Lecture Three - Identities

Section 3.1 – Proving Identities

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Ratio Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos\theta = \pm\sqrt{1-\sin^2\theta}$$

$$\sin\theta = \pm\sqrt{1-\cos^2\theta}$$

$$1+\tan^2\theta = \sec^2\theta$$

$$1+\cot^2\theta = \csc^2\theta$$

Example

Write $\sec\theta\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify.

<u>*Solution*</u>

$$\sec\theta\tan\theta = \frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta}$$

$$= \frac{\sin\theta}{\cos^2\theta}$$

Example

Add $\frac{1}{\sin\theta}+\frac{1}{\cos\theta}$

<u>*Solution*</u>

$$\frac{1}{\sin\theta}+\frac{1}{\cos\theta}=\frac{\cos\theta+\sin\theta}{\sin\theta\cos\theta}$$

$$\frac{1}{\sin\theta}\frac{\cos\theta}{\cos\theta}+\frac{1}{\cos\theta}\frac{\sin\theta}{\sin\theta}$$

Example

Write: $\tan\alpha+\cot\alpha$ in terms of $\sin\alpha$ *and* $\cos\alpha$

<u>*Solution*</u>

$$\begin{aligned}\tan\alpha+\cot\alpha &= \frac{\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha}\\ &= \frac{\sin\alpha}{\cos\alpha}\frac{\sin\alpha}{\sin\alpha}+\frac{\cos\alpha}{\sin\alpha}\frac{\cos\alpha}{\cos\alpha}\\ &= \frac{\sin^2\alpha+\cos^2\alpha}{\cos\alpha\sin\alpha}\\ &= \frac{1}{\cos\alpha\sin\alpha}\end{aligned}$$

Example

Prove: $\tan x+\cos x=\sin x(\sec x+\cot x)$

<u>*Solution*</u>

$$\begin{aligned}\tan x+\cos x &= \frac{\sin x}{\cos x}+\cos x\\ &= \sin x\frac{1}{\cos x}+\cos x\frac{\sin x}{\sin x}\\ &= \sin x\sec x+\sin x\frac{\cos x}{\sin x}\\ &= \sin x(\sec x+\cot x)\end{aligned}$$

$\boxed{\textit{or}}$

$$\begin{aligned}\sin x(\sec x+\cot x) &= \sin x\left(\frac{1}{\cos x}+\frac{\cos x}{\sin x}\right)\\ &= \frac{\sin x}{\cos x}+\sin x\frac{\cos x}{\sin x}\\ &= \tan x+\cos x\end{aligned}$$

Example

Prove: $\cot\alpha + 1 = \csc\alpha\left(\cos\alpha + \sin\alpha\right)$

<u>*Solution*</u>

$$\begin{aligned}\csc\alpha\left(\cos\alpha + \sin\alpha\right) &= \frac{1}{\sin\alpha}\left(\cos\alpha + \sin\alpha\right)\\ &= \frac{1}{\sin\alpha}\cos\alpha + \frac{1}{\sin\alpha}\sin\alpha\\ &= \cot\alpha + 1\end{aligned}$$

Guidelines for Proving Identities

1. Work on the complicated side first (more trigonometry functions)
2. Look for trigonometry substitutions.
3. Look for algebraic operations
4. If not always change everything to sines and cosines
5. Keep an eye on the side you are not working.

Example

Prove $\dfrac{\cos^4 t - \sin^4 t}{\cos^2 t} = 1 - \tan^2 t$

<u>*Solution*</u>

$$\frac{\cos^4 t - \sin^4 t}{\cos^2 t} = \frac{\left(\cos^2 t - \sin^2 t\right)\left(\cos^2 t + \sin^2 t\right)}{\cos^2 t} \qquad a^2 - b^2 = (a-b)(a+b)$$

$$= \frac{\left(\cos^2 t - \sin^2 t\right)(1)}{\cos^2 t} \qquad \cos^2 t + \sin^2 t = 1$$

$$= \frac{\cos^2 t - \sin^2 t}{\cos^2 t}$$

$$= \frac{\cos^2 t}{\cos^2 t} - \frac{\sin^2 t}{\cos^2 t}$$

$$= 1 - \tan^2 t$$

Example

Prove: $1+\cos\theta = \dfrac{\sin^2\theta}{1-\cos\theta}$

<u>*Solution*</u>

$$\frac{\sin^2\theta}{1-\cos\theta} = \frac{1-\cos^2\theta}{1-\cos\theta} \qquad \sin^2\theta = 1-\cos^2\theta$$

$$= \frac{(1-\cos\theta)(1+\cos\theta)}{1-\cos\theta} \qquad a^2-b^2 = (a-b)(a+b)$$

$$= 1+\cos\theta$$

Example

Prove: $\tan^2\alpha\left(1+\cot^2\alpha\right) = \dfrac{1}{1-\sin^2\alpha}$

<u>*Solution*</u>

$$\tan^2\alpha\left(1+\cot^2\alpha\right) = \tan^2\alpha + \tan^2\alpha\cot^2\alpha$$

$$= \tan^2\alpha + \tan^2\alpha\frac{1}{\tan^2\alpha}$$

$$= \tan^2\alpha + 1 \qquad \tan^2\alpha + 1 = \sec^2\alpha$$

$$= \sec^2\alpha$$

$$= \frac{1}{\cos^2\alpha} \qquad \cos^2\alpha = 1-\sin^2\alpha$$

$$= \frac{1}{1-\sin^2\alpha}$$

Example

Prove : $\dfrac{\sin\alpha}{1+\cos\alpha} + \dfrac{1+\cos\alpha}{\sin\alpha} = 2\csc\alpha$

<u>*Solution*</u>

$$\frac{\sin\alpha}{1+\cos\alpha} + \frac{1+\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{\sin\alpha}\cdot\frac{\sin\alpha}{1+\cos\alpha} + \frac{1+\cos\alpha}{\sin\alpha}\cdot\frac{1+\cos\alpha}{1+\cos\alpha}$$

$$= \frac{\sin^2\alpha + (1+\cos\alpha)^2}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{\sin^2\alpha + 1 + \cos^2\alpha + 2\cos\alpha}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{2+2\cos\alpha}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{2(1+\cos\alpha)}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{2}{\sin\alpha}$$

$$= 2\csc\alpha$$

Example

Prove $\quad \dfrac{1+\sin t}{\cos t} = \dfrac{\cos t}{1-\sin t}$

<u>*Solution*</u>

$$\frac{1+\sin t}{\cos t} = \frac{1+\sin t}{\cos t}\cdot\frac{1-\sin t}{1-\sin t}$$

$$= \frac{1-\sin^2 t}{\cos t(1-\sin t)}$$

$$= \frac{\cos^2 t}{\cos t(1-\sin t)}$$

$$= \frac{\cos t}{1-\sin t}$$

Example

Show that $\cot^2\theta + \cos^2\theta = \cot^2\theta\cos^2\theta$ is not an identity by finding a counterexample

<u>*Solution*</u>

$$\cot^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} = \cot^2\frac{\pi}{4}\cos^2\frac{\pi}{4}$$

$$1^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1^2\left(\frac{1}{\sqrt{2}}\right)^2$$

$$1 + \frac{1}{2} = \frac{1}{2}$$

$$\frac{3}{2} \neq \frac{1}{2}$$

Exercises *Section* 3.1 – Proving Identities

1. Prove the identity: $\cos\theta\cot\theta+\sin\theta=\csc\theta$

2. Prove the identity: $\sec\theta\cot\theta-\sin\theta=\dfrac{\cos^2\theta}{\sin\theta}$

3. Prove the identity: $\dfrac{\csc\theta\tan\theta}{\sec\theta}=1$

4. Prove the identity: $(\sin\theta+\cos\theta)^2=1+2\sin\theta\cos\theta$

5. Prove the identity: $\sin\theta(\sec\theta+\cot\theta)=\tan\theta+\cos\theta$

6. Prove the identity: $\cos\theta(\csc\theta+\tan\theta)=\cot\theta+\sin\theta$

7. Prove the identity: $\cot\theta+\tan\theta=\csc\theta\sec\theta$

8. Prove the identity: $\tan x(\cos x+\cot x)=\sin x+1$

9. Prove the identity: $\dfrac{1-\cos^4\theta}{1+\cos^2\theta}=\sin^2\theta$

10. Prove the identity: $\dfrac{1-\sec x}{1+\sec x}=\dfrac{\cos x-1}{\cos x+1}$

11. Prove the identity: $\dfrac{\cos x}{1-\sin x}-\dfrac{1-\sin x}{\cos x}=0$

12. Prove the identity: $\dfrac{1+\cot^3 t}{1+\cot t}=\csc^2 t-\cot t$

13. Prove the identity: $\tan x+\cot x=\sec x\csc x$

14. Prove the identity: $\dfrac{\tan x-\cot x}{\sin x\cos x}=\sec^2 x-\csc^2 x$

15. Prove the identity: $\dfrac{\sec x+\tan x}{\sec x-\tan x}=\dfrac{1+2\sin x+\sin^2 x}{\cos^2 x}$

16. Prove the identity: $\sin^2 x-\cos^2 x=2\sin^2 x-1$

17. Prove the identity: $\sin^4 x-\cos^4 x=\sin^2 x-\cos^2 x$

18. Prove the identity: $\dfrac{\cos\alpha}{1+\sin\alpha}=\sec\alpha-\tan\alpha$

19. Prove the identity: $\dfrac{\sin\alpha}{1-\sin\alpha}-\dfrac{\cos\alpha}{1-\sin\alpha}=\dfrac{1-\cot\alpha}{\csc\alpha-1}$

