Solution   *Section* **6.5 – Inverse Trigonometric Functions**

Exercise

Find the exact value of the expression whenever it is defined: $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Solution

$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arccos\left(\frac{\sqrt{2}}{2}\right)$

Solution

$$\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arctan\left(-\frac{\sqrt{3}}{3}\right)$

Solution

$$\arctan\left(-\frac{\sqrt{3}}{3}\right) = -\arctan\left(\frac{\sqrt{3}}{3}\right)$$

$$= -\frac{\pi}{6}$$

Exercise

Find the exact value of the expression whenever it is defined: $\sin\left(\arcsin\left(-\frac{3}{10}\right)\right)$

Solution

$$\alpha = \arcsin\left(-\frac{3}{10}\right)$$

$$\sin\alpha = -\frac{3}{10}$$

$$\sin\left(\arcsin\left(-\frac{3}{10}\right)\right) = -\frac{3}{10}$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan\left(\arctan\left(14\right)\right)$

Solution

$$\tan\left(\arctan\left(14\right)\right) = 14$$

Exercise

Find the exact value of the expression whenever it is defined: $\sin\left(\sin^{-1}\left(\frac{2}{3}\right)\right)$

Solution

$$\sin\left(\sin^{-1}\left(\frac{2}{3}\right)\right) = \frac{2}{3}$$

Exercise

Find the exact value of the expression whenever it is defined: $\cos^{-1}\left(\cos\left(\frac{5\pi}{6}\right)\right)$

Solution

$$\cos^{-1}\left(\cos\left(\frac{5\pi}{6}\right)\right) = \frac{5\pi}{6} \qquad 0 \le \frac{5\pi}{6} \le \pi$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right)$

Solution

$$\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right) = -\frac{\pi}{6} \qquad -\frac{\pi}{2} \le -\frac{\pi}{6} \le \frac{\pi}{2}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arcsin\left(\sin\left(-\frac{\pi}{2}\right)\right)$

Solution

$$\arcsin\left(\sin\left(-\frac{\pi}{2}\right)\right) = -\frac{\pi}{2} \qquad -\frac{\pi}{2} \le -\frac{\pi}{2} \le \frac{\pi}{2}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arccos(\cos(0))$

<u>Solution</u>

$$\underline{\arccos(\cos(0)) = 0} \qquad 0 \le 0 \le \pi$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right)$

<u>Solution</u>

$$\underline{\tan^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right) = -\frac{\pi}{4}} \qquad -\frac{\pi}{2} \le -\frac{\pi}{4} \le \frac{\pi}{2}$$

Exercise

Find the exact value of the expression whenever it is defined: $\sin\left(\arcsin\left(\frac{1}{2}\right) + \arccos 0\right)$

<u>Solution</u>

$$\sin\left(\arcsin\left(\frac{1}{2}\right) + \arccos 0\right) = \sin\left(\frac{\pi}{6} + 0\right)$$

$$= \sin\left(\frac{\pi}{6}\right)$$

$$= \underline{\frac{1}{2}}$$

Exercise

Find the exact value of the expression whenever it is defined: $\cos\left(\arctan\left(-\frac{3}{4}\right) - \arcsin\frac{4}{5}\right)$

<u>Solution</u>

$$\cos\left(\arctan\left(-\frac{3}{4}\right) - \arcsin\frac{4}{5}\right) = \cos(\alpha - \beta)$$

$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$\alpha = \arctan\left(-\frac{3}{4}\right)$	$\beta = \arcsin\frac{4}{5}$
$\tan\alpha = -\frac{3}{4}$	$\sin\beta = \frac{4}{5}$
$r = \sqrt{3^2 + 4^2} = 5$	$\Rightarrow \cos\beta = \frac{3}{5}$
$\sin\alpha = -\frac{3}{5} \qquad \cos\alpha = \frac{4}{5}$	

$$\cos\left(\arctan\left(-\tfrac{3}{4}\right) - \arcsin\tfrac{4}{5}\right) = \tfrac{4}{5}\tfrac{3}{5} + \left(-\tfrac{3}{5}\right)\tfrac{4}{5}$$

$$\underline{= 0}$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan\left(\cos^{-1}\left(\tfrac{1}{2}\right) + \sin^{-1}\left(-\tfrac{1}{2}\right)\right)$

