Lecture One – Vectors and Vector-Values Functions

Solution *Section* 1.1 – Vectors

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + z^2 = 4, \quad y = 0$

Solution

The circle $x^2 + z^2 = 4$ in the ***xz***-plane

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + y^2 = 4, \quad z = -2$

Solution

The circle $x^2 + y^2 = 4$ in the plane $z = -2$

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + y^2 + z^2 = 1, \quad x = 0$

Solution

The circle $y^2 + z^2 = 1$ in the ***yz***-plane

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + (y-1)^2 + z^2 = 4, \quad y = 0$

Solution

$$x^2 + (0-1)^2 + z^2 = 4 \quad \Rightarrow x^2 + z^2 = 3$$

The circle $x^2 + z^2 = 3$ in the ***xz***-plane

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + y^2 + z^2 = 4, \quad y = x$

Solution

The circle formed by the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $y = x$

Exercise

Find the distance between points $P_1(1,\ 1,\ 1), \quad P_2(3,\ 3,\ 0)$

Solution

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(3-1)^2 + (3-1)^2 + (0-1)^2}$$
$$= \sqrt{4+4+1}$$
$$= \sqrt{9}$$
$$\underline{= 3}$$

Exercise

Find the distance between points $P_1(-1,\ 1,\ 5), \quad P_2(2,\ 5,\ 0)$

Solution

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(2+1)^2 + (5-1)^2 + (0-5)^2}$$
$$= \sqrt{9+16+25}$$
$$= \sqrt{50}$$
$$\underline{= 5\sqrt{2}}$$

Exercise

Find the distance between points $P_1(1,\ 4,\ 5), \quad P_2(4,\ -2,\ 7)$

Solution

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(4-1)^2 + (-2-4)^2 + (7-5)^2}$$
$$= \sqrt{9+36+4}$$
$$\underline{= 7}$$

Exercise

Find the distance between points $P_1(3,\ 4,\ 5),\quad P_2(2,\ 3,\ 4)$

Solution

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(2-3)^2+(3-4)^2+(4-5)^2}$$

$$= \sqrt{1+1+1}$$

$$= \sqrt{3}$$

Exercise

Find the center and radii of the spheres $\qquad x^2+y^2+z^2+4x-4z=0$

Solution

$$\left(x^2+4x\right)+y^2+\left(z^2-4z\right)=0$$

$$\left(x^2+4x+4\right)+y^2+\left(z^2-4z+4\right)=4+4$$

$$(x+2)^2+y^2+(z-2)^2=8$$

The center is at $(-2,\ 0,\ 2)$ and the radius is $\sqrt{8}=2\sqrt{2}$

Exercise

Find the center and radii of the spheres $\qquad x^2+y^2+z^2-6y+8z=0$

Solution

$$x^2+\left(y^2-6y\right)+\left(z^2+8z\right)=0$$

$$x^2+\left(y^2-6y+\left(-\frac{6}{2}\right)^2\right)+\left(z^2+8z+\left(\frac{8}{2}\right)^2\right)=9+16$$

$$x^2+(y-3)^2+(z+4)^2=25$$

The center is at $(0,\ 3,\ -4)$ and the radius is 5

Exercise

Find the center and radii of the spheres $2x^2+2y^2+2z^2+x+y+z=9$

Solution

$$x^2+y^2+z^2+\frac{1}{2}x+\frac{1}{2}y+\frac{1}{2}z=\frac{9}{2}$$

$$\left(x^2+\frac{1}{2}x+\left(\frac{1}{2}\frac{1}{2}\right)^2\right)+\left(y^2+\frac{1}{2}y+\left(\frac{1}{4}\right)^2\right)+\left(z^2+\frac{1}{2}z+\left(\frac{1}{4}\right)^2\right)=\frac{9}{2}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}$$

$$\left(x+\frac{1}{4}\right)^2+\left(y+\frac{1}{4}\right)^2+\left(z+\frac{1}{4}\right)^2=\frac{9}{2}+\frac{3}{16}$$

$$\left(x+\frac{1}{4}\right)^2+\left(y+\frac{1}{4}\right)^2+\left(z+\frac{1}{4}\right)^2=\frac{75}{16}$$

The center is at $\left(-\frac{1}{4},\ -\frac{1}{4},\ -\frac{1}{4}\right)$ and the radius is $\boxed{\frac{5\sqrt{3}}{4}}$

Exercise

Find a formula for the distance from the point $P(x, y, z)$ to *x*-axis

Solution

The distance between (x, y, z) and $(x, 0, 0)$ is:

$$d=\sqrt{(x-x)^2+(y-0)^2+(z-0)^2}$$

$$\boxed{=\sqrt{y^2+z^2}}$$

Exercise

Find a formula for the distance from the point $P(x, y, z)$ to *xy*-plane

Solution

The distance between (x, y, z) and $(x, 0, z)$ is:

$$d=\sqrt{(x-x)^2+(y-0)^2+(z-z)^2}$$

$$\boxed{=y}$$

Exercise

Let $\vec{u} = \langle -3,\ 4\rangle$ *and* $\vec{v} = \langle 2,\ -5\rangle$. Find the component form and the magnitude if the vector

a) $3\vec{u} - 4\vec{v}$ *b)* $-2\vec{u}$ *c)* $\vec{u} + \vec{v}$

Solution

a) $3\vec{u} - 4\vec{v} = 3\langle -3,\ 4\rangle - 4\langle 2,\ -5\rangle$

$$= \langle -17,\ 32\rangle$$

b) $-2\vec{u} = -2\langle -3,\ 4\rangle$

$$= \langle 6,\ -8\rangle$$

c) $\vec{u} + \vec{v} = \langle -3,\ 4\rangle + \langle 2,\ -5\rangle$

$$= \langle -1,\ -1\rangle$$

Exercise

Let $\vec{u} = \langle 3,\ -2\rangle$ *and* $\vec{v} = \langle -2,\ 5\rangle$. Find the component form and the magnitude if the vector

a) $3\vec{u}$ *b)* $\vec{u} - \vec{v}$ *c)* $2\vec{u} - 3\vec{v}$ *d)* $-2\vec{u} + 5\vec{v}$ *e)* $-\frac{5}{13}\vec{u} + \frac{12}{13}\vec{v}$

