Solution *Section* 4.4 – The Binomial Theorem

Exercise

Find the fifth term in the expansion $\left(x^3+\sqrt{y}\right)^{13}$

Solution

$$\binom{13}{4}\left(x^3\right)^9\left(\sqrt{y}\right)^4=\frac{13!}{4!(13-4)!}x^{27}y^2=715x^{27}y^2$$

Exercise

Find the term involving q^{10} in the binomial expansion $\left(\frac{1}{3}p+q^2\right)^{12}$

Solution

Given: $a=\frac{1}{3}p,\quad b=q^2,\quad n=12$

$$q^{10}=\left(q^2\right)^5=b^5$$

$$\binom{n}{k}a^{n-k}b^k=\binom{12}{5}\left(\frac{1}{3}p\right)^{12-5}\left(q^2\right)^5=\frac{12!}{5!(12-5)!}\left(\frac{1}{3}p\right)^7q^{10}=\frac{88}{243}p^7q^{10}$$

Exercise

Use the binomial theorem to expand and simplify: $(4x-y)^3$

Solution

$$\begin{aligned}(4x-y)^3&=\binom{3}{0}(4x)^3(-y)^0+\binom{3}{1}(4x)^2(-y)^1+\binom{3}{2}(4x)^1(-y)^2+\binom{3}{3}(4x)^0(-y)^3\\&=64x^3+3\left(16x^2\right)(-y)+3(4x)y^2-y^3\\&=64x^3-48x^2y+12xy^2-y^3\end{aligned}$$

Exercise

Use the binomial theorem to expand and simplify: $(x+y)^6$

Solution

$$(x+y)^6=x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$$

Exercise

Use the binomial theorem to expand and simplify: $(x-y)^7$

Solution

$$(x-y)^7 = x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$$

Exercise

Use the binomial theorem to expand and simplify: $(3t-5x)^4$

Solution

$$(3t-5x)^4 = (3t)^4 + 4(3t)^3(-5x)^1 + 6(3t)^2(-5x)^2 + 4(3t)^1(-5x)^3 + (-5x)^4$$
$$= 81t^4 - 540t^3x + 1350t^2x^2 - 1500tx^3 + 625x^4$$

Exercise

Use the binomial theorem to expand and simplify: $\left(\frac{1}{3}x + y^2\right)^5$

Solution

$$\left(\frac{1}{3}x + y^2\right)^5 = \left(\frac{1}{3}x\right)^5 + 5\left(\frac{1}{3}x\right)^4 y^2 + 10\left(\frac{1}{3}x\right)^3\left(y^2\right)^2 + 10\left(\frac{1}{3}x\right)^2\left(y^2\right)^3 + 5\frac{1}{3}x\left(y^2\right)^4 + \left(y^2\right)^5$$
$$= \frac{1}{243}x^5 + \frac{5}{81}x^4y^2 + \frac{10}{27}x^3y^4 + \frac{10}{9}x^2y^6 + \frac{5}{3}xy^8 + y^{10}$$

Exercise

Use the binomial theorem to expand and simplify: $\left(\frac{1}{x^2} + 3x\right)^6$

Solution

$$\left(\frac{1}{x^2} + 3x\right)^6 = \left(x^{-2} + 3x\right)^6$$
$$= \left(x^{-2}\right)^6 + 6\left(x^{-2}\right)^5 3x + 15\left(x^{-2}\right)^4(3x)^2 + 20\left(x^{-2}\right)^3(3x)^3 + 15\left(x^{-2}\right)^2(3x)^4 + 15x^{-2}(3x)^5 + (3x)^6$$
$$= x^{-12} + 18x^{-9} + 135x^{-6} + 540x^{-3} + 1215 + 1458x^3 + 729x^6$$

Exercise

Use the binomial theorem to expand and simplify: $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^5$

Solution

$$\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^5=\left(x^{1/2}+x^{-1/2}\right)^5$$

$$=\left(x^{1/2}\right)^5+5\left(x^{1/2}\right)^4x^{-1/2}+10\left(x^{1/2}\right)^3\left(x^{-1/2}\right)^2+10\left(x^{1/2}\right)^2\left(x^{-1/2}\right)^3+5x^{1/2}\left(x^{-1/2}\right)^4+\left(x^{-1/2}\right)^5$$

$$=x^{5/2}+5x^2x^{-1/2}+10x^{3/2}x^{-1}+10xx^{-3/2}+5x^{1/2}x^{-2}+x^{-5/2}$$

$$=\underline{\left|x^{5/2}+5x^{3/2}+10x^{1/2}+10x^{-1/2}+5x^{-3/2}+x^{-5/2}\right|}$$

