Solution Section 3.6 – Newton's Method

Exercise

Use Newton's method to estimate the on real solution of $x^3+3x+1=0$.
Start with $x_0=0$ and then find x_2

Solution

$$y=x^3+3x+1 \quad\to\quad y'=3x^2+3$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{x_n^3+3x_n+1}{3x_n^2+3}$$

$$x_0=0$$

$$x_1=x_0-\frac{x_0^3+3x_0+1}{3x_0^2+3}=0-\frac{0+3(0)+1}{3(0)+3}=\underline{-\frac{1}{3}}$$

$$x_2=x_1-\frac{x_1^3+3x_1+1}{3x_1^2+3}=-\frac{1}{3}-\frac{\left(-\frac{1}{3}\right)^3+3\left(-\frac{1}{3}\right)+1}{3\left(-\frac{1}{3}\right)+3}=\underline{-0.3222}$$

Exercise

Use Newton's method to estimate the on real solution of $x^4+x-3=0$. Start with $x_0=-1$ for the left-hand zero and with $x_0=1$ for the zero on the right. Then, in each case, find x_2

Solution

$$y=x^4+x-3 \quad\to\quad y'=4x^3+1$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{x_n^4+x_n-3}{4x_n^3+1}$$

$$\boxed{x_0=-1}$$

$$x_1=x_0-\frac{x_0^4+x_0-3}{4x_0^3+1}=-1-\frac{(-1)^4+(-1)-3}{4(-1)^3+1}=\underline{-2}$$

$$x_2=x_1-\frac{x_1^4+x_1-3}{4x_1^3+1}=-2-\frac{(-2)^4+(-2)-3}{4(-2)^3+1}=\underline{-1.64516}$$

$\boxed{x_0 = 1}$

$$x_1 = x_0 - \frac{x_0^4 + x_0 - 3}{4x_0^3 + 1} = 1 - \frac{(1)^4 + (1) - 3}{4(1)^3 + 1} = \frac{6}{5}$$

$$x_2 = x_1 - \frac{x_1^4 + x_1 - 3}{4x_1^3 + 1} = \frac{6}{5} - \frac{\left(\frac{6}{5}\right)^4 + \left(\frac{6}{5}\right) - 3}{4\left(\frac{6}{5}\right)^3 + 1} = 1.16542$$

Exercise

Use Newton's method to estimate the on real solution of $2x - x^2 + 1 = 0$. Start with $x_0 = 0$ for the left-hand zero and with $x_0 = 2$ for the zero on the right. Then, in each case, find x_2

Solution

$$y = 2x - x^2 + 1 \quad \rightarrow \quad y' = 2 - 2x$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2x_n - x_n^2 + 1}{2 - 2x_n}$$

$\boxed{x_0 = 0}$

$$x_1 = x_0 - \frac{2x_0 - x_0^2 + 1}{2 - 2x_0} = 0 - \frac{2(0) - (0)^2 + 1}{2 - 2(0)} = -0.5$$

$$x_2 = x_1 - \frac{2x_1 - x_1^2 + 1}{2 - 2x_1} = -0.5 - \frac{2(-0.5) - (-0.5)^2 + 1}{2 - 2(-0.5)} = -0.41667$$

$\boxed{x_0 = 2}$

$$x_1 = x_0 - \frac{2x_0 - x_0^2 + 1}{2 - 2x_0} = 2 - \frac{2(2) - (2)^2 + 1}{2 - 2(2)} = 2.5$$

$$x_2 = x_1 - \frac{2x_1 - x_1^2 + 1}{2 - 2x_1} = 2.5 - \frac{2(2.5) - (2.5)^2 + 1}{2 - 2(2.5)} = 2.41667$$

Exercise

Use Newton's method to estimate the on real solution of $x^4 - 2 = 0$. Start with $x_0 = 1$ and then find x_2

Solution

$$y = x^4 - 2 \quad \to \quad y' = 4x^3$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 2}{4x_n^3}$$

$$\boxed{x_0 = 1}$$

$$x_1 = x_0 - \frac{x_0^4 - 2}{4x_0^3} = 1 - \frac{(1)^4 - 2}{4(1)^3} = \underline{1.25}$$

$$x_2 = x_1 - \frac{x_1^4 - 2}{4x_1^3} = 1.25 - \frac{(1.25)^4 - 2}{4(1.25)^3} \approx \underline{1.1935}$$

Exercise

Use the Newton's method to approximate the roots to ten digits of $f(x) = 3x^3 - 4x^2 + 1$

Solution

By inspection: $x_1 = 1$ (***root***)

$$\begin{array}{c|cccc} 1 & 3 & -4 & 0 & 1 \\ & & 3 & -1 & -1 \\ \hline & 3 & -1 & -1 & 0 \end{array} \quad \to \quad 3x^2 - x - 1$$