20. Prove the identity: $\dfrac{\frac{1}{\tan x}+\cot x}{\frac{1}{\tan x}+\tan x}=\dfrac{2}{\sec^2 x}$

21. Prove the following equation is an identity: $\dfrac{\cot^2\theta+3\cot\theta-4}{\cot\theta+4}=\cot\theta-1$

22. Prove the following equation is an identity: $\dfrac{\sin\theta}{1+\cos\theta}=\dfrac{1-\cos\theta}{\sin\theta}$

23. Prove the following equation is an identity: $\tan x(\csc x-\sin x)=\cos x$

24. Prove the following equation is an identity: $\sin x(\tan x\cos x-\cot x\cos x)=1-2\cos^2 x$

25. Prove the following equation is an identity: $(1+\tan x)^2+(\tan x-1)^2=2\sec^2 x$

26. Prove the following equation is an identity: $\sec x+\tan x=\dfrac{\cos x}{1-\sin x}$

27. Prove the following equation is an identity: $\dfrac{\tan x-1}{\tan x+1}=\dfrac{1-\cot x}{1+\cot x}$

28. Prove the following equation is an identity: $7\csc^2 x-5\cot^2 x=2\csc^2 x+5$

29. Prove the following equation is an identity: $1-\dfrac{\cos^2 x}{1-\sin x}=-\sin x$

30. Prove the following equation is an identity: $\dfrac{1-\cos x}{1+\cos x}=\dfrac{\sec x-1}{\sec x+1}$

31. Prove the following equation is an identity: $\dfrac{\sec x-1}{\tan x}=\dfrac{\tan x}{\sec x+1}$

32. Prove the following equation is an identity: $\dfrac{\cos x}{\cos x-\sin x}=\dfrac{1}{1-\tan x}$

33. Prove the following equation is an identity: $(\sec x+\tan x)^2=\dfrac{1+\sin x}{1-\sin x}$

34. Prove the following equation is an identity: $\dfrac{\cos x}{1+\tan x}-\dfrac{\sin x}{1+\cot x}=\cos x-\sin x$

35. Prove the following equation is an identity: $\dfrac{\cot x+\csc x-1}{\cot x-\csc x+1}=\csc x+\cot x$

36. Prove the following equation is an identity: $\dfrac{\tan x+\cot x}{\tan x-\cot x}=\dfrac{1}{\sin^2 x-\cos^2 x}$

37. Prove the following equation is an identity: $\dfrac{1-\cot^2 x}{1+\cot^2 x}+1=2\sin^2 x$

38. Prove the following equation is an identity: $\dfrac{1+\cos x}{1-\cos x}-\dfrac{1-\cos x}{1+\cos x}=4\cot x\csc x$

39. Prove the following equation is an identity: $\dfrac{\sin^3 x-\cos^3 x}{\sin x-\cos x}=1+\sin x\cos x$

40. Prove the following equation is an identity: $1+\sec^2 x\sin^2 x=\sec^2 x$

41. Prove the following equation is an identity: $\dfrac{1+\csc x}{\sec x}=\cos x+\cot x$

42. Prove the following equation is an identity: $\tan^2 x = \sec^2 x - \sin^2 x - \cos^2 x$

43. Prove the following equation is an identity: $\dfrac{\sin x}{1-\cos x} + \dfrac{\sin x}{1+\cos x} = 2\csc x$

44. Prove the following equation is an identity: $\cos^2(\alpha-\beta) - \cos^2(\alpha+\beta) = \sin^2(\alpha+\beta) - \sin^2(\alpha-\beta)$

45. Prove the following equation is an identity: $\tan x \csc x - \sec^2 x \cos x = 0$

46. Prove the following equation is an identity: $(1+\tan x)^2 - 2\tan x = \dfrac{1}{(1-\sin x)(1+\sin x)}$

47. Prove the following equation is an identity: $\dfrac{3\csc^2 x - 5\csc x - 28}{\csc x - 4} = \dfrac{3}{\sin x} + 7$

48. Prove the following equation is an identity: $\left(\sec^2 x - 1\right)\left(\sec^2 x + 1\right) = \tan^4 x + 2\tan^2 x$

49. Prove the following equation is an identity: $\dfrac{\csc x}{\cot x} - \dfrac{\cot x}{\csc x} = \dfrac{\sin x}{\cot x}$

50. Prove the following equation is an identity: $\dfrac{1-\cos^2 x}{1+\cos x} = \dfrac{\sec x - 1}{\sec x}$

51. Prove the following equation is an identity: $\dfrac{\cos x}{1+\cos x} = \dfrac{\sec x - 1}{\tan^2 x}$

52. Prove the following equation is an identity: $\dfrac{1-2\sin^2 x}{1+2\sin x \cos x} = \dfrac{\cos x - \sin x}{\cos x + \sin x}$

53. Prove the following equation is an identity: $(\cos x - \sin x)^2 + (\cos x + \sin x)^2 = 2$

54. Prove the following equation is an identity: $\dfrac{\sin x}{1+\cos x} + \dfrac{1+\cos x}{\sin x} = 2\csc x$

55. Prove the following equation is an identity: $\dfrac{\sin x + \tan x}{\cot x + \csc x} = \sin x \tan x$

56. Prove the following equation is an identity: $\csc^2 x \sec^2 x = \sec^2 x + \csc^2 x$

57. Prove the following equation is an identity: $\cos^2 x + 1 = 2\cos^2 x + \sin^2 x$

58. Prove the following equation is an identity: $1 - \dfrac{\cos^2 x}{1+\sin x} = \sin x$

59. Prove the following equation is an identity: $\cot^2 x = (\csc x - 1)(\csc x + 1)$

60. Prove the following equation is an identity: $\dfrac{\sec x - 1}{\tan x} = \dfrac{\tan x}{\sec x + 1}$

61. Prove the following equation is an identity: $10\csc^2 x - 6\cot^2 x = 4\csc^2 x + 6$

62. Prove the following equation is an identity: $\dfrac{\csc x + \cot x}{\tan x + \sin x} = \csc x \cot x$

63. Prove the following equation is an identity: $\frac{1-\sec x}{\tan x}+\frac{\tan x}{1-\sec x}=-2\csc x$

64. Prove the following equation is an identity: $\csc x-\sin x=\cos x\cot x$

65. Prove the following equation is an identity: $\frac{\tan x+\sec x}{\sec x}-\frac{\tan x+\sec x}{\tan x}=-\cos x\cot x$

66. Prove the following equation is an identity: $\cot^3 x=\cot x\left(\csc^2 x-1\right)$

67. Prove the following equation is an identity: $\frac{\cot^2 x}{\csc x-1}=\frac{1+\sin x}{\sin x}$

68. Prove the following equation is an identity: $\cot^2 x+\csc^2 x=2\csc^2 x-1$

69. Prove the following equation is an identity: $\frac{\cot^2 x}{1+\csc x}=\csc x-1$

70. Prove the following equation is an identity: $\sec^4 x-\tan^4 x=\sec^2 x+\tan^2 x$

71. Prove the following equation is an identity: $\frac{\cos x}{1+\sin x}+\frac{1+\sin x}{\cos x}=2\sec x$

72. Prove the following equation is an identity: $\frac{\sin x+\cos x}{\sin x-\cos x}=\frac{1+2\sin x\cos x}{2\sin^2 x-1}$