Solution

a) $3\vec{u} = 3\langle 3,\ -2\rangle$

$$= \langle 9,\ -6\rangle$$

b) $\vec{u} - \vec{v} = \langle 3,\ -2\rangle - \langle -2,\ 5\rangle$

$$= \langle 5,\ -7\rangle$$

c) $2\vec{u} - 3\vec{v} = 2\langle 3,\ -2\rangle - 3\langle -2,\ 5\rangle$

$$= \langle 6,\ -4\rangle - \langle -6,\ 15\rangle$$

$$= \langle 12,\ -19\rangle$$

d) $-2\vec{u} + 5\vec{v} = -2\langle 3,\ -2\rangle + 5\langle -2,\ 5\rangle$

$$= \langle -6,\ 4\rangle + \langle -10,\ 25\rangle$$

$$= \langle -14,\ 29\rangle$$

e) $-\frac{5}{13}\vec{u} + \frac{12}{13}\vec{v} = -\frac{5}{13}\langle 3,\ -2\rangle + \frac{12}{13}\langle -2,\ 5\rangle$

$$= \langle -6,\ 4\rangle - \langle -10,\ 25\rangle$$

$$= \langle 4,\ -21\rangle$$

Exercise

Find scalars a, b, and c such that $\langle 2,\ 2,\ 2\rangle = a\langle 1,\ 1,\ 0\rangle + b\langle 0,\ 1,\ 1\rangle + c\langle 1,\ 0,\ 1\rangle$

Solution

$$\begin{aligned}\langle 2,\ 2,\ 2\rangle &= a\langle 1,\ 1,\ 0\rangle + b\langle 0,\ 1,\ 1\rangle + c\langle 1,\ 0,\ 1\rangle \\ &= \langle a+c,\ a+b,\ b+c\rangle\end{aligned}$$

$$\begin{cases} a+c=2 \\ a+b=2 \\ b+c=2 \end{cases} \quad \begin{cases} c=2-a \\ b=2-a \\ \end{cases} \quad \begin{cases} \\ \\ 2a-4=2 \end{cases}$$

$$\underline{a=b=c=1}$$

Exercise

Find the component form of the vector: The sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ where

$$A=(1,\ -1),\quad B=(2,\ 0),\quad C=(-1,\ 3),\quad and\ \ D=(-2,\ 2)$$

Solution

$$\begin{aligned}\overrightarrow{AB} &= \langle 2-1,\ 0-(-1)\rangle \\ &= \underline{\langle 1,\ 1\rangle}\end{aligned}$$

$$\begin{aligned}\overrightarrow{CD} &= \langle -2-(-1),\ 2-3\rangle \\ &= \underline{\langle -1,\ -1\rangle}\end{aligned}$$

$$\begin{aligned}\overrightarrow{AB}+\overrightarrow{CD} &= \langle 1,\ 1\rangle + \langle -1,\ -1\rangle \\ &= \underline{\langle 0,\ 0\rangle}\end{aligned}$$

Exercise

Find the component form of the vector: The unit vector that makes an angle $\theta = \frac{2\pi}{3}$ with the positive x-axis

Solution

$$\left\langle \cos\frac{2\pi}{3},\ \sin\frac{2\pi}{3}\right\rangle = \underline{\left\langle -\frac{1}{2},\ \frac{\sqrt{3}}{2}\right\rangle}$$

Exercise

Find the component form of the vector: The unit vector obtained by rotating the vector $\langle 0,\ 1\rangle$ $120°$ counterclockwise about the origin

Solution

The angle of unit vector $\langle 0,\ 1\rangle$ is 90°, this unit vector rotates 120° which makes an angle of $90°+120°=210°$ with the positive x-axis

$$\langle \cos 210°,\ \sin 210°\rangle = \boxed{\left\langle -\frac{\sqrt{3}}{2},\ -\frac{1}{2}\right\rangle}$$

Exercise

Find the component form of the vector: The unit vector obtained by rotating the vector $\langle 1,\ 0\rangle$ $135°$ counterclockwise about the origin

Solution

The angle of unit vector $\langle 1,\ 0\rangle$ is 0°, this unit vector rotates 135° which makes an angle of $0°+135°=135°$ with the positive x-axis

$$\langle \cos 135°,\ \sin 135°\rangle = \boxed{\left\langle -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right\rangle}$$

Exercise

Find the component form of the vector: The unit vector that makes an angle $\theta=\frac{\pi}{6}$ with the positive x-axis

Solution

$$\boxed{\left\langle \cos \frac{\pi}{6},\ \sin \frac{\pi}{6}\right\rangle = \left\langle \frac{\sqrt{3}}{2},\ \frac{1}{2}\right\rangle}$$

Exercise

Find the component form of the vector: The vector 5 units long in the direction opposite to the direction of $\frac{3}{5}\hat{i}+\frac{4}{5}\hat{j}$

Solution

$$-5\left(\frac{1}{\sqrt{\frac{9}{25}+\frac{16}{25}}}\right)\left(\frac{3}{5}\hat{i}+\frac{4}{5}\hat{j}\right)=-5(1)\left(\frac{3}{5}\hat{i}+\frac{4}{5}\hat{j}\right)$$

$$=-3\hat{i}-4\hat{j}$$

Exercise

Express the velocity vector $\vec{v}=\left(e^t\cos t-e^t\sin t\right)\hat{i}+\left(e^t\cos t+e^t\sin t\right)\hat{j}$ *when* $t=\ln 2$ in terms of its length and direction.

Solution

$$\vec{v}(t=\ln 2)=\left(e^{\ln 2}\cos(\ln 2)-e^{\ln 2}\sin(\ln 2)\right)\hat{i}+\left(e^{\ln 2}\cos(\ln 2)+e^{\ln 2}\sin(\ln 2)\right)\hat{j}$$

$$=\left(2\cos(\ln 2)-2\sin(\ln 2)\right)\hat{i}+\left(2\cos(\ln 2)+2\sin(\ln 2)\right)\hat{j}$$

$$\text{Length}=|\vec{v}|$$

$$=\sqrt{\left(2\cos(\ln 2)-2\sin(\ln 2)\right)^2+\left(2\cos(\ln 2)+2\sin(\ln 2)\right)^2}$$

$$=2\sqrt{\begin{aligned}&\cos^2(\ln 2)-2\cos(\ln 2)\sin(\ln 2)+\sin^2(\ln 2)\\&\quad+\cos^2(\ln 2)+2\cos(\ln 2)\sin(\ln 2)+\sin^2(\ln 2)\end{aligned}}$$

$$=2\sqrt{2}$$

$$\text{Direction}=\frac{\vec{v}}{|\vec{v}|}$$

$$=\frac{2\left(\left(\cos(\ln 2)-\sin(\ln 2)\right)\hat{i}+\left(\cos(\ln 2)+\sin(\ln 2)\right)\hat{j}\right)}{2\sqrt{2}}$$

$$=\frac{\left(\cos(\ln 2)-\sin(\ln 2)\right)}{\sqrt{2}}\hat{i}+\frac{\left(\cos(\ln 2)+\sin(\ln 2)\right)}{\sqrt{2}}\hat{j}$$

