Solution   *Section* **4.1 – System of linear Equations**

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$$

<u>Solution</u>

$$\begin{cases} \; 3x + 2y = -4 \\ 2\times \;\; 2x - y = -5 \end{cases}$$

$$3x + 2y = -4$$
$$\underline{4x - 2y = -10}$$
$$7x = -14$$

$$\underline{x = -2|}$$

$$y = 2x + 5$$
$$ = -4 + 5$$
$$\underline{ = 1|}$$

Solution: $\underline{(-2,\ 1)}$ |

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$$

<u>Solution</u>

$$\begin{cases} -5\times \;\; 2x + 5y = 7 \\ 2\times \;\; 5x - 2y = -3 \end{cases}$$

$$-10x - 25y = -35$$
$$\underline{10x - 4y = -6}$$
$$-29y = -41$$

$$\underline{y = \frac{41}{29}|}$$

$$x = \frac{1}{2}\left(7 - 5\left(\frac{41}{29}\right)\right)$$

$$x = \frac{1}{2}\left(-\frac{2}{29}\right)$$

$$= -\frac{1}{29}$$

$\therefore$ *Solution*: $\left(-\frac{1}{29}, \frac{41}{29}\right)$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$$

Solution

$$\begin{cases} 4x - 7y = -16 \\ -2\times 2x + 5y = 9 \end{cases}$$

$$\begin{aligned} 4x - 7y &= -16 \\ -4x - 10y &= -18 \\ \hline -17y &= -34 \end{aligned}$$

$$y = 2$$

$$\begin{aligned} x &= \frac{9 - 5y}{2} \\ &= \frac{9 - 10}{2} \\ &= -\frac{1}{2} \end{aligned}$$

$\therefore$ *Solution*: $\left(-\frac{1}{2}, 2\right)$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$$

Solution

$$\begin{cases} 3x + 2y = 4 \quad (1) \\ 2x + y = 1 \quad (2) \end{cases}$$

$$(2) \to \quad y = 1 - 2x \quad (3)$$

$$(1) \to \quad 3x + 2 - 4x = 4$$

2

$$x = -2$$

$(3) \rightarrow \quad y = 1 + 4$

$$= 5$$

$\therefore$ *Solution*: $\quad (-2, \ 5)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$$

Solution

$$\begin{cases} -2 \times & 3x + 4y = 2 \\ 3 \times & 2x + 5y = -1 \end{cases}$$

$-6x - 8y = -4$

$\dfrac{6x + 15y = -3}{7y = -7}$

$$y = -1$$

$2x = -1 + 5$

$x = \dfrac{4}{2}$

$$= 2$$

$\therefore$ *Solution*: $\quad (2, \ -1)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method) $\quad \begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$

Solution

$$\begin{cases} 2 \times & 5x - 2y = 4 \\ & -10x + 4y = 7 \end{cases}$$

$10x - 4y = 8$

$\dfrac{-10x + 4y = 7}{0 = 15} \quad (impossible)$

$\therefore$ **No Solution**

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$$

Solution

$$\begin{cases} x - 4y = -8 & (1) \\ 5x - 20y = -40 & (2) \end{cases}$$

$(1) \rightarrow \quad x = 4y - 8$

$(2) \rightarrow \quad 5(4y - 8) - 20y = -40$

$20y - 40 - 20y = -40$

$-40 = -40 \quad (True)$

$\therefore$ *Solution*: $\quad \underline{x - 4y = -8}$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$$

Solution

$$\begin{cases} 2x + y = 3 & (1) \\ x - y = 3 & (2) \end{cases}$$

$(2) \rightarrow \quad x = 3 + y \quad (3)$

$(1) \rightarrow \quad 6 + 2y + y = 3$

$3y = -3$

$\underline{y = -1}$

$(3) \rightarrow \quad \underline{x = 2}$

$\therefore$ *Solution*: $\quad \underline{(2, \ -1)}$

4

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$$

Solution

$$\begin{cases} \quad\quad 2x + 10y = -14 \\ 5\times \quad 7x - 2y = -16 \end{cases}$$

$$2x + 10y = -14$$
$$\underline{35x - 10y = -80}$$
$$\quad 37x = -94$$

$$\underline{x = -\frac{94}{37}}$$

$$2y = 7\left(-\frac{94}{37}\right) + 16$$

$$y = -\frac{329}{37} + 8$$

$$\underline{= -\frac{33}{37}}$$

$\therefore$ *Solution*: $\quad \underline{\left(-\frac{94}{37}, \ -\frac{33}{37}\right)}$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$$

Solution

$$\begin{cases} 3\times \quad 4x - 3y = 24 \\ \quad\quad -3x + 9y = -1 \end{cases}$$

$$12x - 9y = 72$$
$$\underline{-3x + 9y = -1}$$
$$-9x = -71$$

$$\underline{x = \frac{71}{9}}$$

$$3y = 4\left(\frac{71}{9}\right) - 24$$

$$y = \frac{284}{27} - 8$$

5

$$= \frac{68}{27}$$

∴ *Solution*: $\left(\frac{71}{9}, \ \frac{68}{27} \right)$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$$

Solution

$$4x + 2y = 12$$
$$\underline{3x - 2y = 16}$$
$$7x = 28$$

$$\underline{x = 4}$$

$$2y = 12 - 4(4)$$
$$y = -\frac{4}{2}$$
$$\underline{= -2}$$

∴ *Solution*: $(4, \ -2)$

Exercise

Use any method to solve the system equation (*elimination* or *substitution* method)

$$\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$$

Solution

$$x + 2y = -1$$
$$\underline{4x - 2y = 6}$$
$$5x = 5$$

$$\underline{x = 1}$$

$$2y = -x - 1$$
$$y = -\frac{2}{2}$$
$$\underline{= -1}$$

∴ *Solution*: $(1, \ -1)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$$

Solution

$$x - 2y = 5$$
$$\underline{-10x + 2y = 4}$$
$$-9x = 9$$

$$\underline{x = -1|}$$

$$2y = x - 5$$

$$y = -\frac{6}{2}$$

$$\underline{= -3|}$$

$\therefore$ *Solution*: $(-1,\ -3)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$$

Solution

$$12x + 15y = -27$$
$$\underline{30x - 15y = -15}$$
$$42x = -42$$

$$\underline{x = -1|}$$

$$15y = -27 - 12(-1)$$

$$y = -\frac{15}{15}$$

$$\underline{= -1|}$$

$\therefore$ *Solution*: $(-1,\ -1)$

7

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$$

Solution

$4x - 4y = -12$

$\underline{4x + 4y = -20}$

$\quad 8x = -32$

$\underline{x = -4}$

$4y = 4(-4) + 12$

$y = -\dfrac{4}{4}$

$\underline{= -1}$

$\therefore$ ***Solution***: $\underline{(-4, \ -1)}$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\begin{bmatrix} 1 & 4 & | & 7 \\ 3 & 5 & | & 0 \end{bmatrix} \quad R_2 - 3R_1$$

Solution

$\begin{array}{ccc} 3 & 5 & 0 \\ -3 & -12 & -21 \\ \hline 0 & -7 & -21 \end{array}$

$$\begin{bmatrix} 1 & 4 & | & 7 \\ 0 & -7 & | & -21 \end{bmatrix}$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\begin{bmatrix} 1 & -3 & | & 1 \\ 2 & 1 & | & -5 \end{bmatrix} \quad R_2 - 2R_1$$

Solution

8

$$\begin{array}{rrr} 2 & 1 & -5 \\ -2 & 6 & -2 \\ \hline 0 & 7 & -7 \end{array}$$

$$\begin{bmatrix} 1 & -3 & | & 1 \\ 0 & 7 & | & -7 \end{bmatrix}$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\begin{bmatrix} 1 & -3 & | & 3 \\ 5 & 2 & | & 19 \end{bmatrix} \quad R_2 - 5R_1$$

Solution

$$\begin{array}{rrr} 5 & 2 & 19 \\ -5 & 15 & -15 \\ \hline 0 & 17 & -4 \end{array}$$

$$\begin{bmatrix} 1 & -3 & | & 3 \\ 0 & 17 & | & -4 \end{bmatrix}$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\begin{bmatrix} 2 & -3 & | & 8 \\ -6 & 9 & | & 4 \end{bmatrix} \quad R_2 + 3R_1$$

Solution

$$\begin{array}{rrr} -6 & 9 & 4 \\ 6 & -9 & 24 \\ \hline 0 & 0 & 28 \end{array}$$

$$\begin{bmatrix} 2 & -3 & | & 8 \\ 0 & 0 & | & 28 \end{bmatrix}$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\begin{bmatrix} 2 & 3 & | & 11 \\ 1 & 2 & | & 8 \end{bmatrix} \quad 2R_2 - R_1$$

Solution

$$\begin{array}{ccc} 2 & 4 & 16 \\ -2 & -3 & -11 \\ \hline 0 & 1 & 5 \end{array}$$

$$\begin{bmatrix} 2 & 3 & | & 11 \\ 0 & 1 & | & 5 \end{bmatrix}$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\begin{bmatrix} 3 & 5 & | & -13 \\ 2 & 3 & | & -9 \end{bmatrix} \quad 3R_2 - 2R_1$$

Solution

$$\begin{array}{ccc} 6 & 9 & -27 \\ -6 & -10 & 26 \\ \hline 0 & -1 & -1 \end{array}$$

$$\begin{bmatrix} 3 & 5 & | & -13 \\ 0 & -1 & | & -1 \end{bmatrix}$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\begin{bmatrix} 1 & 2 & 2 & | & 2 \\ 0 & 1 & -1 & | & 2 \\ 0 & 5 & 4 & | & 1 \end{bmatrix} \quad R_3 - 5R_2$$

Solution

$$\begin{array}{cccc} 0 & 5 & 4 & 1 \\ 0 & -5 & 5 & -10 \\ \hline 0 & 0 & 9 & -9 \end{array}$$

10

$$\begin{bmatrix} 1 & 2 & 2 & | & 2 \\ 0 & 1 & -1 & | & 2 \\ 0 & 0 & 9 & | & -9 \end{bmatrix}$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\begin{bmatrix} 1 & -1 & 5 & | & -6 \\ 3 & 3 & -1 & | & 10 \\ 1 & 3 & 2 & | & 5 \end{bmatrix} \begin{matrix} R_2 - 3R_1 \\ \\ R_3 - R_1 \end{matrix}$$

