

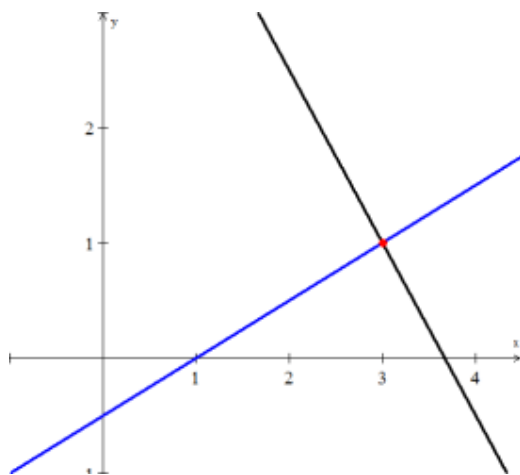
Section 1.2 – Gaussian Elimination

Elimination produces an *upper triangular system*.

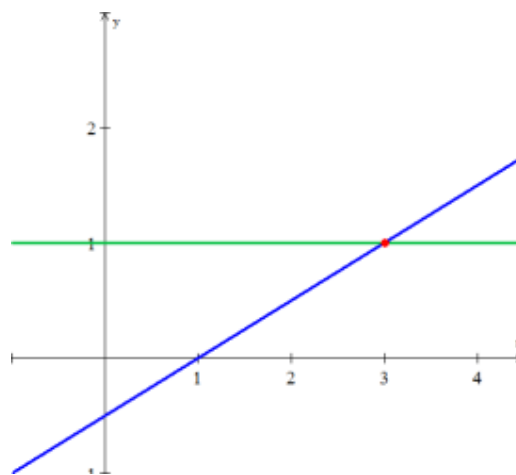
$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases} \Rightarrow \begin{cases} x - 2y = 1 & \text{Multiply by 3} \\ 8y = 8 & \text{and subtract} \end{cases}$$

The equation $8y = 8$ *reveals* $y = 1$

This process is called *back substitution*.



Before elimination



After elimination

Definitions

Pivot: first nonzero in the row that does the elimination

Multiplier: (entry to eliminate) divide by pivot

$$4x - 8y = 4 \quad \text{Multiply equation 1 by } \frac{3}{4} \quad 4x - 8y = 4$$

$$3x + 2y = 11 \quad \text{Subtract from equation 2} \quad 8y = 8$$

The first pivot is 4 (the coefficient of x) and the multiplier is $l = \frac{3}{4}$

The pivots are on the diagonal of the triangle after elimination.

Definition

The operations are the elementary reduction operations, or row operations, or Gaussian operations. They are swapping, multiplying by a scalar or rescaling, and pivoting.

Reduced Row Echelon Form

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

Use the Gaussian elimination method to solve the system

$$3x + y + 2z = 31$$

$$x + y + 2z = 19$$

$$x + 3y + 2z = 25$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 3 & 1 & 2 & 31 \\ 1 & 3 & 2 & 25 \end{array} \right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array} \quad \begin{array}{cccc} 3 & 1 & 2 & 31 \\ -3 & -3 & -6 & -57 \\ 0 & -2 & -4 & -26 \end{array} \quad \begin{array}{cccc} 1 & 2 & 2 & 25 \\ -1 & -1 & -2 & -19 \\ 0 & 2 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -2 & -4 & -26 \\ 0 & 2 & 0 & 6 \end{array} \right] -\frac{1}{2}R_2 \quad \begin{array}{cccc} 0 & 1 & 2 & 13 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 2 & 0 & 6 \end{array} \right] R_3 - 2R_2 \quad \begin{array}{cccc} 0 & 2 & 0 & 6 \\ 0 & -2 & -4 & -26 \\ 0 & 0 & -4 & -20 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -4 & -20 \end{array} \right] -\frac{1}{4}R_3 \quad \begin{array}{cccc} 0 & 0 & 1 & 5 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array} \right] \Rightarrow \begin{array}{l} x + y + 2z = 19 \quad (3) \\ y + 2z = 13 \quad (2) \\ z = 5 \quad (1) \end{array}$$

$$(2) \Rightarrow y = 13 - 2z = 13 - 2(5) = 3$$

$$(3) \Rightarrow x = 19 - y - 2z = 19 - 3 - 10 = 6$$

$$\Rightarrow (6, 3, 5)$$

Example

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$\begin{aligned}x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\5x_3 + 10x_4 + 15x_6 &= 5 \\2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 6\end{aligned}$$

