Lecture Four - Integration

Section 4.1 – Antiderivatives

Antiderivatives

$f(x) = x^3 \quad \Rightarrow \quad f'(x) = 3x^2$

Definition of Antiderivatives

A Function F is an ***antiderivative*** of a function f on an interval I if

$F'(x) = f(x)$ for all x in I.

Theorem

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

Where C is an arbitrary constant.

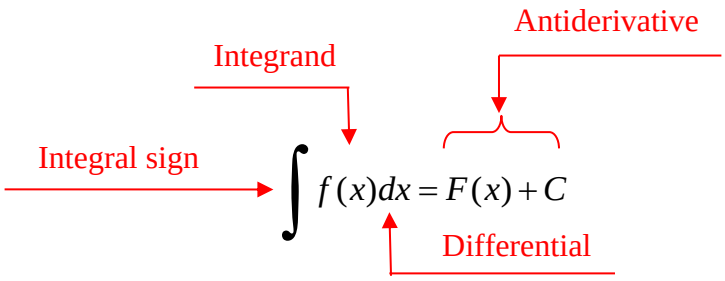
Notation for Antiderivatives and indefinite integrals

The notation $\int f(x)dx = F(x) + C$

where C is an arbitrary constant, means that F is an Antiderivative of f.

That is $F'(x) = f(x)$ for all x in the domain of f.

$\int f(x)dx$ Indefinite integral



Basic Integration Rules

$$\int k\,dx = kx + C$$

$$\int kf(x)\,dx = k\int f(x)\,dx$$

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

Example

Find each indefinite integral.

a) $\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx$

$$= \frac{x^{-2+1}}{-2+1} + C$$

$$= \frac{x^{-1}}{-1} + C$$

$$= -\frac{1}{x} + C$$

b) $\int \sqrt[3]{x}\,dx = \int x^{1/3}\,dx$

$$= \frac{x^{1/3+1}}{1/3+1} + C$$

$$= \frac{x^{4/3}}{4/3} + C$$

$$= \underline{\frac{3}{4}x^{4/3} + C} \quad or \quad = \frac{3}{4}x\sqrt[3]{x} + C$$

Example

Evaluate $\int \left(4x^3 - 5x + 2\right) dx$

Solution

$$\int (4x^3 - 5x + 2)dx = \int 4x^3 dx - \int 5x dx + \int 2dx$$

$$= 4\frac{x^4}{4} - 5\frac{x^2}{2} + 2x + C$$

$$= x^4 - \frac{5}{2}x^2 + 2x + C$$

Example

Evaluate $\int \left(x^2 - 2x + 5\right) dx$

Solution

$$\int \left(x^2 - 2x + 5\right) dx = \frac{x^3}{3} - x^2 + 5x + C$$

Example

Evaluate $\int \sin x \; dx$

Solution

$$\int \sin x dx = -\cos x + C$$

Example

Evaluate $\int \cos 3x \; dx$

Solution

$$\int \cos 3x \; dx = \frac{1}{3}\sin 3x + C$$

Definition

The ***natural logarithm*** is the function given by

$$\ln x = \int_1^x \frac{1}{t}dt, \quad x > 0$$

Zero width: $\ln 1 = \int_1^1 \frac{1}{t}dt = 0$

Definition

The ***number e*** is that number in the domain of the natural logarithm satisfying

$$\ln e = 1 \quad \textit{and} \quad \int_1^e \frac{1}{t}dt = 1$$

Other Indefinite Integrals

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax} \quad \rightarrow \quad \int e^{ax}dx = \frac{1}{a}e^{ax} + C$$

$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x} \quad \rightarrow \quad \int \frac{dx}{x} = \ln|x| + C$$

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2 - x^2}} \quad \rightarrow \quad \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{x^2 + a^2} \quad \rightarrow \quad \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\frac{d}{dx}\left(\sec^{-1}\left|\frac{x}{a}\right|\right) = \frac{a}{x\sqrt{x^2 - a^2}} \quad \rightarrow \quad \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$$

