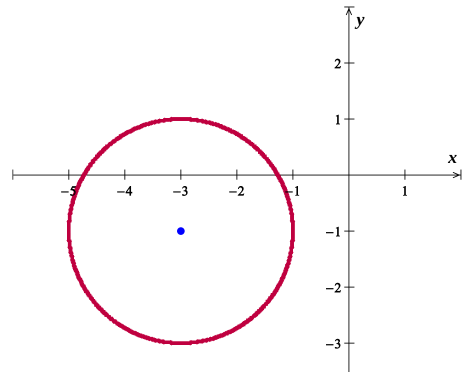
Solution   Section 4.6 – Circles and Parabolas

Exercise

Find the center and the radius of $x^2 + y^2 + 6x + 2y + 6 = 0$



Solution

$$x^2 + 6x + \left(\frac{6}{2}\right)^2 + y^2 + 2y + \left(\frac{2}{2}\right)^2 = -6 + 9 + 1$$

$$(x+3)^2 + (y+1)^2 = 4$$

Center $(-3, -1)$ and $r = 2$

Exercise

Find the center and the radius of $x^2 + y^2 + 8x + 4y + 16 = 0$

Solution

$$x^2 + 8x + \left(\frac{8}{2}\right)^2 + y^2 + 4y + \left(\frac{4}{2}\right)^2 = -16 + 16 + 4$$

$$(x+4)^2 + (y+2)^2 = 4$$

Center $(-4, -2)$ and $r = 2$

Exercise

Find the center and the radius of $x^2 + y^2 - 10x - 6y - 30 = 0$

Solution

$$x^2 - 10x + \left(\frac{-10}{2}\right)^2 + y^2 - 6y + \left(\frac{-6}{2}\right)^2 = 30 + 25 + 9$$

$$(x-5)^2 + (y-3)^2 = 64$$

Center $(5, 3)$ and $r = 8$

Exercise

Find the center and the radius of $x^2 - 6x + y^2 + 10y + 25 = 0$

Solution

$$x^2 - 6x \qquad + y^2 + 10y \qquad = -25$$

$$x^2 - 6x + \left(\frac{1}{2}(-6)\right)^2 + y^2 + 10y + \left(\frac{1}{2}(10)\right)^2 = -25 + \left(\frac{1}{2}(-6)\right)^2 + \left(\frac{1}{2}(10)\right)^2$$

$$(x-3)^2 + (y+5)^2 = -25 + 9 + 25$$

$$(x-3)^2 + (y+5)^2 = 9$$

The equation represents a circle with ***center*** at (3, –5) and ***radius*** 3

Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $20x = y^2$

Solution

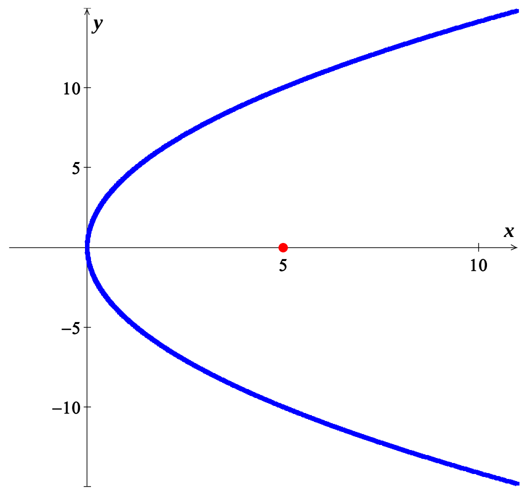
$20x = y^2 \qquad 4px = y^2$

$4p = 20 \quad \Rightarrow \quad \boxed{p = 5}$

Vertex: (0, 0)

Focus $(5,\ 0)$

Directrix: $x = -5$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. ..

Solution

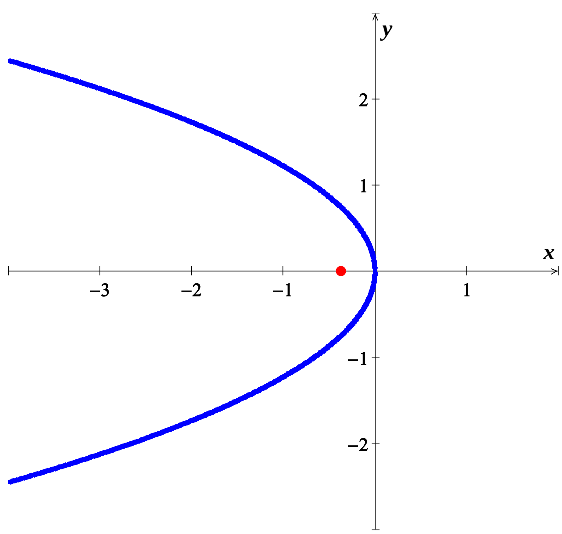
$y^2 = -\frac{3}{2}x = 4px$

$4p = -\frac{3}{2} \quad \Rightarrow \quad \boxed{p = -\frac{3}{8}}$

Vertex: (0, 0)

Focus: $\left(-\frac{3}{8},\ 0\right)$

Directrix: $x = \frac{3}{8}$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $(x+2)^2 = -8(y-1)$

Solution

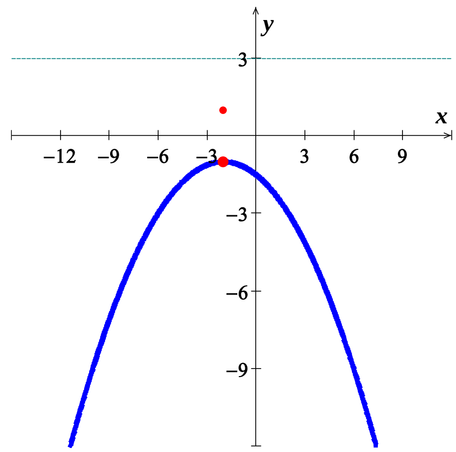
$$(x+2)^2 = 4p(y-1)$$

$$4p = -8 \quad \Rightarrow \quad \boxed{p = -2}$$

Vertex: $(-2,\ 1)$

Focus: $(-2,\ 1-2) = (-2,\ -1)$

Directrix: $y = 1 + 2 = 3$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $(x-3)^2 = \frac{1}{2}(y+1)$

