Lecture Three

Section 3.1 – Mathematical Induction

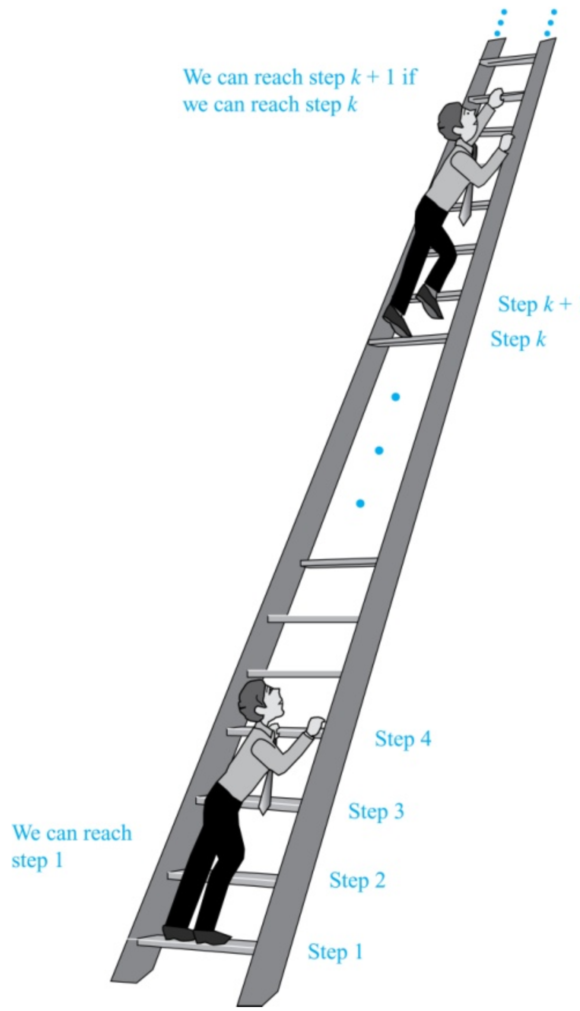
Introduction

Suppose we have an infinite ladder:

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

From (1), we can reach the first rung. Then by applying (2), we can reach the second rung. Applying (2) again, the third rung. And so on. We can apply (2) any number of times to reach any particular rung, no matter how high up.

This example motivates proof by mathematical induction.



Mathematical Induction

Principle of Mathematical Induction

To prove that $P(n)$ is true for all positive integers n, we complete these steps:

- ***Basis Step***: Show that $P(1)$ is true.
- ***Inductive Step***: Show that $P(k) \to P(k+1)$ is true for all positive integers k.

To complete the inductive step, assuming the *inductive hypothesis* that $P(k)$ holds for an arbitrary integer k, show that must $P(k+1)$ be true.

Climbing an Infinite Ladder Example:

BASIS STEP: By (1), we can reach rung 1.

INDUCTIVE STEP: Assume the inductive hypothesis that we can reach rung k. Then by (2), we can reach rung $k + 1$.

Hence, $P(k) \to P(k+1)$ is true for all positive integers k. We can reach every rung on the ladder.

Examples of Proofs by Mathematical induction

Mathematical induction can be expressed as the rule of inference

$$\left(P(1) \wedge \forall k\left(P(k) \rightarrow P(k+1)\right)\right) \rightarrow \forall n P(n)$$

where the domain is the set of positive integers.

In a proof by mathematical induction, we don't assume that $P(k)$ is true for all positive integers! We show that if we assume that $P(k)$ is true, then $P(k + 1)$ must also be true.
Proofs by mathematical induction do not always start at the integer 1. In such a case, the basis step begins at a starting point b where b is an integer.

