Review Cal III

$$\int_1^2 \int_0^{\ln x} x^3 e^y \, dy \, dx = \int_1^2 x^3 e^y \Big|_0^{\ln x} dx$$

$$= \int_1^2 x^3 (x-1)\, dx$$

$$= \int_1^2 (x^4 - x^3)\, dx$$

$$= \frac{1}{5}x^5 - \frac{1}{4}x^4 \Big|_1^2$$

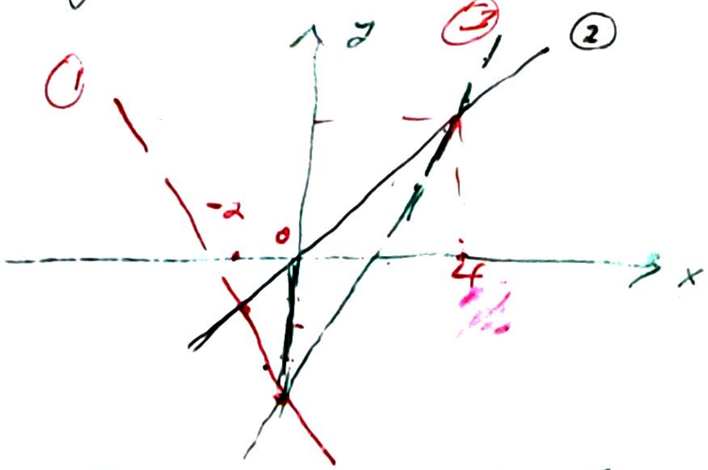
$$= \frac{32}{5} - 4 - \frac{1}{5} + \frac{1}{4}$$

$$= \frac{31}{5} - \frac{15}{4}$$

$$= \frac{49}{20}$$

Ex. 4

$y = -x - 4$ ① $y = x$ ② $y = 2x - 4$ ③



① ∩ ② $\Rightarrow y = -x - 4 = x \Rightarrow x = -2 \rightarrow y = -2$

① ∩ ③ $\rightarrow 2x - 4 = -x - 4 \Rightarrow x = 0 \rightarrow y = -4$

② ∩ ③ $\rightarrow x = 2x - 4 \Rightarrow x = 4 \rightarrow y = 4$

$$\text{Area} = \int_{-2}^{0} \int_{-x-4}^{x} dy\,dx + \int_{0}^{4} \int_{2x-4}^{x} dy\,dx$$

$$= \int_{-2}^{0} (2x+4)\,dx + \int_{0}^{4} (-x+4)\,dx$$

$$= x^2 + 4x \Big|_{-2}^{0} + \left(-\frac{1}{2}x^2 + 4x \Big|_{0}^{4}\right. \qquad \int_a^b d\theta = b - a$$

$$= -4 + 8 - 8 + 16$$

$$= 12 \text{ unit}^2$$

#5 V? $z = xy + 10$, 7

$R = \{(r,\theta) : 2 \le r \le 4,\ 0 \le \theta \le 2\pi\}$

$$V = \int_0^{2\pi}\int_2^4 z\, r\, dr\, d\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

$$= \int_0^{2\pi}\int_2^4 (r^2\cos\theta\sin\theta + 10)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_2^4 \left(\frac{1}{2}r^3\sin 2\theta + 10r\right) dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{8}r^4\sin 2\theta + 5r^2\right)\Big|_2^4\, d\theta$$

$32 - 2$ $\quad$ $80 - 20$

$$= \int_0^{2\pi} (32\sin 2\theta + 80 - 2\sin 2\theta - 20)\, d\theta$$

$$= \int_0^{2\pi} (30\sin 2\theta + 60)\, d\theta$$

$$= -15\cos 2\theta + 60\theta \Big|_0^{2\pi}$$

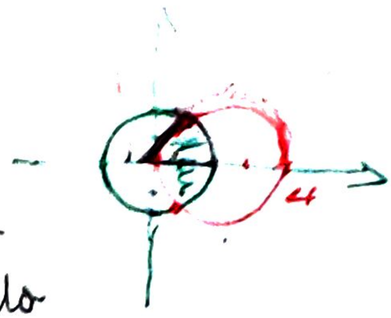
$$= -15 + 120\pi + 15$$

$$= 120\pi \text{ unit}^3$$

406. A? $r = 2$ & $r = 4\cos\theta$

$r = 4\cos\theta = 2 \Rightarrow \cos\theta = \frac{1}{2}$

$\boxed{\theta = \frac{\pi}{3}}$



$$\text{Area} = 2\left[\int_0^{\pi/3} d\theta \int_0^2 r\,dr + \int_{\pi/3}^{\pi/2}\int_0^{4\cos\theta} r\,dr\,d\theta\right]$$