73. Prove the following equation is an identity: $\frac{\csc x-1}{\csc x+1}=\frac{\cot^2 x}{\csc^2 x+2\csc x+1}$

74. Prove the following equation is an identity: $\csc^4 x-\cot^4 x=\csc^2 x+\cot^2 x$

75. Prove the following equation is an identity: $\tan\left(\frac{\pi}{4}+x\right)=\cot\left(\frac{\pi}{4}-x\right)$

76. Prove the identity: $\frac{\sin\theta}{1+\sin\theta}-\frac{\sin\theta}{1-\sin\theta}=-2\tan^2\theta$

77. Prove the identity: $\csc^2 x-\cos^2 x\csc^2 x=1$

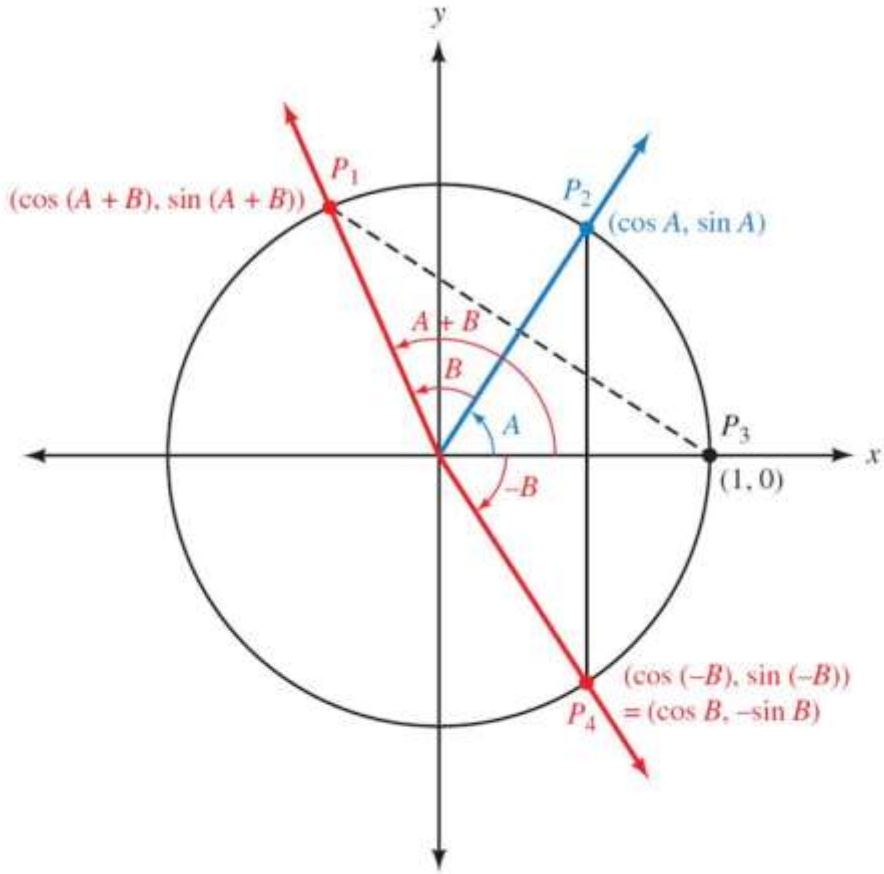
78. Prove the identity: $1-2\sin^2 x=2\cos^2 x-1$

79. Prove the identity: $\csc^2 x-\cos x\sec x=\cot^2 x$

80. Prove the identity: $(\sec x-\tan x)(\sec x+\tan x)=1$

81. Prove the identity: $\left(1+\tan^2 x\right)\left(1-\sin^2 x\right)=1$

Section 3.2 – Sum and Difference Formulas



$$P_1P_3 = P_2P_4$$

$$\left(P_1P_3\right)^2 = \left(P_2P_4\right)^2$$ *Distance between points*

$$[\cos(A+B)-1]^2 + [\sin(A+B)-0]^2 = (\cos A - \cos B)^2 + (\sin A + \sin B)^2$$

$$\cos^2(A+B) - 2\cos(A+B) + 1 + \sin^2(A+B) = (\cos A - \cos B)^2 + (\sin A + \sin B)^2$$

$$1 - 2\cos(A+B) + 1 = \cos^2 A - 2\cos B\cos A + \cos^2 B + \sin^2 A + 2\sin B\sin A + \sin^2 B$$

$$2 - 2\cos(A+B) = \cos^2 A + \sin^2 A + \cos^2 B + \sin^2 B - 2\cos B\cos A + 2\sin B\sin A$$

$$2 - 2\cos(A+B) = 1 + 1 - 2\cos B\cos A + 2\sin B\sin A$$

$$2 - 2\cos(A+B) = 2 - 2\cos B\cos A + 2\sin B\sin A$$

$$-2\cos(A+B) = -2\cos B\cos A + 2\sin B\sin A$$

$$\boxed{\cos(A+B) = \cos B\cos A - \sin B\sin A}$$

Example

Find the exact value for $\cos 75°$

<u>*Solution*</u>

$$\begin{aligned}\cos 75° &= \cos(45° + 30°) \\ &= \cos 45° \cos 30° - \sin 45° \sin 30° \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} \\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Example

Show that $\cos(x + 2\pi) = \cos x$

<u>*Solution*</u>

$$\begin{aligned}\cos(x + 2\pi) &= \cos x \cos 2\pi - \sin x \sin 2\pi \\ &= \cos x \cdot (1) - \sin x \cdot (0) \\ &= \cos x\end{aligned}$$

Example

Simplify: $\cos 3x \cos 2x - \sin 3x \sin 2x$

<u>*Solution*</u>

$$\begin{aligned}\cos 3x \cos 2x - \sin 3x \sin 2x &= \cos(3x + 2x) \\ &= \cos 5x\end{aligned}$$

Example

Show that $\cos(90° - A) = \sin A$

<u>*Solution*</u>

$$\begin{aligned}\cos(90° - A) &= \cos 90° \cos A + \sin 90° \sin A \\ &= 0 \cdot \cos A + 1 \cdot \sin A \\ &= \sin A\end{aligned}$$

$$\boxed{\begin{aligned}&\cos(A+B)=\cos A\cos B-\sin A\sin B\\&\cos(A-B)=\cos A\cos B+\sin A\sin B\\&\sin(A+B)=\sin A\cos B+\cos A\sin B\\&\sin(A-B)=\sin A\cos B-\cos A\sin B\end{aligned}}$$

Example

Find the exact value of $\sin\frac{\pi}{12}$

<u>*Solution*</u>

$$\begin{aligned}\sin\frac{\pi}{12}&=\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\\&=\sin\frac{\pi}{3}\cos\frac{\pi}{4}-\cos\frac{\pi}{3}\sin\frac{\pi}{4}\\&=\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}-\frac{1}{2}\frac{\sqrt{2}}{2}\\&=\frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Example

Find the exact value of $\cos 15^\circ$

<u>*Solution*</u>

$$\begin{aligned}\cos 15^\circ&=\cos\left(45^\circ-30^\circ\right)\\&=\cos\left(45^\circ\right)\cos\left(30^\circ\right)+\sin\left(45^\circ\right)\sin\left(30^\circ\right)\\&=\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\frac{1}{2}\\&=\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}\\&=\frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

Example

If $\sin A = \frac{3}{5}$ with A in QI, and $\cos B = -\frac{5}{13}$ with B in QIII, find $\sin(A+B)$, $\cos(A+B)$, and $\tan(A+B)$

<u>*Solution*</u>

$$\sin A = \frac{3}{5} \to A \in QI$$

$$\begin{aligned}
\to \cos A &= \sqrt{1-\sin^2 A} \\
&= \sqrt{1-\left(\frac{3}{5}\right)^2} \\
&= \sqrt{1-\frac{9}{25}} \\
&= \sqrt{\frac{25-9}{25}} \\
&= \sqrt{\frac{16}{25}} \\
&= \frac{4}{5}
\end{aligned}$$

$$\cos B = -\frac{5}{13} \to B \in QIII$$

$$\begin{aligned}
\to \sin B &= -\sqrt{1-\cos^2 B} \\
&= -\sqrt{1-\left(\frac{5}{13}\right)^2} \\
&= -\frac{12}{13}
\end{aligned}$$

$$\begin{aligned}
\sin(A+B) &= \sin A\cos B + \cos A\sin B \\
&= \frac{3}{5}\left(-\frac{5}{13}\right) + \frac{4}{5}\left(-\frac{12}{13}\right) \\
&= -\frac{15}{65} - \frac{48}{65} \\
&= -\frac{63}{65}
\end{aligned}$$

$$\begin{aligned}
\cos(A+B) &= \cos A\cos B - \sin A\sin B \\
&= \frac{4}{5}\left(-\frac{5}{13}\right) - \frac{3}{5}\left(-\frac{12}{13}\right) \\
&= -\frac{20}{65} + \frac{36}{65} \\
&= \frac{16}{65}
\end{aligned}$$

$$\begin{aligned}
\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\
&= \frac{-\frac{63}{65}}{\frac{16}{65}} \\
&= -\frac{63}{16}
\end{aligned}$$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

$$= \frac{\frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B - \sin A\sin B}{\cos A\cos B}}$$

$$= \frac{\frac{\sin A\cos B}{\cos A\cos B} + \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} - \frac{\sin A\sin B}{\cos A\cos B}}$$

$$= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A}{\cos A}\frac{\sin B}{\cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\boxed{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}}$$

$$\boxed{\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}}$$

Example

If $\sin A = \frac{3}{5}$ with A in QI, and $\cos B = -\frac{5}{13}$ with B in QIII, find $\tan(A+B)$

Solution

$$\tan A = \frac{\sin A}{\cos A} = \frac{3/5}{4/5} = \frac{3}{4}$$

$$\tan B = \frac{\sin B}{\cos B} = \frac{-12/13}{-5/13} = \frac{12}{5}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$= \frac{\frac{3}{4} + \frac{12}{5}}{1 - \frac{3}{4}\frac{12}{5}}$$

$$= \frac{\frac{15}{20} + \frac{48}{20}}{1 - \frac{36}{20}}$$

$$= \frac{\frac{63}{20}}{-\frac{16}{20}}$$

$$= -\frac{63}{16}$$

Example

Common household current is called ***alternating current*** because the current alternates direction within the wires. The voltage V in a typical 115-volt outlet can be expressed by the function $V(t) = 163\sin\omega t$ where ω is the angular speed (in radians per second) of the rotating generator at the electrical plant, and t is time measured in seconds.

