Solution *Section* 4.1 – Matrix Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

Exercise

Find the standard matrix for the transformation defined by the equations

a) $\begin{cases} w_1 = 2x_1 - 3x_2 + 4x_4 \\ w_2 = 3x_1 + 5x_2 - x_4 \end{cases}$

b) $\begin{cases} w_1 = 7x_1 + 2x_2 - 8x_3 \\ w_2 = \quad\quad - x_2 + 5x_3 \\ w_3 = 4x_1 + 7x_2 - x_3 \end{cases}$

c) $\begin{cases} w_1 = x_1 \\ w_2 = x_1 + x_2 \\ w_3 = x_1 + x_2 + x_3 \\ w_4 = x_1 + x_2 + x_3 + x_4 \end{cases}$

Solution

a) $$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 & 4 \\ 3 & 5 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The standard matrix is $\begin{bmatrix} 2 & -3 & 0 & 4 \\ 3 & 5 & 0 & -1 \end{bmatrix}$

b) $$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The standard matrix is $\begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix}$

c) $$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The standard matrix is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Exercise

Find the standard matrix for the operator T defined by the formula

$$T\left(x_1, x_2\right)=\left(2x_1 - x_2, x_1 + x_2\right)$$

Solution

$$T\left(x_1, x_2\right)=\begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$$

$$=\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The standard matrix is $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

Exercise

Find the standard matrix for the operator T defined by the formula

$$T\left(x_1, x_2, x_3\right)=\left(x_1 + 2x_2 + x_3,\ x_1 + 5x_2,\ x_3\right)$$

Solution

$$T\left(x_1,x_2,x_3\right)=\begin{bmatrix} x_1 + 2x_2 + x_3 \\ x_1 + 5x_2 \\ x_3 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The standard matrix is $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Exercise

Find the standard matrix for the operator T defined by the formula

$$T(x_1, x_2, x_3) = (4x_1,\ 7x_2,\ -8x_3)$$

Solution

$$T(x_1, x_2, x_3) = \begin{pmatrix} 4x_1 \\ 7x_2 \\ -8x_3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The standard matrix is $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -8 \end{pmatrix}$

Exercise

Find the standard matrix for the operator T defined by the formula

$$T(x_1, x_2) = (x_2,\ -x_1,\ x_1 + 3x_2,\ x_1 - x_2)$$

Solution

$$T(x_1, x_2) = \begin{pmatrix} x_2 \\ -x_1 \\ x_1 + 3x_2 \\ x_1 - x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The matrix is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{pmatrix}$

Exercise

Find the standard matrix for the operator T defined by the formula

$$T(x_1, x_2, x_3) = (0, \ 0, \ 0, \ 0, \ 0)$$

Solution

$$T(x_1, x_2, x_3) = (0, \ 0, \ 0, \ 0, \ 0)$$

The matrix is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Exercise

Find the standard matrix for the operator T defined by the formula

$$T(x_1, x_2, x_3, x_4) = (x_4, \ x_1, \ x_3, \ x_2, \ x_1 - x_3)$$

Solution

$$T(x_1, x_2, x_3, x_4) = (x_4, \ x_1, \ x_3, \ x_2, \ x_1 - x_3)$$

$$T(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_4 \\ x_1 \\ x_3 \\ x_2 \\ x_1 - x_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

The matrix is $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}$

Exercise

Find the standard matrix for the operator T defined by the formula

$$T\left(x_1, x_2, x_3, x_4\right)=\left(7x_1+2x_2-x_3+x_4,\ x_2+x_3,\ -x_1\right)$$

Solution

$$T\left(x_1, x_2, x_3, x_4\right)=\begin{pmatrix} 7x_1+2x_2-x_3+x_4 \\ x_2+x_3 \\ -x_1 \end{pmatrix}$$

$$=\begin{pmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

The matrix is $\begin{pmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$

Exercise

Plot $\vec{u}=\begin{pmatrix} 5 \\ 2 \end{pmatrix},\quad \vec{v}=\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and their images under the given transformation T

$$T\left(\vec{x}\right)=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

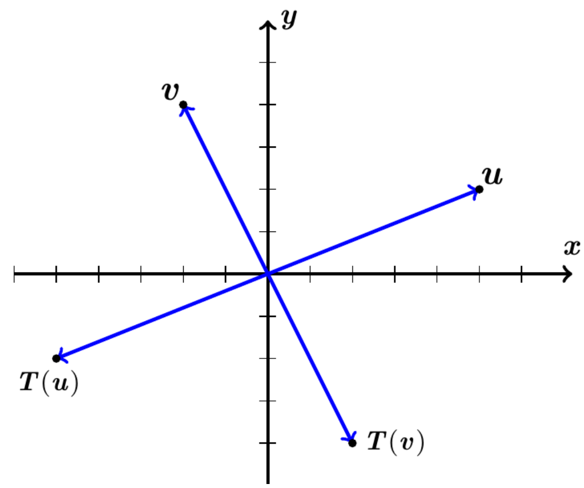
Solution

$$T\left(\vec{u}\right)=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$=\begin{pmatrix} -5 \\ -2 \end{pmatrix}$$

$$T\left(\vec{v}\right)=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$=\begin{pmatrix} 2 \\ -4 \end{pmatrix}$$



$\therefore$ *Reflection* through the origin $\left(180°\right)$

Exercise

Plot $\vec{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and their images under the given transformation T

$$T(\vec{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

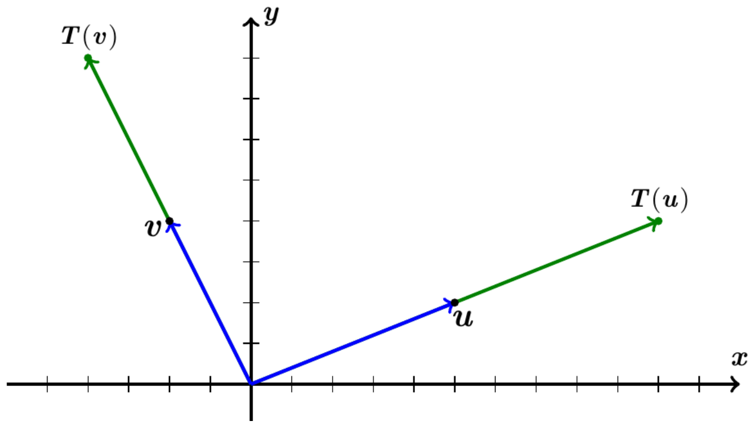
Solution

$$T(\vec{u}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 10 \\ 4 \end{pmatrix}$$

$$T(\vec{v}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ 8 \end{pmatrix}$$



$$\begin{pmatrix} 10 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$T(\vec{u}) = \vec{u}$$

$\therefore$ *Dilation* with factor $(k = 2)$ on $\mathbb{R}^2$

Exercise

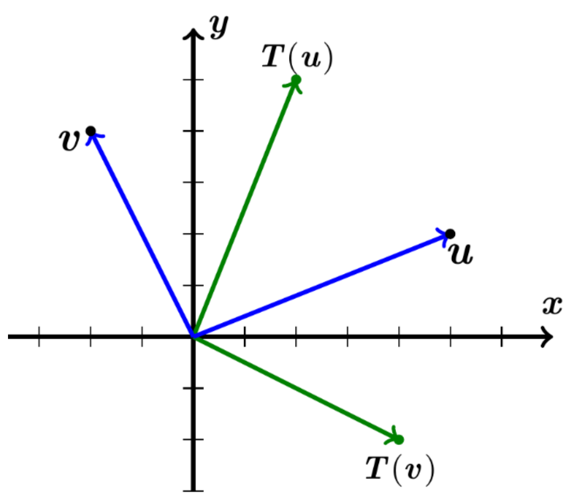
Plot $\vec{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and their images under the given transformation T

$$T(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Solution

$$T(\vec{u}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$



$$T(\vec{v}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$=\begin{pmatrix}4\\-2\end{pmatrix}$$

$$\begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}5\\2\end{pmatrix}$$

$$\vec{x}_1 = \vec{x}_2$$

$\therefore$ *reflection* about the line $\vec{x}_1 = \vec{x}_2$

Exercise

Plot $\vec{u} = \begin{pmatrix}5\\2\end{pmatrix}$, $\vec{v} = \begin{pmatrix}-2\\4\end{pmatrix}$ and their images under the given transformation T

$$T(\vec{x}) = \begin{pmatrix}0 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

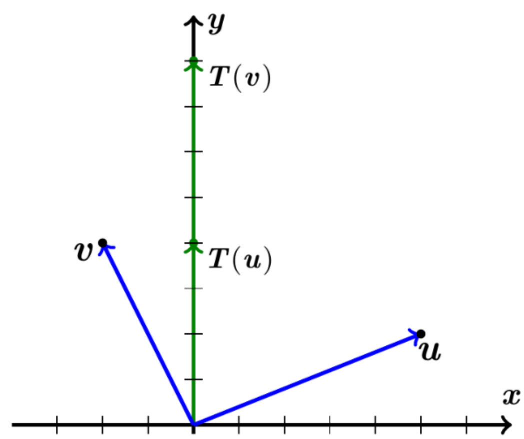
Solution

$$T(\vec{u}) = \begin{pmatrix}0 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}5\\2\end{pmatrix}$$

$$=\begin{pmatrix}0\\4\end{pmatrix}$$

$$T(\vec{v}) = \begin{pmatrix}0 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}-2\\4\end{pmatrix}$$

$$=\begin{pmatrix}0\\8\end{pmatrix}$$



$$\underset{T(\vec{u})}{\begin{pmatrix}0\\4\end{pmatrix}} = 2\underset{\vec{u}}{\begin{pmatrix}*\\2\end{pmatrix}}$$

$$\underset{T(\vec{v})}{\begin{pmatrix}0\\8\end{pmatrix}} = 2\underset{\vec{v}}{\begin{pmatrix}*\\4\end{pmatrix}}$$

$\therefore$ Orthogonal projection on the y-axis and with *Dilation* with factor $(k=2)$

Exercise

Plot $\vec{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and their images under the given transformation T

$$T(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

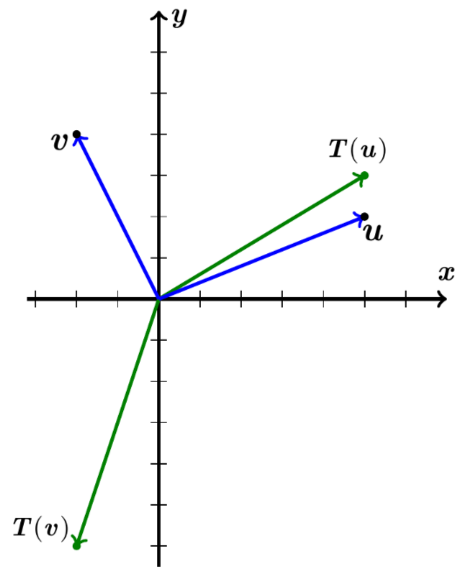
Solution

$$T(\vec{u}) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$T(\vec{v}) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ -6 \end{pmatrix}$$



$$\underset{T(\vec{u})}{\begin{pmatrix} 5 \\ 3 \end{pmatrix}} = \underset{\vec{u}}{\begin{pmatrix} 5 \\ 2 \end{pmatrix}}$$

$$\underset{T(\vec{v})}{\begin{pmatrix} -2 \\ -6 \end{pmatrix}} = \underset{\vec{v}}{\begin{pmatrix} -2 \\ 4 \end{pmatrix}}$$

$\therefore$ *Expansion* of $\vec{u}$ in the y–direction with factor $k = \frac{3}{2}$

$\therefore$ *Expansion* of $\vec{v}$ in the y–direction with factor $k = \frac{3}{2}$ and reflection about x–axis.

Solution *Section* 4.2 – General Linear Transformations

Exercise

The matrix $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ gives a shearing transformation $T(x, y) = (x,\ 3x + y)$.

What happens to (1, 0) and (2, 0) on the x-axis.

What happens to the points on the vertical lines $x = 0$ and $x = a$?.

Solution

The points (1, 0) and (2, 0) on the x-axis transform by T to (1, 3) and (2, 6). The horizontal x-axis transforms to the straight line with slope 3 (going through (0, 0) of course). The points on the y-axis are not moved because $T(0, y) - (0, y)$. The y -axis is the line of eigenvectors of T with $\lambda = 1$.

The vertical line $x = a$ is moved up by $3a$, since $3a$ is added to the y component. This is ***shearing***. Vertical lines slide higher as you go from left to right.

Exercise

A nonlinear transformation T is invertible if every $\vec{b}$ in the output space comes from exactly one x in the input space. $T(\vec{x}) = \vec{b}$ always has exactly one solution. Which of these transformation (on real numbers $\vec{x}$ is invertible and what is T^{-1}? None are linear, not even T_3. When you solve $T(\vec{x}) = \vec{b}$, you are inverting T:

$$T_1(\vec{x}) = x^2 \quad T_2(\vec{x}) = x^3 \quad T_3(\vec{x}) = x + 9 \quad T_4(\vec{x}) = e^x \quad T_5(\vec{x}) = \frac{1}{x} \quad \textit{for nonzero } x's$$

Solution

T_1 is not invertible because

$x^2 = 1 \rightarrow x = \pm 1$ and $x^2 = -1$ has no solution.

T_4 is not invertible because

$e^x = -1$ has no solution.

T_2 is invertible.

The solutions to $x^3 = b \rightarrow x = b^{1/3} = T_2^{-1}(b)$

T_3 is invertible.

The solutions to $x + 9 = b \rightarrow x = b - 9 = T_3^{-1}(b)$

T_5 is invertible.

The solutions to $\frac{1}{x} = b \rightarrow x = \frac{1}{b} = T_5^{-1}(b)$

Exercise

If S and T are linear transformations, is $S(T(\vec{v}))$ linear or quadratic?

a) If $S(\vec{v}) = \vec{v}$ and $T(\vec{v}) = \vec{v}$, then $S(T(\vec{v})) = \vec{v}$ or $\vec{v}^2$?

b) $S(\vec{w}_1 + \vec{w}_2) = S(\vec{w}_1) + S(\vec{w}_2)$ and $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$ combine into

$$S(T(\vec{v}_1 + \vec{v}_2)) = S(\_\_\_\_\_) = \_\_\_\_ + \_\_\_\_\_\_$$

Solution

a) $S(T(\vec{v})) = S(\vec{v})$ $\qquad T(\vec{v}) = \vec{v}$

$= \vec{v}$ $\qquad S(\vec{v}) = \vec{v}$

b) $$S(T(\vec{v}_1 + \vec{v}_2)) = S(T(\vec{v}_1) + T(\vec{v}_2))$$
$$= S(T(\vec{v}_1)) + S(T(\vec{v}_2))$$

It is quadratic.

Exercise

Find the range and kernel (like the column space and nullspace) of T:

a) $T(v_1, v_2) = (v_2, v_1)$

b) $T(v_1, v_2, v_3) = (v_1, v_2)$

c) $T(v_1, v_2) = (0, 0)$

d) $T(v_1, v_2) = (v_1, v_1)$

Solution

a) Range is the line $y = 0$, Kernel is the line $x = y$ in the xy plane.

b) Range is the xy plane, Kernel is the complementary line in $\mathbb{R}^3$.

c) Range is the point (0, 0), Kernel is plane

d) Range is the line $x = y$ in the xy plane, Kernel is the line $x = 0$.

Exercise

M is any 2 by 2 matrix and $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$. The transformation T is defined by $T(M)=AM$. What rules of matrix multiplication show that T is linear?

Solution

The distribution law and the association law for multiplication give the linearity

$$\begin{aligned} A(cM+dN) &= A(cM)+A(dN)\\ &=(Ac)M+(Ad)N\\ &=cA(M)+dA(N)\end{aligned}$$

Exercise

Which of these transformations satisfy $T(\vec{v}+\vec{w})=T(\vec{v})+T(\vec{w})$ and which satisfy $T(c\vec{v})=cT(\vec{v})$?

a) $T(\vec{v})=\dfrac{\vec{v}}{\|\vec{v}\|}$

b) $T(\vec{v})=v_1+v_2+v_3$

c) $T(\vec{v})=\left(v_1,\ 2v_2,\ 3v_3\right)$

d) $T(\vec{v})=$ largest component of $\vec{v}$.

Solution

a) This is scaling the vector into a normal vector. This it is impossible that we get additivity, because the sums of normal vectors don't have to be normal. For example $T(0, 1)$ and $T(1, 0)$ for instance. However, true to its name this does have the scaling property. For ***c*** value, this value will be canceled from $\vec{v}$ and $\|\vec{v}\|$.

b) This satisfies both. One immediate way to see that it is matrix multiplication by [1, 1, 1], which is a linear operation and thus satisfies both properties.

c) This satisfies both. This a matrix multiplication by $\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix}$

d) Doesn't satisfy additivity [(0, 1) and (1, 0) still work]. Scaling doesn't work either, if we scale by -1 we now pick out the negative of the smallest component, which doesn't have to be related in any way to the largest component.

Exercise

Consider the basis $S=\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\}$ for R^3, where $\vec{v}_1=(1,\ 1,\ 1)$ $\vec{v}_2=(1,\ 1,\ 0)$ $\vec{v}_3=(1,\ 0,\ 0)$ and let $T:\ \mathbb{R}^3\to\mathbb{R}^3$ be the linear transformation for which

$$T\left(\vec{v}_1\right)=(2,\ -1,\ 4),\quad T\left(\vec{v}_2\right)=(3,\ 0,\ 1),\quad T\left(\vec{v}_3\right)=(-1,\ 5,\ 1)$$

Find a formula for $T\left(\vec{x}_1, \vec{x}_2, \vec{x}_3\right)$, and then use that formula to compute $T(2,\ 4,\ -1)$

Solution

Assume: $\vec{u}=c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3$

$$\begin{aligned}\left(\vec{x}_1, \vec{x}_2, \vec{x}_3\right)&=c_1(1,\ 1,\ 1)+c_2(1,\ 1,\ 0)+c_3(1,\ 0,\ 0)\\&=\left(c_1+c_2+c_3,\ c_1+c_2,\ c_1\right)\end{aligned}$$

$$\begin{cases}c_1+c_2+c_3=x_1\\ c_1+c_2=x_2\\ c_1=x_3\end{cases}$$

$$\begin{cases}c_3=x_1-x_2\\ c_2=x_2-x_3\\ c_1=x_3\end{cases}$$

$$\begin{aligned}T\left(\vec{x}_1, \vec{x}_2, \vec{x}_3\right)&=x_3T\left(\vec{v}_1\right)+\left(x_2-x_3\right)T\left(\vec{v}_2\right)+\left(x_1-x_2\right)T\left(\vec{v}_3\right)\\&=x_3(2,\ -1,\ 4)+\left(x_2-x_3\right)(3,\ 0,\ 1)+\left(x_1-x_2\right)(-1,\ 5,\ 1)\\&=\left(2x_3+3x_2-3x_3-x_1+x_2,\ -x_3+5x_1-5x_2,\ 4x_3+x_2-x_3+x_1-x_2\right)\\&=\left(-x_1+4x_2-x_3,\ 5x_1-5x_2-x_3,\ x_1+3x_3\right)\end{aligned}$$

$$\begin{aligned}T(2,\ 4,\ -1)&=(-2+16+1,\ 10-20+1,\ 2-3)\\&=(15,\ -9,\ -1)\end{aligned}$$

Exercise

Consider the basis $S = \left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\}$ for R^3, where $\vec{v}_1 = (1,\ 2,\ 1)$ $\vec{v}_2 = (2,\ 9,\ 0)$ $\vec{v}_3 = (3,\ 3,\ 4)$ and let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation for which

$$T\left(\vec{v}_1\right) = (1,\ 0), \quad T\left(\vec{v}_2\right) = (-1,\ 1), \quad T\left(\vec{v}_3\right) = (0,\ 1)$$

Find a formula for $T\left(\vec{x}_1, \vec{x}_2, \vec{x}_3\right)$, and then use that formula to compute $T(7,\ 13,\ 7)$

Solution

Assume: $\quad \vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

$$\left(\vec{x}_1, \vec{x}_2, \vec{x}_3\right) = c_1(1,\ 2,\ 1) + c_2(2,\ 9,\ 0) + c_3(3,\ 3,\ 4)$$

$$= \left(c_1 + 2c_2 + 3c_3,\ 2c_1 + 9c_2 + 3c_3,\ c_1 + 4c_3\right)$$

$$\begin{cases} c_1 + 2c_2 + 3c_3 = x_1 \\ 2c_1 + 9c_2 + 3c_3 = x_2 \\ c_1 \qquad\quad + 4c_3 = x_3 \end{cases}$$

$$\begin{cases} c_1 + 7c_2 = x_2 - x_1 \\ \\ c_1 + 4c_3 = x_3 \end{cases}$$

$$2c_1 + \frac{9}{7}x_2 - \frac{9}{7}x_1 - \frac{9}{7}c_1 + \frac{3}{4}x_3 - \frac{3}{4}c_1 = x_2$$

$$2c_1 - \frac{9}{7}c_1 - \frac{3}{4}c_1 = x_2 - \frac{9}{7}x_2 + \frac{9}{7}x_1 - \frac{3}{4}x_3$$

$$-\frac{1}{28}c_1 = \frac{9}{7}x_1 - \frac{2}{7}x_2 - \frac{3}{4}x_3$$

$$\underline{c_1 = -36x_1 + 8x_2 + 21x_3}$$

$$c_3 = \frac{1}{4}x_3 - \frac{1}{4}c_1$$

$$\underline{c_3 = 9x_1 - 2x_2 - 5x_3}$$

$$c_2 = \frac{1}{7}x_2 - \frac{1}{7}x_1 - \frac{1}{7}c_1$$

$$c_2 = \frac{1}{7}x_2 - \frac{1}{7}x_1 + \frac{36}{7}x_1 - \frac{8}{7}x_2 - 3x_3$$

$$\underline{c_2 = 5x_1 - x_2 - 3x_3}$$

$$T\left(\vec{x}_1,\ \vec{x}_2,\ \vec{x}_3\right)=\left(-36x_1+8x_2+21x_3\right)T\left(\vec{v}_1\right)+\left(5x_1-x_2-3x_3\right)T\left(\vec{v}_2\right)$$
$$+\left(9x_1-2x_2-5x_3\right)T\left(\vec{v}_3\right)$$
$$=\left(-36x_1+8x_2+21x_3\right)(1,\ 0)+\left(5x_1-x_2-3x_3\right)(-1,\ 1)+\left(9x_1-2x_2-5x_3\right)(0,\ 1)$$
$$=\left(36x_1-8x_2+21x_3-5x_1+x_2+3x_3,\ 5x_1-x_2-3x_3+9x_1-2x_2-5x_3\right)$$
$$=\left(41x_1+9x_2+24x_3,\ 14x_1-3x_2-8x_3\right)$$

$$T(7,\ 13,\ 7)=\left(37(7)-13(13)+24(7),\ \ 8(7)+3(13)-8(7)\right)$$
$$\underline{=(-2,\ 3)}$$

Exercise

let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be vectors in a vector space *V*, and let $T: V \to R^3$ be the linear transformation for which

$$T\left(\vec{v}_1\right)=(1,\ -1,\ 2),\quad T\left(\vec{v}_2\right)=(0,\ 3,\ 2),\quad T\left(\vec{v}_3\right)=(-3,\ 1,\ 2).$$

Find $T\left(2\vec{v}_1-3\vec{v}_2+4\vec{v}_3\right)$

Solution

$$T\left(2\vec{v}_1-3\vec{v}_2+4\vec{v}_3\right)=2T\left(\vec{v}_1\right)-3T\left(\vec{v}_2\right)+4T\left(\vec{v}_3\right)$$
$$=2(1,\ -1,\ 2)-3(0,\ 3,\ 2)+4(-3,\ 1,\ 2)$$
$$=(2,\ -2,\ 4)-(0,\ 9,\ 6)+(-12,\ 4,\ 8)$$
$$\underline{=(-10,\ -7,\ 6)}$$

Exercise

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operation given by the formula $T(x,\ y)=(2x-y,\ -8x+4y)$

Which of the following vectors are in $R(T)$

a) $(1,\ -4)$ ***b***) $(5,\ 0)$ ***c***) $(-3,\ 12)$

Solution

a) $T(x,y)=(2x-y,\ -8x+4y)=(1,\ -4)$

$$\begin{cases} 2x-y=1 \\ -8x+4y=-4 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ -8 & 4 & -4 \end{array}\right] \quad R_2 + 4R_1$$

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \quad \tfrac{1}{2}R_1$$

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]$$

This is a consistent system, therefore (1, −4) is in $R(T)$

b) $T(x, y) = (2x - y,\ -8x + 4y) = (5,\ 0)$

$$\begin{cases} 2x - y = 5 \\ -8x + 4y = 0 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & -1 & 5 \\ -8 & 4 & 0 \end{array}\right] \quad R_2 + 4R_1$$

$$\left[\begin{array}{cc|c} 2 & -1 & 5 \\ 0 & 0 & 20 \end{array}\right] \quad \rightarrow\ 0 \neq 20$$

This is an inconsistent system, therefore (5, 0) is not in $R(T)$

c) $T(x, y) = (2x - y,\ -8x + 4y) = (-3,\ 12)$

$$\begin{cases} 2x - y = -3 \\ -8x + 4y = 12 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & -1 & -3 \\ -8 & 4 & 12 \end{array}\right] \quad R_2 + 4R_1$$

$$\left[\begin{array}{cc|c} 2 & -1 & -3 \\ 0 & 0 & 0 \end{array}\right] \quad \tfrac{1}{2}R_1$$

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \end{array}\right]$$

This is a consistent system, therefore (−3, 12) is in $R(T)$

Exercise

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operation given by the formula $T(x, y) = (2x - y, \ -8x + 4y)$

Which of the following vectors are in $\ker(T)$

a) $(5, 10)$ ***b***) $(3, 2)$ ***c***) $(1, 1)$

Solution

a) $T(5, 10) = (10 - 10, \ -40 + 40)$

$= (0, 0)$

Therefore (5, 10) is in $\ker(T)$

b) $T(3, 2) = (6 - 2, \ -24 + 8)$

$= (4, -16)$

Therefore (3, 2) is not in $\ker(T)$

c) $T(1, 1) = (2 - 1, \ -8 + 4)$

$= (1, -4)$

Therefore (1, 1) is not in $\ker(T)$

Exercise

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear operation given by the formula

$$T(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = (4x_1 + x_2 - 2x_3 - 3x_4, \ 2x_1 + x_2 + x_3 - 4x_4, \ 6x_1 - 9x_3 + 9x_4)$$

Which of the following vectors are in $R(T)$

a) $(0, 0, 6)$ ***b***) $(1, 3, 0)$ ***c***) $(2, 4, 1)$

Solution

a) $T(x_1, x_2, x_3, x_4) = (4x_1 + x_2 - 2x_3 - 3x_4, \ 2x_1 + x_2 + x_3 - 4x_4, \ 6x_1 - 9x_3 + 9x_4) = (0, 0, 6)$

$$\begin{cases} 4x_1 + x_2 - 2x_3 - 3x_4 = 0 \\ 2x_1 + x_2 + x_3 - 4x_4 = 0 \\ 6x_1 \qquad - 9x_3 + 9x_4 = 6 \end{cases}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 0 \\ 2 & 1 & 1 & -4 & 0 \\ 6 & 0 & -9 & 9 & 6 \end{array}\right] \quad \begin{array}{l} \\ 2R_2 - R_1 \\ 2R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & -3 & -12 & 27 & 12 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 2 & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & 0 & 0 & 12 & 12 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{12}R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 2 & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 - 2R_3 \\ R_2 + 5R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 0 & -2 \\ 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} \frac{1}{4}R_1 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -\frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

This is a consistent system, therefore (0, 0, 6) is in $R(T)$

b) $T\left(x_1, x_2, x_3, x_4\right) = \left(4x_1 + x_2 - 2x_3 - 3x_4,\ 2x_1 + x_2 + x_3 - 4x_4,\ 6x_1 - 9x_3 + 9x_4\right) = (1,\ 3,\ 0)$

$$\begin{cases} 4x_1 + x_2 - 2x_3 - 3x_4 = 1 \\ 2x_1 + x_2 + x_3 - 4x_4 = 3 \\ 6x_1 \qquad\quad - 9x_3 + 9x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 1 \\ 2 & 1 & 1 & -4 & 3 \\ 6 & 0 & -9 & 9 & 0 \end{array}\right] \begin{array}{l} \\ 2R_2 - R_1 \\ 2R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 1 \\ 0 & 1 & 4 & -5 & 5 \\ 0 & -3 & -12 & 27 & -3 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 2 & -4 \\ 0 & 1 & 4 & -5 & 5 \\ 0 & 0 & 0 & 12 & 12 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{12}R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 2 & -4 \\ 0 & 1 & 4 & -5 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 + 5R_3 \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 0 & -6 \\ 0 & 1 & 4 & 0 & 10 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad \tfrac{1}{4}R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -\frac{3}{2} & 0 & -\frac{3}{2} \\ 0 & 1 & 4 & 0 & 10 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

This is a consistent system, therefore (1, 3, 0) is in $R(T)$

c) $T\left(x_1, x_2, x_3, x_4\right) = \left(4x_1 + x_2 - 2x_3 - 3x_4,\ 2x_1 + x_2 + x_3 - 4x_4,\ 6x_1 - 9x_3 + 9x_4\right) = (2,\ 4,\ 1)$

$$\begin{cases} 4x_1 + x_2 - 2x_3 - 3x_4 = 2 \\ 2x_1 + x_2 + x_3 - 4x_4 = 4 \\ 6x_1 \qquad - 9x_3 + 9x_4 = 1 \end{cases}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 2 \\ 2 & 1 & 1 & -4 & 4 \\ 6 & 0 & -9 & 9 & 1 \end{array}\right] \quad \begin{array}{l} \\ 2R_2 - R_1 \\ 2R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 2 \\ 0 & 1 & 4 & -5 & 6 \\ 0 & -3 & -12 & 27 & -4 \end{array}\right] \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 2 & -4 \\ 0 & 1 & 4 & -5 & 6 \\ 0 & 0 & 0 & 12 & 14 \end{array}\right] \quad \begin{array}{l} \\ \\ \frac{1}{12}R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 2 & -4 \\ 0 & 1 & 4 & -5 & 6 \\ 0 & 0 & 0 & 1 & \frac{7}{6} \end{array}\right] \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 + 5R_3 \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 0 & -\frac{19}{3} \\ 0 & 1 & 4 & 0 & \frac{71}{6} \\ 0 & 0 & 0 & 1 & \frac{7}{6} \end{array}\right] \qquad \tfrac{1}{4}R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -\frac{3}{2} & 0 & -\frac{19}{12} \\ 0 & 1 & 4 & 0 & \frac{71}{6} \\ 0 & 0 & 0 & 1 & \frac{7}{6} \end{array}\right]$$

This is a consistent system, therefore (2, 4, 1) is in $R(T)$

Exercise

Let $T:\ \mathbb{R}^4 \to \mathbb{R}^3$ be the linear operation given by the formula

$$T\left(\vec{x}_1,\ \vec{x}_2,\ \vec{x}_3,\ \vec{x}_4\right) = \left(4x_1 + x_2 - 2x_3 - 3x_4,\ \ 2x_1 + x_2 + x_3 - 4x_4,\ \ 6x_1 - 9x_3 + 9x_4\right)$$

Which of the following vectors are in $\ker(T)$

a) $(3,\ -8,\ 2,\ 0)$ ***b***) $(0,\ 0,\ 0,\ 1)$ ***c***) $(0,\ -4,\ 1,\ 0)$

Solution

a) $T(3,\ -8,\ 2,\ 0) = (12-8-4,\ 6-8+2,\ 18-18)$

$$= (0,\ 0,\ 0)$$

Therefore, (3, –8, 2, 0) is in $\ker(T)$

b) $T(0,\ 0,\ 0,\ 1) = (-3,\ -4,\ 9)$

Therefore, (0, 0, 0, 1) is ***not*** in $\ker(T)$

c) $T(0,\ -4,\ 1,\ 0) = (-4-2,\ -4+1,\ -9)$

$$= (-6,\ -3,\ -9)$$

Therefore, (0, –4, 1, 0) is ***not*** in $\ker(T)$

Exercise

Determine if the given function T is a linear transformation

$$T: M_{22} \to M_{22} \quad by \quad T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2ab & 3cd \\ 0 & 0 \end{bmatrix}$$

Solution

Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$$T(A+B) = T\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2(a_1 + a_2)(b_1 + b_2) & 3(c_1 + c_2)(d_1 + d_2) \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2a_1b_1 + 2a_1b_2 + 2a_2b_1 + 2a_2b_2 & 3c_1d_1 + 3c_1d_2 + 3c_2d_1 + 3c_2d_2 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2a_1b_1 & 3c_1d_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2a_2b_2 & 3c_2d_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2a_1b_2 + 2a_2b_1 & 3c_1d_2 + 3c_2d_1 \\ 0 & 0 \end{bmatrix}$$

$$= T(A) + T(B) + \begin{bmatrix} 2a_1b_2 + 2a_2b_1 & 3c_1d_2 + 3c_2d_1 \\ 0 & 0 \end{bmatrix}$$

$$\neq T(A) + T(B)$$

Function T is NOT a linear transformation.

Exercise

Determine if the given function T is a linear transformation

$$T: M_{22} \to M_{22} \quad by \quad T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+d & 0 \\ 0 & b+c \end{bmatrix}$$

Solution

Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$$T(A+B) = T\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + a_2 + d_1 + d_2 & 0 \\ 0 & b_1 + b_2 + c_1 + c_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + d_1 & 0 \\ 0 & b_1 + c_1 \end{bmatrix} + \begin{bmatrix} a_2 + d_2 & 0 \\ 0 & b_2 + c_2 \end{bmatrix}$$

$$= T(A) + T(B) \quad \checkmark$$

$$T(kA) = T\left(k \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \right)$$

$$= T\left(\begin{bmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} ka_1 + kd_1 & 0 \\ 0 & kb_1 + kc_1 \end{bmatrix}$$

$$= \begin{bmatrix} k(a_1 + d_1) & 0 \\ 0 & k(b_1 + c_1) \end{bmatrix}$$

$$= k \begin{bmatrix} a_1 + d_1 & 0 \\ 0 & b_1 + c_1 \end{bmatrix}$$

$$= kT(A) \quad \checkmark$$

Since $T(A+B) = T(A) + T(B)$ and $T(kA) = kT(A)$, then function T is a linear transformation.

Exercise

Determine if the given function T is a linear transformation where A is fixed 2×3 matrix

$$T : M_{22} \to M_{23} \quad by \quad T(B) = BA$$

Solution

$$T(B+C) = (B+C)A$$

$$= BA + CA$$

$$= T(B) + T(C)$$

$$T(rB) = (rB)A$$

$$= r(BA)$$

$$= rT(B)$$

Function T is a linear transformation

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T = f_A$

$$T(x, y) = (x^2, y)$$

Solution

Let $\vec{u} = (x_1, y_1)$ *and* $\vec{v} = (x_2, y_2)$

$$\begin{aligned} T(\vec{u}+\vec{v}) &= T(x_1 + x_2,\ y_1 + y_2) \\ &= \left((x_1 + x_2)^2,\ y_1 + y_2\right) \\ &= \left(x_1^2 + x_2^2 + 2x_1x_2,\ y_1 + y_2\right) \\ &= \left(x_1^2,\ y_1\right) + \left(x_2^2,\ y_2\right) + \left(2x_1x_2,\ 0\right) \\ &= T(\vec{u}) + T(\vec{v}) + \left(2x_1x_2,\ 0\right) \\ &\neq T(\vec{u}) + T(\vec{v}) \end{aligned}$$

The function T is ***not*** a linear transformation.

Domain: $T:\ \mathbb{R}^2 \to \mathbb{R}^2$

Exercise

Determine if the given function T is a linear transformation. Also, give the domain and range of T; if T is linear, find the A such $T = f_A$.

$$T(x, y, z) = (2x + y,\ x - y + z)$$

Solution

Let $\vec{u} = (x_1, y_1, z_1)$ *and* $\vec{v} = (x_2, y_2, z_2)$

$$\begin{aligned} T(\vec{u}+\vec{v}) &= T(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= \left(2(x_1 + x_2) + y_1 + y_2,\ \ x_1 + x_2 - (y_1 + y_2) + z_1 + z_2\right) \\ &= \left(2x_1 + y_1 + 2x_2 + y_2,\ \ x_1 - y_1 + z_1 + x_2 - y_2 + z_2\right) \\ &= \left(2x_1 + y_1,\ x_1 - y_1 + z_1\right) + \left(2x_2 + y_2,\ x_2 - y_2 + z_2\right) \\ &= T(x_1, y_1, z_1) + T(x_2, y_2, z_2) \end{aligned}$$

$$=T(\vec{u})+T(\vec{v})$$

$$T(r\vec{u})=T\left(rx_1,\ ry_1,\ rz_1\right)$$

$$=\left(2rx_1+ry_1,\ rx_1-ry_1+rz_1\right)$$

$$=r\left(2x_1+y_1,\ x_1-y_1+z_1\right)$$

$$=rT(\vec{u})$$

Since $T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v})$ and $T(r\vec{u})=rT(\vec{u})$, then function T is a linear transformation.

Domain: $T:\ \mathbb{R}^3\to\mathbb{R}^2$

$$T(x,\ y,\ z)=(2x+y,\ \ x-y+z)$$

$$=\begin{pmatrix}2x+y\\ x-y+z\end{pmatrix}$$

$$A=\begin{array}{c}\begin{array}{ccc}x & y & z\end{array}\\ \begin{bmatrix}2 & 1 & 0\\ 1 & -1 & 1\end{bmatrix}\end{array}$$

Exercise

Determine if the given function T is a linear transformation. Also, give the domain and range of T; if T is linear, find the A such $T=f_A$.

$$T(x,\ y,\ z)=(z-x,\ z-y)$$

Solution

Let $\vec{u}=\left(x_1,\ y_1,\ z_1\right)$ *and* $\vec{v}=\left(x_2,\ y_2,\ z_2\right)$

$$T(\vec{u}+\vec{v})=T\left(x_1+x_2,\ y_1+y_2,\ z_1+z_2\right)$$

$$=\left(z_1+z_2-\left(x_1+x_2\right),\ z_1+z_2-\left(y_1+y_2\right)\right)$$

$$=\left(z_1+z_2-x_1-x_2,\ \ z_1+z_2-y_1-y_2\right)$$

$$=\left(z_1-x_1,\ \ z_1-y_1\right)+\left(z_2-x_2,\ \ z_2-y_2\right)$$

$$=T\left(x_1,\ y_1,\ z_1\right)+T\left(x_2,\ y_2,\ z_2\right)$$

$$=T(\vec{u})+T(\vec{v})$$

$$T(r\vec{u})=T\left(rx_1,\ ry_1,\ rz_1\right)$$

$$=\left(rz_1 - rx_1,\ rz_1 - ry_1\right)$$

$$=r\left(z_1 - x_1,\ z_1 - y_1\right)$$

$$=rT(\vec{u})$$

Since $T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v})$ and $T(r\vec{u})=rT(\vec{u})$, then function T is a linear transformation.

Domain: $T:\ \mathbb{R}^3 \to \mathbb{R}^2$

$$T(x,\ y,\ z)=(z-x,\ z-y)$$

$$=\begin{pmatrix} -x+z \\ -y+z \end{pmatrix}$$

$$\begin{array}{cc} & \begin{matrix} x & \ \ y & \ \ z \end{matrix} \\ A= & \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T=f_A$

$$T\left(x_1,\ x_2,\ x_3\right)=\left(2x_1 - x_2 + x_3,\ \ x_2 - 4x_3\right)$$

Solution

Let $\vec{u}=\left(x_1,\ x_2,\ x_3\right)$ *and* $\vec{v}=\left(y_1,\ y_2,\ y_3\right)$

$$T(\vec{u}+\vec{v})=T\left(x_1+y_1,\ x_2+y_2,\ x_3+y_3\right)$$

$$=\left(2\left(x_1+y_1\right)-x_2-y_2+x_3+y_3\ ,\ x_2+y_2-4\left(x_3+y_3\right)\right)$$

$$=\left(2x_1+2y_1-x_2-y_2+x_3+y_3\ ,\ x_2+y_2-4x_3-4y_3\right)$$

$$=\left(2x_1-x_2+x_3\ ,\ x_2-4x_3\right)+\left(2y_1-y_2+y_3\ ,\ y_2-4y_3\right)$$

$$=T(\vec{u})+T(\vec{v})$$

$$T(r\vec{u})=T\left(rx_1,\ rx_2,\ rx_3\right)$$

$$=\left(2rx_1-rx_2+rx_3\ ,\ rx_2-4rx_3\right)$$

$$=r\left(2x_1-x_2+x_3\ ,\ x_2-4x_3\right)$$

$$=rT(\vec{u})$$

Since $T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v})$ and $T(r\vec{u})=rT(\vec{u})$, then function T is a linear transformation.

Domain: $T:\ \mathbb{R}^3 \to \mathbb{R}^2$

$$T\left(x_1,\, x_2,\, x_3\right)=\left(2x_1-x_2+x_3,\ \ x_2-4x_3\right)$$

$$=\begin{pmatrix} 2x_1-x_2+x_3 \\ x_2-4x_3 \end{pmatrix}$$

$$A=\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{pmatrix} 2 & -1 & 1 \\ 0 & -1 & -4 \end{pmatrix} \end{array}$$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T=f_A$

$$T\left(x_1,\, x_2\right)=\left(2x_1-x_2,\ -3x_1+x_2,\ 2x_1-3x_2\right)$$

Solution

Let $\vec{u}=\left(u_1,\, u_2\right)$ *and* $\vec{v}=\left(v_1,\, v_2\right)$

$$\begin{aligned}
T(\vec{u}+\vec{v}) &= T\left(u_1+v_1,\ u_2+v_2\right) \\
&= \left(2\left(u_1+v_1\right)-\left(u_2+v_2\right),\ \ -3\left(u_1+v_1\right)+\left(u_2+v_2\right),\ 2\left(u_1+v_1\right)-3\left(u_2+v_2\right)\right) \\
&= \left(2u_1+2v_1-u_2-v_2,\ \ -3u_1-3v_1+u_2+v_2,\ 2u_1+2v_1-3u_2-3v_2\right) \\
&= \left(\left(2u_1-u_2\right)+\left(2v_1-v_2\right),\ \left(-3u_1+u_2\right)+\left(-3v_1+v_2\right),\ \left(2u_1-3u_2\right)+\left(2v_1-3v_2\right)\right) \\
&= \left(2u_1-u_2,\ \ -3u_1+u_2,\ 2u_1-3u_2\right)+\left(2v_1-v_2,\ \ -3v_1+v_2,\ 2v_1-3v_2\right) \\
&= T(\vec{u})+T(\vec{v})
\end{aligned}$$

$$\begin{aligned}
T(r\vec{u}) &= T\left(r\left(u_1,\, u_2\right)\right) \\
&= T\left(ru_1,\, ru_2\right) \\
&= \left(2ru_1-ru_2,\ -3ru_1+ru_2,\ 2ru_1-3ru_2\right) \\
&= r\left(2u_1-u_2,\ -3u_1+u_2,\ 2u_1-3u_2\right) \\
&= rT(\vec{u})
\end{aligned}$$

Since $T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v})$ and $T(r\vec{u})=rT(\vec{u})$, then function T is a linear transformation.

Domain: $T: \ \mathbb{R}^2 \to \mathbb{R}^3$

$$T(x_1, x_2) = (2x_1 - x_2, \ -3x_1 + x_2, \ 2x_1 - 3x_2)$$

$$= \begin{pmatrix} 2x_1 - x_2 \\ -3x_1 + x_2 \\ 2x_1 - 3x_2 \end{pmatrix}$$

$$\begin{array}{cc} & \begin{array}{cc} x_1 & x_2 \end{array} \\ A = & \begin{pmatrix} 2 & -1 \\ -3 & 1 \\ 2 & -3 \end{pmatrix} \end{array}$$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T = f_A$

$$T(x_1, x_2) = (x_1 + 4x_2, \ 0, \ x_1 - 3x_2, \ x_1)$$

Solution

Let $\vec{u} = (u_1, u_2)$ *and* $\vec{v} = (v_1, v_2)$

$$T(\vec{u} + \vec{v}) = T(u_1 + v_1, \ u_2 + v_2)$$

$$= \big((u_1 + v_1) + 4(u_2 + v_2), \ 0, \ (u_1 + v_1) - 3(u_2 + v_2), \ (u_1 + v_1)\big)$$

$$= (u_1 + v_1 + 4u_2 + 4v_2, \ 0, \ u_1 + v_1 - 3u_2 - 3v_2, \ u_1 + v_1)$$

$$= \big((u_1 + 4u_2) + (v_1 + 4v_2), \ 0, \ (u_1 - 3u_2) + (v_1 - 3v_2), \ u_1 + v_1\big)$$

$$= (u_1 + 4u_2, \ 0, \ u_1 - 3u_2, \ u_1) + (v_1 + 4v_2, \ 0, \ v_1 - 3v_2, \ v_1)$$

$$= T(\vec{u}) + T(\vec{v})$$

$$T(r\vec{u}) = T\big(r(u_1, u_2)\big)$$

$$= T(ru_1, ru_2)$$

$$= (ru_1 + 4ru_2, \ 0, \ ru_1 - 3ru_2, \ ru_1)$$

$$= r(u_1 + 4u_2, \ 0, \ u_1 - 3u_2, \ u_1)$$

$$= rT(\vec{u})$$

Since $T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v})$ and $T(r\vec{u})=rT(\vec{u})$, then function T is a linear transformation.

