Solution Section 2.6 – Chain Rule

Exercise

Find the derivative of $y=\left(3x^4+1\right)^4\left(x^3+4\right)$

Solution

$$y'=\left(3x^4+1\right)^3\left(4\left(12x^3\right)\left(x^3+4\right)+3x^2\left(3x^4+1\right)\right) \qquad \left(U^nV^m\right)'=U^{n-1}V^{m-1}\left(nU'V+mUV'\right)$$

$$=\left(3x^4+1\right)^3\left(48x^6+192x^3+9x^6+3x^2\right)$$

$$\boxed{=\left(3x^4+1\right)^3\left(57x^6+192x^3+3x^2\right)}$$

OR

$$y'=4\left(12x^3\right)\left(3x^4+1\right)^3\left(x^3+4\right)+3x^2\left(3x^4+1\right)^4 \qquad (uv)'=u'v+uv'$$

$$=48x^3\left(3x^4+1\right)^3\left(x^3+4\right)+3x^2\left(3x^4+1\right)^4$$

$$=3x^2\left(3x^4+1\right)^3\left[16x\left(x^3+4\right)+3x^4+1\right]$$

$$=3x^2\left(3x^4+1\right)^3\left(16x^4+64x+3x^4+1\right)$$

$$\boxed{=3x^2\left(3x^4+1\right)^3\left(19x^4+64x+1\right)}$$

Exercise

Find the derivative of $p(t)=\dfrac{(2t+3)^3}{4t^2-1}$

Solution

$$p'(t)=\frac{(2t+3)^2}{\left(4t^2-1\right)^2}\left(6\left(4t^2-1\right)-8t\,(2t+3)\right) \qquad \left(U^nV^m\right)'=U^{n-1}V^{m-1}\left(nU'V+mUV'\right)$$

$$\boxed{=\frac{(2t+3)^2\left(8t^2-24t-6\right)}{\left(4t^2-1\right)^2}}$$

OR

$$P'(t)=\frac{2(3)(2t+3)^2\left(4t^2-1\right)-8t(2t+3)^3}{\left(4t^2-1\right)^2}$$

$$\left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{(2t+3)^2\left[6\left(4t^2-1\right)-8t(2t+3)\right]}{\left(4t^2-1\right)^2}$$

$$=\frac{(2t+3)^2\left[24t^2-6-16t^2-24t\right]}{\left(4t^2-1\right)^2}$$

$$=\frac{(2t+3)^2\left(8t^2-24t-6\right)}{\left(4t^2-1\right)^2}$$

$$=\frac{2(2t+3)^2\left(4t^2-12t-3\right)}{\left(4t^2-1\right)^2}$$

Exercise

Find the derivative of $y=\left(x^3+1\right)^2$

Solution

$$u=x^3+1 \quad \rightarrow y=u^2$$

$$\frac{d}{dx}y=\frac{dy}{du}\frac{du}{dx}$$

$$=2u\left(3x^2\right)$$

$$y'=2\left(x^3+1\right)\left(3x^2\right)$$

$$=6x^2\left(x^3+1\right)$$

Exercise

Find the derivative of $y=\left(x^2+3x\right)^4$

Solution

$u = x^2 + 3x$

$y' = n \quad (u)^{n-1} \quad \frac{d}{dx}[u]$

$= 4\left(x^2 + 3x\right)^3 \frac{d}{dx}[x^2 + 3x]$

$$\boxed{= 4(2x+3)\left(x^2 + 3x\right)^3}$$

Exercise

Find the derivative of $y = \dfrac{4}{2x+1}$

Solution

$$\boxed{y' = \frac{-8}{(2x+1)^2}}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

OR

$y = 4(2x+1)^{-1}$

$y' = -4(2x+1)^{-2}(2)$

$= -8(2x+1)^{-2}$

$$\boxed{= -\frac{8}{(2x+1)^2}}$$

Exercise

Find the derivative of $y = \dfrac{2}{(x-1)^3}$

Solution

$$\boxed{y' = -\frac{6}{(x-1)^4}}$$

$$\left(\frac{1}{U^n}\right)' = -\frac{nU'}{U^{n+1}}$$

Exercise

Find the derivative of $y = x^2\sqrt{x^2+1}$

Solution

$$y' = \frac{x}{\sqrt{x^2+1}}\left(2\left(x^2+1\right)+\frac{1}{2}(2x)x\right)$$

$$\left(U^n V^m\right)' = U^{n-1}V^{m-1}\left(nU'V + mUV'\right)$$

$$= \frac{x\left(3x^2+2\right)}{\sqrt{x^2+1}}$$

OR

$$y = x^2\left(x^2+1\right)^{1/2}$$

$$y' = x^2\,\frac{d}{dx}\left[(x^2+1)^{1/2}\right] + (x^2+1)^{1/2}\,\frac{d}{dx}\left[x^2\right]$$

$$= x^2\left[\frac{1}{2}(x^2+1)^{-1/2}(2x)\right] + (x^2+1)^{1/2}\,[2x]$$

$$= x^3(x^2+1)^{-1/2} + 2x(x^2+1)^{1/2}$$

$$= \frac{(x^2+1)^{1/2}}{(x^2+1)^{1/2}}\left[x^3(x^2+1)^{-1/2} + 2x(x^2+1)^{1/2}\right]$$

$$= \frac{x^3(x^2+1)^{-1/2}(x^2+1)^{1/2} + 2x(x^2+1)^{1/2}(x^2+1)^{1/2}}{(x^2+1)^{1/2}}$$

$$= \frac{x^3+2x(x^2+1)}{(x^2+1)^{1/2}}$$

$$= \frac{x^3+2x^3+2x}{\sqrt{x^2+1}}$$

$$= \frac{3x^3+2x}{\sqrt{x^2+1}}$$

$$= \frac{x\left(3x^2+2\right)}{\sqrt{x^2+1}}$$

Exercise

Find the derivative of $y=\left(\frac{x+1}{x-5}\right)^2$

Solution

$$y'=2\left(\frac{x+1}{x-5}\right)\frac{-5-1}{(x-5)^2} \qquad \left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

$$=-\frac{12(x+1)}{(x-5)^3}$$

OR

$$y'=2\left(\frac{x+1}{x-5}\right)\frac{d}{dx}\left[\frac{x+1}{x-5}\right]$$

$$=2\left(\frac{x+1}{x-5}\right)\left[\frac{(1)(x-5)-(1)(x+1)}{(x-5)^2}\right]$$

$$=2\left(\frac{x+1}{x-5}\right)\left(\frac{x-5-x-1}{(x-5)^2}\right)$$

$$=2\left(\frac{x+1}{x-5}\right)\left(\frac{-6}{(x-5)^2}\right)$$

$$=-\frac{12(x+1)}{(x-5)^3}$$

Exercise

Find the derivative of $s(t)=\sqrt{2t^2+5t+2}$

Solution

$$s(t)=\left(2t^2+5t+2\right)^{1/2} \qquad U=2t^2+5t+2 \to U'=4t+5$$

$$s'(t)=\frac{1}{2}(4t+5)\left(2t^2+5t+2\right)^{-1/2} \qquad \left(U^n\right)'=nU'U^{n-1}$$

