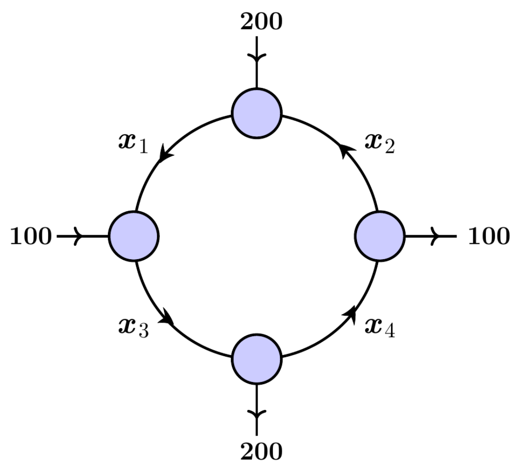
# Solution    *Section* 1.8 – Applications

### *Exercise*

The flow of traffic, in vehicles per hour, through a network of streets as is shown below




*a)* Solve this system for $x_i, \quad i = 1,\ 2,\ 3,\ 4.$

*b)* Find the traffic flow when $x_4 = 0$.

*c)* Find the traffic flow when $x_4 = 100$.

*d)* Find the traffic flow when $x_1 = 2x_2$.

***Solution***

***a)*** $$\begin{cases} x_1 + 100 = x_3 \\ x_2 + 200 = x_1 \\ x_2 + 100 = x_4 \\ x_4 + 200 = x_3 \end{cases}$$

$$\begin{cases} -x_1 + x_3 = 100 \\ x_1 - x_2 = 200 \\ -x_2 + x_4 = 100 \\ x_3 - x_4 = 200 \end{cases}$$

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 1 & -1 & 0 & 0 & 200 \\ 0 & -1 & 0 & 1 & 100 \\ 0 & 0 & 1 & -1 & 200 \end{array}\right) \qquad R_2 + R_1$$

$$\begin{vmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1\begin{vmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 0 & -1 & 1 & 0 & 300 \\ 0 & -1 & 0 & 1 & 100 \\ 0 & 0 & 1 & -1 & 200 \end{array}\right) \quad R_3 - R_2$$

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 0 & -1 & 1 & 0 & 300 \\ 0 & 0 & -1 & 1 & -200 \\ 0 & 0 & 1 & -1 & 200 \end{array}\right) \quad R_4 + R_3$$

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 0 & -1 & 1 & 0 & 300 \\ 0 & 0 & -1 & 1 & -200 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} \to\ -x_1 + x_3 = 100 \\ \to\ -x_2 + x_3 = 100 \\ \to\ -x_3 + x_4 = 100 \end{array}$$

Let $x_4$ be the free variable

$$\begin{cases} \quad x_3 = x_4 + 200 \\ \quad x_2 = x_4 - 100 \\ x_1 = 200 + x_2 = x_4 + 100 \end{cases}$$

***Solution***: $\left(x_4 + 100,\ \ x_4 - 100,\ \ x_4 + 200,\ \ x_4\right)$

***OR***

$$\begin{vmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1\begin{vmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= -1(1) - 1(-1)$$

$$= -1 + 1$$

$$= 0$$

$$\begin{cases} -x_1 + x_3 = 100 & \to\ x_1 = x_3 - 100 = x_4 + 100 \\ x_1 - x_2 = 200 & \\ -x_2 + x_4 = 100 & \to\ x_2 = x_4 - 100 \\ x_3 - x_4 = 200 & \to\ x_3 = x_4 + 200 \end{cases}$$

***b)*** The traffic flow when $x_4 = 0$ is:

$\therefore\ (100,\ \ -100,\ \ 200,\ \ 0)$

***c)*** The traffic flow when $x_4 = 100$ is:

$\therefore\ (200,\quad 0,\quad 300,\quad 100)$

***d)*** The traffic flow when $x_1 = 2x_2$ :

$x_4 + 100 = 2(x_4 - 100)$

$x_4 + 100 = 2x_4 - 200$

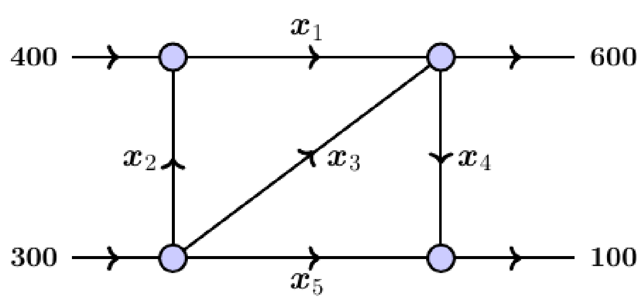
$x_4 = 300$

$\therefore\ (400,\quad 200,\quad 500,\quad 300)$

## *Exercise*

Through a network, Express $x_n$'s in terms of the parameters ***s*** and ***t***.

***Solution***




$$\begin{cases} x_1 = x_2 + 400 \\ x_1 + x_3 = x_4 + 600 \\ x_4 + x_5 = 100 \\ x_2 + x_3 + x_5 = 300 \end{cases}$$

$$\begin{cases} x_1 - x_2 = 400 \\ x_2 + x_3 - x_4 = 600 \\ x_4 + x_5 = 100 \\ x_2 + x_3 + x_5 = 300 \end{cases}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 1 & 0 & 1 & -1 & 0 & 600 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 1 & 1 & 0 & 1 & 300 \end{array}\right) \quad \begin{array}{c} \\ R_2 - R_1 \\ \\ R_3 \leftrightarrow R_4 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 1 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & 1 & 100 \end{array}\right) \quad \begin{array}{c} \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 100 \end{array}\right) \quad \begin{array}{c} \\ \\ \\ R_4 - R_3 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{rl} x_1 - x_2 = 400 & \rightarrow x_1 = 400 + x_2 \\ x_2 + x_3 - x_4 = 200 & \rightarrow x_2 = 200 - x_3 + x_4 \\ x_4 + x_5 = 100 & \rightarrow \underline{x_4 = 100 - t} \end{array}$$

Let $x_5 = t$ & $x_3 = s$

$x_2 = 200 - s + 100 - t \ \underline{= 300 - s - t}$

$x_1 = 400 + 300 - s - t \ \underline{= 700 - s - t}$

## *Exercise*

Water is flowing through a network of pipes. Express $x_n$ 's in terms of the parameters ***s*** and ***t***.

***Solution***

$x_1 + x_3 = 900$

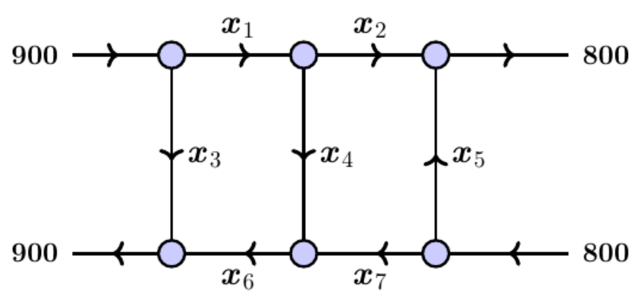
$x_1 = x_2 + x_4 \quad \rightarrow \quad x_1 - x_2 - x_4 = 0$

$x_2 + x_5 = 800$

$x_5 + x_7 = 800$

$x_6 = x_4 + x_7 \quad \rightarrow \quad x_4 - x_6 + x_7 = 0$

$x_3 + x_6 = 900$




$$\begin{array}{ll} \left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 900 \end{array}\right] & \begin{array}{l} \\ R_2 - R_1 \\ \\ \\ \\ \\ \end{array} \end{array}$$

$$\begin{array}{ll} \left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & -900 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 900 \end{array}\right] & \begin{array}{l} \\ \\ R_3 + R_2 \\ R_6 \\ \\ R_4 \end{array} \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & -900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 900 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ R_4 + R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 800 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ R_5 + R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \\ R_6 - R_5 \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{rll} x_2 &= 900 - x_3 & (5) \\ x_2 &= 900 - x_3 - x_4 & (4) \\ x_3 &= 100 - x_4 + x_5 & (3) \\ -x_4 &= 800 - x_5 - x_6 & (2) \\ x_5 &= 800 - x_7 & (1) \end{array}$$

