Section 3.5 – Introduction & Basic Theory of Linear Systems

A system of differential equations is a set of one or more equations, involving one or more differential equations.

There are several physical problems that involve number of separate elements such as an example: electrical networks, mechanical, and more other fields.

Example of *Predator-Prey* Systems (*Ecology*)

The dynamical of biological growth of populations is a branch of ecology.

The growth rate of species is depending on their population. The rate is increased by birth and food supply causes the species to live, and decreased by death, overcrowded, etc….

Consider two species that exist together and interact as an example such as wolves and deer, shark and food fish, etc... Vito Volterra, an Italian mathematician, formulated a predator-prey system model.

Let the *prey* population denoted by $F(t)$.

Let the *predator* population denoted by $G(t)$.

For each $F(t)$ and $G(t)$, we have a reproductive rate which denoted by r_F and r_G for prey and predator respectively.

Therefore, that will imply to: $\begin{cases} F' = r_F F(t) \\ G' = r_G G(t) \end{cases}$

Let assume there is absence of predator, by using *Malthusian*model, the prey population will be given by

$$G = 0 \quad \Rightarrow R_F = a > 0$$

When there are predator activities, then , and the decrease in the reproductive rate would also be proportional to $G(t)$.

$$R_F = a - bG \qquad a,\ b > 0$$

In the absence of prey, by using *Malthusian* model, the predator population will be given by

$$F = 0 \quad \Rightarrow R_G = -c < 0$$

The presence of the prey would decrease in the reproductive rate would be proportional to the size of the prey population..

$$R_G = -c + d\ F \qquad c,\ d > 0$$

That will give us a system of: $\begin{cases} F' = (a - bG)F \\ G' = (-c + d\ F)G \end{cases}$

This model is ***nonlinear*** because the right-side contains the product FG.
It is ***autonomous*** because the right-side doesn't depend explicitly on the independent variable.

Summary of Predator-Prey

The Predator-Prey or ***Lotka–Volterra*** system is given by:

$$\begin{cases} \dot{x} = -ax + bxy \\ \dot{y} = cy - dxy \end{cases}$$

Where x is the predator, their prey is 'y', and the coefficient a, b, c, and d are positive real numbers and they are defined as follow:

a: is the natural decay.

ax: is a rate term, which shows that without prey to eat, the predator population diminishes.

c: is the natural growth coefficient.

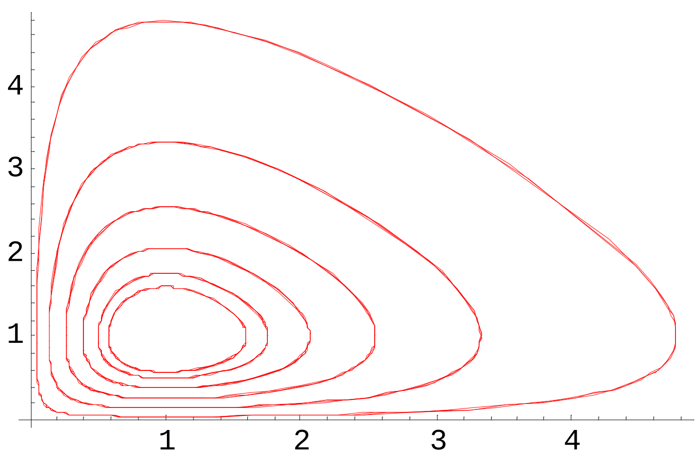
cy: is a rate term, where the prey population increases.

b** and **c: predator efficiency in converting food into fertility and the probability that are predator-prey encounter removes of the prey.

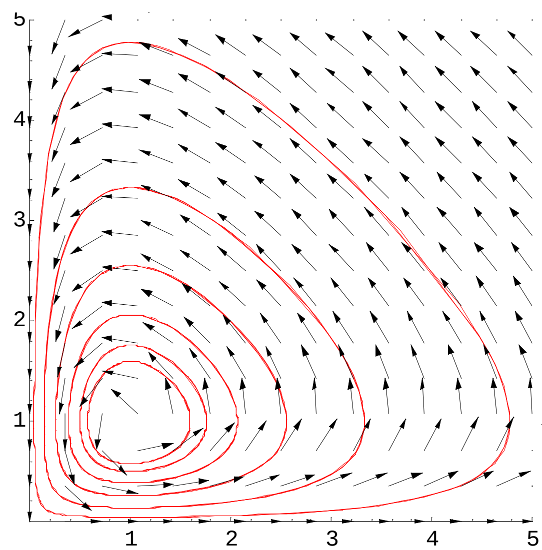
bxy** and **cxy: Food promotes the predator population's growth rate, while serving as food diminishes the prey populations' growth rate.

