Section **1.4 – Exact Differential Equations**

A class of equations known as exact equations for which there is also a well-defined method of solution

Theorem

Let the function M, N, M_y and N_x, where M_y and N_x are partial derivatives, be continuous in the rectangular region $R: \alpha < x < \beta, \ \gamma < y < \delta$ then

$$M(x, y) + N(x, y)y' = 0$$

Is an exact differential equation in R, if and only if

$$M_y(x, y) = N_x(x, y)$$

At each point in R. That is, there exists a function ψ satisfying

$$\psi_y(x, y) = M(x, y) \quad and \quad \psi_x(x, y) = N(x, y)$$

Iff $M_y(x, y) = N_x(x, y)$

Example

Solve the differential equation: $\qquad 2x + y^2 + 2xyy' = 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = 2x + y^2 \quad \Rightarrow \quad M_y = 2y$$

$$\frac{\partial \psi}{\partial y} = N = 2xy \quad \Rightarrow \quad N_x = 2y$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = 2x + y^2 \quad \Rightarrow \quad \psi = \int \left(2x + y^2\right) dx = x^2 + xy^2 + h(y)$$

$$\psi_y = 2xy + h'(y) = 2xy \quad \Rightarrow \quad h'(y) = 0$$

$$\Rightarrow h(y) = C$$

$$\psi(x, y) = x^2 + xy^2 + C$$

$$\boxed{x^2 + xy^2 = C}$$

Example

Solve the differential equation: $\quad y\cos x + 2xe^y + \left(\sin x + x^2 e^y - 1\right)y' = 0$

Solution

$M = y\cos x + 2xe^y = \dfrac{\partial\psi}{\partial x} \implies M_y = \cos x + 2xe^y$

$\dfrac{\partial\psi}{\partial y} = N = \sin x + x^2 e^y - 1 \implies N_x = \cos x + 2xe^y$

$\implies M_y = N_x$

$\psi = \displaystyle\int \left(y\cos x + 2xe^y\right)dx = y\sin x + x^2 e^y + h(y)$

$\psi_y = \sin x + x^2 e^y + h'(y) = \sin x + x^2 e^y - 1 \implies h'(y) = -1$

$\implies h(y) = -y$

$\psi(x,\ y) = y\sin x + x^2 e^y - y = C$

$\boxed{y\sin x + x^2 e^y - y = C}$

Example

Solve the differential equation: $\quad 3xy + y^2 + \left(x^2 + xy\right)y' = 0$

Solution

$M = 3xy + y^2 = \dfrac{\partial\psi}{\partial x} \implies M_y = 3x + 2y$

$N = x^2 + xy = \dfrac{\partial\psi}{\partial y} \implies N_x = 2x + y$

$\implies M_y \neq N_x$

Can be solved by this procedure.

Integrating Factors

It is sometimes possible to convert a differential equation that is not exact equation by multiplying the equation by a suitable integrating factor.

Definition

An integrating factor for the differential equation $\omega = Mdx + Ndy = 0$ is a function $\mu(x, y)$ such that the form $\mu\omega = \mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy$ is exact.

$$\left(\mu M\right)_y = \left(\mu N\right)_x$$

$$M\mu_y - N\mu_x + \left(M_y - N_x\right)\mu = 0$$

Assuming that μ is a function of x only, we have

$$\left(\mu M\right)_y = \mu M_y \quad \& \quad \left(\mu N\right)_x = \mu N_x + N\frac{d\mu}{dx}$$

$$\Rightarrow \mu M_y = \mu N_x + N\frac{d\mu}{dx}$$

$$\boxed{\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu}$$

Example

Find an integrating factor for the equation $\left(3xy + y^2\right) + \left(x^2 + xy\right)y' = 0$

And then solve the equation.

