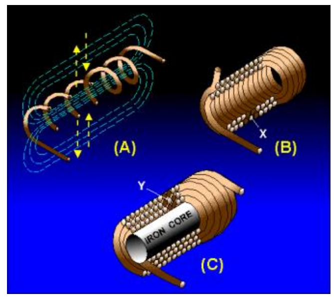
4.4 – Inductance

Self-Induced εmf of an inductor

An inductor is a coil. The circuit symbol for an inductor is

When an inductor is connected to a source, current will flow through it and this current will produce magnetic field inside the coil. That is there is magnetic flux crossing the loops of the coil due to its own magnetic field. If the current changes with time, then the magnetic field inside the coil will change with time which implies the magnetic flux crossing the coil will change with time. According to Faraday's law, this change in magnetic flux will produce induced εmf in the coil. This kind of induced εmf is called self induced εmf because it is caused by the current in the coil itself.





Self-induced εmf is directly proportional to the rate of change of current in the coil. The constant of proportional between the self induced εmf and the rate of change of current in the coil is called ***inductance*** (L) of the coil.

$$\boxed{\varepsilon_{self} = -L\frac{dI}{dt}}$$

$\varepsilon_{self} \rightarrow$ self induced εmf

$L \rightarrow$ Inductance $\frac{dI}{dt}$, rate of change of current with time.

The negative sign indicates that the polarity of the self-induced εmf is in such a way aas to oppose the cause for the change in flux which is the rate of change of current with time. (It essentially represents Lenz's rule)

The unit of measurement for inductance is *volt/(Ampere/second)* which is defined to be the Henry, abreviated as ***H***.

The average induced εmf in a given time interval Δt can be obtained by integrating ε_{self} with time in a time interval Δt & then dividing by Δt

$$\bar{\varepsilon}_{self} = -\frac{L}{\Delta t}\int_0^{\Delta t} \frac{dI}{dt}\, dA$$

$$= -\frac{L}{\Delta t}\int_0^{\Delta I} dI$$

$$= -L\frac{\Delta I}{\Delta t}$$

$$\boxed{\bar{\varepsilon}_{self} = -L\frac{\Delta I}{\Delta t} = -L\frac{\left(I_f - I_i\right)}{\Delta t}}$$

$\bar{\varepsilon}_{self} \rightarrow$ Average induced εmf in a time interval Δt

$\Delta I = I_f - I_i \rightarrow$ Change in current in a time interval Δt

Using Faraday's law, the inductance of a coil can be expresses in terms of the crossing the coil and the current in the coil

Faraday's law $\Rightarrow$ $\varepsilon_{self} = -N\dfrac{d\phi_B}{dt}$

Where N is the number of turns & ϕ is magentic flux per turn. Thus

$$\varepsilon_{self} = -N\frac{d\phi_B}{dt} = -L\frac{dI}{dt}$$

$$\frac{d}{dt}\left(N\phi_B\right) = \frac{d}{dt}\left(LI\right)$$

$$N\phi_B = LI + constant$$

The constant can be replaced without loss of generality because the physical theory depends on derivatives. Therfore the inductance can also be given as

$$\boxed{L = \frac{N\phi_B}{I}}$$

$N \rightarrow$ Number of turns

$\phi_B \rightarrow$ Magnetic flux per turns

$I \rightarrow$ Current in the coil

Example

The current in an inductor of inductance 5H, the current varies with time according to the equation $I = 2\sin 10t \quad (A)$

a) Give a formula for the self induced εmf as a function of time

b) Calculate the value of the induced εmf after $\frac{\pi}{40}$ sec

Solution

a) Given: $L = 5H, \quad I = 2\sin 10t$

$$\varepsilon_{self} = -L\frac{dI}{dt}$$

$$= -5\frac{d}{dt}(2\sin 10t)$$

$$\underline{= -100\cos 10t \ \ V|}$$

b) $\varepsilon_{self}(t) = -100\cos 10t$

$$\varepsilon_{self}\left(t = \frac{\pi}{40}\right) = -100\cos\left(10\frac{\pi}{40}\right)$$

$$\underline{= -70 \ V|}$$

Example

The current is a 2*mH* inductor changed from 10A to 4A in 0.2 seconds. Calculate the average induced εmf in the inductor?

Solution

Given: $L = 5H, \quad I_i = 10A, \quad I_f = 4A, \quad \Delta t = 0.2s$

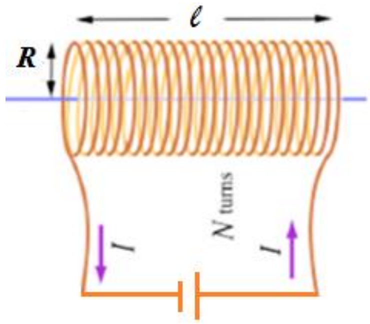
$$\overline{\varepsilon}_{self} = -L\frac{\left(I_f - I_i\right)}{\Delta t}$$

$$= -2\times 10^{-3}\left(\frac{4-10}{0.2}\right)$$

$$\underline{= 0.06 \ V|}$$

Inductance of a Solenoid in terms of its geometry

The inductance of an inductor depends on the geometry of the coil only. Consider a solenoid of length ℓ, radius R and number of turns N.



The magnetic flux crossing a single coil due its own current is $\phi_B = BA \ (\theta = 0)$ inside the solenoid the field is approximately parallel to its axis. But

$$A = \pi R^2 \ \& \ B = \mu_0 \frac{NI}{\ell}$$

Thus $L = \dfrac{N\phi_B}{I}$

$$= \frac{N}{I}\frac{\mu_0 N\pi IR^2}{\ell}$$

$$\boxed{L = \frac{\mu_0 N^2 \pi R^2}{\ell}}$$ Inductance of a solenoid in terms of its geometry

$N \rightarrow$ Number of turns
$R \rightarrow$ Radius of solenoid
$\ell \rightarrow$ Length of solenoid
$\mu_0 = 4\pi \times 10^{-7} \ \frac{TM}{A}$

