

## Section 1.3 – Fractions and Rationalization

### *Fraction (Basic)*

$$\frac{a}{b} = \frac{\text{numerator}}{\text{denominator}}$$

$$\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc \quad \text{Cross multiplication}$$

$$\frac{a}{b} = \frac{na}{nb} = \frac{an}{bn}$$

$$\begin{aligned} a) \quad \frac{5}{6} &= \frac{25}{30} ? \\ \frac{5}{6} &= \frac{5 \cdot 5}{6 \cdot 5} = \frac{25}{30} \end{aligned}$$

$$\begin{aligned} b) \quad \frac{16}{48} &= \frac{1}{3} \\ \frac{16}{48} &= \frac{1}{3} \Leftrightarrow (16)(3) = (1)(48) \\ 48 &= 48 \end{aligned}$$

$$\begin{aligned} \text{Simplify: } \frac{12}{18} &= \frac{2.6}{2.9} \\ &= \frac{2.2.3}{2.3.3} \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \text{Simplify: } \frac{36}{56} &= \frac{2.18}{2.28} \\ &= \frac{18}{28} \\ &= \frac{2.9}{2.14} \\ &= \frac{9}{14} \end{aligned}$$

If the denominators are the same  $\Rightarrow$  add the numerators

$$\frac{3}{5} + \frac{4}{5} = \frac{3+4}{5} = \frac{7}{5}$$

If the denominators are the same  $\Rightarrow$  subtract the numerators

$$\frac{4}{9} - \frac{2}{9} = \frac{4-2}{9} = \frac{2}{9}$$

If the denominators are not the same

$\Rightarrow$  Find Least Common Denominator (LCD) and convert so that the fractions have the same denominators

**LCD:** is the smallest whole number that is a multiple of each

$$\frac{5}{8} + \frac{1}{12}$$

LCD (8, 12)

$$8 = 2^3$$

$$12 = 2^2 \cdot 3$$

$$2^3 \cdot 3 = 24$$

$$\text{LCD}(8, 12) = 24$$

$$\begin{aligned}\frac{5}{8} + \frac{1}{12} &= \frac{5}{8} \cdot \frac{3}{3} + \frac{1}{12} \cdot \frac{2}{2} \\ &= \frac{15}{24} + \frac{2}{24} \\ &= \frac{15+2}{24} \\ &= \frac{17}{24}\end{aligned}$$

$$\frac{69}{75} - \frac{1}{50}$$

LCD (75, 50)

$$75 = 5^3$$

$$50 = 2 \cdot 5^2$$

$$2 \cdot 5^3 = 150$$

$$\text{LCD}(75, 50) = 150$$

$$\begin{aligned}\frac{69}{75} - \frac{1}{50} &= \frac{(69)(2) - (1)(3)}{150} \\ &= \frac{138-3}{150} \\ &= \frac{135}{150} \\ &= \frac{9}{10}\end{aligned}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

$$\begin{aligned}\frac{2}{7} + \frac{3}{5} &= \frac{2(5)+3(7)}{7(5)} \\ &= \frac{10+21}{35} \\ &= \frac{31}{35}\end{aligned}$$

$$\begin{aligned}\text{or } \frac{2}{7} \frac{5}{5} + \frac{3}{5} \frac{7}{7} &= \frac{10}{35} + \frac{21}{35} \\ &= \frac{10+21}{35} \\ &= \frac{31}{35}\end{aligned}$$

$$\begin{aligned}\frac{5}{9} + \frac{3}{4} &= \frac{5(4)+3(9)}{9(4)} \\ &= \frac{20+27}{36} \\ &= \frac{47}{36}\end{aligned}$$

$$\begin{aligned}\frac{17}{15} + \frac{5}{12} &= \frac{17(12)+5(15)}{15(12)} \\ &= \frac{204+75}{180} \\ &= \frac{279}{180} \\ &= \frac{31(9)}{20(9)} \\ &= \frac{31}{20}\end{aligned}$$

$$\begin{aligned}
 \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} &= \frac{5(7)(9) + (3)(7)(9) + (3)(5)(9) + (3)(5)(7)}{(3)(5)(7)(9)} \\
 &= \frac{315 + 189 + 135 + 105}{945} \\
 &= \frac{744}{945} \\
 &= \frac{248}{315} \\
 &= \frac{248}{315}
 \end{aligned}$$

