Solution — Section 2.5 – Numerical Integration

Exercise

Find the Midpoint Rule approximations to: $\int_0^1 \sin \pi x \, dx$ *using* $n = 6$ *subintervals*

Solution

$$\Delta x = \frac{1-0}{6} = \frac{1}{6}$$

$$x_0 = 0, \ x_1 = 0 + \frac{1}{6} = \frac{1}{6}, \ x_2 = \frac{1}{3}, \ x_3 = \frac{1}{2}, \ x_4 = \frac{2}{3}, \ x_5 = \frac{5}{6}, \ x_6 = 1$$

$$m_1 = \frac{1}{2}\left(0 + \frac{1}{6}\right) = \frac{1}{12}, \ m_2 = \frac{1}{4}, \ m_3 = \frac{5}{12}, \ m_4 = \frac{7}{12}, \ m_5 = \frac{9}{12}, \ m_6 = \frac{11}{12}$$

$$M(6) = \left(\sin\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{5\pi}{12}\right) + \sin\left(\frac{7\pi}{12}\right) + \sin\left(\frac{9\pi}{12}\right) + \sin\left(\frac{11\pi}{12}\right)\right)\left(\frac{1}{6}\right)$$

$$\approx 0.6439505509$$

Exercise

Find the Midpoint Rule approximations to: $\int_0^1 e^{-x} dx$ *using* $n = 8$ *subintervals*

Solution

$$\Delta x = \frac{1-0}{8} = \frac{1}{8}$$

$$x_0 = 0, \ x_1 = \frac{1}{8}, \ x_2 = \frac{1}{4}, \ x_3 = \frac{3}{8}, \ x_4 = \frac{1}{2}, \ x_5 = \frac{5}{8}, \ x_6 = \frac{3}{4}, \ x_7 = \frac{7}{8}, \ x_8 = 1$$

$$m_1 = \frac{1}{2}\left(0 + \frac{1}{8}\right) = \frac{1}{16}, \ m_2 = \frac{3}{16}, \ m_3 = \frac{5}{16}, \ m_4 = \frac{7}{16}, \ m_5 = \frac{9}{16}, \ m_6 = \frac{11}{16}, \ m_7 = \frac{13}{16}, \ m_8 = \frac{15}{16}$$

$$M(8) = \frac{1}{8}\left(e^{-1/16} + e^{-3/16} + e^{-5/16} + e^{-7/16} + e^{-9/16} + e^{-11/16} + e^{-13/16} + e^{-15/16}\right)$$

$$\approx 0.6317092095$$

Exercise

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of 10^{-4} by (***a***) the Trapezoid Rule and (***b***) Simpson's Rule. $\int_1^3 (2x-1)\, dx$

Solution

a) ***i***) $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$

$$T = \frac{1}{2}\Delta x\left(m\, f\left(x_i\right)\right)$$

$$= \frac{1}{2}\frac{1}{2}(24) \quad = \underline{6|}$$

$$f(x) = 2x - 1 \quad \Rightarrow \quad f'(x) = 2$$

$$\Rightarrow \quad f''(x) = 0 = M$$

$$\Rightarrow Error = 0$$

	x_i	$f\left(x_i\right) = 2x_i - 1$	m	$mf\left(x_i\right)$
x_0	1	1	1	1
x_1	$1 + \frac{1}{2} = \frac{3}{2}$	2	2	4
x_2	2	3	2	6
x_3	$\frac{5}{2}$	4	2	8
x_4	3	5	1	5
				24

ii) $$\int_1^3 (2x - 1)\, dx = \left[x^2 - x\right]_1^3$$

$$= \left(3^2 - 3\right) - \left(1^2 - 1\right)$$

$$\underline{= 6|}$$

iii) $$Error = \frac{|E_T|}{True\ \ Value} \times 100 = 0\%$$

b) ***i***) $$\underline{|\Delta x} = \frac{b-a}{n} = \frac{3-1}{4} = \underline{\frac{1}{2}|}$$

