Solution   Section 2.6 – Tangent Planes and Linear Approximation

Exercise

Find the tangent plane and normal line of the surface $x^2+y^2+z^2=3$ at the point $P_0\,(1,\ 1,\ 1)$

Solution

$$f(x,y,z)=x^2+y^2+z^2$$

$$f_x=2x,\quad f_y=2y,\quad f_z=2z$$

$$\nabla f=2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$\nabla f\,(1,\ 1,\ 1)=2\hat{i}+2\hat{j}+2\hat{k}$$

Tangent Line: $f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)+f_z(P_0)(z-z_0)=0$

$$2(x-1)+2(y-1)+2(z-1)=0$$

$$2x+2y+2z=6$$

$$\boxed{x+y+z=3}$$

Normal Line: $x=x_0+f_x(P_0)t,\quad y=y_0+f_y(P_0)t,\quad z=z_0+f_z(P_0)t$

$$\boxed{x=1+2t,\quad y=1+2t,\quad z=1+2t}$$

Exercise

Find the tangent plane and normal line of the surface $x^2+2xy-y^2+z^2=7$ at the point $P_0\,(1,\ -1,\ 3)$

Solution

$$f(x,y,z)=x^2+2xy-y^2+z^2$$

$$\rightarrow\quad f_x=2x+2y,\quad f_y=2x-2y,\quad f_z=2z$$

$$\nabla f=(2x+2y)\hat{i}+(2x-2y)\hat{j}+2z\hat{k}$$

$$\nabla f\,(1,\ -1,\ 3)=4\hat{j}+6\hat{k}$$

Tangent Line: $f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)+f_z(P_0)(z-z_0)=0$

$$0(x-1)+4(y+1)+6(z-3)=0$$

$$4y+6z=14$$

$$\boxed{2y+3z=7}$$

Normal Line: $x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$

$$x = 1, \quad y = -1 + 4t, \quad z = 3 + 6t$$

Exercise

Find the tangent plane and normal line of the surface $\cos \pi x - x^2 y + e^{xz} + yz = 4$ at the point $P_0(0,\ 1,\ 2)$

Solution

$$f(x, y, z) = \cos \pi x - x^2 y + e^{xz} + yz$$

$$\rightarrow \quad f_x = -\pi \sin \pi x - 2xy + ze^{xz}, \quad f_y = -x^2 + z, \quad f_z = xe^{xz} + y$$

$$\nabla f = \left(-\pi \sin \pi x - 2xy + ze^{xz}\right)\hat{i} + \left(z - x^2\right)\hat{j} + \left(xe^{xz} + y\right)\hat{k}$$

$$\nabla f(0,\ 1,\ 2) = 2\hat{i} + 2\hat{j} + \hat{k}$$

Tangent Line: $f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$

$$2(x - 0) + 2(y - 1) + (z - 2) = 0$$

$$2x + 2y + z - 4 = 0$$

Normal Line: $x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$

$$x = 2t, \quad y = 1 + 2t, \quad z = 2 + t$$

Exercise

Find the tangent plane and normal line of the surface $x^2 - xy - y^2 - z = 0$ at the point $P_0(1,\ 1,\ -1)$

Solution

$$f(x, y, z) = x^2 - xy - y^2 - z$$

$$\rightarrow \quad f_x = 2x - y, \quad f_y = -x - 2y, \quad f_z = -1$$

$$\nabla f = (2x - y)\hat{i} - (x + 2y)\hat{j} - \hat{k}$$

$$\nabla f(1,\ 1,\ -1) = \hat{i} - 3\hat{j} - \hat{k}$$

Tangent Line: $f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$

$$(x - 1) - 3(y - 1) - (z + 1) = 0$$

$$\boxed{x-3y-z+1=0}$$

Normal Line: $x = x_0 + f_x\left(P_0\right)t,\quad y = y_0 + f_y\left(P_0\right)t,\quad z = z_0 + f_z\left(P_0\right)t$

$$\boxed{x=1+t,\quad y=1-3t,\quad z=-1-t}$$

Exercise

Find the tangent plane and normal line of the surface $x^2+y^2-2xy-x+3y-z=-4$ at the point $P_0\left(2,\ -3,\ 18\right)$

Solution

$$f(x,y,z)=x^2+y^2-2xy-x+3y-z$$

$$\rightarrow\quad f_x=2x-2y-1,\quad f_y=2y-2x+3,\quad f_z=-1$$

$$\nabla f=(2x-2y-1)\hat{i}-(2y-2x+3)\hat{j}-\hat{k}$$

$$\nabla f(2,\ -3,\ 18)=9\hat{i}-7\hat{j}-\hat{k}$$

Tangent Line: $f_x\left(P_0\right)\left(x-x_0\right)+f_y\left(P_0\right)\left(y-y_0\right)+f_z\left(P_0\right)\left(z-z_0\right)=0$

$$9(x-2)-7(y+3)-(z-18)=0$$

$$\boxed{9x-7y-z=21}$$

Normal Line: $x = x_0 + f_x\left(P_0\right)t,\quad y = y_0 + f_y\left(P_0\right)t,\quad z = z_0 + f_z\left(P_0\right)t$

$$\boxed{x=2+9t,\quad y=-3-7t,\quad z=18-t}$$

Exercise

Find an equation for the plane that is tangent to the surface $z=\ln\left(x^2+y^2\right)$ at the point $(1,\ 0,\ 0)$

Solution

$$z=f(x,y)=\ln\left(x^2+y^2\right)$$

$$f_x=\frac{2x}{x^2+y^2}\quad\rightarrow f_x(1,0)=2$$

$$f_y=\frac{2x}{x^2+y^2}\quad\rightarrow f_y(1,0)=0$$

Tangent Line: $2(x-1)-(y-0)-z=0$ $\qquad f_x\left(P_0\right)\left(x-x_0\right)+f_y\left(P_0\right)\left(y-y_0\right)-\left(z-z_0\right)=0$

$$\boxed{2x-z-2=0}$$

Exercise

Find an equation for the plane that is tangent to the surface $z = e^{-x^2 - y^2}$ at the point $(0,\ 0,\ 1)$

Solution

$$z = f(x, y) = e^{-x^2 - y^2}$$

$$f_x = -2xe^{-x^2 - y^2} \quad \rightarrow f_x(0,0) = 0$$

$$f_y = -2ye^{-x^2 - y^2} \quad \rightarrow f_y(0,0) = 0$$

Tangent Line: $-(z-1) = 0$ $\qquad f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) - (z - z_0) = 0$

$$\boxed{z = 1}$$

Exercise

Find an equation for the plane that is tangent to the surface $z = \sqrt{y - x}$ at the point $(1,\ 2,\ 1)$

Solution

$$z = f(x, y) = \sqrt{y - x}$$

$$f_x = -\frac{1}{2}(y - x)^{-1/2} \quad \rightarrow f_x(1,2) = -\frac{1}{2}$$

$$f_y = \frac{1}{2}(y - x)^{-1/2} \quad \rightarrow f_y(1,2) = \frac{1}{2}$$

Tangent Line: $-\frac{1}{2}(x-1) + \frac{1}{2}(y-2) - (z-1) = 0$

$$-\frac{1}{2}x + \frac{1}{2}y - z + \frac{1}{2} = 0$$

$$\boxed{x - y + 2z - 1 = 0}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$z = 2x^2 + y^2; \quad (1,\ 1,\ 3) \ \text{and} \ (0,\ 2,\ 4)$$

Solution

$$f(x, y, z) = 2x^2 + y^2 - z$$

$$\nabla f = 4x\hat{i} + 2y\hat{j} - \hat{k} \qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$\nabla f(1,\ 1,\ 3) = 4\hat{i} + 2\hat{j} - \hat{k}$$

The equation of the tangent plane:

$$4(x-1)+2(y-1)-(z-3)=0$$

$$\boxed{4x+2y-z=3}$$

$$\nabla f(0,\ 2,\ 3)=4\hat{j}-\hat{k}$$

The equation of the tangent plane:

$$4(y-2)-(z-4)=0$$

$$\boxed{4y-z=4}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$x^2+\frac{1}{4}y^2-\frac{1}{9}z^2=1; \quad (0,\ 2,\ 0)\ \ and\ \ \left(1,\ 1,\ \frac{3}{2}\right)$$

Solution

$$f(x,y,z)=x^2+\frac{1}{4}y^2-\frac{1}{9}z^2-1$$

$$\nabla f=2x\hat{i}+\frac{1}{2}y\hat{j}-\frac{2}{9}\hat{k} \qquad\qquad \nabla f=f_x\hat{i}+f_y\hat{j}+f_z\hat{k}$$

$$\nabla f(0,\ 2,\ 0)=\hat{j}$$

The equation of the tangent plane:

$$y-2=0$$

$$\boxed{y=2}$$

$$\nabla f\left(1,\ 1,\ \frac{3}{2}\right)=2\hat{i}+\frac{1}{2}\hat{j}-\frac{1}{3}\hat{k}$$

The equation of the tangent plane:

$$2(x-1)+\frac{1}{2}(y-1)-\frac{1}{3}\left(z-\frac{3}{2}\right)=0$$

$$2x-2+\frac{1}{2}y-\frac{1}{2}-\frac{1}{3}z+\frac{1}{2}=0$$

$$2x+\frac{1}{2}y-\frac{1}{3}z=2$$

$$\boxed{12x+3y-2z=12}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$xy\sin z-1=0; \quad \left(1,\ 2,\ \frac{\pi}{6}\right)\ \ and\ \ \left(-2,\ -1,\ \frac{5\pi}{6}\right)$$

Solution

$$f(x, y, z) = xy\sin z - 1$$

$$\nabla f = y\sin z\hat{i} + x\sin z\hat{j} + xy\cos z\hat{k} \qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$\nabla f\left(1,\ 2,\ \frac{\pi}{6}\right) = 2\sin\frac{\pi}{6}\hat{i} + \sin\frac{\pi}{6}\hat{j} + 2\cos\frac{\pi}{6}\hat{k}$$

$$= \hat{i} + \frac{1}{2}\hat{j} + \sqrt{3}\hat{k}$$

The equation of the tangent plane:

$$(x-1) + \frac{1}{2}(y-2) + \sqrt{3}\left(z - \frac{\pi}{6}\right) = 0$$

$$x + \frac{1}{2}y + \sqrt{3}z = 2 + \frac{\pi}{6}\sqrt{3}$$

$$\boxed{6x + 3y + 6\sqrt{3}z = 12 + \pi\sqrt{3}}$$

$$\nabla f\left(-2,\ -1,\ \frac{5\pi}{6}\right) = -2\sin\frac{5\pi}{6}\hat{i} - \sin\frac{5\pi}{6}\hat{j} + 2\cos\frac{5\pi}{6}\hat{k}$$

$$= -\frac{1}{2}\hat{i} - \hat{j} - \sqrt{3}\hat{k}$$

The equation of the tangent plane:

$$-\frac{1}{2}(x+2) - (y+1) - \sqrt{3}\left(z - \frac{5\pi}{6}\right) = 0$$

$$-\frac{1}{2}x + -y - \sqrt{3}z = 2 - \frac{5\pi}{6}\sqrt{3}$$

$$\boxed{3x + 6y + 6\sqrt{3}z = 5\pi\sqrt{3} - 12}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$yze^{xz} - 8 = 0; \quad (0,\ 2,\ 4) \ \textit{and} \ (0,\ -8,\ -1)$$

