Solution   *Section* 1.2 – Definitions / Techniques of Limits

Exercise

Find the limit: $\lim\limits_{x \to 3} (-1)$

Solution

$$\lim_{x \to 3} (-1) \underline{= -1|}$$

Exercise

Find the limit: $\lim\limits_{x \to -1} (3)$

Solution

$$\lim_{x \to -1} (3) \underline{= 3|}$$

Exercise

Find the limit: $\lim\limits_{x \to 1000} 18\pi^2$

Solution

$$\lim_{x \to 1000} 18\pi^2 \underline{= 18\pi^2|}$$

Exercise

Find the limit: $\lim\limits_{x \to 1} \sqrt{5x+6}$

Solution

$$\lim_{x \to 1} \sqrt{5x+6} \underline{= \sqrt{11}|}$$

Exercise

Find the limit: $\lim\limits_{x \to 9} \sqrt{x}$

Solution

$$\lim_{x \to 9} \sqrt{x} = \sqrt{9} \underline{= 3|}$$

Exercise

Find the limit: $\lim\limits_{x\to -3}\left(x^2+3x\right)$

Solution

$$\lim_{x\to -3}\left(x^2+3x\right)=(-3)^2+3(-3)=9-9\ \underline{=0}$$

Exercise

Find the limit: $\lim\limits_{x\to -4}|x-4|$

Solution

$$\lim_{x\to -4}|x-4|=|-4-4|=|-8|\ \underline{=8}$$

Exercise

Find the limit: $\lim\limits_{x\to 4}(x+2)$

Solution

$$\lim_{x\to 4}(x+2)=4+2\ \underline{=6}$$

Exercise

Find the limit: $\lim\limits_{x\to 4}(x-4)$

Solution

$$\lim_{x\to 4}(x-4)=4-4\ \underline{=0}$$

Exercise

Find the limit: $\lim\limits_{x\to 2}(5x-6)^{3/2}$

Solution

$$\begin{aligned}\lim_{x\to 2}(5x-6)^{3/2}&=(10-6)^{3/2}\\&=\sqrt{4^3}\\&\underline{=8}\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to 9} \dfrac{x-9}{\sqrt{x}-3}$

Solution

$$\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} = \frac{9-9}{3-3} = \frac{0}{0}$$

$$= \lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}-3}$$

$$= \lim_{x\to 9} (\sqrt{x}+3)$$

$$= 6$$

Exercise

Find the limit: $\lim\limits_{x\to 1} (2x+4)$

Solution

$$\lim_{x\to 1} (2x+4) = 2(1)+4 = 6$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{x^2-4}{x-2}$

Solution

$$\lim_{x\to 1} \frac{x^2-4}{x-2} = \frac{1^2-4}{1-2}$$

$$= \frac{-3}{-1}$$

$$= 3$$

Exercise

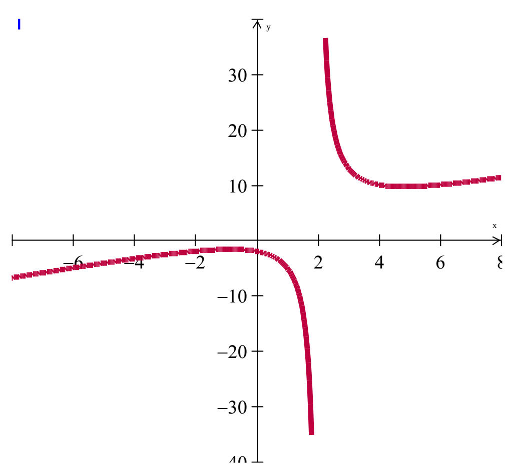
Find the limit: $\lim\limits_{x\to 2} \dfrac{x^2+4}{x-2}$

Solution

$$\lim_{x\to 2} \frac{x^2+4}{x-2} = \frac{2^2+4}{2-2}$$

$$= \frac{8}{0}$$

$= \infty$ (***Doesn't exist***)



Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{|x|}{x}$

Solution

$$\lim_{x\to 0} \frac{|x|}{x} = \frac{0}{0}$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \frac{x}{-x} = -1 \qquad\qquad \lim_{x\to 0^+} \frac{|x|}{x} = \frac{x}{x} = 1$$

Doesn't exist

Exercise

Find: $\lim\limits_{x\to 3} \dfrac{x^2 - x - 1}{\sqrt{x+1}}$

Solution

$$\lim_{x\to 3} \frac{x^2 - x - 1}{\sqrt{x+1}} = \frac{3^2 - 3 - 1}{\sqrt{3+1}}$$

$$= \frac{5}{2}$$

Exercise

Find: $\lim\limits_{x\to 2} \dfrac{x^2 + x - 6}{x - 2}$

Solution

$$\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2} = \frac{2^2 + 2 - 6}{2 - 2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(x+3)(x-2)}{x-2}$$

$$= \lim_{x\to 2} (x+3)$$

$$= 5$$

Exercise

Find the limit: $\lim\limits_{x\to 0} (3x - 2)$

Solution

$$\lim_{x\to 0} (3x - 2) = 3(0) - 2$$

$$= -2$$

Exercise

Find the limit: $\lim\limits_{x\to 1}(2x^2 - x + 4)$

Solution

$$\lim\limits_{x\to 1}(2x^2 - x + 4) = 2(1)^2 - (1) + 4$$

$$= 5$$

Exercise

Find the limit: $\lim\limits_{x\to -2}\left(x^3 - 2x^2 + 4x + 8\right)$

Solution

$$\lim\limits_{x\to -2}\left(x^3 - 2x^2 + 4x + 8\right) = (-2)^3 - 2(-2)^2 + 4(-2) + 8$$

$$= -16$$

Exercise

Find the limit: $\lim\limits_{x\to 2}\dfrac{x^2 - 4}{x - 2}$

Solution

$$\lim\limits_{x\to 2}\frac{x^2-4}{x-2} = \frac{2^2-4}{2-2} = \frac{0}{0}$$

$$= \lim\limits_{x\to 2}\frac{(x-2)(x+2)}{x-2}$$

$$= \lim\limits_{x\to 2}(x+2)$$

$$= 4$$

Exercise

Find the limit: $\lim\limits_{x\to 2}\dfrac{x^3-8}{x-2}$

Solution

$$\lim\limits_{x\to 2}\frac{x^3-8}{x-2} = \frac{0}{0}$$

$$\lim\limits_{x\to 2}\frac{x^3-8}{x-2} = \lim\limits_{x\to 2}\frac{(x-2)(x^2+2x+4)}{x-2}$$

$$= \lim\limits_{x\to 2} x^2 + 2x + 4$$

$$= 2^2 + 2(2) + 4$$

$$= 12$$

Exercise

Find the limit: $\lim\limits_{x\to 3} \frac{x^2+x-12}{x-3}$

Solution

$$\lim_{x\to 3} \frac{x^2+x-12}{x-3} = \frac{0}{0}$$

$$\lim_{x\to 3} \frac{(x-3)(x+4)}{x-3} = \lim_{x\to 3}(x+4)$$

$$\boxed{=7}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \frac{\sqrt{x+4}-2}{x}$

Solution

$$\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} = \frac{\sqrt{4}-2}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$$

$$= \lim_{x\to 0} \frac{x+4-4}{x\left(\sqrt{x+4}+2\right)}$$

$$= \lim_{x\to 0} \frac{x}{x\left(\sqrt{x+4}+2\right)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{x+4}+2}$$

$$= \frac{1}{\sqrt{4}+2}$$

$$\boxed{= \frac{1}{4}}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \frac{3}{\sqrt{3x+1}+1}$

