Multivariable

$$A=\left(x_1,\, y_1,\, z_1\right) \quad B=\left(x_2,\, y_2,\, z_2\right)$$

Distance: $$d=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2}$$

Midpoint: $$M=\left(\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2},\ \frac{z_1+z_2}{2}\right)$$

Sphere Equation: $$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Equation of a line in space: $$\begin{cases} x=x_1+\left(x_2-x_1\right)t \\ y=y_1+\left(y_2-y_1\right)t \\ z=z_1+\left(z_2-z_1\right)t \end{cases}$$

$$u=\left(u_1,\, u_2,\, u_3\right) \quad v=\left(v_1,\, v_2,\, v_3\right)$$

Length (or ***Norm*** or ***Magnitude***): $$\|v\|=|v|=\sqrt{v_1^2+v_2^2+v_3^2}$$

Dot (Inner) Product: $$u\bullet v=|u|\ |v|\ \cos\theta$$

$$u\bullet v=u_1v_1+u_2v_2+u_3v_3$$

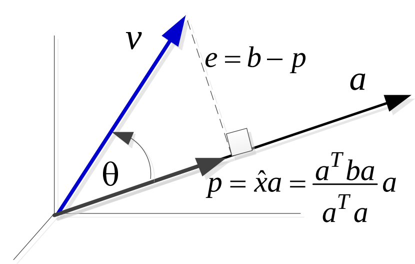
Angle between 2 vectors: $$\theta=\cos^{-1}\left(\frac{u\bullet v}{|u|\ |v|}\right)$$

$$\theta=\cos^{-1}\frac{u_1v_1+u_2v_2+u_3v_3}{\left(\sqrt{u_1^2+u_2^2+u_3^2}\right)\left(\sqrt{v_1^2+v_2^2+v_3^2}\right)}$$

Projection matrix $$p=\hat{x}a=\frac{a^Tv}{a^Ta}a$$

Length $$\|p\|=\|v\|\cos\theta$$

Error $$e=v-p$$



- Two vectors are ***parallel*** *iff* they are scalar multiples of each other.
- Two vectors A and B are ***orthogonal*** (*perpendicular*) *iff* $A \bullet B = 0$
- A set of vectors $v_1, v_2, \cdots, v_n$ are ***linearly dependent*** iff there are scalars $c_1, c_2, \cdots, c_n$ not all zero such that $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$

  They are ***linearly independent*** if no such collection of scalars exist.

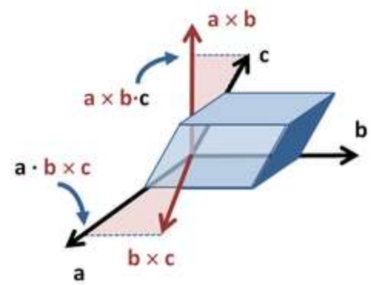
Properties of Vector Product

- ✓ $A \times B$ is a vector perpendicular to both A and B.
- ✓ $A \times B = -(B \times A)$
- ✓ $A \times A = 0$
- ✓ $A \times (B + C) = A \times B + A \times C$
- ✓ $(cA) \times B = A \times cB = c(A \times B)$ for any scalar c.
- ✓ The length of $A \times B$ is the ***area of the parallelogram*** spanned by A and B. This area is $\|A\|\|B\|\sin\theta$
- ✓ If A or B is 0 or A is parallel to B, then $A \times B$ is the vector 0.

Volume of the Parallelepiped is

$$V = (area\ of\ \ base).(height) = |u \bullet (v \times w)|$$

$$V = \left| \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \right|$$

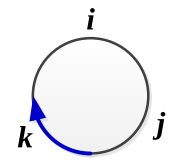


Schwarz Inequality: $\|A.B\| \le \|A\|.\|B\|$

Determinant: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow ad - bc$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Cross Product: $A\times B=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}=\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}\boldsymbol{i}-\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}\boldsymbol{j}+\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\boldsymbol{k}$



$$=\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}\boldsymbol{i}-\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}\boldsymbol{j}+\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\boldsymbol{k}$$

$$\boxed{=\left(y_1z_2-y_2z_1,\ x_1y_2-x_2z_1,\ x_1y_2-x_2z_1\right)}$$

Cross Product Properties

1. $\boldsymbol{u}\times\boldsymbol{v}$ reverses rows 2 and 3 in the determinant so it is equals $-(\boldsymbol{u}\times\boldsymbol{v})$

