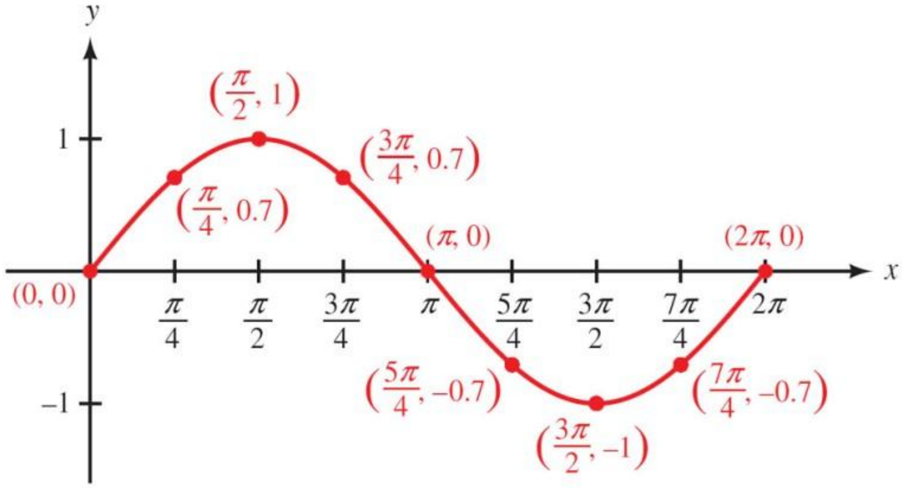
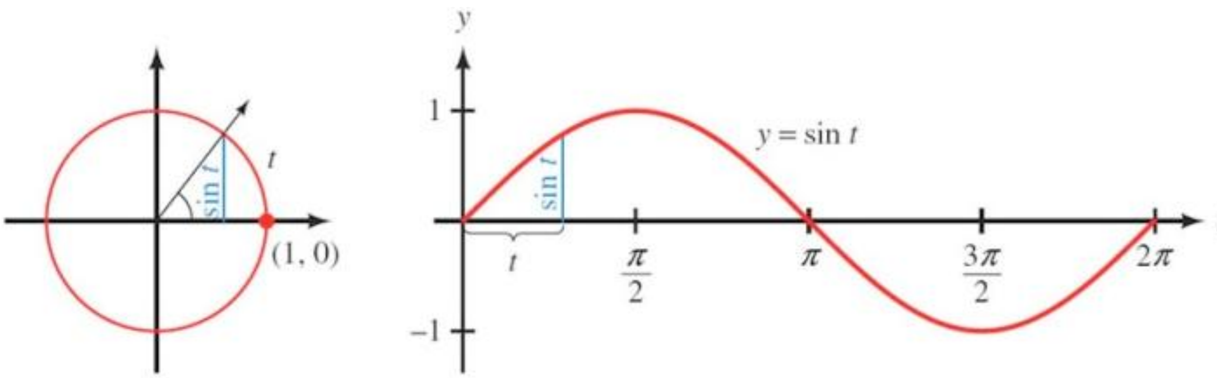
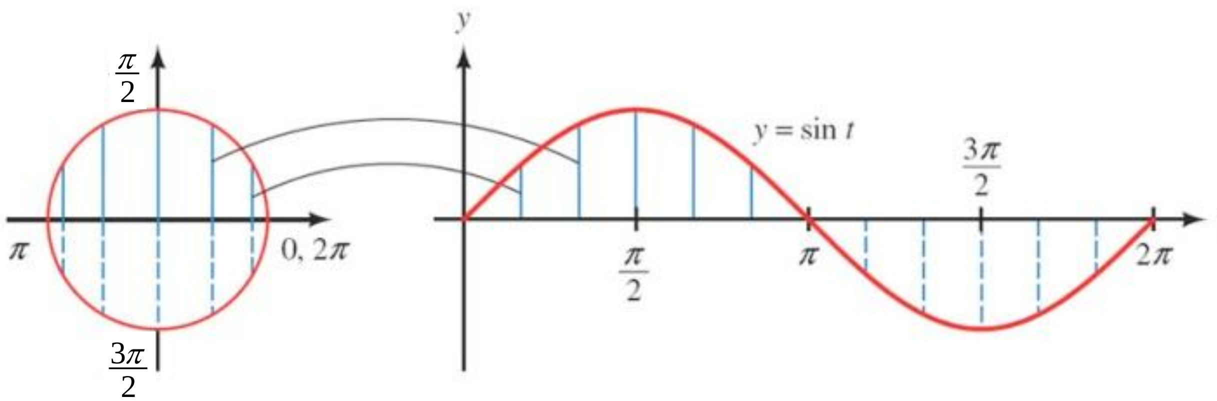
Section 2.4 – Translation of Trigonometric Functions

