Solution   *Section* 3.3 – Applied Optimization

Exercise

Find two nonnegative numbers x and y for which $2x+y=30$, such that xy^2 is maximized.

Solution

$$2x+y=30$$
$$y=30-2x$$

$$\begin{aligned} M &= xy^2 \\ &= x(30-2x)^2 \\ &= x\left(900-120x+4x^2\right) \\ &= 900x-120x^2+4x^3 \end{aligned}$$

$$M' = 900-240x+12x^2$$

$$x=5 \quad \Rightarrow y=30-2(5)=20$$
$$x=15 \quad \Rightarrow y=30-2(15)=0$$

$(0,\ 0)$ $\quad M=xy^2=0\left(0^2\right)=0$

$(5,\ 20)$ $\quad M=xy^2=5\left(20^2\right)=2000$

$(15,\ 0)$ $\quad M=xy^2=15\left(0^2\right)=0$

The values that maximize xy^2 are $\boxed{x=5 \quad and \quad y=20}$

Exercise

A rectangular page will contain 54 in^2 of print. The margins at the top and bottom of the page are 1.5 *inches* wide. The margins on each side are 1 *inch* wide. What should the dimensions of the page be to minimize the amount of paper used?

Solution

$$xy=54 \qquad y=\frac{54}{x}$$

$$\begin{aligned} A &= (x+3)(y+2) \\ &= (x+3)\left(\frac{54}{x}+2\right) \\ &= 54+2x+\frac{162}{x}+6 \end{aligned}$$

$$= 60 + 2x + \frac{162}{x}$$

$$\frac{dA}{dx} = A' = 2 - \frac{162}{x^2}$$

$$2 - \frac{162}{x^2} = 0$$

$$-\frac{162}{x^2} = -2$$

$$162 = 2x^2$$

$$x^2 = 81$$

$$\rightarrow x = \pm 9 \quad \Rightarrow x = 9 \ (only)$$

$$\rightarrow x + 3 = 12$$

$$y = \frac{54}{9} = 6$$

$$\rightarrow y + 2 = 8$$

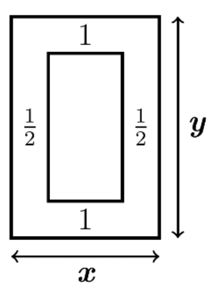
Dimension: 8 by 12 *in.*

Exercise

A rectangular page in a textbook (with width x and length y) has an area of $98\ in^2$, top and bottom margins set at 1 *in.*, left and right margins set at $\frac{1}{2}$ *in.* The printable area of the page is the rectangle that lies within the margins. What are the dimensions of the page that maximize the printable area?

Solution

$$xy = 98\ in^2 \qquad x = \frac{98}{y}$$



$$A = (x-1)(y-2)$$

$$= \left(\frac{98}{y} - 1\right)(y-2)$$

$$= 98 - 196\frac{1}{y} - y + 2$$

$$= 100 - 196\frac{1}{y} - y$$

$$A'(y) = 196\frac{1}{y^2} - 1 = 0$$

$$y^2 = 196 \quad \rightarrow \quad \boxed{y = 14}$$

$$x = \frac{98}{14}\ \boxed{= 7}$$

Dimension: 7 by 14

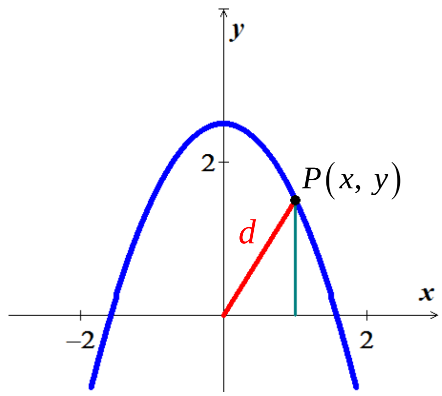
Exercise

What point on the graph of $f(x)=\frac{5}{2}-x^2$ is closest to the origin? (*Hint*: You can minimize the square of the distance.)

Solution

$$f(x)=\frac{5}{2}-x^2$$

Let $P(x,\ y)=P\left(x,\ \frac{5}{2}-x^2\right)$



$$d^2=x^2+\left(\frac{5}{2}-x^2\right)^2=g(x)$$

$$g(x)=x^2+\frac{25}{4}-5x^2+x^4$$

$$=x^4-4x^2+\frac{25}{4}$$

$$g'(x)=4x^3-8x=0$$

$$4x\left(x^2-2\right)=0\quad\rightarrow\quad \underline{x=0,\ \pm\sqrt{2}}\,|$$

$$y=f(0)\ \underline{=\frac{5}{2}}\,|$$

$$y=f\left(\pm\sqrt{2}\right)=\frac{5}{2}-2\ \underline{=\frac{1}{2}}\,|$$

The points closest to the origin on the graph are $\left(\pm\sqrt{2},\ \frac{1}{2}\right)$

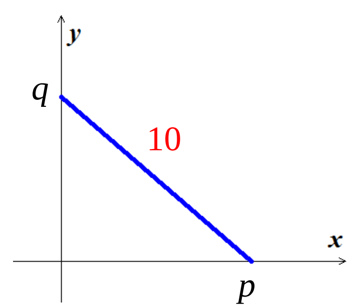
Exercise

A line segment of length 10 joins the points $(0,\ p)$ and $(q,\ 0)$ to form a triangle in the first quadrant. Find the values of p and q that maximize the area of the triangle.

Solution

$$\ell=10$$

$$p^2+q^2=100\quad\Rightarrow\quad p=\sqrt{100-q^2}$$



$$A=\frac{1}{2}pq$$

$$=\frac{1}{2}q\sqrt{100-q^2}$$

$$A'(q)=\frac{1}{2}\sqrt{100-q^2}-\frac{1}{2}q^2\left(100-q^2\right)^{-1/2}$$

$$=\frac{100-q^2-q^2}{2\sqrt{100-q^2}}$$

$$= \frac{50-q^2}{\sqrt{100-q^2}} = 0 \quad \left(q \neq \pm 10\right)$$

$CN: \quad \boxed{q = 5\sqrt{2},\ 10}$

$p = \sqrt{100-50} \ \boxed{= 5\sqrt{2}}$

Therefore, the area of the triangle is maximized when $p = q = 5\sqrt{2}$

Exercise

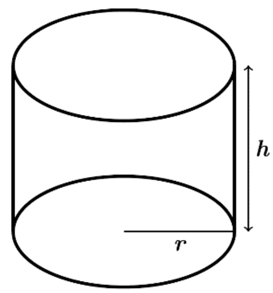
A metal cistern in the shape of a right circular cylinder with volume $V = 50\ m^3$ needs to be painted each year to reduce corrosion. The paint is applied only to surfaces exposed to the elements (the outside cylinder wall and the circular top). Find the dimension r and h of the cylinder that minimize the area of the painted surfaces.

Solution

$V = \pi r^2 h = 50\ m^3$

$h = \frac{50}{\pi r^2}$

Surface:



$$A = 2\pi r h + \pi r^2$$

$$= 2\pi r \frac{50}{\pi r^2} + \pi r^2$$

$$= \frac{100}{r} + \pi r^2$$

$$A'(r) = -\frac{100}{r^2} + 2\pi r = 0$$

$$\frac{2\pi r^3 - 100}{r^2} = 0 \quad r \neq 0$$

$$\boxed{r = \sqrt[3]{\frac{50}{\pi}}}$$

$$h = \frac{50}{\pi}\left(\frac{\pi}{50}\right)^{2/3} \boxed{= \sqrt[3]{\frac{50}{\pi}}}$$

Exercise

The product of two numbers is 72. Minimize the sum of the second number and twice the first number

Solution

$xy = 72 \qquad \Rightarrow y = \frac{72}{x}$

$S = 2x + y$

$$S = 2x + \frac{72}{x}$$

$$\frac{dS}{dx} = S' = 2 - \frac{72}{x^2}$$

$$2 - \frac{72}{x^2} = 0$$

$$-\frac{72}{x^2} = -2$$

$$72 = 2x^2$$

$$\Rightarrow x^2 = 36$$

$$x = \pm 6 \rightarrow x = 6$$

$$y = \frac{72}{x} = \frac{72}{6} = 12$$

Dimension: 6 by 12

Exercise

Verify the function $V = 27x - \frac{1}{4}x^3$ has an absolute maximum when $x = 6$. What is the maximum volume?

Solution

$$V' = 27 - \frac{3}{4}x^2 = 0$$

$$-\frac{3}{4}x^2 = -27$$

$$\Rightarrow x^2 = 27\frac{4}{3} = 36 \;\rightarrow x = \pm 6 \qquad \Rightarrow \boxed{x = 6}\ (only)$$

$$27x - \frac{1}{4}x^3 = 0$$

$$108x - x^3 = 0$$

$$x(108 - x^2) = 0$$

$$\Rightarrow x = 0,\ \sqrt{108}$$

$$V(6) = 27(6) - \frac{1}{4}(6)^3 = 108\ \ unit^3 \text{ is the maximum volume}$$

Exercise

A net enclosure for golf practice is open at one end. The volume of the enclosure is $83\frac{1}{3}$ cubic meters. Find the dimensions that require the least amount of netting.

Solution

$$V = x^2 y = 83\frac{1}{3} = \frac{250}{3}$$

$$y = \frac{250}{3x^2}$$

$$S = x^2 + 3xy$$

$$= x^2 + 3x\frac{250}{3x^2}$$

$$= x^2 + \frac{250}{x}$$

$$= x^2 + 250x^{-1}$$

$$S' = 2x - 250x^{-2} = 2x - \frac{250}{x^2}$$

$$S' = 2x - \frac{250}{x^2} = 0$$

$$x^2\left(2x - \frac{250}{x^2}\right) = 0.x^2$$

$$\Rightarrow 2x^3 - 250 = 0$$

$$2x^3 = 250$$

$$\rightarrow x = 5m$$

$$y = \frac{250}{3x^2}$$

$$= \frac{250}{3(5)^2}$$

$$\approx 3.33m$$

Dimension: 5 by 3.33 *m*

Exercise

Find two numbers x and y such that their sum is 480 and x^2y is maximized.

Solution

Given: $x + y = 480; \Rightarrow y = 480 - x$

$$f(x) = x^2y = x^2(480 - x) = 480x^2 - x^3$$

$$f'(x) = 960x - 3x^2 = 0$$

$$\Rightarrow 3x(320 - x) = 0$$

$$\Rightarrow x = 0, x = 320$$

$$\Rightarrow y = 480, y = 160$$

$x = 0,\ y = 320$ actually gives a minimum for the function, while the solution that maximizes the function is **$x = 320,\ y = 160$**.

Exercise

If the price charged for a candy bar is $p(x)$ cents, then x thousand candy bars will be sold in a certain city, where $p(x)=82-\frac{x}{20}$. How many candy bars must be sold to maximize revenue?

Solution

Recall that: $R=xp=82x-\frac{x^2}{20}$

$$R'(x)=82-\frac{x}{10}=0$$

$$\Rightarrow x=820\ (thousand)$$

x = \$820,000 candy bars give a maximum

Exercise

$S(x)=-x^3+6x^2+288x+4000;\ 4\le x\le 20$ is an approximation to the number of salmon swimming upstream to spawn, where x represents the water temperature in degrees Celsius. Find the temperature that produces the maximum number of salmon.

Solution

$$S'(x)=-3x^2+12x+288=0$$

$$-3(x-12)(x+8)=0 \quad \Rightarrow \quad \underline{x=12,\ \ x=-8}$$

Temperature that maximizes the number of salmon is 12 ℃.

Exercise

A company wishes to manufacture a box with a volume of 52 cubic feet that is open on top and is twice as long as it is wide. Find the width of the box that can be produced using the minimum amount of material.

Solution

Given: $l=2w$, $V=lwh=52$

Substituting: $(2w)wh=52, \Rightarrow 2w^2h=52 \ \Rightarrow h=\frac{26}{w^2}$

Surface Area (2 long sides, 2 short sides, bottom): $A=2lh+2wh+lw$

$$A=2(2w)\left(\frac{26}{w^2}\right)+2w\left(\frac{26}{w^2}\right)+(2w)w$$

$$A=\frac{104}{w}+\frac{52}{w}+2w^2=156w^{-1}+2w^2$$

$$A'(w)=-156w^{-2}+4w=0$$

$$\left(-156w^{-2}+4w\right)w^2=0(w^2)$$

$$\Rightarrow -156+4w^3=0$$

$$-4(39-w^3)=0 \quad \rightarrow \quad \underline{w=\sqrt[3]{39}\ ft}$$

Exercise

A rectangular field is to be enclosed on four sides with a fence. Fencing costs \$3 per *foot* for two opposite sides, and \$4 per *foot* for the other two sides. Find the dimensions of the field of area 730 *square feet* that would be the cheapest to enclose.

Solution

$$A=lw=730 \Rightarrow l=\frac{730}{w}$$

It makes the most sense to let the short sides (w) cost \$4 per foot.

$$C=2l(3)+2w(4)$$

$$C=6\frac{730}{w}+8w$$

$$=4380w^{-1}+8w$$

$$C'(w)=-4380w^{-2}+8=0$$

$$w^2\left(-4380w^{-2}+8\right)=w^2(0)$$

$$-4380+8w^2=0$$

$$8w^2=4380 \Rightarrow w^2=\frac{4380}{8}=547.5$$

$$\underline{w=\sqrt{547.5}\approx 23.4\ ft} \quad @\ \ \$4\ \text{ per } ft$$

$$l=\frac{730}{23.4}\approx \underline{31.2\ ft} \quad @\ \ \$3\ \text{ per } ft$$

Exercise

A manufacturer wants to design an open box that has a square base and a surface area of 108 in^2. What dimensions will produce a box with a maximum volume?

Solution

$$V=x^2h$$

Surface Area (S) = area of base + 4 (area of each side)

$$108=x^2+4xh$$

$$108-x^2=4xh$$

$$h=\frac{108-x^2}{4x}$$

$$V = x^2 h$$
$$= x^2 \frac{108 - x^2}{4x}$$
$$= x \frac{108 - x^2}{4}$$
$$= \frac{108x}{4} - \frac{x^3}{4}$$
$$= 27x - \frac{x^3}{4}$$

$$27x - \frac{1}{4}x^3 = 0 \qquad \Rightarrow 108x - x^3 = 0$$

$$x(108 - x^2) = 0 \quad \to x = 0,\ \sqrt{108}$$

$$V' = 27 - \frac{3}{4}x^2 = 0$$

$$-\frac{3}{4}x^2 = -27$$

$$\Rightarrow x^2 = 27\left(\frac{4}{3}\right) = 36$$

$$\to x = \pm 6 \ \Rightarrow x = 6 \ (only)$$

$$h = \frac{108 - x^2}{4x}$$
$$= \frac{108 - 6^2}{46}$$
$$\underline{= 3 \ \ in}$$

Exercise

A company wants to manufacture cylinder aluminum can with a volume 1000 cm^3. What should the radius and height of the can be to minimize the amount of aluminum used?

Solution

$$V = \pi r^2 h = 1000 \quad \Rightarrow \quad h = \frac{1000}{\pi r^2}$$

Surface Area $S = 2\pi r^2 + 2\pi rh$

$$S = 2\pi r^2 + 2\pi r \frac{1000}{\pi r^2}$$
$$= 2\pi r^2 + \frac{2000}{r}$$

$$S' = 4\pi r - \frac{2000}{r^2} = 0$$

$$4\pi r = \frac{2000}{r^2}$$

$$r^3 = \frac{2000}{4\pi}$$

$$r = \left(\frac{2000}{4\pi}\right)^{1/3}$$

$$\approx 5.419\ cm$$

$$h = \frac{1000}{\pi r^2}$$

$$= \frac{1000}{\pi\ 5.149^2}$$

$$\approx 10.84\ \ cm$$

Exercise

What is the smallest perimeter possible for a rectangle whose area is $16\ in^2$, and what are its dimensions?

Solution

Area: $A = \ell \cdot w = 16 \quad \Rightarrow \quad w = 16 \cdot \ell^{-1}$

Perimeter: $P = 2\ell + 2w$

$$P = 2\ell + 2\left(16 \cdot \ell^{-1}\right)$$

$$P = 2\ell + 32 \cdot \ell^{-1}$$

$$P' = 2 - 32 \cdot \ell^{-2} = 0$$

$$2 = 32 \cdot \ell^{-2}$$

$$2 \cdot \frac{\ell^2}{2} = 32 \cdot \ell^{-2} \cdot \frac{\ell^2}{2}$$

$$\ell^2 = 16 \quad \Rightarrow \quad \ell = \pm 4$$

Since $\ell > 0 \quad \Rightarrow \quad \ell = 4\ \ in$

$$w = 16 \cdot 4^{-1} = 4\ \ in$$

Exercise

You are planning to make an open rectangular box from an 8-*in*. by 15-*in*. piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way, and what is its volume?

Solution

Volume of the box is: $V(x) = x(15 - 2x)(8 - 2x) \qquad 2x < 8 \quad \Rightarrow \quad 0 < x < 4$

$$V(x) = x\left(120 - 46x + 4x^2\right)$$

$$= 120x - 46x^2 + 4x^3$$

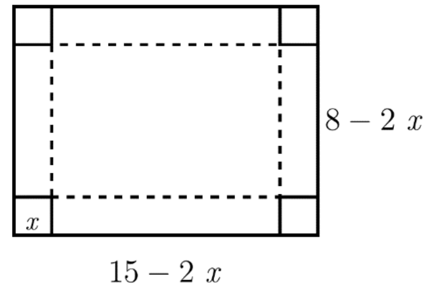
$$V'(x) = 120 - 92x + 12x^2 = 0$$

$$\boxed{x = \frac{5}{3}} \quad \cancel{x = 6} \quad (\textit{not in domain})$$

$$V\left(\frac{5}{3}\right) = 120\left(\frac{5}{3}\right) - 46\left(\frac{5}{3}\right)^2 + 4\left(\frac{5}{3}\right)^3 \approx 91\ in^3$$

$$\begin{cases} 15 - 2\left(\frac{5}{3}\right) = \frac{35}{3} \\ 8 - 2\left(\frac{5}{3}\right) = \frac{14}{3} \end{cases}$$

Box dimensions: $\frac{35}{3} \times \frac{14}{3} \times \frac{5}{3}$ *inches*



Exercise

A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 *m* of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?

Solution

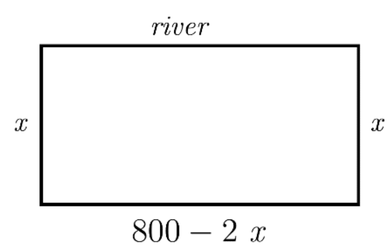
Area: $A(x) = x(800 - 2x) \quad 0 \le x \le 400$

$$A(x) = 800x - 2x^2$$

$$A'(x) = 800 - 4x = 0 \quad \Rightarrow \quad \boxed{x = 200}$$

$$A(200) = 800(200) - 2(200)^2 = 80{,}000\ m^2$$

Dimensions: 200 *m* by **(800-2(200))** = 400 *m*.



Exercise

A piece of cardboard measures 10-*in*. by 15-*in*. Two equal squares are removed from the corners of 10-*in*. side. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.

a) Write a formula $V(x)$ for the volume of the box

b) Find the domain of V for the problem situation and graph V over this domain

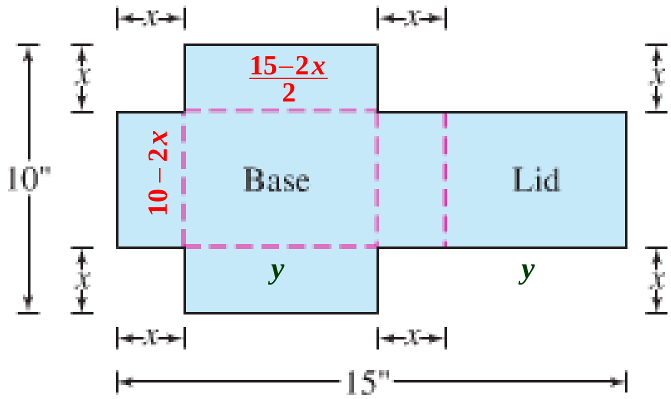
c) Use the graphical or analytically method to find the maximum volume and the value of x that gives it.

