Solution   *Section* 1.4 – Nested Quantifiers

Exercise

Translate these statements into English, where the domain for each variable consists of all real numbers

a) $\forall x \exists y (x < y)$

b) $\exists x \forall y (xy = y)$

c) $\forall x \forall y \left(\left((x \ge 0) \wedge (y < 0) \right) \to (x - y > 0) \right)$

d) $\forall x \forall y \left(\left((x \ge 0) \wedge (y \ge 0) \right) \to (xy \ge 0) \right)$

e) $\forall x \forall y \exists z (xy = z)$

f) $\forall x \forall y \exists z (x = y + z)$

Solution

a) For every real number *x* there exists a real number *y* such that *x* is less than *y*. Basically, there is a larger number.

b) There exists real number *x* such that for every a real number *y*, $xy = y$. This is asserting the existence of a multiplication identity for the real numbers, and the statement is true, since we can take $x = 1$.

c) For every real number *x* and real number *y*, if *x* is nonnegative and *y* is negative, then the difference $x - y$ is positive. More simply, a nonnegative number minus a negative number is positive which is true.

d) For every real number *x* and real number *y*, if *x* is positive and *y* is positive, then the product xy is positive. More simply, a product of 2 positive numbers is positive.

e) For every real number *x* and real number *y*, there exists a real number *z* such that the product $xy = z$. More simply, the real numbers are closed under multiplication.

f) For every real number *x* and real number *y*, there exists a real number *z* such that the product $x = y + z$. This is a true statement, since we can take $z = x - y$ in each case.

Exercise

Let $Q(x, y)$ be the statement "*x* has sent an e-mail message to *y*," where the domain for both *x* and *y* consists of all students in your class. Express each of these quantifications in English

a) $\exists x \exists y Q(x, y)$

b) $\exists x \forall y Q(x, y)$

c) $\forall x \exists y Q(x, y)$

d) $\exists y \forall x Q(x, y)$

e) $\forall y \exists x Q(x, y)$

f) $\forall x \forall y Q(x, y)$

Solution

a) There exist students x and y such that x has sent a message to y.
In other words, there is some student in your class who has sent a message to some student in your class.

b) There exists a student x for every student y such that x has sent a message to every y.
In other words, there is a student in your class who has sent a message to every student in your class.

c) For every student x in your class there exists a student y such that x has sent a message to y.
In other words, every student in your class has sent a message to at least one student in your class.

d) There exists a student y for every student x such that y has sent a message to every x.
In other words, there is a student in your class who has sent a message to every student in your class.

e) For every student y in your class there exists a student x such that y has sent a message to x.
In other words, every student in your class has sent a message to at least one student in your class.

f) Every student in your class has sent a message to every student in your class.

Exercise

Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators where the domain consists of all integers.

a) The product of two negative integers is positive.

b) The average of two positive integers is positive.

c) The difference of two negative integers is not necessarily negative.

d) The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers.

Solution

a) $\forall x \forall y \left((x<0) \wedge (y<0) \rightarrow (xy>0) \right)$

b) $\forall x \forall y \left((x>0) \wedge (y>0) \ \rightarrow \ \frac{x+y}{2}>0 \right)$

c) $\exists x \exists y \left((x<0) \wedge (y<0) \wedge (x-y \geq 0) \right)$

d) $\forall x \forall y \left(|x+y| \leq |x|+|y| \right)$

Exercise

Rewrite these statements so that the negations only appear within the predicates

a) $\neg\exists y\forall xP(x,y)$

b) $\neg\forall x\exists yP(x,y)$

c) $\neg\exists y\left(Q(y)\wedge\forall x\neg R(x,y)\right)$

Solution

a) $$\begin{aligned}\neg\exists y\forall xP(x,y) &\equiv \forall y\neg\forall xP(x,y)\\ &\equiv \forall y\exists x\neg P(x,y)\end{aligned}$$

b) $$\begin{aligned}\neg\forall x\exists yP(x,y) &\equiv \exists x\neg\exists yP(x,y)\\ &\equiv \exists x\forall y\neg P(x,y)\end{aligned}$$

c) $$\begin{aligned}\neg\exists y\left(Q(y)\wedge\forall x\neg R(x,y)\right) &\equiv \forall y\neg\left(Q(y)\wedge\forall x\neg R(x,y)\right)\\ &\equiv \forall y\left(\neg Q(y)\vee\neg\left(\forall x\neg R(x,y)\right)\right)\\ &\equiv \forall y\left(\neg Q(y)\vee\exists xR(x,y)\right)\end{aligned}$$

Exercise

Express the negations of each of these statements so that all negation symbols immediately precede predicates.

