Solution *Section* 3.5 – Least Squares Analysis

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0,\ 2),\ (1,\ 2),\ (2,\ 0)\}$$

Solution

$$\{(0,\ 2),\ (1,\ 2),\ (2,\ 0)\}$$

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

The normal equation formula: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

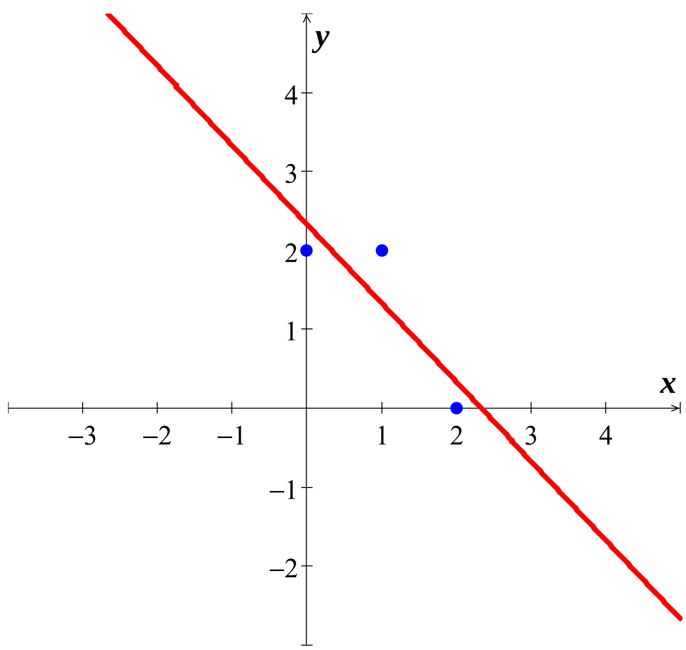
$$m = \frac{\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 3 & 3 \end{vmatrix}}$$

$$= \frac{-6}{6}$$

$$\boxed{= -1}$$

$$b = \frac{\begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix}}{6}$$

$$\boxed{= \frac{7}{3}}$$



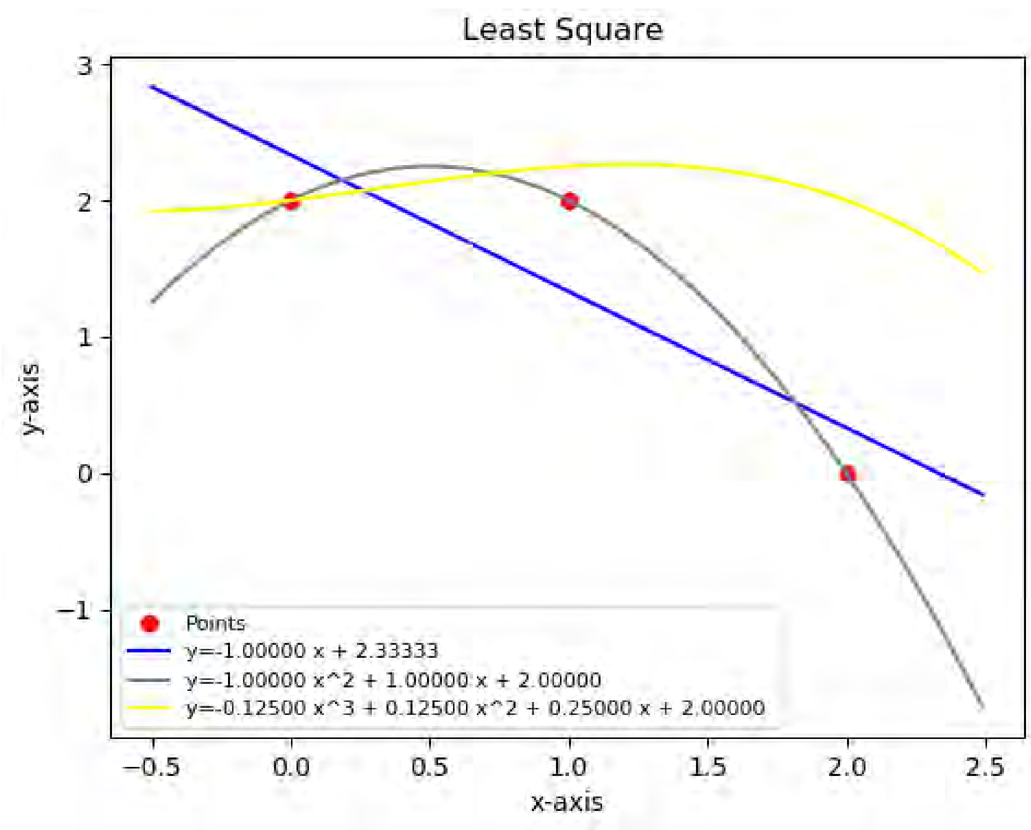
Thus, $\boxed{y = -x + \frac{7}{3}}$

$$A\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ \frac{7}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{7}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{7}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$$



Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}}$$

$$= \frac{\sqrt{6}}{3}$$

$$\approx 0.8164966$$

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The ***second*** *order* equation:

$$y = -x^2 + x + 2$$

Error = 0.00000

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The ***third order*** equation:

$$y = -.1250x^3 - 0.1250x^2 + 0.25x + 2$$

Error = 2.01556

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(1,\ 5),\ (2,\ 4),\ (3,\ 1),\ (4,\ 1),\ (5,-1)\}$$

Solution

$$\{(1,\ 5),\ (2,\ 4),\ (3,\ 1),\ (4,\ 1),\ (5,-1)\}$$

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix} \qquad \text{where} \quad A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

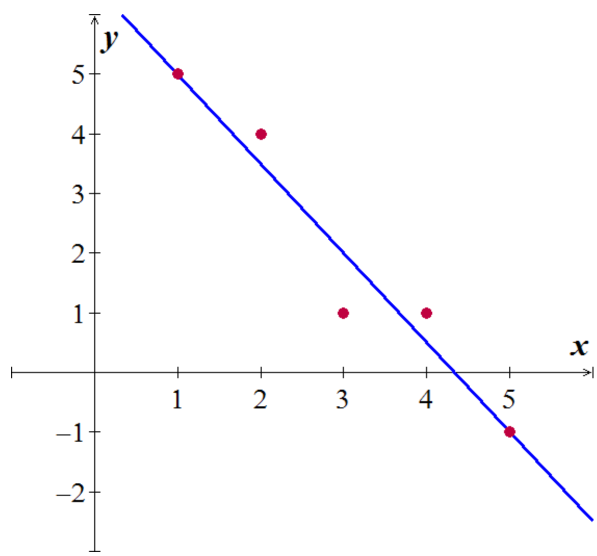
The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 55 & 15 \\ 15 & 5 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$$

$$m = \frac{\begin{vmatrix} 15 & 15 \\ 10 & 5 \end{vmatrix}}{\begin{vmatrix} 55 & 15 \\ 15 & 5 \end{vmatrix}} = \frac{-75}{50} = -\frac{3}{2}$$

$$b = \frac{\begin{vmatrix} 55 & 15 \\ 15 & 10 \end{vmatrix}}{50} = \frac{325}{50} = \frac{13}{2}$$



