SOLUTION *Section* 3.7 – Power Series

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of *x* does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=0}^{\infty} x^n$$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{x^n}\right| = |x| < 1 \quad \Rightarrow \quad -1 < x < 1$$

When $x = 1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} 1$

and $x = -1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} (-1)^n$ the series diverges.

a) The radius is 1; the interval of converges $-1 < x < 1$

b) The interval of absolute convergence is $-1 < x < 1$

c) There are no values for which the series converges conditionally

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of *x* does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=0}^{\infty} (x+5)^n$$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(x+5)^{n+1}}{(x+5)^n}\right|$$

$$= |x+5| < 1$$

$$-6 < x < -4$$

When $x = -6 \quad \Rightarrow \quad \sum_{n=0}^{\infty} (-1)^n$

and $x=-4 \quad \Rightarrow \quad \sum_{n=0}^{\infty} 1$ the series diverges.

a) The radius is 1; the interval of converges $-6 < x < -4$

b) The interval of absolute convergence is $-6 < x < -4$

c) There are no values for which the series converges conditionally.

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of *x* does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(3x-2)^{n+1}}{n+1} \cdot \frac{n}{(3x-2)^n}\right|$$

$$= \frac{n}{n+1}|3x-2| < 1$$

$$\lim_{n\to\infty} \frac{n}{n+1}|3x-2| < 1$$

$$|3x-2| < 1$$

$$-1 < 3x-2 < 1$$

$$1 < 3x < 3$$

$$\boxed{\frac{1}{3} < x < 1}$$

When $x=\frac{1}{3} \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ which is the alternating harmonic series and is conditionally convergent.

$x=1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{1}{n}$ the series diverges harmonic.

a) The radius is $\frac{1}{3}$; the interval of converges $\frac{1}{3} \le x < 1$

b) The interval of absolute convergence is $\frac{1}{3} < x < 1$

c) The series converges conditionally at $x=\frac{1}{3}$

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of x does the series converge
(***b***) absolutely,
(***c***) conditionally?

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(x-2)^{n+1}}{10^{n+1}} \cdot \frac{10^n}{(x-2)^n}\right|$$

$$= \frac{|x-2|}{10} < 1$$

$$-1 < \frac{x-2}{10} < 1$$

$$-10 < x-2 < 10$$

$$-8 < x < 12$$

When $x = -8 \quad \Rightarrow \quad \sum_{n=0}^{\infty} (-1)^n$ which is a divergent series

$x = 12 \quad \Rightarrow \quad \sum_{n=0}^{\infty} 1$ the series diverges

a) The radius is 10; the interval of converges $-8 < x < 12$
b) The interval of absolute convergence is $-8 < x < 12$
c) There are no values for which the series converges conditionally

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of x does the series converge
(***b***) absolutely,
(***c***) conditionally?

$$\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{3^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n x^n}\right|$$

$$= \frac{3}{n+1}\left|x^n\right| < 1$$

$$3|x|\lim_{n\to\infty}\frac{1}{n+1}<1 \quad\Rightarrow\quad \forall x$$

a) The radius is ∞; the series converges for all x.

b) The series convergence absolutely for all x.

c) There are no values for which the series converges conditionally

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of x does the series converge
(***b***) absolutely,
(***c***) conditionally?

$$\sum_{n=0}^{\infty}\frac{x^n}{\sqrt{n^2+3}}$$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt{(n+1)^2+3}}\cdot\frac{\sqrt{n^2+3}}{x^n}\right|$$

$$=|x|\lim_{n\to\infty}\frac{n^2+3}{n^2+2n+4}<1$$

$$|x|<1 \quad\Rightarrow\quad -1<x<1$$

When $x=-1 \quad\Rightarrow\quad \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n^2+3}}$ which is a convergent conditionally series

$x=1 \quad\Rightarrow\quad \sum_{n=0}^{\infty}\frac{1}{\sqrt{n^2+3}}$ the series diverges

a) The radius is 1; the series converges for $-1\le x<1$.

b) The series convergence absolutely for $-1<x<1$.

c) The series convergence conditionally for $x=-1$

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of x does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{\sqrt{n+3}}$$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+2}}{\sqrt{n+1}+3}\cdot\frac{\sqrt{n}+3}{x^{n+1}}\right|$$

$$= |x| \lim_{n\to\infty}\frac{\sqrt{n}+3}{\sqrt{n+1}+3} < 1$$

$|x| < 1 \quad \Rightarrow \quad -1 < x < 1$

When $x = -1 \quad \Rightarrow \quad \sum_{n=0}^{\infty}\frac{1}{\sqrt{n}+3}$ which is a divergent series

$x = 1 \quad \Rightarrow \quad \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}+3}$ the series converges conditionally

a) The radius is 1; the series converges for $-1 < x \le 1$.

b) The series convergence absolutely for $-1 < x < 1$.

c) The series convergence conditionally for $x = 1$

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of x does the series converge (***b***) absolutely, (***c***) conditionally?

$$\sum_{n=1}^{\infty}\sqrt[n]{n}\,(2x+5)^n$$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt[n+1]{n+1}\,(2x+5)^{n+1}}{\sqrt[n]{n}\,(2x+5)^n}\right|$$

$$= |2x+5| \lim_{n\to\infty}\frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}} < 1$$

$$= |2x+5|\frac{\lim_{m\to\infty}\sqrt[m]{m}}{\lim_{n\to\infty}\sqrt[n]{n}} < 1$$

$$= |2x+5| < 1$$

$|2x+5| < 1$

$-1 < 2x+5 < 1$

$-3 < x < -2$

When $x=-3 \quad \Rightarrow \quad \sum_{n=0}^{\infty}(-1)^n \sqrt[n]{n}$ which is a divergent series

$x=-0 \quad \Rightarrow \quad \sum_{n=0}^{\infty}\sqrt[n]{n}$ which is a divergent series

a) The radius is 1; the series converges for $-3 < x < -2$.

b) The series convergence absolutely for $-3 < x < -2$.

c) There are no values for which the series convergence conditionally

Exercise

(***a***) Find the series' radius and interval of convergence. For what values of x does the series converge
(***b***) absolutely,
(***c***) conditionally?

$$\sum_{n=1}^{\infty}\left(2+(-1)^n\right)\cdot(x+1)^{n-1}$$

Solution

$$\sum_{n=1}^{\infty}\left(2+(-1)^n\right)\cdot(x+1)^{n-1}=\sum_{n=1}^{\infty}2(x+1)^{n-1} \;+\; \sum_{n=1}^{\infty}(-1)^n(x+1)^{n-1}$$

