Solution *Section* 4.5 – Partial Fraction Decomposition

Exercise

Write the partial fraction decomposition of each rational expression $\frac{4}{x(x-1)}$

Solution

$$\frac{4}{x(x-1)}=\frac{A}{x}+\frac{B}{x-1}$$

$$4=A(x-1)+Bx$$

$$4=Ax-A+Bx$$

$$4=(A+B)x-A$$

$$\begin{cases} A+B=0 \\ -A=4 \end{cases} \rightarrow \begin{cases} B=-A=4 \\ A=-4 \end{cases}$$

$$\boxed{\frac{4}{x(x-1)}=-\frac{4}{x}+\frac{4}{x-1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{3x}{(x+2)(x-1)}$

Solution

$$\frac{3x}{(x+2)(x-1)}=\frac{A}{x+2}+\frac{B}{x-1}$$

$$3x=A(x-1)+B(x+2)$$

$$3x=Ax-A+Bx+2B$$

$$3x=(A+B)x-A+2B$$

$$\begin{cases} A+B=3 \\ -A+2B=0 \end{cases}$$

$$\begin{array}{r} A+B=3 \\ -A+2B=0 \\ \hline 3B=3 \end{array} \Rightarrow B=1$$

$$\begin{pmatrix} 1 & 1 & 3 \\ -1 & 2 & 0 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$A=2B \rightarrow \begin{array}{c} 2B+B=3 \Rightarrow 3B=3 \Rightarrow B=1 \\ A=2 \end{array}$$

$$\boxed{\frac{3x}{(x+2)(x-1)}=\frac{2}{x+2}+\frac{1}{x-1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{1}{x\left(x^2+1\right)}$

Solution

$$\frac{1}{x\left(x^2+1\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$$

$$1=A\left(x^2+1\right)+x\left(Bx+C\right)$$

$$1=Ax^2+A+Bx^2+Cx$$

$$1=\left(A+B\right)x^2+Cx+A$$

$$\begin{cases} A+B=0 \\ C=0 \\ A=1 \end{cases} \quad \rightarrow B=-A=-1$$

$$\boxed{\frac{1}{x\left(x^2+1\right)}=\frac{1}{x}-\frac{x}{x^2+1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{1}{\left(x+1\right)\left(x^2+4\right)}$

Solution

$$\frac{1}{\left(x+1\right)\left(x^2+4\right)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+4}$$

$$1=A\left(x^2+4\right)+\left(x+1\right)\left(Bx+C\right)$$

$$1=Ax^2+4A+Bx^2+Cx+Bx+C$$

$$1=\left(A+B\right)x^2+\left(B+C\right)x+4A+C$$

$$\begin{cases} A+B=0 \\ B+C=0 \\ 4A+C=1 \end{cases} \rightarrow \begin{cases} A=-B \\ C=-B \\ -4B-B=1 \end{cases} \begin{matrix} \rightarrow A=C=\frac{1}{5} \\ \\ \rightarrow B=-\frac{1}{5} \end{matrix}$$

$$\frac{1}{\left(x+1\right)\left(x^2+4\right)}=\frac{\frac{1}{5}}{x+1}+\frac{\frac{1}{5}x+\frac{1}{5}}{x^2+4}$$

$$\boxed{=\frac{1}{5}\frac{1}{x+1}+\frac{1}{5}\frac{x+1}{x^2+4}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2}{(x-1)^2(x+1)^2}$

Solution

$$\frac{x^2}{(x-1)^2(x+1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}$$

$$x^2=A(x-1)(x+1)^2+B(x+1)^2+C(x-1)^2(x+1)+D(x-1)^2$$

$$=A(x-1)\left(x^2+2x+1\right)+B\left(x^2+2x+1\right)+C\left(x^2-2x+1\right)(x+1)+D\left(x^2-2x+1\right)$$

$$=Ax^3+2Ax^2+Ax-Ax^2-2Ax-A+Bx^2+2Bx+B$$
$$+Cx^3-2Cx^2+Cx+Cx^2-2Cx+C+Dx^2-2Dx+D$$

$$x^2=(A+C)x^3+(A+B-C+D)x^2+(-A+2B-C-2D)x-A+B+C+D$$

$$\begin{cases} A+C=0 \\ A+B-C+D=1 \\ -A+2B-C-2D=0 \\ -A+B+C+D=0 \end{cases} \qquad \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 & 1 \\ -1 & 2 & -1 & -2 & 0 \\ -1 & 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{4} \end{array}\right)$$

$$\frac{x^2}{(x-1)^2(x+1)^2}=\frac{\frac{1}{4}}{x-1}+\frac{\frac{1}{4}}{(x-1)^2}+\frac{-\frac{1}{4}}{x+1}+\frac{\frac{1}{4}}{(x+1)^2}$$

$$=\frac{1}{4}\frac{1}{x-1}+\frac{1}{4}\frac{1}{(x-1)^2}-\frac{1}{4}\frac{1}{x+1}+\frac{1}{4}\frac{1}{(x+1)^2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x+1}{x^2(x-2)^2}$

