SOLUTION Section 3.2 – Infinite Series

Exercise

Find the limit of the following sequences or determine the limit does not exist $a_n = \dfrac{n^3}{n^4+1}$

Solution

$$\lim_{n\to\infty} \frac{n^3}{n^4+1} = \lim_{n\to\infty} \frac{n^3}{n^4}$$

$$= \lim_{n\to\infty} \frac{1}{n}$$

$$\underline{= 0}$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $a_n = n^{1/n}$

Solution

$$\lim_{n\to\infty} n^{1/n} = \infty^0$$

$$y = n^{1/n} \quad \Leftrightarrow \quad \ln y = \frac{1}{n}\ln n$$

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} \qquad \textit{Using L'Hôpital's rule}$$

$$\underline{= 0}$$

$$\ln y = 0 \quad \Rightarrow \quad y = e^0 = 1$$

$$\lim_{n\to\infty} n^{1/n} \ \underline{= 1}$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\left\{ \dfrac{n^{12}}{3n^{12}+4} \right\}$

Solution

$$\lim_{n\to\infty} \frac{n^{12}}{3n^{12}+4} = \lim_{n\to\infty} \frac{n^{12}}{3n^{12}} \ \underline{= \frac{1}{3}}$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\left\{\frac{2e^{n+1}}{e^n}\right\}$

Solution

$$\lim_{n\to\infty}\frac{2e^{n+1}}{e^n}=\lim_{x\to\infty}2e\ \underline{=2e}$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\left\{\frac{\tan^{-1}n}{n}\right\}$

Solution

$$\lim_{n\to\infty}\frac{\tan^{-1}n}{n}=\frac{\frac{\pi}{2}}{\infty}\ \underline{=0}$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\left\{\left(1+\frac{2}{n}\right)^n\right\}$

Solution

$$y=\left(1+\frac{2}{n}\right)^n \quad \Leftrightarrow \quad \ln y=n\ln\left(1+\frac{2}{n}\right)=\frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}}$$

$$\lim_{n\to\infty}\frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\frac{-\frac{2}{n^2}}{1+\frac{2}{n}}}{-\frac{1}{n^2}}$$

Using L'Hôpital's rule

$$=\lim_{n\to\infty}\frac{2}{1+\frac{2}{n}}$$

$$\underline{=2}$$

$$\ln y=2 \quad\to\quad y=e^2$$

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n\ \underline{=e^2}$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\left\{\left(\frac{n}{n+5}\right)^n\right\}$

Solution

$$y=\left(\frac{n}{n+5}\right)^n \quad \Leftrightarrow \quad \ln y = n\ln\left(\frac{n}{n+5}\right)$$

$$\lim_{n\to\infty} n\ln\left(\frac{n}{n+5}\right)=\lim_{n\to\infty}\frac{\ln\left(\frac{n}{n+5}\right)}{\frac{1}{n}}$$ ***Using L'Hôpital's rule***

$$=\lim_{n\to\infty}\frac{\frac{n+5}{n}\frac{n+5-n}{(n+5)^2}}{-\frac{1}{n^2}}$$

$$=\lim_{n\to\infty}\frac{\frac{5}{n^2+5}}{-\frac{1}{n^2}}$$

$$=\lim_{n\to\infty}\frac{-5n^2}{n^2+5}$$

$$\underline{=-5}$$

$$\ln y=-5 \quad \to \quad y=e^{-5}$$

$$\lim_{n\to\infty}\left(\frac{n}{n+5}\right)^n \ \underline{=e^{-5}}$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\left\{\frac{\ln\left(\frac{1}{n}\right)}{n}\right\}$

Solution

$$\lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{1}{n}\right)=\lim_{n\to\infty}\frac{-\ln(n)}{n}$$ ***Using L'Hôpital's rule***

$$=\lim_{n\to\infty}\frac{-\frac{1}{n}}{1}$$

$$\underline{=0}$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $\{\ln \sin(1/n) + \ln n\}$

Solution

$$\lim_{n\to\infty} \left(\ln \sin(1/n) + \ln n\right) = \lim_{n\to\infty} \left(\ln \frac{\sin(1/n)}{n}\right)$$

$$= \ln 1$$

$$\underline{= 0}$$

$$\lim_{x\to\infty} \frac{\sin(1/x)}{x} = 1$$

Exercise

Find the limit of the following sequences or determine the limit does not exist $a_n = \frac{n!}{n^n}$

Solution

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!}$$

$$= \frac{n^n (n+1)}{(n+1)(n+1)^n}$$

$$= \frac{n^n}{(n+1)^n}$$

$$= \left(\frac{n}{n+1}\right)^n$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \quad \underline{= \frac{1}{e}}$$

Exercise

Find a formula for the *n*th term partial sum of the series and use it to find the series' sum if the series converges $\quad 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots$

Solution

$$a = 2, \quad r = \frac{1}{3}$$

$$s_n = a\frac{1-r^n}{1-r} = 2\frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 2\frac{1}{1-\frac{1}{3}} \quad \underline{= 3}$$

Exercise

Find a formula for the nth term partial sum of the series and use it to find the series' sum if the series converges $\quad \frac{9}{100}+\frac{9}{100^2}+\frac{9}{100^3}+\cdots+\frac{9}{100^n}+\cdots$

Solution

$$a=\frac{9}{100}, \quad r=\frac{1}{100}$$

$$s_n=a\frac{1-r^n}{1-r}=\frac{9}{100}\frac{1-\left(\frac{1}{100}\right)^n}{1-\frac{1}{100}}$$

$$\lim_{n\to\infty} s_n=\lim_{n\to\infty}\frac{9}{100}\frac{1-\left(\frac{1}{100}\right)^n}{1-\frac{1}{100}} \qquad \lim_{n\to\infty}\left(\frac{1}{100}\right)^n=0$$

$$=\frac{9}{100}\frac{1}{1-\frac{1}{100}}$$

$$=\frac{9}{100}\frac{1}{\frac{99}{100}}$$

$$=\frac{9}{99}$$

$$\boxed{=\frac{1}{11}}$$

Exercise

Find a formula for the nth term partial sum of the series and use it to find the series' sum if the series converges $\quad 1-2+4-8+\cdots+(-1)^{n-1}2^{n-1}+\cdots$

Solution

$r=-2 \quad \to \quad |r|>1 \qquad$ The series ***diverges***

Exercise

Find a formula for the nth term partial sum of the series and use it to find the series' sum if the series converges $\quad \frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(n+1)(n+2)}+\cdots$

Solution

$$\frac{1}{(n+1)(n+2)}=\frac{1}{n+1}-\frac{1}{n+2}$$

$$s_n=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$

$$=\frac{1}{2}-\frac{1}{n+2} \qquad \lim_{n\to\infty}\frac{1}{n+2}=0$$

$$\lim_{n\to\infty} s_n \boxed{=\frac{1}{2}}$$

Exercise

Find a formula for the nth term partial sum of the series and use it to find the series' sum if the series converges $\frac{5}{1\cdot 2}+\frac{5}{2\cdot 3}+\frac{5}{3\cdot 4}+\cdots+\frac{5}{n(n+1)}+\cdots$

