Math 2318 – Linear Algebra **_Exam 1_** **_Review_**

Instructor: Fred Khoury

1. Evaluate the $\det(A)$: $A=\begin{bmatrix}1 & k & k^2\\ 1 & k & k^2\\ 1 & k & k^2\end{bmatrix}$

2. Find $A^2,\ A^{-2},\ and\ A^{-k}$ by inspection

$a)\ A=\begin{bmatrix}\frac{1}{4} & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}$ $b)\ A=\begin{bmatrix}-3 & 0 & 0 & 0\\ 0 & 6 & 0 & 0\\ 0 & 0 & -3 & 0\\ 0 & 0 & 0 & -2\end{bmatrix}$

3. Find the components of the vector $\overrightarrow{P_1P_2}$ with initial point $P_1(2,\ -1,\ 4)$ and terminal point $P_2(7,\ 5,\ -8)$

4. Find $\boldsymbol{u}\times\boldsymbol{v}$, where $\boldsymbol{u}=(1,\ 2,\ -2)$ *and* $\boldsymbol{v}=(3,\ 0,\ 1)$ and show that $\boldsymbol{u}\times\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ and to $\boldsymbol{v}$.

5. Calculate the scalar triple product $u\bullet(v\times w)$ of the vectors:

a) $\boldsymbol{u}=3\boldsymbol{i}-2\boldsymbol{j}-5\boldsymbol{k}$ $\boldsymbol{v}=\boldsymbol{i}+4\boldsymbol{j}-4\boldsymbol{k}$ $\boldsymbol{w}=3\boldsymbol{j}+2\boldsymbol{k}$

b) $\boldsymbol{u}=(-2,0,6)$ $\boldsymbol{v}=(1,-3,1)$ $\boldsymbol{w}=(-5,-1,1)$

6. Given $\boldsymbol{u}=(3,\ 2,\ -1)$, $\boldsymbol{v}=(0,\ 2,\ -3)$, *and* $\boldsymbol{w}=(2,\ 6,\ 7)$ Compute the vectors

a) $\boldsymbol{u}\times\boldsymbol{v}$

b) $\boldsymbol{v}\times\boldsymbol{w}$

c) $\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w})$

d) $(\boldsymbol{u}\times\boldsymbol{v})\times\boldsymbol{w}$

e) $\boldsymbol{u}\times(\boldsymbol{v}-2\,\boldsymbol{w})$

f) $\|\boldsymbol{u}\|$

g) Unit vector of $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$

h) Angle between $\boldsymbol{v}$, and $\boldsymbol{w}$

i) $\|3\boldsymbol{u}-5\boldsymbol{v}+\boldsymbol{w}\|$

j) $\boldsymbol{u}\cdot\boldsymbol{v}$

k) $\boldsymbol{u}\cdot\boldsymbol{w}$

7. Determine whether the vectors form an orthogonal set

a) $\boldsymbol{v}_1=(2,\ 3)$, $\boldsymbol{v}_2=(-3,\ 2)$

b) $\boldsymbol{v}_1=(-3,\ 4,\ -1)$, $\boldsymbol{v}_2=(1,\ 2,\ 5)$, $\boldsymbol{v}_3=(4,\ -3,\ 0)$

c) $\boldsymbol{v}_1=(2,-2,1)$, $\boldsymbol{v}_2=(2,1,-2)$, $\boldsymbol{v}_3=(1,2,2)$

8. Find the vector component of $\boldsymbol{u}$ along $\boldsymbol{a}$ $\left(proj_{\boldsymbol{a}}\, \boldsymbol{u} = \dfrac{\boldsymbol{u} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} \right)$ and the vector component of $\boldsymbol{u}$ orthogonal to $\boldsymbol{a}$.

a) $\boldsymbol{u} = (-1,\ -2),\quad \boldsymbol{a} = (-2,\ 3)$

b) $v = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix},\quad a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

c) $\boldsymbol{u} = (1,\ 1,\ 1),\quad \boldsymbol{a} = (0,\ 2,\ -1)$

d) $\boldsymbol{u} = (2,\ 0,\ 1),\quad \boldsymbol{a} = (1,\ 2,\ 3)$

9. Find the area of the parallelogram determined by the given vectors $\boldsymbol{u} = (1,\ 1,\ 1),\quad \boldsymbol{v} = (3,\ 2,\ -5)$