Domain: $T: \ \mathbb{R}^2 \to \mathbb{R}^4$

$$T(x_1, x_2)=(x_1+4x_2, 0, x_1-3x_2, x_1)$$
$$=\begin{pmatrix} x_1+4x_2 \\ 0 \\ x_1-3x_2 \\ x_1 \end{pmatrix}$$

$$A=\begin{array}{c} \begin{matrix} x_1 & x_2 \end{matrix} \\ \begin{pmatrix} 1 & 4 \\ 0 & 0 \\ 1 & -3 \\ 1 & 0 \end{pmatrix} \end{array}$$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T = f_A$

$$T(x_1, x_2, x_3)=(x_1-5x_2+4x_3, \ x_2-6x_3)$$

Solution

Let $\vec{u}=(u_1, u_2, u_3)$ *and* $\vec{v}=(v_1, v_2, v_3)$

$$\begin{aligned} T(\vec{u}+\vec{v}) &= T(u_1+v_1, \ u_2+v_2, \ u_3+v_3) \\ &= \big((u_1+v_1)-5(u_2+v_2)+4(u_3+v_3), \ (u_2+v_2)-6(u_3+v_3)\big) \\ &= (u_1+v_1-5u_2-5v_2+4u_3+4v_3, \ u_2+v_2-6u_3-6v_3) \\ &= \big((u_1-5u_2+4u_3)+(v_1-5v_2+4v_3), \ (u_2-6u_3)+(v_2-6v_3)\big) \\ &= (u_1-5u_2+4u_3, \ u_2-6u_3)+(v_1-5v_2+4v_3, \ v_2-6v_3) \\ &= T(\vec{u})+T(\vec{v}) \end{aligned}$$

$$\begin{aligned} T(r\vec{u}) &= T\big(r(u_1, u_2, u_3)\big) \\ &= T(ru_1, ru_2, ru_3) \\ &= (ru_1-5ru_2+4ru_3, \ ru_2-6ru_3) \end{aligned}$$

$$= r\left(u_1 - 5u_2 + 4u_3,\ \ u_2 - 6u_3\right)$$

$$= rT(\vec{u})$$

Since $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{u}) = rT(\vec{u})$, then function T is a linear transformation.

Domain: $T:\ \mathbb{R}^3 \to \mathbb{R}^2$

$$T\left(x_1, x_2, x_3\right) = \left(x_1 - 5x_2 + 4x_3,\ \ x_2 - 6x_3\right)$$

$$= \begin{pmatrix} x_1 - 5x_2 + 4x_3 \\ x_2 - 6x_3 \end{pmatrix}$$

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ A = & \begin{pmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{pmatrix} \end{array}$$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T = f_A$

$$T\left(x_1, x_2, x_3, x_4\right) = \left(x_1 + 2x_2,\ \ 0,\ \ 2x_2 + x_4,\ \ x_2 - x_4\right)$$

Solution

Let $\vec{u} = \left(u_1, u_2, u_3, u_4\right)$ *and* $\vec{v} = \left(v_1, v_2, v_3, v_4\right)$

$$T(\vec{u}+\vec{v}) = T\left(u_1 + v_1,\ \ u_2 + v_2,\ \ u_3 + v_3,\ \ u_4 + v_4\right)$$

$$= \left(\left(u_1 + v_1\right) + 2\left(u_2 + v_2\right),\ \ 0,\ \ 2\left(u_2 + v_2\right) - 3\left(u_4 + v_4\right),\ \ \left(u_2 + v_2\right) - \left(u_4 + v_4\right)\right)$$

$$= \left(u_1 + v_1 + 2u_2 + 2v_2,\ \ 0,\ \ 2u_2 + 2v_2 - 3u_4 - 3v_4,\ \ u_2 + v_2 - u_4 - v_4\right)$$

$$= \left(\left(u_1 + 2u_2\right) + \left(v_1 + 2v_2\right),\ \ 0,\ \ \left(2u_2 - 3u_4\right) + \left(2v_2 - 3v_4\right),\ \ \left(u_2 - u_4\right) + \left(v_2 - v_4\right)\right)$$

$$= \left(u_1 + 2u_2,\ \ 0,\ \ 2u_2 - 3u_4,\ \ u_2 - u_4\right) + \left(v_1 + 2v_2,\ \ 0,\ \ 2v_2 - 3v_4,\ \ v_2 - v_4\right)$$

$$= T(\vec{u}) + T(\vec{v})$$

$$T(r\vec{u}) = T\left(r\left(u_1, u_2, u_3, u_4\right)\right)$$

$$= T\left(ru_1, ru_2, ru_3, ru_4\right)$$

$$= \left(ru_1 + 2ru_2,\ \ 0,\ \ 2ru_2 - 3ru_4,\ \ ru_2 - ru_4\right)$$

$$= r\left(u_1 + 2u_2,\ \ 0,\ \ 2u_2 - 3u_4,\ \ u_2 - u_4\right)$$

$$= rT(\vec{u})$$

Since $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{u}) = rT(\vec{u})$, then function *T* is a linear transformation.

Domain: $T:\ \mathbb{R}^4 \to \mathbb{R}^4$

$$T\left(x_1, x_2, x_3, x_4\right) = \left(x_1 + 2x_2,\ \ 0,\ \ 2x_2 + x_4,\ \ x_2 - x_4\right)$$

$$= \begin{pmatrix} x_1 + 2x_2 \\ 0 \\ 2x_2 + x_4 \\ x_2 - x_4 \end{pmatrix}$$

$$A = \begin{array}{c} \begin{matrix} \ \ x_1 & x_2 & x_3 & x_4 \end{matrix} \\ \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \end{array}$$

Exercise

Determine if the given function *T* is a linear transformation. Also give the domain and range of *T*; if *T* is linear, find the *A* such $T = f_A$

$$T\left(x_1, x_2, x_3, x_4\right) = 3x_1 + 4x_3 - 2x_4$$

Solution

Let $\vec{u} = \left(u_1, u_2, u_3, u_4\right)$ *and* $\vec{v} = \left(v_1, v_2, v_3, v_4\right)$

$$T(\vec{u}+\vec{v}) = T\left(u_1 + v_1,\ \ u_2 + v_2,\ \ u_3 + v_3,\ \ u_4 + v_4\right)$$

$$= 3\left(u_1 + v_1\right) + 4\left(u_3 + v_3\right) - 2\left(u_4 + v_4\right)$$

$$= 3u_1 + 3v_1 + 4u_3 + 4v_3 - 2u_4 - 2v_4$$

$$= \left(3u_1 + 4u_3 - 2u_4\right) + \left(3v_1 + 4v_3 - 2v_4\right)$$

$$= T(\vec{u}) + T(\vec{v})$$

$$T(r\vec{u}) = T\left(r\left(u_1, u_2, u_3, u_4\right)\right)$$

$$= T\left(ru_1, ru_2, ru_3, ru_4\right)$$

$$= 3ru_1 + 4ru_3 - 2ru_4$$
$$= r\left(3u_1 + 4u_3 - 2u_4\right)$$
$$= rT(\vec{u})$$

Since $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{u}) = rT(\vec{u})$, then function T is a linear transformation.

Domain: $T:\ \mathbb{R}^4 \to \mathbb{R}^1$

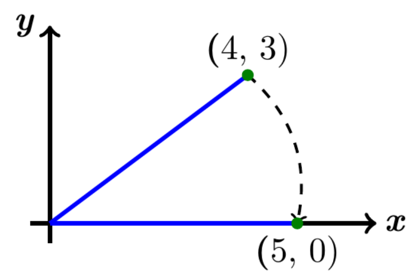
$$T\left(x_1, x_2, x_3, x_4\right) = 3x_1 + 4x_3 - 2x_4$$
$$= \left(3x_1 + 4x_3 - 2x_4\right)$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \end{array}$$
$$A = \begin{pmatrix} 3 & 0 & 4 & -2 \end{pmatrix}$$

Exercise

A Givens rotation is a linear transformation from $\mathbb{R}^n$ *to* $\mathbb{R}^n$ used in computer to create a zero entry in a vector (usually a column of a matrix). The standard matrix of a Givens rotation in $\mathbb{R}^2$ has the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}; \qquad a^2 + b^2 = 1$$



A Givens rotation in $\mathbb{R}^2$

Find a and b that $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is rotated into $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$.

Solution

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$$\begin{cases} 4a - 3b = 5 & (1) \\ 4b + 3a = 0 & \rightarrow \quad b = -\frac{3a}{4} \end{cases}$$

$$(1) \quad \rightarrow \quad 4a - 3\left(-\frac{3a}{4}\right) = 5$$

$$4a+\frac{9a}{4}=5$$

$$\left(4+\frac{9}{4}\right)a=5$$

$$\frac{25}{4}a=5$$

$$a=\frac{4}{5}$$

$$b=-\frac{3}{4}\frac{4}{5}$$

$$b=-\frac{3}{5}$$

$$A=\begin{pmatrix}\frac{4}{5} & \frac{3}{5}\\ -\frac{3}{5} & \frac{4}{5}\end{pmatrix}$$

Solution Section 4.3 – LU−Decompositions

Exercise

What matrix E puts A into triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$$

Solution

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{pmatrix} \quad R_3 - 3R_1 \quad : \ell_{31}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix} = U$$

$$E_{31}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix} = U$$

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

$$L = E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$A = LU$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix}$$

Exercise

Solve $L\vec{c}=\vec{b}$ to find $\vec{c}$. Then solve $U\vec{x}=\vec{c}$ to find $\vec{x}$. What was A?

$$L=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\qquad U=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\qquad b=\begin{bmatrix}4\\5\\6\end{bmatrix}$$

Solution

$$L\vec{c}=\vec{b}$$

$$\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}4\\5\\6\end{pmatrix}$$

$$\begin{cases} \boxed{c_1=4} \\ c_1+c_2=5 \Rightarrow \underline{|c_2} = 5-4 \underline{=1}| \\ c_1+c_2+c_3=6 \Rightarrow \underline{|c_3} = 6-4-1 \underline{=1}| \end{cases}$$

$$\vec{c}=\begin{pmatrix}4\\1\\1\end{pmatrix}$$

$$U\vec{x}=\vec{c}$$

$$\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}4\\1\\1\end{pmatrix}$$

$$\Rightarrow\begin{cases}x+y+z=4\\ y+z=1\\ z=1\end{cases}\rightarrow\begin{cases}x=3\\ y=0\end{cases}$$

$$\vec{x}=\begin{pmatrix}3\\0\\1\end{pmatrix}$$

$$L\vec{c}=\vec{b}$$

$$LU\vec{x}=\vec{b}$$

$$\underbrace{\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}3\\0\\1\end{pmatrix}}_{\vec{x}}=\underbrace{\begin{pmatrix}4\\5\\6\end{pmatrix}}_{\vec{b}}$$

Exercise

Find L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots

Solution

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

Exercise

For which c is $A = LU$ impossible – with three pivots?

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad R_2 - 3R_1$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & c-6 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad R_3 - R_1 \qquad \to c-6 \neq 0 \Rightarrow \boxed{c \neq 6}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & c-6 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \frac{1}{c-6} R_1$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{c-6} \\ 0 & 1 & 1 \end{pmatrix} \quad R_3 - R_2$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{c-6} \\ 0 & 0 & \frac{c-7}{c-6} \end{pmatrix} \quad \to c-7 \neq 0$$

$\Rightarrow \quad \underline{c \neq 7}$

LU will be impossible for $c = 6$ *and* $c = 7$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix} \quad \frac{1}{2}R_1 \quad \boxed{\frac{1}{2} : \ell_{11}}$$

$$\begin{bmatrix} 1 & 4 \\ -1 & -1 \end{bmatrix} \quad R_2 + R_1 \quad \boxed{1 : \ell_{21}}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \quad \frac{1}{3}R_2 \quad \boxed{\frac{1}{3} : \ell_{22}}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \quad \boldsymbol{U}$$

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & \frac{1}{3} \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} \quad \boldsymbol{L}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$\begin{cases} 2y_1 = -2 \\ -y_1 + 3y_2 = -2 \end{cases}$$

$$\begin{cases} y_1 = -1 \\ y_2 = -1 \end{cases}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\begin{cases} x_1 + 4x_2 = -1 \\ \underline{x_2 = -1} \end{cases} \rightarrow \underline{x_1 = 3}$$

The solution: $x_1 = 3$ *and* $x_2 = -1$

Exercise

Find an *LU*-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix}$$

Solution

$$\begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix} \quad -\tfrac{1}{5}R_1 \quad \boxed{-\tfrac{1}{5} : \ell_1}$$

$$\begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix} \quad R_2 - 6R_1 \quad \boxed{-6 : \ell_{21}}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & -7 \end{bmatrix} \quad -\tfrac{1}{7}R_2 \quad \boxed{-\tfrac{1}{7} : \ell_2}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \boldsymbol{U}$$

$$\begin{bmatrix} -\tfrac{1}{5} & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 0 \\ -6 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} -5 & 0 \\ 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 0 \\ 6 & -\frac{1}{7} \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix} \quad \boldsymbol{L}$$

$$\begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix}$$

$$\begin{cases} -5y_1 = -10 & \rightarrow \underline{y_1 = 2} \\ 6y_1 - 7y_2 = 19 & \Rightarrow \underline{y_2 = -1} \end{cases}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{cases} x_1 + 2x_2 = 2 \\ \underline{x_2 = -1} \end{cases} \rightarrow \underline{x_1 = 4}$$

The solution: $\underline{x_1 = 4 \quad and \quad x_2 = -1}$

Exercise

Find an *LU*-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix} \quad \frac{1}{2}R_1 \quad \boxed{\frac{1}{2} : \ell_1}$$

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix} \quad R_3 + R_1 \quad \boxed{1 : \ell_{31}}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ 0 & 4 & 1 \end{bmatrix} \quad -\tfrac{1}{2} R_2 \quad \boxed{-\frac{1}{2} : \ell_{22}}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 4 & 1 \end{bmatrix} \quad R_3 - 4R_2 \quad \boxed{-4 : \ell_{32}}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & -4 & 1 \end{bmatrix} \xrightarrow{E_4^{-1}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} \quad \tfrac{1}{5} R_3 \quad \boxed{\frac{1}{5} : \ell_{32}}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & -4 & \frac{1}{5} \end{bmatrix} \xrightarrow{E_5^{-1}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad U$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix} \quad L$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$$

$$\begin{cases} 2y_1 = -4 \\ -2y_2 = -2 \\ y_1 + 4y_2 + 5y_3 = 6 \end{cases} \Rightarrow \begin{matrix} y_1 = -2 \\ y_2 = 1 \\ y_3 = 0 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{cases} x_1 - x_2 - x_3 = -2 \ \rightarrow \ \boxed{x_1 = -1} \\ \quad x_2 - x_3 = 1 \ \rightarrow \ \boxed{x_2 = 1} \\ \qquad\quad \boxed{x_3 = 0} \end{cases}$$

Solution: $\boxed{x_1 = -1, \quad x_2 = 1, \quad x_3 = 0}$

Exercise

Find an *LU*-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

Solution

$$\begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad -\tfrac{1}{3}R_1 \quad \boxed{-\tfrac{1}{3} : \ell_1} \qquad \begin{bmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad R_2 - R_1 \quad \boxed{-1 : \ell_{21}} \qquad \begin{bmatrix} -3 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} -3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \tfrac{1}{2}R_2 \quad \boxed{\tfrac{1}{2} : \ell_{22}} \qquad \begin{bmatrix} -3 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad R_3 - R_2 \quad \boxed{-1 : \ell_{32}} \qquad \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{E_4^{-1}} \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad U \qquad\qquad \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad L$$

$$\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

$$\begin{cases} -3y_1 = -33 \Rightarrow \underline{y_1 = 11|} \\ y_1 + 2y_2 = 7 \Rightarrow \underline{y_2 = -2|} \\ y_2 + y_3 = -1 \Rightarrow \underline{y_3 = 1|} \end{cases}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{cases} x_1 - 4x_2 + 2x_3 = 11 \Rightarrow \underline{x_1 = 1|} \\ \underline{x_2 = -2|} \\ \underline{x_3 = 1|} \end{cases}$$

Solution: $\underline{x_1 = 1, \quad x_2 = -2, \quad x_3 = 1|}$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$$

<u>Solution</u>

$$\begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} \quad \begin{matrix} R_2 + R_1 & \ell_{21} = 1 \\ R_3 - 2R_1 & \ell_{31} = -2 \end{matrix}$$

$$\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{bmatrix} \quad R_3 + 5R_2 \quad \ell_{32} = 5$$

$$\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}$$

$L\vec{y} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$$

$$\begin{cases} \boxed{y_1 = -7} \\ -y_1 + y_2 = 5 & \rightarrow \boxed{y_2 = -2} \\ 2y_1 - 5y_2 + y_3 = 2 & \rightarrow \boxed{y_3 = 6} \end{cases}$$

$U\vec{x} = \vec{y}$

$$\begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 6 \end{bmatrix}$$

$$\begin{cases} 3x_1 - 7x_2 - 2x_3 = -7 & \rightarrow \boxed{x_1 = 3} \\ -2x_2 - x_3 = -2 & \rightarrow \boxed{x_2 = 4} \\ \boxed{x_3 = -6} \end{cases}$$

Solution: $\boxed{x_1 = 3, \quad x_2 = 4, \quad x_3 = -6}$

Exercise

Find an *LU*-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} 2 & -6 & 4 \\ -4 & 8 & 0 \\ 0 & -4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 2 & -6 & 4 \\ -4 & 8 & 0 \\ 0 & -4 & 6 \end{bmatrix} \quad R_2 + 2R_1 \quad \ell_{21} = 2$$

$$\begin{bmatrix} 2 & -6 & 4 \\ 0 & -4 & 8 \\ 0 & -4 & 6 \end{bmatrix} \quad R_3 - R_2 \quad \ell_{32} = -1$$

$$\begin{bmatrix} 2 & -6 & 4 \\ 0 & -4 & 8 \\ 0 & 0 & -2 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$L\vec{y} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$$

$$\begin{cases} \underline{y_1 = 2|} \\ -2y_1 + y_2 = -4 & \rightarrow \underline{y_2 = 0|} \\ y_2 + y_3 = 6 & \rightarrow \underline{y_3 = 6|} \end{cases}$$

$U\vec{x} = \vec{y}$

$$\begin{bmatrix} 2 & -6 & 4 \\ 0 & -4 & 8 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$$

$$\begin{cases} 3x_1 - 7x_2 - 2x_3 = 2 & \rightarrow \underline{x_1 = -11|} \\ -4x_2 + 8x_3 = 0 & \rightarrow \underline{x_2 = -6|} \\ -2x_3 = 6 & \rightarrow \underline{x_3 = -3|} \end{cases}$$

Solution: $\underline{x_1 = 3, \quad x_2 = 4, \quad x_3 = -6|}$

Exercise

Find an *LU*-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} 2 & -4 & 2 \\ -4 & 5 & 2 \\ 6 & -9 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 2 & -4 & 2 \\ -4 & 5 & 2 \\ 6 & -9 & 1 \end{bmatrix} \quad \begin{matrix} R_2 + 2R_1 & \ell_{21} = 2 \\ R_3 - 3R_1 & \ell_{31} = -3 \end{matrix}$$

$$\begin{bmatrix} 2 & -4 & 2 \\ 0 & -3 & 6 \\ 0 & 3 & -5 \end{bmatrix} \quad R_3 + R_2 \quad \ell_{32} = 1$$

$$\begin{bmatrix} 2 & -4 & 2 \\ 0 & -3 & 6 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$$

$L\vec{y} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$$

$$\begin{cases} \underline{y_1 = 6} \\ -2y_1 + y_2 = 0 & \rightarrow \underline{y_2 = 12} \\ 3y_1 - y_2 + y_3 = 6 & \rightarrow \underline{y_3 = 0} \end{cases}$$

$U\vec{x} = \vec{y}$

$$\begin{bmatrix} 2 & -4 & 2 \\ 0 & -3 & 6 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{cases} 2x_1 - 4x_2 + 2x_3 = 6 & \rightarrow \quad \underline{x_1 = -5} \\ \quad -3x_2 + 6x_3 = 12 & \rightarrow \quad \underline{x_2 = -4} \\ & \rightarrow \quad \underline{x_3 = 0} \end{cases}$$

Solution: $\underline{x_1 = -5, \quad x_2 = -4, \quad x_3 = 0}$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix} \quad \begin{matrix} R_2 - R_1 & \ell_{21} = -1 \\ R_3 - 3R_1 & \ell_{31} = -3 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 10 & -1 \end{bmatrix} \quad R_3 + 5R_2 \quad \ell_{32} = 5$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}$$

$$L\vec{y} = \vec{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$$

$$\begin{cases} \quad \underline{y_1 = 0} \\ y_1 + y_2 = -5 & \rightarrow \quad \underline{y_2 = -5} \\ 3y_1 - 5y_2 + y_3 = 7 & \rightarrow \quad \underline{y_3 = -18} \end{cases}$$

$U\vec{x} = \vec{y}$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -18 \end{bmatrix}$$

$$\begin{cases} x_1 - x_2 + 2x_3 = 0 & \rightarrow \quad \underline{x_1 = -5} \\ -2x_2 - x_3 = -5 & \rightarrow \quad \underline{x_2 = 1} \\ -6x_3 = -18 & \rightarrow \quad \underline{x_3 = 3} \end{cases}$$

Solution: $\underline{x_1 = -5, \quad x_2 = 1, \quad x_3 = 3}$

Exercise

Find an LU-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \\ 7 \end{bmatrix}$$

Solution

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad -R_1 \quad \boxed{-1 : \ell_{11}} \qquad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_1^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad R_2 - 2R_1 \quad \boxed{-2 : \ell_{21}} \qquad \begin{bmatrix} -1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_2^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & 0 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad \tfrac{1}{3}R_2 \quad \boxed{\tfrac{1}{3} : \ell_{22}} \qquad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_3^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad R_3 + R_2 \quad \boxed{1 : \ell_{23}} \qquad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_4^{-1}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix}1&0&-1&0\\0&1&0&2\\0&0&2&2\\0&0&1&5\end{bmatrix}\quad \tfrac{1}{2}R_3 \quad \boxed{\tfrac{1}{2}:\ell_{23}}$$

$$\begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&\tfrac{1}{2}&0\\0&0&0&1\end{bmatrix}\xrightarrow{E_4^{-1}}\begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&2&0\\0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&-1&0\\0&1&0&2\\0&0&1&1\\0&0&1&5\end{bmatrix}\quad R_4-R_3 \quad \boxed{-1:\ell_{43}}$$

$$\begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&2&0\\0&0&-1&1\end{bmatrix}\xrightarrow{E_5^{-1}}\begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&2&0\\0&0&1&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&-1&0\\0&1&0&2\\0&0&1&1\\0&0&0&4\end{bmatrix}\quad \tfrac{1}{4}R_4 \quad \boxed{\tfrac{1}{4}:\ell_{43}}$$

$$\begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&2&0\\0&0&1&\tfrac{1}{4}\end{bmatrix}\xrightarrow{E_6^{-1}}\begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&2&0\\0&0&1&4\end{bmatrix}$$

$$\begin{bmatrix}1&0&-1&0\\0&1&0&2\\0&0&1&1\\0&0&0&1\end{bmatrix}\quad \boldsymbol{U}$$

$$\begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&2&0\\0&0&1&4\end{bmatrix}\quad \boldsymbol{L}$$

For lower triangular: $\begin{bmatrix}-1&0&0&0\\-2&\tfrac{1}{3}&0&0\\0&1&\tfrac{1}{2}&0\\0&0&-1&\tfrac{1}{4}\end{bmatrix}\xrightarrow{E^{-1}}\begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&2&0\\0&0&1&4\end{bmatrix}$

$$\begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&2&0\\0&0&1&4\end{bmatrix}\begin{bmatrix}1&0&-1&0\\0&1&0&2\\0&0&1&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}5\\-1\\3\\7\end{bmatrix}$$

$$\begin{bmatrix}-1&0&0&0\\2&3&0&0\\0&-1&2&0\\0&0&1&4\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=\begin{bmatrix}5\\-1\\3\\7\end{bmatrix}\rightarrow\begin{cases}-y_1=5\rightarrow \underline{y_1=-5|}\\2y_1+3y_2=-1\rightarrow \underline{y_2=3|}\\-y_2+2y_3=3\rightarrow \underline{y_3=3|}\\y_3+4y_4=7\rightarrow \underline{y_4=1|}\end{cases}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 3 \\ 1 \end{bmatrix} \quad \rightarrow \quad \begin{cases} x_1 - x_3 = -5 \rightarrow \underline{x_1 = -3} \\ x_2 + 2x_4 = 3 \rightarrow \underline{x_2 = 1} \\ x_3 + x_4 = 3 \rightarrow \underline{x_3 = 2} \\ \underline{x_4 = 1} \end{cases}$$

Solution: $\underline{x_1 = -3, \quad x_2 = 1, \quad x_3 = 2, \quad x_4 = 3}$

Exercise

Find an *LU*-decomposition of the coefficient matrix, and then use to solve the system

$$\begin{bmatrix} 1 & -2 & -2 & -3 \\ 3 & -9 & 0 & -9 \\ -1 & 2 & 4 & 7 \\ -3 & -6 & 26 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \\ 3 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & -2 & -2 & -3 \\ 3 & -9 & 0 & -9 \\ -1 & 2 & 4 & 7 \\ -3 & -6 & 26 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 & \ell_{21} = -3 \\ R_3 + R_1 & \ell_{31} = 1 \\ R_4 + 3R_1 & \ell_{41} = 3 \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & -12 & 20 & -7 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - 4R_2 & \ell_{42} = -4 \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -7 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 + 2R_3 & \ell_{43} = 2 \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -3 & 4 & -2 & 1 \end{bmatrix}$$

$L\vec{y} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -3 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{cases} \boxed{y_1 = 1} & \\ 3y_1 + y_2 = 6 & \rightarrow \ \boxed{y_2 = 3} \\ -y_1 + y_3 = 0 & \rightarrow \ \boxed{y_3 = 1} \\ -3y_1 + 4y_2 - 2y_3 + y_4 = 3 & \rightarrow \ \boxed{y_4 = -4} \end{cases}$$

$U\vec{x} = \vec{y}$

$$\begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -4 \end{bmatrix}$$

$$\begin{cases} x_1 - x_2 + 2x_3 - 3x_4 = 1 & \rightarrow \ \boxed{x_1 = 38} \\ -3x_2 + 6x_3 = 3 & \rightarrow \ \boxed{x_2 = 16} \\ 2x_3 + 4x_4 = 1 & \rightarrow \ \boxed{x_3 = \frac{17}{2}} \\ & \rightarrow \ \boxed{x_4 = -4} \end{cases}$$

Solution: $\boxed{x_1 = 38, \quad x_2 = 16, \quad x_3 = \frac{17}{2}, \quad x_4 = -4}$

Exercise

Find an LU factorization matrix $\begin{pmatrix} 2 & 5 \\ -3 & -4 \end{pmatrix}$

Solution

$$\begin{pmatrix} 2 & 5 \\ -3 & -4 \end{pmatrix} \quad R_2 + \frac{3}{2}R_1 \quad \ell_{21} = \frac{3}{2}$$

$$\begin{pmatrix} 2 & 5 \\ 0 & \frac{7}{2} \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 \\ -\frac{3}{2} & 1 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 2 & 5 \\ -3 & -4 \end{pmatrix}} = \underset{L}{\begin{pmatrix} 1 & 0 \\ -\frac{3}{2} & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 2 & 5 \\ 0 & \frac{7}{2} \end{pmatrix}}$$

Exercise

Find an LU factorization matrix $\begin{pmatrix} 6 & 4 \\ 12 & 5 \end{pmatrix}$

Solution

$$\begin{pmatrix} 6 & 4 \\ 12 & 5 \end{pmatrix} \quad R_2 - 2R_1 \quad \ell_{21} = -2$$

$$\begin{pmatrix} 6 & 4 \\ 0 & -3 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 6 & 4 \\ 12 & 5 \end{pmatrix}} = \underset{L}{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 6 & 4 \\ 0 & -3 \end{pmatrix}}$$

Exercise

Find an LU factorization matrix $\begin{pmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{pmatrix}$

Solution

$$\begin{pmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{pmatrix} \quad \begin{matrix} R_2 + 3R_1 & \ell_{21} = 3 \\ R_3 - 3R_1 & \ell_{31} = -3 \end{matrix}$$

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 6 & 8 \end{pmatrix} \qquad R_3 - 2R_2 \quad \ell_{32} = -2$$

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{pmatrix}} = \underset{L}{\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}}$$

Exercise

Find an LU factorization matrix $\begin{pmatrix} -5 & 0 & 4 \\ 10 & 2 & -5 \\ 10 & 10 & 16 \end{pmatrix}$

Solution

$$\begin{pmatrix} -5 & 0 & 4 \\ 10 & 2 & -5 \\ 10 & 10 & 16 \end{pmatrix} \qquad \begin{matrix} R_2 + 2R_1 & \ell_{21} = -2 \\ R_3 + 2R_1 & \ell_{31} = -2 \end{matrix}$$

$$\begin{pmatrix} -5 & 0 & 4 \\ 0 & 2 & 3 \\ 0 & 10 & 24 \end{pmatrix} \qquad R_3 - 5R_2 \quad \ell_{31} = -5$$

$$\begin{pmatrix} -5 & 0 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 9 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 5 & 1 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} -5 & 0 & 4 \\ 10 & 2 & -5 \\ 10 & 10 & 16 \end{pmatrix}} = \underset{L}{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 5 & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} -5 & 0 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 9 \end{pmatrix}}$$

Exercise

Find an LU factorization matrix $\begin{pmatrix} 3 & 7 & 2 \\ 6 & 19 & 4 \\ 9 & 9 & 14 \end{pmatrix}$

Solution

$$\begin{pmatrix} 3 & 7 & 2 \\ 6 & 19 & 4 \\ 9 & 9 & 14 \end{pmatrix} \quad \begin{matrix} R_2 - 2R_1 & \ell_{21} = -2 \\ R_3 - 3R_1 & \ell_{31} = -3 \end{matrix}$$

$$\begin{pmatrix} 3 & 7 & 2 \\ 0 & 5 & 0 \\ 0 & -12 & 8 \end{pmatrix} \quad R_3 + \frac{12}{5}R_2 \quad \ell_{32} = \frac{12}{5}$$

$$\begin{pmatrix} 3 & 7 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -\frac{12}{5} & 1 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 3 & 7 & 2 \\ 6 & 19 & 4 \\ 9 & 9 & 14 \end{pmatrix}} = \underset{L}{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -\frac{12}{5} & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 3 & 7 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}}$$

Exercise

Find an LU factorization matrix $\begin{pmatrix} 2 & 3 & 2 \\ 4 & 13 & 9 \\ -6 & 5 & 4 \end{pmatrix}$

Solution

$$\begin{pmatrix} 2 & 3 & 2 \\ 4 & 13 & 9 \\ -6 & 5 & 4 \end{pmatrix} \quad \begin{matrix} R_2 - 2R_1 & \ell_{21} = -2 \\ R_3 + 3R_1 & \ell_{31} = 3 \end{matrix}$$

$$\begin{pmatrix} 2 & 3 & 2 \\ 0 & 7 & 5 \\ 0 & 14 & 10 \end{pmatrix} \quad R_3 - 2R_2 \quad \ell_{32} = -2$$

$$\begin{pmatrix} 2 & 3 & 2 \\ 0 & 7 & 5 \\ 0 & 0 & 0 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 2 & 1 \end{pmatrix}$$

$$\underset{A}{\begin{pmatrix} 2 & 3 & 2 \\ 4 & 13 & 9 \\ -6 & 5 & 4 \end{pmatrix}} = \underset{L}{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 2 & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 2 & 3 & 2 \\ 0 & 7 & 5 \\ 0 & 0 & 0 \end{pmatrix}}$$

Exercise

Find an LU factorization matrix $\begin{pmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{pmatrix}$

Solution

$$\begin{pmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{pmatrix} \qquad \begin{matrix} R_2 + R_1 & \ell_{21} = 1 \\ R_3 - 4R_1 & \ell_{31} = -4 \\ R_4 + 2R_1 & \ell_{41} = 2 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & -10 & 15 & 5 \\ 0 & 2 & -3 & -1 \end{pmatrix} \qquad \begin{matrix} R_3 - 5R_2 & \ell_{32} = -5 \\ R_4 + R_2 & \ell_{42} = 1 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 4 & 5 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 4 & 5 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A \qquad = \qquad L \qquad\qquad U$$

Exercise

Find an LU factorization matrix $\begin{pmatrix} 1 & 3 & 1 & 5 \\ 5 & 20 & 6 & 31 \\ -2 & -1 & -1 & -4 \\ -1 & 7 & 1 & 7 \end{pmatrix}$

Solution

$$\begin{pmatrix} 1 & 3 & 1 & 5 \\ 5 & 20 & 6 & 31 \\ -2 & -1 & -1 & -4 \\ -1 & 7 & 1 & 7 \end{pmatrix} \quad \begin{matrix} R_2 - 5R_1 & \ell_{21} = -5 \\ R_3 + 2R_1 & \ell_{31} = 2 \\ R_4 + R_1 & \ell_{41} = 1 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 1 & 5 \\ 0 & 5 & 1 & 5 \\ 0 & 5 & 1 & 6 \\ 0 & 10 & 2 & 12 \end{pmatrix} \quad \begin{matrix} R_3 - R_2 & \ell_{32} = -1 \\ R_4 - 2R_2 & \ell_{42} = -2 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 1 & 5 \\ 0 & 5 & 1 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 1 & 5 \\ 5 & 20 & 6 & 31 \\ -2 & -1 & -1 & -4 \\ -1 & 7 & 1 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 & 5 \\ 0 & 5 & 1 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$A \qquad = \qquad L \qquad\qquad U$$

Exercise

Find an LU factorization matrix $\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix}$

Solution

$$\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix} \quad \begin{matrix} R_2 + 2R_1 & \ell_{21} = 2 \\ R_3 - R_1 & \ell_{31} = -1 \\ R_4 + 3R_1 & \ell_{41} = 3 \end{matrix}$$

$$\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{pmatrix} \quad \begin{matrix} R_3 + 3R_2 & \ell_{32} = 3 \\ R_4 - 4R_2 & \ell_{42} = -4 \end{matrix}$$

$$\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{pmatrix} \quad R_4 - 2R_4 \quad \ell_{44} = -2$$

$$\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}}_{U}$$

Exercise

Find an LU factorization matrix $\begin{pmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{pmatrix}$

Solution

$$\begin{pmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{pmatrix} \quad \begin{matrix} R_2 - 3R_1 & \ell_{21} = -3 \\ R_3 - R_1 & \ell_{31} = -1 \\ R_4 - 2R_1 & \ell_{41} = -2 \\ R_5 + 3R_1 & \ell_{51} = 3 \end{matrix}$$

$$\begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & -3 & -1 & 6 \\ 0 & 6 & 2 & -7 \\ 0 & -9 & -3 & 13 \end{pmatrix} \quad \begin{matrix} R_3 + R_2 & \ell_{32} = 1 \\ R_4 - 2R_2 & \ell_{42} = -2 \\ R_5 + 3R_2 & \ell_{52} = 3 \end{matrix}$$

$$\begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 10 \end{pmatrix} \quad \begin{matrix} R_4 + R_3 & \ell_{43} = 1 \\ R_5 - 2R_3 & \ell_{53} = -2 \end{matrix}$$

$$\begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 1 & 0 \\ -3 & -3 & 2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 \\ -3 & -3 & 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A \qquad = \qquad L \qquad\qquad U$$

Exercise

Let A be a lower triangular $n \times n$ matrix with nonzero entries on the diagonal. Show that A is invertible and A^{-1} is lower triangular.

Solution

Since A is a lower triangular $n \times n$ matrix with nonzero entries on the diagonal, then the determinant is equal to the products of the main diagonal entries.

Therefore, A^{-1} exists and A is invertible.

To find A^{-1}

$$\begin{bmatrix} A & I \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} a_{11} & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ * & a_{22} & \vdots & \vdots & 0 & 1 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right]$$

To pivot the augmented matrix above and the upper triangular for A and I are zeros. The results from the pivot will not change the zero values in the upper triangular, since we are trying to get one in the main diagonal and zero elsewhere.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & b_{11} & 0 & \cdots & 0 \\ 0 & 1 & \vdots & \vdots & * & b_{22} & \vdots & \vdots \\ \vdots & \cdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & b_{n1} & b_{n2} & \cdots & b_{nn} \end{array}\right]$$

Therefore, A^{-1} is lower triangular

Exercise

Let $A = LU$ be an LU factorization. Explain why A can be row reduced to U using only replacement operations.

Solution

Let $A = LU$ be an LU factorization for A.

Since L is unit lower triangular, from previous problem, A is invertible. So, the matrix L can be row reduced to I by using the appropriate pivots to reduced to zero in the lower entries of the main diagonal which maintain one's.
The row operation done to L are row-replacement operations.

If elementary matrices $E_1, E_2, \cdots, E_n$, then

$$\begin{aligned} E_n \cdots E_2 E_1 A &= \left(E_n \cdots E_2 E_1\right) LU \\ &= IU \\ &= U \end{aligned}$$

That implies that A can be row reduced to U using only row-replacement operations.

Exercise

Suppose an $m \times n$ matrix A admits a factorization $A = CD$ where C is $m \times 4$ and D is $4 \times n$.

a) Show that A is the sum of four outer products.

b) Let $m = 400 \quad and \quad n = 100$. Explain why a computer programmer might prefer to store the data from A in the form of two matrices C and D.

Solution

a) C is $m \times 4$ that implies $C = \begin{bmatrix} c_{i1} & c_{i2} & c_{i3} & c_{i4} \end{bmatrix} \qquad 1 \le i \le m$

D is $4 \times n$ that implies $D = \begin{bmatrix} d_{1j} \\ d_{2j} \\ d_{3j} \\ d_{4j} \end{bmatrix} \qquad 1 \le j \le n$

$$\begin{aligned} A &= CD \\ &= \begin{bmatrix} c_{i1} & c_{i2} & c_{i3} & c_{i4} \end{bmatrix} \begin{bmatrix} d_{1j} \\ d_{2j} \\ d_{3j} \\ d_{4j} \end{bmatrix} \\ &= c_{i1} d_{1j} + c_{i2} d_{2j} + c_{i3} d_{3j} + c_{i4} d_{4j} \end{aligned}$$

= Sum of four outer products

b) ***Given***: $m = 400 \quad and \quad n = 100$

The size of matrix A is $400 \times 100 = 40,000$ *entries*

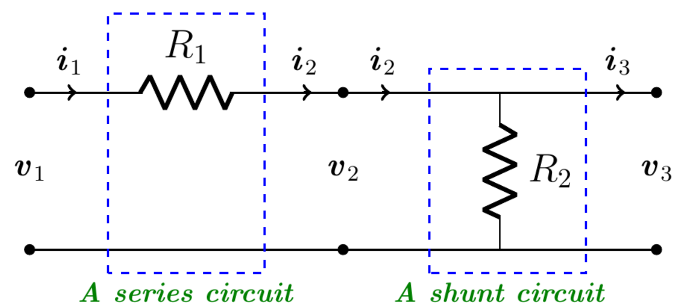
Matrix C has: $400 \times 4 = 1,600$ *entries*

Matrix D has: $100 \times 4 = 400$ *entries*

Both matrices C and D have: $1,600 + 400 = 2,000$ *entries* Only which is lot less than 40,000 entries.

Exercise

A ladder network, where two circuits are connected in series, so that the output of one circuit becomes the input of the next circuit.



The transformation $\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} \longrightarrow \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$ is linear with a transfer matrix A of the ladder network.

Let the transfer matrix A_1 of the series circuit is given by $\begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = A_1 \begin{pmatrix} v_1 \\ i_1 \end{pmatrix}$

Let the transfer matrix A_2 of the shunt circuit is given by $\begin{pmatrix} v_3 \\ i_3 \end{pmatrix} = A_2 \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$

a) Compute the transfer matrix of the ladder network.

b) Design a ladder network whose transfer matrix is $\begin{pmatrix} 1 & -8 \\ -\frac{1}{2} & 5 \end{pmatrix}$

Solution

a) For "series circuit":

The voltage across R_1 is: $v_1 = R_1 i_1$

The current: $i_2 = i_1$

The drop voltage: $v_2 = v_1 - R_1 i_1$

$$\begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = \begin{pmatrix} v_1 - R_1 i_1 \\ i_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ i_1 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix} \qquad \textit{Upper Triangular}$$

For "shunt circuit":

The voltage is: $v_3 = v_2$

Voltage R_2 : $current \times R_2$

The current: $i_2 = \frac{1}{R_2} v_2 + i_3$

$$i_3 = -\frac{1}{R_2} v_2 + i_2$$

$$\begin{pmatrix} v_3 \\ i_3 \end{pmatrix} = \begin{pmatrix} v_2 \\ -\frac{1}{R_2} v_2 + i_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{R_2} & 1 \end{pmatrix} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{R_2} & 1 \end{pmatrix} \qquad \textit{Lower Triangular}$$

The transfer matrix of the ladder network A:

$$A = A_2 A_1$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{R_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -R_1 \\ -\frac{1}{R_2} & \frac{R_1}{R_2} + 1 \end{pmatrix}$$

b) $\begin{pmatrix} 1 & -8 \\ -\frac{1}{2} & 5 \end{pmatrix} = \begin{pmatrix} 1 & -R_1 \\ -\frac{1}{R_2} & \frac{R_1}{R_2}+1 \end{pmatrix}$

$\underline{R_1 = 8}|$

$-\frac{1}{R_2} = -\frac{1}{2} \quad \rightarrow \quad \underline{R_2 = 2}|$

$\frac{R_1}{R_2} + 1 = \frac{8}{2} + 1$

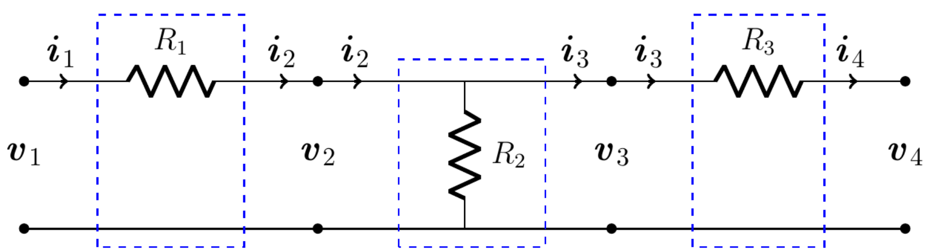
$\qquad = 4 + 1$

$\qquad \underline{= 5}|$ ✓

The given ladder network whose transfer matrix is $\begin{pmatrix} 1 & -8 \\ -.5 & 5 \end{pmatrix}$ has the resistors $R_1 = 8\ \Omega$ and $R_2 = 2\ \Omega$

Exercise

A ladder network, where three circuits are connected in series, so that the output of one circuit becomes the input of the next circuit.



a) Compute the transfer matrix of the ladder network

b) Design a ladder network whose transfer matrix is $\begin{pmatrix} 3 & -12 \\ -\frac{1}{3} & \frac{5}{3} \end{pmatrix}$

Solution

a) Across R_1 is:

The current: $i_2 = i_1$

The drop voltage: $v_2 = v_1 - R_1 i_1$

$$\begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = \begin{pmatrix} v_1 - R_1 i_1 \\ i_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ i_1 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 & -R_1 \\ 0 & 1 \end{pmatrix} \qquad \textit{Upper Triangular}$$

For across R_2:

The voltage is: $v_3 = v_2$

Voltage R_2: *current* $\times R_2$

The current: $i_2 = \frac{1}{R_2} v_2 + i_3$

$$i_3 = -\frac{1}{R_2} v_2 + i_2$$

$$\begin{pmatrix} v_3 \\ i_3 \end{pmatrix} = \begin{pmatrix} v_2 \\ -\frac{1}{R_2} v_2 + i_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{R_2} & 1 \end{pmatrix} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{R_2} & 1 \end{pmatrix} \qquad \textit{Lower Triangular}$$

Across R_3 is:

The current: $i_3 = i_4$

The drop voltage: $v_4 = v_3 - R_3 i_3$

$$\begin{pmatrix} v_4 \\ i_4 \end{pmatrix} = \begin{pmatrix} v_3 - R_3 i_3 \\ i_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -R_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_3 \\ i_3 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & -R_3 \\ 0 & 1 \end{pmatrix} \qquad \textit{Upper Triangular}$$

The transfer matrix of the ladder network A:

$A = A_3 A_2 A_1$

$$=\begin{pmatrix}1 & -R_3\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -\frac{1}{R_2} & 1\end{pmatrix}\begin{pmatrix}1 & -R_1\\ 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1+\frac{R_3}{R_2} & -R_3\\ -\frac{1}{R_2} & 1\end{pmatrix}\begin{pmatrix}1 & -R_1\\ 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1+\frac{R_3}{R_2} & \left(1+\frac{R_3}{R_2}\right)\left(-R_1\right)-R_3\\ -\frac{1}{R_2} & \frac{R_1}{R_2}+1\end{pmatrix}$$

$$=\begin{pmatrix}\frac{R_2+R_3}{R_2} & \frac{-R_1R_2-R_1R_3}{R_2}-R_3\\ -\frac{1}{R_2} & \frac{R_1+R_2}{R_2}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{R_2+R_3}{R_2} & \frac{-R_1R_2-R_1R_3-R_2R_3}{R_2}\\ -\frac{1}{R_2} & \frac{R_1+R_2}{R_2}\end{pmatrix}$$

b) $$\begin{pmatrix}3 & -12\\ -\frac{1}{3} & \frac{5}{3}\end{pmatrix}=\begin{pmatrix}\frac{R_2+R_3}{R_2} & \frac{-R_1R_2-R_1R_3-R_2R_3}{R_2}\\ -\frac{1}{R_2} & \frac{R_1+R_2}{R_2}\end{pmatrix}$$

$$\begin{cases}1+\frac{R_3}{R_2}=3 & \rightarrow \quad \frac{R_3}{3}=2 \quad (1)\\ \frac{-R_1R_2-R_1R_3-R_2R_3}{R_2}=-12 & \rightarrow \quad (2)\\ -\frac{1}{R_2}=-\frac{1}{3} & \rightarrow \quad \boxed{R_2=3}\\ \frac{R_1}{R_2}+1=\frac{5}{3} & \rightarrow \quad \frac{R_1}{3}=\frac{2}{3} \quad (3)\end{cases}$$

$$\begin{cases}(1) \rightarrow \boxed{R_3=6}\\ (3) \rightarrow \boxed{R_1=2}\end{cases}$$

$(2)\quad \dfrac{-(2)(3)-(2)(6)-(3)(6)}{3} \stackrel{?}{=} -12$

$\dfrac{-6-12-18}{3} \stackrel{?}{=} -12$

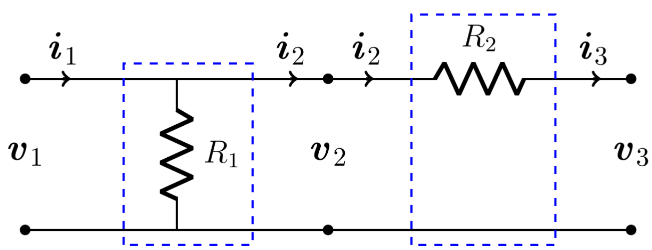
$-\dfrac{36}{3} \stackrel{?}{=} -12$

$-12 = -12 \quad \checkmark$

The given ladder network whose transfer matrix is $\begin{pmatrix} 3 & -12 \\ -\frac{1}{3} & \frac{5}{3} \end{pmatrix}$ has the resistors $R_1 = 2\ \Omega$, $R_2 = 3\ \Omega$, and $R_3 = 6\ \Omega$

Exercise

A ladder network, where two circuits are connected in series, so that the output of one circuit becomes the input of the next circuit.



a) Compute the transfer matrix of the ladder network

b) Find the values of the resistors when the input voltage is 12 volts and current is 6 amps if the output voltage is 9 volts and current is 4 amps

Solution

a) Across R_1 is:

The drop voltage: $current \times R_1$

The current: $i_1 = \dfrac{1}{R_1} v_1 + i_2$

$$i_2 = -\frac{1}{R_1} v_1 + i_1$$

$$\begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -\frac{1}{R_1} v_1 + i_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{R_1} & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ i_1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & 0 \\ -\frac{1}{R_1} & 1 \end{pmatrix} \qquad \textit{Lower Triangular}$$

For across R_2:

The current is: $i_3 = i_2$

The drop voltage: $v_3 = v_3 - R_2 i_2$

$$\begin{pmatrix} v_3 \\ i_3 \end{pmatrix} = \begin{pmatrix} v_2 - R_2 i_2 \\ i_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -R_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 & -R_2 \\ 0 & 1 \end{pmatrix} \qquad \textit{Upper Triangular}$$

The transfer matrix of the ladder network A:

$$A = A_1 A_2$$

$$= \begin{pmatrix} 1 & -R_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{R_1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{R_2}{R_1} & -R_2 \\ -\frac{1}{R_1} & 1 \end{pmatrix}$$

b) $$\begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = \begin{pmatrix} 12 \\ -\frac{12}{R_1} + 6 \end{pmatrix}$$

$$\begin{cases} v_2 = 12 \\ i_2 = -\frac{12}{R_1} + 6 \end{cases}$$

$$\begin{pmatrix} v_3 \\ i_3 \end{pmatrix} = \begin{pmatrix} v_2 - R_2 i_2 \\ i_2 \end{pmatrix}$$

$$\begin{pmatrix} 9 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 - R_2 \left(-\frac{12}{R_1} + 6 \right) \\ -\frac{12}{R_1} + 6 \end{pmatrix}$$

$$\begin{cases} 12 + 12\dfrac{R_2}{R_1} - 6R_2 = 9 \\ -\dfrac{12}{R_1} + 6 = 4 \end{cases}$$

$$\begin{cases} 12\dfrac{R_2}{R_1} - 6R_2 = -3 & (1) \\ -\dfrac{12}{R_1} = -2 & \rightarrow \underline{R_1 = 6} \end{cases}$$

$$(1) \rightarrow \quad \left(4\frac{1}{6} - 2\right)R_2 = -1$$

$$-\frac{8}{6}R_2 = -1$$

$$\underline{R_2 = \frac{3}{4}}$$

The resistors $R_1 = 6\ \Omega$ and $R_2 = \frac{3}{4}\ \Omega$,

Solution Section 4.4 –Eigenvalues & Eigenvectors

Exercise

Find the eigenvalues and eigenvectors of $A,\ A^2,\ A^{-1},\ and\ \ A+4I$:

$$A=\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad and \quad A^2=\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

Check the trace $\lambda_1+\lambda_2$ and the determinant $\lambda_1\lambda_2$ for A and also A^2.

