Solution   *Section* 3.3 – Double-angle Half-Angle Formulas

Exercise

Let $\sin A = -\frac{3}{5}$ with A in Q*III* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

Solution

$\cos A = -\frac{4}{5}$ $A \in QIII \Rightarrow 180° < A < 270° \quad \to 90° < \frac{A}{2} < 135°$

a) $\sin 2A = 2\left(-\frac{4}{5}\right)\left(-\frac{3}{5}\right)$ $\sin 2A = 2\sin A\cos A$

$= \frac{24}{25}$

b) $\cos 2A = \cos^2 A - \sin^2 A$

$= \left(-\frac{4}{5}\right)^2 - \left(-\frac{3}{5}\right)^2$

$= \frac{16}{25} - \frac{9}{25}$

$= \frac{7}{25}$

c) $\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{24}{7}$

d) $\sin\frac{A}{2} = \sqrt{\frac{1}{2}\left(1+\frac{4}{5}\right)}$ $\sin\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1-\cos A)}$

$= \sqrt{\frac{1}{2}\frac{9}{5}}$

$= \frac{3}{\sqrt{10}}$ $\frac{3\sqrt{10}}{10}$

e) $\cos\frac{A}{2} = -\sqrt{\frac{1}{2}\left(1-\frac{4}{5}\right)}$ $\cos\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1+\cos A)}$

$= -\frac{1}{\sqrt{10}}$ $\frac{\sqrt{10}}{10}$

f) $\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = -3$

Exercise

Let $\sin A = \frac{3}{5}$ with A in Q*II* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

Solution

$\cos A = -\frac{4}{5}$ $A \in QII \Rightarrow 90° < A < 180° \quad \to 45° < \frac{A}{2} < 90°$

a) $\sin 2A = 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)$ $\sin 2A = 2\sin A\cos A$

$= -\frac{24}{25}$

b) $\cos 2A = \frac{16}{25} - \frac{9}{25}$ $\cos 2A = \cos^2 A - \sin^2 A$

$= \frac{7}{25}$

c) $\tan 2A = \frac{\sin 2A}{\cos 2A} = -\frac{24}{7}$

d) $\sin\frac{A}{2} = \sqrt{\frac{1}{2}\left(1+\frac{4}{5}\right)}$ $\sin\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1-\cos A)}$

$= \frac{3}{\sqrt{10}}$

e) $\cos\frac{A}{2} = \sqrt{\frac{1}{2}\left(1-\frac{4}{5}\right)}$ $\cos\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1+\cos A)}$

$= \frac{1}{\sqrt{10}}$

f) $\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = 3$

Exercise

Let $\cos A = \frac{3}{5}$ with A in QIV and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin \frac{A}{2}$ *e)* $\cos \frac{A}{2}$ *f)* $\tan \frac{A}{2}$

Solution

$\sin A = -\frac{4}{5}$

$A \in QIV \quad \Rightarrow \; 270° < A < 360° \quad \to 135° < \frac{A}{2} < 180°$

a) $\sin 2A = 2\left(-\frac{3}{5}\right)\left(-\frac{4}{5}\right)$ $\quad \sin 2A = 2\sin A\cos A$

$= \frac{24}{25}$

b) $\cos 2A = \frac{9}{25} - \frac{16}{25}$ $\quad \cos 2A = \cos^2 A - \sin^2 A$

$= -\frac{7}{25}$

c) $\tan 2A = \frac{\sin 2A}{\cos 2A} = -\frac{24}{7}$

d) $\sin \frac{A}{2} = \sqrt{\frac{1}{2}\left(1 - \frac{3}{5}\right)}$ $\quad \sin \frac{A}{2} = \pm\sqrt{\frac{1}{2}(1 - \cos A)}$

$= \frac{1}{\sqrt{5}}$

e) $\cos \frac{A}{2} = -\sqrt{\frac{1}{2}\left(1 + \frac{3}{5}\right)}$ $\quad \cos \frac{A}{2} = \pm\sqrt{\frac{1}{2}(1 + \cos A)}$

$= -\frac{2}{\sqrt{10}}$

f) $\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = -\frac{1}{2}$

Exercise

Let $\cos A = \frac{5}{13}$ with A in QI and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin \frac{A}{2}$ *e)* $\cos \frac{A}{2}$ *f)* $\tan \frac{A}{2}$

Solution

$\sin A = \frac{12}{13}$

$A \in QI \quad \Rightarrow \; 0° < A < 90° \quad \to 0° < \frac{A}{2} < 45°$

a) $\sin 2A = 2\left(\frac{12}{13}\right)\left(\frac{5}{13}\right)$ $\quad \sin 2A = 2\sin A\cos A$

$= \frac{120}{169}$

b) $\cos 2A = \frac{25}{169} - \frac{144}{169}$ $\quad \cos 2A = \cos^2 A - \sin^2 A$

$= -\frac{119}{169}$

c) $\tan 2A = \frac{\sin 2A}{\cos 2A} = -\frac{120}{119}$

d) $\sin \frac{A}{2} = \sqrt{\frac{1}{2}\left(1 - \frac{5}{13}\right)}$ $\quad \sin \frac{A}{2} = \pm\sqrt{\frac{1}{2}(1 - \cos A)}$

$= \sqrt{\frac{1}{2}\frac{8}{13}}$

$= \frac{2}{\sqrt{13}}$

e) $\cos \frac{A}{2} = \sqrt{\frac{1}{2}\left(1 + \frac{5}{13}\right)}$ $\quad \cos \frac{A}{2} = \pm\sqrt{\frac{1}{2}(1 + \cos A)}$

$= \frac{3}{\sqrt{13}}$

f) $\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \frac{2}{3}$

Exercise

Let $\cos A = -\frac{12}{13}$ with A in Q*II* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

Solution

$\sin A = \frac{5}{13}$

$90° < A < 180° \quad \to 45° < \frac{A}{2} < 90°$

a) $\sin 2A = 2\left(\frac{5}{13}\right)\left(-\frac{12}{13}\right)$ $\quad \sin 2A = 2\sin A\cos A$

$= -\frac{120}{169}$

b) $\cos 2A = \frac{144}{169} - \frac{25}{169}$ $\quad \cos 2A = \cos^2 A - \sin^2 A$

