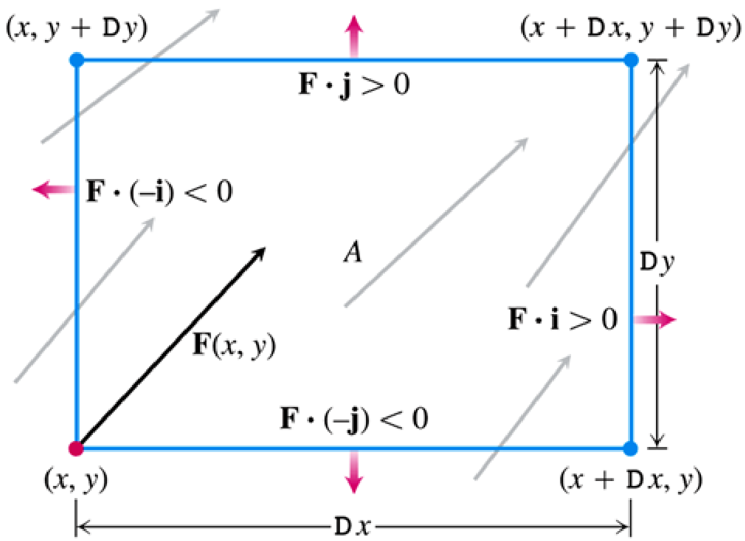
Section 4.4 – Green's Theorem

Green's theorem gives the relationship between a line integral around a simple closed curve *C* and a double integral over the plane region *D* bounded by *C*. It is the two-dimensional special case of the more general ***Stokes' theorem***, and is named after British mathematician *George Green*.

Green's theorem applies to any vector field, independent of any particular interpretation of the field, provided assumptions of the theorem are satisfied, We introduce two new ideas for Green's theorem: *divergence* and *circulation density* around an axis perpendicular to the plane.

Divergence

Suppose that $\vec{F}(x, y) = M(x, y)\hat{i} + N(x, y)\hat{j}$ is the velocity field of fluid flowing in the plane and that the first partial derivatives of *M* and *N* are continuous at each point of a region *R*.



Fluid Flow Rates:

Top: $\vec{F}(x,\ y+\Delta y) \bullet \hat{j}\ \Delta x = N(x, y+\Delta y)\Delta x$

Bottom: $\vec{F}(x,\ y) \bullet (-\hat{j})\ \Delta x = -N(x, y)\Delta x$

Right: $\vec{F}(x+\ \Delta x,\ y) \bullet \hat{i}\ \Delta y = M(x+\Delta x, y)\Delta y$

Left: $\vec{F}(x,\ y) \bullet (-\hat{i}\ )\Delta y = -M(x, y)\Delta y$

Top and Bottom: $\left(N(x,\ y+\Delta y) - N(x,\ y)\right)\Delta x \ \approx\ \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$

Right and Left: $\left(M(x+\Delta x,\ y) - M(x,\ y)\right)\Delta y \ \approx\ \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y$

Adding the last two equations gives the net effect of the flow rates:

$$\textit{Flux across rectangle boundary} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \Delta x \Delta y$$

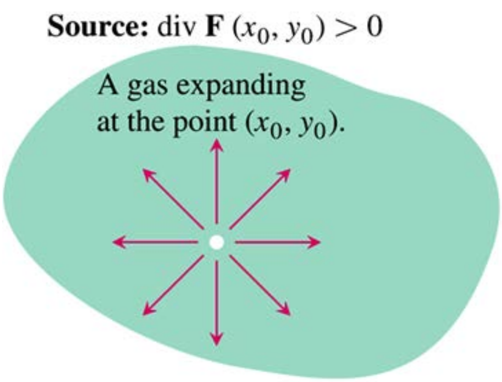
The estimate of the total flux per unit area or flux density for the rectangle:

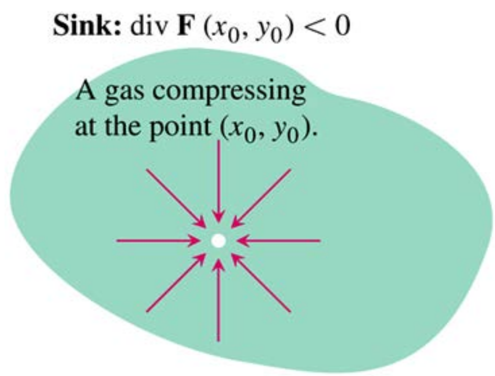
$$\frac{\textit{Flux across rectangle boundary}}{\textit{rectangle area}} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right)$$

