Solution   *Section* 4.6 – Surfaces and Area

Exercise

Find a parametrization of the surface: The paraboloid $z = x^2 + y^2, \ z \le 4$

Solution

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$z = x^2 + y^2 = r^2 \qquad\qquad z \le 4 \ \rightarrow \ r^2 \le 4 \Rightarrow 0 \le r \le 2$$

Then: $\vec{r}(r, \theta) = (r\cos\theta)\,\hat{i} + (r\sin\theta)\,\hat{j} + r^2\,\hat{k} \qquad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi$

Exercise

Find a parametrization of the surface: The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 4$

Solution

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$z = 2\sqrt{x^2 + y^2} = 2r \qquad \begin{aligned} z = 2 &\rightarrow r = 1 \\ z = 4 &\rightarrow r = 2 \end{aligned}$$

Then: $\vec{r}(r, \theta) = (r\cos\theta)\,\hat{i} + (r\sin\theta)\,\hat{j} + 2r\,\hat{k} \qquad 1 \le r \le 2, \quad 0 \le \theta \le 2\pi$

Exercise

Find a parametrization of the surface cut from the sphere $x^2 + y^2 + z^2 = 8$ by the plane $z = -2$

Solution

$$x^2 + y^2 + z^2 = 8 = \rho^2 \quad \rightarrow \quad \rho = 2\sqrt{2}$$

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

$$x = 2\sqrt{2}\sin\phi\cos\theta, \quad y = 2\sqrt{2}\sin\phi\sin\theta, \quad z = 2\sqrt{2}\cos\phi$$

For $z = -2$

$$2\sqrt{2}\cos\phi = -2$$

$$\cos\phi = -\frac{1}{\sqrt{2}}$$

$$\phi = \frac{3\pi}{4}$$

For $z = 2\sqrt{2}$

$$2\sqrt{2}\cos\phi = 2\sqrt{2}$$

$$\cos\phi = 1$$

$$\boxed{\phi = 0}$$

Then: $\vec{r}(\phi,\ \theta) = \left(2\sqrt{2}\sin\phi\cos\theta\right)\hat{i} + \left(2\sqrt{2}\sin\phi\sin\theta\right)\hat{j} + \left(2\sqrt{2}\cos\phi\right)\hat{k}$

$$0 \le \phi \le \frac{3\pi}{4}, \quad 0 \le \theta \le 2\pi$$

Exercise

Find a parametrization of the surface the plane $2x - 4y + 3z = 16$

Solution

$$z = \frac{1}{3}(16 - 2x + 4y)$$

Then: $\vec{r}(u,\ v) = \left\langle u,\ v,\ \frac{1}{3}(16 - 2u + 4v)\right\rangle \qquad u,\ v\ \in\ (-\infty,\ \infty)$

Exercise

Find a parametrization of the surface the cap of the sphere $x^2 + y^2 + z^2 = 16$ for $2\sqrt{2} \le z \le 4$

Solution

$$x^2 + y^2 + z^2 = 16 = \rho^2 \quad \to \quad \rho = 4$$

$$x = 4\sin\phi\cos\theta, \quad y = 4\sin\phi\sin\theta, \quad z = 4\cos\phi$$

For $z = 2\sqrt{2}$

$$4\cos\phi = 2\sqrt{2}$$

$$\cos\phi = \frac{\sqrt{2}}{2}$$

$$\boxed{\phi = \frac{\pi}{4}}$$

For $z = 4$

$$4\cos\phi = 4$$

$$\cos\phi = 1$$

$$\boxed{\phi = 0}$$

Then: $\vec{r}(\phi,\ \theta) = \langle 4\sin\phi\cos\theta,\ 4\sin\phi\sin\theta,\ 4\cos\phi\rangle \quad 0 \le \phi \le \frac{\pi}{4}, \quad 0 \le \theta \le 2\pi$

Exercise

Find a parametrization of the surface the frustum of the cone $z^2 = x^2 + y^2$ for $2 \le z \le 8$

Solution

$x = r\cos\theta, \quad y = r\sin\theta$

$z^2 = x^2 + y^2 = r^2 \quad \rightarrow \quad \underline{z = r}$

$z = 2 = r$

$z = 8 = r$

Then: $\vec{r}(r,\ \theta) = \langle r\cos\theta,\ r\sin\theta,\ r\rangle \qquad 2 \le r \le 8, \quad 0 \le \theta \le 2\pi$

Exercise

Find a parametrization of the surface the cone $z^2 = 4\left(x^2 + y^2\right)$ for $0 \le z \le 4$

Solution

$x = r\cos\theta, \quad y = r\sin\theta$

$z^2 = 4\left(x^2 + y^2\right) = 4r^2$

$\rightarrow \quad \underline{z = 2r}$

$z = 0 = 2r \quad \rightarrow \ r = 0$

$z = 4 = 2r \quad \rightarrow \ r = 2$

Then: $\vec{r}(r,\ \theta) = \left\langle \frac{1}{2}r\cos\theta,\ \frac{1}{2}r\sin\theta,\ r\right\rangle \qquad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi$

Exercise

Find a parametrization of the surface the portion of the cylinder $x^2 + y^2 = 9$ in the first octant, for $0 \le z \le 3$

Solution

$x = 3\cos\theta, \quad y = 3\sin\theta$

$z = r \quad \rightarrow \quad 0 \le r \le 3$

Then: $\vec{r}(r,\ \theta) = \langle 3\cos\theta,\ 3\sin\theta,\ r\rangle \qquad 0 \le r \le 3, \quad 0 \le \theta \le \frac{\pi}{2}$

Exercise

Find a parametrization of the surface the cylinder $y^2+z^2=36$ for $0\le x\le 9$

Solution

$$y=6\cos\theta, \quad z=6\sin\theta$$
$$x=r \quad \rightarrow \quad 0\le r\le 9$$

Then: $\vec{r}(r,\ \theta)=\langle r,\ 6\cos\theta,\ 6\sin\theta\rangle \qquad 0\le r\le 9, \quad 0\le\theta\le 2\pi$

Exercise

Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of the plane $y+2z=2$ inside the cylinder $x^2+y^2=1$

Solution

$$x=r\cos\theta, \quad y=r\sin\theta$$
$$y+2z=2 \quad \rightarrow \quad z=\frac{2-y}{2}=\frac{2-r\sin\theta}{2}$$

Then: $\vec{r}(r,\ \theta)=(r\cos\theta)\ \hat{i}+(r\sin\theta)\ \hat{j}+\left(\frac{2-r\sin\theta}{2}\right)\hat{k} \qquad 0\le r\le 1, \quad 0\le\theta\le 2\pi$

$$\vec{r}_r=(\cos\theta)\ \hat{i}+(\sin\theta)\ \hat{j}-\left(\frac{\sin\theta}{2}\right)\hat{k}$$
$$\vec{r}_\theta=(-r\sin\theta)\ \hat{i}+(r\cos\theta)\ \hat{j}-\left(\frac{r\cos\theta}{2}\right)\hat{k}$$

$$\vec{r}_r\times\vec{r}_\theta=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \cos\theta & \sin\theta & -\frac{1}{2}\sin\theta\\ -r\sin\theta & r\cos\theta & -\frac{1}{2}r\cos\theta\end{vmatrix}$$

$$=\left(-\frac{1}{2}r\cos\theta\sin\theta+\frac{1}{2}r\cos\theta\sin\theta\right)\hat{i}-\left(-\frac{1}{2}r\cos^2\theta-\frac{1}{2}r\sin^2\theta\right)\hat{j}$$
$$+\left(r\cos^2\theta+r\sin^2\theta\right)\hat{k}$$

$$=\frac{1}{2}r\hat{j}+r\hat{k}$$

$$\left|\boldsymbol{r}_r\times\boldsymbol{r}_\theta\right|=\sqrt{\frac{r^2}{4}+r^2}$$
$$=\frac{\sqrt{5}}{2}r$$

$$A = \int_0^{2\pi} \int_0^1 \frac{\sqrt{5}}{2} r \, dr d\theta$$

$$= \frac{\sqrt{5}}{4} \int_0^{2\pi} \left(r^2 \Big|_0^1 \right) d\theta$$

$$= \frac{\sqrt{5}}{4} \int_0^{2\pi} d\theta$$

$$= \frac{\sqrt{5}}{4}(2\pi)$$

$$= \frac{\pi\sqrt{5}}{2} \quad unit^2$$

Exercise

Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of the cone $z = \frac{\sqrt{x^2 + y^2}}{3}$ between the planes $z = 1$ and $z = \frac{4}{3}$

Solution

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$z = \frac{\sqrt{x^2 + y^2}}{3} = \frac{r}{3} \qquad z = 1 \to r = 3 \qquad z = \frac{4}{3} \to r = 4$$

Then: $\vec{r}(r, \theta) = (r\cos\theta)\,\hat{i} + (r\sin\theta)\,\hat{j} + \left(\frac{r}{3}\right)\hat{k} \qquad 3 \le r \le 4, \quad 0 \le \theta \le 2\pi$

$$\vec{r}_r = (\cos\theta)\,\hat{i} + (\sin\theta)\,\hat{j} + \frac{1}{3}\hat{k}$$

$$\vec{r}_\theta = (-r\sin\theta)\,\hat{i} + (r\cos\theta)\,\hat{j}$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & \frac{1}{3} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \left(0 - \frac{1}{3}r\cos\theta\right)\hat{i} - \left(0 + \frac{1}{3}r\sin\theta\right)\hat{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\hat{k}$$

$$= \left(-\frac{1}{3}r\cos\theta\right)\hat{i} - \left(\frac{1}{3}r\sin\theta\right)\hat{j} + r\,\hat{k}$$

$$\left|\vec{r}_r \times \vec{r}_\theta\right| = \sqrt{\frac{1}{9}r^2\cos^2\theta + \frac{1}{9}r^2\sin^2\theta + r^2}$$

$$= \sqrt{\frac{1}{9}r^2 + r^2}$$

$$= \boxed{\frac{\sqrt{10}}{3} r}$$

$$A = \int_0^{2\pi} d\theta \int_3^4 \frac{\sqrt{10}}{3} r\, dr$$

$$= \frac{\pi\sqrt{10}}{3} \left(r^2 \Big|_3^4\right.$$

$$= \frac{\pi\sqrt{10}}{3}(16-9)$$

$$= \boxed{\frac{7\pi\sqrt{10}}{3} \quad unit^2}$$

Exercise

Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of the cylinder $x^2 + z^2 = 10$ between the planes $y = -1$ and $y = 1$

Solution

$$x = u\cos v, \quad z = u\sin v$$

$$x^2 + z^2 = 10 = u^2\cos^2 v + u^2\sin^2 v$$

$$u^2 = 10 \to u = \sqrt{10}$$

Then: $\vec{r}(y, v) = (u\cos v)\,\hat{i} + y\hat{j} + (u\sin v)\,\hat{k}$

$$= \left(\sqrt{10}\cos v\right)\hat{i} + y\hat{j}\left(\sqrt{10}\sin v\right)\hat{k}$$

$$\vec{r}_y = \hat{j}$$

$$\vec{r}_v = \left(-\sqrt{10}\sin v\right)\hat{i} + \left(\sqrt{10}\cos v\right)\hat{k}$$

$$\vec{r}_y \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ -\sqrt{10}\sin v & 0 & \sqrt{10}\cos v \end{vmatrix}$$

$$= \left(\sqrt{10}\cos v\right)\hat{i} + \left(\sqrt{10}\sin v\right)\hat{k}$$

$$\left|\vec{r}_r \times \vec{r}_\theta\right| = \sqrt{10\cos^2 v + 10\sin^2 v}$$

$$= \boxed{\sqrt{10}}$$

$$A = \int_0^{2\pi} dv \int_{-1}^{1} \sqrt{10}\, dy$$

$$= 2\pi\sqrt{10}\left(y\right|_{-1}^{1}$$

$$\underline{= 4\pi\sqrt{10} \quad unit^2}$$

Exercise

Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of the cap cut from the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 4$

Solution

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$z = x^2 + y^2 = r^2$$

$$z = 1 \rightarrow r = 1$$

$$z = 4 \rightarrow r = 2$$

Then: $\vec{r}(r,\ \theta) = (r\cos\theta)\ \hat{i} + (r\sin\theta)\ \hat{j} + r^2\ \hat{k} \qquad 1 \le r \le 2, \quad 0 \le \theta \le 2\pi$

$$\vec{r}_r = (\cos\theta)\ \hat{i} + (\sin\theta)\ \hat{j} + 2r\ \hat{k}$$

$$\vec{r}_\theta = (-r\sin\theta)\ \hat{i} + (r\cos\theta)\ \hat{j}$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \left(0 - 2r^2\cos\theta\right)\hat{i} - \left(0 + 2r^2\sin\theta\right)\hat{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\hat{k}$$

$$= \left(-2r^2\cos\theta\right)\hat{i} - \left(2r^2\sin\theta\right)\hat{j} + r\,\hat{k}$$

$$\left|\vec{r}_r \times \vec{r}_\theta\right| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2}$$

$$= \sqrt{4r^4 + r^2}$$

$$\underline{= r\sqrt{4r^2 + 1}}$$

$$A = \int_0^{2\pi}\int_1^2 r\sqrt{4r^2+1}\ drd\theta \qquad d\left(4r^2+1\right) = 8rdr$$

$$= \frac{1}{8}\int_0^{2\pi} d\theta \int_1^2 \left(4r^2+1\right)^{1/2}\ d\left(4r^2+1\right)$$

$$= \frac{\pi}{6}\left(4r^2+1\right)^{3/2}\Big|_1^2$$

$$= \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)$$

$$= \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) \ \ unit^2$$

Exercise

Find the area of the following surface using a parametric description of the surface:
The half cylinder $\{(r,\ \theta,\ z): \ \ r = 4, \ \ \ 0 \le \theta \le \pi. \ \ \ 0 \le z \le 7\}$

Solution

$x = 4\cos\theta, \ \ \ y = 4\sin\theta$

$z = r$

Then: $\vec{r}(r,\ \theta) = \langle 4\cos\theta,\ 4\sin\theta,\ r\rangle$

$\vec{r}_r = \langle 0,\ 0,\ 1\rangle$

$\vec{r}_\theta = \langle -4\sin\theta,\ 4\cos\theta,\ 0\rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ -4\sin\theta & 4\cos\theta & 0 \end{vmatrix}$$

$$= \langle -4\cos\theta,\ -4\sin\theta,\ 0\rangle$$

$$\left|\vec{r}_r \times \vec{r}_\theta\right| = \sqrt{16\cos^2\theta + 16\sin^2\theta}$$

$$= 4$$

$$Area = \int_0^\pi \int_0^7 4\ dz\ d\theta$$

$$= 4(\pi)(7)$$

$$= 28\pi \ \ unit^2$$

Exercise

Find the area of the following surface using a parametric description of the surface:
The plane $z = 3 - x - 3y$ in the first octant

Solution

$\vec{r}(u,\ v) = \langle u,\ v,\ 3 - u - 3v\rangle$

$\vec{r}_u = \langle 1,\ 0,\ -1\rangle$

$\vec{r}_v = \langle 0,\ 1,\ -3\rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1 & 0 & -1 \\ 0 & 1 & -3 \end{vmatrix}$$

$$= \langle 1,\ 3,\ 1 \rangle$$

$$\left| \vec{r}_u \times \vec{r}_v \right| = \sqrt{1+9+1}$$

$$\boxed{= \sqrt{11}}$$

$$0 = 3 - u - 3v \quad \to \quad u = 3 - 3v$$

$$0 \le u \le 3 - 3v$$

$$u = 0 \quad \to \quad v = 3$$

$$0 \le v \le 3$$

$$Area = \int_0^1 \int_0^{3-3v} \sqrt{11}\ du\ dv$$

$$= \sqrt{11} \int_0^1 \left(u \ \Big|_0^{3-3v} \right. dv$$

$$= 3\sqrt{11} \int_0^1 (1 - v)\, dv$$

$$= 3\sqrt{11} \left(v - \frac{1}{2} v^2 \ \Big|_0^1 \right.$$

$$= 3\sqrt{11} \left(1 - \frac{1}{2} \right)$$

$$\boxed{= \frac{3\sqrt{11}}{2} \quad unit^2}$$

Exercise

Find the area of the following surface using a parametric description of the surface

