Solution Section 1.4 – Limits at Infinity

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $h(x) = \dfrac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}}$

Solution

$$\lim_{x\to\infty} \frac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}} = -\frac{5}{3}$$

$$\lim_{x\to-\infty} \frac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}} = -\frac{5}{3}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{2x+3}{5x+7}$

Solution

$$\lim_{x\to\infty} \frac{2x+3}{5x+7} = \lim_{x\to\infty} \frac{2 + \frac{3}{x}}{5 + \frac{7}{x}} = \frac{2}{5}$$

$$\lim_{x\to-\infty} \frac{2x+3}{5x+7} = \lim_{x\to-\infty} \frac{2 + \frac{3}{x}}{5 + \frac{7}{x}} = \frac{2}{5}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{2x^3+7}{x^3 - x^2 + x + 7}$

Solution

$$\lim_{x\to\infty} \frac{2x^3+7}{x^3 - x^2 + x + 7} = \lim_{x\to\infty} \frac{2 + \frac{7}{x^3}}{1 - \frac{1}{x} + \frac{1}{x^2} + \frac{7}{x^3}} = 2$$

$$\lim_{x\to-\infty} \frac{2x^3+7}{x^3 - x^2 + x + 7} = \lim_{x\to-\infty} \frac{2 + \frac{7}{x^3}}{1 - \frac{1}{x} + \frac{1}{x^2} + \frac{7}{x^3}} = 2$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{x+1}{x^2+3}$

Solution

$$\lim_{x\to\infty} \frac{x+1}{x^2+3} = \lim_{x\to\infty} \frac{\frac{x}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{3}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}} \underline{=0}|$$

$$\lim_{x\to-\infty} \frac{x+1}{x^2+3} = \lim_{x\to-\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}} \underline{=0}|$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{7x^3}{x^3-3x^2+6x}$

Solution

$$\lim_{x\to\infty} \frac{7x^3}{x^3-3x^2+6x} = \lim_{x\to\infty} \frac{7}{1-\frac{3}{x}+\frac{6}{x^2}} \underline{=7}|$$

$$\lim_{x\to-\infty} \frac{7x^3}{x^3-3x^2+6x} = \lim_{x\to-\infty} \frac{7}{1-\frac{3}{x}+\frac{6}{x^2}} \underline{=7}|$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{9x^4+x}{2x^4+5x^2-x+6}$

Solution

$$\lim_{x\to\infty} \frac{9x^4+x}{2x^4+5x^2-x+6} = \lim_{x\to\infty} \frac{\frac{9x^4}{x^4}+\frac{x}{x^4}}{\frac{2x^4}{x^4}+\frac{5x^2}{x^4}-\frac{x}{x^4}+\frac{6}{x^4}}$$

$$= \lim_{x\to\infty} \frac{9+\frac{1}{x^3}}{2+\frac{5}{x^2}-\frac{1}{x^3}+\frac{6}{x^4}}$$

$$\underline{=\frac{9}{2}}|$$

$$\lim_{x\to-\infty} \frac{9x^4+x}{2x^4+5x^2-x+6} = \lim_{x\to-\infty} \frac{9+\frac{1}{x^3}}{2+\frac{5}{x^2}-\frac{1}{x^3}+\frac{6}{x^4}} \underline{=\frac{9}{2}}|$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$

Solution

$$\lim_{x\to\infty} \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x} = \lim_{x\to\infty} \frac{-2 - \frac{2}{x^2} + \frac{3}{x^3}}{3 + \frac{3}{x} - \frac{5}{x^2}} = -\frac{2}{3}$$

$$\lim_{x\to-\infty} \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x} = \lim_{x\to-\infty} \frac{-2 - \frac{2}{x^2} + \frac{3}{x^3}}{3 + \frac{3}{x} - \frac{5}{x^2}} = -\frac{2}{3}$$

Exercise

Find $\displaystyle\lim_{x\to\infty} x^{12}$

Solution

$$\lim_{x\to\infty} x^{12} = \infty$$

Exercise

Find $\displaystyle\lim_{x\to-\infty} 3x^{9}$

Solution

$$\lim_{x\to-\infty} 3x^{9} = -\infty$$

Exercise

Find $\displaystyle\lim_{x\to-\infty} x^{-8}$

Solution

$$\lim_{x\to-\infty} x^{-8} = \frac{1}{(-\infty)^8} = 0$$

Exercise

Find $\displaystyle\lim_{x\to-\infty} x^{-9}$

Solution

$$\lim_{x\to-\infty} x^{-9} = \frac{1}{(-\infty)^9} = 0$$

Exercise

Find $\lim\limits_{x\to-\infty} 2x^{-6}$

Solution

$$\lim_{x\to-\infty} 2x^{-6} = \frac{2}{\infty} = 0$$

Exercise

Find $\lim\limits_{x\to\infty} \left(3x^{12} - 9x^{7}\right)$

Solution

$$\lim_{x\to\infty} \left(3x^{12} - 9x^{7}\right) = \infty$$

Exercise

Find $\lim\limits_{x\to-\infty} \left(3x^{7} + x^{2}\right)$

Solution

$$\lim_{x\to-\infty} \left(3x^{7} + x^{2}\right) = \lim_{x\to-\infty} x^{2}\left(3x^{5} + 1\right)$$

