Solution Section 4.4 – Goodness-of-Fit

Exercise

A poll typically involves the selection of random digits to be used for telephone numbers. The New York Times states that "within each (telephone) exchange, random digits were added to form a complete telephone number, thus permitting access to listed and unlisted numbers. "When such digits are randomly generated, what is the distribution of those digits? Given such randomly generated digits, what is a test for "goodness-off-fit"?

Solution

When digits are randomly generated they should form a uniform distribution – i.e., a distribution in which each of the digits is equally likely. The test for goodness-to-fit is a test of the hypothesis that the sample data fit the uniform distribution.

Exercise

When generating random digits, we can test the generated digits for goodness-of-fit with the distribution in which all of the digits are equally likely. What does an exceptionally large value of the χ^2 test statistic suggest about the goodness-of-fit? What does an exceptionally small value of the χ^2 test statistic (such as 0.002) suggest about the goodness-of-fit?

Solution

The calculated χ^2 is a measure of the discrepancy between the hypothesis distribution and the sample data. An exceptionally large value of the χ^2 test statistic suggests a large discrepancy between the hypothesized distribution and the sample data – that there is not goodness-of-fit, and that the observed and expected frequencies are quite different. An exceptionally small of the χ^2 test statistic suggests an extremely good fir – that the observed and expected values are almost identical.

Exercise

You purchased a slot machine, and tested it by playing it 1197 times. There are 10 different categories of outcome, including no win, win jackpot, win with three bells, and so on. When testing the claim the observed outcomes agree with the expected frequencies, the author obtained a test statistic of $\chi^2 = 8.185$. Use a 0.05 significance level to test the claim that the actual outcomes agree with the expected frequencies. Does the slot machine appear to be functioning as expected?
Conduct the hypothesis test and the test statistic, critical value and/or *P*-value, and state the conclusion.

Solution

Original claim: the actual outcomes agree with the expected frequencies

H_0 : The actual outcomes agree with the expected frequencies

H_1 : At least one outcome is not as expected

$\alpha = 0.05 \quad and \quad df = 9$

$C.V. \ \chi^2 = \chi^2_\alpha = \chi^2_{0.05} = 16.919$

Calculations:

$$\chi^2 = \sum \frac{(O-E)^2}{E} = 8.185$$

P-value = $\chi^2 \ cdf\,(8.185,\ 99,\ 9) = 0.5156$

Conclusion

Do not reject H_0 ; there is not sufficient evidence to reject the claim that the actual outcomes agree with the expected frequencies. There is no reason to say the slot machine is not functioning as expected.

Exercise

Do "*A*" students tend to sit in a particular part of the classroom? The author recorded the locations of the students who received grades *A*, with these results: 17 sat in the front, 9 sat in the middle, and 5 sat in the back of the classroom. When testing the assumption that the "*A*" students are distributed evenly throughout the room, the author obtained the test statistic of $\chi^2 = 7.226$. If using a 0.05 significance level, is there sufficient evidence to support the claim that the "*A*" students are not evenly distributed throughout the classroom? If so, does that mean you can increase your likelihood of getting an *A* by sitting in the front of the room?

Conduct the hypothesis test and the test statistic, critical value and/or *P*-value, and state the conclusion.

Solution

Original claim: "*A*" student are not evenly distributed throughout the classroom

H_0 : "*A*" students are evenly distributed throughout the classroom

H_1 : "*A*" students are not evenly distributed throughout the classroom

$\alpha = 0.05 \quad and \quad df = 2$

Degrees of Freedom	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597

$C.V. \ \chi^2 = \chi^2_\alpha = \chi^2_{0.05} = 5.991$

Calculations:

$$\chi^2 = \sum \frac{(O-E)^2}{E} = 7.226$$

P-value = $\chi^2 \ cdf\,(7.226,\ 99,\ 2) = 0.0270$

Conclusion

Reject H_0; there is sufficient evidence to support the claim "*A*" students are not evenly distributed throughout the classroom.

Exercise

Randomly selected nonfat occupational injuries and illnesses are categorized according to the day of the week that they first occurred, and the results are listed below. Use a 0.05 significance level to test the claim that such injuries and illness occur with equal frequency on the different days of the week Conduct the hypothesis test and the test statistic, critical value and/or *P*-value, and state the conclusion.

Day	*Mon*	*Tues*	*Wed*	*Thurs*	*Fri*
Number	23	23	21	21	19

Solution

Original Claim: The injuries and illnesses occur with equal frequencies on the different days.

