

Solution **Section 1.4 – Lines and Curves in Space**

Exercise

Find the parametric equation for the line through the point $P(3, -4, -1)$ parallel to the vector $\hat{i} + \hat{j} + \hat{k}$

Solution

$$\underline{x = 3 + t, \quad y = -4 + t, \quad z = -1 + t \quad |}$$

Exercise

Find the parametric equation for the line through the points $P(1, 2, -1)$ and $Q(-1, 0, 1)$

Solution

The direction: $\overrightarrow{PQ} = -2\hat{i} - 2\hat{j} + 2\hat{k}$ and $P(1, 2, -1)$

$$\underline{x = 1 - 2t, \quad y = 2 - 2t, \quad z = -1 + 2t \quad |}$$

Exercise

Find the parametric equation for the line through the points $P(-2, 0, 3)$ and $Q(3, 5, -2)$

Solution

The direction: $\overrightarrow{PQ} = 5\hat{i} + 5\hat{j} - 5\hat{k}$ and $P(-2, 0, 3)$

$$\underline{x = -2 + 5t, \quad y = 5t, \quad z = 3 - 5t \quad |}$$

Exercise

Find the parametric equation for the line through the origin parallel to the vector $2\hat{j} + \hat{k}$

Solution

The direction: $2\hat{j} + \hat{k}$ and $P(0, 0, 0)$

$$\underline{x = 0, \quad y = 2t, \quad z = t \quad |}$$

Exercise

Find the parametric equation for the line through the point $P(3, -2, 1)$ parallel to the line

$$x = 1 + 2t, \quad y = 2 - t, \quad z = 3t$$

Solution

The direction: $2\hat{i} - \hat{j} + 3\hat{k}$ and $P(3, -2, 1)$

$$\underline{x = 3 + 2t, \quad y = -2 - t, \quad z = 1 + 3t}$$

Exercise

Find the parametric equation for the line through $(2, 4, 5)$ perpendicular to the plane $3x + 7y - 5z = 21$

Solution

The direction: $3\hat{i} + 7\hat{j} - 5\hat{k}$ and $(2, 4, 5)$

$$\underline{x = 2 + 3t, \quad y = 4 + 7t, \quad z = 5 - 5t}$$

Exercise

Find the parametric equation for the line through $(2, 3, 0)$ perpendicular to the vectors $\vec{u} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{v} = 3\hat{i} + 4\hat{j} + 5\hat{k}$

Solution

The direction:

$$\begin{aligned}\vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} \\ &= -2\hat{i} + 4\hat{j} - 2\hat{k}\end{aligned}$$

The point on the line: $(2, 3, 0)$

$$\underline{x = 2 - 2t, \quad y = 3 + 4t, \quad z = -2t}$$

Exercise

Find the parameterization for the line segment joining the points $(0, 0, 0)$, $\left(1, 1, \frac{3}{2}\right)$.

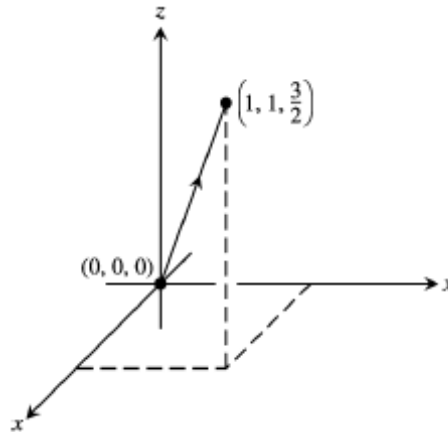
Draw coordinate axes and sketch the segment, indicate the direction on increasing t for the parameterization.

Solution

$$\text{Let: } P = (0, 0, 0) \quad Q = \left(1, 1, \frac{3}{2}\right)$$

$$\text{The direction: } \overrightarrow{PQ} = \hat{i} + \hat{j} + \frac{3}{2}\hat{k}$$

The line is given by: $x = t$, $y = t$, $z = \frac{3}{2}t$, $0 \leq t \leq 1$



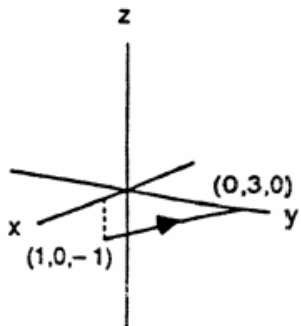
Exercise

Find the parameterization for the line segment joining the points $(1, 0, -1)$, $(0, 3, 0)$. Draw coordinate axes and sketch the segment, indicate the direction on increasing t for the parameterization.

