Solutions Section 8.4 – Solving Trigonometric Equations

Exercise

Find all solutions of the equation: $\sin x = \frac{\sqrt{2}}{2}$

Solution

$$\sin x = \frac{\sqrt{2}}{2}$$

$$\hat{x} = \sin^{-1}\frac{\sqrt{2}}{2}$$

$$= 45^\circ \qquad\qquad x \in QI, QII$$

$$x = 45^\circ \quad \rightarrow \quad \boxed{x = 45^\circ + 360^\circ k}$$

$$x = 180^\circ - 45^\circ = 135^\circ \quad \rightarrow \quad \boxed{x = 135^\circ + 360^\circ k}$$

Exercise

Find all solutions of the equation: $\cos x = -\frac{\pi}{3}$

Solution

$\cos x = -\frac{\pi}{3} < -1$ has ***no solution*** $([-1,1])$

Exercise

Find all solutions of the equation: $2\cos\theta - \sqrt{3} = 0$

Solution

$$2\cos\theta = \sqrt{3}$$

$$\cos\theta = \frac{\sqrt{3}}{2} \qquad\qquad \theta \in QI,\ QIV$$

$$\theta = \frac{\pi}{6} \quad \rightarrow \quad \boxed{\theta = \frac{\pi}{6} + 2\pi k}$$

$$\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \quad \rightarrow \quad \boxed{\theta = \frac{11\pi}{6} + 2\pi k}$$

Exercise

Find all solutions of the equation: $2\cos 2\theta - \sqrt{3} = 0$

Solution

$$2\cos 2\theta = \sqrt{3}$$

$$\cos 2\theta = \frac{\sqrt{3}}{2} \qquad\qquad \theta \in QI, QIV$$

$$2\theta = \frac{\pi}{6} \quad \rightarrow \quad \boxed{\theta = \frac{\pi}{12} + \pi n}$$

$$2\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \quad \rightarrow \quad \boxed{\theta = \frac{11\pi}{12} + \pi n}$$

Exercise

Find all solutions of the equation: $\sqrt{3}\tan\frac{1}{3}x = 1$

Solution

$$\tan\frac{1}{3}x = \frac{1}{\sqrt{3}}$$

$$\frac{1}{3}x = \frac{\pi}{6} \;\; or \;\; \frac{7\pi}{6}$$

$$\frac{1}{3}x = \frac{\pi}{6} + \pi n$$

$$\boxed{x = \frac{\pi}{2} + 3\pi n}$$

Exercise

Find all solutions of the equation: $\cos\left(4x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Solution

$$\cos\left(4x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \cos\frac{\pi}{4}$$

$$4x - \frac{\pi}{4} = \frac{\pi}{4} + 2\pi k$$

$$4x = \frac{\pi}{2} + 2\pi k$$

$$\boxed{x = \frac{\pi}{4} + \frac{\pi}{2}k}$$

$$\cos\left(4x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \cos\frac{7\pi}{4}$$

$$4x - \frac{\pi}{4} = \frac{7\pi}{4} + 2\pi k$$

$$4x = 2\pi + 2\pi k$$

$$\boxed{x = \frac{\pi}{2}k}$$

Exercise

Find all solutions of the equation: $(\cos\theta - 1)(\sin\theta + 1) = 0$

Solution

$$\cos\theta - 1 = 0$$

$$\cos\theta = 1$$

$$\boxed{\theta = 0^\circ + 360^\circ k}$$

$$\sin\theta + 1 = 0$$

$$\sin\theta = -1$$

$$\boxed{\theta = 270^\circ + 360^\circ k}$$

Exercise

Find all solutions of the equation: $\cot^2 x - 3 = 0$

Solution

$$\cot^2 x = 3$$

$$\cot x = \pm\sqrt{3}$$

$$\boxed{x = \frac{\pi}{6} + 2\pi k, \quad \frac{5\pi}{6} + 2\pi k, \quad \frac{7\pi}{6} + 2\pi k, \quad \frac{11\pi}{6} + 2\pi k}$$

Or

$$\boxed{x = \frac{\pi}{6} + \pi n, \quad \frac{5\pi}{6} + \pi n}$$

Exercise

Find all solutions of the equation: $\cos x + 1 = 2\sin^2 x$

Solution

$$\cos x + 1 = 2\left(1 - \cos^2 x\right)$$

$$\cos x + 1 = 2 - 2\cos^2 x$$

$$\cos x + 1 - 2 + 2\cos^2 x = 0$$

$$2\cos^2 x + \cos x - 1 = 0$$

$\cos x = -1$ $x = \pi + 2\pi n$	$\cos x = \frac{1}{2}$ $x = \frac{\pi}{3} + 2\pi n; \quad x = \frac{5\pi}{3} + 2\pi n$