$= \frac{119}{169}$

c) $\tan 2A = \frac{\sin 2A}{\cos 2A} = -\frac{120}{119}$

d) $\sin\frac{A}{2} = \sqrt{\frac{1}{2}\left(1+\frac{12}{13}\right)}$ $\quad \sin\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1-\cos A)}$

$= \frac{5}{\sqrt{26}}$

e) $\cos\frac{A}{2} = \sqrt{\frac{1}{2}\left(1-\frac{5}{13}\right)}$ $\quad \cos\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1+\cos A)}$

$= \frac{2}{\sqrt{13}}$

f) $\tan\frac{A}{2} = \dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{5}{2}$

Exercise

Let $\sin A = -\frac{7}{25}$ with A in Q*III* and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

Solution

$\cos A = -\frac{24}{25}$

$180° < A < 270° \quad \to 90° < \frac{A}{2} < 135°$

a) $\sin 2A = 2\left(-\frac{7}{25}\right)\left(-\frac{24}{25}\right)$ $\quad \sin 2A = 2\sin A\cos A$

$= \frac{336}{625}$

b) $\cos 2A = \frac{576}{625} - \frac{49}{625}$ $\quad \cos 2A = \cos^2 A - \sin^2 A$

$= \frac{527}{625}$

c) $\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{336}{527}$

d) $\sin\frac{A}{2} = \sqrt{\frac{1}{2}\left(1+\frac{24}{25}\right)}$ $\quad \sin\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1-\cos A)}$

$= \sqrt{\frac{1}{2}\frac{49}{25}}$

$= \frac{7}{5\sqrt{2}}$

e) $\cos\frac{A}{2} = \sqrt{\frac{1}{2}\left(1-\frac{24}{25}\right)}$ $\quad \cos\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1+\cos A)}$

$= \frac{1}{5\sqrt{2}}$

f) $\tan\frac{A}{2} = \dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = 7$

Exercise

Let $\sin A = -\frac{24}{25}$ with A in QIV and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

Solution

$\cos A = \frac{7}{25}$

$270° < A < 360° \quad \to 135° < \frac{A}{2} < 180°$

a) $\sin 2A = 2\left(-\frac{24}{25}\right)\left(\frac{7}{25}\right)$ $\quad \sin 2A = 2\sin A\cos A$

$= -\frac{336}{625}$

b) $\cos 2A = \frac{49}{625} - \frac{576}{625}$ $\quad \cos 2A = \cos^2 A - \sin^2 A$

$= -\frac{527}{625}$

c) $\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{336}{527}$

d) $\sin\frac{A}{2} = \sqrt{\frac{1}{2}\left(1 - \frac{7}{25}\right)}$ $\quad \sin\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1-\cos A)}$

$= \frac{3}{5\sqrt{2}}$

e) $\cos\frac{A}{2} = -\sqrt{\frac{1}{2}\left(1 + \frac{7}{25}\right)}$ $\quad \cos\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1+\cos A)}$

$= -\frac{4}{5}$

f) $\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = -\frac{3}{5\sqrt{2}}$

Exercise

Let $\cos A = \frac{15}{17}$ with A in QI and find

a) $\sin 2A$ *b)* $\cos 2A$ *c)* $\tan 2A$ *d)* $\sin\frac{A}{2}$ *e)* $\cos\frac{A}{2}$ *f)* $\tan\frac{A}{2}$

Solution

$\sin A = \frac{8}{17}$

$0° < A < 90° \quad \to 0° < \frac{A}{2} < 45°$

a) $\sin 2A = 2\left(\frac{8}{17}\right)\left(\frac{15}{17}\right)$ $\quad \sin 2A = 2\sin A\cos A$

$= \frac{240}{289}$

b) $\cos 2A = \frac{225}{289} - \frac{64}{289}$ $\quad \cos 2A = \cos^2 A - \sin^2 A$

$= \frac{161}{289}$

c) $\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{240}{289}$

d) $\sin\frac{A}{2} = \sqrt{\frac{1}{2}\left(1 - \frac{15}{17}\right)}$ $\quad \sin\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1-\cos A)}$

$= \frac{1}{\sqrt{17}}$

e) $\cos\frac{A}{2} = \sqrt{\frac{1}{2}\left(1 + \frac{15}{17}\right)}$ $\quad \cos\frac{A}{2} = \pm\sqrt{\frac{1}{2}(1+\cos A)}$

$= \frac{4}{\sqrt{17}}$

f) $\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{2}{4}$

Exercise

Let $\cos x = \frac{1}{\sqrt{10}}$ with x in Q*IV* and find $\cot 2x$

Solution

x in QIV $\Rightarrow \sin x < 0$

$$\begin{aligned}\sin x &= -\sqrt{1-\cos^2 x} \\ &= -\sqrt{1-\frac{1}{10}} \\ &= -\sqrt{\frac{9}{10}} \\ &= -\frac{3}{\sqrt{10}}\end{aligned}$$

$$\begin{aligned}\cot 2x &= \frac{\cos 2x}{\sin 2x} \\ &= \frac{2\cos^2 x - 1}{2\sin x \cos x} \\ &= \frac{2\left(\frac{1}{\sqrt{10}}\right)^2 - 1}{2\frac{1}{\sqrt{10}}\left(-\frac{3}{\sqrt{10}}\right)} \\ &= \frac{2\frac{1}{10} - 1}{-\frac{6}{10}} \\ &= \frac{\frac{2-10}{10}}{-\frac{6}{10}} \\ &= \frac{-8}{-6} \\ &= \frac{4}{3}\end{aligned}$$

Exercise

Verify: $(\cos x - \sin x)(\cos x + \sin x) = \cos 2x$

Solution

$$\begin{aligned}(\cos x - \sin x)(\cos x + \sin x) &= \cos^2 x - \sin^2 x \qquad (a+b)(a-b) = a^2 - b^2 \\ &= \cos 2x\end{aligned}$$

Exercise

Prove: $\cot x \sin 2x = 1 + \cos 2x$

Solution

$$\begin{aligned}\cot x \sin 2x &= \frac{\cos x}{\sin x}(2\sin x \cos x) \\ &= 2\cos^2 x \\ &= \cos 2x + 1\end{aligned}$$