Solution

$$\tan\left(\cos^{-1}\left(\tfrac{1}{2}\right) + \sin^{-1}\left(-\tfrac{1}{2}\right)\right) = \tan\left(\tfrac{\pi}{3} - \tfrac{\pi}{6}\right)$$

$$= \tan\left(\tfrac{\pi}{3}\right)$$

$$\underline{= \sqrt{3}}$$

Exercise

Find the exact value of the expression whenever it is defined: $\sin\left(2\arccos\left(-\tfrac{3}{5}\right)\right)$

Solution

$$\sin\left(2\arccos\left(-\tfrac{3}{5}\right)\right) = \sin 2\alpha$$

$$= 2\sin\alpha\cos\alpha$$

$$\alpha = \arccos\left(-\tfrac{3}{5}\right) \;\rightarrow\; \cos\alpha = -\tfrac{3}{5}$$

$$\sin\alpha = \tfrac{3}{5}$$

$$\sin\left(2\arccos\left(-\tfrac{3}{5}\right)\right) = 2\tfrac{3}{5}\left(-\tfrac{3}{5}\right)$$

$$\underline{= -\tfrac{18}{25}}$$

Exercise

Find the exact value of the expression whenever it is defined: $\cos\left(2\sin^{-1}\left(\tfrac{15}{17}\right)\right)$

Solution

$$\cos\left(2\sin^{-1}\left(\tfrac{15}{17}\right)\right) = \cos 2\alpha$$

$$= 1 - 2\sin^2\alpha$$

$$\alpha = \sin^{-1}\left(\tfrac{15}{17}\right)$$

$$\underline{\sin \alpha = \tfrac{15}{17}}\Big|$$

$$\cos\left(2\sin^{-1}\left(\tfrac{15}{17}\right)\right) = 1 - 2\left(\tfrac{15}{17}\right)^2$$

$$= 1 - \frac{450}{289}$$

$$\underline{= -\frac{161}{289}}\Big|$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan\left(2\tan^{-1}\left(\tfrac{3}{4}\right)\right)$

Solution

$$\tan \tan\left(2\tan^{-1}\left(\tfrac{3}{4}\right)\right) = \tan 2\alpha$$

$$\alpha = \tan^{-1}\left(\tfrac{3}{4}\right)$$

$$\underline{\tan \alpha = \tfrac{3}{4}}\Big|$$

$$\tan\left(2\tan^{-1}\left(\tfrac{3}{4}\right)\right) = \frac{2\tan \alpha}{1 - \tan^2 \alpha}$$

$$= \frac{2\left(\tfrac{3}{4}\right)}{1 - \left(\tfrac{3}{4}\right)^2}$$

$$= \frac{\tfrac{3}{2}}{1 - \tfrac{9}{16}}$$

$$= \frac{3}{2}\frac{16}{7}$$

$$\underline{= \frac{24}{7}}\Big|$$

Exercise

Find the exact value of the expression whenever it is defined: $\cos\left(\frac{1}{2}\tan^{-1}\left(\frac{8}{15}\right)\right)$

Solution

$$\cos\left(\frac{1}{2}\tan^{-1}\left(\frac{8}{15}\right)\right) = \cos\left(\frac{1}{2}\alpha\right)$$

$$\alpha = \tan^{-1}\left(\frac{8}{15}\right)$$

$$\tan\alpha = \frac{8}{15}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1}{2}(1+\cos\alpha)}$$

$$= \sqrt{\frac{1}{2}\left(1+\frac{8}{17}\right)}$$

$$= \sqrt{\frac{25}{34}}$$

$$= \frac{5}{\sqrt{34}} \quad \text{or} \quad \frac{5\sqrt{34}}{34}$$
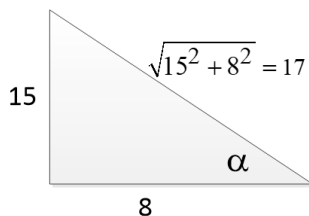
Triangle with horizontal leg 15, vertical leg 8, hypotenuse $\sqrt{15^2+8^2}=17$, angle α.

