

Solution **Section 3.4 – Properties of Logarithms**

Exercise

Express the following in terms of sums and differences of logarithms: $\log_3(ab)$

Solution

$$\log_3(ab) = \log_3 a + \log_3 b$$

Exercise

Express the following in terms of sums and differences of logarithms: $\log_7(7x)$

Solution

$$\begin{aligned}\log_7(7x) &= \log_7 7 + \log_7 x \\ &= 1 + \log_7 x\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms: $\log \frac{x}{1000}$

Solution

$$\begin{aligned}\log \frac{x}{1000} &= \log x - \log 10^3 \\ &= \log x - 3\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_5 \left(\frac{125}{y} \right)$

Solution

$$\begin{aligned}\log_5 \left(\frac{125}{y} \right) &= \log_5 5^3 - \log_5 y \\ &= 3 - \log_5 y\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b x^7$

Solution

$$\log_b x^7 = 7 \log_b x$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln \sqrt[7]{x}$

Solution

$$\begin{aligned} \ln \sqrt[7]{x} &= \ln x^{1/7} \\ &= \frac{1}{7} \ln x \end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \frac{x^2 y}{z^4}$

Solution

$$\begin{aligned} \log_a \frac{x^2 y}{z^4} &= \log_a x^2 y - \log_a z^4 \\ &= \log_a x^2 + \log_a y - \log_a z^4 \\ &= 2 \log_a x + \log_a y - 4 \log_a z \end{aligned}$$

Quotient Rule

Product Rule

Power Rule

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \frac{x^2 y}{b^3}$

Solution

$$\begin{aligned} \log_b \left(\frac{x^2 y}{b^3} \right) &= \log_b x^2 y - \log_b b^3 \\ &= \log_b x^2 + \log_b y - \log_b b^3 \\ &= 2 \log_b x + \log_b y - 3 \log_b b \\ &= 2 \log_b x + \log_b y - 3 \end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \left(\frac{x^3 y}{z^2} \right)$

Solution

$$\begin{aligned} \log_b \left(\frac{x^3 y}{z^2} \right) &= \log_b (x^3 y) - \log_b z^2 \\ &= \log_b x^3 + \log_b y - \log_b z^2 \\ &= \underline{3 \log_b x + \log_b y - 2 \log_b z} \end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \left(\frac{\sqrt[3]{xy^4}}{z^5} \right)$

Solution

$$\begin{aligned} \log_b \left(\frac{\sqrt[3]{xy^4}}{z^5} \right) &= \log_b (\sqrt[3]{xy^4}) - \log_b (z^5) \\ &= \underline{\log_b (x^{1/3}) + \log_b (y^4) - \log_b (z^5)} \end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log \left(\frac{100x^3 \sqrt[3]{5-x}}{3(x+7)^2} \right)$

Solution

$$\begin{aligned} \log \left(\frac{100x^3 \sqrt[3]{5-x}}{3(x+7)^2} \right) &= \log (100x^3 \sqrt[3]{5-x}) - \log (3(x+7)^2) \\ &= \log 10^2 + \log x^3 + \log (5-x)^{1/3} - \left[\log 3 + \log ((x+7)^2) \right] \\ &= 2 \log 10 + 3 \log x + \frac{1}{3} \log (5-x) - \log 3 - 2 \log (x+7) \\ &= \underline{2 + 3 \log x + \frac{1}{3} \log (5-x) - \log 3 - 2 \log (x+7)} \end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \sqrt[4]{\frac{m^8 n^{12}}{a^3 b^5}}$

Solution

$$\begin{aligned}
\log_a \sqrt[4]{\frac{m^8 n^{12}}{a^3 b^5}} &= \log_a \left(\frac{m^8 n^{12}}{a^3 b^5} \right)^{1/4} && \text{Power Rule} \\
&= \frac{1}{4} \log_a \left(\frac{m^8 n^{12}}{a^3 b^5} \right) && \text{Quotient Rule} \\
&= \frac{1}{4} \left[\log_a m^8 n^{12} - \log_a a^3 b^5 \right] && \text{Product Rule} \\
&= \frac{1}{4} \left[\log_a m^8 + \log_a n^{12} - \left(\log_a a^3 + \log_a b^5 \right) \right] && \text{Power Rule} \\
&= \frac{1}{4} \left[8 \log_a m + 12 \log_a n - 3 - 5 \log_a b \right] \\
&= \underline{2 \log_a m + 3 \log_a n - \frac{3}{4} - \frac{5}{4} \log_a b}
\end{aligned}$$

