Section 1.3 – The Algebra of Matrices

Matrices

$$\begin{array}{r c} & \begin{array}{ccc} & \textit{Column} & \\ C_1 & C_2 & C_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{r} Row\ 1 \to R_1 \\ Row\ 2 \to R_2 \\ Row\ 3 \to R_3 \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

This is called Matrix (*Matrices*)

Each number in the array is an ***element*** or ***entry***

The matrix is said to be of order m *x* n

m: numbers of rows,

n: number of columns

When $m = n$, then matrix is said to be ***square***.

Given the system equations

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Write into an ***augmented matrix*** form

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 31 \\ 1 & 1 & 2 & 19 \\ 1 & 3 & 2 & 25 \end{array}\right]$$

The Matrix: $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}$ is called the ***coefficient matrix*** of the system.

The matrix A above has 3 rows and 3 columns, therefore the order of the matrix A is (3 *x* 3)

$$A = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Equality of Matrices

Definition of Equality of Matrices

Two matrices ***A*** and ***B*** are equal if and only if they have the same order (size) $m \times n$ and if each pair corresponding elements is equal

$a_{ij} = b_{ij}$ for $i = 1, 2, \ldots, m$ ***and*** $j = 1, 2, \ldots, n$

Example

Find the values of the variables for which each statement is true, if possible.

a) $\begin{bmatrix} 2 & 1 \\ p & q \end{bmatrix} = \begin{bmatrix} x & y \\ -1 & 0 \end{bmatrix}$

$x = 2,\ y = 1,\ \ p = -1,\ \ q = 0$

b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$

can't be true

c) $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

$\Rightarrow \begin{bmatrix} w = 9 & x = 17 \\ 8 = y & -12 = z \end{bmatrix}$

Addition and *Subtraction* of Matrices

Definition

If $A=\left[a_{ij}\right]$ and $B=\left[b_{ij}\right]$ are $m\times n$ matrices, their sum $A+B$, is the $m\times n$ matrix obtained by adding the corresponding entries; that is

$$\left[a_{ij}\right]+\left[b_{ij}\right]=\left[a_{ij}+b_{ij}\right]$$

Matrices can be added if their shapes are the same, meaning have the same ***order***.

$$\begin{bmatrix}1&2\\3&4\\0&0\end{bmatrix}+\begin{bmatrix}2&2\\4&4\\9&9\end{bmatrix}=\begin{bmatrix}1+2&2+2\\3+4&4+4\\0+9&0+9\end{bmatrix}$$
$$=\begin{bmatrix}3&4\\7&8\\9&9\end{bmatrix}$$

Scalar Multiplication Matrices

Definition

If k is a scalar and $A=\left[a_{ij}\right]$ is an $m\times n$ matrices, then scalar product $\boldsymbol{kA}$ is the $m\times n$ matrix obtained by multiplying each entry of A by k; that is

$$k\left[a_{ij}\right]=\left[ka_{ij}\right]$$

$$kA=k\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}$$
$$=\begin{bmatrix}ka_{11}&ka_{12}\\ka_{21}&ka_{22}\end{bmatrix}$$

Example

$$2\begin{bmatrix}1&2\\3&4\\0&0\end{bmatrix}=\begin{bmatrix}(2)1&(2)2\\(2)3&(2)4\\(2)0&(2)0\end{bmatrix}$$
$$=\begin{bmatrix}2&4\\6&8\\0&0\end{bmatrix}$$

Definition

If $A_1, A_2, \ldots, A_n$ are matrices of the same size, and if $c_1, c_2, \ldots, c_n$ are scalars, then expression of the form

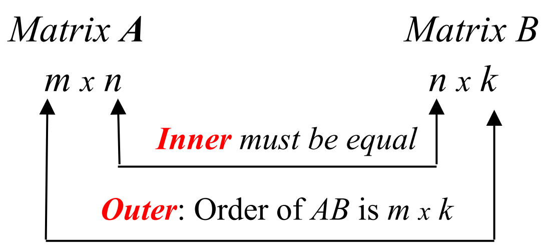
$$c_1A_1 + c_2A_2 + \ldots + c_nA_n$$

Is called a ***linear combination*** of $A_1, A_2, \ldots, A_n$ with *coefficients* $c_1, c_2, \ldots, c_n$.