Solution

$$f(x, y, z) = yze^{xz} - 8$$

$$\nabla f = yz^2e^{xz}\hat{i} + ze^{xz}\hat{j} + (y + xyz)e^{xz}\hat{k} \qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$\nabla f(0,\ 2,\ 4) = 32\hat{i} + 4\hat{j} + 2\hat{k}$$

The equation of the tangent plane:

$$32(x-0) + 4(y-2) + 2(z-4) = 0$$

$$\boxed{32x + 4y + 2z = 16}$$

$$\nabla f(0,\ -8,\ -1) = -8\hat{i} - \hat{j} - 8\hat{k}$$

The equation of the tangent plane:

$$-8(x-0)-(y+8)-8(z+1)=0$$

$$\boxed{8x+y+8z=16}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$z = x^2 e^{x-y}; \quad (2,\ 2,\ 4) \ \text{and} \ (-1,\ -1,\ 1)$$

Solution

$$f(x,y,z) = x^2 e^{x-y} - z$$

$$\nabla f = \left(2x + x^2\right) e^{x-y}\hat{i} - x^2 e^{x-y}\hat{j} - \hat{k} \qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$\nabla f(2,\ 2,\ 4) = 8\hat{i} - 4\hat{j} - \hat{k}$$

The equation of the tangent plane:

$$8(x-2)-4(y-2)-(z-4)=0$$

$$\boxed{8x-4y-z=4}$$

$$\nabla f(-1,\ -1,\ 1) = -\hat{i} - \hat{j} - \hat{k}$$

The equation of the tangent plane:

$$-(x+1)-(y+1)-(z-1)=0$$

$$\boxed{x+y+z=-1}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$z = \ln(1+xy); \quad (1,\ 2,\ \ln 3) \ \text{and} \ (-2,\ -1,\ \ln 3)$$

Solution

$$f(x,y) = \ln(1+xy)$$

$$\nabla f = \frac{y}{1+xy}\hat{i} + \frac{x}{1+xy}\hat{j} \qquad \nabla f = f_x\hat{i} + f_y\hat{j}$$

$$\nabla f(1,\ 2) = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j}$$

The equation of the tangent plane:

$$z = f(1,\ 2) + f_x(1,\ 2)(x-1) + f_y(1,\ 2)(y-2)$$

$$= \ln 3 + \frac{2}{3}(x-1) + \frac{1}{3}(y-2)$$

$$= \ln 3 + \frac{2}{3}x - \frac{2}{3} + \frac{1}{3}y - \frac{2}{3}$$

$$\boxed{= \frac{2}{3}x + \frac{1}{3}y - \frac{4}{3} + \ln 3}$$

$$\nabla f(-2, \ -1,) = -\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j}$$

The equation of the tangent plane:

$$z = f(-2, \ -1) + f_x(-2, \ -1)(x+2) + f_y(-2, \ -1)(y+1)$$

$$= \ln 3 - \frac{1}{3}(x+2) - \frac{2}{3}(y+1)$$

$$= \ln 3 - \frac{1}{3}x - \frac{2}{3} - \frac{2}{3}y - \frac{2}{3}$$

$$\boxed{= -\frac{2}{3}x - \frac{1}{3}y - \frac{4}{3} + \ln 3}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$z = f(x, y) = \frac{1}{x^2 + y^2} \quad \text{at the point } \left(1, \ 1, \ \frac{1}{2}\right)$$

Solution

$$f_x = -\frac{2x}{\left(x^2+y^2\right)^2}\Bigg|_{\left(1, \ 1, \ \frac{1}{2}\right)}$$

$$= -\frac{2}{(1+1)^2}$$

$$\boxed{= -\frac{1}{2}}$$

$$f_y = -\frac{2y}{\left(x^2+y^2\right)^2}\Bigg|_{\left(1, \ 1, \ \frac{1}{2}\right)}$$

$$= -\frac{2}{4}$$

$$\boxed{= -\frac{1}{2}}$$

$$f(x, y, z) = \frac{1}{x^2+y^2} - 1$$

$$\boxed{f_z = -1}$$

Tangent plane:

$$-\frac{1}{2}(x-1)-\frac{1}{2}(y-1)-\left(z-\frac{1}{2}\right)=0$$

$$-x+1-y+1-2z+1=0$$

$$\underline{x+y+2z=3}$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$x^2+y+z=3; \quad (1,\ 1,\ 1) \text{ and } (2,\ 0,\ -1)$$

Solution

$$f(x,\ y,\ z)=x^2+y+z-3$$

$$\nabla f=\langle 2x,\ 1,\ 1\rangle$$

At $(1,\ 1,\ 1)$:

$$\nabla f=\langle 2,\ 1,\ 1\rangle$$

Tangent plane:

$$2(x-1)+(y-1)+(z-1)=0$$

$$\underline{2x+y+z=4}$$

At $(2,\ 0,\ -1)$:

$$\nabla f=\langle 4,\ 1,\ 1\rangle$$

Tangent plane:

$$4(x-2)+y+(z+1)=0$$

$$\underline{4x+y+z=7}$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$x^2+y^3+z^4=2; \quad (1,\ 0,\ 1) \text{ and } (-1,\ 0,\ 1)$$

Solution

$$f(x,\ y,\ z)=x^2+y^3+z^4-2$$

$$\nabla f=\left\langle 2x,\ 3y^2,\ 4z^3\right\rangle$$

At $(1,\ 0,\ 1)$:

$$\nabla f=\langle 2,\ 0,\ 4\rangle$$

Tangent plane:

$$2(x-1)+4(z-1)=0$$

$$2x+4z=6$$

$$\underline{x+2z=3}\rfloor$$

At $(-1,\ 0,\ 1)$:

$$\nabla f=\langle -2,\ 0,\ 4\rangle$$

Tangent plane:

$$-2(x-1)+4(z-1)=0$$

$$\underline{x-2z=-3}\rfloor$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$xy+xz+yz=12; \quad (2,\ 2,\ 2)\ \textit{and}\ (2,\ 0,\ 6)$$

Solution

$$f(x,\ y,\ z)=xy+xz+yz-12$$

$$\nabla f=\langle y+z,\ \ x+z,\ \ x+y\rangle$$

At $(2,\ 2,\ 2)$:

$$\nabla f=\langle 4,\ 4,\ 4\rangle$$

Tangent plane:

$$4(x-2)+4(y-2)+4(z-2)=0$$

$$\underline{x+y+z=6}\rfloor$$

At $(2,\ 0,\ 6)$:

$$\nabla f=\langle 6,\ 8,\ 2\rangle$$

Tangent plane:

$$6(x-2)+8y+2(z-6)=0$$

$$\underline{3x+4y+z=12}\rfloor$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$x^2+y^2-z^2=0; \quad (3,\ 4,\ 5)\ \textit{and}\ (-4,\ -3,\ 5)$$

Solution

$$f(x,\ y,\ z)=x^2+y^2-z^2$$

$\nabla f = \langle 2x, \; 2y, \; -2z \rangle$

At $(3, \; 4, \; 5)$:

$\nabla f = \langle 6, \; 8, \; -10 \rangle$

Tangent plane:

$6(x-3)+8(y-4)-10(z-5)=0$

$\boxed{3x+4y-5z=0}$

At $(-4, \; -3, \; 5)$:

$\nabla f = \langle -8, \; -6, \; -10 \rangle$

Tangent plane:

$-8(x+4)-6(y+3)-10(z-5)=0$

$\boxed{4x+3y+5z=0}$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$xy\sin z = 1; \quad \left(1, \; 2, \; \frac{\pi}{6}\right) \; and \; \left(-2, \; -1, \; \frac{5\pi}{6}\right)$$

Solution

$f(x, \; y, \; z) = xy\sin z - 1$

$\nabla f = \langle y\sin z, \; x\sin z, \; xy\cos z \rangle$

At $\left(1, \; 2, \; \frac{\pi}{6}\right)$:

$\nabla f = \left\langle 1, \; \frac{1}{2}, \; \sqrt{3} \right\rangle$

Tangent plane:

$(x-1)+\frac{1}{2}(y-2)+\sqrt{3}\left(z-\frac{\pi}{6}\right)=0$

$\boxed{x+\frac{1}{2}y+\sqrt{3}z = 2+\frac{\pi\sqrt{3}}{6}}$

At $\left(-2, \; -1, \; \frac{5\pi}{6}\right)$:

$\nabla f = \left\langle -\frac{1}{2}, \; -1, \; -\sqrt{3} \right\rangle$

Tangent plane:

$-\frac{1}{2}\left(x+\frac{1}{2}\right)-(y+1)-\sqrt{3}\left(z-\frac{5\pi}{6}\right)=0$

$\boxed{\frac{1}{2}x+y+\sqrt{3}z = \frac{5\pi\sqrt{3}}{6}-2}$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$yze^{xz}=8;\quad (0,\ 2,\ 4)\ \textit{and}\ (0,\ -8,\ -1)$$

Solution

$$f(x,\ y,\ z)=yze^{xz}-8$$

$$\nabla f=\left\langle yz^2e^{xz},\ ze^{xz},\ e^{xz}(y+xyz)\right\rangle$$

At $(0,\ 2,\ 4)$:

$$\nabla f=\langle 32,\ 4,\ 8\rangle$$

Tangent plane:

$$32x+4(y-2)+8(z-4)=0$$

$$\boxed{8x+y+2z=10}$$

At $(0,\ -8,\ -1)$:

$$\nabla f=\langle -8,\ -1,\ 8\rangle$$

Tangent plane:

$$-8x-(y+8)+8(z+1)=0$$

$$\boxed{8x+y-8z=0}$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$z^2-\frac{x^2}{16}-\frac{y^2}{9}=1;\quad \left(4,\ 3,\ -\sqrt{3}\right)\ \textit{and}\ \left(-8,\ 9,\ \sqrt{14}\right)$$