Exercise

Use the properties of logarithms to rewrite: $\log_p \sqrt[3]{\frac{m^5 n^4}{t^2}}$

Solution

$$\begin{aligned}
\log_p \sqrt[3]{\frac{m^5 n^4}{t^2}} &= \log_p \left(\frac{m^5 n^4}{t^2} \right)^{1/3} && \text{Power Rule} \\
&= \frac{1}{3} \log_p \left(\frac{m^5 n^4}{t^2} \right) && \text{Quotient Rule} \\
&= \frac{1}{3} \left(\log_p m^5 n^4 - \log_p t^2 \right) && \text{Product Rule} \\
&= \frac{1}{3} \left(\log_p m^5 + \log_p n^4 - \log_p t^2 \right) && \text{Power Rule} \\
&= \frac{1}{3} \left(5 \log_p m + 4 \log_p n - 2 \log_p t \right) \\
&= \underline{\frac{5}{3} \log_p m + \frac{4}{3} \log_p n - \frac{2}{3} \log_p t}
\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \sqrt[n]{\frac{x^3 y^5}{z^m}}$

Solution

$$\log_b \sqrt[n]{\frac{x^3 y^5}{z^m}} = \log_b \left(\frac{x^3 y^5}{z^m} \right)^{1/n}$$

$$\begin{aligned}
&= \frac{1}{n} \log_b \left(\frac{x^3 y^5}{z^m} \right) && \text{Power Rule} \\
&= \frac{1}{n} \left(\log_b x^3 y^5 - \log_b z^m \right) && \text{Quotient Rule} \\
&= \frac{1}{n} \left(\log_b x^3 + \log_b y^5 - \log_b z^m \right) && \text{Product Rule} \\
&= \frac{1}{n} \left(3 \log_b x + 5 \log_b y - m \log_b z \right) && \text{Power Rule} \\
&= \underline{\frac{3}{n} \log_b x + \frac{5}{n} \log_b y - \frac{m}{n} \log_b z}
\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \sqrt[3]{\frac{a^2 b}{c^5}}$

Solution

$$\begin{aligned}
\log_a \sqrt[3]{\frac{a^2 b}{c^5}} &= \log_a \left(\frac{a^2 b}{c^5} \right)^{1/3} && \text{Convert the radical to power} \\
&= \frac{1}{3} \log_a \left(\frac{a^2 b}{c^5} \right) && \text{Power Rule} \\
&= \frac{1}{3} \left[\log_a a^2 b - \log_a c^5 \right] && \text{Quotient Rule} \\
&= \frac{1}{3} \left[\log_a a^2 + \log_a b - \log_a c^5 \right] && \text{Product Rule} \\
&= \frac{1}{3} \left[2 \log_a a + \log_a b - 5 \log_a c \right] && \text{Power Rule} \\
&= \frac{2}{3} \log_a a + \frac{1}{3} \log_a b - \frac{5}{3} \log_a c \\
&= \underline{\frac{2}{3} + \frac{1}{3} \log_a b - \frac{5}{3} \log_a c}
\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \left(x^4 \sqrt[3]{y} \right)$

Solution

$$\begin{aligned}
\log_b \left(x^4 \sqrt[3]{y} \right) &= \log_b \left(x^4 \right) + \log_b \left(\sqrt[3]{y} \right) \\
&= \log_b \left(x^4 \right) + \log_b \left(y^{1/3} \right)
\end{aligned}$$

$$= 4 \log_b x + \frac{1}{3} \log_b y$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_5 \left(\frac{\sqrt{x}}{25y^3} \right)$