Solution

For A:

$$\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix}$$

$$=(2-\lambda)^2-1$$

$$=\lambda^2-4\lambda+3=0$$

The eigenvalues of A are $\lambda_1=1 \quad and \quad \lambda_2=3$.

The trace of a square matrix A is the sum of the elements on the main diagonal: 2 + 2 agrees with 1+ 3. The det(A) = 3 agrees with the product $\lambda_1\lambda_2$.

The eigenvectors for A are:

$\lambda_1=1$: $\quad \left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix} 2-1 & -1 \\ -1 & 2-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\to\begin{cases} x-y=0 \\ \end{cases}$$

$$\Rightarrow \ x=y$$

Therefore, the eigenvector $V_1=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2=3$: $\quad \left(A-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\to\begin{cases} -x-y=0 \\ \end{cases}$$

$$\Rightarrow \ x=-y$$

Therefore, the eigenvector $V_2=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For A^2:

$$\det\left(A^2-\lambda I\right)=\begin{vmatrix}5-\lambda & -4\\ -4 & 5-\lambda\end{vmatrix}$$

$$=(5-\lambda)^2-16$$

$$=\lambda^2-10\lambda+9=0$$

The eigenvalues of A^2 are $\lambda_1=1\quad and\quad \lambda_2=9$. **Or** $\lambda_1=1^2=1\quad and\quad \lambda_2=3^2=9$

$$\begin{cases} tr(A)=5+5=10\\ \lambda_1+\lambda_2=1+9=10\end{cases}$$

$$\underline{tr(A)=\lambda_1+\lambda_2}\,\big|$$

$$\begin{cases} \left|A^2\right|=\begin{vmatrix}5 & -4\\ -4 & 5\end{vmatrix}=9\\ \lambda_1\lambda_2=1(9)=9\end{cases}$$

$$\Rightarrow\ \underline{\left|A^2\right|=\lambda_1\lambda_2}\,\big|$$

$\lambda_1=1$: $\quad\left(A^2-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix}4 & -4\\ -4 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\rightarrow\begin{cases}4x-4y=0\end{cases}$$

$$\Rightarrow\ \underline{x=y}\,\big|$$

Therefore; the eigenvector $V_1=\begin{pmatrix}1\\ 1\end{pmatrix}$

$\lambda_2=9$: $\quad\left(A^2-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix}-4 & -4\\ -4 & -4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\rightarrow\begin{cases}-4x-4y=0\end{cases}$$

$$\Rightarrow\ \underline{x=-y}\,\big|$$

Therefore; the eigenvector $V_2=\begin{pmatrix}1\\ -1\end{pmatrix}$

For A^{-1}:

$$A^{-1}=\frac{1}{3}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$$

$$=\begin{pmatrix}\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}\end{pmatrix}$$

$$\det\left(A^{-1}-\lambda I\right)=\begin{vmatrix}\frac{2}{3}-\lambda & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}-\lambda\end{vmatrix}$$

$$=\left(\frac{2}{3}-\lambda\right)^2-\frac{1}{9}$$

$$=\lambda^2-\frac{4}{3}\lambda+\frac{1}{3}=0$$

The eigenvalues of A^{-1} are $\lambda_1=1$ *and* $\lambda_2=\frac{1}{3}$.

$\lambda_1=1$: $\quad \left(A^{-1}-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix}\frac{2}{3}-1 & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}-1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}-\frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & -\frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \begin{cases}-\frac{1}{3}x+\frac{1}{3}y=0\end{cases}$$

$\rightarrow \underline{x=y}$

Therefore; the eigenvector $V_1=\begin{pmatrix}1\\ 1\end{pmatrix}$

$\lambda_2=\frac{1}{3}$: $\quad \left(A^{-1}-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix}\frac{2}{3}-\frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}-\frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \begin{cases}\frac{1}{3}x+\frac{1}{3}y=0\end{cases}$$

$\rightarrow \underline{x=-y}$

Therefore; the eigenvector $V_2=\begin{pmatrix}1\\ -1\end{pmatrix}$

For $A+4I$:

$$A+4I=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}+\begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}$$

$$=\begin{pmatrix}6 & 1\\ 1 & 6\end{pmatrix}$$

$$\det\left(A^{-1}-\lambda I\right)=\begin{vmatrix}6-\lambda & 1\\ 1 & 6-\lambda\end{vmatrix}$$

$$=(6-\lambda)^2-1$$

$$=\lambda^2-12\lambda+35=0$$

The eigenvalues of A^{-1} are $\lambda_1=5$ *and* $\lambda_2=7$.

$\lambda_1=5$: $\left(A+4I-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix}6-5 & 1\\ 1 & 6-5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \begin{cases}x+y=0\\ \end{cases}$$

$\rightarrow \underline{x=-y}$

Therefore; the eigenvector $V_1=\begin{pmatrix}1\\ -1\end{pmatrix}$

$\lambda_2=7$: $\left(A+4I-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \begin{cases}\\ x-y=0\end{cases}$$

$\rightarrow \underline{x=y}$

Therefore; the eigenvector $V_2=\begin{pmatrix}1\\ 1\end{pmatrix}$

The eigenvalues$(A)=\lambda$

The eigenvalues $\left(A^2\right)=\lambda^2$

The eigenvalues $\left(A^{-1}\right)=\frac{1}{\lambda}$

Exercise

Show directly that the given vectors are eigenvectors of the given matrix. What are the corresponding eigenvalues?

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$$

Solution

$$\begin{aligned} A\vec{v}_1 &= \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ -21 \end{bmatrix} \\ &= 7\begin{bmatrix} 1 \\ -3 \end{bmatrix} \\ &= 7\vec{v}_1 \end{aligned}$$

$\vec{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue 7

$$\begin{aligned} A\vec{v}_2 &= \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= 0\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= 0\vec{v}_2 \end{aligned}$$

$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 0

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & -2 \\ -3 & 6-\lambda \end{vmatrix} \\ &= (1-\lambda)(6-\lambda) - 6 \\ &= 6 - 7\lambda + \lambda^2 - 6 \\ &= \lambda^2 - 7\lambda = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = 0 \quad and \quad \lambda_2 = 7$

Exercise

For which real numbers c does this matrix A have

$$A = \begin{pmatrix} 2 & -c \\ -1 & 2 \end{pmatrix}$$

a) Two real eigenvalues and eigenvectors.

b) A repeated eigenvalue with only one eigenvector

c) Two complex eigenvalues and eigenvectors.

Solution

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 2-\lambda & -c \\ -1 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)^2 - c \\ &= \lambda^2 - 4\lambda + 4 - c = 0 \end{aligned}$$

$$\lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-c)}}{2(1)}$$

$$\boxed{\lambda_{1,2} = \frac{4 \pm \sqrt{16+4c}}{2}}$$

a) Two real eigenvalues and eigenvectors, when

$$16 + 4c > 0$$

$$4c > -16$$

$$\boxed{c > -4}$$

b) A repeated eigenvalue with only one eigenvector, when

$$16 + 4c = 0$$

$$\boxed{c = -4}$$

c) Two complex eigenvalues and eigenvectors, when

$$16 + 4c < 0$$

$$\boxed{c < -4}$$

Exercise

Find the eigenvalues of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{AB}$, and $\boldsymbol{BA}$:

$$A=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \quad B=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \quad AB=\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix} \quad BA=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$$

a) The eigenvalues of $\boldsymbol{AB}$ (are equal to) (are not equal to) eigenvalues of $\boldsymbol{A}$ times eigenvalues of $\boldsymbol{B}$.

b) The eigenvalues of $\boldsymbol{AB}$ (are equal to) (are not equal to) eigenvalues of $\boldsymbol{BA}$.

Solution

Since $\boldsymbol{A}$ is a lower triangular, then $\lambda_1=\lambda_2=1$

Since $\boldsymbol{B}$ is an upper triangular, then $\lambda_1=\lambda_2=1$

$$\det(AB-I)=\begin{vmatrix}1-\lambda & 1\\ 1 & 2-\lambda\end{vmatrix}$$

$$=(1-\lambda)(2-\lambda)-1$$

$$=\lambda^2-3\lambda+1=0$$

$$\lambda_{1,2}=\frac{3\pm\sqrt{5}}{2}$$

$$\det(BA-I)=\begin{vmatrix}2-\lambda & 1\\ 1 & 1-\lambda\end{vmatrix}$$

$$=(2-\lambda)(1-\lambda)-1$$

$$=\lambda^2-3\lambda+1=0$$

$$\lambda_{1,2}=\frac{3\pm\sqrt{5}}{2}$$

a) The eigenvalues of $\boldsymbol{AB}$ are ***not*** equal to eigenvalues of $\boldsymbol{A}$ times eigenvalues of $\boldsymbol{B}$.

b) The eigenvalues of $\boldsymbol{AB}$ are equal to the eigenvalues of $\boldsymbol{BA}$.

Exercise

When $a+b=c+d$ show that (1, 1) is an eigenvector and find both eigenvalues of

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Solution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}=\begin{pmatrix} a+b \\ c+d \end{pmatrix}$$

$$=\begin{pmatrix} a+b \\ a+b \end{pmatrix}$$

$$=(a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

If $a+b=c+d=\lambda_1$

$tr(A)=a+d=\lambda_1+\lambda_2$

$$\begin{aligned}\lambda_2 &= (a+d)-\lambda_1 \\ &= a+d-(a+b) \\ &= a+d-a-b \\ &= d-b \qquad\qquad \text{or} \quad = a-c\end{aligned}$$

The eigenvalues for λ_2:

$$\begin{pmatrix} a-\lambda_2 & b \\ c & d-\lambda_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a-(a-c) & b \\ c & d-(d-b) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c & b \\ c & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\to\{cx+by=0$$

$cx=-by$

The eigenvector: $V_2=\begin{pmatrix} b \\ -c \end{pmatrix}$

Exercise

The eigenvalues of A equal to the eigenvalues of A^T. This is because $\det(A-\lambda I)$ equals $\det\left(A^T-\lambda I\right)$.

That is true because ______. Show by an example that the eigenvectors of A and A^T are not the same.

Solution

$$\begin{aligned}\det(A-\lambda I) &= \det(A-\lambda I)^T \\ &= \det\left(A^T-(\lambda I)^T\right) \\ &= \det\left(A^T-\lambda I\right)\end{aligned}$$

Therefore, A and A^T have the same eigenvalues.

Let consider the matrix:

$$A=\begin{pmatrix}0 & 1\\ 4 & 0\end{pmatrix} \Rightarrow A^T=\begin{pmatrix}0 & 4\\ 1 & 0\end{pmatrix}$$

$$\begin{aligned}|A-\lambda I| &= \begin{vmatrix}-\lambda & 1\\ 4 & -\lambda\end{vmatrix} \\ &= \lambda^2-4=0\end{aligned}$$

The eigenvalues of A are: $\lambda_{1,2}=\pm 2$

For $\lambda_1=-2$: $\quad (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}2 & 1\\ 4 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \rightarrow \begin{cases}2x+y=0\end{cases}$$

$y=-2x$

$$V_1=\begin{pmatrix}1\\ -2\end{pmatrix}$$

For $\lambda_2=2$: $\quad (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix}-2 & 1\\ 4 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \rightarrow \begin{cases}-2x+y=0\end{cases}$$

$y=2x$

$$V_2=\begin{pmatrix}1\\ 2\end{pmatrix}$$

For the transpose matrix A^T

$$\left|A^T-\lambda I\right|=\begin{vmatrix}-\lambda & 4\\ 1 & -\lambda\end{vmatrix}$$

$$= \lambda^2 - 4 = 0$$

The eigenvalues of A^T are: $\underline{\lambda_{1,2} = \pm 2}$

For $\lambda_1 = -2$: $\left(A^T - \lambda_1 I\right)V_3 = 0$

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \\ x + 2y = 0 \end{cases}$$

$\underline{x = -2y}$

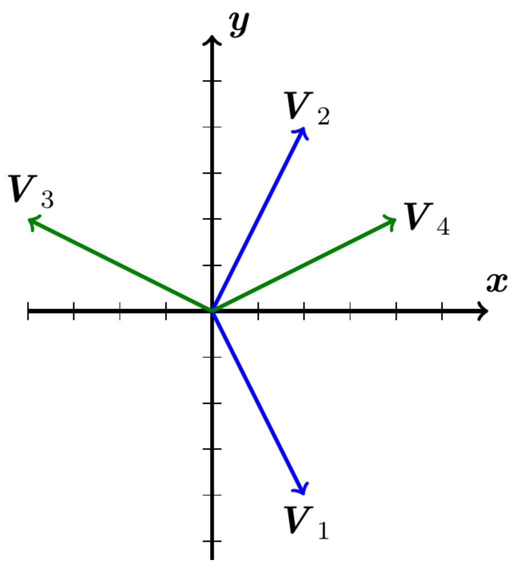
$$V_3 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2$: $\left(A^T - \lambda_2 I\right)V_4 = 0$

$$\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \\ x - 2y = 0 \end{cases}$$

$\underline{x = 2y}$

$$V_4 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$



$$V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad V_3 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad V_4 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The eigenvectors of A and A^T are not the same and from the graph they are not on same line.

Exercise

Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. Compute the eigenvalues and eigenvectors of A.

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^2 + 1 = 0$$

$$(2-\lambda)^2 = -1$$

$$2 - \lambda = \pm\sqrt{-1}$$

$$\underline{= \pm i}$$

The eigenvalues of A are: $\underline{\lambda_{1,2} = 2 \pm i}$

For $\lambda_1 = 2-i \Rightarrow \left(A-\lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2-(2-i) & -1 \\ 1 & 2-(2-i) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \\ x+iy=0 \end{cases}$$

$\underline{x=-iy}$

The eigenvector is: $V_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

For $\lambda_2 = 2+i \Rightarrow \left(A-\lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \\ x-iy=0 \end{cases}$$

$\underline{x=iy}$

The eigenvector is: $V_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

Exercise

Let $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

a) What is the characteristic polynomial for A (i.e. compute $\det\left(A-\lambda I\right)$?

b) Verify that 1 is an eigenvalue of A. What is a corresponding eigenvector?

c) What are the other eigenvalues of A?

Solution

a) $\det\left(A-\lambda I\right) = \begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix}$

$$= (2-\lambda)(1-\lambda)(-1-\lambda) - 2 + 9 - 3(1-\lambda) - 3(2-\lambda) + 2(-1-\lambda)$$

$$= \left(2-3\lambda+\lambda^2\right)(-1-\lambda) + 7 - 3 + 3\lambda - 6 + 3\lambda - 2 - 2\lambda$$

$$= -2 + 3\lambda - \lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3 + 4\lambda - 4$$

$$\underline{= -\lambda^3 + 2\lambda^2 + 5\lambda - 6}$$

b) *If* $\lambda = 1 \quad \to -\lambda^3 + 2\lambda^2 + 5\lambda - 6 \overset{?}{=} 0$

$$-1^3 + 2(1)^2 + 5(1) - 6 \overset{?}{=} 0$$

$$-1 + 2 + 5 - 6 \overset{?}{=} 0$$

$$\boxed{0 = 0}$$

1 is an eigenvalue of A.

$$\begin{pmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \to \begin{cases} x - 2y + 3z = 0 \\ x \qquad + z = 0 \\ x + 3y - 2z = 0 \end{cases}$$

$$\begin{cases} \quad x = -z \\ 3y = 2z - x = 2z + z = 3z \quad \Rightarrow \quad y = z \end{cases}$$

The eigenvector for $\lambda = 1$ is $V = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

c) $-\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$

$$\lambda_1 = 1 \quad \lambda_2 = -2 \quad \lambda_3 = 3$$

Exercise

For the matrix: $\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$$

i. $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix}$

$$= (3-\lambda)(-1-\lambda) - 0$$

$$= \lambda^2 - 2\lambda - 3$$

The characteristic equation: $\underline{\lambda^2 - 2\lambda - 3}$

ii. $\lambda^2 - 2\lambda - 3 = 0$

The eigenvalues are $\lambda_1 = -1$ *and* $\lambda_2 = 3$

iii. $\lambda_1 = -1 \quad \rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 0 \\ 8 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 4x = 0 \\ \end{cases}$$

$\underline{x = 0}$

Therefore, the eigenvector $V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda_2 = 3 \quad \rightarrow \quad (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \\ 8x - 4y = 0 \end{cases}$$

$\underline{2x = y}$

Therefore; the eigenvector $V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

The eigenvectors are given by: $V = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

Exercise

For the matrix: $\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$$

i. $\det(A - \lambda I) = \begin{vmatrix} 10-\lambda & -9 \\ 4 & -2-\lambda \end{vmatrix}$

$$= (10-\lambda)(-2-\lambda)+36$$
$$= \lambda^2 - 8\lambda + 16$$

The characteristic equation: $\boxed{\lambda^2 - 8\lambda + 16 = 0}$

ii. $\lambda^2 - 8\lambda + 16 = 0$

The eigenvalues are $\boxed{\lambda_{1,2} = 4}$

iii. $\lambda_1 = 4 \rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \\ 4x - 6y = 0 \end{cases}$$

$\boxed{2x = 3y}$

Therefore; the eigenvector $\boxed{V_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}}$

For the second eigenvector $V_2 \quad \Rightarrow \quad AV_2 = V_1$

$$\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \\ 4x - 2y = 2 \end{cases}$$

$\boxed{y = 2x - 1}$

If $x = 1 \quad \Rightarrow \quad y = 1$

$\boxed{V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

Exercise

For the matrix: $\begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}$$

i. $\det\left(A - \lambda I\right) = \begin{vmatrix} -\lambda & 3 \\ 4 & -\lambda \end{vmatrix}$
$$= \lambda^2 - 12$$

The characteristic equation: $\boxed{\lambda^2 - 12 = 0}$

ii. $\lambda^2 - 12 = 0$

The eigenvalues are $\boxed{\lambda_{1,2} = \pm\sqrt{12}}$

iii. For $\lambda_1 = -\sqrt{12} \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} \sqrt{12} & 3 \\ 4 & \sqrt{12} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \sqrt{12}x + 3y = 0 \\ \end{cases}$$

$$\rightarrow \boxed{\sqrt{12}x = -3y}$$

Therefore; the eigenvector $V_1 = \begin{pmatrix} -3 \\ \sqrt{12} \end{pmatrix}$

For $\lambda_2 = \sqrt{12} \quad \Rightarrow \quad (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -\sqrt{12} & 3 \\ 4 & -\sqrt{12} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -\sqrt{12}x + 3y = 0 \\ \end{cases}$$

$$\rightarrow \boxed{\sqrt{12}x = 3y}$$

Therefore; the eigenvector $V_2 = \begin{pmatrix} 3 \\ \sqrt{12} \end{pmatrix}$

The vectors are given by: $V = \begin{pmatrix} -3 & 3 \\ \sqrt{12} & \sqrt{12} \end{pmatrix}$

Exercise

For the matrix: $\begin{pmatrix} -2 & -7 \\ 1 & 2 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -2 & -7 \\ 1 & 2 \end{pmatrix}$$

i. $\begin{vmatrix} -2-\lambda & -7 \\ 1 & 2-\lambda \end{vmatrix} = (-2-\lambda)(2-\lambda) + 7$

$$= -4 + \lambda^2 + 7$$
$$= \lambda^2 + 3$$

The characteristic equation: $\boxed{\lambda^2 + 3 = 0}$

ii. $\lambda^2 = -3$

The eigenvalues are: $\boxed{\lambda_{1,2} = \pm i\sqrt{3}}$

iii. For $\lambda_1 = -i\sqrt{3} \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -2 + i\sqrt{3} & -7 \\ 1 & 2 + i\sqrt{3} \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \\ x_1 + \left(2 + i\sqrt{3}\right)y_1 = 0 \end{cases}$$

$$\boxed{x_1 = -\left(2 + i\sqrt{3}\right)y_1}$$

Therefore; the eigenvector $V_1 = \begin{pmatrix} 2 + i\sqrt{3} \\ -1 \end{pmatrix}$

For $\lambda_2 = i\sqrt{3} \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -2 - i\sqrt{3} & -7 \\ 1 & 2 - i\sqrt{3} \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \\ x_2 + \left(2 - i\sqrt{3}\right)y_2 = 0 \end{cases}$$

$$\boxed{x_2 = -\left(2 - i\sqrt{3}\right)y_2}$$

Therefore; the eigenvector $V_2 = \begin{pmatrix} 2 - i\sqrt{3} \\ -1 \end{pmatrix}$

The vectors are given by: $V = \begin{pmatrix} 2 + i\sqrt{3} & 2 - i\sqrt{3} \\ -1 & -1 \end{pmatrix}$

Exercise

For the matrix: $\begin{pmatrix} 12 & 14 \\ -7 & -9 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 12 & 14 \\ -7 & -9 \end{pmatrix}$$

i. $\left|A-\lambda I\right| = \begin{vmatrix} 12-\lambda & 14 \\ -7 & -9-\lambda \end{vmatrix}$

$$= (12-\lambda)(-9-\lambda) - (14)(-7)$$

$$= -108 - 12\lambda + 9\lambda + \lambda^2 + 98$$

$$= \lambda^2 - 3\lambda - 10$$

The characteristic equation: $\boxed{\lambda^2 - 3\lambda - 10 = 0}$

ii. The eigenvalues are: $\boxed{\lambda_1 = -2 \ \ and \ \ \lambda_2 = 5}$

iii. For $\lambda_1 = -2$, we have: $\left(A-\lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 12+2 & 14 \\ -7 & -9+2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 14 \\ -7 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 14x + 14y = 0 \\ \end{cases}$$

$\boxed{x = -y}$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = 5$, we have $\left(A-\lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 7 & 14 \\ -7 & -14 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 7x + 14y = 0 \\ \end{cases}$$

$\boxed{x = -2y}$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$

The vectors are given by: $V = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$

Exercise

For the matrix: $\begin{pmatrix} -4 & 1 \\ -2 & 1 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -4 & 1 \\ -2 & 1 \end{pmatrix}$$

i. $\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix}$

$$= (-4-\lambda)(1-\lambda) + 2$$

$$= \lambda^2 + 3\lambda - 2$$

The characteristic equation: $\boxed{\lambda^2 + 3\lambda - 2 = 0}$

ii. Thus, the eigenvalues are: $\boxed{\lambda_{1,2} = \frac{-3 \pm \sqrt{17}}{2}}$

iii. For $\lambda_1 = \frac{-3-\sqrt{17}}{2}$: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -4 - \frac{-3-\sqrt{17}}{2} & 1 \\ -2 & 1 - \frac{-3-\sqrt{17}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-5+\sqrt{17}}{2} & 1 \\ -2 & \frac{5+\sqrt{17}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} \frac{-5+\sqrt{17}}{2} x + y = 0 \\ -2x + \frac{5+\sqrt{17}}{2} y = 0 \end{cases}$$

$$\boxed{x = \left(\frac{5+\sqrt{17}}{4} \right) y}$$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} \frac{5+\sqrt{17}}{4} \\ 1 \end{pmatrix}}$$

For $\lambda_2 = \frac{-3+\sqrt{17}}{2}$: $\quad \left(A - \lambda_2 I\right) V_2 = 0$

$$\begin{pmatrix} -4 - \frac{-3+\sqrt{17}}{2} & 1 \\ -2 & 1 - \frac{-3+\sqrt{17}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-5-\sqrt{17}}{2} & 1 \\ -2 & \frac{5-\sqrt{17}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} \frac{-5-\sqrt{17}}{2} x + y = 0 \\ -2x + \frac{5-\sqrt{17}}{2} y = 0 \end{cases}$$

$$\boxed{x = \left(\frac{5-\sqrt{17}}{4}\right) y}$$

$$\Rightarrow \quad \boxed{V_2 = \begin{pmatrix} \frac{5-\sqrt{17}}{4} \\ 1 \end{pmatrix}}$$

The eigenvectors can be written: $\begin{pmatrix} \frac{5+\sqrt{17}}{4} & \frac{5-\sqrt{17}}{4} \\ 1 & 1 \end{pmatrix}$

Exercise

For the matrix: $\begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$$

i. $\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 3 \\ -6 & -4-\lambda \end{vmatrix}$

$$= (-4-\lambda)(5-\lambda) + 18$$

$$= \lambda^2 - \lambda - 2$$

The characteristic equation: $\lambda^2 - \lambda - 2 = 0$

ii. Thus, the eigenvalues are: $\lambda_1 = -1 \ \ and \ \ \lambda_2 = 2$

iii. For $\lambda_1 = -1$: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 6 & 3 \\ -6 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 6x + 3y = 0 \\ -6x - 3y = 0 \end{cases}$$

$y = -2x$

$$\Rightarrow \ V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

For $\lambda_2 = 2 \ \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 3 & 3 \\ -6 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 3x + 3y = 0 \\ \end{cases}$$

$y = -x$

$$\Rightarrow \ V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} -2 & 3 \\ 0 & -5 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -2 & 3 \\ 0 & -5 \end{pmatrix}$$

i. $$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 3 \\ 0 & -5-\lambda \end{vmatrix}$$

$$= (-2-\lambda)(-5-\lambda) - 0$$

$$= (2+\lambda)(5+\lambda)$$

The characteristic equation: $\underline{(2+\lambda)(5+\lambda) = 0}$

ii. Thus, the eigenvalues are: $\underline{\lambda_1 = -5 \ \ and \ \ \lambda_2 = -2}$

iii. For $\lambda_1 = -5$: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 3x + 3y = 0 \\ \end{cases}$$

$\underline{y = -x}$

$$\Rightarrow \ \underline{V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

For $\lambda_2 = -2$: $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 3 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 3y = 0 \\ -3y = 0 \end{cases}$$

$\underline{y = 0}$

$$\Rightarrow \ \underline{V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 2 & 0 \\ -4 & -2 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 2 & 0 \\ -4 & -2 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 \\ -4 & -2-\lambda \end{vmatrix}$

$$= (2-\lambda)(-2-\lambda) - 0$$

$$= \lambda^2 - 4$$

The characteristic equation: $\boxed{\lambda^2 - 4 = 0}$

ii. Thus, the eigenvalues are: $\boxed{\lambda_1 = -2 \ \ and \ \ \lambda_2 = 2}$

iii. For $\lambda_1 = -2$: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 0 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 4x = 0 \\ \end{cases}$$

$\boxed{x = 0}$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = 2$: $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 \\ -4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \\ -4x - 4y = 0 \end{cases}$$

$\boxed{x = -y}$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix}$

$$= \lambda^2 - 4\lambda - 5$$

The characteristic equation: $\lambda^2 - 4\lambda - 5 = 0$

ii. The eigenvalues are: $\lambda_1 = -1 \ \ and \ \ \lambda_2 = 5$

iii. For $\lambda_1 = -1$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 2x + 2y = 0 \\ \end{cases}$$

$x = -y$

$$\Rightarrow \ V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = 5$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -4x + 2y = 0 \\ \end{cases}$$

$2x = y$

$$\Rightarrow \ V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix}$

$$= \lambda^2 - 5\lambda + 4$$

The characteristic equation: $\underline{\lambda^2 - 5\lambda + 4 = 0}$

ii. The eigenvalues are: $\underline{\lambda_1 = 1 \;\; and \;\; \lambda_2 = 4}$

iii. For $\lambda_1 = 1$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} x + 2y = 0 \\ \end{cases}$$

$\underline{x = -2y}$

$$\Rightarrow \; \underline{V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = 4 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \\ x - y = 0 \end{cases}$$

$\underline{x = y}$

$$\Rightarrow \; \underline{V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} -4-\lambda & 2 \\ -\frac{5}{2} & 2-\lambda \end{vmatrix}$

$$= \lambda^2 + 2\lambda - 3$$

The characteristic equation: $\lambda^2 + 2\lambda - 3 = 0$

ii. The eigenvalues are: $\lambda_1 = -3 \text{ and } \lambda_2 = 1$

iii. For $\lambda_1 = -3$: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -1 & 2 \\ -\frac{5}{2} & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -x + 2y = 0 \\ \end{cases}$$

$x = 2y$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1$: $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -5 & 2 \\ -\frac{5}{2} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -5x + 2y = 0 \\ \end{cases}$$

$5x = 2y$

$$\Rightarrow \quad V_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} -\frac{5}{2} & 2 \\ \frac{3}{4} & -2 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -\frac{5}{2} & 2 \\ \frac{3}{4} & -2 \end{pmatrix}$$

i. $$|A - \lambda I| = \begin{vmatrix} -\frac{5}{2} - \lambda & 2 \\ \frac{3}{4} & -2 - \lambda \end{vmatrix}$$

$$= \lambda^2 + \frac{9}{2}\lambda + \frac{7}{2}$$

The characteristic equation: $2\lambda^2 + 9\lambda + 7 = 0$

ii. The eigenvalues are: $\lambda_1 = -\frac{7}{2}$ *and* $\lambda_2 = -1$

iii. For $\lambda_1 = -\frac{7}{2}$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 2 \\ \frac{3}{4} & \frac{3}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} x + 2y = 0 \\ \end{cases}$$

$x = -2y$

$$\Rightarrow V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

For $\lambda_1 = -1$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -\frac{3}{2} & 2 \\ \frac{3}{4} & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} -\frac{3}{2}x + 2y = 0 \\ \end{cases}$$

$3x = 4y$

$$\Rightarrow V_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{vmatrix}$

$$= (3-\lambda)(-3-\lambda) + 9$$

$$= \lambda^2$$

The characteristic equation: $\lambda^2 = 0$

ii. The eigenvalues are: $\lambda_{1,2} = 0$

iii. For $\lambda_1 = 0$: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} 3x - y = 0 \\ \end{cases}$$

$3x = y$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

For the second eigenvector V_2 $\quad \Rightarrow \quad AV_2 = V_1$

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \to \quad \begin{cases} 3x - y = 1 \\ \end{cases}$$

$\to$ *if* $x = 1 \quad \Rightarrow \quad y = 2$

$$\Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} -6-\lambda & 5 \\ -5 & 4-\lambda \end{vmatrix}$

$$= -24 + 2\lambda + \lambda^2 + 25$$

$$= \lambda^2 + 2\lambda + 1$$

The characteristic equation: $\lambda^2 + 2\lambda + 1 = 0$

ii. The eigenvalues are: $\lambda_{1,2} = -1$

iii. For $\lambda_1 = 0$: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -5x + 5y = 0 \\ \end{cases}$$

$x = y$

$$\Rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For the second eigenvector $V_2 \quad \Rightarrow AV_2 = V_1$

$$\begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -6x_2 + 5y_2 = 1 \\ \end{cases}$$

$\rightarrow$ *if* $x_2 = 0 \quad \rightarrow \quad y_2 = \frac{1}{5}$

$$\Rightarrow V_2 = \begin{pmatrix} 0 \\ \frac{1}{5} \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix}$$

i. $$|A - \lambda I| = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix}$$

$$= \lambda^2 - 8\lambda + 17$$

The characteristic equation: $\lambda^2 - 8\lambda + 17 = 0$

ii. $$\lambda_{1,2} = \frac{8 \pm \sqrt{64-68}}{2}$$

The eigenvalues are: $\lambda_{1,2} = 4 \pm i$

iii. For $\lambda_1 = 4 - i$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} (2-i)x - y = 0 \end{cases}$$

$$(2-i)x = y$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$$

$$\Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 2+i \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix}$

$$= \lambda^2 + 1$$

The characteristic equation: $\boxed{\lambda^2 + 1 = 0}$

ii. $\lambda^2 = -1$

The eigenvalues are: $\boxed{\lambda_{1,2} = \pm i}$

iii. For $\lambda_1 = -i$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1+i & 1 \\ -2 & -1+i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} (1+i)x + y = 0 \\ \end{cases}$$

$$\boxed{(1+i)x = -y}$$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} -1 \\ 1+i \end{pmatrix}}$$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} -1 \\ 1-i \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}$

- ***i.*** Find the characteristic equation
- ***ii.*** Find the eigenvalues
- ***iii.*** Find the eigenvectors

Solution

$$A = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 5-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix}$

$$= 15 - 8\lambda + \lambda^2 + 2$$

$$= \lambda^2 - 8\lambda + 17$$

The characteristic equation: $\lambda^2 - 8\lambda + 17 = 0$

ii. $\lambda_{1,2} = \dfrac{8 \pm \sqrt{64-68}}{2}$

The eigenvalues are: $\lambda_{1,2} = 4 \pm i$

iii. For $\lambda_1 = 4 - i$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1+i & 1 \\ -2 & -1+i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} (1+i)x + y = 0 \\ \end{cases}$$

$$(1+i)x = -y$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ 1+i \end{pmatrix}$$

$$\Rightarrow \quad V_2 = \begin{pmatrix} -1 \\ 1-i \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 4 & 5 \\ -2 & 6 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 4 & 5 \\ -2 & 6 \end{pmatrix}$$

i. $$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 5 \\ -2 & 6-\lambda \end{vmatrix}$$

$$= 24 - 10\lambda + \lambda^2 + 10$$

$$= \lambda^2 - 10\lambda + 34$$

The characteristic equation: $\lambda^2 - 10\lambda + 34 = 0$

ii. $$\lambda_{1,2} = \frac{10 \pm \sqrt{100-136}}{2}$$

The eigenvalues are: $\lambda_{1,2} = 5 \pm 3i$

iii. For $\lambda_1 = 5 - 3i$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -1+3i & 5 \\ -2 & 1+3i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} (-1+3i)x + 5y = 0 \\ \end{cases}$$

$$(-1+3i)x = -y$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ -1+3i \end{pmatrix}$$

$$\Rightarrow \quad V_2 = \begin{pmatrix} -1 \\ -1-3i \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 5-\lambda & -4 \\ 2 & -1-\lambda \end{vmatrix}$

$$= -5 - 4\lambda + \lambda^2 + 8$$

$$= \lambda^2 - 4\lambda + 3$$

The characteristic equation: $\boxed{\lambda^2 - 4\lambda + 3 = 0}$

ii. The eigenvalues are: $\boxed{\lambda_{1,2} = 1,\ 3}$

iii. For $\lambda_1 = 1$: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 4x - 4y = 0 \\ \end{cases}$$

$\boxed{x = y}$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = 3$: $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x - 4y = 0 \\ \end{cases}$$

$\boxed{x = 2y}$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 6 & -6 \\ 4 & -4 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 6 & -6 \\ 4 & -4 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 6-\lambda & -6 \\ 4 & -4-\lambda \end{vmatrix}$

$$= -24 - 2\lambda + \lambda^2 + 24$$

$$= \lambda^2 - 2\lambda$$

The characteristic equation: $\underline{\lambda^2 - 2\lambda = 0}|$

ii. The eigenvalues are: $\underline{\lambda_{1,2} = 0,\ 2}|$

iii. For $\lambda_1 = 0$: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 6 & -6 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} 6x - 6y = 0 \\ \end{cases}$$

$\underline{x = y}|$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}|$$

For $\lambda_2 = 2$: $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 4 & -6 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} 4x - 6y = 0 \\ \end{cases}$$

$\underline{2x = 3y}|$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}}|$$

Exercise

For the matrix: $\begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 5-\lambda & -3 \\ 2 & -\lambda \end{vmatrix}$

$$= \lambda^2 - 5\lambda + 6$$

The characteristic equation: $\lambda^2 - 5\lambda + 6 = 0$

ii. The eigenvalues are: $\lambda_{1,2} = 2,\ 3$

iii. For $\lambda_1 = 2$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} 3x - 3y = 0 \end{cases}$$

$x = y$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 3$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} 2x - 3y = 0 \end{cases}$$

$2x = 3y$

$$\Rightarrow \quad V_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 5-\lambda & -4 \\ 3 & -2-\lambda \end{vmatrix}$

$$= -10 - 3\lambda + \lambda^2 + 12$$

$$= \lambda^2 - 3\lambda + 2$$

The characteristic equation: $\boxed{\lambda^2 - 3\lambda + 2 = 0}$

ii. The eigenvalues are: $\boxed{\lambda_{1,2} = 1,\ 2}$

iii. For $\lambda_1 = 1$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 4 & -4 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 4x - 4y = 0 \\ \end{cases}$$

$\boxed{x = y}$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

For $\lambda_1 = 2$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 3 & -4 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 3x - 4y = 0 \\ \end{cases}$$

$\boxed{3x = 4y}$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 6 & -10 \\ 2 & -3 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 6 & -10 \\ 2 & -3 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 6-\lambda & -10 \\ 2 & -3-\lambda \end{vmatrix}$

$$= -18 - 3\lambda + \lambda^2 + 20$$

$$= \lambda^2 - 3\lambda + 2$$

The characteristic equation: $\lambda^2 - 3\lambda + 2 = 0$

ii. The eigenvalues are: $\lambda_{1,2} = 1,\ 2$

iii. For $\lambda_1 = 1$: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 5 & -10 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} 2x - 4y = 0 \end{cases}$$

$x = 2y$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_1 = 2$: $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 4 & -10 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} 2x - 5y = 0 \end{cases}$$

$2x = 5y$

$$\Rightarrow \quad V_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 11 & -15 \\ 6 & -8 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 11 & -15 \\ 6 & -8 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 11-\lambda & -15 \\ 6 & -8-\lambda \end{vmatrix}$

$$= -88 - 3\lambda + \lambda^2 + 90$$

$$= \lambda^2 - 3\lambda + 2$$

The characteristic equation: $\underline{\lambda^2 - 3\lambda + 2 = 0}$

ii. The eigenvalues are: $\underline{\lambda_{1,2} = 1,\ 2}$

iii. For $\lambda_1 = 1$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 10 & -15 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} \\ 6x - 9y = 0 \end{cases}$$

$\underline{2x = 3y}$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}}$$

For $\lambda_1 = 2$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 9 & -15 \\ 6 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} \\ 6x - 10y = 0 \end{cases}$$

$\underline{3x = 5y}$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$

$$= 9 - 6\lambda + \lambda^2 - 1$$

$$= \lambda^2 - 6\lambda + 8$$

The characteristic equation: $\boxed{\lambda^2 - 6\lambda + 8 = 0}$

ii. $\lambda_{1,2} = \dfrac{6 \pm \sqrt{36-32}}{2}$

$$= \frac{6 \pm 2}{2}$$

The eigenvalues are: $\boxed{\lambda_1 = 2 \ \ \& \ \ \lambda_2 = 4}$

iii. For $\lambda_1 = 2$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} x + y = 0 \\ \end{cases}$$

$\boxed{x = -y}$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = 4$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -x + y = 0 \\ \end{cases}$$

$\boxed{x = y}$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 9 & 2 \\ 2 & 6 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 9 & 2 \\ 2 & 6 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 9-\lambda & 2 \\ 2 & 6-\lambda \end{vmatrix}$

$$= 54 - 15\lambda + \lambda^2 - 4$$

$$= \lambda^2 - 15\lambda + 50$$

The characteristic equation: $\underline{\lambda^2 - 15\lambda + 50 = 0}$

ii. $\lambda_{1,2} = \dfrac{15 \pm \sqrt{225-200}}{2}$

$$= \frac{15 \pm 5}{2}$$

The eigenvalues are: $\underline{\lambda_1 = 5 \ \ \& \ \ \lambda_2 = 10}$

iii. For $\lambda_1 = 5$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \\ 2x + y = 0 \end{cases}$$

$\underline{2x = -y}$

$$\Rightarrow \ V_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 10$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -x + 2y = 0 \\ \end{cases}$$

$\underline{x = 2y}$

$$\Rightarrow \ V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 13 & 4 \\ 4 & 7 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 13 & 4 \\ 4 & 7 \end{pmatrix}$$

i. $$|A - \lambda I| = \begin{vmatrix} 13-\lambda & 4 \\ 4 & 7-\lambda \end{vmatrix}$$

$$= 91 - 20\lambda + \lambda^2 - 16$$

$$= \lambda^2 - 20\lambda + 75$$

The characteristic equation: $\underline{\lambda^2 - 20\lambda + 75 = 0}$

ii. $$\lambda_{1,2} = \frac{20 \pm \sqrt{400-300}}{2}$$

$$= \frac{20 \pm 10}{2}$$

The eigenvalues are: $\underline{\lambda_1 = 5 \ \ \& \ \ \lambda_2 = 15}$

iii. For $\lambda_1 = 5$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \\ 4x + 2y = 0 \end{cases}$$

$\underline{2x = -y}$

$$\Rightarrow \ V_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 15$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -2x + 4y = 0 \\ \end{cases}$$