Solution

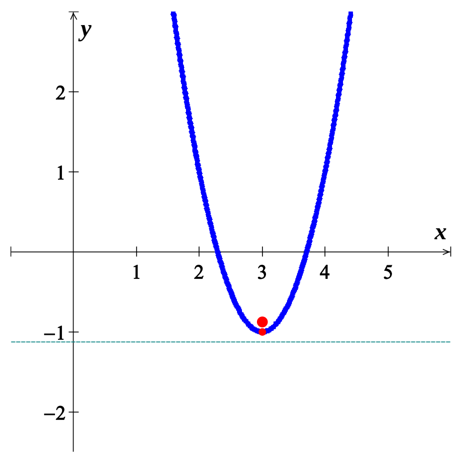
$$(x-3)^2 = 4p(y+1)$$

$$4p = \frac{1}{2} \quad \Rightarrow \quad \boxed{p = \frac{1}{8}}$$

Vertex: $(3,\ -1)$

Focus: $\left(3,\ -1+\frac{1}{8}\right) = \left(3,\ -\frac{7}{8}\right)$

Directrix: $y = -1 - \frac{1}{8} = -\frac{9}{8}$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $(y+1)^2 = -12(x+2)$

Solution

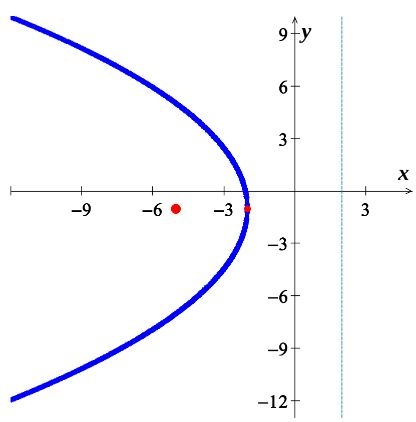
$$(y+1)^2 = 4p(x+2)$$

$$4p = -12 \quad \Rightarrow \quad \boxed{p = -3}$$

Vertex: $(-2,\ -1)$

Focus: $(-2-3,\ -1) = \boxed{(-5,\ -1)}$

Directrix: $x = -1 + 3 = 2$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $y = x^2 - 4x + 2$

Solution

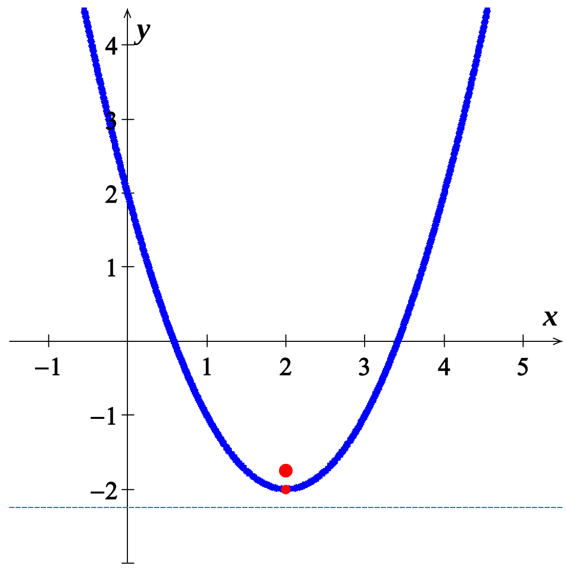
$$y = ax^2 + bx + c \quad \Rightarrow \quad a = 1$$

$$p = \frac{1}{4a} = \frac{1}{4} \quad \Rightarrow \quad \boxed{p = \frac{1}{4}}$$

Vertex: $\begin{cases} h = -\frac{b}{2a} = -\frac{-4}{2(1)} = 2 \\ k = 2^2 - 4(2) + 2 = -2 \end{cases} \rightarrow (2, \ -2)$

Focus: $\left(2, \ -2 + \frac{1}{4}\right) = \left(2, \ -\frac{7}{4}\right)$

Directrix: $\underline{y = -2 - \frac{1}{4} = -\frac{9}{4}}$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $y^2 + 14y + 4x + 45 = 0$

Solution

$$y^2 + 14y = -4x - 45$$

$$y^2 + 14y + (7)^2 = -4x - 45 + (7)^2$$

$$(y + 7)^2 = -4x + 4$$

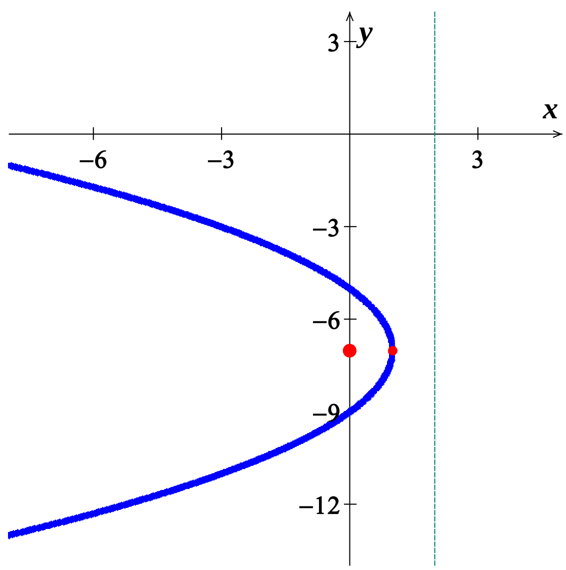
$$(y + 7)^2 = -4(x - 1)$$

$$4p = -4 \quad \Rightarrow \quad \boxed{p = -1}$$

Vertex: $(1, \ -7)$

Focus: $(1 - 1, \ -7) = \boxed{(0, \ -7)}$

Directrix: $\underline{x = 1 + 1 = 2}$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $x^2 + 20y = 10$

Solution

$$x^2 = -20y + 10$$

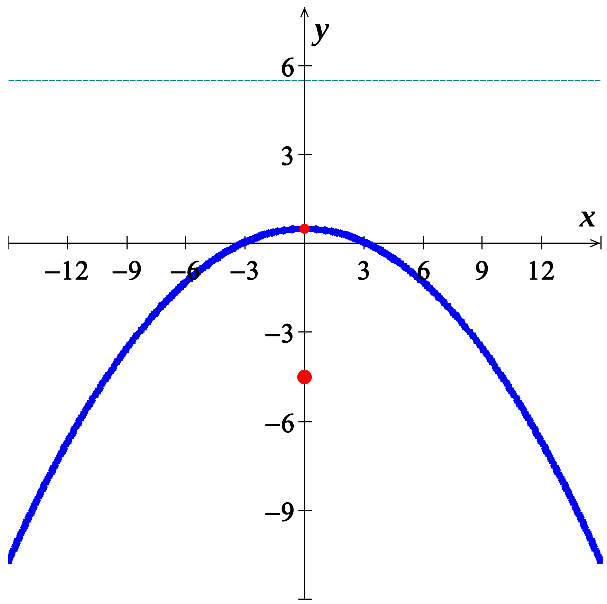
$$x^2 = -20\left(y - \tfrac{1}{2}\right)$$

$$4p = -20 \quad \Rightarrow \quad \boxed{p = -5}$$

Vertex: $\boxed{\left(0,\ \frac{1}{2}\right)}$

Focus: $\left(0,\ \frac{1}{2} - 5\right) = \boxed{\left(0,\ -\frac{9}{2}\right)}$

