# Lecture One

# Section 1.1 – Introduction to System of Linear Equations

Given the linear equations

$$\begin{cases} x - 2y = 1\\ 3x + 2y = 11 \end{cases}$$

The solution to this system is (3, 1), which means that 2 lines meeting at a single point.

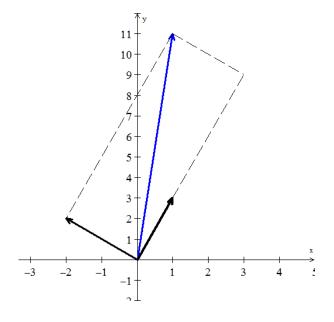
We can rewrite the system equation as linear combination:

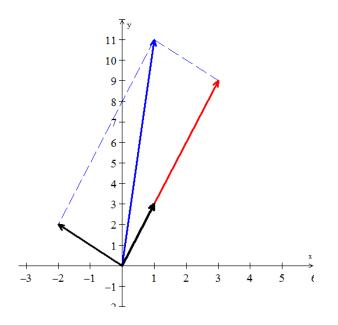
$$x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

$$x.v_1 + y.v_2 = v$$

$$\begin{bmatrix} 1+x \\ 3+y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} \Rightarrow \begin{bmatrix} x=3 \\ y=9 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 9 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$





Therefore, the side vectors are  $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$  and  $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ 

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The diagonal sum is  $\begin{bmatrix} 3-2 \\ 9+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$ 

The linear combination is given by:

$$3\begin{bmatrix}1\\3\end{bmatrix}+1\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$$

Thus, the solution is x = 3 y = 1

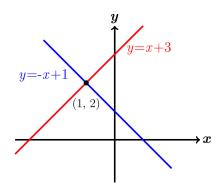
#### <u>Note</u>

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$$
 is called the "coefficient matrix"

The matrix form of the system is written as Ax = b

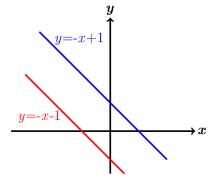
$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

# **Graphically**



One solution (lines intersect)
Consistent

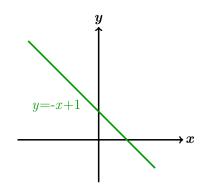
Independent



No Solution (lines //)

Inconsistent

Independent



Infinite solution

Consistent

Dependent

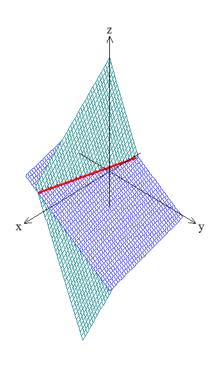
## Three Equations in 3 Unknowns

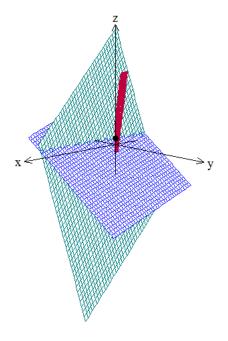
Given the system equations

$$x + 2y + 3z = 6$$

$$2x + 5y + 2z = 4$$

$$6x - 3y + z = 2$$

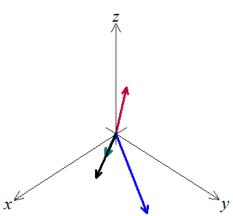




This system can be written as linear combination:

$$x \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + z \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Let 
$$b = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$



We want to multiply the three column vectors by x, y, z to produce  $\boldsymbol{b}$ .

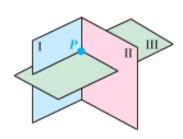
The combination of the three vectors that produces vector b is 2 times the third vector.

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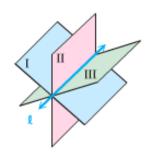
$$2(3, 2, 1) = (6, 4, 2) = b$$

Therefore, the coefficients that we need are x = 0, y = 0, and z = 2.

$$0\begin{bmatrix} 1\\2\\6 \end{bmatrix} + 0\begin{bmatrix} 2\\5\\-3 \end{bmatrix} + 2\begin{bmatrix} 3\\2\\1 \end{bmatrix} = \begin{bmatrix} 6\\4\\2 \end{bmatrix}$$



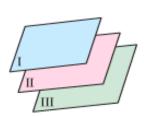




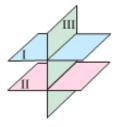
Points of a line in common



All points in common



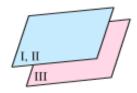
No points in common



No points in common



No points in common



No points in common

# **Exercises** Section 1.1 – Introduction to System of Linear Equations

1. Find a solution for x, y, z to the system of equations

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3e + 2\sqrt{2} + \pi \\ 6e + 5\sqrt{2} + 4\pi \\ 9e + 8\sqrt{2} + 7\pi \end{pmatrix}$$

- 2. Draw the two pictures in two planes for the equations: x 2y = 0, x + y = 6
- 3. Normally 4 planes in 4-dimensional space meet at a \_\_\_\_\_\_. Normally 4 column vectors in 4-deimensional space can combine to produce b. what combinations of (1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1) produces b = (3, 3, 3, 2)? What 4 equations for x, y, z, w are you solving?
- **4.** What 2 by 2 matrix A rotates every vector through 45°? The vector (1, 0) goes to  $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ . The vector (0, 1) goes to  $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ .

Those determine the matrix. Draw these particular vectors is the xy-plane and find A.

5. What two vectors are obtained by rotating the plane vectors  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  by 30° (cw)?

Write a matrix A such that for every vector  $\vec{v}$  in the plane,  $A\vec{v}$  is the vector obtained by rotating  $\vec{v}$  clockwise by 30°.

Find a matrix B such that for every 3-dimensional vector  $\vec{v}$ , the vector  $B\vec{v}$  is the reflection of  $\vec{v}$  through the plane x + y + z = 0. Hint: v = (1, 0, 0)

6. In each part, find a system of linear equation corresponding to the given augmented matrix

a) 
$$\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$$

$$b) \begin{bmatrix} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \\ 5 & -6 & 1 & 1 \\ -8 & 0 & 0 & 3 \end{bmatrix}$$

7. Find the augmented matrix for the given system of linear equations.

a) 
$$\begin{cases} -2x_1 = 6 \\ 3x_1 = 8 \\ 9x_1 = -3 \end{cases}$$

c) 
$$\begin{cases} 2x_1 + 2x_3 = 1 \\ 3x_1 - x_2 + 4x_3 = 7 \\ 6x_1 + x_2 - x_3 = 0 \end{cases}$$

$$b) \begin{cases} 3x_1 - 2x_2 = -1 \\ 4x_1 + 5x_2 = 3 \\ 7x_1 + 3x_2 = 2 \end{cases}$$

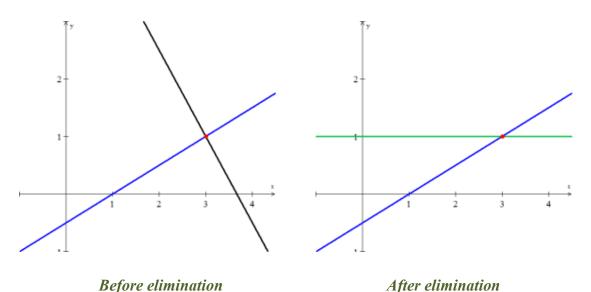
# Section 1.2 - Gaussian Elimination

Elimination produces an upper triangular system.

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases} \Rightarrow \begin{cases} x - 2y = 1 & Mutliply by 3 \\ 8y = 8 & and subtract \end{cases}$$

The equation 8y = 8 reveals y = 1

This process is called back substitution.



# Definitions

**Pivot**: first nonzero in the row that does the elimination

Multiplier: (entry to eliminate) divide by pivot

$$4x-8y=4$$
 Multiply equation 1 by  $\frac{3}{4}$   $4x-8y=4$   
 $3x+2y=11$  Subtract from equation 2  $8y=8$ 

The first pivot is 4 (the coefficient of x) and the multiplier is  $l = \frac{3}{4}$ 

The pivots are on the diagonal of the triangle after elimination.

# Definition

The operations are the elementary reduction operations, or row operations, or Gaussian operations. They are swapping, multiplying by a scalar or rescaling, and pivoting.

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#### Reduced Row Echelon Form

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

### **Example**

Use the Gaussian elimination method to solve the system

$$3x + y + 2z = 31$$
  
 $x + y + 2z = 19$   
 $x + 3y + 2z = 25$ 

#### **Solution**

$$\begin{bmatrix}
1 & 1 & 2 & | 19 \\
3 & 1 & 2 & | 31 \\
1 & 3 & 2 & | 25
\end{bmatrix}
\begin{matrix}
R_2 - 3R_1 \\
R_3 - R_1
\end{matrix}
\qquad
\begin{matrix}
-3 & -3 & -6 & -57 \\
0 & -2 & -4 & -26
\end{matrix}
\qquad
\begin{matrix}
-1 & -1 & -2 & -19 \\
0 & 2 & 0 & 6
\end{matrix}$$

$$\begin{bmatrix}
1 & 1 & 2 & | 19 \\
0 & -2 & -4 & | -26 \\
0 & 2 & 0 & 6
\end{bmatrix}
\begin{matrix}
-\frac{1}{2}R_2
\end{matrix}
\qquad
0 & 1 & 2 & 13 \\
0 & 2 & 0 & 6
\end{matrix}$$

$$\begin{bmatrix}
1 & 1 & 2 & | 19 \\
0 & 1 & 2 & | 13 \\
0 & 2 & 0 & 6
\end{bmatrix}
\begin{matrix}
0 & 2 & 0 & 6 \\
0 & -2 & -4 & -26 \\
0 & 0 & -4 & -20
\end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -4 & -20 \end{bmatrix} \quad 0 \quad 0 \quad 1 \quad 5$$

$$\begin{bmatrix} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{bmatrix} \Rightarrow \begin{array}{c} x + y + 2z = 19 & (3) \\ y + 2z = 13 & (2) \\ z = 5 & (1) \end{array}$$

(2) 
$$\Rightarrow y = 13 - 2z = 13 - 2(5) = 3$$
  
(3)  $\Rightarrow x = 19 - y - 2z = 19 - 3 - 10 = 6$   
 $\Rightarrow$  (6, 3, 5)

### **Example**

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$

$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1$$

$$5x_3 + 10x_4 + 15x_6 = 5$$

$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6$$

#### Solution

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & | & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & | & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & | & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & | & 6 \end{bmatrix} \quad R_2 - 2R_1 \quad Adding \ \ \begin{pmatrix} -2 \end{pmatrix} \ times \ the \ 1st \ row \ to \ the \ 4th$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix} -R_2$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & | & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & | & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & | & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & | & 6 \end{bmatrix} \begin{array}{c} R_3 - 5R_2 \\ R_4 - 4R_2 \end{array}$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_2 - 3R_3$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & | & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & | & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \longrightarrow \begin{cases} x_1 + 3x_2 & +4x_4 + 2x_5 & = 0 \\ x_3 + 2x_4 & = 0 \\ & +x_6 = \frac{1}{3} \end{cases}$$

The general solution of the system:  $\left(-3x_2 - 4x_4 - 2x_5, x_2, -2x_4, x_4, x_5, \frac{1}{3}\right)$ 

## **Example**

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$2x + 8y - z + w = 0$$

$$4x + 16y - 3z - w = -10$$

$$-2x + 4y - z + 3w = -6$$

$$-6x + 2y + 5z + w = 3$$

#### **Solution**

$$\begin{bmatrix} 2 & 8 & -1 & 1 & 0 \\ 4 & 16 & -3 & -1 & -10 \\ -2 & 4 & -1 & 3 & -6 \\ -6 & 2 & 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 26 & 2 & 4 & 3 \end{bmatrix} R_4 - \frac{13}{6} R_2$$

$$\begin{bmatrix} 2 & 8 & -1 & 1 & 0 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 26 & 2 & 4 & 3 \end{bmatrix}$$
 Interchange  $R_2$  and  $R_3$ 

$$\begin{bmatrix} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & \frac{19}{3} & -\frac{14}{3} & 16 \end{bmatrix} R_4 + \frac{19}{3}R_3$$

$$\begin{bmatrix} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & 0 & -\frac{71}{3} & -\frac{142}{3} \end{bmatrix} \xrightarrow{2x+8y-z+w=0} \xrightarrow{2x=-8y+z-w=6} \Rightarrow \boxed{x=3}$$

$$2x+8y-z+w=0 \rightarrow 2x=-8y+z-w=6 \Rightarrow \boxed{x=3}$$

$$12y-2z+4w=-6 \rightarrow 12y=2z-4w-6=-6 \Rightarrow \boxed{y=-\frac{1}{2}}$$

$$-z-3w=-10 \rightarrow \boxed{z=10-3w=4}$$

$$-\frac{71}{3}w=-\frac{142}{3} \rightarrow \boxed{w=2}$$

**Solution**:  $(3, -\frac{1}{2}, 4, 2)$ 

#### **Theorem:** Free Variable Theorem for Homogeneous Systems

If a *homogeneous linear* system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has n - r free variables.

#### **Theorem**

A homogeneous linear system with more unknowns than equations has infinitely many unknowns.

#### **Breakdown Elimination**

#### Permanent failure with no solution

$$x-2y=1$$
 Subtract 3 times  $x-2y=1$   
 $3x-6y=11$  eqn. 1 from eqn. 2  $0y=8$ 

The last equation 0y = 8; therefore, there is *no* solution.

This system has no second pivot, since no zero allowed as a pivot.