The *Sine* Graph

Graphing the function: $y = \sin x$



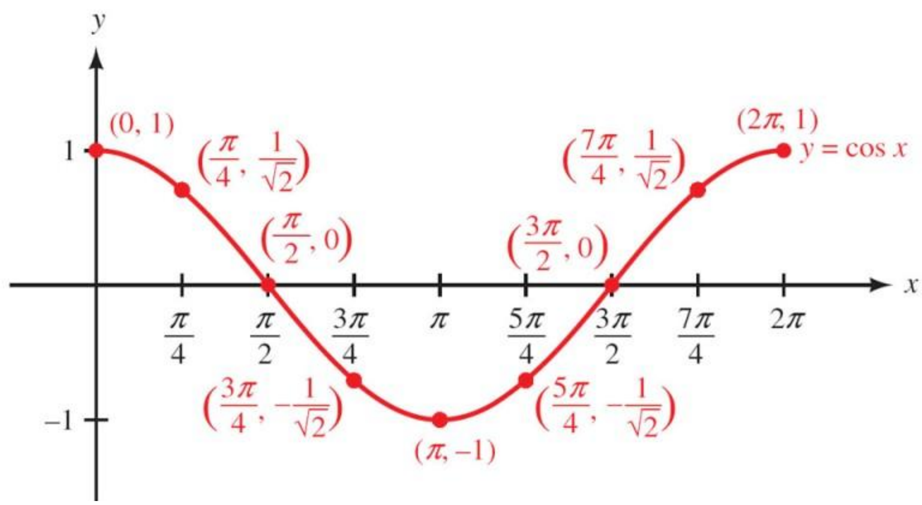




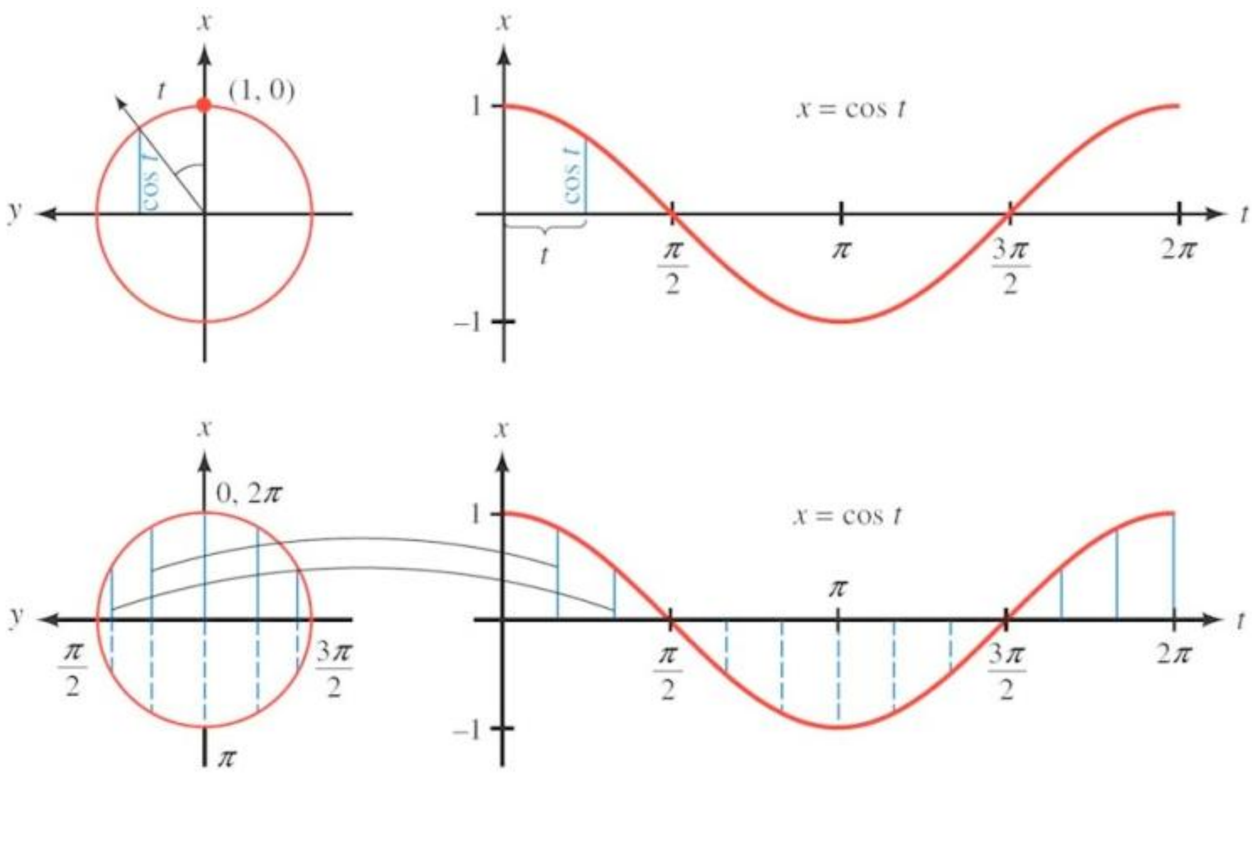
Range: $-1 \leq y \leq 1$ $\boxed{-1 \leq \sin x \leq 1}$