Example

Evaluate $\int e^{-10x}dx$

Solution

$$\int e^{-10x}dx = \boxed{-\frac{1}{10}e^{-10x} + C}$$

Example

Evaluate $\int \frac{5}{x}\,dx$

Solution

$$\int \frac{5}{x}\,dx = \underline{\boxed{5\ln|x| + C}}$$

Example

Evaluate $\int \frac{4}{\sqrt{9-x^2}}\,dx$

Solution

$$\int \frac{4}{\sqrt{9-x^2}}\,dx = \underline{\boxed{4\sin^{-1}\left(\frac{x}{3}\right) + C}} \qquad a^2 = 9 \;\rightarrow\; a = 3$$

Example

Evaluate $\int \frac{dx}{16x^2+1}$

Solution

$$\int \frac{dx}{16x^2+1} = \frac{1}{16}\int \frac{dx}{x^2+\frac{1}{16}}$$

$$= \frac{1}{16}\int \frac{dx}{x^2+\left(\frac{1}{4}\right)^2}$$

$$= \left(\frac{1}{16}\right)4\tan^{-1}(4x) + C$$

$$= \underline{\boxed{\frac{1}{16}\tan^{-1}(4x) + C}}$$

Exercises *Section* 4.1 – Antiderivatives

Find each indefinite integral.

