SOLUTION Section 2.1 – Definitions of 2*nd* and Higher Order Equations

Exercise

Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous. $t^2 y'' = 4y' - \sin t$

Solution

$$y'' - \frac{4}{t^2} y' = -\frac{\sin t}{t^2}$$

$y'' + p(t)y' + q(t)y = g(t)$ It is linear and inhomogeneous

Exercise

Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous. $ty'' + (\sin t)y' = 4y - \cos 5t$

Solution

$$y'' + \left(\frac{\sin t}{t}\right) y' - \frac{4}{t} y = -\frac{\cos 5t}{t}$$

$y'' + p(t)y' + q(t)y = g(t)$ It is linear and inhomogeneous $(g(t) \neq 0)$

Exercise

Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous. $t^2 y'' + 4yy' = 0$

Solution

It is nonlinear $(4yy')$ $(g(t) \neq 0)$

Exercise

Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous. $y'' + 4y' + 7y = 3e^{-t} \sin t$

Solution

Compare to $y'' + p(t)y' + q(t)y = g(t)$

$\Rightarrow p(t) = 4, \quad q(t) = 7, \quad g(t) = 3e^{-t} \sin t$ $(g(t) \neq 0)$

Hence, the equation is linear and inhomogeneous.

Exercise

Show by direct substitution that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation. Then verify by direct substitution, that any linear combination $C_1y_1(t)+C_2y_2(t)$ of the 2 given solutions is also a solution.

$$y''+4y=0 \qquad y_1(t)=\cos 2t \qquad y_2(t)=\sin 2t$$

Solution

$$\begin{aligned} y &= C_1y_1+C_2y_2 \\ &= C_1\cos 2t+C_2\sin 2t \end{aligned}$$

$$y'=-2C_1\sin 2t+2C_2\cos 2t$$

$$y''=-4C_1\cos 2t-4C_2\sin 2t$$

If $C_1y_1(t)+C_2y_2(t)$, then

$$\begin{aligned} y''+4y &= -4C_1\cos 2t-4C_2\sin 2t+4\left(C_1\cos 2t+C_2\sin 2t\right) \\ &= -4C_1\cos 2t-4C_2\sin 2t+4C_1\cos 2t+4C_2\sin 2t \\ &= 0 \end{aligned}$$

Exercise

Show by direct substitution that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation. Then verify by direct substitution, that any linear combination $C_1y_1(t)+C_2y_2(t)$ of the 2 given solutions is also a solution.

$$y''-2y'+2y=0; \quad y_1(t)=e^t\cos t \quad y_2(t)=e^t\sin t$$

Solution

$$y_1(t)=e^t\cos t \quad \Rightarrow \quad y_1'(t)=e^t\cos t-e^t\sin t=e^t\left(\cos t-\sin t\right)$$

$$\begin{aligned} \Rightarrow \quad y_1''(t) &= e^t\left(\cos t-\sin t\right)+e^t\left(-\sin t-\cos t\right) \\ &= e^t\cos t-e^t\sin t-e^t\sin t-e^t\cos t \\ &= -2e^t\sin t \end{aligned}$$

$$\begin{aligned} y_1''-2y_1'+2y_1 &= -2e^t\sin t-2\left(e^t\cos t-e^t\sin t\right)+2e^t\cos t \\ &= -2e^t\sin t-2e^t\cos t+2e^t\sin t+2e^t\cos t \\ &= 0 \end{aligned}$$

$$y_2(t)=e^t\sin t \quad \Rightarrow \quad y_2'(t)=e^t\sin t+e^t\cos t$$

$$\begin{aligned} y_2''(t) &= e^t\sin t+e^t\cos t+e^t\cos t-e^t\sin t \\ &= 2e^t\cos t \end{aligned}$$

$$y_1'' - 2y_1' + 2y_1 = 2e^t \cos t - 2\left(e^t \cos t + e^t \sin t\right) + 2e^t \sin t$$

$$= 2e^t \cos t - 2e^t \cos t - 2e^t \sin t + 2e^t \sin t$$

$$\underline{= 0|}$$

If $y(t) = C_1 e^t \cos t + C_2 e^t \sin t$

$$y'(t) = C_1 e^t \cos t - C_1 e^t \sin t + C_2 e^t \sin t + C_2 e^t \cos t$$

$$= \left(C_1 + C_2\right) e^t \cos t + \left(C_2 - C_1\right) e^t \sin t$$

$$y''(t) = \left(C_1 + C_2\right) e^t \cos t - \left(C_1 + C_2\right) e^t \sin t + \left(C_2 - C_1\right) e^t \sin t + \left(C_2 - C_1\right) e^t \cos t$$

$$= \left(C_1 + C_2 + C_2 - C_1\right) e^t \cos t + \left(C_2 - C_1 - C_1 - C_2\right) e^t \sin t$$

$$= 2C_2 e^t \cos t - 2C_1 e^t \sin t$$

$$y'' - 2y' + 2y = 2C_2 e^t \cos t - 2C_1 e^t \sin t - 2\left(\left(C_1 + C_2\right) e^t \cos t + \left(C_2 - C_1\right) e^t \sin t\right) + 2\left(C_1 e^t \cos t + C_2 e^t \sin t\right)$$

$$= 2C_2 e^t \cos t - 2C_1 e^t \sin t - 2C_1 e^t \cos t - 2C_2 e^t \cos t + 2C_2 e^t \sin t - 2C_1 e^t \sin t + 2C_1 e^t \cos t + 2C_2 e^t \sin t$$

$$\underline{= 0|}$$

Exercise

Explain why $y_1(t)$ and $y_2(t)$ are linearly independent solutions. Calculate Wronskian and use it to explain the independence of the given solutions.

$$y'' + 9y = 0 \qquad y_1(t) = \cos 3t \qquad y_2(t) = \sin 3t$$

Solution

$$w(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix}$$

$$= 3\cos^2 3t + 3\sin^2 3t$$

$$= 3\left(\cos^2 3t + \sin^2 3t\right)$$

$$= 3 \neq 0$$

The solutions $y_1(t)$ & $y_2(t)$ are linearly independent.

Exercise

Show that $y_1(t) = e^t$ and $y_2(t) = e^{-3t}$ form a fundamental set of solutions for $y'' + 2y' - 3y = 0$, then find a solution satisfying $y(0) = 1$ and $y'(0) = -2$.