Magnetic Energy Stored by an inductor

Consider an inductor connected to a source. According to Lenz's rule, the self-induced εmf should oppose the source because it is the cause for the change of flux. Therefore the source has to do work to push a charge through the inductor. This work is stored by the inductor as magnetic energy. The work done by the source in pushing a charge dq across the inducto is

$$dw_{ext} = -dq\ \varepsilon_{self}$$

The negative sign is needed because the work done by the external force (source) is opposite to the self-induced εmf

$$\varepsilon_{self} = -L\frac{dI}{dt}$$

$$\begin{aligned} dw_{ext} &= -dq\left(-L\frac{dI}{dt}\right) \\ &= L\frac{dq}{dt}dI \qquad but\ \frac{dq}{dt} = I \\ &= LI\ dI \end{aligned}$$

dw_{ext} is equal to the amount of magnetic energy $\left(dU_B\right)$ stored in transferred a charge dq

$$dw_{ext} = dU_B = LI\ dI$$

The amount of magnetic energy stored in increasing the current from zero to I is

$$\begin{aligned} U_B &= \int dU_B = \int_0^I LI\ dI \\ &= \frac{1}{2}LI^2 \\ \Delta U_B &= \frac{1}{2}LI^2 \end{aligned}$$

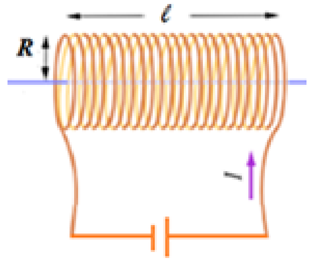
Assuming the magnetic energy at $I = 0$ is zero result.

$$\boxed{U_B = \frac{1}{2}LI^2}$$

$U_B \rightarrow$ Magnetic energy stored in an inductor with current I

Example

Obtain an expression for the magnetic energy density (magnetic energy per a unit volume) of a solenoid in terms of magnetic field



Solution

Magnetic energy density $= \dfrac{U_B}{V}$

Where V is the volume of solenoid

But $U_B = \frac{1}{2}LI^2$ & $V = \ell A = \ell\pi R^2$

$$\frac{U_B}{V} = \frac{\frac{1}{2}LI^2}{\ell\pi R^2} \qquad \text{But } L = \frac{\mu_0 N^2 \pi R^2}{\ell}$$

$$= \frac{1}{2}\frac{\mu_0 N^2 \pi R^2 I^2}{\ell^2 \pi R^2}$$

$$= \frac{\mu_0 N^2 I^2}{2\ell^2}$$

But $B = \dfrac{\mu_0 NI}{\ell}$ *or* $I = \dfrac{B\ell}{\mu_0 N}$

$$\frac{U_B}{V} = \frac{\mu_0 N^2}{2\ell^2}\frac{B^2\ell^2}{\mu_0^2 N^2}$$

$$\boxed{\frac{U_B}{V} = \frac{B^2}{2\mu_0}}$$

$\dfrac{U_B}{V} \rightarrow$ Magnetic energy density

$B \rightarrow$ Magnetic field

Example

A solenoid of length 10 *cm*, number of turns 200 and cross-sectional radius 2 *cm* is carrying a current of 2*A*.

a) Calculate the magnetic energy density inside the solenoid

b) Calculate the total magnetic energy stores inside the solenoid

Solution

a) ***Given***: $\ell = 0.1m, \quad R = 0.02m, \quad N = 200, \quad I = 2A$

$$B = \frac{\mu_0 NI}{\ell} = \frac{\left(4\pi \times 10^{-7}\right)(200)(2)}{0.1} = 16000\pi \times 10^{-7} = 16\pi \times 10^{-4}\ T$$

$$\frac{U_B}{V} = \frac{B^2}{2\mu_0}$$

$$= \frac{\left(16\pi \times 10^{-4}\right)^2}{2\left(4\pi \times 10^{-7}\right)}$$

$$= 32\pi \times 10^{-1}$$

$$\boxed{= 3.2\pi \ \ J / m^3}$$

b) Find U_B?

$$\frac{U_B}{V} = 3.2\pi \quad \Rightarrow \quad U_B = 3.2\pi V$$

But $V = \ell A = \ell \pi R^2 = (0.1)\pi(0.02)^2 = 4\pi \times 10^{-5}\ m^2$

$$U_B = 3.2\pi\left(4\pi \times 10^{-5}\right) \boxed{\approx 1.26 \times 10^{-3}\ J}$$

Or

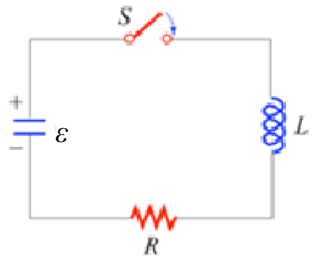
Alternatively, first L can be calculated from

$$L = \frac{\mu_0 N^2 \pi R^2}{\ell} = \frac{\left(4\pi \times 10^{-7}\right)\left(200^2\right)(\pi)(0.02)^2}{0.1} \approx 6.316 \times 10^{-4}$$

$$U_B = \frac{1}{2} LI^2 = \frac{1}{2}\left(6.316 \times 10^{-4}\right)\left(2^2\right) \boxed{\approx 1.26 \times 10^{-3}\ J}$$

A series combination of an inductor and a resistor connected to a *dc* source

Consider a battery of εmf ε connected to a series combination of a resistor of resistance R and and inductor of inductance L.



As the switch is closed, the current will increase from zero to a certain value. That is $\frac{dI}{dt}$ is positive.

Therefore the self-induced εmf $\varepsilon_{self} = -L\frac{dI}{dt}$ is negative which means the potential energy of the charges decrease as they cross the inductor. That means the right end of the inductor is more positive than the left end as shown. As the current crosses the resistor, the electric potential energy of the charge decrease because some of it will be converted to heat energy, which means the right end of the resistor is more positive than its left end as shown.

Now applying Kirchhoff's rule with a clockwise transverse direction

$$\boxed{\varepsilon = -L\frac{dI}{dt} - IR = 0}$$

Obtaining $I(t)$

$$L\frac{dI}{dt} = \varepsilon - IR$$

$$\frac{dI}{dt} = \frac{\varepsilon - IR}{L}$$

$$\int_0^I \frac{dI}{\varepsilon - IR} = \int_0^t \frac{dt}{L} \qquad d(\varepsilon - IR) = -RdI$$

$$-\frac{1}{R}\int_0^I \frac{d(\varepsilon - IR)}{\varepsilon - IR} = \int_0^t \frac{dt}{L}$$

$$-\left[\frac{1}{R}\ln|\varepsilon - IR|\right]_0^I = \frac{1}{L}[t]_0^t$$

$$-\frac{1}{R}\left(\ln|\varepsilon - IR| - \ln\varepsilon\right) = \frac{1}{L}t$$

$$\ln\frac{\varepsilon - IR}{\varepsilon} = -\frac{R}{L}t$$

$$\frac{\varepsilon - IR}{\varepsilon} = e^{-\frac{R}{L}t}$$

$$\varepsilon - IR = \varepsilon e^{-\frac{R}{L}t}$$

$$IR = \varepsilon - \varepsilon e^{-\frac{R}{L}t}$$

$$\boxed{I(t)=\frac{\varepsilon}{R}\left(1-e^{-\frac{R}{L}t}\right)}$$

Dependence of current on time for a series combination of an inductor and a resistor connected to a dc (constant) εmf .

