

## ***Solution***      **Section 4.5 – Working with Integrals**

### ***Exercise***

If  $f$  is an odd function, why is  $\int_{-a}^a f(x) dx = 0$ ?

### **Solution**

If  $f(x)$  is an odd function then it is symmetric about the origin, which the region between  $-a$  and  $a$ , there is as much area above the axis and under  $f$  as there is below the axis and above  $f$ . Therefore, the net area must be 0.

### ***Exercise***

If  $f$  is an even function, why is  $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

### **Solution**

If  $f$  is an even function then it is symmetric about the  $y$ -axis, which the region that between  $-a$  and 0 has the same net area as the region between 0 and  $a$ .

$$\begin{aligned}\text{So } \int_{-a}^a f(x) dx &= \int_{-a}^0 f(x) dx + \int_0^a f(x) dx \\ &= 2 \int_0^a f(x) dx\end{aligned}$$

### ***Exercise***

Is  $x^{12}$  an even or odd function? Is  $\sin(x^2)$  an even or odd function?

### **Solution**

$$\begin{aligned}f(x) &= x^{12} \\ f(-x) &= (-x)^{12} \\ &= x^{12} \\ &= x^{12} = f(x)\end{aligned}$$

Therefore;  $f(x)$  is an *even* function.

$$g(x) = \sin(x^2)$$

$$\begin{aligned}
 g(-x) &= \sin((-x)^2) \\
 &= \sin(x^2) \\
 &= g(x)
 \end{aligned}$$

Therefore;  $g(x)$  is also an even function.

### Exercise

Use symmetry to evaluate the following integrals  $\int_{-2}^2 x^9 dx$

### Solution

Because  $x^9$  is an *odd* function, then

$$\int_{-2}^2 x^9 dx = 0$$

### Exercise

Use symmetry to evaluate the following integrals  $\int_{-200}^{200} 2x^5 dx$

### Solution

Because  $2x^5$  is an *odd* function, then

$$\int_{-200}^{200} 2x^5 dx = 0$$

### Exercise

Use symmetry to evaluate the following integrals  $\int_{-\pi/4}^{\pi/4} \cos x dx$

### Solution

Because  $\cos x$  is an even function, then

$$\begin{aligned}
 \int_{-\pi/4}^{\pi/4} \cos x dx &= 2 \int_0^{\pi/4} \cos x dx \\
 &= 2 \sin x \Big|_0^{\pi/4}
 \end{aligned}$$

$$= 2 \left( \frac{\sqrt{2}}{2} \right)$$

$$= \sqrt{2} \quad |$$

### Exercise

Use symmetry to evaluate the following integrals  $\int_{-2}^2 (x^9 - 3x^5 + 2x^2 - 10) dx$

### Solution

$$\int_{-2}^2 (x^9 - 3x^5 + 2x^2 - 10) dx = \overset{\text{Odd}}{\int_{-2}^2 (x^9 - 3x^5) dx} + \overset{\text{Even}}{\int_{-2}^2 (2x^2 - 10) dx}$$

$$= 0 + 2 \int_0^2 (2x^2 - 10) dx$$

$$= 2 \left( \frac{2}{3} x^3 - 10x \right) \bigg|_0^2$$

$$= 2 \left( \frac{16}{3} - 20 \right)$$

$$= -\frac{88}{3} \quad |$$

### Exercise

Use symmetry to evaluate the following integrals  $\int_{-\pi/2}^{\pi/2} (\cos 2x + \cos x \sin x - 3 \sin x^5) dx$

### Solution

$$\int_{-\pi/2}^{\pi/2} (\cos 2x + \cos x \sin x - 3 \sin x^5) dx = \overset{\text{Even}}{\int_{-\pi/2}^{\pi/2} \cos 2x dx} + \overset{\text{Odd}}{\int_{-\pi/2}^{\pi/2} (\cos x \sin x - 3 \sin x^5) dx}$$

$$= 2 \int_0^{\pi/2} \cos 2x dx$$

$$= \sin 2x \bigg|_0^{\pi/2}$$

$$= 0 \quad |$$

### Exercise

Find the average value of the following functions on the given interval.  $f(x) = x^3$  on  $[-1, 1]$

