Solution *Section* 1.5 – Calculus of Vector-Valued Functions

Exercise

$\boldsymbol{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find an equation in *x* and *y* whose is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of *t*.

$\vec{r}(t)=(t+1)\hat{i}+\left(t^2-1\right)\hat{j}, \quad t=1$

Solution

$$x=t+1, \quad y=t^2-1$$

$$\Rightarrow \boxed{y=(x-1)^2-1=x^2-2x}$$

$$\vec{v}(t)=\vec{r}\,'=\hat{i}+2t\,\hat{j}$$

$$\boxed{\vec{v}(t=1)=\hat{i}+2\hat{j}}$$

$$\vec{a}=\vec{v}\,'=2\hat{j}$$

$$\boxed{\vec{a}(t=1)=2\hat{j}}$$

Exercise

$\vec{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find an equation in *x* and *y* whose is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of *t*.

$\vec{r}(t)=\frac{t}{t+1}\hat{i}+\frac{1}{t}\hat{j}, \quad t=-\frac{1}{2}$

Solution

$$x=\frac{t}{t+1}, \quad y=\frac{1}{t}\to t=\frac{1}{y}$$

$$x=\frac{\frac{1}{y}}{\frac{1}{y}+1}$$

$$=\frac{1}{1+y}$$

$$1+y=\frac{1}{x}$$

$$y=\frac{1}{x}-1$$

$$\left(\frac{t}{t+1}\right)'=\frac{1}{(t+1)^2} \qquad \left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

$$\vec{v}(t) = \frac{1}{(t+1)^2}\hat{i} - \frac{1}{t^2}\hat{j}$$

$$\vec{v}\left(t = -\frac{1}{2}\right) = \frac{1}{\left(-\frac{1}{2}+1\right)^2}\hat{i} - \frac{1}{\frac{1}{4}}\hat{j}$$

$$= 4\hat{i} - 4\hat{j}$$

$$\left(\frac{1}{(t+1)^2}\right)' = \frac{-2}{(t+1)^3} \qquad \left(\frac{1}{U^n}\right)' = -\frac{nU'}{U^{n+1}}$$

$$\vec{a} = \vec{v}' = \frac{-2}{(t+1)^3}\hat{i} + \frac{2}{t^3}\hat{j}$$

$$\vec{a}\left(t = -\frac{1}{2}\right) = \frac{-2}{\left(-\frac{1}{2}+1\right)^3}\hat{i} + \frac{2}{\left(-\frac{1}{2}\right)^3}\hat{j}$$

$$= \frac{-2}{-\frac{1}{8}}\hat{i} + \frac{2}{-\frac{1}{8}}\hat{j}$$

$$= 16\hat{i} - 16\hat{j}$$

Exercise

$\vec{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find an equation in *x* and *y* whose is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of *t*.

$$\vec{r}(t) = e^t\hat{i} + \frac{2}{9}e^{2t}\hat{j}, \quad t = \ln 3$$

Solution

$$x = e^t, \quad y = \frac{2}{9}e^{2t} = \frac{2}{9}\left(e^t\right)^2$$

$$y = \frac{2}{9}x^2$$

$$\vec{v}(t) = e^t\hat{i} + \frac{4}{9}e^{2t}\hat{j}$$

$$\vec{v}(t = \ln 3) = e^{\ln 3}\hat{i} + \frac{4}{9}e^{2\ln 3}\hat{j}$$

$$= 3\hat{i} + \frac{4}{9}e^{\ln 3^2}\hat{j} \qquad e^{\ln 3^2} = e^{\ln 9} = 9$$

$$= 3\hat{i} + 4\hat{j}$$

$$\vec{a}(t) = e^t\hat{i} + \frac{8}{9}e^{2t}\hat{j}$$

$$\vec{a}(t = \ln 3) = e^{\ln 3}\hat{i} + \frac{8}{9}e^{2\ln 3}\hat{j}$$

$$= 3\hat{i} + \frac{8}{9}e^{\ln 9}\hat{j}$$

$$\boxed{= 3\hat{i} + 8\hat{j}}$$

Exercise

$\vec{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find an equation in *x* and *y* whose is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of *t*.

