

Section 4.4 – Determinants

Determinant of a 2 x 2 Matrix

Determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and is define as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example

Let $A = \begin{bmatrix} -3 & 4 \\ 6 & 8 \end{bmatrix}$. Find $|A|$

Solution

$$\begin{aligned} |A| &= \begin{vmatrix} -3 & 4 \\ 6 & 8 \end{vmatrix} \\ &= -3(8) - 4(6) \\ &= -48 \end{aligned}$$

Example

Evaluate: $\begin{vmatrix} 2 & -3 \\ -4 & 1 \end{vmatrix}$

Solution

$$\begin{aligned} \begin{vmatrix} 2 & -3 \\ -4 & 1 \end{vmatrix} &= 2(1) - (-3)(-4) \\ &= 2 - 12 \\ &= -10 \end{aligned}$$

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor

For a square matrix $A = [a_{ij}]$, the minor M_{ij} of an element a_{ij} is the determinant of the matrix formed by deleting the i^{th} row and the j^{th} column of A .

Cofactor: $A_{ij} = (-1)^{i+j} M_{ij}$

$$\begin{aligned} |A| &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

Example

$$A = \begin{pmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{pmatrix} \text{ Find the determinant of } A.$$

Solution

$$\begin{aligned} |A| &= \begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix} \\ &= -8 \begin{vmatrix} -6 & 7 \\ -3 & 5 \end{vmatrix} - 0 \begin{vmatrix} 4 & 7 \\ -1 & 5 \end{vmatrix} + 6 \begin{vmatrix} 4 & -6 \\ -1 & -3 \end{vmatrix} \\ &= -8(-30 - (-21)) - 0 + 6(-12 - 6) \\ &= -8(-9) + 6(-18) \\ &= \underline{-36} \end{aligned}$$

Determinant Using Diagonal Method

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \quad (1)$$

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$$a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} \quad (2)$$

$$\text{Determinant: } D = (1) - (2)$$

Example

Evaluate $\begin{vmatrix} 2 & -3 & -2 \\ -1 & -4 & -3 \\ -1 & 0 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & -3 & -2 \\ -1 & -4 & -3 \\ -1 & 0 & 2 \end{vmatrix} \begin{array}{cc} 2 & -3 \\ -1 & -4 \\ -1 & 0 \end{array} = 2(-4)(2) + (-3)(-3)(-1) + (-2)(-1)(0) - (-2)(-4)(-1) - (2)(-3)(0) - (-3)(-1)(2) \\ = -23$$

Example

Evaluate $\begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix} \begin{array}{cc} -8 & 0 \\ 4 & -6 \\ -1 & -3 \end{array} = (-8)(-6)(5) + 0(7)(-1) + 6(4)(-3) - 6(-6)(-1) - (-8)(7)(-3) - 0(4)(5) \\ = -36$$

Example

Evaluate $\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix} \begin{matrix} x & 0 \\ 2 & x \\ -3 & x \end{matrix} = x^2 + 0 - 2x - (3x) - x^4 - 0$$
$$= -x^4 + x^2 - 5x$$

Exercises *Section 4.4 – Determinants*

(1 – 34) Evaluate

1. $\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix}$

2. $\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix}$

3. $\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix}$

4. $\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix}$

5. $\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix}$

6. $\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix}$

7. $\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}$

8. $\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix}$

9. $\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix}$

10. $\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix}$

11. $\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix}$

12. $\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix}$

13. $\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix}$

14. $\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix}$

15. $\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix}$

16. $\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix}$

17. $\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix}$

18. $\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix}$

19. $\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix}$

20. $\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix}$

21. $\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix}$

22. $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix}$

23. $\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix}$

24. $\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix}$

25. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix}$

26. $\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix}$

27. $\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix}$

28. $\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$

29. $\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix}$

30. $\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix}$

31. $\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix}$

32. $\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix}$

33. $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix}$

34. $\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}$

(35 – 46) Solve for x

$$35. \begin{vmatrix} x & 3 \\ 2 & 1 \end{vmatrix} = 12$$

$$36. \begin{vmatrix} x & 1 \\ 2 & x \end{vmatrix} = -1$$

$$37. \begin{vmatrix} 3 & x \\ x & 4 \end{vmatrix} = -13$$

$$38. \begin{vmatrix} x & 2 \\ 3 & x \end{vmatrix} = x$$

$$39. \begin{vmatrix} 4 & 6 \\ -2 & x \end{vmatrix} = 32$$

$$40. \begin{vmatrix} x+2 & -3 \\ x+5 & -4 \end{vmatrix} = 3x-5$$

$$41. \begin{vmatrix} x+3 & -6 \\ x-2 & -4 \end{vmatrix} = 28$$

$$42. \begin{vmatrix} x & -3 \\ -1 & x \end{vmatrix} \geq 0$$

$$43. \begin{vmatrix} 2 & x & 1 \\ 1 & 2 & -1 \\ 3 & 4 & -2 \end{vmatrix} = -6$$

$$44. \begin{vmatrix} 1 & x & -3 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} = 8$$

$$45. \begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix} = 39$$

$$46. \begin{vmatrix} x & 0 & 0 \\ 7 & x & 1 \\ 7 & 2 & 1 \end{vmatrix} = -1$$