per gallon enters the tank at a rate of 1 *gal/min* and the well-stirred mixture flows out at a rate of 2 *gal/min*.

a) How long does it take for the tank to become empty?

b) How much salt is present when the tank contains 100 *gal* of brine?

c) What is the maximum amount of salt present in the tank during the time interval found in part (*a*)?

d) When is the maximum achieved?

34. A tank, having a capacity of 700 *gal*, initially contains 10 *lb*. of salt dissolved in 100 *gal* of water. At time $t = 0$, a solution containing 0.5 *lb*. of salt per gallon flows into the tank at a rate of 3 *gal/min* and the well-stirred mixture flows out of the tank at a rate of 2 *gal/min*.

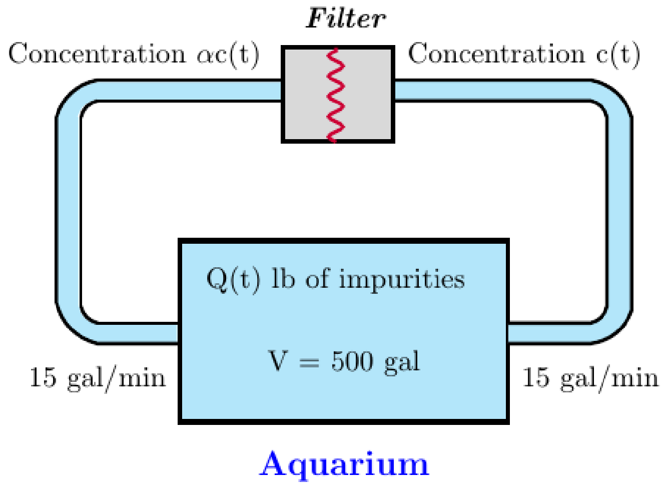
a) How much time will elapse before the tank is filled to capacity?

b) What is the salt concentration in the tank when it contains 400 *gal* of solution?

c) What is the salt concentration at the instant the tank is filled to capacity?

35. A 500-*gal* aquarium is cleansed by the recirculating filter system schematically shown in the figure.

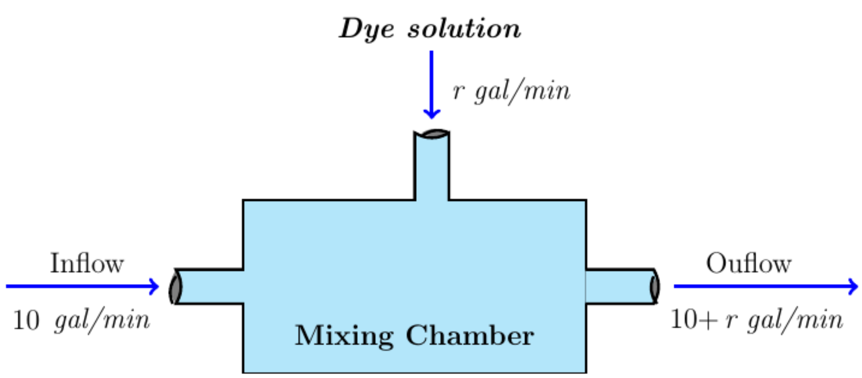
Water containing impurities is pumped out at a rate of 15 *gal/min*, filtered, and returned to the aquarium at the same rate. Assume that passing through the filter reduces the concentration of impurities by a fractional amount α. In the other words, if the impurity concentration upon entering the filter is $c(t)$, the exit concentration is $\alpha\, c(t)$, where $0 < \alpha < 1$.



a) Apply the basic conservation principle $(\textit{rate of change} = \textit{rate in} - \textit{rate out})$ to obtain a differential equation for the amount of impurities present in the aquarium at time *t*. Assume that filtering occurs instantaneously. If the outflow concentration at any time is $c(t)$, assume that the inflow concentration at that same instant is $\alpha\, c(t)$.

b) What value of filtering constant α. will reduce impurity levels to 1% of their original values in a period of 3 *hr*.?

36. A mixing chamber initially contains 2 *gal* of a clear fluid. Clear fluid flows into the chamber at a rate of 10 *gal/min*. A dye solution having a concentration of 4 *oz/gal* is injected into the mixing chamber at a rate of *r gal/min*. When the mixing process is started, the well-mixed mixture is pumped from the chamber at a rate $10+r$ *gal/min*.

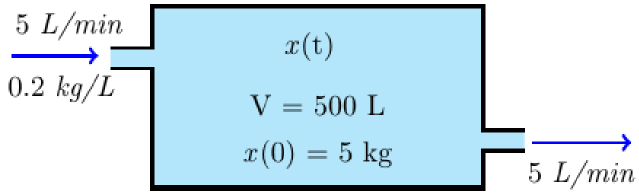


a) Develop a mathematical model for the mixing process.

b) The objective is to obtain a dye concentration in the outflow mixture of 1 *oz/gal*. What injection rate *r* is required to achieve this equilibrium solution? Would this equilibrium value of *r* be different if the fluid in the chamber at time $t=0$ contained some dye?

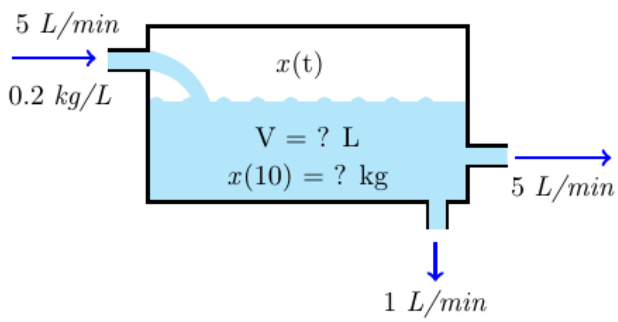
c) Assume the mixing chamber contains 2 *gal* of clear fluid at time $t=0$. How long will it take for the outflow concentration to rise to within 1% of the desired concentration?

37. Suppose a brine containing 0.2 *kg* of salt per liter runs into a tank initially filled with 500 *L* of water containing 5 *kg* of salt The brine enter the tank at a rate of 5 *L/min* . The mixture, kept uniform by stirring, is flowing out at the rate at the same rate.

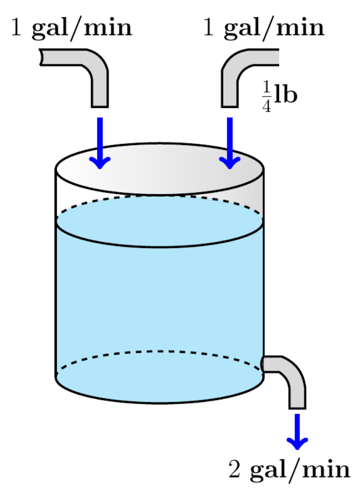


a) Find the concentration, in *kg/L* , of salt in the tank after 10 *min*.

b) After 10 *min*, a leak develops in the tank and an additional liter per minute of mixture flows out of the tank. What will be the concentration, in *kg/L* , of salt in the tank 20 *min* after the leak develops?



38. A tank of 100-*gallon* capacity is initially full of water. Pure water is allowed to run into the tank at the rate of 1 *gal/min* , and at the same time brine containing $\frac{1}{4}$ *lb* of salt per gallon flows into the tank also at the rate of 1 *gal/min* . The mixture flows out at the rate of 2 *gal/min* .

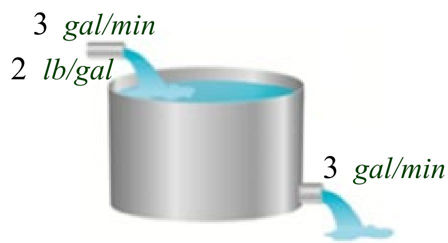


a) Find the amount of salt in the tank after *t minutes*.

b) How much salt is present at the end of 25 *minutes*?

c) How much salt is present after a long time?

39. A tank of 50-*gallon* capacity is initially full of pure water. Starting at time $t = 0$ brine containing $2\ lb$ of salt per gallon flows into the tank also at the rate of $3\ gal/min$. The mixture flows out at the rate of $3\ gal/min$.



a) Find the amount of salt in the tank after *t minutes*.

b) How much salt is present at the end of 25 *minutes*?

c) How much salt is present after a long time?

Section 1.6 – Exact Differential Equations

A class of equations known as exact equations for which there is also a well-defined method of solution
The expression $M(x, y)dx + N(x, y)dy$ is called an exact differential form.

1.6-1 *Theorem*

Let the function M, N, M_y *and* N_x, where M_y *and* N_x are partial derivatives, be continuous in the rectangular region $R: \alpha < x < \beta, \ \gamma < y < \delta$ then

$$M(x, y) + N(x, y)y' = 0$$

Is an exact differential equation in *R*, *iff* $\quad M_y(x, y) = N_x(x, y)$

At each point in *R*. That is, there exists a function ψ satisfying

$$\psi_x(x, y) = M(x, y) \quad \text{and} \quad \psi_y(x, y) = N(x, y) \qquad \text{Iff } M_y(x, y) = N_x(x, y)$$

$$\psi(x, y) = \int M(x, y)dx$$

Example 1

Solve the differential equation: $\quad 2x + y^2 + 2xyy' = 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = 2x + y^2 \quad \Rightarrow \quad M_y = 2y$$

$$\frac{\partial \psi}{\partial y} = N = 2xy \quad \Rightarrow \quad N_x = 2y$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = 2x + y^2$$

$$\psi = \int \left(2x + y^2\right)dx$$

$$= x^2 + xy^2 + h(y)$$

$$\psi_y = 2xy + h'(y) = 2xy \quad \Rightarrow \quad h'(y) = 0 \qquad \textit{Integrate } h'(y)$$

$$\Rightarrow \ h(y) = C$$

$$\psi(x, y) = x^2 + xy^2 + C$$

$$\boxed{x^2 + xy^2 = C}$$

Example 2

Solve the differential equation: $y\cos x + 2xe^y + \left(\sin x + x^2 e^y - 1\right)y' = 0$

Solution

$$M = y\cos x + 2xe^y = \frac{\partial\psi}{\partial x} \quad \Rightarrow \quad M_y = \cos x + 2xe^y$$

$$\frac{\partial\psi}{\partial y} = N = \sin x + x^2 e^y - 1 \quad \Rightarrow \quad N_x = \cos x + 2xe^y$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int\left(y\cos x + 2xe^y\right)dx$$

$$= y\sin x + x^2 e^y + h(y)$$

$$\psi_y = \sin x + x^2 e^y + h'(y)$$

$$= \sin x + x^2 e^y - 1$$

$$\Rightarrow \quad h'(y) = -1$$

$$\Rightarrow h(y) = -y$$

$$\psi(x, y) = y\sin x + x^2 e^y - y = C$$

$$\boxed{y\sin x + x^2 e^y - y = C}$$

Example 3

Solve the differential equation: $3xy + y^2 + \left(x^2 + xy\right)y' = 0$

Solution

$$M = 3xy + y^2 = \frac{\partial\psi}{\partial x} \quad \Rightarrow \quad M_y = 3x + 2y$$

$$N = x^2 + xy = \frac{\partial\psi}{\partial y} \quad \Rightarrow \quad N_x = 2x + y$$

$$\Rightarrow M_y \neq N_x$$

Can be solved by this procedure.

1.6-2 Integrating Factors

It is sometimes possible to convert a differential equation that is not exact equation by multiplying the equation by a suitable integrating factor.

1.6-3 *Definition*

An integrating factor for the differential equation $\omega = Mdx + Ndy = 0$ is a function $\mu(x, y)$ such that the form $\mu\omega = \mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy$ is exact.

$$(\mu M)_y = (\mu N)_x$$

$$M\mu_y - N\mu_x + \left(M_y - N_x\right)\mu = 0$$

Assuming that μ is a function of x only, we have

$$(\mu M)_y = \mu M_y \quad \& \quad (\mu N)_x = \mu N_x + N\frac{d\mu}{dx}$$

$$\Rightarrow \quad \mu M_y = \mu N_x + N\frac{d\mu}{dx}$$

$$\boxed{\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu}$$

$$\int \frac{d\mu}{\mu} = \int \frac{M_y - N_x}{N}dx$$

$$\ln\mu = \int \frac{M_y - N_x}{N}dx$$

$$\boxed{\mu = e^{\int \frac{M_y - N_x}{N}dx}}$$

Example 4

Find an integrating factor for the equation $\left(3xy + y^2\right) + \left(x^2 + xy\right)y' = 0$, and then solve the equation.

Solution

$$M_y = \frac{\partial}{\partial y}\left(3xy + y^2\right) = 3x + 2y$$

$$N_x = \frac{\partial}{\partial x}\left(x^2 + xy\right) = 2x + y$$

$$\Rightarrow M_y \ne N_x$$

$$\frac{M_y - N_x}{N} = \frac{3x + 2y - 2x - y}{x^2 + xy}$$
$$= \frac{x + y}{x(x + y)}$$
$$= \boxed{\frac{1}{x}}$$

$$\frac{d\mu}{dx} = \frac{\mu}{x}$$

$$\int \frac{d\mu}{\mu} = \int \frac{dx}{x}$$

$$\ln \mu = \ln x$$

$$\boxed{\mu = x}$$

$$x\left(3xy + y^2\right) + x\left(x^2 + xy\right)y' = 0$$

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}\left(3x^2y + xy^2\right) = 3x^2 + 2xy \\ N_x &= \frac{\partial}{\partial x}\left(x^3 + x^2y\right) = 3x^2 + 2xy \end{aligned} \quad \Rightarrow M_y = N_x$$

$$\psi = \int \left(3x^2y + xy^2\right)dx$$
$$= x^3y + \frac{1}{2}x^2y^2 + h(y)$$

$$\psi_y = x^3 + x^2y + h'(y) = x^3 + x^2y$$

$$h'(y) = 0$$

$$h(y) = C$$

$$\psi(x, y) = x^3y + \frac{1}{2}x^2y^2 = C$$

$$\boxed{x^3y + \frac{1}{2}x^2y^2 = C}$$

1.6-4 *Bernoulli* Equations

An equation of the form $y' + P(x)y = Q(x)y^n, \quad n \neq 0,\ 1$ is called a ***Bernoulli equation***.

If $n = 0 \quad \Rightarrow \quad y' + Py = Q$ ***First order*** linear differential equation

If $n = 1 \quad \Rightarrow \quad y' + Py = Qy \ \rightarrow \ y' + (P - Q)y = 0$ ***Separable*** equation.

