Section 2.6 – Linear Independence

There are n columns in an m by n matrix, and each column has m components. But the true ***dimension*** of the column space is not necessarily m or n. The dimension is measured by counting ***independent columns***.

- **Independent vectors** (*not too many*)
- **Spanning a space** (*not too few*)

Linear Independence (LI)

The columns of A are ***linearly independent*** when the only solution to $A\vec{x} = 0$ is $\vec{x} = \vec{0}$. ***No other combination*** Ax ***of the columns gives the zero vector***.

Definitions

- A set of two or more vectors is ***linearly dependent*** if one vector in the set is a linear combination of the others. A set of one vector is ***linearly dependent*** if that one vector is the zero vector.

$$\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_k$$

- The sequence of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is ***linearly independent*** if the only combination that gives the zero vector is $0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_k$. Thus, linear independence means that:

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k = 0 \text{ only happens when all } x\text{'s are zero.}$$

- A (nonempty) set of vectors is ***linearly independent*** if it is not linearly dependent.
- If three vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3$ are in the same plane, they are dependent.
- The empty set is linearly independent, for linearly dependent sets must be nonempty.
- A set consisting of a single nonzero vector is linearly independent. For if $\{\vec{v}\}$ is linearly dependent, then $a\vec{v} = \vec{0}$ for some nonzero scalar a. Thus,

$$\vec{v} = a^{-1}(a\vec{v}) = a^{-1}\vec{0} = \vec{0}$$

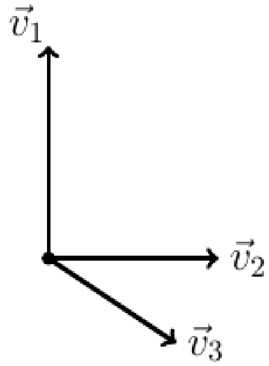
Theorem

A set S with two or more vectors $S = \{\vec{v}_1, \cdots, \vec{v}_k\}$ is

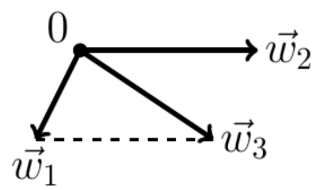
a) Linearly dependent *iff* at least one of the vectors in S is expressible as a linear combination of the other vectors in S. There are numbers $c_1, \ldots, c_k$ at least one of which is nonzero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = 0$$

b) Linearly independent *iff* no vector in S is expressible as a linear combination of the other vectors in S.



Independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$



Dependent vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3$

The combination $\vec{w}_1 - \vec{w}_2 + \vec{w}_3$ is (0, 0, 0)

Example

a) The vectors (1, 0) and (0, 1) are ***independent***.

b) The vectors (1, 1) and (1, 0.0001) are ***independent***.

c) The vectors (1, 1) and (2, 2) are ***dependent***.

d) The vectors (1, 1) and (0, 0) are ***dependent***.

Theorem

a) A finite set that contains $\vec{0}$ is linearly dependent.

b) A set with exactly one vector is linearly independent if and only if that vector is not $\vec{0}$.

c) A set with exactly two vectors is linearly independent *iff* neither vector is a scalar multiple of the other.

Theorem

Let S be a set $\boldsymbol{k}$ vectors in $\mathbb{R}^n$, then if $k > n$, S is ***linearly dependent***.

Example

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are 3 vectors in $\mathbb{R}^2 \Rightarrow$ *Linearly dependent*.

Example

Determine whether the vectors $\vec{v}_1 = (1,\ -2,\ 3)$ $\vec{v}_2 = (5,\ 6,\ -1)$ $\vec{v}_3 = (3,\ 2,\ 1)$ are linearly dependent or linearly independent in $\mathbb{R}^3$

Solution

$$k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \mathbf{0}$$

$$k_1(1,\ -2,\ 3) + k_2(5,\ 6,\ -1) + k_3(3,\ 2,\ 1) = (0,\ 0,\ 0)$$

$$\rightarrow \begin{cases} k_1 + 5k_2 + 3k_3 = 0 \\ -2k_1 + 6k_2 + 2k_3 = 0 \\ 3k_1 - k_2 + k_3 = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{bmatrix} \quad \begin{matrix} R_2 + 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & -16 & -8 & 0 \end{bmatrix} \quad R_3 + R_2$$

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \tfrac{1}{16}R_2$$

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 - 5R_2$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} k_1 + \frac{1}{2}k_3 = 0 \\ k_2 + \frac{1}{2}k_3 = 0 \end{matrix}$$

Solve the system equations: $k_1 = -\frac{1}{2}t, \quad k_2 = -\frac{1}{2}t, \quad k_3 = t$

This shows that the system has nontrivial solutions and hence that the vectors are linearly dependent.