Solution

$$\frac{x+1}{x^2(x-2)^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}+\frac{D}{(x-2)^2}$$

$$x+1=Ax(x-2)^2+B(x-2)^2+Cx^2(x-2)+Dx^2$$

$$=Ax\left(x^2-4x+4\right)+B\left(x^2-4x+4\right)+Cx^3-2Cx^2+Dx^2$$

$$=Ax^3-4Ax^2+4Ax+Bx^2-4Bx+4B+Cx^3-2Cx^2+Dx^2$$

$$=(A+C)x^3+(-4A-B-2C+D)x^2+(4A-4B)x+4B$$

$$\begin{cases} A+C=0 \\ -4A-B-2C+D=0 \\ 4A-4B=1 \\ 4B=1 \end{cases} \qquad \begin{cases} C=-\frac{1}{2} \\ D=2+\frac{1}{4}-1=\frac{5}{4} \\ A=\frac{1}{2} \\ B=\frac{1}{4} \end{cases}$$

$$\boxed{\frac{x+1}{x^2(x-2)^2}=\frac{\frac{1}{2}}{x}+\frac{\frac{1}{4}}{x^2}+\frac{-\frac{1}{2}}{x-2}+\frac{\frac{5}{4}}{(x-2)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x-3}{(x+2)(x+1)^2}$

Solution

$$\frac{x-3}{(x+2)(x+1)^2}=\frac{A}{x+2}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$x-3=A(x+1)^2+B(x+1)(x+2)+C(x+2)$$

$$=Ax^2+2Ax+A+B\left(x^2+3x+2\right)+Cx+2C$$

$$=(A+B)x^2+(2A+3B+C)x+A+2B+2C$$

$$\begin{cases} A+B=0 \\ 2A+3B+C=1 \\ A+2B+2C=-3 \end{cases} \rightarrow \begin{cases} A=-B \\ -2B+3B+C=1 \\ -B+2B+2C=-3 \end{cases} \rightarrow \begin{cases} B+C=1 \\ B+2C=-3 \end{cases}$$

$$C=-4, \quad B=5, \quad A=-5$$

$$\boxed{\frac{x-3}{(x+2)(x+1)^2}=-\frac{5}{x+2}+\frac{5}{x+1}-\frac{4}{(x+1)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2+x}{(x+2)(x-1)^2}$

Solution

$$\frac{x^2+x}{(x+2)(x-1)^2}=\frac{A}{x+2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$x^2+x=A(x-1)^2+B(x-1)(x+2)+C(x+2)$$

$$=Ax^2-2Ax+A+Bx^2+Bx-2B+Cx+2C$$

$$\begin{array}{l} x^2 \\ x \\ x^0 \end{array}\begin{cases} A+B=1 \\ -2A+B+C=1 \\ A-2B+2C=0 \end{cases} \longrightarrow \quad A=\frac{2}{9} \quad B=\frac{7}{9} \quad C=\frac{2}{3}$$

$$\frac{x^2+x}{(x+2)(x-1)^2}=\boxed{\frac{\frac{2}{9}}{x+2}+\frac{\frac{7}{9}}{x-1}+\frac{\frac{2}{3}}{(x-1)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{10x^2+2x}{(x-1)^2\left(x^2+2\right)}$

Solution

$$\frac{10x^2+2x}{(x-1)^2\left(x^2+2\right)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+2}$$

$$10x^2+2x=A(x-1)\left(x^2+2\right)+B\left(x^2+2\right)+(Cx+D)(x-1)^2$$

$$=Ax^3+2Ax-Ax^2-2A+Bx^2+2B+(Cx+D)\left(x^2-2x+1\right)$$

$$=Ax^3+2Ax-Ax^2-2A+Bx^2+2B+Cx^3-2Cx^2+Cx+Dx^2-2Dx+D$$

$$=(A+C)x^3+(B-2A-2C+D)x^2+(2A+C-2D)x-2A+2B+D$$

$$\begin{cases} A+C=0 \\ B-2A-2C+D=10 \\ 2A+C-2D=2 \\ -2A+2B+D=0 \end{cases} \rightarrow \quad \boxed{A=\frac{42}{5}} \quad \boxed{B=\frac{34}{5}} \quad \boxed{C=-\frac{42}{5}} \quad \boxed{D=\frac{16}{5}}$$

$$\frac{10x^2+2x}{(x-1)^2\left(x^2+2\right)}=\boxed{\frac{\frac{42}{5}}{x-1}+\frac{\frac{34}{5}}{(x-1)^2}+\frac{-\frac{42}{5}x+\frac{16}{5}}{x^2+2}}$$

$$=\boxed{\frac{42}{5(x-1)}+\frac{34}{5(x-1)^2}+\frac{-42x+16}{5\left(x^2+2\right)}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2+2x+3}{(x+1)\left(x^2+2x+4\right)}$