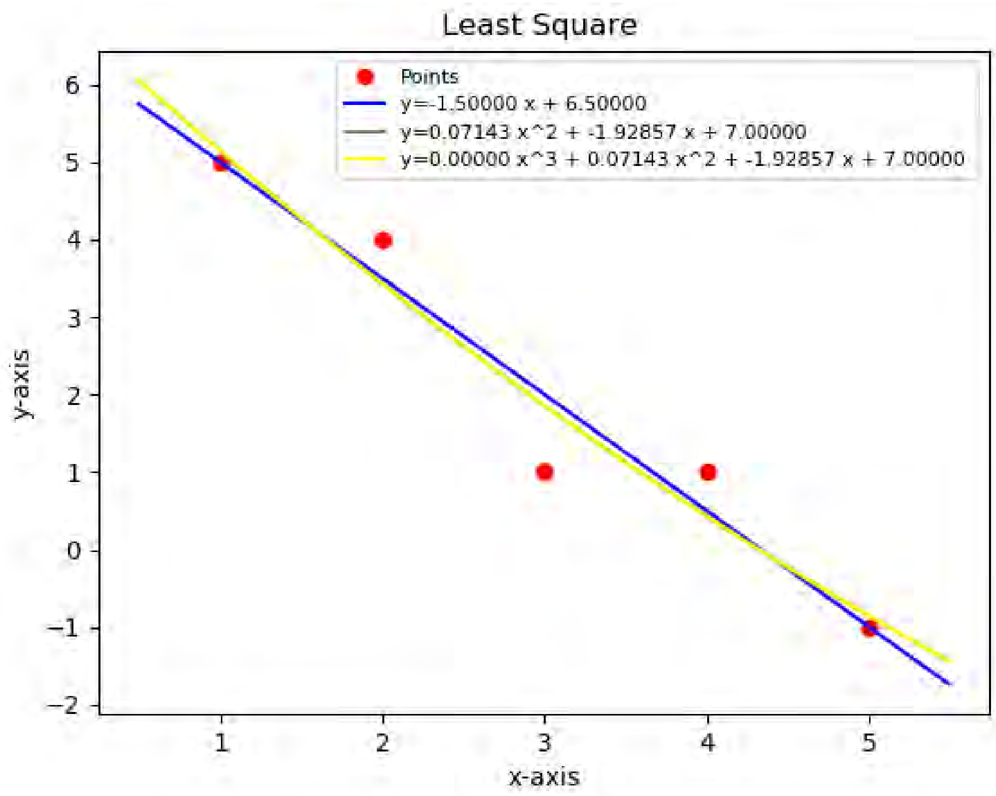
Thus, $y = -\frac{3}{2}x + \frac{13}{2}$ *or* $y = -1.5x + 6.5$

$$A\vec{x} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ \frac{13}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ \frac{7}{2} \\ 2 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ \frac{7}{2} \\ 2 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{1}{2} \\ -1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$



Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{4} + 1 + \frac{1}{4}}$$

$$= \frac{\sqrt{6}}{2}$$

$$\approx 1.224745$$

The ***second order*** equation:

$$y = 0.07143x^2 - 1.92857x + 7$$

Error = 1.19523

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The ***third order*** equation:

$$y = 0.0x^3 + 0.07143x^2 - 1.92857x + 7$$

Error = 1.19523

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

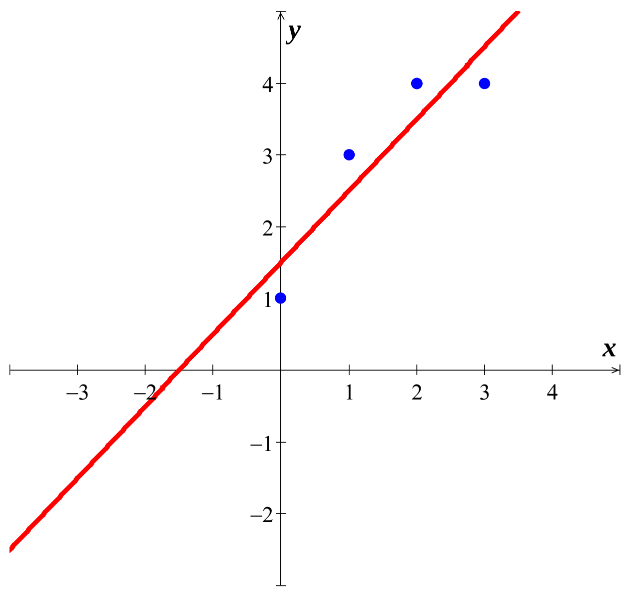
$$\{(0,\ 1),\ (1,\ 3),\ (2,\ 4),\ (3,\ 4)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

where $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$ $\quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\quad \vec{y} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$



The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 23 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 4 & -6 \\ -6 & 14 \end{pmatrix}\begin{pmatrix} 23 \\ 12 \end{pmatrix} \qquad X = A^{-1}B$$

$$= \frac{1}{20}\begin{pmatrix} 20 \\ 30 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$$

We have: $m = 1$ *and* $b = \frac{3}{2}$.

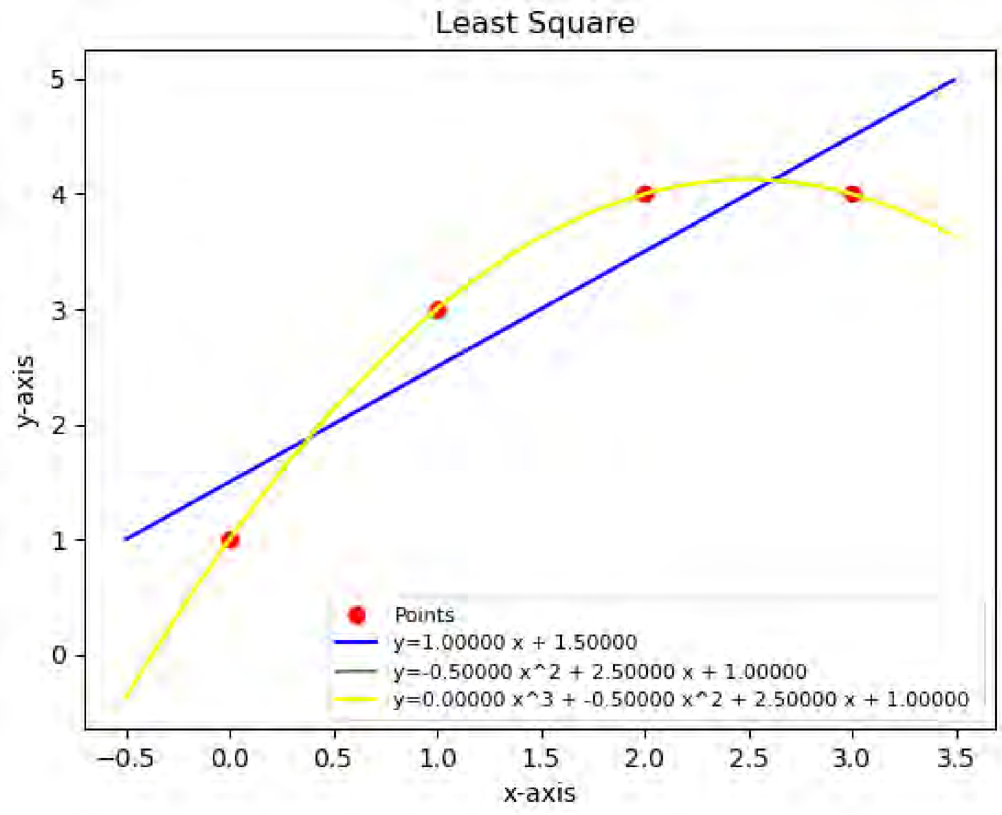
Thus, $\boxed{y = x + \frac{3}{2}}$

$$A\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ \frac{7}{2} \\ \frac{9}{2} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ \frac{7}{2} \\ \frac{9}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$



Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{4\left(\frac{1}{4}\right)}$$

$$= 1$$

The ***second order*** equation:

$$y = -0.50x^2 + 2.5x + 1.0$$

Error = 0.0000

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The ***third order*** equation:

$$y = 0.0x^3 - 0.5x^2 + 2.5x + 1$$

Error = 0.00000

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-2,\ 0),\ (-1,\ 0),\ (0,\ 1),\ (1,\ 3),\ (2,\ 5)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

where $A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$

The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

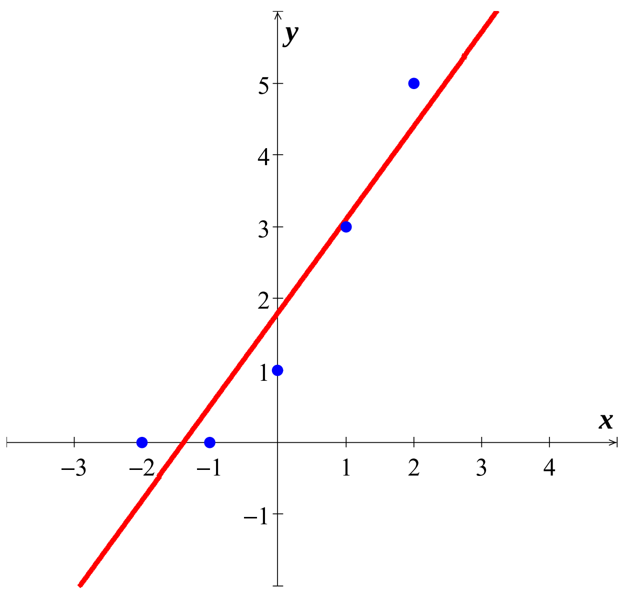
$$\begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{50}\begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}\begin{pmatrix} 13 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{13}{10} \\ \frac{9}{5} \end{pmatrix}$$

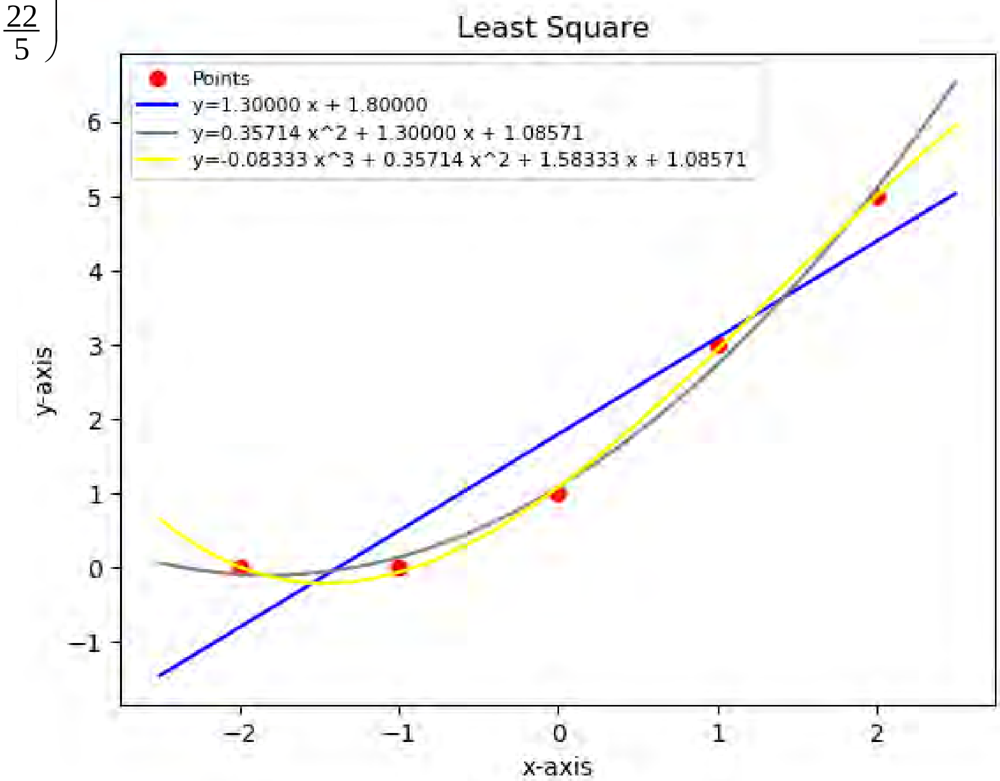
We have: $m = 1.3 \quad \textit{and} \quad b = 1.8$

Thus, $\boxed{y = \frac{13}{10}x + \frac{9}{5}}$



$$A\vec{x} = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{13}{10} \\ \frac{9}{5} \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} \\ \frac{1}{2} \\ \frac{9}{5} \\ \frac{31}{10} \\ \frac{22}{5} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix} - \begin{pmatrix} -\frac{4}{5} \\ \frac{1}{2} \\ \frac{9}{5} \\ \frac{31}{10} \\ \frac{22}{5} \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ -\frac{1}{2} \\ -\frac{4}{5} \\ -\frac{1}{10} \\ \frac{3}{5} \end{pmatrix}$$



Error: $\left\| \vec{y} - A\vec{x} \right\| = \sqrt{\frac{16}{25} + \frac{1}{4} + \frac{16}{25} + \frac{1}{100} + \frac{9}{25}}$

$$= \sqrt{\frac{41}{25} + \frac{26}{100}}$$

$$= \frac{\sqrt{190}}{10}$$

$$\approx 1.37840$$

The *second order* equation:

$y = 0.35714x^2 + 1.30x + 1.08571$

Error = 0.33806

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The *third order* equation:

$y = -0.08333x^3 + 0.35714x^2 + 1.58333x + 1.08571$

Error = 0.11952

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(2,\ 3),\ (3,\ 2),\ (5,\ 1),\ (6,\ 0)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

where $A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix}$ $\quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\quad \vec{y} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$

