Solution Section 2.1 – Integration by Parts

Exercise

Evaluate the integral $\int xe^{2x}dx$

Solution

		$\int e^{2x}\ dx$
+	x	$\frac{1}{2}e^{2x}$
–	1	$\frac{1}{4}e^{2x}$

$$\int xe^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

Let: $u = x \Rightarrow du = dx$

$$dv = e^{2x}dx \Rightarrow v = \int dv = \int e^{2x}dx = \frac{1}{2}e^{2x}$$

$$\int udv = uv - \int vdu$$

$$\int xe^{2x}dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

Exercise

Evaluate the integral $\int x\ln x\ dx$

Solution

Let: $u = \ln x \Rightarrow du = \frac{1}{x}dx$

$$dv = xdx \Rightarrow v = \int dv = \int xdx = \frac{1}{2}x^2$$

$$\int x\ln xdx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x^2\frac{dx}{x}$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int xdx$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$$

Exercise

Evaluate the integral $\int x^3 e^x dx$

Solution

		$\int e^x\, dx$
+	x^3	e^x
−	$3x^2$	e^x
+	$6x$	e^x
−	6	e^x

$$\int x^3 e^x dx = \underline{e^x\left(x^3 - 3x^2 + 6x - 6\right) + C}$$

Let: $u = x^3 \Rightarrow du = 3x^2 dx$

$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$

$$\int x^3 e^x dx = x^3 e^x - \int e^x 3x^2 dx$$

$$= x^3 e^x - 3\int e^x x^2 dx$$

Let: $u = x^2 \Rightarrow du = 2x dx$

$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$

$$\int e^x x^2 dx = x^2 e^x - 2\int x e^x dx$$

$$\int x^3 e^x dx = x^3 e^x - 3\left[x^2 e^x - 2\int x e^x dx\right]$$

$$= x^3 e^x - 3x^2 e^x + 6\int x e^x dx$$

Let: $u = x \Rightarrow du = dx$

$dv = e^x dx \Rightarrow v = \int e^x dx = e^x$

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x$$

$$\int x^3 e^x dx = x^3 e^x - 3x^2 e^x + 6\left[x e^x - e^x\right] + C$$

$$= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$$

$$= \underline{e^x\left(x^3 - 3x^2 + 6x - 6\right) + C}$$

Exercise

Evaluate the integral $\int \ln x^2 dx$

Solution

$$\int \ln x^2 dx = 2\int \ln x dx \qquad u = \ln x \Rightarrow du = \frac{1}{x}dx \qquad v = \int dx = x$$

$$\int \ln x^2 dx = 2\left[x \ln x - \int x \frac{1}{x} dx\right]$$

$$= 2\left[x\ln x - \int dx \right]$$

$$= 2(x\ln x - x) + C$$

$$\boxed{= 2x(\ln x - 1) + C}$$

Exercise

Evaluate the integral $\int \frac{2x}{e^x} dx$

Solution

		$\int e^{-x}\,dx$
+	$2x$	$-e^{-x}$
−	2	e^{-x}

$$\int \frac{2x}{e^x} dx \boxed{= -e^{-x}(2x+2) + C}$$

$u = 2x \Rightarrow du = 2dx$

$dv = e^{-x}dx \Rightarrow v = -e^{-x}$

$$\int \frac{2x}{e^x} dx = 2x(-e^{-x}) - \int -e^{-x}\,2dx$$

$$= -2xe^{-x} + 2\int e^{-x}dx$$

$$= -2xe^{-x} - 2e^{-x} + C$$

$$= -2e^{-x}(x+1) + C$$

$$\boxed{= -\frac{2(x+1)}{e^x} + C}$$

Exercise

Evaluate the integral $\int \ln(3x)dx$

Solution

$u = \ln 3x \Rightarrow du = \frac{3}{3x}dx = \frac{1}{x}dx \qquad dv = dx \Rightarrow v = x$

$$\int \ln(3x)dx = x\ln(3x) - \int x\frac{1}{x}dx$$

$$= x\ln(3x) - \int dx$$

$$= x\ln(3x) - x + C$$

$$\boxed{= x\left[\ln(3x) - 1\right] + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{1}{x\ln x}\,dx$

Solution

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{\ln x}\frac{1}{x}\,dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$$

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du$$

$$= \ln u + C$$

$$= \ln\left|\ln x\right| + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x}{\sqrt{x-1}}\,dx$

Solution

Let: $u = x \ \Rightarrow du = dx$

$$dv = \frac{dx}{\sqrt{x-1}} \Rightarrow v = \int (x-1)^{-1/2}\,d(x-1)$$

$$= \frac{(x-1)^{1/2}}{1/2}$$

$$= 2(x-1)^{1/2}$$

$$\int \frac{x}{\sqrt{x-1}}\,dx = 2x\sqrt{x-1} - 2\int (x-1)^{1/2}\,dx$$

$$= 2x\sqrt{x-1} - 2\frac{(x-1)^{3/2}}{3/2} + C$$

$$= 2x\sqrt{x-1} - \frac{4}{3}(x-1)\sqrt{x-1} + C$$

$$= \sqrt{x-1}\left[2x - \frac{4}{3}x + \frac{4}{3}\right] + C$$

$$= \sqrt{x-1}\left[\frac{6x-4x+4}{3}\right] + C$$

$$= \sqrt{x-1}\left[\frac{2x+4}{3}\right] + C$$

$$= \frac{2}{3}\sqrt{x-1}(x+2) + C$$

Let: $u = x-1 \ \Rightarrow \ x = u+1$
$du = dx$

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int (u+1)u^{-1/2}du$$

$$= \int \left(u^{1/2} + u^{-1/2}\right)du$$

$$= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C$$

$$= (x-1)^{1/2}\left(\frac{2}{3}x - \frac{2}{3} + 2\right) + C$$

$$= \sqrt{x-1}\left[\frac{2x+4}{3}\right] + C$$

$$= \frac{2}{3}\sqrt{x-1}(x+2) + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{x^3 e^{x^2}}{\left(x^2+1\right)^2}dx$

Solution

Let: $u = x^2 e^{x^2} \quad \Rightarrow du = \left(2xe^{x^2} + 2xx^2 e^{x^2}\right)dx$

$$du = 2xe^{x^2}\left(1+x^2\right)dx$$

$$dv = x\left(x^2+1\right)^{-2}dx \qquad \Rightarrow v = \int x(x^2+1)^{-2}dx$$

$$= \frac{1}{2}\int (x^2+1)^{-2}d(x^2+1)$$

$$= \frac{(x^2+1)^{-1}}{-1}$$

$$= -\frac{1}{2(x^2+1)}$$

$$\int \frac{x^3e^{x^2}}{(x^2+1)^2}dx = x^2e^{x^2}\left(-\frac{1}{2(x^2+1)}\right) - \int -\frac{1}{2(x^2+1)}2xe^{x^2}(x^2+1)dx$$

$$= -\frac{x^2e^{x^2}}{2(x^2+1)} + \int xe^{x^2}dx$$

Let: $u = x^2 \quad \Rightarrow du = 2xdx$

$$\int \frac{x^3e^{x^2}}{\left(x^2+1\right)^2}dx = -\frac{x^2e^{x^2}}{2(x^2+1)} + \frac{1}{2}\int e^u du$$

$$= -\frac{x^2e^{x^2}}{2(x^2+1)} + \frac{1}{2}e^u + C$$

$$= -\frac{x^2e^{x^2}}{2(x^2+1)} + \frac{1}{2}e^{x^2} + C$$

$$= \frac{1}{2}e^{x^2}\left[-\frac{x^2}{(x^2+1)} + 1\right] + C$$

$$= \frac{1}{2}e^{x^2}\left[\frac{-x^2+x^2+1}{(x^2+1)}\right] + C$$

$$= \frac{e^{x^2}}{2(x^2+1)} + C$$

Exercise

Evaluate the integral $\int x^2 e^{-3x}dx$

Solution

$u = x^2 \Rightarrow du = 2xdx$

$dv = e^{-3x}dx \Rightarrow v = -\frac{1}{3}e^{-3x}$

$$\int x^2 e^{-3x}dx = -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\int xe^{-3x}dx$$