$H_0 : p_M = p_T = p_W = p_{Th} = p_F = \frac{1}{5} = 0.20$

$H_1 :$ *at* least one $\mathrm{p_i} \neq 0$

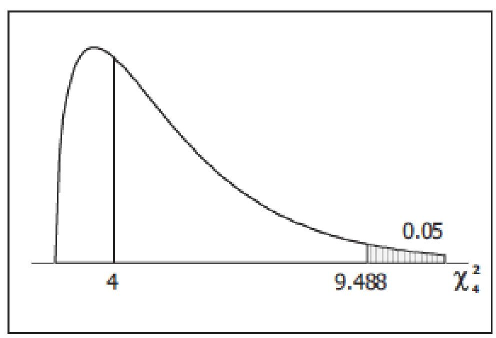
$\alpha = 0.05 \quad and \quad df = 4$

Degrees of Freedom	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860

$C.V.\ \chi^2 = \chi^2_\alpha = \chi^2_{0.05} = 9.488 \quad E = \frac{1}{5}\sum O = \frac{107}{5} = 21.4$

Calculations:

Day	*O*	*E*	$\frac{(O-E)^2}{E}$
M	23	21.4	0.1196
T	23	21.4	0.1196
W	21	21.4	0.0075
Th	21	21.4	0.075
F	19	21.4	0.2693
	107	107	0.5234



$$\chi^2 = \sum \frac{(O-E)^2}{E} = 0.5234$$

P-value = $\chi^2\ cdf\,(0.523,\ 99,\ 4) = 0.9712$

Conclusion

Do not reject H_0; there is not sufficient evidence to reject the claim that $\mathrm{p_i} \neq 0$ for each day.

There is no sufficient evidence to reject the claim that the injuries and illnesses occur with equal frequencies on the different days of the week.

Exercise

Records of randomly selected births were obtained and categorized according to the day of the week that they occurred. Because babies are unfamiliar with our schedule of weekdays, a reasonable claim is that occur on the different days with equal frequency. Use a 0.01 significance level to test that claim. Can you provide an explanation for the result?
Conduct the hypothesis test and the test statistic, critical value and/or *P*-value, and state the conclusion.

Day	*Sun*	*Mon*	*Tues*	*Wed*	*Thurs*	*Fri*	*Sat*
Number of births	77	110	124	122	120	123	97

Solution

Original Claim: births occur on the different days with equal frequency.

$H_0: p_S = p_M = p_T = p_W = p_{Th} = p_F = p_S = \frac{1}{7}$

$H_1: \textit{at} \text{ least one } \mathrm{p_i} \neq \frac{1}{7}$

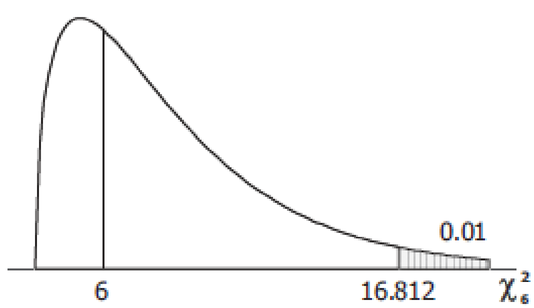
$\alpha = 0.01 \quad \textit{and} \quad df = 6$

Degrees of Freedom	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548

$C.V.\ \chi^2 = \chi^2_\alpha = \chi^2_{0.05} = 16.812$ $\qquad E = \frac{1}{7}\sum O = \frac{773}{7} = 110.43$

Calculations:

Day	*O*	*E*	$\frac{(O-E)^2}{E}$
S	77	110.43	10.119
M	110	110.43	0.0017
T	124	110.43	1.6679
W	122	110.43	1.2125
Th	120	110.43	0.8296
F	123	110.43	1.4312
S	97	110.43	1.6330
	773	773	16.8952



$$\chi^2 = \sum \frac{(O-E)^2}{E} = 16.8952$$

P-value = $\chi^2\ cdf\,(16.895,\ 99,\ 6) = 0.0097$

Conclusion

Reject H_0; there is sufficient evidence to support the claim that $\mathrm{p_i} = \frac{1}{7}$ for each day. There is sufficient evidence to reject the claim that births occur on the different days with equal frequency. Births that do not occur naturally (induced, Caesarean sections) are typically not scheduled for Saturday and Sunday, accounting for the smaller than expected numbers of births on those days.

Exercise

The table below lists the frequency of wins for different post positions in the Kentucky Derby horse race. A post position of 1 is closest to the inside rail, so that horse has the shortest distance to run. (Because the number of horses varies from year to year, only the first ten post positions are included.) Use a 0.05 significance level to test the claim that the likelihood of winning is the same for the different post positions. Based on the result, should bettor consider the post position of a horse racing in the Kentucky Derby?

Conduct the hypothesis test and the test statistic, critical value and/or *P*-value, and state the conclusion.

Post Position	1	2	3	4	5	6	7	8	9	10
Wins	19	14	11	14	14	7	8	11	5	11

Solution

Original Claim: The likelihood of winning is the same for all post positions.

