Section 3.8 – Bayes' Theorem

A continuation of Conditional Probability, try to find the probability of earlier event conditioned on the occurrence of a later event.

Example

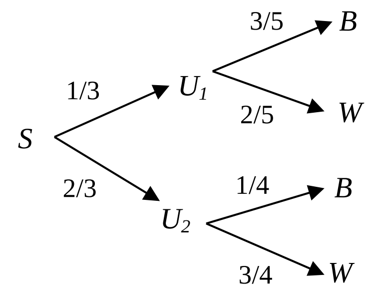
One urn has 3 blue and 2 white balls; a second urn has 1 blue and 3 white balls. A single fair die is rolled and if 1 or 2 comes up, a ball is drawn out of the first urn; otherwise, a ball is drawn out of the second urn. If the drawn ball is blue, what is the probability that it came out of the first urn? Out of the second urn?

<u>Solution</u>

$$\begin{cases} U_1 \to 3\ Blue,\ 2\ White \\ U_2 \to 1\ Blue,\ 3\ White \end{cases}$$

Fair Die

$$\begin{cases} if\ \ 1\ or\ 2 \to (1) & P = \frac{2}{6} = \frac{1}{3} \\ otherwise \to (2) & P = \frac{4}{6} = \frac{2}{3} \end{cases}$$



$$P(1^{st} Urn\ has\ tale\ B) = P\left(U_1 \mid B\right)$$

$$= \frac{P\left(U_1 \cap B\right)}{P(B)}$$

$$P\left(U_1 \mid B\right) = \frac{P\left(U_1 \cap B\right)}{P\left(U_1 \cap B\right) + P\left(U_2 \cap B\right)}$$

$$= \frac{\frac{3}{5}\frac{1}{3}}{\frac{3}{5}\frac{1}{3} + \frac{2}{3}\frac{1}{4}}$$

$$\underline{\approx .55|}$$

$$P\left(U_2 \mid B\right) = \frac{P\left(U_2 \cap B\right)}{P\left(U_2 \cap B\right) + P\left(U_1 \cap B\right)}$$

$$= \frac{\frac{1}{4}\frac{2}{3}}{\frac{3}{5}\frac{1}{3} + \frac{2}{3}\frac{1}{4}}$$

$$\underline{\approx .45|}$$

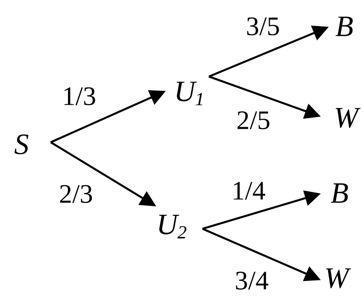
Example

One urn has 3 blue and 2 white balls; a second urn has 1 blue and 3 white balls. A single fair die is rolled and if 1 or 2 comes up, a ball is drawn out of the first urn; otherwise, a ball is drawn out of the second urn. If the drawn ball is white, what is the probability that it came out of the first urn? Out of the second urn?

<u>*Solution*</u>

Fair Die

$$\begin{cases} \text{if } 1 \text{ or } 2 \ (\mathrm{n}=2) \rightarrow P = \frac{2}{6} = \frac{1}{3} \\ \quad \text{otherwise} \rightarrow P = \frac{4}{6} = \frac{2}{3} \end{cases}$$



$$P\left(U_1 \,|\, W\right) = \frac{\frac{1}{3}\ \frac{2}{5}}{\frac{1}{3}\ \frac{2}{5} + \frac{2}{3}\ \frac{3}{4}}$$

$$= \frac{4}{19}$$

$$\underline{\approx .21|}$$

$$P\left(U_2 \,|\, W\right) = \frac{\frac{2}{3}\ \frac{3}{4}}{\frac{1}{3}\ \frac{2}{5} + \frac{2}{3}\ \frac{3}{4}}$$

$$= \frac{15}{19}$$

$$\underline{\approx .79|}$$

Bayes' Formula

$$P\left(U_1 \mid E\right) = \frac{P\left(U_1 \cap E\right)}{P(E)}$$

$$= \frac{P\left(U_1 \cap E\right)}{P\left(U_1 \cap E\right) + P\left(U_2 \cap E\right) + \cdots}$$

$$= \frac{P\left(E \mid U_1\right) P\left(U_1\right)}{P\left(E \mid U_1\right) P\left(U_1\right) + P\left(E \mid U_2\right) P\left(U_2\right) + \cdots}$$

