

Solution **Section 3.6 – Solving Linear Recurrence Relations**

Exercise

Determine which of these are linear and homogeneous recurrence relations with constant coefficients. Also find the degree of those that are

a) $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$

b) $a_n = 2na_{n-1} + a_{n-2}$

c) $a_n = a_{n-1} + a_{n-4}$

d) $a_n = a_{n-1} + 2$

e) $a_n = a_{n-1}^2 + a_{n-2}$

f) $a_n = a_{n-2}$

g) $a_n = a_{n-1} + n$

h) $a_n = 3a_{n-2}$

i) $a_n = 3$

j) $a_n = a_{n-1}^2$

k) $a_n = a_{n-1} + 2a_{n-3}$

l) $a_n = \frac{a_{n-1}}{n}$

Solution

a) Linear (terms a_i all to the first power), has constant coefficients (3, 4 and 5), and is homogeneous (no terms are functions of just n); has degree 3

b) Linear (terms a_i all to the first power), doesn't have constant coefficients ($2n$), and is homogeneous

c) Linear, homogeneous, with constant coefficients; degree 4

d) Linear with constant coefficients, not homogeneous because of 2

e) Not linear since a_{n-1}^2

f) Linear, homogeneous, with constant coefficients; degree 2

g) Linear but not homogeneous because of the n .

h) Linear, homogeneous, with constant coefficients; degree 2

i) Linear with constant coefficients, not homogeneous because of 3

- j) Not linear since a_{n-1}^2
- k) Linear, homogeneous, with constant coefficients; degree 3
- l) Linear with constant coefficients, not homogeneous

Exercise

Solve these recurrence relations together with the initial conditions given

- a) $a_n = 2a_{n-1}$ for $n \geq 1$, $a_0 = 3$
- b) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$
- c) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$
- d) $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 4$
- e) $a_n = \frac{a_{n-2}}{4}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$
- f) $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$
- g) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 2$, $a_1 = 1$
- h) $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = -3$
- i) $a_{n+2} = -4a_{n-1} + 5a_n$ for $n \geq 0$, $a_0 = 2$, $a_1 = 8$

Solution

- a) The characteristic polynomial is $r - 2 = 0 \Rightarrow r = 2$

The general solution: $a_n = \alpha_1 2^n$

$$3 = \alpha_1 2^0 \rightarrow \alpha_1 = 3$$

Therefore, the solution is $a_n = 3 \cdot 2^n$

- b) The characteristic polynomial is $r^2 - 5r + 6 = 0 \Rightarrow r = 2, 3$

The general solution: $a_n = \alpha_1 2^n + \alpha_2 3^n$

$$1 = \alpha_1 2^0 + \alpha_2 3^0 \rightarrow 1 = \alpha_1 + \alpha_2$$

$$0 = \alpha_1 2^1 + \alpha_2 3^1 \rightarrow 0 = 2\alpha_1 + 3\alpha_2$$

$$\Rightarrow \alpha_1 = 3, \alpha_2 = -2$$

Therefore, the solution is $a_n = 3 \cdot 2^n - 2 \cdot 3^n$

- c) The characteristic polynomial is $r^2 - 4r + 4 = 0 \Rightarrow r = 2, 2$

The general solution: $a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n$

$$\begin{aligned} 6 &= \alpha_1 2^0 + \alpha_2 (0) 2^0 \rightarrow 6 = \alpha_1 \\ 8 &= \alpha_1 2^1 + \alpha_2 (1) 2^1 \rightarrow 8 = 2\alpha_1 + 2\alpha_2 \end{aligned} \Rightarrow \alpha_1 = 6, \alpha_2 = -2$$

Therefore, the solution is $a_n = 6 \cdot 2^n - 2n \cdot 2^n = \underline{(6-2n)2^n}$

d) The characteristic polynomial is $r^2 - 4 = 0 \Rightarrow r = \pm 2$

The general solution: $a_n = \alpha_1 (-2)^n + \alpha_2 2^n$

$$\begin{aligned} 0 &= \alpha_1 (-2)^0 + \alpha_2 2^0 \rightarrow 0 = \alpha_1 + \alpha_2 \\ 4 &= \alpha_1 (-2)^1 + \alpha_2 2^1 \rightarrow 4 = -2\alpha_1 + 2\alpha_2 \end{aligned} \Rightarrow \alpha_1 = -1, \alpha_2 = 1$$