Directrix: $\boxed{y = \frac{1}{2} + 5 = \frac{11}{2}}$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $x^2 = 16y$

Solution

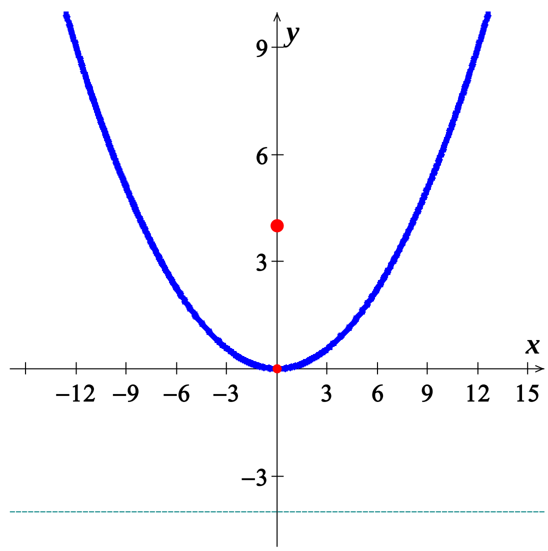
$$x^2 = 16y = 4py$$

$$4p = 16 \quad \Rightarrow \quad \boxed{p = 4}$$

Vertex: (0, 0)

Focus: $(0,\ 4)$

Directrix: $y = -4$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $x^2 = -\frac{1}{2}y$

Solution

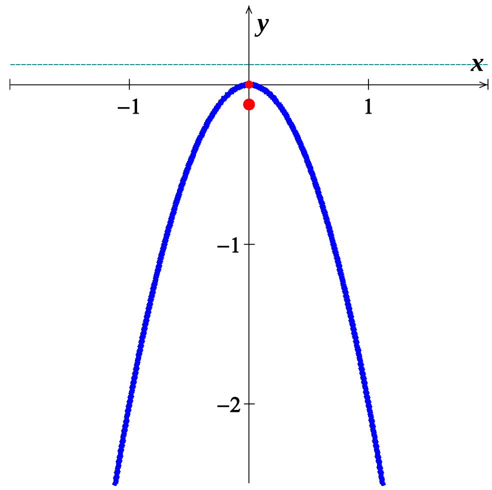
$$x^2 = -\frac{1}{2}y = 4py$$

$$4p = -\frac{1}{2} \quad \Rightarrow \quad \boxed{p = -\frac{1}{8}}$$

Vertex: (0, 0)

Focus: $\left(0,\ -\frac{1}{8}\right)$

Directrix: $y = \frac{1}{8}$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $(y+1)^2 = -4(x-2)$

Solution

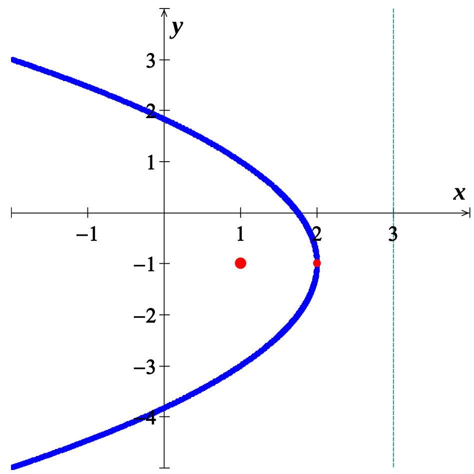
$$(y+1)^2 = 4p(x-2)$$

$$4p = -4 \quad \Rightarrow \quad \boxed{p = -1}$$

Vertex: $(2,\ -1)$

Focus: $(2-1,\ -1) = \boxed{(1,\ -1)}$

Directrix: $\boxed{x = 2+1 = 3}$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $x^2 + 6x - 4y + 1 = 0$

Solution

$$x^2 + 6x + \left(\frac{6}{2}\right)^2 = 4y - 1 + (3)^2$$

$$(x+3)^2 = 4y + 8$$

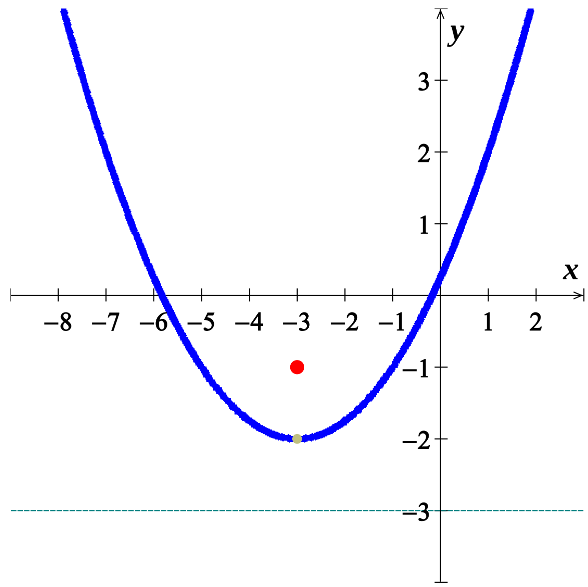
$$(x+3)^2 = 4(y+2)$$

$$4p = 4 \quad \Rightarrow \quad \boxed{p = 1}$$

Vertex: $\boxed{(-3,\ -2)}$

Focus: $(-3,\ -2+1) = \boxed{(-3,\ -1)}$

Directrix: $\boxed{y = -2-1 = -3}$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $y^2 + 2y - x = 0$

Solution

$$y^2 + 2y = x$$

$$y^2 + 2y + \left(\frac{2}{2}\right)^2 = x + (1)^2$$

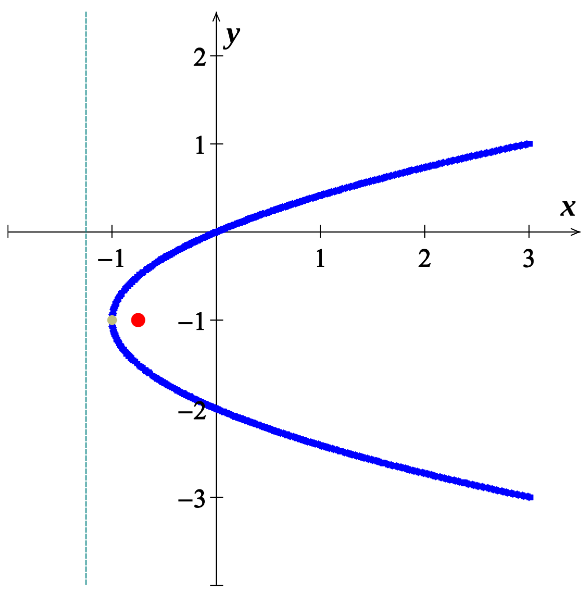
$$(y+1)^2 = (x+1)$$

$$4p = 1 \quad \Rightarrow \quad \boxed{p = \frac{1}{4}}$$

Vertex: $(-1,\ -1)$

Focus: $\left(-1+\frac{1}{4},\ -1\right) = \boxed{\left(-\frac{3}{4},\ -1\right)}$

