Lecture R – Introduction to Differential Equation

Section R.1 – Derivative

Constant Rule

$$\frac{d}{dx}[c] = 0 \quad c \text{ is constant}$$

Example

Find the derivative:

a) $f(x) = -2$ $\qquad f'(x) = 0$

b) $y = \pi$ $\qquad y' = 0$

Power Rule

$$\frac{d}{dx}[x^n] = nx^{n-1} \qquad n \text{ is any real number}$$

Constant Times a Function

$$\frac{d}{dx}\left[cf(x)\right] = cf'(x)$$

Example

Find the derivative each function

a. $y = \frac{9}{4x^2} = \frac{9}{4}x^{-2}$

Solution

$$\rightarrow y' = \frac{9}{4}(-2)x^{-3} = -\frac{9}{2x^3}$$

b. $y = \sqrt[3]{x} = x^{1/3}$

Solution

$$\rightarrow y' = \frac{1}{3}x^{(1/3)-1}$$

$$= \frac{1}{3}x^{-2/3}$$

$$= \frac{1}{3\sqrt[3]{x^2}}$$

The Product Rule

The derivative of the product of two differentiable functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first,

$$(f.g)' = f.g' + f'.g$$

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'gh + fg'h + fgh'$$

Example

Find the derivative of $y = \left(4x + 3x^2\right)(6 - 3x)$

Solution

$$y = 24x + 6x^2 - 9x^3$$

$$\begin{aligned} y' &= \left(4x + 3x^2\right)\frac{d}{dx}(6 - 3x) + (6 - 3x)\frac{d}{dx}\left(4x + 3x^2\right) \\ &= \left(4x + 3x^2\right)(-3) + (6 - 3x)(4 + 6x) \\ &= -12x - 9x^2 + 24 + 36x - 12x - 18x^2 \\ &= -27x^2 + 12x + 24 \end{aligned}$$

Example

Find the derivative of $y = \left(\frac{1}{x} + 1\right)(2x + 1)$

Solution

$$\begin{aligned} y' &= \left(x^{-1} + 1\right)\frac{d}{dx}(2x + 1) + (2x + 1)\frac{d}{dx}\left(x^{-1} + 1\right) \\ &= \left(x^{-1} + 1\right)(2) + (2x + 1)\left(-x^{-2}\right) \\ &= \frac{2}{x} + 2 + (2x + 1)\left(-\frac{1}{x^2}\right) \\ &= \frac{2}{x} + 2 - \frac{2x}{x^2} - \frac{1}{x^2} \\ &= \frac{2}{x} + 2 - \frac{2}{x} - \frac{1}{x^2} \\ &= 2 - \frac{1}{x^2} \\ &= \frac{2x^2 - 1}{x^2} \end{aligned}$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}=\frac{gf'-fg'}{g^2}$$

Example

Find the derivative of $y=\dfrac{x+4}{5x-2}$

Solution

$$y'=\frac{(5x-2)\frac{d}{dx}[(x+4)]-(x+4)\frac{d}{dx}[(5x-2)]}{(5x-2)^2}$$

$$=\frac{(5x-2)(1)-(x+4)(5)}{(5x-2)^2}$$

$$=\frac{5x-2-5x-20}{(5x-2)^2}$$

$$=-\frac{22}{(5x-2)^2}$$

Example

Find the derivative of $y=\dfrac{3-\frac{2}{x}}{x+4}$

Solution

$$y=\frac{\frac{3x-2}{x}}{x+4}=\frac{3x-2}{x}\cdot\frac{1}{x+4}=\frac{3x-2}{x^2+4x}$$

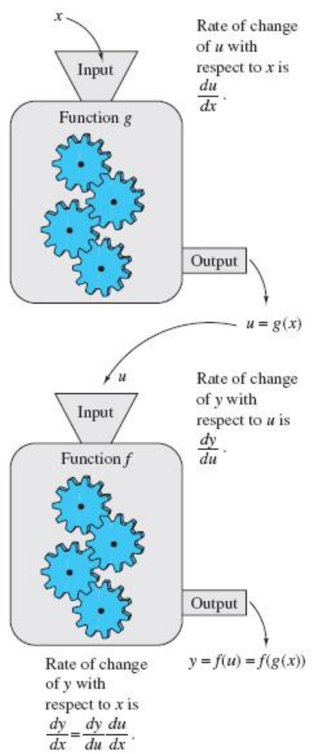
$$y'=\frac{\left(x^2+4x\right)(3)-(3x-2)(2x+4)}{\left[x(x+4)\right]^2}$$

$$\left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{3x^2+12x-6x^2-12x+4x+8}{x^2(x+4)^2}$$

$$=\frac{-3x^2+4x+8}{x^2(x+4)^2}$$

The Chain Rule



The Chain Rule

If $y = f(u)$ is a differentiable function of u, and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