Solution

$$\frac{5}{n(n+1)}=\frac{5}{n}-\frac{5}{n+1}$$

$$s_n=5\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$=5\left(1-\frac{1}{n+1}\right)$$

$$\lim_{n\to\infty}\frac{1}{n+1}=0$$

$$\lim_{n\to\infty} s_n \ \boxed{=5}$$

Exercise

Write out the first few terms of each series to show how the series starts. Then find the sum of the series

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}$$

Solution

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}=1-\frac{1}{4}+\frac{1}{16}-\frac{1}{64}+\cdots$$

The sum of geometric series: $\frac{1}{1-\left(-\frac{1}{4}\right)}=\boxed{\frac{4}{5}}$

Exercise

Write out the first few terms of each series to show how the series starts. Then find the sum of the series

$$\sum_{n=0}^{\infty}\frac{1}{4^n}$$

Solution

$$\sum_{n=0}^{\infty}\frac{1}{4^n}=1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots$$

The sum of geometric series: $\frac{1}{1-\frac{1}{4}}=\boxed{\frac{4}{3}}$

Exercise

Write out the first few terms of each series to show how the series starts. Then find the sum of the series

$$\sum_{n=0}^{\infty}\left(\frac{5}{2^n}-\frac{1}{3^n}\right)$$

Solution

$$\sum_{n=0}^{\infty}\left(\frac{5}{2^n}-\frac{1}{3^n}\right)=(5-1)+\left(\frac{5}{2}-\frac{1}{3}\right)+\left(\frac{5}{4}-\frac{1}{9}\right)+\left(\frac{5}{8}-\frac{1}{27}\right)+\cdots$$

The sum of geometric series: $\frac{5}{1-\left(\frac{1}{2}\right)}-\frac{1}{1-\left(\frac{1}{3}\right)}=10-\frac{3}{2}=\frac{17}{2}$

Exercise

Write out the first few terms of each series to show how the series starts. Then find the sum of the series

$$\sum_{n=0}^{\infty}\frac{2^{n+1}}{5^n}$$

Solution

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{2^{n+1}}{5^n}&=2+\frac{4}{5}+\frac{8}{25}+\frac{16}{125}+\cdots\\&=2\left(1+\frac{2}{5}+\frac{4}{25}+\frac{8}{125}+\cdots\right)\end{aligned}$$

The sum of geometric series: $2\frac{1}{1-\left(\frac{2}{5}\right)}=2\cdot\frac{5}{3}=\frac{10}{3}$

Exercise

Determine if the geometric series converges or diverges. If a series converges, find its sum

$$1+\left(\frac{2}{5}\right)+\left(\frac{2}{5}\right)^2+\left(\frac{2}{5}\right)^3+\left(\frac{2}{5}\right)^4+\cdots$$

Solution

$r=\frac{2}{5}<1$

$\Rightarrow$ The series is geometric ***converges*** to $\frac{1}{1-\left(\frac{2}{5}\right)}=\frac{5}{3}$

Exercise

Determine if the geometric series converges or diverges. If a series converges, find its sum

$$1+(-3)+(-3)^2+(-3)^3+(-3)^4+\cdots$$

Solution

$r=-3 \quad \Rightarrow \quad |r|=|-3|>1$

$\Rightarrow$ The series is geometric and ***diverges***.

Exercise

Determine if the geometric series converges or diverges. If a series converges, find its sum

$$\left(-\frac{2}{3}\right)^2+\left(-\frac{2}{3}\right)^3+\left(-\frac{2}{3}\right)^4+\left(-\frac{2}{3}\right)^5+\cdots$$

Solution

The series is geometric with $r=-\frac{2}{3} \quad \Rightarrow \quad \left|-\frac{2}{3}\right|<1$

Converges to $\dfrac{-\frac{2}{3}}{1-\left(-\frac{2}{3}\right)}=\boxed{-\frac{2}{5}}$

Exercise

Express each of the numbers as the ratio of two integers (fraction) $\quad 0.\overline{23}=0.23\ 23\ 23\cdots$

Solution

$$\begin{aligned}
0.\overline{23} &= 0.23+.0023+.000023+\cdots \\
&= \frac{23}{100}+\frac{23}{10^4}+\frac{23}{10^6}+\cdots \\
&= \sum_{n=0}^{\infty}\frac{23}{100}\left(\frac{1}{10^2}\right)^n \\
&= \frac{\frac{23}{100}}{1-\frac{1}{100}} \\
&= \boxed{\frac{23}{99}}
\end{aligned}$$

Exercise

Express each of the numbers as the ratio of two integers (fraction) $\quad 0.\overline{234}=0.234\ 234\ 234\cdots$

Solution

$$0.\overline{234} = \frac{234}{10^3} + \frac{234}{10^6} + \frac{234}{10^9} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{234}{1000}\left(\frac{1}{10^3}\right)^n$$

$$= \frac{\frac{234}{1000}}{1 - \frac{1}{1000}}$$

$$\boxed{= \frac{234}{999}}$$

Exercise

Express each of the numbers as the ratio of two integers (fraction) $1.\overline{414} = 1.414\ 414\ 414\cdots$

Solution

$$1.\overline{414} = 1 + \frac{414}{10^3} + \frac{414}{10^6} + \frac{414}{10^9} + \cdots$$

$$= 1 + \sum_{n=0}^{\infty} \frac{414}{1000}\left(\frac{1}{10^3}\right)^n$$

$$= 1 + \frac{\frac{414}{1000}}{1 - \frac{1}{1000}}$$

$$= 1 + \frac{414}{999}$$

$$\boxed{= \frac{1413}{999}}$$

Exercise

Express each of the numbers as the ratio of two integers (fraction) $1.24\overline{123} = 1.24\ 123\ 123\ 123\cdots$

Solution

$$1.24\overline{123} = 1.24 + \frac{123}{10^3} + \frac{123}{10^6} + \frac{123}{10^9} + \cdots$$

$$= 1.24 + \sum_{n=0}^{\infty} \frac{123}{10^5}\left(\frac{1}{10^3}\right)^n$$

$$= 1.24 + \frac{\frac{123}{10^5}}{1 - \frac{1}{1000}}$$

$$= 1.24 + \frac{\frac{123}{10^5}}{\frac{10^3 - 1}{10^3}}$$

$$= 1.24 + \frac{123}{10^5}\frac{10^3}{999}$$

$$= \frac{124}{100} + \frac{123}{99,900}$$

$$= \frac{123,999}{99,900}$$

$$= \boxed{\frac{41,333}{33,300}}$$

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$

Solution

Geometric series ***converges***: $\dfrac{1}{1-\frac{1}{\sqrt{2}}} = \dfrac{\sqrt{2}}{\sqrt{2}-1}\cdot\dfrac{\sqrt{2}+1}{\sqrt{2}+1} = \dfrac{2+\sqrt{2}}{2-1} = \boxed{2+\sqrt{2}}$

