Solution Section 1.6 – Precise Definition of Limits

Exercise

Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \;\Rightarrow\; a < x < b$ for $a = 1,\;\; b = 7,\;\; x_0 = 5$

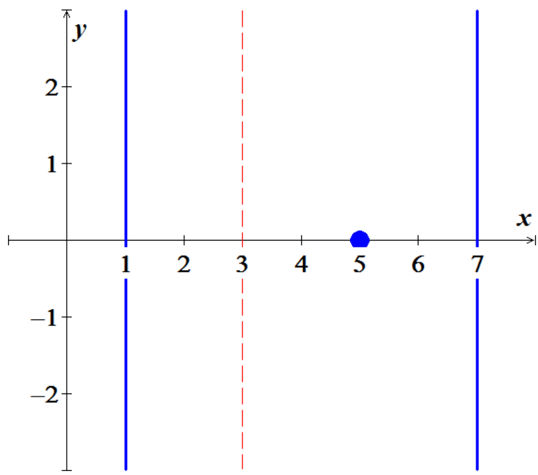
Solution

$$|x-5| < \delta \;\Rightarrow\; -\delta < x - 5 < \delta$$

$$-\delta + 5 < x < \delta + 5$$

$$-\delta + 5 = 1 \;\Rightarrow\; \underline{\delta = 4}$$

$$\delta + 5 = 7 \;\Rightarrow\; \underline{\delta = 2}$$



Exercise

Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \;\Rightarrow\; a < x < b$ for $a = -\frac{7}{2},\;\; b = -\frac{1}{2},\;\; x_0 = -\frac{3}{2}$

Solution

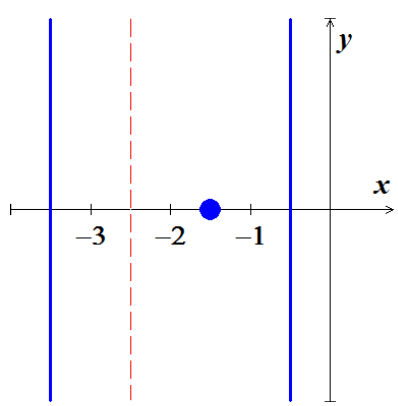
$$\left|x + \frac{3}{2}\right| < \delta$$

$$-\delta < x + \frac{3}{2} < \delta$$

$$-\delta - \frac{3}{2} < x < \delta - \frac{3}{2}$$

$$-\delta - \frac{3}{2} = -\frac{7}{2} \;\Rightarrow\; \delta = \frac{7}{2} - \frac{3}{2} \underline{= 2}$$

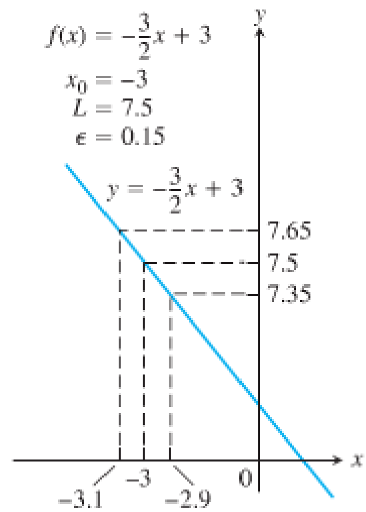
$$\delta - \frac{3}{2} = -\frac{1}{2} \;\Rightarrow\; \delta = \frac{1}{2} - \frac{3}{2} \underline{= -1}$$



Exercise

Use the graph to find a $\delta > 0$ such that for all x $\quad 0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$

Solution



Given: $\quad a = -3.1, \quad b = -2.9, \quad x_0 = -3$

$$|x + 3| < \delta$$

$$-\delta < x + 3 < \delta$$
$$-\delta - 3 < x < \delta - 3$$

$$-\delta - 3 = -3.1$$

$$\Rightarrow \quad \underline{|\delta = 3.1 - 3 = \underline{0.1}|}$$

$$\delta - 3 = -2.9$$

$$\Rightarrow \quad \underline{|\delta = 3 - 2.9 = \underline{0.1}|}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = x + 1, \quad L = 5, \quad x_0 = 4, \quad \varepsilon = 0.01$$

