

**Exercise**

(a) Find the series' radius and interval of convergence. For what values of  $x$  does the series converge (b) absolutely, (c) conditionally?

$$\sum_{n=0}^{\infty} x^n$$

**Solution**

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{x^{n+1}}{x^n} \right| = |x| < 1 \Rightarrow -1 < x < 1$$

$$\text{When } x = 1 \Rightarrow \sum_{n=0}^{\infty} 1$$

$$\text{and } x = -1 \Rightarrow \sum_{n=0}^{\infty} (-1)^n \text{ the series diverges.}$$

- a) The radius is 1; the interval of converges  $-1 < x < 1$
- b) The interval of absolute convergence is  $-1 < x < 1$
- c) There are no values for which the series converges conditionally

**Exercise**

(a) Find the series' radius and interval of convergence. For what values of  $x$  does the series converge (b) absolutely, (c) conditionally?

$$\sum_{n=0}^{\infty} (x+5)^n$$

**Solution**

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{(x+5)^{n+1}}{(x+5)^n} \right| \\ = |x+5| < 1$$

$$-6 < x < -4$$

$$\text{When } x = -6 \Rightarrow \sum_{n=0}^{\infty} (-1)^n$$

and  $x = -4 \Rightarrow \sum_{n=0}^{\infty} 1$  the series diverges.

- a) The radius is 1; the interval of converges  $-6 < x < -4$
- b) The interval of absolute convergence is  $-6 < x < -4$
- c) There are no values for which the series converges conditionally.

### Exercise

(a) Find the series' radius and interval of convergence. For what values of  $x$  does the series converge (b) absolutely, (c) conditionally?

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$$

### Solution

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{(3x-2)^{n+1}}{n+1} \cdot \frac{n}{(3x-2)^n} \right|$$

$$= \frac{n}{n+1} |3x-2| < 1$$

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} |3x-2| < 1$$

$$|3x-2| < 1$$

$$-1 < 3x-2 < 1$$

$$1 < 3x < 3$$

$$\frac{1}{3} < x < 1$$

When  $x = \frac{1}{3} \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$  which is the alternating harmonic series and is conditionally convergent.

$x = 1 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n}$  the series diverges harmonic.

- a) The radius is  $\frac{1}{3}$ ; the interval of converges  $\frac{1}{3} \leq x < 1$
- b) The interval of absolute convergence is  $\frac{1}{3} < x < 1$
- c) The series converges conditionally at  $x = \frac{1}{3}$

### Exercise

- (a) Find the series' radius and interval of convergence. For what values of  $x$  does the series converge  
(b) absolutely,  
(c) conditionally?

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$$

### Solution

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{(x-2)^{n+1}}{10^{n+1}} \cdot \frac{10^n}{(x-2)^n} \right|$$
$$= \frac{|x-2|}{10} < 1$$

$$-1 < \frac{x-2}{10} < 1$$

$$-10 < x-2 < 10$$

$$-8 < x < 12$$

When  $x = -8 \Rightarrow \sum_{n=0}^{\infty} (-1)^n$  which is a divergent series

$x = 12 \Rightarrow \sum_{n=0}^{\infty} 1$  the series diverges

- a) The radius is 10; the interval of converges  $-8 < x < 12$   
b) The interval of absolute convergence is  $-8 < x < 12$   
c) There are no values for which the series converges conditionally

### Exercise

- (a) Find the series' radius and interval of convergence. For what values of  $x$  does the series converge  
(b) absolutely,  
(c) conditionally?

$$\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$$

### Solution

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{3^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n x^n} \right|$$
$$= \frac{3}{n+1} |x| < 1$$

$$3|x| \lim_{n \rightarrow \infty} \frac{1}{n+1} < 1 \Rightarrow \forall x$$

- a) The radius is  $\infty$ ; the series converges for all  $x$ .
- b) The series convergence absolutely for all  $x$ .
- c) There are no values for which the series converges conditionally

### Exercise

- (a) Find the series' radius and interval of convergence. For what values of  $x$  does the series converge
- (b) absolutely,
- (c) conditionally?

$$\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2 + 3}}$$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{\sqrt{(n+1)^2 + 3}} \cdot \frac{\sqrt{n^2 + 3}}{x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \frac{n^2 + 3}{n^2 + 2n + 4} < 1 \end{aligned}$$

$$|x| < 1 \Rightarrow -1 < x < 1$$

$$\text{When } x = -1 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2 + 3}} \text{ which is a convergent conditionally series}$$

$$x = 1 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2 + 3}} \text{ the series diverges}$$

- a) The radius is 1; the series converges for  $-1 \leq x < 1$ .
- b) The series convergence absolutely for  $-1 < x < 1$ .
- c) The series convergence conditionally for  $x = -1$

### Exercise

- (a) Find the series' radius and interval of convergence. For what values of  $x$  does the series converge (b) absolutely, (c) conditionally?

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{\sqrt{n} + 3}$$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+2}}{\sqrt{n+1}+3} \cdot \frac{\sqrt{n}+3}{x^{n+1}} \right| \\ &= |x| \lim_{n \rightarrow \infty} \frac{\sqrt{n}+3}{\sqrt{n+1}+3} < 1\end{aligned}$$

$$|x| < 1 \Rightarrow -1 < x < 1$$

When  $x = -1 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}+3}$  which is a divergent series

$x = 1 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}+3}$  the series converges conditionally

- a) The radius is 1; the series converges for  $-1 < x \leq 1$ .
- b) The series convergence absolutely for  $-1 < x < 1$ .
- c) The series convergence conditionally for  $x = 1$

### Exercise

(a) Find the series' radius and interval of convergence. For what values of  $x$  does the series converge (b) absolutely, (c) conditionally?

$$\sum_{n=1}^{\infty} \sqrt[n]{n} (2x+5)^n$$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{n+1 \sqrt[n+1]{n+1} (2x+5)^{n+1}}{\sqrt[n]{n} (2x+5)^n} \right| \\ &= |2x+5| \lim_{n \rightarrow \infty} \frac{n+1 \sqrt[n+1]{n+1}}{\sqrt[n]{n}} < 1 \\ &= |2x+5| \frac{\lim_{m \rightarrow \infty} \sqrt[m]{m}}{\lim_{n \rightarrow \infty} \sqrt[n]{n}} < 1 \\ &= |2x+5| < 1\end{aligned}$$

$$|2x+5| < 1$$

$$-1 < 2x+5 < 1$$

$$-3 < x < -2$$

When  $x = -3 \Rightarrow \sum_{n=0}^{\infty} (-1)^n \sqrt[n]{n}$  which is a divergent series

$x = -0 \Rightarrow \sum_{n=0}^{\infty} \sqrt[n]{n}$  which is a divergent series

- a) The radius is 1; the series converges for  $-3 < x < -2$ .
- b) The series convergence absolutely for  $-3 < x < -2$ .
- c) There are no values for which the series convergence conditionally

### Exercise

- (a) Find the series' radius and interval of convergence. For what values of  $x$  does the series converge
- (b) absolutely,
- (c) conditionally?

