Solution   Section 3.5 – Determinants and Cramer's Rule

Exercise

Verify that $\det(AB) = \det(A)\det(B)$ when: $A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{pmatrix}$

Solution

$$AB = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{pmatrix}$$

$$\det(AB) = \begin{vmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{vmatrix}\begin{matrix} 9 & -1 \\ 31 & 1 \\ 10 & 0 \end{matrix} = -170$$

$$\det(A) = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 10$$

$$\det(B) = \begin{vmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{vmatrix} = -17$$

$$\det(AB) = \det(A)\det(B) \quad = -170 \quad \checkmark$$

Exercise

For which value(s) of $\boldsymbol{k}$ does A fail to be invertible? $A = \begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix}$

Solution

For $\boldsymbol{A}$ to have an invertible the determinant cannot be equal to zero. To ***fail*** det(A) = 0.

$$|A| = \begin{vmatrix} k-3 & -2 \\ -2 & k-2 \end{vmatrix} = 0$$

$$(k-3)(k-2) - 4 = 0$$

$$k^2 - 5k + 6 - 4 = 0$$

$$k^2 - 5k + 2 = 0 \Rightarrow \boxed{k = \frac{5 \pm \sqrt{17}}{2}}$$

Exercise

Without directly evaluating, show that $\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$

Solution

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \overset{R_2+R_1}{=} \begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$

It is equal to zero, since first row and third row are proportional.

$$\begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \underset{R_3 - \frac{1}{a+b+c}R_1}{=} \begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Exercise

If the entries in every row of A add to zero, solve $\boldsymbol{Ax} = 0$ to prove det $A = 0$. If those entries add to one, show that det $(A - I) = 0$. Does this mean det $A = I$?

Solution

If $\boldsymbol{x} = (1, 1, \dots, 1)$, then $\boldsymbol{Ax}$ = the sums of the rows of $\boldsymbol{A}$. Since every row of A add to zero, that implies $\boldsymbol{Ax} = 0$. Since A has non-zero nullspace, it is not invertible and det $A = 0$. If the entries in every row of A sum to one, then the entries in every row of $A - \mathrm{I}$ sum to zero. A – I has a non-zero nullspace and det $(A - \mathrm{I}) = 0$. This does not mean that det $A = \mathrm{I}$.

Example: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ every row of A add to zero $\Rightarrow \det A = -1 \neq 1 = \det I$

Exercise

Does $\det(AB) = \det(BA)$ in general?

a) True or false if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square n x n matrices?

b) True or false if $\boldsymbol{A}$ is m x n and B is n x m with $m \neq n$?

Solution

a) Matrices A and B are square matrices, then by the property:

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$$

Therefore it is true for any $\boldsymbol{A}$ and $\boldsymbol{B}$ square matrices.

b) False, example if $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \end{pmatrix}$

$$\det AB = \det\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix}\right] = \det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$$

$$\det AB = \det\left[\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] = \det(2) = 2$$

Exercise

True or false, with a reason if true or a counterexample if false:

a) The determinant of $I + A$ is 1 + det $\boldsymbol{A}$.

b) The determinant of ABC is $|A||B||C|$.

c) The determinant of 4A is $4|A|$

d) The determinant of $AB - BA$ is zero. (try an example)

e) If A is not invertible then AB is not invertible.

f) The determinant of $A - B$ equals to det A – det B.

Solution

a) ***False***, if $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \det(I + A) = \det\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

$$\det A = 1 \Rightarrow 1 + \det A = 1 + 1 = 2 \ \underline{\neq \det(I + A)}$$

b) ***True***, $\det(ABC) = \det(A)\det(BC) = \det(A)\det(B)\det(C)$.

c) ***False***, in general $\det(4A) = 4^n \det(A)$ if A is n x n.

d) ***False***, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{aligned} AB - BA &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

$$\det(AB - BA) = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 1 \neq 0$$

e) False, any matrix is invertible, iff its determinant is nonzero. So det A = 0 which

$\det(AB)=\det(A)\det(B)=0$. Therefore, AB can't be invertible.

f) $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}\Rightarrow|A|=0,\quad B=\begin{pmatrix}0&1\\1&0\end{pmatrix}\Rightarrow|B|=-1$