a) It is essential for electric generators to rotate at precisely 60 cycles per second so household appliances and computers will function properly. Determine ω for these electric generators.

b) Determine a value of ϕ so that the graph of $V(t) = 163\cos(\omega t - \phi)$ is the same as the graph of $V(t) = 163\sin\omega t$

<u>*Solution*</u>

a) Each cycle is 2π radians at 60 cycles per second, so the angular speed is $\omega = 60(2\pi) = 120\pi$ radians per second.

b) $\cos\left(x - \frac{\pi}{2}\right) = \cos x \cos\frac{\pi}{2} + \sin x \sin\frac{\pi}{2}$

$$= \cos x(0) + \sin x(1)$$

$$= \sin x$$

If $\phi = \frac{\pi}{2} \rightarrow V(t) = 163\cos\left(\omega t - \frac{\pi}{2}\right) = 163\sin(\omega t)$

Exercises *Section* 3.2 – Sum and Difference Formulas

1. Prove the identity $\cos(A+B)+\cos(A-B)=2\cos A\cos B$

2. Prove the identity $\sin\left(\frac{\pi}{4}+x\right)+\sin\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$

3. Prove the identity $\dfrac{\sin(A-B)}{\cos A\cos B}=\tan A-\tan B$

4. Prove the identity $\sec(A+B)=\dfrac{\cos(A-B)}{\cos^2 A-\sin^2 B}$

5. Prove the identity $\dfrac{\cos 4\alpha}{\sin\alpha}-\dfrac{\sin 4\alpha}{\cos\alpha}=\dfrac{\cos 5\alpha}{\sin\alpha\cos\alpha}$

6. Write the expression as a single trigonometric function $\sin 8x\cos x-\cos 8x\sin x$

7. Show that $\sin\left(x-\frac{\pi}{2}\right)=-\cos x$

8. If $\sin A=\frac{4}{5}$ with A in QII, and $\cos B=-\frac{5}{13}$ with B in QIII, find $\sin(A+B)$, $\cos(A+B)$, and $\tan(A+B)$

9. If $\sin A=\frac{1}{\sqrt{5}}$ with A in QI, and $\tan B=\frac{3}{4}$ with B in QI, find $\sin(A+B)$, $\cos(A+B)$, and $\tan(A+B)$

10. If $\sec A=\sqrt{5}$ with A in QI, and $\sec B=\sqrt{10}$ with B in QI, find $\sec(A+B)$

11. Prove the following equation is an identity: $\sin(x-y)-\sin(y-x)=2\sin x\cos y-2\cos x\sin y$

12. Prove the following equation is an identity: $\cos(x-y)+\cos(y-x)=2\cos x\cos y+2\sin x\sin y$

13. Prove the following equation is an identity: $\tan(x+y)\tan(x-y)=\dfrac{\tan^2 x-\tan^2 y}{1-\tan^2 x\tan^2 y}$

14. Prove the following equation is an identity: $\dfrac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)}=\dfrac{1-\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$

15. Prove the following equation is an identity: $\sec(x+y)=\dfrac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\sin^2 y}$

16. Prove the following equation is an identity: $\csc(x-y)=\dfrac{\sin x\cos y+\cos x\sin y}{\sin^2 x-\sin^2 y}$

17. Prove the following equation is an identity: $\dfrac{\cos(x+y)}{\cos(x-y)}=\dfrac{\cot y-\tan x}{\cot y+\tan x}$

18. Prove the following equation is an identity: $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\cot y + \cot x}{\cot y - \cot x}$

19. Prove the following equation is an identity: $\dfrac{\sin(x-y)}{\sin x \cos y} = 1 - \cot x \tan y$

20. Prove the following equation is an identity: $\dfrac{\sin(x-y)}{\sin x \sin y} = \cot y - \cot x$

21. Prove the following equation is an identity: $\dfrac{\cos(x+y)}{\cos x \sin y} = \cot y - \tan x$

22. Prove the following equation is an identity: $\tan(x+y) + \tan(x-y) = \dfrac{2\tan x}{\cos^2 y\left(1 - \tan^2 x \tan^2 y\right)}$

23. Prove the following equation is an identity: $\dfrac{\sin(x+y)}{\cos(x-y)} = \dfrac{1 + \cot x \tan y}{\cot x + \tan y}$

24. Prove the following equation is an identity: $\dfrac{\cos(x-y)}{\cos(x+y)} = \dfrac{1 + \tan x \tan y}{1 - \tan x \tan y}$

Section 3.3 – Double-angle Formulas

$$\begin{aligned}\sin 2A &= \sin(A+A)\\ &= \sin A\cos A + \cos A\sin A\\ &= 2\sin A\cos A\end{aligned}$$

$$\sin 2A \neq 2\sin A$$

Example

If $\sin A = \frac{3}{5}$ with A in QII, find $\sin 2A$

<u>*Solution*</u>

$$\begin{aligned}\cos A &= \pm\sqrt{1-\sin^2 A}\\ &= -\sqrt{1-\left(\frac{3}{5}\right)^2}\\ &= -\sqrt{1-\frac{9}{25}}\\ &= -\sqrt{\frac{25-9}{25}}\\ &= -\sqrt{\frac{16}{25}}\\ &= -\frac{4}{5}\end{aligned}$$

$$\begin{aligned}\sin 2A &= 2\sin A\cos A\\ &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)\\ &= -\frac{24}{5}\end{aligned}$$

Example

Prove $(\sin\theta+\cos\theta)^2 = 1+\sin 2\theta$

<u>*Solution*</u>

$$\begin{aligned}(\sin\theta+\cos\theta)^2 &= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta\\ &= \sin^2\theta+\cos^2\theta + 2\sin\theta\cos\theta\\ &= 1+2\sin\theta\cos\theta\\ &= 1+\sin 2\theta\end{aligned}$$

$$\begin{aligned}\cos 2A &= \cos(A+A)\\ &= \cos A\cos A - \sin A\sin A\\ &= \cos^2 A - \sin^2 A\end{aligned}$$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= \cos^2 A - \left(1-\cos^2 A\right)\\ &= \cos^2 A - 1 + \cos^2 A\\ &= 2\cos^2 A - 1\end{aligned}$$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= \left(1-\sin^2 A\right) - \sin^2 A\\ &= 1 - \sin^2 A - \sin^2 A\\ &= 1 - 2\sin^2 A\end{aligned}$$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= 2\cos^2 A - 1\\ &= 1 - 2\sin^2 A\end{aligned}$$

Example

If $\sin A = \frac{1}{\sqrt{5}}$, find $\cos 2A$

<u>*Solution*</u>

$$\begin{aligned}\cos 2A &= 1 - 2\sin^2 A\\ &= 1 - 2\left(\frac{1}{\sqrt{5}}\right)^2\\ &= 1 - 2\cdot\frac{1}{5}\\ &= 1 - \frac{2}{5}\\ &= \frac{3}{5}\end{aligned}$$

Example

Prove $\sin 2x = \dfrac{2\cot x}{1+\cot^2 x}$

Solution

$$\frac{2\cot x}{1+\cot^2 x} = \frac{2\frac{\cos x}{\sin x}}{1+\frac{\cos^2 x}{\sin^2 x}}$$

$$= \frac{2\frac{\cos x}{\sin x}}{\frac{\sin^2 x+\cos^2 x}{\sin^2 x}}$$

$$= 2\frac{\cos x}{\sin x}\frac{\sin^2 x}{\sin^2 x+\cos^2 x}$$

$$= 2\frac{\cos x}{1}\frac{\sin x}{1}$$

$$= 2\cos x\sin x$$

$$= \sin 2x$$

Example

Prove $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

Solution

$$\cos 4x = \cos(2.2x)$$

$$= 2\cos^2 2x - 1$$

$$= 2(\cos 2x)^2 - 1$$

$$= 2\left(2\cos^2 x - 1\right)^2 - 1$$

$$= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1$$

$$= 8\cos^4 x - 8\cos^2 x + 2 - 1$$

$$= 8\cos^4 x - 8\cos^2 x + 1$$

$$\tan 2A = \tan(A + A)$$

$$= \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$\boxed{\tan 2A = \frac{2\tan A}{1 - \tan^2 A}}$$

Example

Simplify $\dfrac{2\tan 15°}{1 - \tan^2 15°}$

<u>*Solution*</u>

$$\frac{2\tan 15°}{1 - \tan^2 15°} = \tan(2 \cdot 15°)$$

$$= \tan(30°)$$

$$= \frac{1}{\sqrt{3}}$$

Example

Prove $\tan\theta = \dfrac{1 - \cos 2\theta}{\sin 2\theta}$

<u>*Solution*</u>

$$\frac{1 - \cos 2\theta}{\sin 2\theta} = \frac{1 - (1 - 2\sin^2\theta)}{2\sin\theta\cos\theta}$$

$$= \frac{1 - 1 + 2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$= \frac{2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta}$$