$$\sum_{n=1}^{\infty} (2 + (-1)^n) \cdot (x+1)^{n-1}$$

### Solution

$$\sum_{n=1}^{\infty} (2 + (-1)^n) \cdot (x+1)^{n-1} = \sum_{n=1}^{\infty} 2(x+1)^{n-1} + \sum_{n=1}^{\infty} (-1)^n (x+1)^{n-1}$$

For the series  $\sum_{n=1}^{\infty} 2(x+1)^{n-1}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2(x+1)^n}{2(x+1)^{n-1}} \right|$$

$$= |x+1| < 1$$

$$-1 < x+1 < 1$$

$$-2 < x < 0$$

For the series  $\sum_{n=1}^{\infty} (-1)^n (x+1)^{n-1}$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x+1)^n}{(x+1)^{n-1}} \right|$$

$$= |x+1| < 1$$

$$-1 < x+1 < 1$$

$$-2 < x < 0$$

When  $x = -2 \Rightarrow \sum_{n=0}^{\infty} (2 + (-1)^n) \cdot (-1)^{n-1}$  which is a divergent series

$x = 0 \Rightarrow \sum_{n=0}^{\infty} (2 + (-1)^n)$  which is a divergent series

- a) The radius is  $\frac{1}{2}$ ; the series converges for  $-2 < x < 0$ .
- b) The series convergence absolutely for  $-2 < x < 0$ .
- c) There are no values for which the series convergence conditionally

### Exercise

Find the radius of convergence of the power series  $\sum_{n=0}^{\infty} n! x^n$

#### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} (n+1) \\ &= \infty \end{aligned}$$

$$\rightarrow R = \frac{1}{\infty} = 0$$

By the Ratio Test, the series diverges for  $|x| > 0$  and converges only at its center, 0.

Therefore; the radius of convergence is  $R = 0$ .

### Exercise

Find the radius of convergence of the power series  $\sum_{n=0}^{\infty} 3(x-2)^n$

#### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{3(x-2)^{n+1}}{3(x-2)^n} \right| \\ &= |x-2| \end{aligned}$$

By the Ratio Test, the series converges for  $|x-2| < 1$  and diverges for  $|x-2| > 1$ .

Therefore; the radius of convergence is  $R = 1$ .

### Exercise

Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| \\ &= x^2 \lim_{n \rightarrow \infty} \frac{1}{(2n+2)(2n+3)} \\ &= 0 \end{aligned}$$

$$\rightarrow R = \frac{1}{0} = \infty$$

By the Ratio Test, the series converges for all  $x$ . Therefore; the radius of convergence is  $R = \infty$ .

### Exercise

Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \frac{n+1}{n+2} \\ &= |x| \end{aligned}$$

$$\rightarrow R = 1$$

By the Ratio Test, the series converges for  $|x| < 1$  and diverges for  $|x| > 1$ .

Therefore; the radius of convergence is  $R = 1$ .

### Exercise

Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} (3x)^n$$

### Solution

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(3x)^{n+1}}{(3x)^n} \right|$$



$$\underline{= 3|x|}$$

$$\rightarrow 3|x| < 1 \Rightarrow R = \frac{1}{3}$$

By the Ratio Test, the series converges for  $|x| < \frac{1}{3}$  and diverges for  $|x| > \frac{1}{3}$ .

Therefore; the radius of convergence is  $R = \frac{1}{3}$ .

### Exercise

Find the radius of convergence of the power series  $\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(4x)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(4x)^n} \right| \\ &= |4x| \lim_{n \rightarrow \infty} \left| \frac{n^2}{(n+1)^2} \right| \\ &\underline{= 4|x|} \end{aligned}$$

$$\rightarrow 4|x| < 1 \Rightarrow R = \frac{1}{4}$$

By the Ratio Test, the series converges for  $|x| < \frac{1}{4}$  and diverges for  $|x| > \frac{1}{4}$ .

Therefore; the radius of convergence is  $R = \frac{1}{4}$ .

### Exercise

Find the radius of convergence of the power series  $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{5^n}$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{5^{n+1}} \cdot \frac{5^n}{x^n} \right| \\ &\underline{= \frac{|x|}{5}} \end{aligned}$$

$$\rightarrow \frac{|x|}{5} < 1 \Rightarrow R = 5$$

By the Ratio Test, the series converges for  $|x| < 5$  and diverges for  $|x| > 5$ .

Therefore; the radius of convergence is  $R = 5$ .

### Exercise

Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| \\ &= x^2 \lim_{n \rightarrow \infty} \frac{1}{(2n+1)(2n+1)} \\ &= 0\end{aligned}$$

$$R = \frac{1}{0} = \infty$$

By the Ratio Test, the series converges for all  $x$ . Therefore; the radius of convergence is  $R = \infty$ .

### Exercise

Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(2n)! x^{2n}}{n!}$$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(2n+2)! x^{2n+2}}{(n+1)!} \cdot \frac{n!}{(2n)! x^{2n}} \right| \\ &= x^2 \lim_{n \rightarrow \infty} \left( \frac{(2n+1)(2n+2)}{n+1} \right) \\ &= \infty\end{aligned}$$

$$\rightarrow R = \frac{1}{\infty} = 0$$

By the Ratio Test, the series diverges for  $|x| > 0$  and converges only at its center, 0.

Therefore; the radius of convergence is  $R = 0$ .

### Exercise

Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right) \\ &= \underline{|x|}\end{aligned}$$

$$\rightarrow R = 1$$

So, by the Ratio Test, the radius of convergence is  $R = 1$ .

The series centered at 0, it converges in the interval  $(-1, 1)$

$$\text{When } x = 1 \quad \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots \quad \text{diverges}$$

$$\text{When } x = -1 \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \dots \quad \text{converges}$$

Therefore; the interval of convergence  $[-1, 1)$

### Exercise

Find the interval of convergence of the power series  $\sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^n}{2^n}$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x+1)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(x+1)^n} \right| \\ &= \underline{\frac{1}{2} |x+1|}\end{aligned}$$

$$|x+1| < 2 \rightarrow R = 2$$

$$\begin{cases} x+1 = -2 & x = -3 \\ x+1 = 2 & x = 1 \end{cases}$$

So, by the Ratio Test, the radius of convergence is  $R = 2$ .

The series centered at  $-1$ , it converges in the interval  $(-3, 1)$

$$\text{When } x = -3 \quad \sum_{n=0}^{\infty} \frac{(-1)^n (-2)^n}{2^n} = \sum_{n=0}^{\infty} 1 \quad \text{diverges}$$

$$\text{When } x = 1 \quad \sum_{n=0}^{\infty} \frac{(-1)^n (2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n \quad \text{diverges}$$

Therefore; the interval of convergence  $(-3, 1)$

### Exercise

Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \left| \frac{n^2}{(n+1)^2} \right| \\ &= \underline{|x|}\end{aligned}$$

$$\rightarrow R = 1$$

So, by the Ratio Test, the radius of convergence is  $R = 1$ .