Solution

$$\lim_{x\to 0} \frac{3}{\sqrt{3x+1}+1} = \frac{3}{\sqrt{3(0)+1}+1}$$

$$= \frac{3}{1+1}$$

$$\boxed{= \frac{3}{2}}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} f(x) \qquad f(x) = \begin{cases} x^2+1 & x<0 \\ 2x+1 & x>0 \end{cases}$

Solution

$$\lim_{x\to 0^-} x^2+1 = 1$$

$$\lim_{x\to 0+} 2x+1 = 1$$

$$\lim_{x\to 0} f(x) \ \underline{=1}$$

Exercise

Find the limit: $\lim\limits_{x\to -2} \dfrac{5}{x+2}$

Solution

$$\lim_{x\to -2} \frac{5}{x+2} = \frac{5}{0}$$

$$\underline{=\infty}$$

Exercise

Find the limit: $\lim\limits_{x\to 3} \dfrac{\sqrt{x+1}-1}{x}$

Solution

$$\lim_{x\to 3} \frac{\sqrt{x+1}-1}{x} = \frac{\sqrt{3+1}-1}{3} = \frac{2-1}{3}$$

$$\underline{=\frac{1}{3}}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x\to 1} (x+1)$$

$$\underline{=2}$$

Exercise

Find the limit: $\lim\limits_{x\to -2} \dfrac{|x+2|}{x+2}$

Solution

$$\lim_{x\to -2} \frac{|x+2|}{x+2} = \frac{|-2+2|}{-2+2} = \frac{0}{0}$$

$$\lim_{x\to -2^+} \frac{|x+2|}{x+2} = \frac{(x+2)}{(x+2)} = 1$$

$$\lim_{x\to -2^-} \frac{|x+2|}{x+2} = \frac{(x+2)}{-(x+2)} = -1$$

Doesn't exist

Exercise

Find the limit: $\lim\limits_{x\to 0} (2x-8)^{1/3}$

Solution

$$\begin{aligned}\lim_{x\to 0} (2x-8)^{1/3} &= (2(0)-8)^{1/3} \\ &= (-8)^{1/3} \\ &= -2\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to 2} \dfrac{x^2-7x+10}{x-2}$

Solution

$$\lim_{x\to 2} \frac{x^2-7x+10}{x-2} = \frac{2^2-7(2)+10}{2-2} = \frac{0}{0}$$

$$\begin{aligned}\lim_{x\to 2} \frac{x^2-7x+10}{x-2} &= \lim_{x\to 2} \frac{(x-2)(x-5)}{x-2} \\ &= \lim_{x\to 2} (x-5) \\ &= 2-5 \\ &= -3\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{5x^3+8x^2}{3x^4-16x^2}$

Solution

$$\lim_{x\to 0} \frac{5x^3+8x^2}{3x^4-16x^2} = \frac{0}{0}$$

$$\begin{aligned}\lim_{x\to 0} \frac{5x^3+8x^2}{3x^4-16x^2} &= \lim_{x\to 0} \frac{x^2(5x+8)}{x^2\left(3x^2-16\right)} \\ &= \lim_{x\to 0} \frac{5x+8}{3x^2-16} \\ &= \frac{8}{-16} \\ &= -\frac{1}{2}\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{\frac{1}{x}-1}{x-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$\begin{aligned}\lim_{x\to 1} \frac{\frac{1}{x}-1}{x-1} &= \lim_{x\to 1} \frac{\frac{1-x}{x}}{x-1} \\ &= \lim_{x\to 1} \left(\frac{1-x}{x}\right)\left(\frac{1}{x-1}\right) \\ &= \lim_{x\to 1} \left(\frac{-(x-1)}{x}\right)\left(\frac{1}{x-1}\right) \\ &= \lim_{x\to 1} \frac{-1}{x} \\ &= -1\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{u\to 1} \dfrac{u^4-1}{u^3-1}$

Solution

$$\lim_{u\to 1}\frac{u^4-1}{u^3-1}=\lim_{u\to 1}\frac{\left(u^2-1\right)\left(u^2+1\right)}{(u-1)\left(u^2+u+1\right)}$$

$$=\lim_{u\to 1}\frac{(u-1)(u+1)\left(u^2+1\right)}{(u-1)\left(u^2+u+1\right)}$$

$$=\lim_{u\to 1}\frac{(u+1)\left(u^2+1\right)}{u^2+u+1}$$

$$=\frac{(1+1)\left(1^2+1\right)}{1^2+1+1}$$

$$=\frac{4}{3}$$

Exercise

Find the limit: $\lim_{x\to 1}\dfrac{x-1}{\sqrt{x+3}-2}$

Solution

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2}=\frac{1-1}{\sqrt{1+3}-2}=\frac{0}{\sqrt{4}-2}=\frac{0}{0}$$

$$=\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2}\bullet\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$$

$$=\lim_{x\to 1}\frac{(x-1)\left(\sqrt{x+3}+2\right)}{x+3-4}$$

$$=\lim_{x\to 1}\frac{(x-1)\left(\sqrt{x+3}+2\right)}{x-1}$$

$$=\lim_{x\to 1}\left(\sqrt{x+3}+2\right)$$

$$=\sqrt{1+3}+2$$

$$=4$$

Exercise

Find the limit: $\lim_{x\to -1}\dfrac{\sqrt{x^2+8}-3}{x+1}$

Solution

$$\lim_{x\to -1}\frac{\sqrt{x^2+8}-3}{x+1}=\frac{\sqrt{(-1)^2+8}-3}{-1+1}=\frac{\sqrt{9}-3}{0}=\frac{0}{0}$$

$$= \lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} \bullet \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$$

$$= \lim_{x\to -1} \frac{x^2+8-9}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to -1} \frac{x^2-1}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to -1} \frac{(x-1)(x+1)}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to -1} \frac{(x-1)}{\sqrt{x^2+8}+3}$$

$$= \frac{-2}{\sqrt{9}+3} \quad = \frac{-2}{6}$$

$$\boxed{= -\frac{1}{3}}$$

Exercise

Find the limit: $\displaystyle \lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3}$

Solution

$$\lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3} = \frac{2-\sqrt{(-3)^2-5}}{-3+3} = \frac{2-\sqrt{9-5}}{0} = \frac{2-\sqrt{4}}{0} = \frac{0}{0}$$

$$= \lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3} \bullet \frac{2+\sqrt{x^2-5}}{2+\sqrt{x^2-5}}$$

$$= \lim_{x\to -3} \frac{4-\left(x^2-5\right)}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{4-x^2+5}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{9-x^2}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{(x-3)(x+3)}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{(x-3)}{2+\sqrt{x^2-5}}$$

$$= \frac{-6}{2+\sqrt{9-5}}$$

$$= \frac{-6}{2+\sqrt{4}}$$

$$= -\frac{3}{2}$$

Exercise

Find the limit: $\lim_{x\to 0}(2\sin x - 1)$

Solution

$$\lim_{x\to 0}(2\sin x - 1) = 2\sin(0) - 1$$

$$= 0 - 1$$

$$= -1$$

Exercise

Find the limit: $\lim_{x\to 0}\sin^2 x$

Solution

$$\lim_{x\to 0}\sin^2 x = \sin^2(0)$$

$$= 0$$

Exercise

Find the limit: $\lim_{x\to 0}\sec x$

Solution

$$\lim_{x\to 0}\sec x = \sec(0)$$

$$= \frac{1}{\cos(0)}$$

$$= 1$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{1+x+\sin x}{3\cos x}$

