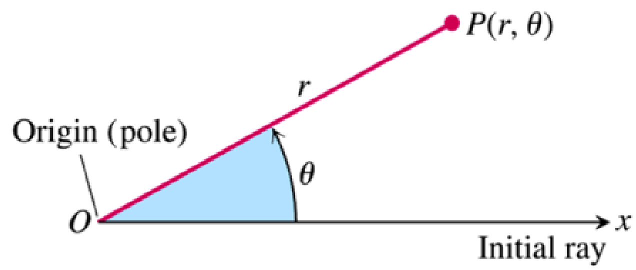
Section 4.6 – Polar Coordinates

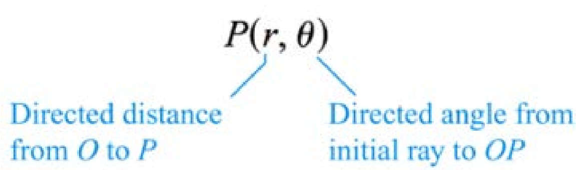
To reach the point whose address is (2, 1), we start from origin and travel 2 units right and then 1 unit up. Another way to get to that point, we can travel $\sqrt{5}$ units on the terminal side of an angle in standard position and this type is called *Polar Coordinates*.

Definition of Polar Coordinates

To define polar coordinates, let an ***origin*** *O* (called the ***pole***) and an ***initial ray*** from *O*. Then each point *P* can be located by assigning to it a ***polar coordinate pair*** $(r,\ \theta)$ in which r gives the directed from *O* to *P* and θ gives the directed angle from the initial ray to yay *OP*.



Polar Coordinates



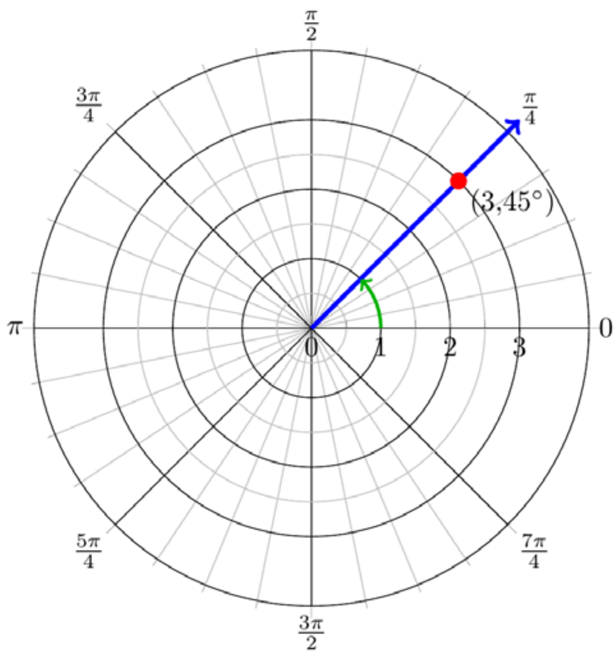
Definition – Relationships between Rectangular and Polar Coordinates

The rectangular coordinates (x, y) and polar coordinates $(r,\ \theta)$ of a point P are related as follows:

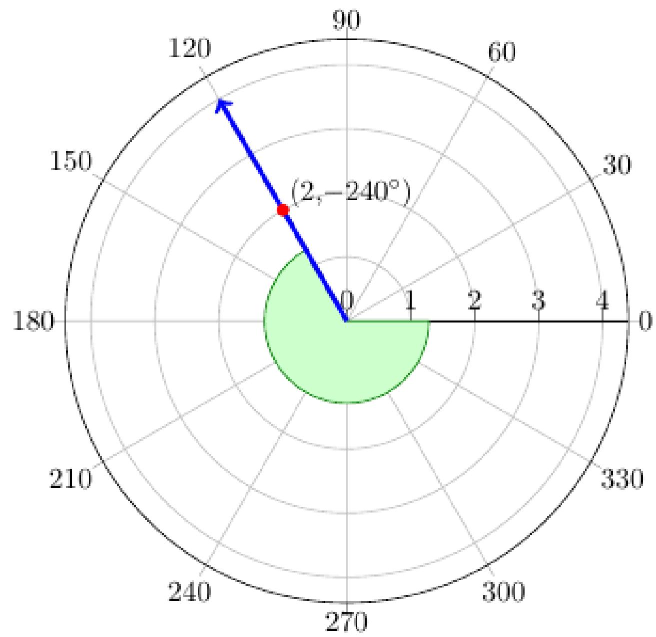
1. $x = r\cos\theta, \quad y = r\sin\theta$

