Solution Section 3.6 – Exponential Growth and Decay

Exercise

Suppose that \$10,000 is invested at interest rate of 5.4% per *year*, compounded continuously.

a) Find the exponential growth function
b) What will the balance be after, 1 *yr* 10 *yrs*?
c) After how long will the investment be double?

Solution

a) $\boxed{P(t) = 10000e^{0.054t}}$

b) $P(t=1) = 10000e^{0.054(1)}$

$\boxed{\approx \$10,555}$

$P(t=10) = 10000e^{0.054(10)}$

$\boxed{\approx \$17,160}$

c) $T = \dfrac{\ln 2}{k}$

$= \dfrac{\ln 2}{0.054}$

$\boxed{\approx 12.8 \ \ yrs}$

Exercise

In 1990, the population of Africa was 643 *million* and by 2000 it had grown to 813 *million*

a) Use the exponential growth function $A(t) = A_0 e^{kt}$, in which t is the number of years after 1990, to find the exponential growth function that models data
b) By which year will Africa's population reach 2000 *million*, or two *billion*?

Solution

a) $A(t) = A_0 e^{kt}$ From 1990 to 2000, is 10 *years*

$813 = 643e^{k(10)}$

$\dfrac{813}{643} = e^{10k}$

$\ln \dfrac{813}{643} = \ln e^{10k}$

$\ln \dfrac{813}{643} = 10k$

$\dfrac{1}{10} \ln \dfrac{813}{643} = k$

$k \approx 0.023$

$\boxed{A(t) = 643e^{0.023t}}$

b) $2000 = 643e^{0.023t}$

$$\frac{2000}{643} = e^{0.023t}$$

$$\ln\frac{2000}{643} = \ln e^{0.023t}$$

$$\ln\frac{2000}{643} = 0.023t$$

$$\frac{\ln\frac{2000}{643}}{0.023} = t$$

$$\boxed{t \approx 49}$$

Africa's population reach 2000 *million* in $\boxed{Year: \ 2039}$

Exercise

The radioactive element carbon-14 has a half-life of 5750 yr. The percentage of carbon-14 present in the remains of organic matter can be used to determine the age of that organic matter. Archaeologists discovered that the linen wrapping from one of the Dead Sea Scrolls had lost 22.3% of its carbon-14 at the time it was found. How old was the linen wrapping?

Solution

When t = 5750 (half-life) $\rightarrow$ P(t) will be half P_0 $\rightarrow P(t) = \frac{1}{2}P_0$

$$P(t) = P_0 e^{-kt}$$

$$\frac{1}{2}P_0 = P_0 e^{-k(5750)}$$

$$\frac{1}{2} = e^{-k(5750)}$$

$$\ln\frac{1}{2} = \ln e^{-k(5750)}$$

$$\ln\frac{1}{2} = -k(5750)$$

$$k = -\frac{\ln\frac{1}{2}}{5750}$$

$$\boxed{\approx 0.00012}$$

$$P(t) = P_0 e^{-(0.00012)t}$$

Lost 22.3% $\Rightarrow$ 100 – 22.3 = 77.7% left from it is original.

$$0.777P_0 = P_0 e^{-0.00012t}$$

$$0.777 = e^{-0.00012t}$$

$$\ln 0.777 = \ln e^{-0.00012t}$$

$$\ln 0.777 = -0.00012t$$

$$-0.00012t = \ln 0.777$$

$$t = \frac{\ln 0.777}{-0.00012}$$

$$\approx 2103$$

Exercise

Suppose that \$2000 is invested at interest rate $\boldsymbol{k}$, compounded continuously, and grows to \$2983.65 in 5 *yrs*.

a) What is the interest rate?

b) Find the exponential growth function

c) What will the balance be after 10 *yrs*.?

d) After how long will the \$2000 have doubled?

Solution

a) $P(t) = P_0 e^{kt}$

$$P(t=5) = P_0 e^{k5} = 2983.65$$

$$2000e^{k5} = 2983.65$$

$$e^{5k} = \frac{2983.65}{2000}$$

$$\ln e^{5k} = \ln\left(\frac{2983.65}{2000}\right)$$

$$5k \ln e = \ln\left(\frac{2983.65}{2000}\right)$$

$$k = \frac{1}{5}\ln\left(\frac{2983.65}{2000}\right)$$

$$\approx 0.08$$

or $k = 8\%$

b) $P(t) = 2000e^{0.08t}$

c) $P(t=10) = 2000e^{0.08(10)}$

$$\approx \$4451.08$$

d) $T = \dfrac{\ln 2}{0.08}$ $\qquad T = \dfrac{\ln 2}{k}$

$\approx 8.7 \ yrs$

Exercise

In 2005, the population of China was about 1.306 *billion*, and the exponential growth rate was 0.6% per *year*.

a) Find the exponential growth function

b) Estimate the population in 2008

c) After how long will the population be double what it was in 2005?