Solution

$$|(x + 1) - 5| < .01$$

$$|x - 4| < .01$$

$$-.01 < x - 4 < .01$$
$$-.01 + 4 < x - 4 + 4 < .01 + 4$$
$$3.99 < x < 4.01$$

$$|x - 4| < \delta$$

$$-\delta < x - 4 < \delta$$
$$-\delta + 4 < x < \delta + 4$$

$$-\delta + 4 = 3.99$$

$$\underline{|\delta = 4 - 3.99 = \underline{0.01}|}$$

$$\delta + 4 = 4.01$$
$$\underline{|\delta = 4.01 - 4 = \underline{0.01}|}$$

$$\Rightarrow \quad \underline{\delta = .01\,|}$$

Exercise

Find an open interval about x_0 on which the inequality $\left|f(x)-L\right|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<\left|x-x_0\right|<\delta$ the inequality $\left|f(x)-L\right|<\varepsilon$ holds.

$$f(x)=\sqrt{x+1}, \quad L=1, \quad x_0=0, \quad \varepsilon=0.1$$

Solution

$$\left|\sqrt{x+1}-1\right|<0.1$$

$$\begin{aligned}
&-0.1<\sqrt{x+1}-1<0.1\\
&-0.1+1<\sqrt{x+1}-1+1<0.1+1\\
&.9<\sqrt{x+1}<1.1\\
&(.9)^2<\left(\sqrt{x+1}\right)^2<(1.1)^2\\
&.81<x+1<1.21\\
&.81-1<x+1-1<1.21-1\\
&-0.19<x<0.21
\end{aligned}$$

$$\left|x-0\right|<\delta \quad\Rightarrow\quad -\delta<x<\delta$$

$$\begin{aligned}
-\delta=-0.19 \ \Rightarrow\ & \underline{\delta=0.19} \\
& \underline{\delta=0.21}
\end{aligned} \quad \rightarrow \quad \underline{\delta=0.19}$$

Exercise

Find an open interval about x_0 on which the inequality $\left|f(x)-L\right|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<\left|x-x_0\right|<\delta$ the inequality $\left|f(x)-L\right|<\varepsilon$ holds.

$$f(x)=\sqrt{x-7}, \quad L=4, \quad x_0=23, \quad \varepsilon=1$$

Solution

$$\left|\sqrt{x-7}-4\right|<1$$

$$\begin{aligned}
&-1<\sqrt{x-7}-4<1\\
&3<\sqrt{x-7}<5\\
&(3)^2<\left(\sqrt{x-7}\right)^2<(5)^2\\
&9<x-7<25\\
&9+7<x-7+7<25+7\\
&16<x<32
\end{aligned}$$

$$\left|x-23\right|<\delta$$

$$-\delta < x - 23 < \delta$$

$$-\delta + 23 < x < \delta + 23$$

$-\delta + 23 = 16 \;\Rightarrow\; \delta = 23 - 16 = \underline{7|}$

$\delta + 23 = 32 \;\Rightarrow\; \delta = 32 - 23 = \underline{9|}$

$\Rightarrow\; \underline{\delta = 7\ |}$

Exercise

Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = x^2, \quad L = 3, \quad x_0 = \sqrt{3}, \quad \varepsilon = 0.1$$

Solution

$|x^2 - 3| < 0.1$

$$-0.1 < x^2 - 3 < 0.1$$

$$2.9 < x^2 < 3.1$$

$$\sqrt{2.9} < x < \sqrt{3.1}$$

$|x - \sqrt{3}| < \delta$

$$-\delta < x - \sqrt{3} < \delta$$

$$-\delta + \sqrt{3} < x < \delta + \sqrt{3}$$

$-\delta + \sqrt{3} = \sqrt{2.9} \;\Rightarrow\; \delta = \sqrt{3} - \sqrt{2.9} = \underline{.029|}$

$\delta + \sqrt{3} = \sqrt{3.1} \;\Rightarrow\; \delta = \sqrt{3.1} - \sqrt{3} = \underline{.029|}$

$\Rightarrow\; \underline{\delta = .029\ |}$

Exercise

Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = \frac{120}{x}, \quad L = 5, \quad x_0 = 24, \quad \varepsilon = 1$$

