# *Section* 2.6 – Tangent Planes and Linear Approximation

## Tangent Planes and Normal Lines

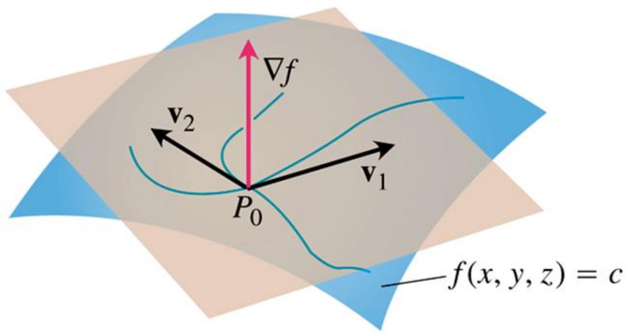
If $\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$ is a smooth curve on the level surface $f(x,y,z) = c$ of a differentiable function *f*, then $f(g(t),h(t),k(t)) = c$.

Differentiating both sides of this equation with respect to *t* leads to

$$\frac{d}{dt} f(g(t),h(t),k(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\hat{i} + \frac{dh}{dt}\hat{j} + \frac{dk}{dt}\hat{k}\right)}_{\frac{d\mathbf{r}}{dt}} = 0$$




### *Definition*

The ***tangent plane*** at the point $P_0(x_0,y_0,z_0)$ on the level surface $f(x,y,z) = c$ of a differentiable function *f* is the plane through $P_0$ normal to $\nabla f\big|_{P_0}$.

The ***normal line*** of the surface at $P_0$ is the line through $P_0$ parallel to $\nabla f\big|_{P_0}$.

***Normal Line*** to $f(x,y,z) = c$ at $P_0(x_0,y_0,z_0)$

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$$

***Tangent Plane*** to $f(x,y,z) = c$ at $P_0(x_0,y_0,z_0)$

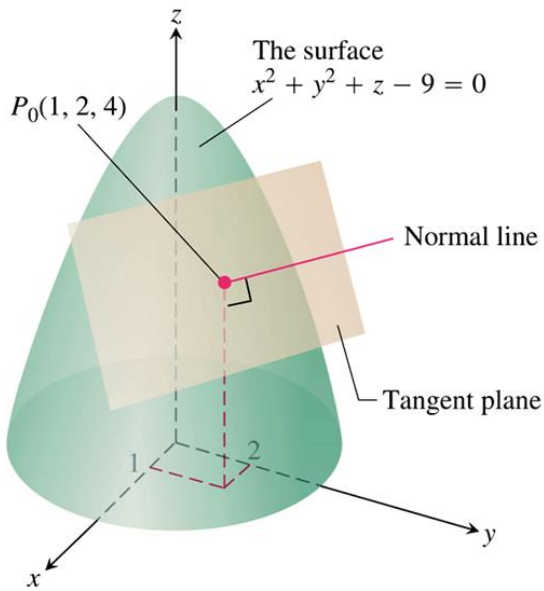
$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

### *Example*

Find the tangent plane and normal line of the surface $f(x,y,z)=x^2+y^2+z-9=0$ at the point $P_0(1,\ 2,\ 4)$

***Solution***

The tangent plane is the plane through $P_0$ perpendicular to the gradient of *f* at $P_0$.




The gradient is:

$$\nabla f\Big|_{P_0}=\left(2x\hat{i}+2y\hat{j}+\hat{k}\right)\Big|_{(1,\ 2,\ 4)}$$

$$=2\hat{i}+4\hat{j}+\hat{k}$$

The tangent plane is the plane

$$2(x-1)+4(y-2)+(z-4)=0$$

$$2x+4y+z=14$$

The line normal to the surface at $P_0$ is

$$x=1+2t,\quad y=2+4t,\quad z=4+t$$

**Plane Tangent to a Surface** $z=f(x,y)$ **at** $\left(x_0,\ y_0,\ f(x_0,y_0)\right)$

The plane tangent to the surface $z=f(x,y)$ of a differentiable function *f* at the point

$P_0\left(x_0,\ y_0,\ z_0\right)=\left(x_0,\ y_0,\ f(x_0,y_0)\right)$ is

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)-(z-z_0)=0$$

### *Example*

Find the plane tangent to the surface $z=x\cos y-ye^x$ at (0, 0, 0)

***Solution***

$$f(x,y)=x\cos y-ye^x$$

$$f_x(0,0)=\left(\cos y-ye^x\right)_{(0,0)}$$

$$=1-0$$

$$=\underline{1}$$

$$f_y(0,0) = \left(-x\sin y - e^x\right)_{(0,0)}$$

$$= 0 - 1$$

$$\boxed{= -1}$$

Therefore, the tangent plane is

$$1(x-0) - (y-0) - (z-0) = 0$$

$$\boxed{x - y - z = 0}$$

### *Example*

The surfaces $f(x,y,z) = x^2 + y^2 - 2 = 0$ and $g(x,y,z) = x + z - 4 = 0$ meet in an ellipse E. Find the parametric equations for the line tangent to E at the point $P_0(1,\ 1,\ 3)$

***Solution***

The tangent line is orthogonal to both $\nabla f$ and $\nabla g$ at $P_0$ and therefore parallel to $\boldsymbol{v} = \nabla f \times \nabla g$.

