

Solution **Section 1.3 – Models of Motions**

Exercise

A body of mass m falls from rest subject to gravity in a medium offering resistance proportional to the square of the velocity. Determine the velocity and position of the body at t seconds.

Solution

The resistance force R , according to the given, is proportional to the square of the velocity. Since this force is in an upward direction. $R = -r.v^2$

$$\text{Let } r = \frac{g}{a^2} \rightarrow R = -\frac{gv^2}{a^2}$$

$$F = mg + F_r$$

$$m \frac{dv}{dt} = mg - mR$$

$$m \frac{dv}{dt} = mg - m \frac{gv^2}{a^2}$$

$$\frac{dv}{dt} = g \left(1 - \frac{v^2}{a^2} \right)$$

$$\int \frac{dv}{1 - \left(\frac{v}{a}\right)^2} = g \int dt$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| \quad \text{or} \quad \frac{1}{a} \tanh^{-1} \frac{x}{a}$$

$$a \tanh^{-1} \frac{v}{a} = gt + C_1$$

$$\tanh^{-1} \frac{v}{a} = \frac{g}{a} t + C_2$$

$$\frac{v}{a} = \tanh \left(\frac{g}{a} t + C_2 \right)$$

$$v(t) = a \tanh \left(\frac{g}{a} t + C_2 \right)$$

$$v(0) = 0 \rightarrow 0 = a \tanh(C_2) \Rightarrow \underline{C_2 = 0}$$

$$\underline{v(t) = a \tanh \left(\frac{g}{a} t \right)}$$

$$v(t) = \frac{dy}{dt} = a \tanh \left(\frac{g}{a} t \right)$$

$$\int dy = a \int \tanh \left(\frac{g}{a} t \right) dt$$

$$\int \tanh ax dx = \frac{1}{a} \ln(\cosh ax) + C$$

$$y(t) = \frac{a^2}{g} \ln \left(\cosh \frac{g}{a} t \right) + C_3$$

$$y(0) = 0 \rightarrow 0 = \frac{a^2}{g} \ln(\cosh 0) + C_3$$

$$0 = \frac{a^2}{g} \ln(1) + C_3 \rightarrow \underline{C_3 = 0}$$

$$\underline{y(t) = \frac{a^2}{g} \ln\left(\cosh \frac{g}{a} t\right)}$$

Exercise

A body of mass m , with initial velocity v_0 , falls vertically. If the initial position is denoted s_0 . Determine the velocity and position of the body at t seconds.

Assume the body acted upon by gravity alone and the air resistance proportional to the square of the velocity.

Solution

The resistance force R , according the given, is proportional to the square of the velocity. $R = -r.v^2$

$$F = mg + F_r$$

$$m \frac{dv}{dt} = mg - mr^2 v^2$$

$$\frac{dv}{dt} = g - r^2 v^2$$

$$\int \frac{dv}{g - (rv)^2} = \int dt$$

$$\frac{1}{g} \int \frac{dv}{1 - \left(\frac{r}{\sqrt{g}} v\right)^2} = \int dt$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| \quad \text{or} \quad \frac{1}{a} \tanh^{-1} \frac{x}{a}$$

$$\frac{1}{g} \frac{\sqrt{g}}{r} \tanh^{-1} \left(\frac{r}{\sqrt{g}} v \right) = t + C_1$$

$$\tanh^{-1} \left(\frac{r}{\sqrt{g}} v \right) = r\sqrt{g}t + C_2$$

$$\frac{r}{\sqrt{g}} v = \tanh(r\sqrt{g}t + C_2)$$

$$v(t) = \frac{\sqrt{g}}{r} \tanh(r\sqrt{g}t + C_2)$$

$$v(0) = v_0 \rightarrow v_0 = \frac{\sqrt{g}}{r} \tanh(C_2) \Rightarrow \underline{C_2 = \tanh^{-1} \left(\frac{rv_0}{\sqrt{g}} \right)}$$

$$\underline{v(t) = \frac{\sqrt{g}}{r} \tanh \left(r\sqrt{g}t + \tanh^{-1} \left(\frac{rv_0}{\sqrt{g}} \right) \right)}$$

$$v(t) = \frac{ds}{dt} = \frac{\sqrt{g}}{r} \tanh(r\sqrt{g}t + C_2)$$

$$\int ds = \frac{\sqrt{g}}{r} \int \tanh(r\sqrt{g}t + C_2) dt \qquad \int \tanh ax dx = \frac{1}{a} \ln(\cosh ax) + C$$

$$s(t) = \frac{\sqrt{g}}{r} \frac{1}{r\sqrt{g}} \ln(\cosh(r\sqrt{g}t + C_2)) + C_3$$

$$= \frac{1}{r^2} \ln(\cosh(r\sqrt{g}t + C_2)) + C_3$$

$$s(0) = s_0 \rightarrow s_0 = \frac{1}{r^2} \ln(\cosh(C_2)) + C_3$$

$$\rightarrow C_3 = s_0 - \frac{1}{r^2} \ln(\cosh(C_2))$$

$$s(t) = \frac{1}{r^2} \ln\left(\cosh\left(r\sqrt{g}t + \tanh^{-1}\left(\frac{rv_0}{\sqrt{g}}\right)\right)\right) + s_0 - \frac{1}{r^2} \ln\left(\cosh\left(\tanh^{-1}\left(\frac{rv_0}{\sqrt{g}}\right)\right)\right)$$

Exercise

A body falls from a height of 300 ft. What distance has it traveled after 4 sec. if subject to g , the earth's acceleration?

Solution

$$a(t) = -g$$

$$v(t) = -\int g dt$$

$$= -gt + C_1$$

$$v(0) = 0 \rightarrow C_1 = 0$$

$$v(t) = -32.2t$$

$$v(t) = \frac{dh}{dt} = -32.2t$$

$$\int dh = -32.2 \int t dt$$

$$h(t) = -16.1t^2 + C_2$$

$$h(0) = 300 \rightarrow C_2 = 300$$

$$h(t) = -16.1t^2 + 300$$

$$h(t = 4) = -16.1(16) + 300$$

$$= 42.4 \text{ ft}$$

Exercise

A body falls from an initial velocity of $1,000 \text{ ft/s}$. What distance has it traveled after 3 sec. if subject to $g = 32 \text{ ft/s}^2$, the earth's acceleration?

Solution

$$a(t) = g$$

$$v(t) = \int g dt$$
$$= gt + C_1$$

$$v(0) = 1000 \rightarrow \underline{C_1 = 1,000}$$

$$v(t) = 32t + 1,000$$

$$v(t) = \frac{dh}{dt} = 32t + 1,000$$

$$\int dh = \int (32t + 1,000) dt$$

$$h(t) = 16t^2 + 1,000t + C_2$$

$$h(0) = 0 \rightarrow \underline{C_2 = 0}$$

$$\underline{h(t) = 16t^2 + 1,000t}$$

$$h(t = 3) = 16(9) + 3,000$$
$$\underline{= 3,144 \text{ ft}}$$

Exercise

A projectile is fired straight upwards with an initial velocity of $1,600 \text{ ft/s}$. What is its velocity at $40,000 \text{ ft}$.
($g = 32 \text{ ft/s}^2$)

Solution

$$a(t) = -g$$

$$v(t) = -\int 32 dt$$
$$= -32t + C_1$$

$$v(0) = 1600 \rightarrow \underline{C_1 = 1,600}$$

$$\underline{v(t) = -32t + 1,600}$$

$$v(t) = \frac{dh}{dt} = -32t + 1,600$$

$$\int dh = \int (-32t + 1,600) dt$$

$$h(t) = -16t^2 + 1,600t + C_2$$

$$h(0) = 0 \rightarrow \underline{C_2 = 0}$$

$$h(t) = -16t^2 + 1,600t$$

$$-16t^2 + 1,600t = 40,000$$

$$t^2 - 100t + 2,500 = 0$$

$$(t - 50)^2 = 0 \rightarrow \underline{t = 50}$$

$$v(50) = -32(50) + 1,600$$

$$\underline{= 0 \text{ ft/sec}}$$

Exercise

A projectile is fired straight upwards with an initial velocity of 1,000 ft/s . What is its velocity at 8,000 ft .

$$(g = 32 \text{ ft/s}^2)$$

Solution

$$v(t) = -\int 32 dt$$

$$= -32t + C_1$$

$$v(0) = 1000 \rightarrow \underline{C_1 = 1,000}$$

$$\underline{v(t) = -32t + 1,000}$$

$$v(t) = \frac{dh}{dt} = -32t + 1,000$$

$$\int dh = \int (-32t + 1,000) dt$$

$$h(t) = -16t^2 + 1,000t + C_2$$

$$h(0) = 0 \rightarrow \underline{C_2 = 0}$$

$$h(t) = -16t^2 + 1,000t$$

$$-16t^2 + 1,000t = 8,000$$

$$2t^2 - 125t + 1,000 = 0$$

$$t = \frac{125 \pm \sqrt{15625 - 8000}}{4} = \frac{125 \pm \sqrt{7,625}}{4} \approx \begin{cases} 9.42 \\ 53.08 \end{cases}$$

$$v(9.42) = -32(9.42) + 1,000 \approx \underline{698.56 \text{ ft/sec}}$$

$$v(53.08) = -32(53.08) + 1,000 \approx \underline{-698.56 \text{ ft/sec}}$$

Exercise

An 8 lb. weight falls from rest toward earth. Assuming that the weight is acted upon by air resistance, numerically equal to $2v$, but measured in pounds, find the velocity and distance fallen after t seconds. (The variable v represents the velocity measured in ft/sec.)

Solution

The resistance force $F = 2v$

$$F = mg + F_r$$

$$m \frac{dv}{dt} = 8 - 2v$$

$$\frac{8}{32} \frac{dv}{dt} = 8 - 2v$$

$$\frac{dv}{dt} = 8(4 - v)$$

$$\int \frac{dv}{4-v} = \int 8dt$$

$$-\ln|4-v| = 8t + C_1$$

$$\ln|4-v| = C_2 - 8t$$

$$4-v = e^{C_2 - 8t}$$

$$v = 4 - e^{C_2} e^{-8t}$$

$$v(t) = 4 - C_3 e^{-8t}$$

$$v(0) = 0 \rightarrow 0 = 4 - C_3 \Rightarrow \underline{C_3 = 4}$$

$$v(t) = \underline{4(1 - e^{-8t})}$$

$$v(t) = \frac{dy}{dt} = 4(1 - e^{-8t})$$

$$\int dy = \int 4(1 - e^{-8t}) dt$$

$$y(t) = 4\left(t + \frac{1}{8}e^{-8t}\right) + C_4$$

$$y(0) = 0 \rightarrow 0 = 4\left(\frac{1}{8}\right) + C_4 \Rightarrow \underline{C_4 = -\frac{1}{2}}$$

$$y(t) = \underline{4\left(t + \frac{1}{8}e^{-8t}\right) - \frac{1}{2}}$$

Exercise

A stone is released from rest and dropped into a deep well. Eight seconds later, the sound of the stone splashing into the water at the bottom of the well returns to the ear of the person who released the stone. How long does it take the stone to drop to the bottom of the well? How deep is the well? Ignore air resistance. The speed of sound is 340 m/s.

Solution

$$\begin{aligned}d &= \frac{1}{2}gt^2 \\&= \frac{1}{2}9.8t^2 \\&= 4.9t^2 \\d &= 340s \\&= 340(8 - t) \\4.9t^2 &= 2720 - 340t \\4.9t^2 + 340t - 2720 &= 0 \\t &= \underline{7.2438 \text{ sec}} \\d &= 340(8 - 7.2438) \\&= \underline{257.1 \text{ m}}\end{aligned}$$

Exercise

A rocket is fired vertically and ascends with constant acceleration $a = 100 \text{ m/s}^2$ for 1.0 min. At that point, the rocket motor shuts off and the rocket continues upward under the influence of gravity. Find the maximum altitude acquired by the rocket and the total time elapsed from the take-off until the rocket returns to the earth. Ignore air resistance.

Solution

$$\begin{aligned}d &= \frac{1}{2}(a - g)t^2 \\&= \frac{1}{2}(100 - 9.8)t^2 \\d(1\text{hr} = 60\text{min}) &= \frac{1}{2}(100 - 9.8)(60)^2 \\&= 162,360 \text{ m} \\v &= d' \\&= \left(\frac{1}{2}(100 - 9.8)t^2\right)' \\&= (100 - 9.8)t \\v(60) &= (100 - 9.8)(60) \\&= 5412 \text{ m/s}\end{aligned}$$

The velocity will be reduced: $5412 - 9.8t = 0$

$$t = 552.2 \text{ s}$$

The altitude: $d(t) = -\frac{9.8}{2}t^2 + 5412t + 162,360$

$$\begin{aligned} d(552.2) &= -\frac{9.8}{2}(552.2)^2 + 5412(552.2) + 162,360 \\ &= 1.657 \times 10^6 \text{ m} \end{aligned}$$

Back to the ground: $4.9t^2 = 1.657 \times 10^6$

$$t_b = 581.5 \text{ s}$$

Total time: $t = 552.2 + 581.5 = 1193.7 \text{ s}$

Exercise

A ball having mass $m = 0.1 \text{ kg}$ falls from rest under the influence of gravity in a medium that provides a resistance that is proportional to its velocity. For a velocity of 0.2 m/s the force due to the resistance of the medium is -1 N . Find the terminal velocity of the ball.

1 N is the force required to accelerate a 1 kg mass at a rate of 1 m/s^2 : $1\text{N} = 1 \text{ kg} \cdot \text{m/s}^2$

Solution

The resistance force: $R = -rv$

$$-1 = -0.2r \Rightarrow r = 5$$

The terminal velocity: $v_{term} = -\frac{mg}{r}$

$$\begin{aligned} &= \frac{0.1(9.8)}{5} \\ &= -0.196 \text{ m/s} \end{aligned}$$

Exercise

A ball is projected vertically upward with initial velocity v_0 from ground level. Ignore air resistance.

- What is the maximum height acquired by the ball?
- How long does it take the ball to reach its maximum height? How long does it take the ball to return to the ground? Are these times identical?
- What is the speed of the ball when it impacts with the ground on its return?

Solution

The position: $x(t) = -\frac{1}{2}gt^2 + v_0t$

a) The maximum height when the velocity is zero

$$v = x' = -gt + v_0 = 0$$

$$t = \frac{v_0}{g}$$

$$\begin{aligned} \text{Maximum height} &= -\frac{1}{2}g\left(\frac{v_0}{g}\right)^2 + v_0 \frac{v_0}{g} \\ &= -\frac{1}{2}\frac{v_0^2}{g} + \frac{v_0^2}{g} \\ &= \frac{v_0^2}{2g} \end{aligned}$$

b) The ball will take to reach the maximum height $t = \frac{v_0}{g}$ and the same to return to the ground, both

$$\text{are equal to } t = \frac{v_0}{g}$$

c) When the ball hits the ground the time is equal to zero.

$$v = -g(0) + v_0 = v_0$$

Exercise

An object having mass 70 kg falls from rest under the influence of gravity. The terminal velocity of the object is -20 m/s . Assume that the air resistance is proportional to the velocity.

a) Find the velocity and distance traveled at the end of 2 seconds .

b) How long does it take the object to reach 80% of its terminal velocity?