For $n \neq 0,\ 1$; the Bernoulli equation can be written as $y^{-n}\dfrac{dy}{dx} + Py^{1-n} = Q \qquad (1)$

Let $u = y^{1-n} \quad \Rightarrow \quad \dfrac{du}{dx} = (1-n)y^{-n}\dfrac{dy}{dx}$

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx}$$

$$(1) \quad \Rightarrow \quad \frac{1}{1-n}\frac{du}{dx} + Pu = Q$$

$\underline{u' + (1-n)Pu = (1-n)Q}$ Which is first-order linear differential equation.

Example 5

Find the general solution $y' - 4y = 2e^x\sqrt{y}$

Solution

$$\sqrt{y} = y^{1/2} \quad \Rightarrow \quad n = \frac{1}{2}$$

Let $u = y^{1-\frac{1}{2}} = y^{1/2} \quad \Rightarrow \quad y = u^2$

$$\frac{du}{dx} = \frac{1}{2}y^{-1/2}\frac{dy}{dx}$$

$$2y^{1/2}\frac{du}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx} - 4y = 2e^x u$$

$$2u\frac{du}{dx} - 4u^2 = 2ue^x \qquad \textit{Divide by } 2u$$

$$u' - 2u = e^x$$

$$e^{\int -2dx} = e^{-2x}$$

$$\int e^x e^{-2x}dx = \int e^{-x}dx = -e^{-x}$$

$$u = \frac{1}{e^{-2x}}\left(-e^{-x} + C\right)$$

$$y^{1/2} = -e^{x} + Ce^{2x}$$

$$\boxed{y(x) = \left(Ce^{2x} - e^{x}\right)^2}$$

Example 6

Find the general solution $xy' + y = 3x^3y^2$

Solution

$$y' + \frac{1}{x}y = 3x^2y^2$$

Let $u = y^{1-2} = y^{-1} \quad \Rightarrow \quad y = \frac{1}{u}$

$$\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$$

$$\Rightarrow \quad y' = -y^2u' = -\frac{1}{u^2}u'$$

$$-\frac{1}{u^2}u' + \frac{1}{x}\frac{1}{u} = 3x^2\frac{1}{u^2} \qquad \textit{Multiply both sides by } -u^2$$

$$u' - \frac{1}{x}u = -3x^2$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$\int -3x^2x^{-1}dx = -3\int xdx = -\frac{3}{2}x^2$$

$$u = x\left(-\frac{3}{2}x^2 + C_1\right)$$

$$\frac{1}{y} = \frac{-3x^3 + 2C_1x}{2}$$

$$\boxed{y(x) = \frac{2}{Cx - 3x^3}}$$

1.6-5 Homogeneous Equations $\frac{dy}{dx}=f(x,y)$

The form of a homogeneous equation suggests that it may be simplified by using a variable denoted by 'v', to represent the ratio of y to x. This

$$y = xv \quad \Rightarrow \quad \frac{dy}{dx}=F(v)$$

Let assume that v is a function of x, then

$$\frac{dy}{dx}=x\frac{dv}{dx}+v \quad \Rightarrow \quad F(v)=x\frac{dv}{dx}+v$$

The most significant fact about this equation is that the variables x & v can always be separated, regardless of the form of the function F.

$$\frac{dx}{x}=\frac{dv}{F(v)-v}$$

Solving this equation and then replacing v by $\frac{y}{x}$ gives the solution of the original equation.

Example 7

Solve the differential equation $\qquad \frac{dy}{dx}=\frac{y^2+2xy}{x^2}$

Solution

$$\frac{dy}{dx}=\left(\frac{y}{x}\right)^2+2\frac{y}{x}=v^2+2v$$

$$x\frac{dv}{dx}+v=v^2+2v \qquad \Rightarrow x\frac{dv}{dx}=v^2+v$$

$$xdv=v(v+1)dx$$

$$\int\frac{dx}{x}=\int\frac{dv}{v(v+1)}$$

$$\int\frac{dx}{x}=\int\left(\frac{1}{v}-\frac{1}{v+1}\right)dv$$

$$\ln x+\ln C=\ln v-\ln(v+1)$$

$$\ln(\mathrm{C}x)=\ln\frac{v}{v+1}$$

$$\mathrm{C}x=\frac{v}{v+1}=\frac{\frac{y}{x}}{\frac{y}{x}+1}$$

$$= \frac{y}{y+x}$$

$$\Rightarrow \; Cxy + Cx^2 = y$$

$$Cx^2 = y - Cxy$$

$$\boxed{y(x) = \frac{Cx^2}{1-Cx}}$$

Example 8

Find the general solution $\quad y' = \dfrac{x^2 e^{y/x} + y^2}{xy}$

Solution

Let $y = xv \quad \Rightarrow \quad y' = v + xv'$

$$v + xv' = \frac{x^2 e^{xv/x} + (xv)^2}{x(xv)}$$

$$xv' = \frac{x^2 e^v + x^2 v^2}{x^2 v} - v$$

$$x\frac{dv}{dx} = \frac{e^v + v^2}{v} - v$$

$$x\frac{dv}{dx} = \frac{e^v}{v}$$

Integrate both sides

$$\int \frac{v}{e^v} dv = \int \frac{dx}{x}$$

$$-ve^{-v} - e^{-v} = \ln x + C$$

$$-e^{-v}(v+1) = \ln x + C$$

$$-e^{-y/x}\left(\frac{y}{x} + 1\right) = \ln x + C$$

$$\boxed{y + x = -xe^{y/x}(\ln x + C)}$$

1.6-6 Equations with Linear Coefficients

For equations with linear coefficients in the form: $\left(a_1x+b_1y+c_1\right)dx+\left(a_2x+b_2y+c_2\right)dy=0$

The general case: $a_1b_2 \neq a_2b_1$

Let consider: $\dfrac{dy}{dx}=G(ax+by)$

If $c_1=c_2=0 \Rightarrow \left(a_1x+b_1y\right)dx+\left(a_2x+b_2y\right)dy=0$

$$\frac{dy}{dx}=-\frac{a_1x+b_1y}{a_2x+b_2y}=-\frac{a_1+b_1\frac{y}{x}}{a_2+b_2\frac{y}{x}}$$

In this case by letting $v=\dfrac{y}{x}$

If $c_1, c_2 \neq 0$, we let $x=u+h$ *and* $y=v+k$

$$\begin{cases} a_1h+b_1k+c_1=0 \\ a_2h+b_2k+c_2=0 \end{cases}$$ has a solution

$$\frac{dv}{du}=-\frac{a_1u+b_1v}{a_2u+b_2v}=-\frac{a_1+b_1\frac{v}{u}}{a_2+b_2\frac{v}{u}}$$

Example 9

Solve $\left(-3x+y+6\right)dx+\left(x+y+2\right)dy=0$

Solution

$$\begin{cases} a_1=-3 & b_1=1 & c_1=6 \\ a_2=1 & b_2=1 & c_2=2 \end{cases}$$

$$\begin{cases} a_1b_2=(-3)(1)=-3 \\ a_2b_1=(1)(1)=1 \end{cases} \rightarrow a_1b_2 \neq a_2b_1$$

$$-\begin{cases} -3h+k=-6 \\ h+k=-2 \end{cases} \qquad \begin{cases} a_1h+b_1k=-c_1 \\ a_2h+b_2k=-c_2 \end{cases}$$

$$4h=4$$

$$h=1, \ k=-3$$

$$\begin{cases} x=u+h=u+1 \\ y=v+k=v-3 \end{cases}$$

$$(-3u-3+v-3+6)du+(u+1+v-3+2)dv=0$$

$$(-3u+v)du+(u+v)dv=0$$

$$\frac{dv}{du}=\frac{3-\frac{v}{u}}{1+\frac{v}{u}}$$

Let $w=\frac{v}{u} \quad \rightarrow \ v=uw$

$$\frac{dv}{du}=w+u\frac{dw}{du}$$

$$w+u\frac{dw}{du}=\frac{3-w}{1+w}$$

$$u\frac{dw}{du}=\frac{3-w}{1+w}-w$$

$$u\frac{dw}{du}=\frac{3-2w-w^2}{1+w}$$

$$\frac{w+1}{w^2+2w-3}dw=-\frac{du}{u}$$

$$\frac{1}{2}\int\frac{1}{w^2+2w-3}d\left(w^2+2w-3\right)=-\int\frac{du}{u}$$

$$\frac{1}{2}\ln\left|w^2+2w-3\right|=-\ln u+\ln C_1$$

$$\ln\sqrt{w^2+2w-3}=\ln C_1\frac{1}{u}$$

$$\sqrt{w^2+2w-3}=C_1\frac{1}{u}$$

$$w^2+2w-3=C\frac{1}{u^2}$$

$$\frac{v^2}{u^2}+2\left(\frac{v}{u}\right)-3=C\frac{1}{u^2}$$

$$v^2+2uv-3u^2=C \qquad x=u+1 \quad y=v-3$$

$$\boxed{(y+3)^2+2(x-1)(y+3)-3(x-1)^2=C}$$

Exercises *Section* 1.6 – Exact Differential Equations

(1 – 61) Solve the differential equation

1. $(2x+y)dx+(x-6y)dy=0$
2. $(2x+3)dx+(2y-2)dy=0$
3. $(1-y\sin x)+(\cos x)y'=0$
4. $\dfrac{dy}{dx}=-\dfrac{ax+by}{bx+cy}$
5. $\dfrac{dy}{dx}=\dfrac{3x^2+y}{3y^2-x}$
6. $2xydx+\left(x^2-1\right)dy=0$
7. $y'=\dfrac{x^2+y^2}{2xy}$
8. $2xyy'=x^2+2y^2$
9. $xy'=y+2\sqrt{xy}$
10. $xy^2y'=x^3+y^3$
11. $x^2y'=xy+x^2e^{y/x}$
12. $x^2y'=xy+y^2$
13. $xyy'=x^2+3y^2$
14. $\left(x^2-y^2\right)y'=2xy$
15. $xyy'=y^2+x\sqrt{4x^2+y^2}$
16. $xy'=y+\sqrt{x^2+y^2}$
17. $y^2y'+2xy^3=6x$
18. $x^2y'+2xy=5y^4$
19. $2xy'+y^3e^{-2x}=2xy$
20. $y^2(xy'+y)\left(1+x^4\right)^{1/2}=x$
21. $3y^2y'+y^3=e^{-x}$
22. $3xy^2y'=3x^4+y^3$
23. $xe^yy'=2\left(e^y+x^3e^{2x}\right)$
24. $(2x\sin y\cos y)y'=4x^2+\sin^2 y$
25. $\left(x+e^y\right)y'=xe^{-y}-1$
26. $\left(x^2+y^2\right)dx+\left(x^2-xy\right)dy=0$
27. $x\dfrac{dy}{dx}+y=x^2y^2$
28. $\left(3x^2-2xy+2\right)+\left(6y^2-x^2+3\right)y'=0$
29. $\left(e^x\sin y-2y\sin x\right)dx+\left(e^x\cos y+2\cos x\right)dy=0$
30. $\left(\dfrac{y}{x}+6x\right)dx+(\ln x-2)dy=0, \qquad x>0$
31. $\left(e^{2y}-y\cos xy\right)dx+\left(2xe^{2y}-x\cos xy+2y\right)dy=0$
32. $\dfrac{xdx}{\left(x^2+y^2\right)^{3/2}}+\dfrac{ydy}{\left(x^2+y^2\right)^{3/2}}=0$
33. $(2x-1)dx+(3y+7)dy=0$
34. $(5x+4y)dx+\left(4x-8y^3\right)dy=0$
35. $(\sin y-y\sin x)dx+(\cos x+x\cos y-y)dy=0$
36. $\left(2xy^2-3\right)dx+\left(2x^2y+4\right)dy=0$
37. $\left(1+\ln x+\dfrac{y}{x}\right)dx-(1-\ln x)dy=0$
38. $\left(x-y^3+y^2\sin x\right)dx-\left(3xy^2+2y\cos x\right)dy=0$
39. $\left(x^3+y^3\right)dx+3xy^2dy=0$
40. $\left(3x^2y+e^y\right)dx+\left(x^3+xe^y-2y\right)dy=0$
41. $xdy+\left(y-2xe^x-6x^2\right)dx=0$
42. $\left(1-\dfrac{3}{y}+x\right)dy+\left(y-\dfrac{3}{x}+1\right)dx=0$
43. $\left(x^2y^3-\dfrac{1}{1+9x^2}\right)\dfrac{dx}{dy}+x^3y^2=0$

44. $(5y-2x)y'-2y=0$

45. $(x-y)dx-xdy=0$

46. $(x+y)dx+xdy=0$

47. $\dfrac{dy}{dx}=-\dfrac{2xy^2+1}{2x^2y}$

48. $\left(1+e^x y+xe^x y\right)dx+\left(xe^x+2\right)dy=0$

49. $\left(2xy^3+1\right)dx+\left(3x^2y^2-\frac{1}{y}\right)dy=0$

50. $(2x+y)dx+(x-2y)dy=0$

51. $e^x(y-x)dx+\left(1+e^x\right)dy=0$

52. $\left(ye^{xy}-\dfrac{1}{y}\right)dx+\left(xe^{xy}+\dfrac{x}{y^2}\right)dy=0$

53. $(\tan x-\sin x\sin y)dx+(\cos x\cos y)dy=0$

54. $\left(2x^3-xy^2-2y+3\right)dx-\left(x^2y+2x\right)dy=0$

55. $(x+\sin y)dx+(x\cos y-2y)dy=0$

56. $\left(x+\dfrac{1}{\sqrt{y^2-x^2}}\right)dx+\left(1-\dfrac{x}{y\sqrt{y^2-x^2}}\right)dy=0$

57. $\left(2x+y^2-\cos(x+y)\right)dx+\left(2xy-\cos(x+y)-e^y\right)dy=0$

58. $\left(\dfrac{2}{\sqrt{1-x^2}}+y\cos(xy)\right)dx+\left(x\cos(xy)-y^{-1/3}\right)dy=0$

59. $(2x+y\cos(xy))dx+(x\cos(xy)-2y)dy=0$

60. $\left(e^x\sin y-3x^2\right)dx+\left(e^x\cos y+\frac{1}{3}y^{-2/3}\right)dy=0$

61. $\left(2y\sin x\cos x-y+2y^2e^{xy^2}\right)dx=\left(x-\sin^2 x-4xye^{xy^2}\right)dy$