Solution

$$y_1(t) = e^t \Rightarrow y'' + 2y' - 3y = e^t + 2e^t - 3e^t = 0$$

$$y_2(t) = e^{-3t} \Rightarrow y'' + 2y' - 3y = 9e^{-3t} - 6e^{-3t} - 3e^{-3t} = 0$$

$$y(t) = C_1 e^t + C_2 e^{-3t} \qquad y(0) = C_1 + C_2 = 1$$

$$y'(t) = C_1 e^t - 3C_2 e^{-3t} \qquad y'(0) = C_1 - 3C_2 = -2 \qquad \Rightarrow C_1 = \frac{1}{4} \quad C_2 = \frac{3}{4}$$

$$\boxed{y(t) = \frac{1}{4}e^t + \frac{3}{4}e^{-3t}}$$

Exercise

Use the Wronskian to show that are linearly independence $\quad y_1(x) = e^{-3x}, \quad y_2(x) = e^{3x}$

Solution

$$W(x) = \begin{vmatrix} e^{-3x} & e^{3x} \\ -3e^{-3x} & 3e^{3x} \end{vmatrix}$$

$$= 3 + 3$$

$\boxed{= 6 \ \neq 0}$ Thus the functions are linearly independent.

Exercise

Use the Wronskian to show that are linearly independence $\quad \mathbf{f}_1 = 1, \quad \mathbf{f}_2 = e^x, \quad \mathbf{f}_3 = e^{2x}$

Solution

The Wronskian is

$$W(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = e^x 4e^{2x} - 2e^{2x}e^x = 2e^{3x} \neq 0$$

Thus the functions are linearly independent.

Exercise

Use the Wronskian to show that are linearly independence $\left\{e^x, \ xe^x, \ (x+1)e^x\right\}$

Solution

$$W=\begin{vmatrix} e^x & xe^x & (x+1)e^x \\ e^x & (x+1)e^x & (x+2)e^x \\ e^x & (x+2)e^x & (x+3)e^x \end{vmatrix}$$

$$=(x+1)(x+3)e^{3x}+x(x+2)e^{3x}+(x+1)(x+2)e^{3x}-(x+1)^2e^{3x}-(x+2)^2e^{3x}-x(x+3)e^{3x}$$

$$=\left(x^2+4x+3+x^2+2x+x^2+3x+2-x^2-2x-1-x^2-4x-4-x^2-3x\right)e^{3x}$$

$$\boxed{=0}$$

Thus the set $\left\{e^x,\ xe^x,\ (x+1)e^x\right\}$ is linearly dependent.

Exercise

Use the Wronskian to show that are linearly independence

$$y_1(x)=e^{-3x},\quad y_2(x)=\cos 2x,\quad y_3(x)=\sin 2x$$

Solution

$$W=\begin{vmatrix} e^{-3x} & \cos 2x & \sin 2x \\ -3e^{-3x} & -2\sin 2x & 2\cos 2x \\ 9e^{-3x} & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$=8e^{-3x}\sin^2 2x+18e^{-3x}\cos^2 2x+12e^{-3x}\sin 2x\cos 2x$$

$$+18e^{-3x}\sin^2 2x+8e^{-3x}\cos^2 2x-12e^{-3x}\sin 2x\cos 2x$$

$$\boxed{=26e^{-3x}\neq 0}$$

$\therefore\ y_1,\ y_2,\ and\ y_3$ are linearly independent.

Exercise

Use the Wronskian to show that are linearly independence $y_1(x)=e^x,\quad y_2(x)=e^{2x},\quad y_3(x)=e^{3x}$

Solution

$$W=\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$=18e^{6x}+3e^{6x}+4e^{6x}-2e^{6x}-12e^{6x}-9e^{6x}$$

$$\boxed{=2e^{6x}\ \neq 0}$$

$\therefore\ y_1,\ y_2,\ and\ y_3$ are linearly independent.

Exercise

Use the Wronskian to show that are linearly independence

$$y_1(x) = \cos^2 x, \quad y_2(x) = \sin^2 x, \quad y_3(x) = \sec^2 x, \quad y_4(x) = \tan^2 x$$

Solution

Since $\cos^2 x + \sin^2 x = 1$ & $\sec^2 x = 1 + \tan^2 x$

$c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x = 0$

Let: $c_1 = c_2 = 0 \quad c_3 = -1 \quad c_4 = 1$

$\cos^2 x + \sin^2 x - \sec^2 x + \tan^2 x = 0$

The set of functions are linearly dependent.

Exercise

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

$$y_1(t) = \cos t \sin t, \quad y_2(t) = \sin 2t$$

Solution

$y_1(t) = cy_2(t)$

$\cos t \sin t = c \sin 2t \quad \rightarrow \quad \boxed{c = \frac{1}{2}}$

The given functions are linearly dependent.

Exercise

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

$$y_1(t) = e^{3t}, \quad y_2(t) = e^{-4t}$$

Solution

$y_1(t) = cy_2(t)$

$e^{3t} = ce^{-4t} \quad \rightarrow \quad e^{7t} = c$

Since an exponential function is strictly monotone, this is a contradiction.

Hence, given functions are linearly independent on $(0,\ 1)$

Exercise

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

$$y_1(t) = te^{2t}, \quad y_2(t) = e^{2t}$$

Solution

$$y_1(t) = cy_2(t)$$

$$te^{2t} = ce^{2t} \quad \rightarrow \quad \underline{c = t}$$

Hence, given functions are linearly independent on $(0,\ 1)$

Exercise

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

$$y_1(t) = t^2\cos(\ln t), \quad y_2(t) = t^2\sin(\ln t)$$

Solution

$$y_1(t) = cy_2(t)$$

$$t^2\cos(\ln t) = ct^2\sin(\ln t) \quad \rightarrow \quad \cos(\ln t) = c\sin(\ln t) \ \Rightarrow \ \underline{c = \cot(\ln t)}$$

Hence, given functions are linearly independent on $(0,\ 1)$

Exercise

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

$$y_1(t) = \tan^2 t - \sec^2 t, \quad y_2(t) = 3$$

Solution

$$y_1(t) = cy_2(t)$$

$$\tan^2 t - \sec^2 t = 3c \quad \rightarrow \quad -1 = 3c \ \Rightarrow \ \underline{c = -\tfrac{1}{3}}$$

The given functions are linearly dependent.

Exercise

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0,\ 1)$

$$y_1(t) \equiv 0, \quad y_2(t) = e^t$$

Solution

$$y_1(t) = cy_2(t)$$

$$0 \equiv ce^t \quad \rightarrow \quad \underline{c \equiv 0}$$

The given functions are linearly dependent.