Solution

$$\begin{aligned} \log_5 \left(\frac{\sqrt{x}}{25y^3} \right) &= \log_5 \left(x^{1/2} \right) - \log_5 \left(25y^3 \right) \\ &= \log_5 \left(x^{1/2} \right) - \left[\log_5 \left(5^2 \right) + \log_5 \left(y^3 \right) \right] \\ &= \log_5 \left(x^{1/2} \right) - \log_5 \left(5^2 \right) - \log_5 \left(y^3 \right) \\ &= \frac{1}{2} \log_5 (x) - 2 \log_5 (5) - 3 \log_5 (y) \\ &= \frac{1}{2} \log_5 (x) - 2 - 3 \log_5 (y) \end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \frac{x^3 w}{y^2 z^4}$

Solution

$$\begin{aligned} \log_a \frac{x^3 w}{y^2 z^4} &= \log_a x^3 w - \log_a y^2 z^4 && \text{Quotient rule} \\ &= \log_a x^3 + \log_a w - \left(\log_a y^2 + \log_a z^4 \right) && \text{Product rule} \\ &= \log_a x^3 + \log_a w - \log_a y^2 - \log_a z^4 && \text{Distribute minus} \\ &= 3 \log_a x + \log_a w - 2 \log_a y - 4 \log_a z && \text{Power rule} \end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \frac{\sqrt{y}}{x^4 \sqrt[3]{z}}$

Solution

$$\begin{aligned} \log_a \frac{\sqrt{y}}{x^4 \sqrt[3]{z}} &= \log_a y^{1/2} - \log_a x^4 z^{1/3} && \text{Quotient rule} \\ &= \log_a y^{1/2} - \left(\log_a x^4 + \log_a z^{1/3} \right) && \text{Product rule} \end{aligned}$$

$$= \log_a y^{1/2} - \log_a x^4 - \log_a z^{1/3}$$

Distribute minus

$$= \frac{1}{2} \log_a y - 4 \log_a x - \frac{1}{3} \log_a z$$

Power rule

Exercise

Express the following in terms of sums and differences of logarithms $\ln \sqrt[4]{\frac{x^7}{y^5 z}}$

Solution

$$\ln \sqrt[4]{\frac{x^7}{y^5 z}} = \ln \left(\frac{x^7}{y^5 z} \right)^{1/4}$$

$$= \frac{1}{4} \ln \left(\frac{x^7}{y^5 z} \right)$$

Power rule

$$= \frac{1}{4} (\ln x^7 - \ln y^5 z)$$

Quotient rule

$$= \frac{1}{4} (\ln x^7 - (\ln y^5 + \ln z))$$

Product rule

$$= \frac{1}{4} (\ln x^7 - \ln y^5 - \ln z)$$

$$= \frac{1}{4} (7 \ln x - 5 \ln y - \ln z)$$

Power rule

$$= \frac{7}{4} \ln x - \frac{5}{4} \ln y - \ln z$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln x \sqrt[3]{\frac{y^4}{z^5}}$

Solution

$$\ln x \sqrt[3]{\frac{y^4}{z^5}} = \ln x + \ln \left(\frac{y^4}{z^5} \right)^{1/3}$$

Product rule

$$= \ln x + \ln \left(\frac{y^{4/3}}{z^{5/3}} \right)$$

$$= \ln x + \ln y^{4/3} - \ln z^{5/3}$$

Quotient rule

$$= \ln x + \frac{4}{3} \ln y - \frac{5}{3} \ln z$$

Power rule

Exercise

Express the following in terms of sums and differences of logarithms

$$\log_b \sqrt[5]{\frac{m^4 n^5}{x^2 ab^{10}}}$$

Solution

$$\begin{aligned}\log_b \sqrt[5]{\frac{m^4 n^5}{x^2 ab^{10}}} &= \log_b \left(\frac{m^4 n^5}{x^2 ab^{10}} \right)^{1/5} \\&= \frac{1}{5} \log_b \left(\frac{m^4 n^5}{x^2 ab^{10}} \right) \\&= \frac{1}{5} \left(\log_b (m^4 n^5) - \log_b (x^2 ab^{10}) \right) \\&= \frac{1}{5} \left(\left(\log_b (m^4) + \log_b (n^5) \right) - \left(\log_b (x^2) + \log_b (a) + \log_b (b^{10}) \right) \right) \\&= \frac{1}{5} \left(4 \log_b m + 5 \log_b n - 2 \log_b x - \log_b a - 10 \right) \\&= \frac{4}{5} \log_b m + \log_b n - \frac{2}{5} \log_b x - \frac{1}{5} \log_b (a) - 2\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms

$$\log_b \frac{a^5 b^{10}}{c^2 \sqrt[4]{d^3}}$$

Solution

$$\begin{aligned}\log_b \frac{a^5 b^{10}}{c^2 \sqrt[4]{d^3}} &= \log_b (a^5 b^{10}) - \log_b (c^2 d^{3/4}) \\&= \log_b (a^5) + \log_b (b^{10}) - \left(\log_b (c^2) + \log_b (d^{3/4}) \right) \\&= 5 \log_b a + 10 - 2 \log_b c - \frac{3}{4} \log_b d\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms

$$\ln \left(x^2 \sqrt{x^2 + 1} \right)$$

Solution

$$\begin{aligned}\ln \left(x^2 \sqrt{x^2 + 1} \right) &= \ln x^2 + \ln (x^2 + 1)^{1/2} \\&= 2 \ln x + \frac{1}{2} \ln (x^2 + 1)\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln \frac{x^2}{x^2 + 1}$

Solution

$$\begin{aligned}\ln \frac{x^2}{x^2 + 1} &= \ln x^2 - \ln(x^2 + 1) \\ &= \underline{2 \ln x - \ln(x^2 + 1)}\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln \left(\frac{x^2 (x+1)^3}{(x+3)^{1/2}} \right)$

Solution

$$\begin{aligned}\ln \left(\frac{x^2 (x+1)^3}{(x+3)^{1/2}} \right) &= \ln(x^2 (x+1)^3) - \ln(x+3)^{1/2} \\ &= \ln x^2 + \ln(x+1)^3 - \frac{1}{2} \ln(x+3) \\ &= \underline{2 \ln x + 3 \ln(x+1) - \frac{1}{2} \ln(x+3)}\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln \sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$

Solution

$$\begin{aligned}\ln \sqrt{\frac{(x+1)^5}{(x+2)^{20}}} &= \ln \left(\frac{(x+1)^5}{(x+2)^{20}} \right)^{1/2} \\ &= \frac{1}{2} \ln \left(\frac{(x+1)^5}{(x+2)^{20}} \right) \\ &= \frac{1}{2} (\ln(x+1)^5 - \ln(x+2)^{20}) \\ &= \frac{1}{2} (5 \ln(x+1) - 20 \ln(x+2)) \\ &= \underline{\frac{5}{2} \ln(x+1) - 10 \ln(x+2)}\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms

$$\ln \frac{(x^2 + 1)^5}{\sqrt{1-x}}$$

Solution

$$\begin{aligned} \ln \frac{(x^2 + 1)^5}{\sqrt{1-x}} &= \ln (x^2 + 1)^5 - \ln (1-x)^{1/2} \\ &= \underline{5 \ln (x^2 + 1) - \frac{1}{2} \ln (1-x)} \end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms

$$\ln \left(\sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}} \right)$$

Solution

$$\begin{aligned} \ln \left(\sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}} \right) &= \ln \left(\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)} \right)^{1/3} \\ &= \frac{1}{3} \ln \left(\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)} \right) \\ &= \frac{1}{3} \left(\ln (x(x+1)(x-2)) - \ln ((x^2+1)(2x+3)) \right) \\ &= \frac{1}{3} \left(\ln x + \ln (x+1) + \ln (x-2) - \left(\ln (x^2+1) + \ln (2x+3) \right) \right) \\ &= \frac{1}{3} \left(\ln x + \ln (x+1) + \ln (x-2) - \ln (x^2+1) - \ln (2x+3) \right) \\ &= \underline{\frac{1}{3} \ln x + \frac{1}{3} \ln (x+1) + \frac{1}{3} \ln (x-2) - \frac{1}{3} \ln (x^2+1) - \frac{1}{3} \ln (2x+3)} \end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms

$$\ln \left(\sqrt{\frac{1}{x(x+1)}} \right)$$

Solution

$$\ln \left(\sqrt{\frac{1}{x(x+1)}} \right) = \ln \left(\frac{1}{x(x+1)} \right)^{1/2}$$

$$\begin{aligned}
&= \frac{1}{2}(\ln 1 - \ln(x(x+1))) \\
&= -\frac{1}{2}(\ln x + \ln(x+1)) \\
&= \underline{-\frac{1}{2}\ln x - \frac{1}{2}\ln(x+1)}
\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln\left(\sqrt{(x^2+1)(x-1)^2}\right)$