$u = x \Rightarrow du = dx$

$dv = e^{-3x}dx \Rightarrow v = -\frac{1}{3}e^{-3x}$

$$\int x^2 e^{-3x}dx = -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\left[-\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}dx\right]$$

$$= -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\left[-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right] + C$$

$$= -\frac{1}{3}x^2 e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C$$

$$= -\frac{9x^2+6x+2}{27}e^{-3x} + C$$

		$\int e^{-3x}$
+	x^2	$-\frac{1}{3}e^{-3x}$
−	$2x$	$\frac{1}{9}e^{-3x}$
+	2	$-\frac{1}{27}e^{-3x}$

$$\int x^2 e^{-3x}dx = -\frac{1}{3}x^2 e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C$$

Exercise

Evaluate the integral $\int \theta \cos \pi\theta d\theta$

Solution

Let: $u = \theta \qquad dv = \cos \pi\theta d\theta$

$du = d\theta \qquad v = \int \cos \pi\theta d\theta = \frac{1}{\pi}\sin \pi\theta$

$$\int \theta \cos \pi\theta d\theta = \frac{\theta}{\pi}\sin \pi\theta - \int \frac{1}{\pi}\sin \pi\theta d\theta$$

$$= \frac{\theta}{\pi}\sin \pi\theta + \frac{1}{\pi}\frac{1}{\pi}\cos \pi\theta + C$$

$$= \frac{\theta}{\pi}\sin \pi\theta + \frac{1}{\pi^2}\cos \pi\theta + C$$

Exercise

Evaluate the integral $\int x^2 \sin x \, dx$

Solution

		$\int \sin x$
x^2	$(+)$	$-\cos x$
$2x$	$(-)$	$-\sin x$
2	$(+)$	$\cos x$
0		

$$\int x^2 \sin x dx = \underline{-x^2 \cos x + 2x \sin x + 2\cos x + C}$$

Exercise

Evaluate the integrals $\int x(\ln x)^2 \, dx$

Solution

$u = \ln x \quad \to \quad x = e^u$

$du = \frac{1}{x}dx \quad \Rightarrow \quad xdu = dx \to dx = e^u du$

$$\int x(\ln x)^2 \, dx = \int e^u u^2 e^u du$$

$$= \int u^2 e^{2u} du$$

$$= \frac{1}{2}u^2 e^{2u} - \frac{1}{2}ue^{2u} + \frac{1}{4}e^{2u} + C$$

$$= \frac{1}{4}e^{2u}\left(2u^2 - 2u + 1\right) + C$$

$$= \frac{1}{4}x^2\left(2(\ln x)^2 - 2\ln x + 1\right) + C$$

	$\int e^{2u} du$
u^2	$\frac{1}{2}e^{2u}$
$2u$	$\frac{1}{4}e^{2u}$
2	$\frac{1}{8}e^{2u}$
0	

2nd *Method*

$u = \ln x \qquad dv = \int (x \ln x) dx$

$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$

$$\int x(\ln x)^2 \, dx = (\ln x)\left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right) - \int \left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right)\frac{1}{x}dx$$

$u = \ln x \Rightarrow du = \frac{1}{x}dx$

$dv = xdx \Rightarrow v = \frac{1}{2}x^2$

$$\int x \ln x dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x^2 \frac{dx}{x}$$

$$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{4}x^2\ln x - \int \left(\frac{1}{2}x\ln x - \frac{1}{4}x\right)dx$$

$$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{4}x^2\ln x - \left(\frac{1}{2}\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right) - \frac{1}{8}x^2\right) + C$$

$$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{4}x^2\ln x - \frac{1}{4}x^2\ln x + \frac{1}{8}x^2 + \frac{1}{8}x^2 + C$$

$$\boxed{= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C}$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int xdx$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$$

3^{nd} Method

$u = (\ln x)^2 \qquad dv = \int xdx$

$du = 2(\ln x)\frac{1}{x}dx \qquad v = \frac{1}{2}x^2$

$$\int x(\ln x)^2\,dx = \frac{1}{2}x^2(\ln x)^2 - \int \frac{1}{2}x^2(2\ln x)\frac{1}{x}dx$$

$$= \frac{1}{2}x^2(\ln x)^2 - \int x\ln xdx$$

$$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{4}x^2\ln x - \left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right) + C$$

$$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{4}x^2\ln x - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C$$

$$\boxed{= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C}$$

$u = \ln x \Rightarrow du = \frac{1}{x}dx$

$dv = xdx \Rightarrow v = \frac{1}{2}x^2$

$$\int x\ln xdx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x^2\frac{dx}{x}$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int xdx$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$$

Exercise

Evaluate the integral $\int\left(x^2 - 2x + 1\right)e^{2x}dx$

Solution

		$\int e^{2x}$
+	$x^2 - 2x + 1$	$\frac{1}{2}e^{2x}$
−	$2x - 2$	$\frac{1}{4}e^{2x}$
+	2	$\frac{1}{8}e^{2x}$

$$\int\left(x^2 - 2x + 1\right)e^{2x}dx = \frac{1}{2}\left(x^2 - 2x + 1\right)e^{2x} - \frac{1}{4}(2x - 2)e^{2x} + \frac{1}{8}(2)e^{2x} + C$$

$$= \left(\frac{1}{2}x^2 - x + \frac{1}{2} - \frac{1}{2}x + \frac{1}{2} + \frac{1}{4}\right)e^{2x} + C$$

$$\boxed{= \left(\frac{1}{2}x^2 - \frac{3}{2}x + \frac{5}{4}\right)e^{2x} + C}$$

Exercise

Evaluate the integral $\int \tan^{-1} y\, dy$

Solution

Let: $u = \tan^{-1} y \quad dv = dy$

$du = \dfrac{dy}{1+y^2} \qquad v = y$

$$\int \tan^{-1} y\, dy = y\tan^{-1} y - \int \frac{ydy}{1+y^2} \qquad d\left(1+y^2\right) = 2ydy \ \rightarrow\ \tfrac{1}{2}d\left(1+y^2\right) = ydy$$

$$= y\tan^{-1} y - \int \frac{\frac{1}{2}d\left(1+y^2\right)}{1+y^2}$$

$$= y\tan^{-1} y - \frac{1}{2}\ln\left(1+y^2\right) + C$$

$$\boxed{= y\tan^{-1} y - \ln\sqrt{1+y^2} + C}$$

Exercise

Evaluate the integral $\int \sin^{-1} y\, dy$

Solution

Let: $u = \sin^{-1} y \quad dv = dy$

$du = \dfrac{dy}{\sqrt{1-y^2}} \qquad v = y$

$$\int \sin^{-1} y\, dy = y\sin^{-1} y - \int \frac{ydy}{\sqrt{1-y^2}} \qquad d\left(1-y^2\right) = -2ydy \ \rightarrow\ -\tfrac{1}{2}d\left(1-y^2\right) = ydy$$

$$= y\sin^{-1} y + \frac{1}{2}\int \left(1-y^2\right)^{-1/2} d\left(1-y^2\right)$$

$$= y\sin^{-1} y + \frac{1}{2}(2)\left(1-y^2\right)^{1/2} + C$$

$$\boxed{= y\sin^{-1} y + \sqrt{1-y^2} + C}$$

Exercise

Evaluate the integral $\displaystyle\int 4x\sec^2 2x\, dx$

Solution

Let: $u = 4x \quad \to du = 4 \qquad dv = \sec^2 2x dx \quad \to v = \frac{1}{2}\tan 2x$

$$\int 4x\sec^2 2x\, dx = 2x\tan 2x - \int 4\left(\frac{1}{2}\tan 2x\right)dx$$

$$= 2x\tan 2x - 2\frac{1}{2}\ln\left|\sec 2x\right| + C$$

$$\boxed{= 2x\tan 2x - \ln\left|\sec 2x\right| + C}$$

Exercise

Evaluate the integral $\displaystyle\int e^{2x}\cos 3x dx$

Solution

		$\int \cos 3x\, dx$
+	e^{2x}	$\frac{1}{3}\sin 3x$
-	$2e^{2x}$	$-\frac{1}{9}\cos 3x$
+	$4e^{2x}$	$-\frac{1}{9}\int \cos 3x\, dx$

$$\int e^{2x}\cos 3x dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x - \frac{4}{9}\int e^{2x}\cos 3x dx$$

$$\int e^{2x}\cos 3x dx + \frac{4}{9}\int e^{2x}\cos 3x dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x$$

$$\frac{13}{9}\int e^{2x}\cos 3x dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x$$

$$\int e^{2x}\cos 3x dx = \boxed{\frac{e^{2x}}{13}\left(3\sin 3x + 2\cos 3x\right) + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$