$\cos 2x = 2\cos^2 x - 1 \Rightarrow 2\cos^2 x = \cos 2x + 1$

Exercise

Prove: $\cot\theta = \dfrac{\sin 2\theta}{1-\cos 2\theta}$

Solution

$$\begin{aligned}\frac{\sin 2\theta}{1-\cos 2\theta} &= \frac{2\sin\theta\cos\theta}{1-\left(1-2\sin^2\theta\right)} \\ &= \frac{2\sin\theta\cos\theta}{1-1+2\sin^2\theta} \\ &= \frac{2\sin\theta\cos\theta}{2\sin^2\theta} \\ &= \frac{\cos\theta}{\sin\theta} \\ &= \cot\theta\end{aligned}$$

Exercise

Simplify $\cos^2 7x - \sin^2 7x$

Solution

$$\cos^2 7x - \sin^2 7x = \cos\left(2(7x)\right) \qquad\qquad \cos 2x = \cos^2 x - \sin^2 x$$
$$= \cos 14x$$

Exercise

Write $\sin 3x$ in terms of $\sin x$

Solution

$$\sin 3x = \sin(2x + x)$$
$$= \sin 2x \cos x + \cos 2x \sin x$$
$$= (2\sin x \cos x)\cos x + \left(1 - 2\sin^2 x\right)\sin x$$
$$= 2\sin x \cos^2 x + \sin x - 2\sin^3 x \qquad\qquad \cos^2 x = 1 - \sin^2 x$$
$$= 2\sin x\left(1 - \sin^2 x\right) + \sin x - 2\sin^3 x$$
$$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$$
$$= 3\sin x - 4\sin^3 x$$

Exercise

Find the values of the six trigonometric functions of θ if $\cos 2\theta = \frac{4}{5}$ *and* $90° < \theta < 180°$

Solution

$\cos^2\theta = \frac{1+\cos 2\theta}{2} = \frac{1+\frac{4}{5}}{2} = \frac{\frac{9}{5}}{2} = \frac{9}{10}$	$\sin^2\theta = \frac{1-\cos 2\theta}{2} = \frac{1-\frac{4}{5}}{2} = \frac{\frac{1}{5}}{2} = \frac{1}{10}$
$\cos\theta = -\sqrt{\frac{9}{10}} = -\frac{3}{\sqrt{10}}\frac{\sqrt{10}}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}$	$\sin\theta = \sqrt{\frac{1}{10}} = \frac{1}{\sqrt{10}}\frac{\sqrt{10}}{\sqrt{10}} = \frac{\sqrt{10}}{10}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $= \dfrac{\frac{\sqrt{10}}{10}}{-\frac{3\sqrt{10}}{10}}$ $= -\dfrac{\sqrt{10}}{10}\dfrac{10}{3\sqrt{10}}$ $= -\dfrac{1}{3}$	$\cot\theta = \dfrac{1}{\tan\theta}$ $= \dfrac{1}{-\frac{1}{3}}$ $= -3$
$\csc\theta = \dfrac{1}{\sin\theta}$ $= \dfrac{1}{\frac{1}{\sqrt{10}}}$ $= \sqrt{10}$	$\sec\theta = \dfrac{1}{\cos\theta}$ $= \dfrac{1}{-\frac{3}{\sqrt{10}}}$ $= -\dfrac{\sqrt{10}}{3}$

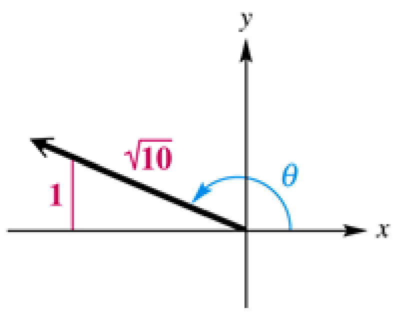
Exercise

Use a right triangle in QII to find the value of $\cos\theta$ *and* $\tan\theta$

Solution

$$r = \sqrt{10},\ y = 1$$

$$\begin{aligned} x &= -\sqrt{r^2 - y^2} \\ &= -\sqrt{\left(\sqrt{10}\right)^2 - 1^2} \\ &= -\sqrt{10-1} \\ &= -\sqrt{9} \\ &= -3 \end{aligned}$$



$$\cos\theta = \frac{-3}{\sqrt{10}}\frac{\sqrt{10}}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}$$

$$\tan\theta = -\frac{1}{3}$$

Exercise

Prove the following equation is an identity: $\sin 3x = \sin x\left(3\cos^2 x - \sin^2 x\right)$

Solution

$$
\begin{aligned}
\sin 3x &= \sin(x + 2x) \\
&= \sin x \cos 2x + \sin 2x \cos x \\
&= \sin x\left(\cos^2 x - \sin^2 x\right) + (2\sin x \cos x)\cos x \\
&= \sin x \cos^2 x - \sin^3 x + 2\sin x \cos^2 x \\
&= 3\sin x \cos^2 x - \sin^3 x \\
&= \sin x\left(3\cos^2 x - \sin^2 x\right)
\end{aligned}
$$

Exercise

Prove the following equation is an identity: $\cos 3x = \cos^3 x - 3\cos x \sin^2 x$

Solution

$$
\begin{aligned}
\cos 3x &= \cos(x + 2x) \\
&= \cos x \cos 2x - \sin x \sin 2x \\
&= \cos x\left(\cos^2 x - \sin^2 x\right) - \sin x(2\sin x \cos x) \\
&= \cos^3 x - \sin^2 x \cos x - 2\sin^2 x \cos x \\
&= \cos^3 x - 3\sin^2 x \cos x
\end{aligned}
$$

Exercise

Prove the following equation is an identity: $\cos^4 x - \sin^4 x = \cos 2x$

Solution

$$
\begin{aligned}
\cos^4 x - \sin^4 x &= \left(\cos^2 x - \sin^2 x\right)\left(\cos^2 x + \sin^2 x\right) \qquad (a-b)(a+b) = a^2 + b^2 \\
&= (\cos 2x)(1) \\
&= \cos 2x
\end{aligned}
$$