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation.

$$y'' + 2y' - 3y = 0$$

Solution

Let $v = y' \quad \Rightarrow \quad v' = y''$

$y'' = -2y' + 3y$

$v' = -2v + 3y$

The following system of the first-order equations: $\begin{cases} y' = v \\ v' = -2v + 3y \end{cases}$

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation.

$$y'' + 3y' + 4y = 2\cos 2t$$

Solution

Let $v = y' \quad \Rightarrow \quad v' = y''$

$y'' = -3y' - 4y + 2\cos 2t$

$v' = -3v - 4y + 2\cos 2t$

The following system of the first-order equations: $\begin{cases} y' = v \\ v' = -3v - 4y + 2\cos 2t \end{cases}$

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation. $\quad y'' + 2y' + 2y = 2\sin 2\pi t$

Solution

Let $v = y' \quad \Rightarrow \quad v' = y''$

$y'' = -2y' - 2y + 2\sin 2\pi t$

$v' = -2v - 2y + 2\sin 2\pi t$

The following system of the first-order equations: $\begin{cases} y' = v \\ v' = -2v - 2y + 2\sin 2\pi t \end{cases}$

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation. $\quad y'' + \mu\left(t^2 - 1\right)y' + y = 0$

Solution

Let $v = y' \quad \Rightarrow \quad v' = y''$

$y'' = -\mu\left(t^2 - 1\right)y' - y$

$v' = -\mu\left(t^2 - 1\right)v - y$

The following system of the first-order equations:

$$\begin{cases} y' = v \\ v' = -\mu\left(t^2 - 1\right)v - y \end{cases}$$

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation. $\quad 4y'' + 4y' + y = 0$

Solution

Let $v = y' \quad \Rightarrow \quad v' = y''$

$4y'' = -4y' - y$

$y'' = -y' - \frac{1}{4}y$

$v' = -v - \frac{1}{4}y$

The following system of the first-order equations: $\begin{cases} y' = v \\ v' = -v - \frac{1}{4}y \end{cases}$

Exercise

Find a particular solution satisfying the given initial conditions

$$y'' - 4y = 0\ ; \quad y_1(t) = e^{2t},\ y_2(t) = 2e^{-2t}; \quad y(0) = 1,\ y'(0) = -2$$

Solution

$$W = \begin{vmatrix} e^{2t} & 2e^{-2t} \\ 2e^{2t} & -4e^{-2t} \end{vmatrix}$$

$\underline{= -8 \ \neq 0}$

$\therefore\ y_1 \ \textit{and}\ y_2$ are linearly independent.

$y(t) = C_1 y_1(t) + C_2 y_2(t)$

$y(t) = C_1 e^{2t} + 2C_2 e^{-2t} \qquad y(0) = 1 \ \rightarrow \ C_1 + 2C_2 = 1$

$$y'(t)=2C_1e^{2t}-4C_2e^{-2t} \qquad y'(0)=-2 \ \rightarrow \ 2C_1-4C_2=-2$$

$$\begin{cases} C_1+2C_2=1 \\ C_1-2C_2=-1 \end{cases} \qquad \Delta=\begin{vmatrix}1 & 2\\ 1 & -2\end{vmatrix}=-4 \quad \Delta_1=\begin{vmatrix}1 & 2\\ -1 & -2\end{vmatrix}=0 \quad \Delta_2=\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}=-2$$

$$\underline{C_1=0, \quad C_2=\tfrac{1}{2}}$$

$$\underline{y(t)=e^{-2t}}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$y''-y=0\ ; \quad y_1(t)=2e^t, \ y_2(t)=e^{-t+3}; \quad y(-1)=1, \ y'(-1)=0$$

Solution

$$W=\begin{vmatrix}2e^t & e^{-t+3}\\ 2e^t & -e^{-t+3}\end{vmatrix}$$

$$\underline{=-4e^3 \ \neq 0}$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$$y(t)=C_1y_1(t)+C_2y_2(t)$$

$$y(t)=2C_1e^t+C_2e^{-t+3} \qquad y(-1)=1 \ \rightarrow \ 2C_1e^{-1}+C_2e^4=1$$

$$y'(t)=2C_1e^t-C_2e^{-t+3} \qquad y'(-1)=0 \ \rightarrow \ 2C_1e^{-1}-C_2e^4=0$$

$$\begin{cases} 2C_1+e^5C_2=e \\ 2C_1-e^5C_2=0 \end{cases} \ \rightarrow \ 4C_1=e$$

$$\underline{C_1=\frac{e}{4}, \quad C_2=\frac{1}{2e^4}}$$

$$\underline{y(t)=\frac{e}{4}e^t+\frac{1}{2e^4}e^{-t+3}}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$y''+y=0\ ; \quad y_1(t)=0, \ y_2(t)=\sin t; \quad y\left(\frac{\pi}{2}\right)=1, \ y'\left(\frac{\pi}{2}\right)=1$$

Solution

$$W = \begin{vmatrix} 0 & \sin t \\ 0 & \cos t \end{vmatrix}$$

$$= 0$$

$\therefore\ y_1$ *and* y_2 are linearly dependent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t) = C_2 \sin t \qquad y\left(\frac{\pi}{2}\right) = 1 \ \rightarrow\ C_2 = 1$$

$$y'(t) = C_2 \cos t \qquad y'\left(\frac{\pi}{2}\right) = 1 \ \rightarrow\ \cancel{0 = 1}$$

$$y(t) = C_2 \sin t$$

Exercise

Find a particular solution satisfying the given initial conditions

$$y'' + y = 0\ ; \quad y_1(t) = \cos t,\ y_2(t) = \sin t; \quad y\left(\frac{\pi}{2}\right) = 1,\ y'\left(\frac{\pi}{2}\right) = 1$$

Solution

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$$

$$= \cos^2 t + \sin^2 t$$

$$= 1 \neq 0$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t) = C_1 \cos t + C_2 \sin t \qquad y\left(\frac{\pi}{2}\right) = 1 \ \rightarrow\ C_2 = 1$$

$$y'(t) = -C_1 \sin t + C_2 \cos t \qquad y'\left(\frac{\pi}{2}\right) = 1 \ \rightarrow\ C_1 = -1$$

$$y(t) = -\cos t + \sin t$$

Exercise

Find a particular solution satisfying the given initial conditions

$$y'' - 4y' + 4y = 0\ ; \quad y_1(t) = e^{2t},\ y_2(t) = te^{2t}; \quad y(0) = 2,\ y'(0) = 0$$

Solution

$$W = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & (1+2t)e^{2t} \end{vmatrix}$$

$$= (1+2t-2t)e^{4t}$$

$$\boxed{= e^{4t} \ \neq 0}$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$y(t) = C_1 e^{2t} + C_2 te^{2t}$ $\qquad y(0) = 2 \ \rightarrow \ \boxed{C_1 = 2}$

$y'(t) = 2C_1 e^{2t} + C_2 (1+2t)e^{2t}$ $\qquad y'(0) = 0 \ \rightarrow \ 2C_1 + C_2 = 0 \ \rightarrow \ \boxed{C_2 = -4}$

$$\boxed{y(t) = 2e^{2t} - 4te^{2t}}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$2y'' - y' = 0\ ; \quad y_1(t) = 1, \ y_2(t) = e^{t/2}; \quad y(2) = 0, \ y'(2) = 2$$