$$= -\infty$$

Exercise

Find $\lim\limits_{x\to-\infty} \left(-2x^{16} + 2\right)$

Solution

$$\lim_{x\to-\infty} \left(-2x^{16} + 2\right) = -\infty$$

Exercise

Find $\lim\limits_{x\to-\infty} \left(2x^{-6} + 4x^{5}\right)$

Solution

$$\lim_{x\to-\infty} \left(2x^{-6} + 4x^{5}\right) = \lim_{x\to-\infty} x^{-6}\left(2 + 4x^{11}\right) \qquad +\infty(-\infty)$$

$$= -\infty$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{\cos x}{3x}$

Solution

$$-\frac{1}{3x} \le \frac{\cos x}{3x} \le \frac{1}{3x}, \quad -1 \le \cos x \le 1$$

$$\lim_{x\to-\infty} \frac{\cos x}{3x} = 0 \qquad \textit{By the Sandwich Theorem}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{x+\sin x}{2x+7-5\sin x}$

Solution

$$\lim_{x\to\infty} \frac{x+\sin x}{2x+7-5\sin x} = \lim_{x\to\infty} \frac{1+\frac{\sin x}{x}}{2+\frac{7}{x}-\frac{5\sin x}{x}}$$

$$= \frac{1+0}{2+0-0}$$

$$= \frac{1}{2}$$

Exercise

Find $\lim\limits_{x\to\infty} \sqrt{\dfrac{8x^2-3}{2x^2+x}}$

Solution

$$\lim_{x\to\infty} \sqrt{\frac{8x^2-3}{2x^2+x}} = \lim_{x\to\infty} \sqrt{\frac{8-\frac{3}{x^2}}{2+\frac{1}{x}}}$$

$$= \sqrt{\frac{8}{2}}$$

$$= 2$$

Exercise

Find $\lim\limits_{x\to-\infty} \left(\dfrac{x^2+x-1}{8x^2-3}\right)^{1/3}$

Solution

$$\lim_{x\to-\infty} \left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3} = \lim_{x\to-\infty} \left(\frac{1+\frac{1}{x}-\frac{1}{x^2}}{8-\frac{3}{x^2}}\right)^{1/3}$$

$$=\left(\frac{1}{8}\right)^{1/3}$$

$$\boxed{=\frac{1}{2}}$$

Exercise

Find $\displaystyle\lim_{x\to\infty}\frac{2\sqrt{x}+x^{-1}}{3x-7}$

Solution

$$\lim_{x\to\infty}\frac{2\sqrt{x}+x^{-1}}{3x-7}=\lim_{x\to\infty}\frac{\frac{2\sqrt{x}}{x}+\frac{x^{-1}}{x}}{3-\frac{7}{x}}$$

$$=\lim_{x\to\infty}\frac{\frac{2}{x^{1/2}}+\frac{1}{x^2}}{3-\frac{7}{x}}$$

$$\boxed{=0}$$

Exercise

Find $\displaystyle\lim_{x\to\infty}\frac{x^{-1}+x^{-4}}{x^{-2}+x^{-3}}$

Solution

$$\lim_{x\to\infty}\frac{x^{-1}+x^{-4}}{x^{-2}+x^{-3}}=\lim_{x\to\infty}\frac{\frac{x^{-1}}{x^{-2}}+\frac{x^{-4}}{x^{-2}}}{\frac{x^{-2}}{x^{-2}}+\frac{x^{-3}}{x^{-2}}}$$

$$=\lim_{x\to\infty}\frac{x+\frac{1}{x^2}}{1+\frac{1}{x}}$$

$$\boxed{=\infty}$$

Exercise

Find $\displaystyle\lim_{x\to-\infty}\frac{4-3x^3}{\sqrt{x^6+9}}$

Solution

$$\lim_{x\to-\infty}\frac{4-3x^3}{\sqrt{x^6+9}}=\lim_{x\to-\infty}\frac{\frac{4-3x^3}{\sqrt{x^6}}}{\frac{\sqrt{x^6+9}}{\sqrt{x^6}}}$$