The series centered at 0, it converges in the interval  $(-1, 1)$

$$\text{When } x = -1 \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \dots \quad \text{converges by alternating series}$$

$$u_{n+1} = \frac{1}{(n+1)^2} < \frac{1}{n^2} = u_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$$

$$\text{When } x = 1 \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \quad \text{converges by } p\text{-series}$$

Therefore; the interval of convergence  $[-1, 1]$

### Exercise

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{4^{n+1}} \cdot \frac{4^n}{x^n} \right| \\ &= \underline{\frac{|x|}{4}}\end{aligned}$$

$$\rightarrow R = 4$$

So, by the Ratio Test, the radius of convergence is  $R = 4$ .

The series centered at 0, it converges in the interval  $(-4, 4)$

When  $x = -4$   $\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + \dots$  *diverges by alternating series*

When  $x = 4$   $\sum_{n=1}^{\infty} 1$  *diverges*

Therefore; the interval of convergence  $(-4, 4)$

### Exercise

Find the interval of convergence of the power series  $\sum_{n=0}^{\infty} (2x)^n$

### Solution

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{(2x)^n} \right|$$

$$= 2|x|$$

$$2|x| = 1 \rightarrow R = \frac{1}{2}$$

So, by the Ratio Test, the radius of convergence is  $R = \frac{1}{2}$ .

The series converges in the interval  $(-\frac{1}{2}, \frac{1}{2})$

When  $x = -\frac{1}{2}$   $\sum_{n=0}^{\infty} (-1)^n = -1 + 1 - 1 + \dots$  *diverges by alternating series*

When  $x = \frac{1}{2}$   $\sum_{n=0}^{\infty} 1$  *diverges*

Therefore; the interval of convergence  $(-\frac{1}{2}, \frac{1}{2})$

### Exercise

Find the interval of convergence of the power series  $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| \\ &= \underline{|x|}\end{aligned}$$

$$|x| = 1 \rightarrow R = 1$$

So, by the Ratio Test, the radius of convergence is  $R = 1$ .

The series converges in the interval  $(-1, 1)$

When  $x = -1$   $\sum_{n=1}^{\infty} \frac{1}{n}$  *diverges* by *p-series*

When  $x = 1$   $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  *converges* by *Alternating Series*

Therefore; the interval of convergence  $(-1, 1]$

### Exercise

Find the interval of convergence of the power series  $\sum_{n=0}^{\infty} (-1)^n (n+1) x^n$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+2)x^{n+1}}{(n+1)x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \left| \frac{n+2}{n+1} \right| \\ &= \underline{|x|}\end{aligned}$$

$$|x| = 1 \rightarrow R = 1$$

The series converges in the interval  $(-1, 1)$

When  $x = -1$   $\sum_{n=0}^{\infty} (n+1)$  *diverges*

When  $x = 1$   $\sum_{n=0}^{\infty} (-1)^n (n+1)$  *diverges*

Therefore; the interval of convergence  $(-1, 1)$

### Exercise

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{5n+5}}{(n+1)!} \cdot \frac{n!}{x^{5n}} \right| \\ &= |x^5| \lim_{n \rightarrow \infty} \left| \frac{1}{n+1} \right| \\ &= \underline{0} \quad \rightarrow \quad R = \infty\end{aligned}$$

The series converges for all  $x$ . Therefore; the interval of convergence  $(-\infty, \infty)$

### Exercise

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(3x)^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{(3x)^n} \right| \\ &= |3x| \lim_{n \rightarrow \infty} \left| \frac{1}{(2n+1)(2n+2)} \right| \\ &= \underline{0}\end{aligned}$$

$$\rightarrow R = \infty$$

The series converges for all  $x$ . Therefore; the interval of convergence  $(-\infty, \infty)$

### Exercise

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} (2n)! \left(\frac{x}{3}\right)^n$$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| (2n+2)! \left(\frac{x}{3}\right)^{n+1} \cdot (2n)! \left(\frac{x}{3}\right)^{-n} \right| \\ &= \left| \frac{x}{3} \right| \lim_{n \rightarrow \infty} |(2n+1)(2n+2)| \\ &= \underline{\infty}\end{aligned}$$

The series converges only for  $x = 0$

### Exercise

Find the interval of convergence of the power series  $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{(n+1)(n+2)}$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+2)(n+3)} \cdot \frac{(n+1)(n+2)}{x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \left| \frac{n+1}{n+3} \right| \\ &= |x|\end{aligned}$$

$$|x| = 1 \rightarrow R = 1$$

The series converges in the interval  $(-1, 1)$

When  $x = -1$   $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$  *converges* by Alternating Series

$$u_{n+1} = \frac{1}{(n+3)(n+2)} < \frac{1}{(n+1)(n+2)} = u_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{(n+1)(n+2)} = 0$$

When  $x = 1$   $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)(n+2)}$  *converges* by Limit Comparison Test to  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Therefore; the interval of convergence  $[-1, 1]$

### Exercise

Find the interval of convergence of the power series  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{6^n}$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{6^{n+1}} \cdot \frac{6^n}{x^n} \right| \\ &= \frac{1}{6} |x|\end{aligned}$$

$$\frac{1}{6} |x| = 1 \rightarrow R = 6$$

The series converges in the interval  $(-6, 6)$



When  $x = -6$   $\sum_{n=1}^{\infty} (-1)^n$  *diverges*

When  $x = 6$   $\sum_{n=1}^{\infty} (-1)^{n+1}$  *diverges*

Therefore; the interval of convergence  $(-6, 6)$

### Exercise

Find the interval of convergence of the power series  $\sum_{n=0}^{\infty} (-1)^n \frac{n!(x-5)^n}{3^n}$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!(x-5)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n!(x-5)^n} \right| \\ &= \left| \frac{x-5}{3} \right| \lim_{n \rightarrow \infty} (n+1) \\ &= \infty \end{aligned}$$

The series converges only for  $x = 5$

### Exercise

Find the interval of convergence of the power series  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-4)^n}{n 9^n}$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-4)^{n+1}}{(n+1)9^{n+1}} \cdot \frac{n9^n}{(x-4)^n} \right| \\ &= \frac{1}{9} |x-4| \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| \\ &= \frac{1}{9} |x-4| \end{aligned}$$

$$\frac{1}{9} |x-4| = 1 \rightarrow R = 9$$

$$|x-4| = 9 \Rightarrow \begin{cases} x-4 = -9 & x = -5 \\ x-4 = 9 & x = 13 \end{cases}$$

The series converges in the interval  $(-5, 13)$  and center  $x = 4$

When  $x = -5$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-9)^n}{n 9^n} = \sum_{n=1}^{\infty} \frac{-1}{n} \text{ diverges}$$

When  $x = 13$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(9)^n}{n 9^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges by Alternating Series}$$

$$u_{n+1} = \frac{1}{n+1} < \frac{1}{n} = u_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Therefore; the interval of convergence  $(-5, 13]$

