Section 3.7 – Power Series

Power Series and Converge

Definitions

A **power series about $x = 0$** is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$$

In which the ***center*** a and the ***coefficients*** $c_0, c_1, c_2, \cdots, c_n, \cdots$ are constants.

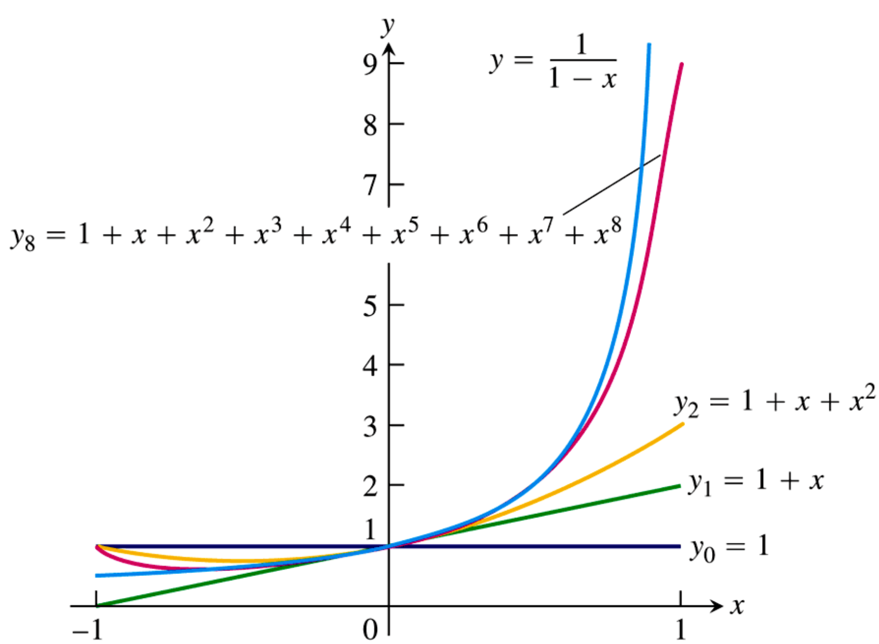
Example

Find the convergence of $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$

Solution

This is the geometric series with first term 1 and ratio x. it converges to $\frac{1}{1-x}$ *for* $|x| < 1$

$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1$



Example

The power series $1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2-\cdots+\left(-\frac{1}{2}\right)^n(x-2)^n+\cdots$

This is the geometric series with first term 1 and ratio $r=-\frac{x-2}{2}$. it converges to

$\left|\frac{x-2}{2}\right|<1 \quad for \quad 0<x<4$. The sum

$$\frac{1}{1-r}=\frac{1}{1+\frac{x-2}{2}}$$
$$=\frac{2}{x}$$

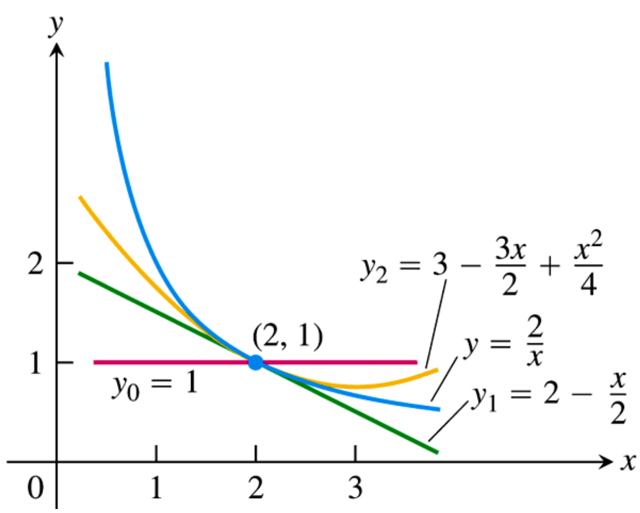
$$\frac{2}{x}=1-\frac{x-2}{2}+\frac{(x-2)^2}{4}-\cdots+\left(-\frac{1}{2}\right)^n(x-2)^n+\cdots$$

The series generates polynomial approximations of $f(x)=\frac{2}{x}$ for values of x near 2:

$P_0(x)=1$

$P_1(x)=1-\frac{1}{2}(x-2)=2-\frac{x}{2}$

$P_2(x)=1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2=3-\frac{3x}{2}+\frac{x^2}{4}$



Example

For what values of *x* do the power series converges? $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x}\right|$$

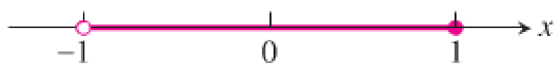
$$=\frac{n}{n+1}|x|\to|x|$$

The series converges absolutely for $|x|<1$. It diverges if $|x|>1$.