$\det(A)-\det(B)=0-(-1)\ =1$

$\det(A-B)=\det\begin{pmatrix}1&-1\\-1&0\end{pmatrix}\ =-1$ $\quad\Rightarrow\det(A-B)\neq\det(A)-\det(B)$

Exercise

Use row operations to show the 3 by 3 "Vandermonde determinant" is

$$\det\begin{bmatrix}1&a&a^2\\1&b&b^2\\1&c&c^2\end{bmatrix}=(b-a)(c-a)(c-b)$$

Solution

$$\det\begin{bmatrix}1&a&a^2\\1&b&b^2\\1&c&c^2\end{bmatrix}=\det\begin{bmatrix}1&a&a^2\\1&b&b^2\\1&c&c^2\end{bmatrix}\begin{matrix}\\R_2-R_1\\R_3-R_1\end{matrix}$$

$$=\det\begin{bmatrix}1&a&a^2\\0&b-a&b^2-a^2\\0&c-a&c^2-a^2\end{bmatrix}\ \ factor\,(b-a)$$

$$=(b-a)\det\begin{bmatrix}1&a&a^2\\0&1&b+a\\0&c-a&c^2-a^2\end{bmatrix}\ R_2-(c-a)R_2$$

$$(c-a)(c+a)-(b+a)(c-a)=(c-a)(c+a-b-a)$$

$$=(b-a)\det\begin{bmatrix}1&a&a^2\\0&1&b+a\\0&0&(c-a)(c-b)\end{bmatrix}\ \ \textit{Multiply the main diagonal by (b - a)}$$

$$=(b-a)(c-a)(c-b)$$

Exercise

The inverse of a 2 by 2 matrix seems to have determinant = 1:

$$\det A^{-1} = \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{ad-bc}{ad-bc} = 1$$

What is wrong with this calculation? What is the correct $\det A^{-1}$

Solution

The $\det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $(ad-bc)$ it is part of the determinant and it is not the solution.

$$\det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\det\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad-bc}\frac{1}{ad-bc}(ad-bc)$$

$$= \frac{1}{ad-bc}$$

Exercise

A *Hessenberg* matrix is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci's rule $|H_4| = |H_3| + |H_2|$. The same rule will continue for all sizes $|H_n| = |H_{n-1}| + |H_{n-2}|$. Which Fibonacci number is $|H_n|$?

$$H_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad H_3 = \begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad H_4 = \begin{bmatrix} 2 & 1 & & \\ \mathbf{1} & 2 & \mathbf{1} & \\ \mathbf{1} & 1 & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

Solution

$$|H_4| = 2C_{11} + 1C_{12}$$

The cofactor C_{11} for H_4 is the determinant $|H_3|$.

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

The cofactor $C_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$

$$= -\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= -\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} \qquad \left|H_2\right| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= -\left|H_3\right| + \left|H_2\right|$$

$$\left|H_4\right| = 2C_{11} + 1C_{12}$$

$$= 2\left|H_3\right| - \left|H_3\right| + \left|H_2\right|$$

$$= \left|H_3\right| + \left|H_2\right|$$

The actual number: $\left|H_2\right| = 3, \quad \left|H_3\right| = 5, \quad H_4 = 8$.

Since $\left|H_n\right|$ follows Fibonacci's rule $\left|H_{n-1}\right| + \left|H_{n-2}\right|$, it must be $\left|H_n\right| = F_{n+2}$.

Exercise

Evaluate the determinant:

a) $\begin{vmatrix} -1 & 7 \\ -8 & -3 \end{vmatrix}$

b) $\begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix}$

c) $\begin{vmatrix} k-1 & 2 \\ 4 & k-3 \end{vmatrix}$

d) $\begin{vmatrix} c & -4 & 3 \\ 2 & 1 & c^2 \\ 4 & c-1 & 2 \end{vmatrix}$

e) $\begin{vmatrix} 1 & 0 & 3 \\ 4 & 0 & -1 \\ 2 & 8 & 6 \end{vmatrix}$

g) $\begin{vmatrix} -3 & 1 & 2 \\ 6 & 2 & 1 \\ -9 & 1 & 2 \end{vmatrix}$

h) $\begin{vmatrix} 1 & 4 & -3 & 1 \\ 2 & 0 & 6 & 3 \\ 4 & -1 & 2 & 5 \\ 1 & 0 & -2 & 4 \end{vmatrix}$