Exercise

Expand and simplify: $(2y-3)^4$

Solution

$$(2y-3)^4=(2y)^4+4(2y)^3(-3)+6(2y)^2(-3)^2+4(2y)(-3)^3+(-3)^4$$

$$=\underline{\left|16y^4-96y^3+216y^2-216y+81\right|}$$

Exercise

Expand and simplify: $(x+2)^5$

Solution

$$(x+2)^5=x^5+5x^4(2)+10x^3(2)^2+10x^2(2)^3+5x(2)^4+(2)^5$$

$$=\underline{\left|x^5+10x^4+40x^3+80x^2+80x+32\right|}$$

Exercise

Expand and simplify: $\left(x^2-y^2\right)^6$

Solution

$$\left(x^2-y^2\right)^6$$

$$=\left(x^2\right)^6+6\left(x^2\right)^5\left(-y^2\right)+15\left(x^2\right)^4\left(-y^2\right)^2+20\left(x^2\right)^3\left(-y^2\right)^3+15\left(x^2\right)^2\left(-y^2\right)^4+15\left(x^2\right)\left(-y^2\right)^5+\left(-y^2\right)^6$$

$$=\underline{\left|x^{12}-6x^{10}y^2+15x^8y^4-20x^6y^6+15x^4y^8-15x^2y^{10}+y^{12}\right|}$$

Exercise

Expand and simplify: $(ax-by)^4$

Solution

$$(ax-by)^4=(ax)^4+4(ax)^3(-by)+6(ax)^2(-by)^2+4(ax)(-by)^3+(-by)^4$$
$$=a^4x^4-4a^3x^3by+6a^2x^2b^2y^2-4axb^3y^3+b^4y^4$$

Exercise

Expand and simplify: $(ax+by)^5$

Solution

$$(ax+by)^5=(ax)^5+5(ax)^4(by)+10(ax)^3(by)^2+10(ax)^2(by)^3+5(ax)(by)^4+(by)^5$$
$$=a^5x^5+5a^4x^4by+10a^3x^3b^2y^2+10a^2x^2b^3y^3+5axb^4y^4+b^5y^5$$

Exercise

Expand and simplify: $\left(\sqrt{x}-\sqrt{3}\right)^4$

Solution

$$\left(\sqrt{x}-\sqrt{3}\right)^4=\left(\sqrt{x}\right)^4+4\left(\sqrt{x}\right)^3\left(-\sqrt{3}\right)+6\left(\sqrt{x}\right)^2\left(-\sqrt{3}\right)^2+4\left(\sqrt{x}\right)\left(-\sqrt{3}\right)^3+\left(-\sqrt{3}\right)^4$$
$$=x^2-4x\sqrt{3x}+18x^2-13\sqrt{3x}+9$$

Exercise

Expand and simplify: $\left(\sqrt{x}-\sqrt{2}\right)^6$

Solution

$$\left(\sqrt{x}-\sqrt{2}\right)^6=\left(\sqrt{x}\right)^6+6\left(\sqrt{x}\right)^5\left(-\sqrt{2}\right)+15\left(\sqrt{x}\right)^4\left(-\sqrt{2}\right)^2+20\left(\sqrt{x}\right)^3\left(-\sqrt{2}\right)^3$$
$$+15\left(\sqrt{x}\right)^2\left(-\sqrt{2}\right)^4+15\left(\sqrt{x}\right)\left(-\sqrt{2}\right)^5+\left(-\sqrt{2}\right)^6$$
$$=x^3-6x^2\sqrt{2x}+30x^2-40x\sqrt{2x}+60x-60\sqrt{2x}+8$$