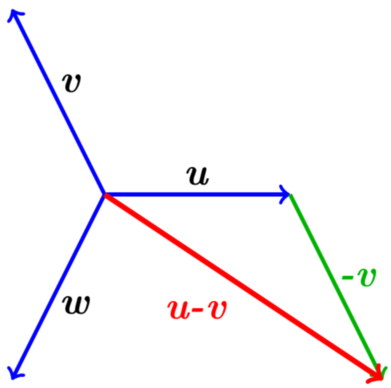
Exercise

Sketch the indicated vector

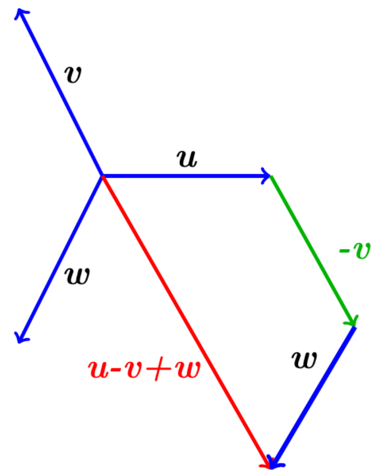
a) $\vec{u}-\vec{v}$ *b)* $2\vec{u}-\vec{v}$ *c)* $\vec{u}-\vec{v}+\vec{w}$

Solution

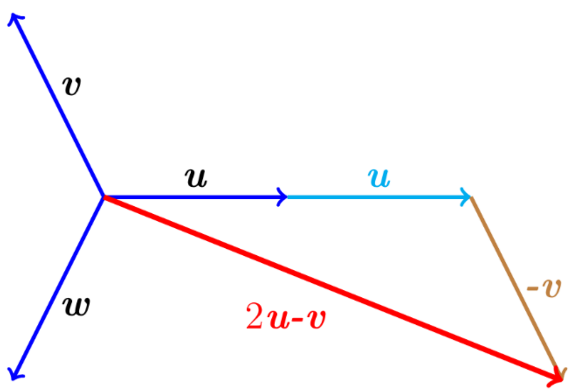
a)



b)



c)



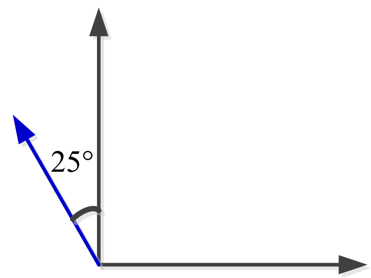
Exercise

An Airplane is flying in the direction 25° west of north at 800 *km/h*. Find the component form of the velocity of the airplane, assuming that the positive *x*-axis represents due east and the positive *y*-axis represents due north.

Solution

25° west of north is 25° + 90° = 115° north of east

$$800\left\langle \cos 115°,\ \sin 115° \right\rangle \approx \left\langle -338.095,\ 725.046 \right\rangle$$



Exercise

A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What speed and direction should the jetliner have in order for the resultant vector to be 500 *mph* due east?

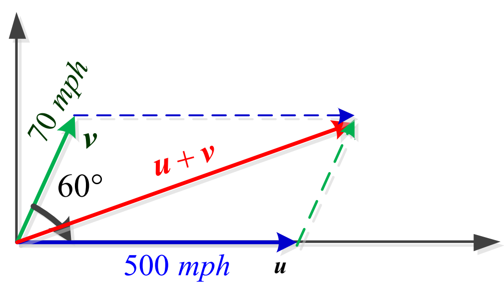
Solution

$\vec{u} = \left\langle x,\ y \right\rangle =$ the velocity of the airplane;

$\vec{v}\ =$ the velocity of the tailwind

$$\vec{v} = \left\langle 70\cos 60°,\ 70\sin 60° \right\rangle$$

$$= \left\langle 35,\ 35\sqrt{3} \right\rangle$$



$$\vec{u} + \vec{v} = \left\langle 500,\ 0 \right\rangle$$

$$\left\langle x,\ y \right\rangle + \left\langle 35,\ 35\sqrt{3} \right\rangle = \left\langle 500,\ 0 \right\rangle$$

$$\left\langle x,\ y \right\rangle = \left\langle 500,\ 0 \right\rangle - \left\langle 35,\ 35\sqrt{3} \right\rangle$$

$$= \left\langle 765,\ -35\sqrt{3} \right\rangle$$

$$\underline{\vec{u} = \left\langle 765,\ -35\sqrt{3} \right\rangle}$$

$$|\vec{u}| = \sqrt{465^2 + \left(-35\sqrt{3}\right)^2}\ \underline{\approx 468.9\ mph}$$

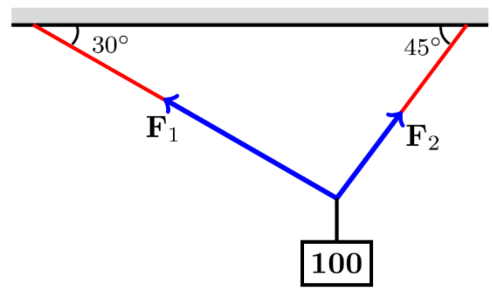
$$\underline{/\theta} = \tan^{-1}\frac{-35\sqrt{3}}{465}\ \underline{\approx -7.4°}$$

The direction is 7.4° *south* of *east*

Exercise

Consider a 100-N weight suspended by two wires. Find the magnitudes and components of the force vectors $\vec{F}_1$ *and* $\vec{F}_2$

Solution



$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos 30°,\ \left|\vec{F}_1\right|\sin 30° \right\rangle$$

