

## *Formulas*



By: *Fred E. Khoury*



## Quadratic equation

$$ax^2 + bx + c = 0$$

$$\text{If } a + b + c = 0 \Rightarrow x = 1, \frac{c}{a}$$

### Proof

$$\begin{aligned}x_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\&= \frac{-(-a-c) \pm \sqrt{(-a-c)^2 - 4ac}}{2a} \\&= \frac{a+c \pm \sqrt{a^2 + 2ac + c^2 - 4ac}}{2a} \\&= \frac{a+c \pm \sqrt{a^2 - 2ac + c^2}}{2a} \\&= \frac{a+c \pm \sqrt{(a-c)^2}}{2a} \\&= \frac{a+c \pm (a-c)}{2a}\end{aligned}$$

$$\begin{aligned}\boxed{x_1} &= \frac{a+c+(a-c)}{2a} \\&= \frac{a+c+a-c}{2a} \\&= \frac{2a}{2a} \\&= \underline{1}\end{aligned}$$

$$a+b+c=0 \rightarrow b=-a-c$$

$$\begin{aligned}\boxed{x_2} &= \frac{a+c-(a-c)}{2a} \\&= \frac{a+c-a+c}{2a} \\&= \frac{2c}{2a} \\&= \underline{\frac{c}{a}}\end{aligned}$$

### Example

$$2x^2 + x - 3 = 0$$

$$2+1-3=0$$

$$\Rightarrow \underline{x=1, -\frac{3}{2}}$$

## Quadratic equation

$$ax^2 + bx + c = 0$$

$$\text{If } a - b + c = 0 \Rightarrow x = -1, -\frac{c}{a}$$

### Proof

$$\begin{aligned}x_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\&= \frac{-(a+c) \pm \sqrt{(a+c)^2 - 4ac}}{2a} \\&= \frac{-a-c \pm \sqrt{a^2 + 2ac + c^2 - 4ac}}{2a} \\&= \frac{-a-c \pm \sqrt{a^2 - 2ac + c^2}}{2a} \\&= \frac{-a-c \pm \sqrt{(a-c)^2}}{2a} \\&= \frac{-a-c \pm (a-c)}{2a}\end{aligned}$$

$$a - b + c = 0 \rightarrow b = a + c$$

$$\begin{cases} \boxed{x_1 = \frac{-a-c+(a-c)}{2a} = \frac{-a-c+a-c}{2a} = \frac{2c}{2a} = -\frac{c}{a}} \\ \boxed{x_2 = \frac{-a-c-(a-c)}{2a} = \frac{-a-c-a+c}{2a} = \frac{-2a}{2a} = -1} \end{cases}$$

### Example

$$2x^2 - x - 3 = 0$$

$$2 - (-1) - 3 = 0$$

$$\Rightarrow \boxed{x = -1, \frac{3}{2}}$$

## Solving Exponential Function

$$a^{mx+n} = b^{px+q} \Rightarrow x = \frac{q \ln b - n \ln a}{m \ln a - p \ln b} \quad \text{coefficient } \frac{\text{no } x's}{x's}$$

**Numerator:** multiply  $q$  with  $\ln b$  **minus** multiply  $n$  with  $\ln a$

**Denominator:** multiply  $m$  with  $\ln a$  **minus** multiply  $p$  with  $\ln b$

### Proof

$$\ln a^{mx+n} = \ln b^{px+q}$$

$$(mx+n) \ln a = (px+q) \ln b$$

$$mx \ln a + n \ln a = px \ln b + q \ln b$$

$$mx \ln a - px \ln b = q \ln b - n \ln a$$

$$x(m \ln a - p \ln b) = q \ln b - n \ln a$$

$$x = \frac{q \ln b - n \ln a}{m \ln a - p \ln b}$$

$$mx \ln a + n \ln a = px \ln b + q \ln b$$

### Example

Solve:  $3^{2x-1} = 7^{x+1}$

#### Solution

$$x = \frac{\ln 3 + \ln 7}{2 \ln 3 - \ln 7}$$

### Example

Solve:  $4^{x+3} = 3^{-x}$

#### Solution

$$x = \frac{-3 \ln 4}{\ln 4 + \ln 3}$$

$$\ln 4^{x+3} = \ln 3^{-x}$$

$$(x+3) \ln 4 = -x \ln 3$$

$$x \ln 4 + x \ln 3 = -3 \ln 4$$

$$x(\ln 4 + \ln 3) = -3 \ln 4$$

### Example

Solve:  $4^{-x} = 3^{x+3}$

#### Solution

$$x = \frac{3 \ln 3}{\ln 3 - \ln 4}$$

## ***Growth & Decay Formula***

$$A = A_0 e^{kt} \Rightarrow \underline{kT = \ln \frac{A}{A_0}}$$

### ***Proof***

$$A = A_0 e^{kt}$$

$$\frac{A}{A_0} = e^{kt}$$

$$\ln \frac{A}{A_0} = \ln e^{kt}$$

$$\boxed{\ln \frac{A}{A_0} = kt} \quad \checkmark$$



## Inverse Functions

$$f(x) = \frac{ax+b}{cx+d} \Rightarrow f^{-1}(x) = \frac{-dx+b}{cx-a}$$

### Proof

$$y = \frac{ax+b}{cx+d}$$

$$x = \frac{ay+b}{cy+d}$$

$$cxy + dx = ay + b$$

$$cxy - ay = -dx + b$$

$$(cx - a)y = -dx + b$$

$$y = \frac{-dx + b}{cx - a}$$

$$\boxed{f^{-1}(x) = \frac{-dx+b}{cx-a}} \quad \checkmark$$

Interchange  $a$  and  $d$  and change there signs.