The components of ***v*** and the coordinates of $P_0$ give us equations for the line.

$$\nabla f\Big|_{(1,\ 1,\ 3)} = \left(2x\hat{i} + 2y\hat{j}\right)\Big|_{(1,\ 1,\ 3)}$$

$$= 2\hat{i} + 2\hat{j}$$

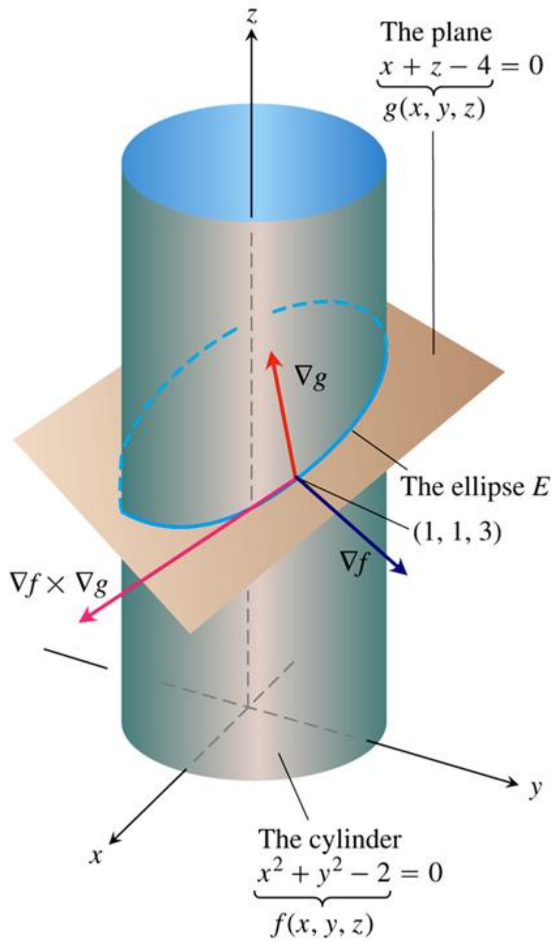
$$\nabla g\Big|_{(1,\ 1,\ 3)} = \left(\hat{i} + \hat{k}\right)\Big|_{(1,\ 1,\ 3)}$$

$$= \hat{i} + \hat{k}$$

$$\vec{v} = \left(2\hat{i} + 2\hat{j}\right) \times \left(\hat{i} + \hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$

$$\boxed{= 2\hat{i} - 2\hat{j} - 2\hat{k}}$$




The tangent line is

$$x = 1 + 2t, \quad y = 1 - 2t, \quad z = 3 - 2t$$

## Estimating Change in a Specific Direction

How much the value of a function *f* changes if we move a small distance *ds* from a point $P_0$ to another point neary.

$$df = f'\left(P_0\right)ds \qquad (\textit{single variable})$$

$$df = \left(\nabla f\Big|_{P_0} \bullet \vec{u}\right)ds \qquad (\textit{two or more variables})$$

$\vec{u}$ is the direction of the motion away from $P_0$.

### Estimating the Change in *f* in a Direction $\vec{u}$

To estimate the change in the value of a differentiable function *f* when we move a small distance *ds* from a point $P_0$ in a paricular direction $\vec{u}$ is given by

$$df = \underbrace{\left(\nabla f\Big|_{P_0} \bullet \vec{u}\right)}_{\text{Directional derivative}} \cdot \underbrace{ds}_{\text{Distance}}$$

### *Example*

Estimate how much the value of $f(x,y,z) = y\sin x + 2yz$ will change if the point $P(x,y,z)$ moves 0.1 unit from $P_0(0,\ 1,\ 0)$ straight toward $P_1(2,\ 2,\ -2)$

***Solution***

$$\overrightarrow{P_0P_1} = 2\hat{i} + \hat{j} - 2\hat{k}$$

The direction of the vector is:

$$\begin{aligned}\vec{u} &= \frac{\overrightarrow{P_0P_1}}{\left|\overrightarrow{P_0P_1}\right|} \\ &= \frac{2\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} \\ &= \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\end{aligned}$$

$$\begin{aligned}\nabla f\Big|_{(0,1,0)} &= \left((y\cos x)\hat{i} + (\sin x + 2z)\hat{j} + 2y\hat{k}\right)\Big|_{(0,1,0)} \\ &= \hat{i} + 2\hat{k}\end{aligned}$$

$$\nabla f\Big|_{P_0} \bullet \vec{u} = \left(\hat{i} + 2\hat{k}\right) \bullet \left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right)$$

$$= \frac{2}{3} - \frac{4}{3}$$

$$\boxed{= -\frac{2}{3}}$$

The change $df$ in $f$ that results from moving $ds = 0.1$ unit away from $P_0$ in the direction of $\vec{u}$ is

$$df = \left(\nabla f\Big|_{P_0} \bullet \vec{u}\right)(ds)$$

$$= \left(-\frac{2}{3}\right)(0.1)$$

$$\boxed{\approx -0.067 \ \ unit}$$

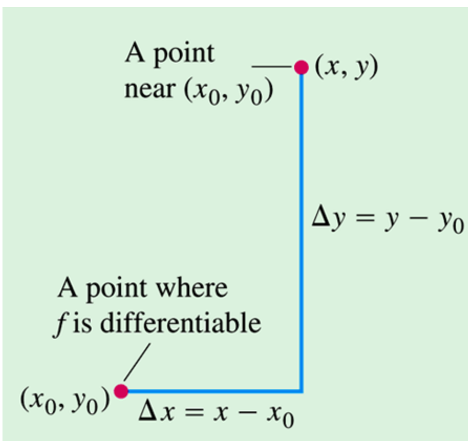
## *Definition*

The ***linearization*** of a function $f(x, y)$ at a point $(x_0, y_0)$ where $f$ is differentiable is the function

$$L(x,y) = f(x_0, y_0) + f_x\Big|_{(x_0, y_0)}(x - x_0) + f_y\Big|_{(x_0, y_0)}(y - y_0)$$