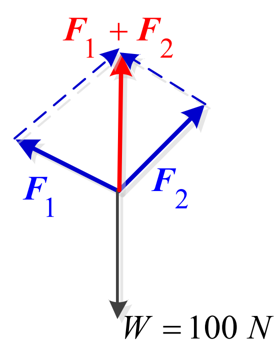
$$= \left\langle -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right|,\ \frac{1}{2}\left|\vec{F}_1\right| \right\rangle$$

$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos 45°,\ \left|\vec{F}_2\right|\sin 45° \right\rangle$$

$$= \left\langle \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| \right\rangle$$

$$\vec{F}_1 + \vec{F}_2 = \langle 0,\ 100 \rangle$$

$$\left\langle -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right|,\ \frac{1}{2}\left|\vec{F}_1\right| \right\rangle + \left\langle \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| \right\rangle = \langle 0,\ 100 \rangle$$



$$\left\langle -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|,\ \frac{1}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| \right\rangle = \langle 0,\ 100 \rangle$$

$$\begin{cases} -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| = 0 \\ \frac{1}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| = 100 \end{cases}$$

$$\Rightarrow \ \left|\vec{F}_1\right| \approx 73.205\ \ N \qquad \left|\vec{F}_2\right| \approx 89.658\ \ N$$

$$\Rightarrow \begin{cases} \left|\vec{F}_1\right| = \dfrac{200\sqrt{2}}{\sqrt{6}+\sqrt{2}} & \approx 73.205\ N \\ \left|\vec{F}_2\right| = \dfrac{200\sqrt{3}}{\sqrt{6}+\sqrt{2}} & \approx 89.658\ N \end{cases}$$

$$\vec{F}_1 = \left\langle -\frac{100\sqrt{6}}{\sqrt{6}+\sqrt{2}},\ \frac{100\sqrt{2}}{\sqrt{6}+\sqrt{2}} \right\rangle$$

$$\approx \langle -63.397,\ 36.603 \rangle$$

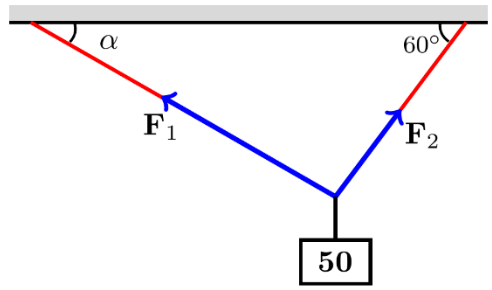
$$\vec{F}_2 = \left\langle \frac{100\sqrt{6}}{\sqrt{6}+\sqrt{2}},\ \frac{100\sqrt{6}}{\sqrt{6}+\sqrt{2}} \right\rangle$$

$$\approx \langle 63.397,\ 63.397 \rangle$$

Exercise

Consider a 50-N weight suspended by two wires, If the magnitude of vector $\vec{F}_1 = 35\ N$, find the angle α and the magnitude of vector $\vec{F}_2$

Solution



$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos\alpha,\ \left|\vec{F}_1\right|\sin\alpha \right\rangle$$
$$= \langle -35\cos\alpha,\ 35\sin\alpha \rangle$$

$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos 60°,\ \left|\vec{F}_2\right|\sin 60° \right\rangle$$
$$= \left\langle \frac{1}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| \right\rangle$$

$$w = \langle 0,\ -50 \rangle \quad \Rightarrow \quad \vec{F}_1 + \vec{F}_2 = \langle 0,\ 50 \rangle$$

$$\langle -35\cos\alpha,\ 35\sin\alpha \rangle + \left\langle \frac{1}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| \right\rangle = \langle 0,\ 50 \rangle$$

$$\left\langle -35\cos\alpha + \frac{1}{2}\left|\vec{F}_2\right|,\ 35\sin\alpha + \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| \right\rangle = \langle 0,\ 50 \rangle$$

$$\rightarrow \begin{cases} -35\cos\alpha + \frac{1}{2}\left|\vec{F}_2\right| = 0 \\ 35\sin\alpha + \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| = 50 \end{cases} \qquad \rightarrow \begin{cases} \left|\vec{F}_2\right| = 70\cos\alpha \\ \end{cases}$$

$$35\sin\alpha + \frac{\sqrt{3}}{2}(70\cos\alpha) = 50$$
$$35\sqrt{3}\cos\alpha = 50 - 35\sin\alpha$$
$$\sqrt{3}\cos\alpha = \frac{10}{7} - \sin\alpha$$

$$\left(\sqrt{3}\cos\alpha\right)^2 = \left(\frac{10}{7} - \sin\alpha\right)^2$$
$$3\cos^2\alpha = \frac{100}{49} - \frac{20}{7}\sin\alpha + \sin^2\alpha$$
$$3\left(1 - \sin^2\alpha\right) = \frac{100}{49} - \frac{20}{7}\sin\alpha + \sin^2\alpha$$
$$3 - 3\sin^2\alpha - \frac{100}{49} + \frac{20}{7}\sin\alpha - \sin^2\alpha = 0$$
$$-4\sin^2\alpha + \frac{20}{7}\sin\alpha + \frac{47}{49} = 0$$

$$-196\sin^2\alpha + 140\sin\alpha + 47 = 0 \quad \Rightarrow \quad \sin\alpha = \frac{5 \pm 6\sqrt{2}}{14}$$

Since $\alpha > 0 \Rightarrow \sin\alpha > 0$

$\rightarrow \sin\alpha = \frac{5+6\sqrt{2}}{14} \approx 0.963$

$\underline{\lvert\alpha} \approx \sin^{-1}(0.963)$

$\approx 74.42^\circ$

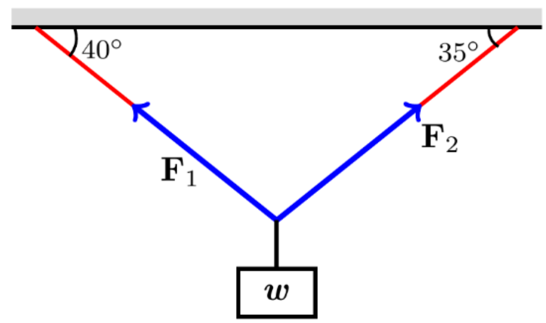
$\left|\vec{F}_2\right| = 70\cos\alpha$

$= 70\cos 74.42^\circ$

$\approx 18.81\ N$

Exercise

Consider a ***w***-N weight suspended by two wires, If the magnitude of vector $\vec{F}_2 = 100\ N$, find ***w*** and the magnitude of vector $\vec{F}_1$



