Section 4.3 – Conservative Vector Fields

Line Integrals of Vector Fields

Assume the vector field $\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$ has a continuous components, and the curve *C* has a smooth parametrization $\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}, \ \ a \le t \le b$. $\vec{r}(t)$ defines along the path *C* which we call the ***forward direction***. At each point along the path *C*, the tangent vector $\vec{T} = \frac{d\vec{r}}{ds} = \frac{\vec{v}}{|\vec{v}|}$ is a unit vector tangent to the path and pointing is this forward direction. The tangential component is given by the dot product

$$\vec{F} \bullet \vec{T} = \vec{F} \bullet \frac{d\vec{r}}{ds}$$

Definition

Let $\vec{F}$ be a vector field with continuous components defined along a smooth curve *C* parametrized by $\vec{r}(t), \ \ a \le t \le b$. Then the line integral of $\vec{F}$ along *C* is

$$\int_C \vec{F} \bullet \vec{T}\ ds = \int_C \left(\vec{F} \bullet \frac{d\vec{r}}{ds}\right) ds = \int_C \vec{F} \bullet d\vec{r}$$

Evaluating the Line Integral of $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ along *C*: $\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$

1. Express the vector field $\vec{F}$ in terms of the parametrized curve *C* as $\vec{F}(\vec{r}(t))$ by substituting the components $x = g(t),\ y = h(t),\ z = k(t)$ of $\vec{r}$ into the scalar components $M(x,y,z),\ N(x,y,z),\ P(x,y,z)$ of $\vec{F}$.
2. Find the derivative (velocity) vector $\frac{d\mathbf{r}}{dt}$.
3. Evaluate the line integral with respect to the parameter *t*, $a \le t \le b$, to obtain

$$\int_C \vec{F} \bullet d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt}\, dt$$

Example

Evaluate $\int_C \vec{F} \bullet d\vec{r}$, where $\vec{F} = z\hat{i} + xy\hat{j} - y^2\hat{k}$ along the curve C given by

$\vec{r}(t) = t^2\hat{i} + t\,\hat{j} + \sqrt{t}\,\hat{k} \quad 0 \le t \le 1.$

Solution

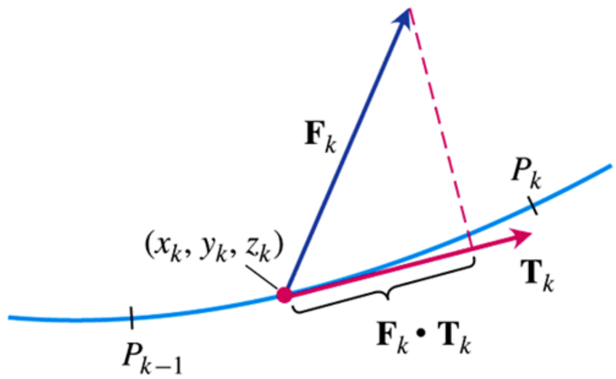
$$\vec{F}(\vec{r}(t)) = \sqrt{t}\,\hat{i} + t^3\hat{j} - t^2\hat{k}$$

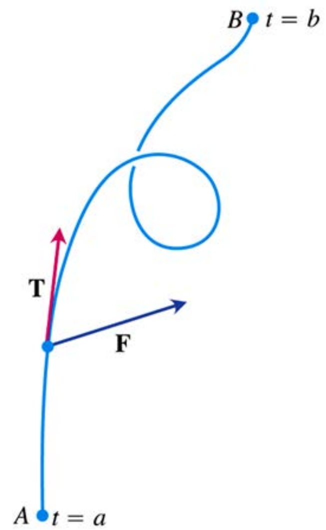
$$\frac{d\vec{r}}{dt} = 2t\,\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k}$$

$$\begin{aligned}
\vec{F}(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt} &= \left(\sqrt{t}\hat{i} + t^3\hat{j} - t^2\hat{k}\right) \bullet \left(2t\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k}\right) \\
&= 2t\sqrt{t} + t^3 - \frac{t^2}{2\sqrt{t}} \\
&= 2t^{3/2} + t^3 - \frac{1}{2}t^{3/2} \\
&= \frac{3}{2}t^{3/2} + t^3
\end{aligned}$$

$$\begin{aligned}
\int_C \vec{F} \bullet d\vec{r} &= \int_0^1 \vec{F}(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt}\, dt \\
&= \int_0^1 \left(\frac{3}{2}t^{3/2} + t^3\right) dt \\
&= \frac{3}{5}t^{5/2} + \frac{1}{4}t^4 \,\Big|_0^1 \\
&= \frac{3}{5} + \frac{1}{4} \\
&= \boxed{\frac{17}{20}}
\end{aligned}$$