Exercise

Find all solutions of the equation: $\cos(\ln x) = 0$

Solution

$$\cos(\ln x) = 0 \quad \rightarrow \quad \begin{cases} \ln x = \frac{\pi}{2} + 2\pi k \\ \ln x = \frac{3\pi}{2} + 2\pi k \end{cases}$$

$$\ln x = \frac{\pi}{2} + \pi n$$

$$x = e^{\pi/2 + \pi n}$$

Exercise

Find the solutions of the equation that are in the interval $[0,\ 2\pi)$: $2\sin^2 x = 1 - \sin x$

Solution

$$2\sin^2 x + \sin x - 1 = 0$$

$\sin x = -1$ $x = \frac{3\pi}{2}$	$\sin x = \frac{1}{2}$ $x = \frac{\pi}{6}; \quad x = \frac{5\pi}{6}$

Exercise

Find the solutions of the equation that are in the interval $[0,\ 2\pi)$: $\tan^2 x \sin x = \sin x$

Solution

$$\tan^2 x \sin x - \sin x = 0$$

$$\sin x\left(\tan^2 x - 1\right) = 0$$

$\sin x = 0$ $x = 0; \quad x = \pi$	$\tan^2 x - 1 = 0 \Rightarrow \tan^2 x = 1$ $\tan x = \pm 1$ $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Exercise

Find the solutions of the equation that are in the interval $[0,\ 2\pi)$: $1-\sin x=\sqrt{3}\cos x$

Solution

$$(1-\sin x)^2=\left(\sqrt{3}\cos x\right)^2$$

$$1-2\sin x+\sin^2 x=3\cos^2 x$$

$$1-2\sin x+\sin^2 x=3\left(1-\sin^2 x\right)$$

$$1-2\sin x+\sin^2 x=3-3\sin^2 x$$

$$1-2\sin x+\sin^2 x-3+3\sin^2 x=0$$

$$4\sin^2 x-2\sin x-2=0$$

$\sin x=1$	$\sin x=-\frac{1}{2}$	
$x=\frac{\pi}{2}\rightarrow(\textit{check})$	$x=\frac{7\pi}{6}$	$x=\frac{11\pi}{6}$
$1-\sin\frac{\pi}{2}=\sqrt{3}\cos\frac{\pi}{2}$	$1-\sin\frac{7\pi}{6}=\sqrt{3}\cos\frac{7\pi}{6}$	$1-\sin\frac{11\pi}{6}=\sqrt{3}\cos\frac{11\pi}{6}$
$1-(1)=\sqrt{3}(0)$	$1-\left(-\frac{1}{2}\right)=\sqrt{3}\left(-\frac{\sqrt{3}}{2}\right)$	$1-\left(-\frac{1}{2}\right)=\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)$
$0=0$ ✓	$\frac{3}{2}=-\frac{3}{2}$	$\frac{3}{2}=\frac{3}{2}$ ✓

The solutions are: $x=\frac{\pi}{2},\ \frac{11\pi}{6}$

Exercise

Find the solutions of the equation that are in the interval $[0,\ 2\pi)$: $\sin x+\cos x\cot x=\csc x$

Solution

$$\sin x+\cos x\frac{\cos x}{\sin x}=\frac{1}{\sin x}$$ ***Multiply by* sinx *both sides*** $(\sin x\neq 0)$

$$\sin^2 x+\cos^2 x=1$$

$1=1$ ***(True)***

The solutions are: $x\in[0,\ 2\pi)$ except 0 and π.

