Section 1.3 – The Algebra of Matrices

Matrices

$$\begin{array}{c|c} & \textit{Column} \\ & \begin{array}{ccc} C_1 & C_2 & C_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{r} Row\ 1 \rightarrow R_1 \\ Row\ 2 \rightarrow R_2 \\ Row\ 3 \rightarrow R_3 \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

This is called Matrix (*Matrices*)

Each number in the array is an ***element*** or ***entry***

The matrix is said to be of order $m\ x\ n$

m: numbers of rows,

n: number of columns

When $m = n$, then matrix is said to be ***square***.

Given the system equations

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Write into an ***augmented matrix*** form

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 31 \\ 1 & 1 & 2 & 19 \\ 1 & 3 & 2 & 25 \end{array}\right]$$

The Matrix: $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}$ is called the ***coefficient matrix*** of the system.

The matrix A above has 3 rows and 3 columns, therefore the order of the matrix A is $(3\ x\ 3)$

$$A = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Equality of Matrices

Definition of Equality of Matrices

Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal if and only if they have the same order (size) $m \times n$ and if each pair corresponding elements is equal

$a_{ij} = b_{ij}$ for $i = 1, 2, \ldots, m$ ***and*** $j = 1, 2, \ldots, n$

Example

Find the values of the variables for which each statement is true, if possible.

a) $\begin{bmatrix} 2 & 1 \\ p & q \end{bmatrix} = \begin{bmatrix} x & y \\ -1 & 0 \end{bmatrix}$

$x = 2,\ y = 1,\ \ p = -1,\ \ q = 0$

b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$

can't be true

c) $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

$\Rightarrow \begin{bmatrix} w = 9 & x = 17 \\ 8 = y & -12 = z \end{bmatrix}$

Addition and *Subtraction* of Matrices

Definition

If $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ are $m\times n$ matrices, their sum $A+B$, is the $m\times n$ matrix obtained by adding the corresponding entries; that is

$$\left[a_{ij}\right]+\left[b_{ij}\right]=\left[a_{ij}+b_{ij}\right]$$

Matrices can be added if their shapes are the same, meaning have the same ***order***.

$$\begin{bmatrix}1&2\\3&4\\0&0\end{bmatrix}+\begin{bmatrix}2&2\\4&4\\9&9\end{bmatrix}=\begin{bmatrix}1+2&2+2\\3+4&4+4\\0+9&0+9\end{bmatrix}$$

$$=\begin{bmatrix}3&4\\7&8\\9&9\end{bmatrix}$$

Scalar Multiplication Matrices

Definition

If k is a scalar and $A=\left[a_{ij}\right]$ is an $m\times n$ matrices, then scalar product ***kA*** is the $m\times n$ matrix obtained by multiplying each entry of A by k; that is

$$k\left[a_{ij}\right]=\left[ka_{ij}\right]$$

$$kA=k\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}$$

$$=\begin{bmatrix}ka_{11}&ka_{12}\\ka_{21}&ka_{22}\end{bmatrix}$$

Example

$$2\begin{bmatrix}1&2\\3&4\\0&0\end{bmatrix}=\begin{bmatrix}(2)1&(2)2\\(2)3&(2)4\\(2)0&(2)0\end{bmatrix}$$

$$=\begin{bmatrix}2&4\\6&8\\0&0\end{bmatrix}$$

Definition

If $A_1, A_2, \ldots, A_n$ are matrices of the same size, and if $c_1, c_2, \ldots, c_n$ are scalars, then expression of the form

$$c_1A_1 + c_2A_2 + \ldots + c_nA_n$$

Is called a ***linear combination*** of $A_1, A_2, \ldots, A_n$ with *coefficients* $c_1, c_2, \ldots, c_n$.

Matrix Multiplication

Product of Two Matrices

Let ***A*** be an *m x n* matrix and let ***B*** be an *n x k* matrix. To find the element in the i^{th} row and j^{th} column of the product matrix ***AB***, multiply each element in the i^{th} row of ***A*** by the corresponding element in the j^{th} column of ***B***, and then add these products. The product matrix ***AB*** is an *m x k* matrix.