2. The cross product $\boldsymbol{u}\times\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$, then $\boldsymbol{u}.(\boldsymbol{u}\times\boldsymbol{v})=0$

3. The cross product $\boldsymbol{u}\times\boldsymbol{v}$ is perpendicular to $\mathbf{v}$, then $(\boldsymbol{u}\times\boldsymbol{v}).\boldsymbol{v}=0$

4. The cross product of any vector with itself (two equal rows) is $\boldsymbol{u}\times\boldsymbol{u}=0$.

5. $\boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{k},\quad \boldsymbol{j}\times\boldsymbol{k}=\boldsymbol{i},\quad \boldsymbol{k}\times\boldsymbol{i}=\boldsymbol{j}$

6. Lagrange's identity: $\|\boldsymbol{u}\times\boldsymbol{v}\|^2=\|\boldsymbol{u}\|^2\ \|\boldsymbol{v}\|^2-(\boldsymbol{u}.\boldsymbol{v})^2$

$$=\|\boldsymbol{u}\|\ \|\boldsymbol{v}\|\ |\sin\theta|$$

$$|\boldsymbol{u}.\boldsymbol{v}|=\|\boldsymbol{u}\|\ \|\boldsymbol{v}\|\ |\cos\theta|$$

Work: $W=\int_a^b f(x)\,dx$

Volume: $V=\int_a^b 2\pi x\left(y_2-y_1\right)dx$

Planes and Surfaces in Space

The plane through $P_0\left(x_0,\ y_0,\ z_0\right)$ normal to $\mathbf{n}=a\boldsymbol{i}+b\boldsymbol{j}+c\boldsymbol{k}$:

Vector Equation: $\mathbf{n}\cdot\overrightarrow{P_0P}=0$

Component equation: $a\left(x-x_0\right)+b\left(y-y_0\right)+c\left(z-z_0\right)=0$

$$ax+by+cz=d,\quad \textit{where}\quad d=ax_0+by_0+cz_0$$

Vector-valued Functions and Motion in Space

Let $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$ be a vector function.

Then $v(t) = \dfrac{dr}{dt}$ is the velocity vector and $|v(t)|$ is the speed.

The acceleration vector is $a(t) = \dfrac{dv}{dt} = \dfrac{d^2r}{dt^2}$.

The unit tangent vector is $T = \dfrac{v}{|v|}$ and the length of $\boldsymbol{r}(t)$ from $t = a$ to $t = b$ is $L = \displaystyle\int_a^b |v|\, dt$

Lines in Space

A vector equation of the line through $P_0(x_0, y_0, z_0)$ parallel to $v = (v_1, v_2, v_3)$ is $\boldsymbol{r}(t) = r_0 + tv$

$$\langle x, y, z\rangle = (x_0, y_0, z_0) + t(v_1, v_2, v_3), \text{ for } -\infty < t < \infty$$

Parametric equations for this line are:

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \text{ for } -\infty < t < \infty$$

Principal Unit Normal: $N = \dfrac{dT/dt}{\left|dT/dt\right|}$

Curvature: $\kappa = \left|\dfrac{d\boldsymbol{T}}{ds}\right| = \dfrac{1}{|v|}\left|\dfrac{d\boldsymbol{T}}{dt}\right| = \dfrac{|a \times v|}{|v|^3}$

Radius of ***Curvature***: $\rho = \dfrac{1}{\kappa}$

Tangential and normal scalar components of acceleration:

$$a = a_T\boldsymbol{T} + a_N\boldsymbol{N} \quad \begin{cases} a_T = \dfrac{d^2s}{dt^2} = \dfrac{a \cdot v}{|v|} \\ a_N = \kappa\left(\dfrac{ds}{dt}\right)^2 = \kappa|v|^2 = \dfrac{|a \times v|}{|v|} \end{cases}$$

