Solution Section 2.3 – Product and Quotient Rules

Exercise

Find the derivative of $y=(x+1)\left(\sqrt{x}+2\right)$

Solution

$$y'=(1)\left(x^{1/2}+2\right)+(x+1)\left(\frac{1}{2}x^{-1/2}\right)$$

$$=x^{1/2}+2+\frac{1}{2}x^{1/2}+\frac{1}{2}x^{-1/2}$$

$$=\frac{3}{2}x^{1/2}+\frac{1}{2}x^{-1/2}+2$$

Exercise

Find the derivative of $y=\left(4x+3x^2\right)(6-3x)$

Solution

$$y'=\left(4x+3x^2\right)\frac{d}{dx}(6-3x)+(6-3x)\frac{d}{dx}\left(4x+3x^2\right)$$

$y=24x+6x^2-9x^3$

$$=\left(4x+3x^2\right)(-3)+(6-3x)(4+6x)$$

$$=-12x-9x^2+24+36x-12x-18x^2$$

$$=-27x^2+12x+24$$

Exercise

Find the derivative of $y=\left(\frac{1}{x}+1\right)(2x+1)$

Solution

$$y'=\left(x^{-1}+1\right)\frac{d}{dx}(2x+1)+(2x+1)\frac{d}{dx}\left(x^{-1}+1\right)$$

$$=\left(x^{-1}+1\right)(2)+(2x+1)\left(-x^{-2}\right)$$

$$=\frac{2}{x}+2+(2x+1)\left(-\frac{1}{x^2}\right)$$

$$=\frac{2}{x}+2-\frac{2}{x}-\frac{1}{x^2}$$

$$=2-\frac{1}{x^2}$$

$$=\frac{2x^2-1}{x^2}$$

Exercise

Find the derivative of $y=\dfrac{3-\frac{2}{x}}{x+4}$

Solution

$$y=\dfrac{\frac{3x-2}{x}}{x+4}$$

$$=\frac{3x-2}{x}\cdot\frac{1}{x+4}$$

$$=\frac{3x-2}{x^2+4x}$$

$$y'=\dfrac{\begin{vmatrix}0&3\\1&4\end{vmatrix}x^2+2\begin{vmatrix}0&-2\\1&0\end{vmatrix}x+\begin{vmatrix}3&-2\\4&0\end{vmatrix}}{\left(x^2+4x\right)^2}$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\dfrac{\begin{vmatrix}a&b\\d&e\end{vmatrix}x^2+2\begin{vmatrix}a&c\\d&f\end{vmatrix}x+\begin{vmatrix}b&c\\e&f\end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

$$=\frac{-3x^2+4x+8}{x^2(x+4)^2}$$

OR

$$y'=\frac{\left(x^2+4x\right)(3)-(3x-2)(2x+4)}{[x(x+4)]^2}$$

$$=\frac{3x^2+12x-6x^2-12x+4x+8}{x^2(x+4)^2}$$

$$=\frac{-3x^2+4x+8}{x^2(x+4)^2}$$

Exercise

Find the derivative of $g(x)=\dfrac{x^2-4x+2}{x^2+3}$

Solution

$$g'(x)=\dfrac{\begin{vmatrix}1&-4\\1&0\end{vmatrix}x^2+2\begin{vmatrix}1&2\\1&3\end{vmatrix}x+\begin{vmatrix}-4&2\\0&3\end{vmatrix}}{\left(x^2+3\right)^2}$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\dfrac{\begin{vmatrix}a&b\\d&e\end{vmatrix}x^2+2\begin{vmatrix}a&c\\d&f\end{vmatrix}x+\begin{vmatrix}b&c\\e&f\end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

$$= \boxed{\frac{4x^2+2x-12}{\left(x^2+3\right)^2}}$$

Or

$$g' = \frac{(2x-4)\left(x^2+3\right)-\left(x^2-4x+2\right)(2x)}{\left(x^2+3\right)^2}$$