Solution

$$W = \begin{vmatrix} 1 & e^{t/2} \\ 0 & \frac{1}{2}e^{t/2} \end{vmatrix}$$

$$\boxed{= \frac{1}{2}e^{t/2} \ \neq 0}$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$y(t) = C_1 + C_2 e^{t/2}$ $\qquad y(2) = 0 \ \rightarrow \ C_1 + eC_2 = 0$

$y'(t) = \frac{1}{2} C_2 e^{t/2}$ $\qquad y'(2) = 2 \ \rightarrow \ \frac{1}{2} eC_2 = 2$

$$\boxed{C_1 = -4, \quad C_2 = \frac{4}{e}}$$

$$\boxed{y(t) = -4 + \frac{4}{e}e^{t/2}}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$y'' - 3y' + 2y = 0\ ; \quad y_1(t) = 2e^t,\ y_2(t) = e^{2t}; \quad y(-1) = 1,\ y'(-1) = 0$$

Solution

$$W = \begin{vmatrix} 2e^t & e^{2t} \\ 2e^t & 2e^{2t} \end{vmatrix}$$

$$= 2e^{3t} \ne 0$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t) = 2C_1 e^t + C_2 e^{2t} \qquad y(-1) = 1 \ \rightarrow\ 2e^{-1}C_1 + e^{-2}C_2 = 1$$

$$y'(t) = 2C_1 e^t + 2C_2 e^{2t} \qquad y'(-1) = 0 \ \rightarrow\ 2e^{-1}C_1 + 2e^{-2}C_2 = 2$$

$$\begin{cases} 2eC_1 + C_2 = e^2 \\ eC_1 + C_2 = e^2 \end{cases} \qquad \Delta = \begin{vmatrix} 2e & 1 \\ e & 1 \end{vmatrix} = e \quad \Delta_1 = \begin{vmatrix} e^2 & 1 \\ e^2 & 1 \end{vmatrix} = 0 \quad \Delta_2 = \begin{vmatrix} 2e & e^2 \\ e & e^2 \end{vmatrix} = e^3$$

$$C_1 = 0, \quad C_2 = e^2$$

$$y(t) = e^{2t+2}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$ty'' + y' = 0\ ; \quad y_1(t) = \ln t,\ y_2(t) = \ln 3t; \quad y(3) = 0,\ y'(3) = 3$$

Solution

$$W = \begin{vmatrix} \ln t & \ln 3t \\ \frac{1}{t} & \frac{1}{t} \end{vmatrix}$$

$$= \frac{1}{t}(\ln t - \ln 3t) \ne 0$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t) = C_1 \ln t + C_2 \ln 3t \qquad y(3) = 0 \ \rightarrow\ (\ln 3)C_1 + (\ln 9)C_2 = 0$$

$$y'(t) = \frac{C_1}{t} + \frac{C_2}{t} \qquad y'(3) = 3 \ \rightarrow\ \frac{1}{3}(C_1 + C_2) = 3$$

$$\begin{cases} (\ln 3)C_1 + (2\ln 3)C_2 = 0 \\ C_1 + C_2 = 9 \end{cases}$$

$$\Delta = \begin{vmatrix} \ln 3 & 2\ln 3 \\ 1 & 1 \end{vmatrix} = -\ln 3 \quad \Delta_1 = \begin{vmatrix} 0 & 2\ln 3 \\ 9 & 1 \end{vmatrix} = -18\ln 3 \quad \Delta_2 = \begin{vmatrix} \ln 3 & 0 \\ 1 & 9 \end{vmatrix} = 9\ln 3$$

$$C_1 = 18, \quad C_2 = -9$$

$$y(t) = 18\ln t - 9\ln 3t$$

Exercise

Find a particular solution satisfying the given initial conditions

$$t^2y'' - ty' - 3y = 0\ ; \quad y_1(t) = t^3,\ y_2(t) = -\frac{1}{t};\quad y(-1) = 0,\ y'(-1) = -2 \quad (t < 0)$$

Solution

$$W = \begin{vmatrix} t^3 & -\frac{1}{t} \\ 3t^2 & \frac{1}{t^2} \end{vmatrix}$$

$$= 4t \neq 0$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$$y(t) = C_1y_1(t) + C_2y_2(t)$$

$$y(t) = C_1t^3 - \frac{C_2}{t} \qquad\qquad y(-1) = 0 \ \rightarrow\ -C_1 + C_2 = 0$$

$$y'(t) = 3C_1t^2 + \frac{C_2}{t^2} \qquad\qquad y'(-1) = -2 \ \rightarrow\ 3C_1 + C_2 = -2$$

$$\begin{cases} -C_1 + C_2 = 0 \\ 3C_1 + C_2 = -2 \end{cases} \qquad \Delta = \begin{vmatrix} -1 & 1 \\ 3 & 1 \end{vmatrix} = -4 \quad \Delta_1 = \begin{vmatrix} 0 & 1 \\ -2 & 1 \end{vmatrix} = 2 \quad \Delta_2 = \begin{vmatrix} -1 & 0 \\ 3 & -2 \end{vmatrix} = 2$$

$$C_1 = -\frac{1}{2}, \quad C_2 = -\frac{1}{2}$$

$$y(t) = -\frac{1}{2}t^3 + \frac{1}{2t}$$

Exercise

Find a particular solution satisfying the given initial conditions

$$y'' + \pi^2y = 0\ ; \quad y_1(t) = \sin \pi t + \cos \pi t,\ y_2(t) = \sin \pi t - \cos \pi t; \quad y\left(\frac{1}{2}\right) = 1,\ y'\left(\frac{1}{2}\right) = 0$$

Solution

$$W = \begin{vmatrix} \sin \pi t + \cos \pi t & \sin \pi t - \cos \pi t \\ \pi \cos \pi t - \pi \sin \pi t & \pi \cos \pi t + \pi \sin \pi t \end{vmatrix}$$

$$= \pi \sin^2 \pi t + \pi \cos^2 \pi t + 2\pi \sin \pi t \cos \pi t - 2\pi \sin \pi t \cos \pi t + \pi \sin^2 \pi t + \pi \cos^2 \pi t$$

$$= 2\pi \left(\sin^2 \pi t + \cos^2 \pi t \right)$$

$$\underline{= 2\pi \quad \neq 0}$$

$\therefore\ y_1$ *and* y_2 are linearly independent.