Solution

$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos 40^\circ,\ \left|\vec{F}_1\right|\sin 40^\circ \right\rangle$

$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos 35^\circ,\ \left|\vec{F}_2\right|\sin 35^\circ \right\rangle$

$= \left\langle 100(0.819),\ 100(0.5736) \right\rangle$

$= \left\langle 81.915,\ 57.358 \right\rangle$

$\vec{F}_1 + \vec{F}_2 = \langle 0,\ w\rangle$

$\left\langle -\left|\vec{F}_1\right|\cos 40^\circ,\ \left|\vec{F}_1\right|\sin 40^\circ \right\rangle + \left\langle 81.915,\ 57.358 \right\rangle = \langle 0,\ w\rangle$

$\left\langle -\left|\vec{F}_1\right|\cos 40^\circ + 81.915,\ \left|\vec{F}_1\right|\sin 40^\circ + 57.358 \right\rangle = \langle 0,\ w\rangle$

$-\left|\vec{F}_1\right|\cos 40^\circ + 81.915 = 0$

$\left|\vec{F}_1\right|\cos 40^\circ = 81.915$

$\left|\vec{F}_1\right| = \frac{81.915}{\cos 40^\circ}$

$\approx 106.933\ N$

$w = \left|\vec{F}_1\right|\sin 40^\circ + 57.358$

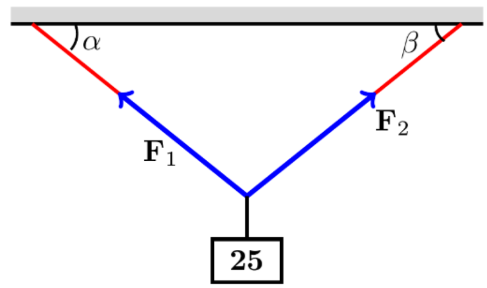
$= 106.933\sin 40^\circ + 57.358$

$\approx 126.093\ N$

Exercise

Consider a 25-N weight suspended by two wires, If the magnitude of vector $\vec{F}_1$ *and* $\vec{F}_2$ are both 75 N, then angles α and β are equal. Find α.

Solution



$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos\alpha,\ \left|\vec{F}_1\right|\sin\alpha \right\rangle$$
$$= \left\langle -75\cos\alpha,\ 75\sin\alpha \right\rangle$$

$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos\beta,\ \left|\vec{F}_2\right|\sin\beta \right\rangle$$
$$= \left\langle 75\cos\beta,\ 75\sin\beta \right\rangle$$

$$w = \langle 0,\ -25\rangle \quad \Rightarrow \quad F_1 + F_2 = \langle 0,\ 25\rangle$$

$$\langle -75\cos\alpha,\ 75\sin\alpha\rangle + \langle 75\cos\beta,\ 75\sin\beta\rangle = \langle 0,\ 25\rangle \qquad \textit{since} \quad \alpha = \beta$$

$$\langle -75\cos\alpha + 75\cos\alpha,\ 75\sin\alpha + 75\sin\alpha\rangle = \langle 0,\ 25\rangle$$

$$-75\cos\alpha + 75\cos\beta = 0 \quad \Rightarrow \quad \cos\alpha = \cos\beta$$

$$150\sin\alpha = 25$$

$$\sin\alpha = \frac{25}{150}$$

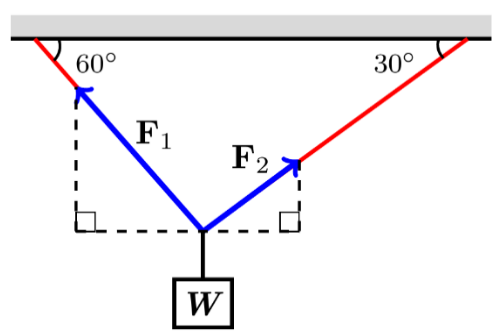
$$\alpha = \sin^{-1}\frac{25}{150}$$

$$\boxed{\approx 9.59^\circ}$$

Exercise

Consider a $W = 100\ N$ weight suspended by two wires. Find the magnitudes and components of the force vectors $\vec{F}_1$ *and* $\vec{F}_2$

Solution



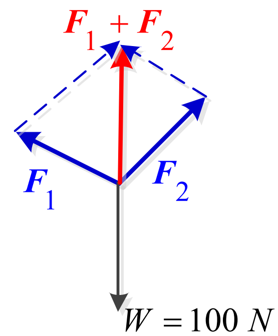
$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos 60^\circ,\ \left|\vec{F}_1\right|\sin 60^\circ \right\rangle$$
$$= \left\langle -\frac{1}{2}\left|\vec{F}_1\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_1\right| \right\rangle$$

$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos 30^\circ,\ \left|\vec{F}_2\right|\sin 30^\circ \right\rangle$$
$$= \left\langle \frac{\sqrt{3}}{2}\left|\vec{F}_2\right|,\ \frac{1}{2}\left|\vec{F}_2\right| \right\rangle$$

$$\vec{F}_1 + \vec{F}_2 = \langle 0,\ 100\rangle$$

$$\left\langle -\frac{1}{2}\left|\vec{F}_1\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_1\right|\right\rangle + \left\langle \frac{\sqrt{3}}{2}\left|\vec{F}_2\right|,\ \frac{1}{2}\left|\vec{F}_2\right|\right\rangle = \langle 0,\ 100\rangle$$

$$\left\langle -\frac{1}{2}\left|\vec{F}_1\right| + \frac{\sqrt{3}}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_1\right| + \frac{1}{2}\left|\vec{F}_2\right|\right\rangle = \langle 0,\ 100\rangle$$



$$\begin{cases} -\frac{1}{2}\left|\vec{F}_1\right| + \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| = 0 \\ \frac{\sqrt{3}}{2}\left|\vec{F}_1\right| + \frac{1}{2}\left|\vec{F}_2\right| = 100 \end{cases}$$

$$\Delta = \begin{vmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix} = -1 \quad \Delta_1 = \begin{vmatrix} 0 & \frac{\sqrt{3}}{2} \\ 100 & \frac{1}{2} \end{vmatrix} = -50\sqrt{3} \quad \Delta = \begin{vmatrix} -\frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 100 \end{vmatrix} = -50$$