Solution

a) In 2005 $\Rightarrow t = 0$

$$k = \frac{0.6}{100}$$

$$= 0.006$$

$$P_0 = 1.306$$

$$P(t) = 1.306e^{0.006t}$$

b) $P(t = 3) = 1.306e^{0.006(3)}$

$$\approx 1.33$$

c) $2(1.306) = 1.306e^{0.006t}$

$$2 = e^{0.006t}$$

$$e^{0.006t} = 2$$

$$\ln e^{0.006t} = \ln 2$$

$$0.006t = \ln 2$$

$$t = \frac{\ln 2}{0.006}$$

$$\approx 116 \ yrs$$

Exercise

How long will it take for the money in an account that is compounded continuously at 3% interest to double?

Solution

$T = \dfrac{\ln 2}{0.03}$ $\qquad T = \dfrac{\ln 2}{k}$

$\approx 23 \ yrs$

Exercise

If 600 *g* of radioactive substance are present initially and 3 *yrs* later only 300 *g* remain, how much of the substance will be present after 6 *yrs*?

Solution

$y(t) = 600e^{kt}$ $\qquad\qquad\qquad y(t) = y_0 e^{kt}$

When $t = 3 \to y = 300$

$300 = 600e^{k(3)}$

$\frac{300}{600} = e^{3k}$

$\ln\frac{300}{600} = \ln e^{3k}$ $\qquad\qquad\qquad \ln e = 1$

$3k = \ln\frac{300}{600}$

$k = \frac{1}{3}\ln\frac{300}{600}$

$\approx -.231$

$y(t) = 600e^{-.231t}$

$y(6) = 600e^{-.231(6)}$

$\approx 150\ g$

Exercise

The population of an endangered species of bird was 4200 in 1990. Thirteen years later, in 2003, the bird population declined to 3000. The population of the birds is decreasing exponentially according to the function $A(t) = 4200e^{kt}$ where *A* is the bird population *t* years after 1990. Use this information to find the value of ***k***.

Solution

$k = \frac{\ln\left(\frac{3000}{4200}\right)}{13}$ $\qquad\qquad kT = \ln\frac{A}{A_0}$

≈ -0.26

Exercise

A city had a population of 21,400 in 2000 and a population of 23,200 in 2005.

a) Find the exponential growth function for the city.
b) Use the growth function to predict the population of the city in 2018.

Solution

a) **Given**: $P(0)=21,400 \quad P(5)=23,200$

$$k=\frac{1}{5}\ln\frac{23,200}{21,400} \qquad kT=\ln\frac{P}{P_0}$$

$$=\frac{1}{5}\ln\frac{116}{107}$$

$$\approx 0.01615$$

$$P(t)=21,400e^{\frac{1}{5}\ln\left(\frac{116}{107}\right)t}$$

$$P(t)=21,400e^{0.01615t}$$

b) $P(18)=21,400e^{0.01615(18)}$

$$\approx 28,620$$

Exercise

A city had a population of 53,700 in 2002 and a population of 58,100 in 2006.

a) Find the exponential growth function for the city.
b) Use the growth function to predict the population of the city in 2013.

Solution

a) **Given**: $P(0)=53,700 \quad P(4)=58,100$

$$k=\frac{1}{4}\ln\frac{58,100}{53,700} \qquad kT=\ln\frac{P}{P_0}$$

$$=\frac{1}{4}\ln\frac{581}{537}$$

$$\approx 0.019688$$

$$P(t)=53,700e^{\frac{1}{4}\ln\left(\frac{581}{537}\right)t} \qquad P(t)=53,700e^{0.019688t}$$

b) $t=2013-2002=11$

$$P(11)=53,700e^{0.019688(11)}$$

$$\approx 66,685$$

Exercise

The population of Charlotte, North Carolina, is growing exponentially. The population of Charlotte was 395,934 in 1990 and 610,949 in 2005. Find the expoenential growth function that models the population of Charlote and use it to predict the popultion of Charlotte in 2017.

Solution

Given: $P(0)=395{,}934 \quad P(15)=610{,}949$

$$k=\frac{1}{15}\ln\frac{610{,}949}{395{,}934} \qquad\qquad kT=\ln\frac{P}{P_0}$$

$$\approx 0.02892$$

$$P(t)=395{,}934e^{0.02892t}$$

$$P(27)=395{,}934e^{0.02892(27)}$$

$$\approx 864{,}392$$

Exercise

The population of Las Vegas, Nevada, is growing exponentially. The population of Las Vegas was 258,295 in 1990 and 545,147 in 2005. Find the expoenential growth function that models the population oc Las Vegas and use it to predict the popultion of Las Vegas in 2017.