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}n$

Solution

$\lim\limits_{n\to\infty}(-1)^{n+1}n \neq 0 \quad \Rightarrow \quad$ *The given series diverges*

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=0}^{\infty}\frac{\cos n\pi}{5^n}$

Solution

$\cos(n\pi) = (-1)^n \quad \Rightarrow$ It is geometric series ***converges*** with sum $\dfrac{1}{1-\left(-\frac{1}{5}\right)} = \boxed{\frac{5}{6}}$

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=0}^{\infty} e^{-2n}$

Solution

$$e^{-2n} = \left(\frac{1}{e^2}\right)^2 \Rightarrow \text{It is geometric series } \textbf{\textit{converges}} \text{ with sum } \frac{1}{1-\left(\frac{1}{e^2}\right)} = \boxed{\frac{e^2}{e^2-1}}$$

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty} \ln\frac{1}{3^n}$

Solution

$$\lim_{n\to\infty} \ln\frac{1}{3^n} \underline{= -\infty \neq 0} \quad \Rightarrow \quad \textit{The given series diverges}$$

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Solution

$$\lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \frac{n\cdot n\cdot\cdots\cdot n}{1\cdot 2\cdot 3\cdots n} > \lim_{n\to\infty} n \underline{= \infty} \quad \Rightarrow \textit{The given series diverges}$$

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n}$

Solution

$$\sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n} = \sum_{n=1}^{\infty} \frac{2^n}{4^n} + \sum_{n=1}^{\infty} \frac{3^n}{4^n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$ ***Both are geometric series.***

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$$\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{\frac{3}{4}}{1-\frac{3}{4}} = 3$$

$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = 3 + 1 = 4$$

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=0}^{\infty} \frac{e^{\pi n}}{\pi^{ne}}$

Solution

$$\sum_{n=0}^{\infty} \frac{e^{\pi n}}{\pi^{ne}} = \sum_{n=0}^{\infty} \left(\frac{e^{\pi}}{\pi^{e}}\right)^n \qquad r = \frac{e^{\pi}}{\pi^{e}} \approx 1.03 > 1$$

The geometric series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=0}^{\infty} (1.075)^n$

Solution

This geometric series: $r = 1.075 > 1$ The geometric series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{1000}$

Solution

This geometric series: $r = 3 > 1$ The geometric series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty} \frac{n+10}{10n+1}$

Solution

$$\lim_{n\to\infty} \frac{n+10}{10n+1} = \frac{1}{10} \neq 0$$

The series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty} \frac{4n+1}{3n-1}$

Solution

$$\lim_{n\to\infty} \frac{4n+1}{3n-1} = \frac{4}{3} \neq 0$$

The series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$

Solution

$$S_n = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{n+1} - \frac{1}{n+2}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{2}$$

The series ***converges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$

Solution

$$S_n=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$

$$=\frac{1}{2}-\frac{1}{n+2}$$

$$\lim_{n\to\infty}S_n \ \boxed{=\frac{1}{2}}$$ The series ***converges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{n^3}$

Solution

$$\lim_{n\to\infty}\frac{3^n}{n^3}=\lim_{n\to\infty}\frac{3^n\ln 2}{3n^2}$$

$$=\lim_{n\to\infty}\frac{3^n(\ln 2)^2}{6n}$$

$$=\lim_{n\to\infty}\frac{3^n(\ln 2)^3}{6}$$

$$\boxed{=\infty}$$ The series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=0}^{\infty}\frac{3}{5^n}$

Solution

$$\sum_{n=0}^{\infty}\frac{3}{5^n}=\sum_{n=0}^{\infty}3\left(\frac{1}{5}\right)^n$$

This geometric series: $r=\frac{1}{5}<1$ The geometric series ***converges***

$$S=\frac{3}{1-\frac{1}{5}}\ \boxed{=\frac{15}{4}}$$

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\ln n}$

Solution

Since $n > \ln n \Rightarrow \dfrac{n}{\ln n}$ do not approach 0 as $n \to \infty$

The series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty} \ln\frac{1}{n}$

Solution

$$\begin{aligned}\sum_{n=1}^{\infty} \ln\frac{1}{n} &= -\sum_{n=1}^{\infty} \ln n \\ &= -\left(0 + \ln 2 + \ln 3 + \cdots\right)\end{aligned}$$

The series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{k}{n}\right)^n$

Solution

$$\sum_{n=1}^{\infty} \left(1+\frac{k}{n}\right)^n = \sum_{n=1}^{\infty} \left(\left(1+\frac{k}{n}\right)^{n/2}\right)^2$$

$$\lim_{n\to\infty} \left(1+\frac{k}{n}\right)^{n/2} = e^k \neq 0$$

The series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

Solution

$$\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$$

This geometric series: $r = \frac{1}{e} < 1$ The geometric series ***converges***

$$S = \frac{1}{1-\frac{1}{e}} = \frac{e}{e-1}$$

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty} \arctan n$

Solution

$\lim\limits_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$ The series ***diverges***

Exercise

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

Solution

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} \left(\ln(n+1) - \ln n\right)$$

$$\begin{aligned} S_n &= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln(n+1) - \ln) \\ &= \ln(n+1) - \ln 1 \\ &= \ln(n+1) \end{aligned}$$

$$\lim_{n\to\infty} S_n = \infty$$

The series ***diverges***

Exercise

Determine if the series converges or diverges and describe whether they do so monotonically or by oscillation. Give the limit when the sequence converges. $a_n = 0.3^n$

Solution

Since $0.3 < 1$, this sequence ***converges*** to 0, and since $0.3 > 0$, this converge is monotone.

Exercise

Determine if the series converges or diverges and describe whether they do so monotonically or by oscillation. Give the limit when the sequence converges. $a_n = 1.3^n$

Solution

Since $1.2 > 1$, this sequence *diverges* monotonically to ∞.

Exercise

Determine if the series converges or diverges and describe whether they do so monotonically or by oscillation. Give the limit when the sequence converges. $a_n = (-0.6)^n$

Solution

Since $|-0.6| < 1$, this sequence ***converges*** to 0, and since $-0.6 < 0$, this converge is not monotone.

Exercise

Determine if the series converges or diverges and describe whether they do so monotonically or by oscillation. Give the limit when the sequence converges. $a_n = (-1.01)^n$

Solution

Since $|-1.1| > 1$, this sequence ***diverges***, and since $-1.1 < 0$, this diverge is not monotone.

Exercise

Determine if the series converges or diverges and describe whether they do so monotonically or by oscillation. Give the limit when the sequence converges. $a_n = 2^n 3^{-n}$

Solution

$a_n = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$; Since $0 < \frac{2}{3} < 1$, the sequence ***converges*** monotonically to zero.