The predator-prey system or model is based on the population Law of Mass Action.
The Law of Mass Action is defined as:

"*The rate of change of one population due to interaction of another is proportional to the product of the two populations*."



(*a*)



(*b*) Flow direction.

$$\begin{cases} \dot{x} = x - xy \\ \dot{y} = -y + xy \end{cases}$$

Where $x(t)$ is the prey,
$y(t)$ is the predator.

Definition

A linear system of differential equations is any set of differential equations having the following ***standard form***:

$$\begin{aligned}
x_1' &= a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + f_1(t) \\
x_2' &= a_{21}(t)x_1 + \cdots + a_{2n}(t)x_n + f_2(t) \\
&\vdots \qquad\qquad \vdots \\
x_n' &= a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + f_n(t)
\end{aligned}$$

Where $x_1, \cdots, and\ x_n$ are the unknown functions. The ***coefficients*** $a_{ij}(t)$ and $f_i(t)$ are known functions of the independent variable $t \in (a,\ b)$ is an interval in **R**.

If all of the $f_i(t) = 0$, the system said to be ***homogeneous***. Otherwise it is ***inhomogeneous***.
The inhomogeneous part is sometimes called the ***forcing term***.

Matrix Notation for Linear Systems

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

In simple form, we can rewrite:

$$x'(t) = A(t)x(t) + f(t)$$

$$x' = Ax + f$$

Example

Given the linear system $\begin{cases} x_1' = x_1 + 2x_2 \\ x_2' = 2x_1 + x_2 \end{cases}$

Write in the form $x' = Ax + f$

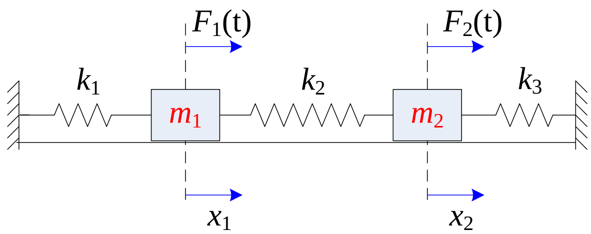
Solution

$$x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Example of a *Spring-Mass* (*mechanical*)

Two masses move on a frictionless surface under the influence of external forces $F_1(t)$ and $F_2(t)$, and they are also constraint by the three springs whose constant are $k_1,\ k_2,\ and\ k_3$



Let examine the forces acting on m_1

The ***first*** spring exerts a force of: $f_1 = -k_1 x_1$ k_1 ***is the spring constant***.

The ***second*** spring exerts a force of: $f_2 = k_2\left(x_2 - x_1\right)$

By Newton's second law:

$$m_1 \frac{d^2 x_1}{dt^2} = \sum forces = f_1 + f_2 + F_1(t)$$

$$m_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2\left(x_2 - x_1\right) + F_1(t)$$

$$= -\left(k_1 + k_2\right) x_1 + k_2 x_2 + F_1(t)$$

The forces acting on m_2

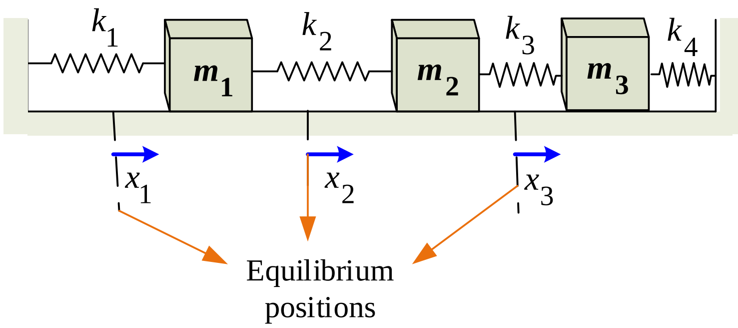
The ***second*** spring exerts a force of: $-f_2 = -k_2\left(x_2 - x_1\right)$

The ***third*** spring exerts a force of: $f_3 = -k_3 x_2$

$$m_2 \frac{d^2 x_2}{dt^2} = -k_2\left(x_2 - x_1\right) - k_3 x_2 + F_2(t)$$

$$= k_2 x_1 - \left(k_2 + k_3\right) x_2 + F_2(t)$$

Example



Consider the mass-and–spring systems, as shown above. Three masses connected to each other and to two walls by 4 indicated springs. Assume the masses slide without friction and each spring obeys Hooke's law $(F = -kx)$.