$$= \frac{2x^3+6x-4x^2-12-2x^3+8x^2-4x}{\left(x^2+3\right)^2}$$

$$= \boxed{\frac{4x^2+2x-12}{\left(x^2+3\right)^2}}$$

Exercise

Find the derivative of $f(x) = \dfrac{(3-4x)(5x+1)}{7x-9}$

Solution

$$f(x) = \frac{-20x^2+11x+3}{7x-9}$$

$$f'(x) = \frac{\begin{vmatrix}-20 & 11\\ 0 & 7\end{vmatrix}x^2 + 2\begin{vmatrix}-20 & 3\\ 0 & -9\end{vmatrix}x + \begin{vmatrix}11 & 3\\ 7 & -9\end{vmatrix}}{(7x-9)^2}$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right) = \frac{\begin{vmatrix}a & b\\ d & e\end{vmatrix}x^2 + 2\begin{vmatrix}a & c\\ d & f\end{vmatrix}x + \begin{vmatrix}b & c\\ e & f\end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

$$= \boxed{\frac{-140x^2+360x-120}{(7x-9)^2}}$$

Or

$$D_x\left[\frac{(3-4x)(5x+1)}{7x-9}\right] = \frac{\left[(-4)(5x+1)+(3-4x)(5)\right](7x-9)-(3-4x)(5x+1)(7)}{(7x-9)^2}$$

$$= \frac{[-20x-4+15-20x](7x-9)-\left(15x+3-20x^2-4x\right)(7)}{(7x-9)^2}$$

$$= \frac{(-40x+11)(7x-9)-7\left(-20x^2+11x+3\right)}{(7x-9)^2}$$

$$= \frac{-280x^2+360x+77x-99+140x^2-77x-21}{(7x-9)^2}$$

$$= \boxed{\frac{-140x^2+360x-120}{(7x-9)^2}}$$

Exercise

Find the derivative of $f(x)=x\left(1-\frac{2}{x+1}\right)$

Solution

$$f(x)=x-\frac{2x}{x+1}$$

$$\boxed{f'(x)=1-\frac{2}{(x+1)^2}} \qquad \left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

OR

$$\left(\frac{2x}{x+1}\right)' \Rightarrow \quad \begin{matrix} f=2x & f'=2 \\ g=x+1 & g'=1 \end{matrix}$$

$$\begin{aligned} f'(x) &= 1-\frac{2(x+1)-2x}{(x+1)^2} \\ &= 1-\frac{2x+2-2x}{(x+1)^2} \\ &= \boxed{1-\frac{2}{(x+1)^2}} \end{aligned}$$

Exercise

Find the derivative of $f(x)=\left(\sqrt{x}+3\right)\left(x^2-5x\right)$

Solution

$$\begin{aligned} f' &= \left(\tfrac{1}{2}x^{-1/2}\right)\left(x^2-5x\right)+\left(\sqrt{x}+3\right)(2x-5) \\ &= \tfrac{1}{2}x^{3/2}-\tfrac{5}{2}x^{1/2}+2x^{3/2}-5x^{1/2}+6x-15 \\ &= \tfrac{5}{2}x^{3/2}-\tfrac{15}{2}x^{1/2}+6x-15 \\ &= \boxed{\tfrac{5}{2}x^{3/2}+6x-\tfrac{15}{2}x^{1/2}-15} \end{aligned}$$

Exercise

Find the derivative of $y=(2x+3)\left(5x^2-4x\right)$

Solution

$$y=(2x+3)\left(5x^2-4x\right)=10x^3-8x^2+15x^2-12x$$

$$=10x^3+7x^2-12x$$

$$\boxed{y'=30x^2+14x-12}$$

Exercise

Find the derivative of $y=\left(x^2+1\right)\left(x+5+\frac{1}{x}\right)$

Solution

$$y=x^3+5x^2+x+x+5+\frac{1}{x}$$

$$=x^3+5x^2+2x+5+x^{-1}$$

$$y'=3x^2+10x+2-x^{-2}$$

$$\boxed{=3x^2+10x+2-\frac{1}{x^2}}$$

Exercise

Find the derivative of $y=\dfrac{x+4}{5x-2}$

Solution

$$\boxed{y'=-\frac{22}{(5x-2)^2}}$$

$$\left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

OR

$$y'=\frac{(5x-2)\frac{d}{dx}[(x+4)]-(x+4)\frac{d}{dx}[(5x-2)]}{(5x-2)^2}$$

$$=\frac{(5x-2)(1)-(x+4)(5)}{(5x-2)^2}$$

$$=\frac{5x-2-5x-20}{(5x-2)^2}$$

$$\boxed{=-\frac{22}{(5x-2)^2}}$$

Exercise

Find the derivative of $z = \dfrac{4-3x}{3x^2+x}$

Solution

$$\boxed{z' = \frac{9x^2-24x-4}{\left(3x^2+x\right)^2}}$$

$$\begin{array}{ccc} 0 & -3 & 4 \\ 3 & 1 & 0 \end{array}$$

$$\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)' = \frac{(ae-bd)x^2+2(af-cd)x+bf-ce}{\left(dx^2+ex+f\right)^2}$$