Solution

$$\begin{array}{cccc} 3 & 3 & -1 & 10 \\ -3 & 3 & -15 & 18 \\ \hline 0 & 6 & -16 & 28 \end{array} \qquad \begin{array}{cccc} 1 & 3 & 2 & 5 \\ -1 & 1 & -5 & 6 \\ \hline 0 & 4 & -3 & 11 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 5 & | & -6 \\ 0 & 6 & -16 & | & 28 \\ 0 & 4 & -3 & | & 11 \end{bmatrix}$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\begin{bmatrix} 3 & 2 & 1 & | & 1 \\ 2 & 4 & 4 & | & 22 \\ -1 & -2 & 3 & | & 15 \end{bmatrix} \begin{matrix} 3R_2 - 2R_1 \\ \\ 3R_3 + R_1 \end{matrix}$$

Solution

$$\begin{array}{cccc} 6 & 12 & 12 & 66 \\ -6 & -4 & -2 & -2 \\ \hline 0 & 8 & 10 & 64 \end{array} \qquad \begin{array}{cccc} -3 & -6 & 9 & 45 \\ 3 & 2 & 1 & 1 \\ \hline 0 & -4 & 10 & 46 \end{array}$$

$$\begin{bmatrix} 3 & 2 & 1 & | & 1 \\ 0 & 8 & 10 & | & 64 \\ 0 & -4 & 10 & | & 46 \end{bmatrix}$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & -4 & 2 & -7 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

Solution

$$\begin{array}{cccc} 2 & 1 & 1 & 3 \\ -2 & -2 & -2 & -4 \\ \hline 0 & -1 & -1 & -1 \end{array} \qquad \begin{array}{cccc} 3 & -4 & 2 & -7 \\ -3 & -3 & -3 & -6 \\ \hline 0 & -7 & -1 & -13 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & -7 & -1 & -13 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ 2 & -3 & 5 & -1 & 0 \\ 1 & 0 & 3 & 1 & -4 \\ -4 & 3 & 2 & -1 & 3 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 + 4R_1 \end{array}$$

Solution

$$\begin{array}{ccccc} 2 & -3 & 5 & -1 & 0 \\ -2 & 4 & -2 & -6 & 4 \\ \hline 0 & 1 & 3 & -7 & 4 \end{array} \qquad \begin{array}{ccccc} 1 & 0 & 3 & 1 & -4 \\ -1 & 2 & -1 & -3 & 2 \\ \hline 0 & 2 & 2 & -2 & -2 \end{array} \qquad \begin{array}{ccccc} -4 & 3 & 2 & -1 & 3 \\ 4 & -8 & 4 & 12 & -8 \\ \hline 0 & -5 & 6 & 11 & -5 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ 0 & 1 & 3 & -7 & 4 \\ 0 & 2 & 2 & -2 & -2 \\ 0 & -5 & 6 & 11 & -5 \end{array}\right]$$

Exercise

Use the Gauss-Jordan method to solve the system

$$x - y + 5z = -6$$
$$3x + 3y - z = 10$$
$$x + 3y + 2z = 5$$

Solution

$$\begin{bmatrix} 1 & -1 & 5 & \big| & -6 \\ 3 & 3 & -1 & \big| & 10 \\ 1 & 3 & 2 & \big| & 5 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{matrix}$$

$$\begin{matrix} 3 & 3 & -1 & 10 \\ -3 & 3 & -15 & 18 \\ \hline 0 & 6 & -16 & 28 \end{matrix} \qquad \begin{matrix} 1 & 3 & 2 & 5 \\ -1 & 1 & -5 & 6 \\ \hline 0 & 4 & -3 & 11 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 5 & \big| & -6 \\ 0 & 6 & -16 & \big| & 28 \\ 0 & 4 & -3 & \big| & 11 \end{bmatrix} \tfrac{1}{6}R_2$$

$$0 \quad 1 \quad -\tfrac{8}{3} \quad \tfrac{14}{3}$$

$$\begin{bmatrix} 1 & -1 & 5 & \big| & -6 \\ 0 & 1 & -\tfrac{8}{3} & \big| & \tfrac{14}{3} \\ 0 & 4 & -3 & \big| & 11 \end{bmatrix} \begin{matrix} R_1 + R_2 \\ \\ R_3 - 4R_2 \end{matrix}$$

$$\begin{matrix} 0 & 4 & -3 & 11 \\ 0 & -4 & \tfrac{32}{3} & -\tfrac{56}{3} \\ \hline 0 & 0 & \tfrac{23}{3} & -\tfrac{23}{3} \end{matrix} \qquad \begin{matrix} 1 & -1 & 5 & -6 \\ 0 & 1 & -\tfrac{8}{3} & \tfrac{14}{3} \\ \hline 1 & 0 & \tfrac{7}{3} & -\tfrac{4}{3} \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & \tfrac{7}{3} & \big| & -\tfrac{4}{3} \\ 0 & 1 & -\tfrac{8}{3} & \big| & \tfrac{14}{3} \\ 0 & 0 & \tfrac{23}{3} & \big| & -\tfrac{23}{3} \end{bmatrix} \tfrac{3}{23}R_3$$

$$0 \quad 0 \quad 1 \quad -1$$

$$\begin{bmatrix} 1 & 0 & \tfrac{7}{3} & \big| & -\tfrac{4}{3} \\ 0 & 1 & -\tfrac{8}{3} & \big| & \tfrac{14}{3} \\ 0 & 0 & 1 & \big| & -1 \end{bmatrix} \begin{matrix} R_1 - \tfrac{7}{3}R_3 \\ R_2 + \tfrac{8}{3}R_3 \\ \\ \end{matrix}$$

$$\begin{matrix} 1 & 0 & \tfrac{7}{3} & -\tfrac{4}{3} \\ 0 & 0 & -\tfrac{7}{3} & \tfrac{7}{3} \\ \hline 1 & 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & 1 & -\tfrac{8}{3} & \tfrac{14}{3} \\ 0 & 0 & \tfrac{8}{3} & -\tfrac{8}{3} \\ \hline 0 & 1 & 0 & 2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \big| & 1 \\ 0 & 1 & 0 & \big| & 2 \\ 0 & 0 & 1 & \big| & -1 \end{bmatrix}$$

Solution: $(1,\ 2,\ -1)$

Exercise

Use the Gauss-Jordan method to solve the system

$$\begin{cases} 2x - y + 4z = -3 \\ x - 2y - 10z = -6 \\ 3x \quad\;\; + 4z = 7 \end{cases}$$

Solution

$$\begin{bmatrix} 2 & -1 & 4 & | & -3 \\ 1 & -2 & -10 & | & -6 \\ 3 & 0 & 4 & | & 7 \end{bmatrix} \begin{matrix} \frac{1}{2}R_1 \\ \\ \\ \end{matrix}$$

$$\begin{matrix} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{matrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 2 & | & -\frac{3}{2} \\ 1 & -2 & -10 & | & -6 \\ 3 & 0 & 4 & | & 7 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\begin{matrix} 1 & -2 & -10 & -6 \\ -1 & \frac{1}{2} & -2 & \frac{3}{2} \\ \hline 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \end{matrix} \qquad \begin{matrix} 3 & 0 & 4 & 7 \\ -3 & \frac{3}{2} & -6 & \frac{9}{2} \\ \hline 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{matrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 2 & | & -\frac{3}{2} \\ 0 & -\frac{3}{2} & -12 & | & -\frac{9}{2} \\ 0 & \frac{3}{2} & -2 & | & \frac{23}{2} \end{bmatrix} \begin{matrix} \\ -\frac{2}{3}R_2 \\ \\ \end{matrix}$$

$$\begin{matrix} 0 & 1 & 8 & 3 \end{matrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 2 & | & -\frac{3}{2} \\ 0 & 1 & 8 & | & 3 \\ 0 & \frac{3}{2} & -2 & | & \frac{23}{2} \end{bmatrix} \begin{matrix} R_1 + \frac{1}{2}R_2 \\ \\ R_3 - \frac{3}{2}R_2 \end{matrix}$$

$$\begin{matrix} 0 & \frac{3}{2} & -2 & \frac{23}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ \hline 0 & 0 & -14 & 7 \end{matrix} \qquad \begin{matrix} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & \frac{1}{2} & 4 & \frac{3}{2} \\ \hline 1 & 0 & 6 & 0 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 6 & | & 0 \\ 0 & 1 & 8 & | & 3 \\ 0 & 0 & -14 & | & 7 \end{bmatrix} \begin{matrix} \\ \\ -\frac{1}{14}R_3 \end{matrix}$$

$$\begin{matrix} 0 & 0 & 1 & -\frac{1}{2} \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 6 & | & 0 \\ 0 & 1 & 8 & | & 3 \\ 0 & 0 & 1 & | & -\frac{1}{2} \end{bmatrix} \begin{matrix} R_1 - 6R_3 \\ R_2 - 8R_3 \\ \\ \end{matrix}$$

$$\begin{matrix} 1 & 0 & 6 & 0 \\ 0 & 0 & -6 & 3 \\ \hline 1 & 0 & 0 & 3 \end{matrix} \qquad \begin{matrix} 0 & 1 & 8 & 3 \\ 0 & 0 & -8 & 4 \\ \hline 0 & 1 & 0 & 7 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & 7 \\ 0 & 0 & 1 & | & -\frac{1}{2} \end{bmatrix}$$

Solution: $\left(3,\ 7,\ -\frac{1}{2}\right)$

Exercise

14

Use the Gauss-Jordan method to solve the system
$$\begin{cases} 4x + 3y - 5z = -29 \\ 3x - 7y - z = -19 \\ 2x + 5y + 2z = -10 \end{cases}$$

