Solution Section 2.2 – Limits and Continuity

Exercise

Find the limits $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$

Solution

$$\lim\limits_{(x,y)\to(0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} = \frac{3(0)^2 - (0)^2 + 5}{(0)^2 + (0)^2 + 2}$$

$$= \frac{5}{2}$$

Exercise

Find the limit $\lim\limits_{(x,y)\to(0,4)} \dfrac{x}{\sqrt{y}}$

Solution

$$\lim\limits_{(x,y)\to(0,4)} \frac{x}{\sqrt{y}} = \frac{0}{\sqrt{4}}$$

$$= 0$$

Exercise

Find the limit $\lim\limits_{(x,y)\to(3,4)} \sqrt{x^2 + y^2 - 1}$

Solution

$$\lim\limits_{(x,y)\to(3,4)} \sqrt{x^2 + y^2 - 1} = \sqrt{3^2 + 4^2 - 1}$$

$$= \sqrt{24}$$

$$= 2\sqrt{6}$$

Exercise

Find the limit $\lim\limits_{(x,y)\to(0,0)} \cos\dfrac{x^2 + y^3}{x + y + 1}$

Solution

$$\lim_{(x,y)\to(0,0)} \cos\frac{x^2+y^3}{x+y+1} = \cos\frac{0^2+0^3}{0+0+1}$$
$$= \cos 0$$
$$\underline{= 1}$$

Exercise

Find the limit $\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x}$

Solution

$$\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x} = e^0 \cdot \lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = 1(1)$$
$$\underline{= 1}$$

Exercise

Find the limit $\lim_{(x,y)\to\left(\frac{\pi}{2},0\right)} \frac{\cos y+1}{y-\sin x}$

Solution

$$\lim_{(x,y)\to\left(\frac{\pi}{2},0\right)} \frac{\cos y+1}{y-\sin x} = \frac{\cos 0+1}{0-\sin\frac{\pi}{2}}$$
$$= \frac{1+1}{-1}$$
$$\underline{= -2}$$

Exercise

Find the limit $\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-2xy+y^2}{x-y}$

Solution

$$\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-2xy+y^2}{x-y} = \frac{1^2-2(1)(1)+1^2}{1-1} = \frac{0}{0}$$

$$\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-2xy+y^2}{x-y} = \lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{(x-y)^2}{x-y}$$

$$= \lim_{\substack{(x,y)\to(1,1) \\ x\neq y}} (x-y)$$

$$= 1 - 1$$

$$\underline{= 0}$$

Exercise

Find the limit $\displaystyle \lim_{\substack{(x,y)\to(1,1) \\ x\neq y}} \frac{x^2 - y^2}{x - y}$

Solution

$$\lim_{\substack{(x,y)\to(1,1) \\ x\neq y}} \frac{x^2 - y^2}{x - y} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$\lim_{\substack{(x,y)\to(1,1) \\ x\neq y}} \frac{x^2 - y^2}{x - y} = \lim_{\substack{(x,y)\to(1,1) \\ x\neq y}} \frac{(x-y)(x+y)}{x-y}$$

$$= \lim_{\substack{(x,y)\to(1,1) \\ x\neq y}} (x+y)$$

$$= 1 + 1$$

$$\underline{= 2}$$

Exercise

Find the limit $\displaystyle \lim_{\substack{(x,y)\to(2,-4) \\ x\neq x^2,\ y\neq -4}} \frac{y+4}{x^2y - xy + 4x^2 - 4x}$

Solution

$$\lim_{\substack{(x,y)\to(2,-4) \\ x\neq x^2,\ y\neq -4}} \frac{y+4}{x^2y - xy + 4x^2 - 4x} = \lim_{\substack{(x,y)\to(2,-4) \\ x\neq x^2,\ y\neq -4}} \frac{y+4}{y\left(x^2 - x\right) + 4\left(x^2 - x\right)}$$

$$= \lim_{\substack{(x,y)\to(2,-4) \\ x\neq x^2,\ y\neq -4}} \frac{y+4}{\left(x^2 - x\right)(y+4)}$$

$$= \lim_{\substack{(x,y)\to(2,-4) \\ x\neq x^2,\ y\neq -4}} \frac{1}{x(x-1)}$$

$$= \frac{1}{2(2-1)}$$

$$= \frac{1}{2}$$

Exercise

Find the limit $\displaystyle \lim_{\substack{(x,y)\to(4,3) \\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1}$