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$y(t) = C_1 (\sin \pi t + \cos \pi t) + C_2 (\sin \pi t - \cos \pi t) \qquad y\left(\tfrac{1}{2}\right) = 1 \ \rightarrow \ C_1 + C_2 = 1$$

$$y'(t) = C_1 (\pi \cos \pi t - \pi \sin \pi t) + C_2 (\pi \cos \pi t + \pi \sin \pi t) \qquad y'\left(\tfrac{1}{2}\right) = 0 \ \rightarrow \ -\pi C_1 + \pi C_2 = 0$$

$$\begin{cases} C_1 + C_2 = 1 \\ -C_1 + C_2 = 0 \end{cases} \quad \rightarrow \quad \underline{C_1 = \tfrac{1}{2}, \quad C_2 = \tfrac{1}{2}}$$

$$y(t) = \tfrac{1}{2}(\sin \pi t + \cos \pi t) + \tfrac{1}{2}(\sin \pi t - \cos \pi t)$$

$$\underline{= \sin \pi t}$$

Exercise

Find a particular solution satisfying the given initial conditions $x^3 y^{(3)} - x^2 y'' + 2xy' - 2y = 0$

$y(1) = 3, \quad y'(1) = 2, \quad y''(1) = 1 \qquad y_1(x) = x, \quad y_2(x) = x \ln x, \quad y_3(x) = x^2$

Solution

$$W = \begin{vmatrix} x & x \ln x & x^2 \\ 1 & \ln x + 1 & 2x \\ 0 & \frac{1}{x} & 2 \end{vmatrix}$$

$$= 2x \ln x + 2x + x - 2x - 2x \ln x$$

$$\underline{= x \quad \neq 0}$$

$\therefore\ y_1, y_2,$ *and* y_3 are linearly independent.

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$$

$$y(x) = C_1 x + C_2 x \ln x + C_3 x^2 \qquad y(1) = 3 \ \rightarrow \ C_1 + C_3 = 3$$

$$y'(x) = C_1 + C_2 (1 + \ln x) + 2C_3 x \qquad y'(1) = 2 \ \rightarrow \ C_1 + C_2 + 2C_3 = 2$$

$$y''(x) = \frac{C_2}{x} + 2C_3 \qquad y''(1) = 1 \ \rightarrow \ C_2 + 2C_3 = 1$$

$$\begin{cases} C_1 + C_3 = 3 \\ C_1 + C_2 + 2C_3 = 2 \\ C_2 + 2C_3 = 1 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{vmatrix} = 1 \quad \Delta_1 = \begin{vmatrix} 3 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{vmatrix} = 1$$

$$\underline{C_1 = 1, \quad C_2 = -3, \quad C_3 = 2}$$

$$\boxed{y(x) = x - 3x\ln x + 2x^2}$$

Exercise

Find a particular solution satisfying the given initial conditions $y^{(3)} + 2y'' - y' - 2y = 0$

$y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 0 \qquad y_1(x) = e^x, \quad y_2(x) = e^{-x}, \quad y_3(x) = e^{-2x}$

Solution

$$W = \begin{vmatrix} e^x & e^{-x} & e^{-2x} \\ e^x & -e^{-x} & -2e^{-2x} \\ e^x & e^{-x} & 4e^{-2x} \end{vmatrix}$$

$$= -4e^{-2x} - 2e^{-2x} + e^{-2x} + e^{-2x} + 2e^{-2x} - 4e^{-2x}$$

$$= -6e^{-2x} \neq 0$$

$\therefore\ y_1, y_2,$ *and* y_3 are linearly independent. $\qquad y(x) = C_1y_1(x) + C_2y_2(x) + C_3y_3(x)$

$$y(x) = C_1e^x + C_2e^{-x} + C_3e^{-2x} \qquad y(0) = 1 \ \rightarrow\ C_1 + C_2 + C_3 = 1$$

$$y'(x) = C_1e^x - C_2e^{-x} - 2C_3e^{-2x} \qquad y'(0) = 2 \ \rightarrow\ C_1 - C_2 - 2C_3 = 2$$

$$y''(x) = C_1e^x + C_2e^{-x} + 4C_3e^{-2x} \qquad y''(0) = 0 \ \rightarrow\ C_1 + C_2 + 4C_3 = 0$$

$$\begin{cases} C_1 + C_2 + C_3 = 1 \\ C_1 - C_2 - 2C_3 = 2 \\ C_1 + C_2 + 4C_3 = 0 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix} = -9 \quad \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 0 & 1 & 4 \end{vmatrix} = -12 \quad \Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -2 \\ 1 & 0 & 4 \end{vmatrix} = 0$$

$$\underline{C_1 = \tfrac{4}{3}, \quad C_2 = 0, \quad C_3 = -\tfrac{1}{3}}$$

$$\boxed{y(x) = \tfrac{4}{3}e^x - \tfrac{1}{3}e^{-2x}}$$

Exercise

Find a particular solution satisfying the given initial conditions $y^{(3)}-6y''+11y'-6y=0$

$y(0)=0, \quad y'(0)=0, \quad y''(0)=3 \qquad y_1(x)=e^x, \quad y_2(x)=e^{2x}, \quad y_3(x)=e^{3x}$

Solution

$$W=\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$=18e^{6x}+3e^{6x}+4e^{6x}-2e^{6x}-12e^{6x}-9e^{6x}$$

$$=2e^{6x} \neq 0$$

$\therefore\ y_1, y_2,$ *and* y_3 are linearly independent.

$$y(x)=C_1y_1(x)+C_2y_2(x)+C_3y_3(x)$$

$$y(x)=C_1e^x+C_2e^{2x}+C_3e^{3x} \qquad y(0)=0 \ \rightarrow\ C_1+C_2+C_3=0$$

$$y'(x)=C_1e^x+2C_2e^{2x}+3C_3e^{3x} \qquad y'(0)=0 \ \rightarrow\ C_1+2C_2+3C_3=0$$

$$y''(x)=C_1e^x+4C_2e^{2x}+9C_3e^{3x} \qquad y''(0)=3 \ \rightarrow\ C_1+4C_2+9C_3=3$$

$$\begin{cases} C_1+C_2+C_3=0 \\ C_1+2C_2+3C_3=0 \\ C_1+4C_2+9C_3=3 \end{cases} \qquad \Delta=\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}=2 \quad \Delta_1=\begin{vmatrix} 0 & 1 & 1 \\ 0 & 2 & 3 \\ 3 & 4 & 9 \end{vmatrix}=3 \quad \Delta_2=\begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 9 \end{vmatrix}=-6$$

$$C_1=\frac{3}{2}, \quad C_2=-3, \quad C_3=\frac{3}{2}$$

$$y(x)=\frac{3}{2}e^x-3e^{2x}+\frac{3}{2}e^{3x}$$

Exercise

Find a particular solution satisfying the given initial conditions $y^{(3)}-3y''+3y'-y=0$

$y(0)=2, \quad y'(0)=0, \quad y''(0)=0 \qquad y_1(x)=e^x, \quad y_2(x)=xe^x, \quad y_3(x)=x^2e^x$

