

## Solution

### Section 1.4 – Inverse Matrices - Finding $A^{-1}$

#### Exercise

Apply Gauss-Jordan method to find the inverse of this triangular “Pascal matrix”

*Triangular Pascal matrix*  $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$

#### Solution

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 3 & 1 & -1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ \\ R_3 - 2R_2 \\ R_4 - 3R_2 \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & 1 & 2 & -3 & 0 & 1 \end{array} \right] \begin{array}{l} \\ \\ \\ R_4 - 3R_3 \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{array} \right]$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$

✚ The inverse matrix  $A^{-1}$  looks like  $A$ , except odd-numbered diagonals are multiplied by -1.

### ***Exercise***

If  $A$  is invertible and  $AB = AC$ , prove that  $B = C$

### **Solution**

$$AB = AC$$

*Multiply by  $A^{-1}$  both sides.*

$$A^{-1}(AB) = A^{-1}(AC)$$

*Multiplication is associative*

$$(A^{-1}A)B = (A^{-1}A)C$$

$$A^{-1}A = I$$

$$IB = IC$$

$$\boxed{B = C}$$

### ***Exercise***

If  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ , find two matrices  $B \neq C$  such that  $AB = AC$

### **Solution**

$$\text{Let } B = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\boxed{B \neq C \Rightarrow AB = AC}$$

### Exercise

If  $A$  has **row** 1 + **row** 2 = **row** 3, show that  $A$  is not invertible

- a) Explain why  $Ax = (1, 0, 0)$  can't have a solution.
- b) Which right sides  $(b_1, b_2, b_3)$  might allow a solution to  $Ax = b$
- c) What happens to **row** 3 in elimination?

### Solution

- a) Let  $A_1, A_2, A_3$  be the row vectors of  $A$  and  $x$  is a solution to  $Ax = (1, 0, 0)$ .

Then  $A_1 \cdot x = 1, A_2 \cdot x = 0, A_3 \cdot x = 0$ .

Since  $A_1 + A_2 = A_3$

Means  $A_1 \cdot x + A_2 \cdot x = A_3 \cdot x$

Implies  $1 + 0 = 0$  a contradiction

- b) If  $Ax = (b_1, b_2, b_3) \Rightarrow A_1 \cdot x = b_1, A_2 \cdot x = b_2, A_3 \cdot x = b_3$

Since  $A_1 + A_2 = A_3$

$A_1 \cdot x + A_2 \cdot x = A_3 \cdot x$

$\Rightarrow b_1 + b_2 = b_3$

- c) In the elimination matrix, the third row will be zero.

### Exercise

True or false (with a counterexample if false and a reason if true):

- a) A 4 by 4 matrix with a row of zeros is not invertible.
- b) A matrix with 1's down the main diagonal is invertible.
- c) If  $A$  is invertible then  $A^{-1}$  is invertible.
- d) If  $A$  is invertible then  $A^2$  is invertible.

### Solution

- a) True, because it can have at most 3 pivots.

- b) False, if the matrix:  $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$  and only has 2 pivots, thus is not invertible.

- c) True, If  $A$  is invertible then necessarily  $A^{-1}$  is invertible.

d) True,  $A^2x = 0$  where  $x$  is nonzero matrix.

$$A^{-1}A^2x = (A^{-1}A)Ax = IAx = Ax = 0$$

Since  $A$  is invertible, this can only be true if  $x$  was zero to begin with. Thus  $A^2$  must also be invertible.

### Exercise

Do there exist 2 by 2 matrices  $A$  and  $B$  with real entries such that  $AB - BA = I$ , where  $I$  is the identity matrix?

### Solution

$$\text{Let } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

$$BA = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{pmatrix}$$

$$\begin{aligned} AB - BA &= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} - \begin{pmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{pmatrix} \\ &= \begin{pmatrix} bg - cf & af + bh - be - df \\ ce + dg - ag - ch & cf - bg \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{cases} bg - cf = 1 \\ af + bh - be - df = 0 \\ ce + dg - ag - ch = 0 \\ cf - bg = 1 \end{cases} \rightarrow \begin{cases} bg - cf = 1 \\ cf - bg = 1 \\ \hline 0 = 2 \end{cases}$$

Therefore,  $AB - BA \neq I$  for any 2 by 2 matrices.

