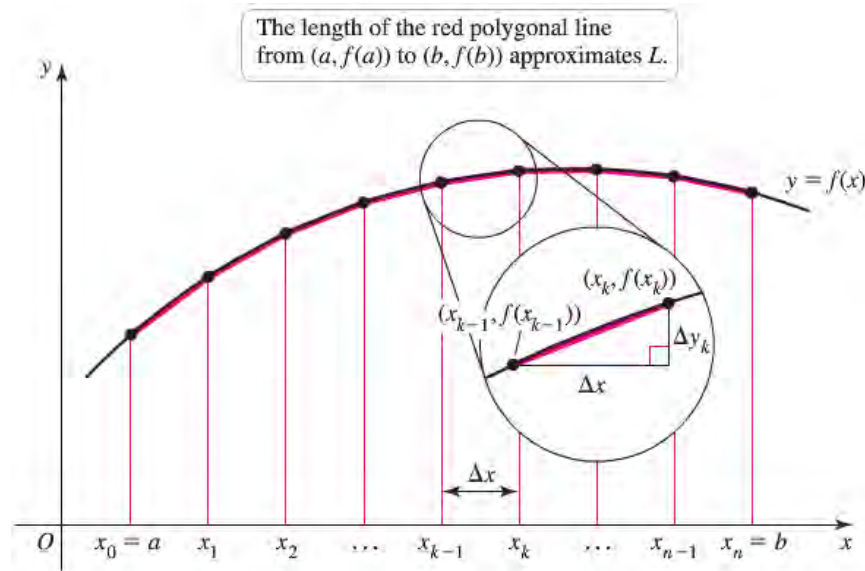


## Section 1.5 – Length of Curves

### Length of a curve $y = f(x)$

We assume that  $f$  has a continuous derivative at every point of  $[a, b]$ . Such function is called **smooth**, and its graph is a **smooth curve** because it doesn't have any breaks, corners, or cusps.



### Definition

If  $f'$  is continuous on  $[a, b]$ , then the length (arc length) of the curve  $y = f(x)$  from the point  $A = (a, f(a))$  to the point  $B = (b, f(b))$  is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

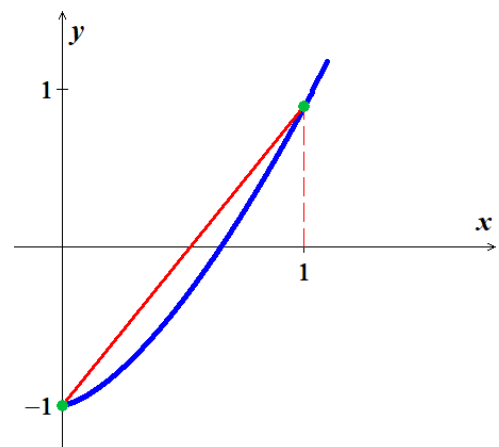
### Example

Find the length of the curve  $y = \frac{4\sqrt{2}}{3}x^{3/2} - 1$ ,  $0 \leq x \leq 1$

### Solution

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2} x^{1/2}$$

$$= 2\sqrt{2}x^{1/2}$$



$$\begin{aligned}\left(\frac{dy}{dx}\right)^2 &= \left(2\sqrt{2}x^{1/2}\right)^2 \\ &= 8x\end{aligned}$$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= \int_0^1 (1 + 8x)^{1/2} \, dx$$

$$\text{or } u = 1 + 8x \quad du = 8dx \rightarrow dx = \frac{du}{8}$$

$$= \frac{1}{8} \int_0^1 (1 + 8x)^{1/2} \, d(1 + 8x)$$

$$= \frac{1}{8} \left( \frac{2}{3} (1 + 8x)^{3/2} \right) \Big|_0^1$$

$$= \frac{1}{12} \left[ (1 + 8(\textcolor{red}{1}))^{3/2} - (1 + 8(\textcolor{blue}{0}))^{3/2} \right]$$

$$= \frac{1}{12} \left( (9)^{3/2} - (1)^{3/2} \right)$$

$$= \frac{1}{12} [27 - 1]$$

$$= \frac{1}{12} (26)$$

$$= \frac{\textcolor{blue}{13}}{\textcolor{blue}{6}} \quad \textcolor{green}{unit} \Big|$$

$$\approx \textcolor{blue}{2.17} \quad \textcolor{green}{unit} \Big|$$

**Length of a curve**  $y = f(x)$ :

If  $f(x) = ax^m + bx^n$ , then

$$\begin{aligned} L &= \int_c^d \sqrt{1 + \left(\frac{df}{dx}\right)^2} dx \\ &= \left(ax^m - bx^n\right) \Big|_c^d \end{aligned}$$