Solution

$$W=\begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & (1+x)e^x & \left(2x+x^2\right)e^x \\ e^x & (2+x)e^x & \left(2+4x+x^2\right)e^x \end{vmatrix}$$

$$=\left(2+6x+5x^2+x^3+2x^2+x^3+2x^2+x^3-x^2-x^3-2x-4x^2-x^3-2x-4x^2-x^3\right)e^{3x}$$

$$=(2+2x)e^{3x} \neq 0$$

$\therefore\ y_1, y_2,$ *and* y_3 are linearly independent.

$$y(x)=C_1y_1(x)+C_2y_2(x)+C_3y_3(x)$$

$y(x)=C_1e^x+C_2xe^x+C_3x^2e^x \qquad y(0)=2 \ \rightarrow\ C_1=2$

$y'(x)=C_1e^x+C_2(1+x)e^x+C_3\left(2x+x^2\right)e^x \qquad y'(0)=0 \ \rightarrow\ C_1+C_2=0$

$y''(x)=C_1e^x+C_2(2+x)e^x+C_3\left(2+4x+x^2\right)e^x \qquad y''(0)=0 \ \rightarrow\ C_1+2C_2+2C_3=0$

$$C_1=2, \quad C_2=-2, \quad C_3=1$$

$$y(x)=2e^x-2xe^x+x^2e^x$$

Exercise

Find a particular solution satisfying the given initial conditions $\quad y^{(3)}-5y''+8y'-4y=0$

$y(0)=1, \quad y'(0)=4, \quad y''(0)=0 \qquad y_1(x)=e^x, \quad y_2(x)=e^{2x}, \quad y_3(x)=xe^{2x}$

Solution

$$W=\begin{vmatrix} e^x & e^{2x} & xe^{2x} \\ e^x & 2e^{2x} & (1+2x)e^{2x} \\ e^x & 4e^{2x} & (4+4x)e^{2x} \end{vmatrix}$$

$$=(8+8x+4+8x+4x-2x-4-8x-4-4x)e^{5x}$$

$$=(6x+4)e^{5x} \neq 0$$

$\therefore\ y_1, y_2,$ *and* y_3 are linearly independent.

$$y(x)=C_1y_1(x)+C_2y_2(x)+C_3y_3(x)$$

$y(x)=C_1e^x+C_2e^{2x}+C_3xe^{2x} \qquad y(0)=1 \ \rightarrow\ C_1+C_2=1$

$y'(x)=C_1e^x+2C_2e^{2x}+C_3(1+2x)e^{2x} \qquad y'(0)=4 \ \rightarrow\ C_1+2C_2+C_3=4$

$y''(x)=C_1e^x+4C_2e^{2x}+(4+4x)C_3e^{2x} \qquad y''(0)=0 \ \rightarrow\ C_1+4C_2+4C_3=0$

$$\begin{cases} C_1 + C_2 = 1 \\ C_1 + 2C_2 + C_3 = 4 \\ C_1 + 4C_2 + 4C_3 = 0 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 4 & 4 \end{vmatrix} = 1 \quad \Delta_1 = \begin{vmatrix} 1 & 1 & 0 \\ 4 & 2 & 1 \\ 0 & 4 & 4 \end{vmatrix} = -12 \quad \Delta_2 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 4 & 1 \\ 1 & 0 & 4 \end{vmatrix} = 13$$

$$\underline{C_1 = -12, \quad C_2 = 13, \quad C_3 = -10}$$

$$\underline{y(x) = -12e^{x} + 13e^{2x} - 10xe^{2x}}$$

Exercise

Find a particular solution satisfying the given initial conditions $y^{(3)} + 9y'' = 0$

$y(0) = 3, \quad y'(0) = -1, \quad y''(0) = 2 \qquad y_1(x) = 1, \quad y_2(x) = \cos 3x, \quad y_3(x) = \sin 3x$

Solution

$$W = \begin{vmatrix} 1 & \cos 3x & \sin 3x \\ 0 & -3\sin 3x & 3\cos 3x \\ 0 & -9\cos 3x & -9\sin 3x \end{vmatrix}$$

$$= 27\sin^2 3x + 27\cos^2 3x$$

$$\underline{= 27 \neq 0}$$

$\therefore\ y_1, y_2,$ *and* y_3 are linearly independent.

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$$

$$y(x) = C_1 + C_2 \cos 3x + C_3 \sin 3x \qquad y(0) = 3 \ \rightarrow\ C_1 + C_2 = 3$$

$$y'(x) = -3C_2 \sin 3x + 3C_3 \cos 3x \qquad y'(0) = -1 \ \rightarrow\ 3C_3 = -1$$

$$y''(x) = -9C_2 \cos 3x - 9C_3 \sin 3x \qquad y''(0) = 0 \ \rightarrow\ -9C_2 = 0$$

$$\underline{C_1 = 3, \quad C_2 = 0, \quad C_3 = -\tfrac{1}{3}}$$

$$\underline{y(x) = 3 - \tfrac{1}{3}\sin 3x}$$

Exercise

Find a particular solution satisfying the given initial conditions $y^{(3)} - 3y'' + 4y' - 2y = 0$

$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0 \qquad y_1(x) = e^{x}, \quad y_2(x) = e^{x}\cos x, \quad y_3(x) = e^{x}\sin x$

Solution

$$W = \begin{vmatrix} e^x & e^x\cos x & e^x\sin x \\ e^x & (\cos x - \sin x)e^x & (\sin x + \cos x)e^x \\ e^x & -2e^x\sin x & 2e^x\cos x \end{vmatrix}$$

$$= \left(2\cos^2 x - \sin x\cos x + \cos^2 x - 2\sin^2 x - \sin x\cos x + \sin^2 x + 2\sin^2 x + 2\sin x\cos x - 2\cos^2 x\right)e^{3x}$$

$$\underline{= e^{3x} \neq 0}$$

$\therefore\ y_1, y_2, \text{ and } y_3$ are linearly independent.

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$$

$y(x) = C_1 e^x + C_2 e^x\cos x + C_3 e^x\sin x$ $\qquad y(0)=1 \ \rightarrow\ C_1 + C_2 = 1$

$y'(x) = C_1 e^x + C_2(\cos x - \sin x)e^x + C_3(\sin x + \cos x)e^x$ $\qquad y'(0)=0 \ \rightarrow\ C_1 + C_2 + C_3 = 0$

$y''(x) = C_1 e^x - 2C_2 e^x\sin x + 2C_3 e^x\cos x$ $\qquad y''(0)=0 \ \rightarrow\ C_1 + 2C_3 = 0$

$$\begin{cases} C_1 + C_2 = 1 \\ C_1 + C_2 + C_3 = 0 \\ C_1 + 2C_3 = 0 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 1 \quad \Delta_1 = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2$$

$$\underline{C_1 = 2, \quad C_2 = -1, \quad C_3 = -1}$$

$$\underline{y(x) = 2e^x - e^x\cos x - e^x\sin x}$$

Exercise

Find a particular solution satisfying the given initial conditions $\qquad x^3y^{(3)} - 3x^2y'' + 6xy' - 6y = 0$

$y(1)=6, \quad y'(1)=14, \quad y''(1)=1 \qquad y_1(x)=x, \quad y_2(x)=x^2, \quad y_3(x)=x^3$

Solution

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$= 12x^3 + 2x^3 - 6x^3 - 6x^3$$

$$\underline{= 2x^3 \neq 0}$$

$\therefore\ y_1, y_2, \text{ and } y_3$ are linearly independent.