The *Cosine* Graph

Graphing the function: $y = \cos x$



Unit Circle (***rotated***)



Amplitude

If the greatest value of y is M and the least value of y is m, then the amplitude of the graph of y is defined to be

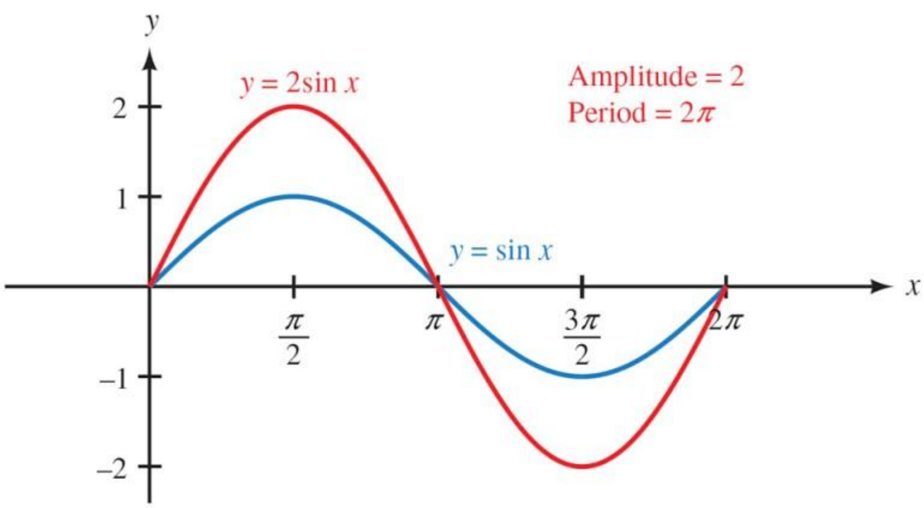
$$A = \frac{1}{2}|M - m|$$

The amplitude is $|A|$.

Example

Identify the amplitude of the graph and then sketch the graph: $y = 2\sin x$ *for* $0 \le x \le 2\pi$

Amplitude: A = 2



Note:

If $A > 0$, then the graph of $y = A\sin x$ and $y = A\cos x$ will have amplitude A and range [-A, A].

Period

For any function $y = f(x)$, the smallest positive number *p* for which

$$f(x+p) = f(x) \qquad \text{for all } x$$

is called the period of $f(x)$

The least possible positive value of *p* is the period of the function.

Many things in daily life repeat with a predictable pattern, such as weather, tides, and hours of daylight.

This periodic graph represents a normal heartbeat.

The graphs $y = A\sin Bx$ and $y = A\cos Bx$ $\qquad \rightarrow Period = \dfrac{2\pi}{|B|}$

One cycle: $0 \le \arg ument \le 2\pi$

$$0 \le Bx \le 2\pi$$

Example

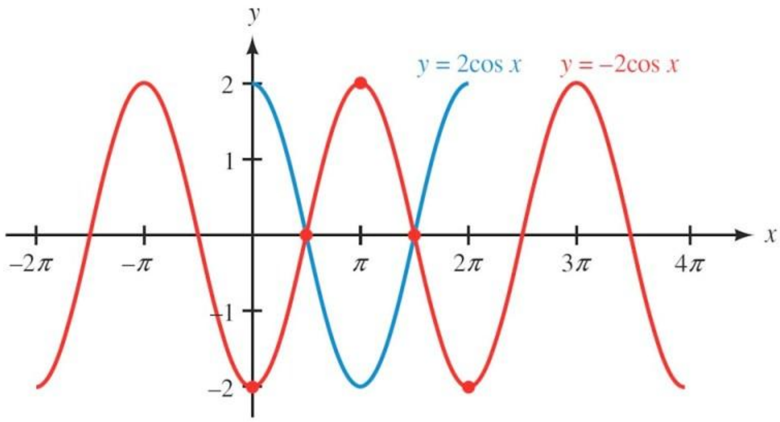
$y = \sin x \qquad \rightarrow P = 2\pi$

$y = \sin 2x \qquad \rightarrow P = \dfrac{2\pi}{2} = \pi$

$y = \sin 3x \qquad \rightarrow P = \dfrac{2\pi}{3}$

$y = \cos \frac{1}{2} x \qquad \rightarrow P = \dfrac{2\pi}{\frac{1}{2}} = 4\pi$

Reflecting About the *x*-axis



Note: The graphs of $y = A\sin x$ and $y = A\cos x$ will be reflected about *x*-axis if $A < 0$. The amplitude will be $|A|$.