#### Permanent failure with infinitely many solutions

$$x-2y=1$$
 Subtract 3 times  $x-2y=1$   
 $3x-6y=3$  eqn. 1 from eqn. 2  $0y=0$ 

Every y satisfies 0y = 0. There is only one equation x - 2y = 1.

There are *unique infinitely* many solutions.

#### **Three Equations in Three Unknowns**

To understand Gaussian elimination, you have to go beyond 2 by 2 systems.

Consider the system equations: 
$$\begin{cases} 2x + 4y - 2z = 2 \\ 4x + 9y - 3z = 8 \\ -2x - 3y + 7z = 10 \end{cases}$$
$$\begin{cases} 2x + 4y - 2z = 2 & \text{subtract 2 times eqn.} 1 \end{cases}$$

$$\begin{cases} 2x + 4y - 2z = 2 & subtract \ 2 \text{ times eqn.} 1 & 2x + 4y - 2z = 2 \\ 4x + 9y - 3z = 8 & from \ eqn. 2 & y + z = 4 \\ -2x - 3y + 7z = 10 & -2x - 3y + 7z = 10 \end{cases}$$

$$\begin{cases} 2x + 4y - 2z = 2 & Add \ eqn. 1 & 2x + 4y - 2z = 2 \\ y + z = 4 & y + z = 4 \\ -2x - 3y + 7z = 10 & and \ eqn. 3 & y + 5z = 12 \end{cases}$$

$$\begin{cases} 2x + 4y - 2z = 2 & \Rightarrow |\underline{x} = 1 - 2y + z = \underline{1}| \\ y + z = 4 & Subtract \ eqn.2 & y + z = 4 \Rightarrow |\underline{y} = 4 - z = \underline{2}| \\ y + 5z = 12 & from \ eqn.3 & 4z = 8 \Rightarrow |\underline{z} = \underline{2}| \end{cases}$$

The solution is (-1, 2, 2)

# Definition

A square matrix is nonsingular if it is the matrix of coefficient of a homogeneous system, with a unique solution. It is singular otherwise, that is, if it is the matrix of coefficients of a homogeneous system, with infinitely many solutions.

# **Exercises** Section 1.2 – Gaussian Elimination

- 1. When elimination is applied to the matrix  $A = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix}$ 
  - a) What are the first and second pivots?
  - b) What is the multiplier  $l_{21}$  in the first step ( $l_{21}$  times row 1 is subtracted from row 2)?
  - c) What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3?
  - d) What is the multiplier  $l_{31} = 0$ , subtracting 0 times row 1 from row 3?
- 2. Use elimination to reach upper triangular matrices U. Solve by back substitution or explain why this impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the -x in equation (3).

$$\begin{cases} x + y + z = 7 \\ x + y - z = 5 \\ x - y + z = 3 \end{cases} \begin{cases} x + y + z = 7 \\ x + y - z = 5 \\ -x - y + z = 3 \end{cases}$$

3. For which numbers a does the elimination break down (1) permanently (2) temporarily

$$ax + 3y = -3$$
$$4x + 6y = 6$$

Solve for x and y after fixing the second breakdown by a row change.

**4.** Find the pivots and the solution for these four equations:

$$2x + y = 0$$

$$x + 2y + z = 0$$

$$y + 2z + t = 0$$

$$z + 2t = 5$$

**5.** Look for a matrix that has row sums 4 and 8, and column sums 2 and s.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{array}{c} a+b=4 & a+c=2 \\ c+d=8 & b+d=s \end{array}$$

The four equations are solvable only if s =\_\_\_\_. Then find two different matrices that have the correct row and column sums.

6. Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of A is a \_\_\_\_\_ of the first two rows. Find a third equation that can't be solved together with x + y + z = 0 and x - 2y - z = 1

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(7-14) Use the Gauss-Jordan method to solve the system

7. 
$$\begin{cases} x - y + 5z = -6 \\ 3x + 3y - z = 10 \\ x + 3y + 2z = 5 \end{cases}$$

10. 
$$\begin{cases} x + 2y - 3z = -15 \\ 2x - 3y + 4z = 18 \\ -3x + y + z = 1 \end{cases}$$

13. 
$$\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$

7. 
$$\begin{cases} x - y + 5z = -6 \\ 3x + 3y - z = 10 \\ x + 3y + 2z = 5 \end{cases}$$
8. 
$$\begin{cases} 2x - y + 4z = -3 \\ x - 2y - 10z = -6 \\ 3x + 4z = 7 \end{cases}$$

$$\begin{cases} 4x + 3y - 5z = -29 \end{cases}$$

10. 
$$\begin{cases} x + 2y - 3z = -15 \\ 2x - 3y + 4z = 18 \\ -3x + y + z = 1 \end{cases}$$
11. 
$$\begin{cases} x + 2y + 3z = 10 \\ 4x + 5y + 6z = 11 \\ 7x + 8y + 9z = 12 \end{cases}$$
12. 
$$\begin{cases} 2x + y + 2z = 4 \\ 2x + 2y = 5 \\ 2x - y + 6z = 2 \end{cases}$$

14. 
$$\begin{cases} x + 2y + z = 8 \\ -x + 3y - 2z = 1 \\ 3x + 4y - 7z = 10 \end{cases}$$

9. 
$$\begin{cases} 4x + 3y - 5z = -29 \\ 3x - 7y - z = -19 \\ 2x + 5y + 2z = -10 \end{cases}$$

12. 
$$\begin{cases} 2x + y + 2z = 4 \\ 2x + 2y = 5 \\ 2x - y + 6z = 2 \end{cases}$$

(15-49) Use augmented elimination to solve linear system

15. 
$$\begin{cases} 2x - 5y + 3z = 1 \\ x - 2y - 2z = 8 \end{cases}$$

nation to solve linear system
$$\begin{aligned}
-2x + 6y + 7z &= 3 \\
-4x + 5y + 3z &= 7 \\
-6x + 3y + 5z &= -4
\end{aligned}$$
23. 
$$\begin{cases}
2x - y + z &= 1 \\
3x - 3y + 4z &= 5 \\
4x - 2y + 3z &= 4
\end{cases}$$
24. 
$$\begin{cases}
3x - 4y + 4z &= 7 \\
x - y - 2z &= 2 \\
2x - 3y + 6z &= 5
\end{cases}$$
25. 
$$\begin{cases}
x - 2y - z &= 2 \\
2x - y + z &= 4 \\
-x + y + z &= 4
\end{cases}$$
26. 
$$\begin{cases}
x + y + z &= 3 \\
-y + 2z &= 1 \\
-x + z &= 0
\end{cases}$$
27. 
$$\begin{cases}
3x + y + 3z &= 14 \\
7x + 5y + 8z &= 37 \\
x + 3y + 2z &= 9
\end{cases}$$
28. 
$$\begin{cases}
4x - 2y + z &= 7 \\
x + y + z &= -2 \\
4x + 2y + z &= 3
\end{cases}$$

29. 
$$\begin{cases} 2x - 2y + z = -4 \\ 6x + 4y - 3z = -24 \\ x - 2y + 2z = 1 \end{cases}$$

16. 
$$\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$$

23. 
$$\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$$

30. 
$$\begin{cases} 9x + 3y + z = 4 \\ 16x + 4y + z = 2 \\ 25x + 5y + z = 2 \end{cases}$$

17. 
$$\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$$

24. 
$$\begin{cases} 3x - 4y + 4z = 7 \\ x - y - 2z = 2 \\ 2x - 3y + 6z = 5 \end{cases}$$

31. 
$$\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$$

18. 
$$\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 1 \end{cases}$$

25. 
$$\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$$

32. 
$$\begin{cases} x - 3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$$

19. 
$$\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$$

26. 
$$\begin{cases} x + y + z = 3 \\ -y + 2z = 1 \\ -x + z = 0 \end{cases}$$

33. 
$$\begin{cases} x + 2y - z = 5 \\ 2x - y + 3z = 0 \\ 2y + z = 1 \end{cases}$$

15. 
$$\begin{cases} 2x - 5y + 3z = 1 \\ x - 2y - 2z = 8 \end{cases}$$
16. 
$$\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$$
17. 
$$\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$$
18. 
$$\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$$
19. 
$$\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$$
20. 
$$\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$$
21. 
$$\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$$

27. 
$$\begin{cases} 3x + y + 3z = 14 \\ 7x + 5y + 8z = 37 \\ x + 3y + 2z = 9 \end{cases}$$

34. 
$$\begin{cases} x + y + z = 6 \\ 3x + 4y - 7z = 1 \\ 2x - y + 3z = 5 \end{cases}$$

21. 
$$\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$$

28. 
$$\begin{cases} 4x - 2y + z = 7 \\ x + y + z = -2 \\ 4x + 2y + z = 3 \end{cases}$$

35. 
$$\begin{cases} 3x + 2y + 3z = 3 \\ 4x - 5y + 7z = 1 \\ 2x + 3y - 2z = 6 \end{cases}$$

36. 
$$\begin{cases} x - 3y + z = 2 \\ 4x - 12y + 4z = 8 \\ -2x + 6y - 2z = -4 \end{cases}$$
37. 
$$\begin{cases} 2x - 2y + z = -1 \\ x + 2y - z = 2 \\ 6x + 4y + 3z = 5 \end{cases}$$

37. 
$$\begin{cases} 2x - 2y + z = -1 \\ x + 2y - z = 2 \\ 6x + 4y + 3z = 5 \end{cases}$$

38. 
$$\begin{cases} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 \\ x_2 - 3x_3 - x_4 = 0 \\ 3x_1 + 2x_3 - x_4 = 6 \\ -4x_1 + x_2 + 4x_3 + 2x_4 = -3 \end{cases}$$

39. 
$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 5 \\ x_1 + 2x_2 - x_3 - 2x_4 = -1 \\ x_1 - 3x_2 - 3x_3 - x_4 = -1 \\ 2x_1 - x_2 + 2x_3 - x_4 = -2 \end{cases}$$

40. 
$$\begin{cases} 2x + 8y - z + w = 0 \\ 4x + 16y - 3z - w = -10 \\ -2x + 4y - z + 3w = -6 \\ -6x + 2y + 5z + w = 3 \end{cases}$$

41. 
$$\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

42. 
$$\begin{cases} 2x + 2y + 4z = 0 \\ -y - 3z + w = 0 \\ 3x + y + z + 2w = 0 \\ x + 3y - 2z - 2w = 0 \end{cases}$$

43. 
$$\begin{cases} 2x + z + w = 5 \\ y - w = -1 \\ 3x - z - w = 0 \\ 4x + y + 2z + w = 9 \end{cases}$$

44. 
$$\begin{cases} 4y + z = 20 \\ 2x - 2y + z = 0 \\ x + z = 5 \\ x + y - z = 10 \end{cases}$$

45. 
$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x - 3w = -3 \end{cases}$$

46. 
$$\begin{cases} 2u - 3v + w - x + y = 0 \\ 4u - 6v + 2w - 3x - y = -5 \\ -2u + 3v - 2w + 2x - y = 3 \end{cases}$$

$$47. \begin{cases} 6x_3 + 2x_4 - 4x_5 - 8x_6 = 8 \\ 3x_3 + x_4 - 2x_5 - 4x_6 = 4 \\ 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 + x_6 = 2 \\ 6x_1 - 9x_2 + 11x_4 - 19x_5 + 3x_6 = 1 \end{cases}$$

48. 
$$\begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases}$$

$$49. \begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ 5x_3 + 10x_4 + 15x_6 = 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$$

**50.** Add 3 times the second row to the first of

$$\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

51. For what value(s) of 
$$k$$
, if any, does the system 
$$\begin{cases} x + y - z = 1 \\ 2x + 3y + kz = 3 \\ x + ky + 3z = 2 \end{cases}$$
 have

- a) A unique solution?
- b) Infinitely many solutions?
- c) No solution?
- **52.** Choose a coefficient b that makes the system singular.

$$\begin{cases} 3x + 4y = 16 \\ 4x + by = g \end{cases}$$

Then choose a right-hand side g that makes it solvable.

Find 2 solutions in that singular case.

53. This system us not linear, in some sense,

$$\begin{cases} 2\sin\alpha - \cos\beta + 3\tan\theta = 3\\ 4\sin\alpha + 2\cos\beta - 2\tan\theta = 10\\ 6\sin\alpha - 3\cos\beta + \tan\theta = 9 \end{cases}$$

Does the system have a solution?

# Section 1.3 – The Algebra of Matrices

#### **Matrices**

This is called Matrix (Matrices)

Each number in the array is an element or entry

The matrix is said to be of order  $m \times n$ 

*m*: numbers of rows,

*n*: number of columns

When m = n, then matrix is said to be **square**.

Given the system equations

$$3x + y + 2z = 31$$

$$x + y + 2z = 19$$

$$x + 3y + 2z = 25$$

Write into an augmented matrix form

The Matrix:  $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}$  is called the *coefficient matrix* of the system.