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$$

$$y(x)=C_1x+C_2x^2+C_3x^3 \qquad y(1)=1 \ \rightarrow \ C_1+C_2+C_3=6$$

$$y'(x)=C_1+2C_2x+3C_3x^2 \qquad y'(1)=14 \ \rightarrow \ C_1+2C_2+3C_3=14$$

$$y''(x)=2C_2+6C_3x \qquad y''(1)=1 \ \rightarrow \ 2C_2+6C_3=1$$

$$\begin{cases} C_1+C_2+C_3=6 \\ C_1+2C_2+3C_3=14 \\ 2C_2+6C_3=1 \end{cases} \qquad \Delta=\begin{vmatrix}1&1&1\\1&2&3\\0&2&6\end{vmatrix}=2 \quad \Delta_1=\begin{vmatrix}6&1&1\\14&2&3\\1&2&6\end{vmatrix}=-19 \quad \Delta_2=\begin{vmatrix}1&6&1\\1&14&3\\0&1&6\end{vmatrix}=46$$

$$\underline{C_1=-\frac{19}{2}, \quad C_2=23, \quad C_3=-\frac{15}{2}}$$

$$\boxed{y(x)=-\frac{19}{2}x+23x^2-\frac{15}{2}x^3}$$

Exercise

Find a particular solution satisfying the given initial conditions $x^3y^{(3)}+6x^2y''+4xy'-4y=0$

$y(1)=1, \quad y'(1)=5, \quad y''(1)=-11 \qquad y_1(x)=x, \quad y_2(x)=x^{-2}, \quad y_3(x)=x^{-2}\ln x$

Solution

$$W=\begin{vmatrix} x & x^{-2} & x^{-2}\ln x \\ 1 & -2x^{-3} & (1-2\ln x)x^{-3} \\ 0 & 6x^{-4} & (-5+6\ln x)x^{-4} \end{vmatrix}$$

$$=(10-12\ln x+6-6+12\ln x+5-6\ln x)x^{-6}$$

$$\boxed{=(15-6\ln x)x^{-6} \neq 0}$$

$\therefore\ y_1, y_2, \textit{and } y_3$ are linearly independent.

$$y(x)=C_1y_1(x)+C_2y_2(x)+C_3y_3(x)$$

$$y(x)=C_1x+C_2x^{-2}+C_3x^{-2}\ln x \qquad y(1)=1 \ \rightarrow \ C_1+C_2=1$$

$$y'(x)=C_1-2C_2x^{-3}+C_3x^{-3}(1-2\ln x) \qquad y'(1)=5 \ \rightarrow \ C_1-2C_2+C_3=5$$

$$y''(x)=6C_2x^{-4}+C_3x^{-4}(-5+6\ln x) \qquad y''(1)=-11 \ \rightarrow \ 6C_2-5C_3=-11$$

$$\begin{cases} C_1+C_2=1 \\ C_1-2C_2+C_3=5 \\ 6C_2-5C_3=-11 \end{cases} \qquad \Delta=\begin{vmatrix}1&1&0\\1&-2&1\\0&6&-5\end{vmatrix}=9 \quad \Delta_1=\begin{vmatrix}1&1&0\\5&-2&1\\-11&6&-5\end{vmatrix}=18 \quad \Delta_2=\begin{vmatrix}1&1&0\\1&5&1\\0&-11&-5\end{vmatrix}=-9$$

$$\boxed{C_1 = 2, \quad C_2 = -1, \quad C_3 = 1}$$

$$\boxed{y(x) = 2x - x^{-2} + x^{-2}\ln x}$$

Exercise

Given the mass, damping, and spring constants of an undriven spring-mass system $my'' + \mu y' + ky = 0$

$$m = 1\ kg, \quad \mu = 0\ kg/s, \quad k = 4kg/s^2, \quad y(0) = -2\ m, \quad y'(0) = -2\ m/s$$

a) Provide separate plots of the position versus time (y vs. t) and the velocity versus time (v vs. t)

b) Provide a combined plot of both position and velocity versus time

c) Provide a plot of the velocity versus position (v vs. y) in the yv phase plane.

Solution

$$my'' = -\mu y' - ky$$

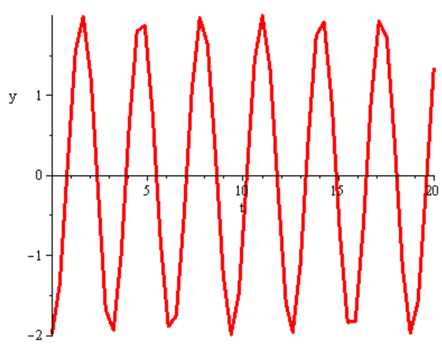
$$y'' = -\frac{\mu}{m}y' - \frac{k}{m}y$$

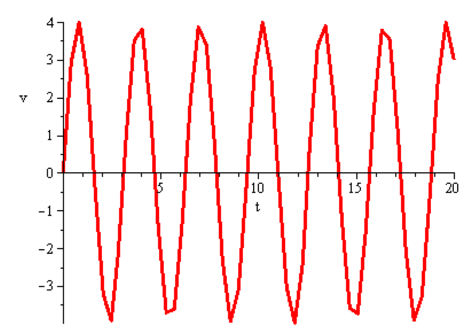
Let $\boxed{v = y'}$ $\Rightarrow v' = -\frac{\mu}{m}v - \frac{k}{m}y$