Exercise

Determine if the series converges or diverges and describe whether they do so monotonically or by oscillation. Give the limit when the sequence converges. $a_n = (-0.003)^n$

Solution

Since $|-0.003| < 1$, this sequence ***converges*** to 0, and since $-0.003 < 0$, this converge is not monotone.

Exercise

Find the limit of the following sequences or state that they diverge $a_n = \frac{\sin n}{2^n}$

Solution

Since $-1 \le \sin n \le 1$ for all *n*, this sequence satisfies $-\frac{1}{2^n} \le \frac{\sin n}{2^n} \le \frac{1}{2^n}$ and

since both $\pm\frac{1}{2^n} \xrightarrow[n\to\infty]{} 0$ this ***converge*** to 0.

Exercise

Find the limit of the following sequences or state that they diverge $a_n = \dfrac{\cos\left(\frac{n\pi}{2}\right)}{\sqrt{n}}$

Solution

Since $-1 \le \cos\left(\frac{n\pi}{2}\right) \le 1$ for all *n*, this sequence satisfies $-\frac{1}{\sqrt{n}} \le \dfrac{\cos\left(\frac{n\pi}{2}\right)}{\sqrt{n}} \le \frac{1}{\sqrt{n}}$ and since both

$\pm\frac{1}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0$ this ***converge*** to 0.

Exercise

Find the limit of the following sequences or state that they diverge $a_n = \dfrac{2\tan^{-1} n}{n^3+4}$

Solution

$$-\frac{\pi}{2} < \tan^{-1} n < \frac{\pi}{2} \quad \Rightarrow \quad -\frac{\pi}{n^3+4} < \frac{2\tan^{-1} n}{n^3+4} < \frac{\pi}{n^3+4}$$

By the Squeeze Theorem, the given sequence ***converges*** to zero.

Exercise

Find the limit of the following sequences or state that they diverge $a_n = \dfrac{n\sin^3 n}{n+1}$

Solution

Let $b_n = \frac{n}{n+1} \xrightarrow[n\to\infty]{} 0$

Then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n \sin^3 n}{n+1} \dfrac{n+1}{n}$

$= \lim\limits_{n\to\infty} \sin^3 n$ doesn't converge.

This sequence ***diverges***.

Exercise

Many people take aspirin on a regular basis as a preventive measure for heart disease. Suppose a person take 80 *mg* of aspirin every 24 *hr*. Assume also that aspirin has a half-life of 24 *hr*; that is, every 24 *hr* half of the drug in the blood is eliminated.

a) Find a recurrence relation for the sequence $\{d_n\}$ that gives the amount of drug in the blood after the n^{th} dose, where $d_1 = 80$

b) Find the limit of $\{d_n\}$

Solution

a) After the n^{th} dose is given, the amount of drug in the bloodstream is $d_n = 0.5d_{n-1} + 80$, since the half-life is one day. The initial value is $d_1 = 80$

b) $Sum = \dfrac{a}{1-r} = \dfrac{80}{1-\frac{1}{2}} = 160$

The limit of this sequence is 160 *mg*.

Exercise

Suppose a tank is filled with 100 *L* of a 40% alcohol solution (by volume). You repeatedly perform the following operation: Remove 2 *L* of the solution from the tank and replace them with 2 *L* of 10% alcohol solution

a) Let C_n be the concentration of the solution in the tank after the n^{th} replacement, where $C_0 = 40\%$.

Write the first five terms of the sequence $\{C_n\}$

b) After how many replacements does the alcohol concentration reach 15%?

c) Determine the limiting (steady-state) concentration of the solution that is approached after many replacements.

Solution

a) Let D_n be the total number of liters of alcohol in the mixture after n^{th} replacement.

Next, 2 liters of the 100 liters is removed left with 98 liters (0.98).

D_n liters of alcohol, and then $0.1\times 2 = 0.2$ liters of alcohol are added.

Thus $D_n = 0.98D_{n-1} + 0.2$ *and* let $C_n = \dfrac{D_n}{100}$

$$100C_n = 0.98\left(100C_{n-1}\right) + 0.2 \quad \Rightarrow \quad \boxed{C_n = 0.98C_{n-1} + 0.002}$$

$$C_0 = 0.4$$

$$C_1 = 0.98C_0 + 0.002 = 0.98(.4) + 0.002 = \boxed{0.394}$$

$$C_2 = 0.98C_1 + 0.002 = 0.98(0.394) + 0.002 = \boxed{0.38812}$$

$$C_3 = 0.98C_2 + 0.002 = 0.98(0.38812) + 0.002 = \boxed{0.382358}$$

$$C_4 = 0.98C_3 + 0.002 = 0.98(0.382358) + 0.002 = \boxed{0.376710}$$

$$C_5 = 0.98C_4 + 0.002 = 0.98(0.376710) + 0.002 = \boxed{0.371176}$$

b) For $C_n < 0.15$

C	0	0.4									
C	1	0.394	C	26	0.277419	C	51	0.207066	C	76	0.164610
C	2	0.38812	C	27	0.273870	C	52	0.204925	C	77	0.163318
C	3	0.382358	C	28	0.270393	C	53	0.202826	C	78	0.162052
C	4	0.376710	C	29	0.266985	C	54	0.200770	C	79	0.160811
C	5	0.371176	C	30	0.263645	C	55	0.198754	C	80	0.159595
C	6	0.365753	C	31	0.260372	C	56	0.196779	C	81	0.158403
C	7	0.360438	C	32	0.257165	C	57	0.194843	C	82	0.157235
C	8	0.355229	C	33	0.254022	C	58	0.192947	C	83	0.156090
C	9	0.350124	C	34	0.250941	C	59	0.191088	C	84	0.154968
C	10	0.345122	C	35	0.247922	C	60	0.189266	C	85	0.153869
C	11	0.340219	C	36	0.244964	C	61	0.187481	C	86	0.152791
C	12	0.335415	C	37	0.242065	C	62	0.185731	C	87	0.151736
C	13	0.330707	C	38	0.239223	C	63	0.184016	C	88	0.150701
C	14	0.326093	C	39	0.236439	C	64	0.182336	**C**	**89**	**0.149687**
C	15	0.321571	C	40	0.233710	C	65	0.180689			
C	16	0.317139	C	41	0.231036	C	66	0.179076			
C	17	0.312797	C	42	0.228415	C	67	0.177494			
C	18	0.308541	C	43	0.225847	C	68	0.175944			
C	19	0.304370	C	44	0.223330	C	69	0.174425			
C	20	0.300282	C	45	0.220863	C	70	0.172937			
C	21	0.296277	C	46	0.218446	C	71	0.171478			
C	22	0.292351	C	47	0.216077	C	72	0.170048			
C	23	0.288504	C	48	0.213756	C	73	0.168648			
C	24	0.284734	C	49	0.211481	C	74	0.167275			
C	25	0.281039	C	50	0.209251	C	75	0.165929			

c) Assume that the $\lim\limits_{n\to\infty} C_n = L$, then $L = 0.98L + 0.002$

$$0.02L = 0.002 \quad \Rightarrow \quad \boxed{L = 0.1 = 10\%}$$

Exercise

The Greeks solved several calculus problems almost 2000 years before the discovery of calculus. One example is Archimedes' calculation of the area of the region *R* bounded by a segment of a parabola, which he did using the "method of exhaustion".
The idea was to fill *R* with an infinite sequence of triangles. Archimedes began with an isosceles triangle inscribed in the parabola, with an area A_1, and proceeded in stages, with the number of new triangles doubling at each stage. He was able to show (the key to the solution) that at each stage, the area of a new triangle is $\frac{1}{8}$ of the area of a triangle at the previous stage; for example, $A_2 = \frac{1}{8}A_1$, and so forth. Show, as Archimedes did, that the area of *R* is $\frac{4}{3}$ times the area of A_1.