For the series $\sum_{n=1}^{\infty}2(x+1)^{n-1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2(x+1)^n}{2(x+1)^{n-1}}\right|$$

$$=|x+1|<1$$

$$-1<x+1<1$$
$$-2<x<0$$

For the series $\sum_{n=1}^{\infty}(-1)^n(x+1)^{n-1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x+1)^n}{(x+1)^{n-1}}\right|$$

$$=|x+1|<1$$

$$-1<x+1<1$$

$-2 < x < 0$

When $x = -2 \Rightarrow \sum_{n=0}^{\infty}\left(2+(-1)^n\right)\cdot(-1)^{n-1}$ which is a divergent series

$x = 0 \Rightarrow \sum_{n=0}^{\infty}\left(2+(-1)^n\right)$ which is a divergent series

a) The radius is $\frac{1}{2}$; the series converges for $-2 < x < 0$.

b) The series convergence absolutely for $-2 < x < 0$.

c) There are no values for which the series convergence conditionally

Exercise

Find the radius of convergence of the power series $\sum_{n=0}^{\infty} n!\,x^n$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right|$$

$$= |x| \lim_{n\to\infty} (n+1)$$

$$\boxed{= \infty}$$

$\rightarrow \quad R = \frac{1}{\infty} = 0$

By the Ratio Test, the series diverges for $|x| > 0$ and converges only at its center, 0.

Therefore; the radius of convergence is $R = 0$.

Exercise

Find the radius of convergence of the power series $\sum_{n=0}^{\infty} 3(x-2)^n$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{3(x-2)^{n+1}}{3(x-2)^n}\right|$$

$$\boxed{= |x-2|}$$

By the Ratio Test, the series converges for $|x-2| < 1$ and diverges for $|x-2| > 1$.

Therefore; the radius of convergence is $R=1$.

Exercise

Find the radius of convergence of the power series $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$$

$$= x^2\lim_{n\to\infty}\frac{1}{(2n+2)(2n+3)}$$

$$\underline{=0}\,\rfloor$$

$$\to\quad R=\frac{1}{0}=\infty$$

By the Ratio Test, the series converges for all *x*. Therefore; the radius of convergence is $R=\infty$.

Exercise

Find the radius of convergence of the power series $\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n+1}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+2}\cdot\frac{n+1}{x^n}\right|$$

$$=|x|\lim_{n\to\infty}\frac{n+1}{n+2}$$

$$\underline{=|x|}\,\rfloor$$

$$\to\quad R=1$$

By the Ratio Test, the series converges for $|x|<1$ and diverges for $|x|>1$.

Therefore; the radius of convergence is $R=1$.

Exercise

Find the radius of convergence of the power series $\sum_{n=0}^{\infty}(3x)^n$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(3x)^n}\right|$$

$$= \boxed{3|x|}$$

$\rightarrow \quad 3|x| < 1 \Rightarrow R = \frac{1}{3}$

By the Ratio Test, the series converges for $|x| < \frac{1}{3}$ and diverges for $|x| > \frac{1}{3}$.

Therefore; the radius of convergence is $R = \frac{1}{3}$.

Exercise

Find the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(4x)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(4x)^n}\right|$$

$$= |4x| \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\right|$$

$$= \boxed{4|x|}$$

$\rightarrow \quad 4|x| < 1 \Rightarrow R = \frac{1}{4}$

By the Ratio Test, the series converges for $|x| < \frac{1}{4}$ and diverges for $|x| > \frac{1}{4}$.

Therefore; the radius of convergence is $R = \frac{1}{4}$.

Exercise

Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{5^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{5^{n+1}}\cdot\frac{5^n}{x^n}\right|$$

$$= \boxed{\frac{|x|}{5}}$$

$\rightarrow \quad \frac{|x|}{5} < 1 \Rightarrow R = 5$

By the Ratio Test, the series converges for $|x| < 5$ and diverges for $|x| > 5$.

Therefore; the radius of convergence is $R = 5$.

Exercise

Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right|$$

$$= x^2 \lim_{n\to\infty} \frac{1}{(2n+1)(2n+1)}$$

$$\underline{=0}$$

$$R = \frac{1}{0} = \infty$$

By the Ratio Test, the series converges for all x. Therefore; the radius of convergence is $R = \infty$.

Exercise

Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{n!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!\,x^{2n+2}}{(n+1)!}\cdot\frac{n!}{(2n)!x^{2n}}\right|$$

$$= x^2 \lim_{n\to\infty}\left(\frac{(2n+1)(2n+2)}{n+1}\right)$$

$$\underline{=\infty}$$

$$\rightarrow \quad R = \frac{1}{\infty} = 0$$

By the Ratio Test, the series diverges for $|x| > 0$ and converges only at its center, 0.

Therefore; the radius of convergence is $R = 0$.

Exercise

Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right|$$

$$= |x| \lim_{n\to\infty}\left(\frac{n}{n+1}\right)$$

$$= |x|$$

$$\to \quad R = 1$$

So, by the Ratio Test, the radius of convergence is $R = 1$.

The series centered at 0, it converges in the interval $(-1,\ 1)$

When $x = 1$ $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ *diverges*

When $x = -1$ $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \cdots$ *converges*

Therefore; the interval of convergence $[-1,\ 1)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(x+1)^n}{2^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{2^{n+1}}\cdot\frac{2^n}{(x+1)^n}\right|$$

$$= \frac{1}{2}|x+1|$$

$$|x+1| < 2 \to \quad R = 2$$

$$\begin{cases} x+1 = -2 & x = -3 \\ x+1 = 2 & x = 1 \end{cases}$$

So, by the Ratio Test, the radius of convergence is $R = 2$.

The series centered at -1, it converges in the interval $(-3,\ 1)$

When $x = -3$ $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(-2)^n}{2^n} = \sum_{n=0}^{\infty}1$ *diverges*

When $x = 1$ $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(2)^n}{2^n} = \sum_{n=0}^{\infty}(-1)^n$ *diverges*

Therefore; the interval of convergence $(-3,\ 1)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{x^n}\right|$$

$$= |x| \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\right|$$

$$\boxed{= |x|}$$

$\to \quad R = 1$

So, by the Ratio Test, the radius of convergence is $R = 1$.

The series centered at 0, it converges in the interval $(-1,\ 1)$

When $x = -1$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \cdots$ *converges by alternating series*

$$u_{n+1} = \frac{1}{(n+1)^2} < \frac{1}{n^2} = u_n$$

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

When $x = 1$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ *converges by p-series*

Therefore; the interval of convergence $[-1,\ 1]$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{4^{n+1}}\cdot\frac{4^n}{x^n}\right|$$

$$\boxed{= \frac{|x|}{4}}$$

$\to \quad R = 4$

So, by the Ratio Test, the radius of convergence is $R = 4$.