i) $\begin{vmatrix} 1 & 3 & 2 & -1 \\ 4 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 5 & 1 & 3 & -2 \end{vmatrix}$

j) $\begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & 1 & -2 & 0 \\ 1 & 0 & 3 & -3 \end{vmatrix}$

f) $\begin{vmatrix} x & -3 & 9 \\ 2 & 4 & x+1 \\ 1 & x^2 & 3 \end{vmatrix}$

Solution

a) $\begin{vmatrix} -1 & 7 \\ -8 & -3 \end{vmatrix} = (-1)(-3)-(7)(-8)\ \boxed{=59}$

b) $\begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix} = (a-3)(a-2)+15$

$$= a^2-5a+6+15$$

$$\boxed{= a^2-5a+21}$$

c) $\begin{vmatrix} k-1 & 2 \\ 4 & k-3 \end{vmatrix} = (k-1)(k-3)-8$

$$= k^2-4k+3-8$$

$$\boxed{= k^2-4k-5}$$

d) $\begin{vmatrix} c & -4 & 3 \\ 2 & 1 & c^2 \\ 4 & c-1 & 2 \end{vmatrix} = 2c-16c^2+6c-6-12-c^4+c^3+16\ \boxed{=-c^4+c^3-16c^2+8c-2}$

e) $\begin{vmatrix} 1 & 0 & 3 \\ 4 & 0 & -1 \\ 2 & 8 & 6 \end{vmatrix} = 0+0+96-0+8-0\ \boxed{=104}$

f) $\begin{vmatrix} x & -3 & 9 \\ 2 & 4 & x+1 \\ 1 & x^2 & 3 \end{vmatrix} = 12x-3(x+1)+18x^2-36-x^3(x+1)+18$

$$= 12x-3x-3+18x^2-36-x^4-x^3+18$$

$$\boxed{= -x^4-x^3+18x^2+9x-21}$$

g) $\begin{vmatrix} -3 & 1 & 2 \\ 6 & 2 & 1 \\ -9 & 1 & 2 \end{vmatrix} = -12-9+12+36+3-12\ \boxed{=18}$

h) $\begin{vmatrix} 1 & 4 & -3 & 1 \\ 2 & 0 & 6 & 3 \\ 4 & -1 & 2 & 5 \\ 1 & 0 & -2 & 4 \end{vmatrix}\ \boxed{=275}$

i) $\begin{vmatrix} 1 & 3 & 2 & -1 \\ 4 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 5 & 1 & 3 & -2 \end{vmatrix} = 0$ Since row 3 has zero.

j) $\begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & 1 & -2 & 0 \\ 1 & 0 & 3 & -3 \end{vmatrix} = (2)(-1)(-2)(-3) \; \underline{= -12}$

Exercise

Find all the values of λ for which det($\boldsymbol{A}$) = 0: $A = \begin{bmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 4 \end{bmatrix}$

Solution

$$\begin{vmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 4 \end{vmatrix} = (\lambda - 1)(\lambda - 4) + 2$$

$$= \lambda^2 - 5\lambda + 4 + 2$$

$$= \lambda^2 - 5\lambda + 6 = 0 \qquad \textit{Solve for } \lambda.$$

$$\boxed{\lambda = -1,\ 6}$$

Exercise

Find all the values of λ for which det($\boldsymbol{A}$) = 0: $A = \begin{bmatrix} \lambda - 6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda - 4 \end{bmatrix}$

Solution

$$\begin{vmatrix} \lambda - 6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda - 4 \end{vmatrix} = \lambda(\lambda - 6)(\lambda - 4) + 4(\lambda - 6)$$

$$= \lambda\left(\lambda^2 - 10\lambda + 24\right) + 4\lambda - 24$$

$$= \lambda^3 - 10\lambda^2 + 24\lambda + 4\lambda - 24$$

$$= \lambda^3 - 10\lambda^2 + 28\lambda - 24$$

$$\lambda^3 - 10\lambda^2 + 28\lambda - 24 = 0 \rightarrow \boxed{\lambda = 2,\ 2,\ 6}$$

