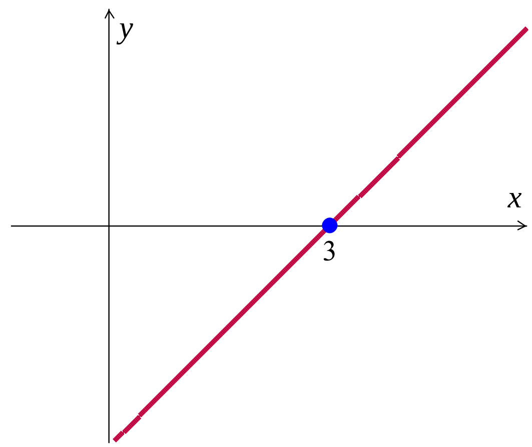
Solution Section 1.10 - Autonomous Equations and Stability

Exercise

The graph of the right-hand side $y' = f(y)$ is shown. Identify the equilibrium points and sketch the equilibrium solutions in the *ty*-plane. Classify each equilibrium point as either unstable or asymptotically stable.

Solution

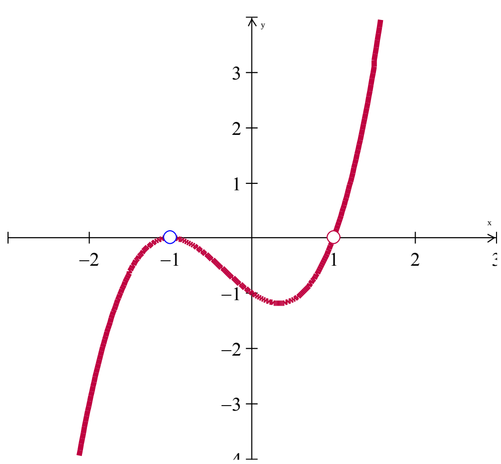


The equilibrium point is: 3 and is stable

Exercise

The graph of the right-hand side $y' = f(y)$ is shown. Identify the equilibrium points and sketch the equilibrium solutions in the *ty*-plane. Classify each equilibrium point as either unstable or asymptotically stable.

Solution

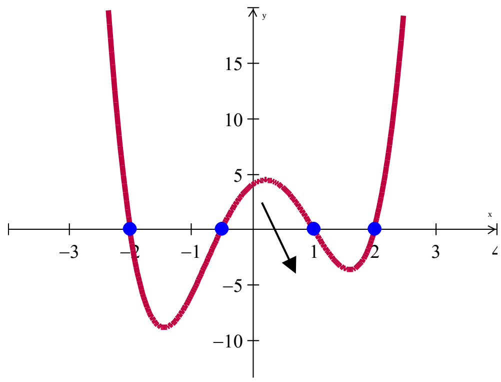


The equilibrium points are: $-1,\ 1$ and both are unstable

Exercise

The graph of the right-hand side $y' = f(y)$ is shown. Identify the equilibrium points and sketch the equilibrium solutions in the *ty*-plane. Classify each equilibrium point as either unstable or asymptotically stable.

Solution



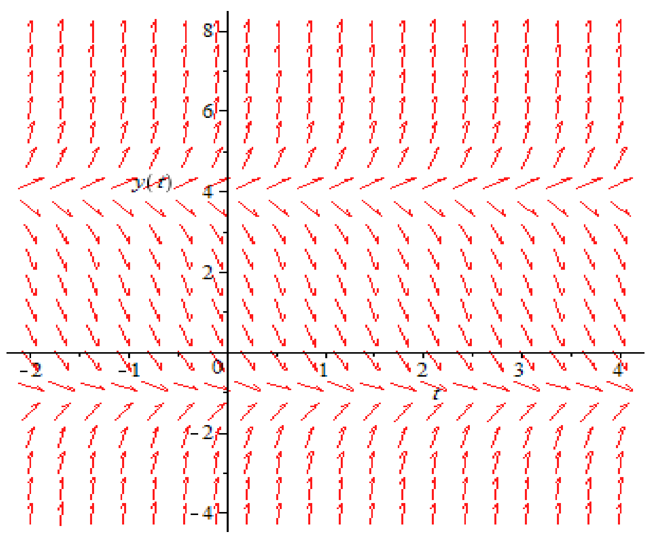
The equilibrium points are: $-2,\ -\frac{1}{2},\ 1,\ 2$

$-2,\ 1$ are asymptotically stable

$-\frac{1}{2},\ 2$ are unstable

Exercise

Impose the equilibrium solution(s), classifying each as either unstable or asymptotically stable



Solution

Because the $y' = f(y)$ is autonomous, the slope at any point $(t,\ x)$ in the direction field does not depend on *t*, only on *y*.

There are two equilibrium points. The smaller of them is unstable and the other is asymptotically stable.

Exercise

An autonomous differential equation is given by $y' = 2 - y$

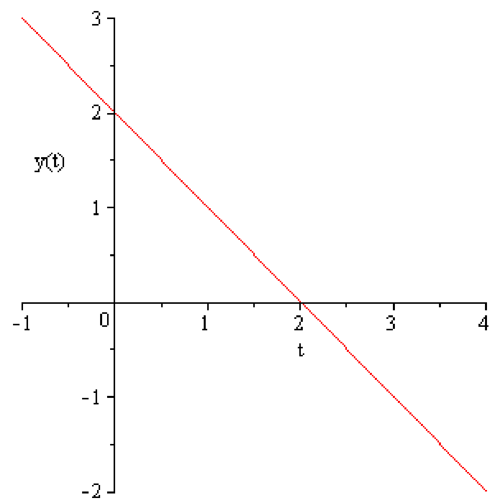
a) Sketch a graph of $f(y)$

b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

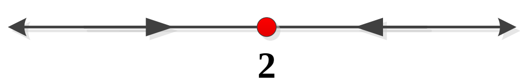
c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

Solution

a) $f(y) = 2 - y$



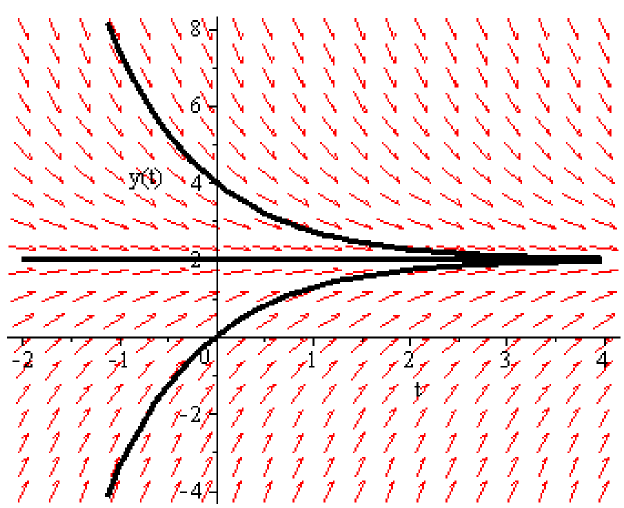
b) The phase line for the autonomous equation is



$y = 2$ is asymptotically stable

c) The phase line indicates that the solutions increase if $y < 2$ and decrease if $y > 2$.