$$=\frac{1}{2}\frac{4t+5}{\sqrt{2t^2+5t+2}}$$

Exercise

Find the derivative of $f(x)=\dfrac{1}{\left(x^2-3x\right)^2}$

Solution

$$f'(x)=-\frac{2(2x-3)}{\left(x^2-3x\right)^3}$$

$$\left(\frac{1}{U^n}\right)'=-\frac{nU'}{U^{n+1}}$$

Exercise

Find the derivative of $y=t^2\sqrt{t-2}$

Solution

$$y'=\frac{t}{\sqrt{t-2}}\left(2(t-2)+\frac{1}{2}t\right)$$

$$\left(U^nV^m\right)'=U^{n-1}V^{m-1}\left(nU'V+mUV'\right)$$

$$=\frac{1}{2}\frac{5t-4}{\sqrt{t-2}}$$

OR

$$f=t^2 \qquad f'=2t$$

$$g=(t-2)^{1/2} \qquad g'=\tfrac{1}{2}(t-2)^{-1/2}$$

$$y'=2t\sqrt{t-2}+t^2\tfrac{1}{2}(t-2)^{-1/2}$$

$$=\left[2t(t-2)^{1/2}+t^2\tfrac{1}{2}(t-2)^{-1/2}\right]\frac{2(t-2)^{1/2}}{2(t-2)^{1/2}}$$

$$=\frac{4t(t-2)+t^2}{2(t-2)^{1/2}}$$

$$=\frac{4t^2-8t+t^2}{2\sqrt{t-2}}$$

$$=\frac{5t^2-4t}{2\sqrt{t-2}}$$

Exercise

Find the derivative of $y=\left(\frac{6-5x}{x^2-1}\right)^2$

Solution

$$f'=2\,\frac{5x^2-12x+5}{\left(x^2-1\right)^2}\left(\frac{6-5x}{x^2-1}\right)$$

$$\begin{array}{ccc} 0 & -5 & 6 \\ 1 & 0 & -1 \end{array}\qquad \left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)'=\frac{(ae-bd)x^2+2(af-cd)x+bf-ce}{\left(dx^2+ex+f\right)^2}$$

$$\boxed{=\frac{2\left(5x^2-12x+5\right)(6-5x)}{\left(x^2-1\right)^3}}$$

OR

$f=6-5x \quad f'=-5$

$g=x^2-1 \quad g'=2x$

$$y'=2\frac{-5\left(x^2-1\right)-(2x)(6-5x)}{\left(x^2-1\right)^2}\left(\frac{6-5x}{x^2-1}\right) \qquad \left(U^n\right)'=nU'U^{n-1}$$

$$=2\frac{-5x^2+5-12x+10x^2}{\left(x^2-1\right)^3}(6-5x)$$

$$\boxed{=\frac{2\left(5x^2-12x+5\right)(6-5x)}{\left(x^2-1\right)^3}}$$

Exercise

Find the derivative of $y=4x(3x+5)^5$

Solution

$$y'=4(3x+5)^5+5(3)(3x+5)^4(4x)$$

$$=4(3x+5)^5+60x(3x+5)^4$$

$$=4(3x+5)^4(3x+5+15x)$$

$$\boxed{=4(3x+5)^4(18x+5)}$$

Exercise

Find the derivative of $y = \left(3x^2 - 5x\right)^{1/2}$

Solution

$$u = 3x^2 - 5x \quad \& \quad y = u^{1/2}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{2}u^{-1/2}(6x-5) \\ &= \frac{1}{2}(6x-5)\left(3x^2-5x\right)^{-1/2} \\ &= \frac{6x-5}{2\left(3x^2-5x\right)^{1/2}} \end{aligned}$$

Exercise

Find the derivative of $D_x\left(x^2 + 5x\right)^8$

Solution

$$\begin{aligned} D_x\left(x^2+5x\right)^8 &= 8\left(x^2+5x\right)^7\left(x^2+5x\right)' \\ &= 8\left(x^2+5x\right)^7(2x+5) \\ &= 8(2x+5)\left(x^2+5x\right)^7 \end{aligned}$$

Exercise

Find the derivative of $y = \dfrac{(3x+2)^7}{x-1}$

Solution

$$\begin{aligned} y' &= \frac{7(3)(3x+2)^6(x-1)-(1)(3x+2)^7}{(x-1)^2} \\ &= \frac{(3x+2)^6(21x-21-3x-2)}{(x-1)^2} \\ &= \frac{(3x+2)^6(18x-23)}{(x-1)^2} \end{aligned}$$

Exercise

Find the derivative of $y=\left(\frac{x^2}{8}+x-\frac{1}{x}\right)^4$

Solution

$$y'=4\left(\frac{x^2}{8}+x-\frac{1}{x}\right)^3\left(\frac{2x}{8}+1-\frac{-1}{x^2}\right)$$

$$=4\left(\frac{x^2}{8}+x-\frac{1}{x}\right)^3\left(\frac{x}{4}+1+\frac{1}{x^2}\right)$$

$$=\left(x+4+\frac{4}{x^2}\right)\left(\frac{x^2}{8}+x-\frac{1}{x}\right)^3$$

Exercise

Find the derivative of $y=\sqrt{3x^2-4x+6}$

Solution

$$y=\left(3x^2-4x+6\right)^{1/2}=u^{1/2} \qquad u=3x^2-4x+6 \Rightarrow u'=6x-4$$

$$y'=\frac{1}{2}u^{1/2}u'$$

$$=\frac{1}{2}\left(3x^2-4x+6\right)^{-1/2}2(3x-42)$$

$$=\frac{3x-2}{\sqrt{3x^2-4x+6}}$$

Exercise

Find the derivative of $y=\cot\left(\pi-\frac{1}{x}\right)$

Solution

$$u=\pi-\frac{1}{x} \rightarrow u'=\frac{1}{x^2}$$

$$y'=-\csc^2\left(\pi-\frac{1}{x}\right)\left(\frac{1}{x^2}\right)$$

$$=-\frac{1}{x^2}\csc^2\left(\pi-\frac{1}{x}\right)$$

Exercise

Find the derivative of $y = 5\cos^{-4} x$

Solution

$$y = 5\cos^{-4} x \qquad\qquad u = \cos x \quad \to \quad u' = -\sin x$$

$$y' = 5u^{-5}u'$$

$$= 5(-4)\cos^{-5} x(-\sin x)$$

$$\boxed{= 20\sin x\ \cos^{-5} x}$$

Exercise

Find the derivative of $y = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right)$

