Solution Section 2.5 – Variation of Parameters

Exercise

$\left\{y_1(x)=e^{2x},\ y_2(x)=e^{-3x}\right\}$ is a fundamental set of solutions of $y''+y'-6y=3e^{2x}$.

Find a particular solution of the equation?

Solution

$$W=\begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x} \end{vmatrix}=-3e^{-x}-2e^{-x}\ \underline{=-5e^{-x}\neq 0}$$

$$v_1(x)=-\int\frac{e^{-3x}\left(3e^{2x}\right)}{-5e^{-x}}dx \qquad\qquad v_1(x)=-\int\frac{y_2\,g(x)}{W}dx$$

$$=\frac{3}{5}\int dx$$

$$=\frac{3}{5}x$$

$$v_2(x)=\int\frac{e^{2x}\left(3e^{2x}\right)}{-5e^{-x}}dx \qquad\qquad v_2(x)=\int\frac{y_1\,g(x)}{W}dx$$

$$=-\frac{3}{5}\int e^{5x}dx$$

$$=-\frac{3}{25}e^{5x}$$

The particular solution:

$$y_p=v_1y_1+v_2y_2$$

$$=\frac{3}{5}xe^{2x}-\frac{3}{25}e^{-3x}e^{5x}$$

$$=\frac{3}{5}xe^{2x}-\frac{3}{25}e^{2x}$$

The general solution:

$$y(x)=C_1e^{2x}+C_2e^{-3x}+\frac{3}{5}xe^{2x}-\frac{3}{25}e^{2x}$$

$$=\left(C_1-\frac{3}{25}\right)e^{2x}+C_2e^{-3x}+\frac{3}{5}xe^{2x}$$

$$=C_3e^{2x}+C_2e^{-3x}+\frac{3}{5}xe^{2x}$$

Exercise

Find a particular solution to: $y'' - y = t + 3$

Solution

The homogeneous equation for the differential equation $y'' - y = 0$

$$\lambda^2 - 1 = 0 \qquad \textit{Solve for } \lambda$$

$$\lambda_1 = -1 \quad \lambda_2 = 1$$

Therefore; $y_1 = e^{-t}$ *and* $y_2 = e^t$

$$W = \begin{vmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{vmatrix} = 1 + 1 = 2 \neq 0$$

$$v_1' = \frac{-y_2}{y_1 y_2' - y_1' y_2} g(t)$$

$$= -\frac{e^t}{2}(t+3)$$

$$v_1(t) = -\frac{1}{2}\int (t+3)e^t dt \qquad \begin{cases} u = t+3 & dv = e^t dt \\ du = dt & v = e^t \end{cases}$$

$$= -\frac{1}{2}\left[e^t(t+3) - \int e^t dt\right]$$

$$= -\frac{1}{2}\left(te^t + 3e^t - e^t\right)$$

$$= -\frac{1}{2}\left(te^t + 2e^t\right)$$

$$= -\left(\frac{1}{2}te^t + e^t\right)$$

$$v_2' = \frac{y_1}{y_1 y_2' - y_1' y_2} g(t)$$

$$= \frac{e^{-t}}{2}(t+3)$$

$$v_1(t) = \frac{1}{2}\int (t+3)e^{-t} dt \qquad \begin{cases} u = t+3 & dv = e^{-t} dt \\ du = dt & v = -e^t \end{cases}$$

$$= \frac{1}{2}\left[-e^{-t}(t+3) + \int e^{-t} dt\right]$$

$$= \frac{1}{2}\left(-te^{-t} - 3e^{-t} - e^{-t}\right)$$

$$= -\frac{1}{2}\left(te^{-t} + 4e^{-t}\right)$$

$$= -\frac{1}{2}te^{-t} - 2e^{-t}$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$= -\left(\frac{1}{2}te^t + e^t\right)e^{-t} - \left(\frac{1}{2}te^{-t} + 2e^{-t}\right)e^t$$

$$= -\frac{1}{2}t - 1 - \frac{1}{2}t - 2$$

$$\underline{= -t - 3}$$

Exercise

Find a particular solution to: $y'' - 2y' + y = e^t$

Solution

The homogeneous equation for the differential equation $y'' - 2y' + y = 0$

$\lambda^2 - 2\lambda + 1 = 0$ ***Solve for λ***

$\lambda_{1,2} = 1$

Therefore; $y_1 = e^t$ *and* $y_2 = te^t$

$$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}$$

$$= e^{2t} + te^{2t} - te^{2t}$$

$$= e^{2t} \neq 0$$

$$v_1' = \frac{-y_2}{y_1 y_2' - y_1' y_2} g(t)$$

$$= -\frac{te^t}{e^t\left(e^t + te^t\right) - e^t . te^t} e^t$$

$$= -\frac{te^{2t}}{e^{2t}}$$

$$= -t$$

$$v_1(t) = \int -t dt$$

$$= -\frac{1}{2}t^2$$

$$v_2' = \frac{y_1}{y_1 y_2' - y_1' y_2} g(t)$$

$$= \frac{e^t}{e^{2t}} e^t$$

$$= 1$$

$$v_2(t) = \int 1\, dt$$

$$= t$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$= -\frac{1}{2}t^2 e^t + t^2 e^t$$

$$\boxed{= \frac{1}{2}t^2 e^t}$$

Exercise

Find a particular solution to: $x'' - 4x' + 4x = e^{2t}$

Solution

The homogeneous equation for the differential equation: $x'' - 4x' + 4x = 0$

$\lambda^2 - 4\lambda + 4 = 0$ *Solve for λ*

$\rightarrow \boxed{\lambda_{1,2} = 2}$

Therefore; $x_1 = e^{2t}$ *and* $x_2 = te^{2t}$

$$W = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t} + 2te^{2t} \end{vmatrix}$$

$$= e^{4t} + 2te^{4t} - 2te^{4t}$$

$$= e^{4t} \neq 0$$

$v_1' = \frac{-y_2}{W} g(t)$ $= -\frac{te^{2t}}{e^{4t}} e^{2t}$ $= -t$ $v_1(t) = \int -t\ dt$ $= -\frac{1}{2}t^2$	$v_2' = \frac{y_1}{W} g(t)$ $= \frac{e^{2t}}{e^{4t}} e^{2t}$ $= 1$ $v_2(t) = \int 1\ dt = t$

$$y_p = v_1 y_1 + v_2 y_2$$

$$= -\frac{1}{2}t^2 e^{2t} + t^2 e^{2t}$$

$$\boxed{= \frac{1}{2}t^2 e^{2t}}$$

Exercise

Find a particular solution to: $x'' + x = \tan^2 t$

Solution

The homogeneous equation for the differential equation: $x'' + x = 0$

$x_1 = \cos t$ *and* $x_2 = \sin t$

$$W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$$

$$=1$$

$$x_p = v_1 x_1 + v_2 x_2$$

$$v_1' = \frac{-\sin t}{1}\tan^2 t$$
$$= -\sin t\left(\sec^2 t - 1\right)$$
$$= -\sin t\left(\frac{1}{\cos^2 t} - 1\right)$$
$$= -\frac{\sin t}{\cos^2 t} + \sin t$$
$$= -\sec t\tan t + \sin t$$

$$v_1 = -\sec t - \cos t$$

$$v_2' = \frac{\cos t}{1}\tan^2 t$$
$$= \cos t\left(\sec^2 t - 1\right)$$
$$= \sec t - \cos t$$
$$v_2 = \ln\left|\sec t + \tan t\right| - \sin t$$

$$x_p = \left(-\sec t - \cos t\right)\cos t + \left(\ln\left|\sec t + \tan t\right| - \sin t\right)\sin t$$
$$= -\sec t\cos t - \cos^2 t + \sin t\ln\left|\sec t + \tan t\right| - \sin^2 t$$
$$= -\sec t\frac{1}{\sec t} - \left(\cos^2 t + \sin^2 t\right) + \sin t\ln\left|\sec t + \tan t\right|$$
$$\boxed{= -2 + \sin t\ln\left|\sec t + \tan t\right|}$$

Exercise

Find a particular solution to the given second-order differential equation $y'' + 25y = -2\tan(5x)$

Solution

$$\lambda^2 + 25 = 0 \quad\Rightarrow\quad \lambda_{1,2} = \pm 5i$$

$$y_p = C_1\cos 5x + C_2\sin 5x$$

$$W = \begin{vmatrix} \cos 5x & \sin 5x \\ -5\sin 5x & 5\cos 5x \end{vmatrix} = 5\cos^2 5x + 5\sin^2 5x \ \boxed{= 5 \ne 0}$$

$$v_1(x) = -\int \frac{\sin 5x(-2\tan 5x)}{5}dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$= \frac{2}{5}\int \frac{\sin^2 5x}{\cos 5x}dx$$

$$=\frac{2}{5}\int\frac{1-\cos^2 5x}{\cos 5x}dx$$

$$=\frac{2}{5}\int(\sec 5x-\cos 5x)\,dx$$

$$=\frac{2}{5}\left[\frac{1}{5}\ln\left|\tan 5x+\sec 5x\right|-\frac{1}{5}\sin 5x\right]$$

$$=\frac{2}{25}\left(\ln\left|\tan 5x+\sec 5x\right|-\sin 5x\right)$$

$$v_2(x)=\int\frac{\cos 5x(-2\tan 5x)}{5}dx \qquad v_2(x)=\int\frac{y_1 g(x)}{W}dx$$

$$=-\frac{2}{5}\int\sin 5x\,dx$$

$$=\boxed{\frac{2}{25}\cos 5x}$$

$$y_p=v_1y_1+v_2y_2$$

$$=\frac{2}{25}\left(\ln\left|\tan 5x+\sec 5x\right|-\sin 5x\right)(\cos 5x)+\frac{2}{25}\cos 5x\sin 5x$$

$$=\boxed{\frac{2}{25}\ln\left|\tan 5x+\sec 5x\right|}$$

Exercise

Find a particular solution to the given second-order differential equation $y''-6y'+9y=5e^{3x}$

Solution

$$\lambda^2-6\lambda+9=(\lambda-3)^2=0\quad\Rightarrow\quad\boxed{\lambda_{1,2}=3}$$

$$y_h=(C_1+C_2x)e^{3x}$$

$$W=\begin{vmatrix}e^{3x} & xe^{3x}\\ 3e^{3x} & e^{3x}+3xe^{3x}\end{vmatrix}=e^{6x}+3xe^{6x}-3xe^{6x}\ \boxed{=e^{6x}\neq 0}$$

$$v_1(x)=-\int\frac{xe^{3x}\left(5e^{3x}\right)}{e^{6x}}dx=-5\int x\,dx\ \boxed{=-\frac{5}{2}x^2} \qquad v_1(x)=-\int\frac{y_2 g(x)}{W}dx$$

$$=-5\int x\,dx$$

$$=\boxed{-\frac{5}{2}x^2}$$

$$v_2(x) = \int \frac{e^{3x}\left(5e^{3x}\right)}{e^{6x}}dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$= 5\int dx$$

$$\underline{= 5x|}$$

$$y_p = -\frac{5}{2}x^2e^{3x} + 5x^2e^{3x} \qquad\qquad y_p = v_1y_1 + v_2y_2$$

$$\boxed{= \frac{5}{2}x^2e^{3x}}$$

Exercise

Find a particular solution to the given second-order differential equation $y'' + 4y = 2\cos 2x$

Solution

$$\lambda^2 + 4 = 0 \quad \Rightarrow \quad \underline{\lambda_{1,2} = \pm 2i|}$$

$$y_h = C_1\cos 2x + C_2\sin 2x$$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x \underline{= 2 \neq 0|}$$

$$v_1(x) = -\int \frac{\sin 2x(2\cos 2x)}{2}dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$= -\frac{1}{2}\int \sin 4x dx$$

$$= \frac{1}{8}\cos 4x$$

$$v_2(x) = \int \frac{\cos 2x(2\cos 2x)}{2}dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$= \int \cos^2 2x dx$$

$$= \frac{1}{2}\int (1 + \cos 4x)dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right)$$

$$\boxed{y_p = \frac{1}{2}x\sin 2x} \qquad\qquad y_p = v_1y_1 + v_2y_2$$

Exercise

Find a particular solution to the given second-order differential equation $y'' - 5y' + 6y = 4e^{2x} + 3$

Solution

$$\lambda^2 - 5\lambda + 6 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = \frac{5 \pm \sqrt{1}}{2} = 2,\ 3}$$

$$y_h = C_1 e^{2x} + C_2 e^{3x}$$

$$W = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x} \neq 0$$

$$v_1(x) = -\int \frac{e^{3x}\left(4e^{2x} + 3\right)}{e^{5x}} dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\int \left(4 + 3e^{-2x}\right) dx$$

$$= -4x + \frac{3}{2} e^{-2x}$$

$$v_2(x) = \int \frac{e^{2x}\left(4e^{2x} + 3\right)}{e^{5x}} dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \int \left(4e^{-x} + 3e^{-3x}\right) dx$$

$$= -4e^{-x} - e^{-3x}$$

$$y_p = \left(-4x + \frac{3}{2} e^{-2x}\right) e^{2x} - \left(4e^{-x} + e^{-3x}\right) e^{3x} \qquad\qquad y_p = v_1 y_1 + v_2 y_2$$

$$= -4xe^{2x} + \frac{3}{2} - 4e^{2x} - 1$$

$$\boxed{= -4xe^{2x} - 4e^{2x} + \frac{1}{2}}$$

Exercise

Verify that $y_1(t) = t$ and $y_2(t) = t^{-3}$ are solution to the homogenous equation

$$t^2 y''(t) + 3ty'(t) - 3y(t) = 0$$

Solution

The homogeneous equation for the differential equation: $y'' + \frac{3}{t} y' - \frac{3}{t^2} y = 0$

For $y_1 = t \quad \to y_1' = 1 \quad \to y_1'' = 0$

$$y'' + \frac{3}{t}y' - \frac{3}{t^2}y = 0 + \frac{3}{t}(1) - \frac{3}{t^2}t$$

$$= \frac{3}{t} - \frac{3}{t}$$

$$= 0$$

$y_1(t)$ is a solution

For $y_2 = t^{-3} \quad \to y_1' = -3t^{-4} \quad \to y_1'' = 12t^{-5}$

$$y'' + \frac{3}{t}y' - \frac{3}{t^2}y = 12t^{-5} + \frac{3}{t}\left(-3t^{-4}\right) - \frac{3}{t^2}t^{-3}$$

$$= 12t^{-5} - 9t^{-5} - 3t^{-5}$$

$$= 0$$

$y_2(t)$ is a solution

Wronskian: $W\left(t, t^{-3}\right) = \begin{vmatrix} t & t^{-3} \\ 1 & -3t^{-4} \end{vmatrix} = -4t^{-3}$

$$v_1' = -\frac{t^{-3}t^{-3}}{-4t^{-3}} = \frac{1}{4}t^{-3} \quad \Rightarrow \quad v_1 = \int \left(\frac{1}{4}t^{-3}\right)dt = -\frac{1}{8}t^{-2}$$

$$v_2' = -\frac{t \cdot t^{-3}}{-4t^{-3}} = -\frac{1}{4}t \quad \Rightarrow \quad v_2 = \int \left(-\frac{1}{4}t\right)dt = -\frac{1}{8}t^2$$

$$y_p = v_1y_1 + v_2y_2$$

$$= -\frac{1}{8}t^{-2}t - \frac{1}{8}t^2t^{-3}$$

$$= -\frac{1}{8}t^{-1} - \frac{1}{8}t^{-1}$$

$$= -\frac{1}{4}t^{-1}$$

Thus, the general solution is: $y(t) = C_1t + \frac{C_2}{t^3} - \frac{1}{4t}$

Exercise

Find the general solution $\quad y'' - y = \frac{1}{x}$

Solution

Characteristic Eqn.: $\lambda^2 - 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 1$

The homogeneous Eqn.: $y_h = C_1e^{-x} + C_2e^{x}$

$$W=\begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x}\end{vmatrix}=1+1=\boxed{2\ \neq 0}$$

$$v_1(x)=-\int\frac{e^{x}\frac{1}{x}}{2}dx \qquad\qquad v_1(x)=-\int\frac{y_2 g(x)}{W}dx$$

$$=-\frac{1}{2}\int\frac{e^{x}}{x}dx$$

$$v_2(x)=\int\frac{e^{-x}\frac{1}{x}}{2}dx \qquad\qquad v_2(x)=\int\frac{y_1 g(x)}{W}dx$$

$$=\frac{1}{2}\int\frac{e^{-x}}{x}dx$$

$$y_p=-\frac{1}{2}e^{-x}\int\frac{e^{x}}{x}dx+\frac{1}{2}e^{x}\int\frac{e^{-x}}{x}dx \qquad\qquad y_p=u_1y_1+u_2y_2$$