Solution

$$\begin{aligned}
\ln\left(\sqrt{(x^2+1)(x-1)^2}\right) &= \ln\left((x^2+1)(x-1)^2\right)^{1/2} \\
&= \frac{1}{2}\ln\left((x^2+1)(x-1)^2\right) \\
&= \frac{1}{2}\left(\ln(x^2+1) + \ln(x-1)^2\right) \\
&= \frac{1}{2}\left(\ln(x^2+1) + 2\ln(x-1)\right) \\
&= \underline{\frac{1}{2}\ln(x^2+1) + \ln(x-1)}
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $\log(x+5) + 2\log x$

Solution

$$\begin{aligned}
\log(x+5) + 2\log x &= \log(x+5) + \log x^2 \\
&= \underline{\log(x^2(x+5))}
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $3\log_b x - \frac{1}{3}\log_b y + 4\log_b z$

Solution

$$\begin{aligned}
3\log_b x - \frac{1}{3}\log_b y + 4\log_b z &= \log_b x^3 + \log_b z^4 - \log_b y^{1/3} \\
&= \log_b (x^3 z^4) - \log_b \sqrt[3]{y}
\end{aligned}$$

$$\underline{= \log_b \left(\frac{x^3 z^4}{\sqrt[3]{y}} \right) \quad |}$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $\frac{1}{2} \log_b (x+5) - 5 \log_b y$

Solution

$$\begin{aligned} \frac{1}{2} \log_b (x+5) - 5 \log_b y &= \log_b (x+5)^{1/2} - \log_b y^5 \\ &= \log_b \left(\frac{\sqrt{x+5}}{y^5} \right) \end{aligned} \quad |$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $\ln(x^2 - y^2) - \ln(x - y)$

Solution

$$\begin{aligned} \ln(x^2 - y^2) - \ln(x - y) &= \ln \frac{x^2 - y^2}{x - y} \\ &= \ln \frac{(x - y)(x + y)}{x - y} \\ &= \ln(x + y) \end{aligned} \quad |$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $\ln(xz) - \ln(x \sqrt{y}) + 2 \ln \frac{y}{z}$

Solution

$$\begin{aligned} \ln(xz) - \ln(x \sqrt{y}) + 2 \ln \frac{y}{z} &= \ln(xz) + \ln \left(\frac{y}{z} \right)^2 - \ln(x \sqrt{y}) \\ &= \ln \left(\frac{xzy^2}{z^2} \right) - \ln(x \sqrt{y}) \\ &= \ln \left(\frac{xy^2}{z} \cdot \frac{1}{x \sqrt{y}} \right) \\ &= \ln \left(\frac{y^{3/2}}{z} \right) \end{aligned} \quad |$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $\log(x^2y) - \log z$

Solution

$$\log(x^2y) - \log z = \log\left(\frac{x^2y}{z}\right)$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $\log(z^2\sqrt{y}) - \log z^{1/2}$

Solution

$$\begin{aligned}\log(z^2\sqrt{y}) - \log z^{1/2} &= \log\left(\frac{z^2\sqrt{y}}{z^{1/2}}\right) \\ &= \log(z^{3/2}\sqrt{y}) \\ &= \log(\sqrt{z^3y})\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$2\log_a x + \frac{1}{3}\log_a(x-2) - 5\log_a(2x+3)$$

Solution

$$\begin{aligned}2\log_a x + \frac{1}{3}\log_a(x-2) - 5\log_a(2x+3) &= \log_a x^2 + \log_a(x-2)^{1/3} - \log_a(2x+3)^5 \\ &= \log_a x^2(x-2)^{1/3} - \log_a(2x+3)^5 \\ &= \log_a \frac{x^2(x-2)^{1/3}}{(2x+3)^5}\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$5\log_a x - \frac{1}{2}\log_a(3x-4) - 3\log_a(5x+1)$$