The approximation $\qquad f(x,y) \approx L(x,y)$

is the ***standard linear*** approximation of $f$ at $(x_0, y_0)$



### *Example*

Find the linearization of $f(x,y)=x^2-xy+\frac{1}{2}y^2+3$ at the point (3, 2)

***Solution***

$$f(3,\ 2)=3^2-(3)(2)+\frac{1}{2}2^2+3$$

$$\underline{=8}\rfloor$$

$$f_x(3,\ 2)=\frac{\partial}{\partial x}\left(x^2-xy+\frac{1}{2}y^2+3\right)\Big|_{(3,2)}$$

$$=2x-y\Big|_{(3,2)}$$

$$=2(3)-2$$

$$\underline{=4}\rfloor$$

$$f_y(3,2)=-x+y\Big|_{(3,2)}$$

$$=-3+2$$

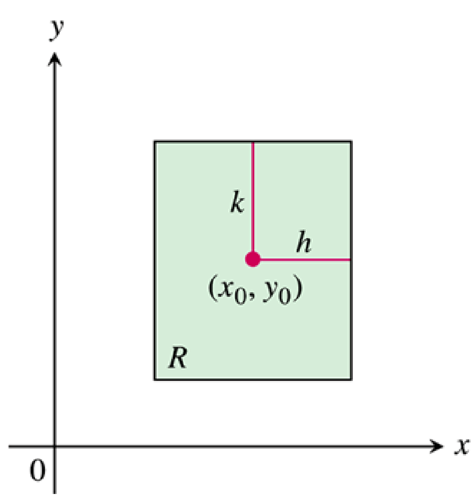
$$\underline{=-1}\rfloor$$

$$L(x,y)=8+4(x-3)-1(y-2)$$

$$\underline{=4x-y-2}\rfloor$$

## The Error in the Standard Linear Approximation

If *f* has continuous first and second partial derivatives throughout an open set containing a rectangle *R* centered at $(x_0,\ y_0)$ and if *M* is any upper bound for the values of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on *R*, then the error $E(x,y)$ incurred in replacing $F(x,y)$ on *R* by its linearization




$$L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

Satisfies the inequality:

$$\left|E(x,y)\right| \le \tfrac{1}{2}M\left(\left|x-x_0\right|+\left|y-y_0\right|\right)^2$$

$$R: \quad \left|x-x_0\right| \le h, \quad \left|y-y_0\right| \le k$$

## *Example*

Find an upper bound for the error in the approximation $f(x,y) \approx L(x,y)$ of $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$ over the rectangle

$$R: \quad |x-3| \le 0.1, \quad |y-2| \le 0.1$$

Express the upper bound as a percentage of $f(3,2)$, the value of *f* at the center of the rectangle.

***Solution***

$$f_{xx} = \frac{\partial}{dx}(2x-y) = 2 \quad \rightarrow \quad \left|f_{xx}\right| = 2$$

$$f_{yy} = \frac{\partial}{dy}(-x+y) = 1 \quad \rightarrow \quad \left|f_{yy}\right| = 1$$

$$f_{xy} = \frac{\partial}{dy}(2x-y) = -1 \quad \rightarrow \quad \left|f_{xy}\right| = |-1| = 1$$

The largest of these is 2, so let $M = 2$.

$$\begin{aligned}\left|E(x,y)\right| &\le \tfrac{1}{2}M\left(\left|x-x_0\right|+\left|y-y_0\right|\right)^2 \\ &= \tfrac{1}{2}(2)\left(|x-3|+|y-2|\right)^2 \\ &= \left(|x-3|+|y-2|\right)^2\end{aligned}$$

Since $|x-3| \le 0.1, \quad |y-2| \le 0.1$

$$\left|E(x,y)\right| \le (0.1+0.1)^2 = \underline{0.04}$$

As a percentage of $f(3,2) = 8$, the error is no greater than

$$\frac{0.04}{8} \times 100 \ \underline{= 0.5\%}$$

## Differentials

### *Definition*

If we move from $(x_0, y_0)$ to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

In the linearization of *f* is called the ***total differential of f***.

### *Example*

Suppose that a cylindrical can is designed to have a radius of 1 *in*. and a height of 5 *in*., but that the radius and height are off the amounts $dr = +0.03$ and $dh = -0.1$. Estimate the resulting absolute change in the volume of the can.