Solution

$$\begin{bmatrix} 4 & 3 & -5 & | & -29 \\ 3 & -7 & -1 & | & -19 \\ 2 & 5 & 2 & | & -10 \end{bmatrix} \quad \frac{1}{4}R_1$$

$$\begin{array}{cccc} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \end{array}$$

$$\begin{bmatrix} 1 & \frac{3}{4} & -\frac{5}{4} & | & -\frac{29}{4} \\ 3 & -7 & -1 & | & -19 \\ 2 & 5 & 2 & | & -10 \end{bmatrix} \quad \begin{array}{c} R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & -7 & -1 & -19 \\ -3 & -\frac{9}{4} & \frac{15}{4} & \frac{87}{4} \\ \hline 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \end{array} \qquad \begin{array}{cccc} 2 & 5 & 2 & -10 \\ -2 & -\frac{3}{2} & \frac{5}{2} & \frac{29}{2} \\ \hline 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}$$

$$\begin{bmatrix} 1 & \frac{3}{4} & -\frac{5}{4} & | & -\frac{29}{4} \\ 0 & -\frac{37}{4} & \frac{11}{4} & | & \frac{11}{4} \\ 0 & \frac{7}{2} & \frac{9}{2} & | & \frac{9}{2} \end{bmatrix} \quad -\frac{4}{37}R_2$$

$$\begin{array}{cccc} 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \end{array}$$

$$\begin{bmatrix} 1 & \frac{3}{4} & -\frac{5}{4} & | & -\frac{29}{4} \\ 0 & 1 & -\frac{11}{37} & | & -\frac{11}{37} \\ 0 & \frac{7}{2} & \frac{9}{2} & | & \frac{9}{2} \end{bmatrix} \quad \begin{array}{c} R_1 - \frac{3}{4}R_2 \\ \\ R_3 - \frac{7}{2}R_2 \end{array}$$

$$\begin{array}{cccc} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{3}{4} & \frac{33}{148} & \frac{33}{148} \\ \hline 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \end{array} \qquad \begin{array}{cccc} 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \\ 0 & -\frac{7}{2} & \frac{77}{72} & \frac{77}{72} \\ \hline 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array}$$

$$\begin{bmatrix} 1 & 0 & -\frac{38}{37} & | & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & | & -\frac{11}{37} \\ 0 & 0 & \frac{401}{72} & | & \frac{401}{72} \end{bmatrix} \quad \frac{72}{401}R_3$$

$$\begin{array}{cccc} 0 & 0 & 1 & 1 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -\frac{38}{37} & | & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & | & -\frac{11}{37} \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \quad \begin{array}{c} R_1 + \frac{38}{37}R_3 \\ R_2 + \frac{11}{37}R_3 \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 0 & \frac{38}{37} & \frac{38}{37} \\ \hline 1 & 0 & 0 & -6 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{11}{37} & \frac{11}{37} \\ \hline 0 & 1 & 0 & 0 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & -6 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$$

Solution: $(-6,\ 0,\ 1)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x + 2y - 3z = -15 \\ 2x - 3y + 4z = 18 \\ -3x + y + z = 1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ \\ R_3 + 3R_1 \end{array}$$

$$\begin{array}{cccc} -2 & -4 & 6 & 30 \\ 2 & -3 & 4 & 18 \\ \hline 0 & -7 & 10 & 48 \end{array} \qquad \begin{array}{cccc} 3 & 6 & -9 & -45 \\ -3 & 1 & 1 & 1 \\ \hline 0 & 7 & -8 & -44 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \end{array}\right] \quad -\frac{1}{7}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 7 & -8 & -44 \end{array}\right] \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 - 7R_2 \end{array}$$

$$\begin{array}{cccc} 1 & 2 & -3 & -15 \\ 0 & -2 & \frac{20}{7} & \frac{96}{7} \\ \hline 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \end{array} \qquad \begin{array}{cccc} 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \\ \hline 0 & 0 & 2 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 2 & 4 \end{array}\right] \quad \frac{1}{2}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 + \frac{1}{7}R_3 \\ \\ R_2 + \frac{10}{7}R_3 \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 0 & \frac{1}{7} & \frac{2}{7} \\ \hline 1 & 0 & 0 & -1 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & \frac{10}{7} & \frac{20}{7} \\ \hline 0 & 1 & 0 & -4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $(-1, \ -4, \ 2)$

Exercise

Use the Gauss-Jordan method to solve the system
$$\begin{cases} x + 2y + 3z = 10 \\ 4x + 5y + 6z = 11 \\ 7x + 8y + 9z = 12 \end{cases}$$

Solution

$$\begin{bmatrix} 1 & 2 & 3 & | & 10 \\ 4 & 5 & 6 & | & 11 \\ 7 & 8 & 9 & | & 12 \end{bmatrix} \begin{array}{l} \\ R_2 - 4R_1 \\ R_3 - 7R_1 \end{array}$$

$$\begin{array}{rrrr} -4 & -8 & -12 & -40 \\ 4 & 5 & 6 & 11 \\ \hline 0 & -3 & -6 & -29 \end{array} \qquad \begin{array}{rrrr} -7 & -14 & -21 & -70 \\ 7 & 8 & 9 & 12 \\ \hline 0 & -6 & -12 & -58 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 & | & 10 \\ 0 & -3 & -6 & | & -29 \\ 0 & -6 & -12 & | & -58 \end{bmatrix} \quad \tfrac{1}{3}R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & | & 10 \\ 0 & 1 & 2 & | & \frac{29}{3} \\ 0 & -6 & -12 & | & -58 \end{bmatrix} R_3 + 6R_2$$

$$\begin{array}{rrrr} 0 & -6 & -12 & -58 \\ 0 & 6 & 12 & 58 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 & | & 10 \\ 0 & 1 & 2 & | & \frac{29}{3} \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

let z be the variable

From Row 1 $\Rightarrow y + 2z = \frac{29}{3}$

$$\underline{y = \frac{29}{3} - 2z}$$

From Row 1 $\Rightarrow x + 2y + 3z = 10$

$$x = 10 - 2y - 3z$$

$$x = 10 - 2\left(\frac{29}{3} - 2z\right) - 3z$$

$$x = 10 - \frac{58}{3} + 4z - 3z$$

$$\underline{x = z - \frac{28}{3}}$$

Solution: $\underline{\left(z - \frac{28}{3},\ \frac{29}{3} - 2z,\ z\right)}$

17

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 2x + y + 2z = 4 \\ 2x + 2y \quad = 5 \\ 2x - y + 6z = 2 \end{cases}$

Solution

$$\begin{bmatrix} 2 & 1 & 2 & | & 4 \\ 2 & 2 & 0 & | & 5 \\ 2 & -1 & 6 & | & 2 \end{bmatrix} \quad \frac{1}{2}R_1$$

$$\begin{array}{cccc} 1 & \frac{1}{2} & 1 & 2 \end{array}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 1 & | & 2 \\ 2 & 2 & 0 & | & 5 \\ 2 & -1 & 6 & | & 2 \end{bmatrix} \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 2 & 2 & 0 & 5 \\ -2 & -1 & -2 & -4 \\ \hline 0 & 1 & -2 & 1 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 6 & 2 \\ -2 & -1 & -2 & -4 \\ \hline 0 & -2 & 4 & -2 \end{array}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 1 & | & 2 \\ 0 & 1 & -2 & | & 1 \\ 0 & -2 & 4 & | & -2 \end{bmatrix} \quad \begin{array}{l} R_1 - \frac{1}{2}R_2 \\ \\ R_3 + 2R_2 \end{array}$$

$$\begin{array}{cccc} 1 & \frac{1}{2} & 1 & 2 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \hline 1 & 0 & 2 & \frac{3}{2} \end{array} \qquad \begin{array}{cccc} 0 & -2 & 4 & -2 \\ 0 & 2 & -4 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 2 & | & \frac{3}{2} \\ 0 & 1 & -2 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

From Row 3: $0 = 0$ is a true statement. Let z be the variable.

From Row 2: $y - 2z = 1$

$$\underline{y = 1 + 2z}$$

From Row 1: $x + 2z = \dfrac{3}{2}$

$$\underline{x = -2z + \dfrac{3}{2}}$$

$\therefore$ **Solution**: $\left(-2z + \dfrac{3}{2},\ 2z + 1,\ z \right)$

Exercise

Use the Gauss-Jordan method to solve the system
$$\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$

Solution

$$\begin{bmatrix} 1 & 1 & 2 & | & 8 \\ -1 & -2 & 3 & | & 1 \\ 3 & -7 & 4 & | & 10 \end{bmatrix} \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} -1 & -2 & 3 & 1 \\ 1 & 1 & 2 & 8 \\ \hline 0 & -1 & 5 & 9 \end{array} \qquad \begin{array}{cccc} 3 & -7 & 4 & 10 \\ -3 & -3 & -6 & -24 \\ \hline 0 & -10 & -2 & -14 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 2 & | & 8 \\ 0 & -1 & 5 & | & 9 \\ 0 & -10 & -2 & | & -14 \end{bmatrix} -R_2$$

$$\begin{bmatrix} 1 & 1 & 2 & | & 8 \\ 0 & 1 & -5 & | & -9 \\ 0 & -10 & -2 & | & -14 \end{bmatrix} \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 10R_2 \end{array} \qquad \begin{array}{cccc} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ \hline 1 & 0 & 7 & 17 \end{array} \qquad \begin{array}{cccc} 0 & -10 & -2 & -14 \\ 0 & 10 & -50 & -90 \\ \hline 0 & 0 & -52 & -104 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 7 & | & 17 \\ 0 & 1 & -5 & | & -9 \\ 0 & 0 & -52 & | & -104 \end{bmatrix} -\dfrac{1}{52}R_3$$

$$\begin{bmatrix} 1 & 0 & 7 & | & 17 \\ 0 & 1 & -5 & | & -9 \\ 0 & 0 & 1 & | & 2 \end{bmatrix} \begin{array}{l} R_1 - 7R_3 \\ R_2 + 5R_3 \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & 0 & 7 & 17 \\ 0 & 0 & -7 & -14 \\ \hline 1 & 0 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -5 & -9 \\ 0 & 0 & 5 & 10 \\ \hline 0 & 1 & 0 & 1 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$$

∴ *Solution*: $(3, \ 1, \ 2)$

Exercise

Use augmented elimination to solve linear system

$$2x - 5y + 3z = 1$$
$$x - 2y - 2z = 8$$

Solution

$$\begin{bmatrix} 1 & -2 & -2 & | & 8 \\ 2 & -5 & 3 & | & 1 \end{bmatrix} \quad R_2 - 2R_1$$

$$\begin{array}{cccc} 2 & -5 & 3 & 1 \\ -2 & 4 & 4 & -16 \\ \hline 0 & -1 & 7 & -15 \end{array}$$

$$\begin{bmatrix} 1 & -2 & -2 & | & 8 \\ 0 & -1 & 7 & | & -15 \end{bmatrix} \quad -R_2$$

$$\begin{bmatrix} 1 & -2 & -2 & | & 8 \\ 0 & 1 & -7 & | & 15 \end{bmatrix} \quad R_1 + 2R_2$$

$$\begin{array}{cccc} 1 & -2 & -2 & 8 \\ 0 & 2 & -14 & 30 \\ \hline 1 & 0 & -16 & 38 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -16 & | & 38 \\ 0 & 1 & -7 & | & 15 \end{bmatrix} \quad \begin{array}{l} \rightarrow \quad x - 16z = 38 \\ \rightarrow \quad y - 7z = 15 \end{array}$$

$$\begin{cases} x = 16z + 38 \\ y = 7z + 15 \end{cases}$$

$\therefore$ *Solution*: $\left(16z + 38, \ 7z + 15, \ z\right)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$