(62 – 68) The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

62. $x^2y^3+x\left(1+y^2\right)y'=0,\qquad \mu(x,y)=\dfrac{1}{xy^3}$

63. $y^2-xy+\left(x^2\right)y'=0,\qquad \mu(x,y)=\dfrac{1}{xy^2}$

64. $x^2y^3-y+x\left(1+x^2y^2\right)y'=0,\qquad \mu(x,y)=\dfrac{1}{xy}$

65. $\left(\dfrac{\sin y}{y}-2e^{-x}\sin x\right)dx+\left(\dfrac{\cos y+2e^{-x}\cos x}{y}\right)dy=0,\qquad \mu(x,y)=ye^x$

66. $(x+2)\sin ydx+x\cos ydy=0,\qquad \mu(x,y)=xe^x$

67. $\left(x^2+y^2-x\right)dx-ydy=0,\qquad \mu(x,y)=\dfrac{1}{x^2+y^2}$

68. $(2y-6x)dx+\left(3x-4x^2y^{-1}\right)dy=0,\qquad \mu(x,y)=xy^2$

(69 – 71) Find the general solution of each homogeneous equation

69. $\left(x^2+y^2\right)dx-2xydy=0$

70. $(x+y)dx+(y-x)dy=0$

71. $\dfrac{dy}{dx}=\dfrac{y\left(x^2+y^2\right)}{xy^2-2x^3}$

(72 – 81) Find an integrating factor and solve the given equation

72. $\left(3x^2y+2xy+y^3\right)dx+\left(x^2+y^2\right)dy=0$

73. $dx+\left(\dfrac{x}{y}-\sin y\right)dy=0$

74. $e^xdx+\left(e^x\cot y+2y\csc y\right)dy=0$

75. $\left(3x+\dfrac{6}{y}\right)dx+\left(\dfrac{x^2}{y}+3\dfrac{y}{x}\right)dy=0$

76. $\left(x+3x^3\sin y\right)dx+\left(x^4\cos y\right)dy=0$

77. $\left(2x^2+y\right)dx+\left(x^2y-x\right)dy=0$

78. $\left(3x^2+y\right)dx+\left(x^2y-x\right)dy=0$

79. $\left(2y^2+2y+4x^2\right)dx+(2xy+x)dy=0$

80. $\left(x^4-x+y\right)dx-xdy=0$

81. $(2xy)dx+\left(y^2-3x^2\right)dy=0$

(82 – 114) Solve the given initial-value problem

82. $\dfrac{dy}{dx}=\dfrac{xy^2-\cos x\sin x}{y\left(1-x^2\right)},\quad y(0)=2$

83. $(x+y)^2\,dx+\left(2xy+x^2-1\right)dy,\quad y(1)=1$

84. $\left(e^x+y\right)dx+\left(2+x+ye^y\right)dy,\quad y(0)=1$

85. $(2x-y)dx+(2y-x)dy,\quad y(1)=3$

86. $\left(9x^2+y-1\right)dx-(4y-x)dy,\quad y(1)=0$

87. $\left(x+y^3\right)y'+y+x^3=0,\quad y(0)=-2$

88. $y'=\left(3x^2+1\right)\left(y^2+1\right),\quad y(0)=1$

89. $\left(y^3+\cos t\right)y'=2+y\sin t,\quad y(0)=-1$

90. $\left(y^3-t^3\right)y'=3t^2y+1,\quad y(-2)=-1$

91. $\dfrac{dy}{dx}=(-2x+y)^2-7,\quad y(0)=0$

92. $(2y-x)y'-y+2x=0,\quad y(1)=0$

93. $\left(e^{2y}+t^2y\right)y'+ty^2+\cos t=0,\quad y\left(\frac{\pi}{2}\right)=0$

94. $y'=-\dfrac{y\cos(ty)+1}{t\cos(ty)+2ye^{y^2}},\quad y(\pi)=0$

95. $\left(2ty+\dfrac{1}{y}\right)y'+y^2=1,\quad y(1)=1$

96. $\left(ye^x+1\right)dx+\left(e^x-1\right)dy=0\quad y(1)=1$

97. $2xy^2+4=2\left(3-x^2y\right)y'\quad y(-1)=8$

98. $y'+\dfrac{4}{x}y=x^3y^2\quad y(2)=-1$

99. $y'=5y+e^{-2x}y^{-2}\quad y(0)=2$

100. $6y'-2y=xy^4\quad y(0)=-2$

101. $y'+\dfrac{y}{x}-\sqrt{y}=0\quad y(1)=0$

102. $xyy'+4x^2+y^2=0\quad y(2)=-7$

103. $xy'=y(\ln x-\ln y)\quad y(1)=4$

104. $y' - (4x - y + 1)^2 = 0 \quad y(0) = 2$

105. $\left(e^{t+y} + 2y\right)y' + \left(e^{t+y} + 3t^2\right) = 0, \quad y(0) = 0$

106. $(4y + 2x - 5)dx + (6y + 4x - 1)dy, \quad y(-1) = 2$

107. $\left(ye^{xy} - \frac{1}{y}\right)dx + \left(xe^{xy} + \frac{x}{y^2}\right)dy = 0 \quad y(1) = 1$

108. $(2y\ln t - t\sin y)y' + \frac{1}{t}y^2 + \cos y = 0, \quad y(2) = 0$

109. $(\tan y - 2)dx + \left(x\sec^2 y + \frac{1}{y}\right)dy = 0 \quad y(0) = 1$

110. $2xy - 9x^2 + \left(2y + x^2 + 1\right)\frac{dy}{dx} = 0 \quad y(0) = -3$

111. $\frac{2t}{t^2+1}y - 2t + \left(2 - \ln\left(t^2 + 1\right)\right)\frac{dy}{dt} = 0 \quad y(5) = 0$

112. $3y^3e^{3xy} - 1 + \left(2ye^{3xy} + 3xy^2e^{3xy}\right)y' = 0 \quad y(0) = 1$

113. $2xydx + \left(1 + x^2\right)dy = 0\ ; \quad y(2) = -5$

114. $\frac{dy}{dx} = -\frac{2x\cos y + 3x^2y}{x^3 - x^2\sin y - y}\ ; \quad y(0) = 2$

(115 – 117) Find an integrating factor of the form $x^n y^m$ and solve the equation

115. $\left(2y^2 - 6xy\right)dx + \left(3xy - 4x^2\right)dy = 0$

116. $(12 + 5xy)dx + \left(6xy^{-1} + 3x^2\right)dy = 0$

117. $\left(3y + 4xy^2\right)dx + \left(2x + 3x^2y\right)dy = 0$

(118 – 123) Find the general solution by using Bernoulli

118. $\frac{dy}{dx} - 5y = -\frac{5}{2}xy^3$

119. $\frac{dy}{dx} + \frac{y}{x} = x^2y^2$

120. $\frac{dy}{dx} - y = e^{2x}y^3$

121. $\frac{dy}{dx} + \frac{y}{x-2} = 5(x-2)y^{1/2}$

122. $\frac{dy}{dx} + y = e^x y^{-2}$

123. $\frac{dy}{dx} + y^3x + y = 0$

(124 – 128) Find the general solution by using homogeneous equations.

124. $\left(xy + y^2\right)dx - x^2dy = 0$

125. $\left(x^2 + y^2\right)dx + 2xydy = 0$

126. $\left(y^2 - xy\right)dx + x^2dy = 0$

127. $\frac{dy}{d\theta} = \frac{\theta\sec\left(\frac{y}{\theta}\right) + y}{\theta}$

128. $\frac{dy}{dx} = \frac{y(\ln y - \ln x + 1)}{x}$

(129 – 132) Find the general solution by using Equation with Linear Coefficients

129. $(-3x+y-1)dx+(x+y+3)dy=0$

130. $(x+y-1)dx+(y-x-5)dy=0$

131. $(2x+y+4)dx+(x-2y-2)dy=0$

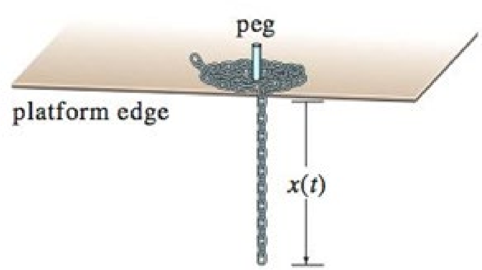
132. $(2x-y)dx+(4x+y-3)dy=0$

133. Prove that $Mdx+Ndy=0$ has an integrating factor that depends only on the sum $x+y$ if and only if the expression

$$\frac{N_x - M_y}{M-N} \text{ depends only on } x+y$$

Use the prove to solve the equation $(3+y+xy)dx+(3+x+xy)dy=0$

134. A portion of a uniform chain of length 8 *feet* is loosely coiled around a peg at the edge of a high horizontal platform, and the remaining portion of the chain hangs at rest over the edge of the platform.



Suppose the length of the overhanging chain is 3 *feet*, that the chain weighs 2 2 *lb/ft*, and that the positive direction is downward. Starting at $t=0$ seconds, the weight of the overhanging portion causes the chain on the table to uncoil smoothly and to fall to the floor. If $x(t)$ denotes the length of the chain overhanging the table at time $t>0$, then $v=\frac{dx}{dt}$ is its velocity. When all resistive forces are ignored, it can be shown that a mathematical model relating *v* to *x* is given by

$$xv\frac{dv}{dx}+v^2=32x$$

a) Rewrite this model in differential form and solve the *DE* for *v* in terms of *x* by finding am appropriate integrating factor. Find an explicit solution $v(x)$.

b) Determine the velocity with which the chain leaves the platform.

Section 1.7 - Modeling Population Growth

1.7-1 Modeling Population Growth

The mathematical model of the growth of a population is given by:

$$P' = rP$$

Where ***r***: reproductive rate.

The natural of the predictions of the model depend on the nature of the reproductive rate *r*.

1.7-2 Malthusian Method

Since *r* is a constant because the birth or death rates do not depend on time or on the size.

$$\frac{dP}{dt} = rP$$

$$\int \frac{dP}{P} = \int r\,dt$$

$$\ln|P| = rt + C_1$$

$$\begin{aligned} P &= e^{rt+C_1} \\ &= e^{rt}e^{C_1} \\ &= \underline{Ce^{rt}} \end{aligned}$$

Therefore, the solution to $P' = rP$ is given by:

$$\begin{aligned} P(t) &= Ce^{rt} \\ &= P_0 e^{rt} \end{aligned}$$

The population at time $t = 0$ is P_0.

Example 1

A biologist starts with 10 *cells* in a culture. Exactly 24 *hrs.* later he counts 25. Assuming a Malthusian model, what the reproductive rate? What will be the number of cells of the end of 10 *days*?

Solution

$$P = P_0 e^{rt}$$

$$P = 10e^{rt}$$

$25 = 10e^{r(1)}$ **24 *hrs.* = 1 *day*** $\boldsymbol{P = 25}$

$$\frac{25}{10} = e^{r}$$

$$\ln\frac{25}{10} = \ln e^{r}$$

$$r = \ln 2.5$$

$$\approx 0.9163$$

$$P(t) = 10e^{0.9163t}$$

$$P(10) = 10e^{0.9163(10)}$$

$$\boxed{\approx 95367 \ \ cells}$$

Example 2

A certain radioactive material is decaying at a rate proportional to the amount present. If a sample of 50 *grams* of the material was present initially and after 2 *hours* the sample lost 10% of its mass, find:

a) An expression for the mass of the material remaining at any time.

b) The mass of the material after 4 *hours*.

c) How long will it take for 75% of the material to decay?

d) The half-life of the material.

Solution

Given: $A_0 = 50g$ $\qquad$ $A(2) = 50 - .1(50) = 45g$

a) $A(2) = 50e^{-2r}$

$$45 = 50e^{-2r}$$

$$e^{-2r} = \frac{45}{50}$$

$$= \frac{9}{10}$$ *Convert to logarithm*

$$-2r = \ln\frac{9}{10}$$

$$r = -\frac{1}{2}\ln\frac{9}{10}$$

$$A(t) = 50e^{\frac{1}{2}\ln\left(\frac{9}{10}\right)t}$$

$$= 50e^{\ln\left(\frac{9}{10}\right)^{t/2}}$$

$$\boxed{= 50\left(\frac{9}{10}\right)^{t/2}}$$

b) $A(4) = 50\left(\frac{9}{10}\right)^{2}$

$$\underline{= 40.5\ g}\,\Big|$$

c) $At\ 75\% \quad \Rightarrow \quad A(t) = 50 \times .25 = 12.5\ g$

$$12.5 = 50\left(\frac{9}{10}\right)^{t/2}$$

$$\left(\frac{9}{10}\right)^{t/2} = \frac{12.5}{50}$$

$$\frac{t}{2}\ln\left(\frac{9}{10}\right) = \ln\left(\frac{12.5}{50}\right)$$

$$t = \frac{2\ln\left(\frac{12.5}{50}\right)}{\ln\left(\frac{9}{10}\right)}$$

$$\underline{\approx 26.32\ \ hours}\,\Big|$$

d) $T = \dfrac{\ln 2}{-\frac{1}{2}\ln 0.9}$

$$\underline{\approx 13.16\ \ hrs}\,\Big|$$

Note

We can this formula to solve most of the questions $\boxed{rT = \ln\dfrac{A}{A_0}}$

1.7-3 Logistic Model of Growth

Logistic population growth occurs when the ***growth*** rate decreases as the population reaches carrying capacity. Carrying capacity is the maximum number of individuals in a population that the environment can support.

Suppose an environment is capable of sustaining no more than a fixed number K of individuals in its populations. The quantity K is called the ***carrying capacity*** of the environment. In reality this model in unrealistic because environments impose limitations to population growth.

