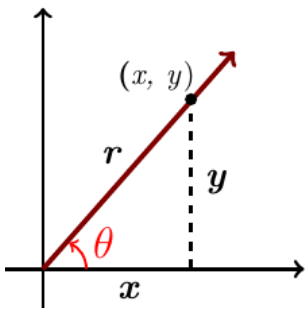
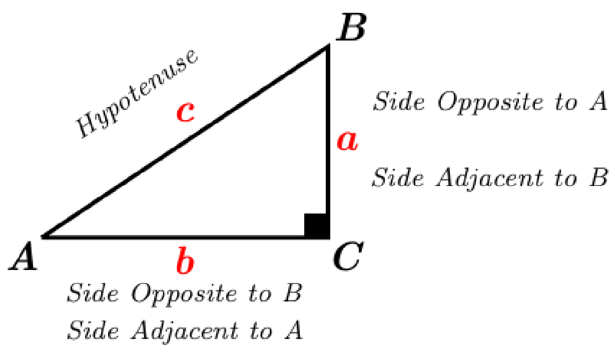
Section 2.2 – Trigonometric Functions

Let (x, y) be a point on the terminal side of an angle θ in standard position

The distance from the point to the origin is given by: $r = \sqrt{x^2 + y^2}$

Six Trigonometry Functions





$$\sin\theta = \frac{Opposite}{Hypotenuse} = \frac{opp}{hyp} = \frac{y}{r}$$

$$\csc\theta = \frac{hyp}{opp} = \frac{1}{\sin\theta} = \frac{r}{y}$$

$$\cos\theta = \frac{Adjacent}{Hypotenuse} = \frac{adj}{hyp} = \frac{x}{r}$$

$$\sec\theta = \frac{hyp}{adj} = \frac{1}{\cos\theta} = \frac{r}{x}$$

$$\tan\theta = \frac{opp}{adj} = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$$

$$\cot\theta = \frac{adj}{opp} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} = \frac{x}{y}$$

$$\sin A = \frac{a}{c} = \cos B$$

$$\csc A = \frac{c}{a} = \sec B$$

$$\cos A = \frac{b}{c} = \sin B$$

$$\sec A = \frac{c}{b} = \csc B$$

$$\tan A = \frac{a}{b} = \cot B$$

$$\cot A = \frac{b}{a} = \tan B$$

Example

Find the six trigonometry functions of θ if θ is in the standard position and the point $(8, 15)$ is on the terminal side of θ.

Solution

$$r = \sqrt{8^2 + 15^2} \qquad r = \sqrt{x^2 + y^2}$$

$$= 17$$

$$\sin\theta = \frac{y}{r} = \frac{15}{17} \qquad \cos\theta = \frac{x}{r} = \frac{8}{17} \qquad \tan\theta = \frac{y}{x} = \frac{15}{8}$$

$$\csc\theta = \frac{r}{y} = \frac{17}{15} \qquad \sec\theta = \frac{r}{x} = \frac{17}{8} \qquad \cot\theta = \frac{x}{y} = \frac{8}{15}$$

Example

Which will be greater, tan 30° or tan 40°? How large could $\tan\theta$ be?

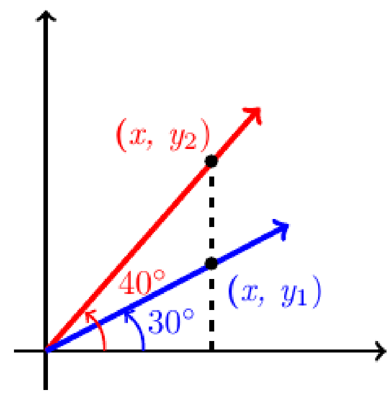
Solution

$$\tan 30^\circ = \frac{y_1}{x}$$

$$\tan 40^\circ = \frac{y_2}{x}$$

$$Ratio: \frac{y_2}{x} > \frac{y_1}{x}$$

$\rightarrow$ tan 40° > tan 30°



No limit as to how large $\tan\theta$ can be.