Solution

At the nth stage, there are 2^{n-1} triangles of area

$$A_n = \frac{1}{8}A_{n-1} = \frac{1}{8^{n-1}}A_1$$

So the total area of the triangles formed at the nth stage is

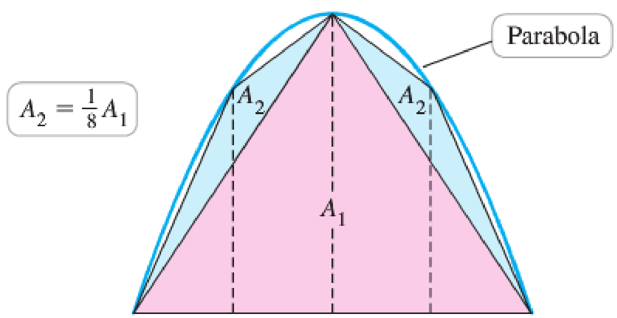
$$\frac{2^{n-1}}{8^{n-1}}A_1 = \left(\frac{1}{4}\right)^{n-1}A_1$$

The total area under the parabola is

$$\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n-1}A_1 = A_1\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n-1}$$

$$= A_1\frac{1}{1-\frac{1}{4}}$$

$$= \frac{4}{3}A_1$$



Exercise

a) Evalute the series $\displaystyle\sum_{k=1}^{\infty}\frac{3^k}{\left(3^{k+1}-1\right)\left(3^k-1\right)}$

b) For what values of a does the the series converge, and in those cases, what is its value?

$$\sum_{k=1}^{\infty}\frac{a^k}{\left(a^{k+1}-1\right)\left(a^k-1\right)}$$

Solution

a) $\displaystyle\frac{3^k}{\left(3^{k+1}-1\right)\left(3^k-1\right)} = \frac{A}{3^{k+1}-1} + \frac{B}{3^k-1}$

$$3^k = A3^k - A + B3^{k+1} - B$$
$$= A3^k + 3B3^k - A - B$$

$$=\frac{1}{2}\left(\frac{1}{3^k-1}-\frac{1}{3^{k+1}-1}\right) \qquad \begin{cases} A+3B=1 \\ -A-B=0 \end{cases} \rightarrow B=\frac{1}{2}=-A$$

$$\sum_{k=1}^{\infty}\frac{3^k}{\left(3^{k+1}-1\right)\left(3^k-1\right)}=\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{3^k-1}-\frac{1}{3^{k+1}-1}\right)$$

This series telescopes to gives

$$S_n=\frac{1}{2}\left(\frac{1}{3-1}-\frac{1}{3^{n+1}-1}\right)$$

$$\lim_{n\to\infty} S_n \ \boxed{=\frac{1}{4}}$$

b) $$\frac{a^k}{\left(a^{k+1}-1\right)\left(a^k-1\right)}=\frac{A}{a^{k+1}-1}+\frac{B}{a^k-1} \qquad \begin{aligned} a^k &= Aa^k-A+Ba^{k+1}-B \\ &= Aa^k+aBa^k-A-B \end{aligned}$$

$$=\frac{1}{a-1}\left(\frac{1}{a^k-1}-\frac{1}{a^{k+1}-1}\right) \quad (a\neq 1) \qquad \begin{cases} A+aB=1 \\ -A-B=0 \end{cases} \rightarrow B=\frac{1}{a-1}=-A$$

$$\sum_{k=1}^{\infty}\frac{a^k}{\left(a^{k+1}-1\right)\left(a^k-1\right)}=\frac{1}{a-1}\sum_{k=1}^{\infty}\left(\frac{1}{a^k-1}-\frac{1}{a^{k+1}-1}\right)$$

This series telescopes to gives $\quad S_n=\frac{1}{a-1}\cdot\left(\frac{1}{a-1}-\frac{1}{a^{n+1}-1}\right)$

$\lim\limits_{n\to\infty}\frac{1}{a^{n+1}-1}$ converges only iff $|a|>1$

Therefore, the series converges to:

$$\lim_{n\to\infty} S_n \ \boxed{=\frac{1}{(a-1)^2}} \quad if \quad |a|>1$$

Exercise

Suppose you borrow $20,000 for a new car at a monthly interest rate of 0.75%. If you make payments of $600/month, after how many months will the loan balance be zero? Estimate the answer by graphing the sequence of loan balances and then obtain an exact answer using infinite series.

Solution

$B_0=20{,}000$

Let B_n be the loan after n months.

$$B_n=1.0075\cdot B_{n-1}-600$$

$$B_{n-1} = 1.0075 \cdot B_{n-2} - 600 \qquad B_n = 1.0075\left(1.0075 \cdot B_{n-2} - 600\right) - 600$$

$$= (1.0075)^2 B_{n-2} - 600(1 + 1.0075)$$

$$B_{n-2} = 1.0075 \cdot B_{n-3} - 600 \qquad B_n = (1.0075)^2 B_{n-2} - 600(1 + 1.0075)$$

$$= (1.0075)^2\left(1.0075 \cdot B_{n-3} - 600\right) - 600(1 + 1.0075)$$

$$= (1.0075)^3 B_{n-3} - 600\left(1 + 1.0075 + 1.0075^2\right)$$

$$\vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots$$

$$= (1.0075)^n B_0 - 600\left(1 + 1.0075 + (1.0075)^2 + \cdots + (1.0075)^{n-1}\right)$$

$$1 + 1.0075 + (1.0075)^2 + \cdots + (1.0075)^{n-1} = \frac{(1.0075)^n - 1}{1.0075 - 1} = \frac{(1.0075)^n - 1}{0.0075}$$