Work Done by a Force over a Curve in Space





Definition

Let C be a smooth curve parametrized by $\boldsymbol{r}(t),\ \ a \le t \le b$, and $\vec{F}$ be a continuous force field over a region containing C. Then the ***work*** done in moving an object from point $A = \vec{r}(a)$ to the point $B = \vec{r}(b)$ along C is

$$W = \int_C \boldsymbol{F} \cdot \boldsymbol{T}\ ds = \int_a^b \boldsymbol{F}\left(\mathbf{r}(t)\right) \cdot \frac{d\mathbf{r}}{dt} dt$$

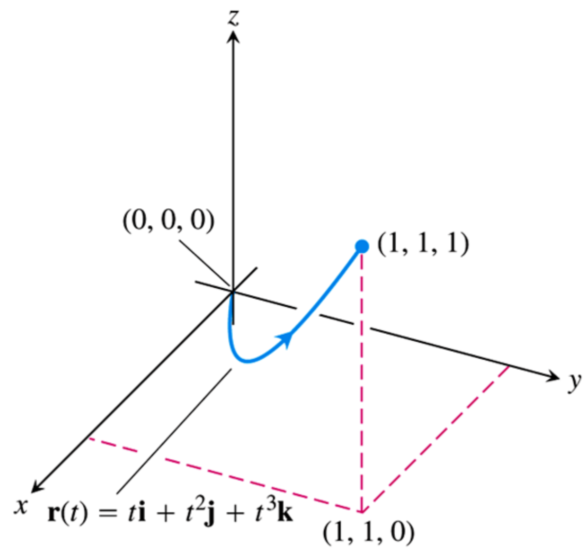
Different ways to write the work integral for $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ ***over the curve C:*** $\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$	
$\boldsymbol{W} = \int_C \vec{F} \cdot \vec{T}\ ds$	***The definition***
$= \int_C \vec{F} \cdot d\vec{r}$	***Vector differential form***
$= \int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt} dt$	***Parametric vector evaluation***
$= \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt} \right) dt$	***Parametric scalar evaluation***
$= \int_C Mdx + Ndy + Pdz$	***Scalar differential form***

Example

Find the work done by the force field $\vec{F} = \left(y - x^2\right)\hat{i} + \left(z - y^2\right)\hat{j} + \left(x - z^2\right)\hat{k}$ along the curve $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k} \quad 0 \le t \le 1$, form (0, 0, 0) to (1, 1, 1).

Solution

$$\begin{aligned}\vec{F} &= \left(y - x^2\right)\hat{i} + \left(z - y^2\right)\hat{j} + \left(x - z^2\right)\hat{k}\\ &= \left(t^2 - t^2\right)\hat{i} + \left(t^3 - t^4\right)\hat{j} + \left(t - t^6\right)\hat{k}\\ &= \left(t^3 - t^4\right)\hat{j} + \left(t - t^6\right)\hat{k}\end{aligned}$$



$$\begin{aligned}\frac{d\vec{r}}{dt} &= \frac{d}{dt}\left(t\hat{i} + t^2\hat{j} + t^3\hat{k}\right)\\ &= \hat{i} + 2t\hat{j} + 3t^2\hat{k}\end{aligned}$$

$$\begin{aligned}\vec{F} \cdot \frac{d\vec{r}}{dt} &= \left[\left(t^3 - t^4\right)\hat{j} + \left(t - t^6\right)\hat{k}\right] \cdot \left(\hat{i} + 2t\hat{j} + 3t^2\hat{k}\right)\\ &= 2t\left(t^3 - t^4\right) + 3t^2\left(t - t^6\right)\\ &= 2t^4 - 2t^5 + 3t^3 - 3t^8\end{aligned}$$

$$\begin{aligned}W &= \int_0^1 \vec{F} \cdot \frac{d\vec{r}}{dt}\,dt\\ &= \int_0^1 \left(2t^4 - 2t^5 + 3t^3 - 3t^8\right)dt\\ &= \frac{2}{5}t^5 - \frac{1}{3}t^6 + \frac{3}{4}t^4 - \frac{1}{3}t^9 \Big|_0^1\\ &= \frac{2}{5} - \frac{1}{3} + \frac{3}{4} - \frac{1}{3}\\ &= \frac{29}{60}\end{aligned}$$

Example

Find the work done by the force field $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ in moving an object along the curve C parametrized by $\vec{r}(t) = \cos(\pi t)\hat{i} + t^2\hat{j} + \sin(\pi t)\hat{k} \quad 0 \le t \le 1$.