***Solution***

To estimate the absolute change in $V = \pi r^2 h$,

$$\Delta V \approx dV = V_r(r_0, h_0)dr + V_h(r_0, h_0)dh$$

$$\begin{aligned} dV &= (2\pi r_0 h_0)(0.03) + (\pi r_0^2)(-0.1) \\ &= 2\pi(1)(5)(0.03) + \pi(1)^2(-0.1) \\ &= 0.2\pi \\ &\approx 0.63\ in^3 \end{aligned}$$

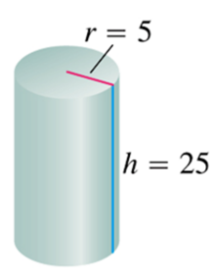
### *Example*

Your company manufactures right circular cylindrical molasses storage tanks that are 25 *ft* with a radius of 5 *ft*. How sensitive are the tanks' volumes to small variations in height and radius?

***Solution***

$$V = \pi r^2 h$$

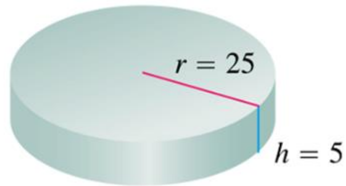
$$\begin{aligned} dV &= V_r(r_0, h_0)dr + V_h(r_0, h_0)dh \\ &= V_r(5, 25)dr + V_h(5, 25)dh \\ &= (2\pi rh)_{(5,\ 25)}dr + (\pi r^2)_{(5,\ 25)}dh \\ &= 250\pi dr + 25\pi dh \end{aligned}$$




A 1-unit change in *r* will change *V* about 250π units.

A 1-unit change in *h* will change *V* about 25π units.

The tanks' volume is 10 times more sensitive to a small change in $r$ than it is to a small change of equal size in $h$.




$$dV = (2\pi rh)_{(25,\ 5)}\,dr + \left(\pi r^2\right)_{(25,\ 5)}\,dh$$
$$= 250\pi dr + 625\pi dh$$

Now the volume is more sensitive to changes in $h$ than to changes in $r$.

The general rule is that functions are most sensitive to small changes in the variables that generated the largest partial derivatives.

### *Example*

The volume $V = \pi r^2 h$ of a right circular cylinder is to be calculated from measured values of $r$ and $h$. Suppose that $r$ is measured with an error of no more than 2% and $h$ with an error of no more than 0.5%. Estimate the resulting possible percentage error in the calculation of $V$.

***Solution***

$$\left|\frac{dr}{r}\times 100\right| \le 2 \qquad \left|\frac{dh}{h}\times 100\right| \le 0.5$$

$$\frac{dV}{V} = \frac{2\pi rhdr + \pi r^2 dh}{\pi r^2 h}$$
$$= \frac{2dr}{r} + \frac{dh}{h}$$

$$\left|\frac{dV}{V}\right| = \left|\frac{2dr}{r} + \frac{dh}{h}\right|$$
$$\le \left|2\frac{dr}{r}\right| + \left|\frac{dh}{h}\right|$$
$$\le 2(0.02) + 0.005$$
$$\underline{= 0.045}$$

The error in the volume is at the most 4.5%

## Functions of More Than Two Variables

- The linearization of $f(x,y,z)$ at a point $P_0(x_0,y_0,z_0)$ is

$$L(x,y,z) = f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0)$$

- Suppose that *R* is a closed rectangular solid centered at $P_0$ and lying in an open region on which the second partial derivatives of *f* are continuous. Suppose also that $|f_{xx}|$, $|f_{yy}|$, $|f_{zz}|$, $|f_{xy}|$, $|f_{xz}|$, and $|f_{yz}|$ are all less than or equal to *M* throughout *R*. Then the error $E(x,y,z) = f(x,y,z) - L(x,y,z)$ in the approximation of *f* by *L* is bounded throughout *R* by the inequality

$$|E(x,y)| \le \tfrac{1}{2}M\left(|x-x_0| + |y-y_0| + |z-z_0|\right)^2$$

- If the second partial derivatives of *f* are continuous and if *x, y*, and *z* change from $x_0, y_0, \textit{and } z_0$ by small amounts *dx, dy*, and *dz*, the total differential

$$df = f_x(P_0)dx + f_y(P_0)dy + f_z(P_0)dz$$

### *Example*

Find the linearization $L(x,y,z)$ of $f(x,y,z) = x^2 - xy + 3\sin z$ at the point (2, 1, 0).

Find the upper bound for the error incurred in replacing *f* by *L* on the rectangle

$$R: \quad |x-2| \le 0.01, \quad |y-1| \le 0.02, \quad |z| \le 0.01$$

***Solution***

$f(2,1,0) = 2^2 - (2)(1) + 3\sin 0 = 2$

| | | |
|---|---|---|
| $f_x(2,1,0) = 2x - y = 3$ | $f_{xx} = 2$ | $f_{xy} = -1$ |
| $f_y(2,1,0) = -x = -2$ | $f_{yy} = 0$ | $f_{xz} = 0$ |
| $f_z(2,1,0) = 3\cos z = 3$ | $f_{zz} = -3\sin z$ | $f_{yz} = 0$ |

$|-3\sin z| \le 3\sin 0.01 \approx 0.03$

Let *M* = 2.

