Solution Section 2.1 – Integration by Parts

Exercise

Evaluate the integral $\int xe^{2x}dx$

Solution

		$\int e^{2x}\, dx$
+	x	$\frac{1}{2}e^{2x}$
–	1	$\frac{1}{4}e^{2x}$

$$\int xe^{2x}dx = \boxed{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}$$

Let: $u = x \Rightarrow du = dx$

$$dv = e^{2x}dx \Rightarrow v = \int dv = \int e^{2x}dx = \frac{1}{2}e^{2x}$$

$$\int udv \;=\; uv - \int vdu$$

$$\int xe^{2x}dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx$$

$$= \boxed{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}$$

Exercise

Evaluate the integral $\int x\ln x\, dx$

Solution

Let: $u = \ln x \Rightarrow du = \frac{1}{x}dx$

$$dv = xdx \Rightarrow v = \int dv = \int xdx = \frac{1}{2}x^2$$

$$\int x\ln xdx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x^2\frac{dx}{x}$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int xdx$$

$$= \boxed{\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C}$$

Exercise

Evaluate the integral $\int x^3 e^x dx$

Solution

		$\int e^x\, dx$
+	x^3	e^x
−	$3x^2$	e^x
+	$6x$	e^x
−	6	e^x

$$\int x^3 e^x dx = \underline{e^x\left(x^3 - 3x^2 + 6x - 6\right) + C}$$

Let: $u = x^3 \ \Rightarrow du = 3x^2 dx$

$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$

$$\int x^3 e^x dx = x^3 e^x - \int e^x 3x^2 dx$$

$$= x^3 e^x - 3\int e^x x^2 dx$$

Let: $u = x^2 \ \Rightarrow du = 2x dx$

$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$

$$\int e^x x^2 dx = x^2 e^x - 2\int x e^x dx$$

$$\int x^3 e^x dx = x^3 e^x - 3\left[x^2 e^x - 2\int x e^x dx\right]$$

$$= x^3 e^x - 3x^2 e^x + 6\int x e^x dx$$

Let: $u = x \ \Rightarrow du = dx$

$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x$$

$$\int x^3 e^x dx = x^3 e^x - 3x^2 e^x + 6\left[x e^x - e^x\right] + C$$

$$= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$$

$$= \underline{e^x\left(x^3 - 3x^2 + 6x - 6\right) + C}$$

Exercise

Evaluate the integral $\int \ln x^2 dx$

Solution

$$\int \ln x^2 dx = 2\int \ln x dx \qquad u = \ln x \Rightarrow du = \frac{1}{x} dx \qquad v = \int dx = x$$

$$\int \ln x^2 dx = 2\left[x \ln x - \int x \frac{1}{x} dx\right]$$

$$= 2\left[x\ln x - \int dx\right]$$

$$= 2(x\ln x - x) + C$$

$$\boxed{= 2x(\ln x - 1) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{2x}{e^x}\,dx$

Solution

		$\int e^{-x}\,dx$
+	$2x$	$-e^{-x}$
−	2	e^{-x}

$$\int \frac{2x}{e^x}\,dx \boxed{= -e^{-x}(2x+2) + C}$$

$$u = 2x \Rightarrow du = 2dx$$

$$dv = e^{-x}dx \Rightarrow v = -e^{-x}$$

$$\int \frac{2x}{e^x}\,dx = 2x(-e^{-x}) - \int -e^{-x}\,2dx$$

$$= -2xe^{-x} + 2\int e^{-x}dx$$

$$= -2xe^{-x} - 2e^{-x} + C$$

$$= -2e^{-x}(x+1) + C$$

$$\boxed{= -\frac{2(x+1)}{e^x} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \ln(3x)\,dx$

Solution

$$u = \ln 3x \Rightarrow du = \frac{3}{3x}dx = \frac{1}{x}dx \qquad dv = dx \Rightarrow v = x$$

$$\int \ln(3x)dx = x\ln(3x) - \int x\frac{1}{x}\,dx$$

$$= x\ln(3x) - \int dx$$

$$= x\ln(3x) - x + C$$

$$\boxed{= x\left[\ln(3x) - 1\right] + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{1}{x\ln x}\,dx$

Solution

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{\ln x}\frac{1}{x}\,dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$$

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du$$

$$= \ln u + C$$

$$\boxed{= \ln\left|\ln x\right| + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x}{\sqrt{x-1}}\,dx$

Solution

Let: $u = x \ \Rightarrow du = dx$

$$dv = \frac{dx}{\sqrt{x-1}} \Rightarrow v = \int (x-1)^{-1/2}\,d(x-1)$$

$$= \frac{(x-1)^{1/2}}{1/2}$$

$$= 2(x-1)^{1/2}$$

$$\int \frac{x}{\sqrt{x-1}}\,dx = 2x\sqrt{x-1} - 2\int (x-1)^{1/2}\,dx$$

$$= 2x\sqrt{x-1} - 2\frac{(x-1)^{3/2}}{3/2} + C$$

$$= 2x\sqrt{x-1} - \frac{4}{3}(x-1)\sqrt{x-1} + C$$

$$= \sqrt{x-1}\left[2x - \frac{4}{3}x + \frac{4}{3}\right] + C$$

$$= \sqrt{x-1}\left[\frac{6x-4x+4}{3}\right] + C$$

$$= \sqrt{x-1}\left[\frac{2x+4}{3}\right] + C$$

$$\boxed{= \frac{2}{3}\sqrt{x-1}(x+2) + C}$$

Let: $u = x-1 \ \Rightarrow \ x = u+1$

$du = dx$

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int (u+1)u^{-1/2}du$$

$$= \int \left(u^{1/2} + u^{-1/2}\right)du$$

$$= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C$$

$$= (x-1)^{1/2}\left(\frac{2}{3}x - \frac{2}{3} + 2\right) + C$$

$$= \sqrt{x-1}\left[\frac{2x+4}{3}\right] + C$$

$$\boxed{= \frac{2}{3}\sqrt{x-1}(x+2) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^3 e^{x^2}}{\left(x^2+1\right)^2}dx$

Solution

Let: $u = x^2 e^{x^2} \quad \Rightarrow du = \left(2xe^{x^2} + 2xx^2 e^{x^2}\right)dx$

$$du = 2xe^{x^2}\left(1+x^2\right)dx$$

$dv = x\left(x^2+1\right)^{-2}dx \qquad \Rightarrow v = \displaystyle\int x(x^2+1)^{-2}dx$

$$= \frac{1}{2}\int (x^2+1)^{-2}d(x^2+1)$$

$$= \frac{(x^2+1)^{-1}}{-1}$$

$$= -\frac{1}{2(x^2+1)}$$

$$\int \frac{x^3 e^{x^2}}{(x^2+1)^2}dx = x^2 e^{x^2}\left(-\frac{1}{2(x^2+1)}\right) - \int -\frac{1}{2(x^2+1)}2xe^{x^2}(x^2+1)dx$$

$$= -\frac{x^2 e^{x^2}}{2(x^2+1)} + \int xe^{x^2}dx$$

Let: $u = x^2 \quad \Rightarrow du = 2xdx$

$$\int \frac{x^3 e^{x^2}}{\left(x^2+1\right)^2}dx = -\frac{x^2 e^{x^2}}{2(x^2+1)} + \frac{1}{2}\int e^u du$$

$$= -\frac{x^2 e^{x^2}}{2(x^2+1)} + \frac{1}{2}e^u + C$$

$$= -\frac{x^2 e^{x^2}}{2(x^2+1)} + \frac{1}{2}e^{x^2} + C$$

$$= \frac{1}{2}e^{x^2}\left[-\frac{x^2}{(x^2+1)} + 1\right] + C$$

$$= \frac{1}{2}e^{x^2}\left[\frac{-x^2+x^2+1}{(x^2+1)}\right] + C$$

$$= \frac{e^{x^2}}{2(x^2+1)} + C$$

Exercise

Evaluate the integral $\int x^2 e^{-3x}dx$

Solution

$u = x^2 \Rightarrow du = 2xdx$

$dv = e^{-3x}dx \Rightarrow v = -\frac{1}{3}e^{-3x}$

$$\int x^2 e^{-3x}dx = -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\int xe^{-3x}dx$$

$u = x \Rightarrow du = dx$

$dv = e^{-3x}dx \Rightarrow v = -\frac{1}{3}e^{-3x}$

$$\int x^2 e^{-3x}dx = -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\left[-\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}dx\right]$$

$$= -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\left[-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right] + C$$

$$= -\frac{1}{3}x^2 e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C$$

$$\boxed{= -\frac{9x^2+6x+2}{27}e^{-3x} + C}$$

		$\int e^{-3x}$
+	x^2	$-\frac{1}{3}e^{-3x}$
−	$2x$	$\frac{1}{9}e^{-3x}$
+	2	$-\frac{1}{27}e^{-3x}$

$$\int x^2 e^{-3x}dx =$$

$$\boxed{-\frac{1}{3}x^2 e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C}$$

Exercise

Evaluate the integral $\int \theta \cos \pi\theta \, d\theta$

Solution

Let: $u = \theta \qquad dv = \cos \pi\theta \, d\theta$

$du = d\theta \qquad v = \int \cos \pi\theta \, d\theta = \frac{1}{\pi}\sin \pi\theta$

$$\int \theta \cos \pi\theta \, d\theta = \frac{\theta}{\pi}\sin \pi\theta - \int \frac{1}{\pi}\sin \pi\theta \, d\theta$$

$$= \frac{\theta}{\pi}\sin \pi\theta + \frac{1}{\pi}\frac{1}{\pi}\cos \pi\theta + C$$

$$\boxed{= \frac{\theta}{\pi}\sin \pi\theta + \frac{1}{\pi^2}\cos \pi\theta + C}$$

Exercise

Evaluate the integral $\int x^2 \sin x \, dx$

Solution

		$\int \sin x$
x^2	$(+)$	$-\cos x$
$2x$	$(-)$	$-\sin x$
2	$(+)$	$\cos x$
0		

$$\int x^2 \sin x dx = \boxed{-x^2 \cos x + 2x \sin x + 2\cos x + C}$$

Exercise

Evaluate the integrals $\int x(\ln x)^2 \, dx$

Solution

$u = \ln x \quad \rightarrow \quad x = e^u$

$du = \frac{1}{x} dx \quad \Rightarrow \quad xdu = dx \rightarrow dx = e^u du$

$$\int x(\ln x)^2 \, dx = \int e^u u^2 e^u du$$

$$= \int u^2 e^{2u} du$$

$$= \frac{1}{2} u^2 e^{2u} - \frac{1}{2} u e^{2u} + \frac{1}{4} e^{2u} + C$$

$$= \frac{1}{4} e^{2u} \left(2u^2 - 2u + 1\right) + C$$

$$= \boxed{\frac{1}{4} x^2 \left(2(\ln x)^2 - 2\ln x + 1\right) + C}$$

	$\int e^{2u} du$
u^2	$\frac{1}{2} e^{2u}$
$2u$	$\frac{1}{4} e^{2u}$
2	$\frac{1}{8} e^{2u}$
0	

2nd Method

$u = \ln x \qquad dv = \int (x \ln x) dx$

$du = \frac{1}{x} dx \qquad v = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2$

$$\int x(\ln x)^2 \, dx = (\ln x)\left(\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2\right) - \int \left(\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2\right) \frac{1}{x} dx$$

$u = \ln x \Rightarrow du = \frac{1}{x} dx$

$dv = xdx \Rightarrow v = \frac{1}{2} x^2$

$$\int x \ln x dx = \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x^2 \frac{dx}{x}$$

$$=\frac{1}{2}x^2(\ln x)^2-\frac{1}{4}x^2\ln x-\int\left(\frac{1}{2}x\ln x-\frac{1}{4}x\right)dx$$

$$=\frac{1}{2}x^2(\ln x)^2-\frac{1}{4}x^2\ln x-\left(\frac{1}{2}\left(\frac{1}{2}x^2\ln x-\frac{1}{4}x^2\right)-\frac{1}{8}x^2\right)+C$$

$$=\frac{1}{2}x^2(\ln x)^2-\frac{1}{4}x^2\ln x-\frac{1}{4}x^2\ln x+\frac{1}{8}x^2+\frac{1}{8}x^2+C$$

$$\boxed{=\frac{1}{2}x^2(\ln x)^2-\frac{1}{2}x^2\ln x+\frac{1}{4}x^2+C}$$

$$=\frac{1}{2}x^2\ln x-\frac{1}{2}\int xdx$$

$$=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2$$

3rd Method

$u=(\ln x)^2 \qquad dv=\int xdx$

$du=2(\ln x)\frac{1}{x}dx \qquad v=\frac{1}{2}x^2$

$$\int x(\ln x)^2\,dx=\frac{1}{2}x^2(\ln x)^2-\int\frac{1}{2}x^2(2\ln x)\frac{1}{x}dx$$

$$=\frac{1}{2}x^2(\ln x)^2-\int x\ln xdx$$

$$=\frac{1}{2}x^2(\ln x)^2-\frac{1}{4}x^2\ln x-\left(\frac{1}{2}x^2\ln x-\frac{1}{4}x^2\right)+C$$

$$=\frac{1}{2}x^2(\ln x)^2-\frac{1}{4}x^2\ln x-\frac{1}{2}x^2\ln x+\frac{1}{4}x^2+C$$

$$\boxed{=\frac{1}{2}x^2(\ln x)^2-\frac{1}{2}x^2\ln x+\frac{1}{4}x^2+C}$$

$u=\ln x\Rightarrow du=\frac{1}{x}dx$

$dv=xdx\Rightarrow v=\frac{1}{2}x^2$

$$\int x\ln xdx=\frac{1}{2}x^2\ln x-\frac{1}{2}\int x^2\frac{dx}{x}$$

$$=\frac{1}{2}x^2\ln x-\frac{1}{2}\int xdx$$

$$=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2$$

Exercise

Evaluate the integral $\int\left(x^2-2x+1\right)e^{2x}dx$

Solution

		$\int e^{2x}$
$+$	x^2-2x+1	$\frac{1}{2}e^{2x}$
$-$	$2x-2$	$\frac{1}{4}e^{2x}$
$+$	2	$\frac{1}{8}e^{2x}$

$$\int\left(x^2-2x+1\right)e^{2x}dx=\frac{1}{2}\left(x^2-2x+1\right)e^{2x}-\frac{1}{4}(2x-2)e^{2x}+\frac{1}{8}(2)e^{2x}+C$$

$$=\left(\frac{1}{2}x^2-x+\frac{1}{2}-\frac{1}{2}x+\frac{1}{2}+\frac{1}{4}\right)e^{2x}+C$$

$$\boxed{=\left(\frac{1}{2}x^2-\frac{3}{2}x+\frac{5}{4}\right)e^{2x}+C}$$

Exercise

Evaluate the integral $\int \tan^{-1} y\, dy$

Solution

Let: $u = \tan^{-1} y \quad dv = dy$

$du = \dfrac{dy}{1+y^2} \quad v = y$

$$\int \tan^{-1} y\, dy = y\tan^{-1} y - \int \frac{ydy}{1+y^2} \qquad d\left(1+y^2\right) = 2ydy \quad \to \quad \tfrac{1}{2}d\left(1+y^2\right) = ydy$$

$$= y\tan^{-1} y - \int \frac{\frac{1}{2}d\left(1+y^2\right)}{1+y^2}$$

$$= y\tan^{-1} y - \tfrac{1}{2}\ln\left(1+y^2\right) + C$$

$$= y\tan^{-1} y - \ln\sqrt{1+y^2} + C$$

Exercise

Evaluate the integral $\int \sin^{-1} y\, dy$

Solution

Let: $u = \sin^{-1} y \quad dv = dy$

$du = \dfrac{dy}{\sqrt{1-y^2}} \quad v = y$

$$\int \sin^{-1} y\, dy = y\sin^{-1} y - \int \frac{ydy}{\sqrt{1-y^2}} \qquad d\left(1-y^2\right) = -2ydy \quad \to \quad -\tfrac{1}{2}d\left(1-y^2\right) = ydy$$

$$= y\sin^{-1} y + \tfrac{1}{2}\int \left(1-y^2\right)^{-1/2} d\left(1-y^2\right)$$

$$= y\sin^{-1} y + \tfrac{1}{2}(2)\left(1-y^2\right)^{1/2} + C$$

$$= y\sin^{-1} y + \sqrt{1-y^2} + C$$

Exercise

Evaluate the integral $\displaystyle\int 4x\sec^2 2x\, dx$

Solution

Let: $u = 4x \quad \to du = 4 \qquad dv = \sec^2 2x dx \quad \to v = \frac{1}{2}\tan 2x$

$$\int 4x\sec^2 2x\, dx = 2x\tan 2x - \int 4\left(\frac{1}{2}\tan 2x\right) dx$$

$$= 2x\tan 2x - 2\frac{1}{2}\ln\left|\sec 2x\right| + C$$

$$\boxed{= 2x\tan 2x - \ln\left|\sec 2x\right| + C}$$

Exercise

Evaluate the integral $\displaystyle\int e^{2x}\cos 3x dx$

Solution

$$\int e^{2x}\cos 3x dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x - \frac{4}{9}\int e^{2x}\cos 3x dx$$

$$\int e^{2x}\cos 3x dx + \frac{4}{9}\int e^{2x}\cos 3x dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x$$

$$\frac{13}{9}\int e^{2x}\cos 3x dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x$$

$$\int e^{2x}\cos 3x dx = \boxed{\frac{e^{2x}}{13}\left(3\sin 3x + 2\cos 3x\right) + C}$$

		$\int \cos 3x\, dx$
+	e^{2x}	$\frac{1}{3}\sin 3x$
-	$2e^{2x}$	$-\frac{1}{9}\cos 3x$
+	$4e^{2x}$	$-\frac{1}{9}\int \cos 3x\, dx$

Exercise

Evaluate the integral $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$

Solution

Let: $u = \sqrt{x} \quad \to \quad du = \frac{1}{2\sqrt{x}}dx \Rightarrow 2du = \frac{1}{\sqrt{x}}dx$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = \int (\cos u)(2du)$$

$$= 2\int \cos u\, du$$

$$= 2\sin u + C$$

$$\boxed{= 2\sin\sqrt{x} + C}$$

Exercise

Evaluate the integral $\int \frac{(\ln x)^3}{x}\,dx$

Solution

$$\int \frac{(\ln x)^3}{x}\,dx = \int (\ln x)^3\, d(\ln x) \qquad\qquad d(\ln x) = \frac{dx}{x}$$

$$= \boxed{\frac{1}{4}(\ln x)^4 + C}$$

Exercise

Evaluate the integral $\int x^5 e^{x^3}\,dx$

Solution

Let: $u = x^3 \qquad dv = x^2 e^{x^3}\,dx = \frac{1}{3}d\left(e^{x^3}\right) \qquad d\left(e^{x^3}\right) = 3x^2 e^{x^3}\,dx$

$du = 3x^2 dx \qquad v = \frac{1}{3}e^{x^3}$

$$\int x^5 e^{x^3}\,dx = x^3 \frac{1}{3}e^{x^3} - \int \frac{1}{3}e^{x^3}\, 3x^2 dx \qquad d\left(e^{x^3}\right) = 3x^2 e^{x^3}\,dx \qquad \int u\,dv = uv - \int v\,du$$

$$= \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}\int d\left(e^{x^3}\right)$$

$$= \boxed{\frac{1}{3}x^3 e^{x^3} - \frac{1}{3}e^{x^3} + C}$$

Exercise

Evaluate the integral $\int x^2 \ln x^3\,dx$

Solution

$$\int x^2 \ln x^3\,dx = \int 3x^2 \ln x\,dx \qquad u = \ln x \quad v = \int 3x^2 dx = x^3 \qquad \int u\,dv = uv - \int v\,du$$

$$du = \frac{1}{x}dx$$

$$= x^3 \ln x - \int x^2 dx$$

$$= \boxed{x^3 \ln x - \frac{1}{3}x^3 + C}$$

Exercise

Evaluate the integral $\int \ln\left(x+x^2\right)dx$

Solution

Let:
$$u = \ln\left(x+x^2\right) \qquad dv = dx$$
$$du = \frac{2x+1}{x+x^2}dx \qquad v = x$$

$$\begin{aligned}
\int \ln\left(x+x^2\right)dx &= x\ln\left(x+x^2\right) - \int x\frac{2x+1}{x+x^2}dx \\
&= x\ln\left(x+x^2\right) - \int \frac{2x+1}{x(1+x)}xdx \\
&= x\ln\left(x+x^2\right) - \int \frac{2x+2-1}{1+x}dx \\
&= x\ln\left(x+x^2\right) - \int \frac{2(x+1)-1}{x+1}dx \\
&= x\ln\left(x+x^2\right) - \int \left(2-\frac{1}{x+1}\right)dx \\
&= x\ln\left(x+x^2\right) - \left(2x-\ln|x+1|\right)+C \\
&= x\ln\left(x+x^2\right) - 2x+\ln|x+1|+C
\end{aligned}$$

Exercise

Evaluate the integral $\int e^{-x}\sin 4x\,dx$

Solution

$$\int e^{-x}\sin 4x\,dx = -\frac{1}{4}e^{-x}\cos 4x - \frac{1}{16}e^{-x}\sin 4x - \frac{1}{16}\int e^{-x}\sin 4x\,dx$$

$$\left(1+\frac{1}{16}\right)\int e^{-x}\sin 4x\,dx = -\frac{1}{4}e^{-x}\cos 4x - \frac{1}{16}e^{-x}\sin 4x$$

$$\frac{17}{16}\int e^{-x}\sin 4x\,dx = -\frac{1}{16}e^{-x}\left(4\cos 4x+\sin 4x\right)$$

$$\int e^{-x}\sin 4x\,dx = -\frac{e^{-x}}{17}\left(4\cos 4x+\sin 4x\right)+C$$

		$\int \sin 4x\,dx$
+	e^{-x}	$-\frac{1}{4}\cos 4x$
-	$-e^{-x}$	$-\frac{1}{16}\sin 4x$
+	e^{-x}	$-\frac{1}{16}\int \sin 4x\,dx$

Exercise

Evaluate the integral $\int e^{-2\theta} \sin 6\theta \, d\theta$

Solution

$$\int e^{-2\theta} \sin 6\theta \, d\theta = -\frac{1}{6} e^{-2\theta} \cos 6\theta - \frac{1}{18} e^{-2\theta} \sin 6\theta - \frac{1}{9} \int e^{-2\theta} \sin 6\theta \, d\theta$$

$$\left(1+\frac{1}{9}\right) \int e^{-2\theta} \sin 6\theta \, d\theta = -\frac{1}{18} e^{-2\theta} \left(3\cos 6\theta + \sin 6\theta\right)$$

$$\frac{10}{9} \int e^{-2\theta} \sin 6\theta \, d\theta = -\frac{1}{18} e^{-2\theta} \left(3\cos 6\theta + \sin 6\theta\right)$$

$$\int e^{-2\theta} \sin 6\theta \, d\theta = -\frac{e^{-2\theta}}{20} \left(3\cos 6\theta + \sin 6\theta\right) + C$$

		$\int \sin 6\theta \, d\theta$
+	$e^{-2\theta}$	$-\frac{1}{6}\cos 6\theta$
−	$-2e^{-2\theta}$	$-\frac{1}{36}\sin 6\theta$
+	$4e^{-2\theta}$	$-\frac{1}{36}\int \sin 6\theta \, d\theta$

Exercise

Evaluate the integral $\int xe^{-4x} dx$

Solution

$$\int xe^{-4x} dx = \left(-\frac{x}{4} - \frac{1}{16}\right) e^{-4x} + C$$

		$\int e^{-4x} \, dx$
+	x	$-\frac{1}{4} e^{-4x}$
−	1	$\frac{1}{16} e^{-4x}$

Exercise

Evaluate the integral $\int x \ln(x+1) dx$

Solution

$$u = \ln(x+1) \Rightarrow du = \frac{1}{x+1} dx$$

$$dv = xdx \Rightarrow v = \frac{1}{2} x^2$$

$$\int x \ln(x+1) dx = \frac{1}{2} x^2 \ln(x+1) - \frac{1}{2} \int \frac{x^2}{x+1} dx$$

$$= \frac{1}{2} x^2 \ln(x+1) - \frac{1}{2} \int \left(x - 1 + \frac{1}{x+1}\right) dx$$

$$= \frac{1}{2} x^2 \ln(x+1) - \frac{1}{2} \left(\frac{1}{2} x^2 - x + \ln(x+1)\right) + C$$

$$= \frac{1}{2} x^2 \ln(x+1) - \frac{1}{4} x^2 + \frac{1}{2} x - \frac{1}{2} \ln(x+1) + C$$

$$= -\frac{1}{4} x^2 + \frac{1}{2} x + \frac{1}{2} \left(x^2 - 1\right) \ln(x+1) + C$$

Exercise

Evaluate the integral $\int \frac{(\ln x)^2}{x}dx$

Solution

$$\int \frac{(\ln x)^2}{x}dx = \int (\ln x)^2 d(\ln x)$$

$$= \frac{1}{3}(\ln x)^3 + C$$

Exercise

Evaluate the integral $\int \frac{xe^{2x}}{(2x+1)^2}dx$

Solution

$$u = xe^{2x} \quad \rightarrow \quad du = (2x+1)e^{2x}dx$$

$$dv = \frac{dx}{(2x+1)^2} = \frac{1}{2}\frac{d(2x+1)}{(2x+1)^2} \quad \rightarrow \quad v = -\frac{1}{2}\frac{1}{2x+1}$$

$$\int \frac{xe^{2x}}{(2x+1)^2}dx = -\frac{xe^{2x}}{4x+2} + \frac{1}{2}\int e^{2x}dx$$

$$= -\frac{x}{4x+2}e^{2x} + \frac{1}{4}e^{2x} + C$$

Exercise

Evaluate the integral $\int \frac{5x}{e^{2x}}dx$

Solution

		$\int e^{-2x}\,dx$
+	$5x$	$-\frac{1}{2}e^{-2x}$
−	5	$\frac{1}{4}e^{-2x}$

$$\int \frac{5x}{e^{2x}}dx = \int 5xe^{-2x}dx$$

$$= \left(-\frac{5}{2}x - \frac{5}{4}\right)e^{-2x} + C$$

Exercise

Evaluate the integral $\int \frac{e^{1/x}}{x^2}dx$

Solution

$$\int \frac{e^{1/x}}{x^2}dx = -\int e^{1/x}d\left(\frac{1}{x}\right)$$

$$= -e^{1/x} + C$$

Exercise

Evaluate the integral $\int x^5 \ln 3x\, dx$

Solution

$u = \ln 3x \quad \to \quad du = \frac{1}{x}dx$

$dv = x^5\, dx \quad \to \quad v = \frac{1}{6}x^6$

$$\int x^5 \ln 3x\, dx = \frac{1}{6}x^6 \ln 3x - \frac{1}{6}\int x^5 dx$$

$$= \frac{1}{6}x^6 \ln 3x - \frac{1}{36}x^6 + C$$

Exercise

Evaluate the integral $\int x\sqrt{x-5}\, dx$

Solution

Let $u = \sqrt{x-5} \quad \to \quad u^2 = x - 5 \;\Rightarrow\; x = u^2 + 5$

$2udu = dx$

$$\int x\sqrt{x-5}\, dx = \int \left(u^2+5\right)u\left(2udu\right)$$

$$= \int \left(2u^4 + 10u^2\right)du$$

$$= \frac{2}{5}u^5 + \frac{10}{3}u^3 + C$$

Exercise

Evaluate the integral $\int \frac{x}{\sqrt{6x+1}}dx$

Solution

$u = x \quad \to \quad du = dx$

$dv = (6x+1)^{-1/2}\, dx = \frac{1}{6}(6x+1)^{-1/2}d(6x+1) \quad \to \quad v = \frac{1}{3}(6x+1)^{1/2}$

$$\int \frac{x}{\sqrt{6x+1}}dx = \frac{1}{3}x\sqrt{6x+1} - \frac{1}{3}\int (6x+1)^{1/2}\, dx$$

$$= \frac{1}{3}x\sqrt{6x+1} - \frac{1}{18}\int (6x+1)^{1/2}\, d(6x+1)$$

$$\underline{= \frac{1}{3}x\sqrt{6x+1} - \frac{1}{27}(6x+1)^{3/2} + C}$$

Exercise

Evaluate the integral $\int x\cos x\, dx$

Solution

$$\int x\cos x\, dx \underline{= x\sin x + \cos x + C}$$

		$\int \cos x$
+	x	$\sin x$
−	1	$-\cos x$

Exercise

Evaluate the integral $\int x\csc x\cot x\, dx$

Solution

$u = x \quad \rightarrow \quad du = dx$

$dv = \csc x\cot x\, dx \quad \rightarrow \quad v = -\csc x$

$$\int x\csc x\cos x\, dx = -x\csc x + \int \csc dx$$

$$\underline{= -x\csc x - \ln\left|\csc x + \cot x\right| + C}$$

Exercise

Evaluate the integral $\int x^3\sin x\, dx$

Solution

$$\int x^3\sin x\, dx \underline{= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C}$$

		$\int \sin x$
+	x^3	$-\cos x$
−	$3x^2$	$-\sin x$
+	$6x$	$\cos x$
−	6	$\sin x$

Exercise

Evaluate the integral $\int x^2\cos x\, dx$

Solution

$$\int x^2\cos x\, dx \underline{= x^2\sin x + 2x\cos x - 2\sin x + C}$$

		$\int \cos x$
+	x^2	$\sin x$
−	$2x$	$-\cos x$
+	2	$-\sin x$

Exercise

Evaluate the integral $\int e^{-3x}\sin 5x\,dx$

Solution

$$\int e^{-3x}\sin 5x\,dx = -\frac{1}{5}e^{-3x}\cos 5x - \frac{3}{25}e^{-3x}\sin 5x - \frac{9}{25}\int e^{-3x}\sin 5x\,dx$$

$$\left(1+\frac{9}{25}\right)\int e^{-3x}\sin 5x\,dx = -\frac{1}{25}(5\cos 5x + 3\sin 5x)e^{-3x}$$

$$\frac{34}{25}\int e^{-3x}\sin 5x\,dx = -\frac{1}{25}(5\cos 5x + 3\sin 5x)e^{-3x}$$

$$\int e^{-3x}\sin 5x\,dx = -\frac{1}{34}(5\cos 5x + 3\sin 5x)e^{-3x} + C$$

		$\int \sin 5x$
+	e^{-3x}	$-\frac{1}{5}\cos 5x$
−	$-3e^{-3x}$	$-\frac{1}{25}\sin 5x$
+	$9e^{-3x}$	$-\int \frac{1}{25}\sin 5x$

Exercise

Evaluate the integral $\int e^{-3x}\sin 4x\,dx$

Solution

$$\int e^{-3x}\sin 4x\,dx = -\frac{1}{4}e^{-3x}\cos 4x - \frac{3}{16}e^{-3x}\sin 4x - \frac{9}{16}\int e^{-3x}\sin 4x\,dx$$

$$\left(1+\frac{9}{16}\right)\int e^{-3x}\sin 4x\,dx = -\frac{1}{16}(4\cos 4x + 3\sin 4x)e^{-3x}$$

$$\frac{25}{16}\int e^{-3x}\sin 4x\,dx = -\frac{1}{16}(4\cos 4x + 3\sin 4x)e^{-3x}$$

$$\int e^{-3x}\sin 4x\,dx = -\frac{1}{25}(4\cos 4x + 3\sin 4x)e^{-3x} + C$$

		$\int \sin 4x$
+	e^{-3x}	$-\frac{1}{4}\cos 4x$
−	$-3e^{-3x}$	$-\frac{1}{16}\sin 4x$
+	$9e^{-3x}$	$-\frac{1}{16}\int \sin 4x$

Exercise

Evaluate the integral $\int e^{4x}\cos 2x\,dx$

Solution

$$\int e^{4x}\cos 2x\,dx = \frac{1}{2}e^{4x}\sin 2x + e^{4x}\cos 2x - 4\int e^{4x}\cos 2x\,dx$$

$$5\int e^{4x}\cos 2x\,dx = \frac{1}{2}(\sin 2x + 2\cos 2x)e^{4x}$$

$$\int e^{4x}\cos 2x\,dx = \frac{1}{10}(\sin 2x + 2\cos 2x)e^{4x} + C$$

		$\int \cos 2x$
+	e^{4x}	$\frac{1}{2}\sin 2x$
−	$4e^{4x}$	$-\frac{1}{4}\cos 2x$
+	$16e^{4x}$	$-\frac{1}{4}\int \cos 2x$

Exercise

Evaluate the integral $\displaystyle\int e^{3x}\cos 3x\,dx$

Solution

		$\int \cos 3x$
$+$	e^{3x}	$\frac{1}{3}\sin 3x$
$-$	$3e^{3x}$	$-\frac{1}{9}\cos 3x$
$+$	$9e^{3x}$	$-\frac{1}{9}\int \cos 3x$

$$\int e^{3x}\cos 3x\,dx = \frac{1}{3}e^{3x}\sin 3x + \frac{1}{3}e^{3x}\cos 3x - \int e^{3x}\cos 3x\,dx$$

$$2\int e^{3x}\cos 3x\,dx = \frac{1}{3}(\sin 3x + \cos 3x)e^{3x}$$

$$\int e^{3x}\cos 3x\,dx = \frac{1}{6}(\sin 3x + \cos 3x)e^{3x} + C$$

Exercise

Evaluate the integral $\displaystyle\int x^2 e^{4x}\,dx$

Solution

$$\int x^2 e^{4x}\,dx = \left(\frac{1}{4}x^2 - \frac{1}{8}x + \frac{1}{32}\right)e^{4x} + C$$

$$\int x^n e^{ax}\,dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!a^{k+1}}x^{n-k}$$

Exercise

Evaluate the integral $\displaystyle\int x^3 e^{-3x}\,dx$

Solution

$$\int x^3 e^{-3x}\,dx = \left(-\frac{1}{3}x^3 + \frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27}\right)e^{-3x} + C$$

$$\int x^n e^{ax}\,dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!a^{k+1}}x^{n-k}$$

Exercise

Evaluate the integral $\displaystyle\int x^3\cos 2x\,dx$

Solution

		$\int \cos 2x$
$+$	x^3	$\frac{1}{2}\sin 2x$
$-$	$3x^2$	$-\frac{1}{4}\cos 2x$
$+$	$6x$	$-\frac{1}{8}\sin 2x$
$-$	6	$\frac{1}{16}\cos 2x$

$$\int x^3\cos 2x\,dx = \frac{1}{2}x^3\sin 2x + \frac{3}{4}x^2\cos 2x - \frac{3}{4}x\sin 2x - \frac{3}{8}\cos 2x + C$$

Exercise

Evaluate the integral $\displaystyle\int x^3 \sin x\, dx$

Solution

		$\int \sin x$
+	x^3	$-\cos x$
–	$3x^2$	$-\sin x$
+	$6x$	$\cos x$
–	6	$\sin x$

$$\int x^3 \sin x\, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6\sin x + C$$

Exercise

Evaluate the integral $\displaystyle\int_0^{1/\sqrt{2}} 2x \sin^{-1}\left(x^2\right) dx$

Solution

$$u = \sin^{-1}\left(x^2\right) \qquad dv = 2x\,dx$$

$$du = \frac{2x}{\sqrt{1-x^4}}dx \qquad v = x^2$$

$$\int u\,dv = uv - \int v\,du$$

$$\int_0^{1/\sqrt{2}} 2x \sin^{-1}\left(x^2\right) dx = \left[x^2 \sin^{-1}\left(x^2\right)\right]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} x^2 \frac{2x}{\sqrt{1-x^4}} dx \qquad d\left(1-x^4\right) = -4x^3 dx$$

$$= \left(\left(\frac{1}{\sqrt{2}}\right)^2 \sin^{-1}\left(\left(\frac{1}{\sqrt{2}}\right)^2\right) - 0\right) + \int_0^{1/\sqrt{2}} \frac{d\left(1-x^4\right)}{2\sqrt{1-x^4}}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) + \left[\sqrt{1-x^4}\right]_0^{1/\sqrt{2}}$$

$$= \frac{1}{2}\frac{\pi}{6} + \left(\sqrt{1-\frac{1}{4}} - 1\right)$$

$$= \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

$$= \frac{\pi + 6\sqrt{3} - 12}{12}$$

Exercise

Evaluate the integral $\displaystyle\int_1^e x^3 \ln x\,dx$

Solution

$$u = \ln x \qquad v = \int x^3 dx = \frac{1}{4}x^4$$

$$du = \frac{1}{x}dx$$

$$\int_1^e x^3 \ln x dx = \left[\frac{1}{4}x^4 \ln x\right]_1^e - \frac{1}{4}\int_1^e x^4 \frac{dx}{x}$$

$$= \frac{1}{4}\left(e^4 \ln e - 1^4 \ln 1\right) - \frac{1}{4}\int_1^e x^3 dx$$

$$= \frac{e^4}{4} - \frac{1}{16}\left[x^4\right]_1^e$$

$$= \frac{e^4}{4} - \frac{1}{16}\left(e^4 - 1\right)$$

$$= \frac{4}{4}\frac{e^4}{4} - \frac{1}{16}e^4 + \frac{1}{16}$$

$$\boxed{= \frac{3e^4 + 1}{16}}$$

Exercise

Evaluate the integral $\int_0^1 x\sqrt{1-x}dx$

Solution

Let:

$u = x \qquad dv = \sqrt{1-x}dx = (1-x)^{1/2}\, dx \qquad d(1-x) = -dx$

$du = dx \qquad v = -\int (1-x)^{1/2}\, d(1-x) = -\frac{2}{3}(1-x)^{2/3}$

$$\int_0^1 x\sqrt{1-x}dx = \left[x\left(-\frac{2}{3}(1-x)^{2/3}\right)\right]_0^1 - \int_0^1 -\frac{2}{3}(1-x)^{2/3}\, dx \qquad \int udv = uv - \int vdu$$

$$= \left[-\frac{2}{3}x(1-x)^{2/3}\right]_0^1 + \frac{2}{3}\int_0^1 (1-x)^{2/3}\left(-d(1-x)\right)$$

$$= -\frac{2}{3}\left[(1)(0)^{2/3} - 0\right] - \left[\frac{2}{3}\left(\frac{2}{5}\right)(1-x)^{5/3}\right]_0^1$$

$$= -\frac{4}{15}\left[0 - (1)^{5/3}\right]$$

$$\boxed{= \frac{4}{15}}$$

Exercise

Evaluate the integral $\int_0^{\pi/3} x \tan^2 x dx$

Solution

$$u = x \quad \to \quad dv = \tan^2 x dx = \frac{\sin^2 x}{\cos^2 x} dx = \frac{1-\cos^2 x}{\cos^2 x} dx$$

$$du = dx \quad \to \quad v = \int \left(\frac{1}{\cos^2 x} - 1 \right) dx = \tan x - x$$

$$\int_0^{\pi/3} x \tan^2 x dx = \left[x(\tan x - x) \right]_0^{\pi/3} - \int_0^{\pi/3} (\tan x - x)\, dx \qquad \int u dv = uv - \int v du$$

$$= \left[\frac{\pi}{3}\left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) - 0 \right] - \left[-\ln|\cos x| - \frac{x^2}{2} \right]_0^{\pi/3}$$

$$= \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \left[\ln\left|\cos\frac{\pi}{3}\right| + \frac{1}{2}\left(\frac{\pi}{3}\right)^2 - \ln|1| - 0 \right]$$

$$= \frac{\pi}{3}\sqrt{3} - \frac{\pi^2}{9} + \ln\left|\frac{1}{2}\right| + \frac{\pi^2}{18}$$

$$= \frac{\pi}{3}\sqrt{3} - \ln 2 - \frac{\pi^2}{18}$$

Exercise

Evaluate the integral $\int_0^{\pi} x \sin x\, dx$

Solution

$$\int_0^{\pi} x \sin x\, dx = -x\cos x + \sin x \Big|_0^{\pi}$$

$$= \pi$$

		$\int \sin x\, dx$
+	x	$-\cos x$
–	1	$-\sin x$

Exercise

Evaluate the integral $\int_1^{e} \ln 2x\, dx$

Solution

$$\int_1^e \ln 2x\, dx = \frac{1}{2}\int_1^e \ln 2x\, d(2x) \qquad \int \ln x\, dx = x\ln x - x$$

$$= x\ln 2x - x \Big|_1^e$$

$$= e\ln 2e - e - \ln 2 + 1$$

$$= e(\ln 2 + \ln e) - e - \ln 2 + 1$$

$$= e\ln 2 - \ln 2 + 1$$

$$= (e-1)\ln 2 + 1$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/2} x\cos 2x\,dx$

Solution

		$\int \cos 2x\,dx$
+	x	$\frac{1}{2}\sin 2x$
−	1	$-\frac{1}{4}\cos 2x$

$$\int_0^{\pi/2} x\cos 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\Big|_0^{\pi/2}$$

$$= -\frac{1}{4} - \frac{1}{4}$$

$$\underline{= -\frac{1}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\ln 2} xe^x\,dx$

Solution

		$\int e^x\,dx$
+	x	e^x
−	1	e^x

$$\int_0^{\ln 2} xe^x\,dx = e^x(x-1)\Big|_0^{\ln 2}$$

$$= 2(\ln 2 - 1) + 1$$

$$\underline{= 2\ln 2 - 1}$$

Exercise

Evaluate the integral $\displaystyle\int_1^{e^2} x^2\ln x\,dx$

Solution

$u = \ln x$

$du = \frac{1}{x}dx \qquad v = \int x^2 dx = \frac{1}{3}x^3$

$$\int x^2\ln x\,dx = \frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2 dx$$

$$\int_1^{e^2} x^2\ln x\,dx = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3\Big|_1^{e^2}$$

$$= \frac{2}{3}e^6 - \frac{1}{9}e^6 + \frac{1}{9}$$

$$\underline{= \frac{5}{9}e^6 + \frac{1}{9}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^3 xe^{x/2}dx$

Solution

$$\int_0^3 xe^{x/2}dx = (2x-4)e^{x/2}\Big|_0^3$$

$$\boxed{= 2e^{3/2} + 4}$$

$$\int x^n e^{ax}\,dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k\,n!}{(n-k)!\,a^{k+1}}x^{n-k}$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 x^2e^{-2x}dx$

Solution

$$\int_0^2 x^2e^{-2x}dx = \left(-\frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{4}\right)e^{-2x}\Big|_0^2$$

$$= \left(-2+1-\frac{1}{4}\right)e^{-4} + \frac{1}{4}$$

$$\boxed{= \frac{1}{4} - \frac{5}{4}e^{-4}}$$

$$\int x^n e^{ax}\,dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k\,n!}{(n-k)!\,a^{k+1}}x^{n-k}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/4} x\cos 2x\,dx$

Solution

$$\int_0^{\pi/4} x\cos 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\Big|_0^{\pi/4}$$

$$\boxed{= \frac{\pi}{8} - \frac{1}{4}}$$

		$\int \cos 2x\,dx$
+	x	$\frac{1}{2}\sin 2x$
−	1	$-\frac{1}{4}\cos 2x$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi} x\sin 2x\,dx$

Solution

$$\int_0^{\pi} x\sin 2x\,dx = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\Big|_0^{\pi}$$

$$\boxed{= -\frac{\pi}{2}}$$

		$\int \sin 2x\,dx$
+	x	$-\frac{1}{2}\cos 2x$
−	1	$-\frac{1}{4}\sin 2x$

Exercise

Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the cure $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$

Solution

$$V = 2\pi \int_0^{\ln 2} (\ln 2 - x) e^x dx$$

$$= 2\pi \int_0^{\ln 2} \left(\ln 2 e^x - xe^x\right) dx$$

$$= 2\pi \ln 2 \left[e^x \right]_0^{\ln 2} - 2\pi \int_0^{\ln 2} xe^x dx$$

$$= 2\pi \ln 2 \left(e^{\ln 2} - e^0\right) - 2\pi \left[xe^x - e^x \right]_0^{\ln 2}$$

$$= 2\pi \ln 2 (2-1) - 2\pi \left[\ln 2 e^{\ln 2} - e^{\ln 2} - (0-1) \right]$$

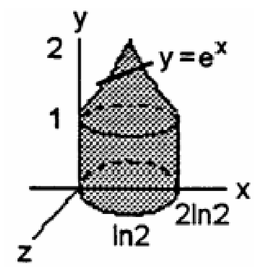
$$= 2\pi \ln 2 - 2\pi [2\ln 2 - 2 + 1]$$

$$= 2\pi \ln 2 - 4\pi \ln 2 + 2\pi$$

$$= -2\pi \ln 2 + 2\pi$$

$$= 2\pi (1 - \ln 2) \quad unit^3$$

		e^x
+	x	e^x
−	1	e^x



Exercise

Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate aces, the cure $y = e^{-x}$, and the line $x = 1$, about

a) the line $y - axis$

b) the line $x = 1$

Solution

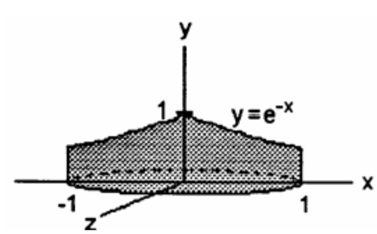
a) $V = 2\pi \int_0^1 xe^{-x} dx$

$$= 2\pi \left(\left[-xe^{-x} - e^{-x} \right]_0^1 \right)$$

$$= 2\pi \left(-e^{-1} - e^{-1} + 0 + 1 \right)$$

$$= 2\pi \left(-\frac{1}{e} - \frac{1}{e} + 1 \right)$$

		e^{-x}
$(+)$	x	$-e^{-x}$
$(-)$	1	e^{-x}



$$= 2\pi\left(-\frac{2}{e}+1\right)$$

$$\boxed{= 2\pi - \frac{4\pi}{e}} \quad unit^3$$

b) $$V = 2\pi\int_0^1 (1-x)e^{-x}dx$$

$$= 2\pi\left(\int_0^1 e^{-x}dx - \int_0^1 xe^{-x}dx\right)$$

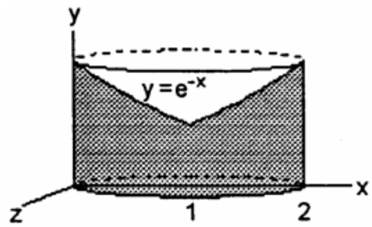
$$= 2\pi\left(\left[-e^{-x} - \left(-xe^{-x} - e^{-x}\right)\right]_0^1\right)$$

$$= 2\pi\left[e^{-x} + xe^{-x} - e^{-x}\right]_0^1$$

$$= 2\pi\left[xe^{-x}\right]_0^1$$

$$= 2\pi\left(e^{-1}\right)$$

$$\boxed{= \frac{2\pi}{e}} \quad unit^3$$



Exercise

Find the volume of the solid that is generated by the region bounded by $f(x) = e^{-x}$, $x = \ln 2$, and the coordinate axes is revolved about the y-$axis$.

Solution

$$V = 2\pi\int_0^{\ln 2} xe^{-x}\, dx$$

$$V = \int_a^b 2\pi(radius)(height)\,dx \quad \text{Shells Method}$$

		$\int e^{-x}\, dx$
+	x	$-e^{-x}$
−	1	e^{-x}

$$= 2\pi\left[e^{-x}(-x-1)\right]\Big|_0^{\ln 2}$$

$$= 2\pi\left(e^{-\ln 2}(-\ln 2 - 1) + 1\right)$$

$$= 2\pi\left(\frac{1}{2}(-\ln 2 - 1) + 1\right)$$

$$= 2\pi\left(-\frac{1}{2}\ln 2 + \frac{1}{2}\right)$$

$$\boxed{= \pi(1 - \ln 2)} \quad unit^3$$

Exercise

Find the volume of the solid that is generated by the region bounded by $f(x) = \sin x$, and the *x-axis* on $[0, \pi]$ is revolved about the *y-axis*.

Solution

$$V = 2\pi\int_0^{\pi} x\sin x\, dx \qquad V = \int_a^b 2\pi(radius)(height)\,dx \quad \textit{Shells Method}$$

$$= 2\pi\left[-x\cos x + \sin x\right]\Big|_0^{\pi}$$

$$\boxed{= 2\pi^2} \quad unit^3$$

		$\int \sin x$
+	x	$-\cos x$
–	1	$-\sin x$

Exercise

Find the area of the region generated when the region bounded by $y = \sin x$ and $y = \sin^{-1} x$ on the interval $\left[0, \ \frac{1}{2}\right]$.

