Section 3.5 – Exponential and logarithmic Equations

Exponential Equations

$$b^M = b^N \quad \leftrightarrow \quad M = N \quad \text{for any b} > 0, \neq 1$$

Example

Solve $\ 5^{3x-6} = 125$

Solution

$5^{3x-6} = 5^3$

$3x - 6 = 3$

$3x = 9$

$\underline{x = 3}$

Example

Solve $\ 8^{x+2} = 4^{x-3}$

Solution

$\left(2^3\right)^{x+2} = \left(2^2\right)^{x-3}$

$2^{3(x+2)} = 2^{2(x-3)}$

$3(x+2) = 2(x-3)$

$3x + 6 = 2x - 6$

$3x - 2x = -6 - 6$

$\underline{x = -12}$

Using *Natural Logarithms*

1. Isolate the exponential expression

2. Take the natural logarithm on both sides of the equation

3. Simplify using one of the following properties: $\ln b^x = x \ln b$ *or* $\ln e^x = x$

4. Solve for the variable

Example

Solve: $7e^{2x} - 5 = 58$

Solution

$7e^{2x} - 5 = 58$	*Isolate the exponential expression*
$7e^{2x} = 63$	*Divide by 7 both sides*
$e^{2x} = 9$	*Natural logarithm on both sides*
$\ln e^{2x} = \ln 9$	*Use inverse Property*
$2x = \ln 9$	
$x = \dfrac{\ln 9}{2} \approx 1.0986$	

Example

Solve: $3^{2x-1} = 7^{x+1}$

Solution

$\ln 3^{2x-1} = \ln 7^{x+1}$	*Natural logarithm on both sides*
$(2x - 1)\ln 3 = (x + 1)\ln 7$	*Power Rule*
$2x \ln 3 - \ln 3 = x \ln 7 + \ln 7$	
$2x \ln 3 - x \ln 7 = \ln 3 + \ln 7$	
$x(2\ln 3 - \ln 7) = \ln 3 + \ln 7$	
$x = \dfrac{\ln 3 + \ln 7}{2\ln 3 - \ln 7} \approx 12.1143$	

Logarithmic Equations

1. Express the equation in the form $\log_b M = c$

2. Use the definition of a logarithm to rewrite the equation in exponential form:

$$\log_b M = c \quad \Rightarrow \quad b^c = M$$

3. Solve for the variable
4. Check proposed solution in the original equation. Include only the set for $M > 0$

Example

Solve: $\log(x) + \log(x-3) = 1$

Solution

$\log(x(x-3)) = 1$	*Product Rule*
$x(x-3) = 10^1$	*Convert to exponential form*
$x^2 - 3x = 10$	
$x^2 - 3x - 10 = 0$	*Solve for x*
$x = -2, 5$	

Check: $x = -2 \Rightarrow \log(-2) + \log(x - 3) = 1$

$x = 5 \Rightarrow \log(5) + \log(5-3) = 1$

$\therefore$ *Solution*: $\underline{x = 5}$

Example

Solve: $\log_6(3x+2) + \log_6(x-1) = 1$

Solution

$\log_6[(3x+2)(x-1)] = 1$	*Product Rule*
$(3x+2)(x-1) = 6^1$	*Convert to exponential form*
$3x^2 - x - 2 = 6$	
$3x^2 - x - 8 = 0$	*Solve for x*

$$x = \frac{1-\sqrt{97}}{6} < 0 \qquad x = \frac{1+\sqrt{97}}{6} > 1$$

$\therefore$ *Solution*: $x = \dfrac{1+\sqrt{97}}{6}$

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Property of Logarithmic Equality

