Solution Section 4.5 – Diagonalization

Exercise

The Lucas numbers are like Fibonacci numbers except they start with $L_1 = 1$ *and* $L_2 = 3$. Following the rule $L_{k+2} = L_{k+1} + L_k$. The next Lucas numbers are 4, 7, 11, 18. Show that the Lucas number $L_{100} = \lambda_1^{100} + \lambda_2^{100}$.

Solution

Let $u_k = \begin{pmatrix} L_{k+1} \\ L_k \end{pmatrix}$, the rule $\begin{matrix} L_{k+2} = L_{k+1} + L_k \\ L_{k+1} = L_{k+1} \end{matrix}$ becomes $u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$. $\Rightarrow A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = -\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1$$

The characteristic equation is $\lambda^2 - \lambda - 1 = 0$ and the solutions are

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618 \quad \textit{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -.618$$

$$(A - \lambda_1 I)v_1 = \begin{bmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to x_1 - \lambda_1 y_1 = 0 \Rightarrow v_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$$

$$(A - \lambda_2 I)v_2 = \begin{bmatrix} 1-\lambda_2 & 1 \\ 1 & -\lambda_2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to x_2 - \lambda_2 y_2 = 0 \Rightarrow v_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

The linear combination: $c_1 v_1 + c_2 v_2 = u_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$\lambda_1 v_1 + \lambda_2 v_2 = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 + \lambda_2^2 \\ \lambda_1 + \lambda_2 \end{pmatrix} = \begin{bmatrix} \textit{trace of } A^2 \\ \textit{trace of } A \end{bmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

The solution $u_{100} = A^{99} u_1$

$$L_{100} = c_1 \lambda_1^{99} + c_2 \lambda_2^{99} = \lambda_1^{100} + \lambda_2^{100}$$

Exercise

Find all eigenvector matrices S that diagonalize A (rank 1) to give $S^{-1}AS = \Lambda$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

What is A^n? Which matrices B commute with A (so that $AB = BA$)

Solution

Since A has rank 1, its nullspace is a two-dimensional plane. Any vector with $x + y + z = 0$ solves $A\mathbf{x} = 0$. So $\lambda = 0$ is an eigenvalue with multiplicity 2. There are two independent eigenvectors. The other eigenvalues must be $\lambda = 3$ because the trace A is $1 + 1 + 1 = 3$.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 + 2 - 3(1-\lambda) = -\lambda^3 + 3\lambda^2$$

The eigenvalues are $\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 3$.

The eigenvectors for $\lambda_3 = 3$ is:

$$(A - \lambda_3 I)v_3 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \quad x_3 + y_3 - 2z_3$$

$$\textit{if } z_3 = 1 \to x_3 + y_3 = 2 \to x_3 = y_3 = 1 \Rightarrow v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The eigenvectors for $\lambda_1 = \lambda_2 = 0$ are any two independent vectors in the plane of $x + y + z = 0$

The possible matrices S:

$$S = \begin{pmatrix} x & X & c \\ y & Y & c \\ -x-y & -X-Y & c \end{pmatrix} \text{ and } S^{-1}AS = \Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

where $c \neq 0$ *and* $xY \neq yX$.

The powers A^n come: $\quad A^2 = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = 3A$ and $A^n = 3^{n-1}A$

If $AB = BA$, all the column and row of $\boldsymbol{B}$ must be the same. One possible $\boldsymbol{B}$ is $\boldsymbol{A}$ itself, since $AA = AA$, $\boldsymbol{B}$ is any linear combination of permutation matrices.