The General Power Rule

$$\frac{dy}{dx} = \frac{d}{dx}\left[u(x)^n\right] = n\ u^{n\text{-}1}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[u^n\right] = \ n\ u^{n\text{-}1}u'$$

Example

Find the derivative of $y = \left(x^2 + 3x\right)^4$

Solution

$$u = x^2 + 3x$$

$$y' = n \quad (u)^{n-1} \quad \frac{d}{dx}[u]$$

$$= 4\left(x^2 + 3x\right)^3 \frac{d}{dx}[x^2 + 3x]$$

$$= 4\left(x^2 + 3x\right)^3 (2x + 3)$$

Derivatives of Trigonometric Functions

$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$	$(\tan x)' = \sec^2 x$
$(\csc x)' = -\csc x \cot x$	$(\sec x)' = \sec x \tan x$	$(\cot x)' = -\csc^2 x$

Example

Find the derivatives

a) $y = \sin x \cos x$

$$y' = \sin x(\cos x)' + \cos x(\sin x)'$$

$$= \sin x(-\sin x) + \cos x(\cos x)$$

$$= \cos^2 x - \sin^2 x$$

b) $y = \dfrac{\cos x}{1-\sin x}$

$$y' = \frac{(1-\sin x)(\cos x)' - \cos x(1-\sin x)'}{(1-\sin x)^2}$$

$$= \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$$

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$

$$= \frac{1-\sin x}{(1-\sin x)^2}$$

$$= \frac{1}{1-\sin x}$$

Derivative of the Natural Exponential Function

$$\frac{d}{dx}\left[e^{x}\right]=e^{x}$$

$$\frac{d}{dx}\left[e^{U}\right]=e^{U}\,\frac{dU}{dx}$$

Differentiate each function.

a) $f(x)=e^{-2x^3}$

$$f'(x)=e^{-2x^3}\,\frac{d}{dx}[-2x^3]$$

$$=e^{-2x^3}\left[-6x^2\right]$$

$$\boxed{=-\frac{6x^2}{e^{2x^3}}}$$

b) $f(x)=4e^{x^2}$

$$f'(x)=4e^{x^2}\,\frac{d}{dx}[x^2]$$

$$=4e^{x^2}(2x)$$

$$\boxed{=8xe^{x^2}}$$

c) $y=10e^{3x^2}$

$$y'=10e^{3x^2}\left(3x^2\right)'$$

$$=10e^{3x^2}\left(6x\right)$$

$$\boxed{=60x\,e^{3x^2}}$$

Derivative of ***ln***

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx} = \frac{u'}{u}$$

Note: $\ln U \Rightarrow U > 0$

Derivative of $\log_a x$

$$\frac{d}{dx}\left[\log_a x\right] = \frac{1}{(\ln a)x}$$

Example

Find the Derivatives

a) $f(x) = \ln(x^2 - 4)$

$Let \quad u = x^2 - 4 \quad \Rightarrow \quad \frac{du}{dx} = 2x$

$f'(x) = \frac{1}{u}\frac{du}{dx}$

$= \frac{1}{x^2-4}(2x)$

$\underline{= \frac{2x}{x^2-4}}$

b) $f(x) = x^2 \ln x$

$f' = x^2 \frac{d}{dx}[\ln x] + \ln x \frac{d}{dx}\left[x^2\right] \qquad (fg)' = f'g + fg'$

$= x^2\left(\frac{1}{x}\right) + 2x\ln x$

$= x + 2x\ln x$

$\underline{= x(1 + 2\ln x)}$

c) $f(x) = -\frac{\ln x}{x^2}$

$f' = -\frac{x^2\frac{d}{dx}[\ln x] - \ln x \frac{d}{dx}[x^2]}{(x^2)^2}$

$= -\frac{x^2\frac{1}{x} - 2x\ln x}{x^4}$

$$= -\frac{x - 2x\ln x}{x^4}$$

$$= -\frac{x(1 - 2\ln x)}{x^4}$$

$$\boxed{= -\frac{1 - 2\ln x}{x^3}}$$

Other Bases and Differentiation

$$\frac{d}{dx}\left[a^x\right] = a^x \ln a$$

$$\frac{d}{dx}\left[a^u\right] = a^u (\ln a)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\log_a x\right] = \left(\frac{1}{\ln a}\right)\frac{1}{x} \quad \frac{d}{dx}\left[\log_a u\right] = \left(\frac{1}{\ln a}\right)\left(\frac{1}{u}\right)\frac{du}{dx}$$