The matrix A above has 3 rows and 3 columns, therefore the order of the matrix A is  $(3 \times 3)$ 

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

#### **Equality of Matrices**

#### **Definition of Equality of Matrices**

Two matrices  $\boldsymbol{A}$  and  $\boldsymbol{B}$  are equal if and only if they have the same order (size)  $m \times n$  and if each pair corresponding elements is equal

$$a_{ij} = b_{ij}$$
 for  $i = 1, 2, ..., m$  and  $j = 1, 2, ..., n$ 

# **Example**

Find the values of the variables for which each statement is true, if possible.

a) 
$$\begin{bmatrix} 2 & 1 \\ p & q \end{bmatrix} = \begin{bmatrix} x & y \\ -1 & 0 \end{bmatrix}$$
$$x = 2, y = 1, p = -1, q = 0$$

$$b) \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

can't be true

c) 
$$\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} w = 9 & x = 17 \\ 8 = y & -12 = z \end{bmatrix}$$

#### Addition and Subtraction of Matrices

#### **Definition**

If  $A = \begin{bmatrix} a_{ij} \end{bmatrix}$  and  $B = \begin{bmatrix} b_{ij} \end{bmatrix}$  are  $m \times n$  matrices, their sum A + B, is the  $m \times n$  matrix obtained by adding the corresponding entries; that is

$$\left[ a_{ij} \right] + \left[ b_{ij} \right] = \left[ a_{ij} + b_{ij} \right]$$

Matrices can be added if their shapes are the same, meaning have the same order.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 4 & 4 \\ 9 & 9 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+2 \\ 3+4 & 4+4 \\ 0+9 & 0+9 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 4 \\ 7 & 8 \\ 9 & 9 \end{bmatrix}$$

### **Scalar** Multiplication Matrices

#### **Definition**

If k is a scalar and  $A = \begin{bmatrix} a_{ij} \end{bmatrix}$  is an  $m \times n$  matrices, then scalar product kA is the  $m \times n$  matrix obtained by multiplying each entry of A by k; that is

$$k \left[ a_{ij} \right] = \left[ k a_{ij} \right]$$

$$kA = k \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
$$= \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$$

# Example

$$\begin{bmatrix}
1 & 2 \\
3 & 4 \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
(2)1 & (2)2 \\
(2)3 & (2)4 \\
(2)0 & (2)0
\end{bmatrix}$$

$$= \begin{bmatrix}
2 & 4 \\
6 & 8 \\
0 & 0
\end{bmatrix}$$

### **Definition**

If  $A_1, A_2, ..., A_n$  are matrices of the same size, and if  $c_1, c_2, ..., c_n$  are scalars, then expression of the form

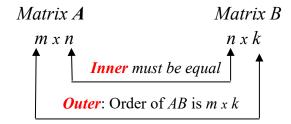
$$c_1 A_1 + c_2 A_2 + \dots + c_n A_n$$

Is called a *linear combination* of  $A_1, A_2, ..., A_n$  with coefficients  $c_1, c_2, ..., c_n$ .

# Matrix Multiplication

#### **Product of Two Matrices**

Let A be an  $m \times n$  matrix and let B be an  $n \times k$  matrix. To find the element in the  $i^{th}$  row and  $j^{th}$  column of the product matrix AB, multiply each element in the  $i^{th}$  row of A by the corresponding element in the  $j^{th}$  column of B, and then add these products. The product matrix AB is an  $m \times k$  matrix.



- $\checkmark$  To multiply AB or dot product, if A has n columns, B must have n rows.
- ✓ Squares matrices can be multiplied if and only if (*iff*) they have the same size.
- ✓ The entry in row *i* and column *j* of AB is (row i of A).(col j of B)

The result: 
$$\sum a_{ik}b_{kj}$$

$$\begin{bmatrix} * & * & * & & & & & & \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{i5} \end{bmatrix} \begin{bmatrix} * & * & b_{1j} & * & * & * \\ & b_{2j} & & & & \\ & \vdots & & & & \\ & & \vdots & & & \\ & & b_{5j} & & & \end{bmatrix} = \begin{bmatrix} * & * & & & & & \\ * & * & (AB)_{ij} & * & * & * \\ & & * & & & \\ & & * & & & \end{bmatrix}$$

$$4 \ bv \ 5 \qquad 5 \ bv \ 6 \qquad 4 \ bv \ 6$$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$2x2 \quad 2x2 \quad \rightarrow \quad 2x2$$

$$a_{11} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & - \\ - & - \end{bmatrix}$$

$$a_{12} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} - & af + bh \\ - & - \end{bmatrix}$$

$$a_{21} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} - & - \\ - & ce + dg \end{bmatrix}$$

$$a_{22} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} - & - \\ - & cf + dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

### Example

Find: 
$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}$$

#### Solution

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1(5) + 1(1) & 1(6) + 1(0) \\ 2(5) - 1(1) & 2(6) - 1(0) \end{bmatrix}$$
$$= \begin{bmatrix} 6 & 6 \\ 9 & 12 \end{bmatrix}$$

### Special Case

When A is a square matrix, then

A times 
$$A = A^2$$
 times  $A = A^3$ 

$$A^p = AA \cdots A \quad (p \text{ factors})$$

$$(A^p)(A^q) = A^{p+q}$$

$$(A^p)^q = A^{pq}$$

### **Block Multiplication**

If the cuts between columns of A match the cuts between rows of B, then the block multiplication of ABallowed.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{21} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{21} + a_{22}b_{22} \end{bmatrix}$$

#### Important special case

$$\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix}$$

### **Matrix Form of the Equations**

The coefficient matrix is  $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}$ 

The equivalent matrix equation is in the form AX = b:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Multiplication by *rows* 

$$AX = \begin{bmatrix} (row\ 1).X \\ (row\ 2).X \\ (row\ 3).X \end{bmatrix}$$

Multiplication by *columns* 
$$AX = x$$
 (*column* 1) + y (*column* 2) + z (*column* 3)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

#### **Identity Matrix**

The identity matrix is given by the form: 
$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \boxed{Ix = x}$$

# **Properties of Matrix**

#### **Addition and Scalar Multiplication**

$$A + B = B + A$$
 Commutative Property of Addition

$$A + (B + C) = (A + B) + C$$
 Associative Property of Addition

$$(kl)A = k(lA)$$
 Associative Property of Scalar Multiplication

$$k(A + B) = kA + kB$$
 Distributive Property

$$k(A-B) = kA - kB$$
 Distributive Property

$$(k+l)A = kA + lA$$
 Distributive Property

$$(k-l)A = kA - lA$$
 Distributive Property

$$A + 0 = 0 + A = A$$
 Additive Identity Property

$$A + (-A) = (-A) + A = 0$$
 Additive Inverse Property

$$k(AB) = kA(B) = A(kB)$$

### Multiplication

$$AB \neq BA$$
 Commutative "law" is usually broken

$$A(BC) = (AB)C$$
 Associative Property of Multiplication (Parentheses not needed)

$$A(B+C) = AB + AC$$
 Distributive Property

$$(B+C)A = BA + CA$$
 Distributive Property

$$A(B-C) = AB - AC$$
 Distributive Property

$$(B-C)A = BA - CA$$
 Distributive Property

\*\*\*\*\*\*

Consider the three vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$ :

$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The linear combinations in three-dimensional space are cu + dv + ew

**Combination** 
$$c \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d - c \\ e - d \end{bmatrix}$$

Combine the three vectors u, v, and w into on matrix A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Multiplies the matrix A by a vector x, where c, d, e are the component of a vector x.

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d - c \\ e - d \end{bmatrix}$$

We can rewrite the form, matrix A times the vector x, as the combination cu + dv + ew

$$Ax = \begin{bmatrix} u & v & w \\ d & e \end{bmatrix} = c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$$

Write the matrix in the form Ax = b

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d - c \\ e - d \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b$$

Where the *x* is the input and *b* is the output.

#### **Cyclic Difference**

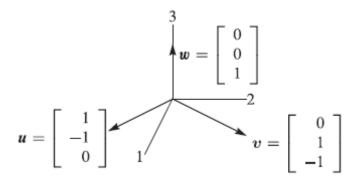
The linear combinations of three vectors u, v, and  $w^*$  lead to a cyclic difference matrix C and is given by:

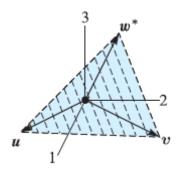
$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad w^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = b$$

The matrix C is not triangular. It is not easy to find the solution to Cx = b, because either we are going to have *infinitely many solution* or *no solution*.

Let looks at these problems geometrically.





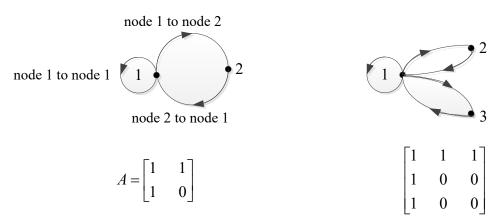
# **Exercises** Section 1.3 – The Algebra of Matrices

- 1. For the matrices:  $A = \begin{bmatrix} p & 0 \\ q & r \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ , when does AB = BA
- (2-8) Find values for the variables so that the matrices are equal.
- $\mathbf{2.} \quad \begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$
- $3. \quad \begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$
- 4.  $\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$
- 5.  $\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$
- **6.**  $\begin{bmatrix} a+11 & 12z+1 & 5m \\ 11k & 3 & 1 \end{bmatrix} + \begin{bmatrix} 9a & 9z & 4m \\ 12k & 5 & 3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$
- 7.  $\begin{bmatrix} x+2 & 3y+1 & 5z \\ 8w & 2 & 3 \end{bmatrix} + \begin{bmatrix} 3x & 2y & 5z \\ 2w & 5 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$
- **8.**  $\begin{bmatrix} 2x-3 & y-2 & 2z+1 \\ 5 & 2w & 7 \end{bmatrix} + \begin{bmatrix} 3x-3 & y+2 & z-1 \\ -5 & 5w+1 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$
- 9. Find a combination  $x_1 w_1 + x_2 w_2 + x_3 w_3$  that gives the zero vector:

$$w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \quad w_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}; \quad w_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

- Those vectors are independent or dependent?
- The vectors lie in a \_\_\_\_\_.
- The matrix W with those columns is not invertible.
- 10. The very last words say that the 5 by 5 centered difference matrix is not invertible, Write down the 5 equations Cx = b. Find a combination of left sides that gives zero. What combination of  $b_1$ ,  $b_2$ ,  $b_3$ ,  $b_4$ ,  $b_5$  must be zero?

11. A direct graph starts with n nodes. There are  $n^2$  possible edges, each edge leaves one of the n nodes and enters one of the n nodes (possibly itself). The n by n adjacency matrix has  $a_{ij} = 1$  when edge leaves node i and enter node j; if no edge then  $a_{ij} = 0$ . Here are directed graphs and their adjacency matrices:



The i, j entry of  $A^2$  is  $a_{i1}a_{1j} + ... + a_{in}a_{nj}$ .

Why does that sum count the two-step paths from i to any node to j?

The i, j entry of  $A^k$  counts k-steps paths:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{array}{c} counts \ the \ paths \\ with \ two \ edges \end{array} \quad \begin{bmatrix} 1 \ to \ 2 \ to \ 1, 1 \ to \ 1 \ to \ 1 \\ 2 \ to \ 1 \ to \ 2 \end{bmatrix}$$

List all 3-step paths between each pair of nodes and compare with  $A^3$ . When  $A^k$  has **no zeros**, that number k is the diameter of the graph – the number of edges needed to connect the most pair of nodes. What is the diameter of the second graph?

- **12.** A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?
  - a) AB
- *b*) *BA*
- c) ABD
- d) DBA

- e) ABC
- f) ABCD
- g) A(B+C)
- **13.** What rows or columns or matrices do you multiply to find.
  - a) The third column of AB?
  - b) The second column of AB?
  - c) The first row of AB?
  - d) The second row of AB?
  - e) The entry in row 3, column 4 of AB?
  - f) The entry in row 2, column 3 of AB?

**14.** Add AB to AC and compare with A(B+C):

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad and \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

15. True or False

- a) If  $A^2$  is defined then A is necessarily square.
- b) If AB and BA are defined then A and B are squares.
- c) If AB and BA are defined then AB and BA are squares.
- d) If AB = B, then A = I

**16.** a) Find a nonzero matrix A such that  $A^2 = 0$ 

b) Find a matrix that has  $A^2 \neq 0$  but  $A^3 = 0$ 

17. Suppose you solve Ax = b for three special right sides b:

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If the three solutions  $x_1, x_2, x_3$  are the columns of a matrix X, what is A times X?