Exercise

Determine whether the matrix is diagonalizable

$a)\ \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ $\qquad b)\ \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$

$c)\ \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$ $\qquad d)\ \begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Solution

a) $\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0$

The only eigenvalue: $\lambda = 2$, the eigenvectors are:

$$(A-\lambda I)V_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \to_{x=0} \Rightarrow V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$S = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and the inverse doesn't exist. Therefore the matrix A is not diagonalizable.

b) $\det(A-\lambda I) = \begin{vmatrix} -3-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (-3-\lambda)(1-\lambda)+4$

$= \lambda^2 + 2\lambda + 1 = 0$

The only eigenvalue: $\lambda = -1$, the eigenvectors are:

$$(A-\lambda I)V_1 = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \to \begin{matrix} -2x+2y=0 \\ 2x-2y=0 \end{matrix} \to x = y \Rightarrow V_{1,2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ (***linearly dependent***)

$S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \to$ the inverse doesn't exist. Therefore the matrix A is not diagonalizable.

This space is 1-dimensional, A does not have 2 linearly independent eigenvectors.

c) $\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 0 & 1 \\ -1 & 3-\lambda & 0 \\ -4 & 13 & -1-\lambda \end{vmatrix}$

$$\begin{aligned}
&= (-1-\lambda)(3-\lambda)(-1-\lambda) - 13 + 4(3-\lambda) \\
&= \left(1+2\lambda+\lambda^2\right)(3-\lambda) - 13 + 12 - 4\lambda \\
&= 3 + 6\lambda + 3\lambda^2 - \lambda - 2\lambda^2 - \lambda^3 - 1 - 4\lambda \\
&= -\lambda^3 + \lambda^2 + \lambda + 2
\end{aligned}$$

The eigenvalues are: $\lambda_1 = 2, \quad \lambda_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \lambda_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

The eigenvector for $\lambda_1 = 2$ is:

$$(A - \lambda_1 I)V_1 = \begin{pmatrix} -3 & 0 & 1 \\ -1 & 1 & 0 \\ -4 & 13 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{matrix} -3x_1 + z_1 = 0 \\ -x_1 + y_1 = 0 \\ -4x_1 + 13y_1 - 3z_1 = 0 \end{matrix} \Rightarrow \begin{matrix} z_1 = 3x_1 \\ y_1 = x_1 \end{matrix}$$

$$if \quad x_1 = \tfrac{1}{3} \Rightarrow v_1 = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix}$$

d) Since the matrix is upper triangular with diagonal entries 2, 2, 3, and 3, the eigenvalues are 2 and 3 (each multiplicity of 2)

For $\lambda = 2 \quad \Rightarrow \quad (A - 2I)V_1 = 0$

$$\begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{cases} -x_2 + x_4 = 0 \\ x_3 - x_4 = 0 \\ x_3 + 2x_4 = 0 \\ x_4 = 0 \end{cases} \Rightarrow x_2 = x_3 = x_4 = 0$$

$$V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ has dimension 1.}$$

For $\lambda = 3 \quad \Rightarrow \quad (A - 3I)V_2 = 0$

$$\begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{cases} -x_1 - x_2 + x_4 = 0 \\ -x_2 + x_3 - x_4 = 0 \\ 2x_4 = 0 \end{cases} \to \begin{cases} x_1 = -x_2 \\ x_3 = x_2 \\ x_4 = 0 \end{cases}$$

$$V_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

The matrix is not diagonalizable since it has only 2 distinct eigenvectors. Note that showing that the geometric multiplicity of either eigenvalue is less than its algebraic multiplicity is sufficient to show that the matrix is not diagonalizable,

Exercise

Find a matrix P that diagonalizes A, and compute $P^{-1}AP$

a) $A=\begin{bmatrix}-14 & 12\\ -20 & 17\end{bmatrix}$ ***b***) $A=\begin{bmatrix}1 & 0\\ 6 & -1\end{bmatrix}$ ***c***) $A=\begin{bmatrix}3 & -2 & 0\\ -2 & 3 & 0\\ 0 & 0 & 5\end{bmatrix}$

Solution

a) $\det(A-\lambda I)=\begin{vmatrix}-14-\lambda & 12\\ -20 & 17-\lambda\end{vmatrix}=\lambda^2-3\lambda+2$