$$= \lim_{x\to-\infty} \frac{\frac{4-3x^3}{x^3}}{\sqrt{\frac{x^6+9}{x^6}}}$$

$$= \lim_{x\to-\infty} \frac{\frac{4}{x3} - 3}{\sqrt{1+\frac{9}{x^6}}}$$

$$= \frac{-3}{\sqrt{1}}$$

$$\underline{= -3|}$$

Exercise

Find $\lim_{x\to-\infty}\left(\sqrt{x^2+3}+x\right)$

Solution

$$\lim_{x\to-\infty}\left(\sqrt{x^2+3}+x\right) = \lim_{x\to-\infty}\left(\sqrt{x^2+3}+x\right)\frac{\sqrt{x^2+3}-x}{\sqrt{x^2+3}-x}$$

$$= \lim_{x\to-\infty}\frac{x^2+3-x^2}{\sqrt{x^2+3}-x}$$

$$= \lim_{x\to-\infty}\frac{3}{\sqrt{x^2+3}-x}$$

$$= \lim_{x\to-\infty}\frac{\frac{3}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{3}{x^2}}-\frac{x}{x}}$$

$$= \lim_{x\to-\infty}\frac{\frac{3}{x}}{\sqrt{1+\frac{3}{x^2}}+1}$$

$$= \frac{0}{\sqrt{1}+1}$$

$$\underline{= 0|}$$

Exercise

Find $\lim_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right)$

Solution

$$\lim_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right) = \lim_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right)\frac{\sqrt{x^2+3x}+\sqrt{x^2-2x}}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to\infty} \frac{\left(x^2+3x\right)-\left(x^2-2x\right)}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to\infty} \frac{x^2+3x-x^2+2x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to\infty} \frac{5x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to\infty} \frac{\frac{5x}{\sqrt{x^2}}}{\sqrt{\frac{x^2}{x^2}+\frac{3x}{x^2}}+\sqrt{\frac{x^2}{x^2}-\frac{2x}{x^2}}}$$

$$= \lim_{x\to\infty} \frac{5}{\sqrt{1+\frac{3}{x}}+\sqrt{1-\frac{2}{x}}}$$

$$= \frac{5}{\sqrt{1}+\sqrt{1}}$$

$$= \frac{5}{2}$$

Exercise

Find $\lim_{x\to\infty} \frac{2x-3}{4x+10}$

Solution

$$\lim_{x\to\infty} \frac{2x-3}{4x+10} = \frac{1}{2}$$

Exercise

Find $\lim_{x\to\infty} \frac{x^4-1}{x^5+2}$

Solution

$$\lim_{x\to\infty} \frac{x^4-1}{x^5+2} = 0$$

Exercise

Find $\lim_{x\to-\infty} \left(-3x^3+5\right)$

Solution

$$\lim_{x\to-\infty} \left(-3x^3+5\right) = \infty$$

Exercise

Find $\lim_{x\to\infty} \left(e^{-2x} + \frac{2}{x}\right)$

Solution

$$\lim_{x\to\infty} \left(e^{-2x} + \frac{2}{x}\right) = e^{-\infty} + 0 \underline{\left. = 0 \right|}$$

Exercise

Find $\lim_{x\to\infty} \frac{1}{\ln x + 1}$

Solution

$$\lim_{x\to\infty} \frac{1}{\ln x + 1} = \frac{1}{\infty} \underline{\left. = 0 \right|}$$

Exercise

Find $\lim_{x\to\infty} \left(3 + \frac{10}{x^2}\right)$

Solution

$$\lim_{x\to\infty} \left(3 + \frac{10}{x^2}\right) = 3 + 0 \underline{\left. = 3 \right|}$$

Exercise

Find $\lim_{x\to\infty} \left(5 + \frac{1}{x} + \frac{10}{x^2}\right)$

Solution

$$\lim_{x\to\infty} \left(5 + \frac{1}{x} + \frac{10}{x^2}\right) = 5 + 0 + 0 \underline{\left. = 5 \right|}$$

Exercise

Find $\lim_{x\to\infty} \frac{4x^2 + 2x + 3}{x^2}$

Solution

$$\lim_{x\to\infty} \frac{4x^2 + 2x + 3}{x^2} = \lim_{x\to\infty} \frac{4x^2}{x^2}$$

$$\underline{\left. = 4 \right|}$$

Exercise

Find $\lim\limits_{x\to\infty}\left(5+\frac{100}{x}+\frac{\sin^4 x^3}{x^2}\right)$