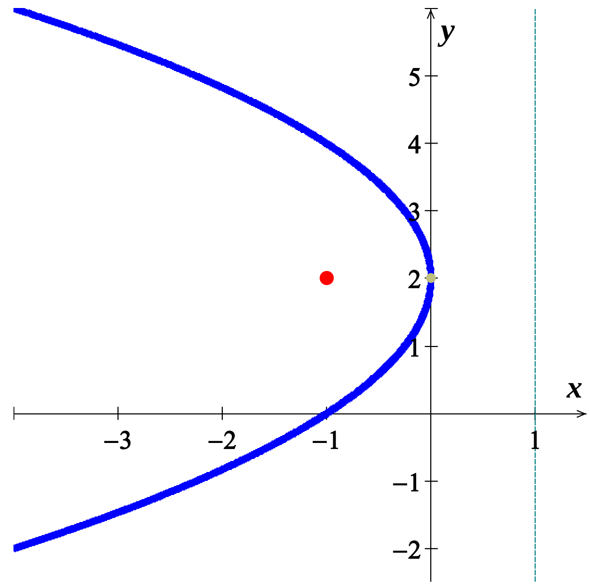
Directrix: $\boxed{x = -1 - \frac{1}{4} = -\frac{5}{4}}$



Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $y^2-4y+4x+4=0$

Solution



$$y^2-4y=-4x-4$$

$$y^2-4y+\left(\frac{-4}{2}\right)^2=-4x-4+(-2)^2$$

$$(y-2)^2=-4x$$

$$4p=-4 \quad \Rightarrow \quad \boxed{p=-1}$$

Vertex: $(0,\ 2)$

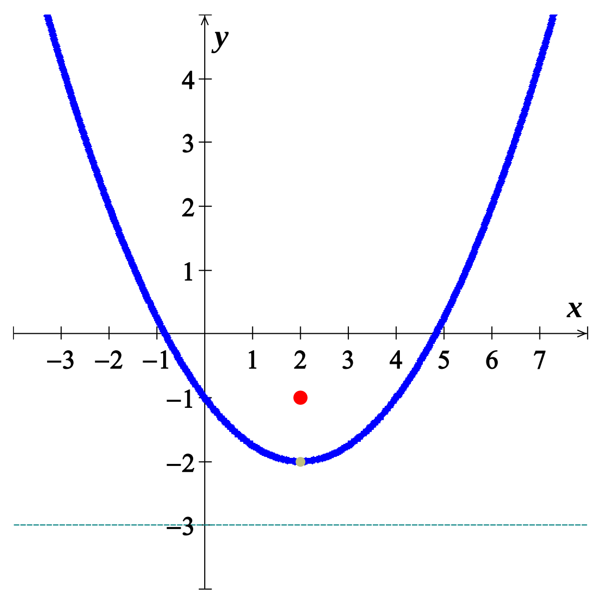
Focus: $=\boxed{(-1,\ 2)}$

Directrix: $\boxed{x=1}$

Exercise

Find the ***vertex***, ***focus***, and ***directrix*** of the parabola. Sketch its graph. $x^2-4x-4y=4$

Solution



$$x^2-4x=4y+4$$

$$x^2-4x+\left(\frac{-4}{2}\right)^2=4y+4+(-2)^2$$

$$(x-2)^2=4(y+2)$$

$$4p=4 \quad \Rightarrow \quad \boxed{p=1}$$

Vertex: $\boxed{(2,\ -2)}$

Focus: $(2,\ -2+1)=\boxed{(2,\ -1)}$

Directrix: $\boxed{y=-2-1=-3}$

Exercise

Find an equation of the parabola that satisfies the given conditions $Focus: F(2,0) \quad directrix: x=-2$

Solution

$$x=-2=-p \quad \to \quad p=2$$

$$y^2=4px$$

$$\boxed{y^2=8x}$$

Exercise

Find an equation of the parabola that satisfies the given conditions $Focus: F(0,-40) \quad directrix: y=4$

Solution

$$y=4=-p \quad \to \quad p=-4$$

$$x^2=4py$$

$$\boxed{x^2=-16y}$$

Exercise

Find an equation of the parabola that satisfies the given conditions $Focus: F(-3,-2) \quad directrix: y=1$

Solution

$$y=1=k-p \quad \to \quad k-p=1$$

$$\begin{cases} \boxed{h=-3} \\ k+p=-2 \end{cases} \to \begin{cases} k+p=-2 \\ k-p=1 \end{cases} \Rightarrow 2k=-1 \to \boxed{k=-\frac{1}{2}}$$

$$k-p=1 \quad \to \quad \boxed{p=k-1=-\frac{1}{2}-1=-\frac{3}{2}}$$

Vertex: $\boxed{\left(-3,\ -\frac{1}{2}\right)}$

$$(x+3)^2=4\left(-\frac{3}{2}\right)\left(y+\frac{1}{2}\right)$$

$$\boxed{(x+3)^2=-6\left(y+\frac{1}{2}\right)}$$

Exercise

Find an equation of the parabola that satisfies the given conditions $Vertex: V(3,-5) \quad directrix: x=2$

Solution

$$Vertex: V(3,-5) \quad \begin{cases} h=3 \\ k=-5 \end{cases}$$

$$directrix: x=2=h-p \quad \Rightarrow \underline{p}=h-2=3-2\ \underline{=1}$$

$$(y-k)^2=4p(x-h)$$

$$\underline{(y+5)^2=4(x-3)}$$

Exercise

Find an equation of the parabola that satisfies the given conditions $Vertex: V(-2,3) \quad directrix: y=5$

Solution

$$Vertex: V(-2,\ 3) \quad \begin{cases} h=-2 \\ k=3 \end{cases}$$

$$directrix: y=5=k-p \quad \Rightarrow \underline{p}=k-5=3-5\ \underline{=-2}$$

$$(x-h)^2=4p(y-k)$$

$$\underline{(x+2)^2=-8(y-3)}$$

Exercise

Find an equation of the parabola that satisfies the given conditions $Vertex: V(-1,0) \quad focus: F(-4,0)$