For any $M > 0, N > 0, b > 0, \neq 1$

$$\log_b M = \log_b N \quad \Rightarrow \quad M = N$$

Example

Solve: $\ln(x-3) = \ln(7x-23) - \ln(x+1)$

Solution

$\ln(x-3) = \ln\left(\dfrac{7x-23}{x+1}\right)$ *Quotient Rule*

$x - 3 = \dfrac{7x-23}{x+1}$

$(x-3)(x+1) = 7x - 23$

$x^2 - 2x - 3 = 7x - 23$

$x^2 - 9x + 20 = 0$

$\underline{x = 4,\ 5}$

         Check: $x = 4 \implies \ln(4-3) = \ln(7(4) - 23) - \ln(4+1)$

                        $x = 5 \implies \ln(5-3) = \ln(7(5) - 23) - \ln(5+1)$

$\therefore$ ***Solution***: $\underline{x = 4,\ 5}$

Example

Solve: $\log(x+6) - \log(x+2) = \log x$

Solution

$\log\dfrac{x+6}{x+2} = \log x$ *Quotient Rule*

$\dfrac{x+6}{x+2} = x$ *Multiply by x + 2*

$x + 6 = x(x+2)$

$x + 6 = x^2 + 2x$

$x^2 + x - 6 = 0$ *Solve for X*

$\underline{x = -3,\ 2}$

       Check: $x = -3 \to \log(-3+6) - \log(-3+2) = \log(-3)$ *Or Domain*

                   $x = 2 \to \log(2+6) - \log(2+2) = \log(2)$

$\therefore$ ***Solution***: $\underline{x = 2}$

Exercises   *Section* **3.5 – Exponential and logarithmic Equations**

(1 – 105)   Solve the equations

1. $2^x = 128$

2. $3^x = 243$

3. $5^x = 70$

4. $6^x = 50$

5. $5^x = 134$

6. $7^x = 12$

7. $9^x = \dfrac{1}{\sqrt[3]{3}}$

8. $49^x = \dfrac{1}{343}$

9. $2^{5x+3} = \dfrac{1}{16}$

10. $\left(\dfrac{2}{5}\right)^x = \dfrac{8}{125}$

11. $2^{3x-7} = 32$

12. $4^{2x-1} = 64$

13. $3^{1-x} = \dfrac{1}{27}$

14. $2^{-x^2} = 5$

15. $2^{-x} = 8$

16. $\left(\dfrac{1}{3}\right)^x = 81$

17. $3^{-x} = 120$

18. $27 = 3^{5x}\,9^{x^2}$

19. $4^{x+3} = 3^{-x}$

20. $2^{x+4} = 8^{x-6}$

21. $8^{x+2} = 4^{x-3}$

22. $7^x = 12$

23. $5^{x+4} = 4^{x+5}$

24. $5^{x+2} = 4^{1-x}$

25. $3^{2x-1} = 0.4^{x+2}$

26. $4^{3x-5} = 16$

27. $4^{x+3} = 3^{-x}$

28. $7^{2x+1} = 3^{x+2}$

29. $3^{x-1} = 7^{2x+5}$

30. $4^{x-2} = 2^{3x+3}$

31. $3^{5x-8} = 9^{x+2}$

32. $3^{x+4} = 2^{1-3x}$

33. $3^{2-3x} = 4^{2x+1}$

34. $4^{x+3} = 3^{-x}$

35. $7^{x+6} = 7^{3x-4}$

36. $2^{-100x} = (0.5)^{x-4}$

37. $4^x \left(\dfrac{1}{2}\right)^{3-2x} = 8.\left(2^x\right)^2$

38. $5^x + 125\left(5^{-x}\right) = 30$

39. $4^x - 3\left(4^{-x}\right) = 8$

40. $5^{3x-6} = 125$

41. $e^x = 15$

42. $e^{x+1} = 20$

43. $9e^x = 107$

44. $e^{x\ln 3} = 27$

45. $e^{x^2} = e^{7x-12}$

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46. $f(x) = xe^x + e^x$