Solution

$$\begin{aligned}
5\log_a x - \frac{1}{2}\log_a (3x-4) - 3\log_a (5x+1) &= \log_a x^5 - \log_a (3x-4)^{1/2} - \log_a (5x+1)^3 \\
&= \log_a x^5 - \left[\log_a (3x-4)^{1/2} + \log_a (5x+1)^3 \right] \\
&= \log_a x^5 - \left[\log_a (3x-4)^{1/2} (5x+1)^3 \right] \\
&= \log_a \frac{x^5}{(3x-4)^{1/2} (5x+1)^3}
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\log(x^3 y^2) - 2\log(x\sqrt[3]{y}) - 3\log\left(\frac{x}{y}\right)$$

Solution

$$\begin{aligned}
\log(x^3 y^2) - 2\log(x\sqrt[3]{y}) - 3\log\left(\frac{x}{y}\right) &= \log(x^3 y^2) - \log(xy^{1/3})^2 - \log(xy^{-1})^3 \\
&= \log(x^3 y^2) - \left[\log(x^2 y^{2/3}) + \log(x^3 y^{-3}) \right] \\
&= \log(x^3 y^2) - \log(x^2 y^{2/3} x^3 y^{-3}) \\
&= \log(x^3 y^2) - \log(x^5 y^{-7/3}) \\
&= \log\left(\frac{x^3 y^2}{x^5 y^{-7/3}}\right) \\
&= \log\left(\frac{y^2 y^{7/3}}{x^2}\right) \\
&= \log\left(\frac{y^{13/3}}{x^2}\right) \\
&= \log\left(\frac{\sqrt[3]{y^{13}}}{x^2}\right) \\
&= \log\left(\frac{y^4 \sqrt[3]{y}}{x^2}\right)
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\ln y^3 + \frac{1}{3} \ln(x^3 y^6) - 5 \ln y$$

Solution

$$\begin{aligned} \ln y^3 + \frac{1}{3} \ln(x^3 y^6) - 5 \ln y &= \ln y^3 + \ln(x^3 y^6)^{1/3} - \ln y^5 \\ &= \ln y^3 + \ln(x^{3/3} y^{6/3}) - \ln y^5 \\ &= \ln y^3 + \ln(xy^2) - \ln y^5 \\ &= \ln(y^3 xy^2) - \ln y^5 \\ &= \ln\left(\frac{y^5 x}{y^5}\right) \\ &= \ln x \end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$2 \ln x - 4 \ln\left(\frac{1}{y}\right) - 3 \ln(xy)$$

Solution

$$\begin{aligned} 2 \ln x - 4 \ln\left(\frac{1}{y}\right) - 3 \ln(xy) &= \ln x^2 - \ln\left(\frac{1}{y}\right)^4 - \ln(xy)^3 \\ &= \ln x^2 - \left[\ln(y^{-4}) + \ln(x^3 y^3)\right] \\ &= \ln x^2 - \ln(y^{-4} x^3 y^3) \\ &= \ln x^2 - \ln(y^{-1} x^3) \\ &= \ln \frac{x^2}{y^{-1} x^3} \\ &= \ln \frac{y}{x} \end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$4 \ln x + 7 \ln y - 3 \ln z$$

Solution

$$\begin{aligned} 4 \ln x + 7 \ln y - 3 \ln z &= \ln x^4 + \ln y^7 - \ln z^3 \\ &= \ln(x^4 y^7) - \ln z^3 \\ &= \ln\left(\frac{x^4 y^7}{z^3}\right) \end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{1}{3} \left[5 \ln(x+6) - \ln x - \ln(x^2 - 25) \right]$$

Solution

$$\begin{aligned} \frac{1}{3} \left[5 \ln(x+6) - \ln x - \ln(x^2 - 25) \right] &= \frac{1}{3} \left[5 \ln(x+6) - (\ln x + \ln(x^2 - 25)) \right] \\ &= \frac{1}{3} \left[\ln(x+6)^5 - \ln x(x^2 - 25) \right] \\ &= \frac{1}{3} \left[\ln \frac{(x+6)^5}{x(x^2 - 25)} \right] \\ &= \ln \left(\frac{(x+6)^5}{x(x^2 - 25)} \right)^{1/3} \end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{2}{3} \left[\ln(x^2 - 4) - \ln(x+2) \right] + \ln(x+y)$$