Exercise

Prove: $\cot\theta = \dfrac{\sin 2\theta}{1-\cos 2\theta}$

Solution

$$\frac{\sin 2\theta}{1-\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1-\left(1-2\sin^2\theta\right)}$$

$$= \frac{2\sin\theta\cos\theta}{1-1+2\sin^2\theta}$$

$$= \frac{2\sin\theta\cos\theta}{2\sin^2\theta}$$

$$= \frac{\cos\theta}{\sin\theta}$$

$$= \cot\theta$$

Exercise

Prove the following equation is an identity: $\sin 2x = -2\sin x\sin\left(x-\frac{\pi}{2}\right)$

Solution

$$\sin 2x = 2\sin x\cos x \qquad \cos x = \sin\left(\tfrac{\pi}{2}-x\right)$$

$$= 2\sin x\sin\left(\tfrac{\pi}{2}-x\right) \qquad \sin(-x) = -\sin x$$

$$= -2\sin x\sin\left(x-\tfrac{\pi}{2}\right)$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin 4t}{4} = \cos^3 t\sin t - \sin^3 t\cos t$

Solution

$$\frac{\sin 4t}{4} = \frac{1}{4}\left(2\sin 2t\cos 2t\right)$$

$$= \frac{1}{2}\left(2\sin t\cos t\right)\left(\cos^2 t-\sin^2 t\right)$$

$$= \sin t\cos t\left(\cos^2 t-\sin^2 t\right)$$

$$= \sin t\cos^3 t-\cos t\sin^3 t$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos 2x}{\sin^2 x} = \csc^2 x - 2$

Solution

$$\frac{\cos 2x}{\sin^2 x} = \frac{1 - 2\sin^2 x}{\sin^2 x}$$

$$= \frac{1}{\sin^2 x} - \frac{2\sin^2 x}{\sin^2 x}$$

$$= \csc^2 x - 2$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos 2x + \cos 2y}{\sin x + \cos y} = 2\cos y - 2\sin x$

Solution

$$\frac{\cos 2x + \cos 2y}{\sin x + \cos y} = \frac{2\cos\left(\frac{2x+2y}{2}\right)\cos\left(\frac{2x-2y}{2}\right)}{\sin x + \cos y}$$

$$= \frac{2\cos(x+y)\cos(x-y)}{\sin x + \cos y}$$

$$= \frac{2(\cos x\cos y - \sin x\sin y)(\cos x\cos y + \sin x\sin y)}{\sin x + \cos y}$$

$$= 2\frac{\cos^2 x\cos^2 y - \sin^2 x\sin^2 y}{\sin x + \cos y}$$

$$= 2\frac{\left(1 - \sin^2 x\right)\cos^2 y - \sin^2 x\left(1 - \cos^2 y\right)}{\sin x + \cos y}$$

$$= 2\frac{\cos^2 y - \sin^2 x\cos^2 y - \sin^2 x + \sin^2 x\cos^2 y}{\sin x + \cos y}$$

$$= 2\frac{\cos^2 y - \sin^2 x}{\sin x + \cos y}$$

$$= 2\frac{(\cos y - \sin x)(\cos y + \sin x)}{\sin x + \cos y}$$

$$= 2(\cos y - \sin x)$$

$$= 2\cos y - 2\sin x$$

$$\frac{\cos 2x + \cos 2y}{\sin x + \cos y} = \frac{\cos^2 x - \sin^2 x + \cos^2 y - \sin^2 y}{\sin x + \cos y}$$

$$= \frac{1 - \sin^2 x - \sin^2 x + \cos^2 y - \left(1 - \cos^2 y\right)}{\sin x + \cos y}$$

$$= \frac{1-2\sin^2 x+\cos^2 y-1+\cos^2 y}{\sin x+\cos y}$$

$$= \frac{2\cos^2 y-2\sin^2 x}{\sin x+\cos y}$$

$$= 2\frac{\cos^2 y-\sin^2 x}{\sin x+\cos y}$$

$$= 2\frac{(\cos y-\sin x)(\cos y+\sin x)}{\sin x+\cos y}$$

$$= 2(\cos y-\sin x)$$

$$\underline{= 2\cos y-2\sin x}$$

Exercise

Prove the following equation is an identity: $\frac{\cos 2x}{\cos^2 x}=\sec^2 x-2\tan^2 x$

Solution

$$\frac{\cos 2x}{\cos^2 x}=\frac{1-2\sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}-\frac{2\sin^2 x}{\cos^2 x}$$

$$\underline{= \sec^2 x-2}$$

Exercise

Prove the following equation is an identity: $\sin 4x=(4\sin x\cos x)\left(2\cos^2 x-1\right)$

Solution

$$\sin 4x=\sin(2(2x))$$

$$= 2\sin 2x\cos 2x$$

$$= 2(2\sin x\cos x)\left(2\cos^2 x-1\right)$$

$$\underline{= (4\sin x\cos x)\left(2\cos^2 x-1\right)}$$

Exercise

Prove the following equation is an identity: $\cos 4x = \cos^4 x - 6\sin^2 x\cos^2 x + \sin^4 x$

Solution

$$\begin{aligned}
\cos 4x &= \cos\big(2(2x)\big) \\
&= \cos^2 2x - \sin^2 2x \\
&= (\cos 2x)^2 - (\sin 2x)^2 \\
&= \left(\cos^2 x - \sin^2 x\right)^2 - (2\sin x\cos x)^2 \\
&= \cos^4 x - 2\sin^2 x\cos^2 x - \sin^4 x - 4\sin^2 x\cos^2 x \\
&= \cos^4 x - 6\sin^2 x\cos^2 x - \sin^4 x
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cos 2y = \dfrac{1-\tan^2 y}{1+\tan^2 y}$

Solution

$$\begin{aligned}
\cos 2y &= \cos^2 y - \sin^2 y \\
&= \frac{\cos^2 y - \sin^2 y}{1} \\
&= \frac{\cos^2 y - \sin^2 y}{\cos^2 y + \sin^2 y} \\
&= \frac{\frac{\cos^2 y}{\cos^2 y} - \frac{\sin^2 y}{\cos^2 y}}{\frac{\cos^2 y}{\cos^2 y} + \frac{\sin^2 y}{\cos^2 y}} \\
&= \frac{1-\tan^2 y}{1+\tan^2 y}
\end{aligned}$$

$$\begin{aligned}
\frac{1-\tan^2 y}{1+\tan^2 y} &= \frac{1-\frac{\sin^2 y}{\cos^2 y}}{1+\frac{\sin^2 y}{\cos^2 y}} \\
&= \frac{\frac{\cos^2 y - \sin^2 y}{\cos^2 y}}{\frac{\cos^2 y + \sin^2 y}{\cos^2 y}} \\
&= \frac{\cos^2 y - \sin^2 y}{\cos^2 y + \sin^2 y} \\
&= \frac{\cos^2 y - \sin^2 y}{1} \\
&= \cos^2 y - \sin^2 y
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\tan^2 x(1+\cos 2x) = 1-\cos 2x$