$$\frac{8}{9} + \frac{1}{12} + \frac{3}{16}$$

$$\frac{8}{9} + \frac{1}{12} + \frac{3}{16} = \frac{8(16) + 1(12) + 3(9)}{144}$$

$$\begin{cases} 9 = 3^2 \\ 12 = 2^2 \cdot 3 \\ 16 = 2^4 \end{cases}$$

$$\text{LCD } 2^4 \cdot 3^2 = 144$$

$$= \frac{128 + 12 + 27}{144}$$

$$= \frac{167}{144}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

$$\frac{2}{7} - \frac{3}{5} = \frac{2(5) - 3(7)}{7(5)} = \frac{10 - 21}{35} = -\frac{11}{35}$$

$$\frac{a}{c} \cdot \frac{b}{d} = \frac{ab}{cd}$$

$$\frac{2}{7} \cdot \frac{3}{5} = \frac{6}{35}$$

$$\frac{a}{c} \div \frac{b}{d} = \frac{a}{c} \times \frac{d}{b} = \frac{ad}{cb}$$

$$\frac{2}{7} \div \frac{3}{5} = \frac{2}{7} \cdot \frac{5}{3} = \frac{10}{21}$$

**Find:**

$$1. \quad \frac{13}{21} + \frac{5}{21} = \frac{13+5}{21} = \frac{6}{7}$$

$$2. \quad \frac{7}{12} - \frac{4}{15} = \frac{7(5) - 4(4)}{60} = \frac{35-16}{60} = \frac{19}{60}$$

$$3. \quad \frac{5}{8} + \frac{1}{2} = \frac{5+4}{8} = \frac{9}{8}$$

$$4. \quad \frac{5}{8} + \frac{1}{2} + \frac{2}{3} = \frac{5(3) + 1(12) + 2(8)}{24} = \frac{43}{24}$$

$$5. \quad \frac{7}{8} - \frac{1}{10} = \frac{7(5) - 1(4)}{40} = \frac{31}{40}$$

$$6. \quad \frac{11}{5} - \frac{31}{7} = -\frac{78}{35}$$

$$7. \quad \frac{3}{4} \cdot \frac{3}{2} = \frac{9}{8}$$

$$8. \quad \frac{3}{4} \cdot \frac{4}{3} \cdot \frac{2}{3} = \frac{2}{3}$$

$$9. \quad \frac{3}{4} \div \frac{3}{2} = \frac{3}{4} \cdot \frac{2}{3} = \frac{2}{4} = \frac{1}{2}$$

$$10. \quad \frac{14}{15} \div \frac{14}{3} = \frac{14}{15} \cdot \frac{3}{14} = \frac{1}{5}$$

## Operations with Fractions

A rational expression is proper if the degree of numerator is less than the degree of denominator

A rational expression is improper if the degrees of numerator is greater than or equal the degree of denominator

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$$

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}$$

$$\frac{a/b}{c/d} = \frac{a}{b} \frac{d}{c} = \frac{ad}{bc}$$

$$\frac{a/b}{c} = \frac{a}{b} \frac{1}{c} = \frac{a}{bc}$$

$$\frac{ab}{ac} = \frac{b}{c}$$

$$\frac{ad+ac}{ad} = \frac{a(d+c)}{ad} = \frac{b+c}{d}$$

$$\frac{ab+cd}{ad} \quad \text{stay}$$

### ***Example***

Perform each indicated operation & simplify

$$a) \quad x + \frac{2}{x} = \frac{x^2 + 2}{x}$$

$$\begin{aligned} b) \quad \frac{2}{x+1} - \frac{1}{2x+1} &= \frac{2(2x+1) - 1(x+1)}{(x+1)(2x+1)} \\ &= \frac{4x+2-x-1}{(x+1)(2x+1)} \\ &= \frac{3x+1}{(x+1)(2x+1)} \end{aligned}$$

