Solution Section 2.7 – Maximum/Minimum Problems

Exercise

Find all the local maxima, local minima, and saddle points of the function

$f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$

Solution

$f_x = 2x + y + 3 = 0 \qquad f_y = x + 2y - 3 = 0$

$$\begin{cases} 2x + y = -3 \\ x + 2y = 3 \end{cases} \rightarrow \quad x = -3 \quad y = 3$$

Therefore, the critical point is $(-3,\ 3)$

$f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = 1$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - 1^2 = 3 > 0 \quad \text{and} \quad f_{xx} = 2 > 0$$

The function *f* has a ***local minimum*** at $(-3,\ 3)$ and the value is

$$f(-3,\ 3) = (-3)^2 + (-3)(3) + 3^2 + 3(-3) - 3(3) + 4$$

$$= -5$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x,y) = x^2 - xy + y^2 + 2x + 2y - 4$$

Solution

$f_x = 2x - y + 2 = 0 \qquad f_y = -x + 2y + 2 = 0$

$$\begin{cases} 2x - y = -2 \\ x - 2y = 2 \end{cases} \qquad \Delta = \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} = -3 \quad \Delta_x = \begin{vmatrix} -2 & -1 \\ 2 & -2 \end{vmatrix} = 6$$

$x = \frac{6}{-3} = -2 \quad y = -4 + 2 = -2$

Therefore, the critical point is $(-2,\ -2)$

$f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = -1$

At $(-2,\ -2)$: $f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = -1$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - 1 = 3 > 0 \quad \text{and} \quad f_{xx} = 2 > 0$$

The function *f* has a ***local minimum*** at $(-2,\ -2)$ and the value is

$$f(-2,\ -2) = 4 - 4 + 4 - 4 - 4 - 4$$

$$\underline{= -8}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x, y) = x^3 + y^3 - 3xy + 15$$

Solution

$$f_x = 3x^2 - 3y = 0 \qquad f_y = 3y^2 - 3x = 0$$

$$\begin{cases} x^2 = y \\ y^2 = x \end{cases} \quad \left(x^2\right)^2 = x \quad \to \quad x^4 = x$$

$$x\left(x^3 - 1\right) = 0 \quad \to \quad \underline{x = 0,\ \ 1}$$

$$\begin{cases} x = 0 & y = 0 \\ x = 1 & y = 1 \end{cases}$$

Therefore, the critical point is $(0,\ 0)$ & $(1,\ 1)$

$$f_{xx} = 6x \quad f_{yy} = 6y \quad f_{xy} = -3$$

At $(0,\ 0)$

$$f_{xx} = 0 \quad f_{yy} = 0 \quad f_{xy} = -3$$

$$f_{xx}f_{yy} - f_{xy}^2 = 0 - 9 = -9 < 0$$

The function *f* has a ***saddle point*** at $(0,\ 0)$ and the value is $\underline{f(0,\ 0) = 15}$

At $(1,\ 1)$

$$f_{xx} = 6 \quad f_{yy} = 6 \quad f_{xy} = -3$$

$$f_{xx}f_{yy} - f_{xy}^2 = 36 - 9 = 27 > 0 \quad \text{and} \quad f_{xx} = 6 > 0$$

The function *f* has a ***local minimum*** at $(1,\ 1)$ and the value is $\underline{f(1,\ 1) = 14}$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x,y) = x^4 - 8x^2 + 3y^2 - 6y$$

Solution

$f_x = 4x^3 - 16x = 0 \qquad f_y = 6y - 6 = 0$

$$\begin{cases} 4x\left(x^2 - 4\right) = 0 & \rightarrow \ \underline{x = 0, \ \pm 2} \\ \quad y = 1 \end{cases}$$

Therefore, the critical point is $(0,\ 1)$ & $(\pm 2,\ 1)$

$f_{xx} = 12x^2 - 16 \quad f_{yy} = 6 \quad f_{xy} = 0$

At $(0,\ 1)$

$f_{xx} = -16 \quad f_{yy} = 6 \quad f_{xy} = 0$

$f_{xx}f_{yy} - f_{xy}^2 = -96 < 0$

The function *f* has a ***saddle point*** at $(0,\ 1)$ and the value is $\underline{f(0,\ 1) = -3}$

At $(2,\ 1)$

$f_{xx} = 32 \quad f_{yy} = 6 \quad f_{xy} = 0$

$f_{xx}f_{yy} - f_{xy}^2 = 192 > 0 \quad \text{and} \quad f_{xx} = 32 > 0$

The function *f* has a ***local minimum*** at $(2,\ 1)$ and the value is $\underline{f(2,\ 1) = -19}$

At $(-2,\ 1)$

$f_{xx} = 32 \quad f_{yy} = 6 \quad f_{xy} = 0$

$f_{xx}f_{yy} - f_{xy}^2 = 192 > 0 \quad \text{and} \quad f_{xx} = 32 > 0$

The function *f* has a ***local minimum*** at $(-2,\ 1)$ and the value is $\underline{f(-2,\ 1) = -19}$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x,y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$$

Solution

$f_x = 2y - 10x + 4 = 0 \qquad f_y = 2x - 4y + 4 = 0$

$$\begin{cases} -5x + y = -2 \\ x - 2y = -2 \end{cases} \rightarrow \quad x = \frac{2}{3} \quad y = \frac{4}{3}$$

Therefore, the critical point is $\left(\frac{2}{3},\ \frac{4}{3}\right)$

$$f_{xx}\Big|_{\left(\frac{2}{3},\ \frac{4}{3}\right)} = -10 \quad f_{yy}\Big|_{\left(\frac{2}{3},\ \frac{4}{3}\right)} = -4 \quad f_{xy}\Big|_{\left(\frac{2}{3},\ \frac{4}{3}\right)} = 2$$

$$f_{xx}f_{yy} - f_{xy}^2 = (-10)(-4) - 2^2 = 36 > 0 \quad \textit{and} \quad f_{xx} = -10 < 0$$

The function f has a local maximum at $\left(\frac{2}{3},\ \frac{4}{3}\right)$ and the value is

$$f\left(\frac{2}{3},\ \frac{4}{3}\right) = 2\left(\frac{2}{3}\right)\left(\frac{4}{3}\right) - 5\left(\frac{2}{3}\right)^2 - 2\left(\frac{4}{3}\right)^2 + 4\left(\frac{2}{3}\right) + 4\left(\frac{4}{3}\right) - 4$$

$$= 0$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$f(x,y) = x^2 - 4xy + y^2 + 6y + 2$

Solution

$$f_x = 2x - 4y = 0 \qquad f_y = -4x + 2y + 6 = 0$$

$$\begin{cases} x - 2y = 0 \\ -2x + y = -3 \end{cases} \rightarrow \quad x = 2 \quad y = 1$$

Therefore, the critical point is $(2,\ 1)$

$$f_{xx}\Big|_{(2,1)} = 2, \quad f_{yy}\Big|_{(2,1)} = 2, \quad f_{xy}\Big|_{(2,1)} = -4$$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - (-4)^2 = -12 < 0 \quad \Rightarrow \quad \textit{Saddle point}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$f(x,y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$

Solution

$$f_x = 4x + 3y - 5 = 0 \qquad f_y = 3x + 8y + 2 = 0$$

$$\begin{cases} 4x+3y=5 \\ 3x+8y=-2 \end{cases} \rightarrow \quad x=2 \quad y=-1$$

Therefore, the critical point is $(2,\ -1)$

$$f_{xx}\Big|_{(2,-1)}=4, \quad f_{yy}\Big|_{(2,-1)}=8, \quad f_{xy}\Big|_{(2,-1)}=3$$

$$f_{xx}f_{yy}-f_{xy}^2=(4)(8)-3^2=23>0 \quad \textit{and} \quad f_{xx}=4>0$$

The function f has a local minimum at $(2,\ -1)$ and the value is

$$f(2,\ -1)=2(2)^2+3(2)(-1)+4(-1)^2-5(2)+2(-1)$$

$$\underline{=-6}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$f(x,y)=x^2-y^2-2x+4y+6$

Solution

$$f_x=2x-2=0 \qquad f_y=-2y+4=0$$

$$\begin{cases} 2x=2 \\ 2y=4 \end{cases} \rightarrow \quad x=1 \quad y=2$$

Therefore, the critical point is $(1,\ 2)$

$$f_{xx}\Big|_{(1,2)}=2, \quad f_{yy}\Big|_{(1,2)}=-2, \quad f_{xy}\Big|_{(1,2)}=0$$

$$f_{xx}f_{yy}-f_{xy}^2=(2)(-2)-0^2=-4<0 \quad \rightarrow \quad \textit{Saddle Point}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$f(x,y)=\sqrt{56x^2-8y^2-16x-31}+1-8x$

Solution

$$f_x=\frac{1}{2}\frac{112x-16}{\sqrt{56x^2-8y^2-16x-31}}-8=0 \qquad f_y=\frac{1}{2}\frac{-16y}{\sqrt{56x^2-8y^2-16x-31}}=0$$

$$\begin{cases} 56x - 8 = 8\sqrt{56x^2 - 8y^2 - 16x - 31} \\ -8y = 0 \end{cases} \rightarrow \begin{matrix} x = \frac{16}{7} & \cancel{x = -2} \\ y = 0 \end{matrix}$$

Therefore, the critical point is $\left(\frac{16}{7},\ 0\right)$

$$f_{xx}\Big|_{\left(\frac{16}{7},\ 0\right)} = \frac{56\sqrt{56x^2 - 8y^2 - 16x - 31} - (56x-8)(56x-8)\left(56x^2 - 8y^2 - 16x - 31\right)^{-1/2}}{56x^2 - 8y^2 - 16x - 31}$$

$$= -\frac{8}{15}$$

$$f_{yy}\Big|_{\left(\frac{16}{7},\ 0\right)} = \frac{-8\sqrt{56x^2 - 8y^2 - 16x - 31} - (-8y)\left(56x^2 - 8y^2 - 16x - 31\right)^{-1/2}(-8y)}{56x^2 - 8y^2 - 16x - 31}$$

$$= -\frac{8}{15}$$

$$f_{xy}\Big|_{\left(\frac{16}{7},\ 0\right)} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = \left(-\frac{8}{15}\right)\left(-\frac{8}{15}\right) - 0 = \frac{34}{225} > 0 \quad \textit{and} \quad f_{xx} = -\frac{8}{15} < 0$$

The function f has a local maximum at $\left(\frac{16}{7},\ 0\right)$ and the value is

$$f\left(\frac{16}{7},\ 0\right) = \sqrt{56\left(\frac{16}{7}\right)^2 - 8(0)^2 - 16\left(\frac{16}{7}\right) - 31} + 1 - 8\left(\frac{16}{7}\right)$$

$$= -\frac{16}{7}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y) = 1 - \sqrt[3]{x^2 + y^2}$

Solution

$$f_x = -\frac{1}{3}2x\left(x^2 + y^2\right)^{-2/3}$$

$$= \frac{-2x}{3\left(x^2 + y^2\right)^{2/3}} = 0$$

$$f_y = -\frac{1}{3}2y\left(x^2 + y^2\right)^{-2/3}$$

$$= \frac{-2y}{3\left(x^2+y^2\right)^{2/3}} = 0$$

There are no solutions to the system $f_x(x,y)=0$ *and* $f_y(x,y)=0$, however, this occurs when $x=0 \quad y=0$. The critical point is $(0,\ 0)$

We cannot use the second derivative test, but this is the only possible local maximum, local minimum, or saddle point. $f(x,y)$ has a local maximum of $f(0,0)=1$ since

$$f(x,y)=1-\sqrt[3]{x^2+y^2} \le 1 \quad \forall (x,y)-\{(0,0)\}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$f(x,y)=x^3+y^3+3x^2-3y^2-8$

Solution

$$f_x = 3x^2+6x = 0 \qquad f_y = 3y^2-6y = 0$$

$$\begin{cases} 3x(x+2)=0 \\ 3y(y-2)=0 \end{cases} \rightarrow \begin{matrix} x=0,\ -2 \\ y=0,\ 2 \end{matrix}$$

Therefore, the critical point is $(0,0)$, $(0,2)$, $(-2,0)$, and $(-2,2)$

$$f_{xx}=6x+6, \quad f_{yy}=6y-6, \quad f_{xy}=0$$

For $(0,0)$ $\left. f_{xx}\right|_{(0,0)}=6, \quad \left. f_{yy}\right|_{(0,0)}=-6, \quad \left. f_{xy}\right|_{(0,0)}=0$

$$f_{xx}f_{yy}-f_{xy}^2=(6)(-6)-0^2=-36<0 \quad \Rightarrow \quad \textit{Saddle Point}$$