$H_0: p_1 = p_2 = \ldots = p_{10} = \frac{1}{10}$

$H_1: at \text{ least one } \mathrm{p_i} \neq \frac{1}{10}$

$\alpha = 0.05 \quad and \quad df = 9$

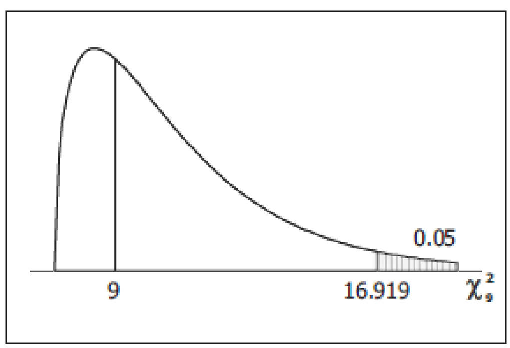
Degrees of Freedom	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589

$C.V.\ \chi^2 = \chi^2_{\alpha} = \chi^2_{0.05} = 16.919$

$E = \frac{1}{10}\sum O = \frac{114}{10} = 11.4$

Calculations:

Position	*O*	*E*	$\frac{(O-E)^2}{E}$
1	19	11.4	5.0667
2	14	11.4	0.5930
3	11	11.4	0.0140
4	14	11.4	0.5930
5	14	11.4	0.5930
6	7	11.4	1.6982
7	8	11.4	1.0140
8	11	11.4	0.0140
9	5	11.4	3.5930
10	11	11.4	0.0140
	114	114.0	13.193



$$\chi^2 = \sum \frac{(O-E)^2}{E} = 13.193$$

P-value = $\chi^2\ cdf\,(13.193,\ 99,\ 9) = 0.1541$

Conclusion

Do not reject H_0; there is not sufficient evidence to reject the claim that $p_i = \frac{1}{10}$ for each position. There is no sufficient evidence to reject the claim that the likelihood of winning is the same for all post positions. Based on these results, post position is not a significant consideration when betting on the Kentucky Derby.

Exercise

The table below lists the cases of violent crimes are randomly selected and categorized by month. Use a 0.01 significance level to test the claim that the rate of violent crime is the same for each month. Can you explain the result?
Conduct the hypothesis test and the test statistic, critical value and/or *P*-value, and state the conclusion.

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Number	786	704	835	826	900	868	920	901	856	862	783	797

Solution

Original Claim: The occurrence of violent crime is the same for each month.

$$H_0 : p_{Jan} = p_{Feb} = \ldots = p_{Dec} = \frac{1}{12}$$

$$H_1 : at \text{ least one } p_i \neq \frac{1}{12}$$

$$\alpha = 0.01 \quad and \quad df = 11$$

Degrees of Freedom	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757

$$C.V. \ \chi^2 = \chi^2_{\alpha} = \chi^2_{0.01} = 24.725$$

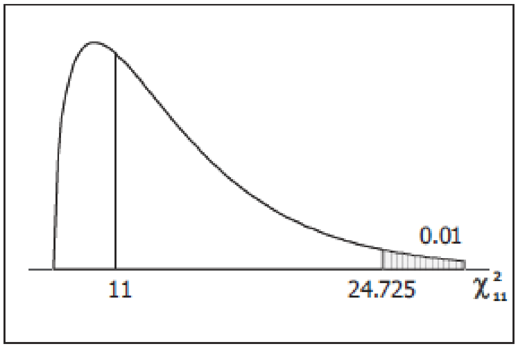
$$E = \frac{1}{12}\sum O$$

$$= \frac{10038}{12}$$

$$= 836.5$$

Calculations:

Month	*O*	*E*	$\frac{(O-E)^2}{E}$
Jan	786	836.5	3.0487
Feb	704	836.5	20.9877
Mar	835	836.5	0.0027
Apr	826	836.5	0.1318
May	900	836.5	4.8204
Jun	868	836.5	1.1862
Jul	920	836.5	8.3350
Aug	901	836.5	4.9734
Sep	856	836.5	0.4546
Oct	862	836.5	0.7773
Nov	783	836.5	3.4217
Dec	797	836.5	1.8652
	10038	10038.0	50.0048



$$\chi^2 = \sum \frac{(O-E)^2}{E} = 50.0048$$

P-value = $\chi^2 \, cdf\,(50.005,\ 99,\ 11) = 0.0000006$

Conclusion

Reject H_0; there is sufficient evidence to support the claim that $p_i = \frac{1}{12}$ for each month. There is sufficient evidence to reject the claim that the occurrence of violent crime is the same for each month. A major factor involved in this conclusion is the large contribution of the month of February to the calculated χ^2 statistic. The comparison of frequencies for each month is not fair because not all months have the same number of days.

