Cal II - Review

① Geom ② Integral ③ Comparison + 6
④ Ratio ⑤ Root ⑥ Alternating + 8

$S_{ca} \to$ any.

(15) centre, radius, interval of convergence

(16) $f(x)$ $a = 1$ $0, 1, 2, \ldots$

$f(x)$ $n = 4$ $a = 0 \to$

$$\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Comparison $\to$ not original - ($S = ?$)

Geometric series $\sum a(r)^n$

$$\sum_{n=0}^{\infty} e^{-3n} = \sum (e^{-3})^n$$

$$= \sum \left(\frac{1}{e^3}\right)^n$$

$$|r| = \frac{1}{e^3} < 1$$

$$S = \frac{1}{1 - \frac{1}{e^3}}$$

$$= \frac{e^3}{e^3 - 1}$$

By the Geometric series, the given series converges of sum $\frac{e^3}{e^3-1}$

Integral Test

$$\sum_{k=2}^{\infty} \frac{4}{k \ln^2 k}$$

$$\int_2^{\infty} \frac{4}{x \ln^2 x}\, dx = 4\int_2^{\infty} \frac{d(\ln x)}{(\ln x)^2} \qquad d(\ln x) = \frac{1}{x}dx$$

$$= -4 \frac{1}{\ln x}\Big|_2^{\infty}$$

$$= -4\left(0 - \frac{1}{\ln 2}\right)$$

$$= \frac{4}{\ln 2}$$

By the Integral Test, the given series converges.

$$\sum 2n^{-3/2} = \sum \frac{2}{n^{+3/2}} \qquad \frac{a}{n^p}$$

$$p = +\frac{3}{2} > 1$$

$\therefore$ By the p-series, the given series converges

Comparison Test

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1}$$

$$2^n+1 > 2^n$$

$$\frac{1}{2^n+1} < \frac{1}{2^n}$$

$$\sum \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n \qquad r = \frac{1}{2} < 1$$

By the geometric series, it converges

$\therefore$ By the Comparison Test, the given series converges

Ratio Test

$$\sum \frac{2^n}{n!}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$$

$$= 2 \lim_{n\to\infty} \frac{1}{n+1}$$

$$= \underline{0} < 1$$

By the Ratio Test, the given series converges

Root Test

$$\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n+1}\right)^n} = \lim_{n\to\infty} \frac{n}{2n+1}$$

$$= \frac{1}{2} < 1.$$

$\therefore$ By the root Test, the given series converges

ream   Root

$$\left(r \right)^n \qquad\qquad \left(a_n \right)^n$$

Alternating Test.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$$

$$e^n < e^{n+1}$$

$$\frac{1}{e^n} > \frac{1}{e^{n+1}}$$

$$u_n > u_{n+1} \checkmark$$

$$\frac{1}{e^n} \longrightarrow 0 \checkmark$$

By the alternating series, the given series converges

$\sum \frac{(-1)^n}{e^n}$ $\int_1^\infty e^{-x}$ abs.

$\sum |(\frac{-1}{e})^n|$ Geom.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

$n < n+1$

$n! < (n+1)!$

$\frac{1}{n!} > \frac{1}{(n+1)!}$

$u_n > u_{n+1}$ ✓

$\frac{1}{n!} \longrightarrow 0$ ✓

$\sum_{n=0}^{\infty} (-1)^n \frac{n! (x-5)^n}{3^n}$ centre, radius, interval

$x - 5 = 0 \Rightarrow x = 5$

Centre : $x = 5$

$R = \lim_{n\to\infty} \left| \frac{u_n}{u_{n+1}} \right|$

$= \lim_{n\to\infty} \frac{n!}{3^n} \cdot \frac{3^{n+1}}{(n+1)!}$

$= 3 \lim_{n\to\infty} \frac{1}{n+1}$

$= 0$

Series converges only @ $x = 5$

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-1)^{n+1}}{n+1}$$

centre : $x = 1$

$$R = \lim_{n \to \infty} \frac{1}{n+1} \cdot \frac{n+2}{1}$$

$$= \underline{1}$$

$$-1 < x-1 < 1$$

$$0 < x < 2$$

@ $x=0 \Rightarrow \displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(-1)^{n+1}}{n+1} = \sum \frac{1}{n+1}$