$A = \frac{1}{2} r^2 \theta$

$\frac{1}{2}\, 4\, \frac{\pi}{3} = \frac{2\pi}{3}$

$$= 2\left[\frac{\pi}{3}\,\frac{1}{2} r^2\Big|_0^2 + \frac{1}{2}\int_{\pi/3}^{\pi/2} r^2\Big|_0^{4\cos\theta} d\theta\right]$$

$$= \frac{4\pi}{3} + \int_{\pi/3}^{\pi/2} 16\cos^2\theta\, d\theta$$

$$= \frac{4\pi}{3} + 8\int_{\pi/3}^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= \frac{4\pi}{3} + 8\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{\pi/3}^{\pi/2}$$

$$= \frac{4\pi}{3} + 8\left(\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

$$= 4\pi - \frac{4\pi}{3} - 2\sqrt{3}$$

$$\boxed{= \frac{8\pi}{3} - 2\sqrt{3} \ \text{unit}^2}$$

#7 V?

$$z = \frac{20}{1+x^2+y^2} - 2$$

$$= \frac{20}{1+r^2} - 2 \geq 0$$

$$\frac{20}{1+r^2} \geq 2$$

$$10 \geq 1 + r^2$$

$$r^2 \leq 9 \rightarrow \cancel{-3}^{\,0} \leq r \leq 3$$

$$V = \int_0^{2\pi} d\theta \int_0^3 \left(\frac{20r}{1+r^2} - 2r\right) dr$$

$$= 2\pi \left[\int_0^3 \frac{10\, d(1+r^2)}{1+r^2} - r^2 \Big|_0^3\right)$$

$$= 2\pi \left[10 \ln(1+r^2)\Big|_0^3 - 9\right]$$

$$= 2\pi \left(10 \ln 10 - 9\right) \text{ unit}^3$$

Assignment 2

$e^{-x}e^{-y}e^{-2z}$

Q1. $\int_1^{\ln 8}\int_0^{\ln 4}\int_0^{\ln 2} e^{-x-y-2z}\,dx\,dy\,dz$

$$= \int_1^{\ln 8} e^{-2z}\,dz \int_0^{\ln 4} e^{-y}\,dy \int_0^{\ln 2} e^{-x}\,dx$$

$$= -\frac{1}{2}e^{-2z}\Big|_1^{\ln 8} \left(-e^{-y}\Big|_0^{\ln 4}\right) \left(-e^{-x}\Big|_0^{\ln 2}\right)$$

$$= -\frac{1}{2}\left(e^{-2\ln 8} - e^{-2}\right)\left(-\frac{1}{4}+1\right)\left(-\frac{1}{2}+1\right)$$

$$= -\frac{1}{2}\left(\frac{1}{64} - \frac{1}{e^2}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)$$

$$= \frac{3}{16}\left(\frac{1}{e^2} - \frac{1}{64}\right)$$

#2 $\int_0^{\pi/2}\int_0^1\int_0^{\pi/2} \sin\pi x \cos y \sin 2z \, dy\, dx\, dz$

$$= \int_0^{\pi/2} \sin 2z \, dz \int_0^1 \sin\pi x \, dx \int_0^{\pi/2} \cos y \, dy$$

$$= -\frac{1}{2}\cos 2z \Big|_0^{\pi/2} \left(\left(-\frac{1}{\pi}\cos\pi x\right)\Big|_0^1 \left(\sin y\Big|_0^{\pi/2}\right.\right.$$