Solution

Given: $P(0)=258{,}295 \quad P(15)=545{,}147$

$$k=\frac{1}{15}\ln\frac{545{,}147}{258{,}295} \qquad\qquad kT=\ln\frac{P}{P_0}$$

$$\approx 0.049797$$

$$P(t)=258{,}295e^{0.049797t}$$

$$P(27)=258{,}295e^{0.049797(27)}$$

$$\approx 990{,}908$$

Exercise

Find the decay function for the amount of Polonium $\left({}^{210}Po\right)$ that remains in a sample after t days.

Solution

$$k=\frac{\ln\frac{1}{2}}{138} \qquad\qquad k=\frac{\ln\frac{1}{2}}{T}$$

$$=-0.005023$$

$$\boxed{A(t) = A_0 e^{-0.005023t}}$$

Exercise

Estimate the percentage of polonium $\left({}^{210}Po\right)$ that remains in a sample after 2 *years*.

Solution

$$k = \frac{\ln\frac{1}{2}}{138} \qquad\qquad k = \frac{\ln\frac{1}{2}}{T}$$

$$= -0.005023$$

$$A(t) = A_0 e^{-0.005023t}$$

$$\ln\frac{A}{A_0} = -\frac{0.005023}{2} \qquad\qquad \ln\frac{A}{A_0} = kT$$

$$= .0025115$$

$$\frac{A}{A_0} = e^{-.0025115}$$

$$\approx 0.9975$$

∴ The percentage of polonium that remains in a sample after 2 *years* is about 99.75%.

Exercise

Estimate the age of a bone if it now contains 65% of its original amount of carbon-14.

Solution

Given: $A = .65A_0$

$$k = \frac{\ln\frac{1}{2}}{5730} \qquad\qquad k = \frac{\ln\frac{1}{2}}{T}$$

$$= -\frac{\ln 2}{5730}$$

$$tk = \ln\frac{0.65A_0}{A_0} \qquad\qquad kT = \ln\frac{A}{A_0}$$

$$t = -\frac{5730\ \ln(0.65)}{\ln 2}$$

$$\approx 3561$$

∴ The age of a bone is approximately 3561 *years* old.

Exercise

Geologists have determined that Crater Lake inn Oregon was formed by a volcanic eruption. Chemical analysis of a wood chip assumed to be from a tree that died during the eruption has shown that it contains approximately 45% of its original carbon-14. Estimate how long ago the volcanic eruption occurred.

Solution

Given: $A = .45A_0$

$$k = \frac{\ln\frac{1}{2}}{5730} \qquad\qquad k = \frac{\ln\frac{1}{2}}{T}$$

$$= -\frac{\ln 2}{5730}$$

$$tk = \ln\frac{0.45A_0}{A_0} \qquad\qquad kT = \ln\frac{A}{A_0}$$

$$t = -\frac{5730\ \ln(0.45)}{\ln 2}$$

$$\approx 6,600$$

∴ The age of a bone is approximately 6,600 *years* old.

Exercise

Lead shielding is used to contain radiation. The percentage of a certain radiation that can penetrate x *millimeters* of lead shielding is given by $I(x) = 100e^{-1.5x}$

a) What percentage of radiation will penetrate a lead shield that is 1 *millimeter* thick?

b) How many *millimeters* of lead shielding penetrates the shielding?

Solution

a) $I(1) = 100e^{-1.5}$

$$\approx 22.313$$

∴ The percentage of radiation will penetrate a lead shield is approximately 22.313%

Exercise

After a race, a runner's pulse rate R, in *beats* per *minute*, decreases according to the function

$$R(t) = 145e^{-0.092t}, \quad 0 \le t \le 15$$

Where t is measured in minutes.

a) Find the runner's pulse rate at the end of the race and 1 *minute* after the end of the race.

b) How long after the end of the race will the runner's pulse rate be 80 *beats* per *minute*?