Solution

The direction: $\overrightarrow{PQ} = -\hat{i} + 3\hat{j} + \hat{k}$ and $(1, 0, -1)$

$$x = 1 - t, \quad y = 3t, \quad z = -1 + t, \quad 0 \leq t \leq 1$$



Exercise

Find equation for the plane through normal to $\vec{n} = 3\hat{i} - 2\hat{j} - \hat{k}$

Solution

$$3(x - 0) - 2(y - 2) - (z + 1) = 0$$

$$3x - 2y + 4 - z - 1 = 0$$

$$\underline{3x - 2y - z = -3}$$

Exercise

Find equation for the plane through $(1, -1, 3)$ parallel to the plane $3x + y + z = 7$

Solution

$$3(x-1) + (y+1) + (z-3) = 0$$

$$3x - 3 + y + 1 + z - 3 = 0$$

$$\underline{3x + y + z = 5}$$

Exercise

Find equation for the plane through $(1, 1, -1)$, $(2, 0, 2)$ and $(0, -2, 1)$

Solution

$$\overrightarrow{PQ} = \hat{i} - \hat{j} + 3\hat{k} \quad \overrightarrow{PS} = -\hat{i} - 3\hat{j} + 2\hat{k}$$

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PS} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} \\ &= 7\hat{i} - 5\hat{j} - 4\hat{k} \quad \text{is normal to the plane.} \end{aligned}$$

$$7(x-2) - 5(y+0) - 4(z-2) = 0$$

$$7x - 14 - 5y - 4z + 8 = 0$$

$$\underline{7x - 5y - 4z = 6}$$

Exercise

Find equation for the plane through $P_0(2, 4, 5)$ perpendicular to the line $x = 5 + t$, $y = 1 + 3t$, $z = 4t$

Solution

$$x = 5 + t, \quad y = 1 + 3t, \quad z = 4t$$

$$\Rightarrow \vec{n} = \hat{i} + 3\hat{j} + 4\hat{k}$$

$$1(x-2) + 3(y-4) + 4(z-5) = 0$$

$$x - 2 + 3y - 12 + 4z - 20 = 0$$

$$\underline{x + 3y + 4z = 34}$$

Exercise

Find equation for the plane through $A(1, -2, 1)$ perpendicular to the vector from the origin to A .

Solution

$$\Rightarrow \vec{n} = \hat{i} - 2\hat{j} + \hat{k}$$

$$1(x-1) - 2(y+2) + 1(z-1) = 0$$

$$x - 1 - 2y - 4 + z - 1 = 0$$

$$\underline{x - 2y + z = 6}$$

Exercise

Find the point of intersection of the lines $x = 2t + 1$, $y = 3t + 2$, $z = 4t + 3$ and $x = s + 2$, $y = 2s + 4$, $z = -4s - 1$, and find the plane determined by these lines.

Solution

$$\begin{cases} x = 2t + 1 = s + 2 \\ y = 3t + 2 = 2s + 4 \\ z = 4t + 3 = -4s - 1 \end{cases}$$

$$\Rightarrow \begin{cases} 2t - s = 1 \\ 3t - 2s = 2 \end{cases} \rightarrow \underline{t = 0} \quad \underline{s = -1}$$

$$z = 4t + 3 = -4s - 1$$

$$4(0) + 3 = -4(-1) - 1$$

$$3 = 3 \quad \checkmark \quad (\text{Satisfied})$$

The lines intersect when $t = 0$ and $s = -1$

\Rightarrow The point of intersection $x = 1$, $y = 2$, $z = 3$

Therefore; the point is $\underline{P(1, 2, 3)}$

The normal vectors: $\vec{n}_1 = 2\hat{i} + 3\hat{j} + 4\hat{k}$ $\vec{n}_2 = \hat{i} + 2\hat{j} - 4\hat{k}$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix}$$

$$= -20\hat{i} + 12\hat{j} + \hat{k}$$

\vec{n}_1 and \vec{n}_2 are directions of the lines.