At $x=1$, we get the alternating harmonic series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$, which converges.

At $x=-1$, we get the alternating harmonic series $-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\cdots$, the negative of the harmonic series; it diverges.

The series ***converges*** for $-1<x\le 1$ and ***diverges*** elsewhere.



Example

For what values of *x* do the power series converges? $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{x^{2n+1}}{2n+1}\cdot\frac{2n-1}{x^{2n-1}}\right|$$

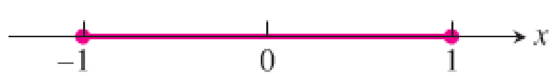
$$=\frac{2n-1}{2n+1}x^2\to x^2$$

The series converges absolutely for $x^2<1$. It diverges if $x^2>1$.

At $x=1$, we get the alternating harmonic series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$, which converges.

At $x=-1$, we get the alternating harmonic series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$, it converges.

The series ***converges*** for $-1\le x\le 1$ and ***diverges*** elsewhere.



Example

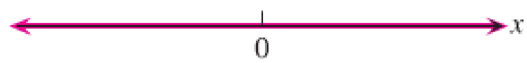
For what values of x do the power series converges? $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$$

$$= \frac{|x|}{n+1} \to 0 \quad (\forall x)$$

The series ***converges absolutely*** for all x.



Example

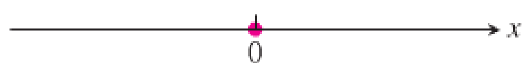
For what values of x do the power series converges? $\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$

Solution

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right|$$

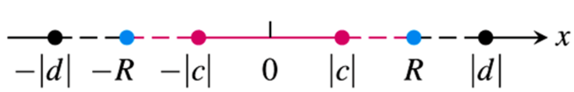
$$= (n+1)|x| \to \infty$$

The series ***diverges absolutely*** for all x except $x = 0$.



Theorem

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.



Radius of Convergence of a Power Series

Corollary to Theorem

The convergence of the series $\sum c_n (x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x $(R = \infty)$.
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$)

R is called the ***radius of convergence*** of the power series, and the interval of radius R centered at $x = a$ is called the ***interval of convergence***.

Definition

Suppose that $L = \lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$ exists or is ∞. Then the power series $\sum c_n (x-a)^n$ has radius of convergence $R = \dfrac{1}{L}$. (If $L = 0$, then $R = \infty$; if $L = \infty$, then $R = 0$) and $R = \lim\limits_{n\to\infty} \left|\dfrac{a_n}{a_{n+1}}\right|$

How to Test a Power Series for Convergence

1. Use the Ratio Test (or Root Test) to find the interval where the series converges. Ordinarily, this is an open interval
$$|x-a| < R \quad \textit{or} \quad a - R < x < a + R$$
2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint. Use the Comparison Test, the Integral Test, or the Alternating Series Test.
3. If the interval of absolute convergence is $a - R < x < a + R$, the series diverges for $|x-a| > R$ (it does not even converge conditionally) because the nth term does not approach zero for those values of x.

Example

Determine the centre, radius, and interval of convergence of $\displaystyle\sum_{n=0}^{\infty}\frac{(2x+5)^n}{(n^2+1)3^n}$

Solution

$$\sum_{n=0}^{\infty}\frac{(2x+5)^n}{(n^2+1)3^n}=\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n\frac{1}{n^2+1}\left(x+\frac{5}{2}\right)^n$$

The centre of convergence is

$$x+\frac{5}{2}=0 \quad\Rightarrow\quad \underline{x=-\frac{5}{2}}$$

$$L=\lim_{n\to\infty}\left|\frac{\left(\frac{2}{3}\right)^{n+1}\frac{1}{(n+1)^2+1}}{\left(\frac{2}{3}\right)^n\frac{1}{n^2+1}}\right|$$

$$=\lim_{n\to\infty}\frac{2}{3}\frac{n^2+1}{(n+1)^2+1}$$

$$\underline{=\frac{2}{3}}$$

$$R=\frac{1}{L}\ \underline{=\frac{3}{2}}$$

The series converges absolutely on ***interval***

$$\left(-\frac{5}{2}-\frac{3}{2},\ -\frac{5}{2}+\frac{3}{2}\right)=\underline{(-4,\ -1)} \qquad a-R<x<a+R$$

It diverges on $(-\infty,\ -4)\cup(-1,\ \infty)$

At $x=-4 \quad\Rightarrow\quad \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1}$

At $x=-1 \quad\Rightarrow\quad \displaystyle\sum_{n=0}^{\infty}\frac{3^n}{(n^2+1)3^n}=\sum_{n=0}^{\infty}\frac{1}{n^2+1}$

Both series ***converge*** (*absolutely*).

