Solution   *Section* 4.3 – Conservative Vector Fields

Exercise

Find the gradient field of the function $f(x,y,z)=\left(x^2+y^2+z^2\right)^{-1/2}$

Solution

$$\frac{\partial f}{\partial x}=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2x)$$
$$=-x\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial f}{\partial y}=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2y)$$
$$=-y\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial f}{\partial z}=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2z)$$
$$=-z\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\nabla f=-x\left(x^2+y^2+z^2\right)^{-3/2}\,\hat{i}-y\left(x^2+y^2+z^2\right)^{-3/2}\,\hat{j}-z\left(x^2+y^2+z^2\right)^{-3/2}\,\hat{k}$$
$$\boxed{=\frac{-x\,\hat{i}-y\hat{j}-z\,\hat{k}}{\left(x^2+y^2+z^2\right)^{3/2}}}$$

Exercise

Find the gradient field of the function $f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}$

Solution

$$f(x,\ y,\ z)=\ln\sqrt{x^2+y^2+z^2}$$
$$=\ln\left(x^2+y^2+z^2\right)^{1/2}$$
$$=\frac{1}{2}\ln\left(x^2+y^2+z^2\right)$$

$$\frac{\partial f}{\partial x}=\frac{1}{2}\frac{2x}{x^2+y^2+z^2}$$

$$= \frac{x}{x^2+y^2+z^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}\frac{2y}{x^2+y^2+z^2}$$

$$= \frac{y}{x^2+y^2+z^2}$$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\frac{2z}{x^2+y^2+z^2}$$

$$= \frac{z}{x^2+y^2+z^2}$$

$$\boxed{\nabla f = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2+y^2+z^2}}$$

Exercise

Find the gradient field of the function $f(x,y,z) = e^z - \ln\left(x^2+y^2\right)$

Solution

$$\frac{\partial f}{\partial x} = -\frac{2x}{x^2+y^2} \qquad \frac{\partial f}{\partial y} = -\frac{2y}{x^2+y^2} \qquad \frac{\partial f}{\partial z} = e^z$$

$$\nabla f = \boxed{-\frac{2x}{x^2+y^2}\hat{i} - \frac{2y}{x^2+y^2}\hat{j} + e^z\hat{k}}$$

Exercise

Find the line integral of $\int_C (x-y)\,dx$ where $C: \ x=t, \quad y=2t+1, \quad for \ 0 \le t \le 3$

Solution

$x=t, \quad y=2t+1, \quad for \ 0 \le t \le 3$

$dx = dt$

$$\int_C (x-y)\,dx = \int_0^3 \left(t-(2t+1)\right)dt$$

$$= \int_0^3 (-t-1)\,dt$$

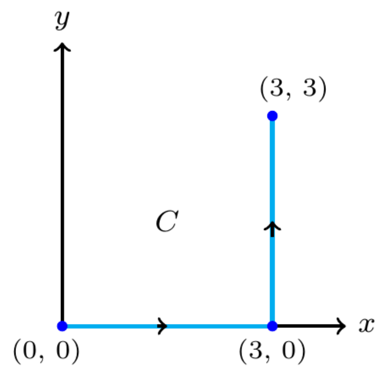
$$= -\left(\frac{1}{2}t^2 + t\right|_0^3$$

$$= -\left(\frac{9}{2} + 3\right)$$

$$\underline{= -\frac{15}{2}}$$

Exercise

Find the line integral of $\int_C \left(x^2 + y^2\right)dy$ where C is



Solution

$C_1: \quad x = t, \quad y = 0, \quad 0 \le t \le 3 \qquad \Rightarrow dy = 0$

$C_2: \quad x = 3, \quad y = t, \quad 0 \le t \le 3 \qquad \Rightarrow dy = dt$

$$\int_C \left(x^2 + y^2\right)dy = \int_{C_1} \left(x^2 + y^2\right)dy + \int_{C_2} \left(x^2 + y^2\right)dy$$

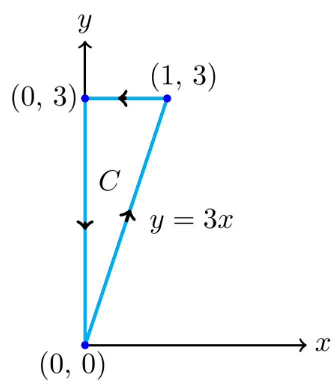
$$= \int_0^3 \left(t^2 + 0\right)(0) + \int_0^3 \left(9 + t^2\right)dt$$

$$= 9t + \frac{1}{3}t^3 \Big|_0^3$$

$$\underline{= 36}$$

Exercise

Find the line integral of $\int_C \sqrt{x + y}\ dx$ where C is



Solution

$C_1: \quad x = t, \quad y = 3t, \quad 0 \le t \le 1$

$\Rightarrow dx = dt$

$C_2: \quad x = 1 - t, \quad y = 3, \quad 0 \le t \le 1$

$\Rightarrow dx = -dt$

$C_3: \quad x = 0, \quad y = 3 - t, \quad 0 \le t \le 3$

$\Rightarrow dx = 0$

$$\int_C \sqrt{x+y}\ dx = \int_{C_1} \sqrt{x+y}\ dx + \int_{C_2} \sqrt{x+y}\ dx + \int_{C_3} \sqrt{x+y}\ dx$$

$$= \int_0^1 \sqrt{t+3t}\,dt + \int_0^1 \sqrt{1-t+3}\ (-dt) + \int_0^3 \sqrt{3-t}\ (0)$$

$$= \int_0^1 2\sqrt{t}\ dt + \int_0^1 \sqrt{4-t}\ d(4-t)$$

$$= 2\left(\frac{2}{3}t^{3/2}\ \Big|_0^1 + \frac{2}{3}\left((4-t)^{3/2}\ \Big|_0^1\right.\right.$$

$$= \frac{4}{3} + \frac{2}{3}\left(3^{3/2} - 4^{3/2}\right)$$

$$= \frac{4}{3} + \frac{2}{3}\left(3\sqrt{3} - 8\right)$$

$$= \frac{4 + 6\sqrt{3} - 16}{3}$$

$$= \frac{6\sqrt{3} - 12}{3}$$

$$= 2\sqrt{3} - 4$$

Exercise

Find the work done by the force field $\vec{F} = xy\hat{i} + y\hat{j} - yz\hat{k}$ over the curve $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t\hat{k}, \quad 0 \le t \le 1$.

Solution

$$\frac{d\vec{r}}{dt} = \hat{i} + 2t\,\hat{j} + \hat{k}$$

$$\vec{F} = xy\hat{i} + y\hat{j} - yz\hat{k}$$
$$= t^3\hat{i} + t^2\hat{j} - t^3\hat{k}$$

$$\vec{F} \cdot \frac{d\vec{r}}{dt} = \left(t^3\hat{i} + t^2\hat{j} - t^3\hat{k}\right) \cdot \left(\hat{i} + 2t\hat{j} + \hat{k}\right)$$
$$= t^3 + 2t^3 - t^3$$
$$= 2t^3$$

$$Work = \int_0^1 \vec{F} \cdot \frac{d\vec{r}}{dt}\,dt$$
$$= \int_0^1 2t^3\ dt$$

$$= \frac{1}{2} t^4 \Big|_0^1$$

$$\boxed{= \frac{1}{2}}$$

Exercise

Find the work done by the force field $\vec{F} = 2y\hat{i} + 3x\hat{j} + (x+y)\hat{k}$ over the curve $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + \frac{t}{6}\hat{k}, \quad 0 \le t \le 2\pi$.

Solution

$$\vec{F} = 2y\hat{i} + 3x\hat{j} + (x+y)\hat{k}$$

$$= (2\sin t)\hat{i} + (3\cos t)\hat{j} + (\cos t + \sin t)\hat{k}$$

$$\frac{d\vec{r}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j} + \frac{1}{6}\hat{k}$$

$$\vec{F} \cdot \frac{d\vec{r}}{dt} = \left((2\sin t)\hat{i} + (3\cos t)\hat{j} + (\cos t + \sin t)\hat{k}\right) \cdot \left((-\sin t)\hat{i} + (\cos t)\hat{j} + \frac{1}{6}\hat{k}\right)$$

$$= -2\sin^2 t + 3\cos^2 t + \frac{1}{6}\cos t + \frac{1}{6}\sin t$$

$$= -2\left(\frac{1-\cos 2t}{2}\right) + 3\left(\frac{1+\cos 2t}{2}\right) + \frac{1}{6}\cos t + \frac{1}{6}\sin t$$

$$= \cos 2t - 1 + \frac{3}{2} + \frac{3}{2}\cos 2t + \frac{1}{6}\cos t + \frac{1}{6}\sin t$$

$$= \frac{1}{2} + \frac{5}{2}\cos 2t + \frac{1}{6}\cos t + \frac{1}{6}\sin t$$

$$Work = \int_0^{2\pi} \left(\frac{1}{2} + \frac{5}{2}\cos 2t + \frac{1}{6}\cos t + \frac{1}{6}\sin t\right) dt \qquad W = \int \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \frac{1}{2}t + \frac{5}{4}\sin 2t + \frac{1}{6}\sin t - \frac{1}{6}\cos t \Big|_0^{2\pi}$$

$$= \left(\pi - \frac{1}{6}\right) - \left(-\frac{1}{6}\right)$$

$$\boxed{= \pi}$$

Exercise

Find the work done by the force field $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ over the curve $\vec{r}(t) = (\sin t)\hat{i} + (\cos t)\hat{j} + t\hat{k}, \quad 0 \le t \le 2\pi$.