Even and ***Odd*** Functions

Definition

An *even function* is a function for which $f(-x) = f(x)$

An *odd function* is a function for which $f(-x) = -f(x)$

Even Functions	***Odd Functions***
$y = \cos(\theta)$, $y = \sec(\theta)$	$y = \sin(\theta)$, $y = \csc(\theta)$ $y = \tan(\theta)$, $y = \cot(\theta)$
Graphs are symmetric about the y-axis	*Graphs are symmetric about the origin*

Vertical Translations

For $d > 0$, $y = f(x) + d \Rightarrow$ The graph shifted up d units

$y = f(x) - d \Rightarrow$ The graph shifted down d units

Example

Sketch the graph $y = -3 - 2\sin \pi x$

Amplitude: $A = 2$

Period: $P = \dfrac{2\pi}{\pi} = 2$

Vertical Shifting: $y = -3$ *Down 3 units*

Phase shift

If we add a term to the argument of the function, the graph will be translated in a ***horizontal direction***.

In the function $y = f(x - c)$, the expression $x - c$ is called the ***argument***.

$$\boxed{\phi = -\frac{C}{B}}$$

Example

Graph $y = \sin\left(x + \frac{\pi}{2}\right), \quad if \quad -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

Amplitude: $A = 1$

Period: $P = \frac{2\pi}{1} = 2\pi$

$x + \frac{\pi}{2} = 0 \to x = -\frac{\pi}{2}$

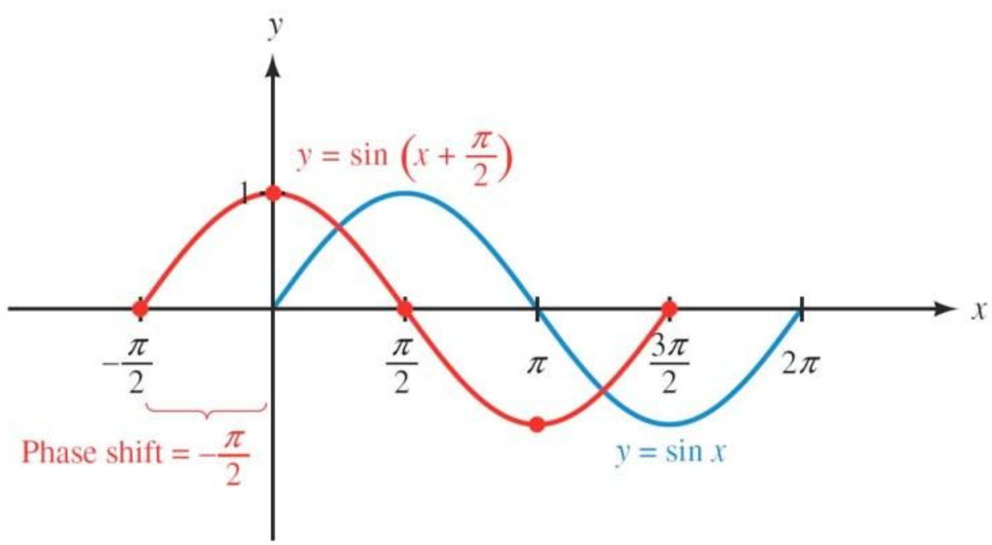
Phase Shift: $\phi = -\frac{\pi}{2}$

$0 \le \arg ument \le 2\pi$

$0 \le x + \frac{\pi}{2} \le 2\pi$

$-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

x	x	x	$y = \sin\left(x + \frac{\pi}{2}\right)$
$\phi + 0$	$-\frac{\pi}{2} + 0$	$-\frac{\pi}{2}$	0
$\phi + \frac{1}{4}P$	$-\frac{\pi}{2} + \frac{1}{2}\pi$	0	1
$\phi + \frac{1}{2}P$	$-\frac{\pi}{2} + \frac{3}{2}\pi$	$\frac{\pi}{2}$	0
$\phi + \frac{3}{4}P$	$-\frac{\pi}{2} + \frac{3}{4}\pi$	π	-1
$\phi + P$	$-\frac{\pi}{2} + 2\pi$	$\frac{3\pi}{2}$	0



Graphing the *Sine* and *Cosine* Functions

The graphs of $y = k + A\sin(Bx + C)$ and $y = k + A\cos(Bx + C)$, where B > 0, will have the following characteristics:

Amplitude $= |A|$ Period: $P = \frac{2\pi}{B}$

Phase Shift: $\phi = -\frac{C}{B}$ Vertical translation: $y = k$

If $A <\ 0$ the graph will be reflected about the x-axis

Example

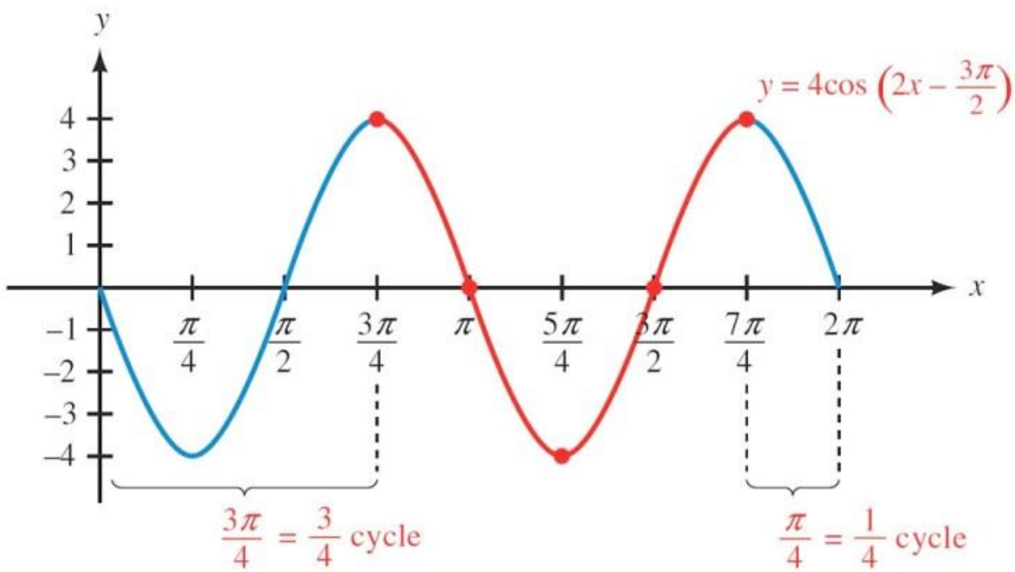
Graph $y = 4\cos\left(2x - \frac{3\pi}{2}\right)$ *for* $0 \le x \le 2\pi$

Amplitude: $A = 4$

Period: $P = \frac{2\pi}{2} = \pi$

Phase Shift: $\phi = \frac{\frac{3\pi}{2}}{2} = \frac{3\pi}{4}$

x	x	$y = 4\cos\left(2x - \frac{3\pi}{2}\right)$
$\frac{3\pi}{4} + 0$	$\frac{3\pi}{4}$	4
$\frac{3\pi}{4} + \frac{1}{4}\pi$	π	0
$\frac{3\pi}{4} + \frac{1}{2}\pi$	$\frac{5\pi}{4}$	-4
$\frac{3\pi}{4} + \frac{3}{4}\pi$	$\frac{3\pi}{2}$	0
$\frac{3\pi}{4} + \pi$	$\frac{7\pi}{4}$	4



