Solution   *Section* 4.1 – System of linear Equations

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$$

Solution

$$\begin{cases} \quad 3x+2y=-4 \\ 2\times \quad 2x-y=-5 \end{cases}$$

$$\begin{array}{r} 3x+2y=-4 \\ \underline{4x-2y=-10} \\ 7x=-14 \end{array}$$

$$\underline{x=-2}|$$

$$\begin{aligned} y &= 2x+5 \\ &= -4+5 \\ &= 1 \end{aligned}$$

Solution: $\boxed{(-2,\ 1)}$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$$

Solution

$$\begin{cases} -5\times \quad 2x+5y=7 \\ 2\times \quad 5x-2y=-3 \end{cases}$$

$$\begin{array}{r} -10x-25y=-35 \\ \underline{10x-4y=-6} \\ -29y=-41 \end{array}$$

$$\underline{y=\frac{41}{29}}|$$

$$x=\frac{1}{2}\left(7-5\left(\frac{41}{29}\right)\right)$$

$$x=\frac{1}{2}\left(-\frac{2}{29}\right)$$

$$=-\frac{1}{29}$$

$\therefore$ ***Solution***: $\left(-\frac{1}{29}, \frac{41}{29}\right)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 4x-7y=-16 \\ 2x+5y=9 \end{cases}$$

Solution

$$\begin{cases} & 4x-7y=-16 \\ -2\times & 2x+5y=9 \end{cases}$$

$$\begin{array}{r} 4x-7y=-16 \\ -4x-10y=-18 \\ \hline -17y=-34 \end{array}$$

$$y=2$$

$$x=\frac{9-5y}{2}$$

$$=\frac{9-10}{2}$$

$$=-\frac{1}{2}$$

$\therefore$ ***Solution***: $\left(-\frac{1}{2}, 2\right)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 3x+2y=4 \\ 2x+y=1 \end{cases}$$

Solution

$$\begin{cases} 3x+2y=4 & (1) \\ 2x+y=1 & (2) \end{cases}$$

$$(2)\to \quad y=1-2x \quad (3)$$

$$(1)\to \quad 3x+2-4x=4$$

$\underline{x = -2}$

$(3) \rightarrow \quad y = 1 + 4$

$\quad\quad\quad\quad \underline{= 5}$

$\therefore$ ***Solution***: $\underline{(-2,\ 5)}$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$$

Solution

$$\begin{cases} -2\times \quad 3x + 4y = 2 \\ \ \ 3\times \quad 2x + 5y = -1 \end{cases}$$

$$\begin{array}{r} -6x - 8y = -4 \\ \underline{6x + 15y = -3} \\ 7y = -7 \end{array}$$

$\underline{y = -1}$

$2x = -1 + 5$

$x = \dfrac{4}{2}$

$\quad \underline{= 2}$

$\therefore$ ***Solution***: $\underline{(2,\ -1)}$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method) $\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$

Solution

$$\begin{cases} 2\times \quad 5x - 2y = 4 \\ \quad\quad -10x + 4y = 7 \end{cases}$$

$$\begin{array}{r} 10x - 4y = 8 \\ \underline{-10x + 4y = 7} \\ 0 = 15 \end{array} \quad (\textit{impossible})$$

$\therefore$ ***No Solution***

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$$

Solution

$$\begin{cases} x-4y=-8 & (1) \\ 5x-20y=-40 & (2) \end{cases}$$

$(1) \to \quad x=4y-8$

$(2) \to \quad 5(4y-8)-20y=-40$

$20y-40-20y=-40$

$-40=-40 \quad (\mathit{True})$

$\therefore$ ***Solution***: $\quad \underline{x-4y=-8}$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$$

Solution

$$\begin{cases} 2x+y=3 & (1) \\ x-y=3 & (2) \end{cases}$$

$(2) \to \quad x=3+y \quad (3)$

$(1) \to \quad 6+2y+y=3$

$3y=-3$

$\underline{y=-1}$

$(3) \to \quad \underline{x=2}$

$\therefore$ ***Solution***: $\quad \underline{(2,\ -1)}$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 2x+10y=-14 \\ 7x-2y=-16 \end{cases}$$

Solution

$$\begin{cases} \quad 2x+10y=-14 \\ 5\times \quad 7x-2y=-16 \end{cases}$$

$$\begin{array}{r} 2x+10y=-14 \\ 35x-10y=-80 \\ \hline 37x=-94 \end{array}$$

$$\underline{x=-\frac{94}{37}}$$

$$2y=7\left(-\frac{94}{37}\right)+16$$

$$y=-\frac{329}{37}+8$$

$$=-\frac{33}{37}$$

$\therefore$ ***Solution***: $\left(-\frac{94}{37},\ -\frac{33}{37}\right)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 4x-3y=24 \\ -3x+9y=-1 \end{cases}$$

Solution

$$\begin{cases} 3\times \quad 4x-3y=24 \\ \quad -3x+9y=-1 \end{cases}$$

$$\begin{array}{r} 12x-9y=72 \\ -3x+9y=-1 \\ \hline -9x=-71 \end{array}$$

$$\underline{x=\frac{71}{9}}$$

$$3y=4\left(\frac{71}{9}\right)-24$$

$$y=\frac{284}{27}-8$$

$$= \frac{68}{27}$$

$\therefore$ ***Solution***: $\left(\frac{71}{9},\ \frac{68}{27}\right)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$$

Solution

$$\begin{array}{r} 4x + 2y = 12 \\ \underline{3x - 2y = 16} \\ 7x = 28 \end{array}$$

$$x = 4$$

$$2y = 12 - 4(4)$$

$$y = -\frac{4}{2}$$

$$= -2$$

$\therefore$ ***Solution***: $(4,\ -2)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$$

Solution

$$\begin{array}{r} x + 2y = -1 \\ \underline{4x - 2y = 6} \\ 5x = 5 \end{array}$$

$$x = 1$$

$$2y = -x - 1$$

$$y = -\frac{2}{2}$$

$$= -1$$

$\therefore$ ***Solution***: $(1,\ -1)$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$$

Solution

$$\begin{array}{r} x - 2y = 5 \\ \underline{-10x + 2y = 4} \\ -9x = 9 \end{array}$$

$\underline{x = -1}|$

$2y = x - 5$

$y = -\frac{6}{2}$

$\quad \underline{= -3}|$

$\therefore$ ***Solution***: $\quad \underline{(-1,\ -3)}|$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$$

Solution

$$\begin{array}{r} 12x + 15y = -27 \\ \underline{30x - 15y = -15} \\ 42x = -42 \end{array}$$

$\underline{x = -1}|$

$15y = -27 - 12(-1)$

$y = -\frac{15}{15}$

$\quad \underline{= -1}|$

$\therefore$ ***Solution***: $\quad \underline{(-1,\ -1)}|$

Exercise

Use any method to solve the system equation (***elimination*** or ***substitution*** method)

$$\begin{cases} 4x-4y=-12 \\ 4x+4y=-20 \end{cases}$$

Solution

$$\begin{array}{r} 4x-4y=-12 \\ \underline{4x+4y=-20} \\ 8x=-32 \end{array}$$

$$\underline{x=-4}$$

$$4y=4(-4)+12$$

$$y=-\frac{4}{4}$$

$$\underline{=-1}$$

$\therefore$ ***Solution***: $\quad \underline{(-4,\ -1)}$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 1 & 4 & 7 \\ 3 & 5 & 0 \end{array}\right] \quad R_2-3R_1$$

Solution

$$\begin{array}{ccc} 3 & 5 & 0 \\ -3 & -12 & -21 \\ \hline 0 & -7 & -21 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 4 & 7 \\ 0 & -7 & -21 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right] \quad R_2-2R_1$$

Solution

$$\begin{array}{ccc} 2 & 1 & -5 \\ -2 & 6 & -2 \\ \hline 0 & 7 & -7 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 7 & -7 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 5 & 2 & 19 \end{array}\right] \quad R_2 - 5R_1$$

Solution

$$\begin{array}{ccc} 5 & 2 & 19 \\ -5 & 15 & -15 \\ \hline 0 & 17 & -4 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 17 & -4 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ -6 & 9 & 4 \end{array}\right] \quad R_2 + 3R_1$$

Solution

$$\begin{array}{ccc} -6 & 9 & 4 \\ 6 & -9 & 24 \\ \hline 0 & 0 & 28 \end{array}$$

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ 0 & 0 & 28 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 2 & 3 & 11 \\ 1 & 2 & 8 \end{array}\right] \quad 2R_2 - R_1$$

Solution

$$\begin{array}{ccc} 2 & 4 & 16 \\ -2 & -3 & -11 \\ \hline 0 & 1 & 5 \end{array}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 11 \\ 0 & 1 & 5 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cc|c} 3 & 5 & -13 \\ 2 & 3 & -9 \end{array}\right] \quad 3R_2 - 2R_1$$

Solution

$$\begin{array}{ccc} 6 & 9 & -27 \\ -6 & -10 & 26 \\ \hline 0 & -1 & -1 \end{array}$$

$$\left[\begin{array}{cc|c} 3 & 5 & -13 \\ 0 & -1 & -1 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 5 & 4 & 1 \end{array}\right] \quad R_3 - 5R_2$$

Solution

$$\begin{array}{cccc} 0 & 5 & 4 & 1 \\ 0 & -5 & 5 & -10 \\ \hline 0 & 0 & 9 & -9 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 2 & | & 2 \\ 0 & 1 & -1 & | & 2 \\ 0 & 0 & 9 & | & -9 \end{bmatrix}$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right] \quad \begin{array}{c} R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

Solution

$$\begin{array}{cccc} 3 & 3 & -1 & 10 \\ -3 & 3 & -15 & 18 \\ \hline 0 & 6 & -16 & 28 \end{array} \qquad \begin{array}{cccc} 1 & 3 & 2 & 5 \\ -1 & 1 & -5 & 6 \\ \hline 0 & 4 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 4 & -3 & 11 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 2 & 4 & 4 & 22 \\ -1 & -2 & 3 & 15 \end{array}\right] \quad \begin{array}{c} 3R_2 - 2R_1 \\ 3R_3 + R_1 \end{array}$$

Solution

$$\begin{array}{cccc} 6 & 12 & 12 & 66 \\ -6 & -4 & -2 & -2 \\ \hline 0 & 8 & 10 & 64 \end{array} \qquad \begin{array}{cccc} -3 & -6 & 9 & 45 \\ 3 & 2 & 1 & 1 \\ \hline 0 & -4 & 10 & 46 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & 8 & 10 & 64 \\ 0 & -4 & 10 & 46 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & -4 & 2 & -7 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

Solution

$$\begin{array}{cccc} 2 & 1 & 1 & 3 \\ -2 & -2 & -2 & -4 \\ \hline 0 & -1 & -1 & -1 \end{array} \qquad \begin{array}{cccc} 3 & -4 & 2 & -7 \\ -3 & -3 & -3 & -6 \\ \hline 0 & -7 & -1 & -13 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & -7 & -1 & -13 \end{array}\right]$$

Exercise

Perform the matrix row operation (or operations) and write the new matrix.

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ 2 & -3 & 5 & -1 & 0 \\ 1 & 0 & 3 & 1 & -4 \\ -4 & 3 & 2 & -1 & 3 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 + 4R_1 \end{array}$$

Solution

$$\begin{array}{ccccc} 2 & -3 & 5 & -1 & 0 \\ -2 & 4 & -2 & -6 & 4 \\ \hline 0 & 1 & 3 & -7 & 4 \end{array} \qquad \begin{array}{ccccc} 1 & 0 & 3 & 1 & -4 \\ -1 & 2 & -1 & -3 & 2 \\ \hline 0 & 2 & 2 & -2 & -2 \end{array} \qquad \begin{array}{ccccc} -4 & 3 & 2 & -1 & 3 \\ 4 & -8 & 4 & 12 & -8 \\ \hline 0 & -5 & 6 & 11 & -5 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & -2 \\ 0 & 1 & 3 & -7 & 4 \\ 0 & 2 & 2 & -2 & -2 \\ 0 & -5 & 6 & 11 & -5 \end{array}\right]$$

Exercise

Use the Gauss-Jordan method to solve the system
$$\begin{aligned} x-y+5z&=-6\\ 3x+3y-z&=10\\ x+3y+2z&=5 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6\\ 3 & 3 & -1 & 10\\ 1 & 3 & 2 & 5 \end{array}\right] \begin{array}{l} \\ R_2-3R_1\\ R_3-R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 3 & -1 & 10\\ -3 & 3 & -15 & 18\\ \hline 0 & 6 & -16 & 28 \end{array} \qquad \begin{array}{cccc} 1 & 3 & 2 & 5\\ -1 & 1 & -5 & 6\\ \hline 0 & 4 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6\\ 0 & 6 & -16 & 28\\ 0 & 4 & -3 & 11 \end{array}\right] \begin{array}{l} \\ \frac{1}{6}R_2\\ \\ \end{array} \qquad 0 \quad 1 \quad -\frac{8}{3} \quad \frac{14}{3}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6\\ 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ 0 & 4 & -3 & 11 \end{array}\right] \begin{array}{l} R_1+R_2\\ \\ R_3-4R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 4 & -3 & 11\\ 0 & -4 & \frac{32}{3} & -\frac{56}{3}\\ \hline 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array} \qquad \begin{array}{cccc} 1 & -1 & 5 & -6\\ 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ \hline 1 & 0 & \frac{7}{3} & -\frac{4}{3} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3}\\ 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array}\right] \begin{array}{l} \\ \\ \frac{3}{23}R_3 \end{array} \qquad 0 \quad 0 \quad 1 \quad -1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3}\\ 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1-\frac{7}{3}R_3\\ R_2+\frac{8}{3}R_3\\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & \frac{7}{3} & -\frac{4}{3}\\ 0 & 0 & -\frac{7}{3} & \frac{7}{3}\\ \hline 1 & 0 & 0 & 1 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{8}{3} & \frac{14}{3}\\ 0 & 0 & \frac{8}{3} & -\frac{8}{3}\\ \hline 0 & 1 & 0 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & -1 \end{array}\right]$$

Solution: $(1,\ 2,\ -1)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 2x - y + 4z = -3 \\ x - 2y - 10z = -6 \\ 3x \qquad + 4z = 7 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & -3 \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array}\right] \begin{array}{l} \frac{1}{2}R_1 \\ \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 3R_1 \end{array} \qquad \begin{array}{cccc} 1 & -2 & -10 & -6 \\ -1 & \frac{1}{2} & -2 & \frac{3}{2} \\ \hline 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \end{array} \qquad \begin{array}{cccc} 3 & 0 & 4 & 7 \\ -3 & \frac{3}{2} & -6 & \frac{9}{2} \\ \hline 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}\right] \begin{array}{l} \\ -\frac{2}{3}R_2 \\ \\ \end{array} \qquad \begin{array}{cccc} 0 & 1 & 8 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & 1 & 8 & 3 \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}\right] \begin{array}{l} R_1 + \frac{1}{2}R_2 \\ \\ R_3 - \frac{3}{2}R_2 \end{array} \qquad \begin{array}{cccc} 0 & \frac{3}{2} & -2 & \frac{23}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ \hline 0 & 0 & -14 & 7 \end{array} \qquad \begin{array}{cccc} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & \frac{1}{2} & 4 & \frac{3}{2} \\ \hline 1 & 0 & 6 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & -14 & 7 \end{array}\right] \begin{array}{l} \\ \\ -\frac{1}{14}R_3 \end{array} \qquad \begin{array}{cccc} 0 & 0 & 1 & -\frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \begin{array}{l} R_1 - 6R_3 \\ R_2 - 8R_3 \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & 0 & 6 & 0 \\ 0 & 0 & -6 & 3 \\ \hline 1 & 0 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 1 & 8 & 3 \\ 0 & 0 & -8 & 4 \\ \hline 0 & 1 & 0 & 7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right]$$

Solution: $\left(3,\ 7,\ -\frac{1}{2}\right)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 4x+3y-5z=-29 \\ 3x-7y-z=-19 \\ 2x+5y+2z=-10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 4 & 3 & -5 & -29 \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array}\right] \frac{1}{4}R_1$$

$$1 \quad \frac{3}{4} \quad -\frac{5}{4} \quad -\frac{29}{4}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3-2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & -7 & -1 & -19 \\ -3 & -\frac{9}{4} & \frac{15}{4} & \frac{87}{4} \\ \hline 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \end{array} \qquad \begin{array}{cccc} 2 & 5 & 2 & -10 \\ -2 & -\frac{3}{2} & \frac{5}{2} & \frac{29}{2} \\ \hline 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}\right] -\frac{4}{37}R_2$$

$$0 \quad 1 \quad -\frac{11}{37} \quad -\frac{11}{37}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}\right] \begin{array}{l} R_1-\frac{3}{4}R_2 \\ \\ R_3-\frac{7}{2}R_2 \end{array}$$

$$\begin{array}{cccc} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{3}{4} & \frac{33}{148} & \frac{33}{148} \\ \hline 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \end{array} \qquad \begin{array}{cccc} 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \\ 0 & -\frac{7}{2} & \frac{77}{72} & \frac{77}{72} \\ \hline 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array}\right] \frac{72}{401}R_3$$

$$0 \quad 0 \quad 1 \quad 1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1+\frac{38}{37}R_3 \\ R_2+\frac{11}{37}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 0 & \frac{38}{37} & \frac{38}{37} \\ \hline 1 & 0 & 0 & -6 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{11}{37} & \frac{11}{37} \\ \hline 0 & 1 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Solution: $(-6,\ 0,\ 1)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x+2y-3z=-15 \\ 2x-3y+4z=18 \\ -3x+y+z=1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3+3R_1 \end{array}$$

$$\begin{array}{cccc} -2 & -4 & 6 & 30 \\ 2 & -3 & 4 & 18 \\ \hline 0 & -7 & 10 & 48 \end{array} \qquad \begin{array}{cccc} 3 & 6 & -9 & -45 \\ -3 & 1 & 1 & 1 \\ \hline 0 & 7 & -8 & -44 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \end{array}\right] \quad -\frac{1}{7}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 7 & -8 & -44 \end{array}\right] \begin{array}{l} R_1-2R_2 \\ \\ R_3-7R_2 \end{array}$$

$$\begin{array}{cccc} 1 & 2 & -3 & -15 \\ 0 & -2 & \frac{20}{7} & \frac{96}{7} \\ \hline 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \end{array} \qquad \begin{array}{cccc} 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \\ \hline 0 & 0 & 2 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 2 & 4 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{2}R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1+\frac{1}{7}R_3 \\ R_2+\frac{10}{7}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 0 & \frac{1}{7} & \frac{2}{7} \\ \hline 1 & 0 & 0 & -1 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & \frac{10}{7} & \frac{20}{7} \\ \hline 0 & 1 & 0 & -4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $(-1,\ -4,\ 2)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x+2y+3z=10 \\ 4x+5y+6z=11 \\ 7x+8y+9z=12 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \end{array}\right] \begin{array}{l} \\ R_2-4R_1 \\ R_3-7R_1 \end{array}$$

$$\begin{array}{cccc} -4 & -8 & -12 & -40 \\ 4 & 5 & 6 & 11 \\ \hline 0 & -3 & -6 & -29 \end{array} \qquad \begin{array}{cccc} -7 & -14 & -21 & -70 \\ 7 & 8 & 9 & 12 \\ \hline 0 & -6 & -12 & -58 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -3 & -6 & -29 \\ 0 & -6 & -12 & -58 \end{array}\right] \ \frac{1}{3}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & \frac{29}{3} \\ 0 & -6 & -12 & -58 \end{array}\right] \ R_3+6R_2$$

$$\begin{array}{cccc} 0 & -6 & -12 & -58 \\ 0 & 6 & 12 & 58 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & \frac{29}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

let z be the variable

From Row 1 $\Rightarrow y+2z=\frac{29}{3}$

$$\boxed{y=\frac{29}{3}-2z}$$

From Row 1 $\Rightarrow x+2y+3z=10$

$$x=10-2y-3z$$

$$x=10-2\left(\frac{29}{3}-2z\right)-3z$$

$$x=10-\frac{58}{3}+4z-3z$$

$$\boxed{x=z-\frac{28}{3}}$$

Solution: $\boxed{\left(z-\frac{28}{3},\ \frac{29}{3}-2z,\ z\right)}$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 2x+y+2z=4 \\ 2x+2y \qquad =5 \\ 2x-y+6z=2 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \quad \tfrac{1}{2}R_1 \qquad \begin{array}{cccc} 1 & \tfrac{1}{2} & 1 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \tfrac{1}{2} & 1 & 2 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-2R_1 \end{array} \qquad \begin{array}{cccc} 2 & 2 & 0 & 5 \\ -2 & -1 & -2 & -4 \\ \hline 0 & 1 & -2 & 1 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 6 & 2 \\ -2 & -1 & -2 & -4 \\ \hline 0 & -2 & 4 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \tfrac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & -2 \end{array}\right] \begin{array}{l} R_1-\tfrac{1}{2}R_2 \\ \\ R_3+2R_2 \end{array} \qquad \begin{array}{cccc} 1 & \tfrac{1}{2} & 1 & 2 \\ 0 & -\tfrac{1}{2} & 1 & -\tfrac{1}{2} \\ \hline 1 & 0 & 2 & \tfrac{3}{2} \end{array} \qquad \begin{array}{cccc} 0 & -2 & 4 & -2 \\ 0 & 2 & -4 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & \tfrac{3}{2} \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From Row 3: 0 = 0 is a true statement. Let ***z*** be the variable.

From Row 2: $y-2z=1$

$$\underline{y=1+2z}$$

From Row 1: $x+2z=\frac{3}{2}$

$$\underline{x=-2z+\frac{3}{2}}$$

$\therefore$ ***Solution***: $\underline{\left(-2z+\frac{3}{2},\ 2z+1,\ z\right)}$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} -1 & -2 & 3 & 1 \\ 1 & 1 & 2 & 8 \\ \hline 0 & -1 & 5 & 9 \end{array} \qquad \begin{array}{cccc} 3 & -7 & 4 & 10 \\ -3 & -3 & -6 & -24 \\ \hline 0 & -10 & -2 & -14 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] -R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 10R_2 \end{array}$$

$$\begin{array}{cccc} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ \hline 1 & 0 & 7 & 17 \end{array} \qquad \begin{array}{cccc} 0 & -10 & -2 & -14 \\ 0 & 10 & -50 & -90 \\ \hline 0 & 0 & -52 & -104 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right] -\frac{1}{52}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 - 7R_3 \\ R_2 + 5R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & 7 & 17 \\ 0 & 0 & -7 & -14 \\ \hline 1 & 0 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -5 & -9 \\ 0 & 0 & 5 & 10 \\ \hline 0 & 1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

∴ ***Solution***: $(3,\ 1,\ 2)$

Exercise

Use augmented elimination to solve linear system $\begin{matrix} 2x-5y+3z=1 \\ x-2y-2z=8 \end{matrix}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 2 & -5 & 3 & 1 \end{array}\right] \quad R_2 - 2R_1$$

$$\begin{array}{cccc} 2 & -5 & 3 & 1 \\ -2 & 4 & 4 & -16 \\ \hline 0 & -1 & 7 & -15 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & -1 & 7 & -15 \end{array}\right] \quad -R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & 1 & -7 & 15 \end{array}\right] \quad R_1 + 2R_2$$

$$\begin{array}{cccc} 1 & -2 & -2 & 8 \\ 0 & 2 & -14 & 30 \\ \hline 1 & 0 & -16 & 38 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -16 & 38 \\ 0 & 1 & -7 & 15 \end{array}\right] \quad \begin{array}{l} \rightarrow \quad x-16z=38 \\ \rightarrow \quad y-7z=15 \end{array}$$

$$\begin{cases} x = 16z+38 \\ y = 7z+15 \end{cases}$$

$\therefore$ ***Solution***: $\underline{(16z+38,\ 7z+15,\ z)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 5 \\ 1 & -1 & 1 & -2 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccc} 2 & 1 & -1 & 5 \\ -2 & -2 & -2 & -4 \\ \hline 0 & -1 & -3 & 1 \end{array} \qquad \begin{array}{cccc} 1 & -1 & 1 & -2 \\ -1 & -1 & -1 & -2 \\ \hline 0 & -2 & 0 & -4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & 1 \\ 0 & -2 & 0 & -4 \end{array}\right] \quad \begin{array}{c} (2) \\ (1) \\ -2y=-4 \end{array}$$

$\underline{y=2}|$

$(1) \rightarrow \quad -y-3z=1$

$3z=-1-2$

$\underline{z=-1}|$

$$(2) \rightarrow \quad x + y + z = 2$$

$$x = 2 - 2 + 1$$

$$\underline{= 1}|$$

$\therefore$ ***Solution***: $\underline{(1,\ 2,\ -1)}\ |$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ -1 & -1 & 1 & 1 \\ 3 & -1 & 1 & 9 \end{array}\right] \begin{array}{c} \\ 2R_2 + R_1 \\ 2R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} -2 & -2 & 2 & 2 \\ 2 & 1 & 1 & 9 \\ \hline 0 & -1 & 3 & 11 \end{array} \qquad \begin{array}{cccc} 6 & -2 & 2 & 18 \\ -6 & -3 & -3 & -27 \\ \hline 0 & -5 & -1 & -9 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \\ 0 & -5 & -1 & -9 \end{array}\right] \begin{array}{c} \\ \\ R_3 - 5R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -5 & -1 & -9 \\ 0 & 5 & -15 & -55 \\ \hline 0 & 0 & -16 & -64 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \\ 0 & 0 & -16 & -64 \end{array}\right] \begin{array}{c} (2) \\ (1) \\ -16z = -64 \end{array}$$

$$\underline{z = 4}|$$

$$(1) \rightarrow \quad -y + 3z = 11$$

$$y = 12 - 11$$

$$\underline{= 1}|$$

$$(2) \rightarrow \quad 2x + y + z = 9$$

$$2x = 9 - 1 - 4$$

$$\underline{x = 2}|$$

$\therefore$ ***Solution***: $\underline{(2,\ 1,\ 4)}\ |$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ -3 & 6 & 2 & 11 \end{array}\right] \quad R_3 + 3R_1 \qquad \begin{array}{cccc} -3 & 6 & 2 & 11 \\ 3 & 15 & -3 & -12 \\ \hline 0 & 21 & -1 & -1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ 0 & 21 & -1 & -1 \end{array}\right] \quad R_3 - 7R_2 \qquad \begin{array}{cccc} 0 & 21 & -1 & -1 \\ 0 & -21 & 7 & 7 \\ \hline 0 & 0 & 6 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ 0 & 0 & 6 & 6 \end{array}\right] \quad \begin{array}{ll} \to x + 5y - z = -4 & (2) \\ \to 3y - z = -1 & (1) \\ \to 6z = 6 & \end{array}$$

$\underline{z = 1}$

$(1) \to \quad 3y = -1 + 1$

$\underline{y = 0}$

$(2) \to \quad x = -4 + 1$

$\underline{x = -3}$

$\therefore$ ***Solution***: $\underline{(-3,\ 0,\ 1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 2 & -3 & 2 & 10 \\ 3 & -1 & 1 & 9 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \qquad \begin{array}{cccc} 2 & -3 & 2 & 10 \\ -2 & -6 & -8 & -28 \\ \hline 0 & -9 & -6 & -18 \end{array} \qquad \begin{array}{cccc} 3 & -1 & 1 & 9 \\ -3 & -9 & -12 & -42 \\ \hline 0 & -10 & -11 & -33 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -9 & -6 & -18 \\ 0 & -10 & -11 & -33 \end{array}\right] \quad 9R_3 - 10R_2 \qquad \begin{array}{cccc} 0 & -90 & -99 & -297 \\ 0 & 90 & 60 & 180 \\ \hline 0 & 0 & -39 & -117 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -9 & -6 & -18 \\ 0 & 0 & -39 & -117 \end{array}\right] \quad \begin{array}{ll} x + 3y + 4z = 14 & (3) \\ -9y - 6z = -18 & (2) \\ -39z = -117 & (1) \end{array}$$

$(1) \to \quad z = \dfrac{117}{39}$

$\underline{= 3}$

$(2) \to \quad 9y = 18 - 6(3)$

$9y = 0$

$\underline{y = 0}|$

$(3) \to \quad x = 14 - 12$

$\underline{x = 2}|$

$\therefore$ ***Solution***: $\underline{(2,\ 0,\ 3)}\ |$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 3 & 2 & 1 & 8 \\ 2 & -3 & 2 & -16 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \qquad \begin{array}{cccc} 3 & 2 & 1 & 8 \\ -3 & -12 & 3 & -60 \\ \hline 0 & -10 & 4 & -52 \end{array} \qquad \begin{array}{cccc} 2 & -3 & 2 & -16 \\ -2 & -8 & 2 & -40 \\ \hline 0 & -11 & 4 & -56 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 0 & -10 & 4 & -52 \\ 0 & -11 & 4 & -56 \end{array}\right] \begin{array}{l} \\ \\ 10R_3 - 11R_2 \end{array} \qquad \begin{array}{cccc} 0 & -110 & 40 & -560 \\ 0 & 110 & -44 & 572 \\ \hline 0 & 0 & -4 & 12 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 0 & -10 & 4 & -52 \\ 0 & 0 & -4 & 12 \end{array}\right] \quad \begin{array}{rl} x + 4y - z = 20 & (3) \\ -10y + 4z = -52 & (2) \\ -4z = 12 & (1) \end{array}$$

$(1) \to \quad \underline{z = -3}|$

$(2) \to \quad -10y = -52 + 12$

$-10y = -40$

$\underline{y = 4}|$

$(3) \to \quad x = 20 - 16 - 3$

$\underline{x = 1}|$

$\therefore$ ***Solution***: $\underline{(1,\ 4,\ -3)}\ |$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 2 & -1 & 7 \\ 2 & -3 & 2 & -1 \end{array}\right] R_3 - 2R_1 \qquad \begin{array}{cccc} 2 & -3 & 2 & -1 \\ -2 & -4 & -2 & -34 \\ \hline 0 & -7 & 0 & -35 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 2 & -1 & 7 \\ 0 & -7 & 0 & -35 \end{array}\right] \quad \begin{array}{rl} x + 2y + z = 17 & (3) \\ 2y - z = 7 & (2) \\ -7y = -35 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{y = 5}$

$(2) \rightarrow \quad z = 10 - 7$

$\qquad \underline{= 3}$

$(3) \rightarrow \quad x = 17 - 10 - 3$

$\qquad \underline{= 4}$

∴ ***Solution***: $\underline{(4,\ 5,\ 3)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ -4 & 5 & 3 & 7 \\ -6 & 3 & 5 & -4 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \qquad \begin{array}{cccc} -4 & 5 & 3 & 7 \\ 4 & -12 & -14 & -6 \\ \hline 0 & -7 & -11 & 1 \end{array} \qquad \begin{array}{cccc} -6 & 3 & 5 & -4 \\ 6 & -18 & -21 & -9 \\ \hline 0 & -15 & -16 & -13 \end{array}$$

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ 0 & -7 & -11 & 1 \\ 0 & -15 & -16 & -13 \end{array}\right] 7R_3 - 15R_1 \qquad \begin{array}{cccc} 0 & -105 & -112 & -91 \\ 0 & 105 & 165 & -15 \\ \hline 0 & 0 & 53 & -106 \end{array}$$

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ 0 & -7 & -11 & 1 \\ 0 & 0 & 53 & -106 \end{array}\right] \quad \begin{array}{rl} -2x + 6y + 7z = 3 & (3) \\ -7y - 11z = 1 & (2) \\ 53z = -106 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z = -2}|$

$(2) \rightarrow \quad -7y = 1 - 22$

$-7y = -21$

$\underline{y = 3}|$

$(3) \rightarrow \quad -2x = 3 - 18 + 14$

$-2x = -1$

$\underline{x = \frac{1}{2}}|$

$\therefore$ ***Solution***: $\underline{\left(\frac{1}{2},\ 3,\ -2\right)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 3 & -3 & 4 & 5 \\ 4 & -2 & 3 & 4 \end{array}\right] \begin{array}{l} \\ 2R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 6 & -6 & 8 & 10 \\ -6 & 3 & -3 & -3 \\ \hline 0 & -3 & 5 & 7 \end{array} \qquad \begin{array}{cccc} 4 & -2 & 3 & 4 \\ -4 & 2 & -2 & -2 \\ \hline 0 & 0 & 1 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 0 & -3 & 5 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{ll} 2x - y + z = 1 & (2) \\ -3y + 5z = 7 & (1) \\ \underline{z = 2}| & \end{array}$$

$(1) \rightarrow \quad -3y = 7 - 10$

$-3y = -3$

$\underline{y = 1}|$

$(3) \rightarrow \quad 2x = 1 + 1 - 2$

$\underline{x = 0}|$

$\therefore$ ***Solution***: $\underline{(0,\ 1,\ 2)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x-4y+4z=7 \\ x-y-2z=2 \\ 2x-3y+6z=5 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 3 & -4 & 4 & 7 \\ 2 & -3 & 6 & 5 \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3-2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & -4 & 4 & 7 \\ -3 & 3 & 6 & -6 \\ \hline 0 & -1 & 10 & 1 \end{array} \qquad \begin{array}{cccc} 2 & -3 & 6 & 5 \\ -2 & 2 & 4 & -4 \\ \hline 0 & -1 & 10 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & -1 & 10 & 1 \\ 0 & -1 & 10 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3=R_2 \end{array} \qquad \begin{array}{ll} \to x-y-2z=2 & (2) \\ \to -y+10z=1 & (1) \end{array}$$

$(1) \to \quad \underline{y=10z-1}$

$(2) \to \quad x=2+10z-1+2z$

$\qquad \underline{=12z+1}$

$\therefore$ ***Solution***: $\underline{(12z+1,\ 10z-1,\ z)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 2 & -1 & 1 & 4 \\ -1 & 1 & 1 & 4 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3+R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 1 & 4 \\ -2 & 4 & 2 & -4 \\ \hline 0 & 3 & 3 & 0 \end{array} \qquad \begin{array}{cccc} -1 & 1 & 1 & 4 \\ 1 & -2 & -1 & 2 \\ \hline 0 & -1 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & 0 & 6 \end{array}\right] \quad \begin{array}{ll} x-2y-z=2 & (3) \\ 3y+3z=0 & (2) \\ -y=6 & (1) \end{array}$$

$(1) \to \quad \underline{y=-6}$

$(2) \to \quad z=-y$

$\qquad \underline{=6}$

$(3) \to \quad x=2-12+6$

$\qquad \underline{=-4}$

$\therefore$ ***Solution***: $\underline{(-4,\ -6,\ 6)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x+y+z=3 \\ -y+2z=1 \\ -x+z=0 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ -1 & 0 & 1 & 0 \end{array}\right] \quad R_3 + R_1 \qquad \begin{array}{cccc} -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 3 \\ \hline 0 & 1 & 2 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right] \quad R_3 + R_2 \qquad \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 0 & -1 & 2 & 1 \\ \hline 0 & 0 & 4 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 4 & 4 \end{array}\right] \quad \begin{array}{rl} x+y+z=3 & (3) \\ -y+2z=1 & (2) \\ 4z=4 & (1) \end{array}$$