$$S = \frac{1}{3}\Delta x\left(\sum m\, f\left(x_i\right)\right)$$

$$= \frac{1}{3}\frac{1}{2}(36) \quad = \underline{6|}$$

$$f(x) = 2x - 1 \quad \Rightarrow \quad f^{(4)}(x) = 0 = M$$

$$\Rightarrow \left|E_s\right| = 0$$

	x_i	$f\left(x_i\right) = 2x_i - 1$	m	$mf\left(x_i\right)$
x_0	1	1	1	1
x_1	$\frac{3}{2}$	2	4	8
x_2	2	3	2	6
x_3	$\frac{5}{2}$	4	4	16
x_4	3	5	1	5
				36

ii) $$\int_1^3 (2x - 1)\, dx = 6$$

$$\left|E_s\right| = \int_1^3 (2x - 1)\, dx - S = 6 - 6 = 0$$

iii) $$Error = \frac{|E_T|}{True\ \ Value} \times 100 = 0\%$$

Exercise

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of 10^{-4} by (***a***) the Trapezoid Rule and (***b***) Simpson's Rule. $\int_{-1}^{1}\left(x^2+1\right)dx$

Solution

a) ***i***) $\Delta x = \dfrac{b-a}{n} = \dfrac{1+1}{4} = \dfrac{1}{2}$

	x_i	$f\left(x_i\right)$	m	$m\,f\left(x_i\right)$
x_0	-1	2	1	2
x_1	$-\frac{1}{2}$	$\frac{5}{4}$	2	$\frac{5}{2}$
x_2	0	1	2	2
x_3	$\frac{1}{2}$	$\frac{5}{4}$	2	$\frac{5}{2}$
x_4	1	2	1	2
				11

$$T = \frac{1}{2}\Delta x\left(m\,f\left(x_i\right)\right) = \frac{1}{2}\frac{1}{2}(11) = \frac{11}{4}$$

$$f(x) = x^2+1 \quad\Rightarrow\quad f'(x) = 2x$$

$$\Rightarrow\quad f''(x) = 2 = M$$

$$\left|E_T\right| = \frac{1-(-1)}{12}\left(\frac{1}{2}\right)^2(2) = 0.0833\ldots$$

ii) $$\int_{-1}^{1}\left(x^2+1\right)dx = \left[\frac{1}{3}x^3+x\right]_{-1}^{1} = \left(\frac{1}{3}+1\right)-\left(-\frac{1}{3}-1\right) = \frac{8}{3}$$

$$E_T = \int_{-1}^{1}\left(x^2+1\right)dx - T = \frac{8}{3}-\frac{11}{4} = -\frac{1}{12}$$

iii) $$Error = \frac{\left|E_T\right|}{True\ \ Value}\times 100 = \frac{\frac{1}{12}}{\frac{8}{3}} \approx 3\%$$

b) ***i***) $\Delta x = \dfrac{b-a}{n} = \dfrac{-1-(-1)}{4} = \dfrac{1}{2}$

	x_i	$f\left(x_i\right)$	m	$m\,f\left(x_i\right)$
x_0	-1	2	1	2
x_1	$-\frac{1}{2}$	$\frac{5}{4}$	4	5
x_2	0	1	2	2
x_3	$\frac{1}{2}$	$\frac{5}{4}$	4	5
x_4	1	2	1	2
				16

$$S = \frac{1}{3}\Delta x\left(\sum m\,f\left(x_i\right)\right) = \frac{1}{3}\frac{1}{2}(16) = \frac{8}{3}$$

$$f(x) = x^2+1 \quad\Rightarrow\quad f^{(4)}(x) = 0 = M$$

$$\Rightarrow \left|E_s\right| = 0$$

ii) $$\int_{-1}^{1}\left(x^2+1\right)dx = \frac{8}{3}$$

$$E_S = \int_{-1}^{1}\left(x^2+1\right)dx - S = \frac{8}{3}-\frac{8}{3} = 0$$

iii) $$Error = \frac{\left|E_T\right|}{True\ \ Value}\times 100 = 0\%$$

Exercise

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of 10^{-4} by (***a***) the Trapezoid Rule and (***b***) Simpson's Rule. $\int_2^4 \frac{1}{(s-1)^2}\,ds$