The stable equilibrium solution is $y = 2$



Exercise

An autonomous differential equation is given by $y' = (y+1)(y-4)$

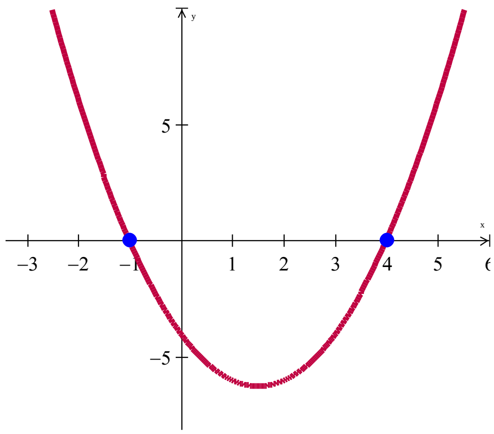
a) Sketch a graph of $f(y)$

b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

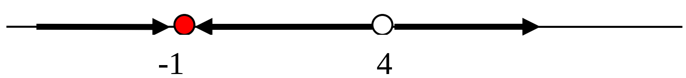
c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

Solution

a) $f(y) = (y+1)(y-4)$

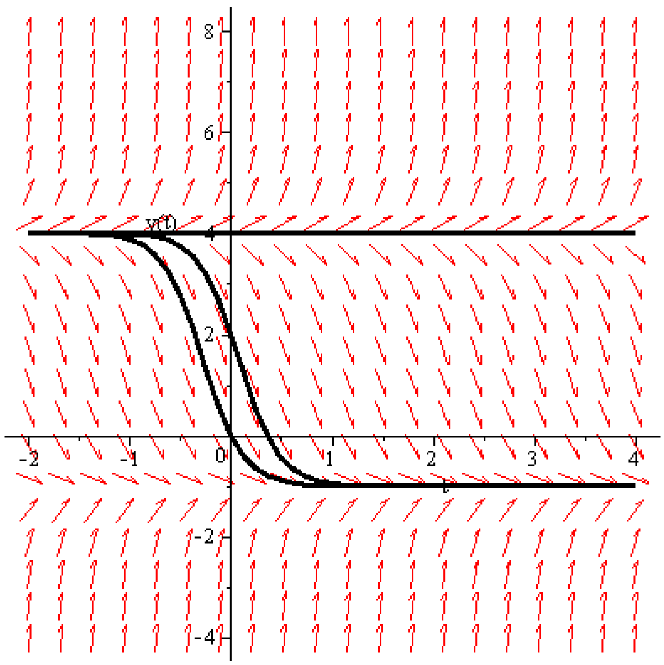


b) The phase line for the autonomous equation is



$y=-1$ is asymptotically stable and $y=4$ is unstable

c)



Exercise

An autonomous differential equation is given by $y' = 9y - y^3$

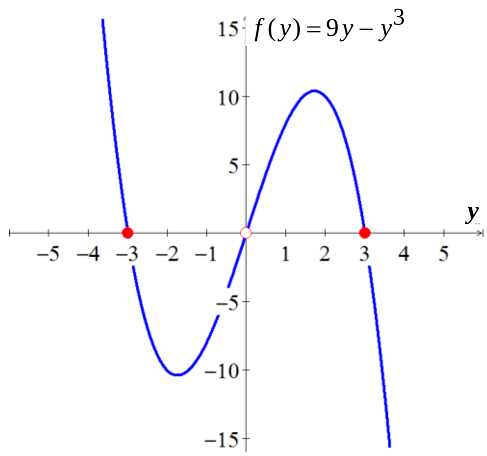
a) Sketch a graph of $f(y)$

b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

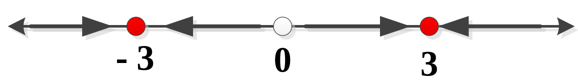
c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

Solution

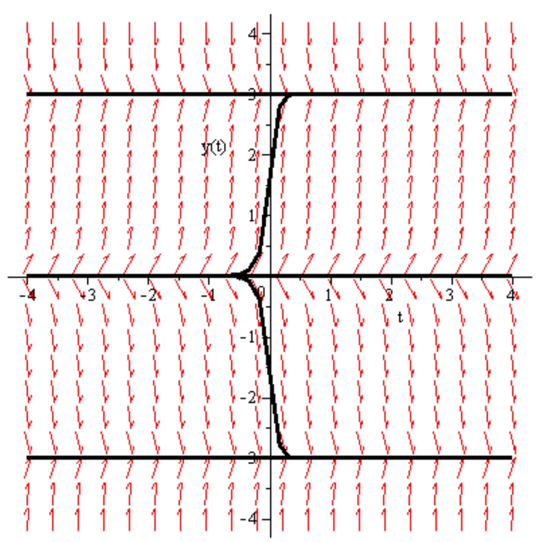
a) $f(y) = 9y - y^3 = y\left(9 - y^2\right)$



b) The phase line for the autonomous equation is



c) The solutions increase if $y < -3$, decrease for $-3 < y < 0$, increase if $0 < y < 3$, and decrease for $y > 3$.



The stable equilibrium solutions at $y(t) = -3, \quad y(t) = 3$ and unstable equilibrium solutions at $y(t) = 0$

Exercise

An autonomous differential equation is given by $y' = \sin y$

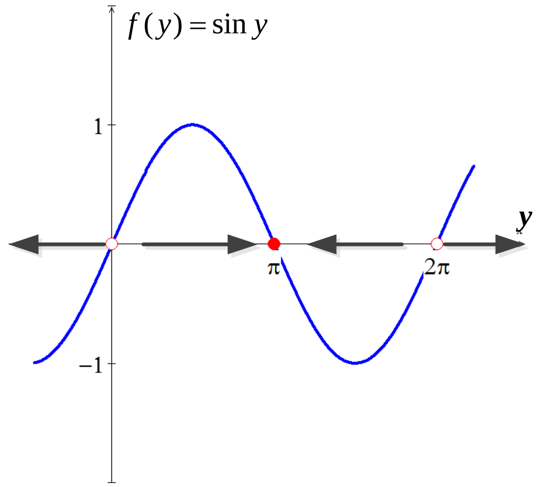
a) Sketch a graph of $f(y)$

b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

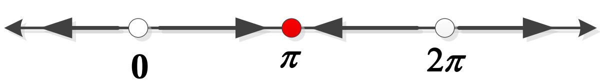
c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

Solution

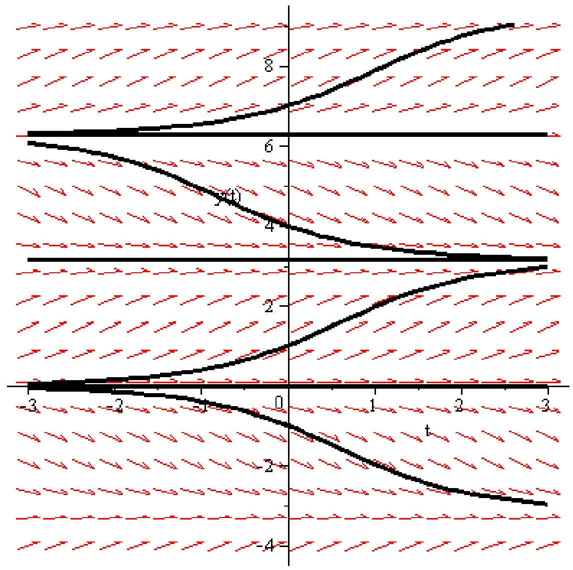
a) $f(y) = \sin y$



b) The phase line for the autonomous equation is



c) The solutions decrease if $-\pi < y < 0$, increase for $0 < y < \pi$, increase if $\pi < y < 2\pi$



The stable equilibrium solutions at $y(t) = \pi$ and unstable equilibrium solutions at $y(t) = 0,\ 2\pi$

Exercise

An autonomous differential equation is given by $y' = y^2 - 3y$

a) Sketch a graph of $f(y)$

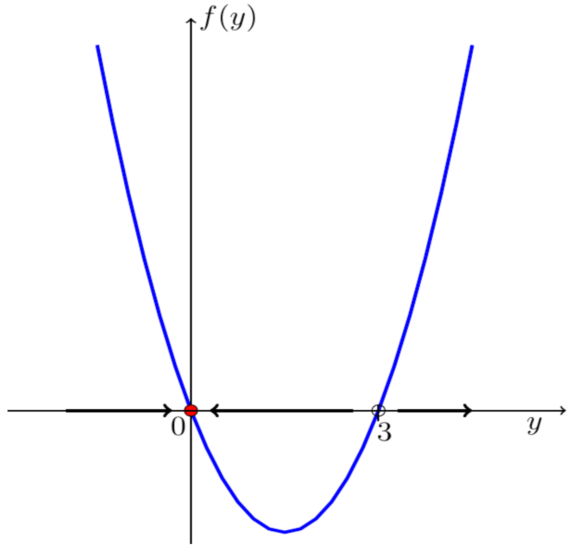
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