$\underline{x = 2y}$

$$\Rightarrow \ V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 5 & -1 \\ 3 & -1 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 5 & -1 \\ 3 & -1 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 5-\lambda & -1 \\ 3 & -1-\lambda \end{vmatrix}$

$$= -5 - 4\lambda + \lambda^2 + 3$$

$$= \lambda^2 - 4\lambda - 2$$

The characteristic equation: $\boxed{\lambda^2 - 4\lambda - 2 = 0}$

ii. $\lambda_{1,2} = \dfrac{4 \pm \sqrt{16+8}}{2}$

$$= \frac{4 \pm 2\sqrt{6}}{2}$$

The eigenvalues are: $\boxed{\lambda_{1,2} = 2 \pm \sqrt{6}}$

iii. For $\lambda_1 = 2 - \sqrt{6}$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3+\sqrt{6} & -1 \\ 3 & -3+\sqrt{6} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \left(3+\sqrt{6}\right)x - y = 0 \\ \end{cases}$$

$$\boxed{\left(3+\sqrt{6}\right)x = y}$$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 1 \\ 3+\sqrt{6} \end{pmatrix}}$$

For $\lambda_2 = 2 + \sqrt{6}$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 3-\sqrt{6} & -1 \\ 3 & -3-\sqrt{6} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \left(3-\sqrt{6}\right)x - y = 0 \\ \end{cases}$$

$$\boxed{\left(3-\sqrt{6}\right)x = y}$$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} 1 \\ 3-\sqrt{6} \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix}$

$$= -2 + \lambda + \lambda^2 - 4$$

$$= \lambda^2 + \lambda - 6$$

The characteristic equation: $\lambda^2 + \lambda - 6 = 0$

ii. $\lambda_{1,2} = \dfrac{-1 \pm \sqrt{1+24}}{2}$

The eigenvalues are: $\lambda_1 = -3$ & $\lambda_2 = 2$

iii. For $\lambda_1 = -3$: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} 4x + y = 0 \\ \end{cases}$$

$4x = -y$

$$\Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} -x + y = 0 \\ \end{cases}$$

$x = y$

$$\Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix}$

$$= 1 + 2\lambda + \lambda^2 + 4$$

$$= \lambda^2 + 2\lambda + 5$$

The characteristic equation: $\lambda^2 + 2\lambda + 5 = 0$

ii. $\lambda_{1,2} = \dfrac{-2 \pm \sqrt{4-20}}{2}$

The eigenvalues are: $\lambda_{1,2} = -1 \pm 2i$

iii. For $\lambda_1 = -1 - 2i$: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2i & -4 \\ 1 & 2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} \\ x + 2iy = 0 \end{cases}$$

$x = -2iy$

$$\Rightarrow V_1 = \begin{pmatrix} -2i \\ 1 \end{pmatrix}$$

$$\Rightarrow V_2 = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -6 \\ -2 & 3 & 4 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -6 \\ -2 & 3 & 4 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} -1-\lambda & 0 & 0 \\ 2 & -5-\lambda & -6 \\ -2 & 3 & 4-\lambda \end{vmatrix}$

$$= (-1-\lambda)(-5-\lambda)(4-\lambda) + 18(-1-\lambda)$$

$$= \left(5 + 6\lambda + \lambda^2\right)(4-\lambda) - 18 - 18\lambda$$

$$= 20 - 5\lambda + 24\lambda - 6\lambda^2 + 4\lambda^2 - \lambda^3 - 18 - 18\lambda$$

$$= -\lambda^3 - 2\lambda^2 + \lambda + 2$$

The characteristic equation: $\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$

ii. $\lambda = 1$

$$\begin{array}{c|cccc} 1 & 1 & 2 & -1 & -2 \\ & & 1 & 3 & 2 \\ \hline & 1 & 3 & 2 & 0 \end{array} \quad \rightarrow \quad \lambda^2 + 3\lambda + 2 = 0$$

The eigenvalues are: $\lambda_1 = -2 \quad \lambda_2 = -1 \quad and \quad \lambda_3 = 1$

iii. For $\lambda_1 = -2$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & -6 \\ -2 & 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} x = 0 \\ 2x - 3y - 6z = 0 & (1) \\ -2x + 3y + 6z = 0 \end{cases}$$

$(1) \rightarrow \quad y = -2z$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -1 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & -4 & -6 \\ -2 & 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} 0 = 0 & \\ 2x - 4y - 6z = 0 & (1) \\ -2x + 3y + 5z = 0 & (2) \end{cases}$$

$$(1) + (2) \to \quad -y - z = 0 \quad \Rightarrow \underline{y = -z}$$

$$(1) \to \quad 2x = -4z + 6z \quad \Rightarrow \underline{x = z}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}$$

For $\lambda_3 = 1 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -2 & 0 & 0 \\ 2 & -6 & -6 \\ -2 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} -2x = 0 & \to \underline{x = 0} \\ 2x - 6y - 6z = 0 & (1) \\ -2x + 3y + 3z = 0 & (2) \end{cases}$$

$$(1) + (2) \to \quad -3y - 3z = 0 \quad \Rightarrow \underline{y = -z}$$

$$\Rightarrow \underline{V_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}$$

The vectors are given by: $V = \begin{pmatrix} 0 & 1 & 0 \\ -2 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

Exercise

For the matrix: $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

i. $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & -1-\lambda \end{vmatrix}$

$$= -\left(1-\lambda^2\right)(2-\lambda)$$

The characteristic equation: $\boxed{\left(1-\lambda^2\right)(2-\lambda) = 0}$

ii. The eigenvalues are: $\boxed{\lambda_1 = -1 \quad \lambda_2 = 1 \quad and \quad \lambda_3 = 2}$

iii. For $\lambda_1 = -1$: $\left(A-\lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x+y-z=0 & (1) \\ \boxed{y=0} \end{cases}$$

$(1) \rightarrow \boxed{2x = z}$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}$$

r $\lambda_2 = 1$: $\left(A-\lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} y-z=0 \quad \boxed{z=0} \\ \boxed{y=0} \end{cases}$$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

For $\lambda_3 = 2$: $\left(A-\lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -x+y-z=0 & (1) \\ \\ \boxed{y=3z} \end{cases}$$

$(1) \rightarrow \boxed{x = 2z}$

$$\Rightarrow \boxed{V_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{pmatrix}$

- ***i.*** Find the characteristic equation
- ***ii.*** Find the eigenvalues
- ***iii.*** Find the eigenvectors

Solution

$$A = \begin{pmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{pmatrix}$$

i. $$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -7 & 0 \\ 5 & 10-\lambda & 4 \\ 0 & 5 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^2(10-\lambda) - 20(2-\lambda) + 35(2-\lambda)$$

$$= (2-\lambda)\big((10-\lambda)(2-\lambda) + 15\big)$$

$$= (2-\lambda)\left(35 - 12\lambda + \lambda^2\right)$$

The characteristic equation: $\boxed{(2-\lambda)\left(\lambda^2 - 12\lambda + 35\right) = 0}$

ii. $$\lambda = \frac{12 \pm \sqrt{144-140}}{2}$$

$$= \frac{12 \pm 2}{2}$$

The eigenvalues are: $\boxed{\lambda_1 = 2 \quad \lambda_2 = 5 \quad and \quad \lambda_3 = 7}$

iii. For $\lambda_1 = 2$: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 0 & -7 & 0 \\ 5 & 8 & 4 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} \boxed{y = 0} \\ 5x + 8y + 4z = 0 \qquad \boxed{5x = -4z} \end{cases}$$

$$\Rightarrow \quad \boxed{V_1 = \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}}$$

For $\lambda_2 = 5$: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -5 & -7 & 0 \\ 5 & 3 & 4 \\ 0 & 5 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} -5x-7y=0 & \rightarrow \underline{-5x=7y} \\ \\ 5y-5z=0 & \rightarrow \underline{z=y} \end{cases}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 7 \\ -5 \\ -5 \end{pmatrix}}$$

For $\lambda_3 = 7$: $\quad (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} 0 & -7 & 0 \\ 5 & 8 & 4 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} \underline{y=0} & \\ 5x+8y+4z=0 & \rightarrow \underline{5x=-4z} \\ & \end{cases}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} -4 \\ 0 \\ 5 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

i. $\quad |A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & -1 \\ 1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix}$

$$= \left(1 - 2\lambda + \lambda^2\right)(3-\lambda) + 2 + 2 - 2\lambda - 3 + \lambda$$

$$= 3 - 7\lambda + 5\lambda^2 - \lambda^3 + 1 - \lambda$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

The characteristic equation: $\quad \underline{-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0}$

ii. $\lambda = 1$

$$\begin{array}{c|cccc} 1 & -1 & 5 & -8 & 4 \\ & & -1 & 4 & -4 \\ \hline & -1 & 4 & -4 & 0 \end{array} \quad \rightarrow \underline{-\lambda^2 + 4\lambda - 4 = 0}$$

The eigenvalues are: $\underline{\lambda_1 = 1 \ \ and \ \ \lambda_{2,3} = 2}$

iii. For $\lambda_1 = 1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 2x - y + z = 0 & \\ x - z = 0 & \rightarrow \underline{x = z} \\ x - y = 0 & \rightarrow \underline{x = y} \end{cases}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} x - y - z = 0 \quad (1) \\ \\ \end{cases}$$

$(1) \rightarrow \ \underline{x = y + z} \quad let \ \ \underline{y = 0}$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$$

$(1) \rightarrow \ \ let \ \ \underline{z = 0} \quad \underline{x = y}$

$$\Rightarrow \underline{V_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$

$$= (-\lambda)(3-\lambda)^2 + 16 + 16 + 16\lambda - 4(3-\lambda) - 4(3-\lambda)$$

$$= -9\lambda + 6\lambda^2 - \lambda^3 + 32 + 16\lambda - 12 + 4\lambda - 12 + 4\lambda$$

$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8$$

The characteristic equation: $\underline{-\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0}$

ii. $\lambda = -1$

-1	-1	6	15	8
		1	-7	-8
	-1	7	8	0

$\rightarrow \underline{-\lambda^2 + 7\lambda + 8 = 0}$

The eigenvalues are: $\underline{\lambda_{1,2} = -1 \ \ and \ \ \lambda_3 = 8}$

iii. For $\lambda_{1,2} = -1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x + y + 2z = 0 & (1) \\ \end{cases}$$

Assume $z = 0 \rightarrow \underline{2x = -y}$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}}$$

Assume $y = 0 \rightarrow \underline{x = -z}$

$$\Rightarrow \; \boxed{V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

For $\lambda_3 = 8 \quad \Rightarrow \; (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 5x - 2y - 4z = 0 & (1) \\ x - 4y + z = 0 & (2) \\ 4x + 2y - 5z = 0 & (3) \end{cases}$$

$(1) + (3) \rightarrow \quad 9x - 9z = 0$

$\boxed{x = z}$

$(2) \rightarrow \quad 4y = 2z$

Assume $\quad z = 2 = x \quad \rightarrow \quad \boxed{y = 1}$

$$\Rightarrow \; \boxed{V_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -8 & -5 & -3-\lambda \end{vmatrix}$

$$= (1 - 2\lambda + \lambda^2)(-3 - \lambda) - 2 + 3 - 3\lambda + 6 + 2\lambda$$

$$= -\lambda^3 - \lambda^2 + 4\lambda + 4$$

The characteristic equation: $\quad \boxed{-\lambda^3 - \lambda^2 + 4\lambda + 4 = 0}$

ii. $\lambda = -1$

$$\begin{array}{r|rrrr} -1 & -1 & -1 & 4 & 4 \\ & & 1 & 0 & -4 \\ \hline & -1 & 0 & 4 & 0 \end{array} \quad \rightarrow \underline{-\lambda^2 + 4 = 0}$$

The eigenvalues are: $\underline{\lambda_1 = -2, \ \lambda_2 = -1, \ and \ \lambda_3 = 2}$

iii. For $\lambda_1 = -2 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & -1 \\ -8 & -5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 3x + y + z = 0 \\ 2x + 3y - z = 0 \\ -8x - 5y - z = 0 \end{cases}$$

Assume $z = 1 \rightarrow \begin{cases} 3x + y = -1 \\ 2x + 3y = 1 \end{cases}$

$\Delta = \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} = 7 \quad \Delta_x = \begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix} = -4 \quad \Delta_y = \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = 5$

$x = -\frac{4}{7} \quad y = \frac{5}{7} \quad z = 1$

$\Rightarrow \underline{V_1 = \begin{pmatrix} -4 \\ 5 \\ 7 \end{pmatrix}}$

For $\lambda_2 = -1 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ -8 & -5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 2x + y + z = 0 \\ 2x + 2y - z = 0 \\ -8x - 5y - 2z = 0 \end{cases}$$

$z = 1 \rightarrow \begin{cases} 2x + y = -1 \\ 2x + 2y = 1 \end{cases}$

$\Delta = \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 2 \quad \Delta_x = \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} = -3 \quad \Delta_y = \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = 4$

$x = -\frac{3}{2} \quad y = 2 \quad z = 1$

$\Rightarrow \underline{V_2 = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}}$

For $\lambda_3 = 2 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -8 & -5 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} -x+y+z=0 \\ 2x-y-z=0 \\ -8x-5y-5z=0 \end{cases}$$

$$x=0 \quad \to \begin{cases} y+z=0 \\ -y-z=0 \end{cases}$$

$$\underline{y=-z}\,\big|$$

$$\Rightarrow \quad V_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

i. $|A-\lambda I| = \begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix}$

$$= -\left(1-\lambda^2\right)(2-\lambda)+14-16+8\lambda+1-\lambda-3-3\lambda$$

$$= -\lambda^3+2\lambda^2+5\lambda-6$$

The characteristic equation: $\underline{-\lambda^3+2\lambda^2+5\lambda-6=0}\,\big|$

ii. $\lambda = 1$

$$\begin{array}{c|rrrr} 1 & -1 & 2 & 5 & -6 \\ & & -1 & 1 & 6 \\ \hline & -1 & 1 & 6 & 0 \end{array} \quad \to \underline{-\lambda^2+\lambda+6=0}\,\big|$$

The eigenvalues are: $\underline{\lambda_1 = -2, \ \lambda_2 = 1, \ \textit{and} \ \ \lambda_3 = 3}\,\big|$

iii. For $\lambda_1 = -2 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} 3x - y + 4z = 0 \\ 3x + 4y - z = 0 \\ 2x + y + z = 0 \end{cases}$$

Assume $z = 1 \quad \to \begin{cases} 3x - y = -4 \\ 3x + 4y = 1 \end{cases}$

$$\Delta = \begin{vmatrix} 3 & -1 \\ 3 & 4 \end{vmatrix} = 15 \quad \Delta_x = \begin{vmatrix} -4 & -1 \\ 1 & 4 \end{vmatrix} = -15 \quad \Delta_y = \begin{vmatrix} 3 & -4 \\ 3 & 1 \end{vmatrix} = 15$$

$x = -1 \quad y = 1$

$$\Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} -y + 4z = 0 \\ 3x + y - z = 0 \\ 2x + y - 2z = 0 \end{cases}$$

Assume $z = 1 \quad \to \begin{cases} y = 4 \\ 3x + y = 1 \end{cases} \quad \to \quad x = -1$

$$\Rightarrow \quad V_2 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 3 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} -2x - y + 4z = 0 \\ 3x - y - z = 0 \\ 2x + y - 4z = 0 \end{cases}$$

Assume $z = 1 \quad \to \begin{cases} -2x - y = -4 \\ 3x - y = 1 \end{cases}$

$$\Delta = \begin{vmatrix} -2 & -1 \\ 3 & -1 \end{vmatrix} = 5 \quad \Delta_x = \begin{vmatrix} -4 & -1 \\ 1 & -1 \end{vmatrix} = 5 \quad \Delta_y = \begin{vmatrix} -2 & -4 \\ 3 & 1 \end{vmatrix} = 10$$

$x = -1 \quad y = 2$

$$\Rightarrow \quad V_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix}$

$$= \left(1 - 2\lambda + \lambda^2\right)(3-\lambda) - (3-\lambda)$$

$$= (3-\lambda)\left(\lambda^2 - 2\lambda\right)$$

The characteristic equation: $(3-\lambda)\left(\lambda^2 - 2\lambda\right) = 0$

ii. The eigenvalues are: $\lambda_1 = 0,\ \lambda_2 = 2,\ and\ \lambda_3 = 3$

iii. For $\lambda_1 = 0 \quad \Rightarrow \ \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} x + y = 0 \\ \\ 3z = 0 \end{cases}$$

$x = -y \quad z = 0$

$$\Rightarrow \ V_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow \ \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} -x + y = 0 & \to \ x = y \\ \\ z = 0 \end{cases}$$

$$\Rightarrow \quad \boxed{V_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

For $\lambda_3 = 3 \quad \Rightarrow \left(A - \lambda_3 I\right) V_3 = 0$

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 2x = y \\ x = 2y \\ \end{cases}$$

$\boxed{x = y = 0}$

$$\Rightarrow \quad \boxed{V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{pmatrix}$$

i. $$\left|A - \lambda I\right| = \begin{vmatrix} -\frac{5}{2} - \lambda & 1 & 1 \\ 1 & -\frac{5}{2} - \lambda & 1 \\ 1 & 1 & -\frac{5}{2} - \lambda \end{vmatrix}$$

$$= -\left(\frac{5}{2} + \lambda\right)^3 + 2 + 3\left(\frac{5}{2} + \lambda\right)$$

$$= -\frac{49}{8} - \frac{63}{4}\lambda - \frac{15}{2}\lambda^2 - \lambda^3$$

$$-\frac{49}{8}-\frac{63}{4}\lambda-\frac{15}{2}\lambda^2-\lambda^3=0$$

The characteristic equation: $\underline{8\lambda^3+60\lambda^2+126\lambda+49=0}$

ii. $\lambda=-\frac{1}{2}$

$$\begin{array}{r|rrrr} -\frac{1}{2} & 8 & 60 & 126 & 49 \\ & & -4 & -28 & -49 \\ \hline & 8 & 56 & 98 & 0 \end{array} \quad \rightarrow \underline{8\lambda^2+56\lambda+98=0}$$

$$4\lambda^2+28\lambda+49=0$$

$$\lambda=\frac{-28\pm\sqrt{784-784}}{8}$$

The eigenvalues are: $\underline{\lambda_1=-\frac{1}{2} \quad \& \quad \lambda_{2,3}=-\frac{7}{2}}$

iii. For $\lambda_1=-\frac{7}{2} \quad \Rightarrow \left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow x+y+z=1$$

Assume $z=0 \quad \rightarrow x+y=1$

$y=1 \quad \Rightarrow \quad \underline{x=-1}$

$$\rightarrow \underline{V_1=\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$$

Assume $y=0 \quad \rightarrow x+z=1$

$z=1 \quad \Rightarrow \quad \underline{x=-1}$

$$\rightarrow \underline{V_2=\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

For $\lambda_3=-\frac{1}{2} \quad \Rightarrow \left(A-\lambda_3 I\right)V_3=0$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{array}{r} -2x+y+z=0 \\ x-2y+z=0 \\ x+y-2z=0 \end{array}$$

Assume $z=1 \quad \rightarrow \begin{cases} -2x+y=-1 \\ x-2y=-1 \end{cases}$

$$\Delta = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 3 \quad \Delta_x = \begin{vmatrix} -2 & -1 \\ 1 & -1 \end{vmatrix} = 3$$

$$\underline{x = 1} \quad y = -1 + 2 \; \underline{= 1}$$

$$\to \; V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

i. $\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix}$

$$= (4-\lambda)(1-\lambda)(1-\lambda) + 2(1-\lambda)$$

$$= (1-\lambda)\left[(4-\lambda)(1-\lambda) + 2\right]$$

$$= (1-\lambda)\left(\lambda^2 - 5\lambda + 6\right)$$

The characteristic equation: $\underline{-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0}$

ii. $-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0 \quad \to \quad \lambda_1 = 1$

$$\begin{array}{c|cccc} 1 & -1 & 6 & -11 & 6 \\ & & -1 & 5 & -6 \\ \hline & -1 & 5 & -6 & 0 \end{array} \quad \to \quad \underline{-\lambda^2 + 5\lambda - 6 = 0}$$

Therefore; the eigenvalues are: $\underline{\lambda_{1,2,3} = 1,\ 2,\ 3}$

iii. For $\lambda_1 = 1 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} 3x_1 + x_3 = 0 & (1) \\ -2x_1 = 0 & (2) \\ \end{cases}$$

$$(2) \quad \Rightarrow \quad \underline{x_1 = 0|}$$

$$(1) \quad \Rightarrow \quad \underline{x_3 = x_1 = 0|}$$

Therefore; the eigenvector $\underline{V_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}|}$

For $\lambda_2 = 2 \quad \Rightarrow \quad (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} 2x_1 + x_3 = 0 & \Rightarrow \quad x_3 = -2x_1 \\ -2x_1 - x_2 = 0 & \Rightarrow \quad x_2 = -2x_1 \\ -2x_1 - x_3 = 0 \end{cases}$$

Therefore; the eigenvector $\underline{V_2 = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}|}$

For $\lambda_3 = 3 \quad \Rightarrow \quad (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} x_1 + x_3 = 0 & \Rightarrow \quad x_3 = -x_1 \\ -2x_1 - 2x_2 = 0 & \Rightarrow \quad x_2 = -x_1 \\ -2x_1 - 2x_3 = 0 \end{cases}$$

Therefore; the eigenvector $\underline{V_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}|}$

Exercise

For the matrix: $\begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$$

i. $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & -5 \\ \frac{1}{5} & -1-\lambda & 0 \\ 1 & 1 & -2-\lambda \end{vmatrix}$

$$= (3-\lambda)(-1-\lambda)(-2-\lambda) - 1 + 5(-1-\lambda)$$

$$= (3-\lambda)\left(\lambda^2 + 3\lambda + 2\right) - 1 - 5 - 5\lambda$$

$$= 3\lambda^2 + 9\lambda + 6 - \lambda^3 - 3\lambda^2 - 2\lambda - 6 - 5\lambda$$

$$= -\lambda^3 + 2\lambda$$

The characteristic equation: $-\lambda^3 + 2\lambda = 0$

ii. $-\lambda\left(\lambda^2 - 2\right) = 0$

Therefore; the eigenvalues are: $\lambda_{1,2,3} = 0,\ \pm\sqrt{2}$

iii. For $\lambda_1 = -\sqrt{2} \quad \Rightarrow \quad \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3+\sqrt{2} & 0 & -5 \\ \frac{1}{5} & -1+\sqrt{2} & 0 \\ 1 & 1 & -2+\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \left(3+\sqrt{2}\right)x_1 - 5x_3 = 0 & \Rightarrow \quad x_3 = \frac{3+\sqrt{2}}{5}x_1 \\ \frac{1}{5}x_1 + \left(-1+\sqrt{2}\right)x_2 = 0 & \Rightarrow \quad x_2 = -\frac{1}{5\left(-1+\sqrt{2}\right)}x_1 \\ x_1 + x_2 + \left(-2+\sqrt{2}\right)x_3 = 0 \end{cases}$$

Therefore; the eigenvector $V_1 = \begin{pmatrix} 1 \\ \frac{1}{5\left(1-\sqrt{2}\right)} \\ \frac{3+\sqrt{2}}{5} \end{pmatrix}$

For $\lambda_2 = 0 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3x_1 - 5x_3 = 0 & \Rightarrow \quad x_3 = \frac{3}{5}x_1 \\ \frac{1}{5}x_1 - x_2 = 0 & \Rightarrow \quad x_2 = \frac{1}{5}x_1 \\ x_1 + x_2 - 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = \frac{3}{5}x_1 \\ x_2 = \frac{1}{5}x_1 \end{cases}$$

Therefore; the eigenvector $V_2 = \begin{pmatrix} 5 \\ \frac{1}{5} \\ \frac{3}{5} \end{pmatrix}$

For $\lambda_3 = \sqrt{2} \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} 3-\sqrt{2} & 0 & -5 \\ \frac{1}{5} & -1-\sqrt{2} & 0 \\ 1 & 1 & -2-\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \left(3-\sqrt{2}\right)x_1 - 5x_3 = 0 & \Rightarrow \quad x_3 = \frac{3-\sqrt{2}}{5}x_1 \\ \frac{1}{5}x_1 + \left(-1-\sqrt{2}\right)x_2 = 0 & \Rightarrow \quad x_2 = \frac{1}{5\left(1+\sqrt{2}\right)}x_1 \\ x_1 + x_2 + \left(-2-\sqrt{2}\right)x_3 = 0 \end{cases}$$

Therefore; the eigenvector $V_3 = \begin{pmatrix} 1 \\ \frac{1}{5\left(1+\sqrt{2}\right)} \\ \frac{3-\sqrt{2}}{5} \end{pmatrix}$

Exercise

For the matrix: $\begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$$

i. $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & -5 \\ \frac{1}{5} & -1-\lambda & 0 \\ 1 & 1 & -2-\lambda \end{vmatrix}$

$$= (3-\lambda)(-1-\lambda)(-2-\lambda) - 1 + 5(-1-\lambda)$$

$$= (3-\lambda)\left(\lambda^2 + 3\lambda + 2\right) - 1 - 5 - 5\lambda$$

$$= 3\lambda^2 + 9\lambda + 6 - \lambda^3 - 3\lambda^2 - 2\lambda - 6 - 5\lambda$$

$$= -\lambda^3 + 2\lambda$$

The characteristic equation: $\boxed{-\lambda^3 \ \ 2\lambda = 0}$

ii. $-\lambda\left(\lambda^2 - 2\right) = 0$

Therefore; the eigenvalues are: $\boxed{\lambda_{1,2,3} = 0, \ \pm\sqrt{2}}$

iv. For $\lambda_1 = -\sqrt{2} \quad \Rightarrow \quad \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3+\sqrt{2} & 0 & -5 \\ \frac{1}{5} & -1+\sqrt{2} & 0 \\ 1 & 1 & -2+\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \left(3+\sqrt{2}\right)x_1 - 5x_3 = 0 & \Rightarrow \quad \boxed{x_3 = \frac{3+\sqrt{2}}{5}x_1} \\ \frac{1}{5}x_1 + \left(-1+\sqrt{2}\right)x_2 = 0 & \Rightarrow \quad \boxed{x_2 = -\frac{1}{5\left(-1+\sqrt{2}\right)}x_1} \\ x_1 + x_2 + \left(-2+\sqrt{2}\right)x_3 = 0 \end{cases}$$

Therefore; the eigenvector $V_1 = \begin{pmatrix} 1 \\ \frac{1}{5\left(1-\sqrt{2}\right)} \\ \frac{3+\sqrt{2}}{5} \end{pmatrix}$

For $\lambda_2 = 0 \quad \Rightarrow \left(A-\lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3x_1 - 5x_3 = 0 & \Rightarrow \quad \underline{x_3 = \frac{3}{5}x_1} \\ \frac{1}{5}x_1 - x_2 = 0 & \Rightarrow \quad \underline{x_2 = \frac{1}{5}x_1} \\ x_1 + x_2 - 2x_3 = 0 \end{cases}$$

Therefore; the eigenvector $V_2 = \begin{pmatrix} 5 \\ \frac{1}{5} \\ \frac{3}{5} \end{pmatrix}$

For $\lambda_3 = \sqrt{2} \quad \Rightarrow \left(A-\lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} 3-\sqrt{2} & 0 & -5 \\ \frac{1}{5} & -1-\sqrt{2} & 0 \\ 1 & 1 & -2-\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \left(3-\sqrt{2}\right)x_1 - 5x_3 = 0 & \Rightarrow \quad \underline{x_3 = \frac{3-\sqrt{2}}{5}x_1} \\ \frac{1}{5}x_1 + \left(-1-\sqrt{2}\right)x_2 = 0 & \Rightarrow \quad \underline{x_2 = \frac{1}{5\left(1+\sqrt{2}\right)}x_1} \\ x_1 + x_2 + \left(-2-\sqrt{2}\right)x_3 = 0 \end{cases}$$

Therefore; the eigenvector $V_3 = \begin{pmatrix} 1 \\ \frac{1}{5\left(1+\sqrt{2}\right)} \\ \frac{3-\sqrt{2}}{5} \end{pmatrix}$

Exercise

For the matrix: $\begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$$

i. $\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 & 1 \\ -1 & 3-\lambda & 0 \\ -4 & 13 & -1-\lambda \end{vmatrix}$

$$= (-1-\lambda)^2(3-\lambda) - 13 + 4(3-\lambda)$$

$$= (\lambda^2 + 2\lambda + 1)(3-\lambda) - 13 + 12 - 4\lambda$$

$$= 3\lambda^2 + 6\lambda + 3 - \lambda^3 - 2\lambda^2 - \lambda - 1 - 4\lambda$$

$$= -\lambda^3 + \lambda^2 + \lambda + 2$$

The characteristic equation: $\underline{-\lambda^3 + \lambda^2 + \lambda + 2 = 0}$

ii. $\lambda = 2$

$$\begin{array}{c|cccc} 2 & -1 & 1 & 1 & 2 \\ & & -2 & -2 & 2 \\ \hline & -1 & -1 & -1 & 0 \end{array} \quad \rightarrow \underline{-\lambda^2 - \lambda - 1 = 0}$$

Therefore; the eigenvalues are: $\underline{\lambda_{1,2,3} = 2,\ -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}}$

iii. For $\lambda_1 = 2$, we have: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -3 & 0 & 1 \\ -1 & 1 & 0 \\ -4 & 13 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -3x_1 + z_1 = 0 & \Rightarrow \quad \underline{z_1 = 3x_1} \\ -x_1 + y_1 = 0 & \Rightarrow \quad \underline{y_1 = x_1} \\ -4x_1 + 13y_1 - 3z_1 = 0 \end{cases}$$

Therefore, the eigenvector $\boxed{V_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}}$

For $\lambda_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, we have: $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -\frac{1}{2} + i\frac{\sqrt{3}}{2} & 0 & 1 \\ -1 & \frac{7}{2} + i\frac{\sqrt{3}}{2} & 0 \\ -4 & 13 & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow$$

$$\begin{cases} \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x_2 + z_2 = 0 & \Rightarrow \quad z_2 = -\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x_2 \\ -x_2 + \left(\frac{7}{2} + i\frac{\sqrt{3}}{2}\right)y_2 = 0 & \Rightarrow \quad y_2 = \left(\frac{2}{7 + i\sqrt{3}}\right)x_2 \\ -4x_2 + 13y_2 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)z_2 = 0 & \end{cases}$$

Therefore; the eigenvector $V_2 = \begin{pmatrix} 1 \\ \frac{1 - i\sqrt{3}}{2} \\ \frac{2}{7 + i\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1 - i\sqrt{3}}{2} \\ \frac{7 - i\sqrt{3}}{26} \end{pmatrix}$

$$V_3 = \begin{pmatrix} 1 \\ \frac{1 + i\sqrt{3}}{2} \\ \frac{2}{7 - i\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1 + i\sqrt{3}}{2} \\ \frac{7 + i\sqrt{3}}{26} \end{pmatrix}$$

Exercise

For the matrix: $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{pmatrix}$

i. Find the characteristic equation
ii. Find the eigenvalues
iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{pmatrix}$$

i. $$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 4 & 3-\lambda & 2 \\ -8 & -4 & -3-\lambda \end{vmatrix}$$

$$= (1-\lambda)(3-\lambda)(-3-\lambda) + 8(1-\lambda)$$

$$= -9 + 9\lambda + \lambda^2 - \lambda^3 + 8 - 8\lambda$$

$$= -\lambda^3 + \lambda^2 + \lambda - 1$$

The characteristic equation: $\boxed{-\lambda^3 + \lambda^2 + \lambda - 1 = 0}$

ii. $\lambda = 1$

$$\begin{array}{c|cccc} 1 & -1 & 1 & 1 & -1 \\ & & -1 & 0 & 1 \\ \hline & -1 & 0 & 1 & 0 \end{array} \quad \rightarrow \quad \boxed{-x^2 + 1 = 0}$$

Thus, the eigenvalues are: $\boxed{\lambda_{1,2,3} = 1,\ 1,\ -1}$

iii. For $\lambda_1 = 1 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 4 & 2 & 2 \\ -8 & -4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 0 = 0 \\ 4x + 2y + 2z = 0 & \Rightarrow \quad 2x = -y - z \\ -8x - 4y - 4z = 0 & \Rightarrow \quad 2x = -y - z \end{cases}$$

If $x = 0 \quad \Rightarrow \quad \boxed{y = -z}$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = -1 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 2 & 0 & 0 \\ 4 & 4 & 2 \\ -8 & -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x = 0 & \Rightarrow \quad \underline{x = 0} \\ 4x + 4y + 2z = 0 & \Rightarrow \quad \underline{z = -2y} \\ -8x - 4y - 2z = 0 \end{cases}$$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}}$$

For $\lambda_3 = -1$

$$AV_3 = V_2$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$$\begin{cases} \underline{x = 0} \\ 4x + 3y + 2z = 1 & \Rightarrow \quad \underline{2z = 1 - 3y} \\ -8x - 4y - 3z = -2 & \Rightarrow \end{cases}$$

$$\Rightarrow \boxed{V_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} -1 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & -12 & 2 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -1 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & -12 & 2 \end{pmatrix}$$

i. $$|A - \lambda I| = \begin{vmatrix} -1-\lambda & -4 & -2 \\ 0 & 1-\lambda & 1 \\ -6 & -12 & 2-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(1-\lambda)(2-\lambda) + 24 - 12(1-\lambda) + 12(-1-\lambda)$$

$$= -\lambda^3 + 2\lambda^2 + \lambda - 2 + 24 - 12 + 12\lambda - 12 - 12\lambda$$

$$= -\lambda^3 + 2\lambda^2 + \lambda - 2$$

The characteristic equation: $\underline{-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0}$

ii. $\lambda = 1$

$$\begin{array}{r|rrrr} 1 & -1 & 2 & 1 & -2 \\ & & -1 & 1 & 2 \\ \hline & -1 & 1 & 2 & 0 \end{array} \quad \rightarrow \quad \underline{-x^2 + x + 2 = 0}$$

Thus, the eigenvalues are: $\underline{\lambda_1 = -1 \quad \lambda_2 = 1 \quad and \quad \lambda_3 = 2}$

iii. For $\lambda_1 = -1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & -4 & -2 \\ 0 & 2 & 1 \\ -6 & -12 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -4y - 2z = 0 & & \\ 2y + z = 0 & \rightarrow 2y = -z & \Rightarrow \underline{y = -\frac{1}{2}z} \\ -6x - 12y + 3z = 0 & \rightarrow -6x = 12y - 3z & \underline{x = \frac{3}{2}z} \end{cases}$$

$$\Rightarrow V_1 = \boxed{\begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}}$$

For $\lambda_2 = 1 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -2 & -4 & -2 \\ 0 & 0 & 1 \\ -6 & -12 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2x - 4y - 2z = 0 & \Rightarrow \quad -2x - 4y = 0 \\ \qquad \underline{z = 0} \\ -6x - 12y + 2z = 0 \end{cases}$$

$\underline{x = -2y}$

$$\Rightarrow V_2 = \boxed{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}$$

For $\lambda_3 = 2 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -3 & -4 & -2 \\ 0 & -1 & 1 \\ -6 & -12 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} -3x - 4y - 2z = 0 & \Rightarrow \quad -3x = 6z \\ \quad -y + z = 0 & \Rightarrow \quad \underline{y = z} \\ -6x - 12y = 0 \end{cases}$$

$\underline{x = -2z}$

$$\Rightarrow V_3 = \boxed{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$$

i. $$\left|A - \lambda I\right| = \begin{vmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(4-\lambda)(-1-\lambda) - 4 - 8 + 4(4-\lambda) + 4(3-\lambda) + 2\lambda + 2$$

$$= -\lambda^3 + 6\lambda^2 - 5\lambda - 12 - 12 + 16 - 4\lambda + 12 - 4\lambda + 2\lambda + 2$$

$$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6$$

The characteristic equation: $\underline{-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0}$

ii. $\lambda = 1$

$$\begin{array}{c|cccc} 1 & -1 & 6 & -11 & 6 \\ & & -1 & 5 & -6 \\ \hline & -1 & 5 & -6 & 0 \end{array} \quad \to \quad \underline{-\lambda^2 + 5\lambda - 6 = 0}$$

Thus, the eigenvalues are: $\underline{\lambda_1 = 1 \quad \lambda_2 = 2 \quad \textit{and} \quad \lambda_3 = 3}$

iii. For $\lambda_1 = 1 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x + 2y + 2z = 0 & (1) \\ x + 3y + z = 0 & (2) \\ -2x - 4y - 2z = 0 & (3) \end{cases}$$

$(1) + (3) \quad \to \quad \underline{y = 0}$

$$\begin{cases} (1) & 2x+2z=0 \\ (2) & \underline{x+z=0} \\ & 3x+3z=0 \end{cases} \Rightarrow \boxed{x=-z}$$

$$\Rightarrow \boxed{V_1=\begin{pmatrix}-1\\0\\1\end{pmatrix}}$$

For $\lambda_2=2 \quad \Rightarrow \left(A-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix}1&2&2\\1&2&1\\-2&-4&-3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{cases} x+2y+2z=0 & (1) \\ x+2y+z=0 & (2) \\ -2x-4y-3z=0 & (3) \end{cases}$$

$$\begin{cases} 2\times(2) & 2x+4y+2z=0 \\ (3) & -2x-4y-3z=0 \end{cases}$$

$$\boxed{z=0}$$

$$(1) \to \quad x+2y=0 \to \boxed{x=-2y}$$

$$\Rightarrow \boxed{V_2=\begin{pmatrix}-2\\1\\0\end{pmatrix}}$$

For $\lambda_3=3 \quad \Rightarrow \left(A-\lambda_3 I\right)V_3=0$

$$\begin{pmatrix}0&2&2\\1&1&1\\-2&-4&-4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{cases} 2y+2z=0 & \to \boxed{y=-z} \\ x+y+z=0 & \qquad \to \boxed{x=0} \\ -2x-4y-4z=0 \end{cases}$$

$$\Rightarrow \boxed{V_3=\begin{pmatrix}0\\-1\\1\end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}$$

i. $|A - \lambda I| = \begin{vmatrix} -6-\lambda & 4 & 4 \\ -4 & 2-\lambda & 4 \\ -10 & 8 & 4-\lambda \end{vmatrix}$

$$= (-6-\lambda)(2-\lambda)(4-\lambda) - 160 - 128 + 40(2-\lambda) + 32(6+\lambda) + 16(4-\lambda)$$

$$= -\lambda^3 + 4\lambda$$

The characteristic equation: $\underline{-\lambda^3 + 4\lambda = 0}$

ii. $-\lambda\left(\lambda^2 - 4\right) = 0$

Thus, the eigenvalues are: $\underline{\lambda_1 = 0 \quad \lambda_2 = -2 \quad and \quad \lambda_3 = 2}$

iii. For $\lambda_1 = 0 \quad \Rightarrow \quad \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -6x + 4y + 4z = 0 & (1) \\ -4x + 2y + 4z = 0 & (2) \\ -10x + 8y + 4z = 0 & (3) \end{cases}$$

$(1) - (2) \quad \rightarrow \quad -2x + 2y = 0$

$\underline{x = y}$

$$\begin{cases} (1) & -2x + 4z = 0 \\ (2) & -2x + 4z = 0 \end{cases} \rightarrow \underline{x = 2z}$$

$$\Rightarrow \quad \boxed{V_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = -2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -4 & 4 & 4 \\ -4 & 4 & 4 \\ -10 & 8 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -4x + 4y + 4z = 0 & (1) \\ -4x + 4y + 4z = 0 & (2) \\ -10x + 8y + 6z = 0 & (3) \end{cases}$$

$$\begin{cases} -x + y + z = 0 & (4) \\ -5x + 4y + 3z = 0 & (5) \end{cases}$$

$$5 \times (4) - (5) \quad \to \quad y + 2z = 0$$

$$\boxed{y = -2z}$$

$$(5) \quad \to \quad -5x - 8z + 3z = 0$$

$$\boxed{x = -z}$$

$$\Rightarrow \quad \boxed{V_2 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}}$$

For $\lambda_3 = 2 \quad \Rightarrow (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -8 & 4 & 4 \\ -4 & 0 & 4 \\ -10 & 8 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -8x + 4y + 4z = 0 \\ -4x + 4z = 0 \\ -10x + 8y + 2z = 0 \end{cases} \Rightarrow \boxed{x = z}$$

$$\begin{cases} (1) & 4y - 4z = 0 \\ (3) & 8y - 8z = 0 \end{cases} \to \boxed{y = z}$$

$$\Rightarrow \quad \boxed{V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

Exercise

For the matrix: $\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

i. $$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & 2 & 0 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -2-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -\lambda & 0 & 2 \\ 1 & -\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left(\lambda^2(-2-\lambda) + 2 + \lambda\right)$$

$$= (1-\lambda)\left(-\lambda^3 - 2\lambda^2 + \lambda + 2\right)$$

$$= \lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2$$

The characteristic equation: $\underline{\lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2 = 0}$

ii. $\lambda = 1$

$$\begin{array}{c|ccccc} 1 & 1 & 1 & -3 & -1 & 2 \\ & & 1 & 2 & -1 & -2 \\ \hline 1 & 1 & 2 & -1 & -2 & 0 \\ & & 1 & 3 & 2 & \\ \hline & 1 & 3 & 2 & 0 & \end{array}$$

$\rightarrow \underline{\lambda^3 + 2\lambda^2 - \lambda + 2 = 0}$

$\rightarrow \underline{\lambda^2 + 3\lambda + 2 = 0}$

Thus, the eigenvalues are: $\underline{\lambda_{1,2,3,4} = -2,\ -1,\ 1,\ 1}$

iii. For $\lambda_1 = -2 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 0 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x_1 + 2x_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \\ x_2 = 0 \\ x_4 = 0 \end{cases}$$

$$\rightarrow \begin{cases} \underline{x_1 = -x_3} \\ \underline{x_2 = 0} \\ \underline{x_4 = 0} \end{cases}$$

Therefore; the eigenvector $V_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_2 = -1 \quad \Rightarrow \quad (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \\ x_2 - x_3 = 0 \\ \underline{x_4 = 0} \end{cases}$$

$$\rightarrow \begin{cases} \underline{x_1 = -2x_3} \\ \underline{x_2 = x_3} \end{cases}$$

Therefore; the eigenvector $V_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_3 = 1 \quad \Rightarrow \quad (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x_1 + 2x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_2 - 3x_3 = 0 \end{cases}$$

$$\to \begin{cases} \underline{x_1 = 2x_3} \\ \underline{x_2 = 3x_3} \\ \forall\, x_4 \end{cases}$$

Therefore; the eigenvector $V_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x_1 + 2x_3 = 2 \\ x_1 - x_2 + x_3 = 3 \\ x_2 - 3x_3 = 1 \end{cases}$$

$$\begin{cases} x_1 = 2x_3 - 2 \\ \\ x_2 = 1 + 3x_3 \end{cases}$$

Therefore; the eigenvector $V_4 = \begin{pmatrix} 0 \\ 4 \\ 1 \\ 1 \end{pmatrix} \qquad V_4 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Exercise

For the matrix: $\begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

i. Find the characteristic equation

ii. Find the eigenvalues

iii. Find the eigenvectors

Solution

$$A = \begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

i. $\det(A - \lambda I) = \begin{vmatrix} 10-\lambda & -9 & 0 & 0 \\ 4 & -2-\lambda & 0 & 0 \\ 0 & 0 & -2-\lambda & -7 \\ 0 & 0 & 1 & 2-\lambda \end{vmatrix}$

$$= (10-\lambda)\begin{vmatrix} -2-\lambda & 0 & 0 \\ 0 & -2-\lambda & -7 \\ 0 & 1 & 2-\lambda \end{vmatrix} + 9\begin{vmatrix} 4 & 0 & 0 \\ 0 & -2-\lambda & -7 \\ 0 & 1 & 2-\lambda \end{vmatrix}$$

$$= (10-\lambda)\left[(-2-\lambda)^2(2-\lambda) + 7(-2-\lambda)\right] + 9\left[(4)(-2-\lambda)(2-\lambda) + 28\right]$$

$$= (10-\lambda)(-2-\lambda)\left(3+\lambda^2\right) + 9\left(4\lambda^2 + 12\right)$$

$$= \left(3+\lambda^2\right)\left(-8\lambda + \lambda^2 + 16\right)$$

$$= \left(3+\lambda^2\right)(\lambda - 4)^2$$

$\Rightarrow$ The characteristic equation: $\left(3+\lambda^2\right)(\lambda-4)^2 = 0$

ii. The eigenvalues are $\lambda_{1,2,3,4} = 4,\ 4,\ \pm i\sqrt{3}$

iii. For $\lambda_1 = 4 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 6 & -9 & 0 & 0 \\ 4 & -6 & 0 & 0 \\ 0 & 0 & -6 & -7 \\ 0 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 6x_1 - 9x_2 = 0 & \rightarrow \quad \boxed{2x_1 = 3x_2} \\ 4x_1 - 6x_2 = 0 \\ -6x_3 - 7x_4 = 0 & (1) \\ x_3 - 2x_4 = 0 & (2) \end{cases}$$

$$(1)\&(2) \quad \rightarrow \quad \boxed{x_3 = x_4 = 0}$$

Therefore; the eigenvector $\boxed{V_1 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}}$

$$\begin{pmatrix} 6 & -9 & 0 & 0 \\ 4 & -6 & 0 & 0 \\ 0 & 0 & -6 & -7 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 6x_1 - 9x_2 = 3 & (3) \\ 4x_1 - 6x_2 = 2 & (4) \\ -6x_3 - 7x_4 = 0 & (5) \\ x_3 - 2x_4 = 0 & (6) \end{cases}$$

$$\begin{cases} 6x_1 - 9x_2 = 3 \\ 4x_1 - 6x_2 = 2 \end{cases} \qquad \Delta = \begin{vmatrix} 6 & -9 \\ 4 & -6 \end{vmatrix} = 0 \quad \Delta_4 = \begin{vmatrix} 6 & 3 \\ 4 & 2 \end{vmatrix} = 0$$

$$(5)\&(6) \quad \rightarrow \quad \boxed{x_3 = x_4 = 0}$$

Therefore; the eigenvector $\boxed{V_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}$