### Solution

$$\begin{aligned}\text{Average value} &= \frac{1}{1 - (-1)} \int_{-1}^1 x^3 dx \\ &= \frac{1}{2} \frac{1}{4} x^4 \Big|_{-1}^1 \\ &= 0\end{aligned}$$

### Exercise

Find the average value of the following functions on the given interval.  $f(x) = \frac{1}{x^2 + 1}$  on  $[-1, 1]$

### Solution

$$\begin{aligned}\text{Average value} &= \frac{1}{1 - (-1)} \int_{-1}^1 \frac{1}{x^2 + 1} dx \\ &= \frac{1}{2} \tan^{-1} x \Big|_{-1}^1 \\ &= \frac{1}{2} \left( \frac{\pi}{4} + \frac{\pi}{4} \right) \\ &= \frac{\pi}{4}\end{aligned}$$

### Exercise

Find the average value of the following functions on the given interval.  $f(x) = \frac{1}{x}$  on  $[1, e]$

### Solution

$$\begin{aligned}\text{Average value} &= \frac{1}{e - 1} \int_1^e \frac{1}{x} dx \\ &= \frac{1}{e - 1} (\ln|x|) \Big|_1^e \\ &= \frac{1}{e - 1} (\ln e - 0) \\ &= \frac{1}{e - 1}\end{aligned}$$

### Exercise

Find the average value of the following functions on the given interval.  $f(x) = e^{2x}$  on  $[0, \ln 2]$

### Solution

$$\begin{aligned}\text{Average value} &= \frac{1}{\ln 2} \int_0^{\ln 2} e^{2x} dx \\&= \frac{1}{2} \frac{1}{\ln 2} e^{2x} \Big|_0^{\ln 2} \\&= \frac{1}{2 \ln 2} (e^{2 \ln 2} - 1) \\&= \frac{1}{2 \ln 2} (4 - 1) \\&= \frac{3}{2 \ln 2}\end{aligned}$$

### Exercise

Suppose that  $\int_0^4 f(x) dx = 10$  and  $\int_0^4 g(x) dx = 20$ . Furthermore, suppose that  $f$  is an even function and  $g$  is an odd function. Evaluate the integral

$$\int_{-4}^4 f(x) dx$$

### Solution

$f$  is an even function.

$$\begin{aligned}\int_{-4}^4 f(x) dx &= 2 \int_0^4 f(x) dx \\&= 2(10) \\&= 20\end{aligned}$$

### Exercise

Suppose that  $\int_0^4 f(x)dx = 10$  and  $\int_0^4 g(x)dx = 20$ . Furthermore, suppose that  $f$  is an even function and  $g$  is an odd function. Evaluate the integral

$$\int_{-4}^4 3g(x)dx$$

### Solution

$g$  is an odd function

$$\int_0^4 g(x)dx = -\int_{-4}^0 g(x)dx$$

$$\begin{aligned}\int_{-4}^4 3g(x)dx &= 3\int_{-4}^0 g(x)dx + 3\int_0^4 g(x)dx \\ &= 3\int_{-4}^0 g(x)dx - 3\int_{-4}^0 g(x)dx \\ &= 0\end{aligned}$$

### Exercise

Suppose that  $\int_0^4 f(x)dx = 10$  and  $\int_0^4 g(x)dx = 20$ . Furthermore, suppose that  $f$  is an even function and  $g$  is an odd function. Evaluate the integral

$$\int_0^1 8xf(4x^2)dx$$

### Solution

$$\int_0^1 8xf(4x^2)dx = \int_0^1 f(4x^2)d(4x^2) \qquad d(4x^2) = 8xdx$$

$$\begin{cases} x=1 & \rightarrow 4x^2 = 4 \\ x=0 & \rightarrow 4x^2 = 0 \end{cases}$$

$$\begin{aligned}\int_0^1 8xf(4x^2)dx &= \int_0^4 f(4x^2)d(4x^2) \\ &= 10\end{aligned}$$