Solution

$$M_y = \frac{\partial}{\partial y}\left(3xy + y^2\right) = 3x + 2y \qquad N_x = \frac{\partial}{\partial x}\left(x^2 + xy\right) = 2x + y$$

$$\Rightarrow M_y \neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{3x + 2y - 2x - y}{x^2 + xy} = \frac{x + y}{x(x + y)} = \frac{1}{x}$$

$$\frac{d\mu}{dx} = \frac{\mu}{x} \quad \Rightarrow \quad \int\frac{d\mu}{\mu} = \int\frac{dx}{x}$$

$$\ln\mu = \ln x$$

$$\mu = x$$

$$x\left(3xy + y^2\right) + x\left(x^2 + xy\right)y' = 0$$

$$M_y = \frac{\partial}{\partial y}\left(3x^2 y + xy^2\right) = 3x^2 + 2xy \qquad N_x = \frac{\partial}{\partial x}\left(x^3 + x^2 y\right) = 3x^2 + 2xy$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int\left(3x^2 y + xy^2\right)dx = x^3 y + \frac{1}{2}x^2 y^2 + h(y)$$

$$\psi_y = x^3 + x^2 y + h'(y) = x^3 + x^2 y \quad \Rightarrow \quad h'(y) = 0$$

$$\Rightarrow h(y) = C$$

$$\psi(x, y) = x^3 y + \frac{1}{2}x^2 y^2 = C$$

$$\boxed{x^3 y + \frac{1}{2}x^2 y^2 = C}$$

Bernoulli Equations

An equation of the form $y' + P(x)y = Q(x)y^n$, $n \neq 0, 1$ is called a **Bernoulli equation**.

If $n = 0 \Rightarrow y' + Py = Q$ First–order linear differential equation

If $n = 1 \Rightarrow y' + Py = Qy \rightarrow y' + (P - Q)y = 0$ Separable equation.

For $n \neq 0, 1$, the Bernoulli equation can be written as $y^{-n}\dfrac{dy}{dx} + Py^{1-n} = Q$ (1)

Let $u = y^{1-n} \Rightarrow \dfrac{du}{dx} = (1-n)y^{-n}\dfrac{dy}{dx}$

$$y^{-n}\dfrac{dy}{dx} = \dfrac{1}{1-n}\dfrac{du}{dx}$$

$(1) \Rightarrow \dfrac{1}{1-n}\dfrac{du}{dx} + Pu = Q$

$\underline{u' + (1-n)Pu = (1-n)Q}$ Which is 1st–order linear differential equation.

Example

Find the general solution $y' - 4y = 2e^x\sqrt{y}$

Solution

$\sqrt{y} = y^{1/2} \Rightarrow n = \dfrac{1}{2}$

Let $u = y^{1-\frac{1}{2}} = y^{1/2} \Rightarrow y = u^2$

$\dfrac{du}{dx} = \dfrac{1}{2}y^{-1/2}\dfrac{dy}{dx} \Rightarrow 2y^{1/2}\dfrac{du}{dx} = \dfrac{dy}{dx}$

$\dfrac{dy}{dx} - 4y = 2e^x u$

$2u\dfrac{du}{dx} - 4u^2 = 2ue^x$ *Divide by $2u$*

$u' - 2u = e^x$

$$e^{\int -2dx} = e^{-2x}$$

$$\int e^x e^{-2x}dx = \int e^{-x}dx = -e^{-x}$$

$u = \dfrac{1}{e^{-2x}}\left(-e^{-x} + C\right)$

$y^{1/2} = -e^x + Ce^{2x}$

$\underline{y = \left(Ce^{2x} - e^x\right)^2}$

Example

Find the general solution $xy' + y = 3x^3 y^2$

Solution

$$y' + \frac{1}{x}y = 3x^2 y^2$$

Let $u = y^{1-2} = y^{-1} \quad \Rightarrow \quad y = \frac{1}{u}$

$$\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \quad \Rightarrow \quad y' = -y^2 u' = -\frac{1}{u^2}u'$$

$$-\frac{1}{u^2}u' + \frac{1}{x}\frac{1}{u} = 3x^2 \frac{1}{u^2} \qquad \text{*Multiply both sides by* } -u^2$$

$$u' - \frac{1}{x}u = -3x^2$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$\int -3x^2 x^{-1}dx = -3\int x\,dx = -\frac{3}{2}x^2$$