Solution

$$\begin{bmatrix} 1 & 1 & 1 & | & 2 \\ 2 & 1 & -1 & | & 5 \\ 1 & -1 & 1 & | & -2 \end{bmatrix} \quad \begin{array}{l} R_2 - 2R_1 \\ \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccc} 2 & 1 & -1 & 5 \\ -2 & -2 & -2 & -4 \\ \hline 0 & -1 & -3 & 1 \end{array} \qquad \begin{array}{cccc} 1 & -1 & 1 & -2 \\ -1 & -1 & -1 & -2 \\ \hline 0 & -2 & 0 & -4 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 2 \\ 0 & -1 & -3 & | & 1 \\ 0 & -2 & 0 & | & -4 \end{bmatrix} \quad \begin{array}{l} (2) \\ (1) \\ -2y = -4 \end{array}$$

$y = 2$

$(1) \rightarrow \quad -y - 3z = 1$

$\qquad 3z = -1 - 2$

$\qquad z = -1$

$(2) \rightarrow \quad x + y + z = 2$

$$x = 2 - 2 + 1$$

$$\underline{= 1}$$

$\therefore$ *Solution*: $(1, \ 2, \ -1)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$

Solution

$$\begin{bmatrix} 2 & 1 & 1 & | & 9 \\ -1 & -1 & 1 & | & 1 \\ 3 & -1 & 1 & | & 9 \end{bmatrix} \quad \begin{matrix} \\ 2R_2 + R_1 \\ 2R_3 - 3R_1 \end{matrix}$$

$$\begin{array}{cccc} -2 & -2 & 2 & 2 \\ 2 & 1 & 1 & 9 \\ \hline 0 & -1 & 3 & 11 \end{array} \qquad \begin{array}{cccc} 6 & -2 & 2 & 18 \\ -6 & -3 & -3 & -27 \\ \hline 0 & -5 & -1 & -9 \end{array}$$

$$\begin{bmatrix} 2 & 1 & 1 & | & 9 \\ 0 & -1 & 3 & | & 11 \\ 0 & -5 & -1 & | & -9 \end{bmatrix} \quad R_3 - 5R_2$$

$$\begin{array}{cccc} 0 & -5 & -1 & -9 \\ 0 & 5 & -15 & -55 \\ \hline 0 & 0 & -16 & -64 \end{array}$$

$$\begin{bmatrix} 2 & 1 & 1 & | & 9 \\ 0 & -1 & 3 & | & 11 \\ 0 & 0 & -16 & | & -64 \end{bmatrix} \quad \begin{matrix} (2) \\ (1) \\ -16z = -64 \end{matrix}$$

$$\underline{z = 4}$$

$(1) \rightarrow \quad -y + 3z = 11$

$$y = 12 - 11$$

$$\underline{= 1}$$

$(2) \rightarrow \quad 2x + y + z = 9$

$$2x = 9 - 1 - 4$$

$$\underline{x = 2}$$

$\therefore$ *Solution*: $(2, \ 1, \ 4)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

Solution

$$\begin{bmatrix} 1 & 5 & -1 & | & -4 \\ 0 & 3 & -1 & | & -1 \\ -3 & 6 & 2 & | & 11 \end{bmatrix} \quad R_3 + 3R_1$$

$$\begin{array}{cccc} -3 & 6 & 2 & 11 \\ 3 & 15 & -3 & -12 \\ \hline 0 & 21 & -1 & -1 \end{array}$$

$$\begin{bmatrix} 1 & 5 & -1 & | & -4 \\ 0 & 3 & -1 & | & -1 \\ 0 & 21 & -1 & | & -1 \end{bmatrix} \quad R_3 - 7R_2$$

$$\begin{array}{cccc} 0 & 21 & -1 & -1 \\ 0 & -21 & 7 & 7 \\ \hline 0 & 0 & 6 & 6 \end{array}$$

$$\begin{bmatrix} 1 & 5 & -1 & | & -4 \\ 0 & 3 & -1 & | & -1 \\ 0 & 0 & 6 & | & 6 \end{bmatrix} \quad \begin{array}{l} \to x + 5y - z = -4 \quad (2) \\ \to 3y - z = -1 \quad\quad (1) \\ \to 6z = 6 \end{array}$$

$\underline{z = 1}$

$(1) \to \quad 3y = -1 + 1$

$\quad\quad \underline{y = 0}$

$(2) \to \quad x = -4 + 1$

$\quad\quad \underline{x = -3}$

∴ *Solution*: $\underline{(-3,\ 0,\ 1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$

Solution

$$\begin{bmatrix} 1 & 3 & 4 & | & 14 \\ 2 & -3 & 2 & | & 10 \\ 3 & -1 & 1 & | & 9 \end{bmatrix} \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -3 & 2 & 10 \\ -2 & -6 & -8 & -28 \\ \hline 0 & -9 & -6 & -18 \end{array} \qquad \begin{array}{cccc} 3 & -1 & 1 & 9 \\ -3 & -9 & -12 & -42 \\ \hline 0 & -10 & -11 & -33 \end{array}$$

$$\begin{bmatrix} 1 & 3 & 4 & | & 14 \\ 0 & -9 & -6 & | & -18 \\ 0 & -10 & -11 & | & -33 \end{bmatrix} \quad 9R_3 - 10R_2$$

$$\begin{array}{cccc} 0 & -90 & -99 & -297 \\ 0 & 90 & 60 & 180 \\ \hline 0 & 0 & -39 & -117 \end{array}$$

$$\begin{bmatrix} 1 & 3 & 4 & | & 14 \\ 0 & -9 & -6 & | & -18 \\ 0 & 0 & -39 & | & -117 \end{bmatrix} \quad \begin{array}{l} x + 3y + 4z = 14 \quad (3) \\ -9y - 6z = -18 \quad (2) \\ -39z = -117 \quad (1) \end{array}$$

$(1) \to \quad z = \dfrac{117}{39}$

$\quad\quad \underline{= 3}$

$$(2) \rightarrow \quad 9y = 18 - 6(3)$$
$$9y = 0$$
$$\underline{y = 0}|$$

$$(3) \rightarrow \quad x = 14 - 12$$
$$\underline{x = 2}|$$

$\therefore$ *Solution*: $(2, \ 0, \ 3)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$

Solution

$$\begin{bmatrix} 1 & 4 & -1 & | & 20 \\ 3 & 2 & 1 & | & 8 \\ 2 & -3 & 2 & | & -16 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\begin{array}{cccc} 3 & 2 & 1 & 8 \\ -3 & -12 & 3 & -60 \\ \hline 0 & -10 & 4 & -52 \end{array}$$

$$\begin{array}{cccc} 2 & -3 & 2 & -16 \\ -2 & -8 & 2 & -40 \\ \hline 0 & -11 & 4 & -56 \end{array}$$

$$\begin{bmatrix} 1 & 4 & -1 & | & 20 \\ 0 & -10 & 4 & | & -52 \\ 0 & -11 & 4 & | & -56 \end{bmatrix} 10R_3 - 11R_2$$

$$\begin{array}{cccc} 0 & -110 & 40 & -560 \\ 0 & 110 & -44 & 572 \\ \hline 0 & 0 & -4 & 12 \end{array}$$

$$\begin{bmatrix} 1 & 4 & -1 & | & 20 \\ 0 & -10 & 4 & | & -52 \\ 0 & 0 & -4 & | & 12 \end{bmatrix} \begin{matrix} x + 4y - z = 20 & (3) \\ -10y + 4z = -52 & (2) \\ -4z = 12 & (1) \end{matrix}$$

$$(1) \rightarrow \quad \underline{z = -3}|$$

$$(2) \rightarrow \quad -10y = -52 + 12$$
$$-10y = -40$$
$$\underline{y = 4}|$$

$$(3) \rightarrow \quad x = 20 - 16 - 3$$
$$\underline{x = 1}|$$

$\therefore$ *Solution*: $(1, \ 4, \ -3)$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$$

Solution

$$\begin{bmatrix} 1 & 2 & 1 & | & 17 \\ 0 & 2 & -1 & | & 7 \\ 2 & -3 & 2 & | & -1 \end{bmatrix} \quad R_3 - 2R_1$$

$$\begin{array}{cccc} 2 & -3 & 2 & -1 \\ -2 & -4 & -2 & -34 \\ \hline 0 & -7 & 0 & -35 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 & | & 17 \\ 0 & 2 & -1 & | & 7 \\ 0 & -7 & 0 & | & -35 \end{bmatrix} \quad \begin{array}{l} x + 2y + z = 17 \quad (3) \\ 2y - z = 7 \quad (2) \\ -7y = -35 \quad (1) \end{array}$$

$(1) \rightarrow \quad \underline{y = 5}$

$(2) \rightarrow \quad z = 10 - 7$

$\qquad \underline{= 3}$

$(3) \rightarrow \quad x = 17 - 10 - 3$

$\qquad \underline{= 4}$

$\therefore$ *Solution*: $\underline{(4,\ 5,\ 3)}$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$$

Solution

$$\begin{bmatrix} -2 & 6 & 7 & | & 3 \\ -4 & 5 & 3 & | & 7 \\ -6 & 3 & 5 & | & -4 \end{bmatrix} \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} -4 & 5 & 3 & 7 \\ 4 & -12 & -14 & -6 \\ \hline 0 & -7 & -11 & 1 \end{array} \qquad \begin{array}{cccc} -6 & 3 & 5 & -4 \\ 6 & -18 & -21 & -9 \\ \hline 0 & -15 & -16 & -13 \end{array}$$

$$\begin{bmatrix} -2 & 6 & 7 & | & 3 \\ 0 & -7 & -11 & | & 1 \\ 0 & -15 & -16 & | & -13 \end{bmatrix} \quad 7R_3 - 15R_1$$

$$\begin{array}{cccc} 0 & -105 & -112 & -91 \\ 0 & 105 & 165 & -15 \\ \hline 0 & 0 & 53 & -106 \end{array}$$

$$\begin{bmatrix} -2 & 6 & 7 & | & 3 \\ 0 & -7 & -11 & | & 1 \\ 0 & 0 & 53 & | & -106 \end{bmatrix} \quad \begin{array}{l} -2x + 6y + 7z = 3 \quad (3) \\ -7y - 11z = 1 \quad (2) \\ 53z = -106 \quad (1) \end{array}$$

$(1) \rightarrow \quad \underline{z = -2}$

$(2) \rightarrow \quad -7y = 1 - 22$

$$-7y = -21$$

$$\underline{y = 3}$$

$(3) \rightarrow \quad -2x = 3 - 18 + 14$

$$-2x = -1$$

$$\underline{x = \frac{1}{2}}$$

$\therefore$ *Solution*: $\left(\frac{1}{2}, \ 3, \ -2\right)$

Exercise

Use augmented elimination to solve linear system $\quad \begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