Exercise

Prove that if a square matrix $\boldsymbol{A}$ has a column of zeros, then det($\boldsymbol{A}$) = 0

Solution

Consider a 3 by 3 matrix with a zero column, however to find the determinant we can interchange any column of that matrix; therefore:

$$\boldsymbol{A}=[a_{ij}]=\begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$$

By definition, the determinant of $\boldsymbol{A}$ using the cofactor:

$$|A|=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$=0\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} 0 & a_{23} \\ 0 & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} 0 & a_{22} \\ 0 & a_{32} \end{vmatrix}$$

$$\underline{=0|}$$

Exercise

With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix}=|A||D| \quad \textit{but} \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix}\neq|A||D|-|C||B|$$

a) Why is the first statement true? Somehow B doesn't enter.

b) Show by example that equality fails (as shown) when C enters.

c) Show by example that the answer $\det(AD-CB)$ is also wrong.

Solution

a) If we don't pick any 0 entries, then the first two columns are picked from $\boldsymbol{A}$ and the last two rows are from D. We can't pick any columns or rows from B, because there aren't any left.

b) $$\left|\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right|=1.$$

and $A=\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}=0, \quad B=\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}=0, \quad C=\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}=0, \quad D=\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}=0$

c) Use the example from part (b): $1\neq 0$ $\quad \begin{vmatrix} A & B \\ C & D \end{vmatrix}\neq|A||D|-|C||B|$

Exercise

Show that the value of the following determinant is independent of θ.

$$\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta-\cos\theta & \sin\theta+\cos\theta & 1 \end{vmatrix}$$

Solution

$$\begin{aligned} \begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta-\cos\theta & \sin\theta+\cos\theta & 1 \end{vmatrix} &= \begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} \\ &= \sin^2\theta+\cos^2\theta \\ &= \sin^2\theta-\left(-\cos^2\theta\right) \\ &= 1 \end{aligned}$$

Therefore, the determinant is independent of θ.

Exercise

Show that the matrices $A=\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ *and* $B=\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$ commute if and only if $\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix}=0$

Solution

$$AB=\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} d & e \\ 0 & f \end{pmatrix}=\begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix}$$

$$BA=\begin{pmatrix} d & e \\ 0 & f \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}=\begin{pmatrix} da & db+ec \\ 0 & fc \end{pmatrix}$$

$$AB=BA \quad \Rightarrow \quad \begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix}=\begin{pmatrix} da & db+ec \\ 0 & fc \end{pmatrix}$$

Iff $ae+bf=db+ec$

$$\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix}=b(d-f)-e(a-c)=bd-bf-ea+ec=0$$

$$\boxed{bd+ec=bf+ae} \quad \checkmark$$

Exercise

Show that $\det(A) = \frac{1}{2}\begin{vmatrix} tr(A) & 1 \\ tr(A^2) & tr(A) \end{vmatrix}$ for every 2×2 matrix A.

Solution

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow tr(A) = a + d$

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} \Rightarrow tr(A^2) = a^2+bc+bc+d^2$$

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{aligned} \frac{1}{2}\begin{vmatrix} tr(A) & 1 \\ tr(A^2) & tr(A) \end{vmatrix} &= \frac{1}{2}\begin{vmatrix} a+d & 1 \\ a^2+bc+bc+d^2 & a+d \end{vmatrix} \\ &= \frac{1}{2}\left[(a+d)^2 - \left(a^2+bc+bc+d^2\right)\right] \\ &= \frac{1}{2}\left(a^2+2ad+d^2-a^2-bc-bc-d^2\right) \\ &= \frac{1}{2}(2ad-2bc) \\ &= ad-bc \\ &= \det(A) \end{aligned}$$

Exercise

What is the maximum number of zeros that a 4×4 matrix can have without a zero determinant? Explain your reasoning.