At $t=0$

$$I(t=0)=\frac{\varepsilon}{R}\left(1-e^{-\frac{R}{L}0}\right)=\frac{\varepsilon}{R}(1-1)\ \underline{=0}\quad \text{(As expected)}$$

The current attains its maximum value at infinitely

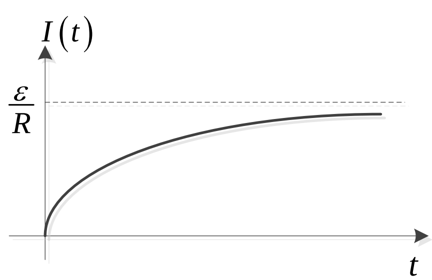
$$I_{\max}=\frac{\varepsilon}{R}\lim_{t\to\infty}\left(1-e^{-\frac{R}{L}t}\right)$$

$$=\frac{\varepsilon}{R}\left(1-\lim_{t\to\infty}e^{-\frac{R}{L}t}\right)\qquad but\ \lim_{t\to\infty}e^{-\frac{R}{L}t}=0$$

$$=\frac{\varepsilon}{R}$$

$$\boxed{I_{\max}=\frac{\varepsilon}{R}}\qquad I_{\max}\to \text{Maximum value of current}$$

The current approached its maximum value asymptotically with time. The following the graph of the current as a function of time.



The value $\frac{R}{L}$ determines how fast the current approaches its maximum value and is called the time constant the current. The greater the time constant the faster the current approaches its maximum value.

The value across the resistor as a function of time is given by

$$V_R(t)=IR=R\frac{\varepsilon}{R}\left(1-e^{-\frac{R}{L}t}\right)$$

$$\Rightarrow\ \boxed{V_R(t)=\varepsilon\left(1-e^{-\frac{R}{L}t}\right)}$$

The self-induced εmf across the inductor as a function of time is given by

$$\varepsilon_{self} = \left| -L\frac{dI}{dt} \right|$$

$$= L\frac{d}{dt}\left\{ \frac{\varepsilon}{R}\left(1 - e^{-\frac{R}{L}t} \right) \right\}$$

$$= L\frac{\varepsilon}{R}\frac{d}{dt}\left\{ -e^{-\frac{R}{L}t} \right\}$$

$$= L\frac{\varepsilon}{R}\frac{R}{L}e^{-\frac{R}{L}t}$$

$$\boxed{\varepsilon_{self}(t) = \varepsilon e^{-\frac{R}{L}t}}$$

The sum of $V_R(t)$ and ε_{self} is always ε as expected

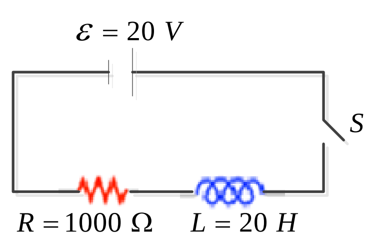
$$V_R + \varepsilon_{self} = \varepsilon\left(1 - e^{-\frac{R}{L}t} \right) + \varepsilon e^{-\frac{R}{L}t} \underline{= \varepsilon}$$

Example

A series combination of a 1000 Ω resistor and a 20H inductor is connected to a battery of εmf 20V.

a) Calculate the maximum current
b) Calculate the time constant of the circuit
c) Calculate the current in the circuit 0.1 second after the switch is closed
d) Calculate the voltages across the resistor and inducator after 0,01 second

Solution



a) $I_{\max} = \frac{\varepsilon}{R} = \frac{20}{1000} \underline{= 0.02A}$

b) $Time\ constant = \frac{R}{L} = \frac{1000}{20} \underline{= 50\ \frac{\Omega}{H}}$

c) $I(t) = \frac{\varepsilon}{R}\left(1 - e^{-\frac{R}{L}t} \right)$

$$= 0.02\left(1 - e^{-50t} \right)$$

$$I(t = 0.1) = 0.02\left(1 - e^{-50(0.1)} \right) \underline{= 0.1987\ A}$$

d) $V_R(t) = \varepsilon\left(1 - e^{-\frac{R}{L}t} \right) = 20\left(1 - e^{-50t} \right)$

$$V_R\left(t=0.01\right)=20\left(1-e^{-50(0.01)}\right)\ \underline{\approx 7.869\ \ V}$$

$$\varepsilon_{self}\left(t\right)=\varepsilon e^{-\frac{R}{L}t}=20e^{-50t}$$

$$\varepsilon_{self}\left(t=0.01\right)=20e^{-50(0.01)}\ \underline{\approx 12.131\ V}$$

Note: $V_R+\varepsilon_{self}=7.869+12.131=20\ \underline{=\varepsilon}\ \ \surd$

Mutual Induction

When two coils carrying time dependent currents are in the vicinity of each other, they will induce εmf on each other. This kind of induction is called ***mutual induction***.

