### *Exercise*

Determine if the system is linear, and if so determine which is homogeneous? or inhomogeneous?

$$\begin{cases} x_1' = -2x_1 + x_1 x_2 \\ x_2' = -3x_1 - x_2 \end{cases}$$

***Solution***

The system is nonlinear because of the term $x_1 x_2$

### *Exercise*

Determine if the system is linear, and if so determine which is homogeneous? or inhomogeneous?

$$\begin{cases} x_1' = -x_2 \\ x_2' = \sin x_1 \end{cases}$$

***Solution***

The system is nonlinear because of the term $\sin x_1$

### *Exercise*

Determine if the system is linear, and if so determine which is homogeneous? or inhomogeneous?

$$\begin{cases} x_1' = x_1 + (\sin t) x_2 \\ x_2' = 2t x_1 - x_2 \end{cases}$$

***Solution***

The system is linear and homogeneous, because $f_1(t) = f_2(t) = 0$

$$x_1' = a_{11}(t) x_1 + a_{12}(t) x_2 + f_1(t)$$
$$x_2' = a_{21}(t) x_1 + a_{2n}(t) x_2 + f_2(t)$$

### *Exercise*

Write the given system of equations in matrix-form then show that the given vector is a solution to the system

$$\begin{cases} x_1' = -3x_1 + x_2 \\ x_2' = -2x_1 \end{cases} \qquad v = \left(-e^{-2t} + e^{-t}, -e^{-2t} + 2e^{-t}\right)^T$$

***Solution***

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$x_1 = -e^{-2t} + e^{-t} \qquad x_2 = -e^{-2t} + 2e^{-t}$

$x_1' = -3x_1 + x_2$

$$2e^{-2t} - e^{-t} = -3\left(-e^{-2t} + e^{-t}\right) + \left(-e^{-2t} + 2e^{-t}\right)$$

$$2e^{-2t} - e^{-t} = 3e^{-2t} - 3e^{-t} - e^{-2t} + 2e^{-t}$$

$$2e^{-2t} - e^{-t} = 2e^{-2t} - e^{-t}$$

$$\begin{aligned} x_2' &= \left(-e^{-2t} + 2e^{-t}\right)' \\ &= 2e^{-2t} - 2e^{-t} \\ &= -2\left(-e^{-2t} + e^{-t}\right) \\ &= -2x_1 \end{aligned}$$

### Exercise

Write the given system of equations in matrix-form then show that the given vector is a solution to the system

$$\begin{cases} x_1' = -x_1 + 4x_2 \\ x_2' = \phantom{-x_1 + {}} 3x_2 \end{cases} \qquad v = \left(e^{3t} - e^{-t}, e^{3t}\right)^T$$

***Solution***

$x_1 = e^{3t} - e^{-t} \qquad x_2 = e^{3t}$

$$\begin{aligned} x_1' &= 3e^{3t} + e^{-t} \\ &= 4e^{3t} - e^{3t} + e^{-t} \\ &= -\left(e^{3t} - e^{-t}\right) + 4e^{3t} \\ &= -x_1 + 4x_2 \end{aligned}$$

$x_2' = 3e^{3t} = 3x_2$

### Exercise

Verify by substitution that $x_1(t)$ and $x_2(t)$ are solutions of the given homogenous equation. Show also that the solutions $x_1(t)$ and $x_2(t)$ are linearly independent. Find the solution of the given homogeneous equation with the initial condition $x(0) = x_0$

$$x_1(t)=\begin{pmatrix}-e^{2t}\\ 2e^{2t}\end{pmatrix}, \qquad x_2(t)=\begin{pmatrix}-e^{-2t}\\ e^{-2t}\end{pmatrix}$$

$$x'=\begin{pmatrix}-6 & -4\\ 8 & 6\end{pmatrix}x \qquad x(0)=\begin{pmatrix}-5\\ 8\end{pmatrix}$$

***Solution***

$$x_1'(t)=\begin{pmatrix}-e^{2t}\\ 2e^{2t}\end{pmatrix}'=\begin{pmatrix}-2e^{2t}\\ 4e^{2t}\end{pmatrix}$$

$$\begin{pmatrix}-6 & -4\\ 8 & 6\end{pmatrix}x_1=\begin{pmatrix}-6 & -4\\ 8 & 6\end{pmatrix}\begin{pmatrix}-e^{2t}\\ 2e^{2t}\end{pmatrix}$$

$$=\begin{pmatrix}-2e^{2t}\\ 4e^{2t}\end{pmatrix} \qquad \Rightarrow \text{Therefore, } x_1 \text{ is a solution.}$$

$$x_2'(t)=\begin{pmatrix}-e^{-2t}\\ e^{-2t}\end{pmatrix}'=\begin{pmatrix}2e^{2t}\\ -2e^{2t}\end{pmatrix}$$

$$\begin{pmatrix}-6 & -4\\ 8 & 6\end{pmatrix}x_2=\begin{pmatrix}-6 & -4\\ 8 & 6\end{pmatrix}\begin{pmatrix}-e^{2t}\\ e^{2t}\end{pmatrix}$$

$$=\begin{pmatrix}-2e^{2t}\\ 2e^{2t}\end{pmatrix} \qquad \Rightarrow x_2 \text{ is also a solution.}$$

$$x_1(0)=\begin{pmatrix}-1\\ 2\end{pmatrix} \qquad x_2(0)=\begin{pmatrix}-1\\ 1\end{pmatrix}$$

$$x(t)=C_1\begin{pmatrix}-e^{2t}\\ 2e^{2t}\end{pmatrix}+C_2\begin{pmatrix}-e^{-2t}\\ e^{-2t}\end{pmatrix}$$

$$x(0)=C_1\begin{pmatrix}-1\\ 2\end{pmatrix}+C_2\begin{pmatrix}-1\\ 1\end{pmatrix}$$

$$\begin{pmatrix}-5\\ 8\end{pmatrix}=\begin{pmatrix}-C_1-C_2\\ 2C_1+C_2\end{pmatrix} \qquad \Rightarrow\begin{cases}-C_1-C_2=-5\\ 2C_1+C_2=8\end{cases}\rightarrow \qquad \boxed{C_1=3} \qquad \boxed{C_2=2}$$

$$x(t)=3\begin{pmatrix}-e^{2t}\\ 2e^{2t}\end{pmatrix}+2\begin{pmatrix}-e^{-2t}\\ e^{-2t}\end{pmatrix}$$

$$=\begin{pmatrix}-3e^{2t}-2e^{-2t}\\ 6e^{2t}+2e^{-2t}\end{pmatrix}$$

## *Exercise*

Verify by substitution that $x_1(t)$ and $x_2(t)$ are solutions of the given homogenous equation. Show also that the solutions $x_1(t)$ and $x_2(t)$ are linearly independent. Find the solution of the given homogeneous equation with the initial condition $x(0) = x_0$

$$x_1(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}, \qquad x_2(t) = \begin{pmatrix} e^{2t}(t+2) \\ e^{2t}(t+1) \end{pmatrix}$$

$$x' = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} x \qquad x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

**Solution**

$$x_1'(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}' = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} x_1(t) = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix} \underline{= x_1'}$$

$$x_2'(t) = \begin{pmatrix} e^{2t}(t+2) \\ e^{2t}(t+1) \end{pmatrix}'$$

$$= \begin{pmatrix} 2e^{2t}(t+2) + e^{2t} \\ 2e^{2t}(t+1) + e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2te^{2t} + 4e^{2t} + e^{2t} \\ 2te^{2t} + 2e^{2t} + e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2te^{2t} + 5e^{2t} \\ 2te^{2t} + 3e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} x_2(t) = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} te^{2t} + 2e^{2t} \\ te^{2t} + e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2te^{2t} + 5e^{2t} \\ 2te^{2t} + 3e^{2t} \end{pmatrix} \underline{= x_2'}$$

$$x_1(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad x_2(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The vectors are independent ( $x_1$ is not a multiple of $x_2$ ), so the $x_1$ and $x_2$ are independent.

Therefore, the general solution is:

$$x(t) = C_1 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} e^{2t}(t+2) \\ e^{2t}(t+1) \end{pmatrix}$$

$$x(0) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 + 2C_2 \\ C_1 + C_2 \end{pmatrix}$$

$$\begin{cases} C_1 + 2C_2 = 0 \\ C_1 + C_2 = 1 \end{cases} \quad \rightarrow \quad \underline{C_1 = 2} \quad \underline{C_2 = -1}$$

$$x(t) = 2\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} - \begin{pmatrix} te^{2t} + 2e^{2t} \\ te^{2t} + e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} -te^{2t} \\ -te^{2t} + e^{2t} \end{pmatrix}$$

## *Exercise*

Verify by substitution that $x_1(t)$ and $x_2(t)$ are solutions of the given homogenous equation. Show also that the solutions $x_1(t)$ and $x_2(t)$ are linearly independent. Find the solution of the given homogeneous equation with the initial condition $x(0) = x_0$

$$x_1(t) = \begin{pmatrix} \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \cos t \end{pmatrix}, \quad x_2(t) = \begin{pmatrix} \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \sin t \end{pmatrix}$$

$$x' = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} x \qquad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

***Solution***

$$x_1'(t) = \begin{pmatrix} \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \cos t \end{pmatrix}' = \begin{pmatrix} -\frac{1}{2}\sin t - \frac{1}{2}\cos t \\ -\sin t \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} x_1 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \cos t \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2}\cos t - \frac{1}{2}\sin t \\ -\sin t \end{pmatrix}$$

$$x_2'(t)=\begin{pmatrix}\frac{1}{2}\cos t+\frac{1}{2}\sin t\\ \sin t\end{pmatrix}' = \begin{pmatrix}-\frac{1}{2}\sin t+\frac{1}{2}\cos t\\ \cos t\end{pmatrix}'$$

$$\begin{pmatrix}1 & -1\\ 2 & -1\end{pmatrix}x_2=\begin{pmatrix}1 & -1\\ 2 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{2}\cos t+\frac{1}{2}\sin t\\ \sin t\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{2}\cos t-\frac{1}{2}\sin t\\ \cos t\end{pmatrix}$$

$$x_1(0)=\begin{pmatrix}\frac{1}{2}\cos(0)-\frac{1}{2}\sin(0)\\ \cos(0)\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\ 1\end{pmatrix} \qquad x_2(0)=\begin{pmatrix}\frac{1}{2}\cos(0)+\frac{1}{2}\sin(0)\\ \sin(0)\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\ 0\end{pmatrix}$$

The vectors are independent ( $x_1$ is not a multiple of $x_2$ ), so the $x_1$ and $x_2$ are independent.

Therefore, the general solution is:

$$x(t)=C_1\begin{pmatrix}\frac{1}{2}\cos t-\frac{1}{2}\sin t\\ \cos t\end{pmatrix}+C_2\begin{pmatrix}\frac{1}{2}\cos t+\frac{1}{2}\sin t\\ \sin t\end{pmatrix}$$

$$x(0)=C_1\begin{pmatrix}\frac{1}{2}\cos(0)-\frac{1}{2}\sin(0)\\ \cos(0)\end{pmatrix}+C_2\begin{pmatrix}\frac{1}{2}\cos(0)+\frac{1}{2}\sin(0)\\ \sin(0)\end{pmatrix}$$

$$\begin{pmatrix}1\\ 0\end{pmatrix}=C_1\begin{pmatrix}\frac{1}{2}\\ 1\end{pmatrix}+C_2\begin{pmatrix}\frac{1}{2}\\ 0\end{pmatrix}$$

$$\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}\frac{1}{2}C_1+\frac{1}{2}C_2\\ C_1\end{pmatrix} \Rightarrow \qquad \underline{C_1=0|} \qquad \underline{C_2=2|}$$

$$x(t)=2\begin{pmatrix}\frac{1}{2}\cos t+\frac{1}{2}\sin t\\ \sin t\end{pmatrix}$$

$$=\begin{pmatrix}\cos t+\sin t\\ 2\sin t\end{pmatrix}$$

## *Exercise*

Rewrite the given equation into a system in normal form with initial value.

$y^{(4)}-y^{(3)}+7y=\cos t\,;\qquad y(0)=y'(0)=1,\quad y''(0)=0,\quad y^{(3)}(0)=2$

***Solution***

$y_1=y$

$y_2=y_1'=y'$

$y_3 = y_2' = y''$

$y_4 = y_3' = y'''$

$y^{(4)} - y^{(3)} + 7y = \cos t$

$\boxed{y_4' = y_4 - 7y_1 + \cos t}$

$y_1(0) = y_2(0) = 1, \quad y_3(0) = 0, \quad y_4(0) = 2$

### Exercise

Rewrite the given equation into a system in normal form with initial value.

$$y^{(4)} + 3y'' - (\sin t)y' + 8y = t^2, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 3, \quad y'''(0) = 4$$

***Solution***

$x_1 = y$

$x_2 = x_1' = y'$

$x_3 = x_2' = y''$

$x_4 = x_3' = y'''$

$x_4' = y^{(4)}$

$\boxed{= -3x_3 + (\sin t)x_2 - 8x_1 + t^2}$

$x_1(0) = 1, \ x_2(0) = 2, \ x_3(0) = 3, \ x_4(0) = 4$

### Exercise

Rewrite the given equation into a system in normal form with initial value.

$$y^{(6)} - (y')^3 = e^{2t} - \sin y; \qquad y(0) = y'(0) = y''(0) = y^{(3)}(0) = y^{(4)}(0) = y^{(5)}(0) = 0$$

***Solution***

$x_1 = y$

$x_2 = x_1' = y'$

$x_3 = x_2' = y''$

$x_4 = x_3' = y'''$

$x_5 = x_4' = y^{(4)}$

$x_6 = x_5' = y^{(5)}$

$$y^{(6)}-(y')^3=e^{2t}-\sin y$$

$$\boxed{x_6'=x_2^3-\sin x_1+e^{2t}}$$

$$x_1(0)=x_2(0)=x_3(0)=x_4(0)=x_5(0)=x_6(0)=0$$

## *Exercise*

Rewrite the given equation into a system in normal form with initial value.

$$\begin{cases}3x''=-5x+2y\\ 4y''=6x-2y\end{cases} \qquad \begin{cases}x(0)=-1, & x'(0)=0\\ y(0)=1, & y'(0)=2\end{cases}$$

***Solution***

$$x_1=x$$

$$x_2=x_1'=x'$$

$$x_3=y$$

$$x_4=x_3'=y'$$

$$\begin{cases}x''=-\frac{5}{3}x+\frac{3}{2}y\\ y''=\frac{3}{2}x-\frac{1}{2}y\end{cases}$$

$$\begin{cases}x_1'=x_2\\ x_2'=-\frac{5}{3}x_1+\frac{3}{2}x_3\\ x_3'=x_4\\ x_4'=\frac{3}{2}x_1-\frac{1}{2}x_3\end{cases} \qquad \begin{cases}x_1(0)=-1, & x_2(0)=0\\ x_3(0)=1, & x_4(0)=2\end{cases}$$

## *Exercise*

Rewrite the given equation into a system in normal form with initial value.

$$\begin{cases}x'''-y=t\\ 2x''+5y''-2y=1\end{cases} \qquad \begin{cases}x(0)=x'(0)=x''(0)=4\\ y(0)=y'(0)=1\end{cases}$$

***Solution***

$$\begin{cases}x'''=y+t\\ 5y''=-2x''+2y+1\end{cases}$$

$$x_1=x \qquad x_2=x_1'=x' \qquad x_3=x_2'=x''$$

$$x_4=y$$

$$x_5=x_4'=y'$$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 + t \\ x_4' = x_5 \\ x_5' = -\frac{2}{5}x_3 + \frac{2}{5}x_4 + \frac{1}{5} \end{cases} \qquad \begin{cases} x_1(0) = x_2(0) = x_3(0) = 4 \\ x_4(0) = x_5(0) = 1 \end{cases}$$

### *Exercise*

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $x'' + 3x' + 7x = t^2$

***Solution***

Let $x_1 = x, \quad x_2 = x' = x_1'$

Yield the system $\begin{cases} x_1' = x_2 \\ x_2' = -7x_1 - 3x_2 + t^2 \end{cases}$

### *Exercise*

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $x^{(4)} + 6x'' - 3x' + x = \cos 3t$

***Solution***

Let $x_1 = x, \quad x_2 = x' = x_1', \quad x_3 = x'' = x_2', \quad x_4 = x''' = x_3'$

Yield the system $\begin{cases} x_1' = x_2, \quad x_2' = x_3, \quad x_3' = x_4 \\ x_4' = -x_1 + 3x_2 - 6x_3 + \cos 3t \end{cases}$

### *Exercise*

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $t^2x'' + tx' + \left(t^2 - 1\right)x = 0$

***Solution***

Let $x_1 = x, \quad x_2 = x' = x_1'$

Yield the system $\begin{cases} x_1' = x_2 \\ t^2x_2' = \left(1 - t^2\right)x_1 - tx_2 \end{cases}$

## Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $\quad t^3x^{(3)}-2t^2x''+3tx'+5x=\ln t$

**Solution**

Let $\quad x_1=x,\quad x_2=x'=x_1',\quad x_3=x''=x_2'$

Yield the system $\begin{cases} x_1'=x_2, \quad x_2'=x_3, \quad x_3'=x_4 \\ t^3x_3'=-5x_1-3tx_2+2t^2x_3+\ln t \end{cases}$

## Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $\quad x''-5x+4y=0,\quad y''+4x-5y=0$

**Solution**

Let $\begin{matrix} x_1=x & x_2=x'=x_1' \\ y_1=y & y_2=y'=y_1' \end{matrix} \Rightarrow \quad \begin{cases} x_1'=x_2 \\ x_2'=5x_1-4y_1 \end{cases}$

$\begin{cases} y_1'=y_2 \\ y_2'=-4x_1+5y_1 \end{cases}$

## Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $\quad x''-3x'+4x-2y=0,\quad y''+2y'-3x+y=\cos t$

**Solution**

Let $\begin{matrix} x_1=x & x_2=x'=x_1' \\ y_1=y & y_2=y'=y_1' \end{matrix} \Rightarrow \begin{cases} x_1'=x_2 \\ x_2'=-4x_1+2y_1+3x_2 \end{cases} \quad \begin{cases} y_1'=y_2 \\ y_2'=3x_1-y_1-2y_2+\cos t \end{cases}$

## Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $\quad x''=3x-y+2z,\quad y''=x+y-4z,\quad z''=5x-y-z$

**Solution**

Let $\begin{matrix} x_1=x & x_2=x'=x_1' \\ y_1=y & y_2=y'=y_1' \\ z_1=z & z_2=z'=z_1' \end{matrix} \quad \Rightarrow \quad \begin{cases} x_1'=x_2, \quad y_1'=y_2, \quad z_1'=z_2 \\ x_2'=3x_1-y_1+2z_1 \\ y_2'=x_1+y_1-4z_1 \\ z_2'=5x_1-y_1-z_1 \end{cases}$

### *Exercise*

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $x'' = (1-y)x, \quad y'' = (1-x)y$

***Solution***

Let $\begin{array}{ll} x_1 = x & x_2 = x' = x_1' \\ y_1 = y & y_2 = y' = y_1' \end{array} \quad \Rightarrow \quad \begin{cases} x_1' = x_2, \quad y_1' = y_2 \\ x_2' = (1-y_1)x_1 \\ y_2' = (1-x_1)y_1 \end{cases}$

### *Exercise*

Prove that the general solution of

$$X' = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} X$$

On the interval $(-\infty, \infty)$ is

$$X = C_1 \begin{pmatrix} 6 \\ -1 \\ -5 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} e^{-2t} + C_3 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} e^{3t}$$

***Solution***

Let $A = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$$X_1 = \begin{pmatrix} 6 \\ -1 \\ -5 \end{pmatrix} e^{-t} \quad \rightarrow \quad X_1' = \begin{pmatrix} -6 \\ 1 \\ 5 \end{pmatrix} e^{-t}$$

$$AX_1 = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ -1 \\ -5 \end{pmatrix} e^{-t} = \begin{pmatrix} -6 \\ 1 \\ 5 \end{pmatrix} e^{-t} \quad \rightarrow \quad X_1' = AX_1$$

$$X_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} e^{-2t} \quad \rightarrow \quad X_2' = \begin{pmatrix} 6 \\ -2 \\ -2 \end{pmatrix} e^{-2t}$$

$$AX_2 = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} e^{-2t} = \begin{pmatrix} 6 \\ -2 \\ -2 \end{pmatrix} e^{-2t} \quad \rightarrow \quad X_2' = AX_2$$

$$X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} e^{3t} \quad \rightarrow \quad X_3' = \begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix} e^{3t}$$

$$AX_3 = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} e^{3t} = \begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix} e^{3t} \quad \rightarrow \quad X_3' = AX_3$$

$$W = \begin{vmatrix} 6 & -3 & 2 \\ -1 & 1 & 1 \\ -5 & 1 & 1 \end{vmatrix} = 20 \neq 0$$

$\therefore\ X_1,\ X_2,\ and\ X_3$ form a fundamental set for $X' = AX$ on $(-\infty,\ \infty)$

## Exercise

Prove that the general solution of

$$X' = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} X + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} 4 \\ -6 \end{pmatrix} t + \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

On the interval $(-\infty,\ \infty)$ is

$$X = C_1 \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix} e^{\sqrt{2}t} + C_2 \begin{pmatrix} 1 \\ -1+\sqrt{2} \end{pmatrix} e^{-\sqrt{2}t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} -2 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

***Solution***

Let $A = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$

$$X_1 = \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix} e^{\sqrt{2}t} \quad \rightarrow \quad X_1' = \begin{pmatrix} \sqrt{2} \\ -\sqrt{2}-2 \end{pmatrix} e^{\sqrt{2}t}$$

$$AX_1 = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix} e^{\sqrt{2}t} = \begin{pmatrix} \sqrt{2} \\ -\sqrt{2}-2 \end{pmatrix} e^{\sqrt{2}t} \quad \rightarrow \quad X_1' = AX_1$$

$$X_2 = \begin{pmatrix} 1 \\ -1+\sqrt{2} \end{pmatrix} e^{-\sqrt{2}t} \quad \rightarrow \quad X_2' = \begin{pmatrix} -\sqrt{2} \\ \sqrt{2}-2 \end{pmatrix} e^{-\sqrt{2}t}$$

$$AX_1 = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix} e^{\sqrt{2}t} = \begin{pmatrix} \sqrt{2} \\ -\sqrt{2}-2 \end{pmatrix} e^{\sqrt{2}t} \quad \rightarrow \quad X_1' = AX_1$$

$$X_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} -2 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \rightarrow \quad X_p' = \begin{pmatrix} 2 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$AX_p = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} -2 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} 4 \\ -6 \end{pmatrix} t + \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

$$=\begin{pmatrix}-1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}t^2-2t+1\\ 4t\end{pmatrix}+\begin{pmatrix}t^2+4t-1\\ t^2-6t+5\end{pmatrix}$$

$$=\begin{pmatrix}-t^2-2t-1\\ -t^2+6t-1\end{pmatrix}+\begin{pmatrix}t^2+4t-1\\ t^2-6t+5\end{pmatrix}$$

$$=\begin{pmatrix}2t-2\\ 4\end{pmatrix}$$

$$=\begin{pmatrix}2\\ 0\end{pmatrix}t+\begin{pmatrix}-2\\ 4\end{pmatrix}$$

$$=X'_p$$

$$W=\begin{vmatrix}1 & 1\\ -1-\sqrt{2} & -1+\sqrt{2}\end{vmatrix}=2\sqrt{2}\neq 0$$

$\therefore\ X_1$ *and* $X_2$ form a fundamental set for $X'=AX$ on $(-\infty,\ \infty)$

## *Exercise*

$$\mathbf{x}'=\begin{bmatrix}4 & 2\\ -3 & -1\end{bmatrix}\mathbf{x};\quad \vec{x}_1=\begin{bmatrix}2e^t\\ -3e^t\end{bmatrix},\quad \vec{x}_2=\begin{bmatrix}e^{2t}\\ -e^{2t}\end{bmatrix}$$

*a)* Verify that the given vectors are solutions of the given system.

*b)* Use the Wronskian to show that they are linearly independent.

*c)* Write the general solution of the system.

***Solution***

***a)*** $\vec{x}_1{}'=\begin{bmatrix}2e^t\\ -3e^t\end{bmatrix}'=\begin{bmatrix}2e^t\\ -3e^t\end{bmatrix}$ $\qquad \mathbf{x}'\,\vec{x}_1=\begin{bmatrix}4 & 2\\ -3 & -1\end{bmatrix}\begin{bmatrix}2e^t\\ -3e^t\end{bmatrix}=\begin{bmatrix}2e^t\\ -3e^t\end{bmatrix}=\vec{x}_1{}'$ $\checkmark$

$\vec{x}_2{}'=\begin{bmatrix}e^{2t}\\ -e^{2t}\end{bmatrix}'=\begin{bmatrix}2e^{2t}\\ -2e^{2t}\end{bmatrix}$ $\qquad \mathbf{x}'\,\vec{x}_2=\begin{bmatrix}4 & 2\\ -3 & -1\end{bmatrix}\begin{bmatrix}e^{2t}\\ -e^{2t}\end{bmatrix}=\begin{bmatrix}2e^{2t}\\ -2e^{2t}\end{bmatrix}=\vec{x}_2{}'$ $\checkmark$

***b)*** $W=\begin{vmatrix}2e^t & e^{2t}\\ -3e^t & -e^{2t}\end{vmatrix}=e^{3t}\neq 0$

The solutions $x_1$ *and* $x_2$ are linearly independent.

***c)*** $\mathbf{x}(t)=C_1\vec{x}_1+C_2\vec{x}_2=C_1\begin{pmatrix}2e^t\\ -3e^t\end{pmatrix}+C_2\begin{pmatrix}e^{2t}\\ -e^{2t}\end{pmatrix}=\begin{pmatrix}2C_1e^t+C_2e^{2t}\\ -3C_1e^t-C_2e^{2t}\end{pmatrix}$

## Exercise

$$\mathbf{x}' = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\mathbf{x}; \quad \vec{x}_1 = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 5 \end{cases}$$

*a)* Verify that the given vectors are solutions of the given system.
*b)* Use the Wronskian to show that they are linearly independent.
*c)* Write the general solution of the system.
*d)* Find the particular solution that satisfies the given initial conditions

**Solution**

**a)** $\vec{x}_1{}' = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix}' = \begin{bmatrix} 3e^{3t} \\ 9e^{3t} \end{bmatrix}$ $\qquad \mathbf{x}' \cdot \vec{x}_1 = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ 9e^{3t} \end{bmatrix} = \vec{x}_1{}' \ \checkmark$

$\vec{x}_2{}' = \begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix}' = \begin{bmatrix} -4e^{-2t} \\ -2e^{-2t} \end{bmatrix}$ $\qquad \mathbf{x}' \cdot \vec{x}_2 = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} -4e^{2t} \\ -2e^{2t} \end{bmatrix} = \vec{x}_2{}' \ \checkmark$

**b)** $W = \begin{vmatrix} e^{3t} & 2e^{-2t} \\ 3e^{3t} & e^{-2t} \end{vmatrix} = -5e^{t} \neq 0$

The solutions $x_1$ *and* $x_2$ are linearly independent.

**c)** $\mathbf{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 = C_1\begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} + C_2\begin{pmatrix} 2e^{-2t} \\ e^{-2t} \end{pmatrix} = \begin{pmatrix} C_1e^{3t} + 2C_2e^{-2t} \\ 3C_1e^{3t} + C_2e^{-2t} \end{pmatrix}$

**d)** $x_1 = C_1e^{3t} + 2C_2e^{-2t} \qquad x_2 = 3C_1e^{3t} + C_2e^{-2t}$

$x_1(0) = C_1 + 2C_2 = 0 \qquad x_2(0) = 3C_1 + C_2 = 5$

$\Rightarrow \quad C_1 = 2 \quad C_2 = -1$

$$\begin{cases} x_1 = 2e^{3t} - 2e^{-2t} \\ x_2 = 6e^{3t} - e^{-2t} \end{cases}$$

## Exercise

$$\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\mathbf{x}; \quad \vec{x}_1 = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}, \quad \begin{cases} x_1(0) = 5 \\ x_2(0) = -3 \end{cases}$$

*a)* Verify that the given vectors are solutions of the given system.
*b)* Use the Wronskian to show that they are linearly independent.
*c)* Write the general solution of the system.
*d)* Find the particular solution that satisfies the given initial conditions

***Solution***

$$\textbf{a)}\quad \vec{x}_1{}' = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}' = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} \end{bmatrix} \qquad \mathbf{x}' \cdot \vec{x}_1 = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} \end{bmatrix} = \vec{x}_1{}' \quad \checkmark$$

$$\vec{x}_2{}' = \begin{bmatrix} e^{-2t} \\ 5e^{-2t} \end{bmatrix}' = \begin{bmatrix} -2e^{-2t} \\ -10e^{-2t} \end{bmatrix} \qquad \mathbf{x}' \cdot \vec{x}_2 = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} e^{-2t} \\ 5e^{-2t} \end{bmatrix} = \begin{bmatrix} -2e^{2t} \\ -10e^{2t} \end{bmatrix} = \vec{x}_2{}' \quad \checkmark$$

$$\textbf{b)}\quad W = \begin{vmatrix} e^{2t} & e^{-2t} \\ e^{2t} & 5e^{-2t} \end{vmatrix} = 4 \neq 0$$

The solutions $x_1$ *and* $x_2$ are linearly independent.

$$\textbf{c)}\quad \mathbf{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 = C_1\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} + C_2\begin{pmatrix} e^{-2t} \\ 5e^{-2t} \end{pmatrix} = \begin{pmatrix} C_1e^{2t} + C_2e^{-2t} \\ C_1e^{2t} + 5C_2e^{-2t} \end{pmatrix}$$

$$\textbf{d)}\quad x_1 = C_1e^{2t} + C_2e^{-2t} \qquad x_2 = C_1e^{2t} + 5C_2e^{-2t}$$

$$x_1(0) = C_1 + C_2 = 5 \qquad x_2(0) = C_1 + 5C_2 = -3$$

$$\Rightarrow \quad C_1 = 7 \quad C_2 = -2$$

$$\begin{cases} x_1 = 7e^{2t} - 2e^{-2t} \\ x_2 = 7e^{2t} - 10e^{-2t} \end{cases}$$

## *Exercise*

$$\mathbf{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\mathbf{x}; \quad \vec{x}_1 = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 8 \\ x_2(0) = 0 \end{cases}$$

*a)* Verify that the given vectors are solutions of the given system.
*b)* Use the Wronskian to show that they are linearly independent.
*c)* Write the general solution of the system.
*d)* Find the particular solution that satisfies the given initial conditions

***Solution***

$$\textbf{a)}\quad \vec{x}_1{}' = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}' = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} \qquad \mathbf{x}' \cdot \vec{x}_1 = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} = \vec{x}_1{}' \quad \checkmark$$

$$\vec{x}_2{}' = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}' = \begin{bmatrix} -5e^{-2t} \\ -15e^{-2t} \end{bmatrix} \qquad \mathbf{x}' \cdot \vec{x}_2 = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} = \begin{bmatrix} -5e^{2t} \\ -15e^{2t} \end{bmatrix} = \vec{x}_2{}' \quad \checkmark$$

***b)*** $W = \begin{vmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{vmatrix} = 7e^{-3t} \neq 0$

The solutions $x_1$ *and* $x_2$ are linearly independent.

***c)*** $\boldsymbol{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 = C_1\begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix} + C_2\begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix} = \begin{pmatrix} 3C_1e^{2t} + C_2e^{-5t} \\ 2C_1e^{2t} + 3C_2e^{-5t} \end{pmatrix}$

***d)*** $x_1 = 3C_1e^{2t} + C_2e^{-5t} \qquad x_2 = 2C_1e^{2t} + 3C_2e^{-5t}$

$x_1(0) = 3C_1 + C_2 = 8 \qquad x_2(0) = 2C_1 + 3C_2 = 0 \Rightarrow \boxed{C_1 = \frac{24}{7} \quad C_2 = -\frac{16}{7}}$

$$\begin{cases} x_1 = \frac{72}{7}e^{2t} - \frac{16}{7}e^{-2t} \\ x_2 = \frac{48}{7}e^{2t} - \frac{48}{7}e^{-2t} \end{cases}$$

## *Exercise*

$$\boldsymbol{x}' = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 0 \\ x_3(0) = 4 \end{cases}$$

*a)* Verify that the given vectors are solutions of the given system.
*b)* Use the Wronskian to show that they are linearly independent.
*c)* Write the general solution of the system.
*d)* Find the particular solution that satisfies the given initial conditions

***Solution***

***a)*** $\vec{x}_1{}' = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}' = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix} \qquad \boldsymbol{x}' \cdot \vec{x}_1 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix} = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix} = \vec{x}_1{}' \quad \checkmark$

$\vec{x}_2{}' = \begin{bmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}' = \begin{bmatrix} -6e^{3t} \\ 0 \\ 3e^{3t} \end{bmatrix} \qquad \boldsymbol{x}' \cdot \vec{x}_2 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix} = \begin{bmatrix} -6e^{3t} \\ 0 \\ 3e^{3t} \end{bmatrix} = \vec{x}_2{}' \quad \checkmark$

$\vec{x}_3{}' = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}' = \begin{bmatrix} 10e^{5t} \\ -10e^{5t} \\ 5e^{5t} \end{bmatrix} \qquad \boldsymbol{x}' \cdot \vec{x}_3 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix} = \begin{bmatrix} 10e^{5t} \\ -10e^{5t} \\ 5e^{5t} \end{bmatrix} = \vec{x}_3{}' \quad \checkmark$

***b)*** $W = \begin{vmatrix} 2e^{t} & -2e^{3t} & 2e^{5t} \\ 2e^{t} & 0 & -2e^{5t} \\ e^{t} & e^{3t} & e^{5t} \end{vmatrix} = 4e^{9t} + 4e^{9t} + 4e^{9t} + 4e^{9t} = 16e^{9t} \neq 0$

The solutions $x_1,\ x_2\ and\ x_3$ are linearly independent.

***c)*** $\mathbf{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 + C_3\vec{x}_3 = C_1\begin{pmatrix} 2e^{t} \\ 2e^{t} \\ e^{t} \end{pmatrix} + C_2\begin{pmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{pmatrix} + C_3\begin{pmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{pmatrix}$

$$= \begin{pmatrix} 2C_1e^{t} - 2C_2e^{3t} + 2C_3e^{5t} \\ 2C_1e^{t} \qquad\qquad - 2C_3e^{5t} \\ C_1e^{t} + C_2e^{3t} + C_3e^{5t} \end{pmatrix}$$

***d)*** $x_1 = 2C_1e^{t} - 2C_2e^{3t} + 2C_3e^{5t} \qquad x_2 = 2C_1e^{t} - 2C_3e^{5t} \qquad x_3 = C_1e^{t} + C_2e^{3t} + C_3e^{5t}$

$$\begin{cases} x_1(0) = 2C_1 - 2C_2 + 2C_3 = 0 \\ x_2(0) = 2C_1 - 2C_3 = 0 \\ x_3(0) = C_1 + C_2 + C_3 = 4 \end{cases} \quad \rightarrow \quad \left[\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & 1 & 1 & 4 \end{array}\right]$$

$$\Rightarrow \quad \underline{C_1 = 1 \quad C_2 = 2 \quad C_3 = 1|}$$

$$\begin{cases} x_1(t) = 2e^{t} - 4e^{3t} + 2e^{5t} \\ x_2(t) = 2e^{t} - 2e^{5t} \\ x_3(t) = e^{t} + 2e^{3t} + e^{5t} \end{cases}$$

## ***Exercise***

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\mathbf{x}; \quad \vec{x}_1 = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 10 \\ x_2(0) = 12 \\ x_3(0) = -1 \end{cases}$$

*a)* Verify that the given vectors are solutions of the given system.

*b)* Use the Wronskian to show that they are linearly independent.

*c)* Write the general solution of the system.

*d)* Find the particular solution that satisfies the given initial conditions

***Solution***

**a)** $\vec{x}_1{}' = \begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix}' = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{pmatrix}$ $\quad$ $\mathbf{x}' \cdot \vec{x}_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{pmatrix} = \vec{x}_1{}' \; \checkmark$

$\vec{x}_2{}' = \begin{pmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix}' = \begin{pmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{pmatrix}$ $\quad$ $\mathbf{x}' \cdot \vec{x}_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{pmatrix} = \vec{x}_2{}' \; \checkmark$

$\vec{x}_3{}' = \begin{pmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{pmatrix}' = \begin{pmatrix} 0 \\ -e^{-t} \\ e^{-t} \end{pmatrix}$ $\quad$ $\mathbf{x}' \cdot \vec{x}_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} 0 \\ -e^{-t} \\ e^{-t} \end{pmatrix} = \vec{x}_3{}' \; \checkmark$

**b)** $W = \begin{vmatrix} e^{2t} & e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & -e^{-t} & -e^{-t} \end{vmatrix} = 3 \neq 0$ $\quad$ The solutions $x_1$, $x_2$ *and* $x_3$ are linearly independent.