**Iff**  $f(x)$  satisfies these 2 conditions:

1.  $m + n = 2$
2.  $abmn = -\frac{1}{4}$

**Proof**

$$f'(x) = max^{m-1} + nbx^{n-1}$$

$$\begin{aligned} 1 + (f')^2 &= 1 + \left(max^{m-1} + nbx^{n-1}\right)^2 \\ &= 1 + m^2 a^2 x^{2m-2} + 2abmnx^{m+n-2} + n^2 b^2 x^{2n-2} \end{aligned}$$

*We need to combined to a perfect square*

$$a^2 - 2ab + b^2 = (a - b)^2$$

$$\text{➤ If } x^{m+n-2} = 1 = x^0 \rightarrow \boxed{m+n=2}$$

$$= m^2 a^2 x^{2m-2} + (1 + 2abmn) + n^2 b^2 x^{2n-2} \quad a^2 - 2ab + b^2 = (a - b)^2$$

$$\text{➤ Let } 1 + 2abmn = -2abmn \rightarrow \boxed{abmn = -\frac{1}{4}}$$

$$= m^2 a^2 x^{2m-2} - 2abmn + n^2 b^2 x^{2n-2} \quad x^{2(m+n-2)} = 1$$

$$= \left(max^{m-1} - nbx^{n-1}\right)^2$$

$$L = \int_c^d \sqrt{\left(max^{m-1} - nbx^{n-1}\right)^2} dx$$

$$= \int_c^d \left(max^{m-1} - nbx^{n-1}\right) dx$$

$$= \left(ax^m - bx^n\right) \Big|_c^d \quad \checkmark$$

### Example

Find the length of the graph of  $f(x) = \frac{x^3}{12} + \frac{1}{x}$ ,  $1 \leq x \leq 4$

### Solution

$$a = \frac{1}{12}, \quad m = 3, \quad b = 1, \quad n = -1$$

$$1. \quad m + n = 3 - 1 = 2 \quad \checkmark$$

$$2. \quad abmn = \frac{1}{12}(1)(3)(-1) = -\frac{1}{4} \quad \checkmark$$

$$\begin{aligned}
 L &= \left( \frac{x^3}{12} - \frac{1}{x} \right) \Big|_1^4 \\
 &= \left( \frac{4^3}{12} - \frac{1}{4} \right) - \left( \frac{1}{12} - \frac{1}{1} \right) \\
 &= \frac{72}{12} \\
 &= \underline{6 \text{ unit}}
 \end{aligned}$$

### Discontinuities in $\frac{dy}{dx}$

**Formula for the length of**  $x = g(y)$ ,  $c \leq y \leq d$

If  $g'$  is continuous on  $[c, d]$ , the length of the curve  $x = g(y)$  from the point  $A = (g(c), c)$  to the point  $B = (g(d), d)$  is the value of the integral

$$L = \int_c^d \sqrt{1 + [g'(y)]^2} \, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

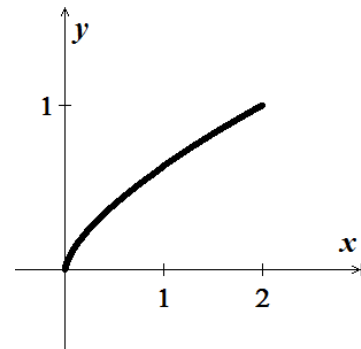
### Example

Find the length of the curve  $y = \left(\frac{x}{2}\right)^{2/3}$  from  $x = 0$  to  $x = 2$ .

### Solution

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{2}{3} \left(\frac{x}{2}\right)^{-1/3} \left(\frac{1}{2}\right) \\
 &= \frac{1}{3} \left(\frac{2}{x}\right)^{1/3} \quad \boxed{x \neq 0} \text{ (CP)}
 \end{aligned}$$

$$y = \left(\frac{x}{2}\right)^{2/3} \quad \text{Raised both sides to the power } 3/2$$



$$y^{3/2} = \frac{x}{2}$$

$$x = 2y^{3/2}$$

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2}$$

$$= 3y^{1/2}$$

$$\rightarrow \begin{cases} x = 0 & \Rightarrow y = 0 \\ x = 2 & \Rightarrow y = \left(\frac{2}{2}\right)^{2/3} = 1 \end{cases}$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$= \int_0^1 \sqrt{1 + \left(3y^{1/2}\right)^2} dy$$