### Exercise

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+2}}{(n+2)4^{n+2}} \cdot \frac{(n+1)4^{n+1}}{(x-3)^{n+1}} \right| \\ &= \frac{1}{4} |x-3| \lim_{n \rightarrow \infty} \left| \frac{n+1}{n+2} \right| \\ &= \frac{1}{4} |x-3| \end{aligned}$$

$$\frac{1}{4} |x-3| = 1 \rightarrow R = 4$$

$$|x-3| = 4 \Rightarrow \begin{cases} x-3 = -4 & x = -1 \\ x-3 = 4 & x = 7 \end{cases}$$

The series converges in the interval  $(-1, 7)$  and center  $x = 3$

When  $x = -1$

$$\sum_{n=0}^{\infty} \frac{(-4)^{n+1}}{(n+1)4^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} \text{ converges by Alternating Series}$$

$$u_{n+1} = \frac{1}{n+2} < \frac{1}{n+1} = u_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$$

When  $x = 4$

$$\sum_{n=0}^{\infty} \frac{4^{n+1}}{(n+1)4^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{n+1} \quad \text{diverges by Integral Test}$$

$$\int_0^{\infty} \frac{dx}{x+1} = \ln(x+1) \Big|_0^{\infty}$$

$$= \infty$$

Therefore; the interval of convergence  $[-1, 7)$

### Exercise

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-1)^{n+1}}{n+1}$$

### Solution

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-1)^{n+2}}{n+2} \cdot \frac{n+1}{(x-1)^{n+1}} \right|$$

$$= |x-1| \lim_{n \rightarrow \infty} \left| \frac{n+1}{n+2} \right|$$

$$= |x-1|$$

$$|x-1| = 1 \rightarrow R = 1$$

$$|x-1| = 1$$

$$\Rightarrow \begin{cases} x-1 = -1 & x = 0 \\ x-1 = 1 & x = 2 \end{cases}$$

The series converges in the interval  $(0, 2)$  and center  $x = 1$

When  $x = 0$

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} \quad \text{diverges by Integral Test}$$

$$\int_0^{\infty} \frac{dx}{x+1} = \ln(x+1) \Big|_0^{\infty}$$

$$= \infty$$

When  $x = 1$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} \quad \text{converges by Alternative Test}$$

$$u_{n+1} = \frac{1}{n+2} < \frac{1}{n+1} = u_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$$

Therefore; the interval of convergence  $(0, 2]$

### Exercise

Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-2)^n}{n2^n}$$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{(x-2)^n} \right| \\ &= \frac{1}{2} |x-2| \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| \\ &= \frac{1}{2} |x-2| \end{aligned}$$

$$|x-2| = 2 \rightarrow R = 2$$

$$|x-2| = 2 \Rightarrow \begin{cases} x-2 = -2 & x = 0 \\ x-2 = 2 & x = 4 \end{cases}$$

The series converges in the interval  $(0, 4)$  and center  $x = 2$

When  $x = 0$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-2)^n}{n2^n} = \sum_{n=0}^{\infty} \frac{-1}{n} \quad \text{diverges by Integral Test}$$

$$\int_0^{\infty} \frac{-dx}{x} = -\ln x \Big|_0^{\infty}$$

$$\underline{= -\infty}$$

When  $x = 4$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \quad \text{converges by Alternative Test}$$

$$u_{n+1} = \frac{1}{n+1} < \frac{1}{n} = u_n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Therefore; the interval of convergence  $(0, 4]$

### Exercise

Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{3^{n-1}}$$

### Solution

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-3)^n}{3^n} \cdot \frac{3^{n-1}}{(x-3)^{n-1}} \right|$$
$$= \frac{1}{3} |x-3|$$

$$\frac{1}{3} |x-3| = 1 \rightarrow R = 3$$

$$|x-3| = 3$$

$$\Rightarrow \begin{cases} x-3 = -3 & x = 0 \\ x-3 = 3 & x = 6 \end{cases}$$

The series converges in the interval  $(0, 6)$

When  $x = 0$

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} (-1) \text{ diverges}$$

When  $x = 6$

$$\sum_{n=1}^{\infty} \frac{3^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} 1 \text{ diverges}$$

Therefore; the interval of convergence  $(0, 6)$

### Exercise

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

### Solution

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{2n+3} \cdot \frac{2n+1}{x^{2n+1}} \right|$$
$$= x^2 \lim_{n \rightarrow \infty} \left| \frac{2n+1}{2n+3} \right|$$
$$= x^2$$

$$\rightarrow R = 1$$

The series converges in the interval  $(-1, 1)$

When  $x = -1$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} \text{ converges by Alternating Series}$$

When  $x = 1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \text{ converges by Alternating Series}$$

Therefore; the interval of convergence  $[-1, 1]$

### Exercise

Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n}{n+1} (-2x)^{n-1}$$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{n+1}{n+2} (-2x)^n \cdot \frac{n+1}{n(-2x)^{n-1}} \right| \\ &= |-2x| \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{n(n+2)} \right| \\ &= \underline{2|x|} \end{aligned}$$

$$2|x| = 1 \rightarrow R = \frac{1}{2}$$

The series converges in the interval  $\left(-\frac{1}{2}, \frac{1}{2}\right)$

When  $x = -\frac{1}{2}$

$$\sum_{n=1}^{\infty} \frac{n}{n+1} \text{ diverges by } nth \text{ Term Test}$$

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$$

When  $x = \frac{1}{2}$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1} \text{ diverges by Alternating Series}$$

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0$$

Therefore; the interval of convergence  $\left(-\frac{1}{2}, \frac{1}{2}\right)$

### Exercise

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{(n+1)!} \cdot \frac{n!}{x^{2n}} \right| \\ &= x^2 \lim_{n \rightarrow \infty} \left| \frac{1}{n+1} \right| \\ &= 0\end{aligned}$$

$$\rightarrow R = \infty$$

Therefore; the interval of convergence  $(-\infty, \infty)$

### Exercise

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!}$$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{3n+4}}{(3n+4)!} \cdot \frac{(3n+1)!}{x^{3n+1}} \right| \\ &= |x^3| \lim_{n \rightarrow \infty} \left| \frac{1}{(3n+2)(3n+3)(3n+4)} \right| \\ &= 0\end{aligned}$$

$$\rightarrow R = \infty$$

Therefore; the interval of convergence  $(-\infty, \infty)$

### Exercise

Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n! x^n}{(2n)!}$$

### Solution

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)! x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n! x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \left| \frac{n+1}{(2n+1)(2n+2)} \right|\end{aligned}$$

$$= 0$$

$$\rightarrow R = \infty$$

Therefore; the interval of convergence  $(-\infty, \infty)$

### Exercise

Find the interval of convergence of the power series  $\sum_{n=1}^{\infty} \frac{2 \cdot 3 \cdot 4 \cdots (n+1)x^n}{n!}$

### Solution

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{2 \cdot 3 \cdot 4 \cdots (n+1)(n+2)x^{n+1}}{(n+1)!} \cdot \frac{n!}{2 \cdot 3 \cdot 4 \cdots (n+1)x^n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \left| \frac{n+2}{n+1} \right| \\ &= |x| \end{aligned}$$

$$|x| = 1 \rightarrow R = 1$$

The series converges in the interval  $(-1, 1)$

When  $x = -1$

$$\sum_{n=1}^{\infty} (-1)^n \frac{2 \cdot 3 \cdot 4 \cdots (n+1)}{n!} = \sum_{n=1}^{\infty} (-1)^n (n+1) \text{ diverges}$$

When  $x = 1$

$$\sum_{n=1}^{\infty} \frac{2 \cdot 3 \cdot 4 \cdots (n+1)}{n!} = \sum_{n=1}^{\infty} (n+1) \text{ diverges}$$

Therefore; the interval of convergence  $(-1, 1)$

### Exercise

Determine the centre, radius, and interval of convergence of the power series  $\sum_{n=0}^{\infty} \frac{x^{2n}}{\sqrt{n+1}}$

### Solution

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \left| \frac{1}{\sqrt{n+2}} \cdot \sqrt{n+1} \right| \\ &= \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n}} \\ &= 1 \end{aligned}$$



The *radius* of convergence is 1.