The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 74 & 16 \\ 16 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 17 \\ 6 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 74 & 16 \\ 16 & 4 \end{vmatrix} = 40 \quad \Delta_m = \begin{vmatrix} 17 & 16 \\ 6 & 4 \end{vmatrix} = -28 \quad \Delta_b = \begin{vmatrix} 74 & 17 \\ 16 & 6 \end{vmatrix} = 172$$

$$m = -\frac{28}{40} = -\frac{7}{10}$$

$$b = \frac{172}{40} = \frac{43}{10}$$

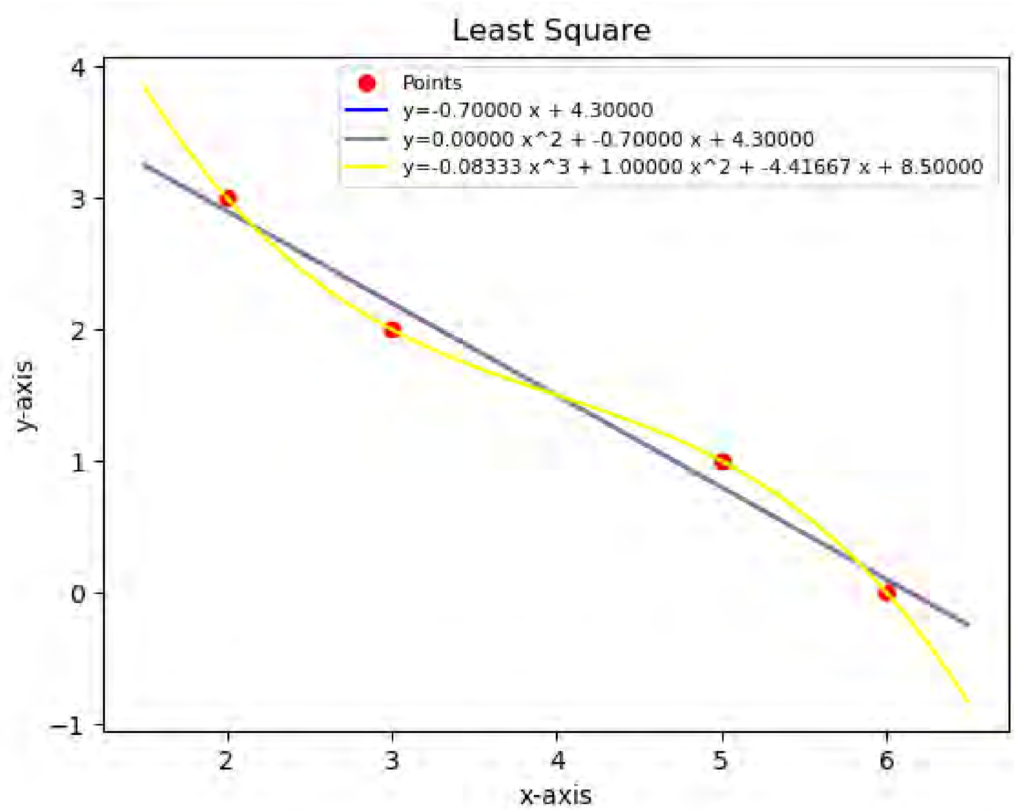
Thus, $y = -\frac{7}{10}x + \frac{43}{10}$

$$A\vec{x} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} -\frac{7}{10} \\ \frac{43}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{29}{10} \\ \frac{22}{10} \\ \frac{8}{10} \\ \frac{1}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{29}{10} \\ \frac{22}{10} \\ \frac{8}{10} \\ \frac{1}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{10} \\ -\frac{1}{5} \\ \frac{1}{5} \\ -\frac{1}{10} \end{pmatrix}$$



Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{2\left(\frac{1}{100} + \frac{1}{25}\right)}$$

$$= \frac{\sqrt{10}}{10}$$

$$= 0.31623$$

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The ***second order*** equation:

$$y = 0.0x^2 - 0.7x + 4.3$$

Error = 0.31623

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The ***third order*** equation:

$$y = -0.08333x^3 + x^2 - 4.41667x + 8.5$$

Error = 0.00000

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-1,\ 0),\ (0,\ 1),\ (1,\ 2),\ (2,\ 4)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

where $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$ $\quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix}$ $\quad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$

The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 10 \\ 7 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 6 & 2 \\ 2 & 4 \end{vmatrix} = 20 \quad \Delta_m = \begin{vmatrix} 10 & 2 \\ 7 & 4 \end{vmatrix} = 26 \quad \Delta_b = \begin{vmatrix} 6 & 10 \\ 2 & 7 \end{vmatrix} = 22$$

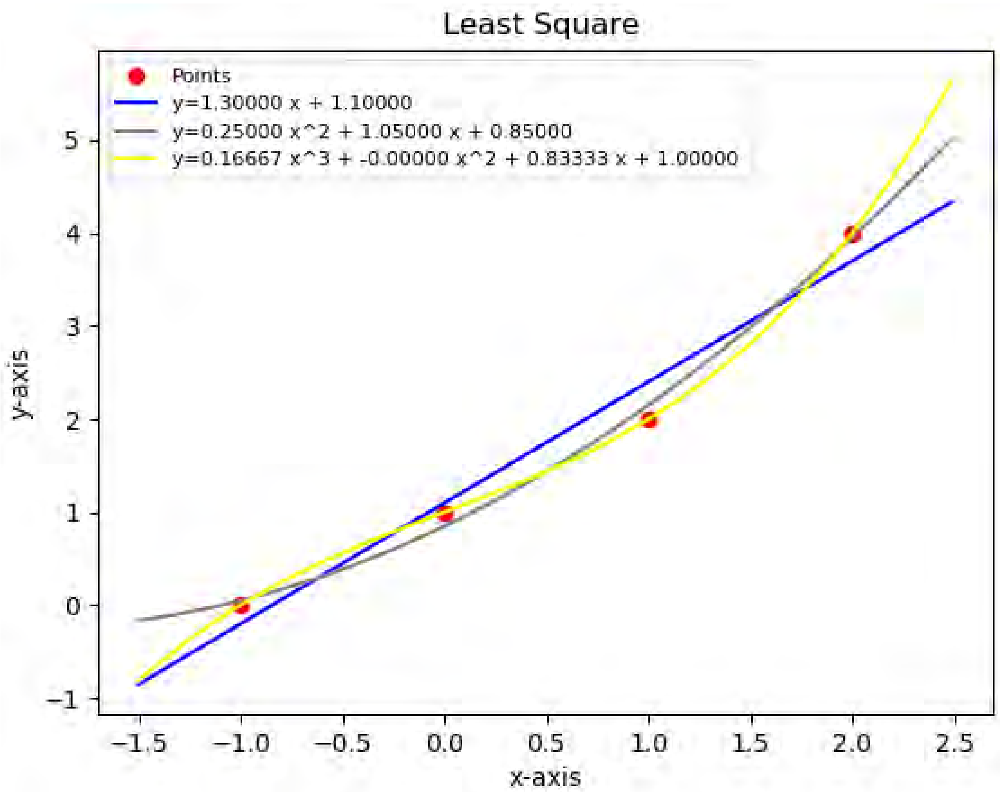
$$\boxed{m = \frac{26}{20} = \frac{13}{10}}$$

$$\boxed{b = \frac{22}{20} = \frac{11}{10}}$$

Thus, $\boxed{y = \frac{13}{10}x + \frac{11}{10}}$

$$A\vec{x} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \frac{13}{10} \\ \frac{11}{10} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{5} \\ \frac{11}{10} \\ \frac{12}{5} \\ \frac{37}{10} \end{pmatrix}$$



$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} -\frac{1}{5} \\ \frac{11}{10} \\ \frac{12}{5} \\ \frac{37}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} \\ -\frac{1}{10} \\ -\frac{2}{5} \\ \frac{3}{10} \end{pmatrix}$$

Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{25} + \frac{1}{100} + \frac{4}{25} + \frac{9}{100}}$$

$$= \sqrt{\frac{4+1+16+9}{100}}$$

$$= \frac{\sqrt{30}}{10}$$

$$= 0.54772$$

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The ***second order*** equation:

$$y = 0.25x^2 + 1.05x + 0.85$$

Error = 0.22361

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The ***third order*** equation:

$$y = 0.16667x^3 + 0.82222x + 1$$

Error = 0.00000

Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(1,\ 0),\ (2,\ 1),\ (4,\ 2),\ (5,\ 3)\}$$

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

where $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$

The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 46 & 12 \\ 12 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 25 \\ 6 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 46 & 12 \\ 12 & 4 \end{vmatrix} = 40 \quad \Delta_m = \begin{vmatrix} 25 & 12 \\ 6 & 4 \end{vmatrix} = 28 \quad \Delta_b = \begin{vmatrix} 46 & 25 \\ 12 & 6 \end{vmatrix} = -24$$

$$m = \frac{28}{40} = \frac{7}{10}$$

$$b = -\frac{24}{40} = -\frac{3}{5}$$

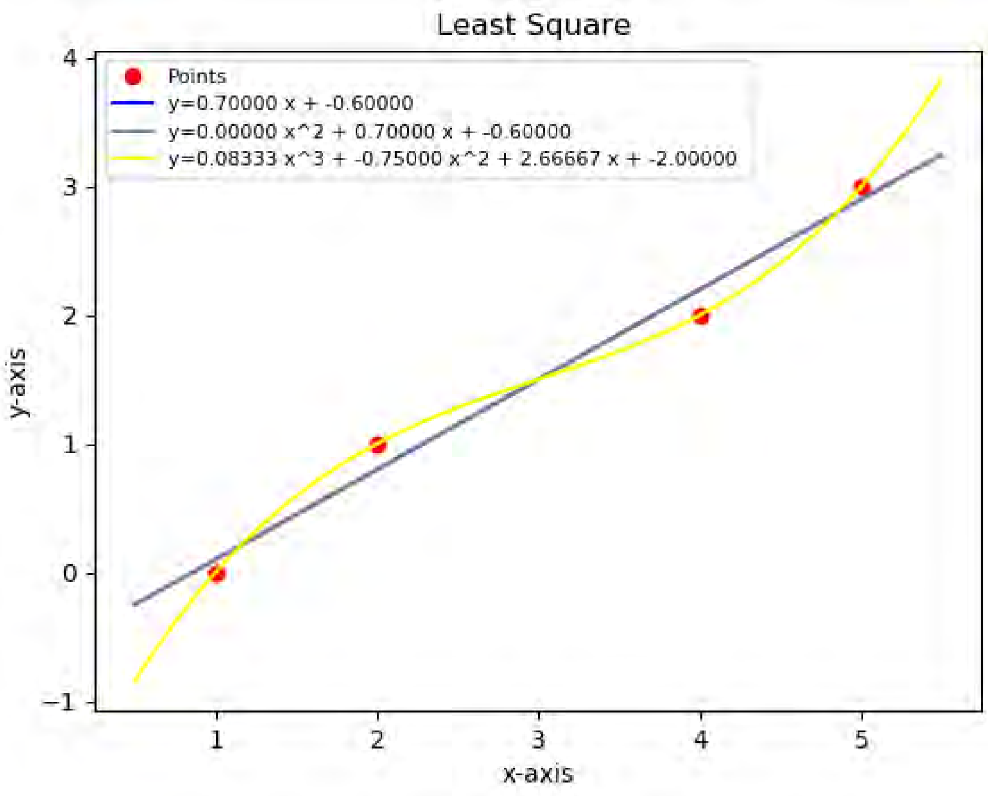
Thus, $y = \frac{7}{10}x - \frac{3}{5}$

$$A\vec{x} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} \frac{7}{10} \\ -\frac{3}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{10} \\ \frac{4}{5} \\ \frac{11}{5} \\ \frac{29}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} \frac{1}{10} \\ \frac{4}{5} \\ \frac{11}{5} \\ \frac{29}{10} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{10} \\ \frac{1}{5} \\ -\frac{1}{5} \\ \frac{1}{10} \end{pmatrix}$$



Error:

$$\|\vec{y} - A\vec{x}\| = \sqrt{2\left(\frac{1}{100} + \frac{1}{25}\right)}$$

$$= \frac{\sqrt{10}}{10}$$

$$= 0.31623$$

****

The ***second order*** equation:

$$y = 0.0x^2 + 0.7x - .6$$

Error = 0.31623

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The ***third order*** equation:

$$y = 0.08333x^3 - 0.75x^2 + 2.66667x - 2$$

Error = 0.00000

Exercise

Find the orthogonal projection of the vector $\vec{u}$ on the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{u} = (-3,\ -3,\ 8,\ 9);\quad \vec{v}_1 = (3,\ 1,\ 0,\ 1),\quad \vec{v}_2 = (1,\ 2,\ 1,\ 1),\quad \vec{v}_3 = (-1,\ 0,\ 2,\ -1)$$

Solution

Let $A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{pmatrix}$$

$$A^T \vec{u} = \begin{pmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -3 \\ 8 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 8 \\ 10 \end{pmatrix}$$

The normal solution is $A^T A \vec{x} = A^T \vec{u}$

$$\begin{pmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ 10 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{vmatrix} = 134 \qquad \Delta_1 = \begin{vmatrix} -3 & 6 & -4 \\ 8 & 7 & 0 \\ 10 & 0 & 6 \end{vmatrix} = -134 \qquad \Delta_2 = \begin{vmatrix} 11 & -3 & -4 \\ 6 & 8 & 0 \\ -4 & 10 & 6 \end{vmatrix} = 268$$

$$x_1 = \frac{-134}{134} = -1 \qquad x_2 = \frac{268}{134} = 2 \qquad x_3 = \frac{134}{134} = 1$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

So $proj_W \vec{u} = A\vec{x}$

$$= \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 3 \\ 4 \\ 0 \end{pmatrix}$$

$proj_W \vec{u} = (-2,\ 3,\ 4,\ 0)$

Exercise

Find the orthogonal projection of the vector $\vec{u}$ on the subspace of $\mathbb{R}^4$ spanned by the vectors

$\vec{u} = (6,\ 3,\ 9,\ 6);\quad \vec{v}_1 = (2,\ 1,\ 1,\ 1),\quad \vec{v}_2 = (1,\ 0,\ 1,\ 1),\quad \vec{v}_3 = (-2,\ -1,\ 0,\ -1)$

Solution

Let $A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix}$$

$$A^T \vec{u} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 9 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 30 \\ 21 \\ -21 \end{pmatrix}$$

The normal solution is $A^T A\vec{x} = A^T \vec{u}$

$$\begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 30 \\ 21 \\ -21 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{vmatrix} = 3 \qquad \Delta_1 = \begin{vmatrix} 30 & 4 & -6 \\ 21 & 3 & -3 \\ -21 & -3 & 6 \end{vmatrix} = 18 \qquad \Delta_2 = \begin{vmatrix} 7 & 30 & -6 \\ 4 & 21 & -3 \\ -6 & -21 & 6 \end{vmatrix} = 9$$

$$\underline{x_1 = \frac{18}{3} = 6} \quad \underline{x_2 = \frac{9}{3} = 3} \quad \underline{x_3 = 4}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

