SOLUTION Section 4.5 – Partial Orderings

Exercise

Which of these relations on $\{0, 1, 2, 3\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.

a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$
b) $\{(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$
c) $\{(0, 0), (1, 1), (1, 2), (2, 2), (3, 3)\}$
d) $\{(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$
e) $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$
f) $\{(0, 0), (2, 2), (3, 3)\}$
g) $\{(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 3)\}$
h) $\{(0, 0), (1, 1), (1, 2), (2, 2), (3, 1), (3, 3)\}$
i) $\{(0, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (2, 3), (3, 0), (3, 3)\}$
j) $\{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 3)\}$

Solution

a) This relation is reflexive because each of 0, 1, 2, 3 is related to itself.
This relation is antisymmetric because a to be related to b is for a to be equal to b
Since a is related to b and b related to c and $a = b = c$, then a is related to c. So the relation is transitive.
The equality relation on any set satisfies all three conditions, therefore is a partial ordering.

b) It is reflexive but it is not antisymmetric since we have $2R3$ and $3R2$ but $2 \neq 3$.
Therefore, this is not a partial ordering.

c) This relation is reflexive because each of 0, 1, 2, 3 is related to itself.
This relation is antisymmetric because a to be related to b is for a to be equal to b
It is transitive for the same reason and $1R1 \text{ and } 1R2 \quad \Rightarrow 1R2$
Therefore, is a partial ordering.

d) This relation is reflexive because each of 0, 1, 2, 3 is related to itself.
This relation is antisymmetric because a to be related to b is for a to be equal to b
It is transitive for the same reason and $1R1 \text{ and } 1R2 \quad \Rightarrow 1R2$, $1R3 \text{ and } 3R3 \quad \Rightarrow 1R3$, and
$2R3 \text{ and } 3R3 \quad \Rightarrow 2R3$
Therefore, is a partial ordering.

e) It is reflexive but it is not antisymmetric since we have $0R1$ and $1R0$ but $0 \neq 1$.
Therefore, this is not a partial ordering.

f) Since 1 is not related to itself, so this relation is not reflexive.
Therefore, R is not a partial ordering.

g) This relation is reflexive because each of 0, 1, 2, 3 is related to itself.
This relation is antisymmetric because a to be related to b is for a to be equal to b
It is transitive for the same reason and $2R0 \text{ and } 0R0 \quad \Rightarrow 2R0$, and $2R3 \text{ and } 3R3 \quad \Rightarrow 2R3$

Therefore, is a partial ordering.

h) Since $3R1 \; and \; 1R2 \quad \Rightarrow \cancel{3R2}$, so this relation is not transitive.
Therefore, R is not a partial ordering.

i) Since $1R2 \; and \; 2R0 \quad \Rightarrow \cancel{1R0}$, so this relation is not transitive.
Therefore, R is not a partial ordering.

j) Since $0R1 \; and \; 1R0$ but $0 \neq 1$, so this relation is not antisymmetric and it is not transitive because $2R0 \; and \; 0R1 \quad \Rightarrow \cancel{2R1}$.
Therefore, R is not a partial ordering.

Exercise

Is (*S, R*) a poset If *S* is the set of all people in the world and $(a, \; b) \in R$, where *a* and *b* are people, if

a) a is a taller than *b*?
b) a is not taller than *b*?
c) $a = b$ or a is an ancestor of *b*?
d) a and *b* have a common friend?
e) a is a shorter than *b*?
f) a weighs more than *b*?
g) $a = b$ or a is a descendant of *b*?
h) a and *b* do not have a common friend?

Solution

a) Since nobody is taller than himself, this relation is not reflexive, so (*S, R*) is not a poset.

b) To be not a taller means exactly the same height or shorter. 2 different people *x* and *y* could have the same height, in which case $xRy \; and \; yRx \; but \; x \neq y$, so *R* is not antisymmetric.
Therefore, this relation is not a poset.

c) The equality clause in the given of *R* guarantees that *R* is reflexive.
If *a* is ancestor to *b*, then *b* can't be ancestor to *a*, so the relation is vacuously antisymmetric.
If *a* is ancestor to *b* and *b* is ancestor to *c*, then *a* is ancestor to *c*, thus *R* is transitive.
Therefore, this relation is a poset.

d) Let *x* and *y* be any 2 distinct friends, $xRy \; and \; yRx \; but \; x \neq y$, so *R* is not antisymmetric.
Therefore, this relation is not a poset.

e) Let 2 people can be the same height since are not the same person, so *R* is not antisymmetric.
Therefore, this relation is not a poset.

f) Since nobody is weight more than himself, this relation is not reflexive, so this relation is not a poset.

g) Since $a = a$, then the *R* is reflexive.

Given that $a = b$ but if a is a descendant of b, then b cannot be a descendant of a. So, the relation is vacuously antisymmetric.
if a is a descendant of b and b is a descendant of c, then a is a descendant of c. So, the R is transitive.
Therefore, this relation is a poset.

h) Since anyone and himself have a common friend, then this relation is not reflexive, so this relation is not a poset.