OR

$u = 4-3x \quad v = 3x^2+x$

$u' = -3 \qquad v' = 6x+1$

$$z' = \frac{-3\left(3x^2+x\right)-(6x+1)(4-3x)}{\left(3x^2+x\right)^2} \qquad z' = \frac{u'v-v'u}{u^2}$$

$$= \frac{-9x^2-3x-\left(24x-18x^2+4-3x\right)}{\left(3x^2+x\right)^2}$$

$$= \frac{-9x^2-3x-21x+18x^2-4}{\left(3x^2+x\right)^2}$$

$$\boxed{= \frac{9x^2-24x-4}{\left(3x^2+x\right)^2}}$$

Exercise

Find the derivative of $y = (2x-7)^{-1}(x+5)$

Solution

$$y = \frac{x+5}{2x-7}$$

$$\boxed{y' = -\frac{17}{(2x-7)^2}} \qquad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

OR

$$y' = -(2x-7)^{-2}(2)(x+5)+(2x-7)^{-1}$$

$$= -(2x-7)^{-2}(2x+10)+(2x-7)^{-1}$$

$$= \left[-(2x-7)^{-2}(2x+10) + (2x-7)^{-1} \right] \frac{(2x-7)^2}{(2x-7)^2}$$

$$= \frac{-2x-10+2x-7}{(2x-7)^2}$$

$$= \boxed{\frac{-17}{(2x-7)^2}}$$

Exercise

Find the derivative of $f(x) = \dfrac{\sqrt{x}-1}{\sqrt{x}+1}$

Solution

$$f'(x) = \frac{\frac{1}{2}(1+1)x^{-1/2}}{\left(\sqrt{x}+1\right)^2} \qquad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{(cx+d)^2}$$

$$= \boxed{\frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)^2}}$$

OR

$$f'(x) = \frac{\frac{1}{2}x^{-1/2}\left(x^{1/2}+1\right) - \frac{1}{2}x^{-1/2}\left(x^{1/2}-1\right)}{\left(\sqrt{x}+1\right)^2} \qquad u = x^{1/2}-1 \quad v = x^{1/2}+1$$

$$u' = \frac{1}{2}x^{-1/2} \quad v' = \frac{1}{2}x^{-1/2}$$

$$= \frac{1}{2}\frac{1+x^{-1/2}-1+x^{-1/2}}{\left(\sqrt{x}+1\right)^2}$$

$$= \frac{1}{2}\frac{2x^{-1/2}}{\left(\sqrt{x}+1\right)^2}$$

$$= \frac{1}{x^{1/2}\left(\sqrt{x}+1\right)^2}$$

$$= \boxed{\frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)^2}}$$

Exercise

Find the derivative of $y = \dfrac{1}{\left(x^2-1\right)\left(x^2+x+1\right)}$

Solution

$$y = \frac{1}{x^4+x^3+x^2-x^2-x-1}$$

$$= \frac{1}{x^4 + x^3 - x - 1}$$

$$y' = \frac{-\left(4x^3 + 3x^2 - 1\right)}{\left(x^4 + x^3 - x - 1\right)^2} \qquad \left(\frac{1}{v}\right)' = -\frac{v'}{v^2}$$

$$\boxed{= \frac{-4x^3 - 3x^2 + 1}{\left(x^4 + x^3 - x - 1\right)^2}}$$

Exercise

Find the derivative of $f(x) = \dfrac{x^{3/2}\left(x^2 + 1\right)}{x + 1}$

Solution

$$f(x) = \frac{x^{7/2} + x^{3/2}}{x + 1}$$

$$u = x^{7/2} + x^{3/2} \qquad v = x + 1$$

$$u' = \frac{7}{2}x^{5/2} + \frac{3}{2}x^{1/2} \qquad v' = 1$$

$$f'(x) = \frac{\frac{7}{2}x^{7/2} + \frac{3}{2}x^{3/2} + \frac{7}{2}x^{5/2} + \frac{3}{2}x^{1/2} - x^{7/2} - x^{3/2}}{(x + 1)^2}$$

$$\boxed{= \frac{1}{2}\frac{5x^{7/2} + x^{3/2} + 7x^{5/2} + 3x^{1/2}}{(x + 1)^2}}$$