Solution

$$y' = \frac{3\pi}{2}\cos\left(\frac{3\pi t}{2}\right) + \frac{3\pi}{2}\left(-\cos\left(\frac{3\pi t}{2}\right)\right)$$

$$= \frac{3\pi}{2}\cos\left(\frac{3\pi t}{2}\right) - \frac{3\pi}{2}\cos\left(\frac{3\pi t}{2}\right)$$

$$\boxed{= \frac{3\pi}{2}\left(\cos\left(\frac{3\pi t}{2}\right) - \cos\left(\frac{3\pi t}{2}\right)\right)}$$

Exercise

Find the derivative of $r = 6(\sec\theta - \tan\theta)^{3/2}$

Solution

$$r = 6(\sec\theta - \tan\theta)^{3/2} = 6u^{3/2} \quad \Rightarrow \quad u = \sec\theta - \tan\theta \ \to \ u' = \sec\theta\tan\theta - \sec^2\theta$$

$$\Rightarrow \ u = \sec\theta - \tan\theta \ \to \ u' = \sec\theta\tan\theta - \sec^2\theta$$

$$r' = 6\left(\frac{3}{2}\right)(\sec\theta - \tan\theta)^{3/2-1}\left(\sec\theta\tan\theta - \sec^2\theta\right)$$

$$= 9(\sec\theta - \tan\theta)^{1/2}\left(\sec\theta\tan\theta - \sec^2\theta\right)$$

$$\boxed{= 9\left(\sec\theta\tan\theta - \sec^2\theta\right)\sqrt{\sec\theta - \tan\theta}}$$

Exercise

Find the derivative of $g(x) = \dfrac{\tan 3x}{(x+7)^4}$

Solution

$$g'(x) = \frac{\left(3\sec^2 3x\right)(x+7)^4 - 4(x+7)^3 \tan 3x}{(x+7)^8}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2} \qquad u = \tan 3x \qquad v = (x+7)^4$$

$$u' = 3\sec^2 3x \qquad v' = 4(x+7)^3$$

$$= \frac{(x+7)^3\left[3(x+7)\sec^2 3x - 4\tan 3x\right]}{(x+7)^8}$$

$$= \frac{3(x+7)\sec^2 3x - 4\tan 3x}{(x+7)^5}$$

Exercise

Find the derivative of $f(\theta) = \left(\dfrac{\sin\theta}{1+\cos\theta}\right)^2$

Solution

$$f'(\theta) = 2\left(\frac{\sin\theta}{1+\cos\theta}\right)\left(\frac{\sin\theta}{1+\cos\theta}\right)'$$

$$= \frac{2\sin\theta}{1+\cos\theta}\left(\frac{\cos\theta(1+\cos\theta) - (-\sin\theta)\sin\theta}{(1+\cos\theta)^2}\right)$$

$$= \frac{2\sin\theta}{1+\cos\theta}\left(\frac{\cos\theta + \cos^2\theta + \sin^2\theta}{(1+\cos\theta)^2}\right)$$

$$= \frac{2\sin\theta}{1+\cos\theta}\left(\frac{\cos\theta + 1}{(1+\cos\theta)^2}\right)$$

$$= \frac{2\sin\theta}{(1+\cos\theta)^2}$$

Exercise

Find the derivative of $y = \sin^2(\pi t - 2)$

Solution

$$y' = 2\sin(\pi t - 2)\left(\sin(\pi t - 2)\right)'$$

$$= 2\sin(\pi t - 2)\left(\pi\cos(\pi t - 2)\right)$$

$$\underline{= 2\pi\sin(\pi t - 2)\cos(\pi t - 2)}$$

Exercise

Find the derivative of $y = (t\tan t)^{10}$

Solution

$$y' = 10(t\tan t)^9 (t\tan t)'$$

$$= 10(t\tan t)^9 \left(\tan t + t\sec^2 t\right)$$

$$= 10(t\tan t)^9 \tan t + 10t(t\tan t)^9 \sec^2 t$$

$$\underline{= 10t^9 \tan^{10} t + 10t^{10} \tan^9 t\sec^2 t}$$

Exercise

Find the derivative of $y = \cos\left(5\sin\left(\frac{t}{3}\right)\right)$

Solution

$$y' = -\sin\left(5\sin\left(\frac{t}{3}\right)\right)\left(5\sin\left(\frac{t}{3}\right)\right)'$$

$$= -\sin\left(5\sin\left(\frac{t}{3}\right)\right)\left(5\frac{1}{3}\cos\left(\frac{t}{3}\right)\right)$$

$$\underline{= -\frac{5}{3}\sin\left(5\sin\left(\frac{t}{3}\right)\right)\cos\left(\frac{t}{3}\right)}$$

Exercise

Find the derivative of $y = 4\sin\left(\sqrt{1+\sqrt{t}}\right)$

Solution

$$y' = 4\cos\left(\sqrt{1+\sqrt{t}}\right)\left(\sqrt{1+\sqrt{t}}\right)'$$

$$\left(\left(1+\sqrt{t}\right)^{1/2}\right)' = \frac{1}{2}\left(1+\sqrt{t}\right)^{-1/2}\left(t^{1/2}\right)'$$

$$= \frac{1}{2}\left(1+\sqrt{t}\right)^{-1/2}\left(\frac{1}{2}t^{-1/2}\right)$$

$$= \frac{1}{4}\frac{1}{\sqrt{t}\sqrt{1+\sqrt{t}}}$$

$$= \frac{1}{4} \frac{1}{\sqrt{t\left(1+\sqrt{t}\right)}}$$

$$y' = 4\cos\left(\sqrt{1+\sqrt{t}}\right)\left(\frac{1}{4}\frac{1}{\sqrt{t+t\sqrt{t}}}\right)$$

$$= \frac{\cos\left(\sqrt{1+\sqrt{t}}\right)}{\sqrt{t+t\sqrt{t}}}$$

Exercise

Find the derivative of $y = \tan^2\left(\sin^3 x\right)$

Solution

$$u = \sin^3 x \quad \Rightarrow \quad u' = 3\sin^2 x(\sin x) = 3\sin^2 x(\cos x)$$

$$y' = 2\tan\left(\sin^3 x\right).\left(\tan\left(\sin^3 x\right)\right)'$$

$$= 2\tan\left(\sin^3 x\right)\cdot\sec^2\left(\sin^3 x\right)\cdot\left(\sin^3 x\right)'$$

$$= 2\tan\left(\sin^3 x\right)\cdot\sec^2\left(\sin^3 x\right)\cdot\left(3\sin^2 x\cos x\right)$$

$$= 6\cos x\sin^2 x\cdot\tan\left(\sin^3 x\right)\cdot\sec^2\left(\sin^3 x\right)$$