Solution

$$\begin{aligned}
\tan^2 x(1+\cos 2x) &= \frac{\sin^2 x}{\cos^2 x}\left(1+2\cos^2 x-1\right) \\
&= \frac{\sin^2 x}{\cos^2 x}\left(2\cos^2 x\right) \\
&= 2\sin^2 x \\
&= 1-1+2\sin^2 x \\
&= 1-\left(1-2\sin^2 x\right) \\
&= 1-\cos 2x
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos 2x}{\sin^2 x} = 2\cot^2 x - \csc^2 x$

Solution

$$\begin{aligned}
\frac{\cos 2x}{\sin^2 x} &= \frac{\cos^2 x-\sin^2 x}{\sin^2 x} \\
&= \frac{\cos^2 x}{\sin^2 x} - \frac{\sin^2 x}{\sin^2 x} \\
&= \cot^2 x - 1 \qquad\qquad \cot^2 x + 1 = \csc^2 x \\
&= \cot^2 x + \cot^2 x - \csc^2 x \\
&= 2\cot^2 x - \csc^2 x
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\tan x + \cot x = 2\csc 2x$

Solution

$$\begin{aligned}
\tan x + \cot x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \\
&= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} \\
&= \frac{1}{\cos x \sin x} \\
&= \frac{1}{\frac{1}{2}\sin 2x}
\end{aligned}$$

$$= 2\frac{1}{\sin 2x}$$

$$\underline{= 2\csc 2x}$$

Exercise

Prove the following equation is an identity: $\tan 2x = \dfrac{2}{\cot x - \tan x}$

Solution

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$= \frac{2\frac{\tan x}{\tan x}}{\frac{1}{\tan x} - \frac{\tan^2 x}{\tan x}}$$

$$\underline{= \frac{2}{\cot x - \tan x}}$$

Exercise

Prove the following equation is an identity: $\dfrac{1-\tan x}{1+\tan x} = \dfrac{1-\sin 2x}{\cos 2x}$

Solution

$$\frac{1-\tan x}{1+\tan x} = \frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}$$

$$= \frac{\frac{\cos x - \sin x}{\cos x}}{\frac{\cos x + \sin x}{\cos x}}$$

$$= \frac{\cos x - \sin x}{\cos x + \sin x}\frac{\cos x - \sin x}{\cos x - \sin x}$$

$$= \frac{\cos^2 x - 2\cos x \sin x + \sin^2 x}{\cos^2 x + \sin^2 x}$$

$$\underline{= \frac{1-\sin 2x}{\cos 2x}}$$

Exercise

Prove the following equation is an identity: $\sin 2\alpha \sin 2\beta = \sin^2(\alpha+\beta) - \sin^2(\alpha-\beta)$

Solution

$$\sin 2\alpha \sin 2\beta = (2\sin\alpha\cos\alpha)(2\sin\beta\cos\beta)$$

$$= (2\sin\alpha\cos\beta)(2\sin\beta\cos\alpha)$$

$$=\left(2\tfrac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]\right)\left(2\tfrac{1}{2}\left[\sin(\beta+\alpha)+\sin(\beta-\alpha)\right]\right)$$

$$=\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)\left(\sin(\alpha+\beta)-\sin(\alpha-\beta)\right)$$

$$=\sin^2(\alpha+\beta)-\sin^2(\alpha-\beta)$$

Exercise

Prove the following equation is an identity: $\cos^2(A-B)-\cos^2(A+B)=\sin 2A\ \sin 2B$

Solution

$$\begin{aligned}\cos^2(A-B)-\cos^2(A+B)&=\left(\cos(A-B)-\cos(A+B)\right)\left(\cos(A-B)+\cos(A+B)\right)\\&=(2\sin A\sin B)(2\cos A\cos B)\\&=(2\sin A\cos A)(2\sin B\cos B)\\&=\sin 2A\ \sin 2B\end{aligned}$$

Exercise

Use half-angle formulas to find the exact value of $\sin 105°$

Solution

$$\begin{aligned}\sin 105°&=\sin\frac{210°}{2}\\&=\sqrt{\frac{1-\cos 210°}{2}}\qquad \textit{reference}: 210°-180°=30°\\&=\sqrt{\frac{1+\cos 30°}{2}}\\&=\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}\\&=\sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}}\\&=\sqrt{\frac{2+\sqrt{3}}{4}}\\&=\frac{\sqrt{2+\sqrt{3}}}{2}\end{aligned}$$

Exercise

Find the exact of $\tan 22.5°$

Solution

$$\tan 22.5° = \tan\frac{45°}{2}$$
$$= \frac{1-\cos 45°}{\sin 45°}$$
$$= \frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$
$$= \frac{\frac{2-\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$
$$= \frac{2-\sqrt{2}}{\sqrt{2}}$$
$$= \frac{2}{\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \boxed{\sqrt{2}-1}$$

Exercise

Given: $\cos x = \frac{2}{3}$, $\frac{3\pi}{2} < x < 2\pi$, find $\cos\frac{x}{2}$, $\sin\frac{x}{2}$, and $\tan\frac{x}{2}$

Solution

$$\frac{3\pi}{2} < x < 2\pi \Rightarrow \frac{3\pi}{4} < \frac{x}{2} < \pi \Rightarrow x \in QII$$