$$\Rightarrow \begin{cases} \boxed{\left|\vec{F}_1\right| = 50\sqrt{3}\ \ N} \\ \boxed{\left|\vec{F}_2\right| = 50\ \ N} \end{cases}$$

$$\vec{F}_1 = \left\langle -\frac{1}{2}\left(50\sqrt{3}\right),\ \frac{\sqrt{3}}{2}\left(50\sqrt{3}\right)\right\rangle$$

$$= \boxed{\left\langle -25\sqrt{3},\ 75\right\rangle}$$

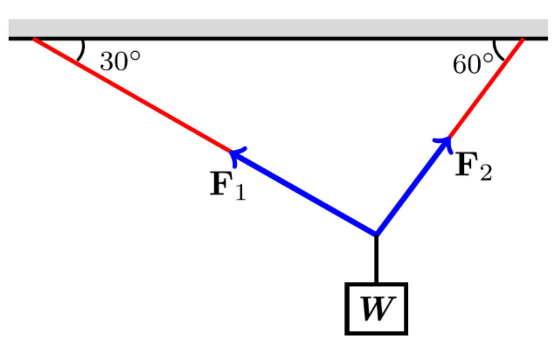
$$\vec{F}_2 = \left\langle \frac{\sqrt{3}}{2}(50),\ \frac{1}{2}(50)\right\rangle$$

$$= \boxed{\left\langle 25\sqrt{3},\ 25\right\rangle}$$

Exercise

Consider a $W = 50\ N$ weight suspended by two wires. Find the magnitudes and components of the force vectors $\vec{F}_1$ *and* $\vec{F}_2$

Solution



$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos 30°,\ \left|\vec{F}_1\right|\sin 30°\right\rangle$$

$$= \left\langle -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right|,\ \frac{1}{2}\left|\vec{F}_1\right|\right\rangle$$

$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos 60°,\ \left|\vec{F}_2\right|\sin 60°\right\rangle$$

$$= \left\langle \frac{1}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| \right\rangle$$

$$\vec{F}_1 + \vec{F}_2 = \langle 0,\ 50 \rangle$$

$$\left\langle -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right|,\ \frac{1}{2}\left|\vec{F}_1\right| \right\rangle + \left\langle \frac{1}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| \right\rangle = \langle 0,\ 50 \rangle$$

$$\left\langle -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right| + \frac{1}{2}\left|\vec{F}_2\right|,\ \frac{1}{2}\left|\vec{F}_1\right| + \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| \right\rangle = \langle 0,\ 50 \rangle$$

$$\begin{cases} -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right| + \frac{1}{2}\left|\vec{F}_2\right| = 0 \\ \frac{1}{2}\left|\vec{F}_1\right| + \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| = 50 \end{cases}$$

$$\Delta = \begin{vmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{vmatrix} = -1 \quad \Delta_1 = \begin{vmatrix} 0 & \frac{1}{2} \\ 50 & \frac{\sqrt{3}}{2} \end{vmatrix} = -25 \quad \Delta = \begin{vmatrix} -\frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 50 \end{vmatrix} = -25\sqrt{3}$$

$$\Rightarrow \begin{cases} \boxed{\left|\vec{F}_1\right| = 25\ N} \\ \boxed{\left|\vec{F}_2\right| = 25\sqrt{3}\ N} \end{cases}$$

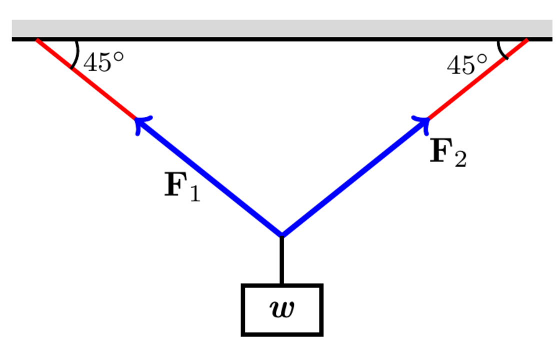
$$\boxed{\vec{F}_1 = \left\langle -\frac{25\sqrt{3}}{2},\ \frac{25}{2} \right\rangle}$$

$$\boxed{\vec{F}_2 = \left\langle \frac{25\sqrt{3}}{2},\ \frac{75}{2} \right\rangle}$$

Exercise

Consider a $W = 100\ N$ weight suspended by two wires. Find the magnitudes and components of the force vectors $\vec{F}_1$ *and* $\vec{F}_2$

Solution



$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos 45°,\ \left|\vec{F}_1\right|\sin 45° \right\rangle$$

$$= \left\langle -\frac{\sqrt{2}}{2}\left|\vec{F}_1\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_1\right| \right\rangle$$

$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos 45°,\ \left|\vec{F}_2\right|\sin 45° \right\rangle$$

$$= \left\langle \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| \right\rangle$$

$$\vec{F}_1 + \vec{F}_2 = \langle 0,\ 100 \rangle$$

$$\left\langle -\frac{\sqrt{2}}{2}\left|\vec{F}_1\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_1\right| \right\rangle + \left\langle \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| \right\rangle = \langle 0,\ 100 \rangle$$

$$\left\langle -\frac{\sqrt{2}}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| \right\rangle = \langle 0,\ 100 \rangle$$

$$\begin{cases} -\frac{\sqrt{2}}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| = 0 \\ \frac{\sqrt{2}}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| = 100 \end{cases}$$

$$\Delta = \begin{vmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{vmatrix} = -1 \quad \Delta_1 = \begin{vmatrix} 0 & \frac{\sqrt{2}}{2} \\ 100 & \frac{\sqrt{2}}{2} \end{vmatrix} = -50\sqrt{2} \quad \Delta = \begin{vmatrix} -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 100 \end{vmatrix} = -50\sqrt{2}$$

$$\Rightarrow \begin{cases} \boxed{\left|\vec{F}_1\right| = 50\sqrt{2}\ N} \\ \boxed{\left|\vec{F}_2\right| = 50\sqrt{2}\ N} \end{cases}$$

$$\vec{F}_1 = \left\langle -\frac{\sqrt{2}}{2}\left(50\sqrt{2}\right),\ \frac{\sqrt{2}}{2}\left(50\sqrt{2}\right) \right\rangle$$

$$\boxed{= \langle -50,\ 50 \rangle}$$

$$\vec{F}_2 = \left\langle \frac{\sqrt{2}}{2}\left(50\sqrt{2}\right),\ \frac{\sqrt{2}}{2}\left(50\sqrt{2}\right) \right\rangle$$