Therefore, the solution is $a_n = \underline{2^n - (-2)^n}$

e) The characteristic polynomial is $r^2 - \frac{1}{4} = 0 \Rightarrow r = \pm \frac{1}{2}$

The general solution: $a_n = \alpha_1 \left(-\frac{1}{2}\right)^n + \alpha_2 \left(\frac{1}{2}\right)^n = \alpha_1 (-2)^{-n} + \alpha_2 (2)^{-n}$

$$\begin{aligned} 1 &= \alpha_1 \left(-\frac{1}{2}\right)^0 + \alpha_2 \left(\frac{1}{2}\right)^0 \rightarrow 1 = \alpha_1 + \alpha_2 \\ 0 &= \alpha_1 \left(-\frac{1}{2}\right)^1 + \alpha_2 \left(\frac{1}{2}\right)^1 \rightarrow 0 = -\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 \end{aligned} \Rightarrow \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{2}$$

Therefore, the solution is $a_n = \frac{1}{2} \left(-\frac{1}{2}\right)^n + \frac{1}{2} \left(\frac{1}{2}\right)^n = \underline{\left(\frac{1}{2}\right)^{n+1} - \left(-\frac{1}{2}\right)^{n+1}}$

f) The characteristic polynomial is $r^2 - r - 6 = 0 \Rightarrow r = -2, 3$

The general solution: $a_n = \alpha_1 (-2)^n + \alpha_2 3^n$

$$\begin{aligned} 3 &= \alpha_1 (-2)^0 + \alpha_2 3^0 \rightarrow 3 = \alpha_1 + \alpha_2 \\ 6 &= \alpha_1 (-2)^1 + \alpha_2 3^1 \rightarrow 6 = -2\alpha_1 + 3\alpha_2 \end{aligned} \Rightarrow \alpha_1 = \frac{3}{5}, \alpha_2 = \frac{12}{5}$$

Therefore, the solution is $a_n = \underline{\frac{3}{5}(-2)^n + \frac{12}{5}3^n}$

g) The characteristic polynomial is $r^2 - 7r + 10 = 0 \Rightarrow r = 2, 5$

The general solution: $a_n = \alpha_1 2^n + \alpha_2 5^n$

$$\begin{aligned} 2 &= \alpha_1 2^0 + \alpha_2 5^0 \rightarrow 2 = \alpha_1 + \alpha_2 \\ 1 &= \alpha_1 2^1 + \alpha_2 5^1 \rightarrow 1 = 2\alpha_1 + 5\alpha_2 \end{aligned} \Rightarrow \alpha_1 = 3, \alpha_2 = -1$$

Therefore, the solution is $a_n = \underline{3 \cdot 2^n - 5^n}$

h) The characteristic polynomial is $r^2 + 6r + 9 = 0 \Rightarrow r = -3, -3$

The general solution: $a_n = \alpha_1 (-3)^n + \alpha_2 n(-3)^n$

$$3 = \alpha_1 (-3)^0 + \alpha_2 (0)(-3)^0 \rightarrow 3 = \alpha_1 \Rightarrow \alpha_1 = 3, \alpha_2 = -2$$

$$-3 = \alpha_1 (-3)^1 + \alpha_2 (1)(-3)^1 \rightarrow -3 = -3\alpha_1 + -3\alpha_2$$

Therefore, the solution is $\boxed{a_n = 3 \cdot (-3)^n - 2n(-3)^n = (3 - 2n)(-3)^n}$

i) The characteristic polynomial is $r^2 + 4r - 5 = 0 \Rightarrow r = -5, 1$

The general solution: $a_n = \alpha_1 (-5)^n + \alpha_2 1^n = \alpha_1 (-5)^n + \alpha_2$

$$2 = \alpha_1 (-5)^0 + \alpha_2 \rightarrow 2 = \alpha_1 + \alpha_2 \Rightarrow \alpha_1 = -1, \alpha_2 = 3$$

$$8 = \alpha_1 (-5)^1 + \alpha_2 \rightarrow 8 = -5\alpha_1 + \alpha_2$$

Therefore, the solution is $\boxed{a_n = -(-5)^n + 3}$

Exercise

How many different messages can be transmitted in n microseconds using three different signals if one signal requires 1 microsecond for transmittal, the other two signals require 2 microseconds each for transmittal, and a signal in a message is followed immediately by the next signal?