$$\vec{r}(t) = (\cos 2t)\hat{i} + (3\sin 2t)\hat{j}, \quad t = 0$$

Solution

$$x = \cos 2t, \quad y = 3\sin 2t \to \sin 2t = \frac{y}{3}$$

$$\cos^2 2t + \sin^2 2t = 1$$

$$x^2 + \frac{y^2}{9} = 1$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -(2\sin 2t)\hat{i} + (6\cos 2t)\hat{j}$$

$$\vec{v}(t=0) = -(2\sin 0)\hat{i} + (6\cos 0)\hat{j}$$

$$\boxed{= 6\hat{j}}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = -(4\cos 2t)\hat{i} - (12\sin 2t)\hat{j}$$

$$\vec{a}(t=0) = -(4\cos 2t)\hat{i} - (12\sin 2t)\hat{j}$$

$$\boxed{= -4\hat{i}}$$

Exercise

Give the position vectors of particles moving along various curves in the *xy*-plane. Find the particle's velocity and acceleration vectors at the stated times and sketch them as vectors on the curve

Motion on the circle $x^2 + y^2 = 1$ $\quad \vec{r}(t) = (\sin t)\hat{i} + (\cos t)\hat{j}, \quad t = \frac{\pi}{4}$ *and* $\frac{\pi}{2}$

Solution

$$\vec{v} = \frac{d\mathbf{r}}{dt} = (\cos t)\hat{i} - (\sin t)\hat{j}$$

$$\vec{v}\left(t = \frac{\pi}{4}\right) = \left(\cos\frac{\pi}{4}\right)\hat{i} - \left(\sin\frac{\pi}{4}\right)\hat{j}$$

$$\boxed{= \frac{\sqrt{2}}{2}\hat{i} - \frac{\sqrt{2}}{2}\hat{j}}$$

$$\vec{v}\left(t=\frac{\pi}{2}\right)=\left(\cos\frac{\pi}{2}\right)\hat{i}-\left(\sin\frac{\pi}{2}\right)\hat{j}$$

$$\boxed{=-\hat{j}}$$

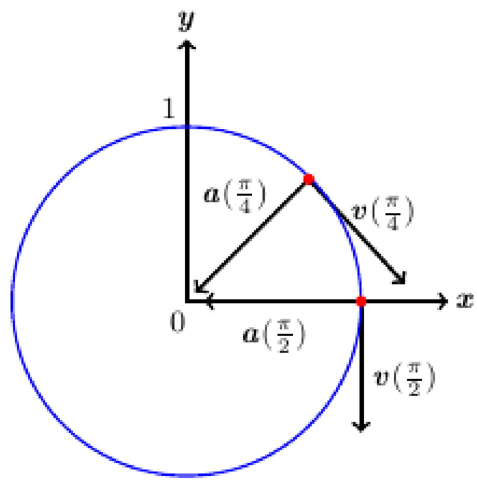
$$\vec{a}=\frac{d\vec{v}}{dt}=-(\sin t)\hat{i}-(\cos t)\hat{j}$$

$$\vec{a}\left(t=\frac{\pi}{4}\right)=-\left(\sin\frac{\pi}{4}\right)\hat{i}-\left(\cos\frac{\pi}{4}\right)\hat{j}$$

$$\boxed{=-\frac{\sqrt{2}}{2}\hat{i}-\frac{\sqrt{2}}{2}\hat{j}}$$

$$\vec{a}\left(t=\frac{\pi}{2}\right)=-\left(\sin\frac{\pi}{2}\right)\hat{i}-\left(\cos\frac{\pi}{2}\right)\hat{j}$$

$$\boxed{=-\hat{i}}$$



Exercise

Give the position vectors of particles moving along various curves in the *xy*-plane. Find the particle's velocity and acceleration vectors at the stated times and sketch them as vectors on the curve

Motion on the cycloid $x=t-\sin t, \quad y=1-\cos t$

$$\vec{\boldsymbol{r}}(t)=(t-\sin t)\hat{i}+(1-\cos t)\hat{j}; \quad t=\pi \ and \ \frac{3\pi}{2}$$

Solution

$$\vec{v}=\frac{d\vec{\boldsymbol{r}}}{dt}=(1-\cos t)\hat{i}+(\sin t)\hat{j}$$

$$\vec{v}(t=\pi)=(1-\cos\pi)\hat{i}+(\sin\pi)\hat{j}$$

$$\boxed{=2\hat{i}}$$

$$\vec{v}\left(t=\frac{3\pi}{2}\right)=\left(1-\cos\frac{3\pi}{2}\right)\hat{i}+\left(\sin\frac{3\pi}{2}\right)\hat{j}$$