Solution

Let: $u = \sqrt{x} \quad \to \quad du = \frac{1}{2\sqrt{x}}dx \Rightarrow 2du = \frac{1}{\sqrt{x}}dx$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = \int (\cos u)(2du)$$

$$= 2\int \cos u\, du$$

$$= 2\sin u + C$$

$$\boxed{= 2\sin\sqrt{x} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{(\ln x)^3}{x}\,dx$

Solution

$$\int \frac{(\ln x)^3}{x}\,dx = \int (\ln x)^3\, d(\ln x) \qquad d(\ln x) = \frac{dx}{x}$$

$$= \boxed{\frac{1}{4}(\ln x)^4 + C}$$

Exercise

Evaluate the integral $\displaystyle\int x^5 e^{x^3}\,dx$

Solution

Let: $u = x^3 \qquad dv = x^2 e^{x^3}\,dx = \frac{1}{3}d\left(e^{x^3}\right) \qquad d\left(e^{x^3}\right) = 3x^2 e^{x^3}\,dx$

$du = 3x^2 dx \qquad v = \frac{1}{3}e^{x^3}$

$$\int x^5 e^{x^3}\,dx = x^3 \frac{1}{3}e^{x^3} - \int \frac{1}{3}e^{x^3} 3x^2 dx \qquad d\left(e^{x^3}\right) = 3x^2 e^{x^3}\,dx \qquad \int udv = uv - \int vdu$$

$$= \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}\int d\left(e^{x^3}\right)$$

$$= \boxed{\frac{1}{3}x^3 e^{x^3} - \frac{1}{3}e^{x^3} + C}$$

Exercise

Evaluate the integral $\displaystyle\int x^2 \ln x^3\,dx$

Solution

$$\int x^2 \ln x^3\,dx = \int 3x^2 \ln x\,dx \qquad u = \ln x \quad v = \int 3x^2 dx = x^3 \qquad \int udv = uv - \int vdu$$

$$du = \frac{1}{x}dx$$

$$= x^3 \ln x - \int x^2 dx$$

$$= \boxed{x^3 \ln x - \frac{1}{3}x^3 + C}$$

Exercise

Evaluate the integral $\int \ln\left(x+x^2\right)dx$

Solution

Let:

$$u=\ln\left(x+x^2\right) \qquad dv=dx$$

$$du=\frac{2x+1}{x+x^2}dx \qquad v=x$$

$$\begin{aligned}
\int \ln\left(x+x^2\right)dx &= x\ln\left(x+x^2\right)-\int x\frac{2x+1}{x+x^2}dx \\
&= x\ln\left(x+x^2\right)-\int \frac{2x+1}{x(1+x)}xdx \\
&= x\ln\left(x+x^2\right)-\int \frac{2x+2-1}{1+x}dx \\
&= x\ln\left(x+x^2\right)-\int \frac{2(x+1)-1}{x+1}dx \\
&= x\ln\left(x+x^2\right)-\int \left(2-\frac{1}{x+1}\right)dx \\
&= x\ln\left(x+x^2\right)-\left(2x-\ln|x+1|\right)+C \\
&= x\ln\left(x+x^2\right)-2x+\ln|x+1|+C
\end{aligned}$$

Exercise

Evaluate the integral $\int e^{-x}\sin 4x\,dx$

Solution

$$\int e^{-x}\sin 4x\,dx=-\frac{1}{4}e^{-x}\cos 4x-\frac{1}{16}e^{-x}\sin 4x-\frac{1}{16}\int e^{-x}\sin 4x\,dx$$

$$\left(1+\frac{1}{16}\right)\int e^{-x}\sin 4x\,dx=-\frac{1}{4}e^{-x}\cos 4x-\frac{1}{16}e^{-x}\sin 4x$$

$$\frac{17}{16}\int e^{-x}\sin 4x\,dx=-\frac{1}{16}e^{-x}\left(4\cos 4x+\sin 4x\right)$$

$$\int e^{-x}\sin 4x\,dx=-\frac{e^{-x}}{17}\left(4\cos 4x+\sin 4x\right)+C$$

		$\int \sin 4x\,dx$
+	e^{-x}	$-\frac{1}{4}\cos 4x$
-	$-e^{-x}$	$-\frac{1}{16}\sin 4x$
+	e^{-x}	$-\frac{1}{16}\int \sin 4x\,dx$

Exercise

Evaluate the integral $\displaystyle\int e^{-2\theta}\sin 6\theta\, d\theta$

Solution

$$\int e^{-2\theta}\sin 6\theta\, d\theta = -\frac{1}{6}e^{-2\theta}\cos 6\theta - \frac{1}{18}e^{-2\theta}\sin 6\theta - \frac{1}{9}\int e^{-2\theta}\sin 6\theta\, d\theta$$

$$\left(1+\frac{1}{9}\right)\int e^{-2\theta}\sin 6\theta\, d\theta = -\frac{1}{18}e^{-2\theta}\left(3\cos 6\theta + \sin 6\theta\right)$$

$$\frac{10}{9}\int e^{-2\theta}\sin 6\theta\, d\theta = -\frac{1}{18}e^{-2\theta}\left(3\cos 6\theta + \sin 6\theta\right)$$

$$\int e^{-2\theta}\sin 6\theta\, d\theta = -\frac{e^{-2\theta}}{20}\left(3\cos 6\theta + \sin 6\theta\right) + C$$

		$\int \sin 6\theta\, d\theta$
+	$e^{-2\theta}$	$-\frac{1}{6}\cos 6\theta$
−	$-2e^{-2\theta}$	$-\frac{1}{36}\sin 6\theta$
+	$4e^{-2\theta}$	$-\frac{1}{36}\int \sin 6\theta\, d\theta$

Exercise

Evaluate the integral $\displaystyle\int xe^{-4x}dx$

Solution

$$\int xe^{-4x}dx = \left(-\frac{x}{4}-\frac{1}{16}\right)e^{-4x} + C$$

		$\int e^{-4x}\, dx$
+	x	$-\frac{1}{4}e^{-4x}$
−	1	$\frac{1}{16}e^{-4x}$

Exercise

Evaluate the integral $\displaystyle\int x\ln(x+1)\,dx$

Solution

$u = \ln(x+1) \Rightarrow du = \frac{1}{x+1}dx$

$dv = xdx \Rightarrow v = \frac{1}{2}x^2$

$$\int x\ln(x+1)\,dx = \frac{1}{2}x^2\ln(x+1) - \frac{1}{2}\int \frac{x^2}{x+1}dx$$

$$= \frac{1}{2}x^2\ln(x+1) - \frac{1}{2}\int\left(x-1+\frac{1}{x+1}\right)dx$$

$$= \frac{1}{2}x^2\ln(x+1) - \frac{1}{2}\left(\frac{1}{2}x^2 - x + \ln(x+1)\right) + C$$

$$= \frac{1}{2}x^2\ln(x+1) - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{2}\ln(x+1) + C$$

$$= -\frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{2}\left(x^2-1\right)\ln(x+1) + C$$