Formula $\quad \left(U^m V^n W^p\right)' = U^{m-1} V^{n-1} W^{p-1}\left(mU'VW + nUV'W + pUVW'\right)$

Proof

$$\left(U^m V^n W^p\right)' = \left(U^m\right)' V^n W^p + U^m \left(V^n\right)' W^p + U^m V^n \left(W^p\right)'$$

$$= mU^{m-1}U'V^nW^p + nU^mV^{n-1}V'W^p + pU^mV^nW^{p-1}W' \qquad \textit{factor} \ \ U^{m-1}V^{n-1}W^{p-1}$$

$$= U^{m-1}V^{n-1}W^{p-1}\left(mU'VW + nUV'W + pUVW'\right)$$

$$\left(U^m V^n\right)' = U^{m-1}V^{n-1}\left(mU'V + nUV'\right)$$

Exercises *Section* R.1 – Derivative

Find the derivative to the following functions

1. $f(t) = -3t^2 + 2t - 4$

2. $g(x) = 4\sqrt[3]{x} + 2$

3. $f(x) = x\left(x^2 + 1\right)$

4. $f(x) = \frac{2x^2 - 3x + 1}{x}$

5. $f(x) = \frac{4x^3 - 3x^2 + 2x + 5}{x^2}$

6. $f(x) = \frac{-6x^3 + 3x^2 - 2x + 1}{x}$

7. $f(x) = x\left(1 - \frac{2}{x+1}\right)$

8. $g(s) = \frac{s^2 - 2s + 5}{\sqrt{s}}$

9. $f(x) = \frac{x+1}{\sqrt{x}}$

10. $f(x) = 3x\left(2x^2 + 5x\right)$

11. $y = 3\left(2x^2 + 5x\right)$

12. $y = \frac{x^2 + 4x}{5}$

13. $y = \frac{3x^4}{5}$

14. $y = \frac{x^2 - 4}{2x + 5}$

15. $y = \frac{(1+x)(2x-1)}{x-1}$

16. $y = \frac{4}{2x+1}$

17. $y = \frac{2}{(x-1)^3}$

Use the General Power Rule to find the derivative of the function

18. $f(x) = \sqrt{2t^2 + 5t + 2}$

19. $f(x) = \frac{1}{\left(x^2 - 3x\right)^2}$

20. $y = t^2\sqrt{t-2}$

21. $y = \left(\frac{6 - 5x}{x^2 - 1}\right)^2$

22. $y = x^2\sqrt{x^2 + 1}$

23. $y = \left(\frac{x+1}{x-5}\right)^2$

24. $y = \sqrt[3]{(x+4)^2}$

Find the derivative of the trigonometric function

25. $y = x^2 \sin x$

26. $y = \frac{\sin x}{x}$

27. $y = \frac{\cot x}{1 + \cot x}$

28. $y = x^2 \sin x + 2x\cos x - 2\sin x$

29. $y = x^3 \sin x \cos x$

30. $y = \frac{4}{\cos x} + \frac{1}{\tan x}$

Differentiate each function.

31. $f(x) = x^2 e^x$

32. $f(x) = \frac{e^x + e^{-x}}{2}$

33. $f(x) = (1 + 2x)e^{4x}$

34. $y = x^2 e^{5x}$

35. $y = e^{x^2+1}\sqrt{5x+2}$

36. $f(x) = \frac{e^x}{x^2}$

37. $f(x) = x^2 e^x - e^x$

38. $f(x) = \ln \sqrt[3]{x+1}$

39. $f(x) = \ln\left[x^2\sqrt{x^2+1}\right]$

40. $y = \ln \dfrac{x^2}{x^2+1}$

41. $y = \ln \dfrac{1+e^x}{1-e^x}$

42. $y = x\ 3^{x+1}$

43. $f(t) = \dfrac{\log_8\left(t^{3/2}+1\right)}{t}$

Section R.2 – Integration

Definition of Antiderivative

A Function F is an Antiderivative of a function f if for every x in the domain of f, it follows that $F'(x) = f(x)$

Notation for Antiderivatives and indefinite integrals

The notation

$$\int f(x)dx = F(x) + C$$

where C is an arbitrary constant, means that F is an Antiderivative of f.
That is $F'(x) = f(x)$ for all x in the domain of f.