So $proj_W \vec{u} = A\vec{x}$

$$= \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 2 \\ 9 \\ 5 \end{pmatrix}$$

$$\underline{proj_W \vec{v} = (7,\ 2,\ 9,\ 5)}$$

Exercise

Find the orthogonal projection of the vector $\vec{u}$ on the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{u} = (-2,\ 0,\ 2,\ 4); \quad v_1 = (1,\ 1,\ 3,\ 0), \quad \vec{v}_2 = (-2,\ -1,\ -2,\ 1), \quad \vec{v}_3 = (-3,\ -1,\ 1,\ 3)$$

Solution

Let $A = \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{pmatrix}$$

$$A^T \vec{u} = \begin{pmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix}$$

The normal solution is $A^T A\vec{x} = A^T \vec{u}$

$$\begin{pmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{vmatrix} = 10 \qquad \Delta_1 = \begin{vmatrix} 4 & -9 & -1 \\ 4 & 10 & 8 \\ 20 & 8 & 20 \end{vmatrix} = -8 \qquad \Delta_2 = \begin{vmatrix} 11 & 4 & -1 \\ -9 & 4 & 8 \\ -1 & 20 & 20 \end{vmatrix} = -16$$

$$x_1 = \frac{-8}{10} = -\frac{4}{5} \qquad x_2 = \frac{-16}{10} = -\frac{8}{5} \qquad x_3 = \frac{8}{5}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} \\ -\frac{8}{5} \\ \frac{8}{5} \end{pmatrix}$$

$$proj_W \vec{u} = A\vec{x}$$

$$= \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} \\ -\frac{8}{5} \\ \frac{8}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{12}{5} \\ -\frac{4}{5} \\ \frac{12}{5} \\ \frac{16}{5} \end{pmatrix}$$

$$proj_W \vec{u} = \left(-\frac{12}{5},\ -\frac{4}{5},\ \frac{12}{5},\ \frac{16}{5}\right)$$

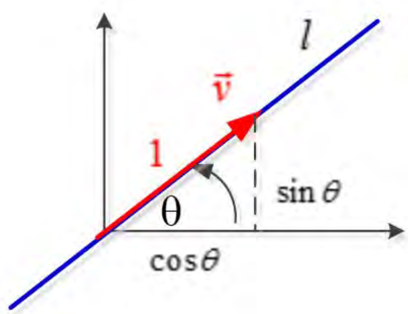
Exercise

Find the standard matrix for the orthogonal projection P of $\mathbb{R}^2$ on the line passes through the origin and makes an angle θ with the positive x-axis.

Solution

Since the line l in 2-dimensional, than we can take $\vec{v} = (\cos\theta,\ \sin\theta)$ as a basis for this subspace

$$A = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$



$$[P] = A^T A$$

$$= [\cos\theta \quad \sin\theta] \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

Exercise

Hooke's law in physics states that the length x of a uniform spring is a linear function of the force y applied to it. If we express the relationship as $y = mx + b$, then the coefficient m is called the spring constant.

Suppose a particular unstretched spring has a measured length of 6.1 *inches*.(i.e., x = 6.1 when y = 0). Forces of 2 pounds, 4 pounds, and 6 pounds are then applied to the spring, and the corresponding lengths are found to be 7.6 inches, 8.7 inches, and 10.4 inches. Find the spring constant.

Solution

$$M = \begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

The normal equation: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 278.82 & 32.8 \\ 32.8 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 112.4 \\ 12 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} m \\ b \end{pmatrix} &= \frac{1}{39.44}\begin{pmatrix} 4 & -32.8 \\ -32.8 & 278.2 \end{pmatrix}\begin{pmatrix} 112.4 \\ 12 \end{pmatrix} \\ &= \frac{1}{39.44}\begin{pmatrix} 56 \\ -348.32 \end{pmatrix} \\ &= \begin{pmatrix} 1.4 \\ -8.8 \end{pmatrix} \end{aligned}$$

Thus, the estimated value of the spring constant is $\approx 1.4\ pounds$

Exercise

Prove:

If A has a linearly independent column vectors, and if $\vec{b}$ is orthogonal to the column space of A, then the least squares solution of $A\vec{x} = \vec{b}$ is $\vec{x} = \vec{0}$.

Solution

If A has linearly independent column vectors, then $A^T A$ is invertible and the least squares solution of $A\vec{x} = \vec{b}$ is the solution of $A^T A\vec{x} = A^T\vec{b}$, but since $\vec{b}$ is orthogonal to the column space of A.

$A^T\vec{b} = 0$, so $\vec{x}$ is a solution of $A^T A\vec{x} = 0$.

Thus $\vec{x} = \vec{0}$ since $A^T A$ is invertible.

Exercise

Let A be an $m \times n$ matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of $\mathbb{R}^n$ onto the row space of A.

Solution

A^T will have linearly independent column vectors, and the column space A^T is the row space of A.

Thus, the standard matrix for the orthogonal projection of $\mathbb{R}^n$ onto the row space of A is

$$[P] = A^T \left[\left(A^T\right)^T A^T \right]^{-1} \left(A^T\right)^T$$

$$= A^T \left(AA^T\right)^{-1} A$$

Exercise

Let W be the line with parametric equations $x = 2t, \quad y = -t, \quad z = 4t$

a) Find a basis for W.

b) Find the standard matrix for the orthogonal projection on W.

c) Use the matrix in part (b) to find the orthogonal projection of a point $P_0\left(x_0, y_0, z_0\right)$ on W.

d) Find the distance between the point $P_0(2, 1, -3)$ and the line W.

Solution

a) $W = span\{(2, -1, 4)\}$

So that the vector $(2, -1, 4)$ forms a basis for W (linear independence)

b) Let $A = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$

$$[P] = A\left(A^T A\right)^{-1} A^T$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \left(\begin{bmatrix} 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} [21]^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \frac{1}{21} \begin{bmatrix} 4 & -2 & 8 \\ -2 & 1 & -4 \\ 8 & -4 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix}$$

c) $\begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \frac{4}{21}x_0 - \frac{2}{21}y_0 + \frac{8}{21}z_0 \\ -\frac{2}{21}x_0 + y_0 - \frac{4}{21}z_0 \\ \frac{8}{21}x_0 - \frac{4}{21}y_0 + \frac{16}{21}z_0 \end{bmatrix}$

d) $\begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{6}{7} \\ \frac{3}{7} \\ -\frac{12}{7} \end{bmatrix}$

The distance between P_0 and W equals to the distance between P_0 and its projection on W.