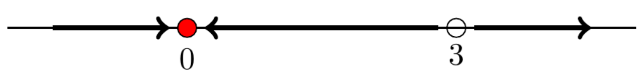
Solution

a) $f(y) = y^2 - 3y = 0 \quad \rightarrow \quad \underline{y = 0,\ 3}$

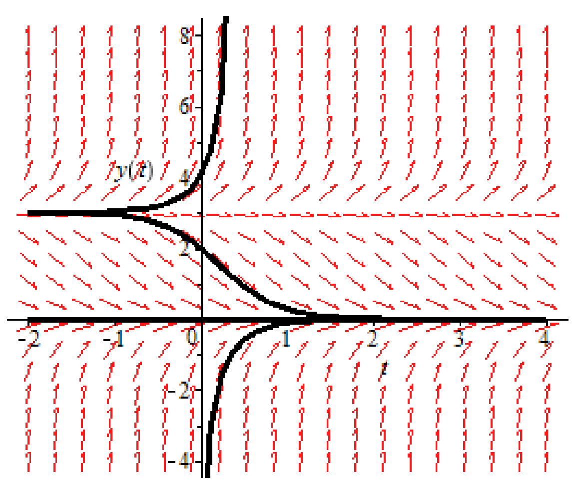
The critical points are 0 and 3.



b) The phase line for the autonomous equation is



c) The solutions increase if $-\infty < y < 0$ and $0 < y < \infty$, decrease $0 < y < 3$



The asymptotically stable equilibrium solution at $y = 0$ (attractor) and unstable equilibrium solutions at $y = 3$ (repeller).

Exercise

An autonomous differential equation is given by $y' = y^2 - y^3$

a) Sketch a graph of $f(y)$

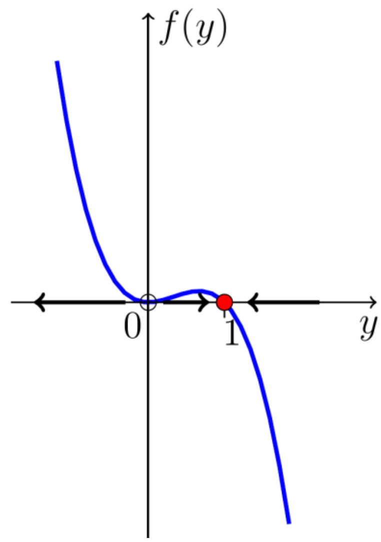
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

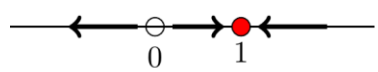
Solution

a) $f(y) = y^2 - y^3 = 0 \quad \to \quad \underline{y = 0,\ 1}$

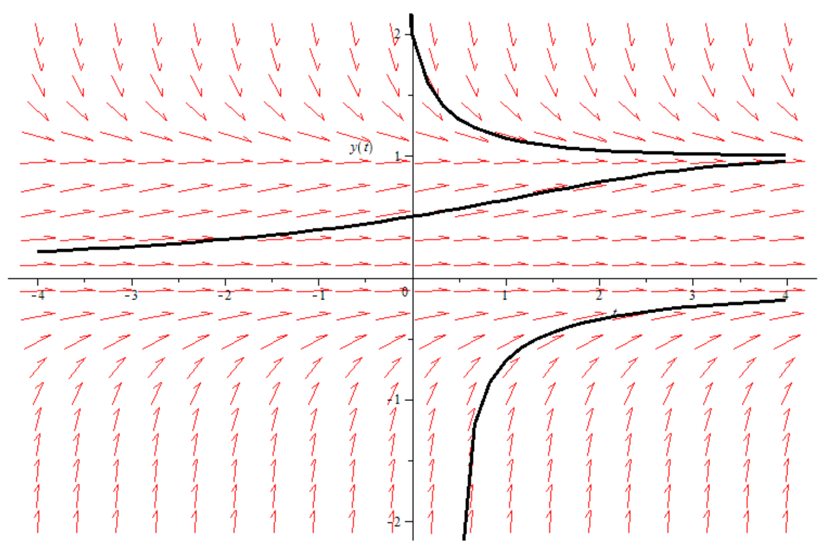
The critical points are 0 and 1.



b) The phase line for the autonomous equation is



c) The solutions increase if $0 < y < 1$ and $1 < y < \infty$, decrease $-\infty < y < 0$



The asymptotically stable at $y = 1$ (attractor) and semi-stable at $y = 0$.

Exercise

An autonomous differential equation is given by $y' = (y-2)^4$

a) Sketch a graph of $f(y)$

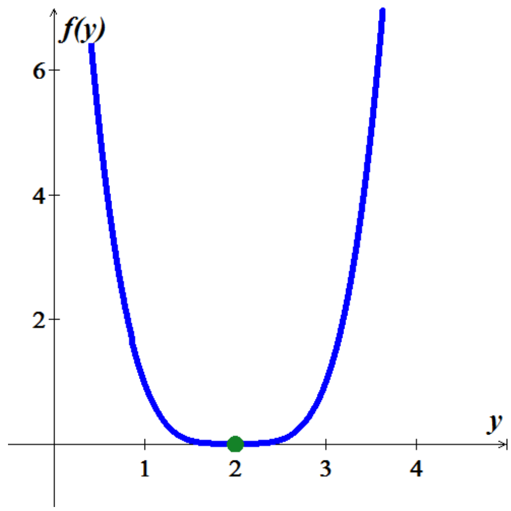
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

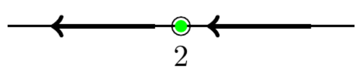
Solution

a) $f(y) = (y-2)^4 = 0 \quad \rightarrow \quad \underline{y=2}|$

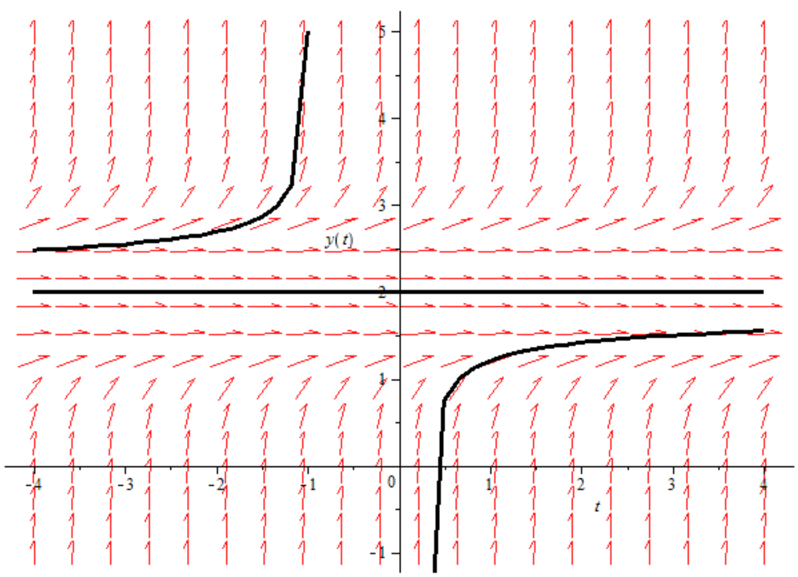
The critical point is 2.



b) The phase line for the autonomous equation is



c) The solutions increase if $-\infty < y < 2$ and $2 < y < \infty$



The semi-stable at $y = 2$.