$(1) \to \quad \underline{z=1}$

$(2) \to \quad -y = 1-2$

$\qquad \underline{y=1}$

$(3) \to \quad x = 3-1-1$

$\qquad \underline{=1}$

$\therefore$ ***Solution***: $\underline{(1,\ 1,\ 1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x+y+3z=14 \\ 7x+5y+8z=37 \\ x+3y+2z=9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 3 & 1 & 3 & 14 \\ 7 & 5 & 8 & 37 \end{array}\right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - 7R_1 \end{array} \qquad \begin{array}{cccc} 3 & 1 & 3 & 14 \\ -3 & -9 & -6 & -27 \\ \hline 0 & -8 & -3 & -13 \end{array} \qquad \begin{array}{cccc} 7 & 5 & 8 & 37 \\ -7 & -21 & -14 & -63 \\ \hline 0 & -16 & -6 & -26 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -8 & -3 & -13 \\ 0 & -16 & -6 & -26 \end{array}\right] \quad R_3 - 2R_2 \qquad \begin{array}{cccc} 0 & -16 & -6 & -26 \\ 0 & 16 & 6 & 26 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -8 & -3 & -13 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{rl} x+3y+2z=9 & (2) \\ -8y-3z=-13 & (1) \end{array}$$

$(1) \rightarrow \quad -8y = 3z - 13$

$$\underline{y = -\frac{3}{8}z + \frac{13}{8}}$$

$(3) \rightarrow \quad x = 9 - 3\left(\frac{13}{8} - \frac{3}{8}z\right) - 2z$

$$= 9 - \frac{39}{8} + \frac{9}{8}z - 2z$$

$$\underline{= \frac{33}{8} - \frac{7}{8}z}$$

$\therefore$ ***Solution***: $\underline{\left(\frac{33}{8} - \frac{7}{8}z,\ \ \frac{13}{8} - \frac{3}{8}z,\ \ z\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 4x - 2y + z = 7 \\ x + y + z = -2 \\ 4x + 2y + z = 3 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 4 & -2 & 1 & 7 \\ 4 & 2 & 1 & 3 \end{array}\right] \begin{array}{l} \\ R_2 - 4R_1 \\ R_3 - 4R_1 \end{array} \qquad \begin{array}{cccc} 4 & -2 & 1 & 7 \\ -4 & -4 & -4 & 8 \\ \hline 0 & -6 & -3 & 15 \end{array} \qquad \begin{array}{cccc} 4 & 2 & 1 & 3 \\ -4 & -4 & -4 & 8 \\ \hline 0 & -2 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 15 \\ 0 & -2 & -3 & 11 \end{array}\right] \begin{array}{l} \\ \\ -3R_3 + R_2 \end{array} \qquad \begin{array}{cccc} 0 & 6 & 9 & -33 \\ 0 & -6 & -3 & 15 \\ \hline 0 & 0 & 6 & -18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 15 \\ 0 & 0 & 6 & -18 \end{array}\right] \quad \begin{array}{rl} x+y+z=-2 & (3) \\ -6y-3z=15 & (2) \\ 6z=-18 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z = -3}$

$(2) \rightarrow \quad -6y = 15 - 9$

$$\underline{y = -1}$$

$(3) \rightarrow \quad x = -2 + 1 + 3$

$$\underline{= 2}$$

$\therefore$ ***Solution***: $\underline{(2,\ -1,\ -3)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 2 & -2 & 1 & -4 \\ 6 & 4 & -3 & -24 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-6R_1 \end{array}$$

$$\begin{array}{rrrr} 2 & -2 & 1 & -4 \\ -2 & 4 & -4 & -2 \\ \hline 0 & 2 & -3 & -6 \end{array} \qquad \begin{array}{rrrr} 6 & 4 & -3 & -24 \\ -6 & 12 & -12 & -6 \\ \hline 0 & 16 & -15 & -30 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 16 & -15 & -30 \end{array}\right] \begin{array}{l} \\ \\ R_3-8R_2 \end{array}$$

$$\begin{array}{rrrr} 0 & 16 & -15 & -30 \\ 0 & -16 & 24 & 48 \\ \hline 0 & 0 & 9 & 18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 0 & 9 & 18 \end{array}\right] \quad \begin{array}{rl} x-2y+2z=1 & (3) \\ 2y-3z=-6 & (2) \\ 9z=18 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z=2}$

$(2) \rightarrow \quad 2y=-6+6$

$\qquad \underline{y=0}$

$(3) \rightarrow \quad x=1-4$

$\qquad \underline{=-3}$

$\therefore$ ***Solution***: $\underline{(-3,\ 0,\ 2)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

Solution

$$\begin{cases} z+9x+3y=4 \\ z+16x+4y=2 \\ z+25x+5y=2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 1 & 16 & 4 & 2 \\ 1 & 25 & 5 & 2 \end{array}\right] \begin{array}{l} \\ R_2-R_1 \\ R_3-R_1 \end{array}$$

$$\begin{array}{rrrr} 1 & 16 & 4 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 7 & 1 & -2 \end{array} \qquad \begin{array}{rrrr} 1 & 25 & 5 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 16 & 2 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 16 & 2 & -2 \end{array}\right] \quad 7R_3 - 16R_3 \qquad \begin{array}{cccc} 0 & 112 & 14 & -14 \\ 0 & -112 & -16 & 32 \\ \hline 0 & 0 & -2 & 18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 0 & -2 & 18 \end{array}\right] \quad \begin{array}{rl} z + 9x + 3y = 4 & (3) \\ 7x + y = -2 & (2) \\ -2y = 18 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{y = -9}|$

$(2) \rightarrow \quad 7x = -2 + 9$

$\qquad \underline{= 1}|$

$(3) \rightarrow \quad z = 4 - 9 + 27$

$\qquad \underline{= 22}|$

∴ ***Solution***: $\underline{(1,\ -9,\ 22)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 2 & -8 \\ 3 & -1 & -4 & 3 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 2 & -8 \\ -2 & -4 & 6 & -18 \\ \hline 0 & -5 & 8 & -26 \end{array} \qquad \begin{array}{cccc} 3 & -1 & -4 & 3 \\ -3 & -6 & 9 & -27 \\ \hline 0 & -7 & 5 & -24 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & -7 & 5 & -24 \end{array}\right] \quad 5R_3 - 7R_2 \qquad \begin{array}{cccc} 0 & -35 & 25 & -120 \\ 0 & 35 & -56 & 182 \\ \hline 0 & 0 & -31 & 62 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & 0 & -31 & 62 \end{array}\right] \quad \begin{array}{rl} x + 2y - 3z = 9 & (3) \\ -5y + 8z = -26 & (2) \\ -31z = 62 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z = -2}|$

$(2) \rightarrow \quad -5y = -26 + 16$

$\qquad -5y = 10$

$\qquad \underline{y = 2}|$

$(3) \rightarrow \quad x = 9 - 4 - 6$

$\underline{=-1}$

$\therefore$ ***Solution***: $\underline{(-1,\ 2,\ -2)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x \qquad -3z=-5 \\ 2x-y+2z=16 \\ 7x-3y-5z=19 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 2 & -1 & 2 & 16 \\ 7 & -3 & -5 & 19 \end{array}\right] \quad \begin{array}{l} \\ R_2-2R_1 \\ R_3-7R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 2 & 16 \\ -2 & 0 & 6 & 10 \\ \hline 0 & -1 & 8 & 26 \end{array} \qquad \begin{array}{cccc} 7 & -3 & -5 & 19 \\ -7 & 0 & 21 & 35 \\ \hline 0 & -3 & 16 & 54 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & -3 & 16 & 54 \end{array}\right] \quad \begin{array}{l} \\ \\ R_3-3R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -3 & 16 & 54 \\ 0 & 3 & -24 & -78 \\ \hline 0 & 0 & -8 & -24 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & 0 & -8 & -24 \end{array}\right] \quad \begin{array}{rl} x-3z=-5 & (3) \\ -y+8z=26 & (2) \\ -8z=-24 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z=3}$

$(2) \rightarrow \quad -y=26-24$

$\qquad \underline{y=-2}$

$(3) \rightarrow \quad x=-5+9$

$\qquad \underline{=4}$

$\therefore$ ***Solution***: $\underline{(4,\ -2,\ 3)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x+2y-z=5 \\ 2x-y+3z=0 \\ 2y+z=1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 2 & -1 & 3 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right] \quad R_2-2R_1$$

$$\begin{array}{cccc} 2 & -1 & 3 & 0 \\ -2 & -4 & 2 & -10 \\ \hline 0 & -5 & 5 & -10 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 2 & 1 & 1 \end{array}\right] \quad 5R_3 + 2R_2$$

$$\begin{array}{cccc} 0 & 10 & 5 & 5 \\ 0 & -10 & 10 & -20 \\ \hline 0 & 0 & 15 & -15 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 0 & 15 & -15 \end{array}\right] \quad \begin{array}{rl} x+2y-z=5 & (3) \\ -5y+5z=-10 & (2) \\ 15z=-15 & (1) \end{array}$$

$(1) \to \quad \underline{z=-1}$

$(2) \to \quad -5y=-10+5$

$\qquad \underline{y=1}$

$(3) \to \quad x=5-2-1$

$\qquad \underline{=2}$

$\therefore$ ***Solution***: $\underline{(2,\ 1,\ -1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x+y+z=6 \\ 3x+4y-7z=1 \\ 2x-y+3z=5 \end{cases}$

<u>Solution</u>

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 4 & -7 & 1 \\ 2 & -1 & 3 & 5 \end{array}\right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 4 & -7 & 1 \\ -3 & -3 & -3 & -18 \\ \hline 0 & 1 & -10 & -17 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 3 & 5 \\ -2 & -2 & -2 & -12 \\ \hline 0 & -3 & 1 & -7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -10 & -17 \\ 0 & -3 & 1 & -7 \end{array}\right] \quad R_3 + 3R_2$$

$$\begin{array}{cccc} 0 & -3 & 1 & -7 \\ 0 & 3 & -30 & -51 \\ \hline 0 & 0 & -29 & -58 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -10 & -17 \\ 0 & 0 & -29 & -58 \end{array}\right] \quad \begin{array}{rl} x+y+z=6 & (3) \\ y-10z=-17 & (2) \\ -29z=-58 & (1) \end{array}$$

$(1) \to \quad \underline{z=2}$

$(2) \to \quad y=-17+20$

$\qquad \underline{=3}$

$(3) \to \quad x=6-3-2$

$\qquad \underline{=1}$

$\therefore$ ***Solution***: $\underline{(1,\ 3,\ 2)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x+2y+3z=3 \\ 4x-5y+7z=1 \\ 2x+3y-2z=6 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 4 & -5 & 7 & 1 \\ 2 & 3 & -2 & 6 \end{array}\right] \begin{array}{l} \\ 3R_2-4R_1 \\ 3R_3-2R_1 \end{array}$$

$$\begin{array}{cccc} 12 & -15 & 21 & 3 \\ -12 & -8 & -12 & -12 \\ \hline 0 & -23 & 9 & -9 \end{array} \qquad \begin{array}{cccc} 6 & 9 & -6 & 18 \\ -6 & -4 & -6 & -6 \\ \hline 0 & 5 & -12 & 12 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 0 & -23 & 9 & -9 \\ 0 & 5 & -12 & 12 \end{array}\right] \begin{array}{l} \\ \\ 23R_3+5R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 115 & -276 & 276 \\ 0 & -115 & 45 & -45 \\ \hline 0 & 0 & -231 & 231 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 0 & -23 & 9 & -9 \\ 0 & 0 & -231 & 231 \end{array}\right] \quad \begin{array}{rl} 3x+2y+3z=3 & (3) \\ -23y+9z=-9 & (2) \\ -231z=231 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z=-1}$

$(2) \rightarrow \quad -23y=-9+9$

$\qquad \underline{y=0}$

$(3) \rightarrow \quad 3x=3+3$

$\qquad \underline{x=2}$

∴ ***Solution***: $\underline{(2,\ 0,\ -1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x-3y+z=2 \\ 4x-12y+4z=8 \\ -2x+6y-2z=-4 \end{cases}$

Solution

$$\begin{cases} & x-3y+z=2 \\ \frac{1}{4}\times & 4x-12y+4z=8 \\ -\frac{1}{2}\times & -2x+6y-2z=-4 \end{cases}$$

$$\begin{cases} x-3y+z=2 \\ x-3y+z=2 \\ x-3y+z=2 \end{cases}$$

Since all three equations are the same.

$\therefore$ ***Solution***: is the plane $\boxed{x-3y+z=2}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x-2y+z=-1 \\ x+2y-z=2 \\ 6x+4y+3z=5 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & -2 & 1 & -1 \\ 6 & 4 & 3 & 5 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-6R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -2 & 1 & -1 \\ -2 & -4 & 2 & -4 \\ \hline 0 & -6 & 3 & -5 \end{array} \qquad \begin{array}{cccc} 6 & 4 & 3 & 5 \\ -6 & -12 & 6 & -12 \\ \hline 0 & -8 & 9 & -7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -6 & 3 & -5 \\ 0 & -8 & 9 & -7 \end{array}\right] \begin{array}{l} \\ \\ 3R_3-4R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -24 & 27 & -21 \\ 0 & 24 & -12 & 20 \\ \hline 0 & 0 & 15 & -1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -6 & 3 & -5 \\ 0 & 0 & 15 & -1 \end{array}\right] \begin{array}{ll} x+2y-z=2 & (3) \\ -6y+3z=-5 & (2) \\ 15z=-1 & (1) \end{array}$$

$(1) \rightarrow \quad \boxed{z=-\frac{1}{15}}$

$(2) \rightarrow \quad -6y=-5+\frac{1}{5}$

$-6y=-\frac{24}{5}$

$\boxed{y=\frac{4}{5}}$

$(3) \rightarrow \quad x=2-\frac{8}{5}-\frac{1}{15}$

$\boxed{=\frac{1}{3}}$

$\therefore$ ***Solution***: $\boxed{\left(\frac{1}{3},\ \frac{4}{5},\ -\frac{1}{15}\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 \\ x_2 - 3x_3 - x_4 = 0 \\ 3x_1 + 2x_3 - x_4 = 6 \\ -4x_1 + x_2 + 4x_3 + 2x_4 = -3 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 3 & 0 & 2 & -1 & 6 \\ -4 & 1 & 4 & 2 & -3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_1 \\ R_4 + 4R_1 \end{array}$$

$$\begin{array}{ccccc} 3 & 0 & 2 & -1 & 6 \\ -3 & 15 & -6 & 6 & -12 \\ \hline 0 & 15 & -4 & 5 & -6 \end{array} \qquad \begin{array}{ccccc} -4 & 1 & 4 & 2 & -3 \\ 4 & -20 & 8 & -8 & 16 \\ \hline 0 & -19 & 12 & -6 & 13 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 15 & -4 & 5 & -6 \\ 0 & -19 & 12 & -6 & 13 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 15R_2 \\ R_4 + 19R_2 \end{array}$$

$$\begin{array}{ccccc} 0 & 15 & -4 & 5 & -6 \\ 0 & -15 & 45 & 15 & 0 \\ \hline 0 & 0 & 41 & 20 & -6 \end{array} \qquad \begin{array}{ccccc} 0 & -19 & 12 & -6 & 13 \\ 0 & 19 & -57 & -19 & 0 \\ \hline 0 & 0 & -45 & -25 & 13 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & -45 & -25 & 13 \end{array}\right] \begin{array}{l} \\ \\ \\ 41R_4 + 45R_2 \end{array}$$

$$\begin{array}{ccccc} 0 & 0 & -1845 & -1025 & 533 \\ 0 & 0 & 1845 & 900 & -270 \\ \hline 0 & 0 & 0 & -125 & 263 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & 0 & -125 & 263 \end{array}\right] \quad \begin{array}{rl} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 & (4) \\ x_2 - 3x_3 - x_4 = 0 & (3) \\ 41x_3 + 20x_4 = -6 & (2) \\ -125x_4 = 263 & (1) \end{array}$$

$(1) \to \quad \boxed{x_4 = -\frac{263}{125}}$

$(2) \to \quad 41x_3 = -6 + \frac{1,052}{25}$

$$= \frac{902}{25}$$

$$\boxed{x_3 = \frac{22}{25}}$$

$(3) \to \quad x_2 = \frac{66}{25} - \frac{263}{125}$

$$\boxed{= \frac{67}{125}}$$

$(4) \to \quad x_1 = 4 + \frac{67}{25} - \frac{44}{25} - \frac{526}{125}$

$$= 4 + \frac{23}{25} - \frac{526}{125}$$

$$= \frac{500 + 115 - 526}{125}$$

$$= \frac{89}{125}$$

$\therefore$ ***Solution***: $\left(\frac{89}{125},\ \frac{67}{125},\ \frac{22}{25},\ -\frac{263}{125} \right)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x_1 + x_2 + x_3 + x_4 = 5 \\ x_1 + 2x_2 - x_3 - 2x_4 = -1 \\ x_1 - 3x_2 - 3x_3 - x_4 = -1 \\ 2x_1 - x_2 + 2x_3 - x_4 = -2 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 1 & 2 & -1 & -2 & -1 \\ 1 & -3 & -3 & -1 & -1 \\ 2 & -1 & 2 & -1 & -2 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - 2R_1 \end{array}$$

$$\begin{array}{ccccc} 1 & 2 & -1 & -2 & -1 \\ -1 & -1 & -1 & -1 & -5 \\ \hline 0 & 1 & -2 & -3 & -6 \end{array} \qquad \begin{array}{ccccc} 1 & -3 & -3 & -1 & -1 \\ -1 & -1 & -1 & -1 & -5 \\ \hline 0 & -4 & -4 & -2 & -6 \end{array} \qquad \begin{array}{ccccc} 2 & -1 & 2 & -1 & -2 \\ -2 & -2 & -2 & -2 & -10 \\ \hline 0 & -3 & 0 & -3 & -12 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & -4 & -4 & -2 & -6 \\ 0 & -3 & 0 & -3 & -12 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 4R_2 \\ R_4 + 3R_2 \end{array}$$

$$\begin{array}{ccccc} 0 & -4 & -4 & -2 & -6 \\ 0 & 4 & -8 & -12 & -24 \\ \hline 0 & 0 & -12 & -14 & -30 \end{array}$$

$$\begin{array}{ccccc} 0 & -3 & 0 & -3 & -12 \\ 0 & 3 & -6 & -9 & -18 \\ \hline 0 & 0 & -6 & -12 & -30 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & -12 & -14 & -30 \\ 0 & 0 & -6 & -12 & -30 \end{array}\right] \begin{array}{l} \\ \\ \\ -2R_4 + R_3 \end{array}$$

$$\begin{array}{ccccc} 0 & 0 & 12 & 24 & 60 \\ 0 & 0 & -12 & -14 & -30 \\ \hline 0 & 0 & 0 & 10 & 30 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & -12 & -14 & -30 \\ 0 & 0 & 0 & 10 & 30 \end{array}\right] \quad \begin{aligned} x_1 + x_2 + x_3 + x_4 &= 5 & (4) \\ x_2 - 2x_3 - 3x_4 &= -6 & (3) \\ -12x_3 - 14x_4 &= -30 & (2) \\ 10x_4 &= 30 & (1) \end{aligned}$$

$(1) \to \quad \underline{x_4 = 3|}$

$(2) \to \quad -12x_3 = -30 + 42$

$\qquad\qquad = 12$

$\qquad \underline{x_3 = -1|}$

$(3) \to \quad x_2 = -6 - 2 + 9$

$\qquad\qquad \underline{= 1|}$

$(4) \to \quad x_1 = 5 - 1 + 1 - 3$

$\qquad\qquad \underline{= 2|}$

∴ ***Solution***: $\underline{(2,\ 1,\ -1,\ 3)\ |}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x + 8y - z + w = 0 \\ 4x + 16y - 3z - w = -10 \\ -2x + 4y - z + 3w = -6 \\ -6x + 2y + 5z + w = 3 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 4 & 16 & -3 & -1 & -10 \\ -2 & 4 & -1 & 3 & -6 \\ -6 & 2 & 5 & 1 & 3 \end{array}\right] \quad \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \quad R_4 - \frac{13}{6}R_2$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \qquad \textit{Interchange } R_2 \textit{ and } R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & \frac{19}{3} & -\frac{14}{3} & 16 \end{array}\right] \quad R_4 + \frac{19}{3}R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & 0 & -\frac{71}{3} & -\frac{142}{3} \end{array}\right] \quad \begin{array}{ll} 2x+8y-z+w=0 & (3) \\ 12y-2z+4w=-6 & (2) \\ -z-3w=-10 & (1) \\ -\frac{71}{3}w=-\frac{142}{3} \quad \to \quad \underline{w=2} & \end{array}$$

$(1) \to \quad z = 10 - 3w \underline{= 4}$

$(2) \to \quad 12y = 2z - 4w - 6$

$\qquad \underline{y = -\frac{1}{2}}$

$(3) \to \quad 2x = -8y + z - w$

$\qquad 2x = 4 + 4 - 2$

$\qquad 2x = 6$

$\qquad \underline{x = 3}$

$\therefore$ ***Solution***: $\underline{\left(3,\ -\frac{1}{2},\ 4,\ 2\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 \qquad = 0 \\ \qquad x_2 + x_3 = 0 \end{cases}$

Solution

$$\begin{cases} x_1 = -2x_2 \\ x_3 = -x_2 \end{cases}$$

$2x_1 + x_2 + 3x_3 = 0$

$-4x_2 + x_2 - 3x_2 = 0 \quad \to \quad \underline{x_2 = 0}$

$\therefore$ ***Solution***: $\underline{(0,\ 0,\ 0)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x+2y+4z \qquad =0 \\ \quad -y-3z+w=0 \\ 3x+y+z+2w=0 \\ x+3y-2z-2w=0 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & -1 & -3 & 1 & 0 \\ 3 & 1 & 1 & 2 & 0 \\ 1 & 3 & -2 & -2 & 0 \end{array}\right] \begin{array}{c} \\ -R_2 \\ 2R_3-3R_1 \\ 2R_4-R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & -4 & -10 & 4 & 0 \\ 0 & 4 & -8 & -4 & 0 \end{array}\right] \begin{array}{c} \\ \\ R_3+4R_2 \\ R_4-4R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -20 & 0 & 0 \end{array}\right] \begin{array}{ll} 2x+2y-4z=0 & (1) \\ y+3z-w=0 & (2) \\ \rightarrow \underline{z=0} & \\ & \end{array}$$

$(2) \quad \rightarrow \quad \underline{y=w}$

$(1) \quad \rightarrow 2x=-2y \quad \underline{x=-w}$

$\therefore$ ***Solution***: $\underline{(-w,\ w,\ 0,\ w)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x \qquad +z+w=5 \\ \qquad y \qquad -w=-1 \\ 3x \qquad -z-w=0 \\ 4x+y+2z+w=9 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 3 & 0 & -1 & -1 & 0 \\ 4 & 1 & 2 & 1 & 9 \end{array}\right] \begin{array}{c} \\ \\ 2R_3-3R_1 \\ 2R_4-4R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 & -15 \\ 0 & 2 & 0 & -2 & -2 \end{array}\right] \quad R_4 - 2R_2$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 & -15 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{rl} 2x + z + w = 5 & (1) \\ y - w = -1 & (2) \\ -5z - 5w = -15 & (3) \end{array}$$

$(2) \rightarrow \quad \underline{y = 1 + w}$

$(3) \rightarrow \quad \underline{z = 3 - w}$

$(1) \rightarrow \quad 2x = 5 - (3 - w) - w \quad \Rightarrow \quad \underline{x = 1}$

∴ ***Solution***: $\underline{(1,\ 1 + w,\ 3 - w,\ w)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 4y + z = 20 \\ 2x - 2y + z = 0 \\ x \quad + z = 5 \\ x + y - z = 10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 1 & 5 \\ 0 & 4 & 1 & 20 \end{array}\right] \quad \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - R_1 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 0 & -4 & 3 & -20 \\ 0 & -1 & 2 & -5 \\ 0 & 4 & 1 & 20 \end{array}\right] \quad \begin{array}{c} \\ \\ 4R_3 - R_2 \\ R_4 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 0 & -4 & 3 & -20 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \quad \begin{array}{r} x + y = 10 \\ \rightarrow -4y = -20 \\ \rightarrow z = 0 \\ \\ \end{array}$$

∴ ***Solution***: $\underline{(5,\ 5,\ 0)}$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \qquad\qquad - 3w = -3 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ -1 & 3 & -2 & 1 \\ 3 & 4 & -7 & 10 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & -2 & -10 & -14 \end{array}\right] \begin{array}{l} \\ \\ 5R_3 + 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & 0 & -52 & -52 \end{array}\right] \quad \begin{array}{rl} x + 2y + z = 8 & (3) \\ 5y - z = 9 & (2) \\ -52z = -52 & (1) \end{array}$$

$(1) \Rightarrow z = 1$

$(2) \Rightarrow 5y = 9 + 1 = 10 \to y = 2$

$(3) \Rightarrow x = 8 - 4 - 1 = 3$

$\therefore$ ***Solution***: $(3,\ 2,\ 1)$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} 2u - 3v + w - x + y = 0 \\ 4u - 6v + 2w - 3x - y = -5 \\ -2u + 3v - 2w + 2x - y = 3 \end{cases}$$

Solution

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 4 & -6 & 2 & -3 & -1 & -5 \\ -2 & 3 & -2 & 2 & -1 & 3 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -3 & -5 \\ 0 & 0 & -1 & 1 & 0 & 3 \end{array}\right] \qquad \begin{array}{rl} 2u-3v+w-x+y=0 & (3) \\ -x-3y=-5 & (2) \\ -w+x=3 & (1) \end{array}$$

$$(2) \quad \Rightarrow \quad x=5-3y$$

$$(1) \quad \Rightarrow \quad w=x-3=2-3y$$

$$(3) \quad \Rightarrow \quad 2u=3v-2+3y+5-3y-y=3v-y+3$$

$$u=\frac{3}{2}v-\frac{1}{2}y+\frac{3}{2}$$

$\therefore$ ***Solution***: $\left(\frac{3}{2}v-\frac{1}{2}y+\frac{3}{2}, v, 2-3y, 5-3y, y\right)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 6x_3+2x_4-4x_5-8x_6=8 \\ 3x_3+x_4-2x_5-4x_6=4 \\ 2x_1-3x_2+x_3+4x_4-7x_5+x_6=2 \\ 6x_1-9x_2+11x_4-19x_5+3x_6=1 \end{cases}$

Solution

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 6 & -9 & 0 & 11 & -19 & 3 & 1 \end{array}\right] \quad R_4-3R_1$$

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & -3 & -1 & 2 & 0 & -5 \end{array}\right] \quad \begin{array}{l} \\ \\ R_3-2R_2 \\ R_4+R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & -1 \end{array}\right] \quad \begin{array}{l} 2x_1-3x_2+x_3+4x_4-7x_5=2-x_6 \\ 3x_3+x_4-2x_5=4+4x_6 \\ \\ \rightarrow x_6=\frac{1}{4} \end{array}$$

$$\begin{cases} 3x_3=5-x_4+2x_5 \\ 2x_1=\frac{7}{4}+3x_2-\frac{1}{3}\left(5-x_4+2x_5\right)-4x_4+7x_5 \end{cases}$$

$$\begin{cases} \boxed{x_3 = \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5} \\ 2x_1 = \frac{1}{12} + 3x_2 - \frac{11}{3}x_4 + \frac{19}{3}x_5 \end{cases}$$

$\therefore$ ***Solution***: $\boxed{\left(\frac{1}{24} + \frac{3}{2}x_2 - \frac{11}{6}x_4 + \frac{19}{6}x_5,\ x_2,\ \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5,\ x_4,\ x_5,\ \frac{1}{4}\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array}\right] \begin{array}{l} \\ 3R_2 - 5R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & -1 & 4 & 26 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{rl} 3x_1 + 2x_2 - x_3 = -15 & (3) \\ -x_2 + 11x_3 = 75 & (2) \\ -7x_3 = -49 & (1) \end{array}$$

$(1) \rightarrow x_3 = 7$

$(2) \rightarrow x_2 = 77 - 75 = 2$

$(1) \rightarrow 3x_1 = -15 - 4 + 7 = 12$

$\boxed{x_1 = -4}$

$\therefore$ ***Solution***: $\boxed{(-4, 2, 7)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ \qquad\qquad\quad 5x_3 + 10x_4 \qquad + 15x_6 = 5 \\ 2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$

Solution

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 5R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right] \begin{array}{l} \\ \\ \\ \frac{1}{6}R_4 \quad \textit{then interchanging row3 and row4} \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{cases} x_1 + 3x_2 \qquad + 4x_4 + 2x_5 \qquad = 0 \\ \qquad\qquad x_3 + 2x_4 \qquad\qquad = 0 \\ \qquad\qquad\qquad\qquad\quad + x_6 = \frac{1}{3} \end{cases}$$

The general solution of the system: $x_6 = \frac{1}{3}, \quad x_3 = -2x_4, \quad x_1 = -3x_2 - 4x_4 - 2x_5$

$\therefore$ ***Solution***: $\left(-3x_2 - 4x_4 - 2x_5,\ x_2,\ -2x_4,\ x_4,\ x_5,\ \frac{1}{3}\right)$

Exercise

At SnackMix, caramel corn worth \$2.50 per *pound* is mixed with honey roasted missed nuts worth \$7.50 per *pound* in order to get 20 *lbs*. of a mixture worth \$4.50 per *pound*. How much of each snack is used?

Solution

$$\begin{aligned} x + y &= 20 \qquad &(1) \\ 2.50x + 7.50y &= 90 \qquad &(2) \end{aligned}$$

(1) $y = 20 - x$

(2) $2.5x + 7.5(20 - x) = 90$

$2.5x + 150 - 7.5x = 90$

$-5x = 90 - 150$

$-5x = -60$

$x = \frac{-60}{-5} = 12$

$y = 20 - x$
$= 20 - 12$
$= 8$

The mixture consists of 12 *lbs*. of caramel and 8 *lbs*. of nuts

Solution Section 4.2 – Matrix operations and Their Applications

Exercise

Find values for the variables so that the matrices are equal. $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

Solution

$$\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$$

$$\Rightarrow \begin{cases} w = 9 & x = 17 \\ y = 8 & z = -12 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal. $\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$

Solution

$$\begin{cases} \underline{x = 12|} \\ y + 3 = 5 \to \underline{y = 2|} \\ 2z = 6 \to \underline{z = 3|} \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal. $\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$

Solution

$$\begin{bmatrix} 5 = y+3 & x-4 = 2 & 9 = 9 \\ 2 = z+4 & -3 = -3 & 8 = 8 \\ 6 = 6 & 0 = 0 & 5 = w \end{bmatrix}$$

$$\to \begin{cases} y = 2 & z = -2 \\ x = 6 & w = 5 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

Solution

$$\begin{bmatrix} a-5 & 5b & 4c+6 \\ -2d & 7f-6 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

$$\begin{cases} a-5=15 & \rightarrow & a=20 \\ 5b=25 & \rightarrow & b=5 \\ 4c+6=6 & \rightarrow & 4c=0 \rightarrow c=0 \\ -2d=-8 & \rightarrow & d=4 \\ 7f-6=1 & \rightarrow & 7f=7 \rightarrow f=1 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+11 & 12z+1 & 5m \\ 11k & 3 & 1 \end{bmatrix} + \begin{bmatrix} 9a & 9z & 4m \\ 12k & 5 & 3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

Solution

$$\begin{bmatrix} a+11+9a & 12z+1+9z & 5m+4m \\ 11k+12k & 3+5 & 1+3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 10a+11 & 21z+1 & 9m \\ 23k & 8 & 4 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

$$10a+11=41 \quad \rightarrow \quad 10a=30$$

$$\underline{a=3}$$

$$21z+1=-62 \quad \rightarrow \quad 21z=-63$$

$$\underline{z=-3}$$

$$9m=72 \quad \rightarrow \quad \underline{m=8}$$

$$23k=92 \quad \rightarrow \quad \underline{k=\frac{92}{23}=4}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} x+2 & 3y+1 & 5z \\ 8w & 2 & 3 \end{bmatrix} + \begin{bmatrix} 3x & 2y & 5z \\ 2w & 5 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 4x+2 & 5y+1 & 10z \\ 10w & 7 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$$

$$\begin{cases} 4x+2=10 & \rightarrow \quad \underline{x=2|} \\ 5y+1=-14 & \rightarrow \quad \underline{y=-3|} \\ 10z=80 & \rightarrow \quad \underline{z=8|} \\ 10w=10 & \rightarrow \quad \underline{w=1|} \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} 2x-3 & y-2 & 2z+1 \\ 5 & 2w & 7 \end{bmatrix} + \begin{bmatrix} 3x-3 & y+2 & z-1 \\ -5 & 5w+1 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 5x-6 & 2y & 3z \\ 0 & 7w+1 & 10 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$$

$$\begin{cases} 5x-6=20 & \rightarrow \quad \underline{x=\frac{26}{5}|} \\ 2y=8 & \rightarrow \quad \underline{y=4|} \\ 3z=9 & \rightarrow \quad \underline{z=3|} \\ 7w+1=8 & \rightarrow \quad \underline{w=1|} \end{cases}$$

Exercise

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix}$$

Solution

$$\begin{aligned} A-B &= \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix} - \begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 & -5 \\ 2 & 1 & 9 \end{bmatrix} \end{aligned}$$

$$3A+2B=3\begin{bmatrix}3 & 1 & 1\\ -1 & 2 & 5\end{bmatrix}+2\begin{bmatrix}2 & -3 & 6\\ -3 & 1 & -4\end{bmatrix}$$

$$=\begin{bmatrix}9 & 3 & 3\\ -3 & 6 & 15\end{bmatrix}+\begin{bmatrix}4 & -6 & 12\\ -6 & 2 & -8\end{bmatrix}$$

$$=\begin{bmatrix}13 & -3 & 15\\ -9 & 8 & 7\end{bmatrix}$$

Exercise

Given $A=\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$ $\quad F=\begin{bmatrix}3 & 3\\ -1 & -1\end{bmatrix}$ $\quad$ Find $\quad 3F+2A$

Solution

$$3F+2A=3\begin{bmatrix}3 & 3\\ -1 & -1\end{bmatrix}+2\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$$

$$=\begin{bmatrix}3(3) & 3(3)\\ 3(-1) & 3(-1)\end{bmatrix}+\begin{bmatrix}2(1) & 2(2)\\ 2(4) & 2(3)\end{bmatrix}$$

$$=\begin{bmatrix}9 & 9\\ -3 & -3\end{bmatrix}+\begin{bmatrix}2 & 4\\ 8 & 6\end{bmatrix}$$

$$=\begin{bmatrix}9+2 & 9+4\\ -3+8 & -3+6\end{bmatrix}$$

$$=\begin{bmatrix}11 & 13\\ 5 & 3\end{bmatrix}$$

Exercise

Evaluate $\quad \begin{bmatrix}2\\ 5\\ 8\end{bmatrix}+\begin{bmatrix}-6\\ 3\\ 12\end{bmatrix}$

Solution

$$\begin{bmatrix}2\\ 5\\ 8\end{bmatrix}+\begin{bmatrix}-6\\ 3\\ 12\end{bmatrix}=\begin{bmatrix}-4\\ 8\\ 20\end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 5 & 8 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 9 & 1 \\ 4 & 2 & 5 \end{bmatrix}$

Solution

$$\begin{bmatrix} 5 & 8 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 9 & 1 \\ 4 & 2 & 5 \end{bmatrix}$$

*It is **impossible**;* 2×2 and 2×3 are not the same size.