Exercise

Find the derivative of $f(x) = \left(\left(x^2+3\right)^5 + x\right)^2$

Solution

$$f'(x) = 2\left(\left(x^2+3\right)^5 + x\right)\left(10x\left(x^2+3\right)^4 + 1\right)$$

Exercise

Find the derivative of $y = \left(\frac{3x-1}{x^2+3}\right)^2$

Solution

$$y = (3x-1)^2\left(x^2+3\right)^{-2} \qquad \left(U^m V^n\right)' = U^{m-1}V^{n-1}\left(mU'V + nUV'\right)$$

$$y' = (3x-1)\left(x^2+3\right)^{-3}\left(6x\left(x^2+3\right)-4x(3x-1)\right)$$

$$= \frac{3x-1}{\left(x^2+3\right)^3}\left(6x^3+18x-12x^2+4x\right)$$

$$\boxed{= \frac{(3x-1)\left(6x^3-12x^2+22x\right)}{\left(x^2+3\right)^3}}$$

Exercise

Find the derivative of $\quad y = \cos\sqrt{\sin(\tan \pi x)}$

Solution

$$y' = -\left(\sin\sqrt{\sin(\tan \pi x)}\right)\left(\frac{1}{2}\frac{\pi\cos(\tan \pi x)\sec^2 \pi x}{\sqrt{\sin(\tan \pi x)}}\right)$$

$$\boxed{= -\frac{\pi}{2}\frac{\sec^2 \pi x\ \cos(\tan \pi x)\ \sin\sqrt{\sin(\tan \pi x)}}{\sqrt{\sin(\tan \pi x)}}}$$

Exercise

Find the derivative of $\quad f(x) = \frac{x}{\sqrt{x^2+1}}$

Solution

$$f(x) = x\left(x^2+1\right)^{-1/2}$$

$$f'(x) = \frac{x^2+1-\frac{1}{2}\left(2x^2\right)}{x^2+1} \qquad \left(U^m V^n\right)' = U^{m-1}V^{n-1}\left(mU'V+nUV'\right)$$

$$\boxed{= \frac{1}{x^2+1}}$$

Exercise

Find the derivative of $\quad y = \cos(1-2x)^2$

Solution

$$y' = -\left(2(-2)(1-2x)\right)\ \sin(1-2x)^2$$

$$\boxed{= 4(1-2x)\ \sin(1-2x)^2}$$

Exercise

Find the derivative of $f(x)=(4x-3)^2$

Solution

$$\boxed{f'(x)=8(4x-3)}$$

$$\left(U^n\right)'=nU'\,U^{n-1}$$

Exercise

Find the derivative of $f(x)=\dfrac{x}{\sqrt[3]{x^2+4}}$

Solution

$$f(x)=x\left(x^2+4\right)^{-1/3}$$

$$f'(x)=\left(x^2+4\right)^{-4/3}\left(x^2+4-\tfrac{1}{3}\left(2x^2\right)\right)$$

$$\left(U^mV^n\right)'=U^{m-1}V^{n-1}\left(mU'V+nUV'\right)$$

$$\boxed{=\frac{1}{3}\frac{x^2+12}{\left(x^2+4\right)^{4/3}}}$$

Exercise

Find the derivative of $f(x)=\left(\dfrac{x^2}{x^3+2}\right)^2$

Solution

$$f(x)=x^4\left(x^3+2\right)^{-2}$$

$$f'(x)=x^3\left(x^3+2\right)^{-3}\left(4x^3+8-2x^3\right)$$

$$\left(U^mV^n\right)'=U^{m-1}V^{n-1}\left(mU'V+nUV'\right)$$

$$\boxed{=\frac{x^3\left(2x^3+8\right)}{\left(x^3+2\right)^3}}$$

Exercise

Find the derivative of $y=\sin\sqrt[3]{x}+\sqrt[3]{\sin x}$

Solution

$$\boxed{y'=\frac{1}{3}x^{-2/3}\cos\sqrt[3]{x}+\frac{1}{3}\cos x(\sin x)^{-2/3}}$$

Exercise

Find the derivative of $f(\theta) = 4\tan\left(\theta^2 + 3\theta + 2\right)$

Solution

$$\underline{f'(\theta) = 4(2\theta + 3)\sec^2\left(\theta^2 + 3\theta + 2\right)}$$

Exercise

Find the derivative of $f(\theta) = \tan(\sin\theta)$

Solution

$$\underline{f'(\theta) = \cos\theta\ \sec^2(\sin\theta)}$$

Exercise

Find the derivative of $y = 5x + \sin^3 x + \sin x^3$

Solution

$$\underline{y' = 5 + 3\cos x \sin^2 x + 3x^2 \cos x^3}$$

Exercise

Find the derivative of $y = \csc^5 3x$

Solution

$$y' = 15\csc^4 3x(-\csc 3x \cot 3x)$$

$$\underline{= -15\cot 3x\ \csc^5 3x}$$

Exercise

Find the derivative of $y = 2x\sqrt{x^2 - 2x + 2}$

Solution

$$y' = 2\sqrt{x^2 - 2x + 2} + 2x(2x - 2)\left(x^2 - 2x + 2\right)^{-1/2}$$

$$\underline{= 2\sqrt{x^2 - 2x + 2} + \frac{4x^2 - 4x}{\sqrt{x^2 - 2x + 2}}}$$

Exercise

Find the derivative of $\dfrac{d}{du}\left(\dfrac{4u^2+u}{8u+1}\right)^3$

Solution

$$\left(U^n\right)' = nU'\,U^{n-1} \qquad \frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix}x^2 + 2\begin{vmatrix} a & c \\ d & f \end{vmatrix}x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

$$\frac{d}{du}\left(\frac{4u^2+u}{8u+1}\right)^3 = 3\left(\frac{4u^2+u}{8u+1}\right)^2 \frac{\begin{vmatrix} 4 & 1 \\ 0 & 8 \end{vmatrix}u^2 + \begin{vmatrix} 4 & 0 \\ 0 & 1 \end{vmatrix}u + \begin{vmatrix} 1 & 0 \\ 8 & 1 \end{vmatrix}}{(8u+1)^2}$$

$$= 3\left(32u^2+4u+1\right)\frac{\left(4u^2+u\right)^2}{(8u+1)^4}$$

Exercise

Find the derivative of $y = \frac{1}{2}x^2\sqrt{16-x^2}$

Solution

$$y = \frac{1}{2}x^2\left(16-x^2\right)^{1/2}$$

$$y' = \frac{1}{2}x\left(16-x^2\right)^{-1/2}\left(32-2x^2-x^2\right) \qquad \left(U^mV^n\right)' = U^{m-1}V^{n-1}\left(mU'V+nUV'\right)$$

$$= \frac{1}{2}\frac{32x-3x^3}{\sqrt{16-x^2}}$$

Exercise

Find the derivative of $y = \left(\dfrac{x-3}{2x+5}\right)^4$

Solution

$$y' = 4\frac{5+6}{(2x+5)^2}\left(\frac{x-3}{2x+5}\right)^3 \qquad \left(U^n\right)' = nU'\,U^{n-1} \quad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

