

Solution **Section 2.8 – Derivatives of Logarithmic & Exponential Functions**

Exercise

Find the derivative of $y = \ln \sqrt{x+5}$

Solution

$$y = \ln(x+5)^{1/2}$$

$$= \frac{1}{2} \ln(x+5)$$

$$\underline{y' = \frac{1}{2(x+5)}} \quad \Bigg|$$

Exercise

Find the Derivatives of $y = (3x+7)\ln(2x-1)$

Solution

$$f = 3x+7 \quad f' = 3$$

$$g = \ln(2x-1) \quad g' = \frac{2}{2x-1}$$

$$\underline{y' = 3\ln(2x-1) + \frac{2(3x+7)}{2x-1}} \quad \Bigg|$$

Exercise

Find the Derivatives of $f(x) = \ln \sqrt[3]{x+1}$

Solution

$$f(x) = \ln(x+1)^{1/3}$$

$$= \frac{1}{3} \ln(x+1)$$

$$u = x+1 \Rightarrow \frac{du}{dx} = 1$$

$$f'(x) = \frac{1}{3} \frac{1}{x+1}$$

$$\underline{= \frac{1}{3(x+1)}} \quad \Bigg|$$

Exercise

Find the Derivatives of $f(x) = \ln\left(x^2 \sqrt{x^2 + 1}\right)$

Solution

$$f(x) = \ln\left(x^2\right) + \ln\sqrt{x^2 + 1} \quad \text{Product Property}$$

$$f(x) = \ln\left(x^2\right) + \ln\left(x^2 + 1\right)^{1/2}$$

$$f(x) = 2 \ln x + \frac{1}{2} \ln\left(x^2 + 1\right) \quad \text{Power Property}$$

$$\begin{aligned} f'(x) &= 2 \frac{1}{x} + \frac{1}{2} \frac{2x}{x^2 + 1} \\ &= \frac{\frac{2}{x} + \frac{x}{x^2 + 1}}{\quad} \quad \text{Differentiate} \end{aligned}$$

Exercise

Find the Derivatives of $y = \ln \frac{x^2}{x^2 + 1}$

Solution

$$y = \ln x^2 - \ln x^2 + 1$$

$$\begin{aligned} y' &= \frac{2x}{x^2} - \frac{2x}{x^2 + 1} \\ &= \frac{\frac{2}{x} - \frac{2x}{x^2 + 1}}{\quad} \end{aligned}$$

Exercise

Find the Derivatives of $y = \ln \left[\frac{x^2(x+1)^3}{(x+3)^{1/2}} \right]$

Solution

$$y = \ln \left[x^2 (x+1)^3 \right] - \ln (x+3)^{1/2} \quad \text{Quotient Rule}$$

$$= \ln x^2 + \ln (x+1)^3 - \ln (x+3)^{1/2} \quad \text{Product Rule}$$

$$= 2 \ln x + 3 \ln (x+1) - \frac{1}{2} \ln (x+3) \quad \text{Power Rule}$$

$$y' = \frac{2}{x} + \frac{3}{x+1} - \frac{1}{2(x+3)}$$

Exercise

Find the Derivatives of $y = \ln(x^2 + 1)$

Solution

$$y' = \frac{2x}{x^2 + 1} \quad (\ln U)' = \frac{U'}{U}$$

Exercise

Find the Derivatives of $f(x) = \ln(x^2 - 4)$

Solution

$$\text{Let } u = x^2 - 4 \Rightarrow \frac{du}{dx} = 2x$$

$$f'(x) = \frac{1}{x^2 - 4} (2x) \\ = \frac{2x}{x^2 - 4}$$

Exercise

Find the derivative $f(x) = 2 \ln(x^2 - 3x + 4)$

Solution

$$f'(x) = 2 \frac{2x - 3}{x^2 - 3x + 4} \\ = \frac{4x - 6}{x^2 - 3x + 4}$$

Exercise

Find the derivative $f(x) = 3 \ln(1 + x^2)$

Solution

$$f'(x) = 3 \frac{2x}{1+x^2}$$

$$= \frac{6x}{1+x^2}$$

Exercise

Find the derivative $f(x) = (1 + \ln x)^3$

Solution

$$f'(x) = 3(1 + \ln x)^2 (1 + \ln x)'$$

$$= 3(1 + \ln x)^2 \left(\frac{1}{x}\right)$$

$$= \frac{3}{x}(1 + \ln x)^2$$

Exercise

Find the derivative $f(x) = (x - 2 \ln x)^4$

Solution

$$f'(x) = 4(x - 2 \ln x)^3 (x - 2 \ln x)'$$

$$= 4(x - 2 \ln x)^3 \left(1 - \frac{2}{x}\right)$$

$$= 4(x - 2 \ln x)^3 \left(\frac{x-2}{x}\right)$$

$$= \frac{4x-8}{x}(x - 2 \ln x)^3$$

Exercise

Find the Derivatives of $f(x) = x^2 \ln x$

Solution

$$f' = x^2 \left(\frac{1}{x}\right) + 2x \ln x$$

$$= x + 2x \ln x$$

$$= x(1 + 2 \ln x)$$

$$(fg)' = f'g + fg'$$

Exercise

Find the Derivatives of $f(x) = -\frac{\ln x}{x^2}$

Solution

$$\begin{aligned} f' &= -\frac{x^2 \frac{d}{dx}(\ln x) - \ln x \frac{d}{dx}(x^2)}{(x^2)^2} \\ &= -\frac{x^2 \frac{1}{x} - 2x \ln x}{x^4} \\ &= -\frac{x - 2x \ln x}{x^4} \\ &= -\frac{x(1 - 2 \ln x)}{x^4} \\ &= -\frac{1 - 2 \ln x}{x^3} \end{aligned}$$

Exercise

Find the derivative of $y = \ln(t^2)$

Solution

$$\begin{aligned} y' &= \frac{(t^2)'}{t^2} \\ &= \frac{2t}{t^2} \\ &= \frac{2}{t} \end{aligned}$$

Exercise

Find the derivative of $y = \ln(2\theta + 2)$

Solution

$$\begin{aligned} y' &= \frac{2}{2\theta + 2} \\ &= \frac{1}{\theta + 1} \end{aligned}$$