Exercise

The table below lists the results of the Advanced Placement Biology class conducted genetics experiments with fruit flies. Use a 0.05 significance level to test the claim that the observed frequencies agree with the proportions that were expected according to principles of genetics

Conduct the hypothesis test and the test statistic, critical value and/or *P*-value, and state the conclusion.

Characteristic	***Red eye / normal wing***	***Sepia eye / normal wing***	***Red eye / vestigial wing***	***Sepia eye / vestigial wing***
Frequency	59	15	2	4
Expected proportion	$\frac{9}{16}$	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{1}{16}$

Solution

Original Claim: Observed frequencies fit the expected proportions.

$H_0: \ p_1 = \frac{9}{16}, \quad p_2 = \frac{3}{16}, \quad p_3 = \frac{3}{16}, \quad p_4 = \frac{1}{16}$

$H_1: at\ least\ one\ p_i$ is not as claimed

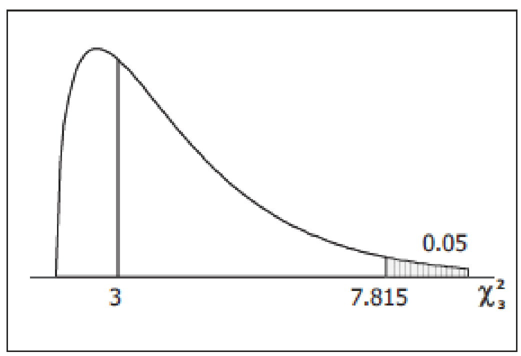
$\alpha = 0.05 \quad and \quad df = 3$

Degrees of Freedom	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838

$C.V.\ \chi^2 = \chi^2_\alpha = \chi^2_{0.05} = 7.815$

Calculations:

Day	*O*	*E*	$\frac{(O-E)^2}{E}$
1	59	$80 \cdot \frac{9}{16} = 45$	4.3556
2	15	$80 \cdot \frac{3}{16} = 15$	0.00
3	2	$80 \cdot \frac{3}{16} = 15$	11.2667
4	4	$80 \cdot \frac{1}{16} = 5$	0.200
	80	80	15.8222



$\chi^2 = \sum \frac{(O-E)^2}{E} = \underline{15.8222}$

P-value = $\chi^2\ cdf\ (15.822,\ 99,\ 3) = \underline{0.0012}$

Conclusion

Reject H_0; there is sufficient evidence to reject the claim that the proportions are as claimed.

There is sufficient evidence to reject the claim that observed frequencies fit the proportions that were expected according to the principles of genetics

Exercise

The table below lists the claims that its M&M plain candies are distributed with the following color percentages: 16% green, 20% orange, 14% yellow, 24% blue, 13% red, and 13% brown. Use a 0.05 significance level to test the claim that the color distribution is as claimed.

Green	Orange	Yellow	Blue	Red	Brown
19	25	8	27	13	8

Solution

Original Claim: The color distribution is as stated

$H_0:\ p_G = .16,\quad p_O = .20,\quad p_Y = .14,\quad p_{Bl} = .24,\quad p_R = .13,\quad p_{BR} = .13$

$H_1:$ *at least one* p_i is not as claimed

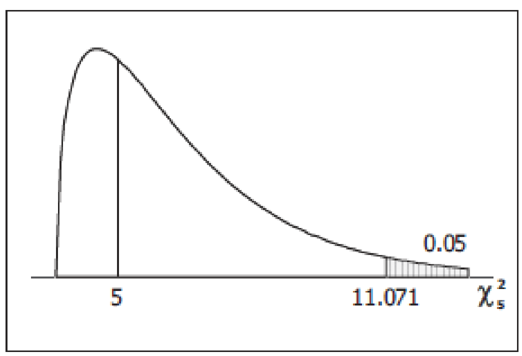
$\alpha = 0.05 \quad and \quad df = 5$

Degrees of Freedom	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
5	0.412	0.554	0.831	1.145	1.610	9.236	11.071	12.833	15.086	16.750

$C.V.\ \ \chi^2 = \chi^2_\alpha = \chi^2_{0.05} = 11.071$

Calculations:

Day	O	E	$\frac{(O-E)^2}{E}$
G	19	$100(.16)=16$	0.5625
O	25	$100(.20)=20$	1.2500
Y	8	$100(.14)=14$	2.5714
Bl	27	$100(.24)=24$	0.3750
R	13	$100(.13)=13$	0.0
Br	8	$100(.13)=13$	1.9231
	100	100	6.6820



$$\chi^2 = \sum \frac{(O-E)^2}{E} = 6.682$$

P-value = $\chi^2\ cdf\,(6.682,\ 99,\ 5) = 0.2454$

Conclusion

Do not reject H_0; there is not sufficient evidence to reject the claim that the proportion are as stated. There is no sufficient evidence to reject the claim that the color distribution is as stated.