Solution

a) $V(x) = x(10 - 2x)\left(\frac{15 - 2x}{2}\right)$

$$= \left(5x - x^2\right)(15 - 2x)$$

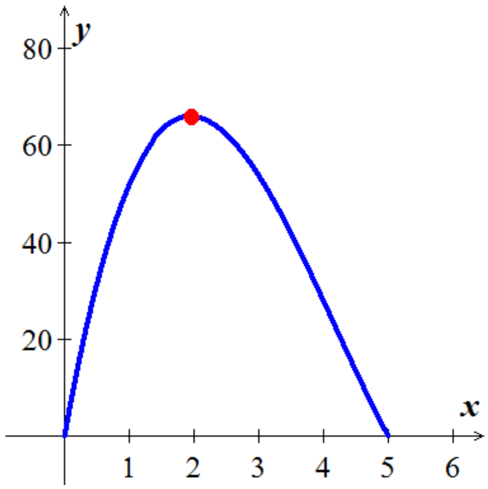
$$= 2x^3 - 25x^2 + 75x$$



b) $x > 0, \quad 2x < 10, \quad 2x < 15$

$$x < 5, \quad x < \frac{15}{2}$$

Domain: $(0,\ 5)$



c) $V' = 6x^2 - 50x + 75 = 0 \quad \Rightarrow \quad x \approx 1.96, \ \cancel{6.37}$

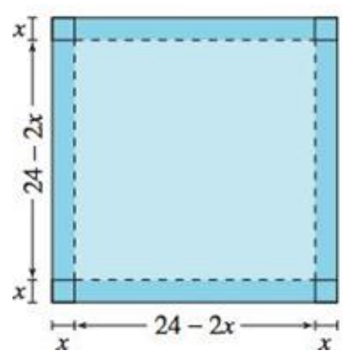
$$V(1.96) = 2(1.96)^3 - 25(1.96)^2 + 75(1.96) = \boxed{66.02\ in^3}$$

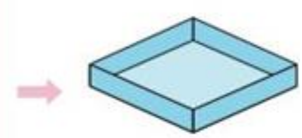
Exercise

An open box of maximum volume is to be made from a square piece of material, 24 *inches* on a side, by cutting equal squares from the corners and turning up the sides.

a) Write the volume V as a function of x.
b) Find the critical number of the function and find the maximum value.
c) Graph the function and verify the maximum volume from the graph.

Solution





a) $V = x(24 - 2x)^2 \quad 0 < x < 12$

b) $V = x\left(576 - 96x + 4x^2\right)$

$$= 4x^3 - 96x^2 + 576x$$

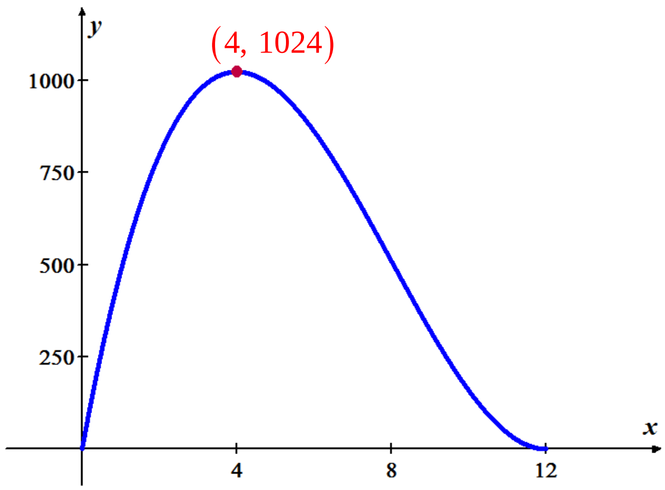
$$\frac{dV}{dx} = 12x^2 - 192x + 576 = 0$$

$$x^2 - 16x + 48 = 0 \quad \to \quad x = 4, \ \cancel{12}$$

The critical number: $x = 4$

$$V(4) = 4\left(576 - 96(4) + 4(4)^2\right) \underline{= 1024 \ in^3}$$

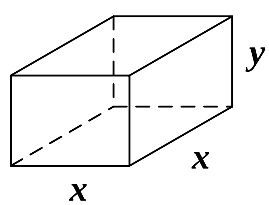
c)



Exercise

A rectangular solid (with a square base) has a surface area of 337.5 cm^2. Find the dimensions that will result in a solid with maximum volume.

Solution



$$S = 2x^2 + 4xy = 337.5$$

$$y = \frac{337.5 - 2x^2}{4x}$$

$$V = x^2 y$$

$$= \frac{1}{4}\left(337.5x - 2x^3\right)$$

$$\frac{dV}{dx} = \frac{1}{4}\left(337.5 - 6x^2\right) = 0$$

$$x^2 = \frac{337.5}{6} \quad \rightarrow \quad \underline{x = 7.5 \ cm}$$

$$y = \frac{337.5 - 2(7.5)^2}{4(7.5)} \underline{= 7.5 \ cm}$$

Exercise

A manufacturer wants to design an open box having square base and a surface area of 108 *square inches*. What dimensions will produce a box with maximum volume?

Solution

$$S = (area \ of \ \ base) \times (area \ of \ \ 4 \ sides)$$

$108 = x^2 + 4xh$

$h = \dfrac{108 - x^2}{4x}$

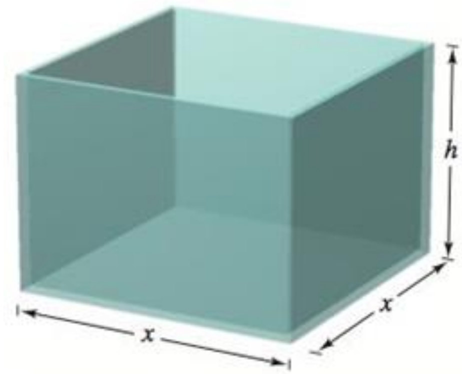
$V = x^2\left(\dfrac{108 - x^2}{4x}\right) \qquad\qquad V = x^2 h$

$= \frac{1}{4}\left(108x - x^3\right)$

$\dfrac{dV}{dx} = \frac{1}{4}\left(108 - 3x^2\right) = 0$

$x^2 = \dfrac{108}{3} = 36 \quad \rightarrow \quad \underline{x = 6,\ \cancel{-6}}$

$h\big|_{x=6} = \dfrac{108 - 36}{24} \ \underline{= 3}$



V is maximum when the dimensions of the box are $6 \times 6 \times 3$ *inches*.

Exercise

A parcel delivery service will deliver a package only if the length plus girth (distance around) does not exceed 108 *inches*.

a) Find the dimensions of a rectangular box with square ends that satisfies the delivery service's restriction and has maximum volume. What is the maximum volume?

b) Find the dimensions (radius and height) of a cylinder container that meets the delivery service's requirement and has maximum volume. What is the maximum volume?

Solution

a) Let x: length of the side of the square.

$L + 4x = 108 \quad \rightarrow \quad L = 108 - 4x$

The volume of the box:

$V = L \cdot x^2 = (108 - 4x) \cdot x^2$

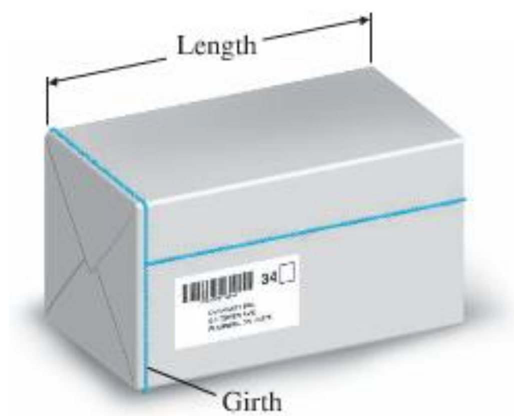
$= 108x^2 - 4x^3$

$V' = 216x - 12x^2 = 0$

$x = 0 \qquad \underline{x = \dfrac{216}{12} = 18}$

$V(18) = 108(18)^2 - 4(18)^3 \ \underline{= 11,664 \ \ in^3}$

$L = 108 - 4(18) \ \underline{= 36 \ \ in}$



Therefore, the volume is maximum at 11,664 in^3 for an 18″ by 18″ by 36″ container.

b) Let x: radius and L: height of the cylindrical container.

$L + 2\pi x = 108 \quad \rightarrow \quad L = 108 - 2\pi x$

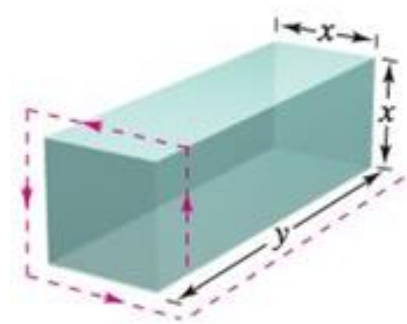
$$V = \pi x^2 L$$
$$= \pi x^2 (108 - 2\pi x)$$
$$= 108\pi x^2 - 2\pi^2 x^3$$
$$V' = 216\pi x - 6\pi^2 x^2$$
$$= 6\pi x(36 - \pi x)$$
$$x = 0 \quad \boxed{x = \frac{36}{\pi}\ \ in}$$
$$V\left(\frac{36}{\pi}\right) = 108\pi\left(\frac{36}{\pi}\right)^2 - 2\pi^2\left(\frac{36}{\pi}\right)^3 \ \underline{= 14{,}851\ \ in^3|}$$
$$L = 108 - 2\pi\left(\frac{36}{\pi}\right) \ \underline{= 36|}$$

Therefore, the volume is maximum at 14,851 in^3 for a container of radius $\frac{36}{\pi}$ and height of 36″.

Exercise

A rectangular package to be sent by a postal service can have a maximum combined length and girth (perimeter of a cross section) of 108 *inches*. Find the dimensions of the package of maximum volume that can be sent.

Solution



$$P = 4x + y = 108 \quad \Rightarrow \quad y = 108 - 4x$$
$$V = x^2 y$$
$$= x^2(108 - 4x)$$
$$\underline{= 108x^2 - 4x^3|}$$
$$\frac{dV}{dx} = 216x - 12x^2 = 0 \quad \rightarrow \quad \underline{|x = \frac{216}{12}} \ \underline{= 18\ \ in|}$$
$$y = 108 - 4(18) \ \underline{= 36\ \ in|}$$

Exercise

A page is to contain 30 *square inches* of print. The margins at the top and bottom of the page are 2 *inches* wide. The margins on the sides are 1 *inch* wide. What dimensions will minimize the amount of paper used?

Solution

Let the dimensions of the original sheet have width *x* and height *y*.

The dimensions of the print area would then be, (*x* - 2) and (*y* - 4) respectively.

Area of the print space: $A_1 = (x-2)(y-4) = 30$

$$y-4=\frac{30}{x-2} \Rightarrow y=\frac{30}{x-2}+4$$

Area of the entire page: $A = xy = x\left(\frac{30}{x-2}+4\right)$

$$A=\frac{30x}{x-2}+4x$$

$$A'=\frac{(x-2)(30)-(30x)}{(x-2)^2}+4$$

$$A'=\frac{-60}{(x-2)^2}+4=0$$

$$\frac{4}{1}=\frac{60}{(x-2)^2}$$

$$\Rightarrow 4(x-2)^2=60$$

$$(x-2)^2=15$$

$$x=2-\sqrt{15}<0$$

$$x=2+\sqrt{15}\ \underline{\approx 5.9\ in}$$

$$y=\frac{30}{x-2}+4\approx\frac{30}{5.9-2}+4\ \underline{\approx 4.8\ in}$$

Dimension: 5.9 *in.* x 4.8 *in.*

Exercise

A rectangular page is to contain 24 squares *inches* of print. The margins at the top and bottom of the page are $1\frac{1}{2}$ *inches*, and the margins on the left and right are to be 1 *inch*. What should the dimensions of the page be so that the least amount of paper used?

Solution

Given: $xy = 24 \quad\to\quad y=\frac{24}{x}$

$$A=(x+3)(y+2)$$

$$=(x+3)\left(\frac{24}{x}+2\right)$$

$$=30+2x+\frac{72}{x}$$

$$\frac{dA}{dx}=2-\frac{72}{x^2}=0$$

$$x^2=36 \Rightarrow x=\pm 6$$

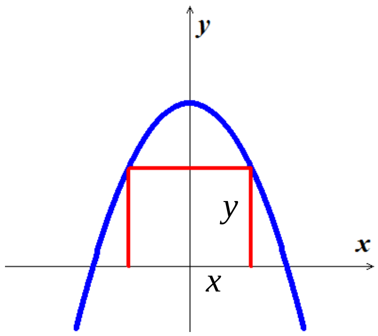
$x = 6 \quad \Rightarrow \quad y = \frac{24}{6} = 4$

$\therefore$ The dimension of paper is: $(6+3) = 9 \times 6 = (4+2)$ *inches*.

Exercise

A rectangle has its base on the x-axis and its upper vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

Solution



Area of the rectangle: $A = 2xy = 2x\left(12 - x^2\right)$

$A = 24x - 2x^3$

$A' = 24 - 6x^2 = 0$

$\rightarrow x^2 = 4 \Rightarrow x = \pm 2 \quad \boxed{x = 2}$

$y = 12 - 2^2 = 8$

$A = 2(2)(8) = 32$

The largest area is 32 square units, and the dimensions are 4 *units* by 8 *units*.

Exercise

Find the points of $y = 4 - x^2$ that are closet to (0, 3)

Solution

$$d = \sqrt{(x-0)^2 + (y-3)^2}$$

$$= \sqrt{x^2 + \left(4 - x^2 - 3\right)^2}$$

$$= \sqrt{x^2 + \left(1 - x^2\right)^2}$$

$$= \sqrt{x^2 + 1 - 2x^2 + x^4}$$

$$= \sqrt{x^4 - x^2 + 1}$$

$f(x) = x^4 - x^2 + 1$

$f'(x) = 4x^3 - 2x = 2x(2x^2 - 1)$

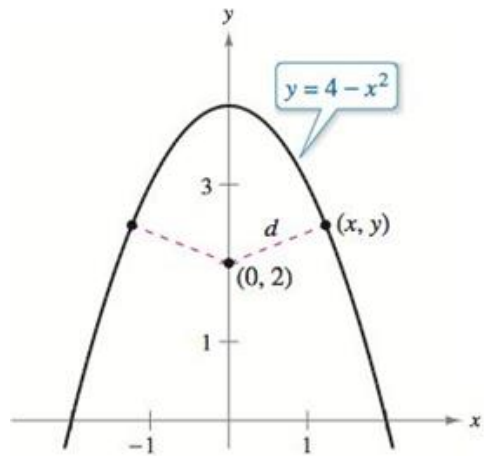
$2x(2x^2 - 1) = 0 \rightarrow x = 0, \pm\sqrt{\frac{1}{2}}$

$\left(\sqrt{\frac{1}{2}}, \frac{7}{2}\right) \quad \left(-\sqrt{\frac{1}{2}}, \frac{7}{2}\right)$

Exercise

Which points on the graph of $y = 4 - x^2$ are closest to the point $(0,\ 2)$?

Solution



$$d = \sqrt{(x-0)^2 + (y-2)^2}$$

$$= \sqrt{x^2 + \left(4 - x^2 - 2\right)^2}$$

$$= \sqrt{x^2 + \left(2 - x^2\right)^2}$$

$$= \sqrt{4 - 3x^2 + x^4}$$

Let $\quad f(x) = x^4 - 3x^2 + 4$

$$f'(x) = 4x^3 - 6x = 0 \quad \rightarrow \quad \underline{x = 0,\ \pm\sqrt{\frac{3}{2}}}$$

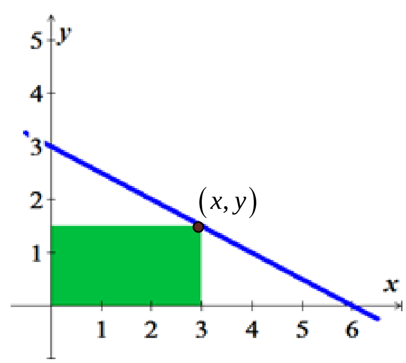
$f(0) = 4$ yields a relative ***maximum***

$f\left(\pm\sqrt{\frac{3}{2}}\right) = \frac{9}{4} - \frac{9}{2} + 4 = \frac{7}{4}$ yield a relative ***minimum***

Exercise

A rectangle is bounded by the x- and y-axes and the graph of $y = \frac{1}{2}(6 - x)$. What length and width should the rectangle have so that its area is a maximum?

Solution



Given: $y = \frac{1}{2}(6 - x)$

$$A = xy = (x)\frac{1}{2}(6 - x)$$

$$= 3x - \frac{1}{2}x^2$$

$$\frac{dA}{dx} = 3 - x = 0 \quad \rightarrow \quad \underline{x = 3}$$

$$y = \frac{1}{2}(6 - 3)\ \underline{= \frac{3}{2}}$$

Exercise

A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(1,\ 2)$.

a) Write the length L of the hypotenuse as a function of x.

b) Graph the function and approximate x graphically such that the length of the hypotenuse is a minimum.

c) Find the vertices of the triangle such that its area is a minimum.

Solution

$$\frac{y-2}{0-1}=\frac{2-0}{1-x} \qquad (slope)$$

$$y-2=\frac{-2}{1-x}$$

$$y=\frac{2}{x-1}+2$$

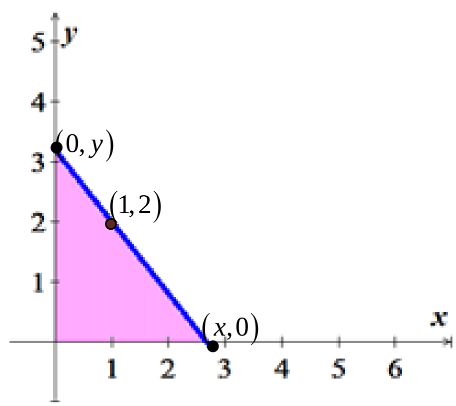
$$\boxed{y=\frac{2x}{x-1}}$$

a) $L=\sqrt{x^2+y^2}$

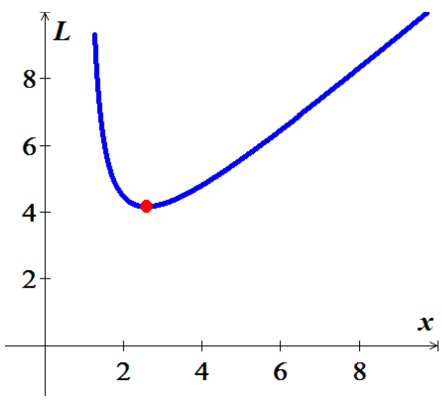
$$=\sqrt{x^2+\left(\frac{2x}{x-1}\right)^2}$$

$$=\sqrt{x^2+\frac{4x^2}{x^2-2x+1}}$$

$$\boxed{=\frac{x}{x-1}\sqrt{x^2-2x+5}} \qquad x>1$$



b) L is minimum when $x=2.587$ *and* $L=4.162$



c) $A=\frac{1}{2}xy$

$$=\frac{1}{2}x\left(\frac{2}{x-1}+2\right)$$

$$=\frac{x}{x-1}+x$$

$$=\frac{x^2}{x-1}$$

$$\frac{dA}{dx}=\frac{x^2-2x}{(x-1)^2}=0 \quad \rightarrow x=\cancel{0},\ \boxed{2}$$

$$\boxed{y=4}$$

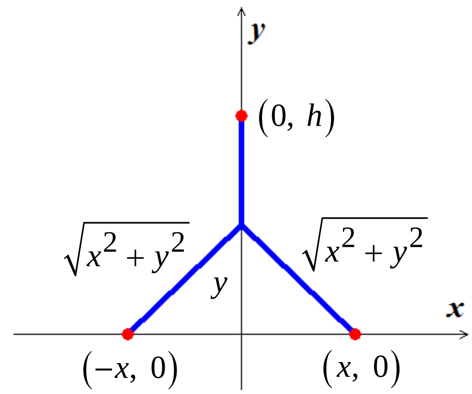
Vertices are: $\boxed{(0,\ 0),\ (2,\ 0),\ (0,\ 4)}$

Exercise

Two factories are located at the coordinates $(-x,\ 0)$ and $(x,\ 0)$, and their power is at $(0,\ h)$. Find y such that the total length of power line from the power supply to the factories is a minimum.