The eigenvalues are: $\lambda_1=2 \quad \lambda_2=1$

For $\lambda_1=2 \quad \rightarrow \quad \begin{pmatrix}-16 & 12\\ -20 & 15\end{pmatrix}\begin{pmatrix}x_1\\ y_1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow\begin{cases}-16x_1+12y_1=0\\ -20x_1+15y_1=0\end{cases} \rightarrow 4x_1=3y_1$

Therefore the eigenvector: $V_1=\begin{pmatrix}3\\ 4\end{pmatrix}$

For $\lambda_2=1 \quad \rightarrow \quad \begin{pmatrix}-15 & 12\\ -20 & 16\end{pmatrix}\begin{pmatrix}x_2\\ y_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow\begin{cases}-15x_2+12y_2=0\\ -20x_2+16y_2=0\end{cases} \rightarrow 5x_2=4y_2$

Therefore the eigenvector: $V_2=\begin{pmatrix}4\\ 5\end{pmatrix}$

The eigenvectors: $P=\begin{pmatrix}3 & 4\\ 4 & 5\end{pmatrix} \quad \rightarrow \quad P^{-1}=\begin{pmatrix}-5 & 4\\ 4 & -3\end{pmatrix}$

$$\begin{aligned}P^{-1}AP&=\begin{pmatrix}-5 & 4\\ 4 & -3\end{pmatrix}\begin{pmatrix}-14 & 12\\ -20 & 17\end{pmatrix}\begin{pmatrix}3 & 4\\ 4 & 5\end{pmatrix}\\ &=\begin{pmatrix}-10 & 8\\ 4 & -3\end{pmatrix}\begin{pmatrix}3 & 4\\ 4 & 5\end{pmatrix}\\ &=\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\end{aligned}$$

b) $\det(A-\lambda I)=\begin{vmatrix}1-\lambda & 0\\ 6 & -1-\lambda\end{vmatrix}=\lambda^2-1=0$

The eigenvalues are: $\lambda_1=-1, \quad \lambda_2=1$

For $\lambda_1=-1 \quad \rightarrow \quad \begin{pmatrix}2 & 0\\ 6 & 0\end{pmatrix}\begin{pmatrix}x_1\\ y_1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow\begin{cases}2x_1=0\\ 6x_1=0\end{cases}$

Therefore the eigenvector: $V_1=\begin{pmatrix}0\\ 1\end{pmatrix}$

For $\lambda_2 = 1 \quad \rightarrow \quad \begin{pmatrix} 0 & 0 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \\ 6x_2 - 2y_2 = 0 \end{cases} \rightarrow 3x_2 = y_2$

Therefore the eigenvector: $V_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

The eigenvectors: $P = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \quad \rightarrow \quad P^{-1} = \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

c) $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (3-\lambda)^2(5-\lambda) - 4(5-\lambda)$

$$\begin{aligned} &= (5-\lambda)\left(\lambda^2 - 6\lambda + 9 - 4\right) \\ &= (5-\lambda)\left(\lambda^2 - 6\lambda + 5\right) \\ &= (5-\lambda)(\lambda - 5)(\lambda - 1) = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = 1, \quad \lambda_2 = 5, \quad \lambda_3 = 5$

The eigenvector for $\lambda_1 = 1$ is:

$$(A - \lambda_1 I)v_1 = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} 2x_1 - 2y_1 = 0 \\ \\ 4z_1 = 0 \end{matrix} \Rightarrow \begin{matrix} x_1 = y_1 \\ \\ z_1 = 0 \end{matrix}$$

$$if\ y_1 = 1 \rightarrow x_1 = 1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

The eigenvectors for $\lambda_{2,3} = 5$ is:

$$(A - \lambda I)v_2 = \begin{pmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{matrix} -2x_2 - 2y_2 = 0 \\ \\ \end{matrix} \Rightarrow \begin{matrix} x_2 = -y_2 \\ \\ 0z_2 = 0 \end{matrix}$$

$$\Rightarrow v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \textit{and} \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow P^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Exercise

