# *Section* 2.5 – Polynomial Functions

## Polynomial Function

A *Polynomial function* $P(x)$ in $x$ is a sum of the form is given by:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

Where the coefficients $a_n$, $a_{n-1}$, $\ldots, a_2$, $a_1, a_0$ are real numbers and the exponents are whole numbers.

***Degree***

$$a_n x^n$$

***Leading Term***

***Leading Coefficient***

Non-polynomial Functions: $\frac{1}{x} + 2x;\quad \sqrt{x^2 - 3} + x;\quad \frac{x-5}{x^2+2}$

| ***Degree of f*** | ***Form of f(x)*** | ***Graph of f(x)*** |
|---|---|---|
| 0 | $f(x) = a_0$ | A horizontal line |
| 1 | $f(x) = a_1 x + a_0$ | A line with slope $a_1$ |
| 2 | $f(x) = a_2 x^2 + a_1 x + a_0$ | A parabola with a vertical axis |

All polynomial functions are ***continuous functions***.

## *End Behavior* $\left(a_n x^n\right)$

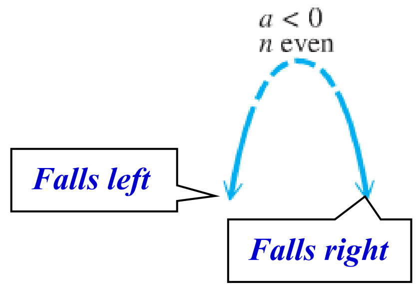
If $n$ (degree) is ***even***:

If $a_n < 0$ (in front $x^n$ is negative).

Then the function falls from the left and right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to -\infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty$$

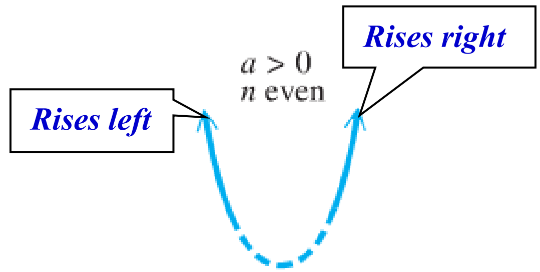



If $a_n > 0$ (in front $x^n$ is positive).

Then the function rises from the left and right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to \infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to \infty$$



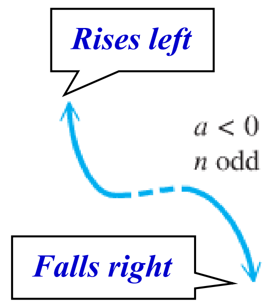

If $n$ (degree) is ***odd***:

If $a_n < 0$ (negative).

Then the function rises from the left side and falls from the right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to \infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty$$

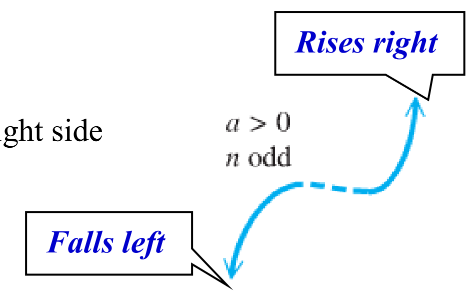



If $a_n > 0$ (positive).

Then the function falls from the left side and rises from the right side

$$x \to -\infty \quad \Rightarrow \quad f(x) \to -\infty$$

$$x \to \infty \quad \Rightarrow \quad f(x) \to \infty$$




## *Example*

Determine the end behavior of the graph of the polynomial function $f(x) = -4x^5 + 7x^2 - x + 9$

***Solution***

Leading term: $-4x^5$ with 5th degree ($n$ is odd)

$x \to -\infty \quad \Rightarrow \quad f(x) = -(-)^5 = (-)(-) = + \to \infty \qquad f(x)$ rises left

$x \to \infty \quad \Rightarrow \quad f(x) \to -\infty \qquad f(x)$ falls right

## The Intermediate Value *Theorem*

For any polynomial function $f(x)$ with real coefficients and $f(a) \neq f(b)$ for $a < b$, then $f$ takes on every value between $f(a)$ and $f(b)$ in the interval $[a, b]$.

$\therefore$ $f(a)$ and $f(b)$ are the ***opposite signs***. Then the function has a real zero between *a* and *b*.

### *Example*

Using the intermediate value theorem, determine, if possible, whether the function has a real zero between *a* and *b*.