Definition

The divergence (flux density) of a vector field $\vec{F} = M\hat{i} + N\hat{j}$ at the point (x, y) is

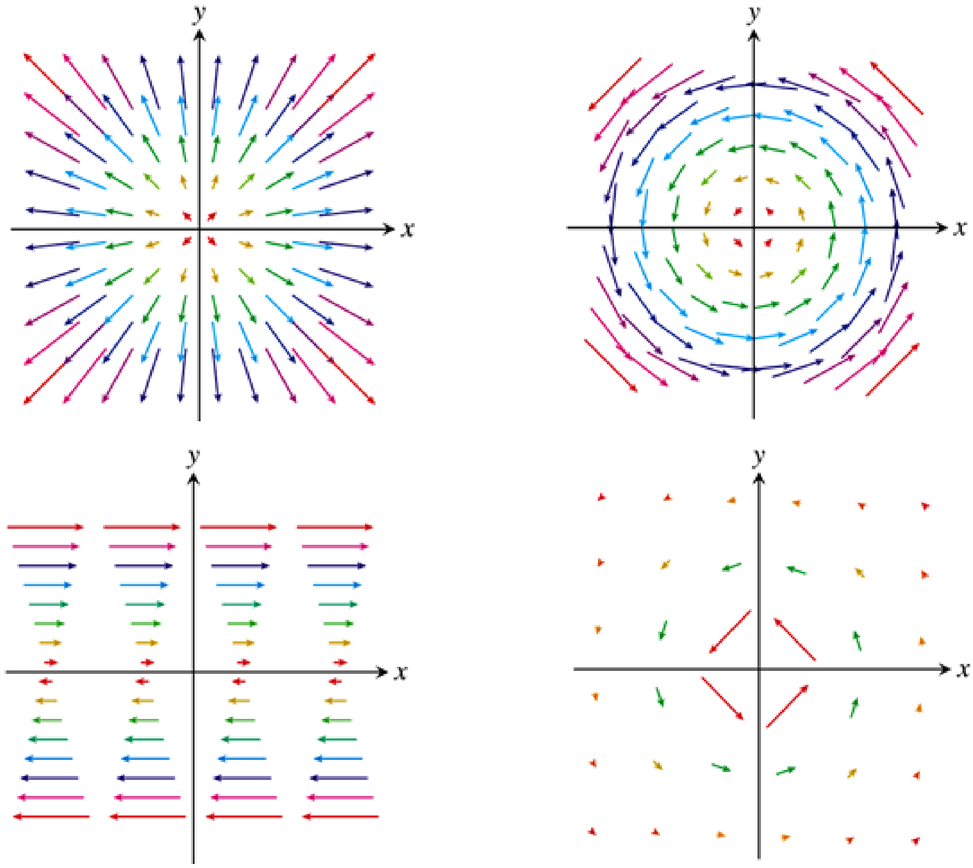
$$div\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$





Example

The following vector fields represent the velocity of a gas flowing in the *xy*-plane. Find the divergence of each vector field and interpret its physical meaning.



a) Uniform expansion or compression: $\vec{F}(x,\ y) = cx\hat{i} + cy\hat{j}$

b) Uniform rotation: $\vec{F}(x,\ y) = -cy\hat{i} + cx\hat{j}$

c) Shearing flow: $\vec{F}(x,\ y) = y\hat{i}$

d) Whirlpool effect: $\vec{F}(x,\ y) = \dfrac{-y}{x^2+y^2}\hat{i} + \dfrac{x}{x^2+y^2}\hat{j}$

Solution

a) $div\vec{F} = \dfrac{\partial}{\partial x}(cx) + \dfrac{\partial}{\partial y}(cy)$

$= c + c$

$= 2c$

If $c > 0,$ the gas is undergoing uniform expansion

If $c < 0,$ the gas is undergoing uniform compression

b) $div\vec{F} = \frac{\partial}{\partial x}(-cy) + \frac{\partial}{\partial y}(cx)$

$\underline{= 0}$

The gas is neither expanding nor compressing.

c) $div\vec{F} = \frac{\partial}{\partial x}(y)$

$\underline{= 0}$

The gas is neither expanding nor compressing.