Solution

$$\vec{F}(\vec{r}(t)) = \cos(\pi t)\hat{i} + t^2\hat{j} + \sin(\pi t)\hat{k}$$

$$\frac{d\vec{r}}{dt} = -\pi\sin(\pi t)\hat{i} + 2t\hat{j} + \pi\cos(\pi t)\hat{k}$$

$$\begin{aligned}\vec{F}(\vec{r}(t)) \bullet \frac{d\vec{r}}{dt} &= \left(\cos(\pi t)\hat{i} + t^2\hat{j} + \sin(\pi t)\hat{k}\right) \bullet \left(-\pi\sin(\pi t)\hat{i} + 2t\hat{j} + \pi\cos(\pi t)\hat{k}\right) \\ &= -\pi\cos(\pi t)\sin(\pi t) + 2t^3 + \pi\cos(\pi t)\sin(\pi t) \\ &= 2t^3\end{aligned}$$

The work done is the line integral

$$\begin{aligned}W &= \int_0^1 2t^3\, dt \\ &= \frac{1}{2}t^4 \Big|_0^1 \\ &= \boxed{\frac{1}{2}}\end{aligned}$$

Flow integrals and Circulation for Velocity Fields

Definitions

If $\vec{r}(t)$ parametrizes a smooth curve C in the domain of a continuous velocity field $\vec{F}$, the ***flow*** along the curve point $A = \vec{r}(a)$ to $B = \vec{r}(b)$ is

$$Flow = \int_C \vec{F} \bullet \vec{T}\, ds$$

The integral in this case is called a ***flow integral***. If the curve starts and ends at the same point, so that $A = B$, the flow is called the ***circulation*** around the curve.

Example

A fluid's velocity field is $\vec{F} = x\hat{i} + z\hat{j} + y\hat{k}$. Find the flow along the helix

$$\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}, \quad 0 \le t \le \frac{\pi}{2}$$

Solution

$$\vec{F} = x\hat{i} + z\hat{j} + y\hat{k}$$
$$= (\cos t)\hat{i} + t\hat{j} + (\sin t)\hat{k}$$

$$\frac{d\vec{r}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j} + \hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left((\cos t)\hat{i} + t\hat{j} + (\sin t)\hat{k}\right) \bullet \left((-\sin t)\hat{i} + (\cos t)\hat{j} + \hat{k}\right)$$
$$= -\cos t \sin t + t\cos t + \sin t$$

$$Flow = \int_0^{\pi/2} (-\cos t \sin t + t\cos t + \sin t)\, dt$$

		$\cos t$
+	t	$\sin t$
–	1	$-\cos t$

$$\int -\cos t \sin t\, dt = \int \cos t\; d(\cos t) = \tfrac{1}{2}\cos^2 t$$

$$= \frac{1}{2}\cos^2 t + t\sin t + \cos t - \cos t \Big|_0^{\pi/2}$$

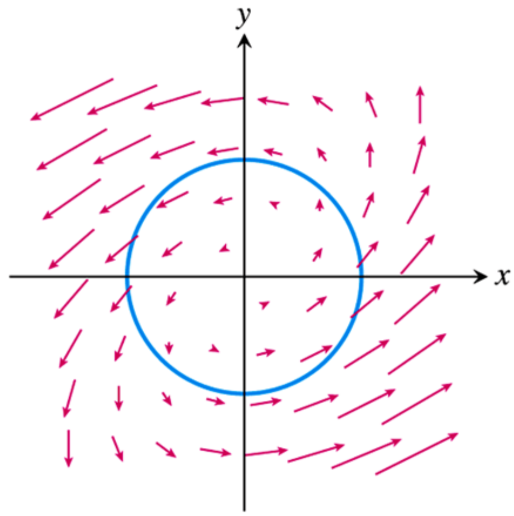
$$= \frac{1}{2}\cos^2 t + t\sin t \Big|_0^{\pi/2}$$

$$\boxed{= \frac{\pi}{2} - \frac{1}{2}}$$

Example

Find the circulation of the field $\vec{F} = (x-y)\hat{i} + x\hat{j}$ around the circle