$$u = x\left(-\frac{3}{2}x^2 + C_1\right)$$

$$\frac{1}{y} = \frac{-3x^3 + 2C_1 x}{2}$$

$$\underline{y = \frac{2}{Cx - 3x^3}}$$

Homogeneous Equations $\frac{dy}{dx} = f(x, y)$

The form of a homogeneous equation suggests that it may be simplified by using a variable denoted by $'v'$, to represent the ratio of y to x. This

$$y = xv \quad \Rightarrow \quad \frac{dy}{dx} = F(v)$$

Let assume that v is a function of x, then

$$\frac{dy}{dx} = x\frac{dv}{dx} + v \quad or \quad y' = xv' + v$$

The most significant fact about this equation is that the variables x & v can always be **separated**, regardless of the form of the function F.

$$\frac{dx}{x} = \frac{dv}{F(v) - v}$$

Solving this equation and then replacing v by $\frac{y}{x}$ gives the solution of the original equation.

Example

Solve the differential equation $\qquad \frac{dy}{dx} = \frac{y^2 + 2xy}{x^2}$

<u>*Solution*</u>

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + 2\frac{y}{x} = v^2 + 2v$$

$$x\frac{dv}{dx} + v = v^2 + 2v \qquad \Rightarrow \quad x\frac{dv}{dx} = v^2 + v$$

$$xdv = v(v + 1)dx$$

$$\int \frac{dx}{x} = \int \frac{dv}{v(v + 1)}$$

$$\int \frac{dx}{x} = \int \left(\frac{1}{v} - \frac{1}{v + 1}\right)dv$$

$$\ln x + \ln C = \ln v - \ln(v + 1)$$

$$\ln(Cx) = \ln\frac{v}{v + 1}$$

$$Cx = \frac{v}{v + 1} = \frac{\frac{y}{x}}{\frac{y}{x} + 1} = \frac{y}{y + x} \quad \Rightarrow \quad Cxy + Cx^2 = y$$