For $(0,2)$ $\left. f_{xx}\right|_{(0,2)}=6, \quad \left. f_{yy}\right|_{(0,2)}=6, \quad \left. f_{xy}\right|_{(0,2)}=0$

$$f_{xx}f_{yy}-f_{xy}^2=(6)(6)-0^2=36>0 \quad \textit{and} \quad f_{xx}>0$$

The function f has a local minimum at $(0,2)$ and the value is $f(0,2)\ \underline{=-12}$

For $(-2,0)$ $\left. f_{xx}\right|_{(-2,0)}=-6, \quad \left. f_{yy}\right|_{(-2,0)}=-6, \quad \left. f_{xy}\right|_{(-2,0)}=0$

$$f_{xx}f_{yy}-f_{xy}^2=(-6)(-6)-0^2=36>0 \quad \textit{and} \quad f_{xx}<0$$

The function f has a local maximum at $(-2,0)$ and the value is $f(-2,0)\ \underline{=-4}$

For $(-2,2)$ $\quad f_{xx}\Big|_{(-2,2)} = 6, \quad f_{yy}\Big|_{(-2,2)} = 6, \quad f_{xy}\Big|_{(-2,2)} = 0$

$$f_{xx}f_{yy} - f_{xy}^2 = (-6)(6) - 0^2 = -36 < 0 \quad \Rightarrow \quad \textit{Saddle Point}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y) = 4xy - x^4 - y^4$

Solution

$f_x = 4y - 4x^3 = 0 \qquad f_y = 4x - 4y^3 = 0$

$$\begin{cases} y - x^3 = 0 \\ x - y^3 = 0 \end{cases} \Rightarrow x = y \to x - x^3 = 0 \to x\left(1 - x^2\right) = 0 \quad \to x = 0, \ \pm 1$$

Therefore, the critical point is $(0,0)$, $(1,1)$, and $(-1,-1)$

$f_{xx} = -12x^2, \quad f_{yy} = -12y^2, \quad f_{xy} = 4$

For $(0,0)$ $\quad f_{xx}\Big|_{(0,0)} = 0, \quad f_{yy}\Big|_{(0,0)} = 0, \quad f_{xy}\Big|_{(0,0)} = 4$

$$f_{xx}f_{yy} - f_{xy}^2 = 0 - 4^2 = -16 < 0 \quad \Rightarrow \quad \textit{Saddle Point}$$

For $(1,1)$ $\quad f_{xx}\Big|_{(1,1)} = -12, \quad f_{yy}\Big|_{(1,1)} = -12, \quad f_{xy}\Big|_{(1,1)} = 4$

$$f_{xx}f_{yy} - f_{xy}^2 = (-12)(-12) - 4^2 = 128 > 0 \quad \textit{and} \quad f_{xx} < 0$$

The function has a local maximum at $(1,1)$ and the value is $f(1,1) \underline{= 2}$

For $(-1,-1)$ $\quad f_{xx}\Big|_{(-1,-1)} = -12, \quad f_{yy}\Big|_{(-1,-1)} = -12, \quad f_{xy}\Big|_{(-1,-1)} = 0$

$$f_{xx}f_{yy} - f_{xy}^2 = (-12)(-12) - 4^2 = 128 > 0 \quad \textit{and} \quad f_{xx} < 0 \ f$$

The function f has a local maximum at $(-1,-1)$ and the value is $f(-1,-1) \underline{= 2}$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y) = \dfrac{1}{x^2 + y^2 - 1}$

Solution

$$f_x = \frac{-2x}{\left(x^2 + y^2 - 1\right)^2} = 0 \qquad f_y = \frac{-2y}{\left(x^2 + y^2 - 1\right)^2} = 0$$

$\Rightarrow x = y = 0$ Therefore, the critical point is $(0,0)$

$$f_{xx} = \frac{-2\left(x^2+y^2-1\right)^2 - (-2x)(4x)\left(x^2+y^2-1\right)}{\left(x^2+y^2-1\right)^4}$$

$$= \frac{-2x^2-2y^2+2+8x^2}{\left(x^2+y^2-1\right)^3}$$

$$= \frac{6x^2-2y^2+2}{\left(x^2+y^2-1\right)^3}$$

$$f_{yy} = \frac{-2\left(x^2+y^2-1\right)^2 - (-2y)(4y)\left(x^2+y^2-1\right)}{\left(x^2+y^2-1\right)^4}$$

$$= \frac{-2x^2-2y^2+2+8y^2}{\left(x^2+y^2-1\right)^3}$$

$$= \frac{-2x^2+6y^2+2}{\left(x^2+y^2-1\right)^3}$$

$$f_{xy} = \frac{-2x(4y)\left(x^2+y^2-1\right)}{\left(x^2+y^2-1\right)^4}$$

$$= \frac{-8xy}{\left(x^2+y^2-1\right)^3}$$

$$f_{xx}\Big|_{(0,0)} = -2, \quad f_{yy}\Big|_{(0,0)} = -2, \quad f_{xy}\Big|_{(0,0)} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - 0^2 = 4 > 0 \quad \textit{and} \quad f_{xx} < 0$$

The function f has a local maximum at $(0,0)$ and the value is $\underline{f(0,0) = -1}$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y) = \frac{1}{x} + xy + \frac{1}{y}$

Solution

$$f_x = -\frac{1}{x^2} + y = 0 \qquad f_y = x - \frac{1}{y^2} = 0$$

$$\rightarrow \begin{cases} y = \frac{1}{x^2} & (x \neq 0) \\ x = \frac{1}{y^2} & (y \neq 0) \end{cases} \quad x = x^4 \quad \Rightarrow x = 1 = y$$

Therefore, the critical point is $(1,1)$

$$f_{xx}\Big|_{(1,1)} = \left(\frac{2}{x^3}\right)\Big|_{(1,1)} = 2, \quad f_{yy}\Big|_{(1,1)} = \left(\frac{2}{y^3}\right)\Big|_{(1,1)} = -2, \quad f_{xy}\Big|_{(1,1)} = (1)\Big|_{(1,1)} = 1$$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - 1^2 = 3 > 0 \quad \textit{and} \quad f_{xx} > 0$$

The function f has a local minimum at $(1,1)$ and the value is $\underline{f(1,\ 1) = 3}$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y) = y\sin x$

Solution

$$f_x = y\cos x = 0 \qquad f_y = \sin x = 0$$

$$\rightarrow \begin{cases} y\cos x = 0 \\ \sin x = 0 \end{cases} \quad x = n\pi \quad y = 0 \quad \text{Therefore, the critical point is } (n\pi, 0)$$

$$f_{xx}\Big|_{(n\pi,0)} = -y\sin x\Big|_{(n\pi,0)} = 0$$

$$f_{yy}\Big|_{(n\pi,0)} = 0$$

$$f_{xy}\Big|_{(n\pi,0)} = \cos x\Big|_{(n\pi,0)} = \pm 1$$

If n is even: $\quad f_{xx}f_{yy} - f_{xy}^2 = 0 - 1^2 = -1 < 0 \quad \Rightarrow \quad$ *Saddle Point*

If n is odd: $\quad f_{xx}f_{yy} - f_{xy}^2 = 0 - (-1)^2 = -1 < 0 \quad \Rightarrow \quad$ *Saddle Point*

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y) = e^{2x}\cos y$

Solution

$$f_x = 2e^{2x}\cos y = 0 \qquad f_y = -e^{2x}\sin y = 0$$

Since $e^{2x} \neq 0 \quad \forall x$, the functions $\cos y$ and $\sin y$ cannot equal to zero for the same y.

$\therefore$ No critical points $\Rightarrow$ no extrema and no saddle points.

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y) = e^y - ye^x$

Solution

$$f_x = -ye^x = 0 \qquad f_y = e^y - e^x = 0$$

$$\rightarrow \begin{cases} -ye^x = 0 \\ e^y - e^x = 0 \end{cases} \quad y = 0 \quad e^x = e^y = 1 = e^0 \Rightarrow x = 0$$

$\therefore$ The critical point is $(0,0)$

$$f_{xx}\Big|_{(0,0)} = -ye^x\Big|_{(0,0)} = 0$$

$$f_{yy}\Big|_{(0,0)} = e^y = 1$$

$$f_{xy}\Big|_{(0,0)} = -e^x\Big|_{(0,0)} = -1$$

$$f_{xx}f_{yy} - f_{xy}^2 = 0(1) - (-1)^2 = -1 < 0 \quad \Rightarrow \quad \textit{Saddle Point}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y) = e^{-y}\left(x^2 + y^2\right)$

Solution

$$f_x = 2xe^{-y} = 0$$

$$f_y = -e^{-y}\left(x^2 + y^2\right) + 2ye^{-y} = e^{-y}\left(2y - x^2 - y^2\right) = 0$$

$$\rightarrow \begin{cases} 2xe^{-y} = 0 & \rightarrow \boxed{x = 0} \\ e^{-y}\left(2y - x^2 - y^2\right) = 0 \end{cases} \quad 2y - x^2 - y^2 = 0 \rightarrow y(2 - y = 0) \quad \boxed{y = 0,\ 2}$$

$\therefore$ The critical point is $(0,0)$ and $(0,2)$

$$f_{xx} = 2e^{-y}$$

$$f_{yy} = -e^{-y}\left(2y - x^2 - y^2\right) + e^{-y}(2 - 2y) = e^{-y}\left(2 - 4y + x^2 + y^2\right)$$

$f_{xy} = -2xye^{-y}$

For $(0,0)$ $\quad f_{xx}\Big|_{(0,0)} = 2, \quad f_{yy}\Big|_{(0,0)} = 2, \quad f_{xy}\Big|_{(0,0)} = 0$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - 0^2 = 4 > 0 \quad \textit{and} \quad f_{xx} > 0$$

The function f has a local minimum at $(0,0)$ and the value is $\underline{f(0,\ 0) = 0}$

For $(0,2)$ $\quad f_{xx}\Big|_{(0,2)} = \frac{2}{e^2}, \quad f_{yy}\Big|_{(0,2)} = -\frac{2}{e^2}, \quad f_{xy}\Big|_{(0,2)} = 0$

$$f_{xx}f_{yy} - f_{xy}^2 = \frac{2}{e^2}\left(-\frac{2}{e^2}\right) - 0^2 = -\frac{4}{e^4} < 0 \quad \Rightarrow \quad \textit{Saddle Point}$$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x,y) = 2\ln x + \ln y - 4x - y$

Solution

$$f_x = \frac{2}{x} - 4 = 0 \qquad f_y = \frac{1}{y} - 1 = 0$$

$$\rightarrow \begin{cases} 2 = 4x \\ 1 = y \end{cases} \quad x = \frac{1}{2}$$

$\therefore$ The critical point is $\left(\frac{1}{2},\ 1\right)$

$$f_{xx}\Big|_{\left(\frac{1}{2},1\right)} = \left(-\frac{2}{x^2}\right)\Big|_{\left(\frac{1}{2},1\right)} = -8$$

$$f_{yy}\Big|_{\left(\frac{1}{2},1\right)} = \left(-\frac{1}{y^2}\right)\Big|_{\left(\frac{1}{2},1\right)} = -1$$

$$f_{xy}\Big|_{\left(\frac{1}{2},1\right)} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = (-8)(-1) - 0^2 = 8 > 0 \quad \textit{and} \quad f_{xx} < 0$$

The function f has a local maximum at $\left(\frac{1}{2},\ 1\right)$ and the value is $\underline{f\left(\frac{1}{2},\ 1\right) = -3 - 2\ln 2}$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x,y)=\ln(x+y)+x^2-y$

Solution

$$f_x=\frac{1}{x+y}+2x=0$$

$$f_y=\frac{1}{x+y}-1=0$$

$$\to\begin{cases}\frac{1}{x+y}=-2x & \to -2x(x+y)=1\\ \frac{1}{x+y}=1 & \to 1=x+y\end{cases}\Rightarrow -2x(1)=1\to x=-\frac{1}{2}\quad y=\frac{3}{2}$$

$\therefore$ The critical point is $\left(-\frac{1}{2},\ \frac{3}{2}\right)$

$$f_{xx}=-\frac{1}{(x+y)^2}+2,\quad f_{yy}=-\frac{1}{(x+y)^2},\quad f_{xy}=-\frac{1}{(x+y)^2}$$

$$f_{xx}\Big|_{\left(-\frac{1}{2},\frac{3}{2}\right)}=1$$

$$f_{yy}\Big|_{\left(-\frac{1}{2},\frac{3}{2}\right)}=-1$$

$$f_{xy}\Big|_{\left(-\frac{1}{2},\frac{3}{2}\right)}=-1$$

$$f_{xx}f_{yy}-f_{xy}^2=(1)(-1)-(-1)^2=-2<0\quad \textit{and}\quad \textit{Saddle Point}$$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x,\ y)=1+x^2+y^2$