Solution

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \quad \text{Adding } (-2) \text{ times the 1st row to the 2nd} \\ R_4 - 2R_1 \quad \text{Adding } (-2) \text{ times the 1st row to the 4th} \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array} \right] \quad -R_2$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array} \right] \quad \begin{array}{l} R_3 - 5R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array} \right] \quad \frac{1}{6}R_4 \quad \text{then interchanging row3 and row4}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad R_2 - 3R_3$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \rightarrow \begin{cases} x_1 + 3x_2 + 4x_4 + 2x_5 = 0 \\ x_3 + 2x_4 = 0 \\ + x_6 = \frac{1}{3} \end{cases}$$

The general solution of the system: $\left(-3x_2 - 4x_4 - 2x_5, x_2, -2x_4, x_4, x_5, \frac{1}{3} \right)$

Example

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$2x + 8y - z + w = 0$$

$$4x + 16y - 3z - w = -10$$

$$-2x + 4y - z + 3w = -6$$

$$-6x + 2y + 5z + w = 3$$

Solution

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 4 & 16 & -3 & -1 & -10 \\ -2 & 4 & -1 & 3 & -6 \\ -6 & 2 & 5 & 1 & 3 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 26 & 2 & 4 & 3 \end{array} \right] R_4 - \frac{13}{6}R_2$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 26 & 2 & 4 & 3 \end{array} \right] \text{Interchange } R_2 \text{ and } R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & \frac{19}{3} & -\frac{14}{3} & 16 \end{array} \right] R_4 + \frac{19}{3}R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & 0 & -\frac{71}{3} & -\frac{142}{3} \end{array} \right] \begin{array}{l} 2x + 8y - z + w = 0 \rightarrow 2x = -8y + z - w = 6 \Rightarrow \boxed{x=3} \\ 12y - 2z + 4w = -6 \rightarrow 12y = 2z - 4w - 6 = -6 \Rightarrow \boxed{y = -\frac{1}{2}} \\ -z - 3w = -10 \\ -\frac{71}{3}w = -\frac{142}{3} \rightarrow \boxed{z = 10 - 3w = 4} \\ \rightarrow \boxed{w = 2} \end{array}$$

$$\text{Solution: } \boxed{\left(3, -\frac{1}{2}, 4, 2 \right)}$$

Theorem: Free Variable Theorem for Homogeneous Systems

If a *homogeneous linear* system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

Theorem

A *homogeneous linear* system with more unknowns than equations has *infinitely many unknowns*.

Breakdown Elimination

Permanent failure with no solution

$$\begin{array}{lll} x - 2y = 1 & \text{Subtract 3 times} & x - 2y = 1 \\ 3x - 6y = 11 & \text{eqn. 1 from eqn. 2} & 0y = 8 \end{array}$$

The last equation $0y = 8$; therefore, there is *no* solution.

This system has no second pivot, since no zero allowed as a pivot.

Permanent failure with infinitely many solutions

$$\begin{array}{lll} x - 2y = 1 & \text{Subtract 3 times} & x - 2y = 1 \\ 3x - 6y = 3 & \text{eqn. 1 from eqn. 2} & 0y = 0 \end{array}$$

Every y satisfies $0y = 0$. There is only one equation $x - 2y = 1$.

There are *unique infinitely* many solutions.

Three Equations in Three Unknowns

To understand Gaussian elimination, you have to go beyond 2 by 2 systems.

Consider the system equations:
$$\begin{cases} 2x + 4y - 2z = 2 \\ 4x + 9y - 3z = 8 \\ -2x - 3y + 7z = 10 \end{cases}$$

$$\begin{cases} 2x + 4y - 2z = 2 & \text{subtract 2 times eqn.1} & 2x + 4y - 2z = 2 \\ 4x + 9y - 3z = 8 & \text{from eqn.2} & y + z = 4 \\ -2x - 3y + 7z = 10 & & -2x - 3y + 7z = 10 \end{cases}$$

$$\begin{cases} 2x + 4y - 2z = 2 & \text{Add eqn.1} & 2x + 4y - 2z = 2 \\ y + z = 4 & & y + z = 4 \\ -2x - 3y + 7z = 10 & \text{and eqn.3} & y + 5z = 12 \end{cases}$$

$$\begin{cases} 2x + 4y - 2z = 2 \\ y + z = 4 \\ y + 5z = 12 \end{cases} \quad \begin{array}{l} \text{Subtract eqn.2} \\ \text{from eqn.3} \end{array}$$

$$\begin{array}{ll} 2x + 4y - 2z = 2 & \Rightarrow \boxed{x = 1 - 2y + z = -1} \\ y + z = 4 & \Rightarrow \boxed{y = 4 - z = 2} \\ 4z = 8 & \Rightarrow \boxed{z = 2} \end{array}$$