Exercise

Find the derivative of $f(x) = \dfrac{x^3 - 4x^2 + x}{x - 2}$

Solution

$$f'(x) = \frac{3x^3 - 8x^2 + x - 6x^2 + 16x - 2 - x^3 + 4x^2 - x}{(x - 2)^2} \qquad u = x^3 - 4x^2 + x \quad v = x - 2$$

$$u' = 3x^2 - 8x + 1 \qquad v' = 1$$

$$\boxed{= \frac{2x^3 - 10x^2 + 16x - 2}{(x - 2)^2}}$$

Exercise

Find the derivative of $g(x)=\dfrac{x(3-x)}{2x^2}$

Solution

$$g(x)=\frac{1}{2}\frac{3-x}{x}$$

$$\boxed{g'=-\frac{3}{2x^2}}$$

$$\left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

OR

$u=3-x \quad v=x$

$u'=-1 \quad v'=1$

$$g'(x)=\frac{1}{2}\frac{-x-3+x}{x^2}$$

$$\boxed{=-\frac{3}{2x^2}}$$

Exercise

Find the derivative of $y=\dfrac{2x^2}{3x+1}$

Solution

$$\boxed{y'=\frac{6x^2+4x}{(3x+1)^2}}$$

$$\begin{matrix} 2 & 0 & 0 \\ 0 & 3 & 1 \end{matrix}$$

$$\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)'=\frac{(ae-bd)x^2+2(af-cd)x+bf-ce}{\left(dx^2+ex+f\right)^2}$$

OR

$u=x^2 \quad v=3x+1$

$u'=2x \quad v'=3$

$$y'=2\frac{6x^2+2x-3x^2}{(3x+1)^2}$$

$$\boxed{=\frac{6x^2+4x}{(3x+1)^2}}$$

Exercise

Find the derivative of $\quad f(x)=\dfrac{x^9+x^8+4x^5-7x}{x^4-3x^2+2x+1}$

Solution

$u=x^9+x^8+4x^5-7x \qquad v=x^4-3x^2+2x+1$

$u'=9x^8+8x^7+20x^4-7 \qquad v'=4x^3-6x+2$

x^{12}	x^{11}	x^{10}	x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x	x^0
9 −12	8 −4	−27 6	18 −24 6 −2	9 16 20 −16 −2	8	−60 24	40 −8	20 −7 −28		21 42	−14 14	−7

$$\boxed{f'(x)=\frac{-3x^{12}+4x^{11}-21x^{10}-2x^9+27x^8+8x^7-36x^6+32x^5-15x^4+63x^2-7}{\left(x^4-3x^2+2x+1\right)^2}}$$

Exercise

Find the derivative of $\quad f(x)=\dfrac{x}{1+x^2}$

Solution

$$\boxed{f'=\frac{1-x^2}{\left(x^2+1\right)^2}}$$

$\begin{matrix}0 & 1 & 0\\ 1 & 0 & 1\end{matrix}$

$$\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)'=\frac{(ae-bd)x^2+2(af-cd)x+bf-ce}{\left(dx^2+ex+f\right)^2}$$

OR

$u=x \qquad v=1+x^2$

$u'=1 \qquad v'=2x$

$$f'(x)=\frac{1+x^2-2x^2}{\left(1+x^2\right)^2}$$

$$\boxed{=\frac{1-x^2}{\left(1+x^2\right)^2}}$$

Exercise

Find the derivative of $\quad y=\dfrac{x^2-2ax+a^2}{x-a}$

Solution

$$y=\frac{(x-a)^2}{x-a}$$

$$=x-a$$

$$\underline{y'=1}$$

Exercise

Find the derivative of $\quad f(x)=\dfrac{x^2+4x^{1/2}}{x^2}$

Solution

$$f(x)=1+4x^{-3/2}$$

$$\underline{f'(x)=-6x^{-5/2}}$$

Exercise

Find the derivative of $\quad f(x)=(2x+1)\left(3x^2+2\right)$

Solution

$$f'(x)=2\left(3x^2+2\right)+(6x)(2x+1)$$

$$=6x^2+4+12x^2+6x$$

$$\underline{=18x^2+6x+4}$$

Exercise

Find the derivative of $\quad f(x)=\dfrac{x^2-1}{x^2+1}$

Solution

$$\underline{f'=\frac{-2x^3+2x^2+4x}{\left(x^2+1\right)^2}}$$

$$\begin{matrix} 1 & 0 & -1 \\ 1 & 0 & 1 \end{matrix}$$

$$\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)'=\frac{(ae-bd)x^2+2(af-cd)x+bf-ce}{\left(dx^2+ex+f\right)^2}$$