1. $\int v^2 dv$

2. $\int x^{1/2} dx$

3. $\int 4y^{-3} dy$

4. $\int (x^3 - 4x + 2) dx$

5. $\int \left(3z^2 - 4z + 5\right) dz$

6. $\int \left(x^2 - 1\right)^2 dx$

7. $\int \frac{x^2 + 1}{\sqrt{x}} dx$

8. $\int \left(\sqrt[4]{x^3} + 1\right) dx$

9. $\int \sqrt{x}(x + 1) dx$

10. $\int (1 + 3t)t^2 dt$

11. $\int \frac{x^2 - 5}{x^2} dx$

12. $\int (-40x + 250) dx$

13. $\int \frac{x + 2}{\sqrt{x}} dx$

14. $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$

15. $\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx$

16. $\int 2x\left(1 - x^{-3}\right) dx$

17. $\int \left(\frac{4 + \sqrt{t}}{t^3}\right) dt$

18. $\int (-2\cos t) dt$

19. $\int 7\sin\frac{\theta}{3} d\theta$

20. $\int \frac{2}{5}\sec\theta\tan\theta d\theta$

21. $\int \left(4\sec x\tan x - 2\sec^2 x\right) dx$

22. $\int (2\cos 2x - 3\sin 3x) dx$

23. $\int \left(1 + \tan^2\theta\right) d\theta$

24. $\int \frac{\csc\theta}{\csc\theta - \sin\theta} d\theta$

25. $\int \left(2e^x - 3e^{-2x}\right) dx$

26. $\int \frac{dx}{\sqrt{9 - x^2}}$

27. $\int \frac{dx}{9 + 3x^2}$

28. $\int \frac{4x^2 - 3x + 2}{x^2} dx$

29. $\int \left(x^8 - 3x^3 + 1\right) dx$

30. $\int (2x + 1)^2 dx$

31. $\int \frac{x + 1}{x} dx$

32. $\int \left(\frac{1}{x^2} - \frac{2}{x^{5/2}}\right) dx$

33. $\int \frac{x^4 - 2\sqrt{x} + 2}{x^2} dx$

34. $\int (1 + \cos 3\theta) d\theta$

35. $\int 2\sec^2\theta\, d\theta$

36. $\int \sec 2x\tan 2x\, dx$

37. $\int 2e^{2x} dx$

38. $\int \frac{12}{x} dx$

39. $\int \frac{dx}{\sqrt{1 - x^2}}$

40. $\int \frac{dx}{x^2 + 1}$

41. $\int \frac{1 + \tan\theta}{\sec\theta} d\theta$

42. $\int \left(\sqrt[4]{x^3} + \sqrt{x^5}\right) dx$

43. $\int \left(\sqrt[4]{x^3} + 1 \right) dx$

44. $\int \left(5x^4 + 3x^2 + 2x + 5 \right) dx$

45. $\int \left(5x^{4/3} + 3x^{2/3} + 2x^{1/3} \right) dx$

46. $\int \left(5x^{-4/3} + 3x^{-2/3} + 2x^{-1/3} \right) dx$

47. $\int \dfrac{x^4 - 3x^2 + 5}{x^4} dx$

48. $\int \left(\dfrac{3}{x^7} - \dfrac{5}{x^6} \right) dx$

49. $\int \dfrac{x+8}{\sqrt{x}} dx$

50. $\int \dfrac{x^2+8}{\sqrt[3]{x}} dx$

51. $\int \cos\left(\dfrac{5\pi}{3} x \right) dx$

52. $\int \sin\left(\dfrac{2x}{3} \right) dx$

53. $\int \left(5\cos x + 4\sin x + 3\sec^2 x \right) dx$

54. $\int \sec\theta \left(\sec\theta + \tan\theta \right) d\theta$

55. $\int \left(\tan^2\theta + 1 \right) d\theta$

56. $\int \left(\cos^4\theta - \sin^4\theta \right) d\theta$

57. $\int \left(\cos^2\theta - \sin^2\theta \right) d\theta$

58. $\int \left(\cos^2\theta + \sin^2\theta \right) d\theta$

59. $\int \left(\cos 2x \cos 4x - \sin 2x \sin 4x \right) dx$

60. $\int \left(\sin 2x \cos 4x - \cos 2x \sin 4x \right) dx$

61. $\int \left(\sin 3x \cos 2x + \cos 3x \sin 2x \right) dx$

62. $\int \cos 2x \sin 2x \, dx$

63. $\int \left(2\cos^2 x - 1 \right) dx$

64. $\int \left(1 - 2\sin^2 x \right) dx$

65. $\int e^{-5x} \, dx$

66. $\int 4e^{4x} \, dx$

67. $\int \left(2\sin\theta - 5e^{\theta} \right) d\theta$

68. $\int \left(\dfrac{3}{x} + \sec^2 x \right) dx$

69. $\int \left(\sin x + 2^x \right) dx$

70. $\int \left(2x - 3^x \right) dx$

71. $\int \left(4x - \dfrac{3}{x} - \csc^2 x \right) dx$

72. $\int \left(e^{4x} - \dfrac{3}{x} + 2\csc x \cot x \right) dx$

73. $\int (a+b) e^{(a+b)x} \, dx$

74. $\int \left(a^2 - b^2 \right) e^{(a-b)x} \, dx$

Find the function with the following property

75. $\frac{dy}{dx} = 2x - 7, \quad y(2) = 0$

76. $\frac{dy}{dx} = \frac{1}{x^2} + x, \quad y(2) = 1; \ x > 0$

77. $\frac{ds}{dt} = 1 + \cos t, \quad s(0) = 4$

78. $\frac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1$

79. $f'(x) = 3x^2 - 1 \quad \& \quad f(0) = 10$

80. $f'(t) = \sin t + 2t \quad \& \quad f(0) = 5$

81. $f'(x) = x^2 + x^{-2} \quad \& \quad f(1) = 1$

82. $f'(x) = \sin^2 x \quad \& \quad f(1) = 1$

83. Find the general solution of $F'(x) = 4x + 2$, and find the particular solution that satisfies the initial condition $F(1) = 8$.

84. Derive the position function if a ball is thrown upward with initial velocity of 32 *feet* per second from an initial height of 48 *feet*. When does the ball hit the ground? With what velocity does the ball hit the ground?

85. Suppose a publishing company has found that the marginal cost at a level of production of x thousand books is given by

$$\frac{dC}{dx} = \frac{50}{\sqrt{x}}$$

And that the fixed cost (the cost before the first book can be produced) is a $25,000. Find the cost function $C(x)$.