Exercise

Evaluate $\begin{bmatrix} -5 & 0 \\ 4 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix}$

Solution

$$\begin{bmatrix} -5 & 0 \\ 4 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -5+6 & 0+(-3) \\ 4+2 & \frac{1}{2}+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 \\ 6 & \frac{7}{2} \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix}$

Solution

$$\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} 5-4 & -6+6 \\ 8+8 & 9-3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 16 & 6 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} -5 & 6 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 5 & -8 \end{bmatrix}$

Solution

$$\begin{bmatrix} -5 & 6 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 5 & -8 \end{bmatrix} = \begin{bmatrix} -5-(-3) & 6-2 \\ 2-5 & 4-(-8) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 4 \\ -3 & 12 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 8 & 6 & -4 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -8 \end{bmatrix}$

Solution

$$\begin{bmatrix} 8 & 6 & -4 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -8 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 4 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1(4)+3(1) & 1(6)+3(0) \\ 2(4)+5(1) & 2(6)+5(0) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 6 \\ 13 & 12 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix}$

Solution

$$\begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -3(-6)+4(2)+2(3) & -3(4)+4(3)+2(-2) \\ 5(-6)+0(2)+4(3) & 5(4)+0(3)+4(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 32 & -4 \\ -18 & 12 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix} . \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix} . \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1(1)-1(1)+4(1) & 1(1)-1(2)+4(-1) & 1(0)-1(4)+4(3) \\ 4(1)-1(1)+3(1) & 4(1)-1(2)+3(-1) & 4(0)-1(4)+3(3) \\ 2(1)+0(1)-2(1) & 2(1)+0(2)-2(-1) & 2(0)+0(4)-2(3) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -5 & 8 \\ 6 & -1 & 5 \\ 0 & 4 & -6 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix} . \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix} . \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1(1)+1(4)+0(2) & 1(-1)+1(-1)+0(0) & 1(4)+1(3)+0(-2) \\ 1(1)+2(4)+4(2) & 1(-1)+2(-1)+4(0) & 1(4)+2(3)+4(-2) \\ 1(1)-1(4)+3(2) & 1(-1)-1(-1)+3(0) & 1(4)-1(3)+3(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2 & 7 \\ 17 & -3 & 2 \\ 3 & 0 & -5 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} -2 & -3 & -4 \\ 2 & -1 & 0 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$

Solution

$$\begin{bmatrix} -2 & -3 & -4 \\ 2 & -1 & 0 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix} = \begin{bmatrix} -3-12 & -2-6-8 & -8+3+8 \\ -1 & 2-2 & 8+1 \\ -2+9 & 4-4+6 & 16+2-6 \end{bmatrix}$$

$$=\begin{bmatrix} -15 & -16 & 3 \\ -1 & 0 & 9 \\ 7 & 6 & 12 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{18} \\ \sqrt{3} & \sqrt{27} & 0 \end{bmatrix}\begin{bmatrix} 8 & -10 \\ 9 & 12 \\ 0 & 2 \end{bmatrix}$

Solution

$$\begin{bmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{18} \\ \sqrt{3} & \sqrt{27} & 0 \end{bmatrix}\begin{bmatrix} 8 & -10 \\ 9 & 12 \\ 0 & 2 \end{bmatrix} = \begin{pmatrix} 17\sqrt{2} & -4\sqrt{2} \\ 35\sqrt{3} & 26\sqrt{3} \end{pmatrix}$$

Exercise

Evaluate $\begin{bmatrix} x & 2x+1 & 4 \\ 5 & x-1 & 8 \\ -2 & 3x & 2x+1 \end{bmatrix} + \begin{bmatrix} 2x-1 & -2x-1 & 4x \\ -5 & 6 & x+1 \\ -5 & 2 & -2x \end{bmatrix}$

Solution

$$\begin{bmatrix} x & 2x+1 & 4 \\ 5 & x-1 & 8 \\ -2 & 3x & 2x+1 \end{bmatrix} + \begin{bmatrix} 2x-1 & -2x-1 & 4x \\ -5 & 6 & x+1 \\ -5 & 2 & -2x \end{bmatrix} = \begin{bmatrix} 3x-1 & 0 & 4x+4 \\ 0 & x+5 & x+9 \\ -7 & 3x+2 & 1 \end{bmatrix}$$

Exercise

Given $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}$. Find AB and BA.

Solution

$$AB = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 13 \\ 4 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} -16 & 29 \\ -4 & 10 \end{bmatrix}$$

$Note: \quad AB \neq BA$

Exercise

Given $A=\begin{pmatrix}2 & -3\\ 1 & 4\end{pmatrix}$ $B=\begin{pmatrix}-2 & 4\\ 2 & -3\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}2 & -3\\ 1 & 4\end{pmatrix}\begin{pmatrix}-2 & 4\\ 2 & -3\end{pmatrix}$$

$$=\begin{pmatrix}-6 & 17\\ 6 & -8\end{pmatrix}$$

$$BA=\begin{pmatrix}-2 & 4\\ 2 & -3\end{pmatrix}\begin{pmatrix}2 & -3\\ 1 & 4\end{pmatrix}$$

$$=\begin{pmatrix}0 & 14\\ 1 & -20\end{pmatrix}$$

Exercise

Given $A=\begin{pmatrix}3 & -2\\ 4 & 1\end{pmatrix}$ $B=\begin{pmatrix}-1 & -1\\ 0 & 4\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}3 & -2\\ 4 & 1\end{pmatrix}\begin{pmatrix}-1 & -1\\ 0 & 4\end{pmatrix}$$

$$=\begin{pmatrix}-3 & -11\\ 4 & 0\end{pmatrix}$$

$$BA=\begin{pmatrix}-1 & -1\\ 0 & 4\end{pmatrix}\begin{pmatrix}3 & -2\\ 4 & 1\end{pmatrix}$$

$$=\begin{pmatrix}-7 & 1\\ 16 & 4\end{pmatrix}$$

Exercise

Given $A=\begin{pmatrix}3 & -1\\ 2 & 3\end{pmatrix}$ $B=\begin{pmatrix}4 & 1\\ 2 & -3\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}3 & -1\\ 2 & 3\end{pmatrix}\begin{pmatrix}4 & 1\\ 2 & -3\end{pmatrix}$$

$$=\begin{pmatrix}10 & 6\\ 14 & -7\end{pmatrix}$$

$$BA = \begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & -1 \\ 0 & -11 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 2 \\ 4 & -4 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -4 \\ 14 & -12 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -4 & 5 \\ 6 & 0 & 3 \\ -3 & -2 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -13 \\ 3 & -4 \end{pmatrix}$$

Exercise

Given $A=\begin{pmatrix}-1 & 3\\ 2 & 1\\ -3 & 2\end{pmatrix}$ $B=\begin{pmatrix}1 & -2 & 3\\ 0 & 1 & 2\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}-1 & 3\\ 2 & 1\\ -3 & 2\end{pmatrix}\begin{pmatrix}1 & -2 & 3\\ 0 & 1 & 2\end{pmatrix}$$

$$=\begin{pmatrix}-1 & 5 & 4\\ 2 & -3 & 8\\ -3 & 8 & -5\end{pmatrix}$$

$$BA=\begin{pmatrix}1 & -2 & 3\\ 0 & 1 & 2\end{pmatrix}\begin{pmatrix}-1 & 3\\ 2 & 1\\ -3 & 2\end{pmatrix}$$

$$=\begin{pmatrix}-14 & 7\\ -4 & 5\end{pmatrix}$$

Exercise

Given $A=\begin{pmatrix}2 & 4\\ 0 & -1\\ -3 & 2\end{pmatrix}$ $B=\begin{pmatrix}3 & 0 & -2\\ -2 & 6 & 2\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}2 & 4\\ 0 & -1\\ -3 & 2\end{pmatrix}\begin{pmatrix}3 & 0 & -2\\ -2 & 6 & 2\end{pmatrix}$$

$$=\begin{pmatrix}-2 & 24 & 4\\ 2 & -6 & -2\\ -13 & 12 & 10\end{pmatrix}$$

$$BA=\begin{pmatrix}3 & 0 & -2\\ -2 & 6 & 2\end{pmatrix}\begin{pmatrix}2 & 4\\ 0 & -1\\ -3 & 2\end{pmatrix}$$

$$=\begin{pmatrix}12 & 8\\ -10 & 10\end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} \quad A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -2 & -6 \\ 0 & -1 & 2 \\ 5 & -3 & -4 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -3 & 9 \\ 2 & -3 & 4 \\ 4 & -6 & 3 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 8 & -2 \\ 3 & -8 & 4 \\ -2 & 13 & -5 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 5 & -1 \\ 11 & -5 & 6 \\ -8 & 7 & -4 \end{pmatrix}$$

Exercise

Given $A=\begin{pmatrix}1 & -2 & 0\\ 2 & 0 & 1\\ 2 & -2 & -1\end{pmatrix}$ $B=\begin{pmatrix}-3 & 1 & 0\\ 1 & 4 & -1\\ 0 & 0 & 2\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}1 & -2 & 0\\ 2 & 0 & 1\\ 2 & -2 & -1\end{pmatrix}\begin{pmatrix}-3 & 1 & 0\\ 1 & 4 & -1\\ 0 & 0 & 2\end{pmatrix}$$

$$=\begin{pmatrix}-5 & -7 & 2\\ -6 & 2 & 2\\ -8 & -6 & 0\end{pmatrix}$$

$$BA=\begin{pmatrix}-3 & 1 & 0\\ 1 & 4 & -1\\ 0 & 0 & 2\end{pmatrix}\begin{pmatrix}1 & -2 & 0\\ 2 & 0 & 1\\ 2 & -2 & -1\end{pmatrix}$$

$$=\begin{pmatrix}-1 & 6 & 1\\ 7 & 0 & 5\\ 4 & -4 & -2\end{pmatrix}$$

Exercise

Given $A=\begin{bmatrix}-3 & 4\\ 2 & -3\\ -1 & 0\end{bmatrix}$ $B=\begin{bmatrix}4 & 1\\ 1 & -2\\ 3 & -4\end{bmatrix}$, Find

a) $A+B$

b) $A-B$

c) $3A$

d) $-2B$

e) $2A+3B$

f) A^2

g) AB

h) BA

Solution

a) $A+B=\begin{bmatrix}-3 & 4\\ 2 & -3\\ -1 & 0\end{bmatrix}+\begin{bmatrix}4 & 1\\ 1 & -2\\ 3 & -4\end{bmatrix}$

$$=\begin{bmatrix}1 & 5\\ 3 & -5\\ 2 & -4\end{bmatrix}$$

b) $A-B=\begin{bmatrix}-3 & 4\\ 2 & -3\\ -1 & 0\end{bmatrix}-\begin{bmatrix}4 & 1\\ 1 & -2\\ 3 & -4\end{bmatrix}$

$$= \begin{bmatrix} -7 & 3 \\ 1 & -1 \\ -4 & 4 \end{bmatrix}$$

c) $3A = 3\begin{bmatrix} -3 & 4 \\ 2 & -3 \\ -1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} -9 & 12 \\ 6 & -9 \\ -3 & 0 \end{bmatrix}$$

d) $-2B = -2\begin{bmatrix} 4 & 1 \\ 1 & -2 \\ 3 & -4 \end{bmatrix}$

$$= \begin{bmatrix} -8 & -2 \\ -2 & 4 \\ -6 & 8 \end{bmatrix}$$

e) $2A + 3B = 2\begin{bmatrix} -3 & 4 \\ 2 & -3 \\ -1 & 0 \end{bmatrix} + 3\begin{bmatrix} 4 & 1 \\ 1 & -2 \\ 3 & -4 \end{bmatrix}$

$$= \begin{bmatrix} -6 & 8 \\ 4 & -6 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 12 & 3 \\ 3 & -6 \\ 9 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 11 \\ 7 & -12 \\ 7 & -12 \end{bmatrix}$$

f) $A^2 = \mathit{doesn't\ exist}$ (not a square matrix)

g) $AB = \nexists$ $\quad (2\times 3 \quad 2\times 3)$ the inner not equal

h) $BA = \nexists$ $\quad (2\times 3 \quad 2\times 3)$ the inner not equal

Exercise

Given $A=\begin{bmatrix}2 & -2\\ 3 & 4\\ 1 & 0\end{bmatrix}$ $B=\begin{bmatrix}-1 & 8\\ 2 & -2\\ -4 & 3\end{bmatrix}$, Find

a) $A+B$
b) $A-B$
c) $3A$
d) $-2B$
e) $2A+3B$
f) A^2
g) AB
h) BA

Solution

a) $A+B=\begin{bmatrix}2 & -2\\ 3 & 4\\ 1 & 0\end{bmatrix}+\begin{bmatrix}-1 & 8\\ 2 & -2\\ -4 & 3\end{bmatrix}$

$$=\begin{bmatrix}1 & 6\\ 5 & 2\\ -3 & 3\end{bmatrix}$$

b) $A-B=\begin{bmatrix}2 & -2\\ 3 & 4\\ 1 & 0\end{bmatrix}-\begin{bmatrix}-1 & 8\\ 2 & -2\\ -4 & 3\end{bmatrix}$

$$=\begin{bmatrix}3 & -10\\ 1 & 6\\ 5 & -3\end{bmatrix}$$

c) $3A=3\begin{bmatrix}2 & -2\\ 3 & 4\\ 1 & 0\end{bmatrix}$

$$=\begin{bmatrix}6 & -6\\ 9 & 12\\ 3 & 0\end{bmatrix}$$

d) $-2B=-2\begin{bmatrix}-1 & 8\\ 2 & -2\\ -4 & 3\end{bmatrix}$

$$=\begin{bmatrix}2 & -16\\ -4 & 4\\ 8 & -6\end{bmatrix}$$

e) $2A+3B=2\begin{bmatrix}2 & -2\\ 3 & 4\\ 1 & 0\end{bmatrix}+3\begin{bmatrix}-1 & 8\\ 2 & -2\\ -4 & 3\end{bmatrix}$

$$= \begin{bmatrix} 4 & -4 \\ 6 & 8 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 24 \\ 6 & -6 \\ -12 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 20 \\ 12 & 2 \\ -10 & 9 \end{bmatrix}$$

f) $A^2 = \textit{doesn't exist}$ (not a square matrix)

g) $AB = \not\exists$ $\quad (2\times 3 \quad 2\times 3)$ the inner not equal

h) $BA = \not\exists$ $\quad (2\times 3 \quad 2\times 3)$ the inner not equal

Exercise

Given $A = \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$, Find

a) $A + B$

b) $A - B$

c) $3A$

d) $-2B$

e) $2A + 3B$

f) A^2

g) AB

h) BA

Solution

a) $A + B = \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 1 & -1 \\ 2 & 2 & 1 \\ -1 & 2 & 5 \end{bmatrix}$$

b) $A - B = \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -3 & 5 & -1 \\ -2 & -4 & 3 \\ -7 & 4 & 1 \end{bmatrix}$$

c) $3A = 3\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}$

$$=\begin{bmatrix} -6 & 9 & -3 \\ 0 & -3 & 6 \\ -12 & 9 & 9 \end{bmatrix}$$

d) $$-2B=-2\begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} -2 & 4 & 0 \\ -4 & -6 & 2 \\ -6 & 2 & -4 \end{bmatrix}$$

e) $$2A+3B=2\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}+3\begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} -4 & 6 & -2 \\ 0 & -2 & 4 \\ -8 & 6 & 6 \end{bmatrix}+\begin{bmatrix} 3 & -6 & 0 \\ 6 & 9 & -3 \\ 9 & -3 & 6 \end{bmatrix}$$

$$=\begin{bmatrix} -1 & 0 & -2 \\ 6 & 7 & 1 \\ 1 & 3 & 12 \end{bmatrix}$$

f) $$A^2=\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}$$

$$=\begin{bmatrix} 4+4 & -6-3-3 & 2+6-3 \\ -8 & 1+6 & -2+6 \\ 8-12 & -12-3+9 & 4+6+9 \end{bmatrix}$$

$$=\begin{bmatrix} 8 & -12 & 5 \\ -8 & 7 & 4 \\ -4 & -6 & 19 \end{bmatrix}$$

g) $$AB=\begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} -2+6-3 & 4+9+1 & -3-2 \\ -2+6 & -3-2 & 1+4 \\ -4+6+9 & 8+9-3 & -3+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 14 & -5 \\ 4 & -5 & 5 \\ 11 & 14 & 3 \end{bmatrix}$$

h) $BA = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 3 & -1 \\ 0 & -1 & 2 \\ -4 & 3 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -2 & 3+2 & -1-4 \\ -4+4 & 6-3-3 & -2+6-3 \\ -6-8 & 9+1+6 & -3-2+6 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 5 & -5 \\ 0 & 0 & 1 \\ -14 & 16 & 1 \end{bmatrix}$$

Exercise

Given $A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}$ $\quad B = \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$, Find

a) $A+B$

b) $A-B$

c) $3A$

d) $-2B$

e) $2A+3B$

f) A^2

g) AB

h) BA

Solution

a) $A+B = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 4 & 4 \\ 4 & 0 & 1 \\ 1 & 8 & 1 \end{bmatrix}$$

b) $A-B = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & -4 \\ -2 & -6 & 5 \\ 9 & 0 & -5 \end{bmatrix}$$

c) $3A = 3\begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}$

$$=\begin{bmatrix} 0 & 6 & 0 \\ 3 & -9 & 9 \\ 15 & 12 & -6 \end{bmatrix}$$

d) $-2B=-2\begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$

$$=\begin{bmatrix} 2 & -4 & -8 \\ -6 & -6 & 4 \\ 8 & -8 & -6 \end{bmatrix}$$

e) $2A+3B=2\begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}+3\begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$

$$=\begin{bmatrix} 0 & 4 & 0 \\ 2 & -6 & 6 \\ 10 & 8 & -4 \end{bmatrix}+\begin{bmatrix} -3 & 6 & 12 \\ 9 & 9 & -6 \\ -12 & 12 & 9 \end{bmatrix}$$

$$=\begin{bmatrix} -3 & 10 & 12 \\ 11 & 3 & 0 \\ -2 & 20 & 5 \end{bmatrix}$$

f) $A^2=\begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}$

$$=\begin{bmatrix} 2 & -6 & 6 \\ -3+15 & 2+9+12 & -9-6 \\ 4-10 & 10-12-8 & 12+4 \end{bmatrix}$$

$$=\begin{bmatrix} 2 & -6 & 6 \\ 12 & 23 & -15 \\ -6 & -10 & 16 \end{bmatrix}$$

g) $AB=\begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}\begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}$

$$=\begin{bmatrix} 6 & 6 & -4 \\ -1-9-12 & 2-9+12 & 4+6+9 \\ -5+12+8 & 10+12-8 & 20-8-6 \end{bmatrix}$$

$$=\begin{bmatrix} 6 & 6 & -4 \\ -22 & 5 & 19 \\ 15 & 14 & 6 \end{bmatrix}$$

h) $BA=\begin{bmatrix} -1 & 2 & 4 \\ 3 & 3 & -2 \\ -4 & 4 & 3 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ 1 & -3 & 3 \\ 5 & 4 & -2 \end{bmatrix}$

$$=\begin{bmatrix} 2+10 & -2-6+16 & 6-8 \\ 3-10 & 6-9-8 & 9+4 \\ 4+15 & -8-12+12 & 12-6 \end{bmatrix}$$

$$=\begin{bmatrix} 12 & 8 & -2 \\ -7 & -11 & 13 \\ 19 & -8 & 6 \end{bmatrix}$$

Exercise

Given $A=\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$ $B=\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$ $C=\begin{pmatrix} 4 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix}$ $D=\begin{pmatrix} -2 & 3 \\ 2 & -1 \\ 3 & 2 \end{pmatrix}$, Find

a) $4A-2B$
b) $3A+C$
c) $3A+B$
d) $2A-3B$
e) AB
f) BA
g) A^2
h) B^3
i) AC
j) CA
k) CD
l) DC

Solution

a) $4A-2B=4\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}-2\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

$$=\begin{pmatrix} -4 & 8 \\ -8 & 4 \end{pmatrix}-\begin{pmatrix} 2 & -4 \\ 4 & -2 \end{pmatrix}$$

$$=\begin{pmatrix} -6 & 12 \\ -12 & 6 \end{pmatrix}$$

b) $3A+C=\nexists$

They are not the same order.

c) $3A+B=3\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}+\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

$$=\begin{pmatrix} -3 & 6 \\ -6 & 3 \end{pmatrix}+\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$=\begin{pmatrix} -2 & 4 \\ -4 & 2 \end{pmatrix}$$

d) $2A-3B=2\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}-3\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}$

$$=\begin{pmatrix}-2 & 4\\ -4 & 2\end{pmatrix}-\begin{pmatrix}3 & -6\\ 6 & -3\end{pmatrix}$$

$$=\begin{pmatrix}-5 & 10\\ -10 & 5\end{pmatrix}$$

e) $AB=\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}$

$$=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$$

f) $BA=\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}$

$$=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$$

g) $A^2=\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}$

$$=\begin{pmatrix}-3 & 0\\ 0 & -3\end{pmatrix}$$

h) $B^3=\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}$

$$=\begin{pmatrix}-3 & 0\\ 0 & -3\end{pmatrix}\begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}$$

$$=\begin{pmatrix}-3 & 6\\ -6 & 3\end{pmatrix}$$

i) $AC=\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}4 & 3 & 2\\ -1 & 2 & 1\end{pmatrix}$ $\qquad 2\times 2 \quad 2\times 3 \quad \to 2\times 3$

$$=\begin{pmatrix}-6 & 1 & 0\\ -9 & -4 & -3\end{pmatrix}$$

j) $CB=\nexists$ $\qquad 2\times 3 \quad 2\times 2$

C and B are not the same order.

k) $CD=\begin{pmatrix}4 & 3 & 2\\ -1 & 2 & 1\end{pmatrix}\begin{pmatrix}-2 & 3\\ 2 & -1\\ 3 & 2\end{pmatrix}$ $\qquad 2\times 3 \quad 3\times 2 \quad \to 2\times 2$

$$=\begin{pmatrix} -8+6+6 & 12-3+4 \\ 2+4+3 & -3+2+2 \end{pmatrix}$$

$$=\begin{pmatrix} 4 & 13 \\ 9 & 1 \end{pmatrix}$$

l) $DC=\begin{pmatrix} -2 & 3 \\ 2 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 4 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix}$ $\qquad 3\times 2 \quad 2\times 3 \quad \to 3\times 3$

$$=\begin{pmatrix} -8-3 & -6+6 & -4+3 \\ 8+1 & 6-2 & 4-1 \\ 12-2 & 9+4 & 6+2 \end{pmatrix}$$

$$=\begin{pmatrix} -11 & 0 & -1 \\ 9 & 4 & 3 \\ 10 & 13 & 8 \end{pmatrix}$$

Exercise

Given $A=\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$ $B=\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$ $C=\begin{pmatrix} 1 & 4 & 5 \\ -2 & 3 & 4 \\ -1 & 0 & -2 \end{pmatrix}$ $D=\begin{pmatrix} 2 & 4 & -2 \\ 0 & 3 & 5 \\ -3 & 1 & 1 \end{pmatrix}$, Find

a) $4A-2B$
b) $3A+C$
c) $3A+B$
d) $2A-3B$
e) AB
f) BA
g) A^2
h) B^3
i) AC
j) CB
k) CD
l) DC

Solution

a) $4A-2B=4\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}-2\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

$$=\begin{pmatrix} 8 & 16 \\ 12 & -4 \end{pmatrix}-\begin{pmatrix} -2 & 6 \\ 4 & -2 \end{pmatrix}$$

$$=\begin{pmatrix} 10 & 10 \\ 8 & -2 \end{pmatrix}$$

b) $3A+C=\nexists$

They are not the same order.

c) $3A+B=3\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}+\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

$$=\begin{pmatrix} 6 & 12 \\ 9 & -3 \end{pmatrix}+\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 15 \\ 11 & -4 \end{pmatrix}$$

d) $2A - 3B = 2\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} - 3\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 4 & 8 \\ 6 & -2 \end{pmatrix} - \begin{pmatrix} -3 & 9 \\ 6 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -1 \\ 0 & 1 \end{pmatrix}$$

e) $AB = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 6 & 2 \\ -5 & 10 \end{pmatrix}$$

f) $BA = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 7 & -7 \\ 1 & 9 \end{pmatrix}$$

g) $A^2 = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 14 & 4 \\ 3 & 13 \end{pmatrix}$$

h) $B^3 = \left(\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}\right)\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 7 & -6 \\ -4 & 7 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -19 & 27 \\ 18 & -19 \end{pmatrix}$$

i) $AC = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix}$ $\qquad 2\times 2 \quad 2\times 3 \quad \rightarrow 2\times 3$

$$= \begin{pmatrix} -6 & 1 & 0 \\ -9 & -4 & -3 \end{pmatrix}$$

j) $CB = \nexists$ $\qquad 2\times 3 \quad 2\times 2$

C and B are not the same order.

k) $CD = \begin{pmatrix} 1 & 4 & 5 \\ -2 & 3 & 4 \\ -1 & 0 & -2 \end{pmatrix}\begin{pmatrix} 2 & 4 & -2 \\ 0 & 3 & 5 \\ -3 & 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} -12 & 21 & 13 \\ -16 & 5 & 23 \\ 4 & -6 & 0 \end{pmatrix}$

l) $DC = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 3 & 5 \\ -3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & 5 \\ -2 & 3 & 4 \\ -1 & 0 & -2 \end{pmatrix}$

$= \begin{pmatrix} 2-8+2 & 8+12 & 10+16+4 \\ -6-5 & 9 & 12-10 \\ -3-2-1 & -12+3 & -15+4-2 \end{pmatrix}$

$= \begin{pmatrix} -4 & 20 & 30 \\ -11 & 9 & 2 \\ -6 & -9 & -12 \end{pmatrix}$

Exercise

A contractor builds three kinds of houses, models *A*, *B*, and *C*, with a choice of two styles, Spanish and contemporary. Matrix *P* shows the number of each kind of house planned for a new 100-home subdivision. The amounts for each of the exterior materials depend primarily on the style of the house. These amounts are shown in matrix *Q*. (concrete is in cubic yards, lumber in units of 1000 board feet, brick in 1000s, and shingles in units of 100 ft^2.) Matrix *R* gives the cost in dollars for each kind of material.

a) What is the total cost of these materials for each model?

b) How much of each of four kinds of material must be ordered

c) What is the total cost for exterior materials?

Solution

$$\begin{array}{ll} & \begin{array}{cc} \textit{Spanish} & \textit{Contemporary} \end{array} \\ \begin{array}{l} \textit{Model A} \\ \textit{Model B} \\ \textit{Model C} \end{array} & \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} = P \end{array}$$

$$\begin{array}{ll} & \begin{array}{cccc} \textit{Concrete} & \textit{Lumber} & \textit{Brick} & \textit{Shingles} \end{array} \\ \begin{array}{l} \textit{Spanish} \\ \textit{Contemporary} \end{array} & \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix} = Q \end{array}$$

$$\begin{array}{ll} & \textit{Cost per unit} \\ \textit{Concrete} & \\ \textit{Lumber} & \\ \textit{Brick} & \\ \textit{Shingles} & \end{array} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} = R$$

a) What is the total cost of these materials for each model?

$$PQ = \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix}$$

$$\begin{array}{cccc} \textit{Concrete} & \textit{Lumber} & \textit{Brick} & \textit{Shingles} \end{array}$$

$$= \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \begin{array}{l} \textit{Model A} \\ \textit{Model B} \\ \textit{Model C} \end{array}$$

$$(PQ)R = \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} 72,900 \\ 54,700 \\ 60,800 \end{bmatrix} \begin{array}{l} \textit{Model A} \\ \textit{Model B} \\ \textit{Model C} \end{array}$$

The total cost of materials is \$72,900 for model *A*, \$54,700 for model *B*, \$60,800 for model *C*.

b) How much of each of four kinds of material must be ordered

$$\begin{array}{cccc} 1500 & 30 & 600 & 60 \\ 100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \\ \hline 3800 & 130 & 1400 & 200 \end{array}$$

$$T = \begin{bmatrix} 3800 & 130 & 1400 & 200 \end{bmatrix}$$

3800 yd^3 of concrete, 130,000 board feet of lumber, 1,400,000 bricks, and 20,000 ft^2 of shingles are needed.

c) What is the total cost for exterior materials?

$$TR = \begin{bmatrix} 3800 & 130 & 1400 & 200 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix}$$

$$= [188,400]$$

The total cost for exterior materials is $188,400.

Exercise

Mitchell Fabricators manufactures three styles of bicycle frames in its two plants. The following table shows the number of each style produced at each plant

	Mountain Bike	*Racing Bike*	*Touring Bike*
North Plant	150	120	100
South Plant	180	90	130

a) Write a 2 *x* 3 matrix ***A*** that represents the information in the table

b) The manufacturer increased production of each style by 20%. Find a Matrix ***M*** that represents the increased production figures.

c) Find the matrix ***A + M*** and tell what it represents

Solution

a) $A = \begin{bmatrix} 150 & 120 & 100 \\ 180 & 90 & 130 \end{bmatrix}$

b) The 20% production will represent

$\boldsymbol{A} + 20\%(\boldsymbol{A})$

$\rightarrow \boldsymbol{A} + .2\boldsymbol{A} = 1.2\boldsymbol{A}$

$$M = (1.2)\begin{bmatrix} 150 & 120 & 100 \\ 180 & 90 & 130 \end{bmatrix}$$

$$= \begin{bmatrix} 180 & 144 & 120 \\ 216 & 108 & 156 \end{bmatrix}$$

c) $A + M = \begin{bmatrix} 150 & 120 & 100 \\ 180 & 90 & 130 \end{bmatrix} + \begin{bmatrix} 180 & 144 & 120 \\ 216 & 108 & 156 \end{bmatrix}$

$$= \begin{bmatrix} 330 & 264 & 220 \\ 396 & 198 & 286 \end{bmatrix}$$

The matrix $A + M$ represents the total production of each style at each plant for the time period (2 months)

Exercise

Sal's Shoes and Fred's Footwear both have outlets in California and Arizona. Sal's sells shoes for \$80, sandals for \$40, and boots for \$120. Fred's prices are \$60, \$30, and \$150 for shoes, sandals and boots, respectively. Half of all sales in California stores are shoes, 1/4 are *sandals*, and 1/4 are *boots*. In Arizona the fractions are 1/5 *shoes*, 1/5 are *sandals*, and 3/5 are *boots*.

a) Write a 2 x 3 matrix called P representing prices for the two stores and three types of footwear.

b) Write a 2 x 3 matrix called F representing fraction of each type of footwear sold in each state.

c) Only one of the two products *PF* and *FP* is meaningful. Determine which one it is, calculate the product, and describe what the entries represent.