---

### Example

Find the inverse function of:  $f(x) = \frac{1}{3x-2}$

### Solution

$$f^{-1}(x) = \frac{2x+1}{3x}$$

$$f(x) = \frac{0x+1}{3x-2}$$

### Example

Find the inverse function of:  $f(x) = \frac{3x+2}{2x-5}$

### Solution

$$f^{-1}(x) = \frac{5x+2}{2x-3}$$

$$f(x) = \frac{3x+2}{2x-5}$$

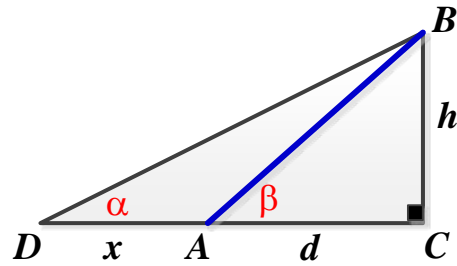
### Example

Find the inverse function of:  $f(x) = \frac{4x}{x+2}$

### Solution

$$f^{-1}(x) = \frac{-2x}{x-4}$$

$$f(x) = \frac{4x}{x+2}$$



### Proof

Triangle  $DCB$ :  $\tan \alpha = \frac{h}{d+x} \Rightarrow h = (d+x) \tan \alpha$

Triangle  $ACB$ :  $\tan \beta = \frac{h}{d} \Rightarrow h = d \tan \beta$

$$h = d \tan \beta = (d+x) \tan \alpha$$

$$d \tan \beta = d \tan \alpha + x \tan \alpha$$

$$d \tan \beta - d \tan \alpha = x \tan \alpha$$

$$d (\tan \beta - \tan \alpha) = x \tan \alpha$$

$$d = \frac{x \tan \alpha}{\tan \beta - \tan \alpha}$$

$$h = \frac{x \tan \alpha \tan \beta}{\tan \beta - \tan \alpha}$$

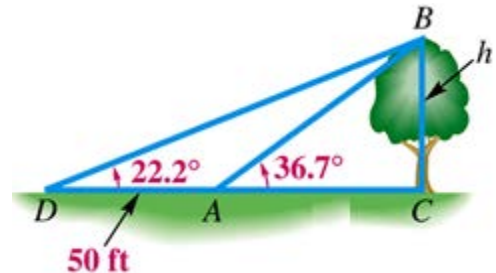
*Height is equal to distance times (tan tan) divides by the (tan(larger angle) – tan) (difference between tangents)*

### Example

From a given point on the ground, the angle of elevation to the top of a tree is  $36.7^\circ$ . From a second point, 50 feet back, the angle of elevation to the top of the tree is  $22.2^\circ$ . Find the height of the tree to the nearest foot.

### Solution

$$h = 50 \frac{\tan 22.2^\circ \tan 36.7^\circ}{\tan 36.7^\circ - \tan 22.2^\circ} \approx 45 \text{ ft}$$





# Integration by Part

Evaluate  $\int x^n e^{ax} dx$

		$\int e^{ax}$
+	$x^n$	$\frac{1}{a} e^{ax}$
-	$nx^{n-1}$	$\frac{1}{a^2} e^{ax}$
+	$n(n-1)x^{n-2}$	$\frac{1}{a^3} e^{ax}$
-	$n(n-1)(n-2)x^{n-3}$	$\frac{1}{a^4} e^{ax}$
	$\vdots \vdots$	$\vdots \vdots$

$$\int x^n e^{ax} dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a^2} x^{n-1} e^{ax} + \frac{n(n-1)}{a^3} x^{n-2} e^{ax} - \frac{n(n-1)(n-2)}{a^4} x^{n-3} e^{ax} + \dots$$

$$= e^{ax} \sum_{k=0}^n (-1)^k \cdot \frac{n!}{(n-k)!} \cdot \frac{1}{a^{k+1}} \cdot x^{n-k}$$

## Jose's Method

Evaluate  $\int e^{ax} \cos bx \, dx$

$$\int e^{ax} \cos bx \, dx = \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx \, dx$$

$$\left(1 + \frac{a^2}{b^2}\right) \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{b^2} (b \sin bx + a \cos bx)$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (b \sin bx + a \cos bx) + C$$

		$\int \cos bx \, dx$
+	$e^{ax}$	$\frac{1}{b} \sin bx$
-	$ae^{ax}$	$-\frac{1}{b^2} \cos bx$
+	$a^2 e^{ax}$	$-\frac{1}{b^2} \int \cos bx \, dx$

## Proof

Find  $\int e^{ax} \cos bx \, dx$

### Solution

Let:  $u = e^{ax} \quad dv = \cos bx \, dx$   
 $du = ae^{ax} \, dx \quad v = \int \cos bx \, dx = \frac{1}{b} \sin bx$

$$\int e^{ax} \cos bx \, dx = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \int e^{ax} \sin bx \, dx \qquad \int u \, dv = uv - \int v \, du$$

Let:  $u = e^{ax} \quad dv = \sin bx \, dx$   
 $du = ae^{ax} \, dx \quad v = \int \sin bx \, dx = -\frac{1}{b} \cos bx$

$$\begin{aligned} \int e^{ax} \cos bx \, dx &= \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \left[ -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int e^{ax} \cos bx \, dx \right] \\ &= \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx \, dx \end{aligned}$$

$$\int e^{ax} \cos bx \, dx + \frac{a^2}{b^2} \int e^{ax} \cos bx \, dx = \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx + C_1$$

$$\frac{a^2 + b^2}{b^2} \int e^{ax} \cos bx \, dx = \frac{1}{b^2} e^{ax} (b \sin bx + a \cos bx) + C_1$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (b \sin bx + a \cos bx) + C$$