### Exercise

Suppose that  $\int_0^4 f(x)dx = 10$  and  $\int_0^4 g(x)dx = 20$ . Furthermore, suppose that  $f$  is an even function and  $g$  is an odd function. Evaluate the integral

$$\int_{-2}^2 3xf(x)dx$$

### Solution

$f$  is an even function, that implies  $xf(x)$  is an odd function.

$$\int_{-2}^2 3xf(x)dx = 0$$

### Exercise

Suppose that  $\int_0^4 f(x)dx = 10$  and  $\int_0^4 g(x)dx = 20$ . Furthermore, suppose that  $f$  is an even function and  $g$  is an odd function. Evaluate the integral

$$\int_{-4}^4 (4f(x) - 3g(x))dx$$

### Solution

$f$  is an even function and  $g$  is an odd function.

$$\begin{aligned} \int_{-4}^4 (4f(x) - 3g(x))dx &= 4 \int_{-4}^4 f(x)dx - 3 \int_{-4}^4 g(x)dx \\ &= 8 \int_0^4 f(x)dx - 3(0) \\ &= 80 \end{aligned}$$

### Exercise

Suppose that  $f$  is an even function with  $\int_0^8 f(x)dx = 9$ . Evaluate the integral  $\int_{-1}^1 xf(x^2)dx$

### Solution

$f$  is an even function, that implies  $xf(x)$  is an odd function.

$$\int_{-1}^1 x f(x^2) dx = 0$$

### Exercise

Suppose that  $f$  is an even function with  $\int_0^8 f(x) dx = 9$ . Evaluate the integral  $\int_{-2}^2 x^2 f(x^3) dx$

### Solution

$$d(x^3) = 3x^2 dx$$

$$\begin{cases} x = 2 & \rightarrow x^3 = 8 \\ x = -2 & \rightarrow x^3 = -8 \end{cases}$$

$$\begin{aligned} \int_{-2}^2 x^2 f(x^3) dx &= \frac{1}{3} \int_{-8}^8 f(x^3) d(x^3) \\ &= \frac{1}{3} \int_{-8}^0 f(x^3) d(x^3) + \frac{1}{3} \int_0^8 f(x^3) d(x^3) \quad f \text{ is an even function} \\ &= \frac{2}{3} \int_0^8 f(x^3) d(x^3) \\ &= \frac{2}{3} \cdot 9 \\ &= 6 \end{aligned}$$

### Exercise

Suppose that  $p$  is a nonzero real number and  $f$  is an odd integrable function with  $\int_0^1 f(x) dx = \pi$ .

Evaluate the integral  $\int_0^{\frac{\pi}{2p}} (\cos px) f(\sin px) dx$

### Solution

$$\begin{cases} x = \frac{\pi}{2p} & \rightarrow \sin px = 1 \\ x = 0 & \rightarrow \sin px = 0 \end{cases}$$

$$\int_0^{\frac{\pi}{2p}} (\cos px) f(\sin px) dx = \frac{1}{p} \int_0^1 f(\sin px) d(\sin px) \quad d(\sin px) = p \cos px dx$$



$$= \frac{\pi}{p} \Big|$$

### Exercise

Suppose that  $p$  is a nonzero real number and  $f$  is an odd integrable function with  $\int_0^1 f(x) dx = \pi$ .