Quadratic Surfaces

Ellipsoid: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Elliptic Paraboloid: $$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

Elliptic Cone: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$$

Hyperboloid of one sheet: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Hyperboloid of two sheets: $$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Hyperbolic Paraboloid: $$z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

Integration in Vector Fields

Line Integrals

To integrate a continuous function $f\left(x,\ y,\ z\right)$ over a curve C:

- $r\left(t\right)=x\left(t\right)\boldsymbol{i}+y\left(t\right)\boldsymbol{j}+z\left(t\right)\boldsymbol{k} \quad a\le t\le b$
- $\displaystyle\int_C f\left(x,\ y,\ z\right)ds=\int_a^b f\left(x\left(t\right),y\left(t\right),z\left(t\right)\right)\cdot\left|v\left(t\right)\right|\ dt$

Where $v\left(t\right)=\dfrac{dr}{dt} \quad and \quad ds=\left|v\left(t\right)\right|dt$

F conservative on D $\quad\Leftrightarrow\quad F=\nabla\varphi$ for some potential function φ

$\Leftrightarrow\quad \displaystyle\oint_C F\cdot dx=0$ over closed paths C in D

$\Leftrightarrow\quad \displaystyle\int_C F\cdot dx$ is independent of path for C in D.

$\Leftrightarrow\quad \nabla\times F=0$ on D.

Work: $\quad W=\displaystyle\int_C \boldsymbol{F}\cdot\boldsymbol{T}\ ds=\int_a^b F\cdot r'\left(t\right)\ dt=\int_a^b F\cdot dx=\int_C\left(fdx+gdy+hdz\right)$

Circulation of F on C: $\displaystyle\oint_C \boldsymbol{F}\cdot\boldsymbol{T}\ ds$

Flux of F across C = $\displaystyle\oint_C \boldsymbol{F}\cdot\boldsymbol{n}\ ds=\oint_C\left(fdy-gdx\right)$

Where $\boldsymbol{F}=\left\langle f,\ g\right\rangle$ and **n** is outward pointing normal along C.

Conservative Vector Field

$\boldsymbol{F}$ is conservative if $\boldsymbol{F} = \nabla\varphi$ for some function $\varphi(x,\ y,\ z)$. If $\boldsymbol{F}$ is conservative, then $\int_A^B \boldsymbol{F} \cdot d\boldsymbol{r}$ is independent of path and $\int_A^B \boldsymbol{F} \cdot d\boldsymbol{r} = \varphi(B) - \varphi(A)$

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \textit{and} \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$$

Fundamental Theorem of Calculus

Fundamental Theorem of Calculus $\quad \int_a^b f'(x)\,dx = f(b) - f(a)$

Fundamental Theorem of Line Integrals $\quad \int_C \nabla f \cdot dr = f(B) - f(A)$

Green's Theorem

$$\oint_C \boldsymbol{F}\cdot\boldsymbol{n}\, ds = \oint_C (f dy - g dx) = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA$$

outward flux *Divergence integral*

$$\oint_C \boldsymbol{F}\cdot\boldsymbol{T}\, ds = \oint_C (f dy + g dx) = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

ccw *circulation* *Curl integral*

$$Area\ of\ R = \oint_C x dy = -\oint_C y dx = \frac{1}{2}\oint_C (x dy - y dx)$$

Circulation: $\oint_C \boldsymbol{F}\cdot dr = \iint_R \nabla\times\boldsymbol{F}\cdot\boldsymbol{k}\, dA$

Flux: $\oint_C \boldsymbol{F}\cdot\boldsymbol{n}\, ds = \iint_R \nabla\cdot\boldsymbol{F}\, dA$

Divergence: $\oint_S \boldsymbol{F}\cdot\boldsymbol{n}\, dS = \iiint_D \nabla\cdot\boldsymbol{F}\, dV$

Stokes' Theorem

$$\oint_C \boldsymbol{F}\cdot dr = \iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\, dS$$

ccw circulation *Curl integral*

Surface Integrals

$$\boldsymbol{r}(u,\ v) = \langle x(u,\ v), y(u,\ v), z(u,\ v)\rangle, \quad a \le u \le b, \quad c \le v \le d$$