Let $\quad x_6 = s \quad \& \quad x_7 = t$

$(1) \rightarrow \quad x_5 = 800 - t$

$(2) \rightarrow \quad x_4 = s - t$

$(3) \rightarrow \quad x_3 = 900 - s$

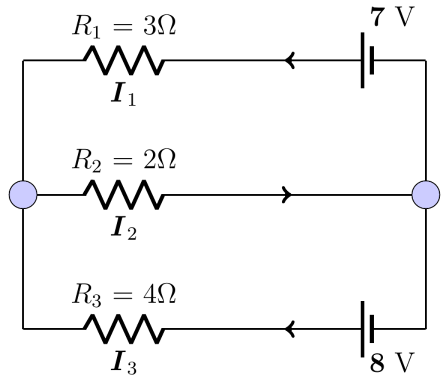
$(2) \rightarrow \quad x_2 = t$

$(1) \rightarrow \quad x_2 = s$

***Solution***: $\underline{(s,\ t,\ 900 - s,\ s - t,\ 800 - t,\ s,\ t)}$

### *Exercise*

Determine the currents $I_1, I_2,$ *and* $I_3$ for the electrical network shown below




***Solution***

$$I_2 = I_1 + I_3$$
$$3I_1 + 2I_2 = 7$$
$$2I_2 + 4I_3 = 8$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 3I_1 + 2I_2 \qquad = 7 \\ \qquad I_2 + 2I_3 = 4 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 3 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 13 \qquad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 7 & 2 & 0 \\ 4 & 1 & 2 \end{vmatrix} = 13 \qquad D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 7 & 0 \\ 0 & 4 & 2 \end{vmatrix} = 26 \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 3 & 2 & 7 \\ 0 & 1 & 4 \end{vmatrix} = 13$$

$\underline{I_1 = 1\ A}$ $\quad$ $\underline{I_2 = 2\ A}$ $\quad$ $\underline{I_3 = 1\ A}$

***OR***

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 3 & 2 & 0 & 7 \\ 0 & 1 & 2 & 4 \end{array}\right) \quad R_2 - 3R_1$$

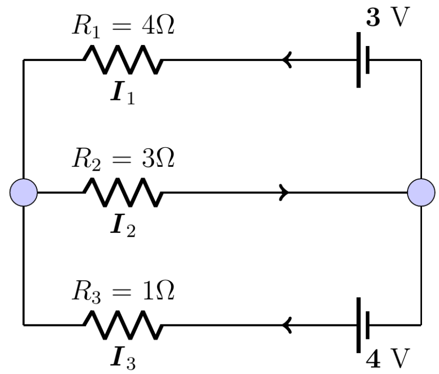
$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -3 & 7 \\ 0 & 1 & 2 & 4 \end{array}\right) \quad -5R_3 + R_2$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -3 & 7 \\ 0 & 0 & -13 & -13 \end{array}\right) \quad \begin{array}{l} I_1 = I_2 - I_3 \\ 5I_2 = 3I_3 + 7 \\ \underline{I_3 = 1} \end{array}$$

$\underline{I_2 = 2}$ $\quad$ $\underline{I_1 = 1}$

## *Exercise*

Determine the currents $I_1, I_2$, *and* $I_3$ for the electrical network shown below




***Solution***

$$I_2 = I_1 + I_3$$

$$4I_1 + 3I_2 = 3$$

$$3I_2 + I_3 = 4$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 4I_1 + 3I_2 \qquad = 3 \\ \qquad 3I_2 + I_3 = 4 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 4 & 3 & 0 \\ 0 & 3 & 1 \end{vmatrix} = 19 \qquad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 3 & 3 & 0 \\ 4 & 3 & 1 \end{vmatrix} = 0 \qquad D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 4 & 3 & 0 \\ 0 & 4 & 1 \end{vmatrix} = 19 \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 4 & 3 & 3 \\ 0 & 3 & 4 \end{vmatrix} = 19$$

$\underline{I_1 = 0\ A}|$ $\underline{I_2 = 1\ A}|$ $\underline{I_3 = 1\ A}|$

***OR***

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 4 & 3 & 0 & 3 \\ 0 & 3 & 1 & 4 \end{array}\right) \quad R_2 - 4R_1$$

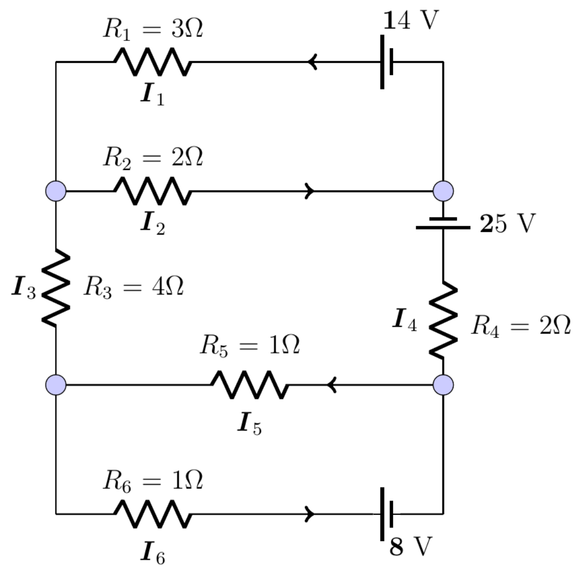
$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 3 \\ 0 & 3 & 1 & 4 \end{array}\right) \quad 7R_3 - 3R_2$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 3 \\ 0 & 0 & 19 & 19 \end{array}\right) \begin{array}{l} \rightarrow\ I_1 = I_2 - I_3 \quad (2) \\ \rightarrow\ 7I_2 = 4I_3 + 3 \quad (1) \\ \quad \underline{I_3 = 1}| \end{array}$$

$\underline{I_2 = 1}|$ $\underline{I_1 = 0}|$

### *Exercise*

Determine the currents $I_1, I_2, I_3, I_4, I_5,$ *and* $I_6$ for the electrical network shown below




## Solution

$$I_1 + I_3 = I_2 \quad \to \quad I_1 - I_2 + I_3 = 0$$

$$I_1 + I_4 = I_2 \quad \to \quad I_1 - I_2 + I_4 = 0$$

$$I_3 + I_6 = I_5 \quad \to \quad I_3 - I_5 + I_6 = 0$$

$$\begin{cases} 3I_1 + 2I_2 = 14 \\ 2I_2 + 4I_3 + I_5 + 2I_4 = 25 \\ I_5 + I_6 = 8 \end{cases}$$

$$\begin{array}{cl} \left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] & \begin{array}{l} \\ R_2 - R_1 \\ \\ R_4 - 3R_1 \\ \\ \\ \end{array} \end{array}$$

$$\begin{array}{cl} \left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] & \begin{array}{l} \\ R_4 \\ \\ R_2 \\ R_3 \\ \\ \end{array} \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \quad \begin{array}{l} \\ \\ 5R_3 - 2R_2 \\ \\ R_5 + R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \quad \begin{array}{l} \\ \\ \\ 26R_4 + R_3 \\ \\ \\ \end{array}$$