Exercise

Find the solutions of the equation that are in the interval $[0,\ 2\pi)$: $2\sin^3 x + \sin^2 x - 2\sin x - 1 = 0$

Solution

$\sin^2 x(2\sin x + 1) - (2\sin x + 1) = 0$ *Factor by grouping*

$(2\sin x + 1)(\sin^2 x - 1) = 0$

$2\sin x + 1 = 0$ $\sin x = -\frac{1}{2}$ $x = \frac{7\pi}{6},\ \frac{11\pi}{6}$	$\sin^2 x = 1$ $\sin x = \pm 1$ $x = \frac{\pi}{2},\ \frac{3\pi}{2}$

Exercise

Find the solutions of the equation that are in the interval $[0,\ 2\pi)$: $2\tan x \csc x + 2\csc x + \tan x + 1 = 0$

Solution

$2\tan x \csc x + \tan x + 2\csc x + 1 = 0$

$\tan x(2\csc x + 1) + (2\csc x + 1) = 0$

$(2\csc x + 1)(\tan x + 1) = 0$

$2\csc x + 1 = 0$ $\csc x = -\frac{1}{2} = \frac{1}{\sin x}$ $\sin x = -2\ \ (impossible)$	$\tan x = -1$ $x = \frac{3\pi}{4},\ \frac{7\pi}{4}$

Exercise

Solve $2\cos\theta + \sqrt{3} = 0$ $\quad$ *if* $0° \le \theta < 360°$

Solution

$2\cos\theta = -\sqrt{3}$

$\cos\theta = -\frac{\sqrt{3}}{2}$

$\hat{\theta} = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$\theta = 150°,\ \ 210°$

Exercise

Solve $5\cos t + \sqrt{12} = \cos t \qquad if \;\; 0 \le t < 2\pi$

Solution

$$5\cos t - \cos t = -\sqrt{12} \qquad\qquad \sqrt{12} = \sqrt{4.3} = 2\sqrt{3}$$

$$4\cos t = -2\sqrt{3}$$

$$4\cos t = -2\sqrt{3}$$

$$\cos t = -\frac{2\sqrt{3}}{4}$$

$$\cos t = -\frac{\sqrt{3}}{2} \quad \Rightarrow \; t = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\underline{\theta = \frac{5\pi}{6}, \;\; \frac{7\pi}{6}}$$

Exercise

Solve $\tan\theta - 2\cos\theta\tan\theta = 0 \qquad if \;\; 0° \le \theta < 360°$

Solution

$$\tan\theta(1 - 2\cos\theta) = 0$$

$$\tan\theta = 0 \qquad\qquad 1 - 2\cos\theta = 0$$

$$\theta = 0°, \; 180° \qquad\qquad 1 = 2\cos\theta$$

$$\cos\theta = \frac{1}{2} \quad \Rightarrow \; \theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = 60°, \; 300°$$

$$\underline{\theta = 0°, \;\; 60°, \;\; 180°, \;\; 300°}$$

Exercise

Solve $2\sin^2\theta - 2\sin\theta - 1 = 0 \qquad if \;\; 0° \le \theta < 360°$

Solution

$$\sin\theta = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)} \qquad\qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{12}}{4}$$

$$= \frac{2 \pm 2\sqrt{3}}{4}$$

$$=\frac{1\pm\sqrt{3}}{2}$$

$$\hat{\theta}=\sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right)$$

$$\underline{=-21.47^\circ}$$

$$\sin\theta=\frac{1+\sqrt{3}}{2}=1.366>1$$

$$\theta=360^\circ-21.47^\circ\ \underline{=338.53^\circ}$$

$$\theta=180^\circ+21.47^\circ\ \underline{=201.47^\circ}$$

The solutions are: $\underline{\theta=338.53^\circ,\ \ 201.47^\circ}$

Exercise

Solve $\cos\left(A-\frac{\pi}{9}\right)=-\frac{1}{2}$

Solution

$-\frac{1}{2}$ *is negative* $\rightarrow$ *cosine is in QII or QIII.*

$$\cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$$

$$\cos\left(A-\frac{\pi}{9}\right)=-\frac{1}{2}=\cos\frac{2\pi}{3}$$

$$A-\frac{\pi}{9}=\frac{2\pi}{3}+2\pi k$$

$$A=\frac{2\pi}{3}+\frac{\pi}{9}+2\pi k$$

$$\underline{A=\frac{7\pi}{9}+2\pi k}$$

$$\cos^{-1}\left(-\frac{1}{2}\right)=\frac{4\pi}{3}$$

$$\cos\left(A-\frac{\pi}{9}\right)=-\frac{1}{2}=\cos\frac{4\pi}{3}$$

$$A-\frac{\pi}{9}=\frac{4\pi}{3}+2\pi k$$

$$A=\frac{4\pi}{3}+\frac{\pi}{9}+2\pi k$$

$$\underline{A=\frac{13\pi}{9}+2\pi k}$$

Exercise

Solve: $4\cos\theta-3\sec\theta=0\ \ if\ \ 0^\circ\le\theta<360^\circ$

Solution

$$4\cos\theta-3\frac{1}{\cos\theta}=0$$

$$\cos\theta\neq0 \quad\rightarrow\quad \underline{\theta\neq\frac{\pi}{2},\ \frac{3\pi}{2}}$$

