Section 4.8 – Connectivity

Paths

A path is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph.

Definitions

Let G be a graph, and let v and w be vertices in G.

A ***walk*** from v to w is a finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n$$

Where the v's represent vertices, the e's represents edges, $v_0 = v$, $v_n = w$ and for all $i = 1, 2, \ldots, n$, v_{i-1} *and* v_i are the endpoints of e_i. The trivial walk from v to v consists of the single vertex v.

A ***trail*** from v to w is a walk from v to w that does not contain a repeated edge.

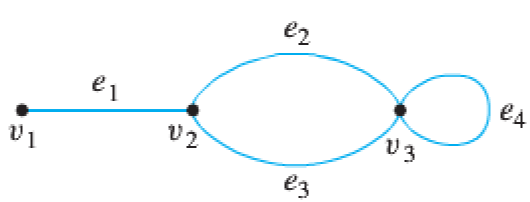
A ***closed walk*** is a walk that starts and ends at the same vertex.

A ***circuit*** is a closed walk that contains at least one edge and does not contain a repeated edge.

A ***simple circuit*** is a circuit that does not any other repeated vertex except first and last.

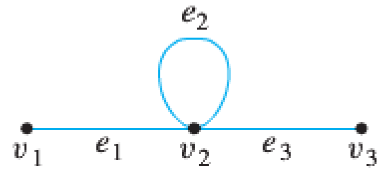
	Repeated Edge?	**Repeated Vertex**	**Starts & Ends at Same Point?**	**Must Contain at Least One Edge?**
Walk	Allowed	Allowed	Allowed	No
Trail	No	Allowed	Allowed	No
Path	No	No	No	No
Closed Walk	Allowed	Allowed	Yes	Yes
Circuit	No	Allowed	Yes	Yes
Simple Circuit	No	First & last only	Yes	Yes

Notation for Walks



The notation $e_1 e_2 e_4 e_3$ refers unambiguous to the following walk: $v_1 e_1 v_2 e_2 e_4 v_3 e_3 v_2$. On the other hand, the notation e_1 is ambiguous if used to refer to a walk. It could mean either $v_1 e_1 v_1$ or $v_2 e_1 v_1$.

The notation $v_2 v_3$ is ambiguous if used to refer to a walk. It could mean $v_2 e_2 v_3$ or $v_2 e_3 v_3$. On the other hand,



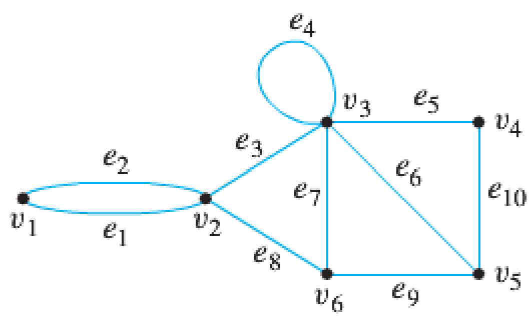
The notation $v_1 v_2 v_2 v_3$ refers unambiguously to the walk $v_1 e_1 v_2 e_2 v_2 e_3 v_3$

Example

Determine which of the following walks are trails, paths, circuits, or simple circuits to the graph below.

a) $v_1 e_1 v_2 e_3 v_3 e_4 v_3 e_5 v_4$ *b)* $e_1 e_3 e_5 e_5 e_6$ *c)* $v_2 v_3 v_4 v_5 v_3 v_6 v_2$

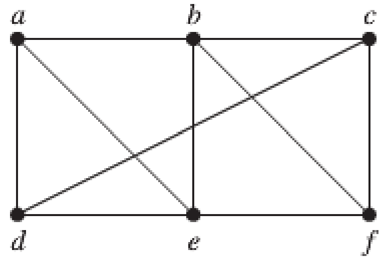
d) $v_2 v_3 v_4 v_5 v_6 v_2$ *e)* $v_1 e_1 v_2 e_1 v_1$ *f)* v_1



