Section 3.4 - Concavity and the Second Derivative Test

Example

Find the second derivative of $f(x)=4x(\ln x)$

<u>*Solution*</u>

$$f'(x)=4\ln x+4x\frac{1}{x}$$
$$=4\ln x+4$$
$$f''(x)=\frac{4}{x}$$

Velocity and acceleration

Example

Suppose a car is moving in a straight line, with its position from a starting point (in feet) at time t (in seconds) given by

$$s(t)=t^3-2t^2-7t+9$$

a) Find the velocity at any time t

$$v(t)=s'(t)=3t^2-4t-7$$

b) Find the acceleration at any time t

$$a(t)=v'(t)=6t-4$$

c) Find the time intervals $(t\geq 0)$ when the car is going forward or backing up

$$v(t)=3t^2-4t-7=0$$
$$t=\frac{7}{3} \quad t=-1$$

The car is backing up first $\left(0,\ \frac{7}{3}\right)$ and forward $\left(\frac{7}{3},\ \infty\right)$

d) Find the time intervals $(t\geq 0)$ when the car is speeding up or slowing down

$$a(t)=6t-4=0 \quad \Rightarrow \ t=\frac{2}{3}$$

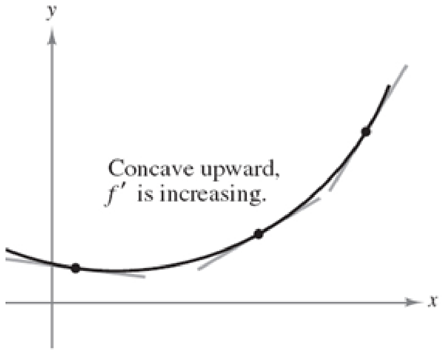
0	$\frac{2}{3}$	$\frac{7}{3}$	
	$v(0.5)<0$ $-$	$v(1)<0$ $-$	$v(3)>0$ $+$
	$a(0.5)<0$ $-$	$a(1)>0$ $+$	$a(3)>0$ $+$
	$+$	$-$	$-$

Concavity

Definition

Let f be differentiable on an open interval I . The graph of f is

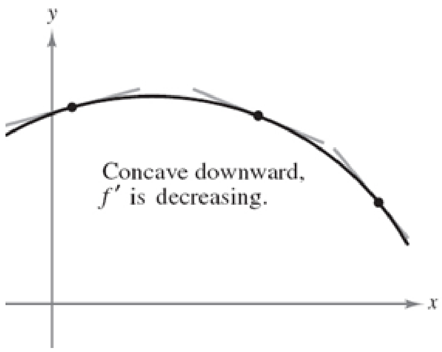
1. **Concave *upward*** on I if f' is increasing on the interval.
2. **Concave *downward*** on I if f' is decreasing on the interval.



Test for Concavity

Let f be function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then f is ***concave upward*** on I.
2. If $f''(x) < 0$ for all x in I, then f is ***concave downward*** on I.



1) Locate the x values @ which $f''(x) = 0$ or undefined
2) Use these test x-value to determine the test intervals
3) Test the sign of $f''(x)$ in each interval

Find the second derivative of $f(x) = -2x^2$ and discuss the concavity of the graph

$f'(x) = -4x$

$\Rightarrow f''(x) = -4 < 0$ for all x

f is concave downward for all x.

Example

Determine the intervals on which the graph of the function is concave upward or concave downward.

$$f(x) = x^4 - 8x^3 + 18x^2$$

Solution

$$f'(x) = 4x^3 - 24x^2 + 36x$$

$$f''(x) = 12x^2 - 48x + 36$$

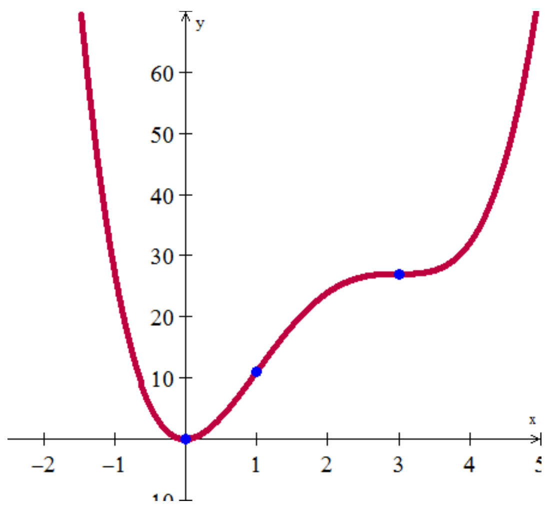
Solve for x:

$$x = 1 \quad x = 3$$

$-\infty$ 1	3	∞
$f''(0) > 0$	$f''(0) < 0$	$f''(2) > 0$
upward	***downward***	***upward***

f is concave upward on $(-\infty, 1)$ *and* $(3, \infty)$

f is concave downward on $(1, 3)$



Second-Derivative Test

Let $f'(c)=0$ and let f'' exist ($\exists$)

1. If $f''(c)>0 \Rightarrow f(c)$ is a relative Minimum
2. If $f''(c)<0 \Rightarrow f(c)$ is a relative Maximum
3. If $f''(c)=0 \Rightarrow$ Test fails $\rightarrow$ use f' to determine Max, Min.