d) $div\vec{F} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$

$= \frac{2xy}{\left(x^2+y^2\right)^2} - \frac{2xy}{\left(x^2+y^2\right)^2}$

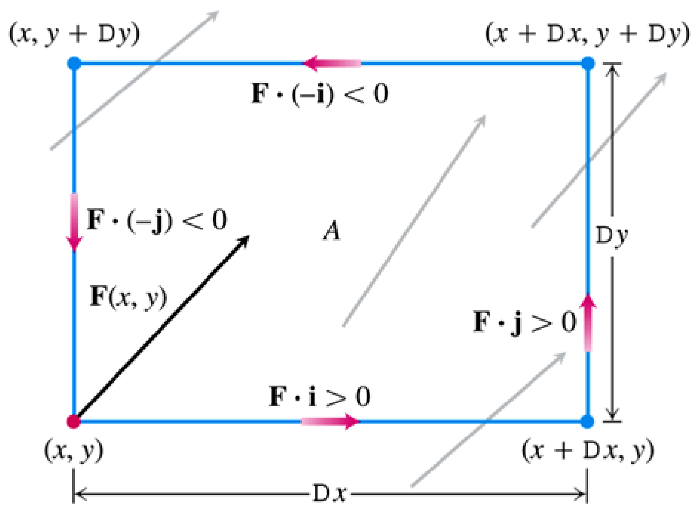
$\underline{= 0}$

The divergence is zero at all points in the domain of the velocity field.

Spin Around an Axis: The $\hat{k}$–Component of Curl

The Green's Theorem has to do with measuring how a floating paddle wheel, with axis perpendicular to the plane, spins at a point in fluid flowing in a plane region. Sometimes refer to *circulation density* of a vector field $\vec{F}$ at a point.

$$\vec{F}(x, y) = M(x, y)\hat{i} + N(x, y)\hat{j}$$



The circulation rate of $\vec{F}$ around the boundary of *A* is the sum of flow rates along the sides in the tangential direction.

Top: $\vec{F}(x, y+\Delta y) \bullet (-\hat{i})\ \Delta x = -M(x, y+\Delta y)\Delta x$

Bottom: $\vec{F}(x, y) \bullet \hat{i}\ \Delta x = M(x, y)\Delta x$

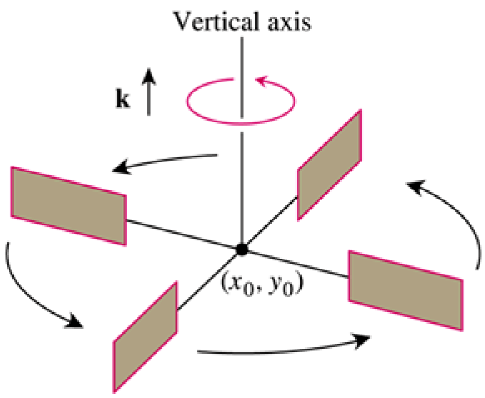
Right: $\vec{F}(x+\ \Delta x, y) \bullet \hat{j}\ \Delta y = N(x+\Delta x, y)\Delta y$

Left: $\vec{F}(x, y) \bullet (-\hat{j}\ )\Delta y = -N(x, y)\Delta y$

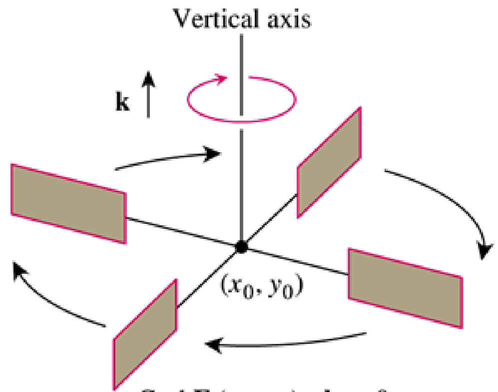
Top and Bottom: $-\big(M(x, y+\Delta y) - M(x, y)\big)\Delta x \approx -\left(\dfrac{\partial M}{\partial y}\Delta y\right)\Delta x$

Right and Left: $\big(N(x+\Delta x, y) - N(x, y)\big)\Delta y \approx \left(\dfrac{\partial N}{\partial x}\Delta x\right)\Delta y$

$$\frac{\textit{Circulation around rectangle}}{\textit{rectangle area}} \approx \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$