Therefore; the interval of convergence of the given power is $[-4,\ -1]$

Example

Determine the radius of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Solution

$$L = \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right|$$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{1}{n+1}$$

$$\underline{= 0|}$$

Thus $\underline{R = \infty}|$

Or $R = \lim_{n\to\infty} \left| \frac{1}{n!} \cdot (n+1)! \right|$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \right|$$

$$= \lim_{n\to\infty} (n+1)$$

$$\underline{= \infty}|$$

This series ***converges*** (*absolutely*) for all x.

Example

Determine the radius of convergence of $\sum_{n=0}^{\infty} n! x^n$

Solution

$$L = \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \right|$$

$$= \lim_{n\to\infty} (n+1)$$

$$\underline{= \infty}|$$

Thus $\underline{R = 0}|$

This series ***converges*** only at its centre of convergence, $x = 0$.

Theorem **– The Series Multiplication Theorem for Power Series**

If $A(x)=\sum_{n=0}^{\infty}a_n x^n$ and $B(x)=\sum_{n=0}^{\infty}b_n x^n$ converge absolutely for $|x|<R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$$

Then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x|<R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\cdot\left(\sum_{n=0}^{\infty} b_n x^n\right)=\sum_{n=0}^{\infty} c_n x^n$$

Finding the coefficients c_n

$$\left(\sum_{n=0}^{\infty} x^n\right)\cdot\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}\right)=\left(1+x+x^2+\cdots\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)$$

$$=\underbrace{\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)}_{by\ 1}+\underbrace{\left(x^2-\frac{x^3}{2}+\frac{x^4}{3}-\cdots\right)}_{by\ x}+\underbrace{\left(x^3-\frac{x^4}{2}+\frac{x^5}{3}-\cdots\right)}_{by\ x^2}+\cdots$$

$$=x+\frac{x^2}{2}+\frac{5x^3}{6}-\frac{x^4}{6}\cdots$$

Theorem

If $\sum_{n=0}^{\infty}a_n x^n$ converges absolutely for $|x|<R$, then $\sum_{n=0}^{\infty}a_n\left(f(x)\right)^n$ converges absolutely for any continuous function f on $|f(x)|<R$

Theorem – The term-by-Term Differentiation Theorem

If $\sum_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence $R > 0$, it defines a function.

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \textit{on the interval} \quad a-R < x < a+R$$

This function f has derivatives of all order inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n (x-a)^{n-2}$$

And so on. Each of these derived series converges at every point of the interval $a-R < x < a+R$

Example

Find the series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

$$= \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1$$

Solution

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} nx^{n-1}$$

$$f''(x) = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + 20x^3 + \cdots + n(n-1)x^{n-1} + \cdots$$

$$= \sum_{n=2}^{\infty} n(n-1)x^{n-2} \qquad \boxed{-1 < x < 1}$$

Theorem – The term-by-Term Integration Theorem

Suppose that $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $a-R<x<a+R \quad (R>0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

Converges $a-R<x<a+R$ and

$$\int f(x)\,dx=\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}+C \quad \textit{for} \quad a-R<x<a+R$$

Example

Identify the function $f(x)=\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots, \quad -1\le x\le 1$

Solution

$f'(x)=1-x^2+x^4-x^6+\cdots, \quad -1\le x\le 1$

This is a geometric series with first term 1 and ratio $-x^2$, so

$f'(x)=\frac{1}{1-\left(-x^2\right)}=\frac{1}{1+x^2}$

$$\int f'(x)\,dx=\int \frac{dx}{1+x^2}=\tan^{-1}x+C$$

The series for $f(x=0)=0$

$$\tan^{-1}0+C=0\to \boxed{C=0}$$

$f(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots$

$\quad =\tan^{-1}x, \quad -1<x<1$

Exercises *Section* 3.7 – Power Series

(**1 – 9**)

(a) Find the series' radius and interval of convergence. For what values of x does the series converge (***b***) absolutely, (***c***) conditionally?