Solution

$$\frac{x^2+2x+3}{(x+1)\left(x^2+2x+4\right)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+2x+4}$$

$$x^2+2x+3=A\left(x^2+2x+4\right)+(Bx+C)(x+1)$$

$$=Ax^2+2Ax+4A+Bx^2+Bx+Cx+C$$

$$=(A+B)x^2+(2A+B+C)x+4A+C$$

$$\begin{cases} A+B=1 \\ 2A+B+C=2 \\ 4A+C=3 \end{cases} \rightarrow \boxed{A=\frac{2}{3}} \quad \boxed{B=\frac{1}{3}} \quad \boxed{C=\frac{1}{3}}$$

$$\underline{\frac{x^2+2x+3}{(x+1)\left(x^2+2x+4\right)}=\frac{\frac{2}{3}}{x+1}+\frac{\frac{1}{3}x+\frac{1}{3}}{x^2+2x+4}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2-11x-18}{x\left(x^2+3x+3\right)}$

Solution

$$\frac{x^2-11x-18}{x\left(x^2+3x+3\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+3x+3}$$

$$x^2-11x-18=Ax^2+3Ax+3A+Bx^2+Cx$$

$$=(A+B)x^2+(3A+C)x+3A$$

$$\begin{cases} A+B=1 \\ 3A+C=-11 \\ 3A=-18 \end{cases} \rightarrow \boxed{A=-6} \quad \boxed{B=7} \quad \boxed{C=7}$$

$$\underline{\frac{x^2-11x-18}{x\left(x^2+3x+3\right)}=-\frac{6}{x}+\frac{7x+7}{x^2+3x+3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{1}{(2x+3)(4x-1)}$

Solution

$$\frac{1}{(2x+3)(4x-1)} = \frac{A}{2x+3} + \frac{B}{4x-1}$$

$$1 = 4Ax - A + 2Bx + 3B$$

$$1 = (4A+2B)x - A + 3B$$

$$\begin{cases} 4A+2B=0 \\ -A+3B=1 \end{cases} \rightarrow \begin{cases} 4A+2B=0 \\ -4A+12B=4 \end{cases} \quad 14B=4 \Rightarrow \underline{B=-\frac{2}{7}} \quad A=3\left(-\frac{2}{7}\right)-1=\underline{\frac{1}{7}}$$

$$\underline{\frac{1}{(2x+3)(4x-1)} = \frac{\frac{1}{7}}{2x+3} - \frac{\frac{2}{7}}{4x-1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2+2x+3}{\left(x^2+4\right)^2}$

Solution

$$\frac{x^2+2x+3}{\left(x^2+4\right)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{\left(x^2+4\right)^2}$$

$$x^2+2x+3 = (Ax+B)\left(x^2+4\right)+Cx+D$$

$$= Ax^3+4Ax+Bx^2+4B+Cx+D$$

$$= Ax^3+Bx^2+(4A+C)x+4B+D$$

$$\begin{cases} A=0 \\ B=1 \\ 4A+C=2 \\ 4B+D=3 \end{cases} \rightarrow \begin{matrix} \\ \\ C=2 \\ D=3-4B=-1 \end{matrix}$$

$$\underline{\frac{x^2+2x+3}{\left(x^2+4\right)^2} = \frac{1}{x^2+4} + \frac{2x-1}{\left(x^2+4\right)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^3+1}{\left(x^2+16\right)^2}$

Solution

$$\frac{x^3+1}{\left(x^2+16\right)^2}=\frac{Ax+B}{x^2+16}+\frac{Cx+D}{\left(x^2+16\right)^2}$$

$$x^3+1=(Ax+B)\left(x^2+16\right)+Cx+D$$

$$=Ax^3+16Ax+Bx^2+16B+Cx+D$$

$$\begin{cases} x^3 & \underline{A=1} \\ x^2 & \underline{B=0} \\ x & 16A+C=0 \\ x^0 & 16B+D=1 \end{cases} \rightarrow \begin{matrix} \underline{C=-16} \\ \underline{D=1} \end{matrix}$$

$$\boxed{\frac{x^3+1}{\left(x^2+16\right)^2}=\frac{x}{x^2+16}+\frac{-16x+1}{\left(x^2+16\right)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{7x+3}{x^3-2x^2-3x}$

Solution

$$\frac{7x+3}{x^3-2x^2-3x}=\frac{7x+3}{x(x+1)(x-3)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-3}$$

$$7x+3=A(x+1)(x-3)+Bx(x-3)+Cx(x+1)$$

$$=Ax^2-2Ax-3A+Bx^2-3B+Cx^2+Cx$$

$$=(A+B+C)x^2+(C-2A)x-3A-3B$$

$$\begin{cases} A+B+C=0 \\ C-2A=7 \\ -3A-3B=3 \end{cases} \rightarrow \boxed{A=-3} \quad \boxed{B=2} \quad \boxed{C=1}$$

$$\boxed{\frac{7x+3}{x^3-2x^2-3x}=\frac{-3}{x}+\frac{2}{x+1}+\frac{1}{x-3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2}{x^3-4x^2+5x-2}$