Evaluate the integral  $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x) f(\sin x) dx$

### Solution

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos px) f(\sin px) dx = \frac{1}{p} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\sin px) d(\sin px) \quad f \text{ is an odd function}$$

$$= 0 \Big|$$

### Exercise

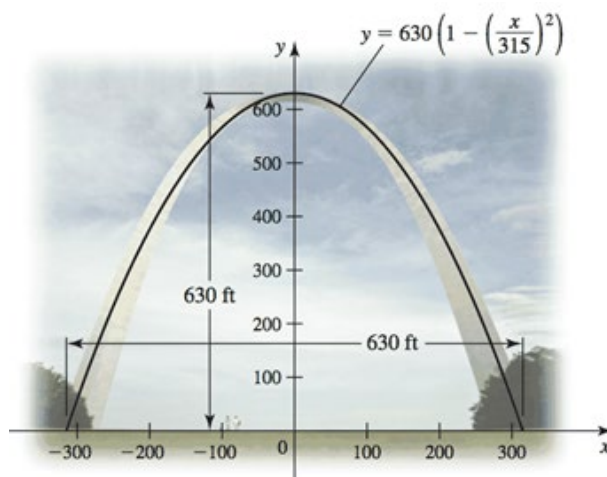
The Gateway Arch in St. Louis is 630 feet high and has a 630-ft base. Its shape can be modeled by the parabola

$$y = 630 \left( 1 - \left( \frac{x}{315} \right)^2 \right)$$

Find the average height of the arch above the ground.

### Solution

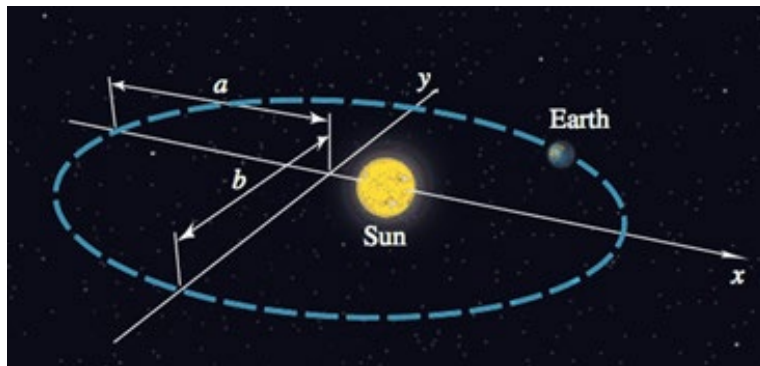
$$\begin{aligned} \text{Average height} &= \frac{1}{630} \int_{-315}^{315} 630 \left( 1 - \frac{1}{315^2} x^2 \right) dx \\ &= x - \frac{1}{315^2} \frac{x^3}{3} \Big|_{-315}^{315} \\ &= 315 - 105 + 315 - 105 \\ &= 420 \text{ ft} \Big| \end{aligned}$$



## Exercise

The planets orbit the Sun in elliptical orbits with the Sun at one focus. The equation of an ellipse whose dimensions are  $2a$  in the  $x$ -direction and  $2b$  in the  $y$ -direction is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



- a) Let  $d^2$  denote the square of the distance from a planet to the center of the ellipse at  $(0, 0)$ . Integrate over the interval  $[-a, a]$  to show that the average value of  $d^2$  is  $\frac{a^2 + 2b^2}{3}$ .
- b) Show that in the case of a circle ( $a = b = R$ ), the average value in part (a) is  $R^2$ .
- c) Assuming  $0 < b < a$ , the coordinates of the Sun are  $(\sqrt{a^2 - b^2}, 0)$ . Let  $D^2$  denote the square of the distance from the planet to the Sun. Integrate over the interval  $[-a, a]$  to show that the average value of  $D^2$  is  $\frac{4a^2 - b^2}{3}$ .