Example

Given $\cos\theta = \frac{3}{5}$ and $\sin\theta < 0$, find $\sin 2\theta,\ \cos 2\theta,\ and\ \tan 2\theta$

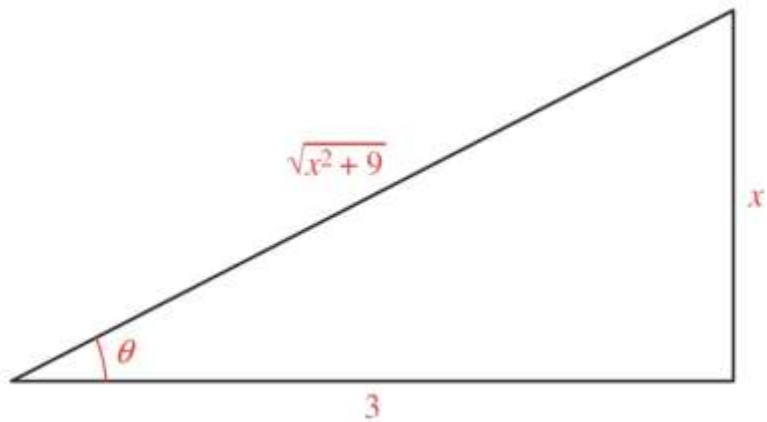
<u>*Solution*</u>

$\sin\theta = -\sqrt{1-\cos^2\theta}$ $= -\sqrt{1-\frac{9}{25}}$ $= -\sqrt{\frac{16}{25}}$ $= -\frac{4}{5}$	$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{3}$ $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ $= \frac{2\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2}$ $= \frac{-\frac{8}{3}}{1-\frac{16}{9}}$ $= \frac{-\frac{8}{3}}{-\frac{7}{9}}$ $= \left(-\frac{8}{3}\right)\left(-\frac{9}{7}\right)$ $= \frac{24}{7}$
$\sin 2\theta = 2\sin\theta\cos\theta$ $= 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right)$ $= -\frac{24}{25}$ $\cos 2\theta = \cos^2\theta - \sin^2\theta$ $= \left(\frac{3}{5}\right)^2 - \left(-\frac{4}{5}\right)^2$ $= \frac{9}{25} - \frac{16}{25}$ $= -\frac{7}{25}$	

Example

If $x = 3\tan\theta$, write the expression $\dfrac{\theta}{2} + \dfrac{\sin 2\theta}{4}$ in terms of just *x*.

<u>*Solution*</u>



$$x = 3\tan\theta \Rightarrow \tan\theta = \frac{x}{3}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{x}{3}\right)$$

$$\frac{\theta}{2} + \frac{\sin 2\theta}{4} = \frac{\theta}{2} + \frac{2\sin\theta\cos\theta}{4}$$

$$= \frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2}$$

$$= \frac{1}{2}(\theta + \sin\theta\cos\theta)$$

$$= \frac{1}{2}\left(\tan^{-1} x + \frac{x}{\sqrt{x^2+9}}\frac{3}{\sqrt{x^2+9}}\right)$$

$$= \frac{1}{2}\left(\tan^{-1} x + \frac{3x}{x^2+9}\right)$$

Exercises *Section* 3.3 – Double-angle Formulas

1. Let $\sin A = -\frac{3}{5}$ with A in QIII and find $\cos 2A$

2. Let $\cos x = \frac{1}{\sqrt{10}}$ with x in QIV and find $\cot 2x$

3. Verify: $(\cos x - \sin x)(\cos x + \sin x) = \cos 2x$

4. Verify: $\cot x \sin 2x = 1 + \cos 2x$

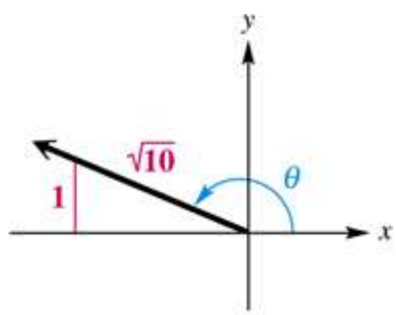
5. Prove: $\cot\theta = \dfrac{\sin 2\theta}{1 - \cos 2\theta}$

6. Simplify $\cos^2 7x - \sin^2 7x$

7. Write $\sin 3x$ in terms of $\sin x$

8. Find the values of the six trigonometric functions of θ if $\cos 2\theta = \frac{4}{5}$ *and* $90° < \theta < 180°$

9. Use a right triangle in QII to find the value of $\cos\theta$ *and* $\tan\theta$



10. Prove the following equation is an identity: $\sin 3x = \sin x\left(3\cos^2 x - \sin^2 x\right)$

11. Prove the following equation is an identity: $\cos 3x = \cos^3 x - 3\cos x \sin^2 x$

12. Prove the following equation is an identity: $\cos^4 x - \sin^4 x = \cos 2x$

13. Prove the following equation is an identity: $\sin 2x = -2\sin x \sin\left(x - \frac{\pi}{2}\right)$

14. Prove the following equation is an identity: $\dfrac{\sin 4t}{4} = \cos^3 t \sin t - \sin^3 t \cos t$

15. Prove the following equation is an identity: $\dfrac{\cos 2x}{\sin^2 x} = \csc^2 x - 2$

16. Prove the following equation is an identity: $\dfrac{\cos 2x + \cos 2y}{\sin x + \cos y} = 2\cos y - 2\sin x$

17. Prove the following equation is an identity: $\dfrac{\cos 2x}{\cos^2 x} = \sec^2 x - 2\tan^2 x$

18. Prove the following equation is an identity: $\sin 4x = (4\sin x\cos x)\left(2\cos^2 x - 1\right)$

19. Prove the following equation is an identity: $\cos 2y = \dfrac{1-\tan^2 y}{1+\tan^2 y}$

20. Prove the following equation is an identity: $\cos 4x = \cos^4 x - 6\sin^2 x\cos^2 x + \sin^4 x$

21. Prove the following equation is an identity: $\tan^2 x(1+\cos 2x) = 1-\cos 2x$

22. Prove the following equation is an identity: $\dfrac{\cos 2x}{\sin^2 x} = 2\cot^2 x - \csc^2 x$

23. Prove the following equation is an identity: $\tan x + \cot x = 2\csc 2x$

24. Prove the following equation is an identity: $\tan 2x = \dfrac{2}{\cot x - \tan x}$

25. Prove the following equation is an identity: $\dfrac{1-\tan x}{1+\tan x} = \dfrac{1-\sin 2x}{\cos 2x}$

26. Prove the following equation is an identity: $\sin 2\alpha \sin 2\beta = \sin^2(\alpha+\beta) - \sin^2(\alpha-\beta)$

27. Prove the following equation is an identity: $\cos^2(A-B) - \cos^2(A+B) = \sin 2A\ \sin 2B$

Section 3.4 – Half-Angle Formulas

$\cos 2A = 2\cos^2 A - 1$

$\cos 2x = 2\cos^2 x - 1$

$\cos 2x + 1 = 2\cos^2 x$

$2\cos^2 x = \cos 2x + 1$

$\cos^2 x = \dfrac{\cos 2x + 1}{2}$ *Divide both sides by 2*

$\cos x = \pm\sqrt{\dfrac{\cos 2x + 1}{2}}$ *Replace x with* $\frac{A}{2}$

$$\Rightarrow \boxed{\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}}$$

$\cos 2A = 1 - 2\sin^2 A$

$\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$\sin^2 x = \dfrac{1 - \cos 2x}{2}$ *Divide both sides by 2*

$\sin x = \pm\sqrt{\dfrac{1 - \cos 2x}{2}}$ *Replace x with* $\frac{A}{2}$

$$\Rightarrow \boxed{\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}}$$

Example

Find the exact value of $\cos 15°$

<u>*Solution*</u>

$$\begin{aligned}\cos 15° &= \cos\left(\tfrac{1}{2}30°\right)\\ &= \sqrt{\frac{1+\cos 30°}{2}}\\ &= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}\\ &= \sqrt{\frac{2+\sqrt{3}}{2}}\end{aligned}$$

Example

If $\cos A = \frac{3}{5}$ with $270° < A < 360°$ find $\sin\frac{A}{2},\ \cos\frac{A}{2},\ and\ \tan\frac{A}{2}$

<u>*Solution*</u>

Since $270° < A < 360°$

$$\frac{270°}{2} < \frac{A}{2} < \frac{360°}{2}$$

$$135° < \frac{A}{2} < 180° \ \Rightarrow \frac{A}{2} \in QII$$

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}$$

$$= \sqrt{\frac{1-\frac{3}{5}}{2}}$$

$$= \sqrt{\frac{5-3}{5}\frac{1}{2}}$$

$$= \sqrt{\frac{2}{5}\frac{1}{2}}$$

$$= \sqrt{\frac{1}{5}}$$

$$= \frac{1}{\sqrt{5}}$$

$$\cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}}$$

$$= -\sqrt{\frac{1+\frac{3}{5}}{2}}$$

$$= -\sqrt{\frac{\frac{8}{5}}{2}}$$

$$= -\sqrt{\frac{4}{5}}$$

$$= -\frac{2}{\sqrt{5}}$$

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$

$$= \frac{\frac{1}{\sqrt{5}}}{-\frac{2}{\sqrt{5}}}$$

$$= -\frac{1}{2}$$

Example

If $\sin A = -\frac{12}{13}$ with $180° < A < 270°$ find the six trigonometric function of $A/2$