OR

$u = x^2 - 1 \quad v = x^2 + 1$

$u' = 2x \qquad v' = 2x$

$$f'(x) = \frac{2x^2 + 2x - 2x^3 + 2x}{\left(x^2+1\right)^2}$$

$$\boxed{= \frac{-2x^3 + 2x^2 + 4x}{\left(x^2+1\right)^2}}$$

Exercise

Find the derivative of $\quad y = \dfrac{4x^3 + 3x + 1}{2x^5}$

Solution

$$y = 2x^{-2} + \frac{3}{2}x^{-4} + \frac{1}{2}x^{-5}$$

$$y' = -4x^{-3} - 6x^{-5} - \frac{5}{2}x^{-6}$$

$$= -\frac{1}{2}x^{-6}\left(8x^3 + 12x + 5\right)$$

$$\boxed{= -\frac{8x^3 + 12x + 5}{2x^6}}$$

Exercise

Find the derivative of $\quad y = \dfrac{4}{3-x}$

Solution

$$\boxed{y' = \frac{4}{(3-x)^2}} \qquad \left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$$

Exercise

Find the derivative of $\quad y = \dfrac{2}{1-x^2}$

Solution

$$\boxed{y' = \frac{4x}{\left(1-x^2\right)^2}} \qquad \left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$$

Exercise

Find the derivative of $f(x) = \dfrac{\pi}{2 - \pi x}$

Solution

$$\boxed{f'(x) = \frac{\pi^2}{(2 - \pi x)^2}}$$

$$\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$$

Exercise

Find the derivative of $y = \dfrac{x - 4}{5x - 2}$

Solution

$$y' = \frac{1(-2) - (-4)(5)}{(5x - 2)^2}$$

$$\left(\frac{ax + b}{cx + d}\right)' = \frac{ad - bc}{(cx + d)^2}$$

$$\boxed{= \frac{18}{(5x - 2)^2}}$$

Exercise

Find the derivative of $y = \dfrac{3x - 4}{2x - 1}$

Solution

$$y' = \frac{-3 + 8}{(2x - 1)^2}$$

$$\left(\frac{ax + b}{cx + d}\right)' = \frac{ad - bc}{(cx + d)^2}$$

$$\boxed{= \frac{5}{(2x - 1)^2}}$$

Exercise

Find the derivative of $y = \dfrac{3x + 4}{2x + 1}$

Solution

$$y' = \frac{3 - 8}{(2x + 1)^2}$$

$$\left(\frac{ax + b}{cx + d}\right)' = \frac{ad - bc}{(cx + d)^2}$$

$$\boxed{= \frac{-5}{(2x + 1)^2}}$$

Exercise

Find the derivative of $y=\dfrac{-3x+4}{2x+1}$

Solution

$$y'=\frac{-3-8}{(2x+1)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

$$=-\frac{11}{(2x+1)^2}$$

Exercise

Find the derivative of $y=\dfrac{-3x-4}{2x-1}$

Solution

$$y'=\frac{3+8}{(2x-1)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

$$=\frac{11}{(2x+1)^2}$$

Exercise

Find the derivative of $y=\dfrac{2x-3}{x+1}$

Solution

$$y'=\frac{2+3}{(x+1)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

$$=\frac{5}{(x+1)^2}$$

Exercise

Find the derivative of $y=\dfrac{3x}{3x-2}$

Solution

$$y'=\frac{-6}{(3x-2)^2}$$

$$\left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y=\dfrac{x-3}{2x+5}$

Solution

$$\boxed{y'=\frac{11}{(2x+5)^2}}$$

$$\left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y=\dfrac{5x-3}{2x+5}$

Solution

$$\boxed{y'=\frac{31}{(2x+5)^2}}$$

$$\left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y=\dfrac{6x-8}{2x-3}$

Solution

$$\boxed{y'=-\frac{2}{(2x-3)^2}}$$

$$\left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $y=\dfrac{x^2-4}{5x^2-2}$