For $\lambda_3 = -i\sqrt{3} \quad \Rightarrow \quad (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} 10 + i\sqrt{3} & -9 & 0 & 0 \\ 4 & -2 + i\sqrt{3} & 0 & 0 \\ 0 & 0 & -2 + i\sqrt{3} & -7 \\ 0 & 0 & 1 & 2 + i\sqrt{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \left(10+i\sqrt{3}\right)x_1 - 9x_2 = 0 & \rightarrow \quad x_1 = \dfrac{9}{10+i\sqrt{3}}x_2 \\ 4x_1 + \left(-2+i\sqrt{3}\right)x_2 = 0 & \rightarrow \quad x_1 = \dfrac{-2+i\sqrt{3}}{4}x_2 \\ \left(-2+i\sqrt{3}\right)x_3 - 7x_4 = 0 & \rightarrow \quad (6) \\ x_3 + \left(2+i\sqrt{3}\right)x_4 = 0 & \rightarrow \quad \underline{x_3 = -\left(2+i\sqrt{3}\right)x_4} \end{cases}$$

$$(6) \quad \rightarrow \quad \frac{7}{-2+i\sqrt{3}}\left(\frac{-2-i\sqrt{3}}{-2-i\sqrt{3}}\right) = -\left(2+i\sqrt{3}\right)$$

$$\frac{9}{10+i\sqrt{3}} \overset{?}{=} \frac{-2+i\sqrt{3}}{4}$$

$$36 \neq \left(-2+i\sqrt{3}\right)\left(10+i\sqrt{3}\right)$$

$$\Rightarrow \quad \underline{x_1 = x_2 = 0}$$

Therefore; the eigenvector $V_3 = \begin{pmatrix} 0 \\ 0 \\ -\left(2+i\sqrt{3}\right) \\ 1 \end{pmatrix}$

Therefore; the eigenvector $V_4 = \begin{pmatrix} 0 \\ 0 \\ -2+i\sqrt{3} \\ 1 \end{pmatrix}$

Exercise

Find the eigenvalues of A^9 for $A = \begin{pmatrix} 1 & 3 & 7 & 11 \\ 0 & \frac{1}{2} & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

Solution

Since the matrix is an upper triangular, then the eigenvalues are: $\lambda = 1,\ \frac{1}{2},\ 0,\ 2$

The eigenvalues of A^9 are: $1^9 \underline{= 1}$ $\quad \left(\frac{1}{2}\right)^9 \underline{= \frac{1}{512}}$

$0^9 \underline{= 0}$ $\quad 2^9 \underline{= 512}$

Exercise

Given: $A=\begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$. Compute A^{11}

Solution

$$\det(A-\lambda I)=\begin{vmatrix} -1-\lambda & 7 & -1 \\ 0 & 1-\lambda & 0 \\ 0 & 15 & -2-\lambda \end{vmatrix}$$

$$=(-1-\lambda)(1-\lambda)(-2-\lambda)$$

The eigenvalues are: $\lambda_{1,2,3}=-1,\ 1,\ -2$

For $\lambda_1=-1$, we have: $\left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix} 0 & 7 & -1 \\ 0 & 2 & 0 \\ 0 & 15 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 7y_1-z_1=0 & \rightarrow & z_1=7y_1 \\ 2y_1=0 & \rightarrow & y_1=0 \\ 15y_1-z_1=0 \end{cases}$$

The eigenvector $V_1=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

For $\lambda_2=1$, we have: $(A-I)V_2=0$

$$\begin{pmatrix} -2 & 7 & -1 \\ 0 & 0 & 0 \\ 0 & 15 & -3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2x_2+7y_2-z_2=0 & \rightarrow & 2x_2=7y_2-z_2 \\ 15y_2-3z_2=0 & \rightarrow & 5y_2=z_2 \end{cases}$$

$$2x_2=7y_2-5y_2$$

$$x_2=y_2$$

The eigenvector $\boxed{V_2 = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}}$

For $\lambda_3 = -2$, we have: $(A + 2I)V_3 = 0$

$$\begin{pmatrix} 1 & 7 & -1 \\ 0 & 3 & 0 \\ 0 & 15 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_3 + 7y_3 - z_3 = 0 & \rightarrow \quad \boxed{x_3 = z_3} \\ 3y_3 = 0 & \rightarrow \quad \boxed{y_3 = 0} \\ 15y_3 = 0 \end{cases}$$

The eigenvector $\boxed{V_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$D^{11} = \begin{pmatrix} (-1)^{11} & 0 & 0 \\ 0 & 1^{11} & 0 \\ 0 & 0 & (-2)^{11} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2048 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 5 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 5R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -5 & 1 \end{array}\right) \begin{array}{l} R_1 - R_3 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 4 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -5 & 1 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$A^{11} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2048 \end{pmatrix}\begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & -2048 \\ 0 & 1 & 0 \\ 0 & 5 & -2048 \end{pmatrix}\begin{pmatrix} 1 & 4 & -1 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 10237 & 2047 \\ 0 & 1 & 0 \\ 0 & 10245 & -2048 \end{pmatrix}$$

Exercise

Find the eigenvalues of the matrices

$$A = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0.61 & 0.52 \\ 0.39 & 0.48 \end{pmatrix}, \quad A^\infty = \begin{pmatrix} 0.57143 & 0.57143 \\ 0.42857 & 0.42857 \end{pmatrix}, \quad and \quad B = \begin{pmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{pmatrix}$$

Solution

The eigenvalues for A:

$$\begin{vmatrix} 0.7-\lambda & 0.4 \\ 0.3 & 0.6-\lambda \end{vmatrix} = (0.7-\lambda)(0.6-\lambda) - .12$$

$$= \lambda^2 - 1.3\lambda + .3 = 0$$

$$\lambda_{1,2} = \frac{1.3 \pm \sqrt{1.69 - 1.2}}{2}$$

$$= .65 \pm \frac{\sqrt{.49}}{2}$$

$$= 0.65 \pm 0.35$$

The eigenvalues are: $\lambda_1 = 1 \quad \lambda_2 = 0.3$

The eigenvalues for A^2:

$\lambda_1 = 1^2 \underline{= 1}$

$\lambda_2 = 0.3^2 \underline{= 0.09}$

The eigenvalues for A^{∞}:

$\lambda^2 - \lambda = 0$

$\lambda_1 = 1^2 \underline{= 1}$

$\lambda_2 = 0.3^{\infty} \underline{= 0}$

The eigenvalues for B:

$$\begin{vmatrix} 0.3-\lambda & 0.6 \\ 0.7 & 0.4-\lambda \end{vmatrix} = \lambda^2 - .7\lambda - .3 = 0$$

$\lambda_{1,2} = 0.35 \pm 0.65$

The eigenvalues are: $\underline{\lambda_1 = 1 \quad \lambda_2 = -0.3}$

Exercise

Given the matrix $\begin{bmatrix} -1 & -3 \\ -3 & 7 \end{bmatrix}$

a) Find the characteristic polynomial.
b) Find the eigenvalues
c) Find the bases for its eigenspaces
d) Graph the eigenspaces
e) Verify directly that $A\vec{v} = \lambda\vec{v}$, for all associated eigenvectors and eigenvalues.

Solution

a) $\begin{vmatrix} -1-\lambda & -3 \\ -3 & 7-\lambda \end{vmatrix} = (-1-\lambda)(7-\lambda) - 9$

$$= -7 - 6\lambda + \lambda^2 - 9$$
$$= \lambda^2 - 6\lambda - 16$$

The characteristic polynomial is $\underline{\lambda^2 - 6\lambda - 16 = 0}$

b) $\lambda^2 - 6\lambda - 16 = 0 \quad \Rightarrow \quad \lambda_1 = -2 \quad \textit{and} \quad \lambda_2 = 8$

c) For $\lambda_1 = -2$, we have: $(A + 2I)V_1 = 0$

$$\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 - 3y_1 = 0 & \Rightarrow \boxed{x_1 = 3y_1} \\ -3x_1 + 9y_1 = 0 \end{cases}$$

Therefore, the eigenvector $\boxed{V_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}}$

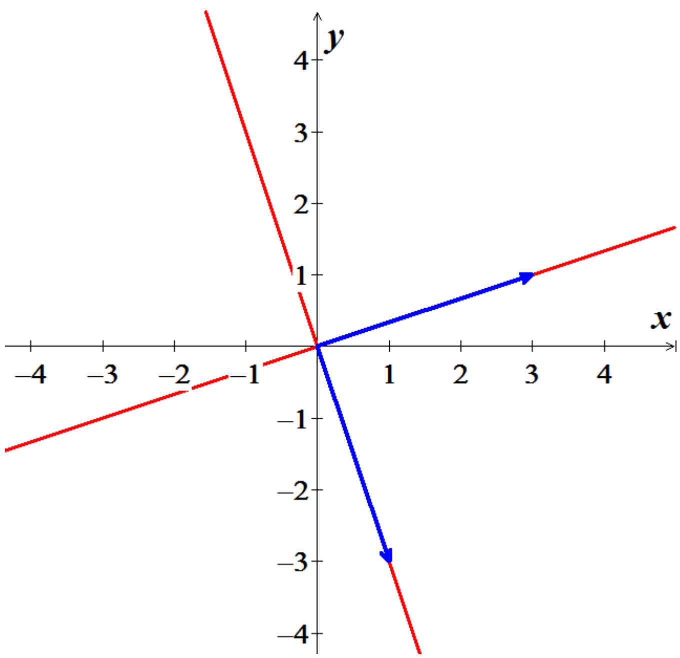
For $\lambda_2 = 8$, we have: $(A + 8I)V_2 = 0$

$$\begin{pmatrix} -9 & -3 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -9x_2 - 3y_2 = 0 & \Rightarrow \boxed{y_2 = -3x_2} \\ -3x_2 - y_2 = 0 \end{cases}$$

Therefore, the eigenvector $\boxed{V_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}}$

d)



e) $AV_1 = \lambda_1 V_1$

$$\begin{pmatrix} -1 & -3 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = -2\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -6 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \end{pmatrix} \checkmark$$

$$AV_2 = \lambda_2 V_2$$

$$\begin{pmatrix} -1 & -3 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = 8\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 8 \\ -24 \end{pmatrix} = \begin{pmatrix} -6 \\ -24 \end{pmatrix} \checkmark$$

Exercise

Given the matrix $\begin{bmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{bmatrix}$

a) Find the characteristic polynomial.

b) Find the eigenvalues

c) Find the bases for its eigenspaces

d) Graph the eigenspaces

e) Verify directly that $A\vec{v} = \lambda\vec{v}$, for all associated eigenvectors and eigenvalues.

Solution

a) $\begin{vmatrix} 5-\lambda & 0 & -4 \\ 0 & -3-\lambda & 0 \\ -4 & 0 & -1-\lambda \end{vmatrix} = (5-\lambda)(-3-\lambda)(-1-\lambda) - 16(-3-\lambda)$

$$= (5-\lambda)\left(3+4\lambda+\lambda^2\right) + 48 + 16\lambda$$

$$= 15 + 20\lambda + 5\lambda^2 - 3\lambda - 4\lambda^2 - \lambda^3 + 48 + 16\lambda$$

$$= -\lambda^3 + \lambda^2 + 33\lambda + 63$$

The characteristic polynomial is $\underline{-\lambda^3 + \lambda^2 + 33\lambda + 63 = 0}$

b) $\lambda = -3$

$$\begin{array}{r|rrrr} -3 & -1 & 1 & 33 & 63 \\ & & 3 & -12 & -63 \\ \hline & -1 & 4 & 21 & 0 \end{array} \quad \rightarrow \quad \underline{-\lambda^2 + 4\lambda + 21 = 0}$$

$$\lambda = \frac{-4 \pm \sqrt{16+84}}{-2}$$

The eigenvalues are: $\underline{\lambda_{1,2,3} = -3, \ -3, \ 7}$

c) For $\lambda_{1,2} = -3$, we have: $(A+3I)V_1 = 0$

$$\begin{pmatrix} 8 & 0 & -4 \\ 0 & 0 & 0 \\ -4 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 8x_1 - 4z_1 = 0 & \rightarrow \quad \underline{z_1 = 2x_1} \\ -4x_1 + 2z_1 = 0 \end{cases}$$

Therefore, the eigenvector $V_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ *and* $V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

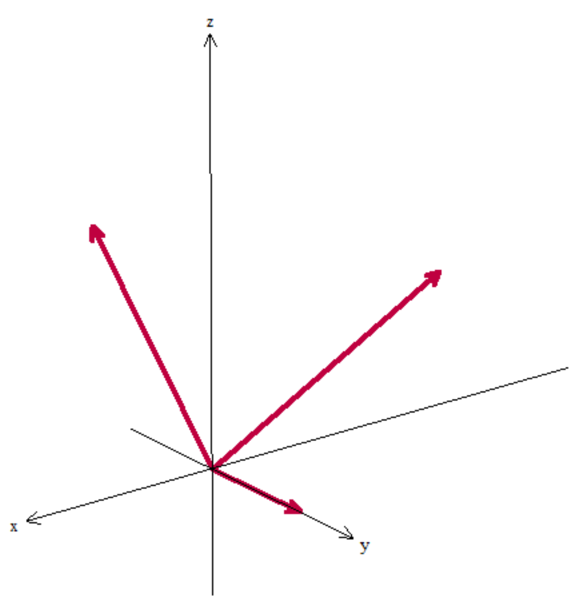
For $\lambda_3 = 7$, we have: $(A - 7I)V_3 = 0$

$$\begin{pmatrix} -2 & 0 & -4 \\ 0 & -10 & 0 \\ -4 & 0 & -8 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2x_3 - 4z_3 = 0 & \rightarrow \quad \underline{x_3 = -2z_3} \\ -10y_3 = 0 & \rightarrow \quad \underline{y_3 = 0} \\ -4x_3 - 8z_3 = 0 \end{cases}$$

Therefore, the eigenvector $\underline{V_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}$

d)



e) $AV_1 = \lambda_1 V_1$

$$\begin{pmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -3 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -6 \end{pmatrix} \checkmark$$

$AV_2 = \lambda_2 V_2$

$$\begin{pmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} \checkmark$$

$AV_3 = \lambda_3 V_3$

$$\begin{pmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 7\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -14 \\ 0 \\ 7 \end{pmatrix} = \begin{pmatrix} -14 \\ 0 \\ 7 \end{pmatrix} \checkmark$$

Exercise

Explain why a 2×2 matrix can have at most two distinct eigenvalues. Explain why an $n \times n$ matrix can have at most n distinct eigenvalues

Solution

A 2×2 matrix has only 2 entries in the main diagonal. Then, Lambda exists only twice in those entries. By using the determinant, the product will produce a power two characteristics equation. A second–degree equation will produce 2 real distinct eigenvalues, or 2 repeated eigenvalues, or 2 complex eigenvalues. Therefore, a 2×2 matrix can have at most two distinct eigenvalues.

The same for an $n \times n$ matrix, the matrix has n entries in the main diagonal with lambda. Then the product of n^{th} lambda will produce a characteristic equation with n power. That means that will have n real distinct eigenvalues, or n repeated eigenvalues, or n complex eigenvalues.
Therefore, a $n \times n$ matrix can have at most n distinct eigenvalues.

Exercise

Construct an example of a 2×2 matrix with only one distinct eigenvalue.

Solution

A 2×2 matrix with only one distinct eigenvalue, which means that we have repeated lambda. To do so, the other diagonal has a zero and the main diagonal has the same value.

Example for one zero in the diagonal.

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

$$\begin{vmatrix} a-\lambda & b \\ 0 & a-\lambda \end{vmatrix} = (a-\lambda)^2 = 0$$

$$\boxed{\lambda_{1,2} = a}$$

Example: $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

Exercise

Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}.

Solution

Since matrix A in invertible, then $AA^{-1} = A^{-1}A = I$

Let λ be an eigenvalue of an invertible matrix A, then there is a nonzero eigenvector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$

$$A^{-1}A\vec{v} = A^{-1}\lambda\vec{v}$$

$$I\vec{v} = \lambda\left(A^{-1}\vec{v}\right)$$

$$\vec{v} = \lambda\left(A^{-1}\vec{v}\right)$$

Since $\vec{v} \neq \vec{0}$ and λ cannot be zero. Then

$$\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$$

$$\lambda^{-1}\vec{v} = A^{-1}\vec{v}$$

That will prove that λ^{-1} is an eigenvalue of A^{-1}

Exercise

Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0

Solution

Assume that A^2 is the zero matrix.

If $A\vec{v} = \lambda\vec{v}$

$AA\vec{v} = A\lambda\vec{v}$

$A^2\,\vec{v} = \lambda\left(A\vec{v}\right)$

$A^2\,\vec{v} = \lambda\left(\lambda\vec{v}\right)$

$A^2\,\vec{v} = \lambda^2\,\vec{v}$

Since $\vec{v} \neq \vec{0}$ and A^2 is the zero matrix. Then λ must be zero.

Therefore, each eigenvalue of A is zero.

Exercise

Show that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T.

Solution

Suppose that λ is an eigenvalue of A, then $A - \lambda I = 0$

$$\begin{aligned}\left(A-\lambda I\right)^T &= A^T - \lambda I^T \\ &= A^T - \lambda I = 0\end{aligned}$$

This will result that matrix and its transpose have the same characteristic equation.

Thus, λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T

Exercise

For $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, find one eigenvalue, without calculation. Justify your answer.

Solution

Since the matrix A has the row then matrix A in not invertible (Columns are linearly dependent).
Therefore, the eigenvalue is zero of the matrix.

Exercise

For $A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$, find one eigenvalue, and two linearly independent eigenvectors, without calculation.

Justify your answer.

Solution

Since the matrix A has the row then matrix A in not invertible (Columns are linearly dependent). Therefore, the eigenvalue is zero of the matrix.

For $\lambda = 0$, then the eigenvector is given by $(A - \lambda I)V = 0$

Since $\lambda = 0$, that implies to $AV = 0$

Since matrix A is nonzero matrix that it will imply to $2x + 2y + 2z = 0$ all rows are the same.

Which it will result to: $x + y + z = 0$

The two linearly independent eigenvectors:

$$V_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad V_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Exercise

Consider an $n \times n$ matrix A with the property that the row sums all equal the same number s. Show that s is an eigenvalue of A.

Solution

Let consider a 2×2 matrix with all ones as entries

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$\underline{s = 1 + 2 = 3}$

$$\begin{vmatrix} 1-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 2$$

$$= \lambda^2 - 3\lambda = 0$$

One of the eigenvalues is: $\underline{\lambda = 3 = s}$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

With $\begin{cases} a + b = s \\ c + d = s \end{cases}$

$$\begin{pmatrix} a \\ c \end{pmatrix} + \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} s \\ s \end{pmatrix}$$

$$1 \cdot \begin{pmatrix} a \\ c \end{pmatrix} + 1 \cdot \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} s \\ s \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A\vec{v} = s\vec{v} \quad \checkmark$$

For $n \times n$ matrix A:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Where $s = \sum_{i=1}^{n} a_{1i} = \ldots = \sum_{i=1}^{n} a_{ni}$

$$\begin{pmatrix} a_{11} + \cdots + a_{1n} \\ \vdots \qquad \vdots \\ a_{n1} + \cdots + a_{nn} \end{pmatrix} = \begin{pmatrix} s \\ \vdots \\ s \end{pmatrix}$$

$$1 \cdot \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} + \cdots + 1 \cdot \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} = s \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = s \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$A\vec{v} = s\vec{v} \quad \checkmark$$

That prove that ***s*** is an eigenvalue of A.

Exercise

Consider an $n \times n$ matrix A with the property that the column sums all equal the same number s. Show that s is an eigenvalue of A.

Solution

Given that the column sums of an $n \times n$ matrix A all equal the same number s.

Then the transpose of the matrix A will imply that A^T has the row sums all equal the same number s. In addition, the matrix A and A^T have the same eigenvalues.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \qquad \text{Where } s = \sum_{i=1}^{n} a_{i1} = \ldots = \sum_{i=1}^{n} a_{in}$$

$$A^T = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{pmatrix} \qquad \text{Where } s = \sum_{i=1}^{n} a_{i1} = \ldots = \sum_{i=1}^{n} a_{in}$$

$$\begin{pmatrix} a_{11} + \cdots + a_{n1} \\ \vdots \qquad \vdots \\ a_{1n} + \cdots + a_{nn} \end{pmatrix} = \begin{pmatrix} s \\ \vdots \\ s \end{pmatrix}$$

$$1 \cdot \begin{pmatrix} a_{11} \\ \vdots \\ a_{1n} \end{pmatrix} + \cdots + 1 \cdot \begin{pmatrix} a_{n1} \\ \vdots \\ a_{nn} \end{pmatrix} = s \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} + \cdots + a_{n1} \\ \vdots \qquad \vdots \\ a_{1n} + \cdots + a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = s \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$A^T \vec{v} = s \vec{v} \quad \checkmark$$

That show that s is an eigenvalue of A^T and since A and A^T have the same eigenvalues.

The prove is completed that s is an eigenvalue of A.

Exercise

Let A be the matrix of the linear transformation T on $\mathbb{R}^2$

T: reflects points across some line through the origin.

Without writing A, find an eigenvalue of A and describe the eigenspace.

Solution

Given T reflects points across some line through the origin in $\mathbb{R}^2$, which implies that the coordinates are equal $(x = \lambda y)$.

The linear transformation can be written in the form: $T(\vec{v}) = \vec{v}$

This line more likely is the scalar nonzero product of the eigenvectors $\vec{v}$.

$$A\vec{v} = \lambda\vec{v}$$

Since A be the matrix of the linear transformation T on $\mathbb{R}^2$, then $A\vec{v} = \vec{v}$.

Thus, the eigenvalue $\lambda = 1$ of the matrix A which will result to the corresponding eigenvector $\vec{v}$.

The other eigenvector $\vec{u}$ can be generated by applying the orthogonal to the line and which leads to the eigenvalue $\lambda = -1$. The result form that each vector on the line through $\vec{u}$ can be transformed into the opposite sign of that vector.

Exercise

Let A be the matrix of the linear transformation T on $\mathbb{R}^2$

T: reflects points about some line through the origin.

Without writing A, find an eigenvalue of A and describe the eigenspace.

Solution

Given T reflects points *about* some line through the origin.

If $\vec{v} \in \mathbb{R}^2$ lines on the line, then the linear transformation can be written in the form:

$$T(\vec{v}) = A\vec{v} = \vec{v}$$

That implies to T rotates points around a given line, the points on the line are not moved at all.

Thus, the eigenvalue $\lambda = 1$ of the matrix A which will result to the corresponding eigenvector $\vec{v}$.

The corresponding eigenspace is either just the line if T doesn't rotate full rotation $(2\pi k)$.

Therefore, the corresponding eigenspace is the line the points are being rotated around.

Exercise

Show that if $\vec{v}$ is an eigenvector of the matrix product AB and $B\vec{v} \neq \vec{0}$, then $B\vec{v}$ is an eigenvector of BA

Solution

Since $\vec{v}$ is an eigenvector of the matrix product AB, that must be some eigenvalue λ to satisfy.

Such that $AB\,\vec{v} = \lambda\,\vec{v}$ and $\vec{v} \neq \vec{0}$.

Since $B\vec{v} \neq \vec{0}$, then we can rewrite

$$AB\,\vec{v} = \lambda\,\vec{v}$$

$$A\left(B\,\vec{v}\right) = \lambda\,\vec{v} \qquad \text{Multiply both sides by matrix } B.$$

$$BA\left(B\,\vec{v}\right) = B\lambda\,\vec{v}$$

$$BA\left(B\,\vec{v}\right) = \lambda\left(B\,\vec{v}\right)$$

Therefore, since $B\vec{v} \neq \vec{0}$, that is clearly that $B\vec{v}$ is an eigenvector of BA.

Exercise

Explain and demonstrate that the eigenspace of a matrix A corresponding to some eigenvalue λ is a subspace.

Solution

λ is an eigenvalue of a square matrix $(n \times n)$, then $A\vec{v} = \lambda\vec{v}$ and $\vec{v}$ is a non-zero vector.

That implies to: $(A - \lambda I)\vec{v} = \vec{0}$.

The eigenspace consists of the zero vector and all the eigenvectors $\vec{v}$ corresponding to the eigenvalue λ.

This is equivalent to the null space of $A - \lambda I$ which includes the trivial (zero vector) solution of $(A - \lambda I)\vec{v} = \vec{0}$ as well as the non-trivial (non-zero) solutions. As the null space is definitely a subspace, and the eigenspace is essentially the same, then the eigenspace is a sunspace too.

Is the eigenspace is closed under addition?

Suppose that $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors corresponding to λ.

Let assume that $A\vec{v}_1 = \lambda\vec{v}_1$ and $A\vec{v}_2 = \lambda\vec{v}_2$

$$\begin{aligned} A\left(\vec{v}_1 + \vec{v}_2\right) &= A\vec{v}_1 + A\vec{v}_2 \\ &= \lambda\vec{v}_1 + \lambda\vec{v}_2 \\ &= \lambda\left(\vec{v}_1 + \vec{v}_2\right) \end{aligned}$$

Therefore, $\left(\vec{v}_1 + \vec{v}_2\right)$ is in the eigenspace of λ under addition.

Is the eigenspace is closed under scalar multiplication?

Let $\vec{v}_1$ be an eigenvector corresponding to λ and c be any real scalar.

$$\begin{aligned} cA\left(\vec{v}_1\right) &= A\left(c\vec{v}_1\right) \\ &= c\left(\lambda\vec{v}_1\right) \\ &= \lambda\left(c\vec{v}_1\right) \end{aligned}$$

Therefore, $c\vec{v}_1$ is in the eigenspace of λ under scalar multiplication.

Therefore, the eigenspace of a matrix A corresponding to some eigenvalue λ is a subspace.

Exercise

If λ is an eigenvalue of the matrix A, prove that λ^2 is an eigenvalue of A^2.

Solution

Since λ is an eigenvalue of the matrix A, then $A\vec{v} = \lambda\vec{v}$ where $\vec{v} \neq \vec{0}$.

$$A\vec{v} = \lambda\vec{v}$$

$$A\left(A\vec{v}\right) = A\left(\lambda\vec{v}\right)$$

$$\begin{aligned} A^2\vec{v} &= \lambda\left(A\vec{v}\right) \qquad\qquad A\vec{v} = \lambda\vec{v} \\ &= \lambda\left(\lambda\vec{v}\right) \\ &= \lambda^2\vec{v} \end{aligned}$$

Therefore, λ^2 is an eigenvalue of A^2

Solution Section 4.5 – Diagonalization

Exercise

The Lucas numbers are like Fibonacci numbers except they start with $L_1 = 1$ *and* $L_2 = 3$. Following the rule $L_{k+2} = L_{k+1} + L_k$. The next Lucas numbers are 4, 7, 11, 18. Show that the Lucas number $L_{100} = \lambda_1^{100} + \lambda_2^{100}$.

Solution

Let $u_k = \begin{pmatrix} L_{k+1} \\ L_k \end{pmatrix}$

the rule $\begin{cases} L_{k+2} = L_{k+1} + L_k \\ \quad L_{k+1} = L_{k+1} \end{cases}$

becomes $\vec{u}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \vec{u}_k$.

$\Rightarrow A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} \\ &= -\lambda(1-\lambda) - 1 \\ &= \lambda^2 - \lambda - 1 \end{aligned}$$

The characteristic equation is $\lambda^2 - \lambda - 1 = 0$ and the solutions are

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618 \quad \textit{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -.618$$

For $\lambda_1 \quad \Rightarrow \quad (A - \lambda_1 I)\vec{v}_1 = 0$

$$\begin{pmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x_1 - \lambda_1 y_1 = 0$$

$x_1 = \lambda_1 y_1$

$\Rightarrow \vec{v}_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$

For $\lambda_2 \Rightarrow \left(A-\lambda_2 I\right)\vec{v}_2 = 0$

$$\begin{pmatrix} 1-\lambda_2 & 1 \\ 1 & -\lambda_2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x_1 - \lambda_2 y_1 = 0$$

$$x_1 = \lambda_2 y_1$$

$$\Rightarrow \vec{v}_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

The linear combination:

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{u}_1$$
$$= \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\lambda_1\vec{v}_1 + \lambda_2\vec{v}_2 = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$
$$= \begin{pmatrix} \lambda_1^2 + \lambda_2^2 \\ \lambda_1 + \lambda_2 \end{pmatrix}$$
$$= \begin{bmatrix} trace\ of\ A^2 \\ trace\ of\ A \end{bmatrix}$$
$$= \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

The solution:

$$\vec{u}_{100} = A^{99}\,\vec{u}_1$$

$$L_{100} = c_1\lambda_1^{99} + c_2\lambda_2^{99}$$
$$= \lambda_1^{100} + \lambda_2^{100}$$

Exercise

Find all eigenvector matrices S that diagonalize A (rank 1) to give $S^{-1}AS = \Lambda$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

What is A^n? Which matrices B commute with A (so that $AB = BA$)

Solution

Since A has rank 1, its nullspace is a two-dimensional plane. Any vector with $x+y+z=0$ solves $A\vec{v} = \vec{0}$. So $\lambda = 0$ is an eigenvalue with multiplicity 2. There are two independent eigenvectors. The other eigenvalues must be $\lambda = 3$ because the trace A is 1 + 1 + 1 = 3.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^3 + 2 - 3(1-\lambda)$$

$$= -\lambda^3 + 3\lambda^2$$

The eigenvalues are $\underline{\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 3}$

For $\lambda_{1,2} = 0 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad x + y + z = 0$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \& \quad V_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $\lambda_3 = 3 \quad \Rightarrow \quad (A - \lambda_3 I)\vec{v}_3 = 0$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to x_3 + y_3 - 2z_3$$

$$\Rightarrow \quad V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The possible matrices S:

$$S = \begin{pmatrix} x & X & c \\ y & Y & c \\ -x-y & -X-Y & c \end{pmatrix}$$

and

$$S^{-1}AS = \Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

where $c \neq 0$ *and* $xY \neq yX$.

The powers A^n come:

$$A^2 = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = 3A$$

and

$$A^n = 3^{n-1}A$$

If $AB = BA$, all the column and row of $\boldsymbol{B}$ must be the same.
One possible $\boldsymbol{B}$ is $\boldsymbol{A}$ itself, since $AA = AA$, $\boldsymbol{B}$ is any linear combination of permutation matrices.

Exercise

Determine whether the matrix is diagonalizable $\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$

Solution

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^2 = 0$$

The eigenvalues are: $\lambda_{1,2} = 2$

For $\lambda_{1,2} = 2 \;\Rightarrow\; (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \; x = 0$$

$$\Rightarrow \quad \underline{V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$S = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$

$$\underline{\det(S) = 0}$$

The inverse doesn't exist.
Therefore, the matrix A is not diagonalizable.

Exercise

Determine whether the matrix is diagonalizable $\begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$

Solution

$$A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix}$$

$$= (-3-\lambda)(1-\lambda) + 4$$

$$= \lambda^2 + 2\lambda + 1 = 0$$

The only eigenvalue: $\underline{\lambda_{1,2} = -1}$

For $\lambda_{1,2} = -1 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad -2x + 2y = 0$$

$$\underline{x = y}$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad V_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$ (***linearly dependent***)

$$S = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$\underline{\det(S) = 0}$$

The inverse doesn't exist.
Therefore, the matrix A is not diagonalizable.

Exercise

Determine whether the matrix is diagonalizable $\begin{pmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{pmatrix}$

Solution

$$A = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 & 1 \\ -1 & 3-\lambda & 0 \\ -4 & 13 & -1-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(3-\lambda)(-1-\lambda) - 13 + 4(3-\lambda)$$

$$= \left(1 + 2\lambda + \lambda^2\right)(3-\lambda) - 13 + 12 - 4\lambda$$

$$= 3 + 6\lambda + 3\lambda^2 - \lambda - 2\lambda^2 - \lambda^3 - 1 - 4\lambda$$

$$= -\lambda^3 + \lambda^2 + \lambda + 2 = 0$$

$$\begin{array}{r|rrrr} 2 & -1 & 1 & 1 & 2 \\ & & -2 & -2 & -2 \\ \hline & -1 & -1 & -1 & 0 \end{array} \quad \to \quad \lambda^2 + \lambda + 1 = 0$$

The eigenvalues are: $\lambda_1 = 2, \quad \lambda_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \lambda_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

For $\lambda_1 = 2 \quad \Rightarrow \quad \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -3 & 0 & 1 \\ -1 & 1 & 0 \\ -4 & 13 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \to \quad -3x_1 + z_1 = 0 \\ \to \quad -x_1 + y_1 = 0 \\ \to \quad -4x_1 + 13y_1 - 3z_1 = 0 \end{array}$$

$$\begin{cases} z_1 = 3x_1 \\ y_1 = x_1 \end{cases}$$

$$\Rightarrow \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

Exercise

Determine whether the matrix is diagonalizable $\begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Solution

Since the matrix is upper triangular with diagonal entries 2, 2, 3, and 3, the eigenvalues are 2 and 3 (each multiplicity of 2)

For $\lambda_1 = 2 \quad \Rightarrow \quad (A - 2I)V_1 = 0$

$$\begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{cases} -x_2 + x_4 = 0 \\ x_3 - x_4 = 0 \\ x_3 + 2x_4 = 0 \\ x_4 = 0 \end{cases}$$

$$\Rightarrow \quad \underline{x_2 = x_3 = x_4 = 0|}$$

$V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ has dimension 1.

For $\lambda_2 = 3 \quad \Rightarrow \quad (A - 3I)V_2 = 0$

$$\begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{cases} -x_1 - x_2 + x_4 = 0 \\ -x_2 + x_3 - x_4 = 0 \\ 2x_4 = 0 \end{cases}$$

$$\begin{cases} \underline{x_1 = -x_2} \\ \underline{x_3 = x_2} \\ \underline{x_4 = 0} \end{cases}$$

$$V_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

The matrix is not diagonalizable since it has only 2 distinct eigenvectors. Note that showing that the geometric multiplicity of either eigenvalue is less than its algebraic multiplicity is sufficient to show that the matrix is not diagonalizable.

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} -14 & 12 \\ -20 & 17 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} -14-\lambda & 12 \\ -20 & 17-\lambda \end{vmatrix}$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

The eigenvalues are: $\boxed{\lambda_1 = 2 \quad \lambda_2 = 1}$

For $\lambda_1 = 1 \quad \Rightarrow \quad (A - I)V_1 = 0$

$$\begin{pmatrix} -15 & 12 \\ -20 & 16 \end{pmatrix}\begin{pmatrix} x_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -15x_1 + 12y_1 = 0 & \rightarrow \boxed{5x_1 = 4y_1} \\ -20x_1 + 16y_1 = 0 \end{cases}$$

Therefore, the eigenvector: $\boxed{V_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}}$

For $\lambda_2 = 3 \quad \Rightarrow \quad (A - 3I)V_2 = 0$

$$\begin{pmatrix} -16 & 12 \\ -20 & 15 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -16x_2 + 12y_2 = 0 \\ -20x_2 + 15y_2 = 0 \end{cases} \quad \to \; 4x_2 = 3y_2$$

Therefore, the eigenvector: $\underline{V_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}}$

The eigenvectors matrix form:

$$P = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 4 & -3 \\ -5 & 4 \end{pmatrix}$$

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} 4 & -3 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} -14 & 12 \\ -20 & 17 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -3 \\ -10 & 8 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \end{aligned}$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix P that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$$

Solution

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{vmatrix} \\ &= \lambda^2 - 1 = 0 \end{aligned}$$

The eigenvalues are: $\underline{\lambda_1 = -1, \quad \lambda_2 = 1}$

For $\lambda_1 = -1 \;\Rightarrow\; (A + I)V_1 = 0$

$$\begin{pmatrix} 2 & 0 \\ 6 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \; 2x_1 = 0$$

$$\Rightarrow \underline{x_1 = 0}$$

Therefore, the eigenvector: $\underline{V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

For $\lambda_2 = 1 \quad \Rightarrow \quad (A-I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow 6x_2 - 2y_2 = 0$$

$$\rightarrow 3x_2 = y_2$$

Therefore, the eigenvector: $V_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

The ***eigenvector matrix*** is given by:

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \checkmark \end{aligned}$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$$

Solution

Upper triangular; the eigenvalues are the main diagonal entries.

The eigenvalues are: $\lambda_{1,2} = 3$

For $\lambda_{1,2} = 3 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow 2y_1 = 0$$

$$\Rightarrow y_1 = 0$$

Therefore, the eigenvector: $\underline{V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

Since, the eigenvalues are repeated, then the matrix A is ***not*** diagonalizable.

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{vmatrix}$$

$$= 8 - 6\lambda + \lambda^2 + 1$$

$$= \lambda^2 - 6\lambda + 9 = 0$$

The eigenvalues are: $\underline{\lambda_{1,2} = 3}$

For $\lambda_{1,2} = 3 \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x_1 + y_1 = 0$$

$\Rightarrow \underline{x_2 = -y_1}$

Therefore, the eigenvector: $\underline{V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

Since, the eigenvalues are repeated, then the matrix A is ***not*** diagonalizable.

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix}$$

$$= 2 - 3\lambda + \lambda^2 - 12$$

$$= \lambda^2 - 3\lambda - 10 = 0$$

The eigenvalues are: $\underline{\lambda_1 = -2, \quad \lambda_2 = 5}$

For $\lambda_1 = -2 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3 & 3 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow 3x_1 + 3y_1 = 0$$

$$\Rightarrow \underline{x_1 = -y_1}$$

Therefore, the eigenvector: $V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For $\lambda_2 = 5 \quad \Rightarrow \quad (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -4 & 3 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow 4x_2 - 3y_2 = 0$$

$$\rightarrow \underline{4x_2 = 3y_2}$$

Therefore, the eigenvector: $V_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

The ***eigenvector matrix*** is given by:

$$P = \begin{pmatrix} -1 & 3 \\ 1 & 4 \end{pmatrix}$$

$$P^{-1} = \frac{1}{-7}\begin{pmatrix} 4 & -3 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{1}{7} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -\frac{4}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{1}{7} \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{1}{7} \end{pmatrix}\begin{pmatrix} 2 & 15 \\ -2 & 20 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix} \qquad \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A=\begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{pmatrix}$$

Solution

$$\det(A-\lambda I)=\begin{vmatrix} -\lambda & 1 & 1 \\ 2 & 1-\lambda & 2 \\ 3 & 3 & 2-\lambda \end{vmatrix}$$

$$=-2\lambda+3\lambda^2-\lambda^3+6+6-3+3\lambda+6\lambda-4+2\lambda$$

$$=-\lambda^3+3\lambda^2+9\lambda+5=0$$

$\lambda=-1$

$$\begin{array}{r|rrrr} -1 & -1 & 3 & 9 & 5 \\ & & 1 & -4 & -5 \\ \hline & -1 & 4 & 5 & 0 \end{array} \quad \rightarrow \quad -\lambda^2+4\lambda+5=0$$

The eigenvalues are: $\lambda_{1,2}=-1, \quad \lambda_3=5$

For $\lambda_{1,2}=-1 \quad \Rightarrow \quad (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow x_1+y_1+z_1=0 \quad (1)$$

Let $z_1=0 \quad (1) \rightarrow \quad x_1=-y_1$

Therefore, the eigenvector: $V_1=\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Let $y_1=0 \quad (1) \rightarrow \quad x_1=-z_1$

Therefore, the eigenvector: $V_2=\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda_3 = 5 \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{pmatrix} \begin{matrix} \\ 5R_2 + 2R_1 \\ 5R_2 + 3R_1 \end{matrix}$$

$$\begin{pmatrix} -5 & 1 & 1 \\ 0 & -18 & 12 \\ 0 & 18 & -12 \end{pmatrix} \begin{matrix} \\ \frac{1}{6}R_2 \\ R_3 + R_2 \end{matrix}$$

$$\begin{pmatrix} -5 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} 3R_1 + R_2 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} -15 & 0 & 5 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \frac{1}{5}R_1 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} -3 & 0 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \rightarrow & -3x_3 + z_3 = 0 \\ \rightarrow & -3y_3 + 2z_3 = 0 \\ & \end{matrix}$$

$$\begin{cases} 3x_3 = z_3 \\ 3y_3 = 2z_3 \end{cases}$$

Therefore, the eigenvector: $V_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

The ***eigenvector matrix*** is given by:

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \begin{matrix} \\ R_2 + R_1 \\ \\ \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} -R_1 + R_2 \\ \\ R_3 + R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 1 & 1 \end{array}\right) \begin{array}{l} 3R_1 - R_3 \\ 2R_2 - R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 0 & -1 & 2 & -1 \\ 0 & -2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 6 & 1 & 1 & 1 \end{array}\right) \begin{array}{l} \frac{1}{3}R_1 \\ -\frac{1}{2}R_2 \\ \frac{1}{6}R_3 \end{array}$$

$$P^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 1 & 1 & 5 \\ -1 & 0 & 10 \\ 0 & -1 & 15 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)^3 + 1 + 1 - 3(3-\lambda)$$

$$= 27 - 27\lambda + 9\lambda^2 - \lambda^3 - 7 + 3\lambda$$

$$= -\lambda^3 + 9\lambda^2 - 24\lambda + 20 = 0$$

$\lambda = 2$

$$\begin{array}{c|cccc} 2 & -1 & 9 & -24 & 20 \\ & & -2 & 14 & -20 \\ \hline & -1 & 7 & -10 & 0 \end{array} \quad \to \quad -\lambda^2 + 7\lambda - 10 = 0$$

The eigenvalues are: $\lambda_{1,2} = 2, \quad \lambda_3 = 5$

For $\lambda_{1,2} = 2 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to x_1 + y_1 + z_1 = 0 \quad (1)$$

Let $z_1 = 0 \quad (1) \to \quad x_1 = -y_1$

Therefore, the eigenvector: $V_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Let $y_1 = 0 \quad (1) \to \quad x_1 = -z_1$

Therefore, the eigenvector: $V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda_3 = 5 \quad \Rightarrow \quad (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{array}{l} \\ 2R_2 + R_1 \\ 2R_3 + R_1 \end{array}$$

$$\begin{pmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \begin{array}{l} 3R_1 + R_2 \\ \\ R_3 + R_2 \end{array}$$

$$\begin{pmatrix} -6 & 0 & 6 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} -\frac{1}{6}R_1 \\ -\frac{1}{3}R_2 \\ \\ \end{array}$$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \rightarrow \; -x_3 + z_3 = 0 \\ \rightarrow \; -y_3 + z_3 = 0 \\ \\ \end{array}$$

$$\begin{cases} x_3 = z_3 \\ y_3 = z_3 \end{cases}$$

Therefore, the eigenvector: $V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

The ***eigenvector matrix*** is given by:

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 + R_1 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} -R_1 + R_2 \\ \\ R_3 + R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right) \quad \begin{array}{l} 3R_1 - R_3 \\ 3R_2 - 2R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 0 & -1 & 2 & -1 \\ 0 & -3 & 0 & 1 & 1 & -2 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right) \quad \begin{array}{l} \frac{1}{3}R_1 \\ -\frac{1}{3}R_2 \\ \frac{1}{3}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} -2 & -2 & 5 \\ 2 & 0 & 5 \\ 0 & 2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A=\begin{pmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix}$$

Solution

$$\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -2 & 2-\lambda \end{vmatrix}$$

$$=(3-\lambda)(2-\lambda)^2+2+2-3+\lambda-4(2-\lambda)$$

$$=(3-\lambda)\left(4-4\lambda+\lambda^2\right)+1+\lambda-8+4\lambda$$

$$=12-16\lambda+7\lambda^2-\lambda^3+5\lambda-7$$

$$\underline{=-\lambda^3+7\lambda^2-11\lambda+5=0}$$

$\lambda=1$

$$\begin{array}{c|cccc} 1 & -1 & 7 & -11 & 5 \\ & & -1 & 6 & -5 \\ \hline & -1 & 6 & -5 & 0 \end{array} \quad \to \quad \underline{-\lambda^2+6\lambda-5=0}$$

The eigenvalues are: $\underline{\lambda_{1,2}=1, \quad \lambda_3=5}$

For $\lambda_{1,2}=1 \quad \Rightarrow \quad \left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{pmatrix} \quad \begin{matrix} R_2-R_1 \ ,,,,,,,, \\ R_3+R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \quad x_1+2y_1-z_1=0$$

Let $z_1=0$ $\quad (1) \to \quad \underline{x_1=-2y_1}$

Therefore, the eigenvector: $\underline{V_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}$

Let $y_1 = 0$ $\quad (1) \rightarrow \quad \underline{x_1 = z_1}$

Therefore, the eigenvector: $\underline{V_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$

For $\lambda_3 = 5 \quad \Rightarrow \quad \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -3 & 2 & -1 \\ 1 & -2 & -1 \\ -1 & -2 & -3 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 2 & -1 \\ 1 & -2 & -1 \\ -1 & -2 & -3 \end{pmatrix} \quad \begin{matrix} \\ 3R_2 + R_1 \\ -3R_3 + R_1 \end{matrix}$$

$$\begin{pmatrix} -3 & 2 & -1 \\ 0 & -4 & -4 \\ 0 & 8 & 8 \end{pmatrix} \quad \begin{matrix} 2R_1 + R_2 \\ \\ R_3 + 2R_2 \end{matrix}$$

$$\begin{pmatrix} -6 & 0 & -6 \\ 0 & -4 & -4 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} -\frac{1}{2}R_1 \\ -\frac{1}{4}R_2 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 3 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \rightarrow & 3x_3 + 3z_3 = 0 \\ \rightarrow & y_3 + z_3 = 0 \\ & \end{matrix}$$

$$\begin{cases} x_3 = -z_3 \\ y_3 = -z_3 \end{cases}$$

Therefore, the eigenvector: $\underline{V_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}}$