The ***general*** solution: $\boxed{y(x)=C_1e^{-x}+C_2e^{x}-\frac{1}{2}e^{-x}\int\frac{e^{x}}{x}dx+\frac{1}{2}e^{x}\int\frac{e^{-x}}{x}dx}$

Exercise

Find the general solution $\quad y''-y=\sinh 2x$

Solution

Characteristic Eqn.: $\quad \lambda^2-1=0 \ \rightarrow\ \boxed{\lambda_{1,2}=\pm 1}$

$$\boxed{y_h=C_1e^{-x}+C_2e^{x}}$$

$$W=\begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x}\end{vmatrix}=1+1=\boxed{2\ \neq 0}$$

$$v_1(x)=-\int\frac{e^{x}\sinh 2x}{2}dx \qquad\qquad v_1(x)=-\int\frac{y_2 g(x)}{W}dx$$

$$=-\frac{1}{4}\int e^{x}\left(e^{2x}-e^{-2x}\right)dx$$

$$=-\frac{1}{4}\int\left(e^{3x}-e^{-x}\right)dx$$

$$\boxed{=-\frac{1}{4}\left(\frac{1}{3}e^{3x}+e^{-x}\right)}$$

$$v_2(x) = \int \frac{e^{-x}\sinh 2x}{2}dx = \frac{1}{4}\int e^{-x}\left(e^{2x} - e^{-2x}\right)dx \qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$= \frac{1}{4}\int \left(e^{x} - e^{-3x}\right)dx$$

$$= \frac{1}{4}\left(e^{x} + \frac{1}{3}e^{-3x}\right)$$

$$y_p = \left(-\frac{1}{12}e^{3x} - \frac{1}{4}e^{-x}\right)e^{-x} + \left(\frac{1}{4}e^{x} + \frac{1}{12}e^{-3x}\right)e^{x} \qquad y_p = u_1y_1 + u_2y_2$$

$$= -\frac{1}{12}e^{2x} - \frac{1}{4}e^{-2x} + \frac{1}{4}e^{2x} + \frac{1}{12}e^{-x}$$

$$= -\frac{1}{4}e^{-2x} + \frac{1}{6}e^{2x} + \frac{1}{12}e^{-x}$$

The ***general*** solution: $y(x) = C_1e^{-x} + C_2e^{x} - \frac{1}{4}e^{-2x} + \frac{1}{6}e^{2x} + \frac{1}{12}e^{-x}$

Exercise

Find the general solution $y'' - y = x$

Solution

Characteristic Eqn.: $\lambda^2 - 1 = 0 \rightarrow \lambda_{1,2} = \pm 1$

$$y_h = C_1e^{-x} + C_2e^{x}$$

$$W = \begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix} = 1 + 1 = 2 \neq 0$$

$$v_1(x) = -\int \frac{e^{x}x}{2}dx \qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$= -\frac{1}{2}(x-1)e^{x}$$

		$\int e^{x}$
+	x	e^{x}
−	1	e^{x}

$$v_2(x) = \int \frac{e^{-x}x}{2}dx \qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$= -\frac{1}{2}(x+1)e^{-x}$$

		$\int e^{-x}$
+	x	$-e^{-x}$
−	1	e^{-x}

$$y_p = -\frac{1}{2}(x-1)e^{x}e^{-x} - \frac{1}{2}(x+1)e^{x}e^{-x} \qquad y_p = u_1y_1 + u_2y_2$$

$$= -x$$

The ***general*** solution: $y(x) = C_1e^{-x} + C_2e^{x} - x$

Exercise

Find the general solution $y'' - y = \cosh x$

Solution

Characteristic Eqn.: $\lambda^2 - 1 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 1}$

$$\boxed{y_h = A_1 e^{-x} + A_2 e^{x}}$$

$$W = \begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix} = 1 + 1 = \boxed{2 \ \neq 0}$$

$$v_1(x) = -\int \frac{e^{x}\cosh x}{2}\,dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W}\,dx$$

$$-\frac{1}{2}\int e^{x}\,\frac{e^{x}+e^{-x}}{2}\,dx$$

$$= -\frac{1}{4}\int \left(e^{2x}+1\right)dx$$

$$= -\frac{1}{4}\left(\frac{1}{2}e^{2x}+x\right)$$

$$\boxed{= -\frac{1}{8}e^{2x} - \frac{1}{4}x}$$

$$v_2(x) = \int \frac{e^{-x}\cosh x}{2}\,dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W}\,dx$$

$$= \frac{1}{4}\int e^{-x}\left(e^{x}+e^{-x}\right)dx$$

$$= \frac{1}{4}\int \left(1+e^{-2x}\right)dx$$

$$\boxed{= \frac{1}{4}x - \frac{1}{8}e^{-2x}}$$

$$y_p = \left(-\frac{1}{8}e^{2x} - \frac{1}{4}x\right)e^{-x} + \left(\frac{1}{4}x - \frac{1}{8}e^{-2x}\right)e^{x} \qquad\qquad y_p = u_1 y_1 + u_2 y_2$$

$$= -\frac{1}{8}e^{x} - \frac{1}{4}xe^{-x} + \frac{1}{4}xe^{x} - \frac{1}{8}e^{-x}$$

$$= -\frac{1}{8}e^{x} - \frac{1}{8}e^{-x} + \frac{1}{2}x\left(\frac{e^{x}-e^{-x}}{2}\right)$$

$$\boxed{= -\frac{1}{8}e^{x} - \frac{1}{8}e^{-x} + \frac{1}{2}x\sinh x}$$

$$y(x) = A_1 e^{-x} + A_2 e^{x} - \frac{1}{8}e^{x} - \frac{1}{8}e^{-x} + \frac{1}{2}x\sinh x$$

$$= \left(A_1 - \frac{1}{8}\right)e^{-x} + \left(A_2 - \frac{1}{8}\right)e^{x} + \frac{1}{2}x\sinh x$$

The ***general*** solution: $\boxed{y(x)=C_1e^{-x}+C_2e^{x}+\frac{1}{2}x\sinh x}$

Or $\boxed{y(x)=C_1e^{-x}+C_2e^{x}-\frac{1}{4}xe^{-x}+\frac{1}{4}xe^{x}}$

Exercise

Find the general solution $y''+y=\sin x$

Solution

Characteristic Eqn.: $\lambda^2+1=0 \Rightarrow \lambda_{1,2}=\pm i$

The homogeneous Eqn.: $\boxed{y_h=C_1\cos x+C_2\sin x}$

$$W=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=\cos^2 x+\sin^2 x=\boxed{1\ \neq 0}$$

$$v_1(x)=-\int\frac{(\sin x)(\sin x)}{1}dx \qquad\qquad v_1(x)=-\int\frac{y_2 g(x)}{W}dx$$

$$=-\int\left(\sin^2 x\right)dx$$

$$=\boxed{-\frac{x}{2}+\frac{1}{4}\sin 2x}$$

$$v_2(x)=\int\frac{(\cos x)(\sin x)}{1}dx \qquad\qquad v_2(x)=\int\frac{y_1 g(x)}{W}dx$$

$$=\int\sin x\, d(\sin x)$$

$$=\boxed{\frac{1}{2}\sin^2 x}$$

$$y_p=\left(-\frac{x}{2}+\frac{1}{4}\sin 2x\right)\cos x+\frac{1}{2}\sin^2 x\sin x \qquad\qquad y_p=u_1y_1+u_2y_2$$

$$=-\frac{1}{2}x\cos x+\frac{1}{4}\sin 2x\cos x+\frac{1}{2}\sin^3 x$$

$$=-\frac{1}{2}x\cos x+\frac{1}{2}\sin x\cos^2 x+\frac{1}{2}\sin^3 x$$

$$=-\frac{1}{2}x\cos x+\frac{1}{2}\sin x\left(\cos^2 x+\sin^2 x\right)$$

$$=-\frac{1}{2}x\cos x+\frac{1}{2}\sin x$$

The ***general*** solution: $\boxed{y(x)=C_1\cos x+C_2\sin x-\frac{1}{2}x\cos x+\frac{1}{2}\sin x}$

Exercise

Find the general solution $y'' - y = e^{x}$

Solution

Characteristic Eqn.: $\lambda^2 - 1 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 1}$

$$\boxed{y_h = C_1 e^{-x} + C_2 e^{x}}$$

$$W = \begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix} = 1 + 1 = \boxed{2 \neq 0}$$

$$v_1(x) = -\int \frac{e^{x} e^{x}}{2} dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= \boxed{-\frac{1}{4} e^{2x}}$$

$$v_2(x) = \int \frac{e^{-x} e^{x}}{2} dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \boxed{\frac{1}{2} x}$$

$$y_p = -\frac{1}{4} e^{2x} e^{-x} + \frac{1}{2} x e^{x} \qquad\qquad y_p = u_1 y_1 + u_2 y_2$$

$$= \boxed{-\frac{1}{4} e^{x} + \frac{1}{2} x e^{x}}$$

The ***general*** solution:

$$\boxed{y(x) = C_1 e^{-x} + C_2 e^{x} - \frac{1}{4} e^{x} + \frac{1}{2} x e^{x}}$$

Exercise

Find the general solution $y'' + y = \sec x$

Solution

Characteristic Eqn.: $\lambda^2 + 1 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm i}$

$$\boxed{y_h = C_1 \cos x + C_2 \sin x}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = \boxed{1 \neq 0}$$

$$v_1(x) = -\int \sin x \sec x \, dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\int \tan x \, dx$$

$$\underline{= -\ln|\sec x|}$$

$$v_2(x) = \int \cos x \sec x \, dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$\underline{= x}$$

$$\underline{y_p = -\cos x \ln|\sec x| + x \sin x} \qquad\qquad y_p = u_1 y_1 + u_2 y_2$$

The ***general*** solution:

$$\underline{y(x) = C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x| + x \sin x}$$

Exercise

Find the general solution $\quad y'' + y = \tan x$

Solution

Characteristic Eqn.: $\quad \lambda^2 + 1 = 0 \quad \to \quad \underline{\lambda_{1,2} = \pm i}$

$$\underline{y_h = C_1 \cos x + C_2 \sin x}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x \ \underline{= 1 \ \neq 0}$$

$$v_1(x) = -\int \sin x \tan x \, dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\int \frac{\sin^2 x}{\cos x} \, dx$$

$$= -\int \frac{1 - \cos^2 x}{\cos x} \, dx$$

$$= -\int (\sec x - \cos x) \, dx$$

$$\underline{= -\ln|\sec x + \tan x| + \sin x}$$

$$v_2(x) = \int \cos x \tan x \, dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \int \sin x \, dx$$

$$\underline{= -\cos x}$$

$$y_p = \left(-\ln|\sec x + \tan x| + \sin x\right)\cos x - \cos x \sin x \qquad\qquad y_p = u_1 y_1 + u_2 y_2$$

$$= -(\cos x)\ln|\sec x + \tan x|$$

The ***general*** solution:

$$y(x) = C_1 \cos x + C_2 \sin x - (\cos x)\ln|\sec x + \tan x|$$

Exercise

Find the general solution $\quad y'' + y = \sin x$

Solution

Characteristic Eqn.: $\quad \lambda^2 + 1 = 0 \rightarrow \lambda_{1,2} = \pm i$

$$y_h = C_1 \cos x + C_2 \sin x$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0$$

$$v_1(x) = -\int \sin^2 x\, dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\frac{1}{2}\int (1 - \cos 2x)\, dx$$

$$= -\frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)$$

$$v_2(x) = \int \cos x \sin x\, dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \frac{1}{2}\int \sin 2x\, dx$$

$$= -\frac{1}{4}\cos 2x$$

$$y_p = \left(-\frac{1}{2}x + \frac{1}{4}\sin 2x\right)\cos x - \frac{1}{4}\cos 2x \sin x \qquad\qquad y_p = u_1 y_1 + u_2 y_2$$

$$= -\frac{1}{2}x\cos x$$

The ***general*** solution:

$$y(x) = C_1 \cos x + C_2 \sin x - \frac{1}{2}x\cos x$$

Exercise

Find the general solution $\quad y'' + y = \csc x$

Solution

$$\textit{Characteristic Eqn.: } \lambda^2+1=0 \quad \Rightarrow \quad \lambda_{1,2}=\pm i$$

$$\boxed{y_h = C_1\cos x + C_2\sin x}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x \ \boxed{=1 \ \neq 0}$$

$$v_1(x) = -\int \frac{\sin x \csc x}{1}dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$= -\int dx$$

$$\boxed{=-x}$$

$$v_2(x) = \int \cos x \csc x \ dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$= \int \cot x \ dx$$

$$\boxed{= \ln|\sin x|}$$

$$\boxed{y_p = -x\cos x + \sin x \ln|\sin x|} \qquad\qquad y_p = u_1y_1 + u_2y_2$$

The ***general*** solution:

$$\boxed{y(x) = C_1\cos x + C_2\sin x - x\cos x + \sin x\ln|\sin x|}$$

Exercise

Find the general solution $\quad y'' + y = \cos^2 x$

Solution

$$\textit{Characteristic Eqn.: } \lambda^2+1=0 \quad \Rightarrow \quad \lambda_{1,2}=\pm i$$

$$\boxed{y_h = C_1\cos x + C_2\sin x}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$\boxed{=1 \ \neq 0}$$

$$v_1(x) = -\int \frac{\sin x\cos^2 x}{1}dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$= \int \cos^2 x \ d(\cos x)$$

$$= \frac{1}{3}\cos^3 x$$

$$v_2(x) = \int \cos x \cos^2 x \, dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \int \cos^3 x \, dx$$

$$= \int \cos^2 x \, d(\sin x)$$

$$= \int \left(1 - \sin^2 x\right) d(\sin x)$$

$$= \sin x - \frac{1}{3}\sin^3 x$$

$$y_p = \frac{1}{3}\cos^4 x + \sin^2 x - \frac{1}{3}\sin^4 x \qquad\qquad y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{1}{3}\left(\cos^4 x - \sin^4 x\right) + \sin^2 x$$

$$= \frac{1}{3}\left(\cos^2 x - \sin^2 x\right)\left(\cos^2 x + \sin^2 x\right) + \sin^2 x$$

$$= \frac{1}{3}\cos 2x + \sin^2 x$$

The ***general*** solution:

$$y(x) = C_1 \cos x + C_2 \sin x + \frac{1}{3}\cos 2x + \sin^2 x$$

Exercise

Find the general solution to the given differential equation. $y'' + y = \csc^2 x$

Solution

Characteristic Eqn.: $\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

$$y_h = C_1 \cos x + C_2 \sin x$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \ \neq 0$$

$$v_1(x) = -\int \sin x \csc^2 x \, dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\int \frac{1}{\sin x} dx$$

$$= -\ln\left|\csc x - \cot x\right|$$

$$v_2(x) = \int \cos x \csc^2 x \, dx \qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \int \frac{1}{\sin^2 x} d(\sin x)$$

$$= -\frac{1}{\sin x}$$

$$\underline{= -\csc x}$$

$$y_p = -\cos x \, \ln|\csc x - \cot x| - \csc x \sin x \qquad y_p = v_1 y_1 + v_2 y_2$$

$$\underline{= -\cos x \, \ln|\csc x - \cot x| - 1}$$

The ***general*** solution:

$$\underline{y(x) = C_1 \cos x + C_2 \sin x - \cos x \, \ln|\csc x - \cot x| - 1}$$

Exercise

Find the general solution to the given differential equation. $y'' + y = \sec^2 x$

Solution

Characteristic Eqn.: $\lambda^2 + 1 = 0 \;\Rightarrow\; \underline{\lambda_{1,2} = \pm i}$

$$\underline{y_h = C_1 \cos x + C_2 \sin x}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x \; \underline{= 1 \;\; \neq 0}$$

$$v_1(x) = -\int \sin x \sec^2 x \, dx \qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= \int \frac{1}{\cos^2 x} d(\cos x)$$

$$= -\frac{1}{\cos x}$$

$$\underline{= -\sec x}$$

$$v_2(x) = \int \cos x \sec^2 x \, dx \qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \int \sec x \, dx$$

$$\underline{= \ln|\sec x + \tan x|}$$

$$y_p = \cos x(-\sec x) + \sin x \ln|\sec x + \tan x| \qquad y_p = v_1 y_1 + v_2 y_2$$

$$= -1 + \sin x \ln\left|\sec x + \tan x\right|$$

The ***general*** solution:

$$y(x) = C_1 \cos x + C_2 \sin x - \cos x(-\sec x) + \sin x \ln\left|\sec x + \tan x\right|$$