## Solution

$$a) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2 - \frac{b^2}{a^2}x^2$$

$$d^2 = x^2 + y^2 = x^2 + b^2 - \frac{b^2}{a^2}x^2$$

$$= b^2 + \left(1 - \frac{b^2}{a^2}\right)x^2$$

$$\text{The average value of } d^2 = \frac{1}{2a} \int_{-a}^a \left( b^2 + \left(1 - \frac{b^2}{a^2}\right)x^2 \right) dx$$

$$\begin{aligned}
&= \frac{1}{2a} \left( b^2 x + \frac{1}{3} \left( 1 - \frac{b^2}{a^2} \right) x^3 \right) \Big|_{-a}^a \\
&= \frac{1}{2a} \left( ab^2 + \frac{1}{3} \left( 1 - \frac{b^2}{a^2} \right) a^3 + ab^2 + \frac{1}{3} \left( 1 - \frac{b^2}{a^2} \right) a^3 \right) \\
&= \frac{1}{2a} \left( 2ab^2 + \frac{2}{3} a^3 - \frac{2}{3} b^2 a \right) \\
&= \frac{\frac{2}{3} b^2 + \frac{a^2}{3}}{1}
\end{aligned}$$

**b)** If  $a = b = R$

$$\begin{aligned}
\text{The average value of } d^2 &= \frac{2R^2}{3} + \frac{R^2}{3} \\
&= R^2
\end{aligned}$$

$$\begin{aligned}
c) \quad D^2 &= \left( x - \sqrt{a^2 - b^2} \right)^2 + y^2 \\
&= x^2 - 2x\sqrt{a^2 - b^2} + a^2 - b^2 + b^2 - \frac{b^2}{a^2} x^2 \\
&= \left( 1 - \frac{b^2}{a^2} \right) x^2 - 2x\sqrt{a^2 - b^2} + a^2
\end{aligned}$$

$$\begin{aligned}
\text{The average value of } D^2 &= \frac{1}{2a} \int_{-a}^a \left( \left( 1 - \frac{b^2}{a^2} \right) x^2 - 2x\sqrt{a^2 - b^2} + a^2 \right) dx \\
&= \frac{1}{2a} \left( \frac{1}{3} \left( 1 - \frac{b^2}{a^2} \right) x^3 - x^2 \sqrt{a^2 - b^2} + a^2 x \right) \Big|_{-a}^a \\
&= \frac{1}{2a} \left( \frac{1}{3} \left( 1 - \frac{b^2}{a^2} \right) a^3 - a^2 \sqrt{a^2 - b^2} + a^3 + \frac{1}{3} \left( 1 - \frac{b^2}{a^2} \right) a^3 + a^2 \sqrt{a^2 - b^2} + a^3 \right) \\
&= \frac{1}{3} \left( 1 - \frac{b^2}{a^2} \right) a^2 + a^2 \\
&= \frac{1}{3} a^2 - \frac{1}{3} b^2 + a^2 \\
&= \frac{\frac{4}{3} a^2 - \frac{1}{3} b^2}{1}
\end{aligned}$$

### Exercise

A particle moves along a line with a velocity given by  $v(t) = 5 \sin \pi t$  starting with an initial position  $s(0) = 0$ . Find the displacement of the particle between  $t = 0$  and  $t = 2$ , which is given by

$$s(t) = \int_0^2 v(t) dt. \text{ Find the distance traveled by the particle during this interval, which is } \int_0^2 |v(t)| dt.$$

### Solution

$$s(t) = \int_0^2 5 \sin \pi t \, dt$$

$$= -\frac{5}{\pi} \cos \pi t \Big|_0^2$$

$$= -\frac{5}{\pi} (1 - 1)$$

$$= 0$$

$$s(t) = \int_0^2 |5 \sin \pi t| \, dt$$

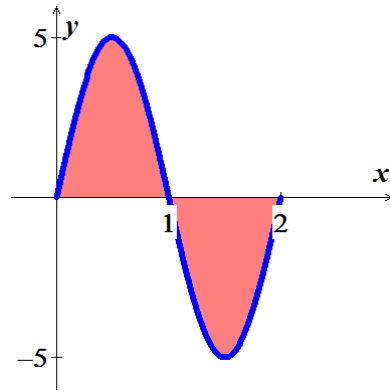
$$= 5 \int_0^1 \sin \pi t \, dt + 5 \int_1^2 (-\sin \pi t) \, dt$$

