Section 4.2 – Series Solutions near Ordinary Points

Example of a First-Order Equation

Find the series solution for the differential equation $y' - 2xy = 0$

Solution

We look for a solution of the form: $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y' - 2xy = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - 2x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

$$a_1 + \sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1} - \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

$$a_1 + \sum_{n=0}^{\infty} \left[(n+2) a_{n+2} - 2a_n \right] x^{n+1} = 0$$

$$\begin{cases} \boxed{a_1 = 0} \\ (n+2) a_{n+2} - 2a_n = 0 \quad \Rightarrow \quad \boxed{a_{n+2} = \frac{2a_n}{n+2}} \end{cases}$$

$\Rightarrow$ *Let* $a_0 = y(0)$ $\qquad a_1 = 0$

$$a_2 = \frac{2a_0}{2} = y(0) \qquad a_3 = \frac{2a_1}{3} = 0$$

$$a_4 = \frac{2a_2}{4} = \frac{1}{2} y(0) \qquad a_5 = \frac{2a_3}{5} = 0$$

$$a_6 = \frac{2a_4}{6} = \frac{1}{6} y(0)$$

$$a_8 = \frac{2a_6}{8} = \frac{1}{2.3.4} y(0)$$

$$y(x)=\sum_{k=0}^{\infty} a_{2k}x^{2k} \ \boxed{= y(0)\sum_{k=0}^{\infty}\frac{x^{2k}}{k!}}$$

Example

Find the general series solution to the equation

$$y''+xy'+y=0$$

Find the particular solution with $y(0)=0$ and $y'(0)=2$

Solution

$$y(x)=\sum_{n=0}^{\infty} a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} na_n x^{n-1}$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \ = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''+xy'+y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} na_n x^{n} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + na_n + a_n\right]x^n = 0$$

$$(n+2)(n+1)a_{n+2} + (n+1)a_n = 0$$

$$(n+2)(n+1)a_{n+2} = -(n+1)a_n \qquad \Rightarrow \qquad a_{n+2} = -\frac{1}{n+2}a_n$$

$$a_0 = y(0) = 0 \qquad\qquad a_1 = y'(0) = 2$$

$$a_2 = -\frac{1}{2}a_0 \qquad\qquad a_3 = -\frac{1}{3}a_1$$

$$a_4 = -\frac{1}{4}a_2 = \frac{1}{2\cdot 4}a_0 \qquad\qquad a_5 = -\frac{1}{5}a_3 = \frac{1}{3\cdot 5}a_1$$

$$a_6 = -\frac{1}{6}a_4 = -\frac{1}{2\cdot 4\cdot 6}a_0 \qquad\qquad a_7 = -\frac{1}{7}a_7 = -\frac{1}{3\cdot 5\cdot 7}a_1$$

The general solution can be written as:

$$y(x) = a_0\left[1 - \frac{1}{2}x^2 + \frac{1}{2\cdot 4}x^4 - \frac{1}{2\cdot 4\cdot 6}x^6 + \cdots\right] + a_1\left[x - \frac{1}{3}x^3 + \frac{1}{3\cdot 5}x^5 - \frac{1}{3\cdot 5\cdot 7}x^7 + \cdots\right]$$

For the given initial $y(0) = 0$ and $y'(0) = 2$, the solution is:

$$\boxed{y(x) = 2\left(x - \frac{1}{3}x^3 + \frac{1}{3\cdot 5}x^5 - \frac{1}{3\cdot 5\cdot 7}x^7 + \cdots\right)}$$

Exercises *Section* 4.2 – Series Solutions near Ordinary Points

Find the series solution.