$$\int_0^{\infty} \frac{dx}{x+1} = \ln|x+1| \Big|_0^{\infty}$$

$$= \infty$$

it diverges @ $x=0$

$x=2$: $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{n+1}$

$$n < n+1$$

$$\frac{1}{n} > \frac{1}{n+1}$$

$$u_n > u_{n+1} \checkmark$$

$$\frac{1}{n+1} \longrightarrow 0 \checkmark$$

it converges @ $x = 2$

Interval of convergence $\underline{0 < x \leq 2}$

$f(x) = 8\sqrt{x}$ $\quad a = 1 \quad n = 0, 1, 2, 3$

$f(x) = 8x^{1/2} \to f(1) = 8$

$f'(x) = 4x^{-1/2} \to f'(1) = 4$

$f''(x) = -2x^{-3/2} \to f''(1) = -2$

$f^{(3)}(x) = 3x^{-5/2} \to f^{(3)}(x) = 3$

$$P_3(x) = 8 + 4(x-1) - \frac{2}{2!}(x-1)^2 + \frac{3}{3!}(x-1)^3$$

$$= 8 + 4(x-1) - (x-1)^2 + \frac{1}{2}(x-1)^3$$

$$\sum_{k=1}^{\infty} \frac{3}{2+e^k}$$

$2+e^k > e^k$

$\frac{3}{2+e^k} < \frac{3}{e^k}$

① $3\left(\frac{1}{e}\right)^k \quad \frac{1}{e} < 1$

Geom.

② $3\int_1^{\infty} e^{-x}dx = -3e^{-x}\Big|_1^{\infty}$

$= \frac{3}{e^x}$

$$\int_1^{\infty} \frac{3}{2+e^x}dx$$

$u = e^x$

$du = e^x dx$

$dx = \frac{1}{u}du$

$$\lim_{k\to\infty} \frac{3}{2+e^{k+1}} \cdot \frac{2+e^k}{3} = \lim_{k\to\infty} \frac{e^k+2}{e^{k+1}+2}$$

$$= \frac{1}{e} < 1$$

$$\sum_{k=1}^{\infty} k \sin\frac{1}{k}$$

$$-1 \le \sin\frac{1}{k} \le 1$$

$$-k \le k\sin\frac{1}{k} \le k$$

$$\lim_{k\to\infty} k = \infty$$

$$\int_1^{\infty} x \sin\left(\frac{1}{x}\right) dx \qquad u = \frac{1}{x} \Rightarrow x = \frac{1}{u}$$

$$du = \frac{-1}{x^2}dx \Rightarrow dx = -x^2 du$$

$$= -\int^{\infty} \frac{1}{u}(\sin u)\frac{1}{u^2}\, du$$

$$= -\int_1^{\infty} \frac{\sin u}{u^3}\, du$$

$$\int \sin u \qquad u^{-3}$$

$$\lim_{k\to\infty} k\sin\frac{1}{k} = \lim_{k\to\infty} \frac{\sin\frac{1}{k}}{\frac{1}{k}} \qquad \frac{1}{k} = x \to 0$$

$$= \lim_{x\to 0} \frac{\sin x}{x}$$

$$= 1 \ne 0$$

$\sin 0 \ \frac{\pi}{2}$

$\cos 0 \ \pi$

$(-1)^n$

$$\sum_{k=4}^{\infty} \frac{1}{k^2-10}$$

$$\int \frac{dx}{x^2-10} \qquad x = \sqrt{10}\sec\theta$$

$$dx = \sqrt{10}\sec\theta\tan\theta\, d\theta$$

$$x^2 - 10 = 10\tan^2\theta.$$

$$\int_4^{\infty} \frac{dx}{x^2-10} = \int_4^{\infty} \frac{\sqrt{10}\sec\theta\tan\theta}{10\tan^2\theta}\, d\theta$$

$$= \frac{\sqrt{10}}{10}\int_4^{\infty} \frac{\sec\theta}{\tan\theta}\, d\theta.$$