Solution

a) $\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2}$

$x_0 = 2 \qquad x_1 = 2 + \frac{1}{2} = \frac{5}{2} \qquad x_2 = 2 + 2\left(\frac{1}{2}\right) = 3 \qquad x_3 = 2 + 3\left(\frac{1}{2}\right) = \frac{7}{2} \qquad x_4 = 4$

$$T = \frac{1}{2}\Delta x\left(m\, f\left(x_i\right)\right)$$

$$= \frac{1}{2}\frac{1}{2}\left(\frac{1}{(2-1)^2} + 2\frac{1}{\left(\frac{5}{2}-1\right)^2} + 2\frac{1}{(3-1)^2} + 2\frac{1}{\left(\frac{7}{2}-1\right)^2} + \frac{1}{(4-1)^2}\right)$$

$$= \frac{1}{4}\left(\frac{1}{1} + \frac{8}{9} + \frac{1}{2} + \frac{8}{25} + \frac{1}{9}\right)$$

$$= \frac{1269}{1800}$$

$$\approx 0.705$$

$$f(s) = (s-1)^{-2} \quad \Rightarrow f'(s) = -2(s-1)^{-3}$$

$$\Rightarrow f''(s) = 6(s-1)^{-4} = \frac{6}{(s-1)^4} \Rightarrow M = 6$$

$$\int_2^4 \frac{1}{(s-1)^2}\,ds = \int_2^4 (s-1)^{-2}\,d(s-1)$$

$$= -\left[(s-1)^{-1}\right]_2^4$$

$$= -\left(3^{-1} - 1^{-1}\right)$$

$$= \frac{2}{3}$$

The percentage error: $\frac{|0.705 - .6667|}{.6667} \approx 0.0575 \quad 5.75\%$

b) $\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2}$

$x_0 = 2 \qquad x_1 = 2 + \frac{1}{2} = \frac{5}{2} \qquad x_2 = 2 + 2\left(\frac{1}{2}\right) = 3 \qquad x_3 = 2 + 3\left(\frac{1}{2}\right) = \frac{7}{2} \qquad x_4 = 4$

$$S = \frac{1}{3}\Delta x\left(m\, f\left(x_i\right)\right)$$

$$=\frac{1}{3}\frac{1}{2}\left(\frac{1}{(2-1)^2}+4\frac{1}{\left(\frac{5}{2}-1\right)^2}+2\frac{1}{(3-1)^2}+4\frac{1}{\left(\frac{7}{2}-1\right)^2}+\frac{1}{(4-1)^2}\right)$$

$$=\frac{1}{6}\left(\frac{1}{1}+\frac{16}{9}+\frac{1}{2}+\frac{16}{25}+\frac{1}{9}\right)$$

$$=\frac{1813}{450}$$

$$\approx 0.67148$$

$$\int_2^4 \frac{1}{(s-1)^2}\,ds=\frac{2}{3}$$

The percentage error: $\frac{|0.67148-.6667|}{.6667}\approx 0.0072$ $\quad 0.72\%$

Exercise

Find the Trapezoid & Simpson's Rule approximations and error: $\int_0^1 \sin \pi x\, dx \quad n=6 \ \ subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m\cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1666666667	0.5000000000	1.0000000000
2	0.3333333333	0.8660254000	1.7320508000
3	0.5000000000	1.0000000000	2.0000000000
4	0.6666666667	0.8660254000	1.7320508000
5	0.8333333333	0.5000000000	1.0000000000
6	1.0000000000	0.0000000000	0.0000000000