Solution

a) The terminal velocity: $v = -\frac{mg}{r}$

$$-20 = -\frac{70(9.8)}{r}$$

$$|r = \frac{70(9.8)}{20} = 34.3|$$

$$v(t) = Ce^{-rt/m} - \frac{mg}{r}$$

$$v(t=0) = Ce^{-r(0)/m} - \frac{mg}{r}$$

$$0 = C - \frac{mg}{r} \Rightarrow \boxed{C = \frac{mg}{r}}$$

$$v(t) = \frac{mg}{r}(e^{-rt/m} - 1)$$

$$|v(t=2) = \frac{70(9.8)}{34.3}\left(e^{-34.3(2)/70} - 1\right) \approx -12.4938|$$

$$\begin{aligned}
x &= \int_0^t v(t) dt \\
&= \frac{mg}{r} \int_0^t (e^{-rs/m} - 1) ds \\
&= \frac{mg}{r} \left[-\frac{m}{r} e^{-rs/m} - s \right]_0^t \\
&= \frac{mg}{r} \left[-\frac{m}{r} e^{-rt/m} - t - \left(-\frac{m}{r} - 0 \right) \right] \\
&= \frac{mg}{r} \left[\frac{m}{r} (1 - e^{-rt/m}) - t \right] \\
\underline{x(2)} &= \frac{70(9.8)}{34.3} \left[\frac{70}{34.3} (1 - e^{-34.3(2)/70}) - 2 \right] \approx -14.5025
\end{aligned}$$

b) The velocity is 80% of its terminal velocity when $.8 = 1 - e^{-rt/m}$

$$e^{-rt/m} = .2$$

$$-\frac{rt}{m} = \ln(.2) \Rightarrow \underline{t = \frac{m}{r} \ln(.2) \approx 3.285 \text{ sec}}$$

Exercise

A lunar lander is falling freely toward the surface of the moon at a speed of 450 m/s . Its retrorockets, when fired, provide a constant deceleration of 2.5 m/s^2 (the gravitational acceleration produced by the moon is assumed to be included in the given acceleration). At What height above the lunar surface should the retrorockets be activated to ensure a “soft touchdown? ($v = 0$ at impact)?

Solution

Given: $t = 0 \rightarrow v_0 = -450 \text{ m/s} \quad a = +2.5$

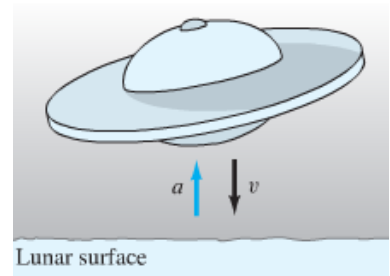
Because an upward thrust increases the velocity v (although decreases the speed $|v|$), then

$$v(t) = 2.5t - 450$$

$$\begin{aligned}
x(t) &= \int v(t) dt \\
&= \int (2.5t - 450) dt \\
&= 1.25t^2 - 450t + x_0
\end{aligned}$$

x_0 : is the height of the lander above the lunar surface at the time $t = 0$ when the retrorockets should be activated.

$$v(t) = 2.5t - 450 = 0$$



$$t = \frac{450}{2.5} = 180 \text{ s}$$

When $x = 0 \rightarrow t = 180$

$$0 = 1.25(180)^2 - 450(180) + x_0$$

$$x_0 = 450(180) - 1.25(180)^2 = 40,500$$

Thus the retrorockets should be activated when the lunar lander is 40.500 m (40,5 km) above the surface of the moon, and it will touch down softly on the lunar surface after 3 minutes of decelerating descent.

Exercise

A body falling in a relatively dense fluid, oil for example, is acted on by three forces: a resistance force R , a buoyant force B , and its weight w due to gravity. The buoyant force is equal to the weight of the fluid displaced by the object. For a slowly moving spherical body of radius a , the resistive force is given by

Stokes's law $R = 6\pi \frac{\mu a}{v}$, where v is the velocity of the body, and μ is the coefficient of viscosity of the surrounding fluid?

- Find the limiting velocity of a solid sphere of radius a and density ρ falling freely in a medium of density ρ' and coefficient of viscosity μ .
- In 1910 R. A. Millikan studied the motion of tiny droplets of oil falling in an electric field. A field of strength E exerts a force Ee on a droplet with charge e . Assume that E has been adjusted so the droplet is held stationary ($v = 0$) and that w and B are as given. Find an expression for e .

Solution

- The equation of motion is $m \frac{dv}{dt} = w - R - B$

$$\frac{4}{3}\pi a^3 \rho \left(\frac{dv}{dt} \right) = \frac{4}{3}\pi a^3 \rho g - 6\pi \mu a v - \frac{4}{3}\pi a^3 \rho' g$$

The limiting velocity occurs when $\frac{dv}{dt} = 0$

- Since the droplet is motionless, $v = \frac{dv}{dt} = 0$, we have the equation of motion

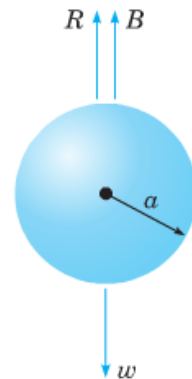
$$0 = \frac{4}{3}\pi a^3 \rho g - Ee - \frac{4}{3}\pi a^3 \rho' g$$

Where ρ is the density of the oil and ρ' is the density of air.

$$Ee = \frac{4}{3}\pi a^3 \rho g - \frac{4}{3}\pi a^3 \rho' g$$

$$Ee = \frac{4}{3}\pi a^3 (\rho - \rho')$$

$$e = \frac{4}{3} \frac{\pi a^3}{E} (\rho - \rho')$$



Exercise

A hemispherical bowl has top radius of 4 ft. and at time $t = 0$ is full of water. At that moment a circular hole with diameter 1 in. is opened in the bottom of the tank. How long will it take for all the water to drain from the tank?

Solution

$$r^2 + (4 - y)^2 = 16 \rightarrow r^2 = 16 - (4 - y)^2$$

(Right Triangle)

$$A(y) = \pi r^2 = \pi [16 - (4 - y)^2] = \pi (8y - y^2)$$

$$a = \pi r^2 = \pi \left(\frac{1}{2} \text{ in.} \cdot \frac{1 \text{ ft.}}{12 \text{ in.}} \right)^2 = \pi \left(\frac{1}{24} \right)^2$$

$$\frac{dV}{dt} = -a\sqrt{2gy}$$

$$\pi (8y - y^2) \frac{dy}{dt} = -\pi \left(\frac{1}{24} \right)^2 \sqrt{2 \cdot 32y}$$

$$(8y - y^2) \frac{dy}{dt} = -\frac{1}{576} (8) \sqrt{y}$$

$$\frac{8y - y^2}{y^{1/2}} dy = -\frac{1}{72} dt$$

$$(8y^{1/2} - y^{3/2}) dy = -\frac{1}{72} dt$$

$$\int (8y^{1/2} - y^{3/2}) dy = -\frac{1}{72} \int dt$$

$$\frac{16}{3} y^{3/2} - \frac{2}{5} y^{5/2} = -\frac{1}{72} t + C$$

$$y(0) = 4$$

$$\frac{16}{3} (4)^{3/2} - \frac{2}{5} (4)^{5/2} = -\frac{1}{72} (0) + C$$

$$\boxed{C = \frac{448}{15}}$$

$$\frac{16}{3} y^{3/2} - \frac{2}{5} y^{5/2} = -\frac{1}{72} t + \frac{448}{15}$$

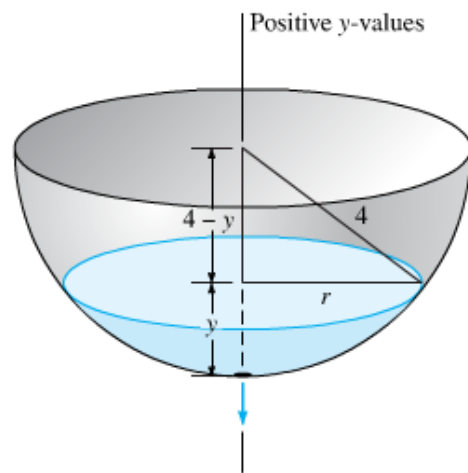
The tank is empty when $y = 0$, thus when

$$0 = -\frac{1}{72} t + \frac{448}{15}$$

$$\frac{1}{72} t = \frac{448}{15}$$

$$t = \frac{448}{15} (72) \approx 2150 \text{ sec}$$

That is about 35 min. 50 s. So it takes slightly less than 36 minutes for the tank to drain.



Exercise

Suppose that the tank has a radius of 3 *feet*. and that its bottom hole is circular with radius 1 *in*. How long will it take the water (initially 9 *ft*. deep) to drain completely?

Solution

$$A = \pi r^2 = \pi(3)^2 = 9\pi$$

$$a = \pi r^2 = \pi\left(1\text{in.} \frac{1\text{ft.}}{12\text{in.}}\right)^2 = \pi\left(\frac{1}{12}\right)^2$$

$$A(y) \frac{dy}{dt} = -a\sqrt{2gy}$$

$$9\pi \frac{dy}{dt} = -\frac{\pi}{144} \sqrt{2(32)y}$$

$$9 \frac{dy}{dt} = -\frac{1}{144} (8\sqrt{y})$$

$$9 \frac{dy}{dt} = -\frac{1}{18} \sqrt{y}$$

$$162 \frac{dy}{\sqrt{y}} = -dt$$

$$162 \int y^{-1/2} dy = - \int dt$$

$$324 y^{1/2} = -t + C$$

$$y(t=0) = 9$$

$$324\sqrt{9} = -0 + C \Rightarrow \underline{C = 972}$$

$$324\sqrt{y} = -t + 972$$

$$\text{Hence } y = 0 \text{ when } t = 972 \text{ sec} = \underline{16 \text{ min } 12 \text{ sec}}$$

Exercise

At time $t = 0$ the bottom plug (at the vertex) of a full conical water tank 16 *ft*. high is removed. After 1 *hr* the water in the tank is 9 *ft*. deep. When will the tank be empty?

Solution

The radius of the cross-section of the cone at height y is proportional to y , so $A(y)$ is proportional to y^2 . Therefore,

$$y^2 y' = -k\sqrt{y}$$

$$A(y) y' = -k\sqrt{y}$$

$$y^{3/2} dy = -k dt$$

$$\int y^{3/2} dy = - \int k dt$$

$$\frac{2}{5}y^{5/2} = -kt + C_1$$

$$2y^{5/2} = -5kt + C$$

With initial condition: $y(0) = 16$

$$2(\textcolor{red}{16})^{5/2} = -5k(\textcolor{red}{0}) + C \Rightarrow \underline{C = 2048}$$

$$2y^{5/2} = -5kt + 2048$$

$$y(1) = 9$$

$$2(\textcolor{red}{9})^{5/2} = -5k(\textcolor{red}{1}) + 2048$$

$$486 - 2048 = -5k \Rightarrow \underline{k = \frac{1562}{5}}$$

$$2y^{5/2} = -1562t + 2048$$

$$\text{Hence } y = 0 \text{ when } t = \frac{2048}{1562} \approx \underline{\textcolor{blue}{1.31} \text{ hr}}$$

Exercise

Suppose that a cylindrical tank initially containing V_0 gallons of water drains (through a bottom hole) in T minutes. Use Torricelli's law to show that the volume of water in the tank after $t \leq T$ minutes is

$$V = V_0 \left(1 - \frac{t}{T}\right)^2$$

Solution

$$\frac{dy}{dt} = -k\sqrt{y}$$

$$A(y)y' = -k\sqrt{y}$$

$$\int y^{-1/2} dy = - \int k dt$$

$$2y^{1/2} = -kt + C$$

$$\text{With initial condition: } y(0) = h \quad 2\sqrt{\textcolor{red}{h}} = -k(\textcolor{red}{0}) + C \rightarrow \underline{C = 2\sqrt{h}}$$

$$2\sqrt{y} = -kt + 2\sqrt{h}$$

$$y(t = T) = 0$$

$$0 = -kT + 2\sqrt{h} \Rightarrow \underline{k = \frac{2\sqrt{h}}{T}}$$

$$2\sqrt{y} = -\frac{2\sqrt{h}}{T}t + 2\sqrt{h}$$

$$\sqrt{y} = \sqrt{h} \left(1 - \frac{t}{T}\right)$$

$$y = h \left(1 - \frac{t}{T}\right)^2$$

If r denotes the radius of the cylinder, then

$$\begin{aligned} V(y) &= \pi r^2 y \\ &= \pi r^2 h \left(1 - \frac{t}{T}\right)^2 \\ &= \underbrace{V_0 \left(1 - \frac{t}{T}\right)^2} \quad V_0 = \pi r^2 h \end{aligned}$$

Exercise

The clepsydra, or water clock – A 12-hr water clock is to be designed with the dimensions, shaped like the surface obtained by revolving the curve $y = f(x)$ around the y -axis. What should be this curve, and what should be the radius of the circular bottom hole, in order that the water level will fall at the constant rate of 4 inches per hour?

Solution

The rate of fall of the water level is

$$\begin{aligned} \frac{dy}{dt} &= -4 \text{ in. / hr} = -\frac{1}{10800} \text{ ft / sec} \\ &= -4 \frac{\text{in.}}{\text{hr}} \cdot \frac{1 \text{ ft}}{12 \text{ in.}} \cdot \frac{1 \text{ hr}}{3600 \text{ sec}} \\ &= -\frac{1}{10800} \text{ ft / sec} \end{aligned}$$

$$A = \pi x^2 \quad \text{and} \quad a = \pi r^2$$

$$A \frac{dy}{dt} = -a \sqrt{2gy}$$

$$\pi x^2 \frac{-1}{10800} = -\pi r^2 \sqrt{2(32)y}$$

$$\frac{x^2}{10800} = 8r^2 \sqrt{y}$$

The curve is of the form $y = kx^4$

$$4 = k(1^4) \rightarrow k = 4$$

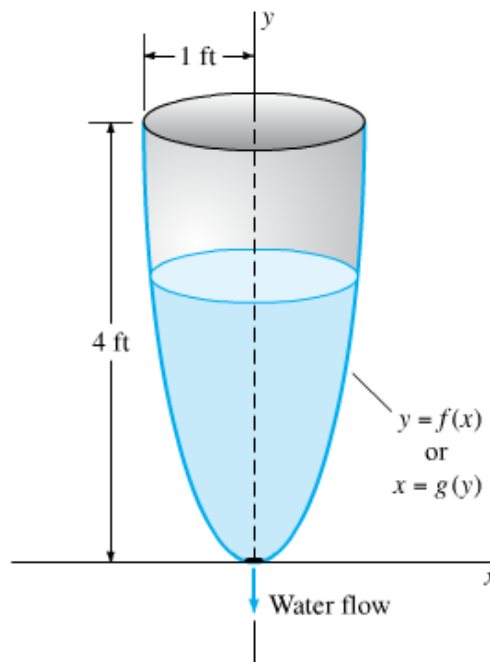
$$y = 4x^4 \rightarrow \sqrt{y} = 2x^2$$

$$\frac{x^2}{10800} = 8r^2 (2x^2)$$

$$\frac{1}{10800} = 16r^2$$

$$r = \frac{1}{\sqrt{172800}}$$

$$\approx 0.0024 \text{ ft} \quad \text{or} \quad 0.028 \text{ in.}$$



Exercise

One of the famous problems in the history of mathematics is the brachistochrone problem: to find the curve along which a particle will slide without friction in the minimum time from one given point P to another point Q , the second point being lower than the first but not directly beneath it.

This problem was posed by Johann Bernoulli in 1696 as a challenge problem to the mathematicians of his day. Correct solutions were found by Johann Bernoulli and his brother Jakob Bernoulli and by Isaac Newton, Gottfried Leibniz, and the Marquis de L'Hospital. The brachistochrone problem is important in the development of mathematics as one of the forerunners of the calculus of variations.