Example

A new, inexpensive skin test is devised for detecting tuberculosis. To evaluate the test before it is put into use, a medical researcher randomly selects 1,000 people. Using the precise but more expensive methods already available, it is found that 8% of the 1,000 people have tested tuberculosis. Now each of the 1,000 subjects is given the new skin test and the following results are recorded: The test indicates tuberculosis in 96% of those who have it and in 2% of those who do not. Based on these results,

a) What is the probability of a randomly chosen person having tuberculosis given that the skin test indicates the disease?

b) What is the probability of a person not having tuberculosis given that the skin test indicates the disease?

c) What is the probability that a person has tuberculosis given that the test indicates no tuberculosis is present?

d) What is the probability that a person does not have tuberculosis given that the test indicates no tuberculosis is present?

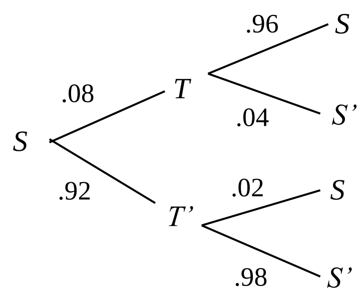
<u>*Solution*</u>

a) $P(T \mid S) = \frac{(.08)(.96)}{(.08)(.96) + (.92)(.02)} = .81$

b) $P(T' \mid S) = 1 - P(T \mid S) = .19$

c) $P(T \mid S') = .004$

d) $P(T' \mid S') = .996$



Example

A company produces 1,000 refrigerators a week at three plants. Plant *A* produces 350 refrigerators a week, plant *B* produces 250 refrigerators a week, and plant *C* produces 400 refrigerators a week. Production records indicate that 5% of the refrigerators produced at plant *A* will be defective, 3% of the refrigerators produced at plant *B* will be defective, 7% of the refrigerators produced at plant *C* will be defective. All the refrigerators are shipped to a central warehouse. If a refrigerator at the warehouse is found to be defective,

a) What is the probability that it was produced at plant *A*?

b) What is the probability that it was produced at plant *B*?

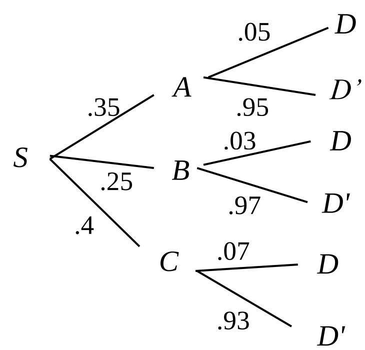
c) What is the probability that it was produced at plant *C*?

<u>*Solution*</u>

a) $P(A\,|\,D) = \dfrac{(.35)(.05)}{(.35)(.05)+(.25)(.03)+(.4)(.07)} \underline{=.33}$

b) $P(B\,|\,D) = \dfrac{(.25)(.03)}{(.35)(.05)+(.25)(.03)+(.4)(.07)} \underline{=.14}$

c) $P(C\,|\,D) = \dfrac{(.4)(.07)}{(.35)(.05)+(.25)(.03)+(.4)(.07)} \underline{=.53}$



Exercises *Section* 3.8 – Bayes' Theorem

1. One urn has 4 red balls and 1 white ball; a second urn has 2 red balls and 3 white balls. A single card is randomly selected from a standard deck. If the card is less than 5 (aces count as 1), a ball is drawn out of the first urn; otherwise a ball is drawn out of the second urn. If the drawn ball is red, what is the probability that it came out of the second urn?