Example

If $\cos\theta = \frac{\sqrt{3}}{2}$, and θ is **Q**IV, find $\sin\theta$ and $\tan\theta$.

Solution

$$\cos\theta = \frac{\sqrt{3}}{2} = \frac{x}{r} \quad \rightarrow x = \sqrt{3}, \quad r = 2$$

$$\begin{aligned} y &= \sqrt{r^2 - x^2} \\ &= \sqrt{2^2 - \left(\sqrt{3}\right)^2} \\ &= \sqrt{4-3} \\ &= 1 \end{aligned}$$

Since θ is Q IV $\Rightarrow$ $y = -1$

$$\sin\theta = -\frac{1}{2} \qquad\qquad \sin\theta = \frac{y}{r}$$

$$\begin{aligned} \tan\theta &= \frac{-1}{\sqrt{3}} \qquad\qquad \tan\theta = \frac{y}{x} \\ &= -\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} \\ &= -\frac{\sqrt{3}}{3} \end{aligned}$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Ratio Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$x^2 + y^2 = r^2$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1 \quad \Rightarrow \quad \boxed{\cos^2\theta + \sin^2\theta = 1}$$

Solving for $\cos\theta$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$$

Solving for $\sin\theta$

$$\sin^2\theta = 1 - \cos^2\theta \quad \Rightarrow \quad \boxed{\sin\theta = \pm\sqrt{1 - \cos^2\theta}}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\left(\frac{\cos\theta}{\cos\theta}\right)^2 + \left(\frac{\sin\theta}{\cos\theta}\right)^2 = \left(\frac{1}{\cos\theta}\right)^2$$

$$\boxed{1 + \tan^2\theta = \sec^2\theta}$$

$\cos^2\theta + \sin^2\theta = 1$
$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$
$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$
$1 + \tan^2\theta = \sec^2\theta$
$1 + \cot^2\theta = \csc^2\theta$

Example

Write $\tan\theta$ in terms of $\sin\theta$.

Solution

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\sin\theta}{\pm\sqrt{1-\sin^2\theta}}$$

$$= \pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$$

Example

Simplify the expression $\sqrt{x^2+9}$ as much as possible after substituting $3\tan\theta$ for x

Solution

$$x = 3\tan\theta$$

$$\sqrt{x^2+9} = \sqrt{(3\tan\theta)^2+9}$$

$$= \sqrt{9\tan^2\theta+9}$$

$$= \sqrt{9\left(\tan^2\theta+1\right)}$$

$$= 3\sqrt{\sec^2\theta}$$

$$\underline{= 3\sec\theta}$$

Example

Triangle *ABC* is a right triangle with $C = 90°$. If $a = 6$ and $c = 10$, find the six trigonometric functions of *A*.

Solution

$$b = \sqrt{c^2 - a^2} = \sqrt{10^2 - 6^2} = 8$$

$$6,\ b \ \to\ 10 \ \to\ 2(3,\ b \ \to\ 5)$$

$$\underline{b = 4 \,|}$$

$\sin A = \frac{a}{c} = \frac{3}{5}$	$\cos A = \frac{b}{c} = \frac{8}{10} = \frac{4}{5}$	$\tan A = \frac{a}{b} = \frac{6}{8} = \frac{3}{4}$
$\csc A = \frac{5}{3}$	$\sec A = \frac{5}{4}$	$\cot A = \frac{4}{3}$

$$if \quad A + B = 90° \quad \Rightarrow \begin{cases} \sin A = \cos B \\ \sec A = \csc B \\ \tan A = \cot B \end{cases}$$

Cofunction Theorem

A trigonometric function of an angle is always equal to the cofunction of the complement of the angle.