In solving this problem, it is convenient to take the origin as the upper point P and to orient the axes as shown. The lower point Q has coordinates (x_0, y_0) . It is then possible to show that the curve of minimum time is given by a function $y = \phi(x)$ that satisfies the differential equation

$$(1 + y'^2)y = k^2 \quad (\text{eq. i})$$

Where k^2 is a certain positive constant to be determined later

- Solve the equation (eq. i) for y' . Why is it necessary to choose the positive square root?
- Introduce the new variable t by the relation

$$y = k^2 \sin^2 t \quad (\text{eq. ii})$$

Show that the equation found in part (a) then takes the form

$$k^2 \sin^2 t \, dt = dx \quad (\text{eq. iii})$$

- Letting $\theta = 2t$, show that the solution of (eq. iii) for which $x = 0$ when $y = 0$ is given by

$$x = k^2 \frac{\theta - \sin \theta}{2}, \quad y = k^2 \frac{1 - \cos \theta}{2} \quad (\text{eq. iv})$$

Equations (iv) are parametric equations of the solution of (eq. i) that passes through $(0, 0)$. The graph of Eqs. (iv) is called a cycloid.

- If we make a proper choice of the constant k , then the cycloid also passes through the point (x_0, y_0) and is the solution of the brachistochrone problem. Find k if $x_0 = 1$ and $y_0 = 2$

Solution

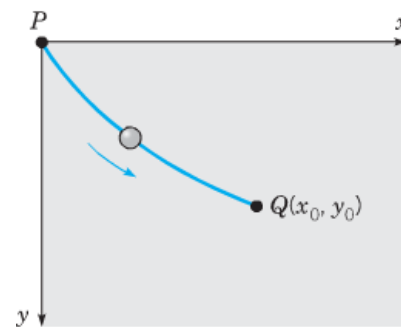
$$a) \quad (1 + y'^2)y = k^2$$

$$1 + y'^2 = \frac{k^2}{y}$$

$$y'^2 = \frac{k^2}{y} - 1$$

$$y' = \pm \sqrt{\frac{k^2 - y}{y}}$$

$$y' = \sqrt{\frac{k^2 - y}{y}}$$



The positive answer is chosen, since y is an increasing function of x .

b) Let $y = k^2 \sin^2 t$

$$dy = 2k^2 \sin t \cos t \, dt$$

$$\left(\sin^2 t \right)' = 2 \sin t (\sin t)' \, dt$$

$$\begin{aligned} y' &= \frac{dy}{dx} = \sqrt{\frac{k^2 - y}{y}} \\ &= \sqrt{\frac{k^2 - k^2 \sin^2 t}{k^2 \sin^2 t}} \\ &= \frac{k \sqrt{1 - \sin^2 t}}{k \sin t} \\ &= \frac{\sqrt{\cos^2 t}}{\sin t} \\ &= \frac{\cos t}{\sin t} \end{aligned}$$

$$\frac{dy}{dx} = \frac{2k^2 \sin t \cos t \, dt}{dx} = \frac{\cos t}{\sin t}$$

$$\underline{2k^2 \sin^2 t \, dt = dx}$$

c) Setting $\theta = 2t$, $2k^2 \sin^2 \left(\frac{\theta}{2} \right) d \left(\frac{\theta}{2} \right) = dx$

$$k^2 \sin^2 \left(\frac{\theta}{2} \right) d\theta = dx$$

$$\int k^2 \sin^2 \left(\frac{\theta}{2} \right) d\theta = \int dx$$

$$\int \sin^2 ax \, dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$k^2 \left(\frac{\theta}{2} - \frac{\sin \theta}{2} \right) = x + C$$

$t = \theta = 0$ at the origin.

$$\underline{C = 0}$$

$$\underline{x(\theta) = k^2 \cdot \frac{\theta - \sin \theta}{2}}$$

$$y = k^2 \sin^2 \left(\frac{\theta}{2} \right)$$

$$\underline{= k^2 \cdot \frac{1 - \cos \theta}{2}}$$

$$\sin^2 \left(\frac{x}{2} \right) = \frac{1 - \cos x}{2}$$

d)
$$\begin{aligned} \frac{y}{x} &= \frac{k^2 \cdot \frac{1 - \cos \theta}{2}}{k^2 \cdot \frac{\theta - \sin \theta}{2}} \\ &= \frac{1 - \cos \theta}{\theta - \sin \theta} \end{aligned}$$

$$\text{Letting } x=1, \quad y=2 \rightarrow \frac{1-\cos\theta}{\theta-\sin\theta}=2$$

$$1-\cos\theta=2\theta-2\sin\theta$$

The solution is: $\theta \approx 1.401$

$$y=k^2 \cdot \frac{1-\cos\theta}{2}$$

$$2y=(1-\cos\theta)k^2$$

$$k^2 = \frac{2y}{1-\cos\theta} = \frac{2(2)}{1-\cos(1.401)} = 4.183$$

$$k \approx 2.194$$

Exercise

Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the substance B at time $t=0$, and if x is the amount of product at time t , then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt} = k(a-x)(b-x) \quad \text{or} \quad \frac{1}{(a-x)(b-x)} \frac{dx}{dt} = k$$

Where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t .

a) If $a=b$

b) If $a \neq b$

Assume in each case that $x=0$ when $t=0$

Solution

$$\frac{1}{(a-x)(b-x)} dx = k dt$$

$$a) \quad a=b \Rightarrow \frac{1}{(a-x)^2} dx = k dt$$

$$\int \frac{1}{(a-x)^2} dx = \int k dt$$

$$\frac{1}{a-x} = kt + C$$

$$x(t=0)=0 \Rightarrow \frac{1}{a} = C$$

$$\frac{1}{a-x} = kt + \frac{1}{a} = \frac{k at + 1}{a}$$

$$a-x = \frac{a}{k at + 1}$$

$$x = a - \frac{a}{k at + 1}$$

$$= \frac{a^2 kt}{kat+1} \Big|$$

$$b) \quad a \neq b \Rightarrow \frac{1}{(a-x)(b-x)} dx = k dt$$

$$\int \frac{1}{(a-x)(b-x)} dx = \int k dt$$

$$\frac{-1}{a-b} \int \frac{1}{a-x} dx + \frac{1}{a-b} \int \frac{1}{b-x} dx = \int k dt$$

$$\frac{1}{a-b} \ln|a-x| - \frac{1}{a-b} \ln|b-x| = kt + C$$

$$\frac{1}{a-b} \ln \left| \frac{a-x}{b-x} \right| = kt + C$$

$$x(0) = 0 \Rightarrow \frac{1}{a-b} \ln \left(\frac{a}{b} \right) = C$$

$$\frac{1}{a-b} \ln \left| \frac{a-x}{b-x} \right| = kt + \frac{1}{a-b} \ln \left(\frac{a}{b} \right)$$

$$\ln \left| \frac{a-x}{b-x} \right| = (a-b)kt + \ln \left(\frac{a}{b} \right)$$

$$\frac{a-x}{b-x} = e^{(a-b)kt + \ln \left(\frac{a}{b} \right)}$$

$$\frac{a-x}{b-x} = \frac{a}{b} e^{(a-b)kt}$$

$$a-x = b \frac{a}{b} e^{(a-b)kt} - x \frac{a}{b} e^{(a-b)kt}$$

$$x \left(\frac{a}{b} e^{(a-b)kt} - 1 \right) = a e^{(a-b)kt} - a$$

$$x = \frac{abe^{(a-b)kt} - ab}{ae^{(a-b)kt} - b}$$

$$\frac{1}{(a-x)(b-x)} = \frac{A}{a-x} + \frac{B}{b-x}$$

$$\begin{cases} -A - B = 0 \\ bA + aB = 1 \end{cases} \rightarrow \begin{cases} B = \frac{1}{a-b} \\ A = -\frac{1}{a-b} \end{cases}$$

Exercise

An open cylindrical tank initially filled with water drains through a hole in the bottom of the tank according to Torricelli's Law. If $h(t)$ is the depth of water in the tank for $t \geq 0$, then Torricelli's Law implies

$h'(t) = -2k\sqrt{h}$, where k is a constant that includes the acceleration due to gravity, the radius of the tank, and the radius of the drain. Assume that the initial depth of the water is $h(0) = H$.

- Find the solution of the initial value problem.
- Find the solution in the case that $k = 0.1$ and $H = 0.5$ m.
- In general, how long does it take the tank to drain in terms of k and H ?

Solution

$$a) \frac{dh}{dt} = -2k\sqrt{h}$$

$$\int \frac{dh}{\sqrt{h}} = -2 \int k dt$$

$$2\sqrt{h} = 2kt + C_1$$

$$h(t) = (kt + C)^2$$

$$h(0) = H \rightarrow H = C^2 \Rightarrow C = \sqrt{H}$$

$$\underline{h(t) = (kt + \sqrt{H})^2}$$

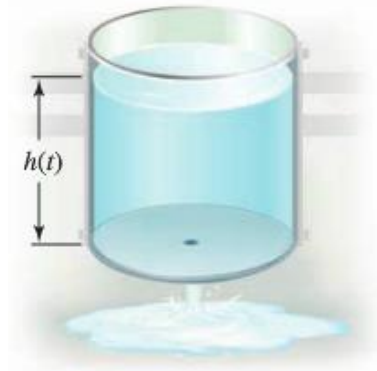
$$b) \text{ Given: } k = 0.1 \quad H = 0.5 \text{ m}$$

$$\underline{h(t) = (0.1t + \sqrt{0.5})^2 = (0.1t + 0.707)^2}$$

$$c) \text{ The tank is drained when } h(t) = 0$$

$$(kt + \sqrt{H})^2 = 0$$

$$kt + \sqrt{H} = 0 \rightarrow \underline{t = -\frac{\sqrt{H}}{k}}$$



Exercise

An object in free fall may be modeled by assuming that the only forces at work are the gravitational force and resistance (friction due to the medium in which the objects falls). By Newton's second law (mass \times acceleration = the sum of the external forces), the velocity of the object satisfies the differential equation

$$\underbrace{m}_{\text{mass}} \cdot \underbrace{v'(t)}_{\text{acceleration}} = \underbrace{mg + f(v)}_{\text{external force}}$$

Where f is a function that models the resistance and the positive direction is downward. One common assumption (often used for motion in air) is that $f(v) = -kv^2$, where $k > 0$ is a drag coefficient.

$$a) \text{ Show that the equation can be written in the form } v'(t) = g - av^2 \text{ where } a = \frac{k}{m}$$

$$b) \text{ For what (positive) value of } v \text{ is } v'(t) = 0? \text{ (This equilibrium solution is called the } \textbf{terminal velocity} \text{.)}$$

$$c) \text{ Find the solution of this separable equation assuming } v(0) = 0 \text{ and } 0 < v(t)^2 < \frac{g}{a} \text{ for } t \geq 0$$

$$d) \text{ Graph the solution found in part (c) with } g = 9.8 \text{ m/s}^2, \quad m = 1 \text{ kg, and } k = 0.1 \text{ kg/m, and verify the terminal velocity agrees with the value found in part (b).}$$

Solution

$$a) \text{ Given: } f(v) = -kv^2$$

$$mv'(t) = mg + f(v)$$

$$mv'(t) = mg - kv^2$$

$$v'(t) = g - \frac{k}{m}v^2$$

$$\underline{v'(t) = g - av^2} \quad \text{where } a = \frac{k}{m}$$

$$b) \quad v'(t) = g - av^2 = 0 \Rightarrow v^2 = \frac{g}{a} \rightarrow \underline{v = \sqrt{\frac{g}{a}}}$$

$$c) \quad \frac{dv}{dt} = g - av^2$$

$$\int \frac{dv}{g - av^2} = \int dt$$

$$-\frac{1}{a} \int \frac{dv}{v^2 - \frac{g}{a}} = \int dt$$

$$-\frac{1}{2a} \sqrt{\frac{a}{g}} \int \frac{dv}{v - \sqrt{\frac{g}{a}}} + \frac{1}{2a} \sqrt{\frac{a}{g}} \int \frac{dv}{v + \sqrt{\frac{g}{a}}} = \int dt$$

$$\frac{1}{2} \sqrt{\frac{1}{ag}} \left(-\ln \left| \sqrt{\frac{g}{a}} - v \right| + \ln \left| \sqrt{\frac{g}{a}} + v \right| \right) = t + C_1$$

$$\ln \frac{\sqrt{\frac{g}{a}} + v}{\sqrt{\frac{g}{a}} - v} = 2\sqrt{agt} + C_2$$

$$\frac{\sqrt{\frac{g}{a}} + v}{\sqrt{\frac{g}{a}} - v} = e^{2\sqrt{agt} + C_2}$$

$$\sqrt{\frac{g}{a}} + v = C e^{2\sqrt{agt}} \left(\sqrt{\frac{g}{a}} - v \right) \quad v(0)=0 \Rightarrow \sqrt{\frac{g}{a}} = \sqrt{\frac{g}{a}} C \rightarrow \underline{C=1}$$

$$v \left(1 + e^{2\sqrt{agt}} \right) = \sqrt{\frac{g}{a}} e^{2\sqrt{agt}} - \sqrt{\frac{g}{a}}$$

$$\underline{v(t) = \frac{e^{2\sqrt{agt}} - 1}{1 + e^{2\sqrt{agt}}} \sqrt{\frac{g}{a}}}$$

$$d) \quad g = 9.8 \text{ m/s}^2, \quad m = 1 \text{ kg}, \quad \text{and } k = 0.1 \text{ kg/m}$$

$$\rightarrow a = \frac{k}{m} = 0.1$$

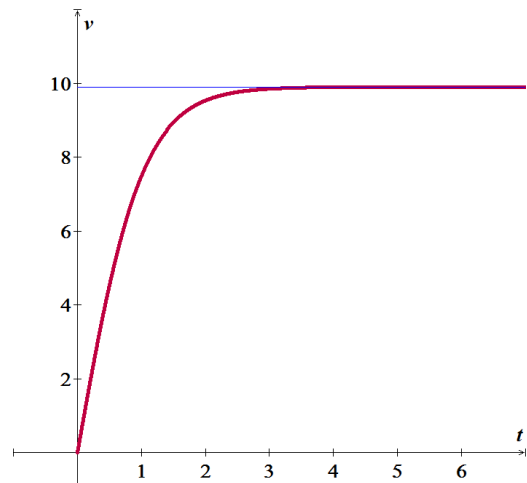
$$v(t) = \sqrt{98} \frac{e^{2\sqrt{.98}t} - 1}{1 + e^{2\sqrt{.98}t}}$$

$$\frac{1}{v^2 - \frac{g}{a}} = \frac{A}{v - \sqrt{\frac{g}{a}}} + \frac{B}{v + \sqrt{\frac{g}{a}}}$$

$$1 = A\sqrt{\frac{g}{a}} + Av + Bv - B\sqrt{\frac{g}{a}}$$

$$\begin{cases} A + B = 0 \rightarrow A = -B \\ A\sqrt{\frac{g}{a}} - B\sqrt{\frac{g}{a}} = 1 \end{cases}$$

$$\underline{A = -B = \frac{1}{2} \sqrt{\frac{a}{g}}}$$



Exercise

Suppose a small cannonball weighing 16 *pounds* is shot vertically upward, with an initial velocity $v_0 = 300 \text{ ft/s}$

The answer to the question “How high does the cannonball go?” depends on whether we take air resistance into account.

- a) Suppose air resistance is ignored. If the positive direction is upward, then a model for the state of the

cannonball is given by $\frac{d^2s}{dt^2} = -g$. Since $\frac{ds}{dt} = v(t)$ the last differential equation is the same as

$\frac{dv}{dt} = -g$, where we take $g = 32 \text{ ft/s}^2$. Find the velocity $v(t)$ of the cannonball at time t .

- b) Use the result in part (a) to determine the height $s(t)$ of the cannonball measured from ground level.

Find the maximum height attained by the cannonball.