Validity of Mathematical Induction

Mathematical induction is valid because of the well ordering property, which states that every nonempty subset of the set of positive integers has a least element. Here is the proof:

- Suppose that $P(1)$ holds and $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.
- Assume there is at least one positive integer n for which $P(n)$ is false. Then the set S of positive integers for which $P(n)$ is false is nonempty.
- By the well-ordering property, S has a least element, say m.
- We know that m cannot be 1 since $P(1)$ holds.
- Since m is positive and greater than 1, $m - 1$ must be a positive integer. Since $m - 1 < m$, it is not in S, so $P(m - 1)$ must be true.
- But then, since the conditional $P(k) \rightarrow P(k + 1)$ for every positive integer k holds, $P(m)$ must also be true. This contradicts $P(m)$ being false.
- Hence, $P(n)$ must be true for every positive integer n

Example

Show that if n is a positive integer, then $1+2+\cdots+n=\dfrac{n(n+1)}{2}$

Solution

Basis Step: $P(1)$ is true since $1=\dfrac{1(1+1)}{2}=1$.

Inductive Step: Assume true for $P(k)$. The inductive hypothesis is $1+2+\cdots+k=\dfrac{k(k+1)}{2}$

Under this assumption, $1+2+\cdots+k+(k+1)=\dfrac{(k+1)\left[(k+1)+1\right]}{2}=\dfrac{(k+1)(k+2)}{2}$

$$\begin{aligned}1+2+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1)\\ &= \frac{k(k+1)+2(k+1)}{2}\\ &= \frac{(k+1)(k+2)}{2} \quad \surd\end{aligned}$$

The last equation shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step.

Example

Conjecture and prove correct a formula for the sum of the first *n* positive odd integers. Then prove your conjecture.

Solution

The sums of the first n positive odd integers for $n = 1, 2, 3, 4, 5$ are

$1 = 1,\ 1 + 3 = 4,\ 1 + 3 + 5 = 9,\ 1 + 3 + 5 + 7 = 16,\ 1 + 3 + 5 + 7 + 9 = 25.$

Basis Step: $P(1)$ is true since $1^2 = 1$.

Inductive Step: $P(k) \to P(k + 1)$ for every positive integer *k*.

Assume the inductive hypothesis holds and then show that $P(k)$ holds has well.

Inductive Hypothesis: $1+3+5+\cdots+(2k-1)=k^2$

So, assuming $P(k)$, it follows that:

$$
\begin{aligned}
1+3+5+\cdots+(2k-1)+(2k+1) &= k^2+(2k+1) \\
&= k^2+2k+1 \\
&= (k+1)^2 \quad \checkmark
\end{aligned}
$$

Hence, we have shown that $P(k + 1)$ follows from $P(k)$. Therefore the sum of the first *n* positive odd integers is n^2.

Example

Use mathematical induction to show that $1+2+2^2+\ldots+2^n=2^{n+1}-1$ for all nonnegative integers *n*.

Solution

Basis Step: For $\boldsymbol{n = 0} \Rightarrow 1 \stackrel{?}{=} 2^{0+1}-1=2-1=1$; hence P_0 is true.

Inductive Step: $1+2+2^2+\ldots+2^k=2^{k+1}-1$ is true for every positive integer *k*.

$$1+2+2^2+\ldots+2^k+2^{k+1}=2^{k+2}-1?$$

$$
\begin{aligned}
1+2+2^2+\ldots+2^k+2^{k+1} &= \left(2^{k+1}-1\right)+2^{k+1} \\
&= 2\cdot 2^{k+1}-1 \\
&= 2^{k+2}-1 \quad \checkmark
\end{aligned}
$$

Hence P_{k+1} is true.

By mathematical induction, the statement $1+2+2^2+\ldots+2^n=2^{n+1}-1$ is true

Example

Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression with initial term a and common ratio r:

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \ldots + ar^n = \frac{ar^{n+1} - a}{r-1} \qquad \text{when} \quad r \neq 1$$

for all nonnegative integers n.

Solution

Basis Step: For $\boldsymbol{n = 0} \Rightarrow \frac{ar^{0+1} - a}{r-1} = \frac{ar - a}{r-1} = \frac{a(r-1)}{r-1} = a$; hence P_0 is true.