**18.** Show that  $(A+B)^2$  is different from  $A^2 + 2AB + B^2$ , when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

Write down the correct rule for  $(A+B)(A+B) = A^2 + \underline{\hspace{1cm}} + B^2$ 

(19-22) Find the product of the 2 matrices by rows or by columns:

**19.** 
$$\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

**21.** 
$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

**20.** 
$$\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{array}{cccc}
\mathbf{22.} & \begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

**23.** Given  $A = \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$   $B = \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix}$  Find A + B, 2A, and -B

(24-37) Find AB and BA, if possible

**24.** 
$$A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}$$

**25.** 
$$A = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$$
  $B = \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}$ 

**26.** 
$$A = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$$
  $B = \begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}$ 

**27.** 
$$A = \begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}$$
  $B = \begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}$ 

**28.** 
$$A = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}$$
  $B = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$ 

**29.** 
$$A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}$$
  $B = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$ 

**30.** 
$$A = \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}$$
  $B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$ 

**31.** 
$$A = \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix}$$
  $B = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix}$ 

**32.** 
$$A = \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix}$$
  $B = \begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ 

**33.** 
$$A = \begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix}$$
  $B = \begin{bmatrix} 4 & -2 \\ -2 & 0 \\ 9 & 1 \end{bmatrix}$ 

**34.** 
$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$
  $B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$ 

**35.** 
$$A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$
  $B = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$ 

**36.** 
$$A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix}$$
  $B = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ 

37. 
$$A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}$$
  $B = \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}$ 

#### Consider the matrices **38.**

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \qquad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Compute the following (where possible):

a) 
$$D + E$$

c) 
$$5A$$
 d)  $-7C$ 

$$e)$$
  $2B-C$ 

e) 
$$2B - C$$
 f)  $-3(D + 2E)$ 

#### **39.** Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} -3 & -1 \\ 2 & 1 \\ 4 & 3 \end{bmatrix} \qquad D = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$$

Compute the following (where possible):

a) 
$$A+B$$
 b)  $A+C$ 

b) 
$$A+C$$

g) 
$$BD$$
 h)  $DB$  i)  $A^2$ 

i) 
$$A^2$$

$$j) B^2$$

$$k) D^2$$

**40.** Let 
$$B = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$
, show that  $B^4 = \begin{pmatrix} a^4 & 0 \\ a^3 + a^2b + ab^2 + b^3 & b^4 \end{pmatrix}$ 

**41.** Let 
$$B = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$
, show that  $B^n = \begin{pmatrix} a^n & 0 \\ \sum_{k=0}^{n-1} a^{n-1-k} b^k & b^n \end{pmatrix}$ 

**42.** Let 
$$A = \begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix}$$
. Prove that  $A^n = \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$  if  $n \ge 1$ 

**43.** Let 
$$A = \begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix}$$
. Prove that  $A^n = \begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix}$  if  $n \ge 1$ 

**44.** The following system of recurrence relations holds for all  $n \ge 0$ 

$$\begin{cases} x_{n+1} = 7x_n + 4y_n \\ y_{n+1} = -9x_n - 5y_n \end{cases}$$

Solve the system for  $x_n$  and  $y_n$  in terms of  $x_0$  and  $y_0$ 

**45.** If 
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
, prove that  $A^2 - (a+d)A + (ad-bc)I_{2\times 2} = 0$ 

**46.** If 
$$A = \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}$$
, use the fact  $A^2 = 4A - 3I$  and mathematical induction, to prove that 
$$A^n = \frac{3^n - 1}{2}A + \frac{3 - 3^n}{2}I \quad \text{if} \quad n \ge 1$$

**47.** A sequence of numbers  $x_1, x_2, ..., x_n, ...$  satisfies the recurrence relation  $x_{n+1} = ax_n + bx_{n-1}$  for  $n \ge 1$ , where a and b are constants. Prove that

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = A \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

Where  $A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$  and hence express  $\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$  in terms of  $\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$ .

If a = 4 and b = -3, use the previous question to find a formula for  $x_n$  in terms  $x_1$  and  $x_0$ 

# Section 1.4 – Inverse Matrices - Finding $A^{-1}$

### **Definition**

The matrix A is invertible if there exists a matrix  $A^{-1}$  such that:

$$A^{-1}A = AA^{-1} = I$$

where  $A^{-1}$  read as "A inverse" and A has to be a square matrix.

#### Not all matrices have inverses.

- 1. The inverse exists *iff* elimination produces *n* pivots (row exchanges allow).
- **2.** The matrix A cannot have two different inverses.
- **3.** If A is invertible, the one and only one solution to Ax = B is  $x = A^{-1}B$

$$AX = B$$

$$A^{-1}(AX) = A^{-1}B$$

$$Associate property$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

$$X$$

- **4.** Suppose there is a *nonzero* vector x such that Ax = 0. Then A cannot have an inverse
- **5.** A 2 by 2 matrix is invertible iff ad bc is not zero.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad Only for 2 by 2 matrices$$

If ad - bc = 0 is the determinant, then  $A^{-1}$  doesn't exist

#### The Inverse of a Product AB

#### **Theorem**

If an  $n \times n$  matrix has an inverse, that inverse is unique.

### Proof

Suppose that A has an inverse  $A^{-1}$  and B is a matrix such that BA = I

$$B = BI$$

$$= B \left( AA^{-1} \right)$$

$$= \left( BA \right) A^{-1}$$

$$= IA^{-1}$$

$$= A^{-1}$$

Therefore, the inverse is unique

#### **Theorem**

If A and B are invertible then so is AB. The inverse of a product AB is  $(AB)^{-1} = B^{-1}A^{-1}$ 

#### **Proof**

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$
$$= (AI)A^{-1}$$
$$= AA^{-1}$$
$$= I$$

#### Reverse Order

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$
  
 $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$ 

#### **Theorem**

If A is invertible and n is a nonnegative integer, then:

- a)  $A^{-1}$  is invertible and  $(A^{-1})^{-1} = A$
- b)  $A^n$  is invertible and  $\left(A^n\right)^{-1} = A^{-n} = \left(A^{-1}\right)^n$
- c) kA is invertible for any nonzero scalar k, and  $(kA)^{-1} = k^{-1}A^{-1}$

#### **Proof**

$$(kA)(k^{-1}A^{-1}) = k^{-1}(kA)A^{-1}$$
$$= (k^{-1}k)AA^{-1}$$
$$= (1)I$$
$$= I$$

$$(k^{-1}A^{-1})(kA) = k^{-1}(kA^{-1})A$$
$$= (k^{-1}k)A^{-1}A$$
$$= (1)I$$
$$= I$$

# Finding $A^{-1}$ using Gauss-Jordan Elimination

$$\left\lceil A\middle|I\right\rceil \to \left\lceil I\middle|A^{-1}\right\rceil$$

Find 
$$A^{-1}$$
 if  $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$ 

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \qquad \frac{2 & -2 & -1 & 0 & 1 & 0}{0 & -2 & -2 & 0 & 0} \qquad \frac{3 & 0 & 0 & 0 & 0 & 1}{0 & 0 & -3 & -3 & 0 & 0} \\ \frac{-2 & 0 & -2 & -2 & 0 & 0}{0 & -2 & -3 & -2 & 1 & 0} \qquad \frac{-3 & 0 & -3 & -3 & 0 & 0}{0 & 0 & -3 & -3 & 0 & 1}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{bmatrix} - \frac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{bmatrix} -\frac{1}{3}R_{3}$$

$$\begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & | & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & | & 1 & 0 & -\frac{1}{3} \end{bmatrix} \quad \begin{bmatrix} R_1 - R_3 & & 1 & 0 & 1 & 1 & 0 & 0 \\ & R_2 - \frac{3}{2} R_3 & & 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ & & & & 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ & 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

- ✓ Matrix A is symmetric across its main diagonal. So is  $A^{-1}$
- Matrix A is *tridiagonal* (only three nonzero diagonals). But  $A^{-1}$  is a full matrix with no zeros. (another reason we don't compute  $A^{-1}$ )

#### Singular versus Invertible

 $A^{-1}$  exists when A has a full set of n pivots. (Row exchanges allowed)

- With *n* pivots, elimination solves all the equations  $Ax_i = b_i$ . The columns  $x_i$  go into  $A^{-1}$ . Then  $AA^{-1} = I$  is at least a *right-inverse*.
- Elimination is really a sequence of multiplications.

#### Conclusion

- If A doesn't have n pivots, elimination will lead to a zero row.
- Elimination steps are taken by an invertible M. So a row of MA is zero.
- If AB = I then MAB = M. The zero row of MA, times B, gives a zero row of M.
- The invertible matrix M can't have a zero row! A must have n pivots if AB = I.

#### Elementary Matrices

#### **Definition**

Let e be an elementary row operation. Then the  $n \times n$  elementary matrix E associated with e is the matrix obtained by applying e to the  $n \times n$  identity matrix. Thus

$$E = eI$$

# Example

a) 
$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$$
  $\rightarrow$  Multiply  $R_2$  of  $I$  by  $-3$ 

**b)** 
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow Multiply the third row by -5$$

c) 
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow Interchange the first and second rows$$

**d)** 
$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow Add - 3 times R_1 to R_2$$

#### **Theorem**

Let e be an elementary operation and let E be the corresponding elementary matrix E = e(I). Then for every  $m \times n$  matrix A

$$e(A) = EA$$

That is, an elementary row operation can be performed on A by multiplying A on the left by the corresponding elementary matrix.

#### **Example** $m \times m$

Let 
$$A = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$
  $M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$   $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$   $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ 

$$MA = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -4 & 6 & 2 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$

This result can be obtained from A by multiplying the first row by 2.

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -2 & 3 & 1 \\ -6 & 2 & 1 & 5 \\ 1 & 0 & 2 & 4 \end{bmatrix}$$

This result can be obtained from A by interchanging rows 2 and 3.

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ 0 & -4 & 10 & 8 \end{bmatrix}$$

This result can be obtained from A by adding 3 times row 1 to row 3.

#### **Uniqueness of Echelon Form**

Two matrices A and B are row-equivalent if and only if they have the same reduced echelon form.

#### Proof

If A and B have the same reduced echelon form E, then A is row-equivalent to E and E is row-equivalent to E. It follows that E is row-equivalent to E.

Now Suppose A and B are row-equivalent. Let  $E_1$  be a reduced echelon form of A and  $E_2$  be a reduced echelon form of B. Then  $E_1$  and  $E_2$  are row equivalent.

Suppose  $E_1 = IF_1$  and  $E_2 = IF_2$ . Since  $E_1$  and  $E_2$  are row equivalent,  $E_2 = CE_1$  for some matrix C. This means I = CI and  $F_2 = CF_1$ . But then C = I and  $F_2 = F_1$ .

### **Example**

Show that the two matrices are row equivalent

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$
 and  $B = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$ 

#### **Solution**

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad \begin{aligned} R_1 + R_2 \\ R_2 - 2R_1 \\ = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix} \\ = B & \begin{vmatrix} 1 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix}$$

## **Definition**

A relationship ~ (equivalent) between elements of a set is called an equivalence relation if

- ✓  $A \sim A$  is always true,
- $\checkmark$   $A \sim B$  always implies  $B \sim A$ ,
- ✓  $A \sim B$  and  $B \sim C$  always implies  $A \sim C$ .

# **Exercises** Section 1.4 – Inverse Matrices – Finding $A^{-1}$

1. Apply Gauss-Jordan method to find the inverse of this triangular "Pascal matrix"

Triangular Pascal matrix 
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

**2.** If A is invertible and AB = AC, prove that B = C

3. If 
$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
, find two matrices  $B \neq C$  such that  $AB = AC$ 

**4.** If A has row 1 + row 2 = row 3, show that A is not invertible

- a) Explain why Ax = (1, 0, 0) can't have a solution.
- b) Which right sides  $(b_1, b_2, b_3)$  might allow a solution to Ax = b
- c) What happens to **row** 3 in elimination?

**5.** True or false (with a counterexample if false and a reason if true):

- a) A 4 by 4 matrix with a row of zeros is not invertible.
- b) A matrix with 1's down the main diagonal is invertible.
- c) If A is invertible then  $A^{-1}$  is invertible.
- d) If A is invertible then  $A^2$  is invertible.

6. Do there exist 2 by 2 matrices A and B with real entries such that AB - BA = I, where I is the identity matrix?

7. If B is the inverse of  $A^2$ , show that AB is the inverse of A.

**8.** Find and check the inverses (assuming they exist) of these block matrices.

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}$$

**9.** For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

10. Find  $A^{-1}$  and  $B^{-1}$  (if they exist) by elimination.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

11. Find  $A^{-1}$  using the Gauss-Jordan method, which has a remarkable inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find the inverse, if exists, of

$$12. \quad \begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$$

**23.** 
$$A = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$$

$$\mathbf{34.} \quad A = \begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$$

$$13. \quad \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$$

**24.** 
$$A = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$$

**35.** 
$$A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$$

$$14. \quad \begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$$

**25.** 
$$A = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$$

**36.** 
$$A = \begin{pmatrix} -3 & \frac{1}{2} \\ 6 & -1 \end{pmatrix}$$

**15.** 
$$A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$$

$$26. \quad A = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$$

$$\mathbf{37.} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$$

$$16. \quad A = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

**28.** 
$$A = \begin{pmatrix} -2 & 7 \\ 0 & 2 \end{pmatrix}$$

**27.**  $A = \begin{pmatrix} -6 & 9 \\ 2 & -3 \end{pmatrix}$ 

**38.** 
$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$$

$$17. \quad A = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$$

**18.**  $A = \begin{bmatrix} a & b \\ 3 & 3 \end{bmatrix}$ 

**29.** 
$$A = \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$$

$$\mathbf{39.} \quad A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

**19.** 
$$A = \begin{bmatrix} -2 & a \\ 4 & a \end{bmatrix}$$

**30.** 
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

**40.** 
$$A = \begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$$

$$20. \quad A = \begin{bmatrix} 4 & 4 \\ b & a \end{bmatrix}$$

$$32. \quad A = \begin{pmatrix} b & 3 \\ b & 2 \end{pmatrix}$$

31.  $A = \begin{pmatrix} 2 & 1 \\ a & a \end{pmatrix}$ 

**41.** 
$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}$$

$$\mathbf{21.} \quad A = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$33. \quad A = \begin{pmatrix} 1 & a \\ 3 & a \end{pmatrix}$$

**22.** 
$$A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

**42.** 
$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$$

**43.** 
$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$$

**44.** 
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 1 & 2 \end{pmatrix}$$

**45.** 
$$A = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

**46.** 
$$A = \begin{pmatrix} -3 & 1 & -1 \\ 1 & -4 & -7 \\ 1 & 2 & 5 \end{pmatrix}$$

**47.** 
$$A = \begin{pmatrix} 1 & 1 & -3 \\ 2 & -4 & 1 \\ -5 & 7 & 1 \end{pmatrix}$$

**48.** 
$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{pmatrix}$$

**49.** 
$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\mathbf{50.} \quad A = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{pmatrix}$$

**51.** 
$$A = \begin{pmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

52. 
$$A = \begin{pmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{pmatrix}$$

53. 
$$A = \begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -6 & 1 \\ -2 & -2 & 5 & 1 \end{bmatrix}$$

$$\mathbf{54.} \quad A = \begin{bmatrix} 1 & -14 & 7 & 38 \\ -1 & 2 & 1 & -2 \\ 1 & 2 & -1 & -6 \\ 1 & -2 & 3 & 6 \end{bmatrix}$$

$$\mathbf{55.} \quad A = \begin{bmatrix} 10 & 20 & -30 & 15 \\ 3 & -7 & 14 & -8 \\ -7 & -2 & -1 & 2 \\ 4 & 4 & -3 & 1 \end{bmatrix}$$

**56.** Show that *A* is not invertible for any values of the entries

$$A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

57. Prove that if A is an invertible matrix and B is row equivalent to A, then B is also invertible.

58. Determine if the given matrix has an inverse, and find the inverse if it exists. Check your answer by multiplying  $A \cdot A^{-1} = I$ 

a) 
$$\begin{bmatrix} 2 & 3 \\ -3 & -5 \end{bmatrix}$$
 b)  $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}$ 

- **59.** Show that the inverse of  $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$  is  $\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$
- **60.** If the product C = AB is invertible (and A & B are square matrices), find a formula for  $A^{-1}$  that involves  $C^{-1}$  and B.