Solution

$$a) \int \frac{d^2s}{dt^2} = \int -g dt$$

$$v(t) = \frac{ds}{dt} = -gt + C$$

$$v_0 = 300 \quad \underline{300 = C}$$

$$\underline{v(t) = -32t + 300}$$

$$b) \int ds = \int (-32t + 300) dt$$

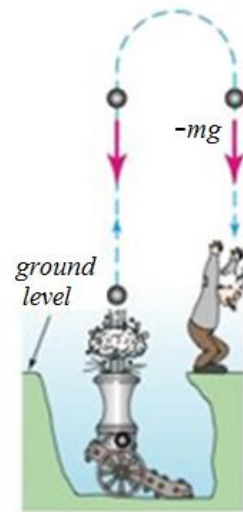
$$s(t) = -16t^2 + 300t + C$$

$$s_0 = 0 \quad \underline{0 = C}$$

$$\underline{s(t) = -16t^2 + 300t}$$

$$v(t) = -32t + 300 = 0 \rightarrow \underline{t = \frac{300}{32} = \frac{75}{8} \approx 9.375}$$

$$\text{The maximum height: } \underline{s(t = 9.375) = 1406.25 \text{ ft}}$$



Exercise

Two chemicals A and B are combined to form a chemical C . The resulting reaction between the two chemicals is such that for each *gram* of A , 4 *grams* of B is used. It is observed that 30 *grams* of the compound C is formed in 10 *minutes*.

- a) Determine the amount of C at time t if the rate of the reaction is proportional to the amounts of A and B remaining and if initially there are 50 *grams* of A and 32 *grams* of B .
- b) How much of the compound C is present at 15 *minutes*.
- c) Interpret the solution as $t \rightarrow \infty$

Solution

a) Let assume that we used a grams of A and b grams of B.

Then there x grams of compound C.

$$a + b = x \quad \text{and} \quad b = 4a$$

$$a + 4a = 5a = x \rightarrow \underline{a = \frac{1}{5}x} \Rightarrow \underline{b = \frac{4}{5}x}$$

The rate at which compound C is formed satisfies:

$$\begin{aligned} \frac{dx}{dt} &= k \left(50 - \frac{1}{5}x \right) \left(32 - \frac{4}{5}x \right) \\ &= \frac{4k}{5} (250 - x)(40 - x) \quad k = \frac{4k}{5} \quad (\text{constant}) \\ &= k(250 - x)(40 - x) \end{aligned}$$

$$\frac{dx}{(250 - x)(40 - x)} = k \, dt$$

$$\frac{1}{(250 - x)(40 - x)} = \frac{A}{250 - x} + \frac{B}{40 - x}$$

$$1 = 40A - Ax + 250B - Bx$$

$$\begin{cases} -A - B = 0 \\ 40A + 250B = 1 \end{cases} \rightarrow A = -\frac{1}{210} \quad B = \frac{1}{210}$$

$$\int \left(-\frac{1}{210} \frac{1}{250 - x} + \frac{1}{210} \frac{1}{40 - x} \right) dx = \int k \, dt$$

$$\frac{1}{210} \ln(250 - x) - \frac{1}{210} \ln(40 - x) = kt + C$$

$$\ln(250 - x) - \ln(40 - x) = 210(kt + C)$$

$$\ln\left(\frac{250 - x}{40 - x}\right) = 210kt + C_1$$

$$\frac{250 - x}{40 - x} = C_2 e^{210kt}$$

$$\underline{x(0) = 0 \rightarrow \frac{25}{4} = C_2}$$

$$\frac{250 - x}{40 - x} = \frac{25}{4} e^{210kt}$$

$$\underline{x(10) = 30 \rightarrow 22 = \frac{25}{4} e^{2100k}}$$

$$e^{2100k} = \frac{88}{25}$$

$$\underline{k = \frac{1}{2100} \ln \frac{88}{25}}$$

$$\frac{250 - x}{40 - x} = \frac{25}{4} e^{\frac{1}{10} t \ln \frac{88}{25}}$$

$$1000 - 4x = (1000 - 25x) e^{0.126t}$$

$$(25e^{0.126t} - 4)x = 1000e^{0.126t} - 1000$$

$$x(t) = 1000 \frac{e^{0.126t} - 1}{25e^{0.126t} - 4}$$

$$b) \quad x(15) = 1000 \frac{e^{0.126(15)} - 1}{25e^{0.126(15)} - 4} \approx 34.78 \text{ g}$$

$$c) \quad x(t) = 1000 \frac{1 - e^{-0.126t}}{25 - 4e^{-0.126t}}$$

$$\begin{aligned} \lim_{t \rightarrow \infty} x(t) &= \lim_{t \rightarrow \infty} 1000 \frac{1 - e^{-0.126t}}{25 - 4e^{-0.126t}} \\ &= \frac{1000}{25} \\ &= 40 \end{aligned}$$

$$a = \frac{1}{5}(40) = 8 \text{ g}$$

$$\text{For chemical A: } 50 - 8 = 42 \text{ g}$$

$$b = \frac{4}{5}(40) = 32 \text{ g}$$

$$\text{For chemical B: } 32 - 32 = 0 \text{ g}$$

Exercise

Two chemicals A and B are combined to form a chemical C. The rate, or velocity, of the reaction is proportional to the product of the instantaneous amounts of A and B not converted to chemical C. Initially, there are 40 grams of A and 50 grams of B, and each gram of B, 2 grams of A is used. It is observed that 10 grams of C is formed in 5 minutes.

- How much is formed in 20 minutes?
- What is the limiting amount of C after a long time?
- How much of chemicals A and B remains after a long time?
- If 100 grams of chemical A is present initially, at what time is chemical C half-formed?

Solution

- Let assume that we used a grams of A and b grams of B.

Then there x grams of compound C.

$$a + b = x \quad \text{and} \quad a = 2b$$

$$2b + b = 3b = x \rightarrow b = \frac{1}{3}x \Rightarrow a = \frac{2}{3}x$$

The rate at which compound C is formed satisfies:

$$\begin{aligned} \frac{dx}{dt} &= k \left(40 - \frac{2}{3}x \right) \left(50 - \frac{1}{3}x \right) \\ &= \frac{k}{3} (120 - 2x)(150 - x) \quad k = \frac{2k}{3} \text{ (constant)} \\ &= k(60 - x)(150 - x) \end{aligned}$$

$$\frac{dx}{(60-x)(150-x)} = k \, dt$$

$$\frac{1}{(60-x)(150-x)} = \frac{A}{60-x} + \frac{B}{150-x}$$

$$1 = 150A - Ax + 60B - Bx$$

$$\begin{cases} -A - B = 0 \\ 150A + 60B = 1 \end{cases} \rightarrow A = \frac{1}{90} \quad B = -\frac{1}{90}$$

$$\int \left(\frac{1}{90} \frac{1}{60-x} - \frac{1}{90} \frac{1}{150-x} \right) dx = \int k \, dt$$

$$-\frac{1}{90} \ln(60-x) + \frac{1}{90} \ln(150-x) = kt + C$$

$$\ln(150-x) - \ln(60-x) = 90(kt + C)$$

$$\ln\left(\frac{150-x}{60-x}\right) = 90kt + C_1$$

$$\frac{150-x}{60-x} = C_2 e^{90kt}$$

$$x(0) = 0 \rightarrow \frac{5}{2} = C_2$$

$$\frac{150-x}{60-x} = \frac{5}{2} e^{90kt}$$

$$x(5) = 10 \rightarrow \frac{14}{5} = \frac{5}{2} e^{450k}$$

$$e^{450k} = \frac{28}{25}$$

$$k = \frac{1}{450} \ln \frac{28}{25} \approx 2.518 \times 10^{-4}$$

$$\frac{150-x}{60-x} = \frac{5}{2} e^{0.02267t}$$

$$250 - 2x = 300e^{0.02267t} - 5xe^{0.02267t}$$

$$(5e^{0.02267t} - 2)x = 300e^{0.02267t} - 250$$

$$x(t) = \frac{300e^{0.02267t} - 250}{5e^{0.02267t} - 2}$$

$$x(20) = \frac{300e^{0.02267(20)} - 250}{5e^{0.02267(20)} - 2} \approx 29.3 \, g$$

$$b) \quad x(t) = \frac{300 - 250e^{-0.02267t}}{5 - 2e^{-0.02267t}}$$

$$\begin{aligned} \lim_{t \rightarrow \infty} x(t) &= \lim_{t \rightarrow \infty} \frac{300 - 250e^{-0.02267t}}{5 - 2e^{-0.02267t}} \\ &= \frac{300}{5} \\ &= 60 \, g \end{aligned}$$

$$c) \quad a = \frac{2}{3}x = \frac{2}{3}60 = \underline{40}$$

$$\text{For chemical A: } 40 - 40 = \underline{0 \text{ g}}$$

$$b = \frac{1}{3}60 = \underline{20}$$

$$\text{For chemical B: } 50 - 20 = \underline{30 \text{ g}}$$

$$2b + b = 3b = x \rightarrow b = \frac{1}{3}x \Rightarrow a = \frac{2}{3}x$$

$$\begin{aligned} \frac{dx}{dt} &= k \left(100 - \frac{2}{3}x \right) \left(50 - \frac{1}{3}x \right) \\ &= \frac{k}{3} (300 - 2x)(150 - x) \quad k = \frac{2k}{3} \text{ (constant)} \\ &= k(150 - x)^2 \end{aligned}$$

$$\int \frac{dx}{(150 - x)^2} = \int k \, dt$$

$$\frac{1}{150 - x} = kt + C_3$$

$$x(0) = 0 \rightarrow \frac{1}{150} = C_3$$

$$\frac{1}{150 - x} = kt + \frac{1}{150}$$

$$x(5) = 10 \rightarrow \frac{1}{140} = 5k + \frac{1}{150}$$

$$5k = \frac{1}{2100} \rightarrow k = \frac{1}{10500}$$

$$\frac{1}{150 - x} = \frac{1}{10500}t + \frac{1}{150} = \frac{t + 70}{10500}$$

$$150 - x = \frac{10500}{t + 70}$$

$$x(t) = 150 - \frac{10500}{t + 70}$$

$$\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} \left(150 - \frac{10500}{t + 70} \right) = \underline{150}$$

When the chemical C is half-formed, that implies to $x(t) = 75$

$$x(t) = 150 - \frac{10500}{t + 70} = 75$$

$$\frac{10500}{t + 70} = 75$$

$$t + 70 = \frac{10500}{75} = 140$$

$$t = 140 - 70 = \underline{70 \text{ min}}$$

Exercise

A skydiver weighs 125 *pounds*, and her parachute and equipment combined weigh another 35 *pounds*. After exiting from a plane at an altitude of 15,000 *feet*, she waits 15 *seconds* and opens her parachute. Assume that the constant of proportionality has the value $k = 0.5$ during free fall and $k = 10$ after the parachute is opened.

Assume that her initial velocity on leaving the plane is *zero*.

- What is her velocity and how far has she traveled 20 *seconds* after leaving the plane?
- How does her velocity at 20 *seconds* compare with her terminal velocity?
- How long does it take her to reach the ground?

Solution

- Assume that the air resistance is proportional to velocity and the positive direction is downward.

$$m \frac{dv}{dt} = mg - kv \quad s(0) = 0$$

$$\int \frac{m}{mg - kv} dv = \int dt$$

$$-\frac{m}{k} \ln(mg - kv) = t + C_1$$

$$\ln(mg - kv) = -\frac{k}{m}t + C_2$$

$$mg - kv = e^{-\frac{k}{m}t + C_2}$$

$$kv = mg - C_3 e^{-\frac{k}{m}t}$$

$$v(t) = \frac{mg}{k} + C e^{-\frac{k}{m}t}$$

$$v(0) = 0 \Rightarrow 0 = \frac{mg}{k} + C \rightarrow C = -\frac{mg}{k}$$

$$v(t) = \frac{mg}{k} \left(1 - e^{-\frac{k}{m}t} \right)$$

Given: $k = 0.5$, $g = 32$, and $m = \frac{w}{g} = \frac{125 + 35}{32} = \frac{160}{32} = 5$

$$v(t) = \frac{160}{0.5} \left(1 - e^{-\frac{0.5}{5}t} \right)$$

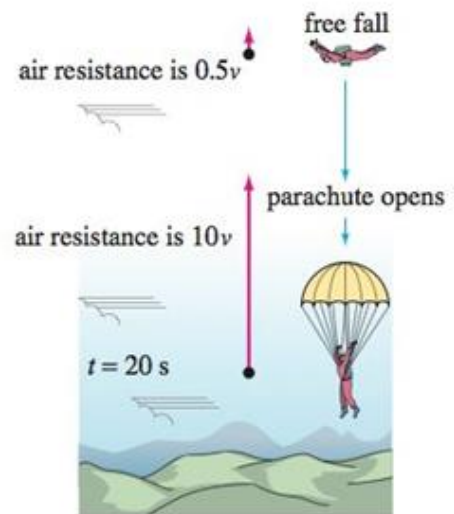
$$= 320 \left(1 - e^{-0.1t} \right)$$

$$s(t) = \int 320 \left(1 - e^{-0.1t} \right) dt$$

$$= 320 \left(t + 10e^{-0.1t} \right) + C$$

$$s(0) = 0 \Rightarrow 0 = 3200 + C \rightarrow C = -3,200$$

$$s(t) = 320t + 3200e^{-0.1t} - 3200$$



When the parachute opens at $t = 15$ sec

$$v(15) = 320(1 - e^{-1.5}) \approx 248.598 \text{ ft/s}^2$$

$$s(15) = 320(15) + 3200e^{-1.5} - 3200 \approx 2314.02 \text{ ft}$$

With the parachute opens at $t = 15$ sec then $t = 0$ (reset) with the given information $k = 10$, $g = 32$, and $m = 5$

$$v(t) = \frac{mg}{k} + Ce^{-\frac{k}{m}t} = 16 + Ce^{-2t}$$

$$v(0) = 248.598 \Rightarrow 248.598 = 16 + C \rightarrow C = 232.598$$

$$v_P(t) = 16 + 232.598e^{-2t}$$

$$s(t) = \int (16 + 232.598e^{-2t}) dt$$

$$= 16t - 116.299e^{-2t} + C$$

$$s(0) = 0 \Rightarrow 0 = -116.299 + C \rightarrow C = 116.299$$

$$s_P(t) = 16t - 116.299e^{-2t} + 116.299$$

After the parachute opens at $t = 5$ sec

$$v_P(5) = 16 + 232.598e^{-10} \approx 16.011 \text{ ft/s}^2$$

$$s_P(5) = 16(5) - 116.299e^{-10} + 116.299 \approx 196.29 \text{ ft}$$

After leaving the plane, at $t = 20$ sec

$$v(20) \approx 16.011 \text{ ft/s}^2$$

$$s(20) = 2314.02 + 196.29 \approx 2510.31 \text{ ft}$$

$$b) \lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} (16 + 232.598e^{-2t}) = 16 \text{ ft/s}^2$$

She has very nearly reached her terminal velocity in 5 seconds after the parachute opens.

c) She exited the plane at an altitude of 15,000 feet.

When she open the parachute the distance to the ground is

$$15,000 - s(15) = 15,000 - 2314.02 \approx 12,686 \text{ ft}$$

$$s_P(t) = 16t - 116.299e^{-2t} + 116.299 = 12,686$$

$$16t - 116.299e^{-2t} - 12,569.701 = 0 \rightarrow t \approx 785.606 \text{ sec}$$

The total time:

$$t = 15 + 785.61 \approx 800.61 \text{ sec}$$

$$\approx 13.34 \text{ min}$$

Exercise

A tank in the form of a right-circular cylinder standing on end is leaking water through a circular hole in its bottom. When friction and contraction of water at the hole are ignored, the height h of water in the tank is described by

$$\frac{dh}{dt} = -\frac{A_h}{A_w} \sqrt{2gh}$$

Where A_w and A_h are the cross-sectional areas of the water and the hole, respectively.