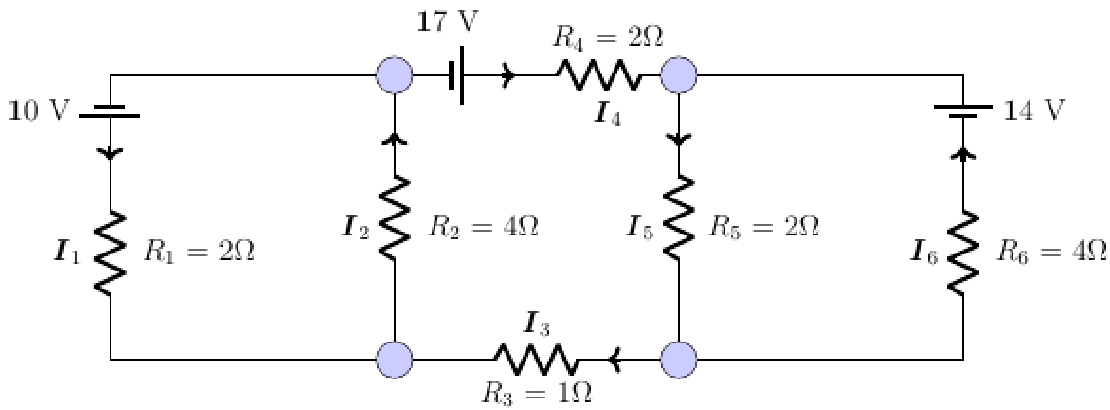
$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & 0 & 36 & 5 & 0 & 97 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \quad \begin{array}{l} \\ \\ \\ \\ 36R_5 - R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & 0 & 36 & 5 & 0 & 97 \\ 0 & 0 & 0 & 0 & -41 & 36 & -97 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \quad \begin{array}{l} \\ \\ \\ \\ \\ 41R_6 + R_5 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & 0 & 36 & 5 & 0 & 97 \\ 0 & 0 & 0 & 0 & -41 & 36 & -97 \\ 0 & 0 & 0 & 0 & 0 & 77 & 231 \end{array}\right] \quad \begin{array}{rcl} I_1 = 4 - 2 & \rightarrow & \boxed{I_1 = 2} \\ 5I_2 = 14 + 3(2) & \rightarrow & \boxed{I_2 = 4} \\ 26I_3 = 97 - 10(2) - 5(5) & \rightarrow & \boxed{I_3 = 2} \\ 36I_4 = 97 - 5(5) & \rightarrow & \boxed{I_4 = 2} \\ -41I_5 = -97 - 36(3) & \rightarrow & \boxed{I_5 = 5} \\ 77I_6 = 231 & \rightarrow & \boxed{I_6 = 3} \end{array}$$

### *Exercise*

Determine the currents $I_1, I_2, I_3, I_4, I_5,$ *and* $I_6$ for the electrical network shown below




***Solution***

$1 \rightarrow \quad I_1 + I_3 = I_2$

$2 \rightarrow \quad I_1 + I_4 = I_2$

$3 \rightarrow \quad I_3 + I_6 = I_5$

$4 \rightarrow \quad I_4 + I_6 = I_5$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ I_1 - I_2 + I_4 = 0 \\ I_3 - I_5 + I_6 = 0 \\ I_4 - I_5 + I_6 = 0 \\ 2I_1 + 4I_2 = 10 \\ 4I_2 + I_3 + 2I_4 + 2I_5 = 17 \\ 2I_5 + 4I_6 = 14 \end{cases}$$

$$\begin{array}{cc} \left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 2 & 0 & 17 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \end{array}\right) & \begin{array}{l} \\ R_2 - R_1 \\ \\ \\ R_5 - R_1 \\ \\ \\ \end{array} \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 2 & 0 & 17 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \end{array}\right) \begin{array}{l} \\ \\ R_3 + R_2 \\ \\ \\ 3R_6 - 4R_5 \\ \\ \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 7 & 6 & 6 & 0 & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \end{array}\right) \begin{array}{l} \\ \\ \\ R_4 - R_3 \\ \\ \\ \\ \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 7 & 6 & 6 & 0 & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \end{array}\right) \begin{array}{l} \\ \\ \\ \\ \\ R_6 + 7R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 13 & 6 & 0 & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \\ \\ \\ \\ R_5 - 13R_4 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 19 & -13 & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \\ \\ \\ \\ \\ 19R_6 - R_5 \\ \\ \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 19 & -13 & 31 \\ 0 & 0 & 0 & 0 & 0 & 51 & 102 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} (4) \\ (3) \\ (2) \\ (1) \\ I_5 = \frac{1}{19}\left(31 + 13I_6\right) \\ \underline{I_6 = 2} \end{array}$$

$\underline{I_5 = 3}$

$(1) \to \quad I_4 = I_5 - I_6 \ \underline{= 1}$

$(2) \to \quad I_3 = I_4 \ \underline{= 1}$

$(3) \to \quad I_2 = \frac{1}{3}\left(I_3 + 5\right) \ \underline{= 2}$

$(4) \to \quad I_1 = I_2 - I_3 \ \underline{= 1}$

## *Exercise*

Consider the invertible matrix: $\quad A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{pmatrix}$

The message: ***ICEBERG DEAD AHEAD***

*a)* Write the uncoded row matrices $1 \times 3$ for the message.
*b)* Use the matrix $A$ to encode the message.
*c)* Decode a message from part *b*) given the matrix $A$.

***Solution***

***a)***

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 = _ | 4 = D | 8 = H | 12 = L | 16 = P | 20 = T | 24 = X |
| 1 = A | 5 = E | 9 = I | 13 = M | 17 = Q | 21 = U | 25 = Y |
| 2 = B | 6 = F | 10 = J | 14 = N | 18 = R | 22 = V | 26 = Z |
| 3 = C | 7 = G | 11 = K | 15 = O | 19 = S | 23 = W | |

$$\begin{array}{cccccc} I\ C\ E & B\ E\ R & G\ \_\ D & E\ A\ D & \_\ A\ H & E\ A\ D \\ [9\ \ 3\ \ 5] & [2\ \ 5\ \ 18] & [7\ \ 0\ \ 4] & [5\ \ 1\ \ 4] & [0\ \ 1\ \ 8] & [5\ \ 1\ \ 4] \end{array}$$

***b)*** Let encode the message ***ICEBERG DEAD AHEAD***

$$\begin{bmatrix}9 & 3 & 5\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 3 & 7 & 9\\ -3 & -2 & 7\end{bmatrix}=\begin{bmatrix}3 & 29 & 80\end{bmatrix}$$

$$\begin{bmatrix}2 & 5 & 18\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 3 & 7 & 9\\ -3 & -2 & 7\end{bmatrix}=\begin{bmatrix}-37 & 3 & 175\end{bmatrix}$$

$$\begin{bmatrix}7 & 0 & 4\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 3 & 7 & 9\\ -3 & -2 & 7\end{bmatrix}=\begin{bmatrix}-5 & 6 & 42\end{bmatrix}$$

$$\begin{bmatrix}5 & 1 & 4\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 3 & 7 & 9\\ -3 & -2 & 7\end{bmatrix}=\begin{bmatrix}-4 & 9 & 47\end{bmatrix}$$

$$\begin{bmatrix}0 & 1 & 8\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 3 & 7 & 9\\ -3 & -2 & 7\end{bmatrix}=\begin{bmatrix}-21 & -5 & 65\end{bmatrix}$$

$$\begin{bmatrix}5 & 1 & 4\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 3 & 7 & 9\\ -3 & -2 & 7\end{bmatrix}=\begin{bmatrix}-4 & 9 & 47\end{bmatrix}$$

The sequence of coded row matrices is

$$\begin{bmatrix}3 & 29 & 80\end{bmatrix}\ \begin{bmatrix}-37 & 3 & 175\end{bmatrix}\ \begin{bmatrix}-5 & 6 & 42\end{bmatrix}\ \begin{bmatrix}-4 & 9 & 47\end{bmatrix}\ \begin{bmatrix}-21 & -9 & 65\end{bmatrix}\ \begin{bmatrix}-4 & 9 & 47\end{bmatrix}$$