Inductive Step: $a + ar + ar^2 + \ldots + ar^k = \frac{ar^{k+1} - a}{r-1}$ is true for every positive integer k.

$$a + ar + ar^2 + \ldots + ar^k + ar^{k+1} = \frac{ar^{k+2} - a}{r-1} \ ?$$

$$\begin{aligned} a + ar + ar^2 + \ldots + ar^k + ar^{k+1} &= \frac{ar^{k+1} - a}{r-1} + ar^{k+1} \\ &= \frac{ar^{k+1} - a + ar^{k+1}(r-1)}{r-1} \\ &= \frac{ar^{k+1} - a + ar^{k+2} - ar^{k+1}}{r-1} \\ &= \frac{-a + ar^{k+2}}{r-1} \\ &= \frac{ar^{k+2} - a}{r-1} \qquad \surd \end{aligned}$$

Hence P_{k+1} is true.

By mathematical induction, the statement

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \ldots + ar^n = \frac{ar^{n+1} - a}{r-1}, \quad \text{when} \quad r \neq 1 \ \text{ is true}$$

Proving Inequalities

Example

Prove that the statement is true for every positive integer n. $n < 2^n$

Solution

Basis Step: For $\boldsymbol{n = 1} \Rightarrow 1 \overset{?}{<} 2^1$ √ $\Rightarrow P_1$ is true.

Inductive Step. Assume that P_k is true $k < 2^k$

We need to prove that P_{k+1} is true, that is $k+1 < 2^{k+1}$

$$\begin{aligned} k+1 &< k+k = 2k \\ &< 2 \cdot 2^k \\ &= 2^{k+1} \quad \surd \end{aligned}$$

Thus, P_{k+1} is true.

By mathematical induction, the statement $n < 2^n$ is true.

Example

Prove that the statement is true for every positive integer n. $2^n < n!$ for every integer n with $n \ge 4$

Solution

Basis Step: For $\boldsymbol{n = 4} \Rightarrow 2^4 \overset{?}{<} 4! \Rightarrow 16 < 24$ √ $\Rightarrow P_4$ is true.

Inductive Step. Assume that P_k is true $2^k < k!$

We need to prove that P_{k+1} is true, that is $2^{k+1} < (k+1)!$

$$\begin{aligned} 2^{k+1} &= 2^k \cdot 2 = 2 \cdot 2^k \\ &< 2 \cdot k! \\ &< (k+1)k! \qquad\qquad 2 < k+1 \\ &= (k+1)! \quad \surd \end{aligned}$$

Thus, P_{k+1} is true.

By mathematical induction, the statement $2^n < n!$ is true.

Harmonic Numbers

Example

The harmonic numbers H_j, $j = 1,\ 2,\ 3,\ \ldots$ are defined by

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}$$

Use the mathematical method to show that $H_{2^n} \geq 1 + \frac{n}{2}$ for all nonnegative integers *n*.

Solution

Basis Step: For ***n* = 0** $\Rightarrow H_{2^0} = H_1 = 1 \geq 1 + \frac{0}{2} = 1$ √ $\Rightarrow P_0$ is true.

Inductive Step. Assume that P_k is true $H_{2^k} \geq 1 + \frac{k}{2}$

We need to prove that P_{k+1} is true, that is $H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$

$$\begin{aligned}
H_{2^{k+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^k + 1} + \cdots + \frac{1}{2^{k+1}} \\
&= H_{2^k} + \frac{1}{2^k + 1} + \cdots + \frac{1}{2^{k+1}} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2^k + 1} + \cdots + \frac{1}{2^{k+1}} \\
&\geq \left(1 + \frac{k}{2}\right) + 2^k \cdot \frac{1}{2^{k+1}} \qquad \left(each\ 2^k\right) \geq \frac{1}{2^{k+1}} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2} \\
&= 1 + \frac{k+1}{2} \quad \surd
\end{aligned}$$

Thus, P_{k+1} is true.