The *centre* of convergence is 0.

The *interval* of convergence is  $(-1, 1)$ .

The series *does not converge* at  $x = -1$  or  $x = 1$

### Exercise

Determine the centre, radius, and interval of convergence of the power series  $\sum_{n=0}^{\infty} 3n(x+1)^n$

### Solution

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \left| \frac{3n}{3(n+1)} \right| & R &= \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \frac{3n}{3n} \\ &= 1 \end{aligned}$$

The radius of convergence is 1, and the centre of convergence is  $-1$ . ( $x+1=0$ )

$$a - R < x < a + R \Rightarrow -1 - 1 < x < -1 + 1$$

Therefore; the given series converges absolutely on  $(-2, 0)$

At  $x = -2$

The series is  $\sum_{n=0}^{\infty} 3n(-1)^n$  which diverges.

At  $x = 0$

The series is  $\sum_{n=0}^{\infty} 3n(1)^n = \sum_{n=0}^{\infty} 3n$  which diverges.

Hence, the interval of convergence is  $(-2, 0)$ .

### Exercise

Determine the centre, radius, and interval of convergence of the power series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} x^n$

### Solution

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^4 2^{2n+2}}{n^4 2^{2n}} \right| & R &= \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \end{aligned}$$

$$= 4 \lim_{n \rightarrow \infty} \left| \left( \frac{n+1}{n} \right)^4 \right|$$

$$= 4$$

The radius of convergence is 4, and the centre of convergence is 0.

$a - R < x < a + R \Rightarrow -4 < x < 4$ , the given series converges absolutely on  $(-4, 4)$

At  $x = -4$ ,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (-1)^n 2^{2n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^4} \text{ which converges (p-series).}$$

At  $x = 4$ ,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} (4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} 2^{2n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \text{ which also converges.}$$

Hence, the interval of convergence is  $[-4, 4]$ .

### Exercise

Determine the centre, radius, and interval of convergence of the power series  $\sum_{n=1}^{\infty} \frac{e^n}{n^3} (4-x)^n$

### Solution

$$R = \lim_{n \rightarrow \infty} \left| \frac{e^n}{n^3} \cdot \frac{(n+1)^3}{e^{n+1}} \right|$$

$$= \frac{1}{e} \lim_{n \rightarrow \infty} \left| \left( \frac{n+1}{n} \right)^3 \right|$$

$$= \frac{1}{e}$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

The radius of convergence is  $\frac{1}{e}$ .

The centre of convergence is 4.  $(4-x=0 \Rightarrow x=4)$

$a - R < x < a + R \Rightarrow 4 - \frac{1}{e} < x < 4 + \frac{1}{e}$ , which the given series converges absolutely

At  $x = 4 - \frac{1}{e}$ ,

the series is  $\sum_{n=1}^{\infty} \frac{e^n}{n^3} \left(\frac{1}{e}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n^3}$  which converges ( $p$ -series).

At  $x = 4 + \frac{1}{e}$ ,

the series is  $\sum_{n=1}^{\infty} \frac{e^n}{n^3} \left(-\frac{1}{e}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$  which also converges ( $p$ -series).

Hence, the interval of convergence is  $\left[4 - \frac{1}{e}, 4 + \frac{1}{e}\right]$ .

### Exercise

Determine the centre, radius, and interval of convergence of the power series  $\sum_{n=0}^{\infty} \frac{1+5^n}{n!} x^n$

#### Solution

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \left| \frac{1+5^n}{n!} \cdot \frac{(n+1)!}{1+5^{n+1}} \right| & R &= \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{1+5^n}{1+5^{n+1}} (n+1) \right| \\ &= \infty \end{aligned}$$

The *radius* of convergence is  $\infty$ .

The *centre* of convergence is 0.

The *interval of convergence* is the real line  $(-\infty, \infty)$

### Exercise

Determine the centre, radius, and interval of convergence of the power series  $\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n}$

#### Solution

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n} &= \sum_{n=1}^{\infty} \frac{4^n \left(x - \frac{1}{4}\right)^n}{n^n} \\ R &= \lim_{n \rightarrow \infty} \left| \frac{4^n}{n^n} \cdot \frac{(n+1)^{n+1}}{4^{n+1}} \right| & R &= \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \end{aligned}$$

$$= \frac{1}{4} \lim_{n \rightarrow \infty} \left| \left( \frac{n+1}{n} \right)^n (n+1) \right|$$

$$= \infty$$

The *radius* of convergence is  $\infty$  .

$$4x - 1 = 0 \Rightarrow x = \frac{1}{4}$$

The *centre* of convergence is  $x = \frac{1}{4}$  |

The *interval of convergence* is the real line  $(-\infty, \infty)$

### Exercise

Determine the centre, radius, and interval of convergence of the power series  $\sum_{n=0}^{\infty} \frac{1+5^n}{n!} x^n$

### Solution

$$a_n = \frac{1+5^n}{n!}$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{(1+5^n)}{n!} \cdot \frac{(n+1)!}{(1+5^{n+1})} \right|$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n \rightarrow \infty} \left| (n+1) \frac{1+5^n}{1+5^{n+1}} \right|$$

$$= \lim_{n \rightarrow \infty} \left| (n+1) \frac{1}{5} \right|$$

$$= \infty$$

The *radius* of convergence is  $\infty$  .

The *centre* of convergence is  $x = 0$  | .

The *interval of convergence* is the real line  $(-\infty, \infty)$

### Exercise

Determine the centre, radius, and interval of convergence of the power series  $\sum \frac{n^2 x^n}{n!}$

### Solution

$$a_n = \frac{n^2}{n!}$$

$$\begin{aligned}
 R &= \lim_{n \rightarrow \infty} \left| \frac{n^2 x^n}{n!} \cdot \frac{(n+1)!}{(n+1)^2 x^{n+1}} \right| \\
 &= \lim_{n \rightarrow \infty} \left| (n+1) \left( \frac{n}{n+1} \right)^2 \right| \\
 &= \infty
 \end{aligned}$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

The *radius* of convergence is  $\infty$  .

The *centre* of convergence is  $x = 0$  .