a) $\forall x\exists y\forall zT(x,y,z)$

b) $\forall x\exists yP(x,y)\vee\forall x\exists yQ(x,y)$

Solution

a) $$\begin{aligned}\neg\left(\forall x\exists y\forall zT(x,y,z)\right) &\equiv \neg\forall x\exists y\forall zT(x,y,z)\\ &\equiv \exists x\neg\exists y\forall zT(x,y,z)\\ &\equiv \exists x\forall y\neg\forall zT(x,y,z)\\ &\equiv \exists x\forall y\exists z\neg T(x,y,z)\end{aligned}$$

b) $$\begin{aligned}\neg\left(\forall x\exists yP(x,y)\vee\forall x\exists yQ(x,y)\right) &\equiv \neg\left(\forall x\exists yP(x,y)\right)\wedge\neg\left(\forall x\exists yQ(x,y)\right)\\ &\equiv \exists x\neg\left(\exists yP(x,y)\right)\wedge\exists x\neg\left(\exists yQ(x,y)\right)\\ &\equiv \exists x\forall y\neg P(x,y)\wedge\exists x\forall y\neg Q(x,y)\end{aligned}$$

Exercise

Let $T(x, y)$ mean that student x likes cuisine y, where the domain for x consists of all students at your school and the domain for y consists of all cuisines. Express each of these statements by a simple English sentence.

a) $\neg T(A, J)$

b) $\exists x T(x, \ Korean) \wedge \forall x Tx(x, \ Mexican)$

c) $\exists y \left(T(Monique, \ y) \vee T(Jay, \ y) \right)$

d) $\forall x \forall z \exists y \left((x \neq z) \rightarrow \neg \left(T(x, y) \wedge T(z, y) \right) \right)$

e) $\exists x \exists z \forall y \left(T(x, y) \leftrightarrow T(z, y) \right)$

f) $\forall x \forall z \exists y \left(T(x, y) \leftrightarrow T(z, y) \right)$

Solution

a) *A* does not like *J* cuisine

b) Some student at your school likes Korean cuisine, and everyone at your school likes Mexican cuisine.

c) There is some cuisine that Monique and Jay likes.

d) For every pair of distinct students at your school, there is some cuisine that at least one them does not like.

e) There are two students at your school who have exactly the same cuisines (tastes).

f) For every pair of students at your school, there is some cuisine about which they have the same opinion (either they both like it or they both do not like it).

Exercise

Let $L(x, y)$ be the statement " x loves y", where the domain for both x and y consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody loves Jerry.

b) Everybody loves somebody.

c) There is somebody whom everybody loves.

d) Nobody loves everybody.

e) There is somebody whom Lois does not love.

f) There is somebody whom no one loves.

g) There is exactly one person whom everybody loves.

h) There are exactly two people whom ***L*** loves.

i) Everyone loves himself or herself.

j) There is someone who loves no one besides himself or herself.

Solution

a) $\forall x \ L(x, \ Jerry)$

b) $\forall x \exists y \ L(x, \ y)$

c) $\exists y \forall x\, L(x,\, y)$

d) $\neg \exists x \forall y\, L(x,\, y)$

e) $\exists x\, \neg L(Lois,\, x)$

f) $\exists x \forall y\, \neg L(x,\, y)$

g) $\exists x \Big(\forall y\, L(y,\, x) \wedge \forall z \big(\big(\forall w\, L(w,\, z) \big) \rightarrow z = x \big) \Big)$

h) $\exists x \exists y \Big(x \neq y \wedge\, L(\mathbf{L},\, x) \wedge\, L(\mathbf{L},\, y) \wedge \forall z \big(L(\mathbf{L},\, z) \rightarrow (z = x \vee z = y) \big) \Big)$

i) $\forall x\, L(x,\, x)$

j) $\exists x \forall y\, \big(L(x,\, y) \leftrightarrow x = y \big)$

Exercise

Let $S(x)$ be the predicate "*x* is a student," $F(x)$ the predicate "*x* is a faculty member," $A(x, y)$ the predicate "*x* has asked *y* a question," where the domain consists of all people associated with your school. Use quantifiers to express each of these statements.

a) Lois asked Professor Fred a question.
b) Every student has asked Professor Fred a question.
c) Every faculty member has either asked Professor Fred a question or been asked a question by Professor Miller.
d) Some student has not asked any faculty member a question.
e) There is a faculty member who has never been asked a question by a student.
f) Some student has asked every faculty member a question.
g) There is a faculty member who has asked every other faculty member a question.
h) Some student has never been asked a question by a faculty member.