$\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}, \quad 0 \le t \le 2\pi$



Solution

$$\vec{F} = (x-y)\hat{i} + x\hat{j}$$

$$= (\cos t - \sin t)\hat{i} + (\cos t)\hat{j}$$

$$\frac{d\vec{r}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left((\cos t - \sin t)\hat{i} + (\cos t)\hat{j}\right) \bullet \left((-\sin t)\hat{i} + (\cos t)\hat{j}\right)$$

$$= -\cos t \sin t + \sin^2 t + \cos^2 t$$

$$= 1 - \cos t \sin t$$

$$Circulation = \int_0^{2\pi} (1 - \cos t \sin t)\, dt$$

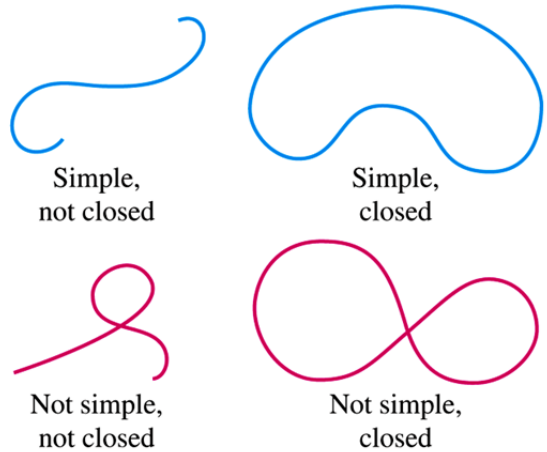
$$= t + \frac{1}{2}\cos^2 t \Big|_0^{2\pi}$$

$$= 2\pi + \frac{1}{2} - \frac{1}{2}$$

$$\underline{= 2\pi}$$

Flux across a Simple Plane Curve

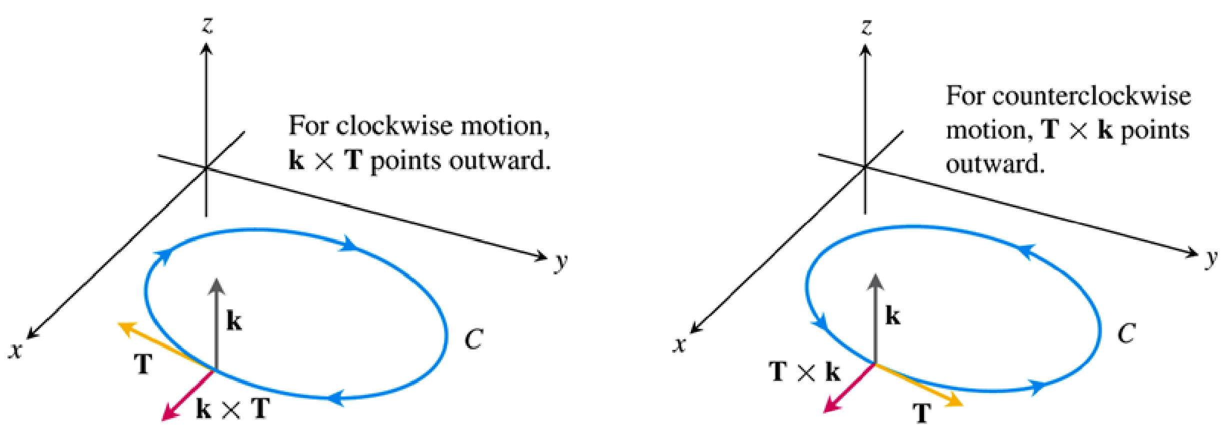
A curve in the *xy*-plane is simple if it does not cross itself. When a curve starts and ends at the same point, it is a ***closed curve*** or ***loop***.