$$4\cos\theta\cos\theta-3\frac{1}{\cos\theta}\cos\theta=0$$

$$4\cos^2\theta-3=0$$

$$4\cos^2\theta=3$$

$\cos^2\theta = \frac{3}{4}$

$\cos\theta = \pm\frac{\sqrt{3}}{2}$

$\theta = \cos^{-1}\left(\pm\frac{\sqrt{3}}{2}\right)$

The solutions are: $\underline{\theta = 30°,\ \ 150°,\ \ 210°,\ \ 330°}$

Exercise

Solve: $2\sin^2 x - \cos x - 1 = 0 \ \ if \ \ 0 \le x < 2\pi$

Solution

$2\left(1-\cos^2 x\right) - \cos x - 1 = 0$

$2 - 2\cos^2 x - \cos x - 1 = 0$

$-2\cos^2 x - \cos x + 1 = 0$

$\cos x = -1 \qquad\qquad \cos x = \frac{1}{2}$

$x = \pi \qquad\qquad x = \frac{\pi}{3},\ \frac{5\pi}{3}$

The solutions are: $\underline{x = \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}}$

Exercise

Solve: $\sin\theta - \sqrt{3}\cos\theta = 1 \qquad if \ \ 0° \le \theta < 360°$

Solution

$\sin\theta - 1 = -\sqrt{3}\cos\theta$

$(\sin\theta - 1)^2 = \left(-\sqrt{3}\cos\theta\right)^2$

$\sin^2\theta - 2\sin\theta + 1 = 3\cos^2\theta \qquad\qquad \cos^2\theta = 1 - \sin^2\theta$

$\sin^2\theta - 2\sin\theta + 1 = 3\left(1 - \sin^2\theta\right)$

$\sin^2\theta - 2\sin\theta + 1 = 3 - 3\sin^2\theta$

$\sin^2\theta - 2\sin\theta + 1 - 3 + 3\sin^2\theta = 0$

$4\sin^2\theta - 2\sin\theta - 2 = 0$

$\sin\theta = 1 \quad \Rightarrow \quad \underline{\theta = 90°}$

$\sin\theta = -\frac{1}{2} \quad \Rightarrow \quad \underline{\theta = 210°,\ 330°}$

Check

$\theta = 90°$	$\theta = 210°$	$\theta = 330°$
$\sin 90° - \sqrt{3}\cos 90° \overset{?}{=} 1$	$\sin 210° - \sqrt{3}\cos 210° \overset{?}{=} 1$	$\sin 330° - \sqrt{3}\cos 330° \overset{?}{=} 1$
$1 - \sqrt{3}(0) \overset{?}{=} 1$	$-\frac{1}{2} - \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) \overset{?}{=} 1$	$-\frac{1}{2} - \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) \overset{?}{=} 1$
$1 = 1$ ✓	$-\frac{1}{2} + \frac{3}{2} \overset{?}{=} 1$	$-\frac{1}{2} - \frac{3}{2} \overset{?}{=} 1$
	$1 = 1$ ✓	$-2 \neq 1$
		(*False statement*)

The solutions are: $\underline{90°,\ 210°}$

Exercise

Solve: $\quad 7\sin^2\theta - 9\cos 2\theta = 0 \qquad$ *if* $\ 0° \le \theta < 360°$

Solution

$$7\sin^2\theta - 9\left(1 - 2\sin^2\theta\right) = 0 \qquad\qquad \cos^2\theta = 1 - 2\sin^2\theta$$

$$7\sin^2\theta - 9 + 18\sin^2\theta = 0$$

$$25\sin^2\theta - 9 = 0$$

$$25\sin^2\theta = 9$$

$$\sin^2\theta = \frac{9}{25} \quad \Rightarrow \quad \sin\theta = \pm\frac{3}{5}$$