Solution

- ***a)*** This walk has a repeated vertex but does not have a repeated edge, so it is a trail from v_1 *to* v_4 but not a path.
- ***b)*** This is just a walk from v_1 *to* v_5. It is not a trail because it has a repeated edge.
- ***c)*** This walk starts and ends at v_2, contains at least one edge, and does not have a repeated edge, so it is a circuit. Since the vertex v_3 is repeated in the middle, it is not a simple circuit.
- ***d)*** This walk starts and ends at v_2, contains at least one edge, and does not have a repeated edge, and does not have a repeated vertex. Thus it is a simple circuit.
- ***e)*** This is just a closed walk starting and ending at v_1. It is not a circuit because edge e_1 is repeated.
- ***f)*** The first vertex of this walk is the same as its last vertex, but it does not contain an edge, and so it is not a circuit, It is a closed walk from v_1 *to* v_1. (It is also a trail from v_1 *to* v_1)

Example



The given graph, *a, d, c, f, e* is a simple path of length 4, because $\{a, d\}$, $\{d, c\}$, $\{c, f\}$, and $\{f, e\}$ are all edges. However, *d, e, c, a* is not a path, because $\{e, c\}$ is not an edge.
Note that *b, c, f, e, b* is a circuit of length 4 because $\{b, c\}$, $\{c, f\}$, $\{f, e\}$, and $\{e, b\}$ are edges, and this path begins and ends at *b*. The path *a, b, e, d, a, b*, which is of length 5, is not simple because it contains the edge $\{a, b\}$ twice.

Connectedness

Definition

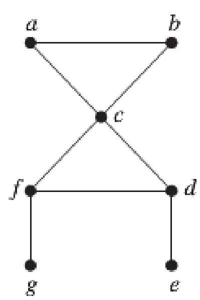
Let *G* be a graph. Two vertices *v* and *w* of *G* are ***connected*** if, and only if, there is a walk from *v* to *w*. The graph *G* is connected if, and only if, given any two vertices *v* and *w* in *G*, there is a walk from *v* to *w*. Symbolically,

$$G \text{ is connected} \quad \Leftrightarrow \quad \forall \text{ vertices } v,\ w \in V(G), \exists \text{ a walk from } v \text{ to } w.$$

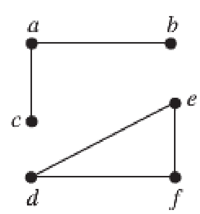
Definition

An undirected graph is called ***connected*** if there is a path between every pair of distinct vertices of the graph. An undirected graph that is not *connected* is called ***disconnected***. We say that we *disconnect* a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.

Example



G_1



G_2

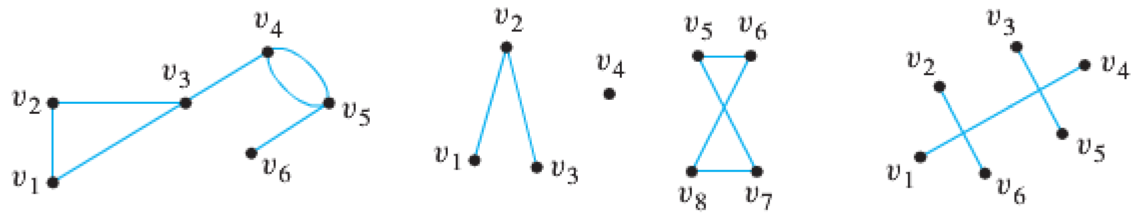
The graph G_1 is connected, because for every pair of distinct vertices there is a path between them.

However, the graph G_2 is not connected. For instance, there is no path in G_2 between vertices *a* and *b*.