Exercise

Evaluate without using a calculator: $\cos\left(\cos^{-1}\frac{3}{5}\right)$

Solution

$$\cos\left(\cos^{-1}\frac{3}{5}\right) = \frac{3}{5}$$

Exercise

Evaluate without using a calculator: $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

Solution

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$= \frac{5\pi}{6}$$

Exercise

Evaluate without using a calculator: $\tan\left(\cos^{-1}\dfrac{3}{5}\right)$

Solution

$\tan\left(\cos^{-1}\dfrac{3}{5}\right)$

$5^2 = 3^2 + a^2 \;\rightarrow\; \underline{a = 4}$

$\tan\left(\cos^{-1}\dfrac{3}{5}\right) = \tan\theta$

$\qquad = \underline{\dfrac{4}{3}}$
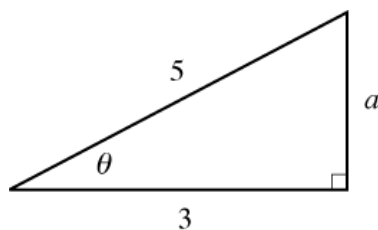
Exercise

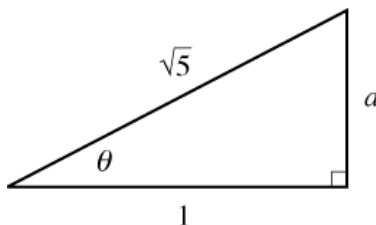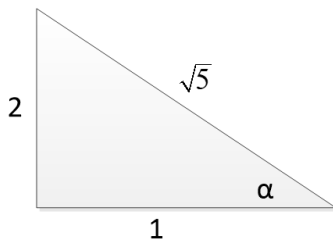Evaluate without using a calculator: $\sin\left(\cos^{-1}\dfrac{1}{\sqrt{5}}\right)$

Solution

$\sin\left(\cos^{-1}\dfrac{1}{\sqrt{5}}\right)$

$\left(\sqrt{5}\right)^2 = 1^2 + a^2$

$a^2 = 5 - 1$

$\underline{a = 2}$

$\sin\left(\cos^{-1}\dfrac{1}{\sqrt{5}}\right) = \sin\theta$

$\qquad = \dfrac{2}{\sqrt{5}}$

Exercise

Evaluate without using a calculator: $\cos\left(\sin^{-1}\dfrac{1}{2}\right)$

Solution

$\cos\left(\sin^{-1}\dfrac{1}{2}\right)$

$\sin\dfrac{\pi}{6} = \dfrac{1}{2}$

$\sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}$

$$\cos\left(\sin^{-1}\frac{1}{2}\right)=\cos\frac{\pi}{6}$$

$$=\frac{\sqrt{3}}{2}$$

Exercise

Evaluate without using a calculator: $\sin\left(\sin^{-1}\frac{3}{5}\right)$

Solution

$$\sin\left(\sin^{-1}\frac{3}{5}\right)=\frac{3}{5}$$

Exercise

Evaluate without using a calculator: $\cos\left(\tan^{-1}\frac{3}{4}\right)$

Solution

$$\alpha=\tan^{-1}\frac{3}{4}$$

$$\tan\alpha=\frac{3}{4}$$

$$r=\sqrt{3^2+4^2}=5$$

$$\cos\left(\tan^{-1}\frac{3}{4}\right)=\frac{4}{5}$$
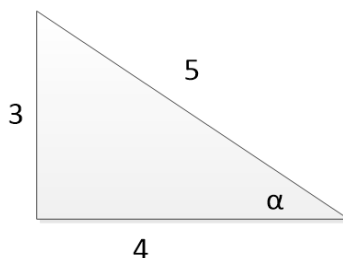
Exercise