$|E| \le \frac{1}{2}2(0.01 + 0.02 + 0.01)^2$

$\quad = 0.0016$

# *Exercises* *Section* 2.6 – Tangent Planes and Linear Approximation

Find the tangent plane and normal line of the surface

**1.** $x^2+y^2+z^2=3$ at the point $P_0\left(1,\ 1,\ 1\right)$

**2.** $x^2+2xy-y^2+z^2=7$ at the point $P_0\left(1,\ -1,\ 3\right)$

**3.** $\cos\pi x-x^2y+e^{xz}+yz=4$ at the point $P_0\left(0,\ 1,\ 2\right)$

**4.** $x^2-xy-y^2-z=0$ at the point $P_0\left(1,\ 1,\ -1\right)$

**5.** $x^2+y^2-2xy-x+3y-z=-4$ at the point $P_0\left(2,\ -3,\ 18\right)$

Find an equation for the plane that is tangent to the surface

**6.** $z=\ln\left(x^2+y^2\right)$ at the point $\left(1,\ 0,\ 0\right)$

**7.** $z=e^{-x^2-y^2}$ at the point $\left(0,\ 0,\ 1\right)$

**8.** $z=\sqrt{y-x}$ at the point $\left(1,\ 2,\ 1\right)$

**9.** $z=2x^2+y^2;\quad \left(1,\ 1,\ 3\right)$ *and* $\left(0,\ 2,\ 4\right)$

**10.** $x^2+\frac{1}{4}y^2-\frac{1}{9}z^2=1;\quad \left(0,\ 2,\ 0\right)$ *and* $\left(1,\ 1,\ \frac{3}{2}\right)$

**11.** $xy\sin z-1=0;\quad \left(1,\ 2,\ \frac{\pi}{6}\right)$ *and* $\left(-2,\ -1,\ \frac{5\pi}{6}\right)$

**12.** $yze^{xz}-8=0;\quad \left(0,\ 2,\ 4\right)$ *and* $\left(0,\ -8,\ -1\right)$

**13.** $z=x^2e^{x-y};\quad \left(2,\ 2,\ 4\right)$ *and* $\left(-1,\ -1,\ 1\right)$

**14.** $z=\ln\left(1+xy\right);\quad \left(1,\ 2,\ \ln 3\right)$ *and* $\left(-2,\ -1,\ \ln 3\right)$

**15.** $z=f\left(x,y\right)=\dfrac{1}{x^2+y^2}$ at the point $\left(1,\ 1,\ \frac{1}{2}\right)$

**16.** $x^2+y+z=3;\quad \left(1,\ 1,\ 1\right)$ *and* $\left(2,\ 0,\ -1\right)$

**17.** $x^2+y^3+z^4=2;\quad \left(1,\ 0,\ 1\right)$ *and* $\left(-1,\ 0,\ 1\right)$

**18.** $xy+xz+yz=12;\quad \left(2,\ 2,\ 2\right)$ *and* $\left(2,\ 0,\ 6\right)$

**19.** $x^2+y^2-z^2=0;\quad \left(3,\ 4,\ 5\right)$ *and* $\left(-4,\ -3,\ 5\right)$

**20.** $xy\sin z=1;\quad \left(1,\ 2,\ \frac{\pi}{6}\right)$ *and* $\left(-2,\ -1,\ \frac{5\pi}{6}\right)$

**21.** $yze^{xz} = 8$; $(0,\ 2,\ 4)$ *and* $(0,\ -8,\ -1)$

**22.** $z^2 - \frac{x^2}{16} - \frac{y^2}{9} = 1$; $\left(4,\ 3,\ -\sqrt{3}\right)$ *and* $\left(-8,\ 9,\ \sqrt{14}\right)$

**23.** $2x + y^2 - z^2 = 0$; $(0,\ 1,\ 1)$ *and* $(4,\ 1,\ -3)$

Find parametric equation for the line tangent to the curve of intersection of the surfaces

**24.** $x + y^2 + 2z = 4, \quad x = 1$ at the point $(1,\ 1,\ 1)$

**25.** $xyz = 1, \quad x^2 + 2y^2 + 3z^2 = 6$ at the point $(1,\ 1,\ 1)$

**26.** $x^3 + 3x^2y^2 + y^3 + 4xy - z^2 = 0, \quad x^2 + y^2 + z^2 = 11$ at the point $(1,\ 1,\ 3)$

**27.** $x^2 + y^2 = 4, \quad x^2 + y^2 - z = 0$ at the point $\left(\sqrt{2},\ \sqrt{2},\ 4\right)$

**28.** Find an equation for the plane tangent to the level surface $f(x,y,z) = x^2 - y - 5z$ at the point $P_0(2,\ -1,\ 1)$. Also, find parametric equations for the line is normal to the surface at $P_0$.

**29.** By about how much will $f(x,y,z) = \ln\sqrt{x^2 + y^2 + z^2}$ change if the point $P(x,y,z)$ moves from $P_0(3,\ 4,\ 12)$ a distance of $ds = 0.1$ unit in the direction of $3\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k}$?

**30.** By about how much will $f(x,y,z) = e^x \cos yz$ change if the point $P(x,y,z)$ moves from the origin at distance of $ds = 0.1$ *unit* in the direction of $2\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$?