Determine if the matrices are diagonalizable. If so, find a matrix $\boldsymbol{P}$ that diagonalizes A and determine $P^{-1}AP$

a) $A = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

b) $A = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix}$

c) $A = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

d) $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Solution

a) $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & -2 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^3$

The eigenvalues are: $\lambda_{1,2,3} = 3$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

b) $A = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix}$

The eigenvalues are: $\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 2$

$$P=\begin{pmatrix}1 & 0 & \frac{3}{4}\\ \frac{4}{3} & 0 & \frac{3}{4}\\ 1 & 0 & 1\end{pmatrix}$$

c) $A=\begin{pmatrix}-2 & 0 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 1 & 3\end{pmatrix}$

The eigenvalues are: $\lambda_{1,2}=-2,\quad \lambda_{3,4}=3$

$$P=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

d) $\det(A-\lambda I)=\begin{vmatrix}-2-\lambda & 0 & 0 & 0\\ 0 & -2-\lambda & 5 & -5\\ 0 & 0 & 3-\lambda & 0\\ 0 & 0 & 0 & 3-\lambda\end{vmatrix}$

Since the matrix A is an upper triangular, then the eigenvalues are: $\lambda_{1,2}=-2\quad \lambda_{3,4}=3$

For $\lambda=-2 \quad\Rightarrow\quad (A+2I)V_1=0$

$$\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 5 & -5\\ 0 & 0 & 5 & 0\\ 0 & 0 & 0 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \qquad \rightarrow \begin{cases}5x_3-5x_4=0\\ 5x_3=0\\ 5x_4=0\end{cases} \quad\Rightarrow\quad x_3=x_4=0$$

$$V_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \textit{and} \quad V_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

For $\lambda=3 \quad\Rightarrow\quad (A-3I)V_2=0$

$$\begin{bmatrix}-5 & 0 & 0 & 0\\ 0 & -5 & 5 & -5\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad \rightarrow \quad \begin{cases}-5x_1=0\\ -5x_2+5x_3-5x_4=0\\ \end{cases} \quad\Rightarrow \begin{cases}x_1=0\\ x_2=x_3-x_4\end{cases}$$

$$V_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad and \quad V_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \rightarrow \quad P^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 2 & -2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \end{aligned}$$

Exercise

The 4 by 4 triangular Pascal matrix P_L and its inverse (alternating diagonals) are

$$P_L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \quad and \quad P_L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Check that P_L and P_L^{-1} have the same eigenvalues. Find a diagonal matrix D with alternating signs that gives $P_L^{-1} = D^{-1}P_L D$, so P_L is similar to P_L^{-1}. Show that $P_L D$ with columns of alternating signs is its own inverse.

Since P_L and P_L^{-1} are similar they have the same Jordan form J. Find J by checking the number of independent eigenvectors of P_L with $\lambda = 1$.

Solution

The triangular matrices P_L and P_L^{-1} both have λ = 1, 1, 1, 1 on their main diagonals. Choose D with alternating 1 and −1 on its diagonal. D equals D^{-1}:

$$D^{-1}P_L D = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = P_L^{-1}$$

Check:

Changing signs in rows 1 and 3 of P_L, and columns 1 and 3, produces the four negative entries in P_L^{-1}. Multiply row i by $(-1)^i$ and column j by $(-1)^j$, which gives the alternating diagonals.

Then $P_L D = pascal(n,\ 1)$ has columns with alternating signs and equals its own inverse!

$$(P_L D)(P_L D) = P_L D^{-1} P_L D = P_L P_L^{-1} = I$$

P_L has only one line of eigenvectors $x = (0,\ 0,\ 0,\ x_4)$ with λ = 1. The rank of $P_L - I$ is certainly 3. So its Jordan form J has only one block (also with λ = 1):

P_L and P_L^{-1} are somehow similar to Jordan's $J = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Exercise

These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors (one from each block). But the block sizes don't match and they are not similar:

$$J = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \textit{and} \quad K = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

For any matrix M compare JM with MK. If they are equal show that M is not invertible. Then $M^{-1}JM = K$ is Impossible; J is not similar to K.