Exercise

Find the general solution to the given differential equation. $\quad y'' + y = \sec x \tan x$

Solution

Characteristic Eqn.: $\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

$$y_h = C_1 \cos x + C_2 \sin x$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0$$

$$v_1(x) = -\int \sin x \sec x \tan x \, dx \qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\int \tan^2 x \, dx$$

$$= \int \left(1 - \sec^2 x\right) dx$$

$$= x - \tan x$$

$$v_2(x) = \int \cos x \sec x \tan x \, dx \qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \int \tan x \, dx$$

$$= \ln\left|\sec x\right|$$

$$y_p = \cos x(x - \tan x) + \sin x \ln\left|\sec x\right| \qquad y_p = v_1 y_1 + v_2 y_2$$

$$= x \cos x - \sin x + \sin x \ln\left|\sec x\right|$$

The ***general*** solution:

$$y(x) = C_1 \cos x + C_3 \sin x + x \cos x + \sin x \ln\left|\sec x\right|$$

Exercise

Find the general solution to the given differential equation. $\quad y'' + y' = x$

Solution

Characteristic Eqn.: $\lambda^2+\lambda=0 \Rightarrow \boxed{\lambda_{1,2}=0,\ -1}$

$$\boxed{y_h=C_1+C_2e^{-x}}$$

$$W=\begin{vmatrix}1 & e^{-x}\\ 0 & -e^{-x}\end{vmatrix}\ \boxed{=-e^{-x}\ \neq 0}$$

$$v_1(x)=\int\frac{e^{-x}x}{e^{-x}}\,dx \qquad\qquad v_1(x)=-\int\frac{y_2g(x)}{W}dx$$

$$=\int x\,dx$$

$$\boxed{=\frac{1}{2}x^2}$$

$$v_2(x)=\int\frac{x}{e^{-x}}\,dx\ \boxed{=\frac{1}{2}x^2} \qquad\qquad v_2(x)=\int\frac{y_1g(x)}{W}dx$$

$$=\int xe^x\,dx$$

$$=(x-1)e^x$$

$$\boxed{y_p=\frac{1}{2}x^2+x-1} \qquad y_p=v_1y_1+v_2y_2$$

The ***general*** solution:

$$y(x)=C_1+C_2e^{-x}+\frac{1}{2}x^2+x-1$$

$$\boxed{=C_3+C_2e^{-x}+\frac{1}{2}x^2+x}$$

Exercise

Find the general solution to the given differential equation. $\quad y''-y'=e^x\cos x$

Solution

Characteristic Eqn.: $\lambda^2-\lambda=0 \Rightarrow \boxed{\lambda_{1,2}=0,\ 1}$

		$\int \cos x$
+	x	$\sin x$
–	1	$-\cos x$

$$\boxed{y_h=C_1+C_2e^{x}}$$

$$W=\begin{vmatrix}1 & e^{x}\\ 0 & e^{x}\end{vmatrix}\ \boxed{=e^{x}\ \neq 0}$$

$$v_1(x)=-\int\frac{e^xe^x\cos x}{e^x}\,dx \qquad\qquad v_1(x)=-\int\frac{y_2g(x)}{W}dx$$

$$= -\int e^x \cos x \, dx$$

$$\int e^x \cos x \, dx = (\sin x + \cos x)e^x - \int e^x \cos x \, dx$$

$$2\int e^x \cos x \, dx = (\sin x + \cos x)e^x$$

$$= -\frac{1}{2}(\sin x + \cos x)e^x$$

		$\int \cos x$
+	e^x	$\sin x$
−	e^x	$-\cos x$
+	e^x	$-\int \cos x$

$$v_2(x) = \int \frac{e^x \cos x}{e^x} \, dx \qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \int \cos x \, dx$$

$$= \sin x$$

$$y_p = -\frac{1}{2}(\sin x + \cos x)e^x + \sin x e^x \qquad y_p = v_1 y_1 + v_2 y_2$$

$$= \cos x e^x + \frac{1}{2}\sin x e^x$$

The ***general*** solution:

$$y(x) = C_1 + C_2 e^x + \cos x e^x + \frac{1}{2}\sin x e^x$$

Exercise

Find the general solution to the given differential equation. $\quad y'' + y' - 2y = xe^x$

Solution

Characteristic Eqn.: $\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda_{1,2} = -2, 1$

$$y_h = C_1 e^{-2x} + C_2 e^x$$

$$W = \begin{vmatrix} e^{-2x} & e^x \\ -2e^{-2x} & e^x \end{vmatrix} = 3e^{-x} \neq 0$$

$$v_1(x) = -\int \frac{xe^x e^x}{3e^{-x}} \, dx \qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\frac{1}{3}\int xe^{3x} \, dx$$

$$= -\frac{1}{3}\left(\frac{1}{3}x - \frac{1}{9}\right)e^{3x}$$

		$\int e^{3x}$
+	x	$\frac{1}{3}e^{3x}$
−	1	$\frac{1}{9}e^{3x}$

$$v_2(x) = \int \frac{xe^x e^{-2x}}{3e^{-x}}\,dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$= \frac{1}{3}\int x\,dx$$

$$= \boxed{\frac{1}{3}x^2}$$

$$y_p = e^{-2x}\left(\frac{1}{27} - \frac{1}{9}x\right)e^{3x} + \frac{1}{3}x^2 e^x \qquad\qquad y_p = v_1 y_1 + v_2 y_2$$

$$= \boxed{\left(\frac{1}{27} - \frac{1}{9}x + \frac{1}{3}x^2\right)e^x}$$

$$y(x) = C_1 e^{-2x} + C_2 e^x + \frac{1}{27}e^x - \frac{1}{9}xe^x + \frac{1}{3}x^2 e^x$$

$$\boxed{= C_1 e^{-2x} + C_3 e^x - \frac{1}{9}xe^x + \frac{1}{3}x^2 e^x}$$

Exercise

Find the general solution $\quad y'' + y' - 2y = e^{3x}$

Solution

Characteristic Eqn.: $\lambda^2 + \lambda - 2 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = 1,\ -2}$

$$\boxed{y_h = C_1 e^{-2x} + C_2 e^x}$$

$$W = \begin{vmatrix} e^{-2x} & e^x \\ -2e^{-2x} & e^x \end{vmatrix} \boxed{= 3e^{-x} \neq 0}$$

$$v_1(x) = -\int \frac{e^x e^{3x}}{3e^{-x}}\,dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$= -\frac{1}{3}\int e^{5x}\,dx$$

$$= \boxed{-\frac{1}{15}e^{5x}}$$

$$v_2(x) = \int \frac{e^{-2x}e^{3x}}{3e^{-x}}\,dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$= \frac{1}{3}\int e^{2x}\,dx$$

$$= \boxed{\frac{1}{6}e^{2x}}$$

$$y_p = -\frac{1}{15}e^{-2x}e^{5x} + \frac{1}{6}e^{x}e^{2x} \qquad\qquad y_p = v_1y_1 + v_2y_2$$

$$= -\frac{1}{15}e^{3x} + \frac{1}{6}e^{3x}$$

$$= \frac{1}{10}e^{3x}$$

$$y(x) = C_1e^{-2x} + C_2e^{x} + \frac{1}{10}e^{3x}$$

Exercise

Find the general solution to the given differential equation. $y'' + 2y' + y = e^{-x}\ln x$

Solution

Characteristic Eqn.: $\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = -1$

$$y_h = C_1e^{-x} + C_2xe^{-x}$$

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-2x} - xe^{-2x} + xe^{-2x} = e^{-2x} \neq 0$$

$$v_1(x) = -\int \frac{xe^{-x}}{e^{-2x}}\left(e^{-x}\ln x\right)dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$= -\int (x\ln x)dx \qquad\qquad u = \ln x \quad dv = x \qquad du = \frac{1}{x}dx \quad v = \int xdx = \frac{1}{2}x^2$$

$$= \frac{1}{4}x^2 - \frac{1}{2}x^2\ln x \qquad\qquad \int x\ln xdx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int xdx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$$

$$v_2(x) = \int \frac{e^{-x}}{e^{-2x}}\left(e^{-x}\ln x\right)dx = \int (\ln x)dx = x\ln x - x \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$u = \ln x \quad dv = 1 \qquad du = \frac{1}{x}dx \quad v = \int dx = x \quad \rightarrow \quad \int \ln xdx = x\ln x - x$$

$$y_p = e^{-x}\left(\frac{1}{4}x^2 - \frac{1}{2}x^2\ln x\right) + xe^{-x}(x\ln x - x) \qquad\qquad y_p = u_1y_1 + u_2y_2$$

$$= \frac{1}{4}x^2e^{-x} - \frac{1}{2}x^2e^{-x}\ln x + x^2e^{-x}\ln x - x^2e^{-x}$$

$$= \frac{1}{2}x^2e^{-x}\ln x - \frac{3}{4}x^2e^{-x}$$

The ***general*** solution: $y(x) = C_1e^{-x} + C_2xe^{-x} + \frac{1}{2}x^2e^{-x}\ln x - \frac{3}{4}x^2e^{-x}$

Exercise

Find the general solution to the given differential equation. $y'' - 2y' + y = \dfrac{e^x}{1+x^2}$

Solution

Characteristic Eqn.: $\lambda^2 - 2\lambda + 1 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = 1}$

$$\boxed{y_h = C_1 e^x + C_2 x e^x}$$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}$$

$$= e^{2x} + xe^{2x} - xe^{2x}$$

$$\boxed{= e^{2x} \neq 0}$$

$$v_1(x) = -\int \frac{xe^x}{e^{2x}}\left(\frac{e^x}{1+x^2}\right)dx \qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$= -\int \frac{x}{1+x^2}dx$$

$$= -\frac{1}{2}\int \frac{1}{1+x^2}d\left(1+x^2\right)$$

$$\boxed{= -\frac{1}{2}\ln\left(1+x^2\right)}$$

$$v_2(x) = \int \frac{e^x}{e^{2x}}\left(\frac{e^x}{1+x^2}\right)dx \qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$= \int \frac{dx}{1+x^2}$$

$$\boxed{= \tan^{-1} x}$$

$$\boxed{y_p = -\frac{1}{2}e^x \ln\left(1+x^2\right) + xe^x \tan^{-1} x} \qquad y_p = u_1 y_1 + u_2 y_2$$

The ***general*** solution:

$$\boxed{y(x) = C_1 e^x + C_2 x e^x - \frac{1}{2}e^x \ln\left(1+x^2\right) + xe^x \tan^{-1} x}$$

Exercise

Find the general solution $y'' + 2y' + y = e^{-x}$

Solution

Characteristic Eqn.: $\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = -1}$

The *homogeneous Eqn.*: $\boxed{y_h = C_1 e^{-x} + C_2 x e^{-x}}$

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix} = e^{-2x} \neq 0$$

$$v_1(x) = -\int \frac{xe^{-x}e^{-x}}{e^{-2x}} dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\int x dx$$

$$\boxed{= -\frac{1}{2}x^2}$$

$$v_2(x) = \int \frac{e^{-x}e^{-x}}{e^{-2x}} dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \int dx$$

$$\boxed{= x}$$

$$y_p = -\frac{1}{2}x^2 e^{-x} + x^2 e^{-x} \boxed{= \frac{1}{2}x^2 e^{-x}} \qquad\qquad y_p = u_1 y_1 + u_2 y_2$$

The ***general*** solution:

$$\boxed{y(x) = \left(C_1 + C_2 x + \frac{1}{2}x^2\right)e^{-x}}$$

Exercise

Find the general solution $\quad y'' - 2y' - 8y = 3e^{-2x}$

Solution

Characteristic Eqn.: $\lambda^2 - 2\lambda - 8 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = -2,\ 4}$

The *homogeneous Eqn.*: $\boxed{y_h = C_1 e^{-2x} + C_2 e^{4x}}$

$$W = \begin{vmatrix} e^{-2x} & e^{4x} \\ -2e^{-2x} & 4e^{4x} \end{vmatrix}$$

$$= 4e^{2x} + 2e^{2x}$$

$$\boxed{= 6e^{2x} \neq 0}$$

$$v_1(x) = -\int \frac{3e^{2x}}{6e^{2x}} dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\frac{1}{2}\int dx$$

$$\boxed{= -\frac{1}{2}x}$$

$$v_2(x) = \int \frac{3e^{-4x}}{6e^{2x}}dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$= \frac{1}{2}\int e^{-6x}dx$$

$$\boxed{= -\frac{1}{12}e^{-6x}}$$

$$y_p = -\frac{1}{2}xe^{-2x} - \frac{1}{12}e^{-6x}e^{4x} \qquad\qquad y_p = v_1y_1 + v_2y_2$$

$$\boxed{= -\frac{1}{2}xe^{-2x} - \frac{1}{12}e^{-2x}}$$

$$y(x) = C_1e^{-2x} + C_2e^{4x} - \frac{1}{2}xe^{-2x} - \frac{1}{12}e^{-2x}$$

$$= \left(C_1 - \frac{1}{12} - \frac{1}{2}x\right)e^{-2x} + C_2e^{4x}$$

$$\boxed{= \left(A_1 - \frac{x}{2}\right)e^{-2x} + C_2e^{4x}}$$

Exercise

Find the general solution to the given differential equation. $y'' + 3y' + 2y = \sin e^x$

Solution

Characteristic Eqn.: $\lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_1 = -2, \ \lambda_2 = -1$

The homogeneous Eqn.: $\boxed{y_h = C_1e^{-2x} + C_2e^{-x}}$

$$W = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} = -e^{-3x} + 2e^{-3x} \ \boxed{= e^{-3x} \ \neq 0}$$

$$v_1(x) = -\int \frac{e^{-x}\sin e^x}{e^{-3x}}dx = -\int \left(e^{2x}\sin e^x\right)dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$u = e^x \qquad dv = e^x \sin e^x$$

$$du = e^x dx \qquad v = \int \sin e^x d\left(e^x\right) = -\cos e^x$$

$$\int \left(e^{2x}\sin e^x\right)dx = -e^x\cos e^x + \int e^x \cos e^x dx$$

$$= -e^x \cos e^x + \int \cos e^x d\left(e^x\right)$$

$$\boxed{= -e^x \cos e^x + \sin e^x}$$

$$v_2(x) = \int \frac{e^{-2x} \sin e^x}{e^{-3x}} dx \qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \int e^x \sin e^x \, dx$$

$$= \int \sin e^x \, d\left(e^x\right)$$

$$\boxed{= -\cos e^x}$$

$$y_p = e^{-2x}\left(e^x \cos e^x - \sin e^x\right) + e^{-x}\left(-\cos e^x\right) \qquad y_p = u_1 y_1 + u_2 y_2$$

$$= e^{-x} \cos e^x - e^{-2x} \sin e^x - e^{-x} \cos e^x$$

$$\boxed{= -e^{-2x} \sin e^x}$$

The ***general*** solution:

$$\boxed{y(x) = C_1 e^{-2x} + C_2 e^{-x} - e^{-2x} \sin e^x}$$

Exercise

Find the general solution $\qquad y'' + 3y' + 2y = 4e^x$

Solution

Characteristic Eqn.: $\lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = -2, \ -1}$

The *homogeneous Eqn.*: $\boxed{y_h = C_1 e^{-2x} + C_2 e^{-x}}$

$$W = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} = -e^{-3x} + 2e^{-3x} \ \boxed{= e^{-3x} \neq 0}$$

$$v_1(x) = -\int \frac{e^{-x}\left(4e^x\right)}{e^{-3x}} dx \qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -4\int e^{3x} dx$$

$$\boxed{= -\frac{4}{3} e^{3x}}$$

$$v_2(x) = \int \frac{e^{-2x} 4e^x}{e^{-3x}} dx \qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= 4\int e^{2x}dx$$

$$\boxed{= 2e^{2x}}$$

The *particular* solution: $y_p = -\frac{4}{3}e^{3x}e^{-2x} + 2e^{2x}e^{-x} \boxed{= \frac{2}{3}e^x}$ $\qquad y_p = v_1y_1 + v_2y_2$