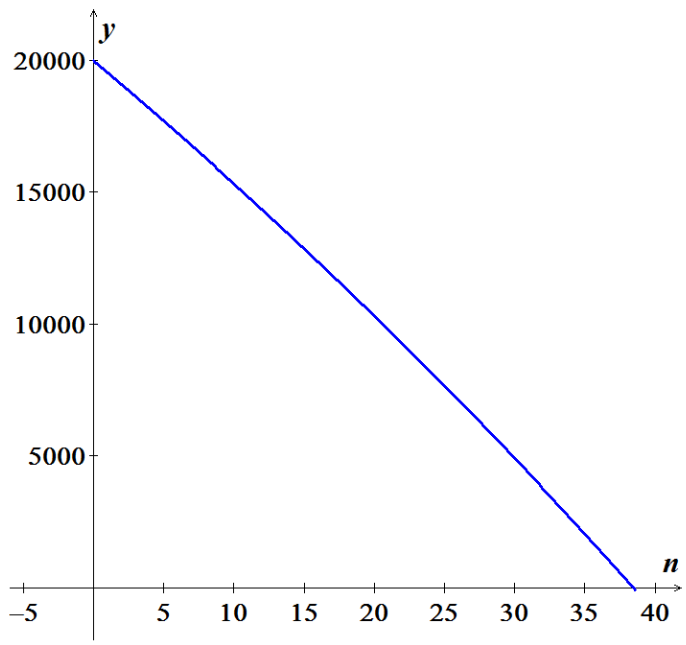
$$B_n = 20{,}000(1.0075)^n - 600\frac{(1.0075)^n - 1}{0.0075} = 0$$

$$150(1.0075)^n - 600(1.0075)^n + 600 = 0$$

$$450(1.0075)^n = 600$$

$$(1.0075)^n = \frac{60}{45} \quad \to \quad \boxed{n = \log_{1.0075}\frac{60}{45} = 38.501}$$

So the loan will be paid off after 39 months.

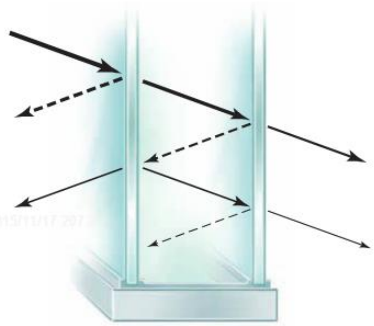


Exercise

An insulated windows consists of two parallel panes of glass with a small spacing between them. Suppose that each pane reflects a fraction p of the incoming light and transmits the remaining light. Considering all reflections of light between the panes, what fraction of the incoming light is ultimately transmitted by the windows? Assume the amount of incoming light is 1.

Solution

Let L_n be the amount of light transmitted through the window the n^{th} time the beam hits the second pane $\Rightarrow \frac{L_n}{1-p}$.



$\frac{pL_n}{1-p}$: is the reflection to the first pane.

$\frac{p^2L_n}{1-p}$: is the reflection back to the second pane.

Thus $\qquad L_{n+1} = (1-p)\frac{p^2L_n}{1-p} = p^2L_n$

The amount of light transmitted through the windows the first time is $(1-p)^2$

Thus the total amount is:

$$\sum_{i=0}^{\infty} p^{2n}(1-p)^2 = \frac{(1-p)^2}{1-p^2}$$

$$= \frac{1-p}{1+p}$$

Exercise

Suppose a rubber ball, when dropped from a given height, returns to a fraction p of that height. In the absence of air resistance, a ball dropped from a height h requires $\sqrt{\frac{2h}{g}}$ seconds to fall to the ground, where $g \approx 9.8\ m/s^2$ is the acceleration due to gravity. The time taken to bounce up to a given to fall from that height to the ground. How long does it take a ball dropped from 10 m to come to rest?

Solution

The height after the n^{th} bounce is: $10p^n$.

The total time spent in that bounce is: $\quad 2\sqrt{\frac{20p^n}{g}}$

The total time before the ball comes to rest:

$$\sqrt{\frac{20}{g}} + 2\sum_{n=1}^{\infty}\sqrt{\frac{20p^n}{g}} = \sqrt{\frac{20}{g}} + 2\sqrt{\frac{20}{g}}\sum_{n=1}^{\infty}\left(\sqrt{p}\right)^n$$

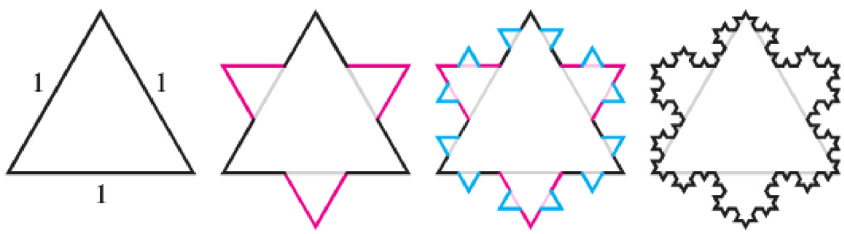
$$= \sqrt{\frac{20}{g}} + 2\sqrt{\frac{20}{g}}\left(\frac{\sqrt{p}}{1-\sqrt{p}}\right)$$

$$= \sqrt{\frac{20}{g}}\left(1 + \frac{2\sqrt{p}}{1-\sqrt{p}}\right)$$

$$= \sqrt{\frac{20}{g}}\left(\frac{1+\sqrt{p}}{1-\sqrt{p}}\right) \text{ sec}$$

Exercise

The fractal called the snowflake island (or Koch island) is constructed as flows: Let I_0 be an equilateral triangle with sides of length 1. The figure I_1 is obtained by replacing the middle third of each side of I_0 with a new outward equilateral triangle with sides of length $\frac{1}{3}$. The process is repeated where I_{n+1} is obtained by replacing the middle third of each side of I_n with a new outward equilateral triangle with sides of length of $\frac{1}{3^{n+1}}$. The limiting figure as $n \to \infty$ is called the snowflake island.



I_0 I_1 I_2 I_3

a) Let L_n be the perimeter of I_n. Show that $\lim\limits_{n\to\infty} L_n = \infty$

b) Let A_n be the area of I_n. Find $\lim\limits_{n\to\infty} A_n$. It exists!

Solution

a) Triangle has 3 equal sizes, from I_0 to I_1 each side turn into 4, and so on.

I_{n+1} is obtained by I_n by dividing each edge into 3 equal parts, removing the middle part, and adding 2 parts equal to it. Thus 3 equal parts turn into 4, so $L_{n+1} = \frac{4}{3}L_n$

This is a geometric series with a ratio greater than 1, so the n^{th} term grows without bound.

$\lim\limits_{n\to\infty} L_n = \infty$

b) From part (*a*) I_n has $3 \cdot 4^n$ sides of length $\frac{1}{3^n}$; each of those sides turns into an added triangle in I_{n+1} of sides length $\frac{1}{3^{n+1}}$.

Thus the added area in I_{n+1} consists of $3\cdot 4^n$ equilateral triangles with side $\dfrac{1}{3^{n+1}}$.