Curl $\mathbf{F}(x_0, y_0) \cdot \mathbf{k} > 0$
Counterclockwise circulation



Curl $\mathbf{F}(x_0, y_0) \cdot \mathbf{k} < 0$
Clockwise circulation

Definition

The ***circulation density*** of a vector field $\vec{F} = M\hat{i} + N\hat{j}$ at the point (x, y) is the scalar expression

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

This expression is also called the ***k-component of the curl***, denoted by $\left(curl\ \vec{F}\right) \cdot \hat{k}$

Example

Find the circulation density, and interpret what it means, for each vector field

Solution

a) *Uniform expansion*:

$$\left(curl\ \vec{F}\right) \cdot \hat{k} = \frac{\partial}{\partial x}(cy) - \frac{\partial}{\partial y}(cx)$$

$$= 0$$

The gas is not circulating at very small scales.

b) *Rotation*:

$$\left(curl\ \vec{F}\right) \cdot \hat{k} = \frac{\partial}{\partial x}(cx) - \frac{\partial}{\partial y}(-cy)$$

$$= 2c$$

The constant circulation density indicates rotation at every point.

If $c > 0$, the rotation is counterclockwise

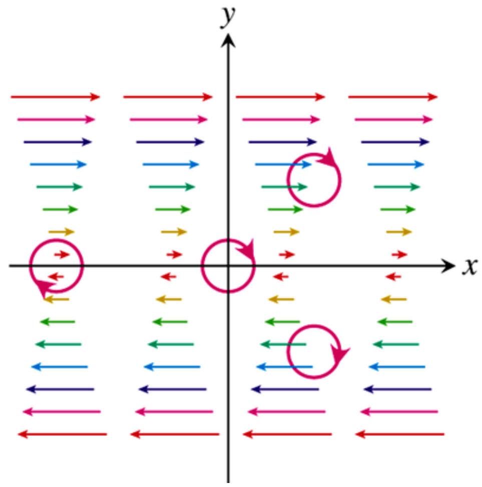
If $c < 0$, the rotation is clockwise

c) *Shear*:

$$\left(curl\ \vec{F}\right) \bullet \hat{k} = -\frac{\partial}{\partial y}(y)$$

$$\underline{= -1}$$

The circulation density is constant and negative, so a paddle wheel floating in water undergoing such a shearing flow spins clockwise. The rate of rotation is the same at each point. The average effect of the fluid flow is to push fluid clockwise around each of the small circles.



d) *Whirlpool*:

$$\left(curl\ \vec{F}\right) \bullet \hat{k} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$$

$$= \frac{y^2-x^2}{\left(x^2+y^2\right)^2} - \frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

$$\underline{= 0}$$

The circulation density is 0 at every point away from the origin (where the vector field is undefined and the whirlpool effect is taking place), and the gas is not circulating at any point for which the vector field is defined.

Theorem – Green's Theorem (Flux-Divergence or Normal Form)

Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\vec{F} = M\hat{i} + N\hat{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the outward flux of $\vec{F}$ across C equals the double integral of div $\vec{F}$ over the region R enclosed by C.

$$\underbrace{\oint_C \vec{F} \cdot \vec{N}\, ds}_{\text{Outward flux}} = \oint_C M dy - N dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy}_{\text{Divergence integral}}$$

Theorem – Green's Theorem (Circulation-Curl or Tangential Form)

Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\vec{F} = M\hat{i} + N\hat{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of $\vec{F}$ around C equals the double integral of $\left(curl\ \vec{F} \right) \cdot \hat{k}$ over R.

$$\underbrace{\oint_C \vec{F} \cdot \vec{T}\, ds = \oint_C M dx + N dy}_{\text{Counterclockwise circulation}} = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy}_{\text{Curl integral}}$$

Example

Verify both forms of Green's Theorem for the vector field $\vec{F}(x, y) = (x - y)\hat{i} + x\hat{j}$

And the region R bounded by the unit circle $C: \quad \vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}, \quad 0 \le t \le 2\pi$