Solution

a) Write a 2 x 3 matrix called *P* representing prices for the two stores and three types of footwear.

$$P = \begin{array}{c c} \begin{bmatrix} 80 & 40 & 120 \\ 60 & 30 & 150 \end{bmatrix} & \begin{array}{l} Sal's \\ Fred's \end{array} \end{array}$$

b) Write a 2 *x* 3 matrix called F representing fraction of each type of footwear sold in each state.

$$F = \begin{array}{c} \begin{array}{cc} CA & AR \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{3}{5} \end{bmatrix} \end{array}$$

c) $$PF = \begin{bmatrix} 80 & 40 & 120 \\ 60 & 30 & 150 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{3}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 80\frac{1}{2}+40\frac{1}{4}+120\frac{1}{4} & 80\frac{1}{5}+40\frac{1}{5}+120\frac{3}{5} \\ 60\frac{1}{2}+30\frac{1}{4}+150\frac{1}{4} & 60\frac{1}{5}+30\frac{1}{5}+150\frac{3}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 96 \\ 75 & 108 \end{bmatrix}$$

Solution Section 4.3 – Multiplicative Inverses of Matrices

Exercise

Show that B is Multiplicative inverse of A

$$A = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

Solution

$$AB = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{pmatrix} -6 & \\ & \end{pmatrix}$$

$$\underline{\neq I}$$

B is not multiplicative inverse of A

Exercise

Show that B is Multiplicative inverse of A

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

Solution

$$AB = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underline{= I}$$

$$BA = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underline{= I}$$

$\therefore B$ is Multiplicative inverse of A

Exercise

Find the inverse, if exists, of $\quad A=\begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

Solution

$$A^{-1}=\frac{1}{-4+6}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} -1 & 3 \\ -\frac{1}{3} & 1 \end{bmatrix}$$

Exercise

Find the inverse, if exists, of $\quad A=\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$

Solution

$$A^{-1}=\frac{1}{10-10}\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

$$=\frac{1}{0}\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A=\begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$

Solution

$$\left[\begin{array}{cc|cc} -2 & 3 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{array}\right] \quad -\tfrac{1}{2}R_1$$

$$\begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ -3 & 4 & 0 & 1 \end{array}\right] \quad R_2+3R_1$$

$$\begin{array}{cccc} -3 & 4 & 0 & 1 \\ 3 & -\frac{9}{2} & -\frac{3}{2} & 0 \\ \hline 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right] \quad -2R_2 \qquad \begin{array}{cccc} 0 & 1 & 3 & -2 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 3 & -2 \end{array}\right] \quad R_1 + \frac{3}{2}R_2 \qquad \begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & \frac{9}{2} & -3 \\ \hline 1 & 0 & 4 & -3 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 4 & -3 \\ 0 & 1 & 3 & -2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} a & b \\ 3 & 3 \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{3a-3b}\begin{bmatrix} 3 & -b \\ -3 & a \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{3(a-b)} & \frac{-b}{3(a-b)} \\ \frac{-3}{3(a-b)} & \frac{a}{3(a-b)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{a-b} & \frac{-b}{3(a-b)} \\ \frac{-1}{a-b} & \frac{a}{3(a-b)} \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} -2 & a \\ 4 & a \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{-2a-4a}\begin{bmatrix} a & -a \\ -4 & -2 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{a}{-6a} & \frac{-a}{-6a} \\ \frac{-4}{-6a} & \frac{-2}{-6a} \end{bmatrix}$$

$$=\begin{bmatrix} -\frac{1}{6} & \frac{1}{6} \\ \frac{2}{3a} & \frac{1}{3a} \end{bmatrix}$$

Exercise

Find the inverse of $A=\begin{bmatrix} 4 & 4 \\ b & a \end{bmatrix}$

Solution

$$A^{-1}=\frac{1}{4a-4b}\begin{bmatrix} a & -4 \\ -b & 4 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{a}{4(a-b)} & \frac{-4}{4(a-b)} \\ \frac{-b}{4(a-b)} & \frac{4}{4(a-b)} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{a}{4(a-b)} & \frac{-1}{a-b} \\ \frac{-b}{4(a-b)} & \frac{1}{a-b} \end{bmatrix}$$

Exercise

Find the inverse of $A=\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$

Solution

$$A^{-1}=\frac{1}{-1+4}\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{3}\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{14}\begin{pmatrix} -1 & -4 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{14} & \frac{2}{7} \\ \frac{3}{14} & -\frac{1}{7} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{5}\begin{pmatrix} -1 & -3 \\ -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{11}\begin{pmatrix} 5 & -3 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{11} & -\frac{3}{11} \\ \frac{2}{11} & \frac{1}{11} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{0}\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} -6 & 9 \\ 2 & -3 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{18-18}\begin{pmatrix} & \\ & \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} -2 & 7 \\ 0 & 2 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -7 \\ 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{7}{4} \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Exercise

Find the inverse of $A=\begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$

Solution

$$A=\frac{1}{-16+16}\begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A=\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Solution

$$A^{-1}=\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Exercise

Find the inverse of $A=\begin{pmatrix} 2 & 1 \\ a & a \end{pmatrix}$

Solution

$$A^{-1}=\frac{1}{a}\begin{pmatrix} a & -1 \\ -a & 2 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & -\frac{1}{a} \\ -1 & \frac{2}{a} \end{pmatrix}$$

Exercise

Find the inverse of $A=\begin{pmatrix} b & 3 \\ b & 2 \end{pmatrix}$

Solution

$$A^{-1}=-\frac{1}{b}\begin{pmatrix} 2 & -3 \\ -b & b \end{pmatrix}$$

$$=\begin{pmatrix} -\frac{2}{b} & \frac{3}{b} \\ 1 & -1 \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & a \\ 3 & a \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{2a}\begin{pmatrix} a & -a \\ -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2a} & -\frac{1}{2a} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{a^2 - 4}\begin{pmatrix} a & -2 \\ -2 & a \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a}{a^2 - 4} & \frac{-2}{a^2 - 4} \\ \frac{-2}{a^2 - 4} & \frac{a}{a^2 - 4} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{0}\begin{pmatrix} \quad \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} -3 & \frac{1}{2} \\ 6 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{0}\begin{pmatrix} & \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find A^{-1} *if* $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\begin{array}{cccccc} 2 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -2 & 0 & 0 \\ \hline 0 & -2 & -3 & -2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 3 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & -3 & 0 & 0 \\ \hline 0 & 0 & -3 & -3 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \; -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \; -\tfrac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \begin{matrix} R_1 - R_3 \\ R_2 - \frac{3}{2}R_3 \\ \\ \end{matrix}$$

$$\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ \hline 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array} \qquad \begin{array}{cccccc} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ \hline 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

Exercise

Find A^{-1}, where $A=\begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3+2R_1 \end{array}$$

$$\begin{array}{cccccc} 3 & 5 & 3 & 0 & 1 & 0 \\ -3 & -6 & 3 & -3 & 0 & 0 \\ \hline 0 & -1 & 6 & -3 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 2 & 4 & 3 & 0 & 0 & 1 \\ -2 & -4 & 2 & -2 & 0 & 0 \\ \hline 0 & 0 & 5 & -2 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} R_1-2R_2 \\ \\ \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 12 & -6 & 2 & 0 \\ \hline 1 & 0 & 11 & -5 & 2 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{5}R_3 \end{array}$$

$$\begin{array}{cccccc} 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right] \begin{array}{l} R_1-11R_3 \\ R_2+6R_3 \\ \\ \end{array}$$

$$\begin{array}{cccccc} 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 6 & -\frac{12}{5} & 0 & \frac{6}{5} \\ \hline 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \end{array} \qquad \begin{array}{cccccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 0 & -11 & \frac{22}{5} & 0 & -\frac{11}{5} \\ \hline 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right]$$

$$A^{-1}=\begin{bmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{bmatrix}$$

Exercise

Find A^{-1}, where $A=\begin{bmatrix}1 & 2 & -1\\ -2 & 0 & 1\\ 1 & -1 & 0\end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ -2 & 0 & 1 & 0 & 1 & 0\\ 1 & -1 & 0 & 0 & 0 & 1\end{array}\right]\begin{array}{l}R_2+2R_1\\ R_3-R_1\end{array}$$

$$\begin{array}{cccccc}-2 & 0 & 1 & 0 & 1 & 0\\ 2 & 4 & -2 & 2 & 0 & 0\\ \hline 0 & 4 & -1 & 2 & 1 & 0\end{array}\qquad\begin{array}{cccccc}1 & -1 & 0 & 0 & 0 & 1\\ -1 & -2 & 1 & -1 & 0 & 0\\ \hline 0 & -3 & 1 & -1 & 0 & 1\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & 4 & -1 & 2 & 1 & 0\\ 0 & -3 & 1 & -1 & 0 & 1\end{array}\right]\ \tfrac{1}{4}R_2$$

$$\begin{array}{cccccc}0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0\\ 0 & -3 & 1 & -1 & 0 & 1\end{array}\right]\begin{array}{l}R_1-2R_2\\ \\ R_3+3R_2\end{array}$$

$$\begin{array}{cccccc}0 & -3 & 1 & -1 & 0 & 1\\ 0 & 3 & -\frac{3}{4} & \frac{3}{2} & \frac{3}{4} & 0\\ \hline 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1\end{array}\qquad\begin{array}{cccccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & -2 & \frac{1}{2} & -1 & -\frac{1}{2} & 0\\ \hline 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0\\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0\\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1\end{array}\right]\ 4R_3$$

$$\begin{array}{cccccc}0 & 0 & 1 & 2 & 3 & 4\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0\\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0\\ 0 & 0 & 1 & 2 & 3 & 4\end{array}\right]\begin{array}{l}R_1+\frac{1}{2}R_3\\ R_2+\frac{1}{4}R_3\\ \\ \end{array}$$

$$\begin{array}{cccccc}1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0\\ 0 & 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2\\ \hline 1 & 0 & 0 & 1 & 1 & 2\end{array}\qquad\begin{array}{cccccc}0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0\\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1\\ \hline 0 & 1 & 0 & 1 & 1 & 1\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 1 & 2\\ 0 & 1 & 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 2 & 3 & 4\end{array}\right]$$

$$A^{-1}=\begin{bmatrix}1 & 1 & 2\\ 1 & 1 & 1\\ 2 & 3 & 4\end{bmatrix}$$

Exercise

Find A^{-1}, where $A=\begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \quad \frac{1}{-2}R_1 \qquad 1 \quad -\frac{5}{2} \quad -\frac{3}{2} \quad -\frac{1}{2} \quad 0 \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \quad R_2-4R_1 \qquad \begin{array}{cccccc} 4 & -1 & 3 & 0 & 1 & 0 \\ -4 & 10 & 6 & 2 & 0 & 0 \\ \hline 0 & 9 & 9 & 2 & 1 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \quad R_3-7R_1 \qquad \begin{array}{cccccc} 7 & -2 & 5 & 0 & 0 & 1 \\ -7 & \frac{35}{2} & \frac{21}{2} & \frac{7}{2} & 0 & 0 \\ \hline 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}\right] \quad \frac{1}{9}R_2 \qquad 0 \quad 1 \quad 1 \quad \frac{2}{9} \quad \frac{1}{9} \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}\right] \quad R_3-\frac{31}{2}R_2 \qquad \begin{array}{cccccc} 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \\ 0 & -\frac{31}{2} & -\frac{31}{2} & -\frac{31}{9} & -\frac{31}{18} & 0 \\ \hline 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array}\right]$$

$\therefore$ The inverse matrix ***doesn't exist***

OR

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{c} R_2+2R_1 \\ -2R_3+7R_1 \end{array} \qquad \begin{array}{cccccc} 4 & -1 & 3 & 0 & 1 & 0 \\ -4 & 10 & 6 & 2 & 0 & 0 \\ \hline 0 & 9 & 9 & 2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} -14 & 4 & -10 & 0 & 0 & -2 \\ 14 & 35 & 21 & 7 & 0 & 0 \\ \hline 0 & 39 & 11 & 7 & 0 & -2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & 39 & 11 & 7 & 0 & -2 \end{array}\right] \begin{array}{l} 9R_1 - 5R_2 \\ \\ 9R_3 - 39R_2 \end{array}$$

$$\begin{array}{cccccc} -18 & 45 & 27 & 9 & 0 & 0 \\ 0 & -45 & -45 & -10 & -5 & 0 \\ \hline -18 & 0 & -18 & -1 & -5 & 0 \end{array}$$

$$\begin{array}{cccccc} 0 & 351 & -99 & 63 & 0 & -18 \\ 0 & -351 & 99 & -78 & -39 & 0 \\ \hline 0 & 0 & 0 & -15 & -39 & -18 \end{array}$$

$$\left[\begin{array}{ccc|ccc} -18 & 0 & -18 & -1 & -5 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & 0 & 0 & -15 & -39 & -18 \end{array}\right]$$

$\therefore$ The inverse matrix ***doesn't exist***

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \quad R_2 + R_1 \qquad \begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ \hline 0 & 4 & 4 & 1 & 1 & 0 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 4 & 1 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \quad \tfrac{1}{4}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{c} R_1 - R_2 \\ \\ R_3 - 4R_2 \end{array} \qquad \begin{array}{cccccc} 0 & 4 & 3 & 0 & 0 & 1 \\ 0 & -4 & -4 & -1 & -1 & 0 \\ \hline 0 & 0 & -1 & -1 & -1 & 1 \end{array} \qquad \begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ \hline 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right) \quad -R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right) \quad \begin{array}{c} R_1 + R_3 \\ R_2 - R_3 \end{array} \qquad \begin{array}{cccccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ \hline 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1 \end{array} \qquad \begin{array}{cccccc} 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \\ \hline 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1 \\ 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{7}{4} & \frac{3}{4} & -1 \\ -\frac{3}{4} & -\frac{3}{4} & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ -2 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \quad R_3+2R_1 \qquad \begin{array}{cccccc} -2 & -3 & 0 & 0 & 0 & 1 \\ 2 & -2 & 2 & 2 & 0 & 0 \\ \hline 0 & -5 & 2 & 2 & 0 & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array}\right) \quad -\tfrac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1+R_2 \\ \\ R_3+5R_2 \end{array} \qquad \begin{array}{cccccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ \hline 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \end{array} \qquad \begin{array}{cccccc} 0 & -5 & 2 & 2 & 0 & 1 \\ 0 & 5 & -\frac{5}{2} & 0 & -\frac{5}{2} & 0 \\ \hline 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \end{array}\right) \quad -2R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -4 & 5 & -2 \end{array}\right) \quad \begin{array}{l} R_1-\frac{1}{2}R_3 \\ R_2+\frac{1}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & -4 & 5 & -2 \end{array}\right)$$

$$A^{-1}=\begin{pmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right) \quad \tfrac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right) \quad R_3 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right) \quad 2R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right) \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 - \frac{5}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right)$$

$$A^{-1}=\begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 1 & 2 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & -1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -4 & -3 & 1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array}\right) \begin{array}{l} \\ -R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 2R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & 0 & 7 & 3 & -2 & 1 \end{array}\right) \begin{array}{l} \\ \\ \frac{1}{7}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{array}\right) \begin{array}{l} R_1 + 3R_3 \\ R_2 - 4R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ 0 & 1 & 0 & \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 3 & 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \frac{1}{3}R_1 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & -3 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{7}{3} & -\frac{5}{3} & 3 & 1 \end{array}\right) \begin{array}{l} \\ \\ \frac{3}{7}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array}\right) \begin{array}{l} R_1 + \frac{1}{3}R_3 \\ R_2 - \frac{2}{3}R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ 0 & 1 & 0 & \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $\qquad A=\begin{pmatrix} -3 & 1 & -1 \\ 1 & -4 & -7 \\ 1 & 2 & 5 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} -3 & 1 & -1 & 1 & 0 & 0 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right) \quad -\frac{1}{3}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2-R_1 \\ R_3-R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{11}{3} & -\frac{22}{3} & \frac{1}{3} & 1 & 0 \\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1 \end{array}\right) \quad -\frac{3}{11}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0 \\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1 \end{array}\right) \quad R_3-\frac{7}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0 \\ 0 & 0 & 0 & & & \end{array}\right)$$

$\therefore$ Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $\quad A=\begin{pmatrix} 1 & 1 & -3 \\ 2 & -4 & 1 \\ -5 & 7 & 1 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 2 & -4 & 1 & 0 & 1 & 0 \\ -5 & 7 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{matrix} R_2-2R_1 \\ R_3+5R_1 \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & -6 & 7 & -2 & 1 & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array}\right) \quad -\tfrac{1}{6}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array}\right) \quad \begin{matrix} R_1-R_2 \\ \\ R_3-12R_2 \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & & & \end{array}\right)$$

$\therefore$ Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $\quad A=\begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \quad \begin{matrix} R_3-3R_1 \\ R_3-2R_1 \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right) \quad -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right) \quad R_1-2R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right) \quad \tfrac{1}{5}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right) \quad \begin{array}{l} R_1 - 11R_3 \\ R_2 + 6R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -6 & 1 \\ -2 & -2 & 5 & 1 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} -2 & -3 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad -\tfrac{1}{2}R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_4 + 2R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_4 - R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\therefore$ Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 1 & -14 & 7 & 38 \\ -1 & 2 & 1 & -2 \\ 1 & 2 & -1 & -6 \\ 1 & -2 & 3 & 6 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & -6 & 0 & 0 & 1 & 0 \\ 1 & -2 & 3 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & -12 & 8 & 36 & 1 & 1 & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array}\right] \quad -\frac{1}{12}R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 + 14R_2 \\ \\ R_3 - 16R_2 \\ R_4 - 12R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & \frac{8}{3} & 4 & \frac{1}{3} & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array}\right] \quad \frac{3}{8}R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + \frac{7}{3}R_3 \\ R_2 + \frac{2}{3}R_3 \\ \\ R_4 - 4R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & -2 & -\frac{1}{2} & -1 & -\frac{3}{2} & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ -\frac{1}{2}R_4 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array}\right] \begin{array}{l} R_1 + \frac{1}{2}R_4 \\ R_2 + 2R_4 \\ R_3 - \frac{3}{2}R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ 0 & 0 & 1 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{bmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 10 & 20 & -30 & 15 \\ 3 & -7 & 14 & -8 \\ -7 & -2 & -1 & 2 \\ 4 & 4 & -3 & 1 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} 10 & 20 & -30 & 15 & 1 & 0 & 0 & 0 \\ 3 & -7 & 14 & -8 & 0 & 1 & 0 & 0 \\ -7 & -2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & -3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad \frac{1}{10}R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 3 & -7 & 14 & -8 & 0 & 1 & 0 & 0 \\ -7 & -2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & -3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 + 7R_1 \\ R_4 - 4R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 0 & -13 & 23 & -\frac{25}{2} & -\frac{3}{10} & 1 & 0 & 0 \\ 0 & 12 & -22 & \frac{25}{2} & \frac{7}{10} & 0 & 1 & 0 \\ 0 & -4 & 9 & -5 & -\frac{2}{5} & 0 & 0 & 1 \end{array}\right] \quad -\frac{1}{13}R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 12 & -22 & \frac{25}{2} & \frac{7}{10} & 0 & 1 & 0 \\ 0 & -4 & 9 & -5 & -\frac{2}{5} & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 - 12R_2 \\ R_4 + 4R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 0 & -\frac{10}{13} & \frac{25}{26} & \frac{11}{26} & \frac{12}{13} & 1 & 0 \\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1 \end{array}\right] \quad -\frac{13}{10}R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 - \frac{7}{13}R_3 \\ R_2 + \frac{23}{13}R_3 \\ \\ R_4 - \frac{25}{13}R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{19}{20} & -\frac{11}{5} & -\frac{23}{10} & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array}\right] \quad \begin{array}{l} \\ R_2 + R_4 \\ R_3 + R_4 \\ \\ \end{array}.$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array}\right] \quad \begin{array}{l} \\ \\ \\ \frac{4}{5}R_4 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array}\right] \quad \begin{array}{l} R_1 - \frac{1}{4}R_4 \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{bmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $\quad A=\begin{bmatrix} x \end{bmatrix}$

Solution

For A^{-1} exists, $x \neq 0$

$$AA^{-1}=I$$

$$\begin{bmatrix} x \end{bmatrix}\begin{bmatrix} a \end{bmatrix}=\begin{bmatrix} 1 \end{bmatrix}$$

$$xa=1$$

$$\boxed{a=\frac{1}{x}}$$

$$A^{-1}=\begin{bmatrix} \frac{1}{x} \end{bmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $\quad A=\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$

Solution

For A^{-1} exists, $\boxed{x, y \neq 0}$

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad AA^{-1}=I$$

$$\begin{cases} ax=1 & bx=0 \\ cy=0 & dy=1 \end{cases}$$

$$\begin{cases} a=\frac{1}{x} & b=0 \\ c=0 & d=\frac{1}{y} \end{cases}$$

$$A^{-1}=\begin{bmatrix} \frac{1}{x} & 0 \\ 0 & \frac{1}{y} \end{bmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $\quad A = \begin{bmatrix} 0 & 0 & x \\ 0 & y & 0 \\ z & 0 & 0 \end{bmatrix}$

Solution

For A^{-1} exists, $x, y, z \neq 0$

$$\begin{pmatrix} 0 & 0 & x \\ 0 & y & 0 \\ z & 0 & 0 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad AA^{-1} = I$$

$$\begin{pmatrix} xg & xh & xi \\ yd & ye & yf \\ za & zb & zc \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{cases} xg = 1 & xh = 0 & xi = 0 \\ yd = 0 & ye = 1 & yf = 0 \\ za = 0 & zb = 0 & zc = 1 \end{cases}$$

$$\begin{cases} g = \frac{1}{x} & h = 0 & i = 0 \\ d = 0 & e = \frac{1}{y} & f = 0 \\ a = 0 & b = 0 & c = \frac{1}{z} \end{cases}$$

$$A^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{z} \\ 0 & \frac{1}{y} & 0 \\ \frac{1}{z} & 0 & 0 \end{pmatrix}$$

Exercise

State the conditions under which A^{-1} exists. Then find a formula for A^{-1} $\quad A = \begin{bmatrix} x & 1 & 1 & 1 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{bmatrix}$

Solution

For A^{-1} exists, $x, y, z, w \neq 0$

$$\begin{pmatrix} x & 1 & 1 & 1 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad AA^{-1} = I$$

$$\begin{cases} xa_{11} + xa_{21} + xa_{31} + xa_{41} = 1 \\ xa_{12} + xa_{22} + xa_{32} + xa_{42} = 0 \\ xa_{13} + xa_{23} + xa_{33} + xa_{43} = 0 \\ xa_{14} + xa_{24} + xa_{34} + xa_{44} = 0 \end{cases}$$

$$\begin{cases} ya_{21} = 0 & a_{21} = 0 \\ ya_{22} = 1 & a_{22} = \dfrac{1}{y} \\ ya_{23} = 0 & a_{23} = 0 \\ ya_{24} = 0 & a_{24} = 0 \end{cases}$$

$$\begin{cases} za_{31} = 0 & a_{31} = 0 \\ za_{32} = 0 & a_{32} = 0 \\ za_{33} = 1 & a_{33} = \dfrac{1}{z} \\ za_{34} = 0 & a_{34} = 0 \end{cases}$$

$$\begin{cases} wa_{41} = 0 & a_{41} = 0 \\ wa_{42} = 0 & a_{42} = 0 \\ wa_{43} = 0 & a_{43} = 0 \\ wa_{44} = 1 & a_{44} = \dfrac{1}{w} \end{cases}$$

$$\to \begin{cases} xa_{11} = 1 & a_{11} = \dfrac{1}{x} \\ xa_{12} + \dfrac{x}{y} = 0 & a_{12} = -\dfrac{1}{y} \\ xa_{13} + \dfrac{x}{z} = 0 & a_{13} = -\dfrac{1}{z} \\ xa_{14} + \dfrac{x}{w} = 0 & a_{14} = -\dfrac{1}{w} \end{cases}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{x} & -\frac{1}{y} & -\frac{1}{z} & -\frac{1}{w} \\ 0 & \frac{1}{y} & 0 & 0 \\ 0 & 0 & \frac{1}{z} & 0 \\ 0 & 0 & 0 & \frac{1}{w} \end{pmatrix}$$

Exercise

Solve the system using A^{-1} $\begin{cases} x \quad\quad + 2z = 6 \\ -x + 2y + 3z = -5 \\ x - y \quad\quad = 6 \end{cases}$ Given $A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

Solution

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3(6) - 2(-5) - 4(6) \\ 3(6) - 2(-5) - 5(6) \\ -1(6) + 1(-5) + 2(6) \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

Solution: $(4, \ -2, \ 1)$

Exercise

Solve the system using A^{-1} $\begin{cases} x + 2y + 5z = 2 \\ 2x + 3y + 8z = 3 \\ -x + \ y + 2z = 3 \end{cases}$

a) Write the linear system as a matrix equation in the form $AX = B$

b) Solve the system using the inverse that is given for the coefficient matrix

the inverse of $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix}$ *is* $\begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}$

Solution

a) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$

b) $\begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$

Exercise

Solve the system using A^{-1} $\quad \begin{cases} x - y + z = 8 \\ \quad 2y - z = -7 \\ 2x + 3y \quad = 1 \end{cases}$

a) Write the linear system as a matrix equation in the form $AX = B$

b) Solve the system using the inverse that is given for the coefficient matrix

the inverse is $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

Solution

a) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -7 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\quad \begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$

Solution

$A = \begin{pmatrix} 3 & 2 \\ 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

$A^{-1} = -\frac{1}{7}\begin{pmatrix} -1 & -2 \\ -2 & 3 \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{pmatrix} \begin{pmatrix} -4 \\ 5 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{6}{7} \\ -\frac{23}{7} \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(\frac{6}{7},\ -\frac{23}{7}\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$

Solution

$$A = \begin{pmatrix} 2 & 5 \\ 5 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{29} \begin{pmatrix} -2 & -5 \\ -5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{29} & \frac{5}{29} \\ \frac{5}{29} & -\frac{2}{29} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{2}{29} & \frac{5}{29} \\ \frac{5}{29} & -\frac{2}{29} \end{pmatrix} \begin{pmatrix} 7 \\ -3 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{29} \\ -\frac{41}{7} \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(-\frac{1}{29},\ -\frac{41}{29}\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$

Solution

$$A = \begin{pmatrix} 4 & -7 \\ 2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} -16 \\ 9 \end{pmatrix}$$

$$A^{-1} = \frac{1}{34}\begin{pmatrix} 5 & 7 \\ -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{34} & \frac{7}{34} \\ -\frac{1}{17} & \frac{2}{17} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{5}{34} & \frac{7}{34} \\ -\frac{1}{17} & \frac{2}{17} \end{pmatrix}\begin{pmatrix} -16 \\ 9 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(-\frac{1}{2},\ 2\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$

Solution

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$A^{-1} = -1\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$$

$$X = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$\therefore$ ***Solution***: $(-2,\ 5)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 3x+4y=2 \\ 2x+5y=-1 \end{cases}$

Solution

$$A=\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} \quad B=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$A^{-1}=\frac{1}{7}\begin{pmatrix} 5 & -4 \\ -2 & 3 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{5}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{3}{7} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{5}{7} & -\frac{4}{7} \\ -\frac{2}{7} & \frac{3}{7} \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$\therefore$ ***Solution***: $(2,\ -1)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 5x-2y=4 \\ -10x+4y=7 \end{cases}$

Solution

$$A=\begin{pmatrix} 5 & -2 \\ -10 & 4 \end{pmatrix} \quad B=\begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$A^{-1}=\frac{1}{0}\begin{pmatrix} 4 & 2 \\ 10 & 5 \end{pmatrix}$$

Inverse matrix ***doesn't exist***.

$$-\frac{1}{2}\begin{cases} 5x-2y=4 \\ 5x-2y=-\frac{7}{2} \end{cases}$$

$$4\neq-\frac{7}{2}$$

$\therefore$ ***No Solution***

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$

Solution

$$A=\begin{pmatrix} 1 & -4 \\ 5 & -20 \end{pmatrix} \quad B=\begin{pmatrix} -8 \\ -40 \end{pmatrix}$$

$$A^{-1}=\frac{1}{0}\begin{pmatrix} & \\ & \end{pmatrix}$$

Inverse matrix ***doesn't exist***.

$$\frac{1}{5}\begin{cases} x-4y=-8 \\ x-4y=-8 \end{cases}$$

∴ ***Solution***: $\quad (4y-8,\ y)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

Solution

$$A=\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \quad B=\begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$A^{-1}=-\frac{1}{3}\begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}\begin{pmatrix} 3 \\ 3 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

∴ ***Solution***: $\quad (2,\ -1)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x+10y=-14 \\ 7x-2y=-16 \end{cases}$

Solution

$$A=\begin{pmatrix} 2 & 10 \\ 7 & -2 \end{pmatrix} \quad B=\begin{pmatrix} -14 \\ -16 \end{pmatrix}$$

$$A^{-1}=-\frac{1}{74}\begin{pmatrix} -2 & -10 \\ -7 & 2 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{37} & \frac{5}{37} \\ \frac{7}{74} & -\frac{1}{37} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{1}{37} & \frac{5}{37} \\ \frac{7}{74} & -\frac{1}{37} \end{pmatrix}\begin{pmatrix} -14 \\ -16 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -\frac{94}{37} \\ -\frac{33}{37} \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(-\frac{94}{37},\ -\frac{33}{37}\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 4x-3y=24 \\ -3x+9y=-1 \end{cases}$

Solution

$$A=\begin{pmatrix} 4 & -3 \\ -3 & 9 \end{pmatrix} \quad B=\begin{pmatrix} 24 \\ -1 \end{pmatrix}$$

$$A^{-1}=\frac{1}{27}\begin{pmatrix} 9 & 3 \\ 3 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{27} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{27} \end{pmatrix}\begin{pmatrix} 24 \\ -1 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{71}{9} \\ \frac{68}{27} \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(\frac{71}{9}, \frac{68}{27}\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 4x+2y=12 \\ 3x-2y=16 \end{cases}$

Solution

$$A = \begin{pmatrix} 4 & 2 \\ 3 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 12 \\ 16 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{14}\begin{pmatrix} -2 & -2 \\ -3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{3}{14} & -\frac{2}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{3}{14} & -\frac{2}{7} \end{pmatrix}\begin{pmatrix} 12 \\ 16 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$\therefore$ ***Solution***: $(4, \ -2)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x+2y=-1 \\ 4x-2y=6 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 2 \\ 4 & -2 \end{pmatrix} \quad B = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{10}\begin{pmatrix} -2 & -2 \\ -4 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{10} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{10} \end{pmatrix}\begin{pmatrix} -1 \\ 6 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\therefore$ ***Solution***: $\underline{(1,\ -1)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x-2y=5 \\ -10x+2y=4 \end{cases}$

Solution

$$\frac{1}{2}\begin{cases} x-2y=5 \\ -5x+y=2 \end{cases}$$

$$A=\begin{pmatrix} 1 & -2 \\ -5 & 1 \end{pmatrix} \quad B=\begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$A^{-1}=-\frac{1}{9}\begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} -\frac{1}{9} & -\frac{2}{9} \\ -\frac{5}{9} & -\frac{1}{9} \end{pmatrix}$$

$$X=\begin{pmatrix} -\frac{1}{9} & -\frac{2}{9} \\ -\frac{5}{9} & -\frac{1}{9} \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$\therefore$ ***Solution***: $\underline{(-1,\ -3)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$

Solution

$$\begin{matrix} \frac{1}{3} \\ \frac{1}{15} \end{matrix} \rightarrow \begin{cases} 4x+5y=-9 \\ 2x-y=-1 \end{cases}$$

$$A=\begin{pmatrix} 4 & 5 \\ 2 & -1 \end{pmatrix} \quad B=\begin{pmatrix} -9 \\ -1 \end{pmatrix}$$

$$A^{-1}=-\frac{1}{14}\begin{pmatrix} -1 & -5 \\ -2 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{14} & \frac{5}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{pmatrix}$$

$$X=\begin{pmatrix} \frac{1}{14} & \frac{5}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{pmatrix}\begin{pmatrix} -9 \\ -1 \end{pmatrix} \qquad X=A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

∴ ***Solution***: $\underline{(-1,\ -1)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 4x-4y=-12 \\ 4x+4y=-20 \end{cases}$

Solution

$$\frac{1}{4} \rightarrow \begin{cases} x-y=-3 \\ x+y=-5 \end{cases}$$

$$A=\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad B=\begin{pmatrix} -3 \\ -5 \end{pmatrix}$$

$$A^{-1}=\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$$

$\therefore$ ***Solution***: $\underline{(-4, \ -1)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} -2x + 3y = 4 \\ -3x + 4y = 5 \end{cases}$

Solution

$$A = \begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$$

$$X = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\therefore$ ***Solution***: $\underline{(1, \ 2)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x - 2y = 6 \\ 4x + 3y = 2 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{11} \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{11} & \frac{2}{11} \\ -\frac{4}{11} & \frac{1}{11} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{3}{11} & \frac{2}{11} \\ -\frac{4}{11} & \frac{1}{11} \end{pmatrix}\begin{pmatrix} 6 \\ 2 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$\therefore$ ***Solution***: $\quad \underline{(2, \ -2)|}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x - 3y = 7 \\ 4x + y = -7 \end{cases}$

Solution

$$A = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 7 \\ -7 \end{pmatrix}$$

$$A^{-1} = \frac{1}{14}\begin{pmatrix} 1 & 3 \\ -4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{14} & \frac{3}{14} \\ -\frac{2}{7} & \frac{1}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{14} & \frac{3}{14} \\ -\frac{2}{7} & \frac{1}{7} \end{pmatrix}\begin{pmatrix} 7 \\ -7 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$\therefore$ ***Solution***: $\quad \underline{(-1, \ -3)|}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right) \quad -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right) \quad \begin{matrix} R_1 - R_2 \\ \\ R_3 + 2R_2 \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & 6 & 3 & -2 & 1 \end{array}\right) \quad \frac{1}{6}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{array}\right) \quad \begin{matrix} R_1 + 2R_3 \\ R_2 - 3R_3 \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$\therefore$ ***Solution***: $(1,\ 2,\ \ -1)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -1 \\ 9 \\ 9 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_2 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -2 & 1 & 0 \\ 0 & -4 & 4 & -3 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \\ -R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & -4 & 4 & -3 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 4R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & -8 & 5 & -4 & 1 \end{array}\right) \quad \begin{array}{l} \\ \\ -\frac{1}{8}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{array}\right) \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 + 3R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ 0 & 0 & 1 & -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & -\frac{3}{8} \\ -\frac{5}{8} & \frac{1}{2} & -\frac{1}{8} \end{pmatrix}\begin{pmatrix} -1 \\ 9 \\ 9 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

$\therefore$ ***Solution***: $(2,\ 1,\ 4)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 5 & -1 \\ 0 & 3 & -1 \\ -3 & 6 & 2 \end{pmatrix} \quad B = \begin{pmatrix} -4 \\ -1 \\ 11 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ -3 & 6 & 2 & 0 & 0 & 1 \end{array}\right) \quad R_3 + 3R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & 21 & -1 & 3 & 0 & 1 \end{array}\right) \quad \frac{1}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 5 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 21 & -1 & 3 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 - 5R_2 \\ \\ R_3 - 21R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & 1 & -\frac{5}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 6 & 3 & -7 & 1 \end{array}\right) \quad \frac{1}{6}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{3} & 1 & -\frac{5}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{array}\right) \quad \begin{array}{l} R_1 - \frac{2}{3}R_3 \\ R_2 + \frac{1}{3}R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & \frac{8}{9} & -\frac{1}{9} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{18} & \frac{1}{18} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{8}{9} & -\frac{1}{9} \\ \frac{1}{6} & -\frac{1}{18} & \frac{1}{18} \\ \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{2}{3} & \frac{8}{9} & -\frac{1}{9} \\ \frac{1}{6} & -\frac{1}{18} & \frac{1}{18} \\ \frac{1}{2} & -\frac{7}{6} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} -4 \\ -1 \\ 11 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

∴ ***Solution***: $(-3,\ 0,\ 1)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x+3y+4z=14 \\ 2x-3y+2z=10 \\ 3x-y+z=9 \end{cases}$

Solution

$$A=\begin{pmatrix} 1 & 3 & 4 \\ 2 & -3 & 2 \\ 3 & -1 & 1 \end{pmatrix} \quad B=\begin{pmatrix} 14 \\ 10 \\ 9 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 2 & -3 & 2 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2-2R_1 \\ R_3-3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -9 & -6 & -2 & 1 & 0 \\ 0 & -10 & -11 & -3 & 0 & 1 \end{array}\right) \quad -\frac{1}{9}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & -\frac{1}{9} & 0 \\ 0 & -10 & -11 & -3 & 0 & 1 \end{array}\right) \begin{array}{l} R_1-3R_2 \\ \\ R_3+10R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & -\frac{1}{9} & 0 \\ 0 & 0 & -\frac{13}{3} & -\frac{7}{9} & -\frac{10}{9} & 1 \end{array}\right) \begin{array}{l} \\ \\ -\frac{3}{13}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{9} & -\frac{1}{9} & 0 \\ 0 & 0 & 1 & \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{array}\right) \begin{array}{l} R_2-2R_3 \\ R_2-\frac{2}{3}R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{39} & -\frac{7}{39} & \frac{6}{13} \\ 0 & 1 & 0 & \frac{4}{39} & -\frac{11}{39} & \frac{2}{13} \\ 0 & 0 & 1 & \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{39} & -\frac{7}{39} & \frac{6}{13} \\ \frac{4}{39} & -\frac{11}{39} & \frac{2}{13} \\ \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{1}{39} & -\frac{7}{39} & \frac{6}{13} \\ \frac{4}{39} & -\frac{11}{39} & \frac{2}{13} \\ \frac{7}{39} & \frac{10}{39} & -\frac{3}{13} \end{pmatrix} \begin{pmatrix} 14 \\ 10 \\ 9 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

∴ ***Solution***: $\underline{(2,\ 0,\ 3)}$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & -1 & 1 & 0 & 0 \\ 0 & -10 & 4 & -3 & 1 & 0 \\ 0 & -11 & 4 & -2 & 0 & 1 \end{array}\right) \begin{array}{l} 5R_1 + 2R_2 \\ \\ 10R_3 - 11R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 5 & 0 & 3 & -1 & 2 & 0 \\ 0 & -10 & 4 & -3 & 1 & 0 \\ 0 & 0 & -4 & 13 & -11 & 10 \end{array}\right) \begin{array}{l} 4R_1 + 3R_3 \\ R_2 + R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 20 & 0 & 0 & 35 & -25 & 30 \\ 0 & -10 & 0 & 10 & -10 & 10 \\ 0 & 0 & -4 & 13 & -11 & 10 \end{array}\right) \begin{array}{l} \frac{1}{20}R_1 \\ -\frac{1}{10}R_2 \\ -\frac{1}{4}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & -\frac{5}{4} & \frac{3}{2} \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{13}{4} & \frac{11}{4} & -\frac{5}{2} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{7}{4} & -\frac{5}{4} & \frac{3}{2} \\ -1 & 1 & -1 \\ -\frac{13}{4} & \frac{11}{4} & -\frac{5}{2} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{7}{4} & -\frac{5}{4} & \frac{3}{2} \\ -1 & 1 & -1 \\ -\frac{13}{4} & \frac{11}{4} & -\frac{5}{2} \end{pmatrix}\begin{pmatrix} 20 \\ 8 \\ -16 \end{pmatrix} \qquad X = A^{-1}B$$

$$= \begin{pmatrix} 35-10-24 \\ -20+8+16 \\ -65+22+40 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