Solution

$$f(x,\ y,\ z)=z^2-\frac{x^2}{16}-\frac{y^2}{9}-1$$

$$\nabla f=\left\langle -\frac{1}{8}x,\ -\frac{2}{9}y,\ 2z\right\rangle$$

At $\left(4,\ 3,\ -\sqrt{3}\right)$:

$$\nabla f=\left\langle -\frac{1}{2},\ -\frac{2}{3},\ -2\sqrt{3}\right\rangle$$

Tangent plane:

$$-\frac{1}{2}(x-4)-\frac{2}{3}(y-3)-2\sqrt{3}\left(z+2\sqrt{3}\right)=0$$

$$\boxed{\frac{1}{2}x+\frac{2}{3}y+2\sqrt{3}z=-2}$$

At $\left(-8,\ 9,\ \sqrt{14}\right)$:

$$\nabla f = \left\langle 1,\ -2,\ 2\sqrt{14}\right\rangle$$

Tangent plane:

$$(x+8)-2(y-9)+2\sqrt{14}\left(z-\sqrt{14}\right)=0$$

$$\underline{x-2y+2\sqrt{14}z=2}$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$2x+y^2-z^2=0;\quad (0,\ 1,\ 1)\ \textit{and}\ (4,\ 1,\ -3)$$

Solution

$$f(x,\ y,\ z)=2x+y^2-z^2$$

$$\nabla f = \langle 2,\ \ 2y,\ \ -2z\rangle$$

At $(0,\ 1,\ 1)$:

$$\nabla f = \langle 2,\ 2,\ -2\rangle$$

Tangent plane:

$$2x+2(y-1)-2(z-1)=0$$

$$\underline{x+y-z=0}$$

At $(4,\ 1,\ -3)$:

$$\nabla f = \langle 2,\ 2,\ 6\rangle$$

Tangent plane:

$$2(x-4)+2(y-1)+6(z+3)=0$$

$$\underline{x-+y+3z=-4}$$

Exercise

Find parametric equation for the line tangent to the curve of intersection of the surfaces

$$x+y^2+2z=4,\quad x=1\ \text{ at the point } (1,\ 1,\ 1)$$

Solution

$$f_x=1,\quad f_y=2y,\quad f_z=2$$

$$\nabla f = \hat{i}+2y\hat{j}+2\hat{k}\Big|_{(1,\ 1,\ 1)}$$

$$=\hat{i}+2\hat{j}+2\hat{k}$$

$$\nabla g = \hat{i}$$

$$\vec{v} = \nabla f \times \nabla g$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 2\hat{j} - 2\hat{k}$$

Tangent Line: $x = 1, \quad y = 1 + 2t, \quad z = 1 - 2t$

Exercise

Find parametric equation for the line tangent to the curve of intersection of the surfaces $xyz = 1, \quad x^2 + 2y^2 + 3z^2 = 6$ at the point $(1,\ 1,\ 1)$

Solution

$$f_x = yz, \quad f_y = xz, \quad f_z = xy$$

$$\nabla f = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$\nabla f\,(1,\ 1,\ 1) = \hat{i} + \hat{j} + \hat{k}$$

$$g_x = 2x, \quad g_y = 4y, \quad g_z = 6z$$

$$\nabla g = 2x\hat{i} + 4y\hat{j} + 6y\hat{k}$$

$$\nabla g\,(1,\ 1,\ 1) = 2\hat{i} + 4\hat{j} + 6\hat{k}$$

$$v = \nabla f \times \nabla g$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 4 & 6 \end{vmatrix}$$

$$= 2\hat{i} - 4\hat{j} + 2\hat{k}$$

Tangent Line: $x = 1 + 2t, \quad y = 1 - 4t, \quad z = 1 + 2t$

Exercise

Find parametric equation for the line tangent to the curve of intersection of the surfaces $x^3 + 3x^2y^2 + y^3 + 4xy - z^2 = 0, \quad x^2 + y^2 + z^2 = 11$ at the point $(1,\ 1,\ 3)$

Solution

$$f_x = 3x^2 + 6xy^2 + 4y \Big|_{(1,\ 1,\ 3)}$$

$$\underline{=13}|$$

$$f_y = 6x^2y + 3y^2 + 4x\Big|_{(1,\ 1,\ 3)}$$

$$\underline{=13}|$$

$$f_z = -2z\Big|_{(1,\ 1,\ 3)}$$

$$\underline{=-6}|$$

$$\nabla f\,(1,\ 1,\ 3) = 13\hat{i} + 13\hat{j} - 6\hat{k}$$

$$g_x = 2x, \quad g_y = 2y, \quad g_z = 2z$$

$$\nabla g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\nabla g\,(1,\ 1,\ 3) = 2\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 13 & 13 & -6 \\ 2 & 2 & 6 \end{vmatrix} \qquad \vec{v} = \nabla f \times \nabla g$$

$$\underline{= 90\hat{i} - 90\hat{j}}|$$

Tangent Line: $\underline{x = 1 + 90t, \quad y = 1 - 90t, \quad z = 3}|$

Exercise

Find parametric equation for the line tangent to the curve of intersection of the surfaces $x^2 + y^2 = 4, \quad x^2 + y^2 - z = 0$ at the point $\left(\sqrt{2},\ \sqrt{2},\ 4\right)$

Solution

$$f_x = 2x, \quad f_y = 2u, \quad f_z = 0$$

$$\nabla f = 2x\hat{i} + 2y\hat{j}$$

$$\nabla f\left(\sqrt{2},\ \sqrt{2},\ 4\right) = 2\sqrt{2}\hat{i} + 2\sqrt{2}\hat{j}$$

$$g_x = 2x, \quad g_y = 2y, \quad g_z = -1$$

$$\nabla g = 2x\hat{i} + 2y\hat{j} - \hat{k} \quad \Rightarrow \quad \nabla g\left(\sqrt{2},\ \sqrt{2},\ 4\right) = 2\sqrt{2}\hat{i} + 2\sqrt{2}\hat{j} - \hat{k}$$

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\sqrt{2} & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 2\sqrt{2} & -1 \end{vmatrix} \qquad \vec{v} = \nabla f \times \nabla g$$

$$\underline{= -2\sqrt{2}\hat{i} + 2\sqrt{2}\hat{j}}|$$

Tangent Line: $\underline{x = \sqrt{2} - 2\sqrt{2}t, \quad y = \sqrt{2} + 2\sqrt{2}t, \quad z = 4}|$

Exercise

Find an equation for the plane tangent to the level surface $f(x,y,z)=x^2-y-5z$ at the point $P_0(2,\ -1,\ 1)$. Also, find parametric equations for the line is normal to the surface at P_0.

Solution

$$\nabla f = 2x\hat{i} - \hat{j} - 5\hat{k}\Big|_{(2,\ -1,\ 1)}$$

$$\underline{= 4\hat{i} - \hat{j} - 5\hat{k}|}$$

Tangent Plane:

$$4(x-2)-(y+1)-5(z-1)=0$$

$$4x-8-y-1-5z+5=0$$

$$\underline{4x-y-5z=4\ |}$$

Normal Line:

$$\begin{cases} x=2+4t \\ y=-1-t \\ z=1-5t \end{cases}$$

Exercise

By about how much will $f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}$ change if the point $P(x,y,z)$ moves from $P_0(3,\ 4,\ 12)$ a distance of $ds=0.1$ unit in the direction of $3\boldsymbol{i}+6\boldsymbol{j}-2\boldsymbol{k}$?

Solution

$$f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}$$

$$(\ln u)' = \frac{u'}{u}$$

$$=\frac{1}{2}\ln\left(x^2+y^2+z^2\right)$$

$$f_x = \frac{x}{x^2+y^2+z^2}\Big|_{(3,4,12)}$$

$$=\frac{3}{9+16+144}$$

$$=\frac{3}{169}$$

$$f_y = \frac{y}{x^2+y^2+z^2}\Big|_{(3,4,12)}$$

$$=\frac{4}{9+16+144}$$

$$=\frac{4}{169}$$

$$f_z = \left.\frac{z}{x^2+y^2+z^2}\right|_{(3,4,12)}$$

$$= \frac{12}{9+16+144}$$

$$= \frac{12}{169}$$

$$\nabla f = \frac{3}{169}\hat{i} + \frac{4}{169}\hat{j} + \frac{12}{169}\hat{k}$$

$$\vec{u} = \frac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{9+36+4}} \qquad\qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}$$

$$\nabla f \bullet \vec{u} = \left(\frac{3}{169}\hat{i} + \frac{4}{169}\hat{j} + \frac{12}{169}\hat{k}\right) \bullet \left(\frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}\right)$$

$$= \frac{9}{1183}$$

$$df = (\nabla f \bullet \vec{u})\, ds$$

$$= \frac{9}{1183}(0.1)$$

$$= \frac{9}{11830}$$

$$\approx 0.0008$$

Exercise

By about how much will $f(x,y,z) = e^x \cos yz$ change if the point $P(x,y,z)$ moves from origin a distance of $ds = 0.1$ unit in the direction of $2\hat{i} + 2\hat{j} - 2\hat{k}$?

