Solution   *Section* 3.1 – Inner Products

Exercise

Let $\langle \mathbf{u},\ \mathbf{v}\rangle$ be the Euclidean inner product on R^2, and let $\mathbf{u}=(1,\ 1)$, $\mathbf{v}=(3,\ 2)$, $\mathbf{w}=(0,\ -1)$, and $k=3$. Compute the following.

a) $\langle \mathbf{u},\ \mathbf{v}\rangle$ *c)* $\langle \mathbf{u}+\mathbf{v},\ \mathbf{w}\rangle$ *e)* $d(\mathbf{u},\ \mathbf{v})$

b) $\langle k\mathbf{v},\ \mathbf{w}\rangle$ *d)* $\|\mathbf{v}\|$ *f)* $\|\mathbf{u}-k\mathbf{v}\|$

Solution

a) $\langle \mathbf{u},\ \mathbf{v}\rangle = 1(3)+1(2)=\underline{5}$

b) $\langle k\mathbf{v},\ \mathbf{w}\rangle = \langle 3v,\ w\rangle$

$$= 9\cdot 0 + 6\cdot(-1)$$

$$= \underline{-6}$$

c) $\langle \mathbf{u}+\mathbf{v},\ \mathbf{w}\rangle = \langle \mathbf{u},\ \mathbf{w}\rangle + \langle \mathbf{v},\ \mathbf{w}\rangle$

$$= 1\cdot 0 + 1\cdot(-1) + 3\cdot 0 + 2\cdot(-1)$$

$$\underline{= -3}$$

d) $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v},\mathbf{v}\rangle} = \sqrt{3^2+2^2} = \underline{\sqrt{13}}$

e) $d(\mathbf{u},\ \mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|$

$$= \|(-2,\ -1)\|$$

$$= \sqrt{(-2)^2+(-1)^2}$$

$$= \underline{\sqrt{5}}$$

f) $\|\mathbf{u}-k\mathbf{v}\| = \|(1,1)-3(3,2)\|$

$$= \|(-8,\ -5)\|$$

$$= \sqrt{(-8)^2+(-5)^2}$$

$$= \underline{\sqrt{89}}$$

Exercise

Let $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u}=(1,\ 1)$, $\boldsymbol{v}=(3,\ 2)$, $\boldsymbol{w}=(0,\ -1)$, and $k=3$. Compute the following for the weighted Euclidean inner product $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 2u_1v_1 + 3u_2v_2$.

a) $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle$

b) $\langle k\boldsymbol{v},\ \boldsymbol{w}\rangle$

c) $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w}\rangle$

d) $\|\boldsymbol{v}\|$

e) $d(\boldsymbol{u},\ \boldsymbol{v})$

f) $\|\boldsymbol{u}-k\boldsymbol{v}\|$

Solution

a) $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 2(1)(3)+3(1)(2)=\underline{12}$

b) $\langle k\boldsymbol{v},\ \boldsymbol{w}\rangle = 2(3\cdot 3)(0)+3(3\cdot 2)(-1)=\underline{-18}$

c) $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w}\rangle = \langle \boldsymbol{u},\ \boldsymbol{w}\rangle + \langle \boldsymbol{v},\ \boldsymbol{w}\rangle$

$$=1\cdot 0+1\cdot(-1)+3\cdot 0+2\cdot(-1)$$

$$\underline{=-3}$$

d) $\|\boldsymbol{v}\| = \sqrt{\langle \boldsymbol{v},\boldsymbol{v}\rangle} = \sqrt{2(3)(3)+3(2)(2)} = \underline{\sqrt{30}}$

e) $d(\boldsymbol{u},\ \boldsymbol{v}) = \|\boldsymbol{u}-\boldsymbol{v}\|$

$$=\|\langle(-2,\ -1)\rangle\|$$

$$=\sqrt{2(-2)(-2)+3(-1)(-1)}$$

$$=\underline{\sqrt{11}}$$

f) $\|\boldsymbol{u}-k\boldsymbol{v}\| = \|(1,1)-3(3,2)\|$

$$=\|\langle(-8,\ -5)\rangle\|$$

$$=\sqrt{2(-8)^2+3(-5)^2}$$

$$=\underline{\sqrt{203}}$$

Exercise

Let $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u}=(3,\ -2)$, $\boldsymbol{v}=(4,\ 5)$, $\boldsymbol{w}=(-1,\ 6)$, and $k=-4$. Verify the following.