Solution

$$\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$$
$$= t\hat{i} + (\sin t)\,\hat{j} + (\cos t)\,\hat{k}$$

$$\frac{d\vec{r}}{dt} = (\cos t)\,\hat{i} + (-\sin t)\,\hat{j} + \hat{k}$$

$$\vec{F} \cdot \frac{d\vec{r}}{dt} = \left(t\hat{i} + (\sin t)\,\hat{j} + (\cos t)\,\hat{k}\right) \cdot \left((\cos t)\,\hat{i} + (-\sin t)\,\hat{j} + \hat{k}\right)$$
$$= t\cos t - \sin^2 t + \cos t$$
$$= t\cos t - \frac{1}{2} + \frac{1}{2}\cos 2t + \cos t$$

$$Work = \int_0^{2\pi} \vec{F} \cdot \frac{d\vec{r}}{dt}\,dt$$
$$= \int_0^{2\pi} \left(t\cos t - \frac{1}{2} + \frac{1}{2}\cos 2t + \cos t\right) dt$$

		$\int \cos t$
+	t	$\sin t$
−	1	$-\cos t$

$$= t\sin t + \cos t - \frac{1}{2}t + \frac{1}{4}\sin 2t + \sin t \,\Big|_0^{2\pi}$$
$$= (1-\pi) - (1)$$
$$\underline{= -\pi}$$

Exercise

Find the work required to move an object with given force field $\vec{F} = \langle -y,\ z,\ x\rangle$ on the path consisting of the line segments from $(0,\ 0,\ 0)$ to $(0,\ 1,\ 0)$ followed by the line segment from $(0,\ 1,\ 0)$ to $(0,\ 1,\ 4)$

Solution

$(0,\ 0,\ 0)$ to $(0,\ 1,\ 0)$ $\rightarrow \vec{r}_1(t) = \langle 0,\ t,\ 0\rangle$

$(0,\ 1,\ 0)$ to $(0,\ 1,\ 4)$ $\rightarrow \vec{r}_2(t) = \langle 0,\ 1,\ 4t\rangle$

$$\vec{r}_1'(t) = \langle 0,\ 1,\ 0\rangle$$
$$\vec{r}_2'(t) = \langle 0,\ 0,\ 4\rangle$$

$$\vec{F} \cdot \vec{r}_1'(t) = \langle -t,\ 0,\ 0\rangle \cdot \langle 0,\ 1,\ 0\rangle = 0$$
$$\vec{F} \cdot \vec{r}_2'(t) = \langle -1,\ 4t,\ 0\rangle \cdot \langle 0,\ 0,\ 4\rangle = 0$$

$$W = \int_0^1 (0+0)\,dt \qquad\qquad W = \int_C \vec{F}\cdot d\vec{r}$$

$$= 0$$

Exercise

Find the work required to move an object with given force field $\vec{F} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}}$ on the path $\vec{r}(t) = \left\langle t^2,\ 3t^2,\ -t^2\right\rangle$ for $1 \le t \le 2$

Solution

$$\frac{d\vec{r}}{dt} = \langle 2t,\ 6t,\ -2t\rangle$$

$$W = \int_1^2 \frac{\left\langle t^2,\ 3t^2,\ -t^2\right\rangle \bullet \langle 2t,\ 6t,\ -2t\rangle}{\left(t^4+9t^4+t^4\right)^{3/2}}\,dt \qquad\qquad W = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_1^2 \frac{2t^3+18t^3+2t^3}{\left(11t^4\right)^{3/2}}\,dt$$

$$= \frac{1}{11\sqrt{11}} \int_1^2 \frac{22t^3}{t^6}\,dt$$

$$= \frac{2}{\sqrt{11}} \int_1^2 t^{-3}\,dt$$

$$= -\frac{1}{\sqrt{11}}\, t^{-2} \Big|_1^2$$

$$= -\frac{1}{\sqrt{11}}\left(\frac{1}{4}-1\right)$$

$$= \frac{3}{4\sqrt{11}}$$

Exercise

Evaluate $\int_C \vec{F}\cdot\vec{T}\,ds$ for the vector field $\vec{F}=x^2\hat{i}-y\hat{j}$ along the curve $x=y^2$ from $(4,\ 2)$ to $(1,\ -1)$

Solution

$$\vec{r}=x\hat{i}+y\hat{j}$$
$$=y^2\hat{i}+y\hat{j} \qquad -1\le y\le 2$$
$$\vec{F}=x^2\hat{i}-y\hat{j}$$
$$=y^4\hat{i}-y\hat{j}$$
$$\frac{d\vec{r}}{dy}=2y\hat{i}+\hat{j}$$
$$\vec{F}\cdot\frac{d\vec{r}}{dy}=\left(y^4\hat{i}-y\hat{j}\right)\cdot\left(2y\hat{i}+\hat{j}\right)$$
$$=2y^5-y$$
$$\int_C \vec{F}\cdot\vec{T}\,ds=\int_2^{-1}\vec{F}\cdot\frac{d\vec{r}}{dy}\,dy$$
$$=\int_2^{-1}\left(2y^5-y\right)dy$$
$$=\frac{1}{3}y^6-\frac{1}{2}y^2\Big|_2^{-1}$$
$$=\left(\frac{1}{3}-\frac{1}{2}\right)-\left(\frac{64}{3}-2\right)$$
$$=-\frac{39}{2}$$

Exercise

Find the circulation and flux of the fields $\vec{F}_1=x\hat{i}+y\hat{j}$ *and* $\vec{F}_2=-y\hat{i}+x\hat{j}$ around and across each of the following curves.

a) The circle $\vec{r}(t)=(\cos t)\hat{i}+(\sin t)\hat{j},\quad 0\le t\le 2\pi$

b) The ellipse $\vec{r}(t)=(\cos t)\hat{i}+(4\sin t)\hat{j},\quad 0\le t\le 2\pi$

Solution

a) $\vec{r}(t)=(\cos t)\hat{i}+(\sin t)\hat{j},\quad 0\le t\le 2\pi$

$$\frac{d\vec{r}}{dt}=(-\sin t)\hat{i}+(\cos t)\hat{j}$$

$$\vec{F}_1 = x\hat{i} + y\hat{j}$$
$$= (\cos t)\hat{i} + (\sin t)\hat{j}$$

$$\vec{F}_1 \bullet \frac{d\vec{r}}{dt} = ((\cos t)\hat{i} + (\sin t)\hat{j}) \bullet ((-\sin t)\hat{i} + (\cos t)\hat{j})$$
$$= -\cos t \sin t + \sin t \cos t$$
$$= 0$$

$$\vec{F}_2 = -y\hat{i} + x\hat{j}$$
$$= -(\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\vec{F}_2 \bullet \frac{d\vec{r}}{dt} = (-(\sin t)\hat{i} + (\cos t)\hat{j}) \bullet ((-\sin t)\hat{i} + (\cos t)\hat{j})$$
$$= \sin^2 t + \cos^2 t$$
$$\underline{= 1}$$

$$Cir_1 = \int_0^{2\pi} \left(\vec{F}_1 \bullet \frac{d\vec{r}}{dt}\right) dt$$
$$= \int_0^{2\pi} 0 dt$$
$$\underline{= 0}$$

$$Cir_2 = \int_0^{2\pi} \left(\vec{F}_2 \bullet \frac{d\vec{r}}{dt}\right) dt$$
$$= \int_0^{2\pi} dt$$
$$\underline{= 2\pi}$$

$$dx = -\sin t \ dt, \quad dy = \cos t \ dt$$
$$M_1 = x = \cos t, \quad N_1 = y = \sin t$$
$$M_2 = -y = -\sin t, \quad N_2 = x = \cos t$$

$$Flux_1 = \int_C M_1 dy - N_1 dx$$
$$= \int_0^{2\pi} \left(\cos^2 t + \sin^2 t\right) dt$$
$$= \int_0^{2\pi} dt$$
$$\underline{= 2\pi}$$