Hence, it is not possible to multiply a non-invertible matrix by another matric and obtain an invertible matrix as a result.

- **61.** Prove that if A is an  $m \times n$  matrix, there is an invertible matrix C such that CA is in reduced rowechelon form.
- 62. Prove that  $2 m \times n$  matrices A and B are row equivalent if and only if there exists a nonsingular matrix P such that B = PA
- **63.** Let *A* and *B* be 2  $m \times n$  matrices. Suppose *A* is row equivalent to *B*. Prove that *A* is nonsingular if and only if *B* is nonsingular.
- **64.** Show that if A and B are two  $n \times n$  invertible matrices then A is row equivalent to B.
- **65.** Prove that a square matrix A is nonsingular if and only if A is a product of elementary matrices.
- **66.** Show that if  $A \sim B$  (that is, if they are row equivalent), then EA = B for some matrix E which is a product of elementary matrices.
- 67. Show that if EA = B for some matrix E which is a product of elementary matrices, then  $AC \sim BC$  for every  $n \times n$  matrix C.
- **68.** Let  $A\vec{x} = 0$  be a homogeneous system of *n* linear equations in *n* unknowns that has only the trivial solution. Show that of *k* is any positive integer, then the system  $A^k \vec{x} = 0$  also has only trivial solution.
- **69.** Let  $A\vec{x} = 0$  be a homogeneous system of *n* linear equations in *n* unknowns, and let *Q* be an invertible  $n \times n$  matrix. Show that  $A\vec{x} = 0$  has just trivial solution if and only if  $(QA)\vec{x} = 0$  has just trivial solution.

- 70. Let  $A\vec{x} = b$  be any consistent system of linear equations, and let  $\vec{x}_1$  be a fixed solution. Show that every solution to the system can be written in the form  $\vec{x} = \vec{x}_1 + \vec{x}_0$  where  $\vec{x}_0$  is a solution to  $A\vec{x} = 0$ . Show also that every matrix of this form is a solution.
- 71. If A and B are  $n \times n$  matrices satisfying  $A^2 = B^2 = (AB)^2 = I_n$ . Prove that AB = BA.
- 72. Let  $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix}$ . Verify that  $A^3 = 5I$ , then find  $A^{-1}$  in term of A.
- 73. Consider B(A, I) = (BA, B), thus if B is the inverse of A, then (BA, B) becomes  $(I, A^{-1})$ . On the other hand B is a product of elementary matrices since it is invertible. This indicates that the inverse of A can be obtained by applying elementary row operations to (A, I) to get  $(I, A^{-1})$ .

Find the inverses of

$$a) \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix}$$

$$b) \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{pmatrix}$$

**74.** Let  $A, B, C, X, Y, Z \in M_n(\mathbb{C}), A$  and C are invertible. Find

$$a) \quad \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}^{-1}$$

$$b) \begin{pmatrix} I & X & Y \\ 0 & I & Z \\ 0 & 0 & I \end{pmatrix}^{-1}$$

75. Suppose that A, B, and A - B are invertible  $n \times n$  matrices. Show that

$$(A-B)^{-1} = A^{-1} + A^{-1} (B^{-1} - A^{-1})^{-1} A^{-1}$$

- **76.** Suppose *P* is invertible and  $A = PBP^{-1}$ . Solve for *B* in terms of *A*.
- 77. Suppose (A-B)C=0, where A and B are  $m \times n$  matrices and C is invertible. Show that A=B.

# Section 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices

## **Transpose**

#### Definition

The transpose of a matrix A is defined as the matrix that is obtained by interchanging the corresponding rows and columns in A. Then the transpose of A, denoted by  $A^T$  or A'.

The columns of  $A^T$  are the rows of A.

When A is an m by n matrix, the transpose is n by m:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \quad then \quad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

The matrix *flips over* the main diagonal. The entry in row i, column j of  $A^T$  comes from row j, column i of the original A.

$$\left(A^T\right)_{ij} = A_{ji}$$

## **Properties of Transpose**

a) 
$$\left(A^T\right)^T = A$$

b) 
$$(A+B)^{T} = A^{T} + B^{T}$$

c) 
$$(A-B)^T = A^T - B^T$$

$$d) \quad (kA)^T = kA^T$$

$$e$$
)  $(AB)^T = B^T A^T$ 

The transpose of a product of any number of matrices is the product of the transposes in the reverse order.

#### **Theorem**

If A is an invertible matrix, then  $A^T$  is also invertible and

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

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#### **Proof**

$$A^{T} (A^{-1})^{T} = (A^{-1}A)^{T}$$

$$= I^{T}$$

$$= I$$

$$(A^{-1})^{T} A^{T} = (AA^{-1})^{T}$$

$$= I^{T}$$

$$= I$$

\*\*\*\*\*\*\*\*\*\*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad and \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$
$$\left(A^T\right)^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$
$$= \begin{bmatrix} \frac{d}{ad - bc} & -\frac{c}{ad - bc} \\ -\frac{b}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix}$$

#### **Trace**

## Definition

If A is a square matrix, then the trace of A, denoted by  $\mathbf{tr}(A)$ , is defined to the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

## **Example**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$
$$tr(A) = a_{11} + a_{22} + a_{33}$$

## **Diagonal**

A square matrix in which all the entries off the main diagonal are zero is called a *diagonal matrix*. A general  $n \times n$  diagonal matrix can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

A diagonal matrix is invertible iff all of its diagonal entries are nonzero; the

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$$

Powers of diagonal matrices are:

$$D^{k} = \begin{bmatrix} d_{1}^{k} & 0 & \cdots & 0 \\ 0 & d_{1}^{k} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{n}^{k} \end{bmatrix}$$

## Triangular Matrices

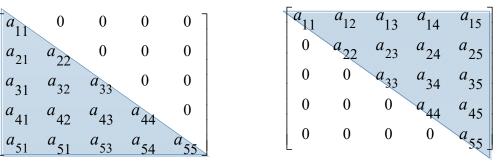
A square matrix in which all the entries above the main diagonal are zero is called *lower diagonal* triangular.

A square matrix in which all the entries below the main diagonal are zero is called *upper diagonal* triangular.

A matrix that is either upper triangular or lower triangular is called *triangular*.

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ a_{51} & a_{51} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

lower diagonal triangular



upper diagonal triangular

#### Theorem

- ✓ The transpose of a lower triangular matrix is upper triangular, and the transpose of a upper triangular matrix is lower triangular.
- ✓ The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.
- ✓ A triangular matrix is invertible iff its diagonal entries are all nonzero.
- ✓ The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

#### **Example**

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

#### Solution

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \qquad AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix} \qquad BA = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

The goal is to describe Gaussian elimination in the most useful way by looking at them closely, which are factorizations of a matrix.

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The factors are triangular matrices.

The factorization that comes from elimination is A = LU.

# Symmetric Matrices

## **Definition**

A square matrix A is said to be **symmetric** if  $A^T = A$ . That means a square matrix must satisfies  $a_{ij} = a_{ji}$ 

## Example

$$A = \begin{pmatrix} 1 & -4 \\ -4 & 1 \end{pmatrix} = A^T$$

$$A = \begin{pmatrix} 6 & 5 & 1 \\ 5 & 0 & 7 \\ 1 & 7 & -1 \end{pmatrix} = A^{T}$$

♣ The *inverse* of a symmetric matrix is also *symmetric*.

#### **Example**

Given  $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ , show that the inverse is symmetric too?

#### Solution

$$A^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

#### **Theorem**

If A and B are symmetric matrices with the same size, and if k is any scalar, then:

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- a)  $A^T$  is symmetric
- b) A + B and A B are symmetric.
- c) kA is symmetric
- $\blacksquare$  If A is an invertible symmetric matrix, then  $A^{-1}$  is symmetric.

## **Proof**

Assume that A is symmetric and invertible then  $A = A^T$ 

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}$$

Which proves that  $A^{-1}$  is symmetric

 $\blacksquare$  Multiplying M by  $M^T$  gives a symmetric matrix.

## **Proof**

The entry (i, j) of  $M^TM$ , it is the dot product of **row** i of  $M^T$  (column i of M) with column j of M. The (i, j) entry is the same dot product, column j with column i. so  $M^TM$  is symmetric. The matrix  $M.M^T$  is also symmetric and  $M^TM$  is a different matrix from  $M.M^T$ .

- $\blacksquare$  If A is an invertible symmetric matrix, then  $AA^T$  and  $A^TA$  are also invertible.
- $\blacksquare$  Matrix A is symmetric across its main diagonal. So is  $A^{-1}$
- $\blacksquare$  Matrix A is tridiagonal (only three nonzero diagonals). But  $A^{-1}$  is a full matrix with no zeros. (another reason we don't compute  $A^{-1}$ )

## Example

Given 
$$M = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
 and  $M^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . Find  $M^T M$  and  $M.M^T$ 

#### **Solution**

$$M^{T}M = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$MM^{T} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 5 \end{bmatrix}$$

## Symmetric in LDU

When elimination is applied to a symmetric matrix,  $A^T = A$  is an advantage.

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$
$$L \qquad D \qquad U$$

If  $A = A^T$  can be factored into LDU with no row exchanges, then  $U = L^T$ . The **symmetric** factorization of a symmetric matrix is  $A = LDL^T$ 

#### **Exercises** Section 1.5 – Transpose, Diagonal, Triangular, and **Symmetric Matrices**

Solve Lc = b to find c. Then solve Ux = c to find x. What was A? 1.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

2. Find L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get A = LU with four pivots

3. Determine whether the given matrix is invertible

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Find  $A^2$ ,  $A^{-2}$ , and  $A^{-k}$  by inspection 4.

$$a) A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$b) A = \begin{vmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{vmatrix}$$

a) 
$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$
 b)  $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$  c)  $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ 

**5.** Decide whether the given matrix is symmetric

$$a) \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$b) \begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix} \qquad c) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

$$c) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

6. Find all values of the unknown constant(s) in order for A to be symmetric

$$A = \begin{bmatrix} 2 & a - 2b + 2c & 2a + b + c \\ 3 & 5 & a + c \\ 0 & -2 & 7 \end{bmatrix}$$

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- 7. Find a diagonal matrix A that satisfies the given condition  $A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- **8.** Let A be an  $n \times n$  symmetric matrix
  - a) Show that  $A^2$  is symmetric
  - b) Show that  $2A^2 3A + I$  is symmetric
- **9.** Prove if  $A^T A = A$ , then A is symmetric and  $A = A^2$
- **10.** A square matrix A is called **skew-symmetric** if  $A^T = -A$ . Prove
  - a) If A is an invertible skew-symmetric matrix, then  $A^{-1}$  is skew-symmetric.
  - b) If A and B are skew-symmetric matrices, then so are  $A^T$ , A + B, A B, and kA for any scalar k.
  - c) Every square matrix A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

$$\left[ Hint: \text{ Note the identity } A = \frac{1}{2} \left( A + A^T \right) + \frac{1}{2} \left( A - A^T \right) \right]$$

- 11. Suppose R is rectangular (m by n) and A is symmetric (m by m)
  - a) Transpose  $R^T AR$  to show its symmetric
  - b) Show why  $R^T R$  has no negative numbers on its diagonal.
- **12.** If L is a lower-triangular matrix, then  $(L^{-1})^T$  is \_\_\_\_\_Triangular
- 13. True or False
  - a) The block matrix  $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$  is automatically symmetric
  - b) If A and B are symmetric then their product is symmetric
  - c) If A is not symmetric then  $A^{-1}$  is not symmetric
  - d) When A, B, C are symmetric, the transpose of ABC is CBA.
  - e) The transpose of a diagonal matrix is a diagonal.
  - f) The transpose of an upper triangular matrix is an upper triangular matrix.
  - g) The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.
  - h) All entries of a symmetric matrix are determined by the entries occurring on and above the main diagonal.
  - *i)* All entries of an upper triangular matrix are determined by the entries occurring on and above the main diagonal.