Solution

$$Vertex: V(-1,\ 0) \quad \begin{cases} h=-1 \\ k=0 \end{cases}$$

$$focus: F(-4,0) \quad \begin{cases} h+p=-4 \\ k=0 \end{cases} \Rightarrow \underline{p}=-4-h=-4+1=\underline{-3}$$

$$(y-k)^2=4p(x-h)$$

$$\underline{y^2=-12(x+1)}$$

Exercise

Find an equation of the parabola that satisfies the given conditions $Vertex: V(1,-2)$ $focus: F(1,0)$

Solution

$$Vertex: V(1,\ -2)\ \begin{cases} h=1 \\ k=-2 \end{cases}$$

$$focus: F(1,\ 0)\ \begin{cases} h=1 \\ k+p=0 \end{cases} \Rightarrow \underline{p=-k=2}$$

$$(x-h)^2=4p(y-k)$$

$$\boxed{(x-1)^2=8(y+2)}$$

Exercise

Find an equation of the parabola that satisfies the given conditions $Vertex: V(0,\ 1)$ $focus: F(0,\ 2)$

Solution

$$Vertex: V(0,\ 1)\ \begin{cases} h=0 \\ k=1 \end{cases}$$

$$focus: F(0,\ 2)\ \begin{cases} h=0 \\ k+p=2 \end{cases} \Rightarrow \underline{p=2-1=1}$$

$$(x-h)^2=4p(y-k)$$

$$\boxed{x^2=4(y-1)}$$

Exercise

Find an equation of the parabola that satisfies the given conditions $Vertex: V(3,\ 2)$ $focus: F(-1,\ 2)$

Solution

$$Vertex: V(3,\ 2)\ \begin{cases} h=3 \\ k=2 \end{cases}$$

$$focus: F(-1,2)\ \begin{cases} h+p=-1 \Rightarrow \underline{p=-1-3=-4} \\ k=2 \end{cases}$$

$$(y-k)^2=4p(x-h)$$

$$\boxed{(y-2)^2=-16(x-3)}$$

Exercise

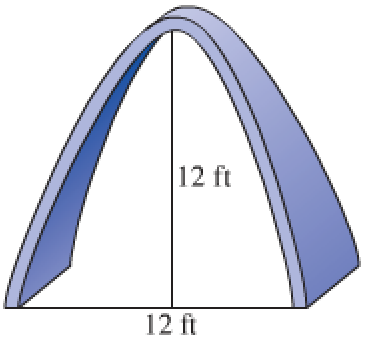
An arch in the shape of a parabola has the dimensions shown in the figure. How wide is the arch 9 *feet* up?

Solution

Vertex: $V(0,\ 12)$

$(x-h)^2 = 4p(y-k)$

$(x-0)^2 = 4p(y-12) \quad \Rightarrow \quad x^2 = 4p(y-12)$



The parabola passes through the point $(6,\ 0) \quad \Rightarrow \quad 6^2 = 4p(0-12)$

$-48p = 36 \rightarrow \boxed{p = -\frac{36}{48} = -\frac{3}{4}}$

The equation is: $x^2 = -3(y-12)$

The arch is 9 feet up that is the *y*-coordinate,

$x^2 = -3(9-12) = 9 \quad \Rightarrow \quad x = 3$

The width is $2(3)\ \boxed{=6\ \ feet}$

Exercise

The cable in the center portion of a bridge is supported as shown in the figure to form a parabola. The center support is 10 *feet* high, the tallest supports are 210 *feet* high, and the distance between the two tallest supports is 400 *feet*. Find the height of the remaining supports if the supports are evenly spaced.

Solution

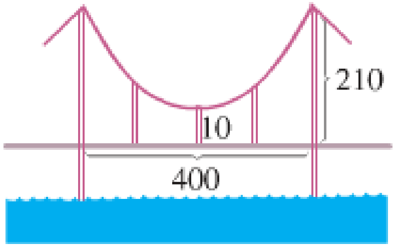
Vertex: $V(0,\ 10)$

$(x-h)^2 = 4p(y-k)$

$(x-0)^2 = 4p(y-10) \quad \Rightarrow \quad x^2 = 4p(y-10)$

The parabola passes through the point $(200,\ 210) \quad \Rightarrow \quad 200^2 = 4p(210-10)$

$800p = 200^2 \rightarrow \boxed{p = \frac{40000}{800} = 50}$



The equation is: $x^2 = 200(y-10)$

The *x*-coordinate of one of the supports is 100.

$100^2 = 200(y-10)$

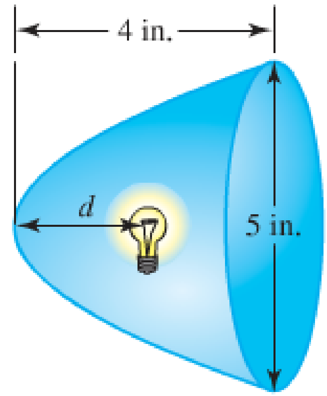
$y - 10 = \frac{10000}{200} = 50$

$\boxed{y = 50+10 = 60\ \ feet}$ The height is 60 *feet*

Exercise

A headlight is being constructed in the shape of a paraboloid with depth 4 *inches* and diameter 5 *inches*. Determine the distance d that the bulb should be form the vertex in order to have the beam of light shine straight ahead.

Solution



Let the vertex be at the origin $V(0,\ 0)$

The equation is: $y^2 = 4px$

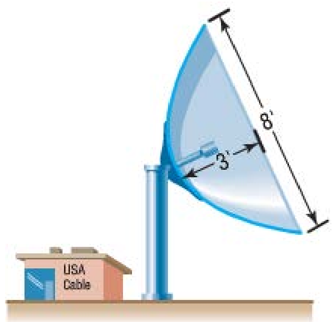
Which it passes through the point $V(4,\ 2.5)$

$(2.5)^2 = 4p(4)$

$p = \dfrac{(2.5)^2}{16} = \dfrac{25}{64}$

The bulb should be $\frac{25}{64} \approx 0.39$ inch from the vertex

Exercise



A satellite dish is shaped like a paraboloid of revolution. The signals that emanate from a satellite strike the surface of the dish and are reflected to a single point, where the receiver is located. If the dish is 8 *feet* across at its opening and 3 *feet* deep at its center, at what position should the receiver be placed? That is, where is the focus?

Solution

From the figure, we can draw the parabola used to form the dish on a rectangular coordinate system so that the vertex of the parabola is at the origin and its focus on the positive y-axis.

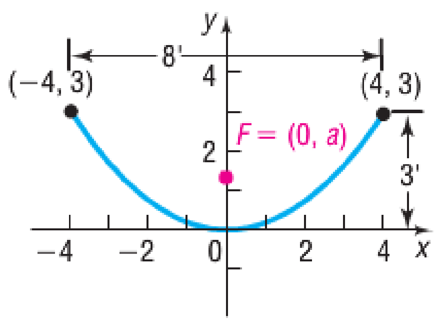
The equation from of the parabola is: $x^2 = 4py$

Since $(4,\ 3)$ is a point on the graph

$4^2 = 4p(3)$

$p = \dfrac{16}{12} = \dfrac{4}{3}$

Therefore, the receiver should be located $\frac{4}{3}\,ft$ from the base of the dish, along its axis of symmetry.



