Section 4.8 – Mathematical Induction

If n is a positive integer and we let P_n denote the mathematical statement $(xy)^n = x^n y^n$, we obtained the following ***infinite sequence*** of statements:

$$
\begin{array}{ll}
\textit{Statement } P_1: & (xy)^1 = x^1 y^1 \\
\textit{Statement } P_2: & (xy)^2 = x^2 y^2 \\
\textit{Statement } P_3: & (xy)^3 = x^3 y^3 \\
\quad\vdots & \quad\vdots \\
\textit{Statement } P_n: & (xy)^n = x^n y^n \\
\quad\vdots & \quad\vdots
\end{array}
$$

Principle of Mathematical Induction

If with each positive integer n there is associated a statement P_n then all the statements P_n are true, provided the following two conditions are satisfied.

1) P_1 is true.

2) Whenever k is a positive integer such that P_k is true, then P_{k+1} is also true.

Steps in Applying the Principle of Mathematical Induction

1) Show that P_1 is true.

2) Assume that P_k is true, and then prove that P_{k+1} is true.

Example

Use the mathematical induction to prove that for every positive integer n, the sum of the first n positive integers is:

$$\frac{n(n+1)}{2}$$

Solution

(1) For $n = 1 \Rightarrow \frac{1(1+1)}{2} = 1$; hence P_1 is true.

(2) Assume that P_k is true.

Thus the induction hypothesis is: $1+2+3+\ldots+k = \frac{k(k+1)}{2}$

For $k + 1$:

$$1+2+3+\ldots+k+(k+1) \stackrel{?}{=} \frac{(k+1)((k+1)+1)}{2}$$

$$\begin{aligned} 1+2+3+\ldots+k+(k+1) &= (1+2+3+\ldots+k)+(k+1) \\ &= \frac{k(k+1)}{2}+(k+1) && \textit{Induction hypothesis} \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} && \textit{Factor out } \mathbf{k+1} \\ &= \frac{(k+1)((k+1)+1)}{2} && \textit{Change form of } \mathbf{k+2} \end{aligned}$$

This shows that P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed

Example

Prove that for every positive integer *n*,

$$1^2+3^2+\ldots+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 1^2=\frac{1(2(1)-1)(2(1)+1)}{3}$

$$1=\frac{3}{3}$$

$1=1$ ✓ hence P_1 is true.

(2) $1^2+3^2+\ldots+(2k-1)^2=\frac{k(2k-1)(2k+1)}{3}$

For $k+1$:

$$1^2+3^2+\ldots+(2k-1)^2+(2(k+1)-1)^2 \overset{?}{=} \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

$$1^2+3^2+\ldots+(2k-1)^2+(2(k+1)-1)^2 \overset{?}{=} \frac{(k+1)(2k+1)(2k+3)}{3}$$

$$1^2+3^2+\ldots+(2k-1)^2+[2k+2-1]^2=\frac{k(2k-1)(2k+1)}{3}+(2k+1)^2$$

$$=\frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}$$

$$=\frac{(2k+1)[k(2k-1)+3(2k+1)]}{3}$$

$$=\frac{(2k+1)(2k^2-k+6k+3)}{3}$$

$$=\frac{(2k+1)(2k^2+5k+3)}{3}$$

$$=\frac{(2k+1)(k+1)(2k+3)}{3} \checkmark$$

This shows that P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed

Example

Prove that 2 is a factor of $n^2 + 5n$ for every positive integer n,

Solution

(1) For $\boldsymbol{n = 1} \Rightarrow n^2 + 5n = 1^2 + 5(1)$

$$= 6$$

$$= 2.3 \quad \checkmark$$

Thus, 2 is a factor of $n^2 + 5n$ for n = 1; hence P_1 is true.

(2) 2 is a factor of $k^2 + 5k \Leftrightarrow k^2 + 5k = 2p$

is 2 a factor of $(k+1)^2 + 5(k+1)$?

$$\begin{aligned}(k+1)^2 + 5(k+1) &= k^2 + 2k + 1 + 5k + 5 \\ &= k^2 + 5k + 2k + 6 \\ &= \left(k^2 + 5k\right) + 2(k+3) \\ &= 2p + 2(k+3) \\ &= 2.(p + k + 3) \quad \checkmark\end{aligned}$$

Thus, 2 is a factor of the last expression; hence P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed

Steps in Applying the Extended Principle of Mathematical Induction

1. Show that P_1 is true.
2. Assume that P_k is true with $k \geq j$, and then prove that P_{k+1} is true.

Example

Let a be a nonzero real number such that $a > -1$. Prove that $(1+a)^n > 1+na$ for every integer $n \geq 2$.

Solution

For $\boldsymbol{n}$ **= 1** $\Rightarrow (1+a)^1 > 1+(1)a \Rightarrow P_1$ is false.

Step **1**. For $\boldsymbol{n}$ **= 2** $\Rightarrow (1+a)^2 \overset{?}{>} 1+(2)a$

$$1+2a+a^2 > 1+a \quad \checkmark$$

$\Rightarrow P_2$ is true.