Exercise

Expand and simplify: $(2x-1)^{12}$

Solution

$$(2x-1)^{12} = (2x)^{12} + 12(2x)^{11}(-1) + 66(2x)^{10}(-1)^2 + 240(2x)^9(-1)^3 + 535(2x)^8(-1)^4$$
$$+ 812(2x)^7(-1)^5 + 924(2x)^6(-1)^6 + 812(2x)^5(-1)^7 + 535(2x)^4(-1)^8$$
$$+ 240(2x)^3(-1)^9 + 66(2x)^2(-1)^{10} + 12(2x)(-1)^{11} + (-1)^{12}$$
$$= 4096x^{12} - 24576x^{11} + 67584x^{10} - 122880x^9 + 136960x^8 - 103936x^7$$
$$+ 59136x^6 - 25984x^5 + 8560x^4 - 1920x^3 + 264x^2 - 24x + 1$$

Exercise

Expand and simplify: $\left(x - \frac{1}{x^2}\right)^9$

Solution

$$\left(x - \frac{1}{x^2}\right)^9 = x^9 + 9x^8\left(-\frac{1}{x^2}\right) + 36x^7\left(-\frac{1}{x^2}\right)^2 + 84x^6\left(-\frac{1}{x^2}\right)^3 + 126x^5\left(-\frac{1}{x^2}\right)^4 + 126x^4\left(-\frac{1}{x^2}\right)^5$$
$$+ 84x^3\left(-\frac{1}{x^2}\right)^6 + 36x^2\left(-\frac{1}{x^2}\right)^7 + 9x\left(-\frac{1}{x^2}\right)^8 + \left(-\frac{1}{x^2}\right)^9$$
$$= x^9 - 9x^6 + 36x^3 - 84 + 126x^{-3} - 126x^{-6} + 84x^{-9} - 36x^{-12} + 9x^{-15} - x^{-18}$$

Exercise

Expand and simplify: $\left(\frac{2}{x} - 3y\right)^5$

Solution

$$\left(\frac{2}{x} - 3y\right)^5 = \left(\frac{2}{x}\right)^5 + 5\left(\frac{2}{x}\right)^4(-3y) + 10\left(\frac{2}{x}\right)^3(-3y)^2 + 10\left(\frac{2}{x}\right)^2(-3y)^3 + 5\left(\frac{2}{x}\right)(-3y)^4 + (-3y)^5$$
$$= \frac{32}{x^5} - 240\frac{y}{x^4} + 720\frac{y^2}{x^3} - 1{,}080\frac{y^3}{x^2} + 810\frac{y^4}{x} - 243y^5$$

Exercise

Expand and simplify: $\left(3\sqrt{x} + \sqrt[4]{x}\right)^4$

Solution

$$\left(3\sqrt{x}+\sqrt[4]{x}\right)^4 = \left(3\sqrt{x}\right)^4 + 4\left(3\sqrt{x}\right)^3\left(\sqrt[4]{x}\right) + 6\left(3\sqrt{x}\right)^2\left(\sqrt[4]{x}\right)^2 + 4\left(3\sqrt{x}\right)\left(\sqrt[4]{x}\right)^3 + \left(\sqrt[4]{x}\right)^4$$

$$= 81x^2 + 108x^{3/2}x^{1/4} + 54x\sqrt{x} + 12x^{1/2}x^{3/4} + x$$

$$= 81x^2 + 108x^{7/4} + 54x\sqrt{x} + 12x^{5/4} + x$$

$$= \underline{\left| 81x^2 + 108x\sqrt[4]{x^3} + 54x\sqrt{x} + 12x\sqrt[4]{x} + x \right|}$$

Exercise

Expand and simplify: $(x+1)^5$

Solution

$$(x+1)^5 \quad \underline{\left| = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 \right|}$$

Exercise

Expand and simplify: $(x-1)^5$

Solution

$$(x-1)^5 \quad \underline{\left| = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1 \right|}$$

Exercise

Expand and simplify: $(x-2)^6$

Solution

$$(x-2)^6 \quad \underline{\left| = x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64 \right|}$$

Exercise

Expand and simplify: $\left(\frac{1}{x^3} - 2x\right)^5$

Solution

$$\left(\frac{1}{x^3} - 2x\right)^5 = \frac{1}{x^{15}} - 10\frac{x}{x^{12}} + 10\frac{4x^2}{x^9} - 10\frac{8x^3}{x^6} + 5\frac{16x^4}{x^3} - 32x^5$$

$$\underline{\left| = \frac{1}{x^{15}} - \frac{10}{x^{11}} + \frac{40}{x^7} - \frac{80}{x^3} + 80x - 32x^5 \right|}$$