The ***eigenvector matrix*** is given by:

$$P = \begin{pmatrix} -2 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} -2 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \quad 2R_2 + R_1$$

$$\left(\begin{array}{ccc|ccc} -2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 - R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} -2 & 0 & 2 & 0 & -2 & 0 \\ 0 & 1 & -3 & 1 & 2 & 0 \\ 0 & 0 & 4 & -1 & -2 & 1 \end{array}\right) \quad \begin{array}{l} -2R_1 + R_3 \\ 4R_2 + 3R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 4 & 0 & 0 & -1 & 2 & 1 \\ 0 & 4 & 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & -1 & -2 & 1 \end{array}\right) \quad \begin{array}{l} \frac{1}{4}R_1 \\ \frac{1}{4}R_2 \\ \frac{1}{4}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & -1 & -\frac{3}{2} \\ -\frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix} \begin{pmatrix} -2 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} -2 & 1 & -5 \\ 1 & 0 & -5 \\ 0 & 1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A=\begin{pmatrix}2&0&-2\\1&3&2\\0&0&3\end{pmatrix}$$

Solution

$$\det(A-\lambda I)=\begin{vmatrix}2-\lambda&0&-2\\1&3-\lambda&2\\0&0&3-\lambda\end{vmatrix}$$

$$=(2-\lambda)(3-\lambda)^2=0$$

The eigenvalues are: $\lambda_1=2,\quad \lambda_{2,3}=3$

For $\lambda_1=2 \;\Rightarrow\; (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}0&0&-2\\1&1&2\\0&0&1\end{pmatrix}\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad \to x_1+y_1+2z_1=0 \quad (1)$$

$$\to z_1=0$$

$(1)\to\; x_1=-y_1$

Therefore, the eigenvector: $V_1=\begin{pmatrix}-1\\1\\0\end{pmatrix}$

For $\lambda_{2,3}=3 \;\Rightarrow\; (A-3I)V_2=0$

$$\begin{pmatrix}-1&0&-2\\1&0&2\\0&0&0\end{pmatrix}\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad \to x_2+2z_2=0$$

$x_2=-2z_2$

$\forall\, y_2\in\mathbb{R}$

Therefore, the eigenvector: $V_2=\begin{pmatrix}-2\\0\\1\end{pmatrix}$ *and* $V_3=\begin{pmatrix}0\\1\\0\end{pmatrix}$

The ***eigenvector matrix*** is given by:

$$P=\begin{pmatrix}-1 & -2 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc}-1 & -2 & 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0 & 1\end{array}\right) \qquad R_2+R_1$$

$$\left(\begin{array}{ccc|ccc}-1 & -2 & 0 & 1 & 0 & 0\\ 0 & -2 & 1 & 1 & 1 & 0\\ 0 & 1 & 0 & 0 & 0 & 1\end{array}\right) \qquad \begin{array}{l}-R_1+R_2\\ \\ 2R_3+R_2\end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & 1 & 0 & 1 & 0\\ 0 & -2 & 1 & 1 & 1 & 0\\ 0 & 0 & 1 & 1 & 1 & 2\end{array}\right) \qquad \begin{array}{l}R_1-R_3\\ R_2-R_3\\ \end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & 0 & -1 & 0 & -2\\ 0 & -2 & 0 & 0 & 0 & -2\\ 0 & 0 & 1 & 1 & 1 & 2\end{array}\right) \qquad -\tfrac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & 0 & -1 & 0 & -2\\ 0 & 1 & 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 1 & 1 & 2\end{array}\right)$$

$$P^{-1}=\begin{pmatrix}-1 & 0 & -2\\ 0 & 0 & 1\\ 1 & 1 & 2\end{pmatrix}$$

$$P^{-1}AP=\begin{pmatrix}-1 & 0 & -2\\ 0 & 0 & 1\\ 1 & 1 & 2\end{pmatrix}\begin{pmatrix}2 & 0 & -2\\ 1 & 3 & 2\\ 0 & 0 & 3\end{pmatrix}\begin{pmatrix}-1 & -2 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

$$=\begin{pmatrix}-2 & 0 & -4\\ 0 & 0 & 3\\ 3 & 3 & 6\end{pmatrix}\begin{pmatrix}-1 & -2 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

$$=\begin{pmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{pmatrix} \qquad \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix}$$

$$= \lambda^2(2-\lambda) + 1 + 1 - 2 + \lambda - 2\lambda$$

$$= -\lambda^3 + 2\lambda^2 - \lambda$$

$$= -\lambda\left(\lambda^2 - 2\lambda + 1\right) = 0$$

The eigenvalues are: $\lambda_1 = 0, \quad \lambda_{2,3} = 1$

For $\lambda_1 = 0 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \rightarrow \ -y_1 - z_1 = 0 \quad (1) \\ \rightarrow \ x_1 + 2y_1 + z_1 = 0 \quad (2) \\ \rightarrow \ -x_1 - y_1 = 0 \quad (3) \end{array}$$

$(1) \rightarrow \ z_1 = -y_1$

$(3) \rightarrow \ x_1 = -y_1$

Therefore, the eigenvector: $V_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

For $\lambda_{2,3} = 1 \quad \Rightarrow \quad (A - I)V_2 = 0$

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow x_2 + y_2 + z_2 = 0$$

Let $z_2 = 0 \quad \rightarrow \quad x_2 = -y_2$

Therefore, the eigenvector: $V_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Let $y_2 = 0 \quad \rightarrow \quad x_2 = -z_2$

Therefore, the eigenvector: $V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

The ***eigenvector matrix*** is given by:

$$P = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 + R_3 \\ R_2 + R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right) \quad R_3 - R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right) \quad \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & -3 \\ 2 & 5-\lambda & -2 \\ 1 & 3 & 1-\lambda \end{vmatrix}$$

$$= \left(1 - 2\lambda + \lambda^2\right)(5-\lambda) - 4 - 18 + 15 - 3\lambda + 6 - 6\lambda - 4 + 4\lambda$$

$$= 5 - 11\lambda + 7\lambda^2 - \lambda^3 - 5\lambda - 5$$

$$= -\lambda^3 + 7\lambda^2 - 16\lambda$$

$$= -\lambda\left(\lambda^2 - 7\lambda + 16\right) = 0$$

$$\lambda = \frac{7 \pm \sqrt{49 - 64}}{2}$$

$$= \frac{7}{2} \pm i\frac{\sqrt{15}}{2}$$

The eigenvalues are: $\lambda_1 = 0, \quad \lambda_{2,3} = \frac{7}{2} \pm i\frac{\sqrt{15}}{2}$

For $\lambda_1 = 0 \quad \Rightarrow \quad \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{pmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ R_2 - R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{pmatrix} \quad \begin{matrix} R_1 - 2R_2 \\ \\ R_3 - R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & -11 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad \begin{matrix} \rightarrow & x_1 - 11z_1 = 0 \\ \rightarrow & y_1 + 4z_1 = 0 \\ \rightarrow & \end{matrix}$$

$$\begin{cases} \underline{x_1 = 11z_1} \\ \underline{y_1 = -4z_1} \end{cases}$$

Therefore, the eigenvector: $V_1 = \begin{pmatrix} 11 \\ -4 \\ 1 \end{pmatrix}$

Since the only real eigenvalue $\lambda = 0$ which has only a one-dimensional eigenspace.

Therefore, the given matrix A is ***not diagonalizable*** over real numbers.

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{pmatrix}$$

Solution

Since the given matrix is a lower triangular, then

The eigenvalues are: $\lambda_{1,2,3} = 2$

For $\lambda_1 = 2 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \\ \rightarrow & 2x = 0 \\ \rightarrow & 2x + 2y = 0 \end{matrix}$$

$$\begin{cases} \underline{x = y = 0} \\ \forall z \in \mathbb{R} \end{cases}$$

Therefore, the eigenvector: $\boxed{V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$

Since the eigenvalue $(\lambda = 2)$ has only one–dimensional eigenspace.

Therefore, the given matrix A is ***not diagonalizable*** over real numbers.

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -2 & -2 \\ 3 & -3-\lambda & -2 \\ 2 & -2 & -2-\lambda \end{vmatrix}$$

$$= \left(4 - \lambda^2\right)(3 + \lambda) + 8 + 12 - 12 - 4\lambda - 8 + 4\lambda - 12 - 6\lambda$$

$$= 12 + 4\lambda - 3\lambda^2 - \lambda^3 - 6\lambda - 12$$

$$= -\lambda^3 - 3\lambda^2 - 2\lambda$$

$$\boxed{= -\lambda\left(\lambda^2 + 3\lambda + 2\right) = 0}$$

The eigenvalues are: $\boxed{\lambda_1 = -2, \quad \lambda_2 = -1 \quad \lambda_3 = 0}$

For $\lambda_1 = -2 \quad \Rightarrow \quad \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 4 & -2 & -2 \\ 3 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 & -2 \\ 3 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix} \quad \begin{matrix} \\ 4R_2 - 3R_1 \\ 2R_3 - R_1 \end{matrix}$$

$$\begin{pmatrix} 4 & -2 & -2 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix} \begin{matrix} R_1 + R_2 \\ \\ R_3 + R_2 \end{matrix}$$

$$\begin{pmatrix} 4 & 0 & -4 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \to & 4x_1 - 4z_1 = 0 \\ \to & 2y_1 - 2z_1 = 0 \\ & \end{matrix}$$

$$\begin{cases} x_1 = z_1 \\ y_1 = z_1 \end{cases}$$

Therefore, the eigenvector: $V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

For $\lambda_2 = -1 \Rightarrow (A + I)V_2 = 0$

$$\begin{pmatrix} 3 & -2 & -2 \\ 3 & -2 & -2 \\ 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -2 & -2 \\ 3 & -2 & -2 \\ 2 & -2 & -1 \end{pmatrix} \begin{matrix} \\ R_2 - R_1 \\ 3R_3 - 2R_1 \end{matrix}$$

$$\begin{pmatrix} 3 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{matrix} R_1 - R_3 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 3 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \to & 3x_2 - 3z_2 = 0 \\ & \\ \to & -2y_2 + z_2 = 0 \end{matrix}$$

$$\to \begin{cases} x_2 = z_2 \\ 2y_2 = z_2 \end{cases}$$

Therefore, the eigenvector: $V_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

For $\lambda_3 = 0 \quad \Rightarrow \quad (A)V_3 = 0$

$$\begin{pmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{pmatrix} \quad \begin{matrix} \\ 2R_2 - 3R_1 \\ R_3 - R_1 \end{matrix}$$

$$\begin{pmatrix} 2 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \to & 2x_3 - 2y_3 - 2z_3 = 0 \\ \to & 2z_3 = 0 \\ & \end{matrix}$$

$$\begin{cases} \boxed{x_3 = y_3} \\ \boxed{z_3 = 0} \end{cases}$$

Therefore, the eigenvector: $\boxed{V_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$

The ***eigenvector matrix*** is given by:

$$P = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \quad \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right) \quad \begin{matrix} R_1 + 2R_2 \\ \\ \\ \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 2 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right) \quad \begin{matrix} R_1 + R_3 \\ \\ \\ \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & 1 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \\ -R_2 \\ -R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} -2 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -2 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -4 & -2 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix}$$

$$= (3-\lambda)^2(5-\lambda) - 4(5-\lambda)$$

$$= (5-\lambda)\left(\lambda^2 - 6\lambda + 9 - 4\right)$$

$$= (5-\lambda)\left(\lambda^2 - 6\lambda + 5\right)$$

$$= (5-\lambda)(\lambda - 5)(\lambda - 1) = 0$$

The eigenvalues are: $\boxed{\lambda_1 = 1, \quad \lambda_{2,3} = 5}$

For $\lambda_1 = 1 \quad \Rightarrow \quad \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \rightarrow 2x_1 - 2y_1 = 0 \\ \\ \rightarrow 4z_1 = 0 \end{matrix}$$

$$\begin{cases} \boxed{x_1 = y_1} \\ \boxed{z_1 = 0} \end{cases}$$

$$\Rightarrow \quad \boxed{V_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

For $\lambda_{2,3} = 5 \quad \Rightarrow \quad \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow -2x_2 - 2y_2 = 0$$

$$\Rightarrow \boxed{x_2 = -y_2}$$

$$\Rightarrow \quad \boxed{V_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

The ***eigenvector matrix*** is:

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \quad R_2 - R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \quad 2R_1 + R_2$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \frac{1}{2}R_1 \\ \frac{1}{2}R_2 \\ \end{array}$$

$$P^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -5 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad \checkmark \end{aligned}$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & -2 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)^3 = 0$$

The eigenvalues are: $\lambda_{1,2,3} = 3$

For $\lambda = 3 \quad \Rightarrow \quad (A - \lambda I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \ 2z = 0$$

$$\Rightarrow \ \underline{z = 0}$$

$$\Rightarrow \ V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad V_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The ***eigenvector matrix*** is:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the one eigenvector has no–dimensional eigenspace.

Therefore, the given matrix A is ***not diagonalizable***

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 19-\lambda & -9 & -6 \\ 25 & -11-\lambda & -9 \\ 17 & -9 & -4-\lambda \end{vmatrix}$$

$$= (19-\lambda)(-11-\lambda)(-4-\lambda) + 1,377 + 1,350 - 102(11+\lambda) - 81(19-\lambda) - 225(4+\lambda)$$

$$= \left(209 + 8\lambda - \lambda^2\right)(4+\lambda) + 2,727 - 1,122 - 102\lambda - 1,539 + 81\lambda - 900 - 225\lambda$$

$$= 836 + 241\lambda + 4\lambda^2 - \lambda^3 - 834 - 246\lambda$$

$$\underline{= -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0}$$

$$\begin{array}{r|rrrr} 1 & -1 & 4 & -5 & 2 \\ & & -1 & 3 & -2 \\ \hline & -1 & 3 & -2 & 0 \end{array} \quad \rightarrow \ \underline{-\lambda^2 + 3\lambda - 2 = 0}$$

The eigenvalues are: $\underline{\lambda_{1,2} = 1,\ 1 \quad \lambda_3 = 2}$

For $\lambda_{1,2} = 1 \quad \Rightarrow \quad \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 18 & -9 & -6 \\ 25 & -12 & -9 \\ 17 & -9 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \rightarrow & 18x - 9y - 6z = 0 \\ \rightarrow & 25x - 12y - 9z = 0 \\ \rightarrow & 17x - 9y - 5z = 0 \end{matrix}$$

$$\begin{pmatrix} 6 & -3 & -2 \\ 25 & -12 & -9 \\ 17 & -9 & -5 \end{pmatrix} \begin{matrix} \\ 6R_2 - 25R_1 \\ 6R_3 - 17R_1 \end{matrix}$$

$$\begin{pmatrix} 6 & -3 & -2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{pmatrix} \begin{matrix} R_1 + R_2 \\ \\ R_3 + R_2 \end{matrix}$$

$$\begin{pmatrix} 6 & 0 & -6 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \rightarrow & 6x - 6z = 0 \\ \rightarrow & 3y - 4z = 0 \\ & \end{matrix}$$

$$\begin{cases} \underline{x = z} \\ \underline{3y = 4z} \end{cases}$$

$$\Rightarrow \quad \boxed{V_1 = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} \quad and \quad V_2 = \begin{pmatrix} 1 \\ \frac{4}{3} \\ 1 \end{pmatrix}}$$

For $\lambda_3 = 2 \quad \Rightarrow \quad \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} 17 & -9 & -6 \\ 25 & -13 & -9 \\ 17 & -9 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \rightarrow & 17x - 9y - 6z = 0 \\ \rightarrow & 25x - 13y - 9z = 0 \\ \rightarrow & 17x - 9y - 6z = 0 \end{matrix}$$

$$\begin{pmatrix} 17 & -9 & -6 \\ 25 & -12 & -9 \\ 17 & -9 & -5 \end{pmatrix} \begin{matrix} \\ 17R_2 - 25R_1 \\ R_3 - R_1 \end{matrix}$$

$$\begin{pmatrix} 17 & -9 & -6 \\ 0 & 21 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} 7R_1 + 3R_2 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 119 & 0 & -6 \\ 0 & 21 & -51 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \rightarrow & 119x - 6z = 0 \\ \rightarrow & 21y - 51z = 0 \\ \rightarrow & \underline{z = 0} \end{matrix}$$

$$\underline{x = y = z = 0}$$

$$\Rightarrow \quad \underline{V_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$$

The ***eigenvector matrix*** is:

$$P = \begin{pmatrix} 3 & 1 & 0 \\ 4 & \frac{4}{3} & 0 \\ 3 & 1 & 0 \end{pmatrix}$$

P^{-1} doesn't exist, one column with zero entries.

Since the eigenvalue $(\lambda = 2)$ has no–dimensional eigenspace.

Therefore, the given matrix A is ***not diagonalizable*** (repeated eigenvalues)

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Solution

Since the matrix A is a lower triangular, then the eigenvalues are the entries values of the main diagonal.

The eigenvalues are: $\underline{\lambda_{1,2} = -2, \quad \lambda_{3,4} = 3}$

For $\lambda_{1,2} = -2 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \qquad \begin{array}{l} \\ \\ \rightarrow \ 5v_3 = 0 \\ \rightarrow \ v_3 + 5v_4 = 0 \end{array}$$

$$\underline{v_3 = v_4 = 0}$$

$$\Rightarrow \quad \boxed{V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad and \quad V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}$$

For $\lambda_{3,4} = 3 \quad \Rightarrow \quad \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -5 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \quad \begin{matrix} \rightarrow & -5v_1 = 0 \\ \rightarrow & -5v_2 = 0 \\ & \\ \rightarrow & \boxed{v_3 = 0} \end{matrix}$$

$$\boxed{v_1 = v_2 = v_3 = 0}$$

$$\Rightarrow \quad \boxed{V_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad and \quad V_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}$$

The ***eigenvector matrix*** is:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

P^{-1} doesn't exist, one row with zero entries

Since the 2 eigenvectors have only the same one–dimensional eigenspace.

Therefore, the given matrix A is ***not diagonalizable***

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

$$A=\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Solution

Since the matrix A is an upper triangular, then the eigenvalues are: $\boxed{\lambda_{1,2}=-2 \quad \lambda_{3,4}=3}$

For $\lambda=-2 \quad \Rightarrow \quad (A+2I)V_1=0$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -5 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{cases} 5x_3-5x_4=0 \\ 5x_3=0 \\ 5x_4=0 \end{cases} \quad \Rightarrow \quad x_3=x_4=0$$

$$\Rightarrow \quad V_1=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad V_2=\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda=3 \quad \Rightarrow \quad (A-3I)V_2=0$

$$\begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -5 & 5 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} \rightarrow & -5x_1=0 \\ \rightarrow & -5x_2+5x_3-5x_4=0 \end{matrix}$$

$$\Rightarrow \begin{cases} \boxed{x_1=0} \\ \boxed{x_2=x_3-x_4} \end{cases}$$

$$V_3=\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad V_4=\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

The ***eigenvector matrix*** is:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \qquad R_2 - R_3$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \qquad R_2 + R_4$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 2 & -2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \qquad \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes A and determine $P^{-1}AP$

$$A=\begin{pmatrix}5 & -3 & 0 & 9\\ 0 & 3 & 1 & -2\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 2\end{pmatrix}$$

Solution

Since the matrix A is an upper triangular, then the eigenvalues are: $\boxed{\lambda_{1,2}=2 \quad \lambda_3=3 \quad \lambda_4=5}$

$$A-\lambda I=\begin{pmatrix}5-\lambda & -3 & 0 & 9\\ 0 & 3-\lambda & 1 & -2\\ 0 & 0 & 2-\lambda & 0\\ 0 & 0 & 0 & 2-\lambda\end{pmatrix}$$

For $\lambda=2 \quad \Rightarrow \quad (A-2I)V_1=0$

$$\begin{pmatrix}3 & -3 & 0 & 9\\ 0 & 1 & 1 & -2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}3 & -3 & 0 & 9\\ 0 & 1 & 1 & -2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \quad R_1+3R_2$$

$$\begin{pmatrix}3 & 0 & 3 & 3\\ 0 & 1 & 1 & -2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\\ 0\end{pmatrix}$$

$$\rightarrow\begin{cases}3x_1+3x_3+3x_4=0\\ x_2+x_3-2x_4=0\end{cases}$$

$$\begin{cases}\boxed{x_1=-x_3-x_4}\\ \boxed{x_2=-x_3+2x_4}\end{cases}$$

$$\Rightarrow \quad \boxed{V_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}}$$

For $\lambda = 3 \quad \Rightarrow \quad (A - 3I)V_3 = 0$

$$\begin{pmatrix} 2 & -3 & 0 & 9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} 2x_1 - 3x_2 + 9x_4 = 0 & (1) \\ x_3 - 2x_4 = 0 \\ \boxed{x_3 = 0} \\ \boxed{x_4 = 0} \end{cases}$$

$$(1) \rightarrow \quad \boxed{2x_1 = 3x_2}$$

$$\Rightarrow \quad \boxed{V_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}}$$

For $\lambda = 5 \quad \Rightarrow \quad (A - 5I)V_4 = 0$

$$\begin{pmatrix} 0 & -3 & 0 & 9 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} -3x_2 + 9x_4 = 0 & (1) \\ -2x_2 + x_3 - 2x_4 = 0 \\ \boxed{x_3 = 0} \\ \boxed{x_4 = 0} \end{cases}$$

$$(1) \rightarrow \quad \boxed{x_2 = 0}$$

$$\Rightarrow \quad V_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The ***eigenvector matrix*** is:

$$P = \begin{pmatrix} -1 & -1 & 3 & 1 \\ -1 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|cccc} -1 & -1 & 3 & 1 & 1 & 0 & 0 & 0 \\ -1 & 2 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \\ R_2 - R_1 \\ R_3 + R_1 \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|cccc} -1 & -1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} 3R_1 + R_2 \\ \\ 3R_3 + R_2 \\ 3R_4 - R_2 \end{array}$$

$$\left(\begin{array}{cccc|cccc} -3 & 0 & 8 & 2 & 2 & 1 & 0 & 0 \\ 0 & 3 & -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 8 & 2 & 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 & 0 & 3 \end{array}\right) \quad \begin{array}{l} R_1 - R_3 \\ 8R_3 + R_3 \\ \\ 8R_4 - R_3 \end{array}$$

$$\left(\begin{array}{cccc|cccc} -3 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 24 & 0 & -6 & -6 & 9 & 3 & 0 \\ 0 & 0 & 8 & 2 & 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 6 & 6 & -9 & -3 & 24 \end{array}\right) \quad \begin{array}{l} \\ R_2 + R_4 \\ 3R_3 - R_4 \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|cccc} -3 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 24 & 0 & 0 & 0 & 0 & 0 & 24 \\ 0 & 0 & 24 & 0 & 0 & 12 & 12 & -24 \\ 0 & 0 & 0 & 6 & 6 & -9 & -3 & 24 \end{array}\right) \quad \begin{array}{l} -\frac{1}{3}R_1 \\ \frac{1}{24}R_2 \\ \frac{1}{24}R_3 \\ \frac{1}{6}R_4 \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 1 & 1 & -\frac{3}{2} & -\frac{1}{2} & 4 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \\ 1 & -\frac{3}{2} & -\frac{1}{2} & 4 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \\ 1 & -\frac{3}{2} & -\frac{1}{2} & 4 \end{pmatrix} \begin{pmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 & 3 & 1 \\ -1 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \\ 1 & -\frac{3}{2} & -\frac{1}{2} & 4 \end{pmatrix} \begin{pmatrix} -2 & -2 & 9 & 5 \\ -2 & 4 & 6 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} \quad \checkmark$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes A and determine $P^{-1}AP$

$$A=\begin{pmatrix}3&0&0&0\\0&2&0&0\\0&0&2&0\\1&0&0&3\end{pmatrix}$$

Solution

Since the matrix A is a lower triangular, then the eigenvalues are: $\underline{\lambda_{1,2}=2 \quad \lambda_{3,4}=3}$

$$A-\lambda I=\begin{pmatrix}3-\lambda&0&0&0\\0&2-\lambda&0&0\\0&0&2-\lambda&0\\1&0&0&3-\lambda\end{pmatrix}$$

For $\lambda_{1,2}=2 \quad \Rightarrow \quad (A-2I)V_1=0$

$$\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\begin{matrix}\to \quad \underline{x_1=0}\\ \\ \\ \to \quad x_1=x_4=0\end{matrix}$$

$$\Rightarrow \underline{x_1=x_4=0}$$

$$\Rightarrow \quad V_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad and \quad V_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

For $\lambda_{3,4}=3 \quad \Rightarrow \quad (A-3I)V_3=0$

$$\begin{pmatrix}0&0&0&0\\0&-1&0&0\\0&0&-1&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\Rightarrow \underline{x_1=x_2=x_3=0}$$

$$\Rightarrow \quad V_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad V_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Since the eigenvalue $(\lambda = 3)$ has only one–dimensional eigenspace.

Therefore, the matrix A is *not diagonalizable*.

Exercise

The 4 by 4 triangular Pascal matrix P_L and its inverse (alternating diagonals) are

$$P_L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \quad \text{and} \quad P_L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Check that P_L and P_L^{-1} have the same eigenvalues. Find a diagonal matrix D with alternating signs that gives $P_L^{-1} = D^{-1}P_L D$, so P_L is similar to P_L^{-1}. Show that $P_L D$ with columns of alternating signs is its own inverse.

Since P_L and P_L^{-1} are similar they have the same Jordan form J. Find J by checking the number of independent eigenvectors of P_L with $\lambda = 1$.

Solution

The triangular matrices P_L and P_L^{-1} both have $\lambda = 1, 1, 1, 1$ on their main diagonals. Choose D with alternating 1 and -1 on its diagonal. D equals D^{-1}:

$$D^{-1}P_L D = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -2 & -1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$

$$= P_L^{-1}$$

Check:

Changing signs in rows 1 and 3 of P_L, and columns 1 and 3, produces the four negative entries in P_L^{-1}. Multiply row i by $(-1)^i$ and column j by $(-1)^j$, which gives the alternating diagonals.

Then $P_L D = pascal(n, 1)$ has columns with alternating signs and equals its own inverse!

$$\begin{aligned}(P_L D)(P_L D) &= P_L D^{-1} P_L D \\ &= P_L P_L^{-1} \\ &= I\end{aligned}$$

P_L has only one line of eigenvectors $x = (0, 0, 0, x_4)$ with $\lambda = 1$. The rank of $P_L - I$ is certainly 3. So its Jordan form J has only one block (also with $\lambda = 1$):

P_L and P_L^{-1} are somehow similar to Jordan's $J = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Exercise

These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors (one from each block). But the block sizes don't match and they are not similar:

$$J = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \textit{and} \quad K = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

For any matrix M compare JM with MK. If they are equal show that M is not invertible. Then $M^{-1}JM = K$ is Impossible; J is not similar to K.

Solution

Let $M = (m_{ij})$, then

$$M=\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix}$$

$$JM=\begin{pmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad and \quad MK=\begin{pmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{pmatrix}$$

If $JM = MK$ then $m_{21}=m_{22}=m_{24}=m_{41}=m_{42}=m_{44}=0$

Which in particular means that the second row is either a multiple of the fourth row, or the fourth row is all 0's. In either of these cases M is not invertible.

Suppose that J were similar to K. Then there would be some invertible matrix M such that $MK=JM$. But we just showed that in this case M is never invertible (contradiction). Thus, J is not similar to K.

Exercise

If $\boldsymbol{x}$ is in the nullspace of A show that $M^{-1}\boldsymbol{x}$ is in the nullspace of $M^{-1}AM$.

The nullspaces of A and $M^{-1}AM$ have the same (vectors) (basis) (dimension)

Solution

$$x \in N(A) \Rightarrow Ax=0 \Rightarrow M^{-1}AM\left(M^{-1}x\right)=0 \Rightarrow M^{-1}x \in N\left(M^{-1}AM\right)$$

$$x \in N\left(M^{-1}AM\right) \Rightarrow M^{-1}AMx=0 \Rightarrow AMx=0 \Rightarrow Mx \in N(A)$$

So, any vector in $N(A)$ *resp.* $N\left(M^{-1}AM\right)$ is a linear combination of those in $N\left(M^{-1}AM\right)$ *resp.* $N(A)$, hence is contained in it. That is, the two vector spaces consist of the same vectors.

Exercise

Prove that A^T is always similar to A (λ's are the same):

a) For one Jordan block J_i, find M_i so that $M_i^{-1}J_iM_i = J_i^T$.

b) For any J with blocks J_i, build M_0 from blocks so that $M_0^{-1}JM_0 = J^T$.

c) For any $A = MJM^{-1}$: Show that A^T is similar to J^T and so to J and so to A.

<u>***Solution***</u>

a) For one Jordan block J_i, then

$$\begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ \cdot^{\cdot^{\cdot}} & & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} \lambda & 1 & 0 & & 0 \\ & \lambda & 1 & 0 & \\ & & \lambda & & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}\begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ \cdot^{\cdot^{\cdot}} & & & \\ 1 & & & \end{pmatrix} = \begin{pmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ 0 & 1 & \lambda & & \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}$$

So, J is similar to J^T

b) For any J with block J_i, that satisfies $J_i^T = M_i^{-1}J_iM_i$

Let M_0 be the block-diagonal matrix consisting of the $M_i^{'}s$ along the diagonal. Then

$$M_0^{-1}JM_0 = \begin{pmatrix} M_1^{-1} & & & \\ & M_2^{-1} & & \\ & & \ddots & \\ & & & M_n^{-1} \end{pmatrix}\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_n \end{pmatrix}\begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_n \end{pmatrix}$$

$$= \begin{pmatrix} M_1^{-1}J_1M_1 & & & \\ & M_2^{-1}J_2M_2 & & \\ & & \ddots & \\ & & & M_n^{-1}J_nM_n \end{pmatrix}$$

$$= \begin{pmatrix} J_1^T & & & \\ & J_2^T & & \\ & & \ddots & \\ & & & J_n^T \end{pmatrix}$$

$$\boxed{= J^T}$$

c) $A^T = \left(MJM^{-1}\right)^T = \left(M^{-1}\right)^T J^T M^T = \left(M^T\right)^{-1} J^T \left(M^T\right)$

So A^T is similar to J^T, which is similar to J, which is similar to A, Thus any matrix is similar to its transpose.

Exercise

Why are these statements all true?

a) If A is similar to B then A^2 is similar to B^2.

b) A^2 and B^2 can be similar when A and B are not similar.

c) $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ is similar to $\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ is not similar to $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

e) If we exchange rows 1 and 2 of A, and then exchange columns 1 and 2 the eigenvalues stay the same. In this case M =?

Solution

a) If A is similar to B then $A = M^{-1}BM$ for some M. Then $A^2 = M^{-1}B^2M$, so A^2 is similar to B^2.

b) Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $A^2 = B^2$ so they are similar but A is not similar to B because nothing but zero matrix.

c) $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

d) They are not similar because the first matrix has a plane of eigenvectors for the eigenvalues 3, while the second only has a line.

e) In order to exchange two rows of A we multiply on the left by

$$M = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

In order to exchange two columns, we multiply on the right by the same M. As $M = M^{-1}$ the new matrix is similar to the old one, so the eigenvalues stay the same.

Exercise

If an n x n matrix A has all eigenvalues $\lambda = 0$ prove that A^n is the zero matrix.

Solution

Suppose that the Jordan Block has a size of $\boldsymbol{i}$ with eigenvalue 0. Then J^2 will have a diagonal of 1's two diagonals above the main diagonal and zeroes elsewhere. J^3 will have a diagonal of 1's three diagonals above the main diagonal and zeroes elsewhere. Therefore $J^i = 0$, since there is no diagonal $\boldsymbol{i}$ diagonals above the main diagonal. If A has all eigenvalues $\lambda = 0$ then A is similar to some matrix with Jordan block $J_1, \ldots, J_k$ with each J_i of size n_i and $\sum_{i=1}^{k} n_i = n$.

Each Jordan block will have eigenvalue of 0, so that $J_i^{n_i} = 0$, and thus $J_i^n = 0$

As A^n is similar to a block-diagonal matrix with blocks $J_1^n, J_2^n, \ldots, J_k^n$ and each of these is 0 we know that $A^n = 0$.

Another way, if A has all eigenvalues 0 this means that the characteristic polynomial of A must be x^n, as this is the only polynomial of degree n all of whose roots are 0. Thus $A^n = 0$ by the Cayley-Hamilton theorem.

Exercise

If A is similar to A^{-1}, must all the eigenvalues equal to 1 or -1?.

Solution

No

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ is similar to $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^{-1}$

Exercise

Determine whether the *two matrices* are similar matrices $A=\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$ $B=\begin{pmatrix}1 & 0\\ 3 & -2\end{pmatrix}$

Solution

$$|A|=\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix}\underline{=1}$$

$$|B|=\begin{vmatrix}1 & 0\\ 3 & -2\end{vmatrix}\underline{=-2}$$

$|A|\neq|B|$; therefore, A and B are ***not*** similar

Exercise

Determine whether the *two matrices* are similar matrices $A=\begin{pmatrix}4 & -1\\ 2 & 4\end{pmatrix}$ $B=\begin{pmatrix}4 & 1\\ 2 & 4\end{pmatrix}$

Solution

$$|A|=\begin{vmatrix}4 & -1\\ 2 & 4\end{vmatrix}\underline{=18}$$

$$|B|=\begin{vmatrix}4 & 1\\ 2 & 4\end{vmatrix}\underline{=14}$$

$|A|\neq|B|$; therefore, A and B are ***not*** similar

Exercise

Determine whether the *two matrices* are similar matrices $A=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$ $B=\begin{pmatrix}1 & 2 & 0\\ \frac{1}{2} & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

Solution

$$|A|=\begin{vmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & 0 & 1\end{vmatrix}\underline{=1}$$

$$|B|=\begin{vmatrix}1 & 2 & 0\\ \frac{1}{2} & 1 & 0\\ 0 & 0 & 1\end{vmatrix}\underline{=0}$$

$|A|\neq|B|$; therefore, A and B are ***not*** similar

Exercise

Determine whether the *two matrices* are similar matrices $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$

Solution

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{vmatrix}$$

$$\underline{= 24}$$

$$|B| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix}$$

$$\underline{= 24}$$

$$|A| = |B|$$

Therefore, A and B are similar

Exercise

Determine whether the *two matrices* are similar matrices $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 0 & 0 \\ 7 & 1 & 0 \\ 8 & 9 & 6 \end{pmatrix}$

Solution

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{vmatrix}$$

$$\underline{= 24}$$

$$|B| = \begin{vmatrix} 4 & 0 & 0 \\ 7 & 1 & 0 \\ 8 & 9 & 6 \end{vmatrix}$$

$$\underline{= 24}$$

$$|A| = |B|$$

Therefore, A and B are similar

Exercise

Determine whether the *two matrices* are similar matrices $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}4&7&0\\0&1&0\\8&9&6\end{pmatrix}$

Solution

$$|A|=\begin{vmatrix}1&0&0\\0&4&0\\0&0&6\end{vmatrix}$$

$$\underline{=24}$$

$$|B|=\begin{vmatrix}4&7&0\\0&1&0\\8&9&6\end{vmatrix}$$

$$\underline{=24}$$

$$|A|=|B|$$

Therefore, A and B are similar

Exercise

Determine whether the *two matrices* are similar matrices $A=\begin{pmatrix}1&0&0\\0&4&0\\0&0&6\end{pmatrix}$ $B=\begin{pmatrix}1&2&0\\3&4&5\\0&7&6\end{pmatrix}$

Solution

$$|A|=\begin{vmatrix}1&0&0\\0&4&0\\0&0&6\end{vmatrix}$$

$$\underline{=24}$$

$$|B|=\begin{vmatrix}1&2&0\\3&4&5\\0&7&6\end{vmatrix}$$

$$\underline{=-47}$$

$$|A|\neq|B|$$

Therefore, A and B are ***not*** similar

Exercise

Determine whether the *two matrices* are similar matrices $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 5 & 6 \end{pmatrix}$

Solution

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{vmatrix}$$

$$\underline{= 24}$$

$$|B| = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 5 & 6 \end{vmatrix}$$

$$\underline{= -12}$$

$$|A| \neq |B|$$

Therefore, A and B are ***not*** similar

Exercise

Determine whether the *two matrices* are similar matrices $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 5 & 1 \\ 3 & 1 & 5 \end{pmatrix}$

Solution

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{vmatrix}$$

$$\underline{= 24}$$

$$|B| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 5 & 1 \\ 3 & 1 & 5 \end{vmatrix}$$

$$\underline{= 24}$$

$$|A| = |B|$$

Therefore, A and B are similar

Exercise

Prove that two similar matrices have the same determinant. Explain geometrically why this is reasonable.

Solution

Suppose that $A = PBP^{-1}$

Then $\det(A) = \det\left(PBP^{-1}\right)$ $\qquad |AB| = |A||B|$

$$= \det(P)\cdot\det(B)\cdot\det\left(P^{-1}\right)$$

$$= \det(B)\cdot\det(P)\cdot\det\left(P^{-1}\right)$$

$$= \det(B)\cdot\det\left(PP^{-1}\right)$$

$$= \det(B)\cdot\det(I)$$

$$= \det(B)$$

Geometric Explanation: The determinant tells us what Factor area changes when using a linear transformation. This "factor" doesn't care about the particular basis you use.

Exercise

Prove that two similar matrices have the same characteristic polynomial and thus the same eigenvalues. Explain geometrically why this is reasonable.

Solution

Suppose that $A = PBP^{-1}$

Then the characteristic polynomial is equal to $\det(A - \lambda I)$.

$$A - \lambda I = PBP^{-1} - \lambda\left(PIP^{-1}\right)$$

$$= P(B - \lambda I)P^{-1}$$

$$\det(A - \lambda I) = \det\left(P(B - \lambda I)P^{-1}\right)$$

$$= \det(P)\cdot\det(B - \lambda I)\cdot\det\left(P^{-1}\right)$$

$$= \det(B - \lambda I)\cdot\det(P)\cdot\det\left(P^{-1}\right)$$

$$= \det(B - \lambda I)\cdot\det\left(PP^{-1}\right) \qquad \det\left(PP^{-1}\right) = \det(I) = 1$$

$$= \det(B - \lambda I)$$

Geometric Explanation: At least in terms of the eigenvalues, these values are numbers λ such that there exists a vector $\vec{v} \neq 0$ such that the linear transformation T satisfies $T(\vec{v}) = \lambda\vec{v}$.

Exercise

Suppose that A is a matrix. Suppose that the linear transformation associated to A has two linearly independent eigenvectors. Prove that A is similar to a diagonal matrix.

Solution

Let T be the linear transformation associated with A. Consider the basis $\vec{v}_1,\ \vec{v}_2$ of the 2 linearly independent eigenvectors of A where $\lambda_1,\ \lambda_2$ the eigenvalues associated with. Then,

$$T\left(\vec{v}_1\right)=\lambda_1\vec{v}_1 \quad and \quad T\left(\vec{v}_2\right)=\lambda_2\vec{v}_2$$

Let T be a matrix with respect to the basis $\vec{v}_1,\ \vec{v}_2$, then we obtain the matrix $\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$

This completes the proof because A is similar to this diagonal matrix by definition.

Exercise

Prove that if A is a 2×2 matrix that has two distinct eigenvalues, then A is similar to a diagonal matrix.

Solution

Suppose A has 2 distinct eigenvalues $\lambda_1,\ \lambda_2$.

Let $\vec{v}_1 \neq 0$ be an eigenvector for λ_1.

Suppose that $\vec{v}_1,\ \vec{v}_2$ are not linearly independent, thus they are scalar multiples of each other.

So, there exists $c\neq 0$ such that $c\vec{v}_1=\vec{v}_2$. Then

$$\begin{aligned}\lambda_2\vec{v}_2 &= A\vec{v}_2\\ &= A\left(c\vec{v}_1\right)\\ &= c\left(A\vec{v}_1\right)\\ &= c\lambda_1\vec{v}_1\\ &= \lambda_1 c\vec{v}_1 && c\vec{v}_1=\vec{v}_2\\ &= \lambda_1\vec{v}_2\end{aligned}$$

So, that $\lambda_2\vec{v}_2-\lambda_1\vec{v}_2=0 \quad\Rightarrow\quad \left(\lambda_2-\lambda_1\right)\vec{v}_2=0$

But then $\lambda_2=\lambda_1$ which contradicts the initial assumption.

Thus $\vec{v}_1,\ \vec{v}_2$ are linearly independent then $T\left(\vec{v}_1\right)=\lambda_1\vec{v}_1 \quad and \quad T\left(\vec{v}_2\right)=\lambda_2\vec{v}_2$

Let T be a matrix with respect to the basis $\vec{v}_1$, $\vec{v}_2$, then we obtain the matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

This completes the proof because A is similar to this diagonal matrix by definition.

Exercise

Suppose that the characteristic polynomial of a matrix has a double root. Is it true that the matrix has two linearly independent eigenvectors? Consider the example $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Is it true that matrices with equal characteristic polynomial are necessarily similar?

Solution

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix}$$
$$= \lambda^2 = 0$$

The characteristic polynomial: $p(x) = x^2$ which has a double root $(eigenvalue:\ \lambda = 0)$.

$$(A-\lambda I)V = AV$$
$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad x = 0$$

Therefore, the eigenvectors are vectors of the form $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which can transform to $\begin{pmatrix} x \\ 0 \end{pmatrix}$

Thus, matrices whose characteristic polynomials have a double root do not necessarily have 2 linear independent.

Let $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then the characteristic polynomial: $p(x) = x^2$ which has a double root $(eigenvalue:\ \lambda = 0)$. But they are not similar. The eigenvector is the $\vec{0}$ vector.

The linear transformation associated to the second matrix send every vector to $\vec{0}$. Thus the 2 matrices can't represent the same linear transformation.

Thus, matrices with equal characteristic polynomial are not necessarily similar.

Exercise

Show that the given matrix is not diagonalizable. $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

Solution

$$\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = 0$$

Since the determinant is 0, the inverse doesn't exist.

Therefore, the matrix is not diagonalizable

Exercise

Determine if the given matrix is diagonalizable. If, so, find matrices S and $\Lambda(D)$ such that the given matrix equals $S\Lambda S^{-1}$

a) $\begin{bmatrix} 3 & 3 \\ 4 & 2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{bmatrix}$

Solution

a) $\begin{vmatrix} 3-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 12$

$$= \lambda^2 - 5\lambda - 6 = 0$$

The eigenvalues $\boxed{\lambda_1 = -1,\ \lambda_2 = 6}$

For $\lambda = -1 \quad \Rightarrow \quad (A+I)V_1 = 0$

$$\begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad 4x_1 + 3y_1 = 0$$

$$\Rightarrow \boxed{4x_1 = -3y_1}$$

Therefore, the eigenvector: $\boxed{V_1 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}}$

For $\lambda = 6 \quad \Rightarrow \quad (A-6I)V_2 = 0$

$$\begin{pmatrix} -3 & 3 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad -3x_2 + 3y_2 = 0$$

$$\Rightarrow \boxed{x_2 = y_2}$$

Therefore, the eigenvector: $\boxed{V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$$S = \begin{pmatrix} -3 & 1 \\ 4 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} -3 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array}\right) \quad 3R_2 + 4R_1$$

$$\left(\begin{array}{cc|cc} -3 & 1 & 1 & 0 \\ 0 & 7 & 4 & 3 \end{array}\right) \quad 7R_1 - R_2$$

$$\left(\begin{array}{cc|cc} -21 & 0 & 3 & -3 \\ 0 & 7 & 4 & 3 \end{array}\right) \quad \begin{array}{c} -\frac{1}{21}R_1 \\ R_2 \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{7} & \frac{1}{7} \\ 0 & 1 & \frac{4}{7} & \frac{3}{7} \end{array}\right)$$

$$S^{-1} = \begin{pmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{3}{7} \end{pmatrix}$$

and $\Lambda = \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}$

$$S\Lambda S^{-1} = \begin{pmatrix} -3 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{3}{7} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 6 \\ -4 & 6 \end{pmatrix}\begin{pmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{3}{7} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 3 \\ 4 & 2 \end{pmatrix}$$

$$\underline{= A}$$

b) $$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & -1-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(-1-\lambda) + (2-\lambda)$$

$$= (2-\lambda)\left((1-\lambda)(-1-\lambda)+1\right)$$

$$= (2-\lambda)\left(-1+\lambda^2+1\right)$$

$$= (2-\lambda)\lambda^2 = 0$$

The eigenvalues $\underline{\lambda_1 = 2,\ \lambda_{2,3} = 0}$

The given matrix is not diagonalizable, since the eigenvalues are not distinct.

Exercise

A is a 5×5 matrix with *two* eigenvalues. One eigenspace is *three*–dimensional, and the other eigenspace is *two*–dimensional. Is *A* diagonalizable? Why?

Solution

Since 5×5 matrix *A* has two eigenvalues with one of the eigenvalues has three linearly independent eigenvectors in the *three*–dimensional and the other eigenvalue has two linearly independent eigenvectors in the *two*–dimensional.

Therefore, since all the ***five*** eigenvectors are linearly independent eigenvectors, that implies that the 5×5 matrix *A* is diagonalizable.

Exercise

A is a 3×3 matrix with *two* eigenvalues. Each eigenspace is *one*–dimensional. Is *A* diagonalizable? Why?

Solution

The given 3×3 matrix *A* has two eigenvalues that implies one of the eigenvalues is repeated value. Since the eigenvectors are in *one*–dimensional, the repeated eigenvalue will result with two eigenvectors linearly dependent.