The cryptogram:

$$3\ \ 29\ \ 80\ \ -37\ \ 3\ \ 175\ \ -5\ \ 6\ \ 42\ \ -4\ \ 9\ \ 47\ \ -21\ \ -9\ \ 65\ \ -4\ \ 9\ \ 47$$

***c)*** To decode a message given the matrix $A$.

$$|A|=\begin{vmatrix}1 & 2 & 2\\ 3 & 7 & 9\\ -3 & -2 & 7\end{vmatrix}=1$$

$$A^{-1}=\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}$$

With the cryptogram:

$$\begin{bmatrix}3 & 29 & 80\end{bmatrix}\ \begin{bmatrix}-37 & 3 & 175\end{bmatrix}\ \begin{bmatrix}-5 & 6 & 42\end{bmatrix}\ \begin{bmatrix}-4 & 9 & 47\end{bmatrix}\ \begin{bmatrix}-21 & -9 & 65\end{bmatrix}\ \begin{bmatrix}-4 & 9 & 47\end{bmatrix}$$

$$\begin{bmatrix} 3 & 29 & 80 \end{bmatrix}\begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} -37 & 3 & 175 \end{bmatrix}\begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 18 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 6 & 42 \end{bmatrix}\begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 9 & 47 \end{bmatrix}\begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -21 & -9 & 65 \end{bmatrix}\begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 8 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 9 & 47 \end{bmatrix}\begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 4 \end{bmatrix}$$

The message is:

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 3 | 5 | 2 | 5 | 18 | 7 | 0 | 1 | 5 | 1 | 4 | 0 | 1 | 8 | 5 | 1 | 4 |
| *I* | *C* | *E* | *B* | *E* | *R* | *G* | _ | *D* | *E* | *A* | *D* | _ | *A* | *H* | *E* | *A* | *D* |

## *Exercise*

You want to send the message: ***LINEAR ALGEBRA*** with a key word ***MATH***

*a)* Write the matrix $A$.

*b)* Write the uncoded row matrices $1\times 2$ for the message.

*c)* Use the matrix $A$ to encode the message.

*d)* Decode a message from part *b*) given the matrix $A$.

***Solution***

***a)***

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 = _ | 4 = D | 8 = H | 12 = L | 16 = P | 20 = T | 24 = X |
| 1 = A | 5 = E | 9 = I | 13 = M | 17 = Q | 21 = U | 25 = Y |
| 2 = B | 6 = F | 10 = J | 14 = N | 18 = R | 22 = V | 26 = Z |
| 3 = C | 7 = G | 11 = K | 15 = O | 19 = S | 23 = W | |

$$\begin{matrix} M & A & T & H \\ 13 & 1 & 20 & 8 \end{matrix}$$

$$A = \begin{pmatrix} 13 & 1 \\ 20 & 8 \end{pmatrix}$$

***b)***

$$\begin{matrix} L & I & N & E & A & R & \_ & A & L & G & E & B & R & A \\ 12 & 9 & 14 & 5 & 1 & 18 & 0 & 1 & 12 & 7 & 5 & 2 & 18 & 1 \end{matrix}$$

$$[12 \quad 9] \quad [14 \quad 5] \quad [1 \quad 18] \quad [0 \quad 1] \quad [12 \quad 7] \quad [5 \quad 2] \quad [18 \quad 1]$$

***c)*** Encoding the message

$$[12 \quad 9]\begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = [336 \quad 84]$$

$$[14 \quad 5]\begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = [282 \quad 54]$$

$$[1 \quad 18]\begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = [373 \quad 145]$$

$$[0 \quad 1]\begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = [20 \quad 8]$$

$$[12 \quad 7]\begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = [296 \quad 68]$$

$$[5 \quad 2]\begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = [105 \quad 21]$$

$$[18 \quad 1]\begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = [254 \quad 26]$$

The cryptogram:

336 84 282 54 373 145 20 8 296 68 105 21 254 26

***d)*** To decode a message given the matrix $A$.

$$A = \begin{pmatrix} 13 & 1 \\ 20 & 8 \end{pmatrix}$$

$$A^{-1} = \frac{1}{84}\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{pmatrix}$$

With the cryptogram:

$$[336 \quad 84] \quad [282 \quad 54] \quad [373 \quad 145] \quad [20 \quad 8] \quad [296 \quad 68] \quad [105 \quad 21] \quad [254 \quad 26]$$

$$[336 \quad 84] \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = [12 \quad 9]$$

$$[282 \quad 54] \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = [14 \quad 5]$$

$$[373 \quad 145] \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = [1 \quad 18]$$

$$[20 \quad 8] \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = [0 \quad 1]$$

$$[296 \quad 68] \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = [12 \quad 7]$$

$$[105 \quad 21] \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = [5 \quad 2]$$

$$[254 \quad 26] \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = [18 \quad 1]$$

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 12 | 9 | 14 | 5 | 1 | 18 | 0 | 1 | 12 | 7 | 5 | 2 | 18 | 1 |
| *L* | *I* | *N* | *E* | *A* | *R* | _ | *A* | *L* | *G* | *E* | *B* | *R* | *A* |

The message is: ***Linear Algebra***

### ***Exercise***

You want to send the message: ***CRYPTOGRAPHY IS A METHOD OF PROTECTING INFORMATIONS*** with a key word ***CODE***

*a)* Write the matrix $A$.
*b)* Write the uncoded row matrices $1\times 2$ for the message.
*c)* Use the matrix $A$ to encode the message.
*d)* Decode a message from part $b$) given the matrix $A$.

***Solution***

***a)***

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 = _ | 4 = D | 8 = H | 12 = L | 16 = P | 20 = T | 24 = X |
| 1 = A | 5 = E | 9 = I | 13 = M | 17 = Q | 21 = U | 25 = Y |
| 2 = B | 6 = F | 10 = J | 14 = N | 18 = R | 22 = V | 26 = Z |
| 3 = C | 7 = G | 11 = K | 15 = O | 19 = S | 23 = W | |

$$\begin{matrix} C & O & D & E \\ 3 & 15 & 4 & 5 \end{matrix}$$

$$A=\begin{pmatrix} 3 & 15 \\ 4 & 5 \end{pmatrix}$$

***b)***

$$\begin{matrix} C & R & Y & P & T & O & G & R & A & P & H & Y & \_ & I & S & \_ & A & \_ \\ 3 & 18 & 25 & 16 & 20 & 15 & 7 & 18 & 1 & 16 & 8 & 25 & 0 & 9 & 19 & 0 & 1 & 0 \end{matrix}$$

$$\begin{matrix} M & E & T & H & O & D & \_ & O & F & \_ & P & R & O & T & E & C & T & I & N & G \\ 13 & 5 & 20 & 8 & 15 & 4 & 0 & 15 & 6 & 0 & 16 & 18 & 15 & 20 & 5 & 3 & 20 & 9 & 14 & 7 \end{matrix}$$

$$\begin{matrix} \_ & I & N & F & O & R & M & A & T & I & O & N & S & \_ \\ 0 & 9 & 14 & 6 & 15 & 18 & 13 & 1 & 20 & 9 & 15 & 14 & 19 & 0 \end{matrix}$$

$$\begin{bmatrix}3 & 18\end{bmatrix} \quad \begin{bmatrix}25 & 16\end{bmatrix} \quad \begin{bmatrix}20 & 15\end{bmatrix} \quad \begin{bmatrix}7 & 18\end{bmatrix} \quad \begin{bmatrix}1 & 16\end{bmatrix} \quad \begin{bmatrix}8 & 25\end{bmatrix} \quad \begin{bmatrix}0 & 9\end{bmatrix} \quad \begin{bmatrix}19 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\end{bmatrix} \quad \begin{bmatrix}13 & 5\end{bmatrix} \quad \begin{bmatrix}20 & 8\end{bmatrix} \quad \begin{bmatrix}15 & 4\end{bmatrix} \quad \begin{bmatrix}0 & 15\end{bmatrix} \quad \begin{bmatrix}6 & 0\end{bmatrix} \quad \begin{bmatrix}16 & 18\end{bmatrix} \quad \begin{bmatrix}15 & 20\end{bmatrix}$$