Solution

$$-1\le \sin\theta \le 1 \;\Rightarrow\; 0\le \sin^4\theta \le 1$$

$$0\le \frac{\sin^4\theta}{x^2}\le \frac{1}{x^2}\;\to 0$$

$$\lim_{x\to\infty}\left(5+\frac{100}{x}+\frac{\sin^4 x^3}{x^2}\right)=5$$

Exercise

Find $\lim\limits_{\theta\to\infty}\frac{\cos\theta}{\theta^2}$

Solution

$$-1\le \cos\theta \le 1 \;\Rightarrow\; -\frac{1}{\theta^2}\le \frac{\cos\theta}{\theta^2}\le \frac{1}{\theta^2}\;\to 0$$

$$\lim_{\theta\to\infty}\frac{\cos\theta}{\theta^2}=0$$

Exercise

Find $\lim\limits_{\theta\to\infty}\frac{\cos\theta^5}{\sqrt{\theta}}$

Solution

$$-1\le \cos\theta^5 \le 1 \;\Rightarrow\; -\frac{1}{\sqrt{\theta}}\le \frac{\cos\theta^5}{\sqrt{\theta}}\le \frac{1}{\sqrt{\theta}}\;\to 0$$

$$\lim_{\theta\to\infty}\frac{\cos\theta^5}{\sqrt{\theta}}=0$$

Exercise

Find $\lim\limits_{x\to\infty}\frac{4x}{20x+1}$

Solution

$$\lim_{x\to\infty}\frac{4x}{20x+1}=\frac{4}{20}=\frac{1}{5}$$

Exercise

Find $\lim\limits_{x\to-\infty} \frac{4x}{20x+1}$

Solution

$$\lim_{x\to-\infty} \frac{4x}{20x+1} = \lim_{x\to-\infty} \frac{4x}{20x}$$

$$= \frac{1}{5}$$

Exercise

Find $\lim\limits_{x\to\infty} \frac{3x^2-7}{x^2+5x}$

Solution

$$\lim_{x\to\infty} \frac{3x^2-7}{x^2+5x} = 3$$

Exercise

Find $\lim\limits_{x\to-\infty} \frac{3x^2-7}{x^2+5x}$

Solution

$$\lim_{x\to-\infty} \frac{3x^2-7}{x^2+5x} = \lim_{x\to-\infty} \frac{3x^2}{x^2}$$

$$= 3$$

Exercise

Find $\lim\limits_{x\to\infty} \frac{6x^2-9x+8}{3x^2+2}$

Solution

$$\lim_{x\to\infty} \frac{6x^2-9x+8}{3x^2+2} = \lim_{x\to\infty} \frac{6x^2}{3x^2}$$

$$= \frac{6}{3}$$

$$= 2$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{6x^2-9x+8}{3x^2+2}$

Solution

$$\lim_{x\to-\infty} \frac{6x^2-9x+8}{3x^2+2} = \lim_{x\to-\infty} \frac{6x^2}{3x^2}$$

$$= \frac{6}{3}$$

$$= 2$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{4x^2-7}{8x^2+5x+2}$

Solution

$$\lim_{x\to\infty} \frac{4x^2-7}{8x^2+5x+2} = \lim_{x\to\infty} \frac{4x^2}{8x^2}$$

$$= \frac{4}{8}$$

$$= \frac{1}{2}$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{4x^2-7}{8x^2+5x+2}$

Solution

$$\lim_{x\to-\infty} \frac{4x^2-7}{8x^2+5x+2} = \lim_{x\to-\infty} \frac{4x^2}{8x^2}$$

$$= \frac{4}{8}$$

$$= \frac{1}{2}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

Solution

$$\lim_{x\to\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4} = \lim_{x\to\infty} \frac{\sqrt{16x^4}+x^2}{2x^2}$$

$$= \lim_{x\to\infty} \frac{4x^2+x^2}{2x^2}$$

$$= \lim_{x\to\infty} \frac{5x^2}{2x^2}$$

$$= \frac{5}{2}$$

Exercise

Find $\displaystyle \lim_{x\to-\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

Solution

$$\lim_{x\to-\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4} = \lim_{x\to-\infty} \frac{\sqrt{16x^4}+x^2}{2x^2}$$

$$= \lim_{x\to-\infty} \frac{4x^2+x^2}{2x^2}$$

$$= \lim_{x\to-\infty} \frac{5x^2}{2x^2}$$

$$= \frac{5}{2}$$

Exercise

Find $\displaystyle \lim_{x\to\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

Solution

$$\lim_{x\to\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144} = \lim_{x\to\infty} \frac{3x^4}{x^4}$$

$$= 3$$

Exercise

Find $\displaystyle \lim_{x\to-\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