Trapezoid Rule approximation ≈ 0.62200847

Simpson's Rule Method

n	x_n	$f(x_n)$	$m\cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1666666667	0.5000000000	2.0000000000
2	0.3333333333	0.8660254000	1.7320508000
3	0.5000000000	1.0000000000	2.0000000000
4	0.6666666667	0.8660254000	1.7320508000
5	0.8333333333	0.5000000000	1.0000000000
6	1.0000000000	0.0000000000	0.0000000000

Simpson's Rule approximation ≈ 0.63689453

	Exact	Trapezoid	Simpson
Value:	0.63661977	0.62200847	0.63689453
Error:		2.2951 %	0.0432 %

Exercise

Find the Trapezoid & Simpson's Rule approximations to and error to $\int_0^1 e^{-x}dx \quad n=8 \ \ subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.1250000000	0.8824969000	1.7649938000
2	0.2500000000	0.7788007800	1.5576015600
3	0.3750000000	0.6872892800	1.3745785600
4	0.5000000000	0.6065306600	1.2130613200
5	0.6250000000	0.5352614300	1.0705228600
6	0.7500000000	0.4723665500	0.9447331000
7	0.8750000000	0.4168620200	0.8337240400
8	1.0000000000	0.3678794400	0.3678794400

Trapezoid Rule approximation ≈ 0.63294342

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.1250000000	0.8824969000	3.5299876000
2	0.2500000000	0.7788007800	1.5576015600
3	0.3750000000	0.6872892800	2.7491571200
4	0.5000000000	0.6065306600	1.2130613200
5	0.6250000000	0.5352614300	2.1410457200
6	0.7500000000	0.4723665500	0.9447331000
7	0.8750000000	0.4168620200	1.6674480800
8	1.0000000000	0.3678794400	0.3678794400

Simpson's Rule approximation ≈ 0.63212141

	Exact	Trapezoid	Simpson
Value:	0.63212056	0.63294342	0.63212141
Error:		0.1302 %	0.0001 %

Exercise

Find the Trapezoid & Simpson's Rule approximations and error to:

$$\int_1^5 \left(3x^2 - 2x\right) dx \quad n = 8 \ \ subintervals$$

Solution

Trapezoid Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.5000000000	3.7500000000	7.5000000000
2	2.0000000000	8.0000000000	16.0000000000
3	2.5000000000	13.7500000000	27.5000000000
4	3.0000000000	21.0000000000	42.0000000000
5	3.5000000000	29.7500000000	59.5000000000
6	4.0000000000	40.0000000000	80.0000000000
7	4.5000000000	51.7500000000	103.500000000
8	5.0000000000	65.0000000000	65.0000000000

Trapezoid Rule approximation ≈ 100.50000000

Simpson's Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.5000000000	3.7500000000	15.0000000000
2	2.0000000000	8.0000000000	16.0000000000
3	2.5000000000	13.7500000000	55.0000000000
4	3.0000000000	21.0000000000	42.0000000000
5	3.5000000000	29.7500000000	119.000000000
6	4.0000000000	40.0000000000	80.0000000000
7	4.5000000000	51.7500000000	207.000000000
8	5.0000000000	65.0000000000	65.0000000000

Simpson's Rule approximation ≈ 100.00000000

	Exact	***Trapezoid***	***Simpson***
Value:	100.000000	100.500000	100.00000000
Error:		0.5000%	0.0000 %

Exercise

Find the Trapezoid & Simpson's Rule approximations and error: $\int_0^{\pi/4} 3\sin 2x\,dx \quad n=8 \;\; \textit{subintervals}$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.0981747704	0.5852709700	1.1705419400
2	0.1963495408	1.1480503000	2.2961006000
3	0.2945243113	1.6667107000	3.3334214000
4	0.3926990817	2.1213203400	4.2426406800
5	0.4908738521	2.4944088400	4.9888176800
6	0.5890486225	2.7716386000	5.5432772000
7	0.6872233930	2.9423558400	5.8847116800
8	0.7853981634	3.0000000000	3.0000000000