$\cos\frac{x}{2} = -\sqrt{\frac{1+\cos x}{2}}$ $= -\sqrt{\frac{1+\frac{2}{3}}{2}}$ $= -\sqrt{\frac{1}{2}\frac{3+2}{3}}$ $= -\sqrt{\frac{5}{6}}$ $= -\frac{\sqrt{5}}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}}$ $= \boxed{-\frac{\sqrt{30}}{6}}$	$\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}$ $= \sqrt{\frac{1-\frac{2}{3}}{2}}$ $= \sqrt{\frac{1}{2}\frac{3-2}{3}}$ $= \sqrt{\frac{1}{6}}$ $= \frac{1}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}}$ $= \boxed{\frac{\sqrt{6}}{6}}$	$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$ $= \frac{\frac{\sqrt{6}}{6}}{-\frac{\sqrt{30}}{6}}$ $= -\frac{\sqrt{6}}{\sqrt{30}}\frac{\sqrt{30}}{\sqrt{30}}$ $= -\frac{6\sqrt{5}}{30}$ $= \boxed{-\frac{\sqrt{5}}{5}}$

Exercise

Prove the identity $2\csc x\cos^2\frac{x}{2}=\frac{\sin x}{1-\cos x}$

Solution

$$2\csc x\cos^2\frac{x}{2}=2\frac{1}{\sin x}\frac{1+\cos x}{2}$$

$$=\frac{1+\cos x}{\sin x}\frac{1-\cos x}{1-\cos x}$$

$(a-b)(a+b)=a^2-b^2$

$$=\frac{1-\cos^2 x}{\sin x(1-\cos x)}$$

$$=\frac{\sin^2 x}{\sin x(1-\cos x)}$$

$$=\frac{\sin x}{1-\cos x}$$

Exercise

Prove the identity $\tan\frac{\alpha}{2}=\sin\alpha+\cos\alpha\cot\alpha-\cot\alpha$

Solution

$$\tan\frac{\alpha}{2}=\frac{1-\cos\alpha}{\sin\alpha}$$

$1=\sin^2\alpha+\cos^2\alpha$

$$=\frac{\sin^2\alpha+\cos^2\alpha-\cos\alpha}{\sin\alpha}$$

$$=\frac{\sin^2\alpha}{\sin\alpha}+\frac{\cos^2\alpha}{\sin\alpha}-\frac{\cos\alpha}{\sin\alpha}$$

$$=\sin\alpha+\cos\alpha\frac{\cos\alpha}{\sin\alpha}-\cot\alpha$$

$$=\sin\alpha+\cos\alpha\cot\alpha-\cot\alpha$$

Exercise

Prove the following equation is an identity: $\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)=\frac{\sin^2 x}{4}$

Solution

$$\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)=\frac{1-\cos x}{2}\cdot\frac{1+\cos x}{2}$$

$(a-b)(a+b)=a^2+b^2$

$$=\frac{1-\cos^2 x}{4}$$

$\cos^2\alpha+\sin^2\alpha=1$

$$=\frac{\sin^2 x}{4}$$

Exercise

Prove the following equation is an identity: $\tan\frac{x}{2}+\cot\frac{x}{2}=2\csc x$

Solution

$$\tan\frac{x}{2}+\cot\frac{x}{2}=\tan\frac{x}{2}+\frac{1}{\tan\frac{x}{2}}$$
$$=\frac{\sin x}{1+\cos x}+\frac{\sin x}{1-\cos x}$$
$$=\sin x\frac{(1-\cos x)+(1+\cos x)}{1-\cos^2 x}$$
$$=\sin x\frac{2}{\sin^2 x}$$
$$=\frac{2}{\sin x}$$
$$=2\csc x$$

Exercise

Prove the following equation is an identity: $2\sin^2\left(\frac{x}{2}\right)=\frac{\sin^2 x}{1+\cos x}$

Solution

$$2\sin^2\left(\frac{x}{2}\right)=2\frac{1-\cos x}{2}$$
$$=1-\cos x\cdot\frac{1+\cos x}{1+\cos x}$$
$$=\frac{1-\cos^2 x}{1+\cos x}$$
$$=\frac{\sin^2 x}{1+\cos x}$$

Exercise

Prove the following equation is an identity: $\tan^2\left(\frac{x}{2}\right)=\frac{\sec x+\cos x-2}{\sec x-\cos x}$

Solution

$$\tan^2\left(\frac{x}{2}\right)=\frac{1-\cos x}{1+\cos x}$$

$\tan^2\alpha=\frac{1-\cos 2\alpha}{1+\cos 2\alpha}$; $x=2\alpha$; $\tan^2\frac{x}{2}=\frac{1-\cos x}{1+\cos x}$

$$=\frac{1-\cos x}{1+\cos x}\frac{1-\cos x}{1-\cos x}$$
$$=\frac{1-2\cos x+\cos^2 x}{1-\cos^2 x}\;\frac{\frac{1}{\cos x}}{\frac{1}{\cos x}}$$

$$= \frac{\frac{1-2\cos x+\cos^2 x}{\cos x}}{\frac{1-\cos^2 x}{\cos x}}$$

$$= \frac{\frac{1}{\cos x} - \frac{2\cos x}{\cos x} + \frac{\cos^2 x}{\cos x}}{\frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}}$$

$$= \frac{\sec x - 2 + \cos x}{\sec x - \cos x}$$

$$\frac{\sec x + \cos x - 2}{\sec x - \cos x} = \frac{\frac{1}{\cos x} + \cos x - 2}{\frac{1}{\cos x} - \cos x}$$

$$= \frac{\frac{1+\cos^2 x - 2\cos x}{\cos x}}{\frac{1-\cos^2 x}{\cos x}}$$

$$= \frac{(1-\cos x)^2}{(1-\cos x)(1+\cos x)}$$

$$= \frac{1-\cos x}{1+\cos x}$$

$$\tan^2 \alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha}; \quad x = 2\alpha; \quad \tan^2 \frac{x}{2} = \frac{1-\cos x}{1+\cos x}$$

$$= \tan^2\left(\frac{x}{2}\right)$$

Exercise

Prove the following equation is an identity: $\sec^2\left(\frac{x}{2}\right) = \frac{2\sec x + 2}{\sec x + 2 + \cos x}$