Solution

$$f_x = e^x \cos yz \quad \Rightarrow \quad f_x(0,0,0) = 1$$

$$f_y = -ze^x \sin yz \quad \Rightarrow \quad f_y(0,0,0) = 0$$

$$f_z = -ze^x \sin yz \quad \Rightarrow \quad f_z(0,0,0) = 0$$

$$\nabla f = \hat{i}$$

$$\vec{u} = \frac{2\hat{i}+2\hat{j}-2\hat{k}}{\sqrt{4+4+4}} \qquad\qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{2}{2\sqrt{3}}\hat{i} + \frac{2}{2\sqrt{3}}\hat{j} - \frac{2}{2\sqrt{3}}\hat{k}$$

$$= \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

$$\nabla f \bullet \vec{u} = (\hat{i}) \bullet \left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}\right)$$

$$= \frac{1}{\sqrt{3}}$$

$$df = (\nabla f \bullet \vec{u})\, ds$$

$$= \frac{1}{\sqrt{3}}(0.1)$$

$$\boxed{= \frac{1}{10\sqrt{3}}} \qquad \boxed{\approx 0.0577}$$

Exercise

Find the linearization $L(x, y)$ of $f(x, y) = x^2 + y^2 + 1$ at the point (0, 0) and (1, 1)

Solution

$$f(0,0) = 1$$

$$f_x = 2x \quad \Rightarrow \quad f_x(0,0) = 0$$

$$f_y = 2y \quad \Rightarrow \quad f_y(0,0) = 0$$

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$L(x, y) = 1 + 0(x - 0) + 0(y - 0) \ \boxed{= 1}$$

$$f(1,1) = 3$$

$$f_x = 2x \quad \Rightarrow \quad f_x(1,1) = 2$$

$$f_y = 2y \quad \Rightarrow \quad f_y(1,1) = 2$$

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$L(x, y) = 3 + 2(x - 1) + 2(y - 1)$$

$$\boxed{= 2x + 2y - 1}$$

Exercise

Find the linearization $L(x, y)$ of $f(x, y) = (x + y + 2)^2$ at the point (0, 0) and (1, 2)

Solution

$$f(0,0) = 4$$

$$f_x = 2(x+y+2) \quad \Rightarrow \quad f_x(0,0) = 4$$

$$f_y = 2(x+y+2) \quad \Rightarrow \quad f_y(0,0) = 4$$

$$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$L(x,y) = 4 + 4(x-0) + 4(y-0)$$

$$\underline{= 4 + 4x + 4y \,|}$$

$$f(1,2) = (1+2+2)^2 = 25$$

$$f_x = 2(x+y+2) \quad \Rightarrow \quad f_x(1,2) = 10$$

$$f_y = 2(x+y+2) \quad \Rightarrow \quad f_y(1,2) = 10$$

$$L(x,y) = 25 + 10(x-1) + 10(y-2)$$

$$\underline{= 10x + 10y - 5 \,|}$$

Exercise

Find the linearization $L(x,y)$ of $f(x,y) = x^3y^4$ at the point (1, 1) and (0, 0)

Solution

$$f(1,\ 1) = 1$$

$$f_x = 3x^2y^4 \quad \Rightarrow \quad f_x(1,1) = 3$$

$$f_y = 4x^2y^3 \quad \Rightarrow \quad f_y(1,1) = 4$$

$$L(x,y) = 1 + 3(x-1) + 4(y-1)$$

$$\underline{= 3x + 4y - 6 \,|}$$

$$f(0,0) = 0$$

$$f_x = 3x^2y^4 \quad \Rightarrow \quad f_x(0,0) = 0$$

$$f_y = 4x^2y^3 \quad \Rightarrow \quad f_y(0,0) = 0$$

$$L(x,\ y) = 0 + 0(x-0) + 0(y-0)$$

$$\underline{= 0 \,|}$$

Exercise

Find the linearization $L(x,y)$ of $f(x,y) = e^{2y-x}$ at the point (0, 0) and (1, 2)

Solution

$$f(0,\ 0) = e^0 = 1$$

$$f_x = -e^{2y-x} \quad \Rightarrow \quad f_x(0,0) = -1$$

$$f_y = 2e^{2y-x} \quad \Rightarrow \quad f_y(0,0) = 2$$

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$L(x,\ y) = 1 - 1(x-0) + 2(y-0)$$

$$\boxed{= 1 - x + 2y}$$

$$f(1,\ 2) = e^3$$

$$f_x = -e^{2y-x} \quad \Rightarrow \quad f_x(1,2) = -e^3$$

$$f_y = 2e^{2y-x} \quad \Rightarrow \quad f_y(0,0) = 2e^3$$

$$L(x,\ y) = e^3 - e^3(x-1) + 2e^3(y-2)$$

$$\boxed{= -e^3x + 2e^3y - 2e^3}$$

Exercise

Find the linearization $L(x,y,z)$ of $f(x,y,z) = x^2 + y^2 + z^2$ at the point (1, 1, 1)

Solution

$$f(1,\ 1,\ 1) = 3$$

$$f_x = 2x \quad \Rightarrow \quad f_x(1,1,1) = 2$$

$$f_y = 2y \quad \Rightarrow \quad f_y(1,1,1) = 2$$

$$f_z = 2z \quad \Rightarrow \quad f_z(1,1,1) = 2$$

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x-x_0) + f_y(y-y_0) + f_z(z-z_0)$$

$$L(x,y,z) = 3 + 2(x-1) + 2(y-1) + 2(z-1)$$

$$\boxed{= 2x + 2y + 2z - 3}$$

Exercise

Find the linearization $L(x,y,z)$ of $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$ at the point (1, 2, 2)

Solution

$$f(1,1,1) = \sqrt{1+4+4} = 3$$

$$f_x = \tfrac{1}{2}\left(x^2 + y^2 + z^2\right)^{-1/2}(2x)$$

$$= \frac{x}{\sqrt{x^2 + y^2 + z^2}}\Bigg|_{(1,2,2)}$$

$$= \tfrac{1}{3}$$

$$f_y = \tfrac{1}{2}\left(x^2 + y^2 + z^2\right)^{-1/2}(2y)$$

$$= \frac{y}{\sqrt{x^2 + y^2 + z^2}}\Bigg|_{(1,2,2)}$$

$$= \tfrac{2}{3}$$

$$f_z = \tfrac{1}{2}\left(x^2 + y^2 + z^2\right)^{-1/2}(2z)$$

$$= \frac{z}{\sqrt{x^2 + y^2 + z^2}}\Bigg|_{(1,2,2)}$$

$$= \tfrac{2}{3}$$

$$L(x, y, z) = f\left(x_0, y_0, z_0\right) + f_x\left(x - x_0\right) + f_y\left(y - y_0\right) + f_z\left(z - z_0\right)$$

$$L(x, y, z) = 3 + \tfrac{1}{3}(x-1) + \tfrac{2}{3}(y-2) + \tfrac{2}{3}(z-2)$$

$$\underline{= \tfrac{1}{3}x + \tfrac{2}{3}y + \tfrac{2}{3}z}$$

Exercise

Find the linearization $L(x, y, z)$ of $f(x, y, z) = \dfrac{\sin xy}{z}$ at the point $\left(\frac{\pi}{2},\ 1,\ 1\right)$

Solution

$$f\left(\tfrac{\pi}{2},\ 1,\ 1\right) = \frac{\sin\frac{\pi}{2}}{1} = 1$$

$$f_x = \frac{y\cos xy}{z}\Bigg|_{\left(\frac{\pi}{2},1,1\right)}$$

$$\underline{= 0}$$

$$f_y = \frac{x\cos xy}{z}\Bigg|_{\left(\frac{\pi}{2},1,1\right)}$$

$$\underline{= 0}$$

$$f_z = -\frac{\sin xy}{z^2}\Big|_{\left(\frac{\pi}{2},1,1\right)}$$

$$\underline{= -1|}$$

$$L(x,y,z) = f\left(x_0, y_0, z_0\right) + f_x\left(x - x_0\right) + f_y\left(y - y_0\right) + f_z\left(z - z_0\right)$$

$$L(x,y,z) = 1 + 0\left(x - \frac{\pi}{2}\right) + 0(y-1) - (z-1)$$

$$\underline{= 2 - z\ |}$$

Exercise

Find the linearization $L(x,y,z)$ of $f(x,y,z) = e^x + \cos(y+z)$ at the point $\left(0,\ \frac{\pi}{4},\ \frac{\pi}{4}\right)$

Solution

$$f\left(0,\ \frac{\pi}{4},\ \frac{\pi}{4}\right) = 1$$

$$f_x = e^x\Big|_{\left(0,\ \frac{\pi}{4},\ \frac{\pi}{4}\right)}$$

$$\underline{= 1|}$$

$$f_y = -\sin(y+z)\Big|_{\left(0,\ \frac{\pi}{4},\ \frac{\pi}{4}\right)}$$

$$\underline{= -1|}$$

$$f_z = -\sin(y+z)\Big|_{\left(0,\ \frac{\pi}{4},\ \frac{\pi}{4}\right)}$$

$$\underline{= -1|}$$

$$L(x,y,z) = f\left(x_0, y_0, z_0\right) + f_x\left(x - x_0\right) + f_y\left(y - y_0\right) + f_z\left(z - z_0\right)$$

$$L(x,y,z) = 1 + x - \left(y - \frac{\pi}{4}\right) - \left(z - \frac{\pi}{4}\right)$$

$$\underline{= x - y - z + 1 + \frac{\pi}{2}\ |}$$

Exercise

Find the linear approximation to the function f at the point $(a,\ b)$ and estimate the given function value

$f(x,\ y) = 4\cos(2x - y)$; $\quad (a,\ b) = \left(\frac{\pi}{4},\ \frac{\pi}{4}\right)$; estimate $f(0.8,\ 0.8)$

Solution

$$f\left(\frac{\pi}{4},\ \frac{\pi}{4}\right)=4\cos\left(\frac{\pi}{2}-\frac{\pi}{4}\right)$$
$$=4\cos\left(\frac{\pi}{4}\right)$$
$$\underline{=2\sqrt{2}}\,|$$

$$f_x(x,\ y)=-8\sin(2x-y)$$

$$f_x\left(\frac{\pi}{4},\ \frac{\pi}{4}\right)=-8\sin\left(\frac{\pi}{4}\right)$$
$$\underline{=-4\sqrt{2}}\,|$$

$$f_y(x,\ y)=4\sin(2x-y)$$

$$f_x\left(\frac{\pi}{4},\ \frac{\pi}{4}\right)=4\sin\left(\frac{\pi}{4}\right)$$
$$\underline{=2\sqrt{2}}\,|$$

$$L(x,y)=2\sqrt{2}-4\sqrt{2}\left(x-\frac{\pi}{4}\right)+2\sqrt{2}\left(y-\frac{\pi}{4}\right) \qquad L(x,y)=f\left(x_0,y_0\right)+f_x\left(x-x_0\right)+f_y\left(y-y_0\right)$$
$$=2\sqrt{2}-4\sqrt{2}x+\pi\sqrt{2}+2\sqrt{2}y-\frac{\pi}{2}\sqrt{2}$$
$$=-4\sqrt{2}x+2\sqrt{2}y+2\sqrt{2}+\frac{\pi}{2}\sqrt{2}$$

$$f(0.8,\ 0.8)=-4\sqrt{2}\,\frac{4}{5}+2\sqrt{2}\,\frac{4}{5}+2\sqrt{2}+\frac{\pi}{2}\sqrt{2}$$
$$=-\frac{8}{5}\sqrt{2}+2\sqrt{2}+\frac{\pi}{2}\sqrt{2}$$
$$=\left(\frac{2}{5}+\frac{\pi}{2}\right)\sqrt{2}$$
$$\underline{=(4+5\pi)\frac{\sqrt{2}}{10}}\,|$$
$$\underline{\approx 2.787}\,|$$

Exercise

Find the linear approximation to the function f at the point $(a,\ b)$ and estimate the given function value