Exercise

Which of these are posets?

a) $(\mathbb{Z}, =)$ *b)* $(Z,\ \neq)$ *c)* $(Z,\ \geq)$ *d)* $(Z,\ \nmid)$

e) $(R,\ =)$ *f)* $(R,\ <)$ *g)* $(R,\ \leq)$ *h)* $(R,\ \neq)$

Solution

a) The equality relation of any set satisfies all three conditions. Therefore, a partial order.

b) This is not a poset since the relation is not reflexive $(a \neq a)$

c) The relation is reflexive since the relation involved the equality sign.

d) This is not a poset since the relation is not reflexive $(2 \nmid 2)$

e) The equality relation of any set satisfies all three conditions. Therefore, a partial order.

f) This is not a poset since the relation is not reflexive $(2 \nless 2)$

g) The relation is reflexive since the relation involved the equality sign.

h) This is not a poset since the relation is not reflexive $(2 = 2)$

It is not antisymmetric since $1R2$ *and* $2R1$ *but* $1 \neq 2$

It is not transitive $1R2$ *and* $2R1$ *but* $1 = 1 \Rightarrow 1 \not{R} 1$

Exercise

Determine whether the relations represented by these zero-one matrices are partial orders

a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *b)* $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *c)* $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *d)* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ *f)* $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

Solution

a) The relation is $\{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (2,\ 1),\ (2,\ 2),\ (3,\ 3)\}$

This is not antisymmetric because $1R2 \quad and \quad 2R1 \quad but\ 1 \neq 2$.

Therefore, this matrix is not a partial order.

b) The relation is $\{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (2,\ 2),\ (3,\ 3)\}$

It is clearly reflexive.

The pairs (1, 2) and (1, 3) are in the relation that neither can be part of a counterexample to antisymmetry or transitivity.

c) The relation is $\{(1,\ 1),\ (1,\ 3),\ (2,\ 1),\ (2,\ 2),\ (3,\ 3)\}$

It is clearly reflexive. The pairs (1, 3) and (2, 1) are in the relation that neither can be part of a counterexample to antisymmetry.

It is not transitive since, (2, 1) and (1, 3) that will lead to (2, 3) which is not in the relation.

Therefore, this matrix is not a partial order.

d) The relation is $\{(1,\ 1),\ (2,\ 2),\ (3,\ 1),\ (3,\ 3)\}$

It is clearly reflexive.

The pair (3, 1) is in the relation that can't be part of a counterexample to antisymmetry or transitivity.

e) The relation is $\{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (2,\ 2),\ (2,\ 3),\ (3,\ 3),\ (3,\ 4),\ (4,\ 1),\ (4,\ 2),\ (4,\ 4)\}$

It is not transitive since, (4, 1) and (1, 3) are in the relation but not (4, 3).

Therefore, this matrix is not a partial order.

f) The relation is $\{(1,\ 1),\ (1,\ 3),\ (2,\ 2),\ (2,\ 3),\ (3,\ 3),\ (3,\ 4),\ (4,\ 1),\ (4,\ 2),\ (4,\ 4)\}$

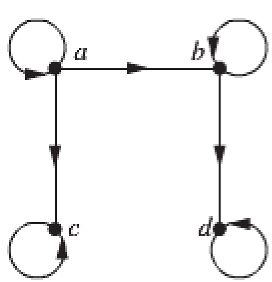
It is not transitive since, (4, 1) and (1, 3) are in the relation but not (4, 3).

Therefore, this matrix is not a partial order.

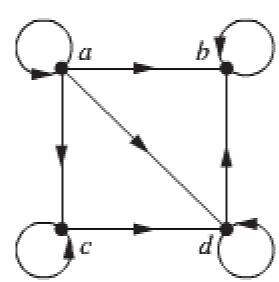
Exercise

Determine whether the relation with the directed graph shown is a partial order.

a)



b)



c)

Solution

a) This is relation is not transitive since there no relation (arrow) between *a* and *d*.

$aRb \text{ and } bRd \Rightarrow \cancel{aRd}$

b) This is relation is not transitive since there no relation (arrow) from *c* and *b*.

c) This relation is reflexive since all points have an arrow to itself.
This relation is antisymmetric since no pair of arrows going in opposite directions between 2 different points.
Therefore, this relation is a partial order.

Exercise

Let $(S,\ R)$ be a poset. Show that $\left(S,\ R^{-1}\right)$ is also a poset, where R^{-1} is the inverse of *R*. The poset $\left(S,\ R^{-1}\right)$ is called the dual of $(S,\ R)$.

Solution

Since *R* is reflexive, then R^{-1} is clearly reflexive.

Suppose that $(a,b) \in R^{-1}$ and $a \neq b$. Then $(b,a) \in R$, so $(a,b) \notin R$, so $(b,a) \notin R^{-1}$

If $(a,b) \in R^{-1}$ and $(b,c) \in R^{-1}$, then $(b,a) \in R$ and $(c,b) \in R$, since *R* is transitive, so $(c,a) \in R$, therefore $(a,c) \in R^{-1}$, thus R^{-1} is transitive.

Therefore $\left(S,\ R^{-1}\right)$ is a poset

Exercise

Draw the Hasse diagram for the "greater than or equal to" relation on {0, 1, 2, 3, 4, 5}

Solution