$$f(x) = (x-1)\left(3x^2 - x - 1\right)$$

We apply Newton's method to $g(x) = 3x^2 - x - 1$

$g(0) = -1 \quad g(1) = 1 \quad g'(x) = 6x - 1$

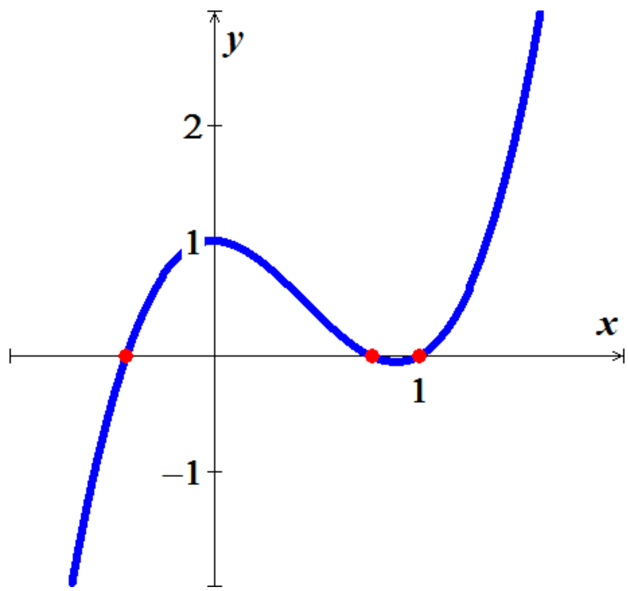
n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	0.0000000000	–1	–1	–1
1	–1	3.000000	–7.000000	–0.5714285714
2	–0.5714285714	0.5510204082	–4.4285714286	–0.4470046083
3	–0.4470046083	0.0464439678	–3.6820276497	–0.4343909149
4	–0.4343909149	0.0004773158	–3.6063454894	–0.4342585605

5	–0.4342585605	0.0000000525	–3.6055513629	–0.4342585459
6	–0.4342585459	0.0000000000	–3.6055512755	–0.4342585459

$x_2 \approx -0.4342585459$

n	x_n	$f\left(x_n\right)$	$f'\left(x_n\right)$	$x_{n+1} = x_n - \frac{f\left(x_n\right)}{f'\left(x_n\right)}$
0	1	1	5	0.8
1	0.8000000000	0.1200000000	3.8000000000	0.7684210526
2	0.7684210526	0.0029916898	3.6105263158	0.7675924505
3	0.7675924505	0.0000020597	3.6055547030	0.7675918792
4	0.7675918792	–0.0000000000	3.6055512755	0.7675918792

$x_3 \approx 0.7675918792$



Exercise

Use the Newton's method to approximate the roots to ten digits of $f(x) = e^{-2x} + 2e^{x} - 6$

Solution

$$f'(x) = -2e^{-2x} + 2e^{x}$$

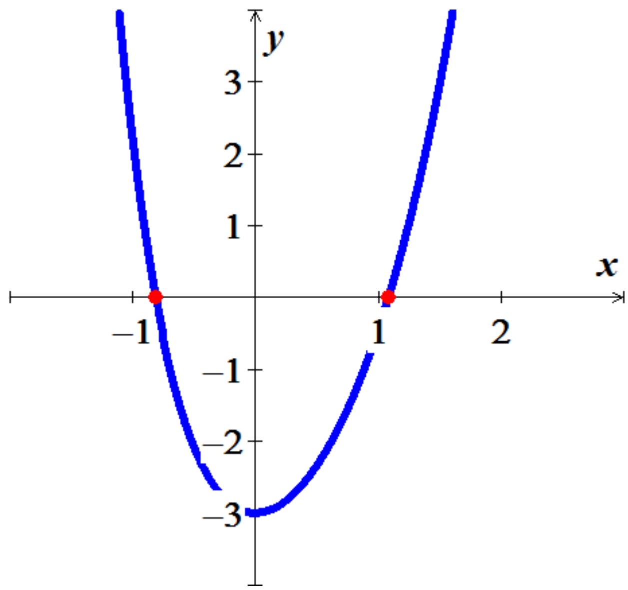
n	x_n	$f\left(x_n\right)$	$f'\left(x_n\right)$	$x_{n+1} = x_n - \frac{f\left(x_n\right)}{f'\left(x_n\right)}$
0	–1.0000000000	2.1248149813	–14.0423533155	–0.8486852642
1	–0.8486852642	0.3155271886	–10.0631909420	–0.8173306780
2	–0.8173306780	0.0109389885	–9.3722247034	–0.8161635070

3	–0.8161635070	0.0000145618	–9.3472814463	–0.8161619491
4	–0.8161619491	0.0000000000	–9.3472481901	–0.8161619491

$\underline{x \approx -0.8161619491}|$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
0	1.0000000000	–0.4281010598	5.1658930904	1.0828706774
1	1.0828706774	0.0209547377	5.6769600505	1.0791794875
2	1.0791794875	0.0000433190	5.6534997356	1.0791718252
3	1.0791718252	0.0000000002	5.6534511061	1.0791718251

$\underline{x \approx 1.0791718251}|$



Exercise

Use the Newton's method to approximate the roots to ten digits of $f(x) = 2x^5 - 6x^3 - 4x + 2$

Solution

$f(x) = 10x^4 - 18x^2 - 4$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
0	0.0000000000	2.0000000000	–4.0000000000	0.5000000000
1	0.5000000000	–0.6875000000	–7.8750000000	0.4126984127
2	0.4126984127	–0.0485945125	–6.7756706818	0.4055265009
3	0.4055265009	–0.0003088207	–6.6896876153	0.4054803372
4	0.4054803372	–0.0000000128	–6.6891368363	0.4054803353

$x \approx 0.4054803353$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	−2.0000000000	−6.0000000000	84.0000000000	−1.9285714286
1	−1.9285714286	−0.6062020289	67.3894731362	−1.9195759282
2	−1.9195759282	−0.0087501134	65.4495366742	−1.9194422357
3	−1.9194422357	−0.0000019108	65.4209537375	−1.9194422065
4	−1.9194422065	0.0000000000	65.4209474938	−1.9194422065

$x \approx -1.9194422065$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	2.0000000000	10.0000000000	84.0000000000	1.8809523810
1	1.8809523810	1.6364873659	57.4894829058	1.8524865245
2	1.8524865245	0.0789319071	51.9953690888	1.8509684681
3	1.8509684681	0.0002159404	51.7110173775	1.8509642921
4	1.8509642921	0.0000000016	51.7102363689	1.8509642921

$x \approx 1.8509642921$