Solution

$\begin{bmatrix} 2 & -1 & 1 & | & 1 \\ 3 & -3 & 4 & | & 5 \\ 4 & -2 & 3 & | & 4 \end{bmatrix} \begin{matrix} \\ 2R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix}$

$$\begin{array}{cccc} 6 & -6 & 8 & 10 \\ -6 & 3 & -3 & -3 \\ \hline 0 & -3 & 5 & 7 \end{array} \qquad \begin{array}{cccc} 4 & -2 & 3 & 4 \\ -4 & 2 & -2 & -2 \\ \hline 0 & 0 & 1 & 2 \end{array}$$

$\begin{bmatrix} 2 & -1 & 1 & | & 1 \\ 0 & -3 & 5 & | & 7 \\ 0 & 0 & 1 & | & 2 \end{bmatrix} \begin{matrix} 2x - y + z = 1 & (2) \\ -3y + 5z = 7 & (1) \\ \underline{z = 2} \end{matrix}$

$(1) \rightarrow \quad -3y = 7 - 10$

$$-3y = -3$$

$$\underline{y = 1}$$

$(3) \rightarrow \quad 2x = 1 + 1 - 2$

$$\underline{x = 0}$$

$\therefore$ *Solution*: $(0, \ 1, \ 2)$

Exercise

Use augmented elimination to solve linear system $\quad\begin{cases}3x-4y+4z=7\\x-y-2z=2\\2x-3y+6z=5\end{cases}$

Solution

$$\begin{bmatrix}1&-1&-2&\bigm|&2\\3&-4&4&\bigm|&7\\2&-3&6&\bigm|&5\end{bmatrix}\begin{matrix}\\R_2-3R_1\\R_3-2R_1\end{matrix}$$

$$\begin{array}{rrrr}3&-4&4&7\\-3&3&6&-6\\\hline 0&-1&10&1\end{array}\qquad\begin{array}{rrrr}2&-3&6&5\\-2&2&4&-4\\\hline 0&-1&10&1\end{array}$$

$$\begin{bmatrix}1&-1&-2&\bigm|&2\\0&-1&10&\bigm|&1\\0&-1&10&\bigm|&1\end{bmatrix}\begin{matrix}\\\\R_3=R_2\end{matrix}\qquad\begin{matrix}\to x-y-2z=2\;\;(2)\\\to -y+10z=1\;\;(1)\end{matrix}$$

$(1)\to\quad \underline{y=10z-1}$

$(2)\to\quad x=2+10z-1+2z$

$\qquad\quad \underline{=12z+1}$

$\therefore$ Solution: $\underline{\left(12z+1,\;10z-1,\;z\right)}$

Exercise

Use augmented elimination to solve linear system $\quad\begin{cases}x-2y-z=2\\2x-y+z=4\\-x+y+z=4\end{cases}$

Solution

$$\begin{bmatrix}1&-2&-1&\bigm|&2\\2&-1&1&\bigm|&4\\-1&1&1&\bigm|&4\end{bmatrix}\begin{matrix}\\R_2-2R_1\\R_3+R_1\end{matrix}$$

$$\begin{array}{rrrr}2&-1&1&4\\-2&4&2&-4\\\hline 0&3&3&0\end{array}\qquad\begin{array}{rrrr}-1&1&1&4\\1&-2&-1&2\\\hline 0&-1&0&6\end{array}$$

$$\begin{bmatrix}1&-2&-1&\bigm|&2\\0&3&3&\bigm|&0\\0&-1&0&\bigm|&6\end{bmatrix}\begin{matrix}x-2y-z=2\;\;(3)\\3y+3z=0\;\;(2)\\-y=6\;\;(1)\end{matrix}$$

$(1)\to\quad \underline{y=-6}$

$(2)\to\quad z=-y$

$\qquad\quad \underline{=6}$

$(3)\to\quad x=2-12+6$

$\qquad\quad \underline{=-4}$

$\therefore$ Solution: $\underline{\left(-4,\;-6,\;6\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + y + z = 3 \\ -y + 2z = 1 \\ -x + z = 0 \end{cases}$

<u>***Solution***</u>

$$\begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 0 & -1 & 2 & | & 1 \\ -1 & 0 & 1 & | & 0 \end{bmatrix} \quad R_3 + R_1$$

$$\begin{array}{cccc} -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 3 \\ \hline 0 & 1 & 2 & 3 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 0 & -1 & 2 & | & 1 \\ 0 & 1 & 2 & | & 3 \end{bmatrix} \quad R_3 + R_2$$

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ 0 & -1 & 2 & 1 \\ \hline 0 & 0 & 4 & 4 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 0 & -1 & 2 & | & 1 \\ 0 & 0 & 4 & | & 4 \end{bmatrix} \quad \begin{array}{ll} x + y + z = 3 & (3) \\ -y + 2z = 1 & (2) \\ 4z = 4 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z = 1}$

$(2) \rightarrow \quad -y = 1 - 2$

$\qquad \underline{y = 1}$

$(3) \rightarrow \quad x = 3 - 1 - 1$

$\qquad \underline{= 1}$

$\therefore$ ***Solution***: $\underline{(1,\ 1,\ 1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x + y + 3z = 14 \\ 7x + 5y + 8z = 37 \\ x + 3y + 2z = 9 \end{cases}$

<u>***Solution***</u>

$$\begin{bmatrix} 1 & 3 & 2 & | & 9 \\ 3 & 1 & 3 & | & 14 \\ 7 & 5 & 8 & | & 37 \end{bmatrix} \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - 7R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 1 & 3 & 14 \\ -3 & -9 & -6 & -27 \\ \hline 0 & -8 & -3 & -13 \end{array} \qquad \begin{array}{cccc} 7 & 5 & 8 & 37 \\ -7 & -21 & -14 & -63 \\ \hline 0 & -16 & -6 & -26 \end{array}$$

$$\begin{bmatrix} 1 & 3 & 2 & | & 9 \\ 0 & -8 & -3 & | & -13 \\ 0 & -16 & -6 & | & -26 \end{bmatrix} \quad R_3 - 2R_2$$

$$\begin{array}{cccc} 0 & -16 & -6 & -26 \\ 0 & 16 & 6 & 26 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\begin{bmatrix} 1 & 3 & 2 & | & 9 \\ 0 & -8 & -3 & | & -13 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad \begin{array}{l} x + 3y + 2z = 9 \quad (2) \\ -8y - 3z = -13 \quad (1) \end{array}$$

$(1) \rightarrow \quad -8y = 3z - 13$

$$\underline{y = -\dfrac{3}{8}z + \dfrac{13}{8}}$$

$(3) \rightarrow \quad x = 9 - 3\left(\dfrac{13}{8} - \dfrac{3}{8}z\right) - 2z$

$$\qquad = 9 - \dfrac{39}{8} + \dfrac{9}{8}z - 2z$$

$$\qquad \underline{= \dfrac{33}{8} - \dfrac{7}{8}z}$$

$\therefore$ *Solution*: $\underline{\left(\dfrac{33}{8} - \dfrac{7}{8}z, \ \dfrac{13}{8} - \dfrac{3}{8}z, \ z\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 4x - 2y + z = 7 \\ x + y + z = -2 \\ 4x + 2y + z = 3 \end{cases}$

Solution

$$\begin{bmatrix} 1 & 1 & 1 & | & -2 \\ 4 & -2 & 1 & | & 7 \\ 4 & 2 & 1 & | & 3 \end{bmatrix} \begin{array}{l} R_2 - 4R_1 \\ R_3 - 4R_1 \end{array} \qquad \begin{array}{cccc} 4 & -2 & 1 & 7 \\ -4 & -4 & -4 & 8 \\ \hline 0 & -6 & -3 & 15 \end{array} \qquad \begin{array}{cccc} 4 & 2 & 1 & 3 \\ -4 & -4 & -4 & 8 \\ \hline 0 & -2 & -3 & 11 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & | & -2 \\ 0 & -6 & -3 & | & 15 \\ 0 & -2 & -3 & | & 11 \end{bmatrix} -3R_3 + R_2 \qquad \begin{array}{cccc} 0 & 6 & 9 & -33 \\ 0 & -6 & -3 & 15 \\ \hline 0 & 0 & 6 & -18 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & | & -2 \\ 0 & -6 & -3 & | & 15 \\ 0 & 0 & 6 & | & -18 \end{bmatrix} \quad \begin{array}{l} x + y + z = -2 \quad (3) \\ -6y - 3z = 15 \quad (2) \\ 6z = -18 \quad (1) \end{array}$$

$(1) \rightarrow \quad \underline{z = -3}$

$(2) \rightarrow \quad -6y = 15 - 9$

$$\qquad \underline{y = -1}$$

$(3) \rightarrow \quad x = -2 + 1 + 3$

$$\qquad \underline{= 2}$$

$\therefore$ *Solution*: $\underline{(2, \ -1, \ -3)}$

28

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x - 2y + z = -4 \\ 6x + 4y - 3z = -24 \\ x - 2y + 2z = 1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 2 & -2 & 1 & -4 \\ 6 & 4 & -3 & -24 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 6R_1 \end{array}$$

$$\begin{array}{rrrr} 2 & -2 & 1 & -4 \\ -2 & 4 & -4 & -2 \\ \hline 0 & 2 & -3 & -6 \end{array} \qquad \begin{array}{rrrr} 6 & 4 & -3 & -24 \\ -6 & 12 & -12 & -6 \\ \hline 0 & 16 & -15 & -30 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 16 & -15 & -30 \end{array}\right] R_3 - 8R_2$$

$$\begin{array}{rrrr} 0 & 16 & -15 & -30 \\ 0 & -16 & 24 & 48 \\ \hline 0 & 0 & 9 & 18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 0 & 9 & 18 \end{array}\right] \begin{array}{l} x - 2y + 2z = 1 \quad (3) \\ 2y - 3z = -6 \quad (2) \\ 9z = 18 \quad (1) \end{array}$$

$(1) \rightarrow \ \underline{z = 2|}$

$(2) \rightarrow \ 2y = -6 + 6$

$\qquad \underline{y = 0|}$

$(3) \rightarrow \ x = 1 - 4$

$\qquad \underline{\underline{= -3|}}$

$\therefore$ **Solution**: $\underline{(-3,\ 0,\ 2)\ |}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 9x + 3y + z = 4 \\ 16x + 4y + z = 2 \\ 25x + 5y + z = 2 \end{cases}$