Matrix Multiplication

Product of Two Matrices

Let ***A*** be an *m x n* matrix and let ***B*** be an *n x k* matrix. To find the element in the i^{th} row and j^{th} column of the product matrix ***AB***, multiply each element in the i^{th} row of ***A*** by the corresponding element in the j^{th} column of ***B***, and then add these products. The product matrix ***AB*** is an *m x k* matrix.



- ✓ To multiply ***AB*** or dot product, if ***A*** has ***n*** columns, ***B*** must have ***n*** rows.
- ✓ Squares matrices can be multiplied if and only if (***iff***) they have the same size.
- ✓ The entry in row *i* and column *j* of ***AB*** is $\boxed{(row\ i\ of\ A).(col\ j\ of\ B)}$

  The result: $\sum a_{ik}b_{kj}$

$$\underset{4\ by\ 5}{\begin{bmatrix} * & * & & & \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{i5} \\ * & & & & \\ * & & & & \end{bmatrix}} \underset{5\ by\ 6}{\begin{bmatrix} * & * & b_{1j} & * & * & * \\ & & b_{2j} & & & \\ & & \vdots & & & \\ & & \vdots & & & \\ & & b_{5j} & & & \end{bmatrix}} = \underset{4\ by\ 6}{\begin{bmatrix} & & * & & & \\ * & * & (AB)_{ij} & * & * & * \\ & & * & & & \\ & & * & & & \end{bmatrix}}$$

$$\boldsymbol{AB} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$2x2 \qquad 2x2 \quad \rightarrow \quad 2x2$$

$$a_{11} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & \_ \\ \_ & \_ \end{bmatrix}$$

$$a_{12} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & af+bh \\ \_ & \_ \end{bmatrix}$$

$$a_{21} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ ce+dg & \_ \end{bmatrix}$$

$$a_{22} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ \_ & cf+dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Example

Find: $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}$

Solution

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} 1(5)+1(1) & 1(6)+1(0) \\ 2(5)-1(1) & 2(6)-1(0) \end{bmatrix} \\ &= \begin{bmatrix} 6 & 6 \\ 9 & 12 \end{bmatrix} \end{aligned}$$

Special Case

When A is a square matrix, then

$$A \quad \textit{times} \quad A = A^2 \quad \textit{times} \quad A \quad \underline{= A^3}$$

$$A^p = AA\cdots A \qquad (p \text{ factors})$$

$$\left(A^p\right)\left(A^q\right) = A^{p+q}$$

$$\left(A^p\right)^q = A^{pq}$$

Block Multiplication

If the cuts between columns of ***A*** match the cuts between rows of ***B***, then the block multiplication of ***AB*** allowed.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \left[\begin{array}{c|c} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{21} + a_{12}b_{22} \\ \hline a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{21} + a_{22}b_{22} \end{array}\right]$$

Important special case

$$\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix}$$

Matrix Form of the Equations

The coefficient matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}$

The equivalent matrix equation is in the form $AX = b$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Multiplication by ***rows*** $\qquad AX = \begin{bmatrix} (row\ 1).X \\ (row\ 2).X \\ (row\ 3).X \end{bmatrix}$

Multiplication by ***columns*** $\qquad AX = x\ (column\ 1) + y\ (column\ 2) + z\ (column\ 3)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Identity Matrix

The identity matrix is given by the form: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \boxed{Ix = x}$

Properties of Matrix

Addition and Scalar Multiplication

$A+B=B+A$	*Commutative Property of Addition*
$A+(B+C)=(A+B)+C$	*Associative Property of Addition*
$(kl)A=k(lA)$	*Associative Property of Scalar Multiplication*
$k(A+B)=kA+kB$	*Distributive Property*
$k(A-B)=kA-kB$	*Distributive Property*
$(k+l)A=kA+lA$	*Distributive Property*
$(k-l)A=kA-lA$	*Distributive Property*
$A+0=0+A=A$	*Additive Identity Property*
$A+(-A)=(-A)+A=0$	*Additive Inverse Property*
$k(AB)=kA(B)=A(kB)$	