Solution

$$\lim_{x\to 0} \frac{1+x+\sin x}{3\cos x} = \frac{1+0+\sin(0)}{3\cos(0)}$$

$$= \frac{1}{3}$$

Exercise

Find the limit: $\lim\limits_{x\to -\pi} \sqrt{x+4}\cos(x+\pi)$

Solution

$$\lim_{x\to -\pi} \sqrt{x+4}\cos(x+\pi) = \sqrt{-\pi+4}\cos(-\pi+\pi)$$

$$= \sqrt{-\pi+4}\cos(0)$$

$$= \sqrt{4-\pi}$$

Exercise

Find $\lim\limits_{x\to -0.5^-} \sqrt{\dfrac{x+2}{x+1}}$

Solution

$$\lim_{x\to -0.5^-} \sqrt{\frac{x+2}{x+1}} = \sqrt{\frac{-0.5+2}{-0.5+1}}$$

$$= \sqrt{\frac{1.5}{0.5}}$$

$$= \sqrt{3}$$

Exercise

Find $\lim\limits_{x\to 1^+} \sqrt{\dfrac{x-1}{x+2}}$

Solution

$$\lim_{x\to 1^+} \sqrt{\frac{x-1}{x+2}} = \sqrt{\frac{1-1}{1+2}}$$

$$= 0$$

Exercise

Find $\lim\limits_{x\to -2^{+}} \left(\dfrac{x}{x+1}\right)\left(\dfrac{2x+5}{x^{2}+x}\right)$

Solution

$$\lim_{x\to -2^{+}} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^{2}+x}\right)=\left(\frac{-2}{-2+1}\right)\left(\frac{2(-2)+5}{(-2)^{2}+(-2)}\right)$$

$$=\left(\frac{-2}{-1}\right)\left(\frac{1}{2}\right)$$

$$=1$$

Exercise

Find $\lim\limits_{x\to 0^{+}} \dfrac{\sqrt{x^{2}+4x+5}-\sqrt{5}}{x}$

Solution

$$\lim_{x\to 0^{+}} \frac{\sqrt{x^{2}+4x+5}-\sqrt{5}}{x}=\frac{\sqrt{5}-\sqrt{5}}{0}=\frac{0}{0}$$

$$=\lim_{x\to 0^{+}} \frac{\sqrt{x^{2}+4x+5}-\sqrt{5}}{x}\,\frac{\sqrt{x^{2}+4x+5}+\sqrt{5}}{\sqrt{x^{2}+4x+5}+\sqrt{5}}$$

$$=\lim_{x\to 0^{+}} \frac{x^{2}+4x+5-5}{x\left(\sqrt{x^{2}+4x+5}+\sqrt{5}\right)}$$

$$=\lim_{x\to 0^{+}} \frac{x^{2}+4x}{x\left(\sqrt{x^{2}+4x+5}+\sqrt{5}\right)}$$

$$=\lim_{x\to 0^{+}} \frac{x(x+4)}{x\left(\sqrt{x^{2}+4x+5}+\sqrt{5}\right)}$$

$$=\lim_{x\to 0^{+}} \frac{x+4}{\sqrt{x^{2}+4x+5}+\sqrt{5}}$$

$$=\frac{0+4}{\sqrt{0^{2}+4(0)+5}+\sqrt{5}}$$

$$=\frac{4}{\sqrt{5}+\sqrt{5}}$$

$$=\frac{4}{2\sqrt{5}}$$

$$=\frac{2}{\sqrt{5}}$$

Exercise

Find $\displaystyle\lim_{x\to -2^+}(x+3)\frac{|x+2|}{x+2}$

Solution

$$\lim_{x\to -2^+}(x+3)\frac{|x+2|}{x+2}=(x+3)\frac{|-2+2|}{-2+2}=\frac{0}{0}$$

Since $x\to -2^+ \quad\Rightarrow\quad x>-2 \Rightarrow\quad |x+2|=(x+2)$

$$\begin{aligned}\lim_{x\to -2^+}(x+3)\frac{|x+2|}{x+2}&=\lim_{x\to -2^+}(x+3)\frac{x+2}{x+2}\\&=\lim_{x\to -2^+}(x+3)\\&=-2+3\\&=1\end{aligned}$$

Exercise

Find $\displaystyle\lim_{x\to 1^+}\frac{\sqrt{2x}(x-1)}{|x-1|}$

Solution

$$\lim_{x\to 1^+}\frac{\sqrt{2x}(x-1)}{|x-1|}=\frac{\sqrt{2(1)}(1-1)}{|1-1|}=\frac{0}{0}$$

Since $x\to 1^+ \quad\Rightarrow\quad x>1 \Rightarrow\quad |x-1|=x-1$

$$\begin{aligned}\lim_{x\to 1^+}\frac{\sqrt{2x}(x-1)}{|x-1|}&=\lim_{x\to 1^+}\frac{\sqrt{2x}(x-1)}{x-1}\\&=\lim_{x\to 1^+}\sqrt{2x}\\&=\sqrt{2}\end{aligned}$$

Exercise

Find $\displaystyle\lim_{\theta\to 0}\frac{\sin\sqrt{2}.\theta}{\sqrt{2}.\theta}$

Solution

Let: $\sqrt{2}\theta=x$

$$\lim_{\theta\to 0}\frac{\sin\sqrt{2}.\theta}{\sqrt{2}.\theta}=\lim_{x\to 0}\frac{\sin x}{x}=1$$

Exercise

Find $\lim\limits_{x\to 0} \dfrac{\sin 3x}{4x}$

Solution

$$\lim_{x\to 0} \frac{\sin 3x}{4x} = \lim_{x\to 0} \frac{\sin 3x}{4x}\frac{3}{3}$$

$$= \frac{3}{4}\lim_{x\to 0} \frac{\sin 3x}{3x} \qquad \text{Let: } 3x = u$$

$$= \frac{3}{4}\lim_{u\to 0} \frac{\sin u}{u} \qquad \textit{By definition: } \lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$= \frac{3}{4}$$

Exercise

Find $\lim\limits_{x\to 0^-} \dfrac{x}{\sin 3x}$

Solution

$$\lim_{x\to 0^-} \frac{x}{\sin 3x} = \lim_{x\to 0^-} \frac{x}{\sin 3x}\left(\frac{3}{3}\right)$$

$$= \frac{1}{3}\lim_{x\to 0^-} \frac{3x}{\sin 3x}$$

$$= \frac{1}{3}\lim_{x\to 0^-} \frac{1}{\frac{\sin 3x}{3x}} \qquad \textit{By definition: } \lim_{u\to 0} \frac{\sin u}{u} = 1$$

$$= \frac{1}{3}$$

Exercise

Find $\lim\limits_{x\to 0} \dfrac{\tan 2x}{x}$

Solution

$$\lim_{x\to 0} \frac{\tan 2x}{x} = \lim_{x\to 0} \frac{\frac{\sin 2x}{\cos 2x}}{x}$$

$$= \lim_{x\to 0} \left(\frac{\sin 2x}{x} \bullet \frac{1}{\cos 2x}\right)$$

$$= \lim_{x\to 0} \left(2\frac{\sin 2x}{2x}\right) \lim_{x\to 0} \left(\frac{1}{\cos 2x}\right)$$

$$= 2 \bullet \frac{1}{\cos 0}$$

$$= 2$$

Exercise

Find $\lim\limits_{x\to 0} 6x^2(\cot x)(\csc 2x)$

Solution

$$\lim_{x\to 0} 6x^2(\cot x)(\csc 2x) = \lim_{x\to 0} 6x^2\left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin 2x}\right)$$

$$= \lim_{x\to 0} 3\cos x\left(\frac{x}{\sin x}\right)\left(\frac{2x}{\sin 2x}\right)$$

$$= 3\lim_{x\to 0}(\cos x)\bullet\lim_{x\to 0}\left(\frac{x}{\sin x}\right)\bullet\lim_{x\to 0}\left(\frac{2x}{\sin 2x}\right) \quad = 3\bullet 1\bullet 1\bullet 1$$

$$= 3$$

Exercise

Find $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\sin 2\theta}$

Solution

$$\lim_{\theta\to 0} \frac{\sin\theta}{\sin 2\theta} = \lim_{\theta\to 0} \frac{\sin\theta}{\sin 2\theta}\frac{2\theta}{2\theta}$$

$$= \frac{1}{2}\lim_{\theta\to 0}\left(\frac{2\theta}{\sin 2\theta}\bullet\frac{\sin\theta}{\theta}\right) \quad = \frac{1}{2}\bullet 1\bullet 1$$

$$= \frac{1}{2}$$

Exercise

Find $\lim\limits_{h\to 0} \dfrac{\sin(\sin h)}{\sin h}$

Solution

Let: $\sin h = \theta$ $\qquad \theta = \sin h \xrightarrow[h\to 0]{} 0$

$$\lim_{h\to 0} \frac{\sin(\sin h)}{\sin h} = \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}$$

$$= 1$$

Exercise

Find $\lim\limits_{\theta\to 0} \dfrac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}$