$$Flux_2 = \int_C M_2 dy - N_2 dx$$

$$= \int_0^{2\pi} (-\sin t \cos t + \sin t \cos t) dt$$

$$= \int_0^{2\pi} (0) dt$$

$$\underline{= 0}$$

b) $\vec{r}(t) = (\cos t)\hat{i} + (4\sin t)\hat{j}, \quad 0 \le t \le 2\pi$

$$\frac{d\vec{r}}{dt} = (-\sin t)\hat{i} + (4\cos t)\hat{j}$$

$$\vec{F}_1 = x\hat{i} + y\hat{j}$$

$$= (\cos t)\hat{i} + (4\sin t)\hat{j}$$

$$\vec{F}_1 \bullet \frac{d\vec{r}}{dt} = \left((\cos t)\hat{i} + (4\sin t)\hat{j}\right) \bullet \left((-\sin t)\hat{i} + (4\cos t)\hat{j}\right)$$

$$= -\cos t \sin t + 16 \sin t \cos t$$

$$= 15 \sin t \cos t$$

$$\vec{F}_2 = -y\hat{i} + x\hat{j}$$

$$= -(4\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\vec{F}_2 \bullet \frac{d\vec{r}}{dt} = \left((-4\sin t)\hat{i} + (\cos t)\hat{j}\right) \bullet \left((-\sin t)\hat{i} + (4\cos t)\hat{j}\right)$$

$$= 4\sin^2 t + 4\cos^2 t$$

$$\underline{= 4}$$

$$Cir_1 = \int_0^{2\pi} \left(\vec{F}_1 \bullet \frac{d\vec{r}}{dt}\right) dt$$

$$= \int_0^{2\pi} 15 \sin t \cos t dt \qquad\qquad d(\sin t) = \cos t \ dt$$

$$= 15 \int_0^{2\pi} \sin t \ d(\sin t)$$

$$= \frac{15}{2} \left(\sin^2 t \right|_0^{2\pi}$$

$$= \frac{15}{2}(1 - 1)$$

$$\underline{= 0}$$

$$Cir_2 = \int_0^{2\pi} \left(\vec{F}_2 \cdot \frac{d\vec{r}}{dt} \right) dt$$

$$= \int_0^{2\pi} 4\, dt$$

$$= 4t \Big|_0^{2\pi}$$

$$\underline{= 8\pi}$$

$$dx = -\sin t\ dt, \quad dy = 4\cos t\ dt$$
$$M_1 = x = \cos t, \quad N_1 = y = 4\sin t$$
$$M_2 = -y = -4\sin t, \quad N_2 = x = \cos t$$

$$Flux_1 = \int_C M_1 dy - N_1 dx$$

$$= \int_0^{2\pi} \left(4\cos^2 t + 4\sin^2 t \right) dt$$

$$= 4\int_0^{2\pi} dt$$

$$\underline{= 8\pi}$$

$$Flux_2 = \int_C M_2 dy - N_2 dx$$

$$= -15 \int_0^{2\pi} (\sin t \cos t)\, dt$$

$$= -15 \int_0^{2\pi} \sin t\ d(\sin t)$$

$$= -15\left(\frac{1}{2}\sin^2 t \right|_0^{2\pi}$$

$$\underline{= 0}$$

Exercise

Find the circulation and flux of the fields $\vec{F}_1 = 2x\hat{i} - 3y\hat{j}$ *and* $\vec{F}_2 = 2x\hat{i} + (x-y)\hat{j}$ across the circle $\vec{r}(t) = (a\cos t)\hat{i} + (a\sin t)\hat{j}, \quad 0 \le t \le 2\pi$

Solution

$$\frac{d\vec{r}}{dt} = (-a\sin t)\hat{i} + (a\cos t)\hat{j}$$

$$\vec{F}_1 = 2x\hat{i} - 3y\hat{j}$$

$$= (2a\cos t)\hat{i} - (3a\sin t)\hat{j}$$

$$\vec{F}_1 \bullet \frac{d\vec{r}}{dt} = \left((2a\cos t)\hat{i} - (3a\sin t)\hat{j}\right) \bullet \left((-a\sin t)\hat{i} + (a\cos t)\hat{j}\right)$$

$$= -5a^2 \cos t \sin t$$

$$\vec{F}_2 = 2x\hat{i} + (x-y)\hat{j}$$

$$= (2a\cos t)\hat{i} + a(\cos t - \sin t)\hat{j}$$

$$\vec{F}_2 \bullet \frac{d\vec{r}}{dt} = \left((2a\cos t)\hat{i} + a(\cos t - \sin t)\hat{j}\right) \bullet \left((-a\sin t)\hat{i} + (a\cos t)\hat{j}\right)$$

$$= -2a^2 \cos t \sin t + a^2\cos^2 t - a^2 \cos t \sin t$$

$$= a^2\left(\cos^2 t - 3\cos t \sin t\right)$$

$$Cir_1 = \int_0^{2\pi} \left(\vec{F}_1 \bullet \frac{d\vec{r}}{dt}\right) dt$$

$$= -5a^2 \int_0^{2\pi} \sin t \cos t dt$$

$$= -5a^2 \int_0^{2\pi} \sin t \; d(\sin t)$$

$$= -5a^2 \left(\sin^2 t \right|_0^{2\pi}$$

$$= 0$$

$$Cir_2 = \int_0^{2\pi} \left(\vec{F}_2 \bullet \frac{d\vec{r}}{dt}\right) dt$$

$$= a^2 \int_0^{2\pi} \left(\cos^2 t - 3\cos t \sin t\right) dt$$

$$= a^2 \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right) dt \; - \; 3a^2 \int_0^{2\pi} (\sin t)\, d(\sin t)$$

$$= a^2 \left(\frac{1}{2}t + \frac{1}{4}\sin 2t - 0 \right|_0^{2\pi}$$

$$\underline{= \pi a^2}$$

$$dx = -a\sin t\ dt, \quad dy = a\cos t\ dt$$

$$M_1 = 2x = 2a\cos t, \quad N_1 = -3y = -3a\sin t$$

$$M_2 = 2a\cos t, \quad N_2 = a\cos t - a\sin t$$

$$Flux_1 = \int_C M_1 dy - N_1 dx$$

$$= \int_0^{2\pi} \left(2a^2\cos^2 t - 3a^2\sin^2 t\right)dt \qquad \cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t, \quad \sin^2 t = \frac{1}{2} - \frac{1}{2}\cos 2t$$

$$= a^2 \int_0^{2\pi} \left(1 + \cos 2t - \frac{3}{2} + \frac{3}{2}\cos 2t\right)dt$$

$$= a^2 \int_0^{2\pi} \left(\frac{5}{2}\cos 2t - \frac{1}{2}\right)dt$$

$$= a^2 \left(\frac{5}{4}\sin 2t - \frac{1}{2}t \right|_0^{2\pi}$$

$$= a^2\left[0 - \frac{1}{2}(2\pi)\right]$$

$$\underline{= -\pi a^2}$$

$$Flux_2 = \int_C M_2 dy - N_2 dx$$

$$= \int_0^{2\pi} \left(2a^2\cos^2 t - a^2\sin^2 t + a^2\cos t\sin t\right)dt$$

$$= a^2 \int_0^{2\pi} \left(1 + \cos 2t - \frac{1}{2} + \frac{1}{2}\cos 2t\right)dt \ + \ a^2 \int_0^{2\pi} (\sin t)\, d(\sin t)$$

$$= a^2\left(\frac{1}{2}t + \frac{3}{4}\sin 2t + \frac{1}{2}\sin^2 t \right|_0^{2\pi}$$

$$= a^2\frac{1}{2}(2\pi)$$

$$\underline{= \pi a^2}$$

Exercise

Find a field $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ in the *xy*-plane with the property that at each point $(x,y) \neq (0,0)$, $\vec{F}$ points toward the origin and $|\vec{F}|$ is

a) The distance from (*x, y*) to the origin
b) Inversely proportional to the distance from (*x, y*) to the origin. (The field is undefined at (0, 0).)

