Section 4.6 - De Moivre's Theorem

De Moivre's *Theorem*

If $r(\cos\theta + i\sin\theta)$ is a complex number, then

$$\left[r(\cos\theta + i\sin\theta)\right]^{n} = r^{n}(\cos n\theta + i\sin n\theta)$$

$$\boxed{\left[rcis\theta\right]^{n} = r^{n}(cisn\theta)}$$

Example

Find $\left(1+i\sqrt{3}\right)^{8}$ and express the result in rectangular form.

<u>*Solution*</u>

$$1+i\sqrt{3} \Rightarrow \begin{cases} x = 1 \\ y = \sqrt{3} \end{cases}$$

$$r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$$

$$\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$$

θ is in QI, that implies: $\theta = 60°$

$$1+i\sqrt{3} = 2cis60°$$

Apply De Moivre's theorem:

$$\begin{aligned}
\left(1+i\sqrt{3}\right)^{8} &= (2cis60°)^{8} \\
&= 2^{8}\left[cis(8.60°)\right] \\
&= 256\left[cis(480°)\right] \qquad 480° - 360° = 120° \\
&= 256\left[cis(120°)\right] \\
&= 256\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\
&= -128 + 128i\sqrt{3}
\end{aligned}$$

n^{th} Root Theorem

For a positive integer n, the complex number $a + bi$ is an $\boldsymbol{n^{th}}$ **root** of the complex number $x + iy$ if

$$(a+bi)^n = x+yi$$

If n is any positive integer, r is a positive real number, and θ is in degrees, then the nonzero complex number $r(\cos\theta + i\sin\theta)$ has exactly n distinct nth roots, given by

$$\sqrt[n]{r}\,(\cos\alpha + i\sin\alpha) \ \text{ or } \ \sqrt[n]{r}\ cis\alpha$$

Where $\alpha = \dfrac{\theta + 360°k}{n}, \quad k = 0, 1, 2, \cdots, n-1$ $\quad \boxed{\alpha = \dfrac{\theta}{n} + \dfrac{360°k}{n}}$

$$\alpha = \frac{\theta + 2\pi k}{n}, \quad k = 0, 1, 2, \cdots, n-1 \qquad \boxed{\alpha = \frac{\theta}{n} + \frac{2\pi k}{n}}$$

Example

Find the two square root of $\boldsymbol{4i}$. Write the roots in rectangular form.

<u>*Solution*</u>

$$4i \Rightarrow \begin{cases} x = 0 \\ y = 4 \end{cases} \qquad \rightarrow r = \sqrt{0^2 + 4^2} = 4$$

$$\tan\theta = \frac{4}{0} = \infty \Rightarrow \boxed{\theta = \frac{\pi}{2}}$$

$$4i = 4cis\frac{\pi}{2}$$

The absolute value: $\sqrt{4} = 2$

Argument: $\alpha = \dfrac{\frac{\pi}{2} + 2\pi k}{2} = \dfrac{\frac{\pi}{2}}{2} + \dfrac{2\pi k}{2} = \dfrac{\pi}{4} + \pi k$

Since there are ***two*** square root, then $k = 0$ and 1.

$$If\ \ k = 0 \Rightarrow \alpha = \frac{\pi}{4} + \pi(0) = \frac{\pi}{4}$$

$$If\ \ k = 1 \Rightarrow \alpha = \frac{\pi}{4} + \pi(1) = \frac{5\pi}{4}$$

The square roots are: $2cis\dfrac{\pi}{4}$ $\ \ and\ \ $ $2cis\dfrac{5\pi}{4}$

$$2cis\frac{\pi}{4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2} + i\sqrt{2}$$

$$2cis\frac{5\pi}{4} = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -\sqrt{2} - i\sqrt{2}$$

Example

Find all fourth roots of $-8+8i\sqrt{3}$. Write the roots in rectangular form.

<u>Solution</u>

$$-8+8i\sqrt{3} \Rightarrow \begin{cases} x=-8 \\ y=8\sqrt{3} \end{cases}$$

$$r=\sqrt{(-8)^2+\left(8\sqrt{3}\right)^2}=16$$

$$\tan\theta=\frac{8\sqrt{3}}{-8}=-\sqrt{3} \Rightarrow \boxed{\theta=120^\circ}$$

$$-8+8i\sqrt{3}=16cis120^\circ$$

The fourth roots have absolute value: $\sqrt[4]{16}=2$

$$\alpha=\frac{120^\circ}{4}+\frac{360^\circ k}{4}=30^\circ+90^\circ k$$

Since there are ***four*** roots, then k = 0, 1, 2, and 3.