**Length of a curve**  $y = f(x)$  is given by the formula:

$$L = \int_c^d \sqrt{1 + [f'(x)]^2} dx = \int_c^d \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

If  $f(x) = ax^m + bx^n$ , then

$$L = \int_c^d \sqrt{1 + \left(\frac{df}{dx}\right)^2} dx = \left[ ax^m - bx^n \right]_c^d$$

**Iff**  $f(x)$  satisfies these 2 conditions:

1.  $m + n = 2$
2.  $abmn = -\frac{1}{4}$

**Proof**

$$f'(x) = max^{m-1} + nbx^{n-1}$$

$$\begin{aligned} 1 + (f')^2 &= 1 + (max^{m-1} + nbx^{n-1})^2 \\ &= 1 + m^2 a^2 x^{2m-2} + 2abmnx^{m+n-2} + n^2 b^2 x^{2n-2} \end{aligned}$$

*We need to combined to a perfect square*

$$a^2 - 2ab + b^2 = (a - b)^2$$

$$\text{➤ If } x^{m+n-2} = 1 = x^0 \rightarrow \boxed{m+n=2}$$

$$= m^2 a^2 x^{2m-2} + (1 + 2abmn) + n^2 b^2 x^{2n-2}$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

$$\text{➤ Let } 1 + 2abmn = -2abmn \rightarrow \boxed{abmn = -\frac{1}{4}}$$

$$= m^2 a^2 x^{2m-2} - 2abmn + n^2 b^2 x^{2n-2}$$

$$x^{2(m+n-2)} = 1$$

$$= (max^{m-1} - nbx^{n-1})^2$$

$$L = \int_c^d \sqrt{(max^{m-1} - nbx^{n-1})^2} dx$$

$$= \int_c^d (max^{m-1} - nbx^{n-1}) dx$$

$$= \left[ ax^m - bx^n \right]_c^d \quad \checkmark$$

### Example

Find the length of the graph of  $f(x) = \frac{x^3}{12} + \frac{1}{x}$ ,  $1 \leq x \leq 4$

### Solution

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

$$\begin{aligned} 1 + [f'(x)]^2 &= 1 + \left( \frac{x^2}{4} - \frac{1}{x^2} \right)^2 \\ &= 1 + \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4} \\ &= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} \\ &= \left( \frac{x^2}{4} + \frac{1}{x^2} \right)^2 \end{aligned}$$

$$\begin{aligned} L &= \int_1^4 \sqrt{1 + (f'(x))^2} \, dx \\ &= \int_1^4 \sqrt{\left( \frac{x^2}{4} + \frac{1}{x^2} \right)^2} \, dx \\ &= \int_1^4 \left( \frac{x^2}{4} + \frac{1}{x^2} \right) \, dx \\ &= \left( \frac{x^3}{12} - \frac{1}{x} \right)_1^4 \\ &= \left( \frac{4^3}{12} - \frac{1}{4} \right) - \left( \frac{1}{12} - \frac{1}{1} \right) \\ &= \frac{72}{12} \\ &= \underline{6 \text{ unit}} \end{aligned}$$

$$a = \frac{1}{12}, \quad m = 3, \quad b = 1, \quad n = -1$$

$$1. \quad m + n = 3 - 1 = 2 \quad \checkmark$$

$$2. \quad abmn = \frac{1}{12}(1)(3)(-1) = -\frac{1}{4} \quad \checkmark$$

$$L = \left( \frac{x^3}{12} - \frac{1}{x} \right)_1^4 = \underline{6 \text{ unit}}$$

### Examples

$$f(x) = \frac{1}{3}x^{3/2} - x^{1/2} \rightarrow L = \frac{1}{3}x^{3/2} + x^{1/2} + C$$

$$f(x) = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2} \rightarrow L = \frac{2}{3}x^{3/2} + \frac{1}{2}x^{1/2}$$

$$f(x) = \frac{1}{6}x^3 + \frac{1}{2x} \rightarrow L = \frac{1}{6}x^3 - \frac{1}{2x} + C$$

$$f(x) = x^3 + \frac{1}{12x} \rightarrow L = x^3 - \frac{1}{12x} + C$$

$$f(x) = \frac{1}{6}x^3 + \frac{1}{2x} \rightarrow L = \frac{1}{6}x^3 - \frac{1}{2x}$$

$$f(x) = \frac{1}{8}x^4 + \frac{1}{4x^2} \rightarrow L = \frac{1}{8}x^4 - \frac{1}{4x^2}$$

$$f(x) = \frac{1}{4}x^4 + \frac{1}{8x^2} \rightarrow L = \frac{1}{4}x^4 - \frac{1}{8x^2}$$

$$f(x) = \frac{1}{10}x^5 + \frac{1}{6x^3} \rightarrow L = \frac{1}{10}x^5 - \frac{1}{6x^3}$$

$$f(x) = \frac{3}{10}x^{1/3} - \frac{3}{2}x^{5/3} \rightarrow L = \frac{3}{10}x^{1/3} + \frac{3}{2}x^{5/3}$$

$$f(x) = x^{1/2} - \frac{1}{3}x^{3/2} \rightarrow L = x^{1/2} + \frac{1}{3}x^{3/2}$$