Example

Find all relative extrema for $f(x)=4x^3+7x^2-10x+8$

<u>*Solution*</u>

$$f'(x)=12x^2+14x-10=0$$

$$x=-\frac{5}{3} \qquad x=\frac{1}{2}$$

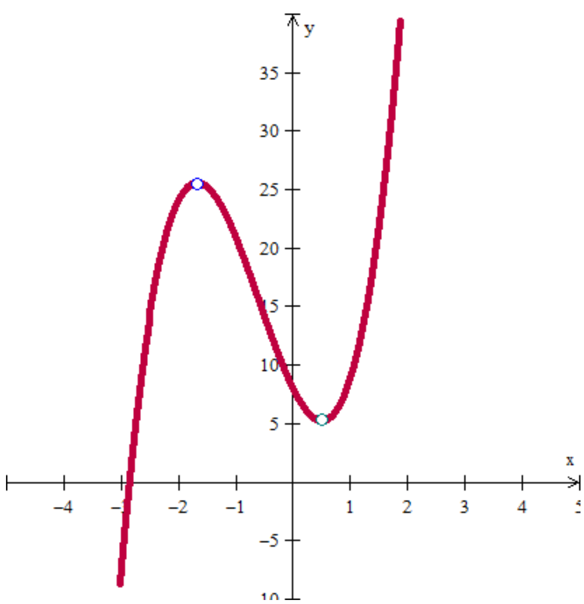
$$f''(x)=24x+14$$

$f''\left(-\frac{5}{3}\right)=24\left(-\frac{5}{3}\right)+14=-26<0$ Leads to relative maximum

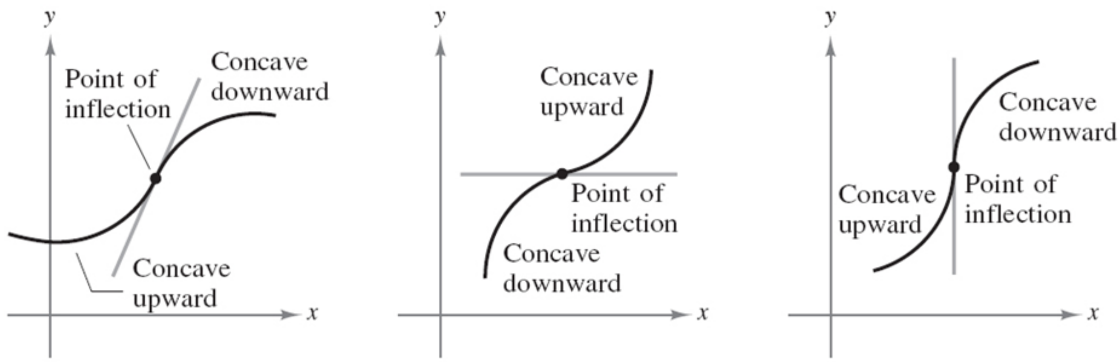
$f''\left(\frac{1}{2}\right)=24\left(\frac{1}{2}\right)+14=26>0$ Leads to relative minimum

RMAX: $\left(-\frac{5}{3},\ \frac{691}{27}\right)$

RMIN: $\left(\frac{1}{2},\ \frac{21}{4}\right)$



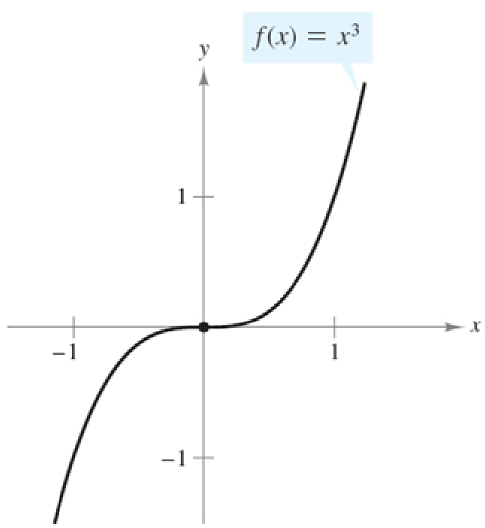
Point of Inflection



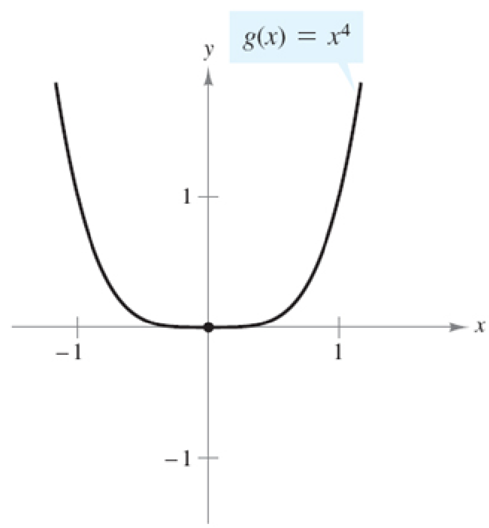
Definition

If the graph of a continuous function has a tangent line @ a point where its concavity changes from upward to downward (or down to upward) then the point is a point of inflection.