### Example

Perform each indicated operation & simplify

$$\begin{aligned} a) \quad \frac{x}{x^2-4} - \frac{1}{x-2} &= \frac{x-1(x+2)}{(x-2)(x+2)} \\ &= \frac{x-x-2}{(x-2)(x+2)} \\ &= \frac{-2}{(x-2)(x+2)} \end{aligned}$$

$$x^2 - 4 = (x-2)(x+2)$$

$$\begin{aligned} b) \quad \frac{1}{3(x^2+2x)} - \frac{1}{3x} &= \frac{1-1(x+2)}{3x(x+2)} \\ &= \frac{1-x-2}{3x(x+2)} \\ &= \frac{-x-1}{3x(x+2)} \end{aligned}$$

$$3(x^2+2x) = 3x(x+2)$$

### Example

Perform each indicated operation & simplify

$$\begin{aligned} a) \quad \frac{\sqrt{x+2} - \frac{x}{4\sqrt{x+2}}}{x+2} &= \left( \sqrt{x+2} - \frac{x}{4\sqrt{x+2}} \right) \div (x+2) \\ &= \left( \frac{4\sqrt{x+2}\sqrt{x+2} - x}{4\sqrt{x+2}} \right) \left( \frac{1}{x+2} \right) \\ &= \frac{4(x+2) - x}{4(x+2)\sqrt{x+2}} \\ &= \frac{4x+8-x}{4(x+2)\sqrt{x+2}} \\ &= \frac{3x+8}{4(x+2)\sqrt{x+2}} \end{aligned}$$

$$\begin{aligned} b) \quad \left( \frac{1}{x+\sqrt{x^2+4}} \right) \left( 1 + \frac{x}{\sqrt{x^2+4}} \right) &= \frac{1}{x+\sqrt{x^2+4}} \cdot \frac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}} \\ &= \frac{1}{\sqrt{x^2+4}} \end{aligned}$$

**Example**

Perform each indicated operation & simplify

$$\begin{aligned}
 & \frac{-x\left(\frac{3x}{3\sqrt{x^2+4}}\right) + \sqrt{x^2+4}}{x^2} + \left(\frac{1}{x+\sqrt{x^2+4}}\right)\left(1 + \frac{3x}{3\sqrt{x^2+4}}\right) \\
 &= \left(-\frac{3x^2}{3\sqrt{x^2+4}} + \sqrt{x^2+4}\right) \frac{1}{x^2} + \left(\frac{1}{x+\sqrt{x^2+4}}\right)\left(\frac{3\sqrt{x^2+4}+3x}{3\sqrt{x^2+4}}\right) \\
 &= \left(\frac{-3x^2+3(\sqrt{x^2+4})^2}{3\sqrt{x^2+4}}\right) \frac{1}{x^2} + \left(\frac{1}{x+\sqrt{x^2+4}}\right)\left(\frac{3(\sqrt{x^2+4}+x)}{3\sqrt{x^2+4}}\right) \\
 &= \left(\frac{-3x^2+3(x^2+4)}{3\sqrt{x^2+4}}\right) \frac{1}{x^2} + \frac{3}{3\sqrt{x^2+4}} \\
 &= \frac{-3x^2+3x^2+12}{3\sqrt{x^2+4}} \frac{1}{x^2} + \frac{3}{3\sqrt{x^2+4}} \\
 &= \frac{12}{3\sqrt{x^2+4}} \frac{1}{x^2} + \frac{3}{3\sqrt{x^2+4}} \\
 &= \frac{12+3x^2}{3x^2\sqrt{x^2+4}} \\
 &= \frac{3(x^2+4)}{3x^2(x^2+4)^{1/2}} \\
 &= \frac{\sqrt{x^2+4}}{x^2}
 \end{aligned}$$