The solution is $\boxed{(-1, 2, 2)}$

Definition

A square matrix is nonsingular if it is the matrix of coefficient of a homogeneous system, with a unique solution. It is singular otherwise, that is, if it is the matrix of coefficients of a homogeneous system, with infinitely many solutions.

Exercises Section 1.2 – Gaussian Elimination

- When elimination is applied to the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix}$
 - What are the first and second pivots?
 - What is the multiplier l_{21} in the first step (l_{21} times row 1 is subtracted from row 2)?
 - What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3?
 - What is the multiplier $l_{31} = 0$, subtracting 0 times row 1 from row 3?
- Use elimination to reach upper triangular matrices U . Solve by back substitution or explain why this is impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the $-x$ in equation (3).

$$\begin{cases} x + y + z = 7 \\ x + y - z = 5 \\ x - y + z = 3 \end{cases} \quad \begin{cases} x + y + z = 7 \\ x + y - z = 5 \\ -x - y + z = 3 \end{cases}$$

- For which numbers a does the elimination break down (1) permanently (2) temporarily

$$ax + 3y = -3$$

$$4x + 6y = 6$$

Solve for x and y after fixing the second breakdown by a row change.

- Find the pivots and the solution for these four equations:

$$2x + y = 0$$

$$x + 2y + z = 0$$

$$y + 2z + t = 0$$

$$z + 2t = 5$$

- Look for a matrix that has row sums 4 and 8, and column sums 2 and s .

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{array}{ll} a + b = 4 & a + c = 2 \\ c + d = 8 & b + d = s \end{array}$$

The four equations are solvable only if $s = \underline{\hspace{1cm}}$. Then find two different matrices that have the correct row and column sums.

- Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of A is a of the first two rows. Find a third equation that can't be solved together with $x + y + z = 0$ and $x - 2y - z = 1$

(7 – 14) Use the Gauss-Jordan method to solve the system

$$7. \begin{cases} x - y + 5z = -6 \\ 3x + 3y - z = 10 \\ x + 3y + 2z = 5 \end{cases}$$

$$10. \begin{cases} x + 2y - 3z = -15 \\ 2x - 3y + 4z = 18 \\ -3x + y + z = 1 \end{cases}$$

$$13. \begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$

$$8. \begin{cases} 2x - y + 4z = -3 \\ x - 2y - 10z = -6 \\ 3x + 4z = 7 \end{cases}$$

$$11. \begin{cases} x + 2y + 3z = 10 \\ 4x + 5y + 6z = 11 \\ 7x + 8y + 9z = 12 \end{cases}$$

$$14. \begin{cases} x + 2y + z = 8 \\ -x + 3y - 2z = 1 \\ 3x + 4y - 7z = 10 \end{cases}$$

$$9. \begin{cases} 4x + 3y - 5z = -29 \\ 3x - 7y - z = -19 \\ 2x + 5y + 2z = -10 \end{cases}$$

$$12. \begin{cases} 2x + y + 2z = 4 \\ 2x + 2y = 5 \\ 2x - y + 6z = 2 \end{cases}$$

(15 – 49) Use augmented elimination to solve linear system

$$15. \begin{cases} 2x - 5y + 3z = 1 \\ x - 2y - 2z = 8 \end{cases}$$

$$22. \begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$$

$$29. \begin{cases} 2x - 2y + z = -4 \\ 6x + 4y - 3z = -24 \\ x - 2y + 2z = 1 \end{cases}$$

$$16. \begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$$

$$23. \begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$$

$$30. \begin{cases} 9x + 3y + z = 4 \\ 16x + 4y + z = 2 \\ 25x + 5y + z = 2 \end{cases}$$

$$17. \begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$$

$$24. \begin{cases} 3x - 4y + 4z = 7 \\ x - y - 2z = 2 \\ 2x - 3y + 6z = 5 \end{cases}$$