If $(c, f(c))$ is a point of inflection of a graph of $f \Rightarrow$ either $f''(c) = 0$ or undefined.



$f''(0) = 0$, and $(0, 0)$ is a point of inflection.



$g''(0) = 0$, but $(0, 0)$ is not a point of inflection.

Extended Applications: Diminishing Returns

$$\underset{input}{x} \rightarrow \underset{output}{y} \qquad \underset{output}{y} = f(\underset{input}{x})$$

Example

Find the point of diminishing returns for the model below, where R is the revenue (in thousands of dollars) and x is the advertising cost (in thousands of dollars).

$$R = \frac{1}{20,000}\left(450x^2 - x^3\right) \qquad 0 \le x \le 300$$

<u>*Solution*</u>

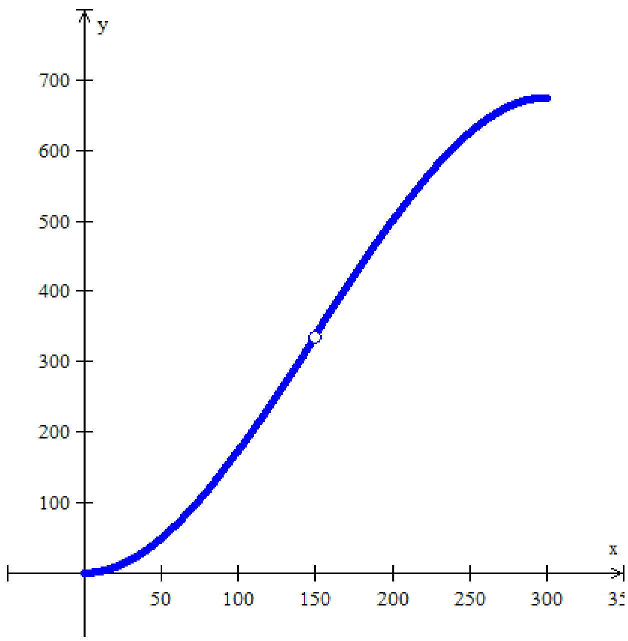
$$R' = \frac{1}{20,000}\left(900x - 3x^2\right)$$

$$R'' = \frac{1}{20,000}(900 - 6x) = 0$$

$$\Rightarrow\ x = \frac{900}{6} = 150$$

x = 150 (or \$150,000) is a ***diminishing point***

An increased investment beyond this point is usually considered a poor use of capital



Exercises *Section* 3.4 - Concavity and the Second Derivative Test

Determine the intervals on which the graph of the function is concave upward or concave downward.

1. $f(x)=\dfrac{x^2-1}{2x+1}$

2. $f(x)=-4x^3-8x^2+32$

3. $f(x)=\dfrac{12}{x^2+4}$

4. Find the largest open interval where the function is concave upward $f(x)=4x-2e^{-x}$

5. Find the points of inflection. $f(x)=x^3-9x^2+24x-18$

6. Find the second derivative of $f(x)=-2\sqrt{x}$ and discuss the concavity of the graph

7. Find the extrema using the second derivative test $f(x)=\dfrac{4}{x^2+1}$

8. Discuss the concavity of the graph of f and find its points of inflection. $f(x)=x^4-2x^3+1$

9. Find all relative extrema of $f(x)=x^4-4x^3+1$

10. The revenue R generated from sales of a certain product is related to the amount x spent on advertising by

$$R(x)=\frac{1}{15{,}000}\left(600x^2-x^3\right), \qquad 0\le x\le 600$$

Where x and R are in thousands of dollars. Is there a point of diminishing returns for this function?

11. Find the point of diminishing returns (x, y) for the function

$$R(x)=-x^3+45x^2+400x+8000, \quad 0\le x\le 20$$

where $R(x)$ represents revenue in thousands of dollars and x represents the amount spent on advertising in tens of thousands of dollars.

12. The population of a certain species of fish introduced into a lake is described by the logistic equation

$$G(t)=\frac{12{,}000}{1+19e^{-1.2t}}$$

where $G(t)$ is the population after t years. Find the point at which the growth rate of this population begins to decline.