$$Cx^2 = y - Cxy$$

$$\boxed{y = \frac{Cx^2}{1 - Cx}}$$

38

Example

Find the general solution $\quad y' = \dfrac{x^2 e^{y/x} + y^2}{xy}$

Solution

Let $y = xv \quad \Rightarrow \quad y' = v + xv'$

$$v + xv' = \frac{x^2 e^{xv/x} + (xv)^2}{x(xv)}$$

$$xv' = \frac{x^2 e^v + x^2 v^2}{x^2 v} - v$$

$$x\frac{dv}{dx} = \frac{e^v + v^2}{v} - v$$

$$x\frac{dv}{dx} = \frac{e^v}{v}$$

$$\int \frac{v}{e^v} dv = \int \frac{dx}{x}$$

$$-ve^{-v} - e^{-v} = \ln x + C$$

$$-e^{-v}(v + 1) = \ln x + C$$

$$-e^{-y/x}\left(\frac{y}{x} + 1\right) = \ln x + C$$

$$\underline{y + x = -xe^{y/x}(\ln x + C)}$$

Exercises   *Section* 1.4 – Exact Differential Equations

Solve the differential equation

1. $(2x + y)dx + (x - 6y)dy = 0$

2. $(2x + 3)dx + (2y - 2)dy = 0$

3. $(1 - y\sin x) + (\cos x)y' = 0$

4. $\dfrac{dy}{dx} = -\dfrac{ax + by}{bx + cy}$

5. $\dfrac{dy}{dx} = \dfrac{3x^2 + y}{3y^2 - x}$

6. $2xydx + (x^2 - 1)dy = 0$

7. $y' = \dfrac{x^2 + y^2}{2xy}$

8. $2xyy' = x^2 + 2y^2$

9. $xy' = y + 2\sqrt{xy}$

10. $xy^2 y' = x^3 + y^3$

11. $x^2 y' = xy + x^2 e^{y/x}$

12. $x^2 y' = xy + y^2$

13. $xyy' = x^2 + 3y^2$

14. $(x^2 - y^2)y' = 2xy$

15. $xyy' = y^2 + x\sqrt{4x^2 + y^2}$

16. $xy' = y + \sqrt{x^2 + y^2}$

17. $y^2 y' + 2xy^3 = 6x$

18. $x^2 y' + 2xy = 5y^4$

19. $2xy' + y^3 e^{-2x} = 2xy$

20. $y^2(xy' + y)(1 + x^4)^{1/2} = x$

21. $3y^2 y' + y^3 = e^{-x}$

22. $3xy^2 y' = 3x^4 + y^3$

23. $xe^y y' = 2(e^y + x^3 e^{2x})$

24. $(2x\sin y\cos y)y' = 4x^2 + \sin^2 y$

25. $(x + e^y)y' = xe^{-y} - 1$

26. $(x^2 + y^2)dx + (x^2 - xy)dy = 0$

27. $x\dfrac{dy}{dx} + y = x^2 y^2$

28. $(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0$

29. $(e^x \sin y - 2y\sin x)dx + (e^x \cos y + 2\cos x)dy = 0$

30. $\left(\dfrac{y}{x} + 6x\right)dx + (\ln x - 2)dy = 0, \qquad x > 0$

31. $(e^{2y} - y\cos x)dx + (2xe^{2y}\cos xy + 2y)dy = 0$

32. $\dfrac{xdx}{(x^2 + y^2)^{3/2}} + \dfrac{ydy}{(x^2 + y^2)^{3/2}} = 0$

33. $(2x - 1)dx + (3y + 7)dy = 0$

34. $(5x + 4y)dx + (4x - 8y^3)dy = 0$

35. $(\sin y - y\sin x)dx + (\cos x + x\cos y - y)dy = 0$

36. $(2xy^2 - 3)dx + (2x^2 y + 4)dy = 0$

37. $\left(1 + \ln x + \dfrac{y}{x}\right)dx - (1 - \ln x)dy = 0$

38. $(x - y^3 + y^2 \sin x)dx - (3xy^2 + 2y\cos x)dy = 0$

39. $(x^3 + y^3)dx + 3xy^2 dy = 0$

40. $(3x^2 y + e^y)dx + (x^3 + xe^y - 2y)dy = 0$

41. $xdy + (y - 2xe^x - 6x^2)dx = 0$

42. $\left(1 - \dfrac{3}{y} + x\right)dy + \left(y - \dfrac{3}{x} + 1\right)dx = 0$

43. $\left(x^2 y^3 - \dfrac{1}{1 + 9x^2}\right)\dfrac{dx}{dy} + x^3 y^2 = 0$

44. $(5y - 2x)y' - 2y = 0$

45. $(x - y)dx - xdy = 0$

40

46. $(x+y)dx + xdy = 0$

47. $\dfrac{dy}{dx} = -\dfrac{2xy^2+1}{2x^2y}$

48. $\left(1+e^x y + xe^x y\right)dx + \left(xe^x + 2\right)dy = 0$

49. $\left(2xy^3 + 1\right)dx + \left(3x^2y^2 - \dfrac{1}{y}\right)dy = 0$

50. $(2x+y)dx + (x-2y)dy = 0$

51. $e^x(y-x)dx + \left(1+e^x\right)dy = 0$

52. $\left(ye^{xy} - \dfrac{1}{y}\right)dx + \left(xe^{xy} + \dfrac{x}{y^2}\right)dy = 0$

53. $(\tan x - \sin x \sin y)dx + (\cos x \cos y)dy = 0$

54. $\left(2x^3 - xy^2 - 2y + 3\right)dx - \left(x^2y + 2x\right)dy = 0$

55. $(x+\sin y)dx + (x\cos y - 2y)dy = 0$

56. $\left(x + \dfrac{1}{\sqrt{y^2-x^2}}\right)dx + \left(1 - \dfrac{1}{y\sqrt{y^2-x^2}}\right)dy = 0$

57. $\left(2x + y^2 - \cos(x+y)\right)dx + \left(2xy - \cos(x+y) - e^y\right)dy = 0$

58. $\left(\dfrac{2}{\sqrt{1-x^2}} + y\cos(xy)\right)dx + \left(x\cos(xy) - y^{-1/3}\right)dy = 0$

