10/ $S = \{(x,y) \mid x^2+y^2=0 \; ; \; x,y \in \mathbb{R}\}$

$x^2+y^2=0 \Rightarrow x=y=0 \quad (x,y \in \mathbb{R})$

a) let $\vec{u} = (x_1, y_1) \ni x_1^2+y_1^2=0$ & $x_1=y_1=0$

$\vec{v} = (x_2, y_2) \ni x_2^2+y_2^2=0$ $\quad x_2=y_2=0$

$\vec{u}+\vec{v} = (x_1, y_1) + (x_2, y_2)$

$= (x_1+x_2, y_1+y_2)$

$\sim (x_1+x_2)^2+(y_1+y_2)^2 = x_1^2+x_2^2+2x_1x_2 + y_1^2+y_2^2+2y_1y_2$

$= (x_1^2+y_1^2)+(x_2^2+y_2^2)+2(x_1x_2+y_1y_2)$

$= 0 + 0 + 0$

$= 0$ ✓

S is closed under addition

b) let $k \in \mathbb{R}$

$k\vec{u} = k(x_1, y_1)$

$= (kx_1, ky_1)$

$= (kx_1 + ky_1)^2$

$= k^2(x_1^2+y_1^2+2x_1y_1)$ $\quad x_1=y_1=0$

$= k^2(0+0+0)$

$= 0$ ✓

S is closed under scalar multiplication.

c) Since S is closed under addition & scalar multiplication, then S is a subspace of $\mathbb{R}^2$

2.6 Linear Independence (LI)

- Independent vectors (not too many)

Columns of A are L.I. when the only soln to $A\vec{x}=\vec{0}$ is $\vec{x}=\vec{0}$, no other combination

Defns

* A set of 2 or more vectors is linearly dependent if 1 vector in the set is a linear combination of others.

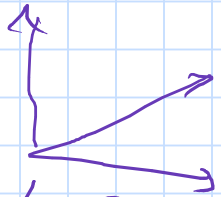
$$\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_n$$

* $\underbrace{\vec{v}_1, \vec{v}_2, \vec{v}_3}_{\text{non zero vectors}}$ L.I. $\quad x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$

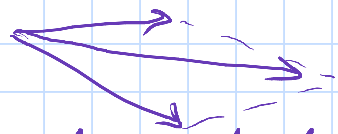
$\boxed{x_1, x_2, x_3 \neq 0}$

* A set of vectors is L.I if it is NOT linearly dependent
* $\vec{w}_1, \vec{w}_2, \vec{w}_3$ are in the same plane $\Rightarrow$ dependent
* The empty set is L.I. for L. dependent sets must be non empty.
* $a\vec{v} = \vec{0} \quad (a \neq 0)$ $\qquad a^{-1} \cdot a = 1$

$$\begin{aligned}\vec{v} &= a^{-1} a \vec{v} \\ &= a^{-1}(a\vec{v}) \\ &= a^{-1}\vec{0} \\ &= \vec{0}\end{aligned}$$



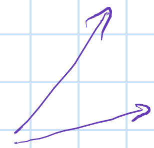
L. I.



dependent

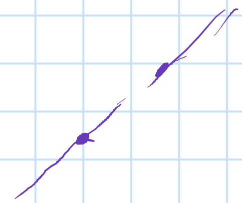
(1, 0) , (0, 1) L. I

(1, 1) (1, .00001) = L I



(1, 1) (2, 2) Linearly dependent

$2(1,1) = (2,2)$



(1, 1) (0, 0) Linearly dependent

<u>Ex</u> $\vec{v}_1 = (1,-2,3)$ $\vec{v}_2 = (5,6,-1)$ $\vec{v}_3 = (3,2,1)$

L. I or dependent

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$$

$x_1, x_2, x_3 \neq 0$ indep.