Solution

$$f_x=2x=0\quad\to\quad x=0$$

$$f_y=2y=0\quad\to\quad y=0$$

$\therefore$ The critical point is $(0,\ 0)$

$$f_{xx}=2,\quad f_{yy}=2,\quad f_{xy}=0$$

$$f_{xx}f_{yy}-f_{xy}^2\Big|_{(0,0)}=4>0\quad \textit{and}\quad f_{xx}>0$$

The function f has a local minimum at $(0,\ 0)$ and the value is $f(0,\ 0)=1$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = x^2 - 6x + y^2 + 8y$

Solution

$$f_x = 2x - 6 = 0 \quad \rightarrow \quad \underline{x = 3}$$

$$f_y = 2y + 8 = 0 \quad \rightarrow \quad \underline{y = -4}$$

$\therefore$ The critical point is $(3, \ -4)$

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 0$$

$$f_{xx} f_{yy} - f_{xy}^2 \Big|_{(3,-4)} = 4 > 0 \quad \textit{and} \quad f_{xx} > 0$$

The function f has a local minimum at $(3, \ -4)$ and the value is

$$\underline{f(3, \ -4) = 9 - 18 + 16 - 32 = -25}$$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = (3x - 2)^2 + (y - 4)^2$

Solution

$$f_x = 6(3x - 2) = 0 \quad \rightarrow \quad \underline{x = \frac{2}{3}}$$

$$f_y = 2(y - 4) = 0 \quad \rightarrow \quad \underline{y = 4}$$

$\therefore$ The critical point is $\left(\frac{2}{3}, \ 4\right)$

$$f_{xx} = 18, \quad f_{yy} = 2, \quad f_{xy} = 0$$

$$f_{xx} f_{yy} - f_{xy}^2 \Big|_{\left(\frac{2}{3}, \ 4\right)} = 36 > 0 \quad \textit{and} \quad f_{xx} > 0$$

The function f has a local minimum at $\left(\frac{2}{3}, \ 4\right)$ and the value is $\underline{f\left(\frac{2}{3}, \ 4\right) = 0}$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = 3x^2 - 4y^2$

Solution

$$f_x = 6x = 0 \quad \rightarrow \quad \underline{x = 0}$$

$$f_y = -8y = 0 \quad \to \quad \boxed{y = 0}$$

$\therefore$ The critical point is $(0,\ 0)$

$$f_{xx} = 6, \quad f_{yy} = -8, \quad f_{xy} = 0$$

$$f_{xx} f_{yy} - f_{xy}^2 \Big|_{(0,\ 0)} = -48 < 0 \quad \textit{and} \quad \textit{Saddle point}$$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x,\ y) = x^4 + y^4 - 16xy$

Solution

$$f_x = 4x^3 - 16y = 0$$

$$f_y = 4y^3 - 16x = 0$$

$$\begin{cases} x^3 = 4y \\ y^3 = 4x \ \to \ x = \dfrac{y^3}{4} \end{cases} \quad \to \quad \left(\frac{y^3}{4}\right)^3 = 4y$$

$$y^9 = 4^4 y$$

$$y\left(y^8 - 2^8\right) = 0 \to y = 0, \ \pm 2$$

$\therefore$ The critical point is $(0,\ 0)$, $(-2,\ -2)$, $(2,\ 2)$

$$f_{xx} = 12x^2 > 0, \quad f_{yy} = 12y^2, \quad f_{xy} = -16$$

$$f_{xx} f_{yy} - f_{xy}^2 = 144x^2y^2 - 256$$

@ $(0,\ 0)$

$$f_{xx} f_{yy} - f_{xy}^2 = -256 < 0 \quad \textit{and} \quad \textit{Saddle point}$$

@ $(-2,\ -2)$

$$f_{xx} f_{yy} - f_{xy}^2 = 2{,}048 > 0 \quad \textit{and} \quad f_{xx} > 0$$

The function f has a local minimum at $(-2,\ -2)$ and the value is $\boxed{f(-2,\ -2) = -32}$

@ $(2,\ 2)$

$$f_{xx} f_{yy} - f_{xy}^2 = 2{,}048 > 0 \quad \textit{and} \quad f_{xx} > 0$$

The function f has a local minimum at $(2,\ 2)$ and the value is $\boxed{f(2,\ 2) = -32}$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = \frac{1}{3}x^3 - \frac{1}{3}y^3 + 3xy$

Solution

$$f_x = x^2 + 3y = 0$$

$$f_y = -y^2 + 3x = 0$$

$$\begin{cases} x^2 = -3y \\ y^2 = 3x \rightarrow x = \dfrac{y^2}{3} \end{cases} \rightarrow \left(\frac{y^2}{3}\right)^2 = -3y$$

$$y^4 = -3^3 y$$

$$y\left(y^3 + 3^3\right) = 0 \rightarrow y = 0, \ -3$$

$\therefore$ The critical point is $(0, \ 0)$, $(3, \ -3)$

$$f_{xx} = 2x, \quad f_{yy} = -2y, \quad f_{xy} = 3$$

$$f_{xx}f_{yy} - f_{xy}^2 = -4xy - 9$$

@ $(0, \ 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = -9 < 0 \quad \textit{and} \quad \textit{Saddle point}$$

@ $(3, \ -3)$

$$f_{xx}f_{yy} - f_{xy}^2 = 36 - 9 = 27 > 0 \quad \textit{and} \quad f_{xx} = 6 > 0$$

The function f has a local minimum at $(3, \ -3)$ and the value is

$$\underline{f(3, \ -3) = 9 + 9 - 27 = -9}$$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = x^4 - 2x^2 + y^2 - 4y + 5$

Solution

$$f_x = 4x^3 - 4x = 4x\left(x^2 - 1\right) = 0 \quad \rightarrow \quad x = 0, \ \pm 1$$

$$f_y = 2y - 4 = 0 \quad \rightarrow \quad \underline{y = 2}$$

$\therefore$ The critical point is $(0, \ 2)$, $(-1, \ 2)$, $(1, \ 2)$

$f_{xx} = 12x^2 - 4, \quad f_{yy} = 2, \quad f_{xy} = 0$

$f_{xx}f_{yy} - f_{xy}^2 = 24x^2 - 8$

@ $(0,\ 2)$

$f_{xx}f_{yy} - f_{xy}^2 = -8 < 0$ *and* *Saddle point*

@ $(-1,\ 2)$

$f_{xx}f_{yy} - f_{xy}^2 = 16 > 0$ *and* $f_{xx} = 8 > 0$

The function f has a local minimum at $(-1,\ 2)$ and the value is

$\underline{f(-1,\ 2) = 1 - 2 + 4 - 8 + 5 = 0}|$

@ $(1,\ 2)$

$f_{xx}f_{yy} - f_{xy}^2 = 16 > 0$ *and* $f_{xx} = 8 > 0$

The function f has a local minimum at $(1,\ 2)$ and the value is

$\underline{f(1,\ 2) = 1 - 2 + 4 - 8 + 5 = 0}|$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x,\ y) = x^2 + xy - 2x - y + 1$

Solution

$f_x = 2x + y - 2 = 0$

$f_y = x - 1 = 0 \quad \rightarrow \quad \underline{x = 1}|$

$y = 2 - 2x \underline{= 0}|$

$\therefore$ The critical point is $(1,\ 0)$

$f_{xx} = 2, \quad f_{yy} = 0, \quad f_{xy} = 1$

$f_{xx}f_{yy} - f_{xy}^2 = -1 < 0$ *and* *Saddle point*

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x,\ y) = x^2 + 6x + y^2 + 8$

Solution

$f_x = 2x + 6 = 0 \quad \to \quad \underline{x = -3}|$

$f_y = 2y = 0 \quad \to \quad \underline{y = 0}|$

$\therefore$ The critical point is $(-3,\ 0)$

$f_{xx} = 2 > 0, \quad f_{yy} = 2, \quad f_{xy} = 0$

$$f_{xx} f_{yy} - f_{xy}^2 \Big|_{(-3,\ 0)} = 4 > 0 \quad \textit{and} \quad f_{xx} > 0$$

The function f has a local minimum at $(-3,\ 0)$ and the value is

$\underline{f(-3,\ 0) = 9 - 18 + 8 = -1}|$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x,\ y) = e^{x^2y^2 - 2xy^2 + y^2}$

Solution

$$f_x = \left(2xy^2 - 2y^2\right)e^{x^2y^2 - 2xy^2 + y^2} = 0 \quad \to \quad 2(x-1)y^2 = 0$$

$$f_y = \left(2x^2y - 4xy + 2y\right)e^{x^2y^2 - 2xy^2 + y^2} = 0 \quad \to \quad 2y\left(x^2 - 2x + 1\right) = 0$$

$$\begin{cases} 2(x-1)y^2 = 0 & \to y = 0,\ x = 1 \\ 2y\left(x^2 - 2x + 1\right) = 0 & \to y = 0,\ x = 1 \end{cases}$$

$\therefore$ The critical point is $(1,\ 0)$, $(x,\ 0)$, $(1,\ y)$

@ $(1,\ 0)$

$$\begin{aligned} f_{xx} &= \left(2y^2 + 2xy^2 - 2y^2\right)e^{x^2y^2 - 2xy^2 + y^2} \\ &= 2xy^2 e^{x^2y^2 - 2xy^2 + y^2}\Big|_{(1,0)} \\ &= 0 \end{aligned}$$

$$\begin{aligned} f_{yy} &= \left(2x^2 - 4x + 2 + 2x^2y - 4xy + 2y\right)e^{x^2y^2 - 2xy^2 + y^2}\Big|_{(1,0)} \\ &= 0 \end{aligned}$$

$$\begin{aligned} f_{xy} &= \left(4xy - 4y + 2x^2y - 4xy + 2y\right)e^{x^2y^2 - 2xy^2 + y^2} \\ &= \left(2x^2y - 2y\right)e^{x^2y^2 - 2xy^2 + y^2}\Big|_{(1,0)} \\ &= 0 \end{aligned}$$

$$f_{xx}f_{yy} - f_{xy}^2 \Big|_{(1,\ 0)} = 0$$

Inconclusive. No extreme values.

Exercise

Identify the critical points of the functions. Then determine whether each critical point corresponds to a local maximum, local minimum, or saddle point. State when your analysis is inconclusive.

$$f(x,\ y) = x^4 + y^4 - 16xy$$

Solution

$$f_x = 4x^3 - 16y = 0 \quad (1)$$

$$f_y = 4y^3 - 16x = 0 \quad (2)$$

$$\begin{cases} (1) \to x^3 = 4y \\ (2) \to y^3 = 4x \end{cases}$$

$$\left(\frac{y^3}{4}\right)^3 = 4y$$

$$y^9 = 4^4 y$$

$$y\left(y^8 - 2^8\right) = 0 \quad \to \quad \underline{y = 0,\ \pm 2}$$

C.P: $(0,\ 0)$, $(2,\ 2)$, $(-2,\ -2)$

$f_{xx} = 12x^2 \qquad f_{yy} = 12y^2 \qquad f_{xy} = -16$

$$f_{xx}f_{yy} - f_{xy}^2 = 144x^2y^2 - 256$$

@ $(0,\ 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = -256 < 0$$

$(0,\ 0)$ is a ***saddle point***.

@ $\pm(2,\ 2)$

$$f_{xx}f_{yy} - f_{xy}^2 = 144(4)(4) - 256 = 2,048 > 0$$

f has a ***local Min*** @ $(2,\ 2)$ & $(-2,\ -2)$

Exercise

Identify the critical points of the functions. Then determine whether each critical point corresponds to a local maximum, local minimum, or saddle point. State when your analysis is inconclusive.

$$f(x, y) = \frac{1}{3}x^3 - \frac{1}{3}y^3 + 2xy$$

Solution

$$f_x = x^2 + 2y = 0 \quad (1)$$

$$f_y = -y^2 + 2x = 0 \quad (2)$$

$$(1) \to y = -\frac{1}{2}x^2$$

$$(2) \to \quad -\left(-\frac{1}{2}x^2\right)^2 + 2x = 0$$

$$-\frac{1}{4}x^4 + 2x = 0$$

$$-\frac{1}{4}x\left(x^3 - 8\right) = 0 \qquad \to \begin{cases} x = 0 & \to y = 0 \\ x = 2 & \to y = -2 \end{cases}$$

C.P: $(0,\ 0)$ & $(2,\ -2)$

$$f_{xx} = 2x \qquad f_{yy} = -2y \qquad f_{xy} = 2 > 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = -4xy - 4$$

@ $(0,\ 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = -4 < 0$$

$(0,\ 0)$ is a ***saddle point***.