Example

Write each function in terms of its cofunction

a) $\cos 52°$

Solution

$$\cos 52° = \sin(90° - 52°) = \sin 38°$$

b) $\tan 71°$

Solution

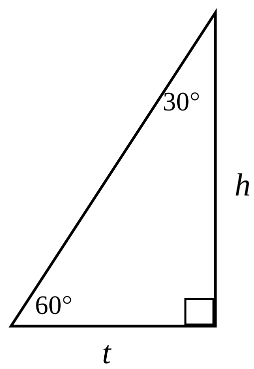
$$\tan 71° = \cot(90° - 71°) = \cot 19°$$

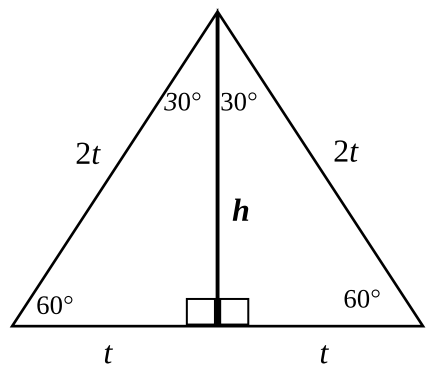
c) $\sec 24°$

Solution

$$\sec 24° = \csc(90° - 24°) = \csc 66°$$

30° - 60° - 90° *Triangle*





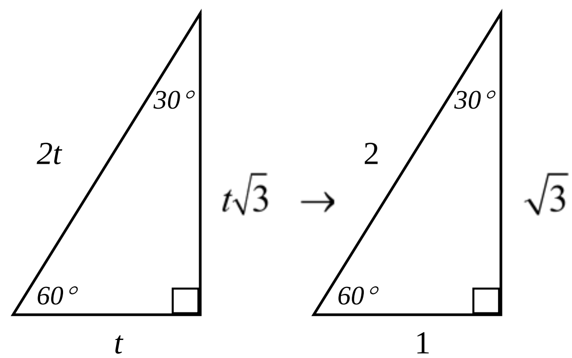
$$t^2 + h^2 = (2t)^2$$

$$t^2 + h^2 = 4t^2$$

$$h^2 = 4t^2 - t^2$$

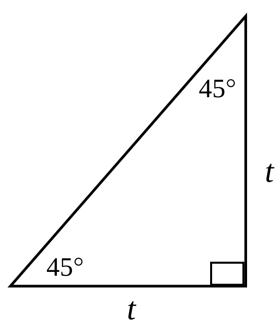
$$h^2 = 3t^2$$

$$h = t\sqrt{3}$$



$$\Rightarrow \boxed{\sin 60° = \frac{\sqrt{3}}{2}}$$

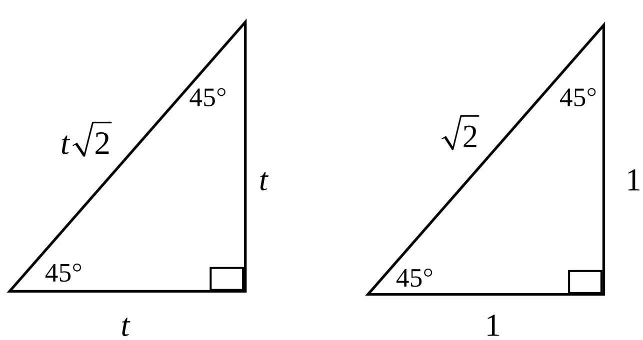
45° - 45° - 90° *Triangle*



$$hypotenuse^2 = t^2 + t^2$$

$$hypotenuse = \sqrt{2t^2}$$

$$hypotenuse = t\sqrt{2}$$

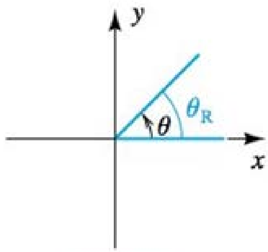


$$\Rightarrow \boxed{\cos 45° = \frac{1}{\sqrt{2}}}$$

Reference Angle

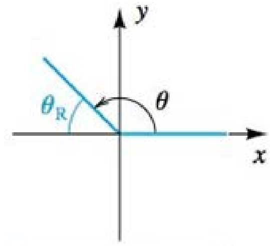
Definition

The reference angle or related angle for any angle θ in standard position ifs the positive acute angle between the terminal side of θ and the x-axis, and it is denoted $\hat{\theta}$