2nd method to determine the linearly is to compute the determinant of the coefficient matrix

$$A = \begin{pmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{pmatrix}$$

$|A| = 0$ Which has nontrivial solutions and the vectors are *linearly dependent*.

Example

Determine whether the vectors are linearly dependent or linearly independent in $\mathbb{R}^4$

$$\vec{v}_1 = (1,\ 2,\ 2,\ -1) \quad \vec{v}_2 = (4,\ 9,\ 9,\ -4) \quad \vec{v}_3 = (5,\ 8,\ 9,\ -5)$$

Solution

$$k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0}$$

$$k_1(1,\ 2,\ 2,\ -1) + k_2(4,\ 9,\ 9,\ -4) + k_3(5,\ 8,\ 9,\ -5) = (0,\ 0,\ 0,\ 0)$$

Which yields the homogeneous linear system

$$\rightarrow \begin{cases} k_1 + 4k_2 + 5k_3 = 0 \\ 2k_1 + 9k_2 + 8k_3 = 0 \\ 2k_1 + 9k_2 + 9k_3 = 0 \\ -k_1 - 4k_2 - 5k_3 = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 4 & 5 & 0 \\ 2 & 9 & 8 & 0 \\ 2 & 9 & 9 & 0 \\ -1 & -4 & -5 & 0 \end{pmatrix} \quad \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \\ R_4 + R_1 \end{matrix}$$

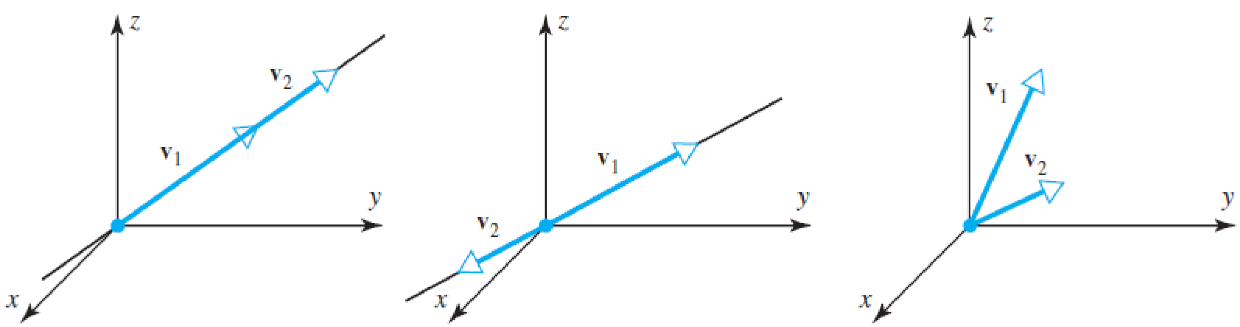
$$\begin{pmatrix} 1 & 4 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} R_1 - 4R_2 \\ \\ R_3 - R_2 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 13 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} R_1 - 13R_3 \\ R_2 + 2R_3 \\ \\ \\ \end{matrix}$$