Exercise

Find the derivative of $y = (\ln x)^3$

Solution

$$\begin{aligned} y' &= 3(\ln x)^2 \frac{1}{x} \\ &= \frac{3(\ln x)^2}{x} \end{aligned}$$

Exercise

Find the derivative of $y = x(\ln x)^2$

Solution

$$\begin{aligned} y' &= (\ln x)^2 + x \left(2(\ln x) \frac{1}{x} \right) \\ &= (\ln x)^2 + 2 \ln x \end{aligned}$$

Exercise

Find the derivative of $y = \frac{x^4}{4} \ln x - \frac{x^4}{16}$

Solution

$$\begin{aligned} y' &= \frac{4x^3}{4} \ln x + \frac{x^4}{4} \frac{1}{x} - \frac{4x^3}{16} \\ &= x^3 \ln x + \frac{1}{4}x^3 - \frac{1}{4}x^3 \\ &= x^3 \ln x \end{aligned}$$

Exercise

Find the derivative of $y = \frac{1 + \ln t}{t}$

Solution

$$\begin{aligned} y' &= \frac{\frac{1}{t} - (1 + \ln t)}{t^2} \\ &= \frac{1 - 1 - \ln t}{t^2} \\ &= -\frac{\ln t}{t^2} \end{aligned}$$

Exercise

Find the derivative $f(x) = \frac{\ln x}{1+x}$

Solution

$$\begin{aligned} f'(x) &= \frac{\left(\frac{1}{x}\right)(1+x) - \ln x}{(1+x)^2} \\ &= \frac{\frac{1}{x} 1 + x - x \ln x}{(1+x)^2} \\ &= \frac{1+x-x \ln x}{x(1+x)^2} \end{aligned}$$

$$u = \ln x \quad v = 1+x$$

$$u' = \frac{1}{x} \quad v' = 1$$

Exercise

Find the derivative $f(x) = \frac{2x}{1+\ln x}$

Solution

$$\begin{aligned} f'(x) &= \frac{2(1+\ln x) - (2x)\frac{1}{x}}{(1+\ln x)^2} \\ &= \frac{2+2\ln x-2}{(1+\ln x)^2} \\ &= \frac{2\ln x}{(1+\ln x)^2} \end{aligned}$$

$$u = 2x \quad v = 1+\ln x$$

$$u' = 2 \quad v' = \frac{1}{x}$$

Exercise

Find the derivative $f(x) = x^3 \ln x$

Solution

$$u = x^3 \quad v = \ln x$$

$$u' = 3x^2 \quad v' = \frac{1}{x}$$

$$\begin{aligned} f'(x) &= 3x^2 \ln x + x^3 \frac{1}{x} \\ &= 3x^2 \ln x + x^2 \\ &= (3\ln x + 1)x^2 \end{aligned}$$

Exercise

Find the derivative $f(x) = 6x^4 \ln x$

Solution

$$u = 6x^4 \quad v = \ln x$$

$$u' = 24x^3 \quad v' = \frac{1}{x}$$

$$\begin{aligned} f'(x) &= 24x^3 \ln x + 6x^4 \frac{1}{x} \\ &= 24x^3 \ln x + 6x^3 \\ &= \underline{6x^3(4 \ln x + 1)} \end{aligned}$$

Exercise

Find the derivative $f(x) = \ln x^8$

Solution

$$f(x) = \ln x^8 = 8 \ln x$$

$$\underline{f'(x) = \frac{8}{x}}$$

Power Rule

$$(\ln x)' = \frac{1}{x}$$

Exercise

Find the derivative $f(x) = 5x - \ln x^5$

Solution

$$\begin{aligned} f(x) &= 5x - \ln x^5 \\ &= 5x - 5 \ln x \end{aligned}$$

$$\underline{f'(x) = 5 - \frac{5}{x}}$$

Power Rule

$$(\ln x)' = \frac{1}{x}$$

Exercise

Find the derivative $f(x) = \ln x^{10} + 2 \ln x$

Solution

$$f(x) = 10 \ln x + 2 \ln x$$

Power Rule

$$= 12 \ln x$$

$$\underline{f'(x) = \frac{12}{x} \quad \Big|}$$

$$(\ln x)' = \frac{1}{x}$$

Exercise

Find the derivative $f(x) = \frac{\ln x}{2x+5}$

Solution

$$u = \ln x \quad v = 2x + 5$$

$$u' = \frac{1}{x} \quad v' = 2$$

$$\begin{aligned} f'(x) &= \frac{\frac{1}{x}(2x+5) - (2)\ln x}{(2x+5)^2} \cdot \frac{x}{x} \\ &= \frac{2x+5-2x\ln x}{x(2x+5)^2} \quad \Big| \end{aligned}$$

Exercise

Find the derivative $f(x) = -2 \ln x + x^2 - 4$

Solution

$$\underline{f'(x) = -\frac{2}{x} + 2x \quad \Big|}$$

Exercise

Find the derivative of $y = \ln\left(\frac{1}{x\sqrt{x+1}}\right)$

Solution

$$\begin{aligned} y &= \ln(1) - \ln(x\sqrt{x+1}) \\ &= -\ln x - \ln(x+1)^{1/2} \\ &= -\ln x - \frac{1}{2} \ln(x+1) \\ y' &= -\frac{1}{x} - \frac{1}{2} \frac{1}{x+1} \\ &= -\frac{3x+2}{2x(x+1)} \quad \Big| \end{aligned}$$

Exercise

Find the derivative of $y = \ln(\ln(\ln x))$

Solution

$$\begin{aligned}y' &= \frac{1}{\ln(\ln x)} \cdot (\ln(\ln x))' \\&= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot (\ln x)' \\&= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{1}{x} \\&= \frac{1}{x(\ln x)(\ln(\ln x))} \quad \Big| \end{aligned}$$