The *interval of convergence* is the real line  $(-\infty, \infty)$

### Exercise

Determine the centre, radius, and interval of convergence of the power series  $\sum \frac{x^{4n}}{n^2}$

### Solution

$$a_n = \frac{1}{n^2} x^{4n}$$

$$\begin{aligned}
 R &= \lim_{n \rightarrow \infty} \left( \frac{1}{n^2} \cdot \frac{(n+1)^2}{1} \right) \left| \frac{x^{4n}}{x^{4n+4}} \right| \\
 &= \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right)^2 \left| \frac{1}{x^4} \right| \\
 &= 1
 \end{aligned}$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

The *radius* of convergence is 1

The *centre* of convergence is  $x = 0$

$$-1 < x < 1$$

$$a - R < x < a + R$$

which the given series converges absolutely

At  $x = -1$ ,

$$\text{the series is } \sum \frac{(-1)^{4n}}{n^2} = \sum \frac{1}{n^2} \text{ which converges (p-series).}$$

At  $x = 1$ ,

$$\text{the series is } \sum \frac{(1)^{4n}}{n^2} = \sum \frac{1}{n^2} \text{ which also converges (p-series).}$$

The interval of convergence is the real line  $[-1, 1]$

### Exercise

Determine the centre, radius, and interval of convergence of the power series  $\sum (-1)^n \frac{(x+1)^{2n}}{n!}$

### Solution

$$a_n = \frac{1}{n!} (x+1)^{2n}$$

$$R = \lim_{n \rightarrow \infty} \left( \frac{1}{n!} \cdot \frac{(n+1)!}{1} \right) \left| \frac{(x+1)^{2n}}{(x+1)^{2n+2}} \right| \qquad R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n \rightarrow \infty} (n+1) \left| \frac{1}{(x+1)^2} \right|$$

$$= \infty$$

The *radius* of convergence is  $\infty$

$$x+1=0 \rightarrow x=-1$$

The *centre* of convergence is  $x=-1$

The *interval* of convergence is the real line  $(-\infty, \infty)$

### Exercise

Determine the centre, radius, and interval of convergence of the power series  $\sum \frac{(x-1)^n}{n \cdot 5^n}$

### Solution

$$a_n = \frac{1}{n \cdot 5^n} (x-1)^n$$

By *Ratio Test*:

$$R = \lim_{n \rightarrow \infty} \left( \frac{1}{n \cdot 5^n} \cdot \frac{(n+1) \cdot 5^{n+1}}{1} \right) \left| \frac{(x-1)^n}{(x-1)^{n+1}} \right| \qquad R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= 5 \lim_{n \rightarrow \infty} \frac{n+1}{n} \left| \frac{1}{x-1} \right|$$

$$= 5$$

The *radius* of convergence is  $5$

$$x-1=0 \rightarrow x=1$$

The *centre* of convergence is  $x=1$

$$-5+1 < x < 5+1$$

$$a-R < x < a+R$$

$$-4 < x < 6$$

which the given series converges absolutely

At  $x = -4$ ,

$$\text{the series is } \sum \frac{(-5)^n}{n \cdot 5^n} = \sum \frac{(-1)^n}{n}$$

$$\frac{1}{n} > \frac{1}{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

which converges *Alternating Harmonic Series*.

At  $x = 6$ ,

$$\text{the series is } \sum \frac{(5)^n}{n \cdot 5^n} = \sum \frac{1}{n} \text{ which diverges (p-series } p = 1 \leq 1)$$

The interval of convergence is the real line  $[-4, 6)$

### Exercise

Determine the centre, radius, and interval of convergence of the power series  $\sum \left(\frac{x}{9}\right)^{3n}$

### Solution

$$a_n = \left(\frac{x}{9}\right)^{3n}$$

By Root Test:

$$R = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{x}{9}\right)^{3n}\right|}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{|x|}{9}\right)^3$$

$$= \frac{1}{729} |x^3| < 1$$

$$R = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

$$\left|\frac{x}{9}\right|^3 < 1$$

$$\left|\frac{x}{9}\right| < 1$$

$$-9 < x < 9$$

The *radius* of convergence is  $9$

The *centre* of convergence is  $x = 0$

At  $x = -9$ ,

the series is  $\sum \left(\frac{-9}{9}\right)^{3n} = \sum (-1)$  which diverges by the *divergence Test*.

At  $x = 9$ ,

the series is  $\sum \left(\frac{9}{9}\right)^{3n} = \sum (1)$  which diverges by the *divergence Test*.

The interval of convergence is the real line  $(-9, 9)$

### Exercise

Determine the centre, radius, and interval of convergence of the power series  $\sum \frac{(x+2)^n}{\sqrt{n}}$

### Solution

$$a_n = \frac{(x+2)^n}{\sqrt{n}}$$

By *Ratio Test*:

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \left( \frac{1}{\sqrt{n}} \cdot \frac{\sqrt{n+1}}{1} \right) \left| \frac{(x+2)^n}{(x+2)^{n+1}} \right| & R &= \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \sqrt{\frac{n+1}{n}} \left| \frac{1}{x+2} \right| \\ &= 1 \end{aligned}$$

The *radius* of convergence is  $1$

$$x+2=0 \rightarrow x=-2$$

The *centre* of convergence is  $x = -2$

$$-2-1 < x < -2+1 \quad a-R < x < a+R$$

$$-3 < x < -1$$

which the given series convergences absolutely

At  $x = -3$ ,

the series is  $\sum \frac{(-1)^n}{\sqrt{n}}$

$$\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

which converges *Alternating Series*.

At  $x = -1$ ,



the series is  $\sum \frac{(1)^n}{\sqrt{n}} = \sum \frac{1}{\sqrt{n}}$  which *diverges* ( $p$ -series  $p = \frac{1}{2} \leq 1$ )

The interval of convergence is the real line  $[-3, -1)$

### Exercise

Determine the centre, radius, and interval of convergence of the power series  $\sum \frac{(x+2)^k}{2^k \ln k}$

### Solution

$$a_k = \frac{(x+2)^k}{2^k \ln k}$$

By *Ratio Test*:

$$\begin{aligned} R &= \lim_{k \rightarrow \infty} \left( \frac{1}{2^k \ln k} \cdot \frac{2^{k+1} \ln(k+1)}{1} \right) \left| \frac{(x+2)^k}{(x+2)^{k+1}} \right| & R &= \lim_{k \rightarrow \infty} \left| \frac{a_k}{a_{k+1}} \right| \\ &= 2 \lim_{n \rightarrow \infty} \frac{\ln(k+1)}{\ln k} \left| \frac{1}{x+2} \right| \\ &= 2 \lim_{n \rightarrow \infty} \frac{\frac{k+1}{k}}{\frac{1}{k}} \\ &= 2 \lim_{n \rightarrow \infty} \frac{k}{k+1} \\ &= 2 \end{aligned}$$

The *radius* of convergence is  $2$

$$x+2=0 \rightarrow x=-2$$

The *centre* of convergence is  $x=-2$

$$-2-2 < x < -2+2 \quad a-R < x < a+R$$

$$-4 < x < 0$$

which the given series converges absolutely.