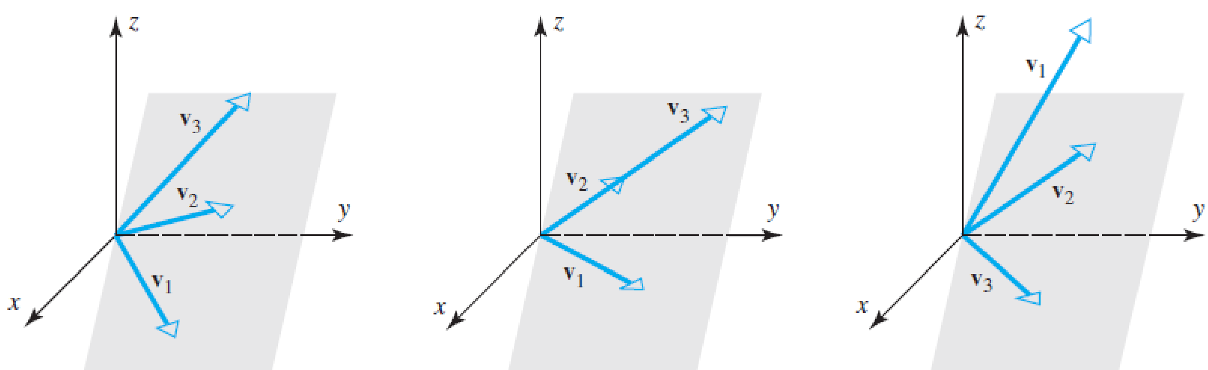
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{matrix} \to & k_1 = 0 \\ \to & k_2 = 0 \\ \to & k_3 = 0 \\ & \end{matrix}$$

Solve the system equations: $k_1 = 0, \quad k_2 = 0, \quad k_3 = 0$ has a trivial solution.

The vectors $\vec{v}_1, \vec{v}_2, and\ \vec{v}_3$ are *linearly independent*.



(a) Linearly dependent (b) Linearly dependent (c) Linearly independent



(a) Linearly dependent (b) Linearly dependent (c) Linearly independent

Linear independence of Functions

Definition

If $\mathbf{f}_1 = f_1(x),\ \mathbf{f}_2 = f_2(x),\ \ldots,\ \mathbf{f}_n = f_n(x)$ are functions that are $n - 1$ times differentiable on the interval $(-\infty, \infty)$, the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the ***Wronskian*** of $f_1, f_2, \ldots, f_n$

$$\begin{cases} if\ \ W = 0 & \Rightarrow \quad Linearly\ Dependent \\ if\ \ W \neq 0 & \Rightarrow \quad Linearly\ Independent \end{cases}$$

Example

Use the Wronskian to show that $\mathbf{f}_1 = x,\quad \mathbf{f}_2 = \sin x$ are linearly independence

Solution

The Wronskian is

$$W(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix}$$

$$= x\cos x - \sin x \neq 0$$

This function is not identically zero. Thus, the functions are linearly independent.

Example

Use the Wronskian to show that $\mathbf{f}_1 = 1, \quad \mathbf{f}_2 = e^x, \quad \mathbf{f}_3 = e^{2x}$ are linearly independence

Solution

The Wronskian is

$$W(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix}$$

$$= e^x 4e^{2x} - 2e^{2x} e^x$$

$$= 2e^{3x} \neq 0$$

Thus, the functions are linearly independent.

Theorem

Let S be a linearly independent subset of a vector space V, and let $\vec{v}$ be a vector in V that is not in S. Then $S \cup \{\vec{v}\}$ is linearly dependent if and only if $\vec{v} \in span(S)$

Proof

If $S \cup \{\vec{v}\}$ is linearly dependent, then there are vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ in $S \cup \{\vec{v}\}$ such that $a_1\vec{u}_1 + a_2\vec{u}_2 + \ldots + a_n\vec{u}_n = \vec{0}$ for some nonzero scalars $a_1, a_2, \ldots, a_n$.

Because S is linearly independent, one of the $\vec{u}_i$'s say $\vec{u}_1$, equal $\vec{v}$. Thus $a_1\vec{v} + a_2\vec{u}_2 + \ldots + a_n\vec{u}_n = \vec{0}$, and so

$$\begin{aligned}
a_1\vec{v} &= -\left(a_2\vec{u}_2 + \ldots + a_n\vec{u}_n\right) \\
\vec{v} &= -a_1^{-1}\left(a_2\vec{u}_2 + \ldots + a_n\vec{u}_n\right) \\
&= -\left(a_1^{-1}a_2\right)\vec{u}_2 - \ldots - \left(a_1^{-1}a_n\right)\vec{u}_n
\end{aligned}$$

Since $\vec{v}$ is linear combination of $\vec{u}_2, \ldots, \vec{u}_n$, which are in S, we have $\vec{v} \in span(S)$.