The plane $z = 10 - x - y$ above the square $|x| \le 2, \quad |y| \le 2$

Solution

$$\vec{r}(u,\ v) = \langle u,\ v,\ 10 - u - v \rangle$$

$$\vec{r}_u = \langle 1,\ 0,\ -1 \rangle$$

$$\vec{r}_v = \langle 0,\ 1,\ -1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= \langle 1,\ 1,\ 1 \rangle$$

$$\left|\vec{r}_u \times \vec{r}_v\right| = \sqrt{1+1+1}$$

$$= \sqrt{3}\ \big|$$

$$|x| \le 2 \quad \to -2 \le u \le 2$$

$$|y| \le 2 \quad \to -2 \le v \le 2$$

$$Area = \int_{-2}^{2}\int_{-2}^{2} \sqrt{3}\ du\ dv$$

$$= \sqrt{3}\int_{-2}^{2} dv \int_{-2}^{2} du$$

$$= \sqrt{3}\ v\Big|_{-2}^{2}\quad u\Big|_{-2}^{2}$$

$$= 16\sqrt{3}\ \ unit^2\ \big|$$

Exercise

Find the area of the following surface using a parametric description of the surface

The hemisphere $x^2 + y^2 + z^2 = 100, \quad z \ge 0$

Solution

$$x^2 + y^2 + z^2 = 100 = \rho^2 \quad \to \quad \rho = 10$$

$$\vec{r} = \langle 10\sin u\cos v,\ 10\sin u\sin v,\ 10\cos u \rangle$$

$$\vec{r}_u = \langle 10\cos u\cos v,\ 10\cos u\sin v,\ -10\sin u \rangle$$

$$\vec{r}_v = \langle -10\sin u\sin v,\ 10\sin u\cos v,\ 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 10\cos u\cos v & 10\cos u\sin v & -10\sin u \\ -10\sin u\sin v & 10\sin u\cos v & 0 \end{vmatrix}$$

$$= \left\langle 100\sin^2 u\cos v,\ 100\sin^2 u\sin v,\ 100\sin u\cos u\cos^2 v + 100\sin u\cos u\sin^2 v \right\rangle$$

$$= \left\langle 100\sin^2 u\cos v,\ 100\sin^2 u\sin v,\ 100\sin u\cos u \right\rangle$$

$$\left|\vec{r}_u \times \vec{r}_v\right| = \sqrt{10^4 \sin^4 u \cos^2 v + 10^4 \sin^4 u \sin^2 v + 10^4 \sin^2 u \cos^2 u}$$

$$= 100\sqrt{\sin^4 u\left(\cos^2 v + \sin^2 v\right) + \sin^2 u \cos^2 u}$$

$$= 100\sqrt{\sin^4 u + \sin^2 u \cos^2 u}$$

$$= 100 \sin u \sqrt{\sin^2 u + \cos^2 u}$$

$$\underline{= 100 \sin u}$$

$$Area = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} 100 \sin u \; du \; dv$$

$$= 100 \int_0^{2\pi} dv \; \int_0^{\frac{\pi}{2}} \sin u \; du$$

$$= -200\pi \left(\cos u \right|_0^{\frac{\pi}{2}}$$

$$= -200\pi(-1)$$

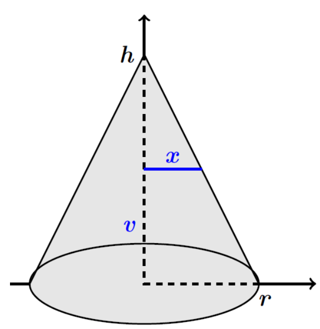
$$\underline{= 200\pi \;\; unit^2}$$

Exercise

Find the area of the following surfaces using a parametric description of the surface
A cone with base radius *r* and height *h*, where *r* and *h* are positive constants.

Solution

Cone equation: $x^2 + y^2 - z = 0$ with $z \le h$



$x^2 + y^2 = r^2$

$\frac{x}{r} = \frac{v}{h} \quad \rightarrow \quad x = \frac{rv}{h}$

$0 \le v \le h, \quad 0 \le u \le 2\pi$

$\vec{r}(u, v) = \left\langle \frac{r}{h} v \cos u, \; \frac{r}{h} v \sin u, \; v \right\rangle$

$\vec{r}_u = \left\langle -\frac{r}{h} v \sin u, \; \frac{r}{h} v \cos u, \; 0 \right\rangle$

$\vec{r}_v = \left\langle \frac{r}{h} \cos u, \; \frac{r}{h} \sin u, \; 1 \right\rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ -\frac{r}{h} v \sin u & \frac{r}{h} v \cos u & 0 \\ \frac{r}{h} \cos u & \frac{r}{h} \sin u & 1 \end{vmatrix}$$

$$= \left\langle \frac{r}{h} v \cos u,\ \frac{r}{h} v \sin u,\ -\frac{r^2}{h^2} v \sin^2 u - \frac{r^2}{h^2} v \cos^2 u \right\rangle$$

$$= \left\langle \frac{r}{h} v \cos u,\ \frac{r}{h} v \sin u,\ -\frac{r^2}{h^2} v \right\rangle$$

$$\left| \vec{r}_u \times \vec{r}_v \right| = \sqrt{\frac{r^2}{h^2} v^2 \cos^2 u + \frac{r^2}{h^2} v^2 \sin^2 u + \frac{r^4}{h^4} v^2}$$

$$= \frac{rv}{h} \sqrt{\cos^2 u + \sin^2 u + \frac{r^2}{h^2}}$$

$$= \frac{rv}{h} \sqrt{1 + \frac{r^2}{h^2}}$$

$$= \frac{rv}{h^2} \sqrt{h^2 + r^2}$$

$$Area = \int_0^{2\pi} \int_0^h \frac{rv}{h^2} \sqrt{h^2 + r^2}\ dvdu$$

$$= \frac{r}{h^2} \sqrt{h^2 + r^2} \left(\frac{1}{2} v^2 \Big|_0^h \right) \int_0^{2\pi} du$$

$$= \frac{r}{h^2} \sqrt{h^2 + r^2} \left(\frac{1}{2} h^2 \right) (2\pi)$$

$$\boxed{= \pi r \sqrt{h^2 + r^2}\ unit^2}$$

Exercise

Find the area of the following surfaces using a parametric description of the surface

The cap of the sphere $x^2 + y^2 + z^2 = 4, \quad 1 \le z \le 2$

Solution

$$\vec{r} = \langle 2 \sin u \cos v,\ 2 \sin u \sin v,\ 2 \cos u \rangle$$

$$\vec{r}_u = \langle 2 \cos u \cos v,\ 2 \cos u \sin v,\ -2 \sin u \rangle$$

$$\vec{r}_v = \langle -2 \sin u \sin v,\ 2 \sin u \cos v,\ 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos u\cos v & 2\cos u\sin v & -2\sin u \\ -2\sin u\sin v & 2\sin u\cos v & 0 \end{vmatrix}$$

$$= \left\langle 4\sin^2 u\cos v,\ 4\sin^2 u\sin v,\ 4\sin u\cos u\cos^2 v + 4\sin u\cos u\sin^2 v \right\rangle$$

$$= \left\langle 4\sin^2 u\cos v,\ 4\sin^2 u\sin v,\ 4\sin u\cos u \right\rangle$$

$$\left|\vec{r}_u \times \vec{r}_v\right| = \sqrt{16\sin^4 u\cos^2 v + 16\sin^4 u\sin^2 v + 16\sin^2 u\cos^2 u}$$

$$= 4\sqrt{\sin^4 u\left(\cos^2 v + \sin^2 v\right) + \sin^2 u\cos^2 u}$$

$$= 4\sqrt{\sin^4 u + \sin^2 u\cos^2 u}$$

$$= 4\sin u\sqrt{\sin^2 u + \cos^2 u}$$

$$= 4\sin u$$

$z = 1 = 2\cos u$

$$\to \cos u = \frac{1}{2} \quad \Rightarrow \quad u = \frac{\pi}{3}$$

$z = 2 = 2\cos u$

$$\to \cos u = 1 \quad \Rightarrow \quad u = 0$$

$$Area = \int_0^{2\pi}\int_0^{\frac{\pi}{3}} 4\sin u\ du\ dv$$

$$= 4\int_0^{2\pi} dv \int_0^{\frac{\pi}{3}} \sin u\ du$$

$$= -8\pi\ \left(\cos u\right|_0^{\frac{\pi}{3}}$$

$$= -8\pi\left(\frac{1}{2} - 1\right)$$

$$= 4\pi\ \ unit^2$$

Exercise

Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 2$.

Solution

$\vec{p} = \hat{k}, \quad \nabla f = 2x\,\hat{i} + 2y\,\hat{j} - \hat{k}$

$$\left|\nabla f\right| = \sqrt{(2x)^2 + (2y)^2 + 1}$$

$$= \sqrt{4x^2 + 4y^2 + 1}$$

$$\left|\nabla f \cdot \vec{p}\right| = 1$$

$$z = 2 \quad \Rightarrow \quad x^2 + y^2 = 2$$

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$r^2 = x^2 + y^2 = 2 \Rightarrow r = \sqrt{2}$$

$$\text{Surface area} = \iint\limits_R \frac{|\nabla F|}{|\nabla F \cdot p|}\, dA$$

$$= \iint\limits_R \sqrt{4x^2 + 4y^2 + 1}\; dxdy$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + 1}\; r\, drd\theta$$

$$= \int_0^{2\pi} d\theta \quad \int_0^{\sqrt{2}} \sqrt{4r^2 + 1}\; r\, dr \qquad\qquad d\left(4r^2 + 1\right) = 8rdr$$

$$= \frac{1}{8}(2\pi) \int_0^{\sqrt{2}} \left(4r^2 + 1\right)^{1/2} d\left(4r^2 + 1\right)$$

$$= \frac{\pi}{6}\left(4r^2 + 1\right)^{3/2} \Big|_0^{\sqrt{2}}$$

$$= \frac{\pi}{6}(27 - 1)$$

$$= \frac{13\pi}{3} \quad unit^2$$

Exercise

Find the area of the portion of the surface $x^2 - 2z = 0$ that lies above the triangle bounded by the lines $x = \sqrt{3}$, $y = 0$, and $y = x$ in the xy-plane.

Solution

$$\vec{p} = \hat{k}$$

$$\nabla f = 2x\,\hat{i} - 2\hat{j}$$

$$\left|\nabla f\right| = \sqrt{4x^2 + 4}$$

$$= 2\sqrt{x^2 + 1}$$

$$\left|\nabla f \cdot \vec{p}\right| = \left|\left(2x\,\hat{i} - 2\,\hat{k}\right) \cdot \left(\hat{k}\right)\right|$$

$$= 2$$

$$Surface\ area = \iint_R \frac{\left|\nabla F\right|}{\left|\nabla F \cdot \vec{p}\right|}\, dA$$

$$= \int_0^{\sqrt{3}} \int_0^{x} \frac{2\sqrt{x^2+1}}{2}\, dydx$$

$$= \int_0^{\sqrt{3}} \sqrt{x^2+1}\ \left(y\ \Big|_0^x\right) dx$$

$$= \int_0^{\sqrt{3}} x\sqrt{x^2+1}\, dx \qquad\qquad d\left(x^2+1\right) = 2xdx$$

$$= \frac{1}{2}\int_0^{\sqrt{3}} \left(x^2+1\right)^{1/2} d\left(x^2+1\right)$$

$$= \frac{1}{2}\left(\frac{2}{3}\left(x^2+1\right)^{3/2}\ \Big|_0^{\sqrt{3}}\right.$$

$$= \frac{1}{3}\left(4^{3/2} - 1\right)$$

$$= \frac{1}{3}(8-1)$$

$$= \frac{7}{3}\quad unit^2$$

Exercise

Find the area of the cap cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$.

Solution

$$\vec{p} = \hat{k}$$

$$\nabla f = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$$

$$\left|\nabla f\right| = \sqrt{4x^2 + 4y^2 + 4z^2}$$

$$= 2\sqrt{x^2 + y^2 + z^2}$$

$$= 2\sqrt{2}$$

$$|\nabla f \cdot \vec{p}| = \left|\left(2x\,\hat{i} + 2y\,\hat{i} + 2z\,\hat{k}\right) \cdot \left(\hat{k}\right)\right|$$

$$= 2z$$

$$z = \sqrt{x^2 + y^2}$$

$$\rightarrow \quad x^2 + y^2 + z^2 = z^2 + z^2$$

$$= 2z^2 = 2$$

$$\Rightarrow \quad z = 1$$

$$x^2 + y^2 + z^2 = 2 \quad \rightarrow \quad z = \sqrt{2 - \left(x^2 + y^2\right)}$$

$$\text{Surface area} = \iint\limits_R \frac{2\sqrt{2}}{2z}\, dydx \qquad\qquad \text{Surface area} = \iint\limits_R \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|}\, dA$$

$$= \sqrt{2} \iint\limits_R \frac{1}{\sqrt{2 - \left(x^2 + y^2\right)}}\, dydx$$

$$= \sqrt{2} \int_0^{2\pi} \int_0^1 \frac{1}{\sqrt{2 - r^2}}\, r\, drd\theta$$

$$= -\frac{\sqrt{2}}{2} \int_0^{2\pi} d\theta \int_0^1 \left(2 - r^2\right)^{-1/2} d\left(2 - r^2\right)$$

$$= -2\pi\sqrt{2} \left(2 - r^2\right)^{1/2} \Big|_0^1$$

$$= -2\pi\sqrt{2}\left(1 - \sqrt{2}\right)$$

$$= 2\pi\left(2 - \sqrt{2}\right) \quad unit^2$$

Exercise

Find the area of the ellipse cut from the plane $z = cx$ (c a constant) by the cylinder $x^2 + y^2 = 1$.

Solution

$$cx - z = 0$$

$$\vec{p} = \hat{k}$$

$$\nabla f = c\,\hat{i} - \hat{k}$$

$$\left|\nabla f\right| = \sqrt{c^2 + 1}$$

$$\left|\nabla f \bullet \vec{p}\right| = \left|\left(c\,\hat{i} - \hat{k}\right) \bullet \left(\hat{k}\right)\right|$$

$$\underline{= 1\,|}$$

$$Surface\ area = \iint_R \sqrt{c^2 + 1}\ dxdy \qquad\qquad Surface\ area = \iint_R \frac{\left|\nabla F\right|}{\left|\nabla F \bullet \vec{p}\right|}\, dA$$

$$= \iint_R \sqrt{c^2 + 1}\ dxdy$$

$$= \int_0^{2\pi} d\theta \int_0^1 \sqrt{c^2 + 1}\ r\, dr$$

$$= \pi\sqrt{c^2 + 1}\left(r^2 \right|_0^1$$

$$\underline{= \pi\sqrt{c^2 + 1} \quad unit^2\ |}$$

Exercise

Find the area of the surface cut from the nose of the paraboloid $x = 1 - y^2 - z^2$ by yz-plane.