@ $(2,\ -2)$

$$f_{xx}f_{yy} - f_{xy}^2 = 16 - 4 = 12 > 0$$

f has a ***local Min*** @ $(2,\ -2)$

Exercise

Identify the critical points of the functions. Then determine whether each critical point corresponds to a local maximum, local minimum, or saddle point. State when your analysis is inconclusive.

$$f(x, y) = xy(2 + x)(y - 3)$$

Solution

$$f(x, y) = 2xy^2 - 6xy + x^2y^2 - 3x^2y$$

$$\begin{aligned} f_x &= 2y^2 - 6y + 2xy^2 - 6xy = 0 \\ &= 2y(y-3) + 2xy(y-3) \\ &= 2y(y-3)(x+1) = 0 \quad (1) \end{aligned}$$

$$\rightarrow \underline{y = 0, \quad y = 3, \quad x = -1}$$

$$\begin{aligned} f_y &= 4xy - 6x + 2x^2y - 3x^2 \\ &= 2x(2y-3) + x^2(2y-3) \\ &= x(2y-3)(x+2) = 0 \quad (2) \end{aligned}$$

$$\rightarrow \underline{x = 0, \quad x = -2, \quad y = \frac{2}{3}}$$

$y = 0 \quad (2) \rightarrow \quad x(-3)(x+2) = 0 \;\Rightarrow \underline{x = 0, -2}$

$y = 3 \quad (2) \rightarrow \quad x(3)(x+2) = 0 \;\Rightarrow \underline{x = 0, -2}$

$x = -1 \quad (2) \rightarrow \quad -(2y-3) = 0 \;\Rightarrow \underline{y = \frac{3}{2}}$

C.P: $(0,\ 0),\ (-2,\ 0),\ (0,\ 3),\ (-2,\ 3),\ \&\ \left(1,\ \frac{3}{2}\right)$

$$f_{xx} = 2y^2 - 6y$$

$$f_{xy} = 4y - 6 + 4xy - 6x$$

$$f_{yy} = 4x + 2x^2$$

$$f_{xx}f_{yy} - f_{xy}^2 = \left(2y^2 - 6y\right)\left(4x + 2x^2\right) - (4y - 6 + 4xy - 6x)^2$$

@ $(0,\ 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = -36 < 0$$

$(0,\ 0)$ is a ***saddle point***.

@ $(-2,\ 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = -36 < 0$$

$(-2,\ 0)$ is a ***saddle point***.

@ $(0,\ 3)$

$$f_{xx}f_{yy} - f_{xy}^2 = -36 < 0$$

$(0,\ 3)$ is a ***saddle point***.

@ $(-2,\ 3)$

$f_{xx}f_{yy} - f_{xy}^2 = -64 < 0$

$(-2,\ 3)$ is a ***saddle point***.

@ $\left(-1,\ \frac{3}{2}\right)$

$f_{xx}f_{yy} - f_{xy}^2 = 9 > 0 \qquad f_{xx} = -\frac{9}{2} < 0$

Function has a ***local max*** @ $\left(-1,\ \frac{3}{2}\right)$

Exercise

Identify the critical points of the functions. Then determine whether each critical point corresponds to a local maximum, local minimum, or saddle point. State when your analysis is inconclusive.

$$f(x,\ y) = 10 - x^3 - y^3 - 3x^2 + 3y^2$$

Solution

$f_x = -3x^2 - 6x = 0$

$= -3x(x+2) = 0 \quad (1)$

$\rightarrow \underline{x = 0,\ \ x = -2}$

$f_y = -3y^2 + 6y$

$= -3y(y-2) = 0 \quad (2)$

$\rightarrow \underline{y = 0,\ \ y = 2}$

C.P: $(0,\ 0),\ (-2,\ 0),\ (0,\ 2),\ \&\ (-2,\ 2)$

$f_{xx} = -6x - 6$

$f_{xy} = 0$

$f_{yy} = -6y + 6$

$f_{xx}f_{yy} - f_{xy}^2 = (-6x-6)(-6y+6)$

$= -36(-xy + x - y + 1)$

@ $(0,\ 0)$

$f_{xx}f_{yy} - f_{xy}^2 = -36 < 0$

$(0,\ 0)$ is a ***saddle point***.

@ $(0,\ 2)$

$f_{xx}f_{yy} - f_{xy}^2 = 36 > 0 \qquad f_{xx} = -6 < 0$

Function has a ***local min*** @ $(0,\ 2)$

@ $(-2,\ 0)$

$f_{xx}f_{yy} - f_{xy}^2 = 36 > 0 \qquad f_{xx} = 6 > 0$

Function has a ***local max*** @ $(-2,\ 0)$

@ $(-2,\ 2)$

$f_{xx}f_{yy} - f_{xy}^2 = -36 < 0$

$(-2,\ 2)$ is a ***saddle point***.

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

$f(x,\ y) = \frac{1}{3}x^3 - \frac{1}{3}y^3 + 2xy$ on the rectangle $R = \{(x,\ y):\ 0 \le x \le 3,\ -1 \le y \le 1\}$

Solution

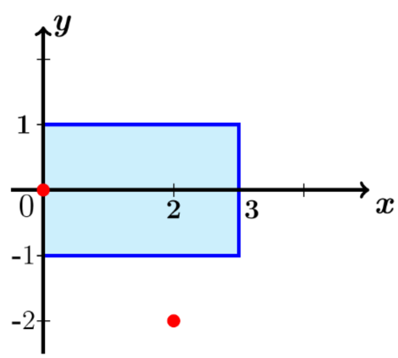
$f_x = x^2 + 2y = 0 \quad \rightarrow \quad y = -\frac{1}{2}x^2$

$f_y = -y^2 + 2x = 0$

$-\frac{1}{4}x^4 + 2x = 0$

$-\frac{1}{4}x\left(x^3 - 8\right) = 0 \qquad \rightarrow \quad \underline{x = 0,\ \ x = 2}$

$\rightarrow \begin{cases} x = 0 & y = 0 \\ x = 2 & y = -2 \end{cases}$



C.P: $(0,\ 0)$ & $(2,\ -2)$ neither in the interior of R.

$0 \le x \le 3 \quad \rightarrow \quad y = -1$

$g(x) = f(x,\ -1) = \frac{1}{3}x^3 - 2x + \frac{1}{3}$

$f_x = x^2 - 2 = 0 \quad \rightarrow \quad \underline{x = \sqrt{2}},\ \ \cancel{x = -\sqrt{2}}$

$g\left(\sqrt{2}\right) = \frac{2}{3}\sqrt{2} - 2\sqrt{2} + \frac{1}{3}$

$$= \boxed{\frac{1-4\sqrt{2}}{3}}$$

$\rightarrow y = 1$

$$g(x) = f(x,\ 1) = \frac{1}{3}x^3 + 2x - \frac{1}{3}$$

$$f_x = x^2 + 2 \ne 0$$

$-1 \le y \le 1 \quad \rightarrow \quad x = 0$

$$h(y) = f(0,\ y) = -\frac{1}{3}y^3$$

$$f_y = y^2 = 0 \quad \rightarrow \quad \boxed{y = 0}$$

$\rightarrow x = 3$

$$h(y) = f(3,\ y) = -\frac{1}{3}y^3 + 6y + 9$$

$$f_y = -y^2 + 6 = 0 \quad \rightarrow \quad \boxed{y = \sqrt{6}}$$

$$h(\sqrt{6}) = f(3,\ \sqrt{6}) \ \boxed{= 4\sqrt{6} + 9}$$

Absolute minimum: $\boxed{f(\sqrt{2},\ 1) = \frac{1-4\sqrt{2}}{3}}$

Absolute maximum: $\boxed{f(3,\ \sqrt{6}) = 4\sqrt{6} + 9}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

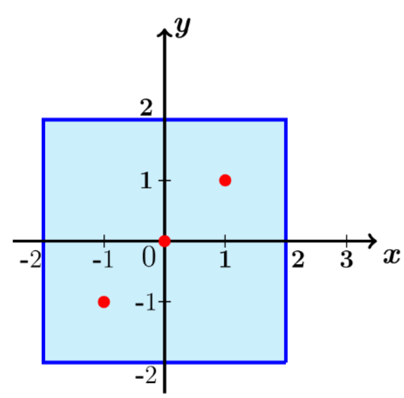
$f(x,\ y) = x^4 + y^4 - 4xy + 1$ on the square $R = \{(x,\ y):\ -2 \le x \le 2,\ -2 \le y \le 2\}$

Solution

$$f_x = 4x^3 - 4y = 0 \quad \rightarrow \quad y = x^3$$

$$f_y = 4y^3 - 4x = 0$$

$$x^9 - x = 0 \quad \rightarrow \quad \boxed{x = 0,\ \pm 1}$$



$x = 0 \quad \rightarrow \quad y = 0$

$x = -1 \quad \rightarrow \quad y = -1$

$x = 1 \quad \rightarrow \quad y = 1$

C.P: $(0,\ 0),\ (1,\ 1)\ \&\ (-1,\ -1)$

	$f(x, y)$
$(0,\ 0)$	1
$(-1,\ -1)$	$1+1-4+1=-1$
$(1,\ 1)$	$1+1-4+1=-1$
$(2,\ 2)$	$16+16-16+1=17$
$(-2,\ 2)$	$16+16+16+1=49$
$(2,\ -2)$	$16+16+16+1=49$
$(-2,\ -2)$	$16+16-16+1=17$

Absolute minimum: $\underline{f(1,\ 1)=f(-1,\ -1)=-1}$

Absolute maximum: $\underline{f(-2,\ 2)=f(2,\ -2)=49}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region *R*.

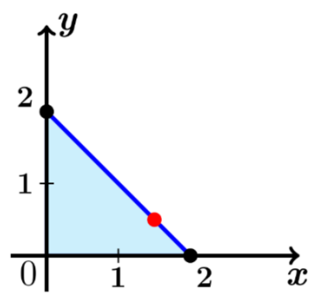
$f(x, y)=x^2y-y^3$ on the triangle $R=\{(x, y):\ 0\le x\le 2,\ 0\le y\le 2-x\}$

Solution

$f_x=2xy=0 \quad (0,\ 0)$

$f_y=x^2-3y^2=0 \quad (2)$

C.P: ***None*** inside the triangle



$y=2-x$

$$\begin{aligned} f(x, y) &= x^2(2-x)-(2-x)^3 \\ &= (2-x)\left(x^2-4+4x-x^2\right) \\ &= (2-x)(4x-4) \\ &= 12x-8-4x^2 \end{aligned}$$

$g'=12-8x=0 \quad \to \quad \underline{x=\frac{3}{2}}$

$x=\frac{3}{2} \quad \to \quad \underline{y=2-\frac{3}{2}=\frac{1}{2}}$

	$f(x, y)$
$(0,\ 0)$	0
$(2,\ 0)$	0
$(0,\ 2)$	-8
$\left(\frac{3}{2},\ \frac{1}{2}\right)$	$\frac{9}{8}-\frac{1}{8}=1$

Absolute minimum: $\underline{f(0,\ 2)=-8}$

Absolute maximum: $\underline{f\left(\frac{3}{2},\ \frac{1}{2}\right)=1}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

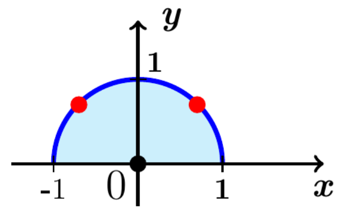
$f(x, y) = xy$ on the semicircular disk $R = \left\{(x, y): \ -1 \le x \le 1, \ 0 \le y \le \sqrt{1-x^2}\right\}$

Solution

$f_x = y = 0$

$f_y = x = 0$

C.P.: $(0, \ 0)$



$y = \sqrt{1-x^2} \quad \to \quad y^2 + x^2 = 1$

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

$$f(x, y) = g(t) = \cos t \sin t$$

$$= \frac{1}{2}\sin 2t$$

$g' = \cos 2t = 0 \quad \to \quad 2t = \frac{\pi}{2}, \ \frac{3\pi}{2}$

$t = \frac{\pi}{4}, \ \frac{3\pi}{4}$

$g\left(\frac{\pi}{4}\right) = \frac{1}{2}$

$g\left(\frac{3\pi}{4}\right) = -\frac{1}{2}$

Absolute minimum: $f\left(-\frac{\sqrt{2}}{2}, \ \frac{\sqrt{2}}{2}\right) = -\frac{1}{2}$