Exercise

Find the derivative of $y = \ln(\sec(\ln x))$

Solution

$$\begin{aligned}y' &= \frac{1}{\sec(\ln x)} \cdot (\sec(\ln x))' \\&= \frac{1}{\sec(\ln x)} \cdot (\sec(\ln x) \tan(\ln x)) \cdot (\ln x)' \\&= \frac{\sec(\ln x)}{\sec(\ln x)} \tan(\ln x) \cdot \frac{1}{x} \\&= \frac{\tan(\ln x)}{x} \quad \Big| \end{aligned}$$

Exercise

Find the derivative of $y = \ln \left(\frac{(x^2 + 1)^5}{\sqrt{1 - x}} \right)$

Solution

$$\begin{aligned}y &= \ln(x^2 + 1)^5 - \ln(1 - x)^{1/2} \\&= 5 \ln(x^2 + 1) - \frac{1}{2} \ln(1 - x) \\y' &= 5 \frac{2x}{x^2 + 1} - \frac{1}{2} \frac{-1}{1 - x}\end{aligned}$$

$$= \frac{10x}{x^2+1} + \frac{1}{2(1-x)} \quad \Bigg|$$

Exercise

Find the derivative of $y = \ln \sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$

Solution

$$\begin{aligned} y &= \frac{1}{2} \ln \left(\frac{(x+1)^5}{(x+2)^{20}} \right) \\ &= \frac{1}{2} \left[\ln(x+1)^5 - \ln(x+2)^{20} \right] \\ &= \frac{1}{2} (5 \ln(x+1) - 20 \ln(x+2)) \end{aligned}$$

$$\begin{aligned} y' &= \frac{1}{2} \left(5 \frac{1}{x+1} - 20 \frac{1}{x+2} \right) \\ &= \frac{5}{2} \left(\frac{1}{x+1} - \frac{4}{x+2} \right) \\ &= \frac{5}{2} \left(\frac{x+2-4x-4}{(x+1)(x+2)} \right) \\ &= \frac{5}{2} \frac{-3x-2}{(x+1)(x+2)} \\ &= -\frac{5}{2} \frac{3x+2}{(x+1)(x+2)} \quad \Bigg| \end{aligned}$$

Exercise

Find the derivative of $f(x) = e^{3x}$

Solution

$$f'(x) = 3e^{3x} \quad \Bigg|$$

Exercise

Find the derivative of $f(x) = e^{-2x^3}$

Solution

$$f'(x) = e^{-2x^3} (-6x^2)$$

$$\underline{= -\frac{6x^2}{e^{2x^3}}}$$

Exercise

Find the derivative of $f(x) = 4e^{x^2}$

Solution

$$\begin{aligned} f'(x) &= 4e^{x^2} (2x) \\ &\underline{= 8xe^{x^2}} \end{aligned}$$

Exercise

Find the derivative of $f(x) = x^2 e^x$

Solution

$$\begin{aligned} f'(x) &= e^x \frac{d}{dx}[x^2] + x^2 \frac{d}{dx}[e^x] \\ &= e^x(2x) + x^2 e^x \\ &\underline{= xe^x(2+x)} \end{aligned}$$

Exercise

Find the derivative $f(x) = 2x^3 e^x$

Solution

$$\begin{aligned} f'(x) &= 6x^2 e^x + 2x^3 e^x \\ &\underline{= 2x^2 e^x (3+x)} \end{aligned}$$

$$\begin{aligned} u &= 2x^3 & v &= e^x \\ u' &= 6x^2 & v' &= e^x \end{aligned}$$

Exercise

Find the derivative $f(x) = \frac{3e^x}{1+e^x}$

Solution

$$f'(x) = \frac{3e^x(1+e^x) - 3e^x e^x}{(1+e^x)^2}$$

$$\begin{aligned} u &= 3e^x & v &= 1+e^x \\ u' &= 3e^x & v' &= e^x \end{aligned}$$

$$= \frac{3e^x + 3e^{2x} - 3e^{2x}}{(1+e^x)^2}$$

$$= \frac{3e^x}{(1+e^x)^2}$$

Exercise

Find the derivative $f(x) = 5e^x + 3x + 1$

Solution

$$f'(x) = 5e^x + 3$$

Exercise

Find the derivative of $f(x) = \frac{e^x + e^{-x}}{2}$

Solution

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f'(x) = \frac{1}{2} \left(\frac{d}{dx} [e^x] + \frac{d}{dx} [e^{-x}] \right)$$

$$= \frac{1}{2} (e^x - e^{-x})$$

Exercise

Find the derivative of $f(x) = \frac{e^x}{x^2}$

Solution

$$f'(x) = \frac{x^2 e^x - e^x (2x)}{x^4}$$

$$= \frac{x^2 e^x - 2x e^x}{x^4}$$

$$= \frac{x e^x (x-2)}{x^4}$$

$$= \frac{e^x (x-2)}{x^3}$$

Exercise

Find the derivative of $f(x) = x^2 e^x - e^x$

Solution

$$\begin{aligned} f'(x) &= e^x \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(e^x) - \frac{d}{dx}(e^x) \\ &= e^x(2x) + x^2 e^x - e^x \\ &= \underline{e^x(x^2 + 2x - 1)} \end{aligned}$$

Exercise

Find the derivative of $f(x) = (1 + 2x)e^{4x}$

Solution

$$\begin{aligned} f'(x) &= (2)e^{4x} + (1 + 2x)(4e^{4x}) \\ &= 2e^{4x} + (1 + 2x)(4e^{4x}) \\ &= 2e^{4x}(1 + 2(1 + 2x)) \\ &= 2e^{4x}(1 + 2 + 4x) \\ &= \underline{2e^{4x}(3 + 4x)} \end{aligned}$$

Exercise

Find the derivative of $y = x^2 e^{5x}$

Solution

$$\begin{aligned} y' &= x^2(5e^{5x}) + 2x(e^{5x}) \\ &= \underline{xe^{5x}(5x + 2)} \end{aligned}$$

Exercise

Find the derivative of $y = x^2 e^{-2x}$

Solution

$$\begin{aligned} y' &= 2xe^{-2x} - 2x^2 e^{-2x} \\ &= \underline{2xe^{-2x}(1 - x)} \end{aligned}$$