Solution

Let L be the amount of the power line.



$$L=h-y+2\sqrt{x^2+y^2}$$

$$\frac{dL}{dy}=-1+\frac{2y}{\sqrt{x^2+y^2}}=0$$

$$\frac{2y}{\sqrt{x^2+y^2}}=1 \ \rightarrow\ 2y=\sqrt{x^2+y^2}$$

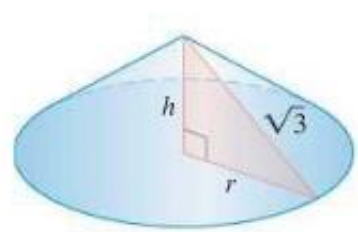
$$4y^2=x^2+y^2$$

$$3y^2=x^2 \quad \rightarrow \boxed{y=\frac{x}{\sqrt{3}}}$$

Exercise

A right triangle whose hypotenuse is $\sqrt{3}\ m$ long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.

Solution



From the figure, the right triangle: $h^2+r^2=3$

Volume: $V=\frac{\pi}{3}r^2h$

$$=\frac{\pi}{3}\left(3-h^2\right)h \qquad 0<h<\sqrt{3}$$

$$=\pi h-\frac{\pi}{3}h^3$$

$$\frac{dV}{dh}=\pi-\pi h^2$$

$$=\pi\left(1-h^2\right)=0$$

0	1	$\sqrt{3}$
	$V'(.5)>0$	$V'(1.1)<0$
	Increasing	***Decreasing***

$$h^2 = 1 \quad \Rightarrow \quad \boxed{h=1} \quad (\boldsymbol{CP})$$

The volume has a maximum at the critical point.

$$r = \sqrt{3-h^2} = \sqrt{2}$$

$$V = \frac{\pi}{3} r^2 h$$

$$= \frac{\pi}{3}\left(\sqrt{2}\right)^2 (1)$$

$$= \frac{2\pi}{3}$$

Therefore, the cone, with a maximum volume, has a radius $\sqrt{2}\ m$, height 1 m, and volume $V = \frac{2\pi}{3}\ m^3$

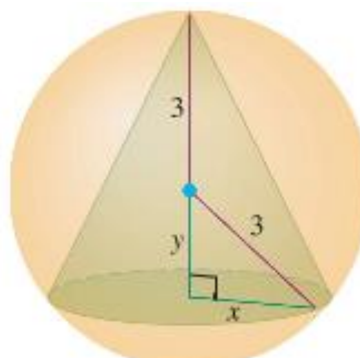
Exercise

Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.

Solution

Volume of the cone is: $V = \frac{1}{3}\pi r^2 h$

$$where \quad r = x = \sqrt{9-y^2} \quad h = y+3$$



$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\sqrt{9-y^2}\right)^2 (y+3)$$

$$= \frac{1}{3}\pi\left(9-y^2\right)(y+3)$$

$$= \frac{1}{3}\pi\left(-y^3 - 3y^2 + 9y + 27\right)$$

$$V'(y) = \frac{1}{3}\pi\left(-3y^2 - 6y + 9\right)$$

$$= \pi\left(-y^2 - 2y + 3\right) = 0$$

Critical points are: $\boxed{y=1}$ $\quad \cancel{y=-3} < 0$

$$V''(y) = \pi(-2y-2)\Big|_{y=1} < 0$$

Using the 2nd derivative test

$$V(y=1) = \frac{1}{3}\pi\left(-1^3 - 3(1)^2 + 9(1) + 27\right)$$

$$= \frac{32\pi}{3}\ unit^3$$

Exercise

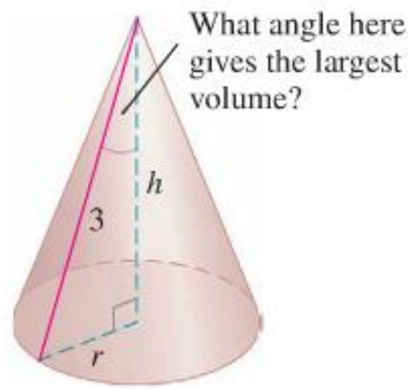
The slant height of the cone is 3 *m*. How large should the indicated angle be to maximize the cone's volume?

Solution

Let θ be the angle of the largest volume

$\sin\theta = \frac{r}{3} \quad \to r = 3\sin\theta$

$\cos\theta = \frac{h}{3} \quad \to h = 3\cos\theta$



$$
\begin{aligned}
V &= \frac{1}{3}\pi r^2 h \\
&= \frac{1}{3}\pi(3\sin\theta)^2(3\cos\theta) \\
&= 9\pi\sin^2\theta\cos\theta \\
&= 9\pi\left(1-\cos^2\theta\right)\cos\theta \\
&= 9\pi\left(\cos\theta-\cos^3\theta\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dV}{d\theta} &= 9\pi\left(-\sin\theta-3\cos^2\theta(-\sin\theta)\right) \\
&= 9\pi\sin\theta\left(-1+3\cos^2\theta\right) = 0
\end{aligned}
$$

$$
\to \begin{cases} \sin\theta = 0 & \theta = 0 \\ -1+3\cos^2\theta & \cos^2\theta = \frac{1}{3} \Rightarrow \cos\theta = \pm\frac{1}{\sqrt{3}} \end{cases}
$$

The critical points are: $0, \quad \cos^{-1}\left(\frac{1}{\sqrt{3}}\right), \quad \cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ *is not in the domain*

When $\theta = 0$, we have a minimum, and when $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.7^\circ$, we have a maximum volume.

Exercise

Find the area of the largest isosceles triangle that can be inscribed in a circle of radius 6.

a) Solve the area as a function of h.

b) Solve the area as a function of α.

c) Identify the type of triangle of maximum area.

Solution

a) $|DC| = \sqrt{36-h^2}$

$A = \frac{1}{2}|AD||BC| \qquad\qquad A = \frac{1}{2}(height)(base)$

$$=\frac{1}{2}(h+6)\left(2\sqrt{36-h^2}\right)$$

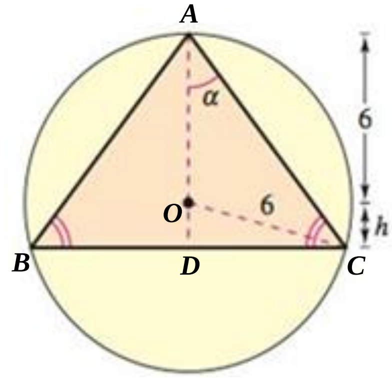
$$\boxed{=(h+6)\sqrt{36-h^2}}$$

$$\frac{dA}{dh}=\sqrt{36-h^2}-\frac{h(h+6)}{\sqrt{36-h^2}}$$

$$=\frac{36-2h^2-6h}{\sqrt{36-h^2}}=0$$

$$h^2+3h-18=0 \quad \rightarrow \quad \boxed{h=3},\ \cancel{-6}$$

$$A(3)=9\sqrt{27}\ \boxed{=27\sqrt{3}\ \ unit^2}$$



b) $A=2\times\frac{1}{2}|AD||DC| \qquad \tan\alpha=\frac{|DC|}{|AD|}$

$$=|AD|^2\tan\alpha \qquad \cos\alpha=\frac{36+|AC|^2-36}{2(6)|AC|}=\frac{|AC|}{12}$$

$$\cos\alpha=\frac{|AD|}{|AC|} \ \Rightarrow\ |AD|=12\cos^2\alpha$$

$$=144\cos^4\alpha\tan\alpha$$

$$\boxed{=144\cos^3\alpha\ \sin\alpha}$$

$$\frac{dA}{d\alpha}=144\left(\cos^4\alpha-3\cos^2\alpha\sin^2\alpha\right)$$

$$=144\cos^2\alpha\left(\cos^2\alpha-3\sin^2\alpha\right)=0$$

$$\begin{cases}\cos^2\alpha=0 & \rightarrow \cancel{\alpha=\pm\frac{\pi}{2}}\\ \cos^2\alpha=3\sin^2\alpha & \cos\alpha=\sqrt{3}\sin\alpha\end{cases} \quad \tan\alpha=\frac{1}{\sqrt{3}} \Rightarrow \boxed{\alpha=30^\circ}$$

$$A(30^\circ)=144\frac{3\sqrt{3}}{8}\ \frac{1}{2}\ \boxed{=27\sqrt{3}\ \ unit^2}$$

c) Since $\alpha=30^\circ \ \Rightarrow\ \angle A=60^\circ=\angle B=\angle C$

$\therefore$ It is an ***equilateral*** triangle.

Exercise

A rectangle is bounded by the *x-axis* and the semicircle $y=\sqrt{25-x^2}$

What length and width should the rectangle have so that its area is a maximum?

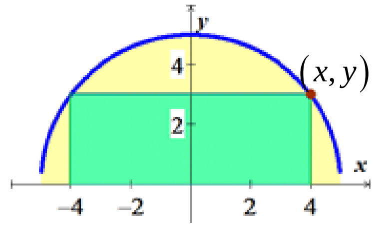
Solution

$$A=(2x)y$$

$$= 2x\sqrt{25-x^2}$$

$$\frac{dA}{dx} = 2\sqrt{25-x^2} - 2\frac{x^2}{\sqrt{25-x^2}}$$

$$= 2\frac{25-2x^2}{\sqrt{25-x^2}} = 0 \quad \rightarrow \quad \boxed{x = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}}$$



$$\boxed{y = \sqrt{25 - \frac{25}{2}} = \frac{5\sqrt{2}}{2}}$$

Dimensions: *length*: $5\sqrt{2}$ *width*: $\frac{5\sqrt{2}}{2}$

Exercise

What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of $1000\ cm^3$?

Solution

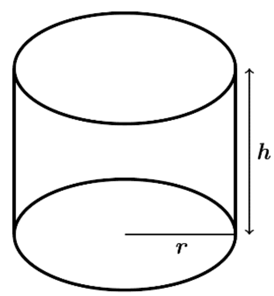
$$V = \pi r^2 h = 1000 \quad \Rightarrow \quad h = \frac{1000}{\pi \cdot r^2}$$

The amount of the material is the surface area by the sides and bottom of the can.

$$S = 2\pi rh + \pi r^2$$

$$= 2\pi r\frac{1000}{\pi r^2} + \pi r^2$$

$$= \frac{2000}{r} + \pi r^2, \quad r > 0$$



$$\frac{dS}{dr} = -\frac{2000}{r^2} + 2\pi r$$

$$= \frac{-2000 + 2\pi r^3}{r^2} = 0$$

$$-2000 + 2\pi r^3 = 0$$

$$2\pi r^3 = 2000$$

$$r^3 = \frac{1000}{\pi} \qquad \Rightarrow \boxed{r = \frac{10}{\sqrt[3]{\pi}}\ unit}$$

$S'' = \frac{4000}{r^3} + 2\pi > 0$, the surface has a minimum area at $r = \frac{10}{\sqrt[3]{\pi}}$

$$h = \frac{1000}{\pi \cdot r^2}$$

$$= \frac{1000}{\pi \cdot \left(\frac{10}{\sqrt[3]{\pi}}\right)^2}$$

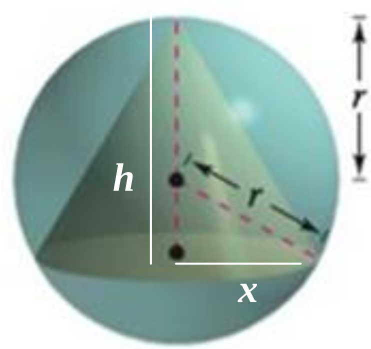
$$= \frac{1000}{100\pi \cdot \pi^{-2/3}}$$

$$\boxed{= \frac{10}{\pi^{1/3}} \ \ unit}$$

Exercise

Find the volume of the largest right circular cone that can be inscribed in a sphere of radius r.

Solution



$$V = \frac{1}{3}\pi x^2 h$$

$$= \frac{1}{3}\pi x^2 \left(r + \sqrt{r^2 - x^2}\right)$$

$$\frac{dV}{dx} = \frac{\pi}{3}\left(2xr + 2x\sqrt{r^2 - x^2} - \frac{x^3}{\sqrt{r^2 - x^2}}\right)$$

$$= \frac{\pi}{3}\left(\frac{2xr\sqrt{r^2 - x^2} + 2xr^2 - 3x^3}{\sqrt{r^2 - x^2}}\right)$$

$$= \frac{x\pi}{3\sqrt{r^2 - x^2}}\left(2r\sqrt{r^2 - x^2} + 2r^2 - 3x^2\right) = 0$$

$$2r\sqrt{r^2 - x^2} + 2r^2 - 3x^2 = 0$$

$$2r\sqrt{r^2 - x^2} = 3x^2 - 2r^2$$

$$4r^2\left(r^2 - x^2\right) = 9x^4 - 12x^2r^2 + 4r^4$$

$$9x^4 - 8x^2r^2 = 0$$

$$x^2\left(9x^2 - 8r^2\right) = 0 \quad \to \quad \boxed{x = 0, \ \frac{2r\sqrt{2}}{3}}$$

$$V = \frac{1}{3}\pi\left(\frac{8r^2}{9}\right)\left(r + \sqrt{r^2 - \frac{8r^2}{9}}\right)$$

$$= \left(\frac{8\pi r^2}{27}\right)\left(r + \frac{r}{3}\right)$$

$$\boxed{= \frac{32\pi r^3}{81} \ \ unit^3}$$

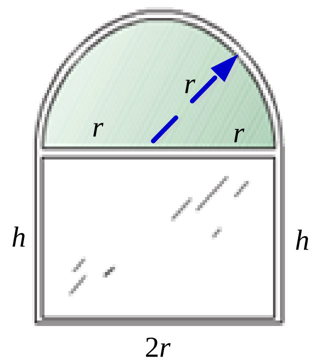
Exercise

A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.

Solution

Perimeter = Perimeter of the rectangle + Perimeter of the semicircle

$$P = 2r + 2h + \pi r \quad \rightarrow \quad h = \frac{1}{2}(P - 2r - \pi r)$$



$$\begin{aligned} A &= 2rh + \frac{1}{4}\pi r^2 \\ &= 2r\frac{1}{2}(P - 2r - \pi r) + \frac{1}{4}\pi r^2 \\ &= rP - 2r^2 - \pi r^2 + \frac{1}{4}\pi r^2 \\ &= rP - 2r^2 - \frac{3}{4}\pi r^2 \end{aligned}$$

$$\frac{dA}{dr} = P - 4r - \frac{3}{2}\pi r = 0$$

$$\left(4 + \frac{3}{2}\pi\right)r = P \quad \rightarrow r = \frac{2P}{8 + 3\pi}$$

$$\begin{aligned} h &= \frac{1}{2}\left(P - 2\frac{2P}{8+3\pi} - \pi\frac{2P}{8+3\pi}\right) \\ &= \frac{1}{2}\left(\frac{8P + 3\pi P - 4P - 2\pi P}{8+3\pi}\right) \\ &= \frac{P}{2}\left(\frac{4+\pi}{8+3\pi}\right) \end{aligned}$$

The proportions that admit the most light:

$$\begin{aligned} \frac{2r}{h} &= 2\cdot\frac{2P}{8+3\pi}\cdot\frac{2}{P}\left(\frac{8+3\pi}{4+\pi}\right) \\ &= \frac{8}{4+\pi} \end{aligned}$$

Exercise

The cost per hour for fuel to run a train is $\frac{v^2}{4}$ dollars, where v is the speed of the train in miles per hour. (Note that the cost goes up as the square of the speed.) Other costs, including labor are \$300 per hour. How fast should the train travel on a 360-*mile* trip to minimize the total cost for the trip?

Solution

x = number of hours it takes the train to travel 360 miles.

Then $360 = xv \quad \rightarrow \quad x = \frac{360}{v}$

Cost: $C=\left(300+\frac{v^2}{4}\right)\left(\frac{360}{v}\right)$

$C(v)=\frac{108,000}{v}+90v$

$C'(v)=-\frac{108,000}{v^2}+90=0$

$\frac{108,000}{v^2}=90 \quad\rightarrow\quad v^2=\frac{108,000}{90}=1,200$

$\underline{v=\sqrt{1200}\approx 34.64}$

$C''(v)=\frac{216,000}{v^3}>0$

The cost has an absolute minimum at $x=34.64$

$C(34.64)=\frac{108,000}{34.64}+90(34.64)\ \underline{=6,235.38}$

Exercise

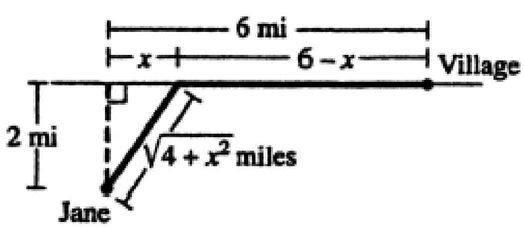
Jane is 2 *mi* offshore in a boat and wishes to reach a coastal village 6 *mi* down a straight shoreline from the point nearest the boat. She can row 2 *mph* and can walk 5 mph. Where should she land her boat to reach the village in the least amount of time?

Solution

x: distance from shoreline to boat.

$row\ \sqrt{4+x^2}\ @\ 2\ mph \quad \rightarrow t_1=\frac{d}{v}=\frac{\sqrt{4+x^2}}{2}$

$walk:\ 6-x\ @\ 5\ mph \quad \rightarrow t_2=\frac{d}{v}=\frac{6-x}{5}$



The total amount of time to reach the village:

$f(x)=\frac{\sqrt{4+x^2}}{2}+\frac{6-x}{5}$

$f'(x)=\frac{1}{2}\frac{2x}{2\sqrt{4+x^2}}+\frac{-1}{5}$

$=\frac{x}{2\sqrt{4+x^2}}-\frac{1}{5}=0$

$\frac{x}{2\sqrt{4+x^2}}=\frac{1}{5}$

$5x=2\sqrt{4+x^2}$

$(5x)^2=\left(2\sqrt{4+x^2}\right)^2$

$$25x^2 = 4\left(4+x^2\right)$$

$$25x^2 = 16 + 4x^2$$

$$21x^2 = 16$$

$$x^2 = \frac{16}{21} \quad \rightarrow \quad x = \frac{4}{\sqrt{21}}$$ ***Distance can't be negative***

$$\rightarrow \begin{cases} x = 0 & \rightarrow f = 2.2 \\ x = \frac{4}{\sqrt{21}} & \rightarrow f \approx 2.12 \\ x = 6 & \rightarrow f \approx 3.16 \end{cases}$$

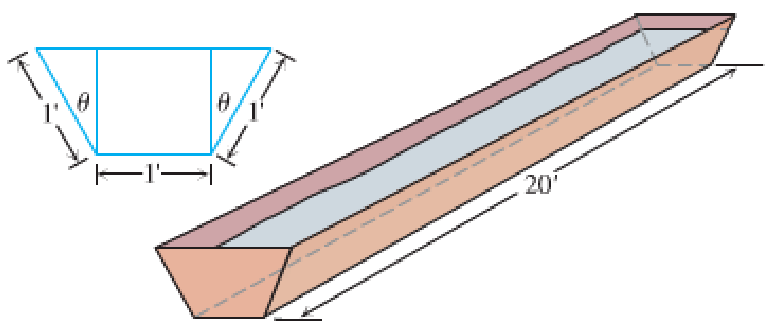
Jane should land her boat $\frac{4}{\sqrt{21}} \approx .87$ miles down the shoreline from the point nearest her boat.