The area of an equilateral triangle with side x is $\dfrac{\sqrt{3}}{4}x^2$.

$$\begin{aligned}\text{Thus } A_{n+1} &= A_n + 3\cdot 4^n \frac{\sqrt{3}}{4}\left(\frac{1}{3^{n+1}}\right)^2 \\ &= A_n + 3\cdot\frac{4^n}{9\cdot 3^{2n}}\frac{\sqrt{3}}{4} \\ &= A_n + \frac{\sqrt{3}}{12}\cdot\left(\frac{4}{9}\right)^n\end{aligned}$$

$$A_0 = \frac{\sqrt{3}}{4}$$

$$A_{n+1} = A_0 + \frac{\sqrt{3}}{12}\sum_{k=0}^{n}\left(\frac{4}{9}\right)^k = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}\sum_{k=0}^{n}\left(\frac{4}{9}\right)^k$$

$$\begin{aligned}\lim_{n\to\infty} A_{n+1} &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}\cdot\frac{1}{1-\frac{4}{9}} \\ &= \frac{\sqrt{3}}{4}\left(1+\frac{1}{3}\cdot\frac{9}{5}\right) \\ &= \boxed{\frac{2\sqrt{3}}{5}}\end{aligned}$$

Exercise

Imagine a stack of hemispherical soap bubbles with decreasing radii $r_1 = 1,\ r_2,\ r_3, \ldots$ Let h_n be the distance between the diameters of bubble n and bubble $n+1$, and let H_n be the total height of the stack with n bubbles.

a) Use the Pythagorean theorem to show that in a stack with n bubbles $h_1^2 = r_1^2 - r_2^2$, $h_2^2 = r_2^2 - r_3^2$, and so forth. Note that for the last bubble $h_n = r_n$.

b) Use part (*a*) to show that the height of a stack with n bubbles is

$$H_n = \sqrt{r_1^2 - r_2^2} + \sqrt{r_2^2 - r_3^2} + \cdots + \sqrt{r_{n-1}^2 - r_n^2} + r_n$$

c) The height of a stack of bubbles depends on how the radii decrease. Suppose that $r_1 = 1,\ r_2 = a,\ r_3 = a^2,\ \ldots,\ r_n = a^{n-1}$ where $0 < a < 1$ is a fixed real number. In terms of *a*, find the height H_n of a stack with *n* bubbles.

d) Suppose the stack in part (*c*) is extended indefinitely $(n \to \infty)$. In terms of *a*, how high would the stack be?

Solution

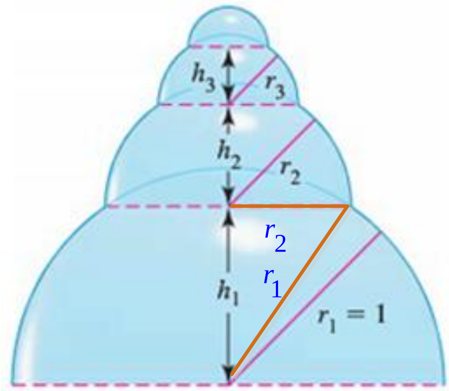
a) Using the Pythagorean theorem; let $k < n$:

$$r_1^2 = h_1^2 + r_2^2 \quad \Rightarrow \quad h_1^2 = r_1^2 - r_2^2$$

$$r_2^2 = h_2^2 + r_3^2 \quad \Rightarrow \quad h_2^2 = r_2^2 - r_3^2$$

$$\vdots \quad \Rightarrow \quad \vdots$$

$$\Rightarrow \quad h_k^2 = r_k^2 - r_{k+1}^2$$



b) Since $h_1^2 = r_1^2 - r_2^2,\ \ h_2^2 = r_2^2 - r_3^2$ *and* $h_k^2 = r_k^2 - r_{k+1}^2$

$$H_n = h_1 + h_2 + \cdots + h_{n-1} + r_n$$

$$\boxed{= \sqrt{r_1^2 - r_2^2} + \sqrt{r_2^2 - r_3^2} + \cdots + \sqrt{r_{n-1}^2 - r_n^2} + r_n}$$

$$\boxed{= r_n + \sum_{i=1}^{n-1} \sqrt{r_i^2 - r_{i+1}^2}}$$

c) ***Given***: $r_1 = 1,\ r_2 = a,\ r_3 = a^2,\ \ldots,\ r_n = a^{n-1}$

$$H_n = r_n + \sum_{i=1}^{n-1} \sqrt{r_i^2 - r_{i+1}^2}$$

Since given $r_i = a^{i-1}$

$$= a^{n-1} + \sum_{i=1}^{n-1} \sqrt{a^{2i-2} - a^{2i}}$$

$$= a^{n-1} + \sum_{i=1}^{n-1} \sqrt{a^{2i-2}\left(1 - a^2\right)}$$

$$= a^{n-1} + \sum_{i=1}^{n-1} a^{i-1}\sqrt{1 - a^2}$$

$$= a^{n-1} + \sqrt{1 - a^2} \sum_{i=1}^{n-1} a^{i-1}$$

Since $0 < a < 1$

$$\boxed{= a^{n-1} + \sqrt{1 - a^2}\left(\frac{1 - a^{n-1}}{1 - a}\right)}$$

$$\sum_{k=1}^{n} r^{k-1} = \frac{1 - r^n}{1 - r}$$

d) $$\lim_{n\to\infty} H_n = \lim_{n\to\infty}\left[a^{n-1} + \sqrt{1 - a^2}\left(\frac{1 - a^{n-1}}{1 - a}\right)\right]$$

$$\lim_{n\to\infty} a^{n-1} = 0 \quad (0 < a < 1)$$

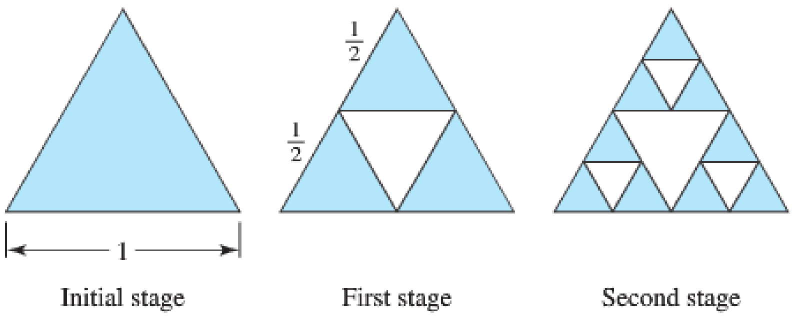
$$= 0 + \sqrt{1-a^2}\left(\frac{1}{1-a}\right)$$

$$= \frac{\sqrt{1-a^2}}{1-a}$$

$$\boxed{= \sqrt{\frac{1+a}{1-a}}}$$

Exercise

The fractal called the *Sierpinski triangle* is the limit of a sequence of figures. Starting with the equilateral triangle with sides of length 1, an inverted equilateral triangle with sides of length $\frac{1}{2}$ is removed. Then, the three inverted equilateral triangles with sides of length $\frac{1}{4}$ are removed from this figure.



The process continues in this way. Let T_n be the total area of the removed triangles after stage *n* of the process. The area on equilateral triangle with side length *L* is $A = \frac{\sqrt{3}}{4}L^2$.

a) Find T_1 and T_2 the total area of the removed triangles after stages 1 and 2, respectively.

b) Find T_n for $n = 1,\ 2,\ 3,\ \ldots$

c) Find $\lim\limits_{n\to\infty} T_n$

d) What is the area of the original triangle that remains as $n \to \infty$?