59. $(2x + y\cos(xy))dx + (x\cos(xy) - 2y)dy = 0$

60. $\left(e^x \sin y - 3x^2\right)dx + \left(e^x \cos y + \dfrac{1}{3}y^{-2/3}\right)dy = 0$

61. $\left(2y\sin x \cos x - y + 2y^2 e^{xy^2}\right)dx = \left(x - \sin^2 x - 4xye^{xy^2}\right)dy$

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

62. $x^2y^3 + x\left(1+y^2\right)y' = 0, \qquad \mu(x,\,y) = \dfrac{1}{xy^3}$

63. $y^2 - xy + \left(x^2\right)y' = 0, \qquad \mu(x,\,y) = \dfrac{1}{xy^2}$

64. $x^2y^3 - y + x\left(1+x^2y^2\right)y' = 0, \qquad \mu(x,\,y) = \dfrac{1}{xy}$

65. $\left(\dfrac{\sin y}{y} - 2e^{-x}\sin x\right)dx + \left(\dfrac{\cos y + 2e^{-x}\cos x}{y}\right)dy = 0, \qquad \mu(x,\,y) = ye^x$

66. $(x+2)\sin ydx + x\cos ydy = 0, \qquad \mu(x,\,y) = xe^x$

67. $\left(x^2 + y^2 - x\right)dx - ydy = 0, \qquad \mu(x,\,y) = \dfrac{1}{x^2+y^2}$

68. $(2y-6x)dx + \left(3x - 4x^2y^{-1}\right)dy = 0, \qquad \mu(x,\,y) = xy^2$

Find the general solution of each homogenous equation

69. $\left(x^2 + y^2\right)dx - 2xydy = 0$

70. $(x+y)dx + (y-x)dy = 0$

71. $\dfrac{dy}{dx} = \dfrac{y\left(x^2 + y^2\right)}{xy^2 - 2x^3}$

Find an integrating factor and solve the given equation

72. $\left(3x^2 y + 2xy + y^3\right)dx + \left(x^2 + y^2\right)dy = 0$

73. $dx + \left(\dfrac{x}{y} - \sin y\right)dy = 0$

74. $e^x dx + \left(e^x \cot y + 2y \csc y\right)dy = 0$

75. $\left(3x + \dfrac{6}{y}\right)dx + \left(\dfrac{x^2}{y} + 3\dfrac{y}{x}\right)dy = 0$

76. $\left(x + 3x^3 \sin y\right)dx + \left(x^4 \cos y\right)dy = 0$

77. $\left(2x^2 + y\right)dx + \left(x^2 y - x\right)dy = 0$

78. $\left(3x^2 + y\right)dx + \left(x^2 y - x\right)dy = 0$

79. $\left(2y^2 + 2y + 4x^2\right)dx + (2xy + x)dy = 0$

80. $\left(x^4 - x + y\right)dx - xdy = 0$

81. $(2xy)dx + \left(y^2 - 3x^2\right)dy = 0$

Solve the given initial-value problem

82. $\dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y\left(1 - x^2\right)}$, $y(0) = 2$

83. $(x+y)^2 dx + \left(2xy + x^2 - 1\right)dy$, $y(1) = 1$

84. $\left(e^x + y\right)dx + \left(2 + x + ye^y\right)dy$, $y(0) = 1$

85. $(2x - y)dx + (2y - x)dy$, $y(1) = 3$

86. $\left(9x^2 + y - 1\right)dx - (4y - x)dy$, $y(1) = 0$

87. $\left(x + y^3\right)y' + y + x^3 = 0$, $y(0) = -2$

88. $y' = \left(3x^2 + 1\right)\left(y^2 + 1\right)$, $y(0) = 1$

89. $\left(y^3 + \cos t\right)y' = 2 + y \sin t$, $y(0) = -1$

90. $\left(y^3 - t^3\right)y' = 3t^2 y + 1$, $y(-2) = -1$

91. $\dfrac{dy}{dx} = (-2x + y)^2 - 7$, $y(0) = 0$

92. $(2y - x)y' - y + 2x = 0$, $y(1) = 0$

93. $\left(e^{2y} + t^2 y\right)y' + ty^2 + \cos t = 0$, $y\left(\dfrac{\pi}{2}\right) = 0$

