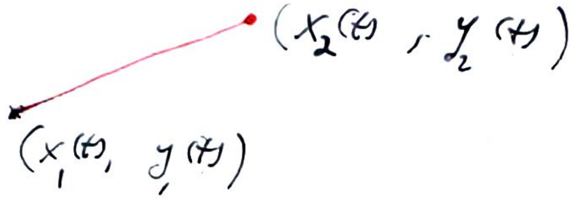
Lect 4. Parametric

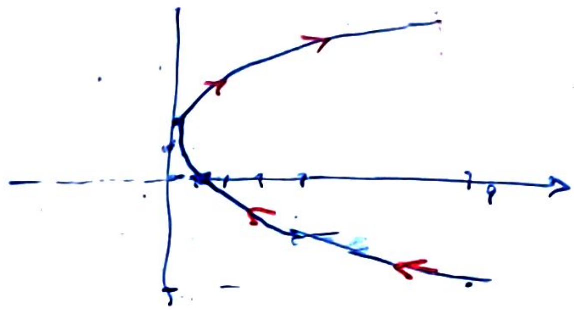
$$x \xrightarrow{f(x)} y$$



Ex

$$\begin{cases} x = t^2 & (1) \\ y = t+1 & (2) \end{cases} \qquad t \in \mathbb{R}$$

t	x	y
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3



$(2) \Rightarrow t = y - 1$

$(1) \Rightarrow x = (y-1)^2$

$= y^2 - 2y + 1$

Ex $\quad x = \cos t \qquad y = \sin t \qquad 0 \le t \le 2\pi$

$\cos^2 t + \sin^2 t = 1$

$x^2 + y^2 = 1$

circle center @ origin w/ $r = 1$

1 rev in ccw

Ex $x = a\cos t \quad y = a\sin t \quad 0 \le t \le 2\pi$

$$\cos^2 t + \sin^2 t = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

$$x^2 + y^2 = a^2$$

it's circle center @ Origin w/ $r = a$, ccw.

Ex $\begin{cases} x = t + \frac{1}{t} & (1) \\ y = t - \frac{1}{t} \end{cases} \quad t > 0$

$$x + y = 2t \rightarrow t = \frac{x+y}{2}$$

$$(1)\ x = \frac{x+y}{2} + \frac{2}{x+y}$$

$$= \frac{(x+y)^2 + 4}{2(x+y)}$$

$$2x(x+y) = (x+y)^2 + 4$$

$$x^2 + y^2 + 2xy + 4 - 2x^2 - 2xy = 0$$

$$y^2 - x^2 = -4$$

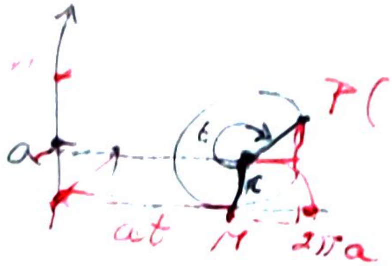
$$x^2 - y^2 = 4$$

$$\frac{x^2}{4} - \frac{y^2}{4} = 1 \quad \text{Hyperbola fctn}$$

$x > 0$,

1.

$$\begin{cases} x = at + a\cos\theta \\ y = a + a\sin\theta \end{cases}$$



$C(at, a)$

$x_1 = a\cos\theta$
$y_1 = a\sin\theta$

$P(at + a\cos\theta,\ a + a\sin\theta)$

$t + \theta = \frac{3\pi}{2} \Rightarrow \theta = \frac{3\pi}{2} - t$

sin cos - cos sin

$x = at + a\cos(\frac{3\pi}{2} - t)$

$= at - a\sin t$

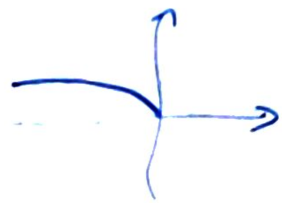
$y = a + a\sin(\frac{3\pi}{2} - t)$

$= a - a\cos t$

#2. $x = -\sqrt{t}$ $\quad y = t \quad t \geq 0$

$x = -\sqrt{y}$

$$\begin{cases} y \geq 0 \\ x \leq 0 \end{cases}$$



#9 $\quad x = \frac{t}{t-1}$ ① $\quad y = \frac{t-2}{t+1}$ ② $\quad -1 < t < 1$

① $\quad xt - x = t$

$xt - t = x$

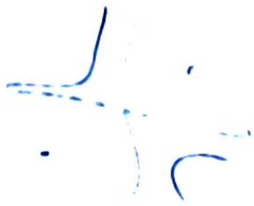
$(x-1)t = x$

$t = \frac{x}{x-1}$

② $\quad y = \left(\frac{x}{x-1} - 2\right)\frac{1}{\frac{x}{x-1} + 1}$ $\qquad (t-2)\frac{1}{t+1}$

$= \frac{-x+2}{x-1} \cdot \frac{x-1}{2x-1}$

$= \frac{2-x}{2x-1}$



4.27 $x = 2\sin t - 3$ (1)

$y = 5 + \cos 2t$.