$$\boxed{=\hat{i}-\hat{j}}$$

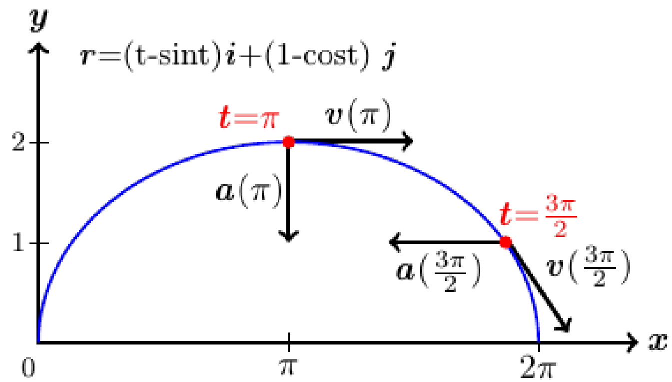
$$\vec{a}=\frac{d\vec{v}}{dt}=(\sin t)\hat{i}+(\cos t)\hat{j}$$

$$\vec{a}(t=\pi)=(\sin\pi)\hat{i}+(\cos\pi)\hat{j}$$

$$\boxed{=-\hat{j}}$$

$$\vec{a}\left(t=\frac{3\pi}{2}\right)=\left(\sin\frac{3\pi}{2}\right)\hat{i}+\left(\cos\frac{3\pi}{2}\right)\hat{j}$$

$$\boxed{=-\hat{i}}$$



Exercise

$\vec{\boldsymbol{r}}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of *t*. Write the particle's velocity at that time as the product of its speed and direction.

$$\vec{\boldsymbol{r}}(t) = (t+1)\hat{i} + \left(t^2 - 1\right)\hat{j} + 2t\,\hat{k}, \quad t = 1$$

Solution

$$\vec{v} = \frac{d\vec{\boldsymbol{r}}}{dt} = \hat{i} + 2t\,\hat{j} + 2\hat{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = 2\,\hat{j}$$

$$\vec{v}(t=1) = \hat{i} + 2\,\hat{j} + 2\hat{k}$$

Speed: $|\vec{v}(1)| = \sqrt{1+4+4} \ \boxed{= 3}$

Direction: $\dfrac{\vec{v}(1)}{|\vec{v}(1)|} = \frac{1}{3}\left(\hat{i} + 2\,\hat{j} + 2\hat{k}\right)$

$$= \frac{1}{3}\hat{i} + \frac{2}{3}\,\hat{j} + \frac{2}{3}\,\hat{k}$$

$$\boxed{\vec{v}(1) = 3\left(\frac{1}{3}\hat{i} + \frac{2}{3}\,\hat{j} + \frac{2}{3}\,\hat{k}\right)}$$

Exercise

$\vec{\boldsymbol{r}}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of *t*. Write the particle's velocity at that time as the product of its speed and direction.

$$\vec{\boldsymbol{r}}(t) = (t+1)\hat{i} + \frac{t^2}{\sqrt{2}}\,\hat{j} + \frac{t^3}{3}\,\hat{k}, \quad t = 1$$

Solution

$$\vec{v} = \frac{d\vec{\boldsymbol{r}}}{dt} = \hat{i} + \frac{2}{\sqrt{2}}t\,\hat{j} + t^2\hat{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{2}{\sqrt{2}}\,\hat{j} + 2t\,\hat{k}$$

$$\vec{v}(t=1) = \hat{i} + \frac{2}{\sqrt{2}}\,\hat{j} + \hat{k}$$

Speed: $|\vec{v}(1)| = \sqrt{1+2+1} \ \boxed{= 2}$

Direction: $\dfrac{\vec{v}(1)}{|\vec{v}(1)|} = \frac{1}{2}\left(\hat{i} + \frac{2}{\sqrt{2}}\,\hat{j} + \hat{k}\right)$

$$=\frac{1}{2}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}+\frac{1}{2}\hat{k}$$

$$\boxed{\vec{v}(1)=2\left(\frac{1}{2}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}+\frac{1}{2}\hat{k}\right)}$$

Exercise

$\vec{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of *t*. Write the particle's velocity at that time as the product of its speed and direction.

$$\vec{r}(t)=(2\cos t)\hat{i}+(3\sin t)\hat{j}+4t\,\hat{k},\quad t=\frac{\pi}{2}$$

Solution

$$\vec{v}=\frac{d\vec{r}}{dt}=-(2\sin t)\hat{i}+(3\cos t)\hat{j}+4\hat{k}$$

$$\vec{a}=\frac{d\vec{v}}{dt}=-(2\cos t)\hat{i}+(3\sin t)\hat{j}$$

$$\vec{v}\left(t=\frac{\pi}{2}\right)=-\left(2\sin\frac{\pi}{2}\right)\hat{i}+\left(3\cos\frac{\pi}{2}\right)\hat{j}+4\hat{k}$$

$$=-2\hat{i}+4\hat{k}$$

Speed: $\left|\vec{v}\left(\frac{\pi}{2}\right)\right|=\sqrt{4+16}\ \boxed{=2\sqrt{5}}$