Therefore, the given 3×3 matrix *A* is ***not*** diagonalizable

Exercise

A is a 4×4 matrix with *three* eigenvalues. One eigenspace is *one*–dimensional, and one of the other eigenspace is *two*–dimensional. Is it possible that *A* is *not* diagonalizable? Justify your answer?

Solution

The given 4×4 matrix *A* has three eigenvalues that implies one of the eigenvalues is repeated value.

However, one eigenspace is *one*–dimensional, and one of the other eigenspace is *two*–dimensional which include that these two eigenvectors are linearly independent.

Since, the other two distinct eigenvalues will result to the linearly independent eigenvectors.

That implies that all the eigenvectors are linearly independent.

Therefore, the given 4×4 matrix *A* is diagonalizable.

Exercise

A is a 7×7 matrix with *three* eigenvalues. One eigenspace is *two*–dimensional, and one of the other eigenspace is *three*–dimensional. Is it possible that A is *not* diagonalizable? Justify your answer?

Solution

The given 7×7 matrix A has three eigenvalues which results to 7 eigenvalues.

Since, one eigenspace is *two*–dimensional, and one of the other eigenspace is *three*–dimensional that will result to 5 linearly independent eigenvectors for that two eigenvalues.

If the third eigenvalue is repeated with *one*–dimensional, it will result to linearly dependent eigenvectors.

Therefore, the given 7×7 matrix A is ***not*** diagonalizable

Exercise

Show that if A is diagonalizable and invertible, then so is A^{-1}.

Solution

Since A is invertible, then:

$$AA^{-1} = A^{-1}A = I$$

And A is diagonalizable:

$$A = PDP^{-1}$$

$$A = PDP^{-1}$$

$$A^{-1} = \left(PDP^{-1}\right)^{-1}$$

$$= \left(P^{-1}\right)^{-1} D^{-1}P^{-1}$$

$$= PD^{-1}P^{-1}$$

Since D is diagonal then D^{-1} is diagonal matrix.

Therefore, A^{-1} is diagonalizable

Exercise

Show that if A has n linearly independent eigenvectors, then so does A^T.

Solution

If A has n linearly independent eigenvectors, then A is diagonalizable.

By the diagonalizable theorem $A = PDP^{-1}$

$$A^T = \left(PDP^{-1}\right)^T$$

$$= \left(P^{-1}\right)^T D^T P^T \qquad \text{Since } D \text{ is diagonal then } D^T = D$$

$$= \left(P^T\right)^{-1} DP^T \qquad \text{Assume that } Q = \left(P^T\right)^{-1}$$

$$= QDQ^{-1}$$

Therefore, A^T is diagonalizable with the columns Q are n linearly independent eigenvectors

Exercise

A factorization $A = PDP^{-1}$ is not unique. Demonstrate this for the matrix $A = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}$. With $D_1 = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, find a matrix P_1 such that $A = P_1 D_1 P_1^{-1}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{vmatrix}$$

$$= 7 - 8\lambda + \lambda^2 + 8$$

$$= \lambda^2 - 8\lambda + 15 = 0$$

The eigenvalues are: $\boxed{\lambda_1 = 3, \quad \lambda_2 = 5}$

For $\lambda_1 = 3 \;\Rightarrow\; (A - 3I)V_1 = 0$

$$\begin{pmatrix} 4 & 2 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \; 4x_1 + 2y_1 = 0$$

$$\Rightarrow \boxed{2x_1 = -y_1}$$

Therefore, the eigenvector: $\boxed{V_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}}$

For $\lambda_2 = 5 \;\Rightarrow\; (A - 5I)V_2 = 0$

$$\begin{pmatrix} 2 & 2 \\ -4 & -4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \; 2x_2 + 2y_2 = 0$$

$$\Rightarrow \boxed{x_2 = -y_2}$$

Therefore, the eigenvector: $\boxed{V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

The ***eigenvector matrix*** is given by: $P = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$

Which implies to: $P_1 = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$

$$P_1^{-1} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$$

$$\begin{aligned} P_1 D_1 P_1^{-1} &= \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -3 & -5 \\ 6 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} \\ &= A \quad \checkmark \end{aligned}$$

However, if we multiply the eigenvector V_1 with 2, it will result $V_1 = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ that implies to:

$$P_2 = \begin{pmatrix} -2 & -1 \\ 4 & 1 \end{pmatrix}$$

$$\begin{aligned} P_2^{-1} &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -4 & -2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -2 & -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} P_2 D_2 P_2^{-1} &= \begin{pmatrix} -2 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -6 & -5 \\ 12 & 5 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} \\ &= A \quad \checkmark \end{aligned}$$

$$A = PDP^{-1} = P_1 D_1 P_1^{-1} = P_2 D_2 P_2^{-1}$$

Therefore, that is shows that matrix A has many different factorizations.

Exercise

Construct a nonzero 2×2 matrix that is invertible but not diagonalizable.

Solution

For a 2×2 invertible matrix A, the eigenvalues must be nonzero and determinant of A is not equal to zero.

Let assume $A=\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$

$\det(A)=ac\ \neq 0$

Matrix A is invertible

Since the matrix A is an upper triangular then the eigenvalues are the main diagonal entries

$\lambda_1 = a \quad \& \quad \lambda_2 = c$

For the matrix A to be not diagonalizable when the eigenvectors are linearly dependents or in one–dimensional.

If we have a repeated eigenvalue that it will result in *one*–dimensional, that it will result that $a=c$.

$A=\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$

$\lambda_{1,2} = a$

For $\lambda_1 = a \quad \Rightarrow \quad (A-aI)V_1 = 0$

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad by_1 = 0$$

$\Rightarrow \quad y_1 = 0$

The eigenvectors are: $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \& \quad V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Therefore, the matrix A to be not diagonalizable since the eigenvectors are linearly dependent in *one*–dimensional

Example: $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Exercise

Construct a nonzero 2×2 matrix that is diagonalizable but not invertible.

Solution

Any 2×2 matrix with 2 distinct eigenvalues is diagonalizable.

Any 2×2 matrix is not invertible when determinant is zero, or either one row or one column is equal to zero.
If one of the eigenvalues is zero, then the matrix is not invertible.

Let assume $A=\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ $\quad a,\ b\ \neq 0$

The eigenvalues are: $\lambda_1 = a$ & $\lambda_2 = 0$

For $\lambda_1 = a \quad \Rightarrow \quad (A-aI)V_1 = 0$

$$\begin{pmatrix} 0 & b \\ 0 & -a \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad by_1 = 0$$

$$\Rightarrow \quad \underline{y_1 = 0|}$$

The eigenvectors are: $\underline{V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}|}$

For $\lambda_2 = 0 \quad \Rightarrow \quad (A)V_2 = 0$

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad ax_1 + by_1 = 0$$

$$\Rightarrow \quad \underline{ax_1 = -by_1|}$$

The eigenvectors are: $\underline{V_1 = \begin{pmatrix} -b \\ a \end{pmatrix}|}$

The ***eigenvector matrix*** is given by:

$$P=\begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix}$$

$$P^{-1}=\frac{1}{a}\begin{pmatrix} a & -b \\ 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & -\frac{b}{a} \\ 0 & \frac{1}{a} \end{pmatrix}$$

$$P^{-1}AP=\begin{pmatrix} 1 & -\frac{b}{a} \\ 0 & \frac{1}{a} \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix}$$

$$=\begin{pmatrix} 1 & -\frac{b}{a} \\ 0 & \frac{1}{a} \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \checkmark$$

Therefore, the result proves that is diagonalizable but not invertible

More Example: $\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$

Exercise

What are the matrices that are similar to themselves only?

Solution

Any matrix to be similar to itself if only if the similar formula

$$A = P^{-1}AP$$
$$PA = PP^{-1}AP$$
$$PA = IAP$$
$$PA = AP$$

One of the matrices that are similar is a scalars matrices (cI).

Exercise

For any scalars *a, b*, and *c*, show that

$$A = \begin{pmatrix} b & c & a \\ c & a & b \\ a & b & c \end{pmatrix}, \quad B = \begin{pmatrix} c & a & b \\ a & b & c \\ b & c & a \end{pmatrix}, \quad C = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$$

are similar.

Moreover, if $BC = CB$, then A has two zero eigenvalues.

Solution

$$\det(A) = \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}$$

$$\underline{= 3abc - a^2 - b^2 - c^2}$$

$$\det(B) = \begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix}$$

$$\underline{= 3abc - a^2 - b^2 - c^2}$$

$$\det(C)=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$\underline{= 3abc - a^2 - b^2 - c^2 \;\Big|}$$

Since $\det(A)=\det(B)=\det(C)$, then the matrices A, B, and C are similars.

$$|A-\lambda I|=\begin{vmatrix} b-\lambda & c & a \\ c & a-\lambda & b \\ a & b & c-\lambda \end{vmatrix}$$

$$=(b-\lambda)(a-\lambda)(c-\lambda)+2abc-a^2(a-\lambda)-b^2(b-\lambda)-c^2(c-\lambda)$$

$$=abc-bc\lambda-ac\lambda+c\lambda^2-ab\lambda+b\lambda^2+a\lambda^2-\lambda^3+2abc-a^3+a^2\lambda-b^3+b^2\lambda-c^3+c^2\lambda$$

$$=-\lambda^3+(c+b+a)\lambda^2+\left(a^2+b^2+c^2-bc-ac-ab\right)\lambda-a^3-b^3-c^3+3abc$$

Given that $BC=CB$

$$BC=\begin{pmatrix} c & a & b \\ a & b & c \\ b & c & a \end{pmatrix}\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$$

$$=\begin{pmatrix} ab+bc+ac & ab+bc+ac & a^2+b^2+c^2 \\ a^2+b^2+c^2 & ab+bc+ac & ab+bc+ac \\ ab+bc+ac & a^2+b^2+c^2 & ab+bc+ac \end{pmatrix}$$

$$CB=\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}\begin{pmatrix} c & a & b \\ a & b & c \\ b & c & a \end{pmatrix}$$

$$=\begin{pmatrix} ab+bc+ac & a^2+b^2+c^2 & ab+bc+ac \\ ab+bc+ac & ab+bc+ac & a^2+b^2+c^2 \\ a^2+b^2+c^2 & ab+bc+ac & ab+bc+ac \end{pmatrix}$$

Since $BC=CB$, then

$$a^2+b^2+c^2=ab+bc+ac$$

$$a^2+b^2+c^2-(ab+bc+ac)=0$$

$$\det(A-\lambda I)=-\lambda^3+(c+b+a)\lambda^2+\left(a^2+b^2+c^2-bc-ac-ab\right)\lambda-a^3-b^3-c^3+3abc$$

$$=-\lambda^3+(c+b+a)\lambda^2-a^3-b^3-c^3+3abc$$

$$\left(a^2+b^2+c^2-ab-bc-ac\right)(a+b+c)=0(a+b+c)$$

$$a^3+ab^2+ac^2-a^2b-abc-a^2c+a^2b+b^3+bc^2-ab^2-b^2c-abc$$
$$+a^2c+b^2c+c^3-abc-bc^2-ac^2=0$$

$$a^3+b^3+c^3-3abc=0$$

So,

$$\det(A-\lambda I)=-\lambda^3+(c+b+a)\lambda^2$$
$$=-\lambda^2(\lambda-(c+b+a))$$

The eigenvalues are: $\underline{\lambda_{1,2}=0 \quad \& \quad \lambda_3=a+b+c}$

Since $BC=CB$, then A has ***two zero*** eigenvalues

Exercise

For positive integer $k \ge 2$, compute $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}^k$

Solution

Let $A=\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$

$$\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix}$$
$$=6-5\lambda+\lambda^2-2$$
$$=\lambda^2-5\lambda+4=0$$

The eigenvalues are: $\underline{\lambda_1=1 \quad \& \quad \lambda_2=4}$

For $\lambda_1=1 \Rightarrow (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x+y=0$$

$\Rightarrow \underline{x=-y}$

$\Rightarrow \underline{V_1=\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

For $\lambda_2=4 \Rightarrow (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad -2x + y = 0$$

$$\Rightarrow \underline{2x = y}$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

The eigenvectors matrix:

$$P = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$P^{-1} = -\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \quad \Rightarrow \quad D^k = \begin{pmatrix} 1^k & 0 \\ 0 & 4^k \end{pmatrix}$$

$$A^k = PD^kP^{-1}$$

$$= \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4^k \end{pmatrix}\begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 4^k \\ 1 & 2\left(4^k\right) \end{pmatrix}\begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2^{2k} \\ 1 & 2\left(2^{2k}\right) \end{pmatrix}\begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2^{2k} \\ 1 & 2^{2k+1} \end{pmatrix}\begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3} + \frac{2^{2k}}{3} & -\frac{1}{3} + \frac{2^{2k}}{3} \\ -\frac{2}{3} + \frac{2^{2k+1}}{3} & \frac{1}{3} + \frac{2^{2k+1}}{3} \end{pmatrix}$$

Exercise

For positive integer $k \geq 2$, compute $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^k$

Solution

Let $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

Since it is an upper triangular, then

The eigenvalues are: $\lambda_{1,2} = \lambda$

For $\lambda_1 = \lambda \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow y = 0$$

$$\Rightarrow V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Since the eigenvalues are repeated and the eigenvectors are one-dimensional, therefore the matrix is not diagonalizable.

To compute $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^k$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^3 = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{pmatrix}$$

$$\vdots \quad \vdots$$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^k = \begin{pmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{pmatrix}$$

Exercise

For positive integer $k \ge 2$, compute $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^k$

Solution

Since the eigenvalues $\left(\lambda_{1,2,3} = 0\right)$ are repeated then it is not diagonalizable, which it will result the matrix doesn't have linearly independent eigenvectors.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore;

If $k = 2 \quad \Rightarrow \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^k = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Otherwise $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Exercise

For positive integer $k \ge 2$, compute $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^k$

Solution

Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 + 1 = 0$$

The eigenvalues are: $\lambda_{1,2,3} = -1$

For $\lambda_1 = -1 \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \to & -x + y = 0 \\ \to & -y + z = 0 \\ \to & x - z = 0 \end{matrix}$$

$$\Rightarrow x = y = z$$

$$\Rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad V_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The given matrix is not diagonalizable, since the matrix doesn't have linearly independent eigenvectors.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{= I}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\underline{= A}$$

When

$$k = 3m+1 \quad \Rightarrow \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^k = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = A$$

$$k = 3m+2 \quad \Rightarrow \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^k = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$k = 3m+3 \quad \Rightarrow \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^k = I_{3\times 3}$$

Exercise

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that A^k is similar to A fro every positive integer k. It is true more generally for any matrix with all eigenvalues equal to 1.

Solution

Since it is an upper triangular, then

The eigenvalues are: $\underline{\lambda_{1,2} = 1}$

For $\lambda_1 = 1 \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow y = 0$$

$$\Rightarrow \quad \boxed{V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

Since the eigenvalues are repeated and the eigenvectors are one-dimensional which are not linearly independent, therefore the matrix is not diagonalizable.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} A^2 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A^3 &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\vdots \qquad \vdots$$

$$A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

Since it is an upper triangular, then

The eigenvalues are: $\boxed{\lambda_{1,2} = 1}$

Therefore, A^k is similar to A fro every positive integer k.

Let A be $n \times n$ matrix with upper triangular and one's in the main diagonal, which implies that all eigenvalues equal to 1. If we use Jordan block, then each A^k block is similar to A.

Exercise

Can a matrix be similar to two different diagonal matrices?

Solution

The matrix can be similar to two different diagonal matrices as long the size is greater or equal to 3. And they the same eigenvalues by changing the entries in the main diagonal.

Example:

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \quad B = \begin{pmatrix} c & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \quad C = \begin{pmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & a \end{pmatrix}$$

Exercise

Prove that if A is diagonalizable, then A^T is diagonalizable.

Solution

If A is diagonalizable, then by the diagonalizable theorem

$$A = PDP^{-1}$$

$$A^T = \left(PDP^{-1}\right)^T$$

$$= \left(P^{-1}\right)^T D^T P^T \qquad \text{Since } D \text{ is diagonal then } D^T = D$$

$$= \left(P^T\right)^{-1} DP^T \qquad \text{Assume that } Q = \left(P^T\right)^{-1}$$

$$= QDQ^{-1}$$

Therefore, A^T is diagonalizable with the columns Q are n linearly independent eigenvectors

Exercise

Prove that if the eigenvalues of a diagonalizable matrix A are all ± 1, then the matrix is equal to its inverse.

Solution

Since the matrix A is diagonalizable with eigenvalues are ± 1, then the diagonal matrix D has ± 1 entries along the main diagonal.

So, $D = D^{-1}$

Matrix A is diagonalizable that implies to $A = PDP^{-1}$

$$A^{-1} = \left(PDP^{-1}\right)^{-1}$$

$$= \left(P^{-1}\right)^{-1} D^{-1} P^{-1}$$

$$= PDP^{-1}$$

$$\underline{= A} \quad \checkmark$$

Therefore, the matrix is equal to its inverse

Exercise

Prove that if A is diagonalizable with n real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $|A| = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$

Solution

If A is diagonalizable with n real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and D is diagonal with the eigenvalues as entries, then

$$D = \begin{bmatrix} \lambda_1 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

$$\underline{|D| = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n}$$

$$D = P^{-1}AP$$

$$\left|P^{-1}AP\right| = |D|$$

$$|A| = \left|P^{-1}AP\right|$$

$$= \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$$

Exercise

If x is a real number, then we can define e^x by the series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

In similar way, If X is a square matrix, then we can define e^X by the series

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \cdots$$

Evaluate e^X, where X is the indicated square matrix.

a) $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

b) $X = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

c) $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

d) $X = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

Solution

a) $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$$e^X = I + I + \frac{1}{2!}I^2 + \frac{1}{3!}I^3 + \frac{1}{4!}I^4 + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots & 0 \\ 0 & 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots \end{pmatrix}$$

$$= \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$$

Where, $e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$

b) $X = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

The eigenvalues are: $\lambda_{1,2} = 0,\ 1$

For $\lambda_1 = 0 \;\Rightarrow\; (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = 0$$

$$\Rightarrow \; V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1 \;\Rightarrow\; (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = y$$

$$\Rightarrow \; V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The eigenvectors matrix:

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \;\Rightarrow\; D^k = \begin{pmatrix} 0 & 0 \\ 0 & 1^k \end{pmatrix}$$

$$X^k = PD^kP^{-1}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{4!}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{4!}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots & 0 \\ 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$$

Given that: $e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$

c) $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$X^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$X^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\vdots \quad \vdots \quad \vdots$$

$$X^{2k+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X^{2k} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{4!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \cdots$$

$$=\begin{pmatrix} 1+\frac{1}{2!}+\frac{1}{4!}+\cdots & 1+\frac{1}{3!}+\frac{1}{5!}+\cdots \\ 1+\frac{1}{3!}+\frac{1}{5!}+\cdots & 1+\frac{1}{2!}+\frac{1}{4!}+\cdots \end{pmatrix}$$

$$=\begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{e+e^{-1}}{2} & \frac{e-e^{-1}}{2} \\ \frac{e-e^{-1}}{2} & \frac{e+e^{-1}}{2} \end{pmatrix}$$

d) $X=\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

$$|X-\lambda I|=\begin{vmatrix} 2-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix}$$

$$=(2-\lambda)(-2-\lambda)=0$$

The eigenvalues are: $\lambda_{1,2}=-2,\ 2$

For $\lambda_1=-2 \quad \Rightarrow \quad (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x=0$$

$$\Rightarrow \quad V_1=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For $\lambda_2=2 \quad \Rightarrow \quad (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to y=0$$

$$\Rightarrow \quad V_2=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The eigenvectors matrix:

$$P=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$P^{-1}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \quad \Rightarrow \quad D^k = \begin{pmatrix} (-2)^k & 0 \\ 0 & 2^k \end{pmatrix}$$

$$\begin{aligned} X^k &= PD^kP^{-1} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (-2)^k & 0 \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2^k \\ (-2)^k & \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2^k & 0 \\ 0 & (-2)^k \end{pmatrix} \end{aligned}$$

$$\begin{aligned} e^X &= I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \cdots \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 2^2 & 0 \\ 0 & (-2)^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 2^3 & 0 \\ 0 & (-2)^3 \end{pmatrix} + \frac{1}{4!}\begin{pmatrix} 2^4 & 0 \\ 0 & (-2)^4 \end{pmatrix} + \cdots \\ &= \begin{pmatrix} 1 + 2 + \frac{1}{2!}2^2 + \frac{1}{3!}2^3 + \frac{1}{4!}2^4 + \cdots & 0 \\ 0 & 1 + (-2) + \frac{1}{2!}(-2)^2 + \frac{1}{3!}(-2)^3 + \frac{1}{4!}(-2)^4 + \cdots \end{pmatrix} \\ &= \begin{pmatrix} e^2 & 0 \\ 0 & e^{-2} \end{pmatrix} \end{aligned}$$

Where, $e^2 = 1 + 2 + \frac{1}{2!}2^2 + \frac{1}{3!}2^3 + \frac{1}{4!}2^4 + \cdots$

$e^{-2} = 1 + (-2) + \frac{1}{2!}(-2)^2 + \frac{1}{3!}(-2)^3 + \frac{1}{4!}(-2)^4 + \cdots$

Solution   *Section* 4.6 – Orthogonal Diagonalization

Exercise

Determine whether the matrix is orthogonal $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Solution

Let $P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\begin{aligned} PP^T &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I_2 \quad \checkmark \end{aligned}$$

Therefore, the given matrix is orthogonal.

Exercise

Determine whether the matrix is orthogonal $\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

Solution

Let $P = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

$$\begin{aligned} PP^T &= \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I_2 \quad \checkmark \end{aligned}$$

Therefore, the given matrix is orthogonal.

Exercise

Determine whether the matrix is orthogonal $\begin{pmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

Solution

Let $P = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

$$PP^T = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{8}{9} & \\ & \end{pmatrix}$$

$$\neq I_2$$

Therefore, the given matrix is ***not*** orthogonal.

Exercise

Determine whether the matrix is orthogonal

$$\begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

Solution

Let $P = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$

$$PP^T = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I_3 \quad \checkmark$$

Therefore, the given matrix is orthogonal.

Exercise

Determine whether the matrix is orthogonal

$$\begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

Solution

Let $P = \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$

$$PP^T = \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix} \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I_3 \quad \checkmark$$

Therefore, the given matrix is orthogonal.

Exercise

Determine whether the matrix is orthogonal

$$\begin{pmatrix} -4 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{pmatrix}$$

Solution

Let $P = \begin{pmatrix} -4 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{pmatrix}$

$$PP^T = \begin{pmatrix} -4 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{pmatrix}\begin{pmatrix} -4 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 25 & & \\ & & \\ & & \end{pmatrix}$$

$$\neq I_3$$

Therefore, the given matrix is ***not*** orthogonal.

Exercise

Determine whether the matrix is orthogonal

$$\begin{pmatrix} 4 & -1 & -4 \\ -1 & 0 & -17 \\ 1 & 4 & -1 \end{pmatrix}$$

Solution

Let $P = \begin{pmatrix} 4 & -1 & -4 \\ -1 & 0 & -17 \\ 1 & 4 & -1 \end{pmatrix}$

$$PP^T = \begin{pmatrix} 4 & -1 & -4 \\ -1 & 0 & -17 \\ 1 & 4 & -1 \end{pmatrix}\begin{pmatrix} 4 & -1 & -4 \\ -1 & 0 & -17 \\ 1 & 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & & \\ & & \\ & & \end{pmatrix}$$

$\neq I_3$

Therefore, the given matrix is ***not*** orthogonal.

Exercise

Determine whether the matrix is orthogonal

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{pmatrix}$$

Solution

$$\text{Let } P = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{pmatrix}$$

$$PP^T = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I_3 \quad \checkmark$$

Therefore, the given matrix is orthogonal.

Exercise

Determine whether the matrix is orthogonal

$$\begin{pmatrix} \frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{5}}{2} \\ 0 & \frac{2\sqrt{5}}{5} & 0 \\ -\frac{\sqrt{2}}{6} & -\frac{\sqrt{5}}{5} & \frac{1}{2} \end{pmatrix}$$

Solution

Let $P = \begin{pmatrix} \frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{5}}{2} \\ 0 & \frac{2\sqrt{5}}{5} & 0 \\ -\frac{\sqrt{2}}{6} & -\frac{\sqrt{5}}{5} & \frac{1}{2} \end{pmatrix}$

$$PP^T = \begin{pmatrix} \frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{5}}{2} \\ 0 & \frac{2\sqrt{5}}{5} & 0 \\ -\frac{\sqrt{2}}{6} & -\frac{\sqrt{5}}{5} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{3} & 0 & -\frac{\sqrt{2}}{6} \\ 0 & \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{53}{36} & & \\ & & \\ & & \end{pmatrix}$$

$$\neq I_3$$

Therefore, the given matrix is ***not*** orthogonal.

Exercise

Determine whether the matrix is orthogonal

$$\begin{pmatrix} \frac{1}{8} & 0 & 0 & \frac{3\sqrt{7}}{8} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{3\sqrt{7}}{8} & 0 & 0 & \frac{1}{8} \end{pmatrix}$$

Solution

Let $P = \begin{pmatrix} \frac{1}{8} & 0 & 0 & \frac{3\sqrt{7}}{8} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{3\sqrt{7}}{8} & 0 & 0 & \frac{1}{8} \end{pmatrix}$

$$PP^T = \begin{pmatrix} \frac{1}{8} & 0 & 0 & \frac{3\sqrt{7}}{8} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{3\sqrt{7}}{8} & 0 & 0 & \frac{1}{8} \end{pmatrix}\begin{pmatrix} \frac{1}{8} & 0 & 0 & \frac{3\sqrt{7}}{8} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{3\sqrt{7}}{8} & 0 & 0 & \frac{1}{8} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I_3 \quad \checkmark$$

Therefore, the given matrix is orthogonal.

Exercise

Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$ $\qquad A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)^2 - 1$$

$$= \lambda^2 - 6\lambda + 8 = 0$$

The eigenvalues are: $\lambda_1 = 2 \quad and \quad \lambda_2 = 4$

For $\lambda_1 = 2$, we have: $(A - \lambda_1 I)\vec{v}_1 = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x + y = 0$$

$$\Rightarrow \quad x = -y$$

Therefore, the eigenvector $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For $\lambda_2 = 4$, we have: $\left(A - \lambda_2 I\right)\vec{v}_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad -x + y = 0$$

$$\Rightarrow \quad \underline{x = y}$$

Therefore, the eigenvector $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(-1,\ 1)}{\sqrt{(-1)^2 + 1^2}}$$

$$= \frac{(-1,\ 1)}{\sqrt{2}}$$

$$= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{(1,\ 1)}{\sqrt{1+1}}$$

$$= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$P^T A P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

Exercise

Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$ $\qquad A=\begin{pmatrix} 6 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{pmatrix}$

Solution

$$\det(A-\lambda I)=\begin{vmatrix} 6-\lambda & 2\sqrt{3} \\ 2\sqrt{3} & 7-\lambda \end{vmatrix}$$

$$=(6-\lambda)(7-\lambda)-12$$

$$=\lambda^2-13\lambda+30=0$$

The eigenvalues are: $\lambda_1=3 \quad and \quad \lambda_2=10$

For $\lambda_1=3$, we have: $(A-\lambda_1 I)\vec{v}_1=0$

$$\begin{pmatrix} 3 & 2\sqrt{3} \\ 2\sqrt{3} & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad 3x+2\sqrt{3}\,y=0$$

$\Rightarrow \ 3x=-2\sqrt{3}\,y$

Therefore, the eigenvector $\vec{v}_1=\begin{pmatrix} -2\sqrt{3} \\ 3 \end{pmatrix}$

For $\lambda_2=10$, we have: $(A-\lambda_2 I)\vec{v}_2=0$

$$\begin{pmatrix} -4 & 2\sqrt{3} \\ 2\sqrt{3} & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad -4x+2\sqrt{3}\,y=0$$

$\Rightarrow \ 2x=\sqrt{3}\,y$

Therefore, the eigenvector $\vec{v}_2=\begin{pmatrix} \sqrt{3} \\ 2 \end{pmatrix}$

$$\vec{u}_1=\frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$=\frac{\left(-\frac{2}{\sqrt{3}},\ 1\right)}{\sqrt{\left(-\frac{2}{\sqrt{3}}\right)^2+1^2}}$$

$$=\frac{\sqrt{3}}{\sqrt{7}}\left(-\frac{2}{\sqrt{3}},\ 1\right)$$

$$=\left(-\frac{2}{\sqrt{7}},\ \frac{\sqrt{3}}{\sqrt{7}}\right)$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{(\sqrt{3},\ 2)}{\sqrt{3+4}}$$

$$= \left(\frac{\sqrt{3}}{\sqrt{7}},\ \frac{2}{\sqrt{7}}\right)$$

$$P = \begin{pmatrix} -\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{pmatrix}$$

$$P^{-1} = P^T$$

$$= \begin{pmatrix} -\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{pmatrix} \begin{pmatrix} 6 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{6}{\sqrt{7}} & \frac{3\sqrt{3}}{\sqrt{7}} \\ \frac{10\sqrt{3}}{\sqrt{7}} & \frac{20}{\sqrt{7}} \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\ \frac{\sqrt{3}}{\sqrt{7}} & \frac{2}{\sqrt{7}} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 10 \end{pmatrix} \qquad \checkmark$$

Exercise

Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$ $\qquad A = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$$

$$= -2 + \lambda + \lambda^2 - 4$$

$$= \lambda^2 + \lambda - 6 = 0$$

The eigenvalues are: $\lambda_1 = -3 \quad and \quad \lambda_2 = 2$

For $\lambda_1 = -3$, we have: $\left(A - \lambda_1 I\right)\vec{v}_1 = 0$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x + 2y = 0$$

$$\Rightarrow \quad \underline{x = -2y}$$

Therefore, the eigenvector $\underline{\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$

For $\lambda_2 = 2$, we have: $\left(A - \lambda_2 I\right)\vec{v}_2 = 0$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad 2x - y = 0$$

$$\Rightarrow \quad \underline{2x = y}$$

Therefore, the eigenvector $\underline{\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{(-2,\ 1)}{\sqrt{4+1}}$$

$$\underline{= \left(-\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right)}$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{(1,\ 2)}{\sqrt{1+4}}$$

$$\underline{= \left(\frac{1}{\sqrt{5}},\ \frac{2}{\sqrt{5}}\right)}$$

$$P = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$P^{-1} = P^T$$

$$P^{-1}AP = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{6}{\sqrt{5}} & -\frac{3}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{4}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} \qquad \checkmark$$

Exercise

Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$ $\qquad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= 1 - 2\lambda + \lambda^2 - 1$$

$$= \lambda^2 - 2\lambda = 0$$

The eigenvalues are: $\lambda_1 = 0 \quad and \quad \lambda_2 = 2$

For $\lambda_1 = 0$, we have: $(A - \lambda_1 I)\vec{v}_1 = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad x + y = 0$$

$$\Rightarrow \ x = -y$$

Therefore, the eigenvector $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For $\lambda_2 = 2$, we have: $(A - \lambda_2 I)\vec{v}_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad -x + y = 0$$

$$\Rightarrow \ x = y$$

Therefore, the eigenvector $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(-1,\ 1)}{\sqrt{2}}$$

$$= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}} \right)$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{(1,\ 1)}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$P^T AP = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \quad \checkmark$$

Exercise

Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$ $\qquad A = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 1-\lambda \end{vmatrix}$$

$$= 2 - 3\lambda + \lambda^2 - 2$$

$$= \lambda^2 - 3\lambda = 0$$

The eigenvalues are: $\lambda_1 = 0 \quad and \quad \lambda_2 = 3$

For $\lambda_1 = 0$, we have: $\left(A - \lambda_1 I\right)\vec{v}_1 = 0$

$$\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad 2x + \sqrt{2}\,y = 0$$

$$\Rightarrow \quad 2x = -\sqrt{2}\,y$$

Therefore, the eigenvector $\boxed{\vec{v}_1 = \begin{pmatrix} -\sqrt{2} \\ 2 \end{pmatrix}}$

For $\lambda_2 = 3$, we have: $\left(A - \lambda_2 I\right)\vec{v}_2 = 0$

$$\begin{pmatrix} -1 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad -x + \sqrt{2}y = 0$$

$\Rightarrow \quad \boxed{x = \sqrt{2}\, y}$

Therefore, the eigenvector $\boxed{\vec{v}_2 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}}$

$$\begin{aligned} \vec{u}_1 &= \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|} \\ &= \frac{\left(-\sqrt{2},\ 2\right)}{\sqrt{2+4}} \\ &= \boxed{\left(-\frac{\sqrt{2}}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)} \end{aligned}$$

$$\begin{aligned} \vec{u}_2 &= \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|} \\ &= \frac{\left(\sqrt{2},\ 1\right)}{\sqrt{3}} \\ &= \boxed{\left(\frac{\sqrt{2}}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)} \end{aligned}$$

$$P = \begin{pmatrix} -\frac{\sqrt{2}}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$\begin{aligned} P^T AP &= \begin{pmatrix} -\frac{\sqrt{2}}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} -\frac{\sqrt{2}}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \frac{3\sqrt{2}}{\sqrt{3}} & \frac{3}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} -\frac{\sqrt{2}}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \quad \checkmark \end{aligned}$$

Exercise

Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 + 2 + 3\lambda$$

$$\underline{= -\lambda^3 + 3\lambda + 2 = 0}$$

$\lambda = 2$

$$\begin{array}{c|cccc} 2 & -1 & 0 & 3 & 2 \\ & & -2 & -4 & -2 \\ \hline & -1 & -2 & -1 & 0 \end{array} \quad \rightarrow \quad \underline{-\lambda^2 - 2\lambda - 1 = 0}$$

The eigenvalues are: $\underline{\lambda_{1,2} = -1 \quad \& \quad \lambda_3 = 2}$

For $\lambda_{1,2} = -1$, we have: $(A - \lambda_1 I)\vec{v}_1 = 0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x + y + z = 0 \quad (1)$$

If $z = 0 \quad \Rightarrow \quad \underline{x = -y}$

Therefore, the eigenvector $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

If $y = 0 \quad \Rightarrow \quad \underline{x = -z}$

Therefore, the eigenvector $\vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda_3 = 2$, we have: $(A - \lambda_3 I)\vec{v}_3 = 0$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{matrix} \\ 2R_2 + R_1 \\ 2R_3 + R_1 \end{matrix}$$

$$\begin{pmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \begin{matrix} 3R_1 + R_2 \\ \\ R_3 + R_2 \end{matrix}$$

$$\begin{pmatrix} -6 & 0 & 6 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} -\frac{1}{6}R_1 \\ -\frac{1}{3}R_2 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \rightarrow & x - z = 0 \\ \rightarrow & y - z = 0 \\ \rightarrow & \end{matrix}$$

$$\Rightarrow \quad \underline{x = y = z}$$

Therefore, the eigenvector $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{(-1,\ 1,\ 0)}{\sqrt{2}}$$

$$= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \bullet \vec{u}_1\right)\vec{u}_1$$

$$= (-1,\ 0,\ 1) - \left((-1,\ 0,\ 1) \bullet \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)\right)\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= (-1,\ 0,\ 1) - \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= (-1,\ 0,\ 1) - \left(-\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$= \left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{\left(-\frac{1}{2}, -\frac{1}{2}, 1\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}$$

$$= \frac{2}{\sqrt{6}}\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)$$

$$= \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{(1, 1, 1)}{\sqrt{3}}$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$P^T AP = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \checkmark$$

Exercise

Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$

$$A=\begin{pmatrix} 0 & 10 & 10 \\ 10 & 5 & 0 \\ 10 & 0 & -5 \end{pmatrix}$$

Solution

$$\det(A-\lambda I)=\begin{vmatrix} -\lambda & 10 & 10 \\ 10 & 5-\lambda & 0 \\ 10 & 0 & -5-\lambda \end{vmatrix}$$

$$=-\lambda\left(-25+\lambda^2\right)-100(5-\lambda)-100(-5-\lambda)$$

$$=25\lambda-\lambda^3-500+100\lambda+500+100\lambda$$

$$=-\lambda^3+225\lambda$$

$$=-\lambda\left(\lambda^2-225\right)=0$$

The eigenvalues are: $\lambda_1=-15 \quad \lambda_2=0 \quad \lambda_3=15$

For $\lambda_1=-15$, we have: $\left(A-\lambda_1 I\right)\vec{v}_1=0$

$$\begin{pmatrix} 15 & 10 & 10 \\ 10 & 20 & 0 \\ 10 & 0 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 15 & 10 & 10 \\ 10 & 20 & 0 \\ 10 & 0 & 10 \end{pmatrix} \quad \begin{matrix} \\ 3R_2-2R_1 \\ 3R_3-2R_1 \end{matrix}$$

$$\begin{pmatrix} 15 & 10 & 10 \\ 0 & 40 & -20 \\ 0 & -20 & 10 \end{pmatrix} \quad \begin{matrix} 4R_1-R_2 \\ \\ 2R_3+R_2 \end{matrix}$$

$$\begin{pmatrix} 60 & 0 & 60 \\ 0 & 40 & -20 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 60 & 0 & 60 \\ 0 & 40 & -20 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \rightarrow & 60x+60z=0 & (1) \\ \rightarrow & 40y-20z=0 & (2) \\ & & \end{matrix}$$

$(1) \Rightarrow x=-z$

$(2) \quad \Rightarrow \quad \underline{2y = z}$

Therefore, the eigenvector $\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

For $\lambda_2 = 0$, we have: $\left(A - \lambda_2 I\right)\vec{v}_2 = 0$

$$\begin{pmatrix} 0 & 10 & 10 \\ 10 & 5 & 0 \\ 10 & 0 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 0 & -5 \\ 10 & 5 & 0 \\ 0 & 10 & 10 \end{pmatrix} \quad R_2 - R_1$$

$$\begin{pmatrix} 10 & 0 & -5 \\ 0 & 5 & 5 \\ 0 & 10 & 10 \end{pmatrix} \quad R_3 - 2R_2$$

$$\begin{pmatrix} 10 & 0 & -5 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{ll} \rightarrow \quad 10x - 5z = 0 & (1) \\ \rightarrow \quad 5y + 5z = 0 & (2) \end{array}$$

$(1) \quad \Rightarrow \quad \underline{2x = z}$

$(2) \quad \Rightarrow \quad \underline{y = -z}$

Therefore, the eigenvector $\vec{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

For $\lambda_3 = 15$, we have: $\left(A - \lambda_3 I\right)\vec{v}_3 = 0$

$$\begin{pmatrix} -15 & 10 & 10 \\ 10 & -10 & 0 \\ 10 & 0 & -20 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -15 & 10 & 10 \\ 10 & -10 & 0 \\ 10 & 0 & -20 \end{pmatrix} \quad \begin{array}{l} 3R_2 + 2R_1 \\ 3R_3 + 2R_1 \end{array}$$

$$\begin{pmatrix} -15 & 10 & 10 \\ 0 & -10 & 20 \\ 0 & 20 & -40 \end{pmatrix} \quad \begin{array}{l} R_1 + R_2 \\ \\ R_3 + 2R_2 \end{array}$$

$$\begin{pmatrix} -15 & 0 & 30 \\ 0 & -10 & 20 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \rightarrow & -x+2z=0 & (1) \\ \rightarrow & -y+2z=0 & (2) \\ & & \end{matrix}$$

$(1) \Rightarrow \underline{x = 2z}$

$(2) \Rightarrow \underline{y = 2z}$

Therefore, the eigenvector $\vec{v}_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(-2,\ 1,\ 2)}{\sqrt{4+1+4}}$$

$$= \left(-\frac{2}{3},\ \frac{1}{3},\ \frac{2}{3}\right)$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{(1,\ -2,\ 2)}{\sqrt{1+4+4}}$$

$$= \left(\frac{1}{3},\ -\frac{2}{3},\ \frac{2}{3}\right)$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{(2,\ 2,\ 1)}{\sqrt{9}}$$

$$= \left(\frac{2}{3},\ \frac{2}{3},\ \frac{1}{3}\right)$$

$$P = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$P^T AP = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 10 & 10 \\ 10 & 5 & 0 \\ 10 & 0 & -5 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 10 & -5 & -10 \\ 0 & 0 & 0 \\ 10 & 10 & 5 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} -15 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 15 \end{pmatrix} \qquad \checkmark$$

Exercise

Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$

$$A = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 3 & 0 \\ 3 & -\lambda & 4 \\ 0 & 4 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 + 16\lambda + 9\lambda$$

$$= -\lambda\left(\lambda^2 - 25\right) = 0$$

The eigenvalues are: $\lambda_1 = -5 \quad \lambda_2 = 0 \quad \lambda_3 = 5$

For $\lambda_1 = -5$, we have: $\left(A - \lambda_1 I\right)\vec{v}_1 = 0$

$$\begin{pmatrix} 5 & 3 & 0 \\ 3 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 3 & 0 \\ 3 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix} \quad 5R_2 - 3R_1$$

$$\begin{pmatrix} 5 & 3 & 0 \\ 0 & 16 & 20 \\ 0 & 4 & 5 \end{pmatrix} \begin{array}{l} 16R_1 - 3R_2 \\ \\ 4R_3 - R_2 \end{array}$$

$$\begin{pmatrix} 80 & 0 & -60 \\ 0 & 16 & 20 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \frac{1}{20}R_1 \\ \frac{1}{4}R_2 \\ \\ \end{array}$$

$$\begin{pmatrix} 4 & 0 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \rightarrow \quad 4x - 3z = 0 \quad (1) \\ \rightarrow \quad 4y + 5z = 0 \quad (2) \\ \\ \end{array}$$

$(1) \quad \Rightarrow \quad \underline{4x = 3z}$

$(2) \quad \Rightarrow \quad \underline{4y = -5z}$

Therefore, the eigenvector $\vec{v}_1 = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$

For $\lambda_2 = 0$, we have: $\left(A - \lambda_2 I\right)\vec{v}_2 = 0$

$$\begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 4 & 0 \end{pmatrix} \begin{array}{l} \\ \\ 3R_3 - 4R_2 \end{array}$$

$$\begin{pmatrix} 3 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \rightarrow \quad 3x + 4z = 0 \quad (1) \\ \rightarrow \quad \underline{y = 0} \\ \\ \end{array}$$

$(1) \quad \Rightarrow \quad \underline{3x = -4z}$

Therefore, the eigenvector $\vec{v}_2 = \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}$

For $\lambda_3 = 5$, we have: $\left(A - \lambda_3 I\right)\vec{v}_3 = 0$

$$\begin{pmatrix} -5 & 3 & 0 \\ 3 & -5 & 4 \\ 0 & 4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 3 & 0 \\ 3 & -5 & 4 \\ 0 & 4 & -5 \end{pmatrix} \begin{array}{l} \\ 5R_2 + 3R_1 \\ \\ \end{array}$$

$$\begin{pmatrix} -5 & 3 & 0 \\ 0 & -16 & 20 \\ 0 & 4 & -5 \end{pmatrix} \begin{array}{l} 16R_1 + 3R_2 \\ \\ 4R_3 + R_2 \end{array}$$

$$\begin{pmatrix} -80 & 0 & 60 \\ 0 & -16 & 20 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} -\frac{1}{20}R_1 \\ -\frac{1}{4}R_2 \\ \\ \end{array}$$

$$\begin{pmatrix} 4 & 0 & -3 \\ 0 & 4 & -5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} \to \quad 4x - 3z = 0 \quad (1) \\ \to \quad 4y - 5z = 0 \quad (2) \\ \\ \end{array}$$

$(1) \Rightarrow \boxed{4x = 3z}$

$(2) \Rightarrow \boxed{4y = 5z}$

Therefore, the eigenvector $\boxed{\vec{v}_3 = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} \quad \boxed{\vec{v}_1 = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}}$$

$$= \frac{(3,\ -5,\ 4)}{\sqrt{9+25+16}}$$

$$= \frac{1}{5\sqrt{2}}(3,\ -5,\ 4)$$

$$\boxed{= \left(\frac{3}{5\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ \frac{4}{5\sqrt{2}}\right)}$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} \quad \boxed{\vec{v}_2 = \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}}$$

$$= \frac{(-4,\ 0,\ 3)}{\sqrt{16+9}}$$

$$\boxed{= \left(-\frac{4}{5},\ 0,\ \frac{3}{5}\right)}$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}\quad \boxed{\vec{v}_3 = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}}$$

$$= \frac{(3,\ 5,\ 4)}{\sqrt{9+25+16}}$$

$$\boxed{= \left(\frac{3}{5\sqrt{2}},\ \frac{1}{\sqrt{2}},\ \frac{4}{5\sqrt{2}}\right)}$$

$$P = \begin{pmatrix} \frac{3}{5\sqrt{2}} & -\frac{4}{5} & \frac{3}{5\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{4}{5\sqrt{2}} & \frac{3}{5} & \frac{4}{5\sqrt{2}} \end{pmatrix}$$

$$P^T AP = \begin{pmatrix} \frac{3}{5\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{4}{5\sqrt{2}} \\ -\frac{4}{5} & 0 & \frac{3}{5} \\ \frac{3}{5\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{4}{5\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{5\sqrt{2}} & -\frac{4}{5} & \frac{3}{5\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{4}{5\sqrt{2}} & \frac{3}{5} & \frac{4}{5\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{\sqrt{2}} & \frac{5}{\sqrt{2}} & -\frac{4}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{3}{\sqrt{2}} & \frac{5}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{3}{5\sqrt{2}} & -\frac{4}{5} & \frac{3}{5\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{4}{5\sqrt{2}} & \frac{3}{5} & \frac{4}{5\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad \checkmark$$