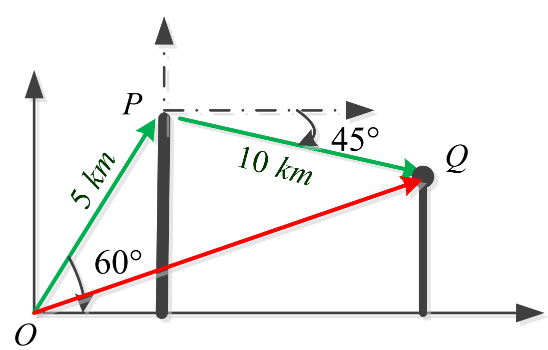
$$\boxed{= \langle 50,\ 50 \rangle}$$

Exercise

A bird flies from its nest 5 km in the direction 60° north east, where it stops to rest on a tree. It then flies 10 km in the direction due southeast and lands atop a telephone pole. Place an *xy*-coordinate system so that the origin is the bird's nest, the *x*-axis points east, and the *y*-axis points north.

a) At what point is the tree located?
b) At what point is the telephone pole?

Solution



a) $\overrightarrow{OP} = \left(5\cos 60°\right)\hat{i} + \left(5\sin 60°\right)\hat{j}$

$$= \frac{5}{2}\hat{i} + \frac{5\sqrt{3}}{2}\hat{j}$$

The tree is located at the point

$$P = \left(\frac{5}{2},\ \frac{5\sqrt{3}}{2}\right)$$

b) $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$

$$= \frac{5}{2}\hat{i} + \frac{5\sqrt{3}}{2}\hat{j} + \left(10\cos 315°\right)\hat{i} + \left(10\sin 315°\right)\hat{j}$$

$$= \frac{5}{2}\hat{i} + \frac{5\sqrt{3}}{2}\hat{j} + \left(10\frac{\sqrt{2}}{2}\right)\hat{i} + \left(10\left(-\frac{\sqrt{2}}{2}\right)\right)\hat{j}$$

$$= \left(\frac{5}{2} + 5\sqrt{2}\right)\hat{i} + \left(\frac{5\sqrt{3}}{2} - \frac{10\sqrt{2}}{2}\right)\hat{j}$$

$$= \left(\frac{5+10\sqrt{2}}{2}\right)\hat{i} + \left(\frac{5\sqrt{3}-10\sqrt{2}}{2}\right)\hat{j}$$

The pole is located at the point $Q = \left(\frac{5+10\sqrt{2}}{2},\ \frac{5\sqrt{3}-10\sqrt{2}}{2}\right)$

Exercise

Suppose that A, B, and C are the corner points of the thin triangular plate of constant density.

a) Find the vector from C to the midpoint M of side AB.

b) Find the vector from C to the point that lies two-thirds of the way from C to M on the median CM.

c) Find the coordinates of the point in which the medians of $\triangle ABC$ intersect (this point is the plate's center of mass).

Solution

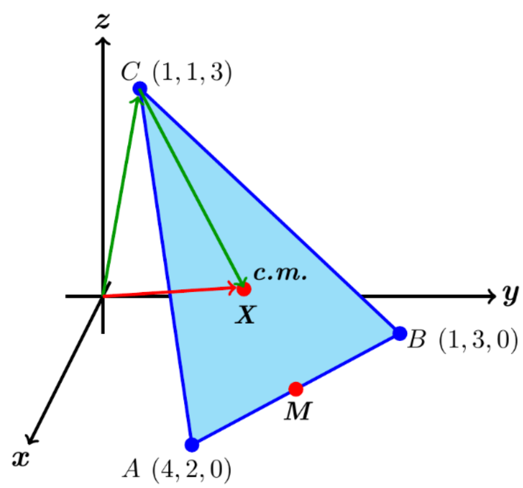
a) The midpoint of AB is:

$$M = \left(\frac{4+1}{2}, \frac{2+3}{2}, 0\right)$$

$$= \left(\frac{5}{2},\ \frac{5}{2},\ 0\right)$$

$$\overrightarrow{CM} = \left(\frac{5}{2} - 1\right)\hat{i} + \left(\frac{5}{2} - 1\right)\hat{j} + (0-3)\hat{k}$$

$$= \frac{3}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$$



b) The desired vector is

$$\overrightarrow{CX} = \frac{2}{3}\overrightarrow{CM}$$

$$= \frac{2}{3}\left(\frac{3}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}\right)$$

$$= \hat{i} + \hat{j} - 2\hat{k}$$

c) The vector whose sum is the vector from the origin to C and the result of part (b) will terminate at the center of mass.

$$\overrightarrow{OX} = \overrightarrow{OC} + \overrightarrow{CX}$$
$$= \hat{i} + \hat{j} + 3\hat{k} + \hat{i} + \hat{j} - 2\hat{k}$$
$$= 2\hat{i} + 2\hat{j} + \hat{k}$$

Therefore; the center of mass point is $(2,\ 2,\ 1)$

Exercise

Show that a unit vector in the plane can be expressed as $\vec{u} = (\cos\theta)\,\hat{i} + (\sin\theta)\,\hat{j}$, obtained by rotating $\hat{i}$ through an angle θ in the counterclockwise direction. Explain why this form gives ***every*** *unit vector* in the plane.

Solution

Let $\vec{u}$ be any unit vector in the plane.

If $\vec{u}$ is positioned so that its initial point and terminal point is at (x, y), then $\vec{u}$ makes an angle θ with $\hat{i}$, measured in the *ccw* direction.

Since $|\vec{u}| = 1 \quad \Rightarrow \quad x = \cos\theta \quad$ *and* $\quad y = \sin\theta$

That implies to: $\vec{u} = (\cos\theta)\hat{i} + (\sin\theta)\hat{j}$

Since $\vec{u}$ is any unit vector in the plane; this holds for every unit vector in the plane.

Exercise

Assume the positive x-*axis* points east and the positive y-*axis* points north.