Solution

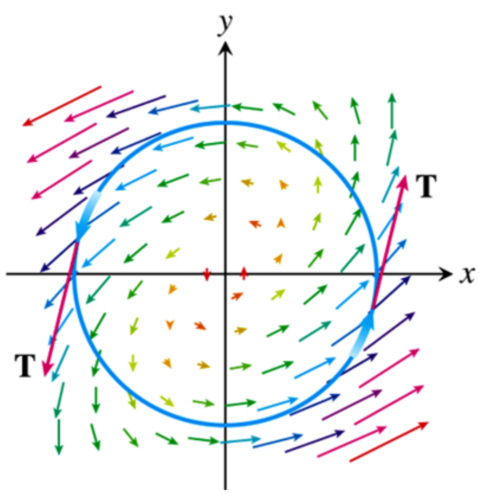
$M = x - y = \cos t - \sin t$

$N = x = \cos t$

$dx = d(\cos t) = -\sin t\ dt$

$dy = d(\sin t) = \cos t\ dt$

$\frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = 0$



1. $$\oint_C M\,dy - N\,dx = \int_0^{2\pi} (\cos t - \sin t)(\cos t\,dt) - (\cos t)(-\sin t\,dt)$$

$$= \int_0^{2\pi} \cos^2 t \; dt$$

$$= \int_0^{2\pi} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2t\right) dt$$

$$= \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t \;\Big|_0^{2\pi}$$

$$\underline{= \pi\,}|$$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy = \iint_R (1+0)\,dxdy$$

$$= \iint_R dxdy$$

$$= \textit{area inside the unit circle}$$

$$\underline{= \pi\,}|$$

2. $$\oint_C M\,dx + N\,dy = \int_0^{2\pi} (\cos t - \sin t)(-\sin t\;dt) + \cos t(\cos t\;dt)$$

$$= \int_0^{2\pi} \left(-\cos t \sin t + \sin^2 t + \cos^2 t\right) dt$$

$$= \int_0^{2\pi} \left(-\tfrac{1}{2}\sin 2t + 1\right) dt$$

$$= \tfrac{1}{4}\cos 2t + t \;\Big|_0^{2\pi}$$

$$\underline{= 2\pi\,}|$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (1-(-1))\,dxdy$$

$$= 2\iint_R dxdy$$

$$\underline{= 2\pi}$$

Example

Evaluate the line integral $\displaystyle\oint_C xydy - y^2 dx$

Where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$

Solution

With the Normal Form Equation: $M = xy \quad N = y^2$

$$\oint_C xydy - y^2 dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy$$

$$= \iint_R (y + 2y)\, dxdy$$

$$= \int_0^1 \int_0^1 3y\ dxdy$$

$$= \int_0^1 \left(3xy \ \Big|_0^1\right) dy$$

$$= 3\int_0^1 y\ dy$$

$$= \frac{3}{2} y^2 \ \Big|_0^1$$

$$\underline{= \frac{3}{2}}$$

With the Tangential Form Equation: $M = -y^2 \quad N = xy$

$$\oint_C -y^2dx + xydy = \iint_R \left(y - (-2y)\right) dxdy$$

$$= \int_0^1 \int_0^1 3y dxdy$$

$$= \frac{3}{2}$$

Example

Calculate the outward flux of the vector field $\vec{F}(x, y) = x\hat{i} + y^2\hat{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$

Solution

$$M = x \quad N = y^2$$

$$Flux = \oint_C \vec{F} \cdot \vec{n}\ ds = \oint_C Mdy - Ndx$$

$$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy \qquad \textbf{Green's Theorem}$$

$$= \int_{-1}^1 \int_{-1}^1 (1 + 2y)\ dxdy$$

$$= \int_{-1}^1 (1 + 2y)x \Big|_{-1}^1 dy$$

$$= \int_{-1}^1 (1 + 2y)(1 - (-1))\ dy$$

$$= 2\int_{-1}^1 (1 + 2y)dy$$

$$= 2\left(y + y^2\right) \Big|_{-1}^1$$

$$= 2\left[1 + 1 - (-1 + 1)\right]$$

$$= 4$$

Exercises Section 4.4 – Green's Theorem

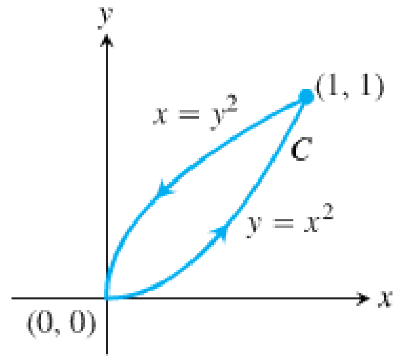
(1–17) Use Green's theorem to find the counterclockwise circulation and outward flux for the field