Solution

$$\left|\frac{120}{x}-5\right|<0.1$$

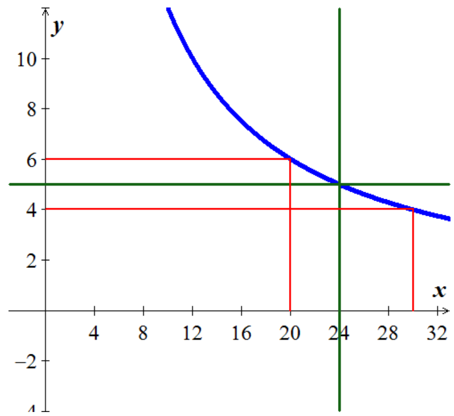
$$-1<\frac{120}{x}-5<1$$

$$4<\frac{120}{x}<6$$

$$\frac{1}{6}<\frac{x}{120}<\frac{1}{4}$$

$$\frac{1}{6}(120)<x<\frac{1}{4}(120)$$

$$20<x<30$$



$$|x-24|<\delta$$

$$-\delta<x-24<\delta$$

$$-\delta+24<x<\delta+24$$

$$-\delta+24=20 \quad\Rightarrow\quad \delta=24-20=\underline{4}$$

$$\delta+24=30 \quad\Rightarrow\quad \delta=30-24=\underline{6}$$

$$\Rightarrow \quad \underline{\delta=4}$$

Exercise

Prove that $\quad \lim\limits_{x\to 4}(9-x)=5$

Solution

$$|(9-x)-5|<\varepsilon$$

$$-\varepsilon<4-x<\varepsilon$$

$$-\varepsilon-4<-x<\varepsilon-4 \qquad\qquad \textit{divide by } (-).$$

$$\varepsilon+4>x>4-\varepsilon$$

$$4-\varepsilon<x<\varepsilon+4$$

$$|x-4|<\delta$$

$$-\delta<x-4<\delta$$

$$-\delta+4<x<\delta+4$$

$$-\delta+4=4-\varepsilon \quad\Rightarrow\quad -\delta=-\varepsilon\Rightarrow \underline{\delta=\varepsilon}$$

$$\delta+4=\varepsilon+4 \quad\Rightarrow\quad \underline{\delta=\varepsilon}$$

$$\Rightarrow \quad \underline{\delta=\varepsilon}$$

Exercise

Prove that $\lim\limits_{x\to 1} \frac{1}{x} = 1$

Solution

$$\left|\frac{1}{x}-1\right| < \varepsilon$$

$$-\varepsilon < \frac{1}{x} - 1 < \varepsilon$$

$$-\varepsilon + 1 < \frac{1}{x} < \varepsilon + 1$$

$$\frac{1}{\varepsilon+1} > x > \frac{1}{-\varepsilon+1}$$

$$\frac{1}{1+\varepsilon} < x < \frac{1}{1-\varepsilon}$$

$$|x-1| < \delta$$

$$-\delta < x-1 < \delta$$

$$1-\delta < x < 1+\delta$$

$$1-\delta = \frac{1}{1+\varepsilon} \quad \Rightarrow \quad \delta = 1 + \frac{1}{1+\varepsilon} = \frac{2+\varepsilon}{1+\varepsilon}$$

$$1+\delta = \frac{1}{1-\varepsilon} \quad \Rightarrow \quad \delta = \frac{1}{1-\varepsilon} - 1 = \frac{\varepsilon}{1-\varepsilon}$$

The smallest : $\delta = \frac{\varepsilon}{1-\varepsilon}$

Exercise

Prove that $\lim\limits_{x\to 5} \frac{x^2-25}{x-5} = 10$

Solution

$$\left|\frac{x^2-25}{x-5} - 10\right| < \varepsilon$$

$$-\varepsilon < \frac{(x-5)(x+5)}{x-5} - 10 < \varepsilon$$

$$-\varepsilon + 10 < x+5 < \varepsilon + 10$$

$$-\varepsilon + 5 < x < \varepsilon + 15$$

$$|x-10| < \delta$$

$$-\delta < x-10 < \delta$$

$$10-\delta < x < 10+\delta$$

$$10-\delta = 5-\varepsilon \quad \Rightarrow \quad \delta = 5+\varepsilon$$

$10+\delta=\varepsilon+15 \quad \Rightarrow \quad \boxed{\delta=\varepsilon+5}$

The smallest : $\boxed{\delta=\varepsilon+5}$

Exercise

Prove that $\quad \lim\limits_{x\to 0} f(x)=0 \quad if \quad f(x)=\begin{cases} 2x, & x<0 \\ \frac{x}{2}, & x\ge 0 \end{cases}$