Solution

$$f_y(y,z) = -2y, \quad f_z(y,z) = -2z$$

$$\sqrt{f_y^2 + f_z^2 + 1} = \sqrt{4y^2 + 4z^2 + 1}$$

$$Area = \iint_R \sqrt{4y^2 + 4z^2 + 1}\ dydz$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{4r^2 + 1}\ r\, dr d\theta \qquad\qquad d\left(4r^2 + 1\right) = 8rdr$$

$$= \frac{1}{8}\int_0^{2\pi} d\theta \int_0^1 \left(4r^2 + 1\right)^{1/2} d\left(4r^2 + 1\right)$$

$$= \frac{\pi}{6}\left(4r^2 + 1\right)^{3/2}\Big|_0^1$$

$$\underline{= \frac{\pi}{6}\left(5\sqrt{5} - 1\right) \quad unit^2\ |}$$

Exercise

Find the area of the surface in the first octant cut from the cylinder $y = \frac{2}{3}z^{3/2}$ by the planes $x = 1$ and $y = \frac{16}{3}$

Solution

$$y = \frac{2}{3}z^{3/2}$$

$$f_x(x,z) = 0, \quad f_z(x,z) = z^{1/2}$$

$$\sqrt{f_x^2 + f_z^2 + 1} = \sqrt{z+1}$$

$$y = \frac{2}{3}z^{3/2}$$
$$= \frac{16}{3}$$

$$\Rightarrow z^{3/2} = 8$$

$$z = 8^{2/3}$$
$$= 4$$

$$Area = \int_0^4 \int_0^1 \sqrt{z+1}\, dxdz$$

$$= \int_0^4 \left(x\sqrt{z+1} \,\Big|_0^1 \right) dz \qquad\qquad d(z+1) = dz$$

$$= \int_0^4 (z+1)^{1/2}\, d(z+1)$$

$$= (z+1)^{3/2} \Big|_0^4$$

$$= \frac{2}{3}\left(5^{3/2} - 1\right)$$

$$= \frac{2}{3}\left(5\sqrt{5} - 1\right) \quad unit^2$$

Exercise

Use a surface integral to find the area of the helicoid

$\vec{r}(r,\ \theta) = (r\cos\theta)\ \hat{i}\ + (r\sin\theta)\ \hat{j}\ + \theta\hat{k},\quad 0 \le \theta \le 2\pi,\quad 0 \le r \le 1$

Solution

$$\vec{r}_r = \cos\theta\,\hat{i}\ + \sin\theta\,\hat{j}$$

$$\vec{r}_\theta = -r\sin\theta\,\hat{i}\ + r\sin\theta\,\hat{j}\ + \hat{k}$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\sin\theta & 1 \end{vmatrix}$$

$$= \sin\theta\,\hat{i}\ - \cos\theta\,\hat{j}\ + r\,\hat{k}$$

$$\left|\vec{r}_r \times \vec{r}_\theta\right| = \sqrt{\sin^2\theta + \cos^2\theta + r^2}$$

$$= \sqrt{1+r^2}$$

$$Area = \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\ drd\theta \qquad\qquad A = \int_c^d\int_a^b \left|\boldsymbol{r}_u \times \boldsymbol{r}_v\right| dudv$$

$$= \int_0^{2\pi} d\theta\ \int_0^1 \left(1+r^2\right)^{1/2}\ dr$$

Let $r = \tan x \quad \to dr = \sec^2 x\ dx$

$\sqrt{1+r^2} = \sec x$

$\int \sqrt{1+r^2}\,dr = \int \sec^3 x\ dx$

Let: $\quad u = \sec x \qquad dv = \sec^2 xdx$

$\quad du = \sec x\tan xdx \qquad v = \tan x$

$$\int \sec^3 xdx = \sec x\tan x - \int \tan x\left(\sec x\tan xdx\right)$$

$$= \sec x\tan x - \int \tan^2 x\sec xdx$$

$$= \sec x\tan x - \int \left(\sec^2 x - 1\right)\sec x\ dx$$

$$= \sec x\tan x - \int \left(\sec^3 x - \sec x\right)\ dx$$

$$= \sec x\tan x - \int \sec^3 x\ dx + \int \sec x\ dx$$

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$
$$= \sec x \tan x + \ln\left|\sec x + \tan x\right| + C_1$$

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C$$

$$r = \tan x \qquad \sec x = \sqrt{1+r^2}$$

$$= 2\pi\left(\frac{r}{2}\sqrt{1+r^2} + \frac{1}{2}\ln\left|r + \sqrt{1+r^2}\right|\right|_0^1$$
$$\boxed{= \pi\left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right) \quad unit^2}$$

Exercise

Use a surface integral to find the area of the surface $f(x, y) = \sqrt{2}\, xy$ above the origin $\{(r, \theta):\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

Solution

$$f_z(x, y) = \sqrt{2}\, y \quad f_y(x, y) = \sqrt{2}\, x$$

$$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{2y^2 + 2x^2 + 1}$$
$$= \sqrt{2\left(y^2 + x^2\right) + 1}$$
$$= \sqrt{2r^2 + 1}$$

$$Area = \int_0^{2\pi}\int_0^2 \sqrt{2r^2+1}\; r\, dr d\theta \qquad\qquad Area = \iint_S 1\, dS$$
$$= \frac{1}{4}\int_0^{2\pi} d\theta \int_0^2 \left(2r^2+1\right)^{1/2} d\left(2r^2+1\right)$$
$$= \frac{1}{4}(2\pi)\ \frac{2}{3}\left(2r^2+1\right)^{3/2}\Big|_0^2$$
$$= \frac{\pi}{3}(27-1)$$
$$\boxed{= \frac{26\pi}{3} \quad unit^2}$$

Exercise

Use a surface integral to find the area of the hemisphere $x^2+y^2+z^2=9$, for $z\ge 0$ (excluding the base).

Solution

$$\vec{r}=\left\langle 3\sin\varphi\cos\theta,\ 3\sin\varphi\sin\theta,\ 3\cos\varphi\right\rangle$$

$$\vec{r}_{\varphi}=\left\langle 3\cos\varphi\cos\theta,\ 3\cos\varphi\sin\theta,\ -3\sin\varphi\right\rangle$$

$$\vec{r}_{\theta}=\left\langle -3\sin\varphi\sin\theta,\ 3\sin\varphi\cos\theta,\ 0\right\rangle$$

$$\vec{r}_{\varphi}\times\vec{r}_{\theta}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3\cos\varphi\cos\theta & 3\cos\varphi\sin\theta & -3\sin\varphi\\ -3\sin\varphi\sin\theta & 3\sin\varphi\cos\theta & 0\end{vmatrix}$$

$$=9\sin^2\varphi\cos\theta\,\hat{i}+9\sin^2\varphi\sin\theta\,\hat{j}+\left(9\sin\varphi\cos\varphi\cos^2\theta+9\sin\varphi\cos\varphi\sin^2\theta\right)\hat{k}$$

$$=9\sin^2\varphi\cos\theta\,\hat{i}+9\sin^2\varphi\sin\theta\,\hat{j}+9\sin\varphi\cos\varphi\,\hat{k}$$

$$\left|\vec{r}_{\varphi}\times\vec{r}_{\theta}\right|=\sqrt{81\sin^4\varphi\cos^2\theta+81\sin^4\varphi\sin^2\theta+81\sin^2\varphi\cos^2\varphi}$$

$$=9\sqrt{\sin^4\varphi\left(\cos^2\theta+\sin^2\theta\right)+\sin^2\varphi\cos^2\varphi}$$

$$=9\sqrt{\sin^4\varphi+\sin^2\varphi\cos^2\varphi}$$

$$=9\sqrt{\sin^2\varphi\left(\sin^2\varphi+\cos^2\varphi\right)}$$

$$=9\sqrt{\sin^2\varphi}$$

$$=9\sin\varphi$$

$$S=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}9\sin\varphi\ d\varphi d\theta$$

$$=-9\ \left(\cos\varphi\ \Big|_0^{\frac{\pi}{2}}\right.\ \int_0^{2\pi}d\theta$$

$$=-9(-1)(2\pi)$$

$$=18\pi\ \ unit^2$$

Exercise

Use a surface integral to find the area of the frustum of the cone $z^2 = x^2 + y^2$, for $2 \le z \le 4$ (excluding the bases).

Solution

$$\vec{r} = \langle v\cos u,\ v\sin u,\ v\rangle$$

$$\vec{r}_u = \langle -v\sin u,\ v\cos u,\ 0\rangle$$

$$\vec{r}_v = \langle \cos u,\ \sin u,\ 1\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix}$$

$$= \langle v\cos u,\ v\sin u,\ -v\rangle$$

$$\left|\vec{r}_u \times \vec{r}_v\right| = \sqrt{v^2\cos^2 u + v^2\sin^2 u + v^2}$$

$$= \sqrt{2v^2}$$

$$= v\sqrt{2}$$

$$S = \sqrt{2}\int_0^{2\pi} du \int_2^4 v\, dv$$

$$= \sqrt{2}(2\pi)\left(\tfrac{1}{2}v^2\right)\Big|_2^4$$

$$= \pi\sqrt{2}(16-4)$$

$$= 12\pi\sqrt{2}\ \ unit^2$$

Exercise

Use a surface integral to find the area of the plane $z = 6 - x - y$ above the square $|x| \le 1$, $|y| \le 1$.

Solution

$$z_x = -1 \quad z_y = -1$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{1+1+1}$$

$$= \sqrt{3}$$

$$Area = \int_{-1}^{1}\int_{-1}^{1} \sqrt{3}\ dxdy$$

$$= \sqrt{3}\int_{-1}^{1} dx \int_{-1}^{1} dy$$

$$= \sqrt{3}\; x \Big|_{-1}^{1} \; y \Big|_{-1}^{1}$$

$$\underline{= 4\sqrt{3} \quad unit^2}$$

Exercise

Use a surface integral to find the area of: The cone $z^2 = 4\left(x^2 + y^2\right), \quad 0 \le z \le 4$

Solution

$$z^2 = 4x^2 + 4y^2$$

$$2zdz = 8xdx \quad \to \quad z_x = \frac{4x}{z}$$

$$2zdz = 8ydy \quad \to \quad z_y = \frac{4y}{z}$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{\frac{16x^2}{z^2} + \frac{16y^2}{z^2} + 1}$$

$$= \sqrt{\frac{16x^2 + 16y^2 + z^2}{z^2}}$$

$$= \sqrt{\frac{16x^2 + 16y^2 + 4x^2 + 4y^2}{4x^2 + 4y^2}}$$

$$= \sqrt{\frac{20\left(x^2 + y^2\right)}{4\left(x^2 + y^2\right)}}$$

$$\underline{= \sqrt{5}}$$

$$Area = \iint_R \sqrt{5}\; dA$$

$$\iint_R dA = area\ of\ the\ circle\ radius = 2$$

$$= \pi\sqrt{5}\left(\pi(2)^2\right)$$

$$\underline{= 4\pi\sqrt{5} \quad unit^2}$$

Exercise

Use a surface integral to find the area of: The paraboloid $z = 2\left(x^2 + y^2\right), \quad 0 \le z \le 8$

Solution

$$z = 2x^2 + 2y^2$$

$$z_x = 4x \quad z_y = 4y$$

$$\begin{aligned}\sqrt{z_x^2 + z_y^2 + 1} &= \sqrt{16x^2 + 16y^2 + 1} \\ &= \sqrt{16\left(x^2 + y^2\right) + 1} \\ &= \sqrt{16r^2 + 1}\end{aligned}$$

$$z = 2\left(x^2 + y^2\right) = 8 \quad \rightarrow \quad x^2 + y^2 = 4 = r^2$$

$$0 \le r \le 2 \quad 0 \le \theta \le 2\pi$$

$$\begin{aligned}Area &= \int_0^{2\pi} d\theta \int_0^2 \sqrt{16r^2 + 1} \ r\, dr \\ &= 2\pi \int_0^2 \frac{1}{32}\left(16r^2 + 1\right)^{1/2} d\left(16r^2 + 1\right) \\ &= \frac{\pi}{24}\left(16r^2 + 1\right)^{3/2} \Big|_0^2 \\ &= \frac{\pi}{24}\left(65^{3/2} - 1\right) \\ &= \frac{\pi}{24}\left(65\sqrt{65} - 1\right) \quad unit^2\end{aligned}$$

Exercise

Use a surface integral to find the area of: The trough $z = x^2, \quad -2 \le x \le 2, \quad 0 \le y \le 4$

Solution

$$z_x = 2x \quad z_y = 0$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{4x^2 + 1}$$

$$Area = \int_0^4 \int_{-2}^2 \sqrt{4x^2 + 1} \ dxdy$$

$$= \int_0^4 dy \int_{-2}^{2} \sqrt{4x^2+1}\ dx \qquad\qquad \int \sqrt{a^2+x^2}\ dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|$$

$$2x = \tan\alpha \qquad \sqrt{4x^2+1} = \sec\alpha$$

$$2dx = \sec^2\alpha\, d\alpha$$

$$\int \sqrt{4x^2+1}\, dx = \int \sec\alpha\ \frac{1}{2}\sec^2\alpha\, d\alpha$$

$$= \frac{1}{2}\int \sec^3\alpha\, d\alpha$$

Let: $\quad u = \sec\alpha \qquad dv = \sec^2\alpha d\alpha$

$\quad du = \sec\alpha\tan\alpha d\alpha \qquad v = \tan\alpha$

$$\int \sec^3\alpha d\alpha = \sec x\tan\alpha - \int \tan\alpha\left(\sec x\tan x dx\right)$$

$$= \sec\alpha\tan\alpha - \int \tan^2\alpha\sec\alpha\, d\alpha$$

$$= \sec\alpha\tan\alpha - \int \left(\sec^2\alpha - 1\right)\sec\alpha\ d\alpha$$

$$= \sec\alpha\tan\alpha - \int \left(\sec^3\alpha - \sec\alpha\right)\ d\alpha$$

$$= \sec\alpha\tan\alpha - \int \sec^3\alpha\ d\alpha + \int \sec\alpha\ d\alpha$$

$$2\int \sec^3\alpha\ d\alpha = \sec\alpha\tan\alpha + \int \sec\alpha\ d\alpha$$

$$= \sec\alpha\tan\alpha + \ln\left|\sec\alpha + \tan\alpha\right|$$

$$\int \sec^3\alpha\ d\alpha = \frac{1}{2}\sec\alpha\tan\alpha + \frac{1}{2}\ln\left|\sec\alpha + \tan\alpha\right|$$

$$= \frac{1}{2}\sqrt{4x^2+1}\ (2x)\ +\ \frac{1}{2}\ln\left|\sqrt{4x^2+1}\ +\ 2x\right|$$

$$= x\sqrt{4x^2+1}\ +\ \frac{1}{2}\ln\left|2x+\sqrt{4x^2+1}\ \right|$$

$$= (4)\ \left(\frac{x}{2}\sqrt{4x^2+1}\ +\ \frac{1}{2}\ln\left|2x+\sqrt{4x^2+1}\right|\ \Big|_{-2}^{2}\right.$$

$$= 4\ \left(\sqrt{17} + \frac{1}{2}\ln\left|4+\sqrt{17}\right| + \sqrt{17} - \frac{1}{2}\ln\left|-4+\sqrt{17}\right|\right)$$

$$= 8\sqrt{17} + 2\ln\left|4+\sqrt{17}\right| - 2\ln\left|-4+\sqrt{17}\right|$$

$$= 8\sqrt{17} + \ln\left(4+\sqrt{17}\right)^2 - \ln\left(\sqrt{17}-4\right)^2$$

$$= 8\sqrt{17} + \ln\left(\frac{4+\sqrt{17}}{2}\right) - \ln\left(\frac{\sqrt{17}-4}{2}\right)$$

$$= 8\sqrt{17} + \ln\left(\frac{\sqrt{17}+4}{\sqrt{17}-4}\right)$$

$$= 8\sqrt{17} + \ln\left(\frac{\sqrt{17}+4}{\sqrt{17}-4}\cdot\frac{\sqrt{17}+4}{\sqrt{17}+4}\right)$$

$$= 8\sqrt{17} + \ln\left(4+\sqrt{17}\right)^2$$

$$= 8\sqrt{17} + 2\ln\left(4+\sqrt{17}\right) \quad unit^2$$

Exercise

Use a surface integral to find the area of: The part of the hyperbolic paraboloid $z = x^2 - y^2$ above the sector $R = \left\{(r,\ \theta):\ \ 0 \le r \le 4,\ \ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}\right\}$