1. $\vec{F} = (x-y)\hat{i} + (y-x)\hat{j}$ and curve C is the square bounded by $x=0,\ x=1,\ y=0,\ y=1$

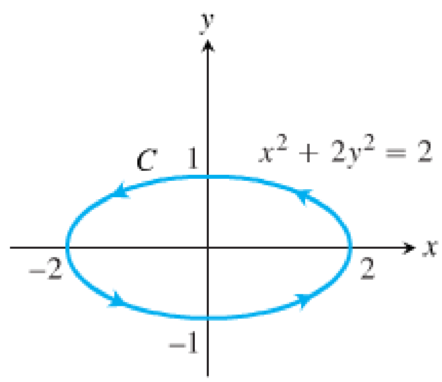
2. $\vec{F} = \left(x^2+4y\right)\hat{i} + \left(x+y^2\right)\hat{j}$ and curve C is the square bounded by $x=0,\ x=1,\ y=0,\ y=1$

3. $\vec{F} = (x+y)\hat{i} - \left(x^2+y^2\right)\hat{j}$ and curve C is the triangle bounded by $y=0,\ x=1,\ y=x$

4. $\vec{F} = \left(xy+y^2\right)\hat{i} + (x-y)\hat{j}$ and curve C



5. $\vec{F} = (x+3y)\hat{i} + (2x-y)\hat{j}$ and curve C



6. $\vec{F} = \left(x+e^x \sin y\right)\hat{i} + \left(x+e^x \cos y\right)\hat{j}$ and curve C is the right-hand loop of the lemniscate $r^2 = \cos 2\theta$

7. *Square*: $\vec{F} = (2xy+x)\hat{i} + (xy-y)\hat{j}$ C: *The square bounded by* $x=0,\ x=1,\ y=0,\ y=1$

8. *Triangle*: $\vec{F} = \left(y-6x^2\right)\hat{i} + \left(x+y^2\right)\hat{j}$

C: *The triangle made by the lines* $y=0,\ y=x,$ *and* $x=1$

9. $\vec{F} = \langle y-x,\ y\rangle$ for the curve $\vec{r}(t) = \langle 2\cos t,\ 2\sin t\rangle,\quad 0 \le t \le 2\pi$

10. $\vec{F} = \langle x,\ y \rangle$; where R is the half-annulus $\{(r,\ \theta):\ \ 1 \le r \le 2,\quad 0 \le \theta \le \pi\}$

11. $\vec{F} = \langle -y,\ x \rangle$; where R is the annulus $\{(r,\ \theta):\ \ 1 \le r \le 3,\quad 0 \le \theta \le 2\pi\}$

12. $\vec{F} = \langle 2x + y,\ x - 4y \rangle$; where R is the quarter-annulus $\left\{(r,\ \theta):\ \ 1 \le r \le 4,\quad 0 \le \theta \le \frac{\pi}{2}\right\}$

13. $\vec{F} = \langle x - y,\ 2y - x \rangle$; where R is the parallelogram $\{(x,\ y):\ \ 1 - x \le y \le 3 - x,\quad 0 \le x \le 1\}$

14. $\vec{F} = \left\langle \ln\left(x^2 + y^2\right),\quad \tan^{-1}\frac{y}{x} \right\rangle$; where R is the annulus $\{(r,\ \theta):\ \ 1 \le r \le 2,\quad 0 \le \theta \le 2\pi\}$

15. $\vec{F} = \nabla\left(\sqrt{x^2 + y^2}\right)$; where R is the half-annulus $\{(r,\ \theta):\ \ 1 \le r \le 3,\quad 0 \le \theta \le \pi\}$

16. $\vec{F} = \langle y\cos x,\ -\sin x \rangle$; where R is the square $\left\{(x,\ y):\ \ 0 \le x \le \frac{\pi}{2},\quad 0 \le y \le \frac{\pi}{2}\right\}$

17. $\vec{F} = \left\langle x + y^2,\ x^2 - y \right\rangle$; where $R = \left\{(x,\ y):\ \ 3y^2 \le x \le 36 - y^2\right\}$

18. Find the outward flux for the field $\vec{F} = \left(3xy - \frac{x}{1 + y^2}\right)\hat{i} + \left(e^x + \tan^{-1} y\right)\hat{j}$ across the cardioid $r = a(1 + \cos\theta),\ \ a > 0$