Solution

For $x<0$: $|2x-0|<\varepsilon$

$$-\varepsilon<2x<0$$

$$-\frac{\varepsilon}{2}<x<0$$

For $x\ge 0$: $\left|\frac{x}{2}-0\right|<\varepsilon$

$$0\le\frac{x}{2}<\varepsilon$$

$$0\le x<2\varepsilon$$

$|x-0|<\delta \quad \Rightarrow \quad -\delta<x<\delta$

$-\delta=-\frac{\varepsilon}{2} \quad \Rightarrow \quad \delta=\frac{\varepsilon}{2}$

$\delta=2\varepsilon$

$\rightarrow$ *the smallest* : $\boxed{\delta=\frac{\varepsilon}{2}}$

Exercise

Prove that $\quad \lim\limits_{x\to 1} (5x-2)=3$

Solution

$|(5x-2)-3|<\varepsilon$

$$-\varepsilon<5x-5<\varepsilon$$

$$5-\varepsilon<5x<\varepsilon+5$$

$$1-\frac{1}{5}\varepsilon<x<1+\frac{1}{5}\varepsilon$$

$|x-3|<\delta$

$$-\delta<x-3<\delta$$

$$3-\delta<x<3+\delta$$

$3-\delta=1-\frac{1}{5}\varepsilon \quad \Rightarrow \quad \delta=\frac{1}{5}\varepsilon+2$

$3+\delta=1+\frac{1}{5}\varepsilon \quad \Rightarrow \quad \delta=\frac{1}{5}\varepsilon-2$

$\rightarrow$ *the smallest* : $\boxed{\delta=\frac{1}{5}\varepsilon-2}$

Exercise

Prove that $\lim\limits_{x\to 2} \dfrac{1}{(x-2)^4} = \infty$

Solution

Let $N > 0$ and let $\delta = \dfrac{1}{\sqrt[4]{N}}$

Suppose that $0 < |x-2| < \delta$

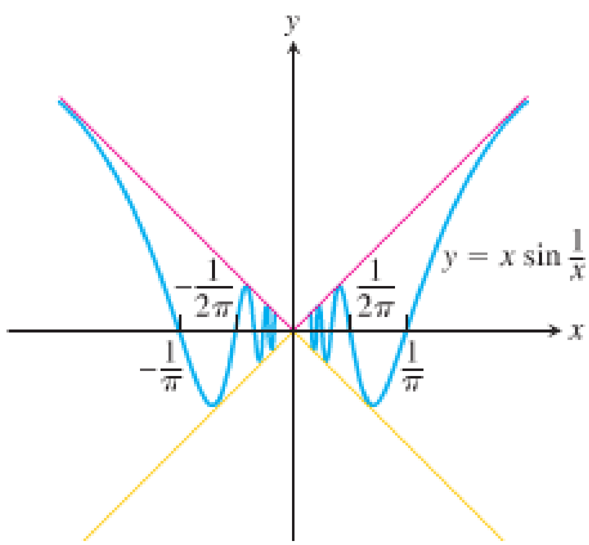
$|x-2| < \delta = \dfrac{1}{\sqrt[4]{N}}$

$\dfrac{1}{|x-2|} > \sqrt[4]{N}$

$\boxed{\dfrac{1}{(x-2)^4} > N}$ ✓

Exercise

Prove that $\lim\limits_{x\to 0} x\dfrac{1}{\sin x} = 0$



Solution

$$\left.\begin{array}{ll} -x \le x\sin\dfrac{1}{x} \le x, & \forall x > 0 \\ -x \ge x\sin\dfrac{1}{x} \ge x, & \forall x < 0 \end{array}\right\} \to \lim_{x\to 0}(-x) = \lim_{x\to 0}(x) = 0$$

Then by the sandwich theorem, $\boxed{\lim\limits_{x\to 0} x\sin\left(\dfrac{1}{x}\right) = 0}$