Exercise

Find the derivative $f(x) = \frac{e^x}{x^2 + 1}$

Solution

$$\begin{aligned} f'(x) &= \frac{e^x(x^2 + 1) - 2xe^x}{(x^2 + 1)^2} \\ &= \frac{(x^2 + 1 - 2x)e^x}{(x^2 + 1)^2} \end{aligned}$$

$$u = e^x \quad v = x^2 + 1$$

$$u' = e^x \quad v' = 2x$$

Exercise

Find the derivative $f(x) = \frac{1 - e^x}{1 + e^x}$

Solution

$$\begin{aligned} f'(x) &= \frac{-e^x(1 + e^x) - e^x(1 - e^x)}{(1 + e^x)^2} \\ &= \frac{-e^x - e^{2x} - e^x + e^{2x}}{(1 + e^x)^2} \\ &= -\frac{2e^x}{(1 + e^x)^2} \end{aligned}$$

$$u = 1 - e^x \quad v = 1 + e^x$$

$$u' = -e^x \quad v' = e^x$$

Exercise

Find the derivative of $y = \frac{e^x + e^{-x}}{x}$

Solution

$$\begin{aligned} y &= \frac{(e^x - e^{-x})x - (e^x + e^{-x})}{x^2} \\ &= \frac{xe^x - xe^{-x} - e^x - e^{-x}}{x^2} \\ &= \frac{(x-1)e^x - (x+1)e^{-x}}{x^2} \end{aligned}$$

$$f = e^x + e^{-x} \quad g = x$$

$$f' = e^x - e^{-x} \quad g' = 1$$

Exercise

Find the derivative of $y = \sqrt{e^{2x^2} + e^{-2x^2}}$

Solution

$$y = \sqrt{e^{2x^2} + e^{-2x^2}} = \left(e^{2x^2} + e^{-2x^2} \right)^{1/2} = U^{1/2}$$

$$U = e^{2x^2} + e^{-2x^2} \quad \left(e^{2x^2} \right)' = \left(2x^2 \right)' e^{2x^2} = 4xe^{2x^2}$$

$$U' = 4xe^{2x^2} - 4xe^{-2x^2}$$

$$y' = \frac{1}{2} \left(4xe^{2x^2} - 4xe^{-2x^2} \right) \left(e^{2x^2} + e^{-2x^2} \right)^{-1/2}$$

$$= \frac{1}{2} \frac{4x \left(e^{2x^2} - e^{-2x^2} \right)}{\left(e^{2x^2} + e^{-2x^2} \right)^{1/2}}$$

$$= \frac{2x \left(e^{2x^2} - e^{-2x^2} \right)}{\sqrt{e^{2x^2} + e^{-2x^2}}}$$

Exercise

Find the derivative of $y = \frac{x}{e^{2x}}$

Solution

$$y' = \frac{1 \left(e^{2x} \right) - x \left(2e^{2x} \right)}{\left(e^{2x} \right)^2}$$

$$= \frac{e^{2x} (1 - 2x)}{\left(e^{2x} \right)^2}$$

$$= \frac{1 - 2x}{e^{2x}}$$

$$f = x \quad g = e^{2x}$$

$$f' = 1 \quad g' = 2e^{2x}$$

Exercise

Find the derivative of $y = 3e^{5x^3+1}$

Solution

$$y' = 3(15x^2)e^{5x^3+1}$$

$$y' = 45x^2e^{5x^3+1}$$

$$f = x^2 \quad g = e^{5x^3+1}$$

$$f' = 2x \quad g' = 15x^2e^{5x^3+1}$$

$$y'' = 45 \left(2xe^{5x^3+1} + (x^2)15x^2e^{5x^3+1} \right)$$

$$= 45e^{5x^3+1} (2x + 15x^4)$$

$$= 45xe^{5x^3+1} (2 + 15x^3) \quad |$$

Exercise

Find the derivative of $(x^2 - 2x + 2)e^x$

Solution

$$y = (x^2 - 2x + 2)e^x$$

$$y' = (2x - 2)e^x + (x^2 - 2x + 2)e^x$$

$$= (2x - 2 + x^2 - 2x + 2)e^x$$

$$= x^2e^x \quad |$$

Exercise

Find the derivative of $e^\theta (\sin \theta + \cos \theta)$

Solution

$$\frac{d}{d\theta} e^\theta (\sin \theta + \cos \theta) = e^\theta (\sin \theta + \cos \theta) + e^\theta (\cos \theta - \sin \theta)$$

$$= e^\theta (\sin \theta + \cos \theta + \cos \theta - \sin \theta)$$

$$= 2e^\theta \cos \theta \quad |$$

Exercise

Find the derivative of $\ln(3\theta e^{-\theta})$

Solution

$$\begin{aligned}\frac{d}{d\theta} \ln(3\theta e^{-\theta}) &= \frac{(3\theta e^{-\theta})'}{3\theta e^{-\theta}} \\ &= 3 \frac{e^{-\theta} - \theta e^{-\theta}}{\theta e^{-\theta}} \\ &= \frac{e^{-\theta}(1 - \theta)}{\theta e^{-\theta}} \\ &= \underline{\underline{\frac{1 - \theta}{\theta}}}\end{aligned}$$

$$\begin{aligned}\ln(3\theta e^{-\theta}) &= \ln(3) + \ln(\theta) + \ln(e^{-\theta}) \\ &= \ln 3 + \ln \theta - \theta \\ \frac{d}{d\theta} \ln(3\theta e^{-\theta}) &= \underline{\underline{\frac{1}{\theta} - 1}}\end{aligned}$$

Exercise

Find the derivative of $\theta^3 e^{-2\theta} \cos 5\theta$

Solution

$$\begin{aligned}\frac{dy}{d\theta} &= (\theta^3)' e^{-2\theta} \cos 5\theta + \theta^3 (e^{-2\theta})' \cos 5\theta + \theta^3 e^{-2\theta} (\cos 5\theta)' \\ &= 3\theta^2 e^{-2\theta} \cos 5\theta - 2\theta^3 e^{-2\theta} \cos 5\theta - 5\theta^3 e^{-2\theta} \sin 5\theta \\ &= \underline{\underline{\theta^3 e^{-2\theta} (3 \cos 5\theta - 2\theta \cos 5\theta - 5\theta \sin 5\theta)}}$$