$$= \int_0^1 \sqrt{1 + 9y} dy$$

$$= \int_0^1 (1 + 9y)^{1/2} dy$$

$$= \frac{1}{9} \frac{2}{3} (1 + 9y)^{3/2} \Big|_0^1$$

$$= \frac{2}{27} \left[ (1 + 9)^{3/2} - (1 + 0)^{3/2} \right]$$

$$= \frac{2}{27} (10^{3/2} - 1)$$

$$= \frac{2}{27} (10\sqrt{10} - 1) \quad \text{unit}$$

$$\approx 2.27 \text{ unit}$$

If  $f(x) = ae^{mx} + be^{nx}$ , then

$$L = \int_c^d \sqrt{1 + \left(\frac{df}{dx}\right)^2} dx$$

$$= \left( ae^{mx} - be^{nx} \right) \Big|_c^d$$

**Iff**  $f(x)$  satisfies these 2 conditions:

1.  $m = -n$
2.  $abmn = -\frac{1}{4}$

**Proof**

$$f'(x) = ame^{mx} + bne^{nx}$$

$$1 + (f')^2 = 1 + (ame^{mx} + bne^{nx})^2$$

$$= 1 + m^2 a^2 e^{2mx} + 2abmne^{(m+n)x} + n^2 b^2 e^{2nx}$$

$$\rightarrow \text{If } e^{(m+n)x} = 1 = e^{(x=0)} \rightarrow \boxed{m = -n}$$

$$= m^2 a^2 x^{2m-2} + (1 + 2abmn) + n^2 b^2 x^{2n-2} \qquad a^2 - 2ab + b^2 = (a - b)^2$$

$$\rightarrow \text{Let } 1 + 2abmn = -2abmn \rightarrow \boxed{abmn = -\frac{1}{4}}$$

$$= m^2 a^2 e^{2mx} - 2abmne^{(m+n)x} + n^2 b^2 e^{2nx} \qquad x^{2(m+n-2)} = 1$$

$$= (ame^{mx} - bne^{nx})^2$$

$$(ame^{mx} - bne^{nx})^2 = m^2 a^2 e^{2mx} - 2abmne^{(m+n)x} + n^2 b^2 e^{2nx}$$

$$L = \int_c^d \sqrt{(ame^{mx} - bne^{nx})^2} dx$$

$$= \int_c^d (ame^{mx} - bne^{nx}) dx$$

$$= \left( ame^{mx} - bne^{nx} \right) \Big|_c^d \quad \checkmark$$

### Example

Find the arc length function for the curve  $f(x) = \ln(x + \sqrt{x^2 - 1})$  on the interval  $[1, \sqrt{2}]$

### Solution

$$\begin{aligned} f'(x) &= \frac{1 + x(x^2 - 1)^{-1/2}}{x + \sqrt{x^2 - 1}} \\ &= \frac{\sqrt{x^2 - 1} + x}{x\sqrt{x^2 - 1} + x^2 - 1} \end{aligned}$$

$$y = \ln(x + \sqrt{x^2 - 1})$$

$$x + \sqrt{x^2 - 1} = e^y$$

$$(\sqrt{x^2 - 1})^2 = (e^y - x)^2$$

$$x^2 - 1 = e^{2y} - 2xe^y + x^2$$

$$2xe^y = e^{2y} + 1$$

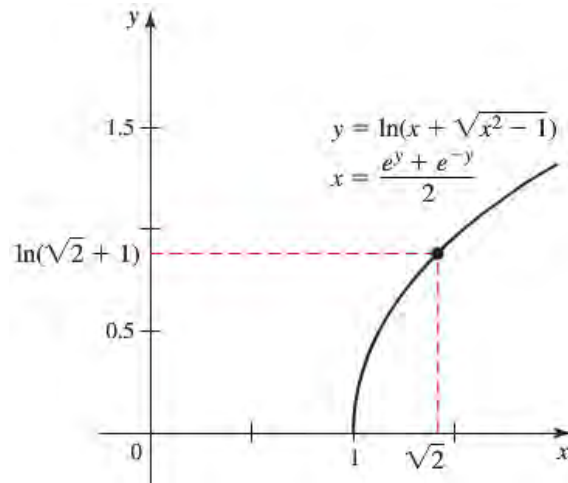
$$\begin{aligned} x &= \frac{e^{2y} + 1}{2e^y} \left( \frac{e^y}{e^y} \right) \\ &= \frac{e^y + e^{-y}}{2} \end{aligned}$$