- j) The inverse of an invertible lower triangular matrix is an upper triangular matrix.
- k) A diagonal matrix is invertible if and only if all of its diagonal entries are positive.
- l) The sum of a diagonal matrix and a lower triangular matrix is a lower triangular matrix.
- m) A matrix that is both symmetric and upper triangular must be a diagonal matrix.
- n) If A and B are  $n \times n$  matrices such that A + B is symmetric, then A and B are symmetric.
- o) If A and B are  $n \times n$  matrices such that A + B is upper triangular, then A and B are upper triangular.
- p) If  $A^2$  is a symmetric matrix, then A is a symmetric matrix.
- q) If kA is a symmetric matrix for some  $k \neq 0$ , then A is a symmetric matrix.
- **14.** Find 2 by 2 symmetric matrices  $A = A^T$  with these properties
  - a) A is not invertible
  - b) A is invertible but cannot be factored into LU (row exchanges needed)
  - c) A can be factored into  $LDL^T$  but not into  $LL^T$  (because of negative D)
- **15.** A group of matrices includes AB and  $A^{-1}$  if it includes A and B. "Products and inverses stay in the group." Which of these sets are groups?

Lower triangular matrices L with 1's on the diagonal, symmetric matrices S, positive matrices M, diagonal invertible matrices D, permutation matrices P, matrices with  $Q^T = Q^{-1}$ . Invent two more matrix groups.

- **16.** Write  $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$  as the product *EH* of an elementary row operation matrix *E* and a symmetric matrix *H*.
- 17. When is the product of two symmetric matrices symmetric? Explain your answer.
- **18.** Express  $\left( \left( AB \right)^{-1} \right)^T$  in terms of  $\left( A^{-1} \right)^T$  and  $\left( B^{-1} \right)^T$
- 19. Find the transpose of the given matrix:  $\begin{bmatrix} 8 & -1 \\ 3 & 5 \\ -2 & 5 \\ 1 & 2 \\ -3 & -5 \end{bmatrix}$
- **20.** Show that if A is symmetric and invertible, then  $A^{-1}$  is also symmetric.
- **21.** Prove that  $(AB)^T = B^T A^T$

**22.** For the given matrix, compute 
$$A^T$$
,  $\left(A^T\right)^{-1}$ ,  $A^{-1}$ , and  $\left(A^{-1}\right)^T$ , then compare  $\left(A^T\right)^{-1}$  and  $\left(A^{-1}\right)^T$ 

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

- 23. Show that a  $2 \times 2$  lower triangular matrix is invertible if and only if  $a_{11}a_{22} \neq 0$  and in this case the inverse is also lower triangular.
- **24.** Let A be any  $2 \times 2$  diagonal matrix. Give a necessary and sufficient condition on the diagonal entries so that A has an inverse. Compute the inverse of any such matrix.

# Section 1.6 – Determinants and Properties

The determinant is a number that contains information about matrix. It is used to find formulas for inverse matrices, pivots, and solutions  $A^{-1}b$ .

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ has inverse } A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Determinant of the matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is written det(A) or |A| and is define as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinant is zero when the matrix has no inverse.

#### **Properties of the Determinants**

By using these property rules, we can compute the determinant of any square matrix.

1. Determinant of the n by n identity matrix is 1.

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad and \quad \begin{vmatrix} 1 \\ & 1 \end{vmatrix} = 1$$

2. Determinant changes sign when 2 rows are exchanged.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad - bc) \quad \Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = - \begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

3. Determinant is a linear function of each row separately.

Multiply row 1 by any number 
$$t$$
:  $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ 

Add row 1 of A to row 1 of A': 
$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

**♣** For 2 by 2 determinants, if you expand to a rectangle, the determinants equal areas.

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**♣** For n-dimensional, the determinants equal volumes.

4. If 2 rows of A are equal, then  $\det A = 0$ .

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$$

5. Subtracting a multiple of one row from another row leaves detA unchanged.

$$\begin{vmatrix} a & b \\ c - ta & d - ta \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

6. A matrix with a row of zeros has  $\det A = 0$ .

$$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0 \quad and \quad \begin{vmatrix} 0 & 0 \\ b & c \end{vmatrix} = 0$$

7. If A is triangular then  $\det A = a_{11}a_{22} \dots a_{nn} = \text{product of diagonal entries.}$ 

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad \quad and \quad \begin{vmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & & \\ 0 & & & a_{nn} \end{vmatrix} = a_{11}a_{22}\dots a_{nn}$$

- 8. If A is singular then det A = 0.
- 9. If A is invertible then  $\det A \neq 0$ .
- **10.** The determinant of AB detA is times detB: |AB| = |A||B|
- 11. The transpose  $A^T$  has the same determinant as A:  $\det(A) = \det(A^T)$

$$\rightarrow$$
 det $(A+B) \neq$  det $(A)$  + det $(B)$ 

## Big Formula for Determinants (Diagonal)

#### **Determinant Using Diagonal Method**

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Determinant: D = (1) + (2)

$$\mathbf{det} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

## **Example**

Evaluate: 
$$\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix}$$

#### **Solution**

$$\begin{vmatrix} x & 0 & -1 \\ 2 & x & x^2 \\ -3 & x & 1 \end{vmatrix} = x = x(x)(1) + 0(x^2)(2) + (-1)(2)(x) - (-1)(x)(-3) - x(x^2)(x) - 0(-3)(1)$$

$$= x^2 - 2x - 3x - x^4$$

$$= x^2 - 5x - x^4$$

## **Determinant by Cofactors**

$$\mathbf{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

## Minor

For a square matrix  $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ , the minor  $M_{ij}$ . Of an element  $a_{ij}$  is the **determinant** of the matrix formed by deleting the  $i^{th}$  row and the  $j^{th}$  column of A.

#### Example

Let 
$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$
 Find  $M_{32}$ 

#### Solution

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix}$$
$$= \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix}$$
$$= 26$$

#### **Theorem**

The determinant is the dot product of any row i of A with its cofactors:

Cofactor Formula: 
$$\begin{aligned} \hline C_{ij} &= (-1)^{i+j} M_{ij} \\ |A| &= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{22} & a_{23} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{21} & a_{23} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{21} & a_{22} \end{vmatrix} \end{aligned}$$

### **Example**

Find the determinant of the matrix:

$$A = \begin{bmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{bmatrix}$$

#### **Solution**

$$|A| = \begin{vmatrix} -8 & 0 & 6 \\ 4 & -6 & 7 \\ -1 & -3 & 5 \end{vmatrix}$$

$$= -8 \begin{vmatrix} -6 & 7 \\ -3 & 5 \end{vmatrix} - 0 \begin{vmatrix} 4 & 7 \\ -1 & 5 \end{vmatrix} + 6 \begin{vmatrix} 4 & -6 \\ -1 & -3 \end{vmatrix}$$

$$= -8(-30 - (-21)) - 0 + 6(-12 - 6)$$

$$= -8(-9) + 6(-18)$$

$$= -36$$

✓ By the property of determinants, If  $\mathbf{A}$  is triangular then  $\det \mathbf{A} = \mathbf{a}_{11} \mathbf{a}_{22} \dots \mathbf{a}_{nn} = \text{product of diagonal}$  entries.

## Example

$$\begin{vmatrix} 2 & 7 & -3 & 8 & 3 \\ 0 & -3 & 7 & 5 & 1 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 9 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix} = (2)(-3)(6)(9)(4)$$

$$= -1296$$

#### **Theorem**

Let A be any n by n matrix.

- a) If A' is the matrix that results when a single row of A is multiplied by a constant k, then  $\det(A') = k \det(A)$ .
- b) If A' is the matrix that results when two rows of A are interchanged, then  $\det(A') = -\det(A)$
- c) If A' is the matrix that results when a multiple of one row of A is added to another row then  $\det(A') = \det(A)$

## Example

Evaluate 
$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix}$$

#### Solution

$$\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

$$= -3(1)(1)(-55)$$

=165

Interchanged 1<sup>st</sup> and 2<sup>nd</sup> row

A common factor of 3 from the first row (no need)

# **Exercises** Section 1.6 – Determinants and Properties

1. Verify that 
$$det(AB) = det(A)det(B)$$
 when:  $A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{pmatrix}$ 

- 2. For which value(s) of k does A fail to be invertible?  $A = \begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix}$
- 3. Without directly evaluating, show that  $\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$
- **4.** If the entries in every row of A add to zero, solve Ax = 0 to prove det(A) = 0. If those entries add to one, show that det(A-I) = 0. Does this mean det(A) = I?
- 5. Does det(AB) = det(BA) in general?
  - a) True or false if  $\boldsymbol{A}$  and  $\boldsymbol{B}$  are square  $n \times n$  matrices?
  - b) True or false if A is  $m \times n$  and B is  $n \times m$  with  $m \neq n$ ?
- **6.** True or false, with a reason if true or a counterexample if false:
  - a) The determinant of I + A is  $1 + \det(A)$ .
  - b) The determinant of ABC is |A||B||C|.
  - c) The determinant of 4A is 4|A|
  - d) The determinant of AB BA is zero. (try an example)
  - e) If A is not invertible then AB is not invertible.
  - f) The determinant of A B equals to det(A) det(B).
- 7. Use row operations to show the 3 by 3 "Vandermonde determinant" is

$$\det\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$$

**8.** The inverse of a 2 by 2 matrix seems to have determinant = 1:

$$\det A^{-1} = \det \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{ad - bc}{ad - bc} = 1$$

What is wrong with this calculation? What is the correct  $\det A^{-1}$ 

9. A *Hessenberg* matrix is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci's rule  $|H_4| = |H_3| + |H_2|$ . The same rule will continue for all sizes  $|H_n| = |H_{n-1}| + |H_{n-2}|$ . Which Fibonacci number is  $|H_n|$ ?

$$H_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad H_3 = \begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad H_4 = \begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

(**10 – 44**) Evaluate

**10.** 
$$\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix}$$

11. 
$$\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix}$$

$$12. \quad \begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix}$$

$$13. \quad \begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix}$$

14. 
$$\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix}$$

$$15. \quad \begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix}$$

**16.** 
$$\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}$$

17. 
$$\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix}$$

18. 
$$\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix}$$

**19.** 
$$\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix}$$

**20.** 
$$\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix}$$

**21.** 
$$\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix}$$

**22.** 
$$\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix}$$

23. 
$$\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix}$$

**24.** 
$$\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix}$$

**25.** 
$$\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix}$$

$$\begin{array}{c|cc} \mathbf{26.} & \begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix}$$

$$\begin{array}{c|c} \mathbf{27.} & \begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix}$$

$$\begin{array}{c|cc}
x^2 & x \\
-3 & 2
\end{array}$$

**29.** 
$$\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix}$$

**30.** 
$$\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix}$$

$$\begin{array}{c|cccc}
\mathbf{31.} & 3 & 0 & 0 \\
2 & 1 & -5 \\
2 & 5 & -1
\end{array}$$

**32.** 
$$\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix}$$

**33.** 
$$\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix}$$

**37.** 
$$\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$$

41. 
$$\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix}$$

$$\begin{array}{c|cccc}
\mathbf{34.} & \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix}$$

$$\begin{array}{c|cccc}
38. & 2 & 1 & -1 \\
4 & 7 & -2 \\
2 & 4 & 0
\end{array}$$

**42.** 
$$\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix}$$

35. 
$$\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix}$$

39. 
$$\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix}$$
40. 
$$\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix}$$

43. 
$$\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}$$

$$\begin{array}{c|cccc}
x & 1 & -1 \\
x^2 & x & x \\
0 & x & 1
\end{array}$$

**40.** 
$$\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix}$$

44. 
$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix}$$

Find all the values of  $\lambda$  for which  $\det(A) = 0$ 

$$a) A = \begin{bmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 4 \end{bmatrix}$$

a) 
$$A = \begin{bmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 4 \end{bmatrix}$$
 b)  $A = \begin{bmatrix} \lambda - 6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda - 4 \end{bmatrix}$ 

Prove that if a square matrix A has a column of zeros, then det(A) = 046.

**47.** With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D| \quad but \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|$$

a) Why is the first statement true? Somehow B doesn't enter.

b) Show by example that equality fails (as shown) when C enters.

Show by example that the answer  $\det(AD - CB)$  is also wrong.

48. Show that the value of the following determinant is independent of  $\theta$ .

$$\begin{vmatrix}
\sin \theta & \cos \theta & 0 \\
-\cos \theta & \sin \theta & 0 \\
\sin \theta - \cos \theta & \sin \theta + \cos \theta & 1
\end{vmatrix}$$

**49.** Show that the matrices 
$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$
 and  $B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$  commute if and only if  $\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$ 

**50.** Show that 
$$\det(A) = \frac{1}{2} \begin{vmatrix} tr(A) & 1 \\ tr(A^2) & tr(A) \end{vmatrix}$$
 for every  $2 \times 2$  matrix  $A$ .

- **51.** What is the maximum number of zeros that a  $4 \times 4$  matrix can have without a zero determinant? Explain your reasoning.
- **52.** Evaluate  $\det(A)$ ,  $\det(E)$ , and  $\det(AE)$ . Then verify that  $\det(A) \cdot \det(E) = \det(AE)$

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & & \\ & 3 & \\ & & 1 \end{bmatrix}$$

53. Show that 
$$\begin{bmatrix} \sin^2 \alpha & \sin^2 \beta & \sin^2 \gamma \\ \cos^2 \alpha & \cos^2 \beta & \cos^2 \gamma \\ 1 & 1 & 1 \end{bmatrix}$$
 is not invertible for any values of  $\alpha$ ,  $\beta$ ,  $\gamma$ 

**54.** The determinant of a 
$$2 \times 2$$
 matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is  $\det(A) = ad - bc$ .