Exercise

A cable TV receiving dish is in the shape of a paraboloid of revolution. Find the location of the receiver, which is placed at the focus, if the dish is 6 *feet* across at its opening and 2 feet deep.

Solution

Given: Parabola is 6 *feet* across and 2 *feet* deep.

Let the vertex of the parabola is at (0, 0) and it opens up, then the equation of the parabola has the form $x^2 = 4ay$

Therefore, the point (3, 2) and (–3, 2) are on the parabola.

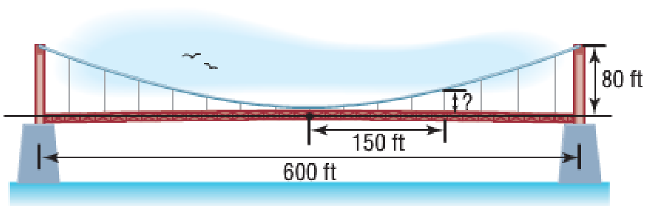
$$3^2 = 4a(2) \quad \rightarrow \quad a = \frac{9}{8} = 1.125$$

Where a is the distance from the vertex to the focus.

Thus, the receiver (located at the focus) is 1.125 *feet* or 13.5 *inches* from the base of the dish, along the axis of the parabola.

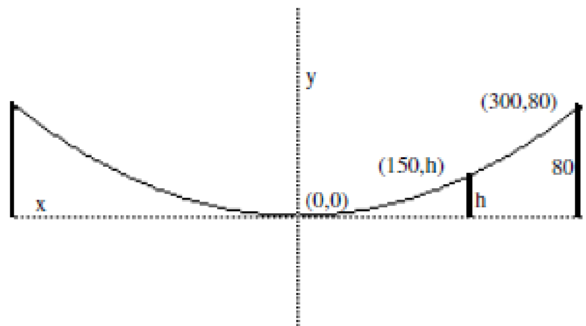
Exercise

The cables of a suspension bridge are in the shape of a parabola, as shown below. The towers supporting the cable are 600 *feet* apart and 80 *feet* high. If the cables touch the road surface midway between the towers, what is the height of the cable from the road at a point 150 *feet* from the center of the bridge?



Solution

Let the vertex of the parabola is at (0, 0) and it opens up, then the equation of the parabola has the form $x^2 = cy$



The point (300, 80) is a point on the parabola.

$$300^2 = c(80) \quad \rightarrow \quad c = \frac{300^2}{80} = 1125$$

$$x^2 = 1125y$$

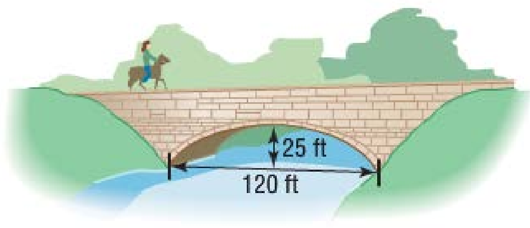
The height of the cable 150 feet from the center is:

$$150^2 = 1125h \quad \rightarrow \quad h = \frac{150^2}{1125} = 20$$

The height of the cable 150 feet from the center is 20 feet.

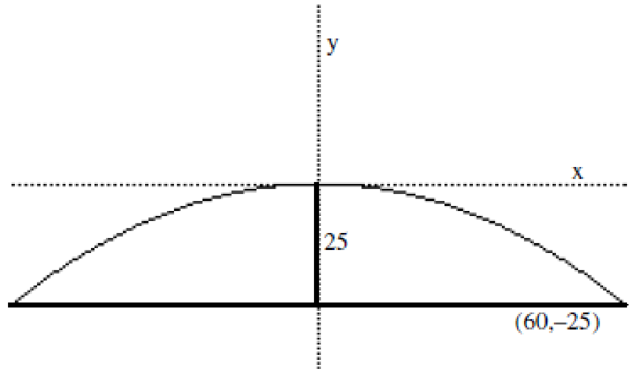
Exercise

A bridge is built in the shape of a parabolic arch. The bridge has a span of 120 *feet* and a maximum height of 25 *feet*. Choose a suitable rectangular coordinate system and find the height of the arch at distances of 10, 30, and 50 *feet* from the center.



Solution

Let the vertex of the parabola is at (0, 0) and it opens down, then the equation of the parabola has the form $x^2 = cy$



The point (60, –25) is a point on the parabola.

$$60^2 = c(-25) \quad \to \quad c = \frac{60^2}{-25} = -144$$

$$x^2 = -144y$$

The height of the arch at

Distance 10:

$$10^2 = -144y \quad \to \quad y = \frac{100}{-144} \approx -0.69$$

The height of the bridge 10 feet from the center is about $25 - 0.69 = 24.31\ ft$

Distance 30:

$$30^2 = -144y \quad \to \quad y = \frac{900}{-144} \approx -6.25$$

The height of the bridge 30 feet from the center is about $25 - 6.25 = 18.75\ ft$

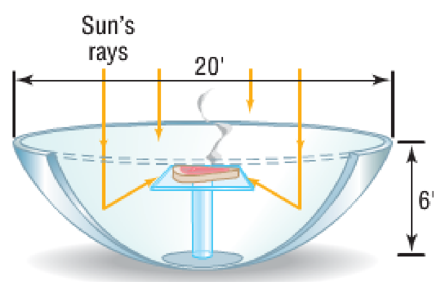
Distance 50:

$$50^2 = -144y \quad \to \quad y = \frac{2500}{-144} \approx -17.36$$

The height of the bridge 10 feet from the center is about $25 - 17.36 = 7.64\ ft$

Exercise

A mirror is shaped like a paraboloid of revolution and will be used to concentrate the rays of the sun at its focus, creating a heat source. If the mirror is 20 *feet* across at its opening and is 6 *feet* deep, where will the heat source be concentrated?



Solution

Let the vertex of the parabola is at (0, 0) and it opens up, then the equation of the parabola has the form $x^2 = 4ay$

Since the parabola is 20 feet across and 6 feet deep.

The points (10, 6) and (–10, 6) are on the parabola.

$$10^2 = 4a(6) \quad \to \quad a = \frac{100}{24} \approx 4.17\ ft$$

The heat will be concentrated about 4.17 *feet* from the base, along the axis of symmetry.

Exercise

A reflecting telescope contains a mirror shaped a paraboloid of revolution. If the mirror is 4 *inches* across at its opening and is 3 *inches* deep, where will the collected light be concentrated?