The ***general*** solution:

$$\boxed{y(x) = C_1e^{-2x} + C_2e^{-x} + \frac{2}{3}e^x}$$

Exercise

Find the general solution $\qquad y'' + 3y' + 2y = \frac{1}{1+e^x}$

Solution

Characteristic Eqn.: $\lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = -2, \ -1}$

$$\boxed{y_h = C_1e^{-2x} + C_2e^{-x}}$$

$$W = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix}$$

$$= -e^{-3x} + 2e^{-3x}$$

$$\boxed{= e^{-3x} \quad \neq 0}$$

$$v_1(x) = -\int \frac{e^{-x}}{e^{-3x}}\frac{1}{1+e^x}dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$= -\int \frac{e^{2x}}{1+e^x}dx$$

$$= -\int \left(\frac{e^x}{1+e^x} - e^x\right)dx$$

$$= \int e^x dx - \int \frac{e^x}{1+e^x}dx$$

$$= \int e^x dx - \int \frac{1}{1+e^x}d\left(1+e^x\right)$$

$$\boxed{= e^x - \ln\left(1+e^x\right)}$$

$$v_2(x) = \int \frac{e^{-2x}}{e^{-3x}}\frac{1}{1+e^x}dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$= \int \frac{e^x}{1+e^x} dx$$

$$= \int \frac{1}{1+e^x} d\left(1+e^x\right)$$

$$= \ln\left(1+e^x\right)$$

$$y_p = e^{-2x}\left(e^x - \ln\left(1+e^x\right)\right) + e^{-x}\ln\left(1+e^x\right) \qquad y_p = v_1 y_1 + v_2 y_2$$

$$= e^{-x}$$

The ***general*** solution:

$$y(x) = C_1 e^{-2x} + C_2 e^{-x} + e^{-x}$$

Exercise

Find the general solution to the given differential equation. $\quad y'' - 4y = \sinh 2x$

Solution

Characteristic Eqn.: $\lambda^2 - 4 = 0 \;\Rightarrow\; \lambda_{1,2} = -2,\ 2$

The *homogeneous Eqn.*: $y_h = C_1 e^{-2x} + C_2 e^{2x}$

$$W = \begin{vmatrix} e^{-2x} & e^{2x} \\ -2e^{-2x} & 2e^{2x} \end{vmatrix} = 2 + 2 = 4 \neq 0$$

$$v_1(x) = -\frac{1}{4}\int e^{2x}\sinh 2x\, dx \qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\frac{1}{8}\int e^{2x}\left(e^{2x} - e^{-2x}\right)dx$$

$$= -\frac{1}{8}\int \left(e^{4x} - 1\right)dx$$

$$= -\frac{1}{8}\left(\frac{1}{4}e^{4x} - x\right)$$

$$v_2(x) = \frac{1}{4}\int e^{-2x}\sinh 2x\, dx \qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= -\frac{1}{8}\int e^{-2x}\left(e^{2x} - e^{-2x}\right)dx$$

$$= \frac{1}{8}\int \left(1 - e^{4x}\right)dx$$

$$=\frac{1}{8}\left(x-\frac{1}{4}e^{4x}\right)$$

$$y_p=\left(\frac{x}{8}-\frac{1}{32}e^{4x}\right)e^{-2x}+\left(\frac{x}{8}-\frac{1}{32}e^{4x}\right)e^{2x} \qquad y_p=v_1y_1+v_2y_2$$

$$=\left(\frac{x}{4}-\frac{1}{16}e^{4x}\right)\left(\frac{e^{-2x}+e^{2x}}{2}\right)$$

$$=\left(\frac{x}{4}-\frac{1}{16}e^{4x}\right)\cosh 2x$$

$$y(x)=C_1e^{-2x}+C_2e^{4x}+\left(\frac{x}{4}-\frac{1}{16}e^{4x}\right)\cosh 2x$$

Exercise

Find the general solution $\quad y''+4y=\sec 2x$

Solution

Characteristic Eqn.: $\lambda^2+4=0 \Rightarrow \lambda_{1,2}=\pm 2i$

$$y_h=C_1\cos 2x+C_2\sin 2x$$

$$W=\begin{vmatrix}\cos 2x & \sin 2x\\ -2\sin 2x & 2\cos 2x\end{vmatrix}$$

$$=2\cos^2 2x+2\sin^2 2x$$

$$=2 \neq 0$$

$$v_1(x)=-\frac{1}{2}\int \sin 2x\sec 2x\, dx \qquad v_1(x)=-\int\frac{y_2g(x)}{W}dx$$

$$=-\frac{1}{2}\int \tan 2x\, dx$$

$$=-\frac{1}{4}\ln|\sec 2x|$$

$$v_2(x)=\frac{1}{2}\int \cos 2x\sec 2x\, dx \qquad v_2(x)=\int\frac{y_1g(x)}{W}dx$$

$$=\frac{1}{2}\int dx$$

$$=\frac{1}{2}x$$

$$y_p=-\frac{1}{4}\ln|\sec x|\cos 2x+\frac{1}{2}x\sin 2x \qquad y_p=v_1y_1+v_2y_2$$

$$y(x)=C_1\cos 2x+C_2\sin 2x-\frac{1}{4}\ln|\sec x|\cos 2x+\frac{1}{2}x\sin 2x$$

Exercise

Find the general solution to the given differential equation. $y'' + 4y = \cos 3x$

Solution

Characteristic Eqn.: $\lambda^2 + 4 = 0 \Rightarrow \boxed{\lambda_{1,2} = \pm 2i}$

The *homogeneous Eqn.*: $\boxed{y_h = C_1 \cos 2x + C_2 \sin 2x}$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$\boxed{= 2 \ne 0}$$

$$v_1(x) = -\frac{1}{2}\int \sin 2x \cos 3x \, dx \qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\frac{1}{4}\int (\sin 5x - \sin x) \, dx$$

$$= -\frac{1}{4}\left(-\frac{1}{5}\cos 5x + \cos x\right) \qquad \cos\alpha \sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

$$\boxed{= \frac{1}{20}\cos 5x - \frac{1}{4}\cos x}$$

$$v_2(x) = \frac{1}{2}\int \cos 2x \cos 3x \, dx \qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \frac{1}{4}\int (\cos 5x + \cos x) \, dx$$

$$\boxed{= \frac{1}{20}\sin 5x + \frac{1}{4}\sin x} \qquad \cos\alpha \cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$y_p = \left(\frac{1}{20}\cos 5x - \frac{1}{4}\cos x\right)\cos 2x + \left(\frac{1}{20}\sin 5x + \frac{1}{4}\sin x\right)\sin 2x \qquad y_p = v_1 y_1 + v_2 y_2$$

$$= \frac{1}{20}\cos 5x \cos 2x - \frac{1}{4}\cos x \cos 2x + \frac{1}{20}\sin 5x \sin 2x + \frac{1}{4}\sin x \sin 2x$$

$$= \frac{1}{40}\cos 5x + \frac{1}{40}\cos 3x - \frac{1}{8}\cos 3x - \frac{1}{8}\cos x + \frac{1}{40}\cos 3x - \frac{1}{40}\cos 2x + \frac{1}{8}\cos x - \frac{1}{8}\cos 3x$$

$$\boxed{= \frac{1}{40}\cos 5x - \frac{1}{5}\cos 3x - \frac{1}{40}\cos 2x}$$

$$y(x) = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{40}\cos 5x - \frac{1}{5}\cos 3x - \frac{1}{40}\cos 2x$$

$$\boxed{= A_1 \cos 2x + C_2 \sin 2x + \frac{1}{40}\cos 5x - \frac{1}{5}\cos 3x}$$

Exercise

Find the general solution to the given differential equation. $y'' + 4y = \sin^2 x$

Solution

Characteristic Eqn.: $\lambda^2 + 4 = 0 \;\Rightarrow\; \lambda_{1,2} = \pm 2i$

The *homogeneous Eqn.*: $\boxed{y_h = C_1 \cos 2x + C_2 \sin 2x}$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$\boxed{= 2 \;\neq 0}$$

$$v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$v_1(x) = -\frac{1}{2}\int \sin 2x \sin^2 x \, dx$$

$$= -\int \sin x \cos x \sin^2 x \, dx$$

$$= -\int \sin^3 x \, d(\sin x)$$

$$\boxed{= -\frac{1}{4}\sin^4 x}$$

$$v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$v_2(x) = \frac{1}{2}\int \cos 2x \sin^2 x \, dx$$

$$= \frac{1}{2}\int \left(1 - 2\sin^2 x\right)\sin^2 x \, dx$$

$$= \frac{1}{2}\int \left(\sin^2 x - 2\sin^4 x\right) dx$$

$$= \frac{1}{2}\int \left(\frac{1}{2} - \frac{1}{2}\cos 2x - 2\left(\frac{1-\cos 2x}{2}\right)^2\right) dx$$

$$= \frac{1}{4}\int \left(1 - \cos 2x - 1 + 2\cos 2x - \cos^2 2x\right) dx$$

$$= \frac{1}{4}\int \left(\cos 2x - \frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$$

$$\boxed{= \frac{1}{8}\sin 2x - \frac{x}{8} - \frac{1}{32}\sin 4x}$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$y_p = -\frac{1}{4}\sin^4 x \cos x + \left(\frac{1}{8}\sin 2x - \frac{x}{8} - \frac{1}{32}\sin 4x\right)\sin x$$

$$y(x) \boxed{= C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4}\sin^4 x \cos x + \left(\frac{1}{8}\sin 2x - \frac{x}{8} - \frac{1}{32}\sin 4x\right)\sin x}$$

Exercise

Find the general solution to the given differential equation. $y'' - 4y = \dfrac{e^x}{x}$

Solution

$$\lambda^2 - 4 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 2$$

$$\boxed{y_h = C_1 e^{-2x} + C_2 e^{2x}}$$

$$W = \begin{vmatrix} e^{-2x} & e^{2x} \\ -2e^{-2x} & 2e^{2x} \end{vmatrix}$$

$$= 2 + 2$$

$$\boxed{= 4 \neq 0}$$

$$v_1(x) = -\int \frac{1}{4} e^{2x} \frac{e^{2x}}{x} dx = -\frac{1}{4} \int \frac{e^{4x}}{x} \qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\frac{1}{4} \int \frac{e^{4x}}{x}$$

$$v_2(x) = \frac{1}{4} \int e^{-2x} \frac{e^{2x}}{x} dx \qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \frac{1}{4} \int \frac{1}{x} dx$$

$$\boxed{= \frac{1}{4} \ln|x|}$$

$$\boxed{y_p = -\frac{1}{4} e^{-2x} \int \frac{e^{4x}}{x} dx + \frac{1}{4} e^{2x} \ln|x|} \qquad y_p = u_1 y_1 + u_2 y_2$$

$$\boxed{y(x) = C_1 e^{-2x} + C_2 e^{2x} + \frac{1}{4} e^{2x} \ln|x| - \frac{1}{4} e^{-2x} \int \frac{e^{4x}}{x} dx}$$

Exercise

Find the general solution to the given differential equation. $y'' - 4y = xe^x$

Solution

$$\lambda^2 - 4 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 2$$

$$\boxed{y_h = C_1 e^{-2x} + C_2 e^{2x}}$$

$$W = \begin{vmatrix} e^{-2x} & e^{2x} \\ -2e^{-2x} & 2e^{2x} \end{vmatrix}$$

$$= 2 + 2$$

$$= 4 \neq 0$$

$$v_1(x) = -\frac{1}{4}\int e^{2x}\left(xe^x\right)dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$= -\frac{1}{4}\int xe^{3x}dx$$

$$= -\frac{1}{4}\left(\frac{1}{3}x - \frac{1}{9}\right)e^{3x}$$

$$v_2(x) = \frac{1}{4}\int e^{-2x}\left(xe^x\right)dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$= \frac{1}{4}\int xe^{-x}dx$$

$$= \frac{1}{4}(-x-1)e^{-x}$$

$$y_p = \frac{1}{36}(1-3x)e^x - \frac{1}{4}(x+1)e^{-x} \qquad\qquad y_p = v_1y_1 + v_2y_2$$

$$y(x) = C_1e^{-2x} + C_2xe^{2x} + \frac{1}{36}(1-3x)e^x - \frac{1}{4}(x+1)e^{-x}$$

Exercise

Find the general solution $\quad y'' + 4y = \sin^2 2t$

Solution

Characteristic Eqn.: $\lambda^2 + 4 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 2i$

$$y_h = C_1 \cos 2t + C_2 \sin 2t$$

$$W = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix}$$

$$= 2\cos^2 t + 2\sin^2 t$$

$$= 2 \neq 0$$

$$v_1(t) = -\int \frac{\sin 2t}{2}\sin^2 2t\ dt \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$= \frac{1}{4}\int\left(1-\cos^2 2t\right)\ d\left(\cos 2t\right)$$

$$= \frac{1}{4}\left(\cos 2t - \frac{1}{3}\cos^3 2t\right)$$

$$v_2(t) = \int \frac{\cos 2t}{2}\sin^2 2t\ dt \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$= \frac{1}{4}\int \sin^2 2t\ d(\sin 2t)$$

$$\boxed{= \frac{1}{12}\sin^3 2t}$$

$$y_p = \left(\frac{1}{4}\cos 2t - \frac{1}{12}\cos^3 2t\right)\cos 2t + \frac{1}{12}\sin^4 2t \qquad\qquad y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{1}{4}\cos^2 2t - \frac{1}{12}\cos^4 2t + \frac{1}{12}\sin^4 2t$$

$$= \frac{1}{4}\cos^2 2t + \frac{1}{12}\left(\sin^4 2t - \cos^4 2t\right)$$

$$= \frac{1}{4}\cos^2 2t + \frac{1}{12}\left(\sin^2 2t - \cos^2 2t\right)\left(\sin^2 2t + \cos^2 2t\right)$$

$$= \frac{1}{4}\cos^2 2t + \frac{1}{12}\sin^2 2t - \frac{1}{12}\cos^2 2t$$

$$\boxed{= \frac{1}{6}\cos^2 2t + \frac{1}{12}\sin^2 2t}$$

The ***general*** solution:

$$\boxed{y(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{6}\cos^2 2t + \frac{1}{12}\sin^2 2t}$$

Exercise

Find the general solution to the given differential equation. $\qquad y'' - 4y' + 4y = 2e^{2x}$

Solution

$$\lambda^2 - 4\lambda + 4 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = 2}$$

$$\boxed{y_h = C_1 e^{2x} + C_2 x e^{2x}}$$

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}$$

$$= e^{4x} + 2xe^{4x} - 2xe^{4x}$$

$$\boxed{= e^{4x} \ \neq 0}$$

$$v_1(x) = -\int \frac{2xe^{4x}}{e^{4x}}dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$= -2\int x dx$$

$$\boxed{= -x^2}$$

$$v_2(x) = \int \frac{2e^{4x}}{e^{4x}} dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= 2\int dx$$

$$\boxed{= 2x}$$

$$y_p = -x^2e^{2x} + 2x^2e^{2x} \ \boxed{= x^2e^{2x}} \qquad\qquad y_p = v_1y_1 + v_2y_2$$

$$\boxed{y(x) = C_1e^{2x} + C_2xe^{2x} + x^2e^{2x}}$$

Exercise

Find the general solution $\quad y'' - 4y' + 4y = (x+1)e^{2x}$

Solution

Characteristic Eqn.: $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0 \quad \Rightarrow \quad \lambda_{1,2} = 2$

The homogeneous Eqn.: $\boxed{y_h = C_1e^{2x} + C_2xe^{2x}}$

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}$$

$$= e^{4x} + 2xe^{4x} - 2xe^{4x}$$

$$\boxed{= e^{4x} \ \neq 0}$$

$$u_1' = -\frac{xe^{2x}(x+1)e^{2x}}{e^{4x}} = -x^2 - x \qquad\qquad u_1' = -\frac{y_2 g(t)}{W}$$

$$u_1 = \int \left(-x^2 - x\right) dx = -\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2$$

$$u_2' = \frac{e^{2x}(x+1)e^{2x}}{e^{4x}} = x + 1 \qquad\qquad u_2' = \frac{y_1 g(t)}{W}$$

$$u_2 = \int (x+1)dx = \tfrac{1}{2}x^2 + x$$

$$y_p = u_1y_1 + u_2y_2 = \left(-\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right)e^{2x} + \left(\tfrac{1}{2}x^2 + x\right)xe^{2x}$$

$$= \left(-\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 + x^2\right)e^{2x}$$

$$\boxed{= \left(\tfrac{1}{6}x^3 + \tfrac{1}{2}x^2\right)e^{2x}}$$

$$\boxed{y(x) = C_1 e^{2x} + C_2 xe^{2x} + \left(\frac{1}{6}x^3 + \frac{1}{2}x^2\right)e^{2x}}$$

Exercise

Find the general solution to the given differential equation. $y'' + 4y' + 5y = 10$