Multiplication

$AB \neq BA$	*Commutative "**law**" is usually broken*
$A(BC)=(AB)C$	*Associative Property of Multiplication* (***Parentheses not needed***)
$A(B+C)=AB+AC$	*Distributive Property*
$(B+C)A=BA+CA$	*Distributive Property*
$A(B-C)=AB-AC$	*Distributive Property*
$(B-C)A=BA-CA$	*Distributive Property*

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Consider the three vectors $\boldsymbol{u}, \quad \boldsymbol{v}, \quad and\ \boldsymbol{w}$:

$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The linear combinations in three-dimensional space are $cu + dv + ew$

Combination
$$c\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}$$

Combine the three vectors $u,\ v,\ and\ w$ into on matrix A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Multiplies the matrix A by a vector x, where $c,\ d,\ e$ are the component of a vector x.

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}$$

We can rewrite the form, matrix A times the vector x, as the combination $cu + dv + ew$

$$Ax = \begin{bmatrix} u & v & w \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$$

Write the matrix in the form $Ax = b$

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b$$

Where the x is the input and b is the output.

Cyclic Difference

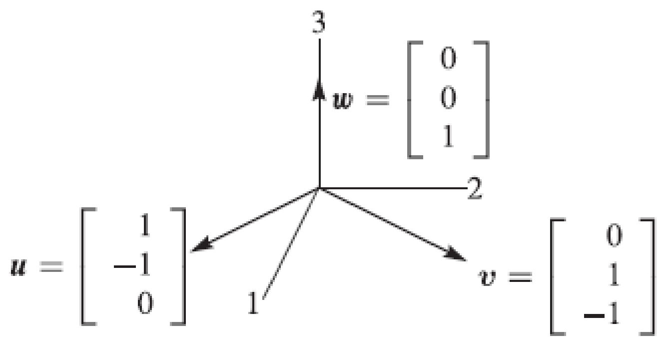
The linear combinations of three vectors $u,\ v,\ and\ w^*$ lead to a cyclic difference matrix C and is given by:

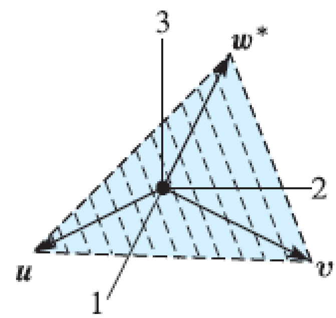
$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad w^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = b$$

The matrix C is not triangular. It is not easy to find the solution to $Cx = b$, because either we are going to have ***infinitely many solution*** or ***no solution***.

Let looks at these problems geometrically.





Exercises *Section* 1.3 – The Algebra of Matrices

1. For the matrices: $A=\begin{bmatrix} p & 0 \\ q & r \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, when does $AB = BA$

(2 – 8) Find values for the variables so that the matrices are equal.

2. $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix}=\begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

3. $\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix}=\begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$

4. $\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix}=\begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$

5. $\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix}+\begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix}=\begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$

6. $\begin{bmatrix} a+11 & 12z+1 & 5m \\ 11k & 3 & 1 \end{bmatrix}+\begin{bmatrix} 9a & 9z & 4m \\ 12k & 5 & 3 \end{bmatrix}=\begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$

7. $\begin{bmatrix} x+2 & 3y+1 & 5z \\ 8w & 2 & 3 \end{bmatrix}+\begin{bmatrix} 3x & 2y & 5z \\ 2w & 5 & -5 \end{bmatrix}=\begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$

8. $\begin{bmatrix} 2x-3 & y-2 & 2z+1 \\ 5 & 2w & 7 \end{bmatrix}+\begin{bmatrix} 3x-3 & y+2 & z-1 \\ -5 & 5w+1 & 3 \end{bmatrix}=\begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$