$\cos 2t = y - 5$ (3)

$c^2 - s^2$
$2c^2 - 1$
$1 - 2s^2$

$\cos^2 t - \sin^2 t$
$2\cos^2 t - 1$
$1 - 2\sin^2 t$

(1) $\sin t = \frac{x+3}{2}$

(3) $y - 5 = 1 - 2\sin^2 t$ $\quad 2 \quad 4$

$y = 6 - \frac{1}{2}(x+3)^2 \rightarrow 6 - \frac{1}{2}(x^2 + 6x + 9)$

$= -\frac{1}{2}x^2 - 3x + \frac{3}{2}$ $\qquad 6 - \frac{9}{2}$

$0 \le t \le 2\pi$ $\quad t = 0 \quad x = -3, \ y = 6$

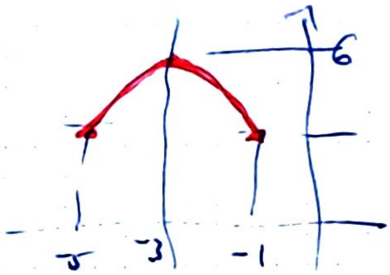
$t = 2\pi \quad x = -3, \ y = 4$

$4 \le y \le 6$

$-1 \le \sin t \le 1$ $\quad t = \frac{3\pi}{2} \Rightarrow x = -5$

$t = \frac{\pi}{2} \Rightarrow x = -1$

$-5 \le x \le -1$



$y = -\frac{1}{2}x^2 - 3x + \frac{3}{2}$

$y' = -x - 3 = 0$ $\quad x = -3$

$-\frac{9}{2} + 9 + \frac{3}{2}$

$4 = -\frac{1}{2}x^2 - 3x + \frac{3}{2}$

$8 = -x^2 - 6x + 3$

$x^2 + 6x + 5 = 0$ $\quad x = -1, -5$

$x = f(t) \qquad y = g(t)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{dy/dt}{dx/dt}\right)$$

$$= \frac{dy'/dt}{dx/dt}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}}$$

Ex $x = \sec t \qquad y = \tan t \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}$

@ $(\sqrt{2}, 1)$ $t = \frac{\pi}{4}$

$$m = \frac{dy/dt}{dx/dt}$$

$$= \frac{\sec^2 t}{\sec t \tan t}$$

$$= \frac{\sec t}{\tan t}\bigg|_{t=\pi/4} \qquad y = m(x - x_1) + y_1$$

$$= \sqrt{2}$$

$$y = \sqrt{2}(x - \sqrt{2}) + 1$$

$$= \sqrt{2}\,x - 1$$

Ex $\frac{d^2y}{dx^2}$ $x = t - t^2$ $y = t - t^3$

$\frac{dx}{dt} = 1 - 2t$ $\frac{dy}{dt} = 1 - 3t^2$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{1-3t^2}{1-2t}$$

$-3 \quad 0 \quad 1$

$0 \quad -2 \quad 1$

$$\frac{dy'}{dt} = \frac{6t^2 - 6t + 2}{(1-2t)^2}$$

$$\frac{d^2y}{dx^2} = \frac{6t^2 - 6t + 2}{(1-2t)^2} \cdot \frac{1}{1-2t}$$

$$= \frac{6t^2 - 6t + 2}{(1-2t)^3}$$

Ex $x = t - t^2$ $y = t - t^3$

$= t(1-t)$ $= t(1-t^2)$

$= t(1-t)(1+t)$

$= x(1+t)$

$$t = \frac{y}{x} - 1$$

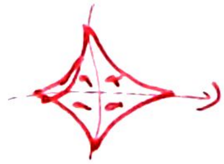
$$x = \frac{y}{x} - 1 - \left(\frac{y^2}{x^2} - 2\frac{y}{x} + 1\right)$$

$$= \frac{y}{x} - 1 - \frac{y^2}{x^2} + \frac{2y}{x} - 1$$

$$= \frac{3y}{x} - \frac{y^2}{x^2} - 1$$

$$x^3 = 3xy - y^2 - x^2$$

Ex Area? $\begin{cases} x = \cos^3 t \\ y = \sin^3 t \end{cases}$ $0 \le t \le 2\pi$