Solution

$$\begin{cases} z + 9x + 3y = 4 \\ z + 16x + 4y = 2 \\ z + 25x + 5y = 2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 1 & 16 & 4 & 2 \\ 1 & 25 & 5 & 2 \end{array}\right] \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{rrrr} 1 & 16 & 4 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 7 & 1 & -2 \end{array} \qquad \begin{array}{rrrr} 1 & 25 & 5 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 16 & 2 & -2 \end{array}$$

$$\begin{bmatrix} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 16 & 2 & -2 \end{bmatrix} \quad 7R_3 - 16R_3$$

$$\begin{array}{cccc} 0 & 112 & 14 & -14 \\ 0 & -112 & -16 & 32 \\ \hline 0 & 0 & -2 & 18 \end{array}$$

$$\begin{bmatrix} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 0 & -2 & 18 \end{bmatrix} \quad \begin{array}{l} z + 9x + 3y = 4 \quad (3) \\ 7x + y = -2 \quad (2) \\ -2y = 18 \quad (1) \end{array}$$

$(1) \rightarrow \quad \underline{y = -9}$

$(2) \rightarrow \quad 7x = -2 + 9$

$\qquad \underline{= 1}$

$(3) \rightarrow \quad z = 4 - 9 + 27$

$\qquad \underline{= 22}$

$\therefore$ *Solution*: $\underline{(1,\ -9,\ 22)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$

Solution

$$\begin{bmatrix} 1 & 2 & -3 & 9 \\ 2 & -1 & 2 & -8 \\ 3 & -1 & -4 & 3 \end{bmatrix} \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 2 & -8 \\ -2 & -4 & 6 & -18 \\ \hline 0 & -5 & 8 & -26 \end{array} \qquad \begin{array}{cccc} 3 & -1 & -4 & 3 \\ -3 & -6 & 9 & -27 \\ \hline 0 & -7 & 5 & -24 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & -7 & 5 & -24 \end{bmatrix} \quad 5R_3 - 7R_2$$

$$\begin{array}{cccc} 0 & -35 & 25 & -120 \\ 0 & 35 & -56 & 182 \\ \hline 0 & 0 & -31 & 62 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & 0 & -31 & 62 \end{bmatrix} \quad \begin{array}{l} x + 2y - 3z = 9 \quad (3) \\ -5y + 8z = -26 \quad (2) \\ -31z = 62 \quad (1) \end{array}$$

$(1) \rightarrow \quad \underline{z = -2}$

$(2) \rightarrow \quad -5y = -26 + 16$

$\qquad -5y = 10$

$\qquad \underline{y = 2}$

$(3) \rightarrow \quad x = 9 - 4 - 6$

$\qquad \underline{= -1}$

$\therefore$ *Solution*: $\underline{(-1,\ 2,\ -2)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x - 3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$

Solution

$$\begin{bmatrix} 1 & 0 & -3 & | & -5 \\ 2 & -1 & 2 & | & 16 \\ 7 & -3 & -5 & | & 19 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 7R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 2 & 16 \\ -2 & 0 & 6 & 10 \\ \hline 0 & -1 & 8 & 26 \end{array}$$

$$\begin{array}{cccc} 7 & -3 & -5 & 19 \\ -7 & 0 & 21 & 35 \\ \hline 0 & -3 & 16 & 54 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -3 & | & -5 \\ 0 & -1 & 8 & | & 26 \\ 0 & -3 & 16 & | & 54 \end{bmatrix} \begin{array}{l} \\ \\ R_3 - 3R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -3 & 16 & 54 \\ 0 & 3 & -24 & -78 \\ \hline 0 & 0 & -8 & -24 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -3 & | & -5 \\ 0 & -1 & 8 & | & 26 \\ 0 & 0 & -8 & | & -24 \end{bmatrix} \quad \begin{array}{l} x - 3z = -5 \quad (3) \\ -y + 8z = 26 \quad (2) \\ -8z = -24 \quad (1) \end{array}$$

$(1) \rightarrow \quad \underline{z = 3}$

$(2) \rightarrow \quad -y = 26 - 24$

$\qquad \underline{y = -2}$

$(3) \rightarrow \quad x = -5 + 9$

$\qquad \underline{= 4}$

$\therefore$ *Solution*: $\underline{(4, \ -2, \ 3)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + 2y - z = 5 \\ 2x - y + 3z = 0 \\ 2y + z = 1 \end{cases}$

Solution

$$\begin{bmatrix} 1 & 2 & -1 & | & 5 \\ 2 & -1 & 3 & | & 0 \\ 0 & 2 & 1 & | & 1 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ \\ \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 3 & 0 \\ -2 & -4 & 2 & -10 \\ \hline 0 & -5 & 5 & -10 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & | & 5 \\ 0 & -5 & 5 & | & -10 \\ 0 & 2 & 1 & | & 1 \end{bmatrix} \begin{array}{l} \\ \\ 5R_3 + 2R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 10 & 5 & 5 \\ 0 & -10 & 10 & -20 \\ \hline 0 & 0 & 15 & -15 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & | & 5 \\ 0 & -5 & 5 & | & -10 \\ 0 & 0 & 15 & | & -15 \end{bmatrix} \quad \begin{matrix} x + 2y - z = 5 & (3) \\ -5y + 5z = -10 & (2) \\ 15z = -15 & (1) \end{matrix}$$

$(1) \rightarrow \quad \underline{z = -1}$

$(2) \rightarrow \quad -5y = -10 + 5$

$\qquad \underline{y = 1}$

$(3) \rightarrow \quad x = 5 - 2 - 1$

$\qquad \underline{= 2}$

∴ *Solution*: $\underline{(2,\ 1,\ -1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + y + z = 6 \\ 3x + 4y - 7z = 1 \\ 2x - y + 3z = 5 \end{cases}$

Solution

$$\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 3 & 4 & -7 & | & 1 \\ 2 & -1 & 3 & | & 5 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\begin{matrix} 3 & 4 & -7 & 1 \\ -3 & -3 & -3 & -18 \\ \hline 0 & 1 & -10 & -17 \end{matrix} \qquad \begin{matrix} 2 & -1 & 3 & 5 \\ -2 & -2 & -2 & -12 \\ \hline 0 & -3 & 1 & -7 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & -10 & | & -17 \\ 0 & -3 & 1 & | & -7 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 3R_2 \end{matrix}$$

$$\begin{matrix} 0 & -3 & 1 & -7 \\ 0 & 3 & -30 & -51 \\ \hline 0 & 0 & -29 & -58 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & -10 & | & -17 \\ 0 & 0 & -29 & | & -58 \end{bmatrix} \quad \begin{matrix} x + y + z = 6 & (3) \\ y - 10z = -17 & (2) \\ -29z = -58 & (1) \end{matrix}$$

$(1) \rightarrow \quad \underline{z = 2}$

$(2) \rightarrow \quad y = -17 + 20$

$\qquad \underline{= 3}$

$(3) \rightarrow \quad x = 6 - 3 - 2$

$\qquad \underline{= 1}$

∴ *Solution*: $\underline{(1,\ 3,\ 2)}$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 3x + 2y + 3z = 3 \\ 4x - 5y + 7z = 1 \\ 2x + 3y - 2z = 6 \end{cases}$$

Solution

$$\begin{bmatrix} 3 & 2 & 3 & | & 3 \\ 4 & -5 & 7 & | & 1 \\ 2 & 3 & -2 & | & 6 \end{bmatrix} \begin{matrix} \\ 3R_2 - 4R_1 \\ 3R_3 - 2R_1 \end{matrix}$$

$$\begin{array}{rrrr} 12 & -15 & 21 & 3 \\ -12 & -8 & -12 & -12 \\ \hline 0 & -23 & 9 & -9 \end{array} \qquad \begin{array}{rrrr} 6 & 9 & -6 & 18 \\ -6 & -4 & -6 & -6 \\ \hline 0 & 5 & -12 & 12 \end{array}$$

$$\begin{bmatrix} 3 & 2 & 3 & | & 3 \\ 0 & -23 & 9 & | & -9 \\ 0 & 5 & -12 & | & 12 \end{bmatrix} \begin{matrix} \\ \\ 23R_3 + 5R_2 \end{matrix}$$

$$\begin{array}{rrrr} 0 & 115 & -276 & 276 \\ 0 & -115 & 45 & -45 \\ \hline 0 & 0 & -231 & 231 \end{array}$$

$$\begin{bmatrix} 3 & 2 & 3 & | & 3 \\ 0 & -23 & 9 & | & -9 \\ 0 & 0 & -231 & | & 231 \end{bmatrix} \begin{matrix} 3x + 2y + 3z = 3 \ (3) \\ -23y + 9z = -9 \ (2) \\ -231z = 231 \quad (1) \end{matrix}$$

$(1) \rightarrow \quad \underline{z = -1|}$

$(2) \rightarrow \quad -23y = -9 + 9$

$\qquad \underline{y = 0|}$

$(3) \rightarrow \quad 3x = 3 + 3$

$\qquad \underline{x = 2|}$

$\therefore$ *Solution*: $\underline{(2, \ 0, \ -1)}$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x - 3y + z = 2 \\ 4x - 12y + 4z = 8 \\ -2x + 6y - 2z = -4 \end{cases}$$

Solution

$$\begin{cases} \qquad x - 3y + z = 2 \\ \frac{1}{4} \times \quad 4x - 12y + 4z = 8 \\ -\frac{1}{2} \times \ -2x + 6y - 2z = -4 \end{cases}$$

$$\begin{cases} x - 3y + z = 2 \\ x - 3y + z = 2 \\ x - 3y + z = 2 \end{cases}$$

33

Since all three equations are the same.