$f(x,\ y)=(x+y)e^{xy};\quad (a,\ b)=(2,\ 0)$; estimate $f(1.95,\ 0.05)$

Solution

$$f(2,\ 0)=2e^0$$
$$\underline{=2}\,|$$

$$f_x(x,\ y)=\left(1+xy+y^2\right)e^{xy}$$

$$\boxed{f_x(2,\ 0)=1}$$

$$f_y(x,\ y)=\left(1+x^2+xy\right)e^{xy}$$

$$\boxed{f_y(2,\ 0)=5}$$

$$L(x,y)=2+(x-2)+5(y-0) \qquad L(x,y)=f\left(x_0,y_0\right)+f_x\left(x-x_0\right)+f_y\left(y-y_0\right)$$

$$=x+5y$$

$$f(1.95,\ 0.05)=\frac{39}{20}+5\left(\frac{1}{20}\right)$$

$$\boxed{=\frac{41}{20}}$$

$$\boxed{\approx 2.205}$$

Exercise

Find the linear approximation to the function f at the point $(a,\ b)$ and estimate the given function value

$f(x,\ y)=xy+x-y;\quad (a,\ b)=(2,\ 3)$; estimate $f(2.1,\ 2.99)$

Solution

$$f(2,\ 3)=6+2-3$$

$$\boxed{=5}$$

$$f_x=y+1\Big|_{(2,\ 3)}$$

$$\boxed{=4}$$

$$f_y=x-1\Big|_{(2,\ 3)}$$

$$\boxed{=1}$$

$$L(x,y)=5+4(x-2)+(y-3) \qquad L(x,y)=f\left(x_0,y_0\right)+f_x\left(x-x_0\right)+f_y\left(y-y_0\right)$$

$$=4x+y-6$$

$$f(2.1,\ 2.99)=4(2.1)+2.99-6$$

$$=4\frac{21}{10}+\frac{299}{100}-6$$

$$=\frac{840+299-600}{100}$$

$$=\frac{539}{100}$$

$$\boxed{=5.39}$$

Exercise

Find the linear approximation to the function f at the point $(a,\ b)$ and estimate the given function value

$$f(x,\ y) = 12 - 4x^2 - 8y^2; \quad (a,\ b) = (-1,\ 4); \text{ estimate } f(-1.05,\ 3.95)$$

Solution

$$f(-1,\ 4) = 12 - 4 - 128$$
$$\underline{= -120|}$$

$$f_x = -8x\Big|_{(-1,\ 4)}$$
$$\underline{= 8|}$$

$$f_y = -16y\Big|_{(-1,\ 4)}$$
$$\underline{= -64|}$$

$$L(x, y) = -120 + 8(x+1) - 64(y-4) \qquad L(x, y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$$
$$\underline{= 8x - 64y + 144|}$$

$$f(-1.05,\ 3.95) = 8(-1.05) - 64(3.95) + 144$$
$$\underline{= -117.2\ |}$$

Exercise

Find the linear approximation to the function f at the point $(a,\ b)$ and estimate the given function value

$$f(x,\ y) = -x^2 + 2y^2; \quad (a,\ b) = (3,\ \ -1); \text{ estimate } f(3.1,\ \ -1.04)$$

Solution

$$f(3,\ \ -1) = -9 + 2$$
$$\underline{= -7|}$$

$$f_x = -2x\Big|_{(3,\ -1)}$$
$$\underline{= -6|}$$

$$f_y = 4y\Big|_{(3,\ -1)}$$
$$\underline{= -4|}$$

$$L(x, y) = -7 - 6(x-3) - 4(y+1) \qquad L(x, y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$$
$$\underline{= -6x - 4y + 7|}$$

$$f(3.1,\ -1.04) = -6(3.1) - 4(-1.04) + 7$$
$$\boxed{= -7.44}$$

Exercise

Find the linear approximation to the function f at the point $(a,\ b)$ and estimate the given function value

$$f(x,\ y) = \sqrt{x^2 + y^2}; \quad (a,\ b) = (3,\ -4); \text{ estimate } f(3.06,\ -3.92)$$

Solution

$$f(3,\ -4) = \sqrt{9+16}$$
$$\boxed{= 5}$$

$$f_x = \frac{x}{\sqrt{x^2 + y^2}}\Bigg|_{(3,\ -4)}$$
$$\boxed{= \frac{3}{5}}$$

$$f_y = \frac{y}{\sqrt{x^2 + y^2}}\Bigg|_{(3,\ -4)}$$
$$\boxed{= -\frac{4}{5}}$$

$$L(x, y) = 5 + \frac{3}{5}(x-3) - \frac{4}{5}(y+4) \qquad L(x, y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$$
$$\boxed{= \frac{3}{5}x - \frac{4}{5}y}$$

$$f(3.06,\ -3.92) = \frac{3}{5}\left(\frac{306}{100}\right) - \frac{4}{5}\left(-\frac{392}{100}\right)$$
$$= \frac{918 + 1568}{500}$$
$$= \frac{1,243}{250}$$
$$\boxed{= 4.972}$$

Exercise

Find the linear approximation to the function f at the point $(a,\ b)$ and estimate the given function value

$$f(x,\ y) = \ln(1 + x + y); \quad (a,\ b) = (0,\ 0); \text{ estimate } f(0.1,\ -0.2)$$

Solution

$$\boxed{f(0,\ 0) = 0}$$

$$f_x = \left.\frac{1}{1+x+y}\right|_{(0,\ 0)}$$
$$= 1$$

$$f_y = \left.\frac{1}{1+x+y}\right|_{(0,\ 0)}$$
$$= 1$$

$$L(x, y) = x + y \qquad\qquad L(x, y) = f\left(x_0, y_0\right) + f_x\left(x - x_0\right) + f_y\left(y - y_0\right)$$

$$f(0.1,\ -0.2) = 0.1 - 0.2$$
$$= -0.1$$

Exercise

Find the linear approximation to the function f at the point $(a,\ b)$ and estimate the given function value

$$f(x,\ y) = \frac{x+y}{x-y}; \quad (a,\ b) = (3,\ 2); \text{ estimate } f(2.95,\ 2.05)$$

Solution

$$f(3,\ 2) = 5$$

$$f_x = \left.\frac{-2y}{(x-y)^2}\right|_{(3,\ 2)}$$
$$= -4$$

$$f_y = \left.\frac{2x}{(x-y)^2}\right|_{(3,\ 2)}$$
$$= 6$$

$$L(x, y) = 5 - 4(x-3) + 6(y-2) \qquad\qquad L(x, y) = f\left(x_0, y_0\right) + f_x\left(x - x_0\right) + f_y\left(y - y_0\right)$$
$$= -4x + 6y + 5$$

$$f(2.95,\ 2.05) = -4(2.95) + 6(2.05) + 5$$
$$= 5.5$$

Exercise

Estimate the change in the function $f(x,\ y) = -2y^2 + 3x^2 + xy$ when $(x,\ y)$ changes from $(1,\ -2)$ to $(1.05,\ -1.9)$.

Solution

$$f_x = 6x + y$$

$$\boxed{f_x(1,\ -2) = 4}$$

$$f_y = -4y + x$$

$$\boxed{f_y(1,\ -2) = 9}$$

$$\begin{aligned}\Delta f &\approx f_x(1,\ -2)\Delta x + f_y(1,\ -2)\Delta y \\ &= 4(1.05 - 1) + 9(-1.9 + 2) \\ &= .2 + .9 \\ &= 1.1\end{aligned}$$

Exercise

What is the largest value that the directional derivative of $f(x, y, z) = xyz$ can have at the point $(1,\ 1,\ 1)$?

Solution

$$\begin{aligned}\nabla f &= yz\hat{i} + xz\hat{j} + xy\hat{k}\Big|_{(1,\ 1,\ 1)} \\ &= \hat{i} + \hat{j} + \hat{k}\end{aligned}$$

The ***maximum value***: $\quad |\nabla f| = \sqrt{1+1+1}$

$$= \sqrt{3}$$

Exercise

You plan to calculate the volume inside a stretch of pipeline that is about 36 *in*. in diameter and 1 *mile* long. With which measurement should you be more careful, the length or the diameter? Why?

Solution

$$1\ mile = 5280\ ft$$

$$r = \frac{36}{2} in \frac{1\ ft}{12\ in} = \frac{3}{2}\ ft$$

$$V = \pi r^2 h$$

$$\begin{aligned}dV &= 2\pi rh dr + \pi r^2 dh \\ &= 2\pi\left(\frac{3}{2}\right)(5280)\, dr + \pi\left(\frac{3}{2}\right)^2 dh \\ &= 15,840\pi dr + \frac{9\pi}{4}\pi dh\end{aligned}$$

We have to be more careful with the diameter, since it has a greater effect on dV.

Exercise

The volume of a cylinder with radius r and height h is $V = \pi r^2 h$. Find the approximate percentage change in the volume when the radius decreases by 3% and the height increases by 2%.

Solution

$$\Delta V = 2\pi rh\Delta r + \pi r^2 \Delta h$$

$$\frac{dV}{V} = \frac{2\pi rh}{\pi r^2 h}dr + \frac{\pi r^2}{\pi r^2 h}dh$$

$$= 2\frac{dr}{r} + \frac{dh}{h}$$

$$= 2(-3\%) + 2\%$$

$\underline{= 4\%}$ Approximate change volume.

Exercise

The volume of an ellipsoid with axes of length $2a$, $2b$, and $2c$ is $V = \pi abc$. Find the percentage change in the volume when a increases by 2%, b increases by 1.5%, and c decreases by 2.5%.

Solution

$$dV = \pi\left(bc\Delta a + ac\Delta b + ab\Delta c\right)$$

$$\frac{dV}{V} = \frac{da}{a} + \frac{db}{b} + \frac{dc}{c}$$

$$= 2\% + 1.5\% - 2.5\%$$

$\underline{= 1\%}$ Approximate change volume.

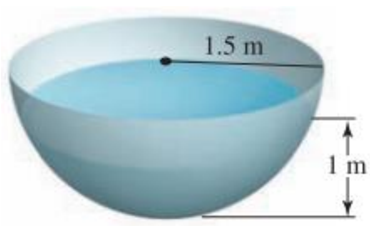
Exercise

A hemispherical tank with a radius of 1.50 m is filled with water to a depth of 1.00 m. Water level drops by 0.05 m (from 1.00 m to 0.95 m)

a) Approximate the change in the volume of water in the tank. The volume of a spherical cap is $V = \frac{1}{3}\pi h^2(3r - h)$, where r is the radius of the sphere and h is the thickness of the cap (in this case, the depth of the water).

b) Approximate the change in the surface area of the water in the tank.