The logistic equation is given by:

$$P' = rP\left(1 - \frac{P}{K}\right) = KrP(K - P) \qquad P' = kP(M - P) \quad where \quad k = \frac{r}{K}\ \ \&\ \ M = K$$

The logistic equation can be solved by separation of variables

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

$$\int \frac{dP}{P\left(1 - \frac{P}{K}\right)} = \int r\,dt \qquad\qquad \frac{1}{P\left(1 - \frac{P}{K}\right)} = \frac{K}{P(K - P)} = \frac{1}{P} + \frac{1}{K - P}$$

$$\int \frac{dP}{P} + \int \frac{dP}{K-P} = \int r\,dt$$

$$\ln|P| - \ln|K-P| = rt + C$$

$$\ln\left|\frac{P}{K-P}\right| = rt + C$$

$$\frac{P}{K-P} = e^{rt+C}$$

$$\frac{P}{K-P} = Ae^{rt}$$

$$t = 0 \quad \Rightarrow \quad A = \frac{P_0}{K-P_0}$$

$$P = KAe^{rt} - PAe^{rt}$$

$$P\left(1 + Ae^{rt}\right) = KAe^{rt}$$

$$P = \frac{KAe^{rt}}{1 + Ae^{rt}}$$

$$= \frac{KA}{e^{-rt} + A}$$

$$= \frac{K\frac{P_0}{K-P_0}}{e^{-rt} + \frac{P_0}{K-P_0}}$$

$$= \frac{KP_0}{\left(K-P_0\right)e^{-rt} + P_0}$$

$$P(t) = \frac{KP_0}{P_0 + \left(K-P_0\right)e^{-r\left(t-t_0\right)}}$$

or $$P(t) = \frac{MP_0}{P_0 + \left(M-P_0\right)e^{-kMt}}$$

***Example* 3**

Suppose we start at time $t_0 = 0$ with a sample of 1000 *cells*. One day later we see that the population has doubled, and sometime later we notice that the population has stabilized at 100,000.

Solution

Given: $K = 100,000 = 10^5 \qquad P_0 = 1000$

$$P(t) = \frac{10^5 10^3}{10^3 + \left(10^5 - 10^3\right)e^{-r(t-0)}}$$

$$= \frac{10^8}{10^3 + 10^3(100-1)e^{-rt}}$$

$$= \frac{10^5}{1+99e^{-rt}}$$

$$2P_0 = \frac{10^5}{1+99e^{-r(1)}}$$

$$2\times 10^3 = \frac{10^5}{1+99e^{-r}}$$

$$1+99e^{-r} = \frac{10^5}{2\times 10^3}$$

$$1+99e^{-r} = 50$$

$$e^{-r} = \frac{49}{99}$$

$$-r = \ln\left(\frac{49}{99}\right)$$

$$\boxed{r \approx 0.7033}$$

$$\boxed{P(t) = \frac{10^5}{1+99e^{-0.7033t}}}$$

1.7-4 Pollution

Consider a lake that has a volume of $V = 100\ km^3$, it is fed by an input river, and there is another river which is fed by the lake at a rate that keeps the volume of the lake constant.

The input rate: $r(t) = 50 + 20\cos\left[2\pi\left(t - \frac{1}{4}\right)\right]$

The maximum flow into the lake occurs when $t = \dfrac{1}{4}$

In addition, there is a factory on the lake that introduces a pollutant into the lake at the rate of $2\ km^3/yr$. Let $x(t)$ denote the total amount of pollution in the lake at time *t*. If we make the assumption that the pollutant is rapidly mixed throughout the lake, then

$$\frac{dx}{dt} = 2 - \left(50 + 20\cos\left[2\pi\left(t - \frac{1}{4}\right)\right]\right) \cdot \frac{x}{100}$$

Exercises *Section* 1.7 - Modeling Population Growth

1. The rate of growth of bacteria in a petri dish is proportional to the number of bacteria in the dish.

2. The rate of growth of a population of field mice is inversely proportional to the square root of the population.

3. A biologist starts with 100 *cells* in a culture. After 24 *hrs.,* he counts 300. Assuming a Malthusian model, what the reproductive rate? What will be the number of cells of the end of 5 *days*?

4. A biologist prepares a culture. After 1 *day* of growth, the biologist counts 1000 *cells*. After 2*days,* he counts 3000. Assuming a Malthusian model, what the reproductive rate and how many cells were present initially?

5. A population of bacteria is growing according to the Malthusian model. If the population is triples in 10 *hrs.*, what is the reproduction rate? How often does the population double itself?

6. Consider a lake that is stocked with walleye pike and that the population of pike is governed by the logistic equation

$$P' = 0.1P\left(1 - \frac{P}{10}\right)$$

Where time is measured in days and P in *thousands* of fish. Suppose that fishing is started in this lake and that 100 *fish* are removed each day.

a) Modify the logistic model to account for the fishing.
b) Find and classify the equilibrium points for your model.
c) Use qualitative analysis to completely discuss the fate of the fish population with this model. In particular, if the initial fish population is 1000, what happens to the fish as time passes? What will happen to an initial population having 2000 fish?

7. Suppose that in 1885 the population of a certain country was 50 million and was growing at the rate of 750,000 people per year at that time. Suppose also that in 1940 its population was 100 million and was then growing at the rate of 1 million per year. Assume that this population satisfies the logistic equation. Determine both the limiting population M and the predicted population for the year 2000.

8. The time rate of change of a rabbit population P is proportional to the square root of P. At time $t = 0$ (months) the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be one year later?

9. Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to reproduce (so that the birth rate is $\beta = 0$) and the death rate δ (deaths per week per fish) is thereafter proportional to $\frac{1}{\sqrt{P}}$. If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long did it take all the fish in the lake to die?

10. Suppose that when a certain lake is stocked with fish, the birth and death rates β and δ are both inversely proportional to $\sqrt{P}$

a) Show that $P(t)=\left(\frac{1}{2}kt+\sqrt{P_0}\right)^2$, where k is a constant.

b) If $P_0=100$ and after 6 months there are 169 fish in the lake, how many will there be after 1 year?

11. The time rate of change of an alligator population P in a swamp is proportional to the square of P. The swamp contained a dozen alligators in 1988, two dozen in 1998.

a) When will there be four dozen alligators in the swamp?

b) What happens thereafter?

12. Consider a prolific breed of rabbits whose birth and death rates, β and δ, are each proportional to the rabbit population $P=P(t)$, with $\beta>\delta$

a) Show that $P(t)=\dfrac{P_0}{1-kP_0t}$, k *constant*

Note that $P(t)\to+\infty \quad as \quad t\to\dfrac{1}{kP_0}$. This is doomsday

b) Suppose that $P_0=6$ and that there are nine rabbits after ten months. When does doomsday occur?

c) With $\beta<\delta$, repeat part (*a*)

d) What now happens to the rabbit population in the long run?

13. Consider a population $P(t)$ satisfying the logistic equation $\dfrac{dP}{dt}=aP-bP^2$, where $B=aP$ is the time rate at which births occur and $D=bP^2$ is the rate at which deaths occur.

a) If the initial population is $P(0)=P_0$, and B_0 births per month and D_0 deaths per month are occurring at time $t=0$, show that the limiting population is $M=\dfrac{B_0P_0}{D_0}$.

b) If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occurring at time $t=0$, how many months does it take for $P(t)$ to reach 95% of the limiting population M?

c) If the initial population is 240 rabbits and there are 9 births per month and 12 deaths per month occurring at time $t=0$, how many months does it take for $P(t)$ to reach 105% of the limiting population M?

14. The amount of drug in the blood of a patient (in *mg*) due to an intravenous line is governed by the initial value problem

$$y'(t) = -0.02y + 3, \quad y(0) = 0 \ \ for\ t \ge 0$$

Where t is measured in hours

a) Find and graph the solution of the initial value problem.

b) What is the steady-state level of the drug?

c) When does the drug level reach 90% of the steady-state value?

15. A fish hatchery has 500 *fish* at time $t = 0$, when harvesting begins at a rate of b *fish/yr*. where $b > 0$. The fish population is modeled by the initial value problem.

$$y'(t) = 0.1y - b, \quad y(0) = 500 \qquad for\ \ t \ge 0$$

Where t is measured in years.

a) Find the fish population for $t \ge 0$ in terms of the harvesting rate b.
b) Graph the solution in the case that $b = 40\ fish\ /\ yr$. Describe the solution.
c) Graph the solution in the case that $b = 60\ fish\ /\ yr$. Describe the solution.

16. A community of hares on an island has a population of 50 when observations begin at $t = 0$. The population for $t \ge 0$ is modeled by the initial value problem.

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{200}\right), \quad P(0) = 50$$

a) Find the solution of the initial value problem.
b) What is the steady-state population?

17. When an infected person is introduced into a closed and otherwise healthy community, the number of people who become infected with the disease (in the absence of any intervention) may be modeled by the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right), \quad P(0) = P_0$$

Where k is a positive infection rate, A is the number of people in the community, and P_0 is the number of infected people at $t = 0$. The model assumes no recovery or intervention.

a) Find the solution of the initial value problem in terms of k, A, and P_0.

b) Graph the solution in the case that $k = 0.025, \quad A = 300, \quad and\ \ P_0 = 1$.

c) For fixed values of k and A, describe the long-term behavior of the solutions for any P_0 with $0 < P_0 < A$

18. The reaction of chemical compounds can often be modeled by differential equations. Let $y(t)$ be the concentration of a substance in reaction for $t \geq 0$ (typical units of *y* are *moles/L*). The change in the concentration of a substance, under appropriate conditions, is $\frac{dy}{dt} = -ky^n$, where $k > 0$ is a rate constant and the positive integer *n* is the order of the reaction.

a) Show that for a first-order reaction $(n = 1)$, the concentration obeys an exponential decay law.

b) Solve the initial value problem for a second-order reaction $(n = 2)$ assuming $y(0) = y_0$

c) Graph and compare the concentration for a first-order and second-order reaction with $k = 0.1$ and $y_0 = 1$

19. The growth of cancer turmors may be modeled by the Gomperts growth equation. Let $M(t)$ be the mass of the tumor for $t \geq 0$. The relevant intial value problem is

$$\frac{dM}{dt} = -aM \ln \frac{M}{K}, \quad M(0) = M_0$$

Where *a* and *K* are positive constants and $0 < M_0 < K$

a) Graph the growth rate function $R(M) = -aM \ln \frac{M}{K}$ assuming $a = 1$ and $K = 4$. For what values of *M* is the growth rate positive? For what values of *M* is maximum?

b) Solve the initial value problem and graph the solution for $a = 1$, $K = 4$, and $M_0 = 1$. Describe the growth pattern of the tumor. Is the growth unbounded? If not, what is the limiting size of the tumor?

c) In the general equation, what is the meaning of *K*?

20. The halibut fishery has been modeled by the differential equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$$

Where $y(t)$ is the biomass (the total mass of the members of the population) in kilograms at time *t* (measured in years), the carrying capacity is estimated to be $M = 8 \times 10^7\ kg$ and $k = 0.71\ per\ year$.

a) If $y(0) = 2 \times 10^7\ kg$, find the biomass a year later.

b) How long will it take for the biomass to reach $4 \times 10^7\ kg$.

21. Suppose a population $P(t)$ satisfies $\frac{dP}{dt} = 0.4P - 0.001P^2, \quad P(0) = 50$, where *t* is measured in years.

a) What is the carrying capacity?

b) What is $P'(0)$?

c) When will the population reach 50% of the carrying capacity?

22. The board of directors of a corporation is calculating the price to pay for a business that is forecast to yield a continuous flow of profit of \$500,000 per *year*. The money will earn a nominal rate of 5% per year compounded continuously. What is the present value of the business?

a) For 20 *years*?

b) Forever (in perpetuity)?

23. The population of a community is known to increase at a rate proportional to the number of people present at a time t. If the population has doubled in 6 *years*, how long it will take to triple?

24. Let population of country be decreasing at the rate proportional to its population. If the population has decreased to 25% in 10 *years*, how long will it take to be half?

25. Suppose that we have an artifact, say a piece of fossilized wood, and measurements show that the ratio of $C-14$ to carbon in the sample is 37% of the current ratio. Let us assume that the wood died at time 0, then compute the time T it would take for one gram of the radioactive carbon to decay this amount.

26. A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there is $50\ mg$ of the material present and after of its original mass, find

a) An expression for the mass of the material remaining at any time t.

b) The mass of the material after 4 *hours*

c) The time at which the material has decayed to one half of its initial mass.

27. The rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in given sample. Half of the original number of radioactive nuclei have undergone disintegration in a period of 1,500 *years*.

a) What percentage of the original radioactive nuclei will remain after 4,500 *years*?

b) In how many years will only one-tenth of the original number remain?

Section 1.8 – Basic Electrical Circuit

An ***electric circuit*** is a path in which electrons from a voltage or current source flow. The point where those electrons enter an ***electrical circuit*** is called the "source" of electrons.

Please refer to *Appendix* ***C*** for more detail about electric definitions'

1.8-1 *Resistor*: (**O**hm's **L**aw)

A ***resistor*** is a component of a circuit that resists the flow of electrical current. It has two terminals across which electricity must pass, and it is designed to drop the voltage of the current as it flows from one terminal to the other. Resistors are primarily used to create and maintain known safe currents within electrical components.

A voltage $V(t)$ across the terminals of a resistor is proportional to the current $I(t)$ in it. The constant proportional $\boldsymbol{R}$ is called the resistance of the resistor in **V**olt/**A**mpere or **O**hms (Ω), and is given by the equation:

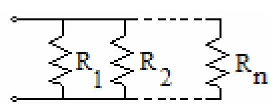
$$V_R = RI$$

For series resistors, the equivalent resistor is:

$R_1 \quad R_2 \qquad R_n$

Then: $R_{eq} = R_1 + R_2 + \cdots + R_n$

For resistors in parallels:

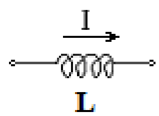


Then: $\dfrac{1}{R_{eq}} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \cdots + \dfrac{1}{R_n}$

1.8-2 *Inductor*: (Faraday's Law)

When a current in a circuit is changing, then the magnetic flux is linking the same circuit changes. This change in flux causes an *emf* ***v*** to be induced in the circuit.

Inductance is symbolized by letter ***L***, is measured in ***h*enrys** (***H***), and is represented graphically as a coiled wire – a reminder that inductance is a consequence of a conductor linking a magnetic field.



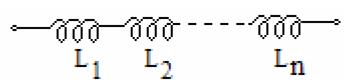
The voltage $V(t)$ is proportional to the time rate of change of the current, and is given by:

$$V_L = L\frac{dI}{dt}$$

and

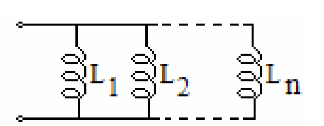
$$I(t) = \frac{1}{L}\int V dt$$

For series inductors, the equivalent inductor is:



$$L_{eq} = L_1 + L_2 + \cdots + L_n$$

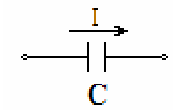
For inductors in parallels:



$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n}$$

1.8-3 *Capacitance*: (Coulomb's Law)

The circuit parameter of ***capacitance*** is represented by letter ***C***, is measured in *farads* (***F***), and is symbolized graphically by two short parallel conductive plates.