Solution

The model is the recurrence relation $a_n = a_{n-1} + a_{n-2} + a_{n-2} = a_{n-1} + 2a_{n-2}$ with $a_0 = a_1 = 1$

The characteristic polynomial is $r^2 - r - 2 = 0$

So, the roots are -1 , and 2

The general solution: $a_n = \alpha_1 (-1)^n + \alpha_2 2^n$

Plugging in initial conditions gives

$$1 = \alpha_1 (-1)^0 + \alpha_2 2^0 \rightarrow 1 = \alpha_1 + \alpha_2$$

$$1 = \alpha_1 (-1)^1 + \alpha_2 2^1 \rightarrow 1 = -\alpha_1 + 2\alpha_2 \Rightarrow \alpha_1 = \frac{1}{3}, \alpha_2 = \frac{2}{3}$$

Therefore, the solution is in n microseconds $\boxed{a_n = \frac{1}{3}(-1)^n + \frac{2}{3}2^n}$ messages can be transmitted.

Exercise

In how many ways can a $2 \times n$ rectangular checkerboard be tiled using 1×2 and 2×2 pieces?

Solution

Let t_n be the number of ways like to tile a $2 \times n$ board with 1×2 and 2×2 pieces. To obtain the recurrence relation, imagine what tiles are placed at the left-hand end of the board. We can place a 2×2 tile there, leaving a $2 \times (n-2)$ board to be tiled, which of course can be done in t_{n-2} ways.

We can place a 1×2 tile at the edge, oriented vertically, leaving $2 \times (n-1)$ board to be tiled, which of course can be done in t_{n-1} ways.

Finally, we can place two 1×2 tiles horizontally, one above the other, leaving a $2 \times (n-2)$ board to be tiled, which of course can be done in t_{n-2} ways. These 3 possibilities are disjoint.

Therefore, our recurrence relation is $t_n = t_{n-1} + 2t_{n-2}$

The initial conditions are $t_0 = t_1 = 1$, since there is only one way to tile as 2×0 board and 2×1 board.

This recurrence relation has characteristic roots -1 and 2 .

So, the general solution is $t_n = \alpha_1 (-1)^n + \alpha_2 2^n$

Plugging in initial conditions gives

$$\begin{aligned} 1 &= \alpha_1 (-1)^0 + \alpha_2 2^0 \rightarrow 1 = \alpha_1 + \alpha_2 \\ 1 &= \alpha_1 (-1)^1 + \alpha_2 2^1 \rightarrow 1 = -\alpha_1 + 2\alpha_2 \end{aligned} \Rightarrow \alpha_1 = \frac{1}{3}, \quad \alpha_2 = \frac{2}{3}$$

Therefore, the solution is $a_n = \frac{1}{3}(-1)^n + \frac{2}{3} \cdot 2^n$

$$= \frac{(-1)^n}{3} + \frac{2^{n+1}}{3}$$

Exercise

Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n \geq 3$, $a_0 = 3$, $a_1 = 6$ and $a_2 = 0$

Solution

$$a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$$

The characteristic polynomial is $r^3 - 2r^2 - r + 2 = 0$

That implies to: $r^2(r-2) - (r-2) = (r-2)(r^2 - 1) = 0$

So, the roots are 1 , -1 , and 2

The general solution:

$$\begin{aligned} a_n &= \alpha_1 1^n + \alpha_2 (-1)^n + \alpha_3 2^n \\ &= \alpha_1 + \alpha_2 (-1)^n + \alpha_3 2^n \end{aligned}$$

Plugging in initial conditions gives

$$3 = \alpha_1 + \alpha_2 (-1)^0 + \alpha_3 2^0 \rightarrow 3 = \alpha_1 + \alpha_2 + \alpha_3$$

$$6 = \alpha_1 + \alpha_2 (-1)^1 + \alpha_3 2^1 \rightarrow 6 = \alpha_1 - \alpha_2 + 2\alpha_3$$

$$\Rightarrow \alpha_1 = 6, \quad \alpha_2 = -2, \quad \alpha_3 = -1$$

$$0 = \alpha_1 + \alpha_2 (-1)^2 + \alpha_3 2^2 \rightarrow 0 = \alpha_1 + \alpha_2 + 4\alpha_3$$

Therefore, the solution is $\underline{a_n = 6 - 2(-1)^n - 2^n}$

Exercise

Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9$, $a_1 = 10$ and $a_2 = 32$

Solution

This is a third-degree recurrence relation.