Solution

$$z_x = 2x \quad z_y = -2y$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1}$$

$$= \sqrt{4r^2 + 1}$$

$$Area = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \int_0^4 \sqrt{4r^2+1}\ r\,dr$$

$$= \frac{\pi}{2}\int_0^4 \frac{1}{8}\left(4r^2+1\right)^{1/2} d\left(4r^2+1\right)$$

$$= \frac{\pi}{24}\left(4r^2+1\right)^{3/2} \Big|_0^4$$

$$= \frac{\pi}{24}\left(65^{3/2} - 1\right)$$

$$= \frac{\pi}{24}\left(65\sqrt{65} - 1\right) \quad unit^2$$

Exercise

Use a surface integral to find the area of: $f(x, y, z) = xy$, where S is the plane $z = 2 - x - y$ in the first octant

Solution

$$z_x = -1 \quad z_y = -1$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{1+1+1}$$

$$= \sqrt{3}$$

$$z = 2 - x - y = 0 \quad \rightarrow \quad y = 2 - x$$

$$y = 2 - x = 0 \quad \rightarrow \quad x = 2$$

First octant: $0 \le y \le 2 - x \quad 0 \le x \le 2$

$$Area = \int_0^2 \int_0^{2-x} \sqrt{3}xy \ dydx$$

$$= \frac{\sqrt{3}}{2} \int_0^2 x\left(y^2 \Big|_0^{2-x} dx\right.$$

$$= \frac{\sqrt{3}}{2} \int_0^2 \left(4x - 4x^2 + x^3\right) dx$$

$$= \frac{\sqrt{3}}{2} \left(2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4 \Big|_0^2\right.$$

$$= \frac{\sqrt{3}}{2} \left(8 - \frac{32}{3} + 4\right)$$

$$= \frac{\sqrt{3}}{2} \left(\frac{4}{3}\right)$$

$$= \frac{2\sqrt{3}}{3} \quad unit^2$$

Exercise

Use a surface integral to find the area of: $f(x, y, z) = x^2 + y^2$, where S is the paraboloid $z = x^2 + y^2, \quad 0 \le z \le 4$

Solution

$$z_x = 2x \quad z_y = 2y$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1}$$

$$= \sqrt{4r^2 + 1}$$

$$z = x^2 + y^2 = r^2 = 0 \quad \to \quad r = 0$$

$$z = x^2 + y^2 = r^2 = 4 \quad \to \quad r = 2$$

$$0 \le r \le 2 \quad 0 \le \theta \le 2\pi$$

$$Area = \iint\limits_R \sqrt{4r^2 + 1}\left(x^2 + y^2\right) dA$$

$$= \int_0^{2\pi} \int_0^2 r^2 \sqrt{4r^2 + 1}\ r\ drd\theta$$

Let $u = 4r^2 + 1 \quad \to \quad du = 8rdr$

$$r^2 = \frac{1}{4}(u - 1)$$

$$\begin{cases} r = 2 & \to u = 17 \\ r = 0 & \to u = 1 \end{cases}$$

$$= \int_0^{2\pi} d\theta \int_1^{17} \frac{1}{4}(u-1)u^{1/2}\frac{1}{8}du$$

$$= \frac{1}{32}(2\pi)\int_1^{17}\left(u^{3/2} - u^{1/2}\right)du$$

$$= \frac{\pi}{16}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right.\Bigg|_1^{17}$$

$$= \frac{\pi}{16}\left(\frac{2}{5}17^2\sqrt{17} - \frac{2}{3}17\sqrt{17} - \frac{2}{5} + \frac{2}{3}\right)$$

$$= \frac{\pi}{16}\frac{1}{15}\left((1734 - 170)\sqrt{17} + 4\right)$$

$$= \frac{\pi}{240}\left(1564\sqrt{17} + 4\right)$$

$$= \frac{\pi}{60}\left(391\sqrt{17} + 1\right) \quad unit^2$$

Exercise

Use a surface integral to find the area of: $f(x, y, z) = 25 - x^2 - y^2$, where S is the hemisphere centered at the origin with radius 5, for $z \geq 0$

Solution

S is the hemisphere with radius 5: $x^2 + y^2 + z^2 = 25$

$2xdx + 2zdz = 0 \quad z_x = -\frac{x}{z}$

$2ydy + 2zdz = 0 \quad z_y = -\frac{y}{z}$

$$\begin{aligned}\sqrt{z_x^2 + z_y^2 + 1} &= \sqrt{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1} \\ &= \sqrt{\frac{x^2 + y^2 + z^2}{z^2}} \\ &= \sqrt{\frac{25}{z^2}} \\ &= \frac{5}{z}\end{aligned}$$

$0 \leq r \leq 5 \quad 0 \leq \theta \leq 2\pi$

$$\begin{aligned}Area &= \iint_R \frac{5}{\sqrt{25 - x^2 - y^2}} \left(25 - x^2 - y^2\right) dA \\ &= 5 \iint_R \sqrt{25 - x^2 - y^2}\ dA \\ &= 5 \int_0^{2\pi} d\theta \int_0^5 \sqrt{25 - r^2}\ r\ dr \\ &= -5\pi \int_0^5 \left(25 - r^2\right)^{1/2} d\left(25 - r^2\right) \\ &= -5\pi \left(\frac{2}{3}\right) \left(25 - r^2\right)^{3/2} \Big|_0^5 \\ &= -\frac{10\pi}{3}(0 - 125) \\ &= \frac{1250\pi}{3} \quad unit^2\end{aligned}$$

Exercise

Use a surface integral to find the area of: $f(x, y, z) = e^x$, where S is the plane $z = 8 - x - 2y$ in the first octant

Solution

$$z_x = -1 \quad z_y = -2$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{1+4+1}$$

$$= \sqrt{6}$$

$$z = 8 - x - 2y = 0 \quad \rightarrow \quad x = 8 - 2y$$

$$x = 8 - 2y = 0 \quad \rightarrow \quad y = 4$$

First octant: $0 \le y \le 4 \quad 0 \le x \le 8 - 2y$

$$Area = \int_0^4 \int_0^{8-2y} \sqrt{6}\, e^x \, dxdy$$

$$= \sqrt{6} \int_0^4 e^x \Big|_0^{8-2y} dy$$

$$= \sqrt{6} \int_0^4 \left(e^{8-2y} - 1\right) dy$$

$$= \sqrt{6} \left(-\frac{1}{2} e^{8-2y} - y \right|_0^4$$

$$= \sqrt{6} \left(-\frac{1}{2} - 4 + \frac{1}{2} e^8\right)$$

$$= \frac{\sqrt{6}}{2}\left(e^8 - 9\right) \quad unit^2$$

Exercise

Use a surface integral to find the area of: $f(x, y, z) = e^z$, where S is the plane $z = 8 - x - 2y$ in the first octant

Solution

$$z_x = -1 \quad z_y = -2$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{1+4+1}$$

$$= \sqrt{6}$$

$z = 8 - x - 2y = 0 \quad \to \quad x = 8 - 2y$

$x = 8 - 2y = 0 \quad \to \quad y = 4$

First octant: $0 \le y \le 4 \quad 0 \le x \le 8 - 2y$

$$Area = \int_0^4 \int_0^{8-2y} \sqrt{6}\, e^z \ dxdy$$

$$= \sqrt{6} \int_0^4 \int_0^{8-2y} e^{8-x-2y} \ dxdy$$

$$= \sqrt{6}\, e^8 \int_0^4 \int_0^{8-2y} e^{-2y}\, e^{-x} \ dxdy$$

$$= -\sqrt{6}\, e^8 \int_0^4 e^{-2y}\, e^{-x} \Big|_0^{8-2y} \ dy$$

$$= -\sqrt{6}\, e^8 \int_0^4 e^{-2y} \left(e^{2y-8} - 1\right) dy$$

$$= -\sqrt{6}\, e^8 \int_0^4 \left(e^{-8} - e^{-2y}\right) dy$$

$$= -\sqrt{6}\, e^8 \left(e^{-8} y + \frac{1}{2} e^{-2y}\right) \Big|_0^4$$

$$= -\sqrt{6}\, e^8 \left(4e^{-8} + \frac{1}{2} e^{-8} - \frac{1}{2}\right)$$

$$= -\sqrt{6} e^8 \left(\frac{9}{2} e^{-8} - \frac{1}{2}\right)$$

$$= \frac{\sqrt{6}}{2}\left(e^8 - 9\right) \quad unit^2$$

Exercise

Evaluate the surface integral $\iint_S (1 + yz)\, dS$; S is the plane $x + y + z = 2$ in the first octant.

Solution

$z = 2 - x - y$

$z_x = -1 \quad z_y = -1$

$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{1 + 1 + 1}$

$= \sqrt{3}$

$$z = 2 - x - y = 0$$

$$\to \begin{cases} y = 2 - x \\ y = 0 \end{cases}$$

$$y = 0 \quad \to \quad 0 \le x \le 2$$

$$\iint_S (1 + yz)\,dS = \sqrt{3}\iint_R (1 + yz)\,dA$$

$$= \sqrt{3}\int_0^2 \int_0^{2-x} \left(1 + y(2 - x - y)\right) dydx$$

$$= \sqrt{3}\int_0^2 \int_0^{2-x} \left(1 + 2y - xy - y^2\right) dydx$$

$$= \sqrt{3}\int_0^2 \left(y + y^2 - \frac{1}{2}xy^2 - \frac{1}{3}y^3 \, \Big|_0^{2-x} \, dx\right.$$

$$= \sqrt{3}\int_0^2 \left(2 - x + 4 - 4x + x^2 - 2x + 2x^2 - \frac{1}{2}x^3 - \frac{8}{3} + 4x - 2x^2 + \frac{1}{3}x^3\right) dx$$

$$= \sqrt{3}\int_0^2 \left(\frac{10}{3} - 3x + x^2 - \frac{1}{6}x^3\right) dx$$

$$= \sqrt{3}\left(\frac{10}{3}x - \frac{3}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{24}x^4 \, \Big|_0^2\right.$$

$$= \sqrt{3}\left(\frac{20}{3} - 6 + \frac{8}{3} - \frac{2}{3}\right)$$

$$= \frac{8\sqrt{3}}{3}$$

Exercise

Evaluate the surface integral $\iint_S \langle 0, y, z\rangle \cdot \vec{n}\; dS$; S is the curve surface of the cylinder $y^2 + z^2 = a^2$, $|x| \le 8$ with outward normal vectors.

Solution

$$\vec{n} = \langle 0,\ y,\ z\rangle$$

$$\iint_S \langle 0,\ y,\ z\rangle \cdot \vec{n}\; dS = a\iint_R \langle 0,\ y,\ z\rangle \cdot \langle 0,\ y,\ z\rangle\, dA$$

$$= a \iint_R \left(y^2 + z^2\right) dA$$

$$= a^3 \iint_R dA$$

$$\iint_R dA = \text{area of the circle radius } \frac{8}{2} = 4$$

$$= a^3 \left(2\pi 4^2\right)$$

$$= 32\pi a^3$$

Exercise

Evaluate the surface integral $\iint\limits_S (x - y + z)\, dS$; S is the entire surface including the base of the hemisphere $x^2 + y^2 + z^2 = 4$, for $z \ge 0$.

Solution

$$\vec{r} = \langle 2\sin\varphi\cos\theta,\ 2\sin\varphi\sin\theta,\ 2\cos\varphi\rangle$$

$$\vec{r}_\varphi = \langle 2\cos\varphi\cos\theta,\ 2\cos\varphi\sin\theta,\ -2\sin\varphi\rangle$$

$$\vec{r}_\theta = \langle -2\sin\varphi\sin\theta,\ 2\sin\varphi\cos\theta,\ 0\rangle$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos\varphi\cos\theta & 2\cos\varphi\sin\theta & -2\sin\varphi \\ -2\sin\varphi\sin\theta & 2\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= 4\sin^2\varphi\cos\theta\ \hat{i} + 4\sin^2\varphi\sin\theta\ \hat{j} + \left(4\sin\varphi\cos\varphi\cos^2\theta + 4\sin\varphi\cos\varphi\sin^2\theta\right)\hat{k}$$

$$= 4\sin^2\varphi\cos\theta\,\hat{i} + 4\sin^2\varphi\sin\theta\,\hat{j} + 4\sin\varphi\cos\varphi\,\hat{k}$$

$$\left|\vec{r}_\varphi \times \vec{r}_\theta\right| = \sqrt{16\sin^4\varphi\cos^2\theta + 16\sin^4\varphi\sin^2\theta + 16\sin^2\varphi\cos^2\varphi}$$

$$= 4\sqrt{\sin^4\varphi\left(\cos^2\theta + \sin^2\theta\right) + \sin^2\varphi\cos^2\varphi}$$

$$= 4\sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi}$$

$$= 4\sqrt{\sin^2\varphi\left(\sin^2\varphi + \cos^2\varphi\right)}$$

$$= 4\sqrt{\sin^2\varphi}$$

$$= 4\sin\varphi$$

$$\iint_S (x-y+z)\,dS = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} (2\sin\varphi\cos\theta - 2\sin\varphi\sin\theta + 2\cos\varphi)(4\sin\varphi)\,d\varphi\,d\theta$$

$$= 8\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left(\sin^2\varphi\cos\theta - \sin^2\varphi\sin\theta + \sin\varphi\cos\varphi\right)d\varphi\,d\theta$$

$$= 8\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left((\cos\theta - \sin\theta)\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\varphi\right) + \tfrac{1}{2}\sin 2\varphi\right)d\varphi\,d\theta$$

$$= 8\int_0^{2\pi} \left((\cos\theta - \sin\theta)\left(\tfrac{1}{2}\varphi - \tfrac{1}{4}\sin 2\varphi\right) - \tfrac{1}{4}\cos 2\varphi \,\Big|_0^{\frac{\pi}{2}}\right) d\theta$$

$$= 8\int_0^{2\pi} \left(\tfrac{\pi}{4}(\cos\theta - \sin\theta) + \tfrac{1}{4} + \tfrac{1}{4}\right)d\theta$$

$$= 8\,\left(\tfrac{\pi}{4}(\sin\theta + \cos\theta) + \tfrac{1}{2}\theta \,\Big|_0^{2\pi}\right.$$

$$= 8\left(\tfrac{\pi}{4} + \pi - \tfrac{\pi}{4}\right)$$

$$= 8\pi$$

Exercise

Evaluate $\iint_S \nabla \ln|\vec{r}| \bullet \vec{n}\; dS$, where S is the hemisphere $x^2 + y^2 + z^2 = a^2$, for $z \ge 0$, and where $\vec{r} = \langle x,\ y,\ z\rangle$. Assume normal vectors point either outward or in the positive z-direction.