Solution

$$A = \int_0^{1/2}\left(\sin^{-1}x - \sin x\right)dx$$

$$u = \sin^{-1}x \qquad du = \frac{dx}{\sqrt{1-x^2}} \qquad v = \int dx = x$$

$$= x\sin^{-1}x\Big|_0^{1/2} - \int_0^{1/2}\frac{x\,dx}{\sqrt{1-x^2}} + \cos x\Big|_0^{1/2}$$

$$= x\sin^{-1}x + \cos x\Big|_0^{1/2} + \frac{1}{2}\int_0^{1/2}\left(1-x^2\right)^{-1/2} d\left(1-x^2\right)$$

$$= x\sin^{-1}x + \cos x + \left(1-x^2\right)^{1/2}\Big|_0^{1/2}$$

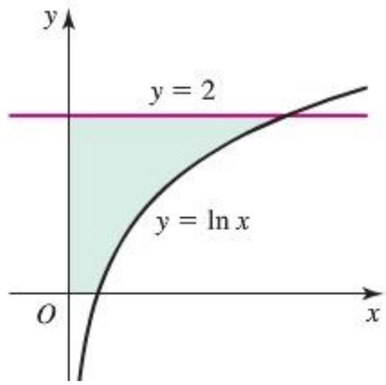
$$= \frac{1}{2}\sin^{-1}\frac{1}{2} + \cos\frac{1}{2} + \left(1-\frac{1}{4}\right)^{1/2} - 1 - 1$$

$$\boxed{= \frac{\pi}{12} + \cos\frac{1}{2} + \frac{\sqrt{3}}{2} - 2} \quad unit^2$$

Exercise

Determine the area of the shaded region bounded by $y = \ln x, \quad y = 2, \quad y = 0, \quad and \ x = 0$

Solution



$$y = \ln x = 0 \quad \rightarrow \quad x = 1$$

$$y = \ln x = 2 \quad \rightarrow \quad x = e^2$$

$$A = 1 \times 2 + \int_1^2 (2 - \ln x)\, dx$$

$$= 2 + (2x - x\ln x + x)\Big|_1^2$$

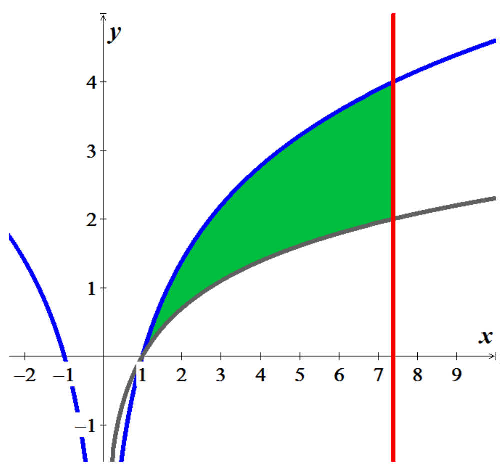
$$= 2 + 4 - 2\ln 2 + 2 - 2 - 1$$

$$\underline{= 5 - 2\ln 2} \quad unit^2$$

Exercise

Find the area between the curves $y = \ln x^2, \quad y = \ln x, \quad and \quad x = e^2$

Solution



$$y = \ln x^2 = \ln x \quad with \quad x > 0$$

$$x^2 = x \Rightarrow \underline{x = 1}$$

$$A = \int_1^{e^2} \left(\ln x^2 - \ln x\right) dx$$

$$= \int_1^{e^2} (2\ln x - \ln x)\, dx$$

$$= \int_1^{e^2} \ln x \ dx \qquad \int \ln x \ dx = x\ln x - x$$

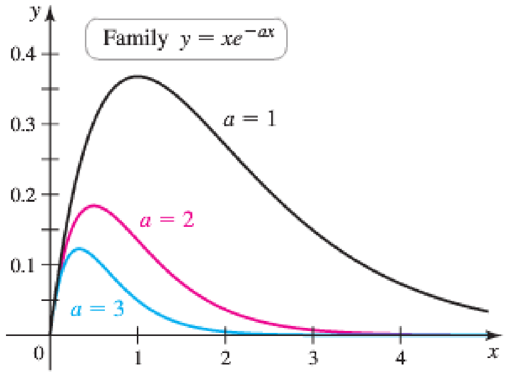
$$= (x\ln x - x)\Big|_1^{e^2}$$

$$= e^2 \ln e^2 - e^2 + 1$$

$$\underline{= e^2 + 1} \quad unit^2$$

Exercise

The curves $y = xe^{-ax}$ are shown in the figure for $a = 1,\ 2,\ and\ 3$.



a) Find the area of the region bounded by $y = xe^{-x}$ and the *x*-axis on the interval [0, 4].

b) Find the area of the region bounded by $y = xe^{-ax}$ and the *x*-axis on the interval [0, 4] where $a > 0$

c) Find the area of the region bounded by $y = xe^{-ax}$ and the *x*-axis on the interval [0, *b*]. Because this area depends on *a* and *b*, we call it $A(a,\ b)$ where $a > 0$ and $b > 0$.

d) Use part (*c*) to show that $A(1,\ \ln b) = 4A\left(2,\ \frac{1}{2}\ln b\right)$

e) Does this pattern continue? Is it true that $A(1,\ \ln b) = a^2 A\left(a,\ \frac{1}{a}\ln b\right)$

Solution

a) $\displaystyle\int_0^4 xe^{-x}dx = e^{-x}(-x-1)\Big|_0^4$

$= e^{-4}(-5) - (-1)$

$= 1 - \dfrac{5}{e^4}$ $\quad$ $unit^2$

		$\int e^{-x}\,dx$
+	x	$-e^{-x}$
−	1	e^{-x}

b) $\displaystyle\int_0^4 xe^{-ax}dx = e^{-ax}\left(-\frac{1}{a}x - \frac{1}{a^2}\right)\Big|_0^4$

$= e^{-4a}\left(-\dfrac{4}{a} - \dfrac{1}{a^2}\right) - \left(-\dfrac{1}{a^2}\right)$

$= \dfrac{1}{a^2} - e^{-4a}\left(\dfrac{4a+1}{a^2}\right)$

$= \dfrac{1}{a^2}\left(1 - \dfrac{4a+1}{e^{-4a}}\right)$ $\quad$ $unit^2$

		$\int e^{-ax}\,dx$
+	x	$-\frac{1}{a}e^{-ax}$
−	1	$\frac{1}{a^2}e^{-ax}$

c) $\displaystyle\int_0^b xe^{-ax}dx = e^{-ax}\left(-\frac{1}{a}x - \frac{1}{a^2}\right)\Big|_0^b$

$$= e^{-ab}\left(-\frac{b}{a}-\frac{1}{a^2}\right)-\left(-\frac{1}{a^2}\right)$$

$$= \frac{1}{a^2} - e^{-ab}\left(\frac{ab+1}{a^2}\right)$$

$$= \frac{1}{a^2}\left(1-\frac{ab+1}{e^{ab}}\right) \quad unit^2$$

d) $A(a,b) = \frac{1}{a^2}\left(1-\frac{ab+1}{e^{ab}}\right)$

$$A(1,\ \ln b) = 1 - \frac{\ln b+1}{e^{\ln b}}$$

$$= 1 - \frac{\ln b+1}{b}$$

$$A\left(2,\ \frac{1}{2}\ln b\right) = \frac{1}{4}\left(1-\frac{\ln b+1}{e^{\ln b}}\right)$$

$$= \frac{1}{4}\left(1-\frac{\ln b+1}{b}\right)$$

$$= \frac{1}{4}A(1,\ \ln b)$$

$$\therefore\quad A(1,\ \ln b) = 4A\left(2,\ \frac{1}{2}\ln b\right)$$

e) $A\left(a,\ \frac{1}{a}\ln b\right) = \frac{1}{a^2}\left(1-\frac{\ln b+1}{e^{\ln b}}\right)$

$$= \frac{1}{a^2}\left(1-\frac{\ln b+1}{b}\right)$$

$$= \frac{1}{a^2}A(1,\ \ln b)$$

Yes, there is a pattern: $A(1,\ \ln b) = a^2 A\left(a,\ \frac{1}{a}\ln b\right)$

Exercise

Suppose a mass on a spring that is slowed by friction has the position function $s(t) = e^{-t}\sin t$

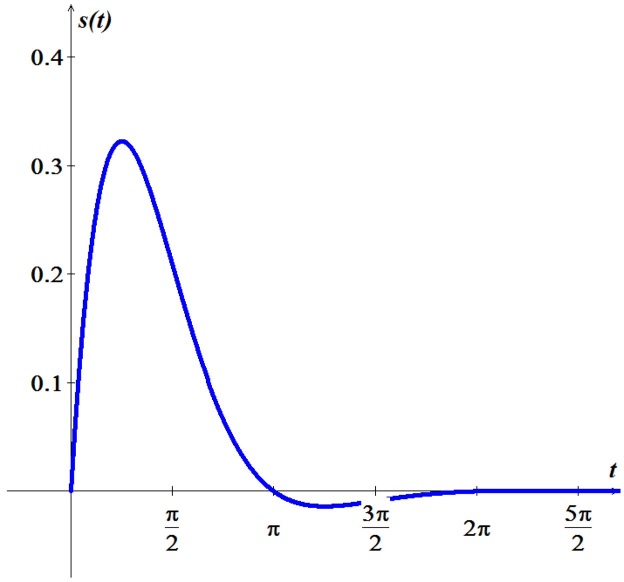
a) Graph the position function. At what times does the oscillator pass through the position $s = 0$?

b) Find the average value of the position on the interval $[0,\ \pi]$.

c) Generalize part (*b*) and find the average value of the position on the interval $[n\pi,\ (n+1)\pi]$, for $n = 0,\ 1,\ 2,\ \dots$

Solution

a) $s(t) = e^{-t}\sin t = 0 \qquad \sin t = 0 \ \rightarrow\ t = n\pi$



b) $\int e^{-t}\sin t\,dt = -e^{-t}(\cos t + \sin t) - \int e^{-t}\sin t\,dt$

$$2\int e^{-t}\sin t\,dt = -e^{-t}(\cos t + \sin t)$$

		$\int \sin t$
+	e^{-t}	$-\cos t$
−	$-e^{-t}$	$-\sin t$
+	e^{-t}	$-\int \sin t\,dt$

$$\begin{aligned} Average &= \frac{1}{\pi}\int_0^{\pi} e^{-t}\sin t\,dt \\ &= -\frac{1}{2\pi}e^{-t}(\cos t - \sin t)\Big|_0^{\pi} \\ &= -\frac{1}{2\pi}\left(-e^{-\pi}-1\right) \\ &= \boxed{\frac{1}{2\pi}\left(e^{-\pi}+1\right)} \end{aligned}$$

c)
$$\begin{aligned} Average &= \frac{1}{\pi}\int_{n\pi}^{(n+1)\pi} e^{-t}\sin t\,dt \\ &= -\frac{1}{2\pi}e^{-t}(\cos t - \sin t)\Big|_{n\pi}^{(n+1)\pi} \\ &= -\frac{1}{2\pi}\left(e^{-(n+1)\pi}\left(\cos((n+1)\pi) - \sin((n+1)\pi)\right) - e^{-n\pi}(\cos n\pi - \sin n\pi)\right) \\ &= -\frac{1}{2\pi}\left(e^{-(n+1)\pi}\cos((n+1)\pi) - e^{-n\pi}\cos n\pi\right) \\ &= \frac{e^{-n\pi}}{2\pi}\left(\cos n\pi - e^{-\pi}\cos(n+1)\pi\right) \\ &= \frac{e^{-n\pi}}{2\pi}\left((-1)^n - e^{-\pi}(-1)^{n+1}\right) \\ &= \boxed{(-1)^n\frac{e^{-n\pi}}{2\pi}\left(1+e^{-\pi}\right)} \end{aligned}$$

Exercise

Given the region bounded by the graphs of $y = x\sin x,\quad y=0,\quad x=0,\quad x=\pi$, find

a) The area of the region.

b) The volume of the solid generated by revolving the region about the *x-axis*

c) The volume of the solid generated by revolving the region about the *y-axis*

d) The centroid of the region

Solution

a) $A = \int_0^\pi x\sin x\, dx$

$= -x\cos x + \sin x \Big|_0^\pi$

$\boxed{= \pi \ \ unit^2}$

		$\int \sin x$
+	x	$-\cos x$
–	1	$-\sin x$

b) $V = \pi \int_0^\pi (x\sin x)^2\ dx$

$= \pi \int_0^\pi x^2 \sin^2 x\, dx$

$= \frac{\pi}{2}\int_0^\pi x^2(1-\cos 2x)\ dx$

$= \frac{\pi}{2}\int_0^\pi \left(x^2 - x^2\cos 2x\right)\ dx$

$= \frac{\pi}{2}\left(\frac{1}{3}x^3 - \frac{1}{2}x^2\sin 2x - \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right)_0^\pi$

$= \frac{\pi}{2}\left(\frac{1}{3}\pi^3 - \frac{\pi}{2}\right)$

$\boxed{= \frac{\pi^4}{6} - \frac{\pi^2}{4} \quad unit^3}$

		$\int \cos 2x$
+	x^2	$\frac{1}{2}\sin 2x$
–	$2x$	$-\frac{1}{4}\cos 2x$
+	2	$-\frac{1}{8}\sin 2x$

c) $V = 2\pi \int_0^\pi x(x\sin x)\ dx$

$= 2\pi \int_0^\pi \left(x^2\sin x\right)\ dx$

$= 2\pi\left(-x^2\cos x + 2x\sin x + 2\cos x\right)_0^\pi$

$= 2\pi\left(\pi^2 - 2 - 2\right)$

$\boxed{= 2\pi^3 - 8\pi \quad unit^3}$

		$\int \sin x$
+	x^2	$-\cos x$
–	$2x$	$-\sin x$
+	2	$\cos x$

d) $m = \int_0^\pi x\sin x\, dx = -x\cos x + \sin x\Big|_0^\pi \underline{= \pi}$ From (***a***)

$M_x = \frac{1}{2}\int_0^\pi (x\sin x)^2\, dx \underline{= \frac{1}{2}\left(\frac{\pi^3}{6} - \frac{\pi}{4}\right)}$ From (***b***)

$M_y = \int_0^\pi x(x\sin x)\, dx = \frac{2\pi^3 - 8\pi}{2\pi} \underline{= \pi^2 - 4}$ From (***c***)

$$\bar{x} = \frac{M_y}{m} = \underline{\frac{\pi^2 - 4}{\pi}} \quad \underline{\approx 1.8684}$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{\pi}\left(\frac{\pi^3}{12} - \frac{\pi}{8}\right) = \underline{\frac{\pi^2}{12} - \frac{1}{8}} \quad \underline{\approx 0.6975}$$

Solution Section 2.2 – Trigonometric Integrals

Exercise

Evaluate the integrals $\int \sin^4 2x \cos 2x dx$

Solution

$$d(\sin 2x) = 2\cos 2x dx \quad \Rightarrow \quad \frac{1}{2} d(\sin 2x) = \cos 2x dx$$

$$\int \sin^4 2x \cos 2x dx = \frac{1}{2}\int \sin^4 2x \, d(\sin 2x)$$

$$= \frac{1}{10}\sin^5 2x + C$$

Exercise

Evaluate the integrals $\int \sin^5 \frac{x}{2} dx$

Solution

$$\sin^5 \frac{x}{2} = \left(\sin^2 \frac{x}{2}\right)^2 \sin \frac{x}{2}$$

$$= \left(1 - \cos^2 \frac{x}{2}\right)^2 \sin \frac{x}{2}$$

$$= \left(1 - 2\cos^2 \frac{x}{2} + \cos^4 \frac{x}{2}\right) \sin \frac{x}{2}$$

$$d\left(\cos \frac{x}{2}\right) = -\frac{1}{2}\sin \frac{x}{2} dx \quad \to \quad -2d\left(\cos \frac{x}{2}\right) = \sin \frac{x}{2} dx$$

$$\int \sin^5 \frac{x}{2} dx = -2\int \left(1 - 2\cos^2 \frac{x}{2} + \cos^4 \frac{x}{2}\right) d\left(\cos \frac{x}{2}\right)$$

$$= -2\left(\cos \frac{x}{2} - \frac{2}{3}\cos^3 \frac{x}{2} + \frac{1}{5}\cos^5 \frac{x}{2}\right) + C$$

$$= -2\cos \frac{x}{2} + \frac{4}{3}\cos^3 \frac{x}{2} - \frac{2}{5}\cos^5 \frac{x}{2} + C$$

Exercise

Evaluate the integrals $\int \cos^3 2x \sin^5 2x \, dx$

Solution

$$\int \cos^3 2x \sin^5 2x \, dx = \int \left(\cos^2 2x\right) \cos 2x \sin^5 2x \, dx$$

$$d(\sin 2x) = 2\cos 2x \, dx$$

$$= \int \left(1-\sin^2 2x\right)\sin^5 2x \, \left(\tfrac{1}{2} d \sin 2x\right)$$

$$= \frac{1}{2}\int \left(\sin^5 2x - \sin^7 2x\right) \, \left(d \sin 2x\right)$$

$$= \frac{1}{2}\left(\frac{1}{6}\sin^6 2x - \frac{1}{8}\sin^8 2x\right) + C$$

$$\boxed{= \frac{1}{12}\sin^6 2x - \frac{1}{16}\sin^8 2x + C}$$

Exercise

Evaluate the integrals $\int 8\cos^4 2\pi x \, dx$

Solution

$$\int 8\cos^4 2\pi x \, dx = 8\int \left(\cos 2\pi x\right)^4 dx \qquad \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$= 8\int \left(\frac{1+\cos 4\pi x}{2}\right)^2 dx$$

$$= 2\int \left(1+\cos 4\pi x\right)^2 dx$$

$$= 2\int \left(1+2\cos 4\pi x+\cos^2 4\pi x\right) dx$$

$$= 2\int dx + 4\int \cos 4\pi x dx + 2\int \cos^2 4\pi x dx$$

$$= 2x + 4\frac{1}{4\pi}\cos 4\pi x + 2\int \frac{1+\cos 8\pi x}{2} dx$$

$$= 2x + \frac{1}{\pi}\cos 4\pi x + \int \left(1+\cos 8\pi x\right) dx$$

$$= 2x + \frac{1}{\pi}\sin 4\pi x + x + \frac{1}{8\pi}\sin 8\pi x + C$$

$$\boxed{= 3x + \frac{1}{\pi}\sin 4\pi x + \frac{1}{8\pi}\sin 8\pi x + C}$$

Exercise

Evaluate the integrals $\int 16\sin^2 x \cos^2 x dx$

Solution

$$\int 16\sin^2 x\cos^2 x dx = 16\int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx \qquad \cos^2\alpha = \frac{1+\cos 2\alpha}{2} \quad \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$= 4\int \left(1-\cos^2 2x\right)dx$$

$$= 4\int \left(1-\frac{1+\cos 4x}{2}\right)dx$$

$$= 4\int \frac{1-\cos 4x}{2}dx$$

$$= 2\left(x-\frac{1}{4}\sin 4x\right)+C$$

$$= 2x-\frac{1}{2}\left(2\sin 2x\cos 2x\right)+C$$

$$= 2x-\left(2\sin x\cos x\right)\left(2\cos^2 x-1\right)+C$$

$$\underline{= 2x-4\sin x\cos^3 x+2\sin x\cos x+C}$$

Exercise

Evaluate the integrals $\int \sec x\tan^2 x\, dx$

Solution

$$\int \sec x\tan^2 x\, dx = \int \sec x\tan x\tan x\, dx \qquad \begin{array}{ll} u = \tan x & dv = \sec x\tan x dx \\ du = \sec^2 x dx & v = \sec x \end{array}$$

$$\int \sec x\tan^2 x\, dx = \tan x\sec x - \int \sec x\sec^2 x\, dx$$

$$= \tan x\sec x - \int \sec x\left(1+\tan^2 x\right)\, dx$$

$$= \tan x\sec x - \left[\int \sec x\, dx + \int \sec x\tan^2 x\, dx\right]$$

$$= \tan x\sec x - \ln\left|\sec x+\tan x\right| - \int \sec x\tan^2 x\, dx$$

$$\int \sec x\tan^2 x\, dx + \int \sec x\tan^2 x\, dx = \tan x\sec x - \ln\left|\sec x+\tan x\right|$$

$$2\int \sec x\tan^2 x\, dx = \tan x\sec x - \ln\left|\sec x+\tan x\right|$$

$$\int \sec x\tan^2 x\, dx = \underline{\frac{1}{2}\tan x\sec x - \frac{1}{2}\ln\left|\sec x+\tan x\right| + C}$$

Exercise

Evaluate the integrals $\int \sec^2 x \tan^2 x\, dx$

Solution

$$\int \sec^2 x \tan^2 x dx = \int \tan^2 x\, d(\tan x) \qquad\qquad d(\tan x) = \sec^2 xdx$$

$$= \boxed{\frac{1}{3}\tan^3 x + C}$$

Exercise

Evaluate the integrals $\int \sec^4 x \tan^2 x\, dx$

Solution

$$\int \sec^4 x \tan^2 x dx = \int \sec^2 x \sec^2 x \tan^2 x\, dx \qquad\qquad d(\tan x) = \sec^2 xdx \quad \sec^2 x = 1 + \tan^2 x$$

$$= \int \left(1 + \tan^2 x\right)\tan^2 x\, d(\tan x)$$

$$= \int \left(\tan^2 x + \tan^4 x\right)\, d(\tan x)$$

$$= \boxed{\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C}$$

Exercise

Evaluate the integrals $\int e^x \sec^3\left(e^x\right)\, dx$

Solution

$$u = \sec\left(e^x\right) \qquad\qquad dv = \sec\left(e^x\right)e^x dx$$

$$du = \sec\left(e^x\right)\tan\left(e^x\right)e^x dx \qquad v = \int \sec\left(e^x\right)d\left(e^x\right) = \tan\left(e^x\right)$$

$$\int e^x \sec^3\left(e^x\right)\, dx = \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec\left(e^x\right)\tan^2\left(e^x\right)e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec\left(e^x\right)\left(\sec^2\left(e^x\right) - 1\right)e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \left(\sec^3\left(e^x\right) - \sec\left(e^x\right)\right)e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec^3\left(e^x\right)e^x dx + \int \sec\left(e^x\right)e^x dx \qquad d\left(e^x\right) = e^x dx$$

$$= \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec^3\left(e^x\right)e^x dx + \int \sec\left(e^x\right)d\left(e^x\right)$$

$$\int \sec^3\left(e^x\right)e^x dx = \sec\left(e^x\right)\tan\left(e^x\right) - \int \sec^3\left(e^x\right)e^x dx + \ln\left|\sec\left(e^x\right) + \tan\left(e^x\right)\right|$$

$$2\int \sec^3\left(e^x\right)e^x dx = \sec\left(e^x\right)\tan\left(e^x\right) + \ln\left|\sec\left(e^x\right) + \tan\left(e^x\right)\right| + C$$

$$\int \sec^3\left(e^x\right)e^x dx = \frac{1}{2}\sec\left(e^x\right)\tan\left(e^x\right) + \frac{1}{2}\ln\left|\sec\left(e^x\right) + \tan\left(e^x\right)\right| + C$$

Exercise

Evaluate $\int \sin^4 x \cos^2 x \, dx$

Solution

$$\int \sin^4 x \cos^2 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{8}\int \left(1 - 2\cos 2x + \cos^2 2x\right)\left(1 + \cos 2x\right) dx$$

$$= \frac{1}{8}\int \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right) dx$$

$$= \frac{1}{8}\int \left(1 - \cos 2x - \frac{1}{2} - \frac{1}{2}\cos 4x\right) dx + \frac{1}{8}\int \cos^2 2x \cos 2x \, dx$$

$$= \frac{1}{8}\int \left(\frac{1}{2} - \cos 2x - \frac{1}{2}\cos 4x\right) dx + \frac{1}{16}\int \left(1 - \sin^2 2x\right) \, d(\sin 2x)$$

$$= \frac{1}{8}\left(\frac{1}{2}x - \frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x\right) + \frac{1}{16}\sin 2x - \frac{1}{48}\sin^3 2x + C$$

$$= \frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C$$

Exercise

Evaluate $\int \tan^3 x \, \sec^4 x \, dx$

Solution

$$\int \tan^3 x \sec^4 x\, dx = \int \tan^3 x\left(1+\tan^2 x\right)\sec^2 x\, dx \qquad \sec^2 x = 1+\tan^2 x$$

$$= \int \left(\tan^3 x + \tan^5 x\right)\, d\left(\tan x\right) \qquad d\left(\tan x\right) = \sec^2 x dx$$

$$\boxed{= \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + C}$$

Exercise

Evaluate $\int \sin 3x \cos 7x\, dx$

Solution

$$\int \sin 3x \cos 7x\, dx = \frac{1}{2}\int \left(\sin\left(-4x\right) + \sin 10x\right) dx \qquad \sin mx \cos nx = \frac{1}{2}\left[\sin\left(m-n\right)x + \sin\left(m+n\right)x\right]$$

$$= \frac{1}{2}\int \left(-\sin 4x + \sin 10x\right) dx$$

$$= \frac{1}{2}\left(\frac{1}{4}\cos 4x - \frac{1}{10}\cos 10x\right) + C$$

$$\boxed{= \frac{1}{8}\cos 4x - \frac{1}{20}\cos 10x + C}$$

Exercise

Evaluate the integrals $\int \sin 2x \cos 3x\, dx$

Solution

$$\int \sin 2x \cos 3x\, dx = \frac{1}{2}\int \left(\sin 5x + \sin\left(-x\right)\right)\, dx \qquad \sin\alpha\cos\beta = \frac{1}{2}\left[\sin\left(\alpha+\beta\right) + \sin\left(\alpha-\beta\right)\right]$$

$$= \frac{1}{2}\int \left(\sin 5x - \sin x\right)\, dx$$

$$= \frac{1}{2}\left(-\frac{1}{5}\cos 5x + \cos x\right) + C$$

$$\boxed{= \frac{1}{2}\cos x - \frac{1}{10}\cos 5x + C}$$

Exercise

Evaluate the integrals $\int \sin^2\theta \cos 3\theta \, d\theta$

Solution

$$\int \sin^2\theta \cos 3\theta \, d\theta = \int \frac{1-\cos 2\theta}{2} \cos 3\theta \, d\theta \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \frac{1}{2}\int \left(\cos 3\theta - \cos 2\theta \cos 3\theta\right) \, d\theta$$

$$= \frac{1}{2}\int \cos 3\theta \, d\theta - \frac{1}{2}\int \cos 2\theta \cos 3\theta \, d\theta$$

$$= \frac{1}{6}\sin 3\theta - \frac{1}{2}\int \frac{1}{2}\left(\cos(5\theta) + \cos(-\theta)\right) \, d\theta \qquad \cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$= \frac{1}{6}\sin 3\theta - \frac{1}{4}\left(\frac{1}{5}\sin 5\theta + \sin\theta\right) + C$$

$$\boxed{= \frac{1}{6}\sin 3\theta - \frac{1}{20}\sin 5\theta - \frac{1}{4}\sin\theta + C}$$

Exercise

Evaluate the integrals $\int \cos^3\theta \sin 2\theta \, d\theta$

Solution

$$\int \cos^3\theta \sin 2\theta \, d\theta = \int \cos^3\theta \left(2\sin\theta\cos\theta\right) \, d\theta \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= -2\int \cos^4\theta \, d(\cos\theta) \qquad d(\cos\theta) = -\sin\theta d\theta$$

$$\boxed{= -\frac{2}{5}\cos^5\theta + C}$$

Exercise

Evaluate the integrals $\int \sin\theta \sin 2\theta \sin 3\theta \, d\theta$

Solution $\qquad \sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

$$\int \sin\theta \sin 2\theta \sin 3\theta \, d\theta = \int \frac{1}{2}\left(\cos(1-2)\theta - \cos(1+2)\theta\right) \sin 3\theta \, d\theta$$

$$= \frac{1}{2}\int \left(\cos(-\theta) - \cos(3\theta)\right) \sin 3\theta \, d\theta$$

$$= \frac{1}{2}\int \cos\theta \;\sin 3\theta \; d\theta - \frac{1}{2}\int \cos 3\theta \;\sin 3\theta \; d\theta$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$= \frac{1}{4}\int (\sin 4\theta + \sin 2\theta)\; d\theta - \frac{1}{4}\int (\sin 6\theta + \sin(0))\; d\theta$$

$$= \frac{1}{4}\left(-\frac{1}{4}\cos 4\theta - \frac{1}{2}\cos 2\theta\right) + \frac{1}{24}\cos 6\theta + C$$

$$= -\frac{1}{16}\cos 4\theta - \frac{1}{8}\cos 2\theta + \frac{1}{24}\cos 6\theta + C$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{\sin^3 x}{\cos^4 x}\,dx$

Solution

$$\int \frac{\sin^3 x}{\cos^4 x}\,dx = \int \frac{\sin^2 x \sin x}{\cos^4 x}\,dx \qquad \cos^2\alpha + \sin^2\alpha = 1$$

$$= \int \frac{\left(1-\cos^2 x\right)\sin x}{\cos^4 x}\,dx$$

$$= -\int \left(\frac{1}{\cos^4 x} - \frac{\cos^2 x}{\cos^4 x}\right) d(\cos x)$$

$$= -\int \left(\cos^{-4} x - \cos^{-2} x\right) d(\cos x)$$

$$= -\left(-\frac{1}{3}\cos^{-3} x + \cos^{-1} x\right) + C$$

$$= \frac{1}{3}\frac{1}{\cos^3 x} - \frac{1}{\cos x} + C$$

$$= \frac{1}{3}\csc^3 x - \csc x + C$$

Exercise

Evaluate the integrals $\displaystyle\int x\cos^3 x\;dx$

Solution

$$\int x\cos^3 x\,dx = \int x\cos^2 x \cos x\;dx$$

$$= \int x\left(1-\sin^2 x\right)\cos x\;dx \qquad \cos^2\alpha + \sin^2\alpha = 1$$

$$= \int x\cos x\,dx - \int x\sin^2 x\cos x\,dx$$

$$u = x \qquad dv = \cos x dx \qquad\qquad u = x \qquad dv = \sin^2 x\cos x dx$$

$$du = dx \qquad v = \sin x \qquad\qquad du = dx \qquad v = \tfrac{1}{3}\sin^3 x$$

$$= x\sin x - \int \sin x\,dx - \left(\tfrac{1}{3}x\sin^3 x - \tfrac{1}{3}\int \sin^3 x\,dx\right)$$

$$= x\sin x + \cos x - \tfrac{1}{3}x\sin^3 x + \tfrac{1}{3}\int \sin^2 x\sin x\,dx$$

$$= x\sin x + \cos x - \tfrac{1}{3}x\sin^3 x - \tfrac{1}{3}\int \left(1-\cos^2 x\right)d\left(\cos x\right)$$

$$= x\sin x + \cos x - \tfrac{1}{3}x\sin^3 x - \tfrac{1}{3}\left(\cos x - \tfrac{1}{3}\cos^3 x\right) + C$$

$$= x\sin x + \cos x - \tfrac{1}{3}x\sin^3 x - \tfrac{1}{3}\cos x + \tfrac{1}{9}\cos^3 x + C$$

$$= x\sin x + \tfrac{2}{3}\cos x - \tfrac{1}{3}x\sin^3 x + \tfrac{1}{9}\cos^3 x + C$$

Exercise

Evaluate the integrals $\int \sin^3 x\cos^4 x\,dx$

Solution

$$\int \sin^3 x\cos^4 x\,dx = \int \sin^2 x\cos^4 x\ \sin x\,dx$$

$$= -\int \left(1-\cos^2 x\right)\cos^4 x\,d\left(\cos x\right)$$

$$= \int \left(\cos^6 x - \cos^4 x\right)\,d\left(\cos x\right)$$

$$= \tfrac{1}{7}\cos^7 x - \tfrac{1}{5}\cos^5 x + C$$

Exercise

Evaluate the integrals $\int \cos^4 x\,dx$

Solution

$$\int \cos^4 x\,dx = \tfrac{1}{4}\int (1+\cos 2x)^2\,dx \qquad\qquad \cos^2 x = \tfrac{1}{2}(1+\cos 2x)$$

$$= \tfrac{1}{4}\int \left(1 + 2\cos 2x + \cos^2 2x\right)dx$$

$$= \frac{1}{4}\int\left(1+2\cos 2x+\frac{1}{2}+\frac{1}{2}\cos 4x\right)dx$$

$$= \frac{1}{4}\int\left(\frac{3}{2}+2\cos 2x+\frac{1}{2}\cos 4x\right)dx$$

$$\boxed{= \frac{1}{4}\left(\frac{3}{2}x+\sin 2x+\frac{1}{8}\sin 4x\right)+C}$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx$

Solution

$$\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx = \int \frac{\tan^2 x \tan x}{(\sec x)^{1/2}}\frac{\sec x}{\sec x}\,dx \qquad 1+\tan^2\alpha = \sec^2\alpha$$

$$= \int (\sec x)^{-3/2}\left(\sec^2 x - 1\right)d(\sec x)$$

$$= \int \left((\sec x)^{1/2} - (\sec x)^{-3/2}\right)d(\sec x)$$

$$\boxed{= \frac{2}{3}(\sec x)^{3/2} + 2(\sec x)^{-1/2} + C}$$

Exercise

Evaluate the integrals $\displaystyle\int \sec^4 3x \tan^3 3x\,dx$

Solution

$$\int \sec^4 3x \tan^3 3x\,dx = \int \sec^2 3x \tan^3 3x\ \sec^2 3x\,dx$$

$$= \frac{1}{3}\int \left(1+\tan^2 3x\right)\tan^3 3x\ d(\tan 3x)$$

$$= \frac{1}{3}\int \left(\tan^3 3x + \tan^5 3x\right)\ d(\tan 3x)$$

$$= \frac{1}{3}\left(\frac{1}{4}\tan^4 3x + \frac{1}{6}\tan^6 3x\right) + C$$

$$\boxed{= \frac{1}{12}\tan^4 3x + \frac{1}{18}\tan^6 3x + C}$$

Exercise

Evaluate the integrals $\displaystyle\int \frac{\sec x}{\tan^2 x}\,dx$

Solution

$$\begin{aligned}\int \frac{\sec x}{\tan^2 x}\,dx &= \int \frac{1}{\cos x}\frac{\cos^2 x}{\sin^2 x}\,dx\\ &= \int \frac{\cos x}{\sin^2 x}\,dx\\ &= \int \frac{1}{\sin^2 x}\,d(\sin x)\\ &= -\frac{1}{\sin x}+C\\ &= -\csc x + C\end{aligned}$$

Exercise

Evaluate the integrals $\displaystyle\int \sin 5x \cos 4x\,dx$

Solution

$$\begin{aligned}\int \sin 5x \cos 4x\,dx &= \frac{1}{2}\int (\sin x + \sin 9x)\,dx\\ &= \frac{1}{2}\left(-\cos x - \frac{1}{9}\cos x9x\right)+C\\ &= \frac{1}{2} - \cos x - \frac{1}{18}\cos x9x + C\end{aligned}$$

$$\sin\alpha\cos\beta=\tfrac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

Exercise

Evaluate the integrals $\displaystyle\int \sin x \cos^5 x\,dx$

Solution

$$\begin{aligned}\int \sin x \cos^5 x\,dx &= -\int \cos^5 x\,d(\cos x)\\ &= -\frac{1}{6}\cos^6 x + C\end{aligned}$$

Exercise

Evaluate the integrals $\displaystyle\int \sin^4 x \cos^3 x\,dx$

Solution

$$\int \sin^4 x \cos^3 x\, dx = \int \sin^4 x\left(1-\sin^2 x\right)\, d(\sin x)$$

$$= \int \left(\sin^4 x - \sin^6 x\right)\, d(\sin x)$$

$$\boxed{= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C}$$

Exercise

Evaluate the integrals $\int \sin^7 2x\ \cos 2x\, dx$

Solution

$$\int \sin^7 2x\ \cos 2x\, dx = \frac{1}{2}\int \sin^7 2x\, d(\sin 2x)$$

$$\boxed{= \frac{1}{16}\sin^8 2x + C}$$

Exercise

Evaluate the integrals $\int \sin^3 2x\sqrt{\cos 2x}\, dx$

Solution

$$\int \sin^3 2x\sqrt{\cos 2x}\, dx = -\frac{1}{2}\int \left(1-\cos^2 2x\right)(\cos 2x)^{1/2}\, d(\cos 2x)$$

$$= -\frac{1}{2}\int \left((\cos 2x)^{1/2} - (\cos 2x)^{5/2}\right)\, d(\cos 2x)$$

$$= -\frac{1}{2}\left(\frac{2}{3}(\cos 2x)^{3/2} - \frac{2}{7}(\cos 2x)^{7/2}\right) + C$$

$$\boxed{= \frac{1}{7}(\cos 2x)^{7/2} - \frac{1}{3}(\cos 2x)^{3/2} + C}$$

Exercise

Evaluate the integrals $\int \frac{\cos^5\theta}{\sqrt{\sin\theta}}\, d\theta$

Solution

$$\int \frac{\cos^5\theta}{\sqrt{\sin\theta}}\, d\theta = \int (\sin\theta)^{-1/2}\left(1-\sin^2\theta\right)^2\, d(\sin\theta)$$

$$= \int (\sin\theta)^{-1/2}\left(1-2\sin^2\theta+\sin^4\theta\right)\, d(\sin\theta)$$

$$= \int \left((\sin\theta)^{-1/2} - 2(\sin\theta)^{3/2} + (\sin\theta)^{7/2} \right) d(\sin\theta)$$

$$\boxed{= 2(\sin\theta)^{1/2} - \tfrac{1}{5}(\sin\theta)^{5/2} + \tfrac{2}{9}(\sin\theta)^{9/2} + C}$$

Exercise

Evaluate the integrals $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\, dx$

Solution

$$\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\, dx = \int_{\pi/6}^{\pi/3} (\sin x)^{-1/2} \left(1 - \sin^2 x\right) d(\sin x)$$

$$= \int_{\pi/6}^{\pi/3} \left((\sin x)^{-1/2} - (\sin x)^{3/2} \right) d(\sin x)$$

$$= 2(\sin x)^{1/2} - \tfrac{2}{5}(\sin x)^{5/2} \Big|_{\pi/6}^{\pi/3}$$

$$= 2\left(\frac{\sqrt{3}}{2}\right)^{1/2} - \frac{2}{5}\left(\frac{\sqrt{3}}{2}\right)^{5/2} - 2\left(\tfrac{1}{2}\right)^{1/2} + \frac{2}{5}\left(\tfrac{1}{2}\right)^{5/2}$$

$$= \sqrt[4]{3}\sqrt{2} - \frac{3}{10}\frac{\sqrt[4]{3}}{\sqrt{2}} - \sqrt{2} + \frac{\sqrt{2}}{20}$$

$$\boxed{= \frac{\sqrt{2}}{20}\left(17\sqrt[4]{3} - 19\right)}$$

Exercise

Evaluate the integrals $\displaystyle\int_{0}^{\pi/4} \tan^4 x dx$

Solution

$$\int_{0}^{\pi/4} \tan^4 x dx = \int_{0}^{\pi/4} \tan^2 x \left(\sec^2 x - 1\right) dx$$

$$= \int_{0}^{\pi/4} \tan^2 x \left(\sec^2 x - 1\right) dx$$

$$= \int_{0}^{\pi/4} \tan^2 x \sec^2 x\, dx - \int_{0}^{\pi/4} \tan^2 x\, dx$$

$$= \int_{0}^{\pi/4} \tan^2 x\, d(\tan x) - \int_{0}^{\pi/4} \left(\sec^2 x - 1\right) dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x \Big|_0^{\pi/4}$$

$$= \frac{1}{3} - 1 + \frac{\pi}{4}$$

$$= \frac{\pi}{4} - \frac{2}{3}$$

Exercise

Evaluate the integrals $\displaystyle\int_0^{\pi/2} \cos^7 x\, dx$

Solution

$$\int_0^{\pi/2} \cos^7 x\, dx = \int_0^{\pi/2} \left(\cos^2 x\right)^3\, d(\sin x)$$

$$= \int_0^{\pi/2} \left(1 - \sin^2 x\right)^3\, d(\sin x)$$

$$= \int_0^{\pi/2} \left(1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x\right)\, d(\sin x)$$

$$= \left(\sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x\right)_0^{\pi/2}$$

$$= \frac{3}{5} - \frac{1}{7}$$

$$= \frac{16}{37}$$

Exercise

Evaluate the integrals $\displaystyle\int_0^{\pi/2} \cos^9 \theta\, d\theta$

Solution

$$\int_0^{\pi/2} \cos^9 \theta\, d\theta = \int_0^{\pi/2} \left(1 - \sin^2 x\right)^4\, d(\sin x)$$

$$= \int_0^{\pi/2} \left(1 - 4\sin^2 x + 6\sin^4 x - 4\sin^6 x + \sin^8 x\right)\, d(\sin x)$$

$$= \left(\sin x - \frac{4}{3}\sin^3 x + \frac{6}{5}\sin^5 x - \frac{4}{7}\sin^7 x + \frac{1}{9}\sin^9 x\right)_0^{\pi/2}$$

$$= 1 - \frac{4}{3} + \frac{6}{5} - \frac{4}{7} + \frac{1}{9}$$

$$= \frac{128}{315}$$

Exercise

Evaluate the integrals $\int_0^{\pi/2} \sin^5 x\, dx$

Solution

$$\int_0^{\pi/2} \sin^5 x\, dx = \int_0^{\pi/2} \left(1-\cos^2 x\right)^2\, d(\cos x)$$

$$= \int_0^{\pi/2} \left(1-2\cos^2 x+\cos^4 x\right)\, d(\cos x)$$

$$= \left(\cos x - \frac{2}{3}\cos^3 x + \frac{1}{5}\cos^5 x\right)_0^{\pi/2}$$

$$= -1+\frac{2}{3}-\frac{1}{5}$$

$$\boxed{= -\frac{8}{15}}$$

Exercise

Evaluate the integrals $\int_0^{\pi/6} 3\cos^5 3x\, dx$

Solution

$$\int_0^{\pi/6} 3\cos^5 3x\, dx = \int_0^{\pi/6} 3\left(\cos^2 3x\right)^2 \cos 3x\, dx$$

$$= \int_0^{\pi/6} \left(1-\sin^2 3x\right)^2\, d(\sin 3x)$$

$$= \int_0^{\pi/6} \left(1-2\sin^2 3x+\sin^4 3x\right) d(\sin 3x)$$

$$= \left[\sin 3x - \frac{2}{3}\sin^2 3x + \frac{1}{5}\sin^4 3x\right]_0^{\pi/6}$$

$$= \sin\frac{\pi}{2} - \frac{2}{3}\sin^2\frac{\pi}{2} + \frac{1}{5}\sin^4\frac{\pi}{2} - 0$$

$$= 1-\frac{2}{3}+\frac{1}{5}$$

$$\boxed{= \frac{8}{15}}$$

Exercise

Evaluate the integrals $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta d\theta$

Solution

$$\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta d\theta = \int_0^{\pi/2} \sin^2 2\theta \left(\cos^2 2\theta\right)\cos 2\theta d\theta \qquad d\left(\sin 2\theta\right) = 2\cos 2\theta d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^2 2\theta\left(1-\sin^2 2\theta\right) d\left(\sin 2\theta\right)$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(\sin^2 2\theta - \sin^4 2\theta\right) d\left(\sin 2\theta\right)$$

$$= \frac{1}{2}\left[\frac{1}{3}\sin^3 2\theta - \frac{1}{5}\sin^5 2\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{1}{3}\sin^3 \pi - \frac{1}{5}\sin^5 \pi - 0\right)$$

$$= 0$$

Exercise

Evaluate the integrals $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}} dx$

Solution

$$\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}} dx = \int_0^{2\pi} \sin\frac{x}{2} dx \qquad \left|\sin\left(\frac{\alpha}{2}\right)\right| = \sqrt{\frac{1-\cos\alpha}{2}}$$

$$= \left[-2\cos\frac{x}{2}\right]_0^{2\pi}$$

$$= -2\left(\cos\pi - \cos 0\right)$$

$$= 2$$

Exercise

Evaluate the integrals $\int_0^{\pi} \sqrt{1-\cos^2\theta} d\theta$

Solution

$$\int_0^{\pi} \sqrt{1-\cos^2\theta} d\theta = \int_0^{\pi} |\sin\theta| d\theta$$

$$= \left[-\cos\theta\right]_0^{\pi}$$

$$= -\cos\pi + \cos 0$$

$$\underline{= 2}$$

Exercise

Evaluate the integrals $\displaystyle\int_0^{\pi/6} \sqrt{1+\sin x}\; dx$

Solution

$$\int_0^{\pi/6} \sqrt{1+\sin x}\; dx = \int_0^{\pi/6} \sqrt{1+\sin x}\,\frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}\; dx$$

$$= \int_0^{\pi/6} \frac{\sqrt{1-\sin^2 x}}{\sqrt{1-\sin x}}\; dx \qquad \cos x = \sqrt{1-\sin^2 x}$$

$$= \int_0^{\pi/6} \frac{\cos x}{\sqrt{1-\sin x}}\; dx \qquad d\left(1-\sin x\right) = -\cos x dx$$

$$= -\int_0^{\pi/6} \left(1-\sin x\right)^{-1/2}\; d\left(1-\sin x\right)$$

$$= -2\left[\left(1-\sin x\right)^{1/2}\right]_0^{\pi/6}$$

$$= -2\left(\sqrt{1-\sin\frac{\pi}{6}} - 1\right)$$

$$= -2\left(\sqrt{1-\frac{1}{2}} - 1\right)$$

$$= -2\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= -2\left(\frac{\sqrt{2}}{2} - 1\right)$$

$$\underline{= 2-\sqrt{2}}$$

Exercise

Evaluate the integrals $\displaystyle\int_{-\pi}^{\pi} \left(1-\cos^2 x\right)^{3/2} dx$

Solution

$$\int_{-\pi}^{\pi}\left(1-\cos^2 x\right)^{3/2}dx = \int_{-\pi}^{\pi}\left(\sin^2 x\right)^{3/2}dx$$

$$= \int_{-\pi}^{\pi}\left|\sin^3 x\right|dx$$

$$= -\int_{-\pi}^{0}\sin^3 x\,dx + \int_{0}^{\pi}\sin^3 x\,dx \qquad \sin^2 x = 1-\cos^2 x$$

$$= -\int_{-\pi}^{0}\left(1-\cos^2 x\right)\sin x\,dx + \int_{0}^{\pi}\left(1-\cos^2 x\right)\sin x\,dx \qquad d(\cos x) = -\sin x dx$$

$$= \int_{-\pi}^{0}\left(1-\cos^2 x\right)d(\cos x) - \int_{0}^{\pi}\left(1-\cos^2 x\right)d(\cos x)$$

$$= \left[\cos x - \tfrac{1}{3}\cos^3 x\right]_{-\pi}^{0} - \left[\cos x - \tfrac{1}{3}\cos^3 x\right]_{0}^{\pi}$$

$$= \left(1-\tfrac{1}{3}-\left(-1+\tfrac{1}{3}\right)\right)-\left(-1+\tfrac{1}{3}-\left(1-\tfrac{1}{3}\right)\right)$$

$$= 1-\tfrac{1}{3}+1-\tfrac{1}{3}+1-\tfrac{1}{3}+1-\tfrac{1}{3}$$

$$= 4-\tfrac{4}{3}$$

$$= \frac{8}{3}$$

Exercise

Evaluate the integrals $\displaystyle\int_{\pi/4}^{\pi/2}\csc^4\theta d\theta$

Solution

$$\int_{\pi/4}^{\pi/2}\csc^4\theta d\theta = \int_{\pi/4}^{\pi/2}\left(1+\cot^2\theta\right)\csc^2\theta d\theta \qquad \csc^2\theta = 1+\cot^2\theta$$

$$= \int_{\pi/4}^{\pi/2}\csc^2\theta d\theta + \int_{\pi/4}^{\pi/2}\cot^2\theta\csc^2\theta d\theta \qquad d(\cot\theta) = -\csc^2\theta d\theta$$

$$= \int_{\pi/4}^{\pi/2}\csc^2\theta d\theta - \int_{\pi/4}^{\pi/2}\cot^2\theta\, d(\cot\theta)$$

$$= \left[-\cot\theta - \tfrac{1}{3}\cot^3\theta\right]_{\pi/4}^{\pi/2}$$

$$= -\left(\cot\frac{\pi}{2} + \frac{1}{3}\cot^3\frac{\pi}{2} - \cot\frac{\pi}{4} - \frac{1}{3}\cot^3\frac{\pi}{4}\right)$$

$$= -\left(0 + \frac{1}{3}(0) - 1 - \frac{1}{3}\right)$$

$$\boxed{= \frac{4}{3}}$$

Exercise

Evaluate the integrals $\displaystyle\int_{-\pi}^{\pi} \sin 3x \sin 3x \; dx$

Solution

$$\int_{-\pi}^{\pi} \sin 3x \sin 3x \; dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos 0 - \cos 6x) \; dx \qquad \sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} (1 - \cos 6x) \; dx$$

$$= \frac{1}{2}\left[x - \frac{1}{6}\sin 6x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left(\pi - \frac{1}{6}\sin 6\pi - \left(-\pi - \frac{1}{6}\sin(-6\pi)\right)\right)$$

$$= \frac{1}{2}(\pi + \pi)$$

$$\boxed{= \pi}$$

Exercise

Evaluate the integrals $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x \cos 7x \; dx$

Solution

$$\int_{-\pi/2}^{\pi/2} \cos x \cos 7x dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (\cos 8x + \cos(-6x)) dx \qquad \cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} (\cos 8x + \cos 6x) dx$$

$$= \frac{1}{2}\left[\frac{1}{6}\sin 6x + \frac{1}{8}\sin 8x\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{1}{6}\sin(3\pi) + \frac{1}{8}\sin(4\pi) - \frac{1}{6}\sin(-3\pi) - \frac{1}{8}\sin(-4\pi)\right)$$

$$\boxed{= 0}$$

Exercise

Evaluate the integrals $\displaystyle\int_0^{\pi} 8\sin^4 y\cos^2 y\,dy$

Solution

$$\int_0^{\pi} 8\sin^4 y\cos^2 y\,dy = 8\int_0^{\pi}\left(\frac{1-\cos 2y}{2}\right)^2\left(\frac{1+\cos 2y}{2}\right)dy$$

$$= \int_0^{\pi}\left(1-2\cos 2y+\cos^2 2y\right)(1+\cos 2y)\,dy$$

$$= \int_0^{\pi}\left(1-2\cos 2y+\cos^2 2y+\cos 2y-2\cos^2 2y+\cos^3 2y\right)dy$$

$$= \int_0^{\pi}\left(1-\cos 2y-\cos^2 2y+\cos^3 2y\right)dy$$

$$= \int_0^{\pi}\left(1-\cos 2y-\frac{1}{2}-\frac{1}{2}\cos 4y\right)dy+\int_0^{\pi}\cos^2 2y\cos 2y\,dy$$

$$= \int_0^{\pi}\left(\frac{1}{2}-\cos 2y-\frac{1}{2}\cos 4y\right)dy+\frac{1}{2}\int_0^{\pi}\left(1-\sin^2 2y\right)d(\sin 2y)$$

$$= \left[\frac{1}{2}y-\frac{1}{2}\sin 2y-\frac{1}{8}\sin 4y+\frac{1}{2}\left(\sin 2y-\frac{1}{3}\sin^3 2y\right)\right]_0^{\pi}$$

$$= \frac{\pi}{2}$$

Exercise

Find the area of the region bounded by the graphs of $y=\tan x$ and $y=\sec x$ on the interval $\left[0,\ \frac{\pi}{4}\right]$