$$= -\frac{5}{\pi} \cos \pi t \Big|_0^1 + \frac{5}{\pi} \cos \pi t \Big|_1^2$$

$$= -\frac{5}{\pi} (-1 - 1) + \frac{5}{\pi} (1 + 1)$$

$$= \frac{10}{\pi} + \frac{10}{\pi}$$

$$= \frac{20}{\pi}$$



### Exercise

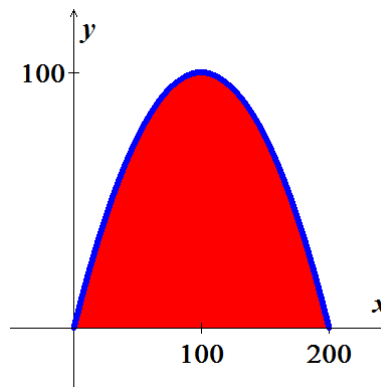
A baseball is launched into the outfield on a parabolic trajectory given by  $y = 0.01x(200 - x)$ . Find the average height of the baseball over the horizontal extent of its flight.

### Solution

$$y = 0.01x(200 - x) = 0 \rightarrow x = 0, 200$$

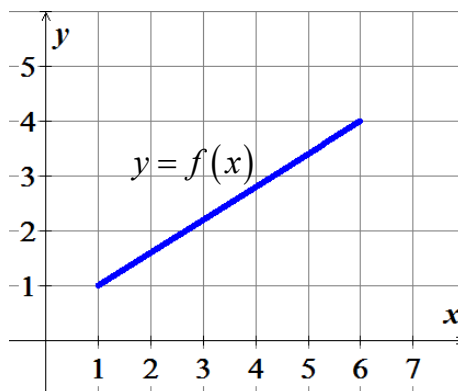
$$Avg = \frac{1}{200} \int_0^{200} (2x - 0.01x^2) \, dx$$

$$\begin{aligned}
 &= \frac{1}{200} \left( x^2 - \frac{1}{300} x^3 \right) \Bigg|_0^{200} \\
 &= \frac{1}{200} \left( 4 \times 10^4 - \frac{1}{300} 8 \times 10^6 \right) \\
 &= \frac{4 \times 10^4}{200} \left( 1 - \frac{2}{3} \right) \\
 &= \frac{200}{3}
 \end{aligned}$$



### Exercise

Find the average value of  $f$  shown in the figure on the interval  $[1, 6]$  and then find the point(s)  $c$  in  $(1, 6)$  guaranteed to exist by the Mean Value Theorem for Integrals



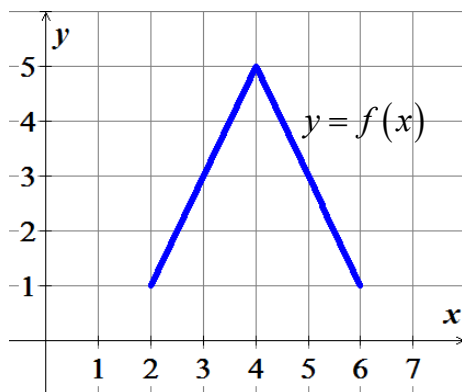
### Solution

Since it is a straight line, then the average value is 2.5

The average value occurs at the midpoint of the interval which is  $(3.5, 2.5)$

### Exercise

Find the average value of  $f$  shown in the figure on the interval  $[2, 6]$  and then find the point(s)  $c$  in  $(2, 6)$  guaranteed to exist by the Mean Value Theorem for Integrals



**Solution**

Over interval  $[2, 4]$ ; it is a straight line, then the average value is 3

Over interval  $[4, 6]$ ; it is a straight line, then the average value is 3 .

Therefore; the overage is 3 over  $[2, 6]$