Consider two coils, coil 1 & coil 2, in the vicinity of each other, the mutual inductance of coil 2 with respect to coil 1 (M_{12}) is defined to be the ratio between the flux crossing coil 2 due to the current in coil 1. If the number of turns of coil 2 is N_2 and the magnetic flux per coil crossing coil 2 is ϕ_{12}, then

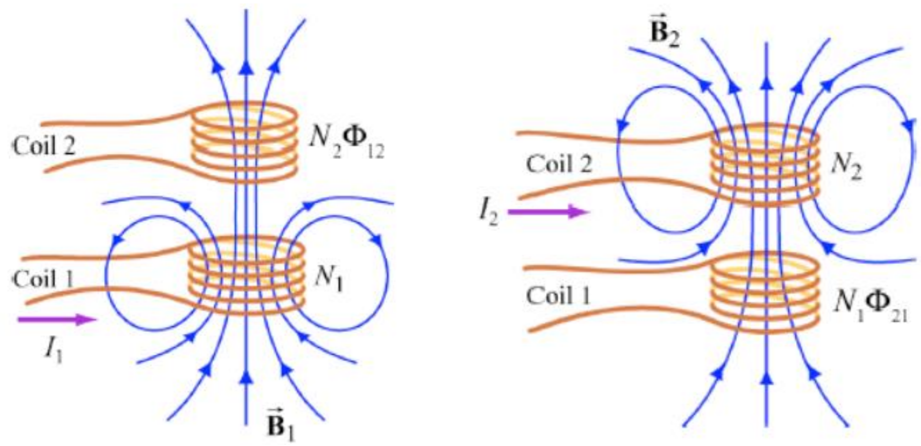
$$\boxed{M_{12} = \frac{N_2 \phi_{12}}{I_1}}$$

$M_{12} \rightarrow$ Mutual inductance of coil 2 with respect to coil 1

$N_2 \rightarrow$ Number of turns of coil 2

$\phi_{12} \rightarrow$ Flux per turn due to current in coil 1 crossing coil 2

$I_1 \rightarrow$ Current in coil 1



Similarly, the mutual inductance of coil 1 with respect to coil 2 is defined to be

$$\boxed{M_{21} = \frac{N_1 \phi_{21}}{I_2}}$$

It can be shown that $M_{12} = M_{21} = M$ where M is called the mutual inductance of the coils.

$$\boxed{M = \frac{N_1 \phi_{21}}{I_2} = \frac{N_2 \phi_{12}}{I_1}}$$

Applying Faraday's law, the εmf induced in coil 2 (ε_2) is given by

$$\varepsilon_2 = -N_2 \frac{d\phi_{12}}{dt} \quad but \quad \phi_{12} = \frac{MI_1}{N_2}$$

$$\varepsilon_2 = -N_2 \frac{d}{dt}\left\{\frac{MI_1}{N_2}\right\}$$

$$\boxed{\varepsilon_2 = M\frac{dI_1}{dt}}$$

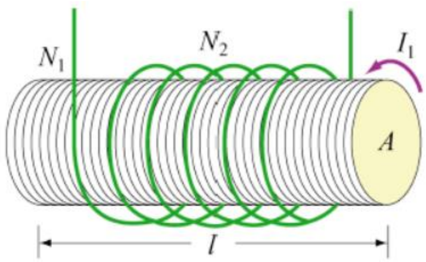
$\varepsilon_2 \to \varepsilon mf$ induced in coil 2 due to current in coil 1

Similarly, the εmf induced in coil 1 (ε_1) due to coil 2 is given by

$$\boxed{\varepsilon_1 = M\frac{dI_2}{dt}}$$

Example

Consider the two concentric solenoid shown with the inner solenoid being coil 1 & the outer being coil 2.



Obtain an expression for the mutual inductance of the coils.

Solution

$$M = M_{12} = \frac{N_2\phi_{12}}{I_1}$$

$$\phi_{12} = B_1 A$$

where A is the area of the inner coil is used because B_1 is approximately zero outside coil 1

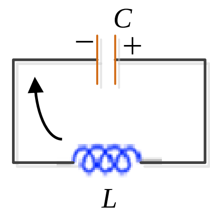
$$B_1 = \frac{\mu_0 N_1 I_1}{\ell}$$

$$\phi_{12} = \frac{\mu_0 N_1 I_1 A}{\ell}$$

$$M = \frac{N_2\phi_{12}}{I_1} = \frac{\mu_0 N_1 N_2 I_1 A}{\ell\, I_1}$$

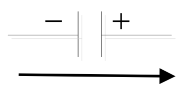
$$\boxed{M = \frac{\mu_0 N_1 N_2 A}{\ell}}$$

A charged Capacitor Connected to an inductor

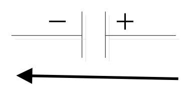


First let's state the sign conventions for the potential differences across a capacitor & an indicator in applying Kirchhoff's loop rule.

Capacitors: As a capacitor is transversed from its negative to its positive terminal, it is taken to be positive if it is being changes and negative if it is being discharge and when it is transversed from its positive to its negative terminal, its potential difference is taken to be negative if being charges and positive if being discharged.



Positve for charging
Negative for discharging



Positve for discharging
Negative for charging

Inductors: When an inductor is transeversed in the direction of the current, its potential field is difference is taken to be $-L\dfrac{dI}{dt}$, and when transerversed opposite to the direction of the current, its potential difference is taken to be $L\dfrac{dI}{dt}$.

Applying these sign conventions to Kirchhoff's loop rule to LC circuit shown using the transersing direction shown, the following equation is obtained

$$-\frac{Q}{C}-L\frac{dI}{dt}=0 \qquad \left(V_C=\frac{Q}{C}\right)$$

$$L\frac{dI}{dt}+\frac{Q}{C}=0$$

$$\frac{dI}{dt}+\frac{1}{LC}Q=0 \qquad \text{but } I=\frac{dQ}{dt}$$

$$\frac{d}{dt}\left(\frac{dQ}{dt}\right)+\frac{1}{LC}Q=0$$

$$\Rightarrow \boxed{\frac{d^2Q}{dt^2}+\frac{1}{LC}Q=0}$$

It can be shown that the general solution of this equation is

$$Q=C_1\cos\left(\sqrt{\frac{1}{LC}}\,t+C_2\right)$$

Where C_1 & C_2 are arbitray constants. C_1 & C_2 can be calculated from intial conditions.