By applying Newton's law $F = ma$ to the 3-masses:

$$m_1 x_1'' = -k_1 x_1 \qquad + k_2 (x_2 - x_1)$$

$$m_2 x_2'' = -k_2 (x_2 - x_1) + k_3 (x_3 - x_2)$$

$$m_3 x_3'' = -k_3 (x_3 - x_2) - k_4 x_3$$

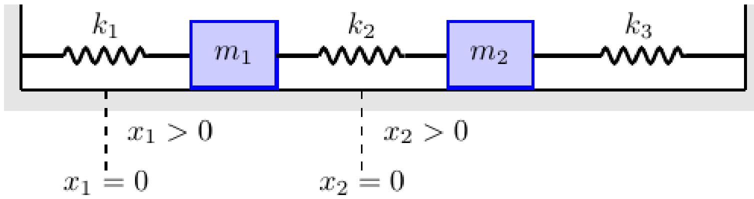
The displacement vector: $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

The mass matrix $M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}$

The stiffness matrix $K = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{pmatrix}$

Example

Consider the spring-mass system consisting of two masses that are constraint by the three springs whose constant are $k_1,\ k_2,\ and\ k_3$. Assume there is no damping and there are no external forces.



Write an equivalent linear system of the first-order differential equations.

Solution

$$m_1 x_1'' = -\left(k_1 + k_2\right)x_1 + k_2 x_2$$

$$m_2 x_2'' = k_2 x_1 - \left(k_2 + k_3\right)x_2$$

$$\begin{cases} x_1'' = -\dfrac{k_1+k_2}{m_1}x_1 + \dfrac{k_2}{m_1}x_2 \\ x_2'' = \dfrac{k_2}{m_2}x_1 - \dfrac{k_2+k_3}{m_2}x_2 \end{cases}$$

To write an equivalent first-order system, let $U = \left(u_1, u_2, u_3, u_4\right)^T$

Where $u_1(t) = x_1(t) \qquad u_2(t) = x_1'(t) \qquad u_3(t) = x_2(t) \qquad u_4(t) = x_2'(t)$

$$\begin{cases} u_2' = x_1'' = -\dfrac{k_1+k_2}{m_1}u_1 + \dfrac{k_2}{m_1}u_3 \\ u_4' = x_2'' = \dfrac{k_2}{m_2}u_1 - \dfrac{k_2+k_3}{m_2}u_3 \end{cases} \quad \rightarrow \quad \begin{cases} u_1' = u_2 \\ u_2' = -\dfrac{k_1+k_2}{m_1}u_1 + \dfrac{k_2}{m_1}u_3 \\ u_3' = u_4 \\ u_4' = \dfrac{k_2}{m_2}u_1 - \dfrac{k_2+k_3}{m_2}u_3 \end{cases}$$

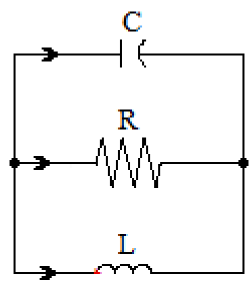
$$\begin{pmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{k_1+k_2}{m_1} & 0 & \dfrac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{k_2}{m_2} & 0 & -\dfrac{k_2+k_3}{m_2} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

The initial conditions for this system involve the initial position and velocity of both masses.

$$x_1(0) = a_1 \qquad x_1'(0) = b_1 \qquad x_2(0) = a_2 \qquad x_2'(0) = b_2$$

Example of a *parallel* LRC circuit

Consider the parallel LRC circuit as shown below



Let V be the voltage drop across the capacitor and I current through the inductor.