$$= \frac{44(x-3)^3}{(2x+5)^5}$$

Exercise

Find the derivative of $y=\left(\frac{5x-3}{2x+5}\right)^5$

Solution

$$y'=5\frac{25+6}{(2x+5)^2}\left(\frac{5x-3}{2x+5}\right)^4 \qquad \left(U^n\right)'=nU'\,U^{n-1} \quad \left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

$$=\boxed{\frac{155(5x-3)^4}{(2x+5)^6}}$$

Exercise

Find the derivative of $y=\left(\frac{6x-8}{2x-3}\right)^6$

Solution

$$y'=6\frac{-18+16}{(2x-3)^2}\left(\frac{6x-8}{2x-3}\right)^5 \qquad \left(U^n\right)'=nU'\,U^{n-1} \quad \left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

$$=\boxed{-\frac{12(6x-8)^5}{(2x-3)^7}}$$

Exercise

Find the derivative of $y=\left(\frac{3x^2-4}{2x^2-1}\right)^3$

Solution

$$y'=3\frac{2(-3+8)x}{\left(2x^2-1\right)^2}\left(\frac{3x^2-4}{2x^2-1}\right)^2 \qquad \left(U^n\right)'=nU'\,U^{n-1} \quad \left(\frac{ax^n+b}{cx^n+d}\right)'=\frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$=\boxed{\frac{30x\left(3x^2-4\right)^2}{\left(2x^2-1\right)^4}}$$

Exercise

Find the derivative of $y=\left(\frac{3x^2+4}{2x^2+1}\right)^{-3}$

Solution

$$y' = (-3)\frac{2(3-8)x}{\left(2x^2+1\right)^2}\left(\frac{3x^2+4}{2x^2+1}\right)^{-4} \qquad \left(U^n\right)' = nU'\,U^{n-1} \quad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$= \frac{15x}{\left(2x^2+1\right)^2}\left(\frac{2x^2+1}{3x^2+4}\right)^4$$

$$= \frac{15x\left(2x^2+1\right)^2}{\left(3x^2+4\right)^4}$$

Exercise

Find the derivative of $\quad y = \left(\frac{2x^2-3}{x^2+1}\right)^{1/3}$

Solution

$$y' = \frac{1}{3}\frac{2(2+3)x}{\left(x^2+1\right)^2}\left(\frac{2x^2-3}{x^2+1}\right)^{-2/3} \qquad \left(U^n\right)' = nU'\,U^{n-1} \quad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$= \frac{10}{3}\frac{x}{\left(x^2+1\right)^2}\left(\frac{x^2+1}{2x^2-3}\right)^{2/3}$$

$$= \frac{10}{3}\frac{x}{\left(x^2+1\right)^{4/3}\left(2x^2-3\right)^{2/3}}$$

Exercise

Find the derivative of $\quad y = \sqrt{\frac{2x^3-3}{2x^3+1}}$

Solution

$$y' = \frac{1}{2}\frac{3(2+6)x^2}{\left(x^3+1\right)^2}\left(\frac{2x^3-3}{x^3+1}\right)^{-1/2} \qquad \left(U^n\right)' = nU'\,U^{n-1} \quad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$= \frac{12x^2}{\left(x^3+1\right)^2}\left(\frac{x^3+1}{2x^3-3}\right)^{1/2}$$

$$= \frac{12x^2}{\left(x^3+1\right)^{3/2}\sqrt{2x^3-3}}$$

Exercise

Find the derivative of $y=\left(\dfrac{2x^4-3}{2x^4+1}\right)^5$

Solution

$$y'=5\frac{4(2+6)x^3}{\left(2x^4+1\right)^2}\left(\frac{2x^4-3}{2x^4+1}\right)^4 \qquad \left(U^n\right)'=nU'\,U^{n-1} \quad \left(\frac{ax^n+b}{cx^n+d}\right)'=\frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$=\frac{160x^3\left(2x^4-3\right)^4}{\left(2x^4+1\right)^6}$$

Exercise

Find the derivative of $y=\left(\dfrac{x^2-4x+1}{5x^2-2x-1}\right)^3$

Solution

$$\left(U^n\right)'=nU'\,U^{n-1} \qquad \frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\frac{\begin{vmatrix}a&b\\d&e\end{vmatrix}x^2+2\begin{vmatrix}a&c\\d&f\end{vmatrix}x+\begin{vmatrix}b&c\\e&f\end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

$$y'=(3)\frac{\begin{vmatrix}1&-4\\5&-2\end{vmatrix}x^2+2\begin{vmatrix}1&1\\5&-1\end{vmatrix}x+\begin{vmatrix}-4&1\\-2&-1\end{vmatrix}}{\left(5x^2-2x-1\right)^2}\left(\frac{x^2-4x+1}{5x^2-2x-1}\right)^2$$

$$=\frac{\left(18x^2-12x+6\right)\left(x^2-4x+1\right)^2}{\left(5x^2-2x-1\right)^4}$$

Exercise

Find the derivative of $y=\left(\dfrac{3x^2-4x+2}{2x^2+x-1}\right)^{2/3}$

Solution

$$\left(U^n\right)'=nU'\,U^{n-1} \qquad \frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\frac{\begin{vmatrix}a&b\\d&e\end{vmatrix}x^2+2\begin{vmatrix}a&c\\d&f\end{vmatrix}x+\begin{vmatrix}b&c\\e&f\end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

$$y' = \frac{2}{3}\frac{\begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix}x^2 + 2\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}x + \begin{vmatrix} -4 & 2 \\ 1 & -1 \end{vmatrix}}{\left(2x^2+x-1\right)^2}\left(\frac{3x^2-4x+2}{2x^2+x-1}\right)^{-1/3}$$

$$= \frac{2}{3}\frac{11x^2-14x+6}{\left(2x^2+x-1\right)^2}\left(\frac{2x^2+x-1}{3x^2-4x+2}\right)^{1/3}$$

$$= \frac{2}{3}\frac{11x^2-14x+6}{\left(2x^2+x-1\right)^{5/3}\left(3x^2-4x+2\right)^{1/3}}$$

Exercise

Find the derivative of $\quad f(x) = \left(\frac{3t^2-1}{3t^2+1}\right)^{-3}$

Solution

$$f(x) = \left(\frac{3t^2+1}{3t^2-1}\right)^3$$

$$f'(x) = 3\frac{3(-3-3)t}{\left(3t^2-1\right)^2}\left(\frac{3t^2+1}{3t^2-1}\right)^2 \qquad \left(U^n\right)' = nU'\,U^{n-1} \quad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$= -\frac{6t\left(3t^2+1\right)^2}{\left(3t^2-1\right)^4}$$