Solution

$$\lim_{\substack{(x,y)\to(4,3) \\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1} = \lim_{\substack{(x,y)\to(4,3) \\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{\left(\sqrt{x}-\sqrt{y+1}\right)\left(\sqrt{x}+\sqrt{y+1}\right)}$$

$$= \lim_{\substack{(x,y)\to(4,3) \\ x\neq y+1}} \frac{1}{\sqrt{x}+\sqrt{y+1}}$$

$$= \frac{1}{\sqrt{4}+\sqrt{3+1}} = \frac{1}{2+2}$$

$$= \frac{1}{4}$$

Exercise

Find the limit $\displaystyle \lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y}$

Solution

$$\lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$\lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y} = \lim_{(x,y)\to(1,-1)} \frac{(x+y)\left(x^2-xy+y^2\right)}{x+y}$$

$$= \lim_{(x,y)\to(1,-1)} \left(x^2-xy+y^2\right)$$

$$= 1^2-(1)(-1)+(-1)^2$$

$$= 3$$

Exercise

Find the limit $\lim\limits_{(x,y)\to(2,2)} \dfrac{x-y}{x^4-y^4}$

Solution

$$\lim_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4} = \lim_{(x,y)\to(2,2)} \frac{x-y}{\left(x^2-y^2\right)\left(x^2+y^2\right)}$$

$$= \lim_{(x,y)\to(2,2)} \frac{x-y}{(x-y)(x+y)\left(x^2+y^2\right)}$$

$$= \lim_{(x,y)\to(2,2)} \frac{1}{(x+y)\left(x^2+y^2\right)}$$

$$= \frac{1}{(2+2)\left(2^2+2^2\right)}$$

$$= \frac{1}{32}$$

Exercise

Find the limit $\lim\limits_{P\to(1,3,4)} \left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)$

Solution

$$\lim_{P\to(1,3,4)} \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{1}{1}+\frac{1}{3}+\frac{1}{4}$$

$$= \frac{19}{12}$$

Exercise

Find the limit $\lim\limits_{P\to(1,-1,-1)} \dfrac{2xy+yz}{x^2+z^2}$

Solution

$$\lim_{P\to(1,-1,-1)} \frac{2xy+yz}{x^2+z^2} = \frac{2(1)(-1)+(-1)(-1)}{1^2+(-1)^2}$$

$$= -\frac{1}{2}$$

Exercise

Find the limit $\lim\limits_{P \to (\pi,0,2)} ze^{-2y} \cos 2x$

Solution

$$\lim_{P \to (\pi,0,2)} ze^{-2y} \cos 2x = 2e^{-2(0)} \cos 2\pi$$

$$= 2$$

Exercise

Find the limit $\lim\limits_{P \to (2,-3,6)} \ln \sqrt{x^2 + y^2 + z^2}$

Solution

$$\lim_{P \to (2,-3,6)} \ln \sqrt{x^2 + y^2 + z^2} = \ln \sqrt{4 + 9 + 36}$$

$$= \ln \sqrt{49}$$

$$= \ln 7$$

Exercise

Find the limit $\lim\limits_{(x,y) \to (4,-2)} (10x - 5y + 6xy)$

Solution

$$\lim_{(x,y) \to (4,-2)} (10x - 5y + 6xy) = 40 + 10 - 48$$

$$= 2$$

Exercise

Find the limit $\lim\limits_{(x,y) \to (1,1)} \frac{xy}{x+y}$

Solution

$$\lim_{(x,y) \to (1,1)} \frac{xy}{x+y} = \frac{1}{2}$$

Exercise

Find the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+y}{xy}$

Solution

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{xy} = \frac{0}{0}$$

Along path $y = x$

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{xy} = \lim_{(x,y)\to(0,0)} \frac{2x}{x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{2}{x}$$

$$= \infty$$

Along path $y = -x$

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{xy} = \lim_{(x,y)\to(0,0)} \frac{0}{-x^2}$$