The current is described by the ***equation***: $\frac{dI}{dt}=\frac{V}{L}$

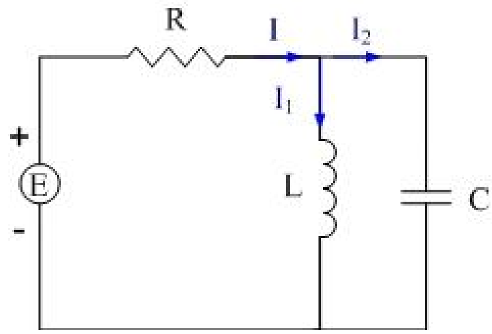
The voltage is described by the ***equation***: $\frac{dV}{dt}=-\frac{1}{C}-\frac{V}{RC}$

Therefore, we can rewrite the equation as ***system equations***

$$\begin{cases} I'=\frac{1}{L}V \\ V'=-\frac{1}{C}-\frac{1}{RC}V \end{cases}$$

Example

Find a first-order system the models the circuit below



Solution

Using Kirchhoff's current law: $I=I_1+I_2$

Kirchhoff's voltage law applied to the loop containing the source and the inductor:

$E=RI+LI_1'$

$E=R\left(I_1+I_2\right)+LI_1'$

$LI_1'=E-R\left(I_1+I_2\right)$

$$I_1' = \frac{1}{L}\left[E - R\left(I_1 + I_2\right)\right]$$

Kirchhoff's voltage law applied to the loop containing the source, resistor and the capacitor:

$$E = RI + \frac{1}{C}Q$$

Differentiate the equation:

$$E' = RI' + \frac{1}{C}Q' \qquad\qquad Q' = I_2$$

$$E' = R\left(I_1' + I_2'\right) + \frac{1}{C}I_2$$

$$E' = RI_1' + RI_2' + \frac{1}{C}I_2$$

$$RI_2' = E' - RI_1' - \frac{1}{C}I_2$$

$$= E' - R\frac{1}{L}\left[E - R\left(I_1 + I_2\right)\right] - \frac{1}{C}I_2$$

$$I_2' = \frac{1}{R}E' - \frac{1}{L}\left[E - R\left(I_1 + I_2\right)\right] - \frac{1}{RC}I_2$$

$$= \frac{1}{R}E' - \frac{1}{L}E + \frac{R}{L}I_1 + \frac{R}{L}I_2 - \frac{1}{RC}I_2$$

$$= \frac{R}{L}I_1 + \left(\frac{R}{L} - \frac{1}{RC}\right)I_2 + \frac{E'}{R} - \frac{E}{L}$$

$$\begin{cases} I_1' = -\frac{R}{L}I_1 - \frac{R}{L}I_2 + \frac{E}{L} & (1) \\ I_2' = \frac{R}{L}I_1 + \left(\frac{R}{L} - \frac{1}{RC}\right)I_2 + \frac{E'}{R} - \frac{E}{L} & (2) \end{cases}$$

Properties of Linear Systems

Properties of Homogeneous Systems

Theorem

Suppose x_1 and x_2 are solution to the homogeneous linear system

$$x' = Ax$$

If C_1 and C_2 are any constants, then $x = C_1 x_1 + C_2 x_2$ is also a solution

Theorem

Suppose $x_1, x_2, \ldots,$ *and* x_n are solution to the homogeneous linear system

If $C_1, C_2, \ldots,$ *and* C_n are any constants, then

$$x(t) = C_1 x_1(t) + C_2 x_2(t) + \cdots + C_n x_n(t)$$

is also a solution to $x' = Ax$

Linearly Independence and Dependence

Proposition

Suppose $y_1, y_2, \ldots,$ *and* y_n are solution to the n-dimensional system $y' = Ay$ defined on the interval $I = (\alpha,\ \beta)$.

1. If the vectors $y_1(0), y_2(0), \ldots,$ *and* $y_n(0)$ are linearly dependent for some $t_0 \in I$, then there are constants $C_1, C_2, \ldots,$ *and* C_n not all zero, such that $C_1 y_1(t) + C_2 y_2(t) + \cdots + C_n y_n(t) = 0$ for all $t \in I$. In particular, $y_1(t), y_2(t), \ldots,$ *and* $y_n(t)$ are linearly dependent for all $t \in I$.