Solution

$$\frac{x^2}{x^3-4x^2+5x-2}=\frac{x^2}{(x-2)(x-1)^2}=\frac{A}{x-2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

$$x^2=A(x-1)^2+B(x-2)(x-1)+C(x-2)$$

$$=Ax^2-2Ax+A+Bx^2-3Bx+2B+Cx-2C$$

$$=(A+B)x^2+(-2A-3B+C)x+A+2B-2C$$

$$\begin{cases} A+B=1 \\ -2A-3B+C=0 \\ A+2B-2C=0 \end{cases} \rightarrow \boxed{A=4} \quad \boxed{B=-3} \quad \boxed{C=-1}$$

$$\boxed{\frac{x^2}{x^3-4x^2+5x-2}=\frac{4}{x-2}-\frac{3}{x-1}-\frac{1}{(x-1)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^3}{\left(x^2+16\right)^3}$

Solution

$$\frac{x^3}{\left(x^2+16\right)^3}=\frac{Ax+B}{x^2+16}+\frac{Cx+D}{\left(x^2+16\right)^2}+\frac{Ex+F}{\left(x^2+16\right)^3}$$

$$x^3=(Ax+B)\left(x^2+16\right)^2+(Cx+D)\left(x^2+16\right)+Ex+F$$

$$=(Ax+B)\left(x^4+32x^2+256\right)+Cx^3+16Cx+Dx^2+16D+Ex+F$$

$$=Ax^5+32Ax^3+256Ax+Bx^4+32Bx^2+256B+Cx^3+Dx^2+(16C+E)x+16D+F$$

$$=Ax^5+Bx^4+(32A+C)x^3+(32B+D)x^2+(256A+16C+E)x+256B+16D+F$$

$$\begin{cases} A=B=0 \\ 32A+C=1 \\ 32B+D=0 \\ 256A+16C+E=0 \\ 256B+16D+F=0 \end{cases} \rightarrow \boxed{A=B=D=F=0} \quad \boxed{C=1} \quad \boxed{E=-16}$$

$$\boxed{\frac{x^3}{\left(x^2+16\right)^3}=\frac{x}{\left(x^2+16\right)^2}+\frac{-16x}{\left(x^2+16\right)^3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{4}{2x^2-5x-3}$

Solution

$$\frac{4}{2x^2-5x-3}=\frac{4}{(2x+1)(x-3)}=\frac{A}{2x+1}+\frac{B}{x-3}$$

$$4=Ax-3A+2Bx+B$$
$$=(A+2B)x-3A+B$$

$$\begin{cases} A+2B=0 \\ -3A+B=4 \end{cases} \rightarrow \boxed{A=-\frac{8}{7}} \quad \boxed{B=\frac{4}{7}}$$

$$\boxed{\frac{4}{2x^2-5x-3}=\frac{-\frac{8}{7}}{2x+1}+\frac{\frac{4}{7}}{x-3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{2x+3}{x^4-9x^2}$

Solution

$$\frac{2x+3}{x^4-9x^2}=\frac{2x+3}{x^2(x-3)(x+3)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-3}+\frac{D}{x+3}$$

$$2x+3=Ax(x^2-9)+B(x^2-9)+Cx^2(x+3)+Dx^2(x-3)$$
$$=Ax^3-9Ax+Bx^2-9B+Cx^3+3Cx^2+Dx^3-3Dx^2$$
$$=(A+C+D)x^3+(B+3C-3D)x^2-9Ax-9B$$

$$\begin{cases} A+C+D=0 \\ B+3C-3D=0 \\ -9A=2 \\ -9B=3 \end{cases} \rightarrow \begin{array}{l} C=\frac{1}{6} \\ D=\frac{1}{18} \\ A=-\frac{2}{9} \\ B=-\frac{1}{3} \end{array}$$

$$\boxed{\frac{2x+3}{x^4-9x^2}=-\frac{\frac{2}{9}}{x}-\frac{\frac{1}{3}}{x^2}+\frac{\frac{1}{6}}{x-3}+\frac{\frac{1}{18}}{x+3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2+9}{x^4-2x^2-8}$

Solution

$$\frac{x^2+9}{x^4-2x^2-8}=\frac{A}{x-2}+\frac{B}{x+2}+\frac{Cx+D}{x^2+2}$$

$$x^2+9=A(x+2)\left(x^2+2\right)+B(x-2)\left(x^2+2\right)+(Cx+D)\left(x^2-4\right)$$

$$=Ax^3+2Ax+2Ax^2+4A+Bx^3+2Bx-2Bx^2-4B+Cx^3-4Cx+Dx^2-4D$$

$$=(A+B+C)x^3+(2A-2B+D)x^2+(2A+2B-4C)x+4A-4B-4D$$

$$\begin{cases} A+B+C=0 \\ 2A-2B+D=1 \\ 2A+2B-4C=0 \\ 4A-4B-4D=9 \end{cases} \rightarrow \boxed{A=\frac{13}{24}} \quad \boxed{B=-\frac{13}{24}} \quad \boxed{C=0} \quad \boxed{D=-\frac{7}{6}}$$

$$\underline{\frac{x^2+9}{x^4-2x^2-8}=\frac{\frac{13}{24}}{x-2}-\frac{\frac{13}{24}}{x+2}-\frac{\frac{7}{6}}{x^2+2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{y}{y^2-2y-3}$