If  $f(x) = ae^{mx} + be^{nx}$ , then

$$L = \int_c^d \sqrt{1 + \left(\frac{df}{dx}\right)^2} dx = \left[ ae^{mx} - be^{nx} \right]_c^d$$

**Iff**  $f(x)$  satisfies these 2 conditions:

1.  $m = -n$
2.  $abmn = -\frac{1}{4}$

**Proof**

$$f'(x) = ame^{mx} + bne^{nx}$$

$$1 + (f')^2 = 1 + (ame^{mx} + bne^{nx})^2$$

$$= 1 + m^2 a^2 e^{2mx} + 2abmne^{(m+n)x} + n^2 b^2 e^{2nx}$$

$$\rightarrow \text{If } e^{(m+n)x} = 1 = e^{(x=0)} \rightarrow \boxed{m+n}$$

$$= m^2 a^2 x^{2m-2} + (1 + 2abmn) + n^2 b^2 x^{2n-2}$$

$$a^2 - 2ab + b^2 = (a-b)^2$$

$$\rightarrow \text{Let } 1 + 2abmn = -2abmn \rightarrow \boxed{abmn = -\frac{1}{4}}$$

$$= m^2 a^2 e^{2mx} - 2abmne^{(m+n)x} + n^2 b^2 e^{2nx}$$

$$x^{2(m+n-2)} = 1$$

$$= (ame^{mx} - bne^{nx})^2$$

$$(ame^{mx} - bne^{nx})^2 = m^2 a^2 e^{2mx} - 2abmne^{(m+n)x} + n^2 b^2 e^{2nx}$$

**Example**

$$f(x) = 2e^x + \frac{1}{8}e^{-x} \rightarrow L = 2e^x - \frac{1}{8}e^{-x}$$

$$f(x) = 2e^{\sqrt{2}x} + \frac{1}{16}e^{-\sqrt{2}x} \rightarrow L = 2e^{\sqrt{2}x} - \frac{1}{16}e^{-\sqrt{2}x}$$

$$y = ax^n + \frac{b}{x}$$

$$y' = nax^{n-1} - \frac{b}{x^2}$$

## Derivative

**Formula**  $\left(U^m V^n W^p\right)' = U^{m-1} V^{n-1} W^{p-1} (mU'VW + nUV'W + pUVW')$

## Proof

$$\begin{aligned}\left(U^m V^n W^p\right)' &= \left(U^m\right)' V^n W^p + U^m \left(V^n\right)' W^p + U^m V^n \left(W^p\right)' \\ &= mU^{m-1} U' V^n W^p + nU^m V^{n-1} V' W^p + pU^m V^n W^{p-1} W' \quad \text{factor } U^{m-1} V^{n-1} W^{p-1} \\ &= U^{m-1} V^{n-1} W^{p-1} (mU'VW + nUV'W + pUVW')\end{aligned}$$

*Derivative:* Rational Function to Power '***n***' in the form  $\frac{ax^n + b}{cx^n + d}$

$$\left( \frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2} = \frac{n \begin{vmatrix} a & b \\ c & d \end{vmatrix} x^{n-1}}{(cx^n + d)^2}$$

***Proof***

$$u = ax^n + b \quad v = cx^n + d$$

$$u' = nax^{n-1} \quad v' = ncx^{n-1}$$

$$\begin{aligned} \left( \frac{ax^n + b}{cx^n + d} \right)' &= \frac{nax^{n-1}(cx^n + d) - ncx^{n-1}(ax^n + b)}{(cx^n + d)^2} & \left( \frac{u}{v} \right)' &= \frac{u'v - v'u}{v^2} \\ &= \frac{nacx^{2n-1} + nadx^{n-1} - nacx^{2n-1} - nbcx^{n-1}}{(cx^n + d)^2} \\ &= \frac{nadx^{n-1} - nbcx^{n-1}}{(cx^n + d)^2} \\ &= \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2} \end{aligned}$$

### ***Example***

Find  $\left(\frac{x+2}{3x-2}\right)'$

#### **Solution**

$$\begin{aligned}\left(\frac{x+2}{3x-2}\right)' &= \frac{-2-6}{(3x-2)^2} \\ &= \frac{-8}{(3x-2)^2}\end{aligned}$$

$$\begin{aligned}\left(\frac{x+2}{3x-2}\right)' &= \frac{3x-2-3(x+2)}{(3x-2)^2} \\ &= \frac{3x-2-3x-6}{(3x-2)^2} \\ &= \frac{-8}{(3x-2)^2}\end{aligned}$$

### ***Example***

Find  $\left(\frac{5x^2-3}{2x^2-4}\right)'$

#### **Solution**

$$\begin{aligned}\left(\frac{5x^2-3}{2x^2-4}\right)' &= \frac{2(-20+6)x}{(2x^2-4)^2} \\ &= \frac{-28x}{(2x^2-4)^2}\end{aligned}$$

$$\begin{aligned}\left(\frac{5x^2-3}{2x^2-4}\right)' &= \frac{10x(2x^2-4)-4x(5x^2-3)}{(3x-2)^2} \\ &= \frac{20x^3-40x-20x^3+12x}{(3x-2)^2} \\ &= \frac{-28x}{(2x^2-4)^2}\end{aligned}$$



Derivative: Rational Function to Power ' $n$ ' in the form  $\left(\frac{ax^n+b}{cx^n+d}\right)^m$

$$\frac{d}{dx}\left(\frac{ax^n+b}{cx^n+d}\right)^m = mn(ad-bc)x^{n-1} \frac{(ax^n+b)^{m-1}}{(cx^n+d)^{m+1}}$$