Solution

$$\sec^2\left(\frac{x}{2}\right) = \frac{1}{\cos^2\left(\frac{x}{2}\right)}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}} \Rightarrow \cos^2\left(\frac{\alpha}{2}\right) = \frac{1+\cos\alpha}{2}$$

$$= \frac{1}{\frac{1+\cos x}{2}}$$

$$= \frac{2}{1+\cos x}\frac{1+\cos x}{1+\cos x}$$

$$= \frac{2+2\cos x}{1+2\cos x+\cos^2 x}$$

$$= \frac{2+2\cos x}{1+2\cos x+\cos^2 x}\frac{\frac{1}{\cos x}}{\frac{1}{\cos x}}$$

$$= \dfrac{\dfrac{2}{\cos x} + 2\dfrac{\cos x}{\cos x}}{\dfrac{1}{\cos x} + \dfrac{2\cos x}{\cos x} + \dfrac{\cos^2 x}{\cos x}}$$

$$= \boxed{\dfrac{2\sec x + 2}{\sec x + 2 + \cos x}}$$

Exercise

Prove the following equation is an identity: $\dfrac{1-\sin^2\left(\frac{x}{2}\right)}{1+\sin^2\left(\frac{x}{2}\right)} = \dfrac{1+\cos x}{3-\cos x}$

Solution

$$\dfrac{1-\sin^2\left(\frac{x}{2}\right)}{1+\sin^2\left(\frac{x}{2}\right)} = \dfrac{1-\frac{1-\cos x}{2}}{1+\frac{1-\cos x}{2}}$$

$$= \dfrac{\frac{2-1-\cos x}{2}}{\frac{2+1-\cos x}{2}}$$

$$= \boxed{\dfrac{1-\cos x}{3-\cos x}}$$

Exercise

Prove the following equation is an identity: $\dfrac{1-\cos^2\left(\frac{x}{2}\right)}{1-\sin^2\left(\frac{x}{2}\right)} = \dfrac{1-\cos x}{1+\cos x}$

Solution

$$\dfrac{1-\cos^2\left(\frac{x}{2}\right)}{1-\sin^2\left(\frac{x}{2}\right)} = \dfrac{1-\frac{1+\cos x}{2}}{1-\frac{1-\cos x}{2}}$$

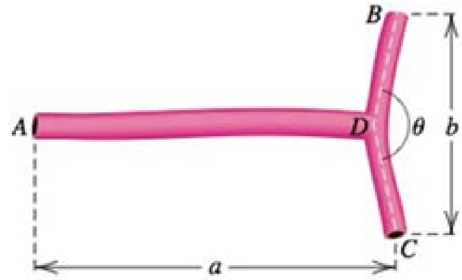
$$= \dfrac{\frac{2-(1+\cos x)}{2}}{\frac{2-(1-\cos x)}{2}}$$

$$= \dfrac{\frac{2-1-\cos x}{2}}{\frac{2-1+\cos x}{2}}$$

$$= \boxed{\dfrac{1-\cos x}{1+\cos x}}$$

Exercise

A common form of cardiovascular branching is bifurcation, in which an artery splits into two smaller blood vessels. The bifurcation angle θ is the angle formed by the two smaller arteries. The line through A and D bisects θ and is perpendicular to the line through B and C.



a) Show that the length l of the artery from A to B is given by $l = a + \frac{b}{2}\tan\frac{\theta}{4}$.

b) Estimate the length l from the three measurements $a = 10\ mm$, $b = 6\ mm$, and $\theta = 156°$.

Solution

a) $\tan\frac{\theta}{2} = \dfrac{\frac{b}{2}}{a - |AD|}$

$$|AD| = a - \frac{b}{2}\frac{1}{\tan\frac{\theta}{2}}$$

$$\sin\frac{\theta}{2} = \frac{b}{2}\frac{1}{|DB|} \Rightarrow |DB| = \frac{b}{2}\frac{1}{\sin\frac{\theta}{2}}$$

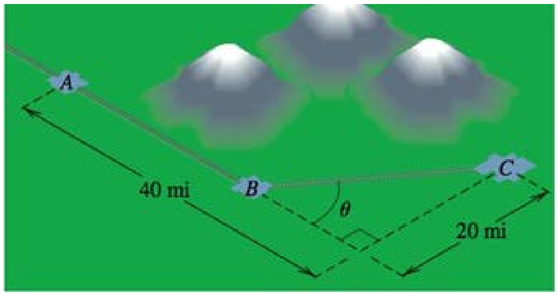
$$\begin{aligned} l &= |AD| + |DB| \\ &= a - \frac{b}{2}\frac{1}{\tan\frac{\theta}{2}} + \frac{b}{2}\frac{1}{\sin\frac{\theta}{2}} \\ &= a + \frac{b}{2}\left(\frac{1}{\sin\frac{\theta}{2}} - \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right) \\ &= a + \frac{b}{2}\left(\frac{1-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right) \qquad \tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} \\ &\boxed{= a + \frac{b}{2}\tan\frac{\theta}{4}} \end{aligned}$$

b) **Given**: $a = 10\ mm,\ b = 6\ mm,\ \theta = 156°$

$$\boxed{l = 10 + \frac{6}{2}\tan\frac{156°}{4}} = 10 + 3\tan 39° \ \boxed{\approx 12.43\ mm}$$

Exercise

A proposed rail road route through three towns located at points *A*, *B*, and *C*. At *B*, the track will turn toward *C* at an angle θ.



a) Show that the total distance *d* from *A* to *C* is given by $d = 20\tan\frac{1}{2}\theta + 40$

b) Because of mountains between *A* and *C*, the turning point *B* must be at least 20 *miles* from *A*.Is there a route that avoids the mountains and measures exactly 50 *miles*?

Solution

a) $d = |AB| + |BC|$

$$\tan\theta = \frac{20}{40 - |AB|} \quad \rightarrow \quad |AB| = 40 - \frac{20}{\tan\theta}$$

$$\sin\theta = \frac{20}{|BC|} \quad \rightarrow \quad |BC| = \frac{20}{\sin\theta}$$

$$d = 40 - \frac{20}{\tan\theta} + \frac{20}{\sin\theta}$$

$$= 40 + 20\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)$$

$$= 40 + 20\left(\frac{1-\cos\theta}{\sin\theta}\right) \qquad \tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$$

$$= 40 + 20\tan\frac{\theta}{2}$$

b) $50 = 40 + 20\tan\frac{\theta}{2}$

$$20\tan\frac{\theta}{2} = 10 \quad \rightarrow \quad \frac{\theta}{2} = \tan^{-1}\frac{1}{2} \approx 25.565° \quad \Rightarrow \quad \theta = 53.13°$$

$$|AB| = 40 - \frac{20}{\tan 53.13°} \approx 25$$

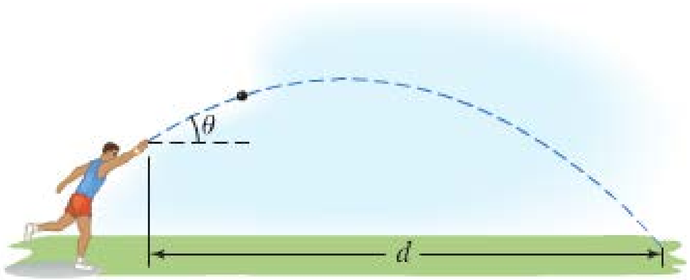
Yes, point *B* is 25 *miles* from *A*.