47. $f(x) = x^3 \left(4e^{4x}\right) + 3x^2 e^{4x}$

48. $e^{2x} - 2e^x - 3 = 0$

49. $e^{0.08t} = 2500$

50. $e^{x^2} = 200$

51. $e^{2x+1} \cdot e^{-4x} = 3e$

52. $e^{2x} - 8e^x + 7 = 0$

53. $e^{2x} + 2e^x - 15 = 0$

54. $e^x + e^{-x} - 6 = 0$

55. $e^{1-3x} \cdot e^{5x} = 2e$

56. $6\ln(2x) = 30$

57. $\log_5(x - 7) = 2$

58. $\log_4(5 + x) = 3$

59. $\log(4x - 18) = 1$

60. $\log_3 x = -2$

61. $\log\left(x^2 + 19\right) = 2$

62. $\ln\left(x^2 - 12\right) = \ln x$

63. $\log\left(2x^2 + 3x\right) = \log(10x + 30)$

64. $\log_5(2x + 3) = \log_5 11 + \log_5 3$

65. $\log_3 x - \log_9(x + 42) = 0$

66. $\log_5 x + \log_5(4x - 1) = 1$

67. $\log x - \log(x + 3) = 1$

68. $\log x + \log(x - 9) = 1$

69. $\log_2(x + 1) + \log_2(x - 1) = 3$

70. $\log_8(x + 1) - \log_8 x = 2$

71. $\ln(x + 8) + \ln(x - 1) = 2\ln x$

72. $\ln(4x + 6) - \ln(x + 5) = \ln x$

73. $\ln(5 + 4x) - \ln(x + 3) = \ln 3$

74. $\ln \sqrt[4]{x} = \sqrt{\ln x}$

75. $\sqrt{\ln x} = \ln \sqrt{x}$

76. $\log x^2 = (\log x)^2$

77. $\log x^3 = (\log x)^2$

78. $\log(\log x) = 1$

79. $\log(\log x) = 2$

80. $\ln(\ln x) = 2$

81. $\ln\left(e^{x^2}\right) = 64$

82. $e^{\ln(x-1)} = 4$

83. $10^{\log(2x+5)} = 9$

84. $\log \sqrt{x^3 - 9} = 2$

85. $\log \sqrt{x^3 - 17} = \frac{1}{2}$

86. $\log_4 x = \log_4(8 - x)$

87. $\log_7(x - 5) = \log_7(6x)$

88. $\ln x^2 = \ln(12 - x)$

89. $\log_2(x + 7) + \log_2 x = 3$

90. $\ln x = 1 - \ln(x + 2)$

91. $\ln x = 1 + \ln(x + 1)$

92. $\log_6(2x - 3) = \log_6 12 - \log_6 3$

93. $\log(3x + 2) + \log(x - 1) = 1$

94. $\log_5(x + 2) + \log_5(x - 2) = 1$

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95. $\log_2 x + \log_2 (x - 4) = 2$

96. $\log_3 x + \log_3 (x + 6) = 3$

97. $\log_3 (x + 3) + \log_3 (x + 5) = 1$

98. $\ln x = \frac{1}{2} \ln \left(2x + \frac{5}{2} \right) + \frac{1}{2} \ln 2$

99. $\ln (-4 - x) + \ln 3 = \ln (2 - x)$

100. $\log_4 x + \log_4 (x - 2) = \log_4 (15)$

101. $\ln (x - 5) - \ln (x + 4) = \ln (x - 1) - \ln (x + 2)$

102. $\ln (4 - x) = \ln (x + 8) + \ln (2x + 13)$

103. $\log \left(x^2 + 4 \right) - \log (x + 2) = 2 + \log (x - 2)$

104. $\log_3 (x - 2) = \log_3 27 - \log_3 (x - 4) - 5^{\log_5 1}$

105. $\log_2 (x + 3) = \log_2 (x - 3) + \log_3 9 + 4^{\log_4 3}$

106. Solve for t using logarithms with base a: $2a^{t/3} = 5$

107. Solve for t using logarithms with base a: $K = H - Ca^t$