Solution

a) $R(15) = 145e^{-0.092(15)}$

≈ 36.48

$R(16) = 145e^{-0.092(16)}$

≈ 33.27

Exercise

A can of soda at $79°F$ is placed in a refrigerator that maintains a constant temperature of $36°F$. The temperature *T* of the soda *t minutes* after it is placed in the refrigerator is given by

$$T(t) = 36 + 43e^{-0.058t}$$

a) Find the temperature of the soda 10 *minutes* after it is placed in the refrigerator.
b) When will the temperature of the soda be $45°F$

Solution

a) $T(10) = 36 + 43e^{-0.058(10)}$

$\approx 60°F$

b) $36 + 43e^{-0.058t} = 45$

$43e^{-0.058t} = 9$

$e^{-0.058t} = \frac{9}{43}$

$-0.058t = \ln\frac{9}{43}$

$t = \frac{-1}{0.058}\ln\frac{9}{43}$

$\approx 27\ \ min$

Exercise

During surgery, a patient's circulatory system requires at least 50 *milligrams* of an anesthetic. The amount of anesthetic present *t hours* after 80 *milligrams* of anesthetic is administered is given by

$$T(t) = 80(0.727)^{t}$$

a) How much of the anesthetic is present in the patient's circulatory system 30 *minutes* after the anesthetic is administered?
b) How long can the operation last if the patient does not receive additional anesthetic?

Solution

a) $T\left(30=\frac{1}{2}hr\right)=80(0.727)^{1/2}$

$$\boxed{\approx 68\ \ mg}$$

b) $T(t)=80(0.727)^{t}=50$

$(0.727)^{t}=\frac{5}{8}$

$t=\log_{.727}\left(\frac{5}{8}\right)$

$\boxed{\approx 1.47\ \ hrs}$

$\boxed{=1\,hr\quad 28'\quad 12''}$

Exercise

The following function models the average typing speed *S*, in *words* per *minute*, for a student who has been typing for *t months*.

$$S(t)=5+29\ln(t+1),\quad 0\le t\le 9$$

Use *S* to determine how long it takes the student to achieve an average speed of 65 *words* per *minute*.

Solution

$S(t)=5+29\ln(t+1)=65$

$29\ln(t+1)=60$

$\ln(t+1)=\frac{60}{29}$

$t+1=e^{\frac{60}{29}}$

$\boxed{t=e^{\frac{60}{29}}-1}$

$\boxed{t\approx 7\ \ months}$

Exercise

The exponential function

$$S(x)=8320(0.73)^{x},\quad 10\le x\le 20$$

models the speed of the dragster during the 10-*second* period immediately following the time when the dragster crosses the finish line. This is the deceleration period.

How long after the start of the race did the dragster attain a speed of 275 *miles* per *hour*?

Solution

$$S(x) = 8320(0.73)^x = 275$$

$$(0.73)^x = \frac{275}{8320}$$

$$\boxed{x = \log_{0.73}\left(\frac{275}{8320}\right) \ \ minutes}$$

$$\boxed{\approx 11 \ \ minutes}$$

Exercise

If $8.000 is invested at an annual interest rate of 5% and compounded annually, find the balance after

a) 4 *years*. *b)* 8 *years*.

Solution

Given: $P = 8,000 \quad r = 0.05 \quad n = 1$

a) $t = 4$

$$A = 8,000\left(1 + \frac{.05}{1}\right)^4 \qquad\qquad A = P\left(1 + \frac{r}{n}\right)^{tn}$$

$$= 8,000(1.05)^4$$

$$\boxed{= \$9,724.05}$$

b) $t = 8$

$$A = 8,000\left(1 + \frac{.05}{1}\right)^4 \qquad\qquad A = P\left(1 + \frac{r}{n}\right)^{tn}$$

$$= 8,000(1.05)^8$$

$$\boxed{\approx \$11,819.64}$$

Exercise

If $20.000 is invested at an annual interest rate of 4.5% and compounded annually, find the balance after

a) 3 *years*. *b)* 5 *years*.

Solution

Given: $P = 20,000 \quad r = 0.045 \quad n = 1$

a) $t = 3$

$$A = 20,000\left(1 + \frac{.045}{1}\right)^3 \qquad\qquad A = P\left(1 + \frac{r}{n}\right)^{tn}$$

$$= 20,000(1.045)^3$$

$$\boxed{= \$22,823.32}$$

b) $t=5$

$$A=20,000\left(1+\frac{.045}{1}\right)^5 \qquad\qquad A=P\left(1+\frac{r}{n}\right)^{tn}$$

$$=20,000(1.045)^5$$

$$\approx \$24,923.64$$

Exercise

If \$10.000 is invested at an annual interest rate of 3% for 5 *years*, find the balance if the interest rate is compounded

a) Annually.
b) Semi-annually.
c) Quarterly
d) Monthly
e) Daily (365)
f) Hourly
g) Continuously