Example

Graph one complete cycle $y = 3 - 5\sin\left(\pi x + \frac{\pi}{4}\right)$

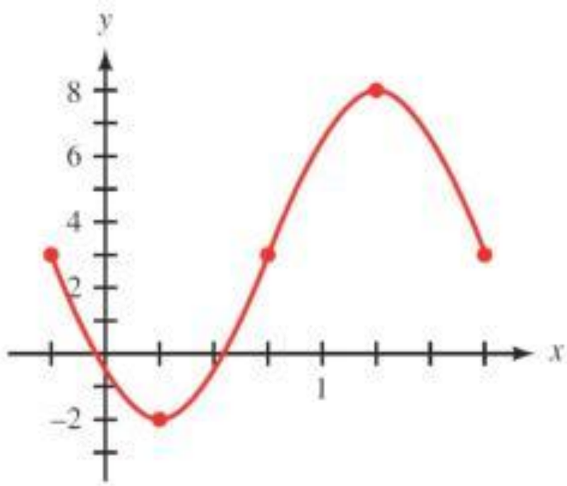
Amplitude: $A = 5$

Period: $P = \frac{2\pi}{\pi} = 2$

Phase Shift: $\phi = -\frac{\frac{\pi}{4}}{\pi} = -\frac{1}{4}$

VT: $y = 3$

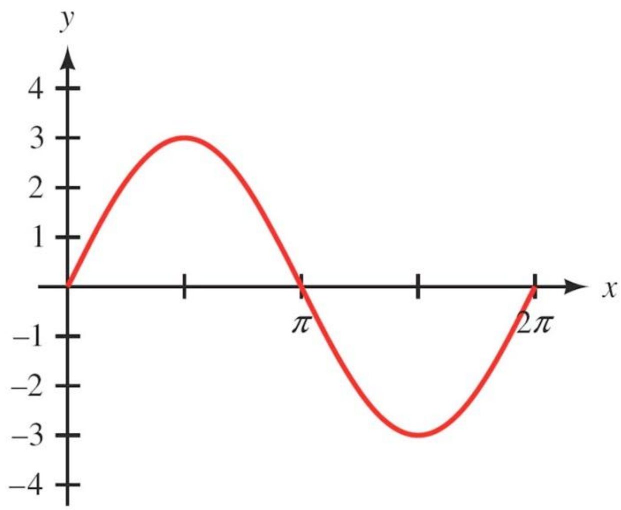
x	x	$y = 3 - 5\sin\left(\pi x + \frac{\pi}{4}\right)$
$-\frac{1}{4} + 0$	$-\frac{1}{4}$	3
$-\frac{1}{4} + \frac{1}{2}$	$\frac{1}{4}$	-2
$-\frac{1}{4} + 1$	$\frac{3}{4}$	3
$-\frac{1}{4} + \frac{3}{2}$	$\frac{5}{4}$	8
$-\frac{1}{4} + 2$	$\frac{7}{4}$	3



Finding the *Sine* and *Cosine* Functions from the Graph

Example

Find an equation $y = k + A\sin(Bx + C)$ or $y = k + A\cos(Bx + C)$ to match the graph



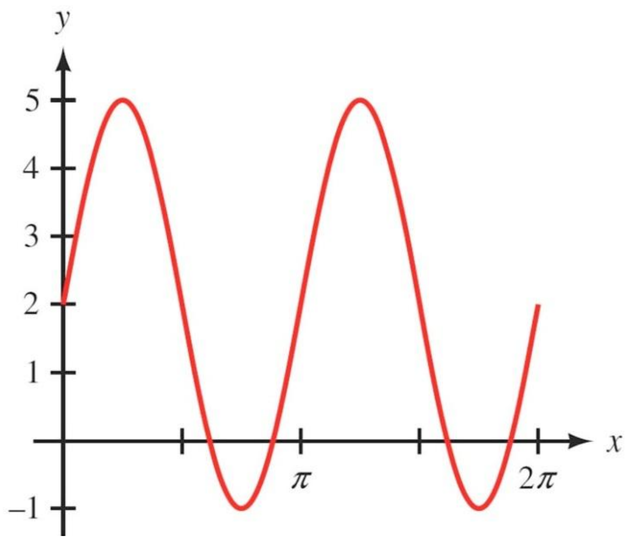
Amplitude = 3

$B = \frac{2\pi}{P} = \frac{2\pi}{2\pi} = 1$

$$y = 3\sin x \qquad 0 \le x \le 2\pi$$

Example

Find an equation $y = k + A\sin(Bx + C)$ or $y = k + A\cos(Bx + C)$ to match the graph



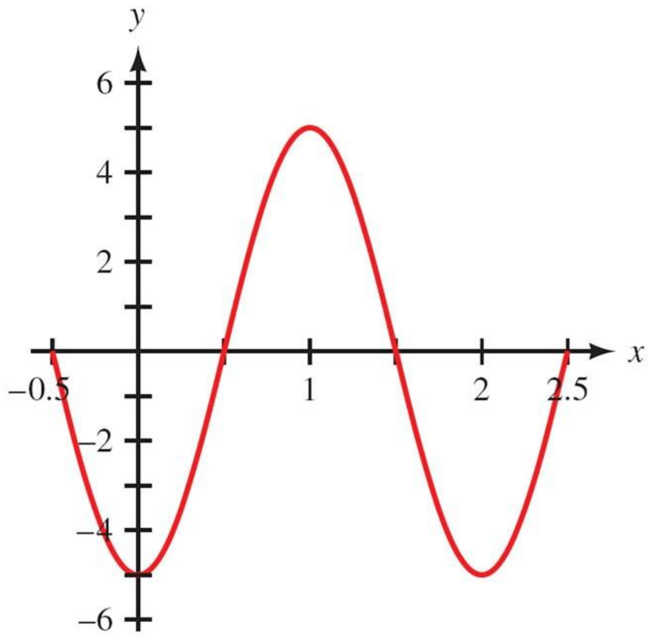
B = $\frac{2\pi}{\pi} = 2$

Amplitude = 3

$$y = 2 + 3\sin 2x \qquad 0 \le x \le 2\pi$$

Example

Find an equation $y = k + A\sin(Bx + C)$ or $y = k + A\cos(Bx + C)$ to match the graph