Solution



a) $V = \frac{1}{3}\pi h^2(3r - h)$

$$= \frac{1}{3}\pi\left(3rh^2 - h^3\right)$$

$$dV = \frac{1}{3}\pi\left(6rh - 3h^2\right)dh$$

$$= \pi\left(2rh - h^2\right)dh$$

$$= \pi\left(2(1.5)(1) - 1^2\right)(-.05)$$

$$\underline{= -0.1\pi \;\; m^3}$$

b) $S = \pi\left(2rh - h^2\right)$

$$dS = \pi(2r - 2h)dh$$

$$= 2\pi(1.5 - 1)(-.05)$$

$$\underline{= -0.05\pi \;\; m^2}$$

Exercise

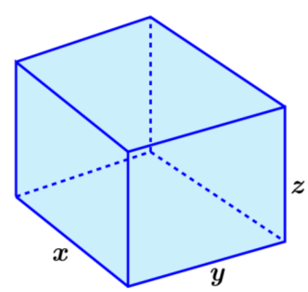
Find the linearization $L(x, y, z)$ of $f(x, y, z) = e^x + \cos(y + z)$ at the point $\left(0, \ \frac{\pi}{4}, \ \frac{\pi}{4}\right)$

Consider a closed rectangular box with a square base. If x is measured with error at most 2% and y is measured with error at most 3% use a differential to estimate the corresponding percentage error in computing the box's

a) Surface area

b) Volume

Solution



Given: $\frac{dx}{x} \le 0.02, \quad \frac{dy}{y} \le 0.03$

a) $S = 2(xx + xy + xy) = 2x^2 + 4xy$

$$dS = (4x + 4y)dx + 4xdy$$

$$= (4x + 4y)\left(x\frac{dx}{x}\right) + 4xy\frac{dy}{y}$$

$$= \left(4x^2 + 4xy\right)\frac{dx}{x} + 4xy\frac{dy}{y}$$

$$\le \left(4x^2 + 4xy\right)(0.02) + 4xy(0.03)$$

$$= 0.02\left(4x^2\right) + 0.02(4xy) + 0.03(4xy)$$

$$= 0.04\left(2x^2\right) + 0.05(4xy)$$

$$\le 0.05\left(2x^2\right) + 0.05(4xy)$$

$$= 0.05\left(2x^2 + 4xy\right)$$

$$\underline{= 0.05\, S}$$

b) $V = x^2 y$

$dV = 2xydx + x^2 dy$

$= 2x^2 y\frac{dx}{x} + x^2 y\frac{dy}{y}$

$\le 2x^2 y(0.02) + x^2 y(.03)$

$= .07\left(x^2 y\right)$

$\underline{= .07\ V}$

Exercise

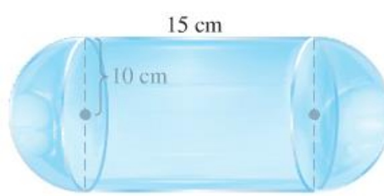
Consider a closed container in the shape of a cylinder of radius 10 *cm* and height 15 *cm* with a hemisphere on each end.

The container is coated with a layer of ice $\frac{1}{2}$ *cm* thick. Use a differential to estimate the total volume of ice.

(*Hint*: assume *r* is radius with $dr = \frac{1}{2}$ and *h* is height with $dh = 0$)

Solution

$$V = \frac{4\pi}{3}r^3 + \pi r^2 h$$



$$dV = 4\pi r^2 dr + 2\pi rhdr + \pi r^2 dh$$

$$= \left(4\pi r^2 + 2\pi rh\right)dr + \pi r^2 dh$$

$$= \left(4\pi(10)^2 + 2\pi(10)(15)\right)\left(\tfrac{1}{2}\right) + \pi(10)^2(0)$$

$$\underline{= 350\pi\ cm^3}$$

Exercise

A standard 12-*fl-oz* can of soda is essentially a cylinder of radius $r = 1\ in$ and height $h = 5\ in.$

a) At these dimensions, how sensitive is the can's volume to a small change in radius versus a small change in height?

b) Could you design a soda can that appears to hold more soda but in fact holds the same 12-fl-oz? What might its dimensions be? (There is more than one correct answer.)

Solution

Given: $r = 1\ in \quad h = 5\ in.$

a) $V = \pi r^2 h \quad \Rightarrow \quad dV = 2\pi rhdr + \pi r^2 dh$

$dV = 10\pi dr + \pi dh$

$$= \pi(10dr + dh)$$

The volume is about 10 times more sensitive to a change in r.

b) $dV = 0 \quad \Rightarrow \quad 2\pi r h dr + \pi r^2 dh = 0$

$2hdr + rdh = 0$

$10dr + dh = 0 \quad \Rightarrow \quad dr = -\frac{1}{10} dh$

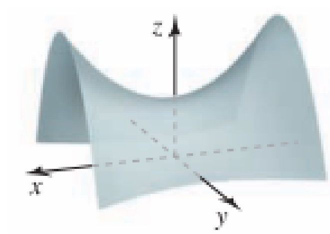
Assume $dh = 1.5$, then $dr = -.15$

$2h(-0.15) + r(1.5) = 0$

$r = 0.85\ in \quad h = 6.5\ in.$ is one solution for $\Delta V \approx dV = 0$

Exercise

Consider the function $f(x, y) = 2x^2 - 4y^2 + 10$, whose graph is shown



a) Fill in the table showing the value of the directional derivative at points (a, b) in the direction given by the unit vectors $\boldsymbol{u}, \boldsymbol{v},$ *and* $\boldsymbol{w}$

b) Interpret each of the directional derivatives computed in part(a) at the point $(2, 0)$

Solution

a) $f_x = 4x \quad f_y = -8y$

$$\nabla f \bullet \vec{u} = (4a\hat{i} - 8b\hat{j}) \bullet (u_x \hat{i} + u_y \hat{j})$$

$$= 4au_x - 8bu_y$$

	$(0, 0)$	$(a, b) = (2, 0)$	$(a, b) = (1, 1)$
$\vec{u} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	0	$4(2)\frac{\sqrt{2}}{2} - 0 = 4\sqrt{2}$	$4(1)\left(\frac{\sqrt{2}}{2}\right) - 8(1)\left(\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$
$\vec{v} = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	0	$4(2)\left(-\frac{\sqrt{2}}{2}\right) - 0 = -4\sqrt{2}$	$4(1)\left(-\frac{\sqrt{2}}{2}\right) - 8(1)\left(\frac{\sqrt{2}}{2}\right) = -6\sqrt{2}$
$\vec{w} = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	0	$4(2)\left(-\frac{\sqrt{2}}{2}\right) - 0 = -4\sqrt{2}$	$4(1)\left(-\frac{\sqrt{2}}{2}\right) - 8(1)\left(-\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$

b) The function is ***increasing*** @ $(2, 0)$ in the direction of $\vec{u}$

The function is ***decreasing*** @ $(2, 0)$ in the direction of $\vec{v} + \vec{w}$

Exercise

Two spheres have the same center and radii *r* and *R*, where $0 < r < R$. The volume of the region between the sphere is $V(r,\ R) = \frac{4\pi}{3}\left(R^3 - r^3\right)$.

a) First use your intuition. If *r* is held fixed, how does *V* change as *R* increases? What is the sign of V_R? If *R* is held fixed, how does *V* change as *r* increases (up to the value of *R*)? What is the sign of V_r?

b) Compute V_r and V_R. Are the results consistent with part (*a*)?

c) Consider spheres with $R = 3$ and $r = 1$. Does the volume change more if *R* is increased by $\Delta R = 0.1$ (with *r* fixed) or if *r* is decreased by $\Delta r = 0.1$ (with *R* fixed)?

Solution

a) r is fixed, then $V_R = 4\pi R^2 > 0$

$\therefore$ If *R* increases then *V* increases.

R is fixed, then $V_r = -4\pi r^2 < 0$

$\therefore$ If *r* increases then *V* decreases.

b) Yes, $V_r = -4\pi r^2 < 0 \quad V_R = 4\pi R^2 > 0$

c) If $R = 3$, $r = 1$, $\Delta r = 0.1$, and $\Delta R = 0.1$

$\Delta R = 0.1 \quad \Rightarrow \quad \Delta V = 4\pi(3)^2(0.1) \underline{= 3.6\pi}$

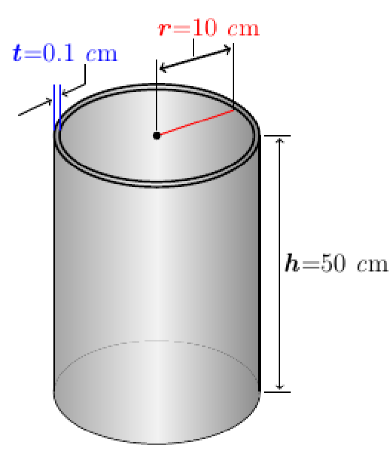
If *r* is decreased by 0.1

$\Delta V = -4\pi(1)^2(-.1) \underline{= 0.4\pi}$

$\therefore$ Volume changes more if *R* is increased.

Exercise

A company manufactures cylindrical aluminum tubes to rigid specifications. The tubes are designed to have an outside radius of $r = 10\ cm$, a height of $h = 50\ cm$, and a thickness of $t = 0.1\ cm$. The manufacturing process produces tubes with a maximum error of $\pm 0.05\ cm$ in the radius and height and a maximum error of $\pm 0.0005\ cm$ in the thickness. The volume of the material used to construct a cylindrical tube is $V(r,h,t) = \pi ht(2r - t)$. Estimate maximum error in the volume of the tube.



Solution

$V(r,h,t) = 2\pi rht - \pi ht^2$

$$dV = 2\pi htdr + \left(2\pi rt - \pi t^2\right)dh + 2\pi h(r-t)dt$$

$$= 2\pi(50)(0.1)(.05) + \left(2\pi(10)(0.1) - \pi(0.1)^2\right)(.05) + 2\pi(50)(10-0.1)(.0005)$$

$$= \pi\left(0.5 + 1.99(.05) + 990(.0005)\right)$$

$$\approx 3.4385$$

The maximum error in the volume is approximately $3.4385\ cm^3$.

