Section 2.4 – Cross Product

The *Cross* Product

To find a vector in 3-space that is perpendicular to two vectors; the type of vector multiplication that facilities this construction is the cross product.

Definition

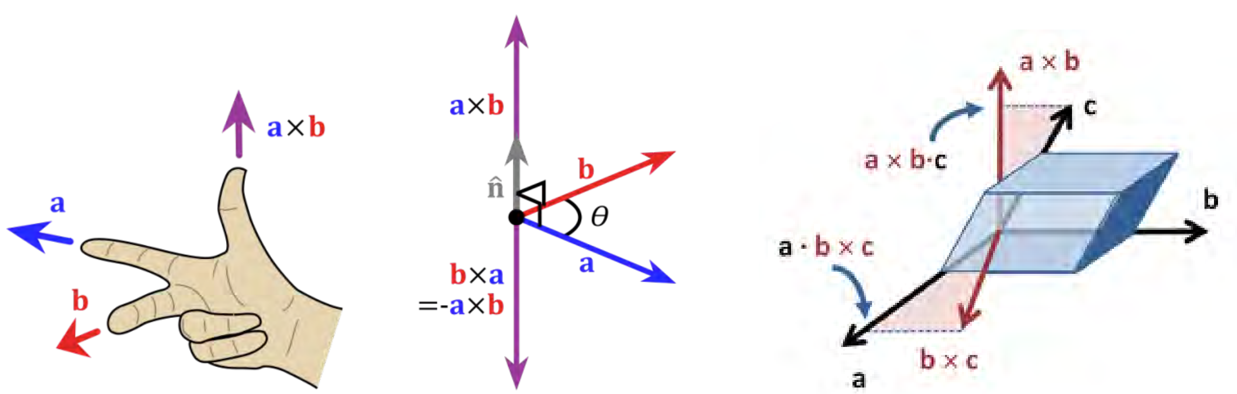
The cross product of $\vec{u} = \left(u_1,\ u_2,\ u_3\right)$ and $\vec{v} = \left(v_1,\ v_2,\ v_3\right)$ is the vector

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \hat{i} - \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} \hat{j} + \begin{vmatrix} u_1 & v_2 \\ u_2 & v_1 \end{vmatrix} \hat{k}$$

$$= \left(u_2 v_3 - u_3 v_2\right)\hat{i} - \left(u_1 v_3 - u_3 v_1\right)\hat{j} + \left(u_1 v_2 - u_2 v_1\right)\hat{k}$$

$$= \left(u_2 v_3 - u_3 v_2,\ \ u_3 v_1 - u_1 v_3,\ \ u_1 v_2 - u_2 v_1\right)$$



- In 1773, ***Joseph Louis Lagrange*** introduced the component form of both the dot and cross products in order to study the tetrahedron in three dimensions. In 1843 the Irish mathematical physicist Sir ***William Rowan Hamilton*** introduced the quaternion product, and with it the terms "*vector*" and "*scalar*". Given two quaternions [0, $\vec{u}$] and [0, $\vec{v}$], where $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^3$, their quaternion product can be summarized as $\left[-\vec{u} \bullet \vec{v},\ \ \vec{u} \times \vec{v}\right]$. ***James Clerk Maxwell*** used Hamilton's quaternion tools to develop his famous ***electromagnetism*** equations, and for this and other reasons quaternions for a time were an essential part of physics education.

Example

Find $\vec{u}\times\vec{v}$, where $\vec{u}=(1,\ 2,\ -2)$ and $\vec{v}=(3,\ 0,\ 1)$

Solution

$$\begin{bmatrix} 1 & 2 & -2 \\ 3 & 0 & 1 \end{bmatrix}$$

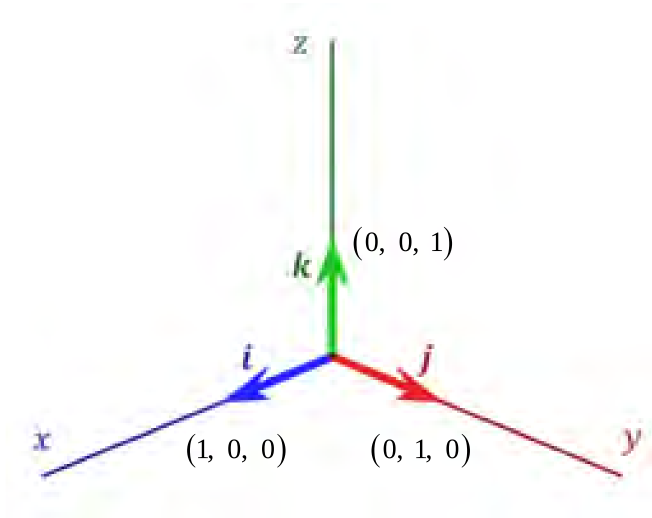
$$\vec{u}\times\vec{v}=\left(\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix},\ -\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix},\ \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}\right)$$