Exercise

An autonomous differential equation is given by $y' = 10 + 3y - y^2$

a) Sketch a graph of $f(y)$

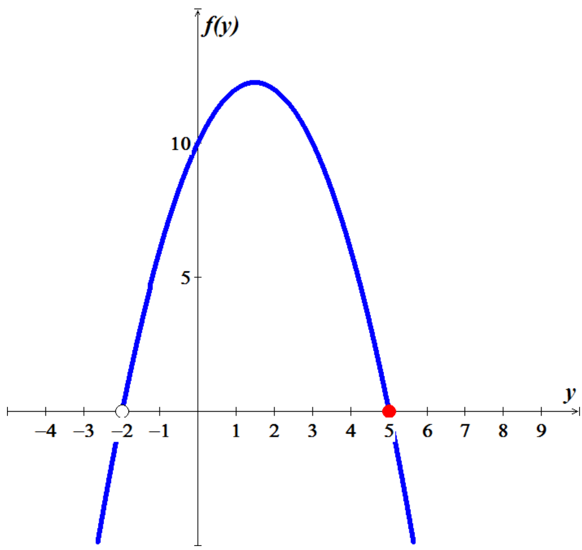
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

Solution

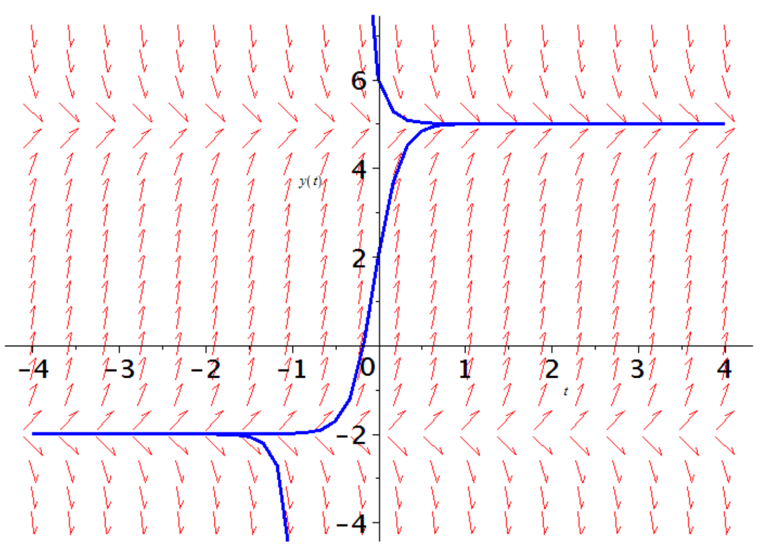
a) $f(y) = 10 + 3y - y^2 = 0 \quad \rightarrow \quad \underline{y = -2,\ 5}|$

The critical points are –2 and 5.



b) The phase line for the autonomous equation is

c) The solutions increase if $-2 < y < 5$, decrease $-\infty < y < -2$ and $5 < y < \infty$



The asymptotically stable at $y = 5$ (attractor) and unstable at $y = -2$ (repeller).

Exercise

An autonomous differential equation is given by $\frac{dy}{dt} = y^2\left(4 - y^2\right)$

a) Sketch a graph of $f(y)$

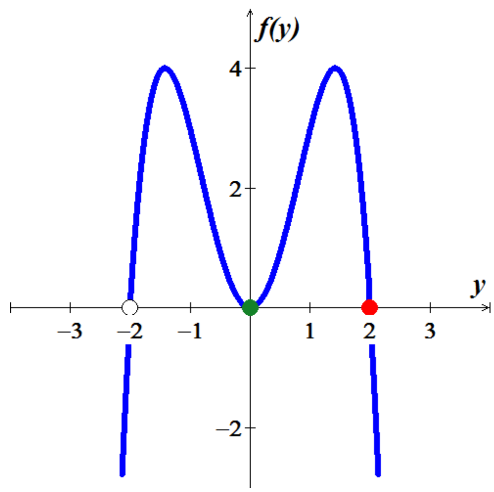
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

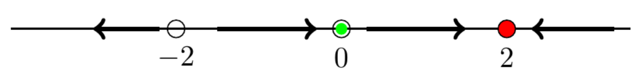
Solution

a) $f(y) = y^2\left(4 - y^2\right) = 0 \quad \rightarrow \quad \underline{y = \pm 2,\ 0}$

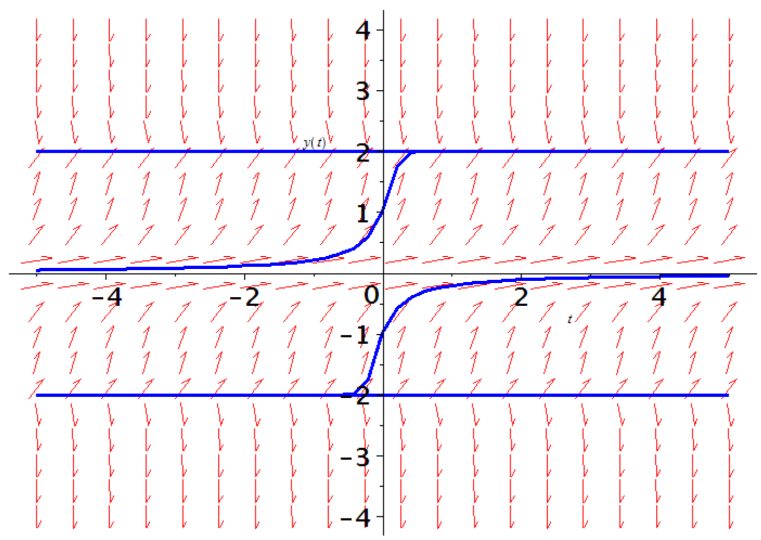
The critical points are ± 2 and 0.



b) The phase line for the autonomous equation is



c) The solutions increase if $-2 < y < 2$, decrease $-\infty < y < -2$ and $2 < y < \infty$



The asymptotically stable at $y = 2$ (attractor), semi-stable at $y = 0$, and unstable at $y = -2$ (repeller).

Exercise

An autonomous differential equation is given by $\frac{dy}{dt} = y(2-y)(4-y)$

a) Sketch a graph of $f(y)$

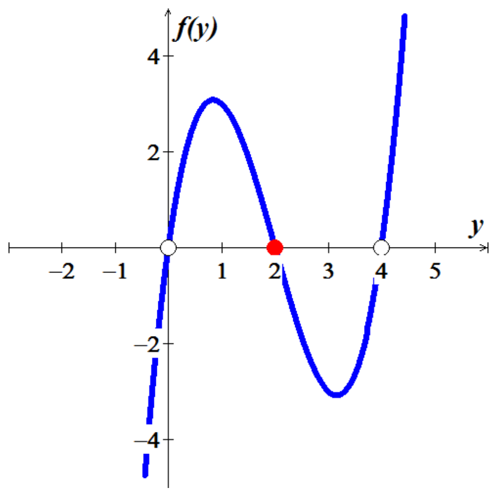
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

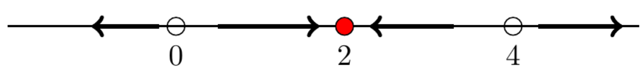
Solution

a) $f(y) = y(2-y)(4-y) = 0 \quad \rightarrow \quad \underline{y = 0,\ 2,\ 4|}$

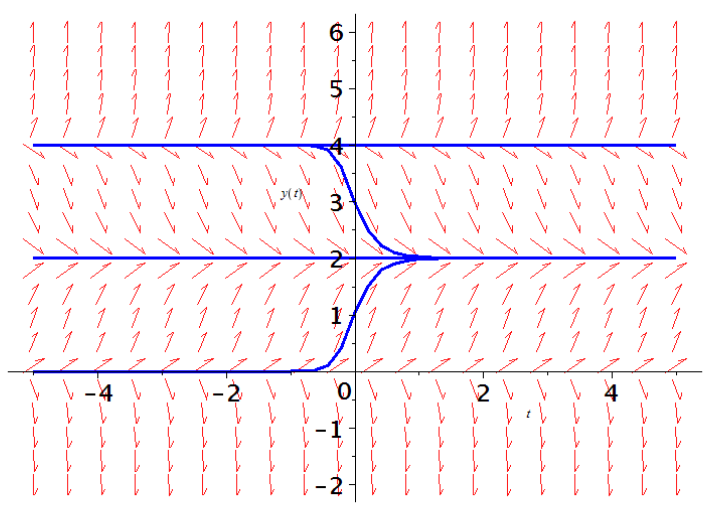
The critical points are $0,\ 2,\ 4$.



b) The phase line for the autonomous equation is



c) The solutions increase if $0 < y < 2$ and $4 < y < \infty$, decrease $-\infty < y < 0$ and $2 < y < 4$



The asymptotically stable at $y = 2$ (attractor) and unstable at $y = 0,\ 4$ (repellers).