**Proof**

$$u = ax^n + b \quad v = cx^n + d$$

$$u' = nax^{n-1} \quad v' = ncx^{n-1}$$

$$\frac{d}{dx}\left(\frac{ax^n+b}{cx^n+d}\right)^m = m \frac{nax^{n-1}(cx^n+d) - ncx^{n-1}(ax^n+b)}{(cx^n+d)^2} \left(\frac{ax^n+b}{cx^n+d}\right)^{m-1} \quad \left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$= \frac{m(nacx^{2n-1} + nadx^{n-1} - nacx^{2n-1} - nbcx^{n-1})(ax^n+b)^{m-1}}{(cx^n+d)^2 (cx^n+d)^{m-1}}$$

$$= \frac{m(nadx^{n-1} - nbcx^{n-1})(ax^n+b)^{m-1}}{(cx^n+d)^{m+1}}$$

$$= \frac{mn(ad-bc)x^{n-1}(ax^n+b)^{m-1}}{(cx^n+d)^{m+1}}$$

**Example**

Find  $\frac{d}{dx} \left( \frac{x+2}{3x-2} \right)^4$

**Solution**

$$\begin{aligned} \frac{d}{dx} \left( \frac{x+2}{3x-2} \right)^4 &= (1)(4)(-2-6) \frac{(x+2)^3}{(3x-2)^5} \\ &= -\frac{32(x+2)^3}{(3x-2)^5} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx} \left( \frac{x+2}{3x-2} \right)^4 &= 4 \left( \frac{x+2}{3x-2} \right)^3 \frac{d}{dx} \left( \frac{x+2}{3x-2} \right) \\ &= 4 \frac{(x+2)^3}{(3x-2)^3} \frac{3x-2-3(x+2)}{(3x-2)^2} \\ &= 4 \frac{(x+2)^3}{(3x-2)^3} \frac{3x-2-3x-6}{(3x-2)^2} \\ &= 4(-8) \frac{(x+2)^3}{(3x-2)^5} \\ &= -\frac{32(x+2)^3}{(3x-2)^5} \end{aligned}$$

**Example**

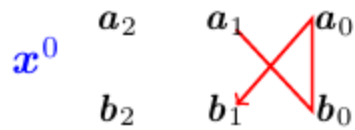
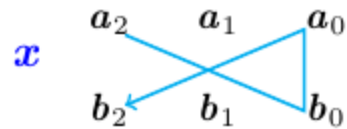
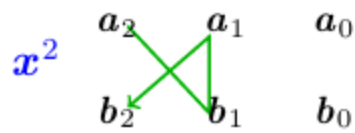
Find  $\frac{d}{dx} \left( \frac{5x^2-3}{2x^2-4} \right)^5$

**Solution**

$$\frac{d}{dx} \left( \frac{5x^2-3}{2x^2-4} \right)^5 = \frac{-140x(5x^2-3)^4}{(2x^2-4)^6}$$

*Derivative:* in the form  $y = \frac{ax^2 + bx + c}{dx^2 + ex + f}$

$$\begin{aligned} \frac{d}{dx} \left( \frac{ax^2 + bx + c}{dx^2 + ex + f} \right) &= \frac{(2ax + b)(dx^2 + ex + f) - (2dx + e)(ax^2 + bx + c)}{(dx^2 + ex + f)^2} \\ &= \frac{2adx^3 + 2aex^2 + 2afx + bdx^2 + bex + bf - 2adx^3 - 2bdx^2 - 2cdx - aex^2 - bex - ce}{(dx^2 + ex + f)^2} \\ &= \frac{(ae - bd)x^2 + 2(af - cd)x + bf - ce}{(dx^2 + ex + f)^2} \\ &= \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2} \end{aligned}$$



### Example

$$f(x) = \frac{x^2 - 6x + 8}{x^2 - 2x + 1}$$

### Solution

$$f'(x) = \frac{\begin{vmatrix} 1 & -6 \\ 1 & -2 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 8 \\ 1 & 1 \end{vmatrix} x + \begin{vmatrix} -6 & 8 \\ -2 & 1 \end{vmatrix}}{(x^2 - 2x + 1)^2}$$

$$= \frac{4x^2 - 14x + 10}{(x^2 - 2x + 1)^2}$$

$$f'(x) = \frac{(2x - 6)(x^2 - 2x + 1) - (2x - 2)(x^2 - 6x + 8)}{(x^2 - 2x + 1)^2}$$

$$= \frac{2x^3 - 4x^2 + 2x - 6x^2 + 12x - 6 - (2x^3 - 12x^2 + 16x - 2x^2 + 12x - 16)}{(x^2 - 2x + 1)^2}$$

$$= \frac{2x^3 - 4x^2 + 2x - 6x^2 + 12x - 6 - 2x^3 + 12x^2 - 16x + 2x^2 - 12x + 16}{(x^2 - 2x + 1)^2}$$

$$= \frac{4x^2 - 14x + 10}{(x^2 - 2x + 1)^2}$$

### Example

$$f(x) = \frac{x + 4}{x^2 + x + 1}$$

$$\begin{matrix} 0 & 1 & 4 \\ 1 & 1 & 1 \end{matrix}$$

### Solution

$$f'(x) = \frac{\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} x^2 + 2 \begin{vmatrix} 0 & 4 \\ 1 & 1 \end{vmatrix} x + \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix}}{(x^2 + x + 1)^2}$$

