

Section 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices

Transpose

Definition

The transpose of a matrix A is defined as the matrix that is obtained by interchanging the corresponding rows and columns in A . Then the transpose of A , denoted by A^T or A' .

The columns of A^T are the rows of A .

When A is an m by n matrix, the transpose is n by m :

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \quad \text{then} \quad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

The matrix **flips over** the main diagonal. The entry in row i , column j of A^T comes from row j , column i of the original A .

$$\left(A^T\right)_{ij} = A_{ji}$$

Properties of Transpose

- a) $\left(A^T\right)^T = A$
- b) $(A + B)^T = A^T + B^T$
- c) $(A - B)^T = A^T - B^T$
- d) $(kA)^T = kA^T$
- e) $(AB)^T = B^T A^T$

The transpose of a product of any number of matrices is the product of the transposes in the reverse order.

Theorem

If A is an invertible matrix, then A^T is also invertible and

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

Proof

$$\begin{aligned}A^T (A^{-1})^T &= (A^{-1} A)^T \\&= I^T \\&= I\end{aligned}$$

$$\begin{aligned}(A^{-1})^T A^T &= (A A^{-1})^T \\&= I^T \\&= I\end{aligned}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\begin{aligned}(A^T)^{-1} &= \frac{1}{ad-bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \\&= \begin{bmatrix} \frac{d}{ad-bc} & -\frac{c}{ad-bc} \\ -\frac{b}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}\end{aligned}$$

Trace

Definition

If A is a square matrix, then the trace of A , denoted by $\mathbf{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A . The trace of A is undefined if A is not a square matrix.

Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\mathbf{tr}(A) = a_{11} + a_{22} + a_{33}$$

Diagonal

A square matrix in which all the entries off the main diagonal are zero is called a **diagonal matrix**. A general $n \times n$ diagonal matrix can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

A diagonal matrix is invertible iff all of its diagonal entries are nonzero; the

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$$

Powers of diagonal matrices are:

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

Triangular Matrices

A square matrix in which all the entries above the main diagonal are zero is called **lower diagonal triangular**.

A square matrix in which all the entries below the main diagonal are zero is called **upper diagonal triangular**.

A matrix that is either upper triangular or lower triangular is called **triangular**.

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

lower diagonal triangular

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix}$$

upper diagonal triangular

Theorem

- ✓ The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.
- ✓ The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.
- ✓ A triangular matrix is invertible iff its diagonal entries are all nonzero.
- ✓ The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

Example

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \quad AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix} \quad BA = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

The goal is to describe Gaussian elimination in the most useful way by looking at them closely, which are factorizations of a matrix.

The factors are triangular matrices.

The factorization that comes from elimination is $A = LU$.

Symmetric Matrices

Definition

A square matrix A is said to be **symmetric** if $A^T = A$. That means a square matrix must satisfies $a_{ij} = a_{ji}$

Example

$$A = \begin{pmatrix} 1 & -4 \\ -4 & 1 \end{pmatrix} = A^T$$

$$A = \begin{pmatrix} 6 & 5 & 1 \\ 5 & 0 & 7 \\ 1 & 7 & -1 \end{pmatrix} = A^T$$

 The **inverse** of a symmetric matrix is also **symmetric**.

Example

Given $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, show that the inverse is symmetric too?


Solution

$$A^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

Theorem

If A and B are symmetric matrices with the same size, and if k is any scalar, then:

- a) A^T is symmetric
- b) $A + B$ and $A - B$ are symmetric.
- c) kA is symmetric

 If A is an invertible symmetric matrix, then A^{-1} is **symmetric**.

Proof

Assume that A is symmetric and invertible then $A = A^T$

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1} = A^{-1}$$

Which proves that A^{-1} is **symmetric**

✚ Multiplying M by M^T gives a symmetric matrix.

Proof

The entry (i, j) of $M^T M$, it is the dot product of **row** i of M^T (column i of M) with column j of M .

The (i, j) entry is the same dot product, column j with column i . so $M^T M$ is symmetric.

The matrix $M.M^T$ is also symmetric and $M^T M$ is a different matrix from $M.M^T$.

✚ If A is an invertible symmetric matrix, then AA^T and $A^T A$ are also invertible.

✚ Matrix A is symmetric across its main diagonal. So is A^{-1}

✚ Matrix A is tridiagonal (only three nonzero diagonals). But A^{-1} is a full matrix with no zeros.
(another reason we don't compute A^{-1})

Example

Given $M = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $M^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $M^T M$ and $M.M^T$

Solution

$$\begin{aligned} M^T M &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} MM^T &= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 \end{bmatrix} \end{aligned}$$

Symmetric in LDU

When elimination is applied to a symmetric matrix, $A^T = A$ is an advantage.