Absolute maximum: $f\left(\frac{\sqrt{2}}{2}, \ \frac{\sqrt{2}}{2}\right) = \frac{1}{2}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

$$f(x, y) = x^2 + y^2 - 2y + 1; \quad R = \left\{(x, y): \ x^2 + y^2 \le 4\right\}$$

Solution

$f_x = 2x = 0 \quad \to \quad x = 0$

$f_y = 2y - 2 = 0 \quad \to \quad y = 1$

C.P.: $(0, \ 1)$

$y^2 + x^2 = 4$

$$\begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}$$

$$f(x, y) = g(t) = 4\cos^2 t + 4\sin^2 t - 4\sin t + 1$$
$$= 5 - 4\sin t$$

$g' = -4\cos t = 0 \quad \rightarrow \quad t = \frac{\pi}{2}, \ \frac{3\pi}{2}$

$g\left(\frac{\pi}{2}\right) = 1 = f(0, 2)$

$g\left(\frac{3\pi}{2}\right) = 9 = f(0, -2)$

$f(0, 1) = 0$

Absolute minimum: $f(0, 1) = 0$

Absolute maximum: $f(0, -2) = 9$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

$$f(x, y) = 2x^2 + y^2; \quad R = \left\{(x, y): x^2 + y^2 \le 16\right\}$$

Solution

$f_x = 4x = 0 \quad \rightarrow \quad x = 0$

$f_y = 2y = 0 \quad \rightarrow \quad y = 0$

C.P.: $(0, 0)$

$y^2 + x^2 = 16 \qquad \begin{cases} x = 4\cos t \\ y = 4\sin t \end{cases}$

$$f(x, y) = g(t) = 32\cos^2 t + 16\sin^2 t$$
$$= 16\cos^2 t + 16$$

$g' = -32\sin t \cos t$

$\quad = -16\sin 2t = 0$

$2t = n\pi \quad \rightarrow \quad t = 0, \ \frac{\pi}{2}, \ \pi, \ \frac{3\pi}{2}$

t	(x, y)	$f(x, y)$
	$(0, 0)$	0
0	$(4, 0)$	32
$\frac{\pi}{2}$	$(0, 4)$	16
π	$(-4, 0)$	32
$\frac{3\pi}{2}$	$(0, -4)$	16

Absolute minimum: $f(0, 0) = 0$

Absolute maximum: $f(\pm 4, 0) = 32$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

$$f(x, y) = 4 + 2x^2 + y^2; \quad R = \{(x, y): \ -1 \le x \le 1, \ -1 \le y \le 1\}$$

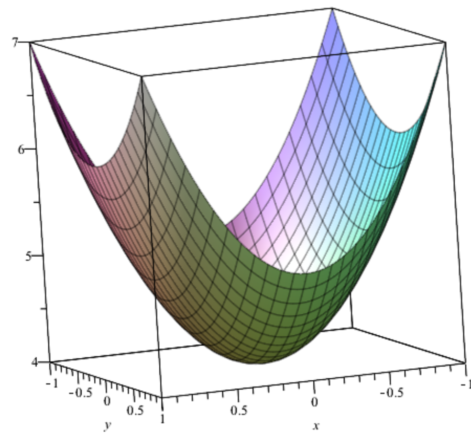
Solution

$f_x = 4x = 0 \quad \to \quad \underline{x = 0}$

$f_y = 2y = 0 \quad \to \quad \underline{y = 0}$

C.P.: $(0, 0)$

	$f(x, y)$
$(0, 0)$	4
$(\pm 1, \pm 1)$	$4+2+1=7$



Absolute minimum: $\quad \underline{f(0, 0) = 4}$

Absolute maximum: $\quad \underline{f(\pm 1, \pm 1) = 7}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

$$f(x, y) = 6 - x^2 - 4y^2; \quad R = \{(x, y): \ -2 \le x \le 2, \ -1 \le y \le 1\}$$

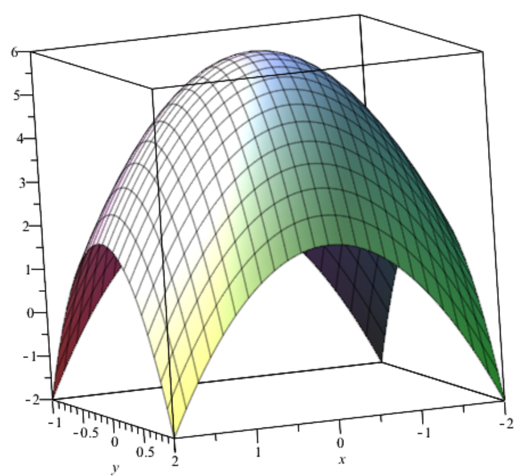
Solution

$f_x = -2x = 0 \quad \to \quad \underline{x = 0}$

$f_y = -8y = 0 \quad \to \quad \underline{y = 0}$

C.P.: $(0, 0)$

	$f(x, y)$
$(0, 0)$	6
$(\pm 2, \pm 1)$	$6-4-4=-2$



Absolute minimum: $\quad \underline{f(\pm 2, \pm 1) = -2}$

Absolute maximum: $\quad \underline{f(0, 0) = 6}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

$$f(x,\ y) = 2x^2 - 4x + 3y^2 + 2; \quad R = \left\{(x,\ y):\ (x-1)^2 + y^2 \le 1\right\}$$

Solution

$$f_x = 4x = 0 \quad \to \quad \underline{x = 0}|$$

$$f_y = 2y = 0 \quad \to \quad \underline{y = 0}|$$

C.P.: $(0,\ 0)$

$$(x-1)^2 + y^2 = 1 \quad \begin{cases} x - 1 = \cos t \to x = 1 + \cos t \\ \qquad\qquad\qquad y = \sin t \end{cases}$$

$$\begin{aligned} f(x,\ y) = g(t) &= 2(1+\cos t)^2 - 4 - 4\cos t + 3\sin^2 t + 2 \\ &= 2\cos^2 t + 3\sin^2 t \\ &= 2 + \sin^2 t \end{aligned}$$

$$\begin{aligned} g' &= 2\sin t\cos t \\ &= \sin 2t = 0 \end{aligned}$$

$$2t = n\pi \quad \to \quad t = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$$

t	(x, y)	$f(x, y)$
	$(0,\ 0)$	2
0	$(2,\ 0)$	2
$\frac{\pi}{2}$	$(1,\ 1)$	3
π	$(0,\ 0)$	2
$\frac{3\pi}{2}$	$(1,\ -1)$	3

Absolute minimum: $\quad \underline{f(0,\ 0) = f(2,\ 0) = 2}|$

Absolute maximum: $\quad \underline{f(1,\ \pm 1) = 3}|$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

$$f(x,\ y) = -2x^2 + 4x - 3y^2 - 6y - 1; \quad R = \left\{(x,\ y):\ (x-1)^2 + (y+1)^2 \le 1\right\}$$

Solution

$$f_x = -4x + 4 = 0 \quad \to \quad \underline{x = 1}|$$

$f_y = -6y - 6 = 0 \quad \to \quad \boxed{y = -1}$

C.P.: $(1,\ -1)$

$(x-1)^2 + (y+1)^2 = 1 \qquad \begin{cases} x - 1 = \cos t \to x = 1 + \cos t \\ y + 1 = \sin t \to y = \sin t - 1 \end{cases}$

$$\begin{aligned} f(x,\ y) = g(t) &= -2(1+\cos t)^2 + 4 + 4\cos t - 3(\sin t - 1)^2 - 6\sin t + 6 - 1 \\ &= 4 - 2\cos^2 t - 3\sin^2 t \\ &= 1 + \cos^2 t \end{aligned}$$

$$\begin{aligned} g' &= -2\sin t \cos t \\ &= -\sin 2t = 0 \end{aligned}$$

$2t = n\pi \quad \to \quad t = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$

t	(x, y)	$f(x, y)$
	$(1,\ -1)$	$-2+4-3+6-1=4$
0	$(2,\ -1)$	$-8+8-3+6-1=2$
$\frac{\pi}{2}$	$(1,\ 0)$	1
π	$(0,\ -1)$	2
$\frac{3\pi}{2}$	$(1,\ -2)$	1

Absolute ***minimum***: $\boxed{f(1,\ 0) = f(1,\ -2) = 1}$

Absolute ***maximum***: $\boxed{f(1,\ -1) = 4}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

$$f(x,\ y) = \sqrt{x^2 + y^2 - 2x + 2}; \quad R = \left\{(x,\ y):\ x^2 + y^2 \le 4,\ y \ge 0\right\}$$

Solution

$g(x,\ y) = x^2 + y^2 - 2x + 2$

$g_x = 2x - 2 = 0 \quad \to \quad \boxed{x = 1}$

$g_y = 2y = 0 \quad \to \quad \boxed{y = 0}$

C.P.: $(1,\ 0)$

$y^2 + x^2 = 4 \qquad \begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}$

$$g(x,\ y) = h(t) = 4 - 4\cos t + 2$$
$$= 6 - \cos t$$

$g' = \sin t = 0$

$t = 0,\ \pi$

t	(x, y)	$f(x, y)$
	$(1,\ 0)$	1
0	$(2,\ 0)$	$\sqrt{2}$
π	$(-2,\ 0)$	$\sqrt{10}$

Absolute ***minimum***: $\quad f(1,\ 0) = 1$

Absolute ***maximum***: $\quad f(-2,\ 0) = \sqrt{10}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region *R*.

$$f(x,\ y) = \frac{-x^2 + 2y^2}{2 + 2x^2y^2};\ R \text{ is the closed region bounded by the lines } y = x,\ y = 2x, \text{ and } y = 2$$

Solution

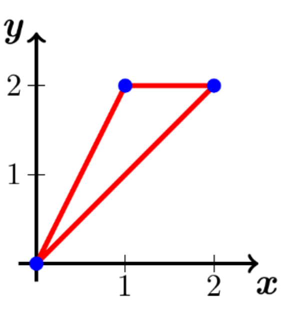
$$f_x = \frac{2\left(-2 - 4y^4\right)x}{\left(2 + 2x^2y^2\right)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d}\right)' = \frac{n(ad - bc)x^{n-1}}{\left(cx^n + d\right)^2}$$

$$= -\frac{\left(2y^4 + 1\right)x}{\left(1 + x^2y^2\right)^2} = 0 \quad \to \quad x = 0$$

$$f_y = \frac{2\left(4 + 2x^4\right)y}{\left(2 + 2x^2y^2\right)^2}$$

$$= \frac{\left(2 + x^4\right)y}{\left(1 + x^2y^2\right)^2} = 0 \quad \to \quad y = 0$$



C.P.: $(0,\ 0)$

@ $y = x \qquad f(x,\ y) = \dfrac{x^2}{2 + 2x^4} = 0$

$$f' = \frac{4x - 4x^5}{\left(2 + 2x^4\right)^2}$$

$$= \frac{4x\left(1 - x^4\right)}{\left(2 + 2x^4\right)^2} = 0 \quad \to \quad x = 0, \pm 1$$

$$y = x = 0 \quad \to \quad f(0,\ 0) = 0$$

$$y = x = \pm 1 \quad \to \quad f(1,\ 1) = \frac{1}{4}$$

@ $y = 2x \quad f(x,\ y) = \dfrac{7x^2}{2 + 8x^4}$

$$f' = \frac{28x - 112x^5}{\left(2 + 8x^4\right)^2}$$

$$= \frac{28x\left(1 - 8x^4\right)}{\left(2 + 8x^4\right)^2} = 0 \quad \to \quad x = 0,\ \pm\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$y = 2x = \sqrt{2} \quad \to \quad f\left(\frac{\sqrt{2}}{2},\ \sqrt{2}\right) = \frac{7}{8}$$

@ $y = 2 \quad f(x,\ y) = \dfrac{-x^2 + 8}{8x^2 + 2}$

$$f' = \frac{-132x}{\left(8x^2 + 2\right)^2} = 0 \quad \to \quad \underline{x = 0|}$$

$$\left(\frac{ax^n + b}{cx^n + d}\right)' = \frac{n(ad - bc)x^{n-1}}{\left(cx^n + d\right)^2}$$

$$y = x = 2 \quad \to \quad f(2,\ 2) = \frac{4}{34} = \frac{2}{17}$$

$$y = 2x = 2 \Rightarrow x = 1 \quad \to \quad f(1,\ 2) = \frac{7}{10}$$

Absolute ***minimum***: $\underline{f(0,\ 0) = 0}$

Absolute ***maximum***: $\underline{f\left(\frac{\sqrt{2}}{2},\ \sqrt{2}\right) = \frac{7}{8}}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

$f(x,\ y) = \sqrt{x^2 + y^2}$; R is the closed region bounded by the ellipse $\dfrac{x^2}{4} + y^2 = 1$