Exercise

An autonomous differential equation is given by $\dfrac{dy}{dt} = y\ln(y+2)$

a) Sketch a graph of $f(y)$

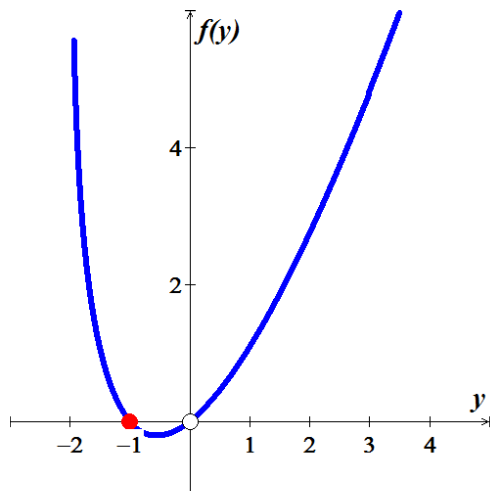
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

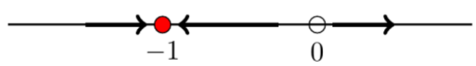
Solution

a) $f(y) = y\ln(y+2) = 0 \quad \rightarrow \quad \underline{y = 0,\ -1}$

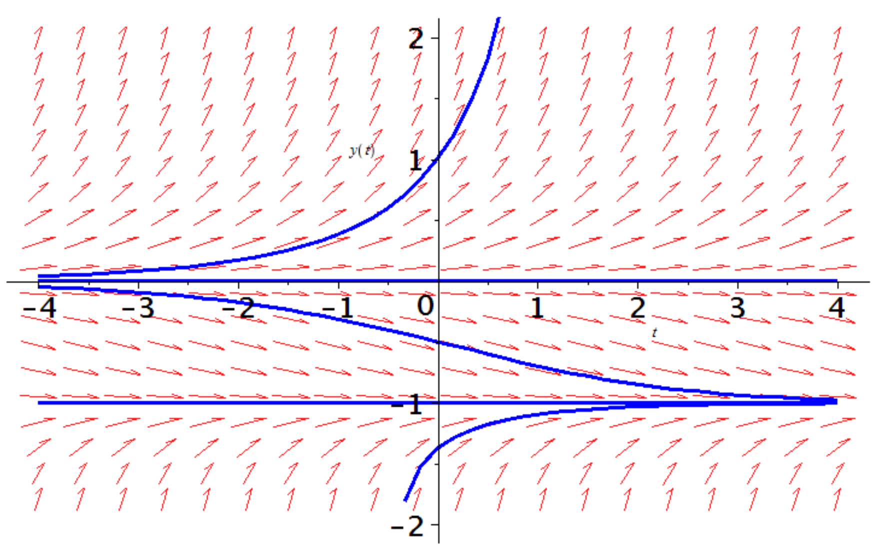
The critical points are $0,\ -1$.



b) The phase line for the autonomous equation is



c) The solutions increase if $-\infty < y < -1$ and $0 < y < \infty$, decrease $-1 < y < 0$



The asymptotically stable at $y = -1$ (attractor) and unstable at $y = 0$ (repeller).

Exercise

An autonomous differential equation is given by $\dfrac{dy}{dt}=\dfrac{ye^y-9y}{e^y}$

a) Sketch a graph of $f(y)$

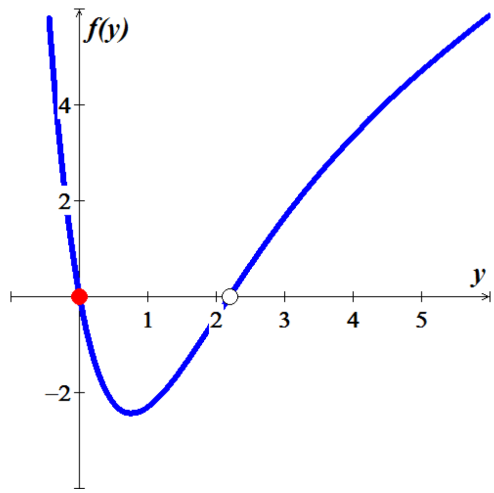
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

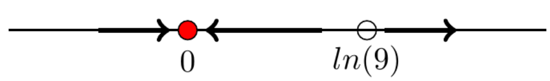
Solution

a) $f(y)=\dfrac{y\left(e^y-9\right)}{e^y}=0 \quad\rightarrow\quad \underline{y=0,\ \ln 9|}$

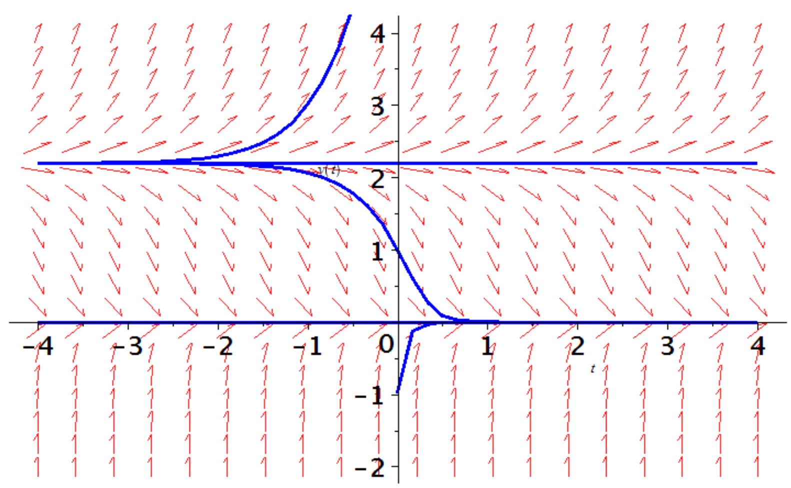
The critical points are $0,\ \ln 9$.



b) The phase line for the autonomous equation is



c) The solutions increase if $-\infty<y<-1$ and $0<y<\infty$, decrease $-1<y<0$



The asymptotically stable at $y=0$ (attractor) and unstable at $y=\ln 9$ (repeller).

Exercise

An autonomous differential equation is given by $y' = \frac{2}{\pi} y - \sin y$

a) Sketch a graph of $f(y)$

b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

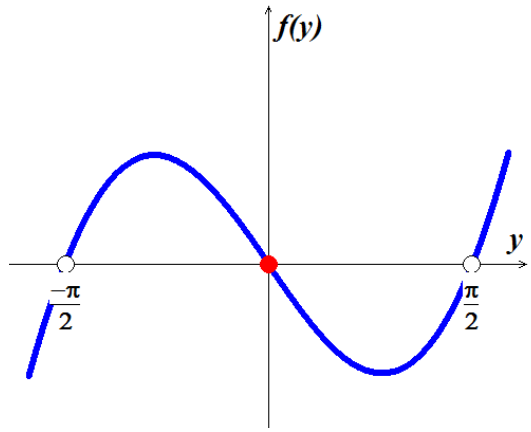
c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

Solution

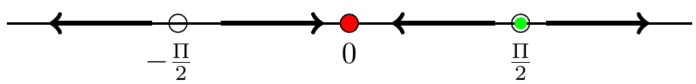
a) $f(y) = \frac{2}{\pi} y - \sin y = 0$

$\frac{2}{\pi} y = \sin y \quad \rightarrow \quad \underline{y = 0, \ \pm \frac{\pi}{2}}$

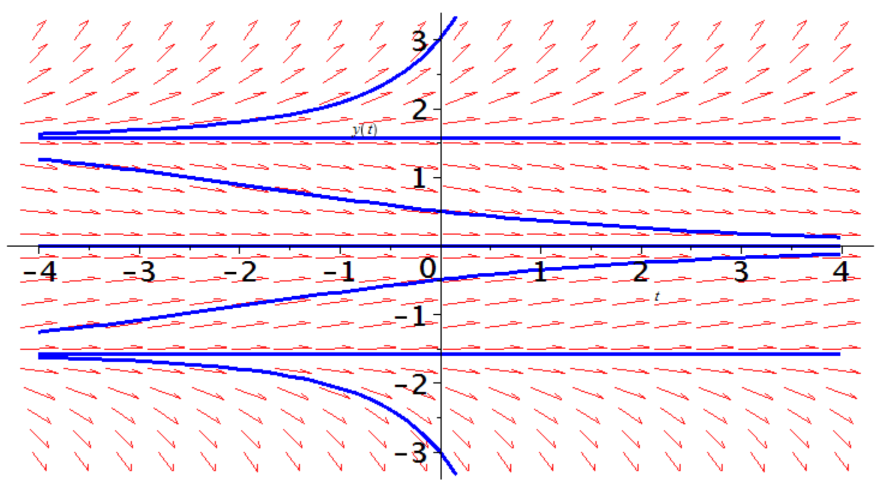
The critical points are $0, \ \pm \frac{\pi}{2}$.



b) The phase line for the autonomous equation is



c) The solutions increase if $-\frac{\pi}{2} < y < 0$ and $y > \frac{\pi}{2}$, decrease $\frac{\pi}{2} < y < 0$ and $y < -\frac{\pi}{2}$



The asymptotically stable at $y = 0$ (attractor) and unstable at $y = \pm \frac{\pi}{2}$ (repeller).