$$= \frac{\sqrt{10}}{10}\left(\ln|\sec\theta + \tan\theta|\right)\Big|_4^{\infty}$$

$$= \frac{\sqrt{10}}{10} \int_4^\infty \csc\theta\, d\theta \qquad \frac{\sec\theta}{\tan\theta} = \csc\theta$$

$$= -\frac{\sqrt{10}}{10} \ln|\csc\theta + \cot\theta| \Big|_4^\infty$$

$$= -\frac{\sqrt{10}}{10} \ln\left|\frac{\sec\theta}{\tan\theta} + \frac{\sqrt{10}}{\sqrt{x^2-10}}\right| \Big|_4^\infty$$

$$= -\frac{\sqrt{10}}{10} \ln\left|\frac{x}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{x^2-10}} + \frac{\sqrt{10}}{\sqrt{x^2-10}}\right| \Big|_4^\infty$$

$$= -\frac{\sqrt{10}}{10} \ln\left|\frac{\sqrt{10}\,x + 10}{\sqrt{10}\sqrt{x^2-10}}\right| \Big|_4^\infty$$

$$= -\frac{\sqrt{10}}{10}\left(\ln 1 - \ln \frac{4\sqrt{10}+10}{4\sqrt{10}}\right)$$

$$= \boxed{\frac{\sqrt{10}}{10} \ln\left(\frac{10+4\sqrt{10}}{4\sqrt{10}}\right)}$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right|$$

$$\sum_{k=4}^{\infty} \frac{1}{k^2 - 10}$$

$$k^2 - 10 > (k-1)^2$$

$$\frac{1}{k^2-10} < \frac{1}{(k-1)^2}$$

$$\frac{1}{(k+1)^2 - 10} \cdot \frac{k^2-10}{1} \longrightarrow 1$$

$$\sum_{k=4}^{\infty} \frac{1}{k^2+10}$$

$$\int_4^\infty \frac{dx}{x^2+10} = \frac{1}{\sqrt{10}} \tan^{-1}\frac{x}{\sqrt{10}} \Big|_4^\infty$$

$$= \frac{1}{\sqrt{10}}\left(\tan^{-1}\infty - \tan^{-1}\frac{4}{\sqrt{10}}\right)$$

$$= \boxed{\frac{1}{\sqrt{10}}\left(\frac{\pi}{2} - \tan^{-1}\frac{4}{\sqrt{10}}\right)}$$

$$k^2 + 10 > k^2$$

$$\frac{1}{k^2+10} < \frac{1}{k^2}$$

$$\sum_{k=1}^{\infty} \frac{\ln(k^2)}{k^2} \qquad \sum \frac{2\ln k}{k^2}$$

$$\lim_{k\to\infty} \frac{a_{n+1}}{a_n} = \lim_{k\to\infty} \frac{2\ln(k+1)}{(k+1)^2} \cdot \frac{k^2}{2\ln(k)}$$

$$= \lim_{k\to\infty} \frac{\ln(k+1)}{\ln(k)} \cdot \underbrace{\left(\frac{k}{k+1}\right)^2}_{\to 1}$$

$$= \lim_{k\to\infty} \frac{\ln(k+1)}{\ln k} \qquad \frac{\infty}{\infty}$$

$$= \lim_{k\to\infty} \frac{\frac{1}{k+1}}{\frac{1}{k}}$$

$$= \lim_{k\to\infty} \frac{k}{k+1}$$

$$= 1$$

$$\frac{\ln k}{k^2} > \frac{1}{k^2}$$

$$2\int_1^{\infty} \frac{\ln x}{x^2}\,dx$$

$$y = \ln x$$
$$x = e^y$$
$$dx = e^y\,dy$$

$$2\int_1^{\infty} \frac{y}{e^{2y}} e^y\,dy = 2\int_1^{\infty} y e^{-y}\,dy$$

		$\int e^{-y}$
+	y	$-e^{-y}$
−	1	e^{-y}

$$= 2e^{-y}(-y-1)$$

$$= 2\,\frac{1}{x}(-\ln x - 1)\Big|_1^{\infty}$$

$$= 2(0-(-1))$$

$$= 2$$