The distance between $(2, 1, -3)$ and $\left(-\frac{6}{7}, \frac{3}{7}, -\frac{12}{7}\right)$ is

$$d = \sqrt{\left(2+\frac{6}{7}\right)^2 + \left(1-\frac{3}{7}\right)^2 + \left(-3+\frac{12}{7}\right)^2}$$

$$= \sqrt{\frac{400}{49} + \frac{16}{49} + \frac{81}{49}}$$

$$= \frac{\sqrt{497}}{7}$$

Exercise

In R^3, consider the line l given by the equations $\quad x = t, \quad y = t, \quad z = t$

And the line m given by the equations $\quad x = s, \quad y = 2s - 1, \quad z = 1$

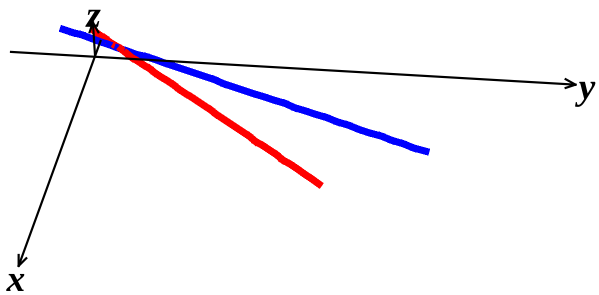
Let P be the point on l, and let Q be a point on m. Find the values of t and s that minimize the distance between the lines by minimizing the squared distance $\|P-Q\|^2$

Solution

When $t = 1 \quad \Rightarrow \quad Let\ P = (1, 1, 1)$ is on line l

When $s = 1 \quad \Rightarrow \quad Let\ Q = (1, 1, 1)$ is on line m

$$\|P-Q\| = \sqrt{(1-1)^2 + (1-1)^2 + (1-1)^2} = 0 \geq 0$$



Thus, these are the values $P=(1,\ 1,\ 1)$ and $Q=(1,\ 1,\ 1)$ are the values for $s=t=1$ that minimize the distance between the lines.

Exercise

Determine whether the statement is true or false,

a) If A is an $m\times n$ matrix, then $A^T A$ is a square matrix.

b) If $A^T A$ is invertible, then A is invertible.

c) If A is invertible, then $A^T A$ is invertible.

d) If $A\vec{x}=\vec{b}$ is a consistent linear system, then $A^T A\vec{x}=A^T\vec{b}$ is also consistent.

e) If $A\vec{x}=\vec{b}$ is an inconsistent linear system, then $A^T A\vec{x}=A^T\vec{b}$ is also inconsistent.

f) Every linear system has a least squares solution.

g) Every linear system has a unique least squares solution.

h) If A is an $m\times n$ matrix with linearly independent columns and $\vec{b}$ is in R^m, then $A\vec{x}=\vec{b}$ has a unique least squares solution.

Solution

a) True; $A^T A$ is an $n\times n$ matrix

b) False; only square matrix has inverses, but $A^T A$ can be invertible when A is not square matrix.

c) True; if A is invertible, so is A^T, so the product $A^T A$ is also invertible

d) True

e) False; the system $A^T A\vec{x}=A^T\vec{b}$ may be consistent

f) True

g) False; the least squares solution may involve a parameter

h) True; if A has linearly independent column vectors; then $A^T A$ is invertible, so $A^T A\vec{x}=A^T\vec{b}$ has a unique solution

Exercise

A certain experiment produces the data $\{(1,\ 1.8),\ (2,\ 2.7),\ (3,\ 3.4),\ (4,\ 3.8),\ (5,\ 3.9)\}$.

Find the function that it will fit these data in the form of $y = \beta_1 x + \beta_2 x^2$

Solution

Given: the equation $y = \beta_1 x + \beta_2 x^2$ that best fits the given points. Then

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$$

where $A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$

The normal equation formula: $A^T A\vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix} \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$$

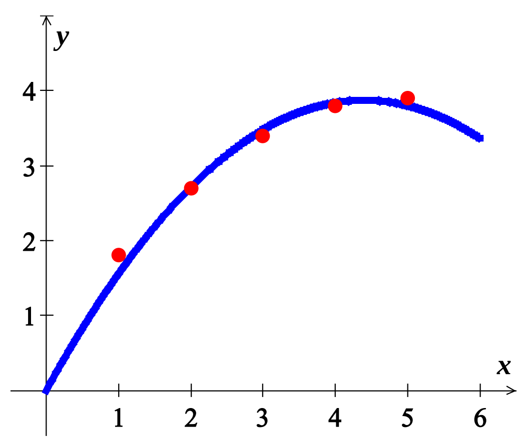
$$\begin{pmatrix} 55 & 225 \\ 225 & 979 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 52.1 \\ 201.5 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 55 & 225 \\ 225 & 979 \end{vmatrix} = 3,220 \qquad \Delta_{\beta_1} = \begin{vmatrix} 52.1 & 225 \\ 201.5 & 979 \end{vmatrix} = 5,668.4 \qquad \Delta_{\beta_2} = \begin{vmatrix} 55 & 52.1 \\ 225 & 201.5 \end{vmatrix} = -640$$

$$\beta_1 = \frac{5,668.4}{3,220} \approx 1.76$$

$$\beta_2 = -\frac{640}{3,220} \approx -0.199$$

$$y = 1.76x - .2x^2$$



Exercise

According to Kepler's first law, a comet should have an ellipse, parabolic, or hyperbolic orbit (with gravitational attractions from the planets ignored). In suitable polar coordinates, the position $(r,\ \upsilon)$ of a comet satisfies an equation of the form

$$r = \beta + e(r \cdot \cos\upsilon)$$

Where β is a constant and e is the eccentricity of the orbit, with $0 \le e < 1$ for an ellipse, $e = 1$ for a parabolic, and $e > 1$ for a hyperbola.

Suppose observations of a newly discovered comet provide the data below.

υ	.88	1.10	1.42	1.77	2.14
r	3.00	2.30	1.65	1.25	1.01

Determine the type of orbit, and predict where the orbit will be when $\upsilon = 4.6$ $(radians)$?

Solution

Given: the equation in the form $r = \beta + e(r \cdot \cos\upsilon)$

$$3 = \beta + e(3 \cdot \cos(.88)) = \beta + 1.911e$$

$$2.3 = \beta + e(2.3\cos(1.1)) = \beta + 1.043e$$

$$1.65 = \beta + e(1.65\cos(1.42)) = \beta + .248e$$

$$1.25 = \beta + e(1.25\cos(1.77)) = \beta - .247e$$

$$1.01 = \beta + e(1.01\cos(2.14)) = \beta - .544e$$

$$\begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix} \begin{pmatrix} \beta \\ e \end{pmatrix} = \begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$$

where $A = \begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix} \quad \vec{\upsilon} = \begin{pmatrix} \beta \\ e \end{pmatrix} \quad \vec{r} = \begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$

The normal equation formula: $A^T A\,\vec{\upsilon} = A^T \vec{r}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1.911 & 1.043 & .248 & -.247 & -.544 \end{pmatrix}\begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix}\begin{pmatrix} \beta \\ e \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1.911 & 1.043 & .248 & -.247 & -.544 \end{pmatrix}\begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2.411 \\ 2.411 & 5.158 \end{pmatrix}\begin{pmatrix} \beta \\ e \end{pmatrix} = \begin{pmatrix} 9.21 \\ 7.683 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 5 & 2.411 \\ 2.411 & 5.158 \end{vmatrix} = 19.98 \qquad \Delta_\beta = \begin{vmatrix} 9.21 & 2.411 \\ 7.683 & 5.158 \end{vmatrix} = 28.98 \qquad \Delta_e = \begin{vmatrix} 5 & 9.21 \\ 2.411 & 7.683 \end{vmatrix} = 16.21$$