Solution

$$\frac{y}{y^2-2y-3}=\frac{A}{y-3}+\frac{B}{y+1}$$

$$y=(A+B)y+A-3B$$

$$\rightarrow \begin{cases} A+B=1 \\ A-3B=0 \end{cases} \Rightarrow \boxed{A=\frac{3}{4}} \quad \boxed{B=\frac{1}{4}}$$

$$\underline{\frac{y}{y^2-2y-3}=\frac{\frac{3}{4}}{y-3}+\frac{\frac{1}{4}}{y+1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x+3}{2x^3-8x}$

Solution

$$\frac{x+3}{2x^3-8x}=\frac{1}{2}\frac{x+3}{x\left(x^2-4\right)}=\frac{1}{2}\left(\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-2}\right)$$

$$=\frac{1}{2}\frac{A(x+2)(x-2)+Bx(x-2)+Cx(x+2)}{x(x+2)(x-2)}$$

$$(A+B+C)x^2+(2C-2B)x-4A=x+3$$

$$\begin{cases} A+B+C=0 \\ 2C-2B=1 \\ -4A=3 \end{cases} \rightarrow \boxed{A=-\frac{3}{4}} \quad \boxed{B=\frac{1}{8}} \quad \boxed{C=\frac{5}{8}}$$

$$\underline{\frac{x+3}{2x^3-8x}=\frac{1}{2}\left(-\frac{\frac{3}{4}}{x}+\frac{\frac{1}{8}}{x+2}+\frac{\frac{5}{8}}{x-2}\right)}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2}{(x-1)\left(x^2+2x+1\right)}$

Solution

$$\frac{x^2}{(x-1)\left(x^2+2x+1\right)}=\frac{x^2}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$x^2=A(x+1)^2+B(x-1)(x+1)+C(x-1)$$

$$=(A+B)x^2+(2A+C)x+A-B-C$$

$$\begin{cases} A+B=1 \\ 2A+C=0 \\ A-B-C=0 \end{cases} \rightarrow \boxed{A=\frac{1}{4}} \quad \boxed{B=\frac{3}{4}} \quad \boxed{C=-\frac{1}{2}}$$

$$\underline{\frac{x^2}{(x-1)\left(x^2+2x+1\right)}=\frac{\frac{1}{4}}{x-1}+\frac{\frac{3}{4}}{x+1}-\frac{\frac{1}{2}}{(x+1)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{3x^2+x+4}{x^3+x}$

Solution

$$\frac{3x^2+x+4}{x^3+x}=\frac{A}{x}+\frac{Bx+C}{x^2+1}=\frac{(A+B)x^2+Cx+A}{x\left(x^2+1\right)}$$

$$3x^2+x+4=(A+B)x^2+Cx+A$$

$$\begin{cases} A+B=3 \\ C=1 \\ A=4 \end{cases} \rightarrow \boxed{A=4} \quad \boxed{B=-1} \quad \boxed{C=1}$$

$$\underline{\frac{3x^2+x+4}{x^3+x}=\frac{4}{x}+\frac{-x+1}{x^2+1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{8x^2+8x+2}{\left(4x^2+1\right)^2}$

Solution

$$\frac{8x^2+8x+2}{\left(4x^2+1\right)^2}=\frac{Ax+B}{4x^2+1}+\frac{Cx+D}{\left(4x^2+1\right)^2}=\frac{(Ax+B)\left(4x^2+1\right)+Cx+D}{\left(4x^2+1\right)^2}$$

$$8x^2+8x+2=(Ax+B)\left(4x^2+1\right)+Cx+D$$

$$=4Ax^3+4Bx^2+(A+C)x+B+D$$

$$\begin{cases} A=0 \\ 4B=8 \\ A+C=8 \\ B+D=2 \end{cases} \rightarrow \boxed{A=0} \quad \boxed{B=2} \quad \boxed{C=8} \quad \boxed{D=0}$$

$$\underline{\frac{8x^2+8x+2}{\left(4x^2+1\right)^2}=\frac{2}{4x^2+1}+\frac{8x}{\left(4x^2+1\right)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{1}{x^2+2x}$

Solution

$$\frac{1}{x^2+2x}=\frac{A}{x}+\frac{B}{x+2}$$

$$1=Ax+2A+Bx$$

$$x \qquad 2A=1 \qquad \rightarrow A=\frac{1}{2}$$

$$x^0 \qquad A+B=0 \quad \rightarrow B=-\frac{1}{2}$$

$$\frac{1}{x^2+2x}=\frac{1}{2}\frac{1}{x}-\frac{1}{2}\frac{1}{x+2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{2x+1}{x^2-7x+12}$