Conversely, let $\vec{v} \in span(S)$.

Then there exist vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ in S and scalars $b_1, b_2, \ldots, b_m$ such that $\vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2 + \ldots + b_m\vec{v}_m$. Hence,

$$b_1\vec{v}_1 + b_2\vec{v}_2 + \ldots + b_m\vec{v}_m + (-1)\vec{v} = \vec{0}$$

Since $\vec{v} \neq \vec{v}_i$ for $i = 1, 2, \ldots, m$, the coefficient of $\vec{v}$ in this linear combination is nonzero, and so the set $\left\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m, \vec{v}\right\}$ is linearly dependent.

Therefore $S \cup \{\vec{v}\}$ is *linearly dependent*.

Exercises *Section* 2.6 – Linear Independence

1. State the following statements as ***true*** or ***false***

a) If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.

b) Any set containing the zero vector is linearly dependent.

c) The empty set is linearly dependent.

d) Subsets of linearly dependent sets are linearly dependent.

e) Subsets of linearly independent sets are linearly independent.

f) If $a_1x_1 + a_2x_2 + \ \ldots + a_nx_n = \vec{0}$ and $x_1, x_2, \ldots, x_n$ are linearly independent, the null the scalars a_i are zero

2. Given three independent vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3$. Take combinations of those vectors to produce $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Write the combinations in a matrix form as $V = WM$.

$$\begin{aligned}\vec{v}_1 &= \vec{w}_1 + \ \vec{w}_2\\ \vec{v}_2 &= \vec{w}_1 + 2\vec{w}_2 + \vec{w}_3\\ \vec{v}_1 &= \qquad\ \vec{w}_2 + c\vec{w}_3\end{aligned}$$

which is $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix}$

What is the test on a matrix **V** to see if its columns are linearly independent?

If $c \neq 1$ show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

If $c = 1$ show that $\vec{v}\,'s$ are linearly *dependent*.

3. Find the largest possible number of independent vectors among

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \vec{v}_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \vec{v}_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad \vec{v}_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

4. Show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent but $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are dependent:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{v}_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

Solve either $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = 0$ *or* $A\vec{x} = 0$. The v's go in the columns of $\boldsymbol{A}$.

5. Decide the dependence or independence of

a) The vectors (1, 3, 2), (2, 1, 3), and (3, 2, 1).

b) The vectors $(1,\ -3,\ 2)$, $(2,\ 1,\ -3)$, and $(-3,\ 2,\ 1)$.

6. Find two independent vectors on the plane $x+2y-3z-t=0$ in $\mathbb{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

7. Determine whether the vectors are linearly dependent or linearly independent in $\mathbb{R}^3$

a) (4, –1, 2), (–4, 10, 2)

b) (8, –1, 3), (4, 0, 1)

c) (–3, 0, 4), (5, –1, 2), (1, 1, 3)

d) (–2, 0, 1), (3, 2, 5), (6, –1, 1), (7, 0, –2)

8. Determine whether the vectors are linearly dependent or linearly independent in $\mathbb{R}^4$

a) $\{(3,\ 8,\ 7,\ -3),\ (1,\ 5,\ 3,\ -1),\ (2,\ -1,\ 2,\ 6),\ (1,\ 4,\ 0,\ 3)\}$

b) $\{(0,\ 0,\ 2,\ 2),\ (3,\ 3,\ 0,\ 0),\ (1,\ 1,\ 0,-1)\}$

c) $\{(0,\ 3,\ -3,\ -6),\ (-2,\ 0,\ 0,\ -6),\ (0,\ -4,\ -2,\ -2),\ (0,\ -8,\ 4,\ -4)\}$

d) $\{(3,\ 0,\ -3,\ 6),\ (0,\ 2,\ 3,\ 1),\ (0,\ -2,\ -2,\ 0),\ (-2,\ 1,\ 2,\ 1)\}$

e) $\{(1,\ 3,\ -4,\ 2),\ (2,\ 2,\ -4,\ 0),\ (2,\ 3,\ 2,\ -4),\ (-1,\ 0,\ 1,\ 0)\}$

f) $\{(1,\ 3,\ -4,\ 2),\ (2,\ 2,\ -4,\ 0),\ (1,\ -3,\ 2,\ -4),\ (-1,\ 0,\ 1,\ 0)\}$

g) $\{(1,\ 0,\ 0,\ -1),\ (0,\ 1,\ 0,\ -1),\ (0,\ 1,\ 0,\ -1),\ (0,\ 0,\ 0,\ 1)\}$