The series centered at 0, it converges in the interval $(-4,\ 4)$

When $x = -4$ $\quad \sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + \cdots$ *diverges by alternating series*

When $x = 4$ $\quad \sum_{n=1}^{\infty} 1$ *diverges*

Therefore; the interval of convergence $(-4,\ 4)$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty} (2x)^n$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{(2x)^n}\right|$$

$$= 2|x|$$

$2|x| = 1 \quad \to \quad R = \frac{1}{2}$

So, by the Ratio Test, the radius of convergence is $R = \frac{1}{2}$.

The series converges in the interval $\left(-\frac{1}{2},\ \frac{1}{2}\right)$

When $x = -\frac{1}{2}$ $\quad \sum_{n=0}^{\infty} (-1)^n = -1 + 1 - 1 + \cdots$ *diverges by alternating series*

When $x = \frac{1}{2}$ $\quad \sum_{n=0}^{\infty} 1$ *diverges*

Therefore; the interval of convergence $\left(-\frac{1}{2},\ \frac{1}{2}\right)$

Exercise

Find the interval of convergence of the power series $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right|$$

$$= |x| \lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$= |x|$$

$|x| = 1 \quad \to \quad R = 1$

So, by the Ratio Test, the radius of convergence is $R = 1$.

The series converges in the interval $(-1,\ 1)$

When $x = -1$ $\quad \sum_{n=1}^{\infty} \frac{1}{n}$ *diverges by p-series*

When $x = 1$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *converges by Alternating Series*

Therefore; the interval of convergence $(-1,\ 1]$

Exercise

Find the interval of convergence of the power series $\quad \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right|$$

$$= |x| \lim_{n\to\infty}\left|\frac{n+2}{n+1}\right|$$

$$= |x|$$

$|x| = 1 \quad \to \quad R = 1$

The series converges in the interval $(-1,\ 1)$

When $x = -1$ $\quad \sum_{n=0}^{\infty} (n+1)$ *diverges*

When $x = 1$ $\quad \sum_{n=0}^{\infty} (-1)^n (n+1)$ *diverges*

Therefore; the interval of convergence $(-1,\ 1)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{5n+5}}{(n+1)!}\cdot\frac{n!}{x^{5n}}\right|$$

$$=\left|x^5\right| \lim_{n\to\infty}\left|\frac{1}{n+1}\right|$$

$$\underline{=0\,|} \quad \to \quad R=\infty$$

The series converges for all x. Therefore; the interval of convergence $(-\infty,\ \infty)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{(3x)^n}\right|$$

$$=|3x| \lim_{n\to\infty}\left|\frac{1}{(2n+1)(2n+2)}\right|$$

$$\underline{=0\,|}$$

$\to \quad R=\infty$

The series converges for all x. Therefore; the interval of convergence $(-\infty,\ \infty)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} (2n)!\left(\frac{x}{3}\right)^n$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|(2n+2)!\left(\frac{x}{3}\right)^{n+1}\cdot(2n)!\left(\frac{x}{3}\right)^{-n}\right|$$

$$=\left|\frac{x}{3}\right| \lim_{n\to\infty}\left|(2n+1)(2n+2)\right|$$

$$\underline{=\infty\,|}$$

The series converges only for $x=0$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{(n+1)(n+2)}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+2)(n+3)}\cdot\frac{(n+1)(n+2)}{x^n}\right|$$

$$=|x|\lim_{n\to\infty}\left|\frac{n+1}{n+3}\right|$$

$$=|x|$$

$|x|=1 \quad\to\quad R=1$

The series converges in the interval $(-1,\ 1)$

When $x=-1$ $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}$ *converges* *by Alternating Series*

$$u_{n+1}=\frac{1}{(n+3)(n+2)}<\frac{1}{(n+1)(n+2)}=u_n$$

$$\lim_{n\to\infty}\frac{1}{(n+1)(n+2)}=0$$

When $x=1$ $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)(n+2)}$ *converges* *by Limit Comparison Test to* $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$

Therefore; the interval of convergence $[-1,\ 1]$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{6^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{6^{n+1}}\cdot\frac{6^n}{x^n}\right|$$

$$=\frac{1}{6}|x|$$

$\frac{1}{6}|x|=1 \quad\to\quad R=6$

The series converges in the interval $(-6,\ 6)$

When $x=-6$ $\sum_{n=1}^{\infty}(-1)^n$ *diverges*

When $x=6$ $\sum_{n=1}^{\infty}(-1)^{n+1}$ *diverges*

Therefore; the interval of convergence $(-6,\ 6)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{n!(x-5)^n}{3^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x-5)^{n+1}}{3^{n+1}}\cdot\frac{3^n}{n!(x-5)^n}\right|$$

$$=\left|\frac{x-5}{3}\right|\lim_{n\to\infty}(n+1)$$

$$=\infty$$

The series converges only for $x=5$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-4)^n}{n\,9^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-4)^{n+1}}{(n+1)9^{n+1}}\cdot\frac{n9^n}{(x-4)^n}\right|$$

$$=\frac{1}{9}|x-4|\lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$=\frac{1}{9}|x-4|$$

$\frac{1}{9}|x-4|=1 \quad\to\quad R=9$

$$|x-4|=9 \;\Rightarrow\; \begin{cases} x-4=-9 & x=-5\\ x-4=9 & x=13\end{cases}$$

The series converges in the interval $(-5,\ 13)$ and center $x=4$

When $x=-5$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-9)^n}{n9^n}=\sum_{n=1}^{\infty}\frac{-1}{n} \quad \textit{diverges}$$

When $x=13$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(9)^n}{n\,9^n}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \quad \textit{converges} \quad \textit{by Alternating Series}$$

$$u_{n+1}=\frac{1}{n+1}<\frac{1}{n}=u_n$$

$$\lim_{n\to\infty}\frac{1}{n}=0$$

Therefore; the interval of convergence $(-5,\ 13]$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}\frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{(x-3)^{n+2}}{(n+2)4^{n+2}}\cdot\frac{(n+1)4^{n+1}}{(x-3)^{n+1}}\right|$$

$$=\frac{1}{4}|x-3|\lim_{n\to\infty}\left|\frac{n+1}{n+2}\right|$$

$$=\frac{1}{4}|x-3|$$

$$\frac{1}{4}|x-3|=1 \ \rightarrow \ R=4$$

$$|x-3|=4 \ \Rightarrow \ \begin{cases} x-3=-4 & x=-1 \\ x-3=4 & x=7 \end{cases}$$