<u>*Solution*</u>

Since $180° < A < 270°$

$$\cos A = -\sqrt{1-\sin^2 A} = -\frac{5}{13}$$

$$90° < \frac{A}{2} < 135° \qquad \Rightarrow \frac{A}{2} \in QII$$

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}$$

$$= \sqrt{\frac{1-\frac{-5}{13}}{2}}$$

$$= \sqrt{\frac{13+5}{13}\frac{1}{2}}$$

$$= \sqrt{\frac{9}{13}}$$

$$= \frac{3}{\sqrt{13}}$$

$$\cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}}$$

$$= -\sqrt{\frac{1+\frac{-5}{13}}{2}}$$

$$= -\sqrt{\frac{\frac{8}{13}}{2}}$$

$$= -\sqrt{\frac{4}{13}}$$

$$= -\frac{2}{\sqrt{13}}$$

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$

$$= \frac{\frac{3}{\sqrt{13}}}{-\frac{2}{\sqrt{13}}}$$

$$= -\frac{3}{2}$$

$$\cot\frac{A}{2} = \frac{1}{\tan\frac{A}{2}}$$

$$= -\frac{2}{3}$$

$$\csc\frac{A}{2} = \frac{1}{\sin\frac{A}{2}}$$

$$= \frac{\sqrt{13}}{3}$$

$$\sec\frac{A}{2} = \frac{1}{\cos\frac{A}{2}}$$

$$= -\frac{\sqrt{13}}{2}$$

$$\boxed{\tan\frac{A}{2}=\frac{1-\cos A}{\sin A}}\qquad\qquad \boxed{\tan\frac{A}{2}=\frac{\sin A}{1+\cos A}}$$

Example

Find the exact of $\tan 15°$

Solution

$$\begin{aligned}\tan 15° &= \tan\frac{30°}{2}\\ &= \frac{1-\cos 30°}{\sin 30°}\\ &= \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}}\\ &= \frac{\frac{2-\sqrt{3}}{2}}{\frac{1}{2}}\\ &= 2-\sqrt{3}\end{aligned}$$

Example

Prove $\sin^2\frac{x}{2}=\frac{\tan x-\sin x}{2\tan x}$

Solution

$$\begin{aligned}\sin^2\frac{x}{2} &= \frac{1-\cos x}{2}\\ &= \frac{\tan x}{\tan x}\frac{1-\cos x}{2}\\ &= \frac{\tan x-\tan x\cos x}{2\tan x}\\ &= \frac{\tan x-\frac{\sin x}{\cos x}\cos x}{2\tan x}\\ &= \frac{\tan x-\sin x}{2\tan x}\end{aligned}$$

Exercises *Section* 3.4 – Half-Angle Formulas

1. Use half-angle formulas to find the exact value of $\sin 105°$

2. Find the exact of $\tan 22.5°$

3. Given: $\cos x = \frac{2}{3}$, $\frac{3\pi}{2} < x < 2\pi$, find $\cos\frac{x}{2}$, $\sin\frac{x}{2}$, and $\tan\frac{x}{2}$

4. Prove the identity $2\csc x \cos^2\frac{x}{2} = \frac{\sin x}{1-\cos x}$

5. Prove the identity $\tan\frac{\alpha}{2} = \sin\alpha + \cos\alpha\cot\alpha - \cot\alpha$

6. Prove the following equation is an identity: $\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right) = \frac{\sin^2 x}{4}$

7. Prove the following equation is an identity: $\tan\frac{x}{2} + \cot\frac{x}{2} = 2\csc x$

8. Prove the following equation is an identity: $2\sin^2\left(\frac{x}{2}\right) = \frac{\sin^2 x}{1+\cos x}$

9. Prove the following equation is an identity: $\tan^2\left(\frac{x}{2}\right) = \frac{\sec x + \cos x - 2}{\sec x - \cos x}$

10. Prove the following equation is an identity: $\sec^2\left(\frac{x}{2}\right) = \frac{2\sec x + 2}{\sec x + 2 + \cos x}$

11. Prove the following equation is an identity: $\dfrac{1-\sin^2\left(\frac{x}{2}\right)}{1+\sin^2\left(\frac{x}{2}\right)} = \frac{1+\cos x}{3-\cos x}$

12. Prove the following equation is an identity: $\dfrac{1-\cos^2\left(\frac{x}{2}\right)}{1-\sin^2\left(\frac{x}{2}\right)} = \frac{1-\cos x}{1+\cos x}$

Section 3.5 – Additional Identities

Identities and Formulas Involving Inverse Functions

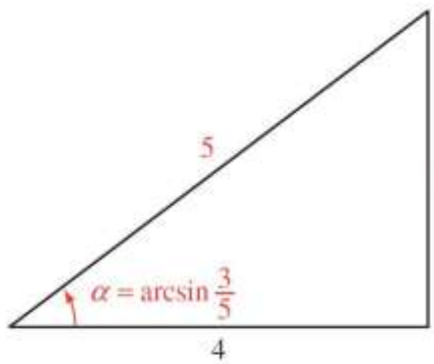
Example

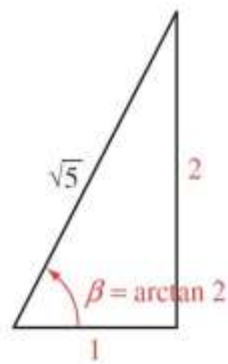
Evaluate $\sin\left(\arcsin\frac{3}{5}+\arctan 2\right)$ without using a calculator.

<u>*Solution*</u>

$$\sin\left(\arcsin\frac{3}{5}+\arctan 2\right)=\sin(\alpha+\beta)$$

$$=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$





$\sin\alpha=\frac{3}{5}$ $\qquad$ $\cos\alpha=\frac{4}{5}$ $\qquad$ $\sin\beta=\frac{2}{\sqrt{5}}$ $\qquad$ $\cos\alpha=\frac{1}{\sqrt{5}}$

$$\sin\left(\arcsin\frac{3}{5}+\arctan 2\right)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$=\frac{3}{5}\frac{1}{\sqrt{5}}+\frac{4}{5}\frac{2}{\sqrt{5}}$$

$$=\frac{3}{5\sqrt{5}}+\frac{8}{5\sqrt{5}}$$

$$=\frac{11}{5\sqrt{5}}$$

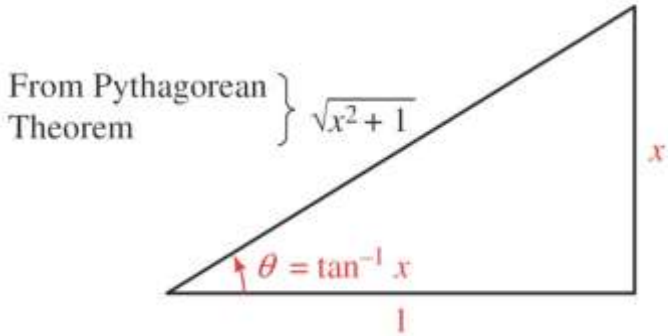
Example

Write $\sin(2\tan^{-1} x)$ as an equivalent expression involving only *x*. (Assume *x* is positive)

<u>*Solution*</u>

Let $\theta = \tan^{-1} x$

$\Rightarrow \tan\theta = x = \frac{x}{1}$



$\sin\theta = \frac{x}{\sqrt{x^2+1}}$ $\qquad \cos\theta = \frac{1}{\sqrt{x^2+1}}$

$$\begin{aligned}\sin(2\tan^{-1} x) &= \sin(2\theta)\\ &= 2\sin\theta\cos\theta\\ &= 2\frac{x}{\sqrt{x^2+1}}\frac{1}{\sqrt{x^2+1}}\\ &= \frac{2x}{x^2+1}\end{aligned}$$

Product to Sum Formulas

$$\begin{array}{lll} \sin A\cos B & +\cos A\sin B & =\sin(A+B) \\ \sin A\cos B & -\cos A\sin B & =\sin(A-B) \\ \hline 2\sin A\cos B & & =\sin(A+B)+\sin(A-B) \end{array}$$

$$\boxed{\begin{aligned} &\sin A\cos B=\tfrac{1}{2}\left[\sin(A+B)+\sin(A-B)\right] \\ &\cos A\sin B=\tfrac{1}{2}\left[\sin(A+B)-\sin(A-B)\right] \\ &\cos A\cos B=\tfrac{1}{2}\left[\cos(A+B)+\cos(A-B)\right] \\ &\sin A\sin B=\tfrac{1}{2}\left[\cos(A-B)-\cos(A+B)\right] \end{aligned}}$$

Example

Verify product formula $\cos A\cos B=\frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right]$ for $A=30°$ and $B=120°$