Solution

$$\lim_{x\to-\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144} = \lim_{x\to-\infty} \frac{3x^4}{x^4}$$

$$= 3$$

Exercise

Find $\lim_{x\to\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right)$

Solution

$$\lim_{x\to\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right) = \infty-\infty$$

$$= \lim_{x\to\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right)\frac{4x^2+\sqrt{16x^4+1}}{4x^2+\sqrt{16x^4+1}}$$

$$= \lim_{x\to\infty} 16x^2\,\frac{16x^4-16x^4-1}{4x^2+\sqrt{16x^4+1}}$$

$$= \lim_{x\to\infty} 16x^2\,\frac{-1}{4x^2+\sqrt{16x^4+1}}$$

$$= \lim_{x\to\infty} \frac{-16x^2}{4x^2+4x^2}$$

$$= \lim_{x\to\infty} \frac{-16x^2}{8x^2}$$

$$= -2$$

Exercise

Find $\lim_{x\to-\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right)$

Solution

$$\lim_{x\to-\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right) = \infty-\infty$$

$$= \lim_{x\to-\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right)\frac{4x^2+\sqrt{16x^4+1}}{4x^2+\sqrt{16x^4+1}}$$

$$= \lim_{x\to-\infty} 16x^2\,\frac{16x^4-16x^4-1}{4x^2+\sqrt{16x^4+1}}$$

$$= \lim_{x\to-\infty} 16x^2\,\frac{-1}{4x^2+\sqrt{16x^4+1}}$$

$$= \lim_{x\to-\infty} \frac{-16x^2}{4x^2+4x^2}$$

$$= \lim_{x\to-\infty} \frac{-16x^2}{8x^2}$$

$$\underline{=-2}$$

Exercise

Find $\lim_{x\to\infty} \frac{x-1}{x^{2/3}-1}$

Solution

$$\lim_{x\to\infty} \frac{x-1}{x^{2/3}-1} = \lim_{x\to\infty} \frac{x}{x^{2/3}}$$

$$= \lim_{x\to\infty} x^{1/3}$$

$$\underline{=\infty}$$

Exercise

Find $\lim_{x\to-\infty} \frac{x-1}{x^{2/3}-1}$

Solution

$$\lim_{x\to-\infty} \frac{x-1}{x^{2/3}-1} = \lim_{x\to-\infty} \frac{x}{x^{2/3}}$$

$$= \lim_{x\to-\infty} x^{1/3}$$

$$\underline{=-\infty}$$

Exercise

Find $\lim_{x\to\infty} \frac{\sqrt{x^2+2x+6}-3}{x-1}$

Solution

$$\lim_{x\to\infty} \frac{\sqrt{x^2+2x+6}-3}{x-1} = \lim_{x\to\infty} \frac{\sqrt{x^2}}{x}$$

$$= \lim_{x\to\infty} \frac{x}{x}$$

$$\underline{=1}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{\left|1-x^2\right|}{x(x+1)}$

Solution

$$\lim_{x\to\infty} \frac{\left|1-x^2\right|}{x(x+1)} = \lim_{x\to\infty} \frac{x^2-1}{x^2+1}$$

$$= \lim_{x\to\infty} \frac{x^2}{x^2}$$

$$= 1$$

Exercise

Find $\lim\limits_{x\to\infty} \left(\sqrt{|x|}-\sqrt{|x-1|}\right)$

Solution

$$\lim_{x\to\infty} \left(\sqrt{|x|}-\sqrt{|x-1|}\right) = \infty - \infty \qquad x\to\infty \Rightarrow |x| = x \ \ \& \ \ |x-1| = x-1$$

$$= \lim_{x\to\infty} \left(\sqrt{x}-\sqrt{x-1}\right)\frac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}}$$

$$= \lim_{x\to\infty} \frac{x-x+1}{\sqrt{x}+\sqrt{x-1}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x}+\sqrt{x-1}}$$

$$= \frac{1}{\infty}$$

$$= 0$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{\tan^{-1}x}{x}$

Solution

$$-\frac{\pi}{2} \le \tan^{-1}x \le \frac{\pi}{2}$$

$$-\frac{\pi}{2x} \le \frac{\tan^{-1}x}{x} \le \frac{\pi}{2x} \quad \to \ 0$$

$$\lim_{x\to\infty} \frac{\tan^{-1}x}{x} = 0$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{\cos x}{e^{3x}}$

Solution

$$-1 \le \cos x \le 1$$

$$-\frac{1}{e^{3x}} \le \frac{\cos x}{e^{3x}} \le \frac{1}{e^{3x}} \rightarrow 0$$

$$\lim_{x\to\infty} \frac{\cos x}{e^{3x}} = 0$$

Exercise

Find $\lim\limits_{x\to 0} \dfrac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}}$