Connected and Disconnected Graphs

Example

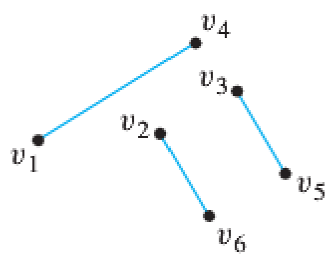
Which of the following graphs are connected?



(*a*) **(*b*)** **(*c*)**

<u>Solution</u>

The graph represented in (*a*) is connected, whereas those of (*b*) and (*c*) are not.
To understand why (*c*) is not connected, two edges may cross at a point that is not a vertex.
Thus the graph in (*c*) can be drawn as follows:



Theorem

There is a simple path between every pair distinct vertices of a connected undirected graph.

Proof

Let u and v be two distinct vertices of the connected undirected graph $G = (V,\ E)$. Because G is connected, there is at least one path between u and v. Let $x_0,\ x_1,\ \ldots,\ x_n$ where $x_0 = u$ and $x_n = v$, be the vertex sequence of a path of least length. This path of least length is simple. To see this, suppose it is not simple. Then $x_i = x_j$ for some i and j with $0 \le i < j$. This means that there is a path from u to v of shorter length with vertex sequence $x_0,\ x_1,\ \ldots,\ x_{i-1}\ x_j,\ \ldots,\ x_n$ obtained by deleting the edges corresponding to the vertex sequence $x_i,\ \ldots,\ x_{j-1}$

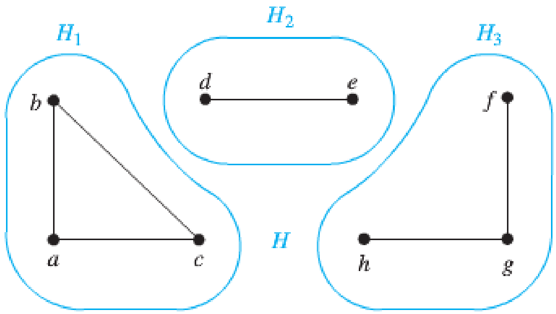
Definition

A graph H is a connected component of a graph G if, and only if,

- H is subgraph of G;
- H is connected; and
- No connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H.

Example

What are the connected components of the graph H shown below?

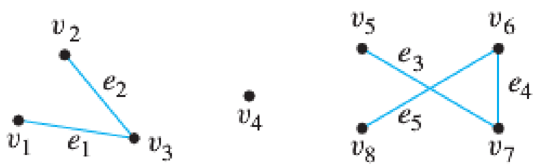


<u>***Solution***</u>

The graph H is the union of the three disjoint connected subgraphs $H_1,\ H_2,\ and\ H_3$. These three subgraphs are the connected components of *H*.

Example

Find all connected components of the following graph *G*.



<u>***Solution***</u>

G has three connected components: $H_1,\ H_2,\ and\ H_3$ with vertex sets $V_1,\ V_2,\ and\ V_3$ and edges $E_1,\ E_2,\ and\ E_3$, where

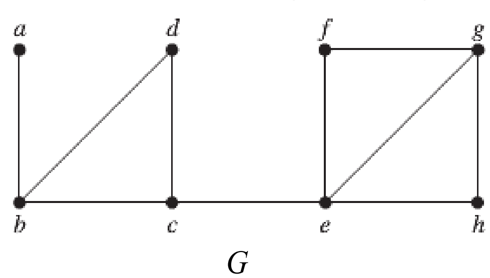
$$V_1 = \{v_1,\ v_2,\ v_3\} \qquad E_1 = \{e_1,\ e_2\}$$

$$V_2 = \{v_4\} \qquad E_2 = \varnothing$$

$$V_3 = \{v_5,\ v_6,\ v_7,\ v_8\} \qquad E_3 = \{e_3,\ e_4,\ e_5\}$$

Example

Find the cut vertices and cut edges in the graph G.