$\int f(x)dx$ Indefinite integral

Integrand   Antiderivative

Integral sign

$$\int f(x)dx = \overbrace{F(x)}^{\text{Antiderivative}} + C$$

Differential

Basic Integration Rules

$$\int kdx = kx + C$$

$$\int kf(x)dx = k\int f(x)dx$$

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

The General Power Rule

The Simple Power Rule is given by:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \mathbf{n \neq -1}$$

$$\int \overbrace{(x^2+1)^3}^{u^3} \underbrace{2x dx}_{du} = \int u^3 du$$

$$= \frac{u^4}{4} + C$$

General Power Rule for Integration

If u is a differentiable function of x, then

$$\int u^n \frac{du}{dx} dx = \int u^n du = \frac{u^{n+1}}{n+1} + C \qquad \mathbf{n \neq -1}$$

Example

Find each indefinite integral.

$$\int 5x dx = \int 5x^1 dx$$

$$= 5\frac{x^{1+1}}{1+1} + C$$

$$= \frac{5}{2}x^2 + C$$

$$\int \sqrt[3]{x} dx = \int x^{1/3} dx$$

$$= \frac{x^{1/3+1}}{1/3+1} + C$$

$$= \frac{x^{4/3}}{4/3} + C$$

$$\boxed{= \frac{3}{4}x^{4/3} + C} \qquad or \qquad = \frac{3}{4}x\sqrt[3]{x} + C$$

Using the Exponential Rule

Let u be a differentiable function of u

$$\int e^u du = e^u + C$$ *General Exponential Rule*

Example

Find the indefinite integral $\int e^{2x+3}dx$

Solution

Let $u = 2x + 3 \rightarrow \mathrm{d}u = 2\mathrm{d}x$

$$\int e^{2x+3}dx = \int e^u \tfrac{1}{2}du$$

$$= \tfrac{1}{2}\int e^u du$$

$$= \tfrac{1}{2}e^u + C$$

$$\boxed{= \tfrac{1}{2}e^{2x+3} + C}$$

Using the Log Rule

Let u be a differentiable function of x.

$$\int \frac{du/dx}{u}dx = \int \frac{1}{u}du = \ln|u| + C$$ *General Logarithmic Rule*

Example

Find the indefinite integral $\int \frac{1}{4x+1}dx$

Solution

Let $u = 4x + 1 \rightarrow du = 4dx \rightarrow \frac{1}{4}du = dx$

$$\int \tfrac{1}{4x+1}dx = \int \tfrac{1}{u}\tfrac{1}{4}du$$

$$= \tfrac{1}{4}\int \tfrac{1}{u}du$$

$$= \tfrac{1}{4}\ln|u| + C$$

$$\boxed{= \tfrac{1}{4}\ln|4x+1| + C}$$

Integration by Parts

Let u and v be differentiable functions of x.

$$\int u\,dv = uv - \int v\,du$$

Guidelines for integration by Parts

1. Let dv be the most complicated portion of the integrand that fits a basic integration formula. Let u be the remaining factor.
2. Let u be the portion of the integrand whose derivative is a function simpler than u. Let dv be the remaining factor.

Example

Find $\int x\cos x\,dx$

Solution

Let:

$$u = x \qquad dv = \cos x\,dx$$

$$du = dx \quad v = \int dv = \int \cos x\,dx = \sin x$$

$$\int u\,dv = uv - \int v\,du$$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + C$$

Example

Evaluate $\int x^2 e^x\,dx$

Solution

$$f(x) = x^2 \quad \text{and} \quad g(x) = e^x$$

$$\int x^2 e^x\,dx = x^2e^x - 2xe^x + 2e^x + C$$

	derivative	$\int e^x\,dx$
$(+)$	x^2	e^x
$(-)$	$2x$	e^x
$(+)$	2	e^x
	0	

Particular Solutions

In many applications of integrations, we are given enough information to determine a particular solution. To do this, we need to know the value of *F(x)* for one value of *x*. This information is called an initial condition.

Example

Solve the differential equation: $y' = te^t$ that satisfies $y(0) = 2$

Solution

$$y = \int te^t dt$$

Integration by part: $\int udv = uv - \int vdu$

$$\begin{cases} u = t \Rightarrow du = dt \\ dv = e^t dt \Rightarrow v = e^t \end{cases}$$

$$y = te^t - \int e^t dt$$

$$= te^t - e^t + C$$

$$y(0) = (0)e^0 - e^0 + C = 2$$

$$-1 + C = 2$$

$$C = 3$$

$$\underline{y(t) = e^t(t-1) + 3}$$

Example

Solve the differential equation: $y' = \frac{1}{x}$ that satisfies $y(1) = 3$

Solution

$$y = \int \frac{1}{x} dx$$

$$= \ln|x| + C$$

$$y(1) = \ln|1| + C = 3$$

$$\boxed{C = 3}$$

$$\underline{y(x) = \ln x + 3} \qquad \textit{with } x > 0$$

Example

Suppose a ball thrown into the air with initial velocity $v_0 = 20\ ft/\sec$. Assuming the ball thrown from a height of $x_0 = 6\ ft$, how long does it take for the ball to hit the ground?