Exercise

The trough in the figure is to be made to the dimensions shown. Only the angle θ can be varied. What value of θ will maximize the trough's volume?

Solution

The area of the cross section:



$$A(\theta) = \cos\theta + \sin\theta\cos\theta \qquad \left(0 < \theta < \frac{\pi}{2}\right)$$

$$A'(\theta) = -\sin\theta + \cos^2\theta - \sin^2\theta$$

$$= -\sin\theta + 1 - \sin^2\theta - \sin^2\theta$$

$$= -2\sin^2\theta - \sin\theta + 1 \ \boxed{= 0}$$

$$\rightarrow \begin{cases} \sin\theta = \frac{1}{2} & \Rightarrow \theta = \frac{\pi}{6} \\ \cancel{\sin\theta = -1} & 0<\theta<\frac{\pi}{2} \end{cases}$$

0	$\frac{\pi}{6}$	$\frac{\pi}{2}$
$A'(\theta) > 0$	$A'(\theta) < 0$	
Increasing	***Decreasing***	

Therefore, there is a maximum value at $\boxed{\theta = \frac{\pi}{6}}$

$$A\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} + \sin\frac{\pi}{6}\cos\frac{\pi}{6}$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\frac{\sqrt{3}}{2}$$

$$\boxed{= \frac{3\sqrt{3}}{4}}$$

Exercise

The height above the ground of an object moving vertically is given by $s(t) = -16t^2 + 96t + 112$

With s in feet and t in seconds. Find

a) The object's velocity when $t = 0$

b) Its maximum height and when it occurs

c) Its velocity when $s = 0$

Solution

a) $v(t) = s'(t) = -32t + 96$

$v(t=0) = \boxed{96\ ft / \sec}$

b) The maximum height occurs when $v(t) = -32t + 96 = 0$

$\boxed{t = \frac{96}{32} = 3\ \sec}$

$\boxed{s(t=3)} = -16(3)^2 + 96(3) + 112$

$= \boxed{256\ ft}$

c) $s = -16t^2 + 96t + 112 = 0 \quad \Rightarrow \quad \begin{cases} t = -1 \\ t = 7 \end{cases}$

$v(t=7) = -32(7) + 96$

$= \boxed{-128\ ft / \sec}$

Exercise

Compare the answers to the following two construction problems.

a) A rectangular sheet of perimeter 36 *cm* and dimensions *x cm* and *y cm* is to be rolled into a cylinder as shown in part (*a*) of the figure. What values of *x* and *y* give the largest volume?

b) The same sheet is to be revolved about one of the sides of length *y* to sweep out the cylinder as shown in part (*b*) of the figure. What values of *x* and *y* give the largest volume?

Solution

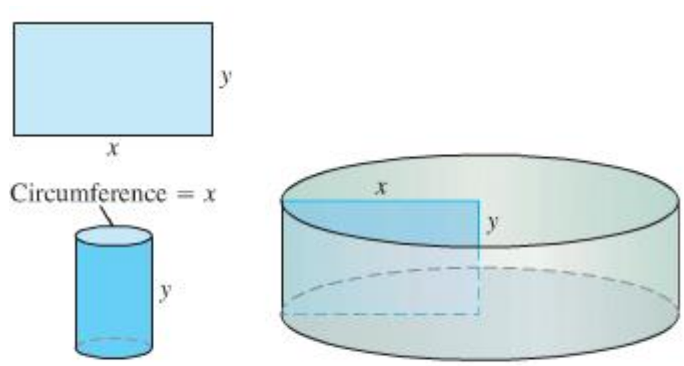
a) $P = 2x + 2y \quad \Rightarrow \quad y = \frac{P}{2} - x$

$P = 36 \quad \Rightarrow \quad y = 18 - x$

When the cylinder is formed: $x = 2\pi r$
and $h = y = 18 - x$

Volume of the cylinder: $V = \pi r^2 h$

$V(x) = \pi \left(\frac{x}{2\pi}\right)^2 (18 - x)$



$$= \pi \frac{x^2}{4\pi^2}(18 - x)$$

$$= \frac{1}{4\pi}\left(18x^2 - x^3\right)$$

$$V'(x) = \frac{1}{4\pi}\left(36x - 3x^2\right) = 0 \quad 3x(12 - x) = 0 \to \begin{cases} \cancel{x = 0} & \textit{no cylinder} \\ x = 12 \end{cases}$$

$$V''(x) = \frac{3}{4\pi}(12 - 2x) \quad \to \quad V''(12) < 0$$

There is a maximum at $x = 12$ *cm*, and $y = 18 - 12 = 6$ *cm*.

b) $V(x) = \pi x^2(18 - x)$

$$V'(x) = 2\pi x(18 - x) - \pi x^2 = 36\pi x - 2\pi x^2 - \pi x^2 = 36\pi x - 3\pi x^2 = 0$$

$$3\pi x(12 - x) = 0 \to \begin{cases} \cancel{x = 0} & \textit{no cylinder} \\ x = 12 \end{cases}$$

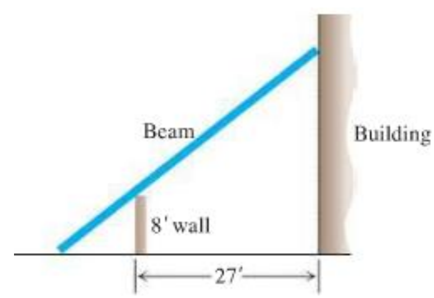
$$V''(x) = 36\pi - 6\pi x \quad \to \quad V''(12) < 0$$

There is a ***maximum*** at $x = 12$ *cm*, and $y = 18 - 12 = 6$ *cm*.

Exercise

The 8-*feet* wall stands 27 *feet* from the building. Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.

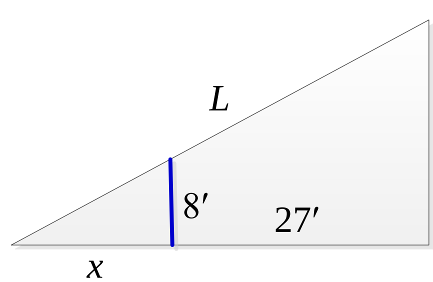
Solution



$$\frac{h}{8} = \frac{x + 27}{x} = 1 + \frac{27}{x} \quad \Rightarrow \quad h = 8 + \frac{216}{x}$$

Using Pythagorean Theorem: $L^2 = h^2 + (x + 27)^2$

$$L(x) = \sqrt{\left(8 + \frac{216}{x}\right)^2 + (x + 27)^2}$$



If we let: $f(x) = \left(8 + \frac{216}{x}\right)^2 + (x + 27)^2$

$$f'(x) = 2\left(8 + \frac{216}{x}\right)\left(-\frac{216}{x^2}\right) + 2(x + 27)(1)$$

$$= 2(8)\left(1 + \frac{27}{x}\right)\left(-\frac{216}{x^2}\right) + 2(x + 27)$$

$$= 2(x + 27)\left(-\frac{1728}{x^3} + 1\right)$$

$$= 2(x + 27)\left(1 - \frac{1728}{x^3}\right) = 0$$

$$\rightarrow \begin{cases} x+27=0 & \rightarrow x=-27 \\ 1-\frac{1728}{x^3}=0 & \rightarrow x^3=1728 \end{cases} \Rightarrow \boxed{x=\sqrt[3]{1728}=12}$$

$$\boxed{L(12)=\sqrt{\left(8+\frac{216}{12}\right)^2+(12+27)^2} \approx 46.87\ \ ft}$$

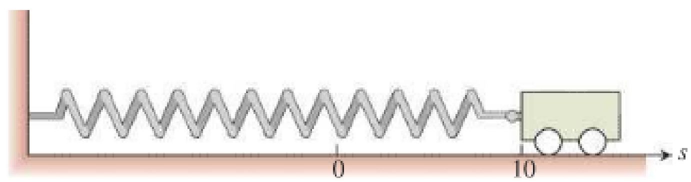
Exercise

A small frictionless cart, attached to the wall by a spring, is pulled 10 *cm* from its rest position and released at time $t = 0$ to roll back and forth for 4 *sec*. Its position at time t is $s=10\cos \pi t$

a) What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?

b) Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?

Solution



a) $s=10\cos \pi t$

$v=s'=-10\pi \sin \pi t$

$speed=|v|=10\pi|\sin \pi t|$

The car moving fast when $|\sin \pi t|=1$

The maximum speed is: $=10\pi \approx 31.42\ cm/\sec$

The cart is moving fastest when $|\sin \pi t|=1 \quad \Rightarrow \quad \pi t=\frac{\pi}{2}+k\pi$

$$t=\frac{1}{2}+k; \quad 0 \le t \le 4$$

$$a=-10\pi^2 \cos \pi t$$

$$\rightarrow \begin{cases} t=0.5 & \rightarrow s=10\cos\frac{\pi}{2}=0 \\ t=1.5 & \rightarrow s=10\cos\frac{3\pi}{2}=0 \\ t=2.5 & \rightarrow s=10\cos\frac{5\pi}{2}=0 \\ t=3.5 & \rightarrow s=10\cos\frac{7\pi}{2}=0 \end{cases} \Rightarrow \boxed{|a|=0\ \ cm/sec^2}$$

b) $|a|=10\pi^2|\cos \pi t|$

The magnitude of the acceleration is greatest when

$$|\cos \pi t|=1 \quad at \quad \boxed{t=0.0,\ 1.0,\ 2.0,\ 3.0,\ 4.0\ \sec}$$

The position of the cart at these times is $|s|=10\ cm$ and the speed is 0 *cm/sec*.

Exercise

The owner of a retail lumber store wants to construct a fence an outdoor storage are adjacent to the store, using all of the store as part of one side of the area. Find the dimensions that will enclose the largest area if

a) 240 feet fencing material are used.

b) 400 feet fencing material are used.

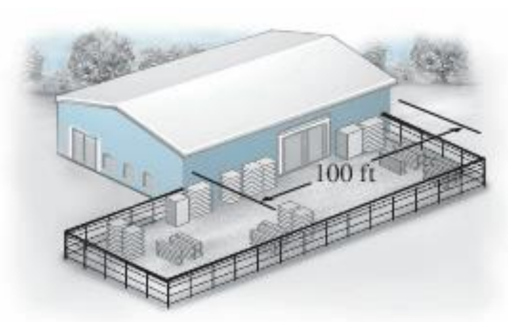
Solution

a) Let x and y be the width and the length of the rectangle respectively.

$$2x+2y-100=240$$

$$2x+2y=340$$

$$x+y=170 \quad \rightarrow \quad y=170-x$$



Area: $A=xy$

$$=x(170-x)$$

$$=170x-x^2 \qquad 100 \le x \le 170$$

$$A'=170-2x=0$$

$x=\frac{170}{2}=85$ which is not in the domain.

$$A(100)=170(100)-(100)^2=7,000$$

$$A(170)=170(170)-(170)^2=0$$

Thus, the maximum occurs when $x=100$ and $y=170-100=70$

b) $2x+2y-100=400$

$$2x+2y=500$$

$$x+y=250 \quad \rightarrow \quad y=250-x$$

$$A=xy=x(250-x)$$

$$=250x-x^2 \qquad 100 \le x \le 250$$

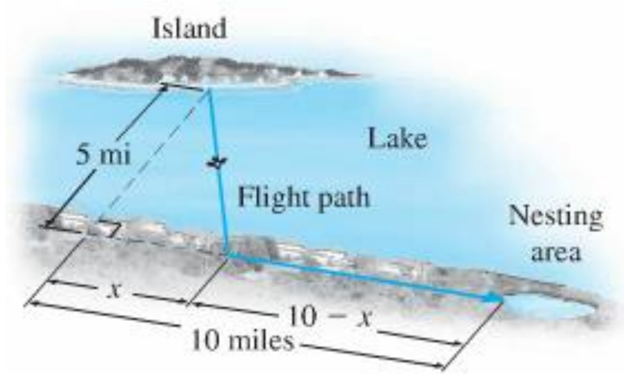
$$A'=250-2x=0 \quad \rightarrow \quad \boxed{x=125}$$

$$A(125)=250(125)-(125)^2=15,625$$

Thus, the maximum occurs when $x=125$ *ft* and $y=250-125=125$ *ft*

Exercise

Some birds tend to avoid flights over large bodies of water during daylight hours. Suppose that an adult bird with this tendency is taken from its nesting area on the edge of a large lake to an island 5 miles offshore and is then release.



a) If it takes only 1.4 times as much energy to fly over water as land, how far up the shore (x, in miles) should the bird head to minimize the total energy expended in returning to the nesting area?

b) If it takes only 1.1 times as much energy to fly over water as land, how far up the shore should the bird head to minimize the total energy expended in returning to the nesting area?

Solution

a) Let the energy to fly over land be 1 *unit*; then the energy to fly over the water is 1.4 *units*.

$$(\text{flight path})^2 = x^2 + 5^2$$

Total Energy:

$$E(x) = (1.4)\sqrt{x^2+25} + (1)(10-x)$$

$$= 1.4\sqrt{x^2+25} + 10 - x$$

$$E'(x) = \frac{1.4(2x)}{2\sqrt{x^2+25}} - 1$$

$$= \frac{1.4x}{\sqrt{x^2+25}} - 1 = 0$$

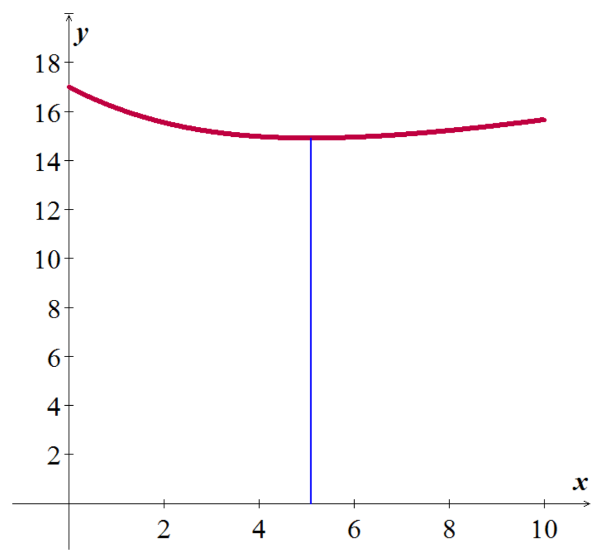
$$\frac{1.4x}{\sqrt{x^2+25}} = 1$$

$$1.4x = \sqrt{x^2+25} \qquad (1.4x)^2 = \left(\sqrt{x^2+25}\right)^2$$

$$1.96x^2 = x^2 + 25$$

$$0.96x^2 = 25$$

$$x^2 = 26.04 \quad \rightarrow \quad x = \pm 5.1$$



0	5.1
$E'(1) < 0$	$E'(6) > 0$
Decreasing	*Increasing*

Thus, the critical value is $\boxed{x = 5.1}$

Thus, the energy will be minimum when $x = 5.1$.

$$E(5.1)=1.4\sqrt{(5.1)^2+25}+10-5.1=\underline{14.9}$$

$$E(10)=1.4\sqrt{(10)^2+25}+10-10=15.65$$

Thus, the absolute minimum occurs when $x=5.1$ *miles*.

b) $E(x)=1.1\sqrt{x^2+25}+10-x \qquad 0\le x\le 10$

$$E'(x)=\frac{1.1x}{\sqrt{x^2+25}}-1=0$$

$$1.1x=\sqrt{x^2+25} \qquad (1.1x)^2=\left(\sqrt{x^2+25}\right)^2$$

$$1.21x^2=x^2+25$$

$$0.21x^2=25$$

$$x^2=119.05 \quad \to \quad x=\pm 10.91$$

The critical value $x=10.91>10$

$$E(0)=1.1\sqrt{(0)^2+25}+10-0=\underline{15.5}$$

$$E(10)=1.1\sqrt{(10)^2+25}+10-10=\underline{12.30}$$

Therefore, the absolute minimum occurs when $x=10$

Exercise

A pharmacy has a uniform annual demand for 200 bottles of a certain antibiotic. It costs \$10 to store one bottle for one year and \$40 to place an order. How many times during the year should the pharmacy order the antibiotic in order to minimize the total storage and reorder costs?

Solution

Let x be the number of times the pharmacy places order.
Let y be the number of demands.

The cost: $C=40x+10\left(\frac{y}{2}\right)=40x+5y$

We also have $xy=200 \quad \Rightarrow \quad y=\frac{200}{x}$

$$C(x)=40x+5\left(\frac{200}{x}\right)=40x+\frac{1,000}{x}$$

$$C'(x)=40-\frac{1,000}{x^2}=0$$

$$\frac{1,000}{x^2}=40 \quad \to \quad x^2=\frac{1,000}{40}=25 \quad \Rightarrow x=\pm 5 \quad \to \quad \boxed{x=5} \text{ since } x>0$$

$C''(x) = \dfrac{2,000}{x^3} > 0$ the function has an absolute minimum at $x = 5$

$C(5) = 40(5) + \dfrac{1,000}{5}$ $\underline{= \$400 \,|}$ is the minimum cost.

Exercise

A 300-room hotel in Las Vegas is filled to capacity every night at \$80 a room. For each \$1 increase in rent, 3 fewer rooms are rented. If each rented room costs \$10 to service per day, how much should be the management charge for each room to maximize gross profit? What is the maximum gross profit?

Solution

Let x: number of dollar increases in the rate per night.

Total number of rooms rented: $300 - 3x$

Rate per night: $80 + x$

Total income = (total number of rooms rented) (rate – 10)

$$y(x) = (300 - 3x)(80 + x - 10) \qquad 0 \le x \le 100$$

$$= (300 - 3x)(70 + x)$$

$$= 21,000 + 90x - 3x^2$$

$$y'(x) = 90 - 6x = 0$$

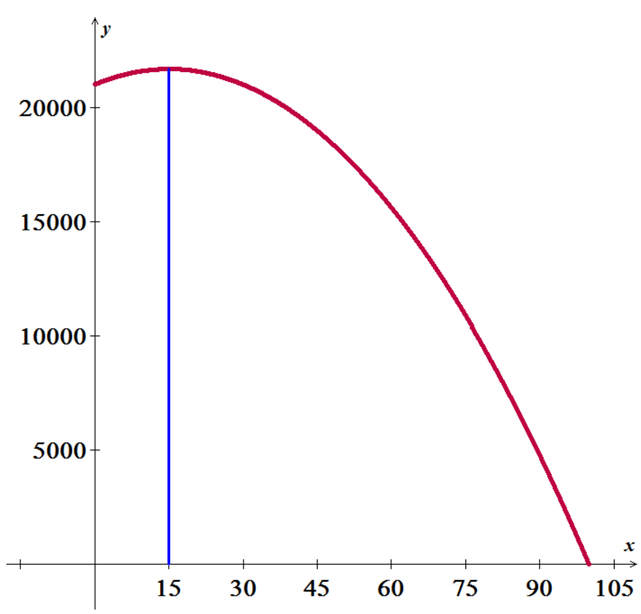
$$6x = 90 \quad \to \quad \boxed{x = 15} \quad (CN)$$

$$y''(x) = -6 < 0.$$

Maximum income: $y(15) = 21,000 + 90(15) - 3(15)^2$

$\underline{= \$21,675.00 \,|}$

The rate per night: $80 + 15$ $\underline{= \$95 \,|}$



Exercise

A multimedia company anticipates that there will be a demand for 20,000 copies of a certain DVD during the next year. It costs the company \$0.50 to store a DVD for one year. Each time it must make additional DVDs, it costs \$200 to set up the equipment. How many DVDs should the company make during each production run to minimize its total storage and setup costs?