Exercise

An autonomous differential equation is given by $y' = 3y - ye^{y^2}$

a) Sketch a graph of $f(y)$

b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

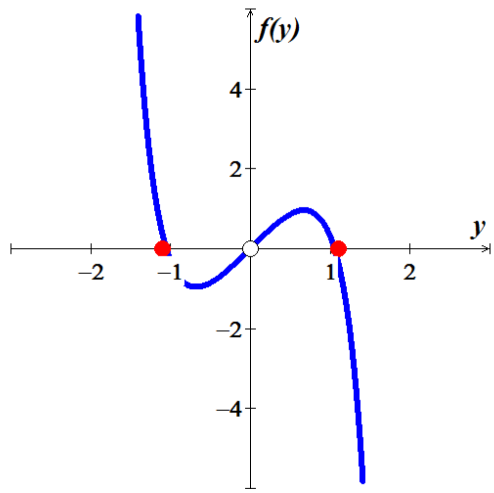
Solution

a) $f(y) = 3y - ye^{y^2} = 0$

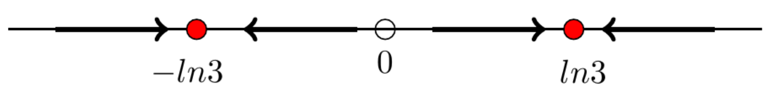
$$y\left(3 - e^{y^2}\right) = 0 \quad \rightarrow \quad y = 0, \ e^{y^2} = 3$$

$$e^{y^2} = 3 \ \rightarrow \ \underline{y = \pm\sqrt{\ln 3}, \ 0}$$

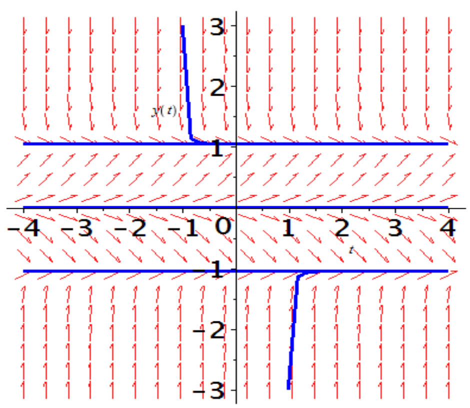
The critical points are $\pm\sqrt{\ln 3}, \ 0$.



b) The phase line for the autonomous equation is



c) The solutions increase if $y < -\ln 3$ and $0 < y < \ln 3$, decrease $-\ln 3 < y < 0$ and $y > \ln 3$



The asymptotically stable at $y = \pm\sqrt{\ln 3}$ (attractor) and unstable at $y = 0$ (repeller).

Exercise

An autonomous differential equation is given by $\quad y' = (1-y)(y+1)^2$

a) Sketch a graph of $f(y)$

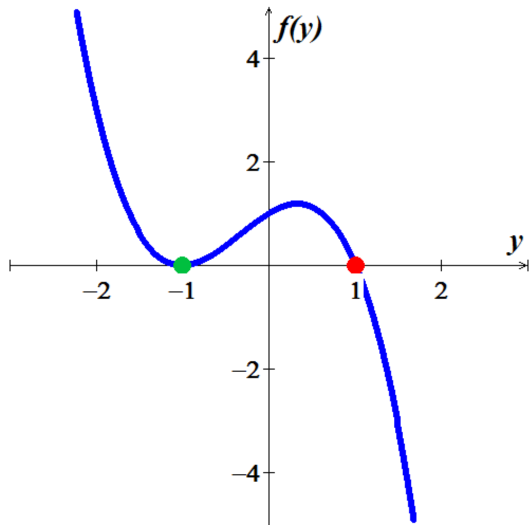
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

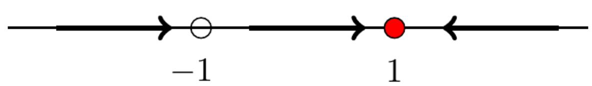
Solution

a) $f(y) = (1-y)(y+1)^2 = 0 \rightarrow \underline{y = -1, \ -1, \ 1}$

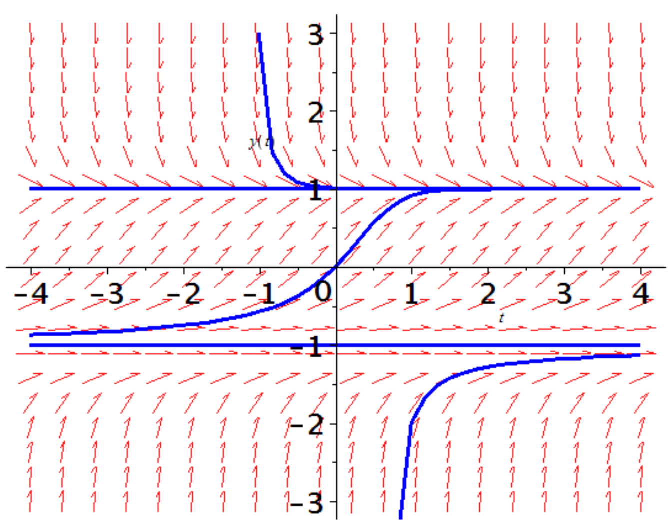
The critical points are ± 1.



b) The phase line for the autonomous equation is



c) The solutions increase if $y < -1$ and $-1 < y < 0$, decrease $y > 1$



The asymptotically stable at $y = 1$ (attractor) and semi-stable at $y = -1$.

Exercise

An autonomous differential equation is given by $y' = \sin\frac{y}{2}$

a) Sketch a graph of $f(y)$

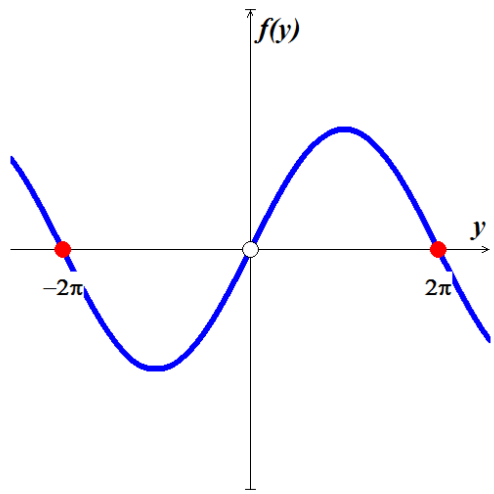
b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

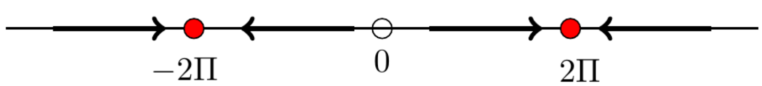
Solution

a) $f(y) = \sin\frac{y}{2} = 0 \rightarrow \underline{y = 2n\pi}$

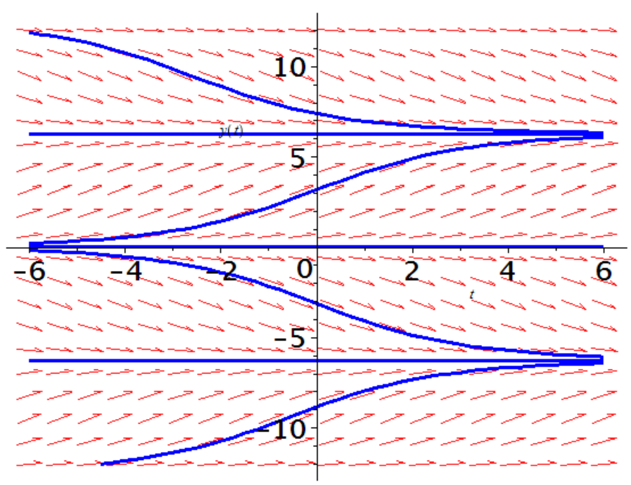
The critical points are $0, \ \pm 2\pi$.



b) The phase line for the autonomous equation is



c) The solutions increase if $0 < y < 2\pi$ and $-3\pi < y < -2\pi$, decrease $-2\pi < y < 0$ and $2\pi < y < 3\pi$



The asymptotically stable at $y = \pm 2\pi$ (attractor) and unstable at $y = 0$ (repeller).