Exercise

Evaluate the integral $\displaystyle\int \frac{(\ln x)^2}{x}dx$

Solution

$$\int \frac{(\ln x)^2}{x}dx = \int (\ln x)^2\, d(\ln x)$$

$$\boxed{= \frac{1}{3}(\ln x)^3 + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{xe^{2x}}{(2x+1)^2}dx$

Solution

$$u = xe^{2x} \quad \to \quad du = (2x+1)e^{2x}dx$$

$$dv = \frac{dx}{(2x+1)^2} = \frac{1}{2}\frac{d(2x+1)}{(2x+1)^2} \quad \to \quad v = -\frac{1}{2}\frac{1}{2x+1}$$

$$\int \frac{xe^{2x}}{(2x+1)^2}dx = -\frac{xe^{2x}}{4x+2} + \frac{1}{2}\int e^{2x}dx$$

$$\boxed{= -\frac{x}{4x+2}e^{2x} + \frac{1}{4}e^{2x} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{5x}{e^{2x}}dx$

Solution

		$\int e^{-2x}\,dx$
+	$5x$	$-\frac{1}{2}e^{-2x}$
−	5	$\frac{1}{4}e^{-2x}$

$$\int \frac{5x}{e^{2x}}dx = \int 5xe^{-2x}dx$$

$$\boxed{= \left(-\frac{5}{2}x - \frac{5}{4}\right)e^{-2x} + C}$$

Exercise

Evaluate the integral $\displaystyle\int \frac{e^{1/x}}{x^2}dx$

Solution

$$\int \frac{e^{1/x}}{x^2}dx = -\int e^{1/x} d\left(\frac{1}{x}\right)$$
$$= -e^{1/x} + C$$

Exercise

Evaluate the integral $\int x^5 \ln 3x\, dx$

Solution

$$u = \ln 3x \quad \to \quad du = \frac{1}{x}dx$$
$$dv = x^5\, dx \quad \to \quad v = \frac{1}{6}x^6$$
$$\int x^5 \ln 3x\, dx = \frac{1}{6}x^6 \ln 3x - \frac{1}{6}\int x^5 dx$$
$$= \frac{1}{6}x^6 \ln 3x - \frac{1}{36}x^6 + C$$

Exercise

Evaluate the integral $\int x\sqrt{x-5}\, dx$

Solution

Let $\quad u = \sqrt{x-5} \quad \to \quad u^2 = x-5 \;\Rightarrow\; x = u^2 + 5$

$2udu = dx$

$$\int x\sqrt{x-5}\, dx = \int \left(u^2+5\right) u\left(2udu\right)$$
$$= \int \left(2u^4 + 10u^2\right) du$$
$$= \frac{2}{5}u^5 + \frac{10}{3}u^3 + C$$

Exercise

Evaluate the integral $\int \frac{x}{\sqrt{6x+1}} dx$

Solution

$$u = x \quad \to \quad du = dx$$
$$dv = (6x+1)^{-1/2}\, dx = \frac{1}{6}(6x+1)^{-1/2} d(6x+1) \quad \to \quad v = \frac{1}{3}(6x+1)^{1/2}$$
$$\int \frac{x}{\sqrt{6x+1}} dx = \frac{1}{3}x\sqrt{6x+1} - \frac{1}{3}\int (6x+1)^{1/2}\, dx$$

$$= \frac{1}{3}x\sqrt{6x+1} - \frac{1}{18}\int (6x+1)^{1/2}\, d(6x+1)$$

$$= \frac{1}{3}x\sqrt{6x+1} - \frac{1}{27}(6x+1)^{3/2} + C$$

Exercise

Evaluate the integral $\int x\cos x\, dx$

Solution

		$\int \cos x$
+	x	$\sin x$
–	1	$-\cos x$

$$\int x\cos x\, dx = x\sin x + \cos x + C$$

Exercise

Evaluate the integral $\int x\csc x\cot x\, dx$

Solution

$u = x \quad \to \quad du = dx$

$dv = \csc x\cot x\, dx \quad \to \quad v = -\csc x$

$$\int x\csc x\cos x\, dx = -x\csc x + \int \csc dx$$

$$= -x\csc x - \ln\left|\csc x + \cot x\right| + C$$

Exercise

Evaluate the integral $\int x^3\sin x\, dx$

Solution

		$\int \sin x$
+	x^3	$-\cos x$
–	$3x^2$	$-\sin x$
+	$6x$	$\cos x$
–	6	$\sin x$

$$\int x^3\sin x\, dx = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$$

Exercise

Evaluate the integral $\int x^2\cos x\, dx$

Solution

		$\int \cos x$
+	x^2	$\sin x$
–	$2x$	$-\cos x$
+	2	$-\sin x$

$$\int x^2\cos x\, dx = x^2\sin x + 2x\cos x - 2\sin x + C$$

Exercise

Evaluate the integral $\displaystyle\int e^{-3x}\sin 5x\,dx$

Solution

$$\int e^{-3x}\sin 5x\,dx = -\frac{1}{5}e^{-3x}\cos 5x - \frac{3}{25}e^{-3x}\sin 5x - \frac{9}{25}\int e^{-3x}\sin 5x\,dx$$

$$\left(1+\frac{9}{25}\right)\int e^{-3x}\sin 5x\,dx = -\frac{1}{25}(5\cos 5x + 3\sin 5x)e^{-3x}$$

$$\frac{34}{25}\int e^{-3x}\sin 5x\,dx = -\frac{1}{25}(5\cos 5x + 3\sin 5x)e^{-3x}$$

$$\int e^{-3x}\sin 5x\,dx = -\frac{1}{34}(5\cos 5x + 3\sin 5x)e^{-3x} + C$$

		$\int \sin 5x$
+	e^{-3x}	$-\frac{1}{5}\cos 5x$
−	$-3e^{-3x}$	$-\frac{1}{25}\sin 5x$
+	$9e^{-3x}$	$-\int \frac{1}{25}\sin 5x$

Exercise

Evaluate the integral $\displaystyle\int e^{-3x}\sin 4x\,dx$

Solution

$$\int e^{-3x}\sin 4x\,dx = -\frac{1}{4}e^{-3x}\cos 4x - \frac{3}{16}e^{-3x}\sin 4x - \frac{9}{16}\int e^{-3x}\sin 4x\,dx$$

$$\left(1+\frac{9}{16}\right)\int e^{-3x}\sin 4x\,dx = -\frac{1}{16}(4\cos 4x + 3\sin 4x)e^{-3x}$$

$$\frac{25}{16}\int e^{-3x}\sin 4x\,dx = -\frac{1}{16}(4\cos 4x + 3\sin 4x)e^{-3x}$$

$$\int e^{-3x}\sin 4x\,dx = -\frac{1}{25}(4\cos 4x + 3\sin 4x)e^{-3x} + C$$

		$\int \sin 4x$
+	e^{-3x}	$-\frac{1}{4}\cos 4x$
−	$-3e^{-3x}$	$-\frac{1}{16}\sin 4x$
+	$9e^{-3x}$	$-\frac{1}{16}\int \sin 4x$

Exercise

Evaluate the integral $\displaystyle\int e^{4x}\cos 2x\,dx$

Solution

$$\int e^{4x}\cos 2x\,dx = \frac{1}{2}e^{4x}\sin 2x + e^{4x}\cos 2x - 4\int e^{4x}\cos 2x\,dx$$

$$5\int e^{4x}\cos 2x\,dx = \frac{1}{2}(\sin 2x + 2\cos 2x)e^{4x}$$

$$\int e^{4x}\cos 2x\,dx = \frac{1}{10}(\sin 2x + 2\cos 2x)e^{4x} + C$$

		$\int \cos 2x$
+	e^{4x}	$\frac{1}{2}\sin 2x$
−	$4e^{4x}$	$-\frac{1}{4}\cos 2x$
+	$16e^{4x}$	$-\frac{1}{4}\int \cos 2x$