Let the initial charge of the capacitor be Q_{max} i.e $Q(t=0)=Q_{max}$

$$Q(t=0)=C_1\cos\left(\sqrt{\frac{1}{LC}}\,(0)+C_2\right)$$

$$Q_{max}=C_1\cos(C_2) \qquad (1)$$

$$I=\frac{dQ}{dt}=-\frac{C_1}{\sqrt{LC}}\sin\left(\sqrt{\frac{1}{LC}}\,t+C_2\right)$$

$$I(t=0)=-\frac{C_1}{\sqrt{LC}}\sin\left(\sqrt{\frac{1}{LC}}\,(0)+C_2\right)$$

$$0=-\frac{C_1}{\sqrt{LC}}\sin(C_2) \qquad (2)$$

$$\frac{(2)}{(1)}\rightarrow \quad -\frac{1}{\sqrt{LC}}\tan(C_2)=0$$

$$\tan(C_2)=0 \quad \Rightarrow \; C_2=0$$

$$(1) \;\Rightarrow\; Q_{max}=C_1\cos(C_2)=C_1\cos(0)$$

$$\Rightarrow \; Q_{max}=C_1$$

Therfore the solution is

$$\boxed{Q(t)=Q_{max}\cos\left(\frac{1}{\sqrt{LC}}\,t\right)}$$

$$\boxed{I(t)=-\frac{Q_{max}}{\sqrt{LC}}\sin\left(\frac{1}{\sqrt{LC}}\,t\right)}$$

The coefficient of time $\left(\frac{1}{\sqrt{LC}}\right)$ represents the angular frequency (number of radians executed per second) & is denoted by ω

$$\boxed{\omega=\frac{1}{\sqrt{LC}}} \qquad \omega \rightarrow \text{Angular frquency of an } LC \text{ circuit}$$

The frequency (Number of cycles per second) is related with angular frequency by $\omega=2\pi f$ (since there are 2π radians in a cycle). Therefore

$$\boxed{f=\frac{1}{2\pi}\sqrt{\frac{1}{LC}}} \qquad f \rightarrow \text{Frquency of an } LC \text{ circuit}$$

The period (time taken for one cycle) is the inverse of the frequency

$$\boxed{T = 2\pi\sqrt{LC}} \qquad T \to \text{Period of an } LC \text{ circiut}$$

Example

Show by substitution that $Q(t) = Q_{\max}\cos\left(\frac{1}{\sqrt{LC}}t\right)$ is a solution to the equation of an LC circuit

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

Solution

$$Q = Q_{\max}\cos\left(\frac{1}{\sqrt{LC}}t\right)$$

$$\frac{dQ}{dt} = -\frac{Q_{\max}}{\sqrt{LC}}\sin\left(\frac{1}{\sqrt{LC}}t\right)$$

$$\frac{d^2Q}{dt^2} = -\frac{Q_{\max}}{LC}\cos\left(\frac{1}{\sqrt{LC}}t\right)$$

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

$$-\frac{Q_{\max}}{LC}\cos\left(\frac{1}{\sqrt{LC}}t\right) + \frac{Q_{\max}}{LC}\cos\left(\frac{1}{\sqrt{LC}}t\right) \overset{?}{=} 0$$

$0 = 0$ $\checkmark$ proved

Example

A 20 F capacitor is connected to a 10 V battery. Then it is disconnected from the battery and connected to a 5H inductor. Then it is disconnected from the battery and connected to a 5 H inductor.

a) Calculate the maximum charge of the capacitor
b) Calculate the angular frequency of the circuit
c) How many cycles does the circuit execute per second
d) Calculate the time taken for one cycle
e) Give expressions for the charge & current as a function of time
f) Show the graph of charge as a function of time & current as a function of time

Solution

Given: $L = 5H, \quad C = 20F, \quad V = 10$

a) $Q_{\max} = CV = (20)(10) \underline{= 200\ C}$

b) $\omega = \dfrac{1}{\sqrt{LC}} = \dfrac{1}{\sqrt{100}} = 0.1\ rad/s$

c) $f = \dfrac{1}{2\pi}\sqrt{\dfrac{1}{LC}} = \dfrac{1}{2\pi}\sqrt{\dfrac{1}{100}} = \dfrac{1}{20\pi}$

d) $T = \dfrac{1}{f} = 20\pi\ s$

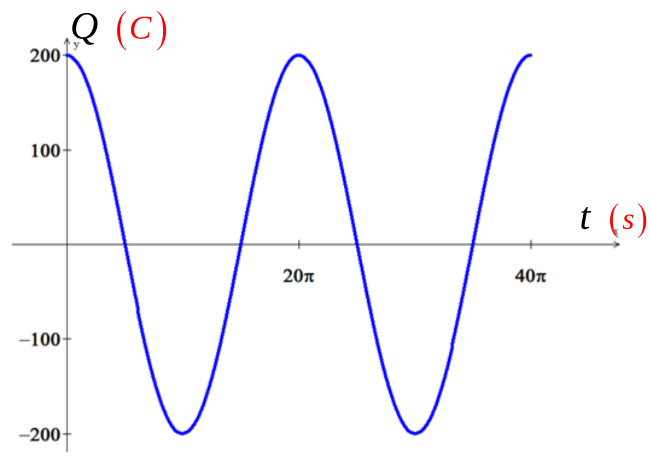
e) $Q = Q_{\max}\cos(\omega\, t)$

$Q = 200\cos(0.1\, t)\ \ C$

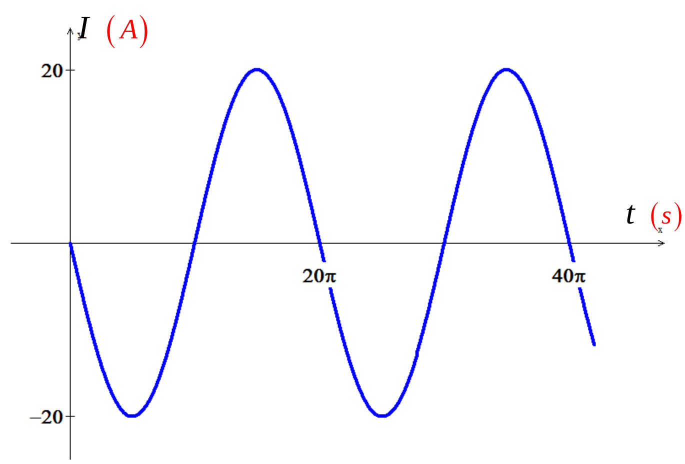
$$I(t) = -Q_{\max}\,\omega\,\sin(\omega t)$$
$$= -(200)(0.1)\sin(0.1t)$$
$$= -20\sin(0.1t)\ \ A$$

f)

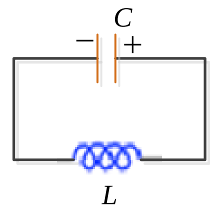
$Q = 200\cos(0.1\, t)$



$I(t) = -20\sin(0.1t)$



Energy of an LC Circuit



From Kirchhoff's loop rule

$$\frac{Q}{C} + L\frac{dI}{dt} = 0$$

$$L\frac{dI}{dt} = -\frac{Q}{C}$$

Myltiply the right side by $\frac{dQ}{dQ}$

$$L\frac{dI}{dt}\frac{dQ}{dQ} = -\frac{Q}{C}$$

$$L\frac{dI}{dQ}\frac{dQ}{dt} = -\frac{Q}{C} \qquad but \quad \frac{dQ}{dt} = I$$

$$LIdI = -\frac{Q}{C}dQ$$

$$\int LIdI = -\int \frac{Q}{C}dQ$$

$$\frac{1}{2}LI^2 = -\frac{1}{2}\frac{1}{C}Q^2 + constant$$

$$\frac{1}{2}LI^2 + \frac{1}{2}\frac{Q^2}{C} = constant$$

But $\frac{1}{2}LI^2$ represents the magnetic energy stored by the capacitor & $\frac{1}{2}\frac{Q^2}{C}$ represents the electrical energy stored by the capacitor.