Exercise

Determine the stability of the equilibrium solutions $x' = 4 - x^2$

Solution

$$f(x) = x' = 4 - x^2 = 0$$

$$\Rightarrow x^2 = 4$$

The equilibrium points $x = \pm 2$

$$f'(x) = -2x$$

$f'(-2) = -2(-2) > 0$ $\quad$ $x = -2$ is unstable

$f'(2) = -2(2) < 0$ $\quad$ $x = 2$ is asymptotically stable

Exercise

Determine the stability of the equilibrium solutions $x' = x(x-1)(x+2)$

Solution

The equation $f(x) = x(x-1)(x+2)$.

$f(x) = 0 \Rightarrow$ The equilibrium points are $x = 0,\ 1,\ -2$.

$$f(x) = x\left(x^2 + x - 2\right) = x^3 + x^2 - 2x$$

$$f'(x) = 3x^2 + 2x - 2$$

$f'(0) = -2 < 0 \quad \Rightarrow \quad x = 0$ ***Asymptotically stable***

$f'(1) = 3 > 0 \quad \Rightarrow \quad x = 1$ ***Unstable***

$f'(-2) = 2 > 0 \quad \Rightarrow \quad x = -2$ ***Unstable***

Exercise

A tank contains 100 *gal* of pure water. A salt solution with concentration 3 *lb/gal* enters the tank at a rate of 2 *gal/min*. Solution drains from the tank at a rate of 2 *gal/min*. Use the qualitative analysis to find the eventual concentration of the salt in the tank.

Solution

Let $x(t)$ represents the amount of salt.

$$Rate\ in = 2\frac{gal}{min} \times 3\frac{lb}{gal} = 6\frac{lb}{\min}$$

$$Rate\ out = 2\frac{gal}{min} \times \frac{x(t)}{100}\frac{lb}{gal} = \frac{x(t)}{50}\frac{lb}{gal}$$

$$\frac{dx}{dt} = 6 - \frac{1}{50}x$$

Let $c(t)$ represents the concentration of salt. Thus, $c(t)=\frac{x(t)}{100} \quad \rightarrow \quad x'=100c'$

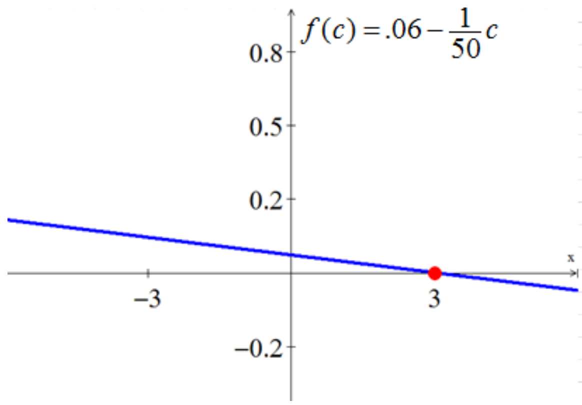
$100c'=6-\frac{1}{50}(100c)$

$100c'=.06-\frac{1}{50}c$

$\Rightarrow f(c)=.06-\frac{1}{50}c=0$

$\frac{1}{50}c=.06 \quad \Rightarrow \quad c=3$

$c=3$ is stable equilibrium point so a trajectory should approach the stable equilibrium solution $c(t)=3$



Exercise

A mathematical model for rate at which a drug disseminates into the bloodstream at time *t*.

$$\frac{dx}{dt}=r-kx$$

Where *r* and *k* are positive constants. The function $x(t)$ describes the concentration of the drug in the bloodstream at time *t*.

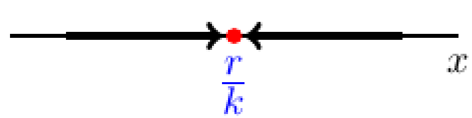
a) Since the DE is autonomous, use the phase portrait concept to find the limiting value of $x(t)$ as $t\to\infty$

b) Solve $x(t)$ subject to $x(0)=0$. Sketch the graph of $x(t)$ and verify your prediction in part (*a*). At what time is the concentration one-half this limiting value?

Solution

a) $\frac{dx}{dt}=r-kx=0 \quad \rightarrow \quad \boxed{x=\frac{r}{k}}$

The equilibrium solution $x=\frac{r}{k}$.



When $x<\frac{r}{k} \quad \Rightarrow \quad \frac{dx}{dt}>0$

When $x>\frac{r}{k} \quad \Rightarrow \quad \frac{dx}{dt}<0$

$\boxed{\lim\limits_{x\to\infty} x(t)=\frac{r}{k}}$

b) $\frac{dx}{dt}+kx=r$

$$e^{\int k\,dt} = e^{kt}$$

$$\int re^{kt}\,dt = \frac{r}{k}e^{kt}$$

$$x(t) = \frac{1}{e^{kt}}\left(\frac{r}{k}e^{kt} + C\right)$$

$$\boxed{= \frac{r}{k} + Ce^{-kt}}$$

$$x(0) = 0 \quad \to 0 = \frac{r}{k} + C \quad \Rightarrow \quad \boxed{C = -\frac{r}{k}}$$

$$\boxed{x(t) = \frac{r}{k} - \frac{r}{k}e^{-kt}}$$

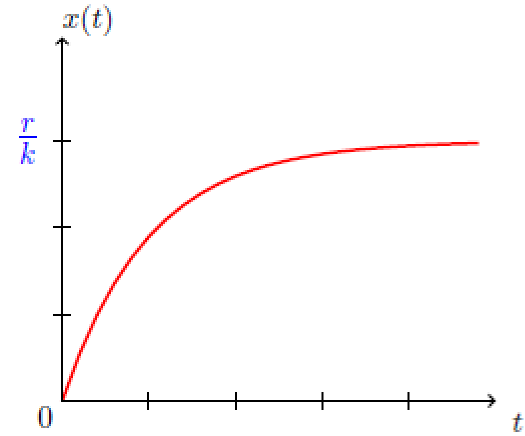
$x \to \frac{r}{k}$ as $t \to \infty$

If $x(T) = \frac{r}{2k}$

$$\frac{r}{2k} = \frac{r}{k} - \frac{r}{k}e^{-kT}$$

$$\frac{r}{2k} = \frac{r}{k}e^{-kT}$$

$$e^{-kT} = \frac{1}{2} \quad \to \quad \boxed{T = \frac{\ln 2}{k}}$$



Exercise

When forgetfulness is taken into account, the rate of memorization of a subject is given by

$$\frac{dA}{dt} = k_1(M - A) - k_2 A$$

Where $k_1 > 0$, $k_2 > 0$, $A(t)$ is the amount memorized in time *t*, *M* is the total amount to be memorized, and $M - A$ is the amount remaining to be memorized.

a) Since the DE is autonomous, use the phase portrait concept to find the limiting value of $A(t)$ as $t \to \infty$. Interpret the result

b) Solve $A(t)$ subject to $A(0) = 0$. Sketch the graph of $A(t)$ and verify your prediction in part (*a*).

