Lecture Three - Multiple Integrals

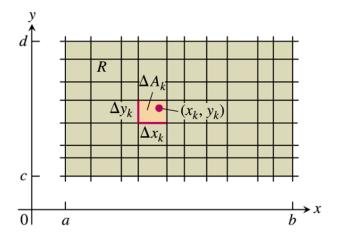
Section 3.1 – Double Integrals over Rectangular Regions

Double Integrals

Consider a function f(x, y) defined on a rectangular region R,

$$R: a \le x \le b, c \le y \le d$$

A small rectangular piece of width Δx and height Δy has area $\Delta A = \Delta x \Delta y$.



To form a Riemann sum over R, select a point (x_k, y_k) in the k^{th} small rectangle, multiply the value of f at that point by the area ΔA_k and add together the products:

$$S_n = \sum_{k=1}^n f(x_k, y_k) \Delta A_k$$

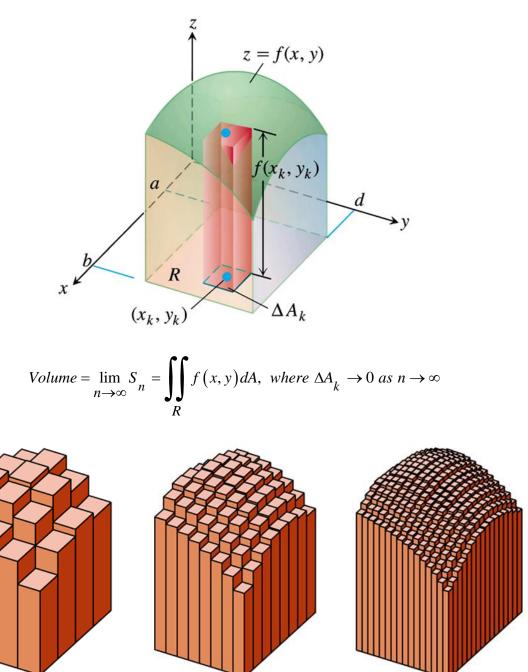
As the rectangles get narrow and short, their number n increases, therefore

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k$$

Then the function f is said to be integrable and the limit is called double integral of f over R,

$$\iint\limits_R f(x,y)dA \quad \text{or} \quad \iint\limits_R f(x,y)dxdy$$

Double Integrals as Volumes



As n increases, the **Riemann** sum approximations approach the total volume of the solid

n = 16

n = 64

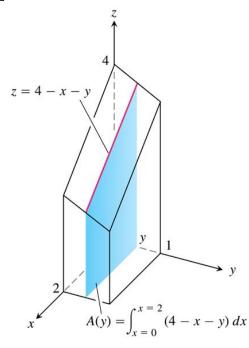
n = 256

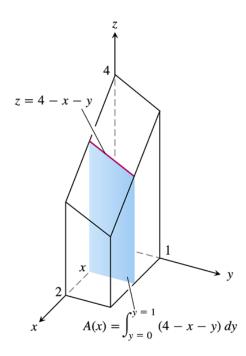
Example

Calculate the volume under the plane z = 4 - x - y over the rectangular region $R: 0 \le x \le 2, 0 \le y \le 1$ in the xy-plane.

Solution

Volume =
$$\int_{x=0}^{x=2} A(x) dx$$
=
$$\int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y) dy dx$$
=
$$\int_{x=0}^{x=2} \left[4y - xy - \frac{1}{2} y^2 \right]_{y=0}^{y=1} dx$$
=
$$\int_{x=0}^{x=2} (4 - x - \frac{1}{2}) dx$$
=
$$\int_{x=0}^{x=2} \left(\frac{7}{2} - x \right) dx$$
=
$$\left[\frac{7}{2} x - \frac{1}{2} x^2 \right]_{0}^{2}$$
=
$$7 - 2$$
=
$$5 \int_{x=0}^{x=2} unit^3$$





$$Volume = \int_0^1 \int_0^2 (4 - x - y) dx dy$$

Theorem – Fubini's Theorem

If f(x, y) is continuous throughout the rectangular region R: $a \le x \le b$, $c \le y \le d$, then

$$\iint_{R} f(x, y) dA = \int_{c}^{d} \int_{a}^{b} f(x, y) dx dy = \int_{a}^{b} \int_{c}^{d} f(x, y) dy dx$$