Find the linearization $L(x,y)$ of

**31.** $f(x,y) = x^2 + y^2 + 1$ at the point (0, 0) and (1, 1)

**32.** $f(x,y) = (x + y + 2)^2$ at the point (0, 0) and (1, 2)

**33.** $f(x,y) = x^3y^4$ at the point (1, 1) and (0, 0)

**34.** $f(x,y) = e^{2y-x}$ at the point (0, 0) and (1, 2)

**35.** $f(x,y,z) = x^2 + y^2 + z^2$ at the point (1, 1, 1)

**36.** $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$ at the point (1, 2, 2)

**37.** $f(x,y,z) = \frac{\sin xy}{z}$ at the point $\left(\frac{\pi}{2},\ 1,\ 1\right)$

**38.** $f(x,y,z) = e^x + \cos(y + z)$ at the point $\left(0,\ \frac{\pi}{4},\ \frac{\pi}{4}\right)$

Find the linear approximation to the function $f$ at the point $(a,\ b)$ and estimate the given function value

**39.** $f(x,\ y) = 4\cos(2x - y)$; $(a,\ b) = \left(\frac{\pi}{4},\ \frac{\pi}{4}\right)$; estimate $f(0.8,\ 0.8)$

**40.** $f(x,\ y) = (x + y)e^{xy}$; $(a,\ b) = (2,\ 0)$; estimate $f(1.95,\ 0.05)$

**41.** $f(x,\ y) = xy + x - y$; $(a,\ b) = (2,\ 3)$; estimate $f(2.1,\ 2.99)$

**42.** $f(x,\ y) = 12 - 4x^2 - 8y^2$; $(a,\ b) = (-1,\ 4)$; estimate $f(-1.05,\ 3.95)$

**43.** $f(x,\ y) = -x^2 + 2y^2$; $(a,\ b) = (3,\ -1)$; estimate $f(3.1,\ -1.04)$

**44.** $f(x,\ y) = \sqrt{x^2 + y^2}$; $(a,\ b) = (3,\ -4)$; estimate $f(3.06,\ -3.92)$

**45.** $f(x,\ y) = \ln(1 + x + y)$; $(a,\ b) = (0,\ 0)$; estimate $f(0.1,\ -0.2)$

**46.** $f(x,\ y) = \dfrac{x + y}{x - y}$; $(a,\ b) = (3,\ 2)$; estimate $f(2.95,\ 2.05)$

**47.** Estimate the change in the function $f(x,\ y) = -2y^2 + 3x^2 + xy$ when $(x,\ y)$ changes from $(1,\ -2)$ to $(1.05,\ -1.9)$.

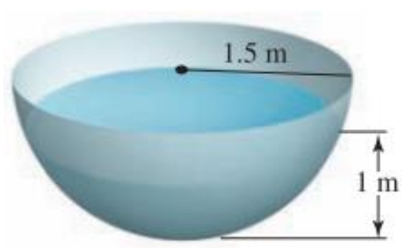
**48.** What is the largest value that the directional derivative of $f(x, y, z) = xyz$ can have at the point $(1,\ 1,\ 1)$?

**49.** You plan to calculate the volume inside a stretch of pipeline that is about 36 *in*. in diameter and 1 *mile* long. With which measurement should you be more careful, the length or the diameter? Why?

**50.** The volume of a cylinder with radius $r$ and height $h$ is $V = \pi r^2 h$. Find the approximate percentage change in the volume when the radius decreases by 3% and the height increases by 2%.

**51.** The volume of an ellipsoid with axes of length $2a$, $2b$, and $2c$ is $V = \pi abc$. Find the percentage change in the volume when $a$ increases by 2%, $b$ increases by 1.5%, and $c$ decreases by 2.5%.

**52.** A hemispherical tank with a radius of 1.50 *m* is filled with water to a depth of 1.00 *m*. Water level drops by 0.05 *m* (from 1.00 *m* to 0.95 *m*)



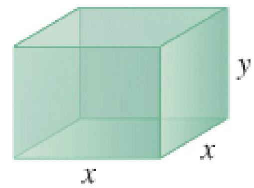

*a)* Approximate the change in the volume of water in the tank. The volume of a spherical cap is $V = \frac{1}{3}\pi h^2(3r - h)$, where $r$ is the radius of the sphere and $h$ is the thickness of the cap (in this case, the depth of the water).

*b)* Approximate the change in the surface area of the water in the tank.

**53.** Consider a closed rectangular box with a square base. If $x$ is measured with error at most 2% and $y$ is measured with error at most 3% use a differential to estimate the corresponding percentage error in computing the box's

*a)* Surface area

*b)* Volume




**54.** Consider a closed container in the shape of a cylinder of radius 10 *cm* and height 15 *cm* with a hemisphere on each end.

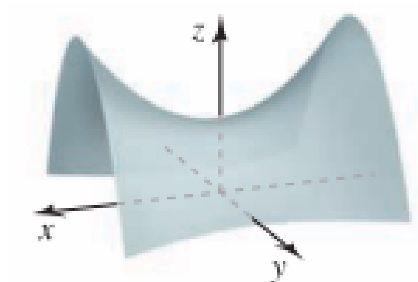

The container is coated with a layer of ice $\frac{1}{2}$ *cm* thick. Use a differential to estimate the total volume of ice. (*Hint*: assume $r$ is radius with $dr = \frac{1}{2}$ and $h$ is height with $dh = 0$)

**55.** A standard 12-*fl-oz* can of soda is essentially a cylinder of radius $r = 1$ *in* and height $h = 5$ *in.*

*a)* At these dimensions, how sensitive is the can's volume to a small change in radius versus a small change in height?