Solution

Let assume that there are 250 working days and then the daily demand is: $\frac{20,000}{250} = 80\ \ DVDs$

Let: x: number of DVDs manufactured during each production run.

y: number of production runs.

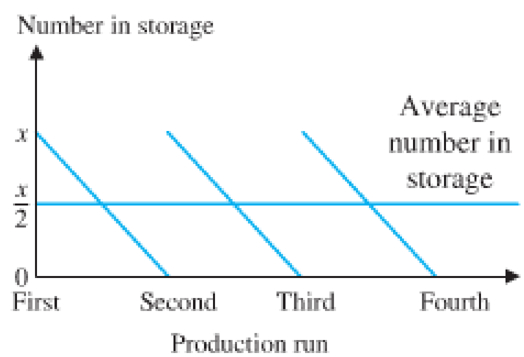
The number of DVDs in storage between production runs will decrease from x to 0, and the average number in storage each day is $\frac{x}{2}$.

Since it costs \$0.50 to store a DVD for one year, the total storage cost is $(0.5)\frac{x}{2} = 0.25x$.

The total cost is:

Total cost = setup cost + storage cost

$$C = 200y + 0.25x$$



The total number of DVDs produced is xy. $xy = 20,000$

$$y = \frac{20,000}{x}$$

$$C(x) = 200\left(\frac{20,000}{x}\right) + 0.25x \qquad 1 \le x \le 20,000$$

$$= \frac{4,000,000}{x} + 0.25x$$

$$C'(x) = -\frac{4,000,000}{x^2} + 0.25 = 0$$

$$0.25 = \frac{4,000,000}{x^2} \quad \rightarrow \quad x^2 = \frac{4,000,000}{0.25}$$

$$\Rightarrow x = \sqrt{\frac{4,000,000}{0.25}} = 4,000$$

0	4,000
$C'(1) < 0$	$C'(10,000) > 0$
Decreasing	***Increasing***

$$C(4,000) = \frac{4,000,000}{4,000} + 0.25(4,000)$$

$$= \$2,000$$

$$y = \frac{20,000}{4,000}$$

$$= 5$$

The company will minimize its total cost by making 4,000 DVDs five times during the year.

Exercise

A university student center sells 1,600 cups of coffee per day at a price of $2.40.

a) A market survey shows that for every $0.05 reduction in price, 50 more cups of coffee will be sold. How much should be the student center charge for a cup of coffee in order to maximize revenue?

b) A different market survey shows that for every $0.10 reduction in the original $2.40 price, 60 more cups of coffee will be sold. Now how much should the student center charge for a cup of coffee in order to maximize revenue?

Solution

a) Let x: number of price reductions

The price of a cup of coffee will be: $p = 2.40 - 0.05x$

The number of cups sold will be: $1,600 + 50x$

Revenue: $R(x) = x \cdot p(x)$

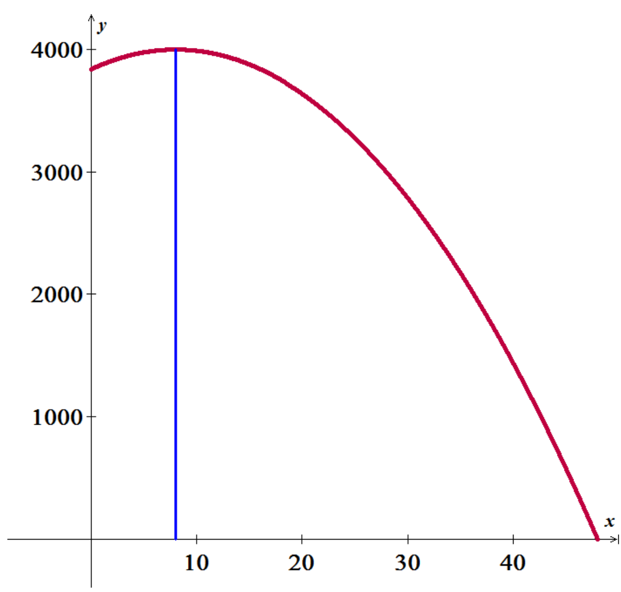
$$= (1,600 + 50x)(2.40 + 0.05x)$$

$$= 3,840 + 40x - 2.5x^2$$

$R'(x) = 40 - 5x = 0$

$$5x = 40 \quad \rightarrow \quad \boxed{x = 8} \quad (CN)$$

$R''(x) = -5 < 0$ that implies R has an absolute maximum.



Maximum revenue: $R(8) = 3,840 + 40(8) - 2.5(8)^2 \ \underline{= \$4,000.00}$ when 1,600 cups of coffee sold at the price $p = 2.40 - 0.05(8) \ \underline{= \$2.00}$ per cup.

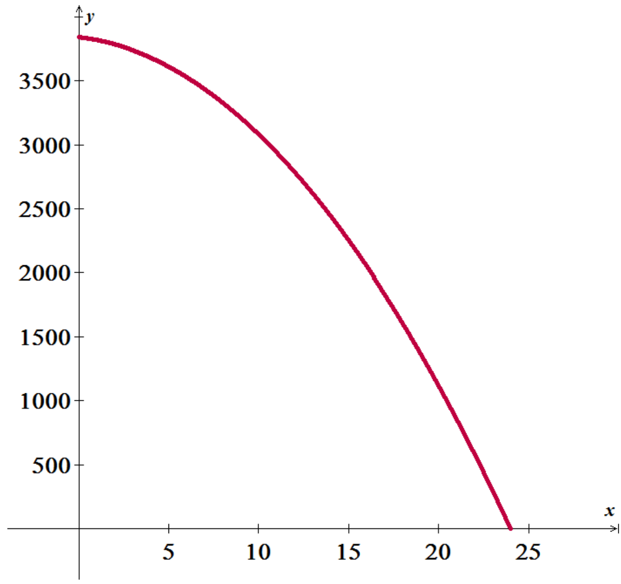
b) *Revenue*: $R(x) = (1,600 + 60x)(2.40 - .10x)$

$$= 3,840 - 16x - 6x^2$$

$R'(x) = -16 - 12x = 0$

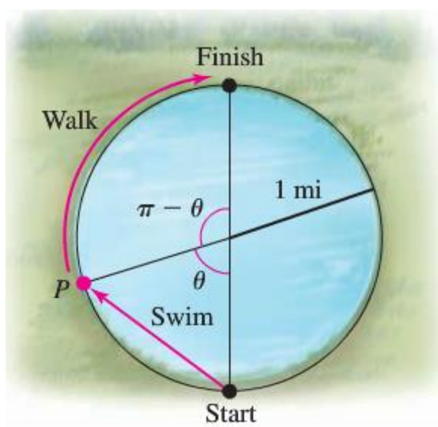
$$12x = -16 \quad \rightarrow \quad \boxed{x = -\frac{4}{3} < 0} \qquad (CN)$$

Thus, $R(x)$ is decreasing and its maximum occurs at $x = 0$ they should charge \$2.40 per cup.



Exercise

Suppose you are standing on the shore of a circular pond with radius 1 *mile* and you want to get to a point on the shore directly opposite your position (on the other end of a diameter). You plan to swim at 2 *mi/hr* from your current position to another point *P* on the shore and then walk at 3 *mi/hr* along the shore to the terminal point.



How should you choose *P* to minimize the total time for the trip?

Solution

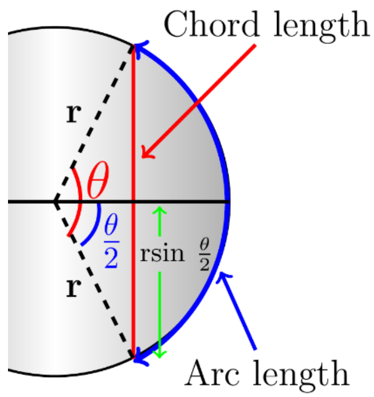
$$\frac{\underline{distance}}{rate} = \frac{2\sin\frac{\theta}{2}}{2} = \sin\frac{\theta}{2}$$

The length of the walking leg is the length of the arc of the circle corresponding to the angle $\pi - \theta$.

For a circle of radius r, the arc length corresponding to an angle θ is $r\theta$.

$$\frac{\underline{distance}}{rate} = \frac{\pi - \theta}{3}$$

The total travel time for the trip is the objective function



$$T(\theta)=\sin\frac{\theta}{2}+\frac{\pi-\theta}{3}; \quad 0\le\theta\le\pi$$

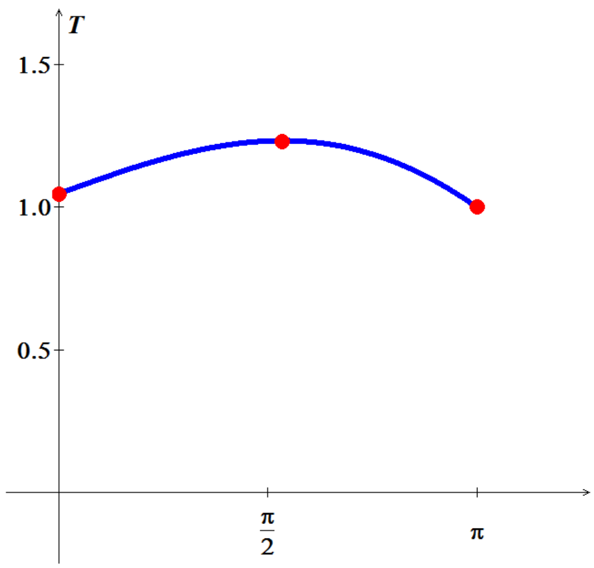
$$T'(\theta)=\frac{1}{2}\cos\frac{\theta}{2}-\frac{1}{3}=0$$

$$\to \ \cos\frac{\theta}{2}=\frac{2}{3} \quad \Rightarrow \quad \theta=2\cos^{-1}\frac{2}{3} \ \underline{\approx 1.68\ rad|} \quad (CN)$$

$T(1.68)\ \underline{\approx 1.23\ hr|}$ ***Max. travel*** time

$T(0)\ \underline{\approx 1.05\ hr|}$

$T(\pi)\ \underline{=1\ hr|}$ ***Min. travel*** time for the entire trip is done swimming.



Exercise

Four *feet* of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?

Solution

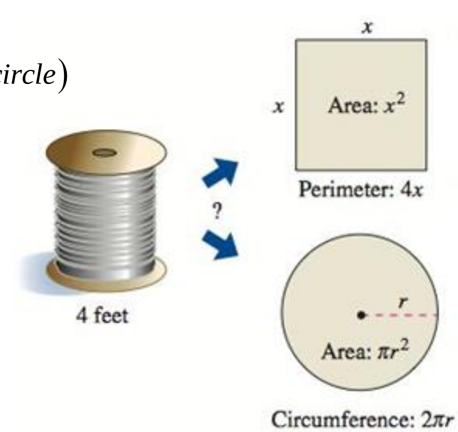
Total Area $A=(\text{area of square})+(\text{area of circle})$

$$A=x^2+\pi r^2$$

Total length of wire $4=(\text{perimeter of square})+(\text{circumference of circle})$

$$4=4x+2\pi r$$

$$\underline{r=\frac{2}{\pi}(1-x)|}$$



$$A=x^2+\frac{4}{\pi}(1-x)^2$$

$$=x^2+\frac{4}{\pi}\left(1-2x+x^2\right)$$

$$= \frac{1}{\pi}\left(\pi x^2 + 4 - 8x + 4x^2\right)$$

$$= \frac{1}{\pi}\left((\pi+4)x^2 - 8x + 4\right)$$

$$\frac{dA}{dx} = \frac{1}{\pi}\left(2(\pi+4)x - 8\right) = 0$$

$$(\pi+4)x = 4$$

$$\boxed{x = \frac{4}{\pi+4}}$$

$$A(0) = \frac{4}{\pi} \qquad A\left(\frac{4}{\pi+4}\right) \approx 0.56 \qquad A(1) = 1$$

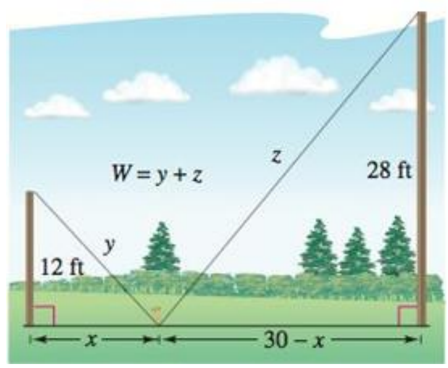
The maximum area occurs when $x = 0$. That is, ***all*** the wire is used for the circle.

Exercise

Two posts, one 12 *feet* high and the other 28 *feet* high, stand 30 *feet* apart. They are to be stayed by two wires, attached to a single stake, running from ground level on the top of each post. Where should the stake be placed to use the least amount of wire?

Solution

Let $W = y + z$ (Total wire length)



$$x^2 + 144 = y^2 \qquad (30-x)^2 + 28^2 = z^2$$

$$y = \sqrt{x^2+144} \qquad z = \sqrt{(30-x)^2 + 28^2}$$

$$W = \sqrt{x^2+144} + \sqrt{x^2 - 60x + 1684}$$

$$\frac{dW}{dx} = \frac{x}{\sqrt{x^2+144}} + \frac{x-30}{\sqrt{x^2-60x+1684}} = 0$$

$$x\sqrt{x^2-60x+1684} = (30-x)\sqrt{x^2+144}$$

$$x^2\left(x^2-60x+1684\right) = (30-x)^2\left(x^2+144\right)$$

$$x^4 - 60x^3 + 1684x^2 = x^4 - 60x^3 + 1044x^2 - 8640x + 129,600$$

$$640x^2 + 8,640x - 129,600 = 0$$

$$2x^2 + 27x - 405 = 0 \qquad x = \frac{-27 \pm \sqrt{729+3240}}{4} = \frac{-27 \pm 63}{4}$$

$$x = 9, \ \cancel{-\frac{45}{2}}$$

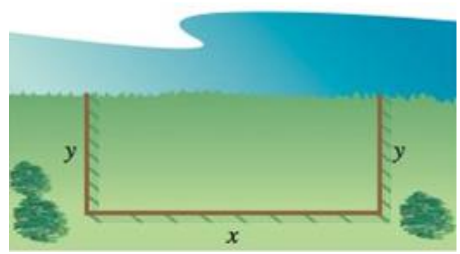
The wire should be staked at least 9 *feet* from the 12-*foot* pole.

Exercise

A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 245,000 m^2 in order to provide enough grass for the herd. No fencing is needed along the river. What dimensions will require the least amount of fencing?

Solution

Given: $xy = 245,000 \rightarrow y = \dfrac{245,000}{x}$



$$S = x + 2y$$
$$= x + \frac{490,000}{x}$$
$$\frac{dS}{dx} = 1 - \frac{490,000}{x^2} = 0$$
$$x^2 = 490,000 \Rightarrow \boxed{x = 700\ m}$$
$$y = \frac{245,000}{700}$$
$$\boxed{= 350\ m}$$

Exercise

A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window. Find the dimensions of a Norman window of maximum area when the total perimeter is 16 *feet*.

Solution

$$P = x + 2y + \pi\frac{x}{2} = 16 \qquad \text{circumference} = \pi r$$
$$2y = 16 - x - \frac{\pi x}{2} \rightarrow \boxed{y = \frac{32 - 2x - \pi x}{4}}$$

$$A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$$
$$= x\frac{32 - 2x - \pi x}{4} + \frac{\pi}{8}x^2$$
$$= 8x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2$$
$$= 8x - \frac{1}{2}x^2 - \frac{\pi}{8}x^2$$
$$\frac{dA}{dx} = 8 - \left(1 + \frac{\pi}{4}\right)x = 0$$
$$\left(\frac{4+\pi}{4}\right)x = 8 \rightarrow \boxed{x = \frac{32}{4+\pi}}$$
$$y = \frac{1}{4}\left(32 - \frac{64}{4+\pi} - \frac{32\pi}{4+\pi}\right)$$
$$= \frac{1}{4}\left(\frac{128 + 32\pi - 64 - 32\pi}{4+\pi}\right)$$

$$= \boxed{\frac{16}{4+\pi}}$$

$$\text{Maximum area } = \frac{32}{4+\pi} \cdot \frac{16}{4+\pi}$$

$$= \boxed{\frac{512}{(4+\pi)^2}}$$

when $x = \frac{32}{4+\pi} \quad y = \frac{16}{4+\pi}$

Exercise

A wooden beam has a rectangular cross section of height h and width w, the strength S of the beam is directly proportional to the width and the square of the height. What are the dimensions of the strongest beam that can be cut from a round log of diameter 20 *inches*?

$S = kwh^2 \quad (k: \text{ proportional constant})$

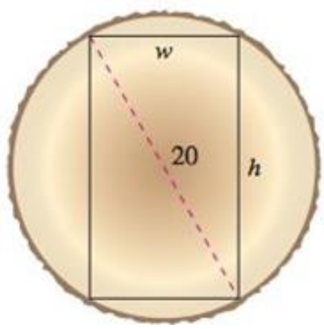
Solution

$$h^2 + w^2 = 400 \rightarrow h^2 = 400 - w^2$$

$$S = kwh^2$$

$$= kw\left(400 - w^2\right)$$

$$= k\left(400w - w^3\right)$$



$$\frac{dS}{dw} = k\left(400 - 3w^2\right) = 0$$

$$w = \frac{20}{\sqrt{3}}$$

$$= \boxed{\frac{20\sqrt{3}}{3} \quad in}$$

$$h = \sqrt{400 - \frac{400}{3}}$$

$$= \frac{20\sqrt{2}}{\sqrt{3}}$$

$$= \boxed{\frac{20\sqrt{6}}{3} \quad in}$$

Exercise

A light source is located over the center of a circular table of diameter 4 *feet*. Find the height h of the light source such that the illumination I at the perimeter of the table is maximum when

$$I = \frac{k\sin\alpha}{s^2}$$

where s is the slant height

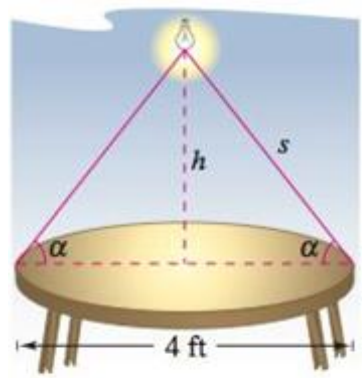
α is the angle at which the light strikes the table

k is a constant.