1. $y' = 3y$

2. $y' = 4y$

3. $y' = x^2 y$

4. $y' + 2xy = 0$

5. $(x-2)y' + y = 0$

6. $(2x-1)y' + 2y = 0$

7. $2(x-1)y' = 3y$

8. $(1+x)y' - y = 0$

9. $(2-x)y' + 2y = 0$

10. $(x-4)y' + y = 0$

11. $x^2 y' = y - x - 1$

12. $(x-3)y' + 2y = 0$

13. $xy' + y = 0$

14. $x^3 y' - 2y = 0$

15. $y'' = 4y$

16. $y'' = 9y$

17. $y'' + y = 0$

18. $y'' - y = 0$

19. $y'' + y = x$

20. $y'' - xy = 0$

21. $y'' + xy = 0$

22. $y'' + xy' + y = 0$

23. $y'' - xy' - y = 0$

24. $y'' + x^2 y = 0$

25. $y'' + k^2 x^2 y = 0$

26. $y'' + 3xy' + 3y = 0$

27. $y'' - 2xy' + y = 0$

28. $y'' - xy' + 2y = 0$

29. $y'' - xy' - x^2 y = 0$

30. $y'' + x^2 y' + xy = 0$

31. $y'' + x^2 y' + 2xy = 0$

32. $y'' - x^2 y' - 3xy = 0$

33. $y'' + 2xy' + 2y = 0$

34. $2y'' + xy' + y = 0$

35. $3y'' + xy' - 4y = 0$

36. $5y'' - 2xy' + 10y = 0$

37. $(x-1)y'' + y' = 0$

38. $(x+2)y'' + xy' - y = 0$

39. $y'' - (x+1)y = 0$

40. $y'' - (x+1)y' - y = 0$

41. $(x^2+1)y'' - 6y = 0$

42. $(x^2+2)y'' + 3xy' - y = 0$

43. $(x^2-1)y'' + xy' - y = 0$

44. $(x^2+1)y'' + xy' - y = 0$

45. $(x^2+1)y'' - xy' + y = 0$

46. $(1-x^2)y'' - 6xy' - 4y = 0$

47. $y'' + (x-1)^2 y' - 4(x-1)y = 0$

48. $(2-x^2)y'' - xy' + 16y = 0$

49. $(x^2+1)y'' + 6xy' + 4y = 0$

50. $(x^2-1)y'' - 6xy' + 12y = 0$

51. $(x^2-1)y'' + 8xy' + 12y = 0$

52. $(x^2-1)y'' + 4xy' + 2y = 0$

53. $(x^2+1)y'' - 4xy' + 6y = 0$

54. $(x^2+2)y'' + 4xy' + 2y = 0$

55. $(x^2-3)y'' + 2xy' = 0$

56. $(x^2+3)y'' - 7xy' + 16y = 0$

Find the series solution to the initial value problems

57. $y'' + 4y = 0$; $y(0) = 0,\quad y'(0) = 3$

58. $y'' + x^2 y = 0$; $y(0) = 1,\quad y'(0) = 0$

59. $y'' - 2xy' + 8y = 0$; $y(0) = 3,\quad y'(0) = 0$

60. $y'' + y' - 2y = 0$; $y(0) = 1,\quad y'(0) = -2$

61. $y'' - 2y' + y = 0$; $y(0) = 0,\quad y'(0) = 1$

62. $y'' + xy' + y = 0 \quad y(0) = 1 \quad y'(0) = 0$

63. $y'' - xy' - y = 0 \quad y(0) = 2 \quad y'(0) = 1$

64. $y'' - xy' - y = 0 \quad y(0) = 1 \quad y'(0) = 0$

65. $y'' + xy' - 2y = 0 \quad y(0) = 1 \quad y'(0) = 0$

66. $y'' + (x-1)y' + y = 0 \quad y(1) = 2 \quad y'(1) = 0$

67. $(x-1)y'' - xy' + y = 0$; $y(0) = -2,\quad y'(0) = 6$

68. $(x+1)y'' - (2-x)y' + y = 0$; $y(0) = 2,\quad y'(0) = -1$

69. $(1-x)y'' + xy' - 2y = 0$; $y(0) = 0,\quad y'(0) = 1$

70. $\left(x^2+1\right)y'' + 2xy' = 0$; $y(0) = 0,\quad y'(0) = 1$

71. $\left(2+x^2\right)y'' - xy' + 4y = 0 \quad y(0) = -1 \quad y'(0) = 3$

72. $\left(2-x^2\right)y'' - xy' + 4y = 0 \quad y(0) = 1 \quad y'(0) = 0$

73. $\left(4-x^2\right)y'' + 2y = 0 \quad y(0) = 0 \quad y'(0) = 1$

74. $\left(x^2-4\right)y'' + 3xy' + y = 0$; $y(0) = 4,\quad y'(0) = 1$

75. $\left(x^2+1\right)y'' + 2xy' - 2y = 0$; $y(0) = 0,\quad y'(0) = 1$

76. $\left(x^2-1\right)y'' + 3xy' + xy = 0$; $y(0) = 4,\quad y'(0) = 6$

77. $\left(2x-x^2\right)y'' - 6(x-1)y' - 4y = 0$; $y(1) = 0,\quad y'(1) = 1$

78. $\left(x^2-6x+10\right)y'' - 4(x-3)y' + 6y = 0$; $y(3) = 2,\quad y'(3) = 0$

79. $\left(4x^2+16x+17\right)y'' - 8y = 0$; $y(-2) = 1,\quad y'(-2) = 0$

80. $\left(x^2+6x\right)y'' + (3x+9)y' - 3y = 0$; $y(-3) = 0,\quad y'(-3) = 2$

Find the series solution near the given value

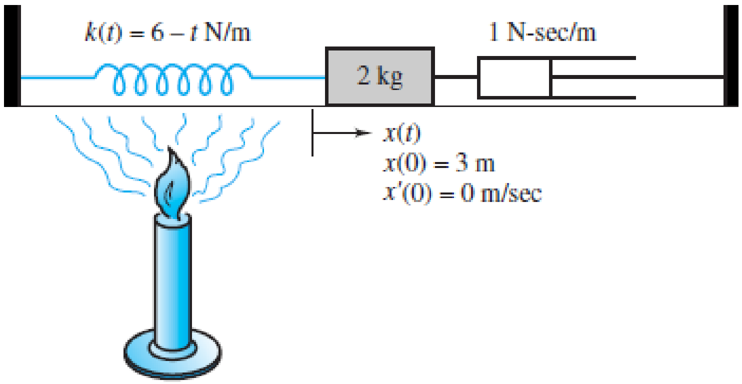
81. $y'' - (x-2)y' + 2y = 0$; *near* $x = 2$

82. $y'' + (x-1)^2 y' - 4(x-1)y = 0$; *near* $x = 1$

83. $y'' + (x-1)y = e^x$; $near\ x = 1$

84. $y'' + xy' + (2x-1)y = 0$; $near\ x = -1$ $y(-1) = 2,$ $y'(-1) = -2$

85. As a spring is heated, its spring "constant" decreases. Suppose the spring is heated so that the spring "constant" at time t is $k(t) = 6 - t$ N/m.



If the unforced mass-spring system has mass $m = 2$ kg and a damping constant $b = 1$ $N\text{-}sec/m$ with initial conditions $x(0) = 3$ m and $x'(0) = 0$ m/sec, then the displacement $x(t)$ is governed by the initial value problem

$$2x''(t) + x'(t) + (6-t)x(t) = 0\ ; \quad x(0) = 3, \quad x'(0) = 0$$

Find at least the first four nonzero terms in a power series expansion about $t = 0$ for the displacement.