Solution

Let $M = (m_{ij})$, then

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix}$$

$$JM = \begin{pmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad and \quad MK = \begin{pmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{pmatrix}$$

If $JM = MK$ then $m_{21} = m_{22} = m_{24} = m_{41} = m_{42} = m_{44} = 0$

Which in particular means that the second row is either a multiple of the fourth row, or the fourth row is all 0's. In either of these cases M is not invertible.

Suppose that J were similar to K. Then there would be some invertible matrix M such that $MK = JM$. But we just showed that in this case M is never invertible (contradiction). Thus J is not similar to K.

Exercise

If $\boldsymbol{x}$ is in the nullspace of A show that $M^{-1}\boldsymbol{x}$ is in the nullspace of $M^{-1}AM$.

The nullspaces of A and $M^{-1}AM$ have the same (vectors) (basis) (dimension)

Solution

$$x \in N(A) \Rightarrow Ax = 0 \Rightarrow M^{-1}AM\left(M^{-1}x\right) = 0 \Rightarrow M^{-1}x \in N\left(M^{-1}AM\right)$$

$$x \in N\left(M^{-1}AM\right) \Rightarrow M^{-1}AMx = 0 \Rightarrow AMx = 0 \Rightarrow Mx \in N(A)$$

So any vector in $N(A)$ *resp.* $N\left(M^{-1}AM\right)$ is a linear combination of those in $N\left(M^{-1}AM\right)$ *resp.* $N(A)$, hence is contained in it. That is, the two vector spaces consists of the same vectors.

Exercise

Prove that A^T is always similar to A (λ's are the same):

a) For one Jordan block J_i, find M_i so that $M_i^{-1}J_iM_i = J_i^T$.

b) For any J with blocks J_i, build M_0 from blocks so that $M_0^{-1}JM_0 = J^T$.

c) For any $A = MJM^{-1}$: Show that A^T is similar to J^T and so to J and so to A.

Solution

a) For one Jordan block J_i, then

$$\begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ \iddots & & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} \lambda & 1 & 0 & & 0 \\ & \lambda & 1 & 0 & \\ & & \lambda & & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}\begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ \iddots & & & \\ 1 & & & \end{pmatrix} = \begin{pmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ 0 & 1 & \lambda & & \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}$$

So J is similar to J^T

b) For any J with block J_i, that satisfies $J_i^T = M_i^{-1}J_iM_i$

Let M_0 be the block-diagonal matrix consisting of the $M_i^{'}s$ along the diagonal. Then

$$\begin{aligned} M_0^{-1}JM_0 &= \begin{pmatrix} M_1^{-1} & & & \\ & M_2^{-1} & & \\ & & \ddots & \\ & & & M_n^{-1} \end{pmatrix}\begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_n \end{pmatrix}\begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_n \end{pmatrix} \\ &= \begin{pmatrix} M_1^{-1}J_1M_1 & & & \\ & M_2^{-1}J_2M_2 & & \\ & & \ddots & \\ & & & M_n^{-1}J_nM_n \end{pmatrix} \\ &= \begin{pmatrix} J_1^T & & & \\ & J_2^T & & \\ & & \ddots & \\ & & & J_n^T \end{pmatrix} \\ &= J^T \end{aligned}$$

c) $A^T = \left(MJM^{-1}\right)^T = \left(M^{-1}\right)^T J^T M^T = \left(M^T\right)^{-1} J^T \left(M^T\right)$

So A^T is similar to J^T, which is similar to J, which is similar to A, Thus any matrix is similar to its transpose.