Solution

$$A=\int_0^{\pi/4}(\sec x-\tan x)\,dx$$

$$= \ln\left|\sec x+\tan x\right|+\ln\left|\cos x\right|\ \Big|_0^{\pi/4}$$

$$= \ln\left(\sqrt{2}+1\right)+\ln\frac{\sqrt{2}}{2}-0$$

$$= \ln\left(\frac{\sqrt{2}}{2}\left(\sqrt{2}+1\right)\right)$$

$$= \ln\left(1+\frac{\sqrt{2}}{2}\right)$$

Solution   Section 2.3 – Trigonometric Substitutions

Exercise

Evaluate the integral $\displaystyle\int \frac{3dx}{\sqrt{1+9x^2}}$

Solution

$$\int \frac{3dx}{\sqrt{1+9x^2}} = \frac{1}{3}\int \frac{\sec^2 t}{3\sec t}dt$$

$$= \int \sec t \, dt$$

$$= \ln|\sec t + \tan t| + C$$

$$= \ln\left|\sqrt{1+u^2} + u\right| + C$$

$$= \ln\left|\sqrt{1+9x^2} + 3x\right| + C$$

$3x = \tan t \quad \Rightarrow dx = \frac{1}{3}\sec^2 t \, dt$

$\sqrt{1+9x^2} = 3\sec^2 t$

Exercise

Evaluate the integral $\displaystyle\int \frac{5dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5} = \sin^{-1}\frac{1}{2} - \sin^{-1}0$

Solution

$$\int \frac{5dx}{\sqrt{25x^2-9}} = \int \frac{5\left(\frac{3}{5}\sec\theta\tan\theta d\theta\right)}{3\tan\theta}$$

$$= \int \sec\theta d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{5}{3}x + \frac{1}{3}\frac{\sqrt{25x^2-9}}{3}\right| + C$$

$5x = 3\sec\theta \quad \rightarrow \quad dx = \frac{3}{5}\sec\theta\tan\theta d\theta$

$\sqrt{25x^2-9} = 3\tan\theta$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sqrt{y^2-49}}{y}dy, \quad y > 7$

Solution

$$\int \frac{\sqrt{y^2-49}}{y}dy = \int \frac{(7\tan\theta)}{7\sec\theta}(7\sec\theta\tan\theta)d\theta$$

$y = 7\sec\theta \quad \rightarrow \quad dy = 7\sec\theta\tan\theta d\theta$

$\sqrt{y^2-49} = 7\tan\theta$

$$= 7\int \tan^2\theta\, d\theta$$

$$= 7\int \left(\sec^2\theta - 1\right) d\theta$$

$$= 7\left(\tan\theta - \theta\right) + C$$

$$\boxed{= 7\left[\frac{\sqrt{y^2-49}}{7} - \sec^{-1}\left(\frac{y}{7}\right)\right] + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{2dx}{x^3\sqrt{x^2-1}}, \quad x > 1$

Solution

$$\int \frac{2dx}{x^3\sqrt{x^2-1}} = \int \frac{2\sec\theta\tan\theta\, d\theta}{\sec^3\theta\tan\theta}$$

$x = \sec\theta \quad dx = \sec\theta\tan\theta\, d\theta$

$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

$$= 2\int \cos^2\theta\, d\theta$$

$$= 2\int \frac{1+\cos 2\theta}{2}\, d\theta$$

$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$= \int \left(1+\cos 2\theta\right) d\theta$$

$$= \theta + \frac{1}{2}\sin 2\theta + C$$

$$= \theta + \sin\theta\cos\theta + C$$

$x = \sec\theta = \dfrac{1}{\cos\theta} \Rightarrow \cos\theta = \dfrac{1}{x}$

$\sin\theta = \tan\theta\cos\theta = \sqrt{x^2-1}\left(\dfrac{1}{x}\right)$

$$\boxed{= \sec^{-1}x + \frac{\sqrt{x^2-1}}{x^2} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^2}{4+x^2}\, dx$

Solution

$$\int \frac{x^2}{4+x^2}\, dx = \int \frac{4\tan^2\theta}{4\sec^2\theta}\, 2\sec^2\theta\, d\theta$$

$x = 2\tan\theta \quad dx = 2\sec^2\theta\, d\theta$

$4 + x^2 = 4 + 4\tan^2\theta = 4\sec^2\theta$

$$= 2\int \tan^2\theta\, d\theta$$

$$= 2\int\left(\sec^2\theta - 1\right)d\theta \qquad\qquad \int \sec^2\theta d\theta = \tan\theta$$

$$= 2\left(\tan\theta - \theta\right) + C$$

$$= 2\left(\frac{x}{2} - \tan^{-1}\left(\frac{x}{2}\right)\right) + C$$

$$\boxed{= x - 2\tan^{-1}\left(\frac{x}{2}\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}}$

Solution

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta d\theta}{\tan^2\theta\sec\theta} \qquad x = \tan\theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

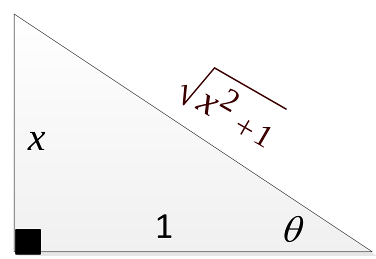
$$dx = \sec^2\theta d\theta$$

$$\sqrt{x^2+1} = \sqrt{\tan^2\theta + 1} = \sec\theta$$

$$= \int \frac{\sec\theta d\theta}{\tan^2\theta}$$

$$= \int \frac{\cos^2\theta d\theta}{\sin^2\theta\cos\theta}$$

$$= \int \frac{\cos\theta d\theta}{\sin^2\theta}$$



$$= \int \sin^{-2}\theta d\left(\sin\theta\right)$$

$$= -\frac{1}{\sin\theta} + C$$

$$\boxed{= -\frac{\sqrt{x^2+1}}{x} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\left(1-x^2\right)^{1/2}}{x^4}dx$

Solution

$$\int \frac{\left(1-x^2\right)^{1/2}}{x^4}dx = \int \frac{\cos\theta}{\sin^4\theta}\cos\theta d\theta \qquad x = \sin\theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$dx = \cos\theta d\theta$$

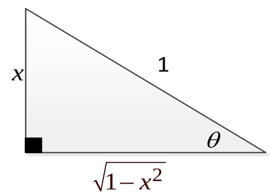
$$\left(1-x^2\right)^{1/2} = \left(1-\sin^2 x\right)^{1/2} = \cos\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\frac{1}{\sin^2\theta}d\theta$$

$$= \int \cot^2 \theta \csc^2 \theta d\theta$$

$$= -\frac{1}{3}\cot^3 \theta + C$$

$$\boxed{= -\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C}$$



Exercise

Evaluate the integral $\displaystyle\int \frac{x^3 dx}{x^2 - 1}$

Solution

$$\int \frac{x^3 dx}{x^2-1} = \int \left(x + \frac{x}{x^2-1}\right)dx$$

$$= \int x dx + \int \frac{x}{x^2-1}dx$$

$$= \int x dx + \frac{1}{2}\int \frac{d\left(x^2-1\right)}{x^2-1}$$

$$\boxed{= \frac{1}{2}x^2 + \frac{1}{2}\ln\left|x^2-1\right| + C}$$

$$\begin{array}{r|l} & x \\ \hline x^2-1 & x^3 \\ & \underline{x^3 - x} \\ & \quad x \end{array}$$

$$d\left(x^2-1\right) = 2xdx \;\Rightarrow\; \frac{1}{2}d\left(x^2-1\right) = xdx$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}dx$

Solution

$$\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}dx = \int \frac{\cos\theta}{\sin\theta}\cos\theta d\theta$$

$$= \int \frac{\cos^2\theta}{\sin\theta}d\theta$$

$$= \int \frac{1-\sin^2\theta}{\sin\theta}d\theta$$

$$= \int \frac{1}{\sin\theta}d\theta - \int \sin\theta d\theta$$

$$= \int \csc\theta d\theta - \int \sin\theta d\theta$$

$$\ln x = \sin\theta \qquad 0 < \theta \le \frac{\pi}{2}$$

$$\frac{1}{x}dx = \cos\theta d\theta$$

$$\sqrt{1-(\ln x)^2} = \sqrt{1-\sin^2\theta} = \cos\theta$$

$$= -\ln\left|\csc\theta + \cot\theta\right| + \cos\theta + C$$

$$\boxed{= -\ln\left|\frac{1}{\ln x} + \frac{\sqrt{1-(\ln x)^2}}{\ln x}\right| + \sqrt{1-(\ln x)^2} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \sqrt{x}\,\sqrt{1-x}\;dx$

Solution

$$\int \sqrt{x}\,\sqrt{1-x}\;dx = \int u\sqrt{1-u^2}\,(2udu)$$

$$= 2\int u^2\sqrt{1-u^2}\,du$$

$u = \sqrt{x} \;\to u^2 = x \quad \Rightarrow dx = 2udu$

$u = \sin\theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$du = \cos\theta d\theta$

$$\int \sqrt{x}\,\sqrt{1-x}\;dx = 2\int u^2\sqrt{1-u^2}\,du = 2\int \sin^2\theta\cos\theta\cos\theta d\theta$$

$\sqrt{1-u^2} = \sqrt{1-\sin^2\theta} = \cos\theta$

$$= 2\int \sin^2\theta\cos^2\theta d\theta$$

$\sin 2\theta = 2\sin\theta\cos\theta \quad \to \sin^2 2\theta = 4\sin^2\theta\cos^2\theta$

$$= \frac{1}{2}\int \sin^2 2\theta d\theta$$

$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$

$$= \frac{1}{2}\int \frac{1-\cos 4\theta}{2}\,d\theta$$

$$= \frac{1}{4}\int d\theta - \frac{1}{4}\int \cos 4\theta d\theta$$

$$= \frac{1}{4}\theta - \frac{1}{16}\sin 4\theta + C$$

$$= \frac{1}{4}\theta - \frac{1}{16}2\sin 2\theta\cos 2\theta + C$$

$$= \frac{1}{4}\theta - \frac{1}{8}2\sin\theta\cos\theta\left(2\cos^2\theta - 1\right) + C$$

$$= \frac{1}{4}\theta - \frac{1}{2}\sin\theta\cos^3\theta + \frac{1}{4}\sin\theta\cos\theta + C$$

$$= \frac{1}{4}\sin^{-1}u - \frac{1}{2}u\left(1-u^2\right)^{3/2} + \frac{1}{4}u\sqrt{1-u^2} + C$$

$$\boxed{= \frac{1}{4}\sin^{-1}\sqrt{x} - \frac{1}{2}\sqrt{x}\,(1-x)^{3/2} + \frac{1}{4}\sqrt{x}\sqrt{1-x} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx$

Solution

$$\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx = \int \frac{\sqrt{u^2-1}}{u}2udu$$

$$= 2\int \sqrt{u^2-1}\;du$$

$$= 2\int \tan\theta\sec\theta\tan\theta d\theta$$

$u = \sqrt{x-1} \quad \rightarrow u^2 = x-1 \quad \Rightarrow 2udu = dx$

$u = \sec\theta \qquad 0 < \theta < \frac{\pi}{2}$

$du = \sec\theta\tan\theta d\theta$

$\sqrt{u^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

$w = \tan\theta \qquad dv = \sec\theta\tan\theta d\theta$

$dw = \sec^2\theta d\theta \qquad v = \sec\theta$

$$2\int \tan\theta\sec\theta\tan\theta d\theta = 2\sec\theta\tan\theta - 2\int \sec^3\theta d\theta$$

$$= 2\sec\theta\tan\theta - 2\int \sec^2\theta\sec\theta d\theta$$

$$= 2\sec\theta\tan\theta - 2\int \left(\tan^2\theta + 1\right)\sec\theta d\theta$$

$$= 2\sec\theta\tan\theta - 2\int \tan^2\theta\sec\theta d\theta - 2\int \sec\theta d\theta$$

$$2\int \tan^2\theta\sec\theta d\theta = 2\sec\theta\tan\theta - 2\int \tan^2\theta\sec\theta d\theta - 2\ln\left|\sec\theta + \tan\theta\right|$$

$$4\int \tan^2\theta\sec\theta d\theta = 2\sec\theta\tan\theta - 2\ln\left|\sec\theta + \tan\theta\right|$$

$$2\int \tan^2\theta\sec\theta d\theta = \sec\theta\tan\theta - \ln\left|\sec\theta + \tan\theta\right|$$

$$\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx = 2\int \tan\theta\sec\theta\tan\theta d\theta$$

$$= \sec\theta\tan\theta - \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= u\sqrt{u^2-1} - \ln\left|u + \sqrt{u^2-1}\right| + C$$

$$= \sqrt{x-1}\sqrt{x-2} - \ln\left|\sqrt{x-1} + \sqrt{x-2}\right| + C$$

Exercise

Evaluate: $\displaystyle\int \frac{2dx}{\sqrt{1-4x^2}}$

Solution

$$\int \frac{2dx}{\sqrt{1-4x^2}} = \int \frac{du}{\sqrt{1-u^2}} \qquad\qquad u = 2x \quad \to \quad du = 2dx$$

$$= \sin^{-1} u + C$$

$$\boxed{= \sin^{-1} 2x + C}$$

Exercise

Evaluate: $\displaystyle\int \frac{dx}{\sqrt{4x^2-49}}$

Solution

$$2x = 7\sec\theta \quad \to dx = \frac{7}{2}\sec\theta\tan\theta\ d\theta$$

$$\sqrt{4x^2-49} = \frac{7}{2}\tan\theta$$

$$\int \frac{dx}{\sqrt{4x^2-49}} = \int \frac{dx}{2\sqrt{x^2-\left(\frac{7}{2}\right)^2}}$$

$$= \frac{1}{2}\int \frac{\frac{7}{2}\sec\theta\tan\theta d\theta}{\frac{7}{2}\tan\theta}$$

$$= \frac{1}{2}\int \sec\theta d\theta$$

$$= \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$\boxed{= \frac{1}{2}\ln\left|\frac{2x}{7} + \frac{\sqrt{4x^2-49}}{7}\right| + C}$$

Exercise

Evaluate: $\displaystyle\int \frac{dx}{\sqrt{x^2+4}}$

Solution

Let: $x = 2\tan\theta \quad \to dx = 2\sec^2\theta d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\sqrt{x^2+4} = 2\left|\sec\theta\right|$$

$$\int \frac{dx}{\sqrt{x^2+4}} = \int \frac{2\sec^2\theta d\theta}{\sqrt{4\sec^2\theta}}$$

$$= \int \frac{2\sec^2\theta d\theta}{2|\sec\theta|}$$

$$= \int \sec\theta d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$\boxed{= \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\left(16-x^2\right)^{3/2}}$

Solution

$$\int \frac{dx}{\left(16-x^2\right)^{3/2}} = \int \frac{4\cos\theta}{(4\cos\theta)^3} d\theta$$

$x = 4\sin\theta \qquad \sqrt{16-x^2} = 4\cos\theta$

$dx = 4\cos\theta d\theta$

$$= \frac{1}{16}\int \frac{1}{\cos^2\theta} d\theta$$

$$= \frac{1}{16}\int \sec^2\theta\, d\theta$$

$$\boxed{= \frac{1}{16}\tan\theta + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\left(1+x^2\right)^2}$

Solution

$$\int \frac{dx}{\left(1+x^2\right)^2} = \int \frac{\sec^2\theta}{\sec^4\theta} d\theta$$

$x = \tan\theta \qquad 1+x^2 = \left(\sec^2\theta\right)^2$

$dx = \sec^2\theta\, d\theta$

$$= \int \frac{1}{\sec^2\theta} d\theta$$

$$= \int \cos^2\theta \, d\theta$$

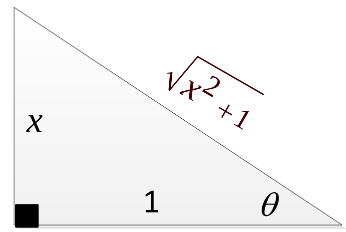
$$= \frac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}\tan^{-1} x + \frac{1}{2}\sin\theta\cos\theta + C$$

$$= \frac{1}{2}\tan^{-1} x + \frac{1}{2}\frac{x}{\sqrt{1+x^2}}\frac{1}{\sqrt{1+x^2}} + C$$

$$= \frac{1}{2}\tan^{-1} x + \frac{x}{2\left(1+x^2\right)} + C$$



Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2+4}}$

Solution

$$\int \frac{dx}{\sqrt{x^2+4}} = \int \frac{2\sec^2\theta}{2\sec\theta}\, d\theta$$

$x = 2\tan\theta \qquad \sqrt{x^2+4} = 2\sec\theta$

$dx = 2\sec^2\theta \; d\theta$

$$= \int \sec\theta \; d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C$$

$$= \ln\left(\sqrt{x^2+4} + x\right) - \ln 2 + C$$

$$= \ln\left(\sqrt{x^2+4} + x\right) + C$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$

Solution

$$\int \frac{dx}{x^2\sqrt{9-x^2}} = \int \frac{3\cos\theta}{9\sin^2\theta(3\cos\theta)}\, d\theta$$

$x = 3\sin\theta \qquad \sqrt{9-x^2} = 3\cos\theta$

$dx = 3\cos\theta d\theta$

$$= \frac{1}{9}\int \csc^2\theta\, d\theta$$

$$= -\frac{1}{9}\cot\theta + C \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{9-x^2}}{3}\cdot\frac{3}{x}$$

$$\boxed{= -\frac{1}{9}\frac{\sqrt{9-x^2}}{x} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}}$

Solution

$$\int \frac{dx}{\sqrt{4x^2+1}} = \frac{1}{2}\int \frac{\sec^2\theta}{\sec\theta}\, d\theta \qquad 2x = \tan\theta \qquad \sqrt{4x^2+1} = \sec\theta$$

$$dx = \frac{1}{2}\sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int \sec\theta\, d\theta$$

$$= \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$\boxed{= \frac{1}{2}\ln\left|\sqrt{4x^2+1} + 2x\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\left(x^2+1\right)^{3/2}}$

Solution

$$\int \frac{dx}{\left(x^2+1\right)^{3/2}} = \int \frac{\sec^2\theta}{(\sec\theta)^3}\, d\theta \qquad x = \tan\theta \qquad \sqrt{x^2+1} = \sec\theta$$

$$dx = \sec^2\theta\, d\theta$$

$$= \int \frac{d\theta}{\sec\theta}$$

$$= \int \cos\theta\, d\theta$$

$$= \sin\theta + C \qquad \sin\theta = \frac{\tan\theta}{\sec\theta} = \frac{x}{\sqrt{x^2+1}}$$

$$\boxed{= \frac{x}{\sqrt{x^2+1}} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{4}{x^2\sqrt{16-x^2}}\,dx$

Solution

$$\int \frac{4}{x^2\sqrt{16-x^2}}\,dx = \int \frac{16\cos\theta}{16\sin^2\theta(4\cos\theta)}\,d\theta$$

$x = 4\sin\theta \qquad \sqrt{16-x^2} = 4\cos\theta$

$dx = 4\cos\theta d\theta$

$$= \frac{1}{4}\int \csc^2\theta\,d\theta$$

$$= -\frac{1}{4}\cot\theta + C$$

Exercise

Evaluate $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\,dx$

Solution

$$\int \frac{x^3}{\sqrt{9-x^2}}\,dx = \int \frac{27\sin^3\theta}{3\cos\theta}\,(3\cos\theta)\,d\theta$$

$x = 3\sin\theta \qquad \sqrt{9-x^2} = 3\cos\theta$

$dx = 3\cos\theta d\theta$

$$= 27\int \sin^3\theta\,d\theta$$

$$= 27\int \left(1-\cos^2\theta\right)\,d\left(\cos\theta\right)$$

$$= 27\left(\cos\theta - \frac{1}{3}\cos^3\theta\right) + C$$

$$= 27\cos\theta - 9\cos^3\theta + C$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2-25}}$

Solution

$$\int \frac{dx}{\sqrt{x^2-25}} = \int \frac{5\sec\theta\tan\theta}{5\tan\theta}\,d\theta$$

$x = 5\sec\theta \qquad \sqrt{x^2-25} = 5\tan\theta$

$dx = 5\sec\theta\tan\theta\,d\theta$

$$= \int \sec\theta\,d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \ln\left|\frac{x}{5} + \frac{1}{5}\sqrt{x^2-25}\right| + C$$

Exercise

Evaluate $\quad \displaystyle\int \frac{\sqrt{x^2-25}}{x}\,dx$

Solution

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \int \frac{5\tan\theta}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta$$

$x = 5\sec\theta \qquad \sqrt{x^2-25} = 5\tan\theta$

$dx = 5\sec\theta\tan\theta\; d\theta$

$$= 5\int \tan^2\theta\; d\theta$$

$$= 5\int \left(\sec^2\theta - 1\right)\, d\theta$$

$$= 5(\tan\theta - \theta) + C$$

$$= \sqrt{x^2-25} - 5\,\text{arcsec}\,\frac{x}{5} + C$$

Exercise

Evaluate $\quad \displaystyle\int \frac{x^3}{\sqrt{x^2-25}}\,dx$

Solution

$$\int \frac{x^3}{\sqrt{x^2-25}}\,dx = \int \frac{5^3\sec^3\theta}{5\tan\theta}(5\sec\theta\tan\theta)\,d\theta$$

$x = 5\sec\theta \qquad \sqrt{x^2-25} = 5\tan\theta$

$dx = 5\sec\theta\tan\theta\; d\theta$

$$= 125\int \sec^4\theta\; d\theta$$

$$= 125\int \left(1+\tan^2\theta\right)\sec^2\theta\; d\theta$$

$$= 125\int \left(1+\tan^2\theta\right)\; d(\tan\theta)$$

$$= 125\left(\tan\theta + \frac{1}{3}\tan^3\theta\right) + C$$

$$= 125\left(\frac{\sqrt{x^2-25}}{5} + \frac{1}{3}\frac{\left(x^2-25\right)^{3/2}}{125}\right) + C$$

$$= \sqrt{x^2-25}\left(25 + \frac{x^2-25}{3}\right) + C$$

$$= \frac{1}{3}\sqrt{x^2-25}\left(x^2+50\right) + C$$

Exercise

Evaluate $\displaystyle\int x^3\sqrt{x^2-25}\,dx$

Solution

$$\int x^3\sqrt{x^2-25}\,dx = \int 5^3\sec^3\theta(5\tan\theta)(5\sec\theta\tan\theta)\,d\theta$$

$x = 5\sec\theta \qquad \sqrt{x^2-25} = 5\tan\theta$

$dx = 5\sec\theta\tan\theta\,d\theta$

$$\begin{aligned}
&= 5^5\int \sec^4\theta\tan^2\theta\,d\theta \\
&= 5^5\int \sec^2\theta\left(1+\tan^2\theta\right)\tan^2\theta\,d\theta \\
&= 5^5\int \left(\tan^2\theta+\tan^4\theta\right)\,d(\tan\theta) \\
&= 5^5\left(\tfrac{1}{3}\tan^3\theta+\tfrac{1}{5}\tan^5\theta\right)+C \\
&= 5^5\left(\frac{1}{3}\frac{1}{5^3}\left(x^2-25\right)^{3/2}+\frac{1}{5^6}\left(x^2-25\right)^{5/2}\right)+C \\
&= \left(x^2-25\right)^{3/2}\left(\frac{25}{3}+\frac{1}{5}\left(x^2-25\right)\right)+C \\
&= \frac{1}{15}\left(x^2-25\right)^{3/2}\left(125+3x^2-75\right)+C \\
&= \frac{1}{15}\left(x^2-25\right)^{3/2}\left(3x^2+50\right)+C
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int x\sqrt{x^2+1}\,dx$

Solution

$$\int x\sqrt{x^2+1}\,dx = \frac{1}{2}\int\left(x^2+1\right)^{1/2}\,d\left(x^2+1\right)$$

$$= \frac{1}{3}\left(x^2+1\right)^{3/2}+C$$

$$\int x\sqrt{x^2+1}\,dx = \int \tan\theta\sec^3\theta\,d\theta$$

$x=\tan\theta \qquad \sqrt{x^2+1}=\sec\theta$

$dx=\sec^2\theta\,d\theta$

$$\begin{aligned}
&= \int \sec^2\theta\,d(\sec\theta) \\
&= \frac{1}{3}\sec^3\theta+C \\
&= \frac{1}{3}\left(x^2+1\right)^{3/2}+C
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int \frac{9x^3}{\sqrt{x^2+1}}\, dx$

Solution

$$\int \frac{9x^3}{\sqrt{x^2+1}}\, dx = \int \frac{9\tan^3\theta}{\sec\theta}\left(\sec^2\theta\right) d\theta$$

$x = \tan\theta \qquad \sqrt{x^2+1} = \sec\theta$

$dx = \sec^2\theta\, d\theta$

$$= 9\int \tan^2\theta \tan\theta \sec\theta\, d\theta$$

$$= 9\int \left(\sec^2\theta - 1\right) d\left(\sec\theta\right)$$

$$= 9\left(\frac{1}{3}\sec^3\theta - \sec\theta\right) + C$$

$$= 3\left(x^2+1\right)\sqrt{x^2+1} - 9\sqrt{x^2+1} + C$$

$$= 3\sqrt{x^2+1}\left(x^2+1-3\right) + C$$

$$\boxed{= 3\sqrt{x^2+1}\left(x^2-2\right) + C}$$

Exercise

Evaluate $\displaystyle\int_0^{\sqrt{3}/2} \frac{x^2}{\left(1-x^2\right)^{3/2}}\, dx$

Solution

$$\int_0^{\sqrt{3}/2} \frac{x^2}{\left(1-x^2\right)^{3/2}}\, dx = \int_0^{\sqrt{3}/2} \frac{\sin^2\theta}{\cos^3\theta}\left(\cos\theta\right) d\theta$$

$x = \sin\theta \qquad \sqrt{1-x^2} = \cos\theta$

$dx = \cos\theta\, d\theta$

$$= \int_0^{\sqrt{3}/2} \tan^2\theta\, d\theta$$

$$= \int_0^{\sqrt{3}/2} \left(\sec^2\theta - 1\right) d\theta$$

$$= \left(\tan\theta - \theta\right)\Big|_0^{\sqrt{3}/2}$$

$$= \left(\frac{x}{\sqrt{1-x^2}} - \arcsin x\right)\Bigg|_0^{\sqrt{3}/2}$$

$$= \frac{\sqrt{3}}{2} \frac{1}{\sqrt{1-\frac{3}{4}}} - \frac{\pi}{3}$$

$$\boxed{= \sqrt{3} - \frac{\pi}{3}}$$

Exercise

Evaluate $\displaystyle\int_0^{\sqrt{3}/2} \frac{1}{\left(1-x^2\right)^{5/2}} dx$

Solution

$$\int_0^{\sqrt{3}/2} \frac{1}{\left(1-x^2\right)^{5/2}} dx = \int_0^{\sqrt{3}/2} \frac{1}{\cos^5 \theta} \cos\theta \, d\theta$$

$x = \sin\theta \qquad \sqrt{1-x^2} = \cos\theta$

$dx = \cos\theta d\theta$

$$= \int_0^{\sqrt{3}/2} \sec^4 \theta \, d\theta$$

$$= \int_0^{\sqrt{3}/2} \left(1 + \tan^2 \theta\right) \sec^2 \theta \, d\theta$$

$$= \int_0^{\sqrt{3}/2} \left(1 + \tan^2 \theta\right) \, d\left(\tan\theta\right)$$

$$= \tan\theta + \frac{1}{3} \tan^3 \theta \Big|_0^{\sqrt{3}/2}$$

$$= \frac{x}{\sqrt{1-x^2}} + \frac{1}{3} \frac{x^3}{\left(1-x^2\right)^{3/2}} \Big|_0^{\sqrt{3}/2}$$

$$= \frac{\sqrt{3}}{2} \frac{1}{\sqrt{1-\frac{3}{4}}} + \frac{\sqrt{3}}{8} \frac{1}{\left(\frac{1}{4}\right)^{3/2}}$$

$$= \sqrt{3} + \sqrt{3}$$

$$\boxed{= 2\sqrt{3}}$$

Exercise

Evaluate $\displaystyle\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx$

Solution

$$\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx = \int_0^3 \frac{27\tan^3\theta}{3\sec\theta} 3\sec^2\theta\; d\theta$$

$x = 3\tan\theta \qquad \sqrt{x^2+9} = 3\sec\theta$

$dx = 3\sec^2\theta\; d\theta$

$$= 27\int_0^3 \tan^2\theta \tan\theta \sec\theta\; d\theta$$

$$= 27\int_0^3 \left(\sec^2\theta - 1\right)\; d\left(\sec\theta\right)$$

$$= 27\left(\tfrac{1}{3}\sec^3\theta - \sec\theta\right)\Big|_0^3$$

$$= 9\sqrt{x^2+9}\left(\frac{x^2+9}{27} - 1\right)\Bigg|_0^3$$

$$= \tfrac{1}{3}\sqrt{x^2+9}\left(x^2 - 18\right)\Big|_0^3$$

$$= -9\sqrt{2} + 18$$

Exercise

Evaluate $\displaystyle\int_0^{3/5} \sqrt{9-25x^2}\; dx$

Solution

$$\int_0^{3/5} \sqrt{9-25x^2}\; dx = \frac{9}{5}\int_0^{3/5} \cos^2\theta\; d\theta$$

$5x = 3\sin\theta \qquad \sqrt{9-25x^2} = 3\cos\theta$

$dx = \frac{3}{5}\cos\theta d\theta$

$$= \frac{9}{10}\int_0^{3/5} \left(1+\cos 2\theta\right)\; d\theta$$

$$= \frac{9}{10}\left(\theta + \frac{1}{2}\sin 2\theta\right)_0^{3/5}$$

$\sin 2\theta = 2\sin\theta\cos\theta = 2\frac{5x}{3}\frac{5\sqrt{9-25x^2}}{3}$

$$= \frac{9}{10}\left(\arcsin\frac{5x}{3} + \frac{25}{9}x\sqrt{9-25x^2}\right)_0^{3/5}$$

$$= \frac{9\pi}{20}$$

Exercise

Evaluate $\displaystyle\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\,dx$

Solution

$$\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\,dx = \int_4^6 \frac{9\sec^2\theta}{3\tan\theta}\,(3\sec\theta\tan\theta)\,d\theta$$

$x = 3\sec\theta \qquad \sqrt{x^2-9} = 3\tan\theta$

$dx = 3\sec\theta\tan\theta\ d\theta$

$$= 9\int_4^6 \sec^3\theta\ d\theta$$

$$= \frac{9}{2}\Big[\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\Big]_4^6$$

$$= \frac{9}{2}\left[\frac{x}{3}\frac{\sqrt{x^2-9}}{3} + \ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right|\right]_4^6$$

$$= \frac{9}{2}\left(2\sqrt{3} + \ln\left(2+\sqrt{3}\right) - \frac{4\sqrt{7}}{9} - \ln\left(\frac{4+\sqrt{7}}{3}\right)\right)$$

$$\boxed{= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{6+3\sqrt{3}}{4+\sqrt{7}}\right)}$$

$u = \sec x \qquad dv = \sec^2 x\,dx$

$du = \sec x\tan x\,dx \qquad v = \tan x$

$$\int \sec^3 x\,dx = \sec x\tan x - \int \tan x\left(\sec x\tan x\,dx\right)$$

$$= \sec x\tan x - \int \tan^2 x\sec x\,dx$$

$$= \sec x\tan x - \int \left(\sec^2 x - 1\right)\sec x\ dx$$

$$= \sec x\tan x - \int \sec^3 x\ dx + \int \sec x\ dx$$

$$2\int \sec^3 x\ dx = \sec x\tan x + \ln\left|\sec x + \tan x\right|$$

$$\int \sec^3 x\ dx = \boxed{\frac{1}{2}\sec x\tan x + \frac{1}{2}\ln\left|\sec x+\tan x\right|}$$

Exercise

Evaluate $\displaystyle\int_{\sqrt{3}}^2 \frac{\sqrt{x^2-3}}{x}dx$

Solution

$$\int_{\sqrt{3}}^2 \frac{\sqrt{x^2-3}}{x}dx = \int_{\sqrt{3}}^2 \frac{\sqrt{3}\tan\theta}{\sqrt{3}\sec\theta}\left(\sqrt{3}\sec\theta\tan\theta\right)d\theta$$

$x = \sqrt{3}\sec\theta \qquad \sqrt{x^2-3} = \sqrt{3}\tan\theta$

$dx = \sqrt{3}\sec\theta\tan\theta\ d\theta$

$$= \sqrt{3}\int_{\sqrt{3}}^2 \tan^2\theta\ d\theta$$

$$= \sqrt{3}\int_{\sqrt{3}}^2 \left(\sec^2\theta - 1\right)\ d\theta$$

$$= \sqrt{3}\left(\tan\theta - \theta\right)\Big|_{\sqrt{3}}^2$$

$$= \sqrt{3}\left(\frac{\sqrt{x^2-3}}{\sqrt{3}} - \operatorname{arcsec}\frac{x}{\sqrt{3}}\right)\Bigg|_{\sqrt{3}}^2$$

$$= \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right)$$

$$\boxed{= 1 - \frac{\pi\sqrt{3}}{6}}$$

Exercise

Evaluate $\displaystyle\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx$

Solution

$$\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx = \int_1^4 \frac{\sqrt{(x+2)^2-9}}{x+2}\,dx$$

$x+2 = 3\sec\theta \qquad \sqrt{(x+2)^2-9} = 3\tan\theta$

$dx = 3\sec\theta\tan\theta\; d\theta$

$$= \int_1^4 \frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta)\; d\theta = 3\int_1^4 \tan^2\theta\; d\theta$$

$$= 3\int_1^4 \left(\sec^2\theta - 1\right)\; d\theta$$

$$= 3(\tan\theta - \theta)\Big|_1^4$$

$$= \sqrt{(x+2)^2-9} - 3\sec^{-1}\left(\frac{x+2}{3}\right)\Big|_1^4$$

$$= \sqrt{27} - 3\sec^{-1}(2) + 3\sec^{-1}(1)$$

$$\boxed{= 3\sqrt{3} - \pi}$$

$\theta = \sec^{-1}\left(\frac{x+2}{3}\right)$

$$\begin{cases} x = 4 \quad \to \quad \theta = \sec^{-1}(2) = \frac{\pi}{3} \\ x = 1 \quad \to \quad \theta = \sec^{-1}(1) = 0 \end{cases}$$

$$= 3(\tan\theta - \theta)\Big|_0^{\pi/3}$$

$$\boxed{= 3\sqrt{3} - \pi}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}}$

Solution

$$\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^{3/2}$$

$$\boxed{= \frac{\pi}{6}}$$

Exercise

Evaluate the integral $\displaystyle\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^x dx}{\left(1+e^{2x}\right)^{3/2}}$

Solution

$$\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^x dx}{\left(1+e^{2x}\right)^{3/2}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{\tan\theta}{\left(\sec^2\theta\right)^{3/2}} \frac{\sec^2\theta}{\tan\theta} d\theta$$

$$e^x = \tan\theta \to x = \ln(\tan\theta)$$

$$dx = \frac{\sec^2\theta}{\tan\theta} d\theta$$

$$\tan^{-1}\left(\frac{3}{4}\right) < \theta < \tan^{-1}\left(\frac{4}{3}\right)$$

$$1+e^{2x} = 1+\tan^2\theta = \sec^2\theta$$

$$= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{\sec^2\theta}{\sec^3\theta} d\theta$$

$$= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{1}{\sec\theta} d\theta$$

$$= \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \cos\theta d\theta$$

$$= \sin\theta \Big|_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)}$$

$$= \sin\left(\tan^{-1}(3/4)\right) - \sin\left(\tan^{-1}(4/3)\right)$$

$$= \frac{4}{5} - \frac{3}{5}$$

$$= \frac{1}{5}$$

Exercise

Evaluate the integral $\displaystyle\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$

Solution

$$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_0^{\pi/4} \frac{e^{\tan\theta}\sec^2\theta}{e^{\tan\theta}\sec\theta} d\theta$$

$$y = e^{\tan\theta} \quad 1 \le y \le e \to \quad 0 \le \theta = \tan^{-1}(\ln y) \le \frac{\pi}{4}$$

$$dy = e^{\tan\theta}\sec^2\theta d\theta$$

$$\sqrt{1+(\ln y)^2} = \sqrt{1+\tan^2\theta} = \sec\theta$$

$$= \int_0^{\pi/4} \sec\theta d\theta$$

$$= \Big[\ln\left|\sec\theta + \tan\theta\right|\Big]_0^{\pi/4}$$

$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln\left|\sec 0 + \tan 0\right|$$

$$\boxed{= \ln\left(1+\sqrt{2}\right)}$$

Exercise

Evaluate $\displaystyle\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}}$

Solution

Let: $u = 3y \quad \Rightarrow \quad du = 3dy \quad \rightarrow \quad \frac{du}{3} = dy$

$$\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}} = \int_{-2/3}^{-\sqrt{2}/3} \frac{\frac{du}{3}}{\frac{u}{3}\sqrt{u^2-1}}$$

$$= \int_{-2/3}^{-\sqrt{2}/3} \frac{du}{u\sqrt{u^2-1}} \qquad\qquad \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|$$

$$= \sec^{-1}|3y|\Big|_{-2/3}^{-\sqrt{2}/3}$$

$$= \sec^{-1}\left|-\sqrt{2}\right| - \sec^{-1}|-2|$$

$$= \frac{\pi}{4} - \frac{\pi}{3}$$

$$\boxed{= -\frac{\pi}{12}}$$

Exercise

Evaluate $\displaystyle\int_0^2 \sqrt{1+4x^2}\,dx$

Solution

$$\int_0^2 \sqrt{1+4x^2}\,dx = \frac{1}{2}\int_0^2 \sec^3\theta\, d\theta$$

$$2x = \tan\theta \qquad \sqrt{1+4x^2} = \sec\theta$$

$$dx = \frac{1}{2}\sec^2\theta d\theta$$

$$\int \sec^3 x dx = \sec x \tan x - \int \tan x\left(\sec x \tan x dx\right)$$

$$u = \sec x \qquad dv = \sec^2 x dx$$

$$du = \sec x \tan x dx \qquad v = \tan x$$

$$= \sec x \tan x - \int \tan^2 x \sec x dx$$

$$= \sec x \tan x - \int \left(\sec^2 x - 1\right)\sec x\, dx$$

$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$2\int \sec^3 x\, dx = \sec x \tan x + \ln\left|\sec x + \tan x\right|$$

$$\int \sec^3 x\, dx = \underline{\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right|}$$

$$\int_0^2 \sqrt{1+4x^2}\,dx = \frac{1}{4}\left(\sec\theta \tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)\Big|_0^2$$

$$= \frac{1}{4}\left(2x\sqrt{1+4x^2} + \ln\left|2x + \sqrt{1+4x^2}\right|\right)\Big|_0^2$$

$$= \frac{1}{4}\left(4\sqrt{17} + \ln\left|4+\sqrt{17}\right|\right)$$

$$\underline{= \sqrt{17} + \frac{1}{4}\ln\left(4+\sqrt{17}\right)}$$

Exercise

Consider the region bounded by the graph $y = \sqrt{x\tan^{-1} x}$ and $y = 0$ for $0 \le x \le 1$. Find the volume of the solid formed by revolving this region about the x-axis.

Solution

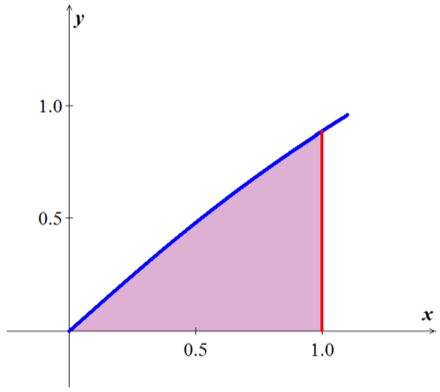
$$V = \pi \int_0^1 \left(\sqrt{x\tan^{-1} x}\right)^2 dx$$

$$= \pi \int_0^1 x\tan^{-1} x dx$$

$u = \tan^{-1} x \qquad dv = xdx$

$du = \dfrac{1}{x^2+1}dx \qquad v = \dfrac{1}{2}x^2$

$$V = \pi\left(\frac{1}{2}\left[x^2 \tan^{-1} x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}dx\right)$$



$$= \frac{\pi}{2}\left(\left(1\tan^{-1}1 - 0\right) - \int_0^1 \left(1 - \frac{1}{1+x^2}\right)dx\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4} - \int_0^1 dx + \int_0^1 \frac{1}{1+x^2}dx\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4} - [x]_0^1 + \left[\tan^{-1} x\right]_0^1\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4} - 1 + \tan^{-1} 1\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4} - 1 + \frac{\pi}{4}\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi}{2} - 1\right)$$

$$= \frac{\pi^2}{4} - \frac{\pi}{2}$$

Exercise

Use two approach to show that the area of a cap (or segment) of a circle of radius r subtended by an angle θ is given by

$$A_{seg} = \frac{1}{2}r^2(\theta - \sin\theta)$$

a) Find the area using geometry (no calculus).

b) Find the area using calculus

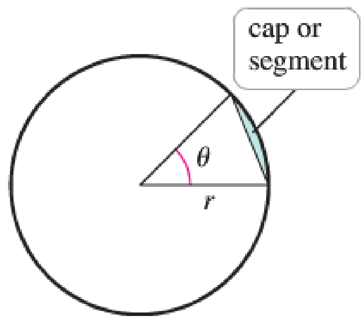
Solution

a) Area of a segment (*cap*) = Area of a sector ***minus*** Area of the isosceles triangle

The area of a sector: $A = \frac{1}{2}\theta r^2$

Area of the isosceles triangle: $A = \frac{1}{2}r^2 \sin\theta$

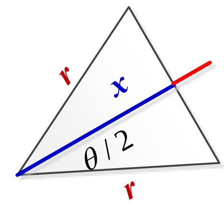
$$A_{seg} = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta = \frac{1}{2}r^2(\theta - \sin\theta)$$



b) $0 \le \theta \le \pi \quad \rightarrow \quad 0 \le \frac{\theta}{2} \le \frac{\pi}{2}$

$x = r\cos\frac{\alpha}{2} \quad \rightarrow \quad dx = -\frac{1}{2}r\sin\frac{\alpha}{2}\, d\alpha$

$\sqrt{r^2 - x^2} = r\sin\frac{\alpha}{2}$



$$A_{cap} = 2\int_{r\cos\theta/2}^{r} \sqrt{r^2 - x^2}\, dx$$

$$= 2\int_{\theta}^{0} \left(r\sin\frac{\alpha}{2}\right)\left(-\frac{1}{2}r\sin\frac{\alpha}{2}\right)d\alpha$$

$$= r^2\int_0^{\theta}\left(\sin^2\frac{\alpha}{2}\right)d\alpha$$

$$= \frac{1}{2}r^2\int_0^{\theta}(1 - \cos\alpha)\, d\alpha$$

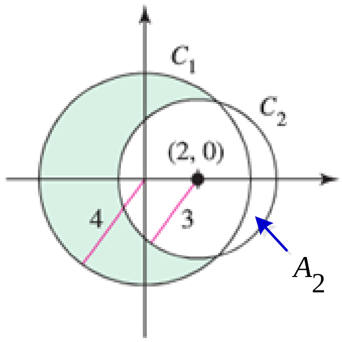
$$= \frac{1}{2}r^2(\alpha - \sin\alpha)\Big|_0^{\theta}$$

$$= \frac{1}{2}r^2(\theta - \sin\theta)$$

Exercise

A lune is a crescent-shaped region bounded by the arcs of two circles. Let C_1 be a circle of radius 4 centered at the origin. Let C_2 be a circle of radius 3 centered at the point $(2,\ 0)$. Find the area of the lune that lies inside C_1 and outside C_2.

Solution



$$C_1 \quad \to \quad x^2+y^2=16 \ \Rightarrow\ y^2=16-x^2$$

$$C_2 \quad \to \quad (x-2)^2+y^2=9 \ \Rightarrow\ y^2=9-(x-2)^2$$

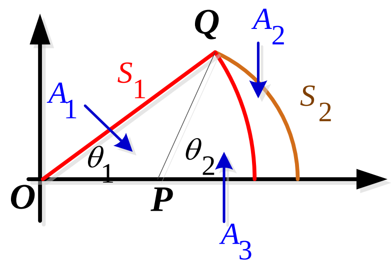
$$16-x^2=9-x^2+4x-4$$

$$11=4x \quad \to \quad \underline{x=\frac{11}{4}|} \ \Rightarrow \lfloor y=\pm\frac{\sqrt{135}}{4}=\pm\frac{3\sqrt{15}}{4}\rfloor$$

For **sector** C_1: $\quad \theta_1=\tan^{-1}\frac{y}{x}=\tan^{-1}\frac{3\sqrt{15}}{11}$

$$\text{Area: } S_1=\frac{1}{2}r^2\theta_1=8\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right)$$

For **sector** C_2: $\quad x_2=\frac{11}{4}-2=\frac{3}{4}$



$$\theta_2=\tan^{-1}\frac{y}{x_2}=\tan^{-1}\sqrt{15}$$

$$\text{Area: } S_2=\frac{1}{2}r_2^2\theta_2=\frac{9}{2}\tan^{-1}\left(\sqrt{15}\right)$$

$OQ=4, \quad PQ=3, \quad OP=2$

$$Area(\Delta APQ)=A_1=\frac{1}{2}(4)(2)\sin\theta_1=4\frac{y}{4}=\underline{\frac{3\sqrt{15}}{4}|}$$

$$A_2=S_2-S_1+A_1$$

$$=\underline{\frac{9}{2}\tan^{-1}\left(\sqrt{15}\right)-8\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right)+\frac{3\sqrt{15}}{4}|}$$

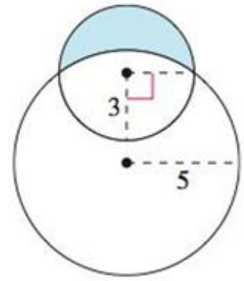
$$A_{lune}=A_{C_1}-A_{C_2}+2A_2$$

$$=16\pi-9\pi+9\tan^{-1}\left(\sqrt{15}\right)-16\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right)+\frac{3\sqrt{15}}{2}$$

$$\underline{=7\pi+9\tan^{-1}\left(\sqrt{15}\right)-16\tan^{-1}\left(\frac{3\sqrt{15}}{11}\right)+\frac{3\sqrt{15}}{2}|} \quad \underline{\approx 26.66|} \quad unit^2$$

Exercise

The crescent-shaped region bounded by two circles forms a lune. Find the area of the lune given that the radius of the smaller circle is 3 and the radius of the larger circle is 5.



Solution

$$\text{Large Circle: } x^2 + y^2 = 25 \quad \to y = \sqrt{25 - x^2}$$

$$\text{Small Circle: } r = 3 \quad \to \quad y = \sqrt{25-9} \; \underline{= 4}$$

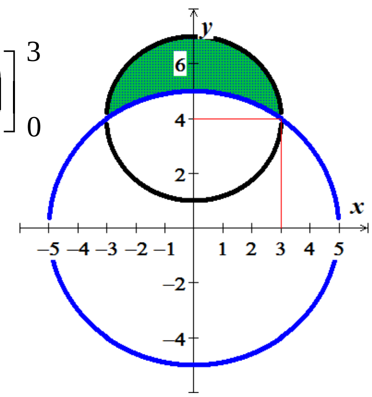
$$x^2 + (y-4)^2 = 9 \quad \to y = 4 + \sqrt{9 - x^2}$$

$$A = 2\int_0^3 \left(4 + \sqrt{9-x^2} - \sqrt{25-x^2}\right)dx$$

$$= 2\left[4x + \frac{1}{2}\left(9\arcsin\left(\frac{x}{3}\right) + x\sqrt{9-x^2}\right) - \frac{1}{2}\left(25\arcsin\left(\frac{x}{5}\right) + x\sqrt{25-x^2}\right)\right]_0^3$$

$$= 2\left[12 + \frac{1}{2}\left(9\frac{\pi}{2}\right) - \frac{1}{2}\left(25\arcsin\left(\frac{3}{5}\right) + 12\right)\right]$$

$$= 12 + \frac{9\pi}{2} - 25\arcsin\left(\frac{3}{5}\right)$$



Exercise

The surface of a machine part is the region between the graphs of $y = |x|$ *and* $x^2 + (y-k)^2 = 25$

a) Find k when the circle is tangent to the graph of $y = |x|$

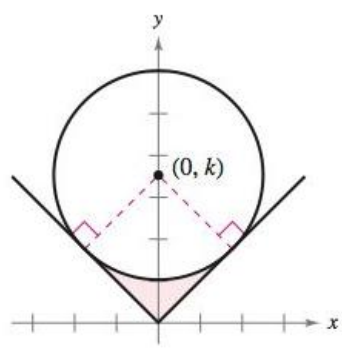
b) Find the area of the surface of the machine part.

c) Find the area of the surface of the machine part as a function of the radius r of the circle.