**c)** $\mathbf{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 + C_3\vec{x}_3 = C_1\begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix} + C_2\begin{pmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix} + C_3\begin{pmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{pmatrix}$

$$= \begin{pmatrix} C_1e^{2t} + C_2e^{-t} \\ C_1e^{2t} \quad + C_3e^{-t} \\ C_1e^{2t} - C_2e^{-t} - C_3e^{-t} \end{pmatrix}$$

**d)** $x_1 = C_1e^{2t} + C_2e^{-t} \qquad x_2 = C_1e^{2t} + C_3e^{-t} \qquad x_3 = C_1e^{2t} - C_2e^{-t} - C_3e^{-t}$

$$\begin{cases} x_1(0) = C_1 + C_2 = 10 \\ x_2(0) = C_1 + C_3 = 12 \\ x_3(0) = C_1 - C_2 - C_3 = -1 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 1 & 0 & 1 & 12 \\ 1 & -1 & -1 & -1 \end{array}\right]$$

$$\Rightarrow \quad C_1 = 7 \quad C_2 = 3 \quad C_3 = 5$$

$$\begin{cases} x_1(t) = 7e^{2t} + 3e^{-t} \\ x_2(t) = 7e^{2t} + 5e^{-t} \\ x_3(t) = 7e^{2t} - 8e^{-t} \end{cases}$$

### *Exercise*

$$\mathbf{x}' = \begin{bmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{bmatrix}\mathbf{x}; \quad \vec{x}_1 = \begin{bmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{bmatrix}, \quad \vec{x}_4 = \begin{bmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{bmatrix}, \quad \begin{cases} x_1(0) = 1 \\ x_2(0) = 3 \\ x_3(0) = 4 \\ x_4(0) = 7 \end{cases}$$

*a)* Verify that the given vectors are solutions of the given system.
*b)* Use the Wronskian to show that they are linearly independent.
*c)* Write the general solution of the system.
*d)* Find the particular solution that satisfies the given initial conditions

***Solution***

***a)*** $$\vec{x}_1{}' = \begin{pmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{pmatrix}' = \begin{pmatrix} -e^{-t} \\ 0 \\ 0 \\ -e^{-t} \end{pmatrix} \qquad \mathbf{x}' \cdot \vec{x}_1 = \begin{pmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ 0 \\ 0 \\ -e^{-t} \end{pmatrix} = \vec{x}_1{}' \ \checkmark$$

$$\vec{x}_2{}' = \begin{pmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{pmatrix}' = \begin{pmatrix} 0 \\ 0 \\ -e^{-t} \\ 0 \end{pmatrix} \qquad \mathbf{x}' \cdot \vec{x}_2 = \begin{pmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -e^{-t} \\ 0 \end{pmatrix} = \vec{x}_2{}' \ \checkmark$$

$$\vec{x}_3{}' = \begin{pmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{pmatrix}' = \begin{pmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{pmatrix} \qquad \mathbf{x}' \cdot \vec{x}_3 = \begin{pmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{pmatrix} = \begin{pmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{pmatrix} = \vec{x}_3{}' \ \checkmark$$

$$\vec{x}_4{}' = \begin{pmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{pmatrix}' = \begin{pmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{pmatrix} \qquad \mathbf{x}' \cdot \vec{x}_3 = \begin{pmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{pmatrix} = \vec{x}_3{}' \ \checkmark$$

***b)*** $$W = \begin{vmatrix} e^{-t} & 0 & 0 & e^{t} \\ 0 & 0 & e^{t} & 0 \\ 0 & e^{-t} & 0 & 3e^{t} \\ e^{-t} & 0 & -2e^{t} & 0 \end{vmatrix} = e^{-t}\begin{vmatrix} 0 & e^{t} & 0 \\ e^{-t} & 0 & 3e^{t} \\ 0 & -2e^{t} & 0 \end{vmatrix} - e^{t}\begin{vmatrix} 0 & 0 & e^{t} \\ 0 & e^{-t} & 0 \\ e^{-t} & 0 & -2e^{t} \end{vmatrix} = 0-(-1) = 1 \neq 0$$

The solutions $x_1,\ x_2\ and\ x_3$ are linearly independent.

**c)** $\mathbf{x}(t)=C_1\vec{x}_1+C_2\vec{x}_2+C_3\vec{x}_3+C_4\vec{x}_4=C_1\begin{pmatrix} e^{-t}\\ 0\\ 0\\ e^{-t}\end{pmatrix}+C_2\begin{pmatrix} 0\\ 0\\ e^{-t}\\ 0\end{pmatrix}+C_3\begin{pmatrix} 0\\ e^{t}\\ 0\\ -2e^{t}\end{pmatrix}+C_4\begin{pmatrix} e^{t}\\ 0\\ 3e^{t}\\ 0\end{pmatrix}$

$$=\begin{pmatrix} C_1e^{-t}+C_4e^{t}\\ C_3e^{t}\\ C_2e^{-t}+3C_4e^{t}\\ C_1e^{-t}-2C_3e^{t}\end{pmatrix}$$

**d)** $x_1(t)=C_1e^{-t}+C_4e^{t},\quad x_2(t)=C_3e^{t},\quad x_3(t)=C_2e^{-t}+3C_4e^{t},\quad x_4(t)=C_1e^{-t}-2C_3e^{t}$

$$\begin{cases} x_1(0)=C_1+C_4=1\\ x_2(0)=C_3=3\\ x_3(0)=C_2+3C_4=4\\ x_4(0)=C_1-2C_3=7\end{cases}\qquad \Rightarrow\ \ \underline{C_1=13\quad C_2=40\quad C_3=3\quad C_4=-12|}$$

$$\begin{cases} x_1(t)=13e^{-t}-12e^{t}\\ x_2(t)=3e^{t}\\ x_3(t)=40e^{-t}-36e^{t}\\ x_3(t)=13e^{-t}-6e^{t}\end{cases}$$

## Exercise

Consider the *RLC* parallel circuit below. Let *V* represent the voltage drop across the capacitor and I represent the current across the inductor.

Show that: $\quad V'=-\dfrac{V}{RC}-\dfrac{1}{C}\qquad\qquad I'=\dfrac{V}{L}$

***Solution***

Using Kirchhoff's current law: $I_1+I_2+I_3=0$

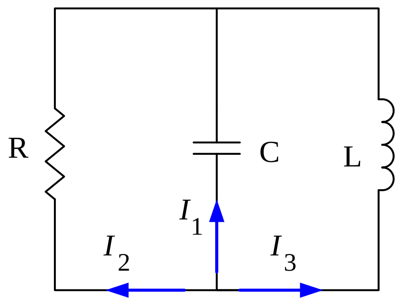
In the RC loop: $\quad V_1-V_2=0$

In the LC loop: $\quad V_2-V_3=0$

$V_2=RI_2,\quad CV_1'=I_1,\quad LI_3'=V_3$

Since the circuit elements are in parallel, therefore $V_1=V_2=V_3=V$

$$LI_3' = V_1 \Rightarrow \boxed{I_3' = \frac{V_1}{L}}$$




$$\begin{aligned} CV_1' &= I_1 \\ &= -I_2 - I_3 && V_2 = RI_2 \\ &= -\frac{V_2}{R} - I_3 && V_2 = V_1 \\ &= -\frac{V_1}{R} - I_3 \end{aligned}$$

$$V_1' = -\frac{V_1}{CR} - \frac{I_3}{C}$$

Since $V_1 = V \quad and \quad I_3 = I$

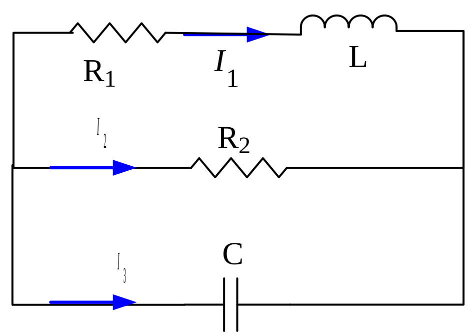
$$\Rightarrow \begin{cases} I' = \dfrac{V}{L} \\ V' = -\dfrac{V}{CR} - \dfrac{I}{C} \end{cases}$$

## *Exercise*

Consider the *RLC* parallel circuit below. Let *V* represent the voltage drop across the capacitor and I represent the current across the inductor.

Show that: $CV' = -I - \dfrac{V}{R_2} \qquad LI' = -R_1 I + V$

***Solution***




Using Kirchhoff's current law: $I + I_2 + I_3 = 0$ **(1)**

In the $R_1LR_2$ loop: $R_1 I + LI' - R_2 I_2 = 0$ **(2)**

In the $R_2C$ loop: $R_2 I_2 - V = 0$ **(3)**

From (3): $V = R_2 I_2 \Rightarrow I_2 = \dfrac{V}{R_2}$

From (2): $LI' = -R_1 I + R_2 I_2 \qquad V = R_2 I_2$

$$\boxed{= -R_1 I + V}$$

From (1): $I_2 = -I - I_3$

$$\frac{V}{R_2} = -I - I_3$$

However, the voltage drop across the capacitor is: $V = \dfrac{q}{C}$

$$\Rightarrow CV = q$$

$$CV' = q' \qquad I_3 = q'$$

$$CV' = I_3$$

$$\frac{V}{R_2} = -I - CV'$$

$$\boxed{CV' = -I - \frac{V}{R_2}}$$

### *Exercise*

Let $I_1$ and $I_2$ represent the current flow across the indicators $L_1$ and $L_2$ respectively. Show that the circuit is modeled by the system

$$\begin{cases} L_1 I_1' = -R_1 I_1 - R_1 I_2 + E \\ L_2 I_2' = -R_1 I_1 - \left(R_1 + R_2\right) I_2 + E \end{cases}$$

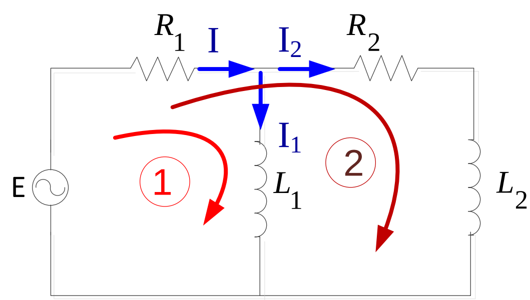
***Solution***

By Kirchhoff's second law:

$$I = I_1 + I_2$$

From loop **1**:

$$-E + R_1 I + L_1 I_1' = 0$$




$$L_1 I_1' = E - R_1 I$$

$$= E - R_1\left(I_1 + I_2\right)$$

$$\boxed{= -R_1 I_1 - R_1 I_2 + E}$$

From loop **2**:

$$-E + R_1 I + R_2 I_2 + L_2 I_2' = 0$$

$$L_2 I_2' = -R_1 I - R_2 I_2 + E$$

$$= -R_1\left(I_1 + I_2\right) - R_2 I_2 + E$$

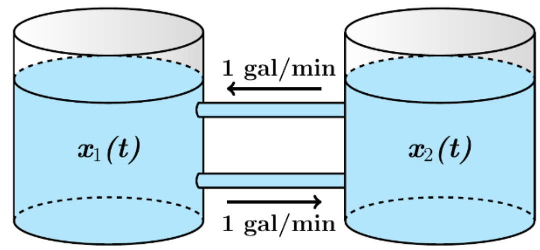
$$\boxed{= -R_1 I_1 - R_1 I_2 - R_2 I_2 + E}$$

## *Exercise*

Two tanks are connected by two pipes. Each tank contains 500 *gallons* of a salt solution. Through on pipe solution is pumped from the first tank to the second at 1 *gal/min*. Through the other pipe, solution is pumped at the same rate from the second to the first tank. Show the salt content in each tank varies with time.

***Solution***

$x_1(t)$ and $x_2(t)$ represent the salt content.




$$\text{Rate out} = 1\ gal\ /\ min \times \frac{x_1}{500}\ lb\ /\ gal = \frac{x_1}{500}\ lb\ /\ min$$

$$\text{Rate in} = 1\ gal\ /\ min \times \frac{x_2}{500}\ lb\ /\ gal = \frac{x_2}{500}\ lb\ /\ min$$

$$\frac{dx_1}{dt} = Rate\ out - Rate\ in = \frac{x_2}{500} - \frac{x_1}{500}$$

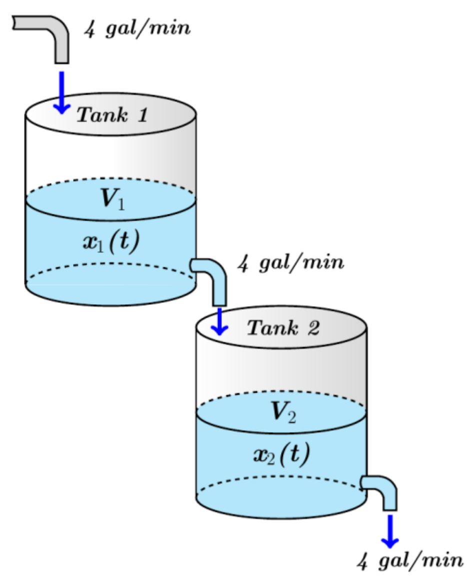
$$\text{And}\quad \frac{dx_2}{dt} = \frac{x_1}{500} - \frac{x_2}{500}$$

$$x' = Ax \quad \rightarrow \quad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -\frac{1}{500} & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

## *Exercise*

Each tank contains 100 *gallons* of a salt solution. Pure water flows into the upper tank at a rate of 4 *gal/min*. Salt solution drains from the upper tank into the lower tank at a rate of 4 *gal/min*. Finally, salt solution drains from the lower tank at a rate of 4 *gal/min*, effectively keeping the volume of solution in each tank at a constant 100 *gal*. If the initial salt content of the upper and lower tanks is 10 and 20 *pounds*, respectively. Set up an initial value problem that models the amount of salt in each tank over time (do not solve).Write the model in matrix-vector form. Is the system homogeneous or inhomogeneous?

***Solution***




$$\text{For the first tank: Rate out } = 4\frac{gal}{min} \times \frac{x_1}{100}\frac{lb}{gal} = \frac{x_1}{25}\ lb\ /\ min$$

$$= \frac{x_1}{25}\ lb\ /\ min$$

$$\frac{dx_1}{dt} = Rate\ out\ - \ Rate\ in$$

$$= -\frac{x_1}{25}$$

$$\text{For the second tank: Rate out } = 4\ \frac{gal}{min} \times \frac{x_2}{100}\frac{lb}{gal}$$

$$= \frac{x_2}{25}\ \frac{lb}{min}$$

$$\frac{dx_2}{dt} = Rate\ out\ -\ Rate\ in$$

$$= \frac{x_1}{25} - \frac{x_2}{25}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -\frac{1}{25} & 0 \\ \frac{1}{25} & -\frac{1}{25} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

## *Exercise*

Two masses on a frictionless tabletop are connected with a spring having spring constant $k_2$. The first mass is connected to a vertical support with a spring having spring constant $k_1$. The second mass is shaken harmonically via a force equaling $F = A\cos\omega t$. Let $x(t)$ and $y(t)$ measure the displacements of the masses $m_1$ and $m_2$, respectively, from their equilibrium positions as a function of time. If both masses start from rest at their equilibrium positions at time $t = 0$.

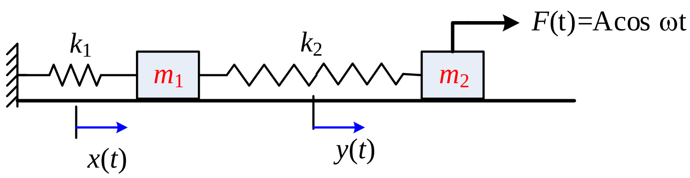
Set up an initial value problem that models the position of the masses over time (do not solve).Write the model in matrix-vector form. Is the system homogeneous or inhomogeneous?

***Solution***

By Newton's Law; the first mass:

$$m_1 x'' = -k_1 x + k_2 (y - x)$$

$$x'' = -\frac{k_1}{m_1} x + \frac{k_2}{m_1}(y - x)$$




The second mass:

$$m_2 y'' = -k_2 (y - x) + A\cos\omega t$$

$$y'' = -\frac{k_2}{m_2}(y - x) + \frac{A}{m_2}\cos\omega t$$

Let assume: $x_1 = x, \quad x_2 = x', \quad x_3 = y, \quad x_4 = y'$

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{k_1}{m_1} x_1 + \frac{k_2}{m_1}(x_3 - x_1) \\ x_3' = x_4 \\ x_4' = -\frac{k_2}{m_2}(x_3 - x_1) + \frac{A}{m_2}\cos\omega t \end{cases} \Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = -\left(\frac{k_1}{m_1} + \frac{k_2}{m_1}\right) x_1 + \frac{k_2}{m_1} x_3 \\ x_3' = x_4 \\ x_4' = \frac{k_2}{m_2} x_1 - \frac{k_2}{m_2} x_3 + \frac{A}{m_2}\cos\omega t \end{cases}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{A}{m_2}\cos \omega t \end{pmatrix}$$

## *Exercise*

Derive the equations $\begin{cases} m_1 x_1'' = -\left(k_1 + k_2\right) x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - \left(k_2 + k_3\right) x_2 \end{cases}$

For the displacements (from equilibrium) of the 2 masses.

***Solution***

First spring is stretched by $x_1$

Second spring is stretched by $x_2 - x_1$

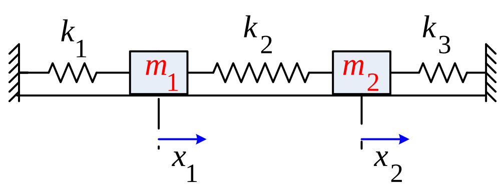
Third spring is stretched by $x_2$

Newton's second law gives:

For $m_1$ $\Rightarrow$ $m_1 x_1'' = -k_1 x_1 + k_2 \left(x_2 - x_1\right)$

For $m_2$ $\Rightarrow$ $m_2 x_2'' = -k_2 \left(x_2 - x_1\right) - k_3 x_2$

That implies to: $\begin{cases} m_1 x_1'' = -\left(k_1 + k_2\right) x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - \left(k_2 + k_3\right) x_2 \end{cases}$




## *Exercise*

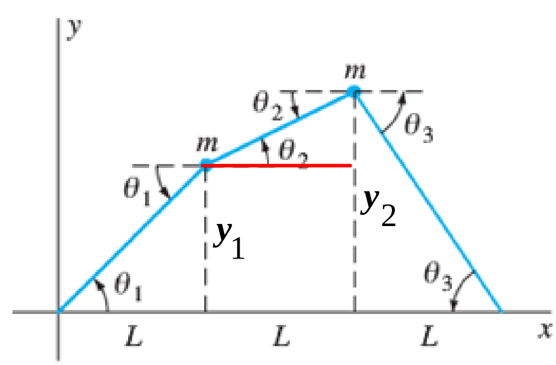
Two particles each of mass *m* are attached to a string under (constant) tension *T*. Assume that the particles oscillate vertically (that is, parallel to the *y*-axis) with amplitudes so small that the sines of the angles shown are accurately approximated by their tangents. Show that the displacement $y_1$ and $y_2$ satisfy the equations

$$\begin{cases} k y_1'' = -2y_1 + y_2 \\ k y_2'' = y_1 - 2y_2 \end{cases} \qquad \textit{where } k = \frac{mL}{T}$$

***Solution***

For the first mass:

$$m y_1'' = -T \sin\theta_1 + T \sin\theta_2$$



$$\approx -T\tan\theta_1 + T\tan\theta_2$$

$$my_1'' = -T\frac{y_1}{L} + T\frac{y_2 - y_1}{L}$$

$$\frac{L}{T}my_1'' = -\frac{L}{T}T\frac{y_1}{L} + \frac{L}{T}T\frac{y_2 - y_1}{L} \qquad \textit{where } k = \frac{mL}{T}$$

$$ky_1'' = -y_1 + y_2 - y_1$$

$$= -2y_1 + y_2$$

For the second mass:

$$my_2'' = -T\sin\theta_2 + T\sin\theta_3$$

$$\approx -T\tan\theta_2 + T\tan\theta_3$$

$$my_2'' = -T\frac{y_2 - y_1}{L} + T\frac{y_2}{L}$$

$$\frac{L}{T}my_2'' = -\frac{L}{T}T\frac{y_2 - y_1}{L} + \frac{L}{T}T\frac{y_2}{L} \qquad \textit{where } k = \frac{mL}{T}$$

$$ky_2'' = -y_2 + y_1 - y_2 = y_1 - 2y_2$$

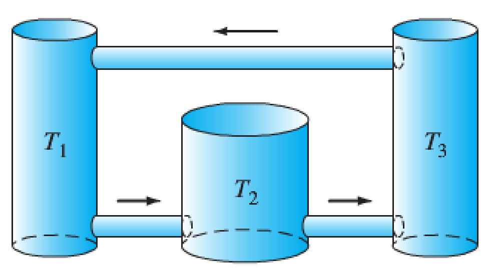
$$\Rightarrow \begin{cases} ky_1'' = -2y_1 + y_2 \\ ky_2'' = y_1 - 2y_2 \end{cases} \qquad \textit{where } k = \frac{mL}{T}$$

### *Exercise*

There 100-gal fermentation vats are connected, and the mixtures in each tank are kept uniform by stirring. Denote by $x_i(t)$ the amount (in pounds) of alcohol in tank $T_i$ at time *t* (*i* = 1, 2, 3). Suppose that the mixture circulates between the tanks at the rate of 10 *gal/min*. Derive the equations

$$\begin{cases} 10x_1' = -x_1 \qquad + x_3 \\ 10x_2' = x_1 - x_2 \\ 10x_3' = \qquad x_2 - x_3 \end{cases}$$

***Solution***




Concentration of salt in each tank is: $c_i = \frac{X}{V} = \frac{x_i}{100}$

Rate in/out = Volume Rate x Concentration

***Rate of change = Rate in – rate out***

For $T_1$: $\quad x_1' = r\frac{x_3}{100} - r\frac{x_1}{100} = \frac{1}{10}\left(x_3 - x_1\right)$

For $T_2$ : $\quad x_2' = r\dfrac{x_1}{100} - r\dfrac{x_2}{100} = \dfrac{1}{10}\left(x_1 - x_2\right)$

For $T_3$ : $\quad x_3' = r\dfrac{x_2}{100} - r\dfrac{x_3}{100} = \dfrac{1}{10}\left(x_2 - x_3\right)$

That implies:

$$\begin{cases} 10x_1' = -x_1 \qquad + x_3 \\ 10x_2' = x_1 - x_2 \\ 10x_3' = \qquad x_2 - x_3 \end{cases}$$

## *Exercise*

Suppose that a particle with mass *m* and electrical charge *q* moves in the *xy*-plane under the influence of the magnetic field $\vec{B} = B\hat{k}$ (thus a uniform field parallel to the *z*-axis), so the force on the particle is $\vec{F} = q\vec{v} \times \vec{B}$ if its velocity is $\vec{v}$. Show that the equations of motion of the particle are

$$mx'' = +qBy', \quad my'' = -qBx'$$

***Solution***

Let $\vec{r} = (x,\ y,\ z)$ be the position vector, then Newton's law

$$\vec{F} = mx''$$

$$\begin{aligned} \vec{F} = mx'' &= q\left(\vec{v} \times \vec{B}\right) \\ &= q\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x' & y' & z' \\ 0 & 0 & B \end{vmatrix} \\ &= qBy'\hat{i} - qBx'\hat{j} \end{aligned}$$

$\Rightarrow mx'' = +qBy', \quad my'' = -qBx'$

### Exercise

Find the eigenvalues and the eigenvectors for each of the matrices. $A=\begin{pmatrix}12 & 14\\ -7 & -9\end{pmatrix}$

**Solution**

$$\det(A-\lambda I)=\begin{vmatrix}12-\lambda & 14\\ -7 & -9-\lambda\end{vmatrix}$$

$$=(12-\lambda)(-9-\lambda)-(14)(-7)$$

$$=-108-12\lambda+9\lambda+\lambda^2+98$$

$$=\lambda^2-3\lambda-10=0$$

Thus, the eigenvalues are: $\lambda_1=-2$ *and* $\lambda_2=5$

For $\lambda_1=-2$, we have: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}12+2 & 14\\ -7 & -9+2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}14 & 14\\ -7 & -7\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \qquad \to \begin{cases}14x+14y=0\\ -7x-7y=0\end{cases} \Rightarrow x=-y$$

$$\Rightarrow V_1=\begin{pmatrix}-1\\ 1\end{pmatrix}$$

For $\lambda_2=5$, we have $(A-\lambda_2 I)V_2=0$

$$\begin{pmatrix}7 & 14\\ -7 & -14\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \qquad \to \begin{cases}7x+14y=0\\ -7x-14y=0\end{cases} \Rightarrow x=-2y$$

$$\Rightarrow V_2=\begin{pmatrix}-2\\ 1\end{pmatrix}$$

### Exercise

Find the eigenvalues and the eigenvectors for each of the matrices. $A=\begin{pmatrix}-4 & 1\\ -2 & 1\end{pmatrix}$

**Solution**

$$\det(A-\lambda I)=\begin{vmatrix}-4-\lambda & 1\\ -2 & 1-\lambda\end{vmatrix}$$

$$=(-4-\lambda)(1-\lambda)+2$$

$$=\lambda^2+3\lambda-2=0$$

Thus, the eigenvalues are: $\lambda_1 = \frac{-3-\sqrt{17}}{2}$ *and* $\lambda_2 = \frac{-3+\sqrt{17}}{2}$

For $\lambda_1 = \frac{-3-\sqrt{17}}{2}$, we have: $(A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -4-\frac{-3-\sqrt{17}}{2} & 1 \\ -2 & 1-\frac{-3-\sqrt{17}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-5+\sqrt{17}}{2} & 1 \\ -2 & \frac{5+\sqrt{17}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} \frac{-5+\sqrt{17}}{2}x + y = 0 \\ -2x + \frac{5+\sqrt{17}}{2}y = 0 \end{cases} \Rightarrow x = \left(\frac{5+\sqrt{17}}{4}\right)y$$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} \frac{5+\sqrt{17}}{4} \\ 1 \end{pmatrix}}$$

For $\lambda_2 = \frac{-3+\sqrt{17}}{2}$, we have: $(A-\lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -4-\frac{-3+\sqrt{17}}{2} & 1 \\ -2 & 1-\frac{-3+\sqrt{17}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-5-\sqrt{17}}{2} & 1 \\ -2 & \frac{5-\sqrt{17}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} \frac{-5-\sqrt{17}}{2}x + y = 0 \\ -2x + \frac{5-\sqrt{17}}{2}y = 0 \end{cases} \Rightarrow x = \left(\frac{5-\sqrt{17}}{4}\right)y$$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} \frac{5-\sqrt{17}}{4} \\ 1 \end{pmatrix}}$$

## *Exercise*

Find the eigenvalues and the eigenvectors for each of the matrices. $A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$

***Solution***

$$\begin{aligned} \det(A-\lambda I) &= \begin{vmatrix} 5-\lambda & 3 \\ -6 & -4-\lambda \end{vmatrix} \\ &= (-4-\lambda)(5-\lambda)+18 \\ &= \lambda^2 - \lambda - 2 = 0 \end{aligned}$$

Thus, the eigenvalues are: $\lambda_1 = -1$ *and* $\lambda_2 = 2$

For $\lambda_1 = -1$, we have: $(A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 6 & 3 \\ -6 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 6x+3y=0 \\ -6x-3y=0 \end{cases} \Rightarrow y=-2x$$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A-\lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 3 & 3 \\ -6 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 3x+3y=0 \\ -6x-6y=0 \end{cases} \Rightarrow y=-x$$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

## *Exercise*

Find the eigenvalues and the eigenvectors for each of the matrices. $A = \begin{pmatrix} -2 & 3 \\ 0 & -5 \end{pmatrix}$

***Solution***

$$\det\left(A-\lambda I\right) = \begin{vmatrix} -2-\lambda & 3 \\ 0 & -5-\lambda \end{vmatrix}$$

$$= \left(-2-\lambda\right)\left(-5-\lambda\right) - 0$$

$$= \left(2+\lambda\right)\left(5+\lambda\right) = 0$$

Thus, the eigenvalues are: $\lambda_1 = -5$ *and* $\lambda_2 = -2$

For $\lambda_1 = -5$, we have: $\left(A-\lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow 3x+3y=0 \Rightarrow y=-x$$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

For $\lambda_2 = -2 \quad \Rightarrow \left(A-\lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 0 & 3 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 3y=0 \\ -3y=0 \end{cases} \Rightarrow y=0$$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

## *Exercise*

Find the eigenvalues and the eigenvectors for each of the matrices. $A=\begin{pmatrix} 6 & 10 \\ -5 & -9 \end{pmatrix}$

***Solution***

$$\det(A-\lambda I)=\begin{vmatrix} 6-\lambda & 10 \\ -5 & -9-\lambda \end{vmatrix}$$
$$=(6-\lambda)(-9-\lambda)+50$$
$$=-54+3\lambda+\lambda^2+50$$
$$=\lambda^2+3\lambda-4=0$$

Thus, the eigenvalues are: $\lambda_1=-4$ *and* $\lambda_2=1$

For $\lambda_1=-4$, we have: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 10 & 10 \\ -5 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 10x+10y=0 \\ -5x-5y=0 \end{cases} \Rightarrow y=-x$$

$$\Rightarrow \boxed{V_1=\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

For $\lambda_2=1 \Rightarrow (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix} 5 & 10 \\ -5 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 5x+10y=0 \\ -5x-10y=0 \end{cases} \Rightarrow x=-2y$$

$$\Rightarrow \boxed{V_2=\begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$

## *Exercise*

Find the eigenvalues and the eigenvectors for each of the matrices. $A=\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

***Solution***

$$\det(A-\lambda I)=\begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix}$$
$$=(3-\lambda)(-1-\lambda)-0$$
$$=\lambda^2-2\lambda-3$$

The characteristic equation: $\lambda^2-2\lambda-3$

$\lambda^2-2\lambda-3=0$ The eigenvalues are $\lambda_1=-1$ *and* $\lambda_2=3$

$\lambda_1=-1 \rightarrow (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 4 & 0 \\ 8 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 4x = 0 \\ 8x = 0 \end{cases} \Rightarrow x = 0$$

Therefore, the eigenvector $V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\lambda_2 = 3 \quad \rightarrow \quad \left(A - \lambda_2 I\right)V_2 = 0$$

$$\begin{pmatrix} 0 & 0 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 0 = 0 \\ 8x - 4y = 0 \end{cases} \Rightarrow 8x = 4y \rightarrow \boxed{2x = y}$$

Therefore, the eigenvector $V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

## *Exercise*

Find the eigenvalues and the eigenvectors for each of the matrices. $A = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

***Solution***

$$\det\left(A - \lambda I\right) = \begin{vmatrix} 10-\lambda & -9 \\ 4 & -2-\lambda \end{vmatrix}$$

$$= (10-\lambda)(-2-\lambda) + 36$$

$$= \lambda^2 - 8\lambda + 16$$

$\Rightarrow$ The characteristic equation: $\lambda^2 - 8\lambda + 16$

$\lambda^2 - 8\lambda + 16 = 0 \quad \Rightarrow$ The eigenvalues are $\lambda_{1,2} = 4$

$$\lambda_1 = 4 \quad \rightarrow \quad \left(A - \lambda_1 I\right)V_1 = 0$$

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 6x - 9y = 0 \\ 4x - 6y = 0 \end{cases} \Rightarrow \boxed{2x = 3y}$$

Therefore the eigenvector $V_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$

## *Exercise*

Find the eigenvalues and the eigenvectors for each of the matrices. $A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

***Solution***

$$\det(A-\lambda I)=\begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix}$$

$$=(4-\lambda)(1-\lambda)(1-\lambda)+2(1-\lambda)$$

$$=(1-\lambda)\left[(4-\lambda)(1-\lambda)+2\right]$$

$$=(1-\lambda)\left(\lambda^2-5\lambda+6\right)$$

$\Rightarrow$ The characteristic equation: $-\lambda^3+6\lambda^2-11\lambda+6$

$-\lambda^3+6\lambda^2-11\lambda+6=0$ The eigenvalues are $\boxed{\lambda=1,\ 2,\ 3}$

$$\lambda_1=1 \quad\rightarrow\quad \begin{pmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad\rightarrow\begin{cases} 3x_1+x_3=0 \\ x_1=0 \end{cases}\Rightarrow x_1=x_3=0$$

Therefore; the eigenvector $V_1=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\lambda_2=2 \quad\rightarrow\quad \begin{pmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad\rightarrow\begin{cases} 2x_1+x_3=0 \\ -2x_1-x_2=0 \\ -2x_1-x_3=0 \end{cases}\Rightarrow\begin{cases} x_3=-2x_1 \\ x_2=-2x_1 \end{cases}$$

Therefore; the eigenvector $V_2=\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$

$$\lambda_3=3 \quad\rightarrow\quad \begin{pmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow\begin{cases} x_1+x_3=0 \\ -2x_1-2x_2=0 \\ -2x_1-2x_3=0 \end{cases}\Rightarrow\begin{cases} x_3=-x_1 \\ x_2=-x_1 \end{cases}$$

Therefore; the eigenvector $V_3=\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

### *Exercise*

Find the eigenvalues and the eigenvectors for each of the matrices. $A=\begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{pmatrix}$

***Solution***

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 4 & 3-\lambda & 2 \\ -8 & -4 & -3-\lambda \end{vmatrix}$$

$$= (1-\lambda)(3-\lambda)(-3-\lambda)+8(1-\lambda)$$

$$= -9+9\lambda+\lambda^2-\lambda^3+8-8\lambda$$

$$= -\lambda^3+\lambda^2+\lambda-1 = 0$$

Thus, the eigenvalues are: $\lambda_1=-1$ *and* $\lambda_{2,3}=1$

For $\lambda_1=1 \quad \Rightarrow (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 4 & 2 & 2 \\ -8 & -4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 0=0 \\ 4x+2y+2z=0 \\ -8x-4y-4z=0 \end{cases} \Rightarrow \begin{cases} \\ 2x=-y-z \\ 2x=-y-z \end{cases} \to \begin{matrix} \text{If } \boxed{x=0} \\ \boxed{y=-z} \end{matrix}$$

$$\Rightarrow V_1=\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda_{2,3}=-1 \quad \Rightarrow (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix} 2 & 0 & 0 \\ 4 & 4 & 2 \\ -8 & -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\to \begin{cases} 2x=0 \\ 4x+4y+2z=0 \\ -8x-4y-2z=0 \end{cases} \Rightarrow \begin{cases} \boxed{x=0} \\ 4y=-2z \to \boxed{z=-2y} \\ 4y=-2z \end{cases}$$

$$\Rightarrow V_2=\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

### *Exercise*

Find the eigenvalues and the eigenvectors for each of the matrices $A=\begin{pmatrix} -1 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & -12 & 2 \end{pmatrix}$

***Solution***

$$\left|A-\lambda I\right|=\begin{vmatrix} -1-\lambda & -4 & -2 \\ 0 & 1-\lambda & 1 \\ -6 & -12 & 2-\lambda \end{vmatrix}$$

$$=(-1-\lambda)(1-\lambda)(2-\lambda)+24-12(1-\lambda)+12(-1-\lambda)$$

$$=-\lambda^3+2\lambda^2+\lambda-2+24-12+12\lambda-12-12\lambda$$

$$=-\lambda^3+2\lambda^2+\lambda-2=0$$

Thus, the eigenvalues are: $\lambda_1=-1 \quad \lambda_2=1 \quad and \quad \lambda_3=2$

For $\lambda_1=-1 \quad \Rightarrow \left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix} 0 & -4 & -2 \\ 0 & 2 & 1 \\ -6 & -12 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -4y-2z=0 \\ 2y+z=0 \\ -6x-12y+3z=0 \end{cases} \Rightarrow \begin{cases} -4y=2z \\ 2y=-z \\ -6x=12y-3z \end{cases} \rightarrow \begin{array}{l} \boxed{y=-\frac{1}{2}z} \\ x=\frac{-9z}{-6}=\frac{3}{2}z \end{array}$$

$$\Rightarrow V_1=\begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

For $\lambda_2=1 \quad \Rightarrow \left(A-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix} -2 & -4 & -2 \\ 0 & 0 & 1 \\ -6 & -12 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} -2x-4y-2z=0 \\ z=0 \\ -6x-12y+2z=0 \end{cases} \Rightarrow \begin{cases} -2x-4y=0 \\ \\ -6x-12y=0 \end{cases} \rightarrow x=-2y$$

$$\Rightarrow V_2=\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_3=2 \quad \Rightarrow \left(A-\lambda_3 I\right)V_3=0$

$$\begin{pmatrix} -3 & -4 & -2 \\ 0 & -1 & 1 \\ -6 & -12 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} -3x - 4y - 2z = 0 \\ -y + z = 0 \\ -6x - 12y = 0 \end{cases} \Rightarrow \underline{y = z} \rightarrow \begin{cases} -3x = 6z \\ \\ -6x = 12z \end{cases} \rightarrow \begin{cases} \\ \\ \underline{x = -2z} \end{cases}$$

$$\Rightarrow \underline{V_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}$$