Definition

If C is a smooth simple closed curve in the domain of a continuous velocity field in $\vec{F} = M(x, y)\hat{i} + N(x, y)\hat{j}$ in the plane, and if $\vec{n}$ is the outward-pointing unit normal vector on C, the flux of $\vec{F}$ across C is

$$\textit{Flux of } \vec{F} \textit{ across } C = \int_C \vec{F} \cdot \vec{n}\, ds$$



$$\begin{aligned}\vec{n} &= \vec{T} \times \hat{k} \\ &= \left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}\right) \times \hat{k} \\ &= \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}\end{aligned}$$

$$\vec{F} \cdot \vec{n} = M(x, y)\frac{dy}{ds} - N(x, y)\frac{dx}{ds}$$

Calculating Flux Across a Smooth Closed Plane Curve

$$\left(\textit{Flux of } \vec{F} = M\hat{i} + N\hat{j} \textit{ across } C\right) = \oint_C M\,dy - N\,dx$$

The integral can be evaluated from any smooth parametrization $x = g(t),\ y = h(t),\ a \le t \le b$, that traces C counterclockwise exactly once.

Example

Find the flux of $\vec{F} = (x - y)\hat{i} + x\hat{j}$ across the circle $x^2 + y^2 = 1$ in the *xy*-plane. (The vector field and curve)

Solution

The parametrization $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}, \quad 0 \le t \le 2\pi$ traces the circle counterclockwise exactly once.

$$M = x - y = \cos t - \sin t, \qquad dy = d(\sin t) = \cos t\ dt$$
$$N = x = \cos t, \qquad dx = d(\cos t) = -\sin t\ dt$$

$$Flux = \int_C M\,dy - N\,dx$$

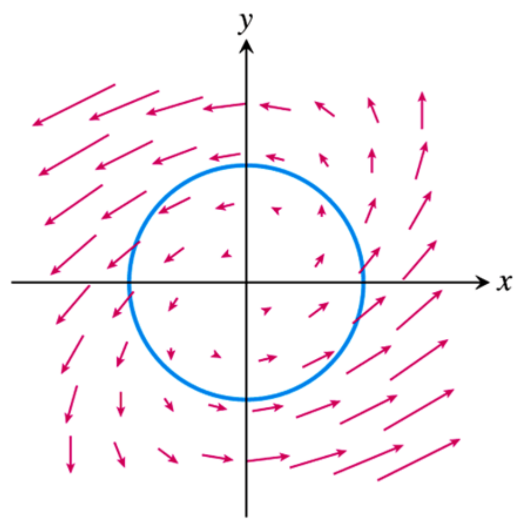
$$= \int_0^{2\pi} \left(\cos^2 t - \sin t \cos t + \cos t \sin t\right) dt$$

$$= \int_0^{2\pi} \cos^2 t\ dt$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right) dt$$

$$= \left(\frac{1}{2}t + \frac{1}{4}\sin 2t\right.\Big|_0^{2\pi}$$

$$\underline{= \pi\ }$$



Exercises *Section* 4.3 – Conservative Vector Fields

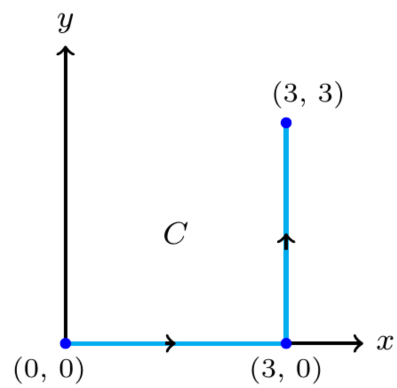
1. Find the gradient field of the function $f(x,y,z)=\left(x^2+y^2+z^2\right)^{-1/2}$

2. Find the gradient field of the function $f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}$

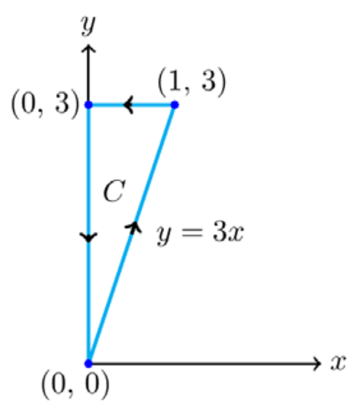
3. Find the gradient field of the function $f(x,y,z)=e^z-\ln\left(x^2+y^2\right)$