Exercise

Find the derivative of $\ln\left(\frac{\sqrt{\theta}}{1 + \sqrt{\theta}}\right)$

Solution

$$\begin{aligned}\frac{d}{d\theta} \ln\left(\frac{\sqrt{\theta}}{1 + \sqrt{\theta}}\right) &= \frac{d}{d\theta} [\ln \theta^{1/2} - \ln(1 + \sqrt{\theta})] \\ &= \frac{d}{d\theta} \left[\frac{1}{2} \ln \theta - \ln(1 + \sqrt{\theta}) \right] \\ &= \frac{1}{2} \frac{1}{\theta} - \frac{\frac{1}{2} \theta^{-1/2}}{1 + \sqrt{\theta}} \\ &= \frac{1}{2\theta} - \frac{1}{2\sqrt{\theta}(1 + \sqrt{\theta})}\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left(\frac{1}{\theta} - \frac{1}{\sqrt{\theta}(1+\sqrt{\theta})} \right) \\
&= \frac{1}{2} \frac{\sqrt{\theta}(1+\sqrt{\theta}) - \theta}{\theta\sqrt{\theta}(1+\sqrt{\theta})} \\
&= \frac{1}{2} \frac{\sqrt{\theta} + \theta - \theta}{\theta\sqrt{\theta}(1+\sqrt{\theta})} \\
&= \frac{1}{2} \frac{\sqrt{\theta}}{\theta\sqrt{\theta}(1+\sqrt{\theta})} \\
&= \frac{1}{2\theta(1+\sqrt{\theta})} \Big|
\end{aligned}$$

Exercise

Find the derivative of $e^{(\cos t + \ln t)}$

Solution

$$\begin{aligned}
e^{(\cos t + \ln t)} &= e^{\cos t} e^{\ln t} \\
&= t e^{\cos t}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} e^{(\cos t + \ln t)} &= \frac{d}{dt} (t e^{\cos t}) \\
&= e^{\cos t} + t e^{\cos t} (-\sin t) \\
&= (1 - t \sin t) e^{\cos t} \Big|
\end{aligned}$$

Exercise

Find the derivative of $e^{\sin t} (\ln t^2 + 1)$

Solution

$$\begin{aligned}
\frac{d}{dt} e^{\overset{u}{\sin t}} (\overset{v}{\ln t^2 + 1}) &= e^{\overset{u'}{\sin t}} \overset{v'}{\cos t} (\ln t^2 + 1) + \frac{v'}{t} e^{\overset{u}{\sin t}} \\
&= e^{\sin t} \left[(\ln t^2 + 1) \cos t + \frac{2}{t} \right] \Big|
\end{aligned}$$

Exercise

Find the Derivatives of $y = e^{x^2} \ln x$

Solution

$$\underline{y' = 2xe^{x^2} \ln x + \frac{e^{x^2}}{x}}$$

$$\begin{aligned} f &= e^{x^2} & g &= \ln x \\ f' &= 2xe^{x^2} & g' &= \frac{1}{x} \end{aligned}$$

Exercise

Find the derivative $f(x) = e^x + x - \ln x$

Solution

$$\underline{f'(x) = e^x + 1 - \frac{1}{x}}$$

Exercise

Find the derivative $f(x) = \ln x + 2e^x - 3x^2$

Solution

$$\underline{f'(x) = \frac{1}{x} + 2e^x - 6x}$$

Exercise

Find the derivative $f(x) = \ln x^2 + 4e^x$

Solution

$$f(x) = 2 \ln x + 4e^x$$

Power Rule

$$\underline{f'(x) = \frac{2}{x} + 4e^x}$$

$$(\ln x)' = \frac{1}{x}$$

Exercise

Find the Derivatives of $y = \ln \frac{1+e^x}{1-e^x}$

Solution

$$y = \ln(1+e^x) - \ln(1-e^x)$$

$$y' = \frac{e^x}{1+e^x} - \frac{-e^x}{1-e^x}$$

$$\begin{aligned}
&= \frac{e^x}{1+e^x} + \frac{e^x}{1-e^x} \\
&= \frac{e^x - e^{2x} + e^x + e^{2x}}{(1+e^x)(1-e^x)} \\
&= \frac{2e^x}{(1+e^x)(1-e^x)} \quad \Bigg|
\end{aligned}$$

Exercise

Find the Derivatives of $y = \frac{\ln x}{e^{2x}}$

Solution

$$\begin{aligned}
y' &= \frac{e^{2x}(1/x) - \ln x(2e^{2x})}{e^{4x}} \\
&= \frac{e^{2x} - 2x \ln x(e^{2x})}{e^{4x}} \\
&= \frac{e^{2x}(1 - 2x \ln x)}{e^{4x}} \quad \Bigg|
\end{aligned}$$

Exercise

Find the Derivatives of $f(x) = e^{2x} \ln(xe^x + 1)$

Solution

$$\begin{aligned}
f &= e^{2x} & U &= 2x \rightarrow U' = 2 & f' &= 2e^{2x} \\
g &= \ln(xe^x + 1) & U &= xe^x + 1 \rightarrow U' = e^x + xe^x & g' &= \frac{e^x + xe^x}{xe^x + 1} \\
f'(x) &= 2e^{2x} \ln(xe^x + 1) + e^{2x} \frac{e^x + xe^x}{xe^x + 1} \\
&= e^{2x} \left[2 \ln(xe^x + 1) + \frac{e^x(1+x)}{xe^x + 1} \right] \quad \Bigg|
\end{aligned}$$

Exercise

Find the Derivatives of $f(x) = \frac{xe^x}{\ln(x^2 + 1)}$

Solution

$$u = xe^x \quad u' = e^x + xe^x$$

$$v = \ln(x^2 + 1) \quad v' = \frac{2x}{x^2 + 1}$$

$$\begin{aligned} f'(x) &= \frac{e^x(1+x)\ln(x^2 + 1) - \frac{2x}{x^2 + 1}xe^x}{\left[\ln(x^2 + 1)\right]^2} \\ &= \frac{e^x \left[(1+x)\ln(x^2 + 1) - \frac{2x^2}{x^2 + 1} \right]}{\left[\ln(x^2 + 1)\right]^2} \\ &= \frac{e^x \left[\frac{(x^2 + 1)(1+x)\ln(x^2 + 1) - 2x^2}{x^2 + 1} \right]}{\left[\ln(x^2 + 1)\right]^2} \\ &= \frac{e^x \left[(x^2 + 1)(1+x)\ln(x^2 + 1) - 2x^2 \right]}{(x^2 + 1) \left[\ln(x^2 + 1)\right]^2} \end{aligned}$$