- Find $h(t)$ if the initial height of the water is H .
- Sketch the graph $h(t)$ and give the interval I of definition in terms of the symbols A_w , A_h , and H .
 $(g = 32 \text{ ft/s}^2)$
- Suppose the tank is 10 feet high and has radius 2 feet and the circular hole has radius $\frac{1}{2}$ inch. If the tank is initially full, how long will it take to empty?

Solution

$$a) \quad \frac{dh}{dt} = -\frac{A_h}{A_w} \sqrt{2gh} = -8 \frac{A_h}{A_w} \sqrt{h}$$

$$\int h^{-1/2} dh = -8 \frac{A_h}{A_w} \int dt$$

$$2\sqrt{h} = -8 \frac{A_h}{A_w} t + C$$

$$h(0) = H \rightarrow 2\sqrt{H} = C$$

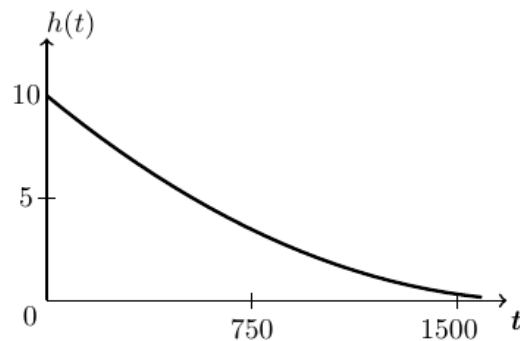
$$2\sqrt{h} = -8 \frac{A_h}{A_w} t + 2\sqrt{H}$$

$$h(t) = \left(\sqrt{H} - 4 \frac{A_h}{A_w} t \right)^2$$

$$b) \quad \sqrt{h} = \sqrt{H} - 4 \frac{A_h}{A_w} t \geq 0$$

$$\sqrt{H} \geq 4 \frac{A_h}{A_w} t$$

$$0 \leq t \leq \frac{A_w \sqrt{H}}{4A_h}$$



$$c) \quad \text{Given: } H = 10 \text{ ft, } R = 2 \text{ ft, } r_h = \frac{1}{2} \frac{1}{12} = \frac{1}{24} \text{ ft}$$

$$A_w = \pi R^2 = 4\pi$$

$$A_h = \pi r_h^2 = \frac{\pi}{576}$$

$$h(t) = \left(\sqrt{10} - 4 \frac{\pi}{576} \left(\frac{1}{4\pi} \right) t \right)^2$$

$$= \left(\sqrt{10} - \frac{1}{576} t \right)^2$$

$$h(t) = \left(\sqrt{10} - \frac{1}{576} t \right)^2 = 0$$

$$\sqrt{10} - \frac{1}{576} t = 0$$

$$t = 576\sqrt{10} \text{ sec} \approx 1821.47 \text{ sec} \approx 30.36 \text{ min}$$

Exercise

A tank in the form of a right-circular cylinder cone standing on end, vertex down, is leaking water through a circular hole in its bottom.

- a) Suppose the tank is 20 feet high and has radius 8 inches. Show that the differential equation governing the height h of water leaking from a tank is

$$\frac{dh}{dt} = -\frac{5}{6h^{3/2}}$$

In this model, friction and contraction of the water at the hole were taken into account with $c = 0.6$ and $g = 32 \text{ ft/s}^2$. If the tank is initially full, how long will it take the tank to empty?

- b) Suppose the tank has a vertex angle of 60° and the circular hole has radius 2 inches. Determine the differential equation governing the height h of water. Use $c = 0.6$ and $g = 32 \text{ ft/s}^2$.
- c) If the height of the water is initially 9 feet, how long will it take the tank to empty?

Solution

$$a) \int 6h^{3/2} dh = -\int 5 dt$$

$$\frac{12}{5} h^{5/2} = -5t + C$$

$$h^{5/2} = -\frac{25}{12} t + C$$

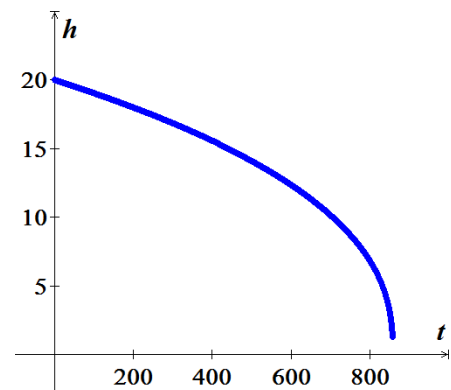
$$h(0) = 20 \rightarrow 20^{5/2} = C \Rightarrow C = 800\sqrt{5}$$

$$h(t) = \left(-\frac{25}{12} t + 800\sqrt{5} \right)^{2/5}$$

$$h(t) = \left(-\frac{25}{12} t + 800\sqrt{5} \right)^{2/5} = 0$$

$$-\frac{25}{12} t + 800\sqrt{5} = 0 \rightarrow t = \frac{9600\sqrt{5}}{25} = 384\sqrt{5} \text{ sec} \approx 858.65 \text{ sec}$$

$$t \approx 14.61 \text{ min}$$



$$b) \tan 30^\circ = \frac{r}{h} \rightarrow r = h \tan 30^\circ$$

$$r = \frac{h}{\sqrt{3}}$$

$$A_w = \pi r^2 = \frac{1}{3} \pi h^2$$

$$\frac{dh}{dt} = -c \frac{A_h}{A_w} \sqrt{2gh}$$

$$= -(0.6) \pi \left(\frac{2}{12} \right)^2 \frac{3}{\pi h^2} \sqrt{64h}$$

$$= -\frac{2}{5h^{3/2}}$$

$$\int h^{3/2} dh = -\frac{2}{5} \int dt$$

$$\frac{2}{5} h^{5/2} = -\frac{2}{5} t + C_2$$

$$h^{5/2} = -t + C_2$$

$$h(0) = 9 \rightarrow 9^{5/2} = C \Rightarrow \underline{C = 243}$$

$$h(t) = (-t + 243)^{2/5}$$

$$h(t) = (-t + 243)^{2/5} = 0 \rightarrow \underline{t = 243 \text{ sec} = 4.05 \text{ min}}$$

Exercise

Suppose that the conical tank is inverted and that water leaks out a circular hole of radius 2 inches in the center of its circular base. Is the time it takes to empty a full tank the same as for the tank with vertex down? Take the friction/contraction coefficient to be $c = 0.6$ and $g = 32 \text{ ft/s}^2$

Solution

$$\frac{r}{8} = \frac{20-h}{20}$$

$$r = \frac{2}{5}(20-h)$$

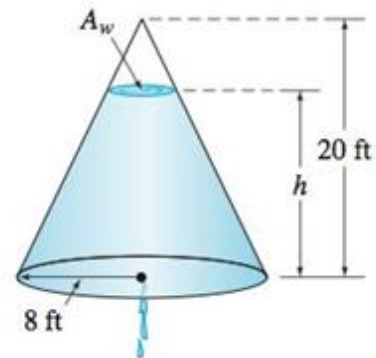
$$A_w = \pi r^2 = \frac{4\pi}{25}(20-h)^2$$

$$\frac{dh}{dt} = -(0.6)(\pi) \left(\frac{2}{12} \right)^2 \frac{25}{4\pi(20-h)^2} \sqrt{64h}$$

$$= -\frac{5}{6} \frac{\sqrt{h}}{(20-h)^2}$$

$$\int \frac{400 - 40h + h^2}{h^{1/2}} dh = -\frac{5}{6} \int dt$$

$$\frac{dh}{dt} = -c \frac{A_h}{A_w} \sqrt{2gh}$$



$$\int (400h^{-1/2} - 40h^{1/2} + h^{3/2}) dh = -\frac{5}{6} \int dt$$

$$800\sqrt{h} - \frac{80}{3}h^{3/2} + \frac{2}{5}h^{5/2} = -\frac{5}{6}t + C$$

$$h(0) = 20 \rightarrow 1600\sqrt{5} - \frac{3200}{3}\sqrt{5} + \frac{1600}{5}\sqrt{5} = C$$

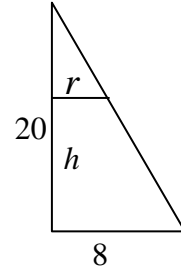
$$C = \frac{2560\sqrt{5}}{3}$$

$$800\sqrt{h} - \frac{80}{3}h^{3/2} + \frac{2}{5}h^{5/2} = -\frac{5}{6}t + \frac{2560\sqrt{5}}{3}$$

$$0 = -\frac{5}{6}t + \frac{2560\sqrt{5}}{3}$$

$$t = 1024\sqrt{5} \text{ sec} \approx 38.16 \text{ min}$$

The tank empties more slowly when the base of the cone is on the bottom.



Exercise

A differential equation for the velocity v of a falling mass m subjected to air resistance proportional to the square of the instantaneous velocity is

$$m \frac{dv}{dt} = mg - kv^2$$

Where $k > 0$ is a constant of proportionality. The positive direction is downward.

- Solve the equation subject to the initial condition $v(0) = v_0$.
- Use the solution in part (a) to determine the limiting, or terminal, velocity of the mass.
- If the distance s , measured from the point where the mass was released above the ground, is related to velocity v by $\frac{ds}{dt} = v(t)$, find an explicit expression for $s(t)$ if $s(0) = 0$

Solution

$$a) \int \frac{m}{mg - kv^2} dv = \int dt \qquad \int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a}$$

$$\frac{1}{\sqrt{k}} \frac{m}{\sqrt{mg}} \tanh^{-1} \left(\frac{\sqrt{k}}{\sqrt{mg}} v \right) = t + C_1$$

$$\tanh^{-1} \left(\sqrt{\frac{k}{mg}} v \right) = \sqrt{\frac{kg}{m}} t + C$$

$$v(0) = v_0 \quad \tanh^{-1} \left(\sqrt{\frac{k}{mg}} v_0 \right) = C$$

$$\sqrt{\frac{k}{mg}} v = \tanh \left(\sqrt{\frac{kg}{m}} t + \tanh^{-1} \left(\sqrt{\frac{k}{mg}} v_0 \right) \right)$$

$$v(t) = \sqrt{\frac{mg}{k}} \tanh \left(\sqrt{\frac{kg}{m}} t + \tanh^{-1} \left(\sqrt{\frac{k}{mg}} v_0 \right) \right)$$

$$b) \quad \tanh t \xrightarrow{t \rightarrow \infty} 1$$

$$\lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} \left(\sqrt{\frac{mg}{k}} \tanh \left(\sqrt{\frac{kg}{m}} t + \tanh^{-1} \left(\sqrt{\frac{k}{mg}} v_0 \right) \right) \right) \\ = \sqrt{\frac{mg}{k}}$$

$$c) \quad s(t) = \sqrt{\frac{mg}{k}} \int \tanh \left(\sqrt{\frac{kg}{m}} t + \tanh^{-1} \left(\sqrt{\frac{k}{mg}} v_0 \right) \right) dt \\ = \sqrt{\frac{mg}{k}} \sqrt{\frac{m}{kg}} \ln \left[\cosh \left(t + \tanh^{-1} \left(\sqrt{\frac{k}{mg}} v_0 \right) \right) \right] + C_2 \\ = \frac{m}{k} \ln \left[\cosh \left(t + \tanh^{-1} \left(\sqrt{\frac{k}{mg}} v_0 \right) \right) \right] + C_2$$

Exercise

An object is dropped from altitude y_0

- Determine the impact velocity if the drag force is proportional to the square of velocity, with drag coefficient κ .
- If the terminal velocity is known to -120 mph and the impact velocity was -90 mph , what was the initial altitude y_0 ?

Solution

$$a) \quad m \frac{dv}{dt} = -mg + \kappa v^2 \quad v(y_0) = 0$$

$$\frac{dv}{dt} = v \frac{dv}{dy}$$

$$mv \frac{dv}{dy} = -mg + \kappa v^2$$

$$\int \frac{mv}{\kappa v^2 - mg} dv = \int dy$$

$$\frac{m}{2\kappa} \int \frac{1}{\kappa v^2 - mg} d(\kappa v^2 - mg) = \int dy$$

$$\frac{m}{2\kappa} \ln |\kappa v^2 - mg| = y + C$$

$$v(y_0) = 0 \rightarrow \frac{m}{2\kappa} \ln mg = y_0 + C \Rightarrow C = \frac{m}{2\kappa} \ln mg - y_0$$

$$\frac{m}{2\kappa} \ln |\kappa v^2 - mg| = y + \frac{m}{2\kappa} \ln mg - y_0$$

$$\ln |\kappa v^2 - mg| = \frac{2\kappa}{m} y + \ln(mg) - \frac{2\kappa}{m} y_0$$

Since the object is falling: $\kappa v^2 - mg < 0$

$$-(\kappa v^2 - mg) = e^{\frac{2\kappa}{m}(y-y_0)} + \ln(mg)$$

$$\kappa v^2 = -e^{\frac{2\kappa}{m}(y-y_0)} \ln(mg) + mg$$

$$v^2 = \frac{mg}{\kappa} - \frac{mg}{\kappa} e^{\frac{2\kappa}{m}(y-y_0)}$$

$$v = \sqrt{\frac{mg}{\kappa} \left(1 - e^{\frac{2\kappa}{m}(y-y_0)} \right)}$$

$$b) \quad v_{\text{terminal}} = 120 \frac{\text{mi}}{\text{hr}} \frac{5280 \text{ ft}}{\text{mi}} \frac{1 \text{ hr}}{3600 \text{ sec}} = \frac{528}{3} = 176 \text{ ft/s}$$

$$v_{\text{impact}} = 90 \frac{\text{mi}}{\text{hr}} \frac{5280 \text{ ft}}{\text{mi}} \frac{1 \text{ hr}}{3600 \text{ sec}} = \frac{528}{4} = 132 \text{ ft/s}$$

$$v_{\text{terminal}} = \sqrt{\frac{mg}{\kappa}} = 176 \rightarrow \frac{32m}{\kappa} = 176^2 \Rightarrow \left[\frac{m}{\kappa} = \frac{176^2}{32} = 968 \right]$$

$$v(0) = \sqrt{\frac{mg}{\kappa} \left(1 - e^{\frac{2\kappa}{m}(y-y_0)} \right)} = 132$$

$$176 \sqrt{1 - e^{-1936y_0}} = 132$$

$$\sqrt{1 - e^{-1936y_0}} = \frac{132}{176} = \frac{3}{4}$$

$$1 - e^{-1936y_0} = \frac{9}{16}$$

$$e^{-1936y_0} = \frac{7}{16}$$

$$-1936y_0 = \ln \frac{7}{16}$$

$$y_0 = -\frac{1}{1936} \ln \frac{7}{16} \approx 400.11 \text{ ft}$$

Exercise

An object is dropped from altitude y_0

- Assume that the drag force is proportional to the velocity, with drag coefficient κ . Obtain an implicit solution relating velocity and altitude.
- If the terminal velocity is known to -120 mph and the impact velocity was -90 mph , what was the initial altitude y_0 ?