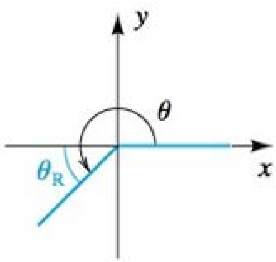
$\theta_R = \theta$

Quadrant **I**



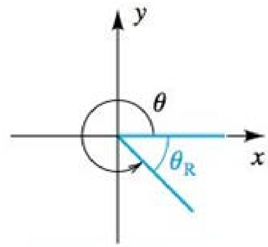
$\theta_R = 180° - \theta$
$= \pi - \theta$

Quadrant **II**



$\theta_R = \theta - 180°$
$= \theta - \pi$

Quadrant **III**



$\theta_R = 360° - \theta$
$= 2\pi - \theta$

Quadrant **IV**

Example

Find the exact value of $\sin 240°$

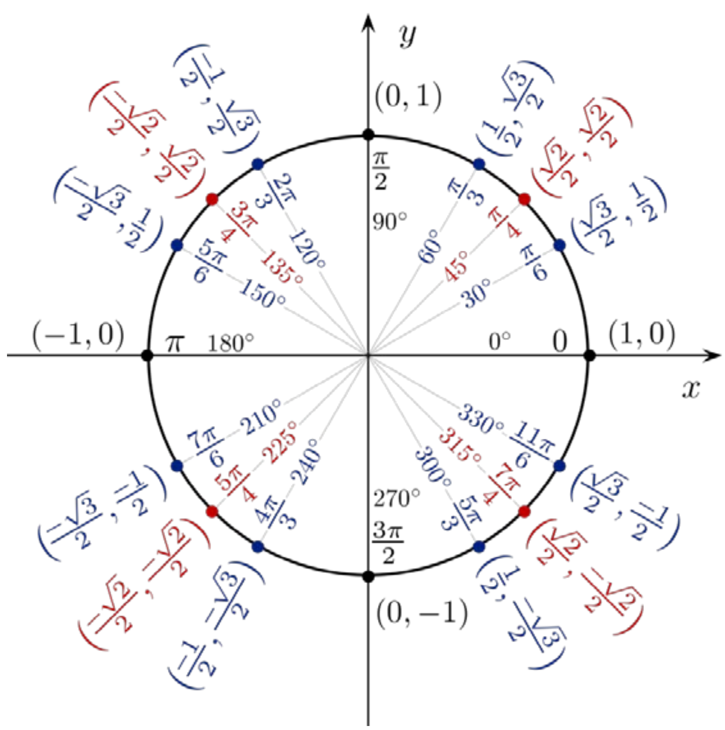
Solution

$\hat{\theta} = 240° - 180° = 60°$

$\rightarrow 240° \in QIII$

$\sin 240° = -\sin 60°$

$= -\frac{\sqrt{3}}{2}$



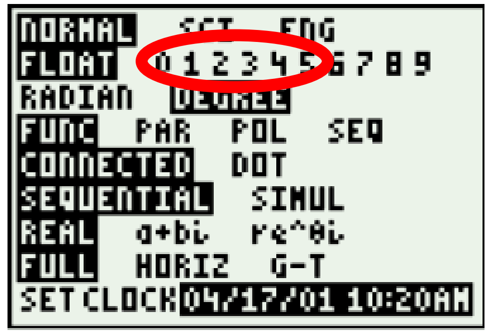
Approximation- Simply using calculator

$\sin 250° \approx -0.9397$

$\cos 250° \approx -0.3420$

$\tan 250° \approx 2.7475$

$\csc 250° = \dfrac{1}{\sin 250°} \approx -1.0642$



*To find the angle by using the inverse trigonometry functions, always enter a **positive** value.*

Example

Find θ if $\sin\theta = -0.5592$ and θ terminates in QIII with $0° \le \theta < 360°$.