a) $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = \langle \boldsymbol{v},\ \boldsymbol{u}\rangle$

b) $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w}\rangle = \langle \boldsymbol{u},\ \boldsymbol{w}\rangle + \langle \boldsymbol{v},\ \boldsymbol{w}\rangle$

c) $\langle \boldsymbol{u},\ \boldsymbol{v}+\boldsymbol{w}\rangle = \langle \boldsymbol{u},\ \boldsymbol{v}\rangle + \langle \boldsymbol{u},\ \boldsymbol{w}\rangle$

d) $\langle k\boldsymbol{u},\ \boldsymbol{v}\rangle = k\langle \boldsymbol{u},\ \boldsymbol{v}\rangle$

e) $\langle \mathbf{0},\ \boldsymbol{v}\rangle = \langle \boldsymbol{v},\ \mathbf{0}\rangle = 0$

Solution

a) $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 3\cdot 4+(-2)\cdot(5)\ \underline{=2}$

$\langle \boldsymbol{v},\ \boldsymbol{u}\rangle = 4\cdot 3+(5)\cdot(-2)\ \underline{=2}$

b) $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w}\rangle = \langle (7,3),\ (-1,6)\rangle = 7(-1)+3(6) = \underline{11}$

$\langle \boldsymbol{u},\ \boldsymbol{w}\rangle + \langle \boldsymbol{v},\ \boldsymbol{w}\rangle = (3)(-1)+(-2)(6)+(4)(-1)+(5)(6) = \underline{11}$

c) $\langle \boldsymbol{u},\ \boldsymbol{v}+\boldsymbol{w}\rangle = \langle (3,-2),\ (3,11)\rangle = 3(3)+(-2)(11) = \underline{-13}$

$\langle \boldsymbol{u},\ \boldsymbol{v}\rangle + \langle \boldsymbol{u},\ \boldsymbol{w}\rangle = (3)(4)+(-2)(5)+(3)(-1)+(-2)(6) = \underline{-13}$

d) $\langle k\boldsymbol{u},\ \boldsymbol{v}\rangle = (-4\cdot 3)\cdot 4+((-4)(-2))\cdot(5) = \underline{-8}$

$k\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = (-4)(3\cdot 4+(-2)\cdot(5)) = \underline{-8}$

e) $\langle \mathbf{0},\ \boldsymbol{v}\rangle = 0\cdot 4+0\cdot(5) = \underline{0}$

$\langle \boldsymbol{v},\ \mathbf{0}\rangle = 4\cdot 0+(5)\cdot(0) = \underline{0}$

Exercise

Let $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle$ be the Euclidean inner product on R^2, and let $\boldsymbol{u}=(3,\ -2)$, $\boldsymbol{v}=(4,\ 5)$, $\boldsymbol{w}=(-1,\ 6)$, and $k=-4$. Verify the following for the weighted Euclidean inner product $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 4u_1v_1 + 5u_2v_2$.

a) $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = \langle \boldsymbol{v},\ \boldsymbol{u}\rangle$

b) $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w}\rangle = \langle \boldsymbol{u},\ \boldsymbol{w}\rangle + \langle \boldsymbol{v},\ \boldsymbol{w}\rangle$

c) $\langle \boldsymbol{u},\ \boldsymbol{v}+\boldsymbol{w}\rangle = \langle \boldsymbol{u},\ \boldsymbol{v}\rangle + \langle \boldsymbol{u},\ \boldsymbol{w}\rangle$

d) $\langle k\boldsymbol{u},\ \boldsymbol{v}\rangle = k\langle \boldsymbol{u},\ \boldsymbol{v}\rangle$

e) $\langle \mathbf{0},\ \boldsymbol{v}\rangle = \langle \boldsymbol{v},\ \mathbf{0}\rangle = 0$

Solution

a) $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 4\cdot 3\cdot 4 + 5\cdot(-2)\cdot(5) = \underline{-2}|$

$\langle \boldsymbol{v},\ \boldsymbol{u}\rangle = 4\cdot 4\cdot 3 + 5\cdot(5)\cdot(-2) = \underline{-2}|$

b) $\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w}\rangle = \langle (7,3),\ (-1,6)\rangle = 4\cdot 7(-1) + 5\cdot 3(6) = \underline{62}|$

$\langle \boldsymbol{u},\ \boldsymbol{w}\rangle + \langle \boldsymbol{v},\ \boldsymbol{w}\rangle = 4\cdot(3)(-1) + 5\cdot(-2)(6) + 4\cdot(4)(-1) + 5\cdot(5)(6) = \underline{62}|$

c) $\langle \boldsymbol{u},\ \boldsymbol{v}+\boldsymbol{w}\rangle = \langle (3,-2),\ (3,11)\rangle = 4\cdot 3(3) + 5\cdot(-2)(11) = \underline{-74}|$