2. $r^2 = x^2 + y^2 \quad \tan\theta = \dfrac{y}{x} \quad if\ \ x \neq 0$

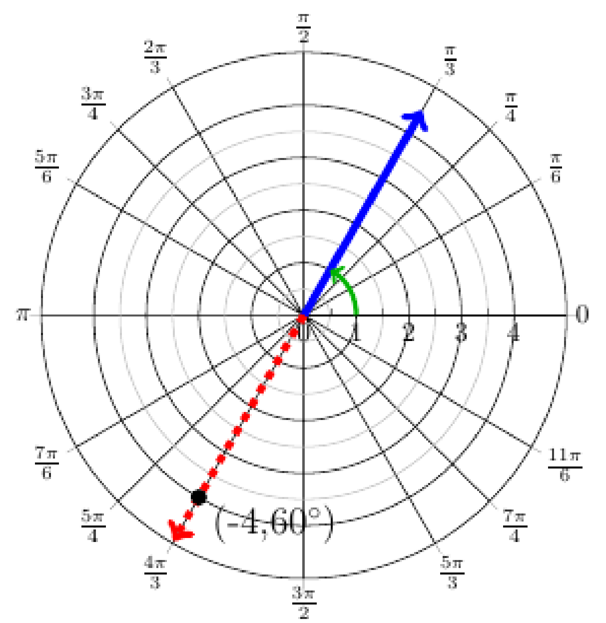
Graphing Polar Coordinates



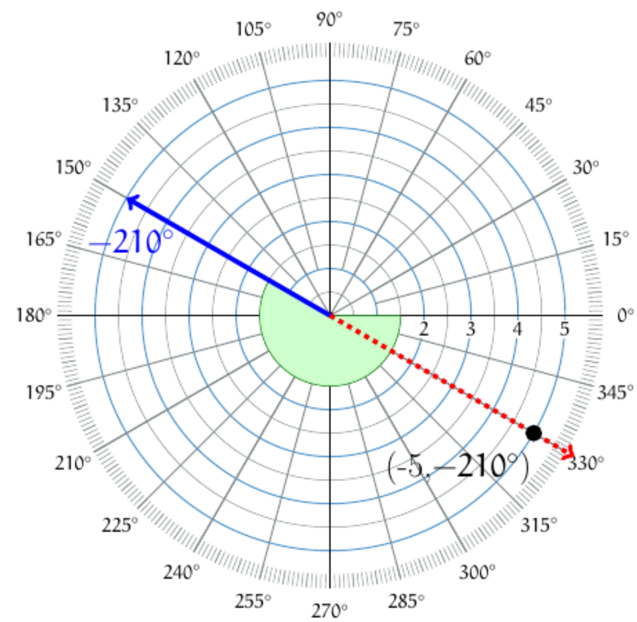
$\left(3,\ 45°\right)$



$\left(2,\ -\frac{4\pi}{3}\right)$



$\left(-4,\ \frac{\pi}{3}\right)$



$\left(-5,\ -210°\right)$

Example

If $(r,\ \theta) = \left(4,\ \frac{7\pi}{6}\right)$ are polar coordinates of a point *P*, find the rectangular coordinates of *P*.

Solution

$$x = r\cos\theta$$
$$= 4\cos\frac{7\pi}{6}$$
$$= 4\left(-\frac{\sqrt{3}}{2}\right)$$
$$\underline{= -2\sqrt{3}}$$

$$y = r\sin\theta$$
$$= 4\sin\frac{7\pi}{6}$$
$$= 4\left(-\frac{1}{2}\right)$$
$$\underline{= -2}$$

The rectangular coordinates of *P* are $\underline{(x,\ y) = \left(-2\sqrt{3},\ -2\right)}$

Example

If $(x,\ y) = \left(-1,\ \sqrt{3}\right)$ are rectangular coordinates of a point *P*, find three different pairs the polar coordinates of *P*.