$$y = \ln(x + \sqrt{x^2 - 1})$$

$$\boxed{x = \frac{e^y + e^{-y}}{2} = g(y)}$$

$$x = 1 \rightarrow y = 0$$

$$x = \sqrt{2} \rightarrow y = \ln(\sqrt{2} + 1)$$



$$a = \frac{1}{2}, \quad m = 1, \quad b = \frac{1}{2}, \quad n = -1$$

$$1. \quad m = -n \quad \checkmark$$

$$2. \quad abmn = -\frac{1}{4} \quad \checkmark$$

$$L = \frac{1}{2} \left( e^y - e^{-y} \right) \Bigg|_0^{\ln(\sqrt{2}+1)}$$

$$= \frac{1}{2} \left( \sqrt{2} + 1 - \frac{1}{\sqrt{2} + 1} - 1 + 1 \right)$$

$$= \frac{1}{2} \left( \frac{3 + 2\sqrt{2} - 1}{\sqrt{2} + 1} \right)$$

$$= \frac{1}{2} \left( \frac{2 + 2\sqrt{2}}{\sqrt{2} + 1} \right)$$

$$= \underline{1 \text{ unit}}$$

**OR** — . . . . .

$$L = \int_0^{\ln(\sqrt{2}+1)} \sqrt{1 + g'(y)^2} \, dy$$

$$= \int_0^{\ln(\sqrt{2}+1)} \sqrt{1 + \left( \frac{e^y - e^{-y}}{2} \right)^2} \, dy$$

$$= \frac{1}{2} \int_0^{\ln(\sqrt{2}+1)} \sqrt{4 + e^{2y} - 2 + e^{-2y}} \, dy$$

$$= \frac{1}{2} \int_0^{\ln(\sqrt{2}+1)} \sqrt{e^{2y} + 2 + e^{-2y}} \, dy$$

$$= \frac{1}{2} \int_0^{\ln(\sqrt{2}+1)} \sqrt{(e^y + e^{-y})^2} \, dy$$

$$= \frac{1}{2} \int_0^{\ln(\sqrt{2}+1)} (e^y + e^{-y}) \, dy$$

$$= \frac{1}{2} (e^y - e^{-y}) \Big|_0^{\ln(\sqrt{2}+1)}$$

$$= \frac{1}{2} \left( \sqrt{2} + 1 - \frac{1}{\sqrt{2} + 1} - 1 + 1 \right)$$

$$= \frac{1}{2} \left( \frac{3 + 2\sqrt{2} - 1}{\sqrt{2} + 1} \right)$$

$$= \frac{1}{2} \left( \frac{2 + 2\sqrt{2}}{\sqrt{2} + 1} \right)$$

$$= \underline{1 \text{ unit}}$$



## The differential Formula for Arc length

If  $y = f(x)$  and if  $f'$  is continuous on  $[a, b]$ , then by the Fundamental Theorem of Calculus, we can define a new function

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} dt$$

$$\frac{ds}{dx} = \sqrt{1 + [f'(t)]^2}$$

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \sqrt{(dx)^2 + (dx)^2 \frac{(dy)^2}{(dx)^2}}$$

$$\boxed{ds = \sqrt{dx^2 + dy^2}}$$

### Example

Find the arc length function for the curve  $f(x) = \frac{x^3}{12} + \frac{1}{x}$  taking  $A = \left(1, \frac{13}{12}\right)$  as the starting point