*b)* Could you design a soda can that appears to hold more soda but in fact holds the same 12-fl-oz? What might its dimensions be? (There is more than one correct answer.)

**56.** Consider the function $f(x, y) = 2x^2 - 4y^2 + 10$, whose graph is shown




*a)* Fill in the table showing the value of the directional derivative at points $(a, b)$ in the direction given by the unit vectors $\boldsymbol{u}, \boldsymbol{v}, and\ \boldsymbol{w}$

| | $(a, b) = (0, 0)$ | $(a, b) = (2, 0)$ | $(a, b) = (1, 1)$ |
|---|---|---|---|
| $\boldsymbol{u} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ | | | |
| $\boldsymbol{v} = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ | | | |

| | | | |
|---|---|---|---|
| $\boldsymbol{w} = \left(-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)$ | | | |

*b)* Interpret each of the directional derivatives computed in part(*a*) at the point $(2,\ 0)$

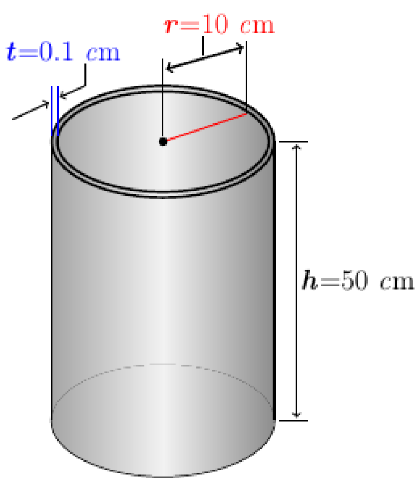
**57.** Two spheres have the same center and radii *r* and *R*, where $0 < r < R$. The volume of the region between the sphere is $V(r,\ R) = \frac{4\pi}{3}\left(R^3 - r^3\right)$.

*a)* First use your intuition. If *r* is held fixed, how does *V* change as *R* increases? What is the sign of $V_R$? If *R* is held fixed, how does *V* change as *r* increases (up to the value of *R*)? What is the sign of $V_r$?

*b)* Compute $V_r$ and $V_R$. Are the results consistent with part (*a*)?

*c)* Consider spheres with $R = 3$ and $r = 1$. Does the volume change more if *R* is increased by $\Delta R = 0.1$ (with *r* fixed) or if *r* is decreased by $\Delta r = 0.1$ (with *R* fixed)?

**58.** A company manufactures cylindrical aluminum tubes to rigid specifications. The tubes are designed to have an outside radius of $r = 10\ cm$, a height of $h = 50\ cm$, and a thickness of $t = 0.1\ cm$. The manufacturing process produces tubes with a maximum error of $\pm 0.05\ cm$ in the radius and height and a maximum error of $\pm 0.0005\ cm$ in the thickness. The volume of the material used to construct a cylindrical tube is $V(r,h,t) = \pi ht(2r - t)$. Estimate maximum error in the volume of the tube.



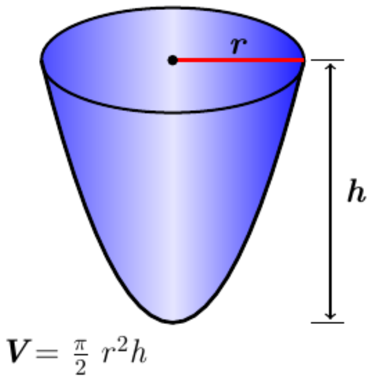

**59.** The volume of a right circular cone with radius *r* and height *h* is $V = \frac{1}{3}\pi hr^2$

*a)* Approximate the change in the volume of the cone when the radius changes from $r = 6.5$ to $r = 6.6$ and the height changes from $h = 4.20$ to $h = 4.15$

*b)* Approximate the change in the volume of the cone when the radius changes from $r = 5.4$ to $r = 5.37$ and the height changes from $h = 12.0$ to $h = 11.96$

**60.** The area of an ellipse with axes of length $2a$ and $2b$ is $A = \pi ab$. Approximate the percent change in the area when $a$ increases by 2% and $b$ increases by 1.5%.

**61.** The Volume of a segment of a circular paraboloid with radius $r$ and height $h$ is $V = \frac{1}{2}\pi h r^2$ .




Approximate the percent change in the volume when the radius decreases by 1.5% and the height increases by 2.2%

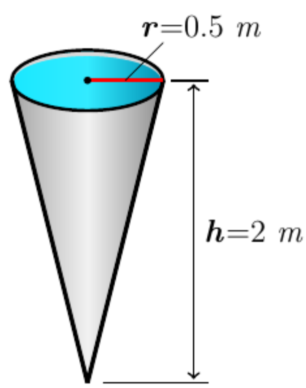
**62.** Batting averages in baseball are defined by $A = \frac{x}{y}$, where $x \ge 0$ is the total number of hits and $y > 0$ is the total number of at-bats. Treat $x$ and $y$ as positive real numbers and note that $0 \le A \le 1$.

*a)* Estimate the change in the batting average if the number of hits increases from 60 to 62 and the number of at-bats increases from 175 to 180.

*b)* If a batter currently has a batting average of $A = 0.35$, does the average decrease if the batter fails to get a hit more than it increases if the batter gets a hit?

