Solution Section 1.3 – Predicates and Quantifiers

Exercise

Let $P(x)$ denote the statement "$x \le 4$". What are these truth values?

$a)\ P(0)$ $b)\ P(4)$ $c)\ P(6)$

Solution

a) True, since $0 \le 4$

b) True, since $4 \le 4$

c) False, since $7 \not\le 4$

Exercise

Let $P(x)$ be the statement "*the word* $\mathbf{x}$ *contains the letter* $\boldsymbol{a}$". What are these truth values?

$a)\ P(orange)$ $b)\ P(lemon)$ $c)\ P(true)$ $d)\ P(false)$

Solution

a) True, since there is an *a* in *orange*.

b) False, since there is no *a* in *lemon*.

c) False, since there is no *a* in *true*.

d) True, since there is an *a* in *false*.

Exercise

State the value of *x* after the statement **if** $P(x)$ **then** $x := 1$ is executed, where $P(x)$ is the statement "$x > 1$", if the value of *x* when the statement is reached is

$a)\ x = 0$ $b)\ x = 1$ $c)\ x = 2$

Solution

a) x is still equal to 0, since the condition is false.

b) x is still equal to 0, since the condition is false.

c) x is still equal to 1 at the end, since the condition is true, so the statement $x := 1$ is executed.

Exercise

Let $P(x)$ be the statement "***x*** *spends more than five hours every weekday in class,*" where the domain for x consists of all students. Express each of these quantifications in English.

$a)\ \exists x\ P(x)$ $\qquad b)\ \forall x\ P(x)$ $\qquad c)\ \exists x\ \neg P(x)$ $\qquad d)\ \forall x\ \neg P(x)$

Solution

a) There is a student who spends more than five hours every weekday in class.

b) Every student spends more than five hours every weekday in class.

c) There is a student who does not spend more than five hours every weekday in class.

d) No student spends more than five hours every weekday in class.

Exercise

Let $N(x)$ be the statement "***x*** *has visited North Dakota,*" where the domain consists of the students in your class. Express each of these quantifications in English.

$a)\ \exists x\ N(x)$ $\qquad b)\ \forall x\ N(x)$ $\qquad c)\ \neg\exists x\ N(x)$ $\qquad d)\ \exists x\ \neg N(x)$

$e)\ \neg\forall x\ N(x)$ $\qquad f)\ \forall x\ \neg N(x)$

Solution

a) Some student in the school has visited North Dakota.
Or, there exists a student in the school who has visited North Dakota

b) Every student in the school has visited North Dakota
Or, all students in the school have visited North Dakota

c) No student in the school has visited North Dakota.
Or, there does not exist a student in the school who has visited North Dakota

d) Some student in the school has not visited North Dakota.
Or, there exists a student in the school who has not visited North Dakota

e) It is not true that every student in the school has visited North Dakota
Or, not all students in the school have visited North Dakota

f) All students in the school has not visited North Dakota.
Or, no student has visited North Dakota

Exercise

Let $C(x)$ be the statement "***x** has a cat,*" let $D(x)$ be the statement "***x** has a dog,*", and let $F(x)$ be the statement "***x** has a ferret.*" Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

a) A student in your class has a cat, a dog, and a ferret.
b) All students in your class have a cat, a dog, or a ferret.
c) Some student in your class has a cat and a ferret, but not a dog.
d) No student in your class has a cat, a dog, and a ferret.
e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

Solution

a) $\exists x\left(C(x)\wedge D(x)\wedge F(x)\right)$

b) $\forall x\left(C(x)\vee D(x)\vee F(x)\right)$

c) $\exists x\left(C(x)\wedge F(x)\wedge \neg D(x)\right)$

d) $\neg\exists x\left(C(x)\wedge D(x)\wedge F(x)\right)$

e) $\left(\exists x C(x)\right)\wedge\left(\exists x D(x)\right)\wedge\left(\exists x F(x)\right)$

Exercise

Let $Q(x)$ be the statement "$x+1>2x$". If the domain consists of all integers, what are these truth values?

a) $Q(0)$ *b)* $Q(-1)$ *c)* $Q(1)$ *d)* $\exists x\ Q(x)$

e) $\forall x\ Q(x)$ *f)* $\exists x\ \neg Q(x)$ *g)* $\forall x\ \neg Q(x)$

Solution

a) Since $0+1>2\cdot 0$, that implies to $Q(0)$ is ***true***.

b) Since $(-1)+1>2\cdot(-1)$, that implies to $Q(-1)$ is ***true***.

c) Since $1+1 \not> 2\cdot 1$, that implies to $Q(1)$ is ***false***.

d) We showed that $Q(0)$ is *true*, therefore there is at least one x that makes $Q(x)$ *true*, so $\exists x\ Q(x)$ is ***true***.

e) We showed that $Q(1)$ is *false*, therefore there is at least one x that makes $Q(x)$ *false*, so $\forall x\ Q(x)$ is ***false***.

f) We showed that $Q(1)$ is *false*, therefore there is at least one x that makes $Q(x)$ *false*, so $\exists x\ \neg Q(x)$ is ***true***.

g) We showed that $Q(0)$ is *true*, therefore there is at least one *x* that makes $Q(x)$ *true*, so $\forall x\ \neg Q(x)$ is ***false***.