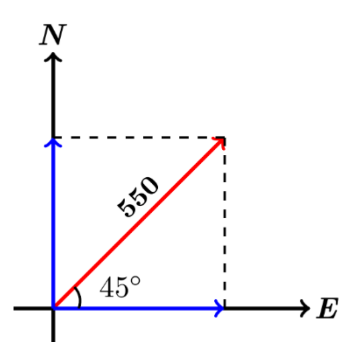
a) An airliner flies northeast at a constant altitude at 550 *mi/hr* in calm air. Find a and b such that its velocity may be expressed in the form $\vec{v} = a\,\hat{i} + b\,\hat{j}$

b) An airliner flies northeast at a constant altitude at 550 *mi/hr* relative to the air in a southerly crosswind $\vec{w} = \langle 0,\ 40\rangle$. Find the velocity of the airliner relative to the ground.

Solution

a) $\vec{v} = 550\langle -\cos 45°,\ \ \sin 45°\rangle$

$$= 550\left\langle -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right\rangle$$
$$= \left\langle -275\sqrt{2},\ 275\sqrt{2}\right\rangle$$



b) $\vec{v} = 550\left\langle -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right\rangle + \langle 0,\ \ 40\rangle$

$$= \left\langle -275\sqrt{2},\ 275\sqrt{2} \right\rangle$$

Exercise

Let $\overrightarrow{PQ}$ extended from $P(2,\ 0,\ 6)$ to $Q(2,\ -8,\ 5)$

a) Find the position vector equal to $\overrightarrow{PQ}$.

b) Find the midpoint M of the line segment PQ. Then find the magnitude of $\overrightarrow{PM}$.

c) Find a vector of length 8 with direction opposite that of $\overrightarrow{PQ}$.

Solution

a) $\overrightarrow{PQ} = \langle 2-2,\ -8-0,\ 5-6 \rangle$

$$= \langle 0,\ -8,\ -1 \rangle$$

b) $M = \left(\frac{2+2}{2},\ \frac{0-8}{2},\ \frac{6+5}{2} \right)$

$$= \left(2,\ -4,\ \frac{11}{2} \right)$$

$$\overrightarrow{PM} = \left\langle 0,\ -4,\ -\frac{1}{2} \right\rangle$$

$$\left|\overrightarrow{PM}\right| = \sqrt{16 + \frac{1}{4}}$$

$$= \frac{1}{2}\sqrt{65}$$

c) $\left|\overrightarrow{PQ}\right| = \sqrt{64+1}$

$$= \sqrt{65}$$

$$vector = \frac{-8}{\sqrt{65}} \langle 0,\ -8,\ -1 \rangle$$

$$= \frac{8}{\sqrt{65}} \langle 0,\ 8,\ 1 \rangle$$

Exercise

An object at the origin is acted on by the forces $\vec{F}_1 = -10\,\hat{i} + 20\hat{k}$, $\vec{F}_2 = 40\,\hat{j} + 10\hat{k}$, and $\vec{F}_3 = -50\,\hat{i} + 20\,\hat{j}$. Find the magnitude of the combined force and use a sketch to illustrate the direction of the combined force.

Solution

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$

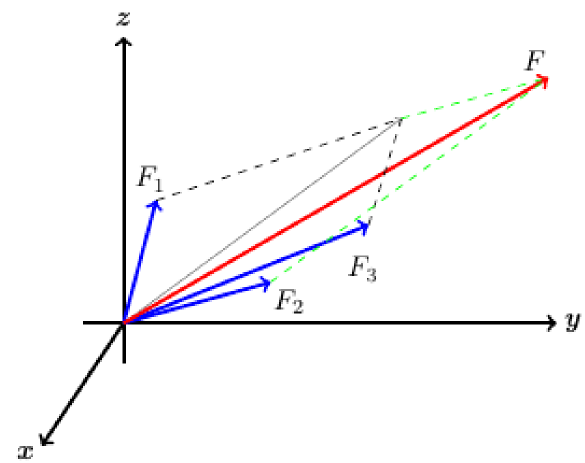
$$= -10\hat{i} + 20\hat{k} + 40\hat{j} + 10\hat{k} - 50\hat{i} + 20\hat{j}$$

$$= -60\hat{i} + 60\hat{j} + 30\hat{k}$$

$$\left|\vec{F}\right| = \sqrt{3600 + 3600 + 900}$$

$$= \sqrt{8100}$$

$$= 90$$



Exercise

A remote sensing probe falls vertically with a terminal of 60 *m/s* when it encounters a horizontal crosswind blowing north at 4 *m/s* and an updraft blowing vertically at 10 *m/s*. find the magnitude and direction of the resulting velocity relative to the ground.

Solution

The velocity relative to the ground is:

$$\langle 0,\ 4,\ 10-60\rangle = \langle 0,\ 4,\ -50\rangle$$

$$\textit{Magnitude}:\ \sqrt{16+2500} = \sqrt{2516}$$

$$= 2\sqrt{629} \qquad \approx 50.16\ \ m/s$$

$$\textit{Direction} = \cos^{-1}\frac{4}{\sqrt{2516}}$$

$$\approx 85.4°$$

Below the horizontal in the northerly horizontal direction.

Exercise

A small plane is flying north in calm air at 250 *mi/hr* when it is hit by a horizontal crosswind blowing northeast at 40 *mi/hr* and a 25 *mi/hr* downdraft. Find the resulting velocity and speed of the plane.

Solution

$$\textit{Velocity vector} = \langle 250,\ 0,\ 0\rangle$$

$$\textit{Crosswind} = \langle 40\cos 45°,\ 40\sin 45°,\ 0\rangle$$

$$= \left\langle 20\sqrt{2},\ 20\sqrt{2},\ 0 \right\rangle$$

$$Downdraft = \langle 0,\ 0,\ -25 \rangle$$

$$Resulting\ velocity = \langle 250,\ 0,\ 0 \rangle + \left\langle 20\sqrt{2},\ 20\sqrt{2},\ 0 \right\rangle + \langle 0,\ 0,\ -25 \rangle$$

$$= \left\langle 250 + 20\sqrt{2},\ 20\sqrt{2},\ -25 \right\rangle$$

$$Speed = \sqrt{\left(250 + 20\sqrt{2}\right)^2 + 800 + 625}$$

$$= \sqrt{62500 + 10^4\sqrt{2} + 800 + 1{,}425}$$

$$= \sqrt{64{,}725 + 10^4\sqrt{2}}$$

$$= 5\sqrt{2{,}589 + 400\sqrt{2}}$$

$$\underline{= 280.83\ \ mph}$$