## *Exercise*

Find the eigenvalues and the eigenvectors for each of the matrices $A = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(4-\lambda)(-1-\lambda) - 4 - 8 + 4(4-\lambda) + 4(3-\lambda) + 2\lambda + 2$$

$$= -\lambda^3 + 6\lambda^2 - 5\lambda - 12 - 12 + 16 - 4\lambda + 12 - 4\lambda + 2\lambda + 2$$

$$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6 \ \underline{= 0}$$

Thus, the eigenvalues are: $\lambda_1 = 1 \quad \lambda_2 = 2 \quad and \quad \lambda_3 = 3$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x + 2y + 2z = 0 \\ x + 3y + z = 0 \\ -2x - 4y - 2z = 0 \end{cases} \Rightarrow (1)\ \&\ (3) \rightarrow \underline{y = 0}$$

$$\rightarrow \begin{cases} 2x + 2z = 0 \\ x + z = 0 \end{cases} \Rightarrow \underline{x = -z}$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x + 2y + 2z = 0 \\ x + 2y + z = 0 \\ -2x - 4y - 3z = 0 \end{cases} \Rightarrow \begin{cases} 2x + 4y + 2z = 0 \\ -2x - 4y - 3z = 0 \end{cases} \to \boxed{z = 0} \to \boxed{x = -2y}$$

$$\Rightarrow V_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_3 = 3 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ -2 & -4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\to \begin{cases} 2y + 2z = 0 \\ x + y + z = 0 \\ -2x - 4y - 4z = 0 \end{cases} \Rightarrow \boxed{y = -z} \to \boxed{x = 0}$$

$$\Rightarrow V_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

## *Exercise*

Find the eigenvalues and the eigenvectors for each of the matrices $A = \begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}$

***Solution***

$$\left|A - \lambda I\right| = \begin{vmatrix} -6-\lambda & 4 & 4 \\ -4 & 2-\lambda & 4 \\ -10 & 8 & 4-\lambda \end{vmatrix}$$

$$= (-6-\lambda)(2-\lambda)(4-\lambda) - 160 - 128 + 40(2-\lambda) + 32(6+\lambda) + 16(4-\lambda)$$

$$= -\lambda^3 + 4\lambda = 0$$

Thus, the eigenvalues are: $\lambda_1 = 0 \quad \lambda_2 = -2 \quad$ *and* $\quad \lambda_3 = 2$

For $\lambda_1 = 0 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -6x+4y+4z=0 \\ -4x+2y+4z=0 \\ -10x+8y+4z=0 \end{cases} \Rightarrow \quad 2x-2y=0 \rightarrow \boxed{x=y}$$

$$\rightarrow \begin{cases} -2x+4z=0 \\ -2x+4z=0 \end{cases} \Rightarrow \boxed{x=2z=y}$$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = -2 \quad \Rightarrow (A-\lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -4 & 4 & 4 \\ -4 & 4 & 4 \\ -10 & 8 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -4x+4y+4z=0 \\ -4x+4y+4z=0 \\ -10x+8y+6z=0 \end{cases} \Rightarrow \quad \begin{matrix} -x+y+z=0 \\ -5x+4y+3z=0 \end{matrix}$$

$$\Rightarrow \begin{cases} 4x-4y=4z \\ -5x+4y=-3z \end{cases} \rightarrow \begin{matrix} \boxed{x=-z} \\ \boxed{y=-2z} \end{matrix}$$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}}$$

For $\lambda_3 = 2 \quad \Rightarrow (A-\lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -8 & 4 & 4 \\ -4 & 0 & 4 \\ -10 & 8 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} -8x+4y+4z=0 \\ -4x+4z=0 \\ -10x+8y+2z=0 \end{cases} \Rightarrow \boxed{x=z} \Rightarrow \begin{matrix} 4y-4z=0 \\ \\ 8y-8z=0 \end{matrix} \Rightarrow \boxed{y=z}$$

$$\Rightarrow \boxed{V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

***Exercise***

Find the eigenvalues and the eigenvectors for each of the matrices. $A=\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

***Solution***

$$\det(A-\lambda I)=\begin{pmatrix} -\lambda & 0 & 2 & 0 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -2-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix}=(1-\lambda)\begin{vmatrix} -\lambda & 0 & 2 \\ 1 & -\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix}$$

$$=(1-\lambda)\left(\lambda^2(-2-\lambda)+2+\lambda\right)$$

$$=(1-\lambda)\left(-\lambda^3-2\lambda^2+\lambda+2\right)$$

$$=\lambda^4+\lambda^3-3\lambda^2-\lambda+2$$

$\Rightarrow$ The characteristic equation: $\lambda^4+\lambda^3-3\lambda^2-\lambda+2$

$\lambda^4+\lambda^3-3\lambda^2-\lambda+2=0 \quad \Rightarrow \quad$ The eigenvalues are $\boxed{\lambda=-2,-1,1,1}$

For $\lambda_1=-2 \quad \Rightarrow \left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix} 2 & 0 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x_1+2x_3=0 \\ x_1+2x_2+x_3=0 \\ x_2=0 \\ x_4=0 \end{cases}$$

$$\rightarrow \begin{cases} x_1=-x_3 \\ x_1=-x_3 \\ x_2=0 \\ x_4=0 \end{cases}$$

Therefore; the eigenvector $V_1=\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_2=-1 \quad \Rightarrow \left(A-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} x_1 + 2x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \\ x_2 - x_3 = 0 \\ x_4 = 0 \end{cases}$$

$$\to \begin{cases} x_1 = -2x_3 \\ x_1 = -x_2 - x_3 \\ x_2 = x_3 \end{cases}$$

Therefore; the eigenvector $V_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda_3 = 1 \quad \Rightarrow (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} -x_1 + 2x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_2 - 3x_3 = 0 \end{cases}$$

$$\to \begin{cases} x_1 = 2x_3 \\ x_1 = x_2 - x_3 \\ x_2 = 3x_3 \end{cases}$$

Therefore; the eigenvector $V_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_4 = 1 \;\to$ Therefore; the eigenvector $V_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

## *Exercise*

Find a fundamental set of solutions for the system $x' = Ax$ , where $A$ is the given matrices.

$$A = \begin{pmatrix} 2 & 0 \\ -4 & -2 \end{pmatrix}$$

***Solution***

$$\left|A - \lambda I\right| = \begin{vmatrix} 2-\lambda & 0 \\ -4 & -2-\lambda \end{vmatrix}$$

$$= (2-\lambda)(-2-\lambda) - 0$$

$$= \lambda^2 - 4 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -2$ *and* $\lambda_2 = 2$

For $\lambda_1 = -2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 0 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \rightarrow \begin{cases} 4x = 0 \\ -4x = 0 \end{cases} \Rightarrow \boxed{x = 0} \quad \boxed{y = 1}$$

The eigenvector is: $V_1 = (0,\ 1)^T$

The solution is: $x_1(t) = e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 \\ -4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \rightarrow \begin{cases} 0 = 0 \\ -4x - 4y = 0 \end{cases} \Rightarrow \boxed{x = -y}$$

The eigenvector is: $V_2 = (-1,\ 1)^T$

The solution is: $x_2(t) = e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Since the vectors $V_1$ and $V_2$ are independent, the solutions $x_1(t)$ and $x_2(t)$ are independent for all $t$ and for a fundamental set of solutions.

## *Exercise*

Find a fundamental set of solutions for the system $x' = Ax$ , where $A$ is the given matrices.

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -6 \\ -2 & 3 & 4 \end{pmatrix}$$

***Solution***

$$|A-\lambda I| = \begin{vmatrix} -1-\lambda & 0 & 0 \\ 2 & -5-\lambda & -6 \\ -2 & 3 & 4-\lambda \end{vmatrix}$$

$$= \lambda^3 + 2\lambda^2 - \lambda - 2 = 0$$

Thus, the eigenvalues are: $\underline{\lambda_1 = -2 \quad \lambda_2 = -1 \quad and \quad \lambda_3 = 1}$

For $\lambda_1 = -2 \quad \Rightarrow (A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & -6 \\ -2 & 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \boxed{x=0} \\ 2x-3y-6z=0 \Rightarrow 3y=-6z \rightarrow \boxed{y=-2z} \\ -2x+3y+6z=0 \end{cases}$$

$$\Rightarrow V_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -1 \quad \Rightarrow (A-\lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & -4 & -6 \\ -2 & 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 0=0 \\ 2x-4y-6z=0 \Rightarrow -4y=6z \rightarrow \boxed{y=-z} \\ -2x+3y+5z=0 \qquad 3y=-5z \end{cases}$$

$$\rightarrow 2x = 4y+6z = 2z \Rightarrow \boxed{x=z}$$

$$\Rightarrow V_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 1 \quad \Rightarrow (A-\lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -2 & 0 & 0 \\ 2 & -6 & -6 \\ -2 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -2x=0 \qquad \boxed{x=0} \\ 2x-6y-6z=0 \Rightarrow -6y=6z \rightarrow \boxed{y=-z} \\ -2x+3y+3z=0 \qquad 3y=-3z \end{cases}$$

$$\Rightarrow V_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

The vectors are given by: $V = \begin{pmatrix} 0 & 1 & 0 \\ -2 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

$$\det(V)=\begin{vmatrix}0&1&0\\-2&-1&-1\\1&1&1\end{vmatrix}=1\ \underline{\neq 0}$$

The solutions are independent for all *t* and form a fundamental set of solutions.

### *Exercise*

Find the general solution of the system $\begin{cases}x_1'(t)=x_1+2x_2\\x_2'(t)=4x_1+3x_2\end{cases}$

***Solution***

$$A=\begin{pmatrix}1&2\\4&3\end{pmatrix}$$

$$|A-\lambda I|=\begin{vmatrix}1-\lambda&2\\4&3-\lambda\end{vmatrix}$$

$$=\lambda^2-4\lambda-5=0$$

Thus, the eigenvalues are: $\underline{\lambda_1=-1\ \ and\ \ \lambda_2=5}$

For $\lambda_1=-1\quad\Rightarrow(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}2&2\\4&4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\rightarrow\ \underline{x=-y}\quad\Rightarrow\ V_1=\begin{pmatrix}1\\-1\end{pmatrix}$$

The solution is: $x_1(t)=\begin{pmatrix}1\\-1\end{pmatrix}e^{-t}$

For $\lambda_2=5\quad\Rightarrow(A-\lambda_2 I)V_2=0$

$$\begin{pmatrix}-4&2\\4&-2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\rightarrow\ \underline{2x=y}\quad\Rightarrow\ V_2=\begin{pmatrix}1\\2\end{pmatrix}$$

The solution is: $x_2(t)=\begin{pmatrix}1\\2\end{pmatrix}e^{5t}$

$$\therefore\ \underline{x(t)=C_1\begin{pmatrix}1\\-1\end{pmatrix}e^{-t}+C_2\begin{pmatrix}1\\2\end{pmatrix}e^{5t}}$$

### *Exercise*

Find the general solution of the system $\begin{cases}x_1'(t)=2x_1+2x_2\\x_2'(t)=x_1+3x_2\end{cases}$

***Solution***

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

$$\left|A - \lambda I\right| = \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= \lambda^2 - 5\lambda + 4 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 1$ *and* $\lambda_2 = 4$

For $\lambda_1 = 1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = -2y} \quad \Rightarrow V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^t$

For $\lambda_2 = 4 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = y} \quad \Rightarrow V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The solution is: $x_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

$$\therefore \ \underline{x(t) = C_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = -4x_1 + 2x_2 \\ x_2'(t) = -\frac{5}{2}x_1 + 2x_2 \end{cases}$

***Solution***

$$A = \begin{pmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{pmatrix}$$

$$\left|A - \lambda I\right| = \begin{vmatrix} -4-\lambda & 2 \\ -\frac{5}{2} & 2-\lambda \end{vmatrix}$$

$$= \lambda^2 + 2\lambda - 3 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 1$ *and* $\lambda_2 = -3$

For $\lambda_1 = 1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -5 & 2 \\ -\frac{5}{2} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{5x = 2y} \quad \Rightarrow \ V_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^t$

For $\lambda_2 = -3 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -1 & 2 \\ -\frac{5}{2} & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = 2y} \quad \Rightarrow \ V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The solution is: $x_2(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-3t}$

$$\therefore \ \underline{x(t) = C_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^t + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-3t}}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = -\frac{5}{2}x_1 + 2x_2 \\ x_2'(t) = \frac{3}{4}x_1 - 2x_2 \end{cases}$

***Solution***

$$A = \begin{pmatrix} -\frac{5}{2} & 2 \\ \frac{3}{4} & -2 \end{pmatrix}$$

$$\left|A - \lambda I\right| = \begin{vmatrix} -\frac{5}{2} - \lambda & 2 \\ \frac{3}{4} & -2 - \lambda \end{vmatrix}$$

$$= \lambda^2 + \frac{9}{2}\lambda + \frac{7}{2} = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1 \ \ and \ \ \lambda_2 = -\frac{7}{2}$

For $\lambda_1 = -1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -\frac{3}{2} & 2 \\ \frac{3}{4} & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{\frac{3}{2}x = 2y} \quad \Rightarrow \ V_1 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

The solution is: $\underline{x_1(t) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{-t}}$

For $\lambda_2 = -\frac{7}{2} \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 1 & 2 \\ \frac{3}{4} & \frac{3}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = -2y} \quad \Rightarrow \quad V_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

The solution is: $x_2(t) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-7t/2}$

$$\therefore \; x(t) = C_1 \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-7t/2}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = 3x_1 - x_2 \\ x_2'(t) = 9x_1 - 3x_2 \end{cases}$

***Solution***

$$A = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{vmatrix}$$

$$= \lambda^2 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = 0$

For $\lambda_1 = 0 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{3x = y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

For the second eigenvector $V_2 \quad \Rightarrow \quad AV_2 = V_1$

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \rightarrow \quad 3x - y = 1$$

$$\rightarrow if \;\; x = 1 \;\; \Rightarrow \;\; y = 2 \qquad V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The solution is: $x_2(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} t$ $\qquad x_2(t) = e^{\lambda t}\left(V_2 + tV_1\right)$

$$\therefore \; x(t) = C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} t\right)$$

### Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = -6x_1 + 5x_2 \\ x_2'(t) = -5x_1 + 4x_2 \end{cases}$

**Solution**

$$A = \begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -6-\lambda & 5 \\ -5 & 4-\lambda \end{vmatrix}$$

$$= \lambda^2 + 2\lambda + 1 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = -1$

For $\lambda_1 = -1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The solution is: $\underline{x_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

For the second eigenvector $V_2 \quad \Rightarrow \quad AV_2 = V_1$

$$\begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \rightarrow \quad -6x + 5y = 1$$

$$\rightarrow if \;\; x = 0 \quad \rightarrow \quad y = \frac{1}{5} \qquad \Rightarrow V_2 = \begin{pmatrix} 0 \\ \frac{1}{5} \end{pmatrix}$$

The solution is: $\underline{x_2(t) = \begin{pmatrix} 0 \\ \frac{1}{5} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t}$ $\qquad x_2(t) = e^{\lambda t}\left(V_2 + tV_1\right)$

$$\therefore \;\; \underline{x(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \left( \begin{pmatrix} 0 \\ \frac{1}{5} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t \right)}$$

### Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 6x_1 - x_2 \\ x_2'(t) = 5x_1 + 2x_2 \end{cases}$

**Solution**

$$A = \begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix}$$

$$\left|A-\lambda I\right| = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix}$$

$$= \lambda^2 - 8\lambda + 17 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = 4 \pm i$

For $\lambda_1 = 4+i \quad \Rightarrow \left(A-\lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad (2-i)x = y \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$

$$z(t) = \begin{pmatrix} 1 \\ 2-i \end{pmatrix} e^{(4+i)t}$$

$$= \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right)(\cos t + i\sin t)e^{4t}$$

$$= \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin t + i\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\sin t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos t\right)\right)e^{4t}$$

$$= \left(\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + i\begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}\right)e^{4t}$$

$$\therefore \quad x(t) = C_1\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix}e^{4t} + C_2\begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}e^{4t}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = x_1 + x_2 \\ x_2'(t) = -2x_1 - x_2 \end{cases}$

***Solution***

$$A = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$$

$$\left|A-\lambda I\right| = \begin{vmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix}$$

$$= \lambda^2 + 1 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = \pm i$

For $\lambda_1 = i \quad \Rightarrow \left(A-\lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{(-1+i)x = y|} \quad \Rightarrow V_1 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}$

$$z(t) = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{it}$$

$$= \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)(\cos t + i\sin t)$$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t + i\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t\right)$$

$$= \begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + i\begin{pmatrix} \sin t \\ -\sin t + \cos t \end{pmatrix}$$

$$\therefore \underline{x(t) = C_1\begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + C_2\begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix}|}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = 5x_1 + x_2 \\ x_2'(t) = -2x_1 + 3x_2 \end{cases}$

***Solution***

$$A = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix}$$

$$= \lambda^2 - 8\lambda + 17 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = 4 \pm i$

For $\lambda_1 = 4 + i \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{(-1+i)x = y|}$$

$$\Rightarrow V_1 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}$$

The solution is: $\underline{x_1(t) = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}|}$

$$z(t)=\begin{pmatrix}1\\-1+i\end{pmatrix}e^{(4+i)t}$$

$$=\left(\begin{pmatrix}1\\-1\end{pmatrix}+i\begin{pmatrix}0\\1\end{pmatrix}\right)(\cos t+i\sin t)e^{4t}$$

$$=\left(\begin{pmatrix}1\\-1\end{pmatrix}\cos t-\begin{pmatrix}0\\1\end{pmatrix}\sin t+i\left(\begin{pmatrix}1\\-1\end{pmatrix}\sin t+\begin{pmatrix}0\\1\end{pmatrix}\cos t\right)\right)e^{4t}$$

$$=\left(\begin{pmatrix}\cos t\\-\cos t-\sin t\end{pmatrix}+i\begin{pmatrix}\sin t\\2\sin t+\cos t\end{pmatrix}\right)e^{4t}$$

$$\therefore\ x(t)=C_1\begin{pmatrix}\cos t\\-\cos t-\sin t\end{pmatrix}e^{4t}+C_2\begin{pmatrix}\sin t\\2\sin t+\cos t\end{pmatrix}e^{4t}$$

## ***Exercise***

Find the general solution of the system $\begin{cases}x_1'(t)=4x_1+5x_2\\x_2'(t)=-2x_1+6x_2\end{cases}$

***Solution***

$$A=\begin{pmatrix}4&5\\-2&6\end{pmatrix}$$

$$|A-\lambda I|=\begin{vmatrix}4-\lambda&5\\-2&6-\lambda\end{vmatrix}$$

$$=\lambda^2-10\lambda+34=0$$

Thus, the eigenvalues are: $\lambda_{1,2}=5\pm 3i$

For $\lambda_1=5+3i \quad\Rightarrow (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}-1-3i&5\\-2&1-3i\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\rightarrow (1+3i)x=5y$$

$$\Rightarrow\quad V_1=\begin{pmatrix}5\\1+3i\end{pmatrix}$$

The solution is: $x_1(t)=\begin{pmatrix}5\\1+3i\end{pmatrix}$

$$z(t)=\begin{pmatrix}5\\1+3i\end{pmatrix}e^{(5+3i)t}$$

$$=\left(\begin{pmatrix}5\\1\end{pmatrix}+i\begin{pmatrix}0\\3\end{pmatrix}\right)(\cos 3t+i\sin 3t)e^{5t}$$

$$= \left( \begin{pmatrix} 5 \\ 1 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \sin 3t + i \left( \begin{pmatrix} 5 \\ 1 \end{pmatrix} \sin 3t + \begin{pmatrix} 0 \\ 3 \end{pmatrix} \cos 3t \right) \right) e^{5t}$$

$$= \left( \begin{pmatrix} 5\cos 3t \\ \cos 3t - 3\sin 3t \end{pmatrix} + i \begin{pmatrix} 5\sin 3t \\ \sin 3t + 3\cos 3t \end{pmatrix} \right) e^{5t}$$

$$\therefore \ x(t) = C_1 \begin{pmatrix} 5\cos 3t \\ \cos 3t - 3\sin 3t \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} 5\sin 3t \\ \sin 3t + 3\cos 3t \end{pmatrix} e^{5t}$$

## Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 5x_1 - 4x_2 \\ x_2'(t) = 2x_1 - x_2 \end{cases}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -4 \\ 2 & -1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$$

$$= \lambda^2 - 4\lambda + 3 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 1$ *and* $\lambda_2 = 3$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = y \quad \Rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 3 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = 2y \quad \Rightarrow V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\therefore \ x(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t}$$

## Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 6x_1 - 6x_2 \\ x_2'(t) = 4x_1 - 4x_2 \end{cases}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 6-\lambda & -6 \\ 4 & -4-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 6 & -6 \\ 4 & -4 \end{pmatrix}$$

$$= \lambda^2 - 2\lambda = 0$$

Thus, the eigenvalues are: $\lambda_1 = 0$ & $\lambda_2 = 2$

For $\lambda_1 = 0 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 6 & -6 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 4 & -6 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{2x = 3y} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\therefore \ \underline{x(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{3t}}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = 5x_1 - 3x_2 \\ x_2'(t) = 2x_1 \end{cases}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -3 \\ 2 & -\lambda \end{vmatrix} \qquad\qquad A = \begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix}$$

$$= \lambda^2 - 5\lambda + 6 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 2$ & $\lambda_2 = 3$

For $\lambda_1 = 2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 3 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{2x = 3y} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\therefore \ \underline{x(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{3t}}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = 5x_1 - 4x_2 \\ x_2'(t) = 3x_1 - 2x_2 \end{cases}$

***Solution***

$$|A-\lambda I| = \begin{vmatrix} 5-\lambda & -4 \\ 3 & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix}$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 1$ & $\lambda_2 = 2$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & -4 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = y|} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 3 & -4 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{3x = 4y|} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\therefore \underline{x(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{2t} \Big|}$$

## ***Exercise***

Find the general solution of the system $\begin{cases} x_1'(t) = 9x_1 - 8x_2 \\ x_2'(t) = 6x_1 - 5x_2 \end{cases}$

***Solution***

$$|A-\lambda I| = \begin{vmatrix} 9-\lambda & -8 \\ 6 & -5-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 9 & -8 \\ 6 & -5 \end{pmatrix}$$

$$= \lambda^2 - 4\lambda + 3 = 0$$

Thus, the eigenvalues are: $\underline{\lambda_1 = 1 \text{ \& } \lambda_2 = 3|}$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 8 & -8 \\ 6 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = y|} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 3 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 6 & -8 \\ 6 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{3x = 4y|} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\therefore \underline{x(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{3t} \Big|}$$

## Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 10x_1 - 6x_2 \\ x_2'(t) = 12x_1 - 7x_2 \end{cases}$

**Solution**

$$|A - \lambda I| = \begin{vmatrix} 10-\lambda & -6 \\ 12 & -7-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 10 & -6 \\ 12 & -7 \end{pmatrix}$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 1$ & $\lambda_2 = 2$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 9 & -6 \\ 12 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \ 3x = 2y \quad \Rightarrow \ V_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 8 & -6 \\ 12 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \ 4x = 3y \quad \Rightarrow \ V_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\therefore \ x(t) = C_1\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{t} + C_2\begin{pmatrix} 3 \\ 4 \end{pmatrix}e^{2t}$$

## Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 6x_1 - 10x_2 \\ x_2'(t) = 2x_1 - 3x_2 \end{cases}$

**Solution**

$$|A - \lambda I| = \begin{vmatrix} 6-\lambda & -10 \\ 2 & -3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 6 & -10 \\ 2 & -3 \end{pmatrix}$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 1$ & $\lambda_2 = 2$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 5 & -10 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \ x = 2y \quad \Rightarrow \ V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 4 & -10 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \ 2x = 5y \quad \Rightarrow \ V_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\therefore\ \underline{x(t) = C_1\begin{pmatrix}2\\1\end{pmatrix}e^{t} + C_2\begin{pmatrix}5\\2\end{pmatrix}e^{2t}}$$

## Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 11x_1 - 15x_2 \\ x_2'(t) = 6x_1 - 8x_2 \end{cases}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 11-\lambda & -15 \\ 6 & -8-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 11 & -15 \\ 6 & -8 \end{pmatrix}$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

Thus, the eigenvalues are: $\underline{\lambda_1 = 1 \ \ \&\ \ \lambda_2 = 2}$

For $\lambda_1 = 1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 10 & -15 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{2x = 3y} \quad \Rightarrow \ V_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 9 & -15 \\ 6 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{3x = 5y} \quad \Rightarrow \ V_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$\therefore\ \underline{x(t) = C_1\begin{pmatrix}3\\2\end{pmatrix}e^{t} + C_2\begin{pmatrix}5\\3\end{pmatrix}e^{2t}}$$

## Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 3x_1 + x_2 \\ x_2'(t) = x_1 + 3x_2 \end{cases}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$= \lambda^2 - 6\lambda + 8 = 0$$

Thus, the eigenvalues are: $\underline{\lambda_1 = 2 \ \ \&\ \ \lambda_2 = 4}$

For $\lambda_1 = 2 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = -y} \quad \Rightarrow \ V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 4 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = y} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \ \underline{x(t) = C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{2t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t}}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = 4x_1 + 2x_2 \\ x_2'(t) = 2x_1 + 4x_2 \end{cases}$

***Solution***

$$\left|A - \lambda I\right| = \begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

$$= \lambda^2 - 8\lambda + 12 = 0$$

Thus, the eigenvalues are: $\underline{\lambda_1 = 2 \ \ \& \ \ \lambda_2 = 6}$

For $\lambda_1 = 2 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = -y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 6 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = y} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \ \underline{x(t) = C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{2t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{6t}}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = 9x_1 + 2x_2 \\ x_2'(t) = 2x_1 + 6x_2 \end{cases}$

***Solution***

$$\left|A - \lambda I\right| = \begin{vmatrix} 9-\lambda & 2 \\ 2 & 6-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 9 & 2 \\ 2 & 6 \end{pmatrix}$$

$$= \lambda^2 - 15\lambda + 50 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 5 \ \ \& \ \ \lambda_2 = 10$

For $\lambda_1 = 5 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{2x = y|} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 10 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = 2y|} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\therefore \ \underline{x(t) = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{10t}}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = 13x_1 + 4x_2 \\ x_2'(t) = 4x_1 + 7x_2 \end{cases}$

***Solution***

$$\left|A - \lambda I\right| = \begin{vmatrix} 13-\lambda & 4 \\ 4 & 7-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 13 & 4 \\ 4 & 7 \end{pmatrix}$$

$$= \lambda^2 - 20\lambda + 75 = 0$$

Thus, the eigenvalues are: $\underline{\lambda_1 = 5 \ \ \& \ \ \lambda_2 = 15|}$

For $\lambda_1 = 5 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{2x = y|} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 15 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = 2y|} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\therefore \ \underline{x(t) = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{15t}}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = 3x_1 - 2x_2 \\ x_2'(t) = 2x_1 - 2x_2 \end{cases}$

***Solution***

$$\left|A-\lambda I\right| = \begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$$

$$= \lambda^2 - \lambda - 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1 \ \ \& \ \ \lambda_2 = 2$

For $\lambda_1 = -1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \ 2x = y \quad \Rightarrow \ V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \ x = 2y \quad \Rightarrow \ V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\therefore \ x(t) = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = 2x_1 - x_2 \\ x_2'(t) = 3x_1 - 2x_2 \end{cases}$

***Solution***

$$\left|A-\lambda I\right| = \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$$

$$= \lambda^2 - 1 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1 \ \ \& \ \ \lambda_2 = 1$

For $\lambda_1 = -1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \ 3x = y \quad \Rightarrow \ V_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

For $\lambda_2 = 1 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \ x = y \quad \Rightarrow \ V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \ x(t) = C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = 5x_1 - x_2 \\ x_2'(t) = 3x_1 - x_2 \end{cases}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -1 \\ 3 & -1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 5 & -1 \\ 3 & -1 \end{pmatrix}$$

$$= \lambda^2 - 4\lambda - 2 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = 2 \pm \sqrt{6}$

For $\lambda_1 = 2 - \sqrt{6} \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3+\sqrt{6} & -1 \\ 3 & -3+\sqrt{6} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \left(3+\sqrt{6}\right)x = y$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 3+\sqrt{6} \end{pmatrix}$$

For $\lambda_2 = 2 + \sqrt{6} \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 3-\sqrt{6} & -1 \\ 3 & -3-\sqrt{6} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \left(3-\sqrt{6}\right)x = y$$

$$\Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 3-\sqrt{6} \end{pmatrix}$$

$$\therefore \ x(t) = C_1 \begin{pmatrix} 1 \\ 3+\sqrt{6} \end{pmatrix} e^{\left(2-\sqrt{6}\right)t} + C_2 \begin{pmatrix} 1 \\ 3-\sqrt{6} \end{pmatrix} e^{\left(2+\sqrt{6}\right)t}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = x_1 + x_2 \\ x_2'(t) = 4x_1 - 2x_2 \end{cases}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$$

$$= \lambda^2 + \lambda - 6 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -3 \ \ \& \ \ \lambda_2 = 2$

For $\lambda_1 = -3 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{4x = y|} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = y|} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \ \underline{x(t) = C_1\begin{pmatrix} 1 \\ 4 \end{pmatrix}e^{-3t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t}|}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = -x_1 - 4x_2 \\ x_2'(t) = x_1 - x_2 \end{cases}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$$

$$= \lambda^2 + 2\lambda + 5 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = -1 \pm 2i$

For $\lambda_1 = -1 - 2i \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2i & -4 \\ 1 & 2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = -2iy|} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 2i \\ -1 \end{pmatrix}$$

$$\underline{x_1(t) = \begin{pmatrix} 2i \\ -1 \end{pmatrix}|}$$

$$z(t) = \begin{pmatrix} 2i \\ -1 \end{pmatrix}e^{(-1-2i)t}$$

$$= \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix} + i\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right)\left(\cos 2t + i\sin 2t\right)e^{-t}$$

$$= \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\sin 2t + i\left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin 2t + \begin{pmatrix} 2 \\ 0 \end{pmatrix}\cos 2t\right)\right)e^{-t}$$

$$= \left(\begin{pmatrix} -2\sin 2t \\ -\cos 2t \end{pmatrix} + i\begin{pmatrix} 2\cos 2t \\ -\sin 2t \end{pmatrix}\right)e^{-t}$$

$$\therefore \ \underline{x(t) = C_1\begin{pmatrix} -2\sin 2t \\ -\cos 2t \end{pmatrix}e^{-t} + C_2\begin{pmatrix} 2\cos 2t \\ -\sin 2t \end{pmatrix}e^{-t}|}$$

### ***Exercise***

Find the general solution of the system $\begin{cases} x_1'(t) = 2x_1 + 3x_2 - 7 \\ x_2'(t) = -x_1 - 2x_2 + 5 \end{cases}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$$

$$= \lambda^2 - 1 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1 \;\; and \;\; \lambda_2 = 1$

For $\lambda_1 = -1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = -y} \quad \Rightarrow V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$

For $\lambda_2 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = -3y} \quad \Rightarrow V_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

The solution is: $x_2(t) = \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t}$

$$x_h(t) = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t}$$

$$\begin{cases} 2a_1 + 3a_2 = 7 \\ -a_1 - 2a_2 = -5 \end{cases} \qquad \Delta = \begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} = -1 \quad \Delta_1 = \begin{vmatrix} 7 & 3 \\ -5 & -2 \end{vmatrix} = 1 \quad \Delta_2 = \begin{vmatrix} 2 & 7 \\ -1 & -5 \end{vmatrix} = -3$$

$$a_1 = -1 \quad a_2 = 3 \quad \to \quad x_p = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\therefore \; x(t) = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

### ***Exercise***

Find the general solution of the system $\begin{cases} x_1'(t) = 5x_1 + 9x_2 + 2 \\ x_2'(t) = -x_1 + 11x_2 + 6 \end{cases}$

***Solution***

$$A = \begin{pmatrix} 5 & 9 \\ -1 & 11 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 9 \\ -1 & 11-\lambda \end{vmatrix}$$

$$= \lambda^2 - 16\lambda - 64 = 0$$

Thus, the eigenvalues are: $\underline{\lambda_{1,2} = 8}$

For $\lambda_1 = 8 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -3 & 9 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = 3y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

The solution is: $\underline{x_1(t) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{8t}}$

For the second eigenvector $V_2 \quad \Rightarrow \quad AV_2 = V_1$

$$\begin{pmatrix} -3 & 9 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \rightarrow \quad -x + 3y = 1$$

$$\rightarrow \mathit{if} \quad y = 1 \quad \Rightarrow \quad x = 2 \qquad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The solution is: $\underline{x_2(t) = \left( \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} t \right) e^{8t}}$ $\qquad x_2(t) = e^{\lambda t}\left(V_2 + tV_1\right)$

$$x_h(t) = C_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{8t} + C_2 \left[ \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} t \right] e^{8t}$$

$$\begin{cases} 5a_1 + 9a_2 = -2 \\ -a_1 11a_2 = -6 \end{cases} \qquad \Delta = \begin{vmatrix} 5 & 9 \\ -1 & 11 \end{vmatrix} = 64 \quad \Delta_1 = \begin{vmatrix} -2 & 9 \\ -6 & 11 \end{vmatrix} = 32 \quad \Delta_2 = \begin{vmatrix} 5 & -2 \\ -1 & -6 \end{vmatrix} = -32$$

$$a_1 = \tfrac{1}{2} \quad a_2 = -\tfrac{1}{2} \quad \rightarrow \quad x_p = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$\therefore \; \underline{x(t) = C_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{8t} + C_2 \left[ \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} t \right] e^{8t} + \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}}$$

## *Exercise*

Find the general solution of the system $\begin{cases} y_1'(t) = 6y_1 + y_2 + 6t \\ y_2'(t) = 4y_1 + 3y_2 - 10t + 4 \end{cases}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 6-\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}$$

$$= \lambda^2 - 9\lambda + 14 = 0$$

The eigenvalues: $\lambda_{1,2} = 2,\ 7$

For $\lambda_1 = 2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to 4x = -y \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

For $\lambda_2 = 7 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = y \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$Y_h = C_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t}$$

$$\varphi(t) = \begin{pmatrix} e^{2t} & e^{7t} \\ -4e^{2t} & e^{7t} \end{pmatrix}$$

$$\varphi^{-1} = \frac{1}{5e^{9t}} \begin{pmatrix} e^{7t} & -e^{7t} \\ 4e^{2t} & e^{2t} \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} e^{-2t} & -e^{-2t} \\ 4e^{-7t} & e^{-7t} \end{pmatrix}$$

$$\begin{cases} 6y_1 + y_2 + 6t \\ 4y_1 + 3y_2 - 10t + 4 \end{cases}$$

| | | $\int e^{-7t}$ |
|---|---|---|
| + | $14t+4$ | $-\frac{1}{7}e^{-7t}$ |
| − | $14$ | $\frac{1}{49}e^{-7t}$ |

| | | $\int e^{-2t}$ |
|---|---|---|
| + | $16t-4$ | $-\frac{1}{2}e^{-2t}$ |
| − | $16$ | $\frac{1}{4}e^{-2t}$ |

$$F(t) = \begin{pmatrix} 6t \\ -10t+4 \end{pmatrix}$$

$$Y_p = \frac{1}{5} \begin{pmatrix} e^{2t} & e^{7t} \\ -4e^{2t} & e^{7t} \end{pmatrix} \int \begin{pmatrix} e^{-2t} & -e^{-2t} \\ 4e^{-7t} & e^{-7t} \end{pmatrix} \begin{pmatrix} 6t \\ -10t+4 \end{pmatrix} dt \qquad X_p = \varphi(t)\int \varphi^{-1}(t)F(t)\,dt$$

$$= \frac{1}{5} \begin{pmatrix} e^{2t} & e^{7t} \\ -4e^{2t} & e^{7t} \end{pmatrix} \int \begin{pmatrix} (16t-4)e^{-2t} \\ (14t+4)e^{-7t} \end{pmatrix} dt$$

$$=\frac{1}{5}\begin{pmatrix} e^{2t} & e^{7t} \\ -4e^{2t} & e^{7t} \end{pmatrix}\begin{pmatrix} (-8t+2-4)e^{-2t} \\ \left(-2t-\frac{4}{7}-\frac{14}{49}\right)e^{-7t} \end{pmatrix}$$

$$=\frac{1}{5}\begin{pmatrix} e^{2t} & e^{7t} \\ -4e^{2t} & e^{7t} \end{pmatrix}\begin{pmatrix} (-8t-2)e^{-2t} \\ \left(-2t-\frac{6}{7}\right)e^{-7t} \end{pmatrix}$$

$$=\frac{1}{5}\begin{pmatrix} -8t-2-2t-\frac{6}{7} \\ 32t+8-2t-\frac{6}{7} \end{pmatrix}$$

$$=\frac{1}{5}\begin{pmatrix} -10t-\frac{20}{7} \\ 30t+\frac{50}{7} \end{pmatrix}$$

$$=\begin{pmatrix} -2t-\frac{4}{7} \\ 6t+\frac{10}{7} \end{pmatrix}$$

$$=\begin{pmatrix} -2 \\ 6 \end{pmatrix}t+\begin{pmatrix} -\frac{4}{7} \\ \frac{10}{7} \end{pmatrix}$$

$$Y(t)=C_1\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{2t}+C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{7t}+\begin{pmatrix} -2 \\ 6 \end{pmatrix}t+\begin{pmatrix} -\frac{4}{7} \\ \frac{10}{7} \end{pmatrix}$$

$$\begin{cases} y_1(t)=C_1e^{2t}+C_2e^{7t}-2t-\frac{4}{7} \\ y_2(t)=-4C_1e^{2t}+C_2e^{7t}+6t+\frac{10}{7} \end{cases}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x'(t)=5x+3y-2e^{-t}+1 \\ y'(t)=-x+y+e^{-t}-5t+7 \end{cases}$