Solution

$$\tan\alpha = \frac{h}{2} \quad \rightarrow \quad h = 2\tan\alpha$$

$$\sin\alpha = \frac{h}{s} \quad \rightarrow \quad \boxed{s = \frac{h}{\sin\alpha} = \frac{2\tan\alpha}{\sin\alpha} = 2\sec\alpha}$$



$$\begin{aligned} I &= \frac{k\sin\alpha}{s^2} \\ &= \frac{k\sin\alpha}{4\sec^2\alpha} \\ &= \frac{k}{4}\sin\alpha\cos^2\alpha \\ &= \frac{k}{4}\sin\alpha\left(1-\sin^2\alpha\right) \\ &= \frac{k}{4}\left(\sin\alpha - \sin^3\alpha\right) \end{aligned}$$

$$\begin{aligned} \frac{dI}{d\alpha} &= \frac{k}{4}\left(\cos\alpha - 3\sin^2\alpha\cos\alpha\right) \\ &= \frac{k}{4}\cos\alpha\left(1-3\sin^2\alpha\right) = 0 \end{aligned}$$

$$\begin{cases} \cos\alpha = 0 & \cancel{\alpha = \frac{\pi}{2}} \\ \sin^2\alpha = \frac{1}{3} & \rightarrow \boxed{\sin\alpha = \frac{1}{\sqrt{3}}} \end{cases} \quad (\alpha \text{ is acute})$$

$$\cos\alpha = \sqrt{1-\frac{1}{3}} = \frac{\sqrt{2}}{\sqrt{3}}$$

$$h = 2\tan\alpha = 2\left(\frac{1}{\sqrt{2}}\right)$$

$$\boxed{= \sqrt{2} \ \ ft}$$

Exercise

When light waves traveling in a transparent medium strike the surface of a second transparent medium, they change direction. This change of direction is called refraction and is defined by ***Snell's Law of Refraction***,

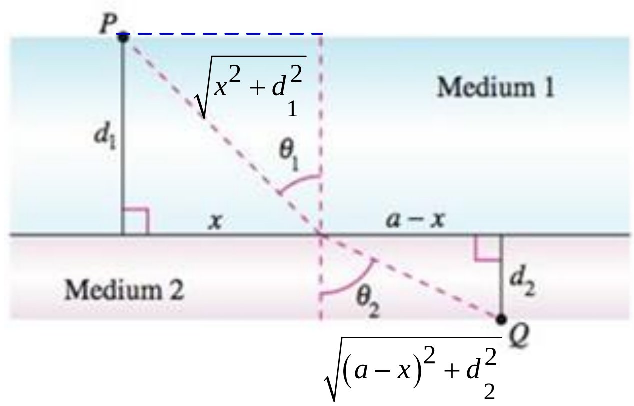
$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

Where θ_1 and θ_2 are the magnitudes of the angles.

v_1 and v_2 are the velocities of light in the two media.

Show that the light waves traveling from P to Q follow the path of the minimum time.

Solution



$$Time = \frac{distance}{speed}$$

$$T(x) = \frac{\sqrt{x^2+d_1^2}}{v_1} + \frac{\sqrt{(a-x)^2+d_2^2}}{v_2}$$

$$T'(x) = \frac{x}{v_1\sqrt{x^2+d_1^2}} + \frac{x-a}{v_2\sqrt{(a-x)^2+d_2^2}} = 0$$

$$\frac{x}{v_1\sqrt{x^2+d_1^2}} = \frac{x-a}{v_2\sqrt{(a-x)^2+d_2^2}}$$

From the graph:

$$\sin\theta_1 = \frac{x}{\sqrt{x^2+d_1^2}} \qquad \sin\theta_2 = \frac{x-a}{\sqrt{(a-x)^2+d_2^2}}$$

$$\Rightarrow \boxed{\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}}$$

$$T'(0) = \frac{-a}{v_2\sqrt{a^2+d_2^2}} < 0$$

$$T'(a) = \frac{a}{v_1\sqrt{a^2+d_1^2}} > 0$$

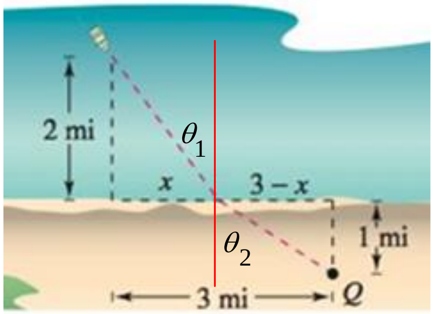
A minimum value of $T(x)$ occurs at $x \in (0,\ a)$ (exactly one critical point)

$\sin\theta_1$ *decreases* $\sin\theta_2$ *increases*, so Snell's Law can hold for at most 1 value of x.

Exercise

You are in a boat 2 *miles* from the nearest point on the coast. You are to go to a point Q that is 3 *miles* down the coast and 1 *mile* inland. You can row at 3 mph and walk 4 mph. Toward what point on the coast should you row in order to reach Q in the least time?

Solution



$$T(x) = \frac{\sqrt{x^2+4}}{v_1} + \frac{\sqrt{(3-x)^2+1}}{v_2} \qquad Time = \frac{distance}{speed}$$

$$T'(x) = \frac{x}{v_1\sqrt{x^2+4}} + \frac{x-3}{v_2\sqrt{x^2-6x+10}} = 0$$

$$\frac{x}{v_1\sqrt{x^2+4}} = \frac{3-x}{v_2\sqrt{x^2-6x+10}}$$

From the graph: $\sin\theta_1 = \frac{x}{\sqrt{x^2+4}}$ $\quad \sin\theta_2 = \frac{x-3}{\sqrt{(3-x)^2+1}}$

$$\Rightarrow \boxed{\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}}$$

$$T'(0) = \frac{-3}{v_2\sqrt{10}} < 0$$

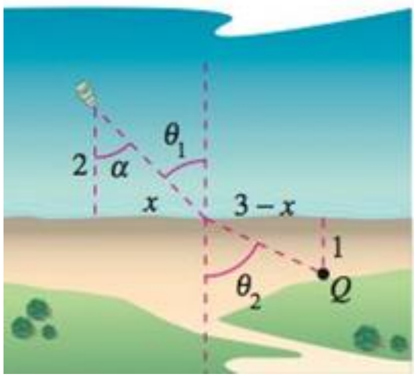
$$T'(3) = \frac{3}{v_1\sqrt{13}} > 0$$

A minimum value of $T(x)$ occurs at $x \in (0,\ a)$ (exactly one critical point)

Exercise

You are in a boat 2 *miles* from the nearest point on the coast. You are to go to a point Q that is 3 *miles* down the coast and 1 *mile* inland. You can row at 2 mph and walk 4 mph. Toward what point on the coast should you row in order to reach Q in the least time?

Solution



Given: $v_1 = 2\ mph \quad v_2 = 4\ mph$

$$Time:\ T(x) = \frac{\sqrt{x^2+4}}{2} + \frac{\sqrt{(3-x)^2+1}}{4}$$

$$T'(x) = \frac{x}{2\sqrt{x^2+4}} + \frac{x-3}{2\sqrt{x^2-6x+10}} = 0$$

$$\frac{x}{\sqrt{x^2+4}} = \frac{3-x}{2\sqrt{x^2-6x+10}}$$

$$\frac{x^2}{x^2+4}=\frac{(3-x)^2}{4\left(x^2-6x+10\right)}$$

$$4x^4-24x^3+40x^2=\left(9-6x+x^2\right)\left(x^2+4\right)$$

$$4x^4-24x^3+40x^2=x^4-6x^3+13x^2-24x+36$$

$$3x^4-18x^3+27x^2+24x-36=0$$

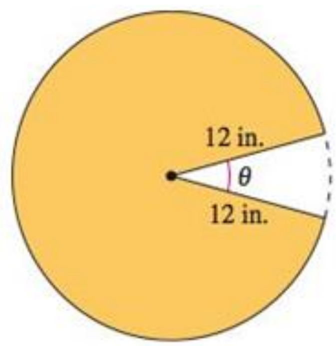
$$x^4-6x^3+9x^2+8x-12=0$$

One of the root $x=1 \in [0,\ 3]$ where minimum value of $T(x)$ occurs

Exercise

A sector with central angle θ is cut from a circle of radius 12 *inches*, and the edges of the sector are brought together to form a cone. Find the magnitude of θ such that the volume of the cone is a maximum.

Solution



$$V=\frac{1}{3}\pi r^2 h$$

$$=\frac{1}{3}\pi r^2\sqrt{144-r^2}$$

$$\frac{dV}{dr}=\frac{2}{3}\pi r\sqrt{144-r^2}-\frac{1}{3}\frac{\pi r^3}{\sqrt{144-r^2}}$$

$$=\frac{\pi r}{3}\left(\frac{288-2r^2-r^2}{\sqrt{144-r^2}}\right)$$

$$=\pi r\left(\frac{96-r^2}{\sqrt{144-r^2}}\right)=0; \quad r=\sqrt{96}\ \underline{=4\sqrt{6}|} \quad r=0$$

$$h=\sqrt{144-96}=\sqrt{48}$$

$$\underline{=4\sqrt{3}\,|}$$

Area of the circle: $A=\pi(12)^2=144\pi$

Lateral surface area of cone:

$$S=\pi\left(4\sqrt{6}\right)\sqrt{\left(4\sqrt{6}\right)^2+\left(4\sqrt{3}\right)^2}$$

$$=4\pi\sqrt{6}\sqrt{96+48}$$

$$\underline{=48\pi\sqrt{6}\,|}$$

Are of the sector: $144\pi-48\pi\sqrt{6}=\frac{1}{2}\theta r^2$

$$48\pi\left(3-\sqrt{6}\right)=\frac{1}{2}\theta(144)$$

$$\theta=\frac{48\pi\left(3-\sqrt{6}\right)}{72}$$

$$\boxed{=\frac{2\pi}{3}\left(3-\sqrt{6}\right)\ rad}$$

Exercise

A rancher has 400 *feet* of fencing with which to enclose two adjacent rectangular corrals. What dimensions should be used so that the enclosed area will be a maximum?

Solution

$$P=4x+3y=400 \quad\rightarrow\quad y=\frac{4}{3}(100-x)$$

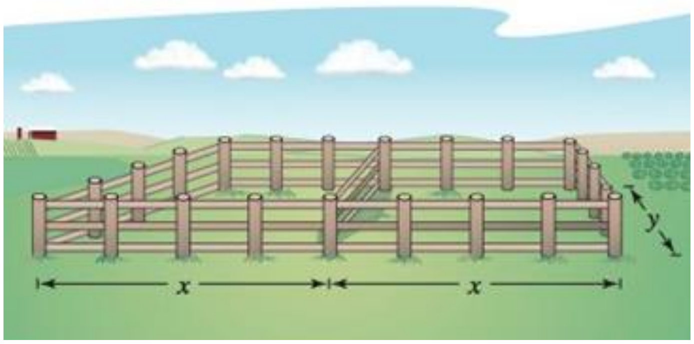
$$A=2xy=\frac{8}{3}x(100-x)$$

$$=\frac{8}{3}\left(100x-x^2\right)$$

$$\frac{dA}{dx}=\frac{8}{3}(100-2x)=0 \quad\rightarrow\quad \boxed{x=50\ \ ft}$$

$$y=\frac{4}{3}(100-50)$$

$$\boxed{=\frac{200}{3}\ \ ft}$$



Exercise

The amount of illumination of a surface is proportional to the intensity of the light source, inversely proportional to the square of the distance from the light source, and proportional to $\sin\theta$, where θ is the angle at which the light strikes the surface. A rectangular room measures 10 *feet* by 24 *feet*, with a 10-*foot* ceiling. Determine the height at which the light should be placed to allow the corners of the floor to receive as much light as possible.

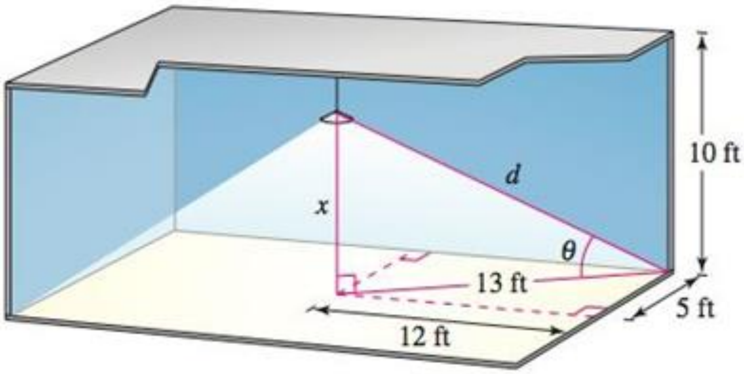
Solution

$$d=\sqrt{x^2+13^2}$$

$$\sin\theta=\frac{x}{d}=\frac{x}{\sqrt{x^2+169}}$$

Let A be the amount of illumination at one of the corners

$$A=\frac{kI}{x^2+169}\sin\theta$$



$$= \frac{kxI}{\left(x^2+169\right)^{3/2}}$$

$$\frac{dA}{dx} = \frac{kI}{\left(x^2+169\right)^{5/2}}\left(x^2+169-3x^2\right)$$

$$= \frac{kI}{\left(x^2+169\right)^{5/2}}\left(169-2x^2\right) = 0$$

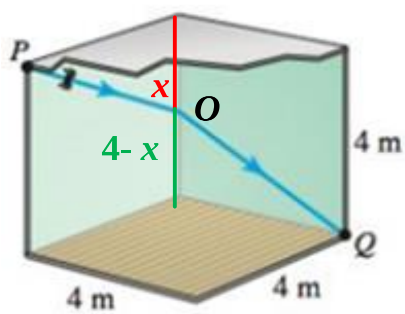
$$x = \frac{13}{\sqrt{2}}\ ft$$

Exercise

Consider a room in the shape of a cube, 4 *meters* on each side. A bug at point *P* wants to walk to point *Q* at the opposite corner. Determine the shortest path.

Solution

$$Distance = |PO| + |OQ|$$



$$f(x) = \sqrt{x^2+16} + \sqrt{(4-x)^2+16}$$

$$f'(x) = \frac{x}{\sqrt{x^2+16}} - \frac{4-x}{\sqrt{(4-x)^2+16}} = 0$$

$$\frac{x}{\sqrt{x^2+16}} = \frac{4-x}{\sqrt{(4-x)^2+16}}$$

$$\left(x\sqrt{(4-x)^2+16}\right)^2 = \left((4-x)\sqrt{x^2+16}\right)^2$$

$$x^2\left(x^2-8x+32\right) = \left(x^2-8x+16\right)\left(x^2+16\right)$$

$$x^4-8x^3+32x^2 = x^4-8x^3+16x^2+16x^2-128x+256$$

$$0 = -128x+256 \quad \to \quad x = 2$$

The bug should head towards the midpoint of the opposite side.

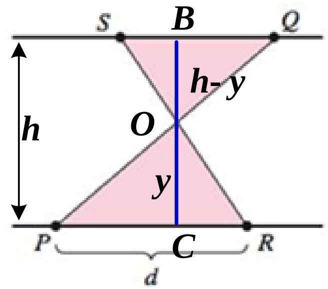
The shortest distance is the line passing through the midpoint.

Exercise

The line joining P and Q crosses the two parallel lines. The point R is d units from P. How far from Q should the point S be positioned so that the sum of the areas of the two shaded triangles is a minimum? So that the sum is a maximum?

Solution

$$Area:\ A(y)=\frac{1}{2}d\,|OC|+\frac{1}{2}|OB||SQ|$$



$$\tan\angle BOQ=\frac{|BQ|}{h-y}=\frac{|PC|}{y}\quad\to\quad |BQ|=\frac{h-y}{y}|PC|$$

$$\tan\angle BOS=\frac{|SB|}{h-y}=\frac{|CR|}{y}\quad\to\quad |SB|=\frac{h-y}{y}|CR|$$

$$\begin{aligned}|SQ|&=|BQ|+|SB|=\frac{h-y}{y}|PC|+\frac{h-y}{y}|CR|\\&=\frac{h-y}{y}\left(|PC|+|CR|\right)\\&=\frac{d(h-y)}{y}\end{aligned}$$

$$\begin{aligned}A(y)&=\frac{1}{2}dy+\frac{1}{2}(h-y)\frac{d(h-y)}{y}\\&=\frac{1}{2}d\left(y+\frac{(h-y)^2}{y}\right)\\&=\frac{1}{2}d\left(\frac{2y^2-2hy+h^2}{y}\right)\end{aligned}$$

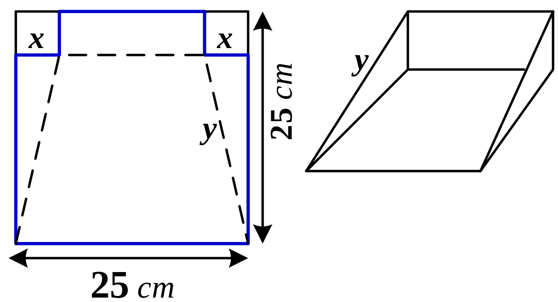
$$\begin{aligned}A'(y)&=\frac{1}{2}d\left(\frac{4y^2-2hy-2y^2+2hy-h^2}{y^2}\right)\\&=\frac{1}{2}d\left(\frac{2y^2-h^2}{y^2}\right)=0\quad\to\quad \boxed{y=\frac{h}{\sqrt{2}}}\end{aligned}$$

$$\begin{aligned}|SQ|&=\frac{h-y}{y}d\\&=\frac{\sqrt{2}}{h}\left(h-\frac{h}{\sqrt{2}}\right)d\\&=\frac{\sqrt{2}}{h}\left(h\frac{1-\sqrt{2}}{\sqrt{2}}\right)d\\&=\boxed{\left(1-\sqrt{2}\right)d}\end{aligned}$$

Exercise

Equal squares are cut out of two adjacent corners of a square of sheet metal having sides of length 25 *cm*. the three resulting flaps are bent up, to form the sides of a dustpan. Find the maximum volume of a dustpan made in this way.

Solution



$$x^2 + y^2 = (25 - x)^2$$

$$y^2 = 625 - 50x + x^2 - x^2 \quad \rightarrow \quad y = 5\sqrt{25 - 2x}$$

$$V = \frac{1}{2}xy(25 - 2x) + 2\left(\frac{1}{3}\right)\left(\frac{1}{2}xy\right)x$$

$$= 5x\sqrt{25 - 2x}\left(\frac{25}{2} - x + \frac{1}{3}x\right)$$

$$= 5x\sqrt{25 - 2x}\left(\frac{25}{2} - \frac{2}{3}x\right)$$

$$= \frac{5}{6}x\sqrt{25 - 2x}(75 - 4x)$$

$$= \frac{5}{6}\sqrt{25 - 2x}\left(75x - 4x^2\right)$$

$$\frac{dV}{dx} = \frac{5}{6}\left(\sqrt{25 - 2x}(75 - 8x) - \frac{75x - 4x^2}{\sqrt{25 - 2x}}\right)$$

$$= \frac{5}{6}\frac{(25 - 2x)(75 - 8x) - 75x + 4x^2}{\sqrt{25 - 2x}}$$

$$= \frac{5}{6}\frac{1{,}875 - 350x + 16x^2 - 75x + 4x^2}{\sqrt{25 - 2x}}$$

$$= \frac{5}{6}\frac{1{,}875 - 425x + 20x^2}{\sqrt{25 - 2x}}$$

$$= \frac{25}{6}\frac{4x^2 - 85x + 375}{\sqrt{25 - 2x}} = 0$$

$$4x^2 - 85x + 375 = 0 \quad \rightarrow \quad x = \frac{85 \pm 35}{8}\begin{cases} x = 15 \\ x = \frac{25}{4} \end{cases}$$

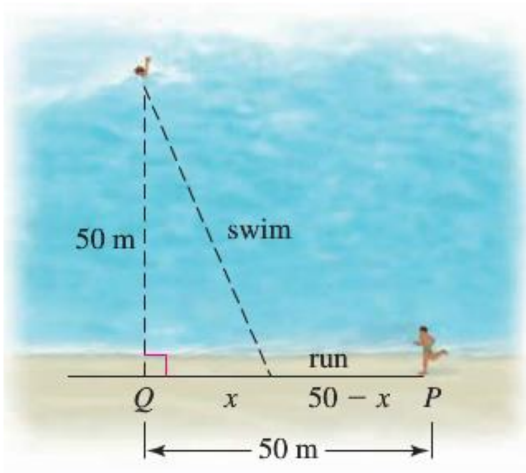
$$V(15) = \times$$

$$V\left(\frac{25}{4}\right) = \frac{5}{6}\sqrt{25 - \frac{25}{2}}\left(\frac{1{,}875}{4} - \frac{625}{4}\right)$$

$$\boxed{= \frac{15{,}625}{12\sqrt{2}} \ cm^3}$$

Exercise

You must get from a point P on the straight shore of a lake to a stranded swimmer who is 50 *m* from a point Q on the shore that is 50 *m* from you. If you can swim at a speed of 2 *m/s* and run at a speed of 4 *m/s*, at what point along the shore, x meters from Q, should you stop running and start swimming if you want to reach the swimmer in the minimum time?



a) Find the function T that gives the travel time as a function of x, where $0 \le x \le 50$

b) Find the critical point of T on (0, 50)

c) Evaluate T at the critical point and the endpoints ($x = 0$ and $x = 50$) to verify that the critical point corresponds to an absolute minimum. What is the minimum travel time?

d) Graph the function T to check your work.