Exercise

Evaluate the integral $\int e^{3x}\cos 3x\,dx$

Solution

		$\int \cos 3x$
+	e^{3x}	$\frac{1}{3}\sin 3x$
−	$3e^{3x}$	$-\frac{1}{9}\cos 3x$
+	$9e^{3x}$	$-\frac{1}{9}\int \cos 3x$

$$\int e^{3x}\cos 3x\,dx = \frac{1}{3}e^{3x}\sin 3x + \frac{1}{3}e^{3x}\cos 3x - \int e^{3x}\cos 3x\,dx$$

$$2\int e^{3x}\cos 3x\,dx = \frac{1}{3}(\sin 3x + \cos 3x)e^{3x}$$

$$\int e^{3x}\cos 3x\,dx = \underline{\frac{1}{6}(\sin 3x + \cos 3x)e^{3x} + C}$$

Exercise

Evaluate the integral $\int x^2 e^{4x}\,dx$

Solution

$$\int x^2 e^{4x}\,dx = \underline{\left(\frac{1}{4}x^2 - \frac{1}{8}x + \frac{1}{32}\right)e^{4x} + C}$$

$$\int x^n e^{ax}\,dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!a^{k+1}}x^{n-k}$$

Exercise

Evaluate the integral $\int x^3 e^{-3x}\,dx$

Solution

$$\int x^3 e^{-3x}\,dx = \underline{\left(-\frac{1}{3}x^3 + \frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27}\right)e^{-3x} + C}$$

$$\int x^n e^{ax}\,dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!a^{k+1}}x^{n-k}$$

Exercise

Evaluate the integral $\int x^3\cos 2x\,dx$

Solution

		$\int \cos 2x$
+	x^3	$\frac{1}{2}\sin 2x$
−	$3x^2$	$-\frac{1}{4}\cos 2x$
+	$6x$	$-\frac{1}{8}\sin 2x$
−	6	$\frac{1}{16}\cos 2x$

$$\int x^3\cos 2x\,dx = \underline{\frac{1}{2}x^3\sin 2x + \frac{3}{4}x^2\cos 2x - \frac{3}{4}x\sin 2x - \frac{3}{8}\cos 2x + C}$$

Exercise

Evaluate the integral $\int x^3 \sin x\, dx$

Solution

$$\int x^3 \sin x\, dx = \underline{-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C}$$

		$\int \sin x$
+	x^3	$-\cos x$
–	$3x^2$	$-\sin x$
+	$6x$	$\cos x$
–	6	$\sin x$

Exercise

Evaluate the integral $\int_0^{1/\sqrt{2}} 2x \sin^{-1}\left(x^2\right) dx$

Solution

$u = \sin^{-1}\left(x^2\right) \qquad dv = 2x dx$

$du = \dfrac{2x}{\sqrt{1-x^4}} dx \qquad v = x^2$

$$\int u dv = uv - \int v du$$

$$\int_0^{1/\sqrt{2}} 2x \sin^{-1}\left(x^2\right) dx = \left[x^2 \sin^{-1}\left(x^2\right)\right]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} x^2 \frac{2x}{\sqrt{1-x^4}} dx \qquad d\left(1-x^4\right) = -4x^3 dx$$

$$= \left(\left(\frac{1}{\sqrt{2}}\right)^2 \sin^{-1}\left(\left(\frac{1}{\sqrt{2}}\right)^2\right) - 0\right) + \int_0^{1/\sqrt{2}} \frac{d\left(1-x^4\right)}{2\sqrt{1-x^4}}$$

$$= \frac{1}{2} \sin^{-1}\left(\frac{1}{2}\right) + \left[\sqrt{1-x^4}\right]_0^{1/\sqrt{2}}$$

$$= \frac{1}{2}\frac{\pi}{6} + \left(\sqrt{1-\frac{1}{4}} - 1\right)$$

$$= \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

$$= \underline{\frac{\pi + 6\sqrt{3} - 12}{12}}$$

Exercise

Evaluate the integral $\int_1^e x^3 \ln x dx$

Solution

$u = \ln x \qquad v = \int x^3 dx = \frac{1}{4} x^4$

$du = \frac{1}{x} dx$

$$\int_1^e x^3 \ln x dx = \left[\frac{1}{4}x^4 \ln x\right]_1^e - \frac{1}{4}\int_1^e x^4 \frac{dx}{x}$$

$$= \frac{1}{4}\left(e^4 \ln e - 1^4 \ln 1\right) - \frac{1}{4}\int_1^e x^3 dx$$

$$= \frac{e^4}{4} - \frac{1}{16}\left[x^4\right]_1^e$$

$$= \frac{e^4}{4} - \frac{1}{16}\left(e^4 - 1\right)$$

$$= \frac{4}{4}\frac{e^4}{4} - \frac{1}{16}e^4 + \frac{1}{16}$$

$$= \frac{3e^4 + 1}{16}$$

Exercise

Evaluate the integral $\int_0^1 x\sqrt{1-x}dx$

Solution

Let:

$$u = x \qquad dv = \sqrt{1-x}dx = (1-x)^{1/2} dx \qquad d(1-x) = -dx$$

$$du = dx \qquad v = -\int (1-x)^{1/2} d(1-x) = -\frac{2}{3}(1-x)^{2/3}$$

$$\int_0^1 x\sqrt{1-x}dx = \left[x\left(-\frac{2}{3}(1-x)^{2/3}\right)\right]_0^1 - \int_0^1 -\frac{2}{3}(1-x)^{2/3} dx \qquad \int udv = uv - \int vdu$$

$$= \left[-\frac{2}{3}x(1-x)^{2/3}\right]_0^1 + \frac{2}{3}\int_0^1 (1-x)^{2/3}(-d(1-x))$$

$$= -\frac{2}{3}\left[(1)(0)^{2/3} - 0\right] - \left[\frac{2}{3}\left(\frac{2}{5}\right)(1-x)^{5/3}\right]_0^1$$

$$= -\frac{4}{15}\left[0 - (1)^{5/3}\right]$$

$$= \frac{4}{15}$$

Exercise

Evaluate the integral $\int_0^{\pi/3} x \tan^2 x dx$

Solution

$$u = x \quad \to \quad dv = \tan^2 x dx = \frac{\sin^2 x}{\cos^2 x} dx = \frac{1-\cos^2 x}{\cos^2 x} dx$$

$$du = dx \quad \to \quad v = \int \left(\frac{1}{\cos^2 x} - 1 \right) dx = \tan x - x$$

$$\int_0^{\pi/3} x \tan^2 x dx = \left[x(\tan x - x) \right]_0^{\pi/3} - \int_0^{\pi/3} (\tan x - x)\, dx \qquad \int u dv = uv - \int v du$$

$$= \left[\frac{\pi}{3}\left(\tan\frac{\pi}{3} - \frac{\pi}{3} \right) - 0 \right] - \left[-\ln|\cos x| - \frac{x^2}{2} \right]_0^{\pi/3}$$

$$= \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3} \right) + \left[\ln\left|\cos\frac{\pi}{3}\right| + \frac{1}{2}\left(\frac{\pi}{3} \right)^2 - \ln|1| - 0 \right]$$

$$= \frac{\pi}{3}\sqrt{3} - \frac{\pi^2}{9} + \ln\left|\frac{1}{2}\right| + \frac{\pi^2}{18}$$

$$\underline{= \frac{\pi}{3}\sqrt{3} - \ln 2 - \frac{\pi^2}{18}}$$

Exercise

Evaluate the integral $\int_0^{\pi} x \sin x\, dx$

Solution

$$\int_0^{\pi} x \sin x\, dx = -x\cos x + \sin x \Big|_0^{\pi}$$

$$\underline{= \pi}$$

		$\int \sin x\, dx$
+	x	$-\cos x$
−	1	$-\sin x$

Exercise

Evaluate the integral $\int_1^{e} \ln 2x\, dx$

Solution

$$\int_1^{e} \ln 2x\, dx = \frac{1}{2}\int_1^{e} \ln 2x\, d(2x) \qquad \int \ln x\, dx = x\ln x - x$$

$$= x\ln 2x - x \Big|_1^{e}$$

$$= e\ln 2e - e - \ln 2 + 1$$

$$= e(\ln 2 + \ln e) - e - \ln 2 + 1$$

$$= e\ln 2 - \ln 2 + 1$$

$$\underline{= (e-1)\ln 2 + 1}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/2} x\cos 2x\,dx$