Solution

$$f_x = \frac{x}{\sqrt{x^2+y^2}} = 0 \quad \to \quad x = 0$$

$$f_y = \frac{y}{\sqrt{x^2+y^2}} = 0 \quad \to \quad y = 0$$

C.P.: $(0,\ 0)$

$$\frac{x^2}{4} + y^2 = 1 \qquad \begin{cases} x = 2\cos t \\ y = \sin t \end{cases}$$

$$f(x,\ y) = g(t) = \sqrt{4\cos^2 t + \sin^2 t}$$
$$= \sqrt{3\cos^2 t + 1}$$

$$g' = \frac{-3\cos t \sin t}{\sqrt{3\cos^2 t + 1}}$$
$$= -\frac{3}{2}\frac{\sin 2t}{\sqrt{3\cos^2 t + 1}} = 0$$

$$2t = n\pi \quad \to \quad t = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$$

t	(x, y)	$f(x, y)$
	$(0,\ 0)$	0
0	$(2,\ 0)$	2
$\frac{\pi}{2}$	$(0,\ 1)$	1
π	$(-2,\ 0)$	2
$\frac{3\pi}{2}$	$(0,\ -1)$	1

Absolute ***minimum***: $f(0,\ 0) = 0$

Absolute ***maximum***: $f(-2,\ 0) = f(2,\ 0) = 2$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

$$f(x,\ y) = x^2 + y^2 - 4; \quad R = \left\{(x,\ y):\ x^2 + y^2 < 4\right\}$$

Solution

$$f_x = 2x = 0 \quad \to \quad x = 0$$
$$f_y = 2y = 0 \quad \to \quad y = 0$$

C.P.: $(0,\ 0)$

$f(0,0)=-4$

$y^2+x^2=4 \qquad \begin{cases} x=2\cos t \\ y=2\sin t \end{cases}$

$f(x,\ y)=g(t)=4\cos^2 t+4\sin^2 t-4$

$=0 \qquad$ No extreme points.

$f(x,\ y)\ge -4$

Absolute ***minimum***: $\underline{f(0,\ 0)=-4}|$

Exercise

Find the absolute maximum and minimum values of the function on the specified region *R*.

$$f(x,\ y)=x+3y; \quad R=\{(x,\ y):\ |x|<1,\ |y|<2\}$$

Solution

$f_x=1\ne 0$

$f_y=3\ne 0$

C.P.: *None*

$-1<x<1 \quad -2<y<2$

(x,y)	$f(x,y)$
$(-1,\ -2)$	-7
$(1,\ -2)$	-5
$(1,\ 2)$	7
$(-1,\ 2)$	5

The range of the function $f(x,\ y)$ on *R* is the interval $(-7,\ 7)$.

$\therefore\ f(x,\ y)$ has ***neither*** an absolute minimum or maximum on *R*.

Exercise

Find the absolute maximum and minimum values of the function on the specified region *R*.

$$f(x,\ y)=2e^{-x-y}; \quad R=\{(x,\ y):\ x\ge 0,\ y\ge 0\}$$

Solution

$f_x=-2e^{-x-y}\ne 0$

$f_y = -2e^{-x-y} \neq 0$

C.P.: *None*

$R = \{(x,\ y):\ x \ge 0,\ y \ge 0\}$

$f(0,\ 0) = 2$

$(x,\ y) \to \infty \quad \Rightarrow \quad f(x,\ y) \to 0$

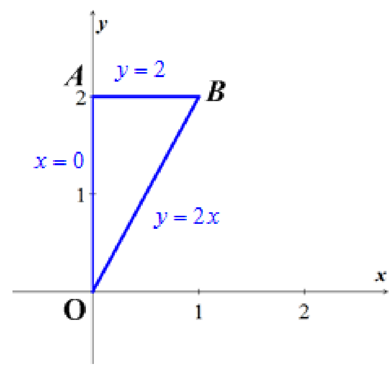
Absolute ***minimum***: *None*

Absolute ***maximum***: $\underline{f(0,\ 0) = 2}|$

Exercise

Find the absolute maxima and minima of the function $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0, \quad y = 2, \quad y = 2x$ in the first quadrant.

Solution



$f_x = 4x - 4 = 0 \quad f_y = 2y - 4 = 0$

$x = 1 \quad y = 2$

The critical point is $(1,\ 2)$ and the value is $f(1,\ 2)\ \underline{= -5}|$

i. On the segment OA. The function $f(0,\ y) = y^2 - 4y + 1$

   This function is defined on the closed interval $0 \le y \le 2$.

$$f'(0,\ y) = 2y - 4 = 0 \quad \to \quad y = 2$$

$$\begin{cases} y = 0 & \to f(0,\ 0) = \underline{1}| \\ y = 2 & \to f(0,\ 2) = \underline{-3}| \end{cases}$$

ii. On the segment OB

$$f(x,\ 2x) = 2x^2 - 4x + (2x)^2 - 4(2x) + 1 = 6x^2 - 12x + 1 \qquad 0 \le x \le 1$$

$$f'(x, 2x) = 12x - 12 = 0 \quad \to \quad x = 1$$

$$\begin{cases} x = 0 & \to f(0,\ 0) = \underline{1}| \\ x = 1 & \to f(1,\ 2) = \underline{-5}| \end{cases} \qquad \therefore\ (1,\ 2) \text{ is not interior point of } OB$$

iii. On the segment AB

$$f(x,\ 2) = 2x^2 - 4x + (2)^2 - 4(2) + 1 = 2x^2 - 4x - 3 \qquad 0 \le x \le 1$$

$$f'(x, 2) = 4x - 4 = 0 \quad \to \quad x = 1$$

$$\begin{cases} x=0 & \rightarrow f(0,\ 2)=\underline{-3}| \\ x=1 & \rightarrow f(1,\ 2)=\underline{-5}| \end{cases}$$

$\Rightarrow (1,\ 2)$ is not interior point of triangular region.

Therefore; the absolute ***maximum*** is 1 at (0, 0) and the absolute ***minimum*** is –5 at $(1,\ 2)$

Exercise

Find the absolute maxima and minima of the function $D(x,y)=x^2-xy+y^2+1$ on the closed triangular plate bounded by the lines $x=0, \quad y=4, \quad y=x$ in the first quadrant.

Solution

$D_x=2x-y=0, \quad D_y=-x+2y=0, \quad \Rightarrow x=y=0$

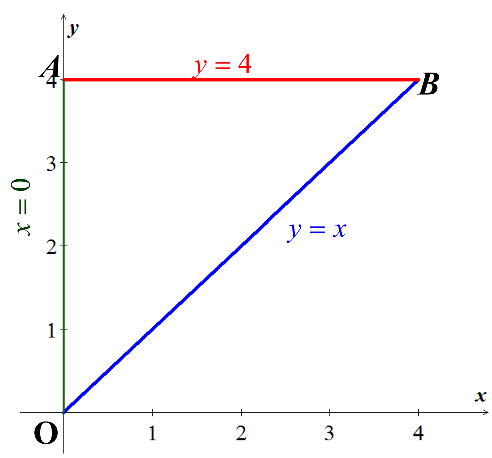
The critical point is $(0,\ 0)$ and the value is $D(0,\ 0)\underline{=1}|$

i. On the segment OA.

$D(0,\ y)=y^2+1, \quad 0\le y\le 4$

$D'(0,\ y)=2y=0 \quad \rightarrow \quad y=0$

$$\begin{cases} y=0 & \rightarrow D(0,\ 0)=\underline{1}| \\ y=4 & \rightarrow D(0,\ 4)=\underline{17}| \end{cases}$$



ii. On the segment OB

$D(x,\ x)=x^2+1 \qquad 0\le x\le 4$

$D'(x,x)=2x=0 \quad \rightarrow \quad x=0$

$x=0 \quad \rightarrow D(0,\ 0)=\underline{1}|$

iii. On the segment AB

$D(x,4)=x^2-4x+17 \qquad 0\le x\le 4$

$D'(x,2)=2x-4=0 \quad \rightarrow \quad x=2$

$$\begin{cases} x=2 & \rightarrow D(2,\ 4)=13 \\ x=4 & \rightarrow D(4,\ 4)=\underline{17}| \end{cases}$$

$\Rightarrow (0,0)$ is not interior point of triangular region.

Therefore; the absolute ***maximum*** is 11 at $(0,4)$ *and* $(4,4)$ and the absolute ***minimum*** is 1 at $(0,0)$

Exercise

Find the absolute maxima and minima of the function $T(x,y) = x^2 + xy + y^2 - 6x + 2$ on the triangular plate $0 \le x \le 5, \quad -3 \le y \le 0$.

Solution

$$T_x = 2x + y - 6 = 0, \quad T_y = x + 2y = 0$$

$$\begin{cases} 2x + y = 6 \\ x + 2y = 0 \end{cases} \rightarrow \boxed{x = 4, \ y = -2}$$

The critical point is $(4,-2)$ and the value is $T(4,-2) = -10$

i. On the segment *OA*.

$$T(0,\ y) = y^2 + 2, \quad -3 \le y \le 0$$

$$T'(0,\ y) = 2y = 0 \quad \rightarrow \quad y = 0$$

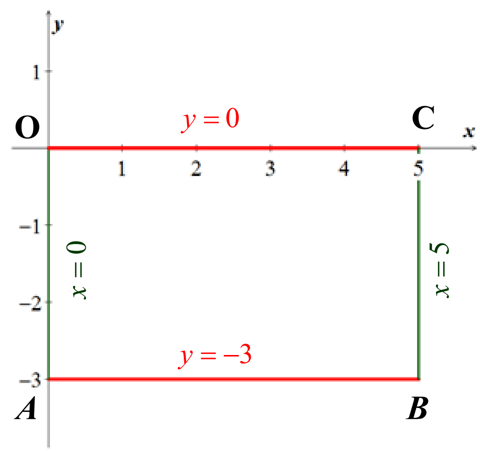
$$\begin{cases} y = 0 & \rightarrow T(0,\ 0) = 2 \\ y = -3 & \rightarrow T(0,-3) = 11 \end{cases}$$

ii. On the segment *AB*

$$T(x,-3) = x^2 - 9x + 11 \qquad 0 \le x \le 5$$

$$T'(x,-3) = 2x - 9 = 0 \quad \rightarrow \quad x = \frac{9}{2}$$

$$\begin{cases} x = \frac{9}{2} & \rightarrow T\left(\frac{9}{2},-3\right) = -\frac{37}{4} \\ x = 0 & \rightarrow T(0,-3) = 11 \end{cases}$$



iii. On the segment *BC*

$$T(5,y) = y^2 + 5y - 3 \qquad -3 \le y \le 0$$

$$T'(5,y) = 2y + 5 = 0 \quad \rightarrow \quad y = -\frac{5}{2}$$

$$\begin{cases} y = 0 & \rightarrow T(5,0) = -3 \\ y = -\frac{5}{2} & \rightarrow T\left(5,-\frac{5}{2}\right) = -\frac{37}{4} \\ y = -3 & \rightarrow T(5,-3) = -9 \end{cases}$$

iv. On the segment *CO*

$$T(x,0) = x^2 - 6x + 2 \qquad 0 \le x \le 5$$

$$T'(x,0) = 2x - 6 = 0 \quad \rightarrow \quad x = 3$$

$$(3,0) \quad \rightarrow T(3,0) = -7$$

Therefore; the absolute ***maximum*** is 11 at $(0,-3)$ and the absolute ***minimum*** is -10 at $(4,-2)$

Exercise

Find the absolute maxima and minima of the function $f(x,y)=\left(4x-x^2\right)\cos y$ on the triangular plate $1\le x\le 3,\quad -\frac{\pi}{4}\le y\le \frac{\pi}{4}$.