Solution

a) $$Time = \frac{distance}{rate}$$

$$= \frac{distance}{rate}(swim) + \frac{distance}{rate}(run)$$

$$= T_s + T_r$$

Swimming distance $= \sqrt{x^2 + 50^2}$

Time swimming: $T_s = \frac{1}{2}\sqrt{x^2 + 2500}$

Running distance $= 50 - x$

Time Running: $T_r = \frac{50 - x}{4}$

$$T(x) = \frac{1}{2}\sqrt{x^2 + 2500} + \frac{50 - x}{4}$$

b) $$T'(x) = \frac{1}{2}\frac{x}{\sqrt{x^2 + 2500}} - \frac{1}{4} = 0$$

$$x = \frac{1}{2}\sqrt{x^2 + 2500}$$

$$4x^2 = x^2 + 2500$$

$$3x^2 = 2500 \quad \rightarrow \quad \boxed{x = \frac{50}{\sqrt{3}}}$$

$$T\left(\frac{50}{\sqrt{3}}\right) = \frac{1}{2}\sqrt{\frac{2500}{3} + 2500} + \frac{1}{4}\left(50 - \frac{50}{\sqrt{3}}\right)$$

$$= 25\sqrt{\frac{4}{3}} + \frac{25}{2} - \frac{25}{2\sqrt{3}}$$

$$= \frac{50}{\sqrt{3}} + \frac{25}{2} - \frac{25}{2\sqrt{3}}$$

$$= \frac{75 + 25\sqrt{3}}{2\sqrt{3}}\,\frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{75\left(1+\sqrt{3}\right)}{6}$$

$$\boxed{= \frac{25\left(1+\sqrt{3}\right)}{2}}$$

Critical point: $\boxed{\left(\frac{50}{\sqrt{3}},\ \frac{25}{2}\left(1+\sqrt{3}\right)\right)}$

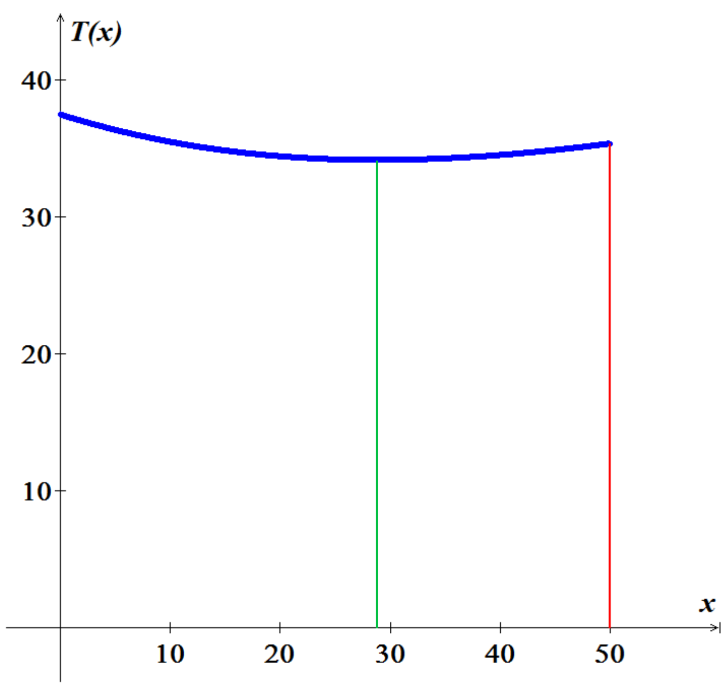
c) $T(0) = \frac{1}{2}50 + \frac{50}{4}\ \boxed{= \frac{75}{2}}\quad \boxed{\approx 37.5\ \ sec}$

$T\left(\frac{50}{\sqrt{3}}\right)\ \boxed{= \frac{25}{2}\left(1+\sqrt{3}\right)}\quad \boxed{\approx 34.151\ \ sec}$

$T(50)\ \boxed{= 25\sqrt{2}}\quad \boxed{\approx 35.36\ \ sec}$

The minimum crossing time is about 34.151 *sec*

d)



Exercise

Consider the function $f(x) = ax^2 + bx + c$ with $a \neq 0$. Explain geometrically why *f* has exactly one absolute extreme value on $(-\infty, \infty)$. Find the critical points to determine the value of x at which f has an extreme value.

Solution

$$f'(x) = 2ax + b = 0$$

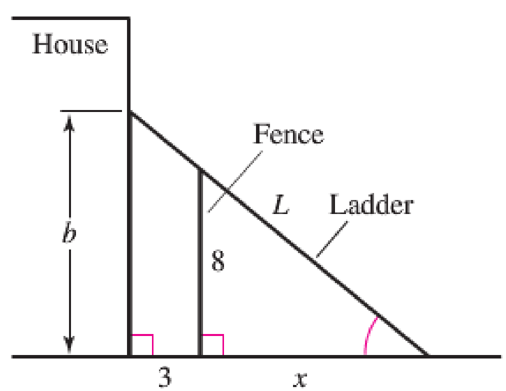
$$x = -\frac{b}{2a} \quad (a \neq 0)$$

There is only one critical point and has a vertex point (extreme point).

Exercise

An 8-*foot*-tall fence runs parallel to the side of a house 3 *feet* away. What is the length of the shortest ladder that clears the fence and reaches the house? Assume that the vertical wall of the house and the horizontal ground have infinite extent.

Solution



$$\frac{8}{b} = \frac{x}{x+3}$$

$$b = \frac{8(x+3)}{x}$$

$$L^2 = b^2 + (x+3)^2$$

$$= \frac{64(x+3)^2}{x^2} + (x+3)^2$$

$$= (x+3)^2\left(\frac{64}{x^2} + 1\right) = f(x)$$

$$f'(x) = (x+3)\left(\frac{128}{x^2} + 2 - \frac{128}{x^3}(x+3)\right)$$

$$= (x+3)\left(2 - \frac{384}{x^3}\right)$$

$$= 2(x+3)\left(\frac{x^3 - 192}{x^3}\right) = 0$$

$$\cancel{x = -3} \qquad x^3 = 192 \rightarrow \boxed{x = \sqrt[3]{192} = 4\sqrt[3]{3}}$$

$$L^2\left(4\sqrt[3]{3}\right) = \left(4\sqrt[3]{3} + 3\right)^2\left(\frac{64}{16\sqrt[3]{9}} + 1\right)$$

$$=\left(4\sqrt[3]{3}+3\right)^2\left(\frac{4}{\sqrt[3]{9}}+1\right)$$

$$\approx 224.765$$

$L \approx 15\ ft$

Exercise

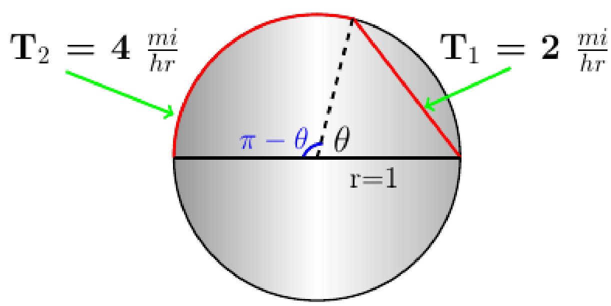
A man wishes to get from an initial point on the shore of a circular pond with radius 1 *mi* to a point on the shore directly opposite (on the other end of the diameter). He plans to swim from the initial point to another point on the shore and then walk along the shore to the terminal point.

a) If he swims at 2 *mi/hr* and walks at 4 *mi/hr*, what are the minimum and maximum times for the trip?

b) If he swims at 2 *mi/hr* and walks at 1.5 *mi/hr*, what are the minimum and maximum times for the trip?

c) If he swims at 2 *mi/hr*, what is the minimum walking speed for which it is quickest to walk the entire distance?

Solution



a) $\dfrac{distance}{rate}=\dfrac{2\sin\frac{\theta}{2}}{2}$

$$=\sin\frac{\theta}{2} \quad (chord\ length)$$

Walk: $Arc\ length=\dfrac{\pi-\theta}{4}$

$$T(\theta)=\sin\frac{\theta}{2}+\frac{\pi-\theta}{4}$$

$$T'(\theta)=\frac{1}{2}\cos\frac{\theta}{2}-\frac{1}{4}=0$$

$$\cos\frac{\theta}{2}=\frac{1}{2} \quad \rightarrow \quad \frac{\theta}{2}=\frac{\pi}{3} \quad \Rightarrow \quad \theta=\frac{2\pi}{3}$$

$$T\left(\frac{2\pi}{3}\right)=\sin\frac{\pi}{3}+\frac{1}{4}\left(\pi-\frac{2\pi}{3}\right)$$

$$=\frac{\sqrt{3}}{2}+\frac{\pi}{12}\ hrs$$

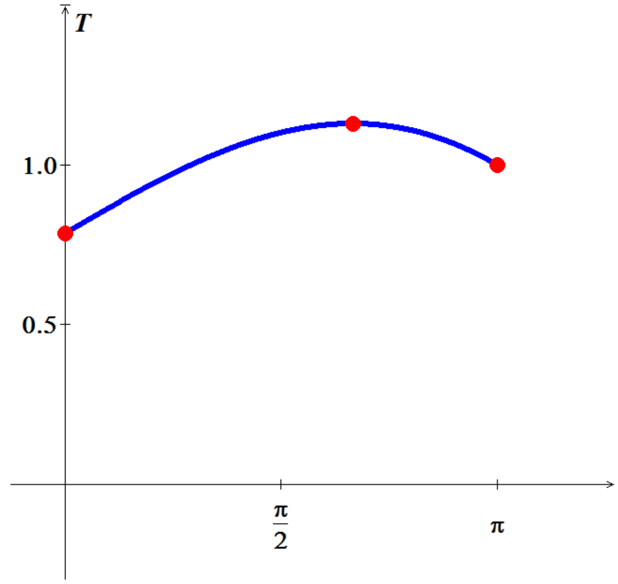
$$T(0)=\sin 0+\frac{1}{4}(\pi-0)$$

$$=\frac{\pi}{4}\ hrs$$

$$T(\pi)=\sin\frac{\pi}{2}+\frac{1}{4}(\pi-\pi)$$

$$=1\ hr$$

Maximum travel time is $\left(\dfrac{\sqrt{3}}{2}+\dfrac{\pi}{12}\ hrs\right) \approx 1.128\ hrs$ at $\theta=\dfrac{2\pi}{3}=120°$



b) ***Given***: swims at 2 *mi/hr*. and walks at 1.5 *mi/hr*.

Swimming time $= \sin\frac{\theta}{2}$

Walking time $= \frac{\pi - \theta}{1.5}$

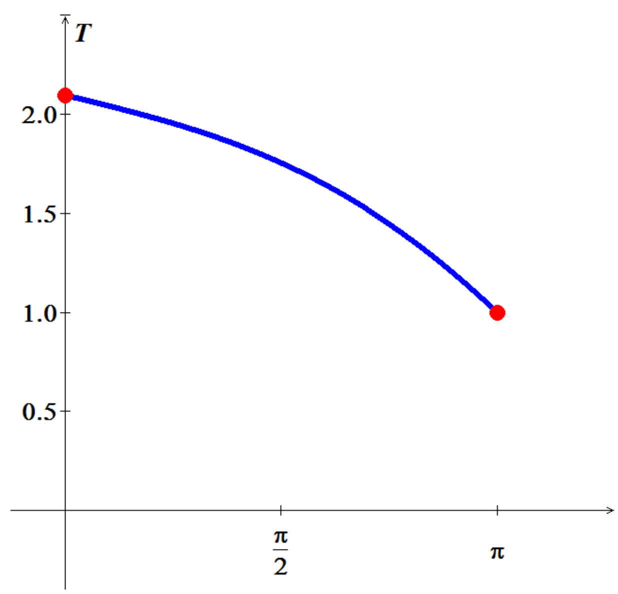
$T(\theta) = \sin\frac{\theta}{2} + \frac{2}{3}(\pi - \theta)$

$T'(\theta) = \frac{1}{2}\cos\frac{\theta}{2} - \frac{2}{3} = 0$

$\cos\frac{\theta}{2} = \frac{4}{3} \cancel{> 1}$ *No Critical Point.*

$T(0) = \frac{2\pi}{3}$ ≈ 2.09 *hrs* $\rightarrow$ Maximum travel time $(\approx 2.09\ hrs)$ is done walking.

$T(\pi) = \sin\frac{\pi}{2} + \frac{2}{3}(\pi - \pi)$ $= 1\ hr$ $\rightarrow$ Minimum travel time $(1\ hr)$ is done swimming.



c) $T_1 = \sin\frac{\theta}{2}$

$T_2 = \frac{\pi - \theta}{v}$

$T(\theta) = \sin\frac{\theta}{2} + \frac{\pi-\theta}{v}$

$T'(\theta) = \frac{1}{2}\cos\frac{\theta}{2} - \frac{1}{v} = 0 \quad \Rightarrow \quad \cos\frac{\theta}{2} = \frac{2}{v}$

$T''(\theta) = -\frac{1}{4}\sin\frac{\theta}{2} < 0 \quad (0 \le \theta \le \pi)$

$\theta = 0 \quad \Rightarrow \quad \boxed{T(0) = \frac{\pi}{v}}$ Minimum all walking

$\theta = \pi \quad \Rightarrow \quad \boxed{T(0) = 1\ hr}$ Minimum all swimming

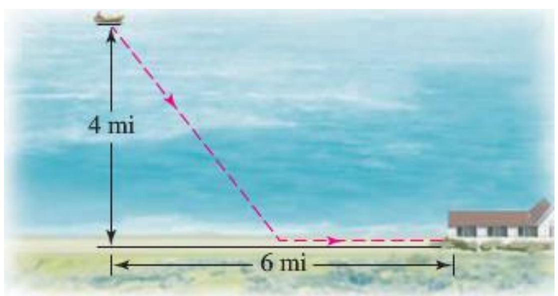
If $\frac{\pi}{v} = 1 \quad \to \quad \pi = v$

If $v > \pi \quad \to \quad T(0) < 1$ Minimum all walking

If $v < \pi \quad \to \quad T(0) > 1$ Minimum all swimming

Exercise

A boat on the ocean is 4 *mi* from the nearest point on a straight shoreline; that point is 6 *mi* from a restaurant on the shore. A woman plans to row the boat straight to a point on the shore and then walk along the shore to the restaurant.



a) If she walks at 3 *mi/hr* and rows at 2 *mi/hr*, at which point on the shore should she land to minimize the total travel time?

b) If she walks at 3 *mi/hr* what is the minimum speed at which she must row so that the quickest way to the restaurant is to row directly (with no walking)?

Solution

$d^2 = x^2 + 16 \quad \Rightarrow \quad d = \sqrt{x^2+16}$

a) Rows Time $= \frac{\sqrt{x^2+16}}{2}$

Walks Time $= \dfrac{6-x}{3}$

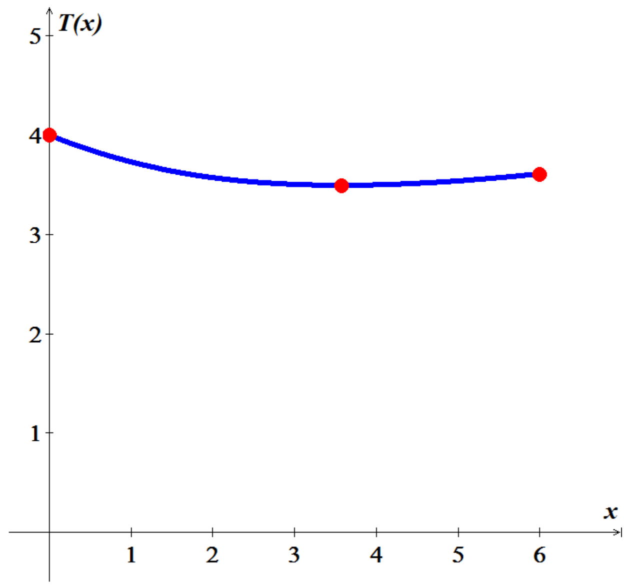
$$T(x) = \frac{1}{2}\sqrt{x^2+16} + 2 - \frac{x}{3}$$

$$T'(x) = \frac{x}{2\sqrt{x^2+16}} - \frac{1}{3} = 0$$

$$(3x)^2 = \left(2\sqrt{x^2+16}\right)^2$$

$$9x^2 = 4x^2 + 64$$

$$5x^2 = 64 \quad \to \quad \boxed{x = \frac{8}{\sqrt{5}} \quad (CN)}$$



T has a local minimum at $x = \dfrac{8}{\sqrt{5}} \in (0,6)$

$$T\left(\frac{8}{\sqrt{5}}\right) = \frac{1}{2}\sqrt{\frac{64}{5}+16} + 2 - \frac{8}{3\sqrt{5}}$$

$$= \frac{6}{\sqrt{5}} - \frac{8}{3\sqrt{5}} + 2$$

$$\boxed{= \frac{10}{3\sqrt{5}} + 2}$$

b) Rows Time $= \dfrac{\sqrt{x^2+16}}{v}$

$$T(x) = \frac{1}{v}\sqrt{x^2+16} + 2 - \frac{x}{3}$$

$$T'(x) = \frac{1}{v}\frac{x}{\sqrt{x^2+16}} - \frac{1}{3} = 0$$

$$\to \quad 3x = v\sqrt{x^2+16}$$

$$T'(0) = -\frac{1}{3} \quad < 0$$

$$T'(6) = \frac{1}{v}\frac{6}{\sqrt{52}} - \frac{1}{3}$$

To have a minimum point between $(0,\ 6)$, then $T'(6)$ should be ≤ 0

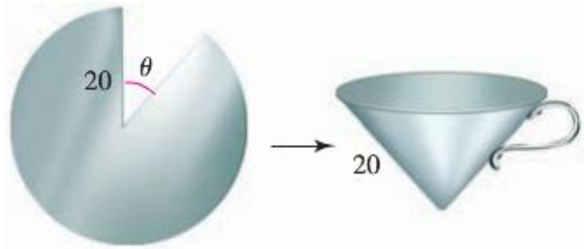
$$T'(6) = \frac{1}{v}\frac{3}{\sqrt{13}} - \frac{1}{3} \leq 0$$

$$\boxed{\frac{9}{\sqrt{13}} \leq v}$$

$v \geq \dfrac{9}{\sqrt{13}}$ $\to$ rows directly to the restaurant.

Exercise

A cone is constructed by cutting a sector of angle θ from a circular sheet of metal with radius 20 *cm*. the cut sheet is then folded up and wheeled. What angle θ maximizes the volume of the cone?