Exercise

Find the Derivatives of $f(x) = xe^{-10x}$

Solution

$$f'(x) = e^{-10x} - 10xe^{-10x}$$

Exercise

Find the Derivatives of $f(x) = x \ln^2 x$

Solution

$$\begin{aligned} f'(x) &= \ln^2 x + x \left(2 \frac{1}{x} \ln x \right) \\ &= \ln^2 x + 2 \ln x \end{aligned}$$

Exercise

Find the Derivatives of $f(x) = e^{-x} \ln x$

Solution

$$\underline{f'(x) = -e^{-x} \ln x + \frac{e^{-x}}{x}}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt{x(x+1)}$

Solution

$$\ln y = \ln(x(x+1))^{1/2} = -\ln x - \frac{1}{2} \ln(x+1)$$

$$\frac{y'}{y} = \frac{1}{2} \left(\frac{1}{x} + \frac{1}{x+1} \right)$$

$$\frac{y'}{y} = \frac{1}{2} \left(\frac{2x+1}{x(x+1)} \right)$$

$$y' = \frac{1}{2} \left(\frac{2x+1}{x(x+1)} \right) \cdot y$$

$$\underline{= \frac{1}{2} \left(\frac{2x+1}{x(x+1)} \right) \sqrt{x(x+1)}}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt{(x^2+1)(x-1)^2}$

Solution

$$\ln y = \ln \left((x^2+1)(x-1)^2 \right)^{1/2}$$

$$= \frac{1}{2} \ln \left((x^2+1)(x-1)^2 \right)$$

$$= \frac{1}{2} \left[\ln(x^2+1) + \ln(x-1)^2 \right]$$

$$= \frac{1}{2} \left[\ln(x^2+1) + 2 \ln(x-1) \right]$$

$$= \frac{1}{2} \ln(x^2+1) + \ln(x-1)$$

$$\frac{y'}{y} = \frac{1}{2} \frac{2x}{x^2+1} + \frac{1}{x-1}$$

$$\begin{aligned}
&= \frac{x}{x^2+1} + \frac{1}{x-1} \\
&= \frac{x(x-1) + (x^2+1)}{(x^2+1)(x-1)} \\
&= \frac{x^2 - x + x^2 + 1}{(x^2+1)(x-1)} \\
&= \frac{2x^2 - x + 1}{(x^2+1)(x-1)}
\end{aligned}$$

$$\begin{aligned}
y' &= \frac{2x^2 - x + 1}{(x^2+1)(x-1)} \cdot y \\
&= \frac{2x^2 - x + 1}{(x^2+1)(x-1)} \sqrt{(x^2+1)(x-1)^2} \\
&= \frac{2x^2 - x + 1}{(x^2+1)(x-1)} |x-1| \sqrt{x^2+1} \\
&= \frac{(2x^2 - x + 1)|x-1|}{(x^2+1)(x-1)} (x^2+1)^{1/2} \\
&= \frac{(2x^2 - x + 1)|x-1|}{\sqrt{x^2+1}(x-1)}
\end{aligned}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt{\frac{1}{t(t+1)}}$

Solution

$$y = \left(\frac{1}{t(t+1)} \right)^{1/2}$$

$$\ln y = \ln \left(\frac{1}{t(t+1)} \right)^{1/2}$$

$$\begin{aligned}
\ln y &= \frac{1}{2} \ln \left(\frac{1}{t(t+1)} \right) \\
&= -\frac{1}{2} \ln(t(t+1)) \\
&= -\frac{1}{2} [\ln t + \ln(t+1)]
\end{aligned}$$

$$\begin{aligned}
\frac{y'}{y} &= -\frac{1}{2}\left(\frac{1}{t} + \frac{1}{t+1}\right) \\
y' &= -\frac{1}{2}\left(\frac{1}{t} + \frac{1}{t+1}\right)y \\
&= -\frac{1}{2}\left(\frac{t+1+t}{t(t+1)}\right)\frac{1}{(t(t+1))^{1/2}} \\
&= -\frac{1}{2}\frac{2t+1}{(t(t+1))^{3/2}} \\
&= -\frac{2t+1}{2(t^2+t)^{3/2}}
\end{aligned}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \frac{\theta+5}{\theta \cos \theta}$

Solution

$$\begin{aligned}
\ln y &= \ln\left(\frac{\theta+5}{\theta \cos \theta}\right) \\
\ln y &= \ln(\theta+5) - \ln(\theta \cos \theta) \\
\ln y &= \ln(\theta+5) - \ln \theta - \ln(\cos \theta) \\
\frac{y'}{y} &= \frac{1}{\theta+5} - \frac{1}{\theta} + \frac{\sin \theta}{\cos \theta} \\
y' &= \left(\frac{1}{\theta+5} - \frac{1}{\theta} + \frac{\sin \theta}{\cos \theta}\right)y \\
y' &= \left(\frac{\theta+5}{\theta \cos \theta}\right)\left(\frac{1}{\theta+5} - \frac{1}{\theta} + \tan \theta\right)
\end{aligned}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

Solution

$$\begin{aligned}
\ln y &= \ln\left(\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}\right)^{1/3} \\
&= \frac{1}{3}\left[\ln x + \ln(x+1) + \ln(x-2) - \ln(x^2+1) - \ln(2x+3)\right] \\
\frac{y'}{y} &= \frac{1}{3}\left(\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right)
\end{aligned}$$

$$y' = \frac{1}{3} \cdot \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}} \cdot \left(\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3} \right)$$