The plane containing the lines is represented by

$$-20(x-1)+12(y-2)+1(z-3)=0$$

$$\Rightarrow \underline{-20x+12y+z=7}$$

Exercise

Find the plane determined by the intersecting lines:

$$L_1 : x = -1 + t, \quad y = 2 + t, \quad z = 1 - t; \quad -\infty < t < \infty$$

$$L_2 : x = 1 - 4s, \quad y = 1 + 2s, \quad z = 2 - 2s; \quad -\infty < s < \infty$$

Solution

$$\text{The normal vectors: } \vec{n}_1 = \hat{i} + \hat{j} - \hat{k} \quad \vec{n}_2 = -4\hat{i} + 2\hat{j} - 2\hat{k}$$

$$\vec{n} = \vec{n}_1 \times \vec{n}_2$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{vmatrix}$$

$$= 6\hat{j} + 6\hat{k}$$

Let $t = 0$

$$L_1 : x = -1, \quad y = 2, \quad z = 1; \Rightarrow \underline{P(-1, 2, 1)}$$

Therefore; the desired plane is:

$$0(x+1)+6(y-2)+6(z-1)=0$$

$$6y-12+6z-6=0$$

$$6y+6z=18$$

$$\underline{y+z=3}$$

Exercise

Find a plane through $P_0(2, 1, -1)$ and perpendicular to the line of intersection of the planes

$$2x + y - z = 3, \quad x + 2y + z = 2$$

Solution

$$\text{The normal vectors: } \vec{n}_1 = 2\hat{i} + \hat{j} - \hat{k} \quad \vec{n}_2 = \hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$= 3\hat{i} - 3\hat{j} + 3\hat{k}$ is the vector in the direction of the line of intersection of the planes.

$$\Rightarrow 3(x-2) - 3(y-1) + 3(z+1) = 0$$

$$3x - 3y + 3z = 0$$

$$\underline{x - y + z = 0} \quad \text{is the desired plane containing } P_0(2, 1, -1)$$

Exercise

Find the distance from the point to the plane $(0, 0, 12)$, $x = 4t$, $y = -2t$, $z = 2t$

Solution

At $t = 0 \Rightarrow P(0, 0, 0)$ and let $S(0, 0, 12)$

$$\overrightarrow{PS} = 12\hat{k} \text{ and } \vec{v} = 4\hat{i} - 2\hat{j} + 2\hat{k}$$

$$\begin{aligned} \overrightarrow{PS} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 12 \\ 4 & -2 & 2 \end{vmatrix} \\ &= 24\hat{i} + 48\hat{j} \end{aligned}$$

$$\begin{aligned} d &= \frac{|\overrightarrow{PS} \times \vec{v}|}{|\vec{v}|} \\ &= \frac{\sqrt{24^2 + 48^2}}{\sqrt{16 + 4 + 4}} \\ &= \frac{24\sqrt{5}}{\sqrt{24}} \\ &= \sqrt{5}\sqrt{24} \\ &= \underline{2\sqrt{30}} \end{aligned}$$

Exercise

Find the distance from the point to the plane $(2, 1, -1)$, $x = 2t$, $y = 1 + 2t$, $z = 2t$

Solution

At $t = 0 \Rightarrow P(0, 1, 0)$ and let $S(2, 1, -1)$

$$\overrightarrow{PS} = 2\hat{i} - \hat{k} \text{ and } \vec{v} = 2\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\begin{aligned}\overrightarrow{PS} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -1 \\ 2 & 2 & 2 \end{vmatrix} \\ &= 2\hat{i} - 6\hat{j} + 4\hat{k}\end{aligned}$$

$$\begin{aligned}d &= \frac{\sqrt{4+36+16}}{\sqrt{4+4+4}} \\ &= \frac{\sqrt{56}}{\sqrt{12}} \\ &= \frac{2\sqrt{14}}{2\sqrt{3}} \\ &= \sqrt{\frac{14}{3}} \text{ unit}\end{aligned}$$

$$d = \frac{|\overrightarrow{PS} \times \vec{v}|}{|\vec{v}|}$$

Exercise

Find the distance from the point to the plane $(3, -1, 4)$, $x = 4 - t$, $y = 3 + 2t$, $z = -5 + 3t$

Solution

At $t = 0 \Rightarrow P(4, 3, -5)$ and let $S(3, -1, 4)$

$$\overrightarrow{PS} = -\hat{i} - 4\hat{j} + 9\hat{k} \text{ and } \vec{v} = -\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\begin{aligned}\overrightarrow{PS} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -4 & 9 \\ -1 & 2 & 3 \end{vmatrix} \\ &= -30\hat{i} - 6\hat{j} - 6\hat{k}\end{aligned}$$

$$\begin{aligned}d &= \frac{\sqrt{900+36+36}}{\sqrt{1+4+9}} \\ &= \sqrt{\frac{972}{14}} \\ &= \sqrt{\frac{486}{7}} \\ &= \frac{9\sqrt{6}}{\sqrt{7}} \frac{\sqrt{7}}{\sqrt{7}} \\ &= \frac{9\sqrt{42}}{7} \text{ unit}\end{aligned}$$