Solution

a) $x^2 + (y-k)^2 = 25 \quad \to \quad \underline{r = 5}$

$k^2 = 5^2 + 5^2 = 50 \quad \to \quad \underline{k = 5\sqrt{2}}$



b) $Area = area\ square - \frac{1}{4}(area\ circle)$

$$= 5^2 - \frac{1}{4}\pi 5^2$$

$$= 25\left(1 - \frac{\pi}{4}\right)\ unit^2$$

c) $Area = area\ square - \frac{1}{4}(area\ circle)$

$$= r^2 - \frac{1}{4}\pi r^2$$

$$= r^2\left(1 - \frac{\pi}{4}\right)\ unit^2$$

Exercise

Consider the function $f(x)=\left(9+x^2\right)^{-1/2}$ and the region ***R*** on the interval [0, 4].

a) Find the area of *R*.

b) Find the volume of the solid generated when *R* is revolved about the *x-axis*.

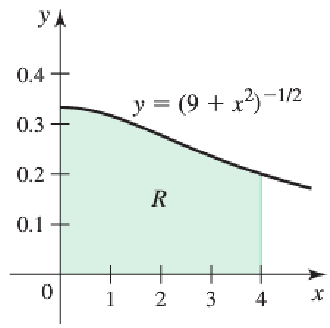
c) Find the volume of the solid generated when *R* is revolved about the *y-axis*.

Solution

a)
$$A=\int_0^4 \frac{dx}{\sqrt{9+x^2}}$$

$x=3\tan\theta \ \rightarrow\ dx=3\sec^2\theta\, d\theta$

$\sqrt{9+x^2}=3\sec\theta$



$$=\int_0^4 \frac{3\sec^2\theta\, d\theta}{3\sec\theta}$$

$$=\int_0^4 \sec\theta\, d\theta$$

$$=\ln\left|\sec\theta+\tan\theta\right|\Big|_0^4$$

$$=\ln\left|\frac{\sqrt{9+x^2}}{3}+\frac{x}{3}\right|\Big|_0^4$$

$$=\ln\left(\frac{5}{3}+\frac{4}{3}\right)-0$$

$$=\ln 3 \quad unit^2$$

b)
$$V=\pi\int_0^4 \frac{dx}{9+x^2}$$

$x=3\tan\theta \ \rightarrow\ dx=3\sec^2\theta\, d\theta$

$9+x^2=9\sec^2\theta$

$$=\pi\int_0^4 \frac{3\sec^2\theta\, d\theta}{9\sec^2\theta}$$

$$=\frac{\pi}{3}\int_0^4 d\theta$$

$$=\frac{\pi}{3}\theta\Big|_0^4$$

$$=\frac{\pi}{3}\tan^{-1}\frac{x}{3}\Big|_0^4$$

$$=\frac{\pi}{3}\tan^{-1}\frac{4}{3} \quad unit^3$$

c)
$$V=2\pi\int_0^4 \frac{x}{\sqrt{9+x^2}}dx$$

$d\left(9+x^2\right)=2xdx$

$$= \pi \int_0^4 \left(9 + x^2\right)^{-1/2} d\left(9 + x^2\right)$$

$$= 2\pi \left(9 + x^2\right)^{1/2} \Big|_0^4$$

$$= 2\pi (5 - 3)$$

$$\underline{= 4\pi|} \quad unit^3$$

Exercise

A total of $\boldsymbol{Q}$ is distributed uniformly on a line segment of length $2L$ along the y-*axis*. The x-component of the electric field at a point $(a,\ 0)$ is given by

$$E_x = \frac{kQa}{2L} \int_{-L}^{L} \frac{dy}{\left(a^2 + y^2\right)^{3/2}}$$

Where k is a physical constant and $a > 0$

a) Confirm that $E_x(a) = \dfrac{kQ}{a\sqrt{a^2 + L^2}}$

b) Letting $\rho = \dfrac{Q}{2L}$ be the charge density on the line segment, show that if $L \to \infty$, *then* $E_x = \dfrac{2k\rho}{a}$

Solution

a) $$E_x = \frac{kQa}{2L} \int_{-L}^{L} \frac{dy}{\left(a^2 + y^2\right)^{3/2}}$$

$$y = a\tan\theta \ \to dy = a\sec^2\theta\, d\theta$$

$$\sqrt{a^2 + y^2} = a\sec\theta$$

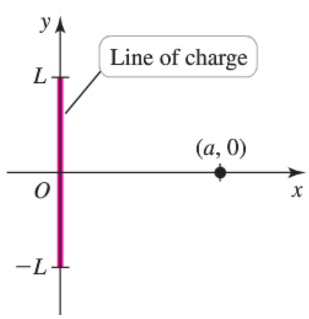
$$= \frac{kQa}{2L} \int_{-L}^{L} \frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta}$$

$$= \frac{kQ}{2aL} \int_{-L}^{L} \frac{d\theta}{\sec\theta}$$

$$= \frac{kQ}{2aL} \int_{-L}^{L} \cos\theta d\theta$$

$$= \frac{kQ}{2aL} \sin\theta \Big|_{-L}^{L}$$

$$= \frac{kQ}{2aL} \left(\frac{y}{\sqrt{a^2 + y^2}}\right) \Big|_{-L}^{L}$$



$$= \frac{kQ}{2aL}\left(\frac{2L}{\sqrt{a^2+L^2}}\right)$$

$$\boxed{= \frac{kQ}{a\sqrt{a^2+L^2}}}$$

b) Let $\rho = \frac{Q}{2L} \quad \rightarrow \quad Q = 2\rho L$

$$E_x(a) = \frac{kQa}{2L} \lim_{L\to\infty} \int_{-L}^{L} \frac{dy}{\left(a^2+y^2\right)^{3/2}}$$

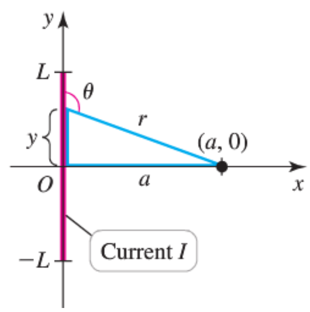
$$= \frac{kQa}{2L} \lim_{L\to\infty}\left(\frac{2L}{a^2\sqrt{a^2+L^2}}\right)$$

$$= k\rho a \ \frac{2}{a^2}$$

$$\boxed{= \frac{2k\rho}{a}}$$

Exercise

A long, straight wire of length $2L$ on the *y-axis* carries a current *I*. according to the Biot-Savart Law, the magnitude of the field due to the current at a point $(a,\ 0)$ is given by



$$B(a) = \frac{\mu_0 I}{4\pi}\int_{-L}^{L} \frac{\sin\theta}{r^2}\,dy$$

Where μ_0 is a physical constant, $a > 0$, and *θ*, *r*, and *y* are related to the figure

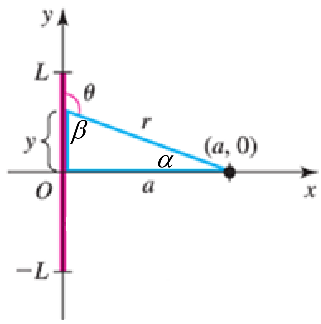
a) Show that the magnitude of the magnetic field at $(a,\ 0)$ is

$$B(a) = \frac{\mu_0 IL}{2\pi a\sqrt{a^2+L^2}}$$

b) What is the magnitude of the magnetic field at $(a,\ 0)$ due to an infinitely long wire $(L\to\infty)$?

Solution

a) $\beta = \pi - \theta \quad \& \quad \alpha + \beta = \frac{\pi}{2}$



$$\sin\theta = \sin(\pi-\beta) = \sin\left(\frac{\pi}{2}+\alpha\right) = \cos\alpha = \frac{a}{r}$$

$$r^2 = y^2 + a^2$$

$$\frac{\sin\theta}{r^2} = \frac{a}{r^3} = \frac{a}{\left(a^2+y^2\right)^{3/2}}$$

$$B(a)=\frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{\sin\theta}{r^2}dy$$

$$y=a\tan u \quad\rightarrow\quad dy=a\sec^2 u\ du$$

$$\sqrt{a^2+y^2}=a\sec u$$

$$=\frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{a}{\left(a^2+y^2\right)^{3/2}}dy$$

$$=\frac{\mu_0 I}{2\pi}\int_{0}^{L}\frac{a^2\sec^2 u\ du}{a^3\sec^3 u}$$

$$=\frac{\mu_0 I}{2a\pi}\int_{0}^{L}\frac{1}{\sec u}du$$

$$=\frac{\mu_0 I}{2a\pi}\int_{0}^{L}\cos u\ du$$

$$=\frac{\mu_0 I}{2a\pi}\sin u\Big|_0^L$$

$$=\frac{\mu_0 I}{2a\pi}\frac{y}{\sqrt{a^2+y^2}}\Bigg|_0^L$$

$$\boxed{=\frac{\mu_0 IL}{2a\pi\sqrt{a^2+L^2}}}$$

b) $$\lim_{L\to\infty}B(a)=\lim_{L\to\infty}\frac{\mu_0 IL}{2a\pi\sqrt{a^2+L^2}}$$

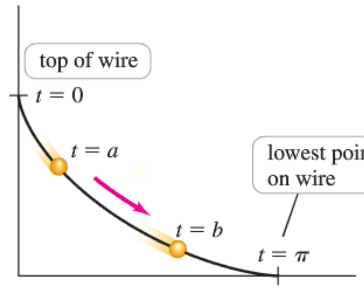
$$=\frac{\mu_0 I}{2a\pi}\lim_{L\to\infty}\frac{L}{\sqrt{a^2+L^2}}$$

$$\lim_{L\to\infty}\frac{L}{\sqrt{a^2+L^2}}=\lim_{L\to\infty}\frac{L}{\sqrt{L^2}}=1$$

$$\boxed{=\frac{\mu_0 I}{2a\pi}}$$

Exercise

The cycloid is the curve traced by a point on the rim of a rolling wheel. Imagine a wire shaped like an inverted cycloid.



A bead sliding down this wire without friction has some remarkable properties. Among all wire shapes, the cycloid is the shape that produces the fastest descent time. It can be shown that the descent time between any two points $0 \le a < b \le \pi$ on the curve is

$$\textit{descent time} = \int_a^b \sqrt{\frac{1-\cos t}{g(\cos a - \cos t)}}\, dt$$

Where g is the acceleration due to gravity, $t = 0$ corresponds to the top of the wire, and $t = \pi$ corresponds to the lowest point on the wire.

a) Find the descent time on the interval $[a,\ b]$.

b) Show that when $b = \pi$, the descent time is the same for all values of a; that is, the descent time to the bottom of the wire is the same for all starting points.

Solution

a) $$\int_a^b \sqrt{\frac{1-\cos t}{g(\cos a - \cos t)}}\, dt = \int_a^b \sqrt{\frac{(1-\cos t)(1+\cos t)}{g(\cos a - \cos t)(1+\cos t)}}\, dt$$

$$= \frac{1}{\sqrt{g}} \int_a^b \sqrt{\frac{\left(1-\cos^2 t\right)}{\cos a + (\cos a - 1)\cos t - \cos^2 t}}\, dt$$

$$= \frac{1}{\sqrt{g}} \int_a^b \frac{\sin t}{\sqrt{\cos a + \left(\frac{\cos a - 1}{2}\right)^2 - \left(\frac{\cos a - 1}{2}\right)^2 + (\cos a - 1)\cos t - \cos^2 t}}\, dt$$

$$= \frac{1}{\sqrt{g}} \int_a^b \frac{\sin t}{\sqrt{\cos a + \left(\frac{\cos a - 1}{2}\right)^2 - \left(\left(\frac{\cos a - 1}{2}\right) - \cos t\right)^2}}\, dt$$

Let: $v = \sqrt{\cos a + \left(\frac{\cos a - 1}{2}\right)^2}$

$$= \frac{1}{2}\sqrt{4\cos a + \cos^2 a - 2\cos a + 1}$$

$$= \frac{1}{2}(\cos a + 1)$$

$$\frac{\cos a - 1}{2} - \cos t = v \sin\theta \quad \to \quad \sin t\ dt = v\cos\theta\ d\theta$$

$$\sqrt{v - \left(\left(\frac{\cos a - 1}{2}\right) - \cos t\right)^2} = v\cos\theta$$

$$= \frac{1}{\sqrt{g}} \int_a^b \frac{v\cos\theta}{v\cos\theta}\, d\theta$$

$$= \frac{1}{\sqrt{g}}\, \theta \,\Big|_a^b$$

$$\theta = \sin^{-1}\left(\frac{\cos a - 1 - 2\cos t}{2}\,\frac{2}{1+\cos a}\right)$$

$$= \frac{1}{\sqrt{g}} \sin^{-1}\left(\frac{\cos a - 1 - 2\cos t}{1+\cos a}\right) \Bigg|_a^b$$

$$= \frac{1}{\sqrt{g}} \left(\sin^{-1}\left(\frac{\cos a - 1 - 2\cos b}{1+\cos a}\right) - \sin^{-1}(-1) \right)$$

$$\boxed{= \frac{1}{\sqrt{g}} \left(\sin^{-1}\left(\frac{\cos a - 1 - 2\cos b}{1+\cos a}\right) + \frac{\pi}{2} \right)}$$

b) $\frac{1}{\sqrt{g}} \left(\sin^{-1}\left(\frac{\cos a - 1 - 2\cos b}{1+\cos a}\right) + \frac{\pi}{2} \right)\Bigg|_{b=\pi} = \frac{1}{\sqrt{g}} \left(\sin^{-1}\left(\frac{\cos a - 1 + 2}{1+\cos a}\right) + \frac{\pi}{2} \right)$

$$= \frac{1}{\sqrt{g}} \left(\sin^{-1}(1) + \frac{\pi}{2} \right)$$

$$\boxed{= \frac{\pi}{\sqrt{g}}}$$

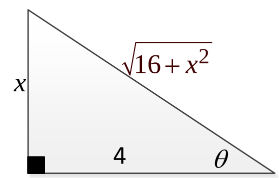
Exercise

Find the area of the region bounded by the curve $f(x) = \left(16 + x^2\right)^{-3/2}$ and the x-*axis* on the interval $[0,\ 3]$

Solution

$$A = \int_0^3 \frac{dx}{\left(16+x^2\right)^{3/2}}$$

$x = 4\tan\theta \quad \to dx = 4\sec^2\theta\, d\theta \qquad 16 + x^2 = 16\sec^2\theta$



$$= \int_0^3 \frac{4\sec^2\theta d\theta}{\left(16\sec^2\theta\right)^{3/2}}$$

$$= \int_0^3 \frac{4\sec^2\theta}{4^3\sec^3\theta} d\theta$$

$$= \frac{1}{16} \int_0^3 \cos\theta\, d\theta$$

$$= \frac{1}{16} \sin\theta \Big|_0^3$$

$$= \frac{1}{16} \frac{x}{\sqrt{16+x^2}} \Bigg|_0^3$$

$$= \frac{1}{16}\left(\frac{3}{5} - 0\right)$$

$$\boxed{= \frac{3}{80}\ \ unit^2}$$

Exercise

Find the length of the curve $y = ax^2$ from $x = 0$ to $x = 10$, where $a > 0$ is a real number.

Solution

$$1+(y')^2 = 1+(2ax)^2$$

$$L = \int_0^{10} \sqrt{1+4a^2x^2}\, dx$$

$$= \int_0^{10} 2a\sqrt{\frac{1}{4a^2}+x^2}\, dx \qquad x=\frac{1}{2a}\tan\theta \quad \frac{1}{4a^2}+x^2=\frac{1}{4a^2}\sec^2\theta$$

$$= \int_0^{10} 2a\frac{1}{2a}\sec\theta\frac{1}{4a^2}\sec^2\theta\, d\theta \qquad dx = \frac{1}{4a^2}\sec^2\theta\, d\theta$$

$$= \frac{1}{2a}\int_0^{10} \sec^3\theta\, d\theta \qquad u=\sec x \quad dv=\sec^2 x dx \quad du=\sec x\tan x dx \quad v=\tan x$$

$$\int \sec^3 x dx = \sec x\tan x - \int \tan x(\sec x\tan x dx)$$

$$= \sec x\tan x - \int \tan^2 x\sec x dx$$

$$= \sec x\tan x - \int \left(\sec^2 x - 1\right)\sec x\, dx$$

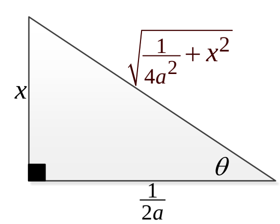
$$= \sec x\tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$\int \sec^3 x\, dx + \int \sec^3 x\, dx = \sec x\tan x - \int \sec^3 x\, dx + \int \sec^3 x\, dx + \int \sec x\, dx$$

$$2\int \sec^3 x\, dx = \sec x\tan x + \int \sec x\, dx$$

$$= \sec x\tan x + \ln\left|\sec x + \tan x\right|$$

$$= \frac{1}{4a}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)\Big|_0^{10}$$



$$= \frac{1}{4a}\left(2a\sqrt{\frac{1}{4a^2}+x^2}\,(2ax) + \ln\left|\sqrt{1+4a^2x^2} + 2ax\right|\right)\Big|_0^{10}$$

$$= \frac{1}{4a}\left((2ax)\sqrt{1+4a^2x^2} + \ln\left|\sqrt{1+4a^2x^2} + 2ax\right|\right)\Big|_0^{10}$$

$$= \frac{1}{4a}\left((20a)\sqrt{1+400a^2} + \ln\left|\sqrt{1+400a^2} + 20a\right|\right)$$

Exercise

Find the arc length of the graph of $f(x)=\frac{1}{2}x^2$ from $x=0$ to $x=1$

Solution

$$1+(f')^2=1+x^2$$

$$L=\int_0^1\sqrt{1+x^2}\,dx \qquad x=\tan\theta \quad \sqrt{x^2+1}=\sec\theta \quad dx=\sec^2\theta\,d\theta$$

$$=\int_0^1\sec^3\theta\,d\theta$$

$$=\frac{1}{2}\left(\sec\theta\tan\theta+\ln\left|\sec\theta+\tan\theta\right|\right)\Big|_0^1$$

$$=\frac{1}{2}\left(x\sqrt{x^2+1}+\ln\left|x+\sqrt{x^2+1}\right|\right)\Big|_0^1$$

$$=\frac{1}{2}\left(\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right)$$

Exercise

A projectile is launched from the ground with an initial speed V at an angle θ from the horizontal. Assume that the *x-axis* is the horizontal ground and y is the height above the ground. Neglecting air resistance and letting g be the acceleration due to gravity, it can be shown that the trajectory of the projectile is given by

$$y=-\frac{1}{2}kx^2+y_{max} \quad \text{where } k=\frac{g}{(V\cos\theta)^2}$$

$$\text{and} \qquad y_{max}=\frac{(V\sin\theta)^2}{2g}$$

a) Note that the high point of the trajectory occurs at $(0,\ y_{max})$. If the projectile is on the ground at $(-a,\ 0)$ and $(a,\ 0)$, what is a?

b) Show that the length of the trajectory (arc length) is $2\int_0^a\sqrt{1+k^2x^2}\,dx$

c) Evaluate the arc length integral and express your result in the terms of V, g, and θ.

d) For fixed value of V and g, show that the launch angle θ that maximizes the length of the trajectory satisfies $(\sin\theta)\ln(\sec\theta+\tan\theta)=1$

Solution

a) At $(\pm a,\ 0)\rightarrow\ y=0=-\frac{1}{2}ka^2+y_{max}$

$$a^2=\frac{2}{k}y_{max} \quad\Rightarrow\quad a=\sqrt{\frac{2y_{max}}{k}}$$

b) $y' = -kx \;\Rightarrow\; 1 + (y')^2 = 1 + k^2x^2$

$$L = \int_{-a}^{a} \sqrt{1+k^2x^2}\, dx$$

since $y(x)$ is an even function

$$= 2\int_0^a \sqrt{1+k^2x^2}\, dx$$

c) $L = 2\int_0^a \sqrt{1+k^2x^2}\, dx$

$x = \frac{1}{k}\tan\theta \;\Rightarrow\; dx = \frac{1}{k}\sec^2\theta\, d\theta; \quad 1+k^2x^2 = \sec^2\theta$

$$= 2\int_0^a \frac{1}{k}\sec\theta \sec^2\theta\, d\theta$$

$$= \frac{2}{k}\int_0^a \sec^3\theta\, d\theta$$

$$= \frac{1}{k}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)\Big|_0^a$$

$$= \frac{1}{k}\left(\sqrt{1+k^2x^2}\,(kx) + \ln\left|\sqrt{1+k^2x^2} + kx\right|\right)\Big|_0^a$$

$$= \frac{1}{k}\left(ak\sqrt{1+k^2a^2} + \ln\left|\sqrt{1+k^2a^2} + ka\right|\right)$$

$$a = \sqrt{\frac{2}{k}\frac{(V\sin\theta)^2}{2g}} = \frac{V\sin\theta}{\sqrt{g\frac{g}{(V\cos\theta)^2}}} = \frac{V^2}{g}\sin\theta\cos\theta$$

$$k = \frac{g}{(V\cos\theta)^2}$$

$$ak = \tan\theta$$

$$L(\theta) = \frac{(V\cos\theta)^2}{g}\left(\tan\theta\sqrt{1+\tan^2\theta} + \ln\left|\sqrt{1+\tan^2\theta} + \tan\theta\right|\right)$$

$$= \frac{V^2\cos^2\theta}{g}\left(\tan\theta\sec\theta + \ln\left|\sec\theta + \tan\theta\right|\right)$$

$$= \frac{V^2}{g}\sin\theta + \frac{V^2}{g}\cos^2\theta\ln\left|\sec\theta + \tan\theta\right|$$

$$= \frac{V^2}{g}\left(\sin\theta + \cos^2\theta\sinh^{-1}(\tan\theta)\right)$$

d) $L'(\theta) = \frac{V^2}{g}\left(\cos\theta - 2\cos\theta\sin\theta\sinh^{-1}(\tan\theta) + \cos^2\theta\frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\right)$

$$= \frac{V^2}{g}\left(\cos\theta - 2\cos\theta\sin\theta\sinh^{-1}(\tan\theta) + \cos^2\theta\sec\theta\right)$$

$$= \frac{2V^2\cos\theta}{g}\left(1 - \sin\theta\sinh^{-1}(\tan\theta)\right) = 0$$

$$\sin\theta\sinh^{-1}(\tan\theta) = 1$$

$$\sin\theta\ln(\sec\theta + \tan\theta) = 1 \quad \checkmark$$

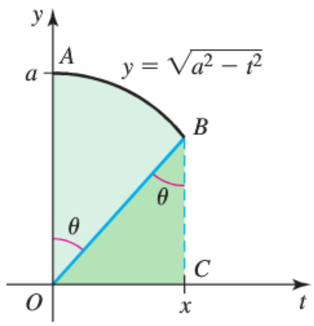
Exercise

Let $F(x)=\int_0^x \sqrt{a^2-t^2}\,dt$. The figure shows that $F(x) = \textit{area of sector } OAB + \textit{area of triangle } OBC$

a) Use the figure to prove that $F(x)=\dfrac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\dfrac{x\sqrt{a^2-x^2}}{2}$

b) Conclude that $\displaystyle\int \sqrt{a^2-x^2}\,dx=\dfrac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\dfrac{x\sqrt{a^2-x^2}}{2}+C$

Solution



a) Area of sector OAB is $\frac{1}{2}\theta a^2$

From the triangle OBC: $\sin\theta=\dfrac{x}{a} \quad\to\quad \theta=\sin^{-1}\dfrac{x}{a}$

$$|BC|=\sqrt{a^2-x^2}$$

Area of sector OAB is $\frac{1}{2}a^2\sin^{-1}\dfrac{x}{a}$

Area of triangle OBC: $\frac{1}{2}x\sqrt{a^2-x^2}$

$F(x) = \textit{area of sector } OAB + \textit{area of triangle } OBC$

$$=\frac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\frac{x\sqrt{a^2-x^2}}{2}$$

b) $$\frac{d}{dx}\left(\frac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\frac{x\sqrt{a^2-x^2}}{2}+C\right)=\frac{a^2}{2}\frac{\frac{1}{a}}{\sqrt{1-\left(\frac{x}{a}\right)^2}}+\frac{1}{2}\sqrt{a^2-x^2}-\frac{1}{2}\frac{x^2}{\sqrt{a^2-x^2}}$$

$$=\frac{1}{2}\frac{a^2}{\sqrt{a^2-x^2}}+\frac{1}{2}\frac{a^2-2x^2}{\sqrt{a^2-x^2}}$$

$$=\frac{1}{2}\frac{2a^2-2x^2}{\sqrt{a^2-x^2}}$$

$$=\frac{a^2-x^2}{\sqrt{a^2-x^2}}$$

$$=\sqrt{a^2-x^2}$$

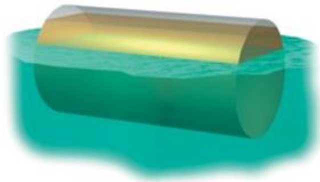
By the antiderivative:

$$\int \sqrt{a^2-x^2}\,dx=\frac{a^2\sin^{-1}\left(\frac{x}{a}\right)}{2}+\frac{x\sqrt{a^2-x^2}}{2}+C \quad \checkmark$$

Exercise

A sealed barrel of oil (weighing 48 pounds per cubic foot) is floating in seawater (weighing 64 pounds per cubic foot). The barrel is not completely full of oil. With the barrel lying on its side, the top 0.2 *foot* of the barrel is empty.

Compare the fluid forces against one end of the barrel from the inside and from the outside.



Solution

$$x^2+y^2=1 \quad\to\quad 2x=2\sqrt{1-y^2}$$

$$F_{inside}=48\int_{-1}^{0.8}(0.8-y)(2)\sqrt{1-y^2}\,dy \qquad\qquad F=w\int_c^d h(y)L(y)dy$$

$$=76.8\int_{-1}^{0.8}\sqrt{1-y^2}\,dy-96\int_{-1}^{0.8}y\sqrt{1-y^2}\,dy$$

$$=76.8\int_{-1}^{0.8}\sqrt{1-y^2}\,dy+48\int_{-1}^{0.8}\left(1-y^2\right)^{1/2}\,d\left(1-y^2\right) \qquad y=\sin\theta \quad \sqrt{1-y^2}=\cos\theta \quad dy=\cos\theta d\theta$$

$$=76.8\int_{-1}^{0.8}\cos^2\theta\,d\theta+32\left(1-y^2\right)^{3/2}\Big|_{-1}^{0.8}$$

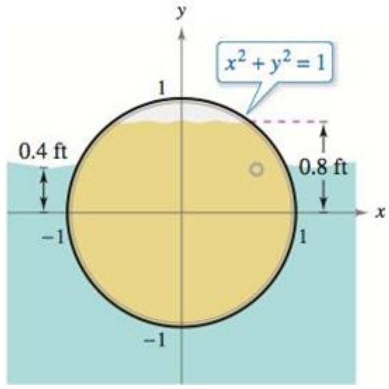
$$=38.4\int_{-1}^{0.8}(1+\cos 2\theta)\,d\theta+32(0.16)^{3/2}$$

$$=38.4\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_{-1}^{0.8}+32(0.4)^3$$

$$=38.4\left(\arcsin y+y\sqrt{1-y^2}\right)\Big|_{-1}^{0.8}+2.048$$

$$=38.4\left(\arcsin 0.8+0.32+\frac{\pi}{2}\right)+2.048$$

$$\underline{\approx 121.3\ \ lbs}$$



$$F_{outside}=64\int_{-1}^{0.4}(0.4-y)(2)\sqrt{1-y^2}\,dy \qquad\qquad F=w\int_c^d h(y)L(y)dy$$

$$=51.2\int_{-1}^{0.4}\sqrt{1-y^2}\,dy-128\int_{-1}^{0.4}y\sqrt{1-y^2}\,dy$$

$$=25.6\left(\arcsin y+y\sqrt{1-y^2}\right)\Big|_{-1}^{0.4}+\frac{128}{3}\left(1-y^2\right)^{3/2}\Big|_{-1}^{0.4}$$

$$\underline{\approx 93.0\ \ lbs}$$

Exercise

The axis of a storage tank in the form of a right circular cylinder is horizontal. The radius and length of the tank are 1 *meter* and 3 *meters*, respectively.

a) Determine the volume of fluid in the tank as a function of its depth *d*.

b) Graph the function in part (a).

c) Design a dip stick for the tank with markings of $\frac{1}{4}$, $\frac{1}{2}$, *and* $\frac{3}{4}$

d) Fluid is entering the tank at a rate of $\frac{1}{4}\ m^3/min$. Determine the rate of change of the depth of the fluid as a function of its depth *d*.

e) Graph the function in part (*d*).\When will the rate of change of the depth be minimum?

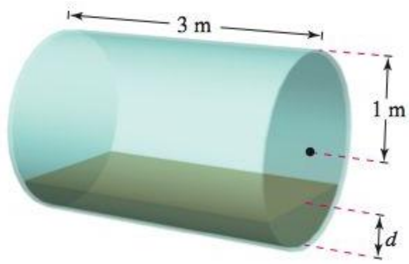
Solution

a) Consider the center at $(0,\ 1)$: $x^2+(y-1)^2=1 \ \to\ x=\sqrt{1-(y-1)^2}$

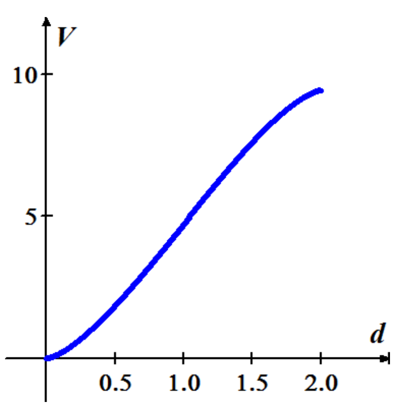
The depth: $0\le d\le 2$

$$y-1=\sin\theta \qquad \sqrt{1-(y-1)^2}=\cos\theta$$
$$d(y-1)=\cos\theta d\theta$$

$$\begin{aligned}V&=\int_0^d (3)\left(2\sqrt{1-(y-1)^2}\right)dy\\
&=6\int_0^d \sqrt{1-(y-1)^2}\ d(y-1)\\
&=6\int_0^d \cos^2\theta\ d\theta\\
&=3\int_0^d (1+\cos 2\theta)\ d\theta\\
&=3\left(\theta+\tfrac{1}{2}\sin 2\theta\right)\Big|_0^d\\
&=3(\theta+\sin\theta\cos\theta)\Big|_0^d\\
&=3\left(\arcsin(y-1)+(y-1)\sqrt{1-(y-1)^2}\right)\Big|_0^d\\
&=\boxed{3\arcsin(d-1)+3(d-1)\sqrt{2d-d^2}+\frac{3\pi}{2}}\end{aligned}$$



b)



c) The full tank holds $3\pi\ m^3$

A dip stick for the tank with markings of $\frac{1}{4},\ \frac{1}{2},\ and\ \frac{3}{4}$

The horizontal lines are: $y=\frac{3\pi}{4}, \quad y=\frac{3\pi}{2}, \quad y=\frac{9\pi}{4}$

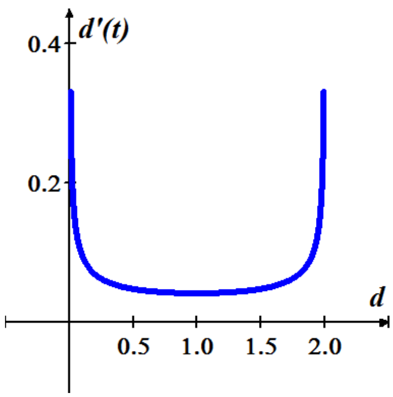
Intersect the curve at $\underline{d=0.596, \quad d=1.0, \quad d=1.404}$

d) $V=6\int_0^d \sqrt{1-(y-1)^2}\, dy \quad \rightarrow \quad \frac{dV}{dt}=\frac{dV}{dd}\frac{dd}{dt}$

$$\frac{dV}{dt}=6\sqrt{1-(d-1)^2}\cdot d'(t)=\frac{1}{4}$$

$$\underline{d'(t)=\frac{1}{24\sqrt{1-(d-1)^2}}}$$

e)



From the graph, the minimum occurs at $d=1$, which is the widest part of the tank.

Exercise

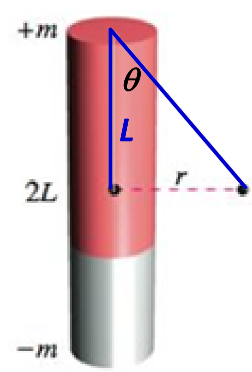
The field strength H of a magnet of length $2L$ on a particle r units from the center of the magnet is

$$H=\frac{2mL}{\left(r^2+L^2\right)^{3/2}}$$

Where $\pm m$ are the poles of the magnet.

Find the average field strength as the particle moves from 0 to R units from the center by evaluating the integral

$$\frac{1}{R}\int_0^R \frac{2mL}{\left(r^2+L^2\right)^{3/2}}dr$$



Solution

$$r=L\tan\theta \quad \rightarrow \quad dr=L\sec^2\theta d\theta$$

$$r^2+L^2=L^2\tan^2\theta+L^2 \underline{=L^2\sec^2\theta}$$

$$\frac{1}{R}\int_0^R \frac{2mL}{\left(r^2+L^2\right)^{3/2}}\,dr = \frac{1}{R}\int_0^R \frac{2mL}{(L\sec\theta)^3}\,L\sec^2\theta\;d\theta$$

$$= \frac{2m}{RL}\int_0^R \frac{1}{\sec\theta}\;d\theta$$

$$= \frac{2m}{RL}\int_0^R \cos\theta\;d\theta$$

$$= \frac{2m}{RL}\sin\theta\Big|_0^R$$

$$= \frac{2m}{RL}\frac{r}{\sqrt{r^2+L^2}}\Bigg|_0^R$$

$$= \boxed{\frac{2m}{L\sqrt{R^2+L^2}}}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{x^2+2x}$

Solution

$$\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} = \frac{Ax+2A+Bx}{x^2+2x}$$

$$1 = (A+B)x + 2A \quad \Rightarrow \begin{cases} 2A = 1 & \rightarrow A = \frac{1}{2} \\ A+B=0 & \rightarrow B = -\frac{1}{2} \end{cases}$$

$$\int \frac{1}{x^2+2x}dx = \frac{1}{2}\int \frac{1}{x}dx - \frac{1}{2}\int \frac{1}{x+2}dx$$

$$= \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + C$$

Exercise

Evaluate $\displaystyle\int \frac{2x+1}{x^2-7x+12}dx$

Solution

$$\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3} = \frac{(A+B)x-3A-4B}{(x-4)(x-3)}$$

$$\rightarrow \begin{cases} A+B=2 \\ -3A-4B=1 \end{cases} \Rightarrow \boxed{A=9} \quad \boxed{B=-7}$$

$$\int \frac{2x+1}{x^2-7x+12}dx = 9\int \frac{dx}{x-4} - 7\int \frac{dx}{x-3}$$

$$= 9\ln|x-4| - 7\ln|x-3| + C$$

$$= \ln\left|\frac{(x-4)^9}{(x-3)^7}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{x+3}{2x^3-8x}dx$

Solution

$$\frac{x+3}{2x^3-8x}=\frac{1}{2}\frac{x+3}{x\left(x^2-4\right)}=\frac{1}{2}\left(\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-2}\right)$$

$$=\frac{1}{2}\frac{A(x+2)(x-2)+Bx(x-2)+Cx(x+2)}{x(x+2)(x-2)}$$

$$(A+B+C)x^2+(2C-2B)x-4A=x+3$$

$$\begin{cases} A+B+C=0 \\ 2C-2B=1 \\ -4A=3 \end{cases} \rightarrow \boxed{A=-\frac{3}{4}} \quad \boxed{B=\frac{1}{8}} \quad \boxed{C=\frac{5}{8}}$$

$$\int \frac{x+3}{2x^3-8x}dx=\frac{1}{2}\int -\frac{3}{4}\frac{dx}{x}+\frac{1}{2}\int \frac{1}{8}\frac{dx}{x+2}+\frac{1}{2}\int \frac{5}{8}\frac{dx}{x-2}$$

$$=-\frac{3}{8}\ln|x|+\frac{1}{16}\ln|x+2|+\frac{5}{16}\ln|x-2|+K$$

$$=\frac{1}{16}\left(\ln|x+2|+5\ln|x-2|-6\ln|x|\right)+K$$

$$\boxed{=\frac{1}{16}\ln\left|\frac{(x+2)(x-2)^5}{x^6}\right|+K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2}{(x-1)\left(x^2+2x+1\right)}dx$

Solution

$$\frac{x^2}{(x-1)\left(x^2+2x+1\right)}=\frac{x^2}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$x^2=A(x+1)^2+B(x-1)(x+1)+C(x-1)$$

$$x^2=(A+B)x^2+(2A+C)x+A-B-C$$

$$\begin{cases} A+B=1 \\ 2A+C=0 \\ A-B-C=0 \end{cases} \rightarrow \boxed{A=\frac{1}{4}} \quad \boxed{B=\frac{3}{4}} \quad \boxed{C=-\frac{1}{2}}$$

$$\int \frac{x^2}{(x-1)\left(x^2+2x+1\right)}dx=\frac{1}{4}\int \frac{dx}{x-1}+\frac{3}{4}\int \frac{dx}{x+1}-\frac{1}{2}\int \frac{dx}{(x+1)^2}$$

$$= \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2}\frac{1}{(x+1)} + K$$

$$= \frac{1}{4}\left(\ln|x-1| + \ln|x+1|^3\right) + \frac{1}{2(x+1)} + K$$

$$\boxed{= \frac{1}{4}\ln\left|(x-1)(x+1)^3\right| + \frac{1}{2(x+1)} + K}$$

Exercise

Evaluate $\displaystyle\int \frac{8x^2+8x+2}{\left(4x^2+1\right)^2}dx$

Solution

$$\frac{8x^2+8x+2}{\left(4x^2+1\right)^2} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{\left(4x^2+1\right)^2} = \frac{(Ax+B)\left(4x^2+1\right)+Cx+D}{\left(4x^2+1\right)^2}$$

$$8x^2+8x+2 = 4Ax^3+4Bx^2+(A+C)x+B+D$$

$$\begin{cases} A=0 \\ 4B=8 \\ A+C=8 \\ B+D=2 \end{cases} \rightarrow \boxed{A=0} \quad \boxed{B=2} \quad \boxed{C=8} \quad \boxed{D=0}$$

$$\int \frac{8x^2+8x+2}{\left(4x^2+1\right)^2}dx = \int \frac{2}{4x^2+1}dx + \int \frac{8x}{\left(4x^2+1\right)^2}dx \qquad d\left(4x^2+1\right) = 8xdx$$

$$= \int \frac{2}{4x^2+1}dx + \int \frac{d\left(4x^2+1\right)}{\left(4x^2+1\right)^2} \qquad \int \frac{du}{u^2} = -\frac{1}{u} \quad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\boxed{= \tan^{-1}2x - \frac{1}{4x^2+1} + K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2+x}{x^4-3x^2-4}dx$

Solution

$$\frac{x^2+x}{x^4-3x^2-4} = \frac{x^2+x}{\left(x^2-4\right)\left(x^2+1\right)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1}$$

$$x^2+x = A(x+2)\left(x^2+1\right) + B(x-2)\left(x^2+1\right) + (Cx+D)\left(x^2-4\right)$$

$$= Ax^3 + Ax + 2Ax^2 + 2A + Bx^3 + Bx - 2Bx^2 - 2B + Cx^3 - 4Cx + Dx^2 - 4D$$

$$= (A+B+C)x^3 + (2A-2B+D)x^2 + (A+B-4C)x + 2A - 2B - 4D$$

$$\begin{cases} A+B+C=0 \\ 2A-2B+D=1 \\ A+B-4C=1 \\ 2A-2B-4D=0 \end{cases} \Rightarrow \boxed{A=\frac{3}{10}} \quad \boxed{B=-\frac{1}{10}} \quad \boxed{C=-\frac{1}{5}} \quad \boxed{D=\frac{1}{5}}$$

$$\int \frac{x^2+x}{x^4-3x^2-4}dx = \frac{3}{10}\int \frac{1}{x-2}dx - \frac{1}{10}\int \frac{1}{x+2}dx + \frac{1}{5}\int \frac{-x+1}{x^2+1}dx$$

$$= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{5}\int \frac{x}{x^2+1}dx + \frac{1}{5}\int \frac{1}{x^2+1}dx \qquad d\left(x^2+1\right) = 2xdx$$

$$= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{10}\int \frac{d\left(x^2+1\right)}{x^2+1} + \frac{1}{5}\int \frac{1}{x^2+1}dx \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{10}\ln\left(x^2+1\right) + \frac{1}{5}\tan^{-1}x + K$$

Exercise

Evaluate $\displaystyle\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{\left(\theta^2+1\right)^3} d\theta$

Solution

$$\frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{\left(\theta^2+1\right)^3} = \frac{A\theta+B}{\theta^2+1} + \frac{C\theta+D}{\left(\theta^2+1\right)^2} + \frac{E\theta+F}{\left(\theta^2+1\right)^3}$$

$$\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1 = (A\theta+B)\left(\theta^2+1\right)^2 + (C\theta+D)\left(\theta^2+1\right) + E\theta + F$$

$$= (A\theta+B)\left(\theta^4+2\theta^2+1\right) + C\theta^3 + C\theta + D\theta^2 + D + E\theta + F$$

$$= A\theta^5 + B\theta^4 + (2A+C)\theta^3 + (2B+D)\theta^2 + (A+C+E)\theta + B + D + F$$

$$\begin{cases} \boxed{A=0} \\ \boxed{B=1} \\ 2A+C=-4 \\ 2B+D=2 \\ A+C+E=-3 \\ B+D+F=1 \end{cases} \rightarrow \boxed{C=-4} \quad \boxed{D=0} \quad \boxed{E=1} \quad \boxed{F=0}$$

$$\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{\left(\theta^2+1\right)^3} d\theta = \int \frac{1}{\theta^2+1}d\theta - 4\int \frac{\theta}{\left(\theta^2+1\right)^2}d\theta + \int \frac{\theta}{\left(\theta^2+1\right)^3}d\theta$$

$$= \int \frac{1}{\theta^2+1}d\theta - 2\int \frac{d\left(\theta^2+1\right)}{\left(\theta^2+1\right)^2} + \frac{1}{2}\int \frac{d\left(\theta^2+1\right)}{\left(\theta^2+1\right)^3} \qquad d\left(\theta^2+1\right) = 2\theta d\theta$$

$$\boxed{= \tan^{-1}\theta + 2\frac{1}{\theta^2+1} - \frac{1}{4}\frac{1}{\left(\theta^2+1\right)^2} + K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^4}{x^2-1}dx$

Solution

$$\frac{x^4}{x^2-1} = x^2+1+\frac{1}{(x-1)(x+1)}$$

$$\begin{array}{r|l} & x^2+1 \\ \hline x^2-1 & x^4 \\ & \underline{x^4-x^2} \\ & x^2 \\ & \underline{x^2-1} \\ & 1 \end{array}$$

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1}+\frac{B}{x+1} = \frac{(A+B)x+A-B}{(x-1)(x+1)}$$

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \rightarrow \boxed{A=\frac{1}{2}} \quad \boxed{B=-\frac{1}{2}}$$

$$\int \frac{x^4}{x^2-1}dx = \int \left(x^2+1\right)dx + \frac{1}{2}\int \frac{1}{x-1}dx - \frac{1}{2}\int \frac{1}{x+1}dx$$

$$= \frac{1}{3}x^3+x+\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

$$= \frac{1}{3}x^3+x+\frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) + C$$

$$\boxed{= \frac{1}{3}x^3+x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{16x^3}{4x^2-4x+1}dx$

Solution

$$\begin{array}{r|l} & 4x+4 \\ \hline 4x^2-4x+1 & 16x^3 \\ & \underline{16x^3-16x^2+4x} \\ & 16x^2-4x \\ & \underline{16x^2-16x+4} \\ & 12x-4 \end{array}$$

$$\frac{16x^3}{4x^2-4x+1} = 4x+4+\frac{12x-4}{(2x-1)^2}$$

$$= 4x+4+\frac{A}{2x-1}+\frac{B}{(2x-1)^2}$$

$$12x-4 = 2Ax-A+B$$

$$\begin{cases} 2A = 12 \\ -A + B = -4 \end{cases} \to \boxed{A = 6} \quad \boxed{B = 2}$$

$$\int \frac{16x^3}{4x^2 - 4x + 1} dx = \int (4x + 4) dx + 6 \int \frac{dx}{2x-1} + 2 \int \frac{dx}{(2x-1)^2}$$

$$= 2x^2 + 4x + 6\left(\frac{1}{2}\right) \ln|2x-1| + 2\left(-\frac{1}{2}\right) \frac{1}{2x-1} + C$$

$$= 2x^2 + 4x + 3\ln|2x-1| - \frac{1}{2x-1} + C$$

Exercise

Evaluate $\displaystyle\int \frac{e^{4x} + 2e^{2x} - e^x}{e^{2x} + 1} dx$

Solution

$$\int \frac{e^{4x} + 2e^{2x} - e^x}{e^{2x} + 1} dx = \int \frac{e^x \left(e^{3x} + 2e^x - 1\right)}{e^{2x} + 1} dx \qquad y = e^x \Rightarrow dy = e^x dx$$

$$= \int \frac{y^3 + 2y - 1}{y^2 + 1} dy$$

$$= \int \left(y + \frac{y-1}{y^2+1} \right) dy$$

$$= \int y dy + \int \frac{y}{y^2+1} dy - \int \frac{1}{y^2+1} dy$$

$$= \int y dy + \frac{1}{2} \int \frac{1}{y^2+1} d\left(y^2+1\right) - \int \frac{1}{y^2+1} dy \qquad d\left(y^2+1\right) = 2y dy$$

$$= \frac{1}{2} y^2 + \frac{1}{2} \ln\left(y^2+1\right) - \tan^{-1} y + C$$

$$= \frac{1}{2} e^{2x} + \frac{1}{2} \ln\left(e^{2x}+1\right) - \tan^{-1} e^x + C$$

Exercise

Evaluate $\displaystyle\int \frac{\sin\theta d\theta}{\cos^2\theta + \cos\theta - 2}$

Solution

Let $y = \cos\theta \Rightarrow dy = -\sin\theta d\theta$

$$\int \frac{\sin\theta d\theta}{\cos^2\theta + \cos\theta - 2} = -\int \frac{dy}{y^2 + y - 2}$$

$$\frac{1}{y^2+y-2}=\frac{1}{(y+2)(y-1)}=\frac{A}{y+2}+\frac{B}{y-1}$$

$$1=(A+B)y-A+2B$$

$$\begin{cases} A+B=0 \\ -A+2B=1 \end{cases} \rightarrow \boxed{A=-\frac{1}{3}} \quad \boxed{B=\frac{1}{3}}$$

$$\int \frac{\sin\theta d\theta}{\cos^2\theta+\cos\theta-2}=-\left(-\frac{1}{3}\int\frac{dy}{y+2}+\frac{1}{3}\int\frac{dy}{y-1}\right)$$

$$=\frac{1}{3}\ln|y+2|-\frac{1}{3}\ln|y-1|+C$$

$$=\frac{1}{3}\left(\ln|y+2|-\ln|y-1|\right)+C$$

$$=\frac{1}{3}\ln\left|\frac{y+2}{y-1}\right|+C$$

$$=\frac{1}{3}\ln\left|\frac{\cos\theta+2}{\cos\theta-1}\right|+C$$

Exercise

Evaluate $\displaystyle\int \frac{(x-2)^2\tan^{-1}(2x)-12x^3-3x}{\left(4x^2+1\right)(x-2)^2}\,dx$

Solution

$$\int \frac{(x-2)^2\tan^{-1}(2x)-12x^3-3x}{\left(4x^2+1\right)(x-2)^2}\,dx=\int\frac{(x-2)^2\tan^{-1}(2x)}{\left(4x^2+1\right)(x-2)^2}\,dx-\int\frac{12x^3+3x}{\left(4x^2+1\right)(x-2)^2}\,dx$$

$$=\int\frac{\tan^{-1}(2x)}{4x^2+1}\,dx-\int\frac{3x\left(4x^2+1\right)}{\left(4x^2+1\right)(x-2)^2}\,dx$$

$$=\int\frac{\tan^{-1}(2x)}{4x^2+1}\,dx-\int\frac{3x}{(x-2)^2}\,dx$$

$$d\left(\tan^{-1}2x\right)=\frac{dx}{(2x)^2+1}=\frac{dx}{4x^2+1}$$

$$\frac{3x}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}=\frac{Ax-2A+B}{(x-2)^2}$$

$$\begin{cases} \boxed{A=3} \\ -2A+B=0 \end{cases} \rightarrow \boxed{B=6}$$

$$\int \frac{(x-2)^2 \tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2} dx = \frac{1}{2}\int \tan^{-1}(2x)\, d\left(\tan^{-1}(2x)\right) - 3\int \frac{dx}{x-2} - 6\int \frac{dx}{(x-2)^2}$$

$$= \frac{1}{4}\left(\tan^{-1}(2x)\right)^2 - 3\int \frac{d(x-2)}{x-2} - 6\int \frac{d(x-2)}{(x-2)^2}$$

$$= \frac{1}{4}\left(\tan^{-1}(2x)\right)^2 - 3\ln|x-2| - \frac{6}{x-2} + C$$

Exercise

Evaluate $\displaystyle\int \frac{\sqrt{x+1}}{x} dx$

Solution

Let $x+1 = u^2 \quad \Rightarrow \quad dx = 2udu$

$$\begin{array}{r} 1 \\ u^2-1 \overline{\smash{)}\, u^2 } \\ \underline{u^2-1} \\ 1 \end{array}$$

$$\int \frac{\sqrt{x+1}}{x} dx = \int \frac{u}{u^2-1} 2udu$$

$$= 2\int \frac{u^2}{u^2-1} du$$

$$= 2\int \left(1 + \frac{1}{u^2-1}\right) du$$

$$= 2\int du + 2\int \frac{1}{u^2-1} du$$

$$\frac{1}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1} = \frac{(A+B)u + A - B}{(u-1)(u+1)}$$