Solution

$$r = \pm\sqrt{x^2 + y^2}$$
$$= \pm\sqrt{(-1)^2 + \left(\sqrt{3}\right)^2}$$
$$= \pm\sqrt{1+3}$$
$$= \pm\sqrt{4}$$
$$\underline{= \pm 2}$$

$$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{-1}$$
$$\underline{= -\sqrt{3}}$$

$$\hat{\theta} = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

$$\theta_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\theta_2 = \frac{2\pi}{3} + 2\pi = \frac{3\pi}{3}$$

$$\theta_3 = -\frac{\pi}{3}$$

The polar coordinates of P are: $\left(2,\ \frac{2\pi}{3}\right)$, $\left(-2,\ \frac{5\pi}{3}\right)$, $\left(2,\ -\frac{4\pi}{3}\right)$, and $\left(-2,\ -\frac{\pi}{3}\right)$

Example

Find a polar equation of an arbitrary line.

Solution

An equation of a line can be written in the form: $ax + by = c$.

$$ax + by = c$$

$$ar\cos\theta + br\sin\theta = c$$

$$r\left(a\cos\theta + b\sin\theta\right) = c$$

$$\boxed{r = \frac{c}{a\cos\theta + b\sin\theta}}$$

Example

Find a polar equation of the hyperbola $x^2 - y^2 = 16$.

Solution

$$(r\cos\theta)^2 - (r\sin\theta)^2 = 16$$

$$r^2\cos^2\theta - r^2\sin^2\theta = 16$$

$$r^2\left(\cos^2\theta - \sin^2\theta\right) = 16$$

$$r^2\left(\cos 2\theta\right) = 16$$

$$\boxed{r^2 = \frac{16}{\cos 2\theta}} \qquad \cos 2\theta \neq 0$$

$$or \quad r^2 = 16\sec 2\theta$$

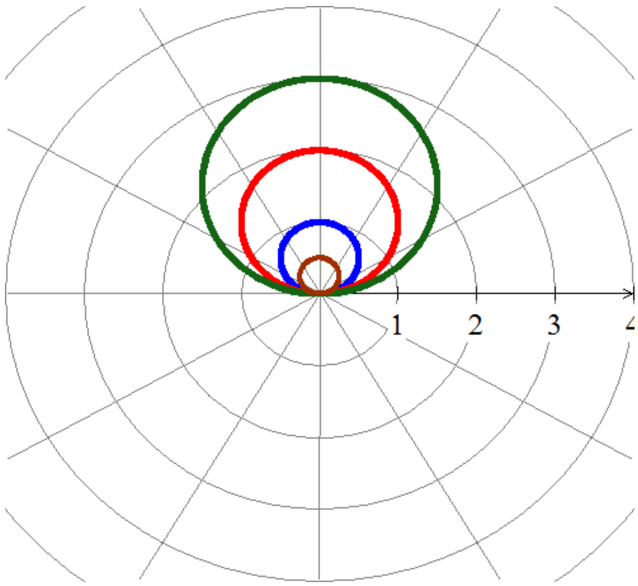
Example

Find an equation in x and y that has the same graph as the polar equation $r = a\sin\theta, \quad a \neq 0$. Sketch the graph.

Solution

$$r^2 = ar\sin\theta$$

$$x^2 + y^2 = ay$$

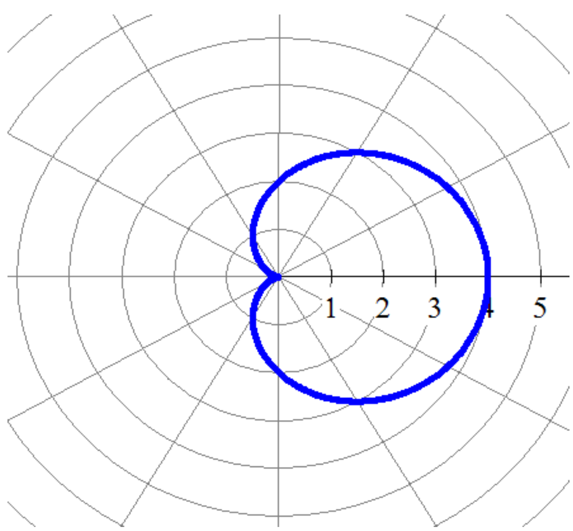


Example

Sketch the graph of the polar equation $r = 2 + 2\cos\theta$.