19. Find the work done by $\vec{F} = 2xy^3\hat{i} + 4x^2y^2\hat{j}$ in moving a particle once counterclockwise around the curve C: The boundary of the triangular region in the first quadrant enclosed by the x-axis, the line $x = 1$ and the curve $y = x^3$

(**20–32**) Apply Green's Theorem to evaluate the integral

20. $\oint_C \left(y^2 dx + x^2 dy\right)$ $\quad C$: The triangle bounded by $x = 0,\ x + y = 1,\ y = 0$

21. $\oint_C (3y\,dx + 2x\,dy)$ $\quad C$: The boundary of $0 \le x \le \pi,\quad 0 \le y \le \sin x$

22. $\oint xy^2 dx + x^2 y\,dy$; C is the triangle with vertices $(0,\ 0)$, $(2,\ 0)$, $(0,\ 2)$ with counterclockwise orientation.

23. $\oint \left(-3y + x^{3/2}\right)dx + \left(x - y^{2/3}\right)dy$; C is the boundary of the half disk $\left\{(x, y):\ x^2 + y^2 \le 2,\ y \ge 0\right\}$ with counterclockwise orientation.

24. $\oint_{(0,\ 1)} \left(2x + e^{y^2}\right)dy - \left(4y^2 + e^{x^2}\right)dx:$ C is the boundary of the square with vertices $(0,\ 0)$, $(1,\ 0)$, $(1,\ 1)$ with counterclockwise orientation.

25. $\oint_C (2x - 3y)\,dy - (3x + 4y)\,dx:$ C is the unit circle

26. $\oint fdy - gdx$; where $\langle f,\ g\rangle = \langle 0,\ xy\rangle$ and C is the triangle with vertices $(0,\ 0)$, $(2,\ 0)$, $(0,\ 4)$ with counterclockwise orientation.

27. $\oint fdy - gdx$; where $\langle f,\ g\rangle = \left\langle x^2,\ 2y^2\right\rangle$ and C is the upper half of the unit circle and the line segment $-1 \le x \le 1$ with clockwise orientation.

28. The circulation line integral of $\vec{F} = \left\langle x^2 + y^2,\ 4x + y^3\right\rangle$, where C is the boundary of $\left\{(x,\ y):\ 0 \le y \le \sin x,\quad 0 \le x \le \pi\right\}$

29. The circulation line integral of $\vec{F} = \left\langle 2xy^2 + x,\ 4x^3 + y\right\rangle$, where C is the boundary of $\left\{(x,\ y):\ 0 \le y \le \sin x,\quad 0 \le x \le \pi\right\}$

30. The flus line integral of $\vec{F} = \left\langle e^{x-y},\ e^{y-x}\right\rangle$, where C is the boundary of $\left\{(x,\ y):\ 0 \le y \le x,\quad 0 \le x \le 1\right\}$

31. $\oint_C \left(3y - e^{\sin x}\right)dx + \left(7x + \sqrt{y^4 + 1}\right)dy:$ where C is the circle $x^2 + y^2 = 9$

32. $\oint_C (3x - 5y)\,dx + (x - 6y)\,dy:$ where C is the ellipse $\frac{x^2}{4} + y^2 = 1$

33. Use either form of Green's Theorem to evaluate the line integral $\oint_C \left(x^3+xy\right)dy+\left(2y^2-2x^2y\right)dx$; C is the square with vertices $(\pm 1,\ \pm 1)$ with *counterclockwise* orientation

34. Use either form of Green's Theorem to evaluate the line integral $\oint_C 3x^3dy-3y^3dx$; C is the circle of radius 4 centered at the origin with *clockwise* orientation.

35. Evaluate $\int_C y^2dx+x^2dy \quad C$ is the circle $x^2+y^2=4$

36. Use the flux form to Green's Theorem to evaluate $\iint_R \left(2xy+4y^3\right)dA$, where R is the triangle with vertices $(0,\ 0)$, $(1,\ 0)$, and $(0,\ 1)$.

37. Show that $\oint_C \ln x \sin y\, dy-\frac{\cos y}{x}dx=0$ for any closed curve C to which Green's Theorem applies.

38. Prove that the radial field $\vec{F}=\frac{\vec{r}}{|\vec{r}|^p}$ where $\vec{r}=\langle x,\ y\rangle$ and p is a real number, is conservative on $\mathbb{R}^2$ with the origin removed. For what value of p is $\vec{F}$ conservative on $\mathbb{R}^2$ (including the origin)?