∴ *Solution*: is the plane $\underline{x - 3y + z = 2}$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 2x - 2y + z = -1 \\ x + 2y - z = 2 \\ 6x + 4y + 3z = 5 \end{cases}$$

Solution

$$\begin{bmatrix} 1 & 2 & -1 & | & 2 \\ 2 & -2 & 1 & | & -1 \\ 6 & 4 & 3 & | & 5 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 6R_1 \end{matrix}$$

$$\begin{array}{cccc} 2 & -2 & 1 & -1 \\ -2 & -4 & 2 & -4 \\ \hline 0 & -6 & 3 & -5 \end{array}$$

$$\begin{array}{cccc} 6 & 4 & 3 & 5 \\ -6 & -12 & 6 & -12 \\ \hline 0 & -8 & 9 & -7 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & | & 2 \\ 0 & -6 & 3 & | & -5 \\ 0 & -8 & 9 & | & -7 \end{bmatrix} 3R_3 - 4R_2$$

$$\begin{array}{cccc} 0 & -24 & 27 & -21 \\ 0 & 24 & -12 & 20 \\ \hline 0 & 0 & 15 & -1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & | & 2 \\ 0 & -6 & 3 & | & -5 \\ 0 & 0 & 15 & | & -1 \end{bmatrix} \begin{matrix} x + 2y - z = 2 \;\;(3) \\ -6y + 3z = -5 \;\;(2) \\ 15z = -1 \;\;\;\;\;(1) \end{matrix}$$

$(1) \rightarrow \;\; \underline{z = -\dfrac{1}{15}}$

$(2) \rightarrow \;\; -6y = -5 + \dfrac{1}{5}$

$\qquad\qquad -6y = -\dfrac{24}{5}$

$\qquad\qquad \underline{y = \dfrac{4}{5}}$

$(3) \rightarrow \;\; x = 2 - \dfrac{8}{5} - \dfrac{1}{15}$

$\qquad\qquad \underline{= \dfrac{1}{3}}$

∴ *Solution*: $\underline{\left(\dfrac{1}{3}, \; \dfrac{4}{5}, \; -\dfrac{1}{15} \right)}$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 \\ x_2 - 3x_3 - x_4 = 0 \\ 3x_1 + 2x_3 - x_4 = 6 \\ -4x_1 + x_2 + 4x_3 + 2x_4 = -3 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 3 & 0 & 2 & -1 & 6 \\ -4 & 1 & 4 & 2 & -3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_1 \\ R_4 + 4R_1 \end{array}$$

$$\begin{array}{ccccc} 3 & 0 & 2 & -1 & 6 \\ -3 & 15 & -6 & 6 & -12 \\ \hline 0 & 15 & -4 & 5 & -6 \end{array} \qquad \begin{array}{ccccc} -4 & 1 & 4 & 2 & -3 \\ 4 & -20 & 8 & -8 & 16 \\ \hline 0 & -19 & 12 & -6 & 13 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 15 & -4 & 5 & -6 \\ 0 & -19 & 12 & -6 & 13 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 15R_2 \\ R_4 + 19R_2 \end{array}$$

$$\begin{array}{ccccc} 0 & 15 & -4 & 5 & -6 \\ 0 & -15 & 45 & 15 & 0 \\ \hline 0 & 0 & 41 & 20 & -6 \end{array} \qquad \begin{array}{ccccc} 0 & -19 & 12 & -6 & 13 \\ 0 & 19 & -57 & -19 & 0 \\ \hline 0 & 0 & -45 & -25 & 13 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & -45 & -25 & 13 \end{array}\right] \begin{array}{l} \\ \\ \\ 41R_4 + 45R_2 \end{array}$$

$$\begin{array}{ccccc} 0 & 0 & -1845 & -1025 & 533 \\ 0 & 0 & 1845 & 900 & -270 \\ \hline 0 & 0 & 0 & -125 & 263 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & 0 & -125 & 263 \end{array}\right] \qquad \begin{array}{ll} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 & \textcolor{red}{(4)} \\ x_2 - 3x_3 - x_4 = 0 & \textcolor{red}{(3)} \\ 41x_3 + 20x_4 = -6 & \textcolor{red}{(2)} \\ -125x_4 = 263 & \textcolor{red}{(1)} \end{array}$$

$(1) \to \quad \underline{x_4 = -\dfrac{263}{125}}$

$(2) \to \quad 41x_3 = -6 + \dfrac{1,052}{25}$

$\qquad\qquad = \dfrac{902}{25}$

$\qquad\quad \underline{x_3 = \dfrac{22}{25}}$

$(3) \to \quad x_2 = \dfrac{66}{25} - \dfrac{263}{125}$

$\qquad\qquad = \underline{\dfrac{67}{125}}$

$(4) \to \quad x_1 = 4 + \dfrac{67}{25} - \dfrac{44}{25} - \dfrac{526}{125}$

$$= 4 + \frac{23}{25} - \frac{526}{125}$$

$$= \frac{500 + 115 - 526}{125}$$

$$= \frac{89}{125}$$

$\therefore$ *Solution*: $\left(\dfrac{89}{125}, \ \dfrac{67}{125}, \ \dfrac{22}{25}, \ -\dfrac{263}{125} \right)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x_1 + x_2 + x_3 + x_4 = 5 \\ x_1 + 2x_2 - x_3 - 2x_4 = -1 \\ x_1 - 3x_2 - 3x_3 - x_4 = -1 \\ 2x_1 - x_2 + 2x_3 - x_4 = -2 \end{cases}$

Solution

$$\begin{bmatrix} 1 & 1 & 1 & 1 & | & 5 \\ 1 & 2 & -1 & -2 & | & -1 \\ 1 & -3 & -3 & -1 & | & -1 \\ 2 & -1 & 2 & -1 & | & -2 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - 2R_1 \end{matrix}$$

$$\begin{array}{ccccc} 1 & 2 & -1 & -2 & -1 \\ -1 & -1 & -1 & -1 & -5 \\ \hline 0 & 1 & -2 & -3 & -6 \end{array} \qquad \begin{array}{ccccc} 1 & -3 & -3 & -1 & -1 \\ -1 & -1 & -1 & -1 & -5 \\ \hline 0 & -4 & -4 & -2 & -6 \end{array} \qquad \begin{array}{ccccc} 2 & -1 & 2 & -1 & -2 \\ -2 & -2 & -2 & -2 & -10 \\ \hline 0 & -3 & 0 & -3 & -12 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & | & 5 \\ 0 & 1 & -2 & -3 & | & -6 \\ 0 & -4 & -4 & -2 & | & -6 \\ 0 & -3 & 0 & -3 & | & -12 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 4R_2 \\ R_4 + 3R_2 \end{matrix}$$

$$\begin{array}{ccccc} 0 & -4 & -4 & -2 & -6 \\ 0 & 4 & -8 & -12 & -24 \\ \hline 0 & 0 & -12 & -14 & -30 \end{array}$$

$$\begin{array}{ccccc} 0 & -3 & 0 & -3 & -12 \\ 0 & 3 & -6 & -9 & -18 \\ \hline 0 & 0 & -6 & -12 & -30 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & | & 5 \\ 0 & 1 & -2 & -3 & | & -6 \\ 0 & 0 & -12 & -14 & | & -30 \\ 0 & 0 & -6 & -12 & | & -30 \end{bmatrix} \begin{matrix} \\ \\ \\ -2R_4 + R_3 \end{matrix}$$

$$\begin{array}{ccccc} 0 & 0 & 12 & 24 & 60 \\ 0 & 0 & -12 & -14 & -30 \\ \hline 0 & 0 & 0 & 10 & 30 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & -12 & -14 & -30 \\ 0 & 0 & 0 & 10 & 30 \end{bmatrix} \quad \begin{array}{l} x_1 + x_2 + x_3 + x_4 = 5 \quad (4) \\ x_2 - 2x_3 - 3x_4 = -6 \quad (3) \\ -12x_3 - 14x_4 = -30 \quad (2) \\ 10x_4 = 30 \quad (1) \end{array}$$

$(1) \rightarrow \quad \underline{x_4 = 3}$

$(2) \rightarrow \quad -12x_3 = -30 + 42$

$\qquad\qquad = 12$

$\qquad \underline{x_3 = -1}$

$(3) \rightarrow \quad x_2 = -6 - 2 + 9$

$\qquad\qquad \underline{= 1}$

$(4) \rightarrow \quad x_1 = 5 - 1 + 1 - 3$

$\qquad\qquad \underline{= 2}$

$\therefore$ *Solution*: $\underline{(2, \ 1, \ -1, \ 3)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x + 8y - z + w = 0 \\ 4x + 16y - 3z - w = -10 \\ -2x + 4y - z + 3w = -6 \\ -6x + 2y + 5z + w = 3 \end{cases}$

Solution

$$\begin{bmatrix} 2 & 8 & -1 & 1 & 0 \\ 4 & 16 & -3 & -1 & -10 \\ -2 & 4 & -1 & 3 & -6 \\ -6 & 2 & 5 & 1 & 3 \end{bmatrix} \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{array}$$

$$\begin{bmatrix} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 26 & 2 & 4 & 3 \end{bmatrix} \quad R_4 - \frac{13}{6}R_2$$

$$\begin{bmatrix} 2 & 8 & -1 & 1 & 0 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 26 & 2 & 4 & 3 \end{bmatrix} \quad \textit{Interchange } R_2 \textit{ and } R_3$$

37

$$\begin{bmatrix} 2 & 8 & -1 & 1 & | & 0 \\ 0 & 12 & -2 & 4 & | & -6 \\ 0 & 0 & -1 & -3 & | & -10 \\ 0 & 0 & \frac{19}{3} & -\frac{14}{3} & | & 16 \end{bmatrix} R_4 + \frac{19}{3}R_3$$

$$\begin{bmatrix} 2 & 8 & -1 & 1 & | & 0 \\ 0 & 12 & -2 & 4 & | & -6 \\ 0 & 0 & -1 & -3 & | & -10 \\ 0 & 0 & 0 & -\frac{71}{3} & | & -\frac{142}{3} \end{bmatrix}$$
$2x + 8y - z + w = 0 \quad (3)$
$12y - 2z + 4w = -6 \quad (2)$
$-z - 3w = -10 \quad (1)$
$-\frac{71}{3}w = -\frac{142}{3} \quad \rightarrow \quad \underline{w = 2}|$

$(1) \rightarrow \quad z = 10 - 3w \underline{= 4}|$

$(2) \rightarrow \quad 12y = 2z - 4w - 6$

$\qquad \underline{y = -\frac{1}{2}}|$

$(3) \rightarrow \quad 2x = -8y + z - w$

$\qquad 2x = 4 + 4 - 2$

$\qquad 2x = 6$

$\qquad \underline{x = 3}|$

$\therefore Solution: \left(3, \ -\frac{1}{2}, \ 4, \ 2\right)|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 \qquad = 0 \\ \qquad x_2 + x_3 = 0 \end{cases}$

Solution

$\begin{cases} x_1 = -2x_2 \\ x_3 = -x_2 \end{cases}$

$2x_1 + x_2 + 3x_3 = 0$

$-4x_2 + x_2 - 3x_2 = 0 \quad \rightarrow \quad \underline{x_2 = 0}|$

$\therefore Solution: (0, 0, 0)|$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 2x + 2y + 4z \quad\;\; = 0 \\ \quad\; -y - 3z + w = 0 \\ 3x + y + z + 2w = 0 \\ x + 3y - 2z - 2w = 0 \end{cases}$$

Solution

$$\begin{bmatrix} 2 & 2 & 4 & 0 & | & 0 \\ 0 & -1 & -3 & 1 & | & 0 \\ 3 & 1 & 1 & 2 & | & 0 \\ 1 & 3 & -2 & -2 & | & 0 \end{bmatrix} \quad \begin{matrix} \\ -R_2 \\ 2R_3 - 3R_1 \\ 2R_4 - R_1 \end{matrix}$$

$$\begin{bmatrix} 2 & 2 & 4 & 0 & | & 0 \\ 0 & 1 & 3 & -1 & | & 0 \\ 0 & -4 & -10 & 4 & | & 0 \\ 0 & 4 & -8 & -4 & | & 0 \end{bmatrix} \quad \begin{matrix} \\ \\ R_3 + 4R_2 \\ R_4 - 4R_2 \end{matrix}$$

$$\begin{bmatrix} 2 & 2 & 4 & 0 & | & 0 \\ 0 & 1 & 3 & -1 & | & 0 \\ 0 & 0 & 2 & 0 & | & 0 \\ 0 & 0 & -20 & 0 & | & 0 \end{bmatrix} \quad \begin{matrix} 2x + 2y - 4z = 0 & (1) \\ y + 3z - w = 0 & (2) \\ \rightarrow \underline{z = 0|} \\ \end{matrix}$$