Exercise

Throwing events in track and field include the shot put, the discus throw, the hammer throw, and the javelin throw. The distance that the athlete can achieve depends on the initial speed of the object thrown and the angle above the horizontal at which the object leaves the hand. This angle is represented by θ. The distance, d, in *feet*, that the athlete throws is modeled by the formula

$$d = \frac{v_0^2}{16}\sin\theta\cos\theta$$

In which v_0 is the initial speed of the object thrown, in *feet* per *second*, and θ is the angle, in *degrees*, at which the object leaves the hand.



a) Use the identity to express the formula so that it contains the since function only.

b) Use the formula from part (*a*) to find the angle, θ, that produces the maximum distance, d, for a given initial speed, v_0.

Solution

a) $d = \frac{v_0^2}{16}\sin\theta\cos\theta$ $\qquad$ $\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{v_0^2}{16}\frac{1}{2}\sin 2\theta$$

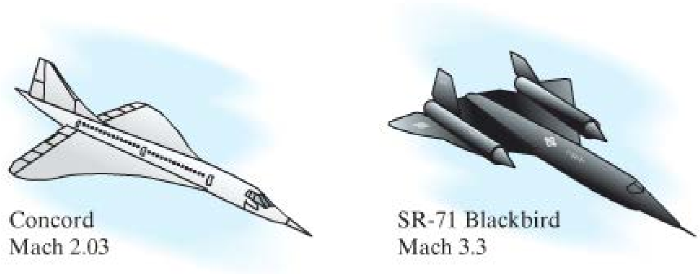
$$= \frac{v_0^2}{32}\sin 2\theta$$

b) The maximum value of a sine function is 1 at $\frac{\pi}{2}$ on the interval $[0,\ 2\pi]$

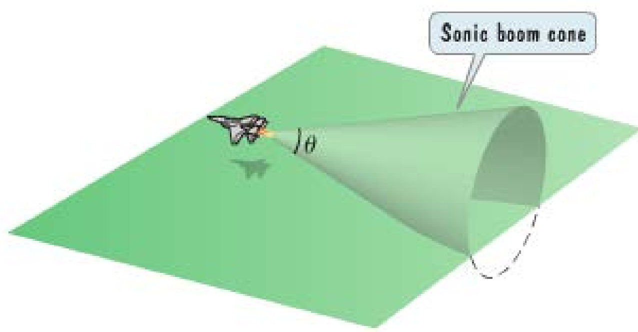
$$2\theta = \frac{\pi}{2} \ \Rightarrow\ \theta = \frac{\pi}{4}$$

Exercise

The speed of a supersonic aircraft is usually represented by a Mach number. A Mach number is the speed of the aircraft, in *miles* per *hour*, divided by the speed of sound, approximately 740 *mph*. Thus, a plane flying at twice the speed of sound has a speed, M, of Mach 2.



If an aircraft has a speed greater than Mach 1, a sonic boom is heard, created by sound waves that form a cone with a vertex angle θ.



The relationship between the cone's vertex angle θ, and the Mach speed, M, of an aircraft that is flying faster than the speed of sound is given by

$$\sin\frac{\theta}{2} = \frac{1}{M}$$

a) If $\theta = \frac{\pi}{6}$, determine the Mach speed, M, of the aircraft. Express the speed as an exact value and as decimal to the nearest tenth.

b) If $\theta = \frac{\pi}{4}$, determine the Mach speed, M, of the aircraft. Express the speed as an exact value and as decimal to the nearest tenth.

Solution

a) At $\theta = \frac{\pi}{6}$

$$\begin{aligned}\sin\frac{\theta}{2} &= \sqrt{\frac{1}{2}(1-\cos\theta)} \\ &= \sqrt{\frac{1}{2}\left(1-\cos\frac{\pi}{6}\right)} \\ &= \sqrt{\frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right)} \\ &= \sqrt{\frac{2-\sqrt{3}}{4}} \\ &= \frac{1}{2}\sqrt{2-\sqrt{3}} = \frac{1}{M}\end{aligned}$$

$$M = \frac{2}{\sqrt{2-\sqrt{3}}}\frac{\sqrt{2-\sqrt{3}}}{\sqrt{2-\sqrt{3}}}$$

$$= \frac{2\sqrt{2-\sqrt{3}}}{2-\sqrt{3}}\frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$= 2\left(2+\sqrt{3}\right)\sqrt{2-\sqrt{3}} \quad \approx 3.9$$

b) At $\theta = \frac{\pi}{4}$

$$\sin\frac{\theta}{2} = \sqrt{\frac{1}{2}(1-\cos\theta)}$$

$$= \sqrt{\frac{1}{2}\left(1-\cos\frac{\pi}{4}\right)}$$

$$= \sqrt{\frac{1}{2}\left(1-\frac{\sqrt{2}}{2}\right)}$$

$$= \sqrt{\frac{2-\sqrt{2}}{4}}$$

$$= \frac{1}{2}\sqrt{2-\sqrt{2}} = \frac{1}{M}$$

$$M = \frac{2}{\sqrt{2-\sqrt{2}}}\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2-\sqrt{2}}}$$

$$= \frac{2\sqrt{2-\sqrt{2}}}{2-\sqrt{2}}\frac{2+\sqrt{2}}{2+\sqrt{2}}$$

$$= \frac{2\left(2+\sqrt{2}\right)\sqrt{2-\sqrt{2}}}{2}$$

$$= \left(2+\sqrt{2}\right)\sqrt{2-\sqrt{2}} \quad \approx 2.6$$