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$
$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_D \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_U$$

🎨 If $A = A^T$ can be factored into LDU with no row exchanges, then $U = L^T$. The ***symmetric factorization of a symmetric matrix is $A = LDL^T$***

Exercises Section 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices

1. Solve $Lc = b$ to find c . Then solve $Ux = c$ to find x . What was A ?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

2. Find L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots

3. Determine whether the given matrix is invertible

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

4. Find A^2 , A^{-2} , and A^{-k} by inspection

$$\begin{array}{lll} \text{a) } A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} & \text{b) } A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} & \text{c) } A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \end{array}$$

5. Decide whether the given matrix is symmetric

$$\begin{array}{lll} \text{a) } \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} & \text{b) } \begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix} & \text{c) } \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \end{array}$$

6. Find all values of the unknown constant(s) in order for A to be symmetric

$$A = \begin{bmatrix} 2 & a - 2b + 2c & 2a + b + c \\ 3 & 5 & a + c \\ 0 & -2 & 7 \end{bmatrix}$$

7. Find a diagonal matrix A that satisfies the given condition $A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
8. Let A be an $n \times n$ symmetric matrix
- Show that A^2 is symmetric
 - Show that $2A^2 - 3A + I$ is symmetric
9. Prove if $A^T A = A$, then A is symmetric and $A = A^2$
10. A square matrix A is called **skew-symmetric** if $A^T = -A$. Prove
- If A is an invertible skew-symmetric matrix, then A^{-1} is skew-symmetric.
 - If A and B are skew-symmetric matrices, then so are A^T , $A + B$, $A - B$, and kA for any scalar k .
 - Every square matrix A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.
- [Hint : Note the identity $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$]
11. Suppose R is rectangular (m by n) and A is symmetric (m by m)
- Transpose $R^T A R$ to show its symmetric
 - Show why $R^T R$ has no negative numbers on its diagonal.
12. If L is a lower-triangular matrix, then $(L^{-1})^T$ is _____ Triangular
13. True or False
- The block matrix $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is automatically symmetric
 - If A and B are symmetric then their product is symmetric
 - If A is not symmetric then A^{-1} is not symmetric
 - When A , B , C are symmetric, the transpose of ABC is CBA .
 - The transpose of a diagonal matrix is a diagonal.
 - The transpose of an upper triangular matrix is an upper triangular matrix.
 - The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.
 - All entries of a symmetric matrix are determined by the entries occurring on and above the main diagonal.
 - All entries of an upper triangular matrix are determined by the entries occurring on and above the main diagonal.

- j) The inverse of an invertible lower triangular matrix is an upper triangular matrix.
- k) A diagonal matrix is invertible if and only if all of its diagonal entries are positive.
- l) The sum of a diagonal matrix and a lower triangular matrix is a lower triangular matrix.
- m) A matrix that is both symmetric and upper triangular must be a diagonal matrix.
- n) If A and B are $n \times n$ matrices such that $A + B$ is symmetric, then A and B are symmetric.
- o) If A and B are $n \times n$ matrices such that $A + B$ is upper triangular, then A and B are upper triangular.
- p) If A^2 is a symmetric matrix, then A is a symmetric matrix.
- q) If kA is a symmetric matrix for some $k \neq 0$, then A is a symmetric matrix.

14. Find 2 by 2 symmetric matrices $A = A^T$ with these properties

- a) A is not invertible
- b) A is invertible but cannot be factored into LU (row exchanges needed)
- c) A can be factored into LDL^T but not into LL^T (because of negative D)

15. A group of matrices includes AB and A^{-1} if it includes A and B . “Products and inverses stay in the group.” Which of these sets are groups?

Lower triangular matrices L with 1's on the diagonal, symmetric matrices S , positive matrices M , diagonal invertible matrices D , permutation matrices P , matrices with $Q^T = Q^{-1}$. ***Invent two more matrix groups.***

16. Write $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$ as the product EH of an elementary row operation matrix E and a symmetric matrix H .

17. When is the product of two symmetric matrices symmetric? Explain your answer.

18. Express $\left((AB)^{-1}\right)^T$ in terms of $\left(A^{-1}\right)^T$ and $\left(B^{-1}\right)^T$

19. Find the transpose of the given matrix: $\begin{bmatrix} 8 & -1 \\ 3 & 5 \\ -2 & 5 \\ 1 & 2 \\ -3 & -5 \end{bmatrix}$

20. Show that if A is symmetric and invertible, then A^{-1} is also symmetric.

21. Prove that $(AB)^T = B^T A^T$

22. For the given matrix, compute A^T , $(A^T)^{-1}$, A^{-1} , and $(A^{-1})^T$, then compare $(A^T)^{-1}$ and $(A^{-1})^T$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

23. Show that a 2×2 lower triangular matrix is invertible if and only if $a_{11}a_{22} \neq 0$ and in this case the inverse is also lower triangular.
24. Let A be any 2×2 diagonal matrix. Give a necessary and sufficient condition on the diagonal entries so that A has an inverse. Compute the inverse of any such matrix.