Solution

		$\int \cos 2x\,dx$
+	x	$\frac{1}{2}\sin 2x$
−	1	$-\frac{1}{4}\cos 2x$

$$\int_0^{\pi/2} x\cos 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\Big|_0^{\pi/2}$$

$$= -\frac{1}{4} - \frac{1}{4}$$

$$\underline{= -\frac{1}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\ln 2} xe^x\,dx$

Solution

		$\int e^x\,dx$
+	x	e^x
−	1	e^x

$$\int_0^{\ln 2} xe^x\,dx = e^x(x-1)\Big|_0^{\ln 2}$$

$$= 2(\ln 2 - 1) + 1$$

$$\underline{= 2\ln 2 - 1}$$

Exercise

Evaluate the integral $\displaystyle\int_1^{e^2} x^2\ln x\,dx$

Solution

$u = \ln x$

$du = \frac{1}{x}dx \qquad v = \int x^2dx = \frac{1}{3}x^3$

$$\int x^2\ln x\,dx = \frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2dx$$

$$\int_1^{e^2} x^2\ln x\,dx = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3\Big|_1^{e^2}$$

$$= \frac{2}{3}e^6 - \frac{1}{9}e^6 + \frac{1}{9}$$

$$\underline{= \frac{5}{9}e^6 + \frac{1}{9}}$$

Exercise

Evaluate the integral $\int_0^3 xe^{x/2}dx$

Solution

$$\int_0^3 xe^{x/2}dx = (2x-4)e^{x/2}\Big|_0^3$$

$$\boxed{= 2e^{3/2} + 4}$$

$$\int x^n e^{ax}\, dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!a^{k+1}}x^{n-k}$$

Exercise

Evaluate the integral $\int_0^2 x^2e^{-2x}dx$

Solution

$$\int_0^2 x^2e^{-2x}dx = \left(-\frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{4}\right)e^{-2x}\Big|_0^2$$

$$= \left(-2+1-\frac{1}{4}\right)e^{-4} + \frac{1}{4}$$

$$\boxed{= \frac{1}{4} - \frac{5}{4}e^{-4}}$$

$$\int x^n e^{ax}\, dx = e^{ax}\sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!a^{k+1}}x^{n-k}$$

Exercise

Evaluate the integral $\int_0^{\pi/4} x\cos 2x\, dx$

Solution

$$\int_0^{\pi/4} x\cos 2x\, dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\Big|_0^{\pi/4}$$

$$\boxed{= \frac{\pi}{8} - \frac{1}{4}}$$

		$\int \cos 2x\, dx$
+	x	$\frac{1}{2}\sin 2x$
−	1	$-\frac{1}{4}\cos 2x$

Exercise

Evaluate the integral $\int_0^{\pi} x\sin 2x\, dx$

Solution

$$\int_0^{\pi} x\sin 2x\, dx = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\Big|_0^{\pi}$$

$$\boxed{= -\frac{\pi}{2}}$$

		$\int \sin 2x\, dx$
+	x	$-\frac{1}{2}\cos 2x$
−	1	$-\frac{1}{4}\sin 2x$

Exercise

Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the cure $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$

Solution

$$V = 2\pi \int_0^{\ln 2} (\ln 2 - x) e^x dx$$

$$= 2\pi \int_0^{\ln 2} \left(\ln 2 e^x - xe^x\right) dx$$

$$= 2\pi \ln 2 \left[e^x \right]_0^{\ln 2} - 2\pi \int_0^{\ln 2} xe^x dx$$

$$= 2\pi \ln 2 \left(e^{\ln 2} - e^0\right) - 2\pi \left[xe^x - e^x \right]_0^{\ln 2}$$

$$= 2\pi \ln 2 (2-1) - 2\pi \left[\ln 2 e^{\ln 2} - e^{\ln 2} - (0-1) \right]$$

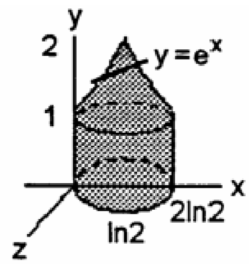
$$= 2\pi \ln 2 - 2\pi [2\ln 2 - 2 + 1]$$

$$= 2\pi \ln 2 - 4\pi \ln 2 + 2\pi$$

$$= -2\pi \ln 2 + 2\pi$$

$$= 2\pi (1 - \ln 2) \quad unit^3$$

	e^x	
+	x	e^x
−	1	e^x



Exercise

Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate aces, the cure $y = e^{-x}$, and the line $x = 1$, about

a) the line $y - axis$

b) the line $x = 1$

Solution

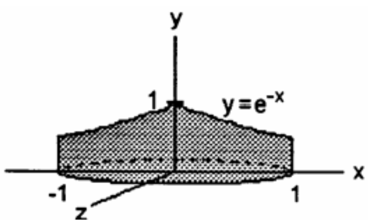
a) $$V = 2\pi \int_0^1 xe^{-x} dx$$

$$= 2\pi \left(\left[-xe^{-x} - e^{-x} \right]_0^1 \right)$$

$$= 2\pi \left(-e^{-1} - e^{-1} + 0 + 1 \right)$$

$$= 2\pi \left(-\frac{1}{e} - \frac{1}{e} + 1 \right)$$

	e^{-x}	
$(+)$	x	$-e^{-x}$
$(-)$	1	e^{-x}



$$= 2\pi\left(-\frac{2}{e}+1\right)$$

$$\boxed{= 2\pi - \frac{4\pi}{e}} \quad unit^3$$

b) $V = 2\pi\int_0^1 (1-x)e^{-x}dx$

$$= 2\pi\left(\int_0^1 e^{-x}dx - \int_0^1 xe^{-x}dx\right)$$

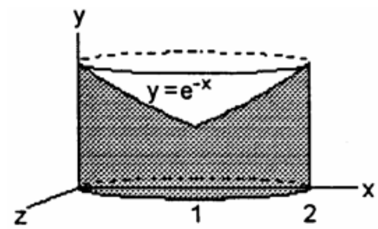
$$= 2\pi\left(\left[-e^{-x} - \left(-xe^{-x} - e^{-x}\right)\right]_0^1\right)$$

$$= 2\pi\left[e^{-x} + xe^{-x} - e^{-x}\right]_0^1$$

$$= 2\pi\left[xe^{-x}\right]_0^1$$

$$= 2\pi\left(e^{-1}\right)$$

$$\boxed{= \frac{2\pi}{e}} \quad unit^3$$



Exercise

Find the volume of the solid that is generated by the region bounded by $f(x) = e^{-x}$, $x = \ln 2$, and the coordinate axes is revolved about the y-$axis$.

Solution

$$V = 2\pi\int_0^{\ln 2} xe^{-x}\, dx$$

$V = \int_a^b 2\pi(radius)(height)\,dx$ *Shells Method*

$$= 2\pi\left[e^{-x}(-x-1)\right]\Big|_0^{\ln 2}$$

		$\int e^{-x}\, dx$
+	x	$-e^{-x}$
−	1	e^{-x}

$$= 2\pi\left(e^{-\ln 2}(-\ln 2 - 1) + 1\right)$$

$$= 2\pi\left(\frac{1}{2}(-\ln 2 - 1) + 1\right)$$

$$= 2\pi\left(-\frac{1}{2}\ln 2 + \frac{1}{2}\right)$$

$$\boxed{= \pi(1 - \ln 2)} \quad unit^3$$

Exercise

Find the volume of the solid that is generated by the region bounded by $f(x) = \sin x$, and the *x-axis* on $[0,\ \pi]$ is revolved about the *y-axis*.