2. If for some $t_0 \in I$ the vectors $y_1(0), y_2(0), \ldots,$ *and* $y_n(0)$ are linearly independent, then $y_1(t), y_2(t), \ldots,$ *and* $y_n(t)$ are linearly independent for all $t \in I$.

Definition

A set of ***n*** solutions to the linear system $x' = Ax$ is linearly independent if it is linearly independent for any one value of *t*.

Example

Given $x_1(t)=\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$ and $x_2(t)=\begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$ are solutions to the homogeneous system $x'(t)=\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}x(t)$

Show that all solutions to this system can be expressed as linear combination of x_1 and x_2

Solution

$$x_1(t)=\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}=e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$x_2(t)=\begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}=e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x(t)=C_1x_1(t)+C_2x_2(t)$$

$$\begin{aligned} x(t=0) &= C_1x_1(0)+C_2x_2(0) \\ &= C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}+C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} C_1+C_2 \\ -C_1+C_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}=\left[x_1(0),\ x_2(0)\right]$$

$\det=\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}=2\neq 0 \Rightarrow$ The matrix is nonsingular and $x_1(0)$, $x_2(0)$ are linearly independent

$$x(0)=\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Example

Consider the system of homogeneous equations

$$x'(t) = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix} x(t)$$

We can show that

$$x_1(t) = \begin{pmatrix} e^t \cos t \\ e^t (\cos t - \sin t) \end{pmatrix} \qquad x_2(t) = \begin{pmatrix} e^t \sin t \\ e^t (\cos t + \sin t) \end{pmatrix}$$

are solutions the given system

$$x_1(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x(t) = C_1 x_1(t) + C_2 x_2(t)$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$\det = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow x_1(0)$ and $x_2(0)$ are linearly independent

Exercises *Section* 3.5 – Introduction & Basic Theory of Linear Systems

For the linear systems which are homogeneous? Which are inhomogeneous?

1. $\begin{cases} x_1' = -2x_1 + x_1x_2 \\ x_2' = -3x_1 - x_2 \end{cases}$ **2.** $\begin{cases} x_1' = -x_2 \\ x_2' = \sin x_1 \end{cases}$ **3.** $\begin{cases} x_1' = x_1 + (\sin t)x_2 \\ x_2' = 2tx_1 - x_2 \end{cases}$

Write the given system of equations in matrix-form then show that the given vector is a solution to the system

4. $\begin{cases} x_1' = -3x_1 + x_2 \\ x_2' = -2x_1 \end{cases}$ $\quad v = \left(-e^{-2t} + e^{-t}, -e^{-2t} + 2e^{-t}\right)^T$

5. $\begin{cases} x_1' = -x_1 + 4x_2 \\ x_2' = \qquad 3x_2 \end{cases}$ $\quad v = \left(e^{3t} - e^{-t}, e^{3t}\right)^T$

Verify by substitution that $x_1(t)$ and $x_2(t)$ are solutions of the given homogenous equation. Show also that the solutions $x_1(t)$ and $x_2(t)$ are linearly independent. Find the solution of the given homogeneous equation with the initial condition $x(0) = x_0$

6. $\begin{cases} x_1(t) = \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix}, & x_2(t) = \begin{pmatrix} -e^{-2t} \\ e^{-2t} \end{pmatrix} \\ x' = \begin{pmatrix} -6 & -4 \\ 8 & 6 \end{pmatrix} x & x(0) = \begin{pmatrix} -5 \\ 8 \end{pmatrix} \end{cases}$

7. $\begin{cases} x_1(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}, & x_2(t) = \begin{pmatrix} e^{2t}(t+2) \\ e^{2t}(t+1) \end{pmatrix} \\ x' = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} x & x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$

8. $\begin{cases} x_1(t) = \begin{pmatrix} \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \cos t \end{pmatrix}, & x_2(t) = \begin{pmatrix} \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \sin t \end{pmatrix} \\ x' = \begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix} x & x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$

Rewrite the given equation into a system in normal form with initial value.