Trapezoid Rule approximation ≈ 1.49517776

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.0981747704	0.5852709700	2.3410838800
2	0.1963495408	1.1480503000	2.2961006000
3	0.2945243113	1.6667107000	6.6668428000
4	0.3926990817	2.1213203400	4.2426406800
5	0.4908738521	2.4944088400	9.9776353600
6	0.5890486225	2.7716386000	5.5432772000
7	0.6872233930	2.9423558400	11.7694233600
8	0.7853981634	3.0000000000	3.0000000000

Simpson's Rule approximation ≈ 1.50001244

	Exact	***Trapezoid***	***Simpson***
Value:	1.500000	1.49517776	1.50001244
Error:		0.3215 %	0.0008 %

Exercise

Find the Trapezoid & Simpson's Rule approximations and error: $\int_0^8 e^{-2x}\,dx \quad n=8\ \ subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.0000000000	0.1353352800	0.2706705600
2	2.0000000000	0.0183156400	0.0366312800
3	3.0000000000	0.0024787500	0.0049575000
4	4.0000000000	0.0003354600	0.0006709200
5	5.0000000000	0.0000454000	0.0000908000
6	6.0000000000	0.0000061400	0.0000122800
7	7.0000000000	0.0000008300	0.0000016600
8	8.0000000000	0.0000001100	0.0000001100

Trapezoid Rule approximation ≈ 0.65651755

Simpson's Rule Method

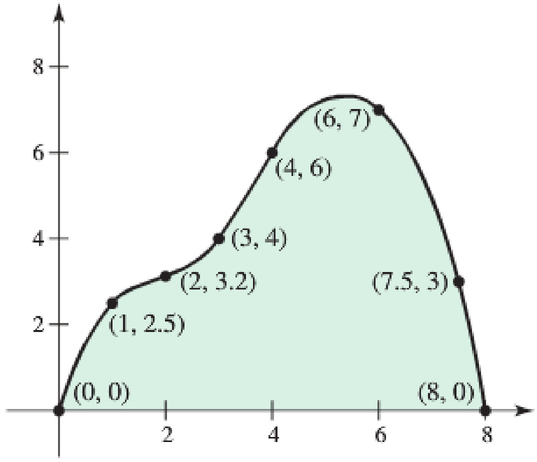
n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.0000000000	0.1353352800	0.5413411200
2	2.0000000000	0.0183156400	0.0366312800
3	3.0000000000	0.0024787500	0.0099150000
4	4.0000000000	0.0003354600	0.0006709200
5	5.0000000000	0.0000454000	0.0001816000
6	6.0000000000	0.0000061400	0.0000122800
7	7.0000000000	0.0000008300	0.0000033200
8	8.0000000000	0.0000001100	0.0000001100

Simpson's Rule approximation ≈ 0.52958521

	Exact	***Trapezoid***	***Simpson***
Value:	0.49999994	0.65651755	0.52958521
Error:		31.3035 %	5.9171 %

Exercise

A piece of wood paneling must be cut in the shape shown below. The coordinates of several point on its curved surface are also shown (with units of inches).



a) Estimate the surface area of the paneling using the Trapezoid Rule

b) Estimate the surface area of the paneling using a left Riemann sum.

c) Could two identical pieces be cut from a 9-in by 9-in piece of wood?

Solution

a) The *trapezoid* Rule gives

$$\frac{(0+.25)\cdot 1}{2}+\frac{(2.5+3.2)\cdot 1}{2}+\frac{(3.2+4)\cdot 1}{2}+\frac{(4+6)\cdot 1}{2}+\frac{(6+7)\cdot 2}{2}+\frac{(7+5.3)\cdot 1.5}{2}+\frac{(3+0)\cdot 0.5}{2}\ \underline{=35.675}$$

b) The left *Riemann* sum gives

$$0\cdot 1+2.5\cdot 1+3.2\cdot 1+4\cdot 1+6\cdot 2+7\cdot 1.5+5.3\cdot 0.5\ \underline{=34.85}$$

c) Although the surface area of the piece appears to be less than half of $81=9^2$ (area of 9×9 piece of wood), the shape prohibits the creation of two identical pieces.