$$\hat{\theta} = \sin^{-1}\left(\frac{3}{5}\right) \approx 36.87°$$

$$\theta \approx 36.87°$$

$$\theta \approx 180° - 36.87° \approx 143.13°$$

$$\theta \approx 180° + 36.87° \approx 216.87°$$

$$\theta \approx 360° - 36.87° \approx 323.13°$$

The solutions are: $\underline{36.87°,\ 143.13°,\ 216.87°,\ 323.13°}$

Exercise

Solve: $2\cos^2 t - 9\cos t = 5$ *if* $0 \le t < 2\pi$

Solution

$$2\cos^2 t - 9\cos t - 5 = 0$$

$$(2\cos t + 1)(\cos t - 5) = 0$$

$2\cos t + 1 = 0$ $\cos t - 5 = 0$

$\cos t = -\frac{1}{2}$ $\cos t = 5$

$\cos t = -\frac{1}{2}$ $\cos t = 5$

$\hat{t} = \cos^{-1}\left(-\frac{1}{2}\right)$ *No solution*

$\hat{t} = \frac{\pi}{3}$

Negative sign $\rightarrow$ *cosine is in QII or QIII*

$t = \pi - \frac{\pi}{3}$ $t = \pi + \frac{\pi}{3}$

$t = \frac{2\pi}{3}$ $t = \frac{4\pi}{3}$

The solutions are: $\frac{2\pi}{3},\ \frac{4\pi}{3}$

Exercise

Solve $\sin\theta\tan\theta = \sin\theta$ *if* $0° \le \theta < 360°$

Solution

$$\sin\theta\tan\theta - \sin\theta = 0$$

$$\sin\theta\left(\tan\theta - 1\right) = 0$$

$\sin\theta = 0$ $\tan\theta - 1 = 0$

$\theta = 0°,\ 180°$ $\tan\theta = 1$

$\theta = 45°,\ 225°$

The solutions are: $0°,\ 45°,\ 180°,\ 225°$

Exercise

Solve $\tan^2 x + \tan x - 2 = 0 \quad if \;\; 0 \le x < 2\pi$

Solution

$$\tan^2 x + \tan x - 2 = 0$$

$\tan x = 1 \qquad\qquad \tan x = -2$

$x = \frac{\pi}{4}, \frac{5\pi}{4} \qquad \hat{x} = \tan^{-1}(2) \approx 1.107 \qquad x \in QII,\ QIV$

$x = 2.034,\ 5.176$

The solutions are: $\frac{\pi}{4},\ \frac{5\pi}{4},\ 2.034,\ 5.176$

Exercise

Solve $\tan x + \sqrt{3} = \sec x \qquad if \;\; 0 \le x < 2\pi$

Solution

$$\left(\tan x + \sqrt{3}\right)^2 = (\sec x)^2$$

$$\tan^2 x + 2\sqrt{3}\tan x + 3 = \sec^2 x$$

$$\tan^2 x + 2\sqrt{3}\tan x + 3 = 1 + \tan^2 x$$

$$\tan^2 x + 2\sqrt{3}\tan x + 3 - 1 - \tan^2 x = 0$$

$$2\sqrt{3}\tan x + 2 = 0$$

$$2\sqrt{3}\tan x = -2$$

$$\tan x = -\frac{2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$$x = \frac{5\pi}{6} \quad or \quad x = \frac{11\pi}{6}$$

$\tan\frac{5\pi}{6} + \sqrt{3} \stackrel{?}{=} \sec\frac{5\pi}{6}$ $-\frac{\sqrt{3}}{3} + \sqrt{3} \stackrel{?}{=} -\frac{2\sqrt{3}}{3}$ $\frac{2\sqrt{3}}{3} \ne -\frac{2\sqrt{3}}{3}$ ***False***	$\tan\frac{11\pi}{6} + \sqrt{3} \stackrel{?}{=} \sec\frac{11\pi}{6}$ $-\frac{\sqrt{3}}{3} + \sqrt{3} \stackrel{?}{=} \frac{2\sqrt{3}}{3}$ $\frac{2\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$

The solutions are: $\frac{11\pi}{6}$

Exercise

Solve $2\cos\theta + \sqrt{3} = 0 \qquad if \;\; 0° \le \theta < 360°$

Solution

$$2\cos\theta = -\sqrt{3}$$

$$\cos\theta = -\frac{\sqrt{3}}{2}$$

$$\hat{\theta} = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\underline{\theta = 150°, \;\; 210° \;}|$$

Exercise

Solve $5\cos t + \sqrt{12} = \cos t \qquad if \;\; 0 \le t < 2\pi$

Solution

$$5\cos t - \cos t = -\sqrt{12}$$

$$4\cos t = -2\sqrt{3}$$

$$4\cos t = -2\sqrt{3}$$

$$\cos t = -\frac{2\sqrt{3}}{4}$$

$$\cos t = -\frac{\sqrt{3}}{2}$$

$$t = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\underline{\theta = \frac{5\pi}{6}, \;\; \frac{7\pi}{6} \;}|$$