The farad is an extremely large quantity of capacitance, practical capacitor values usually lie in the picofarad (*pF*) to microfarad (μF) range.

The graphic symbol for a capacitor is a reminder that capacitance occurs whenever electrical conductors are separated by a dielectric, or insulating, material. This condition implies that electric charge is not transported through the capacitor. Although applying a voltage to the terminals of the capacitor cannot move a charge through the dielectric, it can displace a charge within the dielectric. As the voltage varies with time, the displacement of charge also varies with time, causing what is known as the displacement current.

The potential ***v*** between the terminals of a capacitor is proportional to the charge q on it.

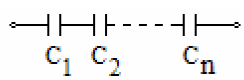
$$Q(t) = Cv(t)$$

$$I = \frac{dq}{dt} = C\frac{dv}{dt}$$

$$\Rightarrow \quad v(t) = \frac{1}{C}\int I\ dt$$

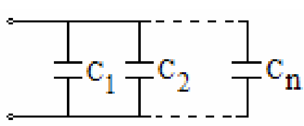
C is **C**oulombs/**V**olts or farads.

For capacitances in series, the equivalent capacitance is given by:



$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}$$

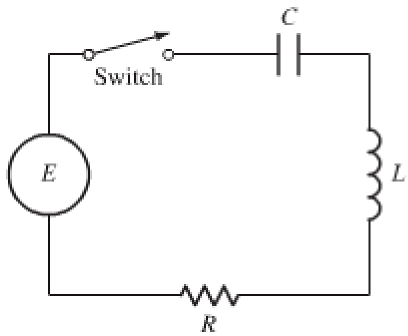
For capacitances in parallels:



$$C_{eq} = C_1 + C_2 + \cdots + C_n$$

1.8-4 *RLC circuit*

RLC circuit is a basic building block in electrical circuits and networks. A second order linear differential equations with constant coefficients is their use as a model of the flow of electric current in the simple series circuit



The current *I*, measured in amperes (***A***), is a function of time *t*.

A ***resistor*** with a resistance of *R ohms* (**Ω**)

An ***inductor*** with an inductance of *L henries* (***H***)

A ***capacitor*** with a capacitance of *C farads* (***F***)

The impressed ***voltage*** *E* in volts (***V***) is a given function of time.

Circuit Element	***Voltage Drop***
Inductor	$L\frac{dI}{dt}$
Resistor	RI
Capacitor	$\frac{1}{C}Q$

In series with a source of electromotive force (such as a battery or a generator) that supplies a voltage of $E(t)$ volts at time *t*. If the switch shown in the circuit is closed, this results in a current of $I(t)$ *amperes* in the circuit and a charge of $Q(t)$ *coulombs* on the capacitor at time *t*. The relation between the functions *Q* and the current *I* is

$$\frac{dQ}{dt} = I(t)$$

We use ***mks*** electric units, in which time is measured in seconds.

According to elementary principles of electricity, the voltage drops across the three circuit elements.

Kirchhoff's Current Law (***KCL***) (also known as Kirchhoff's ***First Law***)

The algebraic sum of all the currents at any node in a circuit equals to zero.

Current is distributed when it reaches a junction: the amount of current entering a junction must equal the amount of current leaving that junction.

Kirchhoff's Voltage Law (***KVL***) (also known as Kirchhoff's ***Second Law***)

The algebraic sum of all the voltages around any closed path in a circuit equals to zero.
In a closed circuit the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit.

According to the elementary laws of electricity, we know that

The voltage drops across the resistor is IR.

The voltage drops across the capacitor is $\frac{Q}{C}$.

The voltage drops across the inductor is $L\frac{dI}{dt}$.

- When the switch is open, no current flows in the circuit; when the switch is closed, there is a current $I(t)$ and a charge $Q(t)$ on the capacitor.

The current and charge in the simple RLC circuit satisfy the basic electrical equation

$$L\frac{dI}{dt}+RI+\frac{1}{C}Q=E(t)$$

The units for voltage, resistance, current, charge, capacitance, inductance, and time are all related:

$$1\ volt\ =\ 1\ ohm\times 1\ ampere=\frac{1\ coulomb}{1\ farad}=\frac{1\ henry\times 1\ ampere}{1\ second}$$

$$\boxed{1\ V\ =\ 1\ \Omega\times 1\ A=\frac{1\ C}{1\ F}=\frac{1\ H\times 1\ A}{1\ sec}}$$

Since $\frac{dQ}{dt}=I(t)\ \Rightarrow\ \frac{d^2Q}{dt^2}=\frac{dI}{dt}$, we can get the second-order linear differential equation

$$LQ''+RQ'+\frac{1}{C}Q=E(t)$$

For the charge $Q(t)$, under the assumption that the voltage $E(t)$ is known.

It is the current, in most problems, rather than the charge Q that us of primary interest, so we differentiate both sides and substitute I for Q' to obtain

$$LI''+RI'+\frac{1}{C}I=E'(t)$$

With initial conditions are

$$Q(t_0)=Q_0,\quad Q'(t_0)=I(t_0)=I_0$$

$$Q''(t_0)=I'_0,\quad I'(t_0)=I'_0$$

$$LQ''+RQ'+\frac{1}{C}Q=E(t)$$

$$LI'_0+RI_0+\frac{1}{C}Q=E(t_0)$$

$$I'_0=\frac{E(t_0)-RI_0-\frac{1}{C}Q}{L}$$

Hence I'_0 is also determined by the initial charge and current, which are physically measurable quantities.

- The most important conclusion is that the flow of current in the circuit is precisely the same form as the one that describes the motion of a spring-mass system.

1.8-5 *Summary*

In *RLC* circuit:

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=E(t)$$

$$I_S(t)=\frac{1}{R}V+C\frac{dV}{dt}+\frac{1}{L}\int V(s)\,ds$$

In terms of current: $I(t)=\dfrac{dQ(t)}{dt}$

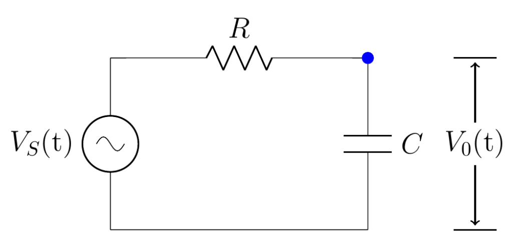
$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}I=\frac{d}{dt}E(t)$$

Without capacitor $\quad L\dfrac{dI}{dt}+RI=E(t)$

Where $Q(t)$ is the change on the capacitor and $E(t)$ is the applied voltage.

1.8-6 Communication Channel

The *RC*-circuit consists of a voltage source, a resistor and a capacitor.



The source voltage $V_S(t)$ is the message sent and the output voltage $V_0(t)$ is the message received.

Kirchhoff's Voltage Law: the source voltage V_S is the sum of the voltage drops across the other two circuit elements.

Voltage drop across the resistor $=V_S-V_0$

The current through the resistor, which by Ohm's Law,

$$\text{Current into node}=\frac{V_S-V_0}{R}$$

$$\text{Current leaving node}=CV_0'$$

Therefore, the model *RC* circuit *ODE*:

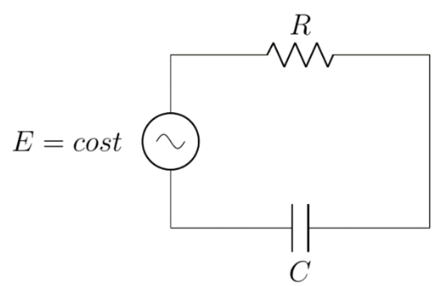
$$CV_0' = \frac{V_S - V_0}{R}$$

$$V_0' + \frac{1}{RC}V_0 = \frac{1}{RC}V_S$$

Example 1

Suppose the electrical circuit has a resistor of $R = 2\Omega$ and a capacitor of $C = \frac{1}{5}F$. Assume the voltage source is $E = \cos t \ (V)$.

If the initial current is 0 *A*, find the resulting current.



Solution

$$2Q' + 5Q = \cos t \ \rightarrow \ Q' + \frac{5}{2}Q = \frac{1}{2}\cos t$$

$$e^{\int \frac{5}{2}dt} = e^{\frac{5}{2}t}$$

$$\int e^{5t/2}\left(\frac{1}{2}\cos t\right)dt = \frac{1}{2}e^{5t/2}\left(\sin t + \frac{5}{2}\cos t\right) - \frac{25}{4}\int \frac{1}{2}e^{5t/2}\cos t \ dt$$

$$\left(1 + \frac{25}{4}\right)\int \frac{1}{2}e^{5t/2}\cos t \ dt = \frac{1}{2}e^{5t/2}\left(\sin t + \frac{5}{2}\cos t\right)$$

$$\frac{29}{4}\int \frac{1}{2}e^{5t/2}\cos t \ dt = \frac{1}{4}e^{5t/2}(2\sin t + 5\cos t)$$

$$\int \frac{1}{2}e^{5t/2}\cos t \ dt = \frac{1}{29}e^{5t/2}(2\sin t + 5\cos t) + K$$

		$\int \cos t \ dt$
+	$\frac{1}{2}e^{5t/2}$	$\sin t$
−	$\frac{5}{4}e^{5t/2}$	$-\cos t$
+	$\frac{25}{8}e^{5t/2}$	

$$Q(t) = \frac{1}{e^{5t/2}}\left[\frac{1}{29}e^{5t/2}(5\cos t + 2\sin t) + K\right]$$

$$= \frac{1}{29}(5\cos t + 2\sin t) + Ke^{-5t/2}$$

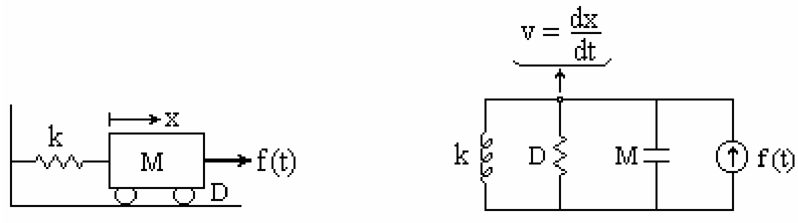
$$I(0) = 0 \ \rightarrow \ 0 = \frac{1}{29}(5) + K$$

$$\boxed{K = -\frac{5}{29}}$$

$$\boxed{Q(t) = \frac{5}{29}\cos t + \frac{2}{29}\sin t - \frac{5}{29}e^{-5t/2}}$$

1.8-7 *Example*

The electrical analog of a carriage on wheels, coupled to the wall through a spring.



a) Mechanical system. ***b)*** Electrical analog.

A mechanical system with a one coordinates movement.

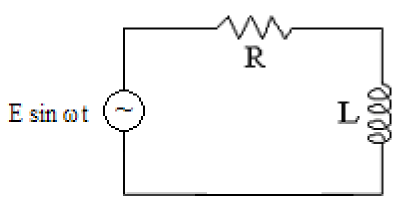
In the case of the electrical network, the equation was obtained by applying Kirchhoff's current law at the node ***v***, and is seen to be identical to the equation that would have been obtained by applying D'Alembert's principle to the mechanical system.

The differential equation for both systems is:

$$M\frac{d^2x}{dt^2}+D\frac{dx}{dt}+kx=f(t)$$

In particular, if one uses the force current analogy (or force-torque for a rational system). The topology of the electrical analog is very similar to that of the mechanical system.

***Example* 2:** Alternating Circuit



Alternating Circuit

Translating the circuit into differential equation:

$$L\frac{di}{dt}+Ri=E\sin\omega t$$

$$\Rightarrow\ \frac{di}{dt}+\frac{R}{L}i=\frac{E}{L}\sin\omega t$$

$$e^{\int\frac{R}{L}dt}=e^{\frac{R}{L}t}$$

$$\int e^{\frac{R}{L}t}\frac{E}{L}\sin\omega t\ dt=\frac{E}{L}e^{\frac{R}{L}t}\left(-\frac{1}{\omega}\cos\omega t+\frac{R}{L\omega^2}\sin\omega t\right)-\frac{R^2}{L^2\omega^2}\int\frac{E}{L}e^{\frac{R}{L}t}\sin\omega t\ dt$$

		$\int \sin\omega t \; dt$
$+$	$\frac{E}{L}e^{\frac{R}{L}t}$	$-\frac{1}{\omega}\cos\omega t$
$-$	$\frac{RE}{L^2}e^{\frac{R}{L}t}$	$-\frac{1}{\omega^2}\sin\omega t$
$+$	$\frac{R^2E}{L^3}e^{\frac{R}{L}t}$	

$$\left(1+\frac{R^2}{L^2\omega^2}\right)\int \frac{E}{L}e^{\frac{R}{L}t}\sin\omega t \; dt = \frac{E}{L^2\omega^2}e^{\frac{R}{L}t}\left(-L\omega\cos\omega t + R\sin\omega t\right)$$

$$\left(\frac{L^2\omega^2+R^2}{L^2\omega^2}\right)\int \frac{E}{L}e^{\frac{R}{L}t}\sin\omega t \; dt = \frac{E}{L^2\omega^2}e^{\frac{R}{L}t}\left(R\sin\omega t - L\omega\cos\omega t\right)$$

$$\int \frac{E}{L}e^{\frac{R}{L}t}\sin\omega t \; dt = \frac{E}{L^2\omega^2+R^2}e^{\frac{R}{L}t}\left(R\sin\omega t - L\omega\cos\omega t\right)+C$$

$$i(t) = \frac{1}{e^{\frac{R}{L}t}}\left[\frac{E}{L^2\omega^2+R^2}e^{\frac{R}{L}t}\left(R\sin\omega t - L\omega\cos\omega t\right)+C\right]$$

$$= \frac{E}{L^2\omega^2+R^2}\left(R\sin\omega t - L\omega\cos\omega t\right)+Ce^{-\frac{R}{L}t}$$

At $t=0;\;\; i=0$

$$0 = \frac{E}{R^2+L^2\omega^2}\left(R\sin\omega 0 - L\omega\cos\omega 0\right)+Ce^{-\frac{R}{L}(0)}$$