Evaluate without using a calculator: $\tan\left(\sin^{-1}\frac{3}{5}\right)$

Solution

$$\sin\alpha=\frac{3}{5}$$

$$\tan\left(\sin^{-1}\frac{3}{5}\right)=\frac{3}{4}$$
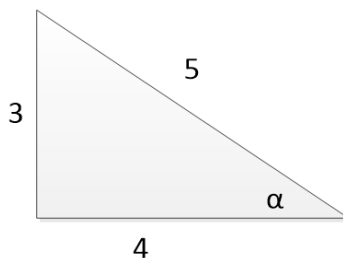
Exercise

Evaluate without using a calculator: $\sec\left(\cos^{-1}\dfrac{1}{\sqrt{5}}\right)$

Solution

$$\alpha = \cos^{-1}\dfrac{1}{\sqrt{5}}$$

$$\cos\alpha = \dfrac{1}{\sqrt{5}}$$

$$\sec\alpha = \dfrac{1}{\dfrac{1}{\sqrt{5}}}$$

$$= \sqrt{5}$$

Exercise

Evaluate without using a calculator: $\cot\left(\tan^{-1}\dfrac{1}{2}\right)$

Solution

$$\alpha = \tan^{-1}\dfrac{1}{2}$$

$$\tan\alpha = \dfrac{1}{2}$$

$$\cot\alpha = \dfrac{1}{\tan\alpha}$$

$$= 2$$

Exercise

Write an equivalent expression that involves x only for $\cos\left(\cos^{-1}x\right)$

Solution

$$\alpha = \cos^{-1}x$$

$$\cos\alpha = x$$

$$\cos\left(\cos^{-1}x\right) = \cos\alpha$$

$$= x$$

Exercise

Write an equivalent expression that involves x only for $\tan\left(\cos^{-1} x\right)$

Solution

$\alpha = \cos^{-1} x$

$\cos\alpha = x = \dfrac{x}{1}$

$x^2 + d^2 = 1 \Rightarrow d^2 = 1 - x^2$

$$d = \sqrt{1-x^2}$$

$\tan\left(\cos^{-1} x\right) = \tan\alpha$

$$= \frac{\sqrt{1-x^2}}{x}$$

(triangle with hypotenuse 1, base x, height $d = \sqrt{1-x^2}$, angle α)
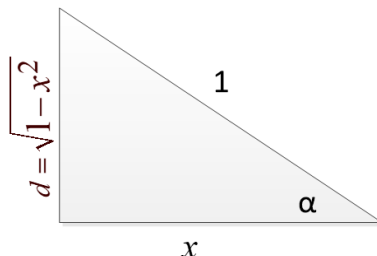
Exercise

Write an equivalent expression that involves x only for $\csc\left(\sin^{-1}\dfrac{1}{x}\right)$

Solution

$\alpha = \sin^{-1}\dfrac{1}{x}$

$\sin\alpha = \dfrac{1}{x}$

$\csc\left(\sin^{-1} x\right) = \csc\alpha = \dfrac{1}{\sin\alpha}$

$$= x$$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\sin\left(\tan^{-1} x\right)$

Solution

$\sin\left(\tan^{-1} x\right) = \sin\alpha$

$\alpha = \tan^{-1} x$

$\tan\alpha = x$

$\sin\left(\tan^{-1} x\right) = \sin\alpha$

$$= \frac{x}{\sqrt{x^2+1}}$$

(triangle with opposite side x, base 1, hypotenuse $\sqrt{x^2+1}$, angle α)
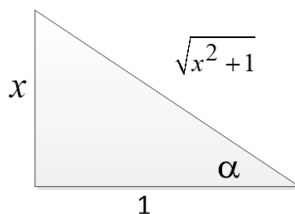
Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\sec\left(\sin^{-1}\dfrac{x}{\sqrt{x^2+4}}\right)$
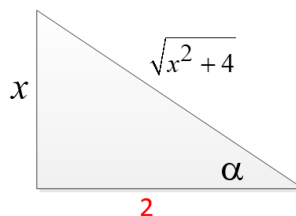
Solution