Exercise

Why are these statements all true?

a) If A is similar to B then A^2 is similar to B^2.

b) A^2 and B^2 can be similar when A and B are not similar.

c) $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ is similar to $\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ is not similar to $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

e) If we exchange rows 1 and 2 of A, and then exchange columns 1 and 2 the eigenvalues stay the same. In this case M =?

Solution

a) If A is similar to B then $A = M^{-1}BM$ for some M. Then $A^2 = M^{-1}B^2M$, so A^2 is similar to B^2.

b) Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $A^2 = B^2$ so they are similar but A is not similar to B because nothing but zero matrix.

c) $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

d) They are not similar because the first matrix has a plane of eigenvectors for the eigenvalues 3, while the second only has a line.

e) In order to exchange two rows of A we multiply on the left by

$$M = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

In order to exchange two columns we multiply on the right by the same M. As $M = M^{-1}$ the new matrix is similar to the old one, so the eigenvalues stay the same.

Exercise

If an n x n matrix A has all eigenvalues $\lambda = 0$ prove that A^n is the zero matrix.

Solution

Suppose that the Jordan Block has a size of $\boldsymbol{i}$ with eigenvalue 0. Then J^2 will have a diagonal of 1's two diagonals above the main diagonal and zeroes elsewhere. J^3 will have a diagonal of 1's three diagonals above the main diagonal and zeroes elsewhere. Therefore $J^i = 0$, since there is no diagonal $\boldsymbol{i}$ diagonals above the main diagonal. If A has all eigenvalues $\lambda = 0$ then A is similar to some matrix with Jordan block $J_1, \ldots, J_k$ with each J_i of size n_i and $\sum_{i=1}^{k} n_i = n$.

Each Jordan block will have eigenvalue of 0, so that $J_i^{n_i} = 0$, and thus $J_i^n = 0$

As A^n is similar to a block-diagonal matrix with blocks $J_1^n, J_2^n, \ldots, J_k^n$ and each of these is 0 we know that $A^n = 0$.

Another way, if A has all eigenvalues 0 this means that the characteristic polynomial of A must be x^n, as this is the only polynomial of degree n all of whose roots are 0. Thus $A^n = 0$ by the Cayley-Hamilton theorem.

Exercise

If A is similar to A^{-1}, must all the eigenvalues equal to 1 or -1?.

Solution

No

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ is similar to $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^{-1}$

Exercise

Show that A and B are not similar matrices

a) $A=\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$ $\quad B=\begin{pmatrix}1 & 0\\ 3 & -2\end{pmatrix}$

b) $A=\begin{pmatrix}4 & -1\\ 2 & 4\end{pmatrix}$ $\quad B=\begin{pmatrix}4 & 1\\ 2 & 4\end{pmatrix}$

c) $A=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$ $\quad B=\begin{pmatrix}1 & 2 & 0\\ \frac{1}{2} & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

Solution

a) $|A|=\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix}=1 \quad |B|=\begin{vmatrix}1 & 0\\ 3 & -2\end{vmatrix}=-2$

$|A|\neq|B|$; therefore A and B are not similar

b) $|A|=\begin{vmatrix}4 & -1\\ 2 & 4\end{vmatrix}=18 \quad |B|=\begin{vmatrix}4 & 1\\ 2 & 4\end{vmatrix}=14$

$|A|\neq|B|$; therefore A and B are not similar

c) $|A|=\begin{vmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & 0 & 1\end{vmatrix}=1 \quad |B|=\begin{vmatrix}1 & 2 & 0\\ \frac{1}{2} & 1 & 0\\ 0 & 0 & 1\end{vmatrix}=0$

$|A|\neq|B|$; therefore A and B are not similar

Exercise

Prove that two similar matrices have the same determinant. Explain geometrically why this is reasonable.