*Matrix **A*** *Matrix B*

m x n *n x k*

Inner *must be equal*

Outer: Order of *AB* is *m x k*

- ✓ To multiply ***AB*** or dot product, if ***A*** has ***n*** columns, ***B*** must have ***n*** rows.
- ✓ Squares matrices can be multiplied if and only if (***iff***) they have the same size.
- ✓ The entry in row *i* and column *j* of AB is $\boxed{(row\ i\ of\ A).(col\ j\ of\ B)}$

  The result: $\sum a_{ik}b_{kj}$

$$\begin{bmatrix} * & * & & & \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{i5} \\ * & & & & \\ * & & & & \end{bmatrix} \begin{bmatrix} * & * & b_{1j} & * & * & * \\ & & b_{2j} & & & \\ & & \vdots & & & \\ & & \vdots & & & \\ & & b_{5j} & & & \end{bmatrix} = \begin{bmatrix} & & * & & & \\ * & * & (AB)_{ij} & * & * & * \\ & & * & & & \\ & & * & & & \end{bmatrix}$$

$$4\ \ by\ \ 5 \qquad\qquad 5\ \ by\ \ 6 \qquad\qquad 4\ \ by\ \ 6$$

$$\boldsymbol{AB}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\cdot\begin{bmatrix} e & f \\ g & h \end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$2x2 \qquad 2x2 \quad \to \quad 2x2$$

$$a_{11} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\cdot\begin{bmatrix} e & f \\ g & h \end{bmatrix}=\begin{bmatrix} ae+bg & \_ \\ \_ & \_ \end{bmatrix}$$

$$a_{12} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\cdot\begin{bmatrix} e & f \\ g & h \end{bmatrix}=\begin{bmatrix} \_ & af+bh \\ \_ & \_ \end{bmatrix}$$

$$a_{21} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\cdot\begin{bmatrix} e & f \\ g & h \end{bmatrix}=\begin{bmatrix} \_ & \_ \\ ce+dg & \_ \end{bmatrix}$$

$$a_{22} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\cdot\begin{bmatrix} e & f \\ g & h \end{bmatrix}=\begin{bmatrix} \_ & \_ \\ \_ & cf+dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\cdot\begin{bmatrix} e & f \\ g & h \end{bmatrix}=\begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Example

Find: $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}$

<u>Solution</u>

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 1(5)+1(1) & 1(6)+1(0) \\ 2(5)-1(1) & 2(6)-1(0) \end{bmatrix}$$

$$=\begin{bmatrix} 6 & 6 \\ 9 & 12 \end{bmatrix}$$

Special Case

When A is a square matrix, then

$$A \quad times \quad A^2=A^2 \quad times \quad A \ \boxed{=A^3}$$

$$A^p=AA\cdots A \qquad (p\ factors)$$

$$\left(A^p\right)\left(A^q\right)=A^{p+q}$$

$$\left(A^p\right)^q=A^{pq}$$

Block Multiplication

If the cuts between columns of ***A*** match the cuts between rows of ***B***, then the block multiplication of ***AB*** allowed.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} . \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \left[\begin{array}{c|c} a_{11}b_{11} + a_{12}b_{21} & a_{12}b_{21} + a_{12}b_{22} \\ \hline a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{21} + a_{22}b_{22} \end{array}\right]$$

Important special case

$$\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix}$$

Matrix Form of the Equations

The coefficient matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}$

The equivalent matrix equation is in the form $AX = b$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Multiplication by ***rows*** $\quad AX = \begin{bmatrix} (row\ 1).X \\ (row\ 2).X \\ (row\ 3).X \end{bmatrix}$

Multiplication by ***columns*** $\quad AX = x\ (column\ 1) + y\ (column\ 2) + z\ (column\ 3)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Identity Matrix

The identity matrix is given by the form: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \boxed{Ix = x}$

Properties of Matrix

Addition and Scalar Multiplication

$A+B=B+A$	*Commutative Property of Addition*
$A+(B+C)=(A+B)+C$	*Associative Property of Addition*
$(kl)A=k(lA)$	*Associative Property of Scalar Multiplication*
$k(A+B)=kA+kB$	*Distributive Property*
$k(A-B)=kA-kB$	*Distributive Property*
$(k+l)A=kA+lA$	*Distributive Property*
$(k-l)A=kA-lA$	*Distributive Property*
$A+0=0+A=A$	*Additive Identity Property*
$A+(-A)=(-A)+A=0$	*Additive Inverse Property*
$k(AB)=kA\left(B\right)=A\left(kB\right)$	

Multiplication

$AB\neq BA$	*Commutative "**law**" is usually broken*
$A(BC)=(AB)C$	*Associative Property of Multiplication* (***Parentheses not needed***)
$A(B+C)=AB+AC$	*Distributive Property*
$(B+C)A=BA+CA$	*Distributive Property*
$A(B-C)=AB-AC$	*Distributive Property*
$(B-C)A=BA-CA$	*Distributive Property*

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Consider the three vectors $\boldsymbol{u}, \quad \boldsymbol{v}, \quad and\ \boldsymbol{w}$:

$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The linear combinations in three-dimensional space are $cu + dv + ew$

Combination $$c\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}$$

Combine the three vectors $u,\ v,\ and\ w$ into on matrix A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Multiplies the matrix A by a vector x, where $c,\ d,\ e$ are the component of a vector x.