Solution

$$a) \quad m \frac{dv}{dt} = -mg - \kappa v$$

$$mv \frac{dv}{dy} = -mg - \kappa v \quad \frac{dv}{dt} = v \frac{dv}{dy}$$

$$m \frac{v}{\kappa v + mg} dv = -dy$$

$$\frac{m}{\kappa} \frac{v}{v + \frac{mg}{\kappa}} dv = -dy$$

$$\frac{m}{\kappa} \int \left(1 - \frac{\frac{mg}{\kappa}}{v + \frac{mg}{\kappa}} \right) dv = - \int dy$$

$$\frac{m}{\kappa} \left(v - \frac{mg}{\kappa} \ln \left| v + \frac{mg}{\kappa} \right| \right) = -y + C$$

$$v(y_0) = 0 \rightarrow -g \left(\frac{m}{\kappa} \right)^2 \ln \left(\frac{mg}{\kappa} \right) = -y_0 + C \Rightarrow C = y_0 - g \left(\frac{m}{\kappa} \right)^2 \ln \left(\frac{mg}{\kappa} \right)$$

$$\frac{m}{\kappa} \left(v - \frac{mg}{\kappa} \ln \left| v + \frac{mg}{\kappa} \right| \right) = -y + y_0 - g \left(\frac{m}{\kappa} \right)^2 \ln \left(\frac{mg}{\kappa} \right)$$

$$y_0 - y = \frac{m}{\kappa} v - \frac{1}{g} \left(\frac{mg}{\kappa} \right)^2 \ln \left| v + \frac{mg}{\kappa} \right| + \frac{1}{g} \left(\frac{mg}{\kappa} \right)^2 \ln \left(\frac{mg}{\kappa} \right)$$

$$= \frac{m}{\kappa} v - \frac{1}{g} \left(\frac{mg}{\kappa} \right)^2 \left(\ln \left| v + \frac{mg}{\kappa} \right| - \ln \left(\frac{mg}{\kappa} \right) \right)$$

$$= \frac{m}{\kappa} v - \frac{1}{g} \left(\frac{mg}{\kappa} \right)^2 \ln \left| \frac{\kappa}{mg} v + 1 \right|$$

$$b) v_{terminal} = -120 \frac{mi}{hr} \frac{5280 ft}{mi} \frac{1 hr}{3600 sec} = -\frac{528}{3} = -176 \frac{ft}{s}$$

$$v_{impact} = -90 \frac{mi}{hr} \frac{5280 ft}{mi} \frac{1 hr}{3600 sec} = -\frac{528}{4} = -132 \frac{ft}{s}$$

$$v_{terminal} = -\frac{mg}{\kappa} = -176 \rightarrow \frac{32m}{\kappa} = 176 \Rightarrow \left| \frac{m}{\kappa} = \frac{176}{32} = 5.5 \right|$$

$$y_0 = 5.5 v_{imp} - \frac{1}{32} (176)^2 \ln \left| \frac{1}{176} v_{imp} + 1 \right|$$

$$= 5.5(-132) - \frac{1}{32} (176)^2 \ln \left| -\frac{132}{176} + 1 \right|$$

$$\approx 615.93 ft$$

Exercise

An object of mass 3 kg is released from rest 500 m above the ground and allowed to fall under the influence of gravity. Assume the gravitational force constant, with $g = 9.81 \text{ m/s}^2$, and the force due to air resistance is proportional to the velocity of the object with proportionality constant $\kappa = 3 \text{ N-sec/m}$. Determine when the object will hit the ground.

Solution

By Newton second law: $m \frac{dv}{dt} = mg - \kappa v$

$$m \frac{dv}{\kappa v - mg} = -dt$$

$$\frac{m}{\kappa} \int \frac{d(\kappa v - mg)}{\kappa v - mg} = - \int dt$$

$$\frac{m}{\kappa} \ln|\kappa v - mg| = -t + C_1$$

$$\ln(\kappa v - mg) = -\frac{\kappa}{m}t + C_2$$

$$\kappa v - mg = Ce^{-\kappa t/m}$$

$$v(t) = \frac{C}{\kappa} e^{-\kappa t/m} + \frac{mg}{\kappa}$$

$$v(0) = 0 \rightarrow \frac{C}{\kappa} + \frac{mg}{\kappa} = 0 \Rightarrow \underline{C = -mg}$$

Given: $m = 3, \quad g = 9.81, \quad \kappa = 3$

$$v(t) = -\frac{mg}{\kappa} e^{-\kappa t/m} + \frac{mg}{\kappa}$$

$$= \frac{3(9.81)}{3} (1 - e^{-t})$$

$$= 9.81(1 - e^{-t})$$

$$x(t) = 9.81 \int (1 - e^{-t}) dt$$

$$= 9.81t + 9.81e^{-t} + C_3$$

$$x(0) = 0 \rightarrow \underline{C_3 = -9.81}$$

$$x(t) = 9.81e^{-t} + 9.81t - 9.81 = 500$$

$$t + e^{-t} = \frac{500 + 9.81}{9.81} = 51.97$$

$$\underline{t = 51.97 \text{ sec}}$$

$$\cancel{t = 4.02}$$

Exercise

A parachutist whose mass is 75 kg drops from helicopter hovering 4000 m above the ground and falls toward the earth under the influence of gravity. Assume the gravitational force is constant. Assume also that the force due to air resistance is proportional to the velocity of the parachutist, with the proportionality constant $\kappa_1 = 15 \text{ N-sec/m}$ when the chute is closed and with constant $\kappa_2 = 105 \text{ N-sec/m}$ when the chute is open. If the chute does not open until 1 min after the parachutist leaves the helicopter, after how many seconds will he hit the ground?

Solution

Given: $m = 75$, $x(t) = 4000$, $g = 9.81$, $\kappa_1 = 15$, $v(0) = 0$

Before the chute is open

By Newton second law: $m \frac{dv}{dt} = mg - \kappa v$

$$m \frac{dv}{\kappa v - mg} = -dt$$

$$\frac{m}{\kappa} \int \frac{d(\kappa v - mg)}{\kappa v - mg} = - \int dt$$

$$\frac{m}{\kappa} \ln |\kappa v - mg| = -t + C_1$$

$$\kappa v - mg = C e^{-\kappa t/m}$$

$$v_1(t) = \frac{C}{\kappa_1} e^{-\kappa_1 t/m} + \frac{mg}{\kappa_1}$$

$$v(0) = 0 \rightarrow \frac{C}{\kappa} + \frac{mg}{\kappa} = 0 \Rightarrow \underline{C = -mg}$$

$$v_1(t) = \frac{75(9.81)}{15} (1 - e^{-t/5})$$

$$= 49.05 (1 - e^{-t/5})$$

$$v_1(t = 1 \text{ min} = 60 \text{ s}) = 49.05 (1 - e^{-60/5}) \approx \underline{49.05 \text{ m/s}}$$

$$x_1(t) = 49.05 \int (1 - e^{-t/5}) dt$$

$$= 49.05t + 245.25 e^{-t/5} + C_2$$

$$x(0) = 0 \rightarrow \underline{C_2 = -245.25}$$

$$x_1(t = 60) = 49.05(60) + 245.25 e^{-12} - 245.25 \approx \underline{2697.75 \text{ m}}$$

When the chute is opened, he is $4000 - 2697.5 \approx 1302.25 \text{ m}$ above the ground.

Given: $m = 75$, $x_2(T) = 1302.25$, $g = 9.81$, $\kappa_2 = 105$, $v(0) = 49.05$

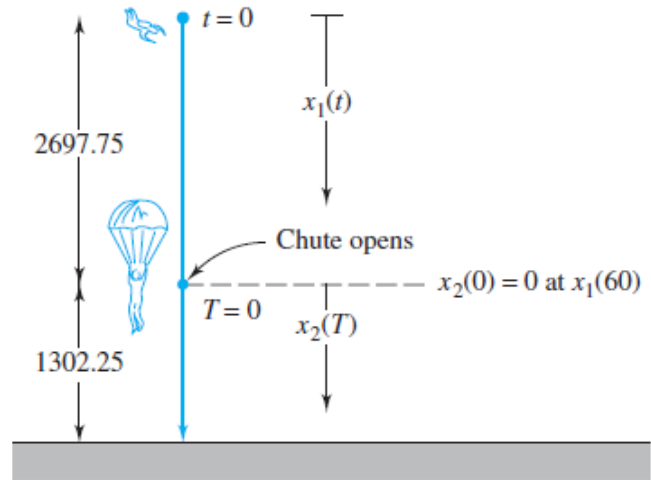
$$v_2(t) = \frac{C}{\kappa_2} e^{-\kappa_2 t/m} + \frac{mg}{\kappa_2}$$

$$= \frac{C}{105} e^{-1.4t} + \frac{75(9.81)}{105}$$

$$v(0) = 49.05 \rightarrow \frac{C}{105} + 7.007 = 49.05 \Rightarrow \underline{C = 4,414.5}$$

$$v_2(t) = 42.043 e^{-1.4t} + 7.007$$

$$x_2(t) = \int (42.043 e^{-1.4t} + 7.01) dt$$



$$= -30.03e^{-1.4t} + 7.01t + C_4$$

$$x(0) = 0 \rightarrow C_4 = 30.03$$

$$x_2(t) = -30.03e^{-1.4t} + 7.01t + 30.03 = 1302.25$$

$$-30.03e^{-1.4t} + 7.01t - 1272.22 = 0 \quad t = 181.49 \text{ sec}$$

Exercise

A parachutist whose mass is 75 kg drops from helicopter hovering 2000 m above the ground and falls toward the earth under the influence of gravity. Assume the gravitational force is constant. Assume also that the force due to air resistance is proportional to the velocity of the parachutist, with the proportionality constant $\kappa_1 = 30 \text{ N-sec/m}$ when the chute is closed and with constant $\kappa_2 = 90 \text{ N-sec/m}$ when the chute is open. If the chute does not open until the velocity of the parachutist reaches 20 m/sec, after how many seconds will he reach the ground?

Solution

Given: $m = 75$, $x(t) = 2000$, $g = 9.8$, $\kappa_1 = 30$, $v(0) = 0$, $v(t_1) = 20$

Before the chute is open

By Newton's second law: $m \frac{dv}{dt} = mg - \kappa_1 v$

$$\frac{dv}{dt} = 9.8 - 0.4v$$

$$\int \frac{dv}{0.4v - 9.8} = - \int dt$$

$$\frac{1}{0.4} \ln|0.4v - 9.8| = -t + C_1$$

$$0.4v - 9.8 = C_2 e^{-0.4t}$$

$$v_1(t) = 2.5C_2 e^{-0.4t} + 24.5$$

$$v(0) = 0 \rightarrow 2.5C_2 + 24.5 = 0 \Rightarrow C_2 = -9.8$$

$$v_1(t) = -24.5e^{-0.4t} + 24.5 = 20$$

$$e^{-0.4t} = \frac{4.5}{24.5} \rightarrow t = -\frac{1}{0.4} \ln \frac{4.5}{24.5} \approx 4.24$$

$$x_1(t) = \int (-24.5e^{-0.4t} + 24.5) dt$$

$$= 61.25e^{-0.4t} + 24.5t + C_3$$

$$x(0) = 0 \rightarrow C_3 = -61.25$$

$$x_1(t) = 61.25e^{-0.4t} + 24.5t - 61.25$$

$$x_1(4.24) = 61.25e^{-0.4(4.24)} + 24.5(4.24) - 61.25 \approx \underline{53.86 \text{ m}}$$

When the chute is opened, he is $2000 - 53.86 \approx 1946.14 \text{ m}$ above the ground.

Given: $m = 75$, $x_2(T) = 1946.14$, $g = 9.81$, $\kappa_2 = 90$ $v(0) = 20$

$$v_2(t) = \frac{1}{\kappa_2} C_4 e^{-\kappa_2 t/m} + \frac{mg}{\kappa_2}$$

$$= \frac{1}{90} C_4 e^{-1.2t} + 8.1667$$

$$v(0) = 20 \rightarrow \frac{1}{90} C_4 + 8.1667 = 20 \Rightarrow \underline{C_4 = 1064.997}$$

$$v_2(t) = 11.833e^{-1.2t} + 8.1667$$

$$x_2(t) = \int (11.833e^{-1.2t} + 8.1667) dt$$

$$= -9.86e^{-1.2t} + 8.1667t + C_5$$

$$x(0) = 0 \rightarrow \underline{C_4 = 9.86}$$

$$x_2(t) = -9.86e^{-1.2t} + 8.1667t + 9.86 = \underline{1946.14}$$

$$= -9.86e^{-1.2t} + 8.1667t - 1936.28 = 0 \quad \underline{t = 237 \text{ sec}}$$

Exercise

An object of mass 5 kg is released from rest 1000 m above the ground and allowed to fall under the influence of gravity. Assume the gravitational force constant, with $g = 9.8 \text{ m/s}^2$, and the force due to air resistance is proportional to the velocity of the object with proportionality constant $\kappa = 50 \text{ N-sec/m}$. Determine when the object will hit the ground.

Solution

By Newton's second law: $m \frac{dv}{dt} = mg - \kappa v$

$$m \frac{dv}{\kappa v - mg} = -dt$$

$$\frac{m}{\kappa} \int \frac{d(\kappa v - mg)}{\kappa v - mg} = - \int dt$$

$$\frac{m}{\kappa} \ln |\kappa v - mg| = -t + C_1$$

$$\kappa v - mg = C e^{-\kappa t/m}$$

Given: $m = 5$, $g = 9.8$, $\kappa = 50$, $v(0) = 0$, $x(0) = 0$

$$v(t) = \frac{C}{50} e^{-\kappa t/m} + \frac{5(9.8)}{50}$$

$$= \frac{C}{50} e^{-10t} + 0.98$$

$$v(0) = 0 \rightarrow \frac{C}{50} + 0.98 = 0 \Rightarrow \underline{C = -49}$$

$$\underline{v(t) = -0.98e^{-10t} + 0.98}$$

$$x(t) = 0.98 \int (1 - e^{-10t}) dt$$

$$= 0.98t + 0.098e^{-10t} + C_2$$

$$x(0) = 0 \rightarrow \underline{C_2 = -0.098}$$

$$x(t) = 0.98t + 0.098e^{-t} - 0.098 = 1000$$

$$0.98t + 0.098e^{-t} - 1000.098 = 0$$

$$\underline{t = 1020 \text{ sec}}$$

Exercise

An object of mass 500 kg is released from rest 1000 m above the ground and allowed to fall under the influence of gravity. Assume the gravitational force constant, with $g = 9.8 \text{ m/s}^2$, and the force due to air resistance is proportional to the velocity of the object with proportionality constant $\kappa = 50 \text{ N-sec/m}$. Determine when the object will hit the ground.

Solution

$$\text{Given: } m = 500, \quad g = 9.8, \quad \kappa = 50, \quad v(0) = 0, \quad x(0) = 0$$

$$\text{By Newton's second law: } m \frac{dv}{dt} = mg - \kappa v$$

$$500 \frac{dv}{dt} = 4,900 - 50v$$

$$\frac{dv}{dt} = 9.8 - 0.1v$$

$$-\int \frac{dv}{0.1v - 9.8} = \int dt$$

$$10 \ln |0.1v - 9.8| = -t + C_1$$

$$0.1v - 9.8 = C_2 e^{-0.1t}$$

$$v(t) = 10C_2 e^{-0.1t} + 98$$

$$v(0) = 0 \rightarrow 10C_2 + 98 = 0 \Rightarrow \underline{C_2 = -9.8}$$

$$\underline{v(t) = 98(1 - e^{-0.1t})}$$

$$\begin{aligned}
 x(t) &= 98 \int (1 - e^{-0.1t}) dt \\
 &= 98t + 980e^{-0.1t} + C_3 \\
 x(0) &= 0 \rightarrow \underline{C_3 = -980}
 \end{aligned}$$

$$x(t) = 98t + 980e^{-0.1t} - 980 = 1000$$

$$98t + 980e^{-0.1t} - 1980 = 0$$

$$\underline{t = 18.7 \text{ sec}}$$

Exercise

A 400-lb object is released from rest 500 ft above the ground and allowed to fall under the influence of gravity. Assuming that the force in pounds due to air resistance is $-10v$, where v is the velocity of the object in ft/s, determine the equation of motion of the object. When will the object hit the ground?