Solution

$$\cos 30°\cos 120° = \frac{1}{2}\left[\cos(30°+120°)+\cos(30°-120°)\right]$$

$$\cos 30°\cos 120° = \frac{1}{2}\left[\cos(150°)+\cos(-90°)\right]$$

$$\frac{\sqrt{3}}{2}\cdot\left(-\frac{1}{2}\right)=\frac{1}{2}\left[-\frac{\sqrt{3}}{2}+0\right]$$

$$-\frac{\sqrt{3}}{4}=-\frac{\sqrt{3}}{4}$$

Example

Write $4\cos 75°\sin 25°$ as a sum or difference

Solution

$$4\cos 75°\sin 25° = 4\tfrac{1}{2}\left[\sin\left(75°+25°\right)-\sin\left(75°-25°\right)\right]$$

$$=2\left[\sin\left(100°\right)-\sin\left(50°\right)\right]$$

Sum to Product Formulas

$$\sin A\cos B = \frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right] \qquad \cos A\sin B = \frac{1}{2}\left[\sin(A+B)-\sin(A-B)\right]$$

$$2\sin A\cos B = \sin(A+B)+\sin(A-B)$$

$$\sin(A+B)+\sin(A-B) = 2\sin A\cos B \qquad \sin(A+B)-\sin(A-B) = 2\cos A\sin B$$

Let $\alpha = A+B$

$$\underline{\beta = A-B}$$

$$\alpha+\beta = 2A \quad \Rightarrow A = \frac{\alpha+\beta}{2}$$

$$\alpha-\beta = 2B \quad \Rightarrow B = \frac{\alpha-\beta}{2}$$

$$\boxed{\begin{aligned}
&\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\
&\sin\alpha-\sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\ \sin\left(\frac{\alpha-\beta}{2}\right)\\
&\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\ \cos\left(\frac{\alpha-\beta}{2}\right)\\
&\cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\ \sin\left(\frac{\alpha-\beta}{2}\right)
\end{aligned}}$$

Example

Verify sum formula $\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ for $\alpha = 30°$ and $\beta = 90°$

<u>*Solution*</u>

$$\cos 30°+\cos 90° = 2\cos\left(\frac{30°+90°}{2}\right)\cos\left(\frac{30°-90°}{2}\right)$$

$$\cos 30°+\cos 90° = 2\cos\left(\frac{120°}{2}\right)\cos\left(\frac{-60°}{2}\right)$$

$$\cos 30°+\cos 90° = 2\cos\left(60°\right)\cos\left(-30°\right)$$

$$\frac{\sqrt{3}}{2}+0 = 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

Example

Verify the identity $-\tan x = \dfrac{\cos 3x - \cos x}{\sin 3x + \sin x}$

<u>*Solution*</u>

$$\frac{\cos 3x - \cos x}{\sin 3x + \sin x} = \frac{-2\sin\frac{3x+x}{2}\sin\frac{3x-x}{2}}{2\sin\frac{3x+x}{2}\cos\frac{3x-x}{2}}$$

$$= -\frac{2\sin 2x \sin x}{2\sin 2x \cos x}$$

$$= -\frac{\sin x}{\cos x}$$

$$= -\tan x$$

Example

Write $\sin 2\theta - \sin 4\theta$ as product of two functions.

<u>*Solution*</u>

$$\sin 2\theta - \sin 4\theta = 2\cos\left(\frac{2\theta+4\theta}{2}\right)\sin\left(\frac{2\theta-4\theta}{2}\right)$$

$$= 2\cos\left(\frac{6\theta}{2}\right)\sin\left(-\frac{2\theta}{2}\right)$$

$$= 2\cos 3\theta \sin(-\theta)$$

$$= -2\cos 3\theta \sin\theta$$

Exercises *Section* 3.5 – Additional Identities

1. Evaluate without using the calculator $\cos\left(\arctan\sqrt{3}+\arcsin\frac{1}{3}\right)$

2. Evaluate without using the calculator $\cos(\arcsin\frac{3}{5}-\arctan 2)$

3. Evaluate without using the calculator $\sin\left(2\cos^{-1}\frac{1}{\sqrt{5}}\right)$

4. Evaluate without using the calculator $\tan\left(2\arcsin\frac{2}{5}\right)$

5. Evaluate without using the calculator $\sin\left(\tan^{-1}u\right)$

6. Write $\sin\left(2\cos^{-1}x\right)$ as an equivalent expression involving only x.

7. Write $\cos\left(2\sin^{-1}u\right)$ as an equivalent expression involving only x.

8. Write $\sec\left(\tan^{-1}\frac{x-2}{2}\right)$ as an equivalent expression involving only x.

9. Write $10\cos 5x\sin 3x$ as a sum or difference

10. Prove the identity: $\dfrac{\sin 3x-\sin x}{\cos 3x-\cos x}=-\cot 2x$

11. Prove the following equation is an identity: $\sin(x+y)\cos(x-y)=\sin x\cos x+\cos y\sin y$

12. Prove the following equation is an identity: $2\sin(x+y)\cos(x-y)=\sin 2x+\sin 2y$

13. Prove the following equation is an identity: $\dfrac{\sin(26k)+\sin(8k)}{\cos(26k)-\cos(8k)}=-\cot(9k)$

14. Prove the following equation is an identity: $\dfrac{\sin(26k)-\sin(12k)}{\sin(26k)+\sin(12k)}=\cot(19k)\tan(7k)$

15. Prove the following equation is an identity: $\sin(x+y)\cos(x-y)=\sin x\cos x+\cos y\sin y$

16. Prove the following equation is an identity: $(\sin\alpha+\cos\alpha)(\sin\beta+\cos\beta)=\sin(\alpha+\beta)+\cos(\alpha-\beta)$

17. Prove the following equation is an identity: $\dfrac{\cos x-\cos 3x}{\cos x+\cos 3x}=\tan 2x\tan x$

18. Prove the following equation is an identity: $\dfrac{\cos 5x+\cos 3x}{\cos 5x-\cos 3x}=-\cot 4x\cot x$

19. Prove the following equation is an identity: $\dfrac{\sin 3t-\sin t}{\cos 3t+\cos t}=\tan t$

20. Prove the following equation is an identity: $\dfrac{\sin 3x+\sin 5x}{\sin 3x-\sin 5x}=-\dfrac{\tan 4x}{\tan x}$

21. Prove the following equation is an identity: $\cos^2 x - \cos^2 y = -\sin(x+y)\sin(x-y)$

22. Prove the following equation is an identity: $\dfrac{\sin 6x + \sin 2x}{2\sin 4x} = \cos 2x$

23. Prove the following equation is an identity: $\dfrac{\cos 8x - \cos 2x}{2\sin 5x} = -\sin 3x$

24. Prove the following equation is an identity: $\dfrac{\sin 9x + \sin 3x}{\cos 9x + \cos 3x} = \tan 6x$

25. Prove the following equation is an identity: $\dfrac{\cos 2x - \cos 6x}{\sin 2x + \sin 6x} = \tan 2x$

26. Prove the following equation is an identity: $\dfrac{\sin 8x + \sin 2x}{\sin 8x - \sin 2x} = \dfrac{\tan 5x}{\tan 3x}$

27. Prove the following equation is an identity: $\dfrac{\cos 6x - \cos 2x}{\cos 6x + \cos 2x} = -\tan 4x \tan 2x$

28. Prove the following equation is an identity: $\sin x(\sin x + \sin 5x) = \cos 2x(\cos 2x - \cos 4x)$

29. Prove the following equation is an identity: $\dfrac{\cos x + \cos y}{\sin x - \sin y} = \cot \dfrac{x-y}{2}$

Section 3.6 – Solving Trigonometry Equations

Addition Property of Equality

For any three algebraic expressions *A*, *B*, and *C*

If $A = B$

Then $A + C = B + C$

Multiplication Property of Equality

For any three algebraic expressions *A*, *B*, and *C*, with $C \neq 0$

If $A = B$

Then $AC = BC$

Example

Solve $2\sin x - 1 = 0$

Solution

$2\sin x = 1$

$\sin x = \frac{1}{2}$

Solutions between (0° and 360°) $x = 30° \text{ or } 150°$

Solutions between (0 and 2π) $x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$

All solutions: $x = 30° + 360°k \quad \text{or} \quad 150° + 360°k$

$x = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad \frac{5\pi}{6} + 2k\pi$