Solution

$$\lambda^2 + 4\lambda + 5 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{-4 \pm 2i}{2} \boxed{= -2 \pm i}$$

$$\boxed{y_h = e^{-2x}\left(C_1 \cos x + C_2 \sin x\right)}$$

$$W = \begin{vmatrix} e^{-2x}\cos x & e^{-2x}\sin x \\ -2e^{-2x}\cos x - e^{-2x}\sin x & -2e^{-2x}\sin x + e^{-2x}\cos x \end{vmatrix}$$

$$= -2e^{-4x}\cos x \sin x + e^{-4x}\cos^2 x + 2e^{-4x}\cos x \sin x + e^{-4x}\sin^2 x$$

$$\boxed{= e^{-4x} \quad \neq 0}$$

$$v_1(x) = -\int \frac{10e^{-2x}\sin x}{e^{-4x}}dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$= -10\int e^{2x}\sin x\, dx$$

		$\int \sin x$
+	e^{2x}	$-\cos x$
−	$2e^{2x}$	$-\sin x$
+	$4e^{2x}$	

$$\int e^{2x}\sin x\, dx = -\cos x e^{2x} + 2\sin x e^{2x} - 4\int e^{2x}\sin x\, dx$$

$$\int e^{2x}\sin x\, dx = \frac{1}{5}\left(-\cos x + 2\sin x\right)e^{2x}$$

$$= (-10)\frac{1}{5}\left(-\cos x + 2\sin x\right)e^{2x}$$

$$\boxed{= \left(2\cos x - 4\sin x\right)e^{2x}}$$

$$v_2(x) = \int \frac{10e^{-2x}\cos x}{e^{-4x}}dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$= 10\int e^{2x}\cos x dx$$

		$\int \cos x$
+	e^{2x}	$\sin x$
−	$2e^{2x}$	$-\cos x$
+	$4e^{2x}$	

$$\int e^{2x}\cos x\, dx = \sin x e^{2x} + 2\cos x e^{2x} - 4\int e^{2x}\cos x\, dx$$

$$\int e^{2x}\cos x\, dx = \frac{1}{5}\left(\sin x + 2\cos x\right)e^{2x}$$

$$= (10)\frac{1}{5}\left(\sin x + 2\cos x\right)e^{2x}$$

$$\boxed{=(2\sin x+4\cos x)e^{2x}}$$

$$y_p=(2\cos x-4\sin x)e^{2x}(\cos x)e^{-2x}+(2\sin x+4\cos x)e^{2x}(\sin x)e^{-2x} \qquad y_p=v_1y_1+v_2y_2$$

$$=2\cos^2 x-4\sin x\cos x+2\sin^2 x+4\cos x\sin x$$

$$\boxed{=2}$$

$$\boxed{y(x)=e^{-2x}\left(C_1\cos x+C_2\sin x\right)+2}$$

Exercise

Find the general solution to the given differential equation. $\quad y''-9y=\dfrac{9x}{e^{3x}}$

Solution

$$\lambda^2-9=0 \quad\Rightarrow\quad \boxed{\lambda_{1,2}=\pm 3}$$

$$\boxed{y_h=C_1e^{-3x}+C_2e^{3x}}$$

$$W=\begin{vmatrix} e^{-3x} & e^{3x} \\ -3e^{-3x} & 3e^{3x} \end{vmatrix}$$

$$=3+3$$

$$\boxed{=6\ \neq 0}$$

$$v_1(x)=-\int \frac{e^{3x}}{6}\frac{9x}{e^{3x}}dx \qquad v_1(x)=-\int\frac{y_2g(x)}{W}dx$$

$$=-\frac{3}{2}\int x\,dx$$

$$\boxed{=-\frac{3}{4}x^2}$$

$$v_2(x)=\int \frac{e^{-3x}}{6}\frac{9x}{e^{3x}}dx \qquad v_2(x)=\int\frac{y_1g(x)}{W}dx$$

$$=\frac{3}{2}\int xe^{-6x}\,dx$$

$$=\frac{3}{2}\left(-\frac{1}{6}x-\frac{1}{36}\right)e^{-6x}$$

$$\boxed{=\left(-\frac{1}{4}x-\frac{1}{24}\right)e^{-6x}}$$

$$y_p=-\frac{3}{4}x^2e^{-3x}-\left(\frac{1}{4}x+\frac{1}{24}\right)e^{-6x}e^{3x} \qquad y_p=v_1y_1+v_2y_2$$

$$\boxed{=-\left(\frac{3}{4}x^2+\frac{1}{4}x+\frac{1}{24}\right)e^{-3x}}$$

$$y(x) = C_1 e^{-3x} + C_2 e^{3x} - \left(\frac{3}{4}x^2 + \frac{1}{4}x + \frac{1}{24}\right)e^{-3x}$$

$$= C_2 e^{3x} - \left(\frac{3}{4}x^2 + \frac{1}{4}x + C_3\right)e^{-3x}$$

Exercise

Find the general solution $\quad y'' + 9y = \csc 3x$

Solution

Characteristic Eqn.: $\lambda^2 + 9 = 0 \quad \Rightarrow \quad \lambda_{1,2} = 3i$

The homogeneous Eqn.: $\quad y_h = C_1 \cos 3x + C_2 \sin 3x$

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$$= 3\cos^2 3x + 3\sin^2 3x$$

$$= 3 \neq 0$$

$$u_1' = -\frac{(\sin 3x)(\csc 3x)}{3} = -\frac{1}{3} \qquad u_1' = -\frac{y_2 g(t)}{W}$$

$$u_1 = \int \left(-\frac{1}{3}\right)dx = -\frac{1}{3}x$$

$$u_2' = \frac{(\cos 3x)(\csc 3x)}{3} = \frac{1}{3}\frac{\cos 3x}{\sin 3x} \qquad u_2' = \frac{y_1 g(t)}{W}$$

$$u_2 = \int \left(\frac{1}{3}\frac{\cos 3x}{\sin 3x}\right)dx$$

$$= \frac{1}{9}\int \frac{1}{\sin 3x}\, d(\sin 3x)$$

$$= \frac{1}{9}\ln|\sin 3x|$$

$$y_p = -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x \ln|\sin 3x| \qquad y_p = u_1 y_1 + u_2 y_2$$

$$y(x) = C_1 \cos 3x + C_2 \sin 3x - \frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x \ln|\sin 3x|$$

Exercise

Find the general solution to the given differential equation. $\quad y'' + 9y = 3\tan 3t$

Solution

Characteristic Eqn.: $\lambda^2 + 9 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 3i$

The *homogeneous Eqn.*: $\boxed{y_h = C_1 \cos 3t + C_2 \sin 3t}$

$$W = \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix}$$

$$= 3\cos^2 3t + 3\sin^2 3t$$

$$= 3 \neq 0$$

$$v_1(t) = -\int \frac{\sin 3t(3\tan 3t)}{3} dt \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\int \frac{\sin^2 3t}{\cos 3t} dt$$

$$= -\int \frac{1-\cos^2 3t}{\cos 3t} dt$$

$$= -\int (\sec 3t - \cos 3t) dt$$

$$\boxed{= -\tfrac{1}{3}\ln|\sec 3t + \tan 3t| + \tfrac{1}{3}\sin 3t}$$

$$v_2(t) = \int \frac{\cos 3t(3\tan 3t)}{3} dt \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \int \sin 3t \; dt$$

$$\boxed{= -\tfrac{1}{3}\cos 3t}$$

$$y_p = -\tfrac{1}{3}\cos 3t \ln|\sec 3t + \tan 3t| + \tfrac{1}{3}\cos 3t \sin 3t - \tfrac{1}{3}\cos 3t \sin 3t \qquad\qquad y_p = u_1 y_1 + u_2 y_2$$

$$\boxed{= -\tfrac{1}{3}(\cos 3t)\ln|\sec 3t + \tan 3t|}$$

The ***general*** solution:

$$\boxed{y(t) = C_1 \cos 3t + C_2 \sin 3t - \tfrac{1}{3}(\cos 3t)\ln|\sec 3t + \tan 3t|}$$

Exercise

Find the general solution to the given differential equation. $\quad y'' + 9y = \sin 3x$

Solution

Characteristic Eqn.: $\lambda^2 + 9 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 3i$

The *homogeneous Eqn.*: $\boxed{y_h = C_1 \cos 3x + C_2 \sin 3x}$

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$$= 3\cos^2 3x + 3\sin^2 3x$$

$$\underline{= 3 \ \neq 0}$$

$$v_1(x) = -\frac{1}{3}\int \sin^2 3x \, dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\frac{1}{6}\int (1 - \cos 6x) \, dx$$

$$= -\frac{1}{6}\left(x - \frac{1}{6}\sin 6x\right)$$

$$v_2(x) = \frac{1}{3}\int \cos 3x \sin 3x \, dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \frac{1}{9}\int \sin 3x \, d(\sin 3x)$$

$$= \frac{1}{18}\sin^2 3x$$

$$y_p = \left(\frac{1}{6}x - \frac{1}{36}\sin 6x\right)\cos 3x + \frac{1}{18}\sin^3 3x \qquad\qquad y_p = v_1 y_1 + v_2 y_2$$

$$y(x) = C_1 \cos 3x + C_2 \sin 3x + \left(\frac{1}{6}x - \frac{1}{36}\sin 6x\right)\cos 3x + \frac{1}{18}\sin^3 3x$$

Exercise

Find the general solution to the given differential equation. $\quad y'' + 9y = \sec 3x$

Solution

Characteristic Eqn.: $\lambda^2 + 9 = 0 \Rightarrow \lambda_{1,2} = \pm 3i$

The *homogeneous Eqn.*: $y_h = C_1 \cos 3x + C_2 \sin 3x$

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$$= 3\cos^2 3x + 3\sin^2 3x$$

$$\underline{= 3 \ \neq 0}$$

$$v_1(x) = -\frac{1}{3}\int \sin 3x \sec 3x \, dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= \frac{1}{9}\int \frac{1}{\cos 3x} d(\cos 3x)$$

$$= \frac{1}{9}\ln|\cos 3x|$$

$$v_2(x) = \frac{1}{3}\int \cos 3x \sec 3x \, dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \frac{1}{3}\int dx$$

$$= \frac{x}{3}$$

$$y_p = \frac{1}{9}\cos 3x \ln|\cos 3x| + \frac{x}{3}\sin 3x \qquad\qquad y_p = v_1 y_1 + v_2 y_2$$

The ***general*** solution:

$$y(x) = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{9}\cos 3x \ln|\cos 3x| + \frac{x}{3}\sin 3x$$

Exercise

Find the general solution to the given differential equation. $\quad y'' + 9y = 2\sec 3x$

Solution

$$\lambda^2 + 9 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 3i$$

$$y_h = C_1 \cos 3x + C_2 \sin 3x$$

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$$= 3\cos^2 3x + 3\sin^2 3x$$

$$= 3 \neq 0$$

$$v_1(x) = -\frac{2}{3}\int \sin 3x \sec 3x \, dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= \frac{2}{9}\int \frac{1}{\cos 3x} d(\cos 3x)$$

$$= \frac{2}{9}\ln|\cos 3x|$$

$$v_2(x) = \frac{2}{3}\int \cos 3x \sec 3x \, dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \frac{2}{3}\int dx$$

$$= \frac{2}{3}x$$

$$\boxed{y_p = \frac{2}{9}\cos 3x \ln|\cos 3x| + \frac{2}{3}x\sin 3x} \qquad\qquad y_p = v_1 y_1 + v_2 y_2$$

The ***general*** solution:

$$\boxed{y(x) = C_1\cos 3x + C_2\sin 3x + \frac{2}{9}\cos 3x \ln|\cos 3x| + \frac{2}{3}x\sin 3x}$$

Exercise

Find the general solution to the given differential equation. $\quad 4y'' + 36y = \csc 3x$

Solution

Characteristic Eqn.: $4\lambda^2 + 36 = 0 \Rightarrow \boxed{\lambda_{1,2} = \pm 3i}$

The *homogeneous Eqn.*: $\boxed{y_h = C_1\cos 3x + C_2\sin 3x}$

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$$= 3\cos^2 3x + 3\sin^2 3x$$

$$\boxed{= 3 \neq 0}$$

$$y'' + 9y = \frac{1}{4}\csc 3x$$

$$v_1(x) = -\frac{1}{12}\int \sin 3x \csc 3x \, dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$= -\frac{1}{12}\int dx$$

$$\boxed{= -\frac{1}{12}x}$$

$$v_2(x) = \frac{1}{12}\int \cos 3x \csc 3x \, dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$= \frac{1}{12}\int \cot 3x \, dx$$

$$\boxed{= \frac{1}{36}\ln|\sin 3x|}$$

$$\boxed{y_p = -\frac{1}{12}x\cos 3x + \frac{1}{36}\sin 3x \ln|\sin 3x|} \qquad\qquad y_p = v_1 y_1 + v_2 y_2$$

$$\boxed{y(x) = C_1\cos 3x + C_2\sin 3x - \frac{1}{12}x\cos 3x + \frac{1}{36}\sin 3x \ln|\sin 3x|}$$

Exercise

Find the general solution $\left(D^2+5D+6\right)y=x^2+2x$

Solution

Characteristic Eqn.: $\lambda^2+5\lambda+6=0 \Rightarrow \boxed{\lambda_{1,2}=-3,\ -2}$

$$\boxed{y_h=C_1e^{-3x}+C_2e^{-2x}}$$

$$W=\begin{vmatrix} e^{-3x} & e^{-2x} \\ -3e^{-3x} & -2e^{-2x} \end{vmatrix}$$

$$=-2e^{-5x}+3e^{-5x}$$

$$\boxed{=e^{-5x}\neq 0}$$

$$v_1(x)=-\int\frac{\left(x^2+2x\right)e^{-2x}}{e^{-5x}}dx \qquad v_1(x)=-\int\frac{y_2g(x)}{W}dx$$

$$=-\int\left(x^2+2x\right)e^{3x}\,dx$$

$$=-\left(\frac{1}{3}x^2+\frac{2}{3}x-\frac{2}{9}x-\frac{2}{9}+\frac{2}{27}\right)e^{3x}$$

$$\boxed{=-\left(\frac{1}{3}x^2+\frac{4}{9}x-\frac{4}{27}\right)e^{3x}}$$

		$\int e^{3x}$
+	x^2+2x	$\frac{1}{3}e^{3x}$
−	$2x+2$	$\frac{1}{9}e^{3x}$
+	2	$\frac{1}{27}e^{3x}$

$$v_2(x)=\int\frac{\left(x^2+2x\right)e^{-3x}}{e^{-5x}}dx \qquad v_2(x)=\int\frac{y_1g(x)}{W}dx$$

$$=\int\left(x^2+2x\right)e^{2x}\,dx$$

$$=-\left(\frac{1}{2}x^2+x-\frac{1}{2}x-\frac{1}{2}+\frac{1}{4}\right)e^{2x}$$

$$\boxed{=\left(\frac{1}{2}x^2+\frac{1}{2}x-\frac{1}{4}\right)e^{2x}}$$

		$\int e^{2x}$
+	x^2+2x	$\frac{1}{2}e^{2x}$
−	$2x+2$	$\frac{1}{4}e^{2x}$
+	2	$\frac{1}{8}e^{2x}$

$$y_p=-\left(\frac{1}{3}x^2+\frac{4}{9}x-\frac{4}{27}\right)e^{3x}e^{-3x}+\left(\frac{1}{2}x^2+\frac{1}{2}x-\frac{1}{4}\right)e^{2x}e^{-2x} \qquad y_p=v_1y_1+v_2y_2$$

$$=-\frac{1}{3}x^2-\frac{4}{9}x+\frac{4}{27}+\frac{1}{2}x^2+\frac{1}{2}x-\frac{1}{4}$$

$$\boxed{=\frac{1}{6}x^2+\frac{1}{18}x-\frac{11}{108}}$$

$$\boxed{y(x)=C_1e^{-3x}+C_2e^{-2x}+\frac{1}{6}x^2+\frac{1}{18}x-\frac{11}{108}}$$