$$= \frac{-x^2 - 8x - 3}{(x^2 + x + 1)^2}$$

$$f'(x) = \frac{x^2 + x + 1 - (x + 4)(2x + 1)}{(x^2 + x + 1)^2}$$

$$= \frac{x^2 + x + 1 - 2x^2 - 9x - 4}{(x^2 + x + 1)^2}$$

$$= \frac{-x^2 - 8x - 3}{(x^2 + x + 1)^2}$$

*Derivative:* in the form  $f(x) = \frac{a_3 x^3 + a_2 x^2 + a_1 x + a_0}{b_3 x^3 + b_2 x^2 + b_1 x + b_0}$

$$u = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \rightarrow u' = 3a_3 x^2 + 2a_2 x + a_1$$

$$v = b_3 x^3 + b_2 x^2 + b_1 x + b_0 \rightarrow v' = 3b_3 x^2 + 2b_2 x + b_1$$

$$u'v - v'u = (3a_3 x^2 + 2a_2 x + a_1)(b_3 x^3 + b_2 x^2 + b_1 x + b_0) - (3b_3 x^2 + 2b_2 x + b_1)(a_3 x^3 + a_2 x^2 + a_1 x + a_0)$$

$x^5$	$x^4$	$x^3$	$x^2$	$x^1$	$x^0$
$3a_3 b_3$	$3a_3 b_2$	$3a_3 b_1$	$3a_3 b_0$		
$-3a_3 b_3$	$2a_2 b_3$	$2a_2 b_2$	$2a_2 b_1$	$2a_2 b_0$	
	$-3a_2 b_3$	$a_1 b_3$	$a_1 b_2$	$a_1 b_1$	$a_1 b_0$
	$-2a_3 b_2$	$-3a_1 b_3$	$-3a_0 b_3$		
		$-2a_2 b_2$	$-2a_1 b_2$	$-2a_0 b_2$	
		$-a_3 b_1$	$-a_2 b_1$	$-a_1 b_1$	$-a_0 b_1$

$$f'(x) = \frac{(a_3 b_2 - a_2 b_3)x^4 + 2(a_3 b_1 - a_1 b_3)x^3 + ((a_2 b_1 - a_1 b_2) + 3(a_3 b_0 - a_0 b_3))x^2 + 2(a_2 b_0 - a_0 b_2)x + a_1 b_0 - a_0 b_1}{(b_3 x^3 + b_2 x^2 + b_1 x + b_0)^2}$$

$$= \frac{\begin{vmatrix} a_3 & a_2 \\ b_3 & b_2 \end{vmatrix} x^4 + 2 \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} x^3 + \left( \begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix} + 3 \begin{vmatrix} a_3 & a_0 \\ b_3 & b_0 \end{vmatrix} \right) x^2 + 2 \begin{vmatrix} a_2 & a_0 \\ b_2 & b_0 \end{vmatrix} x + \begin{vmatrix} a_1 & a_0 \\ b_1 & b_0 \end{vmatrix}}{(b_3 x^3 + b_2 x^2 + b_1 x + b_0)^2}$$

### Example

$$f(x) = \frac{x^3 + 2x^2 - 6x + 2}{2x^3 + x^2 - 2x + 1}$$

$$\begin{array}{cccc} 1 & 2 & -6 & 2 \\ 2 & 1 & -2 & 1 \end{array}$$

### Solution

$$\begin{aligned} f'(x) &= \frac{(1-4)x^4 + 2(10)x^3 + ((-4+6) + 3(1-4))x^2 + 2(2-2)x + (-6+4)}{(2x^3 + x^2 - 2x + 1)^2} \\ &= \frac{-3x^4 + 20x^3 - 7x^2 - 2}{(2x^3 + x^2 - 2x + 1)^2} \end{aligned}$$

$$\begin{array}{cccc} \mathbf{x^4} & a_3 & a_2 & a_1 & a_0 \\ & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$\begin{array}{cccc} \mathbf{x^3} & a_3 & a_2 & a_1 & a_0 \\ & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$\begin{array}{cccc} \mathbf{x^2} & a_3 & a_2 & a_1 & a_0 \\ & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$\begin{array}{cccc} \mathbf{x} & a_3 & a_2 & a_1 & a_0 \\ & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$\begin{array}{cccc} \mathbf{x^0} & a_3 & a_2 & a_1 & a_0 \\ & b_3 & b_2 & b_1 & b_0 \end{array}$$

Derivative: in the form  $f(x) = \frac{a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0}{b_4 x^4 + b_3 x^3 + b_2 x^2 + b_1 x + b_0}$

$$u'v - v'u = \left(4a_4 x^3 + 3a_3 x^2 + 2a_2 x + a_1\right) \left(b_4 x^4 + b_3 x^3 + b_2 x^2 + b_1 x + b_0\right) - \left(4b_4 x^3 + 3b_3 x^2 + 2b_2 x + b_1\right) \left(a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0\right)$$

$$x^7 \quad 4a_4 b_4 - 4a_4 b_4$$

$$x^6 \quad 4a_4 b_3 + 3a_3 b_4 - 4a_3 b_4 - 3a_4 b_3$$

$$x^5 \quad 4a_4 b_2 + 3a_3 b_3 + 2a_2 b_4 - 4a_2 b_4 - 3a_3 b_3 - 2a_4 b_2$$

$$x^4 \quad 4a_4 b_1 + 3a_3 b_2 + 2a_2 b_3 + a_1 b_4 - 4a_1 b_4 - 3a_2 b_3 - 2a_3 b_2 - a_4 b_1$$

$$x^3 \quad 4a_4 b_0 + 3a_3 b_1 + 2a_2 b_2 + a_1 b_3 - 4a_0 b_4 - 3a_1 b_3 - 2a_2 b_2 - a_3 b_1$$