$$\begin{bmatrix}5 & 3\end{bmatrix} \quad \begin{bmatrix}20 & 9\end{bmatrix} \quad \begin{bmatrix}14 & 7\end{bmatrix} \quad \begin{bmatrix}0 & 9\end{bmatrix} \quad \begin{bmatrix}14 & 6\end{bmatrix} \quad \begin{bmatrix}15 & 18\end{bmatrix} \quad \begin{bmatrix}13 & 1\end{bmatrix} \quad \begin{bmatrix}20 & 9\end{bmatrix}$$

$$\begin{bmatrix}15 & 14\end{bmatrix} \quad \begin{bmatrix}19 & 0\end{bmatrix}$$

***c)*** Encoding the message

$$\begin{bmatrix}3 & 18\end{bmatrix}\begin{bmatrix}3 & 15 \\ 4 & 5\end{bmatrix}=\begin{bmatrix}81 & 135\end{bmatrix}$$

$$\begin{bmatrix}25 & 16\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}139 & 455\end{bmatrix}$$

$$\begin{bmatrix}20 & 15\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}120 & 375\end{bmatrix}$$

$$\begin{bmatrix}7 & 18\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}93 & 195\end{bmatrix}$$

$$\begin{bmatrix}1 & 16\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}67 & 95\end{bmatrix}$$

$$\begin{bmatrix}8 & 25\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}124 & 245\end{bmatrix}$$

$$\begin{bmatrix}0 & 9\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}36 & 45\end{bmatrix}$$

$$\begin{bmatrix}19 & 0\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}57 & 285\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}3 & 15\end{bmatrix}$$

$$\begin{bmatrix}13 & 5\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}59 & 220\end{bmatrix}$$

$$\begin{bmatrix}20 & 8\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}92 & 340\end{bmatrix}$$

$$\begin{bmatrix}15 & 4\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}61 & 245\end{bmatrix}$$

$$\begin{bmatrix}0 & 15\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}60 & 75\end{bmatrix}$$

$$\begin{bmatrix}6 & 0\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}18 & 90\end{bmatrix}$$

$$\begin{bmatrix}16 & 18\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}120 & 330\end{bmatrix}$$

$$\begin{bmatrix}15 & 20\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}125 & 325\end{bmatrix}$$

$$\begin{bmatrix}5 & 3\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}27 & 90\end{bmatrix}$$

$$\begin{bmatrix}20 & 9\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}96 & 345\end{bmatrix}$$

$$\begin{bmatrix}14 & 7\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}70 & 245\end{bmatrix}$$

$$\begin{bmatrix}0 & 9\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}36 & 45\end{bmatrix}$$

$$\begin{bmatrix}14 & 6\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}66 & 240\end{bmatrix}$$

$$\begin{bmatrix}15 & 18\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}117 & 315\end{bmatrix}$$

$$\begin{bmatrix}13 & 1\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}43 & 200\end{bmatrix}$$

$$\begin{bmatrix}20 & 9\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}96 & 345\end{bmatrix}$$

$$\begin{bmatrix}15 & 14\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}101 & 295\end{bmatrix}$$

$$\begin{bmatrix}19 & 0\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}57 & 285\end{bmatrix}$$

The cryptogram:

81 135 139 455 120 375 93 195 67 95 124 245 36 45 57 285
3 15 59 220 92 340 61 245 60 75 18 90 120 330 125 325
27 90 96 345 70 245 36 45 66 240 117 315 43 200 96 345
101 295 57 285

***d)*** To decode a message given the matrix $A$.

$$A=\begin{pmatrix}3 & 15\\ 4 & 5\end{pmatrix}$$

$$A^{-1}=-\frac{1}{45}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}81 & 135\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-135 & -810\end{bmatrix}$$

$$=\begin{bmatrix}3 & 18\end{bmatrix}$$

$$-\frac{1}{45}[139 \quad 455]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-1{,}125 \quad -720]$$

$$= [25 \quad 16]$$

$$-\frac{1}{45}[120 \quad 375]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-900 \quad -675]$$

$$= [20 \quad 15]$$

$$-\frac{1}{45}[93 \quad 195]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-315 \quad -810]$$

$$= [7 \quad 18]$$

$$-\frac{1}{45}[67 \quad 95]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-45 \quad -720]$$

$$= [1 \quad 16]$$

$$-\frac{1}{45}[124 \quad 245]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-360 \quad -1{,}125]$$

$$= [8 \quad 25]$$

$$-\frac{1}{45}[36 \quad 45]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[0 \quad -405]$$

$$= [0 \quad 9]$$

$$-\frac{1}{45}[57 \quad 285]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-855 \quad 0]$$

$$= [19 \quad 0]$$

$$-\frac{1}{45}[3 \quad 15]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-45 \quad 0]$$

$$= [1 \quad 0]$$

$$-\frac{1}{45}[59 \quad 220]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-585 \quad -225]$$

$$= [13 \quad 5]$$

$$-\frac{1}{45}[92 \quad 340]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-900 \quad -360]$$

$$= [20 \quad 8]$$

$$-\frac{1}{45}[61 \quad 245]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-675 \quad -180]$$

$$= [15 \quad 4]$$

$$-\frac{1}{45}\begin{bmatrix}60 & 75\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}0 & -675\end{bmatrix}$$
$$=\begin{bmatrix}0 & 15\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}18 & 90\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-270 & 0\end{bmatrix}$$
$$=\begin{bmatrix}6 & 0\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}120 & 330\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-720 & -810\end{bmatrix}$$
$$=\begin{bmatrix}16 & 18\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}125 & 325\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-675 & -900\end{bmatrix}$$
$$=\begin{bmatrix}15 & 20\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}27 & 90\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-225 & -135\end{bmatrix}$$
$$=\begin{bmatrix}5 & 3\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}96 & 345\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-900 & -405\end{bmatrix}$$
$$=\begin{bmatrix}20 & 9\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}70 & 245\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-630 & -315\end{bmatrix}$$
$$=\begin{bmatrix}14 & 7\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}36 & 45\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}0 & -405\end{bmatrix}$$
$$=\begin{bmatrix}0 & 9\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}66 & 240\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-630 & -270\end{bmatrix}$$
$$=\begin{bmatrix}14 & 6\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}117 & 315\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-675 & -810\end{bmatrix}$$
$$=\begin{bmatrix}15 & 18\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}43 & 200\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-585 & -45\end{bmatrix}$$

$$= \begin{bmatrix} 13 & 1 \end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix} 96 & 345 \end{bmatrix}\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}\begin{bmatrix} -900 & -405 \end{bmatrix}$$
$$= \begin{bmatrix} 20 & 9 \end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix} 101 & 295 \end{bmatrix}\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}\begin{bmatrix} -675 & -630 \end{bmatrix}$$
$$= \begin{bmatrix} 15 & 14 \end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix} 57 & 285 \end{bmatrix}\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}\begin{bmatrix} -855 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 19 & 0 \end{bmatrix}$$

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 18 | 25 | 16 | 20 | 15 | 7 | 18 | 1 | 16 | 8 | 25 | 0 | 9 | 19 | 0 | 1 | 0 | 13 | 5 |
| *C* | *R* | *Y* | *P* | *T* | *O* | *G* | *R* | *A* | *P* | *H* | *Y* | _ | *I* | *S* | _ | *A* | _ | *M* | *E* |
| 20 | 8 | 15 | 4 | 0 | 15 | 6 | 0 | 16 | 18 | 15 | 20 | 5 | 3 | 20 | 9 | 14 | 7 | 0 | 9 |
| *T* | *H* | *O* | *D* | _ | *O* | *F* | _ | *P* | *R* | *O* | *T* | *E* | *C* | *T* | *I* | *N* | *G* | _ | *I* |
| 14 | 6 | 15 | 18 | 16 | 1 | 20 | 9 | 15 | 14 | 19 | 0 | | | | | | | | |
| *N* | *F* | *O* | *R* | *M* | *A* | *T* | *I* | *O* | *N* | *S* | _ | | | | | | | | |