9. $y^{(4)} - y^{(3)} + 7y = \cos t;$ $\quad y(0) = y'(0) = 1, \quad y''(0) = 0, \quad y^{(3)}(0) = 2$

10. $y^{(4)} + 3y'' - (\sin t)y' + 8y = t^2,$ $\quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 3, \quad y'''(0) = 4$

11. $y^{(6)} - (y')^3 = e^{2t} - \sin y$; $y(0) = y'(0) = y''(0) = y^{(3)}(0) = y^{(4)}(0) = y^{(5)}(0) = 0$

12. $\begin{cases} 3x'' = -5x + 2y \\ 4y'' = 6x - 2y \end{cases}$ $\begin{cases} x(0) = -1, & x'(0) = 0 \\ y(0) = 1, & y'(0) = 2 \end{cases}$

13. $\begin{cases} x''' - y = t \\ 2x'' + 5y'' - 2y = 1 \end{cases}$ $\begin{cases} x(0) = x'(0) = x''(0) = 4 \\ y(0) = y'(0) = 1 \end{cases}$

Transform the given differential equation or system into an equivalent system of 1st-order differential equation

14. $x'' + 3x' + 7x = t^2$

15. $x^{(4)} + 6x'' - 3x' + x = \cos 3t$

16. $t^2x'' + tx' + \left(t^2 - 1\right)x = 0$

17. $t^3x^{(3)} - 2t^2x'' + 3tx' + 5x = \ln t$

18. $x'' - 5x + 4y = 0, \quad y'' + 4x - 5y = 0$

19. $x'' - 3x' + 4x - 2y = 0, \quad y'' + 2y' - 3x + y = \cos t$

20. $x'' = 3x - y + 2z, \quad y'' = x + y - 4z, \quad z'' = 5x - y - z$

21. $x'' = (1 - y)x, \quad y'' = (1 - x)y$

22. Prove that the general solution of

$$X' = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} X$$

On the interval $(-\infty, \infty)$ is

$$X = C_1 \begin{pmatrix} 6 \\ -1 \\ -5 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} e^{-2t} + C_3 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} e^{3t}$$

23. Prove that the general solution of

$$X' = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} X + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} 4 \\ -6 \end{pmatrix} t + \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

On the interval $(-\infty, \infty)$ is

$$X = C_1 \begin{pmatrix} 1 \\ -1 - \sqrt{2} \end{pmatrix} e^{\sqrt{2}t} + C_2 \begin{pmatrix} 1 \\ -1 + \sqrt{2} \end{pmatrix} e^{-\sqrt{2}t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} -2 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For the systems below:

- ***a)*** Verify that the given vectors are solutions of the given system.
- ***b)*** Use the Wronskian to show that they are linearly independent.
- ***c)*** Write the general solution of the system.
- ***d)*** Find the particular solution that satisfies the given initial conditions

24. $x' = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} x;\quad \vec{x}_1 = \begin{bmatrix} 2e^{t} \\ -3e^{t} \end{bmatrix},\quad \vec{x}_2 = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$

25. $\boldsymbol{x}' = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \boldsymbol{x};\quad \vec{x}_1 = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix},\quad \vec{x}_2 = \begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix},\quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 5 \end{cases}$

26. $\boldsymbol{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix} \boldsymbol{x};\quad \vec{x}_1 = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix},\quad \vec{x}_2 = e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix},\quad \begin{cases} x_1(0) = 5 \\ x_2(0) = -3 \end{cases}$

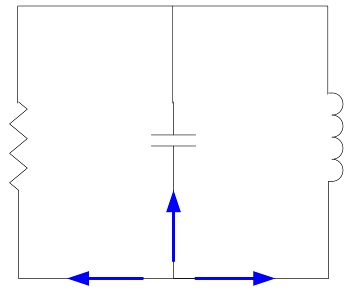
27. $\boldsymbol{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \boldsymbol{x};\quad \vec{x}_1 = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix},\quad \vec{x}_2 = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix},\quad \begin{cases} x_1(0) = 8 \\ x_2(0) = 0 \end{cases}$

28. $\boldsymbol{x}' = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \boldsymbol{x};\quad \vec{x}_1 = \begin{bmatrix} 2e^{t} \\ 2e^{t} \\ e^{t} \end{bmatrix},\quad \vec{x}_2 = \begin{bmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix},\quad \vec{x}_3 = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix},\quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 0 \\ x_3(0) = 4 \end{cases}$