Solution

$$\lim_{\theta\to 0}\frac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}=\lim_{\theta\to 0}\frac{\theta\frac{\cos 4\theta}{\sin 4\theta}}{\sin^2\theta\frac{\cos^2 2\theta}{\sin^2 2\theta}}$$

$$=\lim_{\theta\to 0}\ \theta\frac{\cos 4\theta}{2\sin 2\theta\cos 2\theta}\frac{\sin^2 2\theta}{\sin^2\theta\cos^2 2\theta}$$

$$=\lim_{\theta\to 0}\left(\frac{1}{2}\cdot\theta\cdot\cos 4\theta\bullet\frac{2\sin\theta\cos\theta}{\sin^2\theta}\cdot\frac{1}{\cos^3 2\theta}\right)$$

$$=\lim_{\theta\to 0}\left(\cos 4\theta\bullet\frac{\theta}{\sin\theta}\cdot\cos\theta\cdot\frac{1}{\cos^3 2\theta}\right)$$

$$=\lim_{\theta\to 0}\left(\cos 4\theta\right)\bullet\lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right)\bullet\lim_{\theta\to 0}\left(\frac{\cos\theta}{\cos^3 2\theta}\right)$$

$$=1\bullet 1\bullet 1$$

$$\boxed{=1}$$

Exercise

Find $\displaystyle\lim_{\theta\to\pi/4}\frac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta}$

Solution

$$\lim_{\theta\to\pi/4}\frac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta}=\frac{\frac{1}{2}-\frac{1}{2}}{\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}}=\frac{0}{0}$$

$$=\lim_{\theta\to\pi/4}\frac{(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)}{\sin\theta-\cos\theta}$$

$$=\lim_{\theta\to\pi/4}(\sin\theta+\cos\theta)$$

$$=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}$$

$$\boxed{=\sqrt{2}}$$

Exercise

Find $\displaystyle\lim_{x\to\pi/2}\frac{\frac{1}{\sqrt{\sin x}}-1}{x+\frac{\pi}{2}}$

Solution

$$\lim_{x\to\pi/2}\frac{\frac{1}{\sqrt{\sin x}}-1}{x+\frac{\pi}{2}}=\frac{1-1}{\frac{\pi}{2}+\frac{\pi}{2}}\ \boxed{=0}$$

Exercise

Find $\lim\limits_{x\to 1} \dfrac{x^3-7x^2+12x}{4-x}$

Solution

$$\lim\limits_{x\to 1} \dfrac{x^3-7x^2+12x}{4-x} = \dfrac{1-7+12}{4-1} \underline{=2|}$$

Exercise

Find $\lim\limits_{x\to 4} \dfrac{x^3-7x^2+12x}{4-x}$

Solution

$$\lim\limits_{x\to 4} \dfrac{x^3-7x^2+12x}{4-x} = \dfrac{64-112+48}{4-4} = \dfrac{0}{0}$$

$$= \lim\limits_{x\to 4} \dfrac{x(x-3)(x-4)}{4-x}$$

$$= \lim\limits_{x\to 4} -x(x-3)$$

$$\underline{=-4|}$$

Exercise

Find $\lim\limits_{x\to 1} \dfrac{1-x^2}{x^2-8x+7}$

Solution

$$\lim\limits_{x\to 1} \dfrac{1-x^2}{x^2-8x+7} = \dfrac{0}{0}$$

$$= \lim\limits_{x\to 1} \dfrac{(1-x)(1+x)}{(x-1)(x-7)}$$

$$= \lim\limits_{x\to 1} \dfrac{1+x}{x-7}$$

$$\underline{= -\dfrac{1}{3}|}$$

Exercise

Find $\lim\limits_{x\to 3} \dfrac{\sqrt{3x+16}-5}{x-3}$

Solution

$$\lim\limits_{x\to 3} \dfrac{\sqrt{3x+16}-5}{x-3} = \dfrac{\sqrt{9+16}-5}{3-3} = \dfrac{5-5}{0} = \dfrac{0}{0}$$

$$= \lim_{x\to 3} \frac{\sqrt{3x+16}-5}{x-3}\frac{\sqrt{3x+16}+5}{\sqrt{3x+16}+5}$$

$$= \lim_{x\to 3} \frac{3x+16-25}{(x-3)\left(\sqrt{3x+16}+5\right)}$$

$$= \lim_{x\to 3} \frac{3(x-3)}{(x-3)\left(\sqrt{3x+16}+5\right)}$$

$$= \lim_{x\to 3} \frac{3}{\sqrt{3x+16}+5}$$

$$= \frac{3}{5+5}$$

$$= \frac{3}{10}$$

Exercise

Find $\lim_{x\to 3} \frac{1}{x-3}\left(\frac{1}{\sqrt{x+1}}-\frac{1}{2}\right)$

Solution

$$\lim_{x\to 3} \frac{1}{x-3}\left(\frac{1}{\sqrt{x+1}}-\frac{1}{2}\right) = \frac{1}{0}\left(\frac{1}{2}-\frac{1}{2}\right) = \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{1}{x-3}\left(\frac{2-\sqrt{x+1}}{\sqrt{x+1}}\right)\left(\frac{2+\sqrt{x+1}}{2+\sqrt{x+1}}\right)$$

$$= \lim_{x\to 3} \frac{1}{x-3}\left(\frac{4-x-1}{2\sqrt{x+1}+x+1}\right)$$

$$= \lim_{x\to 3} \frac{x-3}{x-3}\left(\frac{-1}{2\sqrt{x+1}+x+1}\right)$$

$$= \lim_{x\to 3} \frac{-1}{2\sqrt{x+1}+x+1}$$

$$= -\frac{1}{8}$$

Exercise

Find $\lim_{x\to 1/3} \frac{x-\frac{1}{3}}{(3x-1)^2}$

Solution

$$\lim_{x\to 1/3} \frac{x-\frac{1}{3}}{(3x-1)^2} = \frac{\frac{1}{3}-\frac{1}{3}}{\left(3\frac{1}{3}-1\right)^2} = \frac{0}{0}$$

$$= \lim_{x \to 1/3} \frac{x - \frac{1}{3}}{9\left(x - \frac{1}{3}\right)^2}$$

$$= \lim_{x \to 1/3} \frac{1}{9\left(x - \frac{1}{3}\right)} = \frac{1}{0}$$

$$= \infty$$

Exercise

Find $\lim_{x \to 3} \frac{x^4 - 81}{x - 3}$

Solution

$$\lim_{x \to 3} \frac{x^4 - 81}{x - 3} = \frac{81 - 81}{3 - 3} = \frac{0}{0}$$

$$= \lim_{x \to 3} \frac{(x-3)(x+3)\left(x^2+9\right)}{x-3} \qquad a^4 - b^4 = \left(a^2 - b^2\right)\left(a^2 + b^2\right) = (a-b)(a+b)\left(a^2 + b^2\right)$$