$$31. \begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$$

$$18. \begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$$

$$25. \begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$$

$$32. \begin{cases} x - 3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$$

$$19. \begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$$

$$26. \begin{cases} x + y + z = 3 \\ -y + 2z = 1 \\ -x + z = 0 \end{cases}$$

$$33. \begin{cases} x + 2y - z = 5 \\ 2x - y + 3z = 0 \\ 2y + z = 1 \end{cases}$$

$$20. \begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$$

$$27. \begin{cases} 3x + y + 3z = 14 \\ 7x + 5y + 8z = 37 \\ x + 3y + 2z = 9 \end{cases}$$

$$34. \begin{cases} x + y + z = 6 \\ 3x + 4y - 7z = 1 \\ 2x - y + 3z = 5 \end{cases}$$

$$21. \begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$$

$$28. \begin{cases} 4x - 2y + z = 7 \\ x + y + z = -2 \\ 4x + 2y + z = 3 \end{cases}$$

$$35. \begin{cases} 3x + 2y + 3z = 3 \\ 4x - 5y + 7z = 1 \\ 2x + 3y - 2z = 6 \end{cases}$$

$$36. \begin{cases} x - 3y + z = 2 \\ 4x - 12y + 4z = 8 \\ -2x + 6y - 2z = -4 \end{cases}$$

$$37. \begin{cases} 2x - 2y + z = -1 \\ x + 2y - z = 2 \\ 6x + 4y + 3z = 5 \end{cases}$$

$$38. \begin{cases} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 \\ x_2 - 3x_3 - x_4 = 0 \\ 3x_1 + 2x_3 - x_4 = 6 \\ -4x_1 + x_2 + 4x_3 + 2x_4 = -3 \end{cases}$$

$$39. \begin{cases} x_1 + x_2 + x_3 + x_4 = 5 \\ x_1 + 2x_2 - x_3 - 2x_4 = -1 \\ x_1 - 3x_2 - 3x_3 - x_4 = -1 \\ 2x_1 - x_2 + 2x_3 - x_4 = -2 \end{cases}$$

$$40. \begin{cases} 2x + 8y - z + w = 0 \\ 4x + 16y - 3z - w = -10 \\ -2x + 4y - z + 3w = -6 \\ -6x + 2y + 5z + w = 3 \end{cases}$$

$$41. \begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

$$42. \begin{cases} 2x + 2y + 4z = 0 \\ -y - 3z + w = 0 \\ 3x + y + z + 2w = 0 \\ x + 3y - 2z - 2w = 0 \end{cases}$$

$$43. \begin{cases} 2x + z + w = 5 \\ y - w = -1 \\ 3x - z - w = 0 \\ 4x + y + 2z + w = 9 \end{cases}$$

$$44. \begin{cases} 4y + z = 20 \\ 2x - 2y + z = 0 \\ x + z = 5 \\ x + y - z = 10 \end{cases}$$

$$45. \begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x - 3w = -3 \end{cases}$$

$$46. \begin{cases} 2u - 3v + w - x + y = 0 \\ 4u - 6v + 2w - 3x - y = -5 \\ -2u + 3v - 2w + 2x - y = 3 \end{cases}$$

$$47. \begin{cases} 6x_3 + 2x_4 - 4x_5 - 8x_6 = 8 \\ 3x_3 + x_4 - 2x_5 - 4x_6 = 4 \\ 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 + x_6 = 2 \\ 6x_1 - 9x_2 + 11x_4 - 19x_5 + 3x_6 = 1 \end{cases}$$

$$48. \begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases}$$

$$49. \begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ 5x_3 + 10x_4 + 15x_6 = 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$$

50. Add 3 times the second row to the first of

$$\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

51. For what value(s) of k , if any, does the system $\begin{cases} x + y - z = 1 \\ 2x + 3y + kz = 3 \\ x + ky + 3z = 2 \end{cases}$ have

- a) A unique solution?
- b) Infinitely many solutions?
- c) No solution?

52. Choose a coefficient b that makes the system singular.

$$\begin{cases} 3x + 4y = 16 \\ 4x + by = g \end{cases}$$

Then choose a right-hand side g that makes it solvable.

Find 2 solutions in that singular case.

53. This system is not linear, in some sense,

$$\begin{cases} 2\sin \alpha - \cos \beta + 3\tan \theta = 3 \\ 4\sin \alpha + 2\cos \beta - 2\tan \theta = 10 \\ 6\sin \alpha - 3\cos \beta + \tan \theta = 9 \end{cases}$$

Does the system have a solution?