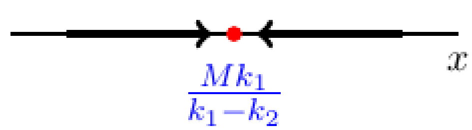
Solution

a) $\frac{dA}{dt} = k_1(M - A) - k_2 A = 0$

$$k_1 M - k_1 A - k_2 A = 0$$

$$A = \frac{k_1 M}{k_1 + k_2}\text{; the equilibrium solution}$$

$$\boxed{\lim_{t\to\infty} A(t) = \frac{k_1 M}{k_1 + k_2}}$$



Since $k_2 > 0$, the material will never be completely memorized and the larger k_2 is, the less the amount of material will be memorized over time.

b) $\frac{dA}{dt} = k_1 M - (k_1 + k_2) A$

$$\frac{dA}{dt} + (k_1 + k_2) A = k_1 M$$

$$e^{\int (k_1 + k_2) dt} = e^{(k_1 + k_2)t}$$

$$\int M k_1 e^{(k_1 + k_2)t}\, dt = \frac{M k_1}{k_1 + k_2} e^{(k_1 + k_2)t}$$

$$A(t) = e^{-(k_1 + k_2)t} \left(\frac{M k_1}{k_1 + k_2} e^{(k_1 + k_2)t} + C \right)$$

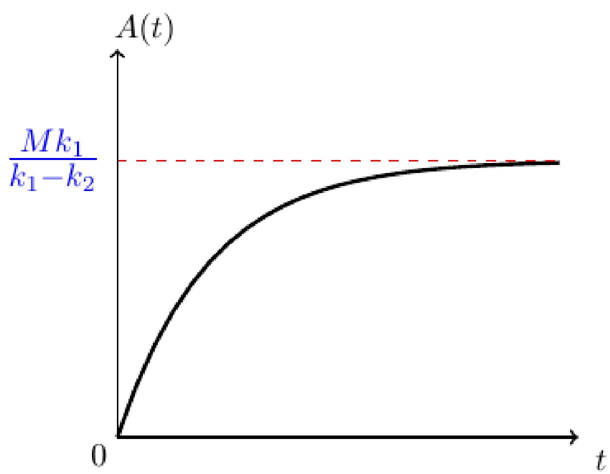
$$\boxed{= \frac{M k_1}{k_1 + k_2} + C e^{-(k_1 + k_2)t}}$$

$$A(0) = 0 \quad \to 0 = \frac{M k_1}{k_1 + k_2} + C \quad \Rightarrow \boxed{C = -\frac{M k_1}{k_1 + k_2}}$$

$$\boxed{A(t) = \frac{M k_1}{k_1 + k_2} - \frac{M k_1}{k_1 + k_2} e^{-(k_1 + k_2)t}}$$

$$\lim_{t \to \infty} A(t) = \lim_{t \to \infty} \left(\frac{M k_1}{k_1 + k_2} - \frac{M k_1}{k_1 + k_2} e^{-(k_1 + k_2)t} \right)$$

$$\boxed{= \frac{M k_1}{k_1 + k_2}}$$



Exercise

The number $N(t)$ of supermarkets throughout the country that are using a computerized checkout system is described by the initial-value problem

$$\frac{dN}{dt} = N(1-0.0005N), \quad N(0)=1$$

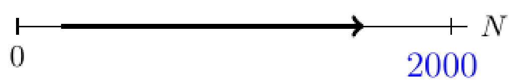
a) Use the phase portrait concept to predict how many supermarkets are expected to adopt the new procedure over a long period of time. Sketch a solution curve of the given initial-value problem.

b) Solve the initial-value problem and then graph it to verify the solution in part (*a*)

c) How many companies are expected to adopt the new technology when $t=10$?

Solution

a) $\frac{dN}{dt} = N(1-0.0005N) = 0$



$N=0 \quad N=\frac{1}{0.0005}=2000$

When $0<N<2000 \quad \Rightarrow \quad \frac{dN}{dt}>0$

From the phase portait:

$\lim\limits_{t\to\infty} N(t) \underline{= 2000}$

b) $\frac{dN}{N(1-0.0005N)} = dt$

$\frac{2000}{N(2000-N)}dN = dt$

$\int\left(\frac{1}{N}-\frac{1}{N-2000}\right)dN = \int dt$

$\ln N - \ln(N-2000) = t + C$

$\ln\frac{N}{N-2000} = t+C$

$\frac{N}{N-2000} = e^{t+C}$

$N = Ne^{t+C} - 2000e^{t+C}$

$N\left(e^{C}e^{t}+1\right) = 2000e^{C}e^{t}$

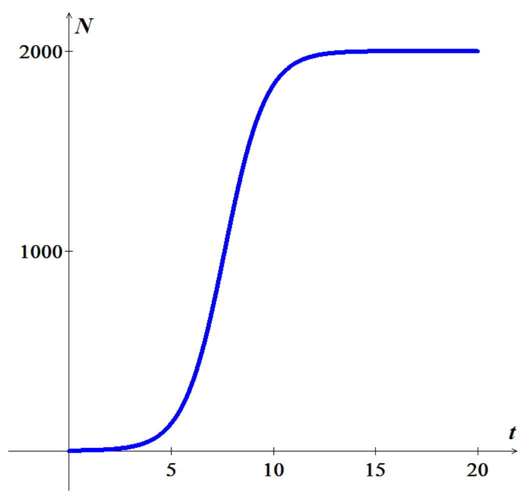
$N(t) = \frac{2000e^{C}e^{t}}{1+e^{C}e^{t}}$

$N(0)=1 \quad 1=\frac{2000e^{C}}{1+e^{C}}$

$1+e^{C} = 2000e^{C}$

$e^{C} = \frac{1}{1999}$

$\boxed{N(t) = \frac{2000e^{t}}{1999+e^{t}}}$



$$N(10) = \frac{2000e^{10}}{1999 + e^{10}} \approx 1833.59$$

About 1834 companies are expected to adopt the new technology when $t = 10$

Exercise

For the linear *ODE* $ty' + y = 2t$

a) Find all solution of the given *DE* equation.

b) Show that the initial value $y(0) = 0$, has exactly one solution.

c) But if $y(0) = y_0 \ne 0$ there is no solution at all. Why doesn't this contradict the Existence and Uniqueness Theorem?

d) Plot several solutions of the *ODE* over the interval $-5 \le t \le 5$

Solution

a) $\frac{1}{t} \times \quad ty' + y = 2t$

$$y' + \frac{1}{t}y = 2 \quad (t \ne 0)$$

$$e^{\int \frac{dt}{t}} = e^{\ln t} = t$$

$$\int 2t\,dt = t^2$$

$$y(t) = \frac{1}{t}\left(t^2 + C\right)$$

$$\boxed{= t + \frac{C}{t}}$$

Each value of *C* gives 2 distinct solutions. On defined $-\infty < t < 0$ and the other on $0 < t < \infty$

b) ***Given***: $y(0) = 0$

$0 = 0 + \frac{C}{0}$, the only solution is when $C = 0$, which gives us the general solution

$$\boxed{y(t) = t}$$

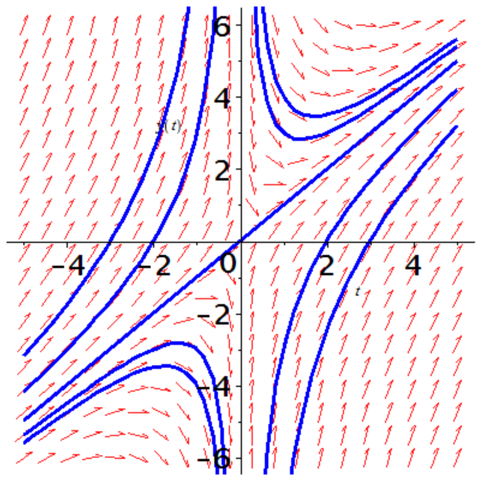
c) $y(0) = y_0 \ne 0 \quad \to \quad y_0 = 0 + \frac{C}{0}$

$\Rightarrow y_0 = 0$ which contradict the given information.

Which contradict the Existence and Uniqueness Theorem of the initial value existence.

d) The solution curve of $y = t$ goes through the origin.

All the other are curves in the shape of hyperbolas and are asymptotic to one end or the other of the *y*-axis



6
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