Solution

a) The slope of the line through the origin and a point (x,y) is: $m = \dfrac{y}{x}$

The vector parallel to the line is given by: $\vec{v} = x\hat{i} + y\hat{j}$

Pointing away from the origin: $\vec{F} = -\dfrac{\vec{v}}{|\vec{v}|} = -\dfrac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}}$ is the unit vector pointing toward the origin.

$$|\vec{F}| = \sqrt{x^2 + y^2}$$

$$\vec{F} = \sqrt{x^2 + y^2}\left(-\frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}}\right)$$

$$= -x\hat{i} - y\hat{j}$$

b) $|\vec{F}| = \dfrac{C}{\sqrt{x^2 + y^2}}, \quad C \neq 0$

$$\vec{F} = \frac{C}{\sqrt{x^2 + y^2}}\left(-\frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}}\right)$$

$$\boxed{= -C\left(\frac{x\hat{i} + y\hat{j}}{x^2 + y^2}\right)}$$

Exercise

A fluid's velocity field is $\vec{F} = -4xy\hat{i} + 8y\hat{j} + 2\hat{k}$. Find the flow along the curve $\vec{r}(t) = t\hat{i} + t^2\hat{j} + \hat{k}, \quad 0 \le t \le 2$

Solution

$$\frac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j}$$

$$\vec{F} = -4xy\hat{i} + 8y\hat{j} + 2\hat{k}$$

$$= -4t^3\hat{i} + 8t^2\hat{j} + 2\hat{k}$$

$$\vec{F} \cdot \frac{d\vec{r}}{dt} = \left(-4t^3\hat{i} + 8t^2\hat{j} + 2\hat{k}\right) \cdot \left(\hat{i} + 2t\,\hat{j}\right)$$

$$= -4t^3 + 16t^3 = 12t^3$$

$$Flow = \int_R \vec{F} \cdot \frac{d\vec{r}}{dt}\,dt$$

$$= \int_0^2 12t^3\,dt$$

$$= 3t^4 \Big|_0^2$$

$$\underline{= 48}$$

Exercise

A fluid's velocity field is $\vec{F} = x^2\hat{i} + yz\hat{j} + y^2\hat{k}$. Find the flow along the curve $\vec{r}(t) = 3t\,\hat{j} + 4t\,\hat{k}, \quad 0 \le t \le 1$

Solution

$$\frac{d\vec{r}}{dt} = 3\hat{i} + 4\hat{j}$$

$$\vec{F} = x^2\,\hat{i} + yz\hat{j} + y^2\,\hat{k}$$

$$= 12t^2\hat{j} + 9t^2\,\hat{k}$$

$$\vec{F} \cdot \frac{d\vec{r}}{dt} = \left(12t^2\hat{j} + 9t^2\hat{k}\right) \cdot \left(3\hat{i} + 4\hat{j}\right)$$

$$= 36t^2 + 36t^2$$

$$\underline{= 72t^2}$$

$$Flow = \int_0^1 72t^2\,dt$$

$$Flow = \int_C \vec{F} \cdot \frac{d\vec{r}}{dt}\,dt$$

$$= 24t^3 \Big|_0^1$$

$$\underline{= 24}$$

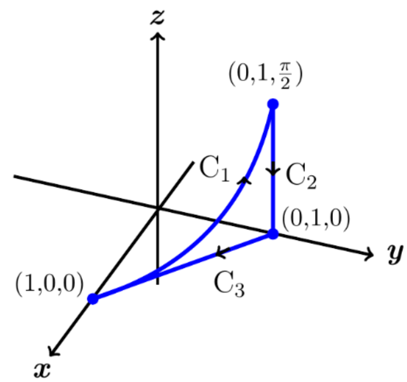
Exercise

Find the circulation of $\vec{F} = 2x\hat{i} + 2z\hat{j} + 2y\hat{k}$ around the closed path consisting of the following three curves traversed in the direction of increasing *t*.

$$C_1 : \ \vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\,\hat{k}, \quad 0 \le t \le \frac{\pi}{2}$$

$$C_2 : \ \vec{r}(t) = \hat{j} + \frac{\pi}{2}(1-t)\hat{k}, \quad 0 \le t \le 1$$

$$C_3 : \ \vec{r}(t) = t\,\hat{i} + (1-t)\hat{j}, \quad 0 \le t \le 1$$



Solution

$$C_1 : \ \vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\,\hat{k}, \quad 0 \le t \le \frac{\pi}{2}$$

$$\frac{d\vec{r}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j} + \hat{k}$$

$$\vec{F} = 2x\hat{i} + 2z\hat{j} + 2y\hat{k}$$

$$= (2\cos t)\hat{i} + 2t\,\hat{j} + (2\sin t)\hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left((2\cos t)\hat{i} + 2t\,\hat{j} + (2\sin t)\hat{k}\right) \bullet \left((-\sin t)\hat{i} + (\cos t)\hat{j} + \hat{k}\right)$$

$$= -2\sin t\cos t + 2t\cos t + 2\sin t$$

$$= -\sin 2t + 2t\cos t + 2\sin t$$

$$Flow_1 = \int_0^{\pi/2} \left(-\sin 2t + 2t\cos t + 2\sin t\right) dt$$

		$\int \cos t$
+	t	$\sin t$
−	1	$-\cos t$

$$= \left(\frac{1}{2}\cos 2t + 2t\sin t + 2\cos t - 2\cos t\right|_0^{\pi/2}$$

$$= \left(\frac{1}{2}\cos 2t + 2t\sin t\right|_0^{\pi/2}$$

$$= \left(-\frac{1}{2} + 2\frac{\pi}{2}\right) - \left(\frac{1}{2}\right)$$

$$= \pi - 1$$

$$C_2 : \ \vec{r}(t) = \hat{j} + \frac{\pi}{2}(1-t)\hat{k}, \quad 0 \le t \le 1$$

$$\frac{d\vec{r}}{dt} = -\frac{\pi}{2}\hat{k}$$

$$\vec{F} = 2x\hat{i} + 2z\hat{j} + 2y\hat{k}$$

$$= \pi(1-t)\hat{j} + 2\hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left(\pi(1-t)\hat{j} + 2\hat{k}\right) \bullet \left(-\frac{\pi}{2}\hat{k}\right)$$
$$= -\pi$$

$$Flow_2 = \int_0^1 (-\pi)\, dt$$
$$= -\pi\ t \Big|_0^1$$
$$\underline{= -\pi|}$$

$C_3: \quad \vec{r}(t) = t\ \hat{i} + (1-t)\hat{j}, \quad 0 \le t \le 1$

$$\frac{d\vec{r}}{dt} = \hat{i} - \hat{j}$$
$$\vec{F} = 2x\hat{i} + 2z\hat{j} + 2y\hat{k}$$
$$= 2t\hat{i} + 2(1-t)\hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left(2t\hat{i} + 2(1-t)\hat{k}\right) \bullet \left(\hat{i} - \hat{j}\right)$$
$$= 2t$$

$$Flow_3 = \int_0^1 (2t)\, dt$$
$$= t^2 \Big|_0^1$$
$$\underline{= 1\ |}$$

$$Circulation = Flow_1 + Flow_2 + Flow_3$$
$$= \pi - 1 - \pi + 1$$
$$\underline{= 0\ |}$$

Exercise

The field $\vec{F} = xy\hat{i} + y\hat{j} - yz\hat{k}$ is the velocity field of a flow in space. Find the flow from $(0,\ 0,\ 0)$ to $(1,\ 1,\ 1)$ along the curve of intersection of the cylinder $y = x^2$ and the plane $z = x$. (***Hint***: Use $t = x$ as the parameter.)

Solution

Let $x = t \quad \Rightarrow \quad y = x^2 = t^2$

$z = x = t$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$= t\hat{i} + t^2\hat{j} + t\hat{k} \qquad 0 \le t \le 1$$

$$\frac{d\vec{r}}{dt} = \hat{i} + 2t\,\hat{j} + \hat{k}$$

$$\vec{F} = xy\hat{i} + y\hat{j} - yz\hat{k}$$

$$= t^3\hat{i} + t^2\hat{j} - t^3\hat{k}$$

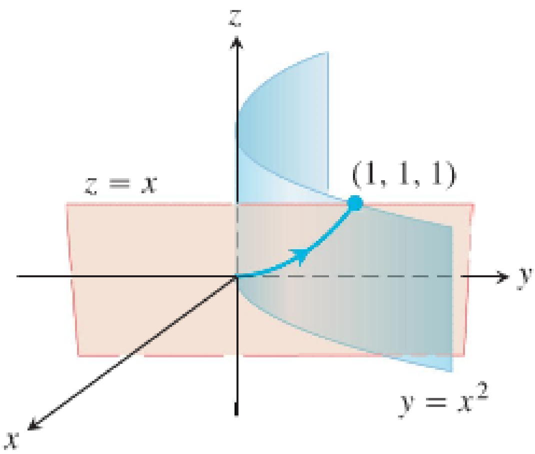
$$\vec{F} \cdot \frac{d\vec{r}}{dt} = \left(t^3\hat{i} + t^2\hat{j} - t^3\hat{k}\right) \cdot \left(\hat{i} + 2t\,\hat{j} + \hat{k}\right)$$

$$= t^3 + 2t^3 - t^3 = 2t^3$$

$$Flow = \int_0^1 \left(2t^3\right) dt$$

$$= \frac{1}{2}\, t^4 \Big|_0^1$$

$$= \frac{1}{2}$$



Exercise

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ for the vector fields $\vec{F}$ and curves C.

$$\vec{F} = \nabla\left(x^2 y\right); \quad C: \vec{r}(t) = \left\langle 9 - t^2,\ t\right\rangle, \quad for\ \ 0 \le t \le 3$$