Solution

$$\lim_{x\to 0} \frac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}} = \frac{2+10}{1+1}$$

$$= 6$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}}$

Solution

$$\lim_{x\to\infty} \frac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}} = \lim_{x\to\infty} \frac{2e^{x}}{e^{x}} \qquad \lim_{x\to\infty} e^{-x} = 0$$

$$= 2$$

Exercise

Find $\lim\limits_{x\to -\infty} \dfrac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}}$

Solution

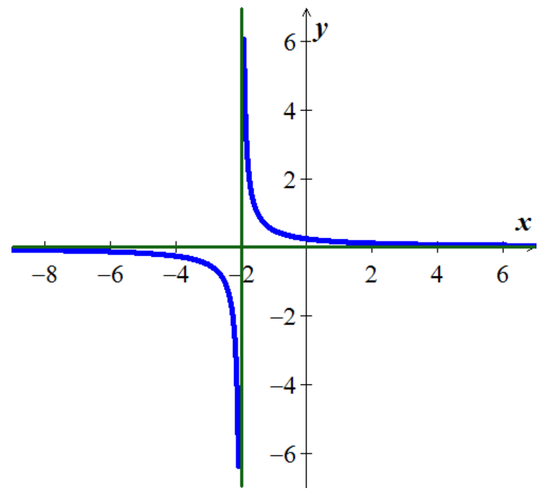
$$\lim_{x\to\infty} \frac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}} = \lim_{x\to\infty} \frac{10e^{-x}}{e^{-x}} \qquad \lim_{x\to -\infty} e^{x} = 0$$

$$= 10$$

Exercise

Graph the rational function $y = \dfrac{1}{2x+4}$. Include the equations of the asymptotes.

Solution



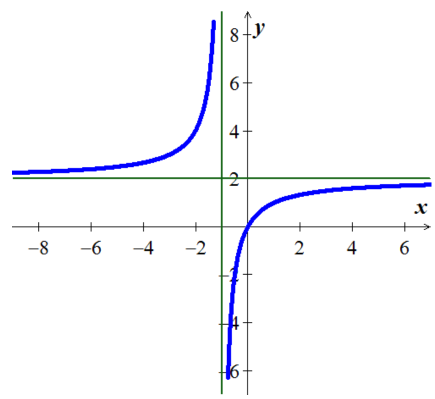
VA: $2x = 4 = 0 \Rightarrow \boxed{x = -2}$

HA: $\boxed{y = 0}$

Exercise

Graph the rational function $y = \dfrac{2x}{x+1}$. Include the equations of the asymptotes.

Solution



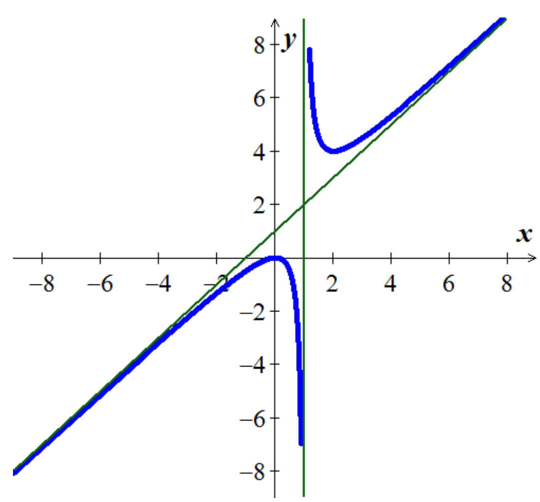
VA: $\boxed{x = -1}$

HA: $\boxed{y = 2}$

Exercise

Graph the rational function $y = \dfrac{x^2}{x-1}$. Include the equations of the asymptotes.

Solution



$$\begin{array}{r|l} & \quad x+1 \\ \hline x-1 & \quad x^2 \\ & \underline{\quad x^2 - x} \\ & \qquad\quad x \\ & \qquad\quad \underline{x-1} \\ & \qquad\qquad \boxed{1} \end{array}$$

$$y = \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$$

VA: $\boxed{x = 1}$

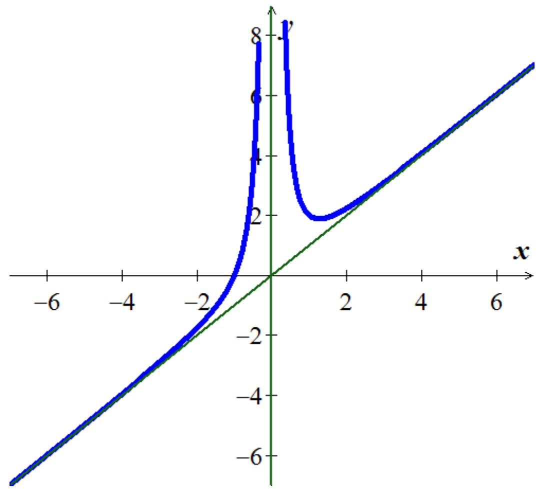
Oblique Asymptote: $\boxed{y = x + 1}$

Exercise

Graph the rational function $y=\dfrac{x^3+1}{x^2}$. Include the equations of the asymptotes.