By mathematical induction, the statement $H_{2^n} \geq 1 + \frac{n}{2}$ is true.

Example

Use mathematical induction to prove that $n^3 - n$ is divisible by 3*,* for every positive integer *n*.

Solution

Let *P*(*n*) be the proposition that $n^3 - n$ is divisible by 3.

Basis Step: For ***n*** **= 1** $\Rightarrow 1^3 - 1 = 0$ which is divisible by 3 $\Rightarrow P_1$ is true.

Inductive Step. Assume that P_k holds $k^3 - k$ is divisible by 3

We need to prove that P_{k+1} is true, that is $(k+1)^3 - (k+1)$ is divisible by 3

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 - k + 3k^2 + 3k \\ &= \left(k^3 - k\right) + 3\left(k^2 + k\right)\end{aligned}$$

$\left(k^3 - k\right)$ is divisible by 3, by the inductive hypothesis,

$3\left(k^2 + k\right)$ is divisible by 3, since it is an integer multiplied by 3.

Thus, P_{k+1} is true.

By mathematical induction, the statement $n^3 - n$ is divisible by 3 is true, for every positive integer *n*.

Example

Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57*,* for every nonnegative integer *n*.

Solution

Basis Step: For ***n*** **= 0** $\Rightarrow 7^2 + 8^1 = 49 + 8 = 57$ which is divisible by 57 $\Rightarrow P_0$ is true.

Inductive Step: Assume that P_k holds $7^{k+2} + 8^{2k+1}$ is divisible by 57

We need to prove that P_{k+1} is true, that is $7^{k+1+2} + 8^{2(k+1)+1} = 7^{k+3} + 8^{2k+3}$ is also divisible by 57

$$\begin{aligned}7^{k+3} + 8^{2k+3} &= 7 \cdot 7^{k+2} + 8^2 \cdot 8^{2k+1} \\ &= 7 \cdot 7^{k+2} + 64 \cdot 8^{2k+1} \\ &= 7 \cdot 7^{k+2} + 7 \cdot 8^{2k+1} + 57 \cdot 8^{2k+1} \\ &= 7 \cdot \left(7^{k+2} + 8^{2k+1}\right) + 57 \cdot 8^{2k+1}\end{aligned}$$

$\left(7^{k+2}+8^{2k+1}\right)$ is divisible by 57, by the inductive hypothesis,

$57 \cdot 8^{2k+1}$ is divisible by 57, since it is an integer multiplied by 57.

Thus, P_{k+1} is true.

By mathematical induction, the statement $7^{n+2}+8^{2n+1}$ is divisible by 57 is true, for every positive integer n.

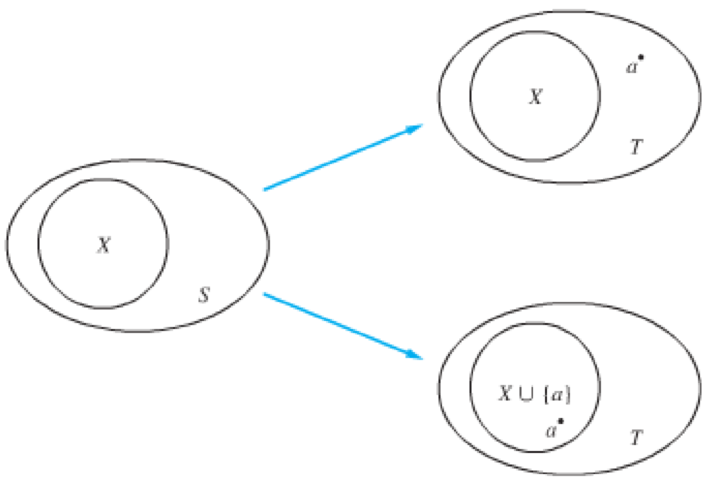
Number of Subsets of a Finite Set

Inductive Hypothesis

For an arbitrary nonnegative integer k, every set with k elements has 2^k subsets.