The volume is far more sensitive to errors in the thickness, since for the thickness 990π is more than for the radius (10π) and height (1.99π)

Exercise

The volume of a right circular cone with radius r and height h is $V = \frac{1}{3}\pi hr^2$

a) Approximate the change in the volume of the cone when the radius changes from $r = 6.5$ to $r = 6.6$ and the height changes from $h = 4.20$ to $h = 4.15$

b) Approximate the change in the volume of the cone when the radius changes from $r = 5.4$ to $r = 5.37$ and the height changes from $h = 12.0$ to $h = 11.96$

Solution

$$V = \frac{1}{3}\pi hr^2$$

$$dV = \frac{1}{3}\pi\left(2rhdr + r^2dh\right)$$

a) $dV = \frac{\pi}{3}\left(2(6.5)(4.2)(6.6-6.5) + (6.5)^2(4.15-4.2)\right)$

$$= \frac{\pi}{3}\left(54.6(0.5) + 42.25(-.05)\right)$$

$$\approx 3.505$$

b) $dV = \frac{\pi}{3}\left(2(5.4)(12)(-.03) + (5.4)^2(-.04)\right)$

$$\approx -5.293$$

Exercise

The area of an ellipse with axes of length $2a$ and $2b$ is $A = \pi ab$. Approximate the percent change in the area when a increases by 2% and b increases by 1.5%.

Solution

$$dA = \pi(b\,da + a\,db)$$

$$\frac{dA}{A} = \frac{\pi}{\pi ab}(b\,da + a\,db)$$

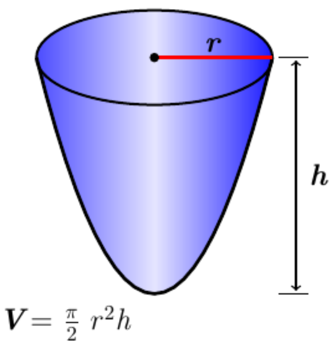
$$\frac{dA}{A} = \frac{da}{a} + \frac{db}{b}$$
$$= 2\% + 1.5\%$$
$$\underline{= 3.5\%}$$

Exercise

The Volume of a segment of a circular paraboloid with radius r and height h is $V = \frac{1}{2}\pi hr^2$.

Approximate the percent change in the volume when the radius decreases by 1.5% and the height increases by 2.2%

Solution



$$dV = \frac{\pi}{2}\left(2rhdr + r^2 dh\right)$$
$$\frac{1}{V}dV = \frac{1}{\frac{1}{2}\pi hr^2}\frac{\pi}{2}\left(2rhdr + r^2 dh\right)$$
$$\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h}$$
$$= 2(-1.5\%) + 2.2\%$$
$$\underline{= -0.8\%}$$

Exercise

Batting averages in baseball are defined by $A = \frac{x}{y}$, where $x \ge 0$ is the total number of hits and $y > 0$ is the total number of at-bats. Treat x and y as positive real numbers and note that $0 \le A \le 1$.

a) Estimate the change in the batting average if the number of hits increases from 60 to 62 and the number of at-bats increases from 175 to 180.

b) If a batter currently has a batting average of $A = 0.35$, does the average decrease if the batter fails to get a hit more than it increases if the batter gets a hit?

c) Does the answer in part (*b*) depend on the current batting average? Explain.

Solution

a) $dA = \frac{1}{y}dx - \frac{x}{y^2}dy$

$$= \frac{1}{175}(62 - 60) - \frac{60}{175^2}(180 - 175)$$
$$= \frac{2}{175} - \frac{300}{175^2}$$
$$= \frac{50}{30,625}$$

$$= \boxed{\frac{2}{1,225}}$$

$$\approx \boxed{0.001633}$$

b) If the batter fails to get a hit, the average decreases by

$$\frac{x}{y} - \frac{x}{y+1} = \frac{x}{y(y+1)}$$

$$= \boxed{\frac{A}{y+1}}$$

If the batter gets a hit, the average increases by

$$\frac{x+1}{y+1} - \frac{x}{y} = \frac{y-x}{y(y+1)}$$

$$= \frac{1 - \frac{x}{y}}{y+1}$$

$$= \boxed{\frac{1-A}{y+1}}$$

If $A = 0.35$, the second of these quantities is larger, therefore the answer is no; the batting average changes more if the batter gets a hit than if he fails to get a hit.

c) The answer depends on whether A is less than or greater than 0.50.

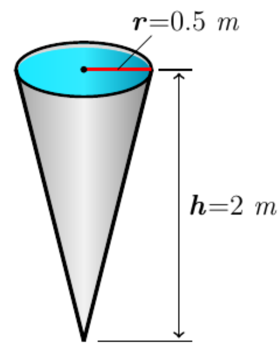
Exercise

A conical tank with radius 0.50 *m* and height 2.0 *m* is filled with water. Water released from the tank, and the water level drops by 0.05 *m* (from 2.0 *m* to 1.95 *m*).

Approximate the change in volume of water in the tank.

(***Hint***: When the water level drops, both the radius and height of the cone of water change).

Solution



$$\frac{x}{r} = \frac{y}{h}$$

$$\frac{x}{.5} = \frac{1.95}{2}$$

$$x = .4875$$

$$dr = 0.4875 - 0.5$$

$$= \boxed{-0.0125}$$

$$V = \frac{1}{3}\pi h r^2$$

$$dV = \frac{1}{3}\pi\left(2rhdr + r^2 dh\right)$$

$$\frac{1}{V}dV = \frac{1}{\frac{1}{3}\pi h r^2}\frac{\pi}{3}\left(2rhdr + r^2 dh\right)$$

$$\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h}$$

$$= -2\frac{0.0125}{0.5} - \frac{.05}{2}$$

$$= -0.05 - \frac{.05}{2}$$

$$\underline{= -0.075|}$$

$$V = \frac{1}{3}\pi(2)(0.5)^2$$

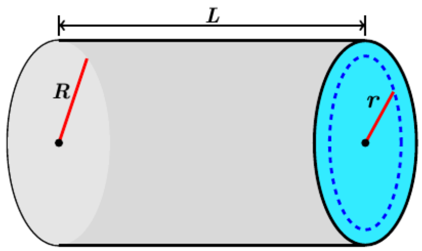
$$\underline{\approx 0.5236|}$$

$$dV = (-.075)(.5236)$$

$$\underline{\approx 0.03927 \ m^3 \ |}$$

Exercise

Poiseuille's law is a fundamental law of fluid dynamics that describes the flow velocity of a viscous incompressible fluid in a cylinder (it is used to model blood flow through veins and arteries). It says that in a cylinder of radius R and length L, the velocity of the fluid $r \le R$ units from the centerline of the cylinder is $V = \frac{P}{4L\upsilon}\left(R^2 - r^2\right)$, where P is the difference in the pressure between the ends of the cylinder and υ is the viscosity of the fluid. Assuming that P and υ are constant, the velocity V along the centerline of the cylinder $(r = 0)$ is $V = \frac{kR^2}{L}$, where k is a constant that we will take to be $k = 1$.



a) Estimate the change in the centerline velocity $(r = 0)$ if the radius of the flow cylinder increases from $R = 3\ cm$ to $R = 3.05\ cm$ and the length increases from $L = 50\ cm$ to $L = 50.5\ cm$.

b) Estimate the percent change in the centerline velocity if the radius of the flow cylinder R decreases by 1% and the length increases by 2%.

Solution

$$k = 1 \quad \rightarrow \quad V = \frac{R^2}{L}$$

a) $$dV = \frac{2R}{L}dR - \frac{R^2}{L^2}dL$$

$$= \frac{2(3)}{50}(3.05 - 3) - \frac{3^2}{50^2}(50.5 - 50)$$

$$= \frac{3}{25}(0.05) - \frac{9}{2500}(0.5)$$

$$= \frac{3}{500} - \frac{9}{5000}$$

$$\boxed{= \frac{21}{5,000}}$$

$$\boxed{= 0.0042 \ \ cm^3}$$

b) $\frac{1}{V} dV = \frac{L}{R^2}\left(\frac{2R}{L} dR - \frac{R^2}{L^2} dL\right)$

$$\frac{dV}{V} = 2\frac{dR}{R} - \frac{dL}{L}$$

R decreases by 1% and the length increases by 2%.

$$= 2(-1\%) - (2\%)$$

$$\boxed{= -4\%}$$

V will decrease by approximately 4%.

Exercise

Suppose that in a large group of people a fraction $0 \le r \le 1$ of the people have flu. The probability that in *n* random encounters, you will meet at least one person with flu is $P = f(n, r) = 1 - (1-r)^n$. Although *n* is a positive integer, regard it as a positive real number.

a) Compute f_r and f_n.

b) How sensitive is the probability *P* to the flu rate *r*? Suppose you meet $n = 20$ people. Approximately how much does the probability *P* increase if the flu rate increases from $r = 0.1$ to $r = 0.11$ (with *n* fixed)?

c) Approximately how much does the probability *P* increase the flu rate increases from $r = 0.9$ to $r = 0.91$

d) Interpret the results of parts (*b*) and (*c*).

Solution

a) $\boxed{f_r = n(1-r)^{n-1}}$

$$f = 1 - (1-r)^n$$

$$f_n = -\frac{\partial}{\partial n}(1-r)^n$$

$$\ln y = \ln (1-r)^n$$

$$\ln y = n \ln (1-r)$$

$$\frac{y_n}{y} = \ln(1-r)$$

$$y_n = (1-r)^n \ln(1-r)$$

$$\boxed{f_n = -(1-r)^n \ln(1-r)}$$

b) $n = 20 \quad r = 0.1$

$$\Delta P \approx f_r(20,\ 0.1)(0.11 - 0.1) \qquad\qquad f_r = n(1-r)^{n-1}$$

$$= 20(1-0.1)^{19}(0.01)$$

$$\boxed{\approx 0.027}$$

c) $n = 20 \quad r = 0.9$

$$\Delta P \approx f_r(20,\ 0.9)(.01) \qquad\qquad f_r = n(1-r)^{n-1}$$

$$= 20(1-0.9)^{19}(0.01)$$

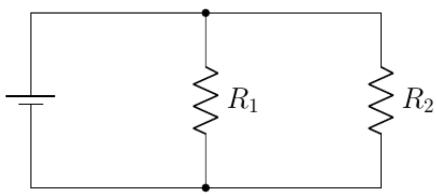
$$\boxed{\approx 2 \times 10^{-20}}$$

d) Small changes in the flu rate have a greater effect on the probability of catching the flu when the flu rate is small compared to when the flu rate is large.

Exercise

When two electrical resistors with resistance $R_1 > 0$ and $R_2 > 0$ are wired in parallel in a circuit, the combined resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$



a) Estimate the change in R if R_1 increases from 2 Ω to 2.05 Ω and R_2 decreases from 3 Ω to 2.95 Ω.

b) Is it true that if $R_1 = R_2$ and R_1 increases by the same small amount as R_2 decreases, then R is approximately unchanged? Explain.

c) Is it true that if R_1 and R_2 increase, then R increases? Explain.

d) Suppose $R_1 > R_2$ and R_1 increases by the same small amount as R_2 decreases. Does R increase or decrease?