Solution

Given: $g = 32$, $m = \frac{400}{32} = 12.5$, $-10v$ $v(0) = 0$, $x(0) = 0$

By Newton second law: $m \frac{dv}{dt} = mg - 10v$

$$12.5 \frac{dv}{dt} = 400 - 10v$$

$$\frac{dv}{dt} = 32 - 0.8v$$

$$-\int \frac{dv}{0.8v - 32} = \int dt$$

$$\frac{1}{0.8} \ln|0.8v - 32| = -t + C_1$$

$$\ln|0.8v - 32| = -0.8t + C_1$$

$$0.8v - 32 = C_1 e^{-0.8t}$$

$$v(t) = 40 - \frac{C_2}{0.8} e^{-0.8t}$$

$$v(0) = 0 \rightarrow -\frac{1}{0.8} C_2 + 40 = 0 \Rightarrow \underline{C_2 = 32}$$

$$\underline{v(t) = 40(1 - e^{-0.8t})}$$

$$x(t) = 40 \int (1 - e^{-0.8t}) dt$$

$$= 40t + 50e^{-0.8t} + C_3$$

$$x(0) = 0 \rightarrow 50 + C_3 = 0 \Rightarrow \underline{C_3 = -50}$$

$$x(t) = 40t + 50e^{-0.8t} - 50 = 500$$

$$4t + 5e^{-0.8t} - 55 = 0 \rightarrow 4t - 55 = 0 \Rightarrow t = 13.75 \text{ sec}$$

Exercise

An object of mass 8 kg is given an upward initial velocity of 20 m/sec and then allowed to fall under the influence of gravity. Assume that the force in Newton due to air resistance is $-16v$, where v is the velocity of the object in m/sec.

- Determine the equation of motion of the object.
- If the object is initially 100 m above the ground, determine when the object will hit the ground.

Solution

Given: $m = 8, \quad g = 9.8, \quad \kappa = 16, \quad v(0) = -20, \quad x(0) = 0$

a) By Newton's second law: $m \frac{dv}{dt} = mg - 16v$

$$\frac{dv}{dt} = 9.8 - 2v$$

$$-\int \frac{dv}{2v - 9.8} = \int dt$$

$$\frac{1}{2} \ln|2v - 9.8| = -t + C_1$$

$$2v - 9.8 = C_2 e^{-2t}$$

$$v(t) = \frac{1}{2} C_2 e^{-2t} + 4.9$$

$$v(0) = -20 \rightarrow \frac{1}{2} C_2 + 4.9 = -20 \Rightarrow C_2 = -49.8$$

$$v(t) = -24.9e^{-2t} + 4.9$$

$$x(t) = \int (-24.9e^{-2t} + 4.9) dt$$

$$= 4.9t + 12.45e^{-2t} + C_3$$

$$x(0) = 0 \rightarrow 12.45 + C_3 = 0 \Rightarrow C_3 = -12.45$$

$$x(t) = 4.9t + 12.45e^{-2t} - 12.45$$

b) $x(t) = 4.9t + 12.45e^{-2t} - 12.45 = 100$

$$4.9t + 12.45e^{-2t} - 112.45 = 0 \rightarrow t \approx 22.9 \text{ sec}$$

Exercise

An object of mass 5 kg is given an downward initial velocity of 50 m/sec and then allowed to fall under the influence of gravity. Assume that the force in Newton due to air resistance is $-10v$, where v is the velocity of the object in m/sec.

- a) Determine the equation of motion of the object.
 b) If the object is initially 100 m above the ground, determine when the object will hit the ground.

Solution

Given: $m = 5, \quad g = 9.8, \quad \kappa = 10, \quad v(0) = 50, \quad x(0) = 0$

a) By Newton's second law: $m \frac{dv}{dt} = mg - 10v$

$$\frac{dv}{dt} = 9.8 - 2v$$

$$-\int \frac{dv}{2v - 9.8} = \int dt$$

$$\frac{1}{2} \ln |2v - 9.8| = -t + C_1$$

$$2v - 9.8 = C_2 e^{-2t}$$

$$v(t) = \frac{1}{2} C_2 e^{-2t} + 4.9$$

$$v(0) = 50 \rightarrow \frac{1}{2} C_2 + 4.9 = 50 \Rightarrow \underline{C_2 = 90.2}$$

$$\underline{v(t) = 45.1e^{-2t} + 4.9}$$

$$x(t) = \int (45.1e^{-2t} + 4.9) dt$$

$$= 4.9t - 22.55e^{-2t} + C_3$$

$$x(0) = 0 \rightarrow \underline{C_3 = 22.55}$$

$$\underline{x(t) = 4.9t - 22.55e^{-2t} + 22.55}$$

b) $x(t) = 4.9t - 22.55e^{-2t} + 22.55 = 100$

$$4.9t - 22.55e^{-2t} - 77.45 = 0 \rightarrow \underline{t \approx 15.81 \text{ sec}}$$

Exercise

A shell of mass 2 kg is shot upward with an initial velocity of 200 m/sec. The magnitude of the force on the shell due to air resistance is $\frac{|v|}{20}$.

- a) When will the shell reach its maximum height above the ground?
 b) What is the maximum height?

Solution

Given: $m = 2, \quad g = 9.81, \quad \frac{|v|}{20}, \quad v(0) = 200, \quad x(0) = 0$

a) $m \frac{dv}{dt} = -mg - \frac{v}{20}$

$$\frac{dv}{dt} = -9.8 - \frac{v}{40}$$

$$\int \frac{dv}{v + 392.4} = -\frac{1}{40} \int dt$$

$$\ln|v + 392.4| = -\frac{1}{40}t + C_1$$

$$v(t) = C_2 e^{-t/40} - 392.4$$

$$v(0) = 200 \rightarrow 200 = C_2 - 392.4 \Rightarrow C_2 = 592.4$$

$$v(t) = 592.4e^{-t/40} - 392.4$$

At maximum height, $v(t) = 0$

$$v(t) = 592.4e^{-t/40} - 392.4 = 0$$

$$t = -40 \ln \frac{392.4}{592.4} \approx 16.476 \text{ sec}$$

$$\begin{aligned} b) \quad x(t) &= \int (592.4e^{-t/40} - 392.4) dt \\ &= -23,696e^{-t/40} - 392.4t + C_3 \end{aligned}$$

$$x(0) = 0 \rightarrow 23,696 = C_3$$

$$x(t) = -23,696e^{-t/40} - 392.4t + 23,696$$

$$\begin{aligned} x(t = 16.476) &= -23,696e^{-16.476/40} - 392.4(16.476) + 23,696 \\ &\approx 1534.81 \text{ m} \end{aligned}$$

Exercise

We need to design a ballistics chamber to decelerate test projectiles fired into it. Assume the resistive force encountered by the projectile is proportional to the square of its velocity and neglect gravity.

The chamber is to be constructed so that the coefficient κ associated with this resistive force is not constant but is, in fact, a linearly increasing function of distance into the chamber:

Let $\kappa(x) = \kappa_0 x$, where κ_0 is a constant; the resistive force then has the form $\kappa(x)v^2 = \kappa_0 x v^2$.

If we use time t as the independent variable, Newton's law of motion leads us to the differential equation

$$m \frac{dv}{dt} + \kappa_0 x v^2 = 0 \quad \text{with} \quad v = \frac{dx}{dt}$$

- Adopt distance x into the chamber as the new independent variable and rewrite the given differential equation as a first order equation in terms of the new independent variable.
- Determine the value κ_0 needed if the chamber is to reduce projectile velocity to 1% of its incoming value within d units of distance.

Solution

$$a) \quad m \frac{dv}{dt} + \kappa_0 x v^2 = 0 \quad \text{with} \quad v = \frac{dx}{dt}$$

$$m \frac{dv}{dx} \frac{dx}{dt} + \kappa_0 x v^2 = 0 \quad v = \frac{dx}{dt}$$

$$\boxed{mv \frac{dv}{dx} + \kappa_0 x v^2 = 0} \quad \text{when } x = 0$$

$$b) \quad \frac{dv}{dx} + \frac{\kappa_0}{m} x v = 0$$

$$\frac{dv}{dx} + \frac{\kappa_0}{m} x v = 0$$

$$e^{\int \frac{\kappa_0}{m} x dx} = e^{\frac{\kappa_0}{2m} x^2}$$

$$v(x) = C e^{-\frac{\kappa_0}{2m} x^2}$$

$$v(0) = v_0 \rightarrow v_0 = C$$

$$\boxed{v(x) = v_0 e^{-\frac{\kappa_0}{2m} x^2}}$$

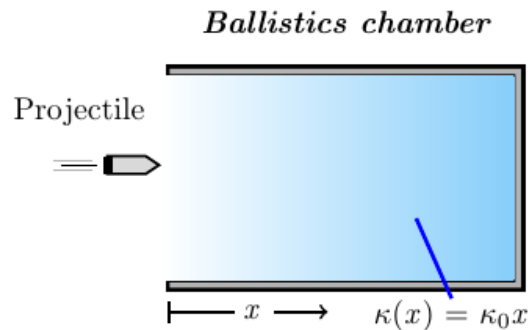
$$\text{Let } v = 0.01v_0 \quad \text{and} \quad x = d$$

$$0.01v_0 = v_0 e^{-\frac{\kappa_0}{2m} d^2}$$

$$e^{-\frac{\kappa_0}{2m} d^2} = 0.01$$

$$-\frac{\kappa_0}{2m} d^2 = \ln(0.01)$$

$$\boxed{\kappa_0 = \frac{2m}{d^2} \ln(100)}$$



Exercise

When the velocity v of an object is very large, the magnitude of the force due to air resistance is proportional to v^2 with the force acting in opposition to the motion of the object. A shell of mass 3 kg is shot upward from the ground with an initial velocity of 500 m/sec . If the magnitude of the force due to air resistance is $(0.1)v^2$.

- When will the shell reach its maximum height above the ground?
- What is the maximum height?

Solution

$$\text{Given: } m = 3, \quad g = 9.81, \quad 0.1v^2, \quad v(0) = 500, \quad x(0) = 0$$

$$a) \quad m \frac{dv}{dt} = -mg - 0.1v^2$$

$$\frac{dv}{dt} = -9.81 - \frac{1}{30}v^2$$

$$\frac{dv}{dt} = -\frac{1}{30}(294.3 + v^2)$$

$$\int \frac{dv}{294.3 + v^2} = -\frac{1}{30} \int dt$$

$$\frac{1}{\sqrt{294.3}} \tan^{-1} \frac{v}{\sqrt{294.3}} = -\frac{1}{30}t + C$$

$$v(0) = 500 \rightarrow 0.058 \tan^{-1}(29.07) = C \Rightarrow \underline{C = 0.089}$$

$$\frac{1}{\sqrt{294.3}} \tan^{-1} \frac{v}{\sqrt{294.3}} = -\frac{1}{30}t + 0.08956$$

$$v(t) = \sqrt{294.3} \tan(-0.572t + 1.54)$$

$$v(t) = 17.155 \tan(-0.572t + 1.54) = 0$$

$$-0.572t + 1.54 = 0 \rightarrow \underline{t = 2.69 \text{ sec}}$$

$$b) \quad x(t) = 17.155 \int \tan(-0.572t + 1.54) dt$$

$$= -30 \ln |\sec(-0.572t + 1.54)| + C$$

$$x(0) = 0 \rightarrow C = 30 \ln |\sec(1.54)| \approx 104.4$$

$$x(t) = -30 \ln |\sec(-0.572t + 1.54)| + 104.4$$

$$\underline{x(2.69) \approx 104.4 \text{ m}}$$

Exercise

A sailboat has been running (on a straight course) under a light wind at 1 m/sec . Suddenly the wind picks up, blowing hard enough to apply a constant force of 600 N to the sailboat. The only other force acting on the boat is water resistance that is proportional to the velocity of the boat. If the proportionality constant for water resistance is $\kappa = 100 \text{ N-sec/m}$ and the mass of the sailboat is 50 kg .

- Find the equation of motion of the sailboat.
- What is the limiting velocity of the sailboat under this wind?
- When the velocity of the sailboat reaches 5 m/sec , the boat begins to rise out of the water and plane. When this happens, the proportionality constant for the water resistance drop to $\kappa = 60 \text{ N-sec/m}$. Find the equation of motion of the sailboat.
- What is the limiting velocity of the sailboat under this wind as it is planning?

Solution

$$\text{Given: } m = 50 \quad F_w = 600 \quad \kappa = 100 \quad v(0) = 1$$

$$a) \quad m \frac{dv}{dt} = F_w - \kappa v$$

$$50 \frac{dv}{dt} = 600 - 100v$$

$$\frac{dv}{dt} = 12 - 2v$$

$$\int \frac{dv}{v-6} = -2 \int dt$$

$$\ln|v-6| = -2t + C_1$$

$$v = C_2 e^{-2t} + 6$$

$$v(0) = 1 \rightarrow C_2 = -5$$

$$v(t) = -5e^{-2t} + 6$$

$$\begin{aligned} x(t) &= \int (-5e^{-2t} + 6) dt \\ &= \frac{5}{2} e^{-12t} + 6t + C_3 \end{aligned}$$

$$x(0) = 0 \rightarrow C_3 = -\frac{5}{2}$$

$$x(t) = \frac{5}{2} e^{-12t} + 6t - \frac{5}{2}$$

$$b) \lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} (-5e^{-2t} + 6) = 6 \text{ m/s}$$

$$c) 50 \frac{dv}{dt} = 600 - 60v$$

$$\frac{dv}{dt} = 12 - 1.2v$$

$$\int \frac{dv}{v-10} = -1.2 \int dt$$

$$\ln|v-10| = -1.2t + C_4$$

$$v(t) = C_5 e^{-6t/5} + 10$$

$$v(0) = 5 \rightarrow C_5 = -5$$

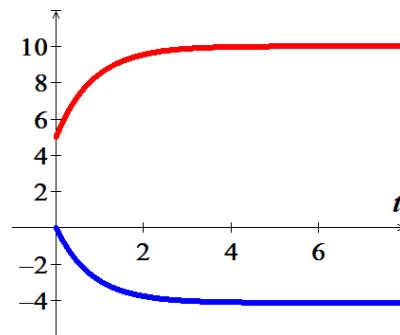
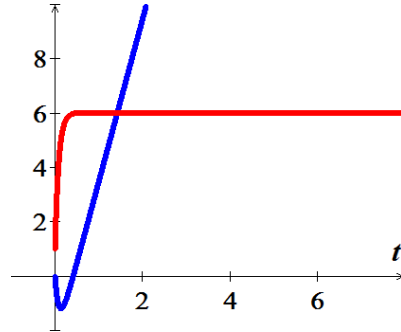
$$v(t) = -5e^{-6t/5} + 10$$

$$\begin{aligned} x(t) &= \int (-5e^{-6t/5} + 10) dt \\ &= \frac{25}{6} e^{-6t/5} + 10t + C_6 \end{aligned}$$

$$x(0) = 0 \rightarrow C_6 = -\frac{85}{6}$$

$$x(t) = \frac{25}{6} e^{-6t/5} + 10t - \frac{85}{6}$$

$$x(t) = \frac{25}{6} e^{-6t/5} - \frac{25}{6}$$



$$d) \lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} \left(-5e^{-6t/5} + 10 \right) = \underline{10 \text{ m/s}}$$

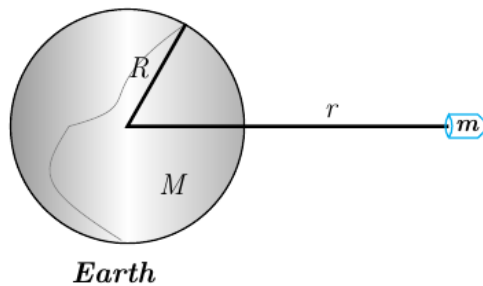
Exercise

According to Newton's law of gravitation, the attractive force between two objects varies inversely as the square of the distances between them. That is, $F_g = \frac{GM_1M_2}{r^2}$

Where M_1 and M_2 are the masses of the objects, r is the distance between them (center to center), F_g is the attractive force, and G is the constant of proportionality.