Solution

$$\vec{F} = \nabla\left(x^2 y\right)$$

$$= \left\langle 2xy,\ x^2\right\rangle$$

$$= \left\langle 18t - 2t^3,\ 81 - 18t^2 + t^4\right\rangle$$

$$\vec{r}'(t) = \langle -2t,\ 1\rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^3 \left\langle 18t - 2t^3,\ 81 - 18t^2 + t^4\right\rangle \cdot \langle -2t,\ 1\rangle\, dt$$

$$= \int_0^3 \left(-36t^2 + 4t^4 +\ 81 - 18t^2 + t^4\right) dt$$

$$= \int_0^3 \left(5t^4 - 54t^2 +\ 81\right) dt$$

$$= t^5 - 18t^3 + 81t \Big|_0^3$$

$$= 243 - 486 + 243$$

$$\underline{= 0}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \nabla(xyz); \quad C: \vec{r}(t) = \left\langle \cos t,\ \sin t,\ \frac{t}{\pi} \right\rangle, \quad for\ \ 0 \le t \le \pi$

Solution

$$\vec{F} = \nabla(xyz)$$

$$= \langle yz,\ xz,\ xy \rangle$$

$$= \left\langle \frac{t}{\pi}\sin t,\ \frac{t}{\pi}\cos t,\ \cos t \sin t \right\rangle$$

$$\vec{r}'(t) = \left\langle -\sin t,\ \cos t,\ \frac{1}{\pi} \right\rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^\pi \left\langle \frac{t}{\pi}\sin t,\ \frac{t}{\pi}\cos t,\ \cos t \sin t \right\rangle \cdot \left\langle -\sin t,\ \cos t,\ \frac{1}{\pi} \right\rangle dt$$

$$= \int_0^\pi \left(-\frac{t}{\pi}\sin^2 t + \frac{t}{\pi}\cos^2 t + \frac{1}{\pi}\cos t \sin t \right) dt$$

$$= \frac{1}{\pi}\int_0^\pi \left(t\cos 2t + \frac{1}{2}\sin 2t \right) dt$$

		$\int \cos 2t$
$+$	t	$\frac{1}{2}\sin 2t$
$-$	1	$-\frac{1}{4}\cos 2t$

$$= \frac{1}{\pi}\left(\frac{1}{2}t\sin 2t + \frac{1}{4}\cos 2t - \frac{1}{4}\cos 2t \right) \Big|_0^\pi$$

$$= \frac{1}{2\pi}\ \left(t\sin 2t \right. \Big|_0^\pi$$

$$\underline{= 0}$$

Or

$$\vec{F} = \nabla(xyz) = \nabla\varphi$$

$$\int_C \vec{F}\cdot d\vec{r} = \varphi(\pi) - \varphi(0)$$

$$= \varphi\left(\cos\pi\sin\pi\left(\frac{\pi}{\pi}\right)\right) - \varphi\left(\cos 0\sin 0\left(\frac{0}{\pi}\right)\right)$$

$$= 0 - 0$$

$$\underline{= 0}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \bullet d\vec{r}$ .for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \langle x,\ -y\rangle$; C is the square with vertices $(\pm 1,\ \pm 1)$ with counterclockwise orientation.

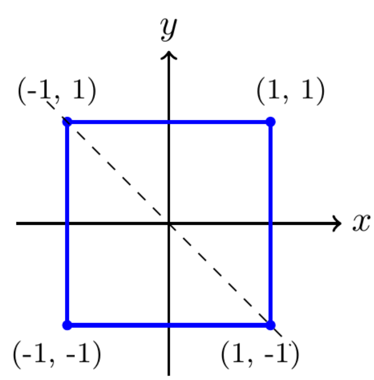
Solution

$(-1,\ -1) \ \rightarrow\ (1,\ -1)$

$$\begin{cases} x = -1 + (1+1)t \\ y = -1 + (-1+1)t \end{cases}$$

$$\vec{r}_1(t) = \langle -1+2t,\ -1\rangle$$

$$\vec{r}_1{}'(t) = \langle 2,\ 0\rangle$$



$(1,\ -1) \ \rightarrow\ (1,\ 1)$

$$\begin{cases} x = 1 + (1-1)t \\ y = -1 + (1+1)t \end{cases}$$

$$\vec{r}_2(t) = \langle 1,\ -1+2t\rangle$$

$$\vec{r}_2{}'(t) = \langle 0,\ 2\rangle$$

$(1,\ 1) \ \rightarrow\ (-1,\ 1)$

$$\vec{r}_3(t) = \langle 1-2t,\ 1\rangle$$

$$\vec{r}_3{}'(t) = \langle -2,\ 0\rangle$$

$(-1,\ 1) \ \rightarrow\ (-1,\ -1)$

$$\vec{r}_4(t) = \langle -1,\ 1-2t\rangle$$

$$\vec{r}_4{}'(t) = \langle 0,\ -2\rangle$$

$$\vec{F}_1 = \langle -1+2t,\ 1\rangle$$

$$\vec{F}_1 \cdot \vec{r}_1{}'(t) = \langle -1+2t,\ 1\rangle \cdot \langle 2,\ 0\rangle$$

$$= 4t-2$$

$$\vec{F}_2 = \langle 1,\ 1-2t\rangle$$

$$\vec{F}_2 \cdot \vec{r}_2{}'(t) = \langle 1,\ 1-2t\rangle \cdot \langle 0,\ 2\rangle$$

$$= 2-4t$$

$$\vec{F}_3 = \langle 1-2t,\ -1\rangle$$

$$\vec{F}_3 \cdot \vec{r}_3{}'(t) = \langle 1-2t,\ -1\rangle \cdot \langle -2,\ 0\rangle$$

$$= 4t-2$$

$$\vec{F}_4 = \langle -1,\ -1+2t\rangle$$

$$\vec{F}_4 \cdot \vec{r}_4{}'(t) = \langle -1,\ -1+2t\rangle \cdot \langle 0,\ -2\rangle$$

$$= 2-4t$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (4t-2+2-4t+4t-2+2-4t)\,dt$$

$$= 0$$

Or

$$\vec{F} = \nabla(xyz) = \nabla\varphi$$

$$= \nabla\left(\frac{1}{2}\left(x^2+y^2\right)\right)$$

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

Since the integral around any closed curve is 0.

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \langle y,\ z,\ -x\rangle;\quad C: \vec{r}(t) = \langle \cos t,\ \sin t,\ 4\rangle,\quad \mathit{for}\ \ 0 \le t \le 2\pi$

Solution

$$\vec{F} = \langle \sin t,\ 4,\ -\cos t\rangle$$

$$\vec{r}'(t) = \langle -\sin t,\ \cos t,\ 0\rangle$$

$$\vec{F} \cdot \vec{r}'(t) = \langle \sin t,\ 4,\ -\cos t\rangle \cdot \langle -\sin t,\ \cos t,\ 0\rangle$$

$$= -\sin^2 t + 4\cos t$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(4\cos t - \sin^2 t\right) dt$$

$$= \int_0^{2\pi} \left(4\cos t - \frac{1}{2} - \frac{1}{2}\cos 2t\right) dt$$

$$= 4\sin t - \frac{1}{2}t - \frac{1}{2}\cos 2t \Big|_0^{2\pi}$$

$$= -\pi$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ for the vector fields $\vec{F}$ and curves C

$\vec{F} = \left\langle y^2,\ x \right\rangle$; where C is the arc of the parabola $x = 4 - y^2$ from $(-5,\ -3)$ to $(0,\ 2)$

Solution

Let $y = t \quad \to \quad -3 \le t \le 2$

$\vec{r}(t) = \left\langle 4 - t^2,\ t \right\rangle$

$\vec{r}'(t) = \langle -2t,\ 1 \rangle$

$\vec{F} = \left\langle t^2,\ 4 - t^2 \right\rangle$

$$\vec{F} \cdot \vec{r}' = \left\langle t^2,\ 4 - t^2 \right\rangle \cdot \langle -2t,\ 1 \rangle$$
$$= -2t^3 + 4 - t^2$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{-3}^{2} \left(-2t^3 + 4 - t^2\right) dt$$

$$= -\frac{1}{2}t^4 + 4t - \frac{1}{3}t^3 \Big|_{-3}^{2}$$

$$= -8 + 8 - \frac{8}{3} + \frac{81}{2} + 12 - 9$$

$$= \frac{-16 + 243 + 18}{6}$$

$$= \frac{245}{6}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \bullet d\vec{r}$ for the vector fields $\vec{F}$ and curves C

$\vec{F} = \left\langle x^2 + y^2,\ 4x + y^2 \right\rangle$; where C is the straight line segment from $(6,\ 3)$ to $(6,\ 0)$

Solution

$$\vec{r}(t) = \langle 6,\ 3 - 3t \rangle$$

$$\vec{r}'(t) = \langle 0,\ -3 \rangle$$

$$\vec{F} = \left\langle 36 + 9 - 18t + 9t^2,\ 24 + 9 - 18t + 9t^2 \right\rangle$$

$$= \left\langle 45 - 18t + 9t^2,\ 33 - 18t + 9t^2 \right\rangle$$

$$\vec{F} \bullet \vec{r}' = \left\langle 45 - 18t + 9t^2,\ 33 - 18t + 9t^2 \right\rangle \bullet \langle 0,\ -3 \rangle$$

$$= -99 + 54t - 27t^2$$

$$\int_C \vec{F} \bullet d\vec{r} = \int_0^1 \left(-99 + 54t - 27t^2 \right) dt$$

$$= -99t + 27t^2 - 9t^3 \Big|_0^1$$

$$= -99 + 27 - 9$$

$$\underline{= -81}$$

OR

$(6,\ 3)$ to $(6,\ 0)$ is just a straight parallel to the x-axis.