Solution

Let the vertex of the parabola is at (0, 0) and it opens up.

Then the equation of the parabola has the form $x^2 = 4ay$

Since the parabola is 4 *inches* across and 3 *inches* deep.

The points (2, 3) and (–2, 3) are on the parabola.

$$2^2 = 4a(3) \quad \to \quad a = \frac{4}{12} \ \boxed{\approx \frac{1}{2} \ in}$$

The collected light will be concentrated 1/3 inch from the base of the mirror along the axis of symmetry.

Exercise

Show that the graph of an equation of the form $Ax^2 + Dx + Ey + F = 0 \qquad A \neq 0$

a) Is a parabola if $E \neq 0$

b) Is a vertical line if $E = 0 \quad and \quad D^2 - 4AF = 0$

c) Is two vertical lines if $E = 0 \quad and \quad D^2 - 4AF > 0$

d) Contains no points if $E = 0 \quad and \quad D^2 - 4AF < 0$

Solution

a) If $E \neq 0 \quad \to \quad Ax^2 + Dx + Ey + F = 0$

The *x*-vertex: $x = -\frac{b}{2a} = -\frac{D}{2A}$

$$A\left(-\frac{D}{2A}\right)^2 + D\left(-\frac{D}{2A}\right) + Ey + F = 0$$

$$\frac{D^2}{4A} - \frac{D^2}{2A} + Ey + F = 0$$

$$Ey = \frac{D^2}{4A} - F$$

$$y = \frac{D^2 - 4AF}{4AE}$$

This is the equation of a parabola whose vertex is: $\left(-\frac{D}{2A}, \frac{D^2 - 4AF}{4AE}\right)$ and whose axis of symmetry is parallel to the *y*-axis.

b) If $E = 0 \quad \to \quad Ax^2 + Dx + F = 0$

$$x = \frac{-D \pm \sqrt{D^2 - 4AF}}{2A}$$

$$= -\frac{D}{2A}$$ *Since* $D^2 - 4AF = 0$

This is a single vertical line.

c) If $E = 0 \rightarrow Ax^2 + Dx + F = 0$

$$x = \frac{-D \pm \sqrt{D^2 - 4AF}}{2A}$$

If $D^2 - 4AF > 0$, then

$x = \frac{-D - \sqrt{D^2 - 4AF}}{2A}$ *and* $x = \frac{-D + \sqrt{D^2 - 4AF}}{2A}$ are two vertical lines.

d) If $E = 0 \rightarrow Ax^2 + Dx + F = 0$

$$x = \frac{-D \pm \sqrt{D^2 - 4AF}}{2A}$$

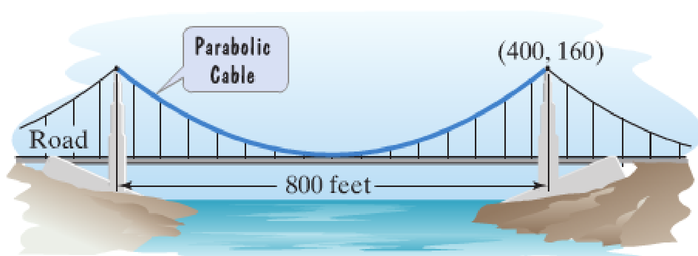
If $D^2 - 4AF < 0$, then there is no real solution. The graph contains no points.

Exercise

The towers of a suspension bridge are 800 *feet* apart and rise 160 *feet* above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 100 *feet* from a tower?

Solution

Given the point: $(400,\ 160)$



$(400)^2 = 4p(160)$ $\qquad x^2 = 4py$

$p = \frac{400^2}{640} = 250$

$x^2 = 1{,}000y$

$x = 400 - 100 = 300$

$(300)^2 = 1{,}000y$ $\qquad x^2 = 4py$

$y = \frac{300^2}{1{,}000} = 90$

The height is 90 *feet*.

Exercise

The cables of a suspension bridge are in the shape of a parabola. The towers supporting the cable are 400 *feet* apart and 100 *feet* high. If the cables are at a height of 10 *feet* midway between the towers, what is the height of the cable at a point 50 *feet* from the center of the bridge?

Solution

Vertex point: $(0,\ 10)$ and the parabola is open up

A point on parabola: $(200,\ 100)$

$$200^2 = c(100-10) \qquad (x-h)^2 = c(y-k)$$

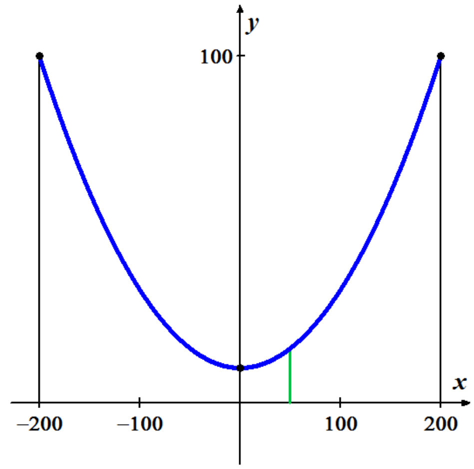
$$c = \frac{40,000}{90} = \frac{4000}{9}$$

$$\boxed{x^2 = \frac{4000}{9}(y-10)}$$

The height of the cable 50 feet from the center – $(50,\ h)$

$$y = \frac{9}{4000}x^2 + 10$$

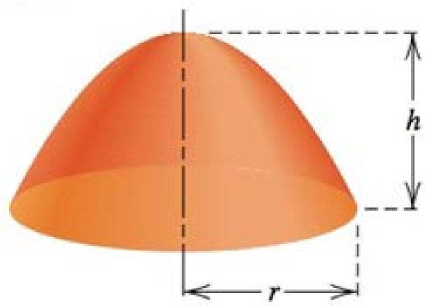
$$h = \frac{9}{4000}(50)^2 + 10 \ \approx 15.625 \ ft$$



The height of the cable 50 *feet* from the center is about 15.625 *feet*.

Exercise

The focal length of the (finite) paraboloid is the distance p between its vertex and focus



a) Express p in terms of r and h.

b) A reflector is to be constructed with a focal length of 10 *feet* and a depth of 5 *feet*. Find the radius of the reflector.