Solution

a) $A(Lois,\; \mathrm{Pr}\, of.\, Fred)$

b) $\forall x \big(S(x) \rightarrow A(x,\; \mathrm{Pr}\, of.\, Fred) \big)$

c) $\forall x \Big(F(x) \rightarrow \big(A(x,\; \mathrm{Pr}\, of.\, Fred) \vee A(\mathrm{Pr}\, of.\, Miller,\, x) \big) \Big)$

d) $\exists x \Big(S(x) \wedge \forall y \big(F(y) \rightarrow \neg A(x,\, y) \big) \Big)$ ***or*** $\exists x \Big(S(x) \wedge \neg \exists y \big(F(y) \rightarrow A(x,\, y) \big) \Big)$

e) $\exists x \Big(F(x) \wedge \forall y \big(S(y) \rightarrow \neg A(y,\, x) \big) \Big)$

f) $\exists x \Big(S(x) \wedge \forall y \big(F(y) \rightarrow A(x,\, y) \big) \Big)$

g) $\exists x \Big(F(x) \wedge \forall y \big(\big(F(y) \wedge y \neq x \big) \rightarrow A(x,\, y) \big) \Big)$

h) $\exists x \Big(S(x) \wedge \forall y \big(F(y) \rightarrow \neg A(y,\, x) \big) \Big)$

Exercise

Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.

a) Every user has access to exactly one mailbox.

b) There is a process that continues to run during all error conditions only if the kernel is working correctly.

c) All users on the campus network can access all websites whose url has a .edu extension.

Solution

a) $\forall y \exists m \left(A(u,\ m) \wedge \forall n \left(n \neq m \rightarrow \neg A(u,\ n) \right) \right)$, where $A(u,\ m)$ means that user *u* has access to mailbox *m*.

b) $\exists p \forall e \left(H(e) \rightarrow S(p,\ running) \right) \rightarrow S(kernel,\ working\ correctly)$, where $H(e)$ means that error condition *e* is in effect and $S(x,\ y)$ means that the status of *x* is *y*.

c) $\forall u \forall s \left(E(s,\ edu) \rightarrow A(u,\ s) \right)$, where $E(s, x)$ means that website *s* has extension *x*, and $A(u,\ s)$ means that user *u* can access website *s*.

Exercise

Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers

a) $\exists x \forall y (x + y = y)$

b) $\forall x \forall y \left(\left((x \geq 0) \wedge (y < 0) \right) \rightarrow (x - y > 0) \right)$

c) $\exists x \exists y \left(\left((x \leq 0) \wedge (y \leq 0) \right) \wedge (x - y > 0) \right)$

d) $\forall x \forall y \left(\left((x \neq 0) \wedge (y \neq 0) \right) \leftrightarrow (xy \neq 0) \right)$

Solution

a) There exists an additive identity for all real numbers

b) A non-negative number minus a negative number is greater than zero.

c) The difference between two non-positive numbers is not necessarily non-positive (i.e. can be positive)

d) The product of two non-zero numbers is non-zero if and only if both factors are non-zero

Exercise

Determine the truth value of each of these statements if the domain for all variables consists of all integers

a) $\forall n \exists m \left(n^2 < m \right)$

b) $\exists n \forall m \left(n < m^2 \right)$

c) $\forall n \exists m \; (n + m = 0)$

d) $\exists n \forall m \; (nm = m)$

e) $\exists n \exists m \; \left(n^2 + m^2 = 5\right)$

f) $\exists n \exists m \; \left(n^2 + m^2 = 6\right)$

g) $\exists n \exists m \; (n + m = 4 \wedge n - m = 1)$

h) $\exists n \exists m \; (n + m = 4 \wedge n - m = 2)$

i) $\forall n \forall m \exists p \; \left(p = \frac{m+n}{2}\right)$

Solution

a) No matter how large *n* might be, we can always find an integer *m* bigger than n^2. This is certainly true, i.e. $m = n^2 + 1$.

b) There is an *n* that is smaller than the square of every integer. This statement is true since we could take $n = -1$, and then *n* would be less than every square, since squares are always greater than or equal to 0.

c) The order of quantifiers: *m* is allowed to depend on *n*. since we can take $m = -n$, this statement is true.

d) Clearly $n = 1$, so the statement is true.

e) $n^2 + m^2 = 5$ has a solution over the integers. This is true statement, since $n = \pm 1, \; m = \pm 2$ and vice versa (8 solutions).

f) $n^2 + m^2 = 6$ there is no integer solution. Therefore this statement is false.

g) There is a unique solution for the statement $\{n + m = 4, \; n - m = 1\}$, namely $n = \frac{5}{2}$ *and* $m = \frac{3}{2}$. Since there do not exist integers that make the equations true, the statement is false.

h) There is a unique solution for the statement $\{n + m = 4, \; n - m = 2\}$, namely $n = 3$ *and* $m = 1$. Therefore the statement is true.

i) If we take $n = 1$ *and* $m = 2$ then $p = \frac{3}{2}$ which is not an integer. Therefore the statement is false.