29. $\boldsymbol{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \boldsymbol{x};\quad \vec{x}_1 = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix},\quad \vec{x}_2 = \begin{bmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{bmatrix},\quad \vec{x}_3 = \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix},\quad \begin{cases} x_1(0) = 10 \\ x_2(0) = 12 \\ x_3(0) = -1 \end{cases}$

30. $\boldsymbol{x}' = \begin{bmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{bmatrix} \boldsymbol{x};\quad \vec{x}_1 = \begin{bmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{bmatrix},\quad \vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{bmatrix},\quad \vec{x}_3 = \begin{bmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{bmatrix},\quad \vec{x}_4 = \begin{bmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{bmatrix},\quad \begin{cases} x_1(0) = 1 \\ x_2(0) = 3 \\ x_3(0) = 4 \\ x_4(0) = 7 \end{cases}$

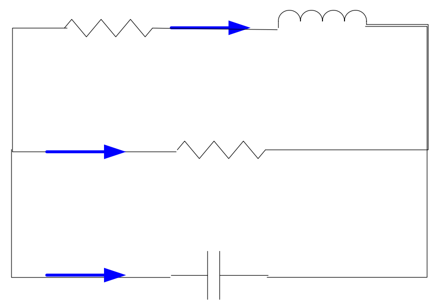
31. Consider the *RLC* parallel circuit below. Let *V* represent the voltage drop across the capacitor and *I* represent the current across the inductor.



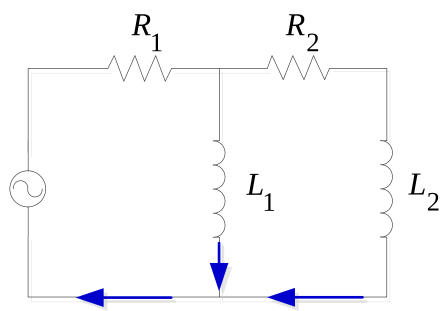
Show that: $\begin{cases} V' = -\dfrac{V}{RC} - \dfrac{1}{C} \\ I' = \dfrac{V}{L} \end{cases}$

32. Consider the *RLC* parallel circuit below. Let *V* represent the voltage drop across the capacitor and *I* represent the current across the inductor.

Show that: $$\begin{cases} CV' = -I - \dfrac{V}{R_2} \\ LI' = -R_1 I + V \end{cases}$$



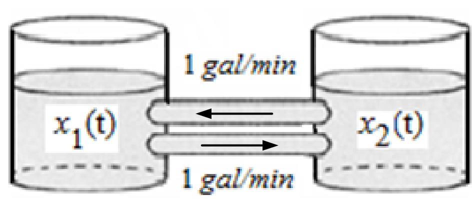
33. Consider the circuit below.



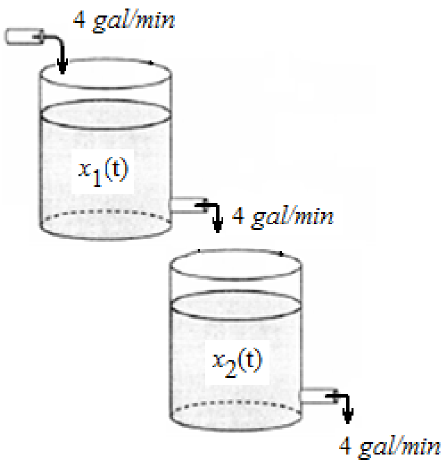
Let I_1 and I_2 represent the current flow across the indicators L_1 and L_2 respectively. Show that the circuit is modeled by the system

$$\begin{cases} L_1 I_1' = -R_1 I_1 - R_1 I_2 + E \\ L_2 I_2' = -R_1 I_1 - \left(R_1 + R_2\right) I_2 + E \end{cases}$$

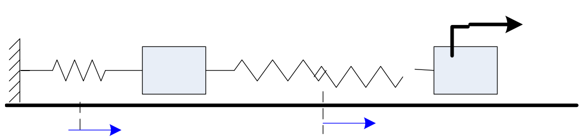
34. Two tanks are connected by two pipes. Each tank contains 500 *gallons* of a salt solution. Through on pipe solution is pumped from the first tank to the second at 1 *gal/min*. Through the other pipe, solution is pumped at the same rate from the second to the first tank. Show the salt content in each tank varies with time.