94. $y' = -\dfrac{y\cos(ty) + 1}{t\cos(ty) + 2ye^{y^2}}$, $y(\pi) = 0$

95. $\left(2ty + \dfrac{1}{y}\right)y' + y^2 = 1$, $y(1) = 1$

96. $\left(ye^x + 1\right)dx + \left(e^x - 1\right)dy = 0$ $y(1) = 1$

97. $2xy^2 + 4 = 2\left(3 - x^2 y\right)y'$ $y(-1) = 8$

98. $y' + \dfrac{4}{x}y = x^3 y^2$ $y(2) = -1$

99. $y' = 5y + e^{-2x}y^{-2}$ $y(0) = 2$

100. $6y' - 2y = xy^4$ $y(0) = -2$

101. $y' + \dfrac{y}{x} - \sqrt{y} = 0$ $y(1) = 0$

102. $xyy' + 4x^2 + y^2 = 0$ $y(2) = -7$

103. $xy' = y(\ln x - \ln y)$ $y(1) = 4$

104. $y' - (4x - y + 1)^2 = 0$ $y(0) = 2$

42

105. $\left(e^{t+y}+2y\right)y'+\left(e^{t+y}+3t^2\right)=0, \quad y(0)=0$

106. $(4y+2x-5)dx+(6y+4x-1)dy, \quad y(-1)=2$

107. $\left(ye^{xy}-\dfrac{1}{y}\right)dx+\left(xe^{xy}+\dfrac{x}{y^2}\right)dy=0 \quad y(1)=1$

108. $(2y\ln t-t\sin y)y'+\dfrac{1}{t}y^2+\cos y=0, \quad y(2)=0$

109. $(\tan y-2)dx+\left(x\sec^2 y+\dfrac{1}{y}\right)dy=0 \quad y(0)=1$

110. $2xy-9x^2+\left(2y+x^2+1\right)\dfrac{dy}{dx}=0 \quad y(0)=-3$

111. $\dfrac{2t}{t^2+1}y-2t+\left(2-\ln\left(t^2+1\right)\right)\dfrac{dy}{dt}=0 \quad y(5)=0$

112. $3y^3e^{3xy}-1+\left(2ye^{3xy}+3xy^2e^{3xy}\right)y'=0 \quad y(0)=1$

113. $2xy\,dx+\left(1+x^2\right)dy=0 \; ; \quad y(2)=-5$

114. $\dfrac{dy}{dx}=-\dfrac{2x\cos y+3x^2 y}{x^3-x^2\sin y-y} \; ; \quad y(0)=2$

Find an integrating factor of the form $x^n y^m$ and solve the equation

115. $\left(2y^2-6xy\right)dx+\left(3xy-4x^2\right)dy=0$

116. $(12+5xy)dx+\left(6xy^{-1}+3x^2\right)dy=0$

117. $\left(3y+4xy^2\right)dx+\left(2x+3x^2 y\right)dy=0$

Find the general solution by using Bernoulli

118. $\dfrac{dy}{dx}-5y=-\dfrac{5}{2}xy^3$

119. $\dfrac{dy}{dx}+\dfrac{y}{x}=x^2 y^2$

120. $\dfrac{dy}{dx}-y=e^{2x}y^3$

121. $\dfrac{dy}{dx}+\dfrac{y}{x-2}=5(x-2)y^{1/2}$

122. $\dfrac{dy}{dx}+y=e^x y^{-2}$

123. $\dfrac{dy}{dx}+y^3 x+y=0$

Find the general solution by using homogeneous equations.

124. $\left(xy+y^2\right)dx-x^2 dy=0$

125. $\left(x^2+y^2\right)dx+2xy\,dy=0$

126. $\left(y^2-xy\right)dx+x^2 dy=0$

127. $\dfrac{dy}{d\theta}=\dfrac{\theta\sec\left(\dfrac{y}{\theta}\right)+y}{\theta}$

128. $\dfrac{dy}{dx}=\dfrac{y(\ln y-\ln x+1)}{x}$

43