∴ ***Solution***: $(1,\ 4,\ -3)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2y-z=7 \\ x+2y+z=17 \\ 2x-3y+2z=-1 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 2 & -3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 17 \\ 7 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array}\right) \quad R_3 - 2R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & -7 & 0 & -2 & 0 & 1 \end{array}\right) \quad \frac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -7 & 0 & -2 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 7R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{7}{2} & -2 & \frac{7}{2} & 1 \end{array}\right) \quad \begin{array}{l} \\ \\ -\frac{2}{7}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{4}{7} & -1 & -\frac{2}{7} \end{array}\right) \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 + \frac{1}{2}R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{7} & 1 & \frac{4}{7} \\ 0 & 1 & 0 & \frac{2}{7} & 0 & -\frac{1}{7} \\ 0 & 0 & 1 & \frac{4}{7} & -1 & -\frac{2}{7} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{7} & 1 & \frac{4}{7} \\ \frac{2}{7} & 0 & -\frac{1}{7} \\ \frac{4}{7} & -1 & -\frac{2}{7} \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{1}{7} & 1 & \frac{4}{7} \\ \frac{2}{7} & 0 & -\frac{1}{7} \\ \frac{4}{7} & -1 & -\frac{2}{7} \end{pmatrix} \begin{pmatrix} 17 \\ 7 \\ -1 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$$

$\therefore$ ***Solution***: $(4,\ 5,\ 3)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} -2x+6y+7z=3 \\ -4x+5y+3z=7 \\ -6x+3y+5z=-4 \end{cases}$

Solution

$$A=\begin{pmatrix} -2 & 6 & 7 \\ -4 & 5 & 3 \\ -6 & 3 & 5 \end{pmatrix} \quad B=\begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} -2 & 6 & 7 & 1 & 0 & 0 \\ -4 & 5 & 3 & 0 & 1 & 0 \\ -6 & 3 & 5 & 0 & 0 & 1 \end{array}\right) \quad -\frac{1}{2}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -\frac{7}{2} & -\frac{1}{2} & 0 & 0 \\ -4 & 5 & 3 & 0 & 1 & 0 \\ -6 & 3 & 5 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2+4R_1 \\ R_3+6R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -\frac{7}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & -7 & -11 & -2 & 1 & 0 \\ 0 & -15 & -16 & -3 & 0 & 1 \end{array}\right) \quad -\frac{1}{7}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -\frac{7}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & -15 & -16 & -3 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1+3R_2 \\ \\ R_3+15R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{17}{14} & \frac{5}{14} & -\frac{3}{7} & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & \frac{53}{7} & \frac{9}{7} & -\frac{15}{7} & 1 \end{array}\right) \quad \frac{7}{53}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{17}{14} & \frac{5}{14} & -\frac{3}{7} & 0 \\ 0 & 1 & \frac{11}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & 1 & \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{array}\right) \quad \begin{array}{l} R_1-\frac{17}{14}R_3 \\ R_2-\frac{11}{7}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{8}{53} & -\frac{9}{106} & -\frac{17}{106} \\ 0 & 1 & 0 & \frac{1}{53} & \frac{16}{53} & -\frac{11}{53} \\ 0 & 0 & 1 & \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{8}{53} & -\frac{9}{106} & -\frac{7}{106} \\ \frac{1}{53} & \frac{16}{53} & -\frac{11}{53} \\ \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{8}{53} & -\frac{9}{106} & -\frac{7}{106} \\ \frac{1}{53} & \frac{16}{53} & -\frac{11}{53} \\ \frac{9}{53} & -\frac{15}{53} & \frac{7}{53} \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ -4 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 3 \\ -2 \end{pmatrix}$$

$\therefore$ ***Solution***: $\left(\frac{1}{2},\ 3,\ -2\right)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

Solution

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -3 & 4 \\ 4 & -2 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 3 & -3 & 4 & 0 & 1 & 0 \\ 4 & -2 & 3 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} 2R_2 - 3R_1 \\ 2R_3 - 4R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 5 & -3 & 2 & 0 \\ 0 & 0 & 2 & -4 & 0 & 2 \end{array}\right) \quad 3R_1 - R_2$$

$$\left(\begin{array}{ccc|ccc} 6 & 0 & -2 & 6 & -2 & 0 \\ 0 & -3 & 5 & -3 & 2 & 0 \\ 0 & 0 & 2 & -4 & 0 & 2 \end{array}\right) \quad \begin{array}{l} R_1 + R_3 \\ 2R_2 - 5R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 6 & 0 & 0 & 2 & -2 & 2 \\ 0 & -6 & 0 & 14 & 4 & -10 \\ 0 & 0 & 2 & -4 & 0 & 2 \end{array}\right) \quad \begin{array}{l} \frac{1}{6}R_1 \\ -\frac{1}{6}R_2 \\ \frac{1}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{7}{6} & -\frac{2}{3} & \frac{5}{3} \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{7}{6} & -\frac{2}{3} & \frac{5}{3} \\ -2 & 0 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{7}{6} & -\frac{2}{3} & \frac{5}{3} \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$\therefore$ ***Solution***: $(0,\ 1,\ 2)$

Exercise

Use the ***inverse*** of the coefficient matrix to solve the linear system $\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{c} R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 3 & -2 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{array}\right) \quad \frac{1}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{array}\right) \quad \begin{array}{c} R_1 + 2R_2 \\ \\ R_3 + R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & 1 \end{array}\right) \quad \begin{array}{c} R_1 - R_3 \\ R_2 - R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & -1 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & -1 \\ -1 & 0 & -1 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & -1 \\ -1 & 0 & -1 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \qquad X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ 6 \end{pmatrix}$$

∴ ***Solution***: $\underline{(-4,\ -6,\ 6)}$

Solution   *Section* 4.4 – Determinants and Cramer's Rule

Exercise

Evaluate $\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix}$

Solution

$$\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix} = -9-(-6)$$

$$\underline{=-3}$$

Exercise

Evaluate $\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix} = -6-(0)$$

$$\underline{=-6}$$

Exercise

Evaluate $\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix} = x(8x)-4x(2x)$$

$$=8x^2-8x^2$$

$$\underline{=0}$$

Exercise

Evaluate $\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix} = 3x - 2x(4)$$

$$= 3x - 8x$$

$$\underline{= -5x}$$

Exercise

Evaluate $\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix}$

Solution

$$\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix} = \underline{-3x^4 - 2x}$$

Exercise

Evaluate $\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix}$

Solution

$$\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix} = \underline{-8a + 5b}$$

Exercise

Evaluate $\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}$

Solution

$$\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix} = 15 - 14$$

$$\underline{= 1}$$

Exercise

Evaluate $\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix} = 5 - 20$$

$$\underline{= -16}$$

Exercise

Evaluate $\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix}$

Solution

$$\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix} = 15 + 6$$

$$\underline{= 21}$$

Exercise

Evaluate $\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix}$

Solution

$$\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix} = -24 + 5$$

$$\underline{= -19}$$

Exercise

Evaluate $\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix}$

Solution

$$\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix} = 3 - 6$$

$$\underline{= -3}$$

Exercise

Evaluate $\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix}$

Solution

$$\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix} = 7 + 18$$

$$\underline{= 25}$$

Exercise

Evaluate $\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix} \underline{= 2\sqrt{5} + 6}$$

Exercise

Evaluate $\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix}$

Solution

$$\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix} = -\frac{3}{8} - \frac{1}{16}$$

$$\underline{= -\frac{7}{16}}$$

Exercise

Evaluate $\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix}$

Solution

$$\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix} = -1 + 1$$

$$\underline{= 0}$$

Exercise

Evaluate $\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix}$

Solution

$$\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix} = \frac{1}{2} + \frac{1}{6}$$

$$= \frac{2}{3}$$

Exercise

Evaluate $\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix}$

Solution

$$\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix} = x^2 - 4x^2$$

$$= -3x^2$$

Exercise

Evaluate $\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix} = 9x - x^3$$

Exercise

Evaluate $\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix} = 2x^2 + 3x$$

Exercise

Evaluate $\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix}$

Solution

$$\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix} = 4(x+2) - 6(x-2)$$

$$= 4x + 8 - 6x + 12$$

$$= -2x + 20$$

Exercise

Evaluate $\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix}$

Solution

$$\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix} = -3x - 3 + 6x + 18$$

$$= -2x + 20$$

Exercise

Evaluate $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix} \begin{matrix} 3 & 0 \\ 2 & 1 \\ 2 & 5 \end{matrix}$$

$$= -3 + 0 + 0 - 0 + 75 - 0$$

$$= 72$$

Exercise

Evaluate $\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix}$

Solution

$$\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix} \begin{matrix} 4 & 0 \\ 3 & -1 \\ 2 & -3 \end{matrix} \qquad \text{or} \quad = 4\begin{vmatrix} -1 & 4 \\ -3 & 6 \end{vmatrix}$$

$$= -24 + 48$$

$$\underline{= 24}$$

Exercise

Evaluate $\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix} \begin{matrix} 3 & 1 \\ -3 & -4 \\ -1 & 3 \end{matrix}$$

$$= -60 + 15$$

$$\underline{= -45}$$

Exercise

Evaluate $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix} \begin{matrix} 1 & 1 \\ 2 & 2 \\ 3 & -4 \end{matrix}$$

$$= 10 + 6 - 8 - 6 + 8 - 10$$

$$\underline{= 0}$$

Exercise

Evaluate $\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix} \begin{matrix} x & 0 \\ 2 & 1 \\ -3 & x \end{matrix}$$

$$= x - 2x - 3 - x^4$$

$$\underline{= -x^4 - x - 3}$$

Exercise

Evaluate $\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix} \begin{matrix} x & 1 \\ x^2 & x \\ 0 & x \end{matrix}$$

$$= x^2 - x^3 - x^3 - x^2$$

$$\underline{= -2x^3}$$

Exercise

Evaluate $\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$

Solution

$$\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix} = 0 + 126 + 48 - (-48 + 36 + 0)$$

$$\underline{= 90}$$

Exercise

Evaluate $\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix} = 0 - 4 - 16 - (-14 - 16 + 0)$$

$$\underline{= 10}$$

Exercise

Evaluate $\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix}$

Solution

$$\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix} \begin{matrix} 3 & 1 \\ -2 & 3 \\ 3 & 4 \end{matrix}$$

$$= -54 + 3 - 16 - 18 - 12 - 12$$

$$\underline{= -109}$$

Exercise

Evaluate $\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix} \begin{matrix} 2x & 1 \\ 0 & 4 \\ 3 & 0 \end{matrix}$$

$$= 16x + 3x + 12$$

$$\underline{= 19x + 12}$$

Exercise

Evaluate $\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix} \begin{matrix} 0 & x \\ x & x^2 \\ x & 7 \end{matrix}$$

$$= 5x^2 + 7x^2 - x^4 + 5x^2$$

$$\underline{= 17x^2 - x^4}$$

Exercise

Evaluate $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix} \begin{matrix} 2 & x \\ -3 & 1 \\ 2 & 1 \end{matrix}$$

$$= 8 - 3 - 2 + 12x$$

$$\underline{= 12x + 3}$$

Exercise

Evaluate $\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} \begin{matrix} 1 & x \\ 3 & 1 \\ 0 & -2 \end{matrix}$$

$$= 2 + 12 + 2 - 6x$$

$$\underline{= -6x + 16}$$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}=-3-4=-7$$

$$D_x=\begin{vmatrix} -4 & 2 \\ -5 & -1 \end{vmatrix}=4-(-10)=14$$

$$D_y=\begin{vmatrix} 3 & -4 \\ 2 & -5 \end{vmatrix}=-15-(-8)=-7$$

$$x=\frac{D_x}{D}=\frac{14}{-7}=-2$$

$$y=\frac{D_y}{D}=\frac{-7}{-7}=1$$

∴ ***Solution***: $(-2,\ 1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix}=-29 \qquad D_x=\begin{vmatrix} 7 & 5 \\ -3 & -2 \end{vmatrix}=1 \qquad D_y=\begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix}=-41$$

$$x=\frac{1}{-29}=-\frac{1}{29} \qquad y=\frac{41}{29}$$

∴ ***Solution***: $\left(-\frac{1}{29},\ \frac{41}{29}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}=-7 \qquad D_x=\begin{vmatrix} -4 & -5 \\ 2 & -1 \end{vmatrix}=14 \qquad D_y=\begin{vmatrix} 3 & -4 \\ 2 & -5 \end{vmatrix}=-7$$

$x = -\frac{14}{7} \ \underline{= -2}\,\rvert$ $\qquad x = \frac{D_x}{D}$

$y = \frac{7}{7} \ \underline{= 1}\,\rvert$ $\qquad y = \frac{D_y}{D}$

Solution: $\underline{(-2,\ 1)}\,\rvert$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$

Solution

$D = \begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix} = -29 \qquad D_x = \begin{vmatrix} 7 & 5 \\ -3 & -2 \end{vmatrix} = 1 \qquad D_y = \begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix} = -41$

$\underline{x = -\frac{1}{29}}\,\rvert \qquad x = \frac{D_x}{D}$

$\underline{y = \frac{41}{29}}\,\rvert \qquad y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{\left(-\frac{1}{29},\ \frac{41}{29}\right)}\,\rvert$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$

Solution

$D = \begin{vmatrix} 4 & -7 \\ 2 & 5 \end{vmatrix} = 34 \qquad D_x = \begin{vmatrix} -16 & -7 \\ 9 & 5 \end{vmatrix} = -17 \qquad D_y = \begin{vmatrix} 4 & -16 \\ 2 & 9 \end{vmatrix} = 68$

$x = -\frac{17}{34} \underline{= -\frac{1}{2}}\,\rvert \qquad x = \frac{D_x}{D}$

$y = \frac{68}{34} \underline{= 2}\,\rvert \qquad y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{\left(-\frac{1}{2},\ 2\right)}\,\rvert$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y=4 \\ 2x+y=1 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}=-1 \qquad D_x=\begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix}=2 \qquad D_y=\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix}=-5$$

$\underline{x=-2}$ $\qquad x=\dfrac{D_x}{D}$

$\underline{y=5}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(-2,\ 5)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+4y=2 \\ 2x+5y=-1 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix}=7 \qquad D_x=\begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix}=14 \qquad D_y=\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}=-7$$

$x=\frac{14}{7}\underline{=2}$ $\qquad x=\dfrac{D_x}{D}$

$y=-\frac{7}{7}\underline{=-1}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(2,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 5x-2y=4 \\ -10x+4y=7 \end{cases}$

Solution

$$D=\begin{vmatrix} 5 & -2 \\ -10 & 4 \end{vmatrix}=0 \qquad D_y=\begin{vmatrix} 5 & 4 \\ -10 & 7 \end{vmatrix}=75\neq 0$$

$\therefore$ ***No Solution***

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & -4 \\ 5 & -20 \end{vmatrix}=0 \qquad D_y=\begin{vmatrix} 1 & -8 \\ 5 & -40 \end{vmatrix}=0$$

$$\frac{1}{5}\times \begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$$

$$\begin{cases} x-4y=-8 \\ x-4y=-8 \end{cases}$$

$\therefore$ ***Solution***: $\quad \underline{(4y-8,\ y)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}=-3 \qquad D_x=\begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix}=-6 \qquad D_y=\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix}=3$$

$$x=\frac{6}{3}=\underline{2} \qquad x=\frac{D_x}{D}$$

$$y=-\frac{3}{3}=\underline{-1} \qquad y=\frac{D_y}{D}$$

$\therefore$ ***Solution***: $\quad \underline{(2,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+10y=-14 \\ 7x-2y=-16 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & 10 \\ 7 & -2 \end{vmatrix}=-74 \qquad D_x=\begin{vmatrix} -14 & 10 \\ -16 & -2 \end{vmatrix}=188 \qquad D_y=\begin{vmatrix} 2 & -14 \\ 7 & -16 \end{vmatrix}=66$$

$$x=-\frac{188}{74}=\underline{-\frac{94}{37}} \qquad x=\frac{D_x}{D}$$

$$y = -\frac{66}{74} = -\frac{33}{37} \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\left(-\frac{94}{37},\ -\frac{33}{37}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$

Solution

$$D = \begin{vmatrix} 4 & -3 \\ -3 & 9 \end{vmatrix} = 27 \qquad D_x = \begin{vmatrix} 24 & -3 \\ -1 & 9 \end{vmatrix} = 213 \qquad D_y = \begin{vmatrix} 4 & 24 \\ -3 & -1 \end{vmatrix} = 68$$

$$x = \frac{213}{27} = \frac{71}{9} \qquad x = \frac{D_x}{D}$$

$$y = \frac{68}{27} \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\left(\frac{71}{9},\ \frac{68}{27}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$

Solution

$$D = \begin{vmatrix} 4 & 2 \\ 3 & -2 \end{vmatrix} = -14 \qquad D_x = \begin{vmatrix} 12 & 2 \\ 16 & -2 \end{vmatrix} = -56 \qquad D_y = \begin{vmatrix} 4 & 12 \\ 3 & 16 \end{vmatrix} = 28$$

$$x = \frac{56}{14} = 4 \qquad x = \frac{D_x}{D}$$

$$y = -\frac{28}{14} = -2 \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $(4,\ -2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+2y=-1 \\ 4x-2y=6 \end{cases}$

Solution

$D=\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix}=-10 \qquad D_x=\begin{vmatrix} -1 & 2 \\ 6 & -2 \end{vmatrix}=-10 \qquad D_y=\begin{vmatrix} 1 & -1 \\ 4 & 6 \end{vmatrix}=10$

$\underline{x=1} \qquad x=\dfrac{D_x}{D}$

$\underline{y=-1} \qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(1,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-2y=5 \\ -10x+2y=4 \end{cases}$

Solution

$D=\begin{vmatrix} 1 & -2 \\ -10 & 2 \end{vmatrix}=-18 \qquad D_x=\begin{vmatrix} 5 & -2 \\ 4 & 2 \end{vmatrix}=18 \qquad D_y=\begin{vmatrix} 1 & 5 \\ -10 & 4 \end{vmatrix}=54$

$\underline{x=-1} \qquad x=\dfrac{D_x}{D}$

$\underline{y=-\frac{54}{18}=-3} \qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(-1,\ -3)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$

Solution

$$\begin{array}{l} \frac{1}{3}\times \\ \frac{1}{15}\times \end{array} \begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$$

$$\begin{cases} 4x+5y=-9 \\ 2x-y=-1 \end{cases}$$

$$D = \begin{vmatrix} 4 & 5 \\ 2 & -1 \end{vmatrix} = -14 \qquad D_x = \begin{vmatrix} -9 & 5 \\ -1 & -1 \end{vmatrix} = 14 \qquad D_y = \begin{vmatrix} 4 & -9 \\ 2 & -1 \end{vmatrix} = 14$$

$\underline{x = -1}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = -1}$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(-1,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$

Solution

$$\begin{matrix} \frac{1}{4}\times \\ \frac{1}{4}\times \end{matrix} \begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$$

$$\begin{cases} x - y = -3 \\ x + y = -5 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \qquad D_x = \begin{vmatrix} -3 & -1 \\ -5 & 1 \end{vmatrix} = -8 \qquad D_y = \begin{vmatrix} 1 & -3 \\ 1 & -5 \end{vmatrix} = -2$$

$\underline{x = -4}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = -1}$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(-4,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + y = 7 \\ x - y = 3 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \qquad D_x = \begin{vmatrix} 7 & 1 \\ 3 & -1 \end{vmatrix} = -10 \qquad D_y = \begin{vmatrix} 1 & 7 \\ 1 & 3 \end{vmatrix} = -4$$

$\underline{x = 5}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y=2}\rfloor \qquad y=\frac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(5,\ 2)}\rfloor$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

Solution

$D=\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}=-3 \qquad D_x=\begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix}=-6 \qquad D_y=\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix}=3$

$\underline{x=2}\rfloor \qquad x=\frac{D_x}{D}$

$\underline{y=-1}\rfloor \qquad y=\frac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(2,\ -1)}\rfloor$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 12x+3y=15 \\ 2x-3y=13 \end{cases}$

Solution

$D=\begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix}=-42 \qquad D_x=\begin{vmatrix} 15 & 3 \\ 13 & -3 \end{vmatrix}=-84 \qquad D_y=\begin{vmatrix} 12 & 15 \\ 2 & 13 \end{vmatrix}=126$

$\underline{x=2}\rfloor \qquad x=\frac{D_x}{D}$

$\underline{y=-3}\rfloor \qquad y=\frac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(2,\ -3)}\rfloor$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-2y=5 \\ 5x-y=-2 \end{cases}$

Solution

$D=\begin{vmatrix} 1 & -2 \\ 5 & -1 \end{vmatrix}=9 \qquad D_x=\begin{vmatrix} 5 & -2 \\ -2 & -1 \end{vmatrix}=-9 \qquad D_y=\begin{vmatrix} 1 & 5 \\ 5 & -2 \end{vmatrix}=-27$

$\underline{x = -1}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = -3}$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(-1,\ -3)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x - 5y = 17 \\ 2x + 3y = 3 \end{cases}$

Solution

$D = \begin{vmatrix} 4 & -5 \\ 2 & 3 \end{vmatrix} = 22 \qquad D_x = \begin{vmatrix} 17 & -5 \\ 3 & 3 \end{vmatrix} = 66 \qquad D_y = \begin{vmatrix} 4 & 17 \\ 2 & 3 \end{vmatrix} = -22$

$\underline{x = 3}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = -1}$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(3,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x + 2y = 2 \\ 2x + 2y = 3 \end{cases}$

Solution

$D = \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 2 \qquad D_x = \begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = -2 \qquad D_y = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5$

$\underline{x = -1}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = \frac{5}{2}}$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{\left(-1,\ \frac{5}{2}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-3y=4 \\ 3x-4y=12 \end{cases}$

Solution

$$D=\begin{vmatrix}1 & -3\\ 3 & -4\end{vmatrix}=5 \qquad D_x=\begin{vmatrix}4 & -3\\ 12 & -4\end{vmatrix}=20 \qquad D_y=\begin{vmatrix}1 & 4\\ 3 & 12\end{vmatrix}=0$$

$\underline{x=4}$ $\qquad x=\dfrac{D_x}{D}$

$\underline{y=0}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(4,\ 0)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x-9y=5 \\ 3x-3y=11 \end{cases}$

Solution

$$D=\begin{vmatrix}2 & -9\\ 3 & -3\end{vmatrix}=21 \qquad D_x=\begin{vmatrix}5 & -9\\ 11 & -3\end{vmatrix}=84 \qquad D_y=\begin{vmatrix}2 & 5\\ 3 & 11\end{vmatrix}=7$$

$\underline{x=4}$ $\qquad x=\dfrac{D_x}{D}$

$\underline{y=\frac{1}{3}}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{\left(4,\ \frac{1}{3}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x-4y=4 \\ x+y=6 \end{cases}$

Solution

$$D=\begin{vmatrix}3 & -4\\ 1 & 1\end{vmatrix}=7 \qquad D_x=\begin{vmatrix}4 & -4\\ 6 & 1\end{vmatrix}=28 \qquad D_y=\begin{vmatrix}3 & 4\\ 1 & 6\end{vmatrix}=14$$

$\underline{x=4}$ $\qquad x=\dfrac{D_x}{D}$

$\underline{y=2}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(4,\ 2)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x = 7y + 1 \\ 2x = 3y - 1 \end{cases}$

Solution

$$\begin{cases} 3x - 7y = 1 \\ 2x - 3y = -1 \end{cases}$$

$$D = \begin{vmatrix} 3 & -7 \\ 2 & -3 \end{vmatrix} = 5 \qquad D_x = \begin{vmatrix} 1 & -7 \\ -1 & -3 \end{vmatrix} = -10 \qquad D_y = \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} = -5$$

$$\underline{x = -2} \qquad x = \frac{D_x}{D}$$

$$\underline{y = -1} \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\underline{(-2,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x = 3y + 2 \\ 5x = 51 - 4y \end{cases}$

Solution

$$\begin{cases} 2x - 3y = 2 \\ 5x + 4y = 51 \end{cases}$$

$$D = \begin{vmatrix} 2 & -3 \\ 5 & 4 \end{vmatrix} = 23 \qquad D_x = \begin{vmatrix} 2 & -3 \\ 51 & 4 \end{vmatrix} = 161 \qquad D_y = \begin{vmatrix} 2 & 2 \\ 5 & 51 \end{vmatrix} = 92$$

$$\underline{x = 7} \qquad x = \frac{D_x}{D}$$

$$\underline{y = 4} \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\underline{(7,\ 4)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} y = -4x + 2 \\ 2x = 3y - 1 \end{cases}$

Solution

$$\begin{cases} 4x + y = 2 \\ 2x - 3y = -1 \end{cases}$$

$$D = \begin{vmatrix} 4 & 1 \\ 2 & -3 \end{vmatrix} = -14 \qquad D_x = \begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix} = -5 \qquad D_y = \begin{vmatrix} 4 & 2 \\ 2 & -1 \end{vmatrix} = -8$$

$$x = \frac{5}{14} \qquad x = \frac{D_x}{D}$$

$$y = \frac{4}{7} \qquad y = \frac{D_y}{D}$$

$\therefore$ **Solution**: $\left(\frac{15}{4}, \ \frac{4}{7}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x = 2 - 3y \\ 2y = 3 - 2x \end{cases}$

Solution

$$\begin{cases} 3x + 3y = 2 \\ 2x + 2y = 3 \end{cases}$$

$$D = \begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix} = 0 \qquad D_y = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5 \neq 0$$

$\therefore$ ***No Solution***

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + 2y - 3 = 0 \\ 12 = 8y + 4x \end{cases}$

Solution

$$\begin{cases} x + 2y = 3 \\ 4x + 8y = 12 \end{cases}$$

$$\begin{cases} x + 2y = 3 \\ x + 2y = 3 \end{cases}$$

$\therefore$ ***Solution:*** $(3 - 2y, \ y)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 7x-2y=3 \\ 3x+y=5 \end{cases}$

Solution

$$D=\begin{vmatrix} 7 & -2 \\ 3 & 1 \end{vmatrix}=13 \qquad D_x=\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix}=13 \qquad D_y=\begin{vmatrix} 7 & 3 \\ 3 & 5 \end{vmatrix}=26$$

$\underline{x=1}$ $\qquad x=\dfrac{D_x}{D}$

$\underline{y=2}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(1,\ 2)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y-z=4 \\ 3x-2y+z=5 \\ 4x-5y-z=-1 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 2 & -1 \\ 3 & -2 & 1 \\ 4 & -5 & -1 \end{vmatrix}=42 \qquad D_x=\begin{vmatrix} 4 & 2 & -1 \\ 5 & -2 & 1 \\ -1 & -5 & -1 \end{vmatrix}=63$$

$$D_y=\begin{vmatrix} 3 & 4 & -1 \\ 3 & 5 & 1 \\ 4 & -1 & -1 \end{vmatrix}=39 \qquad D_z=\begin{vmatrix} 3 & 2 & 4 \\ 3 & -2 & 5 \\ 4 & -5 & -1 \end{vmatrix}=99$$

$x=\dfrac{63}{42}=\underline{\dfrac{3}{2}}$ $\qquad x=\dfrac{D_x}{D}$

$y=\dfrac{39}{42}=\underline{\dfrac{13}{14}}$ $\qquad y=\dfrac{D_y}{D}$

$z=\dfrac{99}{42}=\underline{\dfrac{33}{14}}$ $\qquad z=\dfrac{D_z}{D}$

Solution: $\underline{\left(\dfrac{3}{2},\ \dfrac{13}{14},\ \dfrac{33}{14}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$

Solution

$$D=\begin{vmatrix}1 & 1 & 1\\ 2 & 1 & -1\\ 1 & -1 & 1\end{vmatrix}=-6 \qquad D_x=\begin{vmatrix}2 & 1 & 1\\ 5 & 1 & -1\\ -2 & -1 & 1\end{vmatrix}=-6$$

$$D_y=\begin{vmatrix}1 & 2 & 1\\ 2 & 5 & -1\\ 1 & -2 & 1\end{vmatrix}=-12 \qquad D_z=\begin{vmatrix}1 & 1 & 2\\ 2 & 1 & 5\\ 1 & -2 & 2\end{vmatrix}=6$$

$x=1$ $\qquad x=\frac{D_x}{D}$

$y=2$ $\qquad y=\frac{D_y}{D}$

$z=-1$ $\qquad z=\frac{D_z}{D}$

$\therefore$ ***Solution***: $(1,\ 2,\ -1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+y+z=9 \\ -x-y+z=1 \\ 3x-y+z=9 \end{cases}$

Solution

$$D=\begin{vmatrix}2 & 1 & 1\\ -1 & -1 & 1\\ 3 & -1 & 1\end{vmatrix}\begin{matrix}2 & 1\\ -1 & -1\\ 3 & -1\end{matrix}=-2+3+1+3+2+1$$

$$=8$$

$$D_x=\begin{vmatrix}9 & 1 & 1\\ 1 & -1 & 1\\ 9 & -1 & 1\end{vmatrix}\begin{matrix}9 & 1\\ 1 & -1\\ 9 & -1\end{matrix}=-9+9-1+9+9-1$$

$$=16$$

$$D_y=\begin{vmatrix}2 & 9 & 1\\ -1 & 1 & 1\\ 3 & 9 & 1\end{vmatrix}\begin{matrix}2 & 9\\ -1 & 1\\ 3 & 9\end{matrix}=2+27-9-3-18+9$$

$$=8$$

$$D_z = \begin{vmatrix} 2 & 1 & 9 \\ -1 & -1 & 1 \\ 3 & -1 & 9 \end{vmatrix} \begin{matrix} 2 & 1 \\ -1 & -1 \\ 3 & -1 \end{matrix} = -18+3+9+27+2+9$$

$$= 32$$

$x = 2$ $\qquad\qquad x = \dfrac{D_x}{D}$

$y = 1$ $\qquad\qquad y = \dfrac{D_y}{D}$

$z = \dfrac{32}{8} = 4$ $\qquad\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $(2,\ 1,\ 4)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

Solution

$$D = \begin{vmatrix} 0 & 3 & -1 \\ 1 & 5 & -1 \\ -3 & 6 & 2 \end{vmatrix} \begin{matrix} 0 & 3 \\ 1 & 5 \\ -3 & 6 \end{matrix} = 9-6-15-6$$

$$= -18$$

$$D_x = \begin{vmatrix} -1 & 3 & -1 \\ -4 & 5 & -1 \\ 11 & 6 & 2 \end{vmatrix} \begin{matrix} -1 & 3 \\ -4 & 5 \\ 11 & 6 \end{matrix} = -10-33+24+55-6+24$$

$$= 54$$

$$D_y = \begin{vmatrix} 0 & -1 & -1 \\ 1 & -4 & -1 \\ -3 & 11 & 2 \end{vmatrix} \begin{matrix} 0 & -1 \\ 1 & -4 \\ -3 & 11 \end{matrix} = -3-11+12+2$$

$$= 0$$

$$D_z = \begin{vmatrix} 0 & 3 & -1 \\ 1 & 5 & -4 \\ -3 & 6 & 11 \end{vmatrix} \begin{matrix} 0 & 3 \\ 1 & 5 \\ -3 & 6 \end{matrix} = 36-6-15-33$$

$$= -18$$

$x = -3$ $\qquad\qquad x = \dfrac{D_x}{D}$

$y = 0$ $\qquad\qquad y = \frac{D_y}{D}$

$z = 1$ $\qquad\qquad z = \frac{D_z}{D}$

∴ ***Solution***: $\boxed{(-3,\ 0,\ 1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+3y+4z=14 \\ 2x-3y+2z=10 \\ 3x-y+z=9 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 3 & 4 \\ 2 & -3 & 2 \\ 3 & -1 & 1 \end{vmatrix} \begin{matrix} 1 & 3 \\ 2 & -3 \\ 3 & -1 \end{matrix} = -3+18-8+36+2-6$$

$$= 39$$

$$D_x = \begin{vmatrix} 14 & 3 & 4 \\ 10 & -3 & 2 \\ 9 & -1 & 1 \end{vmatrix} \begin{matrix} 14 & 3 \\ 10 & -3 \\ 9 & -1 \end{matrix} = -42+54-40+108+28-30$$

$$= 78$$

$$D_y = \begin{vmatrix} 1 & 14 & 4 \\ 2 & 10 & 2 \\ 3 & 9 & 1 \end{vmatrix} \begin{matrix} 1 & 14 \\ 2 & 10 \\ 3 & 9 \end{matrix} = 10+84+72-120-18-28$$

$$= 0$$

$$D_z = \begin{vmatrix} 1 & 3 & 14 \\ 2 & -3 & 10 \\ 3 & -1 & 9 \end{vmatrix} \begin{matrix} 1 & 3 \\ 2 & -3 \\ 3 & -1 \end{matrix} = -27+90-28+126+10-54$$

$$= 117$$

$x = \frac{78}{39} = 2$ $\qquad\qquad x = \frac{D_x}{D}$

$y = 0$ $\qquad\qquad y = \frac{D_y}{D}$

$z = \frac{117}{39} = 3$ $\qquad\qquad z = \frac{D_z}{D}$

∴ ***Solution***: $\boxed{(2,\ 0,\ 3)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+4y-z=20 \\ 3x+2y+z=8 \\ 2x-3y+2z=-16 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 4 & -1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix} \begin{matrix} 1 & 4 \\ 3 & 2 \\ 2 & -3 \end{matrix} = 4+8+9+4+3-24$$

$$\underline{= 4}$$

$$D_x = \begin{vmatrix} 20 & 4 & -1 \\ 8 & 2 & 1 \\ -16 & -3 & 2 \end{vmatrix} \begin{matrix} 20 & 4 \\ 8 & 2 \\ -16 & -3 \end{matrix} = 80-64+24-32+60-64$$

$$\underline{= 4}$$

$$D_y = \begin{vmatrix} 1 & 20 & -1 \\ 3 & 8 & 1 \\ 2 & -16 & 2 \end{vmatrix} \begin{matrix} 1 & 20 \\ 3 & 8 \\ 2 & -16 \end{matrix} = 16+40+48+16+16-120$$

$$\underline{= 16}$$

$$D_z = \begin{vmatrix} 1 & 4 & 20 \\ 3 & 2 & 8 \\ 2 & -3 & -16 \end{vmatrix} \begin{matrix} 1 & 4 \\ 3 & 2 \\ 2 & -3 \end{matrix} = -32+64-180-80+24+192$$

$$\underline{= -12}$$

$x = \frac{4}{4} \ \underline{= 1}$ $\qquad$ $x = \frac{D_x}{D}$

$y = \frac{16}{4} \ \underline{= 4}$ $\qquad$ $y = \frac{D_y}{D}$

$z = -\frac{12}{4} \ \underline{= -3}$ $\qquad$ $z = \frac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(1,\ 4,\ -3)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} -2x+6y+7z=3 \\ -4x+5y+3z=7 \\ -6x+3y+5z=-4 \end{cases}$