Exercise

Find the derivative of $\quad f(x) = \left(\frac{x}{3x^2+2x+1}\right)^{1/3}$

Solution

$$\left(U^n\right)' = nU'\,U^{n-1} \qquad \frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix}x^2 + 2\begin{vmatrix} a & c \\ d & f \end{vmatrix}x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

$$f'(x) = \frac{1}{3}\frac{\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix}x^2 + 2\begin{vmatrix} 0 & 0 \\ 3 & 1 \end{vmatrix}x + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}}{\left(3x^2+2x+1\right)^2}\left(\frac{x}{3x^2+2x+1}\right)^{-2/3}$$

$$=\frac{1}{3}\frac{-3x^2+1}{\left(3x^2+2x+1\right)^2}\left(\frac{3x^2+2x+1}{x}\right)^{2/3}$$

$$\boxed{=\frac{-3x^2+1}{3x^{2/3}\left(3x^2+2x+1\right)^{4/3}}}$$

Exercise

Find the derivative of $f(x)=\left(x^2+2x-3\right)^5(2x+3)^6$

Solution

$$\left(U^mV^n\right)'=U^{m-1}V^{n-1}\left(mU'V+nUV'\right)$$

$$f'(x)=\left(x^2+2x-3\right)^4(2x+3)^5\left[5(2x+2)(2x+3)+12\left(x^2+2x-3\right)\right]$$

$$=\left(x^2+2x-3\right)^4(2x+3)^5\left(20x^2+50x+30+12x^2+24x-36\right)$$

$$\boxed{=\left(x^2+2x-3\right)^4(2x+3)^5\left(32x^2+74x-6\right)}$$

Exercise

Find the derivative of $f(x)=\left(2x^2-4x+3\right)^4(3x-5)^5$

Solution

$$\left(U^mV^n\right)'=U^{m-1}V^{n-1}\left(mU'V+nUV'\right)$$

$$f'(x)=\left(2x^2-4x+3\right)^3(3x-5)^4\left[4(4x-4)(3x-5)+15\left(2x^2-4x+3\right)\right]$$

$$=\left(2x^2-4x+3\right)^3(3x-5)^4\left(48x^2-128x+80+30x^2-60x+45\right)$$

$$\boxed{=\left(2x^2-4x+3\right)^3(3x-5)^4\left(88x^2-188x+135\right)}$$

Exercise

Find the derivative of $f(x)=\left(x^2+2x-3\right)^4\left(x^2+3x+5\right)^6$

Solution

$$\left(U^m V^n\right)' = U^{m-1}V^{n-1}\left(mU'V + nUV'\right)$$

$$f'(x) = \left(x^2+2x-3\right)^3\left(x^2+3x+5\right)^5\left[4(2x+2)\left(x^2+3x+5\right)+6(2x+3)\left(x^2+2x-3\right)\right]$$

$$= \left(x^2+2x-3\right)^3\left(x^2+3x+5\right)^5\left(8x^3+32x^2+64x+40+12x^3+42x^2-54\right)$$

$$= \left(x^2+2x-3\right)^3\left(x^2+3x+5\right)^5\left(20x^3+74x^2+64x-14\right)$$

Exercise

Find the derivative of $f(x) = \left(2x^3-5x\right)^3\left(x^2+2x+1\right)^4(2x-3)^5$

Solution

$$\left(U^m V^n W^p\right)' = U^{m-1}V^{n-1}W^{p-1}\left(mU'VW + nUV'W + pUVW'\right)$$

$$f'(x) = \left(2x^3-5x\right)^2\left(x^2+2x+1\right)^3(2x-3)^4\left[3\left(6x^2-5\right)\left(x^2+2x+1\right)(2x-3)\right.$$

$$\left.+4(2x+2)\left(2x^3-5x\right)(2x-3)+5(2)\left(2x^3-5x\right)\left(x^2+2x+1\right)\right]$$

$$= \left(2x^3-5x\right)^2\left(x^2+2x+1\right)^3(2x-3)^4\left[\left(18x^2-15\right)\left(2x^3+x^2-4x-3\right)\right.$$

$$\left.+(8x+8)\left(4x^4-6x^3-10x^2+15x\right)+\left(20x^5+40x^4-20x^3-100x^2-50x\right)\right]$$

$$= \left(2x^3-5x\right)^2\left(x^2+2x+1\right)^3(2x-3)^4$$

x^5	x^4	x^3	x^2	x	x^0
36	18	−72	−54	−60	45
32	−48	−30	−15	120	
20	32	−80	120	50	
	40	−48	−80		
		−20	−100		

$$= \left(2x^3-5x\right)^2\left(x^2+2x+1\right)^3(2x-3)^4\left(88x^5+42x^4-250x^3-129x^2+110x+45\right)$$

Exercise

Find the derivative of $f(x)=\left(x^4+3x\right)^4\left(x^3+2x\right)^5(2x-3)^6$

Solution

$$\left(U^m V^n W^p\right)' = U^{m-1}V^{n-1}W^{p-1}\left(mU'VW + nUV'W + pUVW'\right)$$

$$f'(x)=\left(x^4+3x\right)^3\left(x^3+2x\right)^4(2x-3)^5$$
$$\left[4\left(4x^3+3\right)\left(x^3+2x\right)(2x-3)+5\left(x^4+3x\right)\left(3x^2+2\right)(2x-3)+12\left(x^4+3x\right)\left(x^3+2x\right)\right]$$

$$f'(x)=\left(x^4+3x\right)^3\left(x^3+2x\right)^4(2x-3)^5\ \left[\left(16x^3+9\right)\left(8x^4-9x^3+16x^2-18x\right)\right.$$
$$\left.+\left(5x^4+15x\right)\left(6x^3-9x^2+4x-6\right)+\left(12x^4+36x\right)\left(x^3+2x\right)\right]$$

x^7	$128+30+12$
x^6	$-144-45$
x^5	$256+20+24$
x^4	$-288+72-30+90+36$
x^3	$-81-135$
x^2	$144+60+72$
x^1	$-162-90$

$$f'(x)=\left(x^4+3x\right)^3\left(x^3+2x\right)^4(2x-3)^5\left(170x^7-189x^6+300x^5-120x^4-216x^3+206x^2-252x\right)$$

Exercise

Find the derivative of $f(x)=\dfrac{\left(x^2-6x\right)^5}{\left(3x^2+5x-2\right)^4}$