2. A small manufacturing company has rated 75% of its employees as satisfactory (S) and 25% as unsatisfactory (S'). Personnel records show that 80% of the satisfactory workers had previous work experience (E) in the job they are now doing, while 15% of the unsatisfactory workers had no work experience (E') in the job they are now doing. If a person who has had previous work experience is hired, what is the approximate empirical probability that this person will be an unsatisfactory employee?

3. A basketball team is to play two games in a tournament. The probability of winning the first game is .10. If the first game is won, the probability of winning the second game is .15. If the first game is lost, the probability of winning the second game is .25. What is the probability the first game was won if the second game is lost?

4. To evaluate a new test for detecting Hansen's disease, a group of people 5% of which are known to have Hansen's disease are tested. The test finds Hansen's disease among 98% of those with the disease and 3% of those who don't. What is the probability that someone testing positive for Hansen's disease under this new test actually has it?

5. An urn contains 4 red and 5 white balls. Two balls are drawn in succession without replacement. If the second ball is white, what is the probability that the first ball was white?

6. An urn contains 4 red and 5 white balls. Two balls are drawn in succession without replacement. If the second ball is red, what is the probability that the first ball was red?

7. Urn 1 contains 7 red and 3 white balls. Urn 2 contains 4 red and 5 white balls. A ball is drawn from urn 1 and placed in urn 2. Then a ball is drawn from urn2. If the ball drawn from urn 2 is red, what is the probability that the ball drawn from urn 1 was red?

8. Urn 1 contains 7 red and 3 white balls. Urn 2 contains 4 red and 5 white balls. A ball is drawn from urn 1 and placed in urn 2. Then a ball is drawn from urn2. If the ball drawn from urn 2 is white, what is the probability that the ball drawn from urn 1 was white?

9. A company has rated 75% of its employees as satisfactory and 25% as unsatisfactory. Personnel records indicate that 80% of the satisfactory workers had previous work experience, while only 40% of the unsatisfactory workers had any previous work experience. If a person with previous work experience is hired, what is the probability that this person will be a satisfactory employee? If a person with no previous work experience is hired, what is the probability that this person will be a satisfactory employee?

10. A manufacturer obtains clock-radios from three different subcontractors: 20% from A, 40% from B, and 40% from C. The defective rates for these subcontractors are 1%, 3%, and 2%, respectively. If a defective clock-radio is returned by a customer, what is the probability that it came from subcontractor A? From B? From C?

11. A computer store sells three types of microcomputer, brand A, brand B, brand C. Of the computers sell, 60% are brands A, 25% are brand B, 15% are brand C. They have found that 20% of the brand A computers, 15% of the brand B computers, and 5% of the brand C computers are returned for service during the warranty period. If a computer is returned for service during the warranty period, what is the probability that it is a brand A computer, A brand B computer? A brand C computer?

12. A new, simple test has been developed to detect a particular type of cancer. The test must be evaluated before it is put into use. A medical researcher selects a random sample of 1,000 adults and finds (by other means) that 2% have this type of cancer. Each of the 1,000 adults is given test, and it is found that the test indicates cancer in 98% of those who have it and in 1% of those who do not. Based on these results, what is the probability of a randomly chosen person having cancer given that the test indicates cancer? Of a person having cancer given that the test does not indicate cancer?

13. In a random sample of 200 women who suspect that they are pregnant, 100 turn out to be pregnant. A new pregnancy test given to these women indicated pregnancy in 92 of the 100 pregnant women and in 12 of the 100 non-pregnant women. If a woman suspects she is pregnant and this test indicates that she is pregnant, what is the probability that she is pregnant? If the test indicates that she is not pregnant, what is the probability that she is not pregnant?

14. One of two urns is chosen at random with one as likely to be chosen as the other. Then a ball is drawn from the chosen urn, Urn 1 contains 1 white and 4 red balls, and urn 2 has 3 white and 2 red balls.

a) If a white ball is drawn, what is the probability that it came from urn 1?
b) If a white ball is drawn, what is the probability that it came from urn 2?
c) If a red ball is drawn, what is the probability that it came from urn 2?
d) If a red ball is drawn, what is the probability that it came from urn 1?