THerofre the expression $\frac{1}{2}LI^2 + \frac{1}{2}\frac{1}{C}Q^2$ represents the total energy of the circuit. Thus it follows that the total energy of the circuit is a constant. The process involves interchange between magnetic and electrical energy but bot less of energy.

Since the energy is a constant, it should be equal to the initial energy stored in the capacitor which is $\frac{Q^2_{max}}{2C}$

$$E = \frac{1}{2}LI^2 + \frac{1}{2}\frac{Q^2}{C} = \frac{Q^2_{max}}{2C} \qquad E \rightarrow \text{Total energy of the circuit.}$$

Again since the energy is constant, the rate of dissipation of energy (power) for an LC circuit is zero

$$p = \frac{dE}{dt} = \frac{d}{dt}\left(\frac{Q^2_{max}}{2C}\right) = 0$$

Unlike a resistor, there is no dissipation of energy in a capacitor & an inductor.

Example

Use the expressions for the charge and the current to show that the energy of an LC circuit is always equal to $\frac{Q_{max}^2}{2C}$, independent of time.

Solution

$$Q = Q_{max}\cos\left(\frac{1}{\sqrt{LC}}\,t\right)$$

$$I = -\frac{Q_{max}}{\sqrt{LC}}\sin\left(\frac{1}{\sqrt{LC}}\,t\right)$$

$$E = \frac{1}{2}LI^2 + \frac{1}{2}\frac{Q^2}{C}$$

$$= \frac{1}{2}L\left[-\frac{Q_{max}}{\sqrt{LC}}\sin\left(\frac{1}{\sqrt{LC}}\,t\right)\right]^2 + \frac{1}{2C}\left[Q_{max}\cos\left(\frac{1}{\sqrt{LC}}\,t\right)\right]^2$$

$$= \frac{1}{2C}Q_{max}^2\sin^2\left(\frac{1}{\sqrt{LC}}\,t\right) + \frac{1}{2C}Q_{max}^2\cos^2\left(\frac{1}{\sqrt{LC}}\,t\right)$$

$$= \frac{1}{2C}Q_{max}^2\left\{\sin^2\left(\frac{1}{\sqrt{LC}}\,t\right) + \cos^2\left(\frac{1}{\sqrt{LC}}\,t\right)\right\} \qquad \sin^2\theta + \cos^2\theta = 1$$

$$= \frac{Q_{max}^2}{2C} \quad \checkmark$$

Example

A 16F capacitor is connected to an 8V battery and disconnected from the battery. Then it is connected to a 4H inductor.

a) Calculate the total energy of the circuit at any time
b) Calculate the current in the circuit when the charge is half of the maximum value
c) Calculate the maximum value of the current

Solution

Given: $L = 4H, \quad C = 16F, \quad V = 8V$

a) $E = \frac{Q_{max}^2}{2C}$

$$Q_{max} = Q(t=0) = VC = (8)(16) \underline{= 128\ C}$$

$$E = \frac{128^2}{2(16)} \underline{= 512\ \ J}$$

b) $Q=\frac{1}{2}Q_{max}=\frac{1}{2}(128)\ \underline{=64\ C|}$

$$E=\frac{1}{2}LI^2+\frac{1}{2}\frac{Q^2}{C}=512\ J$$

$$\frac{1}{2}(4)I^2+\frac{1}{2}\frac{64^2}{16}=512$$

$$\frac{1}{2}(4)I^2=384$$

$$I^2=\frac{384}{2}=192$$

$$I=\sqrt{192}\ \underline{=8\sqrt{3}\ \ A|}$$

c) The maximum value of the current occures when $Q=0$

$$\frac{1}{2}LI^2+\frac{1}{2}\frac{Q^2}{C}=512\ J$$

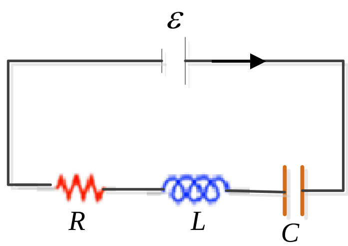
$$\frac{1}{2}(4)I_{max}^2=512$$

$$I_{max}^2=\frac{512}{2}=256$$

$$I_{max}=\sqrt{256}\ \underline{=16\ A|}$$

Series *RLC* Circuit connected to a *DC* source

Consider a battery of εmf ε connected to a series combination of a resistor of resistance *R*, an inductor of inductance *L* and a capacitor of capacitance *C*.



Applying Kichhoff's loop rule, the following equation holds

$$\varepsilon - IR - L\frac{dI}{dt} - \frac{1}{C}Q = 0$$

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = \varepsilon \qquad but \quad I = \frac{dQ}{dt}$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = \varepsilon$$

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = \frac{\varepsilon}{L}$$

But $\frac{R}{L} = 2\beta$ where β is the damping constant

And Let $\frac{1}{LC} = \omega_0^2$ where ω_0 is the natural frequency which is the angular frequency of an LC circuit.

$$\frac{d^2Q}{dt^2} + 2\beta\frac{dQ}{dt} + \omega_0^2 Q = \frac{\varepsilon}{L}$$

The general solution of this equation is the sum of the general solution of the homogeneous equation

$$\frac{d^2Q}{dt^2} + 2\beta\frac{dQ}{dt} + \omega_0^2 Q = 0$$

And a particular solution to the inhomogeneous equation (the actual equation)

The particular solution can be cahnge that is independent of time. If Q_P is independent of time, then

$$\frac{d^2Q_P}{dt^2} = 0 \text{ and } \frac{dQ_P}{dt} = 0.$$

It follows that

$$\omega_0^2 Q_P = \frac{\varepsilon}{L}$$

$$Q_P = \frac{\varepsilon}{\omega_0^2 L} \qquad but \quad \omega_0 = \frac{1}{LC}$$

$\boxed{Q_P = \varepsilon C}$ → A particular solution to the inhomogeneous equation

The homogeneous equation is a 2nd order differential equation with constant coefficients. Its solution has been discussed in detail when leading with a damped harmonic motion. There are 3 tpes of solutions based on the value of $\beta^2 - \omega_0^2$

1- *Underdamped oscillations*

Under damped oscillation occures when $\beta^2 - \omega_0^2 < 0$; and the solution to the homogeneous equation, Q_h is given as

$$Q_h = Ae^{-\beta t}\cos(\omega t - \phi)$$

Where A & ϕ are constants and $\omega = \sqrt{\beta^2 - \omega_0^2}$.