$$= \lim_{x \to 3} (x+3)\left(x^2+9\right) = 6(18)$$

$$= 108$$

Exercise

Find $\lim_{x \to 1} \frac{x^5 - 1}{x - 1}$

Solution

$$\lim_{x \to 1} \frac{x^5 - 1}{x - 1} = \frac{1-1}{1-1} = \frac{0}{0} \qquad \left(a^5 - b^5\right) = (a-b)\left(a^4 + a^3 b + a^2 b^2 + ab^3 + b^4\right)$$

$$= \lim_{x \to 1} \frac{(x-1)\left(x^4 + x^3 + x^2 + x + 1\right)}{x-1}$$

$$= \lim_{x \to 1} \left(x^4 + x^3 + x^2 + x + 1\right)$$

$$= 5$$

Exercise

Find $\lim_{x \to 81} \frac{\sqrt[4]{x} - 3}{x - 81}$

Solution

$$\lim_{x\to 81}\frac{\sqrt[4]{x}-3}{x-81}=\frac{3-3}{81-81}=\frac{0}{0}$$

$$=\lim_{x\to 81}\frac{\sqrt[4]{x}-3}{\left(\sqrt{x}+9\right)\left(\sqrt{x}-9\right)}$$

$$=\lim_{x\to 81}\frac{\sqrt[4]{x}-3}{\left(\sqrt{x}+9\right)\left(\sqrt[4]{x}+3\right)\left(\sqrt[4]{x}-3\right)}$$

$$=\lim_{x\to 81}\frac{1}{\left(\sqrt{x}+9\right)\left(\sqrt[4]{x}+3\right)}$$

$$=\frac{1}{(18)(6)}$$

$$=\frac{1}{108}$$

Exercise

Find the limit: $\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}$

Solution

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}=\frac{0}{0}$$

$$=\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\left(\sqrt[3]{x}\right)^3-1^3}$$

$$=\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\left(\sqrt[3]{x}-1\right)\left(x^{2/3}+\sqrt[3]{x}+1\right)}$$

$$=\lim_{x\to 1}\frac{1}{x^{2/3}+\sqrt[3]{x}+1}$$

$$=\frac{1}{3}$$

Exercise

Find the limit: $\lim_{x\to 2}\frac{x^5-32}{x-2}$

Solution

$$\lim_{x\to 2}\frac{x^5-32}{x-2}=\frac{2^5-32}{2-2}=\frac{0}{0}$$

$$\begin{array}{c|cccccc} 2 & 1 & 0 & 0 & 0 & 0 & -32 \\ & & 2 & 4 & 8 & 16 & 32 \\ \hline & 1 & 2 & 4 & 8 & 16 & 0 \end{array}$$

$$=\lim_{x\to 2}\frac{(x-2)\left(x^4+2x^3+4x^2+8x+16\right)}{x-2}$$

$$= \lim_{x\to 2} \left(x^4 + 2x^3 + 4x^2 + 8x + 16\right)$$

$$= 16 + 16 + 16 + 16 + 16$$

$$\underline{= 80}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{x^6 - 1}{x - 1}$

Solution

$$\begin{array}{c|ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ & & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}$$

$$\lim_{x\to 1} \frac{x^6 - 1}{x - 1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(x^5 + x^4 + x^3 + x^2 + x + 1\right)}{x - 1}$$

$$= \lim_{x\to 1} \left(x^5 + x^4 + x^3 + x^2 + x + 1\right)$$

$$\underline{= 6}$$

Exercise

Find the limit: $\lim\limits_{x\to -1} \dfrac{x^7 + 1}{x + 1}$

Solution

$$\begin{array}{c|cccccccc} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 \end{array}$$

$$\lim_{x\to -1} \frac{x^7 + 1}{x + 1} = \frac{-1+1}{-1+1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{(x+1)\left(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1\right)}{x + 1}$$

$$= \lim_{x\to 1} \left(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1\right)$$

$$\underline{= 1}$$

Exercise

Find the limit: $\lim\limits_{x\to a} \dfrac{x^5 - a^5}{x - a}$

Solution

$$\begin{array}{c|cccccc} a & 1 & 0 & 0 & 0 & 0 & -a^5 \\ & & a & a^2 & a^3 & a^4 & a^5 \\ \hline & 1 & a & a^2 & a^3 & a^4 & 0 \end{array}$$

$$\lim_{x\to a} \frac{x^5 - a^5}{x - a} = \frac{a^5 - a^5}{a - a} = \frac{0}{0}$$

$$= \lim_{x \to a} \frac{(x-a)\left(x^4 + ax^3 + a^2x^2 + a^3x + a^4\right)}{x-a}$$

$$= \lim_{x \to a} \left(x^4 + ax^3 + a^2x^2 + a^3x + a^4\right)$$

$$= a^4 + a^4 + a^4 + a^4 + a^4$$

$$\underline{= 5a^4}$$

Exercise

Find the limit: $\lim_{x \to a} \frac{x^n - a^n}{x-a} \quad n \in \mathbb{Z}^+$

$$\begin{array}{c|ccccccc}
a & 1 & 0 & 0 & 0 & \cdots & 0 & -a^n \\
 & & a & a^2 & a^3 & \cdots & a^{n-1} & a^n \\
\hline
 & 1 & a & a^2 & a^3 & \cdots & a^{n-1} & 0
\end{array}$$

Solution

$$\lim_{x \to a} \frac{x^n - a^n}{x-a} = \frac{a^n - a^n}{a-a} = \frac{0}{0}$$

$$= \lim_{x \to a} \frac{(x-a)\left(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-2}x + a^{n-1}\right)}{x-a}$$

$$= \lim_{x \to a} \left(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-2}x + a^{n-1}\right)$$

$$= a^{n-1} + a^{n-1} + a^{n-1} + \ldots + a^{n-1} + a^{n-1}$$

$$\underline{= na^{n-1}}$$

Exercise

Find the limit: $\lim_{h \to 0} \frac{100}{(10h-1)^{11} + 2}$

Solution

$$\lim_{h \to 0} \frac{100}{(10h-1)^{11} + 2} = \frac{100}{(-1)^{11} + 2}$$

$$= \frac{100}{-1+2}$$

$$\underline{= 100}$$

Exercise

Find the limit: $\lim_{h \to 0} \frac{(5+h)^2 - 25}{h}$

Solution

$$\lim_{h\to 0} \frac{(5+h)^2 - 25}{h} = \frac{5^2 - 25}{0} = \frac{0}{0}$$

$$= \lim_{h\to 0} \frac{((5+h)-5)((5+h)+5)}{h}$$

$$= \lim_{h\to 0} \frac{h(h+10)}{h}$$

$$= \lim_{h\to 0} (h+10)$$

$$\boxed{=10}$$

Exercise

Find the limit: $\lim_{x\to 3} \dfrac{\frac{1}{x^2+2x} - \frac{1}{15}}{x-3}$

Solution

$$\lim_{x\to 3} \frac{\frac{1}{x^2+2x} - \frac{1}{15}}{x-3} = \frac{\frac{1}{15} - \frac{1}{15}}{0} = \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{1}{x-3}\left(\frac{1}{x(x+2)} - \frac{1}{15}\right)$$

$$= \lim_{x\to 3} \frac{1}{x-3}\left(\frac{15 - x^2 - 2x}{15x(x+2)}\right)$$

$$= \lim_{x\to 3} \frac{-(x-3)(x+5)}{15x(x+2)(x-3)}$$

$$= \lim_{x\to 3} \frac{-(x+5)}{15x(x+2)}$$

$$= -\frac{8}{15(3)(5)}$$

$$\boxed{= -\frac{8}{225}}$$

Exercise

Find the limit: $\lim_{x\to 1} \dfrac{\sqrt{10x-9} - 1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{\sqrt{10x-9} - 1}{x-1} = \frac{1-1}{0} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{\sqrt{10x-9}-1}{x-1} \cdot \frac{\sqrt{10x-9}+1}{\sqrt{10x-9}+1}$$

$$= \lim_{x\to 1} \frac{10x-9-1}{(x-1)\left(\sqrt{10x-9}+1\right)}$$

$$= \lim_{x\to 1} \frac{10(x-1)}{(x-1)\left(\sqrt{10x-9}+1\right)}$$

$$= \lim_{x\to 1} \frac{10}{\sqrt{10x-9}+1}$$

$$= \frac{10}{2}$$

$$\underline{= 5}$$

Exercise

Find the limit: $\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2x}\right)$