The message is: ***Cryptography is a Method of Protecting Informations***

## *Exercise*

Write the matrix $A$ with a key word ***MATH,*** then decode the cryptogram

117 9 456 132 386 62 260 104 413 161 104 8

***Solution***

| | | | |
|---|---|---|---|
| *M* | *A* | *T* | *H* |
| 13 | 1 | 20 | 8 |

$$A = \begin{pmatrix} 13 & 1 \\ 20 & 8 \end{pmatrix}$$

$$A^{-1} = \frac{1}{84}\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix}$$

With the cryptogram:

$$\begin{bmatrix} 117 & 9 \end{bmatrix} \begin{bmatrix} 456 & 132 \end{bmatrix} \begin{bmatrix} 386 & 62 \end{bmatrix} \begin{bmatrix} 260 & 104 \end{bmatrix} \begin{bmatrix} 413 & 161 \end{bmatrix} \begin{bmatrix} 104 & 8 \end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}117 & 9\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}=\frac{1}{84}\begin{bmatrix}756 & 0\end{bmatrix}$$
$$=\begin{bmatrix}9 & 0\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}456 & 132\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}=\frac{1}{84}\begin{bmatrix}1,008 & 1,260\end{bmatrix}$$
$$=\begin{bmatrix}12 & 15\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}386 & 62\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}=\frac{1}{84}\begin{bmatrix}1,848 & 420\end{bmatrix}$$
$$=\begin{bmatrix}22 & 5\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}260 & 104\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}=\frac{1}{84}\begin{bmatrix}0 & 1,092\end{bmatrix}$$
$$=\begin{bmatrix}0 & 13\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}413 & 161\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}=\frac{1}{84}\begin{bmatrix}84 & 1,680\end{bmatrix}$$
$$=\begin{bmatrix}1 & 20\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}104 & 8\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}=\frac{1}{84}\begin{bmatrix}672 & 0\end{bmatrix}$$
$$=\begin{bmatrix}8 & 0\end{bmatrix}$$

$$\begin{array}{cccccccccccc} 9 & 0 & 12 & 15 & 22 & 5 & 0 & 13 & 1 & 20 & 8 & 0\\ I & - & L & O & V & E & - & M & A & T & H & - \end{array}$$

The message is: ***I love math***

## *Exercise*

Write the matrix $A$ with a key word ***MATH,*** then decode the cryptogram

438 150 145 37 240 96 635 191 445 157 260 104 413 161 104 8

***Solution***

$$\begin{array}{cccc} M & A & T & H\\ 13 & 1 & 20 & 8 \end{array}$$

$$A=\begin{pmatrix}13 & 1\\ 20 & 8\end{pmatrix}$$

$$A^{-1}=\frac{1}{84}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}$$

With the cryptogram:

$$[438 \quad 150] \quad [145 \quad 37] \quad [240 \quad 96] \quad [635 \quad 191] \quad [445 \quad 157]$$

$$[260 \quad 104] \quad [413 \quad 161] \quad [104 \quad 8]$$

$$\frac{1}{84}[438 \quad 150]\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}[504 \quad 1{,}512]$$
$$= [6 \quad 18]$$

$$\frac{1}{84}[145 \quad 37]\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}[420 \quad 336]$$
$$= [5 \quad 4]$$

$$\frac{1}{84}[240 \quad 96]\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}[0 \quad 1{,}008]$$
$$= [0 \quad 12]$$

$$\frac{1}{84}[635 \quad 191]\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}[1{,}260 \quad 1{,}848]$$
$$= [15 \quad 22]$$

$$\frac{1}{84}[445 \quad 157]\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}[420 \quad 1{,}596]$$
$$= [5 \quad 19]$$

$$\frac{1}{84}[260 \quad 104]\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}[0 \quad 1{,}092]$$
$$= [0 \quad 13]$$

$$\frac{1}{84}[413 \quad 161]\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}[84 \quad 1{,}680]$$
$$= [1 \quad 20]$$

$$\frac{1}{84}[104 \quad 8]\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}[672 \quad 0]$$
$$= [8 \quad 0]$$

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 18 | 5 | 4 | 0 | 12 | 15 | 22 | 5 | 19 | 0 | 13 | 1 | 20 | 8 | 0 |
| *F* | *R* | *E* | *D* | – | *L* | *O* | *V* | *E* | *S* | – | *M* | *A* | *T* | *H* | – |

The message is: ***Fred loves math***

## ***Exercise***

Consider the invertible matrix: $A=\begin{pmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{pmatrix}$

Decode the cryptogram

$$\begin{matrix}1 & -5 & 11 & 19 & -25 & -45 & 11 & -16 & -28 & 20 & -29 & -27\\ 12 & -12 & -53 & 40 & -61 & -35 & 8 & -17 & 7\end{matrix}$$

***Solution***

$|A|=1$

$$A^{-1}=\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}$$

With the cryptogram:

$$\begin{bmatrix}1 & -5 & 11\end{bmatrix}\ \begin{bmatrix}19 & -25 & -45\end{bmatrix}\ \begin{bmatrix}11 & -16 & -28\end{bmatrix}\ \begin{bmatrix}20 & -29 & -27\end{bmatrix}$$
$$\begin{bmatrix}12 & -12 & -53\end{bmatrix}\ \begin{bmatrix}40 & -61 & -35\end{bmatrix}\ \begin{bmatrix}8 & -17 & 7\end{bmatrix}$$

$$\begin{bmatrix}1 & -5 & 11\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}4 & 9 & 6\end{bmatrix}$$

$$\begin{bmatrix}19 & -25 & -45\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}6 & 5 & 18\end{bmatrix}$$

$$\begin{bmatrix}11 & -16 & -28\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}5 & 14 & 20\end{bmatrix}$$

$$\begin{bmatrix}20 & -29 & -27\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}9 & 1 & 12\end{bmatrix}$$

$$\begin{bmatrix}12 & -12 & -53\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}0 & 5 & 17\end{bmatrix}$$

$$\begin{bmatrix}40 & -61 & -35\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}21 & 1 & 20\end{bmatrix}$$

$$\begin{bmatrix} 8 & -17 & 7 \end{bmatrix}\begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = \begin{bmatrix} 9 & 15 & 14 \end{bmatrix}$$

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 9 | 6 | 6 | 5 | 18 | 5 | 14 | 20 | 9 | 1 | 12 | 0 | 5 | 17 | 21 | 1 | 20 | 9 | 15 | 14 |
| *D* | *I* | *F* | *F* | *E* | *R* | *E* | *N* | *T* | *I* | *A* | *L* | _ | *E* | *Q* | *U* | *A* | *T* | *I* | *O* | *N* |

The message is: ***Differential Equation***.