$\langle \boldsymbol{u},\ \boldsymbol{v}\rangle + \langle \boldsymbol{u},\ \boldsymbol{w}\rangle = 4\cdot(3)(4) + 5\cdot(-2)(5) + 4\cdot(3)(-1) + 5\cdot(-2)(6) = \underline{-74}|$

d) $\langle k\boldsymbol{u},\ \boldsymbol{v}\rangle = 4\cdot(-4\cdot 3)\cdot 4 + 5\cdot((-4)(-2))\cdot(5) = \underline{8}|$

$k\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = (-4)(4\cdot 3\cdot 4 + 5\cdot(-2)\cdot(5)) = \underline{8}|$

e) $\langle \boldsymbol{0},\ \boldsymbol{v}\rangle = 4\cdot 0\cdot 4 + 5\cdot 0\cdot(5) = \underline{0}|$

$\langle \boldsymbol{v},\ \boldsymbol{0}\rangle = 4\cdot 4\cdot 0 + 5\cdot(5)\cdot(0) = \underline{0}|$

Exercise

Let $\boldsymbol{u} = (u_1,\ u_2)$ and $\boldsymbol{v} = (v_1,\ v_2)$. Show that the following are inner product on R^3 by verifying that the inner product axioms hold. $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 3u_1v_1 + 5u_2v_2$

Solution

Axiom 1: $\langle \boldsymbol{u},\ \boldsymbol{v}\rangle = 3u_1v_1 + 5u_2v_2 = 3v_1u_1 + 5v_2u_2 = \langle \boldsymbol{v},\ \boldsymbol{u}\rangle$

Axiom 2:
$$\begin{aligned}\langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{w}\rangle &= 3(u_1+v_1)w_1 + 5(u_2+v_2)w_2\\ &= 3(u_1w_1+v_1w_1) + 5(u_2w_2+v_2w_2)\\ &= 3u_1w_1 + 3v_1w_1 + 5u_2w_2 + 5v_2w_2\\ &= (3u_1w_1 + 5u_2w_2) + (3v_1w_1 + 5v_2w_2)\\ &= \langle \boldsymbol{u},\ \boldsymbol{w}\rangle + \langle \boldsymbol{v},\ \boldsymbol{w}\rangle\end{aligned}$$

Axiom 3:
$$\begin{aligned}\langle k\boldsymbol{u},\ \boldsymbol{v}\rangle &= 3(ku_1)v_1 + 5(ku_2)v_2\\ &= k(3u_1v_1 + 5u_2v_2)\\ &= k\langle \boldsymbol{u},\ \boldsymbol{v}\rangle\end{aligned}$$

Axiom 4:
$$\begin{aligned}\langle \boldsymbol{v},\ \boldsymbol{v}\rangle &= 3v_1v_1 + 5v_2v_2\\ &= 3v_1^2 + 5v_2^2 \ge 0\end{aligned}$$

$v_1 = v_2 = 0 \quad iff \quad \boldsymbol{v} = \boldsymbol{0}$

Exercise

Show that the following identity holds for the vectors in any inner product space

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2+\|\boldsymbol{u}-\boldsymbol{v}\|^2=2\|\boldsymbol{u}\|^2+2\|\boldsymbol{v}\|^2$$

Solution

$$\begin{aligned}
\|\boldsymbol{u}+\boldsymbol{v}\|^2+\|\boldsymbol{u}-\boldsymbol{v}\|^2 &= \langle \boldsymbol{u}+\boldsymbol{v},\ \boldsymbol{u}+\boldsymbol{v}\rangle+\langle \boldsymbol{u}-\boldsymbol{v},\ \boldsymbol{u}-\boldsymbol{v}\rangle \\
&= \langle \boldsymbol{u},\boldsymbol{u}+\boldsymbol{v}\rangle+\langle \boldsymbol{v},\boldsymbol{u}+\boldsymbol{v}\rangle+\langle \boldsymbol{u},\boldsymbol{u}-\boldsymbol{v}\rangle-\langle \boldsymbol{v},\boldsymbol{u}-\boldsymbol{v}\rangle \\
&= \langle \boldsymbol{u},\boldsymbol{u}\rangle+\langle \boldsymbol{u},\boldsymbol{v}\rangle+\langle \boldsymbol{v},\boldsymbol{u}\rangle+\langle \boldsymbol{v},\boldsymbol{v}\rangle+\langle \boldsymbol{u},\boldsymbol{u}\rangle-\langle \boldsymbol{u},\boldsymbol{v}\rangle-\langle \boldsymbol{v},\boldsymbol{u}\rangle+\langle \boldsymbol{v},\boldsymbol{v}\rangle \\
&= 2\langle \boldsymbol{u},\boldsymbol{u}\rangle+2\langle \boldsymbol{v},\boldsymbol{v}\rangle \\
&= 2\|\boldsymbol{u}\|^2+2\|\boldsymbol{v}\|^2 \quad \checkmark
\end{aligned}$$