$$d = \frac{|\overrightarrow{PS} \times \vec{v}|}{|\vec{v}|}$$

Exercise

Find the distance from the point to the plane $(2, -3, 4)$, $x + 2y + 2z = 13$

Solution

$$\Rightarrow P(13, 0, 0) \text{ and let } S(2, -3, 4)$$

$$\overrightarrow{PS} = -11\hat{i} - 3\hat{j} + 4\hat{k} \text{ and } \vec{n} = \hat{i} + 2\hat{j} + 2\hat{k}$$

$$|\vec{n}| = \sqrt{1 + 4 + 4} = 3$$

$$d = \left| \left(-11\hat{i} - 3\hat{j} + 4\hat{k} \right) \cdot \left(\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k} \right) \right| \quad d = \left| \overrightarrow{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right|$$

$$= \left| -\frac{11}{3} - \frac{6}{3} + \frac{8}{3} \right|$$

$$= 3 \text{ unit}$$

Exercise

Find the distance from the point to the plane $(0, 0, 0)$, $3x + 2y + 6z = 6$

Solution

$$3x + 2y + 6z = 6$$

$$3x + 2(0) + 6(0) = 6 \rightarrow x = 2$$

$$\Rightarrow P(2, 0, 0) \text{ and let } S(0, 0, 0)$$

$$\overrightarrow{PS} = -2\hat{i} \text{ and } \vec{n} = 3\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\rightarrow |\vec{n}| = \sqrt{9 + 4 + 36} = 7$$

$$d = \left| \left(-2\hat{i} \right) \cdot \left(\frac{3}{7}\hat{i} + \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k} \right) \right| \quad d = \frac{|\overrightarrow{PS} \cdot \vec{n}|}{|\vec{n}|}$$

$$= \frac{6}{7} \text{ unit}$$

Exercise

Find the distance from the point to the plane $(0, 1, 1)$, $4y + 3z = -12$

Solution

$$\Rightarrow P(0, -3, 0) \text{ and let } S(0, 1, 1)$$

$$\overrightarrow{PS} = 4\hat{j} + \hat{k} \text{ and } \vec{n} = 4\hat{j} + 3\hat{k}$$

$$\rightarrow |\vec{n}| = \sqrt{16 + 9} = 5$$

$$\begin{aligned}
 d &= \left| \left(4\hat{j} + \hat{k} \right) \cdot \left(\frac{4}{5}\hat{j} + \frac{3}{5}\hat{k} \right) \right| & d &= \frac{|\overrightarrow{PS} \cdot \vec{n}|}{|\vec{n}|} \\
 &= \left| \frac{16}{5} + \frac{3}{5} \right| \\
 &= \frac{19}{5} \text{ unit}
 \end{aligned}$$

Exercise

Find the distance from the point to the plane $(6, 0, -6)$, $x - y = 4$

Solution

Let $y = 0$, then the point $P(4, 0, 0)$ lies on the line $x - y = 4$

$$\overrightarrow{PS} = 2\hat{i} - 6\hat{k} \quad \text{and} \quad \vec{n} = \hat{i} - \hat{j}$$

$$\begin{aligned}
 d &= \frac{|2 + 0 + 0|}{\sqrt{1+1}} & d &= \frac{|\overrightarrow{PS} \cdot \vec{n}|}{|\vec{n}|} \\
 &= \frac{2}{\sqrt{2}} \\
 &= \sqrt{2} \text{ unit}
 \end{aligned}$$

Exercise

Find the distance from the point to the plane $(3, 0, 10)$, $2x + 3y + z = 2$

Solution

Let $y = z = 0$, then the point $P(1, 0, 0)$ lies on the line $2x + 3y + z = 2$

$$\overrightarrow{PS} = 2\hat{i} + 10\hat{k} \quad \text{and} \quad \vec{n} = 2\hat{i} + 3\hat{j} + \hat{k}$$

$$\begin{aligned}
 d &= \frac{|4 + 0 + 10|}{\sqrt{4 + 9 + 1}} & d &= \frac{|\overrightarrow{PS} \cdot \vec{n}|}{|\vec{n}|} \\
 &= \frac{14}{\sqrt{14}} \\
 &= \sqrt{14} \text{ unit}
 \end{aligned}$$

Exercise

Find the distance from the point to the line $(2, 2, 0)$; $x = -t$, $y = t$, $z = -1 + t$