Solution

$$\begin{aligned} \frac{2}{3} \left[\ln(x^2 - 4) - \ln(x+2) \right] + \ln(x+y) &= \frac{2}{3} \left[\ln \frac{x^2 - 4}{x+2} \right] + \ln(x+y) \\ &= \frac{2}{3} \left[\ln \frac{(x+2)(x-2)}{x+2} \right] + \ln(x+y) \\ &= \frac{2}{3} \ln(x-2) + \ln(x+y) \end{aligned}$$

$$\begin{aligned}
&= \ln(x-2)^{2/3} + \ln(x+y) \\
&= \ln(x-2)^{2/3}(x+y) \\
&= \ln(x+y) \sqrt[3]{(x-2)^2}
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{1}{2}\log_b m + \frac{3}{2}\log_b 2n - \log_b m^2 n$$

Solution

$$\begin{aligned}
\frac{1}{2}\log_b m + \frac{3}{2}\log_b 2n - \log_b m^2 n &= \log_b m^{1/2} + \log_b (2n)^{3/2} - \log_b m^2 n \\
&= \log_b \left(m^{1/2} (2n)^{3/2} \right) - \log_b m^2 n \\
&= \log_b \frac{m^{1/2} 2^{3/2} n^{3/2}}{m^2 n} \\
&= \log_b \frac{2^{3/2} n^{1/2}}{m^{3/2}} \\
&= \log_b \left(\frac{2^3 n}{m^3} \right)^{1/2} \\
&= \log_b \sqrt{\frac{8n}{m^3}}
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{1}{2}\log_y p^3 q^4 - \frac{2}{3}\log_y p^4 q^3$$

Solution

$$\begin{aligned}
\frac{1}{2}\log_y p^3 q^4 - \frac{2}{3}\log_y p^4 q^3 &= \log_y \left(p^3 q^4 \right)^{1/2} - \log_y \left(p^4 q^3 \right)^{2/3} \\
&= \log_y \frac{\left(p^3 q^4 \right)^{1/2}}{\left(p^4 q^3 \right)^{2/3}} \\
&= \log_y \frac{\left(p^3 \right)^{1/2} \left(q^4 \right)^{1/2}}{\left(p^4 \right)^{2/3} \left(q^3 \right)^{2/3}}
\end{aligned}$$

$$\begin{aligned}
&= \log_y \frac{p^{3/2} q^2}{p^{8/3} q^2} \\
&= \log_y \frac{p^{3/2}}{p^{8/3}} \\
&= \log_y \frac{1}{p^{8/3-3/2}} \\
&= \log_y \frac{1}{p^{7/6}}
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{1}{2} \log_a x + 4 \log_a y - 3 \log_a x$$

Solution

$$\begin{aligned}
\frac{1}{2} \log_a x + 4 \log_a y - 3 \log_a x &= 4 \log_a y - \frac{5}{2} \log_a x \\
&= \log_a y^4 - \log_a x^{5/2} \\
&= \log_a \frac{y^4}{\sqrt{x^5}}
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{2}{3} \left[\ln(x^2 - 9) - \ln(x + 3) \right] + \ln(x + y)$$

Solution

$$\begin{aligned}
\frac{2}{3} \left[\ln(x^2 - 9) - \ln(x + 3) \right] + \ln(x + y) &= \frac{2}{3} \ln \frac{x^2 - 9}{x + 3} + \ln(x + y) \\
&= \frac{2}{3} \ln \frac{(x + 3)(x - 3)}{x + 3} + \ln(x + y) \\
&= \frac{2}{3} \ln(x - 3) + \ln(x + y) \\
&= \ln(x - 3)^{2/3} + \ln(x + y) \\
&= \ln \left((x - 3)^{2/3} (x + y) \right) \\
&= \ln \left((x + y) \sqrt[3]{(x - 3)^2} \right)
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y$$

Solution

$$\begin{aligned}\frac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y &= \log_b x^{1/4} - \log_b 5^2 - \log_b y^{10} \\ &= \log_b x^{1/4} - \left[\log_b 5^2 + \log_b y^{10} \right] \\ &= \log_b x^{1/4} - \log_b (5^2 y^{10}) \\ &= \log_b \frac{\sqrt[4]{x}}{25y^{10}}\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$2\ln(x+4) - \ln x - \ln(x^2 - 3)$$