Example

Calculate

$$\iint_{R} f(x, y) dA \text{ for } f(x, y) = 100 - 6x^{2}y \text{ and } R: 0 \le x \le 2, -1 \le y \le 1$$

Solution

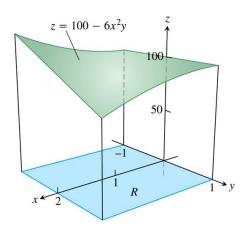
$$\int_{-1}^{1} \int_{0}^{2} (100 - 6x^{2}y) dx dy = \int_{-1}^{1} \left[100x - 2x^{3}y \right]_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1} (200 - 16y) dy$$

$$= \left[200y - 8y^{2} \right]_{-1}^{1}$$

$$= 200 - 8 - (-200 - 8)$$

$$= 400$$

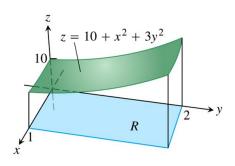


Example

Find the volume of the region bounded above the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below the rectangle $R: 0 \le x \le 1, 0 \le y \le 2$

Solution

Volume =
$$\int_{0}^{1} \int_{0}^{2} (10 + x^{2} + 3y^{2}) dy dx$$
=
$$\int_{0}^{1} [10y + yx^{2} + y^{3}]_{0}^{2} dx$$
=
$$\int_{0}^{1} (2x^{2} + 28) dx$$
=
$$\left[\frac{2}{3}x^{3} + 28x\right]_{0}^{1} = \frac{2}{3} + 28$$
=
$$\frac{86}{3}$$



Exercises Section 3.1 – Double Integrals over Rectangular Regions

Evaluate the iterated integral

$$1. \qquad \int_1^2 \int_0^4 2xy \ dy dx$$

2.
$$\int_{0}^{2} \int_{-1}^{1} (x - y) \, dy dx$$

3.
$$\int_0^1 \int_0^1 \left(1 - \frac{x^2 + y^2}{2}\right) dx dy$$

4.
$$\int_{0}^{3} \int_{-2}^{0} \left(x^{2}y - 2xy \right) dy dx$$

$$5. \qquad \int_0^1 \int_0^1 \frac{y}{1+xy} dx dy$$

6.
$$\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y} dy dx$$

7.
$$\int_0^1 \int_1^2 xye^x dy dx$$

8.
$$\int_{\pi}^{2\pi} \int_{0}^{\pi} (\sin x + \cos y) dx dy$$

9.
$$\int_{1}^{2} \int_{1}^{4} \frac{xy}{\left(x^2 + y^2\right)^2} dxdy$$

$$10. \quad \int_{1}^{3} \int_{1}^{e^{x}} \frac{x}{y} \, dy dx$$

$$11. \quad \int_1^2 \int_0^{\ln x} x^3 e^y dy dx$$

12.
$$\int_{1}^{10} \int_{0}^{1/y} y e^{xy} dx dy$$

$$13. \quad \int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} xy \ dxdy$$

$$14. \quad \int_0^1 \int_{x^2}^x \sqrt{x} \ dy dx$$

15.
$$\int_{0}^{3/2} \int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y dx dy$$

$$16. \quad \int_0^2 \int_0^{4-x^2} 2x \ dy dx$$

17.
$$\int_0^1 \int_{2y}^2 4\cos(x^2) \, dx dy$$

18.
$$\int_0^1 \int_{\sqrt[3]{y}}^1 \frac{2\pi \sin \pi x^2}{x^2} \, dx dy$$

5

Evaluate the double integral over the given region R.