### Solution

$$1 + [f'(x)]^2 = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

$$s(x) = \int_1^x \sqrt{1 + [f'(t)]^2} dt$$

$$= \int_1^x \left(\frac{t^2}{4} + \frac{1}{t^2}\right) dt$$

$$= \left(\frac{t^3}{12} - \frac{1}{t}\right) \Big|_1^x$$

$$= \left(\frac{x^3}{12} - \frac{1}{x}\right) - \left(\frac{1}{12} - 1\right)$$

$$= \frac{x^3}{12} - \frac{1}{x} + \frac{11}{12}$$

$$s(4) = \frac{4^3}{12} - \frac{1}{4} + \frac{11}{12}$$

$$= \underline{6 \text{ unit}}$$

## Exercises      Section 1.5 – Length of Curves

(1 – 30) Find the length of the curve of

1.  $y = \frac{1}{3}(x^2 + 2)^{3/2}$  from  $x = 0$  to  $x = 3$

2.  $y = (x)^{3/2}$  from  $x = 0$  to  $x = 4$

3.  $x = \frac{y^{3/2}}{3} - y^{1/2}$  from  $y = 1$  to  $y = 9$

4.  $x = \frac{y^3}{6} + \frac{1}{2y}$  from  $y = 2$  to  $y = 3$

5.  $f(x) = x^3 + \frac{1}{12x}$  for  $\frac{1}{2} \leq x \leq 2$

6.  $f(x) = \frac{1}{5}x^5 + \frac{1}{12x^3}$   $1 \leq x \leq 2$

7.  $y = \frac{1}{3}x^{1/2} - x^{3/2}$ ,  $0 \leq x \leq \frac{1}{3}$

8.  $y = \frac{1}{3}x^3 + \frac{1}{4x}$ ,  $1 \leq x \leq 2$

9.  $y = 2e^x + \frac{1}{8}e^{-x}$   $0 \leq x \leq \ln 2$

10.  $y = e^{2x} + \frac{1}{16}e^{-2x}$   $0 \leq x \leq \ln 3$

11.  $y = \ln(\cos x)$   $0 \leq x \leq \frac{\pi}{4}$

12.  $f(y) = 2e^{\sqrt{2}y} + \frac{1}{16}e^{-\sqrt{2}y}$   $0 \leq y \leq \frac{\ln 2}{\sqrt{2}}$

13.  $y = \frac{x^3}{3} + x^2 + x + 1 + \frac{1}{4x+4}$   $0 \leq x \leq 2$

14.  $y = \frac{x^3}{3} + x^2 + x + 1 + \frac{1}{4x+4}$   $0 \leq x \leq 4$

15.  $y = \ln(e^x - 1) - \ln(e^x + 1)$   $\ln 2 \leq x \leq \ln 3$

16.  $f(x) = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}$   $1 \leq x \leq 4$

17.  $f(x) = x^3 + \frac{1}{12x}$   $1 \leq x \leq 4$

18.  $f(x) = \frac{1}{8}x^4 + \frac{1}{4x^2}$   $1 \leq x \leq 10$

19.  $f(x) = \frac{1}{4}x^4 + \frac{1}{8x^2}$   $3 \leq x \leq 8$

20.  $f(x) = \frac{1}{10}x^5 + \frac{1}{6x^3}$   $1 \leq x \leq 7$

21.  $f(x) = \frac{3}{10}x^{1/3} - \frac{3}{2}x^{5/3}$   $0 \leq x \leq 12$

22.  $f(x) = x^{1/2} - \frac{1}{3}x^{3/2}$   $2 \leq x \leq 9$

23.  $y = x^{1/2} - \frac{1}{3}x^{3/2}$   $1 \leq x \leq 4$

24.  $x = y^{2/3}$ ,  $1 \leq y \leq 8$

25.  $y = 2x + 4$   $-2 \leq x \leq 2$

26.  $y = \frac{x^3}{6} + \frac{1}{2x}$   $x \in [1, 2]$

27.  $f(x) = x^{1/2} - \frac{1}{3}x^{3/2}$   $1 \leq x \leq 3$

28.  $y = \frac{3}{4}x^{4/3} - \frac{3}{8}x^{2/3} + 5$ ,  $1 \leq x \leq 8$

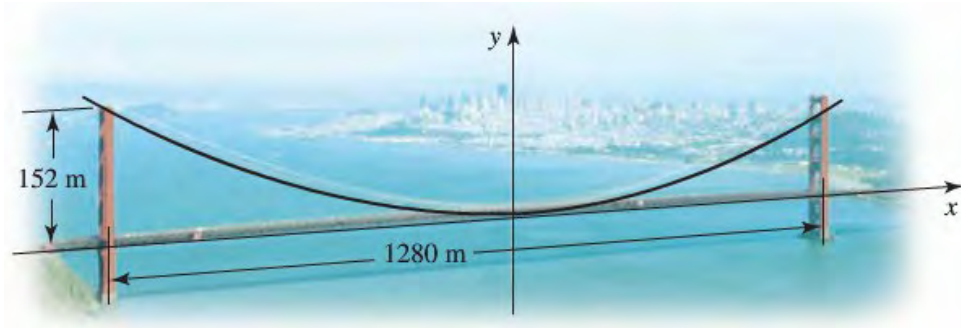
29.  $y = \ln x - \frac{1}{8}x^2$ ;  $1 \leq x \leq 2$