Solution

$$\frac{2x+1}{x^2-7x+12}=\frac{A}{x-4}+\frac{B}{x-3}$$

$$2x+1=Ax-3A+Bx-4B$$

$$x \qquad A+B=2$$

$$x^0 \qquad -3A-4B=1$$

$$A=\frac{\begin{vmatrix}2 & 1\\ 1 & -4\end{vmatrix}}{\begin{vmatrix}1 & 1\\ -3 & -4\end{vmatrix}}=\frac{-9}{-1}=9 \qquad B=\frac{\begin{vmatrix}1 & 2\\ -3 & 1\end{vmatrix}}{-1}=\frac{7}{-1}=-7$$

$$\frac{2x+1}{x^2-7x+12}=\frac{9}{x-4}-\frac{7}{x-3}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2+x}{x^4-3x^2-4}$

Solution

$$\frac{x^2+x}{x^4-3x^2-4}=\frac{x^2+x}{\left(x^2-4\right)\left(x^2+1\right)}$$

$$=\frac{A}{x-2}+\frac{B}{x+2}+\frac{Cx+D}{x^2+1}$$

$$x^2+x=A(x+2)\left(x^2+1\right)+B(x-2)\left(x^2+1\right)+(Cx+D)\left(x^2-4\right)$$

$$=Ax^3+Ax+2Ax^2+2A+Bx^3+Bx-2Bx^2-2B+Cx^3-4Cx+Dx^2-4D$$

$$=(A+B+C)x^3+(2A-2B+D)x^2+(A+B-4C)x+2A-2B-4D$$

$$\begin{cases} x^3 & A+B+C=0 \quad (1) \\ x^2 & 2A-2B+D=1 \quad (2) \\ x & A+B-4C=1 \quad (3) \\ x^0 & 2A-2B-4D=0 \quad (4) \end{cases}$$

$(1)-(3)\rightarrow \quad 5C=-1 \quad \underline{C=-\frac{1}{5}}$

$(2)-(4)\rightarrow \quad 5D=1 \quad \underline{D=\frac{1}{5}}$

$$\begin{cases} A+B=\frac{1}{5} \\ 2A-2B=\frac{4}{5} \end{cases} \rightarrow \begin{cases} 2A+2B=\frac{2}{5} \\ 2A-2B=\frac{4}{5} \end{cases}$$

$4A=\frac{6}{5} \quad \rightarrow \quad \underline{A=\frac{3}{10}}$

$B=\frac{1}{5}-\frac{3}{10} \quad \rightarrow \quad \underline{B=-\frac{1}{10}}$

$$\boxed{\frac{x^2+x}{x^4-3x^2-4}=\frac{3}{10}\frac{1}{x-2}-\frac{1}{10}\frac{1}{x+2}+\frac{1}{5}\frac{-x+1}{x^2+1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{\left(\theta^2+1\right)^3}$

Solution

$$\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{\left(\theta^2+1\right)^3}=\frac{A\theta+B}{\theta^2+1}+\frac{C\theta+D}{\left(\theta^2+1\right)^2}+\frac{E\theta+F}{\left(\theta^2+1\right)^3}$$

$$\theta^4-4\theta^3+2\theta^2-3\theta+1=(A\theta+B)\left(\theta^2+1\right)^2+(C\theta+D)\left(\theta^2+1\right)+E\theta+F$$

$$=(A\theta+B)\left(\theta^4+2\theta^2+1\right)+C\theta^3+C\theta+D\theta^2+D+E\theta+F$$

$$=A\theta^5+B\theta^4+(2A+C)\theta^3+(2B+D)\theta^2+(A+C+E)\theta+B+D+F$$

$$\begin{cases} \boxed{A=0} \\ \boxed{B=1} \\ 2A+C=-4 \\ 2B+D=2 \\ A+C+E=-3 \\ B+D+F=1 \end{cases} \rightarrow \boxed{C=-4} \quad \boxed{D=0} \quad \boxed{E=1} \quad \boxed{F=0}$$

$$\boxed{\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{\left(\theta^2+1\right)^3}=\frac{1}{\theta^2+1}-4\frac{\theta}{\left(\theta^2+1\right)^2}+\frac{\theta}{\left(\theta^2+1\right)^3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{3x^2+7x-2}{x^3-x^2-2x}$

Solution

$$\frac{3x^2+7x-2}{x^3-x^2-2x}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-2}$$

$$3x^2+7x-2=A(x+1)(x-2)+Bx(x-2)+Cx(x+1)$$

$$=Ax^2-Ax-2A$$

$$Bx^2-2Bx$$

$$Cx^2+Cx$$

$$\begin{cases} A+B+C=3 \\ -A-2B+C=7 \\ -2A=-2 \quad \rightarrow \underline{A=1} \end{cases}$$

$$\begin{cases} B+C=2 \\ -2B+C=8 \end{cases} \rightarrow \underline{B=-2} \quad \underline{C=4}$$

$$\boxed{\frac{3x^2+7x-2}{x^3-x^2-2x}=\frac{1}{x}-\frac{2}{x+1}+\frac{4}{x-2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}$