Solution

$$\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2x}\right) = \frac{1}{0} - \frac{2}{0} = \infty - \infty$$

$$= \lim_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x(x-2)}\right)$$

$$= \lim_{x\to 2} \frac{x-2}{x(x-2)}$$

$$= \lim_{x\to 2} \frac{1}{x}$$

$$\underline{= \frac{1}{2}}$$

Exercise

Find the limit: $\lim_{x\to c} \frac{x^2-2cx+c^2}{x-c}$

Solution

$$\lim_{x\to c} \frac{x^2-2cx+c^2}{x-c} = \frac{c^2-2c^2+c^2}{0} = \frac{0}{0}$$

$$= \lim_{x\to c} \frac{(x-c)^2}{x-c}$$

$$= \lim_{x\to c} (x-c)$$

$$\underline{= 0}$$

Exercise

Find the limit: $\displaystyle\lim_{x\to -c} \frac{x^2+5cx+4c^2}{x^2+cx}$

Solution

$$\lim_{x\to -c} \frac{x^2+5cx+4c^2}{x^2+cx} = \frac{c^2-5c^2+4c^2}{c^2-c^2} = \frac{0}{0}$$

$$= \lim_{x\to -c} \frac{(x+c)(x+4c)}{x(x+c)}$$

$$= \lim_{x\to -c} \frac{x+4c}{x}$$

$$= \frac{-c+4c}{-c}$$

$$= \frac{3c}{-c}$$

$$= -3$$

Exercise

Find the limit: $\displaystyle\lim_{x\to 16} \frac{\sqrt[4]{x}-2}{x-16}$

Solution

$$\lim_{x\to 16} \frac{\sqrt[4]{x}-2}{x-16} = \frac{\sqrt[4]{16}-2}{16-16} = \frac{2-2}{0} = \frac{0}{0}$$

$$\lim_{x\to 16} \frac{\sqrt[4]{x}-2}{\left(\sqrt[4]{x}\right)^4-2^4}$$

$$a^4-b^4=\left(a^2+b^2\right)(a-b)(a+b)$$

$$= \lim_{x\to 16} \frac{\sqrt[4]{x}-2}{\left(\sqrt{x}+2^2\right)\left(\sqrt[4]{x}+2\right)\left(\sqrt[4]{x}-2\right)}$$

$$= \lim_{x\to 16} \frac{1}{\left(\sqrt{x}+4\right)\left(\sqrt[4]{x}+2\right)}$$

$$= \frac{1}{\left(\sqrt{16}+4\right)\left(\sqrt[4]{16}+2\right)}$$

$$= \frac{1}{(4+4)(2+2)}$$

$$= \frac{1}{(8)(4)}$$

$$= \frac{1}{32}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$

Solution

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{\sqrt{x}-1}$$

$$= \lim_{x\to 1} \left(\sqrt{x}+1\right)$$

$$\underline{= 2}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{4x+5}-3}$

Solution

$$\lim_{x\to 1} \frac{x-1}{\sqrt{4x+5}-3} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{x-1}{\sqrt{4x+5}-3} \bullet \frac{\sqrt{4x+5}+3}{\sqrt{4x+5}+3}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{4x+5}+3\right)}{4x+5-9}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{4x+5}+3\right)}{4(x-1)}$$

$$= \frac{1}{5} \lim_{x\to 1} \left(\sqrt{4x+5}+3\right)$$

$$\underline{= \frac{6}{5}}$$

Exercise

Find the limit: $\lim\limits_{x\to 4} \dfrac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}$

Solution

$$\lim_{x\to 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}} = \frac{0}{3-3} = \frac{0}{0}$$

$$= \lim_{x\to 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}} \bullet \frac{3+\sqrt{x+5}}{3+\sqrt{x+5}}$$

$$= 3 \lim_{x \to 4} \frac{(x-4)\left(3+\sqrt{x+5}\right)\sqrt{x+5}}{9-(x+5)}$$

$$= 3 \lim_{x \to 4} \frac{(x-4)\left(3+\sqrt{x+5}\right)\sqrt{x+5}}{4-x}$$

$$= -3 \lim_{x \to 4} \left(3+\sqrt{x+5}\right)\sqrt{x+5}$$

$$= -3\,(6)(3)$$

$$\boxed{= -54}$$

Exercise

Find the limit: $\lim\limits_{x \to 0} \dfrac{x}{\sqrt{ax+1}-1} \quad (a \neq 0)$

Solution

$$\lim_{x \to 0} \frac{x}{\sqrt{ax+1}-1} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{x}{\sqrt{ax+1}-1} \bullet \frac{\sqrt{ax+1}+1}{\sqrt{ax+1}+1}$$

$$= \lim_{x \to 0} \frac{x\left(\sqrt{ax+1}+1\right)}{ax+1-1}$$

$$= \lim_{x \to 0} \frac{x\left(\sqrt{ax+1}+1\right)}{ax}$$

$$= \frac{1}{a} \lim_{x \to 0} \left(\sqrt{ax+1}+1\right)$$

$$\boxed{= \frac{2}{a}}$$

Exercise

Find the limit: $\lim\limits_{x \to \pi} \dfrac{\cos^2 x + 3\cos x + 2}{\cos x + 1}$

Solution

$$\lim_{x \to \pi} \frac{\cos^2 x + 3\cos x + 2}{\cos x + 1} = \frac{1-3+2}{-1+1} = \frac{0}{0}$$

$$= \lim_{x \to \pi} \frac{(\cos x + 1)(\cos x + 2)}{\cos x + 1}$$

$$= \lim_{x \to \pi} (\cos x + 2)$$

$$= -1 + 2$$

$$\boxed{= 1}$$

Exercise

Find the limit: $\displaystyle\lim_{x\to\frac{3\pi}{2}} \frac{\sin^2 x+6\sin x+5}{\sin^2 x-1}$

Solution

$$\lim_{x\to\frac{3\pi}{2}} \frac{\sin^2 x+6\sin x+5}{\sin^2 x-1} = \frac{1-6+5}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to\frac{3\pi}{2}} \frac{(\sin x+1)(\sin x+5)}{(\sin x-1)(\sin x+1)}$$

$$= \lim_{x\to\frac{3\pi}{2}} \frac{\sin x+5}{\sin x-1}$$

$$= \frac{-1+5}{-1-1}$$

$$= -2$$

Exercise

Find the limit: $\displaystyle\lim_{x\to\frac{\pi}{2}} \frac{\sin x-1}{\sqrt{\sin x}-1}$