Solution

The maximum number of zeros that a 4×4 matrix can have without a zero determinant is 12 zeros. If the main diagonal has nonzero entries and the rest are zero, then the determinant of the matrix is equal to the product of the main diagonal entries.

Exercise

Evaluate *det* ***A***, *det* ***E***, and *det* (***AE***). Then verify that (*det* ***A***)(*det* ***E***) = *det*(***AE***)

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & & \\ & 3 & \\ & & 1 \end{bmatrix}$$

Solution

$$\det(A)=\begin{vmatrix}4&1&3\\0&-2&0\\3&1&5\end{vmatrix}=-40+18\ \underline{=-22}$$

$$\det(E)=\begin{vmatrix}1&&\\&3&\\&&1\end{vmatrix}\ \underline{=3}$$

$$AE=\begin{bmatrix}4&1&3\\0&-2&0\\3&1&5\end{bmatrix}\begin{bmatrix}1&0&0\\0&3&0\\0&0&1\end{bmatrix}=\begin{bmatrix}4&3&3\\0&-6&0\\3&3&5\end{bmatrix}$$

$$\det(AE)=\begin{vmatrix}4&3&3\\0&-6&0\\3&3&5\end{vmatrix}=-120+54\ \underline{=-66}$$

$$\det(A)\det(E)=(-22)(3)\ \underline{=-66}$$

$$\det(A)\det(E)=\det(AE)\qquad \checkmark$$

Exercise

Show that $\begin{bmatrix}\sin^2\alpha&\sin^2\beta&\sin^2\gamma\\ \cos^2\alpha&\cos^2\beta&\cos^2\gamma\\ 1&1&1\end{bmatrix}$ is not invertible for any values of $\alpha,\ \beta,\ \gamma$

Solution

$$\begin{vmatrix}\sin^2\alpha&\sin^2\beta&\sin^2\gamma\\ \cos^2\alpha&\cos^2\beta&\cos^2\gamma\\ 1&1&1\end{vmatrix}=\sin^2\alpha\cos^2\beta+\sin^2\beta\cos^2\gamma+\sin^2\gamma\cos^2\alpha$$
$$-\sin^2\gamma\cos^2\beta-\sin^2\alpha\cos^2\gamma-\cos^2\alpha\sin^2\beta$$

$$=\sin^2\alpha\left(\cos^2\beta-\cos^2\gamma\right)+\cos^2\alpha\left(\sin^2\gamma-\sin^2\beta\right)+\sin^2\beta\cos^2\gamma-\sin^2\gamma\cos^2\beta$$

$$=\sin^2\alpha\left(\cos^2\beta-\cos^2\gamma\right)+\cos^2\alpha\left(1-\cos^2\gamma-1+\cos^2\beta\right)+\left(1-\cos^2\beta\right)\cos^2\gamma-\left(1-\cos^2\gamma\right)\cos^2\beta$$

$$=\sin^2\alpha\left(\cos^2\beta-\cos^2\gamma\right)+\cos^2\alpha\left(\cos^2\beta-\cos^2\gamma\right)+\cos^2\gamma-\cos^2\gamma\cos^2\beta-\cos^2\beta+\cos^2\gamma\cos^2\beta$$

$$=\left(\sin^2\alpha++\cos^2\alpha\right)\left(\cos^2\beta-\cos^2\gamma\right)+\cos^2\gamma-\cos^2\beta$$

$$=\cos^2\beta-\cos^2\gamma+\cos^2\gamma-\cos^2\beta$$

$\underline{=0}$ Therefore, this matrix in not invertible.

Exercise

Use Cramer's Rule with ratios $\frac{\det B_j}{\det A}$ to solve $A\boldsymbol{x} = b$. Also find the inverse matrix $A^{-1} = \frac{C^T}{\det A}$. Why is the solution $\boldsymbol{x}$ is the first part the same as column 3 of A^{-1}? Which cofactors are involved in computing that column $\boldsymbol{x}$?

$$Ax = b \quad is \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Find the volumes of the boxes whose edges are columns of $\boldsymbol{A}$ and then rows of A^{-1}.