Solution

Suppose that $A=PBP^{-1}$

Then $\det(A)=\det\left(PBP^{-1}\right)$ $\qquad |AB|=|A||B|$

$$=\det(P)\cdot\det(B)\cdot\det\left(P^{-1}\right)$$

$$=\det(B)\cdot\det(P)\cdot\det\left(P^{-1}\right)$$

$$=\det(B)\cdot\det\left(PP^{-1}\right)$$

$$=\det(B)\cdot\det(I)$$

$$= \det(B)$$

Geometric Explanation: The determinant tells us what Factor area changes when using a linear transformation. This "factor" doesn't care about the particular basis you use.

Exercise

Prove that two similar matrices have the same characteristic polynomial and thus the same eigenvalues. Explain geometrically why this is reasonable.

Solution

Suppose that $A = PBP^{-1}$

Then the characteristic polynomial is equal to $\det(A - \lambda I)$.

$$A - \lambda I = PBP^{-1} - \lambda\left(PIP^{-1}\right)$$

$$= P(B - \lambda I)P^{-1}$$

$$\det(A - \lambda I) = \det\left(P(B - \lambda I)P^{-1}\right)$$

$$= \det(P) \cdot \det(B - \lambda I) \cdot \det\left(P^{-1}\right)$$

$$= \det(B - \lambda I) \cdot \det(P) \cdot \det\left(P^{-1}\right)$$

$$= \det(B - \lambda I) \cdot \det\left(PP^{-1}\right) \qquad \det\left(PP^{-1}\right) = \det(I) = 1$$

$$= \det(B - \lambda I)$$

Geometric Explanation: At least in terms of the eigenvalues, these values are numbers λ such that there exists a vector $\boldsymbol{v} \neq 0$ such that the linear transformation T satisfies $T(\boldsymbol{v}) = \lambda \boldsymbol{v}$.

Exercise

Suppose that A is a matrix. Suppose that the linear transformation associated to A has two linearly independent eigenvectors. Prove that A is similar to a diagonal matrix.

Solution

Let T be the linear transformation associated with A. Consider the basis $\boldsymbol{v}_1, \boldsymbol{v}_2$ of the 2 linearly independent eigenvectors of A where λ_1, λ_2 the eigenvalues associated with. Then,

$$T\left(\boldsymbol{v}_1\right) = \lambda_1 \boldsymbol{v}_1 \quad \textit{and} \quad T\left(\boldsymbol{v}_2\right) = \lambda_2 \boldsymbol{v}_2$$

Let T be a matrix with respect to the basis $\boldsymbol{v}_1, \boldsymbol{v}_2$, then we obtain the matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

This completes the proof because A is similar to this diagonal matrix by definition.

Exercise

Prove that if A is a 2×2 matrix that has two distinct eigenvalues, then A is similar to a diagonal matrix.

Solution

Suppose A has 2 distinct eigenvalues $\lambda_1,\ \lambda_2$.

Let $\boldsymbol{v}_1 \neq 0$ be an eigenvector for λ_1.

Suppose that $\boldsymbol{v}_1,\ \boldsymbol{v}_2$ are not linearly independent, thus they are scalar multiples of each other.

So there exists $c\neq 0$ such that $c\boldsymbol{v}_1 = \boldsymbol{v}_2$. Then

$$\lambda_2\boldsymbol{v}_2 = A\boldsymbol{v}_2 = A\left(c\boldsymbol{v}_1\right) = c\left(A\boldsymbol{v}_1\right) = c\lambda_1\boldsymbol{v}_1 = \lambda_1 c\boldsymbol{v}_1 = \lambda_1\boldsymbol{v}_2$$

So that $\lambda_2\boldsymbol{v}_2 - \lambda_1\boldsymbol{v}_2 = 0 \quad\Rightarrow\quad \left(\lambda_2-\lambda_1\right)\boldsymbol{v}_2 = 0$

But then $\lambda_2 = \lambda_1$ which contradicts the initial assumption.