Exercise

Find the derivative of $y = t^{1-e}$

Solution

$$y' = (1-e)t^{1-e-1}$$

$$= (1-e)t^{-e}$$

Exercise

Find the derivative of $y = 2^{\sin 3t}$

Solution

$$y = a^u \Rightarrow y' = a^u \ln a \cdot (u')$$

$$y' = (2^{\sin 3t} \ln 2)(\cos 3t)(3)$$

$$= 3(\ln 2) \cos 3t (2^{\sin 3t})$$

Exercise

Find the derivative of $y = \log_3 (1 + \theta \ln 3)$

Solution

$$y = \frac{\ln(1 + \theta \ln 3)}{\ln 3}$$

$$y' = \frac{1}{\ln 3} \cdot \frac{\ln 3}{1 + \theta \ln 3}$$

$$= \frac{1}{1 + \theta \ln 3}$$

$$y = \ln u \Rightarrow y' = \frac{u'}{u}$$

Exercise

Find the derivative of $y = \log_{25} e^x - \log_5 \sqrt{x}$

Solution

$$\begin{aligned}
 y &= \frac{\ln e^x}{\ln 25} - \frac{\ln x^{1/2}}{\ln 5} \\
 &= \frac{x}{2 \ln 5} - \frac{1}{2} \frac{\ln x}{\ln 5} \\
 &= \frac{1}{2 \ln 5} (x - \ln x)
 \end{aligned}$$

$$\begin{aligned}
 y' &= \frac{1}{2 \ln 5} \left(1 - \frac{1}{x} \right) \\
 &= \frac{x-1}{2x \ln 5}
 \end{aligned}$$

Exercise

Find the derivative of $y = \log_3 r \cdot \log_9 r$

Solution

$$\begin{aligned}
 y &= \frac{\ln r}{\ln 3} \cdot \frac{\ln r}{\ln 9} \\
 &= \frac{1}{\ln 3 \cdot \ln 9} \cdot \ln^2 r
 \end{aligned}$$

$$\begin{aligned}
 y' &= \frac{1}{\ln 3 \cdot \ln 9} \cdot (2 \ln r) \left(\frac{1}{r} \right) \\
 &= \frac{2 \ln r}{r \cdot \ln 3 \cdot \ln 9}
 \end{aligned}$$

Exercise

Find the derivative of $y = \log_7 \left(\frac{\sin \theta \cos \theta}{e^\theta 2^\theta} \right)$

Solution

$$\begin{aligned}
 y &= \frac{\ln(\sin \theta) + \ln(\cos \theta) - \ln(e^\theta) - \ln(2^\theta)}{\ln 7} \\
 &= \frac{1}{\ln 7} [\ln(\sin \theta) + \ln(\cos \theta) - \theta - \theta \ln(2)]
 \end{aligned}$$

$$\begin{aligned}
 y' &= \frac{1}{\ln 7} \left[\frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} - 1 - \ln(2) \right] \\
 &= \frac{1}{\ln 7} (\cot \theta - \tan \theta - 1 - \ln 2)
 \end{aligned}$$

Exercise

Find the derivative of $y = 3 \log_8 \left(\log_2 t \right)$

Solution

$$y = 3 \frac{\ln \left(\log_2 t \right)}{\ln 8} = \frac{3}{\ln 8} \ln \left(\frac{\ln t}{\ln 2} \right)$$

$$\begin{aligned} y' &= \frac{3}{\ln 8} \left(\frac{1}{\frac{\ln t}{\ln 2}} \right) \left(\frac{1}{\ln 2} \cdot \frac{1}{t} \right) \\ &= \frac{3}{2 \ln 2} \left(\frac{\ln 2}{\ln t} \right) \left(\frac{1}{t \ln 2} \right) \\ &= \frac{1}{t (\ln t) (\ln 2)} \end{aligned}$$

Exercise

Find the derivative of $y = t \log_3 \left(e^{(\sin t)(\ln 3)} \right)$

Solution

$$\begin{aligned} y &= t \frac{\ln e^{(\sin t)(\ln 3)}}{\ln 3} \\ &= \frac{1}{\ln 3} t (\sin t) (\ln 3) \\ &= t \sin t \\ y' &= \sin t + t \cos t \end{aligned}$$

Exercise

Find the derivative of $f(x) = \log_3 (x + 8)$

Solution

$$f'(x) = \frac{1}{\ln 3} \left(\frac{1}{x+8} \right) \quad \frac{d}{dx} \left[\log_a u \right] = \left(\frac{1}{\ln a} \right) \left(\frac{1}{u} \right) \frac{du}{dx}$$

Exercise

Find the derivative of $f(x) = 2^{x^2-x}$

Solution

$$f'(x) = (2x-1)(\ln 2) 2^{x^2-x} \quad \frac{d}{dx} [a^u] = a^u \ln(a) \frac{du}{dx}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = (x+1)^x$

Solution

$$\ln y = \ln(x+1)^x = x \cdot \ln(x+1)$$

$$\frac{y'}{y} = \ln(x+1) + x \cdot \frac{1}{x+1}$$

$$\underline{y' = (x+1)^x \left(\ln(x+1) + \frac{x}{x+1} \right)}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = x^2 + x^{2x}$

Solution

$$y - x^2 = x^{2x}$$

$$\ln(y - x^2) = \ln x^{2x} = 2x \ln x$$

$$\frac{1}{y - x^2} (y' - 2x) = 2 \ln x + 2x \frac{1}{x}$$

$$y' - 2x = (y - x^2)(2 \ln x + 2)$$

$$y' - 2x = (x^2 + x^{2x} - x^2)(2 \ln x + 2)$$

$$y' = 2x^{2x}(\ln x + 1) + 2x$$

$$\underline{= 2(x^{2x} \ln x + x^{2x} + x)}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = (\sin x)^x$

Solution

$$\ln y = \ln(\sin x)^x$$

$$\ln y = x \ln(\sin x)$$

$$u = x \quad v = \ln(\sin x)$$

$$u' = 1 \quad v' = \frac{\cos x}{\sin x}$$

$$\frac{y'}{y} = \ln(\sin x) + x \frac{\cos x}{\sin x}$$

$$y' = y(\ln(\sin x) + x \cot x)$$

$$\underline{= (\sin x)^x [\ln(\sin x) + x \cot x]}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = x^{\sin x}$