Exercise

Solve $\tan\theta - 2\cos\theta\tan\theta = 0 \qquad if \;\; 0° \le \theta < 360°$

Solution

$$\tan\theta(1 - 2\cos\theta) = 0$$

$\tan\theta = 0$

$\theta = 0°, \; 180°$

$1 - 2\cos\theta = 0$

$1 = 2\cos\theta$

$\cos\theta = \frac{1}{2} \;\; \Rightarrow \;\; \theta = \cos^{-1}\left(\frac{1}{2}\right)$

$\theta = 60°, \; 300°$

$$\underline{\theta = 0°, \;\; 60°, \;\; 180°, \;\; 300° \;}|$$

Exercise

Solve $2\sin^2\theta - 2\sin\theta - 1 = 0 \qquad if \ \ 0° \le \theta < 360°$

Solution

$$\sin\theta = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{2 \pm \sqrt{12}}{4}$$

$$= \frac{2 \pm 2\sqrt{3}}{4}$$

$$= \frac{1 \pm \sqrt{3}}{2}$$

$$\hat{\theta} = \sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right) = -21.47° \qquad\qquad \sin\theta = \frac{1+\sqrt{3}}{2} = 1.366 > 1$$

$$\theta = 360° - 21.47° \ \underline{= 338.53°}$$

$$\theta = 180° + 21.47° \ \underline{= 201.47°}$$

Exercise

Solve $\cos\left(A - \frac{\pi}{9}\right) = -\frac{1}{2}$

Solution

$-\frac{1}{2}$ *is negative* $\rightarrow$ *cosine is in QII or QIII.*

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \qquad\qquad \cos^{-1}\left(-\frac{1}{2}\right) = \frac{4\pi}{3}$$

$$\cos\left(A - \frac{\pi}{9}\right) = -\frac{1}{2} = \cos\frac{2\pi}{3} \qquad\qquad \cos\left(A - \frac{\pi}{9}\right) = -\frac{1}{2} = \cos\frac{4\pi}{3}$$

$$A - \frac{\pi}{9} = \frac{2\pi}{3} + 2\pi k \qquad\qquad A - \frac{\pi}{9} = \frac{4\pi}{3} + 2\pi k$$

$$A = \frac{2\pi}{3} + \frac{\pi}{9} + 2\pi k \qquad\qquad A = \frac{4\pi}{3} + \frac{\pi}{9} + 2\pi k$$

$$\underline{A = \frac{7\pi}{9} + 2\pi k} \qquad\qquad \underline{A = \frac{13\pi}{9} + 2\pi k}$$

Exercise

Solve: $4\cos\theta - 3\sec\theta = 0 \quad if \quad 0° \le \theta < 360°$

Solution

$$4\cos\theta - 3\frac{1}{\cos\theta} = 0 \qquad \boxed{\cos\theta \ne 0}$$

$$4\cos\theta\cos\theta - 3\frac{1}{\cos\theta}\cos\theta = 0$$

$$4\cos^2\theta - 3 = 0$$

$$4\cos^2\theta = 3$$

$$\cos^2\theta = \frac{3}{4}$$

$$\cos\theta = \pm\frac{\sqrt{3}}{2}$$

$$\theta = \cos^{-1}\left(\pm\frac{\sqrt{3}}{2}\right)$$

The solutions are: $\theta = 30°,\ 150°,\ 210°,\ 330°$

Exercise

Solve: $2\sin^2 x - \cos x - 1 = 0 \quad if \quad 0 \le x < 2\pi$

Solution

$$2\left(1 - \cos^2 x\right) - \cos x - 1 = 0$$

$$2 - 2\cos^2 x - \cos x - 1 = 0$$

$$-2\cos^2 x - \cos x + 1 = 0$$

$$\cos x = -1 \qquad \cos x = \frac{1}{2}$$

$$x = \pi \qquad x = \frac{\pi}{3},\ \frac{5\pi}{3}$$

The solutions are: $x = \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}$

Exercise

Solve: $\sin\theta - \sqrt{3}\cos\theta = 1$ *if* $0° \le \theta < 360°$

Solution

$$\sin\theta - 1 = -\sqrt{3}\cos\theta$$

$$(\sin\theta - 1)^2 = \left(-\sqrt{3}\cos\theta\right)^2$$

$$\sin^2\theta - 2\sin\theta + 1 = 3\cos^2\theta \qquad \cos^2\theta = 1 - \sin^2\theta$$