Exercise

Find the general solution $\left(D^2-3D+2\right)y=\frac{1}{1+e^{-x}}$

Solution

Characteristic Eqn.: $\lambda^2-3\lambda+2=0 \Rightarrow \boxed{\lambda_{1,2}=1,\ 2}$

$$\boxed{y_h=C_1e^x+C_2e^{2x}}$$

$$W=\begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}$$

$$=2e^{3x}-e^{3x}$$

$$\boxed{=e^{3x}\neq 0}$$

$$v_1(x)=-\int \frac{e^{2x}}{e^{3x}}\frac{1}{1+e^{-x}}\,dx \qquad v_1(x)=-\int \frac{y_2 g(x)}{W}dx$$

$$=-\int \frac{e^{-x}}{1+e^{-x}}\,dx$$

$$=\int \frac{1}{1+e^{-x}}\,d\left(1+e^{-x}\right)$$

$$\boxed{=\ln\left(1+e^{-x}\right)}$$

$$v_2(x)=\int \frac{e^{x}}{e^{3x}}\frac{1}{1+e^{-x}}\,dx \qquad v_2(x)=\int \frac{y_1 g(x)}{W}dx$$

$$=\int \frac{1}{e^{2x}+e^x}\,dx$$

$$=\int \frac{1}{e^x\left(e^x+1\right)}\,dx \qquad \frac{1}{e^x\left(e^x+1\right)}=\frac{1}{e^x}-\frac{1}{e^x+1}$$

$$=\int\left(\frac{1}{e^x}-\frac{1}{e^x+1}\frac{e^{-x}}{e^{-x}}\right)dx$$

$$=\int\left(e^{-x}-\frac{e^{-x}}{1+e^{-x}}\right)dx$$

$$=-e^{-x}-\int \frac{1}{1+e^{-x}}\,d\left(1+e^{-x}\right)$$

$$\boxed{=-e^{-x}-\ln\left(1+e^{-x}\right)}$$

$$y_p=e^x\ln\left(1+e^{-x}\right)+\left(-e^{-x}-\ln\left(1+e^{-x}\right)\right)e^{2x} \qquad y_p=v_1y_1+v_2y_2$$

$$= e^{x}\ln\left(1+e^{-x}\right) - e^{x} - e^{2x}\ln\left(1+e^{-x}\right)$$

$$y(x) = C_1 e^{x} + C_2 e^{2x} + e^{x}\ln\left(1+e^{-x}\right) - e^{x} - e^{2x}\ln\left(1+e^{-x}\right)$$

Exercise

Find the general solution $\quad y''' + y' = \sec x$

Solution

Characteristic Eqn.: $\quad \lambda^3 + \lambda = \lambda\left(\lambda^2+1\right) = 0 \quad \to \quad \lambda_{1,2,3} = 0, \ \pm i$

$$y_h = C_1 + C_2\cos x + C_3\sin x$$

$$W = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = \sin^2 x + \cos^2 x = 1 \ \neq 0$$

$$W_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \sec x & -\cos x & -\sin x \end{vmatrix} = \cos^2 x\sec x + \sin^2 x\sec x = \sec x$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \sec x & -\sin x \end{vmatrix} = -1$$

$$W_3 = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \sec x \end{vmatrix} = -\sin x\sec x = -\tan x$$

$$u_1(x) = \int \sec x\, dx = \ln\left|\sec x + \tan x\right| \qquad u_1 = \int \frac{W_1}{W}$$

$$u_2(x) = -\int dx = -x \qquad u_2 = \int \frac{W_2}{W}$$

$$u_3(x) = \int -\tan x\, dx = \ln\left|\cos x\right| \qquad u_3 = \int \frac{W_3}{W}$$

$$y_p = \ln\left|\sec x + \tan x\right| - x\cos x + (\sin x)\ln\left|\cos x\right| \qquad y_p = u_1y_1 + u_2y_2 + u_3y_3$$

$$y(x) = C_1 + C_2\cos x + C_3\sin x + \ln\left|\sec x + \tan x\right| - x\cos x + (\sin x)\ln\left|\cos x\right|$$

Exercise

Find the general solution $\qquad y''' - 3y'' + 2y' = \dfrac{e^x}{1+e^{-x}}$

Solution

Characteristic Eqn.: $\qquad \lambda^3 - 3\lambda^2 + 2\lambda = \lambda\left(\lambda^2 - 3\lambda + 2\right) = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2,3} = 0,\ 1,\ 2}$

$$\boxed{y_h = C_1 + C_2 e^x + C_3 e^{2x}}$$

$$W = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix}$$

$$= 4e^{3x} - 2e^{3x}$$

$$\boxed{= 2e^{3x} \quad \neq 0}$$

$$W_1 = \begin{vmatrix} 0 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ \frac{e^x}{1+e^{-x}} & e^x & 4e^{2x} \end{vmatrix}$$

$$= \frac{2e^{4x}}{1+e^{-x}} - \frac{e^{4x}}{1+e^{-x}}$$

$$\boxed{= \frac{e^{4x}}{1+e^{-x}}}$$

$$W_2 = \begin{vmatrix} 1 & 0 & e^{2x} \\ 0 & 0 & 2e^{2x} \\ 0 & \frac{e^x}{1+e^{-x}} & 4e^{2x} \end{vmatrix}$$

$$\boxed{= -\frac{2e^{3x}}{1+e^{-x}}}$$

$$W_3 = \begin{vmatrix} 1 & e^x & 0 \\ 0 & e^x & 0 \\ 0 & e^x & \frac{e^x}{1+e^{-x}} \end{vmatrix}$$

$$\boxed{= \frac{e^{2x}}{1+e^{-x}}}$$

$$u_1(x) = \frac{1}{2}\int \frac{e^x}{1+e^{-x}}\frac{e^x}{e^x}\, dx \qquad\qquad u_1 = \int \frac{W_1}{W}$$

$$= \frac{1}{2}\int \frac{e^{2x}}{e^x+1}\, dx$$

$$= \frac{1}{2}\int \left(e^x - \frac{e^x}{e^x+1}\right) dx$$

$$= \frac{1}{2}\int e^x\, dx - \frac{1}{2}\int \frac{1}{e^x+1}\, d\left(e^x+1\right)$$

$$= \frac{1}{2}e^x - \frac{1}{2}\ln\left(e^x+1\right)$$

$$u_2(x) = -\int \frac{1}{1+e^{-x}}\frac{e^x}{e^x}\, dx \qquad\qquad u_2 = \int \frac{W_2}{W}$$

$$= -\int \frac{e^x}{e^x+1}\, dx$$

$$= -\int \frac{1}{e^x+1}\, d\left(e^x+1\right)$$

$$= -\ln\left(e^x+1\right)$$

$$u_3(x) = \int \frac{1}{2e^{3x}}\frac{e^{2x}}{1+e^{-x}}\, dx \qquad\qquad u_3 = \int \frac{W_3}{W}$$

$$= \frac{1}{2}\int \frac{e^{-x}}{1+e^{-x}}\, dx$$

$$= -\frac{1}{2}\int \frac{1}{1+e^{-x}}\, d\left(1+e^{-x}\right)$$

$$= -\frac{1}{2}\ln\left(1+e^{-x}\right)$$

$$y_p = \frac{1}{2}e^x - \frac{1}{2}\ln\left(e^x+1\right) - e^x\ln\left(e^x+1\right) - \frac{1}{2}e^{2x}\ln\left(1+e^{-x}\right) \qquad\qquad y_p = u_1y_1 + u_2y_2 + u_3y_3$$

$$y(x) = C_1 + C_2e^x + C_3e^{2x} + \frac{1}{2}e^x - \left(\frac{1}{2} + e^x\right)\ln\left(e^x+1\right) - \frac{1}{2}e^{2x}\ln\left(1+e^{-x}\right)$$

Exercise

Find the general solution $\qquad y''' - 6y'' + 11y' - 6y = e^x$

Solution

Characteristic Eqn.: $\qquad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \quad \rightarrow \quad \lambda_1 = 1$

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array} \quad \rightarrow \lambda^2 - 5\lambda + 6 = 0 \quad \Rightarrow \boxed{\lambda_{2,3} = 2,\ 3}$$

$$\boxed{y_h = C_1 e^x + C_2 e^{2x} + C_3 e^{3x}}$$

$$W = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= 18e^{6x} + 3e^{6x} + 4e^{6x} - 2e^{6x} - 12e^{6x} - 9e^{6x}$$

$$\boxed{= 2e^{6x} \neq 0}$$

$$W_1 = \begin{vmatrix} 0 & e^{2x} & e^{3x} \\ 0 & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} \boxed{= e^{6x}}$$

$$W_2 = \begin{vmatrix} e^x & 0 & e^{3x} \\ e^x & 0 & 3e^{3x} \\ e^x & e^x & 9e^{3x} \end{vmatrix} \boxed{= -2e^{5x}}$$

$$W_3 = \begin{vmatrix} e^x & e^{2x} & 0 \\ e^x & 2e^{2x} & 0 \\ e^x & 4e^{2x} & e^x \end{vmatrix} \boxed{= e^{4x}}$$

$$u_1(x) = \int \frac{e^{6x}}{2e^{6x}}\, dx \qquad\qquad u_1 = \int \frac{W_1}{W}$$

$$= \frac{1}{2}\int dx$$

$$\boxed{= \frac{1}{2}x}$$

$$u_2(x) = -\int \frac{2e^{5x}}{2e^{6x}}\, dx \qquad\qquad u_2 = \int \frac{W_2}{W}$$

$$= -\int e^{-x}\, dx$$

$$\boxed{= e^{-x}}$$

$$u_3(x) = \int \frac{e^{4x}}{2e^{6x}} dx \qquad\qquad u_3 = \int \frac{W_3}{W}$$

$$=\frac{1}{2}\int e^{-2x}dx$$

$$\boxed{=-\frac{1}{4}e^{-2x}}$$

$$y_p=\frac{1}{2}xe^x+e^{-x}e^{2x}-\frac{1}{4}e^{-2x}e^{3x} \qquad y_p=u_1y_1+u_2y_2+u_3y_3$$

$$=\frac{1}{2}xe^x+e^x-\frac{1}{4}e^x$$

$$\boxed{=\frac{1}{2}xe^x+\frac{3}{4}e^x}$$

$$y(x)=C_1e^x+C_2e^{2x}+C_3e^{3x}+\frac{1}{2}xe^x+\frac{3}{4}e^x \qquad \left(C_1+\frac{3}{4}\right)e^x=C_4e^x$$

$$\boxed{=C_4e^x+C_2e^{2x}+C_3e^{3x}+\frac{1}{2}xe^x}$$

Exercise

Find the general solution $\quad x^3y^{(3)}-4x^2y''+8xy'-8y=4\ln x$

Solution

Characteristic Eqn.: $\quad \lambda(\lambda-1)(\lambda-2)-4\lambda(\lambda-1)+8\lambda-8=0$

$$\lambda^3-3\lambda^2+2\lambda-4\lambda^2+4\lambda+8\lambda-8=0$$

$$\lambda^3-7\lambda^2+14\lambda-8=0$$

$$\begin{array}{c|cccc} 1 & 1 & -7 & 14 & -8 \\ & & 1 & -6 & 8 \\ \hline & 1 & -6 & 8 & 0 \end{array} \quad \rightarrow \lambda^2-6\lambda+8=0 \quad \lambda=\frac{6\pm 2}{2}$$

The roots are: $\boxed{\lambda_1=1,\ \lambda_2=2,\ \lambda_3=4}$

$$\boxed{y_h=C_1x+C_2x^2+C_3x^4}$$

$$W=\begin{vmatrix} x & x^2 & x^4 \\ 1 & 2x & 4x^3 \\ 0 & 2 & 16x^2 \end{vmatrix}$$

$$=32x^4+2x^4-8x^4-16x^4$$

$$\boxed{=6x^4 \quad \neq 0}$$

$$y^{(3)}-\frac{4}{x}y''+\frac{8}{x^2}y'-\frac{8}{x^3}y=\frac{4\ln x}{x^3}$$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^4 \\ 0 & 2x & 4x^3 \\ \frac{4\ln x}{x^3} & 2 & 16x^2 \end{vmatrix}$$

$$= \frac{4\ln x}{x^3}\left(4x^5 - 2x^5\right)$$

$$\boxed{= 8x^2 \ln x}$$

$$W_2 = \begin{vmatrix} x & 0 & x^4 \\ 1 & 0 & 4x^3 \\ 0 & \frac{4\ln x}{x^3} & 16x^2 \end{vmatrix}$$

$$= -\frac{4\ln x}{x^3}\left(4x^4 - x^4\right)$$

$$\boxed{= -12x \ln x}$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & \frac{4\ln x}{x^3} \end{vmatrix}$$

$$= \frac{4\ln x}{x^3}\left(2x^2 - x^2\right)$$

$$\boxed{= \frac{4\ln x}{x}}$$

$$u_1(x) = \int \frac{8x^2 \ln x}{6x^4}\, dx \qquad u_1 = \int \frac{W_1}{W}$$

$$= \frac{4}{3}\int \frac{\ln x}{x^2}\, dx$$

$u = \ln x$	$dv = x^{-2}dx$
$du = \frac{dx}{x}$	$v = -\frac{1}{x}$

$$= \frac{4}{3}\left(-\frac{\ln x}{x} + \int \frac{1}{x^2}\, dx\right)$$

$$= \frac{4}{3}\left(-\frac{\ln x}{x} - \frac{1}{x}\right)$$

$$\boxed{= -\frac{4\ln x}{3x} - \frac{4}{3x}}$$

$$u_2(x) = \int \frac{-12x \ln x}{6x^4}\, dx \qquad u_2 = \int \frac{W_2}{W}$$

$$= -2\int \frac{\ln x}{x^3}\, dx$$

$$= -2\left(-\frac{\ln x}{2x^2} + \frac{1}{2}\int x^{-3}dx\right)$$

$u = \ln x$	$dv = x^{-3}dx$
$du = \frac{dx}{x}$	$v = -\frac{1}{2}x^{-2}$

$$= -2\left(-\frac{\ln x}{2x^2} - \frac{1}{4x^2}\right)$$

$$= \frac{\ln x}{x^2} + \frac{1}{2x^2}$$

$$u_3(x) = \int \frac{4\ln x}{x}\frac{1}{6x^4}\, dx \qquad u_3 = \int \frac{W_3}{W}$$

$$= \frac{2}{3}\int \frac{\ln x}{x^5}\, dx$$

$u = \ln x$	$dv = x^{-5}dx$
$du = \frac{dx}{x}$	$v = -\frac{1}{4}x^{-4}$

$$= \frac{2}{3}\left(-\frac{\ln x}{4x^4} + \frac{1}{4}\int \frac{1}{x^5}dx\right)$$

$$= \frac{2}{3}\left(-\frac{\ln x}{4x^4} - \frac{1}{16x^4}\right)$$

$$= -\frac{\ln x}{6x^4} - \frac{1}{24x^4}$$

$$y_p = \left(-\frac{4\ln x}{3x} - \frac{4}{3x}\right)x + \left(\frac{\ln x}{x^2} + \frac{1}{2x^2}\right)x^2 + \left(-\frac{\ln x}{6x^4} - \frac{1}{24x^4}\right)x^4 \qquad y_p = u_1y_1 + u_2y_2 + u_3y_3$$

$$= -\frac{4}{3}\ln x - \frac{4}{3} + \ln x + \frac{1}{2} - \frac{\ln x}{6} - \frac{1}{24}$$

$$= -\frac{1}{2}\ln x - \frac{7}{8}$$

$$y(x) = C_1x + C_2x^2 + C_3x^4 - \frac{1}{2}\ln x - \frac{7}{8}$$

Exercise

Find the general solution by to ***variation of parameters*** with the give n initial conditions.

$$y'' + y = \sec t\ ; \quad y(0) = 1, \quad y'(0) = 2$$

Solution

$$\lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm i$$

$$y_h = C_1\cos t + C_2\sin t$$

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1 \neq 0$$

$$v_1(t) = -\int \frac{\sin t}{1}\sec t\ dt \qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$= -\int \tan t \, dt$$

$$\boxed{= \ln|\cos t|}$$

$$v_2(t) = \int \frac{\cos t}{1} \sec t \, dt \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \int dt$$

$$\boxed{= t}$$

$$\boxed{y_p = \ln|\cos t| \cos t + t \sin t} \qquad\qquad y_p = v_1 y_1 + v_2 y_2$$

$$y(t) = C_1 \cos t + C_2 \sin t + \ln|\cos t| \cos t + t \sin t$$

$$y(0) = 1 \quad \to \quad \boxed{C_1 = 1}$$

$$y' = -C_1 \sin t + C_2 \cos t - \sin t - \ln|\cos t| \sin t + \sin t + t \cos t$$

$$y'(0) = 2 \quad \to \quad \boxed{C_2 = 2}$$

$$\boxed{y(t) = \cos t + 2\sin t + \ln|\cos t| \cos t + t \sin t}$$

Exercise

Find the general solution by to ***variation of parameters*** with the give n initial conditions.