The series converges in the interval $(-1,\ 7)$ and center $x=3$

When $x=-1$

$$\sum_{n=0}^{\infty}\frac{(-4)^{n+1}}{(n+1)4^{n+1}}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1} \quad \textit{converges} \quad \textit{by Alternating Series}$$

$$u_{n+1}=\frac{1}{n+2}<\frac{1}{n+1}=u_n$$

$$\lim_{n\to\infty}\frac{1}{n+1}=0$$

When $x = 4$

$$\sum_{n=0}^{\infty} \frac{4^{n+1}}{(n+1)4^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{n+1} \quad \textit{diverges} \ \ \textit{by Integral Test}$$

$$\int_0^{\infty} \frac{dx}{x+1} = \ln(x+1) \Big|_0^{\infty}$$

$$= \infty$$

Therefore; the interval of convergence $[-1, \ 7)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-1)^{n+1}}{n+1}$

Solution

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(x-1)^{n+2}}{n+2} \cdot \frac{n+1}{(x-1)^{n+1}} \right|$$

$$= |x-1| \lim_{n\to\infty} \left| \frac{n+1}{n+2} \right|$$

$$= |x-1|$$

$|x-1| = 1 \quad \to \quad R = 1$

$|x-1| = 1$

$$\Rightarrow \begin{cases} x-1=-1 & x=0 \\ x-1=1 & x=2 \end{cases}$$

The series converges in the interval $(0, \ 2)$ and center $x = 1$

When $x = 0$

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} \quad \textit{diverges} \ \ \textit{by Integral Test}$$

$$\int_0^{\infty} \frac{dx}{x+1} = \ln(x+1) \Big|_0^{\infty}$$

$$= \infty$$

When $x = 1$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} \quad \textit{converges} \ \ \textit{by Alternative Test}$$

$$u_{n+1} = \frac{1}{n+2} < \frac{1}{n+1} = u_n$$

$$\lim_{n\to\infty} \frac{1}{n+1} = 0$$

Therefore; the interval of convergence $(0,\ 2]$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-2)^n}{n2^n}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{(x-2)^n}\right|$$

$$= \frac{1}{2}|x-2| \lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$= \frac{1}{2}|x-2|$$

$|x-2| = 2 \;\rightarrow\; R = 2$

$$|x-2| = 2 \;\Rightarrow\; \begin{cases} x-2=-2 & x=0 \\ x-2=2 & x=4 \end{cases}$$

The series converges in the interval $(0,\ 4)$ and center $x=2$

When $x=0$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-2)^n}{n2^n} = \sum_{n=0}^{\infty}\frac{-1}{n} \quad \textit{diverges by Integral Test}$$

$$\int_0^{\infty}\frac{-dx}{x} = -\ln x \Big|_0^{\infty}$$

$$= -\infty$$

When $x=4$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \quad \textit{converges by Alternative Test}$$

$$u_{n+1} = \frac{1}{n+1} < \frac{1}{n} = u_n$$

$$\lim_{n\to\infty}\frac{1}{n} = 0$$

Therefore; the interval of convergence $(0,\ 4]$

Exercise

Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{3^{n-1}}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^n}{3^n}\cdot\frac{3^{n-1}}{(x-3)^{n-1}}\right|$$

$$=\frac{1}{3}|x-3|$$

$$\frac{1}{3}|x-3| = 1 \quad \to \quad R = 3$$

$$|x-3| = 3$$

$$\Rightarrow \begin{cases} x-3=-3 & x=0 \\ x-3=3 & x=6 \end{cases}$$

The series converges in the interval $(0,\ 6)$

When $x=0$

$$\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty}(-1) \quad \textit{diverges}$$

When $x=6$

$$\sum_{n=1}^{\infty}\frac{3^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty}1 \quad \textit{diverges}$$

Therefore; the interval of convergence $(0,\ 6)$

Exercise

Find the interval of convergence of the power series $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{2n+3}\cdot\frac{2n+1}{x^{2n+1}}\right|$$

$$= x^2\lim_{n\to\infty}\left|\frac{2n+1}{2n+3}\right|$$

$$= x^2$$

$$\to \quad R = 1$$

The series converges in the interval $(-1,\ 1)$

When $x=-1$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} \quad \textit{converges} \quad \textit{by Alternating Series}$$

When $x=1$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n}}{2n+1} \quad \textit{converges} \quad \textit{by Alternating Series}$$

Therefore; the interval of convergence $[-1,\ 1]$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}(-2x)^{n-1}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n+2}(-2x)^{n}\cdot\frac{n+1}{n(-2x)^{n-1}}\right|$$

$$=|-2x|\lim_{n\to\infty}\left|\frac{(n+1)^2}{n(n+2)}\right|$$

$$\underline{=2|x|}$$

$2|x|=1 \quad\rightarrow\quad R=\frac{1}{2}$

The series converges in the interval $\left(-\frac{1}{2},\ \frac{1}{2}\right)$

When $x=-\frac{1}{2}$

$$\sum_{n=1}^{\infty} \frac{n}{n+1} \quad \textit{diverges} \quad \textit{by nth Term Test}$$

$$\lim_{n\to\infty} \frac{n}{n+1}=1\neq 0$$

When $x=\frac{1}{2}$

$$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{n}{n+1} \quad \textit{diverges} \quad \textit{by Alternating Series}$$

$$\lim_{n\to\infty} \frac{n}{n+1}=1\neq 0$$

Therefore; the interval of convergence $\left(-\frac{1}{2},\ \frac{1}{2}\right)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(n+1)!}\cdot\frac{n!}{x^{2n}}\right|$$

$$= x^2 \lim_{n\to\infty}\left|\frac{1}{n+1}\right|$$

$$= 0$$

$\to \quad R = \infty$

Therefore; the interval of convergence $(-\infty,\ \infty)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{3n+4}}{(3n+4)!}\cdot\frac{(3n+1)!}{x^{3n+1}}\right|$$

$$= \left|x^3\right| \lim_{n\to\infty}\left|\frac{1}{(3n+2)(3n+3)(3n+4)}\right|$$

$$= 0$$

$\to \quad R = \infty$

Therefore; the interval of convergence $(-\infty,\ \infty)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{n!x^n}\right|$$

$$= |x| \lim_{n\to\infty}\left|\frac{n+1}{(2n+1)(2n+2)}\right|$$

$$=0$$

$$\to \quad R=\infty$$

Therefore; the interval of convergence $(-\infty,\ \infty)$

Exercise

Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{2\cdot 3\cdot 4\cdots(n+1)x^n}{n!}$