### Exercise

If  $B$  is the inverse of  $A^2$ , show that  $AB$  is the inverse of  $A$ .

### Solution

Since  $B$  is the inverse of  $A^2$  that implies:  $B = (A^2)^{-1} = (AA)^{-1} = A^{-1}A^{-1}$

Show that  $AB$  is the inverse of  $A$

$$\begin{aligned}(AB)A &= \left( A \left( A^{-1}A^{-1} \right) \right) A \\ &= \left( (AA^{-1})A^{-1} \right) A \\ &= (IA^{-1})A \\ &= A^{-1}A \\ &= I\end{aligned}$$

Therefore,  $AB$  is the inverse of  $A$ .

### Exercise

Find and check the inverses (assuming they exist) of these block matrices.

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}$$

### Solution

$$\begin{aligned}\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ \begin{bmatrix} I & 0 \\ C+A & I \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \Rightarrow C+A=0 \rightarrow A = -C \\ \begin{pmatrix} I & 0 \\ C & I \end{pmatrix}^{-1} &= \begin{pmatrix} I & 0 \\ -C & I \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \begin{bmatrix} E & 0 \\ F & G \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ \begin{bmatrix} AE & 0 \\ CE+DF & DG \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\end{aligned}$$

$$\Rightarrow \begin{cases} AE = I \\ CE + DF = 0 \\ DG = I \end{cases} \Rightarrow \begin{cases} E = A^{-1} \\ G = D^{-1} \end{cases}$$

$$\begin{aligned} CE + DF = 0 &\rightarrow CA^{-1} + DF = 0 \\ DF &= -CA^{-1} \\ D^{-1}DF &= -D^{-1}CA^{-1} \\ IF &= -D^{-1}CA^{-1} \\ F &= -D^{-1}CA^{-1} \end{aligned}$$

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{pmatrix}$$

$$\begin{bmatrix} 0 & I \\ I & D \end{bmatrix} \begin{bmatrix} A & I \\ I & B \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & B \\ A + D & I + DB \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\rightarrow \begin{cases} B = 0 \\ A + D = 0 \\ I + DB = I \end{cases} \Rightarrow \begin{cases} A = -D \\ DB = 0 \end{cases}$$

$$\begin{pmatrix} 0 & I \\ I & D \end{pmatrix}^{-1} = \begin{pmatrix} -D & I \\ I & 0 \end{pmatrix}$$

### ***Exercise***

For which three numbers  $c$  is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

### **Solution**

$$c = 0, A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 7 & 0 \end{bmatrix} \text{ (zero column 2 / row 2)}$$

$$c=2, A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 8 & 7 & 2 \end{bmatrix} \text{ (equal rows)}$$

$$c=7, A = \begin{bmatrix} 2 & 7 & 7 \\ 7 & 7 & 7 \\ 8 & 7 & 7 \end{bmatrix} \text{ (equal columns)}$$

### ***Exercise***

Find  $A^{-1}$  and  $B^{-1}$  (if they exist) by elimination.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

### **Solution**

$$\left( \begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array} \right) \frac{1}{2}R_1$$

$$\left( \begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array} \right) \frac{2}{3}R_2$$

$$\left( \begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - \frac{1}{2}R_2 \\ R_3 - \frac{1}{2}R_2 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array} \right) \frac{3}{4}R_3$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array} \right) \begin{array}{l} R_1 - \frac{1}{3}R_3 \\ R_2 - \frac{1}{3}R_3 \\ \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$\left( \begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array} \right) \frac{1}{2}R_1$$

$$\left( \begin{array}{ccc|ccc} 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 + R_1 \\ R_3 + R_1 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array} \right) R_3 + R_2$$

$$\left( \begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ \textcolor{red}{0} & \textcolor{red}{0} & \textcolor{red}{0} & 0 & 1 & 1 \end{array} \right)$$

$B^{-1}$  doesn't exist, and if we add the columns in  $B$ , the result is zero.