Solution

The line passes through the point $P(0, 0, -1)$ parallel to $\vec{v} = -\hat{i} + \hat{j} + \hat{k}$

$$\overrightarrow{PS} = 2\hat{i} + 2\hat{j} + \hat{k}$$

$$\begin{aligned}\overrightarrow{PS} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ -1 & 1 & 1 \end{vmatrix} \\ &= \hat{i} - 3\hat{j} + 4\hat{k}\end{aligned}$$

$$\begin{aligned}d &= \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}} \\ &= \frac{\sqrt{26}}{\sqrt{3}} \\ &= \frac{\sqrt{78}}{3} \text{ unit}\end{aligned}$$

$$d = \frac{|\overrightarrow{PS} \times \vec{v}|}{|\vec{v}|}$$

Exercise

Find the distance from the point to the line $(0, 4, 1); \quad x = 2 + t, \quad y = 2 + t, \quad z = t$

Solution

The line passes through the point $P(2, 2, 0)$ parallel to $\vec{v} = \hat{i} + \hat{j} + \hat{k}$

$$\overrightarrow{PS} = -2\hat{i} + 2\hat{j} + \hat{k}$$

$$\begin{aligned}\overrightarrow{PS} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} \\ &= \hat{i} + 3\hat{j} - 4\hat{k}\end{aligned}$$

$$\begin{aligned}d &= \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}} \\ &= \frac{\sqrt{26}}{\sqrt{3}} \\ &= \frac{\sqrt{78}}{3} \text{ unit}\end{aligned}$$

$$d = \frac{|\overrightarrow{PS} \times \vec{v}|}{|\vec{v}|}$$

Exercise

Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$

Solution

$$x + 2y + 6z = 1 \Rightarrow P(1, 0, 0)$$

$$x + 2y + 6z = 10 \Rightarrow S(10, 0, 0)$$

$$\overrightarrow{PS} = 9\hat{i} \text{ and } \vec{n} = \hat{i} + 2\hat{j} + 6\hat{k}$$

$$\rightarrow |\vec{n}| = \sqrt{1+4+36} = \sqrt{41}$$

$$d = \left| \left(9\hat{i} \right) \cdot \frac{1}{\sqrt{41}} (\hat{i} + 2\hat{j} + 6\hat{k}) \right| \quad d = \left| \overrightarrow{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right|$$

$$= \frac{1}{\sqrt{41}} |9|$$

$$= \frac{9}{\sqrt{41}} \text{ unit}$$

Exercise

Find the angle between the planes $x + y = 1$, $2x + y - 2z = 2$

Solution

The vectors: $\vec{n}_1 = \hat{i} + \hat{j}$, $\vec{n}_2 = 2\hat{i} + \hat{j} - 2\hat{k}$ are normal to the planes.

The angle between them is:

$$\theta = \cos^{-1} \left(\frac{2+1}{\sqrt{1+1} \sqrt{4+1+4}} \right) \quad \theta = \cos^{-1} \left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} \right)$$

$$= \cos^{-1} \left(\frac{3}{3\sqrt{2}} \right) = \cos^{-1} \left(\frac{1}{\sqrt{2}} \right)$$

$$= \frac{\pi}{4}$$

Exercise

Find the angle between the planes $5x + y - z = 10$, $x - 2y + 3z = -1$

Solution

The vectors: $\vec{n}_1 = 5\hat{i} + \hat{j} - \hat{k}$, $\vec{n}_2 = \hat{i} - 2\hat{j} + 3\hat{k}$ are normal to the planes.

The angle between them is:

$$\theta = \cos^{-1} \left(\frac{5-2-3}{\sqrt{25+1+1} \sqrt{1+4+9}} \right) \quad \theta = \cos^{-1} \left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} \right)$$

$$= \cos^{-1}(0)$$

$$= \frac{\pi}{2}$$

Exercise

Find the angle between the planes $x = 7$, $x + y + \sqrt{2}z = -3$

Solution

The vectors: $\vec{n}_1 = \hat{i}$, $\vec{n}_2 = \hat{i} + \hat{j} + \sqrt{2}\hat{k}$ are normal to the planes.

$$\begin{aligned}\theta &= \cos^{-1} \left(\frac{1+0+0}{1 \cdot \sqrt{1+1+2}} \right) & \theta &= \cos^{-1} \left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} \right) \\ &= \cos^{-1} \left(\frac{1}{2} \right) \\ &= \frac{\pi}{3} \quad \left| \right.\end{aligned}$$