1. $\displaystyle\sum_{n=0}^{\infty} x^n$

2. $\displaystyle\sum_{n=0}^{\infty} (x+5)^n$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$

5. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{\sqrt{n}+3}$

8. $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{n}\,(2x+5)^n$

9. $\displaystyle\sum_{n=1}^{\infty} \left(2+(-1)^n\right)\cdot(x+1)^{n-1}$

(**10 – 18**) Find the radius of convergence of the power series

10. $\displaystyle\sum_{n=0}^{\infty} n!x^n$

11. $\displaystyle\sum_{n=0}^{\infty} 3(x-2)^n$

12. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

13. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$

14. $\displaystyle\sum_{n=0}^{\infty} (3x)^n$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$

16. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{5^n}$

17. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

18. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{n!}$

(**19 – 42**) Find the interval of convergence of the power series

19. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

20. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^n}{2^n}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

22. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$

23. $\displaystyle\sum_{n=0}^{\infty} (2x)^n$

24. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$

25. $\displaystyle\sum_{n=0}^{\infty} (-1)^n (n+1)x^n$

26. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$

27. $\displaystyle\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$

28. $\displaystyle\sum_{n=0}^{\infty} (2n)!\left(\frac{x}{3}\right)^n$

29. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{(n+1)(n+2)}$

30. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{6^n}$

31. $\sum_{n=0}^{\infty}(-1)^n \frac{n!(x-5)^n}{3^n}$

32. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{(x-4)^n}{n9^n}$

33. $\sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$

34. $\sum_{n=0}^{\infty}(-1)^{n+1} \frac{(x-1)^{n+1}}{n+1}$

35. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{(x-2)^{n+1}}{n2^n}$

36. $\sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{3^{n-1}}$

37. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

38. $\sum_{n=1}^{\infty} \frac{n}{n+1}(-2x)^{n-1}$

39. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$

40. $\sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!}$

41. $\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$

42. $\sum_{n=1}^{\infty} \frac{2\cdot 3\cdot 4\cdots(n+1)x^n}{n!}$

(**43 – 56**) Determine the centre, radius, and interval of convergence of each of the power series

43. $\sum_{n=0}^{\infty} \frac{x^{2n}}{\sqrt{n+1}}$

44. $\sum_{n=0}^{\infty} 3n(x+1)^n$

45. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} x^n$

46. $\sum_{n=1}^{\infty} \frac{e^n}{n^3}(4-x)^n$

47. $\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n}$

48. $\sum_{n=0}^{\infty} \frac{1+5^n}{n!} x^n$

49. $\sum \frac{n^2 x^n}{n!}$

50. $\sum \frac{x^{4n}}{n^2}$

51. $\sum (-1)^n \frac{(x+1)^{2n}}{n!}$

52. $\sum \frac{(x-1)^n}{n\cdot 5^n}$

53. $\sum \left(\frac{x}{9}\right)^{3n}$

54. $\sum \frac{(x+2)^n}{\sqrt{n}}$

55. $\sum \frac{(x+2)^k}{2^k \ln k}$

56. $x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots$

57. For what values of x does the series $1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2+\cdots+\left(-\frac{1}{2}\right)^n(x-3)^n+\cdots$ converges? What is its sum? What series do you get if you differentiate the given series term by term? For what values of x does the new series converge? What is its sum?

58. The series $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\frac{x^{11}}{11!}+\cdots$ converges to sin$x$ for all x.

a) Find the first six terms of a series for cos$x$. For what values of x should the series converge?

b) By replacing x by $2x$ in the series for sin$x$, find a series that converges to sin$2x$ for all x.

c) Using the result in part (a) and series multiplication, calculate the first six term of a series for $2\sin x\cos x$. Compare your answer with the answer in part (b).

59. Find the sum of the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ by the first finding the sum of the power series

$$\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \cdots$$

60. Find a series representation of $f(x) = \frac{1}{2+x}$ in powers of $x-1$. What is the interval of convergence of this series?

61. Determine the Cauchy product of the series $1 + x + x^2 + x^3 + \cdots$ *and* $-x + x^2 - x^3 + \cdots$. On what interval and to what function does the product series converge?

62. Determine the power series expansion of $\frac{1}{(1-x)^2}$ by formally dividing $1 - 2x + x^2$ into 1.

Use the power series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad -1 < x < 1$

(63 – 65) Determine the interval of convergence and the sum of each of the series

63. $1 - 4x + 16x^2 - 64x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n (4x)^n$

64. $3 + 4x + 5x^2 + 6x^3 + \cdots = \sum_{n=0}^{\infty} (n+3) x^n$

65. $\frac{1}{3} + \frac{x}{4} + \frac{x^2}{5} + \frac{x^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n+3}$