$$=(2,\ -7,\ -6)$$

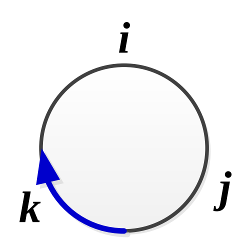
Example

Consider the vectors $\hat{\boldsymbol{i}}=(1,\ 0,\ 0)\quad \hat{\boldsymbol{j}}=(0,\ 1,\ 0)\quad \hat{\boldsymbol{k}}=(0,\ 0,\ 1)$

These vectors each have length of 1 and lie along the coordinate axes. They are called the ***standard unit vectors*** in 3-space.



For example: $(2,\ 3,\ -4)=2\hat{i}+3\hat{j}-4\hat{k}$



Note:

- ✓ $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{\mathbf{0}}$
- ✓ $\hat{j}\times\hat{i}=-\hat{k},\quad \hat{k}\times\hat{j}=-\hat{i},\quad \hat{i}\times\hat{k}=-\hat{j}$
- ✓ $\hat{i}\times\hat{j}=\hat{k},\quad \hat{j}\times\hat{k}=\hat{i},\quad \hat{k}\times\hat{i}=\hat{j}$

Properties

1. $\vec{u}\times\vec{v}$ reverses rows 2 and 3 in the determinant so it is equals $-(\vec{u}\times\vec{v})$

2. The cross product $\vec{u}\times\vec{v}$ is perpendicular to $\vec{u}$, then $\vec{u}\bullet(\vec{u}\times\vec{v})=0$

3. The cross product $\vec{u}\times\vec{v}$ is perpendicular to $\vec{v}$, then $(\vec{u}\times\vec{v})\bullet\vec{v}=0$

4. The cross product of any vector with itself (two equal rows) is $\vec{u}\times\vec{u}=0$.

5. Lagrange's identity: $\|\vec{u}\times\vec{v}\|^2=\|\vec{u}\|^2\ \|\vec{v}\|^2-(\vec{u}\bullet\vec{v})^2$

$$=\|\vec{u}\|\ \|\vec{v}\|\ |\sin\theta|$$

$$|\vec{u}\bullet\vec{v}|=\|\vec{u}\|\ \|\vec{v}\|\ |\cos\theta|$$

Theorem

a) $\vec{u}\times\vec{v}=-(\vec{v}\times\vec{u})$

b) $\vec{u}\times(\vec{v}+\vec{w})=(\vec{u}\times\vec{v})+(\vec{u}\times\vec{w})$

c) $(\vec{u}+\vec{v})\times\vec{w}=(\vec{u}\times\vec{w})+(\vec{v}\times\vec{w})$

d) $k(\vec{u}\times\vec{v})=(k\vec{u})\times\vec{v}=\vec{u}\times(k\vec{v})$

e) $\vec{u}\times\vec{0}=\vec{0}\times\vec{u}=\vec{0}$

f) $\vec{u}\times\vec{u}=0$

Definition

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in 3-space, then $\boxed{\vec{u}\bullet(\vec{v}\times\vec{w})}$ is called the ***scalar triple product*** of $\vec{u}$, $\vec{v}$, and $\vec{w}$.

Example

Calculate the scalar triple product $\vec{u}\bullet(\vec{u}\times\vec{v})$ of the vectors:

$$\vec{u}=-2\hat{i}+6\hat{k}\quad \vec{v}=\hat{i}-3\hat{j}+\hat{k}\quad \vec{w}=-5\hat{i}-\hat{j}+\hat{k}$$

Solution

$$\vec{u}\bullet(\vec{u}\times\vec{v})=\begin{vmatrix}-2&0&6\\1&-3&1\\-5&-1&1\end{vmatrix}$$

$$\underline{=-92}$$

Area of a Parallelogram

Theorem

If $\vec{u}$ and $\vec{v}$ are vectors in 3-space, then $\|\vec{u}\times\vec{v}\|$ is equal to the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.

Example

Find the area of the triangle determined by the points $P_1(2,\ 2,\ 0),\ P_2(-1,\ 0,\ 2),\ and\ P_3(0,\ 4,\ 3)$.