Solution

a) $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}.$

$$\frac{1}{R} = \frac{R_2 + R_1}{R_1 R_2}$$

$$R = \frac{R_1 R_2}{R_2 + R_1}$$

$$dR = \frac{R_2^2}{\left(R_2 + R_1\right)^2} dR_1 + \frac{R_1^2}{\left(R_2 + R_1\right)^2} dR_2 \qquad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

$$= \frac{R_2^2}{\left(R_2 + R_1\right)^2} \frac{R_1^2}{R_1^2} dR_1 + \frac{R_1^2}{\left(R_2 + R_1\right)^2} \frac{R_2^2}{R_2^2} dR_2$$

$$= \left(\frac{R_1 R_2}{R_2 + R_1}\right)^2 \frac{dR_1}{R_1^2} + \left(\frac{R_1 R_2}{R_2 + R_1}\right)^2 \frac{dR_2}{R_2^2}$$

$$= R^2 \left(\frac{dR_1}{R_1^2} + \frac{dR_2}{R_2^2}\right)$$

$$= \left(\frac{6}{5}\right)^2 \left(\frac{2.05-2}{4} + \frac{2.95-3}{9}\right)$$

$$= \frac{36}{25}\left(\frac{.05}{4} - \frac{.05}{9}\right)$$

$$= \frac{36}{500}\left(\frac{5}{36}\right)$$

$$= \frac{1}{100}$$

$$\underline{= 0.01\ \ \Omega}$$

b) If $R_1 = R_2$

R_1 increases by the same small amount as R_2 decreases.

$dR_1 = -dR_2$

$$dR = R^2 \left(\frac{dR_1}{R_1^2} + \frac{dR_2}{R_2^2}\right)$$

$$= R^2 \left(-\frac{dR_2}{R_2^2} + \frac{dR_2}{R_2^2}\right)$$

$$\underline{= 0}$$

c) If R_1 and R_2 increase

$$dR = R^2 \left(\frac{dR_1}{R_1^2} + \frac{dR_2}{R_2^2}\right) > 0$$

Therefore, R increases

d) *Given*: $R_1 > R_2$

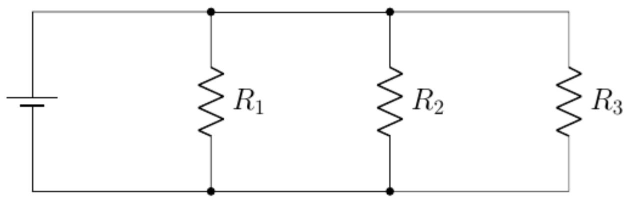
$$R = \frac{R_1 R_2}{R_1 + R_2}$$

R is more sensitive to changes in R_2, so if R_1 increases by the same small amount as R_2 decreases, then R will decrease.

Exercise

When three electrical resistors with resistance $R_1 > 0$, $R_2 > 0$ and $R_3 > 0$ are wired in parallel in a circuit, the combined resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$



Estimate the change in R if R_1 increases from 2 Ω to 2.05 Ω, R_2 decreases from 3 Ω to 2.95 Ω, and R_3 increases from 1.5 Ω to 1.55 Ω

Solution

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

$$\frac{1}{R} = \frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_1 R_2 R_3}$$

$$R = \frac{R_1 R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

$$-\frac{1}{R^2} dR = -\frac{1}{R_1^2} dR_1 - \frac{1}{R_2^2} dR_2 - \frac{1}{R_3^2} dR_3$$

$$dR = R^2 \left(\frac{1}{R_1^2} dR_1 + \frac{1}{R_2^2} dR_2 + \frac{1}{R_3^2} dR_3 \right)$$

$$= \left(\frac{2(3)(1.5)}{6+3+4.5} \right)^2 \left(\frac{1}{4}(2.05-2) + \frac{1}{9}(2.95-3) + \frac{100}{225}(1.55-1.5) \right)$$

$$= \left(\frac{9}{13.5}\right)^2 \left(\frac{1}{4}(.05) - \frac{1}{9}(.05) + \frac{100}{225}(.05)\right)$$

$$= \left(\frac{90}{135}\right)^2 \left(\frac{5}{400} - \frac{5}{900} + \frac{1}{45}\right)$$

$$= \left(\frac{2}{3}\right)^2 \left(\frac{1}{80} - \frac{1}{180} + \frac{1}{45}\right)$$

$$= \frac{4}{9}\left(\frac{9-4+16}{720}\right)$$

$$= \frac{1}{9}\left(\frac{21}{180}\right)$$

$$\boxed{= \frac{7}{540}\ \Omega}$$

$$\boxed{\approx 0.013\ \Omega}$$

Exercise

Consider the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and the plane P given by $Ax + By + Cz + 1 = 0$. Let $h = \frac{1}{\sqrt{A^2 + B^2 + C^2}}$ and $m = \sqrt{a^2A^2 + b^2B^2 + c^2C^2}$

a) Find the equation of the plane tangent to the ellipsoid at the point $(p,\ q,\ r)$.

b) Find the two points on the ellipsoid at which the tangent plane parallel to P and find equations of the tangent planes.

c) Show that the distance between the origin and the plane P is *h*.

d) Show that the distance between the origin and the tangent planes is *hm*.

e) Find a condition that guarantees the plane P does not intersect the ellipsoid.

Solution

a) $f(x,\ y,\ z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

$$\nabla f = \left\langle \frac{2x}{a^2},\ \frac{2y}{b^2},\ \frac{2z}{c^2} \right\rangle \Big|_{(p,\ q,\ r)}$$

$$= \left\langle \frac{2p}{a^2},\ \frac{2q}{b^2},\ \frac{2r}{c^2} \right\rangle$$

Equation of the ***plane tangent*** to the ellipsoid at the point $(p,\ q,\ r)$ is:

$$\frac{2p}{a^2}(x-p) + \frac{2q}{b^2}(y-q) + \frac{2r}{c^2}(z-r) = 0$$

$$\frac{p}{a^2}x + \frac{q}{b^2}y + \frac{r}{c^2}z = \frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2}$$

$$\left.\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\right|_{(p,\ q,\ r)}$$

$$\frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}=1$$

$$\therefore\ \ \boxed{\frac{p}{a^2}x+\frac{q}{b^2}y+\frac{r}{c^2}z=1}$$

b) ***Given***: $Ax+By+Cz+1=0 \rightarrow$ The vector will be $\langle A,\ B,\ C\rangle$ and $\left\langle \frac{p}{a^2},\ \frac{q}{b^2},\ \frac{r}{c^2}\right\rangle$ must be proportional.

$$\left\langle \frac{p}{a^2},\ \frac{q}{b^2},\ \frac{r}{c^2}\right\rangle=\lambda\langle A,\ B,\ C\rangle$$

$$\begin{cases}\dfrac{p}{a^2}=\lambda A & \rightarrow\quad p=\lambda Aa^2\\ \dfrac{q}{b^2}=\lambda B & \rightarrow\quad q=\lambda Bb^2\\ \dfrac{r}{c^2}=\lambda C & \rightarrow\quad r=\lambda Cc^2\end{cases}$$

$$\langle p,\ q,\ r\rangle=\lambda\left\langle Aa^2,\ Bb^2,\ Cc^2\right\rangle$$

$$\frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}=1$$

$$\frac{\lambda^2A^2a^4}{a^2}+\frac{\lambda^2B^2b^4}{b^2}+\frac{\lambda^2C^2c^4}{c^2}=1$$

$$\lambda^2\left(A^2a^2+B^2b^2+C^2c^2\right)=1 \qquad\qquad m=\sqrt{a^2A^2+b^2B^2+c^2C^2}$$

$$\lambda^2m^2=1$$

$$\lambda=\pm\frac{1}{m}$$

$$\langle p,\ q,\ r\rangle=\pm\frac{1}{m}\left\langle Aa^2,\ Bb^2,\ Cc^2\right\rangle$$

Equations of the tangent planes: $\boxed{(p,\ q,\ r)=\pm\left(Aa^2,\ Bb^2,\ Cc^2\right)}$

c) The distance between the plane $Ax+By+Cz+1=0$ to the origin:

Let $S=(x,\ y,\ z)$ be the point on the plane, then

$$\overrightarrow{OS}=\langle x,\ y,\ z\rangle$$

$$\vec{n}=\langle A,\ B,\ C\rangle$$

$$|\vec{n}|=\sqrt{A^2+B^2+C^2}$$

$$d = \left| \frac{\langle x, y, z\rangle \bullet \langle A, B, C\rangle}{\sqrt{A^2+B^2+C^2}} \right|$$

Distance from a Point to a Plane: $d = \left| \frac{\overrightarrow{OS} \bullet \vec{n}}{|\vec{n}|} \right|$

$$= \left| \frac{Ax+By+Cz}{\sqrt{A^2+B^2+C^2}} \right|$$

$$Ax + By + Cz + 1 = 0$$

$$= \left| \frac{-1}{\sqrt{A^2+B^2+C^2}} \right|$$

$$= \frac{1}{\sqrt{A^2+B^2+C^2}}$$

$$= h \quad \checkmark$$

d) The tangent plane at $Q(p,\ q,\ r) = \pm\left(Aa^2,\ \ Bb^2,\ \ Cc^2\right)$ has an equation $Ax + By + Cz = \pm m$

$$\overrightarrow{OQ} = \left\langle Aa^2,\ \ Bb^2,\ \ Cc^2 \right\rangle$$

$$\vec{n} = \langle A,\ B,\ C\rangle$$

$$|\vec{n}| = \sqrt{A^2 + B^2 + C^2}$$

$$d = \left| \frac{\left\langle Aa^2,\ Bb^2,\ Cc^2 \right\rangle \bullet \langle A, B, C\rangle}{\sqrt{A^2+B^2+C^2}} \right| \qquad d = \left| \frac{\overrightarrow{OQ} \bullet \vec{n}}{|\vec{n}|} \right|$$

$$= \frac{a^2A^2+b^2B^2+c^2C^2}{\sqrt{A^2+B^2+C^2}} \qquad m = \sqrt{a^2A^2 + b^2B^2 + c^2C^2} \qquad h = \frac{1}{\sqrt{A^2 + B^2 + C^2}}$$

$$\underline{= hm \,|} \quad \checkmark$$

e) For the plane *P* does not intersect the ellipsoid if and only if the 2 tangent planes parallel to *P* are closer to the origin than *P*; this is equivalent to the condition $m < 1$.