Solution

$$\frac{dv}{dt} = -g \Rightarrow dv = -gdt$$

$$v(t) = -gt + C_1$$

$$v(t=0) = -g(0) + C_1 = 20$$

$$C_1 = 20$$

$$v(t) = -32t + 20$$

$$\frac{dx}{dt} = v \Rightarrow dx = vdt$$

$$\int dx = \int vdt$$

$$x(t) = \int (-32t + 20)dt$$

$$= -16t^2 + 20t + C_2$$

$$x(t=0) = -16(0)^2 + 20(0) + C_2 = 6$$

$$C_2 = 6$$

$$\underline{x(t) = -16t^2 + 20t + 6}$$

Theorem – The Fundamental Theorem of Calculus, P-2

If f is continuous at every point in $[a, b]$, then F is any antiderivative of f on $[a, b]$, then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Example

a) $\displaystyle\int_0^\pi \cos x\; dx = \sin x \Big|_0^\pi$

$$= \sin\pi - \sin 0$$
$$= \underline{0}$$

b) $\displaystyle\int_{-\frac{\pi}{4}}^{0} \sec x \tan x\; dx = \sec x \Big|_{-\frac{\pi}{4}}^{0}$

$$= \sec 0 - \sec\left(-\frac{\pi}{4}\right)$$
$$= \underline{1 - \sqrt{2}}$$

c) $\displaystyle\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4$

$$= \left((4)^{3/2} + \frac{4}{4}\right) - \left((1)^{3/2} + \frac{4}{1}\right)$$
$$= (9) - (5)$$
$$= \underline{4}$$

Exercises *Section* **R.2 – Integration**

Find each indefinite integral.

1. $\int \frac{x+2}{\sqrt{x}}dx$

2. $\int 4y^{-3}dy$

3. $\int (x^3-4x+2)dx$

4. $\int \left(\sqrt[4]{x^3}+1\right)dx$

5. $\int \sqrt{x}(x+1)dx$

6. $\int (1+3t)t^2 dt$

7. $\int \frac{x^2-5}{x^2}dx$

8. $\int (-40x+250)dx$

9. $\int \left(7-3x-3x^2\right)(2x+1)\ dx$

10. $\int xe^{2x}dx$

11. $\int x\ln x dx$

12. $\int \left(x^2-2x+1\right)e^{2x}dx$

13. $\int e^{2x}\cos 3x dx$

Find the general solution of the differential equation

14. $y' = 2t+3$

15. $y' = 3t^2+2t+3$

16. $y' = \sin 2t + 2\cos 3t$

17. $y' = x^3(3x^4+1)^2$

18. $y' = 5x\sqrt{x^2-1}$

19. $y' = x\sqrt{x^2+4}$

20. $y' = (2x+1)e^{x^2+x}$

21. $y' = \frac{1}{6x-5}$

22. $y' = \frac{x^2+2x+3}{x^3+3x^2+9x+1}$

23. $y' = \frac{1}{x(\ln x)^2}$

Evaluate the integrals

24. $\int_{-2}^{2}\left(x^3-2x+3\right)dx$

25. $\int_0^1\left(x^2+\sqrt{x}\right)dx$

26. $\int_0^{\pi/3} 4\sec u \tan u\ du$

27. $\int_{\pi/4}^{3\pi/4} \csc\theta \cot\theta d\theta$

28. $\int_{-\pi/3}^{-\pi/4}\left(4\sec^2 t+\frac{\pi}{t^2}\right)dt$

29. $\int_{-3}^{-1}\frac{y^5-2y}{y^3}dy$

30. $\int_1^8 \frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}}dx$

31. $\int_0^1 (2t+3)^3 dt$

32. $\int_{-1}^{1} r\sqrt{1-r^2}\ dr$

33. Find the general solution of $F'(x) = 4x + 2$, and find the particular solution that satisfies the initial condition $F(1) = 8$.

34. Find the general solution of the differential equation: $y' = t\cos 3t$

35. A ball is thrown into the air from an initial height of 6 *m* with an initial velocity of 120 *m/s*. What will be the maximum height of the ball and at what time will this event occur?

36. Derive the position function if a ball is thrown upward with initial velocity of 32 *ft* per *second* from an initial height of 48 *ft*. When does the ball hit the ground? With what velocity does the ball hit the ground?