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}$$

We can rewrite the form, matrix A times the vector x, as the combination $cu + dv + ew$

$$Ax = \begin{bmatrix} u & v & w \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$$

Write the matrix in the form $Ax = b$

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b$$

Where the x is the input and b is the output.

Cyclic Difference

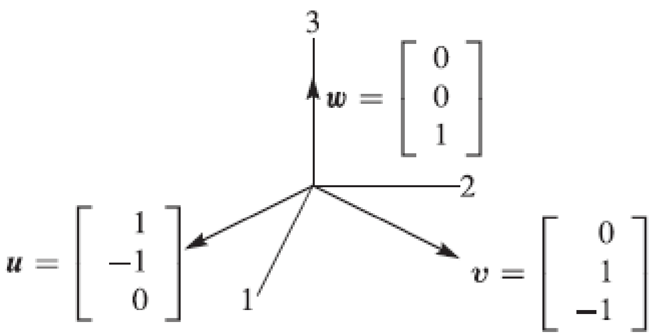
The linear combinations of three vectors $u, v,$ *and* w^* lead to a cyclic difference matrix C and is given by:

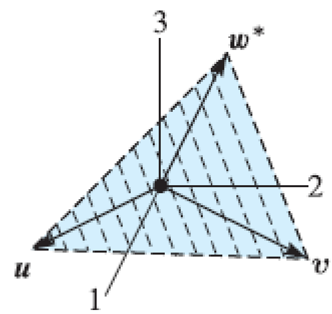
$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad w^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = b$$

The matrix C is not triangular. It is not easy to find the solution to $Cx = b$, because either we are going to have ***infinitely many solution*** or ***no solution***..

Let looks at these problems geometrically.





Exercises   *Section* 1.3 – The Algebra of Matrices

1. For the matrices: $A=\begin{bmatrix} p & 0 \\ q & r \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, when does $AB=BA$

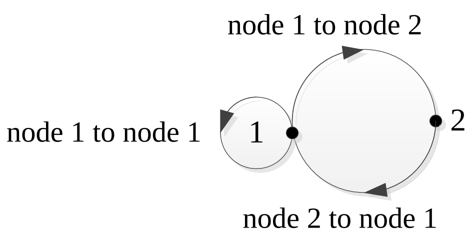
2. Find a combination $x_1w_1+x_2w_2+x_3w_3$ that gives the zero vector:

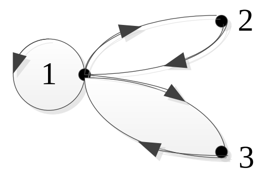
$$w_1=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix};\quad w_2=\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix};\quad w_3=\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are independent or dependent?
The vectors lie in a ______.
The matrix W with those columns is not invertible.

3. The very last words say that the 5 by 5 centered difference matrix is not invertible, Write down the 5 equations $Cx=b$. Find a combination of left sides that gives zero. What combination of $b_1,\, b_2,\, b_3,\, b_4,\, b_5$ must be zero?

4. A direct graph starts with *n* nodes. There are n^2 possible edges, each edge leaves one of the *n* nodes and enters one of the *n* nodes (possibly itself). The *n* by *n* adjacency matrix has $a_{ij}=1$ when edge leaves node *i* and enter node *j*; if no edge then $a_{ij}=0$. Here are directed graphs and their adjacency matrices:





$$A=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The i, j entry of A^2 is $a_{i1}a_{1j}+\ldots+a_{in}a_{nj}$.

Why does that sum count the two-step paths from *i* to any node to *j*?

The $i,\ j$ entry of A^k counts *k*-steps paths:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{matrix} \textit{counts the paths} \\ \textit{with two edges} \end{matrix} \quad \begin{bmatrix} 1 \textit{ to } 2 \textit{ to } 1, 1 \textit{ to } 1 \textit{ to } 1 & 1 \textit{ to } 1 \textit{ to } 2 \\ 2 \textit{ to } 1 \textit{ to } 1 & 2 \textit{ to } 1 \textit{ to } 2 \end{bmatrix}$$

List all 3-step paths between each pair of nodes and compare with A^3. When A^k has ***no zeros***, that number k is the diameter of the graph – the number of edges needed to connect the most pair of nodes. What is the diameter of the second graph?