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \Rightarrow \boxed{A = \frac{1}{2}} \quad \boxed{B = -\frac{1}{2}}$$

$$= 2\int du + 2\int \left(\frac{1}{2}\frac{1}{u-1} - \frac{1}{2}\frac{1}{u+1}\right) du$$

$$= 2u + \int \frac{1}{u-1} du - \int \frac{1}{u+1} du$$

$$= 2u + \ln|u-1| - \ln|u+1| + C$$

$$= 2\sqrt{x+1} + \ln\left|\sqrt{x+1} - 1\right| - \ln\left|\sqrt{x+1} + 1\right| + C$$

$$= 2\sqrt{x+1} + \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{x^3-2x^2+3x-4}{x^2+1}dx$

Solution

$$\int \frac{x^3-2x^2+3x-4}{x^2+1}dx = \int\left(x-2+\frac{2x-2}{x^2+1}\right)dx$$

$$= \int (x-2)\,dx + \int \frac{2x}{x^2+1}dx - 2\int \frac{1}{x^2+1}dx$$

$$= \int (x-2)\,dx + \int \frac{d\left(x^2+1\right)}{x^2+1} - 2\int \frac{1}{x^2+1}dx$$

$$= \frac{1}{2}x^2 - 2x + \ln\left(x^2+1\right) - 2\tan^{-1}(x) + C$$

$$\begin{array}{r|l} & x-2 \\ \hline x^2+1 & x^3-2x^2+3x-4 \\ & \underline{x^3 \qquad\quad +x} \\ & -2x^2+2x-4 \\ & \underline{-2x^2 \qquad\quad -2} \\ & 2x-2 \end{array}$$

Exercise

Evaluate $\displaystyle\int \frac{4x^2+2x+4}{x+1}dx$

Solution

$$\int \frac{4x^2+2x+4}{x+1}dx = \int\left(4x+2+\frac{6}{x+1}\right)dx$$

$$= \int (4x-2)dx + \int \frac{6}{x+1}dx$$

$$= \int (4x-2)dx + 6\int \frac{d(x+1)}{x+1}$$

$$\int \frac{d(U)}{U} = \ln|U|$$

$$= 2x^2 - 2x + 6\ln|x+1| + C$$

Exercise

Evaluate $\displaystyle\int \frac{3x^2+7x-2}{x^3-x^2-2x}dx$

Solution

$$\frac{3x^2+7x-2}{x^3-x^2-2x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

$$3x^2+7x-2 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1)$$

$$= Ax^2 - Ax - 2A$$

$$Bx^2 - 2Bx$$

$$Cx^2 + Cx$$

$$\begin{cases} A+B+C=3 \\ -A-2B+C=7 \\ -2A=-2 \end{cases} \rightarrow \boxed{A=1} \quad \begin{cases} B+C=2 \\ -2B+C=8 \end{cases} \rightarrow \underline{B=-2} \quad \underline{C=4}$$

$$\int \frac{3x^2+7x-2}{x^3-x^2-2x}dx = \int \left(\frac{1}{x} - \frac{2}{x+1} + \frac{4}{x-2}\right)dx$$

$$= \ln|x| - 2\ln|x+1| + 4\ln|x-2| + K$$

$$= \ln\frac{|x|(x-2)^4}{(x+1)^2} + K$$

Exercise

Evaluate $\displaystyle\int \frac{3x^2+2x+5}{(x-1)(x^2-x-20)}dx$

Solution

$$\frac{3x^2+2x+5}{(x-1)(x^2-x-20)} = \frac{A}{x-1} + \frac{B}{x-5} + \frac{C}{x+4}$$

$$3x^2+2x+5 = (A+B+C)x^2 + (-A+3B-6C)x - 20A - 4B + 5C$$

$$\begin{cases} A+B+C=3 \\ -A+3B-6C=2 \\ -20A-4B+5C=5 \end{cases} \rightarrow A=\frac{1}{2}, \quad B=\frac{5}{2}, \quad C=1$$

$$\int \frac{3x^2+2x+5}{(x-1)(x^2-x-20)}dx = \int \left(\frac{1}{2}\frac{1}{x-1} + \frac{5}{2}\frac{1}{x-5} + \frac{1}{x+4}\right)dx$$

$$= \frac{1}{2}\ln|x-1| + \frac{5}{2}\ln|x-5| + \ln|x+4| + K$$

Exercise

Evaluate $\displaystyle\int \frac{5x^2-3x+2}{x^3-2x^2}dx$

Solution

$$\frac{5x^2-3x+2}{x^3-2x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$5x^2-3x+2 = Ax^2 - 2Ax + Bx - 2B + Cx^2$$

$$\begin{cases} A+C=5 \\ -2A+B=-3 \\ -2B=2 \end{cases} \rightarrow \underline{B=-1; \ A=1; \ C=4}$$

$$\int \frac{5x^2 - 3x + 2}{x^3 - 2x^2}\,dx = \int \frac{dx}{x} - \int \frac{dx}{x^2} + 4\int \frac{dx}{x-2}$$

$$= \ln|x| + \frac{1}{x} + 4\ln|x-2| + K$$

Exercise

Evaluate $\displaystyle\int \frac{7x^2 - 13x + 13}{(x-2)\left(x^2 - 2x + 3\right)}\,dx$

Solution

$$\frac{7x^2 - 13x + 13}{(x-2)\left(x^2 - 2x + 3\right)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 - 2x + 3}$$

$$7x^2 - 13x + 13 = Ax^2 - 2Ax + 3A + Bx^2 - 2Bx + Cx - 2C$$

$$\begin{cases} A + B = 7 \\ -2A - 2B + C = -13 \\ 3A - 2C = 13 \end{cases} \rightarrow \quad A = 5;\ \ B = 2;\ C = 1$$

$$\int \frac{7x^2 - 13x + 13}{(x-2)\left(x^2 - 2x + 3\right)}\,dx = \int \frac{5dx}{x-2} + \int \frac{2x+1}{x^2 - 2x + 3}\,dx$$

$$= 5\ln|x-2| + \int \frac{2x - 2 + 3}{x^2 - 2x + 3}\,dx$$

$$= 5\ln|x-2| + \int \frac{2x-2}{x^2 - 2x + 3}\,dx + \int \frac{3}{(x-1)^2 + 3}\,dx$$

$$= 5\ln|x-2| + \ln\left(x^2 - 2x + 3\right) + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + K$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{1+\sin x}$

Solution

$$\int \frac{dx}{1+\sin x} = \int \frac{1}{1+\frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2}du$$

$$= \int \frac{2}{u^2+2u+1}\, du$$

$$= \int \frac{2}{(u+1)^2}\, d(u+1)$$

$$= -\frac{2}{u+1} + C$$

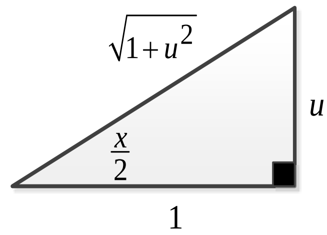
$$\boxed{= -\frac{2}{\tan\left(\frac{x}{2}\right)+1} + C}$$

Let $u = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1}u \quad \rightarrow dx = \frac{2du}{1+u^2}$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}}$$

$$= \frac{2u}{1+u^2}$$



Exercise

Evaluate $\displaystyle\int \frac{dx}{2+\cos x}$

Solution

$$\int \frac{dx}{2+\cos x} = \int \frac{1}{2+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}du$$

$$= 2\int \frac{1}{u^2+3}\, du$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C$$

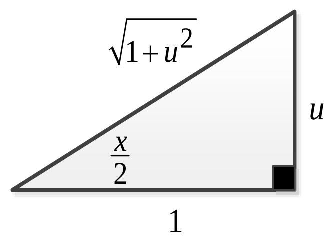
$$\boxed{= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) + C}$$

Let $u = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1}u \quad \rightarrow dx = \frac{2du}{1+u^2}$

$$\cos x = 2\cos^2\frac{x}{2} - 1$$

$$= 2\frac{1}{1+u^2} - 1$$

$$= \frac{1-u^2}{1+u^2}$$



$$\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{1-\cos x}$

Solution

$$\int \frac{dx}{1-\cos x} = \int \frac{1}{1-\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}du$$

$$= \int \frac{1}{u^2}\, du$$

$$= -\frac{1}{u} + C$$

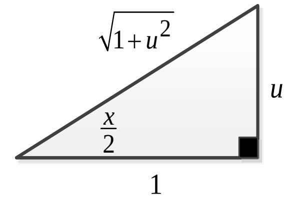
$$= -\frac{1}{\tan\frac{x}{2}} + C \qquad \boxed{= -\cot\frac{x}{2} + C}$$

Let $u = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1} u \quad \to dx = \frac{2du}{1+u^2}$

$$\cos x = 2\cos^2\frac{x}{2} - 1$$

$$= 2\frac{1}{1+u^2} - 1$$

$$= \frac{1-u^2}{1+u^2}$$



Exercise

Evaluate $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

Solution

$$\int \frac{dx}{1+\sin x+\cos x} = \int \frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} du$$

$$= 2\int \frac{1}{2+2u}\, du$$

$$= \int \frac{1}{1+u}\, d(1+u)$$

$$= \ln|1+u| + C$$

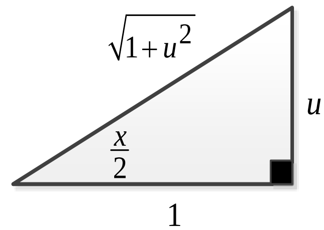
$$\boxed{= \ln\left|1+\tan\frac{x}{2}\right| + C}$$

$u = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1} u \quad \to dx = \frac{2du}{1+u^2}$

$$\cos x = 2\cos^2\frac{x}{2} - 1$$

$$= 2\frac{1}{1+u^2} - 1$$

$$= \frac{1-u^2}{1+u^2}$$



$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2}$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^2-5x+6}\, dx$

Solution

$$\frac{1}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$Ax - 3A + Bx - 2B = 1 \quad \to \begin{cases} A+B=0 \\ -3A-2B=1 \end{cases} \quad \to \quad A=-1 \quad B=1$$

$$\int \frac{1}{x^2-5x+6}\, dx = \int \left(\frac{-1}{x-2} + \frac{1}{x-3}\right) dx$$

$$= \ln|x-3| - \ln|x-2| + C$$

$$\boxed{= \ln\left|\frac{x-3}{x-2}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^2-5x+5}\,dx$

Solution

$$\frac{1}{x^2-5x+5}=\frac{A}{x-\frac{5+\sqrt{5}}{2}}+\frac{B}{x-\frac{5-\sqrt{5}}{2}} \qquad\qquad x=\frac{5\pm\sqrt{5}}{2}$$

$$Ax-\left(\frac{5-\sqrt{5}}{2}\right)A+Bx-\left(\frac{5+\sqrt{5}}{2}\right)B=1$$

$$\begin{cases} A+B=0 \\ -\frac{5-\sqrt{5}}{2}A-\frac{5+\sqrt{5}}{2}B=1 \end{cases} \to \begin{matrix} \frac{5-\sqrt{5}}{2}A+\frac{5-\sqrt{5}}{2}B=0 \\ -\frac{5-\sqrt{5}}{2}A-\frac{5+\sqrt{5}}{2}B=1 \end{matrix}$$

$$-\sqrt{5}B=1 \to B=-\frac{1}{\sqrt{5}} \quad\Rightarrow\quad A=\frac{1}{\sqrt{5}}$$

$$\int \frac{1}{x^2-5x+5}\,dx=\int\left(\frac{\sqrt{5}}{5}\frac{2}{2x-5-\sqrt{5}}-\frac{\sqrt{5}}{5}\frac{2}{2x-5+\sqrt{5}}\right)dx$$

$$=\frac{\sqrt{5}}{5}\ln\left|2x-5-\sqrt{5}\right|-\frac{\sqrt{5}}{5}\ln\left|2x-5+\sqrt{5}\right|+C$$

Exercise

Evaluate $\displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$

Solution

$$\frac{5x^2+20x+6}{x^3+2x^2+x}=\frac{5x^2+20x+6}{x(x+1)^2}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$Ax^2+2Ax+A+Bx^2+Bx+Cx=5x^2+20x+6$$

$$\begin{cases} A+B=5 \\ 2A+B+C=20 \quad \to \quad \underline{B=-1} \quad \underline{C=9} \\ \underline{A=6} \end{cases}$$

$$\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx=\int\left(\frac{6}{x}-\frac{1}{x+1}+\frac{9}{(x+1)^2}\right)dx$$

$$=6\ln|x|-\ln|x+1|-\frac{9}{x+1}+C$$

$$=\ln\frac{x^6}{|x+1|}-\frac{9}{x+1}+C$$

Exercise

Evaluate $\displaystyle\int \frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}\,dx$

Solution

$$\frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}=\frac{2x^3-4x-8}{x(x-1)\left(x^2+4\right)}=\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+4}$$

$$Ax^3-Ax^2+4Ax-4A+Bx^3+4Bx+Cx^3-Cx^2+Dx^2-Dx=2x^3-4x-8$$

$$\begin{cases} x^3 & A+B+C=2 \\ x^2 & -A-C+D=0 \\ x^1 & 4A+4B-D=-4 \\ x^0 & -4A=-8 \end{cases} \to \begin{cases} B+C=0 \\ -C+D=2 \\ 4B-D=-12 \\ \underline{A=2} \end{cases} \Rightarrow \begin{cases} B+D=2 \\ 4B-D=-12 \end{cases} \to \begin{cases} A=2 \\ B=-2 \\ C=2 \\ D=4 \end{cases}$$

$$\int \frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}\,dx=\int\left(\frac{2}{x}-\frac{2}{x-1}+\frac{2x}{x^2+4}+\frac{4}{x^2+4}\right)dx \qquad \int\frac{dx}{x^2+a^2}=\frac{1}{a}\arctan\frac{x}{a}$$

$$\boxed{=2\ln|x|-2\ln|x-1|+\ln\left(x^2+4\right)+2\tan^{-1}\frac{x}{2}+C}$$

Exercise

Evaluate $\displaystyle\int \frac{8x^3+13x}{\left(x^2+2\right)^2}\,dx$

Solution

$$\frac{8x^3+13x}{\left(x^2+2\right)^2}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{\left(x^2+2\right)^2}$$

$$Ax^3+2Ax+Bx^2+2B+Cx+D=8x^3+13x \qquad \begin{cases} x^3 & A=8 \\ x^2 & B=0 \\ x^1 & 2A+C=13 \\ x^0 & D=0 \end{cases} \to C=-3$$

$$\int \frac{8x^3+13x}{\left(x^2+2\right)^2}\,dx=\int\frac{8x}{x^2+2}dx-\int\frac{3x}{\left(x^2+2\right)^2}dx$$

$$=2\int\frac{1}{x^2+2}d\left(x^2+2\right)-\frac{3}{2}\int\frac{1}{\left(x^2+2\right)^2}d\left(x^2+2\right)$$

$$\boxed{=2\ln\left(x^2+2\right)+\frac{3}{2}\frac{1}{x^2+2}+C}$$

Exercise

Evaluate $\displaystyle\int \frac{\sin x}{\cos x+\cos^2 x}\,dx$

Solution

$$\frac{\sin x}{\cos x+\cos^2 x}=\frac{A}{\cos x}+\frac{B}{1+\cos x}$$

$$A+A\cos x+B\cos x=\sin x \qquad \begin{cases} \underline{A=\sin x} \\ A+B=0 \end{cases} \to \underline{B=-\sin x}$$

$$\begin{aligned}
\int \frac{\sin x}{\cos x+\cos^2 x}\,dx &= \int \frac{\sin x}{\cos x}dx-\int \frac{\sin x}{1+\cos x}dx \\
&= -\int \frac{1}{\cos x}\,d(\cos x)+\int \frac{1}{1+\cos x}\,d(1+\cos x) \\
&= -\ln|\cos x|+\ln|1+\cos x|+C \\
&= \ln\left|\frac{1+\cos x}{\cos x}\right|+C \qquad = \ln|\sec x+1|+C
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int \frac{5\cos x}{\sin^2 x+3\sin x-4}\,dx$

Solution

$$\frac{5\cos x}{\sin^2 x+3\sin x-4}=\frac{A}{\sin x-1}+\frac{B}{\sin x+4}$$

$$A\sin x+4A+B\sin x-B=5\cos x \qquad \begin{cases} 4A-B=5\cos x \\ A+B=0 \end{cases} \quad \underline{A=\cos x} \quad \underline{B=-\cos x}$$

$$\begin{aligned}
\int \frac{5\cos x}{\sin^2 x+3\sin x-4}\,dx &= \int \frac{\cos x}{\sin x-1}dx-\int \frac{\cos x}{\sin x+4}dx \\
&= \int \frac{1}{\sin x-1}d(\sin x-1)-\int \frac{1}{\sin x+4}d(\sin x+4) \\
&= \ln|\sin x-1|-\ln|\sin x+4|+C \\
&= \ln\left|\frac{\sin x-1}{\sin x+4}\right|+C
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int \frac{e^x}{\left(e^x-1\right)\left(e^x+4\right)}\,dx$

Solution

Let $u=e^x \quad\to\quad du=e^x dx$

$$\int \frac{e^x}{\left(e^x-1\right)\left(e^x+4\right)}\,dx = \int \frac{du}{(u-1)(u+4)}$$

$$\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}$$

$$Au + 4A + Bu - B = 1 \quad \Rightarrow \begin{cases} A+B=0 \\ 4A-B=1 \end{cases} \quad \to \quad \boxed{A=\frac{1}{5},\ \ B=-\frac{1}{5}}$$

$$\int \frac{du}{(u-1)(u+4)} = \frac{1}{5}\int \frac{1}{u-1}\,du + \frac{4}{5}\int \frac{1}{u+4}\,du$$

$$= \frac{1}{5}\int \frac{1}{u-1}\,d(u-1) + \frac{4}{5}\int \frac{1}{u+4}\,d(u+4)$$

$$= \frac{1}{5}\ln\left|e^x-1\right| - \frac{1}{5}\ln\left(e^x+4\right) + C$$

$$\boxed{= \frac{1}{5}\ln\left|\frac{e^x-1}{e^x+4}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{e^x}{\left(e^{2x}+1\right)\left(e^x-1\right)}\,dx$

Solution

Let $u = e^x \quad \to \quad du = e^x dx$

$$\int \frac{e^x}{\left(e^{2x}+1\right)\left(e^x-1\right)}\,dx = \int \frac{du}{\left(u^2+1\right)(u-1)}$$

$$\frac{1}{\left(u^2+1\right)(u-1)} = \frac{Au+B}{u^2+1} + \frac{C}{u-1}$$

$$Au^2 - Au + Bu - B + Cu^2 + C = 1$$

$$\begin{cases} u^2 & A+C=0 \\ u^1 & -A+B=0 \to \begin{cases} B+C=0 \\ -B+C=1 \end{cases} \\ u^0 & -B+C=1 \end{cases} \qquad \boxed{C=\frac{1}{2} \quad B=-\frac{1}{2} \quad A=-\frac{1}{2}}$$

$$\int \frac{du}{\left(u^2+1\right)(u-1)} = -\frac{1}{2}\int \frac{u}{u^2+1}\,du - \frac{1}{2}\int \frac{du}{u^2+1} + \frac{1}{2}\int \frac{du}{u-1}$$

$$= -\frac{1}{4}\int \frac{1}{u^2+1}\,d\left(u^2+1\right) - \frac{1}{2}\arctan u + \frac{1}{2}\ln|u-1|$$

$$\boxed{= -\frac{1}{4}\ln\left(e^{2x}+1\right) - \frac{1}{2}\arctan e^x + \frac{1}{2}\ln\left|e^x-1\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{\sqrt{x}}{x-4}\,dx$

Solution

Let $u=\sqrt{x} \quad \to \quad du=\frac{1}{2\sqrt{x}}dx \;\Rightarrow\; 2udu=dx$

$$\int \frac{\sqrt{x}}{x-4}\,dx=\int \frac{u}{u^2-4}\,2u\,du$$

$$=\int \frac{2u^2}{u^2-4}\,du$$

$$=\int \left(2+\frac{8}{u^2-4}\right)du$$

$$\frac{8}{u^2-4}=\frac{A}{u-2}+\frac{B}{u+2}$$

$$Au+2A+Bu-2B=8 \qquad \to \begin{cases} A+B=0 \\ 2A-2B=8 \end{cases} \Rightarrow \underline{A=2 \quad B=-2}$$

$$=\int \left(2+\frac{2}{u-2}-\frac{2}{u+2}\right)du$$

$$=2\sqrt{x}+2\ln\left|\sqrt{x}-2\right|-2\ln\left|\sqrt{x}+2\right|+C$$

$$=2\sqrt{x}+2\ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right|+C$$

Exercise

Evaluate $\displaystyle\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$

Solution

Let $u=x^{1/6} \to u^6=x \quad \to \quad 6u^5du=dx$

$u^2=x^{1/3} \quad u^3=x^{1/2}$

$$\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx=\int \frac{6u^5}{u^3-u^2}\,du$$

$$=\int \frac{6u^3}{u-1}\,du$$

$$\begin{array}{r|l} & 6u^2+6u+6 \\ \hline u-1 & 6u^3 \\ & \underline{-6u^3+6u^2} \\ & 6u^2 \\ & \underline{-6u^2+6u} \\ & 6u \\ & \underline{-6u+6} \\ & 6 \end{array}$$

$$=\int \left(6u^2+6u+6+\frac{6}{u-1}\right)du$$

$$=2u^3+3u^2+6u+6\ln|u-1|+C$$

$$=2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+6\ln\left|\sqrt[6]{x}-1\right|+C$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^2-9}\,dx$

Solution

$$\frac{1}{x^2-9}=\frac{A}{x-3}+\frac{B}{x+3}$$

$$Ax+3A+Bx-3B=1 \qquad \Rightarrow \begin{cases} A+B=0 \\ 3A-3B=1 \end{cases} \rightarrow \underline{A=\frac{1}{6} \quad B=-\frac{1}{6}}$$

$$\int \frac{1}{x^2-9}\,dx=\frac{1}{6}\int \frac{1}{x-3}\,dx-\frac{1}{6}\int \frac{1}{x+3}\,dx$$

$$=\frac{1}{6}\ln|x-3|-\frac{1}{6}\ln|x+3|+C$$

$$=\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|+C$$

Exercise

Evaluate $\displaystyle\int \frac{2}{9x^2-1}\,dx$

Solution

$$\frac{2}{9x^2-1}=\frac{A}{3x-1}+\frac{B}{3x+1}$$

$$3Ax+A+3Bx-B=2 \qquad \Rightarrow \begin{cases} 3A+3B=0 \\ A-B=2 \end{cases} \rightarrow \underline{A=1 \quad B=-1}$$

$$\int \frac{2}{9x^2-1}\,dx=\int \frac{1}{3x-1}\,dx-\int \frac{1}{3x+1}\,dx$$

$$=\frac{1}{3}\ln|3x-1|-\frac{1}{3}\ln|3x+1|+C$$

$$=\frac{1}{3}\ln\left|\frac{3x-1}{3x+1}\right|+C$$

Exercise

Evaluate $\displaystyle\int \frac{5}{x^2+3x-4}\,dx$

Solution

$$\frac{5}{x^2+3x-4}=\frac{A}{x-1}+\frac{B}{x+4}$$

$$Ax+4A+Bx-B=5 \qquad \Rightarrow \begin{cases} A+B=0 \\ 4A-B=5 \end{cases} \rightarrow \underline{A=1 \quad B=-1}$$

$$\int \frac{5}{x^2+3x-4}\,dx=\int \frac{1}{x-1}\,dx-\int \frac{1}{x+4}\,dx$$

$$= \ln|x-1| - \ln|x+4| + C$$

$$\boxed{= \ln\left|\frac{x-1}{x+4}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{3-x}{3x^2-2x-1}\, dx$

Solution

$$\frac{3-x}{3x^2-2x-1} = \frac{A}{x-1} + \frac{B}{3x+1}$$

$$3Ax + A + Bx - B = 3 - x \qquad \Rightarrow \begin{cases} 3A + B = -1 \\ A - B = 3 \end{cases} \to \boxed{A = \frac{1}{2} \quad B = -\frac{5}{2}}$$

$$\int \frac{3-x}{3x^2-2x-1}\, dx = \frac{1}{2}\int \frac{1}{x-1}\, dx - \frac{5}{2}\int \frac{1}{3x+1}\, dx$$

$$\boxed{= \frac{1}{2}\ln|x-1| - \frac{5}{6}\ln|3x+1| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2+12x+12}{x^3-4x}\, dx$

Solution

$$\frac{x^2+12x+12}{x^3-4x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

$$Ax^2 - 4A + Bx^2 + 2Bx + Cx^2 - 2Cx = x^2 + 12x + 12$$

$$\begin{cases} x^2 & A + B + C = 1 \\ x^1 & 2B - 2C = 12 \\ x^0 & -4A = 12 \end{cases} \to A = -3 \quad B = 5 \quad C = -1$$

$$\int \frac{x^2+12x+12}{x^3-4x}\, dx = -\frac{3}{x} + \frac{5}{x-2} - \frac{1}{x+2}$$

$$\boxed{= -3\ln|x| + 5\ln|x-2| - \ln|x+2| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x^3-x+3}{x^2+x-2}\, dx$

Solution

$$\frac{x^3-x+3}{x^2+x-2}=x-1+\frac{2x+1}{x^2+x-2}$$

$$\frac{2x+1}{x^2+x-2}=\frac{A}{x-1}+\frac{B}{x+2}$$

$$Ax+2A+Bx-B=2x+1 \quad \Rightarrow \begin{cases} A+B=2 \\ 2A-B=1 \end{cases} \to \underline{A=1 \quad B=1}$$

$$\begin{array}{r} x-1 \\ x^2+x-2 \overline{\smash{)}\, x^3-x+3} \\ \underline{-x^3-x^2+2x} \\ -x^2+x+3 \\ \underline{x^2+x-2} \\ 2x-1 \end{array}$$

$$\int \frac{x^3-x+3}{x^2+x-2}\,dx=\int\left(x-1+\frac{1}{x-1}+\frac{1}{x+2}\right)dx$$

$$=\frac{1}{2}x^2-x+\ln|x-1|+\ln|x+2|+C$$

Exercise

Evaluate $\displaystyle\int \frac{5x-2}{(x-2)^2}\,dx$

Solution

$$\frac{5x-2}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}$$

$$Ax-2A+B=5x-2 \qquad \Rightarrow \begin{cases} A=5 \\ -2A+B=-2 \end{cases} \to \underline{B=8}$$

$$\int \frac{5x-2}{(x-2)^2}\,dx=\frac{5}{x-2}+\frac{8}{(x-2)^2}$$

$$=5\ln|x-2|-\frac{8}{x-2}+C$$

Exercise

Evaluate $\displaystyle\int \frac{2x^3-4x^2-15x+4}{x^2-2x-8}\,dx$

Solution

$$\begin{array}{r} 2x \\ x^2-2x-8 \overline{\smash{)}\, 2x^3-4x^2-15x+4} \\ \underline{2x^3-4x^2-16x} \\ x+4 \end{array}$$

$$\int \frac{2x^3-4x^2-15x+4}{x^2-2x-8}\,dx=\int 2x\,dx+\int\frac{x+4}{x^2-2x-8}\,dx$$

$$\frac{x+4}{x^2-2x-8}=\frac{A}{x-4}+\frac{B}{x+2}$$

$$Ax+2A+Bx-4B=x+4 \quad \Rightarrow \begin{cases} A+B=1 \\ 2A-4B=4 \end{cases} \to \underline{A=\tfrac{4}{3} \quad B=-\tfrac{1}{3}}$$

$$\int \frac{2x^3-4x^2-15x+4}{x^2-2x-8}\,dx=x^2+\frac{4}{3}\int\frac{1}{x-4}\,dx-\frac{1}{3}\int\frac{1}{x+2}\,dx$$

$$=x^2+\frac{4}{3}\ln|x-4|-\frac{1}{3}\ln|x+2|+C$$

Exercise

Evaluate $\displaystyle\int \frac{x+2}{x^2+5x}\,dx$

Solution

$$\frac{x+2}{x^2+5x}=\frac{A}{x}+\frac{B}{x+5}$$

$$Ax+5A+Bx=x+2 \qquad \Rightarrow \begin{cases} A+B=1 \\ 5A=2 \end{cases} \rightarrow \underline{A=\frac{2}{5} \quad B=\frac{3}{5}}$$

$$\int \frac{x+2}{x^2+5x}\,dx=\frac{2}{5}\int \frac{1}{x}\,dx+\frac{3}{5}\int \frac{1}{x+5}\,dx$$

$$=\frac{2}{5}\ln|x|+\frac{3}{5}\ln|x+5|+C$$

Exercise

Evaluate $\displaystyle\int_0^2 \frac{3}{4x^2+5x+1}\,dx$

Solution

$$\frac{3}{4x^2+5x+1}=\frac{A}{x+1}+\frac{B}{4x+1}$$

$$4Ax+A+Bx+B=3 \qquad \Rightarrow \begin{cases} 4A+B=0 \\ A+B=3 \end{cases} \rightarrow \underline{A=-1 \quad B=4}$$

$$\int_0^2 \frac{3}{4x^2+5x+1}\,dx=-\int_0^2 \frac{1}{x+1}\,dx+\int_0^2 \frac{4}{4x+1}\,dx$$

$$=-\ln(x+1)+\ln(4x+1)\Big|_0^2$$

$$=\ln\frac{4x+1}{x+1}\Big|_0^2$$

$$=\ln 3$$

Exercise

Evaluate $\displaystyle\int_1^5 \frac{x-1}{x^2(x+1)}\,dx$

Solution

$$\frac{x-1}{x^2(x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$$

$$Ax^2+Ax+Bx+B+Cx^2=x-1$$

$$\begin{cases} x^2 & A+C=0 \\ x^1 & A+B=1 \quad \rightarrow A=2 \quad C=-2 \\ x^0 & \underline{B=-1} \end{cases}$$

$$\int_1^5 \frac{x-1}{x^2(x+1)}\,dx = \int_1^5 \left(\frac{2}{x} - \frac{1}{x^2} - \frac{2}{x+1}\right) dx$$

$$= 2\ln x + \frac{1}{x} - 2\ln(x+1)\Big|_1^5$$

$$= 2\ln 5 + \frac{1}{5} - 2\ln 6 - 1 + 2\ln 2$$

$$\boxed{= 2\ln\frac{5}{3} - \frac{4}{5}}$$

Exercise

Evaluate $\displaystyle\int_1^2 \frac{x+1}{x(x^2+1)}\,dx$

Solution

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$Ax^2 + A + Bx^2 + Cx = x+1$$

$$\begin{cases} x^2 & A+B=0 \\ x^1 & \underline{C=1} \quad \rightarrow \underline{B=-1} \\ x^0 & \underline{A=1} \end{cases}$$

$$\int_1^2 \frac{x+1}{x(x^2+1)}\,dx = \int_1^2 \frac{1}{x}\,dx - \int_1^2 \frac{x}{x^2+1}\,dx + \int_1^2 \frac{1}{x^2+1}\,dx$$

$$= \int_1^2 \frac{1}{x}\,dx - \frac{1}{2}\int_1^2 \frac{1}{x^2+1}\,d\left(x^2+1\right) + \int_1^2 \frac{1}{x^2+1}\,dx$$

$$= \ln x - \frac{1}{2}\ln\left(x^2+1\right) + \arctan x\Big|_1^2$$

$$= \ln 2 - \frac{1}{2}\ln 5 + \arctan 2 + \frac{1}{2}\ln 2 - \frac{\pi}{4}$$

$$= \frac{1}{2}(3\ln 2 - \ln 5) - \frac{\pi}{4} + \arctan 2$$

$$\boxed{= \frac{1}{2}\ln\frac{8}{5} - \frac{\pi}{4} + \arctan 2}$$

Exercise

Evaluate $\displaystyle\int_0^1 \frac{x^2 - x}{x^2 + x + 1}\, dx$

Solution

$$\int_0^1 \frac{x^2 - x}{x^2 + x + 1}\, dx = \int_0^1 \left(1 - \frac{2x+1}{x^2 + x + 1}\right) dx$$

$$= \int_0^1 dx - \int_0^1 \frac{1}{x^2 + x + 1}\, d\left(x^2 + x + 1\right)$$

$$= x - \ln\left(x^2 + x + 1\right)\Big|_0^1$$

$$\underline{= 1 - \ln 3}$$

Exercise

Evaluate $\displaystyle\int_4^8 \frac{y\,dy}{y^2 - 2y - 3}$

Solution

$$\frac{y}{y^2 - 2y - 3} = \frac{A}{y-3} + \frac{B}{y+1} = \frac{(A+B)y + A - 3B}{(y-3)(y+1)} \qquad \to \begin{cases} A + B = 1 \\ A - 3B = 0 \end{cases} \Rightarrow \boxed{A = \frac{3}{4}} \quad \boxed{B = \frac{1}{4}}$$

$$\int_4^8 \frac{y\,dy}{y^2 - 2y - 3} = \frac{3}{4}\int_4^8 \frac{dy}{y-3} + \frac{1}{4}\int_4^8 \frac{dy}{y+1}$$

$$= \left[\frac{3}{4}\ln|y-3| + \frac{1}{4}\ln|y+1|\right]_4^8$$

$$= \frac{3}{4}\ln|5| + \frac{1}{4}\ln|9| - \left(\frac{3}{4}\ln|1| + \frac{1}{4}\ln|5|\right)$$

$$= \frac{3}{4}\ln 5 + \frac{1}{4}\ln 9 - \frac{1}{4}\ln 5$$

$$= \frac{1}{2}\ln 5 + \frac{1}{4}\ln 3^2 \qquad \textit{Power Rule}$$

$$= \frac{1}{2}\ln 5 + \frac{1}{2}\ln 3$$

$$= \frac{1}{2}(\ln 5 + \ln 3) \qquad \textit{Product Rule}$$

$$\underline{= \frac{1}{2}\ln 15}$$

Exercise

Evaluate $\displaystyle\int_1^{\sqrt{3}} \frac{3x^2+x+4}{x^3+x}\,dx$

Solution

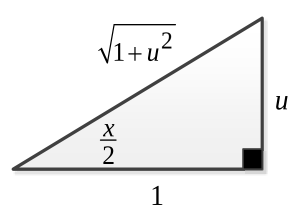
$$\frac{3x^2+x+4}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{(A+B)x^2+Cx+A}{x\left(x^2+1\right)} \qquad \begin{cases} A+B=3 \\ C=1 \\ A=4 \end{cases} \rightarrow \boxed{A=4} \quad \boxed{B=-1} \quad \boxed{C=1}$$

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{3x^2+x+4}{x^3+x}\,dx &= \int_1^{\sqrt{3}} \frac{4}{x}\,dx + \int_1^{\sqrt{3}} \frac{-x+1}{x^2+1}\,dx \\
&= 4\int_1^{\sqrt{3}} \frac{1}{x}\,dx - \int_1^{\sqrt{3}} \frac{x}{x^2+1}\,dx + \int_1^{\sqrt{3}} \frac{1}{x^2+1}\,dx \qquad d\left(x^2+1\right)=2xdx \\
&= 4\int_1^{\sqrt{3}} \frac{1}{x}\,dx - \frac{1}{2}\int_1^{\sqrt{3}} \frac{d\left(x^2+1\right)}{x^2+1} + \int_1^{\sqrt{3}} \frac{1}{x^2+1}\,dx \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} \\
&= \left[4\ln|x| - \tfrac{1}{2}\ln\left(x^2+1\right) + \tan^{-1}x \right]_1^{\sqrt{3}} \\
&= 4\ln\sqrt{3} - \tfrac{1}{2}\ln 4 + \tan^{-1}\sqrt{3} - \left(4\ln 1 - \tfrac{1}{2}\ln 2 + \tan^{-1}1\right) \\
&= 4\ln 3^{1/2} - \tfrac{1}{2}\ln 2^2 + \tfrac{\pi}{3} + \tfrac{1}{2}\ln 2 - \tfrac{\pi}{4} \\
&= 2\ln 3 - \ln 2 + \tfrac{\pi}{12} + \tfrac{1}{2}\ln 2 \\
&= 2\ln 3 - \tfrac{1}{2}\ln 2 + \tfrac{\pi}{12} \\
&= \boxed{\ln\left(\frac{9}{\sqrt{2}}\right) + \frac{\pi}{12}}
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$

Solution



$$\begin{aligned}
\int_0^{\pi/2} \frac{dx}{\sin x + \cos x} &= \int_0^{\pi/2} \frac{1}{\frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}\,du \\
&= 2\int_0^{\pi/2} \frac{du}{2u+1-u^2}
\end{aligned}$$

$$u = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1}u \quad \rightarrow dx = \frac{2du}{1+u^2}$$

$$= -2\int_0^{\pi/2} \frac{du}{u^2 - 2u - 1}$$

$$\cos x = 2\cos^2\frac{x}{2} - 1 = 2\frac{1}{1+u^2} - 1 = \frac{1-u^2}{1+u^2}$$

$$\sin x = 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2}$$

$$= -\frac{1}{\sqrt{2}}\int_0^{\pi/2}\left(\frac{1}{u-1-\sqrt{2}} - \frac{1}{u-1+\sqrt{2}}\right)du$$

$$\frac{2}{u^2 - 2u - 1} = \frac{A}{u-1-\sqrt{2}} + \frac{B}{u-1+\sqrt{2}}$$

$$2 = Au + \left(-1+\sqrt{2}\right)A + Bu + \left(-1-\sqrt{2}\right)B$$

$$\begin{cases} A + B = 0 \\ \left(-1+\sqrt{2}\right)A - \left(1+\sqrt{2}\right)B = 2 \end{cases} \to \begin{cases} B = -A = -\frac{1}{\sqrt{2}} \\ 2\sqrt{2}A = 2 \end{cases}$$

$$= -\frac{1}{\sqrt{2}}\left(\ln\left|\frac{1}{u-1-\sqrt{2}}\right| - \ln\left|\frac{1}{u-1+\sqrt{2}}\right|\right)\Bigg|_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}}\left(\ln\left|\frac{u-1+\sqrt{2}}{u-1-\sqrt{2}}\right|\right)\Bigg|_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}}\left(\ln\left|\frac{\tan\frac{x}{2}-1+\sqrt{2}}{\tan\frac{x}{2}-1-\sqrt{2}}\right|\right)\Bigg|_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}}\left(\ln|-1| - \ln\left|\frac{-1+\sqrt{2}}{-1-\sqrt{2}}\right|\right)$$

$$= \frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

Exercise

Evaluate $\displaystyle\int_0^{\pi/3}\frac{\sin\theta}{1-\sin\theta}d\theta$

Solution

$$\int_0^{\pi/3}\frac{\sin\theta}{1-\sin\theta}d\theta = \int_0^{\pi/3}\frac{1}{\csc\theta - 1}d\theta$$

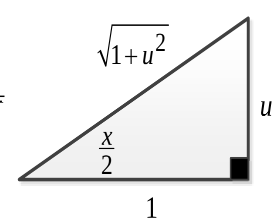
$$= \int_0^{\pi/3}\frac{1}{\frac{1+u^2}{2u}-1}\cdot\frac{2}{1+u^2}du$$

$$u = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1}u \quad \to dx = \frac{2du}{1+u^2}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}}$$

$$= \frac{2u}{1+u^2}$$



$$= \int_0^{\pi/3} \frac{4u}{\left(1+u^2-2u\right)\left(1+u^2\right)}\,du$$

$$= \int_0^{\pi/3} \frac{4u}{(u-1)^2\left(1+u^2\right)}\,du$$

$$\frac{4u}{(u-1)^2\left(1+u^2\right)} = \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{Cu+D}{1+u^2}$$

$$4u = Au + Au^3 - A - Au^2 + B + Bu^2 + Cu^3 - 2Cu^2 + Cu + Du^2 - 2Du + D$$

$$\begin{cases} A + C = 0 \\ -A + B - 2C + D = 0 \\ C - 2D = 4 \\ -A + B + D = 0 \end{cases} \rightarrow \begin{cases} A = 0; \quad B = 2 \\ C = 0; \quad D = -2 \end{cases}$$

$$= \int_0^{\pi/3} \left(\frac{2}{(u-1)^2} - \frac{2}{1+u^2} \right) du$$

$$= \frac{-2}{u-1} - 2\tan^{-1} u \Big|_0^{\pi/3}$$

$$= \frac{-2}{\tan\frac{x}{2} - 1} - 2\tan^{-1}\left(\tan\frac{x}{2}\right)\Big|_0^{\pi/3}$$

$$= \frac{-2}{\tan\frac{x}{2} - 1} - x \Big|_0^{\pi/3}$$

$$= \frac{-2}{\frac{1}{\sqrt{3}} - 1} - \frac{\pi}{3} - 2$$

$$= \frac{-2\sqrt{3}}{1-\sqrt{3}} - \frac{\pi}{3} - 2$$

$$= \frac{-2}{1-\sqrt{3}} - \frac{\pi}{3} \qquad\qquad = \frac{-2}{1-\sqrt{3}}\,\frac{1+\sqrt{3}}{1+\sqrt{3}} - \frac{\pi}{3}$$

$$\boxed{= 1 + \sqrt{3} - \frac{\pi}{3}}$$

Exercise

Find the volume of the solid generated by the revolving the shaded region about x-axis

Solution

$$V = \pi \int_{0.5}^{2.5} y^2 dx$$

$$= \pi \int_{0.5}^{2.5} \frac{9}{3x - x^2} dx$$

$$= 9\pi \int_{0.5}^{2.5} \frac{1}{3x - x^2} dx$$

$$= 9\pi \int_{0.5}^{2.5} \frac{1}{3}\left(\frac{1}{x} + \frac{1}{3-x}\right) dx$$

$$= 3\pi \int_{0.5}^{2.5} \left(\frac{1}{x} - \frac{1}{x-3}\right) dx$$

$$= 3\pi \left[\int_{0.5}^{2.5} \frac{1}{x} dx - \int_{0.5}^{2.5} \frac{1}{x-3} dx\right]$$

$$= 3\pi \left[\ln|x| - \ln|x-3|\right]_{0.5}^{2.5}$$

$$= 3\pi \left[\ln\left|\frac{x}{x-3}\right|\right]_{0.5}^{2.5}$$

$$= 3\pi \left[\ln\left|\frac{2.5}{-.5}\right| - \ln\left|\frac{0.5}{-2.5}\right|\right]$$

$$= 3\pi \left[\ln 5 - \ln \frac{1}{5}\right]$$

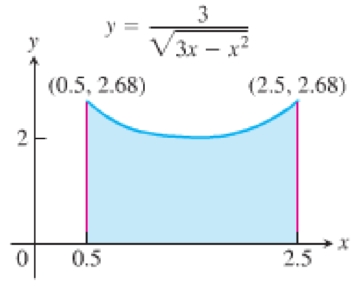
$$= 3\pi \left[\ln 5 + \ln 5\right]$$

$$= 3\pi \left[2 \ln 5\right]$$

$$= 3\pi \ln 25$$

$$\frac{1}{3x - x^2} = \frac{1}{x(3-x)} = \frac{A}{x} + \frac{B}{3-x} = \frac{(B-A)x + 3A}{x(3-x)}$$

$$\begin{cases} B - A = 0 \\ 3A = 1 \end{cases} \Rightarrow \boxed{A = \frac{1}{3}} \quad \boxed{B = \frac{1}{3}}$$



Solution   Section 2.5 – Numerical Integration

Exercise

Find the Midpoint Rule approximations to: $\int_0^1 \sin \pi x \, dx$ *using* $n = 6$ *subintervals*

Solution

$$\Delta x = \frac{1-0}{6} = \frac{1}{6}$$

$$x_0 = 0, \ x_1 = 0 + \frac{1}{6} = \frac{1}{6}, \ x_2 = \frac{1}{3}, \ x_3 = \frac{1}{2}, \ x_4 = \frac{2}{3}, \ x_5 = \frac{5}{6}, \ x_6 = 1$$

$$m_1 = \frac{1}{2}\left(0 + \frac{1}{6}\right) = \frac{1}{12}, \ m_2 = \frac{1}{4}, \ m_3 = \frac{5}{12}, \ m_4 = \frac{7}{12}, \ m_5 = \frac{9}{12}, \ m_6 = \frac{11}{12}$$

$$M(6) = \left(\sin\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{5\pi}{12}\right) + \sin\left(\frac{7\pi}{12}\right) + \sin\left(\frac{9\pi}{12}\right) + \sin\left(\frac{11\pi}{12}\right)\right)\left(\frac{1}{6}\right)$$

$$\approx 0.6439505509$$

Exercise

Find the Midpoint Rule approximations to: $\int_0^1 e^{-x} dx$ *using* $n = 8$ *subintervals*

Solution

$$\Delta x = \frac{1-0}{8} = \frac{1}{8}$$

$$x_0 = 0, \ x_1 = \frac{1}{8}, \ x_2 = \frac{1}{4}, \ x_3 = \frac{3}{8}, \ x_4 = \frac{1}{2}, \ x_5 = \frac{5}{8}, \ x_6 = \frac{3}{4}, \ x_7 = \frac{7}{8}, \ x_8 = 1$$

$$m_1 = \frac{1}{2}\left(0 + \frac{1}{8}\right) = \frac{1}{16}, \ m_2 = \frac{3}{16}, \ m_3 = \frac{5}{16}, \ m_4 = \frac{7}{16}, \ m_5 = \frac{9}{16}, \ m_6 = \frac{11}{16}, \ m_7 = \frac{13}{16}, \ m_8 = \frac{15}{16}$$

$$M(8) = \frac{1}{8}\left(e^{-1/16} + e^{-3/16} + e^{-5/16} + e^{-7/16} + e^{-9/16} + e^{-11/16} + e^{-13/16} + e^{-15/16}\right)$$

$$\approx 0.6317092095$$

Exercise

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of 10^{-4} by (***a***) the Trapezoid Rule and (***b***) Simpson's Rule. $\int_1^3 (2x-1)\,dx$

Solution

a) ***i***) $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$

$$T = \frac{1}{2}\Delta x\left(m\,f\left(x_i\right)\right)$$

$$= \frac{1}{2}\frac{1}{2}(24) \;= \underline{6}|$$

$$f(x) = 2x - 1 \quad \Rightarrow \quad f'(x) = 2$$

$$\Rightarrow \quad f''(x) = 0 = M$$

$$\Rightarrow Error = 0$$

	x_i	$f\left(x_i\right) = 2x_i - 1$	m	$mf\left(x_i\right)$
x_0	1	1	1	1
x_1	$1 + \frac{1}{2} = \frac{3}{2}$	2	2	4
x_2	2	3	2	6
x_3	$\frac{5}{2}$	4	2	8
x_4	3	5	1	5
				24

ii) $$\int_1^3 (2x-1)\,dx = \left[x^2 - x\right]_1^3$$

$$= \left(3^2 - 3\right) - \left(1^2 - 1\right)$$

$$\underline{= 6}|$$

iii) $$Error = \frac{|E_T|}{True\ \ Value} \times 100 = 0\%$$

b) ***i***) $$\underline{|\Delta x} = \frac{b-a}{n} = \frac{3-1}{4} = \underline{\frac{1}{2}}|$$

$$S = \frac{1}{3}\Delta x\left(\sum m\,f\left(x_i\right)\right)$$

$$= \frac{1}{3}\frac{1}{2}(36) \;= \underline{6}|$$

$$f(x) = 2x - 1 \quad \Rightarrow \quad f^{(4)}(x) = 0 = M$$

$$\Rightarrow \left|E_s\right| = 0$$

	x_i	$f\left(x_i\right) = 2x_i - 1$	m	$mf\left(x_i\right)$
x_0	1	1	1	1
x_1	$\frac{3}{2}$	2	4	8
x_2	2	3	2	6
x_3	$\frac{5}{2}$	4	4	16
x_4	3	5	1	5
				36

ii) $$\int_1^3 (2x-1)\,dx = 6$$

$$\left|E_s\right| = \int_1^3 (2x-1)\,dx - S = 6 - 6 = 0$$

iii) $$Error = \frac{|E_T|}{True\ \ Value} \times 100 = 0\%$$

Exercise

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of 10^{-4} by (***a***) the Trapezoid Rule and (***b***) Simpson's Rule. $\int_{-1}^{1}\left(x^2+1\right)dx$