$$= +\frac{1}{2\pi}(-1-1)\;(-1-1)\;(1)$$

$$= \frac{2}{\pi}$$

#3 $\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\varphi} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/2} \frac{1}{3}\sin\varphi \, \rho^3\Big|_0^{2\cos\varphi} d\varphi$$

$$= \frac{2\pi}{3}\int_0^{\pi/2} \sin\varphi\,(8\cos^3\varphi)\, d\varphi$$

$$= \frac{-16\pi}{3}\int_0^{\pi} \cos^3\varphi \; d(\cos\varphi)$$

$$= -\frac{4\pi}{3} \cos^4\varphi\Big|_0^{\pi}$$

$$= -\frac{4\pi}{3}(1-1)$$

$$= 0$$

#4/ $\displaystyle\int_{-2}^{2}\int_{-1}^{1}\int_{0}^{\sqrt{1-z^2}} \frac{1}{(1+x^2+y^2)^2}\,dx\,dz\,dy$

$0 \le x \le \sqrt{1-z^2}$ $\quad \boxed{-1 \le z \le 1} \quad$ $-2 \le y \le 2$

$$= \int_{-1}^{1}\int_{-\pi/2}^{\pi/2}\int^{1} \frac{1}{(1+r^2)^2}\, r\, dr\, d\theta\, dz$$

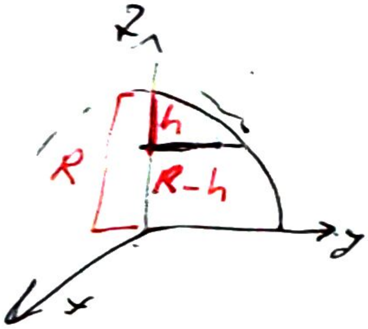
$$= \frac{1}{2}\int_{-1}^{1} dz \int_{-\frac{\pi}{2}}^{\pi/2} d\theta \int_{0}^{1} \frac{d(1+r^2)}{(1+r^2)^2}$$

$$= \frac{1}{2}(1+1)\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\left(\frac{-1}{1+r^2}\Big|_{0}^{1}\right.$$

$$= -\pi\left(\frac{1}{2}-1\right)$$

$$= \frac{\pi}{2}$$

#5/ sphere



$$x^2+y^2+z^2=R^2$$
$$z^2=R^2-x^2-y^2$$
$$z=\sqrt{R^2-x^2-y^2}$$
$$\boxed{R-h \le z \le \sqrt{R^2-x^2-y^2}}$$

$$\cancel{0 = R^2-x^2-y^2}$$
$$\cancel{y^2=R^2-x^2}$$

$$\begin{cases} (R-h)^2 = R^2-x^2-y^2 \\ R^2-2Rh+h^2 = R^2-x^2-y^2 \end{cases}$$
$$y^2 = 2Rh-h^2-x^2$$

$$\boxed{-\sqrt{2Rh-h^2-x^2} \le y \le \sqrt{2Rh-h^2-x^2}}$$

$$y=0 \Rightarrow z^2=R^2-x^2$$
$$x^2=R^2-(R-h)^2$$
$$=2Rh-h^2 \Rightarrow -\sqrt{2Rh-h^2} \le x \le \sqrt{2Rh-h^2}$$

$$V=\int_{-\sqrt{2Rh-h^2}}^{\sqrt{2Rh-h^2}} \int_{-\sqrt{2Rh-h^2-x^2}}^{\sqrt{2Rh-h^2-x^2}} \int_{R-h}^{\sqrt{R^2-x^2-y^2}} dz\,dy\,dx$$

$$=\int_{-\sqrt{2Rh-h^2}}^{\sqrt{2Rh-h^2}} \int_{-\sqrt{\underbrace{2Rh-h^2}_{r^2}-x^2}}^{\sqrt{\underbrace{2Rh-h^2}_{r^2}-x^2}} \left(\sqrt{R^2-(x^2+y^2)}-R+h\right) dy\,dx$$

$$=\int_0^{2\pi} d\theta \int_0^{\sqrt{2Rh-h^2}} \left(\sqrt{R^2-r^2}-R+h\right) r\,dr\,d\theta$$