## *Exercise*

Determine the key word, then decode the given cryptogram

6 18 5 4 15 13 1 20 8

102 649 238 57 324 112 128 622 207

180 613 290 102 360 259 151 580 297

*Hint*: First row is the key

***Solution***

The key word from the first row is

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 6 | 18 | 5 | 4 | 15 | 13 | 1 | 20 | 8 |
| *f* | *r* | *e* | *d* | *o* | *m* | *a* | *t* | *h* |

Since it has 9 numbers, then the matrix is $9 = 3^2$ which is $3\times3$

$$A = \begin{pmatrix} 6 & 18 & 5 \\ 4 & 15 & 13 \\ 1 & 20 & 8 \end{pmatrix}$$

$$|A| = -857$$

$$A^{-1} = -\frac{1}{857}\begin{pmatrix} -140 & -44 & 159 \\ -19 & 43 & -58 \\ 65 & -102 & 18 \end{pmatrix}$$

$$= \frac{1}{857}\begin{pmatrix} 140 & 44 & -159 \\ 19 & -43 & 58 \\ -65 & 102 & -18 \end{pmatrix}$$

With the cryptogram:

$$\begin{bmatrix} 102 & 649 & 238 \end{bmatrix} \quad \begin{bmatrix} 57 & 324 & 112 \end{bmatrix} \quad \begin{bmatrix} 128 & 622 & 207 \end{bmatrix}$$

$$\begin{bmatrix} 180 & 613 & 290 \end{bmatrix} \quad \begin{bmatrix} 102 & 360 & 259 \end{bmatrix} \quad \begin{bmatrix} 151 & 580 & 297 \end{bmatrix}$$

$$\frac{1}{857}\begin{bmatrix}102 & 649 & 238\end{bmatrix}\begin{pmatrix}140 & 44 & -159\\ 19 & -43 & 58\\ -65 & 102 & -18\end{pmatrix}=\frac{1}{857}\begin{bmatrix}11,141 & 857 & 17,140\end{bmatrix}$$

$$=\begin{bmatrix}13 & 1 & 20\end{bmatrix}$$

$$\frac{1}{857}\begin{bmatrix}57 & 324 & 112\end{bmatrix}\begin{pmatrix}140 & 44 & -159\\ 19 & -43 & 58\\ -65 & 102 & -18\end{pmatrix}=\frac{1}{857}\begin{bmatrix}6,856 & 0 & 7,713\end{bmatrix}$$

$$=\begin{bmatrix}8 & 0 & 9\end{bmatrix}$$

$$\frac{1}{857}\begin{bmatrix}128 & 622 & 207\end{bmatrix}\begin{pmatrix}140 & 44 & -159\\ 19 & -43 & 58\\ -65 & 102 & -18\end{pmatrix}=\frac{1}{857}\begin{bmatrix}16,283 & 0 & 11,998\end{bmatrix}$$

$$=\begin{bmatrix}19 & 0 & 14\end{bmatrix}$$

$$\frac{1}{857}\begin{bmatrix}180 & 613 & 290\end{bmatrix}\begin{pmatrix}140 & 44 & -159\\ 19 & -43 & 58\\ -65 & 102 & -18\end{pmatrix}=\frac{1}{857}\begin{bmatrix}17,997 & 11,141 & 1,714\end{bmatrix}$$

$$=\begin{bmatrix}21 & 13 & 2\end{bmatrix}$$

$$\frac{1}{857}\begin{bmatrix}102 & 360 & 259\end{bmatrix}\begin{pmatrix}140 & 44 & -159\\ 19 & -43 & 58\\ -65 & 102 & -18\end{pmatrix}=\frac{1}{857}\begin{bmatrix}4,285 & 15,426 & 0\end{bmatrix}$$

$$=\begin{bmatrix}5 & 18 & 0\end{bmatrix}$$

$$\frac{1}{857}\begin{bmatrix}151 & 580 & 297\end{bmatrix}\begin{pmatrix}140 & 44 & -159\\ 19 & -43 & 58\\ -65 & 102 & -18\end{pmatrix}=\frac{1}{857}\begin{bmatrix}12,855 & 11,998 & 4,285\end{bmatrix}$$

$$=\begin{bmatrix}15 & 14 & 5\end{bmatrix}$$

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 13 | 1 | 20 | 8 | 0 | 9 | 19 | 0 | 14 | 21 | 13 | 2 | 5 | 18 | 0 | 15 | 14 | 5 |
| *M* | *A* | *T* | *H* | – | *I* | *S* | – | *N* | *U* | *M* | *B* | *E* | *R* | – | *O* | *N* | *E* |

The message is: ***Math is number one***

### *Exercise*

Determine the key word, then decode the given cryptogram

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5 | 17 | 21 | 1 | 20 | 9 | 15 | 14 | 19 |
| 259 | 863 | 783 | 77 | 378 | 357 | 301 | 448 | 565 |
| 106 | 266 | 318 | 325 | 365 | 485 | 301 | 522 | 653 |
| 326 | 653 | 738 | 10 | 566 | 495 | 115 | 640 | 555 |
| 290 | 791 | 762 | 115 | 474 | 507 | 119 | 332 | 279 |
| 305 | 454 | 513 | 339 | 645 | 611 | 226 | 341 | 426 |
| 260 | 338 | 368 | 406 | 657 | 830 | 270 | 649 | 590 |
| 110 | 337 | 418 | 74 | 318 | 330 | 261 | 561 | 469 |
| 114 | 426 | 390 | 160 | 543 | 372 | 89 | 535 | 441 |
| 323 | 842 | 783 | 97 | 344 | 245 | 84 | 601 | 444 |
| 424 | 851 | 944 | 175 | 262 | 339 | 379 | 698 | 755 |
| 226 | 341 | 426 | 37 | 454 | 217 | 156 | 694 | 536 |

***Solution***

The key word from the first row, because all the numbers are between 0 and 26, alphabetic letter.

Since it has 9 numbers, then the matrix is $9 = 3^2$ which is $3\times 3$.

Therefore,

$$A = \begin{pmatrix} 5 & 17 & 21 \\ 1 & 20 & 9 \\ 15 & 14 & 19 \end{pmatrix}$$

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 = _ | 4 = D | 8 = H | 12 = L | 16 = P | 20 = T | 24 = X |
| 1 = A | 5 = E | 9 = I | 13 = M | 17 = Q | 21 = U | 25 = Y |
| 2 = B | 6 = F | 10 = J | 14 = N | 18 = R | 22 = V | 26 = Z |
| 3 = C | 7 = G | 11 = K | 15 = O | 19 = S | 23 = W | |

The key word is:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5 | 17 | 21 | 1 | 20 | 9 | 15 | 14 | 19 |
| *E* | *Q* | *U* | *A* | *T* | *I* | *O* | *N* | *S* |

$$A = \begin{pmatrix} 5 & 17 & 21 \\ 1 & 20 & 9 \\ 15 & 14 & 19 \end{pmatrix}$$

$$|A| = -2,764$$

$$A^{-1} = -\frac{1}{2,764}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix}$$

With the cryptogram:

$$[259 \quad 863 \quad 783] \; [77 \quad 378 \quad 357] \; [301 \quad 448 \quad 565] \; [106 \quad 266 \quad 318]$$

$$[325 \quad 365 \quad 485] \; [301 \quad 522 \quad 653] \; [326 \quad 653 \quad 738] \; [103 \quad 566 \quad 495]$$

$$[115 \quad 640 \quad 555] \; [290 \quad 791 \quad 762] \; [115 \quad 474 \quad 507] \; [119 \quad 332 \quad 279]$$

$$[305 \quad 454 \quad 513] \; [339 \quad 645 \quad 611] \; [226 \quad 341 \quad 426] \; [260 \quad 338 \quad 368]$$

$$[406 \quad 657 \quad 830] \; [270 \quad 649 \quad 590] \; [110 \quad 337 \quad 418] \; [74 \quad 318 \quad 330]$$

$$[261 \quad 561 \quad 469] \; [114 \quad 426 \quad 390] \; [160 \quad 543 \quad 372] \; [89 \quad 535 \quad 441]$$

$$[323 \quad 842 \quad 783] \; [97 \quad 344 \quad 245] \; [84 \quad 601 \quad 444] \; [424 \quad 851 \quad 944]$$

$$[175 \quad 262 \quad 339] \; [379 \quad 698 \quad 755] \; [226 \quad 341 \quad 426] \; [37 \quad 454 \quad 217]$$

$$[156 \quad 694 \quad 536]$$

To decode a message given the matrix $A$.