Solution

Given: $P=10,000 \quad r=0.03 \quad t=5$

a) Annually: $n=1$

$$A=10,000\left(1+\frac{.03}{1}\right)^5 \qquad\qquad A=P\left(1+\frac{r}{n}\right)^{tn}$$

$$=10,000(1.03)^5$$

$$\approx \$11,592.74$$

b) Semi-annually: $n=2$

$$A=10,000\left(1+\frac{.03}{2}\right)^{10} \qquad\qquad A=P\left(1+\frac{r}{n}\right)^{tn}$$

$$=10,000(1.015)^{10}$$

$$\approx \$11,605.41$$

c) Quarterly: $n=4$

$$A=10,000\left(1+\frac{.03}{4}\right)^{20} \qquad\qquad A=P\left(1+\frac{r}{n}\right)^{tn}$$

$$\approx \$11,611.84$$

d) Monthly: $n=12$

$$A=10,000\left(1+\frac{.03}{12}\right)^{60} \qquad\qquad A=P\left(1+\frac{r}{n}\right)^{tn}$$

$$\approx \$11,616.17$$

e) Daily: $n=365$

$$A=10,000\left(1+\frac{.03}{365}\right)^{365(5)} \qquad\qquad A=P\left(1+\frac{r}{n}\right)^{tn}$$

$$\approx \$11,618.27$$

f) Hourly: $n = 365 \times 24 = 8,760$

$$A = 10,000\left(1 + \frac{.03}{8,760}\right)^{43,800} \qquad A = P\left(1 + \frac{r}{n}\right)^{tn}$$

$$\approx \$11,618.34$$

g) Continuously

$$A = 10,000e^{(.03)(5)} \qquad A = Pe^{rt}$$

$$\approx \$11,618.34$$

Exercise

If \$20.000 is invested at an annual interest rate of 2% for 10 *years*, find the balance if the interest rate is compounded

a) Annually. *c)* Quarterly *e)* Daily (365) *g)* Continuously

b) Semi-annually. *d)* Monthly *f)* Hourly

Solution

Given: $P = 20,000 \quad r = 0.02 \quad t = 10$

a) Annually: $n = 1$

$$A = 20,000\left(1 + \frac{.02}{1}\right)^{10} \qquad A = P\left(1 + \frac{r}{n}\right)^{tn}$$

$$= 10,000(1.02)^{10}$$

$$\approx \$24,379.89$$

b) Semi-annually: $n = 2$

$$A = 20,000\left(1 + \frac{.02}{2}\right)^{20} \qquad A = P\left(1 + \frac{r}{n}\right)^{tn}$$

$$\approx \$24,403.80$$

c) Quarterly: $n = 4$

$$A = 20,000\left(1 + \frac{.02}{4}\right)^{40} \qquad A = P\left(1 + \frac{r}{n}\right)^{tn}$$

$$\approx \$24,416.88$$

d) Monthly: $n = 12$

$$A = 20,000\left(1 + \frac{.02}{12}\right)^{120} \qquad A = P\left(1 + \frac{r}{n}\right)^{tn}$$

$$\approx \$24,423.99$$

e) Daily: $n = 365$

$$A = 20,000\left(1 + \frac{.02}{365}\right)^{3650} \qquad A = P\left(1 + \frac{r}{n}\right)^{tn}$$

$$\boxed{\approx \$24,427.92}$$

f) Hourly: $n = 365 \times 24 = 8,760$

$$A = 20,000\left(1 + \frac{.02}{8,760}\right)^{87,600} \qquad A = P\left(1 + \frac{r}{n}\right)^{tn}$$

$$\boxed{\approx \$24,428.05}$$

g) Continuously

$$A = 20,000e^{(.02)(10)} \qquad A = Pe^{rt}$$

$$\boxed{\approx \$24,428.05}$$

Exercise

Find the accumulated value of an investment of $10,000 for 5 *years* at an interest rate of 5.5% if the money is

a) Compounded *semiannually*

b) Compounded *quarterly*

c) Compounded *monthly*

d) Compounded *Continuously*

Solution

Given: P = 10000

$t = 5$

$r = 0.055$

a) Semiannually: $n = 2$

$$A = 10000\left(1 + \frac{0.055}{2}\right)^{2(5)}$$

$$\boxed{= \$13,116.51}$$

b) Quarterly: $n = 4$

$$A = 10000\left(1 + \frac{0.055}{4}\right)^{4(5)}$$

$$\boxed{= \$13,140.67}$$

c) Monthly: $n = 12$

$$A = 10000\left(1 + \frac{0.055}{12}\right)^{12(5)}$$

$$\boxed{= \$13,157.04}$$

d) $A = 10000e^{(0.055)(5)}$

$$\boxed{= \$13,165.31}$$

Exercise

Suppose \$1,000 is deposited in an account paying 4% interest per *year* compounded *quarterly*.

a) Find the amount in the account after 10 *years* with no withdraws.

b) How much interest is earned over the 10 *years* period?