$$\sin^2\theta - 2\sin\theta + 1 = 3\left(1 - \sin^2\theta\right)$$

$$\sin^2\theta - 2\sin\theta + 1 - 3 + 3\sin^2\theta = 0$$

$$4\sin^2\theta - 2\sin\theta - 2 = 0$$

$\sin\theta = 1$ $\qquad$ $\sin\theta = -\frac{1}{2}$

$\theta = 90°$ $\qquad$ $\theta = 210°,\ 330°$

Check

$\theta = 90°$	$\theta = 210°$	$\theta = 330°$
$\sin 90° - \sqrt{3}\cos 90° \stackrel{?}{=} 1$	$\sin 210° - \sqrt{3}\cos 210° \stackrel{?}{=} 1$	$\sin 330° - \sqrt{3}\cos 330° \stackrel{?}{=} 1$
$1 - \sqrt{3}(0) \stackrel{?}{=} 1$	$-\frac{1}{2} - \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) \stackrel{?}{=} 1$	$-\frac{1}{2} - \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) \stackrel{?}{=} 1$
$1 = 1$	$-\frac{1}{2} + \frac{3}{2} \stackrel{?}{=} 1$	$-\frac{1}{2} - \frac{3}{2} \stackrel{?}{=} 1$
	$1 = 1$	$-2 \ne 1$ (***False statement***)

The solutions are: $\underline{90°,\ 210°}$

Exercise

Solve: $7\sin^2\theta - 9\cos 2\theta = 0$ *if* $0° \le \theta < 360°$

Solution

$$7\sin^2\theta - 9\left(1 - 2\sin^2\theta\right) = 0 \qquad \cos^2\theta = 1 - 2\sin^2\theta$$

$$7\sin^2\theta - 9 + 18\sin^2\theta = 0$$

$$25\sin^2\theta - 9 = 0$$

$$25\sin^2\theta = 9$$

$$\sin^2\theta = \frac{9}{25}$$

$$\sin\theta = \pm\frac{3}{5}$$

$\hat{\theta} = \sin^{-1}\left(\frac{3}{5}\right)$

$\quad \approx 36.87°$

$\theta \approx 36.87°$

$\theta \approx 180° - 36.87° \approx 143.13°$

$\theta \approx 180° + 36.87° \approx 216.87°$

$\theta \approx 360° - 36.87° \approx 323.13°$

The solutions are: $36.87°,\ 143.13°,\ 216.87°,\ 323.13°$

Exercise

Solve: $2\cos^2 t - 9\cos t = 5 \qquad \text{if } 0 \le t < 2\pi$

Solution

$2\cos^2 t - 9\cos t - 5 = 0$

$(2\cos t + 1)(\cos t - 5) = 0$

$2\cos t + 1 = 0 \qquad \cos t - 5 = 0$

$\cos t = -\frac{1}{2} \qquad \cos t = 5$

$\cos t = -\frac{1}{2} \qquad \cos t = 5$

$\hat{t} = \cos^{-1}\left(-\frac{1}{2}\right) \qquad$ *No solution*

$\hat{t} = \frac{\pi}{3}$

Negative sign $\rightarrow$ cosine is in QII or QIII

$t = \pi - \frac{\pi}{3} \qquad t = \pi + \frac{\pi}{3}$

$t = \frac{2\pi}{3} \qquad t = \frac{4\pi}{3}$

The solutions are: $\frac{2\pi}{3},\ \frac{4\pi}{3}$

Exercise

Solve $\sin\theta\tan\theta = \sin\theta \qquad \text{if } 0° \le \theta < 360°$

Solution

$\sin\theta\tan\theta - \sin\theta = 0$

$\sin\theta(\tan\theta - 1) = 0$

$$\sin\theta = 0 \qquad\qquad \tan\theta - 1 = 0$$
$$\theta = 0°,\ 180° \qquad\qquad \tan\theta = 1$$
$$\theta = 45°,\ 225°$$