Solution

$$D=\begin{vmatrix} -2 & 6 & 7 \\ -4 & 5 & 3 \\ -6 & 3 & 5 \end{vmatrix}\begin{matrix} -2 & 6 \\ -4 & 5 \\ -6 & 3 \end{matrix} = -50-108-84+210+18+120$$

$$=106$$

$$D_x=\begin{vmatrix} 3 & 6 & 7 \\ 7 & 5 & 3 \\ -4 & 3 & 5 \end{vmatrix}\begin{matrix} 3 & 6 \\ 7 & 5 \\ -4 & 3 \end{matrix} = 75-72+147+140-27-210$$

$$=53$$

$$D_y=\begin{vmatrix} -2 & 3 & 7 \\ -4 & 7 & 3 \\ -6 & -4 & 5 \end{vmatrix}\begin{matrix} -2 & 3 \\ -4 & 7 \\ -6 & -4 \end{matrix} = -70-54+112+294-24+60$$

$$=318$$

$$D_z=\begin{vmatrix} -2 & 6 & 3 \\ -4 & 5 & 7 \\ -6 & 3 & -4 \end{vmatrix}\begin{matrix} -2 & 6 \\ -4 & 5 \\ -6 & 3 \end{matrix} = 40-252-36+90+42-96$$

$$=-212$$

$x=\frac{53}{106}=\frac{1}{2}$ $\qquad$ $x=\frac{D_x}{D}$

$y=\frac{318}{106}=3$ $\qquad$ $y=\frac{D_y}{D}$

$z=-\frac{212}{106}=-2$ $\qquad$ $z=\frac{D_z}{D}$

$\therefore$ ***Solution***: $\left(\frac{1}{2},\ 3,\ -2\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -3 & 4 \\ 4 & -2 & 3 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & -3 \\ 4 & -2 \end{matrix} = -18 - 16 - 6 + 12 + 16 + 9$$

$$= -3$$

$$D_x = \begin{vmatrix} 1 & -1 & 1 \\ 5 & -3 & 4 \\ 4 & -2 & 3 \end{vmatrix} \begin{matrix} 1 & -1 \\ 5 & -3 \\ 4 & -2 \end{matrix} = -9 - 16 - 10 + 12 + 8 + 15$$

$$= 0$$

$$D_y = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 5 & 4 \\ 4 & 4 & 3 \end{vmatrix} \begin{matrix} 2 & 1 \\ 3 & 5 \\ 4 & 4 \end{matrix} = 30 + 16 + 12 - 20 - 32 - 9$$

$$= -3$$

$$D_z = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -3 & 5 \\ 4 & -2 & 4 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & -3 \\ 4 & -2 \end{matrix} = -24 - 20 - 6 + 12 + 20 + 12$$

$$= -6$$

$x = -\frac{0}{3} = 0$ $\qquad$ $x = \frac{D_x}{D}$

$y = \frac{-3}{-3} = 1$ $\qquad$ $y = \frac{D_y}{D}$

$z = \frac{-6}{-3} = 2$ $\qquad$ $z = \frac{D_z}{D}$

$\therefore$ ***Solution***: $(0,\ 1,\ 2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x-4y+4z=7 \\ x-y-2z=2 \\ 2x-3y+6z=5 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & -4 & 4 \\ 1 & -1 & -2 \\ 2 & -3 & 6 \end{vmatrix}\begin{matrix} 3 & -4 \\ 1 & -1 \\ 2 & -3 \end{matrix} =-18+16-12+8-18+24$$

$$\underline{=0}$$

$$D_z=\begin{vmatrix} 3 & -4 & 7 \\ 1 & -1 & 2 \\ 2 & -3 & 5 \end{vmatrix}\begin{matrix} 3 & -4 \\ 1 & -1 \\ 2 & -3 \end{matrix} =-15-16-21+14+18+20$$

$$\underline{=0}$$

$$-3\times(2)\quad \begin{cases} -3x+3y+6z=-6 \\ 2x-3y+6z=5 \end{cases}$$

$$-x+12z=-1$$

$$\underline{x=12z+1}$$

$$(2)\rightarrow \quad y=12z+1-2z-2$$

$$\underline{=10z-1}$$

$\therefore$ ***Solution***: $\underline{(12z+1,\ \ 10z-1,\ \ z)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix}\begin{matrix} 1 & -2 \\ 2 & -1 \\ -1 & 1 \end{matrix} =-1+2-2+1-1+4$$

$$\underline{=3}$$

$$D_x=\begin{vmatrix} 2 & -2 & -1 \\ 4 & -1 & 1 \\ 4 & 1 & 1 \end{vmatrix}\begin{matrix} 2 & -2 \\ 4 & -1 \\ 4 & 1 \end{matrix} =-2-8-4-4-2+8$$

$$\underline{=-12}$$

$$D_y = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 4 & 1 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & 4 \\ -1 & 4 \end{matrix} = 4-2-8-4-4-4$$

$$\underline{=-18|}$$

$$D_z = \begin{vmatrix} 1 & -2 & 2 \\ 2 & -1 & 4 \\ -1 & 1 & 4 \end{vmatrix} \begin{matrix} 1 & -2 \\ 2 & -1 \\ -1 & 1 \end{matrix} = -4+8+4-2-4+16$$

$$\underline{=18|}$$

$$x = -\frac{12}{3} \underline{=-4|} \qquad x = \frac{D_x}{D}$$

$$y = -\frac{18}{3} \underline{=-6|} \qquad y = \frac{D_y}{D}$$

$$z = \frac{18}{3} \underline{=6|} \qquad z = \frac{D_z}{D}$$

$\therefore$ ***Solution***: $\underline{(-4,\ -6,\ 6)\ |}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+y+z=3 \\ -y+2z=1 \\ -x+z=0 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & 1 \end{vmatrix} \begin{matrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{matrix} \underline{=-4|}$$

$$D_x = \begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} 3 & 1 \\ 1 & -1 \\ 0 & 0 \end{matrix} \underline{=-4|}$$

$$D_y = \begin{vmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{vmatrix} \begin{matrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{matrix} \underline{=-4|}$$

$$D_z = \begin{vmatrix} 1 & 1 & 3 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \end{vmatrix} \begin{matrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{matrix} \underline{=-4|}$$

$$x = \frac{4}{4} \underline{=1|} \qquad x = \frac{D_x}{D}$$

$y = \frac{4}{4} = 1$ $\qquad\qquad y = \frac{D_y}{D}$

$z = \frac{4}{4} = 1$ $\qquad\qquad z = \frac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(1,\ 1,\ 1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x + y + 3z = 14 \\ 7x + 5y + 8z = 37 \\ x + 3y + 2z = 9 \end{cases}$

Solution

$$D = \begin{vmatrix} 3 & 1 & 3 \\ 7 & 5 & 8 \\ 1 & 3 & 2 \end{vmatrix} \begin{matrix} 3 & 1 \\ 7 & 5 \\ 1 & 3 \end{matrix} = 30 + 8 + 62 - 15 - 72 - 14$$

$$= 0$$

$$D_z = \begin{vmatrix} 3 & 1 & 14 \\ 7 & 5 & 37 \\ 1 & 3 & 9 \end{vmatrix} \begin{matrix} 3 & 1 \\ 7 & 5 \\ 1 & 3 \end{matrix} = 135 + 37 + 294 - 70 - 333 - 63$$

$$= 0$$

$$\begin{array}{r} -3\times(1) \\ (3) \end{array} \begin{cases} -9x - 3y - 9z = -42 \\ \quad x + 3y + 2z = 9 \end{cases}$$
$$\overline{\qquad -8x - 7z = -33}$$

$$x = -\frac{7}{8}z + \frac{33}{8}$$

$$(1) \to \quad y = 14 - 3z - 3\left(-\frac{7}{8}z + \frac{33}{8}\right)$$

$$= \frac{13}{8} - \frac{3}{8}z$$

$\therefore$ ***Solution***: $\underline{\left(\frac{33}{8} - \frac{7}{8}z,\ \ \frac{13}{8} - \frac{3}{8}z,\ \ z\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x-2y+z=7 \\ x+y+z=-2 \\ 4x+2y+z=3 \end{cases}$

Solution

$$D=\begin{vmatrix} 4 & -2 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix}\begin{matrix} 4 & -2 \\ 1 & 1 \\ 4 & 2 \end{matrix}$$

$$=-12$$

$$D_x=\begin{vmatrix} 7 & -2 & 1 \\ -2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix}\begin{matrix} 7 & -2 \\ -2 & 1 \\ 3 & 2 \end{matrix}$$

$$=-24$$

$$D_y=\begin{vmatrix} 4 & 7 & 1 \\ 1 & -2 & 1 \\ 4 & 3 & 1 \end{vmatrix}\begin{matrix} 4 & 7 \\ 1 & -2 \\ 4 & 3 \end{matrix}$$

$$=12$$

$$D_z=\begin{vmatrix} 4 & -2 & 7 \\ 1 & 1 & -2 \\ 4 & 2 & 3 \end{vmatrix}\begin{matrix} 4 & -2 \\ 1 & 1 \\ 4 & 2 \end{matrix}$$

$$=36$$

$x=\frac{24}{12}=2$ $\qquad$ $x=\frac{D_x}{D}$

$y=-\frac{12}{12}=-1$ $\qquad$ $y=\frac{D_y}{D}$

$z=-\frac{36}{12}=-3$ $\qquad$ $z=\frac{D_z}{D}$

$\therefore$ ***Solution***: $(2,\ -1,\ -3)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

Solution

$$D = \begin{vmatrix} 0 & 2 & -1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix} \begin{matrix} 0 & 2 \\ 1 & 2 \\ 2 & 3 \end{matrix}$$

$$= 1$$

$$D_x = \begin{vmatrix} 7 & 2 & -1 \\ 17 & 2 & 1 \\ -1 & 3 & 2 \end{vmatrix} \begin{matrix} 7 & 2 \\ 17 & 2 \\ -1 & 3 \end{matrix}$$

$$= -116$$

$$D_y = \begin{vmatrix} 0 & 7 & -1 \\ 1 & 17 & 1 \\ 2 & -1 & 2 \end{vmatrix} \begin{matrix} 0 & 7 \\ 1 & 17 \\ 2 & -1 \end{matrix}$$

$$= 35$$

$$D_z = \begin{vmatrix} 0 & 2 & 7 \\ 1 & 2 & 17 \\ 2 & 3 & -1 \end{vmatrix} \begin{matrix} 0 & 2 \\ 1 & 2 \\ 2 & 3 \end{matrix}$$

$$= 63$$

$x = -116$ $\qquad x = \dfrac{D_x}{D}$

$y = 35$ $\qquad y = \dfrac{D_y}{D}$

$z = 63$ $\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $(-116,\ 35,\ 63)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & -2 & 1 \\ 6 & 4 & -3 \\ 1 & -2 & 2 \end{vmatrix}\begin{matrix} 2 & -2 \\ 6 & 4 \\ 1 & -2 \end{matrix}$$

$$=18$$

$$D_x=\begin{vmatrix} -4 & -2 & 1 \\ -24 & 4 & -3 \\ 1 & -2 & 2 \end{vmatrix}\begin{matrix} -4 & -2 \\ -24 & 4 \\ 1 & -2 \end{matrix}$$

$$=-54$$

$$D_y=\begin{vmatrix} 2 & -4 & 1 \\ 6 & -24 & -3 \\ 1 & 1 & 2 \end{vmatrix}\begin{matrix} 2 & -4 \\ 6 & -24 \\ 1 & 1 \end{matrix}$$

$$=0$$

$$D_z=\begin{vmatrix} 2 & -2 & -4 \\ 6 & 4 & -24 \\ 1 & -2 & 1 \end{vmatrix}\begin{matrix} 2 & -2 \\ 6 & 4 \\ 1 & -2 \end{matrix}$$

$$=36$$

$x=-\dfrac{54}{18}$ $\qquad$ $x=\dfrac{D_x}{D}$

$=-3$

$y=0$ $\qquad$ $y=\dfrac{D_y}{D}$

$z=2$ $\qquad$ $z=\dfrac{D_z}{D}$

∴ ***Solution***: $(-3,\ 0,\ 2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

Solution

$$D = \begin{vmatrix} 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{vmatrix} \begin{matrix} 9 & 3 \\ 16 & 4 \\ 25 & 5 \end{matrix}$$

$$= -2$$

$$D_x = \begin{vmatrix} 4 & 3 & 1 \\ 2 & 4 & 1 \\ 2 & 5 & 1 \end{vmatrix} \begin{matrix} 4 & 3 \\ 2 & 4 \\ 2 & 5 \end{matrix}$$

$$= -2$$

$$D_y = \begin{vmatrix} 9 & 4 & 1 \\ 16 & 2 & 1 \\ 25 & 2 & 1 \end{vmatrix} \begin{matrix} 9 & 4 \\ 16 & 2 \\ 25 & 2 \end{matrix}$$

$$= 18$$

$$D_z = \begin{vmatrix} 9 & 3 & 4 \\ 16 & 4 & 2 \\ 25 & 5 & 2 \end{vmatrix} \begin{matrix} 9 & 3 \\ 16 & 4 \\ 25 & 5 \end{matrix}$$

$$= -44$$

$$x = \frac{-2}{-2} \qquad x = \frac{D_x}{D}$$

$$= 1$$

$$y = \frac{18}{-2} \qquad y = \frac{D_y}{D}$$

$$= -9$$

$$z = \frac{-44}{-2} \qquad z = \frac{D_z}{D}$$

$$= 22$$

$\therefore$ ***Solution***: $(1,\ -9,\ 22)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & -3 \\ 3 & -1 & -4 \end{vmatrix} \begin{matrix} 2 & -1 \\ 1 & 2 \\ 3 & -1 \end{matrix}$$

$$= -31$$

$$D_x = \begin{vmatrix} -8 & -1 & 2 \\ 9 & 2 & -3 \\ 3 & -1 & -4 \end{vmatrix} \begin{matrix} -8 & -1 \\ 9 & 2 \\ 3 & -1 \end{matrix}$$

$$= 31$$

$$D_y = \begin{vmatrix} 2 & -8 & 2 \\ 1 & 9 & -3 \\ 3 & 3 & -4 \end{vmatrix} \begin{matrix} 2 & -8 \\ 1 & 9 \\ 3 & 3 \end{matrix}$$

$$= -62$$

$$D_z = \begin{vmatrix} 2 & -1 & -8 \\ 1 & 2 & 9 \\ 3 & -1 & 3 \end{vmatrix} \begin{matrix} 2 & -1 \\ 1 & 2 \\ 3 & -1 \end{matrix}$$

$$= 62$$

$x = -\frac{31}{31}$ $\qquad x = \frac{D_x}{D}$

$= -1$

$y = \frac{62}{31}$ $\qquad y = \frac{D_y}{D}$

$= 2$

$z = -\frac{62}{31}$ $\qquad z = \frac{D_z}{D}$

$= -2$

$\therefore$ ***Solution***: $(-1,\ 2,\ -2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x \quad\quad -3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 0 & -3 \\ 2 & -1 & 2 \\ 7 & -3 & -5 \end{vmatrix} \begin{matrix} 1 & 0 \\ 2 & -1 \\ 7 & -3 \end{matrix}$$

$$= 8$$

$$D_x = \begin{vmatrix} -5 & 0 & -3 \\ 16 & -1 & 2 \\ 19 & -3 & -5 \end{vmatrix} \begin{matrix} -5 & 0 \\ 16 & -1 \\ 19 & -3 \end{matrix}$$

$$= 32$$

$$D_y = \begin{vmatrix} 1 & -5 & -3 \\ 2 & 16 & 2 \\ 7 & 19 & -5 \end{vmatrix} \begin{matrix} 1 & -5 \\ 2 & 16 \\ 7 & 19 \end{matrix}$$

$$= -16$$

$$D_z = \begin{vmatrix} 1 & 0 & -5 \\ 2 & -1 & 16 \\ 7 & -3 & 19 \end{vmatrix} \begin{matrix} 1 & 0 \\ 2 & -1 \\ 7 & -3 \end{matrix}$$

$$= 24$$

$x = \frac{32}{8}$ $\qquad\qquad$ $x = \frac{D_x}{D}$

$= 4$

$y = -\frac{16}{8}$ $\qquad\qquad$ $y = \frac{D_y}{D}$

$= -2$

$z = \frac{24}{8}$ $\qquad\qquad$ $z = \frac{D_z}{D}$

$= 3$

$\therefore$ ***Solution***: $(4,\ -2,\ 3)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+2y-z=5 \\ 2x-y+3z=0 \\ \quad 2y+z=1 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 0 & 2 & 1 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{matrix}$$

$$\underline{=-15}$$

$$D_x=\begin{vmatrix} 5 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 2 & 1 \end{vmatrix} \begin{matrix} 5 & 2 \\ 0 & -1 \\ 1 & 2 \end{matrix}$$

$$\underline{=-30}$$

$$D_y=\begin{vmatrix} 1 & 5 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix} \begin{matrix} 1 & 5 \\ 2 & 0 \\ 0 & 1 \end{matrix}$$

$$\underline{=-15}$$

$$D_z=\begin{vmatrix} 1 & 2 & 5 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{matrix}$$

$$\underline{=15}$$

$$x=\frac{30}{15} \qquad x=\frac{D_x}{D}$$

$$\underline{=2}$$

$$y=\frac{15}{15} \qquad y=\frac{D_y}{D}$$

$$\underline{=1}$$

$$z=-\frac{15}{15} \qquad z=\frac{D_z}{D}$$

$$\underline{=-1}$$

$\therefore$ ***Solution***: $\underline{(2,\ 1,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+y+z=6 \\ 3x+4y-7z=1 \\ 2x-y+3z=5 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & -7 \\ 2 & -1 & 3 \end{vmatrix}\begin{matrix} 1 & 1 \\ 3 & 4 \\ 2 & -1 \end{matrix}$$

$$=-29$$

$$D_x=\begin{vmatrix} 6 & 1 & 1 \\ 1 & 4 & -7 \\ 5 & -1 & 3 \end{vmatrix}\begin{matrix} 6 & 1 \\ 1 & 4 \\ 5 & -1 \end{matrix}$$

$$=-29$$

$$D_y=\begin{vmatrix} 1 & 6 & 1 \\ 3 & 1 & -7 \\ 2 & 5 & 3 \end{vmatrix}\begin{matrix} 1 & 6 \\ 3 & 1 \\ 2 & 5 \end{matrix}$$

$$=-87$$

$$D_z=\begin{vmatrix} 1 & 2 & 6 \\ 2 & -1 & 1 \\ 0 & 2 & 5 \end{vmatrix}\begin{matrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{matrix}$$

$$=-58$$

$$x=\frac{29}{29} \qquad x=\frac{D_x}{D}$$

$$=1$$

$$y=\frac{87}{29} \qquad y=\frac{D_y}{D}$$

$$=3$$

$$z=\frac{58}{29} \qquad z=\frac{D_z}{D}$$

$$=2$$

$\therefore$ ***Solution***: $(1,\ 3,\ 2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y+3z=3 \\ 4x-5y+7z=1 \\ 2x+3y-2z=6 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 2 & 3 \\ 4 & -5 & 7 \\ 2 & 3 & -2 \end{vmatrix}\begin{matrix} 3 & 2 \\ 4 & -5 \\ 2 & 3 \end{matrix}$$

$$\underline{=77}$$

$$D_x=\begin{vmatrix} 3 & 2 & 3 \\ 1 & -5 & 7 \\ 6 & 3 & -2 \end{vmatrix}\begin{matrix} 3 & 2 \\ 1 & -5 \\ 6 & 3 \end{matrix}$$

$$\underline{=154}$$

$$D_y=\begin{vmatrix} 3 & 3 & 3 \\ 4 & 1 & 7 \\ 2 & 6 & -2 \end{vmatrix}\begin{matrix} 3 & 3 \\ 4 & 1 \\ 2 & 6 \end{matrix}$$

$$\underline{=0}$$

$$D_z=\begin{vmatrix} 3 & 2 & 3 \\ 4 & -5 & 1 \\ 2 & 3 & 6 \end{vmatrix}\begin{matrix} 3 & 2 \\ 4 & -5 \\ 2 & 3 \end{matrix}$$

$$\underline{=-77}$$

$x=\frac{154}{77}\underline{=2}$ $\qquad x=\frac{D_x}{D}$

$\underline{=2}$

$\underline{y=0}$ $\qquad y=\frac{D_y}{D}$

$z=-\frac{77}{77}$ $\qquad z=\frac{D_z}{D}$

$\underline{=-1}$

$\therefore$ ***Solution***: $\underline{(2,\ 0,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x+5y \qquad =2 \\ 11x+y+2z=3 \\ x+5y+2z=1 \end{cases}$

Solution

$$D = \begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \begin{matrix} 4 & 5 \\ 11 & 1 \\ 1 & 5 \end{matrix}$$

$$= -132$$

$$D_x = \begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \begin{matrix} 2 & 5 \\ 3 & 1 \\ 1 & 5 \end{matrix}$$

$$= -36$$

$$D_y = \begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} \begin{matrix} 4 & 2 \\ 11 & 3 \\ 1 & 1 \end{matrix}$$

$$= -24$$

$$D_z = \begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} \begin{matrix} 4 & 5 \\ 11 & 1 \\ 1 & 5 \end{matrix}$$

$$= 12$$

$$x = \frac{36}{132} \qquad\qquad x = \frac{D_x}{D}$$

$$= \frac{3}{11}$$

$$y = \frac{24}{132} \qquad\qquad y = \frac{D_y}{D}$$

$$= \frac{2}{11}$$

$$z = -\frac{12}{132} \qquad\qquad z = \frac{D_z}{D}$$

$$= -\frac{1}{11}$$

$\therefore$ ***Solution***: $\left(\frac{3}{11},\ \frac{2}{11},\ -\frac{1}{11}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-4y+z=6 \\ 4x-y+2z=-1 \\ 2x+2y-3z=-20 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix}\begin{matrix} 1 & -4 \\ 4 & -1 \\ 2 & 2 \end{matrix}$$

$$=-55$$

$$D_x=\begin{vmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix}\begin{matrix} 6 & -4 \\ -1 & -1 \\ -20 & 2 \end{matrix}$$

$$=144$$

$$D_y=\begin{vmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix}\begin{matrix} 1 & 6 \\ 4 & -1 \\ 2 & -20 \end{matrix}$$

$$=61$$

$$D_z=\begin{vmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix}\begin{matrix} 1 & -4 \\ 4 & -1 \\ 2 & 2 \end{matrix}$$

$$=-230$$

$$x=-\frac{144}{55} \qquad x=\frac{D_x}{D}$$

$$y=-\frac{61}{55} \qquad y=\frac{D_y}{D}$$

$$z=\frac{230}{55} \qquad z=\frac{D_z}{D}$$

$$=\frac{46}{11}$$

$\therefore$ ***Solution***: $\left(-\frac{144}{55},\ -\frac{61}{55},\ \frac{46}{11}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x - \ y + \ z = -1 \\ 3x + 4y - z = -1 \\ 4x - \ y + 2z = -1 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 4 & -1 \\ 4 & -1 & 2 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & 4 \\ 4 & -1 \end{matrix}$$

$$= 5$$

$$D_x = \begin{vmatrix} -1 & -1 & 1 \\ -1 & 4 & -1 \\ -1 & -1 & 2 \end{vmatrix} \begin{matrix} -1 & -1 \\ -1 & 4 \\ -1 & -1 \end{matrix}$$

$$= -5$$

$$D_y = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -1 & -1 \\ 4 & -1 & 2 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & -1 \\ 4 & -1 \end{matrix}$$

$$= 5$$

$$D_z = \begin{vmatrix} 2 & -1 & -1 \\ 3 & 4 & -1 \\ 4 & -1 & -1 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & 4 \\ 4 & -1 \end{matrix}$$

$$= 10$$

$$x = \frac{-5}{5} \qquad\qquad x = \frac{D_x}{D}$$

$$= -1$$

$$y = \frac{5}{5} \qquad\qquad y = \frac{D_y}{D}$$

$$= 1$$

$$z = \frac{10}{5} \qquad\qquad z = \frac{D_z}{D}$$

$$= 2$$

$\therefore$ ***Solution***: $(-1,\ 1,\ 2)$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$$

Solution

$$D = \begin{vmatrix} -1 & -4 & 2 & 1 \\ 2 & -1 & 7 & 9 \\ -1 & 1 & 3 & 1 \\ -1 & -2 & 1 & -4 \end{vmatrix}$$

$$= -243$$

$$D_1 = \begin{vmatrix} -32 & -4 & 2 & 1 \\ 14 & -1 & 7 & 9 \\ 11 & 1 & 3 & 1 \\ -4 & -2 & 1 & -4 \end{vmatrix}$$

$$= -2115$$

$$D_2 = \begin{vmatrix} -1 & -32 & 2 & 1 \\ 2 & 14 & 7 & 9 \\ -1 & 11 & 3 & 1 \\ -1 & -4 & 1 & -4 \end{vmatrix}$$

$$= -1834$$

$$D_3 = \begin{vmatrix} -1 & -4 & -32 & 1 \\ 2 & -1 & 14 & 9 \\ -1 & 1 & 11 & 1 \\ -1 & -2 & -4 & -4 \end{vmatrix}$$

$$= -1279$$

$$D_4 = \begin{vmatrix} -1 & -4 & 2 & -32 \\ 2 & -1 & 7 & 14 \\ -1 & 1 & 3 & 11 \\ -1 & -2 & 1 & -4 \end{vmatrix}$$

$$= 883$$

$$x_1 = \frac{-2115}{-243}$$

$$= \frac{235}{27}$$

$$x_2 = \frac{-1834}{-243}$$

$$= \boxed{\frac{1834}{243}}$$

$$x_3 = \frac{-1279}{-243}$$

$$= \boxed{\frac{1279}{243}}$$

$$\boxed{x_4 = -\frac{883}{243}}$$

∴ ***Solution***: $\boxed{\left(\frac{235}{27}, \frac{1834}{243}, \frac{1279}{243}, -\frac{883}{243}\right)}$

Exercise

Solve for x. $\begin{vmatrix} x & 3 \\ 2 & 1 \end{vmatrix} = 12$

Solution

$$\begin{vmatrix} x & 3 \\ 2 & 1 \end{vmatrix} = x - 6 = 12$$

∴ ***Solution***: $\boxed{x = 18}$

Exercise

Solve for x. $\begin{vmatrix} x & 1 \\ 2 & x \end{vmatrix} = -1$

Solution

$$\begin{vmatrix} x & 1 \\ 2 & x \end{vmatrix} = x^2 - 2 = -1$$

$$x^2 = 1$$

∴ ***Solution***: $\boxed{x = \pm 1}$

Exercise

Solve for x. $\begin{vmatrix} 3 & x \\ x & 4 \end{vmatrix} = -13$

Solution

$$\begin{vmatrix} 3 & x \\ x & 4 \end{vmatrix} = 12 - x^2 = -13$$

$x^2 = 25$

$\therefore$ ***Solution***: $\underline{x = \pm 5}$

Exercise

Solve for x. $\begin{vmatrix} x & 2 \\ 3 & x \end{vmatrix} = x$

Solution

$$\begin{vmatrix} x & 2 \\ 3 & x \end{vmatrix} = x^2 - 6 = x$$

$x^2 - x - 6 = 0$

$\therefore$ ***Solution***: $\underline{x = -2,\ 3}$

Exercise

Solve for x. $\begin{vmatrix} 4 & 6 \\ -2 & x \end{vmatrix} = 32$

Solution

$$\begin{vmatrix} 4 & 6 \\ -2 & x \end{vmatrix} = 4x + 12 = 32$$

$4x = 20$

$\therefore$ ***Solution***: $\underline{x = 5}$

Exercise

Solve for x. $\begin{vmatrix} x+2 & -3 \\ x+5 & -4 \end{vmatrix} = 3x - 5$

Solution

$$\begin{vmatrix} x+2 & -3 \\ x+5 & -4 \end{vmatrix} = -4x - 8 + 3x + 15 = 3x - 5$$

$-4x = -12$

$\therefore$ ***Solution***: $\underline{x = 3}$

Exercise

Solve for x. $\begin{vmatrix} x+3 & -6 \\ x-2 & -4 \end{vmatrix} = 28$

Solution

$$\begin{vmatrix} x+3 & -6 \\ x-2 & -4 \end{vmatrix} = -4x - 12 + 6x - 12 = 28$$

$$2x = 52$$

∴ ***Solution***: $\underline{x = 26}$

Exercise

Solve for x. $\begin{vmatrix} x & -3 \\ -1 & x \end{vmatrix} \ge 0$

Solution

$$\begin{vmatrix} x & -3 \\ -1 & x \end{vmatrix} = x^2 - 3 \ge 0$$

$$x^2 \ge 3$$

∴ ***Solution***: $\underline{x \le -\sqrt{3} \quad x \ge \sqrt{3}}$

Exercise

Solve for x. $\begin{vmatrix} 2 & x & 1 \\ 1 & 2 & -1 \\ 3 & 4 & -2 \end{vmatrix} = -6$

Solution

$$\begin{vmatrix} 2 & x & 1 \\ 1 & 2 & -1 \\ 3 & 4 & -2 \end{vmatrix} = -8 - 3x + 4 - 6 + 8 + 2x = -6$$

$$-x = -4$$

∴ ***Solution***: $\underline{x = 4}$

Exercise

Solve for x. $\begin{vmatrix} 1 & x & -3 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} = 8$

Solution

$$\begin{vmatrix} 1 & x & -3 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} = 2 + 18 + 2 - 6x = 8$$

$-6x = -14$

∴ ***Solution***: $\boxed{x = \frac{7}{3}}$

Exercise

Solve for x. $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix} = 39$

Solution

$$\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix} = 8 - 3 - 2 + 12x = 39$$

$12x = 36$

∴ ***Solution***: $\boxed{x = 3}$

Exercise

Solve for x. $\begin{vmatrix} x & 0 & 0 \\ 7 & x & 1 \\ 7 & 2 & 1 \end{vmatrix} = -1$

Solution

$$\begin{vmatrix} x & 0 & 0 \\ 7 & x & 1 \\ 7 & 2 & 1 \end{vmatrix} = x^2 - 2x = -1$$

$x^2 - 2x + 1 = 0$

∴ ***Solution***: $\boxed{x = 1}$

Exercise

Find the quadratic function $f(x) = ax^2 + bx + c$ for which $f(1) = -10, \quad f(-2) = -31, \quad f(2) = -19$. What is the function?

Solution

$$f(1) = a(1)^2 + b(1) + c \quad \Rightarrow \quad -10 = a + b + c$$

$$f(-2) = a(-2)^2 + b(-2) + c \quad \Rightarrow \quad -31 = 4a - 2b + c$$

$$f(2) = a(2)^2 + b(2) + c \quad \Rightarrow \quad -19 = 4a + 2b + c$$

$$\begin{cases} a + b + c = -10 \\ 4a - 2b + c = -31 \\ 4a + 2b + c = -19 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 4 & -2 & 1 \\ 4 & 2 & 1 \end{vmatrix} = 12 \qquad D_a = \begin{vmatrix} -10 & 1 & 1 \\ -31 & -2 & 1 \\ -19 & 2 & 1 \end{vmatrix} = -48$$

$$D_b = \begin{vmatrix} 1 & -10 & 1 \\ 4 & -31 & 1 \\ 4 & -19 & 1 \end{vmatrix} = 36 \qquad D_c = \begin{vmatrix} 1 & 1 & -10 \\ 4 & -2 & -31 \\ 4 & 2 & -19 \end{vmatrix} = -108$$

$$a = \frac{D_a}{D} = \frac{-48}{12} = -4$$

$$b = \frac{D_b}{D} = \frac{36}{12} = 3$$

$$c = \frac{D_c}{D} = \frac{-108}{12} = -9$$

∴ **Solution**: $f(x) = -x^2 + 3x - 9$

Exercise

You wish to mix candy worth \$3.44 per pound with candy worth \$9.96 per pound to form 24 pounds of a mixture worth \$8.33 per pound.

a) Write the system equations?

b) How many pounds of each candy should you use?

Solution

Let x: total pounds of \$3.44 candy
y: total pounds of \$9.96 candy

a) $\begin{cases} x+y=24 \\ 3.44x+9.96y=8.33(24) \end{cases}$

$\begin{cases} x+y=24 \\ 344x+996y=19,992 \end{cases}$

$\begin{cases} x+y=24 \\ 86x+249y=4,998 \end{cases}$

b) $D=\begin{vmatrix} 1 & 1 \\ 86 & 249 \end{vmatrix}=163$

$D_x=\begin{vmatrix} 24 & 1 \\ 4998 & 249 \end{vmatrix}=978$ $\qquad D_y=\begin{vmatrix} 1 & 24 \\ 86 & 4998 \end{vmatrix}=2,934$

Total pounds of \$3.44 candy: $\frac{978}{163}$ $\underline{=6\ \ lbs|}$

Total pounds of \$9.96 candy: $\frac{2,934}{163}$ $\underline{=18\ \ lbs|}$

Exercise

Anne and Nancy use a metal alloy that is 17.76% copper to make jewelry. How many ounces of a 15% alloy must be mixed with a 19% alloy to form 100 ounces of the desired alloy?

Solution

Let *x*: total ounces 15%

y: total ounces of 19%

$\begin{cases} x+y=100 \\ 15x+19y=17.76(100) \end{cases}$

$\begin{cases} x+y=100 \\ 15x+19y=1776 \end{cases}$

$D=\begin{vmatrix} 1 & 1 \\ 15 & 19 \end{vmatrix}=4$ $\qquad D_x=\begin{vmatrix} 100 & 1 \\ 1776 & 19 \end{vmatrix}=124$

$\therefore$ Total ounces 15%: $\frac{124}{4}$ $\underline{=31\ \ ounces|}$

Exercise

A company makes 3 types of cable. Cable ***A*** requires 3 black, 3 white, and 2 red wires. ***B*** requires 1 black, 2 white, and 1 red. ***C*** requires 2 black, 1 white, and 2 red. They used 95 black, 100 white and 80 red wires.

a) Write the system equations?

b) How many of each cable were made?