Solution

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2\cdot 3\cdot 4\cdots(n+1)(n+2)x^{n+1}}{(n+1)!}\cdot\frac{n!}{2\cdot 3\cdot 4\cdots(n+1)x^n}\right|$$

$$=|x|\lim_{n\to\infty}\left|\frac{n+2}{n+1}\right|$$

$$=|x|$$

$|x|=1 \quad\to\quad R=1$

The series converges in the interval $(-1,\ 1)$

When $x=-1$

$$\sum_{n=1}^{\infty}(-1)^n\ \frac{2\cdot 3\cdot 4\cdots(n+1)}{n!}=\sum_{n=1}^{\infty}\ (-1)^n\,(n+1) \quad \textit{diverges}$$

When $x=1$

$$\sum_{n=1}^{\infty}\frac{2\cdot 3\cdot 4\cdots(n+1)}{n!}=\sum_{n=1}^{\infty}(n+1) \quad \textit{diverges}$$

Therefore; the interval of convergence $(-1,\ 1)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n}}{\sqrt{n+1}}$

Solution

$$R=\lim_{n\to\infty}\left|\frac{1}{\sqrt{n+2}}\cdot\sqrt{n+1}\right|$$

$$=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}}$$

$$=1$$

The *radius* of convergence is 1.

The *centre* of convergence is 0.

The *interval* of convergence is $(-1, 1)$.

The series *does not converge* at $x=-1$ or $x=1$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} 3n(x+1)^n$

Solution

$$R = \lim_{n\to\infty}\left|\frac{3n}{3(n+1)}\right| \qquad\qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty}\frac{3n}{3n}$$

$$\underline{=1}$$

The radius of convergence is 1, and the centre of convergence is -1. $(x+1=0)$

$a-R<x<a+R \quad\Rightarrow\quad -1-1<x<-1+1$

Therefore; the given series convergences absolutely on $(-2,\ 0)$

At $x=-2$

The series is $\displaystyle\sum_{n=0}^{\infty} 3n(-1)^n$ which diverges.

At $x=0$

The series is $\displaystyle\sum_{n=0}^{\infty} 3n(1)^n = \sum_{n=0}^{\infty} 3n$ which diverges.

Hence, the interval of convergence is $(-2,\ 0)$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} x^n$

Solution

$$R = \lim_{n\to\infty}\left|\frac{(n+1)^4\, 2^{2n+2}}{n^4 2^{2n}}\right| \qquad\qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= 4 \lim_{n\to\infty} \left| \left(\frac{n+1}{n} \right)^4 \right|$$

$$\boxed{= 4}$$

The radius of convergence is 4, and the centre of convergence is 0.

$a - R < x < a + R \quad \Rightarrow \quad -4 < x < 4$, the given series convergences absolutely on $(-4,\ 4)$

At $x = -4$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}}(-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}}(-1)^n\, 2^{2n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^4} \text{ which converges (}p\text{-series).}$$

At $x = 4$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}}(4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} 2^{2n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \text{ which also converges.}$$

Hence, the interval of convergence is $[-4,\ 4]$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{e^n}{n^3}(4-x)^n$

Solution

$$R = \lim_{n\to\infty} \left| \frac{e^n}{n^3} \cdot \frac{(n+1)^3}{e^{n+1}} \right| \qquad\qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \frac{1}{e} \lim_{n\to\infty} \left| \left(\frac{n+1}{n} \right)^3 \right|$$

$$\boxed{= \frac{1}{e}}$$

The *radius* of convergence is $\frac{1}{e}$.

The *centre* of convergence is 4. $(4 - x = 0 \ \Rightarrow \ x = 4)$

$a - R < x < a + R \quad \Rightarrow \quad 4 - \frac{1}{e} < x < 4 + \frac{1}{e}$, which the given series convergences absolutely

At $x = 4 - \frac{1}{e}$,

the series is $\sum_{n=1}^{\infty} \frac{e^n}{n^3}\left(\frac{1}{e}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ which converges (p-series).

At $x = 4 + \frac{1}{e}$,

the series is $\sum_{n=1}^{\infty} \frac{e^n}{n^3}\left(-\frac{1}{e}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ which also converges (p-series).

Hence, the interval of convergence is $\left[4 - \frac{1}{e},\ 4 + \frac{1}{e}\right]$.

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{1+5^n}{n!} x^n$

Solution

$$R = \lim_{n\to\infty} \left| \frac{1+5^n}{n!} \cdot \frac{(n+1)!}{1+5^{n+1}} \right| \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1+5^n}{1+5^{n+1}} (n+1) \right|$$

$$= \infty$$

The *radius* of convergence is ∞ .

The *centre* of convergence is 0.

The *interval of convergence* is the real line $(-\infty,\ \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n}$

Solution

$$\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n} = \sum_{n=1}^{\infty} \frac{4^n\left(x - \frac{1}{4}\right)^n}{n^n}$$

$$R = \lim_{n\to\infty} \left| \frac{4^n}{n^n} \cdot \frac{(n+1)^{n+1}}{4^{n+1}} \right| \qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \frac{1}{4} \lim_{n\to\infty} \left| \left(\frac{n+1}{n} \right)^n (n+1) \right|$$

$$\boxed{= \infty}$$

The *radius* of convergence is ∞ .

$4x - 1 = 0 \ \Rightarrow \ x = \frac{1}{4}$

The *centre* of convergence is $\boxed{x = \frac{1}{4}}$

The *interval of convergence* is the real line $(-\infty, \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{1+5^n}{n!} x^n$

Solution

$$a_n = \frac{1+5^n}{n!}$$

$$R = \lim_{n\to\infty} \left| \frac{\left(1+5^n\right)}{n!} \cdot \frac{(n+1)!}{\left(1+5^{n+1}\right)} \right| \qquad\qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| (n+1) \frac{1+5^n}{1+5^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| (n+1) \frac{1}{5} \right|$$

$$\boxed{= \infty}$$

The *radius* of convergence is ∞ .

The *centre* of convergence is $\boxed{x = 0}$.

The *interval of convergence* is the real line $(-\infty, \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\displaystyle\sum \frac{n^2 x^n}{n!}$

Solution

$$a_n = \frac{n^2}{n!}$$

$$R = \lim_{n\to\infty} \left| \frac{n^2 x^n}{n!} \cdot \frac{(n+1)!}{(n+1)^2 x^{n+1}} \right| \qquad\qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| (n+1)\left(\frac{n}{n+1}\right)^2 \right|$$

$$= \infty$$

The *radius* of convergence is ∞ .

The *centre* of convergence is $x = 0$.