Solution

$$\lim_{x\to\frac{\pi}{2}} \frac{\sin x-1}{\sqrt{\sin x}-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\left(\sqrt{\sin x}-1\right)\left(\sqrt{\sin x}+1\right)}{\sqrt{\sin x}-1}$$

$$= \lim_{x\to\frac{\pi}{2}} \left(\sqrt{\sin x}+1\right)$$

$$= 2$$

Exercise

Find the limit: $\displaystyle\lim_{x\to 0} \frac{\frac{1}{2+\sin x}-\frac{1}{2}}{\sin x}$

Solution

$$\lim_{x\to 0} \frac{\frac{1}{2+\sin x}-\frac{1}{2}}{\sin x} = \frac{\frac{1}{2}-\frac{1}{2}}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1}{\sin x}\bullet\frac{2-\sin x-2}{2(2+\sin x)}$$

$$= \frac{1}{2}\lim_{x\to 0} \frac{1}{\sin x}\bullet\frac{-\sin x}{(2+\sin x)}$$

$$= -\frac{1}{2}\lim_{x\to 0}\frac{1}{2+\sin x}$$

$$= -\frac{1}{2}\left(\frac{1}{2}\right)$$

$$\boxed{= -\frac{1}{4}}$$

Exercise

Find the limit: $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{e^{x}-1}$

Solution

$$\lim_{x\to 0}\frac{e^{2x}-1}{e^{x}-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{\left(e^{x}-1\right)\left(e^{x}+1\right)}{e^{x}-1}$$

$$= \lim_{x\to 0}\left(e^{x}+1\right)$$

$$\boxed{= 2}$$

Exercise

Find the limit: $\lim\limits_{x\to \frac{\pi}{4}}\csc x$

Solution

$$\lim_{x\to \frac{\pi}{4}}\csc x = \csc\frac{\pi}{4}$$

$$= \frac{1}{\cos\frac{\pi}{4}}$$

$$\boxed{= \sqrt{2}}$$

Exercise

Find the limit: $\lim\limits_{x\to 4}\dfrac{x-5}{\left(x^2-10x+24\right)^2}$

Solution

$$\lim_{x\to 4}\frac{x-5}{\left(x^2-10x+24\right)^2} = \frac{-1}{(16-41+24)^2}$$

$$\boxed{= -1}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{\sin^2 x}$

Solution

$$\lim_{x\to 0} \frac{\cos x - 1}{\sin^2 x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\cos x - 1}{(1-\cos x)(1+\cos x)}$$

$$= -\lim_{x\to 0} \frac{1}{1+\cos x}$$

$$\boxed{= -\frac{1}{2}}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{1-\cos^2 x}{\sin x}$

Solution

$$\lim_{x\to 0} \frac{1-\cos^2 x}{\sin x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{\sin x}$$

$$= \lim_{x\to 0} \sin x$$

$$\boxed{= 0}$$

Exercise

Find $\lim\limits_{x\to 0} \dfrac{x^3 - 5x^2}{x^2}$

Solution

$$\lim_{x\to 0} \frac{x^3 - 5x^2}{x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} (x-5)$$

$$\boxed{= -5}$$

Exercise

Find $\lim\limits_{x\to 5}\dfrac{4x^2-100}{x-5}$

Solution

$$\lim_{x\to 5}\frac{4x^2-100}{x-5}=\frac{0}{0}$$

$$=\lim_{x\to 5}\frac{4(x-5)(x+5)}{x-5}$$

$$=\lim_{x\to 5}4(x+5)$$

$$=40$$

Exercise

For the function $f(t)$ graphed, find the following limits or explain why they do not exist.

$a)\ \lim\limits_{t\to -2}f(t)$ $\quad b)\ \lim\limits_{t\to -1}f(t)$ $\quad c)\ \lim\limits_{t\to 0}f(t)$ $\quad d)\ \lim\limits_{t\to -0.5}f(t)$

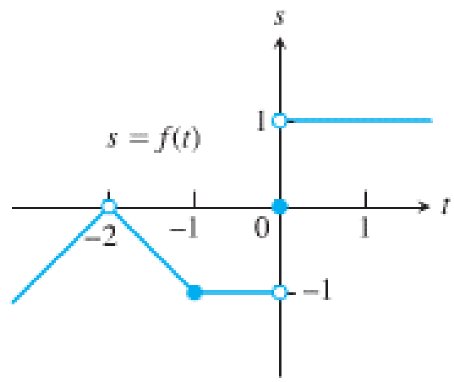
Solution

a) $\lim\limits_{t\to -2}f(t)=0$

b) $\lim\limits_{t\to -1}f(t)=-1$

c) $\lim\limits_{t\to 0}f(t)=$ *doesn't exist*

d) $\lim\limits_{t\to -.5}f(t)=-1$



Exercise

Suppose $\lim\limits_{x\to c}f(x)=5$ *and* $\lim\limits_{x\to c}g(x)=-2$. Find

$a)\ \lim\limits_{x\to c}f(x)g(x)$ $\qquad b)\ \lim\limits_{x\to c}2f(x)g(x)$

$c)\ \lim\limits_{x\to c}\left(f(x)+3g(x)\right)$ $\qquad d)\ \lim\limits_{x\to c}\dfrac{f(x)}{f(x)-g(x)}$

Solution

a) $\lim\limits_{x\to c}f(x)g(x)=\lim\limits_{x\to c}f(x)\bullet\lim\limits_{x\to c}g(x)=(5)(-2)=-10$

b) $\lim\limits_{x\to c}2f(x)g(x)=2\lim\limits_{x\to c}f(x)\bullet\lim\limits_{x\to c}g(x)=2(-10)=-20$

c) $\lim\limits_{x\to c}\left(f(x)+3g(x)\right)=\lim\limits_{x\to c} f(x)+3\lim\limits_{x\to c} g(x)$

$$=5+3(-2)$$

$$\underline{=-1|}$$

d) $\lim\limits_{x\to c}\dfrac{f(x)}{f(x)-g(x)}=\dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} f(x)-\lim\limits_{x\to c} g(x)}$

$$=\frac{5}{5-(-2)}$$

$$\underline{=\frac{5}{7}|}$$

Exercise

Explain why the limits do not exist for $\lim\limits_{x\to 0}\dfrac{x}{|x|}$

Solution

$$\lim_{x\to 0}\frac{x}{|x|}=\frac{0}{0}$$

$$\lim_{x\to 0^-}\frac{x}{|x|}=\frac{-x}{x}=-1$$

$$\lim_{x\to 0^+}\frac{x}{|x|}=\frac{x}{x}=1$$

Doesn't exist

Exercise

Evaluate the limit using the form $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ for $f(x)=x^2,\quad x=1$

Solution

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

$$=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

$$=\lim_{h\to 0}\left(\frac{2xh}{h}+\frac{h^2}{h}\right)$$

$$=\lim_{h\to 0}(2x+h)$$

$$\underline{=2x|}$$

Exercise

Evaluate the limit using the form $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ for $f(x)=\sqrt{3x+1}, \quad x=0$