Solution

a) ***i***) $\Delta x = \frac{b-a}{n} = \frac{1+1}{4} = \frac{1}{2}$

	x_i	$f\left(x_i\right)$	m	$m\,f\left(x_i\right)$
x_0	-1	2	1	2
x_1	$-\frac{1}{2}$	$\frac{5}{4}$	2	$\frac{5}{2}$
x_2	0	1	2	2
x_3	$\frac{1}{2}$	$\frac{5}{4}$	2	$\frac{5}{2}$
x_4	1	2	1	2
				11

$$T = \frac{1}{2}\Delta x\left(m\,f\left(x_i\right)\right) = \frac{1}{2}\frac{1}{2}(11) = \frac{11}{4}$$

$$f(x) = x^2+1 \quad\Rightarrow\quad f'(x) = 2x$$

$$\Rightarrow\quad f''(x) = 2 = M$$

$$\left|E_T\right| = \frac{1-(-1)}{12}\left(\frac{1}{2}\right)^2(2) = 0.0833\ldots$$

ii) $\int_{-1}^{1}\left(x^2+1\right)dx = \left[\frac{1}{3}x^3+x\right]_{-1}^{1} = \left(\frac{1}{3}+1\right)-\left(-\frac{1}{3}-1\right) = \frac{8}{3}$

$$E_T = \int_{-1}^{1}\left(x^2+1\right)dx - T = \frac{8}{3}-\frac{11}{4} = -\frac{1}{12}$$

iii) $Error = \frac{\left|E_T\right|}{True\ \ Value}\times 100 = \frac{\frac{1}{12}}{\frac{8}{3}} \approx 3\%$

b) ***i***) $\Delta x = \frac{b-a}{n} = \frac{-1-(-1)}{4} = \frac{1}{2}$

	x_i	$f\left(x_i\right)$	m	$m\,f\left(x_i\right)$
x_0	-1	2	1	2
x_1	$-\frac{1}{2}$	$\frac{5}{4}$	4	5
x_2	0	1	2	2
x_3	$\frac{1}{2}$	$\frac{5}{4}$	4	5
x_4	1	2	1	2
				16

$$S = \frac{1}{3}\Delta x\left(\sum m\,f\left(x_i\right)\right) = \frac{1}{3}\frac{1}{2}(16) = \frac{8}{3}$$

$$f(x) = x^2+1 \quad\Rightarrow\quad f^{(4)}(x) = 0 = M$$

$$\Rightarrow \left|E_s\right| = 0$$

ii) $\int_{-1}^{1}\left(x^2+1\right)dx = \frac{8}{3}$

$$E_S = \int_{-1}^{1}\left(x^2+1\right)dx - S = \frac{8}{3}-\frac{8}{3} = 0$$

iii) $Error = \frac{\left|E_T\right|}{True\ \ Value}\times 100 = 0\%$

Exercise

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of 10^{-4} by (***a***) the Trapezoid Rule and (***b***) Simpson's Rule. $\int_2^4 \frac{1}{(s-1)^2}\,ds$

Solution

a) $\boxed{\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2}}$

$x_0 = 2 \qquad x_1 = 2 + \frac{1}{2} = \frac{5}{2} \qquad x_2 = 2 + 2\left(\frac{1}{2}\right) = 3 \qquad x_3 = 2 + 3\left(\frac{1}{2}\right) = \frac{7}{2} \qquad x_4 = 4$

$$T = \frac{1}{2}\Delta x\left(m\,f\left(x_i\right)\right)$$

$$= \frac{1}{2}\frac{1}{2}\left(\frac{1}{(2-1)^2} + 2\frac{1}{\left(\frac{5}{2}-1\right)^2} + 2\frac{1}{(3-1)^2} + 2\frac{1}{\left(\frac{7}{2}-1\right)^2} + \frac{1}{(4-1)^2}\right)$$

$$= \frac{1}{4}\left(\frac{1}{1} + \frac{8}{9} + \frac{1}{2} + \frac{8}{25} + \frac{1}{9}\right)$$

$$= \frac{1269}{1800}$$

$$\boxed{\approx 0.705}$$

$f(s) = (s-1)^{-2} \quad \Rightarrow f'(s) = -2(s-1)^{-3}$

$\qquad \Rightarrow f''(s) = 6(s-1)^{-4} = \frac{6}{(s-1)^4} \Rightarrow M = 6$

$$\int_2^4 \frac{1}{(s-1)^2}\,ds = \int_2^4 (s-1)^{-2}\,d(s-1)$$

$$= -\left[(s-1)^{-1}\right]_2^4$$

$$= -\left(3^{-1} - 1^{-1}\right)$$

$$\boxed{= \frac{2}{3}}$$

The percentage error: $\frac{|0.705 - .6667|}{.6667}$ $\boxed{\approx 0.0575}$ $\boxed{5.75\%}$

b) $\boxed{\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2}}$

$x_0 = 2 \qquad x_1 = 2 + \frac{1}{2} = \frac{5}{2} \qquad x_2 = 2 + 2\left(\frac{1}{2}\right) = 3 \qquad x_3 = 2 + 3\left(\frac{1}{2}\right) = \frac{7}{2} \qquad x_4 = 4$

$$S = \frac{1}{3}\Delta x\left(m\,f\left(x_i\right)\right)$$

$$= \frac{1}{3}\frac{1}{2}\left(\frac{1}{(2-1)^2} + 4\frac{1}{\left(\frac{5}{2}-1\right)^2} + 2\frac{1}{(3-1)^2} + 4\frac{1}{\left(\frac{7}{2}-1\right)^2} + \frac{1}{(4-1)^2} \right)$$

$$= \frac{1}{6}\left(\frac{1}{1} + \frac{16}{9} + \frac{1}{2} + \frac{16}{25} + \frac{1}{9}\right)$$

$$= \frac{1813}{450}$$

$$\approx 0.67148$$

$$\int_2^4 \frac{1}{(s-1)^2}\,ds = \frac{2}{3}$$

The percentage error: $\frac{|0.67148 - .6667|}{.6667} \approx 0.0072$ $\quad 0.72\%$

Exercise

Find the Trapezoid & Simpson's Rule approximations and error: $\int_0^1 \sin \pi x \, dx \quad n = 6$ *subintervals*

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1666666667	0.5000000000	1.0000000000
2	0.3333333333	0.8660254000	1.7320508000
3	0.5000000000	1.0000000000	2.0000000000
4	0.6666666667	0.8660254000	1.7320508000
5	0.8333333333	0.5000000000	1.0000000000
6	1.0000000000	0.0000000000	0.0000000000

Trapezoid Rule approximation ≈ 0.62200847

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1666666667	0.5000000000	2.0000000000
2	0.3333333333	0.8660254000	1.7320508000
3	0.5000000000	1.0000000000	2.0000000000
4	0.6666666667	0.8660254000	1.7320508000
5	0.8333333333	0.5000000000	1.0000000000
6	1.0000000000	0.0000000000	0.0000000000

Simpson's Rule approximation ≈ 0.63689453

	Exact	Trapezoid	Simpson
Value:	0.63661977	0.62200847	0.63689453
Error:		2.2951 %	0.0432 %

Exercise

Find the Trapezoid & Simpson's Rule approximations to and error to $\int_0^1 e^{-x}dx \quad n=8 \ \ subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.1250000000	0.8824969000	1.7649938000
2	0.2500000000	0.7788007800	1.5576015600
3	0.3750000000	0.6872892800	1.3745785600
4	0.5000000000	0.6065306600	1.2130613200
5	0.6250000000	0.5352614300	1.0705228600
6	0.7500000000	0.4723665500	0.9447331000
7	0.8750000000	0.4168620200	0.8337240400
8	1.0000000000	0.3678794400	0.3678794400

Trapezoid Rule approximation ≈ 0.63294342

Simpson's Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.1250000000	0.8824969000	3.5299876000
2	0.2500000000	0.7788007800	1.5576015600
3	0.3750000000	0.6872892800	2.7491571200
4	0.5000000000	0.6065306600	1.2130613200
5	0.6250000000	0.5352614300	2.1410457200
6	0.7500000000	0.4723665500	0.9447331000
7	0.8750000000	0.4168620200	1.6674480800
8	1.0000000000	0.3678794400	0.3678794400

Simpson's Rule approximation ≈ 0.63212141

	Exact	Trapezoid	Simpson
Value:	0.63212056	0.63294342	0.63212141
Error:		0.1302 %	0.0001 %

Exercise

Find the Trapezoid & Simpson's Rule approximations and error to:

$$\int_1^5 \left(3x^2 - 2x\right) dx \quad n = 8 \;\; subintervals$$

Solution

Trapezoid Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.5000000000	3.7500000000	7.5000000000
2	2.0000000000	8.0000000000	16.0000000000
3	2.5000000000	13.7500000000	27.5000000000
4	3.0000000000	21.0000000000	42.0000000000
5	3.5000000000	29.7500000000	59.5000000000
6	4.0000000000	40.0000000000	80.0000000000
7	4.5000000000	51.7500000000	103.500000000
8	5.0000000000	65.0000000000	65.0000000000

Trapezoid Rule approximation ≈ 100.50000000

Simpson's Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.5000000000	3.7500000000	15.0000000000
2	2.0000000000	8.0000000000	16.0000000000
3	2.5000000000	13.7500000000	55.0000000000
4	3.0000000000	21.0000000000	42.0000000000
5	3.5000000000	29.7500000000	119.000000000
6	4.0000000000	40.0000000000	80.0000000000
7	4.5000000000	51.7500000000	207.000000000
8	5.0000000000	65.0000000000	65.0000000000

Simpson's Rule approximation ≈ 100.00000000

	Exact	***Trapezoid***	***Simpson***
Value:	100.000000	100.500000	100.00000000
Error:		0.5000%	0.0000 %

Exercise

Find the Trapezoid & Simpson's Rule approximations and error: $\int_0^{\pi/4} 3\sin 2x\, dx \quad n = 8 \;\; subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.0981747704	0.5852709700	1.1705419400
2	0.1963495408	1.1480503000	2.2961006000
3	0.2945243113	1.6667107000	3.3334214000
4	0.3926990817	2.1213203400	4.2426406800
5	0.4908738521	2.4944088400	4.9888176800
6	0.5890486225	2.7716386000	5.5432772000
7	0.6872233930	2.9423558400	5.8847116800
8	0.7853981634	3.0000000000	3.0000000000

Trapezoid Rule approximation ≈ 1.49517776

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.0981747704	0.5852709700	2.3410838800
2	0.1963495408	1.1480503000	2.2961006000
3	0.2945243113	1.6667107000	6.6668428000
4	0.3926990817	2.1213203400	4.2426406800
5	0.4908738521	2.4944088400	9.9776353600
6	0.5890486225	2.7716386000	5.5432772000
7	0.6872233930	2.9423558400	11.7694233600
8	0.7853981634	3.0000000000	3.0000000000

Simpson's Rule approximation ≈ 1.50001244

	Exact	***Trapezoid***	***Simpson***
Value:	1.500000	1.49517776	1.50001244
Error:		0.3215 %	0.0008 %

Exercise

Find the Trapezoid & Simpson's Rule approximations and error: $\int_0^8 e^{-2x}\,dx \quad n=8 \;\; subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.0000000000	0.1353352800	0.2706705600
2	2.0000000000	0.0183156400	0.0366312800
3	3.0000000000	0.0024787500	0.0049575000
4	4.0000000000	0.0003354600	0.0006709200
5	5.0000000000	0.0000454000	0.0000908000
6	6.0000000000	0.0000061400	0.0000122800
7	7.0000000000	0.0000008300	0.0000016600
8	8.0000000000	0.0000001100	0.0000001100

Trapezoid Rule approximation ≈ 0.65651755

Simpson's Rule Method

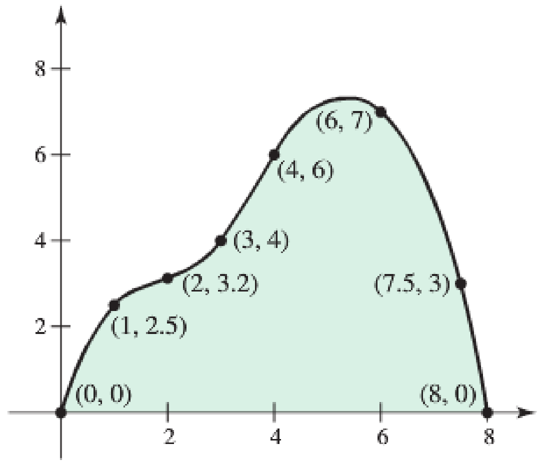
n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.0000000000	0.1353352800	0.5413411200
2	2.0000000000	0.0183156400	0.0366312800
3	3.0000000000	0.0024787500	0.0099150000
4	4.0000000000	0.0003354600	0.0006709200
5	5.0000000000	0.0000454000	0.0001816000
6	6.0000000000	0.0000061400	0.0000122800
7	7.0000000000	0.0000008300	0.0000033200
8	8.0000000000	0.0000001100	0.0000001100

Simpson's Rule approximation ≈ 0.52958521

	Exact	***Trapezoid***	***Simpson***
Value:	0.49999994	0.65651755	0.52958521
Error:		31.3035 %	5.9171 %

Exercise

A piece of wood paneling must be cut in the shape shown below. The coordinates of several point on its curved surface are also shown (with units of inches).



a) Estimate the surface area of the paneling using the Trapezoid Rule

b) Estimate the surface area of the paneling using a left Riemann sum.

c) Could two identical pieces be cut from a 9-in by 9-in piece of wood?

Solution

a) The *trapezoid* Rule gives

$$\frac{(0+.25)\cdot 1}{2}+\frac{(2.5+3.2)\cdot 1}{2}+\frac{(3.2+4)\cdot 1}{2}+\frac{(4+6)\cdot 1}{2}+\frac{(6+7)\cdot 2}{2}+\frac{(7+5.3)\cdot 1.5}{2}+\frac{(3+0)\cdot 0.5}{2}\ \underline{=35.675}$$

b) The left *Riemann* sum gives

$$0\cdot 1+2.5\cdot 1+3.2\cdot 1+4\cdot 1+6\cdot 2+7\cdot 1.5+5.3\cdot 0.5\ \underline{=34.85}$$

c) Although the surface area of the piece appears to be less than half of $81=9^2$ (area of 9×9 piece of wood), the shape prohibits the creation of two identical pieces.

Solution Section 2.6 – Improper Integrals

Exercise

Evaluate the integral $\displaystyle\int_0^{\infty} \frac{dx}{x^2+1}$

Solution

$$\int_0^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_0^{b} \frac{dx}{x^2+1}$$

$$= \lim_{b\to\infty} \left[\tan^{-1} x\right]_0^b$$

$$= \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 0\right)$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{4} \frac{dx}{\sqrt{4-x}}$

Solution

$$\int_0^{4} \frac{dx}{\sqrt{4-x}} = \lim_{b\to 4^-} \int_0^{b} (4-x)^{-1/2}\, dx$$

$$= \lim_{b\to 4^-} \int_0^{b} -(4-x)^{-1/2}\, d(4-x)$$

$$= -2 \lim_{b\to 4^-} \left[(4-x)^{1/2}\right]_0^b$$

$$= -2 \lim_{b\to 4^-} \left[(4-b)^{1/2} - (4)^{1/2}\right]$$

$$= -2(0-2)$$

$$= 4$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{2} \frac{2dx}{x^2+4}$

Solution

$$\int_{-\infty}^{2} \frac{2dx}{x^2+4} = 2\lim_{b\to-\infty}\int_{b}^{2} \frac{dx}{x^2+2^2}$$

$$= 2\lim_{b\to-\infty}\frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_{b}^{2}$$

$$= \lim_{b\to-\infty}\left[\tan^{-1}1-\tan^{-1}\frac{b}{2}\right]$$

$$= \frac{\pi}{4}-\left(-\frac{\pi}{2}\right)$$

$$= \frac{3\pi}{4}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{xdx}{\left(x^2+4\right)^{3/2}}$

Solution

$$\int_{-\infty}^{\infty} \frac{xdx}{\left(x^2+4\right)^{3/2}} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{d\left(x^2+4\right)}{\left(x^2+4\right)^{3/2}} \qquad u = x^2+4 \quad \to du = 2xdx$$

$$= \frac{1}{2}\left[-2\left(x^2+4\right)^{-1/2}\right]_{-\infty}^{\infty}$$

$$= -\left[\frac{1}{\sqrt{x^2+4}}\right]_{-\infty}^{\infty}$$

$$= -(0-0)$$

$$= 0$$

Exercise

Evaluate the integral $\displaystyle\int_{1}^{\infty} \frac{dx}{x\sqrt{x^2-1}}$

Solution

$$\int_{1}^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \int_{1}^{2} \frac{dx}{x\sqrt{x^2-1}} + \int_{2}^{\infty} \frac{dx}{x\sqrt{x^2-1}}$$

$$= \lim_{b \to 1^+} \int_b^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{c \to \infty} \int_2^c \frac{dx}{x\sqrt{x^2-1}}$$

$$= \lim_{b \to 1^+} \left[\sec^{-1}|x| \right]_b^2 + \lim_{c \to \infty} \left[\sec^{-1}|x| \right]_2^c$$

$$= \lim_{b \to 1^+} \left(\sec^{-1} 2 - \sec^{-1} b \right) + \lim_{c \to \infty} \left(\sec^{-1} c - \sec^{-1} 2 \right)$$

$$= \left(\frac{\pi}{3} - 0 \right) + \left(\frac{\pi}{2} - \frac{\pi}{3} \right)$$

$$= \frac{\pi}{2}$$

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx$

Solution

$$\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx = \int_{-\infty}^{0} 2xe^{-x^2}\,dx + \int_{0}^{\infty} 2xe^{-x^2}\,dx$$

$$d\left(-x^2\right) = -2xdx$$

$$= -\lim_{b \to -\infty} \int_b^0 e^{-x^2}\,d\left(-x^2\right) - \lim_{c \to \infty} \int_0^c e^{-x^2}\,d\left(-x^2\right)$$

$$= -\lim_{b \to -\infty} \left[e^{-x^2} \right]_b^0 - \lim_{c \to \infty} \left[e^{-x^2} \right]_0^c$$

$$= -\lim_{b \to -\infty} \left(1 - e^{-b^2} \right) - \lim_{c \to \infty} \left(e^{-c^2} - 1 \right) \qquad = -(1-0) - (0-1)$$

$$= 0$$

Exercise

Evaluate the integral $\int_0^1 (-\ln x)\,dx$

Solution

$$\int_0^1 (-\ln x)\,dx = -\lim_{b \to 0^+} \int_b^1 (\ln x)\,dx$$

$$= -\lim_{b \to 0^+} \left[x\ln x - x \right]_b^1$$

$$= -\lim_{b \to 0^+} \left(\ln 1 - 1 - (b\ln b - b) \right)$$

$$= -(0 - 1 - 0 + 0)$$

$$\underline{\underline{= 1}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{4} \frac{dx}{\sqrt{|x|}}$

Solution

$$\int_{-1}^{4} \frac{dx}{\sqrt{|x|}} = \lim_{b \to 0^-} \int_{-1}^{b} \frac{dx}{\sqrt{-x}} + \lim_{c \to 0^+} \int_{c}^{4} \frac{dx}{\sqrt{x}}$$

$$= \lim_{b \to 0^-} \left[-2\sqrt{-x}\right]_{-1}^{b} + \lim_{c \to 0^+} \left[2\sqrt{x}\right]_{c}^{4}$$

$$= \lim_{b \to 0^-} \left(-2\sqrt{-b} + 2\right) + \lim_{c \to 0^+} \left(2\sqrt{4} - 2\sqrt{c}\right)$$

$$= 2 + 4$$

$$\underline{\underline{= 6}}$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{\infty} e^{-3x}\, dx$

Solution

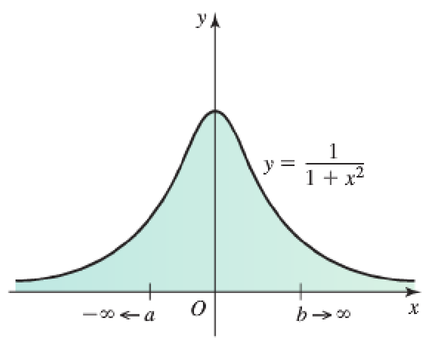
$$\int_{0}^{\infty} e^{-3x}\, dx = -\frac{1}{3} e^{-3x} \Big|_{0}^{\infty}$$

$$= -\frac{1}{3}\left(e^{-\infty} - 1\right)$$

$$\underline{\underline{= \frac{1}{3}}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$



Area of region under the curve $y = \frac{1}{1+x^2}$ on $(-\infty, \infty)$ has finite value π.

Solution

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \tan^{-1} x \Big|_{-\infty}^{\infty}$$

$$= \tan^{-1} \infty - \tan^{-1}(-\infty)$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$\underline{\underline{= \pi}}$$

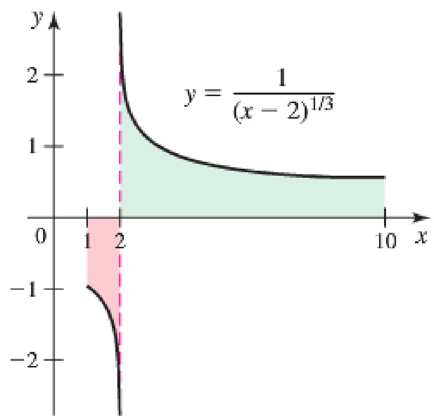
Exercise

Evaluate the integral $\displaystyle\int_1^{10} \frac{dx}{(x-2)^{1/3}}$

Solution

$$\begin{aligned}\int_1^{10} (x-2)^{-1/3}\,dx &= \frac{3}{2}(x-2)^{2/3}\Big|_1^{10}\\ &= \frac{3}{2}\left(8^{2/3}-(-1)^{2/3}\right)\\ &= \frac{3}{2}(4-1)\\ &= \frac{9}{2}\end{aligned}$$

$$\begin{aligned}\int_1^{10} (x-2)^{-1/3}\,dx &= \int_1^{2} (x-2)^{-1/3}\,dx + \int_2^{10} (x-2)^{-1/3}\,dx\\ &= \frac{3}{2}(x-2)^{2/3}\Big|_1^{2} + (x-2)^{2/3}\Big|_2^{10}\\ &= \frac{3}{2}\left(0-(-1)^{2/3}\right)+\frac{3}{2}\left(8^{2/3}-0\right)\\ &= \frac{3}{2}(-1+4)\\ &= \frac{9}{2}\end{aligned}$$



Exercise

Evaluate the integral $\displaystyle\int_1^{\infty} \frac{dx}{x^2}$

Solution

$$\begin{aligned}\int_1^{\infty} \frac{dx}{x^2} &= -\frac{1}{x}\Big|_1^{\infty}\\ &= -\left(\frac{1}{\infty}-1\right)\\ &= -(0-1)\\ &= 1\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\infty} \frac{dx}{(x+1)^3}$

Solution

$$\int_0^{\infty} (x+1)^{-3} dx = -\frac{2}{(x+1)^2}\Big|_0^{\infty}$$

$$= -2\left(\frac{1}{\infty} - 1\right)$$

$$= -2(0-1)$$

$$\underline{=2}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{0} e^x\, dx$

Solution

$$\int_{-\infty}^{0} e^x\, dx = e^x\Big|_{-\infty}^{0}$$

$$= \left(1 - e^{-\infty}\right)$$

$$\underline{=1}$$

Exercise

Evaluate the integral $\displaystyle\int_1^{\infty} 2^{-x}\, dx$

Solution

$$\int_1^{\infty} 2^{-x}\, dx = -\int_1^{\infty} 2^{-x}\, d(-x)$$

$$\int a^x dx = \frac{a^x}{\ln a}$$

$$= -\frac{2^{-x}}{\ln 2}\Big|_1^{\infty}$$

$$= -\frac{1}{\ln 2}\left(0 - \frac{1}{2}\right)$$

$$\underline{= \frac{1}{2\ln 2}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{0} \frac{dx}{\sqrt[3]{2-x}}$

Solution

$$\int_{-\infty}^{0} \frac{dx}{\sqrt[3]{2-x}} = -\int_{-\infty}^{0} (2-x)^{-1/3}\, d(2-x)$$

$$= -\frac{3}{2}(2-x)^{2/3}\Big|_{-\infty}^{0}$$

$$= -\frac{3}{2}\left(2^{2/3} - \infty\right)$$

$$\underline{= \infty} \quad \textit{diverges}$$

Exercise

Evaluate the integral $\displaystyle\int_{4/\pi}^{\infty} \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)dx$

Solution

$$\int_{4/\pi}^{\infty} \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)\, dx = -\int_{4/\pi}^{\infty} \sec^2\left(\frac{1}{x}\right)\, d\left(\frac{1}{x}\right) \qquad d\left(\frac{1}{x}\right) = -\frac{1}{x^2}\,dx$$

$$= -\tan\left(\frac{1}{x}\right)\Big|_{4/\pi}^{\infty}$$

$$= -\left(\tan 0 - \tan\frac{\pi}{4}\right)$$

$$\underline{= 1}$$

Exercise

Evaluate the integral $\displaystyle\int_{e^2}^{\infty} \frac{dx}{x\ln^p x} \quad p > 1$

Solution

$$\int_{e^2}^{\infty} \frac{dx}{x\ln^p x} = \int_{e^2}^{\infty} (\ln x)^{-p}\, d(\ln x)$$

$$= \frac{1}{1-p}(\ln x)^{1-p}\Big|_{e^2}^{\infty}$$

$$= \frac{1}{1-p}\left((\ln x)^{-\infty} - \left(\ln e^2\right)^{1-p}\right)$$

$$= \frac{-1}{1-p} 2^{1-p}$$

$$\boxed{= \frac{1}{(p-1)2^{p-1}}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\infty} \frac{p}{\sqrt[5]{p^2+1}}\, dp$

Solution

$$\int_0^{\infty} \frac{p}{\sqrt[5]{p^2+1}}\, dp = \frac{1}{2}\int_0^{\infty} \left(p^2+1\right)^{-1/5}\ d\left(p^2+1\right) \qquad d\left(p^2+1\right) = 2pdp$$

$$= \frac{5}{8}\left(p^2+1\right)^{4/5}\Big|_0^{\infty}$$

$$\boxed{= \infty} \quad \textit{diverges}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{1} \ln y^2\ dy$

Solution

$$\int_{-1}^{1} \ln y^2\ dy = 2\int_0^1 \ln y^2\ dy \qquad \int \ln x^2 dx = 2\int \ln x dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx \qquad v = \int dx = x$$

$$= 4\left(y\ln y - y\right)\Big|_0^1 \qquad = 2\left[x\ln x - \int dx\right]$$

$$= 4\left[-1-0\right] \qquad \boxed{= 2(x\ln x - x) + C}$$

$$\boxed{= -4}$$

Exercise

Evaluate the integral $\displaystyle\int_{-2}^{6} \frac{dx}{\sqrt{|x-2|}}$

Solution

$$\int_{-2}^{6} \frac{dx}{\sqrt{|x-2|}} = \int_{-2}^{2} \frac{dx}{\sqrt{2-x}} + \int_2^6 \frac{dx}{\sqrt{x-2}}$$

$$= -\int_{-2}^{2} (2-x)^{-1/2}\, d(2-x) + \int_2^6 (x-2)^{-1/2}\, d(x-2)$$

$$= -2\sqrt{2-x}\,\Big|_{-2}^{2} + 2\sqrt{x-2}\,\Big|_{2}^{6}$$

$$= -2(0-2) + 2(2-0)$$

$$\underline{= 8}$$

Exercise

Evaluate $\displaystyle\int_0^{\infty} xe^{-x}dx$

Solution

$$\int_0^{\infty} xe^{-x}dx = -xe^{-x} - e^{-x}\Big|_0^{\infty}$$

$$= 0 - (-1)$$

$$\underline{= 1}$$

		$\int e^{-x}$
+	x	$-e^{-x}$
−	1	e^{-x}

Exercise

Evaluate $\displaystyle\int_0^{1} x\ln x\, dx$

Solution

$u = \ln x \quad dv = x\, dx$

$du = \dfrac{dx}{x} \quad v = \dfrac{1}{2}x^2$

$$\int x\ln x\, dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int xdx \;\; \underline{= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2}$$

$$\int_0^1 x\ln x\, dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2\Big|_0^1$$

$$\underline{= -\frac{1}{4}}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}dx$

Solution

$u = \ln x \quad dv = \dfrac{1}{x^2}\, dx$

$du = \dfrac{dx}{x} \quad v = -\dfrac{1}{x}$

$$\int \frac{\ln x}{x^2}\, dx = -\frac{1}{x}\ln x + \int \frac{1}{x^2}dx$$

$$= -\frac{1}{x}\ln x - \frac{1}{x}$$

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = -\frac{1}{x}(\ln x + 1)\Big|_1^{\infty}$$

$$\underline{= 1}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} (1-x)e^{-x}\,dx$

Solution

$$\int_1^{\infty} (1-x)e^{-x}\,dx = \left[-e^{-x} - (-x-1)e^{-x}\right]_1^{\infty}$$

$$= \left[xe^{-x}\right]_1^{\infty}$$

$$= 0 - e^{1}$$

$$\underline{= \frac{1}{e}}$$

Exercise

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$

Solution

$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \int_{-\infty}^{\infty} \frac{du}{1+u^2} \qquad u = e^x \to du = e^x dx$$

$$= \arctan e^x\Big|_{-\infty}^{\infty} \qquad = \arctan\infty - \arctan 0$$

$$\underline{= \frac{\pi}{2}}$$

Exercise

Evaluate $\displaystyle\int_0^{1} \frac{dx}{\sqrt[3]{x}}$

Solution

$$\int_0^{1} x^{-1/3}dx = \frac{3}{2}x^{2/3}\Big|_0^{1} \quad \underline{= \frac{3}{2}}$$

Exercise

Evaluate $\displaystyle\int_1^\infty \frac{4}{\sqrt[4]{x}}\, dx$

Solution

$$\int_1^\infty 4x^{-1/4}dx = \frac{16}{3}x^{3/4}\Big|_1^\infty$$

$$\underline{= \infty} \qquad \textit{Diverges}$$

Exercise

Evaluate $\displaystyle\int_0^2 \frac{dx}{x^3}$

Solution

$$\int_0^2 \frac{dx}{x^3} = -\frac{1}{2x^2}\Big|_0^2$$

$$= -\frac{1}{8} + \infty$$

$$\underline{= \infty} \qquad \textit{Diverges}$$

Exercise

Evaluate $\displaystyle\int_1^\infty \frac{dx}{x^3}$

Solution

$$\int_1^\infty \frac{dx}{x^3} = -\frac{1}{2x^2}\Big|_1^\infty \quad \underline{= \frac{1}{2}}$$

Exercise

Evaluate $\displaystyle\int_1^\infty \frac{6}{x^4}\, dx$

Solution

$$\int_1^\infty 6x^{-4}\, dx = -2\frac{1}{x^3}\Big|_1^\infty \quad \underline{= 2}$$

Exercise

Evaluate $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$

Solution

$u=\sqrt{x} \ \to\ u^2 = x \quad \Rightarrow dx = 2udu$

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = \int_0^\infty \frac{2u}{u\left(u^2+1\right)}du$$

$$= 2\int_0^\infty \frac{1}{u^2+1}du$$

$$= 2\arctan\sqrt{x}\Big|_0^\infty$$

$$= 2\left(\frac{\pi}{2}-0\right)$$

$$= \pi$$

Exercise

Evaluate $\displaystyle\int_{-\infty}^0 xe^{-4x}dx$

Solution

$$\int_{-\infty}^0 xe^{-4x}dx = \left(-\frac{x}{4}-\frac{1}{16}\right)e^{-4x}\Big|_{-\infty}^0$$

$$= -\frac{1}{16}-\infty$$

$$= -\infty$$ ***Diverges***

Exercise

Evaluate $\displaystyle\int_0^\infty xe^{-x/3}dx$

Solution

$$\int_0^\infty xe^{-x/3}dx = (-3x-9)e^{-x/3}\Big|_0^\infty$$

$$= 9$$

Exercise

Evaluate $\displaystyle\int_0^{\infty} x^2 e^{-x} dx$

Solution

$$\int_0^{\infty} x^2 e^{-x} dx = \left(-x^2 - 2x - 2\right)e^{-x}\Big|_0^{\infty} \underline{\underline{= 2}}$$

Exercise

Evaluate $\displaystyle\int_0^{\infty} e^{-x} \cos x \, dx$

Solution

$$\int e^{-x} \cos x \, dx = e^{-x}(\sin x - \cos x) - \int e^{-x} \cos x \, dx$$

$$2\int e^{-x} \cos x \, dx = e^{-x}(\sin x - \cos x)$$

$$\int_0^{\infty} e^{-x} \cos x \, dx = \frac{1}{2} e^{-x}(\sin x - \cos x)\Big|_0^{\infty}$$

$$= \frac{1}{2}(0 - (-1))$$

$$\boxed{= \frac{1}{2}}$$

		$\int \cos x$
+	e^{-x}	$\sin x$
−	$-e^{-x}$	$-\cos x$
+	e^{-x}	$-\int \cos x$

Exercise

Evaluate $\displaystyle\int_4^{\infty} \frac{1}{x(\ln x)^3} dx$

Solution

$$\int_4^{\infty} \frac{1}{x(\ln x)^3} dx = \int_4^{\infty} (\ln x)^{-3} \, d(\ln x)$$

$$= -\frac{1}{2} \frac{1}{(\ln x)^2}\Big|_4^{\infty}$$

$$= \frac{1}{2}\left(0 - \frac{1}{(\ln 4)^2}\right)$$

$$\boxed{= \frac{1}{2(\ln 4)^2}}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} \frac{\ln x}{x}\, dx$

Solution

$$\int_1^{\infty} \frac{\ln x}{x}\, dx = \int_1^{\infty} \ln x\; d(\ln x)$$

$$= \frac{1}{2}(\ln x)^2 \Big|_1^{\infty}$$

$$= \infty \qquad \textbf{\textit{Diverges}}$$

Exercise

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{4}{16+x^2}\, dx$

Solution

$$\int_{-\infty}^{\infty} \frac{4}{16+x^2}\, dx = \arctan\left(\frac{x}{4}\right)\Big|_{-\infty}^{\infty}$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$

$$= \pi$$

Exercise

Find the area of the region *R* between the graph of $f(x) = \dfrac{1}{\sqrt{9-x^2}}$ and the *x-axis* on the interval $(-3,\ 3)$ (if it exists)

Solution

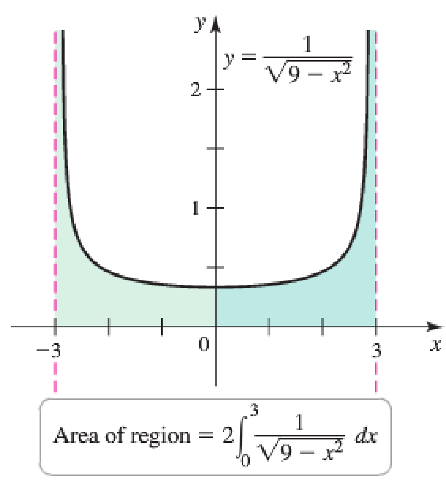
$$A = \int_{-3}^{3} \frac{dx}{\sqrt{9-x^2}}$$

$$= 2\int_0^{3} \frac{dx}{\sqrt{9-x^2}}$$

$$= 2\sin^{-1}\frac{x}{3}\Big|_0^3$$

$$= 2\left(\sin^{-1}1 - \sin^{-1}0\right)$$

$$= \pi \ \ unit^2$$



Exercise

Find the volume of the region bounded by $f(x)=\left(x^2+1\right)^{-1/2}$ and the *x-axis* on the interval $[2, \infty)$ is revolved about the *x-axis*.

Solution

$$V=\pi\int_2^{\infty}\frac{1}{x^2+1}dx \qquad V=\pi\int_a^b (f(x))^2\,dx$$

$$=\pi\tan^{-1}x\Big|_2^{\infty}$$

$$=\pi\left(\tan^{-1}\infty-\tan^{-1}2\right)$$

$$=\pi\left(\frac{\pi}{2}-\tan^{-1}2\right)\ unit^3$$

Exercise

Find the volume of the region bounded by $f(x)=\sqrt{\frac{x+1}{x^3}}$ and the *x-axis* on the interval $[1, \infty)$ is revolved about the *x-axis*.

Solution

$$V=\pi\int_1^{\infty}\frac{x+1}{x^3}\,dx \qquad V=\pi\int_a^b (f(x))^2\,dx$$

$$=\pi\int_1^{\infty}\left(\frac{1}{x^2}+x^{-3}\right)dx$$

$$=\pi\left(-\frac{1}{x}-\frac{1}{2}\frac{1}{x^2}\right)\Big|_1^{\infty}$$

$$=\pi\left(1+\frac{1}{2}\right)$$

$$=\frac{3\pi}{2}\ unit^3$$

Exercise

Find the volume of the region bounded by $f(x)=(x+1)^{-3}$ and the *x-axis* on the interval $[0, \infty)$ is revolved about the *y-axis*.

Solution

$$V=2\pi\int_0^{\infty}x\frac{1}{(x+1)^3}dx \qquad V=2\pi\int_a^b x\cdot f(x)\,dx \quad (\text{Shell method})$$

$$\frac{x}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$

$$x=Ax^2+2Ax+A+Bx+B+C$$

$$=2\pi\int_0^{\infty}\left(\frac{1}{(x+1)^2}-\frac{1}{(x+1)^3}\right)d(x+1)$$

$$= 2\pi\left(\frac{-1}{x+1}+\frac{1}{2}\frac{1}{(x+1)^2}\right)\Bigg|_0^{\infty}$$

$$\begin{cases} \boxed{A=0} \\ 2A+B=1 \ \rightarrow \boxed{B=1} \quad \boxed{C=-1} \\ B+C=0 \end{cases}$$

$$= 2\pi\left(1-\frac{1}{2}\right)$$

$$\boxed{= \pi \ \ unit^3}$$

Exercise

Find the volume of the region bounded by $f(x)=\dfrac{1}{\sqrt{x}\ln x}$ and the *x-axis* on the interval $[2, \infty)$ is revolved about the *x-axis*.

Solution

$$V = \pi\int_2^{\infty}\frac{1}{x\ln^2 x}\,dx \qquad\qquad V=\pi\int_a^b (f(x))^2\,dx$$

$$= \pi\int_2^{\infty}\frac{1}{\ln^2 x}\,d(\ln x)$$

$$= \pi\left(-\frac{1}{\ln x}\right)\Bigg|_2^{\infty}$$

$$= \pi\left(-0+\frac{1}{\ln 2}\right)$$

$$\boxed{= \frac{\pi}{\ln 2} \ \ unit^3}$$

Exercise

Find the volume of the region bounded by $f(x)=\dfrac{\sqrt{x}}{\sqrt[3]{x^2+1}}$ and the *x-axis* on the interval $[0, \infty)$ is revolved about the *x-axis*.

Solution

$$V = \pi\int_0^{\infty}\frac{x}{\left(x^2+1\right)^{2/3}}\,dx \qquad\qquad V=\pi\int_a^b (f(x))^2\,dx$$

$$= \frac{\pi}{2}\int_0^{\infty}\left(x^2+1\right)^{-2/3}\,d\left(x^2+1\right)$$

$$= \frac{3\pi}{2}\left(x^2+1\right)^{1/3}\Bigg|_0^{\infty}$$

$$= \frac{3\pi}{2}(\infty-1)$$

$\boxed{= \infty \quad diverges}$ So the volume doesn't exist

Exercise

Find the volume of the region bounded by $f(x)=\left(x^2-1\right)^{-1/4}$ and the *x-axis* on the interval $(1,\ 2]$ is revolved about the *y-axis*.

Solution

$$V=2\pi\int_1^2 x\left(x^2-1\right)^{-1/4}dx \qquad V=2\pi\int_a^b x\cdot f(x)\,dx \quad (\textit{Shell method})$$

$$=\pi\int_1^2\left(x^2-1\right)^{-1/4}d\left(x^2-1\right)$$

$$=\frac{4\pi}{3}\left(x^2-1\right)^{3/4}\Big|_1^2$$

$$=\frac{4\pi}{3}(3)^{3/4}$$

$$=\frac{4\pi}{3^{1/4}}\ \ unit^3$$

Exercise

Find the volume of the region bounded by $f(x)=\tan x$ and the *x-axis* on the interval $\left[0,\ \frac{\pi}{2}\right)$ is revolved about the *x-axis*.

Solution

$$V=\pi\int_0^{\pi/2}\tan^2 x\,dx \qquad V=\pi\int_a^b (f(x))^2\,dx$$

$$=\pi\int_0^{\pi/2}\left(\sec^2 x-1\right)dx$$

$$=\pi(\tan x-x)\Big|_0^{\pi/2} \qquad \left(\tan\frac{\pi}{2}=\infty\right)$$

$=\infty$ *diverges* So the volume doesn't exist

Exercise

Find the volume of the region bounded by $f(x)=-\ln x$ and the *x-axis* on the interval $(0,\ 1]$ is revolved about the *x-axis*.

Solution

$$V=\pi\int_0^1 \ln^2 x\,dx \qquad V=\pi\int_a^b (f(x))^2\,dx$$

$$u = \ln x \qquad dv = \ln x\, dx \qquad\qquad u = \ln x \qquad dv = dx$$
$$du = \frac{dx}{x} \qquad v = x\ln x - x \qquad\qquad du = \frac{dx}{x} \qquad v = x \qquad \to \int \ln x dx = x\ln x - \int dx = x\ln x - x$$

$$\int \ln^2 x\, dx = \ln x\left(x\ln x - x\right) - \int (\ln x - 1)\, dx$$
$$= x\ln^2 x - x\ln x - \left(x\ln x - x - x\right)$$
$$= x\ln^2 x - 2x\ln x + 2x$$

$$V = \pi\left(x\ln^2 x - 2x\ln x + 2x\right)\Big|_0^1$$
$$= 2\pi \ \ unit^3$$

Exercise

Let R be the region bounded by the graph of $f(x) = x^{-p}$ and the $x\text{-}axis$

a) Let S be the solid generated when R is revolved about the $x\text{-}axis$. For what values of p is the volume of S finite for $0 < x \le 1$?

b) Let S be the solid generated when R is revolved about the $y\text{-}axis$. For what values of p is the volume of S finite for $0 < x \le 1$?

c) Let S be the solid generated when R is revolved about the $x\text{-}axis$. For what values of p is the volume of S finite for $x \ge 1$?

d) Let S be the solid generated when R is revolved about the $y\text{-}axis$. For what values of p is the volume of S finite for $x \ge 1$?

Solution

a) $$V = \pi\int_0^1 \left(x^{-p}\right)^2 dx \qquad\qquad V = \pi\int_a^b f(x)^2\, dx$$
$$= \pi\int_0^1 x^{-2p} dx$$
$$= \pi \frac{x^{-2p+1}}{1-2p}\Bigg|_0^1$$
$$= \frac{\pi}{1-2p}\left(1 - 0^{-2p+1}\right)$$

The volume of S finite when $1 - 2p > 0 \ \Rightarrow \ p < \frac{1}{2}$

b) $$V = 2\pi\int_0^1 x \cdot x^{-p} dx \qquad\qquad V = 2\pi\int_a^b xf(x)\, dx$$
$$= 2\pi\int_0^1 x^{1-p} dx$$

$$= \frac{2\pi}{2-p} x^{2-p} \Big|_0^1$$

$$= \frac{2\pi}{2-p}\left(1 - 0^{2-p}\right)$$

The volume of S finite when $2 - p > 0 \;\Rightarrow\; \underline{p < 2}$

c) $V = \pi \int_1^{\infty} \left(x^{-p}\right)^2 dx$ $\qquad V = \pi \int_a^b f(x)^2\, dx$

$$= \pi \int_1^{\infty} x^{-2p} dx$$

$$= \pi \frac{x^{-2p+1}}{1-2p} \Big|_1^{\infty}$$

$$= \frac{\pi}{1-2p}\left(\infty^{1-2p} - 1\right)$$

The volume of S finite when $1 - 2p < 0 \;\Rightarrow\; \underline{p > \frac{1}{2}}$ $\quad \left(\frac{1}{\infty} = 0\right)$

d) $V = 2\pi \int_0^1 x \cdot x^{-p} dx$ $\qquad V = 2\pi \int_a^b x f(x)\, dx$

$$= 2\pi \int_0^1 x^{1-p} dx$$

$$= \frac{2\pi}{2-p} x^{2-p} \Big|_0^1$$

$$= \frac{2\pi}{2-p}\left(1 - 0^{2-p}\right)$$

The volume of S finite when $2 - p > 0 \;\Rightarrow\; \underline{p < 2}$

Exercise

The magnetic potential P at a point on the axis of a circular coil is given by

$$P = \frac{2\pi NIr}{k} \int_c^{\infty} \frac{1}{\left(r^2 + x^2\right)^{3/2}} dx$$

Where N, I, r, k, and c are constants. Find P.

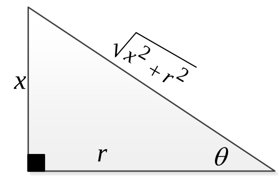
Solution

$$P = \frac{2\pi NIr}{k} \int_c^{\infty} \frac{1}{\left(r^2 + x^2\right)^{3/2}} dx$$

$x = r \tan\theta \qquad x^2 + r^2 = (r \sec\theta)^2$

$dx = r \sec^2\theta \, d\theta$

$$= \frac{2\pi NIr}{k} \int_{c}^{\infty} \frac{1}{r^3 \sec^3 \theta} r \sec^2 \theta \, d\theta$$



$$= \frac{2\pi NI}{kr} \int_{c}^{\infty} \cos\theta \, d\theta$$

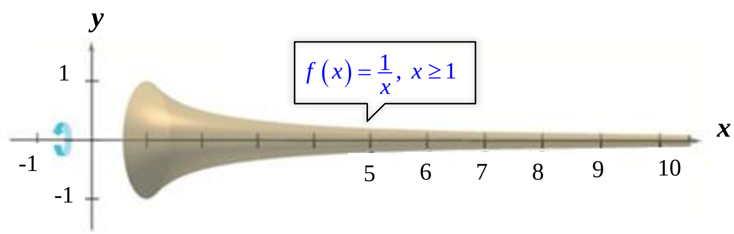
$$= \frac{2\pi NI}{kr} \sin\theta \Big|_{c}^{\infty}$$

$$= \frac{2\pi NI}{kr} \frac{x}{\sqrt{x^2+r^2}} \Big|_{c}^{\infty}$$

$$= \frac{2\pi NI}{kr} \left(1 - \frac{c}{\sqrt{c^2+r^2}}\right)$$

Exercise

The solid formed by revolving (about the *x-axis*) the unbounded region lying between the graph of $f(x) = \frac{1}{x}$ and the *x-axis* $(x \ge 1)$ is called ***Gabriel's Horn***.



Show that this solid has a finite volume and an infinite surface area

Solution

$$V = \pi \int_{1}^{\infty} \frac{1}{x^2} dx \qquad V = \pi \int_{x}^{b} \left(f(x)\right)^2 dx \ \ (\textit{disk method})$$

$$= -\pi \frac{1}{x} \Big|_{1}^{\infty}$$

$$= -\pi (0-1)$$

$$= \pi \ \ unit^3$$

$$f'(x) = -\frac{1}{x^2}$$

$$S = 2\pi \int_{1}^{\infty} \frac{1}{x} \sqrt{1+\frac{1}{x^4}} \, dx \qquad S = 2\pi \int_{a}^{b} f(x)\sqrt{1+\left(f'(x)\right)^2} \, dx$$

Since $1 + \frac{1}{x^4} > 1$ and $\int_{1}^{\infty} \frac{1}{x} dx$ diverges

Therefore the surface area in infinite.

Exercise

Water is drained from a 3000-*gal* tank at a rate that starts at 100 *gal/hr*. and decreases continuously by 5% /*hr*. If the drain left open indefinitely, how much water drains from the tank? Can a full tank be emptied at this rate?

Solution

Rate of the drain water: $r(t) = 100(1 - .05)^t$

$$= 100(0.95)^t$$

$$= 100e^{(\ln 0.95)t}$$

Total water amount drained:

$$D = \int_0^\infty 100e^{(\ln 0.95)t}\,dt$$

$$= \frac{100}{\ln 0.95} e^{(\ln 0.95)t} \Big|_0^\infty$$

$$= \frac{100}{\ln 0.95}(0 - 1) \qquad \ln 0.95 < 0 \xrightarrow[t\to\infty]{} e^{(\ln 0.95)t} = e^{-\infty} = 0$$

$$= -\frac{100}{\ln 0.95} \approx 1950 \ \ gal$$

Since 1950 *gal* $<$ 3000 *gal* which it takes infinite time.

Therefore, the full 3,000–gallon tank cannot be emptied at this rate.