The *interval of convergence* is the real line $(-\infty, \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum \frac{x^{4n}}{n^2}$

Solution

$$a_n = \frac{1}{n^2} x^{4n}$$

$$R = \lim_{n\to\infty} \left(\frac{1}{n^2} \cdot \frac{(n+1)^2}{1} \right) \left| \frac{x^{4n}}{x^{4n+4}} \right| \qquad\qquad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^2 \left| \frac{1}{x^4} \right|$$

$$= 1$$

The *radius* of convergence is 1

The *centre* of convergence is $x = 0$

$$-1 < x < 1 \qquad\qquad a - R < x < a + R$$

which the given series convergences absolutely

At $x = -1$,

the series is $\sum \frac{(-1)^{4n}}{n^2} = \sum \frac{1}{n^2}$ which converges (p-series).

At $x = 1$,

the series is $\sum \frac{(1)^{4n}}{n^2} = \sum \frac{1}{n^2}$ which also converges (p-series).

The interval of convergence is the real line $[-1, 1]$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum (-1)^n \frac{(x+1)^{2n}}{n!}$

Solution

$$a_n = \frac{1}{n!}(x+1)^{2n}$$

$$R = \lim_{n\to\infty} \left(\frac{1}{n!}\cdot\frac{(n+1)!}{1}\right)\left|\frac{(x+1)^{2n}}{(x+1)^{2n+2}}\right| \qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty} (n+1)\left|\frac{1}{(x+1)^2}\right|$$

$$= \infty$$

The *radius* of convergence is ∞

$x+1=0 \quad \to \quad x=-1$

The *centre* of convergence is $x=-1$

The *interval* of convergence is the real line $(-\infty,\ \infty)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum \frac{(x-1)^n}{n\cdot 5^n}$

Solution

$$a_n = \frac{1}{n\cdot 5^n}(x-1)^n$$

By *Ratio Test*:

$$R = \lim_{n\to\infty} \left(\frac{1}{n\cdot 5^n}\cdot\frac{(n+1)\cdot 5^{n+1}}{1}\right)\left|\frac{(x-1)^n}{(x-1)^{n+1}}\right| \qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= 5\lim_{n\to\infty} \frac{n+1}{n}\left|\frac{1}{x-1}\right|$$

$$= 5$$

The *radius* of convergence is 5

$x-1=0 \quad \to \quad x=1$

The *centre* of convergence is $x=1$

$-5+1 < x < 5+1 \qquad a-R < x < a+R$

$-4 < x < 6$

which the given series convergences absolutely

At $x=-4$,

the series is $\sum \frac{(-5)^n}{n\cdot 5^n} = \sum \frac{(-1)^n}{n}$

$$\frac{1}{n} > \frac{1}{n+1}$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

which converges *Alternating Harmonic Series*.

At $x=6$,

the series is $\sum \frac{(5)^n}{n\cdot 5^n} = \sum \frac{1}{n}$ which *diverges* (p-series $p=1\le 1$)

The interval of convergence is the real line $[-4,\ 6)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum \left(\frac{x}{9}\right)^{3n}$

Solution

$$a_n = \left(\frac{x}{9}\right)^{3n}$$

By Root Test:

$$R = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{x}{9}\right)^{3n}\right|} \qquad\qquad R = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

$$= \lim_{n\to\infty} \left(\frac{|x|}{9}\right)^3$$

$$= \frac{1}{729}\left|x^3\right| < 1$$

$$\left|\frac{x}{9}\right|^3 < 1$$

$$\left|\frac{x}{9}\right| < 1$$

$$-9 < x < 9$$

The *radius* of convergence is 9

The *centre* of convergence is $x=0$

At $x=-9$,

the series is $\sum \left(\frac{-9}{9}\right)^{3n} = \sum (-1)$ which diverges by the *divergence Test*.

At $x = 9$,

the series is $\sum \left(\frac{9}{9}\right)^{3n} = \sum (1)$ which diverges by the *divergence Test*.

The interval of convergence is the real line $(-9,\ 9)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum \frac{(x+2)^n}{\sqrt{n}}$

Solution

$$a_n = \frac{(x+2)^n}{\sqrt{n}}$$

By *Ratio Test*:

$$R = \lim_{n\to\infty} \left(\frac{1}{\sqrt{n}} \cdot \frac{\sqrt{n+1}}{1}\right)\left|\frac{(x+2)^n}{(x+2)^{n+1}}\right| \qquad\qquad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty} \sqrt{\frac{n+1}{n}}\left|\frac{1}{x+2}\right|$$

$$= 1$$

The *radius* of convergence is 1

$x + 2 = 0 \quad \rightarrow \quad x = -2$

The *centre* of convergence is $x = -2$

$-2 - 1 < x < -2 + 1 \qquad\qquad a - R < x < a + R$

$-3 < x < -1$

which the given series convergences absolutely

At $x = -3$,

the series is $\sum \frac{(-1)^n}{\sqrt{n}}$

$$\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

which converges *Alternating Series*.

At $x = -1$,

the series is $\sum \frac{(1)^n}{\sqrt{n}} = \sum \frac{1}{\sqrt{n}}$ which *diverges* (p-series $p = \frac{1}{2} \le 1$)

The interval of convergence is the real line $[-3, \ -1)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $\sum \frac{(x+2)^k}{2^k \ln k}$

Solution

$$a_k = \frac{(x+2)^k}{2^k \ln k}$$

By *Ratio Test*:

$$R = \lim_{k\to\infty} \left(\frac{1}{2^k \ln k} \cdot \frac{2^{k+1} \ln(k+1)}{1} \right) \left| \frac{(x+2)^k}{(x+2)^{k+1}} \right| \qquad R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right|$$

$$= 2 \lim_{n\to\infty} \frac{\ln(k+1)}{\ln k} \left| \frac{1}{x+2} \right|$$

$$= 2 \lim_{n\to\infty} \frac{\frac{1}{k+1}}{\frac{1}{k}}$$

$$= 2 \lim_{n\to\infty} \frac{k}{k+1}$$

$$= 2$$

The *radius* of convergence is 2

$x + 2 = 0 \quad \to \quad x = -2$

The *centre* of convergence is $x = -2$

$$-2 - 2 < x < -2 + 2 \qquad a - R < x < a + R$$

$$-4 < x < 0$$

which the given series convergences absolutely.

At $x = -4$,

the series is $\sum \frac{(-2)^k}{2^k \ln k} = \sum \frac{(-1)^k}{\ln k}$

$$\frac{1}{\ln k} > \frac{1}{\ln(k+1)}$$

$$\lim_{n\to\infty} \frac{1}{\ln k} = 0$$

which converges *Alternating Series*.