Consider a projectile of constant mass m being fired vertically from Earth.

Let t represent time and v the velocity of the projectile.



- a) Show that the motion of the projectile, under Earth's gravitational force, is governed by the equation

$$\frac{dv}{dt} = -\frac{gR^2}{r^2},$$

Where r is the distance between the projectile and the center of Earth, R is the radius of Earth, M is the mass of Earth, and $g = \frac{GM}{R^2}$.

- b) Use the fact the $\frac{dr}{dt} = v$ to obtain $v \frac{dv}{dr} = -\frac{gR^2}{r^2}$

- c) If the projectile leaves Earth's surface with velocity v_0 , show that

$$v^2 = \frac{2gR^2}{r} + v_0^2 - 2gR$$

- d) Use the result of part (c) to show that the velocity of the projectile remains positive if and only if $v_0^2 - 2gR > 0$. The velocity $v_e = \sqrt{2gR}$ is called the escape velocity?

- e) If $g = 9.81 \text{ m/sec}^2$ and $R = 6370 \text{ km}$ for Earth, what is Earth's escape velocity?

- f) If the acceleration due to gravity for the Moon is $g_m = \frac{g}{6}$ and the radius of the Moon is $R_m = 1738 \text{ km}$, what is the escape velocity of the Moon?

Solution

$$a) \quad F_g = \frac{GmM}{r^2} \quad g = \frac{GM}{R^2} \rightarrow gR^2 = GM$$

$$= \frac{mgR^2}{r^2}$$

$$m \frac{dv}{dt} = -F_g$$

$$m \frac{dv}{dt} = -\frac{mgR^2}{r^2}$$

$$\boxed{\frac{dv}{dt} = -\frac{gR^2}{r^2}}$$

$$b) \text{ Given: } \frac{dr}{dt} = v$$

$$\frac{dv}{dt} = \frac{dv}{dr} \frac{dr}{dt} = -\frac{gR^2}{r^2}$$

$$\boxed{v \frac{dv}{dr} = -\frac{gR^2}{r^2}}$$

$$c) \quad \int v dv = \int -\frac{gR^2}{r^2} dr$$

$$\frac{1}{2} v^2 = \frac{gR^2}{r} + C_1$$

$$v^2 = \frac{2gR^2}{r} + C_2$$

$$v(R) = v_0 \rightarrow v_0^2 - 2gR = C_1$$

$$\boxed{v^2 = \frac{2gR^2}{r} + v_0^2 - 2gR}$$

$$d) \quad v^2 = \frac{2gR^2}{r} + v_0^2 - 2gR > 0$$

$$\text{Since } \frac{2gR^2}{r} > 0 \Rightarrow v_0^2 - 2gR > 0$$

$$v_e = \sqrt{2gR}$$

$$e) \text{ Given: } g = 9.81 \quad R = 6370$$

$$v_e = \sqrt{2 \left(9.81 \frac{1 \text{ km}}{1000 \text{ m}} \right) (6370)} \quad v_e = \sqrt{2gR}$$

$$\approx 11.18 \text{ km/sec}$$

$$f) \text{ Given: } g_m = \frac{g}{6} = \frac{9.81}{6} \quad R_m = 1738$$

$$v_e = \sqrt{2 \left(\frac{9.81}{6} \frac{1 \text{ km}}{1000 \text{ m}} \right) (1738)} \quad v_e = \sqrt{2gR}$$

$$\approx \underline{2.38 \text{ km/sec}}$$

Exercise

A 180-*lb* skydiver drops from a hot-air balloon. After 10 *sec* of free fall, a parachute is opened. The parachute immediately introduces a drag force proportional to velocity. After an additional 4 *sec*, the parachutist reaches the ground. Assume that air resistance is negligible during free fall and that the parachute is designed so that a 200-*lb* person will reach a terminal velocity of -10 mph .

- What is the speed of the skydiver immediately before the parachute is opened?
- What is the parachutist's impact velocity?
- At what altitude was the parachute opened?
- What is the balloon's altitude?

Solution

Given: $mg = 180 \quad g = 32$

a) $0 \leq t \leq 10 \quad v(0) = 0$

$$v' = -g \rightarrow v(t) = -gt$$

$$v(10) = -(32)(10) = \underline{-320 \text{ ft/sec}}$$

b) $10 \leq t \leq 14 \quad mg = 180 \rightarrow m = \frac{180}{32} = \frac{45}{8}, \quad y(14) = 0, \quad v(0) = -320$

$$mv' + kv = -mg$$

$$\frac{mg}{k} = \frac{200}{k} = 10 \frac{\text{mi}}{\text{hr}} \frac{5280 \text{ ft}}{1 \text{ mi}} \frac{1 \text{ hr}}{3600 \text{ sec}}$$

$$\underline{k = \frac{(200)(3600)}{52800} = \frac{450}{33}}$$

$$v' + \frac{k}{m}v = -g$$

$$v' + \frac{450}{33} \left(\frac{8}{45} \right) v = -32$$

$$\begin{aligned} \frac{dv}{dt} &= - \left(\frac{80}{33} v + 32 \right) \\ &= - \frac{80}{33} \left(v + \frac{66}{5} \right) \end{aligned}$$

$$\int \frac{dv}{v + \frac{66}{5}} = - \frac{80}{33} \int dt$$

$$\ln \left| v + \frac{66}{5} \right| = - \frac{80}{33} t + C_1$$

$$v(t) = C_2 e^{-80t/33} - \frac{66}{5}$$

$$v(0) = -320 \rightarrow -320 = C_2 - \frac{66}{5} \Rightarrow C_2 = -\frac{1534}{5}$$

$$v(t) = -\frac{1534}{5}e^{-80t/33} - \frac{66}{5}$$

$$v(t=4) = -\frac{1534}{5}e^{-320/33} - \frac{66}{5} \approx 13.219 \text{ ft/sec}$$

$$\begin{aligned} c) \quad y(t) &= -\int \left(-\frac{1534}{5}e^{-80t/33} - \frac{66}{5} \right) dt \\ &= -\frac{25,311}{200}e^{-80t/33} + \frac{66}{5}t + C_3 \end{aligned}$$

$$y(0) = 0 \rightarrow C_3 = \frac{25,311}{200}$$

$$y(t) = \frac{25,311}{200}e^{-80t/33} - \frac{66}{5}t - \frac{25,311}{200}$$

$$\begin{aligned} y(4) &= \frac{25,311}{200}e^{-320/33} - \frac{66}{5}(4) - \frac{25,311}{200} \\ &\approx 179.347 \text{ ft} \end{aligned}$$

$$\begin{aligned} d) \quad h_{\text{balloon}} &= y(4) + \frac{1}{2}32(10)^2 \\ &= 179.347 + 1600 \\ &\approx 1779.347 \text{ ft} \end{aligned}$$

Exercise

Suppose water is leaking from a tank through a circular hole of area A_h at its bottom. When water leaks through a hole, friction and contraction of the stream near the hole reduce the volume of water leaving the tank per second to $cA_h\sqrt{2gh}$, where c ($0 < c < 1$) is an empirical constant.

Determine a differential equation for the height h of water at time t for the cubical tank. The radius of the hole is 2 in., $g = 32 \text{ ft/s}^2$, and the friction/contraction factor is $c = 0.6$

Solution

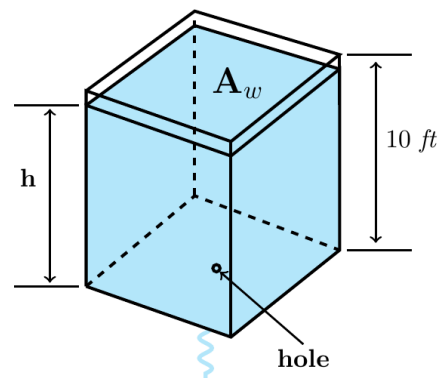
Given: $\frac{dV}{dt} = -cA_h\sqrt{2gh}$

$$A_h = \pi \left(2 \text{ in} \frac{1 \text{ ft}}{12 \text{ in}} \right)^2 = \frac{\pi}{36}$$

$$A_w = 10^2 = 100$$

Volume: $V = hA_w$

$$h = \frac{1}{A_w}V$$



$$\begin{aligned}
 \frac{dh}{dt} &= \frac{1}{A_w} \frac{dV}{dt} \\
 &= \frac{1}{A_w} \left(-cA_h \sqrt{2gh} \right) \\
 &= -\frac{A_h}{A_w} \left(c\sqrt{2gh} \right) \\
 &= -\frac{1}{100} \frac{\pi}{36} \left(c\sqrt{64h} \right) \\
 &= -\frac{\pi}{750} \sqrt{h}
 \end{aligned}$$

The negative sign indicates downward.

$$\begin{aligned}
 h^{-1/2} dh &= \frac{\pi}{750} dt \\
 \int h^{-1/2} dh &= -\frac{\pi}{750} \int dt \\
 2\sqrt{h} &= -\frac{\pi}{750} t + C \\
 h(t) &= \frac{1}{4} \left(C - \frac{\pi}{750} t \right)^2
 \end{aligned}$$

Exercise

The right-circular tank loses water out of a circular hole at its bottom.

The radius of the hole is 2 in., and $g = 32 \text{ ft/s}^2$, and the friction/contraction factor is $c = 0.6$.

- Determine a differential equation for the height h of water at time t for the cubical tank.
- Find the height in function of time.

Solution

- The volume of the tank at time t is

$$V = \frac{1}{3} \pi r^2 h$$

$$\frac{r}{h} = \frac{8}{20}$$

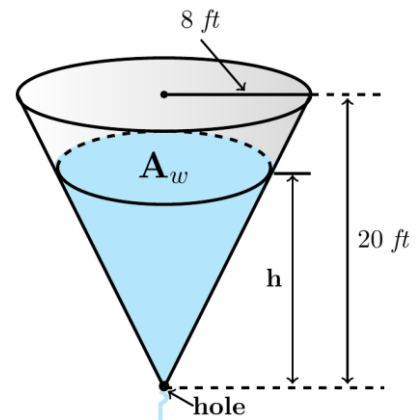
$$r = \frac{2}{5} h$$

$$\begin{aligned}
 V &= \frac{1}{3} \pi \left(\frac{2}{5} h \right)^2 h \\
 &= \frac{4}{75} \pi h^3
 \end{aligned}$$

$$\frac{dV}{dt} = \frac{4}{25} \pi h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{25}{4\pi h^2} \frac{dV}{dt}$$

$$\frac{dV}{dt} = -cA_h \sqrt{2gh}$$



$$\begin{aligned}
&= -\frac{25c}{4\pi h^2} A_h \sqrt{2gh} \\
&= -\frac{25}{4\pi h^2} \frac{6}{10} \frac{\pi}{36} \sqrt{64h} \\
&= -\frac{5}{6\pi h^{3/2}}
\end{aligned}$$

$$A_h = \pi \left(2 \text{in} \frac{1 \text{ft}}{12 \text{in}} \right)^2 = \frac{\pi}{36}$$

$$b) \frac{dh}{dt} = -\frac{5}{6\pi h^{3/2}}$$

$$\int h^{3/2} dh = -\frac{5}{6\pi} \int dt$$

$$\frac{2}{5} h^{5/2} = -\frac{5}{6\pi} t + C_1$$

$$h^{5/2} = C - \frac{25}{12\pi} t$$

$$h(t) = \left(C - \frac{25}{12\pi} t \right)^{2/5}$$

Exercise

In meteorology, the term virga refers to falling raindrops or ice particles that evaporate before they reach the ground. Assume that a typical raindrop is spherical. Starting at some time, which we can designate as $t = 0$, the raindrop of radius r_0 falls from rest from a cloud and begins to evaporate.

- If it is assumed that a raindrop evaporates in such a manner that its shape remains spherical, then it also makes sense to assume that the rate at which the raindrop evaporates – that is, the rate at which it loses mass – is proportional to its surface area. Show that this latter assumption implies that the rate at which the radius r of the raindrop decreases is a constant. Find $r(t)$.
- If the positive direction is downward, construct a mathematical model for the velocity v of the falling raindrop at time $t > 0$. Ignore air resistance.

Solution

- If ρ is the mass density of the raindrop, then $m = \rho V$

$$\begin{aligned}
\frac{dm}{dt} &= \rho \frac{dV}{dt} \\
&= \rho \frac{d}{dt} \left(\frac{4}{3} \pi r^3 \right) \\
&= 4\rho\pi r^2 \frac{dr}{dt}
\end{aligned}$$

$$\begin{aligned}
\frac{dm}{dt} &= kS = k(4\pi r^2) \\
4\pi r^2 k &= 4\rho\pi r^2 \frac{dr}{dt}
\end{aligned}$$

$$k = \rho \frac{dr}{dt}$$

$$\int dr = \frac{k}{\rho} \int dt$$

$$r(t) = \frac{k}{\rho} t + C$$

$$r(0) = r_0 \quad r_0 = C$$

$$r(t) = \frac{k}{\rho} t + r_0$$

b) From Newton's second law

$$\frac{d}{dt}(mv) = mg$$

$$m \frac{dv}{dt} + v \frac{dm}{dt} = mg$$

$$\rho \left(\frac{4}{3} \pi r^3 \right) \frac{dv}{dt} + v \left(k 4 \pi r^2 \right) = \rho \left(\frac{4}{3} \pi r^3 \right) g$$

$$\rho r \frac{dv}{dt} + 3kv = \rho r g$$

$$\frac{dv}{dt} + \frac{3k}{r\rho} v = g \quad r = \frac{k}{\rho} t + r_0 = \frac{kt + \rho r_0}{\rho}$$

$$\frac{dv}{dt} + \frac{3k}{\rho r_0 + kt} v = g$$

Exercise

A horizontal cylindrical tank of length 9 ft, and radius 5 ft, is filled with oil. At $t = 0$ a plug at the lowest point of the tank is removed and a flow results.

Find y the depth of the oil in the tank at any time t while the tank is draining. The constriction coefficient is

$$k = \frac{1}{15}$$

Solution

$$x^2 + (y - 5)^2 = 25$$

$$x^2 = 25 - (y^2 - 10y + 25)$$

$$x = \pm \sqrt{10y - y^2}$$

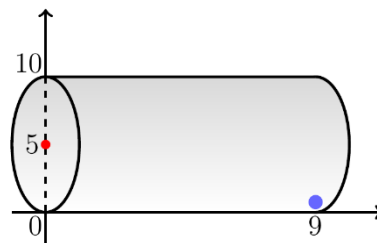
$$dV = \text{Surface} \times dy$$

$$\text{Surface} = (9) 2\sqrt{10y - y^2} = 18\sqrt{10y - y^2}$$

When the liquid of volume V flows out the rate at which it flows through that opening is $k\sqrt{y}$

$$-\frac{dV}{dt} = k\sqrt{y}$$

$$18\sqrt{10y - y^2} \frac{dy}{dt} = -\frac{1}{15}\sqrt{y}$$



$$18 \frac{\sqrt{10y - y^2}}{\sqrt{y}} dy = -\frac{1}{15} dt$$

$$18\sqrt{10-y} dy = -\frac{1}{15} dt$$

$$-\int (10-y)^{1/2} d(10-y) = -\frac{1}{270} \int dt$$

$$\frac{2}{3}(10-y)^{3/2} = \frac{1}{270}t + C$$

$$(10-y)^{3/2} = \frac{1}{180}t + C_1$$

$$y(0)=10 \rightarrow \underline{0=C_1}$$

$$(10-y)^{3/2} = \frac{1}{180}t$$

$$10-y = \left(\frac{1}{180}t\right)^{2/3}$$

$$\underline{y(t) = 10 - \frac{1}{1,080\sqrt{5}}t^{3/2}}$$