5. A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

a) AB *b*) BA *c*) ABD *d*) DBA

e) ABC *f*) ABCD *g*) A(B + C)

6. What rows or columns or matrices do you multiply to find.

a) The third column of AB?

b) The second column of AB?

c) The first row of AB?

d) The second row of AB?

e) The entry in row 3, column 4 of AB?

f) The entry in row 2, column 3 of AB?

7. Add AB to AC and compare with $A(B+C)$:

$$A=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix} \quad \textit{and} \quad B=\begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix} \quad \textit{and} \quad C=\begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix}$$

8. True or False

a) If A^2 is defined then A is necessarily square.

b) If AB and BA are defined then A and B are square.

c) If AB and BA are defined then AB and BA are square.

d) If $AB=B$, then $A=I$

9. *a*) Find a nonzero matrix A such that $A^2=0$

b) Find a matrix that has $A^2 \neq 0$ but $A^3=0$

10. Suppose you solve $Ax=b$ for three special right sides b:

$$Ax_1=\begin{bmatrix}1\\0\\0\end{bmatrix} \quad Ax_2=\begin{bmatrix}0\\1\\0\end{bmatrix} \quad Ax_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

If the three solutions x_1, x_2, x_3 are the columns of a matrix X, what is A times X?

11. Show that $(A+B)^2$ is different from $A^2+2AB+B^2$, when

$$A=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix} \quad \textit{and} \quad B=\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}$$

Write down the correct rule for $(A+B)(A+B)=A^2+$ __________ $+B^2$

12. Find the product of the 2 matrices by rows or by columns:

a) $\begin{bmatrix}2 & 3\\ 5 & 1\end{bmatrix}\begin{bmatrix}4\\ 2\end{bmatrix}$

b) $\begin{bmatrix}1 & 2 & 4\\ 2 & 0 & 1\end{bmatrix}\begin{bmatrix}3\\ 1\\ 1\end{bmatrix}$

c) $\begin{bmatrix}3 & 6\\ 6 & 12\end{bmatrix}\begin{bmatrix}2\\ -1\end{bmatrix}$

d) $\begin{bmatrix}1 & 2 & 4\\ -2 & 3 & 1\\ -4 & 1 & 2\end{bmatrix}\begin{bmatrix}2\\ 2\\ 3\end{bmatrix}$

13. Given $A=\begin{bmatrix}4 & 1 & 3\\ 3 & -1 & -2\\ 0 & 0 & 4\end{bmatrix}$ $\quad B=\begin{bmatrix}-3 & -2 & -3\\ -1 & 0 & 0\\ 8 & -2 & -4\end{bmatrix}$ $\quad$ Find $A+B,\quad 2A,\quad \textit{and} \quad -B$

14. Given $A=\begin{bmatrix}3 & 2 & -3\\ 0 & 1 & 0\end{bmatrix}$ $\quad B=\begin{bmatrix}3 & -4\\ 0 & 1\\ 1 & 0\end{bmatrix}$ $\quad$ Find $AB \quad \textit{and} \quad BA \quad \textit{if possible}$

15. Given $A=\begin{bmatrix}5 & 3\\ -1 & 0\end{bmatrix}$ $\quad B=\begin{bmatrix}4 & -2\\ -2 & 0\\ 9 & 1\end{bmatrix}$ $\quad$ Find $AB \quad \textit{and} \quad BA \quad \textit{if possible}$

16. Given $A=\begin{bmatrix}3 & 0\\ -1 & 2\\ 1 & 1\end{bmatrix}$ $\quad B=\begin{bmatrix}4 & -1\\ 0 & 2\end{bmatrix}$ $\quad$ Find $AB \quad \textit{and} \quad BA \quad \textit{if possible}$

17. Consider the matrices

$$A=\begin{bmatrix}3 & 0\\ -1 & 2\\ 1 & 1\end{bmatrix} \quad B=\begin{bmatrix}4 & -1\\ 0 & 2\end{bmatrix} \quad C=\begin{bmatrix}1 & 4 & 2\\ 3 & 1 & 5\end{bmatrix} \quad D=\begin{bmatrix}1 & 5 & 2\\ -1 & 0 & 1\\ 3 & 2 & 4\end{bmatrix} \quad E=\begin{bmatrix}6 & 1 & 3\\ -1 & 1 & 2\\ 4 & 1 & 3\end{bmatrix}$$

Compute the following (where possible):

a) $D+E$ ***b***) $D-E$ ***c***) $5A$ ***d***) $-7C$ ***e***) $2B-C$ ***g***) $-3(D+2E)$