At $x=0$,

the series is $\sum \frac{(2)^k}{2^k \ln k} = \sum \frac{1}{\ln k}$

$\ln k < k$

$\frac{1}{\ln k} > \frac{1}{k}$

$\frac{1}{k}$ *diverges* (p-series $p=1\leq 1$)

$\therefore$ Which *diverges by Comprison Test.*

The *interval* of convergence is the real line $[-4,\ 0)$

Exercise

Determine the centre, radius, and interval of convergence of the power series $x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots$

Solution

$$x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$$

$$a_k = \frac{x^{2k+1}}{2k+1}$$

By *Ratio Test*:

$$R = \lim_{k\to\infty} \left(\frac{1}{2k+1}\cdot\frac{2k+3}{1}\right)\left|\frac{x^{2k+1}}{x^{2k+3}}\right| \qquad R = \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|$$

$$= \lim_{n\to\infty} \frac{2k+3}{2k+1}\left(\frac{1}{x^2}\right)$$

$$\underline{=1}$$

The *radius* of convergence is $\underline{1}$

The *centre* of convergence is $\underline{x=0}$

$-1<x<1$

which the given series convergences absolutely

At $x=-1$,

the series is $\sum \frac{(-1)^{2k+1}}{2k+1} = \sum \frac{-1}{2k+1}$

$$\int_0^\infty \frac{-1}{2x+1}dx = -\frac{1}{2}\int_0^\infty \frac{1}{2x+1}d(2x+1)$$

$$= -\frac{1}{2}\ln(2x+1)\Big|_0^{\infty}$$

$$= -\frac{1}{2}(\ln\infty - \ln 1)$$

$$\underline{= -\infty}$$

which diverges *Integral Test*.

At $x = 1$,

the series is $\sum \frac{(1)^{2k+1}}{2k+1} = \sum \frac{1}{2k+1}$

$$\int_0^{\infty} \frac{1}{2x+1}\,dx = \frac{1}{2}\int_0^{\infty} \frac{1}{2x+1}\,d(2x+1)$$

$$= \frac{1}{2}\ln(2x+1)\Big|_0^{\infty}$$

$$= \frac{1}{2}(\ln\infty - \ln 1)$$

$$\underline{= \infty}$$

which diverges *Integral Test*.

The *interval* of convergence is the real line $(-1,\ 1)$

Exercise

For what value of x does the series $1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots$ converges? What is its sum? What series do you get if you differentiate the given series term by term? For what value of x does the new series converge? What is its sum?

Solution

$$1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n$$

$$\lim_{b\to\infty}\left|\frac{(x-3)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(x-3)^n}\right| = \left|\frac{x-3}{2}\right| < 1$$

$$\Rightarrow\quad -1 < \frac{x-3}{2} < 1$$

$$-2 < x-3 < 2$$

$$\underline{1 < x < 5}$$

When $x = 1$,

$$\sum_{n=1}^{\infty}(1)^n \text{ which is a divergent series}$$

When $x=5$,

$$\sum_{n=1}^{\infty}(-1)^n \text{ the series diverges}$$

The series is a geometric series, the sum is

$$\frac{1}{1+\frac{x-3}{2}}=\frac{2}{x-1}$$

If $f(x)=1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2+\cdots+\left(-\frac{1}{2}\right)^n(x-3)^n+\cdots$

$$=\frac{2}{x-1}$$

Then $f'(x)=-\frac{1}{2}+\frac{1}{2}(x-3)+\cdots+\left(-\frac{1}{2}\right)^n n(x-3)^{n-1}+\cdots$

$f'(x)$ is convergent when $1<x<5$ and divergent when $x=1$ *or* 5

The sum for $f'(x)$ is $\frac{-2}{(x-1)^2}$

Exercise

The series $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\frac{x^{11}}{11!}+\cdots$ converges to sin$x$ for all x.

a) Find the first six terms of a series for cos$x$. For what values of x should the series converge?

b) By replacing x by $2x$ in the series for sin$x$, find a series that converges to sin$2x$ for all x.

c) Using the result in part (a) and series multiplication, calculate the first six term of a series for $2\sin x\cos x$. Compare your answer with the answer in part (b).

Solution

a) $(\sin x)'=\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\cdots$

$$\cos x=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{(2n)!}$$

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right|=x^2\lim_{n\to\infty}\left|\frac{1}{(2n+1)(2n+2)}\right|=0<1 \quad (\forall x)$$

The series converges for all values of x.

b) $\sin 2x = 2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \frac{2^7x^7}{7!} + \frac{2^9x^9}{9!} - \frac{2^{11}x^{11}}{11!} + \cdots$

$$= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \frac{512x^9}{9!} - \frac{2048x^{11}}{11!} + \cdots$$

c) $2\sin x\cos x = 2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right)$

$$= 2x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right) - 2\frac{x^3}{3!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right)$$

$$+ 2\frac{x^5}{5!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right) - 2\frac{x^7}{7!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right)$$

$$+ 2\frac{x^9}{9!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right) - 2\frac{x^{11}}{11!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots\right)$$

$$= 2x - \frac{2x^3}{2!} + \frac{2x^5}{4!} - \frac{2x^7}{6!} + \frac{2x^9}{8!} - \frac{2x^{11}}{10!} - \frac{2x^3}{3!} + \frac{2x^5}{2!3!} - \frac{2x^7}{4!3!} + \frac{2x^9}{6!3!} - \frac{2x^{11}}{8!3!}$$

$$+ \frac{2x^5}{5!} - \frac{2x^7}{5!2!} + \frac{2x^9}{5!4!} - \frac{2x^{11}}{5!6!} - \frac{2x^7}{7!} + \frac{2x^9}{7!2!} - \frac{2x^{11}}{7!4!} + \frac{2x^9}{9!} - \frac{2x^{11}}{9!2!} - \frac{2x^{11}}{11!} + \cdots$$

$$= 2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \frac{2^7x^7}{7!} + \frac{2^9x^9}{9!} - \frac{2^{11}x^{11}}{11!} + \cdots$$

Exercise

Find the sum of the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ by the first finding the sum of the power series

$$\sum_{n=1}^{\infty} n^2x^n = x + 4x^2 + 9x^3 + 16x^4 + \cdots$$

Solution

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}$$

$$\sum_{n=1}^{\infty} nx^n = x + x^2 + 3x^3 + 4x^4 + \cdots$$

$$= x\left(1 + x + 3x^2 + 4x^3 + \cdots\right)$$

$$= x\frac{d}{dx}\left(1 + x + x^2 + \cdots + x^n + \cdots\right)$$

$$= x\left(\frac{1}{1-x}\right)'$$

$$= \frac{x}{(1-x)^2}$$

$$\frac{d}{dx}\sum_{n=1}^{\infty} nx^n = \left(\frac{x}{(1-x)^2}\right)'$$

$$= \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4}$$

$$= \frac{1+x}{(1-x)^3}$$

$$\frac{d}{dx}\sum_{n=1}^{\infty} nx^n = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

$$= \frac{1+x}{(1-x)^3}$$

Multiply by x both sides

$$x\sum_{n=1}^{\infty} n^2 x^{n-1} = x\frac{1+x}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \cdots$$

$$\boxed{= \frac{x(1+x)}{(1-x)^3}}$$

Let $x = \frac{1}{2}$

$$\sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}\frac{3}{2}}{\left(\frac{1}{2}\right)^3}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} \boxed{= 6}$$

Exercise

Find a series representation of $f(x)=\frac{1}{2+x}$ in powers of $x-1$. What is the interval of convergence of this series?