B = $\frac{2\pi}{2} = \pi$

Amplitude = 5

$$y = -5\cos \pi x \quad -0.5 \le x \le 2.5$$

Or

Phase shift = $-0.5 = -\frac{C}{B}$

$$0.5 = \frac{C}{\pi}$$

$$0.5\pi = C$$

$$y = -5\sin(\pi x + \frac{\pi}{2}) \qquad -0.5 \le x \le 2.5$$

Exercises *Section* 2.4 – Translation of Trigonometric Functions

Find the amplitude, the period, any vertical translation, and any phase shift of

1. $y = 2\sin(x - \pi)$

2. $y = \frac{2}{3}\sin\left(x + \frac{\pi}{2}\right)$

3. $y = 4\cos\left(\frac{1}{2}x + \frac{\pi}{2}\right)$

4. $y = \frac{1}{2}\sin\left(\frac{1}{2}x + \pi\right)$

5. $y = 3\cos\frac{\pi}{2}\left(x - \frac{1}{2}\right)$

6. $y = -\cos\pi\left(x - \frac{1}{3}\right)$

7. $y = 2 - \sin\left(3x - \frac{\pi}{5}\right)$

8. $y = -\frac{2}{3}\sin\left(3x - \frac{\pi}{2}\right)$

9. $y = -1 + \frac{1}{2}\cos(2x - 3\pi)$

10. $y = 2 - \frac{1}{3}\cos\left(\pi x + \frac{3\pi}{2}\right)$

11. $y = \frac{5}{2} - 3\cos\left(\pi x - \frac{\pi}{4}\right)$

12. $y = \cos\frac{1}{2}x$

13. $y = -3 + \sin\left(\pi x + \frac{\pi}{2}\right)$

14. $y = \frac{2}{3} - \frac{4}{3}\cos(3x - \pi)$

Find the amplitude, the period, the phase shift, and the vertical translation and sketch the graph of the equation

15. $y = 2\sin\left(x - \frac{\pi}{3}\right)$

16. $y = 4\cos\left(x - \frac{\pi}{4}\right)$

17. $y = -\sin(3x + \pi) - 1$

18. $y = \cos(2x - \pi) + 2$

19. $y = \sin\left(\frac{1}{2}x - \frac{\pi}{3}\right)$

20. $y = 5\sin\left(3x - \frac{\pi}{2}\right)$

21. $y = 3\cos\left(\frac{1}{2}x - \frac{\pi}{4}\right)$

22. $y = -5\cos\left(\frac{1}{3}x + \frac{\pi}{6}\right)$

23. $y = -2\sin(2\pi x + \pi)$

24. $y = -2\sin(2x - \pi) + 3$

25. $y = 3\cos(x + 3\pi) - 2$

26. $y = 5\cos(2x + 2\pi) + 2$

27. $y = -4\sin(3x - \pi) - 3$

Graph

28. $y = 2\sin(-\pi x)$ *for* $-3 \le x \le 3$

29. $y = 4\cos\left(-\frac{2}{3}x\right)$ *for* $-\frac{15\pi}{4} \le x \le \frac{15\pi}{4}$

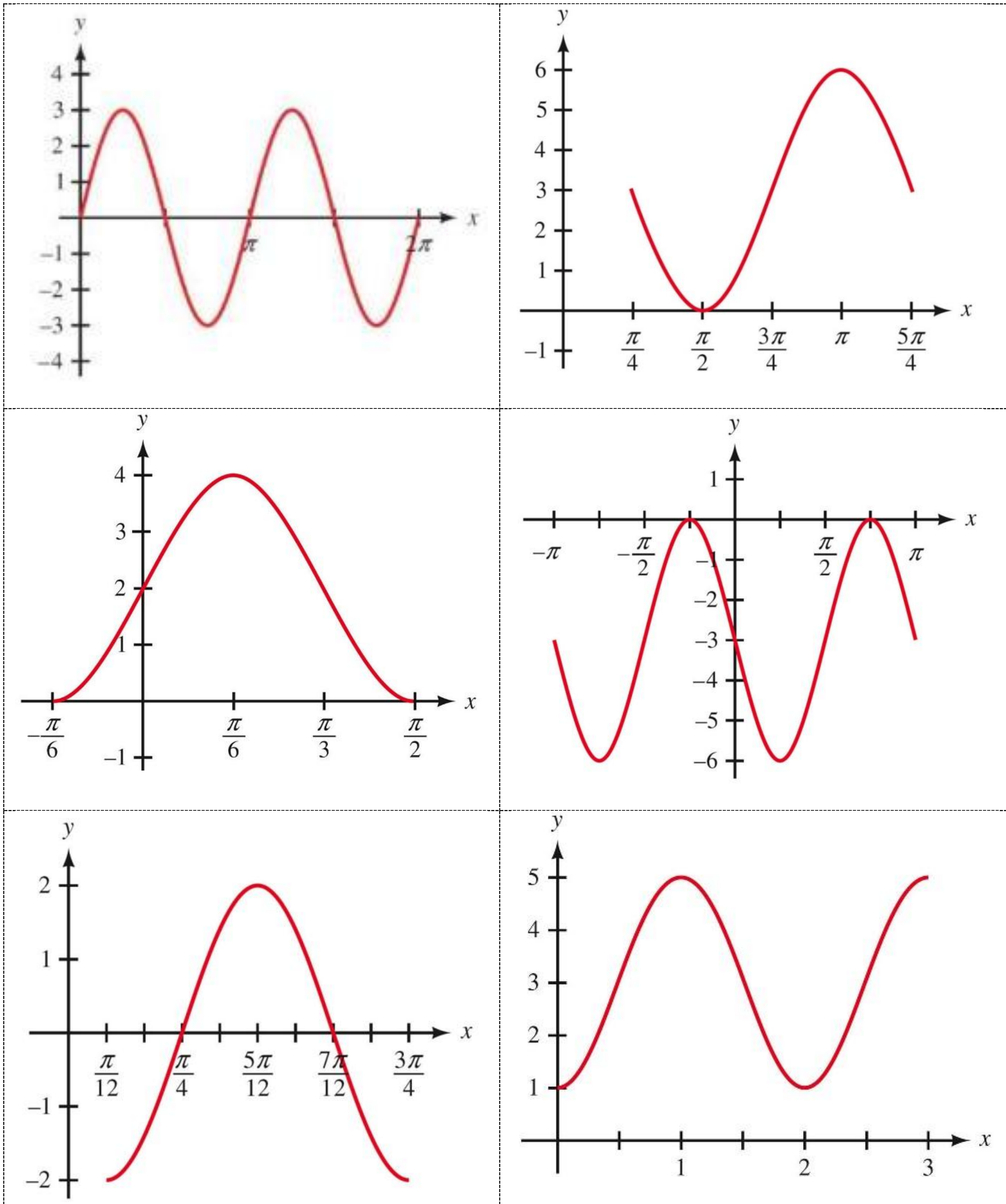
30. $y = \cos\left(x - \frac{\pi}{6}\right)$ for one complete cycle