$$x^2 \quad 3a_3 b_0 + 2a_2 b_1 + a_1 b_2 - 3a_0 b_3 - 2a_1 b_2 - a_2 b_1$$

$$x^1 \quad 2a_2 b_0 + a_1 b_1 - 2a_0 b_2 - a_1 b_1$$

$$x^0 \quad a_1 b_0 - a_0 b_1$$

$$\begin{aligned} & (a_4 b_3 - a_3 b_4) x^6 + 2(a_4 b_2 - a_2 b_4) x^5 + \left(3(a_4 b_1 - a_1 b_4) + (a_3 b_2 - a_2 b_3)\right) x^4 \\ & + \left(4(a_4 b_0 - a_0 b_4) + 2(a_3 b_1 - a_1 b_3)\right) x^3 \end{aligned}$$

$$f'(x) = \frac{\begin{aligned} & + \left((a_2 b_1 - a_1 b_2) + 3(a_3 b_0 - a_0 b_3)\right) x^2 + 2(a_2 b_0 - a_0 b_2) x + a_1 b_0 - a_0 b_1 \end{aligned}}{\left(b_4 x^4 + b_3 x^3 + b_2 x^2 + b_1 x + b_0\right)^2}$$

$$x^6 \quad \begin{array}{ccccc} a_4 & a_3 & a_2 & a_1 & a_0 \\ b_4 & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$x^5 \quad \begin{array}{ccccc} a_4 & a_3 & a_2 & a_1 & a_0 \\ b_4 & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$x^4 \quad \begin{array}{ccccc} a_4 & a_3 & a_2 & a_1 & a_0 \\ b_4 & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$x^3 \quad \begin{array}{ccccc} a_4 & a_3 & a_2 & a_1 & a_0 \\ b_4 & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$x^2 \quad \begin{array}{ccccc} a_4 & a_3 & a_2 & a_1 & a_0 \\ b_4 & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$x \quad \begin{array}{ccccc} a_4 & a_3 & a_2 & a_1 & a_0 \\ b_4 & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$x^0 \quad \begin{array}{ccccc} a_4 & a_3 & a_2 & a_1 & a_0 \\ b_4 & b_3 & b_2 & b_1 & b_0 \end{array}$$

Derivative: in the form  $f(x) = \frac{a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0}{b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0}$

$$u = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \rightarrow u' = 5a_5x^4 + 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$$

$$v = b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0 \rightarrow v' = 5b_5x^4 + 4b_4x^3 + 3b_3x^2 + 2b_2x + b_1$$

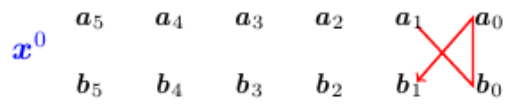
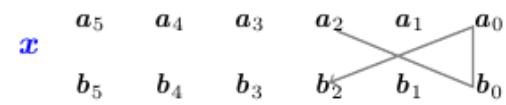
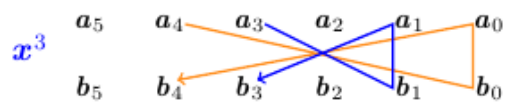
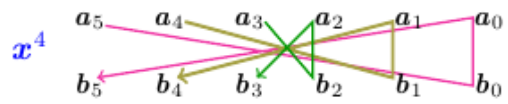
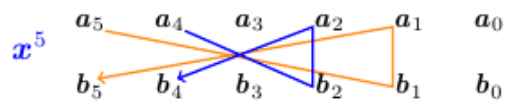
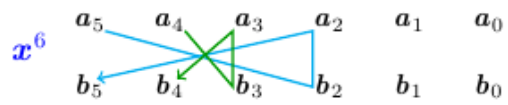
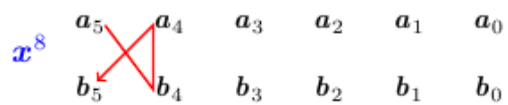
$$u'v - v'u = \left(5a_5x^4 + 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1\right)\left(b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0\right) - \left(5b_5x^4 + 4b_4x^3 + 3b_3x^2 + 2b_2x + b_1\right)\left(a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0\right)$$

$x^9$	$x^8$	$x^7$	$x^6$	$x^5$	$x^4$	$x^3$	$x^2$	$x^1$	$x^0$
$5a_5b_5$	$5a_5b_4$	$5a_5b_3$	$5a_5b_2$	$5a_5b_1$	$5a_5b_0$				
$-5a_5b_5$	$4a_4b_5$	$4a_4b_4$	$4a_4b_3$	$4a_4b_2$	$4a_4b_1$	$4a_4b_0$			
	$-5a_4b_5$	$3a_3b_5$	$3a_3b_4$	$3a_3b_3$	$3a_3b_2$	$3a_3b_1$	$3a_3b_0$		
	$-4a_5b_4$	$-5a_3b_5$	$2a_2b_5$	$2a_2b_4$	$2a_2b_3$	$2a_2b_2$	$2a_2b_1$	$2a_2b_0$	
		$-4a_4b_4$	$-5a_2b_5$	$a_1b_5$	$a_1b_4$	$a_1b_3$	$a_1b_2$	$a_1b_1$	$a_1b_0$
		$-3a_5b_3$	$-4a_3b_4$	$-5a_1b_5$	$-5a_0b_5$	$-4a_0b_4$	$-3a_0b_3$	$-2a_0b_2$	$-a_0b_1$
			$-3a_4b_3$	$-4a_2b_4$	$-4a_1b_4$	$-3a_1b_3$	$-2a_1b_2$	$-a_1b_1$	
			$-2a_5b_2$	$-3a_3b_3$	$-3a_2b_3$	$-2a_2b_2$	$-a_2b_1$		
				$-2a_4b_2$	$-2a_3b_2$	$-a_3b_1$			
				$-a_5b_1$	$-a_4b_1$				