Solution

Given: $P = 1000 \qquad r = .04 \qquad n = 4$

a) $t = 10$

$$A = 1000\left(1 + \frac{.04}{4}\right)^{10(4)} \qquad\qquad A = P\left(1+\frac{r}{n}\right)^{tn}$$

$$\underline{= \$1,488.86}$$

b) The interest earned: $\$1488.86 - \$1000 \ \underline{= \$488.86}$

Exercise

Becky must pay a lump sum of \$6000 in 5 *yrs*.

a) What amount deposited today at 3.1% compounded annually will grow to \$6000 in 5 *yrs*.?

b) If only \$5000 is available to deposit now, what annual interest rate is necessary for the money to increase to \$6000 in 5 *yrs*.?

Solution

a) $A = P\left(1+\frac{r}{n}\right)^{tn}$

$$6000 = P\left(1 + \frac{.031}{1}\right)^{5(1)}$$

$$6000 = P(1.031)^5$$

$$\frac{6000}{(1.031)^5} = P \qquad\qquad 6000 / 1.031 \wedge 5$$

$$\underline{P \approx \$5,150.60}$$

b) $A = P\left(1+\frac{r}{n}\right)^{tn}$

$$6000 = 5000\left(1 + \frac{r}{1}\right)^{5(1)}$$

$$\frac{6000}{5000} = (1+r)^5$$

$$\frac{6}{5} = (1+r)^5$$

$$\left(\frac{6}{5}\right)^{1/5} = 1 + r$$

$$r = \left(\frac{6}{5}\right)^{1/5} - 1 \qquad (6/5)\wedge(1/5)-1$$

$$\approx .0371$$

The interest rate of 3.71% will produce enough to increase the $5,000 to $6,000 by the end of 5 *yrs*.

Exercise

An investment of 1,000 increased to $13,464 in 20 *years*. If the interest was compounded continuously, find the interest rate.

Solution

$$A = Pe^{rt}$$

$$13464 = 1000e^{20r}$$

$$13.464 = e^{20r}$$

$$\ln(13.464) = \ln e^{20r}$$

$$20r = \ln 13.464$$

$$r = \frac{\ln 13.464}{20}$$

$$\approx 0.13$$

The interest rate is 13%.

Exercise

Find the present value of $4,000 if the annual interest rate is 3.5% compounded *quarterly* for 6 *years*.

Solution

Given: $A = 4000.00, \quad r = 0.035, \quad t = 6, \quad n = 4$

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$4000 = P\left(1 + \frac{0.035}{4}\right)^{4(6)}$$

$$P = \frac{4000}{\left(1 + \frac{0.035}{4}\right)^{4(6)}}$$

$$= \$3,245.30$$

Exercise

How much money will there be in an account at the end of 8 *years* if \$18,000 is deposited at 3% interest compounded *semi-annually*?

Solution

$$A = 18000\left(1+\frac{0.03}{2}\right)^{2(8)} \qquad\qquad A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$\underline{= \$22,841.74}$$

Exercise

The function defined by $P(x) = 908e^{-0.0001348x}$ approximates the atmospheric pressure (in *millibars*) at an altitude of x meters. Use P to predict the pressure:

a) At 0 *meters*

b) At 12,000 *meters*

Solution

a) At 0 *meters*

$$P(x=0) = 908e^{-0.0001348(0)}$$

$$\underline{= 908 \ \ millibars}$$

b) At 12,000 *meters*

$$P(x=12,000) = 908e^{-0.0001348(12,000)}$$

$$\underline{\approx 180 \ \ millibars}$$

Exercise

How long, to the nearest tenth of a year, will it take \$1000 to grow to \$3600 at 8% annual interest compounded quarterly?

Solution

Given: A = \$3600

P = \$1000

r = 8% = 0.08

n = 4

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$3600 = 1000\left(1+\frac{0.08}{4}\right)^{4t}$$

$$3.6 = (1.02)^{4t}$$

$$\ln 3.6 = \ln(1.02)^{4t}$$

$$\ln 3.6 = 4t \ln(1.02)$$

$$\frac{\ln 3.6}{4 \ln 1.02} = t$$

$$\underline{t \approx 16.2 \ \ yrs}|$$

Exercise

The annual revenue *R*, in *dollars*, of a new company can be closely modeled by the logistic function

$$R(t) = \frac{625,000}{1 + \frac{3}{10} e^{-.045t}}$$

Where the natural number *t* is the time, in *years*, since the company was founded.

a) According to the model, what will the company's annual revenue for its first year and its second year?

b) According to the model, what will the company's annual revenue approach in the long-term future?