4. Find the line integral of $\int_C (x-y)\,dx$ where $C: \ x=t, \quad y=2t+1, \quad$ *for* $0\le t\le 3$

5. Find the line integral of $\int_C \left(x^2+y^2\right)dy$ where C is



6. Find the line integral of $\int_C \sqrt{x+y}\ dx$ where C is



7. Find the work done by the force field $\vec{F}=xy\hat{i}+y\hat{j}-yz\hat{k}$ over the curve $\vec{r}(t)=t\hat{i}+t^2\hat{j}+t\hat{k}, \quad 0\le t\le 1.$

8. Find the work done by the force field $\vec{F} = 2y\hat{i} + 3x\hat{j} + (x+y)\hat{k}$ over the curve $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + \frac{t}{6}\hat{k}, \quad 0 \le t \le 2\pi$

9. Find the work done by the force field $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ over the curve $\vec{r}(t) = (\sin t)\hat{i} + (\cos t)\hat{j} + t\hat{k}, \quad 0 \le t \le 2\pi$.

10. Find the work required to move an object with given force field $\vec{F} = \langle -y,\ z,\ x \rangle$ on the path consisting of the line segments from $(0,\ 0,\ 0)$ to $(0,\ 1,\ 0)$ followed by the line segment from $(0,\ 1,\ 0)$ to $(0,\ 1,\ 4)$

11. Find the work required to move an object with given force field $\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ on the path $\mathbf{r}(t) = \left\langle t^2,\ 3t^2,\ -t^2 \right\rangle$ for $1 \le t \le 2$

12. Evaluate $\displaystyle\int_C \vec{F} \bullet \vec{T}\, ds$ for the vector field $\vec{F} = x^2\hat{i} - y\hat{j}$ along the curve $x = y^2$ from $(4,\ 2)$ to $(1,\ -1)$

13. Find the circulation and flux of the fields $\vec{F}_1 = x\hat{i} + y\hat{j}$ *and* $\vec{F}_2 = -y\hat{i} + x\hat{j}$ around and across each of the following curves.

a) The circle $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}, \quad 0 \le t \le 2\pi$

b) The ellipse $\vec{r}(t) = (\cos t)\hat{i} + (4\sin t)\hat{j}, \quad 0 \le t \le 2\pi$

14. Find the circulation and flux of the fields $\vec{F}_1 = 2x\hat{i} - 3y\hat{j}$ *and* $\vec{F}_2 = 2x\hat{i} + (x - y)\hat{j}$ across the circle $\vec{r}(t) = (a\cos t)\hat{i} + (a\sin t)\hat{j}, \quad 0 \le t \le 2\pi$

15. Find a field $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ in the *xy*-plane with the property that at each point $(x,y) \ne (0,0)$, $\vec{F}$ points toward the origin and $|\vec{F}|$ is

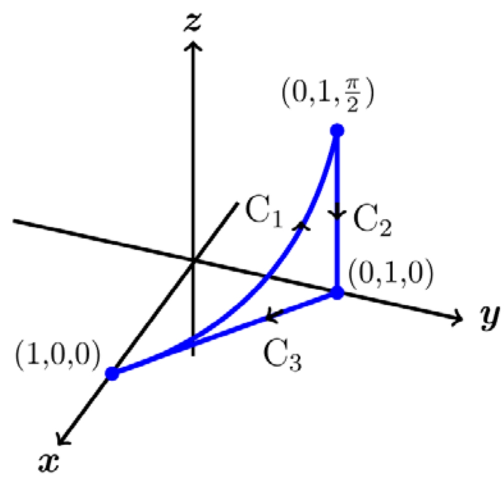
a) The distance from (x, y) to the origin

b) Inversely proportional to the distance from (x, y) to the origin.
(The field is undefined at (0, 0).)

16. A fluid's velocity field is $\vec{F} = -4xy\hat{i} + 8y\hat{j} + 2\hat{k}$. Find the flow along the curve $\vec{r}(t) = t\hat{i} + t^2\hat{j} + \hat{k}, \quad 0 \le t \le 2$

17. A fluid's velocity field is $\vec{F} = x^2\hat{i} + yz\hat{j} + y^2\hat{k}$. Find the flow along the curve $\vec{r}(t) = 3t\hat{j} + 4t\hat{k}, \quad 0 \le t \le 1$

18. Find the circulation of $\vec{F} = 2x\hat{j} + 2z\hat{j} + 2y\hat{k}$ around the closed path consisting of the following three curves traversed in the direction of increasing *t*.