Assuming no rows swaps are required, perform elimination on A and show explicitly that ad - bc is the product of the pivots.

**55.** If A is a 
$$7 \times 7$$
 matrix and let  $\det(A) = 17$ . What is  $\det(3A^2)$ ?

56. Explain without computations why the following determinant is equal to zero

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & 0 & 0 & 0 \\ d_1 & d_2 & 0 & 0 & 0 \\ e_1 & e_2 & 0 & 0 & 0 \end{vmatrix}$$

- **57.** Let A be  $n \times n$  real matrix.
  - a) Show that if  $A^t = -A$  and n is odd, then |A| = 0.
  - b) Show that if  $A^2 + I = 0$ , then *n* must be even.
  - c) Does part (b) remain true for complex matrices?
- **58.** Let A and C be  $m \times m$  and  $n \times n$  matrices, respectively.
  - a) Show that  $\begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = \begin{vmatrix} A & 0 \\ B & C \end{vmatrix} = |A||C|$
  - b) Evaluate

$$i. \quad \begin{vmatrix} I_m & 0 \\ 0 & I_n \end{vmatrix}$$

$$ii. \quad \begin{vmatrix} 0 & I_m \\ I_n & 0 \end{vmatrix}$$

iii. 
$$\begin{vmatrix} I_m & B \\ 0 & I_n \end{vmatrix}$$

*iv.* 
$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}_{n \times n}$$

- c) Find a formula for  $\begin{vmatrix} 0 & A \\ C & B \end{vmatrix}_{n \times n}$
- **59.** Let  $f(x) = (p_1 x)(p_2 x)...(p_n x)$  and let

$$\Delta_n = \begin{vmatrix} p_1 & a & a & \dots & a & a \\ b & p_2 & a & \dots & a & a \\ b & b & p_3 & \dots & a & a \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b & b & b & \dots & p_{n-1} & a \\ b & b & b & \dots & b & p_n \end{vmatrix}$$

a) Show that, if  $a \neq b$ ,

$$\Delta_n = \frac{bf(a) - af(b)}{b - a}$$

b) Show that, if a = b,

$$\Delta_n = a \sum_{i=1}^n f_i(a) + p_n f_n(a)$$

Where  $f_i(a)$  means f(a) with factor  $(p_i - a)$  missing.

c) Use part (b) to evaluate

- **60.** Let A, B, C,  $D \in M_n(\mathbb{C})$ 
  - a) Show that when A is invertible:  $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A| |D CA^{-1}B|$
  - b) Show that when AC = CA:  $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD CB|$
  - c) Can B and C on the right-hand side of the identity be switched?
  - d) Does part (b) remain true if the condition AC = CA is dropped?

# Section 1.7 - Cramer's Rule

#### **Cramer's Rule**

#### **Theorem**

If AX = B is a system of a linear equations in n unknowns such that  $det(A) \neq 0$ , then the system has a unique solution. This solution is

$$x_1 = \frac{\det(B_1)}{\det(A)}$$

$$x_2 = \frac{\det(B_2)}{\det(A)}$$

: :

$$x_n = \frac{\det(B_n)}{\det(A)}$$

Where 
$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & & & & \\ \vdots & & & & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\det \begin{pmatrix} B_1 \end{pmatrix} = \begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & & & & \\ \vdots & & & & \\ b_n & a_{n2} & & a_{nn} \end{vmatrix}$$

## Example

Use Cramer's rule to solve

$$x_1 + x_2 + x_3 = 1$$
  
 $-2x_1 + x_2 = 0$   
 $-4x_1 + x_3 = 0$ 

#### Solution

$$\begin{vmatrix} A \\ A \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 7$$

$$\begin{vmatrix} B_1 \\ A \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$\begin{vmatrix} B_2 \\ A \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 2$$

$$\begin{vmatrix} B_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 0 \end{vmatrix} = 4$$

$$x_1 = \frac{\left|B_1\right|}{\left|A\right|} = \frac{1}{7} \qquad x_1 = \frac{\det(A_1)}{\det(A)}$$

$$x_2 = \frac{\left|B_2\right|}{\left|A\right|} = \frac{2}{7}$$

$$x_3 = \frac{\left|B_3\right|}{\left|A\right|} = \frac{4}{7}$$

**Solution**:  $\left(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right)$ 

### **Example**

Use Cramer's Rule to solve.

$$x_1 + 2x_3 = 6$$
  
 $-3x_1 + 4x_2 + 6x_3 = 30$   
 $-x_1 - 2x_2 + 3x_3 = 8$ 

#### **Solution**

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \implies \det(A) = 44$$

$$\det(A_1) = \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix} = -40$$

$$\det\left(A_{2}\right) = \begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix} = 72$$

$$\det\left(A_{3}\right) = \begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix} = 152$$

$$x_{1} = \frac{-40}{44}$$

$$x_{1} = \frac{\det(A_{1})}{\det(A)}$$

$$= -\frac{10}{11}$$

$$x_{2} = \frac{72}{44}$$

$$x_{2} = \frac{\det(A_{2})}{\det(A)}$$

$$= \frac{18}{11}$$

$$x_3 = \frac{152}{44}$$

$$= \frac{38}{11}$$

$$x_3 = \frac{\det(A_3)}{\det(A)}$$

**Solution**: 
$$\left(-\frac{10}{11}, \frac{18}{11}, \frac{38}{11}\right)$$

## A Formula for $A^{-1}$

## Theorem: Inverse of a matrix using its Adjoint

The i, j entry of  $A^{-1}$  is the cofactor  $C_{ji}$  (not  $C_{ij}$ ) divided by det(A):

Formula for 
$$A^{-1}$$
:  $\left(A^{-1}\right)_{ij} = \frac{C_{ji}}{|A|}$  and  $A^{-1} = \frac{C^T}{|A|}$ 

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

## Example

Find the inverse matrix of  $A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$  using its adjoint.

#### Solution

$$C_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1; \qquad C_{12} = -\begin{vmatrix} -2 & 0 \\ -4 & 1 \end{vmatrix} = 2; \qquad C_{13} = \begin{vmatrix} -2 & 1 \\ -4 & 0 \end{vmatrix} = 4$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1; \qquad C_{22} = \begin{vmatrix} 1 & 1 \\ -4 & 1 \end{vmatrix} = 5; \qquad C_{23} = -\begin{vmatrix} 1 & 1 \\ -4 & 0 \end{vmatrix} = -4$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1; \qquad C_{32} = -\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = -2; \qquad C_{33} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3$$

$$C = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 5 & -2 \\ 4 & -4 & 3 \end{bmatrix}$$

$$\det(A) = \frac{1}{7}$$

$$\Rightarrow A^{-1} = \frac{1}{7} \begin{bmatrix} 1 & -1 & -1 \\ 2 & 5 & -2 \\ 4 & -4 & 3 \end{bmatrix}$$

#### **Theorem**

If A is an  $n \times n$  matrix, then the following statements are equivalent

- a) A is invertible
- **b)** Ax = 0 has only the trivial solution
- c) The reduced row echelon form of A is  $I_n$
- d) A can be expressed as a product of elementary matrices
- e) Ax = b is consistent for every  $n \times 1$  matrix b
- f)  $\det(A) \neq 0$

# **Exercises** Section 1.7 – Cramer's Rule

1. Use Cramer's Rule with ratios  $\frac{\det B_j}{\det A}$  to solve Ax = b. Also find the inverse matrix  $A^{-1} = \frac{C^T}{\det A}$ .

Why is the solution x is the first part the same as column 3 of  $A^{-1}$ ? Which cofactors are involved in computing that column x?

$$Ax = b \quad is \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2. Verify that det(AB) = det(BA) and determine whether the equality det(A+B) = det(A) + det(B) holds

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

3. Verify that  $det(kA) = k^n det(A)$ 

$$a) \quad A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}, \quad k = 2$$

$$b) \quad A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}, \quad k = -2$$

c) 
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}$$
,  $k = 3$ 

$$(4-58)$$
 Use Cramer's rule to solve the system

4. 
$$\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$$

$$\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$$

12. 
$$\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$$

5. 
$$\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$$

$$9. \begin{cases}
5x - 2y = 4 \\
-10x + 4y = 7
\end{cases}$$

13. 
$$\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$$

6. 
$$\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$$

10. 
$$\begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$$

14. 
$$\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$$

$$2x + 3y = 9$$
7. 
$$\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$$

$$5x - 20y = -40$$
11. 
$$\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$$

15. 
$$\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$$

16. 
$$\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$$

$$\begin{cases} x + 2y - 3 = 0 \\ 12 = 8y + 4x \end{cases}$$

13. 
$$\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$$

16. 
$$\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$$
17. 
$$\begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$$
18. 
$$\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$$
19. 
$$\begin{cases} x + y = 7 \\ x - y = 3 \end{cases}$$
20. 
$$\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$$

33. 
$$\begin{cases} 7x - 2y = 3 \\ 3x + y = 5 \end{cases}$$

44. 
$$\begin{cases} x + y + z = 3 \\ -y + 2z = 1 \\ -x + z = 0 \end{cases}$$

18. 
$$\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$$

34. 
$$\begin{cases} 3x + 2y - z = 4 \\ 3x - 2y + z = 5 \\ 4x - 5y - z = -1 \end{cases}$$

$$\begin{cases}
-y + 2z = 1 \\
-x + z = 0
\end{cases}$$

$$\begin{cases} x + y = 7 \\ x - y = 3 \end{cases}$$

35. 
$$\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$$

45. 
$$\begin{cases} 3x + y + 3z = 14 \\ 7x + 5y + 8z = 37 \\ x + 3y + 2z = 9 \end{cases}$$

$$20. \quad \begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$$

35. 
$$\begin{cases} 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$$

46. 
$$\begin{cases} 4x - 2y + z = 7 \\ x + y + z = -2 \\ 4x + 2y + z = 3 \end{cases}$$

21. 
$$\begin{cases} 12x + 3y = 15 \\ 2x - 3y = 13 \end{cases}$$

36. 
$$\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$$

47. 
$$\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$$

22. 
$$\begin{cases} x - 2y = 5 \\ 5x - y = -2 \end{cases}$$

32. 
$$\begin{cases} x+2y-3=0\\ 12=8y+4x \end{cases}$$
33. 
$$\begin{cases} 7x-2y=3\\ 3x+y=5 \end{cases}$$
34. 
$$\begin{cases} 3x+2y-z=4\\ 3x-2y+z=5\\ 4x-5y-z=-1 \end{cases}$$
35. 
$$\begin{cases} x+y+z=2\\ 2x+y-z=5\\ x-y+z=-2 \end{cases}$$
36. 
$$\begin{cases} 2x+y+z=9\\ -x-y+z=1\\ 3x-y+z=9 \end{cases}$$
37. 
$$\begin{cases} 3y-z=-1\\ x+5y-z=-4\\ -3x+6y+2z=11 \end{cases}$$
38. 
$$\begin{cases} x+3y+4z=14\\ 2x-3y+2z=10\\ 3x-y+z=9 \end{cases}$$

$$\begin{cases} x+4y-z=20\\ 3x+2y+z=8 \end{cases}$$

47. 
$$\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$$
48. 
$$\begin{cases} 2x - 2y + z = -4 \\ 6x + 4y - 3z = -24 \\ x - 2y + 2z = 1 \end{cases}$$

$$\begin{cases} 9x + 3y + z = 4 \end{cases}$$

22. 
$$\begin{cases} x - 2y = 5 \\ 5x - y = -2 \end{cases}$$
23. 
$$\begin{cases} 3x + 2y = 2 \\ 2x + 2y = 3 \end{cases}$$

38. 
$$\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$$

$$\begin{cases}
9x + 3y + z = 4 \\
16x + 4y + z = 2 \\
25x + 5y + z = 2
\end{cases}$$

**24.** 
$$\begin{cases} 4x - 5y = 17 \\ 2x + 3y = 3 \end{cases}$$

39. 
$$\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$$

$$25x + 5y + z = 2$$

$$2x - y + 2z = -8$$

25. 
$$\begin{cases} x - 3y = 4 \\ 3x - 4y = 12 \end{cases}$$
26. 
$$\begin{cases} 2x - 9y = 5 \\ 3x - 3y = 11 \end{cases}$$

40. 
$$\begin{cases}
-2x + 6y + 7z = 3 \\
-4x + 5y + 3z = 7 \\
-6x + 3y + 5z = -4
\end{cases}$$

50. 
$$\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$$

$$27. \quad \begin{cases} 3x - 4y = 4 \\ x + y = 6 \end{cases}$$

40. 
$$\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$$

51. 
$$\begin{cases} x - 3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$$

28. 
$$\begin{cases} 3x = 7y + 1 \\ 2x = 3y - 1 \end{cases}$$
29. 
$$\begin{cases} 2x = 3y + 2 \\ 5x = 51 - 4y \end{cases}$$

41. 
$$\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$$

52. 
$$\begin{cases} x + 2y - z = 5 \\ 2x - y + 3z = 0 \end{cases}$$

$$5x = 51 - 4y$$

$$y = -4x + 2$$

42. 
$$\begin{cases} 3x - 4y + 4z = 7 \\ x - y - 2z = 2 \\ 2x - 3y + 6z = 5 \end{cases}$$