Unit normal to the surface: $\dfrac{t_u \times t_v}{|t_u \times t_v|} = \boldsymbol{n}$

Area: Area of surface S = $\displaystyle\int_c^d \int_a^b |t_u \times t_v|\, du\, dv$

$$\iint_S f(x, y, z)\, dS = \int_c^d \int_a^b f\big(x(u,\ v), y(u,\ v), z(u,\ v)\big)\ |t_u \times t_v|\, du\, dv$$

Formulas from Vector Calculus

Assume $F(x,\ y,\ z) = f(x,\ y,\ z)\boldsymbol{i} + g(x,\ y,\ z)\boldsymbol{j} + h(x,\ y,\ z)\boldsymbol{k}$, where $f,\ g,\ and\ h$ are differentiable on a region D of $\boldsymbol{R}^3$.

Gradient: $\nabla f(x,\ y,\ z) = \dfrac{\partial f}{\partial x}\boldsymbol{i} + \dfrac{\partial g}{\partial y}\boldsymbol{j} + \dfrac{\partial h}{\partial z}\boldsymbol{k}$

Curl:

$$\begin{aligned} curl\ \boldsymbol{F} &= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\boldsymbol{k} \\ &= \nabla \times \boldsymbol{F} \end{aligned}$$

$$\nabla \times F(x,\ y,\ z) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

Divergence: $div\ \boldsymbol{F} = \nabla \bullet F(x,\ y,\ z) = \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} + \dfrac{\partial h}{\partial z}$

$$\oint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iint_D \nabla \cdot \boldsymbol{F}\ dV$$

outward flux *Divergence integral*

$$\iint_S F \cdot \boldsymbol{n}\, dS = \oint_D \nabla \cdot F\ dV$$

$$\nabla \times (\nabla f) = 0$$

$$\nabla \bullet (\nabla \times f) = 0$$

Multiple Integrals

Double Integrals as Volumes

When $f(x, y)$ is a positive function over a region R in the xy-plane, we may interpret the double integral of f over R as the volume of the 3-dimensional solid region over the xy-plane bounded below by R and above the surface $z = f(x, y)$.

This volume can be evaluated by computing an iterated integral

Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b, \quad g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on [a, b], then

$$\iint\limits_R f(x, y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dydx$$

2. If R is defined by $c \le y \le d, \quad h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on [c, d], then

$$\iint\limits_R f(x, y)\,dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x, y)\,dxdy$$

Area: $A = \iint\limits_R dA = \iint\limits_R dx\,dy = \iint\limits_R dy\,dx$

$$A = \iint\limits_R r\,dr\,d\theta \qquad (\textit{Polar Coordinates})$$

Triple Integrals

$$\iiint\limits_R f(x,y,z)\,dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x,y,z)\,dzdydx$$

$$a \le x \le b, \quad g(x) \le y \le h(x), \quad G(x,y) \le z \le H(x,y)$$

Cylindrical Coordinates $(r,\ \theta,\ z)$

Equations Relating Rectangular (x, y, z) and Cylindrical $(r,\ \theta,\ z)$ Coordinates

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

$$\iiint\limits_D f(r,\ \theta,\ z)\,dV = \iiint\limits_D f(r,\ \theta,\ z)\,dz\ r dr\ d\theta$$

Spherical Coordinates $(\rho,\ \varphi,\ \theta)$

$$\begin{aligned} x &= r\cos\theta = \rho\sin\varphi\cos\theta \\ y &= r\sin\theta = \rho\sin\varphi\sin\theta \\ z &= \rho\cos\varphi \\ r &= \rho\sin\varphi \end{aligned} \qquad \rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$$

$$\iiint\limits_D f(\rho,\ \varphi,\ \theta)\,dV = \iiint\limits_D f(\rho,\ \varphi,\ \theta)\ \rho^2\sin\varphi\ d\rho\ d\varphi\ d\theta$$

Change of Variables Formula for Double Integrals

$$\iint\limits_R f(x, y)\,dydx = \iint\limits_S f\big(g(u,v),\ h(u,v)\big)\ \left|J(u,v)\right|\ dA$$

***Jacobian* determinant**: $$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$