At  $x=-4$ ,

$$\text{the series is } \sum \frac{(-2)^k}{2^k \ln k} = \sum \frac{(-1)^k}{\ln k}$$

$$\frac{1}{\ln k} > \frac{1}{\ln(k+1)}$$

$$\lim_{n \rightarrow \infty} \frac{1}{\ln k} = 0$$

which converges *Alternating Series*.

At  $x = 0$ ,

$$\text{the series is } \sum \frac{(2)^k}{2^k \ln k} = \sum \frac{1}{\ln k}$$

$$\ln k < k$$

$$\frac{1}{\ln k} > \frac{1}{k}$$

$$\frac{1}{k} \text{ diverges (p-series } p = 1 \leq 1)$$

$\therefore$  Which diverges by Comparison Test.

The interval of convergence is the real line  $[-4, 0)$

### Exercise

Determine the centre, radius, and interval of convergence of the power series  $x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots$

### Solution

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$$

$$a_k = \frac{x^{2k+1}}{2k+1}$$

By Ratio Test:

$$R = \lim_{k \rightarrow \infty} \left( \frac{1}{2k+1} \cdot \frac{2k+3}{1} \right) \left| \frac{x^{2k+1}}{x^{2k+3}} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{2k+3}{2k+1} \left( \frac{1}{x^2} \right)$$

$$= 1$$

$$R = \lim_{k \rightarrow \infty} \left| \frac{a_k}{a_{k+1}} \right|$$

The radius of convergence is  $1$

The centre of convergence is  $x = 0$

$$-1 < x < 1$$

which the given series converges absolutely

At  $x = -1$ ,

$$\text{the series is } \sum \frac{(-1)^{2k+1}}{2k+1} = \sum \frac{-1}{2k+1}$$

$$\int_0^{\infty} \frac{-1}{2x+1} dx = -\frac{1}{2} \int_0^{\infty} \frac{1}{2x+1} d(2x+1)$$

$$\begin{aligned}
&= -\frac{1}{2} \ln(2x+1) \Big|_0^{\infty} \\
&= -\frac{1}{2} (\ln \infty - \ln 1) \\
&= \underline{-\infty}
\end{aligned}$$

which diverges *Integral Test*.

At  $x=1$ ,

$$\begin{aligned}
\text{the series is } \sum \frac{(1)^{2k+1}}{2k+1} &= \sum \frac{1}{2k+1} \\
\int_0^{\infty} \frac{1}{2x+1} dx &= \frac{1}{2} \int_0^{\infty} \frac{1}{2x+1} d(2x+1) \\
&= \frac{1}{2} \ln(2x+1) \Big|_0^{\infty} \\
&= \frac{1}{2} (\ln \infty - \ln 1) \\
&= \underline{\infty}
\end{aligned}$$

which diverges *Integral Test*.

The *interval* of convergence is the real line  $(-1, 1)$

### Exercise

For what value of  $x$  does the series  $1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \dots + \left(-\frac{1}{2}\right)^n (x-3)^n + \dots$  converges? What is its sum? What series do you get if you differentiate the given series term by term? For what value of  $x$  does the new series converge? What is its sum?

### Solution

$$1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \dots + \left(-\frac{1}{2}\right)^n (x-3)^n + \dots = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n$$

$$\lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(x-3)^n} \right| = \left| \frac{x-3}{2} \right| < 1$$

$$\Rightarrow -1 < \frac{x-3}{2} < 1$$

$$-2 < x-3 < 2$$

$$\underline{1 < x < 5}$$

When  $x=1$ ,

$$\sum_{n=1}^{\infty} (1)^n \text{ which is a divergent series}$$

When  $x = 5$ ,

$$\sum_{n=1}^{\infty} (-1)^n \text{ the series diverges}$$

The series is a geometric series, the sum is

$$\frac{1}{1 + \frac{x-3}{2}} = \frac{2}{x-1}$$

$$\begin{aligned} \text{If } f(x) &= 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \dots + \left(-\frac{1}{2}\right)^n (x-3)^n + \dots \\ &= \frac{2}{x-1} \end{aligned}$$

$$\text{Then } f'(x) = -\frac{1}{2} + \frac{1}{2}(x-3) + \dots + \left(-\frac{1}{2}\right)^n n(x-3)^{n-1} + \dots$$

$f'(x)$  is convergent when  $1 < x < 5$  and divergent when  $x = 1$  or  $5$

$$\text{The sum for } f'(x) \text{ is } \frac{-2}{(x-1)^2}$$

### Exercise

The series  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots$  converges to  $\sin x$  for all  $x$ .

- Find the first six terms of a series for  $\cos x$ . For what values of  $x$  should the series converge?
- By replacing  $x$  by  $2x$  in the series for  $\sin x$ , find a series that converges to  $\sin 2x$  for all  $x$ .
- Using the result in part (a) and series multiplication, calculate the first six term of a series for  $2 \sin x \cos x$ . Compare your answer with the answer in part (b).

### Solution

$$a) (\sin x)' = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n}}{(2n)!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| = x^2 \lim_{n \rightarrow \infty} \left| \frac{1}{(2n+1)(2n+2)} \right| = 0 < 1 \quad (\forall x)$$

The series converges for all values of  $x$ .

$$b) \quad \sin 2x = 2x - \frac{2^3 x^3}{3!} + \frac{2^5 x^5}{5!} - \frac{2^7 x^7}{7!} + \frac{2^9 x^9}{9!} - \frac{2^{11} x^{11}}{11!} + \dots$$

$$= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \frac{512x^9}{9!} - \frac{2048x^{11}}{11!} + \dots$$

$$\begin{aligned} c) \quad 2 \sin x \cos x &= 2 \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots \right) \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \right) \\ &= 2x \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \right) - 2 \frac{x^3}{3!} \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \right) \\ &\quad + 2 \frac{x^5}{5!} \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \right) - 2 \frac{x^7}{7!} \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \right) \\ &\quad + 2 \frac{x^9}{9!} \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \right) - 2 \frac{x^{11}}{11!} \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \right) \\ &= 2x - \frac{2x^3}{2!} + \frac{2x^5}{4!} - \frac{2x^7}{6!} + \frac{2x^9}{8!} - \frac{2x^{11}}{10!} - \frac{2x^3}{3!} + \frac{2x^5}{2!3!} - \frac{2x^7}{4!3!} + \frac{2x^9}{6!3!} - \frac{2x^{11}}{8!3!} \\ &\quad + \frac{2x^5}{5!} - \frac{2x^7}{5!2!} + \frac{2x^9}{5!4!} - \frac{2x^{11}}{5!6!} - \frac{2x^7}{7!} + \frac{2x^9}{7!2!} - \frac{2x^{11}}{7!4!} + \frac{2x^9}{9!} - \frac{2x^{11}}{9!2!} - \frac{2x^{11}}{11!} + \dots \\ &= 2x - \frac{2^3 x^3}{3!} + \frac{2^5 x^5}{5!} - \frac{2^7 x^7}{7!} + \frac{2^9 x^9}{9!} - \frac{2^{11} x^{11}}{11!} + \dots \end{aligned}$$