Let T be a set with $k + 1$ elements. Then $T = S \cup \{a\}$, where $a \in T$ and $S = T - \{a\}$. Hence $|T| = k$.

For each subset X of S, there are exactly two subsets of T, i.e., X and $X \cup \{a\}$.



By the inductive hypothesis S has 2^k subsets. Since there are two subsets of T for each subset of S, the number of subsets of T is $2 \cdot 2^k = 2^{k+1}$

Example

Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes.
A right triomino is an L-shaped tile which covers three squares at a time.

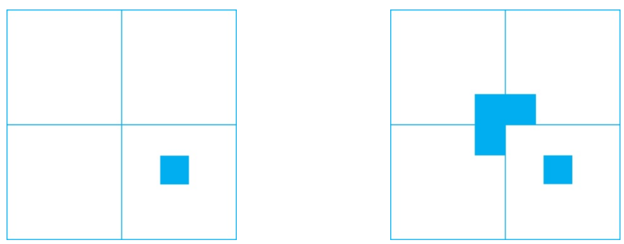
Solution

Let $P(n)$ be the proposition that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes. Use mathematical induction to prove that $P(n)$ is true for all positive integers n.

Basis Step: $P(1)$ is true, because each of the four 2×2 checkerboards with one square removed can be tiled using one right triomino.

Inductive Step: Assume that $P(k)$ is true for every $2^k \times 2^k$ checkerboard, for some positive integer k. with one square removed can be tiled using right triominoes.

Consider a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed. Split this checkerboard into four checkerboards of size $2^k \times 2^k$,by dividing it in half in both directions.



Remove a square from one of the four $2^k \times 2^k$ checkerboards. By the inductive hypothesis, this board can be tiled. Also by the inductive hypothesis, the other three boards can be tiled with the square from the corner of the center of the original board removed. We can then cover the three adjacent squares with a triominoe.

Hence, the entire $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed can be tiled using right triominoes.

Guidelines: Mathematical Induction Proofs

Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words "Basis Step" or "step 1". Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
4. State and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
6. Prove the statement $P(k+1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step. State the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.

Exercises *Section* 3.1 – Mathematical Induction

1. Prove that $1^2+3^2+5^2+\cdots+(2n+1)^2=\frac{1}{3}(n+1)(2n+1)(2n+3)$ whenever n is a nonnegative integer.

2. Prove that $1\cdot 1!+2\cdot 2!+\cdots+n\cdot n!=(n+1)!-1$ whenever n is a positive integer.

3. Prove that $3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^n=\frac{3}{4}\left(5^{n+1}-1\right)$ whenever n is a nonnegative integer.

4. Prove that $2-2\cdot 7+2\cdot 7^2-\cdots+2\cdot(-7)^n=\dfrac{1-(-7)^{n+1}}{4}$ whenever n is a nonnegative integer.

5. Find a formula for the sum of the first n even positive integers. Prove the formula.

6. *a*) Find a formula for $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{n(n+1)}$ by examining the values of this expression for values of this expression for small values of n.
b) Prove the formula.

7. Prove that $1^2-2^2+3^2-\cdots+(-1)^{n-1}n^2=(-1)^{n-1}\dfrac{n(n+1)}{2}$ whenever n is a positive integer.

8. Prove that for very positive integer *n,* $\displaystyle\sum_{k=1}^{n}k2^k=(n-1)2^{n+1}+2$.

9. Prove that for very positive integer *n,* $1\cdot 2+2\cdot 3+\cdots+n(n+1)=\frac{1}{3}n(n+1)(n+2)$.

10. Prove that for very positive integer *n,*
$1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+n(n+1)(n+2)=\frac{1}{4}n(n+1)(n+2)(n+3)$

11. Let $P(n)$ be the statement that $1+\dfrac{1}{4}+\dfrac{1}{9}+\cdots+\dfrac{1}{n^2}<2-\dfrac{1}{n}$ where n is an integer greater than 1.

a) Show is the statement $P(2)$?

b) Show that $P(2)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step.