$x = 6 \quad \&\ dx = 0$

$$\int_C \vec{F} \bullet d\vec{r} = \oint_C \left(x^2 + y^2 \right) dx + \left(4x + y^2 \right) dy$$

$$= \oint_C 0 + \left(24 + y^2 \right) dy$$

$$= \int_3^0 \left(24 + y^2 \right) dy$$

$$= 24y + \frac{1}{3} y^3 \Big|_3^0$$

$$= -72 - 9$$

$$\underline{= -81}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot \vec{T}\, ds$ for the vector fields $\vec{F}$ and curves C.

$$\vec{F} = \langle x,\ y \rangle \text{ on the parabola } \vec{r}(t) = \langle 4t,\ t^2 \rangle \quad 0 \le t \le 1$$

Solution

$$\vec{F} = \langle 4t,\ t^2 \rangle$$

$$\vec{r}' = \langle 4,\ 2t \rangle$$

$$\vec{F} \cdot \vec{r}' = \langle 4t,\ t^2 \rangle \cdot \langle 4,\ 2t \rangle$$

$$= 16t + 2t^3$$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_0^1 \left(16t + 2t^3\right) dt$$

$$= 8t^2 + \frac{1}{2}t^4 \Big|_0^1$$

$$= 8 + \frac{1}{2}$$

$$= \frac{17}{2}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot \vec{T}\, ds$ for the vector fields $\vec{F}$ and curves C.

$$\vec{F} = \langle -y,\ x \rangle \text{ on the semicircle } \vec{r}(t) = \langle 4\cos t,\ 4\sin t \rangle \quad 0 \le t \le \pi$$

Solution

$$\vec{F} = \langle -4\sin t,\ 4\cos t \rangle$$

$$\vec{r}' = \langle -4\sin t,\ 4\cos t \rangle$$

$$\vec{F} \cdot \vec{r}' = \langle -4\sin t,\ 4\cos t \rangle \cdot \langle -4\sin t,\ 4\cos t \rangle$$

$$= 16\sin^2 t + 16\cos^2 t$$

$$= 16$$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_0^{\pi} 16\, dt$$

$$= 16t \Big|_0^{\pi}$$

$$= 16\pi$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot \vec{T}\, ds$ for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \langle y,\ x \rangle$ on the line segment from $(1,\ 1)$ to $(5,\ 10)$

Solution

$$\vec{r}(t) = \langle (5-1)t+1,\ (10-1)t+1 \rangle$$
$$= \langle 4t+1,\ 9t+1 \rangle$$

$$\vec{F} = \langle 9t+1,\ 4t+1 \rangle$$

$$\vec{r}' = \langle 4,\ 9 \rangle$$

$$\vec{F} \cdot \vec{r}' = \langle 9t+1,\ 4t+1 \rangle \cdot \langle 4,\ 9 \rangle$$
$$= 36t + 4 + 36t + 9$$
$$= 72t + 13$$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_0^1 (72t+13)\, dt$$
$$= 36t^2 + 13t \Big|_0^1$$
$$= 36 + 13$$
$$\underline{= 49}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot \vec{T}\, ds$ for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \langle -y,\ x \rangle$ on the parabola $y = x^2$ from $(0,\ 0)$ to $(1,\ 1)$

Solution

$$\vec{r}(t) = \left\langle t,\ t^2 \right\rangle \qquad \langle x = t,\ y \rangle$$

$$\vec{F} = \left\langle -t^2,\ t \right\rangle$$

$$\vec{r}' = \langle 1,\ 2t \rangle$$

$$\vec{F} \cdot \vec{r}' = \left\langle -t^2,\ t \right\rangle \cdot \langle 1,\ 2t \rangle$$
$$= t^2$$

$$\int_C \vec{F} \bullet \vec{T}\, ds = \int_0^1 t^2 dt$$

$$= \frac{1}{3} t^3 \Big|_0^1$$

$$\underline{= \frac{1}{3}}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \bullet \vec{T}\, ds$ for the vector fields $\vec{F}$ and curves C.

$$\vec{F} = \frac{\langle x, y\rangle}{\left(x^2 + y^2\right)^{3/2}} \text{ on the curve } \vec{r}(t) = \left\langle t^2,\ 3t^2 \right\rangle \quad 1 \le t \le 2$$

Solution

$$\vec{F} = \frac{\left\langle t^2,\ 3t^2 \right\rangle}{\left(t^4 + 9t^4\right)^{3/2}}$$

$$= \frac{\left\langle t^2,\ 3t^2 \right\rangle}{\left(10t^4\right)^{3/2}}$$

$$= \frac{1}{10\sqrt{10}} \frac{\left\langle t^2,\ 3t^2 \right\rangle}{t^6}$$

$$= \frac{1}{10\sqrt{10}} \left\langle \frac{1}{t^4},\ \frac{3}{t^4} \right\rangle$$

$$\vec{r}\,' = \langle 2t,\ 6t\rangle$$

$$\vec{F} \bullet \vec{r}\,' = \frac{1}{10\sqrt{10}} \left\langle \frac{1}{t^4},\ \frac{3}{t^4} \right\rangle \bullet \langle 2t,\ 6t\rangle$$

$$= \frac{1}{10\sqrt{10}} \left(\frac{2}{t^3} + \frac{18}{t^3} \right)$$

$$= \frac{2}{\sqrt{10}} \frac{1}{t^3}$$

$$\int_C \vec{F} \bullet \vec{T}\, ds = \frac{2}{\sqrt{10}} \int_1^2 t^{-3} dt$$

$$= -\frac{1}{\sqrt{10}}\, t^{-2} \Big|_1^2$$

$$= -\frac{\sqrt{10}}{10}\left(\frac{1}{4} - 1\right)$$

$$= \frac{3\sqrt{10}}{40}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \bullet \vec{T}\, ds$ for the vector fields $\vec{F}$ and curves C.

$$\vec{F} = \frac{\langle x,\ y\rangle}{x^2 + y^2} \text{ on the line } \vec{r}(t) = \langle t,\ 4t\rangle \quad 1 \le t \le 10$$

Solution

$$\vec{F} = \frac{\langle t,\ 4t\rangle}{t^2 + 16t^2}$$

$$= \frac{1}{17}\left\langle \frac{1}{t},\ \frac{4}{t}\right\rangle$$

$$\vec{r}\,' = \langle 1,\ 4\rangle$$

$$\vec{F} \bullet \vec{r}\,' = \frac{1}{17}\left\langle \frac{1}{t},\ \frac{4}{t}\right\rangle \bullet \langle 1,\ 4\rangle$$

$$= \frac{1}{17}\left(\frac{1}{t} + \frac{16}{t}\right)$$

$$= \frac{1}{t}$$

$$\int_C \vec{F} \bullet \vec{T}\, ds = \int_1^{10} \frac{1}{t}\, dt$$

$$= \ln t \Big|_1^{10}$$

$$= \ln 10$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \langle y,\ -x \rangle$ on the path consisting of the line segment from $(1,\ 2)$ to $(0,\ 0)$ followed by the line segment from $(0,\ 0)$ to $(0,\ 4)$

Solution

$(1,\ 2)$ to $(0,\ 0)$

$$\vec{r}_1(t) = \langle 1-t,\ 2-2t \rangle$$

$$\vec{r}_1{}'(t) = \langle -1,\ -2 \rangle$$

$$\vec{F} = \langle 2-2t,\ t-1 \rangle$$

$$\begin{aligned} \vec{F} \cdot \vec{r}_1{}'(t) &= \langle 2-2t,\ t-1 \rangle \cdot \langle -1,\ -2 \rangle \\ &= -2+2t-2t+2 \\ &= 0 \end{aligned}$$

$(0,\ 0)$ to $(0,\ 4)$

$$\vec{r}_2(t) = \langle 0,\ 4t \rangle$$

$$\vec{r}_2{}'(t) = \langle 0,\ 4 \rangle$$

$$\vec{F} = \langle 4t,\ 0 \rangle$$

$$\begin{aligned} \vec{F} \cdot \vec{r}_2{}'(t) &= \langle 4t,\ 0 \rangle \cdot \langle 0,\ 4 \rangle \\ &= 0 \end{aligned}$$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \vec{F} \cdot \vec{r}_1{}'\ dt + \int_0^1 \vec{F} \cdot \vec{r}_2{}'\ dt \\ &= 0 \end{aligned}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \langle x,\ y \rangle$ on the path consisting of the line segment from $(-1,\ 0)$ to $(0,\ 8)$ followed by the line segment from $(0,\ 8)$ to $(2,\ 8)$