Exercise

Find the angle between the planes $x + y = 1$, $y + z = 1$

Solution

The vectors: $\vec{n}_1 = \hat{i} + \hat{j}$, $\vec{n}_2 = \hat{j} + \hat{k}$ are normal to the planes.

$$\begin{aligned}\theta &= \cos^{-1} \left(\frac{0+1+1}{\sqrt{1+1} \cdot \sqrt{1+1}} \right) & \theta &= \cos^{-1} \left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} \right) \\ &= \cos^{-1} \left(\frac{1}{2} \right) \\ &= \frac{\pi}{3} \quad \left| \right.\end{aligned}$$

Exercise

Find the point in which the line meets the plane $x = 1 - t$, $y = 3t$, $z = 1 + t$; $2x - y + 3z = 6$

Solution

$$2(1-t) - 3t + 3(1+t) = 6$$

$$2 - 2t - 3t + 3 + 3t = 6$$

$$-2t = 1$$

$$t = -\frac{1}{2} \quad \left| \right.$$

$$x = 1 - \left(-\frac{1}{2}\right) = \frac{3}{2}, \quad y = 3\left(-\frac{1}{2}\right) = -\frac{3}{2}, \quad z = 1 + \left(-\frac{1}{2}\right) = \frac{1}{2}$$

$$\Rightarrow P\left(\frac{3}{2}, -\frac{3}{2}, \frac{1}{2}\right) \quad \left| \right.$$

Exercise

Find the point in which the line meets the plane $x = 2$, $y = 3 + 2t$, $z = -2 - 2t$; $6x + 3y - 4z = -12$

Solution

$$12 + 3(3 + 2t) - 4(-2 - 2t) = -12$$

$$12 + 9 + 6t + 8 + 8t = -12$$

$$14t = -41$$

$$t = -\frac{41}{14}$$

$$x = 2, \quad y = 3 + 2\left(-\frac{41}{14}\right) = -\frac{20}{7}, \quad z = -2 - 2\left(-\frac{41}{14}\right) = \frac{27}{7}$$

$$\underline{P\left(2, -\frac{20}{7}, \frac{27}{7}\right)}$$

Exercise

Find an equation of the line through the point $(0, 1, 1)$ and parallel to the line

$$\mathbf{R}(t) = \langle 1 + 2t, 3 - 5t, 7 + 6t \rangle$$

Solution

$$\text{Direction: } \vec{v} = \langle 2, -5, 6 \rangle$$

$$\begin{aligned} \text{Line: } \langle 0, 1, 1 \rangle + t \langle 2, -5, 6 \rangle \\ = \underline{\langle 2t, 1 - 5t, 1 + 6t \rangle} \end{aligned}$$

Exercise

Find an equation of the line through the point $(0, 1, 1)$ that is orthogonal to both $\langle 0, -1, 3 \rangle$ and $\langle 2, -1, 2 \rangle$

Solution

$$\begin{aligned} \text{Direction: } \langle 0, -1, 3 \rangle \times \langle 2, -1, 2 \rangle &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 3 \\ 2 & -1 & 2 \end{vmatrix} \\ &= \langle 1, 6, 2 \rangle \end{aligned}$$

$$\text{Line through } (0, 1, 1): \begin{cases} x = t \\ y = 1 + 6t \\ z = 1 + 2t \end{cases}$$

Exercise

Find an equation of the line through the point $(0, 1, 1)$ that is orthogonal to the vector $\langle -2, 1, 7 \rangle$ and the y -axis

Solution

$$(0, 1, 1) \perp \langle -2, 1, 7 \rangle \text{ \& } y\text{-axis}$$

$$\begin{aligned} \text{Direction: } \langle -2, 1, 7 \rangle \times \langle 0, 1, 0 \rangle &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 7 \\ 0 & 1 & 0 \end{vmatrix} \\ &= \langle -7, 0, -2 \rangle \end{aligned}$$

$$\text{Line through } (0, 1, 1): \begin{cases} x = -7t \\ y = 1 \\ z = 1 - 2t \end{cases}$$

Exercise

Suppose that \vec{n} is normal to a plane and that \vec{v} is parallel to the plane. Describe how you would find a vector \vec{n} that is both perpendicular to \vec{v} and parallel to the plane.