Solution

Let x: Cable ***A***

y: Cable ***B***

z: Cable ***C***

a) $$\begin{cases} 3x + y + 2z = 95 \\ 3x + 2y + z = 100 \\ 2x + y + 2z = 80 \end{cases}$$

b) $$D = \begin{vmatrix} 3 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix} \begin{matrix} 3 & 1 \\ 3 & 2 \\ 2 & 1 \end{matrix} \underline{= 3|}$$

$$D_x = \begin{vmatrix} 95 & 1 & 2 \\ 100 & 2 & 1 \\ 80 & 1 & 2 \end{vmatrix} \begin{matrix} 95 & 1 \\ 100 & 2 \\ 80 & 1 \end{matrix} \underline{= 45|}$$

$$D_y = \begin{vmatrix} 3 & 95 & 2 \\ 3 & 100 & 1 \\ 2 & 80 & 2 \end{vmatrix} \begin{matrix} 3 & 95 \\ 3 & 100 \\ 2 & 80 \end{matrix} \underline{= 60|}$$

$$D_z = \begin{vmatrix} 3 & 1 & 95 \\ 3 & 2 & 100 \\ 2 & 1 & 80 \end{vmatrix} \begin{matrix} 3 & 1 \\ 3 & 2 \\ 2 & 1 \end{matrix} \underline{= 45|}$$

$$x = \frac{45}{3} \underline{= 15|} \qquad x = \frac{D_x}{D}$$

$$y = \frac{60}{3} \underline{= 20|} \qquad y = \frac{D_y}{D}$$

$$z = \frac{45}{3} \underline{= 15|} \qquad z = \frac{D_z}{D}$$

∴ ***Solution***: 15 cable ***A*** 20 cable ***B*** 15 cable ***C***

Exercise

A basketball fieldhouse seats 15,000. Courtside seats sell for \$8.00, end zone for \$6.00, and balcony for \$5.00. Total for a sell–out is \$86,000. If half the courtside and balcony and all end zone seats are sold, ticket sales total \$49,000.

a) Write the system equations?

b) How many of each type of seat are there?

Solution

Let x: Courtside seats

y: end zone

z: balcony

a) $$\begin{cases} x+y+z=15,000 \\ 8x+6y+5z=86,000 \\ \frac{1}{2}(8x)+6y+\frac{1}{2}(5z)=49,000 \end{cases}$$

$$\begin{cases} x+y+z=15,000 \\ 8x+6y+5z=86,000 \\ 8x+12y+5z=98,000 \end{cases}$$

b) $$D=\begin{vmatrix} 1 & 1 & 1 \\ 8 & 6 & 5 \\ 8 & 12 & 5 \end{vmatrix}\begin{matrix} 1 & 1 \\ 8 & 6 \\ 8 & 12 \end{matrix}=18$$

$$D_x=\begin{vmatrix} 15,000 & 1 & 1 \\ 86,000 & 6 & 5 \\ 98,000 & 12 & 5 \end{vmatrix}\begin{matrix} 15,000 & 1 \\ 86,000 & 6 \\ 98,000 & 12 \end{matrix}=54,000$$

$$D_y=\begin{vmatrix} 1 & 15,000 & 1 \\ 8 & 86,000 & 5 \\ 8 & 98,000 & 5 \end{vmatrix}\begin{matrix} 1 & 15,000 \\ 8 & 86,000 \\ 8 & 98,000 \end{matrix}=36,000$$

$$D_z=\begin{vmatrix} 1 & 1 & 15,000 \\ 8 & 6 & 86,000 \\ 8 & 12 & 98,000 \end{vmatrix}\begin{matrix} 1 & 1 \\ 8 & 6 \\ 8 & 12 \end{matrix}=180,000$$

$$x=\frac{54,000}{18}=3,000 \qquad x=\frac{D_x}{D}$$

$$y=\frac{36,000}{18}=2,000 \qquad y=\frac{D_y}{D}$$

$$z=\frac{180,000}{18}=10,000 \qquad z=\frac{D_z}{D}$$

∴ ***Solution***: **3,000** Courtside **2,000** End zone **10,000** Balcony

Exercise

A movie theater charges \$9.00 for adults and \$7.00 for senior citizens. On a day when 325 people paid admission, the total receipts were \$2,495.

a) Write the system equations?

b) How many who paid were adults? How many were seniors?

Solution

Let x: Adults

y: Senior citizens

a) $\begin{cases} x+y=325 \\ 9x+7y=2,495 \end{cases}$

b) $D=\begin{vmatrix} 1 & 1 \\ 9 & 7 \end{vmatrix} = -2$

$D_x=\begin{vmatrix} 325 & 1 \\ 2,495 & 7 \end{vmatrix} = -220$ $\qquad$ $D_y=\begin{vmatrix} 1 & 325 \\ 9 & 2,495 \end{vmatrix} = 430$

$x=\dfrac{220}{2}=110$ $\qquad$ $x=\dfrac{D_x}{D}$

$y=\dfrac{430}{2}=215$ $\qquad$ $y=\dfrac{D_y}{D}$

∴ ***Solution***: **110** Adults $\qquad$ **215** Senior citizens

Exercise

A Broadway theater has 500 seats, divided into orchestra, main, and balcony seating. Orchestra seats sell for \$150, main seats for \$135, and balcony seats for \$110. If all the seats are sold, the gross revenue to the theater is \$64,250. If all the main and balcony seats are sold, but only half the orchestra seats are sold, the gross revenue is \$56,750.

a) Write the system equations?

b) How many of each kind of seat are there?

Solution

Let x: Numbers of orchestra seats

y: Numbers of main seats

z: Numbers of balcony seats

a) $\begin{cases} x+y+z=500 \\ 150x+135y+110z=64,250 \\ \frac{1}{2}(150)x+135y+110z=56,750 \end{cases}$

$\begin{cases} x+y+z=500 \\ 30x+27y+22z=12,850 \\ 15x+27y+22z=11,350 \end{cases}$

b) $D=\begin{vmatrix} 1 & 1 & 1 \\ 30 & 27 & 22 \\ 15 & 27 & 22 \end{vmatrix} \underline{=75|}$

$D_x=\begin{vmatrix} 500 & 1 & 1 \\ 12850 & 27 & 22 \\ 11350 & 27 & 22 \end{vmatrix} \underline{=7,500|}$

$D_y=\begin{vmatrix} 1 & 500 & 1 \\ 30 & 12,850 & 22 \\ 15 & 11,350 & 22 \end{vmatrix} \underline{=15,750|}$

$D_z=\begin{vmatrix} 1 & 1 & 500 \\ 30 & 27 & 12,850 \\ 15 & 27 & 11,350 \end{vmatrix} \underline{=14,250|}$

$x=\frac{7,500}{75} \underline{=100|}$ $\qquad x=\frac{D_x}{D}$

$y=\frac{15,750}{75} \underline{=210|}$ $\qquad y=\frac{D_y}{D}$

$z=\frac{14,250}{75} \underline{=190|}$ $\qquad z=\frac{D_z}{D}$

∴ ***Solution***: There are **100** orchestra seats, **210** main seats, and **190** balcony seats.

Exercise

A movie theater charges \$11 for adults, \$6.50 for children, and \$9 for senior citizens. One day the theater sold 405 tickets and collected \$3,315 in receipts. Twice as many children's tickets were sold as adult tickets.

a) Write the system equations?

b) How many adults, children, and senior citizens went to the theater that day?

Solution

Let x: Numbers of adults

y: Numbers of children

z: Numbers of senior citizens

a) $$\begin{cases} x+y+z=405 \\ 11x+6.5y+9z=3,315 \\ y=2x \end{cases}$$

b) $$\begin{cases} 3x+z=405 \\ 24x+9z=3,315 \end{cases}$$

$$D=\begin{vmatrix} 3 & 1 \\ 24 & 9 \end{vmatrix} \underline{=3|}$$

$$D_x=\begin{vmatrix} 405 & 1 \\ 3,315 & 9 \end{vmatrix} \underline{=330|}$$

$$D_y=\begin{vmatrix} 3 & 405 \\ 24 & 3,315 \end{vmatrix} \underline{=225|}$$

$$x=\frac{330}{3} \underline{=110|} \qquad x=\frac{D_x}{D}$$

$$z=\frac{225}{3} \underline{=75|} \qquad z=\frac{D_z}{D}$$

$$y=2(110) \underline{=220|}$$

∴ ***Solution***: There are **110** adults, **220** children, and **75** senior citizens.

Exercise

Emma has \$20,000 to invest. As her financial planner, you recommend that she diversify into three investements: Treasure bills that yield 5% simple interest. Treasury bonds tht yield 7% simple interest, and corporate bonds that yield 10% simple interest. Emma wishes to earn \$1,390 per year in income. Also, Emma wants her investment in Treasury bills to be \$3,000 more than her investment in corporate bonds. How much money should Emma place in each investement?

Solution

Let x: Amount in Treasure bills.

y: Amount in Treasury bonds.

z: Amount in corporate bonds.

$$\begin{cases} x+y+z=20,000 \\ .05x+.07y+.1z=1,390 \\ x=3,000+z \end{cases}$$

$$\begin{cases} x+y+z=20,000 \\ 5x+7y+10z=139,000 \\ x-z=3,000 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 1 \\ 5 & 7 & 10 \\ 1 & 0 & -1 \end{vmatrix} \underline{=1|}$$

$$D_x=\begin{vmatrix} 20,000 & 1 & 1 \\ 139,000 & 7 & 10 \\ 3,000 & 0 & -1 \end{vmatrix} \underline{=8,000|}$$

$$D_y=\begin{vmatrix} 1 & 20,000 & 1 \\ 5 & 139,000 & 10 \\ 1 & 3,000 & -1 \end{vmatrix} \underline{=7,000|}$$

$$D_z=\begin{vmatrix} 1 & 1 & 20,000 \\ 5 & 7 & 139,000 \\ 1 & 0 & 3,000 \end{vmatrix} \underline{=5,000|}$$

∴ ***Solution***: Emma should invest **\$8,000** in Treasure bills

\$7,000 in Treasury bonds

\$5,000 in corporate bonds.

Exercise

A person invested \$17,000 for one year, part at 10%, part at 12%, and the remainder at 15%. The total annual income from these investements was \$2,110. The amount of money invested at 12% was \$1,000 less than the amounts invested at 10% and 15% combined. Find the amount invested at each rate.

Solution

Let x = Amount invested at 10%
Let y = Amount invested at 12%
Let z = Amount invested at 15%

$$\begin{cases} x+y+z=17{,}000 \\ .1x+.12y+.15z=2{,}110 \\ y=x+z-1{,}000 \end{cases}$$

$$\begin{cases} x+y+z=17{,}000 \\ 10x+12y+15z=211{,}000 \\ x-y+z=1{,}000 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 1 \\ 10 & 12 & 15 \\ 1 & -1 & 1 \end{vmatrix} \underline{=10|}$$

$$D_x=\begin{vmatrix} 17{,}000 & 1 & 1 \\ 211{,}000 & 12 & 15 \\ 1{,}000 & -1 & 1 \end{vmatrix} \underline{=40{,}000|}$$

$$D_y=\begin{vmatrix} 1 & 17{,}000 & 1 \\ 10 & 211{,}000 & 15 \\ 1 & 1{,}000 & 1 \end{vmatrix} \underline{=80{,}000|}$$

$$D_z=\begin{vmatrix} 1 & 1 & 17{,}000 \\ 10 & 12 & 211{,}000 \\ 1 & -1 & 1{,}000 \end{vmatrix} \underline{=50{,}000|}$$

$$x=\frac{40{,}000}{10} \underline{=4{,}000|} \qquad x=\frac{D_x}{D}$$

$$y=\frac{80{,}000}{10} \underline{=8{,}000|} \qquad y=\frac{D_y}{D}$$

$$z=\frac{50{,}000}{10} \underline{=5{,}000|} \qquad z=\frac{D_z}{D}$$

∴ ***Solution***: should invest **\$4,000** invested at 10%
\$8,000 invested at 12%
\$5,000 invested at 15%.

Exercise

At a production, 400 tickets were sold. The ticket prices were \$8, \$10, and \$12, and the total income from ticket sales was \$3,700. How many tickets of each type were sold if the combined number of \$8 and \$10 tickets sold was 7 times the number of \$12 tickets sold?

Solution

Let x = Numbers of tickets sold at \$8
Let y = Numbers of tickets sold at \$10
Let z = Numbers of tickets sold at \$12

$$\begin{cases} x+y+z=400 \\ 8x+10y+12z=3,700 \\ x+y=7z \end{cases}$$

$$\begin{cases} x+y+z=400 \\ 4x+5y+6z=1,850 \\ x+y-7z=0 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 1 \\ 4 & 5 & 6 \\ 1 & 1 & -7 \end{vmatrix} \underline{=-8|}$$

$$D_x=\begin{vmatrix} 400 & 1 & 1 \\ 1,850 & 5 & 6 \\ 0 & 1 & -7 \end{vmatrix} \underline{=-1,600|}$$

$$D_y=\begin{vmatrix} 1 & 400 & 1 \\ 4 & 1850 & 6 \\ 1 & 0 & -7 \end{vmatrix} \underline{=-1,200|}$$

$$D_z=\begin{vmatrix} 1 & 1 & 400 \\ 4 & 5 & 1,850 \\ 1 & 1 & 0 \end{vmatrix} \underline{=-400|}$$

$$x=\frac{1600}{8} \underline{=200|} \qquad x=\frac{D_x}{D}$$

$$y=\frac{1200}{8} \underline{=150|} \qquad y=\frac{D_y}{D}$$

$$z=\frac{400}{8} \underline{=50|} \qquad z=\frac{D_z}{D}$$

∴ ***Solution***: **200** tickets sold at \$8
150 tickets sold at \$10
50 tickets sold at \$12

Exercise

A certain brand of razor blades comes in packages if 6, 12, and 24 blades, costing \$2, \$3, and \$4 per package, respectively. A store sold 12 packages containing a total of 162 razor blades and took in \$35. How many packages of each type were sold?

Solution

Let x = Numbers of packages sold at \$2
Let y = Numbers of packages sold at \$3
Let z = Numbers of packages sold at \$4

$$\begin{cases} x+y+z=12 \\ 2x+3y+4z=35 \\ 6x+12y+24z=162 \end{cases}$$

$$\begin{cases} x+y+z=12 \\ 2x+3y+4z=35 \\ x+2y+4z=27 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & 2 & 4 \end{vmatrix} \underline{=1|}$$

$$D_x=\begin{vmatrix} 12 & 1 & 1 \\ 35 & 3 & 4 \\ 27 & 2 & 4 \end{vmatrix} \underline{=5|}$$

$$D_y=\begin{vmatrix} 1 & 12 & 1 \\ 2 & 35 & 4 \\ 1 & 27 & 4 \end{vmatrix} \underline{=3|}$$

$$D_z=\begin{vmatrix} 1 & 1 & 12 \\ 2 & 3 & 35 \\ 1 & 2 & 27 \end{vmatrix} \underline{=4|}$$

$$x=\frac{5}{1} \underline{=5|} \qquad x=\frac{D_x}{D}$$

$$y=\frac{3}{1} \underline{=3|} \qquad y=\frac{D_y}{D}$$

$$z=\frac{4}{1} \underline{=4|} \qquad z=\frac{D_z}{D}$$

∴ ***Solution***: **5** packages sold at \$2
3 packages sold at \$3
4 packages sold at \$4

Exercise

A store sells cashews for \$5.00 per pound and peanuts for \$1.50 per pound. The manager decides to mix 30 pounds of peanuts with some cashews and sell the mixture for \$3.00 per pound.

a) Write the system equations?

b) How many pounds of cashews should be mixed with peanuts so that the mixture will produce the same revenue as selling the nuts separately?

Solution

Let x: pounds of cashews

y: pounds of in the mixture

a) $\begin{cases} x+30=y \\ 5x+\frac{3}{2}(30)=3y \end{cases}$

$\begin{cases} x-y=-30 \\ 5x-3y=-45 \end{cases}$

b) $D=\begin{vmatrix} 1 & -1 \\ 5 & -3 \end{vmatrix} \underline{=2|}$

$D_x=\begin{vmatrix} -30 & -1 \\ -45 & -3 \end{vmatrix} \underline{=45|}$

$x=\frac{90}{7}$ $\qquad x=\frac{D_x}{D}$

$y=\frac{120}{7}$ $\qquad y=\frac{D_y}{D}$

∴ ***Solution***: $\frac{45}{2}=22.5$ pounds of cashews

Exercise

A wireless store takes presale orders for a new smartphone and tablet. He gets 340 preorders for the smartphone and 250 preorders for the tablet. The combined value of the preorders is \$270,500.00. If the price of a smartphone and tablet together is \$965, how much does each device cost?

Solution

Let x: Cost of a smartphone

y: Cost of a tablet

$\begin{cases} 340x+250y=270,500 \\ x+y=965 \end{cases}$

$\begin{cases} 34x+25y=27,050 \\ x+y=965 \end{cases}$

$$D = \begin{vmatrix} 34 & 25 \\ 1 & 1 \end{vmatrix} \underline{= 9|}$$

$$D_x = \begin{vmatrix} 27,050 & 25 \\ 965 & 1 \end{vmatrix} \underline{= 2,925|} \qquad D_y = \begin{vmatrix} 34 & 27,050 \\ 1 & 965 \end{vmatrix} \underline{= 5,760|}$$

$$x = \frac{2,925}{9} \underline{= \$325|} \qquad x = \frac{D_x}{D}$$

$$y = \frac{5,760}{9} \underline{= \$640|} \qquad y = \frac{D_y}{D}$$

∴ ***Solution***: Cost of a smartphone is $\underline{\$325\ |}$

Cost of a tablet is $\underline{\$640\ |}$

Exercise

A restaurant manager wants to purchase 200 sets of dishes. One design costs \$25 per set, and another costs \$45 per set. If she has only \$7400 to spend, how many sets of each design should be order?

Solution

Let x: Number of sets for \$25 set.
y: Number of sets for \$45 set.

$$\begin{cases} 25x + 45y = 7,400 \\ x + y = 200 \end{cases}$$

$$\begin{cases} 5x + 9y = 1,480 \\ x + y = 200 \end{cases}$$

$$D = \begin{vmatrix} 5 & 9 \\ 1 & 1 \end{vmatrix} \underline{= -4|}$$

$$D_x = \begin{vmatrix} 1480 & 9 \\ 200 & 1 \end{vmatrix} \underline{= -320|} \qquad D_y = \begin{vmatrix} 5 & 1480 \\ 1 & 200 \end{vmatrix} \underline{= -480|}$$

$$x = \frac{320}{4} \underline{= 80|} \qquad x = \frac{D_x}{D}$$

$$y = \frac{480}{4} \underline{= 120|} \qquad y = \frac{D_y}{D}$$

∴ ***Solution***: 80 sets for \$25 set.

120 sets for \$45 set.

Exercise

One group of people purchased 10 hot dogs and 5 soft drinks at a cost of \$35.00. A second bought 7 hot dogs and 4 soft drinks at a cost of \$25.25. What is the cost of a single hot dog and a single soft drink?

Solution

Let x: Cost of a hot dog.
y: Cost of a drink

$$\begin{cases} 10x + 5y = 35 \\ 7x + 4y = 25.25 \end{cases}$$

$$\begin{cases} 2x + y = 7 \\ 700x + 400y = 2,525 \end{cases}$$

$$D = \begin{vmatrix} 2 & 1 \\ 700 & 400 \end{vmatrix} = 100$$

$$D_x = \begin{vmatrix} 7 & 1 \\ 2,525 & 400 \end{vmatrix} = 275$$

$$D_y = \begin{vmatrix} 2 & 7 \\ 700 & 2,525 \end{vmatrix} = 150$$

$$x = \frac{275}{100} = 2.75 \qquad x = \frac{D_x}{D}$$

$$y = \frac{150}{100} = 1.5 \qquad y = \frac{D_y}{D}$$

∴ ***Solution***: Cost of a hot dog is \$2.75
Cost of a soft drink is \$1.50

Exercise

The sum of three times the first number, plus the second number, and twice the third number is 5. If 3 times the second number is subtracted from the sum of the first number and 3 times the third number, the result is 2. If the third number is subtracted from the sum of 2 times the first number and 3 times the second number, the result is 1. Find the three numbers.

Solution

Let x: be the first number.
y: be the second number.
z: be the third number.

$$\begin{cases} 3x + y + 2z = 5 \\ (x + 3z) - 3y = 2 \\ 2x + 3y - z = 1 \end{cases}$$

$$\begin{cases} 3x + y + 2z = 5 \\ x - 3y + 3z = 2 \\ 2x + 3y - z = 1 \end{cases}$$

$$D = \begin{vmatrix} 3 & 1 & 2 \\ 1 & -3 & 3 \\ 2 & 3 & -1 \end{vmatrix} \underline{= 7|}$$

$$D_x = \begin{vmatrix} 5 & 1 & 2 \\ 2 & -3 & 3 \\ 1 & 3 & -1 \end{vmatrix} \underline{= -7|}$$

$$D_y = \begin{vmatrix} 3 & 5 & 2 \\ 1 & 2 & 3 \\ 2 & 1 & -1 \end{vmatrix} \underline{= 14|}$$

$$D_z = \begin{vmatrix} 3 & 1 & 5 \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} \underline{= 21|}$$

$x = -\frac{7}{7} \underline{= -1|}$ $\qquad x = \frac{D_x}{D}$

$y = \frac{14}{7} \underline{= 2|}$ $\qquad y = \frac{D_y}{D}$

$z = \frac{21}{7} \underline{= 3|}$ $\qquad z = \frac{D_z}{D}$

∴ ***Solution***: The three numbers are: $-1,\ 2,\ and\ \ 3$

Exercise

The sum of three numbers is 16. The sum of twice the first number, 3 times the second number, and 4 times the third number is 46. The difference between 5 times the first number and the second number is 31. Find the three numbers.

Solution

Let x: be the first number.

y: be the second number.

z: be the third number.

$$\begin{cases} x+y+z=16 \\ 2x+3y+4z=46 \\ 5x-y=31 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 5 & -1 & 0 \end{vmatrix} \underline{=7|}$$

$$D_x=\begin{vmatrix} 16 & 1 & 1 \\ 46 & 3 & 4 \\ 31 & -1 & 0 \end{vmatrix} \underline{=49|}$$

$$D_y=\begin{vmatrix} 1 & 16 & 1 \\ 2 & 46 & 4 \\ 5 & 31 & 0 \end{vmatrix} \underline{=28|}$$

$$D_z=\begin{vmatrix} 1 & 1 & 16 \\ 2 & 3 & 46 \\ 5 & -1 & 31 \end{vmatrix} \underline{=35|}$$

$$x=\frac{49}{7} \underline{=7|} \qquad x=\frac{D_x}{D}$$

$$y=\frac{28}{7} \underline{=4|} \qquad y=\frac{D_y}{D}$$

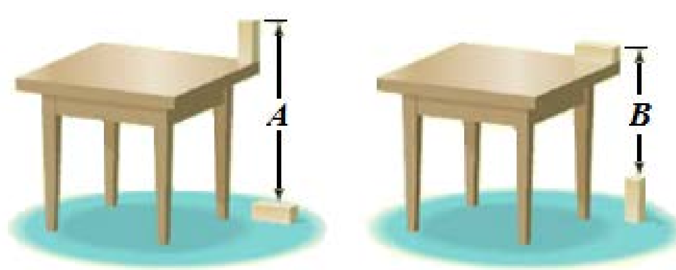
$$z=\frac{35}{7} \underline{=5|} \qquad z=\frac{D_z}{D}$$

∴ ***Solution***: The three numbers are: 7, 4, *and* 5

Exercise

Two blocks of wood having the same length and width are placed on the top and bottom of a table. Length *A* measure 32 *cm*. The blocks are rearranged. Length *B* measures 28 *cm*. Determine the height of the table.

Solution



Let *h*: height of the table.

l: length of the block

w: width of the block

$$\begin{cases} (A) & h-w+l=32 \\ (B) & h-l+w=28 \end{cases}$$

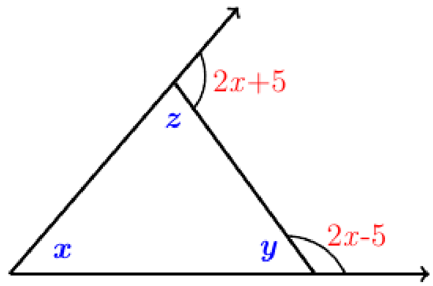
$$\overline{\quad 2h=60 \quad}$$

∴ ***Solution***: The height of the table is **30** *cm*

Exercise

In the following triangle, the degree measures of the three interior angles and two of the exterior angles are represented with variables. Find the measure of each interior angle.

Solution



$$\begin{cases} x + y + z = 180 \\ z + 2x + 5 = 180 \\ y + 2x - 5 = 180 \end{cases}$$

$$\begin{cases} x + y + z = 180 \\ 2x \quad + z = 175 \\ 2x + y \quad = 185 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 3$$

$$D_x = \begin{vmatrix} 180 & 1 & 1 \\ 175 & 0 & 1 \\ 185 & 1 & 0 \end{vmatrix} = 180$$

$$D_y = \begin{vmatrix} 1 & 180 & 1 \\ 2 & 175 & 1 \\ 2 & 185 & 0 \end{vmatrix} = 195$$

$$D = \begin{vmatrix} 1 & 1 & 180 \\ 2 & 0 & 175 \\ 2 & 1 & 185 \end{vmatrix} = 165$$

$$x = \frac{180}{3} = 60^\circ \qquad x = \frac{D_x}{D}$$

$$y = \frac{195}{3} = 65^\circ \qquad y = \frac{D_y}{D}$$

$$z = \frac{165}{3} = 55^\circ \qquad z = \frac{D_z}{D}$$

Exercise

Three painters (Beth, Bill, and Edie), working together, can paint the exterior of a home in 10 *hours*. Bill and Edie together have painted similar house in 15 *hours*. One day, all three worked on this same kind of house for 4 *hours*, after which Edie left. Beth and Bill required 8 more *hours* to finish. Assuming no gain or loss in efficiency, how long should it take each person to complete such a job alone?

Solution

Let x: Beth's time

y: Bill's time

z: Edie's time

Let $\frac{1}{x} = a$: Beth's part of the job done in 1 *hour*.

$\frac{1}{y} = b$: Bill's part of the job done in 1 *hour*.

$\frac{1}{z} = c$: Edie's part of the job done in 1 *hour*.

All completed 1 job in 10 *hours*: $10\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=1$

Bill and Edie 1 job in 15 *hours*: $15\left(\frac{1}{y}+\frac{1}{z}\right)=1$

All worked 1 job in 4 *hours* Beth and Bill required 8 *hours*: $4\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+8\left(\frac{1}{x}+\frac{1}{y}\right)=1$

$$\begin{cases} 10a+10b+10c=1 \\ 15b+15c=1 \\ 4a+4b+4c+8a+8b=1 \end{cases}$$

$$\begin{cases} 10a+10b+10c=1 \\ 15b+15c=1 \\ 12a+12b+4c=1 \end{cases}$$

$$D = \begin{vmatrix} 10 & 10 & 10 \\ 0 & 15 & 15 \\ 12 & 12 & 4 \end{vmatrix} = -1200$$

$$D_a = \begin{vmatrix} 1 & 10 & 10 \\ 1 & 15 & 15 \\ 1 & 12 & 4 \end{vmatrix} = -40$$

$$D_b = \begin{vmatrix} 10 & 1 & 10 \\ 0 & 1 & 15 \\ 12 & 1 & 4 \end{vmatrix} = -50$$

$$D_c = \begin{vmatrix} 10 & 10 & 1 \\ 0 & 15 & 1 \\ 12 & 12 & 1 \end{vmatrix} \underline{= -30|}$$

$$a = \frac{40}{1200} = \frac{1}{30} \quad \rightarrow \quad x = \frac{1}{a} \underline{= 30|}$$

$$b = \frac{50}{1200} = \frac{1}{24} \quad \rightarrow \quad y = \frac{1}{b} \underline{= 24|}$$

$$c = \frac{30}{1200} = \frac{1}{40} \quad \rightarrow \quad z = \frac{1}{c} \underline{= 40|}$$

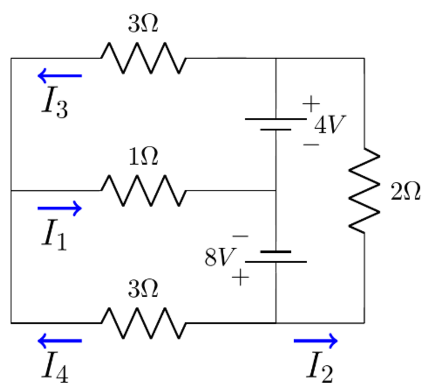
∴ ***Solution***: Took alone to complete a job: Beth **30** *hours*, Bill **24** *hours*, and Eddie **40** *hours*

Exercise

An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:

$$\begin{cases} I_1 = I_3 + I_4 \\ I_1 + 5I_4 = 8 \\ I_1 + 3I_3 = 4 \\ 8 - 4 - 2I_2 = 0 \end{cases}$$

Find the currents I_1, I_2, I_3, and I_4



Solution

$$\begin{cases} I_1 - I_3 - I_4 = 0 \\ I_1 + 5I_4 = 8 \\ I_1 + 3I_3 = 4 \\ \underline{I_2 = 2|} \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & -1 \\ 1 & 0 & 5 \\ 1 & 3 & 0 \end{vmatrix} \underline{= -23|}$$

$$D_1 = \begin{vmatrix} 0 & -1 & -1 \\ 8 & 0 & 5 \\ 4 & 3 & 0 \end{vmatrix} \underline{= -44|} \qquad D_3 = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 8 & 5 \\ 1 & 4 & 0 \end{vmatrix} \underline{= -16|} \qquad D_4 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & 8 \\ 1 & 3 & 4 \end{vmatrix} \underline{= -28|}$$

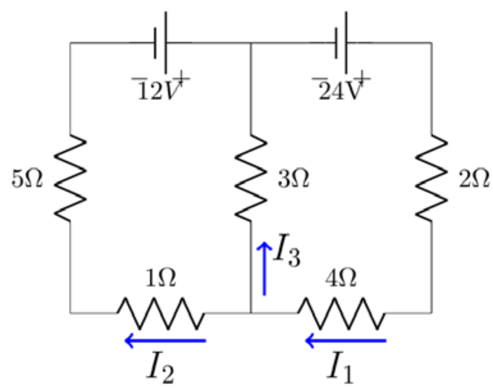
∴ ***Solution***: $\underline{I_1 = \frac{44}{23}|}$ $\underline{I_2 = 2|}$ $\underline{I_3 = \frac{16}{23}|}$ $\underline{I_4 = \frac{28}{23}|}$

Exercise

An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:

$$\begin{cases} I_1 = I_2 + I_3 \\ 24 - 6I_1 - 3I_3 = 0 \\ 12 + 24 - 6I_1 - 6I_2 = 0 \end{cases}$$

Find the currents I_1, I_2, and I_3



Solution

$$\begin{cases} I_1 - I_2 - I_3 = 0 \\ 2I_1 + I_3 = 8 \\ I_1 + I_2 = 6 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \underline{= -4|}$$

$$D_1 = \begin{vmatrix} 0 & -1 & -1 \\ 8 & 0 & 1 \\ 6 & 1 & 0 \end{vmatrix} \underline{= -14|}$$

$$D_2 = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 8 & 1 \\ 1 & 6 & 0 \end{vmatrix} \underline{= -10|}$$

$$D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 0 & 8 \\ 1 & 1 & 6 \end{vmatrix} \underline{= -4|}$$

∴ ***Solution***: $\underline{I_1 = \frac{7}{2}|}$

$\underline{I_2 = \frac{5}{2}|}$

$\underline{I_3 = 1|}$

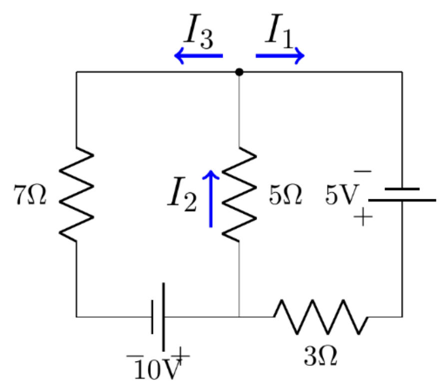
Exercise

An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:

$$\begin{cases} I_2 = I_1 + I_3 \\ 5 - 3I_1 - 5I_2 = 0 \\ 10 - 5I_2 - 7I_3 = 0 \end{cases}$$

Find the currents I_1, I_2, and I_3

Solution



$$\begin{cases} -I_1 + I_2 - I_3 = 0 \\ 3I_1 + 5I_2 = 5 \\ 5I_2 + 7I_3 = 10 \end{cases}$$

$$D = \begin{vmatrix} -1 & 1 & -1 \\ 3 & 5 & 0 \\ 0 & 5 & 7 \end{vmatrix} = -71$$

$$D_1 = \begin{vmatrix} 0 & 1 & -1 \\ 5 & 5 & 0 \\ 10 & 5 & 7 \end{vmatrix} = -10$$

$$D_2 = \begin{vmatrix} -1 & 0 & -1 \\ 3 & 5 & 0 \\ 0 & 10 & 7 \end{vmatrix} = -65$$

$$D_3 = \begin{vmatrix} -1 & 1 & 0 \\ 3 & 5 & 5 \\ 0 & 5 & 10 \end{vmatrix} = -55$$

∴ ***Solution***: $I_1 = \frac{10}{71}$

$I_2 = \frac{65}{71}$

$I_3 = \frac{55}{71}$

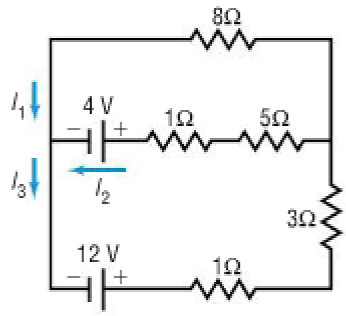
Exercise

An application of Kirchhoff's Rules to the circuit shown results in the following system of equations:

$$\begin{cases} I_3 = I_1 + I_2 \\ 6I_2 + 4I_3 = 8 \\ 8I_1 = 4 + 6I_2 \end{cases}$$

Find the currents I_1, I_2, and I_3

Solution



$$\begin{cases} I_1 + I_2 - I_3 = 0 \\ 3I_2 + 2I_3 = 4 \\ 4I_1 - 3I_2 = 2 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 3 & 2 \\ 4 & -3 & 0 \end{vmatrix} = 26$$

$$D_1 = \begin{vmatrix} 0 & 1 & -1 \\ 4 & 3 & 2 \\ 2 & -3 & 0 \end{vmatrix} = 22$$

$$D_2 = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 4 & 2 \\ 4 & 2 & 0 \end{vmatrix} = 12$$

$$D_3 = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 3 & 4 \\ 4 & -3 & 2 \end{vmatrix} = 34$$

∴ ***Solution***: $I_1 = \frac{22}{26} = \frac{11}{13}$

$I_2 = \frac{12}{26} = \frac{6}{13}$

$I_3 = \frac{34}{26} = \frac{17}{13}$

Solution Section 4.5 – Partial Fraction Decomposition

Exercise

Write the partial fraction decomposition of each rational expression $\frac{4}{x(x-1)}$

Solution

$$\frac{4}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$4 = A(x-1) + Bx$$

$$4 = Ax - A + Bx$$

$$4 = (A+B)x - A$$

$$\begin{cases} A+B=0 \\ -A=4 \end{cases} \rightarrow \begin{cases} B=-A=4 \\ A=-4 \end{cases}$$

$$\boxed{\frac{4}{x(x-1)} = -\frac{4}{x} + \frac{4}{x-1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{3x}{(x+2)(x-1)}$