The characteristic polynomial is $r^3 - 7r - 6 = 0$

By the rational root test, the possible rational roots are $\pm \left\{ \frac{6}{1} \right\} = \pm \{1, 2, 3, 6\}$

We find that $r = -1$ (using calculator).

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array} \quad Q(x) = r^2 - r - 6 = (r+2)(r-3)$$

$$r^3 - 6r^2 + 12r - 8 = (r+1)(r+2)(r-3) = 0$$

So, the roots are $-2, -1$, and 3 .

The general solution:

$$a_n = \alpha_1 (-2)^n + \alpha_2 (-1)^n + \alpha_3 3^n$$

Plugging in initial conditions gives

$$a_0 = 9 = \alpha_1 (-2)^0 + \alpha_2 (-1)^0 + \alpha_3 3^0 \rightarrow 9 = \alpha_1 + \alpha_2 + \alpha_3$$

$$a_1 = 10 = \alpha_1 (-2)^1 + \alpha_2 (-1)^1 + \alpha_3 3^1 \rightarrow 10 = -2\alpha_1 - \alpha_2 + 3\alpha_3$$

$$a_2 = 32 = \alpha_1 (-2)^2 + \alpha_2 (-1)^2 + \alpha_3 3^2 \rightarrow 32 = 4\alpha_1 + \alpha_2 + 9\alpha_3$$

The solution to the system of equations is $\alpha_1 = -3$, $\alpha_2 = 8$ and $\alpha_3 = 4$

Therefore, the specific solution is $\underline{a_n = -3(-2)^n + 8(-1)^n + 4 \cdot 3^n}$

Exercise

Find the solution to $a_n = 5a_{n-2} - 4a_{n-4}$ with $a_0 = 3, a_1 = 2, a_2 = 6$ and $a_3 = 8$

Solution

This is a fourth-degree recurrence relation.

The characteristic polynomial is $r^4 - 5r^2 - 4 = 0$

That implies to: $(r^2 - 1)(r^2 - 4) = (r-1)(r+1)(r-2)(r+2) = 0$

So, the roots are 1, -1, 2, -2

The general solution: $a_n = \alpha_1 + \alpha_2 (-1)^n + \alpha_3 2^n + \alpha_4 (-2)^n$

Plugging in initial conditions gives

$$3 = \alpha_1 + \alpha_2 (-1)^0 + \alpha_3 2^0 + \alpha_4 (-2)^0 \rightarrow 3 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

$$10 = \alpha_1 + \alpha_2 (-1)^1 + \alpha_3 2^1 + \alpha_4 (-2)^1 \rightarrow 10 = \alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4$$

$$6 = \alpha_1 + \alpha_2 (-1)^2 + \alpha_3 2^2 + \alpha_4 (-2)^2 \rightarrow 6 = \alpha_1 + \alpha_2 + 4\alpha_3 + 4\alpha_4$$

$$8 = \alpha_1 + \alpha_2 (-1)^3 + \alpha_3 2^3 + \alpha_4 (-2)^3 \rightarrow 8 = \alpha_1 - \alpha_2 + 8\alpha_3 - 8\alpha_4$$

The solution to the system of equations is $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $\alpha_4 = 0$

Therefore, the solution is $a_n = 1 + (-1)^n + 2^n$

Exercise

Find the recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ with $a_0 = -5, a_1 = 4$ and $a_2 = 88$

Solution

This is a third-degree recurrence relation.

The characteristic polynomial is $r^3 - 6r^2 + 12r - 8 = 0$

By the rational root test, the possible rational roots are $\pm 1, \pm 2, \pm 4, \pm 8$

We find that $r = 2$ (using calculator).

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 12 & -8 \\ & & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$$Q(x) = r^2 - 4r + 4 = (r-2)^2$$

$$r^3 - 6r^2 + 12r - 8 = (r-2)^3 = 0$$

Hence the only root is 2, with multiplicity 3.