$$y'' + y = \sec^3 t \; ; \quad y(0) = 1, \quad y'(0) = \frac{1}{2}$$

Solution

$$\lambda^2 + 1 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = \pm i}$$

$$\boxed{y_h = C_1 \cos t + C_2 \sin t}$$

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t \; \boxed{= 1 \; \neq 0}$$

$$v_1(t) = -\int \frac{\sin t}{1} \sec^3 t \, dt \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\int \tan t \sec^2 t \, dt$$

$$= -\int \sec t \, d(\sec t)$$

$$\boxed{= -\frac{1}{2} \sec^2 t}$$

$$v_2(t)=\int \frac{\cos t}{1}\sec^3 t\ dt \qquad\qquad v_2(x)=\int \frac{y_1 g(x)}{W}dx$$

$$=\int \sec^2 t\ dt$$

$$\underline{=\tan t}$$

$$y_p=-\frac{1}{2}\sec^2 t\cos t+\tan t\sin t \qquad\qquad y_p=v_1y_1+v_2y_2$$

$$=-\frac{1}{2\cos t}+\frac{\sin^2 t}{\cos t}$$

$$=\frac{1}{2}\frac{2\sin^2 t-1}{\cos t}$$

$$\boxed{=-\frac{1}{2}\frac{\cos 2t}{\cos t}}$$

$$\boxed{y(t)=C_1\cos t+C_2\sin t-\frac{1}{2}\frac{\cos 2t}{\cos t}}$$

$$y(0)=1\quad \to C_1-\frac{1}{2}=1\quad \boxed{C_1=\frac{3}{2}}$$

$$y'=-C_1\sin t+C_2\cos t-\frac{1}{2}\frac{-2\sin 2t\cos 2t+\sin t\cos 2t}{\cos^2 t}$$

$$y'(0)=\frac{1}{2}\quad\to\quad \boxed{C_2=\frac{1}{2}}$$

$$\boxed{y(t)=\frac{3}{2}\cos t+\frac{1}{2}\sin t-\frac{1}{2}\frac{\cos 2t}{\cos t}}$$

Exercise

Find the general solution by to ***variation of parameters*** with the give n initial conditions.

$$y''-y=t+\sin t\ ;\quad y(0)=2,\quad y'(0)=3$$

Solution

$$\lambda^2-1=0\quad\Rightarrow\quad \boxed{\lambda_{1,2}=\pm 1}$$

$$\boxed{y_h=C_1e^{-t}+C_2e^{t}}$$

$$W=\begin{vmatrix} e^{-t} & e^{t}\\ -e^{-t} & e^{t}\end{vmatrix}\ \boxed{=2\ \neq 0}$$

		$\int e^t$
+	t	e^t
−	1	e^t

$$v_1(t)=-\frac{1}{2}\int e^t(t+\sin t)\ dt \qquad\qquad v_1(x)=-\int\frac{y_2 g(x)}{W}dx$$

$$= -\frac{1}{2}\int \left(te^t + e^t \sin t\right)\,dt$$

		$\int \sin t$
+	e^t	$-\cos t$
−	e^t	$-\sin t$
+	e^t	

$$\int \left(e^t \sin t\right)\,dt = e^t\left(-\cos t + \sin t\right) - \int \left(e^t \sin t\right)\,dt$$

$$2\int \left(e^t \sin t\right)\,dt = e^t\left(\sin t - \cos t\right)$$

$$= -\frac{1}{2}\left[(t-1)e^t + \frac{1}{2}e^t\left(\sin t - \cos t\right)\right]$$

$$\boxed{= -\frac{1}{2}\left(t - 1 + \frac{1}{2}\sin t - \frac{1}{2}\cos t\right)e^t}$$

		$\int e^{-t}$
+	t	$-e^{-t}$
−	1	e^{-t}

$$v_2(t) = \frac{1}{2}\int e^{-t}\left(t + \sin t\right)\,dt \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$= \frac{1}{2}\int \left(te^{-t} + e^{-t}\sin t\right)\,dt$$

		$\int \sin t$
+	e^{-t}	$-\cos t$
−	$-e^{-t}$	$-\sin t$
+	e^{-t}	

$$\int \left(e^{-t}\sin t\right)\,dt = e^{-t}\left(-\cos t - \sin t\right) - \int \left(e^{-t}\sin t\right)\,dt$$

$$2\int \left(e^{-t}\sin t\right)\,dt = -e^{-t}\left(\sin t + \cos t\right)$$

$$= \frac{1}{2}\left[(-t-1)e^{-t} - \frac{1}{2}e^{-t}\left(\sin t + \cos t\right)\right]$$

$$\boxed{= -\frac{1}{2}\left(t + 1 + \frac{1}{2}\sin t + \frac{1}{2}\cos t\right)e^{-t}}$$

$$y_p = -\frac{1}{2}\left(t - 1 + \frac{1}{2}\sin t - \frac{1}{2}\cos t\right)e^t e^{-t} - \frac{1}{2}\left(t + 1 + \frac{1}{2}\sin t + \frac{1}{2}\cos t\right)e^{-t}e^t \qquad y_p = v_1y_1 + v_2y_2$$

$$= -\frac{1}{2}t + \frac{1}{2} - \frac{1}{4}\sin t + \frac{1}{4}\cos t - \frac{1}{2}t - \frac{1}{2} - \frac{1}{4}\sin t - \frac{1}{4}\cos t$$

$$\boxed{= -t - \frac{1}{2}\sin t}$$

$$\boxed{y(t) = C_1e^{-t} + C_2e^t - t - \frac{1}{2}\sin t}$$

$$y(0) = 2 \quad \to \quad C_1 + C_2 = 2$$

$$y' = -C_1e^{-t} + C_2e^t - 1 - \frac{1}{2}\cos t$$

$$y'(0) = 3 \quad \to -C_1 + C_2 - 1 - \frac{1}{2} = 3 \quad \Rightarrow -C_1 + C_2 = \frac{9}{2}$$

$$\begin{cases} C_1 + C_2 = 2 \\ -C_1 + C_2 = \frac{9}{2} \end{cases} \qquad\qquad \boxed{C_2 = \frac{13}{4} \quad C_1 = -\frac{5}{4}}$$

$$\boxed{y(t) = -\frac{5}{4}e^{-t} + \frac{13}{4}e^t - t - \frac{1}{2}\sin t}$$

Exercise

Find the general solution by to ***variation of parameters*** with the give n initial conditions.

$$y'' - 2y' + y = \frac{e^x}{x}; \quad y(1) = 0, \quad y'(1) = 0$$

Solution

$$\lambda^2 - 2\lambda + 1 = 0 \quad \to \quad \boxed{\lambda_{1,2} = 1}$$

$$\boxed{y_h = C_1 e^x + C_2 x e^x}$$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}$$

$$= e^{2x} + xe^{2x} - xe^{2x}$$

$$\boxed{= e^{2x} \neq 0}$$

$$v_1(x) = -\int \frac{xe^x}{e^{2x}} \frac{e^x}{x}\, dx \qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\int dx$$

$$\boxed{= -x}$$

$$v_2(x) = \int \frac{e^x}{e^{2x}} \frac{e^x}{x}\, dx \qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \int \frac{dx}{x}$$

$$\boxed{= \ln|x|}$$

$$\boxed{y_p = -xe^x + xe^x \ln|x|} \qquad y_p = v_1 y_1 + v_2 y_2$$

$$y(x) = C_1 e^x + C_2 x e^x - xe^x + xe^x \ln|x|$$

$$y(1) = 0 \quad \to eC_1 + eC_2 - e = 0 \quad \Rightarrow \quad C_1 + C_2 = 1$$

$$y' = C_1 e^x + C_2 e^x + C_2 xe^x - xe^x - e^x + e^x \ln|x| + xe^x \ln|x| + e^x$$

$$= C_1 e^x + C_2 e^x + C_2 xe^x - xe^x + e^x \ln|x| + xe^x \ln|x|$$

$$y'(1) = 0 \quad \to eC_1 + 2eC_2 - e = 0 \quad \Rightarrow \quad C_1 + 2C_2 = 1$$

$$\begin{cases} C_1 + C_2 = 1 \\ -C_1 - 2C_2 = -1 \end{cases} \quad \boxed{C_2 = 0} \quad \boxed{C_1 = 1}$$

$$\boxed{y(x) = e^x - xe^x + xe^x \ln|x|}$$

Exercise

Find the general solution by to ***variation of parameters*** with the given initial conditions.

$$y'' + 2y' - 8y = 2e^{-2x} - e^{-x} \ ; \quad y(0) = 1, \quad y'(0) = 0$$

Solution

$$\lambda^2 + 2\lambda - 8 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = -4,\ 2}$$

$$\boxed{y_h = C_1 e^{-4x} + C_2 e^{2x}}$$

$$W = \begin{vmatrix} e^{-4x} & e^{2x} \\ -4e^{-4x} & 2e^{2x} \end{vmatrix}$$

$$= 2e^{-2x} + 4e^{-2x}$$

$$\boxed{= 6e^{-2x} \neq 0}$$

$$v_1(x) = -\int \frac{e^{2x}}{6e^{-2x}} \left(2e^{-2x} - e^{-x}\right) dx \qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\frac{1}{6} \int \left(2e^{2x} - e^{3x}\right) dx$$

$$\boxed{= -\frac{1}{6}\left(e^{2x} - \frac{1}{3}e^{3x}\right)}$$

$$v_2(x) = \frac{1}{6} \int \frac{e^{-4x}}{e^{-2x}} \left(2e^{-2x} - e^{-x}\right) dx \qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \frac{1}{6} \int \left(2e^{-4x} - e^{-3x}\right) dx$$

$$\boxed{= \frac{1}{6}\left(-\frac{1}{2}e^{-4x} + \frac{1}{3}e^{-3x}\right)}$$

$$y_p = \left(-\frac{1}{6}e^{2x} + \frac{1}{18}e^{3x}\right)e^{-4x} + \left(-\frac{1}{12}e^{-4x} + \frac{1}{18}e^{-3x}\right)e^{2x} \qquad y_p = v_1 y_1 + v_2 y_2$$

$$= -\frac{1}{6}e^{-2x} + \frac{1}{18}e^{-x} - \frac{1}{12}e^{-2x} + \frac{1}{18}e^{-x}$$

$$\boxed{= -\frac{1}{4}e^{-2x} + \frac{1}{9}e^{-x}}$$

$$\boxed{y(x) = C_1 e^{-4x} + C_2 e^{2x} - \frac{1}{4}e^{-2x} + \frac{1}{9}e^{-x}}$$

$$y(0) = 1 \quad \rightarrow C_1 + C_2 - \frac{1}{4} + \frac{1}{9} = 1 \quad \Rightarrow \quad C_1 + C_2 = \frac{41}{36}$$

$$y'(x) = -4C_1 e^{-4x} + 2C_2 e^{2x} + \frac{1}{2}e^{-2x} - \frac{1}{9}e^{-x}$$

$$y'(0) = 0 \quad \to -4C_1 + 2C_2 + \frac{1}{2} - \frac{1}{9} = 0 \quad \Rightarrow \quad -4C_1 + 2C_2 = -\frac{7}{18}$$

$$\begin{cases} C_1 + C_2 = \frac{41}{36} \\ -4C_1 + 2C_2 = -\frac{7}{18} \end{cases} \quad \to \quad \boxed{C_1 = \frac{4}{9},\ C_2 = \frac{25}{36}}$$

$$\boxed{y(x) = \frac{4}{9}e^{-4x} + \frac{25}{36}e^{2x} - \frac{1}{4}e^{-2x} + \frac{1}{9}e^{-x}}$$

Exercise

Find the general solution by to ***variation of parameters*** with the given initial conditions.

$$y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x \ ; \quad y(0) = 1, \quad y'(0) = 2$$

Solution

$$\lambda^2 - 3\lambda + 2 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = 1,\ 2}$$

$$\boxed{y_h = C_1 e^{x} + C_2 e^{2x}}$$

$$W = \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix}$$

$$= 2e^{3x} - e^{3x}$$

$$\boxed{= e^{3x} \neq 0}$$

$$v_1(x) = -\int \frac{e^{2x}}{e^{3x}}\left(3e^{-x} - 10\cos 3x\right) dx \qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= -\int e^{-x}\left(3e^{-x} - 10\cos 3x\right) dx$$

$$= \int \left(10e^{-x}\cos 3x - 3e^{-2x}\right) dx$$

$$= 10\int e^{-x}\cos 3x \, dx - \int 3e^{-2x} \, dx$$

		$\int \cos 3x$
+	e^{-x}	$\frac{1}{3}\sin 3x$
−	$-e^{-x}$	$-\frac{1}{9}\cos 3x$
+	e^{-x}	

$$\int e^{-x}\cos 3x \, dx = e^{-x}\left(\frac{1}{3}\sin 3x - \frac{1}{9}\cos 3x\right) - \frac{1}{9}\int e^{-x}\cos 3x \, dx$$

$$\frac{10}{9}\int e^{-x}\cos 3x \, dx = \frac{1}{9}e^{-x}(3\sin 3x - \cos 3x)$$

$$\int e^{-x}\cos 3x \, dx = \frac{1}{10}e^{-x}(3\sin 3x - \cos 3x)$$

$$= e^{-x}(3\sin 3x - \cos 3x) + \frac{3}{2}e^{-2x}$$

$$v_2(x) = \int \frac{e^x}{e^{3x}}\left(3e^{-x} - 10\cos 3x\right) dx \qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \int e^{-2x}\left(3e^{-x} - 10\cos 3x\right) dx$$

$$= 3\int e^{-3x}\, dx - 10\int e^{-2x}\cos 3x\, dx$$

$$\int e^{-2x}\cos 3x\, dx = e^{-2x}\left(\frac{1}{3}\sin 3x - \frac{2}{9}\cos 3x\right) - \frac{4}{9}\int e^{-2x}\cos 3x\, dx$$

$$\frac{13}{9}\int e^{-2x}\cos 3x\, dx = \frac{1}{9}e^{-2x}(3\sin 3x - 2\cos 3x)$$

$$\int e^{-2x}\cos 3x\, dx = \frac{1}{13}e^{-2x}(3\sin 3x - 2\cos 3x)$$

$$= -e^{-3x} - \frac{10}{13}e^{-2x}(3\sin 3x - 2\cos 3x)$$

		$\int \cos 3x$
+	e^{-2x}	$\frac{1}{3}\sin 3x$
−	$-2e^{-2x}$	$-\frac{1}{9}\cos 3x$
+	$4e^{-2x}$	

$$y_p = \left(e^{-x}(3\sin 3x - \cos 3x) + \frac{3}{2}e^{-2x}\right)e^x + \left(-e^{-3x} - \frac{10}{13}e^{-2x}(3\sin 3x - 2\cos 3x)\right)e^{2x}$$

$$= 3\sin 3x - \cos 3x + \frac{3}{2}e^{-x} - e^{-x} - \frac{30}{13}\sin 3x + \frac{20}{13}\cos 3x$$

$$= \frac{1}{2}e^{-x} + \frac{9}{13}\sin 3x + \frac{7}{13}\cos 3x$$

$$y(x) = C_1 e^x + C_2 e^{2x} + \frac{1}{2}e^{-x} + \frac{9}{13}\sin 3x + \frac{7}{13}\cos 3x$$

$$y(0) = 1 \quad \to C_1 + C_2 + \frac{1}{2} + \frac{7}{13} = 1 \quad \Rightarrow \quad C_1 + C_2 = -\frac{1}{26}$$

$$y'(x) = C_1 e^x + 2C_2 e^{2x} - \frac{1}{2}e^{-x} + \frac{27}{13}\cos 3x - \frac{21}{13}\sin 3x$$

$$y'(0) = 2 \quad \to \quad C_1 + 2C_2 - \frac{1}{2} + \frac{27}{13} = 2 \quad \Rightarrow \quad C_1 + 2C_2 = \frac{11}{26}$$

$$\begin{cases} C_1 + C_2 = -\frac{1}{26} \\ C_1 + 2C_2 = \frac{11}{26} \end{cases} \to \quad \Delta = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \quad \Delta_1 = \begin{vmatrix} -\frac{1}{26} & 1 \\ \frac{11}{26} & 2 \end{vmatrix} = -\frac{1}{2} \quad \Delta_2 = \begin{vmatrix} 1 & -\frac{1}{26} \\ 1 & \frac{11}{26} \end{vmatrix} = \frac{6}{13}$$

$$C_1 = -\frac{1}{2},\ C_2 = \frac{6}{13}$$

$$y(x) = -\frac{1}{2}e^x + \frac{6}{13}e^{2x} + \frac{1}{2}e^{-x} + \frac{9}{13}\sin 3x + \frac{7}{13}\cos 3x$$