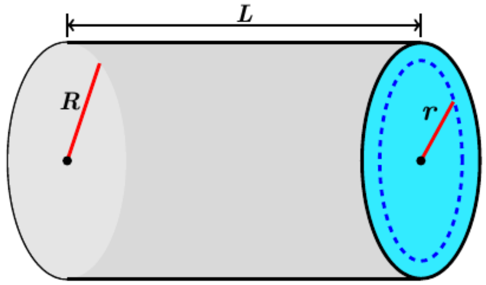
*c)* Does the answer in part (*b*) depend on the current batting average? Explain.

**63.** A conical tank with radius 0.50 *m* and height 2.0 *m* is filled with water.




Water released from the tank, and the water level drops by 0.05 *m* (from 2.0 *m* to 1.95 *m*). Approximate the change in volume of water in the tank.
(***Hint***: When the water level drops, both the radius and height of the cone of water change).

**64.** Poiseuille's law is a fundamental law of fluid dynamics that describes the flow velocity of a viscous incompressible fluid in a cylinder (it is used to model blood flow through veins and arteries). It says that in a cylinder of radius $R$ and length $L$, the velocity of the fluid $r \le R$ units from the centerline of the cylinder is $V = \frac{P}{4L\upsilon}\left(R^2 - r^2\right)$, where $P$ is the difference in the pressure between the ends of the cylinder and $\upsilon$ is the viscosity of the fluid. Assuming that $P$ and $\upsilon$ are constant, the velocity $V$ along the centerline of the cylinder $(r=0)$ is $V=\dfrac{kR^2}{L}$, where $k$ is a constant that we will take to be $k=1$.




*a)* Estimate the change in the centerline velocity $(r=0)$ if the radius of the flow cylinder increases from $R=3\ cm$ to $R=3.05\ cm$ and the length increases from $L=50\ cm$ to $L=50.5\ cm$.

*b)* Estimate the percent change in the centerline velocity if the radius of the flow cylinder $R$ decreases by 1% and the length increases by 2%.

**65.** Suppose that in a large group of people a fraction $0 \le r \le 1$ of the people have flu. The probability that in $n$ random encounters, you will meet at least one person with flu is $P=f(n,\ r)=1-(1-r)^n$ . although $n$ is a positive integer, regard it as a positive real number.

*a)* Compute $f_r$ and $f_n$ .

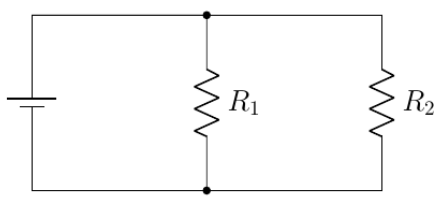
*b)* How sensitive is the probability $P$ to the flu rate $r$? Suppose you meet $n=20$ people. Approximately how much does the probability $P$ increase if the flu rate increases from $r=0.1$ to $r=0.11$ (with $n$ fixed)?

*c)* Approximately how much does the probability $P$ increase the flu rate increases from $r=0.9$ to $r=0.91$

*d)* Interpret the results of parts (*b*) and (*c*).

**66.** When two electrical resistors with resistance $R_1>0$ and $R_2>0$ are wired in parallel in a circuit, the combined resistance $R$ is given by

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$$




*a)* Estimate the change in $R$ if $R_1$ increases from 2 Ω to 2.05 Ω and $R_2$ decreases from 3 Ω to 2.95 Ω.

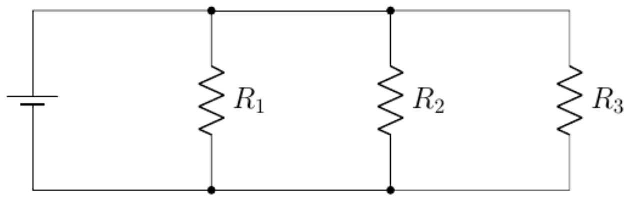
*b)* Is it true that if $R_1=R_2$ and $R_1$ increases by the same small amount as $R_2$ decreases, then $R$ is approximately unchanged? Explain.

*c)* Is it true that if $R_1$ and $R_2$ increase, then $R$ increases? Explain.

*d)* Suppose $R_1 > R_2$ and $R_1$ increases by the same small amount as $R_2$ decreases. Does *R* increase or decrease?

**67.** When three electrical resistors with resistance $R_1 > 0$, $R_2 > 0$ and $R_3 > 0$ are wired in parallel in a circuit, the combined resistance *R* is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$




Estimate the change in *R* if $R_1$ increases from 2 Ω to 2.05 Ω, $R_2$ decreases from 3 Ω to 2.95 Ω, and $R_3$ increases from 1.5 Ω to 1.55 Ω

**68.** Consider the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and the plane *P* given by $Ax + By + Cz + 1 = 0$. Let $h = \frac{1}{\sqrt{A^2 + B^2 + C^2}}$ and $m = \sqrt{a^2 A^2 + b^2 B^2 + c^2 C^2}$

*a)* Find the equation of the plane tangent to the ellipsoid at the point $(p,\ q,\ r)$.

*b)* Find the two points on the ellipsoid at which the tangent plane parallel to *P* and find equations of the tangent planes.

*c)* Show that the distance between the origin and the plane *P* is *h*.

*d)* Show that the distance between the origin and the tangent planes is *hm*.

*e)* Find a condition that guarantees the plane *P* does not intersect the ellipsoid.