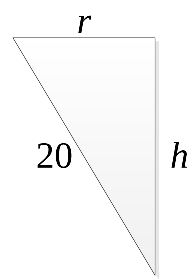
Find the radius and height of the cone with maximum volume that can be formed in this way

Solution

Circumference of the sector $= r\theta \underline{= 20\theta}$

Remaining $= 2\pi r - 20\theta$

$= 20(2\pi - \theta)$



Cone radius: $2\pi r = 20(\pi - \theta)$

$\boxed{\pi r = 10(\pi - \theta)}$

$r^2 + h^2 = 20^2 \quad \Rightarrow \quad r^2 = 400 - h^2$

$$V = \frac{\pi}{3}r^2 h$$

$$= \frac{\pi}{3}\left(400 - h^2\right)h$$

$$= \frac{\pi}{3}\left(400h - h^3\right)$$

$$\frac{d}{dh}V(h) = \frac{\pi}{3}\left(400 - 3h^2\right) = 0$$

$$h^2 = \frac{400}{3} \quad \rightarrow \quad \boxed{h = \frac{20}{\sqrt{3}}}$$

$$r^2 = 400 - \frac{400}{3} = \frac{2}{3}400 \quad \rightarrow \quad \boxed{r = 20\sqrt{\frac{2}{3}}}$$

Since $\pi r = 10(\pi - \theta)$

$$\theta = \pi - \frac{\pi r}{10}$$

$$= \pi - \frac{\pi}{10}20\sqrt{\frac{2}{3}}$$

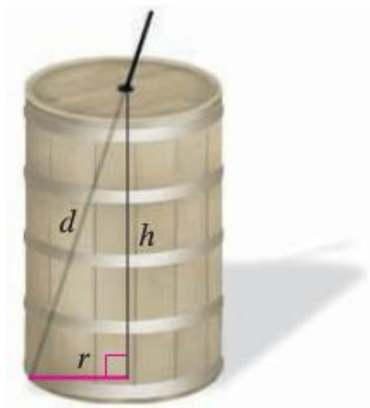
$$\boxed{= \pi\left(1 - 2\sqrt{\frac{2}{3}}\right) \ rad}$$

Exercise

Several mathematical stories originated with the second wedding of the mathematician and astronomer Johannes Kepler. Here is one: while shopping for wine for his wedding, Kepler noticed that the price of a barrel of wine (here assumed to be a cylinder) was determined solely by the length d of a dipstick that was inserted diagonally through a hole in the top of the barrel to the edge of the base of the barrel.

Kepler realized that this measurement does not determine the volume of the barrel and that for a fixed value of d, the volume varies the radius r and height h of the barrel. For a fixed value of d, what is the ratio r/h that maximizes the volume of the barrel?

Solution



$$r^2 + h^2 = d^2 \quad \Rightarrow \quad r^2 = d^2 - h^2$$

$$V = \pi r^2 h$$

$$\begin{aligned} V(h) &= \pi\left(d^2 - h^2\right)h \\ &= \pi\left(d^2 h - h^3\right) \end{aligned}$$

$$V'(h) = \pi\left(d^2 - 3h^2\right) = 0 \quad \rightarrow \quad h^2 = \frac{1}{3}d^2$$

$$\Rightarrow \quad \boxed{h = \frac{d}{\sqrt{3}}}$$

$$\begin{aligned} r^2 &= d^2 - h^2 \\ &= d^2 - \frac{1}{3}d^2 \\ &= \frac{2}{3}d^2 \end{aligned}$$

$$r = d\sqrt{\frac{2}{3}}$$

$$\begin{aligned} \frac{r}{h} &= d\sqrt{\frac{2}{3}}\frac{\sqrt{3}}{d} \\ &= \boxed{\sqrt{2}} \end{aligned}$$

$$V(h) = \pi\left(d^2 h - h^3\right)$$

$$\boxed{V(0) = 0}$$

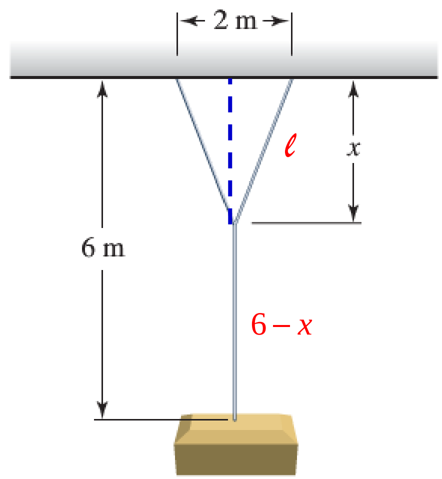
$$\boxed{V(d) = 0}$$

$$\begin{aligned} V\left(\frac{d}{\sqrt{3}}\right) &= \pi\left(\frac{d^3}{\sqrt{3}} - \frac{d^3}{3\sqrt{3}}\right) \\ &= \boxed{\frac{2\pi}{3\sqrt{3}}d^3} \end{aligned}$$

Exercise

A load must be suspended 6 *m* below a high ceiling using cables attached to two supports that are 2 *m* apart. How far below the ceiling (x) should the cables be joined to minimize the total length of cable used?

Solution



$\ell^2 = x^2 + 1 \quad \to \quad \ell = \sqrt{x^2+1}$

Total Length:

$L(x) = 6 - x + 2\sqrt{x^2+1} \quad (0 \le x \le 6)$

$L'(x) = -1 + \dfrac{2x}{\sqrt{x^2+1}} = 0$

$\dfrac{2x}{\sqrt{x^2+1}} = 1$

$(2x)^2 = \left(\sqrt{x^2+1}\right)^2$

$4x^2 = x^2 + 1$

$x^2 = \frac{1}{3} \quad \Rightarrow \quad \boxed{x = \frac{1}{\sqrt{3}} \quad (CN)}$

$L(0) = 6 + 2 \boxed{= 8} \quad \Rightarrow \quad \boxed{x \ne 0}$

$L(6) \boxed{= 2\sqrt{37}} \quad \Rightarrow \quad \boxed{x \ne 6}$

$$\begin{aligned} L\left(\frac{1}{\sqrt{3}}\right) &= 6 - \frac{1}{\sqrt{3}} + 2\sqrt{\frac{1}{3}+1} \\ &= \frac{6\sqrt{3}-1}{\sqrt{3}} + \frac{4}{\sqrt{3}} \\ &\boxed{= \frac{3+6\sqrt{3}}{\sqrt{3}}} \end{aligned}$$

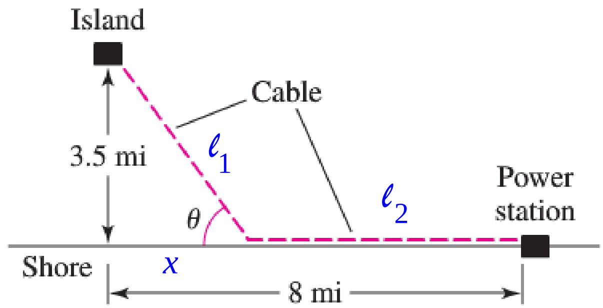
The cables should join at distance $x = \frac{\sqrt{3}}{3}\ m$

Exercise

An island is 3.5 *mi* from the nearest point on a straight shoreline; that point is 8 *mi* from a power station. A utility company plans to lay electrical cable underwater from the island to the shore and then underground along the shore to the power station. Assume that is costs \$2,400/*mi* to lay underwater cable and \$1,200/mi to lay underground cable. At what point should the underwater cable meet the shore in order to minimize the cost of the project?

Solution

$\ell_1 = \sqrt{x^2 + 3.5^2} \quad \ell_2 = 8 - x$



Cost:

$$C(x) = 2{,}400\sqrt{x^2 + 3.5^2} + 1{,}200(8 - x)$$

$$C'(x) = \frac{2{,}400x}{\sqrt{x^2 + 3.5^2}} - 1{,}200 = 0$$

$$(2x)^2 = \left(\sqrt{x^2 + 3.5^2}\right)^2$$

$$4x^2 = x^2 + 3.5^2$$

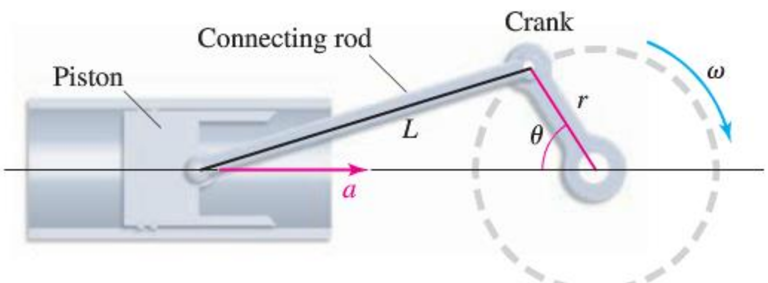
$$x^2 = \frac{3.5^2}{3} \quad \rightarrow \quad \underline{x = \frac{3.5}{\sqrt{3}} = \frac{7}{2\sqrt{3}} < 8 \; (CN)}$$

The optimal point on shore has distance $x = \frac{7\sqrt{3}}{6}$ *mi* from the point on shore nearest the island in the direction of the power station.

Exercise

A crank of radius r rotates with an angular frequency ω. It is connected to a piston by a connecting rod of length L. the acceleration of the piston varies with the position of the crank according to the function

$$a(\theta) = \omega^2 r\left(\cos\theta + \frac{r\cos 2\theta}{L}\right)$$



For fixed ω and r, find the values of θ, with $0 \le \theta \le 2\pi$, for which the acceleration of the piston is a maximum and minimum.

Solution

$$a(\theta) = \omega^2 r\left(\cos\theta + \frac{r\cos 2\theta}{L}\right)$$

$$a'(\theta) = \omega^2 r\left(-\sin\theta - \frac{2r\sin 2\theta}{L}\right) \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= -\omega^2 r\left(\sin\theta + \frac{4r\sin\theta\cos\theta}{L}\right) = 0$$

$$\sin\theta\left(1 + \frac{4r}{L}\cos\theta\right) = 0$$

$\sin\theta = 0$ $\boxed{\theta = 0,\ \pi,\ 2\pi}$	$\cos\theta = -\frac{L}{4r}$ $\theta = \begin{cases} \cos^{-1}\left(-\frac{L}{4r}\right) \\ 2\pi - \cos^{-1}\left(-\frac{L}{4r}\right) \end{cases}$

$$a(\theta = 0) = \boxed{\omega^2 r\left(1 + \frac{r}{L}\right)}$$

$$a(\theta = \pi) = \boxed{\omega^2 r\left(-1 + \frac{r}{L}\right)}$$

$$a(\theta = 2\pi) = \boxed{\omega^2 r\left(1 + \frac{r}{L}\right)}$$

$$\cos\theta = -\frac{L}{4r} \quad \rightarrow \quad \cos 2\theta = 2\cos^2\theta - 1 = \frac{L^2}{8r^2} - 1$$

$$\begin{aligned} a\left(\theta = \cos^{-1}\left(-\frac{L}{4r}\right)\right) &= \omega^2 r\left(-\frac{L}{4r} + \frac{r}{L}\left(\frac{L^2}{8r^2} - 1\right)\right) \\ &= \omega^2 r\left(-\frac{L}{4r} + \frac{L}{8r} - \frac{r}{L}\right) \\ &= \omega^2 r\left(-\frac{L}{8r} - \frac{r}{L}\right) \\ &= \boxed{-\omega^2\left(\frac{L^2 - 8r^2}{8L}\right)} \end{aligned}$$

$$\cos(2\pi - \theta) = \cos\theta$$

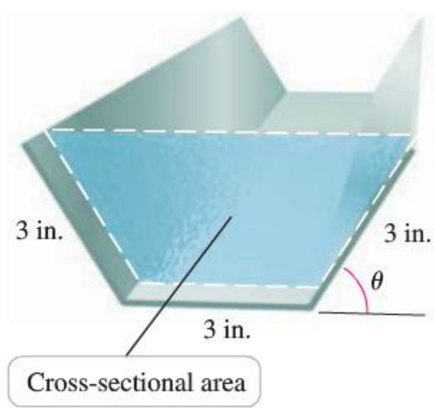
Maximum acceleration occurs at $\boxed{\theta = 0,\ \ 2\pi}$

Maximum acceleration occurs at $\theta = \cos^{-1}\left(-\frac{L}{4r}\right),\quad 2\pi - \cos^{-1}\left(-\frac{L}{4r}\right)$

Exercise

A rain gutter is made from sheets of metal 9 *in* wide. The gutters have a 3-*in* base and two 3-*in* sides, folded up at an angle θ.

What angle θ maximizes the cross-sectional area of the gutter?



Solution

$$\sin\theta = \frac{h}{3} \quad \rightarrow \quad h = 3\sin\theta$$

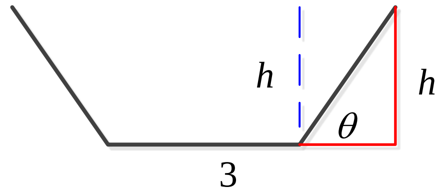
$$\begin{aligned} A &= 3h + (3\sin\theta)(3\cos\theta) \\ &= 9\sin\theta + 9\sin\theta\cos\theta \end{aligned}$$

$$A(\theta) = 9\left(\sin\theta + \frac{1}{2}\sin 2\theta\right)$$

$$A'(\theta) = 9(\cos\theta + \cos 2\theta) = 0$$

$$\cos\theta + \cos 2\theta = 0$$

$$\cos\theta + 2\cos^2\theta - 1 = 0 \quad \rightarrow \quad \cos\theta = -1, \ \frac{1}{2}$$



$$\cos\theta = -1 \qquad \cos\theta = \frac{1}{2}$$

$$\theta = \cancel{\pi} \qquad \theta = \frac{\pi}{3} \quad \cancel{\theta = \frac{5\pi}{3}}$$

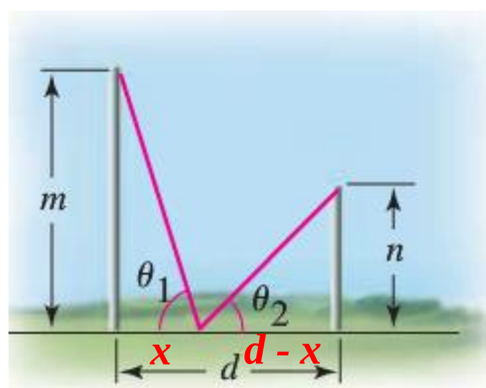
$$A\left(\theta = \frac{\pi}{3}\right) = 9\left(\sin\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right)$$

$$= 9\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\right)$$

$$\boxed{= \frac{27\sqrt{3}}{4}}$$

Exercise

Two poles of heights m and n are separated by a horizontal distance d. A rope is stretched from the top of one pole to the ground and then to the top of the other pole. Show that the configuration that requires the least amount of rope occurs when $\theta_1 = \theta_2$



Solution

Length of the rope is given: $L(x)$

$$L(x) = \sqrt{x^2 + m^2} + \sqrt{(d-x)^2 + n^2}$$

$$L'(x) = \frac{x}{\sqrt{x^2 + m^2}} - \frac{d-x}{\sqrt{(d-x)^2 + n^2}} = 0$$

$$\frac{x}{\sqrt{x^2 + m^2}} = \frac{d-x}{\sqrt{(d-x)^2 + n^2}} \qquad \text{From the graph;}$$

$$\cos\theta_1 = \cos\theta_2$$

$$\therefore \ \boxed{\theta_1 = \theta_2}$$

$$L'(0) = -\frac{d}{\sqrt{d^2 + n^2}} < 0 \quad \searrow$$

$$L'(d) = \frac{d}{\sqrt{d^2 + n^2}} > 0 \quad \nearrow$$

Therefore, a minimum value of $L(x)$ when $x \in (0, d)$

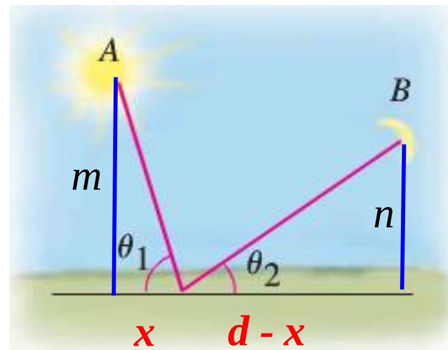
θ_1 *decreases* & θ_2 *increases*

Exercise

Fermat's principle states that when light travels between two points in the same medium (at a constant speed), it travels on the path that minimizes the travel time. Show that when light from a source *A* reflects off of a surface and is received at point *B*, the angle of incidence equals the angle of reflection, or $\theta_1 = \theta_2$

Solution

Speed of light is constant; travel time is minimized when the distance is minimized occurs when $\theta_1 = \theta_2$



$$L(x) = \sqrt{x^2 + m^2} + \sqrt{(d-x)^2 + n^2}$$

$$L'(x) = \frac{x}{\sqrt{x^2 + m^2}} - \frac{d-x}{\sqrt{(d-x)^2 + n^2}} = 0$$

$$\frac{x}{\sqrt{x^2 + m^2}} = \frac{d-x}{\sqrt{(d-x)^2 + n^2}}$$ From the graph;

$$\cos\theta_1 = \cos\theta_2$$

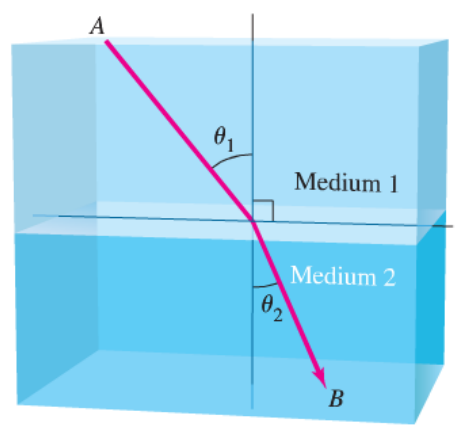
$$\therefore \quad \boxed{\theta_1 = \theta_2}$$

Exercise

Suppose that a light at *A* is in a medium in which light travels at a speed v_1 and the point *B* is in a medium in which light travels at a speed v_2. Using Fermat's Principle, which states that light travels along the path that requires the minimum travel, show that the path taken between points *A* and *B* satisfies

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \quad (\textit{Snell's Law})$$

Solution



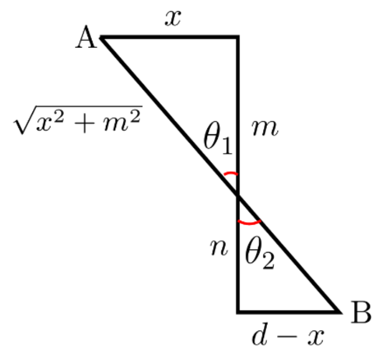
$$Time = T(x) = \frac{distance}{speed}$$

$$T(x) = \frac{\sqrt{x^2 + m^2}}{v_1} + \frac{\sqrt{(d-x)^2 + n^2}}{v_2}$$

$$T'(x) = \frac{x}{v_1\sqrt{x^2 + m^2}} - \frac{d-x}{v_2\sqrt{(d-x)^2 + n^2}} = 0$$

$$\frac{x}{v_1\sqrt{x^2 + m^2}} = \frac{d-x}{v_2\sqrt{(d-x)^2 + n^2}}$$

From the graph: $\sin\theta_1 = \frac{x}{\sqrt{x^2+m^2}}$ & $\sin\theta_2 = \frac{d-x}{\sqrt{(d-x)^2+n^2}}$



$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$ ✓

$T'(0) = -\frac{d}{v_2\sqrt{d^2+n^2}} < 0$ ↘

$T'(d) = \frac{d}{v_1\sqrt{d^2+m^2}} > 0$ ↗

Therefore, a minimum value of $T(x)$ when $x \in (0,\ d)$ with exactly one critical point.

θ_1 *decreases* & θ_2 *increases*, so Snell's law can hold for at most 1 value of x.