Solution

$$\nabla \ln|\vec{r}| = \nabla \ln\sqrt{x^2 + y^2 + z^2}$$

$$= \frac{1}{x^2 + y^2 + z^2}\langle x,\ y,\ z\rangle \qquad x^2 + y^2 + z^2 = a^2$$

$$= \frac{1}{a^2}\langle x,\ y,\ z\rangle$$

$$2zdz + 2xdx = 0 \quad \rightarrow \quad z_x = -\frac{x}{z}$$

$$2zdz = 2ydy \quad \rightarrow \quad z_y = -\frac{y}{z}$$

Since the normal vector point either outward or in the positive z-direction

$$\vec{n} = \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle$$

$$\iint_S \nabla \ln|\vec{r}| \cdot \vec{n}\ dS = \iint_R \frac{1}{a^2}\langle x,\ y,\ z\rangle \cdot \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle dA$$

$$= \frac{1}{a^2} \iint_R \left(\frac{x^2}{z} + \frac{y^2}{z} + z \right) dA$$

$$= \frac{1}{a^2} \iint_R \left(\frac{x^2 + y^2 + z^2}{z} \right) dA$$

$$= \frac{1}{a^2} \iint_R \left(\frac{a^2}{z} \right) dA$$

$$= \iint_R \frac{1}{z}\ dA$$

$$= \iint_R \frac{1}{\sqrt{a^2 - x^2 - y^2}}\ dA$$

$$= \int_0^{2\pi} \int_0^a \frac{1}{\sqrt{a^2 - r^2}}\ r\ dr d\theta$$

$$= -\frac{1}{2} \int_0^{2\pi} d\theta \int_0^a \left(a^2 - r^2\right)^{-1/2} d\left(a^2 - r^2\right)$$

$$= -\pi(2) \left(a^2 - r^2\right)^{1/2} \Big|_0^a$$

$$= -2\pi(0 - a)$$

$$\underline{= 2\pi a}$$

Exercise

Evaluate $\iint_S |\vec{r}|\ dS$, where S is the cylinder $x^2 + y^2 = 4$, for $0 \le z \le 8$, and where $\vec{r} = \langle x,\ y,\ z \rangle$

Assume normal vectors point either outward or in the positive z-direction.

Solution

Parametrize the surface:

$$\vec{r}(u,\ v) = \langle 2\cos u,\ 2\sin u,\ v\rangle$$

$$\vec{r}_u = \langle -2\sin u,\ 2\cos u,\ 0\rangle$$

$$\vec{r}_v = \langle 0,\ 0,\ 1\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle 2\cos u,\ 2\sin u,\ 0\rangle$$

$$\left|\vec{r}_u \times \vec{r}_v\right| = \sqrt{4\cos^2 u + 4\sin^2 u}$$

$$\underline{= 2\,|}$$

$$0 \le z = v \le 8 \quad 0 \le u \le 2\pi$$

$$\iint_S |\vec{r}|\ dS = 2\iint_R \sqrt{x^2 + y^2 + z^2}\ dA$$

$$= 2\iint_R \sqrt{4 + z^2}\ dA$$

$$= 2\int_0^{2\pi} du \int_0^8 \sqrt{4 + v^2}\ dv$$

$$x = 2\tan\alpha \qquad \sqrt{v^2 + 4} = 2\sec\alpha$$

$$dx = 2\sec^2\alpha\, d\alpha$$

$$\int \sqrt{v^2 + 4}\, dx = \int 2\sec\alpha\ \left(2\sec^2\alpha\right)\ d\alpha$$

$$= 4\int \sec^3\alpha\ d\alpha$$

Let: $\quad u = \sec\alpha \qquad dv = \sec^2\alpha\, d\alpha$

$\quad du = \sec\alpha\tan\alpha\, d\alpha \qquad v = \tan\alpha$

$$\int \sec^3\alpha\, d\alpha = \sec x\tan\alpha - \int \tan\alpha\left(\sec x\tan x\, dx\right)$$

$$= \sec\alpha\tan\alpha - \int \tan^2\alpha\sec\alpha\, d\alpha$$

$$= \sec\alpha\tan\alpha - \int \left(\sec^2\alpha - 1\right)\sec\alpha\ d\alpha$$

$$= \sec\alpha\tan\alpha - \int \left(\sec^3\alpha - \sec\alpha\right)\ d\alpha$$

$$= \sec\alpha\tan\alpha - \int \sec^3\alpha\, d\alpha + \int \sec\alpha\, d\alpha$$

$$2\int \sec^3\alpha\, d\alpha = \sec\alpha\tan\alpha + \int \sec\alpha\, d\alpha$$

$$= \sec\alpha\tan\alpha + \ln\left|\sec\alpha + \tan\alpha\right|$$

$$\int \sec^3\alpha\, d\alpha = \frac{1}{2}\sec\alpha\tan\alpha + \frac{1}{2}\ln\left|\sec\alpha + \tan\alpha\right|$$

$$= 4\pi\ (4)\left(\frac{1}{2}\frac{v}{2}\frac{\sqrt{4+v^2}}{2} + \frac{1}{2}\ln\left|\frac{1}{2}v + \frac{1}{2}\sqrt{4+v^2}\right|\ \right|_0^8$$

$$= 8\pi\ \left(\frac{1}{4}v\sqrt{4+v^2} + \ \ln\left|\frac{1}{2}v + \frac{1}{2}\sqrt{4+v^2}\right|\ \right|_0^8$$

$$= 8\pi\ \left(2\sqrt{68} + \ln\left(4 + \frac{1}{2}\sqrt{68}\right) - \ln 1\right)$$

$$= 8\pi\left(4\sqrt{17} + \ln\left(4 + \sqrt{17}\right)\right)$$

Exercise

Evaluate $\iint_S xyz\ dS$, where S is the part of the plane $z = 6 - y$ that lies on the cylinder $x^2 + y^2 = 4$

Assume normal vectors point either outward or in the positive z-direction.

Solution

$$z = 6 - y$$

$$z_x = 0 \quad z_y = -1$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{0+1+1}$$

$$= \sqrt{2}$$

$$\iint_S xyz\ dS = \sqrt{2}\iint_R xyz\ dA$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^2 (r\cos\theta)(r\sin\theta)(6 - r\sin\theta)\ r dr d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^2 \left(6r^3\cos\theta\sin\theta - r^4\cos\theta\sin^2\theta\right)\ dr d\theta$$

$$=\sqrt{2}\int_0^{2\pi}\left(\frac{3}{2}r^4\cos\theta\sin\theta-\frac{1}{5}r^5\cos\theta\sin^2\theta\ \Big|_0^2\ d\theta\right.$$

$$=\sqrt{2}\int_0^{2\pi}\left(12\sin 2\theta-\frac{32}{5}\cos\theta\sin^2\theta\right)d\theta$$

$$=12\sqrt{2}\int_0^{2\pi}\sin 2\theta\,d\theta\ -\ \frac{32\sqrt{2}}{5}\int_0^{2\pi}\sin^2\theta\,d(\sin\theta)$$

$$=-2\sqrt{2}\left(3\cos 2\theta+\frac{16}{15}\sin^3\theta\ \Big|_0^{2\pi}\right.$$

$$=-2\sqrt{2}(3-3)$$

$$\underline{=0\ }|$$

Exercise

Evaluate $\displaystyle\iint_S \frac{\langle x,\ 0,\ z\rangle}{\sqrt{x^2+z^2}}\bullet\vec{n}\ dS$, where S is the cylinder $x^2+z^2=a^2,\quad |y|\le 2$. Assume normal vectors point either outward or in the positive z-direction.

Solution

$$\vec{n}=\langle x,\ 0,\ z\rangle$$

$$\iint_S \frac{\langle x,\ 0,\ z\rangle}{\sqrt{x^2+z^2}}\bullet\vec{n}\ dS=\iint_S \frac{\langle x,\ 0,\ z\rangle\bullet\langle x,\ 0,\ z\rangle}{\sqrt{x^2+z^2}}\,dS$$

$$=\iint_R \frac{x^2+z^2}{\sqrt{x^2+z^2}}\,dA$$

$$=\iint_R \sqrt{x^2+z^2}\ dA$$

$$=\iint_R a\ dA$$

$$=a\int_0^{2\pi}du\int_{-2}^{2}dv$$

$$=a(2\pi)(4)$$

$$\underline{=8\pi a\ }|$$

Exercise

Evaluate the surface integral $\iint_S f(x, y, z)\,dS$: $f(x, y, z) = x^2 + y^2$, where S is the hemisphere $x^2 + y^2 + z^2 = 36, \quad z \ge 0$

Solution

$$\vec{r} = \langle 6\sin\varphi\cos\theta,\ 6\sin\varphi\sin\theta,\ 6\cos\varphi \rangle$$

$$\vec{r}_\varphi = \langle 6\cos\varphi\cos\theta,\ 6\cos\varphi\sin\theta,\ -6\sin\varphi \rangle$$

$$\vec{r}_\theta = \langle -6\sin\varphi\sin\theta,\ 6\sin\varphi\cos\theta,\ 0 \rangle$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6\cos\varphi\cos\theta & 6\cos\varphi\sin\theta & -6\sin\varphi \\ -6\sin\varphi\sin\theta & 6\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= 36\sin^2\varphi\cos\theta\ \hat{i} + 36\sin^2\varphi\sin\theta\ \hat{j} + \left(36\sin\varphi\cos\varphi\cos^2\theta + 36\sin\varphi\cos\varphi\sin^2\theta\right)\hat{k}$$

$$= 36\sin^2\varphi\cos\theta\ \hat{i} + 36\sin^2\varphi\sin\theta\ \hat{j} + 36\sin\varphi\cos\varphi\,\hat{k}$$

$$\left|\vec{r}_\varphi \times \vec{r}_\theta\right| = \sqrt{36^2\sin^4\varphi\cos^2\theta + 36^2\sin^4\varphi\sin^2\theta + 36^2\sin^2\varphi\cos^2\varphi}$$

$$= 36\sqrt{\sin^4\varphi\left(\cos^2\theta + \sin^2\theta\right) + \sin^2\varphi\cos^2\varphi}$$

$$= 36\sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi}$$

$$= 36\sqrt{\sin^2\varphi\left(\sin^2\varphi + \cos^2\varphi\right)}$$

$$= 36\sqrt{\sin^2\varphi}$$

$$= 36\sin\varphi$$

$$\iint_S f(x, y, z)\,dS = \iint_S \left(x^2 + y^2\right)dS$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left(36\sin^2\varphi\cos^2\theta + 36\sin^2\varphi\sin^2\theta\right)\left(36\sin\varphi\right)\,d\varphi\,d\theta$$

$$= 1,296\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin^2\varphi\left(\cos^2\theta + \sin^2\theta\right)\left(\sin\varphi\right)\,d\varphi\,d\theta$$

$$= 1,296\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin^3\varphi\,d\varphi$$

$$= 1,296\pi \int_0^{\frac{\pi}{2}} -\left(1 - \cos^2\varphi\right) d\left(\cos\varphi\right)$$

$$= 1,296\pi \left(\frac{1}{3}\cos^3\varphi - \cos\varphi\right|_0^{\frac{\pi}{2}}$$

$$= 1,296\pi \left(\frac{1}{3} - 1\right)$$

$$= 1,728\pi$$

Exercise

Evaluate the surface integral $\iint_S f(x, y, z)\, dS$; $f(x, y, z) = y$, where S is the cylinder $x^2 + y^2 = 9, \quad 0 \le z \le 3$

Solution

Parametrize the surface:

$$\vec{r}(u, v) = \langle 3\cos u,\ 3\sin u,\ v\rangle$$

$$\vec{r}_u = \langle -3\sin u,\ 3\cos u,\ 0\rangle$$

$$\vec{r}_v = \langle 0,\ 0,\ 1\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\sin u & 3\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle 3\cos u,\ 3\sin u,\ 0\rangle$$

$$\left|\vec{r}_u \times \vec{r}_v\right| = \sqrt{9\cos^2 u + 9\sin^2 u}$$

$$= 3$$

$0 \le z = v \le 3 \quad 0 \le u \le 2\pi$

$$\iint_S f(x, y, z)\, dS = \iint_S y\, dS$$

$$= \int_0^3 dv \int_0^{2\pi} 3(3\sin u)\, du$$

$$= -9(3)\left(\cos u\right|_0^{2\pi}$$

$$= 0$$

Exercise

Evaluate the surface integral $\iint_S f(x, y, z)\,dS$; $f(x, y, z) = x$, where S is the cylinder $x^2 + z^2 = 1, \quad 0 \le y \le 3$

Solution

$$\vec{r}(u, v) = \langle \cos u,\ v,\ \sin u \rangle$$

$$\vec{r}_u = \langle -\sin u,\ 0,\ \cos u \rangle$$

$$\vec{r}_v = \langle 0,\ 1,\ 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin u & 0 & \cos u \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \langle -\cos u,\ 0,\ -\sin u \rangle$$

$$\left| \vec{r}_u \times \vec{r}_v \right| = \sqrt{\cos^2 u + \sin^2 u}$$

$$\underline{= 1|}$$

$$0 \le y = v \le 3 \quad 0 \le u \le 2\pi$$

$$\iint_S f(x, y, z)\,dS = \iint_S x\,dS$$

$$= \int_0^3 dv \int_0^{2\pi} \cos u\,du$$

$$= 3\left(\sin u \,\Big|_0^{2\pi}\right.$$

$$\underline{= 0\ |}$$

Exercise

Evaluate the surface integral $\iint_S f(x, y, z)\,dS$; $f(\rho, \varphi, \theta) = \cos\varphi$, where S is the part of the unit shpere in the first octant

Solution

$$x^2 + y^2 + z^2 = 1, \quad x, y, z \ge 0$$

$$\vec{r} = \langle \sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi \rangle$$

$$\vec{r}_\varphi = \langle \cos\varphi\cos\theta,\ \cos\varphi\sin\theta,\ -\sin\varphi \rangle$$

$$\vec{r}_\theta = \langle -\sin\varphi\sin\theta,\ \sin\varphi\cos\theta,\ 0 \rangle$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= \sin^2\varphi\cos\theta\ \hat{i} + \sin^2\varphi\sin\theta\ \hat{j} + \left(\sin\varphi\cos\varphi\cos^2\theta + \sin\varphi\cos\varphi\sin^2\theta\right)\hat{k}$$

$$= \sin^2\varphi\cos\theta\ \hat{i} + \sin^2\varphi\sin\theta\ \hat{j} + \sin\varphi\cos\varphi\,\hat{k}$$

$$\left|\vec{r}_\varphi \times \vec{r}_\theta\right| = \sqrt{\sin^4\varphi\cos^2\theta + \sin^4\varphi\sin^2\theta + \sin^2\varphi\cos^2\varphi}$$

$$= \sqrt{\sin^4\varphi\left(\cos^2\theta + \sin^2\theta\right) + \sin^2\varphi\cos^2\varphi}$$

$$= \sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi}$$

$$= \sqrt{\sin^2\varphi\left(\sin^2\varphi + \cos^2\varphi\right)}$$

$$= \sin\varphi$$

$$\iint_S f(x, y, z)\,dS = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} (\cos\varphi)(\sin\varphi)\ d\varphi d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta\ \int_0^{\frac{\pi}{2}} \sin 2\varphi\ d\varphi$$

$$= -\frac{\pi}{4}\ \cos 2\varphi \Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{\pi}{4}(-1-1)$$

$$= \frac{\pi}{4}$$

Exercise

Find the flux of $\vec{F} = \frac{\vec{r}}{|\vec{r}|}$ across the sphere of radius a centered at the origin, where $\vec{r} = \langle x, y, z \rangle$. Assume the normal vectors to the surface point outward.