Solution

$$|A| = \begin{vmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{vmatrix} = 2 \qquad |B_1| = \begin{vmatrix} 0 & 6 & 2 \\ 0 & 4 & 2 \\ 1 & 9 & 0 \end{vmatrix} = 4 \qquad |B_2| = \begin{vmatrix} 2 & 0 & 2 \\ 1 & 0 & 2 \\ 5 & 1 & 0 \end{vmatrix} = -2 \qquad |B_1| = \begin{vmatrix} 2 & 6 & 0 \\ 1 & 4 & 0 \\ 5 & 9 & 1 \end{vmatrix} = 2$$

$$x = \frac{4}{2} = 2; \quad y = \frac{-2}{2} = -1; \quad z = \frac{2}{2} = 1$$

The solution is: $(2, \ -1, \ 1)$

$$C_{11} = \begin{vmatrix} 4 & 2 \\ 9 & 0 \end{vmatrix} = -18 \quad C_{12} = -\begin{vmatrix} 1 & 2 \\ 5 & 0 \end{vmatrix} = 10 \quad C_{13} = \begin{vmatrix} 1 & 4 \\ 5 & 9 \end{vmatrix} = -11$$

$$C_{21} = -\begin{vmatrix} 6 & 2 \\ 9 & 0 \end{vmatrix} = 18 \quad C_{22} = \begin{vmatrix} 2 & 2 \\ 5 & 0 \end{vmatrix} = -10 \quad C_{23} = -\begin{vmatrix} 2 & 6 \\ 5 & 9 \end{vmatrix} = 12$$

$$C_{31} = \begin{vmatrix} 6 & 2 \\ 4 & 2 \end{vmatrix} = 4 \quad C_{32} = -\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = -2 \quad C_{33} = \begin{vmatrix} 2 & 6 \\ 1 & 4 \end{vmatrix} = 2$$

$$C = \begin{pmatrix} -18 & 10 & -11 \\ 18 & -10 & 12 \\ 4 & -2 & 2 \end{pmatrix} \Rightarrow C^T = \begin{pmatrix} -18 & 18 & 4 \\ 10 & -10 & -2 \\ -11 & 12 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{C^T}{\det A} = \frac{1}{2}\begin{pmatrix} -18 & 18 & 4 \\ 10 & -10 & -2 \\ -11 & 12 & 2 \end{pmatrix} = \begin{pmatrix} -9 & 9 & 2 \\ 5 & -5 & -1 \\ -\frac{11}{2} & 6 & 1 \end{pmatrix}$$

The solution $\boldsymbol{x}$ is the third column of A^{-1} because $\boldsymbol{b} = (0, 0, 1)$ is the third column of I.

The volume of the boxes whose edges are columns of $\boldsymbol{A}$ = det($\boldsymbol{A}$) = 2.

Since $\left|A^T\right| = |A|$. The box from rows of A^{-1} has volume $\left|A^{-1}\right| = \frac{1}{|A|} \boxed{= \frac{1}{2}}$

Exercise

Verify that $\det(AB)=\det(BA)$ and determine whether the equality $\det(A+B)=\det(A)+\det(B)$ holds

$$A=\begin{bmatrix}2&1&0\\3&4&0\\0&0&2\end{bmatrix}\quad \textit{and}\quad B=\begin{bmatrix}1&-1&3\\7&1&2\\5&0&1\end{bmatrix}$$

Solution

$$AB=\begin{bmatrix}2&1&0\\3&4&0\\0&0&2\end{bmatrix}\begin{bmatrix}1&-1&3\\7&1&2\\5&0&1\end{bmatrix}=\begin{bmatrix}9&-1&8\\31&1&17\\10&0&2\end{bmatrix}\qquad \det(AB)=\begin{vmatrix}9&-1&8\\31&1&17\\10&0&2\end{vmatrix}=-170$$

$$BA=\begin{bmatrix}1&-1&3\\7&1&2\\5&0&1\end{bmatrix}\begin{bmatrix}2&1&0\\3&4&0\\0&0&2\end{bmatrix}=\begin{bmatrix}-1&-3&6\\17&11&4\\10&5&2\end{bmatrix}\qquad \det(BA)=\begin{vmatrix}-1&-3&6\\17&11&4\\10&5&2\end{vmatrix}=-170$$