***Solution***

$$|A-\lambda I|=\begin{vmatrix} 5-\lambda & 3 \\ -1 & 1-\lambda \end{vmatrix} \qquad A=\begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix}$$

$$=\lambda^2-6\lambda+8=0$$

The eigenvalues: $\lambda_{1,2}=2,\ 4$

For $\lambda_1=2 \quad \Rightarrow (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x=-y \quad \Rightarrow \quad V_1=\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $\lambda_2=4 \quad \Rightarrow \left(A-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x=-3y \quad \Rightarrow \quad V_2=\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$Y_h=C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{2t}+C_2\begin{pmatrix} -3 \\ 1 \end{pmatrix}e^{4t}$$

$$\varphi(t)=\begin{pmatrix} -e^{2t} & -3e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix}$$

$$\varphi^{-1}=\frac{1}{2e^{6t}}\begin{pmatrix} e^{4t} & 3e^{4t} \\ -e^{2t} & -e^{2t} \end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix} e^{-2t} & 3e^{-2t} \\ -e^{-4t} & -e^{-4t} \end{pmatrix}$$

$$F(t)=\begin{pmatrix} 1-2e^{-t} \\ e^{-t}-5t+7 \end{pmatrix}$$

$$\varphi^{-1}(t)F(t)=\frac{1}{2}\begin{pmatrix} e^{-2t} & 3e^{-2t} \\ -e^{-4t} & -e^{-4t} \end{pmatrix}\begin{pmatrix} 1-2e^{-t} \\ e^{-t}-5t+7 \end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix} e^{-2t}-2e^{-3t}+3e^{-3t}-15te^{-2t}+21e^{-2t} \\ -e^{-4t}+2e^{-5t}-e^{-5t}+5te^{-4t}-7e^{-4t} \end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix} e^{-3t}+(-15t+22)e^{-2t} \\ e^{-5t}+(5t-8)e^{-4t} \end{pmatrix}$$

| | | $\int e^{-4t}$ |
|---|---|---|
| + | $5t-8$ | $-\frac{1}{4}e^{-4t}$ |
| − | $5$ | $\frac{1}{16}e^{-4t}$ |

$$X_p=\varphi(t)\int \varphi^{-1}(t)F(t)\,dt$$

$$Y_p=\frac{1}{2}\begin{pmatrix} -e^{2t} & -3e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix}\int\begin{pmatrix} e^{-3t}+(-15t+22)e^{-2t} \\ e^{-5t}+(5t-8)e^{-4t} \end{pmatrix}dt$$

$$=\frac{1}{2}\begin{pmatrix} -e^{2t} & -3e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix}\begin{pmatrix} -\frac{1}{3}e^{-3t}+\left(\frac{15}{2}t-11+\frac{15}{4}\right)e^{-2t} \\ -\frac{1}{5}e^{-5t}+\left(-\frac{5}{4}t+2-\frac{5}{16}\right)e^{-4t} \end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix} -e^{2t} & -3e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix}\begin{pmatrix} -\frac{1}{3}e^{-3t}+\left(\frac{15}{2}t-\frac{29}{4}\right)e^{-2t} \\ -\frac{1}{5}e^{-5t}+\left(-\frac{5}{4}t+\frac{27}{16}\right)e^{-4t} \end{pmatrix}$$

| | | $\int e^{-2t}$ |
|---|---|---|
| + | $-15t+22$ | $-\frac{1}{2}e^{-2t}$ |
| − | $-15$ | $\frac{1}{4}e^{-2t}$ |

$$=\frac{1}{2}\begin{pmatrix}\frac{1}{3}e^{-t}-\frac{15}{2}t+\frac{29}{4}+\frac{3}{5}e^{-t}+\frac{15}{4}t-\frac{81}{16}\\ -\frac{1}{3}e^{-t}+\frac{15}{2}t-\frac{29}{4}-\frac{1}{5}e^{-t}-\frac{5}{4}t+\frac{27}{16}\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}\frac{14}{15}e^{-t}-\frac{15}{4}t+\frac{35}{16}\\ -\frac{8}{15}e^{-t}+\frac{25}{4}t-\frac{89}{16}\end{pmatrix}$$

$$\boxed{=\begin{pmatrix}\frac{14}{30}e^{-t}-\frac{15}{8}t+\frac{35}{32}\\ -\frac{8}{30}e^{-t}+\frac{25}{8}t-\frac{89}{32}\end{pmatrix}}$$

$$Y(t)=C_1\begin{pmatrix}-1\\1\end{pmatrix}e^{2t}+C_2\begin{pmatrix}-3\\1\end{pmatrix}e^{4t}+\begin{pmatrix}\frac{14}{30}\\-\frac{8}{30}\end{pmatrix}e^{-t}+\begin{pmatrix}-\frac{15}{4}\\\frac{25}{8}\end{pmatrix}t+\begin{pmatrix}\frac{35}{32}\\-\frac{89}{32}\end{pmatrix}$$

$$\boxed{\begin{cases}y_1(t)=-C_1e^{2t}-3C_2e^{4t}+\frac{14}{30}e^{-t}-\frac{15}{4}t+\frac{35}{32}\\ y_2(t)=C_1e^{2t}+C_2e^{4t}-\frac{8}{30}e^{-t}+\frac{25}{8}t-\frac{89}{32}\end{cases}}$$

## *Exercise*

Find the general solution of the system $\begin{cases}x'(t)=-3x+y+3t\\ y'(t)=2x-4y+e^{-t}\end{cases}$

***Solution***

$$|A-\lambda I|=\begin{vmatrix}-3-\lambda & 1\\ 2 & -4-\lambda\end{vmatrix}\qquad A=\begin{pmatrix}-3 & 1\\ 2 & -4\end{pmatrix}$$

$$=\lambda^2+7\lambda+10=0$$

The eigenvalues: $\boxed{\lambda_{1,2}=-2,\ -5}$

For $\lambda_1=-2 \quad \Rightarrow (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}-1 & 1\\ 2 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad \rightarrow x=y \quad \Rightarrow \quad V_1=\begin{pmatrix}1\\1\end{pmatrix}$$

For $\lambda_2=-5 \quad \Rightarrow (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix}2 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad \rightarrow 2x=-y \quad \Rightarrow \quad V_2=\begin{pmatrix}1\\-2\end{pmatrix}$$

$$\boxed{Y_h=C_1\begin{pmatrix}1\\1\end{pmatrix}e^{-2t}+C_2\begin{pmatrix}1\\-2\end{pmatrix}e^{-5t}}$$

$$\varphi(t)=\begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}$$

$$\varphi^{-1}=-\frac{1}{3e^{-7t}}\begin{pmatrix} -2e^{-5t} & -e^{-5t} \\ -e^{-2t} & e^{-2t} \end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix} 2e^{2t} & e^{2t} \\ e^{5t} & -e^{5t} \end{pmatrix}$$

$$F(t)=\begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}$$

$$\varphi^{-1}(t)F(t)=\frac{1}{3}\begin{pmatrix} 2e^{2t} & e^{2t} \\ e^{5t} & -e^{5t} \end{pmatrix}\begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix} 6te^{2t}+e^{t} \\ 3te^{5t}-e^{4t} \end{pmatrix}$$

| | | $\int e^{2t}$ |
|---|---|---|
| + | $6t$ | $\frac{1}{2}e^{2t}$ |
| − | $6$ | $\frac{1}{4}e^{2t}$ |

$$X_p=\varphi(t)\int\varphi^{-1}(t)F(t)\,dt$$

$$Y_p=\frac{1}{3}\begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\int\begin{pmatrix} 6te^{2t}+e^{t} \\ 3te^{5t}-e^{4t} \end{pmatrix}dt$$

$$=\frac{1}{3}\begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\begin{pmatrix} \left(3t-\frac{3}{2}\right)e^{2t}+e^{t} \\ \left(\frac{3}{5}t-\frac{3}{25}\right)e^{5t}-\frac{1}{4}e^{4t} \end{pmatrix}$$

| | | $\int e^{5t}$ |
|---|---|---|
| + | $3t$ | $\frac{1}{5}e^{5t}$ |
| − | $3$ | $\frac{1}{25}e^{5t}$ |

$$=\frac{1}{3}\begin{pmatrix} 3t-\frac{3}{2}+e^{-t}+\frac{3}{5}t-\frac{3}{25}-\frac{1}{4}e^{-t} \\ 3t-\frac{3}{2}+e^{-t}-\frac{6}{5}t+\frac{6}{25}+\frac{1}{2}e^{-t} \end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix} \frac{3}{4}e^{-t}+\frac{18}{5}t-\frac{81}{50} \\ \frac{3}{2}e^{-t}+\frac{9}{5}t-\frac{63}{50} \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{4}e^{-t}+\frac{6}{5}t-\frac{27}{50} \\ \frac{1}{2}e^{-t}+\frac{3}{5}t-\frac{21}{50} \end{pmatrix}$$

$$Y(t)=C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-2t}+C_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-5t}+\begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix}e^{-t}+\begin{pmatrix} \frac{6}{5} \\ \frac{3}{5} \end{pmatrix}t-\begin{pmatrix} \frac{27}{50} \\ \frac{21}{50} \end{pmatrix}$$

$$\boxed{\begin{cases} y_1(t)=C_1e^{-2t}+C_2e^{-5t}+\frac{1}{4}e^{-t}+\frac{6}{5}t-\frac{27}{50} \\ y_2(t)=C_1e^{-2t}-2C_2e^{-5t}+\frac{1}{2}e^{-t}+\frac{3}{5}t-\frac{21}{50} \end{cases}}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x'(t) = 2x - y + (\sin 2t)e^{2t} \\ y'(t) = 4x + 2y + (2\cos 2t)e^{2t} \end{cases}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ 4 & 2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}$$

$$= \lambda^2 - 4\lambda + 8 = 0$$

The eigenvalues: $\lambda_{1,2} = 2 \pm 2i$

For $\lambda_1 = 2 - 2i \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2i & -1 \\ 4 & 2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to 2ix = y \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

$$z(t) = \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{(-2-2i)t}$$

$$= \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 2 \end{pmatrix} \right)(\cos 2t - i\sin 2t)e^{-2t}$$

$$= \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos 2t + \begin{pmatrix} 0 \\ 2 \end{pmatrix}\sin 2t + i\left( \begin{pmatrix} 0 \\ 2 \end{pmatrix}\cos 2t - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin 2t \right) \right) e^{-2t}$$

$$= \left( \begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix} + i\begin{pmatrix} -\sin 2t \\ 2\cos 2t \end{pmatrix} \right) e^{2t}$$

$$Y_h = C_1 \begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} -\sin 2t \\ 2\cos 2t \end{pmatrix} e^{2t}$$

$$\varphi(t) = \begin{pmatrix} e^{2t}\cos 2t & -e^{2t}\sin 2t \\ 2e^{2t}\sin 2t & 2e^{2t}\cos 2t \end{pmatrix}$$

$$\varphi^{-1} = \frac{1}{2e^{4t}} \begin{pmatrix} 2e^{2t}\cos 2t & e^{2t}\sin 2t \\ -2e^{2t}\sin 2t & e^{2t}\cos 2t \end{pmatrix}$$

$$= \begin{pmatrix} e^{-2t}\cos 2t & \frac{1}{2}e^{-2t}\sin 2t \\ -e^{-2t}\sin 2t & \frac{1}{2}e^{-2t}\cos 2t \end{pmatrix}$$

$$F(t) = \begin{pmatrix} (\sin 2t)e^{2t} \\ (2\cos 2t)e^{2t} \end{pmatrix}$$

$$\varphi^{-1}(t)F(t)=\begin{pmatrix} e^{-2t}\cos 2t & \frac{1}{2}e^{-2t}\sin 2t \\ -e^{-2t}\sin 2t & \frac{1}{2}e^{-2t}\cos 2t \end{pmatrix}\begin{pmatrix} (\sin 2t)e^{2t} \\ (2\cos 2t)e^{2t} \end{pmatrix}$$

$$=\begin{pmatrix} 2\cos 2t\sin 2t \\ \cos^2 2t-\sin^2 2t \end{pmatrix}$$

$$=\begin{pmatrix} \sin 4t \\ \cos 4t \end{pmatrix}$$

$$Y_p=\begin{pmatrix} e^{2t}\cos 2t & -e^{2t}\sin 2t \\ 2e^{2t}\sin 2t & 2e^{2t}\cos 2t \end{pmatrix}\int\begin{pmatrix} \sin 4t \\ \cos 4t \end{pmatrix}dt \qquad X_p=\varphi(t)\int\varphi^{-1}(t)F(t)\,dt$$

$$=\begin{pmatrix} e^{2t}\cos 2t & -e^{2t}\sin 2t \\ 2e^{2t}\sin 2t & 2e^{2t}\cos 2t \end{pmatrix}\begin{pmatrix} -\frac{1}{4}\cos 4t \\ \frac{1}{4}\sin 4t \end{pmatrix}$$

$$=\begin{pmatrix} -\frac{1}{4}\cos 2t\cos 4t-\frac{1}{4}\sin 2t\sin 4t \\ -\frac{1}{2}\sin 2t\cos 4t+\frac{1}{2}\cos 2t\sin 4t \end{pmatrix}e^{2t}$$

$$Y(t)=C_1\begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix}e^{2t}+C_2\begin{pmatrix} -\sin 2t \\ 2\cos 2t \end{pmatrix}e^{2t}+\begin{pmatrix} -\frac{1}{4}\cos 2t\cos 4t-\frac{1}{4}\sin 2t\sin 4t \\ -\frac{1}{2}\sin 2t\cos 4t+\frac{1}{2}\cos 2t\sin 4t \end{pmatrix}e^{2t}$$

$$\begin{cases} x(t)=\left(C_1\cos 2t-C_2\sin 2t-\frac{1}{4}\cos 2t\cos 4t-\frac{1}{4}\sin 2t\sin 4t\right)e^{2t} \\ y(t)=\left(2C_1\sin 2t+2C_2\cos 2t-\frac{1}{2}\sin 2t\cos 4t+\frac{1}{2}\cos 2t\sin 4t\right)e^{2t} \end{cases}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x'(t)=2y+e^t \\ y'(t)=-x+3y-e^t \end{cases}$

***Solution***

$$|A-\lambda I|=\begin{vmatrix} -\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} \qquad A=\begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$$

$$=\lambda^2-3\lambda+2=0$$

The eigenvalues: $\lambda_{1,2}=1,\ 2$

For $\lambda_1=1 \quad \Rightarrow (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x=2y \qquad \Rightarrow \quad V_1=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = y \qquad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{Y_h = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^t + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t}}$$

$$\varphi(t) = \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}$$

$$\varphi^{-1} = \frac{1}{e^{3t}}\begin{pmatrix} e^{2t} & -e^{2t} \\ -e^t & 2e^t \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}$$

$$F(t) = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

$$\varphi^{-1}(t)F(t) = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}\begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -3e^{-t} \end{pmatrix}$$

$$Y_p = \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}\int\begin{pmatrix} 2 \\ -3e^{-t} \end{pmatrix}dt \qquad X_p = \varphi(t)\int \varphi^{-1}(t)F(t)\,dt$$

$$= \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}\begin{pmatrix} 2t \\ 3e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} 4te^t + 3 \\ 2te^t + 3 \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} 4 \\ 2 \end{pmatrix}te^t + \begin{pmatrix} 3 \\ 3 \end{pmatrix}}$$

$$Y(t) = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^t + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t} + \begin{pmatrix} 4 \\ 2 \end{pmatrix}te^t + \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\boxed{\begin{cases} x(t) = 2C_1e^t + C_2e^{2t} + 4te^t + 3 \\ y(t) = C_1e^t + C_2e^{2t} + 2te^t + 3 \end{cases}}$$

### Exercise

Find the general solution of the system $\begin{cases} x'(t) = \qquad 2y+2 \\ y'(t) = -x+3y+e^{-3t} \end{cases}$

***Solution***

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} \qquad\qquad A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

The eigenvalues: $\underline{\lambda_{1,2} = 1,\ 2}$

For $\lambda_1 = 1 \quad \Rightarrow (A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = 2y \qquad \Rightarrow \quad V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A-\lambda_2 I)V_2 = 0$

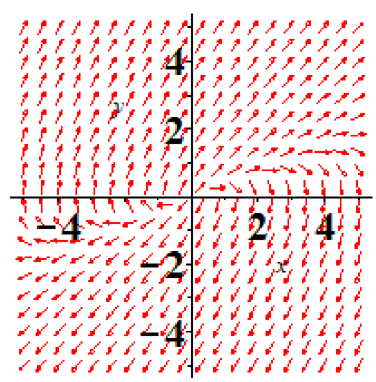
$$\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = y \qquad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\underline{Y_h = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^t + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t}}$$

$$\varphi(t) = \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}$$

$$\varphi^{-1} = \frac{1}{e^{3t}}\begin{pmatrix} e^{2t} & -e^{2t} \\ -e^t & 2e^t \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}$$




$$F(t) = \begin{pmatrix} 2 \\ e^{-3t} \end{pmatrix}$$

$$\varphi^{-1}(t)F(t) = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}\begin{pmatrix} 2 \\ e^{-3t} \end{pmatrix}$$

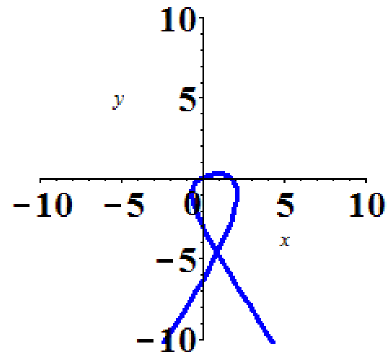
$$= \begin{pmatrix} 2e^{-t} - e^{-4t} \\ -2e^{-2t} + 2e^{-5t} \end{pmatrix}$$

$$Y_p = \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}\int\begin{pmatrix} 2e^{-t} - e^{-4t} \\ -2e^{-2t} + 2e^{-5t} \end{pmatrix}dt \qquad\qquad X_p = \varphi(t)\int \varphi^{-1}(t)F(t)\,dt$$

$$=\begin{pmatrix} 2e^{t} & e^{2t} \\ e^{t} & e^{2t} \end{pmatrix}\begin{pmatrix} -2e^{-t}+\frac{1}{4}e^{-4t} \\ e^{-2t}-\frac{2}{5}e^{-5t} \end{pmatrix}$$

$$=\begin{pmatrix} -4+\frac{1}{2}e^{-3t}+1-\frac{2}{5}e^{-3t} \\ -2+\frac{1}{4}e^{-3t}+1-\frac{2}{5}e^{-3t} \end{pmatrix}$$

$$=\begin{pmatrix} -3+\frac{1}{10}e^{-3t} \\ -1-\frac{3}{20}e^{-3t} \end{pmatrix}$$




$$Y(t)=C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{t}+C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t}+\begin{pmatrix} \frac{1}{10} \\ -\frac{3}{20} \end{pmatrix}e^{-3t}-\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\boxed{\begin{cases} x(t)=2C_1e^{t}+C_2e^{2t}+\frac{1}{10}e^{-3t}-3 \\ y(t)=C_1e^{t}+C_2e^{2t}-\frac{3}{20}e^{-3t}-1 \end{cases}}$$

### *Exercise*

Find the general solution of the system $\begin{cases} x'(t)=x+8y+12t \\ y'(t)=x-y+12t \end{cases}$

***Solution***

$$|A-\lambda I|=\begin{vmatrix} 1-\lambda & 8 \\ 1 & -1-\lambda \end{vmatrix} \qquad A=\begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix}$$

$$=\lambda^2-9=0$$

The eigenvalues: $\boxed{\lambda_{1,2}=\pm 3}$

For $\lambda_1=-3 \quad \Rightarrow (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 4 & 8 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x=-2y \quad \Rightarrow \quad V_1=\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

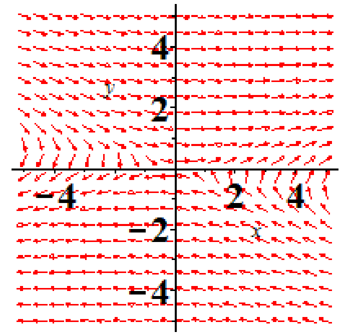
For $\lambda_2=3 \quad \Rightarrow (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix} -2 & 8 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x=4y \quad \Rightarrow \quad V_2=\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\boxed{Y_h=C_1\begin{pmatrix} -2 \\ 1 \end{pmatrix}e^{-3t}+C_2\begin{pmatrix} 4 \\ 1 \end{pmatrix}e^{3t}}$$

$$\varphi(t)=\begin{pmatrix}-2e^{-3t} & 4e^{3t}\\ e^{-3t} & e^{3t}\end{pmatrix}$$

$$\varphi^{-1}=-\frac{1}{6}\begin{pmatrix}e^{3t} & -4e^{3t}\\ -e^{-3t} & -2e^{-3t}\end{pmatrix}$$




$$F(t)=\begin{pmatrix}12t\\ 12t\end{pmatrix}$$

$$\varphi^{-1}(t)F(t)=-\frac{1}{6}\begin{pmatrix}e^{3t} & -4e^{3t}\\ -e^{-3t} & -2e^{-3t}\end{pmatrix}\begin{pmatrix}12t\\ 12t\end{pmatrix}$$

$$=\begin{pmatrix}-e^{3t} & 4e^{3t}\\ e^{-3t} & 2e^{-3t}\end{pmatrix}\begin{pmatrix}2t\\ 2t\end{pmatrix}$$

$$=\begin{pmatrix}6te^{3t}\\ 6te^{-3t}\end{pmatrix}$$

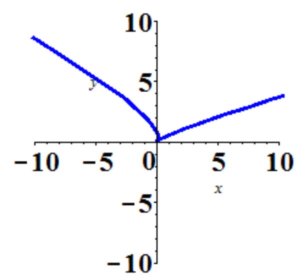
$$Y_p=\begin{pmatrix}-2e^{-3t} & 4e^{3t}\\ e^{-3t} & e^{3t}\end{pmatrix}\int\begin{pmatrix}6te^{3t}\\ 6te^{-3t}\end{pmatrix}dt$$

$$X_p=\varphi(t)\int\varphi^{-1}(t)F(t)\,dt$$

$$=\begin{pmatrix}-2e^{-3t} & 4e^{3t}\\ e^{-3t} & e^{3t}\end{pmatrix}\begin{pmatrix}\left(2t-\frac{2}{3}\right)e^{3t}\\ \left(-2t-\frac{2}{3}\right)e^{-3t}\end{pmatrix}$$

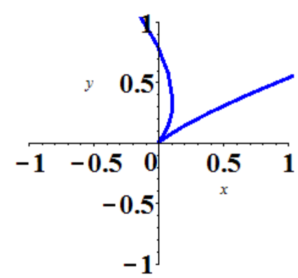
| | | $\int e^{3t}$ |
|---|---|---|
| + | $6t$ | $\frac{1}{3}e^{3t}$ |
| − | $6$ | $\frac{1}{9}e^{3t}$ |

$$=\begin{pmatrix}-4t+\frac{4}{3}-8t-\frac{8}{3}\\ 2t-\frac{2}{3}-2t-\frac{2}{3}\end{pmatrix}$$




$$=\boxed{\begin{pmatrix}-12t-\frac{4}{3}\\ -\frac{4}{3}\end{pmatrix}}$$

$$Y(t)=C_1\begin{pmatrix}-2\\ 1\end{pmatrix}e^{-3t}+C_2\begin{pmatrix}4\\ 1\end{pmatrix}e^{3t}+\begin{pmatrix}-12\\ 0\end{pmatrix}t-\begin{pmatrix}\frac{4}{3}\\ \frac{4}{3}\end{pmatrix}$$




$$\boxed{\begin{cases}x(t)=-2C_1e^{-3t}+4C_2e^{3t}-12t-\frac{4}{3}\\ y(t)=C_1e^{-3t}+C_2e^{3t}-\frac{4}{3}\end{cases}}$$

## ***Exercise***

Find the general solution of the system $\begin{cases} x_1'(t) = x_1 + x_2 - x_3 \\ x_2'(t) = 2x_2 \\ x_3'(t) = x_2 - x_3 \end{cases}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & -1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= -\left(1-\lambda^2\right)(2-\lambda) = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1, \quad \lambda_2 = 1 \ \ and \ \ \lambda_2 = 2$

For $\lambda_1 = -1 \quad \Rightarrow \left(A - \lambda_1 I\right) V_1 = 0$

$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 2x + y - z = 0 \\ \quad y = 0 \\ \end{cases} \quad 2x - z = 0 \quad \Rightarrow V_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 1 \quad \Rightarrow \left(A - \lambda_2 I\right) V_2 = 0$

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} y - z = 0 \quad z = 0 \\ \quad y = 0 \\ \end{cases} \quad \Rightarrow V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda_3 = 2 \quad \Rightarrow \left(A - \lambda_3 I\right) V_3 = 0$

$$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} -x + y - z = 0 \\ \\ \quad y = 3z \end{cases} \quad \Rightarrow V_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$\boxed{x(t) = C_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + C_3 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} e^{2t}}$$

## ***Exercise***

Find the general solution of the system $\begin{cases} x_1'(t) = 2x_1 - 7x_2 \\ x_2'(t) = 5x_1 + 10x_2 + 4x_3 \\ x_3'(t) = 5x_2 + 2x_3 \end{cases}$

***Solution***

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & -7 & 0 \\ 5 & 10-\lambda & 4 \\ 0 & 5 & 2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{pmatrix}$$

$$= (2-\lambda)^2(10-\lambda) - 20(2-\lambda) + 35(2-\lambda)$$

$$= (2-\lambda)\big((10-\lambda)(2-\lambda)+15\big)$$

$$= (2-\lambda)\left(35-12\lambda+\lambda^2\right) = 0$$

Thus, the eigenvalues are: $\lambda_1 = 2, \ \ \lambda_2 = 5 \ \ and \ \ \lambda_2 = 7$

For $\lambda_1 = 2 \quad \Rightarrow (A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & -7 & 0 \\ 5 & 8 & 4 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} y = 0 \\ 5x = -4z \end{cases} \quad \Rightarrow \ V_1 = \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix}$$

For $\lambda_2 = 5 \quad \Rightarrow (A-\lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -5 & -7 & 0 \\ 5 & 3 & 4 \\ 0 & 5 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 5x = -7y \\ \\ y = z \end{cases} \quad \Rightarrow \ V_2 = \begin{pmatrix} -7 \\ 5 \\ 5 \end{pmatrix}$$

For $\lambda_3 = 7 \quad \Rightarrow (A-\lambda_3 I)V_3 = 0$

$$\begin{pmatrix} 0 & -7 & 0 \\ 5 & 8 & 4 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} y = 0 \\ 5x = -4z \end{cases} \quad \Rightarrow \ V_1 = \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix}$$

$$x(t) = C_1\begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}e^{2t} + C_2\begin{pmatrix} -7 \\ 3 \\ 5 \end{pmatrix}e^{5t} + C_3\begin{pmatrix} -7 \\ 5 \\ 5 \end{pmatrix}e^{7t}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = 3x_1 - x_2 - x_3 \\ x_2'(t) = x_1 + x_2 - x_3 \\ x_3'(t) = x_1 - x_2 + x_3 \end{cases}$

***Solution***

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & -1 & -1 \\ 1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \left(1-2\lambda+\lambda^2\right)(3-\lambda) + 2 + 2 - 2\lambda - 3 + \lambda$$

$$= 3 - 7\lambda + 5\lambda^2 - \lambda^3 + 1 - \lambda$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

$$\begin{array}{r|rrrr} 1 & -1 & 5 & -8 & 4 \\ & & -1 & 4 & -4 \\ \hline & -1 & 4 & -4 & 0 \end{array} \quad \rightarrow \boxed{-\lambda^2 + 4\lambda - 4 = 0}$$

Thus, the eigenvalues are: $\boxed{\lambda_1 = 1 \ \ and \ \ \lambda_{2,3} = 2}$

For $\lambda_1 = 1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} y = 0 \\ x = y \\ x = z \end{cases} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} x = y + z \\ \\ \end{cases} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} x = y + z \\ \\ \end{cases} \quad \Rightarrow \quad V_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\boxed{x(t) = C_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^t + C_2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}e^{2t} + C_3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{2t}}$$

## Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 3x_1 + 2x_2 + 4x_3 \\ x_2'(t) = 2x_1 + 2x_3 \\ x_3'(t) = 4x_1 + 2x_2 + 3x_3 \end{cases}$

***Solution***

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

$$\begin{array}{r|rrrr} -1 & -1 & 6 & 15 & 8 \\ & & 1 & -7 & -8 \\ \hline & -1 & 7 & 8 & 0 \end{array} \quad \rightarrow \boxed{-\lambda^2 + 7\lambda + 8 = 0}$$

$$\left|A - \lambda I\right| = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$$

$$= -9\lambda + 6\lambda^2 - \lambda^3 + 32 + 16\lambda - 12 + 4\lambda - 12 + 4\lambda$$

$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 8 \ \ and \ \ \lambda_{2,3} = -1$

For $\lambda_1 = 8 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 5x - 2y - 4z = 0 \\ x - 4y + z = 0 \\ 4x + 2y - 5z = 0 \end{cases} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = -1 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 2x + y + 2z = 0 \\ 2x + y + 2z = 0 \\ 4x + 2y + 4z = 0 \end{cases} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad V_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$x(t) = C_1\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}e^{8t} + C_2\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}e^{-t} + C_3\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}e^{-t}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = x_1 + x_2 + x_3 \\ x_2'(t) = 2x_1 + x_2 - x_3 \\ x_3'(t) = -8x_1 - 5x_2 - 3x_3 \end{cases}$

***Solution***

$$\left|A - \lambda I\right| = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -8 & -5 & -3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}$$

$$= \left(1 - 2\lambda + \lambda^2\right)(-3 - \lambda) - 2 + 3 - 3\lambda + 6 + 2\lambda$$

$$= -\lambda^3 - \lambda^2 + 4\lambda + 4 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1, \;\; \lambda_2 = -2, \;\; and \;\; \lambda_3 = 2$

For $\lambda_1 = -1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ -8 & -5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 2x + y + z = 0 \\ 2x + 2y - z = 0 \\ -8x - 5y - 2z = 0 \end{cases}$$

$$z = 1 \quad \rightarrow \begin{cases} 2x + y = -1 \\ 2x + 2y = 1 \end{cases} \qquad \Delta = \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 2 \quad \Delta_x = \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} = -3 \quad \Delta_y = \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = 4$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & -1 \\ -8 & -5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} 3x + y + z = 0 \\ 2x + 3y - z = 0 \\ -8x - 5y - z = 0 \end{cases}$$

$$z = 1 \quad \to \begin{cases} 3x + y = -1 \\ 2x + 3y = 1 \end{cases} \qquad \Delta = \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} = 7 \quad \Delta_x = \begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix} = -4 \quad \Delta_y = \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = 5$$

$$\Rightarrow \quad V_2 = \begin{pmatrix} -\frac{4}{7} \\ \frac{5}{7} \\ 1 \end{pmatrix}$$

For $\lambda_3 = 2 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -8 & -5 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} -x + y + z = 0 \\ 2x - y - z = 0 \\ -8x - 5y - 5z = 0 \end{cases}$$

$$x = 0 \quad \to \begin{cases} y + z = 0 \\ -y - z = 0 \end{cases} \qquad \Rightarrow \quad V_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$x(t) = C_1 \begin{pmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -\frac{4}{7} \\ \frac{5}{7} \\ 1 \end{pmatrix} e^{-2t} + C_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{2t}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = x_1 - x_2 + 4x_3 \\ x_2'(t) = 3x_1 + 2x_2 - x_3 \\ x_3'(t) = 2x_1 + x_2 - x_3 \end{cases}$

***Solution***

$$\left|A - \lambda I\right| = \begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

$$= -\left(1-\lambda^2\right)(2-\lambda) + 14 - 16 + 8\lambda + 1 - \lambda - 3 - 3\lambda$$

$$= -\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$$

$$\begin{array}{c|cccc} 1 & -1 & 2 & 5 & -6 \\ & & -1 & 1 & 6 \\ \hline & -1 & 1 & 6 & 0 \end{array} \quad \to \quad -\lambda^2 + \lambda + 6 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -2, \ \lambda_2 = 1, \ and \ \lambda_3 = 3$

For $\lambda_1 = -2 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} 3x - y + 4z = 0 \\ 3x + 4y - z = 0 \\ 2x + y + z = 0 \end{cases}$$

$$z = 1 \quad \to \begin{cases} 3x - y = -4 \\ 3x + 4y = 1 \end{cases} \qquad \Delta = \begin{vmatrix} 3 & -1 \\ 3 & 4 \end{vmatrix} = 15 \quad \Delta_x = \begin{vmatrix} -4 & -1 \\ 1 & 4 \end{vmatrix} = -15 \quad \Delta_y = \begin{vmatrix} 3 & -4 \\ 3 & 1 \end{vmatrix} = 15$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} -y + 4z = 0 \\ 3x + y - z = 0 \\ 2x + y - 2z = 0 \end{cases}$$

$$z = 1 \quad \to \begin{cases} y = 4 \\ 3x + y = 1 \end{cases} \qquad \Rightarrow \quad V_2 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 3 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} -2x - y + 4z = 0 \\ 3x - y - z = 0 \\ 2x + y - 4z = 0 \end{cases}$$

$$z = 1 \quad \to \begin{cases} -2x - y = -4 \\ 3x - y = 1 \end{cases} \qquad \Delta = \begin{vmatrix} -2 & -1 \\ 3 & -1 \end{vmatrix} = 5 \quad \Delta_x = \begin{vmatrix} -4 & -1 \\ 1 & -1 \end{vmatrix} = 5 \quad \Delta_y = \begin{vmatrix} -2 & -4 \\ 3 & 1 \end{vmatrix} = 10$$

$$\Rightarrow \quad V_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\boxed{x(t) = C_1\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}e^{-2t} + C_2\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}e^{t} + C_3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}e^{3t}}$$

### *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = x_1 + x_2 + e^t \\ x_2'(t) = x_1 + x_2 + e^{2t} \\ x_3'(t) = \qquad\quad 3x_3 + te^{3t} \end{cases}$