Solution

$$\frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}=\frac{A}{x-1}+\frac{B}{x-5}+\frac{C}{x+4}$$

$$3x^2+2x+5=(A+B+C)x^2+(-A+3B-6C)x-20A-4B+5C$$

$$\begin{cases} x^2 & A+B+C=3 \\ x & -A+3B-6C=2 \\ x^0 & -20A-4B+5C=5 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 1 \\ -1 & 3 & -6 \\ -20 & -4 & 5 \end{vmatrix}=180 \qquad D_A=\begin{vmatrix} 3 & 1 & 1 \\ 2 & 3 & -6 \\ 5 & -4 & 5 \end{vmatrix}=-90$$

$$D_B=\begin{vmatrix} 1 & 3 & 1 \\ -1 & 2 & -6 \\ -20 & 5 & 5 \end{vmatrix}=450 \qquad D_C=\begin{vmatrix} 1 & 1 & 3 \\ -1 & 3 & 2 \\ -20 & -4 & 5 \end{vmatrix}=180$$

$$A=\frac{1}{2},\quad B=\frac{5}{2},\quad C=1$$

$$\frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}=\frac{1}{2}\frac{1}{x-1}+\frac{5}{2}\frac{1}{x-5}+\frac{1}{x+4}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{5x^2-3x+2}{x^3-2x^2}$

Solution

$$\frac{5x^2-3x+2}{x^3-2x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}$$

$$5x^2-3x+2=Ax^2-2Ax+Bx-2B+Cx^2$$

$$\begin{cases} x^2 & A+C=5 & C=4 \\ x & -2A+B=-3 & A=1 \\ x^0 & -2B=2 & \rightarrow B=-1 \end{cases}$$

$$\frac{5x^2-3x+2}{x^3-2x^2}=\frac{1}{x}-\frac{1}{x^2}+\frac{4}{x-2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}$

Solution

$$\frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}=\frac{A}{x-2}+\frac{Bx+C}{x^2-2x+3}$$

$$7x^2-13x+13=Ax^2-2Ax+3A+Bx^2-2Bx+Cx-2C$$

$$\begin{cases} x^2 & A+B=7 \\ x^1 & -2A-2B+C=-13 \\ x^0 & 3A-2C=13 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 0 \\ -2 & -2 & 1 \\ 3 & 0 & -2 \end{vmatrix}=3 \qquad D_A=\begin{vmatrix} 7 & 1 & 0 \\ -13 & -2 & 1 \\ 13 & 0 & -2 \end{vmatrix}=15$$

$$D_B=\begin{vmatrix} 1 & 7 & 0 \\ -2 & -13 & 1 \\ 3 & 13 & -2 \end{vmatrix}=6 \qquad D_C=\begin{vmatrix} 1 & 1 & 7 \\ -2 & -2 & -13 \\ 3 & 0 & 13 \end{vmatrix}=3$$

$$A=5;\ \ B=2;\ \ C=1$$

$$\boxed{\frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}=\frac{5}{x-2}+\frac{2x+1}{x^2-2x+3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{1}{x^2-5x+6}$

Solution

$$\frac{1}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3}$$

$$Ax-3A+Bx-2B=1$$

$$\rightarrow\begin{cases} A+B=0 \\ -3A-2B=1 \end{cases}\rightarrow\quad A=-1\quad B=1$$

$$\boxed{\frac{1}{x^2-5x+6}=\frac{-1}{x-2}+\frac{1}{x-3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{1}{x^2-5x+5}$

Solution

$$\frac{1}{x^2-5x+5}=\frac{A}{x-\frac{5+\sqrt{5}}{2}}+\frac{B}{x-\frac{5-\sqrt{5}}{2}}$$

$$Ax-\left(\frac{5-\sqrt{5}}{2}\right)A+Bx-\left(\frac{5+\sqrt{5}}{2}\right)B=1$$

$$\begin{cases} A+B=0 \\ -\frac{5-\sqrt{5}}{2}A-\frac{5+\sqrt{5}}{2}B=1 \end{cases} \rightarrow \begin{matrix} \frac{5-\sqrt{5}}{2}A+\frac{5-\sqrt{5}}{2}B=0 \\ -\frac{5-\sqrt{5}}{2}A-\frac{5+\sqrt{5}}{2}B=1 \end{matrix}$$

$$-\sqrt{5}B=1\rightarrow \underline{B=-\frac{1}{\sqrt{5}}\Big|} \quad \Rightarrow \quad \underline{A=\frac{1}{\sqrt{5}}\Big|}$$

$$\underline{\frac{1}{x^2-5x+5}=\frac{\sqrt{5}}{5}\frac{2}{2x-5-\sqrt{5}}-\frac{\sqrt{5}}{5}\frac{2}{2x-5+\sqrt{5}}\Big|}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{5x^2+20x+6}{x^3+2x^2+x}$

Solution

$$\frac{5x^2+20x+6}{x^3+2x^2+x}=\frac{5x^2+20x+6}{x(x+1)^2}$$

$$=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$Ax^2+2Ax+A+Bx^2+Bx+Cx=5x^2+20x+6$$

$$\begin{cases} A+B=5 \\ 2A+B+C=20 \\ A=6 \end{cases} \rightarrow \underline{B=-1\Big|} \quad \underline{C=9\Big|}$$

$$\underline{\frac{5x^2+20x+6}{x^3+2x^2+x}=\frac{6}{x}-\frac{1}{x+1}+\frac{9}{(x+1)^2}\Big|}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}$