## Rationalization Techniques

1. If the denominator is  $\sqrt{a}$ , multiply by  $\frac{\sqrt{a}}{\sqrt{a}}$
2. If the denominator is  $\sqrt{a} - \sqrt{b}$ , multiply by  $\frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}}$
3. If the denominator is  $\sqrt{a} + \sqrt{b}$ , multiply by  $\frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}}$

$$(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = a - b$$

### Example

Simplify by rationalizing the denominator

$$\begin{aligned} a) \quad \frac{4}{\sqrt{3}} &= \frac{4}{\sqrt{3}} \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{4\sqrt{3}}{3} \end{aligned}$$

$$\begin{aligned} b) \quad \frac{2}{\sqrt[3]{x}} &= \frac{2}{\sqrt[3]{x}} \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} \\ &= \frac{2\sqrt[3]{x^2}}{x} \end{aligned}$$

$$\begin{aligned} c) \quad \frac{1}{1-\sqrt{2}} &= \frac{1}{1-\sqrt{2}} \frac{1+\sqrt{2}}{1+\sqrt{2}} \\ &= \frac{1+\sqrt{2}}{1-2} \\ &= \frac{1+\sqrt{2}}{-1} \\ &= \underline{-1-\sqrt{2}} \end{aligned}$$

**Example**

Simplify  $\sqrt{27}\sqrt{3}$

$$\begin{aligned}\sqrt{27}\sqrt{3} &= \sqrt{27(3)} \\ &= \sqrt{81} \\ &= 9\end{aligned}$$

**Example**

Simplify  $\sqrt[4]{x^8y^7z^{11}}$

$$\sqrt[4]{x^8y^7z^{11}} = x^2yz^2\sqrt[4]{y^3z^3}$$

**Example**

Simplify  $\frac{5}{\sqrt{10}}$

$$\begin{aligned}\frac{5}{\sqrt{10}} &= \frac{5}{\sqrt{10}} \frac{\sqrt{10}}{\sqrt{10}} \\ &= \frac{5\sqrt{10}}{10} \\ &= \frac{\sqrt{10}}{2}\end{aligned}$$

**Example**

Simplify  $\frac{5}{2-\sqrt{6}}$

$$\begin{aligned}\frac{5}{2-\sqrt{6}} &= \frac{5}{2-\sqrt{6}} \frac{2+\sqrt{6}}{2+\sqrt{6}} \\ &= \frac{5(2+\sqrt{6})}{4-6} \\ &= -\frac{5}{2}(2+\sqrt{6})\end{aligned}$$

**Example**

Simplify  $\frac{1}{\sqrt{r}-\sqrt{3}}$

$$\begin{aligned}\frac{1}{\sqrt{r}-\sqrt{3}} &= \frac{1}{\sqrt{r}-\sqrt{3}} \frac{\sqrt{r}+\sqrt{3}}{\sqrt{r}+\sqrt{3}} \\ &= \frac{\sqrt{r}+\sqrt{3}}{r-3}\end{aligned}$$

### Example

Rationalize the denominator or numerator

$$\begin{aligned} a) \quad \frac{5}{\sqrt{8}} &= \frac{5}{\sqrt{8}} \frac{\sqrt{8}}{\sqrt{8}} \\ &= \frac{5\sqrt{8}}{8} \end{aligned}$$

$$\begin{aligned} b) \quad \frac{1}{\sqrt{6}-\sqrt{3}} &= \frac{1}{\sqrt{6}-\sqrt{3}} \frac{\sqrt{6}+\sqrt{3}}{\sqrt{6}+\sqrt{3}} \\ &= \frac{\sqrt{6}+\sqrt{3}}{(\sqrt{6})^2 - (\sqrt{3})^2} \\ &= \frac{\sqrt{6}+\sqrt{3}}{6-3} = \frac{\sqrt{6}+\sqrt{3}}{3} \\ &= \frac{\sqrt{6}+\sqrt{3}}{3} \end{aligned}$$

$$\begin{aligned} c) \quad \frac{1}{\sqrt{x}+\sqrt{x+2}} &= \frac{1}{\sqrt{x}+\sqrt{x+2}} \frac{\sqrt{x}-\sqrt{x+2}}{\sqrt{x}-\sqrt{x+2}} \\ &= \frac{\sqrt{x}-\sqrt{x+2}}{x-(x+2)} \\ &= \frac{\sqrt{x}-\sqrt{x+2}}{x-x-2} \\ &= \frac{\sqrt{x}-\sqrt{x+2}}{-2} \\ &= \frac{\sqrt{x+2}-\sqrt{x}}{2} \end{aligned}$$