Solution

Let $t=x-1 \quad \Rightarrow \quad x=t+1$, we have

$$\frac{1}{2+x}=\frac{1}{3+t}$$
$$=\frac{1}{3}\frac{1}{1+\frac{t}{3}}$$
$$=\frac{1}{3}\left(1-\frac{t}{3}+\frac{t^2}{3^2}-\frac{t^3}{3^3}+\cdots\right) \qquad \left(-1<\frac{t}{3}<1\right)$$
$$=\sum_{n-0}^{\infty}(-1)^n\frac{t^n}{3^{n+1}} \qquad (-3<t<3)$$
$$=\sum_{n-0}^{\infty}(-1)^n\frac{(x-1)^n}{3^{n+1}} \qquad (-2<x<4)$$

$$R=\lim_{n\to\infty}\frac{3^{n+2}}{3^{n+1}}$$
$$=3$$

The *radius* of convergence of this series is 3.

The distance from the centre of convergence $x-1=0 \Rightarrow x=1$, to the point -2 where the denominator is 0.

Exercise

Determine the Cauchy product of the series $1+x+x^2+x^3+\cdots$ *and* $-x+x^2-x^3+\cdots$. On what interval and to what function does the product series converge?

Solution

$$1+x+x^2+x^3+\cdots=\frac{1}{1-x}$$
$$=\sum_{n=0}^{\infty}x^n$$

$$-x+x^2-x^3+\cdots=\frac{1}{1+x}$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n$$

Let $a_n = 1$ *and* $b_n = (-1)^n$, then the series holds for $-1 < x < 1$

We have

$$c_n = \sum_{j=0}^{n} (-1)^{n-j}$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

Then the Cauchy product is

$$1 + x^2 + x^4 + \cdots = \sum_{n=0}^{\infty} x^{2n}$$

$$= \frac{1}{1-x} \cdot \frac{1}{1+x}$$

$$\boxed{= \frac{1}{1-x^2}} \quad \text{for} \quad -1 < x < 1$$

Exercise

Determine the power series expansion of $\frac{1}{(1-x)^2}$ by formally dividing $1 - 2x + x^2$ into 1.

Use the power series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad -1 < x < 1$

Solution

$$\begin{array}{r l}
 & 1 + 2x + 3x^2 + 4x^3 + \cdots \\
1 - 2x + x^2 \, \big) & 1 \\
 & \underline{1 - 2x + x^2} \\
 & \quad 2x - x^2 \\
 & \quad \underline{2x - 4x^2 + 2x^3} \\
 & \qquad 3x^2 - 2x^3 \\
 & \qquad \underline{3x^2 - 6x^3 + 3x^4} \\
 & \qquad\quad 4x^3 + 3x^4 \\
 & \qquad\quad \underline{4x^3 - 8x^4 + 4x^5} \\
 & \qquad\qquad 11x^4 - \cdots
\end{array}$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n \quad \text{for} \quad -1 < x < 1$$

Exercise

Determine the interval of convergence and the sum of the series

$$1-4x+16x^2-64x^3+\cdots=\sum_{n=0}^{\infty}(-1)^n(4x)^n$$

Solution

$$1-4x+16x^2-64x^3+\cdots=1+(-4x)+(-4x)^2+(-4x)^3+\cdots \qquad \frac{1}{1-u}=1+u+u^2+u^3+\cdots$$

$$=\frac{1}{1-(-4x)}$$

$$=\frac{1}{1+4x}$$

Therefore; the interval of convergence is $-\frac{1}{4}<x<\frac{1}{4}$

Exercise

Determine the interval of convergence and the sum of the series

$$3+4x+5x^2+6x^3+\cdots=\sum_{n=0}^{\infty}(n+3)x^n$$

Solution

$$\sum_{n=0}^{\infty}(n+3)x^n=\sum_{n=0}^{\infty}nx^n+\sum_{n=0}^{\infty}3x^n$$

$$\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\cdots=\frac{1}{1-x}$$

$$\left(\sum_{n=0}^{\infty}x^n\right)'=1+2x+3x^2+4x^3+\cdots \qquad \textit{Differentiate}$$

$$=\frac{1}{(1-x)^2}$$

$$x\left(1+2x+3x^2+4x^3+\cdots\right)=\frac{x}{(1-x)^2} \qquad \textit{Multiply by } x$$

$$\sum_{n=0}^{\infty}nx^n=x+2x^2+3x^3+\cdots$$

$$=\frac{x}{(1-x)^2}$$

Then,

$$\sum_{n=0}^{\infty}(n+3)x^n = \sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} 3x^n$$

$$= \frac{x}{(1-x)^2} + 3\frac{1}{1-x}$$

$$= \boxed{\frac{3-2x}{(1-x)^2}} \qquad (-1 < x < 1)$$

Exercise

Determine the interval of convergence and the sum of the series

$$\frac{1}{3}+\frac{x}{4}+\frac{x^2}{5}+\frac{x^3}{6}+\cdots = \sum_{n=0}^{\infty} \frac{x^n}{n+3}$$

Solution

$$\frac{1}{3}+\frac{x}{4}+\frac{x^2}{5}+\frac{x^3}{6}+\cdots = \frac{1}{x^3}\left(\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\frac{x^6}{6}+\cdots\right)$$

$$= \frac{1}{x^3}\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\frac{x^6}{6}+\cdots - x - \frac{x^2}{2}\right) \qquad \ln(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots$$

$$= \frac{1}{x^3}\left(-\ln(1-x) - x - \frac{x^2}{2}\right)$$

$$= \boxed{-\frac{1}{x^3}\ln(1-x) - \frac{1}{x^2} - \frac{1}{2x}} \qquad (-1 \le x < 1, \;\; x \ne 0)$$