Solution

$$V = 2\pi \int_0^{\pi} x \sin x\, dx \qquad\qquad V = \int_a^b 2\pi (radius)(height)\, dx \quad \textit{Shells Method}$$

$$= 2\pi \left[-x\cos x + \sin x\right]\Big|_0^{\pi}$$

$$\underline{= 2\pi^2} \quad unit^3$$

		$\int \sin x$
+	x	$-\cos x$
−	1	$-\sin x$

Exercise

Find the area of the region generated when the region bounded by $y = \sin x$ and $y = \sin^{-1} x$ on the interval $\left[0,\ \frac{1}{2}\right]$.

Solution

$$A = \int_0^{1/2} \left(\sin^{-1} x - \sin x\right) dx$$

$$u = \sin^{-1} x \qquad du = \frac{dx}{\sqrt{1-x^2}} \qquad v = \int dx = x$$

$$= x\sin^{-1} x \Big|_0^{1/2} - \int_0^{1/2} \frac{x\, dx}{\sqrt{1-x^2}} + \cos x \Big|_0^{1/2}$$

$$= x\sin^{-1} x + \cos x \Big|_0^{1/2} + \frac{1}{2}\int_0^{1/2} \left(1-x^2\right)^{-1/2} d\left(1-x^2\right)$$

$$= x\sin^{-1} x + \cos x + \left(1-x^2\right)^{1/2} \Big|_0^{1/2}$$

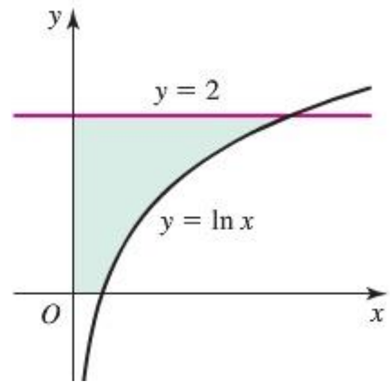
$$= \frac{1}{2}\sin^{-1}\frac{1}{2} + \cos\frac{1}{2} + \left(1-\frac{1}{4}\right)^{1/2} - 1 - 1$$

$$\underline{= \frac{\pi}{12} + \cos\frac{1}{2} + \frac{\sqrt{3}}{2} - 2} \quad unit^2$$

Exercise

Determine the area of the shaded region bounded by $y = \ln x, \quad y = 2, \quad y = 0, \quad and \;\; x = 0$

Solution



$$y = \ln x = 0 \quad \to \quad x = 1$$

$$y = \ln x = 2 \quad \to \quad x = e^2$$

$$A = 1 \times 2 + \int_1^2 (2 - \ln x)\,dx$$

$$= 2 + (2x - x\ln x + x)\Big|_1^2$$

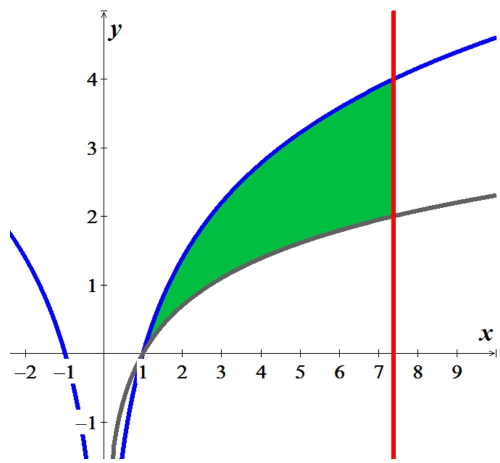
$$= 2 + 4 - 2\ln 2 + 2 - 2 - 1$$

$$= 5 - 2\ln 2 \quad unit^2$$

Exercise

Find the area between the curves $y = \ln x^2, \quad y = \ln x, \quad and \;\; x = e^2$

Solution



$$y = \ln x^2 = \ln x \quad with \quad x > 0$$

$$x^2 = x \Rightarrow x = 1$$

$$A = \int_1^{e^2} \left(\ln x^2 - \ln x\right) dx$$

$$= \int_1^{e^2} (2\ln x - \ln x)\,dx$$

$$= \int_1^{e^2} \ln x \; dx \qquad\qquad \int \ln x \; dx = x\ln x - x$$

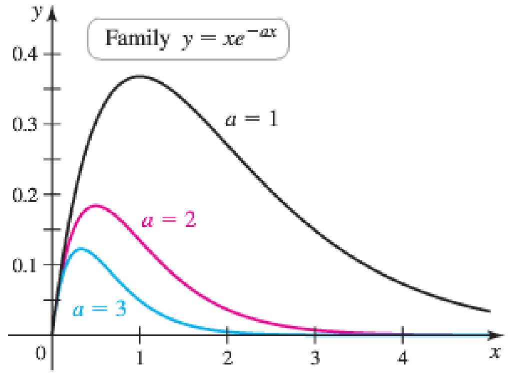
$$= (x\ln x - x)\Big|_1^{e^2}$$

$$= e^2 \ln e^2 - e^2 + 1$$

$$= e^2 + 1 \quad unit^2$$

Exercise

The curves $y = xe^{-ax}$ are shown in the figure for $a = 1,\ 2,\ and\ 3$.



a) Find the area of the region bounded by $y = xe^{-x}$ and the x-axis on the interval [0, 4].

b) Find the area of the region bounded by $y = xe^{-ax}$ and the x-axis on the interval [0, 4] where $a > 0$

c) Find the area of the region bounded by $y = xe^{-ax}$ and the x-axis on the interval [0, b]. Because this area depends on a and b, we call it $A(a,\ b)$ where $a > 0$ and $b > 0$.

d) Use part (c) to show that $A(1,\ \ln b) = 4A\left(2,\ \frac{1}{2}\ln b\right)$

e) Does this pattern continue? Is it true that $A(1,\ \ln b) = a^2 A\left(a,\ \frac{1}{a}\ln b\right)$

Solution

a) $$\int_0^4 xe^{-x}dx = e^{-x}(-x-1)\Big|_0^4$$

$$= e^{-4}(-5) - (-1)$$

$$= 1 - \frac{5}{e^4} \quad unit^2$$

		$\int e^{-x}\ dx$
+	x	$-e^{-x}$
−	1	e^{-x}

b) $$\int_0^4 xe^{-ax}dx = e^{-ax}\left(-\frac{1}{a}x - \frac{1}{a^2}\right)\Big|_0^4$$

$$= e^{-4a}\left(-\frac{4}{a} - \frac{1}{a^2}\right) - \left(-\frac{1}{a^2}\right)$$

$$= \frac{1}{a^2} - e^{-4a}\left(\frac{4a+1}{a^2}\right)$$

$$= \frac{1}{a^2}\left(1 - \frac{4a+1}{e^{-4a}}\right) \quad unit^2$$

		$\int e^{-ax}\ dx$
+	x	$-\frac{1}{a}e^{-ax}$
−	1	$\frac{1}{a^2}e^{-ax}$

c) $$\int_0^b xe^{-ax}dx = e^{-ax}\left(-\frac{1}{a}x - \frac{1}{a^2}\right)\Big|_0^b$$

$$= e^{-ab}\left(-\frac{b}{a}-\frac{1}{a^2}\right)-\left(-\frac{1}{a^2}\right)$$

$$= \frac{1}{a^2} - e^{-ab}\left(\frac{ab+1}{a^2}\right)$$

$$\boxed{= \frac{1}{a^2}\left(1-\frac{ab+1}{e^{ab}}\right)} \quad unit^2$$

d) $A(a,b) = \frac{1}{a^2}\left(1-\frac{ab+1}{e^{ab}}\right)$

$$A(1,\ \ln b) = 1-\frac{\ln b+1}{e^{\ln b}}$$

$$\boxed{= 1-\frac{\ln b+1}{b}}$$

$$A\left(2,\ \tfrac{1}{2}\ln b\right) = \frac{1}{4}\left(1-\frac{\ln b+1}{e^{\ln b}}\right)$$

$$= \frac{1}{4}\left(1-\frac{\ln b+1}{b}\right)$$

$$= \frac{1}{4}A(1,\ \ln b)$$

$$\therefore \quad \boxed{A(1,\ \ln b) = 4A\left(2,\ \tfrac{1}{2}\ln b\right)}$$

e) $A\left(a,\ \frac{1}{a}\ln b\right) = \frac{1}{a^2}\left(1-\frac{\ln b+1}{e^{\ln b}}\right)$

$$= \frac{1}{a^2}\left(1-\frac{\ln b+1}{b}\right)$$

$$= \frac{1}{a^2}A(1,\ \ln b)$$

Yes, there is a pattern: $\boxed{A(1,\ \ln b) = a^2 A\left(a,\ \tfrac{1}{a}\ln b\right)}$