Solution

The desired vector is $\vec{n} \times \vec{v}$ or $\vec{v} \times \vec{n}$, since $\vec{n} \times \vec{v}$ is perpendicular to both \vec{n} and \vec{v} , therefore, also parallel to the plane

Exercise

Given a point $(x_0, y_0, 0)$ and a vector $\mathbf{v} = \langle a, b, 0 \rangle$ in \mathbb{R}^3 , describe the set of points that satisfy the equation $\langle a, b, 0 \rangle \times \langle x - x_0, y - y_0, 0 \rangle = \mathbf{0}$. Use this result to determine an equation of a line in \mathbb{R}^2 passing through (x_0, y_0) parallel to the vector $\langle a, b \rangle$

Solution

$$\begin{aligned} \langle a, b, 0 \rangle \times \langle x - x_0, y - y_0, 0 \rangle &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & 0 \\ x - x_0 & y - y_0 & 0 \end{vmatrix} \\ &= \langle 0, 0, a(y - y_0) - b(x - x_0) \rangle \\ &= \langle 0, 0, 0 \rangle \end{aligned}$$

$$a(y - y_0) - b(x - x_0) = 0$$

$$a(y - y_0) = b(x - x_0)$$

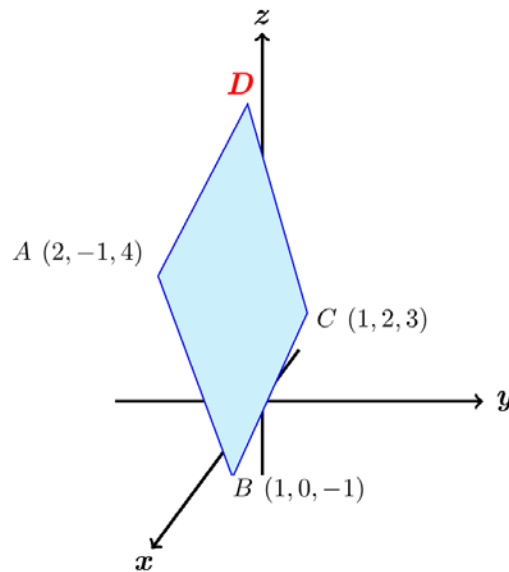
$$\frac{y - y_0}{x - x_0} = \frac{b}{a} = m \quad (\text{slope})$$

Equation of a line passing through (x_0, y_0) parallel to the vector $\langle a, b \rangle$

$$\underline{ay - bx = +ay_0 - bx_0}$$

Exercise

The parallelogram has vertices at $A(2, -1, 4)$, $B(1, 0, -1)$, $C(1, 2, 3)$ and D . Find



- The coordinates of D ,
- The cosine of the interior angle of B
- The vector projection of \overrightarrow{BA} onto \overrightarrow{BC} ,
- The area of the parallelogram,
- An equation for the plane of the parallelogram,
- The areas of the orthogonal projection of the parallelogram on the three coordinate planes.

Solution

$$\begin{aligned} \text{a) } \overrightarrow{AB} &= \langle 1 - 2, 0 + 1, -1 - 4 \rangle \\ &= \langle -1, 1, -5 \rangle \end{aligned}$$

$$\overrightarrow{DC} = \langle 1 - x, 2 - y, 3 - z \rangle$$

$$\overrightarrow{DC} = \overrightarrow{AB}$$

$$\langle 1-x, 2-y, 3-z \rangle = \langle -1, 1, -5 \rangle$$

$$\begin{cases} 1-x=-1 & \rightarrow x=2 \\ 2-y=1 & \rightarrow y=1 \\ 3-z=-5 & \rightarrow z=8 \end{cases}$$

$$\Rightarrow \underline{D = (2, 1, 8)}$$

$$b) \quad \overrightarrow{BA} = \langle 1, -1, 5 \rangle$$

$$\begin{aligned} \overrightarrow{BC} &= \langle 1-1, 2-0, 3+1 \rangle \\ &= \langle 0, 2, 4 \rangle \end{aligned}$$

$$\begin{aligned} \cos B &= \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}| |\overrightarrow{BC}|} \\ &= \frac{\langle 1, -1, 5 \rangle \cdot \langle 0, 2, 4 \rangle}{\sqrt{1+1+25} \sqrt{4+16}} \\ &= \frac{0-2+20}{\sqrt{27} \sqrt{20}} \\ &= \frac{18}{3\sqrt{3} \cdot 2\sqrt{5}} \\ &= \underline{\underline{\frac{3}{\sqrt{15}}}} \end{aligned}$$

$$\begin{aligned} c) \quad \text{proj}_{\overrightarrow{BC}} \overrightarrow{BA} &= \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BC}|^2} \overrightarrow{BC} \\ &= \frac{\langle 1, -1, 5 \rangle \cdot \langle 0, 2, 4 \rangle}{4+16} \langle 0, 2, 4 \rangle \\ &= \frac{18}{20} \langle 0, 2, 4 \rangle \\ &= \frac{9}{10} \langle 0, 2, 4 \rangle \\ &= \underline{\underline{\langle 0, \frac{9}{5}, \frac{18}{5} \rangle}} \end{aligned}$$