### ***Exercise***

Find  $A^{-1}$  using the Gauss-Jordan method, which has a remarkable inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

### **Solution**

$$\left( \begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) R_1 + R_2$$

$$\left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) R_2 + R_3$$

$$\left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) R_3 + R_4$$

$$\left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

### Exercise

Find the inverse.

$$a) \begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$$

$$b) \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$$

$$c) \begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$$

$$d) \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$e) \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$$

$$f) \begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$g) \begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}$$

### Solution

$$a) \begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{12} & \frac{1}{6} \\ \frac{1}{8} & \frac{1}{4} \end{bmatrix}$$

$$\begin{aligned} b) \quad A^{-1} &= \frac{1}{7-8} \begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix} \\ &= - \begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} c) \quad A^{-1} &= \frac{1}{-15-24} \begin{bmatrix} 5 & -6 \\ -4 & -3 \end{bmatrix} \\ &= -\frac{1}{39} \begin{bmatrix} 5 & -6 \\ -4 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{5}{39} & \frac{2}{13} \\ \frac{4}{39} & \frac{1}{13} \end{bmatrix} \end{aligned}$$

$$d) \left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad R_3 - R_1$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array} \right] \quad R_3 - R_2$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array} \right] \quad -\frac{1}{2}R_3$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array} \right] \quad \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$e) \left[ \begin{array}{ccc} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{array} \right]^{-1} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{pmatrix}$$

$$f) \left[ \begin{array}{ccc} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{array} \right]^{-1} = \begin{pmatrix} \frac{\sqrt{2}}{26} & -\frac{3\sqrt{2}}{26} & 0 \\ \frac{2\sqrt{2}}{13} & \frac{\sqrt{2}}{26} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$g) \left[ \begin{array}{cccc} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{array} \right]^{-1} = \text{doesn't exist} \quad \text{This matrix is *singular*}$$

### Exercise

Show that  $A$  is not invertible for any values of the entries

$$A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

### Solution

Since the matrix  $A$  had zero's on its diagonals, therefore  $A$  is not invertible.

### Exercise

Prove that if  $A$  is an invertible matrix and  $B$  is row equivalent to  $A$ , then  $B$  is also invertible.

### Solution

Since  $B$  is row equivalent to  $A$ , there exist some elementary matrices  $E_1, E_2, \dots, E_n$  such that  $B = E_n \dots E_1 A$ . Because  $E_1, E_2, \dots, E_n$  and  $A$  are invertible, then  $B$  is also invertible.

### Exercise

Determine if the given matrix has an inverse, and find the inverse if it exists. Check your answer by multiplying  $A \cdot A^{-1} = I$

$$a) \begin{bmatrix} 2 & 3 \\ -3 & -5 \end{bmatrix} \quad b) \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}$$

### Solution

$$a) \quad 2(-5) - 3(-3) = -10 + 9 = -1$$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} -5 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ -3 & -2 \end{bmatrix}$$

$$AA^{-1} = \begin{pmatrix} 2 & 3 \\ -3 & -5 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$b) \quad \left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array} \right] \quad R_3 - 2R_1$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 3 & 3 & -2 & 0 & 1 \end{array} \right] \quad R_3 - 3R_2$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & * & * & * \end{array} \right]$$

The inverse matrix doesn't exist

### ***Exercise***

Show that the inverse of  $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$  is  $\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$

### **Solution**

$$\begin{aligned} & \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} \\ &= \begin{bmatrix} (\cos \theta)\cos(-\theta) - (\sin \theta)\sin(-\theta) & (\cos \theta)\sin(-\theta) - (\sin \theta)\cos(-\theta) \\ (-\sin \theta)\cos(-\theta) - (\cos \theta)\sin(-\theta) & (-\sin \theta)\sin(-\theta) + (\cos \theta)\cos(-\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta \cos \theta + \sin \theta \sin \theta & -\cos \theta \sin \theta - \sin \theta \cos \theta \\ -\sin \theta \cos \theta + \cos \theta \sin \theta & \sin \theta \sin \theta + \cos \theta \cos \theta \end{bmatrix} \quad \begin{cases} \cos(-\theta) = \cos \theta & (\text{even}) \\ \sin(-\theta) = -\sin \theta & (\text{odd}) \end{cases} \\ &= \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & 0 \\ 0 & \sin^2 \theta + \cos^2 \theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$