Solution

$\hat{\theta} = \sin^{-1} 0.5592 \approx 34°$

$\theta \in$ QIII

$\Rightarrow \theta = 180° + 34° = 214°$

Example

Find θ to the nearest degree if $\cot\theta = -1.6003$ and θ terminates in QII with $0° \le \theta < 360°$.

Solution

$\cot\theta = -1.6003 = \dfrac{1}{\tan\theta}$

$\tan\theta = \dfrac{1}{-1.6003}$

$\hat{\theta} = \tan^{-1}\dfrac{1}{1.6003} = 32°$

$\theta \in$ QII $\quad \Rightarrow \theta = 180° - 32° = 148°$

Exercise ***Section* 2.2 – Trigonometric Functions**

(**1 – 16**) Find the ***six*** trigonometry functions of θ if θ is in the standard position and the given point is on the terminal side of θ.

1. $(-2,\ 3)$
2. $(-3,\ -4)$
3. $(-3,\ 0)$
4. $(12,\ -5)$
5. $(5,\ -12)$
6. $(9,\ -12)$
7. $(16,\ -12)$
8. $(15,\ -8)$
9. $(-6,\ 8)$
10. $(-15,\ 8)$
11. $(-7,\ 24)$
12. $(10,\ -24)$
13. $(7,\ 24)$
14. $(-7,\ -24)$
15. $(-24,\ -7)$
16. $(24,\ -10)$

17. Find the values of the six trigonometric functions for an angle of 90°.

18. Indicate the two quadrants θ could terminate in if $\cos\theta = \frac{1}{2}$

19. Indicate the two quadrants θ could terminate in if $\csc\theta = -2.45$

(**20 – 38**) Find the remaining trigonometric function of θ if

20. $\sin\theta = \frac{12}{13}$ and θ terminates in QI.

21. $\cot\theta = -2$ and θ terminates in QII.

22. $\tan\theta = \frac{3}{4}$ and θ terminates in $QIII$.

23. $\cos\theta = \frac{24}{25}$ and θ terminates in QIV.

24. $\cos\theta = \frac{\sqrt{3}}{2}$ and θ is terminates in QIV.