## Solution

$$\left|A - \lambda I\right| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \left(1 - 2\lambda + \lambda^2\right)(3-\lambda) - (3-\lambda)$$

$$= (3-\lambda)\left(\lambda^2 - 2\lambda\right) = 0$$

The eigenvalues: $\underline{\lambda_{1,2,3} = 0,\ 2,\ 3|}$

For $\lambda_1 = 0 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} x = -y \\ z = 0 \end{cases} \quad \Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} x = y \\ z = 0 \end{cases} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_3 = 3 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} 2x = y \\ x = 2y \end{cases} \quad \Rightarrow \quad V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{X_h = C_1\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{2t} + C_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}e^{3t}|}$$

$$\varphi(t) = \begin{pmatrix} -1 & e^{2t} & 0 \\ 1 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}$$

$$\varphi^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2}e^{-2t} & \frac{1}{2}e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{pmatrix}$$

$$F(t) = \begin{pmatrix} e^{t} \\ e^{2t} \\ te^{3t} \end{pmatrix}$$

$$\varphi^{-1}(t)F(t) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2}e^{-2t} & \frac{1}{2}e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{pmatrix}\begin{pmatrix} e^{t} \\ e^{2t} \\ te^{3t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}e^{t} - \frac{1}{2}e^{2t} \\ \frac{1}{2}e^{-t} + \frac{1}{2} \\ t \end{pmatrix}$$

$$\int \left(\frac{1}{2}e^{t} - \frac{1}{2}e^{2t}\right)dt = \underline{\frac{1}{2}e^{t} - \frac{1}{4}e^{2t}}\Big|$$

$$\int \left(\frac{1}{2}e^{-t} + \frac{1}{2}\right)dt = \underline{-\frac{1}{2}e^{-t} + \frac{1}{2}t}\Big|$$

$$\int t\,dt = \underline{\frac{1}{2}t^{2}}\Big|$$

$$X_p = \begin{pmatrix} -1 & e^{2t} & 0 \\ 1 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}\begin{pmatrix} \frac{1}{2}e^{t} - \frac{1}{4}e^{2t} \\ -\frac{1}{2}e^{-t} + \frac{1}{2}t \\ \frac{1}{2}t^{2} \end{pmatrix} \qquad X_p = \varphi(t)\int \varphi^{-1}(t)F(t)\,dt$$

$$= \begin{pmatrix} -\frac{1}{2}e^{t} + \frac{1}{4}e^{2t} - \frac{1}{2}e^{t} + \frac{1}{2}te^{2t} \\ \frac{1}{2}e^{t} - \frac{1}{4}e^{2t} - \frac{1}{2}e^{t} + \frac{1}{2}te^{2t} \\ \frac{1}{2}t^{2}e^{3t} \end{pmatrix}$$

$$= \begin{pmatrix} -e^{t} + \frac{1}{4}e^{2t} + \frac{1}{2}te^{2t} \\ \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} \\ \frac{1}{2}t^{2}e^{3t} \end{pmatrix}$$

$$X(t)=C_1\begin{pmatrix}-1\\1\\0\end{pmatrix}+C_2\begin{pmatrix}1\\1\\0\end{pmatrix}e^{2t}+C_3\begin{pmatrix}1\\2\\1\end{pmatrix}e^{3t}+\begin{pmatrix}-1\\0\\0\end{pmatrix}e^{t}+\begin{pmatrix}\frac{1}{2}t+\frac{1}{4}\\\frac{1}{2}t-\frac{1}{4}\\0\end{pmatrix}e^{2t}+\begin{pmatrix}0\\0\\\frac{1}{2}t^2\end{pmatrix}e^{3t}$$

$$\boxed{\begin{cases}x_1(t)=-C_1-e^t+\left(C_2+\frac{1}{2}t+\frac{1}{4}\right)e^{2t}+C_3e^{3t}\\x_2(t)=C_1+\left(C_2+\frac{1}{2}t-\frac{1}{4}\right)e^{2t}+2C_3e^{3t}\\x_3(t)=\left(C_3+\frac{1}{2}t^2\right)e^{3t}\end{cases}}$$

## *Exercise*

Find the general solution of the system $y'=Ay$ $\begin{cases}y_1'(t)=-y_1+6y_2\\y_2'(t)=-3y_1+8y_2\end{cases}$ $y(0)=\begin{pmatrix}1\\-2\end{pmatrix}$

***Solution***

$$|A-\lambda I|=\begin{vmatrix}-1-\lambda & 6\\-3 & 8-\lambda\end{vmatrix} \qquad A=\begin{pmatrix}-1 & 6\\-3 & 8\end{pmatrix}$$

$$=(-1-\lambda)(8-\lambda)+18$$

$$=-8-7\lambda+\lambda^2+18$$

$$=\lambda^2-7\lambda+10$$

Thus, the eigenvalues are: $\lambda_1=2$ *and* $\lambda_2=5$

For $\lambda_1=2 \quad \Rightarrow (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}-3 & 6\\-3 & 6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \to \begin{cases}-3x+6y=0\\-3x+6y=0\end{cases} \Rightarrow \boxed{x=2y} \qquad V_1=(2,\ 1)^T$$

The solution is: $\boxed{y_1(t)=e^{2t}\begin{pmatrix}2\\1\end{pmatrix}}$

For $\lambda_2=5 \quad \Rightarrow (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix}-6 & 6\\-3 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \to \begin{cases}-6x+6y=0\\-3x+3y=0\end{cases} \Rightarrow \boxed{x=y} \qquad V_2=(1,\ 1)^T$$

The solution is: $\boxed{y_2(t)=e^{5t}\begin{pmatrix}1\\1\end{pmatrix}}$

The general solution is given by: $y(t)=C_1y_1(t)+C_2y_2(t)$

$$y(t)=C_1e^{2t}\begin{pmatrix}2\\1\end{pmatrix}+C_2e^{5t}\begin{pmatrix}1\\1\end{pmatrix}$$

$$y(0)=C_1\begin{pmatrix}2\\1\end{pmatrix}+C_2\begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\-2\end{pmatrix}=\begin{pmatrix}2C_1+C_2\\C_1+C_2\end{pmatrix}\quad\to\begin{cases}2C_1+C_2=1\\C_1+C_2=-2\end{cases}\to\boxed{C_1=3}\qquad\boxed{C_2=-5}$$

The particular solution is:

$$y(t)=3e^{2t}\begin{pmatrix}2\\1\end{pmatrix}-5e^{5t}\begin{pmatrix}1\\1\end{pmatrix}\quad\to\quad\begin{cases}y_1(t)=6e^{2t}-5e^{5t}\\y_2(t)=3e^{2t}-5e^{5t}\end{cases}$$

## *Exercise*

Find the general solution of the system $y'=Ay$ $\quad\begin{cases}y_1'(t)=y_1+2y_2\\y_2'(t)=-y_1+4y_2\end{cases}\quad y(0)=\begin{pmatrix}3\\2\end{pmatrix}$

***Solution***

$$A=\begin{pmatrix}1&2\\-1&4\end{pmatrix}\qquad\qquad |A-\lambda I|=\begin{vmatrix}1-\lambda&2\\-1&4-\lambda\end{vmatrix}$$

$$=(1-\lambda)(4-\lambda)+2$$

$$=4-5\lambda+\lambda^2+2$$

$$=\lambda^2-5\lambda+6$$

Thus, the eigenvalues are: $\lambda_1=2$ *and* $\lambda_2=3$

For $\lambda_1=2\quad\Rightarrow(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}-1&2\\-1&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\to\begin{cases}-x+2y=0\\-x+2y=0\end{cases}\Rightarrow\boxed{x=2y}$$

The eigenvector is: $V_1=(2,\ 1)^T$

The solution is: $y_1(t)=e^{2t}\begin{pmatrix}2\\1\end{pmatrix}$

For $\lambda_2=3\quad\Rightarrow(A-\lambda_2 I)V_2=0$

$$\begin{pmatrix}-2&2\\-1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\to\begin{cases}-2x+2y=0\\-x+y=0\end{cases}\quad\Rightarrow\quad\boxed{x=y}$$

The eigenvector is: $V_2=(1,\ 1)^T$

The solution is: $y_2(t)=e^{3t}\begin{pmatrix}1\\1\end{pmatrix}$

The general solution is given by: $y(t)=C_1y_1(t)+C_2y_2(t)$

$$y(t) = C_1 e^{2t}\begin{pmatrix}2\\1\end{pmatrix} + C_2 e^{3t}\begin{pmatrix}1\\1\end{pmatrix}$$

$$y(0) = C_1\begin{pmatrix}2\\1\end{pmatrix} + C_2\begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}2C_1 + C_2\\ C_1 + C_2\end{pmatrix} \quad \to \begin{cases}2C_1 + C_2 = 3\\ C_1 + C_2 = 2\end{cases} \to \boxed{C_1 = 1} \qquad \boxed{C_2 = 1}$$

The particular solution is: $y(t) = e^{2t}\begin{pmatrix}2\\1\end{pmatrix} + e^{3t}\begin{pmatrix}1\\1\end{pmatrix}$

## *Exercise*

Find the general solution of the system $y' = Ay$ $\quad \begin{cases}y_1'(t) = -4y_1 - 8y_2\\ y_2'(t) = 4y_1 + 4y_2\end{cases} \quad y(0) = \begin{pmatrix}0\\2\end{pmatrix}$

***Solution***

$$A = \begin{pmatrix}-4 & -8\\ 4 & 4\end{pmatrix} \qquad |A - \lambda I| = \begin{vmatrix}-4-\lambda & -8\\ 4 & 4-\lambda\end{vmatrix}$$

$$= (-4-\lambda)(4-\lambda) + 32$$

$$= -16 + \lambda^2 + 32$$

$$= \lambda^2 + 16 = 0$$

$$\lambda^2 = -16 \Rightarrow \lambda = \pm 4i$$

Thus, the eigenvalues are: $\lambda_1 = -4i$ *and* $\lambda_2 = 4i$

For $\lambda = 4i \quad \Rightarrow (A - \lambda I)V = 0$

$$\begin{pmatrix}-4-4i & -8\\ 4 & 4-4i\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{cases}(-4-4i)x - 8y = 0\\ 4x + (4-4i)y = 0\end{cases} \Rightarrow \textit{divide by } 4\begin{cases}-x - ix - 2y = 0\\ x + y - iy = 0\end{cases}$$

$$-ix - y - iy = 0$$

$$ix = (-1-i)y \Rightarrow x = \frac{-1-i}{i}y\frac{i}{i} = \frac{-i+1}{-1}y = (-1+i)y$$

The eigenvector is: $V = (-1+i,\ 1)^T$

$$z(t) = e^{4it}\begin{pmatrix}-1+i\\1\end{pmatrix}$$

$$= (\cos 4t + i\sin 4t)\left[\begin{pmatrix}-1\\1\end{pmatrix} + i\begin{pmatrix}1\\0\end{pmatrix}\right]$$

$$= \cos 4t\begin{pmatrix}-1\\1\end{pmatrix} - \sin 4t\begin{pmatrix}1\\0\end{pmatrix} + i\left(\sin 4t\begin{pmatrix}-1\\1\end{pmatrix} + \cos 4t\begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$= \begin{pmatrix} -\cos 4t - \sin 4t \\ \cos 4t \end{pmatrix} + i \begin{pmatrix} -\sin 4t + \cos 4t \\ \sin 4t \end{pmatrix}$$

$$y_1(t) = \begin{pmatrix} -\cos 4t - \sin 4t \\ \cos 4t \end{pmatrix} \;\&\; y_2(t) = \begin{pmatrix} -\sin 4t + \cos 4t \\ \sin 4t \end{pmatrix}$$

The general solution is given by: $y(t) = C_1 y_1(t) + C_2 y_2(t)$

$$y(t) = C_1 \begin{pmatrix} -\cos 4t - \sin 4t \\ \cos 4t \end{pmatrix} + C_2 \begin{pmatrix} \cos 4t - \sin 4t \\ \sin 4t \end{pmatrix}$$

$$y(0) = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -C_1 + C_2 \\ C_1 \end{pmatrix} \Rightarrow \underline{C_1 = 2} \quad \underline{C_2 = 2}$$

$$y(t) = 2 \begin{pmatrix} -\cos 4t - \sin 4t \\ \cos 4t \end{pmatrix} + 2 \begin{pmatrix} \cos 4t - \sin 4t \\ \sin 4t \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} -4\sin 4t \\ 2\cos 4t + 2\sin 4t \end{pmatrix}}$$

## Exercise

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = -y_1 - 2y_2 \\ y_2'(t) = 4y_1 + 3y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$

***Solution***

$$A = \begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix} \qquad |A - \lambda I| = \begin{vmatrix} -1-\lambda & -2 \\ 4 & 3-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(3-\lambda) + 8$$

$$= -3 - 2\lambda + \lambda^2 + 8$$

$$= \lambda^2 - 2\lambda + 5 = 0$$

$$\Rightarrow \lambda = 1 \pm 2i$$

For $\lambda = 1 + 2i \quad \Rightarrow (A - \lambda I)V = 0$

$$\begin{pmatrix} -2-2i & -2 \\ 4 & 2-2i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} (-2-2i)x - 2y = 0 \\ 4x + (2-2i)y = 0 \end{cases} \Rightarrow \text{divide by } 2 \begin{cases} -x - ix - y = 0 \\ 2x + y - iy = 0 \end{cases}$$

$$x - ix - iy = 0$$

$$(1-i)x = iy \Rightarrow \frac{i}{i}\frac{1-i}{i} x = y$$

$$\Rightarrow y = -(i+1)x$$

The eigenvector is: $V = (1,\ -1-i)^T$

$$z(t) = e^{(1+2i)t}\begin{pmatrix} 1 \\ -1-i \end{pmatrix}$$
$$= e^t e^{2it}\begin{pmatrix} 1 \\ -1-i \end{pmatrix}$$
$$= e^t(\cos 2t + i\sin 2t)\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right)$$
$$= e^t\left[\cos 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\cos 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix} + i\sin 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]$$
$$= e^t\left[\cos 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] + ie^t\left[\left(\cos 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)\right]$$
$$= e^t\begin{pmatrix} \cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix} + ie^t\begin{pmatrix} \sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix}$$

$$y_1(t) = e^t\begin{pmatrix} \cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix} \qquad \& \qquad y_2(t) = e^t\begin{pmatrix} \sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix}$$

Form a fundamental equation:

$$y(t) = C_1 e^t\begin{pmatrix} \cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix} + C_2 e^t\begin{pmatrix} \sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix}$$
$$y(0) = C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 \\ -C_1 - C_2 \end{pmatrix} \quad \Rightarrow \quad \underline{C_1 = 0} \quad \underline{C_2 = -1}$$
$$\underline{y(t) = -e^t\begin{pmatrix} \sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix}}$$

## *Exercise*

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = 3y_1 - y_2 \\ y_2'(t) = y_1 + y_2 \end{cases} \quad y(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$
$$= (3-\lambda)(1-\lambda) + 1$$
$$= 3 - 4\lambda + \lambda^2 + 1$$

$$= \lambda^2 - 4\lambda + 4 \underline{= 0}$$

Thus, the eigenvalues are: $\lambda_1 = \lambda_2 = 2$

For $\lambda = 2 \quad \Rightarrow (A - 2I)V_1 = 0$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} x - y = 0 \\ x - y = 0 \end{cases} \Rightarrow \underline{x = y}$$

The eigenvector is: $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The solution is: $\underline{y_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}}$

For the second eigenvector $V_2 \quad \Rightarrow (A - 2I)V_2 = V_1$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \rightarrow \begin{cases} x - y = 1 \\ x - y = 1 \end{cases} \Rightarrow \ if\ y = 0 \Rightarrow \underline{x = 1}$$

The eigenvector is: $V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The solution is: $y_2(t) = e^{2t}\left(V_2 + tV_1\right)$

$$\underline{= e^{2t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)}$$

Therefore, the final solution can be written as: $y(t) = C_1 y_1(t) + C_2 y_2(t)$

$$y(t) = C_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$$= e^{2t}\left( C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$$y(0) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} C_1 + C_2 \\ C_1 \end{pmatrix} \quad \rightarrow \begin{cases} C_1 + C_2 = 2 \\ \underline{C_1 = -1} \end{cases} \Rightarrow \underline{C_2 = 2 - C_1 = 3}$$

$$y(t) = e^{2t}\left( -\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$$\underline{= e^{2t}\begin{pmatrix} 2 + 3t \\ -1 + 3t \end{pmatrix}}$$

***Exercise***

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = -3y_1 + y_2 \\ y_2'(t) = -y_1 - y_2 \end{cases} \quad y(0) = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

***Solution***

$$A = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \qquad |A - \lambda I| = \begin{vmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix}$$

$$= (-3-\lambda)(-1-\lambda) + 1$$

$$= 3 + 4\lambda + \lambda^2 + 1$$

$$= \lambda^2 + 4\lambda + 4 = 0$$

Thus, the eigenvalues are: $\lambda_1 = \lambda_2 = -2$

For $\lambda = -2 \quad \Rightarrow (A + 2I)V_1 = 0$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} -x + y = 0 \\ -x + y = 0 \end{cases} \Rightarrow \underline{x = y}$$

The eigenvector is: $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\Rightarrow \underline{y_1(t) = e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

For the second eigenvector $V_2 \quad \Rightarrow (A + 2I)V_2 = V_1$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \to \begin{cases} -x + y = 1 \\ -x + y = 1 \end{cases} \Rightarrow if\ y = 0 \Rightarrow \underline{x = -1}$$

The eigenvector is: $V_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

The solution is: $y_2(t) = e^{-2t}\left(V_2 + tV_1\right)$

$$\underline{= e^{-2t}\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)}$$

Therefore, the final solution can be written as: $y(t) = C_1 y_1(t) + C_2 y_2(t)$

$$y(t) = C_1 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t}\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$$= e^{-2t}\left(C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ 0 \end{pmatrix} + C_2 t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$$y(0) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} C_1 - C_2 \\ C_1 \end{pmatrix} \quad \rightarrow \begin{cases} C_1 - C_2 = 0 \\ \underline{C_1 = -3} \end{cases} \quad \Rightarrow \underline{C_2 = C_1 = -3}$$

$$y(t) = e^{-2t}\left( -3\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 0 \end{pmatrix} - 3t\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$

$$\underline{= e^{-2t}\begin{pmatrix} -3t \\ -3-3t \end{pmatrix}}$$

## *Exercise*

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = 2y_1 + 4y_2 \\ y_2'(t) = -y_1 + 6y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

***Solution***

$$A = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} \qquad |A - \lambda I| = \begin{vmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{vmatrix}$$

$$= (2-\lambda)(6-\lambda) + 4$$

$$= 12 - 2\lambda - 6\lambda + \lambda^2 + 4$$

$$= \lambda^2 - 8\lambda + 16 \underline{= 0}$$

Thus, the eigenvalues are: $\underline{\lambda_1 = \lambda_2 = 4}$

For $\lambda = 4 \quad \Rightarrow (A - 4I)V_1 = 0$

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} -2x + 4y = 0 \\ -x + 2y = 0 \end{cases} \Rightarrow \underline{x = 2y}$$

The eigenvector is: $V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\Rightarrow \quad \underline{y_1(t) = e^{4t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

For the second eigenvector $V_2 \quad \Rightarrow (A - 4I)V_2 = V_1$

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \rightarrow \begin{cases} -2x + 4y = 2 \\ -x + 2y = 1 \end{cases} \Rightarrow \ if \ y = 0 \ \Rightarrow \ \underline{x = -1}$$

The eigenvector is: $V_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

The solution is: $y_2(t) = e^{4t}\left(V_2 + tV_1\right)$

$$\underline{= e^{4t}\left( \begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \end{pmatrix} \right)}$$

Therefore, the final solution can be written as: $y(t)=C_1y_1(t)+C_2y_2(t)$

$$y(t)=C_1e^{-2t}\begin{pmatrix}1\\1\end{pmatrix}+C_2e^{-2t}\left(\begin{pmatrix}-1\\0\end{pmatrix}+t\begin{pmatrix}1\\1\end{pmatrix}\right)$$

$$=e^{4t}\left(C_1\begin{pmatrix}2\\1\end{pmatrix}+C_2\begin{pmatrix}-1\\0\end{pmatrix}+C_2t\begin{pmatrix}2\\1\end{pmatrix}\right)$$

$$y(0)=C_1\begin{pmatrix}2\\1\end{pmatrix}+C_2\begin{pmatrix}-1\\0\end{pmatrix}$$

$$\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}2C_1-C_2\\C_1\end{pmatrix}\quad\to\begin{cases}2C_1-C_2=3\\ \quad C_1=1\end{cases}$$

$$\Rightarrow \boxed{C_2=2C_1-3=-1}$$

$$y(t)=e^{4t}\left(\begin{pmatrix}2\\1\end{pmatrix}-\begin{pmatrix}-1\\0\end{pmatrix}-t\begin{pmatrix}2\\1\end{pmatrix}\right)$$

$$\boxed{=e^{4t}\begin{pmatrix}3-2t\\1-t\end{pmatrix}}$$

## *Exercise*

Find the general solution of the system $y'=Ay$ $\quad\begin{cases}y_1'(t)=-8y_1-10y_2\\ y_2'(t)=5y_1+7y_2\end{cases}\quad y(0)=\begin{pmatrix}3\\1\end{pmatrix}$

***Solution***

$$A=\begin{pmatrix}-8&-10\\5&7\end{pmatrix}\qquad |A-\lambda I|=\begin{vmatrix}-8-\lambda&-10\\5&7-\lambda\end{vmatrix}$$

$$=(-8-\lambda)(7-\lambda)+50$$

$$=-56+8\lambda-7\lambda+\lambda^2+50$$

$$=\lambda^2+\lambda-6=0$$

Thus, the eigenvalues are: $\lambda_1=-3$ *and* $\lambda_2=2$

For $\lambda_1=-3\quad\Rightarrow(A+3I)V_1=0$

$$\begin{pmatrix}-5&-10\\5&10\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\to\begin{cases}-5x-10y=0\\5x+10y=0\end{cases}\quad\Rightarrow 5x=-10y\to \underline{x=-2y}$$

The eigenvector is: $V_1=\begin{pmatrix}-2\\1\end{pmatrix}$

$$\Rightarrow\quad \boxed{y_1(t)=e^{-3t}\begin{pmatrix}-2\\1\end{pmatrix}}$$

For $\lambda_2 = 2 \quad \Rightarrow (A-2I)V_2 = 0$

$$\begin{pmatrix} -10 & -10 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \to \begin{cases} -10x-10y=0 \\ 5x+5y=0 \end{cases} \Rightarrow 5x=-5y \to \underline{x=-y}$$

The eigenvector is: $V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\Rightarrow \quad \underline{y_2(t) = e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

Therefore, the final solution can be written as: $y(t) = C_1 y_1(t) + C_2 y_2(t)$

$$y(t) = C_1 e^{-3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$y(0) = C_1\begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2C_1 - C_2 \\ C_1 + C_2 \end{pmatrix}$$

$$\begin{cases} -2C_1 - C_2 = 3 \\ \underline{C_1 + C_2 = 1} \end{cases} \qquad \to \underline{C_1 = -4}$$

$$-C_1 = 4$$

$$\Rightarrow \underline{C_2 = 1 - C_1 = 5}$$

$$y(t) = -4e^{-3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + 5e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \underline{\begin{pmatrix} 8e^{-3t} - 5e^{2t} \\ -4e^{-3t} + 5e^{2t} \end{pmatrix}}$$

## *Exercise*

Find the general solution of the system $y' = Ay$ $\qquad \begin{cases} y_1'(t) = -3y_1 + 2y_2 \\ y_2'(t) = -3y_1 + 4y_2 \end{cases} \qquad y(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

***Solution***

$$|A-\lambda I| = \begin{vmatrix} -3-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix} \qquad\qquad A = \begin{pmatrix} -3 & 2 \\ -3 & 4 \end{pmatrix}$$

$$= \lambda^2 - \lambda - 6 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -2$ *and* $\lambda_2 = 3$

For $\lambda_1 = -2 \quad \Rightarrow (A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -1 & 2 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = 2y \qquad V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 3 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -6 & 2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to 3x = y \qquad V_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\boxed{y(t) = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t}}$$

$$y(0) = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\begin{cases} 2C_1 + C_2 = 0 \\ C_1 + 3C_2 = 2 \end{cases} \qquad \Delta = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5 \quad \Delta_1 = \begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} = -2 \quad \Delta_2 = \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = 4$$

$$\boxed{C_1 = -\frac{2}{5}, \quad C_2 = \frac{4}{5}}$$

$$y(t) = -\frac{2}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + \frac{4}{5}\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t}$$

$$\boxed{= \begin{pmatrix} -\frac{4}{5} \\ -\frac{2}{5} \end{pmatrix} e^{-2t} + \begin{pmatrix} \frac{4}{5} \\ \frac{12}{5} \end{pmatrix} e^{3t}}$$

## *Exercise*

Find the general solution of the system $y' = Ay \qquad \begin{cases} y_1'(t) = 3y_1 - y_2 \\ y_2'(t) = 5y_1 - 3y_2 \end{cases} \qquad y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix}$$

$$= \lambda^2 - 4 = 0$$

Thus, the eigenvalues are: $\boxed{\lambda_{1,2} = \pm 2}$

For $\lambda_1 = -2 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 5 & -1 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to 5x = y \qquad V_1 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 1 & -1 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = y \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{y(t) = C_1\begin{pmatrix} 5 \\ 1 \end{pmatrix}e^{-2t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t}}$$

$$y(0) = C_1\begin{pmatrix} 5 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} 5C_1 + C_2 = 1 \\ C_1 + C_2 = -1 \end{cases} \qquad \Delta = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} = 4 \quad \Delta_1 = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 \quad \Delta_2 = \begin{vmatrix} 5 & 1 \\ 1 & -1 \end{vmatrix} = -6$$

$$\boxed{C_1 = \frac{1}{2}, \quad C_2 = -\frac{3}{2}}$$

$$y(t) = \frac{1}{2}\begin{pmatrix} 5 \\ 1 \end{pmatrix}e^{-2t} - \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t}$$

$$\boxed{= \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \end{pmatrix}e^{-2t} - \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \end{pmatrix}e^{2t}}$$

## Exercise

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = y_1 + 9y_2 \\ y_2'(t) = -2y_1 - 5y_2 \end{cases} \quad y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

**Solution**

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 9 \\ -2 & -5-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & 9 \\ -2 & -5 \end{pmatrix}$$

$$= \lambda^2 + 4\lambda + 13 = 0$$

Thus, the eigenvalues are: $\boxed{\lambda_{1,2} = -2 \pm 3i}$

For $\lambda_1 = -2 - 3i \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3+3i & 9 \\ -2 & -3+3i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to (1+i)x = -3y \quad V_1 = \begin{pmatrix} -3 \\ 1+i \end{pmatrix}$$

$$z(t) = \begin{pmatrix} -3 \\ 1+i \end{pmatrix}e^{(-2-3i)t}$$

$$= \left(\begin{pmatrix} -3 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)(\cos 3t - i\sin 3t)e^{-2t}$$

$$= \left[\begin{pmatrix} -3 \\ 1 \end{pmatrix}\cos 3t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin 3t + i\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos 3t - \begin{pmatrix} -3 \\ 1 \end{pmatrix}\sin 3t\right)\right]e^{-2t}$$

$$= \left[ \begin{pmatrix} -3\cos 3t \\ \cos 3t + \sin 3t \end{pmatrix} + i \begin{pmatrix} 3\sin 3t \\ \cos 3t - \sin 3t \end{pmatrix} \right] e^{-2t}$$

$$\boxed{y(t) = C_1 \begin{pmatrix} -3\cos 3t \\ \cos 3t + \sin 3t \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} 3\sin 3t \\ \cos 3t - \sin 3t \end{pmatrix} e^{-2t}}$$

$$y(0) = C_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{cases} -3C_1 = 3 \\ C_1 + C_2 = 2 \end{cases} \qquad \boxed{C_1 = -1, \quad C_2 = 3}$$

$$y(t) = \begin{pmatrix} 3\cos 3t \\ -\cos 3t - \sin 3t \end{pmatrix} e^{-2t} + \begin{pmatrix} 9\sin 3t \\ 3\cos 3t - 3\sin 3t \end{pmatrix} e^{-2t}$$

$$\boxed{= \begin{pmatrix} 3\cos 3t + 9\sin 3t \\ 2\cos 3t - 4\text{in}\, 3t \end{pmatrix} e^{-2t}}$$

## *Exercise*

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) = 4y_1 + y_2 \\ y_2'(t) = -2y_1 + y_2 \end{cases}$ $y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}$$

$$= \lambda^2 - 5\lambda + 6 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 2$ & $\lambda_2 = 3$

For $\lambda_1 = 2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to 2x = -y \qquad V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

For $\lambda_2 = 3 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = -y \qquad V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\boxed{y(t) = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t}}$$

$$y(0) = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} C_1 - C_2 = 1 \\ -2C_1 + C_2 = 0 \end{cases} \qquad \boxed{C_1 = -1, \quad C_2 = -2}$$

$$\boxed{\begin{cases} y_1(t) = -e^{2t} + 2e^{3t} \\ y_2(t) = 2e^{2t} - 2e^{3t} \end{cases}}$$

### *Exercise*

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = 2y_1 + y_2 - e^{2t} \\ y_2'(t) = y_1 + 2y_2 \end{cases} \qquad y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \qquad\qquad A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \lambda^2 - 4\lambda + 3 = 0$$

Thus, the eigenvalues are: $\boxed{\lambda_1 = 1 \quad \& \quad \lambda_2 = 3}$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = -y \qquad V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 3 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = y \qquad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{y_h = C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^t + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t}}$$

$$\varphi(t) = \begin{pmatrix} -e^t & e^{3t} \\ e^t & e^{3t} \end{pmatrix}$$

$$\varphi^{-1}(t) = \frac{1}{-2e^{4t}}\begin{pmatrix} e^{3t} & -e^{3t} \\ -e^t & -e^t \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -e^{-t} & e^{-t} \\ e^{-3t} & e^{-3t} \end{pmatrix}$$

$$\varphi^{-1}\cdot\begin{pmatrix} -e^{2t} \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -e^{-t} & e^{-t} \\ e^{-3t} & e^{-3t} \end{pmatrix}\begin{pmatrix} -e^{2t} \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^t \\ -e^{-t} \end{pmatrix}$$

$$X = \frac{1}{2}\int \begin{pmatrix} e^t \\ -e^{-t} \end{pmatrix} dt$$

$$= \frac{1}{2}\begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}$$

$$X_p(t) = \varphi X = \frac{1}{2}\begin{pmatrix} -e^t & e^{3t} \\ e^t & e^{3t} \end{pmatrix}\begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$$

$$y(t) = \begin{pmatrix} -C_1 e^t + C_2 e^{3t} \\ C_1 e^t + C_2 e^{3t} \end{pmatrix} + \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$$

$$y(0) = \begin{pmatrix} -C_1 + C_2 \\ C_1 + C_2 + 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} -C_1 + C_2 = 1 \\ C_1 + C_2 = -2 \end{cases} \quad \rightarrow \quad \underline{C_1 = -\frac{3}{2}, \quad C_2 = -\frac{1}{2}}$$

$$\boxed{\begin{cases} y_1(t) = \frac{3}{2}e^t - \frac{1}{2}e^{3t} \\ y_1(t) = -\frac{3}{2}e^t - \frac{1}{2}e^{3t} + e^{2t} \end{cases}}$$

## Exercise

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) + 2y_2'(t) = 4y_1 + 5y_2 \\ 2y_1'(t) - y_2'(t) = 3y_1 \end{cases}$ $y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

**Solution**

$$\begin{aligned} &y_1'(t) + 2y_2'(t) = 4y_1 + 5y_2 \\ &4y_1'(t) - 2y_2'(t) = 6y_1 \end{aligned} \quad \rightarrow \quad \begin{cases} y_1'(t) = 2y_1 + y_2 \\ y_2'(t) = y_1 + 2y_2 \end{cases}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \lambda^2 - 4\lambda + 3 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 1 \quad and \quad \lambda_2 = 3$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x = -y \qquad V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x = y \qquad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{y(t) = C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t}}$$

$$y(0) = C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} -C_1 + C_2 = 1 \\ C_1 + C_2 = -1 \end{cases} \qquad \boxed{C_1 = -1, \quad C_2 = 0}$$

$$\boxed{y(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{t}}$$

## *Exercise*

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) = 3y_1 - 2y_2 \\ y_2'(t) = 2y_1 - 2y_2 \end{cases}$ $y(0) = \begin{pmatrix} 3 \\ \frac{1}{2} \end{pmatrix}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$$

$$= \lambda^2 - \lambda - 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1$ *and* $\lambda_2 = 2$

For $\lambda_1 = -1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow 2x = y \qquad V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x = 2y \qquad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\boxed{y(t) = C_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-t} + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{2t}}$$

$$y(0)=C_1\begin{pmatrix}1\\2\end{pmatrix}+C_2\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}3\\ \frac{1}{2}\end{pmatrix}$$

$$\begin{cases}C_1+2C_2=3\\ 2C_1+C_2=\frac{1}{2}\end{cases}\qquad \Delta=\begin{vmatrix}1&2\\2&1\end{vmatrix}=-3\quad \Delta_1=\begin{vmatrix}3&2\\ \frac{1}{2}&1\end{vmatrix}=2\quad \Delta_2=\begin{vmatrix}1&3\\2&\frac{1}{2}\end{vmatrix}=-\frac{11}{2}$$

$$\underline{C_1=-\frac{2}{3},\quad C_2=\frac{11}{6}}$$

$$\begin{cases}y_1(t)=-\frac{2}{3}e^{-t}+\frac{11}{3}e^{2t}\\ y_2(t)=-\frac{4}{3}e^{-t}+\frac{11}{6}e^{2t}\end{cases}$$

### *Exercise*

Find the general solution of the system $y'=Ay$ $\begin{cases}y_1'(t)=y_1-2y_2\\ y_2'(t)=3y_1-4y_2\end{cases}$ $y(0)=\begin{pmatrix}-1\\2\end{pmatrix}$

***Solution***

$$|A-\lambda I|=\begin{vmatrix}1-\lambda&-2\\3&-4-\lambda\end{vmatrix}\qquad A=\begin{pmatrix}1&-2\\3&-4\end{pmatrix}$$

$$=\lambda^2+3\lambda+2=0$$

Thus, the eigenvalues are: $\lambda_1=-1$ *and* $\lambda_2=-2$

For $\lambda_1=-1 \quad \Rightarrow (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}2&-2\\3&-3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\to x=y\qquad V_1=\begin{pmatrix}1\\1\end{pmatrix}$$

For $\lambda_2=2 \quad \Rightarrow (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix}3&-2\\3&-2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\to 3x=2y\qquad V_2=\begin{pmatrix}2\\3\end{pmatrix}$$

$$y(t)=C_1\begin{pmatrix}1\\1\end{pmatrix}e^{-t}+C_2\begin{pmatrix}2\\3\end{pmatrix}e^{-2t}$$

$$y(0)=C_1\begin{pmatrix}1\\1\end{pmatrix}+C_2\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}-1\\2\end{pmatrix}$$

$$\begin{cases}C_1+2C_2=-1\\ C_1+3C_2=2\end{cases}\qquad \Delta=\begin{vmatrix}1&2\\1&3\end{vmatrix}=1\quad \Delta_1=\begin{vmatrix}-1&2\\2&3\end{vmatrix}=-7\quad \Delta_2=\begin{vmatrix}1&-1\\1&2\end{vmatrix}=3$$

$$\underline{C_1=-7,\quad C_2=3}$$

$$y(t) = -7\begin{pmatrix}1\\1\end{pmatrix}e^{-t} + 3\begin{pmatrix}2\\3\end{pmatrix}e^{-2t}$$

$$\boxed{\begin{cases} y_1(t) = -7e^{-t} + 6e^{-2t} \\ y_2(t) = -7e^{-t} + 9e^{-2t} \end{cases}}$$

### Exercise

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) = y_1 - 4y_2 \\ y_2'(t) = 4y_1 - 7y_2 \end{cases}$ $y(0) = \begin{pmatrix}3\\2\end{pmatrix}$

**Solution**

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}$$

$$= \lambda^2 + 6\lambda + 9 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = -3$

For $\lambda_1 = -3 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = y \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \boxed{y_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-3t}}$$

For $V_2 \quad \Rightarrow (A - \lambda I)V_2 = V_1$

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \to 4x - 4y = 1 \quad V_2 = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$$

$$y_2(t) = \left(V_2 + tV_1\right)e^{-3t}$$

$$= \boxed{\begin{pmatrix} \frac{1}{4} + t \\ t \end{pmatrix}e^{-3t}}$$

$$y(t) = \left(C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} \frac{1}{4} + t \\ t \end{pmatrix}\right)e^{-3t}$$

$$y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \to \quad C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{cases} C_1 + \frac{1}{4}C_2 = 3 & \boxed{C_2 = 4} \\ \boxed{C_1 = 2} \end{cases}$$

$$y(t) = \left( \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 1+4t \\ 4t \end{pmatrix} \right) e^{-3t}$$

$$\boxed{\begin{cases} y_1(t) = (3+4t)e^{-3t} \\ y_2(t) = (2+4t)e^{-3t} \end{cases}}$$

### Exercise

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = 3y_1 + 9y_2 \\ y_2'(t) = -y_1 - 3y_2 \end{cases} \quad y(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 9 \\ -1 & -3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}$$

$$= \lambda^2 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = 0$

For $\lambda_1 = 0 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = -3y \qquad V_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \boxed{y_1(t) = \begin{pmatrix} -3 \\ 1 \end{pmatrix}}$$