Thus $\boldsymbol{v}_1,\ \boldsymbol{v}_2$ are linearly independent then $T\left(\boldsymbol{v}_1\right) = \lambda_1\boldsymbol{v}_1 \quad \textit{and} \quad T\left(\boldsymbol{v}_2\right) = \lambda_2\boldsymbol{v}_2$

Let T be a matrix with respect to the basis $\boldsymbol{v}_1,\ \boldsymbol{v}_2$, then we obtain the matrix $\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$

This completes the proof because A is similar to this diagonal matrix by definition.

Exercise

Suppose that the characteristic polynomial of a matrix has a double root. Is it true that the matrix has two linearly independent eigenvectors? Consider the example $\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$. Is it true that matrices with equal characteristic polynomial are necessarily similar?

Solution

$$\left|A-\lambda I\right| = \begin{vmatrix}-\lambda & 0\\ 1 & -\lambda\end{vmatrix} = \lambda^2$$

The characteristic polynomial: $p(x) = x^2$ which has a double root $\left(\textit{eigenvalue}:\ \lambda = 0\right)$.

$$(A-\lambda I)V = AV = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \quad\Rightarrow\quad x = 0$$

Therefore, the eigenvectors are vectors of the form $\begin{pmatrix}0\\ 1\end{pmatrix}$ which can transform to $\begin{pmatrix}x\\ 0\end{pmatrix}$

Thus matrices whose characteristic polynomials have a double root do not necessarily have 2 linear independent.

Let $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then the characteristic polynomial: $p(x) = x^2$ which has a double root $(eigenvalue: \lambda = 0)$. But they are not similar. The eigenvector is the $\mathbf{0}$ vector.

The linear transformation associated to the second matrix send every vector to $\mathbf{0}$. Thus the 2 matrices can't represent the same linear transformation.

Thus, matrices with equal characteristic polynomial are not necessarily similar.

Exercise

Show that the given matrix is not diagonalizable. $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

Solution

$$\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = 0$$

Since the determinant is 0, the inverse doesn't exist; therefore the matrix is not diagonalizable

Exercise

Determine if the given matrix is diagonalizable. If, so, find matrices S and $\Lambda(D)$ such that the given matrix equals $S\Lambda S^{-1}$

a) $\begin{bmatrix} 3 & 3 \\ 4 & 2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{bmatrix}$

Solution

a) $\begin{vmatrix} 3-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 12 = \lambda^2 - 5\lambda - 6 = 0$

The eigenvalues $\lambda_1 = -1, \ \lambda_2 = 6$

For $\lambda = -1 \quad \Rightarrow \quad (A + I)V_1 = 0$

$$\begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad 4x_1 + 3y_1 = 0 \quad \Rightarrow x_1 = -\frac{3}{4}y_1$$

Therefore the eigenvector: $V_1 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

For $\lambda = 6 \quad \Rightarrow \quad (A - 6I)V_2 = 0$

$$\begin{pmatrix} -3 & 3 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad -3x_2 + 3y_2 = 0 \quad \Rightarrow x_2 = \frac{3}{4}y_2$$

Therefore the eigenvector: $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$S = \begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix} \quad \rightarrow \quad S^{-1} = \begin{bmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{3}{7} \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\begin{aligned} S\Lambda S^{-1} &= \begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{3}{7} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 6 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{4}{7} & \frac{3}{7} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 \\ 4 & 2 \end{bmatrix} \\ &= A \end{aligned}$$

b) $$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & -1-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(-1-\lambda) + (2-\lambda)$$

$$\begin{aligned} &= (2-\lambda)\big((1-\lambda)(-1-\lambda)+1\big) \\ &= (2-\lambda)\left(-1+\lambda^2+1\right) \\ &= (2-\lambda)\lambda^2 = 0 \end{aligned}$$

The eigenvalues $\lambda_1 = 2, \ \lambda_{2,3} = 0$

The given matrix is not diagonalizable, since the eigenvalues are not distinct.