10. Use the cross product to find a vector that is orthogonal to both $\boldsymbol{u} = (3,\ 3,\ 1),\quad \boldsymbol{v} = (0,\ 4,\ 2)$

11. Find the area of the triangle with the given vertices:

a) $A(2,0)\quad B(3,4)\quad C(-1,2)$

b) $A(2,6,-1)\quad B(1,1,1)\quad C(4,6,2)$

12. Find the volume of the parallelepiped with sides $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

$\boldsymbol{u} = (2,-6,2),\quad \boldsymbol{v} = (0,4,-2),\quad \boldsymbol{w} = (2,2,-4)$

13. Express $\left((AB)^{-1}\right)^T$ in terms of $\left(A^{-1}\right)^T$ and $\left(B^{-1}\right)^T$

***Prove*:**

a) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

b) $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$

c) If A is invertible and $AB = AC$, prove that $B = C$

d) Prove if $A^T A = A$, then A is symmetric and $A = A^2$

e) $\det(A+B) \neq \det(A) + \det(B)$

f) $\det(AB) = \det(A)\det(B)$

g) $\det(kA) = k^n \det\left(A^T\right)$

h) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero vectors such that $\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$, then $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal.

i) Prove that $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ *iff* $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel vectors.

j) Lagrange's identity: $\|\boldsymbol{u}\times\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2\ \|\boldsymbol{v}\|^2 - (\boldsymbol{u}.\boldsymbol{v})^2$

Solution

1. $\det(A) = 0$

2. $a)\ A^2 = \begin{bmatrix} \frac{1}{16} & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \quad A^{-2} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad A^{-k} = \begin{bmatrix} 4^k & 0 & 0 \\ 0 & 3^{-k} & 0 \\ 0 & 0 & 2^k \end{bmatrix}$

$b)\ A^2 = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 36 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad A^{-2} = \begin{bmatrix} \frac{1}{9} & 0 & 0 & 0 \\ 0 & \frac{1}{36} & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \quad A^{-k} = \begin{bmatrix} (-3)^{-k} & 0 & 0 & 0 \\ 0 & (6)^{-k} & 0 & 0 \\ 0 & 0 & (-3)^{-k} & 0 \\ 0 & 0 & 0 & (-2)^{-k} \end{bmatrix}$

3. $(5, 6, -12)$

4. $(2, -7, -6)$, $\boldsymbol{u} \times \boldsymbol{v}$ is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$.

5. $a)$ 49 $\quad b)$ -92

6. $a)\ (-4,\ 9,\ 6)$ $\quad b)\ (32,\ -6,\ -4)$ $\quad c)\ (-14,\ -20,\ -82)$

$d)\ (27,\ 40,\ -42)$ $\quad e)\ (-44,\ 47,\ -22)$ $\quad e)\ \sqrt{14}$

$g)\ \left(\frac{3}{\sqrt{14}},\ \frac{2}{\sqrt{14}},\ -\frac{1}{\sqrt{14}}\right),\ \left(0,\ \frac{2}{\sqrt{13}},\ -\frac{3}{\sqrt{13}}\right),\ \left(\frac{2}{\sqrt{89}},\ \frac{6}{\sqrt{89}},\ \frac{7}{\sqrt{89}}\right)$

$h)\ 105.343°$ $\quad i)\ 22.045$ $\quad j)\ 7$ $\quad k)\ 11$

7. $a)$ *Yes* $\quad b)$ *No* $\quad c)$ *Yes*

8. $a)\ \left(\frac{8}{13},\ -\frac{12}{13}\right)\ \left(-\frac{21}{13},\ -\frac{14}{13}\right)$ $\quad b)\ (\cos\theta,\ 0)\ \ (0,\ \sin\theta)$

$c)\ \left(0,\ \frac{2}{5},\ \frac{-1}{5}\right)\ \left(1,\ \frac{3}{5},\ \frac{6}{5}\right)$ $\quad d)\ \left(\frac{5}{14},\ \frac{5}{7},\ \frac{15}{14}\right)\ \left(\frac{23}{14},\ -\frac{5}{7},\ -\frac{1}{14}\right)$

9. $\sqrt{114}$

10. $(2,\ -6,\ 12)$

11. $a)\ 7 \quad b)\ \frac{\sqrt{374}}{2}$

12. 16

13. $\left(A^{-1}\right)^T \left(B^{-1}\right)^T$