### Exercise

Find the sum of the series  $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$  by the first finding the sum of the power series

$$\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \dots$$

### Solution

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots = \frac{1}{1-x}$$

$$\sum_{n=1}^{\infty} nx^n = x + x^2 + 3x^3 + 4x^4 + \dots$$

$$= x(1 + x + 3x^2 + 4x^3 + \dots)$$

$$= x \frac{d}{dx} (1 + x + x^2 + \dots + x^n + \dots)$$

$$= x \left( \frac{1}{1-x} \right)'$$

$$= \frac{x}{(1-x)^2}$$

$$\frac{d}{dx} \sum_{n=1}^{\infty} nx^n = \left( \frac{x}{(1-x)^2} \right)'$$

$$= \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4}$$

$$= \frac{1+x}{(1-x)^3}$$

$$\frac{d}{dx} \sum_{n=1}^{\infty} nx^n = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

$$= \frac{1+x}{(1-x)^3}$$

Multiply by  $x$  both sides

$$\textcolor{red}{x} \sum_{n=1}^{\infty} n^2 x^{n-1} = \textcolor{red}{x} \frac{1+x}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \dots$$

$$= \frac{x(1+x)}{(1-x)^3} \Bigg|$$

Let  $\textcolor{red}{x} = \frac{1}{2}$

$$\sum_{n=1}^{\infty} n^2 \left( \frac{1}{2} \right)^n = \frac{\frac{1}{2} \frac{3}{2}}{\left( \frac{1}{2} \right)^3}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \underline{\underline{\textcolor{blue}{6}}}$$

### Exercise

Find a series representation of  $f(x) = \frac{1}{2+x}$  in powers of  $x-1$ . What is the interval of convergence of this series?

### Solution

Let  $t = x-1 \Rightarrow x = t+1$ , we have

$$\begin{aligned}\frac{1}{2+x} &= \frac{1}{3+t} \\ &= \frac{1}{3} \frac{1}{1+\frac{t}{3}} \\ &= \frac{1}{3} \left( 1 - \frac{t}{3} + \frac{t^2}{3^2} - \frac{t^3}{3^3} + \dots \right) \quad \left( -1 < \frac{t}{3} < 1 \right) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{3^{n+1}} \quad (-3 < t < 3) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{3^{n+1}} \quad (-2 < x < 4)\end{aligned}$$

$$\begin{aligned}R &= \lim_{n \rightarrow \infty} \frac{3^{n+2}}{3^{n+1}} \\ &= 3\end{aligned}$$

The *radius* of convergence of this series is 3.

The distance from the centre of convergence  $x-1=0 \Rightarrow x=1$ , to the point  $-2$  where the denominator is 0.

### Exercise

Determine the Cauchy product of the series  $1+x+x^2+x^3+\dots$  and  $-x+x^2-x^3+\dots$ . On what interval and to what function does the product series converge?

### Solution

$$\begin{aligned}1+x+x^2+x^3+\dots &= \frac{1}{1-x} \\ &= \sum_{n=0}^{\infty} x^n \\ -x+x^2-x^3+\dots &= \frac{1}{1+x}\end{aligned}$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n$$

Let  $a_n = 1$  and  $b_n = (-1)^n$ , then the series holds for  $-1 < x < 1$

We have

$$c_n = \sum_{j=0}^n (-1)^{n-j}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

Then the Cauchy product is

$$1 + x^2 + x^4 + \dots = \sum_{n=0}^{\infty} x^{2n}$$

$$= \frac{1}{1-x} \cdot \frac{1}{1+x}$$

$$= \frac{1}{1-x^2} \quad \text{for } -1 < x < 1$$

### Exercise

Determine the power series expansion of  $\frac{1}{(1-x)^2}$  by formally dividing  $1 - 2x + x^2$  into 1.

Use the power series  $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$   $-1 < x < 1$

### Solution

$$1 - 2x + x^2 \overline{) 1}$$

$$\underline{1 - 2x + x^2}$$

$$2x - 4x^2 + 2x^3$$

$$\underline{3x^2 - 2x^3}$$

$$3x^2 - 6x^3 + 3x^4$$

$$\underline{4x^3 + 3x^4}$$

$$4x^3 - 8x^4 + 4x^5$$

$$\underline{11x^4 - \dots}$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n \quad \text{for } -1 < x < 1$$



### Exercise

Determine the interval of convergence and the sum of the series

$$1 - 4x + 16x^2 - 64x^3 + \dots = \sum_{n=0}^{\infty} (-1)^n (4x)^n$$

### Solution

$$\begin{aligned} 1 - 4x + 16x^2 - 64x^3 + \dots &= 1 + (-4x) + (-4x)^2 + (-4x)^3 + \dots \\ &= \frac{1}{1 - (-4x)} \\ &= \frac{1}{1 + 4x} \end{aligned}$$

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots$$

Therefore; the interval of convergence is  $-\frac{1}{4} < x < \frac{1}{4}$

### Exercise

Determine the interval of convergence and the sum of the series

$$3 + 4x + 5x^2 + 6x^3 + \dots = \sum_{n=0}^{\infty} (n+3)x^n$$

### Solution

$$\begin{aligned} \sum_{n=0}^{\infty} (n+3)x^n &= \sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} 3x^n \\ \sum_{n=0}^{\infty} x^n &= 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \end{aligned}$$

$$\begin{aligned} \left( \sum_{n=0}^{\infty} x^n \right)' &= 1 + 2x + 3x^2 + 4x^3 + \dots \\ &= \frac{1}{(1-x)^2} \end{aligned}$$

*Differentiate*

$$x(1 + 2x + 3x^2 + 4x^3 + \dots) = \frac{x}{(1-x)^2}$$

*Multiply by x*

$$\begin{aligned} \sum_{n=0}^{\infty} nx^n &= x + 2x^2 + 3x^3 + \dots \\ &= \frac{x}{(1-x)^2} \end{aligned}$$

Then,

$$\begin{aligned}
\sum_{n=0}^{\infty} (n+3)x^n &= \sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} 3x^n \\
&= \frac{x}{(1-x)^2} + 3 \frac{1}{1-x} \\
&= \frac{3-2x}{(1-x)^2} \quad (-1 < x < 1)
\end{aligned}$$

### Exercise

Determine the interval of convergence and the sum of the series

$$\frac{1}{3} + \frac{x}{4} + \frac{x^2}{5} + \frac{x^3}{6} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n+3}$$

### Solution

$$\begin{aligned}
\frac{1}{3} + \frac{x}{4} + \frac{x^2}{5} + \frac{x^3}{6} + \dots &= \frac{1}{x^3} \left( \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \dots \right) \\
&= \frac{1}{x^3} \left( x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \dots - x - \frac{x^2}{2} \right) \quad \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \\
&= \frac{1}{x^3} \left( -\ln(1-x) - x - \frac{x^2}{2} \right) \\
&= -\frac{1}{x^3} \ln(1-x) - \frac{1}{x^2} - \frac{1}{2x} \quad (-1 \leq x < 1, x \neq 0)
\end{aligned}$$