Solution

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{3x+3h+1}-\sqrt{3x+1}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{3x+3h+1}-\sqrt{3x+1}}{h}\bullet\frac{\sqrt{3x+3h+1}+\sqrt{3x+1}}{\sqrt{3x+3h+1}+\sqrt{3x+1}}$$

$$=\lim_{h\to 0}\frac{3x+3h+1-(3x+1)}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}$$

$$=\lim_{h\to 0}\frac{3x+3h+1-3x-1}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}$$

$$=\lim_{h\to 0}\frac{3h}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}$$

$$=\lim_{h\to 0}\frac{3}{\sqrt{3x+3h+1}+\sqrt{3x+1}}$$

$$=\frac{3}{\sqrt{3(0)+1}+\sqrt{3(0)+1}} \qquad Given: x=0$$

$$\boxed{=\frac{3}{2}}$$

Exercise

If $\lim\limits_{x\to 4}\dfrac{f(x)-5}{x-2}=1$, find $\lim\limits_{x\to 4}f(x)$

Solution

$$\lim_{x\to 4}\frac{f(x)-5}{x-2}=1$$

$$\frac{\lim\limits_{x\to 4}f(x)-5}{4-2}=1$$

$$\frac{\lim\limits_{x\to 4}f(x)-5}{2}=1 \qquad \textit{Multiply both sides by 2}$$

$$\lim_{x\to 4}f(x)-5=2 \qquad \textit{Add 5 on both sides}$$

$$\lim_{x\to 4}f(x)=7$$

Exercise

If $\lim\limits_{x\to 0}\frac{f(x)}{x^2}=1$, find $\lim\limits_{x\to 0} f(x)$ *and* $\lim\limits_{x\to 0}\frac{f(x)}{x}$

Solution

$$\lim_{x\to 0}\frac{f(x)}{x^2}=1$$

$$\frac{\lim\limits_{x\to 0} f(x)}{\lim\limits_{x\to 0} x^2}=1$$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0} x^2 \,\boxed{=0}$$

$$\lim_{x\to 0}\frac{f(x)}{x}=\lim_{x\to 0}\left(\frac{f(x)}{x^2}\bullet x\right)$$

$$=\lim_{x\to 0}\frac{f(x)}{x^2}\bullet\lim_{x\to 0} x$$

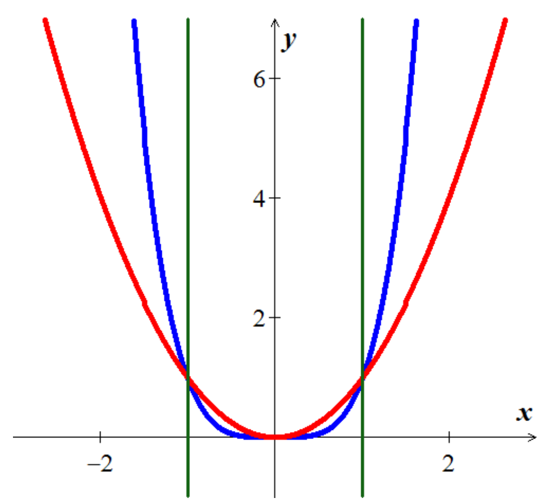
$$=1\bullet 0$$

$$\boxed{=0}$$

Exercise

If $x^4\le f(x)\le x^2;\quad -1\le x\le 1$ and $x^2\le f(x)\le x^4;\quad x<-1$ *and* $x>1$. At what points *c* do you automatically know $\lim\limits_{x\to c} f(x)$? What can you say about the value of the limits at these points?

Solution



$$\lim_{x\to c} x^4=\lim_{x\to c} x^2 \quad\Rightarrow\quad c^4=c^2$$

$$c^4-c^2=0$$

$$c^2\left(c^2-1\right)=0$$

$$c^2=0 \qquad\qquad c^2-1=0$$

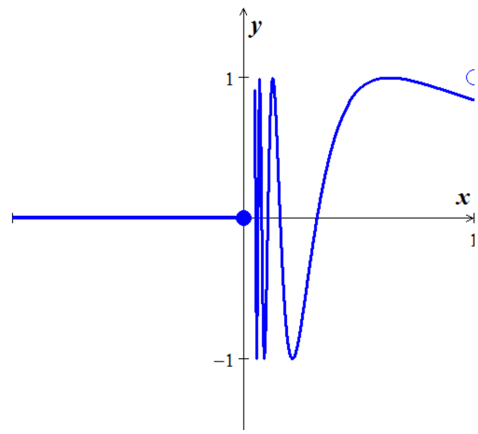
$$\boxed{c=0} \qquad\qquad \boxed{c=\pm 1}$$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0} x^2 \,\boxed{=0}$$

$$\lim_{x\to -1} f(x)=\lim_{x\to 1} f(x)\,\boxed{=1}$$

Exercise

Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$



a) Does $\lim\limits_{x \to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x \to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x \to 0} f(x)$ exist? If so, what is it? If not, why not?

Solution

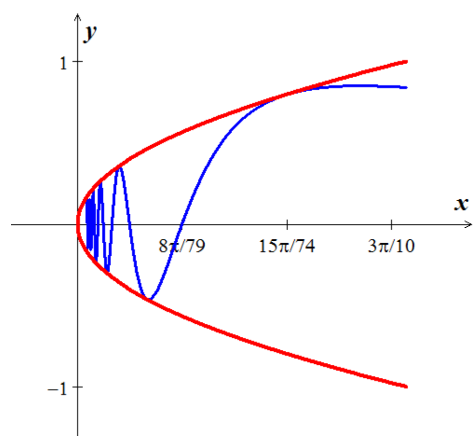
a) $\lim\limits_{x \to 0^+} f(x)$ doesn't exist, since $\sin\left(\frac{1}{x}\right)$ doesn't approach any single value as $x \to 0$

b) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} 0 = 0$

c) $\lim\limits_{x \to 0} f(x)$ doesn't exist, since $\lim\limits_{x \to 0^+} f(x)$ doesn't exist

Exercise

Let $g(x) = \sqrt{x}\ \sin\frac{1}{x}$



a) Does $\lim\limits_{x \to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x \to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x \to 0} g(x)$ exist? If so, what is it? If not, why not?

Solution

a) $\lim\limits_{x \to 0^+} g(x)$ exists, by the sandwich theorem $-\sqrt{x} \le g(x) \le \sqrt{x}$. *for* $x > 0$

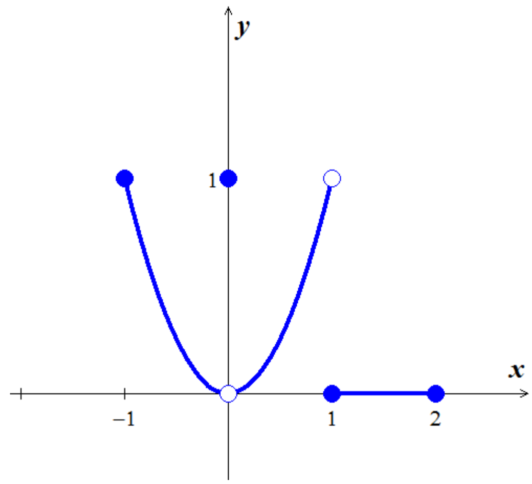
b) $\lim\limits_{x \to 0^-} g(x)$ doesn't exist, since $\sqrt{x}$ is not defined for $x < 0$

c) $\lim\limits_{x \to 0} g(x)$ doesn't exist, since $\lim\limits_{x \to 0^-} g(x)$ doesn't exist.

Exercise

Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

Solution



a) $\lim\limits_{x\to -1^+} f(x) = 1$ ***True***

b) $\lim\limits_{x\to 0^-} f(x) = 0$ ***True***

c) $\lim\limits_{x\to 0^-} f(x) = 1$ ***False***

d) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$ ***True***

e) $\lim\limits_{x\to 0} f(x)$ *exists* ***True***

f) $\lim\limits_{x\to 0} f(x) = 0$ ***True***

g) $\lim\limits_{x\to 0} f(x) = 1$ ***False***

h) $\lim\limits_{x\to 1} f(x) = 1$ ***False***

i) $\lim\limits_{x\to 1} f(x) = 0$ ***False***

j) $\lim\limits_{x\to 2^-} f(x) = 2$ ***False***

k) $\lim\limits_{x\to -1^-} f(x) = 0$ *does not exist* ***True***

l) $\lim\limits_{x\to 2^+} f(x) = 0$ ***False***