Therefore the general solution to the inhomogeneous equation $Q = Q_h + Q_P$ is given by

$$Q = Ae^{-\beta t}\cos(\omega t - \phi) + \varepsilon C$$

And

$$\begin{aligned} I = \frac{dQ}{dt} &= -\beta Ae^{-\beta t}\cos(\omega t - \phi) - A\omega e^{-\beta t}\sin(\omega t - \phi) \\ &= -Ae^{-\beta t}\left[\beta\cos(\omega t - \phi) + \omega\sin(\omega t - \phi)\right] \end{aligned}$$

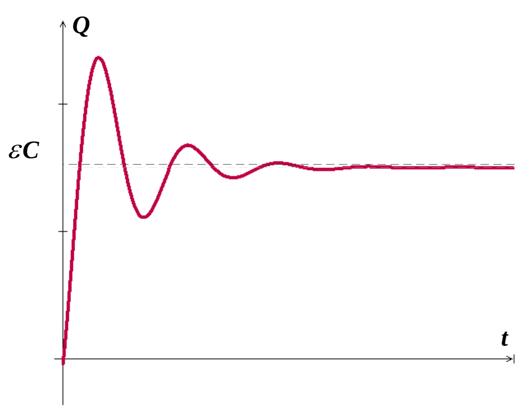
The cosine and sine can be combined nto a single cosine with thetransformation equation for β & ω as $\beta = c_1\cos\delta$ & $\omega = c_1\sin\delta$ where c_1 & δ are constants.

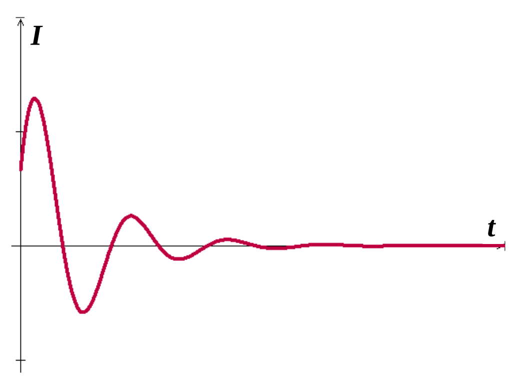
$$\begin{aligned} I &= -Ae^{-\beta t}\left[c_1\cos\delta\cos(\omega t - \phi) + c_1\sin\delta\sin(\omega t - \phi)\right] \\ &= -Ac_1e^{-\beta t}\left[\cos\delta\cos(\omega t - \phi) + \sin\delta\sin(\omega t - \phi)\right] \\ &= -Ac_1e^{-\beta t}\cos(\omega t - \phi - \delta) \end{aligned}$$

And with $-Ac_1 = A'$ & $\phi + \delta = \phi'$

$$\boxed{I = A'e^{-\beta t}\cos(\omega t - \phi')}$$

Note that the charge and the current have the same frequency and the same damping. The charge approaches εC as $t \to \infty$ & the current approaches 0 as $t \to \infty$.





2. Overdamped Oscillation

Overdamped oscillation occurs when $\beta^2 - \omega_0^2 > 0$; and the solution to the homogeneous equation, Q_h is given as

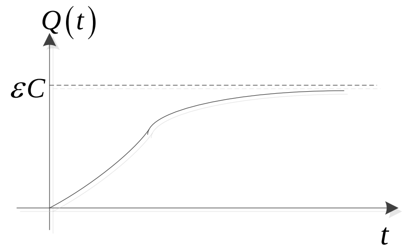
$$Q_h = c_1 e^{\sqrt{\left(-\beta+\beta^2-\omega_0^2\right)}t} + c_2 e^{\left(-\beta-\sqrt{\beta^2-\omega_0^2}\right)t}$$

Where c_1 & c_2 are arbitrary constants.

The general solution to the inhomogenous equation $\left(Q = Q_h + Q_P\right)$ is given as

$$Q = c_1 e^{\sqrt{\left(-\beta+\beta^2-\omega_0^2\right)}t} + c_2 e^{\left(-\beta-\sqrt{\beta^2-\omega_0^2}\right)t} + \varepsilon C$$

The charge approaches the value εC & slowly as time increases without oscillation



3. Critically damped oscillation

Critically damped oscillation occurs when $\beta^2 - \omega_0^2 = 0$; and the solution to the homogeneous equation, Q_h is given as

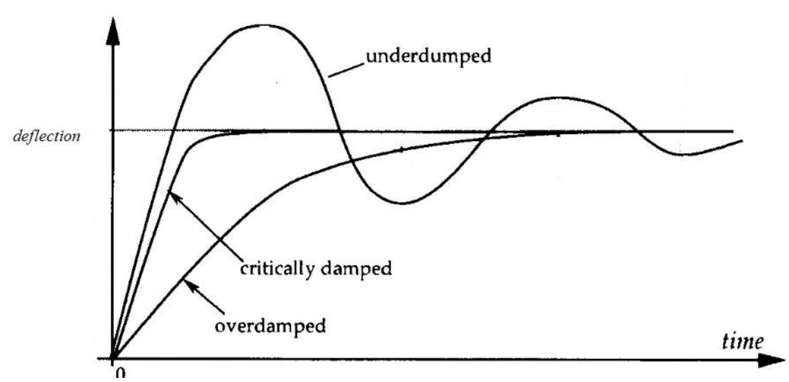
$$Q_h = \left(c_1 + c_2 t\right)e^{-\beta t}$$

The general solution to the inhomogenous equation $\left(Q = Q_h + Q_P\right)$ is given as

$$Q = \left(c_1 + c_2 t\right)e^{-\beta t} + \varepsilon C$$

Where c_1 & c_2 are arbitrary constants.