19.
$$\iint_{R} (6y^2 - 2x) dA \quad R: \quad 0 \le x \le 1, \quad 0 \le y \le 2$$

20.
$$\iint_{R} \left(\frac{\sqrt{x}}{y^2} \right) dA \quad R: \quad 0 \le x \le 4, \quad 1 \le y \le 2$$

21.
$$\iint_{R} y \sin(x+y) dA \quad R: \quad -\pi \le x \le 0, \quad 0 \le y \le \pi$$

22.
$$\iint_{R} e^{x-y} dA \quad R: \quad 0 \le x \le \ln 2, \quad 0 \le y \le \ln 2$$

23.
$$\iint_{R} \frac{y}{x^2 y^2 + 1} dA \quad R: \quad 0 \le x \le 1, \quad 0 \le y \le 1$$

24.
$$\iint_{R} x^{-1/2} e^{y} dA$$
; R is the region bounded by $x = 1$, $x = 4$, $y = \sqrt{x}$, and $y = 0$

25.
$$\iint_{R} (x^2 + y^2) dA \; ; \; R \text{ is the region } \{(x, y) : 0 \le x \le 2, \ 0 \le y \le x\}$$

26.
$$\iint_R \frac{2y}{\sqrt{x^4 + 1}} dA$$
; R is the region bounded by $x = 1$, $x = 2$, $y = x^{3/2}$, $y = 0$

27. Integrate
$$f(x, y) = \frac{1}{xy}$$
 over the *square* $1 \le x \le 2$, $1 \le y \le 2$

28. Integrate
$$f(x, y) = y \cos xy$$
 over the *rectangle* $0 \le x \le \pi$, $0 \le y \le 1$

29. Find the volume of the region bounded above the paraboloid $z = x^2 + y^2$ and below by the square $R: -1 \le x \le 1, -1 \le y \le 1$

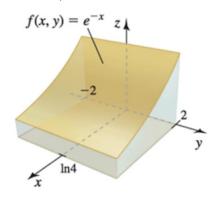
30. Find the volume of the region bounded above the plane
$$z = \frac{y}{2}$$
 and below by the rectangle $R: 0 \le x \le 4, 0 \le y \le 2$

31. Find the volume of the region bounded above the surface
$$z = 4 - y^2$$
 and below by the rectangle $R: 0 \le x \le 1, 0 \le y \le 2$

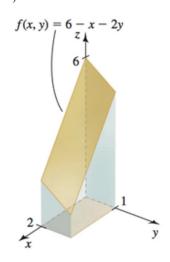
32. Find the volume of the region bounded above the ellipitical paraboloid
$$z = 16 - x^2 - y^2$$
 and below by the square $R: 0 \le x \le 2, 0 \le y \le 2$

33. Evaluate
$$\int_0^{1/2} \left(\sin^{-1} [2x] - \sin^{-1} x \right) dx$$
 by converting it to a double integral.

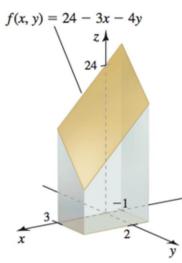
34. Find the volume of the solid beneath the cylinder $f(x, y) = e^{-x}$ and above the region $R = \{(x, y): 0 \le x \le \ln 4, -2 \le y \le 2\}$



35. Find the volume of the solid beneath the plane f(x, y) = 6 - x - 2y and above the region $R = \{(x, y): 0 \le x \le 2, 0 \le y \le 1\}$



36. Find the volume of the solid beneath the plane f(x, y) = 24 - 3x - 4y and above the region $R = \{(x, y): -1 \le x \le 3, 0 \le y \le 2\}$



37. Find the volume of the solid beneath the paraboloid $f(x, y) = 12 - x^2 - 2y^2$ and above the region $R = \{(x, y): 1 \le x \le 2, 0 \le y \le 1\}$