Solution

a) $T_1 = \frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^2$

$$\boxed{= \frac{\sqrt{3}}{16}}$$

$$T_2 = T_1 + 3A = \frac{\sqrt{3}}{16} + 3\frac{\sqrt{3}}{4}\left(\frac{1}{4}\right)^2$$

$$\boxed{= \frac{7\sqrt{3}}{64}}$$

b) At stage *n*, there are 3^{n-1} triangles of side length $\frac{1}{2^n}$ are removed.

Each of those triangles has an area of $\frac{\sqrt{3}}{4\cdot 4^n}=\frac{\sqrt{3}}{4^{n+1}}$

$$\text{Total } = 3^{n-1}\frac{\sqrt{3}}{4^{n+1}} = \frac{\sqrt{3}}{16}\left(\frac{3}{4}\right)^{n-1}$$

$$T_n = \frac{\sqrt{3}}{16}\sum_{k=1}^{n}\left(\frac{3}{4}\right)^{k-1}$$

$$= \frac{\sqrt{3}}{16}\sum_{k=0}^{n}\left(\frac{3}{4}\right)^{k}$$

$$= \frac{\sqrt{3}}{16}\cdot\frac{1-\left(\frac{3}{4}\right)^n}{1-\frac{3}{4}}$$

$$\boxed{= \frac{\sqrt{3}}{4}\left(1-\left(\frac{3}{4}\right)^n\right)}$$

c) $\lim\limits_{n\to\infty} T_n = \lim\limits_{n\to\infty}\frac{\sqrt{3}}{4}\left(1-\left(\frac{3}{4}\right)^n\right)$ $\qquad\left(\frac{3}{4}\right)^n \xrightarrow[n\to\infty]{} 0$

$$\boxed{= \frac{\sqrt{3}}{4}}$$

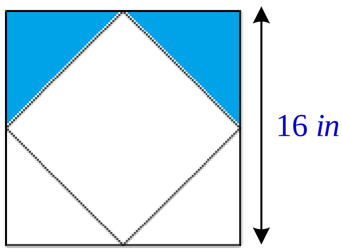
d) The area of the triangle was originally $\frac{\sqrt{3}}{4}$, so none of the original area is left.

Exercise

The sides of a ***square*** are 16 *inches* in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the triangles outside the second square are shaded.
Determine the area of the shaded regions

a) When this process is continued five more times

b) When this pattern of shading is continued infinitely.



Solution

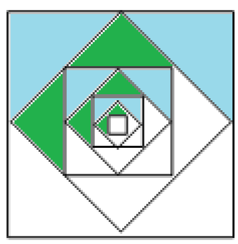
a) The first process: $A_1 = 2\left(\frac{1}{2}8^2\right) = 64 = 2^6$

$$A_2 = 2\left(\frac{1}{2}\left(4\sqrt{2}\right)^2\right) = 32 = 2^5$$

$$A_3 = 2\left(\frac{1}{2}(4)^2\right) = 16 = 2^4$$

$$A_4 = 2\left(\frac{1}{2}\left(2\sqrt{2}\right)^2\right) = 8 = 2^3$$

$$A_k = 64(2)^{-k}$$

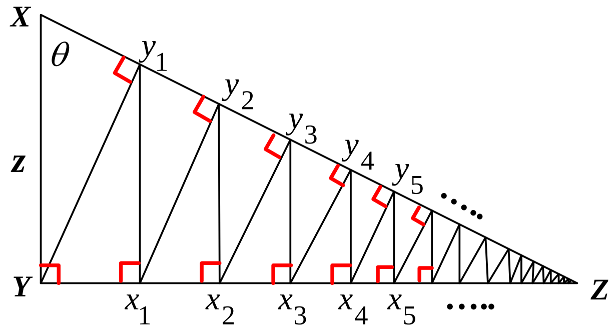


$$A = 64 + 32 + 16 + 8 + 4 + 2$$
$$\boxed{= 126\ in^2}$$

b) $\sum_{n=0}^{\infty} 64\left(\frac{1}{2}\right)^n = \frac{64}{1-\frac{1}{2}}$ $\boxed{= 128\ in^2}$ $\qquad S = \frac{a_0}{1-r}$

Exercise

A right triangle XYZ is shown below where $|XY| = z$ and $\angle X = \theta$. Line segments are continually drawn to be perpendicular to the triangle.



a) Find the total length of the perpendicular line segments $|Yy_1| + |x_1y_1| + |x_1y_2| + \cdots$ in terms of z and θ.

b) Find the total length of the perpendicular line segments when $z = 1$ and $\theta = \frac{\pi}{6}$

Solution

a) $\sin\theta = \frac{|Yy_1|}{z} \Rightarrow |Yy_1| = z\sin\theta$

$\sin\theta = \frac{|x_1y_1|}{|Yy_1|} \Rightarrow |x_1y_1| = |Yy_1|\sin\theta = z\sin^2\theta$

$\sin\theta = \frac{|x_1y_2|}{|x_1y_1|} \Rightarrow |x_1y_2| = |x_1y_1|\sin\theta = z\sin^3\theta$

$\vdots \qquad \vdots$

$$\text{Total Length } = z\sin\theta + z\sin^2\theta + z\sin^3\theta + \cdots$$
$$= z\left(\sin\theta + \sin^2\theta + \sin^3\theta + \cdots\right) \qquad |\sin\theta| < 1 \quad S = \frac{a_0}{1-r}$$
$$\boxed{= z\,\frac{\sin\theta}{1-\sin\theta}}$$

b) ***Given***: $z = 1$ and $\theta = \frac{\pi}{6}$

$$\text{Total Length } = 1\,\frac{\sin\frac{\pi}{6}}{1-\sin\frac{\pi}{6}}$$

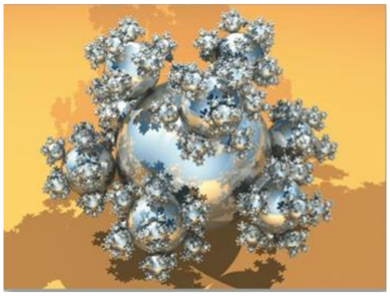
$$= 1\,\frac{\frac{1}{2}}{1-\frac{1}{2}}$$

$$\underline{= 1}$$

Exercise

The sphereflake is a computer–generated fractal that was created by Eric Haines. The radius of the large sphere is 1. To the large sphere, nine spheres of radius $\frac{1}{3}$ are attached. To each of these, nine spheres of radius $\frac{1}{9}$ are attached. This process is continued infinitely.

Prove that the sphereflake has an infinite surface area.



Solution

$$\text{Surface area } = 4\pi(1)^2 + 9\times 4\pi\left(\frac{1}{3}\right)^2 + 9^2\times 4\pi\left(\frac{1}{9}\right)^2 + \cdots$$

$$= 4\pi(1+1+1+\cdots)$$

$$\underline{= \infty}$$