The solutions are: $\underline{0°,\ \ 45°,\ \ 180°,\ \ 225°}$

Exercise

Solve $\tan^2 x + \tan x - 2 = 0 \quad if \ \ 0 \le x < 2\pi$

Solution

$$\tan^2 x + \tan x - 2 = 0$$

$$\tan x = 1 \qquad\qquad \tan x = -2$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4} \qquad\qquad \hat{x} = \tan^{-1}(2) \approx 1.107 \qquad\qquad x \in QII,\ QIV$$

$$x = 2.034,\ 5.176$$

The solutions are: $\underline{\frac{\pi}{4},\ \ \frac{5\pi}{4},\ \ 2.034,\ \ 5.176}$

Exercise

Solve $\tan x + \sqrt{3} = \sec x \qquad\qquad if \ \ 0 \le x < 2\pi$

Solution

$$\left(\tan x + \sqrt{3}\right)^2 = (\sec x)^2$$

$$\tan^2 x + 2\sqrt{3}\tan x + 3 = \sec^2 x$$

$$\tan^2 x + 2\sqrt{3}\tan x + 3 = 1 + \tan^2 x$$

$$\tan^2 x + 2\sqrt{3}\tan x + 3 - 1 - \tan^2 x = 0$$

$$2\sqrt{3}\tan x + 2 = 0$$

$$2\sqrt{3}\tan x = -2$$

$$\tan x = -\frac{2}{2\sqrt{3}}$$

$$= -\frac{1}{\sqrt{3}}$$

$$x = \frac{5\pi}{6} \quad or \quad x = \frac{11\pi}{6}$$

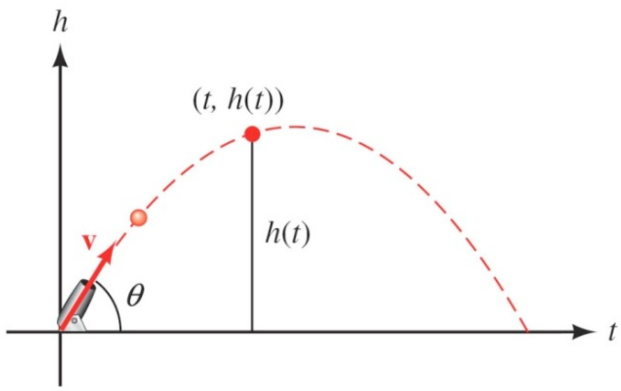
$\tan\frac{5\pi}{6}+\sqrt{3}\overset{?}{=}\sec\frac{5\pi}{6}$	$\tan\frac{11\pi}{6}+\sqrt{3}\overset{?}{=}\sec\frac{11\pi}{6}$
$-\frac{\sqrt{3}}{3}+\sqrt{3}\overset{?}{=}-\frac{2\sqrt{3}}{3}$	$-\frac{\sqrt{3}}{3}+\sqrt{3}\overset{?}{=}\frac{2\sqrt{3}}{3}$
$\frac{2\sqrt{3}}{3}\neq-\frac{2\sqrt{3}}{3}$	$\frac{2\sqrt{3}}{3}=\frac{2\sqrt{3}}{3}$
False	

The solutions are: $\underline{\frac{11\pi}{6}}$

Exercise

If a projectile (such as a bullet) is fired into the air with an initial velocity $\boldsymbol{v}$ at an angle of elevation θ, then the height h of the projectile at time t is given by:

$$h(t)=-16t^2+vt\sin\theta$$



a) Give the equation for the height, if $\boldsymbol{v}$ is 600 *ft./sec* and $\theta=45°$.

b) Use the equation in part (*a*) to find the height of the object after $\sqrt{3}$ *seconds*.

c) Find the angle of elevation of θ of a rifle barrel, if a bullet fired at 1,500 *ft./sec* takes 3 *seconds* to reach a height of 750 *feet*. Give your answer in the nearest of a degree.

Solution

a) $h(t)=-16t^2+600t\sin 45°$

$$=-16t^2+600t\frac{\sqrt{2}}{2}$$

$$=-16t^2+300\sqrt{2}\,t$$

b) $h\left(t=\sqrt{3}\right)=-16\left(\sqrt{3}\right)^2+300\sqrt{2}\,\sqrt{3}$

$$\underline{\approx 686.8\ ft}$$

c) $h(t) = -16t^2 + vt\sin\theta$

$750 = -16(3)^2 + 1500(3)\sin\theta$

$750 = -144 + 4500\sin\theta$

$750 + 144 = 4500\sin\theta$

$\frac{894}{4500} = \sin\theta$

$\theta = \sin^{-1}\left(\frac{894}{4500}\right)$

$\approx 11.5^\circ$