Solution

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle x,\ y,\ z\rangle}{\sqrt{x^2 + y^2 + z^2}}$$

Using spherical to parametrize the sphere.

$$\sqrt{x^2 + y^2 + z^2} = a$$

$$\vec{F} = \frac{1}{a}\langle a\sin u\cos v,\ \ a\sin u\sin v,\ a\cos u\rangle$$
$$= \langle \sin u\cos v,\ \ \sin u\sin v,\ \ \cos u\rangle$$

Using the table

$$\vec{n} = \left\langle a^2\sin^2 u\cos v,\ \ a^2\sin^2 u\sin v,\ a^2\sin u\cos u\right\rangle$$

$$\vec{F}\bullet\vec{n} = \langle \sin u\cos v,\ \ \sin u\sin v,\ \ \cos u\rangle \bullet \left\langle a^2\sin^2 u\cos v,\ \ a^2\sin^2 u\sin v,\ a^2\sin u\cos u\right\rangle$$
$$= a^2\sin^3 u\cos^2 v + a^2\sin^3 u\sin^2 v + a^2\sin u\cos^2 u$$
$$= a^2\sin^3 u\left(\cos^2 v + \sin^2 v\right) + a^2\sin u\cos^2 u$$
$$= a^2\sin^3 u + a^2\sin u\cos^2 u$$
$$= a^2\sin u\left(\sin^2 u + \cos^2 u\right)$$
$$= a^2\sin u$$

$$\iint_S \vec{F}\bullet\vec{n}\ dS = a^2\int_0^{2\pi}\int_0^{\pi}\sin u\ dudv$$
$$= a^2\int_0^{2\pi} dv\quad \left(-\cos u\ \Big|_0^{\pi}\right.$$
$$= a^2(2\pi)(1+1)$$
$$= 4\pi a^2$$

Exercise

Find the flux of the vector field $\vec{F} = \langle x,\ y,\ z\rangle$ across the curved surface of the cylinder $x^2 + y^2 = 1$ for $|z| \le 8$

Solution

$$\vec{n} = \langle x,\ y,\ 0\rangle$$

$$|\vec{n}| = \sqrt{x^2 + y^2}$$
$$= 1$$

$$\iint_S \vec{F} \bullet \vec{n}\, dS = \iint_R \langle x,\ y,\ z\rangle \bullet \langle x,\ y,\ 0\rangle\, dA$$

$$= \iint_R \left(x^2 + y^2\right) dA$$

$$= \iint_R dA$$

$$= \text{area of the circle radius } \frac{8}{2} = 4$$

$$= 2\pi(4)^2$$

$$= 32\pi$$

Exercise

Find the flux of the vector fields across the given surface with the specified orientation $\vec{F} = \langle 0,\ 0,\ -1\rangle$ across the slanted face of the tetrahedron $z = 4 - x - y$ in the first octant; normal vectors point upward

Solution

$z_x = -1, \quad z_y = -1$

Normal vectors point upward & first octant.

$\vec{n} = \langle 1,\ 1,\ 1\rangle$

$z = 4 - x - y = 0 \quad \rightarrow \quad y = 4 - x$

$y = 4 - x = 0 \quad \rightarrow \quad x = 4$

$0 \le x \le 4 \quad 0 \le y \le 4 - x$

$$\iint_S \vec{F} \bullet \vec{n}\, dS = \iint_R \langle 0,\ 0,\ -1\rangle \bullet \langle 1,\ 1,\ 1\rangle\, dA$$

$$= \int_0^4 \int_0^{4-x} (0 + 0 - 1)\, dy dx$$

$$= -\int_0^4 y \Big|_0^{4-x} dx$$

$$= -\int_0^4 (4 - x)\, dx$$

$$= -\left(4x - \frac{1}{2}x^2 \Big|_0^4\right.$$

$= -(16-8)$

$\underline{= -8}$

Exercise

Find the flux of the vector fields across the given surface with the specified orientation

$\vec{F} = \langle x,\ y,\ z \rangle$ across the slanted face of the tetrahedron $z = 10 - 2x - 5y$ in the first octant; normal vectors point upward

Solution

$z_x = -2, \quad z_y = -5$

Normal vectors point upward & first octant. $\vec{n} = \langle 2,\ 5,\ 1 \rangle$

$z = 10 - 2x - 5y = 0 \quad \rightarrow \quad y = \frac{1}{5}(10 - 2x)$

$y = \frac{1}{5}(10 - 2x) = 0 \quad \rightarrow \quad x = 5$

$0 \le x \le 5 \quad 0 \le y \le y = \frac{1}{5}(10 - 2x)$

$$\iint_S \vec{F} \cdot \vec{n}\ dS = \iint_R \langle x,\ y,\ z \rangle \cdot \langle 2,\ 5,\ 1 \rangle\, dA$$

$$= \int_0^5 \int_0^{\frac{1}{5}(10-2x)} (2x + 5y + z)\ dydx$$

$$= \int_0^5 \int_0^{\frac{1}{5}(10-2x)} (2x + 5y + 10 - 2x - 5y)\ dydx$$

$$= 10 \int_0^5 \int_0^{\frac{1}{5}(10-2x)} dydx$$

$$= 10 \int_0^5 y \Big|_0^{\frac{1}{5}(10-2x)} dx$$

$$= 10 \int_0^5 \frac{1}{5}(10 - 2x)\, dx$$

$$= 2\left(10x - x^2\right|_0^5$$

$$= 2(50 - 25)$$

$\underline{= 50}$

Exercise

Find the flux of the vector fields across the given surface with the specified orientation

$\vec{F} = \langle x,\ y,\ z \rangle$ across the slanted face of the cone $z^2 = x^2 + y^2$ for $0 \le z \le 1$; normal vectors point upward

Solution

$$2zdz + 2xdx = 0 \quad \rightarrow \quad z_x = -\frac{x}{z}$$

$$2zdz = 2ydy \quad \rightarrow \quad z_y = -\frac{y}{z}$$

Normal vectors point upward: $\quad \vec{n} = \left\langle -\frac{x}{z},\ -\frac{y}{z},\ 1 \right\rangle$

$$\iint_S \vec{F} \cdot \vec{n}\ dS = \iint_R \langle x,\ y,\ z \rangle \cdot \left\langle -\frac{x}{z},\ -\frac{y}{z},\ 1 \right\rangle dA$$

$$= \iint_R \left(-\frac{x^2}{z} - \frac{y^2}{z} + z \right) dA$$

$$= \iint_R \left(\frac{-x^2 - y^2 + z^2}{z} \right) dA$$

$$= \iint_R \left(\frac{-x^2 - y^2 + x^2 + y^2}{z} \right) dA$$

$$\underline{= 0}$$

Exercise

Find the flux of the vector fields across the given surface with the specified orientation

$\vec{F} = \left\langle e^{-y},\ 2z,\ xy \right\rangle$ across the curved sides of the surface

$S = \{(x,\ y,\ z):\quad z = \cos y,\quad -\pi \le y \le \pi,\quad 0 \le x \le 4\}$; normal vectors point upward

Solution

$$z_x = 0 \quad z_y = -\sin y$$

Normal vectors point upward: $\quad \vec{n} = \langle 0,\ -\sin y,\ 1 \rangle$

$$\iint_S \vec{F} \cdot \vec{n}\ dS = \iint_R \left\langle e^{-y},\ 2z,\ xy \right\rangle \cdot \langle 0,\ -\sin y,\ 1 \rangle dA$$

$$= \iint_R \left(-2z\sin y + xy\right) dA$$

$$= \iint_R \left(-2\cos y \sin y + xy\right) dA$$

$$= \int_0^4 \int_{-\pi}^{\pi} \left(-\sin 2y + xy\right) dy dx$$

$$= \int_0^4 \left(\frac{1}{2}\cos 2y + \frac{1}{2}xy^2 \Big|_{-\pi}^{\pi} dx\right.$$

$$= \frac{1}{2}\int_0^4 \left(1 + \pi^2 x - 1 - \pi^2 x\right) dx$$

$$\underline{= 0}$$

Exercise

Find the flux of the vector fields across the given surface with the specified orientation

$\vec{F} = \frac{\vec{r}}{|\vec{r}|^3}$ across the sphere of radius a centered at the origin, where $\vec{r} = \langle x,\ y,\ z\rangle$; normal vectors point outward

Solution

$\vec{n} = \frac{\vec{r}}{|\vec{r}|}$ pointing outward

$$\iint_S \vec{F} \bullet \vec{n}\ dS = \iint_S \frac{\vec{r}}{|\vec{r}|^3} \bullet \frac{\vec{r}}{|\vec{r}|} dS$$

$$= \iint_S \frac{\vec{r}^{\,2}}{|\vec{r}|^4} dS$$

$$= \iint_S \frac{1}{|\vec{r}|^2} dS$$

$$= \iint_S \frac{1}{a^2} dS$$

$$= \frac{1}{a^2} \times \left(Area\ of\ a\ sphere\right)$$

$$=\frac{1}{a^2}\left(4\pi a^2\right)$$

$$\underline{=4\pi}$$

Exercise

Find the flux of the vector fields across the given surface with the specified orientation

$\vec{F}=\langle -y,\ x,\ 1\rangle$ across the cylinder $y=x^2$ for $0\le x\le 1,\quad 0\le z\le 4$; normal vectors point in the general direction of the positive y-axis

Solution

$$\vec{r}(u,\ v)=\left\langle u,\ u^2,\ v\right\rangle$$

$$\vec{r}_u=\langle 1,\ 2u,\ 0\rangle$$

$$\vec{r}_v=\langle 0,\ 0,\ 1\rangle$$

$$\vec{n}=\vec{r}_u\times\vec{r}_v=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2u & 0\\ 0 & 0 & 1\end{vmatrix}$$

$$=\langle 2u,\ -1,\ 0\rangle$$

Normal vectors point in the general direction of the positive y-axis, then:

$$\vec{n}=\langle -2u,\ 1,\ 0\rangle$$

$$0\le u\le 1,\quad 0\le v\le 4$$

$$\iint_S \vec{F}\bullet\vec{n}\ dS=\iint_S \langle -y,\ x,\ 1\rangle\bullet\langle -2u,\ 1,\ 0\rangle dS$$

$$=\iint_R \left\langle -u^2,\ u,\ 1\right\rangle\bullet\langle -2u,\ 1,\ 0\rangle dA$$

$$=\int_0^4 dv\ \int_0^1\left(2u^3+u\right)du$$

$$=4\ \left(\frac{1}{2}u^4+\frac{1}{2}u^2\ \right|_0^1$$

$$=4\left(\frac{1}{2}+\frac{1}{2}\right)$$

$$\underline{=4}$$

Exercise

Consider the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, where a ,b, and c are positive real numbers.

a) Show that the surface is described by the parametric equations

$$\vec{r}(u,v)=\langle a\cos u\sin v,\ \ b\sin u\sin v,\ \ c\cos v\rangle \quad \text{for } 0\le u\le 2\pi,\ \ 0\le v\le \pi$$

b) ***Write*** an integral for the surface area of the ellipsoid.

Solution

a) $\vec{r}(u,v)=\langle a\cos u\sin v,\ \ b\sin u\sin v,\ \ c\cos v\rangle$

$$\begin{aligned}
\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} &= \frac{a^2\cos^2 u\sin^2 v}{a^2}+\frac{b^2\sin^2 u\sin^2 v}{b^2}+\frac{c^2\cos^2 v}{c^2}\\
&=\cos^2 u\sin^2 v+\sin^2 u\sin^2 v+\cos^2 v\\
&=\left(\cos^2 u+\sin^2 u\right)\sin^2 v+\cos^2 v\\
&=\sin^2 v+\cos^2 v\\
&=1
\end{aligned}$$

b) $\vec{r}_u=\langle -a\sin u\sin v,\ \ b\cos u\sin v,\ \ 0\rangle$

$\vec{r}_v=\langle a\cos u\cos v,\ \ b\sin u\cos v,\ \ -c\sin v\rangle$

$$\vec{n}=\vec{r}_u\times\vec{r}_v=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -a\sin u\sin v & b\cos u\sin v & 0\\ b\cos u\sin v & b\sin u\cos v & -c\sin v\end{vmatrix}$$

$$=\left\langle -b\cos u\sin^2 v,\ \ ac\sin u\sin^2 v,\ \ -ab\sin v\cos v\right\rangle$$

$$\begin{aligned}
|\vec{n}| &= \sqrt{b^2\cos^2 u\sin^4 v+a^2c^2\sin^2 u\sin^4 v+a^2b^2\sin^2 v\cos^2 v}\\
&=|\sin v|\sqrt{\left(b^2\cos^2 u+a^2c^2\sin^2 u\right)\sin^2 v+a^2b^2\cos^2 v}
\end{aligned}$$

$$\iint_S 1\,dS=\int_0^{2\pi}\int_0^{\pi}|\sin v|\ \sqrt{\left(b^2\cos^2 u+a^2c^2\sin^2 u\right)\sin^2 v+a^2b^2\cos^2 v}\ dudv$$

Exercise

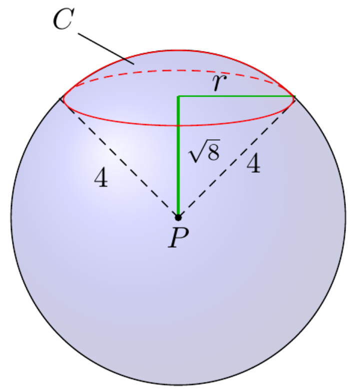
The cone $z^2 = x^2 + y^2, \quad z \ge 0$, cuts the sphere $x^2 + y^2 + z^2 = 16$ along a curve *C*.

a) Find the surface area of the sphere below *C*, for $z \ge 0$

b) Find the surface area of the sphere above *C*.

c) Find the surface area of the cone below *C*, for $z \ge 0$

Solution



$$\begin{cases} z^2 = x^2 + y^2 \\ x^2 + y^2 + z^2 = 16 \end{cases} \rightarrow 2\left(x^2 + y^2\right) = 16$$

$$x^2 + y^2 = 8$$

$$8 + z^2 = 16 \quad \rightarrow \quad \underline{z = 2\sqrt{2}}$$

$$2zdz = 2xdx \quad \rightarrow \quad z_x = \frac{x}{z}$$

$$2zdz = 2ydy \quad \rightarrow \quad z_y = \frac{y}{z}$$

Since the normal vector point outward & in the positive *z*-direction

$$\vec{n} = \left\langle \frac{x}{z}, \ \frac{y}{z}, \ 1 \right\rangle$$

$$\begin{aligned} \sqrt{z_x^2 + z_y^2 + 1} &= \sqrt{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1} \\ &= \sqrt{\frac{x^2 + y^2 + z^2}{z^2}} \\ &= \sqrt{\frac{16}{z^2}} \\ &= \frac{4}{z} \\ &= \frac{4}{\sqrt{16 - x^2 - y^2}} \end{aligned}$$

a) $$\text{Surface of } C = \int_0^{2\pi} \int_{\sqrt{8}}^{4} \frac{4}{\sqrt{16 - r^2}} \, r \, dr d\theta$$

$$\begin{aligned} &= -2\int_0^{2\pi} d\theta \int_{\sqrt{8}}^{4} \left(16 - r^2\right)^{-1/2} dr\left(16 - r^2\right) \\ &= -2(2\pi)(2)\left(16 - r^2\right)^{1/2} \Big|_{\sqrt{8}}^{4} \\ &= -8\pi\left(0 - \sqrt{8}\right) \end{aligned}$$

$$=16\pi\sqrt{2}$$

The total surface area of the sphere: $\pi r^3 = 64\pi$

Since the cone in the positive z-direction, then

Surface area of the sphere below $C = \frac{1}{2}64\pi + 16\pi\sqrt{2}$

$$=16\pi\left(2+\sqrt{2}\right) \quad unit^2$$

b) $$\iint_S 1\, dS = \iint_R \frac{4}{\sqrt{16-x^2-y^2}}\, dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{8}} \frac{4}{\sqrt{16-r^2}}\, r\, dr d\theta$$

$$= -2\int_0^{2\pi} d\theta \int_0^{\sqrt{8}} \left(16-r^2\right)^{-1/2}\, dr\left(16-r^2\right)$$

$$= -2(2\pi)\ (2)\ \left(16-r^2\right)^{1/2}\Big|_0^{\sqrt{8}}$$

$$= -8\pi\left(\sqrt{8}-4\right)$$

$$= 8\pi\left(4-2\sqrt{2}\right)$$

$$= 16\pi\left(2-\sqrt{2}\right)$$

c) $$\iint_S 1\, dS = \iint_R \sqrt{2}\, dA$$

$$= \sqrt{2}\pi\ \left(\sqrt{8}\right)^2$$

$$= 8\pi\sqrt{2}$$

Exercise

Consider the sphere $x^2+y^2+z^2=4$ and the cylinder $(x-1)^2+y^2=1$ for $z\ge 0$.

a) Find the surface area of the cylinder inside the sphere

b) Find the surface area of the sphere inside the cylinder.