Solution

a) *First* year:

$$R(1) = \frac{625,000}{1 + \frac{3}{10} e^{-.045}}$$

$$\underline{\approx \$485,701.25}|$$

Second year:

$$R(1) = \frac{625,000}{1 + \frac{3}{10} e^{-.045(2)}}$$

$$\underline{\approx \$490,511.79}|$$

b)

$$\lim_{t \to \infty} R(t) = \lim_{t \to \infty} \frac{625,000}{1 + \frac{3}{10} e^{-.045t}}$$

$$= \frac{625,000}{1 + \frac{3}{10} \lim_{t \to \infty} e^{-.045t}}$$

$$= \frac{625,000}{1 + \frac{3}{10}(0)} \qquad \lim_{t \to \infty} e^{-\infty} = 0$$

$$\underline{= \$625,000}|$$

Exercise

The number of cars *A* sold annually by an automobile dealership can be closely modeled by the logistic function

$$A(t)=\frac{1,650}{1+\frac{12}{5}e^{-.055t}}$$

a) According to the model, what number of cars will the dealership sell during its first year and its second year?

b) According to the model, what will the dealership's car sales approach in the long-term future?

Solution

a) *First* year:

$$A(1)=\frac{1,650}{1+\frac{12}{5}e^{-.055}}$$

$$\approx 504 \ \ cars$$

Second year:

$$A(2)=\frac{1,650}{1+\frac{12}{5}e^{-.055(2)}}$$

$$\approx 524 \ \ cars$$

b)

$$\lim_{t\to\infty} A(t)=\lim_{t\to\infty}\frac{1,650}{1+\frac{12}{5}e^{-.055t}}$$

$$=\frac{1,650}{1+\frac{12}{5}\lim\limits_{t\to\infty}e^{-.055t}}$$

$$=\frac{1,650}{1+\frac{12}{5}(0)} \qquad \lim_{t\to\infty} e^{-\infty}=0$$

$$=1,650 \ \ cars$$

Exercise

The population of wolves in a preserve satisfies a logistic model in which $P_0=312$ in 2008, $c=1,600$, and $P(6)=416$.

a) Determine the logistic model for this population, where *t* is the number of years after 2008.

b) Use the logistic model from part (*a*) to predict the size of the groundhog population in 2014.

Solution

Given: $P_0=312 \quad c=1,600 \quad P(6)=416$

a) $P(t)=\dfrac{1,600}{1+ae^{-bt}}$ $\qquad P(t)=\dfrac{c}{1+ae^{-bt}}$

$P(0)=\dfrac{1,600}{1+a}=312$

$1+a=\dfrac{1,600}{312}$

$a=\dfrac{200}{39}-1$

$\boxed{=\dfrac{161}{39}}$

$P(t)=\dfrac{1,600}{1+\frac{161}{39}e^{-bt}}$

$P(6)=\dfrac{1,600}{1+\frac{161}{39}e^{-6b}}=416$

$1+\dfrac{161}{39}e^{-6b}=\dfrac{1,600}{416}$

$\dfrac{161}{39}e^{-6b}=\dfrac{50}{13}-1$

$\dfrac{161}{39}e^{-6b}=\dfrac{37}{13}$

$e^{-6b}=\dfrac{37}{13}\dfrac{39}{161}$

$e^{-6b}=\dfrac{111}{161}$

$-6b=\ln\dfrac{111}{161}$

$b=-\dfrac{1}{6}\ln\left(\dfrac{111}{161}\right)$

$\boxed{\approx .062}$

$\boxed{P(t)=\dfrac{1,600}{1+\frac{161}{39}e^{-.062t}}}$

b) $t=2014-2008\ \boxed{=6}$

$P(6)=\dfrac{1,600}{1+\frac{161}{39}e^{-.062(6)}}$

$\boxed{\approx 416.04}$

Exercise

The population of walruses on an island satisfies a logistic model in which $P_0 = 800$ in 2006, $c = 5{,}500$, and $P(1) = 900$.

a) Determine the logistic model for this population, where *t* is the number of years after 2006.

b) Use the logistic model from part (*a*) to predict the year in which the walrus population will first exceed 2000.