Solution

The area of the triangle is $\frac{1}{2}$ the area of the parallelogram determined by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$

$$\begin{aligned}\overrightarrow{P_1P_2} &= (-1,\ 0,\ 2)-(2,\ 2,\ 0)\\ &= (-3,\ -2,\ 2)\end{aligned}$$

$$\begin{aligned}\overrightarrow{P_1P_3} &= (0,\ 4,\ 3)-(2,\ 2,\ 0)\\ &= (-2,\ 2,\ 3)\end{aligned}$$

$$\begin{aligned}\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3} &= \left(\begin{vmatrix}-2 & 2\\ 2 & 3\end{vmatrix},\ -\begin{vmatrix}-3 & 2\\ -2 & 3\end{vmatrix},\ \begin{vmatrix}-3 & -2\\ -2 & 2\end{vmatrix}\right)\\ &= (-10,\ 5,\ -10)\end{aligned}$$

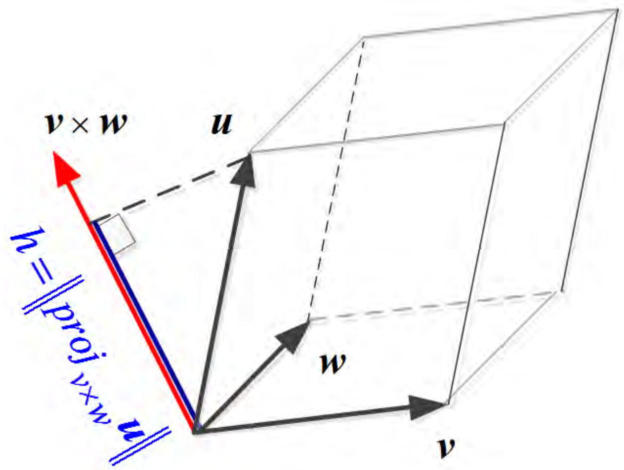
$$\begin{aligned}Area &= \frac{1}{2}\left\|\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}\right\|\\ &= \frac{1}{2}\sqrt{(-10)^2+5^2+(-10)^2}\\ &= \frac{15}{2}\end{aligned}$$

Volume

The Volume of the Parallelepiped is

$$V = (area\ of\ \ base).(height) = \|\vec{v}\times\vec{w}\| \frac{|\vec{u}\bullet(\vec{v}\times\vec{w})|}{\|\vec{v}\times\vec{w}\|} = |\vec{u}\ \bullet\ (\vec{u}\times\vec{v})|$$

$$V = \left| \ \det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \right|$$



Theorem

If the vectors $\vec{u} = (u_1,\ u_2,\ u_3)$, $\vec{v} = (v_1,\ v_2,\ v_3)$, and $\vec{w} = (w_1,\ w_2,\ w_3)$ have the initial point, then they lie in the same plane if and only if

$$\vec{u}\bullet(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$$

Example

Find the volume of the parallelepiped with sides $\vec{u} = (2,\ -6,\ 2)$, $\vec{v} = (0,\ 4,\ -2)$, and $\vec{w} = (2,\ 2,\ -4)$

Solution

$$V = \left| \ \det\begin{bmatrix} 2 & -6 & 2 \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{bmatrix} \right|$$

$= 16$

Exercises   *Section* 2.4 – Cross Product

1. Prove when the cross product $\vec{u}\times\vec{v}$ is perpendicular to $\vec{u}$, then $\vec{u}\bullet(\vec{u}\times\vec{v})=0$

2. Find $\vec{u}\times\vec{v}$, where $\vec{u}=(1,\ 2,\ -2)$ *and* $\vec{v}=(3,\ 0,\ 1)$ and show that $\vec{u}\times\vec{v}$ is perpendicular to $\vec{u}$ and to $\vec{v}$.