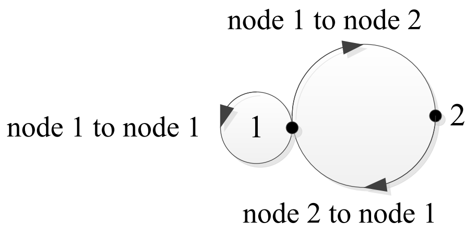
9. Find a combination $x_1 w_1 + x_2 w_2 + x_3 w_3$ that gives the zero vector:

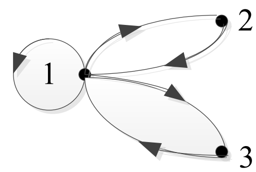
$$w_1=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \quad w_2=\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}; \quad w_3=\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are independent or dependent?
The vectors lie in a ______.
The matrix W with those columns is not invertible.

10. The very last words say that the 5 by 5 centered difference matrix is not invertible, Write down the 5 equations $Cx = b$. Find a combination of left sides that gives zero. What combination of b_1, b_2, b_3, b_4, b_5 must be zero?

11. A direct graph starts with n nodes. There are n^2 possible edges, each edge leaves one of the n nodes and enters one of the n nodes (possibly itself). The n by n adjacency matrix has $a_{ij} = 1$ when edge leaves node i and enter node j; if no edge then $a_{ij} = 0$. Here are directed graphs and their adjacency matrices:





$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The i, j entry of A^2 is $a_{i1}a_{1j} + \ldots + a_{in}a_{nj}$.

Why does that sum count the two-step paths from i to any node to j?

The i, j entry of A^k counts k-steps paths:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{matrix} \textit{counts the paths} \\ \textit{with two edges} \end{matrix} \quad \begin{bmatrix} 1 \textit{ to } 2 \textit{ to } 1, 1 \textit{ to } 1 \textit{ to } 1 & 1 \textit{ to } 1 \textit{ to } 2 \\ 2 \textit{ to } 1 \textit{ to } 1 & 2 \textit{ to } 1 \textit{ to } 2 \end{bmatrix}$$

List all 3-step paths between each pair of nodes and compare with A^3. When A^k has ***no zeros***, that number k is the diameter of the graph – the number of edges needed to connect the most pair of nodes. What is the diameter of the second graph?

12. A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

a) *AB*   *b*) *BA*   *c*) *ABD*   *d*) *DBA*

e) *ABC*   *f*) *ABCD*   *g*) *A*(*B* + *C*)

13. What rows or columns or matrices do you multiply to find.

a) The third column of *AB*?
b) The second column of *AB*?
c) The first row of *AB*?
d) The second row of *AB*?
e) The entry in row 3, column 4 of *AB*?
f) The entry in row 2, column 3 of *AB*?

14. Add AB to AC and compare with $A(B+C)$:

$$A=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix} \quad and \quad B=\begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix} \quad and \quad C=\begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix}$$

15. True or False

a) If A^2 is defined then A is necessarily square.

b) If AB and BA are defined then A and B are squares.

c) If AB and BA are defined then AB and BA are squares.

d) If $AB=B$, then $A=I$

16. *a*) Find a nonzero matrix A such that $A^2=0$

b) Find a matrix that has $A^2\neq 0$ but $A^3=0$

17. Suppose you solve $Ax=b$ for three special right sides b:

$$Ax_1=\begin{bmatrix}1\\0\\0\end{bmatrix} \quad Ax_2=\begin{bmatrix}0\\1\\0\end{bmatrix} \quad Ax_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

If the three solutions x_1, x_2, x_3 are the columns of a matrix *X*, what is A times *X*?