Solution Section 2.7 – First-Order Linear Equations

Exercise

Write an equivalent first-order differential equation and initial condition for y. $y=\int_1^x \frac{1}{t}\,dt$

Solution

$$\int_1^x \frac{1}{t}\,dt \quad\Rightarrow\quad \frac{dy}{dx}=\frac{1}{x}$$

$$y(1)=\int_1^1 \frac{1}{t}\,dt=\ln t\Big|_1^1=\ln 1-\ln 1=0 \qquad\qquad \int_a^a f(x)dx=0$$

$$\boxed{\frac{dy}{dx}=\frac{1}{x};\quad y(1)=0}$$

Exercise

Write an equivalent first-order differential equation and initial condition for $y=2-\int_0^x (1+y(t))\sin t\,dt$

Solution

$$y=2-\int_0^x (1+y(t))\sin t\ dt \quad\Rightarrow\quad \frac{dy}{dx}=-(1+y(x))\sin x$$

$$y(0)=2-\int_0^0 (1+y(t))\sin t\ dt=2 \qquad\qquad \int_a^a f(x)dx=0$$

$$\boxed{\frac{dy}{dx}=-(1+y(x))\sin x;\quad y(0)=2}$$

Exercise

Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

$$y'=1-\frac{y}{x},\quad y(2)=-1,\quad dx=0.5$$

Solution

$$y_1=y_0+\left(1-\frac{y_0}{x_0}\right)dx=-1+\left(1-\frac{-1}{2}\right)(0.5)=-0.25$$

$$y_2=y_1+\left(1-\frac{y_1}{x_1}\right)dx=-0.25+\left(1-\frac{-0.25}{2.5}\right)(0.5)=0.3$$

$$y_3 = y_2 + \left(1 - \frac{y_2}{x_2}\right)dx = 0.3 + \left(1 - \frac{0.3}{3}\right)(0.5) = 0.75$$

$$y' + \frac{1}{x}y = 1 \qquad P(x) = \frac{1}{x}, \quad Q(x) = 1$$

$$y_h = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int (1) e^{\int \frac{1}{x}dx} dx = \int x dx = \frac{1}{2}x^2$$

$$y(x) = \frac{1}{x}\left(\frac{1}{2}x^2 + C\right) = \frac{1}{2}x + \frac{C}{x}$$

$$y(2) = \frac{1}{2}(2) + \frac{C}{2} = -1$$

$$1 + \frac{C}{2} = -1$$

$$\frac{C}{2} = -2 \qquad \underline{\rightarrow C = -4}$$

$$\underline{y(x) = \frac{x}{2} - \frac{4}{x}}$$

$$y(3.5) = \frac{3.5}{2} - \frac{4}{3.5} \ \underline{\approx 0.6071}$$

Exercise

Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

$y' = x(1-y), \quad y(1) = 0, \quad dx = 0.2$

Solution

$$y_1 = y_0 + x_0\left(1 - y_0\right)dx = 0 + 1(1-0)(0.2) = 0.2$$

$$y_2 = y_1 + x_1\left(1 - y_1\right)dx = 0.2 + 1.2(1-0.2)(0.2) = 0.392$$

$$y_3 = y_2 + x_2\left(1 - y_2\right)dx = 0.392 + 1.4(1-0.392)(0.2) = .5622$$

$$\frac{y'}{1-y} = xdx \quad \Rightarrow \quad \int \frac{dy}{1-y} = \int xdx$$

$$\ln|1-y| = \frac{1}{2}x^2 + C$$

$$1 - y = e^{\frac{1}{2}x^2 + C}$$

$$y = 1 - e^{\frac{1}{2}x^2 + C}$$

$$y(1) = 1 - e^{\frac{1}{2}1^2 + C} = 0$$

$$e^{\frac{1}{2}+C} = 1$$

$$\frac{1}{2} + C = 0 \Rightarrow \underline{C = -\frac{1}{2}}$$

$$\boxed{y(x) = 1 - e^{\frac{1}{2}\left(x^2-1\right)}}$$

$$y(1.6) = 1 - e^{\frac{1}{2}\left(1.6^2-1\right)} \approx \underline{0.5416}$$

Exercise

Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

$$y' = y^2(1+2x), \quad y(-1) = 1, \quad dx = 0.5$$

Solution

$$y_1 = y_0 + y_0^2\left(1+2x_0\right)dx = 1 + 1^2(1+2(-1))(0.5) = .5$$

$$y_2 = y_1 + y_1^2\left(1+2x_1\right)dx = 0.5 + 0.5^2(1+2(-0.5))(0.5) = .5$$

$$y_3 = y_2 + y_2^2\left(1+2x_2\right)dx = .5 + .5^2(1+2(0))(0.5) = .625$$

$$\frac{dy}{y^2} = (1+2x)dx \Rightarrow \int \frac{dy}{y^2} = \int (1+2x)dx$$

$$-\frac{1}{y} = x + x^2 + C$$

$$y = -\frac{1}{x+x^2+C}$$

$$y(-1) = -\frac{1}{-1+(-1)^2+C}$$

$$1 = -\frac{1}{C} \Rightarrow \underline{C = -1}$$

$$y(x) = -\frac{1}{x+x^2-1} = \boxed{\frac{1}{1-x-x^2}}$$

$$y(.5) = \frac{1}{1-.5-.5^2} \underline{= 4}$$

Exercise

Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

$y' = ye^x, \quad y(0) = 2, \quad dx = 0.5$

Solution

$$y_1 = y_0 + \left(y_0 e^{x_0}\right)dx = 2 + \left(2e^0\right)(0.5) = 3$$

$$y_2 = y_1 + \left(y_1 e^{x_1}\right)dx = 3 + \left(3e^{0.5}\right)(0.5) = 5.47308$$

$$y_3 = y_2 + \left(y_2 e^{x_2}\right)dx = 5.47308 + \left(5.47308e^1\right)(0.5) = 12.9118$$

$$\frac{dy}{dx} = ye^x \quad \Rightarrow \quad \int \frac{dy}{y} = \int e^x dx$$

$$\ln y = e^x + C$$

$$\ln 2 = e^0 + C \Rightarrow \boxed{C = \ln 2 - 1}$$

$$\ln|y| = e^x + \ln 2 - 1$$

$$y = e^{e^x + \ln 2 - 1}$$

$$= e^{\ln 2} e^{e^x - 1}$$

$$= 2e^{e^x - 1}$$

$$y(1.5) = 2e^{e^{1.5} - 1} \approx 65.0292$$

Exercise

Use the Euler method with $dx = 0.2$ to estimate $y(2)$ if $y' = \frac{y}{x}$ and $y(1) = 2$. What is the exact value of $y(2)$?

Solution

$$y_1(1) = y_0 + \left(\frac{y_0}{x_0}\right)dx = 2 + \left(\frac{2}{1}\right)(0.2) = 2.4$$

$$y_2(1.2) = y_1 + \left(\frac{y_1}{x_1}\right)dx = 2.4 + \left(\frac{2.4}{1.2}\right)(0.2) = 2.8$$

$$y_3 = y_2 + \left(\frac{y_2}{x_2}\right)dx = 2.8 + \left(\frac{2.8}{1.4}\right)(0.2) = 3.2$$

$$y_4 = y_3 + \left(\frac{y_3}{x_3}\right)dx = 3.2 + \left(\frac{3.2}{1.6}\right)(0.2) = 3.6$$

$$y_5 = y_4 + \left(\frac{y_4}{x_4}\right)dx = 3.6 + \left(\frac{3.6}{1.8}\right)(0.2) = 4$$

$$\frac{dy}{dx} = \frac{y}{x} \quad \Rightarrow \quad \int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + C$$

$$\ln 2 = \ln 1 + C \quad \to \quad \boxed{C = \ln 2}$$

$$\ln y = \ln x + \ln 2 = \ln 2x$$

$$\boxed{y = 2x} \qquad \underline{y(2)} = 2(2) \ \underline{= 4}$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y = Ce^{-5t};\quad y'(t) + 5y = 0$

Solution

$$y = Ce^{-5t} \quad \Rightarrow \quad y' = -5Ce^{-5t} = -5y$$

$$y'(t) + 5y = -5y + 5y \ \underline{= 0} \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y = Ct^{-3};\quad ty'(t) + 3y = 0$

Solution

$$y = Ct^{-3} \quad \Rightarrow \quad y' = -3Ct^{-4}$$

$$t\left(-3Ct^{-4}\right) + 3Ct^{-3} = -3Ct^{-3} + 3Ct^{-3} \ \underline{= 0} \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y = C_1 \sin 4t + C_2 \cos 4t;\quad y''(t) + 16y = 0$

Solution

$$y' = 4C_1 \cos 4t - 4C_2 \sin 4t$$

$$y'' = -16C_1 \sin 4t - 16C_2 \cos 4t$$

$$y''(t) + 16y = -16C_1 \sin 4t - 16C_2 \cos 4t + 16C_1 \sin 4t + 16C_2 \cos 4t \ \underline{= 0} \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y = C_1 e^{-x} + C_2 e^{x};\quad y''(x) - y = 0$

Solution

$$y' = -C_1 e^{-x} + C_2 e^{x}$$

$$y'' = C_1 e^{-x} + C_2 e^{x}$$

$$y''(x) - y = C_1 e^{-x} + C_2 e^{x} - C_1 e^{-x} - C_2 e^{x} \underline{= 0} \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants.

$y' + 4y = \cos t,\quad y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t + Ce^{-4t},\quad y(0) = -1$

Solution

$$y(0) = \frac{4}{17}\cos(0) + \frac{1}{17}\sin(0) + Ce^{-4(0)}$$

$$-1 = \frac{4}{17} + C$$

$$C = -1 - \frac{4}{17} = -\frac{21}{17}$$

$$\boxed{y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t - \frac{21}{17}e^{-4t}}$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad ty' + (t+1)y = 2te^{-t},\quad y(t) = e^{-t}\left(t + \frac{C}{t}\right),\quad y(1) = \frac{1}{e}$

Solution

$$y(1) = \frac{1}{e} = e^{-1}$$

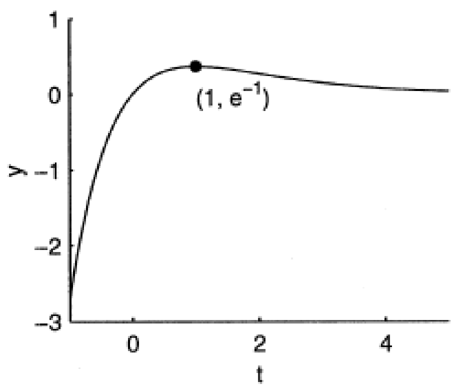
$$y(1) = e^{-1}\left(1 + \frac{C}{1}\right)$$

$$e^{-1} = e^{-1}(1 + C) \Rightarrow 1 = 1 + C$$

Hence, $C = 0$

The solution is: $y(t) = te^{-t}$

This function is defined and differentiable on the whole real line. Hence, the interval of existence is the whole real line.



Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y' = y(2+y), \quad y(t) = \dfrac{2}{-1+Ce^{-2t}}, \quad y(0) = -3$

Solution

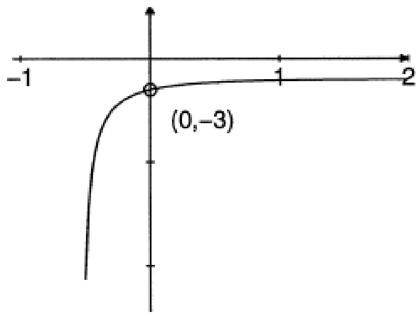
$$y(0) = \frac{2}{-1+Ce^{-2(0)}}$$

$$-3 = \frac{2}{-1+C}$$

$$3-3C = 2$$

$$-3C = -1$$

$$\boxed{C = \frac{1}{3}}$$



The solution is:

$$y(t) = \frac{2}{-1+\frac{1}{3}e^{-2t}}$$

$$= \frac{6}{-3+e^{-2t}}$$

Exercise

Verify that the given function y is a solution of the initial value problem that follows it.

$$y = 16e^{2t} - 10; \quad y' - 2y = 20, \quad y(0) = 6$$

Solution

$$y(0) = 6 \quad \rightarrow \quad y(0) = 16 - 10 = 6 \quad \checkmark$$

$$y = 16e^{2t} - 10 \quad \rightarrow \quad y' = 32e^{2t}$$

$$y' - 2y = 32e^{2t} - 32e^{2t} + 20 = 20 \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the initial value problem that follows it.

$$y = 8t^6 - 3; \quad ty' - 6y = 18, \quad y(1) = 5$$

Solution

$$y = 8t^6 - 3 \quad \rightarrow \quad y(1) = 8 - 3 = 5 \quad \checkmark$$

$$y' = 48t^5$$

$$ty' - 6y = 48t^6 - 48t^6 + 18 = 18 \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the initial value problem that follows it.

$$y=-3\cos 3t; \quad y''+9y=0, \quad y(0)=-3, \quad y'(0)=0$$

Solution

$$y=-3\cos 3t \quad \to \quad y(0)=-3\cos 0 \ \underline{=-3|} \quad \checkmark$$

$$y'=9\sin 3t \quad \to \quad \underline{y(0)=0|} \quad \checkmark$$

$$y''=27\cos 3t$$

$$y''+9y=27\cos 3t-27\cos 3t \ \underline{=0|}$$

Exercise

Verify that the given function y is a solution of the initial value problem that follows it.

$$y=\frac{1}{4}\left(e^{2x}-e^{-2x}\right); \quad y''-4y=0, \quad y(0)=0, \quad y'(0)=1$$

Solution

$$y=\frac{1}{4}\left(e^{2x}-e^{-2x}\right) \quad \to \quad y(0)=\frac{1}{4}(1-1) \ \underline{=0|} \quad \checkmark$$

$$y'=\frac{1}{2}\left(e^{2x}+e^{-2x}\right) \quad \to \quad y'(0)=\frac{1}{2}(1+1) \ \underline{=1|} \quad \checkmark$$

$$y''=e^{2x}-e^{-2x}$$

$$y''-4y=e^{2x}-e^{-2x}-e^{2x}+e^{-2x} \ \underline{=0|} \quad \checkmark$$

Exercise

Find the general solution of the differential equation $y'=xy$

Solution

$$\frac{dy}{dx}=xy$$

$$\frac{dy}{y}=xdx$$

$$\int\frac{dy}{y}=\int xdx$$

$$\ln|y|=\frac{1}{2}x^2+C$$

$$|y|=e^{x^2/2+C}$$

$$y(x)=\pm e^{x^2/2}e^C$$

$$\underline{=Ae^{x^2/2}|} \qquad \text{Where } A=\pm e^C$$

Exercise

Find the general solution of the differential equation $xy' = 2y$

Solution

$$x\frac{dy}{dx} = 2y$$

$$\frac{dy}{y} = 2\frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{2}{x}dx$$

$$\ln|y| = 2\ln|x| + C$$

$$= \ln x^2 + C$$

$$y(x) = \pm e^{\ln x^2 + C}$$

$$= \pm e^C x^2$$

$$\underline{= Ax^2}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = e^{x-y}$

Solution

$$\frac{dy}{dx} = e^x e^{-y}$$

$$\frac{dy}{e^{-y}} = e^x dx$$

$$\int e^y dy = \int e^x dx$$

$$e^y = e^x + C$$

$$\underline{y(x) = \ln\left(e^x + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \left(1 + y^2\right)e^x$

Solution

$$\frac{dy}{dx} = \left(1 + y^2\right)e^x$$

$$\frac{dy}{1 + y^2} = e^x dx$$

$$\int \frac{dy}{1+y^2} = \int e^x dx$$

$$\tan^{-1} y = e^x + C$$

$$\boxed{y(x) = \tan\left(e^x + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = xy + y$

Solution

$$\frac{dy}{dx} = (x+1)y$$

$$\frac{dy}{y} = (x+1)dx$$

$$\int \frac{dy}{y} = \int (x+1)dx$$

$$\ln y = \frac{1}{2}x^2 + x + C$$

$$\boxed{y = e^{x^2/2+x+C}}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$y' = ye^x - 2e^x + y - 2$$

Solution

$$\frac{dy}{dx} = (y-2)e^x + y - 2$$

$$\frac{dy}{dx} = (y-2)\left(e^x + 1\right)$$

$$\frac{dy}{y-2} = \left(e^x + 1\right)dx$$

$$\int \frac{dy}{y-2} = \int \left(e^x + 1\right)dx$$

$$\ln|y-2| = e^x + x + C$$

$$y - 2 = \pm e^{e^x + x + C}$$

$$y - 2 = \pm e^C e^{e^x + x}$$

$$\boxed{y(x) = De^{e^x + x} + 2} \quad D = \pm e^C$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \dfrac{x}{y+2}$

Solution

$$\frac{dy}{dx} = \frac{x}{y+2}$$

$$(y+2)\,dy = xdx$$

$$\int (y+2)\,dy = \int xdx$$

$$\frac{1}{2}y^2 + 2y = \frac{1}{2}x^2 + C$$

$$\boxed{y^2 + 4y = x^2 + 2C}$$

$$y^2 + 4y - x^2 - D = 0, \quad (D = 2C)$$

$$y = \frac{-4 \pm \sqrt{16 - 4\left(-x^2 - D\right)}}{2} = \frac{-4 \pm \sqrt{16 + 4x^2 + 4D}}{2}$$

$$= \frac{-4 \pm 2\sqrt{x^2 + (4+D)}}{2} \qquad E = 4 + D$$

$$= -2 \pm \sqrt{x^2 + E}$$

$$\boxed{y(x) = -2 \pm \sqrt{x^2 + E}}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \dfrac{xy}{x-1}$

Solution

$$\frac{dy}{dx} = y\left(\frac{x}{x-1}\right)$$

$$\frac{dy}{y} = \left(\frac{x}{x-1}\right)dx$$

$$\int \frac{dy}{y} = \int \left(1 + \frac{1}{x-1}\right)dx$$

$$\ln|y| = x + \ln|x-1| + C$$

$$y(x) = \pm e^{x + \ln|x-1| + C}$$

$$= \pm e^C e^x e^{\ln|x-1|}$$

$$\boxed{= De^x\,|x-1|}$$

Exercise

Solve the differential equations: $x\dfrac{dy}{dx}+y=e^x, \quad x>0$

Solution

$$\frac{dy}{dx}+\frac{1}{x}y=\frac{e^x}{x}$$

$$y_h=e^{\int\frac{1}{x}dx}=e^{\ln x}=x$$

$$\int\frac{e^x}{x}e^{\int\frac{1}{x}dx}dx=\int x\frac{e^x}{x}dx=\int e^x dx=e^x$$

$$\boxed{y(x)=\frac{1}{x}\left(e^x+C\right)}, \quad x>0$$

Exercise

Solve the differential equations: $y'+(\tan x)y=\cos^2 x, \quad -\frac{\pi}{2}<x<\frac{\pi}{2}$

Solution

$$y'+(\tan x)y=\cos^2 x$$

$$y_h=e^{\int\tan x dx}=e^{\ln(\cos x)^{-1}}=(\cos x)^{-1} \qquad \int\tan x dx=-\ln|\cos x|=\ln(\cos x)^{-1}$$

$$\int\cos^2 x(\cos x)^{-1}dx=\int\cos x dx=\sin x$$

$$y(x)=\frac{1}{(\cos x)^{-1}}(\sin x+C)$$

$$y(x)=\cos x(\sin x+C)$$

$$\boxed{y(x)=\cos x\sin x+C\cos x}$$

Exercise

Solve the differential equations: $x\dfrac{dy}{dx}+2y=1-\dfrac{1}{x}, \quad x>0$

Solution

$$y'+\frac{2}{x}y=\frac{1}{x}-\frac{1}{x^2}$$

$$y_h=e^{\int\frac{2}{x}dx}=e^{2\ln x}=e^{\ln x^2}=x^2$$

$$\int\left(\frac{1}{x}-\frac{1}{x^2}\right)x^2dx=\int(x-1)dx=\frac{1}{2}x^2-x$$

$$y(x)=\frac{1}{x^2}\left(\frac{1}{2}x^2-x+C\right)$$

$$y(x)=\frac{1}{2}-\frac{1}{x}+\frac{C}{x^2}, \quad x>0$$

Exercise

Solve the differential equations: $(1+x)y'+y=\sqrt{x}$

Solution

$$y'+\frac{1}{1+x}y=\frac{\sqrt{x}}{1+x}$$

$$e^{\int\frac{1}{1+x}dx}=e^{\ln(1+x)}=1+x$$

$$\int\frac{\sqrt{x}}{1+x}(1+x)\,dx=\int x^{1/2}dx=\frac{2}{3}x^{3/2}$$

$$y(x)=\frac{1}{1+x}\left(\frac{2}{3}x^{3/2}+C\right)$$

$$=\frac{2x^{3/2}}{3(1+x)}+\frac{C}{1+x}$$

Exercise

Solve the differential equations: $e^{2x}y'+2e^{2x}y=2x$

Solution

$$y'+2y=2xe^{-2x}$$

$$e^{\int 2dx}=e^{2x}$$

$$\int 2xe^{-2x}\left(e^{2x}\right)dx=2\int xdx=x^2$$

$$y(x)=\frac{1}{e^{2x}}\left(x^2+C\right)$$

$$=x^2e^{-2x}+Ce^{-2x}$$

Exercise

Solve the differential equations: $(t+1)\frac{ds}{dt}+2s=3(t+1)+\frac{1}{(t+1)^2}, \quad t>-1$

Solution

$$s' + \frac{2}{t+1}s = 3 + \frac{1}{(t+1)^3}$$

$$e^{\int \frac{2}{t+1}dt} = e^{2\ln(t+1)} = e^{\ln(t+1)^2} = (t+1)^2$$

$$\int \left(3 + \frac{1}{(t+1)^3}\right)(t+1)^2\, dt = \int \left(3(t+1)^2 + \frac{1}{t+1}\right)dt \qquad d(t+1) = dt$$

$$= 3\int (t+1)^2\, d(t+1) + \int \frac{1}{t+1}\, d(t+1)$$

$$= (t+1)^3 + \ln(t+1)$$

$$s(t) = \frac{1}{(t+1)^2}\left((t+1)^3 + \ln(t+1) + C\right)$$

$$= t + 1 + \frac{\ln(t+1)}{(t+1)^2} + \frac{C}{(t+1)^2}, \quad t > -1$$

Exercise

Solve the differential equations: $\tan\theta \frac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \frac{\pi}{2}$

Solution

$$\frac{dr}{d\theta} + \frac{1}{\tan\theta}r = \frac{\sin^2\theta}{\tan\theta}$$

$$\frac{dr}{d\theta} + \frac{1}{\tan\theta}r = \sin^2\theta\frac{\cos\theta}{\sin\theta}$$

$$\frac{dr}{d\theta} + \frac{1}{\tan\theta}r = \sin\theta\cos\theta$$

$$e^{\int \cot\theta d\theta} = e^{\ln|\sin\theta|} = \sin\theta, \quad 0 < \theta < \frac{\pi}{2}$$

$$\int (\sin\theta\cos\theta)(\sin\theta)\, d\theta = \int \left(\sin^2\theta\cos\theta\right)d\theta \qquad d(\sin\theta) = \cos\theta d\theta$$

$$= \int \sin^2\theta d(\sin\theta)$$

$$= \frac{1}{3}\sin^3\theta$$

$$r(\theta) = \frac{1}{\sin\theta}\left(\frac{1}{3}\sin^3\theta + C\right)$$

$$= \frac{1}{3}\sin^2\theta + \frac{C}{\sin\theta}$$

Exercise

Find the general solution of $y' = \cos x - y\sec x$

Solution

$$y' + (\sec x)y = \cos x$$

$$e^{\int \sec x dx} = e^{\ln|\sec x + \tan x|} = \sec x + \tan x$$

$$\begin{aligned}\int \cos x(\sec x + \tan x)dx &= \int \cos x\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)dx \\ &= \int (1 + \sin x)dx \\ &= x - \cos x\end{aligned}$$

$$\boxed{y(x) = \frac{1}{\sec x + \tan x}(x - \cos x + C)}$$

Exercise

Find the general solution of $\left(1 + x^3\right)y' = 3x^2 y + x^2 + x^5$

Solution

$$y' - \frac{3x^2}{1 + x^3}y = \frac{x^2\left(1 + x^3\right)}{1 + x^3} = x^2$$

$$e^{\int -\frac{3x^2}{1+x^3}dx} = e^{-\int \frac{d\left(1+x^3\right)}{1+x^3}} = e^{-\ln\left(1+x^3\right)} = e^{\ln\left(1+x^3\right)^{-1}} = \frac{1}{1 + x^3}$$

$$\int \frac{1}{1 + x^3} \cdot x^2 dx = \frac{1}{3}\int \frac{d\left(1 + x^3\right)}{1 + x^3} = \frac{1}{3}\ln\left(1 + x^3\right)$$

$$y(x) = \left(1 + x^3\right)\left(\frac{1}{3}\ln\left(1 + x^3\right) + C\right)$$

$$\boxed{= \frac{1}{3}\left(1 + x^3\right)\ln\left(1 + x^3\right) + C\left(1 + x^3\right)}$$

Exercise

Find the general solution of $\frac{dy}{dt} - 2y = 4 - t$

Solution

$$e^{\int -2dt} = e^{-2t}$$

$$\int (4-t)e^{-2t}\,dt = \int \left(4e^{-2t} - te^{-2t}\right)\,dt$$

$$= -2e^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t}$$

$$= -\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t}$$

		$\int e^{-2t}$
+	t	$-\frac{1}{2}e^{-2t}$
−	1	$\frac{1}{4}e^{-2t}$

$$y(t) = \frac{1}{e^{-2t}}\left(-\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t} + C\right)$$

$$y(t) = \frac{1}{2}t - \frac{7}{4} + Ce^{2t}$$

Exercise

Find the general solution of $y' + y = \dfrac{1}{1+e^t}$

Solution

$$e^{\int dt} = e^t$$

$$\int \frac{1}{1+e^t}e^t\,dt = \int \frac{1}{1+e^t}d\left(1+e^t\right) = \ln\left(1+e^t\right)$$

$$y(t) = \frac{1}{e^t}\left(\ln\left(1+e^t\right) + C\right)$$

$$y(t) = e^{-t}\ln\left(1+e^t\right) + Ce^{-t}$$

Exercise

Solve the differential equation $\quad y' = 3y - 4$

Solution

$$y' - 3y = -4$$

$$e^{\int -3dx} = e^{-3x}$$

$$\int -4e^{-3x}dx = \frac{4}{3}e^{-3x}$$

$$y(x) = \frac{1}{e^{-3x}}\left(\frac{4}{3}e^{-3x} + C\right)$$

$$= \frac{4}{3} + Ce^{3x}$$

Exercise

Solve the differential equation $\quad y' = -2y - 4$

Solution

$$y' + 2y = -4$$

$$e^{\int 2dx} = e^{2x}$$

$$\int -4e^{2x}dx = 2e^{2x}$$

$$y(x) = \frac{1}{e^{2x}}\left(2e^{2x} + C\right)$$

$$\underline{= 2 + Ce^{-2x}}$$

Exercise

Solve the differential equation $\quad y' = -y + 2$

Solution

$$y' + y = 2$$

$$e^{\int dx} = e^{x}$$

$$\int 2e^{x}dx = 2e^{x}$$

$$y(x) = \frac{1}{e^{x}}\left(2e^{x} + C\right)$$

$$\underline{= 2 + Ce^{-x}}$$

Exercise

Solve the differential equation $\quad y' = 2y + 6$

Solution

$$y' - 2y = 6$$

$$e^{\int -2dx} = e^{-2x}$$

$$\int 6e^{-2x}dx = -3e^{-2x}$$

$$y(x) = e^{2x}\left(-3e^{-2x} + C\right)$$

$$\underline{= -3 + Ce^{2x}}$$

Exercise

Solve the initial value problem: $t\frac{dy}{dt}+2y=t^3, \quad t>0, \quad y(2)=1$

Solution

$$y'+\frac{2}{t}y=t^2$$

$$e^{\int\frac{2}{t}dt}=e^{2\ln t}=e^{\ln t^2}=t^2$$

$$\int t^2t^2dt=\int t^4dt=\frac{1}{5}t^5$$

$$y(t)=\frac{1}{t^2}\left(\frac{1}{5}t^5+C\right)=\frac{1}{5}t^3+\frac{C}{t^2}$$

$$y(2)=\frac{1}{5}2^3+\frac{C}{2^2}$$

$$1=\frac{8}{5}+\frac{C}{4}$$

$$\frac{C}{4}=1-\frac{8}{5}=-\frac{3}{5}\Rightarrow \boxed{C=-\frac{12}{5}}$$

$$y(t)=\frac{1}{5}t^3-\frac{12}{5t^2}$$

Exercise

Solve the initial value problem: $\theta\frac{dy}{d\theta}+y=\sin\theta, \quad \theta>0, \quad y\left(\frac{\pi}{2}\right)=1$

Solution

$$y'+\frac{1}{\theta}y=\frac{\sin\theta}{\theta}$$

$$e^{\int\frac{1}{\theta}d\theta}=e^{\ln|\theta|}=\theta \quad (>0)$$

$$\int\frac{\sin\theta}{\theta}\theta d\theta=\int\sin\theta d\theta=-\cos\theta$$

$$y(\theta)=\frac{1}{\theta}(-\cos\theta+C)$$

$$y\left(\frac{\pi}{2}\right)=\frac{2}{\pi}\left(-\cos\frac{\pi}{2}+C\right)$$

$$1=\frac{2}{\pi}(0+C)$$

$$1=\frac{2}{\pi}C \qquad C=\frac{\pi}{2}$$

$$y(\theta)=-\frac{\cos\theta}{\theta}+\frac{\pi}{2\theta}$$

Exercise

Solve the initial value problem: $\frac{dy}{dx}+xy=x, \quad y(0)=-6$

Solution

$$y'+xy=x$$

$$e^{\int x\,dx}=e^{x^2/2}$$

$$\int xe^{x^2/2}dx=\int e^{x^2/2}d\left(\frac{x^2}{2}\right)=e^{x^2/2} \qquad d\left(\frac{x^2}{2}\right)=x\,dx$$

$$y(x)=\frac{1}{e^{x^2/2}}\left(e^{x^2/2}+C\right)$$

$$y(0)=\frac{1}{e^{0^2/2}}\left(e^{0^2/2}+C\right)$$

$$-6=1(1+C)$$

$$-6=1+C \;\to\; \underline{C=-7}$$

$$y(x)=\frac{1}{e^{x^2/2}}\left(e^{x^2/2}-7\right)$$

$$\boxed{=1-\frac{7}{e^{x^2/2}}}$$

Exercise

Solve the initial value problem $y'=\frac{y}{x}, \quad y(1)=-2$

Solution

$$\frac{dy}{dx}=\frac{y}{x}$$

$$\int\frac{dy}{y}=\int\frac{dx}{x}$$

$$\ln|y|=\ln|x|+C$$

$$y=\pm e^{\ln|x|+C}$$

$$=\pm e^{C}e^{\ln|x|}$$

$$=Dx$$

$$y=Dx \;\Rightarrow\; D=\frac{y}{x}=\frac{-2}{1}=-2$$

$$\boxed{y=-2x}$$

Exercise

Solve the initial value problem $y' = \dfrac{\sin x}{y}, \quad y\left(\frac{\pi}{2}\right) = 1$

Solution

$$\frac{dy}{dx} = \frac{\sin x}{y}$$

$$ydy = \sin x dx$$

$$\int ydy = \int \sin x dx$$

$$\frac{1}{2}y^2 = -\cos x + C_1$$

$$y^2 = -2\cos x + C \quad \left(C = 2C_1\right)$$

$$y(x) = \pm\sqrt{-2\cos x + C}$$

$$y\left(\frac{\pi}{2}\right) = \sqrt{-2\cos\frac{\pi}{2} + C} \quad 1 = \sqrt{C} \quad \Rightarrow \quad \boxed{C = 1}$$

$$\boxed{y(x) = \sqrt{1 - 2\cos x}}$$

The interval of existence will be the interval containing $\frac{\pi}{2}$ and $1 - 2\cos x > 0$

$$\cos x < \frac{1}{2} \quad \Rightarrow \quad \boxed{\frac{\pi}{3} < x < \frac{5\pi}{3}}$$

Exercise

Find the general solution of $y' = y + 2xe^{2x}; \quad y(0) = 3$

Solution

$$y' - y = 2xe^{2x}$$

$$e^{\int -1dx} = e^{-x}$$

$$\int 2xe^{2x}\left(e^{-x}\right)dx = 2\int xe^{x}dx = 2\left(xe^{x} - e^{x}\right)$$

$$y(x) = \frac{1}{e^{-x}}\left(2xe^{x} - 2e^{x} + C\right)$$

$$= e^{x}\left(2xe^{x} - 2e^{x} + C\right)$$

$$\boxed{= 2xe^{2x} - 2e^{2x} + Ce^{x}}$$

$$y(x=0) = 2(0)e^{2(0)} - 2e^{2(0)} + Ce^{(0)}$$

$$3 = -2 + C \quad \to \quad \boxed{C = 5}$$

$$\boxed{y(x) = 2xe^{2x} - 2e^{2x} + 5e^{x}}$$

Exercise

Find the general solution of $\left(x^2+1\right)y'+3xy=6x; \quad y(0)=-1$

Solution

$$y'+\frac{3x}{x^2+1}y=\frac{6x}{x^2+1}$$

$$e^{\int \frac{3x}{x^2+1}dx}=e^{\frac{3}{2}\ln\left(x^2+1\right)}=e^{\ln\left(x^2+1\right)^{\frac{3}{2}}}=\underline{\left(x^2+1\right)^{\frac{3}{2}}}$$

$$\int \left(x^2+1\right)^{\frac{3}{2}}\frac{6x}{x^2+1}dx=3\int \left(x^2+1\right)^{\frac{1}{2}}d\left(x^2+1\right)$$

$$=2\left(x^2+1\right)^{\frac{3}{2}}$$

$$y(x)=2+C\left(x^2+1\right)^{-\frac{3}{2}} \qquad y(0)=2+C\left((0)^2+1\right)^{-\frac{3}{2}}$$

$$-1=2+C(1)^{-\frac{3}{2}} \qquad \rightarrow \underline{C=-3}$$

$$\underline{y(x)=2-3\left(x^2+1\right)^{-\frac{3}{2}}}$$

Exercise

Solve the initial value problem $\quad y'=\left(4t^3+1\right)y, \quad y(0)=4$

Solution

$$\frac{dy}{dt}=\left(4t^3+1\right)y$$

$$\int \frac{dy}{y}=\int \left(4t^3+1\right)dt$$

$$\ln y=t^4+t+C$$

$$y(t)=e^{t^4+t+C}$$

$$\underline{=Ae^{t^4+t}}$$

$$y(0)=4 \quad \underline{\rightarrow 4=A}$$

$$y(t)\ \underline{=4e^{t^4+t}}$$

Exercise

Solve the initial value problem $y' = \frac{e^t}{2y}, \quad y(\ln 2) = 1$

Solution

$$\int 2y\,dy = \int e^t\,dt$$

$$y^2 = e^t + C$$

$$y(\ln 2) = 1, \quad \to 1 = 2 + C \Rightarrow \underline{C = -1}$$

$$\underline{y^2 = e^t - 1}$$

Exercise

Solve the initial value problem $(\sec x)\,y' = y^3, \quad y(0) = 3$

Solution

$$\int y^{-3}\,dy = \int \frac{dx}{\sec x} = \int \cos x\,dx$$

$$-\frac{1}{2}\frac{1}{y^2} = \sin x + C_1$$

$$y^2 = \frac{1}{-2\sin x + C}$$

$$y = \pm\sqrt{\frac{1}{-2\sin x + C}} \qquad \textit{Since the initial value is positive}$$

$$y = \frac{1}{\sqrt{-2\sin x + C}}$$

$$3 = \sqrt{\frac{1}{C}} \Rightarrow \underline{C = \frac{1}{9}}$$

$$y = \frac{1}{\sqrt{-2\sin x + \frac{1}{9}}}$$

$$\underline{= \frac{3}{\sqrt{-2\sin x + 1}}}$$

Exercise

Solve the initial value problem $\frac{dy}{dx} = e^{x-y}, \quad y(0) = \ln 3$

Solution

$$dy = \left(e^x e^{-y}\right)dx$$

$$\int e^y\,dy = \int e^x\,dx$$

$$e^{y}=e^{x}+C$$

$$y=\ln\left(e^{x}+C\right)$$

$$y(0)=\ln 3\rightarrow \quad \ln 3=\ln(1+C)$$

$$1+C=3 \quad \Rightarrow \underline{C=2}$$

$$\underline{y(x)=\ln\left(e^{x}+2\right)}$$

Exercise

Solve the initial value problem $\quad y'=2e^{3y-t}, \quad y(0)=0$

Solution

$$\frac{dy}{dt}=2e^{3y}e^{-t}$$

$$\int e^{-3y}dy=\int 2e^{-t}dt$$

$$-\frac{1}{3}e^{-3y}=-2e^{-t}+C_1$$

$$e^{-3y}=6e^{-t}+C$$

$$y(0)=0 \rightarrow \quad 1=6+C \Rightarrow \underline{C=-5}$$

$$e^{-3y}=6e^{-t}-5$$

$$-3y=\ln\left(6e^{-t}-5\right)$$

$$\underline{y(t)=-\frac{1}{3}\ln\left(6e^{-t}-5\right)}$$

Exercise

Solve the initial value problem $\quad y'=3y-6, \quad y(0)=9$

Solution

$$y'-3y=-6$$

$$e^{\int -3dx}=e^{-3x}$$

$$\int -6e^{-3x}dx=2e^{-3x}$$

$$y=\frac{1}{e^{-3x}}\left(2e^{-3x}+C\right)$$

$$=2+Ce^{3x}$$

$$y(0)=9 \qquad 9=2+C\rightarrow \underline{C=7}$$

$$\underline{y=7e^{3x}+2}$$

Exercise

Solve the initial value problem $y' = -y + 2, \quad y(0) = -2$

Solution

$$y' + y = 2$$

$$e^{\int dx} = e^{x}$$

$$\int 2e^{x}dx = 2e^{x}$$

$$y = \frac{1}{e^{x}}\left(2e^{x} + C\right) \underline{= 2 + Ce^{-x}}$$

$$y(0) = -2 \qquad -2 = 2 + C \rightarrow \underline{C = -4}$$

$$\underline{y(x) = 2 - 4e^{-x}}$$

Exercise

Solve the initial value problem $y' = -2y - 4, \quad y(0) = 0$

Solution

$$y' + 2y = -4$$

$$e^{\int 2dx} = e^{2x}$$

$$\int -4e^{2x}dx = -2e^{2x}$$

$$y = \frac{1}{e^{2x}}\left(-2e^{2x} + C\right) \underline{= -2 + Ce^{-2x}}$$

$$y(0) = 0 \qquad 0 = -2 + C \rightarrow \underline{C = 2}$$

$$\underline{y(x) = 2e^{-2x} - 2}$$

Solution   Section 2.8 – Applications

Exercise

A 66-*kg* cyclist on a 7-*kg* bicycle starts coasting on level ground at 9 *m/sec*. The $k \approx 3.9\ kg/\sec$

a) About how far will the cyclist coast before reaching a complete stop?

b) How long will it take the cyclist's speed to drop to 1 *m/sec*?

Solution

Mass: $m = 66 + 7 = 73\ kg$

$$v = v_0 e^{-(k/m)t} = 9e^{-(3.9/73)t}$$

a) $s(t) = \int v(t)\,dt = \int 9e^{-(3.9/73)t}\,dt$

$$= 9\left(-\frac{73}{3.9}\right)e^{-(3.9/73)t} + C$$

$$= -\frac{219}{1.3}e^{-(3.9/73)t} + C$$

$$= -\frac{2190}{13}e^{-(3.9/73)t} + C$$

$$s(0) = -\frac{2190}{13}e^{-(3.9/73)(0)} + C$$

$$0 = -\frac{2190}{13} + C$$

$$\boxed{C = \frac{2190}{13}}$$

$$s(t) = -\frac{2190}{13}e^{-(3.9/73)t} + \frac{2190}{13} = \frac{2190}{13}\left(1 - e^{-(3.9/73)t}\right)$$

$$\lim_{t\to\infty} s(t) = \frac{2190}{13}\lim_{t\to\infty}\left(1 - e^{-(3.9/73)t}\right)$$

$$= \frac{2190}{13}(1-0)$$

$$\approx 168.5$$

The cyclist coast about 168.5 meters.

b) $1 = 9e^{-(3.9/76)t}$

$$\frac{1}{9} = e^{-(3.9/73)t} \quad\Rightarrow\quad -\frac{3.9}{73}t = \ln\frac{1}{9}$$

$$t = -\frac{73}{3.9}\ln\frac{1}{9} \approx 41.13\ \sec$$

It will take about 41.13 *seconds*.

Exercise

Suppose that an Iowa class battleship has mass 51,000 metric tons (51,000,000 *kg*) and $k \approx 59,000\ kg/\sec$. Assume that the ship loses power when it is moving at a speed of 9 *m/sec*.

a) About how far will the ship coast before it is dead in the water?

b) About how long will it take the ship's speed to drop to 1 *m/sec*?

Solution

$$v = v_0 e^{-(k/m)t} = 9e^{-(59{,}000/51{,}000{,}000)t} = 9e^{-(59/51{,}000)t}$$

a) $s(t) = \int v(t)\,dt = \int 9e^{-(59/51{,}000)t}\,dt$

$$= 9\left(-\frac{51000}{59}\right)e^{-(59/51{,}000)t} + C$$

$$= -\frac{459,000}{59}e^{-(59/51{,}000)t} + C$$

$$s(0) = -\frac{51,000}{59}e^{-(59/51{,}000)(0)} + C$$

$$0 = -\frac{51,000}{59} + C$$

$$\boxed{C = \frac{51{,}000}{59}}$$

$$s(t) = -\frac{459,000}{59}e^{-(59/51{,}000)t} + \frac{459,000}{59}$$

$$= \frac{459,000}{59}\left(1 - e^{-(59/51{,}000)t}\right)$$

$$\lim_{t\to\infty} s(t) = \frac{459,000}{59}\lim_{t\to\infty}\left(1 - e^{-(59/51{,}000)t}\right)$$

$$= \frac{51,000}{59}(1-0)$$

$$\approx 7780\ m$$

The ship will coast about 7780 meters or 7.78 km.

b) $1 = 9e^{-(59/51{,}000)t}$

$$e^{-(59/51{,}000)t} = \frac{1}{9}$$

$$-\frac{59}{51000}t = \ln\frac{1}{9}$$

$$t = -\frac{51000}{59}\ln\frac{1}{9} \approx 1899.3\ \sec$$

It will take about $\frac{1899.3}{60} \approx 61.65\ \ minutes$

Exercise

A 200-*gal* tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 *lb./gal* of concentrate enters the tank at the rate of 5 *gal/min*, and the well-stirred mixture is withdrawn at the rate of 3 *gal/min*.

a) At what time will the tank be full?

b) At the time the tank is full, how many pounds of concentrate will it contain?

Solution

a) $V(t) = 100 + \left(5\frac{gal}{\text{min}} - 3\frac{gal}{\text{min}}\right)(t \ \text{min}) = 100 + 2t$

$$200 = 100 + 2t$$

$$100 = 2t \quad \Rightarrow \quad \boxed{t = 50 \ \text{min}}$$

b) Let $y(t)$ be the amount of concentrate in the tank at time t.

$$\frac{dy}{dt} = Rate\ in - \ Rate\ out$$

$$\frac{dy}{dt} = \left(0.5 \ \frac{lb}{gal}\right)\left(5 \ \frac{gal}{\text{min}}\right) - \left(\frac{y}{100+2t} \ \frac{lb}{gal}\right)\left(3 \ \frac{gal}{\text{min}}\right)$$

$$= \frac{5}{2} - \frac{3y}{100+2t}$$

$$\frac{dy}{dt} + \frac{3}{100+2t}y = \frac{5}{2} \quad \rightarrow \quad P(t) = \frac{3}{100+2t} \quad Q(t) = \frac{5}{2}$$

$$e^{\int \frac{3dt}{100+2t}} = e^{\frac{3}{2}\int \frac{dt}{50+t}} = e^{\frac{3}{2}\ln(50+t)} = e^{\ln(50+t)^{3/2}} = (50+t)^{3/2}$$

$$\int \frac{5}{2}(50+t)^{3/2}dt = (t+50)^{5/2}$$

$$y(t) = \frac{1}{(t+50)^{3/2}}\left[(t+50)^{5/2} + C\right]$$

$$= t + 50 + \frac{C}{(t+50)^{3/2}}$$

$$y(0) = 0 + 50 + \frac{C}{(0+50)^{3/2}}$$

$$0 = 50 + \frac{C}{50^{3/2}} \quad \rightarrow \frac{C}{50^{3/2}} = -50 \quad \Rightarrow \quad \boxed{C = -50^{5/2}}$$

$$y(t) = t + 50 - \frac{50^{5/2}}{(t+50)^{3/2}}$$

$$y(t=50) = 50 + 50 - \frac{50^{5/2}}{(50+50)^{3/2}} \approx 83.22 \ \ lb\ of\ \ concentrate$$

Exercise

A tank contains 100 *gal* of fresh water. A solution containing 1 *lb./gal* of soluble lawn fertilizer runs into the tank at the rate of 1 *gal/min*, and the mixture is pumped out of the tank at a rate of 3 *gal/min*. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.

Solution

Volume of the tank at time t is:

$$V(t) = 100\ gal + \left(1\frac{gal}{\min} - 3\frac{gal}{\min}\right)(t\ \ \min) = 100 - 2t$$

$$\frac{dy}{dt} = Rate\ in - \ Rate\ out$$

$$\frac{dy}{dt} = \left(1\ \frac{lb}{gal}\right)\left(1\ \frac{gal}{\min}\right) - \left(\frac{y}{100-2t}\ \frac{lb}{gal}\right)\left(3\ \frac{gal}{\min}\right)$$

$$\frac{dy}{dt} = 1 - \frac{3y}{100-2t}$$

$$\frac{dy}{dt} + \frac{3}{100-2t}y = 1 \quad \to \quad P(t) = \frac{3}{100-2t} \quad Q(t) = 1$$

$$e^{\int \frac{3dt}{100-2t}} = e^{\frac{3}{2}\int \frac{-dt}{100-2t}} = e^{-\frac{3}{2}\ln(100-2t)} = e^{\ln(100-2t)^{-3/2}} = (100-2t)^{-3/2}$$

$$\int 1(100-2t)^{-3/2}dt = -\frac{1}{2}\int (100-2t)^{-3/2}d(100-2t) = (100-2t)^{-1/2}$$

$$y(t) = \frac{1}{(100-2t)^{-3/2}}\left[(100-2t)^{-1/2} + C\right]$$

$$y(t) = 100 - 2t + C(100-2t)^{3/2}$$

$$y(0) = 100 - 2(0) + C(100 - 2(0))^{3/2}$$

$$0 = 100 + C(100)^{3/2}$$

$$\boxed{C = -100^{-1/2} = -\frac{1}{10}}$$

$$y(t) = 100 - 2t - 0.1(100-2t)^{3/2}$$

$$\frac{dy}{dx} = -2 - 0.1\frac{3}{2}(100-2t)^{1/2}(-2)$$

$$\frac{dy}{dx} = -2 + 0.3(100-2t)^{1/2} = 0$$

$$(100-2t)^{1/2} = \frac{2}{0.3} \quad \Rightarrow \quad 100 - 2t = \left(\frac{2}{0.3}\right)^2 = \frac{4}{0.09} = \frac{400}{9}$$

$$2t = 100 - \frac{400}{9} = \frac{500}{9}$$

$$\underline{\left\lfloor t = \frac{500}{18} \approx 12.78\ \text{min} \right\rfloor}$$

The maximum amount is:

$$y(t = 12.78) = 100 - 2(12.78) - 0.1(100 - 2(12.78))^{3/2}$$

$$\boxed{y \approx 14.8\ lb}$$

Exercise

An Executive conference room of a corporation contains 4500 ft^3 of air initially free of carbon monoxide. Starting at time $t = 0$, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 ft^3/min. A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 ft^3/min. Find the time when the concentration of carbon monoxide in the room reaches 0.01%.

Solution

Let $y(t)$ be the amount of carbon monoxide (CO) in the room at time t.

$$\frac{dy}{dt} = Rate\ in - \ Rate\ out$$

$$\frac{dy}{dt} = (0.04)(0.3) - \left(\frac{y}{4500}\right)(0.3)$$

$$\frac{dy}{dt} = \frac{12}{1000} - \frac{y}{15,000}$$

$$\frac{dy}{dt} + \frac{1}{15,000}y = \frac{12}{1000} \quad \rightarrow \quad P(t) = \frac{1}{15,000} \quad Q(t) = \frac{12}{1000}$$

$$e^{\int \frac{dt}{15000}} = e^{\frac{1}{15000}t}$$

$$\int \frac{12}{1000} e^{\frac{1}{15000}t} dt = \frac{12}{1000} 15000 e^{\frac{1}{15000}t} = 180 e^{\frac{1}{15000}t}$$

$$y(t) = \frac{1}{e^{\frac{1}{15000}t}} \left[180 e^{\frac{1}{15000}t} + C \right]$$

$$y(t) = 180 + Ce^{\frac{-1}{15000}t}$$

$$y(0) = 180 + Ce^{\frac{-1}{15000}0}$$

$$0 = 180 + C \quad \Rightarrow \quad \boxed{C = -180}$$

$$y(t) = 180 - 180 e^{\frac{-1}{15000}t}$$

When the concentration of CO is 0.01% in the room, the amount of CO satisfies

$$\frac{y}{4500}=\frac{.01}{100} \quad \Rightarrow \quad y=0.45\ ft^3$$

When the room contains the amount $y=0.45\ ft^3$

$$0.45=180-180e^{\frac{-1}{15000}t}$$

$$180e^{\frac{-1}{15000}t}=179.55$$

$$e^{\frac{-1}{15000}t}=\frac{179.55}{180}$$

$$\frac{-1}{15000}t=\ln\left(\frac{179.55}{180}\right)$$

$$t=-15000\ln\left(\frac{179.55}{180}\right)$$

$$\boxed{t\approx 37.55\ \text{min}}$$

Exercise

Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the substance B at time $t = 0$, and if x is the amount of product at time t, then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt}=k(a-x)(b-x) \quad \text{or} \quad \frac{1}{(a-x)(b-x)}\frac{dx}{dt}=k$$

Where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t.

a) If $a=b$

b) If $a\neq b$

Assume in each case that $x=0$ when $t=0$

Solution

$$\frac{1}{(a-x)(b-x)}dx=kdt$$

a) $a=b \quad \Rightarrow \quad \frac{1}{(a-x)^2}dx=kdt$

$$\int\frac{1}{(a-x)^2}dx=\int kdt$$

$$\frac{1}{a-x}=kt+C$$

$$x(t=0)=0 \quad \Rightarrow \quad \frac{1}{a}=C$$

$$\frac{1}{a-x}=kt+\frac{1}{a}=\frac{kat+1}{a}$$

$$a-x=\frac{a}{kat+1}$$

$$x=a-\frac{a}{kat+1}$$

$$\boxed{=\frac{a^2kt}{kat+1}}$$

b) $a\neq b \quad\Rightarrow\quad \dfrac{1}{(a-x)(b-x)}dx=kdt$

$$\int\frac{1}{(a-x)(b-x)}dx=\int kdt$$

$$\frac{1}{(a-x)(b-x)}=\frac{A}{a-x}+\frac{B}{b-x}$$

$$\begin{cases}-A-B=0\\ bA+aB=1\end{cases}\rightarrow\begin{cases}B=\dfrac{1}{a-b}\\ A=-\dfrac{1}{a-b}\end{cases}$$

$$\frac{-1}{a-b}\int\frac{1}{a-x}dx+\frac{1}{a-b}\int\frac{1}{b-x}dx=\int kdt$$

$$\frac{1}{a-b}\ln|a-x|-\frac{1}{a-b}\ln|b-x|=kt+C$$

$$\frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right|=kt+C$$

$$x(0)=0\quad\Rightarrow\quad\boxed{\frac{1}{a-b}\ln\left(\frac{a}{b}\right)=C}$$

$$\frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right|=kt+\frac{1}{a-b}\ln\left(\frac{a}{b}\right)$$

$$\ln\left|\frac{a-x}{b-x}\right|=(a-b)kt+\ln\left(\frac{a}{b}\right)$$

$$\frac{a-x}{b-x}=e^{(a-b)kt+\ln\left(\frac{a}{b}\right)}$$

$$\frac{a-x}{b-x}=\frac{a}{b}e^{(a-b)kt}$$

$$a-x=b\frac{a}{b}e^{(a-b)kt}-x\frac{a}{b}e^{(a-b)kt}$$

$$x\left(\frac{a}{b}e^{(a-b)kt}-1\right)=ae^{(a-b)kt}-a$$

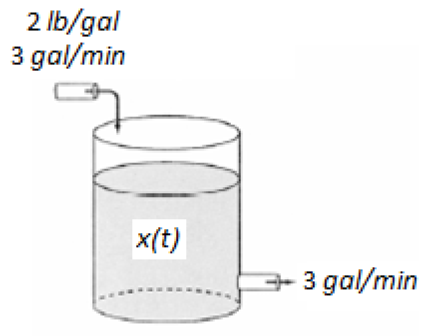
$$\boxed{x=\frac{abe^{(a-b)kt}-ab}{ae^{(a-b)kt}-b}}$$

Exercise

The tank initially holds 100 *gal* of pure water. At time $t=0$, a solution containing 2 *lb* of salt per gallon begins to enter the tank at the rate of 3 gallons per minute. At the same time a drain is opened at the bottom of the tank so that the volume of solution in the tank remains constant.
How much salt is in the tank after 60 *min*?
What will be the eventual salt content in the tank?