Solution

$$\begin{aligned}2\ln(x+4) - \ln x - \ln(x^2 - 3) &= \ln(x+4)^2 - (\ln x + \ln(x^2 - 3)) \\ &= \ln(x+4)^2 - \ln(x(x^2 - 3)) \\ &= \ln \frac{(x+4)^2}{x(x^2 - 3)}\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\ln x + \ln(y+3) + \ln(y+2) - \ln(y^2 + 5y + 6)$$

Solution

$$\begin{aligned}\ln x + \ln(y+3) + \ln(y+2) - \ln(y^2 + 5y + 6) &= \ln(x(y+3)(y+2)) - \ln((y+3)(y+2)) \\ &= \ln\left(\frac{x(y+3)(y+2)}{(y+3)(y+2)}\right) \\ &= \ln x\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\ln x + \ln(x+4) + \ln(x+1) - \ln(x^2 + 5x + 4)$$

Solution

$$\begin{aligned}\ln x + \ln(x+4) + \ln(x+1) - \ln(x^2 + 5x + 4) &= \ln(x(x+4)(x+1)) - \ln((x+4)(x+1)) \\ &= \ln\left(\frac{x(x+4)(x+1)}{(x+4)(x+1)}\right) \\ &= \ln x\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\ln(x^2 - 25) - 2\ln(x+5) + \ln(x-5)$$

Solution

$$\begin{aligned}\ln(x^2 - 25) - 2\ln(x+5) + \ln(x-5) &= \ln(x^2 - 25) + \ln(x-5) - \ln(x+5)^2 \\ &= \ln\frac{(x-5)(x+5)(x-5)}{(x+5)^2} \\ &= \ln\left(\frac{(x-5)^2}{x+5}\right)\end{aligned}$$

Exercise

Assume that $\log_{10} 2 = .3010$. Find each logarithm $\log_{10} 4$, $\log_{10} 5$

Solution

$$\begin{aligned}a) \quad \log_{10} 4 &= \log_{10} 2^2 \\ &= 2\log_{10} 2 \\ &= 2(.301) \\ &= .6020\end{aligned}$$

$$\begin{aligned}b) \quad \log_{10} 5 &= \log_{10} \frac{10}{2} \\ &= \log_{10} 10 - \log_{10} 2 \\ &= 1 - .3010\end{aligned}$$

$$= .6990 \mid$$

Exercise

Given that: $\log_a 2 \approx 0.301$, $\log_a 7 \approx 0.845$, and $\log_a 11 \approx 1.041$ find each of the following:

$$a) \log_a \frac{2}{11}$$

$$c) \log_a 98$$

$$e) \log_a 9$$

$$b) \log_a 14$$

$$d) \log_a \frac{1}{7}$$

$$f) \log_a \frac{77}{8}$$

Solution

$$\begin{aligned} a) \log_a \frac{2}{11} &= \log_a 2 - \log_a 11 \\ &= 0.301 - 1.041 \\ &\approx 1.342 \mid \end{aligned}$$

$$\begin{aligned} b) \log_a 14 &= \log_a 2(7) \\ &= \log_a 2 + \log_a 7 \\ &= 0.301 + 0.845 \\ &\approx 1.146 \mid \end{aligned}$$

$$\begin{aligned} c) \log_a 98 &= \log_a 2(7^2) \\ &= \log_a 2 + \log_a 7^2 \\ &= \log_a 2 + 2\log_a 7 \\ &= 0.301 + 2(0.845) \\ &\approx 1.991 \mid \end{aligned}$$

$$\begin{aligned} d) \log_a \frac{1}{7} &= \log_a 1 - \log_a 7 \\ &\approx 0 - 0.845 \\ &\approx -0.845 \mid \end{aligned}$$

$$e) \log_a 9 \text{ Can't be found from the given information}$$

$$\begin{aligned} f) \log_a \frac{77}{8} &= \log_a 77 - \log_a 8 \\ &= \log_a (7 \times 11) - \log_a 2^3 \\ &= \log_a 7 + \log_a 11 - 3\log_a 2 \\ &\approx 0.845 + 1.041 - 3(0.301) \\ &\approx 1.886 - 0.903 \\ &\approx 0.983 \mid \end{aligned}$$