18. Show that $(A+B)^2$ is different from $A^2+2AB+B^2$, when

$$A=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix} \quad and \quad B=\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}$$

Write down the correct rule for $(A+B)(A+B)=A^2+$ __________ $+B^2$

(19 – 22) Find the product of the 2 matrices by rows or by columns:

19. $\begin{bmatrix}2 & 3\\ 5 & 1\end{bmatrix}\begin{bmatrix}4\\ 2\end{bmatrix}$

20. $\begin{bmatrix}3 & 6\\ 6 & 12\end{bmatrix}\begin{bmatrix}2\\ -1\end{bmatrix}$

21. $\begin{bmatrix}1 & 2 & 4\\ 2 & 0 & 1\end{bmatrix}\begin{bmatrix}3\\ 1\\ 1\end{bmatrix}$

22. $\begin{bmatrix}1 & 2 & 4\\ -2 & 3 & 1\\ -4 & 1 & 2\end{bmatrix}\begin{bmatrix}2\\ 2\\ 3\end{bmatrix}$

23. Given $A=\begin{bmatrix}4 & 1 & 3\\ 3 & -1 & -2\\ 0 & 0 & 4\end{bmatrix}$ $\quad B=\begin{bmatrix}-3 & -2 & -3\\ -1 & 0 & 0\\ 8 & -2 & -4\end{bmatrix}$ $\quad$ Find $A+B$, $2A$, *and* $-B$

(24 – 37) Find AB and BA, if possible

24. $A=\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ $B=\begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}$

25. $A=\begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$ $B=\begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}$

26. $A=\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$ $B=\begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}$

27. $A=\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}$ $B=\begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}$

28. $A=\begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}$ $B=\begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$

29. $A=\begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}$ $B=\begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$

30. $A=\begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}$ $B=\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$

31. $A=\begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix}$ $B=\begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix}$

32. $A=\begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix}$ $B=\begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

33. $A=\begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix}$ $B=\begin{bmatrix} 4 & -2 \\ -2 & 0 \\ 9 & 1 \end{bmatrix}$

34. $A=\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$ $B=\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$

35. $A=\begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ $B=\begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$

36. $A=\begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix}$ $B=\begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

37. $A=\begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}$ $B=\begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}$

38. Consider the matrices

$$A=\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad B=\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \quad C=\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \quad D=\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \quad E=\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Compute the following (where possible):

a) $D+E$ *b)* $D-E$ *c)* $5A$ *d)* $-7C$ *e)* $2B-C$ *f)* $-3(D+2E)$

39. Consider the matrices

$$A=\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad B=\begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix} \quad C=\begin{bmatrix} -3 & -1 \\ 2 & 1 \\ 4 & 3 \end{bmatrix} \quad D=\begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$$

Compute the following (where possible):

a) $A+B$ *b)* $A+C$ *c)* AB *d)* BA *e)* CD *f)* DC

g) BD *h)* DB *i)* A^2 *j)* B^2 *k)* D^2

40. Let $B=\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$, show that $B^4=\begin{pmatrix} a^4 & 0 \\ a^3+a^2b+ab^2+b^3 & b^4 \end{pmatrix}$

41. Let $B=\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$, show that $B^n=\begin{pmatrix} a^n & 0 \\ \sum_{k=0}^{n-1} a^{n-1-k}b^k & b^n \end{pmatrix}$

42. Let $A=\begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix}$. Prove that $A^n=\begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$ if $n\geq 1$

43. Let $A=\begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix}$. Prove that $A^n=\begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix}$ if $n\geq 1$

44. The following system of recurrence relations holds for all $n\geq 0$

$$\begin{cases} x_{n+1}=7x_n+4y_n \\ y_{n+1}=-9x_n-5y_n \end{cases}$$

Solve the system for x_n and y_n in terms of x_0 and y_0

45. If $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, prove that $A^2-(a+d)A+(ad-bc)I_{2\times 2}=0$

46. If $A=\begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}$, use the fact $A^2=4A-3I$ and mathematical induction, to prove that

$$A^n=\frac{3^n-1}{2}A+\frac{3-3^n}{2}I \quad if \quad n\geq 1$$

47. A sequence of numbers $x_1, x_2, \ldots, x_n, \ldots$ satisfies the recurrence relation $x_{n+1}=ax_n+bx_{n-1}$ for $n\geq 1$, where *a* and *b* are constants. Prove that

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}=A\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

Where $A=\begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$ and hence express $\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$ in terms of $\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$.

If $a=4$ and $b=-3$, use the previous question to find a formula for x_n in terms x_1 and x_0