$$= -\infty$$

$\therefore$ Limit ***doesn't exist***

Exercise

Find the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin xy}{x^2+y^2}$

Solution

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{x^2+y^2} = \frac{0}{0}$$

Along path $y = x$

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{\sin x^2}{2x^2} = \frac{0}{0}$$

$$= \lim_{(x,y)\to(0,0)} \frac{2x\cos x^2}{4x}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\cos x^2}{2}$$

$$= \frac{1}{2}$$

Along path $y = -x$

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{-\sin x^2}{2x^2} = \frac{0}{0}$$

$$= \lim_{(x,y)\to(0,0)} \frac{-2x\cos x^2}{4x}$$

$$= \lim_{(x,y)\to(0,0)} \frac{-\cos x^2}{2}$$

$$= -\frac{1}{2}$$

$\therefore$ Limit ***doesn't exist***

Exercise

Find the limit $\displaystyle\lim_{(x,y)\to(-1,1)} \frac{x^2-y^2}{x^2-xy-2y^2}$

Solution

$$\lim_{(x,y)\to(-1,1)} \frac{x^2-y^2}{x^2-xy-2y^2} = \frac{0}{0}$$

$$= \lim_{(x,y)\to(-1,1)} \frac{(x-y)(x+y)}{(x-2y)(x+y)}$$

$$= \lim_{(x,y)\to(-1,1)} \frac{x-y}{x-2y}$$

$$= \frac{-1-1}{-1-2}$$

$$= \frac{2}{3}$$

Exercise

Find the limit $\displaystyle\lim_{(x,y)\to(1,2)} \frac{x^2y}{x^4+2y^2}$

Solution

$$\lim_{(x,y)\to(1,2)} \frac{x^2y}{x^4+2y^2} = \frac{2}{9}$$

Exercise

Find the limit $\lim\limits_{(x,y,z)\to\left(\frac{\pi}{2},0,\frac{\pi}{2}\right)} 4\cos y\sin\sqrt{xz}$

Solution

$$\lim\limits_{(x,y,z)\to\left(\frac{\pi}{2},0,\frac{\pi}{2}\right)} 4\cos y\sin\sqrt{xz} = 4(\cos 0)\sin\sqrt{\frac{\pi^2}{4}}$$

$$= 4\sin\frac{\pi}{2}$$

$$\boxed{= 4}$$

Exercise

Find the limit $\lim\limits_{(x,y,z)\to(5,2,-3)} \tan^{-1}\left(\frac{x+y^2}{z^2}\right)$

Solution

$$\lim\limits_{(x,y,z)\to(5,2,-3)} \tan^{-1}\left(\frac{x+y^2}{z^2}\right) = \tan^{-1}\left(\frac{9}{9}\right)$$

$$= \tan^{-1}(1)$$

$$\boxed{= \frac{\pi}{4}}$$

Exercise

At what points (*x, y, z*) in space are the functions continuous $f(x,y,z) = x^2 + y^2 - 2z^2$

Solution

All (x,y,z)

Exercise

At what points (*x, y, z*) in space are the functions continuous $f(x,y,z) = \sqrt{x^2+y^2-1}$

Solution

$x^2+y^2-1 \ge 0 \quad \to \quad x^2+y^2 \ge 1$. All (x,y,z) except the interior of the cylinder $x^2+y^2=1$

Exercise

At what points (*x, y, z*) in space are the functions continuous $f(x,y,z) = \ln(xyz)$

Solution

All (x, y, z) so that $xyz > 0$

Exercise

At what points (*x, y, z*) in space are the functions continuous $f(x, y, z) = e^{x+y} \cos z$

Solution

All (x, y, z)

Exercise

At what points (*x, y, z*) in space are the functions continuous $h(x, y, z) = \dfrac{1}{|y| + |z|}$

Solution

All (x, y, z) except $(x,\ 0,\ 0)$

Exercise

At what points (*x, y, z*) in space are the functions continuous $h(x, y, z) = \dfrac{1}{z - \sqrt{x^2 + y^2}}$

Solution

All (x, y, z) except $z \neq \sqrt{x^2 + y^2}$