For $V_2 \quad \Rightarrow (A - \lambda I)V_2 = V_1$

$$\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \to x + 3y = -1 \qquad V_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$y_2(t) = V_2 + tV_1$$

$$= \boxed{\begin{pmatrix} -1-3t \\ t \end{pmatrix}}$$

$$y(t) = C_1\begin{pmatrix} -3 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} -1-3t \\ t \end{pmatrix}$$

$$y(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \quad \to \quad C_1\begin{pmatrix} -3 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\begin{cases} -3C_1 - C_2 = 2 & \boxed{C_2 = -14} \\ \boxed{C_1 = 4} \end{cases}$$

$$y(t)=\begin{pmatrix}-12\\4\end{pmatrix}+\begin{pmatrix}14+42t\\-14t\end{pmatrix}$$

$$\boxed{\begin{cases}y_1(t)=2+42t\\y_2(t)=4-14t\end{cases}}$$

### *Exercise*

Find the general solution of the system $y'=Ay$ $\begin{cases}y_1'(t)=2y_1+\frac{3}{2}y_2\\y_2'(t)=-\frac{3}{2}y_1-y_2\end{cases}$ $y(0)=\begin{pmatrix}3\\-2\end{pmatrix}$

***Solution***

$$|A-\lambda I|=\begin{vmatrix}2-\lambda & \frac{3}{2}\\-\frac{3}{2} & -1-\lambda\end{vmatrix} \qquad A=\begin{pmatrix}2 & \frac{3}{2}\\-\frac{3}{2} & -1\end{pmatrix}$$

$$=\lambda^2-\lambda+\frac{1}{4}=0 \quad\rightarrow\quad \left(\lambda-\frac{1}{2}\right)^2=0$$

Thus, the eigenvalues are: $\lambda_{1,2}=\frac{1}{2}$

For $\lambda_1=\frac{1}{2}$ $\Rightarrow\left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix}\frac{3}{2} & \frac{3}{2}\\-\frac{3}{2} & -\frac{3}{2}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \quad\rightarrow x=-y \qquad V_1=\begin{pmatrix}-1\\1\end{pmatrix}$$

$$\Rightarrow \boxed{y_1(t)=\begin{pmatrix}-1\\1\end{pmatrix}e^{t/2}}$$

For $V_2$ $\Rightarrow(A-\lambda I)V_2=V_1$

$$\begin{pmatrix}\frac{3}{2} & \frac{3}{2}\\-\frac{3}{2} & -\frac{3}{2}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix} \quad\rightarrow \frac{3}{2}x+\frac{3}{2}y=-1 \qquad V_2=\begin{pmatrix}-\frac{2}{3}\\0\end{pmatrix}$$

$$y_2(t)=\left(V_2+tV_1\right)e^{t/2}$$

$$=\boxed{\begin{pmatrix}-\frac{2}{3}-t\\t\end{pmatrix}e^{t/2}}$$

$$y(t)=\left(C_1\begin{pmatrix}-1\\1\end{pmatrix}+C_2\begin{pmatrix}-\frac{2}{3}-t\\t\end{pmatrix}\right)e^{t/2}$$

$$y(0)=\begin{pmatrix}3\\-2\end{pmatrix} \quad \rightarrow \quad C_1\begin{pmatrix}-1\\1\end{pmatrix}+C_2\begin{pmatrix}-\frac{2}{3}\\0\end{pmatrix}=\begin{pmatrix}3\\-2\end{pmatrix}$$

$$\begin{cases}-C_1-\frac{2}{3}C_2=3 & \underline{C_2=-\frac{3}{2}|}\\ \quad C_1=-2 & \end{cases}$$

$$\underline{C_1=-2|}$$

$$y(t)=\left(\begin{pmatrix}2\\-2\end{pmatrix}+\begin{pmatrix}1+\frac{3}{2}t\\-\frac{3}{2}t\end{pmatrix}\right)e^{t/2}$$

$$\begin{cases}y_1(t)=\left(3+\frac{3}{2}t\right)e^{t/2}\\ y_2(t)=-\left(2+\frac{3}{2}t\right)e^{t/2}\end{cases}$$

### Exercise

Find the general solution of the system $y'=Ay$ $\begin{cases}y_1'(t)=-5y_1+12y_2\\ y_2'(t)=-2y_1+5y_2\end{cases}$ $\quad y(0)=\begin{pmatrix}8\\3\end{pmatrix}$

***Solution***

$$|A-\lambda I|=\begin{vmatrix}-5-\lambda & 12\\ -2 & 5-\lambda\end{vmatrix} \qquad A=\begin{pmatrix}-5 & 12\\ -2 & 5\end{pmatrix}$$

$$=\lambda^2-1=0$$

Thus, the eigenvalues are: $\lambda_{1,2}=\pm 1$

For $\lambda_1=-1 \quad \Rightarrow (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}-4 & 12\\ -2 & 6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \quad \rightarrow x=3y \qquad V_1=\begin{pmatrix}3\\1\end{pmatrix}$$

For $\lambda_2=1 \quad \Rightarrow (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix}-6 & 12\\ -2 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \quad \rightarrow x=2y \qquad V_1=\begin{pmatrix}2\\1\end{pmatrix}$$

$$y(t)=C_1\begin{pmatrix}3\\1\end{pmatrix}e^{-t}+C_2\begin{pmatrix}2\\1\end{pmatrix}e^{t}$$

$$y(0)=\begin{pmatrix}8\\3\end{pmatrix} \quad \rightarrow \quad C_1\begin{pmatrix}3\\1\end{pmatrix}+C_2\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}8\\3\end{pmatrix}$$

$$\begin{cases}3C_1+2C_2=8\\ C_1+C_2=3\end{cases} \rightarrow \underline{C_1=2,\ C_2=1|}$$

$$y(t) = \begin{pmatrix} 6 \\ 2 \end{pmatrix} e^{-t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t}$$

$$\boxed{\begin{cases} y_1(t) = 6e^{-t} + 2e^{t} \\ y_2(t) = 2e^{-t} + e^{t} \end{cases}}$$

### Exercise

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = -4y_1 + 6y_2 \\ y_2'(t) = -3y_1 + 5y_2 \end{cases} \qquad y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

**Solution**

$$|A - \lambda I| = \begin{vmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{vmatrix} \qquad\qquad A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$$

$$= \lambda^2 - \lambda - 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1$ & $\lambda_2 = 2$

For $\lambda_1 = -1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x = 2y \quad V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x = y \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$y(t) = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

$$y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \rightarrow \quad C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{cases} 2C_1 + C_2 = 3 \\ C_1 + C_2 = 2 \end{cases} \quad \rightarrow \quad \underline{C_1 = 1, \ C_2 = 1|}$$

$$y(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

$$\boxed{\begin{cases} y_1(t) = 2e^{-t} + e^{2t} \\ y_2(t) = e^{-t} + e^{2t} \end{cases}}$$

### *Exercise*

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = y_1 + 2y_2 \\ y_2'(t) = 3y_1 + 2y_2 \end{cases} \quad y(0) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} \qquad\qquad A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

$$= \lambda^2 - 3\lambda - 4 = 0$$

Thus, the eigenvalues are: $\underline{\lambda_{1,2} = -1,\ 4|}$

For $\lambda_1 = -1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = -y \quad V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 4 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to 3x = 2y \quad V_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\underline{y(t) = C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{-t} + C_2\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{4t}|}$$

$$y(0) = C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

$$\begin{cases} -C_1 + 2C_2 = 0 \\ C_1 + 3C_2 = -4 \end{cases} \qquad \Delta = \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} = -5 \quad \Delta_1 = \begin{vmatrix} 0 & 2 \\ -4 & 3 \end{vmatrix} = 8 \quad \Delta_2 = \begin{vmatrix} -1 & 0 \\ 1 & -4 \end{vmatrix} = 4$$

$$\underline{C_1 = -\tfrac{8}{5}, \quad C_2 = -\tfrac{4}{5}|}$$

$$y(t) = -\frac{8}{5}\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{-t} - \frac{4}{5}\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{4t}$$

$$\underline{\begin{cases} y_1(t) = \frac{8}{5}e^{-t} - \frac{8}{5}e^{4t} \\ y_2(t) = -\frac{8}{5}e^{-t} - \frac{12}{5}e^{4t} \end{cases}|}$$

### *Exercise*

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = -5y_1 + y_2 \\ y_2'(t) = 4y_1 - 2y_2 \end{cases} \quad y(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

***Solution***

$$|A-\lambda I| = \begin{vmatrix} -5-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix}$$

$$= \lambda^2 + 7\lambda + 6 = 0$$

Thus, the eigenvalues are: $\boxed{\lambda_{1,2} = -1,\ -6}$

For $\lambda_1 = -1 \quad \Rightarrow (A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -4 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \ 4x = y \qquad V_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

For $\lambda_2 = -6 \quad \Rightarrow (A-\lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \ x = -y \qquad V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\boxed{y(t) = C_1\begin{pmatrix} 1 \\ 4 \end{pmatrix}e^{-t} + C_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{-6t}}$$

$$y(0) = C_1\begin{pmatrix} 1 \\ 4 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{cases} C_1 - C_2 = 1 \\ 4C_1 + C_2 = 2 \end{cases} \quad \to \ \boxed{C_1 = \frac{3}{5}, \quad C_2 = -\frac{2}{5}}$$

$$y(t) = \frac{3}{5}\begin{pmatrix} 1 \\ 4 \end{pmatrix}e^{-t} - \frac{2}{5}\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{-6t}$$

$$\boxed{\begin{cases} y_1(t) = \frac{3}{5}e^{-t} + \frac{2}{5}e^{-6t} \\ y_2(t) = \frac{12}{5}e^{-t} - \frac{2}{5}e^{-6t} \end{cases}}$$

## *Exercise*

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = 3y_1 - 9y_2 \\ y_2'(t) = 4y_1 - 3y_2 \end{cases} \qquad y(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

***Solution***

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & -9 \\ 4 & -3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & -9 \\ 4 & -3 \end{pmatrix}$$

$$= \lambda^2 + 27 = 0$$

Thus, the eigenvalues are: $\boxed{\lambda_{1,2} = \pm 3\sqrt{3}i}$

For $\lambda_1 = 3\sqrt{3}\,i \quad \Rightarrow (A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3-3\sqrt{3}\,i & -9 \\ 4 & -3-3\sqrt{3}\,i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{\left(1-\sqrt{3}\,i\right)x = 3y}$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 3 \\ 1-\sqrt{3}\,i \end{pmatrix}$$

$$y(t) = \begin{pmatrix} 3 \\ 1-\sqrt{3}\,i \end{pmatrix} e^{3\sqrt{3}\,it}$$

$$= \begin{pmatrix} 3 \\ 1-\sqrt{3}\,i \end{pmatrix}\left(\cos\left(3\sqrt{3}\,t\right) + i\sin\left(3\sqrt{3}\,t\right)\right)$$

$$= \begin{pmatrix} 3\cos 3\sqrt{3}t + 3i\sin 3\sqrt{3}t \\ \cos 3\sqrt{3}t + \sqrt{3}\sin 3\sqrt{3}t + i\left(\sin 3\sqrt{3}t - \sqrt{3}\cos 3\sqrt{3}t\right) \end{pmatrix}$$

$$\begin{cases} y_1(t) = 3C_1\cos 3\sqrt{3}t + 3C_2\sin 3\sqrt{3}t \\ y_2(t) = C_1\left(\cos 3\sqrt{3}t + \sqrt{3}\sin 3\sqrt{3}t\right) + C_2\left(\sin 3\sqrt{3}t - \sqrt{3}\cos 3\sqrt{3}t\right) \end{cases}$$

***Given***: $y_1(0) = 2, \quad y_2(0) = -4$

$$\begin{cases} y_1(0) = 3C_1 = 2 & \rightarrow \underline{C_1 = \frac{2}{3}} \\ y_2(0) = C_1 - \sqrt{3}C_2 = -4 & \rightarrow \underline{C_2 = \frac{14}{3\sqrt{3}}} \end{cases}$$

$$\begin{cases} y_1(t) = 2\cos 3\sqrt{3}t + \frac{14}{\sqrt{3}}\sin 3\sqrt{3}t \\ y_2(t) = \frac{2}{3}\cos 3\sqrt{3}t + \frac{2\sqrt{3}}{3}\sin 3\sqrt{3}t + \frac{14}{\sqrt{3}}\sin 3\sqrt{3}t - 14\cos 3\sqrt{3}t \end{cases}$$

$$\underline{\begin{cases} y_1(t) = 2\cos 3\sqrt{3}t + \frac{14}{\sqrt{3}}\sin 3\sqrt{3}t \\ y_2(t) = \frac{16\sqrt{3}}{3}\sin 3\sqrt{3}t - 40\cos 3\sqrt{3}t \end{cases}}$$

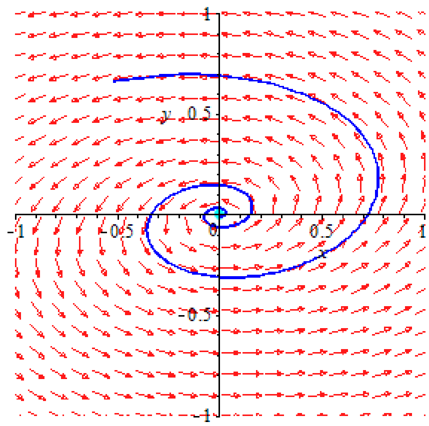
## *Exercise*

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = 3y_1 - 13y_2 \\ y_2'(t) = 5y_1 + y_2 \end{cases} \quad y(0) = \begin{pmatrix} 3 \\ -10 \end{pmatrix}$

***Solution***

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & -13 \\ 5 & 1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix}$$

$$= \lambda^2 - 4\lambda + 68 = 0$$



Thus, the eigenvalues are: $\boxed{\lambda_{1,2} = 2 \pm 8i}$

For $\lambda_1 = 2 + 8i \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1-8i & -13 \\ 5 & -1-8i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \boxed{(1-8i)x = 13y}$$

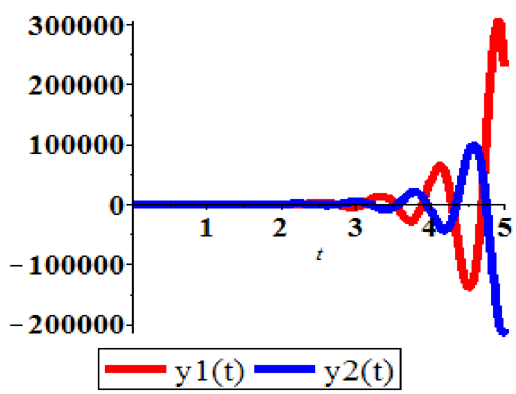
$$\Rightarrow \quad V_1 = \begin{pmatrix} 13 \\ 1-8i \end{pmatrix}$$

$$y(t) = \begin{pmatrix} 13 \\ 1-8i \end{pmatrix} e^{(2+8i)t}$$

$$= \begin{pmatrix} 13 \\ 1-8i \end{pmatrix} (\cos 8t + i \sin 8t) e^{2t}$$

$$= \begin{pmatrix} 13\cos 8t + 13i \sin 8t \\ \cos 8t + 8\sin 8t + i(\sin 8t - 8\cos 8t) \end{pmatrix} e^{2t}$$

$$\begin{cases} y_1(t) = (13C_1 \cos 8t + 13C_2 \sin 8t) e^{2t} \\ y_2(t) = \left(C_1(\cos 8t + 8\sin 8t) + C_2(\sin 8t - 8\cos 8t)\right) e^{2t} \end{cases}$$

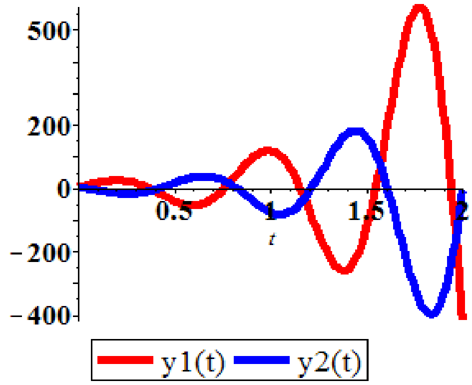



***Given***: $y_1(0) = 3, \quad y_2(0) = -10$

$$\begin{cases} y_1(0) = 13C_1 = 3 & \rightarrow \boxed{C_1 = \frac{3}{13}} \\ y_2(0) = C_1 - 8C_2 = -10 & \rightarrow \boxed{C_2 = \frac{133}{104}} \end{cases}$$

$$\begin{cases} y_1(t) = \left(3\cos 8t + \frac{133}{8}\sin 8t\right) e^{2t} \\ y_2(t) = \left(\frac{3}{13}\cos 8t + \frac{24}{13}\sin 8t + \frac{133}{104}\sin 8t - \frac{133}{13}\cos 8t\right) e^{2t} \end{cases}$$

$$\begin{cases} y_1(t) = \left(3\cos 8t + \frac{133}{8}\sin 8t\right)e^{2t} \\ y_2(t) = \left(\frac{325}{104}\sin 8t - 10\cos 8t\right)e^{2t} \end{cases}$$




## Exercise

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) = 7y_1 + y_2 \\ y_2'(t) = -4y_1 + 3y_2 \end{cases}$ $y(0) = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$

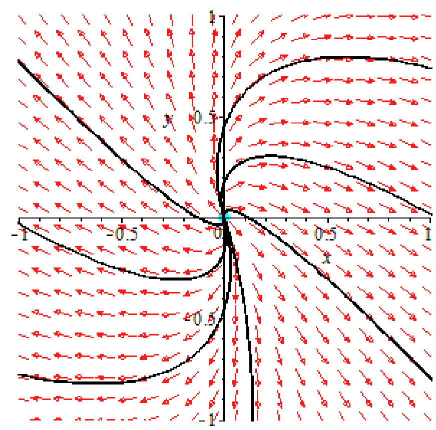
**Solution**

$$|A - \lambda I| = \begin{vmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}$$

$$= \lambda^2 - 10\lambda + 25 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = 5$

For $\lambda_1 = 5 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$



$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad 2x = -y$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \to \quad y_1(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{5t}$$

For $V_2 \quad \Rightarrow (A - \lambda I)V_2 = V_1$

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \to 2x + y = 1 \quad V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$y_2(t) = V_2 + tV_1$$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+t \\ -1-2t \end{pmatrix}e^{5t}$$

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$= C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} 1+t \\ -1-2t \end{pmatrix} e^{5t}$$

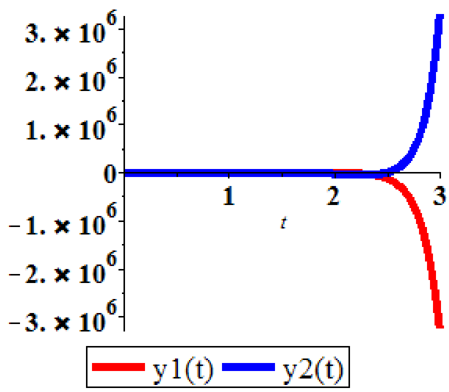
$$y(0) = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

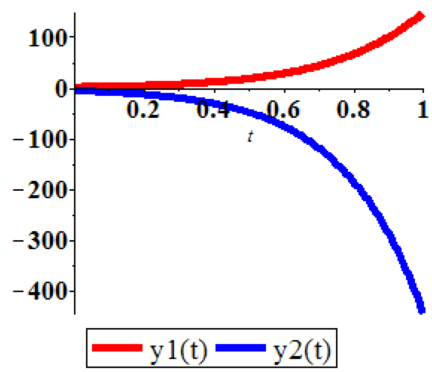
$$\begin{cases} C_1 + C_2 = 2 \\ -2C_1 - C_2 = -5 \end{cases} \quad \rightarrow \quad \boxed{C_1 = 3, \ \ C_2 = -1}$$

$$y(t) = \left( \begin{pmatrix} 3 \\ -6 \end{pmatrix} + \begin{pmatrix} -1-t \\ 1+2t \end{pmatrix} \right) e^{5t}$$

$$\boxed{= \begin{pmatrix} 2-t \\ -5+2t \end{pmatrix} e^{5t}}$$

$$\boxed{\begin{cases} y_1(t) = (2-t)e^{5t} \\ y_2(t) = (-5+2t)e^{5t} \end{cases}}$$




## *Exercise*

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = -y_1 + \frac{3}{2} y_2 \\ y_2'(t) = -\frac{1}{6} y_1 - 2y_2 \end{cases} \quad y(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & \frac{3}{2} \\ -\frac{1}{6} & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} -1 & \frac{3}{2} \\ -\frac{1}{6} & -2 \end{pmatrix}$$

$$= \lambda^2 + 3\lambda + \frac{9}{4} = \left(\lambda + \frac{3}{2}\right)^2 = 0$$

Thus, the eigenvalues are: $\boxed{\lambda_{1,2} = -\frac{3}{2}}$

For $\lambda_1 = -\frac{3}{2} \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = -3y}$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \rightarrow \quad y_1(t) = \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-3t/2}$$

For $V_2 \quad \Rightarrow (A - \lambda I)V_2 = V_1$

$$\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \rightarrow x + 3y = -6 \quad V_2 = \begin{pmatrix} -9 \\ 1 \end{pmatrix}$$

$$y_2(t) = V_2 + tV_1$$

$$= \begin{pmatrix} -9 \\ 1 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -9 - 3t \\ 1 + t \end{pmatrix} e^{-3t/2}$$

$$y(t) = C_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-3t/2} + C_2 \begin{pmatrix} -9 - 3t \\ 1 + t \end{pmatrix} e^{-3t/2} \qquad y(t) = C_1 y_1(t) + C_2 y_2(t)$$

***Given***: $y(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$y(2) = C_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-3} + C_2 \begin{pmatrix} -15 \\ 3 \end{pmatrix} e^{-3} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} \left(-3C_1 - 15C_2\right)e^{-3} = 1 \\ \left(C_1 + 3C_2\right)e^{-3} = 0 \end{cases} \rightarrow \begin{cases} 3C_1 + 15C_2 = -e^3 \\ C_1 + 3C_2 = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 3 & 15 \\ 1 & 3 \end{vmatrix} = -6 \quad \Delta_1 = \begin{vmatrix} -e^3 & 15 \\ 0 & 3 \end{vmatrix} = -3e^3 \quad \Delta_2 = \begin{vmatrix} 3 & -e^3 \\ 1 & 0 \end{vmatrix} = e^3$$

$$\rightarrow \underline{C_1 = \frac{-3e^3}{-6} = \frac{1}{2}e^3, \ C_2 = -\frac{e^3}{6}}$$

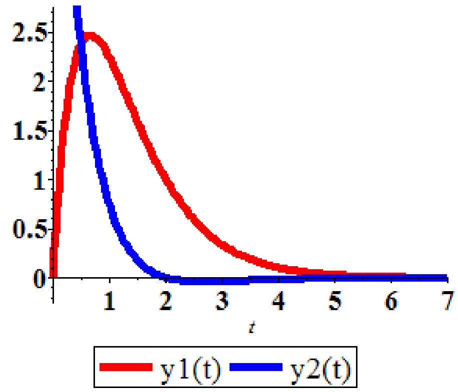
$$y(t) = \frac{1}{2}e^3 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-3t/2} - \frac{1}{6}e^3 \begin{pmatrix} -9 - 3t \\ 1 + t \end{pmatrix} e^{-3t/2}$$

$$= \begin{pmatrix} -\frac{3}{2} + \frac{3}{2} + \frac{1}{2}t \\ \frac{1}{2} - \frac{1}{6} - \frac{1}{6}t \end{pmatrix} e^{-\frac{3t}{2} + 3}$$

$$= \begin{pmatrix} \frac{1}{2}t \\ \frac{1}{3} - \frac{1}{6}t \end{pmatrix} e^{-\frac{3t}{2}+3}$$

$$= \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix} e^{-\frac{3t}{2}+3} + t \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{6} \end{pmatrix} e^{-\frac{3t}{2}+3}$$

$$\begin{cases} y_1(t) = \frac{1}{2} t e^{-\frac{3t}{2}+3} \\ y_2(t) = \left(\frac{1}{3} - \frac{1}{6}t\right) e^{-\frac{3t}{2}+3} \end{cases}$$




## Exercise

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) = 3y_1 - 3y_2 + 2 \\ y_2'(t) = -6y_1 - t \end{cases}$ $y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -3 \\ -6 & -\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & -3 \\ -6 & 0 \end{pmatrix}$$

$$= \lambda^2 - 3\lambda - 18 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = -3,\ 6$

For $\lambda_1 = -3 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 6 & -3 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad 2x = y \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 6 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -3 & -3 \\ -6 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad x = -y \quad \Rightarrow \quad V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$y_h = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-3t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{6t}$$

$$\varphi(t) = \begin{pmatrix} e^{-3t} & -e^{6t} \\ 2e^{-3t} & e^{6t} \end{pmatrix}$$

$$\varphi^{-1}(t) = \frac{1}{3e^{3t}} \begin{pmatrix} e^{6t} & e^{6t} \\ -2e^{-3t} & e^{-3t} \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} e^{3t} & e^{3t} \\ -2e^{-6t} & e^{-6t} \end{pmatrix}$$

$$\varphi^{-1}\cdot\begin{pmatrix}2\\-t\end{pmatrix}=\frac{1}{3}\begin{pmatrix}e^{3t}&e^{3t}\\-2e^{-6t}&e^{-6t}\end{pmatrix}\begin{pmatrix}2\\-t\end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix}2e^{3t}-te^{3t}\\-4e^{-6t}-te^{-6t}\end{pmatrix}$$

| | | $\int e^{3t}$ |
|---|---|---|
| + | $t$ | $\frac{1}{3}e^{3t}$ |
| − | 1 | $\frac{1}{9}e^{3t}$ |

| | | $\int e^{-6t}$ |
|---|---|---|
| + | $t$ | $-\frac{1}{6}e^{-6t}$ |
| − | 1 | $\frac{1}{36}e^{-6t}$ |

$$X=\frac{1}{3}\int\begin{pmatrix}2e^{3t}-te^{3t}\\-4e^{-6t}-te^{-6t}\end{pmatrix}dt$$

$$=\frac{1}{3}\begin{pmatrix}\frac{2}{3}e^{3t}-\frac{1}{3}te^{3t}+\frac{1}{9}e^{3t}\\\frac{2}{3}e^{-6t}+\frac{1}{6}te^{-6t}+\frac{1}{36}e^{-6t}\end{pmatrix}$$

$$i_p(t)=\varphi X=\frac{1}{3}\begin{pmatrix}e^{-3t}&-e^{6t}\\2e^{-3t}&e^{6t}\end{pmatrix}\begin{pmatrix}\frac{2}{3}e^{3t}-\frac{1}{3}te^{3t}+\frac{1}{9}e^{3t}\\\frac{2}{3}e^{-6t}+\frac{1}{6}te^{-6t}+\frac{1}{36}e^{-6t}\end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix}\frac{2}{3}-\frac{1}{3}t+\frac{1}{9}-\frac{2}{3}-\frac{1}{6}t-\frac{1}{36}\\\frac{4}{3}-\frac{2}{3}t+\frac{2}{9}+\frac{2}{3}+\frac{1}{6}t+\frac{1}{36}\end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix}-\frac{1}{2}t+\frac{1}{12}\\-\frac{1}{2}t+\frac{81}{36}\end{pmatrix}$$

$$y(t)=\begin{pmatrix}C_1e^{-3t}-C_2e^{6t}\\2C_1e^{-3t}+C_2e^{6t}\end{pmatrix}+\begin{pmatrix}-\frac{1}{6}t+\frac{1}{36}\\-\frac{1}{6}t+\frac{81}{108}\end{pmatrix}$$

$$=\begin{pmatrix}C_1e^{-3t}-C_2e^{6t}-\frac{1}{6}t+\frac{1}{36}\\2C_1e^{-3t}+C_2e^{6t}-\frac{1}{6}t+\frac{81}{108}\end{pmatrix}$$

***Given***: $y(0)=\begin{pmatrix}1\\-1\end{pmatrix}$

$$y(0)=\begin{pmatrix}C_1-C_2+\frac{1}{36}\\2C_1+C_2+\frac{81}{108}\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\begin{cases}C_1-C_2+\frac{1}{36}=1\\2C_1+C_2+\frac{81}{108}=-1\end{cases}\rightarrow\begin{cases}C_1-C_2=\frac{35}{36}\\2C_1+C_2=-\frac{189}{108}\end{cases}$$

$$\Delta=\begin{vmatrix}1&-1\\2&1\end{vmatrix}=3\quad\Delta_1=\begin{vmatrix}\frac{35}{36}&-1\\-\frac{189}{108}&1\end{vmatrix}=-\frac{84}{108}\quad\Delta_2=\begin{vmatrix}1&\frac{35}{36}\\2&-\frac{189}{108}\end{vmatrix}=-\frac{133}{36}$$

$$\boxed{C_1 = -\frac{7}{27} \quad C_2 = -\frac{133}{108}}$$

$$y(t) = \begin{pmatrix} -\frac{7}{27}e^{-3t} - \frac{133}{108}e^{6t} - \frac{1}{6}t + \frac{1}{36} \\ -\frac{14}{27}e^{-3t} + \frac{133}{108}e^{6t} - \frac{1}{6}t + \frac{81}{108} \end{pmatrix}$$

$$\boxed{\begin{cases} y_1(t) = -\frac{7}{27}e^{-3t} - \frac{133}{108}e^{6t} - \frac{1}{6}t + \frac{1}{36} \\ y_2(t) = -\frac{14}{27}e^{-3t} + \frac{133}{108}e^{6t} - \frac{1}{6}t + \frac{81}{108} \end{cases}}$$

## *Exercise*

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = -5y_1 + y_2 + 6e^{2t} \\ y_2'(t) = 4y_1 - 2y_2 - e^{2t} \end{cases} \quad y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} -5-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix}$$

$$= \lambda^2 + 7\lambda + 6 = 0$$

Thus, the eigenvalues are: $\boxed{\lambda_{1,2} = -1, \ -6}$

For $\lambda_1 = -1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -4 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \boxed{4x = y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

For $\lambda_2 = -6 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \boxed{x = -y} \quad \Rightarrow \quad V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\boxed{y_h = C_1\begin{pmatrix} 1 \\ 4 \end{pmatrix}e^{-t} + C_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{-6t}}$$

$$\varphi(t) = \begin{pmatrix} e^{-t} & -e^{-6t} \\ 4e^{-t} & e^{-6t} \end{pmatrix}$$

$$\varphi^{-1}(t) = \frac{1}{5e^{-7t}}\begin{pmatrix} e^{-6t} & e^{-6t} \\ -4e^{-t} & e^{-t} \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} e^{t} & e^{t} \\ -4e^{6t} & e^{6t} \end{pmatrix}$$

$$\varphi^{-1}\cdot\begin{pmatrix}6e^{2t}\\ -e^{2t}\end{pmatrix}=\frac{1}{5}\begin{pmatrix}e^{t} & e^{t}\\ -4e^{6t} & e^{6t}\end{pmatrix}\begin{pmatrix}6e^{2t}\\ -e^{2t}\end{pmatrix}$$

$$=\frac{1}{5}\begin{pmatrix}5e^{3t}\\ -25e^{8t}\end{pmatrix}$$

$$=\begin{pmatrix}e^{3t}\\ -5e^{8t}\end{pmatrix}$$

$$X=\int\begin{pmatrix}e^{3t}\\ -5e^{8t}\end{pmatrix}dt$$

$$=\begin{pmatrix}\frac{1}{3}e^{3t}\\ -\frac{5}{8}e^{8t}\end{pmatrix}$$

$$i_p(t)=\varphi X=\begin{pmatrix}e^{-t} & -e^{-6t}\\ 4e^{-t} & e^{-6t}\end{pmatrix}\begin{pmatrix}\frac{1}{3}e^{3t}\\ -\frac{5}{8}e^{8t}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{23}{24}e^{2t}\\ \frac{17}{24}e^{2t}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{23}{24}\\ \frac{17}{24}\end{pmatrix}e^{2t}$$

$$\boxed{y(t)=C_1\begin{pmatrix}1\\ 4\end{pmatrix}e^{-t}+C_2\begin{pmatrix}-1\\ 1\end{pmatrix}e^{-6t}+\begin{pmatrix}\frac{23}{24}\\ \frac{17}{24}\end{pmatrix}e^{2t}}$$

***Given***: $y(0)=\begin{pmatrix}1\\ -1\end{pmatrix}$

$$y(0)=\begin{pmatrix}C_1-C_2+\frac{23}{24}\\ 4C_1+C_2+\frac{17}{24}\end{pmatrix}=\begin{pmatrix}1\\ -1\end{pmatrix}$$

$$\begin{cases}C_1-C_2+\frac{23}{24}=1\\ 4C_1+C_2+\frac{17}{24}=-1\end{cases}\quad\rightarrow\quad\begin{cases}C_1-C_2=\frac{1}{24}\\ 4C_1+C_2=-\frac{41}{24}\end{cases}$$

$$\Delta=\begin{vmatrix}1 & -1\\ 4 & 1\end{vmatrix}=5\qquad\Delta_1=\begin{vmatrix}\frac{1}{24} & -1\\ -\frac{41}{24} & 1\end{vmatrix}=-\frac{5}{3}\qquad\Delta_2=\begin{vmatrix}1 & \frac{1}{24}\\ 4 & -\frac{41}{24}\end{vmatrix}=-\frac{15}{8}$$

$$\boxed{C_1 = -\frac{1}{3} \quad C_2 = -\frac{3}{8}}$$

$$y(t) = -\frac{1}{3}\begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-t} - \frac{3}{8}\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-6t} + \begin{pmatrix} \frac{23}{24} \\ \frac{17}{24} \end{pmatrix} e^{2t}$$

$$\boxed{\begin{cases} y_1(t) = -\frac{1}{3}e^{-t} + \frac{3}{8}e^{-6t} + \frac{23}{24}e^{2t} \\ y_2(t) = -\frac{4}{3}e^{-t} - \frac{3}{8}e^{-6t} + \frac{17}{24}e^{2t} \end{cases}}$$