30.  $y = \frac{1}{2}x^2 - \frac{1}{4}\ln x$ ;  $1 \leq x \leq 3$

31. Find the length of the curve  $y = \int_{-2}^x \sqrt{2t^4 - 1} dt$   $-2 \leq x \leq -1$

32. Find the length of the curve  $x = \int_0^y \sqrt{\sec^4 t - 1} dt$   $-\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$

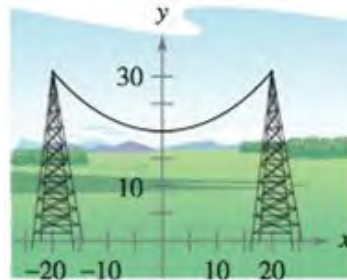
33. Find the length of the curve  $y = 3 - 2x$   $0 \leq x \leq 2$ . Check your answer by finding the length of the segment as the hypotenuse of a right triangle.
34. The profile of the cables on a suspension bridge may be modeled by a parabola. The central span of the Golden Gate Bridge is 1280 m long and 152 m high. The parabola  $y = 0.00037x^2$  gives a good fit to the shape of the cables, where  $|x| \leq 640$ , and  $x$  and  $y$  are measured in meters. Approximate the length of the cables that stretch between the tops of the two towers.



35. Find a curve through the origin in the  $xy$ -plane whose length from  $x = 0$  to  $x = 1$  is

$$L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x} dx$$

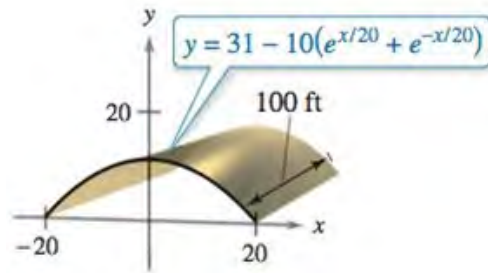
36. Confirm that the circumference of a circle of radius  $a$  is  $2\pi a$
37. Electrical wires suspended between two towers form a catenary modeled by the equation



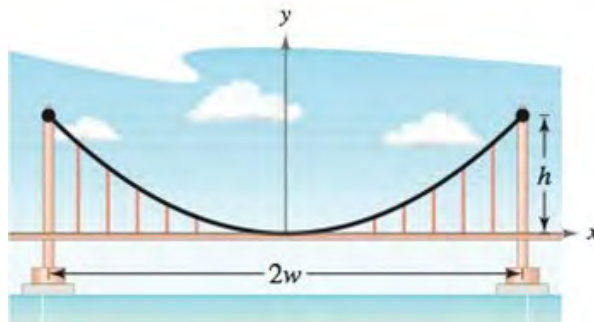
$$y = 20 \cosh \frac{x}{20}, \quad -20 \leq x \leq 20$$

Where  $x$  and  $y$  are measured in meters. The towers are 40 meters apart. Find the length of the suspended cable.

38. A barn is 100 feet long and 40 feet wide. A cross section of the roof is the inverted catenary  $y = 31 - 10(e^{x/20} + e^{-x/20})$ . Find the number of square feet of roofing on the barn.



39. A cable for a suspension bridge has the shape of a parabola with equation  $y = kx^2$ . Let  $h$  represent the height of the cable from its lowest point to its highest point and let  $2w$  represent the total span of the bridge.



Show that the length  $C$  of the cable is given by  $C = 2 \int_0^w \sqrt{1 + \frac{4h^2}{w^4} x^2} dx$

40. Find the total length of the graph of the astroid  $x^{2/3} + y^{2/3} = 4$
41. Find the arc length from  $(0, 3)$  clockwise to  $(2, \sqrt{5})$  along the circle  $x^2 + y^2 = 9$
42. Find the arc length from  $(-3, 4)$  clockwise to  $(4, 3)$  along the circle  $x^2 + y^2 = 25$ . Show that the result is one-fourth the circumference of the circle.
43.  $y = \ln x$  between  $x = 1$  and  $x = b > 1$  that

$$\int \frac{\sqrt{x^2 + a^2}}{x} dx = \sqrt{x^2 + a^2} - a \ln \left( \frac{a + \sqrt{x^2 + a^2}}{x} \right) + C$$

Use any means to approximate the value of  $b$  for which the curve has length 2.