Solution

a) $(x-1)^2+y^2=1 \quad\rightarrow\quad \begin{cases} x-1=\cos u & x=1+\cos u \\ y=\sin u \end{cases}$

$$\vec{r}(u,\ v) = \langle 1+\cos u,\ \sin u,\ v\rangle$$

$$\vec{r}_u = \langle -\sin u,\ \cos u,\ 0 \rangle$$

$$\vec{r}_v = \langle 0,\ 0,\ 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle \cos u,\ \sin u,\ 0 \rangle$$

$$\left| \vec{r}_u \times \vec{r}_v \right| = \sqrt{\cos^2 u + \sin^2 u}$$

$$\underline{= 1}$$

$$z^2 = 4 - x^2 - y^2$$

$$z = \sqrt{4 - \left(1 + 2\cos u + \cos^2 u\right) - \sin^2 u}$$

$$= \sqrt{3 - 2\cos u - \cos^2 u - \sin^2 u}$$

$$= \sqrt{2 - 2\cos u}$$

$$0 \le z = v \le \sqrt{2 - 2\cos u} \quad 0 \le u \le 2\pi$$

$$\iint_S 1\, dS = \iint_R 1\, dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2-2\cos u}} dv du$$

$$= \int_0^{2\pi} v \Big|_0^{\sqrt{2-2\cos u}} du$$

$$= \sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos u}\ du \qquad \sin^2 \alpha = \frac{1 - \cos 2\alpha}{2} \quad \to \quad 1 - \cos u = 2\sin^2 \frac{u}{2}$$

$$= \sqrt{2} \int_0^{2\pi} \sqrt{2\sin^2 \frac{u}{2}}\ du$$

$$= 2 \int_0^{2\pi} \sin \frac{u}{2}\ du$$

$$= -4\ \cos \frac{u}{2} \Big|_0^{2\pi}$$

$$= -4(-1 - 1)$$

$$\underline{= 8}$$

b) $\vec{r} = \left\langle 2\sin\varphi\cos\theta,\ 2\sin\varphi\sin\theta,\ 2\cos\varphi \right\rangle$

$$\vec{r}_{\varphi} = \left\langle 2\cos\varphi\cos\theta,\ 2\cos\varphi\sin\theta,\ -2\sin\varphi \right\rangle$$

$$\vec{r}_{\theta} = \left\langle -2\sin\varphi\sin\theta,\ 2\sin\varphi\cos\theta,\ 0 \right\rangle$$

$$\vec{r}_{\varphi} \times \vec{r}_{\theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\cos\varphi\cos\theta & 2\cos\varphi\sin\theta & -2\sin\varphi \\ -2\sin\varphi\sin\theta & 2\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$= 4\sin^2\varphi\cos\theta\ \hat{i} + 4\sin^2\varphi\sin\theta\,\hat{j} + \left(4\sin\varphi\cos\varphi\cos^2\theta + 4\sin\varphi\cos\varphi\sin^2\theta\right)\hat{k}$$

$$= 4\sin^2\varphi\cos\theta\ \hat{i} + 4\sin^2\varphi\sin\theta\,\hat{j} + 4\sin\varphi\cos\varphi\,\hat{k}$$

$$\left|\vec{r}_{\varphi} \times \vec{r}_{\theta}\right| = \sqrt{16\sin^4\varphi\cos^2\theta + 16\sin^4\varphi\sin^2\theta + 16\sin^2\varphi\cos^2\varphi}$$

$$= 4\sqrt{\sin^4\varphi\left(\cos^2\theta + \sin^2\theta\right) + \sin^2\varphi\cos^2\varphi}$$

$$= 4\sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi}$$

$$= 4\sqrt{\sin^2\varphi\left(\sin^2\varphi + \cos^2\varphi\right)}$$

$$\underline{= 4\sin\varphi}$$

$$(2\sin\varphi\cos\theta - 1)^2 + 4\sin^2\varphi\sin^2\theta = 1$$

$$4\sin^2\varphi\cos^2\theta - 4\sin\varphi\cos\theta + 1 + 4\sin^2\varphi\sin^2\theta = 1$$

$$4\sin^2\varphi\left(\cos^2\theta + \sin^2\theta\right) - 4\sin\varphi\cos\theta = 0$$

$$4\sin\varphi\left(\sin\varphi - \cos\theta\right) = 0$$

$$\begin{cases} \sin\varphi = 0 & \varphi = 0,\ \pi \ \Rightarrow\ \underline{0 \le u \le \pi} \\ \cos\theta = \sin\varphi & \underline{\theta =} \cos^{-1}(\sin\varphi) \ \underline{= \frac{\pi}{2} - \varphi} \end{cases}$$

$$\iint_S 1\, dS = \int_0^{\pi}\int_0^{\frac{\pi}{2}-\varphi} 4\sin\varphi\ d\theta d\varphi$$

$$= 4\int_0^{\pi} (\sin\varphi)\ \theta \Big|_0^{\frac{\pi}{2}-\varphi}\ d\varphi$$

$$= 4\int_0^{\pi} \left(\frac{\pi}{2}\sin\varphi - \varphi\sin\varphi\right) d\varphi$$

$$= 4\left(-\frac{\pi}{2}\cos\varphi + \varphi\cos\varphi - \sin\varphi\right|_0^{\pi}$$

$$= 4\left(\frac{\pi}{2} - \pi + \frac{\pi}{2}\right)$$

$$\underline{= 0}$$

Since it cannot be zero, we have to change $0 \le u \le \pi$ to half and multiply by 2.

$\therefore\ 0 \le u \le \frac{\pi}{2}$

$$\iint_S 1\, dS = 2 \times 4\left(-\frac{\pi}{2}\cos\varphi + \varphi\cos\varphi - \sin\varphi\right|_0^{\frac{\pi}{2}}$$

$$= 8\left(-1 + \frac{\pi}{2}\right)$$

$$\underline{= 4\pi - 8}$$

Exercise

Find the upward flux of the field $\vec{F} = \langle x,\ y,\ z\rangle$ across the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ in the first octant. Show that the flux equals c times the area if the base of the origin.

Solution

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \rightarrow \quad z = c - \frac{c}{a}x - \frac{c}{b}y$$

$$\frac{1}{a}dx + \frac{1}{c}dz = 0 \quad \rightarrow \quad z_x = -\frac{c}{a}$$

$$\frac{1}{b}dy + \frac{1}{c}dz = 0 \quad \rightarrow \quad z_y = -\frac{c}{b}$$

First octant $\vec{n} = \left\langle \frac{c}{a},\ \frac{c}{b},\ 1\right\rangle$

$$\iint_S \vec{F} \bullet \vec{n}\ dS = \iint_R \langle x,\ y,\ z\rangle \bullet \left\langle \frac{c}{a},\ \frac{c}{b},\ 1\right\rangle dA$$

$$= \iint_R \left(\frac{c}{a}x + \frac{c}{b}y + z\right) dA$$

$$= \iint_R \left(\frac{c}{a}x + \frac{c}{b}y + c - \frac{c}{a}x - \frac{c}{b}y\right) dA$$

$$= \iint_R c\ dA$$

$$\underline{= c \times (\text{Area of } A)}$$

As c increases, the slope of the plane gets closer to vertical, so that the x and y components of the vector field $\vec{F} = \langle x, y, z\rangle$ contribute more to the flux; also, the values of z get larger. This the flux increases as c does.

Exercise

Consider the field $\vec{F} = \langle x, y, z\rangle$ and the cone $z^2 = \dfrac{x^2+y^2}{a^2}$, for $0 \le z \le 1$

a) Show that when $a = 1$, the outward flux across the cone is zero.

b) Find the outward flux (away from the z-axis); for any $a > 0$.

Solution

$$2zdz = 2\frac{x}{a^2}dx \quad \rightarrow \quad z_x = \frac{x}{a^2 z}$$

$$2zdz = 2\frac{y}{a^2}dy \quad \rightarrow \quad z_y = \frac{y}{a^2 z}$$

Since the normal is outward: $\quad \vec{n} = \left\langle -\dfrac{x}{a^2 z},\ -\dfrac{y}{a^2 z},\ 1\right\rangle$

a) $a = 1 \ \rightarrow \vec{n} = \left\langle -\dfrac{x}{z},\ -\dfrac{y}{z},\ 1\right\rangle$

$$\iint_S \vec{F}\cdot\vec{n}\ dS = \iint_R \langle x, y, z\rangle \cdot \left\langle -\frac{x}{z},\ -\frac{y}{z},\ 1\right\rangle dA$$

$$= \iint_R \left(-\frac{x^2}{z} - \frac{y^2}{z} + z\right) dA$$

$$= \iint_R \left(-\frac{x^2+y^2}{z} + z\right) dA \qquad\qquad z^2 = \frac{x^2+y^2}{a^2}$$

$$= \iint_R \left(-\frac{z^2}{z} + z\right) dA$$

$$= \iint_R 0\ dA$$

$$= 0$$

b) $$\iint_S \vec{F}\cdot\vec{n}\ dS = \iint_R \langle x, y, z\rangle \cdot \left\langle -\frac{x}{a^2 z},\ -\frac{y}{a^2 z},\ 1\right\rangle dA$$

$$= \iint_R \left(-\frac{x^2}{a^2 z} - \frac{y^2}{a^2 z} + z\right) dA$$

$$= \iint_R \left(-\frac{\left(x^2+y^2\right)}{a^2}\frac{1}{z} + z \right) dA$$

$$= \iint_R \left(-z^2 \frac{1}{z} + z \right) dA$$

$$= \iint_R (-z+z)\, dA$$

$$\underline{= 0}$$

The flow is a radial flow, so it is always tangent to the surface.

Exercise

A sphere of radius a is sliced parallel to the equatorial plane at a distance $a-h$ from the equatorial plane. Find the general formula for the surface area of the resulting spherical cap (excluding the base) with thickness h.

Solution

The sphere equation is: $x^2+y^2+z^2=a^2$

$$2zdz = 2xdx \quad \rightarrow \quad z_x = \frac{x}{z}$$

$$2zdz = 2ydy \quad \rightarrow \quad z_y = \frac{y}{z}$$

$$\vec{n} = \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle$$

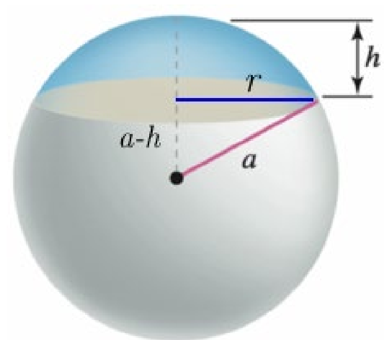
$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1}$$

$$= \sqrt{\frac{x^2+y^2+z^2}{z^2}}$$

$$= \sqrt{\frac{a^2}{z^2}}$$

$$= \frac{a}{z}$$

$$= \frac{a}{\sqrt{a^2-x^2-y^2}}$$



$$r^2 + (a-h)^2 = a^2$$

$$r^2 = a^2 - a^2 + 2ah - h^2$$

$$0 \le r \le \sqrt{2ah-h^2}$$

$$\iint_S 1\, dS = \iint_R \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2ah-h^2}} \frac{a}{\sqrt{a^2 - r^2}}\, r\, dr d\theta$$

$$= -\frac{a}{2} \int_0^{2\pi} d\theta \int_0^{\sqrt{2ah-h^2}} \left(a^2 - r^2\right)^{-1/2} d\left(a^2 - r^2\right)$$

$$= -2a\pi \left(a^2 - r^2\right)^{1/2} \Big|_0^{\sqrt{2ah-h^2}}$$

$$= -2a\pi \left(\sqrt{a^2 - \left(2ah - h^2\right)} - a\right)$$

$$= -2a\pi \left(\sqrt{a^2 - 2ah + h^2} - a\right)$$

$$= -2a\pi \left(\sqrt{(a-h)^2} - a\right)$$

$$= -2a\pi (a - h - a)$$

$$= 2a\pi h$$

Exercise

Consider the radial field $\vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$, where $\vec{r} = \langle x,\ y,\ z\rangle$ and p is a real number. Let S be he sphere of radius a centered at the origin. Show that the outward flux of $\vec{F}$ across the sphere is $\frac{4\pi}{a^{p-3}}$. It is instructive to do the calculation using both an explicit and parametric description of the sphere.

Solution

$$\vec{r} = \langle a\sin u \cos v,\ a\sin u \sin v,\ a\cos u\rangle$$

$$\vec{r}_u = \langle a\cos u \cos v,\ a\cos u \sin v,\ -a\sin u\rangle$$

$$\vec{r}_v = \langle -a\sin u \sin v,\ a\sin u \cos v,\ 0\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos u \cos v & a\cos u \sin v & -a\sin u \\ -a\sin u \sin v & a\sin u \cos v & 0 \end{vmatrix}$$

$$= \left\langle a^2 \sin^2 u \cos v,\ a^2 \sin^2 u \sin v,\ a^2 \sin u \cos u \cos^2 v + a^2 \sin u \cos u \sin^2 v\right\rangle$$

$$=\left\langle a^2\sin^2 u\cos v,\ a^2\sin^2 u\sin v,\ a^2\sin u\cos u\right\rangle$$

$$\iint_S \vec{F}\bullet\vec{n}\ dS = \iint_R \frac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{p/2}}\bullet\left\langle a^2\sin^2 u\cos v,\ a^2\sin^2 u\sin v,\ a^2\sin u\cos u\right\rangle\ dA$$

$$\langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle\bullet\left\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\sin u\cos u\right\rangle$$

$$=\frac{1}{a^p}\iint_R\left(a^3\sin^3 u\cos^2 v+a^3\sin^3 u\sin^2 v+a^3\sin u\cos^2 u\right)dA$$

$$=\frac{1}{a^{p-3}}\iint_R \sin u\left(\sin^2 u\left(\cos^2 v+\sin^2 v\right)+\cos^2 u\right)dA$$

$$=\frac{1}{a^{p-3}}\iint_R \sin u\left(\sin^2 u+\cos^2 u\right)dA$$

$$=\frac{1}{a^{p-3}}\int_0^{2\pi}dv\int_0^{\pi}\sin u\ du$$

$$=\frac{2\pi}{a^{p-3}}\left(-\cos u\ \Big|_0^{\pi}\right.$$

$$\boxed{=\frac{4\pi}{a^{p-3}}}$$

Parametric

$$2zdz+2xdx=0\quad\rightarrow\quad z_x=-\frac{x}{z}$$

$$2zdz+2ydy=0\quad\rightarrow\quad z_y=-\frac{y}{z}$$

$$\vec{n}=\left\langle -\frac{x}{z},\ -\frac{y}{z},\ 1\right\rangle$$

$$\iint_S \vec{F}\bullet\vec{n}\ dS=\iint_R \frac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{p/2}}\bullet\left\langle \frac{x}{z},\ \frac{y}{z},\ 1\right\rangle\ dA$$

$$=\frac{1}{\left(a^2\right)^{p/2}}\iint_R\left(\frac{x^2}{z}+\frac{y^2}{z}+z\right)\ dA$$

$$=\frac{1}{a^p}\iint_R\left(\frac{x^2+y^2+z^2}{z}\right)\ dA$$

$$= \frac{1}{a^p} \iint\limits_R \left(\frac{a^2}{z}\right) dA$$

$$= a^{2-p} \int_0^{2\pi} \int_0^a \frac{rdrd\theta}{\sqrt{a^2 - r^2}}$$

$$= -\frac{1}{2} a^{2-p} \int_0^{2\pi} d\theta \int_0^a \left(a^2 - r^2\right)^{-1/2} d\left(a^2 - r^2\right)$$

$$= -\frac{2\pi}{a^{p-2}} \left(a^2 - r^2\right)^{1/2} \Big|_0^a$$

$$= -\frac{2\pi}{a^{p-2}}(-a)$$

$$\boxed{= \frac{2\pi}{a^{p-3}}}$$

$$\boxed{2 \times \frac{2\pi}{a^{p-3}} = \frac{4\pi}{a^{p-3}}}$$

Exercise

The heat flow vector field for conducting objects is $\vec{F} = -k\nabla T$, where $T(x, y, z)$ is the temperature in the object and $k > 0$ is a constant that depends on the material. Compute the outward flux of $\vec{F}$ across the following surface *S* for the given temperature distributions. Assume $k = 1$.