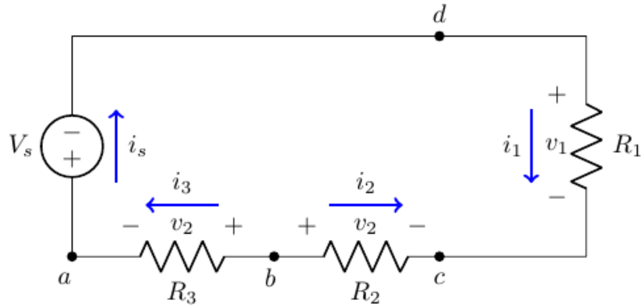
$$0 = \frac{EL}{R^2+L^2\omega^2}(-\omega)+C$$

$$C = \frac{EL\omega}{R^2+L^2\omega^2}$$

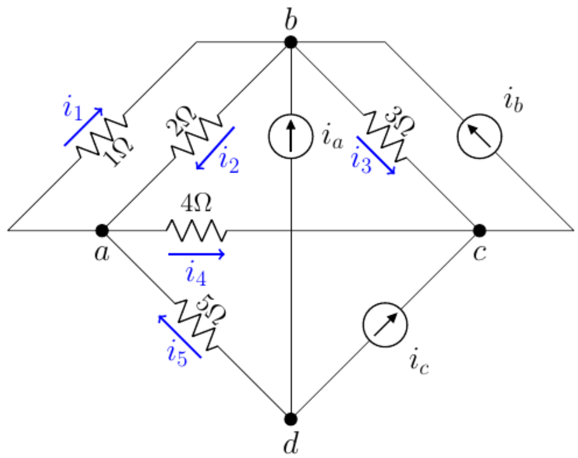
$$i(t) = \frac{E}{R^2+L^2\omega^2}\left(R\sin\omega t - L\omega\cos\omega t\right)+\frac{EL\omega}{R^2+L^2\omega^2}e^{-\frac{R}{L}t}$$

Exercises *Section* 1.8 - Basic Electrical Circuit

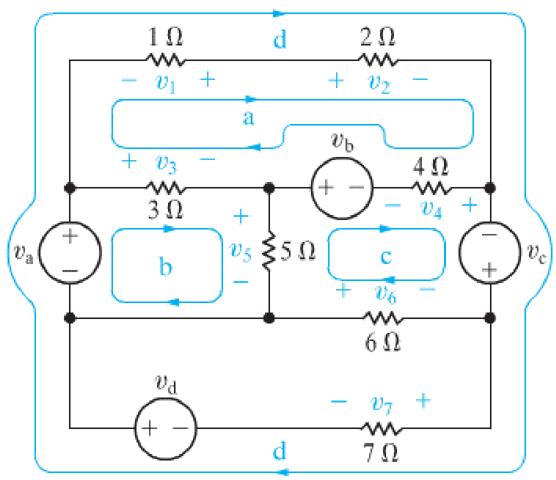
1. Sum the currents at each node in the circuit



2. Sum the currents at each node in the circuit



3. Sum the voltges around rach designated path in the circuit



(4 – 7) A resistor $R = 20\ \Omega$ and a capacitor of $C = 0.1\ F$ are joined in series with an electronic force (*emf*) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge on the capacitor at time t for the given:

4. $E(t) = 100\sin 2t$

5. $E(t) = 100e^{-0.1t}$

6. $E(t) = 100\left(1 - e^{-0.1t}\right)$

7. $E(t) = 100\cos 3t$

(8 – 10) An inductor $(L = 1\ H)$ and a resistor $(R = 0.1\ \Omega)$ are joined in series with an electronic force (*emf*) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing current in the current at time t for the given:

8. $E(t) = 10 - 2t$ **9.** $E(t) = 4\cos 3t$ **10.** $E(t) = 4\sin 2\pi t$

11. An RL circuit with a $1-\Omega$ resistor and a $0.1\text{-}H$ inductor is driven by a voltage $E(t) = \sin 100t\ V$. If the initial inductor current is zero, determine the subsequence resistor and inductor current and the voltages.

12. An RL circuit with a $1-\Omega$ resistor and a $0.01\text{-}H$ inductor is driven by a voltage $E(t) = \sin 100t\ V$. If the initial inductor current is zero, determine the subsequence resistor and inductor current and the voltages.

13. An RL circuit with a $5-\Omega$ resistor and a $0.05\text{-}H$ inductor is driven by a voltage $E(t) = 5\cos 120t\ V$. If the initial inductor current is 1 A, determine the subsequence resistor and inductor current and the voltages.

14. An RC circuit with a $1-\Omega$ resistor and a $10^{-6}\text{-}F$ capacitor is driven by a voltage $E(t) = \sin 100t\ V$. If the initial capacitor current is zero, determine the subsequence resistor and capacitor current and the voltages.

15. Solve the general initial value problem modeling the RC circuit

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E, \quad Q(0) = 0$$

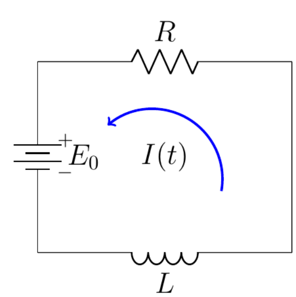
Where E is a constant source of *emf*

16. Solve the general initial value problem modeling the LR circuit

$$L\frac{dI}{dt} + RI = E, \quad I(0) = I_0$$

Where E is a constant source of *emf*

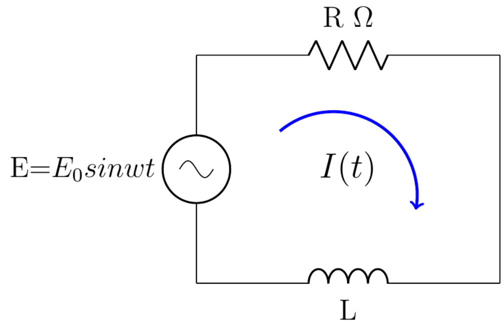
17. For the given RL–circuit



Where E_0 is a constant source of *emf* at time $t = 0$.

Find the current $I(t)$ flowing in the circuit.

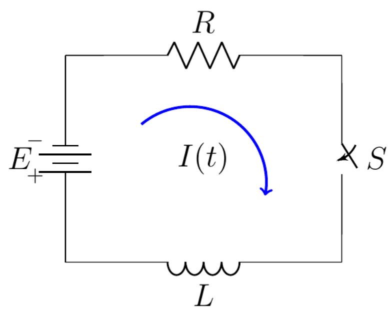
18. For the given $RL-$circuit



Where $E = E_0 \sin \omega t$ is the impressed voltage.

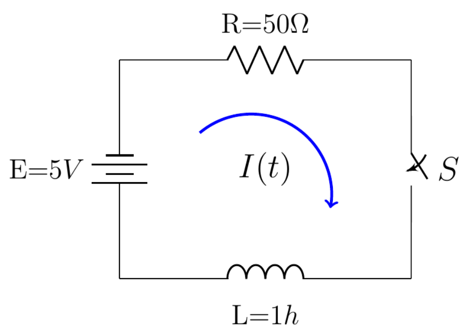
Find the current $I(t)$ flowing in the circuit.

19. For the given $RL-$circuit



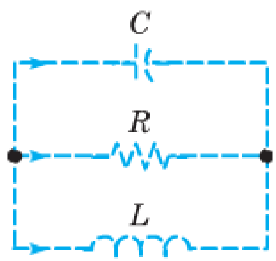
Which has a constant impressed voltage E, a resistor of resistance R, and a coil of impedance L. Find the current $I(t)$ flowing in the circuit.

20. For the given $RL-$circuit



Which has a constant impressed voltage E, a resistor of resistance R, and a coil of impedance L. Find the current $I(t)$ flowing in the circuit.

21. Consider the circuit shown and let I_1, I_2, and I_3 be the currents through the capacitor, resistor, and inductor, respectively. Let V_1, V_2, and V_3 be the corresponding voltage drops. The arrows denote the arbitrary chosen directions in which currents and voltage drops will be taken to be positive.



a) Applying Kirchhoff's second Law to the upper loop in the circuit, show that

$V_1 - V_2 = 0 \quad and \quad V_2 - V_3 = 0$

b) Applying Kirchhoff's first Law to either node in the circuit, show that

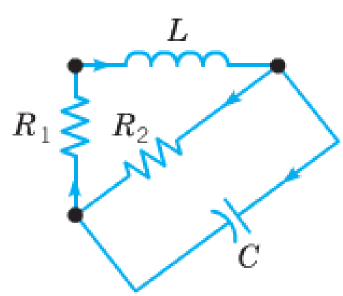
$$I_1 + I_2 + I_3 = 0$$

c) Use the current-voltage relation through each element in the circuit to obtain the equations

$$CV_1' = I_1, \quad V_2 = RI_2, \quad LI_3' = V_3$$

d) Eliminate V_2, V_3, I_1 and I_2 to obtain $CV_1' = -I_3 - \frac{V_1}{R}, \quad LI_3' = V_1$

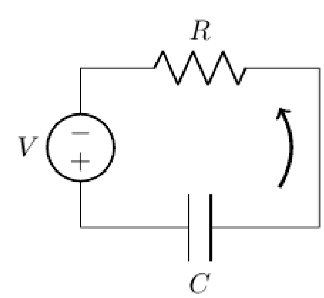
22. Consider the circuit. Use the method outlined to show that the current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations.



$$L\frac{dI}{dt} = -R_1 I - V, \quad C\frac{dV}{dt} = I - \frac{V}{R_2}$$

23. Consider an electric circuit containing a capacitor, resistor, and battery. The charge $Q(t)$ on the capacitor satisfies the equation

$$R\frac{dQ}{dt} + \frac{Q}{C} = V$$



Where R is the resistance, C is the capacitance, and V is the constant voltage supplied by the battery.

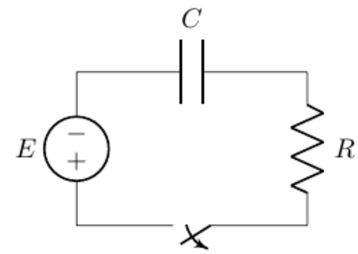
a) If $Q(0) = 0$, find $Q(t)$ at time t.

b) Find the limiting value Q_L that $Q(t)$ approaches after a long time.

c) Suppose that $Q(t_1) = Q(t)$ and that at time $t = t_1$ the battery is removed and the circuit is closed again. Find $Q(t)$ for $t > t_1$.

24. A circuit containing an electromotive force, a capacitor with a capacitance of C farads (F), and a resistor with a resistance of R ohms (Ω). The voltage drop across the capacitor is $\frac{Q}{C}$, where Q is the charge (in coulombs), so in this case ***Kirchhoff's Law*** gives

$$RI + \frac{Q}{C} = E(t)$$



But $I = \frac{dQ}{dt}$, so we have $\quad R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$

Find the charge and the current at time t

a) Suppose the resistance is $5\ \Omega$, the capacitance is 0.05 F, a battery gives voltage of 60 V and initial charge is $Q(0) = 0\ C$

b) Suppose the resistance is $2\ \Omega$, the capacitance is 0.01 F, $E(t) = 10\sin 60t$ and initial charge is $Q(0) = 0\ C$

25. A heart pacemaker consists of a switch, a battery voltage E_0, a capacitor with constant capacitance C, and the heart as a resistor with constant resistance R. When the switch is closed, the capacitor charges; when the switch is open, the capacitor discharges, sending an electrical stimulus to the heart. During the time the heart is being stimulated, the voltage E across the heart satisfies the linear differential equation

$$\frac{dE}{dt} = -\frac{1}{RC}E$$

Solve the DE, subject to $E(4) = E_0$

26. A 30–volt electromotive force is applied to an LR-series circuit in which the inductance is 0.1 *henry* and the resistance is 50 *ohms*.

a) Find the current $i(t)$ if $i(0) = 0$

b) Determine the current as $t \to \infty$

c) Solve the equation when $E(t) = E_0 \sin \omega t$ and $i(0) = i_0$

27. A 100–volt electromotive force is applied to an RC-series circuit in which the resistance is 200 *ohms* and the capacitance is 10^{-4} *farad*.

a) Find the charge $q(t)$ if $q(0) = 0$

b) Find the current as $i(t)$

28. A 200–volt electromotive force is applied to an RC-series circuit in which the resistance is 1000 *ohms* and the capacitance is 5×10^{-6} *farad*.

a) Find the charge $q(t)$ if $i(0) = 0.4$

b) Determine the charge as $t \to \infty$

29. An electromotive force

$$E(t)=\begin{cases}120, & 0\le t\le 20\\ 0, & t>20\end{cases}$$

Is applied to an *LR*-series circuit in which the inductance is 20 *henries* and resistance is 2 *ohms*. Find the current $i(t)$ if $i(0)=0$

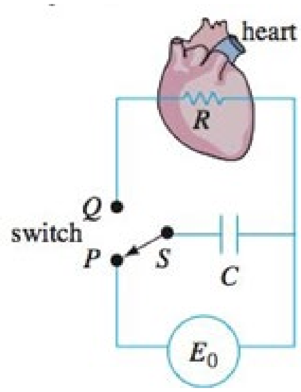
30. Suppose an *RC*-series circuit has a variable resistor. If the resistance at time t is given by $R=k_1+k_2t$, where k_1 and k_2 are known positive constants, then

$$\left(k_1+k_2t\right)\frac{dq}{dt}+\frac{1}{C}q=E(t)$$

If $E(t)=E_0$ and $q(0)=q_0$, where E_0 and q_0 are constants, show that

$$q(t)=E_0C+\left(q_0-E_0C\right)\left(\frac{k_1}{k_1+k_2t}\right)^{1/Ck_2}$$

31. A heart pacemaker, consists of a switch, a battery, a capacitor, and the heart as a resistor.



When the switch *S* is at *P*, the capacitor charges; when *S* is at *Q*, the capacitor discharges, sending an electrical stimulus to the heart. The electrical stimulus is being applied to the heart, the voltage *E* across the heart satisfies the linear DE.

$$\frac{dE}{dt}=-\frac{1}{RC}E$$

a) Let assume that over the time interval of length t_1, $0<t<t_1$, the switch *S* is at position *P* and the capacitor is being charges. When the switch is moved to position *Q* at time t_1 the capacitor discharges, sending an impulse to the heart over the time interval of length t_2: $t_1\le t<t_1+t_2$. Thus, over the initial charging/discharging interval $0<t<t_1+t_2$ the voltage to the heart is actually modeled by the piecewise-defined differential equation

$$\frac{dE}{dt}=\begin{cases}0, & 0<t<t_1\\ -\frac{1}{RC}E, & t_1\le t<t_1+t_2\end{cases}$$

By moving *S* between *P* and *Q*, the charging and discharging over time intervals of lengths t_1 and t_2 is repeated indefinitely. Suppose $t_1 = 4\ s$, $t_2 = 2\ s$. $E_0 = 12\ V$, and $E(0) = 0$, $E(4) = 12$, $E(6) = 0$, $E(10) = 12$, $E(12) = 0$, and so on.