Exercise

Suppose a mass on a spring that is slowed by friction has the position function $s(t) = e^{-t}\sin t$

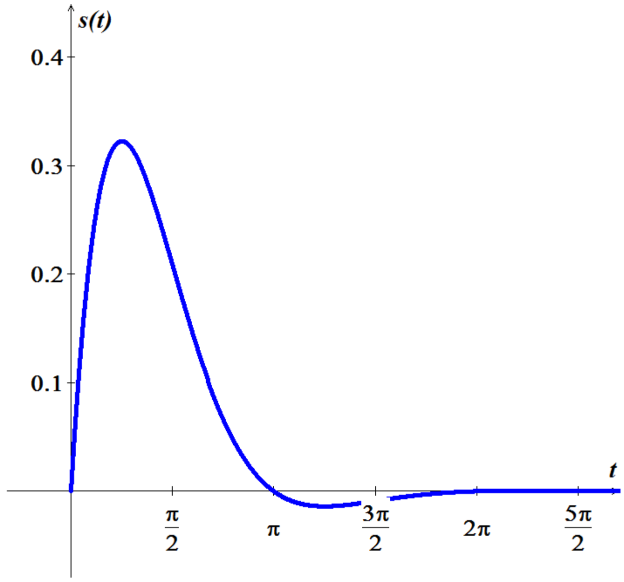
a) Graph the position function. At what times does the oscillator pass through the position $s=0$?

b) Find the average value of the position on the interval $[0,\ \pi]$.

c) Generalize part (*b*) and find the average value of the position on the interval $[n\pi,\ (n+1)\pi]$, for $n = 0,\ 1,\ 2,\ \dots$

Solution

a) $s(t) = e^{-t}\sin t = 0 \qquad \sin t = 0 \quad \to \quad \boxed{t = n\pi}$



b) $$\int e^{-t}\sin t\,dt = -e^{-t}(\cos t+\sin t) - \int e^{-t}\sin t\,dt$$

$$2\int e^{-t}\sin t\,dt = -e^{-t}(\cos t+\sin t)$$

		$\int \sin t$
+	e^{-t}	$-\cos t$
−	$-e^{-t}$	$-\sin t$
+	e^{-t}	$-\int \sin t\,dt$

$$\begin{aligned} Average &= \frac{1}{\pi}\int_0^{\pi} e^{-t}\sin t\,dt \\ &= -\frac{1}{2\pi}e^{-t}(\cos t-\sin t)\Big|_0^{\pi} \\ &= -\frac{1}{2\pi}\left(-e^{-\pi}-1\right) \\ &= \boxed{\frac{1}{2\pi}\left(e^{-\pi}+1\right)} \end{aligned}$$

c) $$\begin{aligned} Average &= \frac{1}{\pi}\int_{n\pi}^{(n+1)\pi} e^{-t}\sin t\,dt \\ &= -\frac{1}{2\pi}e^{-t}(\cos t-\sin t)\Big|_{n\pi}^{(n+1)\pi} \\ &= -\frac{1}{2\pi}\left(e^{-(n+1)\pi}\left(\cos((n+1)\pi)-\sin((n+1)\pi)\right)-e^{-n\pi}(\cos n\pi-\sin n\pi)\right) \\ &= -\frac{1}{2\pi}\left(e^{-(n+1)\pi}\cos((n+1)\pi)-e^{-n\pi}\cos n\pi\right) \\ &= \frac{e^{-n\pi}}{2\pi}\left(\cos n\pi-e^{-\pi}\cos(n+1)\pi\right) \\ &= \frac{e^{-n\pi}}{2\pi}\left((-1)^n-e^{-\pi}(-1)^{n+1}\right) \\ &= \boxed{(-1)^n\frac{e^{-n\pi}}{2\pi}\left(1+e^{-\pi}\right)} \end{aligned}$$

Exercise

Given the region bounded by the graphs of $y = x\sin x, \quad y=0, \quad x=0, \quad x=\pi$, find

a) The area of the region.

b) The volume of the solid generated by revolving the region about the *x-axis*

c) The volume of the solid generated by revolving the region about the *y-axis*

d) The centroid of the region

Solution

a) $A = \int_0^\pi x\sin x\, dx$

$= -x\cos x + \sin x \Big|_0^\pi$

$\boxed{= \pi \ \ unit^2}$

		$\int \sin x$
+	x	$-\cos x$
–	1	$-\sin x$

b) $V = \pi\int_0^\pi (x\sin x)^2\, dx$

$= \pi\int_0^\pi x^2\sin^2 x\, dx$

$= \frac{\pi}{2}\int_0^\pi x^2(1-\cos 2x)\, dx$

$= \frac{\pi}{2}\int_0^\pi \left(x^2 - x^2\cos 2x\right)\, dx$

$= \frac{\pi}{2}\left(\frac{1}{3}x^3 - \frac{1}{2}x^2\sin 2x - \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right)_0^\pi$

$= \frac{\pi}{2}\left(\frac{1}{3}\pi^3 - \frac{\pi}{2}\right)$

$\boxed{= \frac{\pi^4}{6} - \frac{\pi^2}{4} \quad unit^3}$

		$\int \cos 2x$
+	x^2	$\frac{1}{2}\sin 2x$
–	$2x$	$-\frac{1}{4}\cos 2x$
+	2	$-\frac{1}{8}\sin 2x$

c) $V = 2\pi\int_0^\pi x(x\sin x)\, dx$

$= 2\pi\int_0^\pi \left(x^2\sin x\right)\, dx$

$= 2\pi\left(-x^2\cos x + 2x\sin x + 2\cos x\right)_0^\pi$

$= 2\pi\left(\pi^2 - 2 - 2\right)$

$\boxed{= 2\pi^3 - 8\pi \quad unit^3}$

		$\int \sin x$
+	x^2	$-\cos x$
–	$2x$	$-\sin x$
+	2	$\cos x$

d) $m = \int_0^{\pi} x\sin x\, dx = -x\cos x + \sin x\Big|_0^{\pi} \underline{= \pi}$ From (***a***)

$M_x = \frac{1}{2}\int_0^{\pi} (x\sin x)^2\, dx \ \underline{= \frac{1}{2}\left(\frac{\pi^3}{6} - \frac{\pi}{4}\right)}$ From (***b***)

$M_y = \int_0^{\pi} x(x\sin x)\, dx = \frac{2\pi^3 - 8\pi}{2\pi} \ \underline{= \pi^2 - 4}$ From (***c***)

$$\bar{x} = \frac{M_y}{m} = \underline{\frac{\pi^2 - 4}{\pi}} \qquad \underline{\approx 1.8684}$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{\pi}\left(\frac{\pi^3}{12} - \frac{\pi}{8}\right) = \underline{\frac{\pi^2}{12} - \frac{1}{8}} \qquad \underline{\approx 0.6975}$$