Step **2**. Assume that P_k is true $(1+a)^k > 1+ka$

We need to prove that P_{k+1} is true, that is $(1+a)^{k+1} > 1+(k+1)a$

$$\begin{aligned}(1+a)^{k+1} &= (1+a)^k (1+a)^1 \\ &> (1+ka)(1+a)\end{aligned}$$

$$\begin{aligned}(1+ka)(1+a) &= 1+a+ka+ka^2 \\ &= 1+(a+ka)+ka^2 \\ &= 1+a(k+1)+ka^2 \\ &> 1+(k+1)a\end{aligned}$$

$$\begin{aligned}(1+a)^{k+1} &> (1+ka)(1+a) \\ &> 1+(k+1)a \quad \checkmark\end{aligned}$$

Thus, P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed

Exercises Section 4.8 – Mathematical Induction

1. Find all positive integers n for which the given statement is not true

a) $3^n > 6n$ *b)* $3^n > 2n+1$ *c)* $2^n > n^2$ *d)* $n! > 2n$

2. Prove that the statement is true for every positive integer n. $2+4+6+\ldots+2n = n(n+1)$

3. Prove that the statement is true for every positive integer n. $1+3+5+\ldots+(2n-1) = n^2$

4. Prove that the statement is true for every positive integer n. $2+7+12+\ldots+(5n-3) = \frac{1}{2}n(5n-1)$

5. Prove that the statement is true: $1+2\bullet 2+3\bullet 2^2+\ldots+n\bullet 2^{n-1} = 1+(n-1).2^n$

6. Prove that the statement is true: $1^2+2^2+3^2+\ldots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

7. Prove that the statement is true: $\dfrac{1}{1\bullet 2}+\dfrac{1}{2\bullet 3}+\dfrac{1}{3\bullet 4}+\ldots+\dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

8. Prove that the statement is true: $\dfrac{1}{2}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+\ldots+\dfrac{1}{2^n} = 1-\dfrac{1}{2^n}$

9. Prove that the statement is true: $\dfrac{1}{1\cdot 4}+\dfrac{1}{4\cdot 7}+\dfrac{1}{7\cdot 10}+\cdots+\dfrac{1}{(3n-2)\cdot(3n+1)} = \dfrac{n}{3n+1}$

10. Prove that the statement is true: $\dfrac{4}{5}+\dfrac{4}{5^2}+\dfrac{4}{5^3}+\cdots+\dfrac{4}{5^n} = 1-\dfrac{1}{5^n}$

11. Prove that the statement is true: $1^3+2^3+3^3+\cdots+n^3 = \dfrac{n^2(n+1)^2}{4}$

12. Prove that the statement is true: $3+3^2+3^3+\ldots+3^n = \frac{3}{2}\left(3^n-1\right)$

13. Prove that the statement is true: $x^{2n}+x^{2n-1}y+\cdots+xy^{2n-1}+y^{2n} = \dfrac{x^{2n+1}-y^{2n+1}}{x-y}$

14. Prove that the statement is true: $5\bullet 6+5\bullet 6^2+5\bullet 6^3+\cdots+5\bullet 6^n = 6\left(6^n-1\right)$

15. Prove that the statement is true: $7\bullet 8+7\bullet 8^2+7\bullet 8^3+\cdots+7\bullet 8^n = 8\left(8^n-1\right)$

16. Prove that the statement is true: $3+6+9+\cdots+3n = \dfrac{3n(n+1)}{2}$

17. Prove that the statement is true: $5+10+15+\cdots+5n = \dfrac{5n(n+1)}{2}$

18. Prove that the statement is true: $1+3+5+\cdots+(2n-1) = n^2$

19. Prove that the statement is true: $4+7+10+\cdots+(3n+1) = \dfrac{n(3n+5)}{2}$

20. Prove that the statement is true for every positive integer n. $n < 2^n$

21. Prove that the statement is true for every positive integer n. 3 is a factor of $n^3 - n + 3$

22. Prove that the statement is true for every positive integer n. 4 is a factor of $5^n - 1$

23. Prove that the statement by mathematical induction: $\left(a^m\right)^n = a^{mn}$ (a and m are constant)

24. Prove that the statement by mathematical induction: $2^n > 2n \quad \textit{if} \quad n \geq 3$

25. Prove that the statement by mathematical induction: If $0 < a < 1$, then $a^n < a^{n-1}$

26. Prove that the statement by mathematical induction: If $n \geq 4$, then $n! > 2^n$

27. Prove that the statement by mathematical induction: $3^n > 2n + 1 \quad \textit{if} \quad n \geq 2$

28. Prove that the statement by mathematical induction: $2^n > n^2 \quad \textit{for} \quad n > 4$

29. Prove that the statement by mathematical induction: $4^n > n^4 \quad \textit{for} \quad n \geq 5$

30. A pile of n rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must moved one at a time, to a different peg with no ring ever placed on top of a smaller ring. Find the least number of moves that would be required. Prove your result by mathematical induction.