Solution

$$f_x=(4-2x)\cos y=0,\quad f_y=\left(x^2-4x\right)\sin y=0$$

$$\begin{cases}(4-2x)\cos y=0 & \to x=2,\ y=\frac{(n+1)\pi}{2}\\ x(x-4)\sin y=0 & \to x=0,4,\ y=n\pi\end{cases}$$

$\boxed{x=2,\ y=0}$ *because* $1\le x\le 3,\quad -\frac{\pi}{4}\le y\le \frac{\pi}{4}$

The critical point is $(2,0)$ and the value is $f(2,0)\ \underline{=4}$

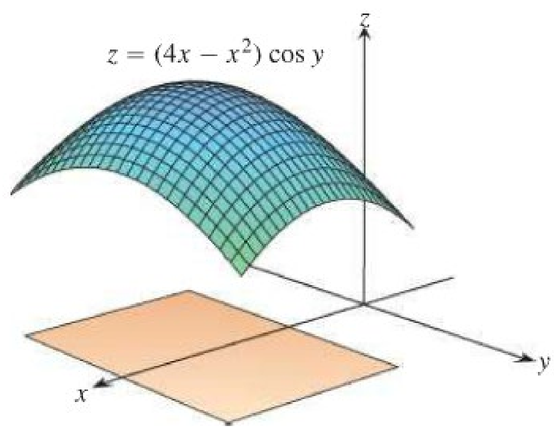
Values of all 4 corner points:

$A\left(1,-\frac{\pi}{4}\right)\ \to f\left(1,-\frac{\pi}{4}\right)=\frac{3\sqrt{2}}{2}$

$B\left(1,\frac{\pi}{4}\right)\ \to f\left(1,\frac{\pi}{4}\right)=\frac{3\sqrt{2}}{2}$

$C\left(3,\frac{\pi}{4}\right)\ \to f\left(3,\frac{\pi}{4}\right)=\frac{3\sqrt{2}}{2}$

$A\left(3,-\frac{\pi}{4}\right)\ \to f\left(3,-\frac{\pi}{4}\right)=\frac{3\sqrt{2}}{2}$



i. On the segment AB

$f(1,y)=3\cos y \qquad -\frac{\pi}{4}\le y\le \frac{\pi}{4}$

$f'(1,y)=-3\sin y=0\ \to\ y=0$

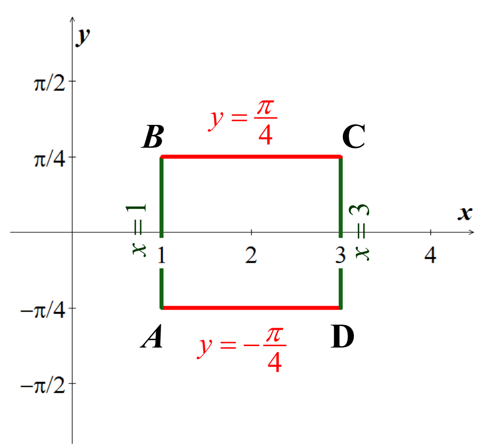
$x=1\ \to f(1,0)=3$

ii. On the segment BC

$f\left(x,\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\left(4x-x^2\right) \qquad 1\le x\le 3$

$f'\left(x,\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}(4-2x)=0\ \Rightarrow\ x=2$

$x=2\ \to f\left(2,\frac{\pi}{4}\right)=2\sqrt{2}$



iii. On the segment CD

$f(3,y)=3\cos y \qquad -\frac{\pi}{4}\le y\le \frac{\pi}{4}$

$f'(3,y)=-3\sin y=0\ \to\ y=0$

iv. On the segment DA

$$f\left(x,-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\left(4x-x^2\right) \qquad 1\le x\le 3$$

$$f'\left(x,-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}(4-2x)=0 \quad\Rightarrow\quad x=2$$

Therefore; the absolute ***maximum*** is 4 at $(2,0)$ and the absolute ***minimum*** is $\frac{3\sqrt{2}}{2}$ at $\left(1,-\frac{\pi}{4}\right)$, $\left(1,\frac{\pi}{4}\right)$, $\left(3,-\frac{\pi}{4}\right)$, *and* $\left(3,\frac{\pi}{4}\right)$

Exercise

Find the point on the graph of $z=x^2+y^2+10$ nearest the plane $x+2y-z=0$

Solution

The point on $z=x^2+y^2+10$ where the tangent plane is parallel to the plane $x+2y-z=0$.

Let $w=z-x^2-y^2-10 \quad\to\quad \nabla w=-2x\boldsymbol{i}-2y\boldsymbol{j}+\boldsymbol{k}$ is normal to $z=x^2+y^2+10$ at (x,y).

The vector ∇w is parallel to $\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$ which is normal to the plane if $x=\frac{1}{2}$ *and* $y=1$

$(-2x=-1 \ \ and \ \ -2y=-2)$, $z=\left(\frac{1}{2}\right)^2+1^2+10=\frac{45}{4}$

Thus, the point $\left(\frac{1}{2},\ 1,\ \frac{45}{4}\right)$ is the point on the surface $z=x^2+y^2+10$ nearest the plane $x+2y-z=0$

Exercise

Find the minimum distance from the point $(2,\ -1,\ 1)$ to the plane $x+y-z=2$

Solution

$$d(x,y,z)=\sqrt{(x-2)^2+(y+1)^2+(z-1)^2}$$

$$x+y-z=2 \quad\Rightarrow\quad z=x+y-2$$

Let: $D(x,y,z)=(x-2)^2+(y+1)^2+(z-1)^2$

$$D(x,y)=(x-2)^2+(y+1)^2+(x+y-2-1)^2$$
$$=(x-2)^2+(y+1)^2+(x+y-3)^2$$

$$D_x=2(x-2)+2(x+y-3)$$
$$=4x+2y-10=0$$

$$D_y=2(y+1)+2(x+y-3)$$
$$=2x+4y-4=0$$

$$\begin{cases} 4x+2y=10 \\ 2x+4y=4 \end{cases} \Rightarrow \boxed{x=\frac{8}{3},\ \ y=-\frac{1}{3}}$$

$\therefore$ The critical point is $\left(\frac{8}{3},\ -\frac{1}{3}\right)$.

$\underline{\left|z=\frac{8}{3}-\frac{1}{3}-2=\frac{1}{3}\right|}$

$$D_{xx}\Big|_{\left(\frac{8}{3},-\frac{1}{3}\right)}=4, \quad D_{yy}\Big|_{\left(\frac{8}{3},-\frac{1}{3}\right)}=4, \quad D_{xy}\Big|_{\left(\frac{8}{3},-\frac{1}{3}\right)}=2$$

$$D_{xx}D_{yy}-D_{xy}^2=(4)(4)-2^2=12>0 \quad \text{and} \quad D_{xx}>0$$

Therefore, the local ***minimum*** of the distance is

$$d\left(\frac{8}{3},-\frac{1}{3},\frac{1}{3}\right)=\sqrt{\left(\frac{8}{3}-2\right)^2+\left(-\frac{1}{3}+1\right)^2+\left(\frac{1}{3}-1\right)^2}$$

$$=\frac{2}{\sqrt{3}}$$

Exercise

Find the maximum value of $s=xy+yz+xz$ where $x+y+z=6$

Solution

$$x+y+z=6 \quad \Rightarrow \quad z=6-x-y$$

$$s(x,y,z)=xy+yz+xz$$

$$\begin{aligned} s(x,y)&=xy+y(6-x-y)+x(6-x-y) \\ &=xy+6y-xy-y^2+6x-x^2-xy \\ &=-x^2-y^2+6y+6x-xy \end{aligned}$$

$$s_x=-2x+6-y=0 \qquad s_y=-2y+6-x=0$$

$$\begin{cases} 2x+y=6 \\ x+2y=6 \end{cases} \Rightarrow \boxed{x=2,\ \ y=2}$$

$\therefore$ The critical point is $(2,\ 2)$.

$\underline{\left|z=6-2-2=2\right|}$

$$s_{xx}\Big|_{(2,2)}=-2, \quad s_{yy}\Big|_{(2,2)}=-2, \quad s_{xy}\Big|_{(2,2)}=-1$$

$$s_{xx}s_{yy}-s_{xy}^2=(-2)(-2)-(-1)^2=3>0 \quad \text{and} \quad s_{xx}<0$$

Therefore, the local *maximum* of the distance is

$$s(2,\ 2,\ 2) = (2)(2) + (2)(2) + (2)(2)$$

$$= 12$$

Exercise

Among all triangles with a perimeter of 9 *units*, find the dimensions of the triangle with the maximum area. It may be easiest to use Heron's formula, which states that the area of a triangle with side length a, b, and c is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $2s$ is the perimeter of the triangle.

Solution

The semi-perimeter is: $\quad s = \dfrac{a+b+c}{2}$

$$c = 2s - a - b$$

$$A^2 = s(s-a)(s-b)(s-2s+a+b)$$

$$= s(s-a)(s-b)(a+b-s)$$

$$f(a,\ b) = s(s-a)(s-b)(a+b-s)$$

$$f_a = s(s-b)(-a-b+s) + s(s-a)(s-b)$$

$$= s(s-b)(-a-b+s+s-a)$$

$$= s(s-b)(2s-2a-b) = 0$$

$$f_a = s(s-a)(-a-b+s) + s(s-a)(s-b)$$

$$= s(s-a)(-a-b+s+s-b)$$

$$= s(s-a)(2s-a-2b) = 0$$

$$\begin{cases} (s-b)(2s-2a-b) = 0 & \to b = s \quad 2a+b = 2s \\ (s-a)(2s-a-2b) = 0 & \to a = s \quad a+2b = 2s \end{cases}$$

$$\begin{cases} 2a+b = 2s \\ a+2b = 2s \end{cases} \qquad \Delta = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 \quad \Delta_a = \begin{vmatrix} 2s & 1 \\ 2s & 2 \end{vmatrix} = 2s \quad \Delta_b = \begin{vmatrix} 2 & 2s \\ 1 & 2s \end{vmatrix} = 2s$$

$$\to \quad a = b = \frac{2}{3}s$$

$$c = 2s - 2\frac{2}{3}s$$

$$= \frac{2}{3}s$$

$a = b = c = \dfrac{2}{3}s$ $\quad \therefore$ Equilateral triangle

The maximum area is obtained when all three sides are equal with each side length is 3 *units* (since the perimeter is 9 *units*).

Exercise

Let P be a plane tangent to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at a point in the first octane. Let T be the tetrahedron in the first octant bounded by P and the coordinate planes $x=0$, $y=0$, and $z=0$. Find the minimum volume T. (the volume of a tetrahedron in one-third the area of the base times the height.)

Solution

Let $Q\left(x_0, y_0, z_0\right)$ be a point on the ellipsoid.

The tangent plane P at the point Q has an equation:

$$\frac{x_0}{a^2}x+\frac{y_0}{b^2}y+\frac{z_0}{c^2}z=1$$

The intersection points of the plane with the axes are:

$$\left(\frac{a^2}{x_0},\ 0,\ 0\right),\ \left(0,\ \frac{b^2}{y_0},\ 0\right),\ \text{and}\ \left(0,\ 0,\ \frac{c^2}{z_0}\right)$$

$\therefore$ The tetrahedron T has base area

$$A=\frac{a^2b^2}{2x_0y_0}$$

Height: $h=\frac{c^2}{z_0}$

$$\therefore\ V=\frac{1}{3}\frac{a^2b^2c^2}{2x_0y_0z_0}$$

$$=\frac{1}{6}\frac{a^2b^2c^2}{x_0y_0z_0}$$

$$\frac{x_0}{a^2}x+\frac{y_0}{b^2}y+\frac{z_0}{c^2}z=1\ \rightarrow\ \frac{1}{a^2}x_0^2+\frac{1}{b^2}y_0^2+\frac{1}{c^2}z_0^2=1$$

$$z_0^2=c^2\left(1-\frac{1}{a^2}x_0^2-\frac{1}{b^2}y_0^2\right)$$

$$V=\frac{1}{6}\frac{a^2b^2c^2}{\left(x_0y_0c\right)\sqrt{\frac{a^2b^2-b^2x_0^2-a^2y_0^2}{a^2b^2}}}$$

$$=\frac{a^3b^3c}{6}\frac{1}{x_0y_0\sqrt{a^2b^2-b^2x_0^2-a^2y_0^2}}$$

$$=\frac{a^3b^3c}{6}\left(a^2b^2x_0^2y_0^2-b^2x_0^4y_0^2-a^2x_0^2y_0^4\right)^{-1/2}$$

$$V_{x_0} = -\frac{a^3b^3c}{12}\frac{2a^2b^2x_0y_0^2 - 4b^2x_0^3y_0^2 - 2a^2x_0y_0^4}{\left(a^2b^2x_0^2y_0^2 - b^2x_0^4y_0^2 - a^2x_0^2y_0^4\right)^{3/2}} = 0$$

$$V_{y_0} = -\frac{a^3b^3c}{12}\frac{2a^2b^2x_0^2y_0 - 2b^2x_0^4y_0 - 4a^2x_0^2y_0^3}{\left(a^2b^2x_0^2y_0^2 - b^2x_0^4y_0^2 - a^2x_0^2y_0^4\right)^{3/2}} = 0$$

$$\begin{cases} 2x_0y_0^2\left(a^2b^2 - 2b^2x_0^2 - a^2y_0^2\right) = 0 \\ 2x_0^2y_0\left(a^2b^2 - b^2x_0^2 - 2a^2y_0^2\right) = 0 \end{cases}$$