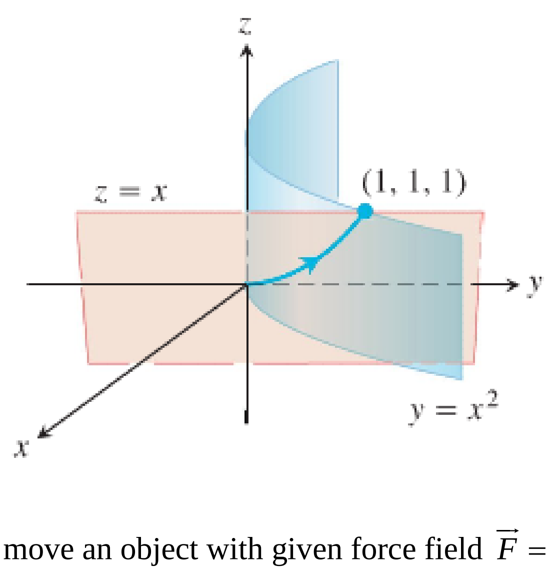


C_1 : $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}, \quad 0 \le t \le \frac{\pi}{2}$

C_2 : $\vec{r}(t) = \hat{j} + \frac{\pi}{2}(1-t)\hat{k}, \quad 0 \le t \le 1$

C_3 : $\vec{r}(t) = t\hat{i} + (1-t)\hat{j}, \quad 0 \le t \le 1$

19. The field $\vec{F} = xy\hat{i} + y\hat{j} - yz\hat{k}$ is the velocity field of a flow in space. Find the flow from $(0,\ 0,\ 0)$ to $(1,\ 1,\ 1)$ along the curve of intersection of the cylinder $y = x^2$ and the plane $z = x$.
(***Hint***: Use $t = x$ as the parameter.)



20. Find the work required to move an object with given force field $\vec{F} = \langle -y,\ z,\ x \rangle$ on the path consisting of the line segments from $(0,\ 0,\ 0)$ to $(0,\ 1,\ 0)$ followed by the line segment from $(0,\ 1,\ 0)$ to $(0,\ 1,\ 4)$

21. Find the work required to move an object with given force field $\vec{F} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{3/2}}$ on the path $\boldsymbol{r}(t) = \left\langle t^2,\ 3t^2,\ -t^2\right\rangle$ for $1 \le t \le 2$

22. Evaluate $\displaystyle\int_C (x-y)\,dx + (x+y)\,dy$ counterclockwise around the triangle with vertices (0, 0), $(1,\ 0)$ and $(0,\ 1)$

(23–28) Evaluate the line integral $\displaystyle\int_C \vec{F} \bullet d\vec{r}$ for the vector fields $\vec{F}$ and curves C.

23. $\vec{F} = \nabla\left(x^2 y\right);\quad C:\ \vec{r}(t) = \left\langle 9-t^2,\ t\right\rangle,\quad \textit{for}\ \ 0 \le t \le 3$

24. $\vec{F} = \nabla(xyz);\quad C:\ \vec{r}(t) = \left\langle \cos t,\ \sin t,\ \frac{t}{\pi}\right\rangle,\quad \textit{for}\ \ 0 \le t \le \pi$

25. $\vec{F} = \langle x,\ -y\rangle$; C is the square with vertices $(\pm 1,\ \pm 1)$ with counterclockwise orientation.

26. $\vec{F} = \langle y,\ z,\ -x\rangle;\quad C:\ \vec{r}(t) = \langle \cos t,\ \sin t,\ 4\rangle,\quad \textit{for}\ \ 0 \le t \le 2\pi$

27. $\vec{F} = \left\langle y^2,\ x\right\rangle$; where C is the arc of the parabola $x = 4 - y^2$ from $(-5,\ -3)$ to $(0,\ 2)$

28. $\vec{F} = \left\langle x^2+y^2,\ 4x+y^2\right\rangle$; where C is the straight line segment from $(6,\ 3)$ to $(6,\ 0)$

(29–34) Evaluate the line integral $\displaystyle\int_C \vec{F} \bullet \vec{T}\ ds$ for the vector fields $\vec{F}$ and curves C.