35. Each tank contains 100 *gallons* of a salt solution. Pure water flows into the upper tank at a rate of 4 *gal/min*. Salt solution drains from the upper tank into the lower tank at a rate of 4 *gal/min*. Finally, salt solution drains from the lower tank at a rate of 4 *gal/min*, effectively keeping the volume of solution in each tank at a constant 100 *gal*. If the initial salt content of the upper and lower tanks is 10 and 20 pounds, respectively. Set up an initial value problem that models the amount of salt in each tank over time (do not solve).Write the model in matrix-vector form. Is the system homogeneous or inhomogeneous?



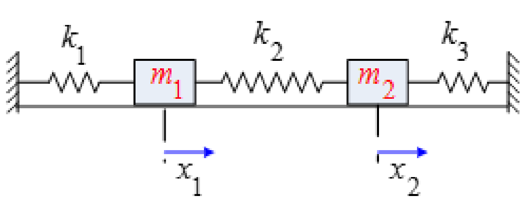
36. Two masses on a frictionless tabletop are connected with a spring having spring constant k_2. The first mass is connected to a vertical support with a spring having spring constant k_1. The second mass is shaken harmonically via a force equaling $F = A\cos\omega t$. Let $x(t)$ and $y(t)$ measure the displacements of the masses m_1 and m_2, respectively, from their equilibrium positions as a function of time. If both masses start from rest at their equilibrium positions at time $t = 0$.



Set up an initial value problem that models the position of the masses over time (do not solve). Write the model in matrix-vector form. Is the system homogeneous or inhomogeneous?

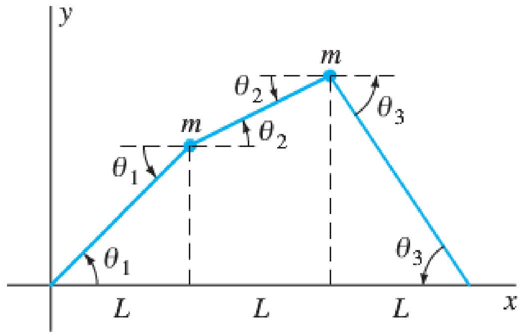
37. Derive the equations

$$\begin{cases} m_1 x_1'' = -\left(k_1 + k_2\right)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - \left(k_2 + k_3\right)x_2 \end{cases}$$



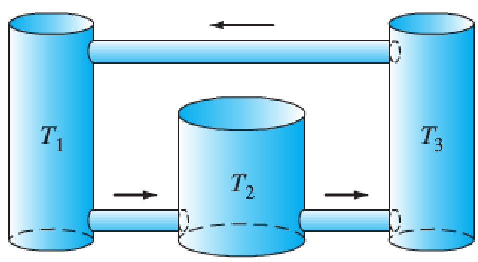
For the displacements (from equilibrium) of the 2 masses.

38. Two particles each of mass m are attached to a string under (constant) tension T. Assume that the particles oscillate vertically (that is, parallel to the y-axis) with amplitudes so small that the sines of the angles shown are accurately approximated by their tangents. Show that the displacement y_1 and y_2 satisfy the equations



$$\begin{cases} ky_1'' = -2y_1 + y_2 \\ ky_2'' = y_1 - 2y_2 \end{cases} \qquad \text{where } k = \frac{mL}{T}$$

39. There 100-gal fermentation vats are connected, and the mixtures in each tank are kept uniform by stirring. Denote by $x_i(t)$ the amount (in pounds) of alcohol in tank T_i at time t (i = 1, 2, 3). Suppose that the mixture circulates between the tanks at the rate of 10 gal/min. Derive the equations



$$\begin{cases} 10x_1' = -x_1 \qquad + x_3 \\ 10x_2' = x_1 - x_2 \\ 10x_3' = \qquad x_2 - x_3 \end{cases}$$

40. Suppose that a particle with mass m and electrical charge q moves in the xy-plane under the influence of the magnetic field $\vec{B} = B\hat{k}$ (thus a uniform field parallel to the z-axis), so the force on the particle is $\vec{F} = q\vec{v} \times \vec{B}$ if its velocity is $\vec{v}$. Show that the equations of motion of the particle are

$$mx'' = +qBy', \quad my'' = -qBx'$$