Solve for $E(t)$ where $0 \le t \le 24$

b) Suppose for the sake of illustration that $R = C = 1$. Graph the solution in part (*a*) for $0 \le t \le 24$

Section 1.9 - Existence and Uniqueness of Solutions

Existence and uniqueness theorem is the tool which makes it possible for us to conclude that there exists only one solution to a first order differential equation which satisfies a given initial condition

The questions of existence and uniqueness

- When can we be sure that a solution exists?
- How many different solutions are there

- ✓ ***Existence***: Under what conditions does the Initial Value Problem (**IVP**) have at least one solution?
- ✓ ***Uniqueness***: Under what conditions does the **IVP** have at most one solution?
- ✓ ***Extension and Long-Term Behavior***: How far ahead into the future and back into the past does a solution extend? How does a solution behave as t gets large?
- ✓ ***Continuity***: Suppose the data f and y_0 change. Can the corresponding change in solution be limited by limiting the change in the data f and y_0.
- ✓ ***Description***: How can a solution and its behavior be described?

1.9-1 Existence of Solutions

Example 1

Consider the initial value problem: $tx' = x + 3t^2$ with $x(0) = 1$

Solution

$$x' = \frac{1}{t}x + 3t$$

$$x' = \frac{1}{t}x + 3t \qquad t \neq 0$$

There is ***no solution*** to the given initial value

$$u(t) = e^{-\int \frac{1}{t}dt}$$

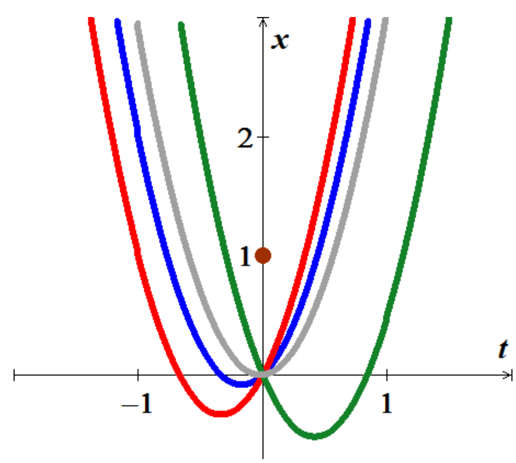
$$= e^{-\ln t}$$

$$= \frac{1}{t}$$

$$\left(\frac{x}{t}\right)' = 3$$

$$\frac{x}{t} = \int 3dt = 3t + C$$

$$\boxed{x(t) = 3t^2 + Ct}$$



1.9-2 *Theorem:* Existence of Solutions

Suppose the function $f(t,x)$ is defined and continuous on the rectangle $\boldsymbol{R}$ in the tx-plane. Then given any point $(t_0, x_0) \in \boldsymbol{R}$, the initial value problem

$$x' = f(t,x) \quad \textit{and} \quad x(t_0) = x_0$$

has a solution $x(t)$ defined in an interval containing x_0. Furthermore, the solution will be defined at least until the solution curve $t \to (t,\ x(t))$ leaves the rectangle $\boldsymbol{R}$.

1.9-3 Interval of Existence of a Solution

Example 2

Consider the initial value problem $x' = 1 + x^2$ with $x(0) = 0$. Find the solution and its interval of existence.

Solution

The right-hand side is $f(t,x) = 1 + x^2$ which is continuous on the entire tx-plane.

The solution to the initial value problem is:

$$\frac{dx}{dt} = 1 + x^2$$

$$\frac{dx}{1+x^2} = dt$$

$$\int \frac{dx}{1+x^2} = \int dt$$

$$\tan^{-1} x = t$$

$$x(t) = \tan t$$

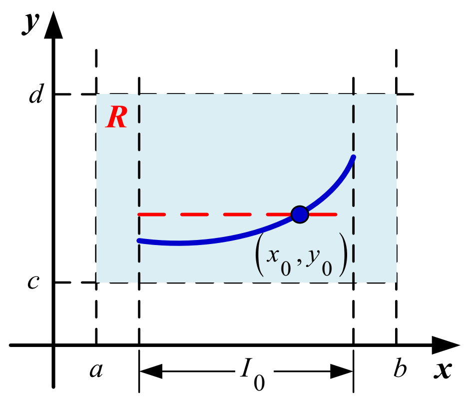
$x(t)$ is discontinuous at $t = \pm\frac{\pi}{2}$.

Hence the solution to the initial value problem is defined only for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

The interval: $\boxed{\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)}$

1.9-4 *Theorem*: Existence of a Unique Solution

Let ***R*** be a rectangular region in the *xy*-plane defined by $a \le x \le b,\ c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on ***R***, then there exists some interval $I_0 : (x_0 - h,\ x_0 + h),\ h > 0$, contained in [*a, b*], and a unique function $y(x)$, defined on I_0 that is a solution of the initial-value problem (IVP)



1.9-5 Mathematics & Theorems

Any theorem is a logical statement which has hypotheses (when it's true) and conclusions (true)

The Hypotheses of the Uniqueness of Solutions Theorem

1. The equation is in normal form $y' = f(t, y)$
2. The right-hand side $f(t, y)$ and its derivative $\frac{\partial f}{\partial y}$ are both continuous in the rectangle ***R***.
3. The initial point (t_0, y_0) is in the rectangle ***R***.

For the Uniqueness Theorem, the conclusions are as follows:

1- There is one and only one solution to the initial value problem.
2- The solution exists until the solution curve $t \to (t,\ y(t))$ leaves the rectangle ***R***.

Example 3

Consider the initial value problem $tx' = x + 3t^2$. Is there a solution to this equation with initial condition $x(1) = 2$? If so, is the solution unique?

Solution

$$x' = \frac{x}{t} + 3t$$

The right-hand side: $f(t,x) = \frac{x}{t} + 3t$ is continuous except where $t = 0$.

We can take ***R*** to be any rectangle which contains the point $(1,\ 2)$ to avoid $t = 0$, we can choose $\frac{1}{2} < t < 2$ and $0 < x < 4$

Then *f* is continuous everywhere in ***R***

$\Rightarrow$ hypotheses of the existence theorem are satisfied.

Since $\frac{\partial f}{\partial x} = \frac{1}{t}$ is also continuous in ***R***.

There is only one solution.

Exercises *Section* 1.9 - Existence and Uniqueness of Solutions

(1 – 12) Which of the initial value problems are guaranteed a unique solution

1. $y' = 4 + y^2, \quad y(0) = 1$

2. $y' = \sqrt{y}, \quad y(4) = 0$

3. $y' = t\tan^{-1} y, \quad y(0) = 2$

4. $\omega' = \omega\sin\omega + s, \quad \omega(0) = -1$

5. $x' = \dfrac{t}{x+1}, \quad x(0) = 0$

6. $y' = \dfrac{1}{x}y + 2, \quad y(0) = 1$

7. $y' = e^t y - y^3, \quad y(0) = 0$

8. $y' = ty^2 - \dfrac{1}{3y+t}, \quad y(0) = 1$

9. $y' = xy, \quad y(0) = 1$

10. $y' = -\dfrac{t^2}{1-y^2}, \quad y(-1) = \dfrac{1}{2}$

11. $y' = \dfrac{y}{\sin t}, \quad y\left(\dfrac{\pi}{2}\right) = 1$

12. $y' = \sqrt{1-y^2}, \quad y(0) = 1$

13. Show that $y(t) = 0$ and $y(t) = t^3$ are both solutions of the initial value problem $y' = 3y^{2/3}$, where $y(0) = 0$. Explain why this fact doesn't contradict Theorem

14. Use a numerical solver to sketch the solution of the given initial value problem

$$\frac{dy}{dt} = \frac{t}{y+1}, \qquad y(2) = 0$$

a) Where does your solver experience difficulty? why? Use the image of your solution to estimate the interval of existence.

b) Find an explicit solution; then use your formula to determine the interval of existence. How does it compare with the approximation found in part (*a*)?

Section 1.10 - Autonomous Equations and Stability

First order autonomous equations, Equilibrium solutions, Stability, Long-term behavior of solutions, direction fields.

A differential equation where the independent variable does not explicitly appear in its expression.

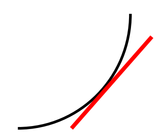
A first-order autonomous equation is an equation of the form

$$x' = f(x)$$

$$\frac{dy}{dx} = f(x, y)$$

1.10-1 *Definition*

The value $f(x, y)$ where the function f assigns to the point represent the slope of a line (*line segment*) call ***a lineal element***.



Example:

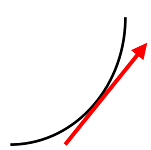
Given $\frac{dy}{dx} = 0.2xy$ and consider the point $(2,\ 3)$

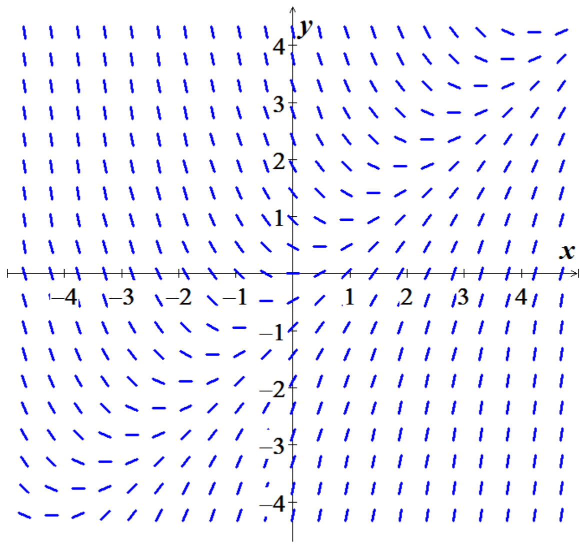
The slope of the lineal element is

$$\begin{aligned}\frac{dy}{dx} &= 0.2xy \\ &= 0.2(2)(3) \\ &= 1.2 \quad (\textit{positive sign})\end{aligned}$$

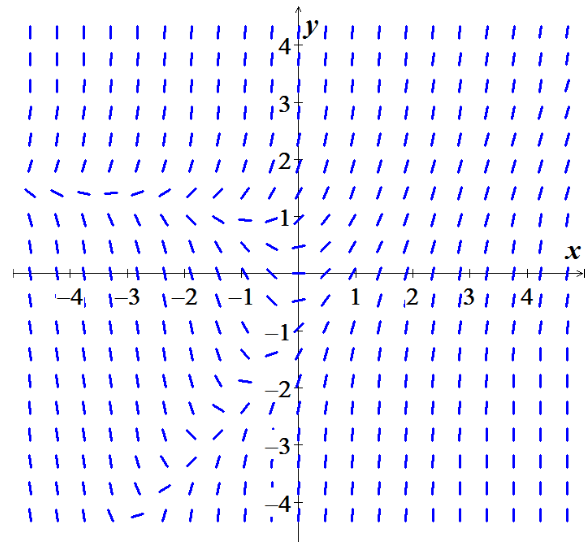
1.10-2 The Direction Fields

What we draw a lineal element at each point (x, y) with slope $f(x, y)$ then the collection of these lineal elements is called a ***direction field*** or a ***slope field*** of the differential equation $\frac{dy}{dx} = f(x, y)$.

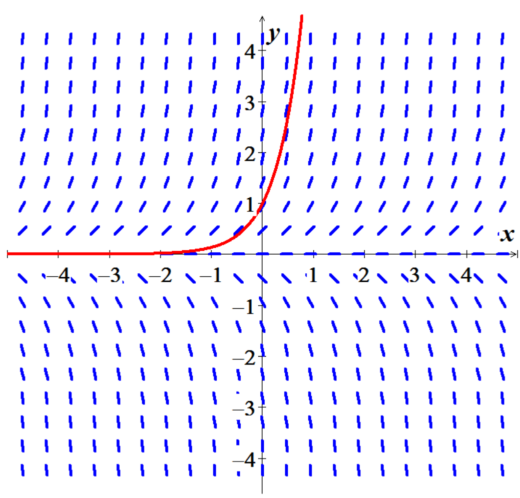




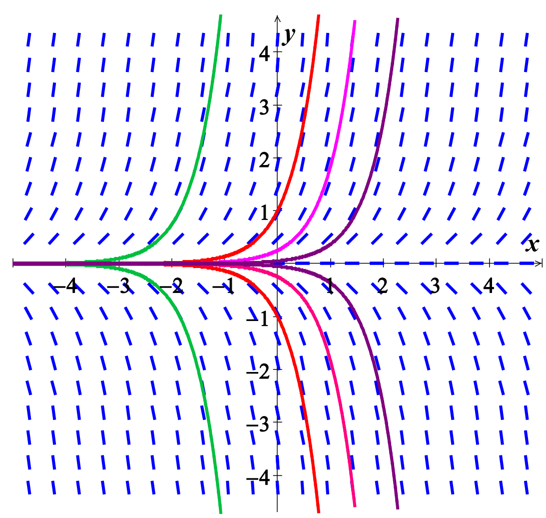
$y' = x - y$



$y' = y^2 - xy + 2x$



$y' = 2y, \quad with\ y(0) = 1 \quad \Rightarrow \quad y = e^{2x}$



Example 1

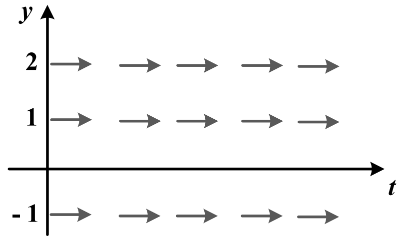
Sketch the direction field for the following differential equation. Sketch the set of integral curves for this differential equation, how the solutions behave as $t \to \infty$ and if this behavior depends on the value of $y(0)$ describe this dependency

$$y' = \left(y^2 - y - 2\right)(1 - y)^2$$

Solution

$$y' = 0 \quad \Rightarrow \quad \left(y^2 - y - 2\right)(1 - y)^2 = 0$$

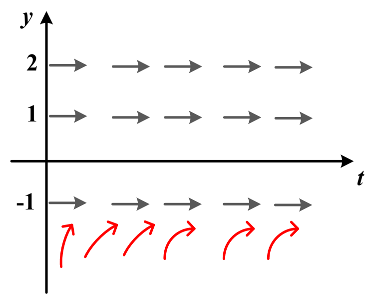
$y = \pm 1,\ 2$ | *Slope of the tangent lines*



This divided into 4 regions.