53. 
$$\begin{cases} x + y + z = 6 \\ 3x + 4y - 7z = 1 \\ 2x - y + 3z = 5 \end{cases}$$

30. 
$$\begin{cases} y = -4x + 2 \\ 2x = 3y - 1 \end{cases}$$
31. 
$$\begin{cases} 3x = 2 - 3y \\ 2y = 3 - 2x \end{cases}$$

54. 
$$\begin{cases} 3x + 2y + 3z = 3 \\ 4x - 5y + 7z = 1 \\ 2x + 3y - 2z = 6 \end{cases}$$

56. 
$$\begin{cases} x - 4y + z = 6 \\ 4x - y + 2z = -1 \\ 2x + 2y - 3z = -20 \end{cases}$$

54. 
$$\begin{cases} 3x + 2y + 3z = 3 \\ 4x - 5y + 7z = 1 \\ 2x + 3y - 2z = 6 \end{cases}$$
56. 
$$\begin{cases} x - 4y + z = 6 \\ 4x - y + 2z = -1 \\ 2x + 2y - 3z = -20 \end{cases}$$
58. 
$$\begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$$
55. 
$$\begin{cases} 4x + 5y = 2 \\ 11x + y + 2z = 3 \\ x + 5y + 2z = 1 \end{cases}$$
57. 
$$\begin{cases} 2x - y + z = -1 \\ 3x + 4y - z = -1 \\ 4x - y + 2z = -1 \end{cases}$$

55. 
$$\begin{cases} 4x + 5y = 2\\ 11x + y + 2z = 3\\ x + 5y + 2z = 1 \end{cases}$$

57. 
$$\begin{cases} 2x - y + z = -1 \\ 3x + 4y - z = -1 \\ 4x - y + 2z = -1 \end{cases}$$

Show that the matrix A is invertible for all values of  $\theta$ , then find  $A^{-1}$  using  $A^{-1} = \frac{1}{\det(A)} adj(A)$ 

$$A = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Section 1.8 – Applications

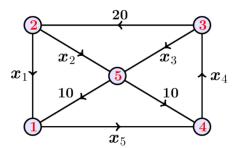
# Network Analysis

Networks composed of branches and junctions are used as models in such fields as economics, traffic analysis, and electrical engineering.

In a network model, you assume that the total flow into a junction is equal to the total flow out of the junction

## **Example**

Set up a system of linear equations to represent the network shown below. Then solve the system for  $x_i$ , i = 1, 2, 3, 4, 5.



#### Solution

$$1 \rightarrow x_1 + 10 = x_5 \implies x_1 - x_5 = -10$$

$$2 \rightarrow x_1 + x_2 = 20$$

$$3 \rightarrow x_4 = x_3 + 20 \implies -x_3 + x_4 = 20$$

$$4 \rightarrow x_4 = x_5 + 10 \implies x_4 - x_5 = 10$$

$$5 \rightarrow x_2 + x_3 = 10 + 10 = 20$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & | & -10 \\ 1 & 1 & 0 & 0 & 0 & | & 20 \\ 0 & 0 & -1 & 1 & 0 & | & 20 \\ 0 & 0 & 0 & 1 & -1 & | & 10 \\ 0 & 1 & 1 & 0 & 0 & | & 20 \\ \end{pmatrix} \quad R_2 - R_1$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & | & -10 \\ 0 & 1 & 0 & 0 & 1 & | & 30 \\ 0 & 0 & -1 & 1 & 0 & | & 20 \\ 0 & 0 & 0 & 1 & -1 & | & 10 \\ 0 & 1 & 1 & 0 & 0 & | & 20 \end{pmatrix} \quad R_5 - R_2$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & | & -10 \\ 0 & 1 & 0 & 0 & 1 & | & 30 \\ 0 & 0 & -1 & 1 & 0 & | & 20 \\ 0 & 0 & 0 & 1 & -1 & | & 10 \\ 0 & 0 & 1 & 0 & -1 & | & -10 \end{pmatrix} \quad R_5 + R_3$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & | & -10 \\ 0 & 1 & 0 & 0 & 1 & | & 30 \\ 0 & 0 & 1 & 1 & 0 & | & 20 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & | & -10 \\ 0 & 1 & 0 & 0 & 1 & | & 30 \\ 0 & 0 & -1 & 1 & 0 & | & 20 \\ 0 & 0 & 0 & 1 & -1 & | & 10 \\ 0 & 0 & 0 & 1 & -1 & | & 10 \end{pmatrix} \quad R_5 - R_4$$

**Solution**: 
$$(x_5 - 10, 30 - x_5, x_5 - 10, 10 + x_5, x_5)$$

### 2<sup>nd</sup> Method

$$\begin{vmatrix} 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{vmatrix} - 1 \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= -1 + 1$$

$$= 0$$

#### Infinite solution:

$$\begin{array}{cccc}
1 & \xrightarrow{x_1 = x_5 - 10} \\
2 & \xrightarrow{x_2 = 20 - x_1} & = 30 - x_5 \\
4 & \xrightarrow{x_3 = x_3 + 10}
\end{array}$$

$$4 \rightarrow x_4 = x_5 + 10$$

$$3 \rightarrow x_3 = x_4 - 20 = x_5 - 10$$

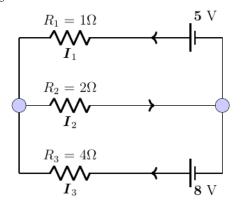
### Electrical network

An electrical network is another type of network where analysis is commonly applied. An analysis of such a system uses two properties of electrical networks known as Kirchhoff's Laws.

- All the current flowing into a junction must flow out of it.
- The sum of the products IR (I is current and R is resistance) around a closed path is equal to the total voltage in the path.

## **Example**

Determine the currents  $I_1$ ,  $I_2$ , and  $I_3$  for the electrical network

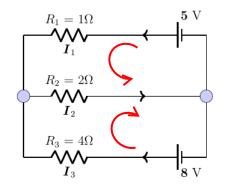


#### Solution

$$\begin{split} I_2 &= I_1 + I_3 \\ I_1 + 2I_2 &= 5 \\ 2I_2 + 4I_3 &= 8 \\ \begin{cases} I_1 - I_2 + I_3 &= 0 \\ I_1 + 2I_2 &= 5 \\ I_2 + 2I_3 &= 4 \end{cases} \end{split}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 7 \qquad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 5 & 2 & 0 \\ 4 & 1 & 2 \end{vmatrix} = 7 \qquad D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 5 & 0 \\ 0 & 4 & 2 \end{vmatrix} = 14 \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 5 \\ 0 & 1 & 4 \end{vmatrix} = 7$$

 $I_1 = 1 A \mid I_2 = 2 A \mid I_3 = 1 A \mid$ 



$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 5 & 0 \\ 0 & 4 & 2 \end{vmatrix} = 14$$
  $D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 5 \\ 0 & 1 & 4 \end{vmatrix} = 7$ 

## Cryptography

A *cryptogram* is a message written according to a secret code (the Greek word *kryptos* means "hidden"). One method of using matrix multiplication to *encode* and *decode* messages.

Let assign a number to each letter in the alphabet (with 0 assigned to a blank space), as shown

5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
7 = G	11 = K	15 = O	19 = S	23 = W	
	5 = E $6 = F$	5 = E $9 = I$ $10 = J$	5 = E $9 = I$ $13 = M$ $6 = F$ $10 = J$ $14 = N$	5 = E $9 = I$ $13 = M$ $17 = Q$ $6 = F$ $10 = J$ $14 = N$ $18 = R$	

77

## **Example**

Consider the invertible matrix: 
$$A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix}$$

The message: **MEET ME MONDAY** 

- a) Write the uncoded row matrices  $1 \times 3$  for the message.
- b) Use the matrix A to encode the message.
- c) Decode a message from part b) given the matrix A.

#### Solution

b) Let encode the message **MEET ME MONDAY** 

$$\begin{bmatrix} 13 & 5 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -26 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 33 & -53 & -12 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 18 & -23 & -42 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 14 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 5 & -20 & 56 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 25 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} -24 & 23 & 77 \end{bmatrix}$$

The sequence of coded row matrices is

$$\begin{bmatrix} 13 & -26 & -21 \end{bmatrix}$$
  $\begin{bmatrix} 33 & -53 & -12 \end{bmatrix}$   $\begin{bmatrix} 18 & -23 & -42 \end{bmatrix}$   $\begin{bmatrix} 5 & -20 & 56 \end{bmatrix}$   $\begin{bmatrix} -24 & 23 & 77 \end{bmatrix}$ 

The cryptogram:

$$13 - 26 - 21 \ 33 - 53 - 12 \ 18 - 23 - 42 \ 5 - 20 \ 56 - 24 \ 23 \ 77$$

c) To decode a message given the matrix A.

$$|A| = \begin{vmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{vmatrix} = 1$$

$$A^{-1} = \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$$

With the cryptogram:

$$\begin{bmatrix} 13 & -26 & -21 \end{bmatrix} \ \begin{bmatrix} 33 & -53 & -12 \end{bmatrix} \ \begin{bmatrix} 18 & -23 & -42 \end{bmatrix} \ \begin{bmatrix} 5 & -20 & 56 \end{bmatrix} \ \begin{bmatrix} -24 & 23 & 77 \end{bmatrix}$$

$$\begin{bmatrix} 13 & -26 & 21 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 13 & 5 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 33 & -53 & -1 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 18 & -23 & -42 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 13 \end{bmatrix}$$

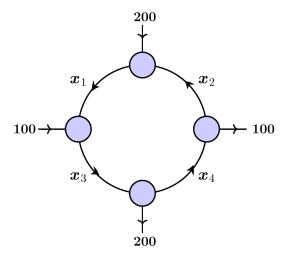
$$\begin{bmatrix} 5 & -20 & 56 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 15 & 14 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -24 & 23 & 77 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 25 & 0 \end{bmatrix}$$

The message is:

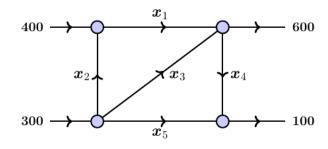
# **Exercises** Section 1.8 – Applications

1. The flow of traffic, in vehicles per hour, through a network of streets as is shown below

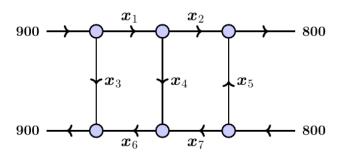


- a) Solve this system for  $x_i$ , i = 1, 2, 3, 4.
- b) Find the traffic flow when  $x_4 = 0$ .
- c) Find the traffic flow when  $x_4 = 100$ .
- d) Find the traffic flow when  $x_1 = 2x_2$ .

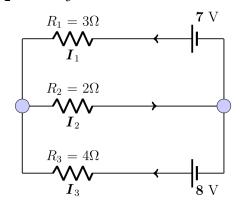
2. Through a network, Express  $x_n$ 's in terms of the parameters s and t.



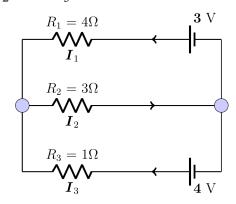
3. Water is flowing through a network of pipes. Express  $x_n$ 's in terms of the parameters s and t.



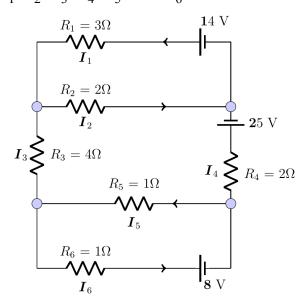
**4.** Determine the currents  $I_1$ ,  $I_2$ , and  $I_3$  for the electrical network shown below



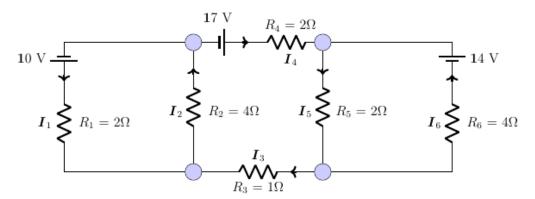
5. Determine the currents  $I_1$ ,  $I_2$ , and  $I_3$  for the electrical network shown below



**6.** Determine the currents  $I_1$ ,  $I_2$ ,  $I_3$ ,  $I_4$ ,  $I_5$ , and  $I_6$  for the electrical network shown below



7. Determine the currents  $I_1$ ,  $I_2$ ,  $I_3$ ,  $I_4$ ,  $I_5$ , and  $I_6$  for the electrical network shown below



8. Consider the invertible matrix:  $A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -2 & -2 & 7 \end{pmatrix}$ 

The message: ICEBERG DEAD AHEAD

- a) Write the uncoded row matrices  $1 \times 3$  for the message.
- b) Use the matrix A to encode the message.
- c) Decode a message from part b) given the matrix A.
- 9. You want to send the message: LINEAR ALGEBRA with a key word MATH
  - a) Write the matrix A.
  - b) Write the uncoded row matrices  $1 \times 2$  for the message.
  - c) Use the matrix A to encode the message.
  - d) Decode a message from part b) given the matrix A.
- 10. Write the matrix A with a key word MATH, then decode the cryptogram

117 9 456 132 386 62 260 104 413 161 104 8

11. Write the matrix A with a key word MATH, then decode the cryptogram

438 150 145 37 240 96 635 191 445 157 260 104 413 161 104 8

**12.** Consider the invertible matrix:  $A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix}$ 

Decode the cryptogram

13. Determine the key word, then decode the given cryptogram

6 18 5 4 15 13 1 20 8 102 649 238 57 324 112 128 622 207 180 613 290 102 360 259 151 580 297

*Hint*: First row is the key