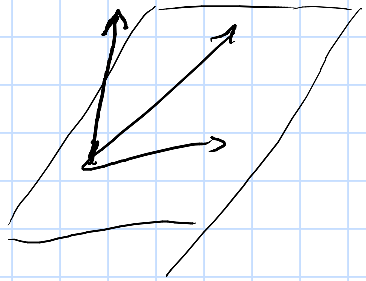
$$x_1\begin{pmatrix}1\\-2\\3\end{pmatrix} + x_2\begin{pmatrix}5\\6\\-1\end{pmatrix} + x_3\begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{aligned} x_1 + 5x_2 + 3x_3 &= 0 \\ -2x_1 + 6x_2 + 2x_3 &= 0 \\ 3x_1 - x_2 + x_3 &= 0 \end{aligned}$$

$A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$ $\det(A) \neq 0$ L. I

$$\begin{vmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{vmatrix} = 0$$

$\therefore$ $\vec{v}_1, \vec{v}_2$ & $\vec{v}_3$ are linearly dependent



Wronskian

$$W = \begin{vmatrix} f_1 & f_2 & \cdots & f_n(x) \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \vdots \\ f_1^{(n-1)} & & & f_n^{(n-1)} \end{vmatrix} =$$

$$|W| \begin{cases} = 0 & \text{linearly dependent} \\ \neq 0 & \text{L.I} \end{cases}$$

Ex $f_1 = x \qquad f_2 = \sin x$

$$W = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix}$$

$$= x\cos x - \sin x \neq 0$$

(L. I)

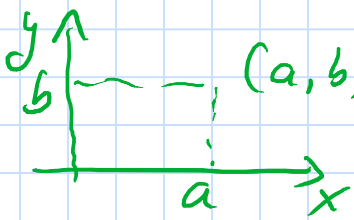
$f_1 = 1 \qquad f_2 = e^x \qquad f_3 = e^{2x}$

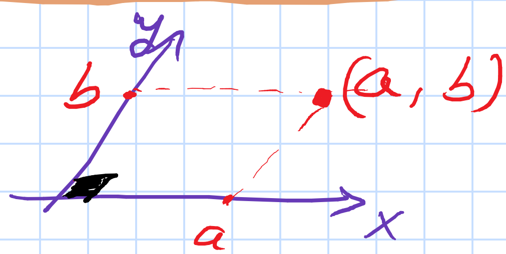
$$W = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix}$$

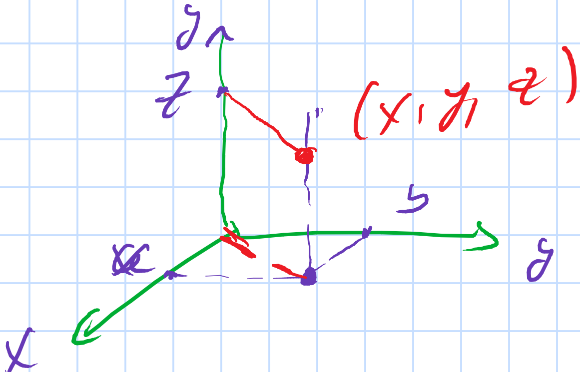
$$= 4e^{3x} - 2e^{3x}$$
$$= 2e^{3x} \neq 0 \qquad \text{(L. I)}$$

Theorem S has a L.I subset of a vector space V & let $\vec{v}$ be a vector in V that is not in S. Then $S \cup \{\vec{v}\}$ is linearly dependent iff $\vec{v} \in \operatorname{span}(S)$

2.7







Basis if V is any vector space and $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a finite set of vectors in V

. S is called Basis

$$\begin{cases} S \text{ is L.I} \\ S \text{ spans } V. \end{cases}$$

ex

$$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad 2x \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Basis   Basis   not

$$\begin{matrix} 1 & 2 \\ 0 & 0 \end{matrix}$$

All standard unit vectors are Basis

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{matrix}$$

L.I $\Rightarrow$ $n \times n$ $\Rightarrow$ $\det \neq 0$ $\quad$ n vectors n - components

pivots columns of A are Basis for column space

Ex

$$R = \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \left[\begin{matrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{matrix}\right] \end{array}$$