Solution

θ	r
0	4
$\frac{\pi}{4}$	$2+\sqrt{2}$
$\frac{\pi}{2}$	2
$\frac{3\pi}{4}$	$2-\sqrt{2}$
π	0
$\frac{3\pi}{2}$	2
2π	4



Exercises *Section* 4.6 – Polar Coordinates

(**1 – 6**) Convert to rectangular coordinates

1. $(4,\ 30°)$

2. $\left(-\sqrt{2},\ \frac{3\pi}{4}\right)$

3. $(3,\ 270°)$

4. $(2,\ 60°)$

5. $\left(\sqrt{2},\ -225°\right)$

6. $\left(4\sqrt{3},\ -\frac{\pi}{6}\right)$

7. Change the polar coordinates to rectangular coordinates $\left(-2,\ \frac{7\pi}{6}\right)$

8. Change the polar coordinates to rectangular coordinates $\left(6,\ \arctan\frac{3}{4}\right)$

9. Change the polar coordinates to rectangular coordinates $\left(10,\ \arccos\left(-\frac{1}{3}\right)\right)$

(**10 – 16**) Convert to polar coordinates

10. $(3,\ 3)$

11. $(-2,\ 0)$

12. $(-1,\ \sqrt{3})$

13. $(-3,\ -3)$ $\quad r \ge 0 \quad 0° \le \theta < 360°$

14. $\left(2,\ -2\sqrt{3}\right)$ $\quad r \ge 0 \quad 0° \le \theta < 360°$

15. $(-2,\ 0)$ $\quad r \ge 0 \quad 0 \le \theta < 2\pi$

16. $\left(-1,\ -\sqrt{3}\right)$ $\quad r \ge 0 \quad 0 \le \theta < 2\pi$

17. Change the rectangular coordinates to polar coordinates $\left(7,\ -7\sqrt{3}\right)$ $\quad r > 0 \quad 0 \le \theta < 2\pi$

18. Change the rectangular coordinates to polar coordinates $\left(-2\sqrt{2},\ -2\sqrt{2}\right)$ $\quad r > 0 \quad 0 \le \theta < 2\pi$

19. The point $(0,\ -3)$ in rectangular coordinates is equivalent to $(3,\ 270°)$ in polar coordinates.

20. The point $(1,\ -1)$ in rectangular coordinates is equivalent to $\left(-\sqrt{2},\ \frac{3\pi}{4}\right)$ in polar coordinates.

21. A point lies at (4, 4) on a rectangular coordinate system. Give its address in polar coordinates $(r,\ \theta)$

(**22 – 34**) Write the equation in rectangular coordinates

22. $r^2 = 4$

23. $r = 6\cos\theta$

24. $r^2 = 4\cos 2\theta$

25. $r(\cos\theta - \sin\theta) = 2$

26. $r^2 = 4\sin 2\theta$

27. $r\sin\theta = -2$

28. $\theta = \frac{\pi}{4}$

29. $r^2\left(4\sin^2\theta - 9\cos^2\theta\right) = 36$

30. $r^2\left(\cos^2\theta + 4\sin^2\theta\right) = 16$

31. $r(\sin\theta - 2\cos\theta) = 6$

32. $r = 8\sin\theta - 2\cos\theta$

33. $r = \tan\theta$

34. $r\left(\sin\theta + r\cos^2\theta\right) = 1$

(**35 – 38**) Find a polar equation that has the same graph as the equation in x and y

35. $y^2 = 6x$

36. $xy = 8$

37. $(x+2)^2 + (y-3)^2 = 13$

38. $y^2 - x^2 = 4$

(**39 – 42**) Write the equation in polar coordinates

39. $x + y = 5$

40. $x^2 + y^2 = 9$

41. $x^2 + y^2 = 4x$

42. $y = -x$

43. $x + y = 4$

(**44 – 54**) Sketch the graph of the polar equation

44. $r = 5$

45. $\theta = \frac{\pi}{4}$

46. $r = 4\cos\theta + 2\sin\theta$

47. $r = 2 + 4\sin\theta$

48. $r = 2 - \cos\theta$

49. $r = 4\csc\theta$

50. $r^2 = 4\cos 2\theta$

51. $r = 2^{\theta} \quad \theta \ge 0$

52. $r = e^{2\theta} \quad \theta \ge 0$

53. $r\theta = 1 \quad \theta > 0$

54. $r = 2 + 2\sec\theta$