29. $\vec{F} = \langle x,\ y\rangle$ on the parabola $\vec{r}(t) = \left\langle 4t,\ t^2\right\rangle \quad 0 \le t \le 1$

30. $\vec{F} = \langle -y,\ x\rangle$ on the semicircle $\vec{r}(t) = \langle 4\cos t,\ 4\sin t\rangle \quad 0 \le t \le \pi$

31. $\vec{F} = \langle y,\ x\rangle$ on the line segment from $(1,\ 1)$ to $(5,\ 10)$

32. $\vec{F} = \langle -y,\ x\rangle$ on the parabola $y = x^2$ from $(0,\ 0)$ to $(1,\ 1)$

33. $\vec{F} = \dfrac{\langle x,\ y\rangle}{\left(x^2+y^2\right)^{3/2}}$ on the curve $\vec{r}(t) = \left\langle t^2,\ 3t^2\right\rangle \quad 1 \le t \le 2$

34. $\vec{F} = \dfrac{\langle x,\ y\rangle}{x^2+y^2}$ on the line $\vec{r}(t) = \langle t,\ 4t\rangle \quad 1 \le t \le 10$

(35–45) Find the work required to move an object on the given oriented curve

35. $\vec{F} = \langle y,\ -x \rangle$ on the path consisting of the line segment from $(1,\ 2)$ to $(0,\ 0)$ followed by the line segment from $(0,\ 0)$ to $(0,\ 4)$

36. $\vec{F} = \langle x,\ y \rangle$ on the path consisting of the line segment from $(-1,\ 0)$ to $(0,\ 8)$ followed by the line segment from $(0,\ 8)$ to $(2,\ 8)$

37. $\vec{F} = \left\langle x^2,\ -xy \right\rangle$ on runs from $(1,\ 0)$ to $(0,\ 1)$ along the unit circle and then from $(0,\ 1)$ to $(0,\ 0)$ along the *y*-axis.

38. $\vec{F} = \langle y,\ x \rangle$ on the parabola $y = 2x^2$ from $(0,\ 0)$ to $(2,\ 8)$

39. $\vec{F} = \langle y,\ -x \rangle$ on the line $y = 10 - 2x$ from $(1,\ 8)$ to $(3,\ 4)$

40. $\vec{F} = \langle x,\ y,\ z \rangle$ on the tilted ellipse $\vec{r}(t) = \langle 4\cos t,\ 4\sin t,\ 4\cos t \rangle \quad 0 \le t \le 2\pi$

41. $\vec{F} = \langle -y,\ x,\ z \rangle$ on the helix $\vec{r}(t) = \left\langle 2\cos t,\ 2\sin t,\ \frac{t}{2\pi} \right\rangle \quad 0 \le t \le 2\pi$

42. $\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ on the line segment from $(1,\ 1,\ 1)$ to $(10,\ 10,\ 10)$

43. $\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ on the path $\vec{r}(t) = \left\langle t^2,\ 3t^2,\ -t^2 \right\rangle, \quad 1 \le t \le 2$

44. $\vec{F} = \dfrac{\langle x,\ y \rangle}{\left(x^2 + y^2\right)^{3/2}}$ over the plane curve $\vec{r}(t) = \left\langle e^t \cos t,\ e^t \sin t \right\rangle$ from the point $(1,\ 0)$ to the point $\left(e^{2\pi},\ 0\right)$ by using the parametrization of the curve to evaluate the work integral

45. $\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{x^2 + y^2 + z^2}$ on the line segment from $(1,\ 1,\ 1)$ to $(8,\ 4,\ 2)$

46. Let *C* be the circle of radius 2 centered at the origin with counterclockwise orientation

a) Give the unit outward vector at any point $(x,\ y)$ on *C*.

b) Find the normal component of the vector field $\vec{F} = 2\langle y,\ -x \rangle$ at any point on *C*.

c) Find the normal component of the vector field $\vec{F} = \dfrac{\langle x,\ y \rangle}{x^2 + y^2}$ at any point on *C*.

47. Find the flow of the field $\vec{F} = \nabla\left(x^2 z e^y\right)$

a) Once around the ellipse C in which the plane $x + y + z = 1$ intersects the cylinder $x^2 + z^2 = 25$, clockwise as viewed from the positive y-axis.

b) Along the curved boundary of the helicoid $\vec{r}(r,\ \theta) = (r\cos\theta)\hat{i} + (r\sin\theta)\hat{j} + \theta\,\hat{k}$ from $(1,\ 0,\ 0)$ to $(1,\ 0,\ 2\pi)$