Solution

$x(t)$: number of pounds of salt in the tank after t min.



Volume: $V(t)=100+(3-3)t=100$

Concentration at time t: $c(t)=\dfrac{x(t)}{V(t)}=\dfrac{x(t)}{100}\ \ lb\,/\,gal$

Rate in = Volume Rate *x* Concentration

$$=3\frac{gal}{min}\times 2\frac{lb}{gal}$$

$$=6\ \ lb\,/\,\text{min}$$

Rate out = Volume Rate *x* Concentration

$$=3\frac{gal}{\text{min}}\times\frac{x(t)}{100}\frac{lb}{gal}$$

$$=\frac{3x(t)}{100}\ \ lb\,/\,\text{min}$$

$\dfrac{dx}{dt}=$ rate of change

$\quad$ = rate in − rate out

$$=6-\frac{3x}{100}$$

$$\frac{dx}{dt}+\frac{3}{100}x=6$$

$$u(t)=e^{\int\left(\frac{3}{100}\right)dt}=e^{0.03t}$$

$$\int 6e^{0.03t}dt=\frac{6}{0.03}e^{0.03t}=200e^{0.03t}$$

$$x(t)=e^{-0.03t}\left(200e^{0.03t}+C\right)$$

$$\boxed{x(t)=200+Ce^{-0.03t}}$$

Since there was no salt present in the tank initially, the initial condition is $x(0)=0$

$$x(t=0)=200+Ce^{-0.03(0)}=0$$

$$200+C=0\quad\rightarrow \boxed{C=-200}$$

$$\boxed{x(t)=200-200e^{-0.03t}}$$

After 60 min: $\quad x(60) = 200 - 200e^{-0.03(60)} \underline{\approx 167\ lb}$

As $t \to \infty$ then $x(t) = \lim\limits_{t\to\infty}\left(200 - 200e^{-0.03t}\right)$

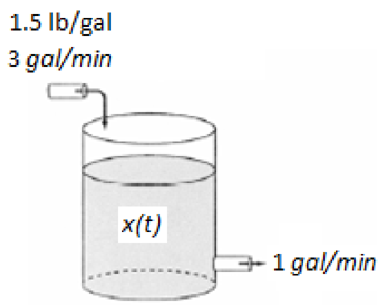
$$= 200 - 200\lim_{t\to\infty}\left(e^{-0.03t}\right) \qquad\qquad \lim_{t\to\infty}\left(e^{-0.03t}\right) = e^{-\infty} = 0$$

$$\underline{= 200\ \ lb}$$

Exercise

The 600-*gal* tank is filled with 300 *gal* of pure water. A spigot is opened above the tank and a salt solution containing 1.5 *lb*. of salt per gallon of solution begins flowing into the tank at the rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave tank at a rate of 1 *gal/min*. What will be the salt content in the tank at the precise moment that the volume of solution in the tank is equal to the tank's capacity (600 *gal*)?

Solution



$$V(t) = 300 + (3-1)t = 300 + 2t$$

$$c(t) = \frac{x(t)}{300+2t}$$

$$\text{Rate in } = 3\frac{gal}{min} \times 1.5\frac{lb}{gal} = 4.5\ \ lb/min$$

$$\text{Rate out } = 1 \times \frac{x}{300+2t} = \frac{x}{300+2t}\ \ lb/\min$$

$$\frac{dx}{dt} = 4.5 - \frac{x}{300+2t}$$

$$\frac{dx}{dt} + \frac{1}{300+2t}x = 4.5$$

$$u(t) = e^{\int \frac{1}{300+2t}dt} \qquad d\left(300+2t\right) = 2dt$$

$$= e^{\frac{1}{2}\int \frac{1}{300+2t}d\left(300+2t\right)}$$

$$= e^{\frac{1}{2}\ln\left(300+2t\right)}$$

$$= e^{\ln\left(300+2t\right)^{1/2}}$$

$$= \sqrt{300+2t}$$

$$\int 4.5\sqrt{300+2t}\ dt = 4.5\frac{1}{2}\frac{2}{3}\left(300+2t\right)^{2/3}$$

$$x(t) = \frac{1}{\sqrt{300+2t}}\left(1.5\left(300+2t\right)^{3/2} + C\right)$$

$$= 1.5\left(300+2t\right) + \frac{C}{\sqrt{300+2t}}$$

$$= 450 + 3t + \frac{C}{\sqrt{300+2t}}$$

$$x(0) = 450 + 3(0) + \frac{C}{\sqrt{300+2(0)}} = 0$$

$$450 + \frac{C}{\sqrt{300}} = 0$$

$$C = -450\sqrt{300} = -4500\sqrt{3}$$

$$x(t) = 450 + 3t - \frac{4500\sqrt{3}}{\sqrt{300+2t}}$$

$$V = 300 + 2t = 600$$

$$t = 150\ min$$

$$x(t=150) = 450 + 3(150) - \frac{4500\sqrt{3}}{\sqrt{300+2(150)}}$$

$$\approx 582\ lb$$

Exercise

The amount of drug in the blood of a patient (in *mg*) due to an intravenous line is governed by the initial value problem

$$y'(t) = -0.02y + 3, \quad y(0) = 0 \quad for\ t \geq 0$$

Where t is measured in hours

a) Find and graph the solution of the initial value problem.
b) What is the steady-state level of the drug?
c) When does the drug level reach 90% of the steady-state value?

Solution

a) $y' + 0.02y = 3$

$$e^{\int 0.02dt} = e^{0.02t}$$

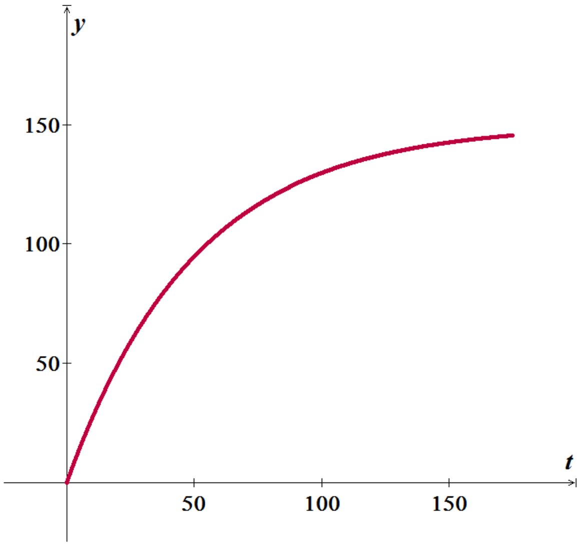
$$\int 3e^{0.02t}dt = 150e^{0.02t}$$

$$y = \frac{1}{e^{0.02t}}\left(150e^{0.02t} + C\right)$$

$$= 150 + Ce^{-0.02t}$$

$$y(0) = 0 \qquad 0 = 150 + C \to C = -150$$

$$y(t) = 150\left(1 - e^{-0.02t}\right)$$



b) The steady-state level is

$$\lim_{t\to\infty} 150\left(1-e^{-0.02t}\right) = 150\ mg$$

c) $150\left(1-e^{-0.02t}\right) = 0.9(150)$

$1-e^{-0.02t} = 0.9$

$e^{-0.02t} = 0.1$

$-0.02t = \ln 0.1$

$t = \dfrac{\ln 0.1}{-0.02} \approx 115\ hrs$

Exercise

A fish hatchery has 500 *fish* at time $t=0$, when harvesting begins at a rate of *b fish/yr*. where $b>0$. The fish population is modeled by the initial value problem.

$$y'(t) = 0.1y - b, \quad y(0) = 500 \qquad for\ \ t \ge 0$$

Where *t* is measured in years.

a) Find the fish population for $t \ge 0$ in terms of the harvesting rate *b*.

b) Graph the solution in the case that $b = 40\ fish\,/\,yr$. Describe the solution.

c) Graph the solution in the case that $b = 60\ fish\,/\,yr$. Describe the solution.

Solution

a) $y' - 0.1y = -b$

$$e^{\int -0.1dt} = e^{-0.1t}$$

$$\int -be^{-0.1t}dt = 10be^{-0.1t}$$

$y(t) = e^{0.1t}\left(10be^{-0.1t} + C\right)$

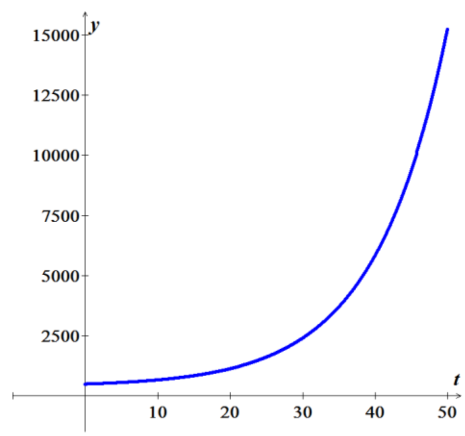
$= 10b + Ce^{0.1t}$

$y(0) = 500 \quad \to \quad 500 = 10b + C \ \Rightarrow\ C = 500 - 10b$

$y(t) = 10b + (500-10b)e^{0.1t}$

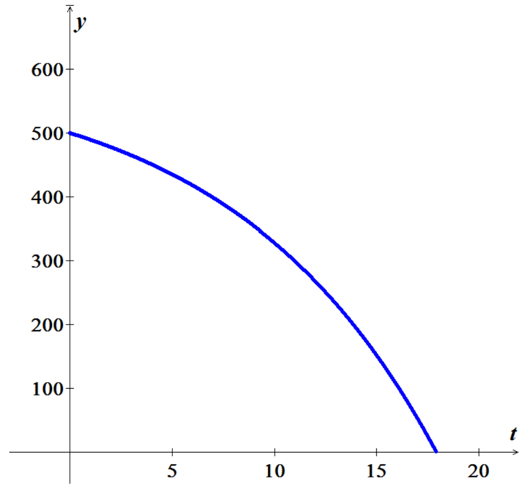
b) For $b = 40$

$y(t) = 400 + 100e^{0.1t}$



c) For $b = 60$

$y(t) = 600 - 100e^{0.1t}$



Exercise

A community of hares on an island has a population of 50 when observations begin at $t = 0$. The population for $t \ge 0$ is modeled by the initial value problem.

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{200}\right), \quad P(0) = 50$$

d) Find the solution of the initial value problem.

e) What is the steady-state population?

Solution

a) $\displaystyle\int \frac{200}{P(200-P)}dP = \int 0.08dt$

$\displaystyle\int \left(\frac{1}{P} + \frac{1}{200-P}\right)dP = \int 0.08dt$

$\ln P + \ln|200 - P| = 0.08t + C$

$\ln\left|\dfrac{P}{200-P}\right| = 0.08t + C$

$P(0) = 50 \quad \to \ln\dfrac{50}{150} == C \Rightarrow \underline{C = -\ln 3}$

$\ln\left|\dfrac{P}{200-P}\right| = 0.08t - \ln 3$

$\dfrac{P}{200-P} = e^{0.08t - \ln 3}$

$\dfrac{P}{200-P} = e^{0.08t}e^{\ln 3^{-1}}$

$\dfrac{P}{200-P} = \dfrac{1}{3}e^{0.08t}$

$3P = 200e^{0.08t} - Pe^{0.08t}$

$P(t) = \dfrac{200e^{0.08t}}{3 + e^{0.08t}}$

$$= \boxed{\frac{200}{3e^{-0.08t}+1}}$$

b) $\lim\limits_{t\to\infty} P(t) = \lim\limits_{t\to\infty} \frac{200}{3e^{-0.08t}+1} \boxed{= 200}$

Exercise

When an infected person is introduced into a closed and otherwise healthy community, the number of people who become infected with the disease (in the absence of any intervention) may be modeled by the logistic equation

$$\frac{dP}{dt} = kP\left(1-\frac{P}{A}\right), \quad P(0) = P_0$$

Where k is a positive infection rate, A is the number of people in the community, and P_0 is the number of infected people at $t=0$. The model assumes no recovery or intervention.

a) Find the solution of the initial value problem in terms of k, A, and P_0.

b) Graph the solution in the case that $k = 0.025, \quad A = 300, \quad and \ P_0 = 1$.

c) For fixed values of k and A, describe the long-term behavior of the solutions for any P_0 with $0 < P_0 < A$

Solution

a) $\frac{dP}{dt} = kP\left(\frac{A-P}{A}\right)$

$$\int \frac{A}{P(A-P)}dP = \int kdt$$

$$\int \left(\frac{1}{P}+\frac{1}{A-P}\right)dP = \int kdt$$

$$\ln P - \ln|A-P| = kt + C_1$$

$$\ln\left|\frac{P}{A-P}\right| = kt + C_1$$

$$\frac{P}{A-P} = Ce^{kt}$$

$$P(0) = P_0 \rightarrow \quad \frac{P_0}{A-P_0} = C$$

$$\frac{P}{A-P} = \frac{P_0}{A-P_0}e^{kt}$$

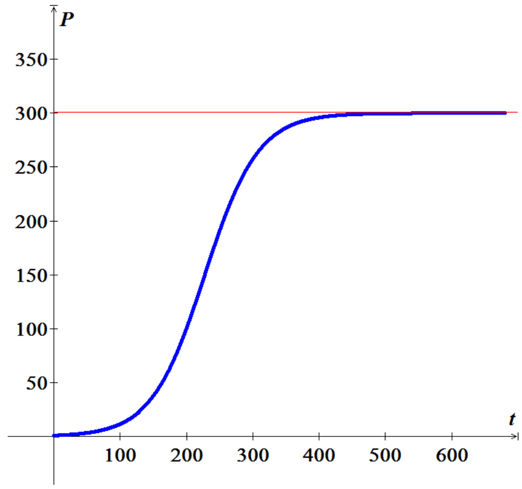
$$P = (A-P)\frac{P_0}{A-P_0}e^{kt}$$

$$\left(A-P_0+P_0e^{kt}\right)P = AP_0e^{kt}$$

$$P(t) = \boxed{\frac{AP_0 e^{kt}}{A - P_0 + P_0 e^{kt}}} = \frac{AP_0}{P_0 + (A - P_0)e^{-kt}}$$

b) $k = 0.025, \quad A = 300, \quad and \quad P_0 = 1$

$$P(t) = \frac{300}{1 + 299e^{-0.025t}}$$



c) $\lim\limits_{t\to\infty} P(t) = \lim\limits_{t\to\infty} \frac{AP_0}{P_0 + (A - P_0)e^{-kt}}$

$$= \frac{AP_0}{P_0}$$

$\underline{= A|}$ Which is the steady-state solution

Exercise

An object in free fall may be modeled by assuming that the only forces at work are the gravitational force and resistance (friction due to the medium in which the objects falls). By Newton's second law (mass × acceleration = the sum of the external forces), the velocity of the object satisfies the differential equation

$$\underset{mass}{m} \cdot \underset{acceleration}{v'(t)} = \underbrace{mg + f(v)}_{external\ force}$$

Where f is a function that models the resistance and the positive direction is downward. One common assumption (often used for motion in air) is that $f(v) = -kv^2$, where $k > 0$ is a drag coefficient.

a) Show that the equation can be written in the form $v'(t) = g - av^2$ where $a = \frac{k}{m}$

b) For what (positive) value of v is $v'(t) = 0$? (This equilibrium solution is called the ***terminal velocity***.)

c) Find the solution of this separable equation assuming $v(0) = 0$ *and* $0 < v(t)^2 < \frac{g}{a}$ for $t \ge 0$

d) Graph the solution found in part (***c***) with $g = 9.8\ m/s^2, \quad m = 1\ kg, \quad and \quad k = 0.1\ kg/m$, and verify the terminal velocity agrees with the value found in part (***b***).

Solution

a) Given: $f(v) = -kv^2$

$$mv'(t) = mg + f(v)$$

$$mv'(t) = mg - kv^2$$

$$v'(t) = g - \frac{k}{m}v^2$$

$$\boxed{v'(t) = g - av^2} \qquad \text{where } a = \frac{k}{m}$$

b) $v'(t) = g - av^2 = 0 \quad \Rightarrow \quad v^2 = \frac{g}{a} \quad \rightarrow \quad \boxed{v = \sqrt{\frac{g}{a}}}$

c) $\frac{dv}{dt} = g - av^2$

$$\int \frac{dv}{g - av^2} = \int dt$$

$$-\frac{1}{a}\int \frac{dv}{v^2 - \frac{g}{a}} = \int dt$$

$$-\frac{1}{2a}\sqrt{\frac{a}{g}}\int \frac{dv}{v - \sqrt{\frac{g}{a}}} + \frac{1}{2a}\sqrt{\frac{a}{g}}\int \frac{dv}{v + \sqrt{\frac{g}{a}}} = \int dt$$

$$\frac{1}{v^2 - \frac{g}{a}} = \frac{A}{v - \sqrt{\frac{g}{a}}} + \frac{B}{v + \sqrt{\frac{g}{a}}}$$

$$1 = A\sqrt{\frac{g}{a}} + Av + Bv - B\sqrt{\frac{g}{a}}$$

$$\begin{cases} A + B = 0 \ \rightarrow \boxed{A = -B} \\ A\sqrt{\frac{g}{a}} - B\sqrt{\frac{g}{a}} = 1 \end{cases}$$

$$\boxed{A = -B = \frac{1}{2}\sqrt{\frac{a}{g}}}$$

$$\frac{1}{2}\sqrt{\frac{1}{ag}}\left(-\ln\left|\sqrt{\frac{g}{a}} - v\right| + \ln\left|\sqrt{\frac{g}{a}} + v\right|\right) = t + C_1$$

$$\ln\frac{\sqrt{\frac{g}{a}} + v}{\sqrt{\frac{g}{a}} - v} = 2\sqrt{ag}\,t + C_2$$

$$\frac{\sqrt{\frac{g}{a}} + v}{\sqrt{\frac{g}{a}} - v} = e^{2\sqrt{ag}t + C_2}$$

$$\sqrt{\frac{g}{a}} + v = Ce^{2\sqrt{ag}t}\left(\sqrt{\frac{g}{a}} - v\right) \qquad v(0)=0 \ \Rightarrow \ \sqrt{\frac{g}{a}} = \sqrt{\frac{g}{a}}C \rightarrow \boxed{C=1}$$

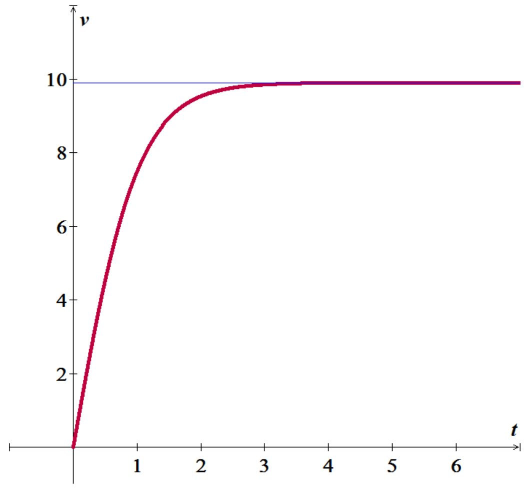
$$v\left(1 + e^{2\sqrt{ag}t}\right) = \sqrt{\frac{g}{a}}e^{2\sqrt{ag}t} - \sqrt{\frac{g}{a}}$$

$$\boxed{v(t) = \frac{e^{2\sqrt{ag}t} - 1}{1 + e^{2\sqrt{ag}t}}\sqrt{\frac{g}{a}}}$$

d) $g = 9.8\ m/s^2, \quad m = 1\ kg, \quad and \quad k = 0.1\ kg/m$

$$\rightarrow a = \frac{k}{m} \ \boxed{= 0.1}$$

$$v(t) = \sqrt{98}\,\frac{e^{2\sqrt{.98}t} - 1}{1 + e^{2\sqrt{.98}t}}$$

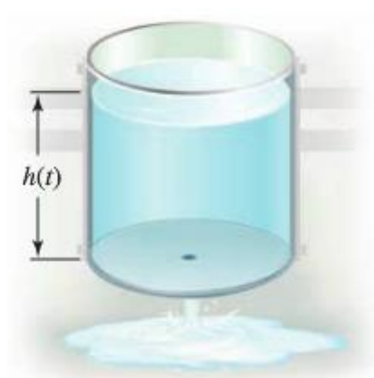


Exercise

An open cylindrical tank initially filled with water drains through a hole in the bottom of the tank according to Torricelli's Law. If $h(t)$ is the depth of water in the tank for $t \ge 0$, then Torricelli's Law implies $h'(t) = -2k\sqrt{h}$, where k is a constant that includes the acceleration due to gravity, the radius of the tank, and the radius of the drain. Assume that the initial depth of the water is $h(0) = H$.

a) Find the solution of the initial value problem.
b) Find the solution in the case that $k = 0.1$ and $H = 0.5\ m$.
c) In general, how long does it take the tank to drain in terms of k and H?

Solution



a) $\frac{dh}{dt} = -2k\sqrt{h}$

$$\int \frac{dh}{\sqrt{h}} = -2\int k\,dt$$

$$2\sqrt{h} = 2kt + C_1$$

$$h(t) = (kt + C)^2$$

$$h(0)=H \quad \to H=C^2 \Rightarrow C=\sqrt{H}$$

$$\boxed{h(t) = \left(kt + \sqrt{H}\right)^2}$$

b) Given: $k = 0.1 \quad H = 0.5\ m$

$$\boxed{h(t) = \left(0.1t + \sqrt{0.5}\right)^2 = (0.1t + 0.707)^2}$$

c) The tank is drained when $h(t) = 0$

$$\left(kt + \sqrt{H}\right)^2 = 0$$

$$kt + \sqrt{H} = 0 \quad \to \quad \boxed{t = -\frac{\sqrt{H}}{k}}$$

Exercise

The reaction of chemical compounds can often be modeled by differential equations. Let $y(t)$ be the concentration of a substance in reaction for $t \ge 0$ (typical units of y are *moles/L*). The change in the concentration of a substance, under appropriate conditions, is $\frac{dy}{dt} = -ky^n$, where $k > 0$ is a rate constant and the positive integer n is the order of the reaction.

a) Show that for a first-order reaction $(n = 1)$, the concentration obeys an exponential decay law.

b) Solve the initial value problem for a second-order reaction $(n = 2)$ assuming $y(0) = y_0$

c) Graph and compare the concentration for a first-order and second-order reaction with $k = 0.1$ and $y_0 = 1$

Solution

a) $\displaystyle\int \frac{dy}{y} = -\int k\,dt$

$\ln|y| = -kt + C_1$

$\boxed{y(t) = Ce^{-kt}}$

b) $n = 2 \quad \rightarrow \quad \frac{dy}{dt} = -ky^2$

$-\displaystyle\int \frac{dy}{y^2} = \int k\,dt$

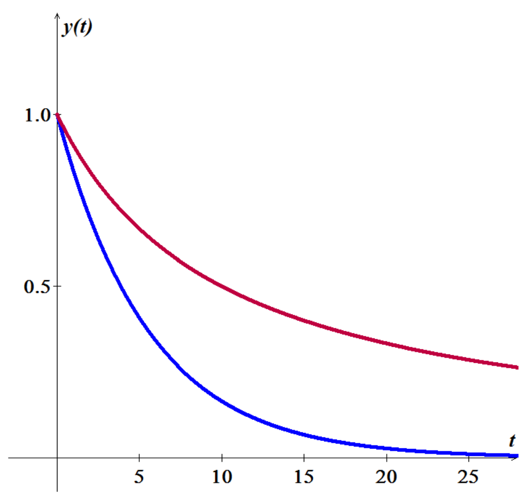
$\frac{1}{y} = kt + C \qquad y(0) = y_0 \quad \rightarrow \quad \boxed{\frac{1}{y_0} = C}$

$\frac{1}{y} = kt + \frac{1}{y_0}$

$\boxed{y(t) = \dfrac{y_0}{1 + ky_0 t}}$

c) $y(t) = \dfrac{1}{1 + 0.1t}$

$y_0 = 1 \quad \rightarrow \quad C = 1 \quad \Rightarrow \quad \boxed{y(t) = e^{-0.1t}}$



Exercise

The growth of cancer turmors may be modeled by the Gomperts growth equation. Let $M(t)$ be the mass of the tumor for $t \ge 0$. The relevant intial value problem is

$$\frac{dM}{dt} = -aM\ln\frac{M}{K}, \quad M(0) = M_0$$

Where a and K are positive constants and $0 < M_0 < K$

a) Graph the growth rate function $R(M) = -aM\ln\frac{M}{K}$ assuming $a = 1$ and $K = 4$. For what values of M is the growth rate positive? For what values of M is maximum?

b) Solve the initial cvalue problem and graph the solution for $a = 1$, $K = 4$, and $M_0 = 1$. Describe the groath pattern of the tumor. Is the growth unbounded? If not, what is the limiting size of the tumor?

c) In the general equation, what is the meaning of K?

Solution

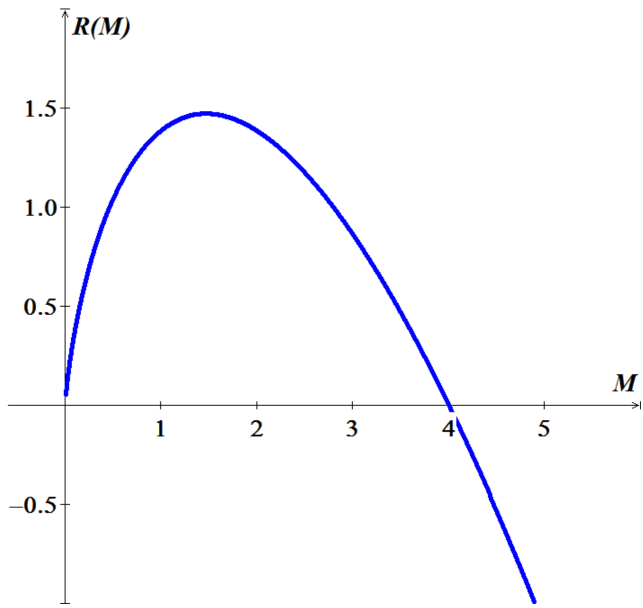
a) $R'(M) = -a\left(\ln\frac{M}{K} + M\frac{1}{K}\frac{K}{M}\right)$

$$= -a\left(\ln\frac{M}{K} + 1\right) = 0$$

$$\Rightarrow \ln\frac{M}{K} = -1 \quad \rightarrow \quad \boxed{M = Ke^{-1} = \frac{K}{e}}$$

For $a = 1$ and $K = 4$

$$\rightarrow \boxed{R(M) = -M\ln\frac{M}{4}}$$



b) $\displaystyle\int\frac{dM}{M(\ln M - \ln K)} = -\int a\,dt$

$$d(\ln M - \ln K) = \frac{1}{M}dM$$

$$\int\frac{d(\ln M - \ln K)}{\ln M - \ln K} = -\int a\,dt$$

$$\ln|\ln M - \ln K| = -at + C_1$$

$$\ln\frac{M}{K} = Ce^{-at}$$

$$\boxed{M(t) = Ke^{Ce^{-at}}}$$

For $a = 1$, $K = 4$, and $M_0 = 1$

$$M(0) = 4e^{C} = 1 \quad \Rightarrow \quad C = \ln\frac{1}{4} = -\ln 4$$

$$\boxed{M(t) = 4e^{-(\ln 4)e^{-t}}}$$

$$\lim_{t\to\infty} M(t) = \lim_{t\to\infty} 4e^{-(\ln 4)e^{-t}} = 4$$

So the limiting size of the tumor is 4.

c) $\displaystyle\lim_{t\to\infty} M(t) = \lim_{t\to\infty} Ke^{Ce^{-at}} = K$ since $a > 0$

Exercise

An endowment is an investment account in which the balance ideally remains constant and withdrawals are made on the interest earned by the account. Such an account may be modeled by the initial value problem $B'(t) = aB - m$ for $t \ge 0$, with $B(0) = B_0$. The constant a reflects the annual interest rate, m is the annual rate of withdrawal, and B_0 is the initial balance in the account.

a) Solve the initial value problem with $a = 0.05$, $m = \$1000/yr.$ and $B_0 = \$15{,}000$. Does the balance in the account increase or decrease?

b) If $a = 0.05$ and $B_0 = \$50{,}000$, what is the annual withdrawal rate m that ensures a constant balance in the account? What is the constant balance?

Solution

a) $B'(t) - aB = -m$

$$e^{\int -adt} = e^{-at}$$

$$\int -me^{-at}dt = \frac{m}{a}e^{-at}$$

$$B(t) = \frac{1}{e^{-at}}\left(\frac{m}{a}e^{-at} + C\right)$$

$$= \frac{m}{a} + Ce^{at}$$

Given: $a = 0.05,\ m = \$1000/yr.\ B_0 = \$15{,}000$

$$B(0) = \frac{1000}{.05} + C = 15{,}000 \quad \Rightarrow \quad C = 15{,}000 - 20{,}000 = -5{,}000$$

$$B(t) = 20{,}000 - 5{,}000\, e^{0.05t}$$

The balance decreases since the exponential increases with time and subtract from 20,000.

b) ***Given***: $a = 0.05\ \ B_0 = \$50{,}000$

$$B = \frac{m}{a} = 50{,}000 \quad \Rightarrow \quad m = 0.05 \times 50{,}000 = 2{,}500$$

Exercise

The halibut fishery has been modeled by the differential equation $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{M}\right)$

Where $y(t)$ is the biomass (the total mass of the members of the population) in kilograms at time t (measured in years), the carrying capacity is estimated to be $M = 8 \times 10^7\ kg$ and $k = 0.71\ per\ year$.

a) If $y(0) = 2 \times 10^7\ kg$, find the biomass a year later.

b) How long will it take for the biomass to reach $4 \times 10^7\ kg$.

Solution

a) $\dfrac{M}{ky(M-y)}dy = dt \quad \rightarrow \quad \dfrac{M}{k}\dfrac{1}{y(M-y)}dy = dt$

$$\frac{1}{y(M-y)} = \frac{A}{y} + \frac{B}{M-y}$$

$$AM - Ay + By = 1 \quad \rightarrow \quad \begin{cases} AM = 1 \Rightarrow A = \frac{1}{M} \\ -A + B = 0 \Rightarrow B = A = \frac{1}{M} \end{cases}$$

$$\frac{M}{k}\frac{1}{M}\int \left(\frac{1}{y} + \frac{1}{M-y}\right)dy = \int dt$$

$$\frac{1}{k}\left(\ln y - \ln(M-y)\right) = t + C_1$$

$$\ln\frac{y}{M-y} = kt + C_2$$

$$\frac{y}{M-y} = e^{kt+C_2}$$

$$y = Me^{kt}e^{C_2} - ye^{kt}e^{C_2} \qquad C = e^{C_2}$$

$$y\left(1 + Ce^{kt}\right) = MCe^{kt}$$

$$y = \frac{MCe^{kt}}{1 + Ce^{kt}}$$

$$= \frac{M}{1 + Ce^{-kt}}$$

$$= \frac{8\times 10^7}{1 + Ce^{-0.71t}}$$

$$y(0) = \frac{8\times 10^7}{1 + C} = 2\times 10^7 \quad \Rightarrow \quad C = \frac{8\times 10^7}{2\times 10^7} - 1 = 3$$

$$y(t) = \frac{8\times 10^7}{1 + 3e^{-0.71t}}$$

$$y(1) = \frac{8\times 10^7}{1 + 3e^{-0.71}} \approx 3.23\times 10^7 \ kg$$

b) $y(t) = \dfrac{8\times 10^7}{1 + 3e^{-0.71t}} = 4\times 10^7$

$$1 + 3e^{-0.71t} = \frac{8\times 10^7}{4\times 10^7} = 2$$

$$3e^{-0.71t} = 1$$

$$e^{-0.71t} = \frac{1}{3}$$

$$-.071t = \ln\frac{1}{3}$$

$$t = \frac{\ln 3}{0.71} \approx 1.55 \ years$$

Exercise

Suppose a population $P(t)$ satisfies $\quad \dfrac{dP}{dt} = 0.4P - 0.001P^2, \quad P(0) = 50$

Where *t* is measured in years.

a) What is the carrying capacity?

b) What is $P'(0)$?

c) When will the population reach 50% of the carrying capacity?

Solution

a) $\dfrac{1}{0.4P(1-0.0025P)}dP = dt$

$$\frac{1}{P(1-0.0025P)} = \frac{A}{P} + \frac{B}{1-0.0025P}$$

$$A - .0025PA + PB = 1 \quad \to \begin{cases} \underline{A=1} \\ -.0025A + B = 0 \quad \underline{B=.0025} \end{cases}$$

$$\int \left(\frac{1}{P} + \frac{.0025}{1-.0025P}\right)dP = 0.4\int dt$$

$$\ln P - \ln(1-.0025P) = 0.4t + C_1$$

$$\ln \frac{P}{1-.0025P} = 0.4t + C_1$$

$$\frac{P}{1-.0025P} = e^{0.4t+C_1} = Ce^{0.4t} \qquad C = e^{C_1}$$

$$Ce^{-0.4t}P = 1 - .0025P$$

$$Ce^{-0.4t}P + .0025P = 1$$

$$\left(Ce^{-0.4t} + .0025\right)P = 1$$

$$P(t) = \frac{1}{Ce^{-0.4t} + .0025}$$

$$P(0) = \frac{1}{C + .0025} = 50 \qquad \underline{C = \frac{1}{50} - .0025 = .0175}$$

$$P(t) = \frac{1}{.0175e^{-0.4t} + .0025}$$

$$\boxed{P(t) = \frac{400}{7e^{-0.4t} + 1}}$$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{400}{1 + 7e^{-0.4t}} \ \underline{= 400}$$

The carrying capacity is 400.

b) $P'(0) = \dfrac{dP}{dt}\Big|_{t=0} = 0.4(50) - 0.001(50)^2 \ \underline{= 17.5}$

c) $P(t) = \dfrac{400}{7e^{-0.4t} + 1} = 200$

$$7e^{-0.4t} + 1 = 2$$

$$e^{-0.4t} = \frac{1}{7}$$

$$-0.4t = \ln\left(\frac{1}{7}\right)$$

$$t = \frac{\ln\left(\frac{1}{7}\right)}{-0.4} \approx 4.86 \ \ years$$

Exercise

Let $P(t)$ be the performance level of someone learning a skill as a function of the training time *t*. The graph of *P* is called a ***learning curve***. We proposed the differential equation

$$\frac{dP}{dt} = k\left(M - P(t)\right)$$

As a reasonable model for learning, where *k* is a positive constant. Solve it as a linear differential equation and use your solution to graph the learning curve.

Solution

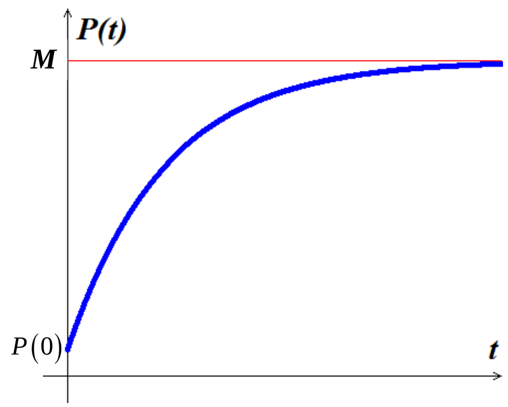
$$\frac{dP}{dt} + kP = kM$$

$$e^{\int kdt} = e^{kt}$$

$$\int kMe^{kt}dt = Me^{kt}$$

$$P(t) = \frac{1}{e^{kt}}\left(Me^{kt} + C\right)$$

$$= M + Ce^{-kt} \quad k > o$$



Exercise

A circuit containing an electromotive force, a capacitor with a capacitance of *C* farads (*F*), and a resistor with a resistance of *R* ohms (Ω). The voltage drop across the capacitor is $\frac{Q}{C}$, where *Q* is the charge (in coulombs), so in this case ***Kirchhoff's Law*** gives

$$RI + \frac{Q}{C} = E(t)$$

But $I = \frac{dQ}{dt}$, so we have $$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

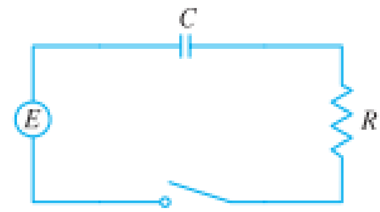
Find the charge and the current at time *t*

a) Suppose the resistance is $5\ \Omega$, the capacitance is 0.05 *F*, a battery gives voltage of 60 *V* and initial charge is $Q(0) = 0\ C$

b) Suppose the resistance is $2\ \Omega$, the capacitance is 0.01 *F*, $E(t) = 10\sin 60t$ and initial charge is $Q(0) = 0\ C$

Solution

a) $5\frac{dQ}{dt}+\frac{1}{.05}Q=60 \quad\rightarrow\quad \frac{dQ}{dt}+4Q=12$



$$e^{\int 4dt}=e^{4t}$$

$$\int 12e^{4t}dt=3e^{4t}$$

$$Q(t)=\frac{1}{e^{4t}}\left(3e^{4t}+C\right)\ \underline{=3+Ce^{-4t}}$$

$$Q(0)=3+C=0 \ \Rightarrow\ \underline{C=-3}$$

$$\underline{Q(t)=3\left(1-e^{-4t}\right)}$$

$$I=\frac{dQ}{dt}\ \underline{=12e^{-4t}}$$

b) $2\frac{dQ}{dt}+\frac{1}{.01}Q=10\sin 60t \quad\rightarrow\quad \frac{dQ}{dt}+50Q=5\sin 60t$

$$e^{\int 50dt}=e^{50t}$$

$$5\int e^{50t}(\sin 60t)\,dt=$$

$$\int e^{50t}(\sin 60t)\,dt=\left(-\frac{1}{60}\cos 60t+\frac{1}{72}\sin 60t\right)e^{50t}-\frac{25}{36}\int e^{50t}(\sin 60t)\,dt$$

$$\frac{61}{36}\int e^{50t}(\sin 60t)\,dt=\left(-\frac{1}{60}\cos 60t+\frac{1}{72}\sin 60t\right)e^{50t}$$

$$\int e^{50t}(\sin 60t)\,dt=\frac{36}{21{,}960}(-6\cos 60t+5\sin 60t)e^{50t}$$

$$5\int e^{50t}(\sin 60t)\,dt=\frac{1}{122}(-6\cos 60t+5\sin 60t)e^{50t}$$

		$\int \sin 60t$
+	e^{50t}	$-\frac{1}{60}\cos 60t$
−	$50e^{50t}$	$-\frac{1}{3600}\sin 60t$
+	$2500e^{50t}$	$-\frac{1}{3600}\int \sin 60t$

$$Q(t)=\frac{1}{e^{50t}}\left(\frac{1}{122}(-6\cos 60t+5\sin 60t)e^{50t}+C\right)$$

$$=\frac{1}{122}(-6\cos 60t+5\sin 60t)+Ce^{-50t}$$

$$Q(0)=-\frac{6}{122}+C=0 \ \Rightarrow\ \underline{C=\frac{3}{61}}$$

$$\underline{Q(t)=\frac{1}{122}\left(-5\cos 60t+6\sin 60t+6e^{-50t}\right)}$$

$$I=\frac{dQ}{dt}=\frac{1}{122}\left(300\sin 60t+360\cos 60t-300e^{-50t}\right)$$

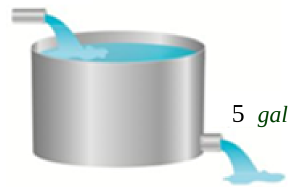
$$\underline{=\frac{30}{61}\left(5\sin 60t+6\cos 60t-5e^{-50t}\right)}$$

Exercise

A tank contains 50 *gallons* of a solution composed of 90% water and 10% alcohol. A second solution containing 50% water and 50% alcohol is added to the tank at the rate of $4 \ gal / \min$. As the second solution is being added, the tank is being drained at a rate of $5 \ gal / \min$. The solution in the tank is stirred constantly. How much alcohol is in the tank after 10 *minutes*?

Solution

Let *y* be the amount (in *lb.*) of additive in the tank at time *t* and $y(0) = 100$



$$V(t) = 50 + \left(4\frac{gal}{\min} - 5\frac{gal}{\min}\right)(t \ \min)$$

$$= 50 - t$$

$$Rate\ out = \frac{y}{50-t}(5) = \frac{5y}{50-t} \ \frac{lb}{\min}$$

$$Rate\ in = \left(\frac{1}{2}\ \frac{lb}{gal}\right)\left(4\ \frac{gal}{\min}\right) = 2\ \frac{lb}{\min}$$

$$\frac{dy}{dt} = 2 - \frac{5}{50-t}y$$

$$\frac{dy}{dt} + \frac{5}{50-t}y = 2$$

$$e^{\int pdt} = e^{\int \frac{5}{50-t}dt} = e^{\int \frac{-5}{50-t}d(50-t)} = e^{-5\ln|50-t|} = (50-t)^{-5}$$

$$\int 2(50-t)^{-5}dt = -2\int (50-t)^{-5}d(50-t) = \frac{1}{2}(50-t)^{-4}$$

$$y(t) = \frac{1}{(50-t)^{-5}}\left(\frac{1}{2}(50-t)^{-4} + C\right)$$

$$= \frac{1}{2}(50-t) + C(50-t)^5$$

$$y(0) = \frac{1}{2}(50) + C(50)^5 = 5 \quad \to \quad C = -\frac{20}{50^5}$$

$$y(t) = \frac{1}{2}(50-t) - \frac{20}{50^5}(50-t)^5$$

$$y(t=20) = \frac{1}{2}(30) - \frac{20}{50^5}(30)^5$$

$$= 15 - \frac{20}{5^5}3^5$$

$$\approx 13.45\ gal$$

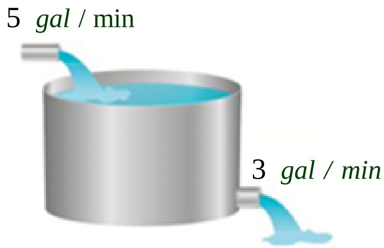
Exercise

A 200-*gallon* tank is half full of distilled water. At time $t=0$, a concentrate solution containing $0.5\ lb/gal$ enters the tank at the rate of $5\ gal/\min$, and well-stirred mixture is withdrawn at the rate of $3\ gal/\min$.

a) At what time will the tank be full?

b) At the time the tank is full, how many pounds of concentrate will it contain?

Solution



a) $V(t)=100+(5-3)t=200$

$2t=100 \quad \Rightarrow \quad \underline{t=50\ min}$

b) $Rate\ out=\dfrac{y}{100+2t}(3)=\dfrac{3y}{100+2t}\ \dfrac{lb}{\min}$

$Rate\ in=\left(0.5\ \dfrac{lb}{gal}\right)\left(5\ \dfrac{gal}{\min}\right)=\underline{2.5\ \dfrac{lb}{\min}}$

$$\frac{dy}{dt}=2.5-\frac{3y}{100+2t}$$

$$\frac{dy}{dt}+\frac{3}{100+2t}y=2.5$$

$$e^{\int\frac{3}{100+2t}dt}=e^{\frac{3}{2}\int\frac{1}{50+t}d(50+t)}=e^{\frac{3}{2}\ln|50+t|}=(50+t)^{3/2}$$

$$2.5\int(50+t)^{3/2}dt=\frac{5}{2}\int(50+t)^{3/2}d(50+t)=(50+t)^{5/2}$$

$$y(t)=\frac{1}{(50+t)^{3/2}}\left((50+t)^{5/2}+C\right)$$

$$\underline{=50+t+C(50+t)^{-3/2}}$$

$$y(0)=50+C(50)^{-3/2}=0$$

$$\underline{C=-(50)^{5/2}}$$

$$y(t)\ \underline{=50+t-(50)^{5/2}(50+t)^{-3/2}}$$

$$y(50)=50+50-(50)^{5/2}(100)^{-3/2}$$

$$\underline{\approx 82.32\ lb}$$

Exercise

A 200-*gallon* tank is half full of distilled water. At time $t=0$, a concentrate solution containing 1 *lb/gal* enters the tank at the rate of $5\ gal/\min$, and well-stirred mixture is withdrawn at the rate of $3\ gal/\min$.

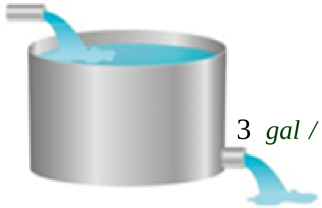
a) At what time will the tank be full?

b) At the time the tank is full, how many pounds of concentrate will it contain?

Solution

c) $V(t)=100+(5-3)t=200$

$2t=100 \quad \Rightarrow \quad \boxed{t=50\ \ min}$



d) $Rate\ out=\dfrac{y}{100+2t}(3)=\dfrac{3y}{100+2t}\ \dfrac{lb}{\min}$

$Rate\ in=\left(1\ \dfrac{lb}{gal}\right)\left(5\ \dfrac{gal}{\min}\right)=\boxed{5\ \dfrac{lb}{\min}}$

$$\frac{dy}{dt}=5-\frac{3y}{100+2t}$$

$$\frac{dy}{dt}+\frac{3}{100+2t}y=5$$

$$e^{\int\frac{3}{100+2t}dt}=e^{\frac{3}{2}\int\frac{1}{50+t}d(50+t)}=e^{\frac{3}{2}\ln|50+t|}=(50+t)^{3/2}$$

$$5\int(50+t)^{3/2}dt=5\int(50+t)^{3/2}d(50+t)=2(50+t)^{5/2}$$

$$y(t)=\frac{1}{(50+t)^{3/2}}\left(2(50+t)^{5/2}+C\right)$$

$$\boxed{=100+2t+C(50+t)^{-3/2}}$$

$$y(0)=100+C(50)^{-3/2}=0$$

$$\rightarrow C=-(100)(25\times 2)^{3/2}\boxed{=25000\sqrt{2}}$$

$$y(t)\ \boxed{=100+2t-25{,}000\sqrt{2}(50+t)^{-3/2}}$$

$$y(50)=100+100-25{,}000\sqrt{2}(100)^{-3/2}$$

$$=200-25\sqrt{2}$$

$$\boxed{\approx 164.64\ \ lb}$$

Exercise

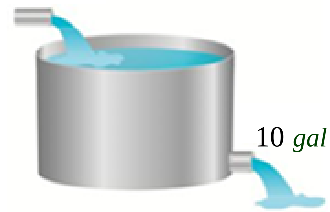
A 200-*gallon* tank is full of a concentrate solution containing $25\ lb$. Starting at time $t=0$, distilled water is admitted to the tank at the rate of $10\ gal/\min$, and well-stirred mixture is withdrawn at the same rate.

a) Find the amount of concentrate in the solution as a function of *t*.

b) Find the time at which the amount of concentrate in the tank reaches 15 *pounds*.

c) Find the quantity of the concentrate in the solution as $t \to \infty$.

Solution



a) $V(t) = 200 + (10-10)t \ \underline{= 200}|$

$$Rate\ out = \frac{10y}{200} = \frac{y}{20}\ \frac{lb}{\min}$$

$$Rate\ in = \underline{0}|$$

$$\frac{dy}{dt} = -\frac{y}{20}$$

$$\int \frac{dy}{y} = -\frac{1}{20}\int dt$$

$$\ln y = -\frac{1}{20}t + C_1 \quad \to \quad \underline{y(t) = Ce^{-t/20}}|$$

$$y(0) = \underline{C = 25}|$$

$$\underline{y(t) = 25e^{-t/20}}|$$

b) $y(t) = 25e^{-t/20} = 15$

$$e^{-t/20} = \frac{3}{5}$$

$$-\frac{t}{20} = \ln\left(\frac{3}{5}\right)$$

$$t = -20\ln\left(\frac{3}{5}\right) \ \underline{\approx 10.2\ min}|$$

c) $\lim\limits_{t\to\infty} y(t) = \lim\limits_{t\to\infty} 25e^{-t/20} \ \underline{= 0}|$