$x'' + 16x = \cos 4t \qquad x(0) = 0 \; ; \; x'(0) = 1$ ①

1st Method

$\mathcal{L}\{x'' + 16x\} = \mathcal{L}\{\cos 4t\}$

$s^2 Y(s) - s\underbrace{y(0)}_{} - \underbrace{y'(0)}_{=1} + 16\,Y(s) = \dfrac{s}{s^2+16}$

$(s^2+16)Y(s) = \dfrac{s}{s^2+16} + 1 = \dfrac{s + s^2 + 16}{s^2+16}$

$Y(s) = \dfrac{s^2+s+16}{(s^2+16)^2} = \dfrac{As+B}{s^2+16} + \dfrac{Cs+D}{(s^2+16)^2}$

$s^2 + s + 16 = As^3 + 16As + Bs^2 + 16B + Cs + D$

$s^3 \mid A = 0$

$s^2 \mid B = 1$

$s^1 \mid 16A + C = 1 \Rightarrow C = 1$

$s^0 \mid 16B + D = 16 \Rightarrow D = 0.$

$Y(s) = \dfrac{1}{s^2+16} + \dfrac{s}{(s^2+16)^2} = \dfrac{1}{4}\dfrac{4}{s^2+16} + \dfrac{1}{8}\dfrac{2(4)s}{(s^2+16)^2}$

$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\dfrac{4}{s^2+16}\right\} + \frac{1}{8}\mathcal{L}^{-1}\left\{\dfrac{8s}{(s^2+16)^2}\right\}$

$\boxed{y(t) = \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t}$

2nd Method

$\lambda^2 + 16 = 0 \Rightarrow \lambda = \pm 4i$

$x_h = e^0(C_1\cos 4t + C_2 \sin 4t)$

$= C_1 \cos 4t + C_2 \sin 4t.$

$x(0) = C_1 = 0$

$x_h' = -4C_1 \sin 4t + 4C_2\cos 4t$

$x_h'(0) = 4C_2 = 1 \Rightarrow \boxed{C_2 = 1/4}$

$\Rightarrow \boxed{x_h = \frac{1}{4}\sin 4t}$

Assume: $x_p = A\cos 4t + B\sin 4t$

$x_p' = -4A\sin 4t + 4B\cos 4t$

$x_p'' = -16A\cos 4t - 16B\sin 4t$

$x'' + 16x = \cos t \Rightarrow -16A\cos 4t - 16B\sin 4t + 16A\cos 4t + 16B\sin 4t = \cos 4t$

$0 = \cos 4t$ #

let: $x_p = At\cos 4t + Bt\sin 4t$

$x' = A\cos 4t - 4At\sin 4t + B\sin 4t + 4Bt\cos 4t$

$x'' = -4A\sin 4t - 4A\sin 4t - 16At\cos 4t + 4B\cos 4t + 4B\cos 4t - 16Bt\sin 4t$

$x'' + 16x = \cos 4t$

$-4A\sin 4t - 4A\sin 4t - 16At\cos 4t + 4B\cos 4t + 4B\cos 4t - 16Bt\sin 4t$

$+16At\cos 4t \qquad + 16Bt\sin 4t = \cos 4t$

$\Rightarrow -8A\sin 4t + 8B\cos 4t = \cos 4t + 0\sin 4t$

$$\begin{cases} -8A = 0 \Rightarrow A = 0 \\ 8B = 1 \Rightarrow B = \frac{1}{8} \end{cases}$$

$$x_p = \frac{1}{8} t \sin 4t$$

$$\Rightarrow \boxed{X(t) = \frac{1}{4} \sin 4t + \frac{1}{8} t \sin 4t}$$

$x = x_h + x_p$

$y'' + y = \cos t \Rightarrow y(t) = \sin t + \frac{1}{2} t \sin t$

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$ / $y(0) = 0$, $y'(0) = 1$

$y_h = e^0 (C_1 \cos t + C_2 \sin t)$

$C_1 \cos t + C_2 \sin t$

$y(0) = \underline{C_1 = 0} \Rightarrow$

$y' = -C_1 \sin t + C_2 \cos t \Rightarrow y'(0) = \underline{C_2 = 1}$

$\underline{y_h = \sin t}$

$y_p = A \cos t + B \sin t$

$y' = -A \sin t + B \cos t$

$y'' = -A \cos t - B \sin t$

$y'' + y = 0 = \cos t$ #

let

$y = At \cos t + Bt \sin t$

$y' = A \cos t - At \sin t + B \sin t + Bt \cos t$

$y'' = -A \sin t - A \sin t - At \cos t + B \cos t + B \cos t - Bt \sin t.$

$y'' + y = 2B \cos t = \cos t \Rightarrow B = \frac{1}{2}$

$y = \frac{1}{2} t \sin t$

$y = \sin t + \frac{1}{2} t \sin t.$

$s^2 Y(s) - s\, y(0) - y'(0) + Y(s) = \frac{s}{s^2+1}$

$(s^2+1) Y(s) = \frac{s}{s^2+1} + 1 = \frac{s^2+s+1}{s^2+1}$

$Y(s) = \frac{s^2+s+1}{(s^2+1)^2} = \frac{As+B}{s^2+1} + \frac{Cs+D}{(s^2+1)^2}$

$s^2 + s + 1 = As^3 + As + Bs^2 + B + Cs + D$

$$\begin{cases} A = 0 \\ B = 1 \\ A + C = 1 \Rightarrow C = 1 \\ B + D = 1 \Rightarrow D = 0. \end{cases}$$

$Y(s) = \frac{1}{s^2+1} + \frac{s}{(s^2+1)^2}$