$$\begin{aligned} d) \quad \text{Area} &= |\overrightarrow{BA} \times \overrightarrow{BC}| \\ &= \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 5 \\ 0 & 2 & 4 \end{vmatrix} \right\| \\ &= \left| -14\hat{i} - 4\hat{j} + 2\hat{k} \right| \\ &= \sqrt{196+16+4} \\ &= \sqrt{216} \end{aligned}$$

$$= 6\sqrt{6} \quad |$$

$$e) \quad \overrightarrow{BA} \times \overrightarrow{BC} = -14\hat{i} - 4\hat{j} + 2\hat{k} = \vec{n}$$

$$-14(x-1) - 4y + 2(z+1) = 0$$

$$-14x + 14 - 4y + 2z + 2 = 0$$

$$-14x - 4y + 2z = -16$$

$$7x + 2y - z = 8 \quad |$$

$$f) \quad \vec{n} = -14\hat{i} - 4\hat{j} + 2\hat{k}$$

$$\text{Area of the projection on } yz\text{-plane} \quad |\vec{n} \cdot \hat{i}| = 14 \quad |$$

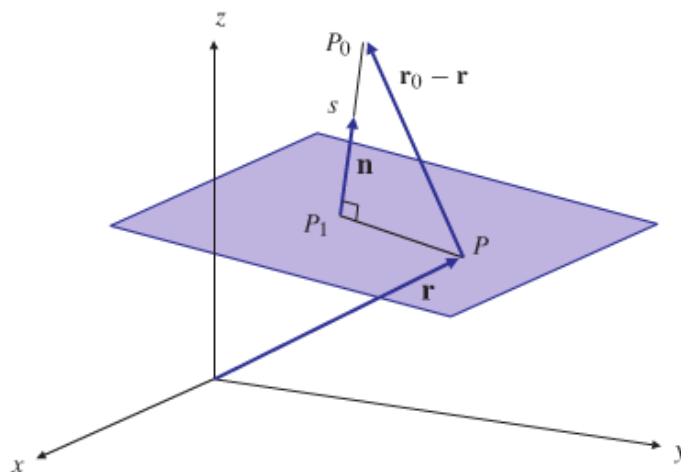
$$\text{Area of the projection on } xz\text{-plane} \quad |\vec{n} \cdot \hat{j}| = 4 \quad |$$

$$\text{Area of the projection on } xy\text{-plane} \quad |\vec{n} \cdot \hat{k}| = 2 \quad |$$

Exercise

a) Find the distance from the point $P_0(x_0, y_0, z_0)$ to the plane P having equation

$$Ax + By + Cz = D$$



b) What is the distance from $(2, -1, 3)$ to the plane $2x - 2y - z = 9$?

Solution

$$a) \quad \vec{n} = A\hat{i} + B\hat{j} + C\hat{k}$$

$$|\vec{n}| = \sqrt{A^2 + B^2 + C^2}$$

$$\text{Let } y = z = 0 \rightarrow P = \left(\frac{D}{A}, 0, 0\right)$$

$$\overrightarrow{PP_0} = \left\langle x_0 - \frac{D}{A}, y_0, z_0 \right\rangle$$

$$\begin{aligned}
 d &= \left(\left(x_0 - \frac{D}{A} \right) \hat{i} + y_0 \hat{j} + z_0 \hat{k} \right) \cdot \frac{A\hat{i} + B\hat{j} + C\hat{k}}{\sqrt{A^2 + B^2 + C^2}} \\
 &= \frac{1}{\sqrt{A^2 + B^2 + C^2}} \left(\left(x_0 - \frac{D}{A} \right) A + y_0 B + z_0 C \right) \\
 &= \frac{Ax_0 + By_0 + Cz_0 - D}{\sqrt{A^2 + B^2 + C^2}}
 \end{aligned}$$

$$d = \overrightarrow{PP_0} \cdot \frac{\vec{n}}{|\vec{n}|}$$

b) Distance $(2, -1, 3)$ to $2x - 2y - z = 9$

$$\begin{aligned}
 d &= \frac{|2(2) - 2(-1) + (-1)(3) - 9|}{\sqrt{4 + 4 + 1}} \\
 &= \frac{|-6|}{3} \\
 &= \underline{2 \text{ units}}
 \end{aligned}$$