## *Exercise*

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = y_1 + 2y_2 + 2t \\ y_2'(t) = 3y_1 + 2y_2 - 4t \end{cases} \quad y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

$$= \lambda^2 - 3\lambda - 4 = 0$$

Thus, the eigenvalues are: $\boxed{\lambda_{1,2} = -1,\ 4}$

For $\lambda_1 = -1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \boxed{x = -y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 4 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \boxed{3x = 2y} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\boxed{y_h = C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + C_2\begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t}}$$

$$\varphi(t) = \begin{pmatrix} -e^{-t} & 2e^{4t} \\ e^{-t} & 3e^{4t} \end{pmatrix}$$

$$\varphi^{-1}(t) = -\frac{1}{5e^{3t}}\begin{pmatrix} 3e^{4t} & -2e^{4t} \\ -e^{-t} & -e^{-t} \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} -3e^{t} & 2e^{t} \\ e^{-4t} & e^{-4t} \end{pmatrix}$$

$$\varphi^{-1}\cdot\begin{pmatrix}2t\\-4t\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-3e^{t} & 2e^{t}\\ e^{-4t} & e^{-4t}\end{pmatrix}\begin{pmatrix}2t\\-4t\end{pmatrix}$$

$$=\frac{1}{5}\begin{pmatrix}-14te^{t}\\-2te^{-4t}\end{pmatrix}$$

$$=-\frac{2}{5}\begin{pmatrix}7te^{t}\\te^{-4t}\end{pmatrix}$$

| | | $\int e^{t}$ |
|---|---|---|
| + | $t$ | $e^{t}$ |
| − | 1 | $e^{t}$ |

$$X=-\frac{2}{5}\int\begin{pmatrix}7te^{t}\\te^{-4t}\end{pmatrix}dt$$

| | | $\int e^{-4t}$ |
|---|---|---|
| + | $t$ | $-\frac{1}{4}e^{-4t}$ |
| − | 1 | $\frac{1}{16}e^{-4t}$ |

$$=-\frac{2}{5}\begin{pmatrix}7\left(te^{t}-1\right)e^{t}\\-\left(\frac{1}{4}t+\frac{1}{16}\right)e^{-4t}\end{pmatrix}$$

$$i_p(t)=\varphi X=-\frac{2}{5}\begin{pmatrix}-e^{-t} & 2e^{4t}\\ e^{-t} & 3e^{4t}\end{pmatrix}\begin{pmatrix}7\left(te^{t}-1\right)e^{t}\\-\left(\frac{1}{4}t+\frac{1}{16}\right)e^{-4t}\end{pmatrix}$$

$$=-\frac{2}{5}\begin{pmatrix}-7t+7-\frac{1}{2}t-\frac{1}{8}\\7t-7-\frac{3}{4}t-\frac{3}{16}\end{pmatrix}$$

$$=-\frac{2}{5}\begin{pmatrix}-\frac{15}{2}t+\frac{55}{8}\\\frac{25}{4}t-\frac{115}{16}\end{pmatrix}$$

$$=\begin{pmatrix}3t-\frac{11}{4}\\-\frac{5}{2}t+\frac{23}{8}\end{pmatrix}$$

$$y(t)=C_1\begin{pmatrix}-1\\1\end{pmatrix}e^{-t}+C_2\begin{pmatrix}2\\3\end{pmatrix}e^{4t}+\begin{pmatrix}3t-\frac{11}{4}\\-\frac{5}{2}t+\frac{23}{8}\end{pmatrix}$$

$$y(0)=\begin{pmatrix}1\\1\end{pmatrix}$$

$$y(0)=C_1\begin{pmatrix}-1\\1\end{pmatrix}+C_2\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}-\frac{11}{4}\\\frac{23}{8}\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{cases}-C_1+2C_2-\frac{11}{4}=1\\C_1+3C_2+\frac{23}{8}=1\end{cases}\rightarrow\begin{cases}-C_1+2C_2=\frac{15}{4}\\C_1+3C_2=-\frac{15}{8}\end{cases}$$

$$\Delta = \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} = -5 \quad \Delta_1 = \begin{vmatrix} \frac{15}{4} & 2 \\ -\frac{15}{8} & 3 \end{vmatrix} = 15 \quad \Delta_2 = \begin{vmatrix} -1 & \frac{15}{4} \\ 1 & -\frac{15}{8} \end{vmatrix} = -\frac{15}{8}$$

$$\boxed{C_1 = -3 \quad C_2 = \frac{3}{8}}$$

$$y(t) = -3\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + \frac{3}{8}\begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t} + \begin{pmatrix} 3t - \frac{11}{4} \\ -\frac{5}{2}t + \frac{23}{8} \end{pmatrix}$$

$$\boxed{\begin{cases} y_1(t) = -3e^{-t} + \frac{3}{4}e^{4t} + 3t - \frac{11}{4} \\ y_2(t) = 3e^{-t} + \frac{9}{8}e^{4t} - \frac{5}{2}t + \frac{23}{8} \end{cases}}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = 3x_1 - x_2 + 4e^{2t} \\ x_2'(t) = -x_1 + 3x_2 + 4e^{4t} \end{cases}$ $\quad X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

$$= \lambda^2 - 6\lambda + 8 = 0$$

The eigenvalues: $\boxed{\lambda_{1,2} = 2,\ 4}$

For $\lambda_1 = 2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = y \qquad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 4 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = -y \qquad \Rightarrow \quad V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\boxed{X_h = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + C_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t}}$$

$$\varphi(t) = \begin{pmatrix} e^{2t} & -e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix}$$

$$\varphi^{-1} = \frac{1}{2e^{6t}}\begin{pmatrix} e^{4t} & e^{4t} \\ -e^{2t} & e^{2t} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e^{-2t} & e^{-2t} \\ -e^{-4t} & e^{-4t} \end{pmatrix}$$

$$F(t) = \begin{pmatrix} 4e^{2t} \\ 4e^{4t} \end{pmatrix}$$

$$\varphi^{-1}(t)F(t) = \frac{1}{2}\begin{pmatrix} e^{-2t} & e^{-2t} \\ -e^{-4t} & e^{-4t} \end{pmatrix}\begin{pmatrix} 4e^{2t} \\ 4e^{4t} \end{pmatrix}$$

$$= \begin{pmatrix} 2+2e^{2t} \\ 2-2e^{-2t} \end{pmatrix}$$

$$X_p = \begin{pmatrix} e^{2t} & -e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix}\int\begin{pmatrix} 2+2e^{2t} \\ 2-2e^{-2t} \end{pmatrix}dt \qquad X_p = \varphi(t)\int\varphi^{-1}(t)F(t)\,dt$$

$$= \begin{pmatrix} e^{2t} & -e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix}\begin{pmatrix} 2t+e^{2t} \\ 2t+e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2te^{2t}+e^{4t}-2te^{4t}-e^{2t} \\ 2te^{2t}+e^{4t}+2te^{4t}+e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \end{pmatrix}te^{2t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{2t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t}$$

$$X(t) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t} + C_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{4t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}te^{2t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{2t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t}$$

$$X(0) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{cases} C_1 - C_2 = 1 \\ C_1 + C_2 = -1 \end{cases} \rightarrow \quad C_1 = 0, \ C_2 = -1$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = -\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{4t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}te^{2t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{2t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t}$$

$$\begin{cases} x_1(t) = 2e^{4t} - 2te^{4t} + 2te^{2t} - e^{2t} \\ x_2(t) = 2te^{4t} + 2te^{2t} + e^{2t} \end{cases}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x_1'(t) = x_1 - x_2 + \frac{1}{t} \\ x_2'(t) = x_1 - x_2 + \frac{1}{t} \end{cases}$ $\quad X(1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$= \lambda^2 = 0$$

The eigenvalues: $\boxed{\lambda_{1,2} = 0,\ 0}$

For $\lambda_1 = 0 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to a_1 = b_1 \qquad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $V_2 \quad \Rightarrow (A - \lambda I)V_2 = V_1$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \to a_2 - b_2 = 1 \quad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$y_2(t) = V_2 + tV_1$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}t$$

$$= \begin{pmatrix} 1+t \\ t \end{pmatrix}$$

$$\boxed{X_h = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1+t \\ t \end{pmatrix}}$$

$$\varphi(t) = \begin{pmatrix} 1 & 1+t \\ 1 & t \end{pmatrix}$$

$$\varphi^{-1} = -\begin{pmatrix} t & -1-t \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -t & 1+t \\ 1 & -1 \end{pmatrix}$$

$$F(t) = \begin{pmatrix} \frac{1}{t} \\ \frac{1}{t} \end{pmatrix}$$

$$\varphi^{-1}(t)F(t)=\begin{pmatrix}-t & 1+t\\ 1 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{t}\\ \frac{1}{t}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{t}\\ 0\end{pmatrix}$$

$$X_p=\begin{pmatrix}1 & 1+t\\ 1 & t\end{pmatrix}\int\begin{pmatrix}\frac{1}{t}\\ 0\end{pmatrix}dt \qquad X_p=\varphi(t)\int\varphi^{-1}(t)F(t)\,dt$$

$$=\begin{pmatrix}1 & 1+t\\ 1 & t\end{pmatrix}\begin{pmatrix}\ln t\\ 0\end{pmatrix}$$

$$=\begin{pmatrix}\ln t\\ \ln t\end{pmatrix}$$

$$X(t)=C_1\begin{pmatrix}1\\ 1\end{pmatrix}+C_2\begin{pmatrix}1+t\\ t\end{pmatrix}+\begin{pmatrix}\ln t\\ \ln t\end{pmatrix}$$

$$X(1)=C_1\begin{pmatrix}1\\ 1\end{pmatrix}+C_2\begin{pmatrix}2\\ 1\end{pmatrix}+\begin{pmatrix}0\\ 0\end{pmatrix}=\begin{pmatrix}2\\ -1\end{pmatrix}$$

$$\begin{cases}C_1+2C_2=2\\ C_1+C_2=-1\end{cases}\quad\rightarrow\quad C_1=\frac{4}{-1}=-4,\ \ C_2=3$$

$$\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}=-4\begin{pmatrix}1\\ 1\end{pmatrix}+3\begin{pmatrix}1+t\\ t\end{pmatrix}+\begin{pmatrix}\ln t\\ \ln t\end{pmatrix}$$

$$\begin{cases}x_1(t)=-1+3t+\ln t\\ x_2(t)=-4+3t+\ln t\end{cases}$$

## Exercise

Find the general solution of the system $\begin{cases}x_1'(t)=3x_1-2x_2-2e^{-t}\\ x_2'(t)=x_1\quad -2e^{-t}\end{cases}$ $\quad X(0)=\begin{pmatrix}2\\ -1\end{pmatrix}$

**Solution**

$$|A-\lambda I|=\begin{vmatrix}3-\lambda & -2\\ 1 & -\lambda\end{vmatrix} \qquad A=\begin{pmatrix}3 & -2\\ 1 & 0\end{pmatrix}$$

$$=\lambda^2-3\lambda+2=0$$

The eigenvalues: $\lambda_{1,2}=1,\ 2$

For $\lambda_1=1 \quad \Rightarrow (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x = y \qquad \Rightarrow \ V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x = 2y \qquad \Rightarrow \ V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\boxed{X_h = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t} + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{2t}}$$

$$\varphi(t) = \begin{pmatrix} e^{t} & 2e^{2t} \\ e^{t} & e^{2t} \end{pmatrix}$$

$$\varphi^{-1} = -\frac{1}{e^{3t}}\begin{pmatrix} e^{2t} & -2e^{2t} \\ -e^{t} & e^{t} \end{pmatrix}$$

$$= \begin{pmatrix} -e^{-t} & 2e^{-t} \\ e^{-2t} & -e^{-2t} \end{pmatrix}$$

$$F(t) = \begin{pmatrix} -2e^{-t} \\ -2e^{-t} \end{pmatrix}$$

$$\varphi^{-1}(t)F(t) = \begin{pmatrix} -e^{-t} & 2e^{-t} \\ e^{-2t} & -e^{-2t} \end{pmatrix}\begin{pmatrix} -2e^{-t} \\ -2e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} -2e^{-2t} \\ 0 \end{pmatrix}$$

$$X_p = \begin{pmatrix} e^{t} & 2e^{2t} \\ e^{t} & e^{2t} \end{pmatrix}\int\begin{pmatrix} -2e^{-2t} \\ 0 \end{pmatrix}dt \qquad\qquad X_p = \varphi(t)\int\varphi^{-1}(t)F(t)\,dt$$

$$= \begin{pmatrix} e^{t} & 2e^{2t} \\ e^{t} & e^{2t} \end{pmatrix}\begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}}$$

$$X(t) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t} + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{2t} + \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$$

$$X(0) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{cases} C_1 + 2C_2 = 1 \\ C_1 + C_2 = -2 \end{cases} \quad \to \quad \boxed{C_1 = \frac{5}{-1} = -5, \ \ C_2 = 3}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = -5\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + 3\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$$

$$\boxed{\begin{cases} x_1(t) = -5e^t + 6e^{2t} + e^{-t} \\ x_2(t) = -5e^t + 3e^{2t} + e^{-t} \end{cases}}$$

## *Exercise*

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) = y_1 \\ y_2'(t) = -4y_1 + y_2 \\ y_3'(t) = 3y_1 + 6y_2 + 2y_3 \end{cases}$ $\quad y(0) = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -4 & 1-\lambda & 0 \\ 3 & 6 & 2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix}$$

$$= (1-\lambda)^2 (2-\lambda) = 0$$

Thus, the eigenvalues are: $\boxed{\lambda_1 = 2 \ \ \& \ \ \lambda_{2,3} = 1}$

For $\lambda_1 = 2 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & 6 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to x = y = 0 \quad \Rightarrow V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{matrix} x = 0 \\ 6y = -z \end{matrix} \quad \Rightarrow V_2 = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}$$

For $V_3 \quad \Rightarrow (A - \lambda I)V_3 = V_2$

$$\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} \quad \to \begin{matrix} -4x = 1 \\ 6y + z = -\frac{21}{4} \end{matrix} \quad \Rightarrow V_3 = \begin{pmatrix} -\frac{1}{4} \\ -\frac{21}{24} \\ 0 \end{pmatrix}$$

$$y_3(t) = \left(V_3 + tV_2\right)e^t$$

$$= \begin{pmatrix} -\frac{1}{4} \\ -\frac{21}{24}+t \\ -6t \end{pmatrix} e^{t}$$

$$y(t) = C_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t} + \left( C_2 \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} + C_3 \begin{pmatrix} -\frac{1}{4} \\ -\frac{21}{24}+t \\ -6t \end{pmatrix} \right) e^{t}$$

$$y(0) = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix} \quad \to \quad \begin{pmatrix} -\frac{1}{4}C_3 \\ C_2 - \frac{21}{24}C_3 \\ C_1 - 6C_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix} \qquad C_3 = 4, \quad C_2 = \frac{33}{6}, \quad C_1 = 3$$

$$y(t) = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} e^{2t} + \begin{pmatrix} -1 \\ 2+4t \\ -33-24t \end{pmatrix} e^{t}$$

$$\begin{cases} y_1(t) = -e^{t} \\ y_2(t) = (2+4t)e^{t} \\ y_2(t) = 3e^{2t} - (33+24t)e^{t} \end{cases}$$

### *Exercise*

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) = -\frac{5}{2}y_1 + y_2 + y_3 \\ y_2'(t) = y_1 - \frac{5}{2}y_2 + y_3 \\ y_3'(t) = y_1 + y_2 - \frac{5}{2}y_3 \end{cases}$ $y(0) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

***Solution***

$$|A - \lambda I| = \begin{vmatrix} -\frac{5}{2}-\lambda & 1 & 1 \\ 1 & -\frac{5}{2}-\lambda & 1 \\ 1 & 1 & -\frac{5}{2}-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{pmatrix}$$

$$= -\left(\frac{5}{2}+\lambda\right)^3 + 2 + 3\left(\frac{5}{2}+\lambda\right)$$

$$= -\frac{49}{8} - \frac{63}{4}\lambda - \frac{15}{2}\lambda^2 - \lambda^3 = 0$$

$$8\lambda^3 + 60\lambda^2 + 126\lambda + 49 = 0$$

$$\begin{array}{c|cccc} -\frac{1}{2} & 8 & 60 & 126 & 49 \\ & & -4 & -28 & -49 \\ \hline & 8 & 56 & 98 & 0 \end{array} \quad \to \quad 8\lambda^2 + 56\lambda + 98 = 0$$

Thus, the eigenvalues are: $\underline{\lambda_1 = -\frac{1}{2} \quad \& \quad \lambda_{2,3} = -\frac{7}{2}}$

For $\lambda_1 = -\frac{1}{2} \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{array}{r} -2x + y + z = 0 \\ x - 2y + z = 0 \\ x + y - 2z = 0 \end{array}$$

$$z = 1 \quad \rightarrow \begin{cases} -2x + y = -1 \\ x - 2y = -1 \end{cases} \quad \Delta = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 3 \quad \Delta_x = \begin{vmatrix} -2 & -1 \\ 1 & -1 \end{vmatrix} = 3$$

$$\rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_1 = -\frac{7}{2} \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow x + y + z = 1$$

$$z = 0 \quad \rightarrow x + y = 1 \quad y = 1 \Rightarrow x = -1 \qquad \rightarrow V_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$y = 0 \quad \rightarrow x + z = 1 \quad z = 1 \Rightarrow x = -1 \qquad \rightarrow V_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$y(t) = C_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^{-t/2} + \left(C_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + C_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right)e^{-7t/2}$$

$$y(0) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} C_1 - C_2 - C_3 \\ C_1 + C_2 \\ C_1 + C_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$\begin{cases} C_1 - C_2 - C_3 = 2 \\ C_1 + C_2 = 3 \\ C_1 + C_3 = -1 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 3 \quad \Delta_1 = \begin{vmatrix} 2 & -1 & -1 \\ 3 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 4$$

$\underline{C_1 = \frac{4}{3}, \quad C_2 = \frac{5}{3}, \quad C_3 = -\frac{7}{3}}$

$$y(t)=\frac{4}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}e^{-t/2}+\left(\frac{5}{3}\begin{pmatrix}-1\\1\\0\end{pmatrix}-\frac{7}{3}\begin{pmatrix}-1\\0\\1\end{pmatrix}\right)e^{-7t/2}$$

$$\boxed{\begin{cases}y_1(t)=\frac{4}{3}e^{-t/2}+\frac{2}{3}e^{-7t/2}\\ y_2(t)=\frac{4}{3}e^{-t/2}+\frac{5}{3}e^{-7t/2}\\ y_2(t)=\frac{4}{3}e^{-t/2}-\frac{7}{3}e^{-7t/2}\end{cases}}$$

## *Exercise*

Find the general solution of the system $\begin{cases}x_1'(t)=3x_1-x_2-x_3\\ x_2'(t)=x_1+x_2-x_3+t\\ x_3'(t)=x_1-x_2+x_3+2e^t\end{cases}$ $\quad X(0)=\begin{pmatrix}1\\1\\-1\end{pmatrix}$

***Solution***

$$|A-\lambda I|=\begin{vmatrix}3-\lambda & -1 & -1\\ 1 & 1-\lambda & -1\\ 1 & -1 & 1-\lambda\end{vmatrix} \qquad A=\begin{pmatrix}3 & -1 & -1\\ 1 & 1 & -1\\ 1 & -1 & 1\end{pmatrix}$$

$$=\left(1-2\lambda+\lambda^2\right)(3-\lambda)+2+2(1-\lambda)-(3-\lambda)$$

$$=3-6\lambda+3\lambda^2-\lambda+2\lambda^2-\lambda^3+1-\lambda$$

$$=-\lambda^3+5\lambda^2-8\lambda+4=0$$

$$\begin{array}{r|rrrr} 1 & -1 & 5 & -8 & 4\\ & & -1 & 4 & -4\\ \hline & -1 & 4 & -4 & 0\end{array} \quad \to\ -\lambda^2+4\lambda-4=0$$

The eigenvalues: $\underline{\lambda_{1,2,3}=1,\ 2,\ 2|}$

For $\lambda_1=1 \quad \Rightarrow (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}2 & -1 & -1\\ 1 & 0 & -1\\ 1 & -1 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \quad \to \begin{cases}2x=y+z\\ x=z\\ x=y\end{cases} \quad \Rightarrow \quad V_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

For $\lambda_2=2 \quad \Rightarrow (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix}1 & -1 & -1\\ 1 & -1 & -1\\ 1 & -1 & -1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \quad \to \begin{cases}x=y+z\end{cases} \quad \Rightarrow \quad V_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$$

For $V_3 \quad \Rightarrow (A-\lambda_2 I)V_3=V_2$

$$\begin{pmatrix}1 & -1 & -1\\ 1 & -1 & -1\\ 1 & -1 & -1\end{pmatrix}V_3 = \begin{pmatrix}1\\1\\0\end{pmatrix} \qquad x-y-z=0 \to \begin{cases} y=0\\ x=z\end{cases} \quad \Rightarrow \quad V_3 = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$\underline{X_h = C_1\begin{pmatrix}1\\1\\1\end{pmatrix}e^t + C_2\begin{pmatrix}1\\1\\0\end{pmatrix}e^{2t} + C_3\begin{pmatrix}1\\0\\1\end{pmatrix}e^{2t}}$$

$$\varphi(t) = \begin{pmatrix}e^t & e^{2t} & e^{2t}\\ e^t & e^{2t} & 0\\ e^t & 0 & e^{2t}\end{pmatrix}$$

$$\varphi^{-1} = \begin{pmatrix}-e^{-t} & e^{-t} & e^{-t}\\ e^{-2t} & 0 & -e^{-2t}\\ e^{-2t} & -e^{-2t} & 0\end{pmatrix}$$

$$F(t) = \begin{pmatrix}0\\ t\\ 2e^t\end{pmatrix}$$

$$\varphi^{-1}(t)F(t) = \begin{pmatrix}-e^{-t} & e^{-t} & e^{-t}\\ e^{-2t} & 0 & -e^{-2t}\\ e^{-2t} & -e^{-2t} & 0\end{pmatrix}\begin{pmatrix}0\\ t\\ 2e^t\end{pmatrix}$$

$$= \begin{pmatrix}te^{-t}+2\\ -2e^{-t}\\ -te^{-2t}\end{pmatrix}$$

$$\int\left(te^{-t}+2\right)dt = \underline{(-t-1)e^{-t}+2t}$$

$$\int -2e^{-t}dt = \underline{2e^{-t}}$$

$$\int -te^{-2t}dt = \underline{\left(\frac{1}{2}t+\frac{1}{4}\right)e^{-2t}}$$

$$X_p = \begin{pmatrix}e^t & e^{2t} & e^{2t}\\ e^t & e^{2t} & 0\\ e^t & 0 & e^{2t}\end{pmatrix}\begin{pmatrix}(-t-1)e^{-t}+2t\\ 2e^{-t}\\ \left(\frac{1}{2}t+\frac{1}{4}\right)e^{-2t}\end{pmatrix} \qquad X_p = \varphi(t)\int\varphi^{-1}(t)F(t)\,dt$$

$$=\begin{pmatrix} -t-1+(2t+2)e^t+\frac{1}{2}t+\frac{1}{4} \\ -t-1+2te^t+2e^t \\ -t-1+2te^t+\frac{1}{2}t+\frac{1}{4} \end{pmatrix}$$

$$=\begin{pmatrix} -\frac{1}{2} \\ -1 \\ -\frac{1}{2} \end{pmatrix}t+\begin{pmatrix} -\frac{3}{4} \\ -1 \\ -\frac{3}{4} \end{pmatrix}+\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}e^t+\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}te^t$$

$$X(t)=C_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^t+C_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{2t}+C_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}e^{2t}+\begin{pmatrix} -\frac{1}{2} \\ -1 \\ -\frac{1}{2} \end{pmatrix}t+\begin{pmatrix} -\frac{3}{4} \\ -1 \\ -\frac{3}{4} \end{pmatrix}+\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}e^t+\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}te^t$$

$$X(0)=C_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}+C_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}+C_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}+\begin{pmatrix} -\frac{3}{4} \\ -1 \\ -\frac{3}{4} \end{pmatrix}+\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} C_1+C_2+C_3-\frac{3}{4}+2=1 \\ C_1+C_2-1+2=1 \\ C_1+C_3-\frac{3}{4}=1 \end{cases}$$

$$\begin{cases} C_1+C_2+C_3=-\frac{1}{4} \\ C_1+C_2=0 \\ C_1+C_3=\frac{7}{4} \end{cases} \qquad \Delta=\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}=-1 \quad \Delta_1=\begin{vmatrix} -\frac{1}{4} & 1 & 1 \\ 0 & 1 & 0 \\ \frac{7}{4} & 0 & 1 \end{vmatrix}=-2 \quad \Delta_2=\begin{vmatrix} 1 & -\frac{1}{4} & 1 \\ 1 & 0 & 0 \\ 1 & \frac{7}{4} & 1 \end{vmatrix}=2$$

$$\boxed{C_1=2,\quad C_2=-2,\quad C_3=-\frac{1}{4}}$$

$$X(t)=\begin{pmatrix} 2+2 \\ 2+2 \\ 2 \end{pmatrix}e^t+\begin{pmatrix} -2-\frac{1}{4} \\ 2 \\ -\frac{1}{4} \end{pmatrix}e^{2t}+\begin{pmatrix} -\frac{1}{2} \\ -1 \\ -\frac{1}{2} \end{pmatrix}t+\begin{pmatrix} -\frac{3}{4} \\ -1 \\ -\frac{3}{4} \end{pmatrix}+\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}te^t$$

$$\boxed{\begin{cases} x_1(t)=4e^t-\frac{9}{4}e^{2t}-\frac{3}{4}-\frac{1}{2}t+2te^t \\ x_2(t)=4e^t+2e^{2t}-1-t+2te^t \\ x_3(t)=2e^t-\frac{1}{4}e^{2t}-\frac{3}{4}-\frac{1}{2}t+2te^t \end{cases}}$$

### ***Exercise***

Find the general solution of the system $\begin{cases} x_1'(t) = x_1 + x_2 \ + e^t \\ x_2'(t) = x_1 + x_2 \ + e^{2t} \\ x_3'(t) = \qquad\quad 3x_3 + te^{3t} \end{cases}$ $\quad X(0) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

**Solution**

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \left(1 - 2\lambda + \lambda^2\right)(3-\lambda) - (3-\lambda)$$

$$= (3-\lambda)\left(\lambda^2 - 2\lambda\right) = 0$$

The eigenvalues: $\lambda_{1,2,3} = 0,\ 2,\ 3$

For $\lambda_1 = 0 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} x = -y \\ z = 0 \end{cases} \quad \Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} x = y \\ z = 0 \end{cases} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_3 = 3 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} 2x = y \\ x = 2y \end{cases} \quad \Rightarrow \quad V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$X_h = C_1\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{2t} + C_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}e^{3t}$$

$$\varphi(t) = \begin{pmatrix} -1 & e^{2t} & 0 \\ 1 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}$$

$$\varphi^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2}e^{-2t} & \frac{1}{2}e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{pmatrix}$$

$$F(t) = \begin{pmatrix} e^{t} \\ e^{2t} \\ te^{3t} \end{pmatrix}$$

$$\varphi^{-1}(t)F(t) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2}e^{-2t} & \frac{1}{2}e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} e^{t} \\ e^{2t} \\ te^{3t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}e^{t} - \frac{1}{2}e^{2t} \\ \frac{1}{2}e^{-t} + \frac{1}{2} \\ t \end{pmatrix}$$

$$\int \left(\frac{1}{2}e^{t} - \frac{1}{2}e^{2t}\right)dt = \frac{1}{2}e^{t} - \frac{1}{4}e^{2t}$$

$$\int \left(\frac{1}{2}e^{-t} + \frac{1}{2}\right)dt = -\frac{1}{2}e^{-t} + \frac{1}{2}t$$

$$\int t\,dt = \frac{1}{2}t^{2}$$

$$X_p = \begin{pmatrix} -1 & e^{2t} & 0 \\ 1 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix} \begin{pmatrix} \frac{1}{2}e^{t} - \frac{1}{4}e^{2t} \\ -\frac{1}{2}e^{-t} + \frac{1}{2}t \\ \frac{1}{2}t^{2} \end{pmatrix} \qquad X_p = \varphi(t)\int \varphi^{-1}(t)F(t)\,dt$$

$$= \begin{pmatrix} -\frac{1}{2}e^{t} + \frac{1}{4}e^{2t} - \frac{1}{2}e^{t} + \frac{1}{2}te^{2t} \\ \frac{1}{2}e^{t} - \frac{1}{4}e^{2t} - \frac{1}{2}e^{t} + \frac{1}{2}te^{2t} \\ \frac{1}{2}t^{2}e^{3t} \end{pmatrix}$$

$$= \begin{pmatrix} -e^{t} + \frac{1}{4}e^{2t} + \frac{1}{2}te^{2t} \\ \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} \\ \frac{1}{2}t^{2}e^{3t} \end{pmatrix}$$

$$X(t)=C_1\begin{pmatrix}-1\\1\\0\end{pmatrix}+C_2\begin{pmatrix}1\\1\\0\end{pmatrix}e^{2t}+C_3\begin{pmatrix}1\\2\\1\end{pmatrix}e^{3t}+\begin{pmatrix}-1\\0\\0\end{pmatrix}e^{t}+\begin{pmatrix}\frac{1}{2}t+\frac{1}{4}\\ \frac{1}{2}t-\frac{1}{4}\\0\end{pmatrix}e^{2t}+\begin{pmatrix}0\\0\\\frac{1}{2}t^2\end{pmatrix}e^{3t}$$

$$X(0)=C_1\begin{pmatrix}-1\\1\\0\end{pmatrix}+C_2\begin{pmatrix}1\\1\\0\end{pmatrix}+C_3\begin{pmatrix}1\\2\\1\end{pmatrix}+\begin{pmatrix}-1\\0\\0\end{pmatrix}+\begin{pmatrix}\frac{1}{4}\\-\frac{1}{4}\\0\end{pmatrix}=\begin{pmatrix}2\\3\\-1\end{pmatrix}$$

$$\begin{cases}-C_1+C_2+C_3=\frac{11}{4}\\ C_1+C_2+2C_3=\frac{13}{4}\\ \quad\boxed{C_3=-1}\end{cases}\rightarrow\begin{cases}-C_1+C_2=\frac{15}{4}\\ C_1+C_2=\frac{21}{4}\end{cases}\quad \boxed{C_1=\frac{3}{4},\ C_2=\frac{9}{2}}$$

$$\begin{cases}x_1(t)=-C_1-e^t+\left(C_2+\frac{1}{2}t+\frac{1}{4}\right)e^{2t}+C_3e^{3t}\\ x_2(t)=C_1+\left(C_2+\frac{1}{2}t-\frac{1}{4}\right)e^{2t}+2C_3e^{3t}\\ x_3(t)=\left(C_3+\frac{1}{2}t^2\right)e^{3t}\end{cases}$$

$$\boxed{\begin{cases}x_1(t)=-\frac{3}{4}-e^t+\left(\frac{1}{2}t+\frac{19}{4}\right)e^{2t}-e^{3t}\\ x_2(t)=\frac{3}{4}+\left(\frac{1}{2}t+\frac{17}{4}\right)e^{2t}-2e^{3t}\\ x_3(t)=\left(\frac{1}{2}t^2-1\right)e^{3t}\end{cases}}$$

## *Exercise*

Find the general solution of the system $x''+x=3$ ; $x(\pi)=1,\quad x'(\pi)=2$

***Solution***

Let $x_1=x \quad x_2=x'=x_1'$

$$\begin{cases}x_1'=x_2\\ x_2'=-x_1+3\end{cases}\rightarrow x(\pi)=x_1(\pi)=1,\quad x'(\pi)=x_2(\pi)=2$$

$$|A-\lambda I|=\begin{vmatrix}-\lambda & 1\\ -1 & -\lambda\end{vmatrix}\qquad A=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

$$=\lambda^2+1=0$$

The eigenvalues: $\boxed{\lambda_{1,2}=\pm i}$

For $\lambda_1 = i \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to ix = y \quad \Rightarrow V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} &= \begin{pmatrix} 1 \\ i \end{pmatrix}(\cos t + i\sin t) \\ &= \begin{pmatrix} \cos t + i\sin t \\ -\sin t + i\cos t \end{pmatrix} \\ &= \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \end{aligned}$$

$$X_h = C_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$$\begin{matrix} x_1' \\ x_2' \end{matrix}\begin{cases} a_2 = 0 \\ -a_1 - 3 = 0 \end{cases} \to \begin{cases} a_2 = 0 \\ a_1 = 3 \end{cases} \Rightarrow \boxed{X_p = \begin{pmatrix} 3 \\ 0 \end{pmatrix}}$$

$$X(t) = C_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad x_1(\pi) = 1, \quad x_2(\pi) = 2$$

$$X(\pi) = C_1\begin{pmatrix} -1 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{cases} -C_1 + 3 = 1 & \to \boxed{C_1 = 2} \\ -C_2 = 2 & \to \boxed{C_2 = -2} \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) = 2\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} - 2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1(t) = 2\cos t - 2\sin t + 3 \\ x_2(t) = -2\sin t - 2\cos t \end{cases}$$

$$\boxed{x(t) = x_1(t) = 2\cos t - 2\sin t + 3}$$

## Exercise

Find the general solution of the system $\begin{cases} x'' = x - y \\ y'' = x - y \end{cases} \qquad \begin{cases} x(3) = 5, & x'(3) = 2 \\ y(3) = 1, & y'(3) = -1 \end{cases}$

***Solution***

$$\begin{aligned} \left|A - \lambda^2 I\right| &= \begin{vmatrix} 1 - \lambda^2 & -1 \\ 1 & -1 - \lambda^2 \end{vmatrix} \\ &= \lambda^4 = 0 \end{aligned}$$

Thus, the eigenvalues are: $\boxed{\lambda_{1,2,3,4} = 0}$

$$x(t) = C_1 + C_2 t + C_3 t^2 + C_4 t^3$$

$$x(3) = \boxed{C_1 + 3C_2 + 9C_3 + 27C_4 = 5}$$

$$x' = C_2 + 2C_3 t + 3C_4 t^2$$

$$x'(3) = \boxed{C_2 + 6C_3 + 27C_4 = 2}$$

$$x'' = x - y \quad \rightarrow \quad y = x - x''$$

$$x'' = 2C_3 + 6C_4 t$$

$$y(t) = C_1 - 2C_3 + \left(C_2 - 6C_4\right)t + C_3 t^2 + C_4 t^3$$

$$y(3) = \boxed{C_1 + 3C_2 + 7C_3 + 9C_4 = 1}$$

$$y' = C_2 - 6C_4 + 2C_3 t + 3C_4 t^2$$

$$y'(3) = \boxed{C_2 + 6C_3 + 21C_4 = -1}$$

$$\Delta = \begin{vmatrix} 1 & 3 & 9 & 27 \\ 0 & 1 & 6 & 27 \\ 1 & 3 & 7 & 9 \\ 0 & 1 & 6 & 21 \end{vmatrix} = 12 \quad \Delta_1 = \begin{vmatrix} 5 & 3 & 9 & 27 \\ 2 & 1 & 6 & 27 \\ 1 & 3 & 7 & 9 \\ -1 & 1 & 6 & 21 \end{vmatrix} = 42 \quad \Delta_2 = \begin{vmatrix} 1 & 5 & 9 & 27 \\ 0 & 2 & 6 & 27 \\ 1 & 1 & 7 & 9 \\ 0 & -1 & 6 & 21 \end{vmatrix} = 42$$

$$\boxed{C_1 = \frac{42}{12} = \frac{7}{2}, \ C_2 = \frac{7}{2}, \ C_3 = -\frac{5}{2}, \ C_4 = \frac{1}{2}}$$

$$\boxed{\begin{cases} x(t) = \frac{7}{2} + \frac{7}{2}t - \frac{5}{2}t^2 + \frac{1}{2}t^3 \\ y(t) = \frac{17}{2} + \frac{1}{2}t - \frac{5}{2}t^2 + \frac{1}{2}t^3 \end{cases}}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x'' = x - y \\ y'' = -x + y \end{cases}$ $\begin{cases} x(0) = -1, & x'(0) = 0 \\ y(0) = 1, & y'(0) = 0 \end{cases}$

***Solution***

$$\left|A - \lambda^2 I\right| = \begin{vmatrix} 1-\lambda^2 & -1 \\ -1 & 1-\lambda^2 \end{vmatrix}$$

$$= \lambda^4 - 2\lambda^2 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = 0$ & $\lambda_{3,4} = \pm\sqrt{2}$

$$x(t) = C_1 + C_2 t + C_3 e^{-\sqrt{2}t} + C_4 e^{\sqrt{2}t}$$

$$x(0) = \boxed{C_1 + C_3 + C_4 = -1}$$

$$x' = C_2 - \sqrt{2}C_3 e^{-\sqrt{2}t} + \sqrt{2}C_4 e^{\sqrt{2}t}$$

$$x'(0) = \boxed{C_2 - \sqrt{2}C_3 + \sqrt{2}C_4 = 0}$$

$$x'' = x - y \quad \rightarrow \quad y = x - x''$$

$$x'' = 2C_3 e^{-\sqrt{2}t} + 2C_4 e^{\sqrt{2}t}$$

$$y(t) = C_1 + C_2 t - C_3 e^{-\sqrt{2}t} - C_4 e^{\sqrt{2}t}$$

$$y(0) = \boxed{C_1 - C_3 - C_4 = 1}$$

$$y' = C_2 + \sqrt{2}C_3 e^{-\sqrt{2}t} - \sqrt{2}C_4 e^{\sqrt{2}t}$$

$$y'(0) = \boxed{C_2 + \sqrt{2}C_3 - \sqrt{2}C_4 = 0}$$

$$\Delta = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -\sqrt{2} & \sqrt{2} \\ 1 & 0 & -1 & -1 \\ 0 & 1 & \sqrt{2} & -\sqrt{2} \end{vmatrix} = 8\sqrt{2} \quad \Delta_1 = \begin{vmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & -\sqrt{2} & \sqrt{2} \\ 1 & 0 & -1 & -1 \\ 0 & 1 & \sqrt{2} & -\sqrt{2} \end{vmatrix} = 0 \quad \Delta_2 = \begin{vmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & -\sqrt{2} & \sqrt{2} \\ 1 & 1 & -1 & -1 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{vmatrix} = 0$$

$$\begin{cases} C_3 + C_4 = -1 \\ \sqrt{2}C_3 - \sqrt{2}C_4 = 0 \end{cases}$$

$$\boxed{C_1 = 0, \ C_2 = 0, \ C_3 = -\frac{\sqrt{2}}{2\sqrt{2}} = -\frac{1}{2}, \ C_4 = -\frac{1}{2}}$$

$$\boxed{\begin{cases} x(t) = -\frac{1}{2}e^{-\sqrt{2}t} - \frac{1}{2}e^{\sqrt{2}t} \\ y(t) = \frac{1}{2}e^{-\sqrt{2}t} + \frac{1}{2}e^{\sqrt{2}t} \end{cases}}$$

## *Exercise*

Find the general solution of the system

$$\begin{cases} \dfrac{d^2x}{dt^2} = y; & x(0) = 3, & x'(0) = 1 \\ \dfrac{d^2y}{dt^2} = x; & y(0) = 1, & y'(0) = -1 \end{cases}$$