The charge qpproaches the value εC as time increases faster then an overdamped oscillation does.



Example

A $2\mu F$ capacitor, a $4mH$ inducatorand a variable resistor are connected to a DC source.

a) Calculate the natural frequency of the circuit
b) What should be the value of the resistance if the oscillation is to be critically damped.
c) Find the range of resistance that will result in a damped harmonic oscillation.
d) If the resistance is half of the resistance that leads to critical damping, calculate the frequency of the resulting damped harmonic oscillation

Solution

Given: $C = 2\times10^{-6}F \quad L = 4\times10^{-3}H$

a) $\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(4\times10^{-3})(2\times10^{-6})}} \approx 11180 \ rad/s$

$$f_0 = \frac{\omega_0}{2\pi} = \frac{11180}{2\pi} = 1779 \ Hz$$

b) Critically damped $\Rightarrow \beta^2 = \omega_0^2$ but $\omega_0^2 = \frac{1}{LC}$ & $\beta = \frac{R}{2L}$

$$\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$$

$$\frac{R}{2L} = \sqrt{\frac{1}{LC}}$$

$$R = 2L\sqrt{\frac{1}{LC}} = 2(4\times10^{-3})(11180) = 89.44 \ \Omega$$

c) Damped oscillation $\Rightarrow \beta^2 - \omega_0^2 < 0 \quad \to \quad \beta < \omega_0$

$$\frac{R}{2L} < \omega_0$$

$$R < 2L\omega_0 = 2(4\times10^{-3})(11180) = 89.44 \ \Omega$$

d) $R = \frac{1}{2}(89.44) = 44.72 \ \Omega$ & $\omega = \sqrt{\omega_0^2 - \beta^2}$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\omega_0^2 - \beta^2}$$

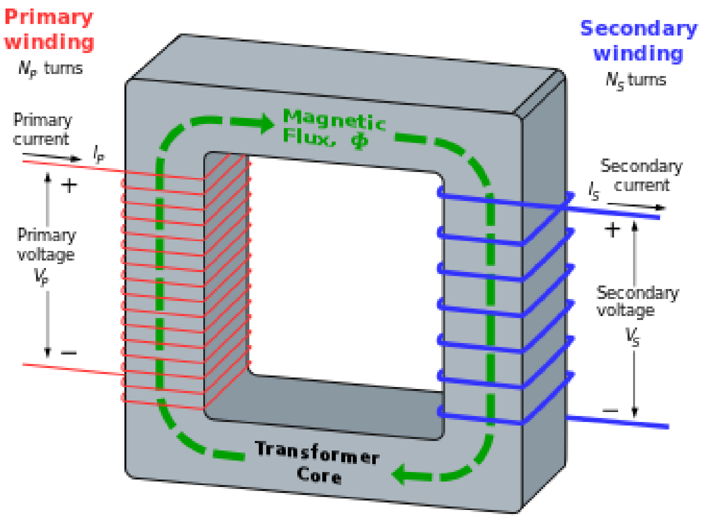
$$= \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{(4\times10^{-3})(2\times10^{-6})} - \left(\frac{44.72}{2(4\times10^{-3})}\right)^2}$$

$$= 1541 \ Hz$$

Transformer

The transformer is a device used to increase or decrease the amplitude of *ac* voltages. It consists two coils wound on a magentic matrial such as iron as shown



One of the coils called the primary coil is connected to an ac source. (An ac voltage is time varying voltage the varies with time typically like a sine or cosine)
The second coil which gives the output is called the secondary coil.

The current in the primary coil produces magnetic field in the magnetic material uniformly. Thus this field crosses both coils. Since the current is changing with time (and hence the magnetic field), according to Faraday's law, there will be induced εmfs in both coils given by

$$\varepsilon_1 = -N_1 \frac{d\phi_1}{dt} \quad and \quad \varepsilon_2 = -N_2 \frac{d\phi_2}{dt}$$

Where N_1 & N_2 are number of turns in the primary and secondary coils respectively; and ε_1 & ε_2 are induced voltages in coil 1 & coil 2 resprectively.
From Kirchhoff's rule ε_1 is equal to the input voltage 1 ϕ_1 & ϕ_2 are fluxes crossing coil 1 & coil 2 respectively.
But since both coils are crossed by the same magnetic field & the cross-sectional area of the magnetic material is the same magnetic material is the same for both, ϕ_1 & ϕ_2 are equal $(\phi_1 = \phi_2 = BA)$. Hence the $\varepsilon mf's$ may be written as

$$\frac{\varepsilon_1}{N_1} = -\frac{d\phi_1}{dt} \quad and \quad \frac{\varepsilon_2}{N_2} = -\frac{d\phi_2}{dt}$$

Which implies $\frac{\varepsilon_1}{N_1} = \frac{\varepsilon_2}{N_2}$ which is customarily written as

$$\boxed{\frac{\varepsilon_1}{\varepsilon_2} = \frac{N_1}{N_2}}$$

$$\Rightarrow \quad \varepsilon_2 = \frac{N_2}{N_1}\varepsilon_1$$

This implies that the output voltage can be adjusted to a desired voltage by the coil of a suitable $\frac{N_2}{N_1}$ ratio.

A transformer whose effect is to increase voltage with $N_2 > N_1$ is called a setup transformer; and a transformer whose effect is to decrease voltage with $N_2 < N_1$ is called s stepdoen transfomer.

Example

The primary coil and the secondary coil of a transformer have 50 and 125 number of turns respectively. The primary coil is connected to an ac voltage of amplitude 10*V*.

a) Calculate the amplitude of the voltage obtained from the secondary coil.

b) Is this a setp up or step down transformer?

Solution

Given: $N_1 = 50, \quad N_2 = 125, \quad \varepsilon_1 = 10$

a) $\varepsilon_2 = \frac{N_2}{N_1}\varepsilon_1 = \frac{125}{50}(10) \underline{= 25 \ \ V}$

b) Since its effect is to increase the amplitude of the voltage, it is a setp up transformer.

	RLC Circuit	Damped Harmonic Oscillator
Variable s	Q	x
Variable ds/dt	$\pm I$	v
Coefficient of s	$1/C$	k
Coefficient of ds/dt	R	b
Coefficient of d^2s/dt^2	L	m
Energy	$LI^2/2$	$mv^2/2$
	$Q^2/2C$	$kx^2/2$