$\alpha = \sin^{-1}\dfrac{x}{\sqrt{x^2+4}}$

$\sin\alpha = \dfrac{x}{\sqrt{x^2+4}}$

$\sqrt{\left(\sqrt{x^2+4}\right)^2 - x^2} = \sqrt{x^2+4-x^2} = \sqrt{4} = 2$

$\sec\left(\sin^{-1}\dfrac{x}{\sqrt{x^2+4}}\right) = \dfrac{1}{\cos\alpha}$
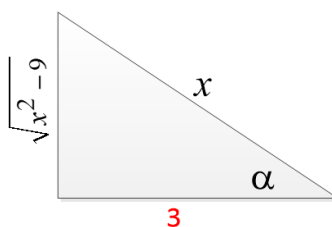
$= \dfrac{2}{\sqrt{x^2+4}}$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\cot\left(\sin^{-1}\dfrac{\sqrt{x^2-9}}{x}\right)$

Solution

$\alpha = \sin^{-1}\dfrac{\sqrt{x^2-9}}{x}$

$\sin\alpha = \dfrac{\sqrt{x^2-9}}{x}$

$\cot\left(\sin^{-1}\dfrac{\sqrt{x^2-9}}{x}\right) = \cot\alpha$

$= \dfrac{3}{\sqrt{x^2-9}}$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\sin\left(2\sin^{-1}x\right)$

Solution

$\alpha = \sin^{-1}x$

$\sin\alpha = x$

$$\sin\left(2\sin^{-1}x\right) = \sin 2\alpha$$

$$= 2\sin\alpha\cos\alpha$$

$$= 2x\sqrt{1-x^2}$$
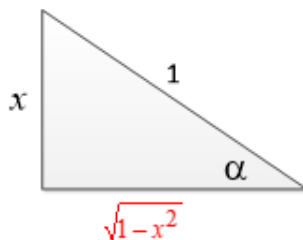
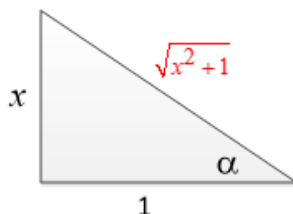## *Exercise*

Write the expression as an algebraic expression in x for $x > 0$: $\cos\left(2\tan^{-1}x\right)$

Solution

$$\alpha = \tan^{-1}x$$
$$\tan\alpha = x$$

$$\cos\left(2\tan^{-1}x\right) = \cos(2\alpha)$$

$$= 2\cos^2\alpha - 1$$

$$= 2\left(\frac{1}{\sqrt{x^2+1}}\right)^2 - 1$$

$$= \frac{2}{x^2+1} - 1$$

$$= \frac{-x^2+1}{x^2+1}$$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\cos\left(\frac{1}{2}\arccos x\right)$

Solution

$$\alpha = \arccos x$$
$$\cos\alpha = x$$

$$\cos\left(\frac{1}{2}\arccos x\right) = \cos\left(\frac{\alpha}{2}\right)$$

$$= \sqrt{\frac{1+\cos\alpha}{2}}$$

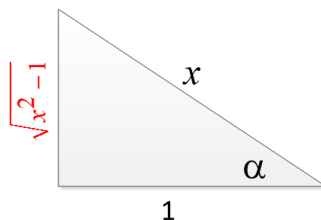$$= \sqrt{\frac{1+x}{2}}$$

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Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\quad \tan\left(\frac{1}{2}\cos^{-1}\frac{1}{x}\right)$