$T(x, y, z) = 100e^{-x-y}$; *S* consists of the faces of the cube $|x| \le 1, \quad |y| \le 1, \quad |z| \le 1$

Solution

$$\vec{F} = -\nabla T$$
$$= -\nabla\left(100e^{-x-y}\right)$$
$$= \left\langle 100e^{-x-y},\ 100e^{-x-y},\ 0\right\rangle$$

Thus, the flow is parallel to the 2 sides where $z = \pm 1$, so the flus is zero.

For the side: $x = -1 \quad \rightarrow \langle -1,\ 0,\ 0\rangle \quad S_1 : \langle -1,\ y,\ z\rangle$

$$\vec{t}_y = \langle 0,\ 1,\ 0\rangle \quad \vec{t}_z = \langle 0,\ 0,\ 1\rangle$$

$$\vec{t}_y \times \vec{t}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= \langle 1,\ 0,\ 0\rangle$$

$$\iint_S \vec{F}\cdot\vec{n}\,dS_1 = \iint_R \left\langle 100e^{-x-y},\ 100e^{-x-y},\ 0\right\rangle \cdot \langle -1,\ 0,\ 0\rangle\, dA$$

$$= -\iint_R 100\,e^{-x-y}\, dA$$

$$= -100\int_{-1}^{1}\int_{-1}^{1} e^{1-y}\, dydz$$

$$= 100\int_{-1}^{1} dz\ \int_{-1}^{1} e^{1-y}\, d(1-y)$$

$$= 100\ z\Big|_{-1}^{1}\ \ e^{1-y}\Big|_{-1}^{1}$$

$$= 100(2)\left(1-e^2\right)$$

$$\underline{= 200\left(1-e^2\right)\,\Big|}$$

For the side: $x=1$ $\rightarrow\langle 1,\ 0,\ 0\rangle$ $S_2:\ \langle 1,\ y,\ z\rangle$

$\vec{t}_y = \langle 0,\ 1,\ 0\rangle$ $\vec{t}_z = \langle 0,\ 0,\ 1\rangle$

$$\vec{t}_y\times\vec{t}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle 1,\ 0,\ 0\rangle$$

$$\iint_S \vec{F}\cdot\vec{n}\,dS_2 = \iint_R \left\langle 100e^{-x-y},\ 100e^{-x-y},\ 0\right\rangle \cdot \langle -1,\ 0,\ 0\rangle\, dA$$

$$= -\iint_R 100e^{-x-y}\, dA$$

$$= 100\int_{-1}^{1}\int_{-1}^{1} e^{-1-y}\, dydz$$

$$= -100\int_{-1}^{1} dz\ \int_{-1}^{1} e^{-1-y}\, d(-1-y)$$

$$= -100\ \ z\Big|_{-1}^{1}\ \ e^{-1-y}\Big|_{-1}^{1}$$

$$= -100(2)\left(e^{-2} - 1\right)$$

$$\boxed{= 200\left(1 - e^{-2}\right)}$$

For the side: $y = -1 \quad \to \langle 0,\ -1,\ 0 \rangle \quad S_3 : \langle x,\ -1,\ z \rangle$

$$\vec{t}_x = \langle 1,\ 0,\ 0 \rangle \quad \vec{t}_z = \langle 0,\ 0,\ 1 \rangle$$

$$\vec{t}_x \times \vec{t}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle 0,\ -1,\ 0 \rangle$$

$$\iint_S \vec{F} \cdot \vec{n}\ dS_3 = \iint_R \left\langle 100e^{-x-y},\ 100e^{-x-y},\ 0 \right\rangle \cdot \langle 0,\ -1,\ 0 \rangle\ dA$$

$$= -\iint_R 100e^{-x-y}\ dA$$

$$= -100 \int_{-1}^{1} \int_{-1}^{1} e^{1-x}\ dxdz$$

$$= 100 \int_{-1}^{1} dz \int_{-1}^{1} e^{1-x}\ d(1-x)$$

$$= 100\ \ z\Big|_{-1}^{1} \quad e^{1-x}\Big|_{-1}^{1}$$

$$= 100(2)\left(1 - e^{2}\right)$$

$$\boxed{= 200\left(1 - e^{2}\right)}$$

For the side: $y = 1 \quad \to \langle 0,\ 1,\ 0 \rangle \quad S_4 : \langle x,\ 1,\ z \rangle$

$$\vec{t}_x = \langle 1,\ 0,\ 0 \rangle \quad \vec{t}_z = \langle 0,\ 0,\ 1 \rangle$$

$$\vec{t}_x \times \vec{t}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle 0,\ -1,\ 0 \rangle$$

$$\iint_S \vec{F} \cdot \vec{n}\ dS_4 = \iint_R \left\langle 100e^{-x-y},\ 100e^{-x-y},\ 0 \right\rangle \cdot \langle 0,\ -1,\ 0 \rangle\ dA$$

$$= -\iint_R 100e^{-x-y}\, dA$$

$$= 100\int_{-1}^{1}\int_{-1}^{1} e^{-1-x}\, dxdz$$

$$= -100\int_{-1}^{1} dz \int_{-1}^{1} e^{-1-x}\, d(-1-x)$$

$$= -100\ \ z\Big|_{-1}^{1}\ \ e^{-1-x}\Big|_{-1}^{1}$$

$$\boxed{= 200\left(1-e^{-2}\right)}$$

The total flux: $= 200 - 200e^{2} + 200 - 200e^{-2} + 200 - 200e^{2} + 200 - 200e^{-2}$

$$= 800 - 400e^{2} - 400e^{-2}$$

$$= -100\left(e^{2} + e^{-2} - 2\right)$$

$$= -100\left(e - e^{-1}\right)^{2}$$

$$\boxed{= -100\left(e - \frac{1}{e}\right)^{2}}$$

Exercise

The heat flow vector field for conducting objects is $\vec{F} = -k\nabla T$, where $T(x,\ y,\ z)$ is the temperature in the object and $k > 0$ is a constant that depends on the material. Compute the outward flux of $\vec{F}$ across the following surface S for the given temperature distributions. Assume $k = 1$.

$T(x,\ y,\ z) = 100e^{-x^2-y^2-z^2}$; S cis the sphere $x^2 + y^2 + z^2 = a^2$

Solution

$$\vec{F} = -\nabla T$$

$$= -\nabla\left(100e^{-x^2-y^2-z^2}\right)$$

$$= \left\langle 200xe^{-x^2-y^2-z^2},\ 200ye^{-x^2-y^2-z^2},\ 200ze^{-x^2-y^2-z^2}\right\rangle$$

$$x^2 + y^2 + z^2 = a^2 \quad \to \quad z = \sqrt{a^2 - x^2 - y^2}$$

$$2xdx + 2zdz = 0 \quad \to \quad z_x = -\frac{x}{z}$$

$$2ydy + 2zdz = 0 \quad \to \quad z_y = -\frac{y}{z}$$

$$\vec{n} = \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle$$

$$\iint_S \vec{F} \cdot \vec{n}\ dS = 200 \iint_R \left\langle xe^{-x^2-y^2-z^2},\ ye^{-x^2-y^2-z^2},\ ze^{-x^2-y^2-z^2} \right\rangle \cdot \left\langle \frac{x}{z},\ \frac{y}{z},\ 1 \right\rangle dA$$

$$= 200 \iint_R \left(\frac{x^2}{z} + \frac{y^2}{z} + z \right) e^{-\left(x^2+y^2+z^2\right)}\ dA$$

$$= 200 \iint_R \left(\frac{x^2+y^2+z^2}{z} \right) e^{-a^2}\ dA$$

$$= 200a^2 e^{-a^2} \iint_R \left(\frac{1}{z} \right) dA$$

$$= 200a^2 e^{-a^2} \int_0^{2\pi} \int_0^a \frac{1}{\sqrt{a^2-r^2}}\ r\ drd\theta$$

$$= -100a^2 e^{-a^2} \int_0^{2\pi} d\theta \int_0^a \left(a^2-r^2\right)^{-1/2} d\left(a^2-r^2\right)$$

$$= -400\pi a^2 e^{-a^2} \left(a^2-r^2\right)^{1/2} \Big|_0^a$$

$$= 400\pi a^3 e^{-a^2}$$

Because the vector field is symmetric, then the outward flux of $\vec{F}$ across is

$$2 \times 400\pi a^3 e^{-a^2} \ \ = 800\pi a^3 e^{-a^2}$$

Exercise

The heat flow vector field for conducting objects is $\vec{F} = -k\nabla T$, where $T(x,\ y,\ z)$ is the temperature in the object and $k > 0$ is a constant that depends on the material. Compute the outward flux of $\vec{F}$ across the following surface S for the given temperature distributions. Assume $k = 1$.

$T(x,\ y,\ z) = -\ln\left(x^2+y^2+z^2\right)$; S cis the sphere $x^2+y^2+z^2 = a^2$

Solution

$$\vec{F} = -\nabla T$$

$$= -\nabla\left(-\ln\left(x^2+y^2+z^2\right)\right)$$

$$= \left\langle \frac{2x}{x^2+y^2+z^2},\ \frac{2y}{x^2+y^2+z^2},\ \frac{2z}{x^2+y^2+z^2} \right\rangle$$

$$= \frac{2}{x^2+y^2+z^2}\langle x,\ y,\ z\rangle$$

$$x^2+y^2+z^2=a^2 \quad \to \quad z=\sqrt{a^2-x^2-y^2}$$

$$2xdx+2zdz=0 \quad \to \quad z_x=-\frac{x}{z}$$

$$2ydy+2zdz=0 \quad \to \quad z_y=-\frac{y}{z}$$

$$\vec{n}=\left\langle \frac{x}{z},\ \frac{y}{z},\ 1\right\rangle$$

$$\iint_S \vec{F}\bullet\vec{n}\ dS = 2\iint_R \frac{\langle x,\ y,\ z\rangle}{x^2+y^2+z^2}\bullet\left\langle \frac{x}{z},\ \frac{y}{z},\ 1\right\rangle dA$$

$$= \frac{2}{a^2}\iint_R \left(\frac{x^2}{z}+\frac{y^2}{z}+z\right) dA$$

$$= \frac{2}{a^2}\iint_R \left(\frac{x^2+y^2+z^2}{z}\right) dA$$

$$= 2\iint_R \left(\frac{1}{z}\right) dA$$

$$= 2\int_0^{2\pi}\int_0^a \frac{1}{\sqrt{a^2-r^2}}\ r\ drd\theta$$

$$= -\int_0^{2\pi} d\theta \quad \int_0^a \left(a^2-r^2\right)^{-1/2} d\left(a^2-r^2\right)$$

$$= -4\pi\ \left(a^2-r^2\right)^{1/2}\Big|_0^a$$

$$\underline{= 4\pi a}$$

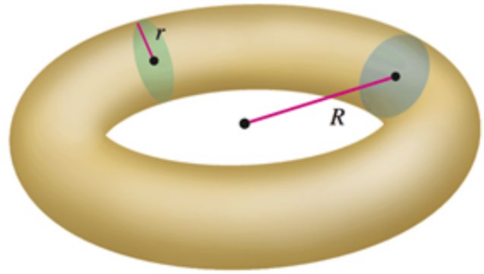
Because the vector field is symmetric, then the outward flux of $\vec{F}$ across is

$2\times 4\pi a\ \underline{= 8\pi a}$

Exercise

Given: $\vec{r}(u,\ v)=\langle (R+r\cos u)\cos v,\ \ (R+r\cos u)\sin v,\ \ r\sin u\rangle$

a) Show that a torus with radii $R>r$ may be described parametrically by $\vec{r}(u,\ v)$ for $0\le u\le 2\pi,\quad 0\le v\le 2\pi$



b) Show that the surface area of the torus is $4\pi^2 Rr$

Solution

a) If we let $\langle R\cos v,\ \ R\sin v,\ \ 0\rangle$ the parametrized for the (outer) circle of radius R.

For the inner circle, that includes the z-axis, we can write the parametrization as: $\langle r\cos u\cos v,\ r\cos u\sin v,\ r\sin u\rangle$.

Therefore, the set of points on the torus can be parametrized by the sum of the se 2 vectors.

$$\langle R\cos v,\ R\sin v,\ 0\rangle+\langle r\cos u\cos v,\ r\cos u\sin v,\ r\sin u\rangle$$

$$=\langle (R+r\cos u)\cos v,\ (R+r\cos u)\sin v,\ r\sin u\rangle$$

b) $\vec{r}(u,\ v)=\langle (R+r\cos u)\cos v,\ \ (R+r\cos u)\sin v,\ \ r\sin u\rangle$

$\vec{t}_u=\langle -r\sin u\cos v,\ \ -r\,\mathrm{sins}\,u\sin v,\ \ r\cos u\rangle$

$\vec{t}_v=\langle -(R+r\cos u)\sin v,\ \ (R+r\cos u)\cos v,\ \ 0\rangle$

$$\vec{t}_u\times\vec{t}_v=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -r\sin u\cos v & -r\,\mathrm{sins}\,u\sin v & r\cos u\\ -(R+r\cos u)\sin v & (R+r\cos u)\cos v & 0\end{vmatrix}$$

$$=\left(-r(R+r\cos u)\cos u\cos v\right)\ \hat{i}$$

$$\left(-r(R+r\cos u)\cos u\sin v\right)\hat{j}$$

$$\left(-r(R+r\cos u)\sin u\cos^2 v-r(R+r\cos u)\sin u\sin^2\right)\hat{k}$$

$$=-r(R+r\cos u)\left\langle \cos u\cos v,\ \ \cos u\sin v,\ \ \sin u\left(\cos^2 v+\sin^2 v\right)\right\rangle$$

$$=-r(R+r\cos u)\langle \cos u\cos v,\ \ \cos u\sin v,\ \ \sin u\rangle$$

$$\left|\vec{t}_u\times\vec{t}_v\right|=r(R+r\cos u)\sqrt{\cos^2 u\cos^2 v+\cos^2 u\sin^2 v+\sin^2 u}$$

$$= r\left(R + r\cos u\right)\sqrt{\cos^2 u\left(\cos^2 v + \sin^2 v\right) + \sin^2 u}$$

$$= r\left(R + r\cos u\right)\sqrt{\cos^2 u + \sin^2 u}$$

$$= r\left(R + r\cos u\right)$$

$$Area\ of\ the\ torus = \int_0^{2\pi}\int_0^{2\pi} r\left(R + r\cos u\right)du\,dv$$

$$= r\int_0^{2\pi}\left(Ru + r\sin u\ \Big|_0^{2\pi}\ dv\right.$$

$$= 2\pi rR\int_0^{2\pi} dv$$

$$\underline{= 4\pi^2 rR \quad unit^2}$$