$$\begin{cases} 2b^2x_0^2 + a^2y_0^2 = a^2b^2 \\ b^2x_0^2 + 2a^2y_0^2 = a^2b^2 \end{cases}$$

$$\Delta = \begin{vmatrix} 2b^2 & a^2 \\ b^2 & 2a^2 \end{vmatrix} = 3a^2b^2 \quad \Delta_x = \begin{vmatrix} a^2b^2 & a^2 \\ a^2b^2 & 2a^2 \end{vmatrix} = a^4b^2 \quad \Delta_y = \begin{vmatrix} 2b^2 & a^2b^2 \\ b^2 & a^2b^2 \end{vmatrix} = a^2b^4$$

$$x_0^2 = \frac{a^4b^2}{3a^2b^2} = \frac{a^2}{3}, \quad y_0^2 = \frac{a^2b^4}{3a^2b^2} = \frac{b^2}{3}$$

$$\begin{aligned} z_0^2 &= c^2\left(1 - \frac{1}{a^2}\frac{a^2}{3} - \frac{1}{b^2}\frac{b^2}{3}\right) \\ &= c^2\left(1 - \frac{1}{3} - \frac{1}{3}\right) \\ &= \frac{1}{3}c^2 \end{aligned}$$

$$\boxed{x_0 = \frac{a}{\sqrt{3}}, \quad y_0 = \frac{b}{\sqrt{3}}, \quad z_0 = \frac{c}{\sqrt{3}}}$$

$$V = \frac{1}{6}\frac{a^2b^2c^2}{\frac{a}{\sqrt{3}}\frac{b}{\sqrt{3}}\frac{c}{\sqrt{3}}}$$

$$\boxed{= \frac{\sqrt{3}}{2}abc}$$

Exercise

Given three distinct noncollinear points *A*, *B*, and *C* in the plane, find the point *P* in the plane such the sum of the distances $|AP|+|BP|+|CP|$ is a minimum. Here is how to procees with three points, assuming that the triangle formed by the three points has no angle greater than $\left(120° = \frac{2\pi}{3}\right)$

a) Assume the coordinates of the threee given points are $A\left(x_1, y_1\right)$, $B\left(x_2, y_2\right)$, and $C\left(x_3, y_3\right)$. Let $d_1(x, y)$ be the distance between $A\left(x_1, y_1\right)$ and a variable point $P(x, y)$. Compute the gradient of d_1 and show that it is a unit vector pointing along the line between the two points.

b) Define d_2 and d_3 in a similar way and show that ∇d_2 and ∇d_3 are also unit vectors in the direction of line between the two points.

c) The goal is to minimize $f(x, y) = d_1 + d_2 + d_3$. Show that the condition $f_x = f_y = 0$ implies that $\nabla d_1 + \nabla d_2 + \nabla d_3 = 0$.

d) Explain why part (*c*) implies that the optimal point *P* has the property the three line segments AP, BP, and CP all intersect symmetrically in angles of $\frac{2\pi}{3}$.

e) What is the optimal solution if one of the angles in the triangle is greater than $\frac{2\pi}{3}$ (draw a picture)?

f) Estimate the Steiner point for the three points $(0, 0)$, $(0, 1)$, $(2, 0)$

Solution

a) $d_1(x, y) = \sqrt{\left(x - x_1\right)^2 + \left(y - y_1\right)^2}$

$$\nabla d_1(x, y) = \frac{x - x_1}{\sqrt{\left(x - x_1\right)^2 + \left(y - y_1\right)^2}}\hat{i} + \frac{y - y_1}{\sqrt{\left(x - x_1\right)^2 + \left(y - y_1\right)^2}}\hat{j}$$

$$= \frac{x - x_1}{d_1(x, y)}\hat{i} + \frac{y - y_1}{d_1(x, y)}\hat{j}$$

$$\left|\nabla d_1(x, y)\right| = \frac{1}{d_1(x, y)}\sqrt{\left(x - x_1\right)^2 + \left(y - y_1\right)^2}$$

$$= \frac{d_1(x, y)}{d_1(x, y)}$$

$$= 1$$

$\therefore$ The gradient of d_1 is a unit vector

b) $d_2(x, y) = \sqrt{\left(x - x_2\right)^2 + \left(y - y_2\right)^2}$

$$\nabla d_2(x, y) = \boxed{\frac{x - x_2}{d_2(x, y)}\hat{i} + \frac{y - y_2}{d_2(x, y)}\hat{j}}$$

$$\begin{aligned} \left|\nabla d_2(x, y)\right| &= \frac{1}{d_2(x, y)}\sqrt{\left(x - x_2\right)^2 + \left(y - y_2\right)^2} \\ &= \frac{d_2(x, y)}{d_2(x, y)} \\ &= \boxed{1} \end{aligned}$$

$\therefore\ \nabla d_2$ is a unit vector

$$d_3(x, y) = \sqrt{\left(x - x_3\right)^2 + \left(y - y_3\right)^2}$$

$$\nabla d_3(x, y) = \boxed{\frac{x - x_3}{d_3(x, y)}\hat{i} + \frac{y - y_3}{d_3(x, y)}\hat{j}}$$

$$\begin{aligned} \left|\nabla d_3(x, y)\right| &= \frac{1}{d_3(x, y)}\sqrt{\left(x - x_3\right)^2 + \left(y - y_3\right)^2} \\ &= \frac{d_3(x, y)}{d_3(x, y)} \\ &= \boxed{1} \end{aligned}$$

$\therefore\ \nabla d_3$ is a unit vector

c) $f(x, y) = d_1 + d_2 + d_3$

$\nabla f = \nabla d_1 + \nabla d_2 + \nabla d_3$

Given that $f_x = f_y = 0$

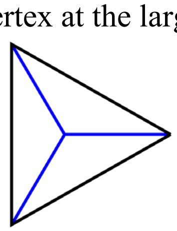
$$\begin{aligned} \nabla f &= f_x\hat{i} + f_y\hat{j} = 0 \\ &= \nabla d_1 + \nabla d_2 + \nabla d_3 \end{aligned}$$

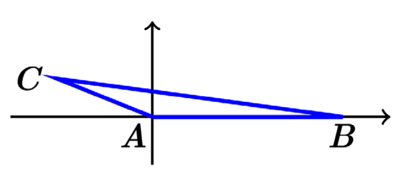
$\nabla d_1 + \nabla d_2 + \nabla d_3 = 0$ ✓

d) Since $\nabla d_1 + \nabla d_2 + \nabla d_3 = 0$ that implies all the 3 *unit* vectors add to 0.

Therefore, all three divide the unit circle into 3 equal sectors, they must make angles of $\pm\frac{2\pi}{3}$.

e) The optimal point is the vertex at the large angle.





f) Three points $(0,\ 0)$, $(0,\ 1)$, $(2,\ 0)$

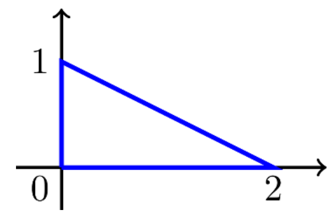
$$f_x = f_y = 0$$

$$f(x,\ y) = d_1 + d_2 + d_3$$

$$= \sqrt{\left(x - x_1\right)^2 + \left(y - y_1\right)^2} + \sqrt{\left(x - x_2\right)^2 + \left(y - y_2\right)^2} + \sqrt{\left(x - x_3\right)^2 + \left(y - y_3\right)^2}$$

$$= \sqrt{x^2 + y^2} + \sqrt{x^2 + (y-1)^2} + \sqrt{(x-2)^2 + y^2}$$

$$f_x = \frac{x}{\sqrt{x^2 + y^2}} + \frac{x}{\sqrt{x^2 + (y-1)^2}} + \frac{x-2}{\sqrt{(x-2)^2 + y^2}} = 0$$



$$f_y = \frac{y}{\sqrt{x^2 + y^2}} + \frac{y-1}{\sqrt{x^2 + (y-1)^2}} + \frac{y}{\sqrt{(x-2)^2 + y^2}} = 0$$

Using maple:

$$solvefor\left(\left\{\frac{x}{\sqrt{x^2+y^2}} + \frac{x}{\sqrt{x^2+(y-1)^2}} + \frac{x-2}{\sqrt{(x-2)^2+y^2}} = 0,\ \frac{y}{\sqrt{x^2+y^2}} + \frac{y-1}{\sqrt{x^2+(y-1)^2}} + \frac{y}{\sqrt{(x-2)^2+y^2}} = 0\right\}\right)$$

$$x = \frac{1}{13} + \frac{4\sqrt{3}}{39} \approx 0.25456931$$

$$y = \frac{8}{13} - \frac{7\sqrt{3}}{39} \approx 0.30450371$$

Exercise

Show that the following two functions have two local maxima but no other extreme points (thus no saddle or basin between the mountains).

$$f(x,\ y) = -\left(x^2 - 1\right)^2 - \left(x^2 - e^y\right)^2$$

Solution

$$f_x = -4x\left(x^2 - 1\right) - 4x\left(x^2 - e^y\right)$$

$$= -4x^3 + 4x - 4x^3 + 4xe^y$$

$$= -8x^3 + 4x\left(1 + e^y\right) = 0$$

$$f_y = 2e^y\left(x^2 - e^y\right) = 0$$

$$\begin{cases} -4x\left(2x^2-1-e^y\right)=0 & \rightarrow x=0, \quad e^y=2x^2-1 \ (1) \\ 2e^y\left(x^2-e^y\right)=0 & \qquad e^y=x^2 \ (2) \end{cases}$$

$$(1) \rightarrow e^y = 2x^2-1\Big|_{x=0} = \cancel{-1}$$

$$\text{from } (2) \rightarrow (1):\ x^2 = 2x^2-1$$

$$x^2 = 1 \quad \rightarrow \quad \underline{x=\pm 1}$$

$$e^y = x^2\Big|_{x=\pm 1} = 1 \quad \rightarrow \quad \underline{y=0}$$

$\therefore$ ***C.P.***: $(\pm 1,\ 0)$

$$f_{xx} = -24x^2 + 4\left(1+e^y\right)$$

$$f_{yy} = 2x^2e^y - 4e^{2y}$$

$$f_{xy} = 4xe^y$$

$$f_{xx}f_{yy} - f_{xy}^2 = \left(-24x^2+4+4e^y\right)\left(2x^2e^y-4e^{2y}\right) - 16x^2e^{2y}$$

@ $(-1,\ 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = (-24+4+4)(2-4)-16 = 16 > 0$$

$$f_{xx} = -24+8 = -16 < 0$$

f has a ***local Max*** @ $(-1,\ 0)$

@ $(1,\ 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = (-24+4+4)(2-4)-16 = 16 > 0$$

$$f_{xx} = -24+8 = -16 < 0$$

f has a ***local Max*** @ $(1,\ 0)$

Exercise

Show that the following two functions have two local maxima but no other extreme points (thus no saddle or basin between the mountains).

$$f(x,\ y) = 4x^2e^y - 2x^4 - e^{4y}$$

Solution

$$f_x = 8xe^y - 8x^3 = 0$$

$f_y = 4x^2e^y - 4e^{4y} = 0$

$$\begin{cases} 8x\left(e^y - x^2\right) = 0 & \rightarrow x = 0, \quad e^y = x^2 \quad (1) \\ 4e^y\left(x^2 - e^{3y}\right) = 0 & \qquad e^{3y} = x^2 \quad (2) \end{cases}$$

$$(1) \rightarrow \; e^y = x^2\Big|_{x=0} = \cancel{0}$$

from $(2) \rightarrow (1)$: $e^{3y} = e^y \;\Rightarrow\; 3y = y$

$$\underline{y = 0}\Big| \quad \rightarrow e^y = x^2 = 1 \quad \underline{x = \pm 1}\Big|$$

$\therefore$ ***C.P.***: $(\pm 1,\ 0)$

$f_{xx} = 8e^y - 24x^2$

$f_{yy} = 4x^2e^y - 16e^{4y}$

$f_{xy} = 8xe^y$

$f_{xx}f_{yy} - f_{xy}^2 = \left(8e^y - 24x^2\right)\left(4x^2e^y - 16e^{4y}\right) - 64x^2e^{2y}$

@ $(-1,\ 0)$

$f_{xx}f_{yy} - f_{xy}^2 = (8-24)(4-16) - 64 = 128 > 0$

$f_{xx} = 8 - 24 = -16 < 0$

f has a ***local Max*** @ $(-1,\ 0)$

@ $(1,\ 0)$

$f_{xx}f_{yy} - f_{xy}^2 = (8-24)(4-16) - 64 = 128 > 0$

$f_{xx} = 8 - 24 = -16 < 0$

f has a ***local Max*** @ $(1,\ 0)$