9. *a*) Show that the three vectors $\vec{v}_1=(1,\ 2,\ 3,\ 4)$ $\vec{v}_2=(0,\ 1,\ 0,\ -1)$ $\vec{v}_3=(1,\ 3,\ 3,\ 3)$ form a linearly dependent set in $\mathbb{R}^4$.

b) Express each vector in part (*a*) as a linear combination of the other two.

10. For which real values of λ do the following vectors form a linearly dependent set in $\mathbb{R}^3$

$$\vec{v}_1=\left(\lambda,\ -\tfrac{1}{2},\ -\tfrac{1}{2}\right)\quad \vec{v}_2=\left(-\tfrac{1}{2},\ \lambda,\ -\tfrac{1}{2}\right)\quad \vec{v}_3=\left(-\tfrac{1}{2},\ -\tfrac{1}{2},\ \lambda\right)$$

11. Show that if $S=\{\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n\}$ is a linearly independent set of vectors, then so is every nonempty subset of S.

12. Show that if $S=\{\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_r\}$ is a linearly dependent set of vectors in a vector space *V*, and if $\vec{v}_{r+1},\ \ldots,\ \vec{v}_n$ are vectors in *V* that are not in *S*, then $\{\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_r,\ \vec{v}_{r+1},\ \ldots,\ \vec{v}_n\}$ is also linearly dependent.

13. Show that $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent and $\vec{v}_3$ does not lie in span $\{\vec{v}_1, \vec{v}_2\}$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent.

14. By using the appropriate identities, where required, determine $F(-\infty, \infty)$ are linearly dependent.

a) $6,\ 3\sin^2 x,\ 2\cos^2 x$

b) $x,\ \cos x$

c) $1,\ \sin x,\ \sin 2x$

d) $(3-x)^2,\ x^2-6x,\ 5$

e) $\cos 2x,\ \sin^2 x,\ \cos^2 x$

15. $f_1(x)=\sin x,\quad f_2(x)=\cos x$ are linearly independent in $F(-\infty, \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using Wrońskian's test.

16. Show $f_1(x)=e^x,\quad f_2(x)=xe^x\quad f_3(x)=x^2e^x$ are linearly independent in $F(-\infty, \infty)$.

17. Use the Wronskian to show that $f_1(x)=\sin x,\quad f_2(x)=\cos x,\quad f_3(x)=x\cos x$ span a three-dimensional subspace of $F(-\infty, \infty)$

18. Show by inspection that the vectors are linearly dependent.

$\vec{v}_1(4,\ -1,\ 3),\ \ \vec{v}_2(2,\ 3,\ -1),\ \ \vec{v}_3(-1,\ 2,\ -1),\ \ \vec{v}_4(5,\ 2,\ 3),\ \ in\ \mathbb{R}^3$

(**19 – 37**) Determine if the given vectors are linearly dependent or independent, (any method)

19. $(2,\ -1,\ 3),\ \ (3,\ 4,\ 1),\ \ (2,\ -3,\ 4),\ \ in\ \mathbb{R}^3$

20. $(1,\ 0,\ 0,\ 0),\ \ (1,\ 1,\ 0,\ 0),\ \ (1,\ 1,\ 1,\ 0),\ \ (1,\ 1,\ 1,\ 1),\ \ in\ \mathbb{R}^4$

21. $A_1\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix},\ \ A_2\begin{bmatrix} -2 & 4 \\ 1 & 0 \end{bmatrix},\ \ A_3\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix},\ \ in\ M_{22}$