Solution

$$\alpha = \cos^{-1}\frac{1}{x}$$

$$\cos\alpha = \frac{1}{x}$$

$$\tan\left(\frac{1}{2}\cos^{-1}\frac{1}{x}\right) = \tan\left(\frac{\alpha}{2}\right)$$

$$= \frac{1 - \cos\alpha}{\sin\alpha}$$

$$= \frac{1 - \frac{1}{x}}{\frac{\sqrt{x^2-1}}{x}}$$

$$= \frac{\frac{x-1}{x}}{\frac{\sqrt{x^2-1}}{x}}$$

$$= \left.\frac{x-1}{\sqrt{x^2-1}}\right|$$

Exercise

Write the expression as an algebraic expression in x: $\quad \sec\left(\tan^{-1}\frac{2}{\sqrt{x^2-4}}\right) \quad x > 0$

Solution

$$\tan\alpha = \frac{2}{\sqrt{x^2-4}}$$

$$\sec\left(\tan^{-1}\frac{2}{\sqrt{x^2-4}}\right) = \sec\alpha$$
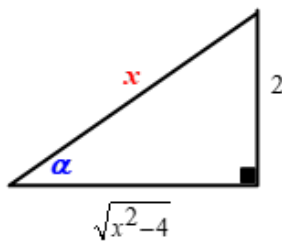
$$= \left.\frac{x}{\sqrt{x^2-4}}\right|$$
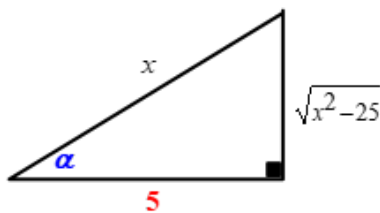
Exercise

Write the expression as an algebraic expression in x: $\sec\left(\sin^{-1}\dfrac{\sqrt{x^2-25}}{x}\right)$ $x>0$

Solution

$$\sin\alpha = \frac{\sqrt{x^2-25}}{x}$$

$$\sec\left(\sin^{-1}\frac{\sqrt{x^2-25}}{x}\right) = \sec\alpha$$
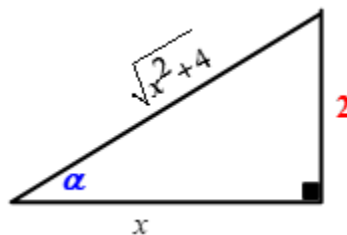
$$= \frac{x}{5}$$

Exercise

Write the expression as an algebraic expression in x: $\sin\left(\cos^{-1}\dfrac{x}{\sqrt{x^2+4}}\right)$ $x>0$

Solution

$$\cos\alpha = \frac{x}{\sqrt{x^2+4}}$$

$$\sin\left(\cos^{-1}\frac{x}{\sqrt{x^2+4}}\right) = \sin\alpha$$
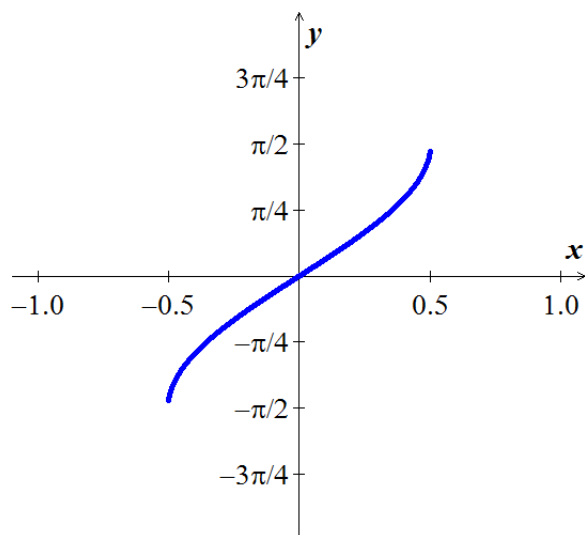
$$= \frac{2}{\sqrt{x^2+4}}$$