Solution

$(-1,\ 0)$ to $(0,\ 8)$

$$\vec{r}_1(t) = \langle t-1,\ 8t \rangle$$

$$\vec{r}_1{}'(t) = \langle 1,\ 8 \rangle$$

$$\vec{F} = \langle t-1,\ 8t\rangle$$

$$\vec{F} \cdot \vec{r}_1{}'(t) = \langle t-1,\ 8t\rangle \cdot \langle 1,\ 8\rangle$$
$$= t - 1 + 64t$$
$$\underline{= 65t - 1}$$

$(0,\ 8)$ to $(2,\ 8)$

$$\vec{r}_2(t) = \langle 2t,\ 8\rangle$$

$$\vec{r}_2{}'(t) = \langle 2,\ 0\rangle$$

$$\vec{F} = \langle 2t,\ 8\rangle \text{o}$$

$$\vec{F} \cdot \vec{r}_2{}'(t) = \langle 2t,\ 8\rangle \cdot \langle 2,\ 0\rangle$$
$$\underline{= 4t}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F} \cdot \vec{r}_1{}'\ dt + \int_0^1 \vec{F} \cdot \vec{r}_2{}'\ dt$$

$$= \int_0^1 (65t - 1 + 4t)\,dt$$

$$= \frac{69}{2}t^2 - t \Big|_0^1$$

$$= \frac{69}{2} - 1$$

$$\underline{= \frac{67}{2}}$$

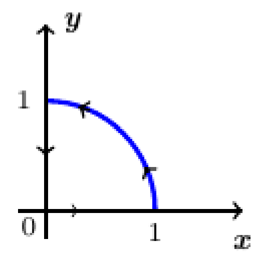
Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \left\langle x^2,\ -xy\right\rangle$ on runs from $(1,\ 0)$ to $(0,\ 1)$ along the unit circle and then from $(0,\ 1)$ to $(0,\ 0)$ along the y-axis.

Solution

Along the unit circle: $\left(0 \le t \le \frac{\pi}{2}\right)$



$$\vec{r}_1(t) = \langle \cos t,\ \sin t\rangle$$

$$\vec{r}_1{}'(t) = \langle -\sin t,\ \cos t\rangle$$

$$\vec{F}_1 = \left\langle \cos^2 t,\ -\cos t \sin t\right\rangle$$

$$\overrightarrow{F_1} \bullet \vec{r}_1{}'(t) = \left\langle \cos^2 t,\ -\cos t \sin t \right\rangle \bullet \left\langle -\sin t,\ \cos t \right\rangle$$

$$= -\sin t \cos^2 t - \sin t \cos^2 t$$

$$\underline{= -2 \sin t \cos^2 t}$$

$(0,\ 1)$ to $(0,\ 0)$: $(0 \le t \le 1)$

$$\vec{r}_2(t) = \langle 0,\ t \rangle$$

$$\vec{r}_2{}'(t) = \langle 0,\ 1 \rangle$$

$$\vec{F}_1 = \langle 0,\ 0 \rangle$$

$$\overrightarrow{F_1} \bullet \vec{r}_1{}'(t) \ \underline{= 0}$$

$$W = \int_0^{\frac{\pi}{2}} \left(-2 \sin t \cos^2 t \right) dt \ + \ 0$$

$$W = \int_{C_1} \vec{F} \bullet d\vec{r} + \int_{C_2} \vec{F} \bullet d\vec{r}$$

$$= 2 \int_0^{\frac{\pi}{2}} \cos^2 t \ d(\cos t)$$

$$= \frac{2}{3} \cos^3 t \Big|_0^{\frac{\pi}{2}}$$

$$\underline{= -\frac{2}{3}}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \langle y,\ x \rangle$ on the parabola $y = 2x^2$ from $(0,\ 0)$ to $(2,\ 8)$

Solution

$$\vec{r}(t) = \left\langle x,\ 2x^2 \right\rangle$$

$$= \left\langle 2t,\ 8t^2 \right\rangle \qquad 0 \le t \le 1$$

$$\vec{F} = \left\langle 8t^2,\ 2t \right\rangle$$

$$\vec{r}' = \langle 2,\ 16t \rangle$$

$$\vec{F} \bullet \vec{r}'(t) = \left\langle 8t^2,\ 2t \right\rangle \bullet \langle 2,\ 16t \rangle$$

$$= 16t^2 + 32t^2$$

$$= 48t^2$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 48t^2\, dt$$

$$= 16\, t^3 \Big|_0^1$$

$$\underline{= 16\,|}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \langle y,\ -x \rangle$ on the line $y = 10 - 2x$ from $(1,\ 8)$ to $(3,\ 4)$

Solution

$$\vec{r}(t) = \langle 2t + 1,\ -4t + 8 \rangle$$

$$\vec{F} = \langle 8 - 4t,\ -2t - 1 \rangle$$

$$\vec{r}' = \langle 2,\ -4 \rangle$$

$$\vec{F} \bullet \vec{r}'(t) = \langle 8 - 4t,\ -2t - 1 \rangle \bullet \langle 2,\ -4 \rangle$$

$$= 16 - 8t + 8t + 4$$

$$= 20$$

$$\int_C \vec{F} \bullet d\vec{r} = \int_0^1 20\, dt$$

$$\underline{= 20\,|}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \langle x,\ y,\ z \rangle$ on the tilted ellipse $\vec{r}(t) = \langle 4\cos t,\ 4\sin t,\ 4\cos t \rangle \quad 0 \le t \le 2\pi$

Solution

$$\vec{F} = \langle 4\cos t,\ 4\sin t,\ 4\cos t \rangle$$

$$\vec{r}' = \langle -4\sin t,\ 4\cos t,\ -4\sin t \rangle$$

$$\vec{F} \bullet \vec{r}' = \langle 4\cos t,\ 4\sin t,\ 4\cos t \rangle \bullet \langle -4\sin t,\ 4\cos t,\ -4\sin t \rangle$$

$$= -16\cos t \sin t + 16 \sin t \cos t - 16 \cos t \sin t$$

$$= -16\cos t \sin t$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-16\cos t \sin t)\ dt$$

$$= \int_0^{2\pi} 16\sin t\ d(\cos t)$$

$$= 8\sin^2 t \Big|_0^{2\pi}$$

$$\underline{= 0}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \langle -y,\ x,\ z \rangle$ on the helix $\vec{r}(t) = \left\langle 2\cos t,\ 2\sin t,\ \frac{t}{2\pi} \right\rangle \quad 0 \le t \le 2\pi$

Solution

$$\vec{F} = \left\langle -2\sin t,\ 2\cos t,\ \frac{t}{2\pi} \right\rangle$$

$$\vec{r}' = \left\langle -2\sin t,\ 2\cos t,\ \frac{1}{2\pi} \right\rangle$$

$$\vec{F} \cdot \vec{r}' = \left\langle -2\sin t,\ 2\cos t,\ \frac{t}{2\pi} \right\rangle \cdot \left\langle -2\sin t,\ 2\cos t,\ \frac{1}{2\pi} \right\rangle$$

$$= 4\sin^2 t + 4\cos^2 t + \frac{t}{4\pi^2}$$

$$= 4 + \frac{1}{4\pi^2} t$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(4 + \frac{1}{4\pi^2} t \right) dt$$

$$= 4t + \frac{1}{8\pi^2} t^2 \Big|_0^{2\pi}$$

$$\underline{= 8\pi + \frac{1}{2}}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ on the line segment from $(1,\ 1,\ 1)$ to $(10,\ 10,\ 10)$

Solution

$$\vec{r}(t) = \langle t+1,\ t+1,\ t+1 \rangle \quad 0 \le t \le 9$$

$$\vec{r}' = \langle 1,\ 1,\ 1 \rangle$$

$$\vec{F} = \frac{\langle t+1,\ t+1,\ t+1 \rangle}{\left(3(t+1)^2\right)^{3/2}}$$

$$= \frac{1}{3\sqrt{3}} \frac{\langle t+1,\ t+1,\ t+1 \rangle}{(t+1)^3}$$

$$= \frac{1}{3\sqrt{3}} \left\langle \frac{1}{(t+1)^2},\ \frac{1}{(t+1)^2},\ \frac{1}{(t+1)^2} \right\rangle$$

$$\vec{F} \cdot \vec{r}' = \frac{1}{3\sqrt{3}} \left\langle \frac{1}{(t+1)^2},\ \frac{1}{(t+1)^2},\ \frac{1}{(t+1)^2} \right\rangle \cdot \langle 1,\ 1,\ 1 \rangle$$

$$= \frac{1}{\sqrt{3}} \frac{1}{(t+1)^2}$$

$$\int_C \vec{F} \cdot d\vec{r} = \frac{1}{\sqrt{3}} \int_0^9 \frac{1}{(t+1)^2}\, dt$$

$$= \frac{1}{\sqrt{3}} \int_0^9 \frac{1}{(t+1)^2}\, d(t+1)$$

$$= -\frac{1}{\sqrt{3}} \left. \frac{1}{t+1} \right|_0^9$$

$$= -\frac{\sqrt{3}}{3}\left(\frac{1}{10} - 1\right)$$

$$= \frac{3\sqrt{3}}{10}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ on the path $\vec{r}(t) = \left\langle t^2,\ 3t^2,\ -t^2 \right\rangle, \quad 1 \le t \le 2$