25. $\tan\theta = -\frac{1}{2}$ and $\cos\theta > 0$

26. $\cos\theta = \frac{3}{5}$ & $\theta \in QI$

27. $\cos\theta = -\frac{4}{5}$ & $\theta \in QII$

28. $\sin\theta = -\frac{3}{5}$ & $\theta \in QIII$

29. $\sin\theta = -\frac{3}{5}$ & $\theta \in QIV$

30. $\cos\theta = -\frac{12}{13}$ & $\theta \in QIII$

31. $\cos\theta = -\frac{5}{13}$ & $\theta \in QII$

32. $\cos\theta = \frac{12}{13}$ & $\theta \in QIV$

33. $\sin\theta = -\frac{8}{17}$ & $\theta \in QIII$

34. $\cos\theta = -\frac{15}{17}$ & $\theta \in QII$

35. $\cos\theta = -\frac{8}{17}$ & $\theta \in QII$

36. $\cos\theta = -\frac{7}{25}$ & $\theta \in QII$

37. $\sin\theta = -\frac{7}{25}$ & $\theta \in QIII$

38. $\sin\theta = -\frac{24}{25}$ & $\theta \in QIV$

39. If $\sin\theta = -\frac{5}{13}$, and θ is QIII, find $\cos\theta$ and $\tan\theta$.

40. If $\cos\theta = \frac{3}{5}$, and θ is QIV, find $\sin\theta$ and $\tan\theta$.

41. Use the reciprocal identities if $\cos\theta = \frac{\sqrt{3}}{2}$ find $\sec\theta$

42. Find $\cos\theta$, given that $\sec\theta = \frac{5}{3}$

43. Find $\sin\theta$, given that $\csc\theta = -\frac{\sqrt{12}}{2}$

44. Use a ratio identity to find $\tan\theta$ if $\sin\theta = \frac{3}{5}$ and $\cos\theta = -\frac{4}{5}$

45. If $\cos\theta = -\frac{1}{2}$ and θ terminates in QII, find $\sin\theta$

46. If $\sin\theta = \frac{3}{5}$ and θ terminated in QII, find $\cos\theta$ and $\tan\theta$.

47. Find $\tan\theta$ if $\sin\theta = \frac{1}{3}$ and θ terminates in QI

48. Find the remaining trigonometric ratios of θ, if $\sec\theta = -3$ and $\theta \in QIII$

49. Using the calculator and rounding your answer to the nearest hundredth, find the remaining trigonometric ratios of θ if $\csc\theta = -2.45$ and $\theta \in QIII$.

50. Write $\frac{\sec\theta}{\csc\theta}$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

51. Write $\cot\theta - \csc\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

52. Write $\frac{\sin\theta}{\cos\theta} + \frac{1}{\sin\theta}$ in terms of $\sin\theta$ and/or $\cos\theta$, and then simplify if possible.

53. Write $\sin\theta\cot\theta + \cos\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

54. Multiply $(1-\cos\theta)(1+\cos\theta)$

55. Multiply $(\sin\theta + 2)(\sin\theta - 5)$

56. Simplify the expression $\sqrt{25 - x^2}$ as much as possible after substituting $5\sin\theta$ for x.

57. Simplify the expression $\sqrt{4x^2 + 16}$ as much as possible after substituting $2\tan\theta$ for x

(**58 – 60**) Simplify by using the table

58. $5\sin^2 30°$ **59.** $\sin^2 60° + \cos^2 60°$ **60.** $(\tan 45° + \tan 60°)^2$

61. Find θ if $\sin\theta = -\frac{1}{2}$ and θ terminates in QIII with $0° \le \theta \le 360°$.

62. Find θ to the nearest degree if $\sec\theta = 3.8637$ and θ terminates in ***Q***IV with $0° \le \theta < 360°$.

(**63 – 67**) Find the exact value of

63. $\cos 225°$ **64.** $\csc 300°$ **65.** $\tan 315°$ **66.** $\cos 420°$ **67.** $\cot 480°$

(**68 – 70**) Use the calculator to find the value of

68. $\csc 166.7°$ **69.** $\sec 590.9°$ **70.** $\tan 195°\,10'$

71. Use the calculator to find θ to the nearest degree if $\sin\theta = -0.3090$ with $\theta \in \boldsymbol{Q}\text{IV}$ with $0° \le \theta < 360°$

72. Use the calculator to find θ to the nearest degree if $\cos\theta = -0.7660$ with $\theta \in \boldsymbol{Q}\text{III}$ with $0° \le \theta < 360°$

73. Use the calculator to find θ to the nearest degree if $\sec\theta = -3.4159$ with $\theta \in \boldsymbol{Q}\text{II}$ with $0° \le \theta < 360°$

74. Find θ to the nearest tenth of a degree if $\tan\theta = -0.8541$ and θ terminates in $\boldsymbol{Q}\text{IV}$ with $0° \le \theta < 360°$

75. Use the calculator to find θ to the nearest degree if $\sin\theta = 0.49368329$ with $\theta \in \boldsymbol{Q}\text{II}$ with $0° \le \theta < 360°$