*a)* $f(x) = x^3 + x^2 - 6x; \quad a = -4, \ b = -2$

*b)* $f(x) = x^3 + x^2 - 6x; \quad a = -1, \ b = 3$

***Solution***

***a)*** $f(x) = x^3 + x^2 - 6x; \quad a = -4, \ b = -2$

$$f(-4) = (-4)^3 + (-4)^2 - 6(-4)$$
$$= -24$$
$$f(-2) = (-2)^3 + (-2)^2 - 6(-2)$$
$$= 8$$

$\therefore$ $f(x)$ has a zero between $-4$ and $-2$

***b)*** $f(x) = x^3 + x^2 - 6x; \quad a = -1, \ b = 3$

$$f(-1) = (-1)^3 + (-1)^2 - 6(-1)$$
$$= 6$$
$$f(3) = (3)^3 + (3)^2 - 6(3) = 18$$
$$= 18$$

$\therefore$ $f(x)$ *zeros* ***can't be determined***

### *Example*

Show that $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3$ has a zero between 1 and 2.

***Solution***

$$f(1) = 1 + 2 - 6 + 2 - 3$$
$$= -4$$
$$f(2) = (2)^5 + 2(2)^4 - 6(2)^3 + 2(2) - 3$$

$= 17$

Since $f(1)$ *and* $f(2)$ have opposite signs.

Therefore, $f(c) = 0$ for at least one real number $c$ between 1 and 2.

# *Exercises* *Section* 2.5 – Polynomial Functions

Determine the end behavior of the graph of the polynomial function

1. $f(x) = 5x^3 + 7x^2 - x + 9$
2. $f(x) = 11x^3 - 6x^2 + x + 3$
3. $f(x) = -11x^3 - 6x^2 + x + 3$
4. $f(x) = 2x^3 + 3x^2 - 23x - 42$
5. $f(x) = 5x^4 + 7x^2 - x + 9$
6. $f(x) = 11x^4 - 6x^2 + x + 3$
7. $f(x) = -5x^4 + 7x^2 - x + 9$
8. $f(x) = -11x^4 - 6x^2 + x + 3$
9. $f(x) = 5x^5 - 16x^2 - 20x + 64$
10. $f(x) = -5x^5 - 16x^2 - 20x + 64$
11. $f(x) = -3x^6 - 16x^3 + 64$
12. $f(x) = 3x^6 - 16x^3 + 4$

Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers.

13. $f(x) = x^3 - x - 1;\quad between\ \ 1\ and\ \ 2$
14. $f(x) = x^3 - 4x^2 + 2;\quad between\ \ 0\ and\ \ 1$
15. $f(x) = 2x^4 - 4x^2 + 1;\quad between\ \ -1\ and\ \ 0$
16. $f(x) = x^4 + 6x^3 - 18x^2;\quad between\ \ 2\ and\ \ 3$
17. $f(x) = x^3 + x^2 - 2x + 1;\quad between\ \ -3\ and\ \ -2$
18. $f(x) = x^5 - x^3 - 1;\quad between\ \ 1\ and\ \ 2$
19. $f(x) = 3x^3 - 10x + 9;\quad between\ \ -3\ and\ \ -2$
20. $f(x) = 3x^3 - 8x^2 + x + 2;\quad between\ \ 2\ and\ \ 3$
21. $f(x) = 3x^3 - 8x^2 + x + 2;\quad between\ \ 1\ and\ \ 2$
22. $f(x) = x^5 + 2x^4 - 6x^3 + 2x - 3;\quad between\ \ 0\ and\ \ 1$
23. $P(x) = 2x^3 + 3x^2 - 23x - 42,\quad a = 3,\quad b = 4$
24. $P(x) = 4x^3 - x^2 - 6x + 1,\quad a = 0,\quad b = 1$
25. $P(x) = 3x^3 + 7x^2 + 3x + 7,\quad a = -3,\quad b = -2$
26. $P(x) = 2x^3 - 21x^2 - 2x + 25,\quad a = 1,\quad b = 2$
27. $P(x) = 4x^4 + 7x^3 - 11x^2 + 7x - 15,\quad a = 1,\quad b = \frac{3}{2}$
28. $P(x) = 5x^3 - 16x^2 - 20x + 64,\quad a = 3,\quad b = \frac{7}{2}$
29. $P(x) = x^4 - x^2 - x - 4,\quad a = 1,\quad b = 2$

**30.** $P(x) = x^3 - x - 8, \quad a = 2, \quad b = 3$

**31.** $P(x) = x^3 - x - 8, \quad a = 0, \quad b = 1$

**32.** $P(x) = x^3 - x - 8, \quad a = 2.1, \quad b = 2.2$