Thus, $\boxed{\det(AB)=\det(BA)}$

$$\det(A)=\begin{vmatrix}2&1&0\\3&4&0\\0&0&2\end{vmatrix}=10$$

$$\det(B)=\begin{vmatrix}1&-1&3\\7&1&2\\5&0&1\end{vmatrix}=-17$$

$$A+B=\begin{bmatrix}2&1&0\\3&4&0\\0&0&2\end{bmatrix}+\begin{bmatrix}1&-1&3\\7&1&2\\5&0&1\end{bmatrix}=\begin{bmatrix}3&0&3\\10&5&2\\5&0&3\end{bmatrix}\qquad \det(A+B)=\begin{vmatrix}3&0&3\\10&5&2\\5&0&3\end{vmatrix}=-30$$

$$\det(A)+\det(B)=10-17=-7$$

$$\neq \det(A+B)$$

Exercise

Verify that $\det(kA) = k^n \det(A) \quad A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}, \quad k = 2$

Solution

$$\det(A) = \begin{vmatrix} -1 & 2 \\ 3 & 4 \end{vmatrix} = -10$$

$$\begin{aligned} \det(2A) &= \begin{vmatrix} -2 & 4 \\ 6 & 8 \end{vmatrix} \\ &= -40 \\ &= 4(-10) \\ &= 2^2(-10) \\ &= k^2 \det(A) \end{aligned}$$

Exercise

Verify that $\det(kA) = k^n \det(A) \quad A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}, \quad k = -2$

Solution

$$\det(A) = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{vmatrix} = 56$$

$$\begin{aligned} \det(-2A) &= \begin{vmatrix} -4 & 2 & -6 \\ -6 & -4 & -2 \\ -2 & -8 & -0 \end{vmatrix} \\ &= -448 \\ &= (-2)^3(56) \\ &= k^3 \det(A) \end{aligned}$$

Exercise

Verify that $\det(kA) = k^n \det(A)$ $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}, \quad k = 3$

Solution

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{vmatrix} = -7$$

$$\begin{aligned} \det(3A) &= \begin{vmatrix} 3 & 3 & 3 \\ 0 & 6 & 9 \\ 0 & 3 & -6 \end{vmatrix} \\ &= -189 \\ &= 3^3(-7) \\ &= k^3 \det(A) \end{aligned}$$

Exercise

Solve by using Cramer's rule

a) $\begin{cases} 7x - 2y = 3 \\ 3x + y = 5 \end{cases}$

b) $\begin{cases} 4x + 5y \qquad = 2 \\ 11x + y + 2z = 3 \\ x + 5y + 2z = 1 \end{cases}$

c) $\begin{cases} x - 4y + z = 6 \\ 4x - y + 2z = -1 \\ 2x + 2y - 3z = -20 \end{cases}$

d) $\begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$

e) $\begin{cases} 2x - y + z = -1 \\ 3x + 4y - z = -1 \\ 4x - y + 2z = -1 \end{cases}$

Solution

a) $\begin{cases} 7x - 2y = 3 \\ 3x + y = 5 \end{cases}$

$D = \begin{vmatrix} 7 & -2 \\ 3 & 1 \end{vmatrix} = 13 \qquad D_x = \begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 13 \qquad D_y = \begin{vmatrix} 7 & 3 \\ 3 & 5 \end{vmatrix} = 26$

$x = \dfrac{D_x}{D} = \dfrac{13}{13} = 1 \qquad y = \dfrac{D_y}{D} = \dfrac{26}{13} = 2$

Solution: $\boxed{(1,\ 2)}$

b) $\begin{cases} 4x+5y \qquad = 2 \\ 11x+y+2z=3 \\ x+5y+2z=1 \end{cases}$

$D = \begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} = -132$ $\qquad D_x = \begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} = -36$

$D_y = \begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} = -24$ $\qquad D_z = \begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} = 12$

$x = \frac{D_x}{D} = \frac{-36}{-132} = \frac{3}{11}$ $\quad y = \frac{D_y}{D} = \frac{-24}{-132} = \frac{2}{11}$ $\quad z = \frac{D_z}{D} = \frac{12}{-132} = -\frac{1}{11}$