$(2) \;\rightarrow\; \underline{y = w|}$

$(1) \;\rightarrow 2x = -2y \quad \underline{x = -w|}$

∴ *Solution*: $\underline{(-w,\; w,\; 0,\; w)}$ |

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 2x \qquad + z + w = 5 \\ \quad\; y \qquad - w = -1 \\ 3x \qquad - z - w = 0 \\ 4x + y + 2z + w = 9 \end{cases}$$

Solution

$$\begin{bmatrix} 2 & 0 & 1 & 1 & | & 5 \\ 0 & 1 & 0 & -1 & | & -1 \\ 3 & 0 & -1 & -1 & | & 0 \\ 4 & 1 & 2 & 1 & | & 9 \end{bmatrix} \quad \begin{matrix} \\ \\ 2R_3 - 3R_1 \\ 2R_4 - 4R_1 \end{matrix}$$

$$\begin{bmatrix} 2 & 0 & 1 & 1 & | & 5 \\ 0 & 1 & 0 & -1 & | & -1 \\ 0 & 0 & -5 & -5 & | & -15 \\ 0 & 2 & 0 & -2 & | & -2 \end{bmatrix} \quad R_4 - 2R_2$$

$$\begin{bmatrix} 2 & 0 & 1 & 1 & | & 5 \\ 0 & 1 & 0 & -1 & | & -1 \\ 0 & 0 & -5 & -5 & | & -15 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \quad \begin{aligned} 2x + z + w &= 5 \quad (1) \\ y - w &= -1 \quad (2) \\ -5z - 5w &= -15 \quad (3) \end{aligned}$$

$(2) \rightarrow \qquad \underline{y = 1 + w}$

$(3) \rightarrow \qquad \underline{z = 3 - w}$

$(1) \rightarrow \quad 2x = 5 - (3 - w) - w \quad \Rightarrow \quad \underline{x = 1}$

$\therefore$ *Solution*: $\underline{(1,\ 1 + w,\ 3 - w,\ w)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 4y + z = 20 \\ 2x - 2y + z = 0 \\ x \qquad + z = 5 \\ x + y - z = 10 \end{cases}$

Solution

$$\begin{bmatrix} 1 & 1 & -1 & | & 10 \\ 2 & -2 & 1 & | & 0 \\ 1 & 0 & 1 & | & 5 \\ 0 & 4 & 1 & | & 20 \end{bmatrix} \quad \begin{aligned} R_2 - 2R_1 \\ R_3 - R_1 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -1 & | & 10 \\ 0 & -4 & 3 & | & -20 \\ 0 & -1 & 2 & | & -5 \\ 0 & 4 & 1 & | & 20 \end{bmatrix} \quad \begin{aligned} 4R_3 - R_2 \\ R_4 + R_2 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -1 & | & 10 \\ 0 & -4 & 3 & | & -20 \\ 0 & 0 & 5 & | & 0 \\ 0 & 0 & 4 & | & 0 \end{bmatrix} \quad \begin{aligned} x + y &= 10 \\ \rightarrow -4y &= -20 \\ \rightarrow z &= 0 \end{aligned}$$

$\therefore$ *Solution*: $\underline{(5,\ 5,\ 0)}$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \qquad\qquad - 3w = -3 \end{cases}$$

Solution

$$\begin{bmatrix} 1 & 2 & 1 & \big| & 8 \\ -1 & 3 & -2 & \big| & 1 \\ 3 & 4 & -7 & \big| & 10 \end{bmatrix} \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 & \big| & 8 \\ 0 & 5 & -1 & \big| & 9 \\ 0 & -2 & -10 & \big| & -14 \end{bmatrix} \begin{array}{l} \\ \\ 5R_3 + 2R_2 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 & \big| & 8 \\ 0 & 5 & -1 & \big| & 9 \\ 0 & 0 & -52 & \big| & -52 \end{bmatrix} \begin{array}{l} x + 2y + z = 8 \quad (3) \\ 5y - z = 9 \quad\;\; (2) \\ -52z = -52 \quad (1) \end{array}$$

(1) $\Rightarrow$ $z = 1$

(2) $\Rightarrow$ $5y = 9 + 1 = 10 \rightarrow y = 2$

(3) $\Rightarrow$ $x = 8 - 4 - 1 = 3$

$\therefore$ *Solution*: $(3,\ 2,\ 1)$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} 2u - 3v + w - x + y = 0 \\ 4u - 6v + 2w - 3x - y = -5 \\ -2u + 3v - 2w + 2x - y = 3 \end{cases}$$

Solution

$$\begin{bmatrix} 2 & -3 & 1 & -1 & 1 & \big| & 0 \\ 4 & -6 & 2 & -3 & -1 & \big| & -5 \\ -2 & 3 & -2 & 2 & -1 & \big| & 3 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\begin{bmatrix} 2 & -3 & 1 & -1 & 1 & \big| & 0 \\ 0 & 0 & 0 & -1 & -3 & \big| & -5 \\ 0 & 0 & -1 & 1 & 0 & \big| & 3 \end{bmatrix} \begin{array}{l} 2u - 3v + w - x + y = 0 \quad (3) \\ -x - 3y = -5 \qquad\qquad\;\; (2) \\ -w + x = 3 \qquad\qquad\quad\; (1) \end{array}$$

$(2) \quad \Rightarrow \quad x = 5 - 3y$

$(1) \quad \Rightarrow \quad w = x - 3 = 2 - 3y$

$(3) \quad \Rightarrow \quad 2u = 3v - 2 + 3y + 5 - 3y - y = 3v - y + 3$

$$u = \frac{3}{2}v - \frac{1}{2}y + \frac{3}{2}$$

$\therefore$ *Solution*: $\left(\frac{3}{2}v - \frac{1}{2}y + \frac{3}{2}, v, 2 - 3y, 5 - 3y, y\right)$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 6x_3 + 2x_4 - 4x_5 - 8x_6 = 8 \\ 3x_3 + x_4 - 2x_5 - 4x_6 = 4 \\ 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 + x_6 = 2 \\ 6x_1 - 9x_2 + 11x_4 - 19x_5 + 3x_6 = 1 \end{cases}$$

Solution

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 6 & -9 & 0 & 11 & -19 & 3 & 1 \end{array}\right] \quad R_4 - 3R_1$$

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & -3 & -1 & 2 & 0 & -5 \end{array}\right] \quad \begin{array}{l} R_3 - 2R_2 \\ R_4 + R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & -1 \end{array}\right]$$

$2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 = 2 - x_6$

$3x_3 + x_4 - 2x_5 = 4 + 4x_6$

$\rightarrow \underline{x_6 = \frac{1}{4}}$

$$\begin{cases} 3x_3 = 5 - x_4 + 2x_5 \\ 2x_1 = \frac{7}{4} + 3x_2 - \frac{1}{3}\left(5 - x_4 + 2x_5\right) - 4x_4 + 7x_5 \end{cases}$$

$$\begin{cases} \underline{x_3 = \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5} \\ 2x_1 = \frac{1}{12} + 3x_2 - \frac{11}{3}x_4 + \frac{19}{3}x_5 \end{cases}$$

$\therefore$ *Solution*: $\quad \left(\frac{1}{24} + \frac{3}{2}x_2 - \frac{11}{6}x_4 + \frac{19}{6}x_5, \ x_2, \ \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5, \ x_4, \ x_5, \ \frac{1}{4}\right)$

Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array}\right] \begin{array}{l} \\ 3R_2 - 5R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & -1 & 4 & 26 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{ll} 3x_1 + 2x_2 - x_3 = -15 & (3) \\ -x_2 + 11x_3 = 75 & (2) \\ -7x_3 = -49 & (1) \end{array}$$

$(1) \rightarrow x_3 = 7$

$(2) \rightarrow x_2 = 77 - 75 = 2$

$(1) \rightarrow 3x_1 = -15 - 4 + 7 = 12$

$$\underline{x_1 = -4}$$

$\therefore$ **Solution**: $\underline{(-4, 2, 7)}$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} x_1 + 3x_2 - 2x_3 \qquad\qquad + 2x_5 \qquad\quad = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ \qquad\qquad 5x_3 + 10x_4 \qquad + 15x_6 = 5 \\ 2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$$

Solution

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \quad \begin{array}{l} \\ -R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \quad \begin{array}{l} \\ \\ R_3 - 5R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right] \quad \tfrac{1}{6}R_4 \;\; then\, interchanging\, row3\, and\, row4$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \tfrac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_2 - 3R_3$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \tfrac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \rightarrow \begin{cases} x_1 + 3x_2 \qquad + 4x_4 + 2x_5 \qquad = 0 \\ \qquad\qquad x_3 + 2x_4 \qquad\qquad = 0 \\ \qquad\qquad\qquad\qquad\qquad + x_6 = \tfrac{1}{3} \end{cases}$$

The general solution of the system: $\;\; x_6 = \tfrac{1}{3}, \;\; x_3 = -2x_4, \;\; x_1 = -3x_2 - 4x_4 - 2x_5$

$\therefore$ *Solution*: $\;\; \left(-3x_2 - 4x_4 - 2x_5, \;\; x_2, \;\; -2x_4, \;\; x_4, \;\; x_5, \;\; \tfrac{1}{3} \right)$

Exercise

At SnackMix, caramel corn worth $2.50 per *pound* is mixed with honey roasted missed nuts worth $7.50 per *pound* in order to get 20 *lbs*. of a mixture worth $4.50 per *pound*. How much of each snack is used?

Solution

$$x + y = 20 \qquad \text{(1)}$$
$$2.50x + 7.50y = 90 \qquad \text{(2)}$$

(1) $y = 20 - x$

(2) $\quad 2.5x + 7.5\,(20 - x) = 90$

$\quad\quad 2.5x + 150 - 7.5x = 90$

$\quad\quad -5x = 90 - 150$

$\quad\quad -5x = -60$

$\quad\quad x = \dfrac{-60}{-5} = 12$

$y = 20 - x$

$\quad = 20 - 12$

$\quad = 8$

The mixture consists of 12 *lbs*. of caramel and 8 *lbs*. of nuts