Exercise

Sketch he graph of the equation: $y = \sin^{-1}2x$

Solution

$$-\frac{\pi}{2} \le y \le \frac{\pi}{2} \quad and \quad -1 \le 2x \le 1$$

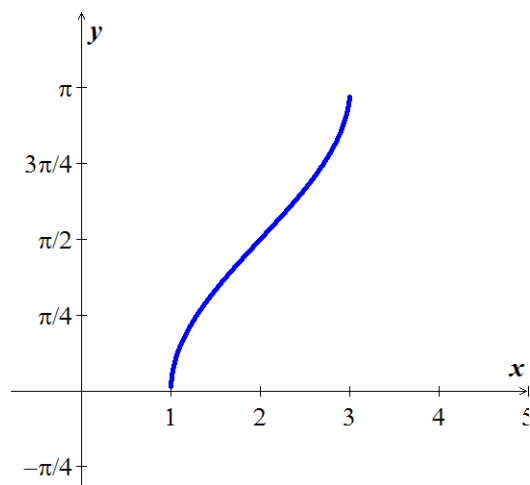$$-\frac{1}{2} \le x \le \frac{1}{2}$$

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Exercise

Sketch he graph of the equation: $y = \sin^{-1}(x-2) + \frac{\pi}{2}$

Solution

$-\frac{\pi}{2} + \frac{\pi}{2} \le y \le \frac{\pi}{2} + \frac{\pi}{2}$ and $-1 \le x - 2 \le 1$

$0 \le y \le \pi$ and $1 \le x \le 3$

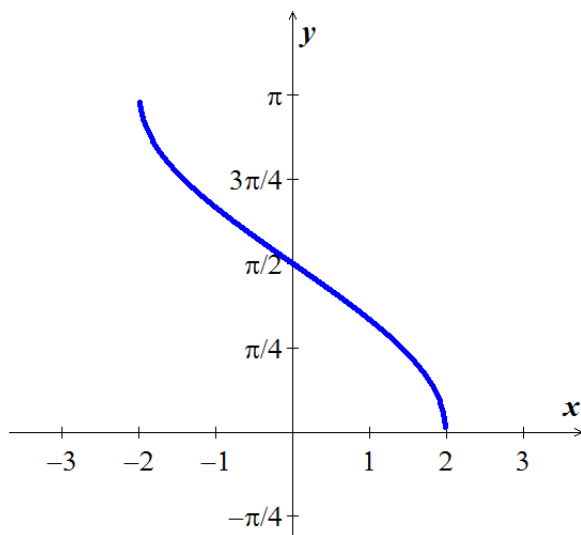## *Exercise*

Sketch he graph of the equation: $y = \cos^{-1}\frac{1}{2}x$

Solution

$0 \le y \le \pi$ and $-1 \le \frac{1}{2}x \le 1$

$-2 \le x \le 2$

Exercise

Evaluate $\sin\left(\tan^{-1}\frac{3}{4}\right)$ without using a calculator
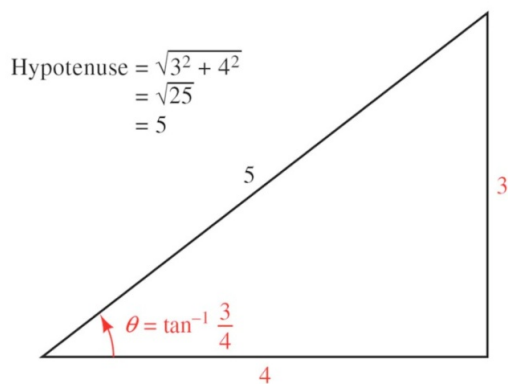
Solution

$\theta = \tan^{-1}\frac{3}{4}$

$\tan\theta = \frac{3}{4}$ $\rightarrow 0° < \theta < 90°$

$\sin\left(\tan^{-1}\frac{3}{4}\right) = \sin\theta$

$= \frac{3}{5}$

$\text{Hypotenuse} = \sqrt{3^2 + 4^2}$
$= \sqrt{25}$
$= 5$

$\theta = \tan^{-1}\frac{3}{4}$

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Exercise

Evaluate $\sin(\cos^{-1} x)$ as an equivalent expression in x only

Solution

$$\sin\theta = \frac{y}{r}$$

$$= \frac{\sqrt{1-x^2}}{1}$$

$$= \sqrt{1-x^2}$$

$$\sin(\cos^{-1} x) = \sin\theta$$

$$= \sqrt{1-x^2}$$