31. $y = \frac{2}{3} - \frac{4}{3}\cos(3x - \pi)$ for one complete cycle

32. $y = -3 + \sin\left(\pi x + \frac{\pi}{2}\right)$ for one complete cycle

33. $y = -1 + 2\sin(4x + \pi)$ over two periods.

34. Find an equation $y = k + A\sin(Bx + C)$ or $y = k + A\cos(Bx + C)$ to match the graph



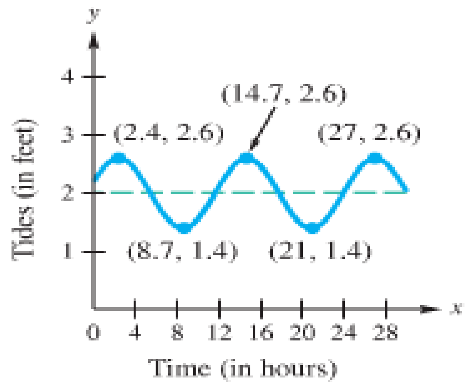
35. The figure shows a function *f* that models the tides in feet at Clearwater Beach, *x* hours after midnight starting on Aug. 26,

a) Find the time between high tides.

b) What is the difference in water levels between high tide and low tide?

c) The tides can be modeled by $f(x) = 0.6\cos[0.511x - 2.4] + 2$

Estimate the tides when $x = 10$.

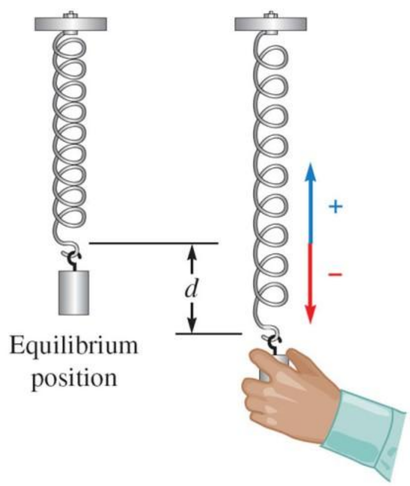


36. The maximum afternoon temperature in a given city might be modeled by $t = 60 - 30\cos\frac{\pi x}{6}$

Where *t* represents the maximum afternoon temperature in month *x*, with *x* = 0 representing January, *x* = 1 representing February, and so on.. Find the maximum afternoon temperature to the nearest degree for each month.

a) Jan. *b)* Apr. *c)* May. *d)* Jun. *e)* Oct.

37. A mass attached to a spring oscillates upward and downward. The length *L* of the spring after *t* seconds is given by the function $L = 15 - 3.5\cos(2\pi t)$, where *L* is measured in cm.



a) Sketch the graph of this function for $0 \le t \le 5$

b) What is the length the spring when it is at equilibrium?

c) What is the length the spring when it is shortest?

d) What is the length the spring when it is longest?

38. The diameter of the Ferris wheel is 250 *ft*, the distance from the ground to the bottom of the wheel is 14 *ft.* We found the height of a rider on that Ferris wheel was given by the function:

$$H = 139 - 125\cos\left(\frac{\pi}{10}t\right)$$

Where t is the number of minutes from the beginning of a ride. Graph a complete cycle of this function.