39. Find the area of the elliptical region cut from the plane $x+y+z=1$ by the cylinder $x^2+y^2=1$

40. Find the area of the cap cut from the paraboloid $x^2+y^2+z^2=1$ by the plane $z=\frac{\sqrt{2}}{2}$

(**41–46**) Evaluate both integrals in Green's theorem of the vector field. Is the vector field conservative?

41. $\vec{F}=\langle x,\ y\rangle;\quad R=\left\{(x,\ y):\ \ x^2+y^2\le 2\right\}$

42. $\vec{F}=\langle y,\ x\rangle$; R is the square with vertices $(0,\ 0)$, $(1,\ 0)$, $(1,\ 1)$, $(0,\ 1)$

43. $\vec{F}=\langle 2y,\ -2x\rangle$; R is the region bounded by $y=\sin x$ and $y=0$ for $0\le x\le \pi$

44. $\vec{F}=\langle -3y,\ 3x\rangle$; R is the triangle with vertices $(0,\ 0)$, $(1,\ 0)$, $(0,\ 2)$

45. $\vec{F}=\left\langle 2xy,\ x^2-y^2\right\rangle$; R is the region bounded by $y=x(2-x)$ and $y=0$

46. $\vec{F}=\left\langle 0,\ x^2+y^2\right\rangle;\quad R=\left\{(x,\ y):\ \ x^2+y^2\le 1\right\}$

(47–55) Find the area of the regions using line integral

47. The region enclosed by the ellipse $x^2 + 4y^2 = 16$

48. The region bounded by the hypocycloid $\mathbf{r}(t) = \left\langle \cos^3 t,\ \sin^3 t \right\rangle$ for $0 \le t \le 2\pi$.

49. The region enclosed by a disk of radius 5

50. A region bounded by an ellipse with semi-major and semi-minor axes of length 12 and 8, respectively.

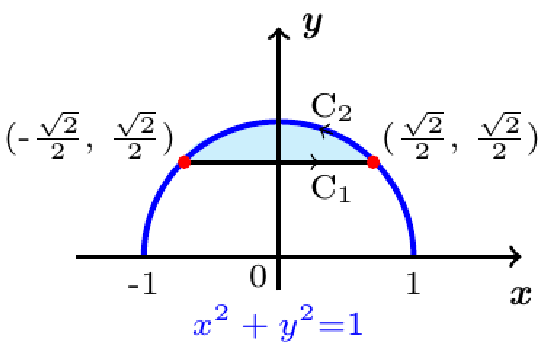
51. The region bounded by an ellipse $9x^2 + 25y^2 = 225$

52. $\left\{(x, y):\ x^2 + y^2 \le 16\right\}$

53. The region bounded by the parabolas $\vec{r}(t) = \left\langle t,\ 2t^2 \right\rangle$ and $\vec{r}(t) = \left\langle t,\ 12 - t^2 \right\rangle$ for $-2 \le t \le 2$

54. The region bounded by the curve $\vec{r}(t) = \left\langle t\left(1 - t^2\right),\ 1 - t^2 \right\rangle$ for $-1 \le t \le 1$

55. The shaded region



56. Prove the identity $\oint_C dx = \oint_C dy = 0$, where C is a simple closed smooth oriented curve.

57. Prove the identity $\oint_C f(x)\,dx + g(y)\,dy = 0$, where f and g have continuous derivatives on the region enclosed by C (is a simple closed smooth oriented curve)

58. Show that the value of $\oint_C xy^2 dx + \left(x^2 y + 2x\right) dy$ depends only on the area of the region enclosed by C.

59. In terms of the parameters a and b, how is the value of $\oint_C aydx + bxdy$ related to the area of the region enclosed by C, assuming counterclockwise orientation of C?

60. Show that if the circulation form of Green's Theorem is applied to the vector field $\left\langle 0,\ \dfrac{f(x)}{c} \right\rangle$ and $R = \{(x, y):\ \ a \le x \le b,\ \ 0 \le y \le c\}$, then the result is the Fundamental Theorem of Calculus,

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

61. Show that if the flux form of Green's Theorem is applied to the vector field $\left\langle \dfrac{f(x)}{c},\ 0 \right\rangle$ and $R = \{(x, y):\ \ a \le x \le b,\ \ 0 \le y \le c\}$, then the result is the Fundamental Theorem of Calculus,

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$