Solution

$$\begin{array}{r} x \phantom{{}+1} \\ x^2 \overline{\big)\, x^3+1} \\ \underline{x^3 \phantom{{}+1}} \\ \boxed{1} \end{array}$$



$$y=\frac{x^3+1}{x^2}=x+\frac{1}{x^2}$$

VA: $\underline{x=0}$

Oblique Asymptote: $\underline{y=x}$

Exercise

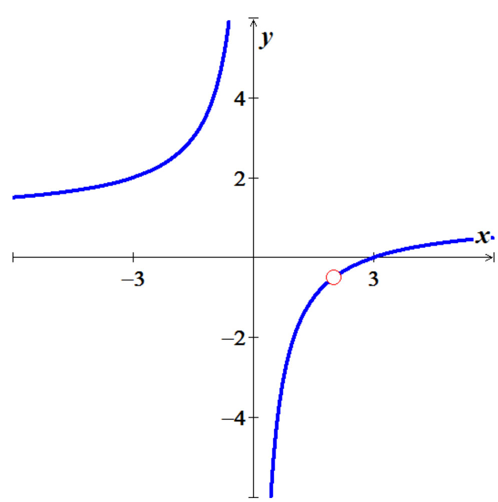
Let $\quad f(x)=\dfrac{x^2-5x+6}{x^2-2x}$

a) Analyze $\lim\limits_{x\to 0^-} f(x)$, $\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 2^-} f(x)$, and $\lim\limits_{x\to 2^+} f(x)$

b) Does the graph of f have any vertical asymptotes? Explain?

Solution

$$f(x)=\frac{x^2-5x+6}{x^2-2x}=\frac{(x-2)(x-3)}{x(x-2)}=\frac{x-3}{x}$$



a) $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\dfrac{x-3}{x}=\dfrac{-3}{0^-}\;\underline{=\infty}$

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}\dfrac{x-3}{x}=\dfrac{-3}{0^+}\;\underline{=-\infty}$

$\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^-}\dfrac{x-3}{x}=\dfrac{2-3}{2}\;\underline{=-\dfrac{1}{2}}$

$\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+}\dfrac{x-3}{x}=\dfrac{2-3}{2}\;\underline{=-\dfrac{1}{2}}$

b) ***VA***: $x=0$ ***Hole***: $x=2\to f(2)=-\frac{1}{2}$

HA: $y=1$ ***OA***: n/a

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{3x}{1-x}$

Solution

$\boldsymbol{VA}: x=1, \quad \boldsymbol{Hole}:\ n/a, \quad \boldsymbol{HA}: y=-3, \quad \boldsymbol{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{x^2}{x^2+9}$

Solution

$\boldsymbol{VA}:\ n/a; \quad \boldsymbol{Hole}:\ n/a; \quad \boldsymbol{HA}: y=1; \quad \boldsymbol{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{x-2}{x^2-4x+3}$

Solution

$\boldsymbol{VA}: x=1,\ 3; \quad \boldsymbol{Hole}:\ n/a; \quad \boldsymbol{HA}: y=0; \quad \boldsymbol{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{5x-1}{1-3x}$

Solution

$\boldsymbol{VA}:\ x=\frac{1}{3}; \quad \boldsymbol{Hole}:\ n/a; \quad \boldsymbol{HA}:\ y=-\frac{5}{3}; \quad \boldsymbol{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{3}{x-5}$

Solution

$\boldsymbol{VA}: x=5, \quad \boldsymbol{Hole}:\ n/a, \quad \boldsymbol{HA}: y=0, \quad \boldsymbol{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{x^3-1}{x^2+1}$

Solution

$$\begin{array}{r|l} & x \\ \hline x^2+1 & x^3-1 \\ & \underline{x^3+x} \\ & \boxed{-x-1} \end{array}$$

$$y=\frac{x^3-1}{x^2+1}=x+\frac{-x-1}{x^2+1}=x-\frac{x+1}{x^2+1}$$

VA: n/a, ***Hole***: n/a, ***HA***: n/a, ***OA***: $y=x$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{3x^2-27}{(x+3)(2x+1)}$

Solution

VA: $x=-3,\ -\frac{1}{2}$; ***Hole***: n/a; ***HA***: $y=\frac{3}{2}$; ***OA***: n/a