$$A = \int |y\,dx|$$

$$\frac{dx}{dt} = \frac{d(\cos^3 t)}{dt} = -3\cos^2 t \sin t$$

$$= 3\int_0^{2\pi} \sin^3 t \cos^2 t \sin t \, dt$$

$$= 3\int_0^{2\pi} \sin^4 t \cos^2 t \, dt$$

$$= 3\int_0^{2\pi} \frac{(1-\cos 2t)^2}{4} \, \frac{(1+\cos 2t)}{2} \, dt$$

$$= \frac{3}{8}\int_0^{2\pi} (1 - 2\cos 2t + \cos^2 2t)(1+\cos 2t)\, dt$$

$$= \frac{3}{8}\int_0^{2\pi} (1 - \cos 2t - \cos^2 2t + \cos^3 2t)\, dt$$

$$= \frac{3}{8}\int_0^{2\pi} \left(\frac{1}{2} - \cos 2t - \frac{1}{2}\cos 4t\right) dt + \frac{3}{8}\int_0^{2\pi} \cos^2 2t \cos 2t \, dt$$

$$= \frac{3}{8}\left[\frac{1}{2}t - \frac{1}{2}\sin 2t - \frac{1}{8}\sin 4t\right]_0^{2\pi} + \frac{3}{16}\int_0^{2\pi} (1 - \sin^2 2t)\, d(\sin 2t)$$

$$= \frac{3}{8}(\pi) + \frac{3}{16}\left(\sin 2t - \frac{1}{3}\sin^3 2t\right)\Big|_0^{2\pi}$$

$$= \frac{3\pi}{8} \text{ unit}^2$$

$$\ldots = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex L? $x = r\cos t \quad y = r\sin t \quad 0 \le t \le 2\pi$

$$\frac{dx}{dt} = -r\sin t \qquad \frac{dy}{dt} = r\cos t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{r^2\sin^2 t + r^2\cos^2 t}$$

$$= r\sqrt{\sin^2 t + \cos^2 t} \quad \#$$

$$= r$$

$$L = \int_0^{2\pi} r\, dt$$

$$= r\, t \Big|_0^{2\pi}$$

$$= r(2\pi - 0)$$

$$= 2\pi r \text{ unit}$$

$$\int_a^b dx = b - a$$

$$\int_0^{2\pi} d\theta = 2\pi$$

Ex L ? $x = \cos^3 t \quad y = \sin^3 t \quad 0 \le t \le 2\pi$

$$\frac{dx}{dt} = -3\cos^2 t \sin t \qquad \frac{dy}{dt} = 3\sin^2 t \cos t$$

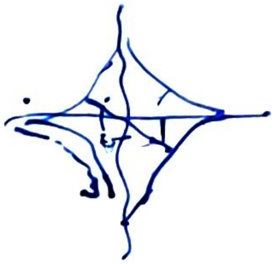
$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}$$

$$= 3\sqrt{\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t)}$$

$$= 3\cos t \sin t$$

$$= \frac{3}{2}\sin 2t$$

$$L = 4\frac{3}{2}\int_0^{\pi/2} \sin 2t \, dt$$



$$= -3\cos 2t \Big|_0^{\pi/2}$$

$$= -3(-1-1)$$

$$= 6 \text{ units}$$

$$S = 2\pi \int_a^b y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

(x – axis)

y – axis   x

Ex $x = \cos t$ $y = 1 + \sin t$ $0 \le t \le 2\pi$

$\frac{dx}{dt} = -\sin t$ $\frac{dy}{dt} = \cos t$ (x – axis)

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\sin^2 t + \cos^2 t}$$
$$= 1$$

$$S = 2\pi \int_0^{2\pi} (1 + \sin t)\, dt$$
$$= 2\pi \left(t - \cos t\right|_0^{2\pi}$$
$$= 2\pi (2\pi - 1 + 1)$$
$$= 4\pi^2 \text{ unit}^2$$

1.2 # 51 $x = t - t^2 = 0$ $y = 1 + e^{-t}$ $x \geq 0$

$t = 0, 1$

$A = \int_0^1 x\, dy$

$\int_0^1 y\, dx$

$\int_0^1 (1+e^{-t})(1-2t)\, dt$

$= \int_0^1 (t - t^2)(-e^{-t})\, dt$

$= \int_0^1 (t^2 - t)\, e^{-t}\, dt$

		$\int e^{-t} dt$
$+$	$t^2 - t$	$-e^{-t}$
$-$	$2t - 1$	e^{-t}
$+$	2	$-e^{-t}$

$= (-t^2 + t - 2t + 1 - 2)\, e^{-t} \Big|_0^1$

$= (-t^2 - t - 1)\, e^{-t} \Big|_0^1$

$= -3e^{-1} - (-1)$

$= \left| 1 - \frac{3}{e} \right|$ unit^2

$= \frac{3}{e} - 1$ unit^2

$\sin 30^\circ = \frac{1}{2}$

$\cos 60^\circ = \frac{1}{2}$

$\pm 1, 0$