Solution

$$f(x)=\left(x^2-6x\right)^5\left(3x^2+5x-2\right)^{-4} \qquad\qquad \left(U^m V^n\right)' = U^{m-1}V^{n-1}\left(mU'V + nUV'\right)$$

$$f'(x)=\left(x^2-6x\right)^4\left(3x^2+5x-2\right)^{-5}\left[5(2x-6)\left(3x^2+5x-2\right)-4\left(x^2-6x\right)(6x+5)\right]$$

$$=\left(x^2-6x\right)^4\left(3x^2+5x-2\right)^{-5}\left[(10x-30)\left(3x^2+5x-2\right)-4\left(6x^3-31x^2-30x\right)\right]$$

$$=\left(x^2-6x\right)^4\left(3x^2+5x-2\right)^{-5}$$

$$\begin{array}{ll} x^3 & 30-24 \\ x^2 & 50-90+124 \\ x & -20-150+120 \\ x^0 & 60 \end{array}$$

$$\boxed{=\frac{\left(x^2-6x\right)^4\left(6x^3+84x^2-50x+60\right)}{\left(3x^2+5x-2\right)^5}}$$

Exercise

Find the derivative of $\quad f(x)=\dfrac{\left(2x^2+3x+1\right)^4}{\left(x^2+5x-6\right)^5}$

Solution

$$f(x)=\left(2x^2+3x+1\right)^4\left(x^2+5x-6\right)^{-5} \qquad \left(U^mV^n\right)'=U^{m-1}V^{n-1}\left(mU'V+nUV'\right)$$

$$f'(x)=\left(2x^2+3x+1\right)^3\left(x^2+5x-6\right)^{-6}\left[4(4x+3)\left(x^2+5x-6\right)-5\left(2x^2+3x+1\right)(2x+5)\right].$$

$$=\frac{\left(2x^2+3x+1\right)^3}{\left(x^2+5x-6\right)^6}\left[(16x+12)\left(x^2+5x-6\right)-\left(2x^2+3x+1\right)(10x+25)\right]$$

$$\begin{array}{ll} x^3 & 16-20 \\ x^2 & 80+12-50-30 \\ x & -96+60-75 \\ x^0 & -7-25 \end{array}$$

$$\boxed{f'(x)=\frac{\left(2x^2+3x+1\right)^3}{\left(x^2+5x-6\right)^6}\left(-4x^3+12x^2-111x-32x\right)}$$

Exercise

Find the derivative of $f(x)=\dfrac{\left(x^3-3x\right)^3\left(x^2+4x\right)^4}{\left(x^2+4x+1\right)^2}$

Solution

$$\left(U^m V^n W^p\right)'=U^{m-1}V^{n-1}W^{p-1}\left(mU'VW+nUV'W+pUVW'\right)$$

$$f'(x)=\frac{\left(x^3-3x\right)^2\left(x^2+4x\right)^3}{\left(x^2+4x+1\right)^3}\begin{bmatrix}3\left(3x^2-3\right)\left(x^2+4x\right)\left(x^2+4x+1\right)\\+3\left(x^3-3x\right)(2x+4)\left(x^2+4x+1\right)-2(2x+4)\left(x^3-3x\right)\left(x^2+4x\right)\end{bmatrix}$$

$$=\frac{\left(x^3-3x\right)^2\left(x^2+4x\right)^3}{\left(x^2+4x+1\right)^3}\begin{bmatrix}\left(9x^2-9\right)\left(x^4+8x^3+17x^2+4x\right)\\+\left(3x^3-9x\right)\left(2x^3+12x^2+18x+4\right)\\-(4x+8)\left(x^5+4x^4-3x^3-12x^2\right)\end{bmatrix}$$

x^6	$9+6-4$
x^5	$72+36-16-16-8$
x^4	$153-9+54-18+12-32$
x^3	$36-72+12-108+48+24$
x^2	$-153-162+96$
x^1	$-36-36$

$$f'(x)=\frac{\left(x^3-3x\right)^2\left(x^2+4x\right)^3}{\left(x^2+4x+1\right)^3}\left(11x^6+68x^5+160x^4-60x^3-219x^2-72x\right)$$

Exercise

Find the derivative of $f(x)=\dfrac{x^2+3}{(2x-1)^3(3x+1)^4}$

Solution

$$f(x)=\left(x^2+3\right)(2x-1)^{-3}(3x+1)^{-4}\qquad\left(U^m V^n W^p\right)'=U^{m-1}V^{n-1}W^{p-1}\left(mU'VW+nUV'W+pUVW'\right)$$

$$f'(x)=(2x-1)^{-4}(3x+1)^{-5}$$
$$\left[2x(2x-1)(3x+1)-6\left(x^2+3\right)(3x+1)-12\left(x^2+3\right)(2x-1)\right]$$

$$=\frac{1}{(2x-1)^4(3x+1)^5}\left(\left(4x^2-2x\right)(3x+1)-6\left(3x^3+x^2+9x+3\right)-12\left(2x^3-x^2+6x-3\right)\right)$$

$x^3 \quad 12-18-24$

$x^2 \quad 4-6-6+12$

$x \quad -2-54-72$

$x^0 \quad -18+36$

$$\boxed{f'(x)=\frac{-30x^3+4x^2-128x+18}{(2x-1)^4(3x+1)^5}}$$

Exercise

Find the derivative of $\quad f(x)=\dfrac{\left(x^3-3x\right)^3\left(x^2+4x\right)^4}{\left(x^2+4x+1\right)^2}$

Solution

$$f(x)=\left(x^3-3x\right)^3\left(x^2+4x\right)^4\left(x^2+4x+1\right)^{-2}$$

$$\left(U^mV^nW^p\right)'=U^{m-1}V^{n-1}W^{p-1}\left(mU'VW+nUV'W+pUVW'\right)$$

$$f'(x)=\left(x^3-3x\right)^2\left(x^2+4x\right)^3\left(x^2+4x+1\right)^{-3}\left[3\left(3x^2-3\right)\left(x^2+4x\right)\left(x^2+4x+1\right)\right.$$
$$\left.+4\left(x^3-3x\right)(2x+4)\left(x^2+4x+1\right)-2\left(x^3-3x\right)\left(x^2+4x\right)(2x+4)\right]$$

$$f'(x)=\left(x^3-3x\right)^2\left(x^2+4x\right)^3\left(x^2+4x+1\right)^{-3}\left[\left(9x^2-9\right)\left(x^4+8x^3+9x^2+4x\right)\right.$$
$$\left.+\left(4x^3-12x\right)\left(2x^3+12x^2+18x+4\right)+\left(-2x^3+6x\right)\left(2x^3+12x^2+16x\right)\right]$$

$x^6 \quad 9+8-4$

$x^5 \quad 72+48-24$

$x^4 \quad 81-9+72-24-32+12$

$x^3 \quad 36-72+16-144+72$

$x^2 \quad -81-216+96$

$x^1 \quad -36-48$

$$\boxed{f'(x)=\frac{\left(13x^6+96x^5+100x^4-92x^3-201x^2-84x\right)\left(x^3-3x\right)^2\left(x^2+4x\right)^3}{\left(x^2+4x+1\right)^3}}$$