**Example**

$$\begin{aligned} -\frac{\sqrt{x^2+1}}{x^2} - \frac{1}{\sqrt{x^2+1}} &= \frac{-\sqrt{x^2+1}\sqrt{x^2+1} - x^2}{x^2\sqrt{x^2+1}} & -\frac{\sqrt{x^2+1}}{x^2} \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} - \frac{1}{\sqrt{x^2+1}} \frac{x^2}{x^2} \\ &= \frac{-(x^2+1) - x^2}{x^2\sqrt{x^2+1}} \\ &= \frac{-x^2-1-x^2}{x^2\sqrt{x^2+1}} \\ &= \frac{-2x^2-1}{x^2\sqrt{x^2+1}} \\ &= -\frac{2x^2+1}{x^2\sqrt{x^2+1}} \end{aligned}$$

**Example**

$$\begin{aligned} \left( \sqrt{x^2+1} - \frac{3x^3}{2\sqrt{x^2+1}} \right) \div (x^3+1) &= \left( \frac{\sqrt{x^2+1}(2\sqrt{x^2+1}) - 3x^3}{2\sqrt{x^2+1}} \right) \cdot \frac{1}{x^3+1} \\ &= \frac{2(x^2+1) - 3x^3}{2(x^3+1)\sqrt{x^2+1}} \\ &= \frac{-3x^3+2x^2+2}{2(x^3+1)\sqrt{x^2+1}} \end{aligned}$$

## Exercises    **Section 1.3 – Fractions and Rationalization**

1. Perform the operation and simplify  $\frac{2}{x^2 - 4} - \frac{1}{x - 2}$
2. Perform each indicated operation & simplify:  $\frac{A}{x+1} - \frac{B}{x-1} + \frac{C}{x+2}$
3. Perform the operation and simplify:  $-\frac{\sqrt{x^2+1}}{x^2} - \frac{1}{\sqrt{x^2+1}}$
4. Perform the operation and simplify:  $\left( \sqrt{x^2+1} - \frac{3x^3}{2\sqrt{x^2+1}} \right) \div (x^3 + 1)$
5. Perform the operation and simplify:  $\frac{6}{x(3x-2)} + \frac{5}{3x-2} - \frac{2}{x^2}$
6. Simplify the fraction:  $\frac{\frac{2}{x+3} - \frac{2}{a+3}}{x-a}$
7. Simplify:  $\frac{3x^2(2x+5)^{1/2} - x^3\left(\frac{1}{2}\right)(2x+5)^{-1/2}(2)}{\left[(2x+5)^{1/2}\right]^2}$
8. Simplify the expression:  $\frac{\left(4x^2+9\right)^{1/2}(2) - (2x+3)\left(\frac{1}{2}\right)\left(4x^2+9\right)^{-1/2}(8x)}{\left[\left(4x^2+9\right)^{1/2}\right]^2}$
9. Simplify the expression:  $\frac{\left(1-x^2\right)^{1/2}(2x) - x^2\left(\frac{1}{2}\right)\left(1-x^2\right)^{-1/2}(-2x)}{\left[\left(1-x^2\right)^{1/2}\right]^2}$
10. Simplify the expression:  $\frac{\left(x^2+4\right)^{1/3}(3) - (3x)\left(\frac{1}{3}\right)\left(x^2+4\right)^{-2/3}(2x)}{\left[\left(x^2+4\right)^{1/3}\right]^2}$

11. Simplify the expression: 
$$\frac{(x^2 - 5)^4 (3x^2) - x^3 (4) (x^2 - 5)^3 (2x)}{\left[ (x^2 - 5)^4 \right]^2}$$

12. Simplify the expression: 
$$\frac{(3x + 2)^{1/2} \left( \frac{1}{3} \right) (2x + 3)^{-2/3} (2) - (2x + 3)^{1/3} \left( \frac{1}{2} \right) (3x + 2)^{-1/2} (3)}{\left[ (3x + 2)^{1/2} \right]^2}$$

13. Simplify the expression: 
$$\frac{(x^2 + 2)^3 (2x) - x^2 (3) (x^2 + 2)^2 (2x)}{\left[ (x^2 + 2)^3 \right]^2}$$