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{x^3+3x^2-2}{x^2-4}$

Solution

$$\begin{array}{r|l} & x+3 \\ \hline x^2-4 & x^3+3x^2-2 \\ & \underline{x^3\qquad -4x} \\ & \quad 3x^2+4x-2 \end{array}$$

$$y=\frac{x^3+3x^2-2}{x^2-4}=x+3+\frac{4x+10}{x^2-4}$$

VA: $x=\pm 2$, ***Hole***: n/a, ***HA***: n/a, ***OA***: $y=x+3$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{x-3}{x^2-9}$

Solution

VA: $x=-3$; ***Hole***: $x=3$; ***HA***: $y=0$; ***OA***: n/a

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{6}{\sqrt{x^2-4x}}$

Solution

VA: $x=0,\ 4$; ***Hole***: n/a; ***HA***: $y=0$; ***OA***: n/a

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{4x^3+1}{1-x^3}$

Solution

$\boldsymbol{VA}: x=1; \quad \boldsymbol{Hole}: n/a; \quad \boldsymbol{HA}: y=-4; \quad \boldsymbol{OA}: n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{x+1}{\sqrt{9x^2+x}}$

Solution

$\boldsymbol{VA}: x=0, -\frac{1}{9}; \quad \boldsymbol{Hole}: n/a; \quad \boldsymbol{HA}: y=\frac{1}{3}; \quad \boldsymbol{OA}: n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=1-e^{-2x}$

Solution

$\boldsymbol{VA}: n/a; \quad \boldsymbol{Hole}: n/a; \quad \boldsymbol{HA}: y=1; \quad \boldsymbol{OA}: n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{1}{\ln x^2}$

Solution

$\boldsymbol{VA}: x=0; \quad \boldsymbol{Hole}: n/a; \quad \boldsymbol{HA}: y=0; \quad \boldsymbol{OA}: n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{1}{\tan^{-1}x}$

Solution

$\boldsymbol{VA}: x=0; \quad \boldsymbol{Hole}: n/a; \quad \boldsymbol{HA}: y=\dfrac{1}{\frac{\pi}{2}}=\dfrac{2}{\pi}; \quad \boldsymbol{OA}: n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{2x^2+6}{2x^2+3x-2}$

Solution

$\boldsymbol{VA}: x=-2, \frac{1}{2}; \quad \boldsymbol{Hole}: n/a; \quad \boldsymbol{HA}: y=1; \quad \boldsymbol{OA}: n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{3x^2+2x-1}{4x+1}$

Solution

$$\begin{array}{r|l} & \frac{3}{4}x+\frac{5}{16} \\ \hline 4x+1 & 3x^2+2x-1 \\ & \underline{3x^2+\frac{3}{4}x} \\ & \qquad \frac{5}{4}x-1 \end{array}$$

$$\textbf{\textit{VA}}:\ x=-\frac{1}{4};\quad \textbf{\textit{Hole}}:\ n/a;\quad \textbf{\textit{HA}}:\ n/a;\quad \textbf{\textit{OA}}:\ y=\frac{3}{4}x+\frac{5}{16}$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{9x^2+4}{(2x-1)^2}$

Solution

$$\textbf{\textit{VA}}:\ x=\frac{1}{2};\quad \textbf{\textit{Hole}}:\ n/a;\quad \textbf{\textit{HA}}:\ y=\frac{9}{4};\quad \textbf{\textit{OA}}:\ n/a$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{1+x-2x^2-x^3}{x^2+1}$

Solution

$$\begin{array}{r|l} & \qquad -x-2 \\ \hline x^2+1 & -x^3-2x^2+x+1 \\ & \underline{-x^3 \qquad\quad -x} \\ & \quad -2x^2+2x \end{array}$$

$$\textbf{\textit{VA}}:\ n/a;\quad \textbf{\textit{Hole}}:\ n/a;\quad \textbf{\textit{HA}}:\ n/a;\quad \textbf{\textit{OA}}:\ y=-x-2$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{x(x+2)^3}{3x^2-4x}$

Solution

$$f(x)=\frac{x\left(x^3+6x^2+12x+8\right)}{x(3x-4)}=\frac{x^3+6x^2+12x+8}{3x-4}$$

$$
\begin{array}{r|l}
 & \frac{1}{3}x^2+\frac{22}{9}x+\frac{196}{27} \\
\hline
3x-4 & x^3+6x^2+12x+8 \\
 & \underline{x^3-\frac{4}{3}x^2} \\
 & \quad \frac{22}{3}x^2+12x \\
 & \quad \underline{\frac{22}{3}x^2-\frac{88}{9}x} \\
 & \qquad\qquad \frac{196}{9}x
\end{array}
$$

VA: $x=\frac{4}{3}$; ***Hole***: $(0,\ -2)$; ***HA***: n/a; ***OA***: $y=\frac{1}{3}x^2+\frac{22}{9}x+\frac{196}{27}$