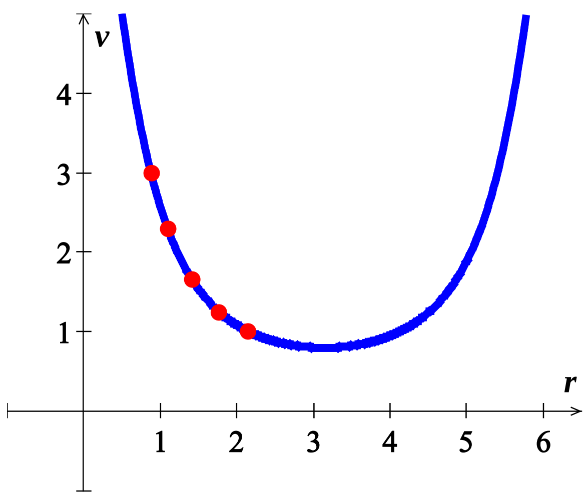
$$\beta = \frac{28.98}{19.98}$$
$$\underline{\approx 1.45}$$

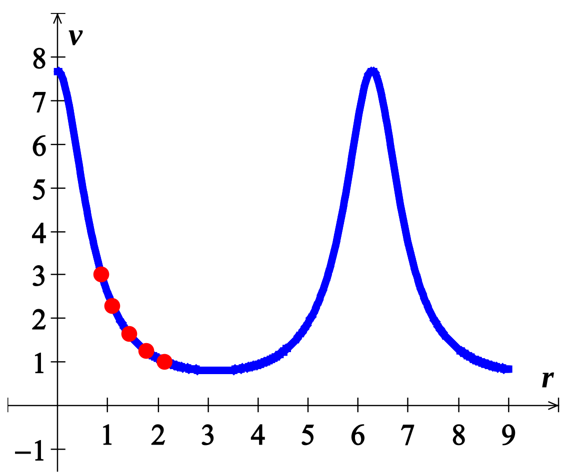
$$e = \frac{16.21}{19.98}$$
$$\underline{\approx 0.811 \quad < 1}$$

Therefore, the orbit is an ***ellipse*** type since $e \approx 0.811 \quad < 1$

Since $r = \beta + e(r \cdot \cos \upsilon)$

Then, $r(\upsilon) = \dfrac{1.45}{1 - 0.811 \cdot \cos \upsilon}$





$$r(4.6) = \frac{1.45}{1 - 0.811 \cdot \cos 4.6}$$
$$\underline{\approx 1.329}$$

Exercise

To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second, from $t = 0$ to $t = 12$

The position (in *feet*) were:

$0,\ 8.8,\ 29.9,\ 62.0,\ 104.7,\ 159.1,\ 222.0,\ 294.5,\ 380.4,\ 471.1,\ 571.7,\ 686.8,\ \textit{and}\ 809.2$

a) Find the least square cubic curve $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$ for these data.

b) Estimate the velocity of the plane when $t = 4.5\ \textit{sec}$, using the result from part (*a*).

Solution

Given: the equation is in form $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \\ 1 & 6 & 36 & 216 \\ 1 & 7 & 49 & 343 \\ 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \\ 1 & 12 & 144 & 1728 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}}_{\vec{t}} \underbrace{=}_{=} \underbrace{\begin{pmatrix} 0 \\ 8.8 \\ 29.9 \\ 62 \\ 104.7 \\ 159.1 \\ 222.0 \\ 294.5 \\ 380.4 \\ 471.1 \\ 571.7 \\ 686.8 \\ 809.2 \end{pmatrix}}_{\vec{y}}$$

The normal equation formula: $A^T A\,\vec{t} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 & 100 & 121 & 144 \\ 0 & 1 & 8 & 27 & 64 & 125 & 216 & 343 & 512 & 729 & 1000 & 1331 & 1728 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \\ 1 & 6 & 36 & 216 \\ 1 & 7 & 49 & 343 \\ 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \\ 1 & 12 & 144 & 1728 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 & 100 & 121 & 144 \\ 0 & 1 & 8 & 27 & 64 & 125 & 216 & 343 & 512 & 729 & 1000 & 1331 & 1728 \end{pmatrix} \begin{pmatrix} 0 \\ 8.8 \\ 29.9 \\ 62 \\ 104.7 \\ 159.1 \\ 222.0 \\ 294.5 \\ 380.4 \\ 471.1 \\ 571.7 \\ 686.8 \\ 809.2 \end{pmatrix}$$

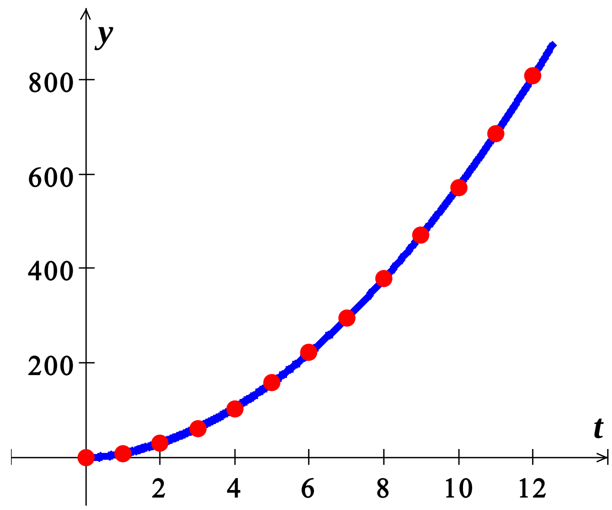
$$\begin{pmatrix} 13 & 78 & 650 & 6,084 \\ 78 & 650 & 6,084 & 60,710 \\ 650 & 6,084 & 60,710 & 630,708 \\ 6,084 & 60,710 & 630,708 & 6,735,950 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 3,800.2 \\ 35,127.7 \\ 348,063.9 \\ 3,599,800.9 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 13 & 78 & 650 & 6,084 \\ 78 & 650 & 6,084 & 60,710 \\ 650 & 6,084 & 60,710 & 630,708 \\ 6,084 & 60,710 & 630,708 & 6,735,950 \end{vmatrix} = 97,538,785,344$$

$$\Delta_0 = \begin{vmatrix} 3800.2 & 78 & 650 & 6,084 \\ 35,127.7 & 650 & 6,084 & 60,710 \\ 348,063.9 & 6,084 & 60,710 & 630,708 \\ 3,599,800.9 & 60,710 & 630,708 & 6,735,950 \end{vmatrix} = -83,470,691,303.8916$$

Or I use my program to find the values

```
rref = (Matrix([
[1, 0, 0, 0, -0.855769230765803],
[0, 1, 0, 0, 4.70248501498163],
[0, 0, 1, 0, 5.55536963037029],
[0, 0, 0, 1, -0.0273601398601744]]))
```



$\boxed{\beta_0 \approx -0.855769}$

$\boxed{\beta_1 \approx 4.702485}$

$\boxed{\beta_2 \approx 5.55537}$

$\boxed{\beta_3 \approx -0.02736}$

$$\boxed{y(t) = -0.855769 + 4.702485\,t + 5.55537\,t^2 - 0.02736\,t^3}$$

Error = 3.9734