f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

12. Prove that $3^n<n!$ if n is an integer greater than 6.

13. Prove that $2^n > n^2$ if n is an integer greater than 4.

14. Prove that for every positive integer n, $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1}-1\right)$.

15. Use mathematical induction to prove that 2 divides n^2+n whenever n is a positive integer.

16. Use mathematical induction to prove that 3 divides n^3+2n whenever n is a positive integer.

17. Use mathematical induction to prove that 5 divides n^5-n whenever n is a positive integer.

18. Use mathematical induction to prove that n^2-1 is divisible by 8 whenever n is an odd positive integer.

19. Use mathematical induction to prove that 21 divides $4^{n+1}+5^{2n-1}$ whenever n is a positive integer.

(20 – 50) Prove that the statement is true by the mathematical induction

20. $1+2\bullet 2+3\bullet 2^2+\ldots+n\bullet 2^{n-1}=1+(n-1).2^n$

21. $1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$

22. $\frac{1}{1\bullet 2}+\frac{1}{2\bullet 3}+\frac{1}{3\bullet 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

23. $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$

24. $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n-2)\cdot(3n+1)}=\frac{n}{3n+1}$

25. $\frac{4}{5}+\frac{4}{5^2}+\frac{4}{5^3}+\cdots+\frac{4}{5^n}=1-\frac{1}{5^n}$

26. $1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$

27. $3+3^2+3^3+\ldots+3^n=\frac{3}{2}\left(3^n-1\right)$

28. $x^{2n}+x^{2n-1}y+\cdots+xy^{2n-1}+y^{2n}=\frac{x^{2n+1}-y^{2n+1}}{x-y}$

29. $5\bullet 6+5\bullet 6^2+5\bullet 6^3+\cdots+5\bullet 6^n=6\left(6^n-1\right)$

30. $7\bullet 8+7\bullet 8^2+7\bullet 8^3+\cdots+7\bullet 8^n=8\left(8^n-1\right)$

31. $3+6+9+\cdots+3n=\frac{3n(n+1)}{2}$

32. $5+10+15+\cdots+5n=\dfrac{5n(n+1)}{2}$

33. $1+3+5+\cdots+(2n-1)=n^2$

34. $4+7+10+\cdots+(3n+1)=\dfrac{n(3n+5)}{2}$

35. $2+4+6+\cdots+2(n-1)+2n=n(n+1)$

36. $1+(1+2)+(1+2+3)+\ \cdots\ +(1+2+\cdots+n)=\dfrac{n(n+1)(n+2)}{6}\quad=\sum_{k=1}^{n}\left(\sum_{i=1}^{k}i\right)$

37. $1+2+3+\ \cdots\ +n<\dfrac{(2n+3)^2}{7}$

38. $\dfrac{1}{2n}\le\dfrac{1\cdot3\cdot5\cdots(2n-3)\cdot(2n-1)}{2\cdot4\cdot6\cdots(2n-2)\cdot(2n)}$

39. $\dfrac{2n+1}{2n+2}\le\dfrac{\sqrt{n+1}}{\sqrt{n+2}}$

40. $n!<n^n\quad for\quad n>1$

41. For every positive integer *n*. $\quad n<2^n$

42. For every positive integer *n*. $\quad$ 3 is a factor of n^3-n+3

43. For every positive integer *n*. $\quad$ 4 is a factor of 5^n-1

44. $(a^m)^n=a^{mn}$ (*a* and *m* are constant)

45. $2^n>2n\quad if\quad n\ge3$

46. If $0<a<1$, then $a^n<a^{n-1}$

47. If $n\ge4$, then $n!>2^n$

48. $3^n>2n+1\quad if\quad n\ge2$

49. $2^n>n^2\quad for\quad n>4$

50. $4^n>n^4\quad for\quad n\ge5$

51. A pile of *n* rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must be moved, one at a time, to a different peg with no ring ever placed on top of a smaller ring. Find the least number of moves that would be required. Prove your result by mathematical induction.