3. Given $\vec{u}=(3,\ 2,\ -1)$, $\vec{v}=(0,\ 2,\ -3)$, *and* $\vec{w}=(2,\ 6,\ 7)$ Compute the vectors

a) $\vec{u}\times\vec{v}$

b) $\vec{v}\times\vec{w}$

c) $\vec{u}\times(\vec{v}\times\vec{w})$

d) $(\vec{u}\times\vec{v})\times\vec{w}$

e) $\vec{u}\times(\vec{v}-2\vec{w})$

4. Use the cross product to find a vector that is orthogonal to both

a) $\vec{u}=(-6,\ 4,\ 2)$, $\vec{v}=(3,\ 1,\ 5)$

b) $\vec{u}=(1,\ 1,\ -2)$, $\vec{v}=(2,\ -1,\ 2)$

c) $\vec{u}=(-2,\ 1,\ 5)$, $\vec{v}=(3,\ 0,\ -3)$

5. Find the area of the parallelogram determined by the given vectors

a) $\vec{u}=(1,\ -1,\ 2)$ *and* $\vec{v}=(0,\ 3,\ 1)$

b) $\vec{u}=(3,\ -1,\ 4)$ *and* $\vec{v}=(6,\ -2,\ 8)$

c) $\vec{u}=(2,\ 3,\ 0)$ *and* $\vec{v}=(-1,\ 2,\ -2)$

6. Find the area of the parallelogram with the given vertices

$P_1(3,\ 2)$, $P_2(5,\ 4)$, $P_3(9,\ 4)$, $P_4(7,\ 2)$

7. Find the area of the triangle with the given vertices:

a) $A(2,\ 0)$ $B(3,\ 4)$ $C(-1,\ 2)$

b) $A(1,\ 1)$ $B(2,\ 2)$ $C(3,\ -3)$

c) $P(2,\ 6,\ -1)$ $Q(1,\ 1,\ 1)$ $R=(4,\ 6,\ 2)$

8. *a*) Find the area of the parallelogram with edges $\vec{v}=(3,\ 2)$ and $\vec{w}=(1,\ 4)$

b) Find the area of the triangle with sides $\vec{v}$, $\vec{w}$, and $\vec{v}+\vec{w}$. Draw it.

c) Find the area of the triangle with sides $\vec{v}$, $\vec{w}$, and $\vec{v}-\vec{w}$. Draw it.

9. Find the volume of the parallelepiped with sides $\vec{u}$, $\vec{v}$, and $\vec{w}$.

a) $\vec{u}=(2,\ -6,\ 2)$, $\vec{v}=(0,\ 4,\ -2)$, $\vec{w}=(2,\ 2,\ -4)$

b) $\vec{u}=(3,\ 1,\ 2)$, $\vec{v}=(4,\ 5,\ 1)$, $\vec{w}=(1,\ 2,\ 4)$

10. Compute the scalar triple product $\vec{u} \bullet (\vec{v} \times \vec{w})$

a) $\vec{u} = (-2,\ 0,\ 6),\quad \vec{v} = (1,\ -3,\ 1),\quad \vec{w} = (-5,\ -1,\ 1)$

b) $\vec{u} = (-1,\ 2,\ 4),\quad \vec{v} = (3,\ 4,\ -2),\quad \vec{w} = (-1,\ 2,\ 5)$

c) $\vec{u} = (a,\ 0,\ 0),\quad \vec{v} = (0,\ b,\ 0),\quad \vec{w} = (0,\ 0,\ c)$

d) $\vec{u} = 3\hat{i} - 2\hat{j} - 5\hat{k},\quad \vec{v} = \hat{i} + 4\hat{j} - 4\hat{k},\quad \vec{w} = 3\hat{j} + 2\hat{k}$

e) $\vec{u} = (3,\ -1,\ 6)\quad \vec{v} = (2,\ 4,\ 3)\quad \vec{w} = (5,\ -1,\ 2)$

11. Use the cross product to find the sine of the angle between the vectors
$\vec{u} = (2,\ 3,\ -6),\quad \vec{v} = (2,\ 3,\ 6)$

12. Simplify $(\vec{u} + \vec{v}) \times (\vec{u} - \vec{v})$

13. Prove Lagrange's identity: $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2\ \|\vec{v}\|^2 - (\vec{u} \bullet \vec{v})^2$

14. Polar coordinates satisfy $x = r\cos\theta$ *and* $y = \sin\theta$. Polar area $J\ dr\ d\theta$ includes J:

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

The two columns are orthogonal. Their lengths are ______. Thus J = ______.

15. Prove that $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$ if and only if $\vec{u}$ and $\vec{v}$ are parallel vectors.

16. State the following statements as True or False

a) The cross product of two nonzero vectors $\vec{u}$ and $\vec{v}$ is a nonzero vector if and only if $\vec{u}$ and $\vec{v}$ are not parallel.

b) A normal vector to a plane can be obtained by taking the cross product of two nonzero and noncollinear vectors lying in the plane.

c) The scalar triple product of $\vec{u}$, $\vec{v}$, and $\vec{w}$ determines a vector whose length is equal to the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$.

d) If $\vec{u}$ and $\vec{v}$ are vectors in 3-space, then $\|\vec{u} \times \vec{v}\|$ is equal to the area of the parallelogram determine by $\vec{u}$ and $\vec{v}$.

e) For all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in R^3, the vectors $(\vec{u} \times \vec{v}) \times \vec{w}$ and $\vec{u} \times (\vec{v} \times \vec{w})$ are the same.

f) If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in R^3, where $\vec{u}$ is nonzero and $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, then $\vec{v} = \vec{w}$