Exercise

Find the ***second*** derivative $\quad y=\dfrac{x^2+3}{(x-1)^3+(x+1)^3}$

Solution

$$(x-1)^3+(x+1)^3=x^3-3x^2+3x-1+x^3+3x^2+3x+1$$
$$=2x^3+6x$$

$$y=\frac{x^2+3}{2x^3+6x}\qquad u=x^2+3\quad v=2x^3+6x$$
$$u'=2x\qquad v'=6x^2+6$$

$$y'=\frac{4x^4+12x^2-6x^4-18x^2-6x^2-18}{\left(2x^3+6x\right)^2}\qquad \left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{-2x^4-12x^2-18}{\left(2x^3+6x\right)^2}$$

$$\boxed{=-2\frac{x^4+6x^2+9}{\left(2x^3+6x\right)^2}}$$

$$u=x^4++6x^2+9\qquad v=\left(2x^3+6x\right)^2$$
$$u'=4x^3+12x\qquad v'=2\left(2x^3+6x\right)\left(6x^2+6\right)$$
$$=4x\left(x^2+3\right)$$

$$y''=-2\frac{4x\left(x^2+3\right)\left(2x^3+6x\right)^2-2\left(2x^3+6x\right)\left(6x^2+6\right)\left(x^4+6x^2+9\right)}{\left(2x^3+6x\right)^4}$$

$$=-4\left(2x^3+6x\right)\frac{2x\left(2x^5+6x^3+6x^3+18x\right)-\left(6x^6+36x^4+54x^2+x^4+36x^2+54\right)}{\left(2x^3+6x\right)^4}$$

$$=-4\frac{4x^5+24x^3+36x^2-6x^6-37x^4-90x^2-54}{\left(2x^3+6x\right)^3}$$

$$\boxed{=-4\frac{-6x^6+4x^5-37x^4+24x^3-54x^2-54}{\left(2x^3+6x\right)^3}}$$

Exercise

Find the ***second*** derivative of $y = \left(1+\frac{1}{x}\right)^3$

Solution

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$\begin{aligned} y' &= 3\left(1+\frac{1}{x}\right)^2\left(1+\frac{1}{x}\right)' \\ &= 3\left(1+\frac{1}{x}\right)^2\left(-\frac{1}{x^2}\right) \\ &= -\frac{3}{x^2}\left(1+\frac{1}{x}\right)^2 \end{aligned}$$

$$\begin{aligned} y'' &= \left(-\frac{3}{x^2}\right)'\left(1+\frac{1}{x}\right)^2 + \left(-\frac{3}{x^2}\right)\left(\left(1+\frac{1}{x}\right)^2\right)' \\ &= \left(-\frac{-3(2x)}{x^4}\right)\left(1+\frac{1}{x}\right)^2 + \left(-\frac{3}{x^2}\right)\left(2\left(1+\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)\right) \\ &= \frac{6}{x^3}\left(1+\frac{1}{x}\right)^2 + \frac{6}{x^4}\left(1+\frac{1}{x}\right) \\ &= \frac{6}{x^3}\left(1+\frac{1}{x}\right)\left(1+\frac{1}{x}+\frac{1}{x}\right) \\ &= \frac{6}{x^3}\left(1+\frac{1}{x}\right)\left(1+\frac{2}{x}\right) \end{aligned}$$

Exercise

Find the ***second*** derivative of $y = 9\tan\left(\frac{x}{3}\right)$

Solution

$$\begin{aligned} y' &= 9\sec^2\left(\frac{x}{3}\right)\cdot\left(\frac{x}{3}\right)' \\ &= 9\sec^2\left(\frac{x}{3}\right)\cdot\left(\frac{1}{3}\right) \\ &= 3\sec^2\left(\frac{x}{3}\right) \end{aligned}$$

$$\begin{aligned} y'' &= 6\sec\left(\frac{x}{3}\right)\cdot\left(\sec\left(\frac{x}{3}\right)\right)' \\ &= 6\sec\left(\frac{x}{3}\right)\cdot\frac{1}{3}\sec\left(\frac{x}{3}\right)\cdot\tan\left(\frac{x}{3}\right) \\ &= 2\sec^2\left(\frac{x}{3}\right)\cdot\tan\left(\frac{x}{3}\right) \end{aligned}$$

Exercise

Find the tangent line to the graph of $y = \sqrt[3]{(x+4)^2}$ when $x = 4$.

Solution

$$y = (x+4)^{2/3}$$

$$y' = \frac{2}{3}(x+4)^{-1/3}$$

$$= \frac{2}{3}\frac{1}{(x+4)^{1/3}}$$

$$= \boxed{\frac{2}{3\sqrt[3]{x+4}}}$$

$$x = 4 \quad \to \quad m = y' = \frac{2}{3\sqrt[3]{4+4}}$$

$$= \frac{2}{3\sqrt[3]{2^3}}$$

$$= \frac{2}{3(2)}$$

$$= \boxed{\frac{1}{3}}$$

$$x = 4 \to y = \sqrt[3]{(4+4)^2} = 4$$

$$y = \frac{1}{3}(x-4)+4$$

$$y = \frac{1}{3}x - \frac{4}{3} + 4$$

$$\boxed{y = \frac{1}{3}x + \frac{8}{3}}$$

Exercise

Evaluate the limit $\displaystyle\lim_{h\to 0} \frac{\sin^2\left(\frac{\pi}{4}+h\right)-\frac{1}{2}}{h}$

Solution

$$\lim_{h\to 0} \frac{\sin^2\left(\frac{\pi}{4}+h\right)-\frac{1}{2}}{h} = \frac{\frac{1}{2}-\frac{1}{2}}{0} = \frac{0}{0}$$

$$f\left(\frac{\pi}{4}\right) = \sin^2\frac{\pi}{4} = \frac{1}{2}$$

$$\lim_{h\to 0} \frac{\sin^2\left(\frac{\pi}{4}+h\right)-\frac{1}{2}}{h} = f'\left(\frac{\pi}{4}\right)$$

$$= 2\sin\frac{\pi}{4}\ \cos\frac{\pi}{4}$$

$$= 2\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}$$

$$= 1$$

Exercise

Evaluate the limit $\displaystyle \lim_{x\to 5} \frac{\tan\left(\pi\sqrt{3x-11}\right)}{x-5}$

Solution

$$\lim_{x\to 5} \frac{\tan\left(\pi\sqrt{3x-11}\right)}{x-5} = \frac{\tan 2\pi}{0} = \frac{0}{0}$$

$$f(x) = \tan\left(\pi\sqrt{3x-11}\right)$$

$$\lim_{x\to 5} \frac{f(x)-f(5)}{x-5} = f'(5)$$

$$= \frac{3\pi}{2\sqrt{3x-11}}\ \sec^2\left(\pi\sqrt{3x-11}\right)\Big|_{x=5}$$

$$= \frac{3\pi}{4}\sec^2(2\pi)$$

$$= \frac{3\pi}{4}$$