If $k=0 \Rightarrow \alpha=30^\circ+90^\circ(0)=30^\circ$

If $k=1 \Rightarrow \alpha=30^\circ+90^\circ(1)=120^\circ$

If $k=2 \Rightarrow \alpha=30^\circ+90^\circ(2)=210^\circ$

If $k=3 \Rightarrow \alpha=30^\circ+90^\circ(3)=300^\circ$

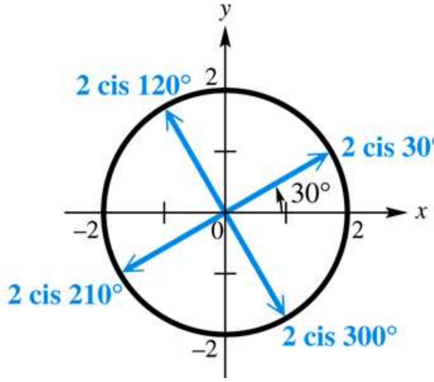
The fourth roots are: $2cis30^\circ,\ \ 2cis120^\circ,\ \ 2cis210^\circ,\ and\ \ 2cis300^\circ$

$$2cis30^\circ=2\left(\cos 30^\circ+i\sin 30^\circ\right)=2\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)=\sqrt{3}+i$$

$$2cis120^\circ=2\left(\cos 120^\circ+i\sin 120^\circ\right)=2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=-1+i\sqrt{3}$$

$$2cis210^\circ=2\left(\cos 210^\circ+i\sin 210^\circ\right)=2\left(-\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)=-\sqrt{3}-i$$

$$2cis300^\circ=2\left(\cos 300^\circ+i\sin 300^\circ\right)=2\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=1-i\sqrt{3}$$



Example

Find all complex number solutions of $x^5 - 1 = 0$. Graph them as vectors in the complex plane.

<u>*Solution*</u>

$x^5 - 1 = 0 \Rightarrow x^5 = 1$

There is one real solution, 1, while there are five complex solutions.

$1 = 1 + 0i$

$r = \sqrt{1^2 + 0^2} = 1$

$\tan\theta = \frac{0}{1} = 0 \Rightarrow \boxed{\theta = 0°}$

$1 = 1cis0°$

The fifth roots have absolute value: $\sqrt[5]{1} = 1$

$\underline{|\alpha =} \frac{0°}{5} + \frac{360°k}{5} = 0° + 72°k \underline{= 72°k|}$

Since there are ***fifth*** roots, then k = 0, 1, 2, 3, and 4.

If $k = 0 \Rightarrow \alpha = 72°(0) = 0°$

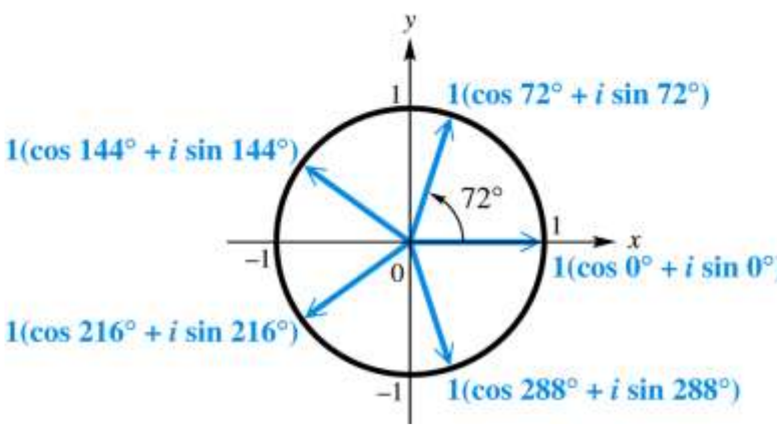
If $k = 1 \Rightarrow \alpha = 72°(0) = 72°$

If $k = 2 \Rightarrow \alpha = 72°(2) = 144°$

If $k = 3 \Rightarrow \alpha = 72°(3) = 216°$

If $k = 4 \Rightarrow \alpha = 72°(4) = 288°$

Solution: $cis0°,\ cis72°,\ cis144°,\ cis216°,\ and\ \ cis288°$



The graphs of the roots lie on a unit circle. The roots are equally spaced about the circle, 72° apart.

Exercises *Section* 4.6 - De Moivre's Theorem

1. Find $(1+i)^8$ and express the result in rectangular form.

2. Find $(1+i)^{10}$ and express the result in rectangular form.

3. Find fifth roots of $z = 1 + i\sqrt{3}$ and express the result in rectangular form.

4. Find the fourth roots of $z = 16 cis 60°$

5. Find the cube roots of 27.

6. Find all complex number solutions of $x^3 + 1 = 0$.

7. Find $(2 cis 30°)^5$