The general solution: $a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n + \alpha_3 n^2 \cdot (-2)^n$

Plugging in initial conditions gives

$$\underline{-5 = a_0 = \alpha_1}$$

$$4 = a_1 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 2 \rightarrow \alpha_2 + \alpha_3 = 7$$

$$88 = a_2 = 4\alpha_1 + 8\alpha_2 + 16\alpha_3$$

$$\alpha_1 + 2\alpha_2 + 4\alpha_3 = 22 \rightarrow 2\alpha_2 + 4\alpha_3 = 27$$

$$\rightarrow \begin{cases} \alpha_2 + \alpha_3 = 7 \\ 2\alpha_2 + 4\alpha_3 = 27 \end{cases} \Rightarrow \begin{cases} \alpha_2 = \frac{1}{2} \\ \alpha_3 = \frac{13}{2} \end{cases}$$

Therefore, the solution:
$$a_n = -5 \cdot 2^n + \frac{1}{2}n \cdot 2^n + \frac{13}{2}n^2 \cdot (-2)^n$$

$$\underline{= -5 \cdot 2^n + n \cdot 2^{n-1} + 13n^2 \cdot (-2)^{n-1}}$$

Exercise

Find the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 5$, $a_1 = -9$ and $a_2 = 15$

Solution

This is a third-degree recurrence relation.

The characteristic polynomial is $r^3 + 3r^2 + 3r + 1 = 0$

$$r^3 + 3r^2 + 3r + 1 = 0 = (r+1)^3 = 0$$

Hence the only root is -1 , with multiplicity 3.

The general solution:
$$\underline{a_n = \alpha_1(-1)^n + \alpha_2 n \cdot (-1)^n + \alpha_3 n^2 \cdot (-1)^n}$$

Plugging in initial conditions gives

$$\underline{5 = a_0 = \alpha_1}$$

$$a_1 = -9 = -\alpha_1 - \alpha_2 - \alpha_3 \rightarrow \alpha_2 + \alpha_3 = 9 - \alpha_1 = 4$$

$$a_2 = 15 = \alpha_1 + 2\alpha_2 + 4\alpha_3 \rightarrow 2\alpha_2 + 4\alpha_3 = 15 - \alpha_1 = 10$$

$$\rightarrow \begin{cases} \alpha_2 + \alpha_3 = 4 \\ 2\alpha_2 + 4\alpha_3 = 10 \end{cases} \Rightarrow \begin{cases} \alpha_2 = 3 \\ \alpha_3 = 1 \end{cases}$$

Therefore, the specific solution is
$$a_n = 5(-1)^n + 3n \cdot (-1)^n + n^2 \cdot (-1)^n$$

$$\underline{= (n^2 + 3n + 5)(-1)^n}$$

Exercise

Find the general form of the solutions of the recurrence relation $a_n = 8a_{n-2} - 16a_{n-4}$

Solution

This is a fourth-degree recurrence relation.

The characteristic polynomial is $r^4 - 8r^2 + 16 = (r^2 - 4)^2$

$$(r^2 - 4)^2 = (r - 2)^2 (r + 2)^2 = 0$$

The roots are -2 and 2 , each with multiplicity 2.

The general solution:
$$a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n + \alpha_3 (-2)^n + \alpha_4 n \cdot (-2)^n$$

Exercise

What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots $1, 1, 1, 1, -2, -2, -2, 3, 3, -4$?

Solution

There are 4 distinct roots, so $t = 4$. The multiplicities are 4, 3, 2, and 1.

The general solution:

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2 + \alpha_{1,3}n^3) + (\alpha_{2,0} + \alpha_{2,1}n + \alpha_{2,2}n^2)(-2)^n + (\alpha_{3,0} + \alpha_{3,1}n)3^n + \alpha_{4,0}(-4)^n$$

Exercise

What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots $-1, -1, -1, 2, 2, 5, 5, 7$?

Solution

There are 4 distinct roots, so $t = 4$. The multiplicities are 3, 2, 2, and 1.

The general solution:

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)(-1)^n + (\alpha_{2,0} + \alpha_{2,1}n)2^n + (\alpha_{3,0} + \alpha_{3,1}n)5^n + \alpha_{4,0}7^n$$