***Solution***

$$\left|A - \lambda^2 I\right| = \begin{vmatrix} -\lambda^2 & 1 \\ 1 & -\lambda^2 \end{vmatrix}$$

$$= \lambda^4 - 1 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = \pm 1$ & $\lambda_{3,4} = \pm i$

$$x(t) = C_1 e^{-t} + C_2 e^{t} + C_3 \cos t + C_4 \sin t$$

$$x(0) = \boxed{C_1 + C_2 + C_3 = 3}$$

$$x' = -C_1 e^{-t} + C_2 e^{t} - C_3 \sin t + C_4 \cos t$$

$$x'(0) = \boxed{-C_1 + C_2 + C_4 = 1}$$

$$x'' = y$$

$$y(t) = C_1 e^{-t} + C_2 e^{t} - C_3 \cos t - C_4 \sin t$$

$$y(0) = \boxed{C_1 + C_2 - C_3 = 1}$$

$$y' = -C_1 e^{-t} + C_2 e^{t} + C_3 \sin t - C_4 \cos t$$

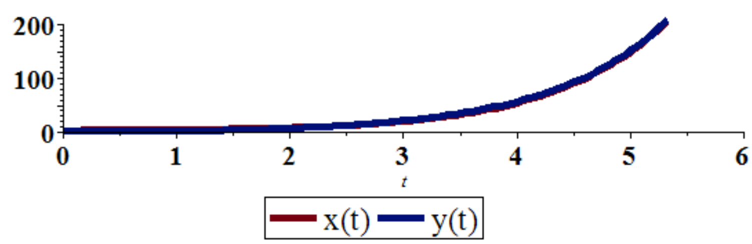
$$y'(0) = \boxed{-C_1 + C_2 - C_4 = -1}$$

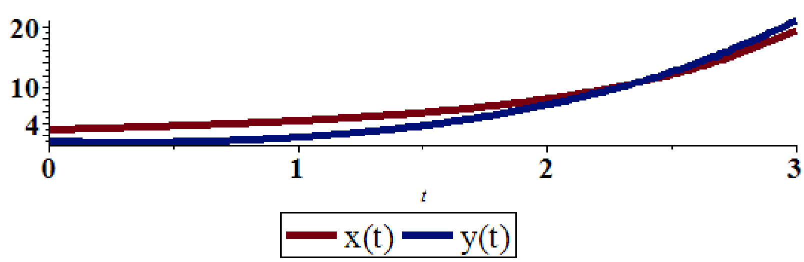
$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & -1 \end{vmatrix} = 8 \quad \Delta_1 = \begin{vmatrix} 3 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & -1 \end{vmatrix} = 8 \quad \Delta_2 = \begin{vmatrix} 1 & 3 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & 0 & -1 \end{vmatrix} = 8$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 3 & 0 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ -1 & 1 & -1 & -1 \end{vmatrix} = 8 \quad \Delta_4 = \begin{vmatrix} 1 & 1 & 1 & 3 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 0 & -1 \end{vmatrix} = 8$$

$$\boxed{C_1 = 1, \quad C_2 = 1, \quad C_3 = 1, \quad C_4 = 1}$$

$$\boxed{\begin{cases} x(t) = e^{-t} + e^{t} + \cos t + \sin t \\ y(t) = e^{-t} + e^{t} - \cos t - \sin t \end{cases}}$$



## *Exercise*

Find the general solution of the system
$$\begin{cases} x'' + 5x - 2y = 0 & x(0) = x'(0) = 0 \\ y'' + 2y - 2x = 3\sin 2t & y(0) = 1, \quad y'(0) = 0 \end{cases}$$

## ***Solution***

$$\begin{cases} x'' = -5x + 2y \\ y'' = 2x - 2y + 3\sin 2t \end{cases}$$

$$\left|A - \lambda^2 I\right| = \begin{vmatrix} -5-\lambda^2 & 2 \\ 2 & -2-\lambda^2 \end{vmatrix} \qquad A = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$$

$$= \lambda^4 + 7\lambda^2 + 6 = 0 \qquad \lambda^2 = -1, \ -6$$

The eigenvalues: $\underline{\lambda_{1,2} = i \ \ \& \ \ \lambda_{3,4} = \pm i\sqrt{6}}$

$$x_h(t) = C_1 \cos t + C_2 \sin t + C_3 \cos\sqrt{6}t + C_4 \sin\sqrt{6}t$$

$$\begin{cases} x_p = A\sin 2t \\ y_p = B\sin 2t \end{cases} \to \begin{cases} x_p'' = -4A\sin 2t \\ y_p'' = -4B\sin 2t \end{cases}$$

$$\begin{cases} -4A\sin 2t + 5A\sin 2t - 2B\sin 2t = 0 \\ -4B\sin 2t + 2B\sin t - 2A\sin 2t = 3\sin 2t \end{cases}$$

$$\begin{cases} A - 2B = 0 \\ -2A - 2B = 3 \end{cases} \to \underline{A = -1, \ \ B = -\tfrac{1}{2}}$$

$$\begin{cases} x_p = -\sin 2t \\ y_p = -\frac{1}{2}\sin 2t \end{cases}$$

$$x(t) = C_1 \cos t + C_2 \sin t + C_3 \cos\sqrt{6}t + C_4 \sin\sqrt{6}t - \sin 2t$$

$$x(0) = \underline{C_1 + C_3 = 0} \quad (1)$$

$$x' = -C_1 \sin t + C_2 \cos t - \sqrt{6}C_3 \sin\sqrt{6}t + \sqrt{6}C_4 \cos\sqrt{6}t - 2\cos 2t$$

$$x'(0) = \underline{C_2 + \sqrt{6}C_4 - 2 = 0} \quad (2)$$

$$x'' + 5x - 2y = 0 \quad \to \quad y = \frac{1}{2}\left(x'' + 5x\right)$$

$$x'' = -C_1 \cos t - C_2 \sin t - 6C_3 \cos\sqrt{6}t - 6C_4 \sin\sqrt{6}t + 4\sin 2t$$

$$y(t) = \frac{1}{2}\left(x'' + 5x\right)$$

$$= \frac{1}{2}\left(4C_1 \cos t + 4C_2 \sin t - C_3 \cos\sqrt{6}t - C_4 \sin\sqrt{6}t - \sin 2t\right)$$

$$= 2C_1 \cos t + 2C_2 \sin t - \frac{1}{2}C_3 \cos\sqrt{6}t - \frac{1}{2}C_4 \sin\sqrt{6}t - \frac{1}{2}\sin 2t$$

$$y(0) = \boxed{2C_1 - \frac{1}{2}C_3 = 1} \quad (3)$$

$$y' = -2C_1 \sin t + 2C_2 \cos t + \frac{\sqrt{6}}{2}C_3 \sin\sqrt{6}t - \frac{\sqrt{6}}{2}C_4 \cos\sqrt{6}t - \cos 2t$$

$$y'(0) = \boxed{2C_2 - \frac{\sqrt{6}}{2}C_4 - 1 = 0} \quad (4)$$

$$\begin{cases} (1) & C_1 + C_3 = 0 \\ (3) & 4C_1 - C_3 = 2 \end{cases} \qquad \boxed{C_1 = \frac{2}{5}, \quad C_3 = -\frac{2}{5}}$$

$$\begin{cases} (2) & C_2 + \sqrt{6}C_4 = 2 \\ (4) & 4C_2 - \sqrt{6}C_4 = 2 \end{cases} \qquad \boxed{C_2 = \frac{4\sqrt{6}}{5\sqrt{6}} = \frac{4}{5}, \quad C_4 = \frac{6}{5\sqrt{6}} = \frac{\sqrt{6}}{5}}$$

$$\boxed{\begin{cases} x(t) = \frac{2}{5}\cos t + \frac{4}{5}\sin t - \frac{2}{5}\cos\sqrt{6}t + \frac{\sqrt{6}}{5}\sin\sqrt{6}t - \sin 2t \\ y(t) = \frac{4}{5}\cos t + \frac{8}{5}\sin t + \frac{1}{5}\cos\sqrt{6}t - \frac{\sqrt{6}}{10}\sin\sqrt{6}t - \frac{1}{2}\sin 2t \end{cases}}$$

## Exercise

Find the general solution of the system $\begin{cases} x'' = -2x' - 5y + 3 \\ y' = x' + 2y \end{cases}$ $\quad x(0) = 0,\ x'(0) = 0,\ y(0) = 1$

***Solution***

Let $\quad \begin{matrix} x_1 = x & x_2 = x' = x_1' \\ y_1 = y & y_2 = y' = y_1' \end{matrix} \qquad \begin{cases} x(0) = x_1(0) = 0 \\ x'(0) = x_2(0) = 0 \\ y(0) = y_1(0) = 1 \end{cases}$

$$\begin{cases} x_1' = x_2 \\ x_2' = -2x_2 - 5y_1 + 3 \\ y_1' = x_2 + 2y_1 \end{cases}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -2-\lambda & -5 \\ 0 & 1 & 2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & -5 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= \lambda\left(4 - \lambda^2\right) - 5\lambda$$

$$= -\lambda^3 - \lambda = 0$$

The eigenvalues: $\boxed{\lambda_1 = 0, \quad \lambda_{2,3} = \pm i}$

For $\lambda_1 = 0 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & -5 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} y = 0 \\ z = 0 \\ \end{cases} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda_2 = -i \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} i & 1 & 0 \\ 0 & -2+i & -5 \\ 0 & 1 & 2+i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} ix = -y \\ (-2+i)y = 5z \\ y = -(2+i)z \end{cases} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ -i \\ \frac{1}{5}(1+2i) \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -i \\ \frac{1}{5}(1+2i) \end{pmatrix} e^{-it} = \begin{pmatrix} 1 \\ -i \\ \frac{1}{5}(1+2i) \end{pmatrix}(\cos t - i\sin t)$$

$$= \begin{pmatrix} \cos t - i\sin t \\ -\sin t - i\cos t \\ \frac{1}{5}\left(\cos t + 2\sin t + i(2\cos t - \sin t)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \cos t \\ -\sin t \\ \frac{1}{5}(\cos t + 2\sin t) \end{pmatrix} + i \begin{pmatrix} -\sin t \\ -\cos t \\ \frac{1}{5}(2\cos t - \sin t) \end{pmatrix}$$

$$X_h = C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} \cos t \\ -\sin t \\ \frac{1}{5}(\cos t + 2\sin t) \end{pmatrix} + C_3 \begin{pmatrix} -\sin t \\ -\cos t \\ \frac{1}{5}(2\cos t - \sin t) \end{pmatrix}$$

$$\begin{cases} -2a_2 - 5a_3 = -3 \\ a_2 + 2a_3 = 0 \end{cases} \qquad \Delta = \begin{vmatrix} -2 & -5 \\ 1 & 2 \end{vmatrix} = 1 \quad \Delta_1 = \begin{vmatrix} -3 & -5 \\ 0 & 2 \end{vmatrix} = -6 \quad \Delta_2 = \begin{vmatrix} -2 & -3 \\ 1 & 0 \end{vmatrix} = 3$$

$\Rightarrow \quad a_2 = -6 \quad a_3 = 3$

$x_1' = x_2 \quad \rightarrow \quad a_1' = -6 \;\Rightarrow\; \underline{a_1 = -6t}|$

$$X_P = \begin{pmatrix} -6t \\ -6 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \end{pmatrix}(t) = C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} \cos t \\ -\sin t \\ \frac{1}{5}(\cos t + 2\sin t) \end{pmatrix} + C_3 \begin{pmatrix} -\sin t \\ -\cos t \\ \frac{1}{5}(2\cos t - \sin t) \end{pmatrix} + \begin{pmatrix} -6t \\ -6 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \end{pmatrix}(0) = C_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 0 \\ \frac{1}{5} \end{pmatrix} + C_3\begin{pmatrix} 0 \\ -1 \\ \frac{2}{5} \end{pmatrix} + \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} C_1 + C_2 = 0 & \rightarrow \underline{C_1 = -2|} \\ -C_3 - 6 = 0 & \rightarrow \underline{C_3 = -6|} \\ \frac{1}{5}C_2 + \frac{2}{5}C_3 + 3 = 1 & \rightarrow \underline{C_2 = 2|} \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \end{pmatrix}(t) = -2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} \cos t \\ -\sin t \\ \frac{1}{5}(\cos t + 2\sin t) \end{pmatrix} - 6\begin{pmatrix} -\sin t \\ -\cos t \\ \frac{1}{5}(2\cos t - \sin t) \end{pmatrix} + \begin{pmatrix} -6t \\ -6 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 + 2\cos t + 6\sin t - 6t \\ -2\sin t + 6\cos t - 6 \\ \frac{2}{5}\cos t + \frac{4}{5}\sin t - \frac{12}{5}\cos t + \frac{6}{5}\sin t + 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2\cos t + 6\sin t - 6t - 2 \\ -2\sin t + 6\cos t - 6 \\ 2\sin t - 2\cos t + 3 \end{pmatrix}$$

$$\boxed{\begin{cases} x(t) = x_1(t) = 2\cos t + 6\sin t - 6t - 2 \\ y(t) = y_1(t) = 2\sin t - 2\cos t + 3 \end{cases}}$$

## *Exercise*

Find the general solution of the system $\begin{cases} x'' = 2x' + 5y + 3 \\ y' = -x' - 2y \end{cases}$ $\quad x(0) = 0,\ x'(0) = 0,\ y(0) = 1$

***Solution***

Let $\quad x_1 = x \quad x_2 = x' = x_1'$
$\quad\quad y_1 = y \quad y_2 = y' = y_1'$

$$\begin{cases} x(0) = x_1(0) = 0 \\ x'(0) = x_2(0) = 0 \\ y(0) = y_1(0) = 1 \end{cases}$$

$$\begin{cases} x_1' = x_2 \\ x_2' = 2x_2 5y_1 + 3 \\ y_1' = -x_2 - 2y_1 \end{cases}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & 2-\lambda & 5 \\ 0 & -1 & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 5 \\ 0 & -1 & -2 \end{pmatrix}$$

$$= \lambda\left(4-\lambda^2\right)-5\lambda$$

$$= -\lambda^3-\lambda = 0$$

The eigenvalues: $\boxed{\lambda_1 = 0, \ \ \lambda_{2,3} = \pm i}$

For $\lambda_1 = 0 \quad \Rightarrow \left(A-\lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 5 \\ 0 & -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} y = 0 \\ z = 0 \end{cases} \quad \Rightarrow \ V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda_2 = -i \quad \Rightarrow \left(A-\lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} i & 1 & 0 \\ 0 & 2+i & 5 \\ 0 & -1 & -2+i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} ix = -y \\ (2+i)y = -5z \\ y = (-2+i)z \end{cases}$$

$$x = 1 \quad \rightarrow y = -i \quad \Rightarrow -i = (-2+i)z$$

$$z = -\frac{i}{-2+i}\frac{-2-i}{-2-i} = \frac{-1+2i}{5}$$

$$\Rightarrow \ V_2 = \begin{pmatrix} 1 \\ -i \\ \frac{1}{5}(-1+2i) \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -i \\ \frac{1}{5}(-1+2i) \end{pmatrix} e^{-it} = \begin{pmatrix} 1 \\ -i \\ \frac{1}{5}(-1+2i) \end{pmatrix}(\cos t - i\sin t)$$

$$= \begin{pmatrix} \cos t - i\sin t \\ -\sin t - i\cos t \\ \frac{1}{5}\left(-\cos t + 2\sin t + i\left(2\cos t + \sin t\right)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \cos t \\ -\sin t \\ \frac{1}{5}(-\cos t + 2\sin t) \end{pmatrix} + i\begin{pmatrix} -\sin t \\ -\cos t \\ \frac{1}{5}(2\cos t + \sin t) \end{pmatrix}$$

$$X_h = C_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} \cos t \\ -\sin t \\ \frac{1}{5}(-\cos t + 2\sin t) \end{pmatrix} + C_3\begin{pmatrix} -\sin t \\ -\cos t \\ \frac{1}{5}(2\cos t + \sin t) \end{pmatrix}$$

$$\begin{cases} 2a_2 + 5a_3 = -3 \\ -a_2 - 2a_3 = 0 \end{cases} \quad \Delta = \begin{vmatrix} 2 & 5 \\ -1 & -2 \end{vmatrix} = 1 \quad \Delta_1 = \begin{vmatrix} -3 & 5 \\ 0 & -2 \end{vmatrix} = 6 \quad \Delta_2 = \begin{vmatrix} 2 & -3 \\ -1 & 0 \end{vmatrix} = -3$$

$$\Rightarrow \quad \underline{a_2 = 6 \quad a_3 = -3}$$

$$x_1' = x_2 \quad \to \quad a_1' = 6 \Rightarrow \underline{a_1 = 6t}$$

$$\underline{X_P = \begin{pmatrix} 6t \\ 6 \\ -3 \end{pmatrix}}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \end{pmatrix}(t) = C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} \cos t \\ -\sin t \\ \frac{1}{5}(-\cos t + 2\sin t) \end{pmatrix} + C_3 \begin{pmatrix} -\sin t \\ -\cos t \\ \frac{1}{5}(2\cos t + \sin t) \end{pmatrix} + \begin{pmatrix} 6t \\ 6 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \end{pmatrix}(\color{red}{0}) = C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{5} \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ -1 \\ \frac{2}{5} \end{pmatrix} + \begin{pmatrix} 0 \\ 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} C_1 + C_2 = 0 & \to \underline{C_1 = 8} \\ -C_3 + 6 = 0 & \to \underline{C_3 = 6} \\ -\frac{1}{5}C_2 + \frac{2}{5}C_3 - 3 = 1 & \to \underline{C_2 = -8} \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \end{pmatrix}(t) = 8 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 8 \begin{pmatrix} \cos t \\ -\sin t \\ \frac{1}{5}(-\cos t + 2\sin t) \end{pmatrix} + 6 \begin{pmatrix} -\sin t \\ -\cos t \\ \frac{1}{5}(2\cos t + \sin t) \end{pmatrix} + \begin{pmatrix} 6t \\ 6 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 8 - 8\cos t - 6\sin t + 6t \\ 8\sin t - 6\cos t + 6 \\ \frac{8}{5}\cos t - \frac{16}{5}\sin t + \frac{12}{5}\cos t + \frac{6}{5}\sin t - 3 \end{pmatrix}$$

$$= \begin{pmatrix} -8\cos t - 6\sin t + 6t + 8 \\ 8\sin t - 6\cos t + 6 \\ 4\cos t - 2\sin t - 3 \end{pmatrix}$$

$$\underline{\begin{cases} x(t) = x_1(t) = \color{blue}{-8\cos t - 6\sin t + 6t + 8} \\ y(t) = y_1(t) = \color{blue}{4\cos t - 2\sin t - 3} \end{cases}}$$

### Exercise

Find the real and imaginary part of $z(t) = e^{2it}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$

***Solution***

$$z(t) = (\cos 2t + i\sin 2t)\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2t + i\sin 2t \\ \cos 2t + i\sin 2t + i\cos 2t - \sin 2t \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2t + i\sin 2t \\ \cos 2t - \sin 2t + i(\sin 2t + \cos 2t) \end{pmatrix}$$

The real part is: $\boxed{(\cos 2t,\ \cos 2t - \sin 2t)^T}$

The imaginary part is: $\boxed{(\sin 2t,\ \sin 2t + \cos 2t)^T}$

### Exercise

Two tanks, each containing 360 *liters* of a salt solution. Pure water pours into tank *A* at a rate of 5 *L/min*. There are two pipes connecting tank *A* to tank *B*. The first pumps salt solution from tank *B* into tank *A* at a rate of 4 *L/min*. The second pumps salt solution from tank *A* into tank *B* at a rate of 9 *L/min*. Finally, there is a drain on tank *B* from which salt solution drains at a rate of 5 *L/min*. Thus, each tank maintains a constant volume of 360 *liters* of salt solution. Initially, there are 60 *kg* of salt present in tank *A*, but tank *B* contains pure water.

*a)* Set up, in matrix-vector form, an initial value problem that models the salt content in each tank over time.

*b)* Find the eigenvalues and eigenvectors of the coefficient matrix in part (*a*), then find the general solution in vector form. Find the solution that satisfies the initial conditions posed in part (*a*).

*c)* Plot each component of your solution in part (*b*) over a period of four time constants $[0,\ 4T_c]$. What is the eventual salt content in each tank? Give both a physical and a mathematical reason for your answer.

***Solution***

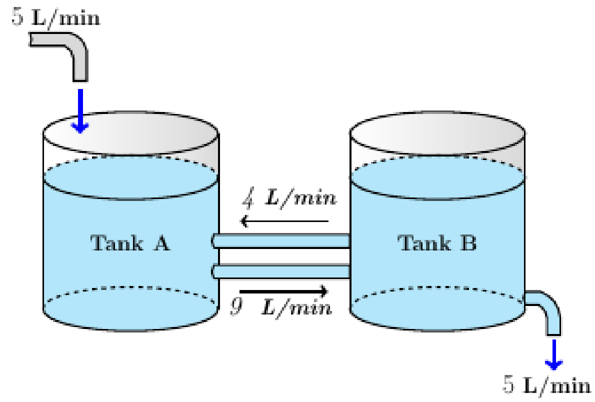
***a)*** Let $x_A(t)$ and $x_A(t)$ represent the number of pounds of salt as a function of time.

*Tank **A***:

$$\text{Rate in } = (5+4)\ \frac{L}{\min}\frac{x_A}{360}\frac{kg}{L} = \frac{x_A}{40}\ kg/min$$

$$Rate\ out\ = 4\ \frac{L}{\min}\frac{x_B}{360}\frac{kg}{L} = \frac{x_B}{90}\ kg\ /\ min$$

$$\frac{dx_A}{dt} = Rate\ in - Rate\ out = -\frac{x_A}{40} + \frac{x_B}{90}$$




*Tank* ***B***:

$$Rate\ in\ = 9\ \frac{L}{\min}\frac{x_A}{360}\frac{kg}{L} = \frac{x_A}{40} kg\ /\ min$$

$$Rate\ out\ = (5+4)\ \frac{L}{\min}\frac{x_B}{360}\frac{kg}{L} = \frac{x_B}{40}\ kg\ /\ min$$

$$\frac{dx_B}{dt} = Rate\ in - Rate\ out = \frac{x_A}{40} - \frac{x_B}{40}$$

$$\begin{cases} x'_A = -\frac{x_A}{40} + \frac{x_B}{90} \\ x'_B = \frac{x_A}{40} - \frac{x_B}{40} \end{cases}$$

The system is: $\begin{pmatrix} x_A \\ x_B \end{pmatrix}' = \begin{pmatrix} -\frac{1}{40} & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{40} \end{pmatrix}\begin{pmatrix} x_A \\ x_B \end{pmatrix}$ $\qquad x' = Ax(t)$

With initial 60 kg of salt in tank A; $\begin{pmatrix} x_A(0) \\ x_B(0) \end{pmatrix} = \begin{pmatrix} 60 \\ 0 \end{pmatrix}$

**b)** $\det(A - \lambda I) = \begin{vmatrix} -\frac{1}{40} - \lambda & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{40} - \lambda \end{vmatrix}$

$$= \left(-\frac{1}{40} - \lambda\right)\left(-\frac{1}{40} - \lambda\right) - \frac{1}{90}\frac{1}{40}$$

$$= \frac{1}{1600} + \frac{1}{20}\lambda + \lambda^2 - \frac{1}{3600}$$

$$= \lambda^2 + \frac{1}{20}\lambda + \frac{5}{14400}$$

$\therefore$ The eigenvalues are: $\boxed{\lambda_1 = -\frac{1}{120}}$ *and* $\boxed{\lambda_2 = -\frac{1}{24}}$

For $\lambda_1 = -\frac{1}{120} \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$, we have

$$\begin{pmatrix} -\frac{1}{40} + \frac{1}{120} & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{40} + \frac{1}{120} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{60} & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{60} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{60} & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{60} \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad x - \frac{2}{3}y = 0$$

$$V_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \to \quad \boxed{x_1(t) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-t/120}}$$

For $\lambda_2 = -\frac{1}{24} \quad \Rightarrow \quad (A - \lambda_2 I)V_2 = 0$, we have

$$\begin{pmatrix} -\frac{1}{40} + \frac{1}{24} & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{40} + \frac{1}{24} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{60} & \frac{1}{90} \\ \frac{1}{40} & \frac{1}{60} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\to \begin{pmatrix} \frac{1}{60} & \frac{1}{90} \\ \frac{1}{40} & \frac{1}{60} \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & \frac{2}{3} \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad x + \frac{2}{3}y = 0$$

$$V_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad \to \quad \boxed{x_2(t) = \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{-t/24}}$$

$$x(t) = C_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-t/120} + C_2 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{-t/24}$$

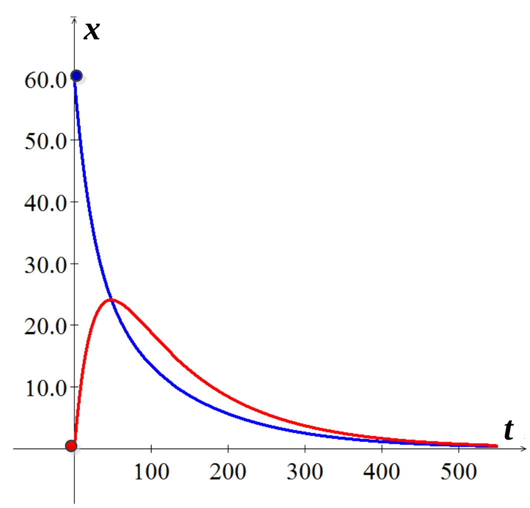
***Given*** $\begin{pmatrix} x_A(0) \\ x_B(0) \end{pmatrix} = \begin{pmatrix} 60 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 60 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 60 \\ 0 \end{pmatrix} = \begin{pmatrix} 2C_1 - 2C_2 \\ 3C_1 + 3C_2 \end{pmatrix}$$

$$\to \begin{cases} 2C_1 - 2C_2 = 60 \\ 3C_1 + 3C_2 = 0 \end{cases} \quad \to \quad \boxed{C_1 = 15} \quad \boxed{C_2 = -15}$$

$$\boxed{x(t) = 15 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-t/120} - 15 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{-t/24}}$$




***c)*** $x(t) = \begin{pmatrix} 30 & 30 \\ 45 & -45 \end{pmatrix} \begin{pmatrix} e^{-t/120} \\ e^{-t/24} \end{pmatrix}$

$$\begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} 30e^{-t/120} + 30e^{-t/24} \\ 45e^{-t/120} - 45e^{-t/24} \end{pmatrix}$$

The time constant on $e^{-t/120}$ is $T_c = 120$

The time constant on $e^{-t/24}$ is $T_c = 24$

If we choose the larger of these two time constants over a period of four time constants

$\left[0,\ 4T_c\right] = [0,\ 480]$.

This allows enough time to show both components decaying to zero.

Physically, if we keep pouring pure water into the tank *B*, eventually the system will purge itself of all salt content.

Mathematically:
$$\begin{cases} 30e^{-t/120} + 30e^{-t/24} \xrightarrow[t\to\infty]{} 0 \\ 45e^{-t/120} - 45e^{-t/24} \xrightarrow[t\to\infty]{} 0 \end{cases}$$

## *Exercise*

Consider the *RLC* parallel circuit below. Let *V* represent the voltage drop across the capacitor and *I* represent the current across the inductor that satisfied the system.

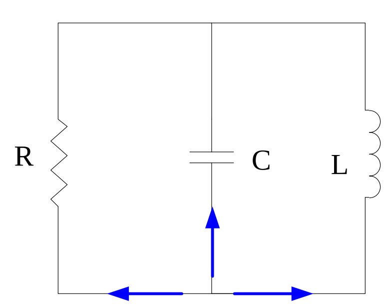
$$\begin{cases} V' = -\dfrac{V}{RC} - \dfrac{1}{C} \\ I' = \dfrac{V}{L} \end{cases}$$

Suppose that the resistance is $R = \frac{1}{2}\Omega$, the capacitor is $C = 1$ *farad*, and the inductance is $L = \frac{1}{2}$ *henry*. If the initial voltage across the capacitor is $V(0) = 10$ *volts* and there is no initial current across the inductor, solve the system to determine the voltage and current as a function of time. Plot the voltage and current as a function of time. Assume current flows in the directions indicated.

***Solution***

$$\begin{cases} V' = -2V - 1 \\ I' = 2V \end{cases}$$

$$\begin{pmatrix} V \\ I \end{pmatrix}' = \begin{pmatrix} -2 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} V \\ I \end{pmatrix}$$




$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & -1 \\ 2 & -\lambda \end{vmatrix}$$

$$= \lambda^2 + 2\lambda + 2 = 0$$

$\therefore$ The *eigenvalues* are: $\lambda = -1 \pm i$

For $\lambda_1 = -1 + i \quad \Rightarrow (A - \lambda_1 I)V = 0$

$$\begin{pmatrix} -1-i & -1 \\ 2 & 1-i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} -x-y-ix=0 \\ 2x+y-iy=0 \end{cases} \rightarrow 2x=(-1+i)y$$

$$V=\begin{pmatrix} -1+i \\ 2 \end{pmatrix} \quad \rightarrow \quad z(t)=\begin{pmatrix} -1+i \\ 2 \end{pmatrix} e^{(-1+i)t}$$

$$z(t)=\begin{pmatrix} -1+i \\ 2 \end{pmatrix} e^{-t}e^{it}$$

$$=e^{-t}(\cos t+i\sin t)\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix}+i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$=e^{t}\left[\cos t\begin{pmatrix} -1 \\ 2 \end{pmatrix}-\sin t\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]+ie^{t}\left[\left(\sin t\begin{pmatrix} -1 \\ 2 \end{pmatrix}+\cos\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\right]$$

$$=e^{-t}\begin{pmatrix} -\cos t-\sin t \\ 2\cos t \end{pmatrix}+ie^{-t}\begin{pmatrix} \cos t-\sin t \\ 2\sin t \end{pmatrix}$$

$$\boxed{x(t)=C_1e^{-t}\begin{pmatrix} -\cos t-\sin t \\ 2\cos t \end{pmatrix}+C_2e^{-t}\begin{pmatrix} \cos t-\sin t \\ 2\sin t \end{pmatrix}}$$

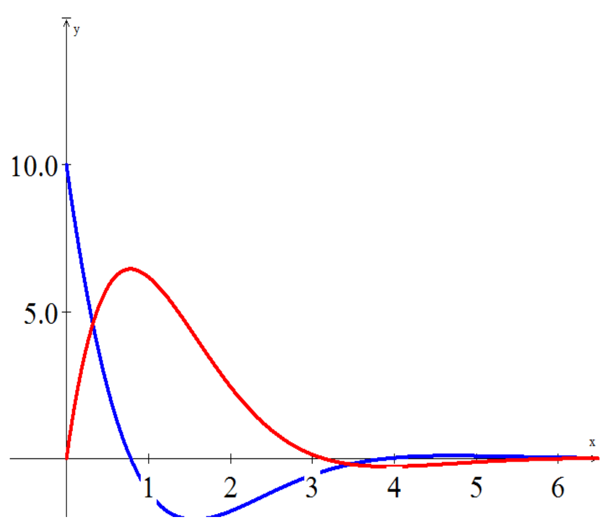
$$x(0)=(1)\begin{pmatrix} -1-0 \\ 2(1) \end{pmatrix}+i(1)\begin{pmatrix} 1-0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 10 \\ 0 \end{pmatrix}=C_1\begin{pmatrix} -1 \\ 2 \end{pmatrix}+C_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 10 \\ 0 \end{pmatrix}=\begin{pmatrix} -1C_1+C_2 \\ 2C_1 \end{pmatrix} \quad \Rightarrow \quad C_1=0 \quad C_2=10$$

$$\boxed{x(t)=10e^{-t}\begin{pmatrix} \cos t-\sin t \\ 2\sin t \end{pmatrix}}$$

$$\begin{pmatrix} V(t) \\ I(t) \end{pmatrix}=x(t)=\boxed{\begin{pmatrix} 10e^{-t}(\cos t-\sin t) \\ 20e^{-t}\sin t \end{pmatrix}}$$



### *Exercise*

Show that the voltage $V$ across the capacitor and the current $I$ through the inductor satisfy the system

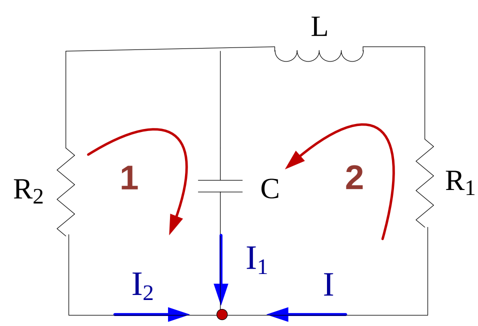
$$\begin{cases} I' = -\frac{R_1}{L} I + \frac{1}{L} V \\ V' = -\frac{1}{C} I - \frac{1}{R_2 C} V \end{cases}$$

Suppose that the capacitance is $C = 1$ *farad*, the inductance is $L = 1$ *henry*, the leftmost resistor has resistance $R_2 = 1\,\Omega$, and the rightmost resistor has resistance $R_1 = 5\,\Omega$. If the initial voltage across the capacitor is 12 volts and the initial current through the inductor is zero, determine the voltage $V$ across the capacitor and the current $I$ through the inductor as functions of time. Plot the voltage and current as functions of time. Assume current flows in the directions indicated.

**Solution**

The current coming into the node at a must equal the current coming out,

$$I + I_1 + I_2 = 0$$




$$-R_2 I_2 + V = 0$$

$$-R_2 \left(-I - I_1\right) + V = 0$$

$$R_2 I + R_2 I_1 = -V$$

The voltage across the capacitor follows the law

$V = \frac{1}{C} q_1$, where $q_1$ is the charge in the capacitor.

$$CV = q_1$$

$$(CV)' = \left(q_1\right)'$$

$$CV' = q_1{}' = I_1$$

$$R_2 I + R_2 I_1 = -V \quad \rightarrow \quad R_2 I + R_2 \left(CV'\right) = -V$$

$$R_2 CV' = -V - R_2 I$$

$$\boxed{V' = -\frac{1}{R_2 C} V - \frac{1}{C} I}$$

*Loop* **2**:

$$-V + LI' + R_1 I = 0$$

$$LI' = V - R_1 I$$

$$\boxed{I' = \frac{1}{L} V - \frac{R_1}{L} I}$$

$$\begin{pmatrix} V \\ I \end{pmatrix}' = \begin{pmatrix} -\frac{1}{R_2C} & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R_1}{L} \end{pmatrix}\begin{pmatrix} V \\ I \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{(1)(1)} & -\frac{1}{1} \\ \frac{1}{1} & -\frac{5}{1} \end{pmatrix}\begin{pmatrix} V \\ I \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} V \\ I \end{pmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & -1 \\ 1 & -5-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(-5-\lambda)+1$$

$$= \lambda^2 + 6\lambda + 6 = 0$$

$\therefore$ The eigenvalues are: $\lambda = -3 \pm \sqrt{3}$

For $\lambda_1 = -3+\sqrt{3} \quad \Rightarrow (A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2-\sqrt{3} & -1 \\ 1 & -2-\sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} (2-\sqrt{3})x - y = 0 \\ x - (2+\sqrt{3})y = 0 \end{cases}$$

$$V_1 = \begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} \quad \rightarrow \quad x_1(t) = \begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} e^{(-3+\sqrt{3})t}$$

For $\lambda_2 = -3-\sqrt{3}$

$$V_2 = \begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix} \quad \rightarrow \quad x_2(t) = \begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix} e^{(-3-\sqrt{3})t}$$

$$\mathbf{x}(t) = C_1 e^{(-3+\sqrt{3})t}\begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} + C_2 e^{(-3-\sqrt{3})t}\begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix}$$

***Given***: $\quad V_0 = 12\ V \quad I_0 = 0A$

$$\begin{pmatrix} 12 \\ 0 \end{pmatrix} = C_1\begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix}$$

$$\begin{cases} (2+\sqrt{3})C_1 + (2-\sqrt{3})C_2 = 12 \\ C_1 + C_2 = 0 \end{cases} \quad \Rightarrow \quad C_1 = 2\sqrt{3}, \ C_2 = -2\sqrt{3}$$

$$\mathbf{x}(t) = 2\sqrt{3}e^{(-3+\sqrt{3})t}\begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} - 2\sqrt{3}e^{(-3-\sqrt{3})t}\begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} \left(4\sqrt{3}+6\right)e^{\left(-3+\sqrt{3}\right)t} - \left(4\sqrt{3}-6\right)e^{\left(-3-\sqrt{3}\right)t} \\ 2\sqrt{3}e^{\left(-3+\sqrt{3}\right)t} - 2\sqrt{3}e^{\left(-3-\sqrt{3}\right)t} \end{pmatrix}$$

Which leads to the solutions

$$V(t) = \left(4\sqrt{3}+6\right)e^{\left(-3+\sqrt{3}\right)t} - \left(4\sqrt{3}-6\right)e^{\left(-3-\sqrt{3}\right)t}$$

$$I(t) = 2\sqrt{3}e^{\left(-3+\sqrt{3}\right)t} - 2\sqrt{3}e^{\left(-3-\sqrt{3}\right)t}$$