Solution

$$y'=\frac{2(-2+20)x}{\left(5x^2-2\right)^2}$$

$$\left(\frac{ax^n+b}{cx^n+d}\right)'=\frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$\boxed{=\frac{36x}{\left(5x^2-2\right)^2}}$$

Exercise

Find the derivative of $y=\dfrac{3x^2-4}{2x^2-1}$

Solution

$$y' = \frac{2(-3+8)x}{\left(2x^2-1\right)^2} \qquad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$= \frac{10x}{\left(2x^2-1\right)^2}$$

Exercise

Find the derivative of $y = \dfrac{3x^2+4}{2x^2+1}$

Solution

$$y' = \frac{2(3-8)x}{\left(2x^2+1\right)^2} \qquad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$= -\frac{10x}{\left(2x^2+1\right)^2}$$

Exercise

Find the derivative of $y = \dfrac{2x^2-3}{x^2+1}$

Solution

$$y' = \frac{2(2+3)x}{\left(x^2+1\right)^2} \qquad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$= \frac{10x}{\left(x^2+1\right)^2}$$

Exercise

Find the derivative of $y = \dfrac{3x^2}{3x^2-2}$

Solution

$$y' = -\frac{12x}{\left(3x^2-2\right)^2} \qquad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

Exercise

Find the derivative of $y = \dfrac{5x^2 - 3}{2x^2 + 5}$

Solution

$$y' = \frac{2(25+6)x}{\left(2x^2+5\right)^2} \qquad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$= \boxed{\frac{62x}{\left(2x^2+5\right)^2}}$$

Exercise

Find the derivative of $y = \dfrac{6x^2 - 8}{2x^2 + 1}$

Solution

$$y' = \frac{2(6+16)x}{\left(2x^2+1\right)^2} \qquad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$= \boxed{\frac{44x}{\left(2x^2+1\right)^2}}$$

Exercise

Find the derivative of $y = \dfrac{6x^3 + 8}{2x^3 + 1}$

Solution

$$y' = \frac{3(6-16)x^2}{\left(2x^3+1\right)^2} \qquad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$= \boxed{-\frac{30x^2}{\left(2x^3+1\right)^2}}$$

Exercise

Find the derivative of $y = \dfrac{5x^3 - 3}{2x^3 + 5}$

Solution

$$y' = \frac{3(25+6)x^2}{\left(2x^3+5\right)^2} \qquad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$= \boxed{\frac{93x^2}{\left(2x^3+1\right)^2}}$$

Exercise

Find the derivative of $y = \dfrac{x^3}{3x^3-2}$

Solution

$$\boxed{y' = -\frac{6x^2}{\left(3x^3-2\right)^2}} \qquad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

Exercise

Find the derivative of $y = \dfrac{2x^3-3}{2x^3+1}$

Solution

$$\boxed{y' = \frac{24x^2}{\left(2x^3+1\right)^2}} \qquad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

Exercise

Find the derivative of $y = \dfrac{2x^4-3}{2x^4+1}$

Solution

$$y' = \frac{4(2+6)x^3}{\left(2x^4+1\right)^2} \qquad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$= \boxed{\frac{32x^3}{\left(2x^4+1\right)^2}}$$

Exercise

Find the derivative of $y=\dfrac{x^2-4x+1}{5x^2-2x-1}$

Solution

$$y'=\frac{\begin{vmatrix}1&-4\\5&-2\end{vmatrix}x^2+2\begin{vmatrix}1&1\\5&-1\end{vmatrix}x+\begin{vmatrix}-4&1\\-2&-1\end{vmatrix}}{\left(5x^2-2x-1\right)^2}$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\frac{\begin{vmatrix}a&b\\d&e\end{vmatrix}x^2+2\begin{vmatrix}a&c\\d&f\end{vmatrix}x+\begin{vmatrix}b&c\\e&f\end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

$$=\frac{18x^2-12x+6}{\left(5x^2-2x-1\right)^2}$$

Exercise

Find the derivative of $y=\dfrac{3x^2-4x+2}{2x^2+x-1}$

Solution

$$y'=\frac{\begin{vmatrix}3&-4\\2&1\end{vmatrix}x^2+2\begin{vmatrix}3&2\\2&-1\end{vmatrix}x+\begin{vmatrix}-4&2\\1&-1\end{vmatrix}}{\left(2x^2+x-1\right)^2}$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\frac{\begin{vmatrix}a&b\\d&e\end{vmatrix}x^2+2\begin{vmatrix}a&c\\d&f\end{vmatrix}x+\begin{vmatrix}b&c\\e&f\end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