Example

Solve $2\sin\theta - 3 = 0$, *if* $0° \le \theta < 360°$

Solution

$2\sin\theta = 3$

$\sin\theta = \frac{3}{2}$

$\sin\theta$ *can't be greater than 1*

No solution

Example

Solve $\cos(A-25°) = -\frac{1}{\sqrt{2}}$

<u>*Solution*</u>

$$\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$$

$-\frac{1}{\sqrt{2}}$ *is negative* $\rightarrow$ *cosine is in QII or QIII.*

$$\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = 135° \quad or \quad 225°$$

$$\cos(A-25°) = -\frac{1}{\sqrt{2}} = \cos(135°)$$

$$A-25° = 135° + 360°k$$

$$A = 25° + 135° + 360°k$$

$$A = 160° + 360°k$$

$$\cos(A-25°) = -\frac{1}{\sqrt{2}} = \cos(225°)$$

$$A-25° = 225° + 360°k$$

$$A = 25° + 225° + 360°k$$

$$A = 250° + 360°k$$

Example

Solve $3\sin\theta - 2 = 7\sin\theta - 1 \quad if \quad 0° \le \theta < 360°$

<u>*Solution*</u>

$$3\sin\theta - 7\sin\theta = 2 - 1$$

$$-4\sin\theta = 1$$

$$\sin\theta = -\frac{1}{4}$$

$$\hat{\theta} = \sin^{-1}\left(-\frac{1}{4}\right) = 14.5°$$

Negative sign $\rightarrow$ *sine is in QIII or QIV*

$$\theta = 14.5° + 180°$$

$$\theta = 194.5°$$

$$\theta = 360° - 14.5°$$

$$\theta = 345.5°$$

Example

Solve $2\sin^2\theta + 2\sin\theta - 1 = 0 \qquad \textit{if} \;\; 0 \le \theta < 2\pi$

<u>*Solution*</u>

$$\sin\theta = \frac{-2 \pm \sqrt{2^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{-2 \pm \sqrt{12}}{4}$$

$$= \frac{-2 \pm 2\sqrt{3}}{4}$$

$$= \frac{2(-1 \pm \sqrt{3})}{4}$$

$$= \frac{-1 \pm \sqrt{3}}{2}$$

$$\sin\theta = \frac{-1-\sqrt{3}}{2} < -1$$

$$\sin\theta = \frac{-1+\sqrt{3}}{2} = 0.3661$$

$$\theta = \sin^{-1}(0.3661)$$

$\hat{\theta} = 0.37$ *(QI or QII)*

$$\theta = 0.37 \qquad \theta = \pi - 0.37 = 2.77$$

Example

Solve: $2\cos x - 1 = \sec x$, *if* $0 \le x < 2\pi$

<u>*Solution*</u>

$$2\cos x - 1 = \frac{1}{\cos x}$$

$$2\cos^2 x - \cos x = 1$$

$$2\cos^2 x - \cos x - 1 = 0$$

$$(2\cos x + 1)(\cos x - 1) = 0$$

$$2\cos x + 1 = 0 \qquad \cos x - 1 = 0$$

$$\cos x = -\frac{1}{2} \qquad \cos x = 1$$

$$x = \frac{2\pi}{3}, \frac{4\pi}{3} \qquad x = 0$$

The solutions are: $\boxed{0,\ \frac{2\pi}{3},\ \frac{4\pi}{3}}$

Example

Solve: $\cos 2\theta + 3\sin\theta - 2 = 0$, *if* $0° \le \theta < 360°$

Solution

$$1 - 2\sin^2\theta + 3\sin\theta - 2 = 0$$

$$-2\sin^2\theta + 3\sin\theta - 1 = 0$$

$$2\sin^2\theta - 3\sin\theta + 1 = 0$$

$$(2\sin\theta - 1)(\sin\theta - 1) = 0$$

$2\sin\theta - 1 = 0$	$\sin\theta - 1 = 0$
$\sin\theta = \frac{1}{2}$	$\sin\theta = 1$
$\theta = 30°, 150°$	$\theta = 90°$

The solutions are: $\boxed{\theta = 30°,\ 90°,\ 150°}$

Example

Solve: $4\cos^2 x + 4\sin x - 5 = 0$, *if* $0 \le x < 2\pi$

Solution

$$4(1 - \sin^2 x) + 4\sin x - 5 = 0$$

$$4 - 4\sin^2 x + 4\sin x - 5 = 0$$

$$-4\sin^2 x + 4\sin x - 1 = 0$$

$$4\sin^2 x - 4\sin x + 1 = 0$$

$$(2\sin x - 1)^2 = 0$$

$$2\sin x - 1 = 0$$

$$\sin x = \frac{1}{2}$$

The solutions are: $\boxed{x = \frac{\pi}{6},\ \frac{5\pi}{6}}$

Example

Solve: $\sin 2\theta + \sqrt{2}\cos\theta = 0,\ \ if\ \ 0° \le \theta < 360°$

<u>*Solution*</u>

$$2\sin\theta\cos\theta + \sqrt{2}\cos\theta = 0$$

$$\cos\theta\left(2\sin\theta + \sqrt{2}\right) = 0$$

$\cos\theta = 0$	$2\sin\theta + \sqrt{2} = 0$
$\cos\theta = 0$	$\sin\theta = -\frac{\sqrt{2}}{2}$
	$\hat{\theta} = \sin^{-1}\frac{1}{\sqrt{2}} = 45°$
$\theta = 90°, 270°$	$\theta = 225°, 315°$

Example

Solve: $\sin\theta - \cos\theta = 1,\ if\ \ 0 \le \theta < 2\pi$

<u>*Solution*</u>

$$\sin\theta = \cos\theta + 1$$

$$\sin^2\theta = (\cos\theta + 1)^2$$

$$1 - \cos^2\theta = \cos^2\theta + 2\cos\theta + 1$$

$$0 = \cos^2\theta + 2\cos\theta + 1 - 1 + \cos^2\theta$$

$$0 = 2\cos^2\theta + 2\cos\theta$$

$$2\cos^2\theta + 2\cos\theta = 0$$

$$2\cos\theta(\cos\theta + 1) = 0$$

$2\cos\theta = 0$	$\cos\theta + 1 = 0$
$\cos\theta = 0$	$\cos\theta = -1$
$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$	$\theta = \pi$

Check

$\theta = \frac{\pi}{2}$ $\sin\frac{\pi}{2} - \cos\frac{\pi}{2} ? = ?1$ $1 - 0 = 1$	$\theta = \frac{3\pi}{2}$ $\sin\frac{3\pi}{2} - \cos\frac{3\pi}{2} = 1$ $-1 - 0 = 1$ **(*False statement*)**	$\theta = \pi$ $\sin\pi - \cos\pi = ?1$ $0 - (-1) = 1$

The solutions are: $\boxed{\frac{\pi}{2},\ \pi}$

Exercises Section 3.6 – Solving Trigonometry Equations

1. Solve $2\cos\theta + \sqrt{3} = 0$ *if* $0° \le \theta < 360°$

2. Solve $5\cos t + \sqrt{12} = \cos t$ *if* $0 \le t < 2\pi$

3. Solve $\tan\theta - 2\cos\theta\tan\theta = 0$ *if* $0° \le \theta < 360°$

4. Solve $2\sin^2\theta - 2\sin\theta - 1 = 0$ *if* $0° \le \theta < 360°$

5. Solve: $4\cos\theta - 3\sec\theta = 0$ *if* $0° \le \theta < 360°$

6. Solve: $2\sin^2 x - \cos x - 1 = 0$ *if* $0 \le x < 2\pi$

7. Solve: $\sin\theta - \sqrt{3}\cos\theta = 1$ *if* $0° \le \theta < 360°$

8. Solve: $7\sin^2\theta - 9\cos 2\theta = 0$ *if* $0° \le \theta < 360°$

9. Solve $2\cos^2 t - 9\cos t = 5$ *if* $0 \le t < 2\pi$

10. Solve $\sin\theta\tan\theta = \sin\theta$ *if* $0° \le \theta < 360°$

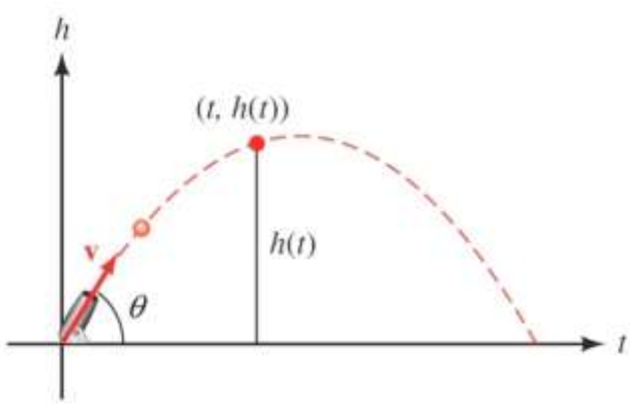
11. Solve $\tan^2 x + \tan x - 2 = 0$ *if* $0 \le x < 2\pi$

12. Solve $\tan x + \sqrt{3} = \sec x$ *if* $0 \le x < 2\pi$

13. Solve $\cos\left(A - \frac{\pi}{9}\right) = -\frac{1}{2}$

14. If a projectile (such as a bullet) is fired into the air with an initial velocity $\boldsymbol{v}$ at an angle of elevation θ, then the height h of the projectile at time t is given by:

$$h(t) = -16t^2 + vt\sin\theta$$



a) Give the equation for the height, if $\boldsymbol{v}$ is 600 *ft./sec* and θ = 45°.

b) Use the equation in part (*a*) to find the height of the object after $\sqrt{3}$ seconds.

c) Find the angle of elevation of θ of a rifle barrel, if a bullet fired at 1,500 ft./sec takes 3 seconds to reach a height of 750 feet. Give your answer in the nearest of a degree.