Solution

a) The point $(r,\ h)$ is on the parabola.

$$r^2 = 4p(h) \qquad x^2 = 4py$$

$$\boxed{p = \frac{r^2}{4h}}$$

b) Given: $p = 10; \quad h = 5$

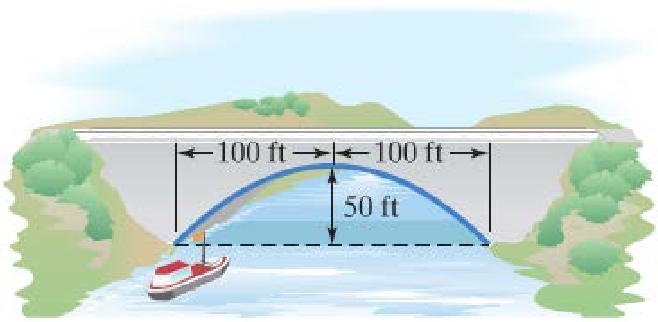
$$r = \sqrt{4(10)(5)} \ = 10\sqrt{2}$$

Exercise

The parabolic arch is 50 *feet* above the water at the center and 200 *feet* wide at the base. Will a boat that is 30 *feet* tall clear the arch 30 *feet* from the center?

Solution

$\left(\frac{200}{2}\right)^2 = 4p(-50)$ $\qquad x^2 = 4py$



$p = \frac{200^2}{-200}$

$\quad = -200$

$\boxed{x^2 = -200y}$

Given the boat tall: $x = 30$

$(30)^2 = -200y$ $\qquad x^2 = 4py$

$y = \frac{900}{-200}$

$\quad = -4.5$

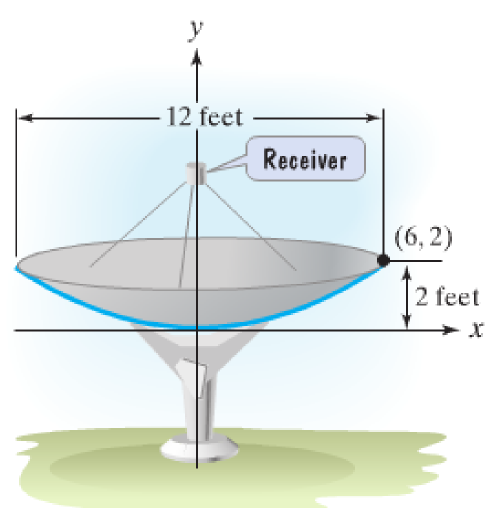
Height of bridge $= 50 - 4.5 = 45.5\ ft$

Yes, the boat will clear the arch.

Exercise

A satellite dish, as shown below, is in the shape of a parabolic surface. Signals coming from a satellite strike the surface of the dish and are reflected to the focus, where the receiver if located. The satellite dish shown has a diameter of 12 *feet* and a depth of 2 *feet*. How far from the base of the dish should the receiver be placed?

Solution



$6^2 = 4p(2)$ $\qquad x^2 = 4py$

$p = \frac{36}{8}$

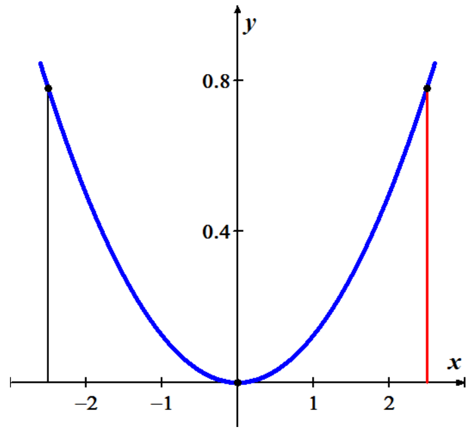
$\quad = 4.5$

The receiver should be located 4.5 *feet* from the base of the dish.

Exercise

A searchlight is shaped like a paraboloid of revolution. If the light source is located 2 *feet* from the base along the axis of symmetry and the opening is 5 *feet* across, how deep should the searchlight be?

Solution



Vertex point: $(0,\ 0)$ and the parabola is open up.

Given: $p = 2$

$x^2 = 8y \qquad\qquad x^2 = 4py$

The opening is 5 *feet* across – $(2.5,\ y)$

$$y = \frac{x^2}{8}$$

$$= \frac{2.5^2}{8}$$

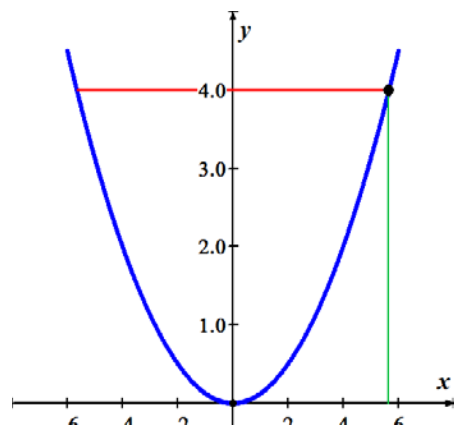
$$= 0.78125\ \ ft$$

The depth of the searchlight should be 0.78125 *feet*.

Exercise

A searchlight is shaped like a paraboloid of revolution. If the light source is located 2 *feet* from the base along the axis of symmetry and the depth of the searchlight is 4 *feet* across, how deep should the opening be?

Solution



Vertex point: $(0,\ 0)$ and the parabola is open up.

Given: $p = 2$

$x^2 = 8y \qquad\qquad x^2 = 4py$

The depth is 4 *feet* – $(x,\ 4)$

$$x^2 = 8(4)$$

$$= 32$$

$$x = \pm 4\sqrt{2}\ \ ft$$

The width of the opening of the searchlight should be $2\left(4\sqrt{2}\right) = 11.31\ \ feet$.

Exercise

A searchlight is shaped like a paraboloid, with the light source at the focus. If the reflector is 3 *feet* across at the opening and 1 *foot* deep, where is the focus?

Solution

Vertex point: $(0,\ 0)$ and the parabola is open up.

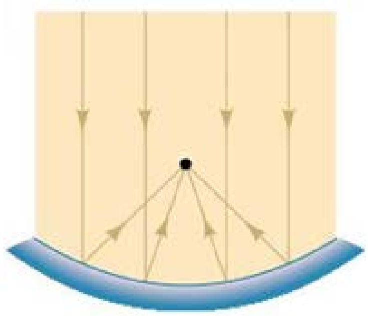
$2x = 3 \ \rightarrow \ x = \frac{3}{2}$

$1 = \frac{1}{4p}\left(\frac{3}{2}\right)^2 \qquad\qquad y = \frac{1}{4p}x^2$

$\underline{p = \frac{9}{16}\ ft|}$

Exercise

A mirror for a reflecting telescope has the shape of a (finite) paraboloid of diameter 8 *inches* and depth 1 *inch*. How far from the center the mirror will the incoming light collect?



Solution

Vertex point: $(0,\ 0)$ and passing through $P\left(\frac{8}{2},\ 1\right) = (4,\ 1)$

$1 = \frac{1}{4p}(4)^2 \qquad\qquad y = \frac{1}{4p}x^2$

$p = \frac{16}{4}\ \underline{= 4|}$

The light will collect 4 *inches* from the center of the mirror.