$$\begin{aligned}
& \left(a_5b_4 - a_4b_5\right)x^8 + 2\left(a_5b_3 - a_3b_5\right)x^7 \\
& + \left(3\left(a_5b_2 - a_2b_5\right) + \left(a_4b_3 - a_3b_4\right)\right)x^6 \\
& + \left(4\left(a_5b_1 - a_1b_5\right) + 2\left(a_4b_2 - a_2b_4\right)\right)x^5 \\
& + \left(5\left(a_5b_0 - a_0b_5\right) + 3\left(a_4b_1 - a_1b_4\right) + \left(a_3b_2 - a_2b_3\right)\right)x^4 \\
& + \left(4\left(a_4b_0 - a_0b_4\right) + 2\left(a_3b_1 - a_1b_3\right)\right)x^3 \\
& + \left(3\left(a_3b_0 - a_0b_3\right) + \left(a_2b_1 - a_1b_2\right)\right)x^2 \\
& + 2\left(a_2b_0 - a_0b_2\right)x + \left(a_1b_0 - a_0b_1\right)
\end{aligned}$$

$$f'(x) = \frac{\text{above expression}}{\left(b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0\right)^2}$$





$\sqrt{3} \approx 1.732050807568877293$

<div><div><div>1</div><div>2</div><div>3</div></div><div><div>1</div><div>2</div><div>3</div><div>4</div></div></div> <div><div>1.732050807568877293</div><div><div>1</div><div>200</div><div>189</div><div>1100</div><div>1029</div><div>7100</div><div>6924</div><div>17600</div><div>1560000</div><div>1732025</div><div>2797500</div><div>279750000</div><div>277128064</div><div>262193600</div><div>26219360000</div><div>2424871169</div><div>197064875100</div><div>173205080725</div><div>2385979437500</div><div>2078460969036</div><div>30751846846400</div><div>27712812921024</div><div>303903392537600</div><div>277128129210944</div><div>2677526332665600</div><div>2424871130596369</div><div>25265520206923100</div><div>24248711305964229</div><div>101680890095887100</div><div>69282032302755084</div><div>3239885779313201600</div><div>3117691453623979041</div><div>12219432568922255900</div><div>1039230845413263263779</div><div>1827127723508992121</div></div></div> <tr><td><div><div>1</div><div>2</div><div>3</div><div>4</div></div><div><div>1 × 1 = 1</div><div>27 × 7 = 189</div><div>343 × 3 = 1029</div><div>3462 × 2 = 6924</div><div>3464 × = 34,64 &gt; 15,600</div><div>346405 × 5 = 1,732,025</div><div>346410 × = ?</div><div>34641008 × 8 = 277128064</div><div>34641016 × = ?</div><div>34641016 × 5 = 173205080725</div><div>34641016506 × 6 = 2078460969036</div><div>3464101615128 × 8 = 2771281291024</div><div>34641016151368 × 8 = 27712812910944</div><div>346410161513767 × 7 = 2424871130596369</div><div>3464101615137747 × 7 = 24248711305964229</div><div>34641016151377542 × 2 = 69282032302755084</div><div>346410161513775449 × 9 = 3117691453623979041</div><div>3464101615137754593 × 3 = 1039230845413263263779</div></div></td></tr>	<div><div>1</div><div>2</div><div>3</div><div>4</div></div> <div><div>1 × 1 = 1</div><div>27 × 7 = 189</div><div>343 × 3 = 1029</div><div>3462 × 2 = 6924</div><div>3464 × = 34,64 &gt; 15,600</div><div>346405 × 5 = 1,732,025</div><div>346410 × = ?</div><div>34641008 × 8 = 277128064</div><div>34641016 × = ?</div><div>34641016 × 5 = 173205080725</div><div>34641016506 × 6 = 2078460969036</div><div>3464101615128 × 8 = 2771281291024</div><div>34641016151368 × 8 = 27712812910944</div><div>346410161513767 × 7 = 2424871130596369</div><div>3464101615137747 × 7 = 24248711305964229</div><div>34641016151377542 × 2 = 69282032302755084</div><div>346410161513775449 × 9 = 3117691453623979041</div><div>3464101615137754593 × 3 = 1039230845413263263779</div></div>
<div><div>1</div><div>2</div><div>3</div><div>4</div></div> <div><div>1 × 1 = 1</div><div>27 × 7 = 189</div><div>343 × 3 = 1029</div><div>3462 × 2 = 6924</div><div>3464 × = 34,64 &gt; 15,600</div><div>346405 × 5 = 1,732,025</div><div>346410 × = ?</div><div>34641008 × 8 = 277128064</div><div>34641016 × = ?</div><div>34641016 × 5 = 173205080725</div><div>34641016506 × 6 = 2078460969036</div><div>3464101615128 × 8 = 2771281291024</div><div>34641016151368 × 8 = 27712812910944</div><div>346410161513767 × 7 = 2424871130596369</div><div>3464101615137747 × 7 = 24248711305964229</div><div>34641016151377542 × 2 = 69282032302755084</div><div>346410161513775449 × 9 = 3117691453623979041</div><div>3464101615137754593 × 3 = 1039230845413263263779</div></div>	