Solution: $\boxed{\left(\frac{3}{11}, \frac{2}{11}, -\frac{1}{11}\right)}$

c) $\begin{cases} x-4y+z=6 \\ 4x-y+2z=-1 \\ 2x+2y-3z=-20 \end{cases}$

$$D = \begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix} = 3-16+8+2-4-48 = -55$$

$$D_x = \begin{vmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix} = 18+160-2-20-24+12 = 144$$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix} = 3+24-80+2+40+72 = 61$$

$$D_z = \begin{vmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix} = 20+8+48+12+2-320 = -230$$

$x = \frac{D_x}{D} = -\frac{144}{55}$, $\quad y = \frac{D_y}{D} = -\frac{61}{55}$, $\quad z = \frac{D_z}{D} = \frac{-230}{-55} = \frac{46}{11}$

Solution: $\left(-\frac{144}{55}, -\frac{61}{55}, \frac{46}{11}\right)$

d) $\begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$

$D = -423 \quad D_{x_1} = -2115 \quad D_{x_2} = -3384 \quad D_{x_3} = -1269 \quad D_{x_4} = 423$

$x_1 = \dfrac{D_{x_1}}{D} = \dfrac{-2115}{-423} = \underline{5}$ $\qquad$ $x_2 = \dfrac{D_{x_2}}{D} = \dfrac{-3384}{-423} = \underline{8}$

$x_3 = \dfrac{D_{x_3}}{D} = \dfrac{-1269}{-423} = \underline{3}$ $\qquad$ $x_4 = \dfrac{D_{x_4}}{D} = \dfrac{423}{-423} = \underline{-1}$

Solution: $\boxed{(5,\ 8,\ 3,\ -1)}$

e) $\begin{cases} 2x - y + z = -1 \\ 3x + 4y - z = -1 \\ 4x - y + 2z = -1 \end{cases}$

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 4 & -1 \\ 4 & -1 & 2 \end{vmatrix} = 16 + 4 - 3 - 16 - 2 + 6 \underline{= 5}$$

$$D_x = \begin{vmatrix} -1 & -1 & 1 \\ -1 & 4 & -1 \\ -1 & -1 & 2 \end{vmatrix} = -8 - 1 + 1 + 4 + 1 - 2 \underline{= -5}$$

$$D_y = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -1 & -1 \\ 4 & -1 & 2 \end{vmatrix} = -4 + 4 - 3 + 4 - 2 + 6 \underline{= 5}$$

$$D_z = \begin{vmatrix} 2 & -1 & -1 \\ 3 & 4 & -1 \\ 4 & -1 & -1 \end{vmatrix} = -8 + 4 + 3 + 16 - 2 - 3 \underline{= 10}$$

$x = \dfrac{D_x}{D} = \dfrac{-5}{5} = \underline{-1}$, $\quad y = \dfrac{D_y}{D} = \dfrac{5}{5} = \underline{1}$, $\quad z = \dfrac{D_z}{D} = \dfrac{10}{5} = \underline{2}$

$\therefore$ Solution: $\underline{(-1,\ 1,\ 2)}$

Exercise

Show that the matrix A is invertible for all values of θ, then find A^{-1} using $A^{-1} = \frac{1}{\det(A)} adj(A)$

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

$$\det(A) = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1 \qquad \Rightarrow \boldsymbol{A} \text{ is invertible}$$

$$C_{11} = \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta; \quad C_{12} = -\begin{vmatrix} -\sin\theta & 0 \\ 0 & 1 \end{vmatrix} = \sin\theta; \quad C_{13} = \begin{vmatrix} -\sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{21} = -\begin{vmatrix} \sin\theta & 0 \\ 0 & 1 \end{vmatrix} = -\sin\theta; \quad C_{22} = \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta; \quad C_{23} = -\begin{vmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{31} = \begin{vmatrix} \sin\theta & 0 \\ \cos\theta & 0 \end{vmatrix} = 0; \quad C_{32} = -\begin{vmatrix} \cos\theta & 0 \\ -\sin\theta & 0 \end{vmatrix} = 0; \quad C_{33} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1$$

$$adj(A) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

$$= \frac{1}{1} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$