$$=\frac{11x^2-14x+6}{\left(2x^2+x-1\right)^2}$$

Exercise

Find the derivative of $y=\dfrac{3x^2+x-4}{2x^2+1}$

Solution

$$y'=\frac{\begin{vmatrix}3&-1\\2&0\end{vmatrix}x^2+2\begin{vmatrix}3&-4\\2&1\end{vmatrix}x+\begin{vmatrix}1&-4\\0&1\end{vmatrix}}{\left(2x^2+1\right)^2}$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\frac{\begin{vmatrix}a&b\\d&e\end{vmatrix}x^2+2\begin{vmatrix}a&c\\d&f\end{vmatrix}x+\begin{vmatrix}b&c\\e&f\end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

$$=\frac{2x^2+22x+1}{\left(2x^2+1\right)^2}$$

Exercise

Find the derivative of $y = \dfrac{2x^2 - 3}{x^2 + 5x + 1}$

Solution

$$y' = \frac{\begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix} x^2 + 2\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} x + \begin{vmatrix} 0 & -3 \\ 5 & 1 \end{vmatrix}}{\left(x^2 + 5x + 1\right)^2}$$

$$\frac{d}{dx}\left(\frac{ax^2 + bx + c}{dx^2 + ex + f}\right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2\begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{\left(dx^2 + ex + f\right)^2}$$

$$= \frac{10x^2 + 10x + 15}{\left(x^2 + 5x + 1\right)^2}$$

Exercise

Find the derivative of $y = \dfrac{3x^2}{3x^2 + 6x - 8}$

Solution

$$y' = \frac{\begin{vmatrix} 3 & 0 \\ 3 & 6 \end{vmatrix} x^2 + 2\begin{vmatrix} 3 & 0 \\ 3 & -8 \end{vmatrix} x + \begin{vmatrix} 0 & 0 \\ 6 & -8 \end{vmatrix}}{\left(3x^2 + 6x - 8\right)^2}$$

$$\frac{d}{dx}\left(\frac{ax^2 + bx + c}{dx^2 + ex + f}\right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2\begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{\left(dx^2 + ex + f\right)^2}$$

$$= \frac{18x^2 - 48x}{\left(3x^2 + 6x - 8\right)^2}$$

Exercise

Find the derivative of $y = \dfrac{x^2 + 2x}{2x^2 + x - 5}$

Solution

$$y' = \frac{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} x^2 + 2\begin{vmatrix} 1 & 0 \\ 2 & -5 \end{vmatrix} x + \begin{vmatrix} 2 & 0 \\ 1 & -5 \end{vmatrix}}{\left(2x^2 + x - 5\right)^2}$$

$$\frac{d}{dx}\left(\frac{ax^2 + bx + c}{dx^2 + ex + f}\right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2\begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{\left(dx^2 + ex + f\right)^2}$$

$$= \frac{-3x^2 - 10x - 10}{\left(2x^2 + x - 5\right)^2}$$

Exercise

Find the derivative of $y = \dfrac{x^2+5x+1}{x^2}$

Solution

$$y' = \dfrac{\begin{vmatrix}1 & 5\\ 1 & 0\end{vmatrix}x^2 + 2\begin{vmatrix}1 & 1\\ 1 & 0\end{vmatrix}x + \begin{vmatrix}5 & 1\\ 0 & 0\end{vmatrix}}{x^4}$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right) = \frac{\begin{vmatrix}a & b\\ d & e\end{vmatrix}x^2 + 2\begin{vmatrix}a & c\\ d & f\end{vmatrix}x + \begin{vmatrix}b & c\\ e & f\end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

$$= \frac{-5x^2-4x}{x^4}$$

$$= \frac{-5x-4}{x^3}$$

Exercise

Find the derivative of $y = \dfrac{x^2-3x+1}{x^2-8x+5}$

Solution

$$y' = \dfrac{\begin{vmatrix}1 & -3\\ 1 & -8\end{vmatrix}x^2 + 2\begin{vmatrix}1 & 1\\ 1 & 5\end{vmatrix}x + \begin{vmatrix}-3 & 1\\ -8 & 5\end{vmatrix}}{\left(x^2-8x+5\right)^2}$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right) = \frac{\begin{vmatrix}a & b\\ d & e\end{vmatrix}x^2 + 2\begin{vmatrix}a & c\\ d & f\end{vmatrix}x + \begin{vmatrix}b & c\\ e & f\end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