Solution

$$\frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}=\frac{2x^3-4x-8}{x(x-1)\left(x^2+4\right)}=\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+4}$$

$$Ax^3-Ax^2+4Ax-4A+Bx^3+4Bx+Cx^3-Cx^2+Dx^2-Dx=2x^3-4x-8$$

$$\begin{cases} x^3 & A+B+C=2 \\ x^2 & -A-C+D=0 \\ x^1 & 4A+4B-D=-4 \\ x^0 & -4A=-8 \end{cases} \rightarrow \begin{cases} B+C=0 \\ -C+D=2 \\ 4B-D=-12 \\ \underline{A=2} \end{cases}$$

$$\Rightarrow \begin{cases} B+D=2 \\ 4B-D=-12 \end{cases}$$

$$\underline{A=2 \quad B=-2 \quad C=2 \quad D=4}$$

$$\boxed{\frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}=\frac{2}{x}-\frac{2}{x-1}+\frac{2x}{x^2+4}+\frac{4}{x^2+4}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{8x^3+13x}{\left(x^2+2\right)^2}$

Solution

$$\frac{8x^3+13x}{\left(x^2+2\right)^2}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{\left(x^2+2\right)^2}$$

$$Ax^3+2Ax+Bx^2+2B+Cx+D=8x^3+13x$$

$$\begin{cases} x^3 & A=8 \\ x^2 & B=0 \\ x^1 & 2A+C=13 \\ x^0 & D=0 \end{cases} \rightarrow \underline{C=-3}$$

$$\boxed{\frac{8x^3+13x}{\left(x^2+2\right)^2}=\frac{8x}{x^2+2}-\frac{3x}{\left(x^2+2\right)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{1}{x^2-9}$

Solution

$$\frac{1}{x^2-9}=\frac{A}{x-3}+\frac{B}{x+3}$$

$$Ax+3A+Bx-3B=1$$

$$\Rightarrow\begin{cases}A+B=0\\3A-3B=1\end{cases}\rightarrow \underline{A=\frac{1}{6}\quad B=-\frac{1}{6}}$$

$$\underline{\frac{1}{x^2-9}=\frac{1}{6}\frac{1}{x-3}-\frac{1}{6}\frac{1}{x+3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{2}{9x^2-1}$

Solution

$$\frac{2}{9x^2-1}=\frac{A}{3x-1}+\frac{B}{3x+1}$$

$$3Ax+A+3Bx-B=2$$

$$\Rightarrow\begin{cases}3A+3B=0\\A-B=2\end{cases}\rightarrow \underline{A=1\quad B=-1}$$

$$\underline{\frac{2}{9x^2-1}=\frac{1}{3x-1}-\frac{1}{3x+1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{5}{x^2+3x-4}$

Solution

$$\frac{5}{x^2+3x-4}=\frac{A}{x-1}+\frac{B}{x+4}$$

$$Ax+4A+Bx-B=5$$

$$\Rightarrow\begin{cases}A+B=0\\4A-B=5\end{cases}\rightarrow \underline{A=1\quad B=-1}$$

$$\frac{5}{x^2+3x-4}=\frac{1}{x-1}-\frac{1}{x+4}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{3-x}{3x^2-2x-1}$

Solution

$$\frac{3-x}{3x^2-2x-1}=\frac{A}{x-1}+\frac{B}{3x+1}$$

$$3Ax+A+Bx-B=3-x$$

$$\Rightarrow\begin{cases}3A+B=-1\\ A-B=3\end{cases}\rightarrow A=\frac{1}{2}\quad B=-\frac{5}{2}$$

$$\frac{3-x}{3x^2-2x-1}=\frac{1}{2}\frac{1}{x-1}-\frac{5}{2}\frac{1}{3x+1}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2+12x+12}{x^3-4x}$

Solution

$$\frac{x^2+12x+12}{x^3-4x}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2}$$

$$Ax^2-4A+Bx^2+2Bx+Cx^2-2Cx=x^2+12x+12$$

$$\begin{cases}x^2 & A+B+C=1\\ x^1 & 2B-2C=12\\ x^0 & -4A=12\end{cases}\rightarrow A=-3\quad B=5\quad C=-1$$

$$\frac{x^2+12x+12}{x^3-4x}=-\frac{3}{x}+\frac{5}{x-2}-\frac{1}{x+2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{5x-2}{(x-2)^2}$

Solution

$$\frac{5x-2}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}$$

$$Ax-2A+B=5x-2$$

$$\Rightarrow\begin{cases}\underline{A=5}\\ -2A+B=-2\end{cases}\rightarrow \underline{B=8}$$

$$\underline{\frac{5x-2}{(x-2)^2}=\frac{5}{x-2}+\frac{8}{(x-2)^2}}$$