Solution

Given: $P_0 = 800 \quad c = 5{,}500 \quad P(1) = 900$

a) $P(t) = \dfrac{5{,}500}{1 + ae^{-bt}}$ $\qquad P(t) = \dfrac{c}{1 + ae^{-bt}}$

$$P(0) = \frac{5{,}500}{1 + a} = 800$$

$$1 + a = \frac{5{,}500}{800}$$

$$a = \frac{55}{8} - 1$$

$$= \frac{47}{8}$$

$$P(t) = \frac{5{,}500}{1 + \frac{47}{8}e^{-bt}}$$

$$P(1) = \frac{5{,}500}{1 + \frac{47}{8}e^{-b}} = 900$$

$$1 + \frac{47}{8}e^{-b} = \frac{5{,}500}{900}$$

$$\frac{47}{8}e^{-b} = \frac{55}{9} - 1$$

$$e^{-b} = \frac{46}{9}\frac{8}{47}$$

$$e^{-b} = \frac{368}{423}$$

$$-b = \ln\left(\frac{368}{423}\right)$$

$$b = -\ln\left(\frac{368}{423}\right)$$

$$\approx 0.139$$

$$P(t) = \frac{5{,}500}{1 + \frac{47}{8}e^{-0.139t}}$$

b) $P(t)=\dfrac{5,500}{1+\frac{47}{8}e^{-0.139t}}=2,000$

$$1+\frac{47}{8}e^{-0.139t}=\frac{5,500}{2,000}$$

$$\frac{47}{8}e^{-0.139t}=\frac{11}{4}-1$$

$$e^{-0.139t}=\frac{7}{4}\frac{8}{47}$$

$$-0.139t=\ln\left(\frac{14}{47}\right)$$

$$t=-\frac{1}{0.139}\ln\left(\frac{14}{47}\right)$$

$$\approx 8.8\ \ years$$

$\therefore$ The walrus population will first exceed 2000 in *year* 2015

Exercise

Newton's Law of Cooling states that is an object at temperature T_0 is placed into an environment at constant temperature *A*, then the temperature of the object, $T(t)$ (in degrees Fahrenheit), after *t minutes* is given by $T(t)=A+\left(T_0-A\right)e^{-kt}$, where *k* is a constant that depends on the object.

a) Determine the constant *k* for a canned soda drink that takes 5 *minutes* to cool from $75°F$ to $65°F$ after being placed in a refrigerator that maintains a constant temperature of $34°F$

b) What will be the temperature of the soda after 30 *minutes*?

c) When will the temperature of the soda drink be $36°F$?

Solution

a) $T(5)=34+(75-34)e^{-5k}=65$

$$41e^{-5k}=31$$

$$e^{-5k}=\frac{31}{41}$$

$$-5k=\ln\left(\frac{31}{41}\right)$$

$$k=-\frac{1}{5}\ln\left(\frac{31}{41}\right)$$

$$\approx 0.0559$$

b) $T(t)=34+41e^{-0.0559t}$

$$T(30)=34+41e^{-0.0559(30)}$$

$$\approx 42°F$$

c) $T(t) = 34 + 41e^{-0.0559t} = 36$

$41e^{-0.0559t} = 2$

$e^{-0.0559t} = \frac{2}{41}$

$-0.0559t = \ln\left(\frac{2}{41}\right)$

$t = -\frac{1}{0.0559}\ln\left(\frac{2}{41}\right)$

$\approx 54 \ \ min$

Exercise

According to a software company, the users of its typing tutorial can expect to type $N(t)$ words per minute after *t hours* of practice with the product, according to the function $N(t) = 100\left(1.04 - 0.99^t\right)$

a) How many words per minute can a student expect to type after 2 *hours* of practice?
b) How many words per minute can a student expect to type after 40 *hours* of practice?
c) How many hours of practice will be required before a student can expect to type 60 words per minute?

Solution

a) $N(2) = 100\left(1.04 - 0.99^2\right)$

≈ 6 *words per minute*

b) $N(40) = 100\left(1.04 - 0.99^{40}\right)$

≈ 70 *words per minute*

c) $N(t) = 100\left(1.04 - 0.99^t\right) = 60$

$1.04 - 0.99^t = \frac{60}{100}$

$-0.99^t = 0.6 - 1.04$

$0.99^t = 0.44$

$t = \log_{.99}(.44)$

$\approx 82 \ \ hours$

Exercise

A lawyer has determined that the number of people $P(t)$ in a city of 1.2 *million* people who have been exposed to a news item after *t days* is given by the function

$$P(t) = 1,200,000\left(1 - e^{-0.03t}\right)$$

a) How many days after a major crime has been reported has 40% of the population heard of the crime?

b) A defense lawyer knows it will be difficult to pick an unbiased jury after 80% of the population has heard of the crime. After how many days will 80% of the population have heard of the crime?

Solution

a) $P(t) = 1,200,000\left(1 - e^{-0.03t}\right) = .4(1,200,000)$

$$1 - e^{-0.03t} = 0.4$$

$$e^{-0.03t} = 0.6$$

$$-0.03t = \ln(0.6)$$

$$t = -\frac{\ln(0.6)}{0.03}$$

$$\approx 17 \;\; days$$