$$= \frac{-5x^2+8x-7}{\left(x^2-8x+5\right)^2}$$

Exercise

Find the ***first*** and ***second*** derivative $y = \dfrac{x^2+5x-1}{x^2}$

Solution

$$y' = \frac{(2x+5)x^2 - 2x\left(x^2+5x-1\right)}{x^4}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2} \qquad u = x^2+5x-1 \quad v = x^2$$

$$u' = 2x+5 \qquad v' = 2x$$

$$= \frac{(2x+5)x^2 - 2x\left(x^2+5x-1\right)}{x^4}$$

$$= x\frac{(2x+5)x - 2\left(x^2+5x-1\right)}{x^4}$$

$$= \frac{2x^2 + 5x - 2x^2 - 10x + 2}{x^3}$$

$$\boxed{= \frac{-5x+2}{x^3}}$$

$u = -5x + 2 \qquad v = x^3$

$u' = -5 \qquad v' = 3x^2$

$$y'' = \frac{(-5)x^3 - 3x^2(-5x+2)}{x^6}$$

$$= x^2 \frac{-5x^3 + 15x - 6}{x^6}$$

$$\boxed{= \frac{-5x^3 + 15x - 6}{x^4}}$$

Exercise

Find y', y'', y''': $\qquad y = (x-3)\sqrt{x+2}$

Solution

$$\boxed{y' = \sqrt{x+2} + \frac{1}{2}(x-3)(x+2)^{-1/2}}$$

$$y'' = \frac{1}{2}(x+2)^{-1/2} + \frac{1}{2}(x+2)^{-1/2} - \frac{1}{4}(x-3)(x+2)^{-3/2}$$

$$\boxed{= (x+2)^{-1/2} - \frac{1}{4}(x-3)(x+2)^{-3/2}}$$

$$y''' = -\frac{1}{2}(x+2)^{-3/2} - \frac{1}{4}(x+2)^{-3/2} + \frac{3}{8}(x-3)(x+2)^{-5/2}$$

$$\boxed{= -\frac{3}{4}(x+2)^{-3/2} + \frac{3}{8}(x-3)(x+2)^{-5/2}}$$

Exercise

For what value(s) of x is the line tangent to the curve $y = x\sqrt{6-x}$ horizontal? Vertical?

Solution

$$y' = \sqrt{6-x} - \frac{x}{2\sqrt{6-x}}$$

$$= \frac{12-3x}{2\sqrt{6-x}} = 0$$

$12 - 3x = 0$

$\boxed{x = 4, \quad y = 4\sqrt{2}}$

$\therefore$ Point $\left(4,\ 4\sqrt{2}\right)$ is a horizontal tangent line.

$\therefore$ The vertical tangent line inside the square root of y. $\Rightarrow$ $\boxed{x = 6}$

$$\lim_{x\to 6} y' = \lim_{x\to 6} \frac{12-3x}{2\sqrt{6-x}}$$
$$= \frac{-6}{0}$$
$$\boxed{= -\infty}$$

Exercise

Find an equation of the tangent line to the graph of $y = \dfrac{x^2-4}{2x+5}$ when $x = 0$

Solution

$$\boxed{f' = \frac{2x^2+10x+8}{(2x+5)^2}}$$

$$\begin{matrix} 1 & 0 & -4 \\ 0 & 2 & 5 \end{matrix}$$

$$\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)' = \frac{(ae-bd)x^2+2(af-cd)x+bf-ce}{\left(dx^2+ex+f\right)^2}$$

OR

$$y' = \frac{(2x+5)(2x)-(x^2-4)(2)}{(2x+5)^2}$$
$$= \frac{4x^2+10x-2x^2+8}{(2x+5)^2}$$
$$\boxed{= \frac{2x^2+10x+8}{(2x+5)^2}}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v-v'u}{v^2}$$

$$\Rightarrow x = 0 \rightarrow y' = \frac{8}{25} = m$$
$$x = 0 \rightarrow y = \frac{x^2-4}{2x+5} = -\frac{4}{5}$$

$$y = \tfrac{8}{25}(x-0) - \tfrac{4}{5}$$

$$y = m\left(x - x_1\right) + y_1$$

$$\boxed{y = \frac{8}{25}x - \frac{4}{5}}$$