Solution

$$\vec{r}' = \langle 2t,\ 6t,\ -2t \rangle$$

$$\vec{F} = \frac{\langle t^2,\ 3t^2,\ -t^2\rangle}{\left(t^4+9t^4+t^4\right)^{3/2}}$$

$$= \frac{1}{11\sqrt{11}}\ \frac{\langle t^2,\ 3t^2,\ -t^2\rangle}{t^6}$$

$$= \frac{1}{11\sqrt{11}}\left\langle \frac{1}{t^4},\ \frac{3}{t^4},\ -\frac{1}{t^4}\right\rangle$$

$$\vec{F}\bullet\vec{r}\,' = \frac{1}{11\sqrt{11}}\left\langle \frac{1}{t^4},\ \frac{3}{t^4},\ -\frac{1}{t^4}\right\rangle \bullet \langle 2t,\ 6t,\ -2t\rangle$$

$$= \frac{1}{11\sqrt{11}}\left(\frac{2}{t^3}+\frac{18}{t^3}+\frac{2}{t^3}\right)$$

$$= \frac{2\sqrt{11}}{11}\ \frac{1}{t^3}$$

$$W = \frac{2\sqrt{11}}{11}\int_1^2 t^{-3}\ dt \qquad\qquad W = \int_C \vec{F}\bullet d\vec{r}$$

$$= -\frac{\sqrt{11}}{11}\ t^{-2}\Big|_1^2$$

$$= -\frac{\sqrt{11}}{11}\left(\frac{1}{4}-1\right)$$

$$\boxed{= \frac{3\sqrt{11}}{44}}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \dfrac{\langle x,\ y\rangle}{\left(x^2+y^2\right)^{3/2}}$ over the plane curve $\vec{r}(t) = \left\langle e^t\cos t,\ e^t\sin t\right\rangle$ from the point $(1,\ 0)$ to the point $\left(e^{2\pi},\ 0\right)$ by using the parametrization of the curve to evaluate the work integral

Solution

$$x = e^t\cos t \quad y = e^t\sin t$$

$$(1,\ 0) \ \Rightarrow\ \begin{cases} 1 = e^t\cos t \\ 0 = e^t\sin t \quad \to t = 0 \end{cases}$$

$$\left(e^{2\pi},\ 0\right) \Rightarrow \begin{cases} e^{2\pi} = e^t \cos t & \to\ t = 2\pi \\ 0 = e^t \sin t \end{cases}$$

$\underline{0 \le t \le 2\pi|}$

$$\vec{r}' = \left\langle e^t\left(\cos t - \sin t\right),\ e^t\left(\cos t + \sin t\right)\right\rangle$$

$$\vec{F} = \frac{\left\langle e^t \cos t,\ e^t \sin t\right\rangle}{\left(e^{2t}\cos^2 t + e^{2t}\sin^2 t\right)^{3/2}}$$

$$= \frac{\left\langle e^t \cos t,\ e^t \sin t\right\rangle}{e^{3t}}$$

$$= \left\langle \frac{\cos t}{e^{2t}},\ \frac{\sin t}{e^{2t}}\right\rangle$$

$$\vec{F} \bullet \vec{r}' = \left\langle \frac{\cos t}{e^{2t}},\ \frac{\sin t}{e^{2t}}\right\rangle \bullet \left\langle e^t\left(\cos t - \sin t\right),\ e^t\left(\cos t + \sin t\right)\right\rangle$$

$$= e^{-t}\left(\cos^2 t - \cos t \sin t + \sin^2 t + \cos t \sin t\right)$$

$$\underline{= e^{-t}\ |}$$

$$W = \int_0^{2\pi} e^{-t}\ dt \qquad\qquad W = \int_C \vec{F} \bullet d\vec{r}$$

$$= -e^{-t}\Big|_0^{2\pi}$$

$$\underline{= 1 - e^{-2\pi}\ |}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \dfrac{\langle x,\ y,\ z\rangle}{x^2 + y^2 + z^2}$ on the line segment from $(1,\ 1,\ 1)$ to $(8,\ 4,\ 2)$

Solution

$$\vec{r}(t) = \langle 7t + 1,\ 3t + 1,\ t + 1\rangle \quad 0 \le t \le 1$$

$$\vec{r}' = \langle 7,\ 3,\ 1\rangle$$

$$\vec{F} = \frac{\langle 7t + 1,\ 3t + 1,\ t + 1\rangle}{(7t + 1)^2 + (3t + 1)^2 + (t + 1)^2}$$

$$= \frac{\langle 7t+1,\ 3t+1,\ t+1\rangle}{49t^2+14t+1+9t^2+6t+1+t^2+2t+1}$$

$$= \frac{\langle 7t+1,\ 3t+1,\ t+1\rangle}{59t^2+22t+3}$$

$$\vec{F}\cdot\vec{r}\,' = \frac{\langle 7t+1,\ 3t+1,\ t+1\rangle}{59t^2+22t+3}\cdot\langle 7,\ 3,\ 1\rangle$$

$$= \frac{49t+7+9t+3+t+1}{59t^2+22t+3}$$

$$= \frac{59t+11}{59t^2+22t+3}$$

$$W = \int_0^1 \frac{59t+11}{59t^2+22t+3}\,dt \qquad\qquad W = \int_C \vec{F}\cdot d\vec{r}$$

$$= \frac{1}{2}\int_0^1 \frac{1}{59t^2+22t+3}\,d\left(59t^2+22t+3\right)$$

$$= \frac{1}{2}\ln\left(59t^2+22t+3\right)\Big|_0^1$$

$$= \frac{1}{2}\left(\ln 84 - \ln 3\right)$$

$$= \frac{1}{2}\ln\frac{84}{3}$$

$$= \frac{1}{2}\ln 28$$

$$= \ln\left(2\sqrt{7}\right)$$

Exercise

Let C be the circle of radius 2 centered at the origin with counterclockwise orientation

a) Give the unit outward vector at any point $(x,\ y)$ on C.

b) Find the normal component of the vector field $\vec{F} = 2\langle y,\ -x\rangle$ at any point on C.

c) Find the normal component of the vector field $\vec{F} = \dfrac{\langle x,\ y\rangle}{x^2+y^2}$ at any point on C.

Solution

$r = 2$ @ origin, *ccw*.

a) $\langle x,\ y\rangle$ outward normal

$$\left|\langle x,\ y\rangle\right| = \sqrt{x^2+y^2}$$

$= r$

$\underline{= 2}$

$\therefore$ unit outward normal: $\frac{1}{2}\langle x,\ y\rangle$

b) Normal component is:

$$\vec{F} \bullet \vec{n} = 2\langle y,\ -x\rangle \bullet \frac{1}{2}\langle x,\ y\rangle$$

$$= xy - xy$$

$$\underline{= 0}$$

c) Normal component is:

$$\vec{F} \bullet \vec{n} = \frac{\langle x,\ y\rangle}{x^2 + y^2} \bullet \frac{1}{2}\langle x,\ y\rangle$$

$$= \frac{1}{2}\frac{x^2 + y^2}{x^2 + y^2}$$

$$\underline{= \frac{1}{2}}$$

Exercise

Find the flow of the field $\vec{F} = \nabla\left(x^2 z e^y\right)$

a) Once around the ellipse C in which the plane $x + y + z = 1$ intersects the cylinder $x^2 + z^2 = 25$, clockwise as viewed from the positive y-axis.

b) Along the curved boundary of the helicoid $\vec{r}(r,\ \theta) = (r\cos\theta)\hat{i} + (r\sin\theta)\hat{j} + \theta\,\hat{k}$ from $(1,\ 0,\ 0)$ to $(1,\ 0,\ 2\pi)$

Solution

a) For any closed path C.

$$\int_C \vec{F} \bullet d\vec{r}\ \underline{= 0}$$

$\vec{F}$ is conservative.

b) $$\int_C \vec{F} \bullet d\vec{r} = \int_{(1,\ 0,\ 0)}^{(1,\ 0,\ 2\pi)} \nabla\left(x^2 z e^y\right) dr$$

$$= \varphi(1,\ 0,\ 2\pi) - \varphi(1,\ 0,\ 0)$$

$$= x^2 z e^y \Big|_{(1,\ 0,\ 2\pi)} - x^2 z e^y \Big|_{(1,\ 0,\ 0)}$$

$$= 2\pi - 0$$

$$\underline{= 2\pi}$$