Exercise

Determine the truth value of each of these statements if the domain consists of all integers

a) $\forall n(n+1>n)$ *b*) $\exists n(2n=3n)$ *c*) $\exists n(n=-n)$ *d*) $\forall n(3n\le 4n)$

Solution

a) True, since adding 1 to a number makes it larger.
b) True, since $2\cdot 0=3\cdot 0$
c) True, since $0=-0$
d) True for all integers, $3n\le 4n \ \Rightarrow\ 3\le 4$

Exercise

Determine the truth value of each of these statements if the domain consists of all real numbers

a) $\exists x\left(x^3=-1\right)$ *b*) $\exists x\left(x^4<x^2\right)$ *c*) $\forall x\left((-x)^2=x^2\right)$ *d*) $\forall x(2x>x)$

Solution

a) Since $(-1)^3=-1$, the statement $\exists x\left(x^3=-1\right)$ is ***true***.

b) Since $\left(\frac{1}{2}\right)^4<\left(\frac{1}{2}\right)^2$, the statement $\exists x\left(x^4<x^2\right)$ is ***true***.

c) Since $(-x)^2=(-1)^2x^2=x^2$, the statement $\forall x\left((-x)^2=x^2\right)$ is ***true***.

d) Since $2(-1)\not> -1$, the statement $\forall x(2x>x)$ is ***false***.

Exercise

Suppose that the domain of the propositional function $P(x)$ consists of the integers 1, 2, 3, 4, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

a) $\exists x\ P(x)$ *b*) $\forall x\ P(x)$ *c*) $\neg\exists x\ P(x)$ *d*) $\neg\forall x\ P(x)$

e) $\forall x\ \left((x\ne 3)\to P(x)\right)\ \vee\ \exists x\ \neg P(x)$

Solution

a) The statement to be true, so either $P(1)$ is true or $P(2)$ is true or $P(3)$ is true or $P(4)$ is true or $P(5)$ is true. Thus, $P(1)\vee P(2)\vee P(3)\vee P(4)\vee P(5)$

b) $P(1)\wedge P(2)\wedge P(3)\wedge P(4)\wedge P(5)$

c) $\neg\big(P(1)\vee P(2)\vee P(3)\vee P(4)\vee P(5)\big)$

d) $\neg\big(P(1)\wedge P(2)\wedge P(3)\wedge P(4)\wedge P(5)\big)$

e) $\big((1\neq 3)\to P(1)\wedge\big((2\neq 3)\to P(2)\big)\wedge\big((3\neq 3)\to P(3)\big)\wedge\big((4\neq 3)\to P(4)\big)\wedge\big((5\neq 3)\to P(5)\big)\big)$

$\vee\big(\neg\big(P(1)\vee\neg P(2)\vee\neg P(3)\vee\neg P(4)\vee\neg P(5)\big)\big)$

Since the hypothesis $x\neq 3$ is false when $x=3$ and true when x is anything other than 3, we have

$\big(P(1)\wedge P(2)\wedge P(3)\wedge P(4)\wedge P(5)\big)\vee\big(\neg P(1)\vee\neg P(2)\vee\neg P(3)\vee\neg P(4)\vee\neg P(5)\big)$

This statement is always true, since the first part is not true, then the second part must be true.

Exercise

For each of these statements find a domain for which the statement is true and a domain for which the statement is false.

a) Everyone is studying discrete mathematics.
b) Everyone is older than 21 years.
c) Every two people have the same mother.
d) No Two different people have the same grandmother.

Solution

Let $A(x)$ *be "x everyone at the school"*

a) Let $B(x)$ *be "x is studying discrete mathematics"*. Then we have $\forall x\ B(x)$, or

$\forall x\ \big(A(x)\to B(x)\big)$

b) Let $C(x)$ *be "x is older than 21 years"*. Then we have $\forall x\ C(x)$, or $\forall x\ \big(A(x)\to C(x)\big)$

c) Let $D(x)$ *be "x has the same mother,"* $E(x)$ *two people*. $\forall x\ \big(E(x)\to D(x)\big)$

d) Let $D(x)$ *be "x has the same grandmother,"* $E(x)$ *two people*. $\neg\forall x\ \big(E(x)\to D(x)\big)$

Exercise

Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.
b) Not everyone is perfect.
c) All your friends are perfect.
d) At least one of your friends is perfect.
e) Everyone is your friend and is perfect.
f) Not everybody is your friend or someone is not perfect.

Solution

b) $\forall x\ \neg X(x)$

c) $\forall x\ \left(Y(x) \to X(x)\right)$

d) $\exists x\ \left(Y(x) \land X(x)\right)$

e) $\forall x\ \left(Y(x) \land X(x)\right)$

f) This is a disjunction. The expression is $\left(\forall x\ \neg Y(x)\right) \lor \left(\exists x\ \neg X(x)\right)$

Exercise

Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) Something is not in the correct place.
b) All tools are in the correct place and are in excellent condition.
c) Everything is in the correct place and in excellent condition.
d) Nothing is in the correct place and is in excellent condition.
e) One of your tools is not in the correct place, but it is in excellent condition.

Solution

Let $A(x)$ *be "x in the correct place"*; let $B(x)$ *be "x is in excellent condition"*; let $C(x)$ *be "x is a tool"*

a) $\exists x\ \neg A(x)$ *There exists something is not in the correct place.*

b) $\forall x\left(C(x) \to \left(A(x) \land B(x)\right)\right)$

c) $\forall x\left(A(x) \land B(x)\right)$

d) $\forall x \neg\left(A(x) \land B(x)\right)$

e) $\exists x\left(C(x) \land \neg A(x) \land B(x)\right)$