22. $\left\{\begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix}, \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix}, \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix}\right\}$ in $M_{2\times 3}(\mathbb{R})$

23. $\left\{\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix}\right\}$ in $M_{2\times 2}(\mathbb{R})$

24. $\left\{\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}\right\}$ in $M_{2\times 2}(\mathbb{R})$

25. $\left\{\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix}\right\}$ $in\ \ M_{2\times 2}(\mathbb{R})$

26. $\left\{\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}-1 & 2\\ 1 & 0\end{pmatrix}, \begin{pmatrix}2 & 1\\ 2 & -2\end{pmatrix}\right\}$ *in* $M_{2\times 2}(\mathbb{R})$

27. $\left\{e^x,\ \ln x\right\}$ *in* $\mathbb{R}$

28. $\left\{x,\ \frac{1}{x}\right\}$ *in* $\mathbb{R}$

29. $\{1+x,\ 1-x\}$ *in* $\mathbb{P}_2(\mathbb{R})$

30. $\left\{9x^2-x+3,\ 3x^2-6x+5,\ -5x^2+x+1\right\}$ *in* $\mathbb{P}_3(\mathbb{R})$

31. $\left\{-x^2,\ 1+4x^2\right\}$ *in* $\mathbb{P}_3(\mathbb{R})$

32. $\left\{7x^2+x+2,\ 2x^2-x+3,\ -3x^2 4\right\}$ *in* $\mathbb{P}_3(\mathbb{R})$

33. $\left\{3x^2+3x+8,\ 2x^2+x,\ 2x^2+2x+2,\ 5x^2-2x+8\right\}$ *in* $\mathbb{P}_3(\mathbb{R})$

34. $\left\{x^3+2x^2,\ -x^2+3x+1,\ x^3-x^2+2x-1\right\}$ *in* $\mathbb{P}_3(\mathbb{R})$

35. $\left\{x^3-x,\ 2x^2+4,\ -2x^3+3x^2+2x+6\right\}$ *in* $\mathbb{P}_3(\mathbb{R})$

36. $\left\{\begin{matrix}x^4-x^3+5x^2-8x+6,\quad -x^4+x^3-5x^2+5x-3,\quad x^4+3x^2-3x+5,\\ 2x^4+3x^3+4x^2-x+1,\ x^3-x+2\end{matrix}\right\}$ *in* $\mathbb{P}_4(\mathbb{R})$

37. $\left\{\begin{matrix}x^4-x^3+5x^2-8x+6,\quad -x^4+x^3-5x^2+5x-3,\\ x^4+3x^2-3x+5,\quad 2x^4+x^3+4x^2+8x\end{matrix}\right\}$ *in* $\mathbb{P}_4(\mathbb{R})$

38. Suppose that the vectors $\vec{u}_1, \vec{u}_2,$ *and* $\vec{u}_3$ are linearly dependent. Are the vectors $\vec{v}_1=\vec{u}_1+\vec{u}_2$, $\vec{v}_2=\vec{u}_1+\vec{u}_3$, and $\vec{v}_3=\vec{u}_2+\vec{u}_3$ also linearly dependent?

(***Hint***: Assume that $a_1\vec{v}_1+a_2\vec{v}_2+a_3\vec{v}_3=0$, and see what the a_i's can be.)

39. Show that the set $F=\left\{1+t,\ t^2,\ t-2\right\}$ is a linearly independent subset of $\mathbb{P}_2$

40. Suppose that *A* is linearly dependent set of vectors and *B* is any set containing *A*. Show that *B* must be linearly dependent.

41. Show that $\{\sin t,\ \sin 2t,\ \cos t\}$ is a linearly independent, subset of $C[0,\ 1]$. Does it span $C[0,\ 1]$

42. Show that the set $\{\sin(t+a),\ \sin(t+b),\ \sin(t+c)\}$ is linearly dependent on $C[0,\ 1]$

43. Show that if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent and $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta$ are linearly dependent, then β can be uniquely expressed as a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$.

44. Show that if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly dependent with $(\alpha_1 \neq 0)$ if and only if there exists an integer k $(1 < k \leq n)$, such that α_k is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$