<u>***Solution***</u>

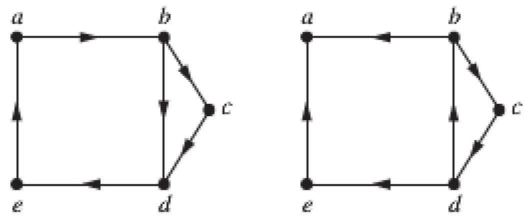
The cut vertices of *G* are *b, c*, and *e*. The removal of one of these vertices (and its adjacent edges) disconnects the graph. The cut edges are {*a, b*} and {*c, e*}. Removing either one of these edges disconnects *G*.

Definition

A directed graph is weakly connected of there is a path between every two vertices in the underlying undirected graph.

Example

Are the directed graphs *G* and *H* shown below strongly connected? Are they weakly connected?



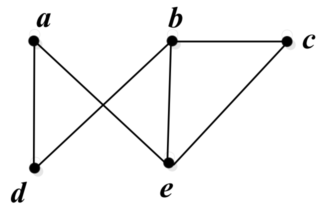
G *H*

Solution

G is strongly connected because there is a path between any two vertices in this directed graph. Hence, *G* is also weakly connected.
The graph *H* is not strongly connected. There is no directed path from *a* to *b* in this graph. However, *H* is weakly connected, because there is a path between any 2 vertices in the underlying undirected graph of *H*.

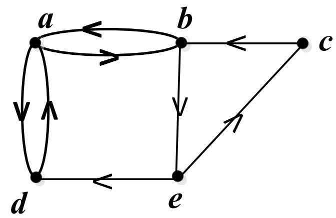
Exercises *Section* 4.8 – Connectivity

1. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? Which are the lengths of those that are paths?



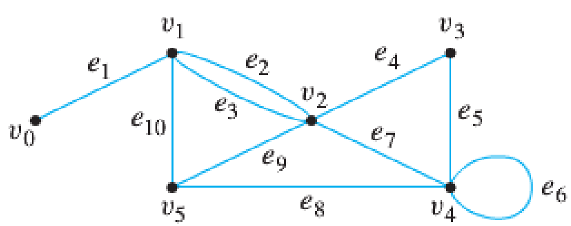
a) *a, e, b, c, b* ***b)*** *a, e, a, d, b, c, a* ***c)*** *e, b, a, d, b, e* ***d)*** *c, b , d, a, e, c*

2. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? Which are the lengths of those that are paths?



a) *a, b, e, c, b* ***b)*** *a, d, a, d, a* ***c)*** *a, d, b, e, a* ***d)*** *a, b , e, c, b, d, a*

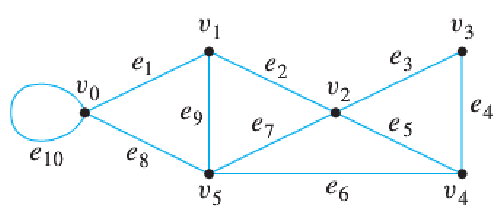
3. Determine whether of the following walks are trails, paths, circuits, or simple circuits or just walk to the graph below.



a) $v_0e_1v_1e_{10}v_5e_9v_2e_2v_1$ *b)* $v_4e_7v_2e_9v_5e_{10}v_1e_3v_2e_9v_5$ *c)* v_2

d) $v_5v_2v_3v_4v_4v_5$ *e)* $v_2v_3v_4v_5v_2v_4v_3v_2$ *f)* $e_5e_8e_{10}e_3$

4. Determine whether of the following walks are trails, paths, circuits, or simple circuits or just walk to the graph below.



a) $v_1e_2v_2e_3v_3e_4v_4e_5v_2e_2v_1e_1v_0$ *b)* $v_2v_3v_4v_5v_2$ *c)* $v_4v_2v_3v_4v_5v_2v_4$

d) $v_2v_1v_5v_2v_3v_4v_2$ *e)* $v_0v_5v_2v_3v_4v_2v_1$ *f)* $v_5v_4v_2v_1$