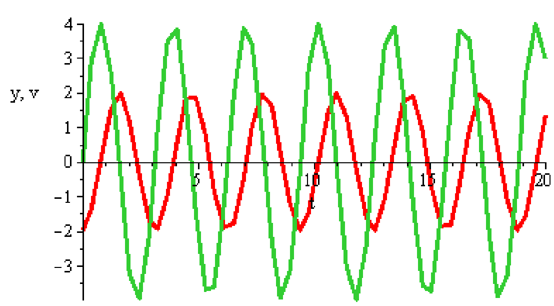
$$= -\frac{0}{1}v - \frac{4}{1}y$$

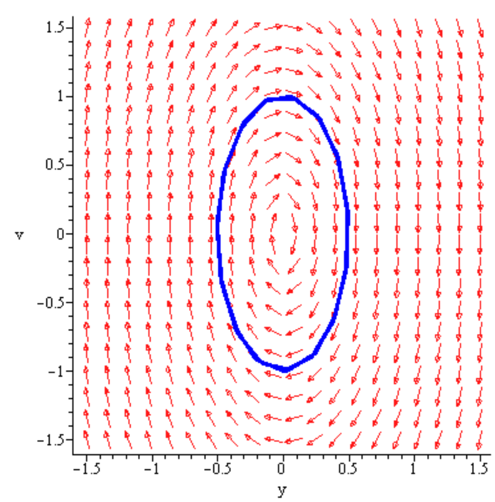
$$\boxed{v' = -4y}$$

$$y(0) = -2, \quad y'(0) = -2 = v(0)$$









Exercise

Given the mass, damping, and spring constants of an undriven spring-mass system $my'' + \mu y' + ky = 0$

$$m = 1\ kg, \quad \mu = 2\ kg/s, \quad k = 1 kg/s^2, \quad y(0) = -3\ m, \quad y'(0) = -2\ m/s$$

a) Provide separate plots of the position versus time (y vs. t) and the velocity versus time (v vs. t)

b) Provide a combined plot of both position and velocity versus time

c) Provide a plot of the velocity versus position (v vs. y) in the yv phase plane.

Solution

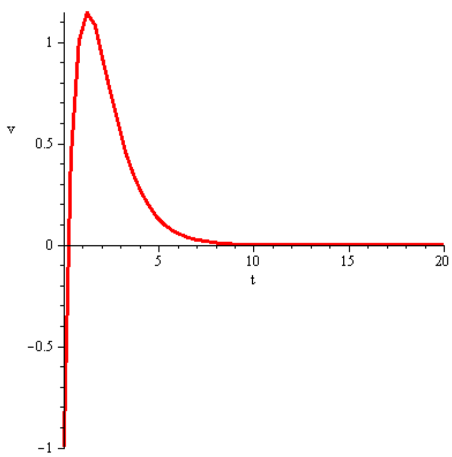
$$my'' = -\mu y' - ky \quad \Rightarrow \quad y'' = -\frac{\mu}{m} y' - \frac{k}{m} y$$

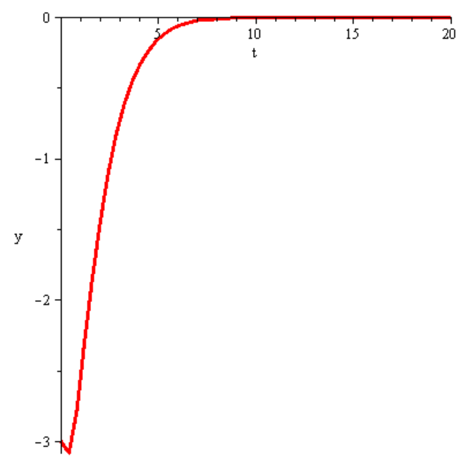
Let $v = y'$ $\Rightarrow v' = -\frac{\mu}{m} v - \frac{k}{m} y$

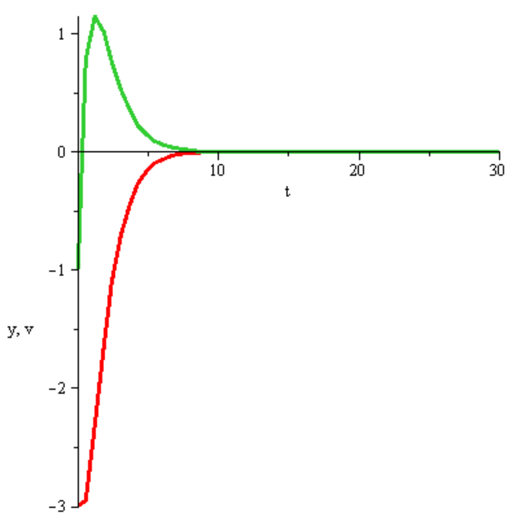
$$= -\frac{2}{1} v - \frac{1}{1} y$$

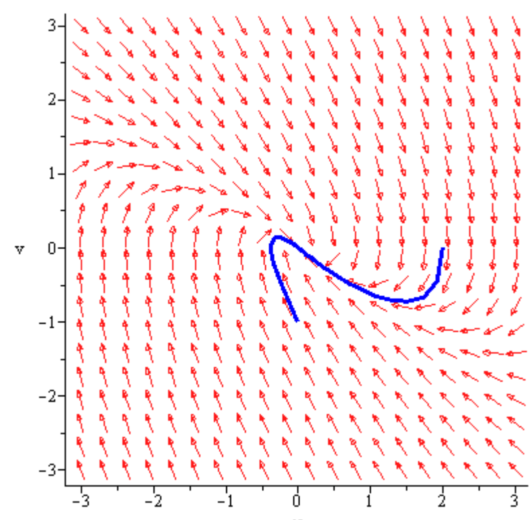
$$= -2v - y$$

The following system of the first-order equations: $\begin{cases} y' = v \\ v' = -2v - y \end{cases}$ *with* $\begin{matrix} y(0) = -3 \\ y'(0) = -2 = v(0) \end{matrix}$









Exercise

When the values of a solution to a differential equation are specified at two different points, these conditions. (In contrast, initial conditions specify the values of a function and its derivative at the same point). The purpose of this is to show that for boundary value problems there is no existence-uniqueness theorem. Given that every solution to

$$y'' + y = 0 \qquad \text{is of the form} \qquad y(t) = c_1 \cos t + c_2 \sin t$$

Where c_1 and c_2 are arbitrary constants, show that

a) There is a unique solution to the given differential equation that satisfies the boundary conditions $y(0) = 2$ and $y\left(\frac{\pi}{2}\right) = 0$

b) There is no solution to given equation that satisfies $y(2) = 0$ and $y(\pi) = 0$

c) There are infinitely many solution to the given DE equation that satisfy $y(0) = 2$ and $y(\pi) = -2$

Solution

a) $\lambda^2 + 1 = 0 \quad \to \quad \boxed{\lambda = \pm i}$

$y(t) = c_1 \cos t + c_2 \sin t$

$$y(0) = 2 \quad \to \quad \boxed{2 = c_1}$$

$$y\left(\frac{\pi}{2}\right) = 0 \quad \to \quad \boxed{0 = c_2}$$

$\boxed{y(t) = 2\cos t}$

b) $y(0) = 2 \quad \to \quad \boxed{2 = c_1}$

$y(\pi) = 0 \quad \to \quad \boxed{0 = -c_1}$

This system is inconsistent, so there is no solution satisfying the given boundary.

c) $y(0) = 2 \quad \to \quad \boxed{2 = c_1}$

$y(\pi) = -2 \quad \to \quad \boxed{-2 = -c_1}$

$\boxed{y(t) = 2\cos t + c_2 \sin t}$

Which has infinitely many solutions given c_2 is an arbitrary constant.