$$V = 2\pi\left[\int_0^{\sqrt{2Rh-h^2}} r\,(R^2-r^2)^{1/2}\,dr + \int_0^{\sqrt{2Rh-h^2}} (h-R)\,r\,dr\right]$$

$$-\tfrac{1}{2}(R-h)$$

$$= 2\pi\left[\int_0^{\sqrt{2Rh-h^2}} -\frac{1}{2}(R^2-r^2)^{1/2}\,d(R^2-r^2) + \frac{1}{2}(h-R)\,r^2\Big|_0^{\sqrt{2Rh-h^2}}\right]$$

$$= -\pi\left[\frac{2}{3}(R^2-r^2)^{3/2}\Big|_0^{\sqrt{2Rh-h^2}} + (R-h)(2Rh-h^2)\right]$$

$$= -\pi\left[\frac{2}{3}\big(\underbrace{R^2-2Rh+h^2}_{(R-h)^2}\big)^{3/2} - \frac{2}{3}R^3 + h(R-h)(2R-h)\right]$$

$$= -\pi\left(\frac{2}{3}(R-h)^3 - \frac{2}{3}R^3 + 2R^2h \underbrace{- Rh^2 - 2Rh^2} + h^3\right)$$

$$= -\pi\left(\cancel{\frac{2}{3}R^3} - 2R^2h + 2Rh^2 - \frac{2}{3}h^3 - \cancel{\frac{2}{3}R^3} + 2R^2h - 3Rh^2 + h^3\right)$$

$$= -\pi\left(-Rh^2 + \frac{1}{3}h^3\right)$$

$$\boxed{= \pi h^2\left(R - \frac{h}{3}\right)}$$

#8/ $\rho = 1 + x^3 \qquad 0 \le x \le 1$

$$m = \int_0^1 (1+x^3)\,dx$$

$$= x + \frac{1}{4}x^4 \Big|_0^1$$

$$= 1 + \frac{1}{4}$$

$$= \frac{5}{4}$$

$$M_{\bar{x}} = \int_0^1 (1+x^3)\,x\,dx$$

$$= \int_0^1 (x + x^4)\,dx$$

$$= \frac{1}{2}x^2 + \frac{1}{5}x^5 \Big|_0^1$$

$$= \frac{1}{2} + \frac{1}{5}$$

$$= \frac{7}{10}$$

$$\bar{x} = \frac{4}{5}\int_0^1 (x + x^4)\,dx$$

$$\bar{x} = \frac{7}{\cancel{10}_{5}} \cdot \frac{\cancel{4}^{2}}{5} = \boxed{\frac{14}{25}}$$

$$\frac{32,\ 29,\ 33}{36^{*}}$$

$$\iiint_D xz\,dV$$

$$\begin{cases} y = x \to y - x = 0 \\ y = x+2 \to y - x = 2 \end{cases} \qquad u = y - x, \quad 0 \le u \le 2$$

$$\left.\begin{cases} x - z = 0 \\ z = x + 3 \to z - x = 3 \end{cases}\right\} \qquad v = x - z, \quad 0 \le v \le 3$$

$$\begin{cases} z = 0 \\ z = 3 \end{cases} \to \quad \underline{w = z}$$

$$0 \le w \le 3$$

$$y - x = u \to y = u + v + w$$

$$x - z = v \to x = v + w$$

$$|J| = \left\| \begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \right\| = |-1| = 1$$

$$\iiint_D xz\,dv = \int_0^3 \int_0^3 \int_0^2 (v + w)w \,du\,dv\,dw$$

$$= 2\int_0^3 \int_0^3 (vw + w^2)\,dv\,dw$$

$$= 2\int_0^3 \left(\frac{1}{2} w v^2 + w^2 v \Big|_0^3 dw\right.$$

$$= 2\int_0^3 \left(\frac{9}{2} w + 3w^2\right) dw$$

$$= 2 \left(\frac{9}{4} w^2 + w^3 \Big|_0^3\right.$$

$$= 2\left(\frac{81}{4} + 27\right)$$

$$= 58\left(\frac{3}{4} + 1\right) \quad \frac{7}{4}$$

$$= 2 \times 27$$