$$-\frac{1}{2,764}[259 \quad 863 \quad 783]\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764}[-58,044 \quad -52,516 \quad -24,876]$$

$$= [21 \quad 19 \quad 9]$$

$$-\frac{1}{2,764}[77 \quad 378 \quad 357]\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764}[-38,696 \quad -19,348 \quad 0]$$

$$= [14 \quad 7 \quad 0]$$

$$-\frac{1}{2,764}[301 \quad 448 \quad 565]\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764}[-33,168 \quad -2,764 \quad -44,224]$$

$$= [12 \quad 1 \quad 16]$$

$$-\frac{1}{2,764}[106 \quad 266 \quad 318]\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764}[-33,168 \quad -2,764 \quad -8,292]$$

$$= [12 \quad 1 \quad 3]$$

$$-\frac{1}{2,764}[325 \quad 365 \quad 485]\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764}[-13,820 \quad 0 \quad -55,280]$$

$$= \begin{bmatrix} 5 & 0 & 20 \end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix} 301 & 522 & 653 \end{bmatrix}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix} -49,752 & -2,764 & -38,696 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 1 & 14 \end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix} 326 & 653 & 738 \end{bmatrix}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix} -52,516 & -16,584 & -41,460 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 6 & 15 \end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix} 103 & 566 & 495 \end{bmatrix}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix} -49,752 & -35,932 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 13 & 0 \end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix} 115 & 640 & 555 \end{bmatrix}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix} -55,280 & -41,460 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 15 & 0 \end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix} 290 & 791 & 762 \end{bmatrix}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix} -52,516 & -41,460 & -33,168 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 15 & 12 \end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix} 115 & 474 & 507 \end{bmatrix}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix} -60,808 & -13,820 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 5 & 0 \end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix} 119 & 332 & 279 \end{bmatrix}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix} -11,056 & -24,876 & -16,584 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 9 & 6 \end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix} 305 & 454 & 513 \end{bmatrix}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix} -16,584 & -13,820 & -49,752 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 5 & 18 \end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}339 & 645 & 611\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}-13{,}820 & -38{,}696 & -55{,}280\end{bmatrix}$$

$$= \begin{bmatrix}5 & 14 & 20\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}226 & 341 & 426\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}-24{,}876 & -2{,}764 & -33{,}168\end{bmatrix}$$

$$= \begin{bmatrix}9 & 1 & 12\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}260 & 338 & 368\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}0 & 13{,}820 & -46{,}988\end{bmatrix}$$

$$= \begin{bmatrix}0 & 5 & 17\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}406 & 657 & 830\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}-58{,}044 & -2{,}764 & -55{,}280\end{bmatrix}$$

$$= \begin{bmatrix}21 & 1 & 20\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}270 & 649 & 590\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}-24{,}876 & -41{,}460 & -38{,}696\end{bmatrix}$$

$$= \begin{bmatrix}9 & 15 & 14\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}110 & 337 & 418\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}-52{,}516 & 0 & -2{,}764\end{bmatrix}$$

$$= \begin{bmatrix}19 & 0 & 1\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}74 & 318 & 330\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}-38{,}696 & -11{,}056 & 0\end{bmatrix}$$

$$= \begin{bmatrix}14 & 4 & 0\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}261 & 561 & 469\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}-2{,}764 & -44{,}224 & -44{,}224\end{bmatrix}$$

$$= \begin{bmatrix}1 & 16 & 16\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}114 & 426 & 390\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-33,168 & -24,876 & -8,292\end{bmatrix}$$

$$= \begin{bmatrix}12 & 9 & 3\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}160 & 543 & 372\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-2,764 & -55,280 & -24.876\end{bmatrix}$$

$$= \begin{bmatrix}1 & 20 & 9\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}89 & 535 & 441\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-41,460 & -38,696 & 0\end{bmatrix}$$

$$= \begin{bmatrix}15 & 14 & 0\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}323 & 842 & 783\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-44,224 & -49,752 & -41,460\end{bmatrix}$$

$$= \begin{bmatrix}16 & 18 & 15\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}97 & 344 & 245\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-5,528 & -33,168 & -13,820\end{bmatrix}$$

$$= \begin{bmatrix}2 & 12 & 5\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}84 & 601 & 444\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-35,932 & -52,516 & 0\end{bmatrix}$$

$$= \begin{bmatrix}13 & 19 & 0\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}424 & 851 & 944\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-63,572 & -24,876 & -55,280\end{bmatrix}$$

$$= \begin{bmatrix}23 & 9 & 20\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}175 & 262 & 339\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-22,112 & 0 & -24,876\end{bmatrix}$$

$$= \begin{bmatrix}8 & 0 & 9\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}379 & 698 & 755\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}-38{,}696 & -24{,}876 & -55{,}280\end{bmatrix}$$
$$= \begin{bmatrix}14 & 9 & 20\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}226 & 341 & 426\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}-24{,}876 & -2{,}764 & -33{,}168\end{bmatrix}$$
$$= \begin{bmatrix}9 & 1 & 12\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}37 & 454 & 217\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}0 & -60{,}808 & -2{,}764\end{bmatrix}$$
$$= \begin{bmatrix}0 & 22 & 1\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}156 & 694 & 536\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}-33{,}168 & -58{,}044 & -13{,}820\end{bmatrix}$$
$$= \begin{bmatrix}12 & 21 & 5\end{bmatrix}$$

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 = _ | 4 = D | 8 = H | 12 = L | 16 = P | 20 = T | 24 = X |
| 1 = A | 5 = E | 9 = I | 13 = M | 17 = Q | 21 = U | 25 = Y |
| 2 = B | 6 = F | 10 = J | 14 = N | 18 = R | 22 = V | 26 = Z |
| 3 = C | 7 = G | 11 = K | 15 = O | 19 = S | 23 = W | |

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 21 | 19 | 9 | 14 | 7 | 0 | 12 | 1 | 16 | 12 | 1 | 3 | 5 | 0 | 20 | 18 | 1 | 14 | 19 | 6 | 15 | |
| *U* | *S* | *I* | *N* | *G* | – | *L* | *A* | *P* | *L* | *A* | *C* | *E* | – | *T* | *R* | *A* | *N* | *S* | *F* | *O* | |
| 18 | 13 | 0 | 20 | 15 | 0 | 19 | 15 | 12 | 22 | 5 | 0 | 4 | 9 | 6 | 6 | 5 | 18 | 5 | 14 | 20 | |
| *R* | *M* | – | *T* | *O* | – | *S* | *O* | *L* | *V* | *E* | – | *D* | *I* | *F* | *F* | *E* | *R* | *E* | *N* | *T* | |
| 9 | 1 | 12 | 0 | 5 | 17 | 21 | 1 | 20 | 9 | 15 | 14 | 19 | 0 | 1 | 14 | 4 | 0 | 1 | 16 | 16 | |
| *I* | *A* | *L* | – | *E* | *Q* | *U* | *A* | *T* | *I* | *O* | *N* | *S* | – | *A* | *N* | *D* | – | *A* | *P* | *P* | |
| 12 | 9 | 3 | 1 | 20 | 9 | 15 | 14 | 0 | 16 | 18 | 15 | 2 | 12 | 5 | 13 | 19 | 0 | 23 | 9 | 20 | |
| *L* | *I* | *C* | *A* | *T* | *I* | *O* | *N* | – | *P* | *R* | *O* | *B* | *L* | *E* | *M* | *S* | – | *W* | *I* | *T* | |
| 8 | 0 | 9 | 14 | 9 | 20 | 9 | 1 | 12 | 0 | 22 | 1 | 12 | 21 | 5 | | | | | | | |
| *H* | – | *I* | *N* | *I* | *T* | *I* | *A* | *L* | – | *V* | *A* | *L* | *U* | *E* | | | | | | | |

The message is:

***Using Laplace Transform to Solve Differential Equations and Application Problems with Initial Value***