Exercise

Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$

$$A=\begin{pmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{pmatrix}$$

Solution

$$\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & 2 & -2 \\ 2 & -1-\lambda & 4 \\ -2 & 4 & -1-\lambda \end{vmatrix}$$

$$=(2-\lambda)\left(1+2\lambda+\lambda^2\right)-32+4+4\lambda-32+16\lambda+4+4\lambda$$

$$=2+3\lambda-\lambda^3+24\lambda-56$$

$$=-\lambda^3+27\lambda-54=0$$

$\lambda=3$

$$\begin{array}{r|rrrr} 3 & -1 & 0 & 27 & -54 \\ & & -3 & -9 & 54 \\ \hline & -1 & -3 & 18 & 0 \end{array} \quad \rightarrow \quad -\lambda^2-3\lambda+18=0$$

The eigenvalues are: $\lambda_{1,2}=3 \quad \& \quad \lambda_3=-6$

For $\lambda_{1,2}=3$, we have: $\left(A-\lambda_1 I\right)\vec{v}_1=0$

$$\begin{pmatrix} -1 & 2 & -2 \\ 2 & -4 & 4 \\ -2 & 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad -x+2y-2z=0 \quad (1)$$

If $z=0 \quad \Rightarrow \quad x=2y$

Therefore, the eigenvector $\vec{v}_1=\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

If $y=0 \quad \Rightarrow \quad x=-2z$

Therefore, the eigenvector $\vec{v}_2=\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda_3=-6$, we have: $\left(A-\lambda_3 I\right)\vec{v}_3=0$

$$\begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix} \begin{array}{l} \\ 4R_2 - R_1 \\ 4R_3 + R_1 \end{array}$$

$$\begin{pmatrix} 8 & 2 & -2 \\ 0 & 18 & 18 \\ 0 & 18 & 18 \end{pmatrix} \begin{array}{l} 9R_1 - R_2 \\ \\ R_3 - R_2 \end{array}$$

$$\begin{pmatrix} 72 & 0 & -36 \\ 0 & 18 & 18 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \frac{1}{36}R_1 \\ \frac{1}{18}R_2 \\ \\ \end{array}$$

$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \rightarrow \ 2x - z = 0 \quad (1) \\ \rightarrow \ y + z = 0 \quad (2) \\ \rightarrow \end{array}$$

$(1) \Rightarrow \underline{2x = z}$

$(2) \Rightarrow \underline{y = -z}$

Therefore, the eigenvector $\vec{v}_3 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(2,\ 1,\ 0)}{\sqrt{4+1}}$$

$$= \left(\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0 \right)$$

$$\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \bullet \vec{u}_1\right)\vec{u}_1$$

$$= (-2,\ 0,\ 1) - \left((-2,\ 0,\ 1) \bullet \left(\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0 \right) \right)\left(\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0 \right)$$

$$= (-2,\ 0,\ 1) + \frac{4}{\sqrt{5}}\left(\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}},\ 0 \right)$$

$$= (-2,\ 0,\ 1) + \left(\frac{8}{5},\ \frac{4}{5},\ 0 \right)$$

$$=\left(-\frac{2}{5},\ \frac{4}{5},\ 1\right)$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$$

$$= \frac{\left(-\frac{2}{5},\ \frac{4}{5},\ 1\right)}{\sqrt{\frac{4}{25}+\frac{16}{25}+1}}$$

$$= \frac{5}{\sqrt{45}}\left(-\frac{2}{5},\ \frac{4}{5},\ 1\right)$$

$$= \frac{5}{3\sqrt{5}}\left(-\frac{2}{5},\ \frac{4}{5},\ 1\right)$$

$$= \left(-\frac{2}{3\sqrt{5}},\ \frac{4}{3\sqrt{5}},\ \frac{5}{3\sqrt{5}}\right)$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \frac{(1,\ -2,\ 2)}{\sqrt{1+4+4}}$$

$$= \left(\frac{1}{3},\ -\frac{2}{3},\ \frac{2}{3}\right)$$

$$P = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} & \frac{1}{3} \\ \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} & -\frac{2}{3} \\ 0 & \frac{5}{3\sqrt{5}} & \frac{2}{3} \end{pmatrix}$$

$$P^T AP = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3\sqrt{5}} & \frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} & \frac{1}{3} \\ \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} & -\frac{2}{3} \\ 0 & \frac{5}{3\sqrt{5}} & \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{6}{\sqrt{5}} & \frac{3}{\sqrt{5}} & 0 \\ -\frac{2}{\sqrt{5}} & \frac{4}{\sqrt{5}} & \frac{5}{\sqrt{5}} \\ -2 & 4 & -4 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} & \frac{1}{3} \\ \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} & -\frac{2}{3} \\ 0 & \frac{5}{3\sqrt{5}} & \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix} \quad \checkmark$$

Exercise

Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$

$$A=\begin{pmatrix}-2&0&-36\\0&-3&0\\-36&0&-23\end{pmatrix}$$

Solution

$$\det(A-\lambda I)=\begin{vmatrix}-2-\lambda&0&-36\\0&-3-\lambda&0\\-36&0&-23-\lambda\end{vmatrix}$$

$$=(-2-\lambda)(-3-\lambda)(-23-\lambda)-(36)(-3-\lambda)(36)$$

$$=-\left(6+5\lambda+\lambda^2\right)(23+\lambda)+3888+1296\lambda$$

$$=-138-115\lambda-23\lambda-6\lambda-5\lambda^2-\lambda^3+3888+1296\lambda$$

$$=-\lambda^3-28\lambda^2+1175\lambda+3750=0$$

$\lambda=-3$

-3	-1	-28	1175	3750
		3	75	-3750
	-1	-25	1250	0

$\rightarrow \ -\lambda^2-25\lambda+1250=0$

$$\lambda=\frac{25\pm\sqrt{625+5,000}}{-2}$$

$$=\frac{25\pm75}{-2}$$

The eigenvalues are: $\lambda_1=25,\quad \lambda_2=-3,\quad \textit{and}\quad \lambda_3=-50$

For $\lambda_1=25$, we have: $\left(A-\lambda_1 I\right)\vec{v}_1=0$

$$\begin{pmatrix}-27&0&-36\\0&-28&0\\-36&0&-48\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad\rightarrow\quad\begin{cases}-27x-36z=0\\ y=0\\ -36x-48z=0\end{cases}$$

$$\Rightarrow\begin{cases}3x+4z=0\\3x+4z=0\end{cases}$$

$$\Rightarrow 3x=-4z$$

Therefore, the eigenvector $\vec{v}_1=\begin{pmatrix}-4\\0\\3\end{pmatrix}$

For $\lambda_2=-3$, we have: $\left(A-\lambda_2 I\right)\vec{v}_2=0$

$$\begin{pmatrix} 1 & 0 & -36 \\ 0 & 0 & 0 \\ -36 & 0 & -20 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \to \quad x - 36z = 0 \\ \\ \to \quad -36x - 20z = 0 \end{array}$$

$$\Rightarrow \begin{cases} x - 36z = 0 \\ 9x + 5z = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & -36 \\ 9 & 5 \end{vmatrix} = 329 \neq 0$$

$$\Rightarrow \underline{x = z = 0}\,\rfloor$$

Therefore, the eigenvector $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_3 = -25$, we have: $\left(A - \lambda_3 I\right)\vec{v}_3 = 0$

$$\begin{pmatrix} 48 & 0 & -36 \\ 0 & 47 & 0 \\ -36 & 0 & 27 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \to \quad 48x - 36z = 0 \\ \to \quad \underline{y = 0}\,\rfloor \\ \to \quad -36x + 27z = 0 \end{array}$$

$$\Rightarrow \begin{cases} 4x - 3z = 0 \\ -4x + 3z = 0 \end{cases}$$

$$\Rightarrow \underline{4x = 3z}\,\rfloor$$

Therefore, the eigenvector $\vec{v}_3 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{(-4,\ 0,\ 3)}{\sqrt{16+9}}$$

$$= \frac{(-4,\ 0,\ 3)}{5}$$

$$= \left(-\frac{4}{5},\ 0,\ \frac{3}{5}\right)$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{(0,1,\ 0)}{\sqrt{1^2}}$$

$$= (0,\ 1,\ 0)$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$
$$= \frac{(3,\ 0,\ 4)}{\sqrt{3^2+4^2}}$$
$$= \frac{(3,\ 0,\ 4)}{5}$$
$$= \left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)$$

$$P = \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

$$P^T AP = \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix} \begin{pmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -20 & 0 & 15 \\ 0 & -3 & 0 \\ -30 & 0 & -40 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 25 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -50 \end{pmatrix}$$

Exercise

Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix}$$
$$= -\lambda\left[(1-\lambda)^2 - 1\right]$$

$$\boxed{= -\lambda\left(\lambda^2 - 2\lambda\right) = 0}$$

The eigenvalues are: $\boxed{\lambda_{1,2} = 0 \quad and \quad \lambda_3 = 2}$

For $\lambda_{1,2} = 0$, we have: $\left(A - \lambda_1 I\right)\vec{v}_1 = 0$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x + y = 0$$

$$\boxed{\Rightarrow \ x = -y}$$

Therefore, the eigenvector $\boxed{\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$

For $\lambda_3 = 2$, we have: $\left(A - \lambda_3 I\right)\vec{v}_3 = 0$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \rightarrow & -x + y = 0 \ (1) \\ \\ \rightarrow & \boxed{z = 0} \end{matrix}$$

$$(1) \Rightarrow \ \boxed{x = y}$$

Therefore; the eigenvector $\boxed{\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\left\|\vec{v}_1\right\|}$$

$$= \frac{(-1,\ 1,\ 0)}{\sqrt{1^2 + 1^2}}$$

$$\boxed{= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)}$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{(0,\ 0,\ 1)}{\sqrt{1^2}}$$

$$\boxed{= (0,\ 0,\ 1)}$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \frac{(1,\ 1,\ 0)}{\sqrt{1^2+1^2}}$$

$$\underline{= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0 \right)}\Bigg|$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$P^{-1} = P^T = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \checkmark$$

Exercise

Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$

$$A=\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Solution

$$\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix}$$

$$=(2-\lambda)^3-1-1-(2-\lambda)-(2-\lambda)-(2-\lambda)$$

$$=8-12\lambda+6\lambda^2-\lambda^3-2-6+3\lambda$$

$$=-\lambda^3+6\lambda^2-9\lambda=0$$

The eigenvalues are: $\lambda_1=0 \quad and \quad \lambda_{2,3}=3$

For $\lambda_1=0$, we have: $(A-\lambda_1 I)\vec{v}_1=0$

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \quad \begin{matrix} 2R_2+R_1 \\ 2R_3+R_1 \end{matrix}$$

$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{pmatrix} \quad \begin{matrix} 3R_1+R_2 \\ \\ R_3+R_2 \end{matrix}$$

$$\begin{pmatrix} 6 & 0 & -6 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} \frac{1}{6}R_1 \\ \frac{1}{3}R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \rightarrow & x_1-z_1=0 \\ \rightarrow & y_1-z_1=0 \end{matrix}$$

$\Rightarrow \underline{x_1 = z_1 = y_1}$

Therefore; the eigenvector $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

For $\lambda_{2,3} = 3$, we have: $(A - \lambda_1 I)\vec{v}_1 = 0$

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \; -x_1 - y_1 - z_1 = 0$$

$\Rightarrow \underline{x_1 = -y_1 - z_1}$

Therefore; the eigenvector $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{aligned} \vec{u}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ &= \frac{(1,\ 1,\ 1)}{\sqrt{3}} \\ &= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) \end{aligned}$$

$$\begin{aligned} \vec{w}_2 &= \vec{v}_2 - (\vec{v}_2 \bullet \vec{u}_1)\vec{u}_1 \\ &= (-1,\ 1,\ 0) - \left[(-1,\ 1,\ 0) \bullet \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)\right]\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) \\ &= (-1,\ 1,\ 0) - 0\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right) \\ &= (-1,\ 1,\ 0) \end{aligned}$$

$$\begin{aligned} \vec{u}_2 &= \frac{\vec{w}_2}{\|\vec{w}_2\|} \\ &= \frac{(-1,\ 1,\ 0)}{\sqrt{2}} \\ &= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right) \end{aligned}$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \bullet \vec{u}_2)\vec{u}_2$$

$$=(-1,\ 0,\ 1)-\left[(-1,\ 0,\ 1)\bullet\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)\right]\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$=(-1,\ 0,\ 1)-\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$=(-1,\ 0,\ 1)-\left(-\frac{1}{2},\ \frac{1}{2},\ 0\right)$$

$$=\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)$$

$$\vec{u}_3=\frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$

$$=\frac{\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}$$

$$=\frac{1}{\frac{\sqrt{6}}{2}}\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)$$

$$=\left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)$$

$$P=\begin{pmatrix}\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}\end{pmatrix}$$

$$P^T AP=\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0 & 0\\ -\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0\\ -\frac{3}{\sqrt{6}} & -\frac{3}{\sqrt{6}} & \frac{4}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 2\end{pmatrix}\qquad\checkmark$$

Exercise

Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$

$$A = \begin{pmatrix} -7 & 24 & 0 & 0 \\ 24 & 7 & 0 & 0 \\ 0 & 0 & -7 & 27 \\ 0 & 0 & 24 & 7 \end{pmatrix}$$

<u>***Solution***</u>

$$\det(A - \lambda I) = \begin{vmatrix} -7-\lambda & 24 & 0 & 0 \\ 24 & 7-\lambda & 0 & 0 \\ 0 & 0 & -7-\lambda & 24 \\ 0 & 0 & 24 & 7-\lambda \end{vmatrix}$$

$$= (-7-\lambda)(7-\lambda)\left[(-7-\lambda)(7-\lambda) - 24^2\right] - 24^2\left[(-7-\lambda)(7-\lambda) - 24^2\right]$$

$$= \left(\lambda^2 - 49\right)\left(\lambda^2 - 625\right) - 576\left(\lambda^2 - 625\right)$$

$$= \left(\lambda^2 - 625\right)\left(\lambda^2 - 49 - 576\right)$$

$$= \left(\lambda^2 - 625\right)^2 = 0$$

The eigenvalues are: $\lambda_{1,2} = 25 \quad and \quad \lambda_{3,4} = -25$

For $\lambda_{1,2} = 25$, we have:

$$\begin{pmatrix} -32 & 24 & 0 & 0 \\ 24 & -18 & 0 & 0 \\ 0 & 0 & -32 & 24 \\ 0 & 0 & 24 & -18 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -32 & 24 & 0 & 0 \\ 24 & -18 & 0 & 0 \\ 0 & 0 & -32 & 24 \\ 0 & 0 & 24 & -18 \end{pmatrix} \begin{array}{l} \\ 4R_2 + 3R_1 \\ \\ 4R_4 + 3R_3 \end{array}$$

$$\begin{pmatrix} -32 & 24 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -32 & 24 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} -\frac{1}{8}R_1 \\ \\ -\frac{1}{8}R_3 \\ \\ \end{array}$$

$$\begin{pmatrix} 4 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} \to \quad 4x_1 - 3x_2 = 0 \\ \\ \to \quad 4x_3 - 3x_4 = 0 \end{array}$$

$$\Rightarrow \begin{cases} \boxed{4x_1 = 3x_2} \\ \boxed{4x_3 = 3x_4} \end{cases}$$

Therefore; the eigenvector $\boxed{\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 4 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 4 \\ 0 \\ 0 \end{pmatrix}}$

For $\lambda_{3,4} = -25$, we have:

$$\begin{pmatrix} 18 & 24 & 0 & 0 \\ 24 & 32 & 0 & 0 \\ 0 & 0 & 18 & 24 \\ 0 & 0 & 24 & 32 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 18 & 24 & 0 & 0 \\ 24 & 32 & 0 & 0 \\ 0 & 0 & 18 & 24 \\ 0 & 0 & 24 & 32 \end{pmatrix} \quad \begin{array}{l} \\ 3R_2 - 4R_1 \\ \\ 3R_4 - 4R_3 \end{array}$$

$$\begin{pmatrix} 18 & 24 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 18 & 24 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \frac{1}{6}R_1 \\ \\ \frac{1}{6}R_3 \\ \\ \end{array}$$

$$\begin{pmatrix} 3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} \to \quad 3x_1 + 4x_2 = 0 \\ \\ \to \quad 3x_3 + 4x_4 = 0 \end{array}$$

$$\Rightarrow \begin{cases} \boxed{3x_1 = 4x_2} \\ \boxed{3x_3 = 4x_4} \end{cases}$$

Therefore; the eigenvector $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ -4 \\ 3 \end{pmatrix} \quad \vec{v}_4 = \begin{pmatrix} -4 \\ 3 \\ 0 \\ 0 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(0,\ 0,\ 3,\ 4)}{\sqrt{3^2+4^2}}$$

$$= \left(0,\ 0,\ \frac{3}{5},\ \frac{4}{5}\right)$$

$$\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{(3,\ 4,\ 0,\ 0)}{\sqrt{3^2+4^2}}$$

$$= \left(\frac{3}{5},\ \frac{4}{5},\ 0,\ 0\right)$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$$

$$= \frac{(0,\ 0,\ -4,\ 3)}{\sqrt{(-4)^2+3^2}}$$

$$= \left(0,\ 0,\ -\frac{4}{5},\ \frac{3}{5}\right)$$

$$\vec{u}_4 = \frac{\vec{v}_4}{\|\vec{v}_4\|}$$

$$= \frac{(-4,\ 3,\ 0,\ 0)}{\sqrt{25}}$$

$$= \left(-\frac{4}{5},\ \frac{3}{5},\ 0,\ 0\right)$$

$$P = \begin{pmatrix} 0 & \frac{3}{5} & 0 & -\frac{4}{5} \\ 0 & \frac{4}{5} & 0 & \frac{3}{5} \\ \frac{3}{5} & 0 & -\frac{4}{5} & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} & 0 & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

$$P^T AP = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} & 0 & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} -7 & 24 & 0 & 0 \\ 24 & 7 & 0 & 0 \\ 0 & 0 & -7 & 24 \\ 0 & 0 & 24 & 7 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} & 0 & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 20 & -15 & 0 & 0 \\ 15 & 20 & 0 & 0 \\ 0 & 0 & 20 & -15 \\ 0 & 0 & 15 & 20 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} & 0 & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -25 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & -25 & 0 \\ 0 & 0 & 0 & 25 \end{pmatrix} \quad \checkmark$$

Exercise

Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 0 & 0 \\ 1 & 3-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix}$$

$$= \lambda^2 (3-\lambda)^2 - \lambda^2$$

$$= \lambda^2\left(9-6\lambda+\lambda^2-1\right)$$

$$= \lambda^2\left(\lambda^2-6\lambda+8\right)$$

$$\underline{= \lambda^2(\lambda-2)(\lambda-4)=0}\,\Big|$$

Therefore, the matrix has eigenvalues $\underline{\lambda_{1,2,3,4} = 0,\ 0,\ 2,\ 4}\,\Big|$

For $\lambda_{1,2} = 0$, then $(A-0)\vec{v}_1 = 0$

$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \rightarrow & 3x_1 + x_2 = 0 \\ \rightarrow & x_1 + 3x_2 = 0 \\ & \\ & \end{matrix}$$

$$\Rightarrow \underline{x_1 = x_2 = 0}\,\Big|$$

The eigenvectors are: $\underline{\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\quad \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}\,\Bigg|$

For $\lambda_3 = 2$, then $(A-2I)\vec{v}_3 = 0$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x_3 + y_3 = 0 \\ x_3 + y_3 = 0 \\ 2z_3 = 0 \\ 2w_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_3 = -y_3 \\ z_3 = w_3 = 0 \end{cases}$$

The eigenvectors are: $V_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ or $V_3 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}$

For $\lambda_4 = 4$, then $(A-4I)V_4 = 0$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -x_4 + y_4 = 0 \\ x_4 - y_4 = 0 \\ -4z_4 = 0 \\ -4w_4 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_4 = y_4 \\ z_4 = w_4 = 0 \end{cases}$$

The eigenvectors are: $V_4 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}$

$$P = \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$P^{-1} = P^T = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} & 0 & 0 \\ \frac{4}{\sqrt{2}} & \frac{4}{\sqrt{2}} & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad \checkmark$$

Exercise

Find the eigenvalues of A and B and check the Orthogonality of their first two eigenvectors. Graph these eigenvectors to see the discrete sines and cosines:

$$A=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad B=\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The $-1, 2, -1$ pattern in both matrices is a "second derivative. Then $A\vec{x}=\lambda\vec{x}$ and $B\vec{x}=\lambda\vec{x}$ are like $\frac{d^2\vec{x}}{dt^2}=\lambda\vec{x}$. This has eigenvectors $x=\sin kt$ and $x=\cos kt$ that are the bases for Fourier series. The matrices lead to "discrete sines" and "discrete cosines" that are the bases for the discrete Fourier Transform. This DFT is absolutely central to all areas of digital signal processing.

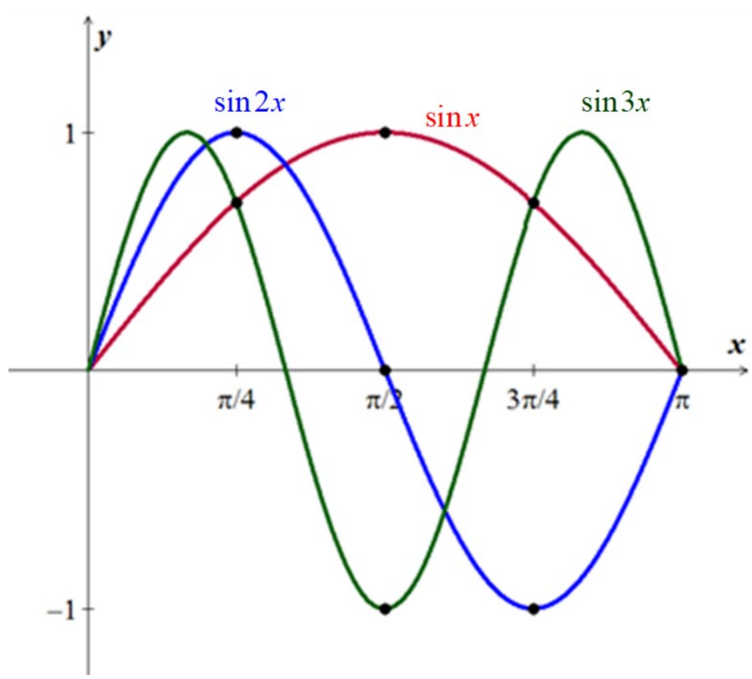
Solution

The eigenvalues of A are $\lambda = 2\pm\sqrt{2}$ *and* 2.

Their sum is 6 (the trace of A) and their product is 4 (the determinant).

The eigenvector matrix S gives the "Discrete Sine Transform".

$$S=\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$



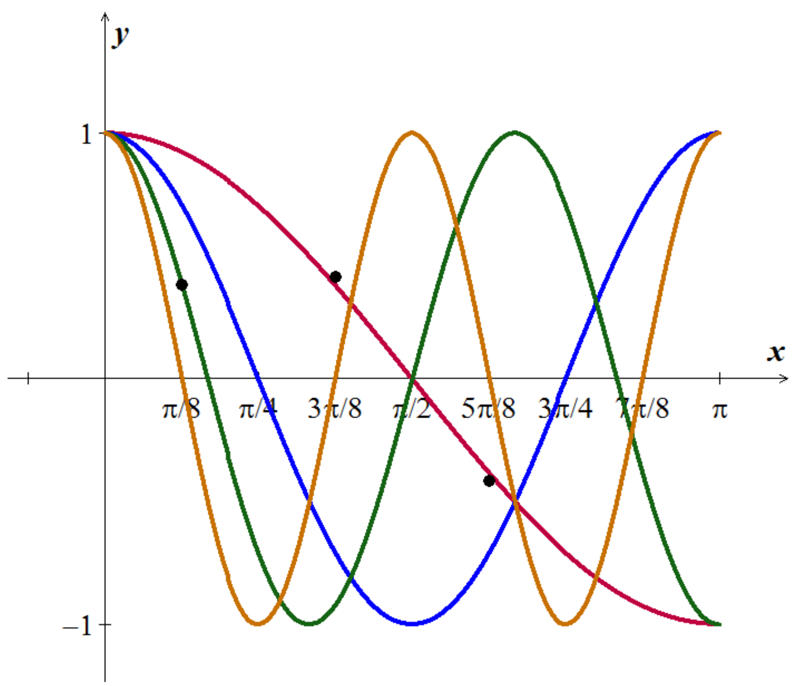
$$V_1 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} \frac{\sqrt{2}}{2} \\ 1 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$V_2 = \begin{pmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} \frac{\sqrt{2}}{2} \\ -1 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

The eigenvalues of B are $\lambda = 2 \pm \sqrt{2},\ \ 2,\ \ 0$.

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \sqrt{2}-1 & -1 & 1-\sqrt{2} \\ 1 & 1-\sqrt{2} & -1 & \sqrt{2}-1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$



Exercise

Suppose $A\vec{x} = \lambda\vec{x}$ and $A\vec{y} = 0\vec{y}$ and $\lambda \neq 0$. Then y is in the nullspace and $\vec{x}$ is in the column space. They are perpendicular because ________. (why are these subspaces orthogonal?) If the second eigenvalue is a nonzero number β, apply this argument to $A - \beta I$. The eigenvalue moves to zero and the eigenvectors stay the same – so they are perpendicular.

Solution

Suppose that $A = A^T$ and $Ax = \lambda x$, $Ay = 0y$, and $\lambda \neq 0$. Then x is in the column space of A, and y is in the left nullspace of A since $N(A) = N\left(A^T\right)$. But $C(A)$ and $N\left(A^T\right)$ are orthogonal complements, so x and y are perpendicular.

If $Ay = \beta y$ with $\beta \neq \lambda$ then $(A - \beta I)x = (\lambda - \beta)x$ and $(A - \beta I)y = 0$. Since $\lambda - \beta \neq 0$ it follows that x is in the column space A- βI and y is in the nullspace of A- βI, and $(A - \beta I)^T = A^T - \beta I^T = A - \beta I$, Therefore we can replace A with $A - \beta I$ in the argument of previous paragraph and it follows that x and y are perpendicular.

Exercise

Which of these classes of matrices do A and B belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad B = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Which of these factorizations are possible for A and B: LU, QR, ADP^{-1}, QDQ^T?

Solution

Matrix A is invertible, orthogonal, a permutation matrix, diagonalizable, and Markov! (Everything but a projection).

Matrix A satisfies $A^2 = I$, $A = A^T$, and also $AA^T = I$, This means it is invertible, symmetric, and orthogonal. Since it is symmetric, it is diagonalizable (with real eigenvalues!). It is a permutation matrix by just looking at it. It is Markov since the columns add to 1. It is not a projection since $A^2 = I \neq A$.

All of the factorization are possible for it: LU and QR are always possible. PDP^{-1} is possible since it is diagonalizable, and QDQ^T is possible since it is symmetric.

Matrix B is a projection, diagonalizable, and Markov. It is not invertible, not orthogonal, and not a permutation.

B is a projection since $B^2 = B$, it is symmetric and thus diagonalizable, and it is Markov since the columns add to 1. It is not invertible since the columns are visibly linearly dependent, it is not orthogonal since the columns are far from orthonormal, and it's clearly not a permutation.

All the factorizations are possible for it: LU and QR are always possible. PDP^{-1} is possible since it is diagonalizable, and QDQ^T is possible since it is symmetric.

Exercise

True or false. Give a reason or a counterexample.

a) A matrix with real eigenvalues and eigenvectors is symmetric.
b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.
c) The inverse of a symmetric matrix is symmetric
d) The eigenvector matrix S of a symmetric matrix is symmetric.
e) A complex symmetric matrix has real eigenvalues.
f) If A is symmetric, then e^{iA} is symmetric.
g) If A is Hermitian, then e^{iA} is Hermitian.
h) An $n \times n$ matrix that is orthogonally diagonalizable must be symmetric.
i) If $A^T = A$ and if vectors $\vec{u}$ and $\vec{v}$ satisfy $A\vec{u} = 3\vec{u}$ and $A\vec{v} = 4\vec{v}$, then $\vec{u} \bullet \vec{v} = 0$
j) An $n \times n$ symmetric matrix has n distinct real eigenvalues.
k) For nonzero $\vec{v}$ in $\mathbb{R}^n$, the matrix $\vec{v}\vec{v}^T$ is called a projection matrix.
l) Every symmetric matrix is orthogonally diagonalizable
m) If $B = PDP^T$, where $P^T = P^{-1}$ and D is a diagonal matrix, then B is a symmetric matrix.
n) An orthogonal matrix is orthogonally diagonalizable.
o) The dimension of an eigenspace of a symmetric matrix equals the multiplicity of the corresponding eigenvalue.

Solution

a) ***False***. Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

Then $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = 0$

So, A has eigenvalues $\lambda_1 = -1 \quad \lambda_2 = 2$

The eigenvectors are: $V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ so both the eigenvalues and eigenvectors are real but A is not symmetric.

b) True. If the matrix A has orthogonal eigenvectors $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n$ with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, we can define $\vec{s}_i = \frac{\vec{x}_i}{\|\vec{x}_i\|}$ for all i; then $A\vec{s}_i = \lambda_i \vec{s}_i$ for all i and the s_i are orthonormal. Then we can diagonalize A as: $A = S\Lambda S^{-1}$ where the i^{th} column of S is $\vec{s}_i$, and Λ is the diagonal matrix, so $S^T = S^{-1}$ and $A = S\Lambda S^T$.

$$A^T = \left(S^T\right)^T \Lambda^T S^T$$
$$= S\Lambda S^T$$
$$= A$$

So, A is symmetric.

c) True. If A is symmetric then it can be diagonalized by an orthogonal matrix Q, $A = QDQ^{-1}$, and then $A^{-1} = QD^{-1}Q^{-1} = QD^{-1}Q^T$. Since D^{-1} is still a diagonal matrix, it follows:

$$\left(A^{-1}\right)^T = QD^{-1}Q^T$$
$$= A^{-1}$$

d) False. The eigenvalues of $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ are: $\lambda_1 = 0 \quad \lambda_2 = 5$ and the eigenvectors are:

$$V_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We can diagonalize A with eigenvector matrix $S = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ which is not symmetric.

e) False. For example, $A = (i)$, the 1 by 1 matrix. The eigenvalue is ***i***, it is not a real number.

f) True. $\left(e^{iA}\right)^T = e^{(iA)^T} = e^{iA}$

g) False. $\left(e^{iA}\right)^H = e^{(iA)^H} = e^{-iA^H} = e^{-iA}$. It is typically not the same as e^{iA}.

Taking $A = (1)$, the 1 by 1 matrix, would be an enough example because $e^{iA} = e^i$ which is not a real number.

p) An $n \times n$ matrix that is orthogonally diagonalizable must be symmetric.

a. True. See Theorem 2 and the paragraph preceding the theorem.

q) If $A^T = A$ and if vectors $\vec{u}$ and $\vec{v}$ satisfy $A\vec{u} = 3\vec{u}$ and $A\vec{v} = 4\vec{v}$, then $\vec{u} \bullet \vec{v} = 0$

r) An $n \times n$ symmetric matrix has *n* distinct real eigenvalues.

s) For nonzero $\vec{v}$ in $\mathbb{R}^n$, the matrix $\vec{v}\vec{v}^T$ is called a projection matrix.

t) Every symmetric matrix is orthogonally diagonalizable

u) If $B = PDP^T$, where $P^T = P^{-1}$ and *D* is a diagonal matrix, then *B* is a symmetric matrix.

v) An orthogonal matrix is orthogonally diagonalizable.

w) The dimension of an eigenspace of a symmetric matrix equals the multiplicity of the corresponding eigenvalue.

b. True. This is a particular case of the statement in Theorem 1, where **u** and **v** are nonzero.

c. False. There are *n* real eigenvalues (Theorem 3), but they need not be distinct (Example 3).

d. False. See the paragraph following formula (2), in which each **u** is a unit vector.

a. True. See Theorem 2.

b. True. See the displayed equation in the paragraph before Theorem 2.

c. False. An orthogonal matrix can be symmetric (and hence orthogonally diagonalizable), but not every orthogonal matrix is symmetric. See the matrix *P* in Example 2.

d. True. See Theorem 3(b).

Exercise

Find a symmetric matrix $\begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$ that has a negative eigenvalue.

a) How do you know it must have a negative pivot?

b) How do you know it can't have two negative eigenvalues?

Solution

a) The eigenvalues of that matrix are $1 \pm b$ / so take any $b < -1 \quad b > 1$. In this case, the determinant is $1 - b^2 < 0$.

b) The signs of the pivots coincide with the signs of the eigenvalues. Alternatively, the product of the pivots is the determinant, which is negative in this case. So, one of the two pivots must be negative.

c) The product of the eigenvalues equals the determinant, which is negative in this case. So, precisely one numbers cannot have a negative product.

Exercise

Prove that *A* is any $m \times n$ matrix, then $A^T A$ has an orthonormal set of *n* eigenvectors

Solution

$\left(A^T A\right)^T = A^T\left(A^T\right)^T = A^T A$, then $A^T A$ is symmetric, therefore there is an eigenvector $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ for $A^T A$.

Let $A\vec{v}_1 = \lambda_1 \vec{v}_1 \quad and \quad A\vec{v}_2 = \lambda_2 \vec{v}_2$

$A\vec{v}_1 \bullet \vec{v}_2 = \vec{v}_1 \bullet A^T \vec{v}_2$ $\qquad A\vec{x} \bullet \vec{y} = \vec{x} \bullet A^T \vec{y}$

$A\vec{v}_1 \bullet \vec{v}_2 = \vec{v}_1 \bullet A\vec{v}_2$ $\qquad A^T = A \ \ (Since\ A\ is\ a\ symmetric)$

$\lambda_1 \vec{v}_1 \bullet \vec{v}_2 = \vec{v}_1 \bullet \lambda_2 \vec{v}_2$

$\lambda_1\left(\vec{v}_1 \bullet \vec{v}_2\right) = \lambda_2\left(\vec{v}_1 \bullet \vec{v}_2\right)$

$\left(\lambda_1 - \lambda_2\right)\left(\vec{v}_1 \bullet \vec{v}_2\right) = 0$ $\qquad$ Since $\lambda_1 \neq \lambda_2$

Therefore; $\vec{v}_1 \bullet \vec{v}_2 = 0$

Then the vectors $A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_n$ are orthogonal

$$
\begin{aligned}
A\vec{v}_1 \bullet A\vec{v}_2 &= \left(A\vec{v}_i\right)^T A\vec{v}_j \\
&= \vec{v}_i^{\,T} A^T A\vec{v}_j \\
&= \vec{v}_i \bullet \left(A^T A\vec{v}_j\right) \\
&= \vec{v}_i \bullet \left(\lambda_j \vec{v}_j\right) \\
&= \lambda_j\left(\vec{v}_i \bullet \vec{v}_j\right) = 0
\end{aligned}
$$

Example

Construct a 3 by 3 matrix A with no zero entries whose columns are mutually perpendicular. Compute $A^T A$. Why is it a diagonal matrix?

Solution

Consider the matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ to be columns mutual perpendicular

Let assume $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{pmatrix}$ $\quad then \quad A^T = \begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \\ 2 & 1 & 2 \end{pmatrix}$ $\qquad or \ \ A = \begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & -1 \\ 2 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$\left(A^T A\right)_{ij} = (\text{column } i \text{ of } A)(\text{column } j \text{ of } A)$$

Exercise

Assuming that $b \neq 0$, find a matrix that orthogonally diagonalizes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix}$$

$$= (a-\lambda)^2 - b^2$$

$$= (a-\lambda-b)(a-\lambda+b)$$

$$= (a-b-\lambda)(a+b-\lambda) = 0$$

Therefore; the eigenvalues are: $\lambda_1 = a-b \quad and \quad \lambda_2 = a+b$

Assume that $b \neq 0$.

For $\lambda_1 = a-b$, then $\left(A - (a-b)I\right)V_1 = 0$

$$\begin{pmatrix} b & b \\ b & b \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad bx_1 = -by_1$$

The eigenvectors are: $V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For $\lambda_2 = a+b$, then $\left(A - (a+b)I\right)V_2 = 0$

$$\begin{pmatrix} -b & b \\ b & -b \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad bx_2 = by_2$$

The eigenvectors are: $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Applying the Gram Schmidt process.

$$\vec{u}_1 = \frac{V_1}{\left\|V_1\right\|}$$

$$= \frac{(-1,\ 1)}{\sqrt{2}}$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\vec{w}_2 = V_2 - \frac{\left\langle \vec{u}_1, V_2 \right\rangle}{\left\|V_2\right\|^2}$$

$$= (1,\ 1) - \frac{(-1,\ 1) \bullet (1,\ 1)}{2}(1,\ 1)$$

$$= (1,\ 1)$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$$

$$= \frac{(1,\ 1)}{\sqrt{2}}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Exercise

Suppose A is a symmetric $n \times n$ matrix and B is any $n \times m$ matrix. Show that $B^T AB$, $B^T B$, and BB^T are symmetric matrices.

Solution

A is a symmetric, that implies to $A = A^T$

$$\left(B^T AB\right)^T = B^T A^T \left(B^T\right)^T$$

$$= B^T AB \quad \checkmark$$

Since, $\left(B^T AB\right)^T = B^T AB$, then $B^T AB$ is symmetric.

$$\left(B^T B\right)^T = B^T\left(B^T\right)^T$$

$$\underline{= B^T B} \quad \checkmark$$

Therefore, $B^T B$ is symmetric.

$$\left(BB^T\right)^T = \left(B^T\right)^T B^T$$

$$\underline{= BB^T} \quad \checkmark$$

Therefore, BB^T is symmetric.

Exercise

Show that if A is an $n \times n$ symmetric matrix, then $(A\vec{x}) \bullet \vec{y} = \vec{x} \bullet (A\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$

Solution

A is a symmetric, that implies to $A = A^T$

$$(A\vec{x}) \bullet \vec{y} = (A\vec{x})^T \bullet \vec{y}$$

$$= \vec{x}^T A^T \bullet \vec{y}$$

$$= \vec{x}^T A \bullet \vec{y}$$

$$\underline{= \vec{x} \bullet (A\vec{y})} \quad \checkmark$$

Exercise

Suppose A is invertible and orthogonally diagonalizable. Explain why A^{-1} is also orthogonally diagonalizable.

Solution

Since A is invertible, then $AA^{-1} = A^{-1}A = I$

And A is orthogonally diagonalizable, then $A = PDP^{-1}$

$$(A)^{-1} = \left(PDP^{-1}\right)^{-1}$$

$$A^{-1} = \left(P^{-1}\right)^{-1} D^{-1}P^{-1} \qquad D^{-1} = D \ \text{(is a diagonal matrix)}$$

$$= PDP^{-1} \quad \checkmark$$

Therefore, A^{-1} is also orthogonally diagonalizable.

Exercise

Suppose A and B are both orthogonally diagonalizable and $AB = BA$. Explain why AB is also orthogonally diagonalizable

Solution

Since A and B are both orthogonally diagonalizable, and A and B are symmetric, then

$$A = A^T \quad \& \quad B = B^T$$

If $AB = BA$, then

$$\begin{aligned}(AB)^T &= (BA)^T \\ &= A^T B^T \qquad \text{A and B are symmetric} \\ &= AB \quad \checkmark\end{aligned}$$

Therefore, AB is also orthogonally diagonalizable

Exercise

Let $A = PDP^{-1}$, where P is orthogonal and D is diagonal, let λ be an eigenvalue of A of multiplicity k. Then λ appears k times on the diagonal of D. Explain why the dimension of the eigenspace for λ is k.

Solution

The columns of P are linearly independent eigenvectors by the ***diagonalization*** theorem corresponding to the eigenvalues λ of A.

Since D is a diagonal with the eigenvalues λ. when the eigenvalues λ is of multiplicity k, then λ appears k times on the diagonal of D.

So, P has exactly k columns of eigenvectors corresponding to the eigenvalues λ.

Therefore, the k columns form a basis for the eigenspace.

Exercise

Suppose $A = PUP^{-1}$, where P is orthogonal and U is an upper triangular. Show that if A is symmetric, then U is symmetric and hence is actually a diagonal matrix.

Solution

Given: $A = PUP^{-1}$

If A is symmetric, then $A = A^T$

$$A = PUP^{-1}$$

$$P^{-1}AP = P^{-1}PUP^{-1}P$$

$$U = P^{-1}AP$$

Since P is orthogonal, then

$$U = P^{-1}AP = P^T AP$$

$$U^T = \left(P^T AP\right)^T$$

$$= P^T A^T \left(P^T\right)^T \qquad \textit{A is symmetric}$$

$$= P^T AP$$

That implies that U is symmetric

Since U is an upper triangular and symmetric, then the entries above and below the main diagonal must be equal to zeros.

Therefore, U is a diagonal matrix.

Exercise

Let $\vec{u}$ be a unit vector in $\mathbb{R}^n$, and let $B = \vec{u}\,\vec{u}^T$.

a) Given $\vec{x} \in \mathbb{R}^n$, compute $B\vec{x}$ and show that $B\vec{x}$ is the orthogonal projection of $\vec{x}$ onto $\vec{u}$.

b) Show that B is a symmetric matrix and $B^2 = B$.

c) Show that $\vec{u}$ is an eigenvector of B. What is the corresponding eigenvalue?

Solution

a) Given $\vec{x} \in \mathbb{R}^n$

$$B\vec{x} = \left(\vec{u}\,\vec{u}^T\right)\vec{x}$$

$$= \vec{u}\left(\vec{u}^T\vec{x}\right)$$

$$= \left(\vec{u}^T\vec{x}\right)\vec{u} \qquad \vec{u}^T\vec{x} : \textit{scalar}$$

$$= \left(\vec{x} \bullet \vec{u}\right)\vec{u}$$

Given that $\vec{u}$ is a unit vector in $\mathbb{R}^n$, then $B\vec{x}$ is the orthogonal projection of $\vec{x}$ onto $\vec{u}$.

b) $B^T = \left(\vec{u}\,\vec{u}^T\right)^T$

$$= \left(\vec{u}^T\right)^T \vec{u}^T$$

$$= \vec{u}\,\vec{u}^T$$

$$= B \qquad \checkmark$$

Therefore, B is symmetric.

$$B^2 = \left(\vec{u}\vec{u}^T\right)^2$$
$$= \left(\vec{u}\vec{u}^T\right)\left(\vec{u}\vec{u}^T\right)$$
$$= \vec{u}\left(\vec{u}^T\vec{u}\right)\vec{u}^T \qquad\qquad \vec{u}^T\vec{u} = 1$$
$$= \vec{u}\,\vec{u}^T$$
$$\underline{= B} \qquad \checkmark$$

Therefore, B^2 is symmetric.

c) Since $\vec{u}^T\vec{u} = 1$, then

$$B\vec{u} = \left(\vec{u}\,\vec{u}^T\right)\vec{u}$$
$$B\vec{u} = \vec{u}\left(\vec{u}^T\,\vec{u}\right)$$
$$= \vec{u}\,(1)$$
$$\underline{= \vec{u}}$$

Therefore, $\vec{u}$ is an eigenvector of B with the corresponding eigenvalue 1.

Exercise

Let B be an $n \times n$ symmetric matrix such that $B^2 = B$. Any such matrix is called a ***projection matrix*** (or an *orthogonal projection matrix*). Given any $\vec{y} \in \mathbb{R}^n$, let $\hat{y} = B\vec{y}$ and $\vec{z} = \vec{y} - \hat{y}$.

a) Show that $\vec{z}$ is orthogonal to $\hat{y}$.

b) Let W be the column space of B. Show that $\vec{y}$ is the sum of a vector in W and a vector in $W^{\perp}$. Why does this prove that $B\vec{y}$ is the orthogonal projection of $\vec{y}$ onto the column space of B?

Solution

Since B is symmetric, then $B = B^T$

Given too, that $B^2 = B$ which is symmetric.

$\hat{y} = B\vec{y}$

a) $\vec{z} \cdot \hat{y} = \left(\vec{y} - \hat{y}\right) \cdot B\vec{y}$

$$= \vec{y} \cdot \left(B\vec{y}\right) - \hat{y} \cdot \left(B\vec{y}\right)$$
$$= \vec{y}^T \cdot \left(B\vec{y}\right) - \left(B\vec{y}\right)^T \cdot \left(B\vec{y}\right)$$
$$= \vec{y}^T \cdot B\vec{y} - \vec{y}^T B^T \cdot B\vec{y}$$
$$= \vec{y}^T \cdot B\vec{y} - \vec{y}^T B \cdot B\vec{y} \qquad\qquad B^2 = B$$

$= \vec{y}^T B\vec{y} - \vec{y}^T B\vec{y}$

$\underline{= 0}$

Therefore, $\vec{z}$ is orthogonal to $\hat{y}$

b) Since W be the column space of B, then $W = Col(B)$ has the form $B\vec{u}$ (for some $\vec{u}$)

$$\begin{aligned}(\vec{y} - \hat{y}) \cdot (B\vec{u}) &= B(\vec{y} - \hat{y}) \cdot \vec{u} \\ &= (B\vec{y} - B\hat{y}) \cdot \vec{u} \\ &= (B\vec{y} - BB\vec{y}) \cdot \vec{u} \qquad\qquad B^2 = B \\ &= (B\vec{y} - B\vec{y}) \cdot \vec{u} \\ &\underline{= 0}\end{aligned}$$

Therefore, $(\vec{y} - \hat{y})$ is in $W^{\perp}$, and the decomposition $\vec{y} = \hat{y} + (\vec{y} - \hat{y})$ expresses $\vec{y}$ as the sum of a vector in W and a vector in $W^{\perp}$.

By the orthogonal Decomposition, this decomposition is unique, and so $\hat{y}$ must be orthogonal projection of $\vec{y}$ onto the column space of B (W)