Direction: $\dfrac{\vec{v}\left(\frac{\pi}{2}\right)}{\left|\vec{v}\left(\frac{\pi}{2}\right)\right|}=\dfrac{1}{2\sqrt{5}}\left(-2\hat{i}+4\hat{k}\right)$

$$=-\frac{1}{\sqrt{5}}\hat{i}+\frac{2}{\sqrt{5}}\hat{k}$$

$$\boxed{\vec{v}\left(\frac{\pi}{2}\right)=2\sqrt{5}\left(-\frac{1}{\sqrt{5}}\hat{i}+\frac{2}{\sqrt{5}}\hat{k}\right)}$$

Exercise

$\vec{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of *t*. Write the particle's velocity at that time as the product of its speed and direction.

$$\vec{r}(t)=(2\ln(t+1))\hat{i}+t^2\hat{j}+\frac{t^2}{2}\hat{k},\quad t=1$$

Solution

$$\vec{v}=\frac{d\vec{r}}{dt}=\frac{2}{t+1}\hat{i}+2t\,\hat{j}+t\,\hat{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{-2}{(t+1)^2}\hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{v}(t=1) = \hat{i} + 2\hat{j} + \hat{k}$$

Speed: $|\vec{v}(1)| = \sqrt{1+4+1} \ \underline{= \sqrt{6}}$

Direction: $\frac{\vec{v}(1)}{|\vec{v}(1)|} = \frac{1}{\sqrt{6}}(\hat{i} + 2\hat{j} + \hat{k})$

$$= \frac{1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} + \frac{1}{\sqrt{6}}\hat{k}$$

$$\vec{v}(1) = \sqrt{6}\left(\frac{1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} + \frac{1}{\sqrt{6}}\hat{k}\right)$$

Exercise

$\vec{\boldsymbol{r}}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of *t*. Write the particle's velocity at that time as the product of its speed and direction.

$$\vec{\boldsymbol{r}}(t) = \left(e^{-t}\right)\hat{i} + (2\cos 3t)\hat{j} + (2\sin 3t)\hat{k}, \quad t = 0$$

Solution

$$\vec{v} = \frac{d\vec{\boldsymbol{r}}}{dt} = -e^{-t}\hat{i} - 6\sin 3t\,\hat{j} + 6\cos 3t\,\hat{k} \qquad \boldsymbol{v}(0) = -\boldsymbol{i} + 6\boldsymbol{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = e^{-t}\hat{i} - 18\cos 3t\,\hat{j} - 18\sin 3t\,\hat{k}$$

Speed: $|\vec{v}(0)| = 1 + 36 \ \underline{= \sqrt{37}}$

Direction: $\frac{\vec{v}(0)}{|\vec{v}(0)|} = \frac{1}{\sqrt{37}}(-\hat{i} + 6\hat{k})$

$$= -\frac{1}{\sqrt{37}}\hat{i} + \frac{6}{\sqrt{37}}\hat{k}$$

$$\vec{v}(1) = \sqrt{37}\left(-\frac{1}{\sqrt{37}}\hat{i} + \frac{6}{\sqrt{37}}\hat{k}\right)$$

Exercise

Find all points on the ellipse $\vec{r}(t) = \langle 1,\ 8\sin t,\ \cos t\rangle$, for $0 \le t \le 2\pi$, at which $\vec{r}(t)$ and $\vec{r}\,'(t)$ are orthogonal.

Solution

$\vec{r}'(t) = \langle 0,\ 8\cos t,\ \ -\sin t \rangle$

$\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal that implies to $\vec{r}(t) \bullet \vec{r}'(t) = 0$

$$\begin{aligned}\vec{r}(t) \bullet \vec{r}'(t) &= \langle 1,\ 8\sin t,\ \ \cos t \rangle \bullet \langle 0,\ 8\cos t,\ \ -\sin t \rangle \\ &= 64\sin t\cos t - \cos t\sin t \\ &= 63\sin t\cos t = 0\end{aligned}$$

$$\rightarrow \begin{cases} \sin t = 0 & \Rightarrow\ t = 0,\ \pi,\ 2\pi \\ \cos t = 0 & \Rightarrow\ t = \frac{\pi}{2},\ \frac{3\pi}{2} \end{cases}$$

$t = 0,\ 2\pi \quad \rightarrow \quad (1,\ 0,\ 1)$

$t = \frac{\pi}{2} \quad \rightarrow \quad (1,\ 8,\ 0)$

$t = \pi \quad \rightarrow \quad (1,\ 0,\ -1)$

$t = \frac{3\pi}{2} \quad \rightarrow \quad (1,\ -8,\ 0)$