Exercise

Find the general solution by to ***variation of parameters*** with the give n initial conditions.

$$y'' + 4y = \sin^2 2t \ ; \quad y\left(\frac{\pi}{8}\right) = 0, \quad y'\left(\frac{\pi}{8}\right) = 0$$

Solution

$$\lambda^2 + 4 = 0 \quad \rightarrow \quad \lambda_{1,2} = \pm 2i$$

$$y_h = C_1 \cos 2t + C_2 \sin 2t$$

$$W = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix}$$

$$= 2\cos^2 2t + 2\sin^2 2t$$

$$= 2 \neq 0$$

$$v_1(t) = -\int \frac{\sin 2t \sin^2 2t}{2}\, dt \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= \frac{1}{4}\int \left(1 - \cos^2 2t\right) d(\cos 2t)$$

$$= \frac{1}{4}\left(\cos 2t - \frac{1}{3}\cos^3 2t\right)$$

$$v_2(t) = \frac{1}{2}\int \cos 2t \sin^2 2t\, dt \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \frac{1}{4}\int \sin^2 2t\, d(\sin 2t)$$

$$= \frac{1}{12}\sin^3 2t$$

$$y_p = \frac{1}{4}\left(\cos 2t - \frac{1}{3}\cos^3 2t\right)\cos 2t + \frac{1}{12}\sin^3 2t(\sin 2t) \qquad\qquad y_p = v_1 y_1 + v_2 y_2$$

$$= \frac{1}{4}\cos^2 2t - \frac{1}{12}\cos^4 2t + \frac{1}{12}\sin^4 2t$$

$$= \frac{1}{4}\cos^2 2t - \frac{1}{12}\left(\cos^4 2t - \sin^4 2t\right)$$

$$= \frac{1}{4}\cos^2 2t - \frac{1}{12}\left(\cos^2 2t - \sin^2 2t\right)\left(\cos^2 2t + \sin^2 2t\right)$$

$$= \frac{1}{8}(1 + \cos 4t) - \frac{1}{12}\cos 4t$$

$$= \frac{1}{8} + \frac{1}{24}\cos 4t$$

$$y(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{8} + \frac{1}{24}\cos 4t$$

$$y\left(\frac{\pi}{8}\right) = 0 \quad \rightarrow \frac{\sqrt{2}}{2}C_1 + \frac{\sqrt{2}}{2}C_2 + \frac{1}{8} = 0 \quad \Rightarrow \quad \sqrt{2}C_1 + \sqrt{2}C_2 = -\frac{1}{4}$$

$$y' = -2C_1 \sin 2t + 2C_2 \cos 2t - \frac{1}{6}\sin 4t$$

$$y'\left(\frac{\pi}{8}\right) = 0 \quad \rightarrow \quad -\sqrt{2}C_1 + \sqrt{2}C_2 = \frac{1}{6}$$

$$\begin{cases} \sqrt{2}C_1 + \sqrt{2}C_2 = -\frac{1}{4} \\ -\sqrt{2}C_1 + \sqrt{2}C_2 = \frac{1}{6} \end{cases} \quad \boxed{C_2 = -\frac{1}{24\sqrt{2}}} \quad \sqrt{2}C_1 = -\frac{1}{4} + \frac{1}{24} \Rightarrow \boxed{C_1 = -\frac{5}{24\sqrt{2}}}$$

$$\boxed{y(t) = -\frac{5\sqrt{2}}{48}\cos 2t - \frac{\sqrt{2}}{48}\sin 2t + \frac{1}{8} + \frac{1}{24}\cos 4t}$$

Exercise

Find the general solution by to ***variation of parameters*** with the give n initial conditions.

$$y'' + 4y = \sin^2 2t \ ; \quad y(0) = 0, \quad y'(0) = 0$$

Solution

$$\lambda^2 + 4 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 2i}$$

$$\boxed{y_h = C_1 \cos 2t + C_2 \sin 2t}$$

$$W = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix}$$

$$= 2\cos^2 2t + 2\sin^2 2t$$

$$\boxed{= 2 \ \neq 0}$$

$$v_1(t) = -\int \frac{\sin 2t \sin^2 2t}{2}\, dt \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= \frac{1}{4}\int \left(1 - \cos^2 2t\right)\, d(\cos 2t)$$

$$\boxed{= \frac{1}{4}\left(\cos 2t - \frac{1}{3}\cos^3 2t\right)}$$

$$v_2(t) = \frac{1}{2}\int \cos 2t \sin^2 2t\, dt \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

$$= \frac{1}{4}\int \sin^2 2t\, d(\sin 2t)$$

$$\boxed{= \frac{1}{12}\sin^3 2t}$$

$$y_p = \frac{1}{4}\left(\cos 2t - \frac{1}{3}\cos^3 2t\right)\cos 2t + \frac{1}{12}\sin^3 2t(\sin 2t) \qquad\qquad y_p = v_1 y_1 + v_2 y_2$$

$$= \frac{1}{4}\cos^2 2t - \frac{1}{12}\cos^4 2t + \frac{1}{12}\sin^4 2t$$

$$=\frac{1}{4}\cos^2 2t-\frac{1}{12}\left(\cos^4 2t-\sin^4 2t\right)$$

$$=\frac{1}{4}\cos^2 2t-\frac{1}{12}\left(\cos^2 2t-\sin^2 2t\right)\left(\cos^2 2t+\sin^2 2t\right)$$

$$=\frac{1}{8}(1+\cos 4t)-\frac{1}{12}\cos 4t$$

$$\boxed{=\frac{1}{8}+\frac{1}{24}\cos 4t}$$

$$y(t)=C_1\cos 2t+C_2\sin 2t+\frac{1}{8}+\frac{1}{24}\cos 4t$$

$$y(0)=0\quad\to C_1+\frac{1}{8}+\frac{1}{24}=0\quad\Rightarrow\quad \boxed{C_1=-\frac{1}{6}}$$

$$y'=-2C_1\sin 2t+2C_2\cos 2t-\frac{1}{6}\sin 4t$$

$$y'(0)=0\quad\to\quad \boxed{C_2=0}$$

$$\boxed{y(t)=-\frac{1}{6}\cos 2t+\frac{1}{8}+\frac{1}{24}\cos 4t}$$

Exercise

Find the general solution by to ***variation of parameters*** with the give n initial conditions.

$$y''-4y'+4y=\left(12x^2-6x\right)e^{2x}\ ;\quad y(0)=1,\quad y'(0)=0$$

Solution

$$\lambda^2-4\lambda+4=(\lambda-2)^2=0\quad\Rightarrow\quad \boxed{\lambda_{1,2}=2}$$

$$\boxed{y_h=C_1e^{2x}+C_2xe^{2x}}$$

$$W=\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{vmatrix}$$

$$=e^{4x}+2xe^{4x}-2xe^{4x}$$

$$\boxed{=e^{4x}\neq 0}$$

$$v_1(x)=-\int\frac{xe^{2x}}{e^{4x}}\left(12x^2-6x\right)e^{2x}\,dx \qquad\qquad v_1(x)=-\int\frac{y_2g(x)}{W}dx$$

$$=-\int\left(12x^3-6x^2\right)\,dx$$

$$\boxed{=2x^3-3x^4}$$

$$v_2(x)=\int\frac{e^{2x}}{e^{4x}}\left(12x^2-6x\right)e^{2x}\,dx \qquad\qquad v_2(x)=\int\frac{y_1g(x)}{W}dx$$

$$= \int \left(12x^2 - 6x\right) dx$$

$$\boxed{= 4x^3 - 3x^2}$$

$$y_p = \left(2x^3 - 3x^4\right)e^{2x} + \left(4x^3 - 3x^2\right)xe^{2x} \qquad y_p = v_1 y_1 + v_2 y_2$$

$$= \left(2x^3 - 3x^4 + 4x^4 - 3x^3\right)e^{2x}$$

$$\boxed{= \left(x^4 - x^3\right)e^{2x}}$$

$$y(x) = \left(C_1 + C_2 x - x^3 + x^4\right)e^{2x}$$

$$y(0) = 1 \quad \rightarrow \quad \boxed{C_1 = 1}$$

$$y'(x) = \left(C_2 - 3x^2 + 4x^3 + 2C_1 + 2C_2 x - 2x^3 + 2x^4\right)e^{2x}$$

$$y'(0) = 0 \quad \rightarrow C_2 + 2C_1 = 0 \quad \Rightarrow \quad \boxed{C_2 = -2}$$

$$\boxed{y(x) = \left(1 - 2x - x^3 + x^4\right)e^{2x}}$$

Exercise

Find the general solution by to ***variation of parameters*** with the given initial conditions.

$2y'' + y' - y = x + 1\,; \quad y(0) = 1, \quad y'(0) = 0$

Solution

$$2\lambda^2 + \lambda - 1 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = -1,\ \frac{1}{2}}$$

$$\boxed{y_h = C_1 e^{-x} + C_2 e^{x/2}}$$

$$W = \begin{vmatrix} e^{-x} & e^{x/2} \\ -e^{-x} & \frac{1}{2}e^{x/2} \end{vmatrix}$$

$$= \frac{1}{2}e^{-x/2} + e^{-x/2}$$

$$\boxed{= \frac{3}{2}e^{-x/2} \quad \neq 0}$$

		$\int e^x$
+	$x+1$	e^x
−	1	e^x

$$y'' + \frac{1}{2}y' - \frac{1}{2}y = \frac{1}{2}(x+1)$$

$$v_1(x) = -\int \frac{2}{3}\frac{e^{x/2}}{e^{-x/2}}\frac{1}{2}(x+1)\ dx \qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$= -\frac{1}{3}\int (x+1)e^{x}\, dx$$

$$\boxed{= -\frac{1}{3}xe^{x}}$$

$$v_2(x) = \frac{1}{3}\int \frac{e^{-x}}{e^{-x/2}}(x+1)\, dx \qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$= \frac{1}{3}\int (x+1)e^{-x/2}\, dx$$

$$\boxed{= \frac{1}{3}(-2x-6)e^{-x/2}}$$

		$\int e^{-x/2}$
+	$x+1$	$-2e^{-x/2}$
−	1	$4e^{-x/2}$

$$y_p = -\frac{1}{3}xe^{x}e^{-x} - \frac{2}{3}(x+3)e^{-x/2}e^{x/2} \qquad y_p = v_1y_1 + v_2y_2$$

$$= -\frac{1}{3}x - \frac{2}{3}x - 2$$

$$\boxed{= -x-2}$$

$$y(x) = C_1e^{-x} + C_2e^{x/2} - x - 2$$

$$y(0)=1 \quad \rightarrow \quad \boxed{C_1 + C_2 = 3}$$

$$y'(x) = -C_1e^{-x} + \frac{1}{2}C_2e^{x/2} - 1$$

$$y'(0)=0 \quad \rightarrow -C_1 + \frac{1}{2}C_2 - 1 = 0$$

$$\boxed{-2C_1 + C_2 = 2}$$

$$\begin{cases} C_1 + C_2 = 3 \\ -2C_1 + C_2 = 2 \end{cases} \quad \rightarrow \quad \boxed{C_1 = \frac{1}{3},\ C_2 = \frac{8}{3}}$$

$$\boxed{y(x) = \frac{1}{3}e^{-x} + \frac{8}{3}e^{x/2} - x - 2}$$

Exercise

Find the general solution by to ***variation of parameters*** with the given initial conditions.

$$4y'' - y = xe^{x/2}; \quad y(0)=1, \quad y'(0)=0$$

Solution

$$4\lambda^2 - 1 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = \pm\frac{1}{2}}$$

$$\boxed{y_h = C_1e^{-x/2} + C_2e^{x/2}}$$

$$W = \begin{vmatrix} e^{-x/2} & e^{x/2} \\ -\frac{1}{2}e^{-x/2} & \frac{1}{2}e^{x/2} \end{vmatrix}$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$\boxed{= 1 \neq 0}$$

$$y'' - \frac{1}{4}y = \frac{1}{4}xe^{x/2}$$

$$v_1(x) = -\int \frac{1}{4}xe^{x/2}e^{x/2}\,dx \qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$= -\frac{1}{4}\int xe^x\,dx$$

$$\boxed{= -\frac{1}{4}(x-1)e^x}$$

$$v_2(x) = \int \frac{1}{4}xe^{x/2}e^{-x/2}\,dx \qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$= \frac{1}{4}\int x\,dx$$

$$\boxed{= \frac{1}{8}x^2}$$

$$y_p = -\frac{1}{4}(x-1)e^x e^{-x/2} + \frac{1}{8}x^2 e^{x/2} \qquad y_p = v_1 y_1 + v_2 y_2$$

$$\boxed{= \left(\frac{1}{8}x^2 - \frac{1}{4}x + \frac{1}{4}\right)e^{x/2}}$$

$$y(x) = C_1 e^{-x/2} + C_2 e^{x/2} + \left(\frac{1}{8}x^2 - \frac{1}{4}x + \frac{1}{4}\right)e^{x/2}$$

$$= C_1 e^{-x/2} + \left(\frac{1}{8}x^2 - \frac{1}{4}x + C_3\right)e^{x/2}$$

$$y(0) = 1 \quad \rightarrow \quad \boxed{C_1 + C_3 = 1}$$

$$y' = -\frac{1}{2}C_1 e^{-x/2} + \left(\frac{1}{4}x - \frac{1}{4} + \frac{1}{16}x^2 - \frac{1}{8}x + \frac{1}{2}C_3\right)e^{x/2}$$

$$y'(0) = 0 \quad \rightarrow -\frac{1}{2}C_1 - \frac{1}{4} + \frac{1}{2}C_3 = 0$$

$$\boxed{-2C_1 + 2C_3 = 1}$$

$$\begin{cases} C_1 + C_3 = 1 \\ -2C_1 + 2C_3 = 1 \end{cases} \quad \rightarrow \quad \boxed{C_3 = \frac{3}{4},\ C_1 = \frac{1}{4}}$$

$$\boxed{y(x) = \frac{1}{4}e^{-x/2} + \left(\frac{1}{8}x^2 - \frac{1}{4}x + \frac{3}{4}\right)e^{x/2}}$$

Exercise

Find the general solution $t^2 y'' - ty' + y = t$; $y(1) = 1,\quad y'(1) = 4$

Solution

Characteristic Eqn.: $\lambda(\lambda - 1) - \lambda + 1 = 0$

$\lambda^2 - 2\lambda + 1 = 0$

The roots are: $\boxed{\lambda_{1,2} = 1}$

$$y_h = \left(C_1 + C_2 \ln t\right)e^{\ln t}$$

$$= \left(C_1 + C_2 \ln t\right)t$$

$$\boxed{= C_1 t + C_2 t \ln t}$$

$$W = \begin{vmatrix} t & t\ln t \\ 1 & 1 + \ln t \end{vmatrix}$$

$$= t + t\ln t - t\ln t$$

$$\boxed{= t \neq 0}$$

$$y'' - \frac{1}{t}y' - \frac{1}{t^2}y = \frac{1}{t}$$

$$W_1 = \begin{vmatrix} 0 & t\ln t \\ \frac{1}{t} & 1 + \ln t \end{vmatrix} \quad \boxed{= -\ln t}$$

$$W_2 = \begin{vmatrix} t & 0 \\ 1 & \frac{1}{t} \end{vmatrix} \quad \boxed{= 1}$$

$$u_1(t) = \int \frac{-\ln t}{t}\, dt \qquad\qquad u_1 = \int \frac{W_1}{W}$$

$$= -\int \ln t \; d(\ln t)$$

$$\boxed{= -\frac{1}{2}(\ln t)^2}$$

$$u_2(t) = \int \frac{1}{t}\, dt \qquad\qquad u_2 = \int \frac{W_2}{W}$$

$$\boxed{= \ln t}$$

$$y_p = -\frac{1}{2}(\ln t)^2 t + (\ln t)t\ln t \qquad\qquad y_p = u_1 y_1 + u_2 y_2$$

$$\boxed{= \frac{1}{2}t(\ln t)^2}$$

$$\boxed{y(x) = C_1 t + C_2 t\ln t + \frac{1}{2}t(\ln t)^2}$$

$$y(1) = 1 \quad \rightarrow \quad \boxed{C_1 = 1}$$

$$y'(t) = C_1 + C_2(1 + \ln t) + \frac{1}{2}\left((\ln t)^2 + 2\ln t\right)$$

$$y'(1) = 4 \quad \rightarrow C_1 + C_2 = 4 \quad \boxed{C_2 = 3}$$

$$\boxed{y(x) = t + 3t\ln t + \frac{1}{2}t(\ln t)^2}$$