Solution

$$\ln y = \ln x^{\sin x}$$

$$\ln y = \sin x \ln x$$

$$\frac{y'}{y} = \cos x \ln x + \frac{\sin x}{x}$$

$$\frac{y'}{y} = \frac{x \cos x \ln x + \sin x}{x}$$

$$y' = y \frac{x \cos x \ln x + \sin x}{x} \\ = x^{\sin x} \frac{\sin x + x(\ln x)(\cos x)}{x}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = (\ln x)^{\ln x}$

Solution

$$\ln y = \ln (\ln x)^{\ln x}$$

$$\ln y = (\ln x) \ln (\ln x)$$

$$\frac{y'}{y} = \frac{1}{x} \ln (\ln x) + \ln x \frac{1}{\ln x}$$

$$y' = \left(\frac{1}{x} \ln (\ln x) + \frac{1}{x} \right) y$$

$$= \left(\frac{\ln (\ln x) + 1}{x} \right) (\ln x)^{\ln x}$$

Exercise

Find the second derivative of $y = 3 e^{5x^3+1}$

Solution

$$y' = 45x^2 e^{5x^3+1}$$

$$y'' = (90x + 675x^5) e^{5x^3+1}$$

Exercise

Find the equations of the tangent lines to $f(x) = e^x$ at the points (0, 1)

Solution

$$f'(x) = e^x$$

$$\begin{aligned}(0, 1) \Rightarrow m &= f'(x=0) \\ &= e^0 \\ &= 1\end{aligned}$$

$$y - 1 = 1(x - 0) + 1$$

$$\underline{y = x + 1}$$

$$y = m(x - x_1) + y_1$$

Exercise

Find the equations of the tangent lines to $f(x) = e^x$ at the points (1, e)

Solution

$$f'(x) = e^x$$

$$(1, e) \Rightarrow m = f'(x=1) = e^1 = e$$

$$y = e(x - 1) + e$$

$$\underline{y = ex}$$

$$y = m(x - x_1) + y_1$$

Exercise

Find the equations of the tangent lines to $y = 4xe^{-x} + 5$ at $x = 1$

Solution

$$y' = 4e^{-x} - 4xe^{-x} = 4e^{-x}(1 - x)$$

$$= 4e^{-x}(1 - x)$$

$$m = y'(x=1)$$

$$= 4e^{-1}(1 - 1) = 0$$

$$\Rightarrow x = 1 \rightarrow y = 4e^{-1} + 5 \quad \left(1, 4e^{-1} + 5\right)$$

$$y = 0(x - 1) + 4e^{-1} + 5$$

$$\underline{y = 4e^{-1} + 5}$$

$$y = m(x - x_1) + y_1$$

Exercise

Find the equation of the tangent lines to $f(x) = 4e^{-8x}$ at the points (0, 4)

Solution

$$f'(x) = -32e^{-8x}$$

$$m = f'(0) = -32e^{-8(0)} = -32$$

$$y = -32(x - 0) + 4$$

$$\underline{y = -32x + 4}$$

$$y = m(x - x_1) + y_1$$

Exercise

The following formula accurately models the relationship between the size of a certain type of tumor and the amount of time that it has been growing:

$$V(t) = 450(1 - e - 0.0022t)^3$$

where t is in months and $V(t)$ is measured in cubic centimeters. Calculate the rate of change of tumor volume at 80 months.

Solution

$$U = 1 - e - 0.0022t \quad V = 450U^3$$

$$U' = -.0022 \quad V' = 450(3)U^2U'$$

$$V'(t) = 450(3)(1 - e - 0.0022t)^2(-.0022)$$

$$\underline{= 2.97(1 - e - 0.0022t)^2}$$

$$V'(t = 80) = 2.97(1 - e - 0.0022(80))^2$$

$$\underline{\approx 10.66}$$

Exercise

A yeast culture at room temperature ($68^\circ F$) is placed in a refrigerator set at a constant temperature of $38^\circ F$. After t hours, the temperature T of the culture is given approximately by

$$T = 30e^{-0.58t} + 38 \quad t \geq 0$$

What is the rate of change of temperature of the culture at the end of 1 hour? At the end of 4 hours?

Solution

$$T' = 30(-0.58)e^{-0.58t} \underline{= -17.4e^{-0.58t}}$$

$$T'(1) = -17.4e^{-0.58(1)} \approx -9.74^\circ F / hr$$

$$T'(4) = -17.4e^{-0.58(4)} \approx -1.71^\circ F / hr$$

Exercise

A mathematical model for the average age of a group of people learning to type is given by

$$N(t) = 10 + 6 \ln t \quad t \geq 1$$

Where $N(t)$ is the number of words per minute typed after t hours of instruction and practice (2 hours per day, 5 days per week). What is the rate of learning after 10 hours of instruction and practice? After 100 hours?

Solution

$$N'(t) = \frac{6}{t}$$

$$N'(10) = \frac{6}{10} = 0.6$$

After 10 hours of instruction and practice, the rate of learning is 0.6 words/minute per hour of instruction and practice.

$$N'(100) = \frac{6}{100} = 0.06$$

After 100 hours of instruction and practice, the rate of learning is 0.06 words/minute per hour of instruction and practice.

Exercise

The population of coyotes in the northwestern portion of Alabama is given by the formula

$P(t) = (t^2 + 100) \ln(t + 2)$, where t represents the time in years since 2000 (the year 2000 corresponds to $t = 0$). Find the rate of change of the coyote population in 2013 ($t = 13$).

Solution

$$P'(t) = 2t \ln(t + 2) + \frac{1}{t + 2} (t^2 + 100)$$

$$P' = f'g + g'f$$

$$f = t^2 + 100 \quad g = \ln(t + 2)$$

$$f' = 2t \quad g' = \frac{1}{t + 2}$$

$$= 2t \ln(t + 2) + \frac{t^2 + 100}{t + 2}$$

$$P'(t = 13) = 2(13) \ln(13 + 2) + \frac{13^2 + 100}{13 + 2} \approx 88.34$$

$$2 * 13 \ln(13 + 2) + (13^2 + 100) / (13 + 2)$$