For $y < -1$, assume $y = -2 \Rightarrow y' = \left(4^2 + 2 - 2\right)(1+2)^2 = 36 > \mathbf{0}$ $(\nearrow)$

$y = -1$, the slopes will flatten out while staying positive

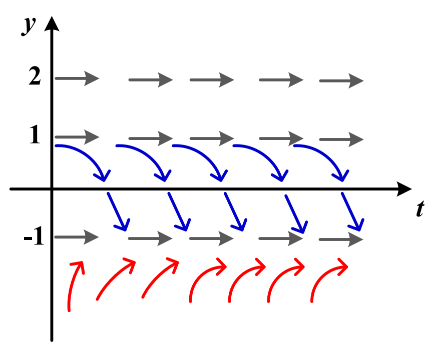


For $-1 < y < 1$, assume $y = 0 \Rightarrow y' = (-2)(1)^2 = -2 < \mathbf{0}$ $(\searrow)$

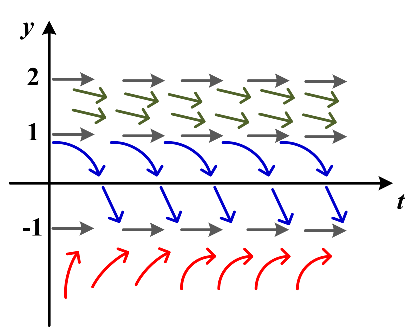
Therefore, tangent lines in this region will have negative slopes and apparently not very steep.

$y = .9 \quad \Rightarrow \quad y' = -.0209$

$y = -.9 \quad \Rightarrow \quad y' = -1.0469$ (Steeper)

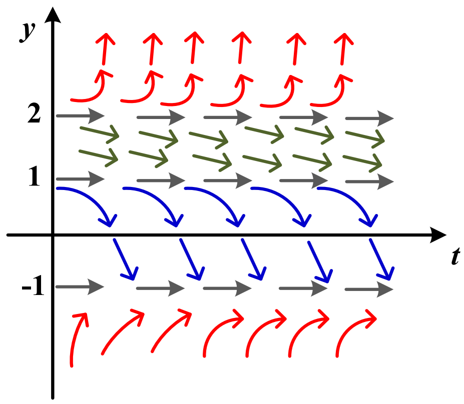


For $1 < y < 2$, assume $y = 1.5 \Rightarrow y' = \left(1.5^2 - 1.5 - 2\right)(-.5)^2 = -0.3125 < \mathbf{0}$ $(\searrow)$ Not to steep

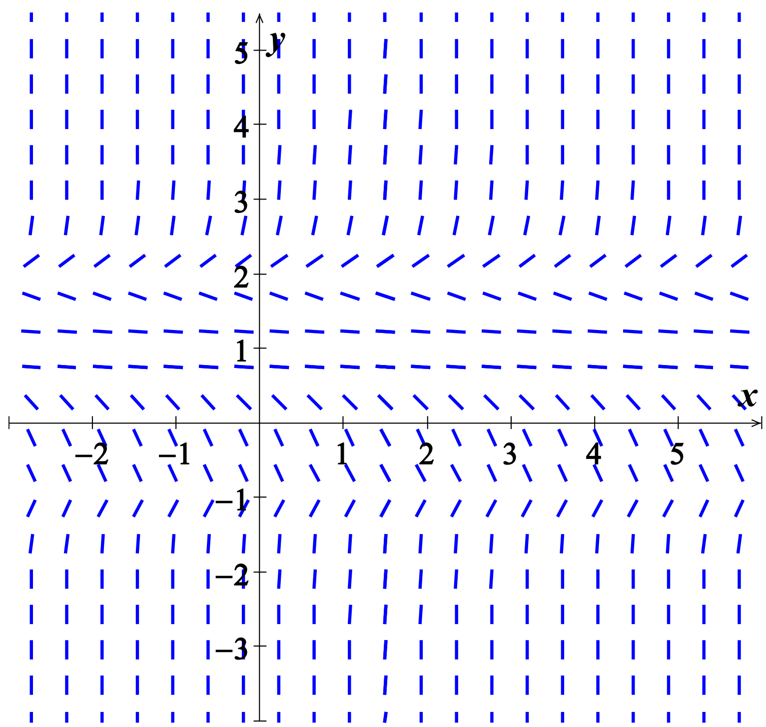


For $y > 2$, assume $y = 3 \Rightarrow y' = (4)(-2)^2 = 16 > \mathbf{0}$ $(\nearrow)$

Start out fairly flat nearly = 2, then will get fairly steep.



Value of $y(0)$	$t \to \infty$
$y(0) < -1$	$y \to -1$
$-1 \le y(0) < 2$	$y \to 1$
$y(0) = 2$	$y \to 2$
$y(0) > 2$	$y \to \infty$



1.10-3 Autonomous first- order DE

A system $\dot{y} = rx - y - xz = 0$, which does not explicitly contain the independent variable t is called an ***autonomous system***. Otherwise, the system is called *non-autonomous system*.

Autonomous	***Non–Autonomous***
$x' = \sin x$	$x' = \sin(tx)$
$y' = y^2 + 1$	$y' = y^2 + t$
$z' = e^z$	$z' = t^2$

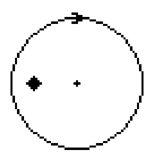
1.10-4 *Equilibrium* Points & Solutions

$x'(t) = 0 = f(x_0) \Rightarrow x_0$ *is an equilibrium point* and also, called a ***critical point***.

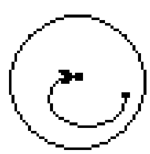
$x'(t) = x_0$ *called equilibrium solution*

From these equilibrium points, we can determine the stability of the system.

- An equilibrium point is ***stable*** if all nearby solutions stay nearby.



- An equilibrium point is ***asymptotically stable*** if all nearby solutions not only stay nearby, but also tend to the equilibrium point. An equilibrium point is stable if all nearby solutions.



1. If $f'(x_0) < 0$, then f is ***decreasing*** at x_0 and x_0 is asymptotically stable.
2. If $f'(x_0) > 0$, then f is ***increasing*** at x_0 and x_0 is unstable.
3. If $f'(x_0) = 0$, no conclusion can be drawn.

The family of all solution curves without the presence of the independent variable is called the ***phase portrait***.

When an independent variable ***t*** is interpreted as time and the solution curve $-P_+ < x < P_+$ could be thought of as the path of a particle moving in the solution space, then the system $f_\mu(x)$ is considered as a ***dynamical system***, where the solution curves are called ***trajectories*** or ***orbits***.

Example 2

Discover the behavior as $t \to \infty$ of all solutions to the differential equation

$$x' = f(x) = \left(x^2 - 1\right)\left(x - 2\right)$$

Solution

The equilibrium points: $f(x) = 0$

$\left(x^2 - 1\right)\left(x - 2\right) = 0$

$\Rightarrow x_1 = -1,\ x_2 = 1,\ x_3 = 2$ are equilibrium.

$f' = 3x^2 - 4x - 1$

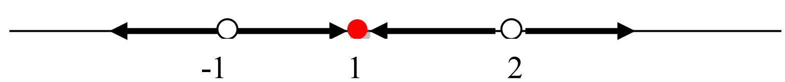
$f'(-1) = 3(-1)^2 - 4(-1) - 1 = 6 > 0$ ***unstable***

$f'(1) = 3(1)^2 - 4(1) - 1 = -2 < 0$ is ***asymptotically stable***

$f'(2) = 3(2)^2 - 4(2) - 1 = 3 > 0$ ***unstable***

$x(t) = -1,\ x(t) = 1,\ x(t) = 2$

These are constant functions, the position of the point the phase line modeled by them is also constant

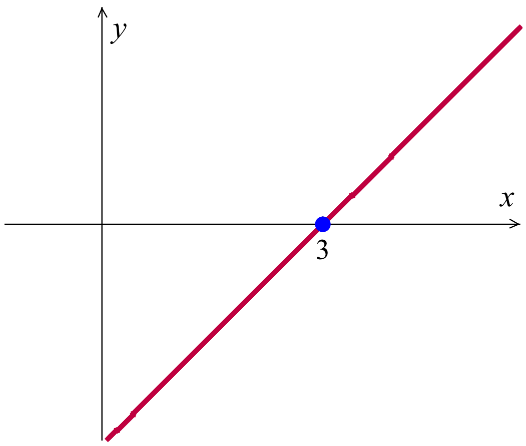


Phase Portrait

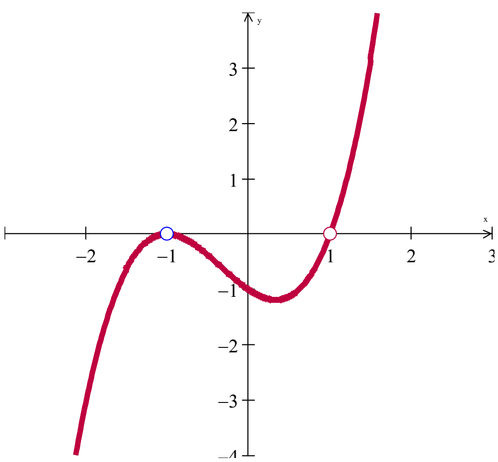
Exercises *Section* 1.10 - Autonomous Equations and Stability

(1 – 4) The graph of the right-hand side $y' = f(y)$ is shown. Identify the equilibrium points and sketch the equilibrium solutions in the *ty*-plane. Classify each equilibrium point as either unstable or asymptotically stable.

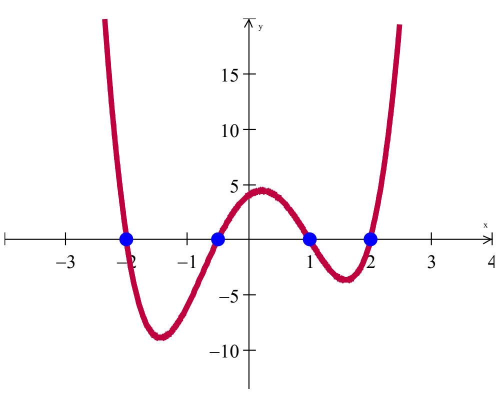
1.



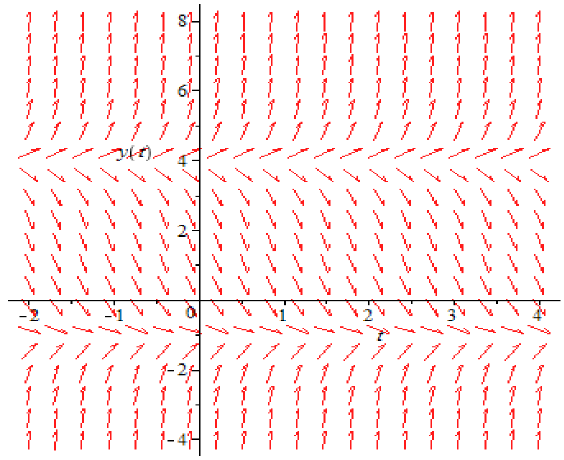
2.



3.



4. Impose the equilibrium solution(s), classifying each as either unstable or asymptotically stable



(5 – 20) An autonomous differential equation is given. Perform each of the following exercises

a) Sketch a graph of $f(y)$

b) Use the graph of f to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

5. $y' = 2 - y$

6. $y' = (y+1)(y-4)$

7. $y' = 9y - y^3$

8. $y' = \sin y$

9. $y' = y^2 - 3y$

10. $y' = y^2 - y^3$

11. $y' = (y-2)^4$

12. $y' = 10 + 3y - y^2$

13. $\dfrac{dy}{dt} = y^2\left(4 - y^2\right)$

14. $\dfrac{dy}{dt} = y(2-y)(4-y)$

15. $\dfrac{dy}{dt} = y\ln(y+2)$

16. $\dfrac{dy}{dt} = \dfrac{ye^y - 9y}{e^y}$

17. $y' = \dfrac{2}{\pi}y - \sin y$

18. $y' = 3y - ye^{y^2}$

19. $y' = (1-y)(y+1)^2$

20. $y' = \sin\dfrac{y}{2}$

(21 – 22) Determine the stability of the equilibrium solutions

21. $x' = 4 - x^2$

22. $x' = x(x-1)(x+2)$

23. A tank contains 100 *gal* of pure water. A salt solution with concentration 3 *lb./gal* enters the tank at a rate of 2 *gal/min*. Solution drains from the tank at a rate of 2 *gal/min*. Use the qualitative analysis to find the eventual concentration of the salt in the tank.

24. A mathematical model for rate at which a drug disseminates into the bloodstream at time *t*.

$$\frac{dx}{dt} = r - kx$$

Where *r* and *k* are positive constants. The function $x(t)$ describes the concentration of the drug in the bloodstream at time *t*.

a) Since the *DE* is autonomous, use the phase portrait concept to find the limiting value of $x(t)$ as $t \to \infty$

b) Solve $x(t)$ subject to $x(0)=0$. Sketch the graph of $x(t)$ and verify your prediction in part (*a*). At what time is the concentration one-half this limiting value?

25. When forgetfulness is taken into account, the rate of memorization of a subject is given by

$$\frac{dA}{dt} = k_1(M - A) - k_2 A$$

Where $k_1 > 0$, $k_2 > 0$, $A(t)$ is the amount memorized in time *t*, *M* is the total amount to be memorized, and $M - A$ is the amount remaining to be memorized.

a) Since the *DE* is autonomous, use the phase portrait concept to find the limiting value of $A(t)$ as $t \to \infty$. Interpret the result

b) Solve $A(t)$ subject to $A(0)=0$. Sketch the graph of $A(t)$ and verify your prediction in part (*a*).

26. The number $N(t)$ of supermarkets throughout the country that are using a computerized checkout system is described by the initial-value problem

$$\frac{dN}{dt} = N(1 - 0.0005N), \quad N(0) = 1$$

a) Use the phase portrait concept to predict how many supermarkets are expected to adopt the new procedure over a long period of time. Sketch a solution curve of the given initial-value problem.

b) Solve the initial-value problem and then graph it to verify the solution in part (*a*)

c) How many companies are expected to adopt the new technology when $t = 10$?

27. For the linear ODE $\qquad ty' + y = 2t$

a) Find all solution of the given *DE* equation.

b) Show that the initial value $y(0)=0$, has exactly one solution.

c) But if $y(0) = y_0 \neq 0$ there is no solution at all. Why doesn't this contradict the Existence and Uniqueness Theorem?

d) Plot several solutions of the *ODE* over the interval $-5 \le t \le 5$