$\uparrow \qquad \uparrow$

Coln: 1, 3 are pivots columns
Basis for coln space

col: 2 & 4 are basis $\Rightarrow$

Coordinates Relative to Basis (Uniqueness)

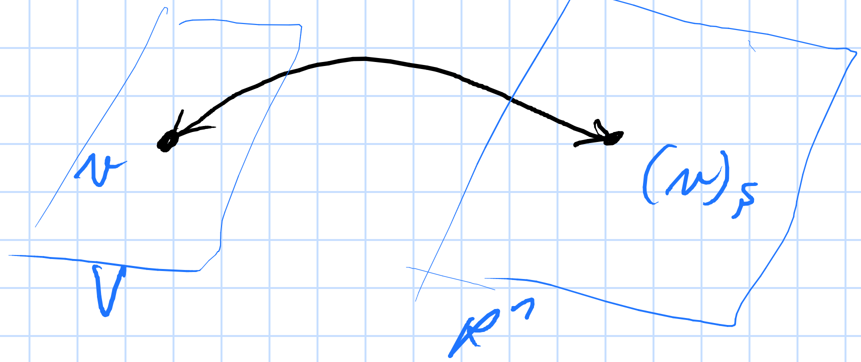
If $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis for a vector V every $\vec{v}_i$ in V can be expressed.

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$$

$$\begin{aligned} - \quad \vec{v} &= c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n \\ \vec{v} &= k_1\vec{v}_1 + k_2\vec{v}_2 + \dots + k_n\vec{v}_n \\ \hline \vec{0} &= (c_1-k_1)\vec{v}_1 + (c_2-k_2)\vec{v}_2 + \dots + (c_n-k_n)\vec{v}_n \end{aligned}$$

$c_1 - k_1 = 0 \qquad c_2 - k_2 = 0 \quad \dots \qquad c_n - k_n = 0$

$c_1 = k_1 \qquad c_2 = k_2 \quad \dots \qquad c_n = k_n$



Ex $\vec{v}_1\ (1,2,1)$ $\quad \vec{v}_2 = (2,9,0)$ $\quad \vec{v}_3 = (3,3,4)$ $\quad \mathbb{R}^3$

Find coordinate vector $\vec{v} = (5,-1,9)$ relative

$S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$$

$$(5,-1,9) = c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4)$$

$(c_1 \cancel{+2c_2, c_1})$

$$\begin{cases} c_1 + 2c_2 + 3c_3 = 5 \\ 2c_1 + 9c_2 + 3c_3 = -1 \\ c_1 + 4c_3 = 9 \end{cases}$$

$c_1 = 1, \quad c_2 = -1, \quad c_3 = 2$

$\therefore (\vec{v})_S = (1,-1,2)$

b) $\vec{v} \quad (\vec{v})_S = (-1,3,2)$

$$\vec{v} = (-1)\vec{v}_1 + 3\vec{v}_2 + 2\vec{v}_3$$
$$= -(1,2,1) + 3(2,9,0) + 2(3,3,4)$$
$$= (11, 31, 7)$$

Dimension

Dim of a finite dimensional V is $\dim(V)$ is number of vectors in a basis of V, zero vector space $\Rightarrow$ $\dim = 0$

+ $\mathrm{Dim}(V) = \#$ elts V is finite

$V = \{\vec{0}\}$ $\dim(V) = 0$

$\mathrm{Dim}(V) = \infty$

$\dim(\mathbb{R}^n) = n$

$\dim(P_n) = n+1$

$\dim(M_{mn}) = mn$

$P_3 = \underset{1}{a_3 x^3} + \underset{1}{a_2 x^2} + \underset{1}{a_1 x} + \underset{1}{a_0}$