Solution

$$\frac{3x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$3x = A(x-1) + B(x+2)$$

$$3x = Ax - A + Bx + 2B$$

$$3x = (A+B)x - A + 2B$$

$$\begin{cases} A+B=3 \\ -A+2B=0 \end{cases}$$

$$3B=3$$

$$\underline{B=1} \rightarrow \underline{A=2}$$

$$\boxed{\frac{3x}{(x+2)(x-1)} = \frac{2}{x+2} + \frac{1}{x-1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{1}{x\left(x^2+1\right)}$

Solution

$$\frac{1}{x\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A\left(x^2+1\right) + x(Bx+C)$$

$$1 = Ax^2 + A + Bx^2 + Cx$$

$$1 = (A+B)x^2 + Cx + A$$

$$\begin{cases} A+B=0 & \rightarrow \quad \underline{B=-1} \\ \underline{C=0} \\ \underline{A=1} \end{cases}$$

$$\boxed{\frac{1}{x\left(x^2+1\right)} = \frac{1}{x} - \frac{x}{x^2+1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{1}{(x+1)\left(x^2+4\right)}$

Solution

$$\frac{1}{(x+1)\left(x^2+4\right)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$$

$$1 = A\left(x^2+4\right) + (x+1)(Bx+C)$$

$$1 = Ax^2 + 4A + Bx^2 + Cx + Bx + C$$

$$1 = (A+B)x^2 + (B+C)x + 4A + C$$

$$\begin{cases} A+B=0 \\ B+C=0 \\ 4A+C=1 \end{cases}$$

$$\rightarrow \begin{cases} A=-B & \rightarrow \quad \underline{A=\frac{1}{5}} \\ C=-B & \rightarrow \quad \underline{C=\frac{1}{5}} \\ -4B-B=1 & \rightarrow \quad \underline{B=-\frac{1}{5}} \end{cases}$$

$$\frac{1}{(x+1)\left(x^2+4\right)}=\frac{\frac{1}{5}}{x+1}+\frac{\frac{1}{5}x+\frac{1}{5}}{x^2+4}$$

$$=\frac{1}{5}\frac{1}{x+1}+\frac{1}{5}\frac{x+1}{x^2+4}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2}{(x-1)^2(x+1)^2}$

Solution

$$\frac{x^2}{(x-1)^2(x+1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}$$

$$x^2=A(x-1)(x+1)^2+B(x+1)^2+C(x-1)^2(x+1)+D(x-1)^2$$

$$=A(x-1)\left(x^2+2x+1\right)+B\left(x^2+2x+1\right)+C\left(x^2-2x+1\right)(x+1)+D\left(x^2-2x+1\right)$$

$$=Ax^3+2Ax^2+Ax-Ax^2-2Ax-A+Bx^2+2Bx+B$$
$$+Cx^3-2Cx^2+Cx+Cx^2-2Cx+C+Dx^2-2Dx+D$$

$$x^2=(A+C)x^3+(A+B-C+D)x^2+(-A+2B-C-2D)x-A+B+C+D$$

$$\begin{cases} A+C=0 & \rightarrow A=-C \\ A+B-C+D=1 \\ -A+2B-C-2D=0 \\ -A+B+C+D=0 \end{cases}$$

$$\begin{cases} B-2C+D=1 \\ B-D=0 \\ B+2C+D=0 \end{cases}$$

$$\Delta=\begin{vmatrix}1 & -2 & 1\\ 1 & 0 & -1\\ 1 & 2 & 1\end{vmatrix}=8 \qquad \Delta_B=\begin{vmatrix}1 & -2 & 1\\ 0 & 0 & -1\\ 0 & 2 & 1\end{vmatrix}=2$$

$$\Delta_C=\begin{vmatrix}1 & 1 & 1\\ 1 & 0 & -1\\ 1 & 0 & 1\end{vmatrix}=-2 \qquad \Delta_D=\begin{vmatrix}1 & -2 & 1\\ 1 & 0 & 0\\ 1 & 2 & 0\end{vmatrix}=2$$

$$B=\frac{1}{4}\quad C=-\frac{1}{4}\quad D=\frac{1}{4}\quad A=\frac{1}{4}$$

$$\boxed{\frac{x^2}{(x-1)^2(x+1)^2}=\frac{1}{4}\frac{1}{x-1}+\frac{1}{4}\frac{1}{(x-1)^2}-\frac{1}{4}\frac{1}{x+1}+\frac{1}{4}\frac{1}{(x+1)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x+1}{x^2(x-2)^2}$

Solution

$$\frac{x+1}{x^2(x-2)^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}+\frac{D}{(x-2)^2}$$

$$\begin{aligned}x+1&=Ax(x-2)^2+B(x-2)^2+Cx^2(x-2)+Dx^2\\&=Ax\left(x^2-4x+4\right)+B\left(x^2-4x+4\right)+Cx^3-2Cx^2+Dx^2\\&=Ax^3-4Ax^2+4Ax+Bx^2-4Bx+4B+Cx^3-2Cx^2+Dx^2\\&=(A+C)x^3+(-4A-B-2C+D)x^2+(4A-4B)x+4B\end{aligned}$$

$$\begin{cases}A+C=0\\-4A-B-2C+D=0\\4A-4B=1\\4B=1\end{cases}$$

$$\begin{cases}C=-\frac{1}{2}\\D=2+\frac{1}{4}-1=\frac{5}{4}\\A=\frac{1}{2}\\B=\frac{1}{4}\end{cases}$$

$$\boxed{\frac{x+1}{x^2(x-2)^2}=\frac{1}{2}\frac{1}{x}+\frac{1}{4}\frac{1}{x^2}-\frac{1}{2}\frac{1}{x-2}+\frac{5}{4}\frac{1}{(x-2)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x-3}{(x+2)(x+1)^2}$

Solution

$$\frac{x-3}{(x+2)(x+1)^2}=\frac{A}{x+2}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$x-3 = A(x+1)^2 + B(x+1)(x+2) + C(x+2)$$
$$= Ax^2 + 2Ax + A + B\left(x^2 + 3x + 2\right) + Cx + 2C$$
$$= (A+B)x^2 + (2A+3B+C)x + A + 2B + 2C$$

$$\begin{cases} A+B=0 \\ 2A+3B+C=1 \\ A+2B+2C=-3 \end{cases}$$

$$\begin{cases} A=-B \\ -2B+3B+C=1 \\ -B+2B+2C=-3 \end{cases}$$

$$\begin{cases} B+C=1 \\ B+2C=-3 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \qquad \Delta_B = \begin{vmatrix} 1 & 1 \\ -3 & 2 \end{vmatrix} = 5 \qquad \Delta_C = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -4$$

$$\boxed{C=-4, \quad B=5, \quad A=-5}$$

$$\boxed{\frac{x-3}{(x+2)(x+1)^2} = -\frac{5}{x+2} + \frac{5}{x+1} - \frac{4}{(x+1)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2+x}{(x+2)(x-1)^2}$

Solution

$$\frac{x^2+x}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$x^2 + x = A(x-1)^2 + B(x-1)(x+2) + C(x+2)$$
$$= Ax^2 - 2Ax + A + Bx^2 + Bx - 2B + Cx + 2C$$

$$\begin{array}{l} x^2 \\ x \\ x^0 \end{array} \begin{cases} A+B=1 \\ -2A+B+C=1 \\ A-2B+2C=0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ -2 & 1 & 1 \\ 1 & -2 & 2 \end{vmatrix} = 9 \qquad \Delta_A = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} = 2$$

$$\Delta_B = \begin{vmatrix} 1 & 1 & 0 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 7 \qquad \Delta_C = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 1 \\ 1 & -2 & 0 \end{vmatrix} = 6$$

$$\underline{A = \frac{2}{9} \quad B = \frac{7}{9} \quad C = \frac{2}{3}}$$

$$\frac{x^2+x}{(x+2)(x-1)^2} = \frac{\frac{2}{9}}{x+2} + \frac{\frac{7}{9}}{x-1} + \frac{\frac{2}{3}}{(x-1)^2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{10x^2+2x}{(x-1)^2(x^2+2)}$

Solution

$$\frac{10x^2+2x}{(x-1)^2(x^2+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2}$$

$$\begin{aligned} 10x^2+2x &= A(x-1)(x^2+2) + B(x^2+2) + (Cx+D)(x-1)^2 \\ &= Ax^3 + 2Ax - Ax^2 - 2A + Bx^2 + 2B + (Cx+D)(x^2-2x+1) \\ &= Ax^3 + 2Ax - Ax^2 - 2A + Bx^2 + 2B + Cx^3 - 2Cx^2 + Cx + Dx^2 - 2Dx + D \\ &= (A+C)x^3 + (B-2A-2C+D)x^2 + (2A+C-2D)x - 2A + 2B + D \end{aligned}$$

$$\begin{cases} A+C=0 & \to A=-C \\ B-2A-2C+D=10 \\ 2A+C-2D=2 \\ -2A+2B+D=0 \end{cases}$$

$$\begin{cases} B+D=10 \\ -C-2D=2 \\ 2B+2C+D=0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 2 & 2 & 1 \end{vmatrix} = 5 \qquad \Delta_B = \begin{vmatrix} 10 & 0 & 1 \\ 2 & -1 & -2 \\ 0 & 2 & 1 \end{vmatrix} = 34$$

$$\Delta_C = \begin{vmatrix} 1 & 10 & 1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{vmatrix} = -42 \qquad \Delta_D = \begin{vmatrix} 1 & 0 & 10 \\ 0 & -1 & 2 \\ 2 & 2 & 0 \end{vmatrix} = 16$$

$$\boxed{B=\frac{34}{5}\quad C=-\frac{42}{5}\quad D=\frac{16}{5}\quad A=\frac{42}{5}}$$

$$\boxed{\frac{10x^2+2x}{(x-1)^2\left(x^2+2\right)}=\frac{42}{5(x-1)}+\frac{34}{5(x-1)^2}+\frac{-42x+16}{5\left(x^2+2\right)}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2+2x+3}{(x+1)\left(x^2+2x+4\right)}$

Solution

$$\frac{x^2+2x+3}{(x+1)\left(x^2+2x+4\right)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+2x+4}$$

$$x^2+2x+3=A\left(x^2+2x+4\right)+(Bx+C)(x+1)$$

$$=Ax^2+2Ax+4A+Bx^2+Bx+Cx+C$$

$$=(A+B)x^2+(2A+B+C)x+4A+C$$

$$\begin{cases} A+B=1 \\ 2A+B+C=2 \\ 4A+C=3 \end{cases}$$

$$\Delta=\begin{vmatrix}1&1&0\\2&1&1\\4&0&1\end{vmatrix}=3 \qquad \Delta_A=\begin{vmatrix}1&1&0\\2&1&1\\3&0&1\end{vmatrix}=2$$

$$\Delta_B=\begin{vmatrix}1&1&0\\2&2&1\\4&3&1\end{vmatrix}=1 \qquad \Delta_C=\begin{vmatrix}1&1&1\\2&1&2\\4&0&3\end{vmatrix}=1$$

$$\boxed{A=\frac{2}{3}\quad B=\frac{1}{3}\quad C=\frac{1}{3}}$$

$$\boxed{\frac{x^2+2x+3}{(x+1)\left(x^2+2x+4\right)}=\frac{\frac{2}{3}}{x+1}+\frac{\frac{1}{3}x+\frac{1}{3}}{x^2+2x+4}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2-11x-18}{x\left(x^2+3x+3\right)}$

Solution

$$\frac{x^2-11x-18}{x\left(x^2+3x+3\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+3x+3}$$

$$x^2-11x-18=Ax^2+3Ax+3A+Bx^2+Cx$$

$$=(A+B)x^2+(3A+C)x+3A$$

$$\begin{cases} A+B=1 & \rightarrow \quad \underline{B=7} \\ 3A+C=-11 & \rightarrow \quad \underline{C=7} \\ 3A=-18 \quad \rightarrow \quad \underline{A=-6} \end{cases}$$

$$\boxed{\frac{x^2-11x-18}{x\left(x^2+3x+3\right)}=-\frac{6}{x}+\frac{7x+7}{x^2+3x+3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{1}{(2x+3)(4x-1)}$

Solution

$$\frac{1}{(2x+3)(4x-1)}=\frac{A}{2x+3}+\frac{B}{4x-1}$$

$$1=4Ax-A+2Bx+3B$$

$$1=(4A+2B)x-A+3B$$

$$\begin{cases} 4A+2B=0 \\ -A+3B=1 \end{cases}$$

$$\Delta=\begin{vmatrix} 4 & 2 \\ -1 & 3 \end{vmatrix}=14 \quad \Delta_A=\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}=-2 \quad \Delta_B=\begin{vmatrix} 4 & 0 \\ -1 & 1 \end{vmatrix}=4$$

$$\underline{A=-\frac{1}{7}} \qquad \underline{B=\frac{2}{7}}$$

$$\boxed{\frac{1}{(2x+3)(4x-1)}=-\frac{\frac{1}{7}}{2x+3}+\frac{\frac{2}{7}}{4x-1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2+2x+3}{\left(x^2+4\right)^2}$

Solution

$$\frac{x^2+2x+3}{\left(x^2+4\right)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{\left(x^2+4\right)^2}$$

$$x^2+2x+3=(Ax+B)\left(x^2+4\right)+Cx+D$$

$$=Ax^3+4Ax+Bx^2+4B+Cx+D$$

$$=Ax^3+Bx^2+(4A+C)x+4B+D$$

$$\begin{cases} \underline{A=0} \\ \underline{B=1} \\ 4A+C=2 \\ 4B+D=3 \end{cases} \rightarrow \begin{array}{l} \\ \\ \underline{C=2} \\ D=3-4B\ \underline{=-1} \end{array}$$

$$\boxed{\frac{x^2+2x+3}{\left(x^2+4\right)^2}=\frac{1}{x^2+4}+\frac{2x-1}{\left(x^2+4\right)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^3+1}{\left(x^2+16\right)^2}$

Solution

$$\frac{x^3+1}{\left(x^2+16\right)^2}=\frac{Ax+B}{x^2+16}+\frac{Cx+D}{\left(x^2+16\right)^2}$$

$$x^3+1=(Ax+B)\left(x^2+16\right)+Cx+D$$

$$=Ax^3+16Ax+Bx^2+16B+Cx+D$$

$$\begin{cases} x^3 & \underline{A=1} \\ x^2 & \underline{B=0} \\ x & 16A+C=0 \\ x^0 & 16B+D=1 \end{cases} \rightarrow \begin{array}{l} \underline{C=-16} \\ \underline{D=1} \end{array}$$

$$\boxed{\frac{x^3+1}{\left(x^2+16\right)^2}=\frac{x}{x^2+16}+\frac{-16x+1}{\left(x^2+16\right)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{7x+3}{x^3-2x^2-3x}$

Solution

$$\frac{7x+3}{x^3-2x^2-3x}=\frac{7x+3}{x(x+1)(x-3)}$$
$$=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-3}$$

$$7x+3=A(x+1)(x-3)+Bx(x-3)+Cx(x+1)$$
$$=Ax^2-2Ax-3A+Bx^2-3B+Cx^2+Cx$$
$$=(A+B+C)x^2+(C-2A)x-3A-3B$$

$$\begin{cases} A+B+C=0 \\ C-2A=7 \\ -3A-3B=3 \end{cases}$$

$$\Delta=\begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ -3 & -3 & 0 \end{vmatrix}=6 \qquad \Delta_A=\begin{vmatrix} 0 & 1 & 1 \\ 7 & 0 & 1 \\ 3 & -3 & 0 \end{vmatrix}=-18$$

$$\Delta_B=\begin{vmatrix} 1 & 0 & 1 \\ -2 & 7 & 1 \\ -3 & 3 & 0 \end{vmatrix}=12 \qquad \Delta_C=\begin{vmatrix} 1 & 1 & 0 \\ -2 & 0 & 7 \\ -3 & -3 & 3 \end{vmatrix}=6$$

$$\boxed{A=-3 \quad B=2 \quad C=1}$$

$$\boxed{\frac{7x+3}{x^3-2x^2-3x}=\frac{-3}{x}+\frac{2}{x+1}+\frac{1}{x-3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2}{x^3-4x^2+5x-2}$

Solution

$$\frac{x^2}{x^3-4x^2+5x-2}=\frac{x^2}{(x-2)(x-1)^2}$$

$$= \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$x^2 = A(x-1)^2 + B(x-2)(x-1) + C(x-2)$$

$$= Ax^2 - 2Ax + A + Bx^2 - 3Bx + 2B + Cx - 2C$$

$$= (A+B)x^2 + (-2A - 3B + C)x + A + 2B - 2C$$

$$\begin{cases} A + B = 1 \\ -2A - 3B + C = 0 \\ A + 2B - 2C = 0 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 0 \\ -2 & -3 & 1 \\ 1 & 2 & -2 \end{vmatrix} = 1 \qquad D_A = \begin{vmatrix} 1 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 2 & -2 \end{vmatrix} = 4$$

$$D_B = \begin{vmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & -2 \end{vmatrix} = -3 \qquad D_C = \begin{vmatrix} 1 & 1 & 1 \\ -2 & -3 & 0 \\ 1 & 2 & 0 \end{vmatrix} = -1$$

$$A = 4 \quad B = -3 \quad C = -1$$

$$\frac{x^2}{x^3 - 4x^2 + 5x - 2} = \frac{4}{x-2} - \frac{3}{x-1} - \frac{1}{(x-1)^2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^3}{\left(x^2+16\right)^3}$

Solution

$$\frac{x^3}{\left(x^2+16\right)^3} = \frac{Ax+B}{x^2+16} + \frac{Cx+D}{\left(x^2+16\right)^2} + \frac{Ex+F}{\left(x^2+16\right)^3}$$

$$x^3 = (Ax+B)\left(x^2+16\right)^2 + (Cx+D)\left(x^2+16\right) + Ex + F$$

$$= (Ax+B)\left(x^4 + 32x^2 + 256\right) + Cx^3 + 16Cx + Dx^2 + 16D + Ex + F$$

$$= Ax^5 + 32Ax^3 + 256Ax + Bx^4 + 32Bx^2 + 256B + Cx^3 + Dx^2 + (16C + E)x + 16D + F$$

$$= Ax^5 + Bx^4 + (32A + C)x^3 + (32B + D)x^2 + (256A + 16C + E)x + 256B + 16D + F$$

$x^5 \qquad \boxed{A = 0}$

$x^4 \qquad \boxed{B = 0}$

$x^3 \qquad 32A + C = 1 \quad \rightarrow \quad \boxed{C = 1}$

$x^2 \qquad 32B + D = 0 \quad \rightarrow \quad \boxed{D = 0}$

$x^1 \qquad 256A + 16C + E = 0 \quad \rightarrow \quad \boxed{E = -16}$

$x^0 \qquad 256B + 16D + F = 0 \quad \rightarrow \quad \boxed{F = 0}$

$$\boxed{\frac{x^3}{\left(x^2+16\right)^3} = \frac{x}{\left(x^2+16\right)^2} + \frac{-16x}{\left(x^2+16\right)^3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{4}{2x^2-5x-3}$

Solution

$$\frac{4}{2x^2-5x-3} = \frac{4}{(2x+1)(x-3)}$$

$$= \frac{A}{2x+1} + \frac{B}{x-3}$$

$$4 = Ax - 3A + 2Bx + B$$

$$= (A+2B)x - 3A + B$$

$$\begin{cases} A + 2B = 0 \\ -3A + B = 4 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} = 7 \qquad \Delta_A = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8 \qquad \Delta_B = \begin{vmatrix} 1 & 0 \\ -3 & 4 \end{vmatrix} = 4$$

$$\boxed{A = -\frac{8}{7} \quad B = \frac{4}{7}}$$

$$\boxed{\frac{4}{2x^2-5x-3} = \frac{-\frac{8}{7}}{2x+1} + \frac{\frac{4}{7}}{x-3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{2x+3}{x^4-9x^2}$

Solution

$$\frac{2x+3}{x^4-9x^2} = \frac{2x+3}{x^2(x-3)(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3} + \frac{D}{x+3}$$

$$2x+3 = Ax\left(x^2-9\right) + B\left(x^2-9\right) + Cx^2(x+3) + Dx^2(x-3)$$

$$= Ax^3 - 9Ax + Bx^2 - 9B + Cx^3 + 3Cx^2 + Dx^3 - 3Dx^2$$

$$= (A+C+D)x^3 + (B+3C-3D)x^2 - 9Ax - 9B$$

$$\begin{cases} A+C+D=0 & & C+D=\frac{2}{9} \\ B+3C-3D=0 & & 3C-3D=\frac{1}{3} \\ -9A=2 & \rightarrow \boxed{A=-\frac{2}{9}} & \\ -9B=3 & \rightarrow \boxed{B=-\frac{1}{3}} & \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 3 & -3 \end{vmatrix} = -6 \qquad \Delta_C = \begin{vmatrix} \frac{2}{9} & 1 \\ \frac{1}{3} & -3 \end{vmatrix} = -1 \qquad \Delta_D = \begin{vmatrix} 1 & \frac{2}{9} \\ 3 & \frac{1}{3} \end{vmatrix} = -\frac{1}{3}$$

$$\boxed{C=\frac{1}{6} \quad D=\frac{1}{18}}$$

$$\boxed{\frac{2x+3}{x^4-9x^2} = -\frac{2}{9}\frac{1}{x} - \frac{1}{3}\frac{1}{x^2} + \frac{1}{6}\frac{1}{x-3} + \frac{1}{18}\frac{1}{x+3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2+9}{x^4-2x^2-8}$

Solution

$$\frac{x^2+9}{x^4-2x^2-8} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+2}$$

$$x^2+9 = A(x+2)\left(x^2+2\right) + B(x-2)\left(x^2+2\right) + (Cx+D)\left(x^2-4\right)$$

$$= Ax^3 + 2Ax + 2Ax^2 + 4A + Bx^3 + 2Bx - 2Bx^2 - 4B + Cx^3 - 4Cx + Dx^2 - 4D$$

$$= (A+B+C)x^3 + (2A-2B+D)x^2 + (2A+2B-4C)x + 4A - 4B - 4D$$

$$\begin{cases} A+B+C=0 \\ 2A-2B+D=1 \\ 2A+2B-4C=0 \\ 4A-4B-4D=9 \end{cases}$$

$$\boxed{A = \frac{13}{24} \quad B = -\frac{13}{24} \quad C = 0 \quad D = -\frac{7}{6}}$$

$$\boxed{\frac{x^2+9}{x^4-2x^2-8} = \frac{\frac{13}{24}}{x-2} - \frac{\frac{13}{24}}{x+2} - \frac{\frac{7}{6}}{x^2+2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{y}{y^2-2y-3}$

Solution

$$\frac{y}{y^2-2y-3} = \frac{A}{y-3} + \frac{B}{y+1}$$

$$y = (A+B)y + A - 3B$$

$$\begin{cases} A+B=1 \\ A-3B=0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -4 \qquad \Delta_A = \begin{vmatrix} 1 & 1 \\ 0 & -3 \end{vmatrix} = -3 \qquad \Delta_B = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$\boxed{A = \frac{3}{4} \quad B = \frac{1}{4}}$$

$$\boxed{\frac{y}{y^2-2y-3} = \frac{\frac{3}{4}}{y-3} + \frac{\frac{1}{4}}{y+1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x+3}{2x^3-8x}$

Solution

$$\frac{x+3}{2x^3-8x} = \frac{1}{2}\frac{x+3}{x\left(x^2-4\right)}$$

$$= \frac{1}{2}\left(\frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}\right)$$

$$= \frac{1}{2}\frac{A(x+2)(x-2) + Bx(x-2) + Cx(x+2)}{x(x+2)(x-2)}$$

$$(A+B+C)x^2 + (2C-2B)x - 4A = x+3$$

$$\begin{cases} A+B+C=0 & B+C=\frac{3}{4} \\ 2C-2B=1 & -B+C=\frac{1}{2} \\ -4A=3 & \to A=-\frac{3}{4} \end{cases}$$

$$B=\frac{1}{8} \quad C=\frac{5}{8}$$

$$\frac{x+3}{2x^3-8x}=\frac{1}{2}\left(-\frac{\frac{3}{4}}{x}+\frac{\frac{1}{8}}{x+2}+\frac{\frac{5}{8}}{x-2}\right)$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{x^2}{(x-1)\left(x^2+2x+1\right)}$

Solution

$$\frac{x^2}{(x-1)\left(x^2+2x+1\right)}=\frac{x^2}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$x^2=A(x+1)^2+B(x-1)(x+1)+C(x-1)$$

$$=(A+B)x^2+(2A+C)x+A-B-C$$

$$\begin{cases} A+B=1 \\ 2A+C=0 \\ A-B-C=0 \end{cases}$$

$$\Delta=\begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix}=4 \qquad \Delta_A=\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{vmatrix}=1$$

$$\Delta_B=\begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & -1 \end{vmatrix}=3 \qquad \Delta_C=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 1 & -1 & 0 \end{vmatrix}=-2$$

$$A=\frac{1}{4} \quad B=\frac{3}{4} \quad C=-\frac{1}{2}$$

$$\frac{x^2}{(x-1)\left(x^2+2x+1\right)}=\frac{\frac{1}{4}}{x-1}+\frac{\frac{3}{4}}{x+1}-\frac{\frac{1}{2}}{(x+1)^2}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{3x^2+x+4}{x^3+x}$

Solution

$$\frac{3x^2+x+4}{x^3+x}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$$

$$=\frac{(A+B)x^2+Cx+A}{x\left(x^2+1\right)}$$

$$3x^2+x+4=(A+B)x^2+Cx+A$$

$$\begin{cases} A+B=3 & \to \ \underline{B=-1} \\ \underline{C=1} & \\ \underline{A=4} & \end{cases}$$

$$\boxed{\frac{3x^2+x+4}{x^3+x}=\frac{4}{x}+\frac{-x+1}{x^2+1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{8x^2+8x+2}{\left(4x^2+1\right)^2}$

Solution

$$\frac{8x^2+8x+2}{\left(4x^2+1\right)^2}=\frac{Ax+B}{4x^2+1}+\frac{Cx+D}{\left(4x^2+1\right)^2}$$

$$=\frac{(Ax+B)\left(4x^2+1\right)+Cx+D}{\left(4x^2+1\right)^2}$$

$$8x^2+8x+2=(Ax+B)\left(4x^2+1\right)+Cx+D$$

$$=4Ax^3+4Bx^2+(A+C)x+B+D$$

$$\begin{cases} \underline{A=0} & \\ 4B=8 & \to \ \underline{B=2} \\ A+C=8 & \to \ \underline{C=8} \\ B+D=2 & \to \ \underline{D=0} \end{cases}$$

$$\boxed{\frac{8x^2+8x+2}{\left(4x^2+1\right)^2}=\frac{2}{4x^2+1}+\frac{8x}{\left(4x^2+1\right)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{1}{x^2+2x}$

Solution

$$\frac{1}{x^2+2x}=\frac{A}{x}+\frac{B}{x+2}$$

$$1=Ax+2A+Bx$$

$$x \qquad 2A=1 \qquad \to A=\frac{1}{2}$$

$$x^0 \qquad A+B=0 \quad \to B=-\frac{1}{2}$$

$$\boxed{\frac{1}{x^2+2x}=\frac{1}{2}\frac{1}{x}-\frac{1}{2}\frac{1}{x+2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{2x+1}{x^2-7x+12}$

Solution

$$\frac{2x+1}{x^2-7x+12}=\frac{A}{x-4}+\frac{B}{x-3}$$

$$2x+1=Ax-3A+Bx-4B$$

$$x \qquad A+B=2$$

$$x^0 \qquad -3A-4B=1$$

$$A=\frac{\begin{vmatrix}2 & 1\\ 1 & -4\end{vmatrix}}{\begin{vmatrix}1 & 1\\ -3 & -4\end{vmatrix}}=\frac{-9}{-1}\ \underline{=9|} \qquad B=\frac{\begin{vmatrix}1 & 2\\ -3 & 1\end{vmatrix}}{-1}=\frac{7}{-1}\ \underline{=-7|}$$

$$\boxed{\frac{2x+1}{x^2-7x+12}=\frac{9}{x-4}-\frac{7}{x-3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2+x}{x^4-3x^2-4}$

Solution

$$\frac{x^2+x}{x^4-3x^2-4}=\frac{x^2+x}{\left(x^2-4\right)\left(x^2+1\right)}$$

$$=\frac{A}{x-2}+\frac{B}{x+2}+\frac{Cx+D}{x^2+1}$$

$$x^2+x=A(x+2)\left(x^2+1\right)+B(x-2)\left(x^2+1\right)+(Cx+D)\left(x^2-4\right)$$

$$=Ax^3+Ax+2Ax^2+2A+Bx^3+Bx-2Bx^2-2B+Cx^3-4Cx+Dx^2-4D$$

$$=(A+B+C)x^3+(2A-2B+D)x^2+(A+B-4C)x+2A-2B-4D$$

$$\begin{cases} x^3 & A+B+C=0 & (1)\\ x^2 & 2A-2B+D=1 & (2)\\ x & A+B-4C=1 & (3)\\ x^0 & 2A-2B-4D=0 & (4)\end{cases}$$

$(1)-(3)\rightarrow\quad 5C=-1\quad \underline{C=-\frac{1}{5}}$

$(2)-(4)\rightarrow\quad 5D=1\quad \underline{D=\frac{1}{5}}$

$$\begin{cases} A+B=\frac{1}{5}\\ 2A-2B=\frac{4}{5}\end{cases}\rightarrow\begin{cases} 2A+2B=\frac{2}{5}\\ 2A-2B=\frac{4}{5}\end{cases}$$

$4A=\frac{6}{5}\quad\rightarrow\quad \underline{A=\frac{3}{10}}$

$B=\frac{1}{5}-\frac{3}{10}\quad\rightarrow\quad \underline{B=-\frac{1}{10}}$

$$\frac{x^2+x}{x^4-3x^2-4}=\frac{3}{10}\frac{1}{x-2}-\frac{1}{10}\frac{1}{x+2}+\frac{1}{5}\frac{-x+1}{x^2+1}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{\left(\theta^2 + 1\right)^3}$

Solution

$$\frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{\left(\theta^2 + 1\right)^3} = \frac{A\theta + B}{\theta^2 + 1} + \frac{C\theta + D}{\left(\theta^2 + 1\right)^2} + \frac{E\theta + F}{\left(\theta^2 + 1\right)^3}$$

$$\begin{aligned}
\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1 &= (A\theta + B)\left(\theta^2 + 1\right)^2 + (C\theta + D)\left(\theta^2 + 1\right) + E\theta + F \\
&= (A\theta + B)\left(\theta^4 + 2\theta^2 + 1\right) + C\theta^3 + C\theta + D\theta^2 + D + E\theta + F \\
&= A\theta^5 + B\theta^4 + (2A + C)\theta^3 + (2B + D)\theta^2 + (A + C + E)\theta + B + D + F
\end{aligned}$$

$$\begin{cases} \boxed{A = 0} \\ \boxed{B = 1} \\ 2A + C = -4 \\ 2B + D = 2 \\ A + C + E = -3 \\ B + D + F = 1 \end{cases} \quad \rightarrow \boxed{C = -4} \quad \boxed{D = 0} \quad \boxed{E = 1} \quad \boxed{F = 0}$$

$$\frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{\left(\theta^2 + 1\right)^3} = \frac{1}{\theta^2 + 1} - 4\frac{\theta}{\left(\theta^2 + 1\right)^2} + \frac{\theta}{\left(\theta^2 + 1\right)^3}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{3x^2 + 7x - 2}{x^3 - x^2 - 2x}$

Solution

$$\frac{3x^2 + 7x - 2}{x^3 - x^2 - 2x} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{x - 2}$$

$$\begin{aligned}
3x^2 + 7x - 2 &= A(x + 1)(x - 2) + Bx(x - 2) + Cx(x + 1) \\
&= Ax^2 - Ax - 2A \\
&\quad\; Bx^2 - 2Bx \\
&\quad\; Cx^2 + Cx
\end{aligned}$$

$$\begin{cases} A+B+C=3 \\ -A-2B+C=7 \\ -2A=-2 & \rightarrow \underline{A=1|} \end{cases}$$

$$\begin{cases} B+C=2 \\ -2B+C=8 \end{cases} \rightarrow \underline{B=-2|} \quad \underline{C=4|}$$

$$\boxed{\frac{3x^2+7x-2}{x^3-x^2-2x} = \frac{1}{x} - \frac{2}{x+1} + \frac{4}{x-2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}$

Solution

$$\frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)} = \frac{A}{x-1} + \frac{B}{x-5} + \frac{C}{x+4}$$

$$3x^2+2x+5 = (A+B+C)x^2 + (-A+3B-6C)x - 20A - 4B + 5C$$

$$\begin{cases} x^2 & A+B+C=3 \\ x & -A+3B-6C=2 \\ x^0 & -20A-4B+5C=5 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 3 & -6 \\ -20 & -4 & 5 \end{vmatrix} = 180 \qquad D_A = \begin{vmatrix} 3 & 1 & 1 \\ 2 & 3 & -6 \\ 5 & -4 & 5 \end{vmatrix} = -90$$

$$D_B = \begin{vmatrix} 1 & 3 & 1 \\ -1 & 2 & -6 \\ -20 & 5 & 5 \end{vmatrix} = 450 \qquad D_C = \begin{vmatrix} 1 & 1 & 3 \\ -1 & 3 & 2 \\ -20 & -4 & 5 \end{vmatrix} = 180$$

$$\boxed{A=\frac{1}{2}, \quad B=\frac{5}{2}, \quad C=1}$$

$$\boxed{\frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)} = \frac{1}{2}\frac{1}{x-1} + \frac{5}{2}\frac{1}{x-5} + \frac{1}{x+4}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{5x^2-3x+2}{x^3-2x^2}$

Solution

$$\frac{5x^2-3x+2}{x^3-2x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}$$

$$5x^2-3x+2=Ax^2-2Ax+Bx-2B+Cx^2$$

$$\begin{cases} x^2 & A+C=5 & C=4 \\ x & -2A+B=-3 & A=1 \\ x^0 & -2B=2 & \rightarrow \underline{B=-1} \end{cases}$$

$$\boxed{\frac{5x^2-3x+2}{x^3-2x^2}=\frac{1}{x}-\frac{1}{x^2}+\frac{4}{x-2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}$

Solution

$$\frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}=\frac{A}{x-2}+\frac{Bx+C}{x^2-2x+3}$$

$$7x^2-13x+13=Ax^2-2Ax+3A+Bx^2-2Bx+Cx-2C$$

$$\begin{cases} x^2 & A+B=7 \\ x^1 & -2A-2B+C=-13 \\ x^0 & 3A-2C=13 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 0 \\ -2 & -2 & 1 \\ 3 & 0 & -2 \end{vmatrix}=3 \qquad D_A=\begin{vmatrix} 7 & 1 & 0 \\ -13 & -2 & 1 \\ 13 & 0 & -2 \end{vmatrix}=15$$

$$D_B=\begin{vmatrix} 1 & 7 & 0 \\ -2 & -13 & 1 \\ 3 & 13 & -2 \end{vmatrix}=6 \qquad D_C=\begin{vmatrix} 1 & 1 & 7 \\ -2 & -2 & -13 \\ 3 & 0 & 13 \end{vmatrix}=3$$

$$\boxed{A=5;\ \ B=2;\ \ C=1}$$

$$\frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}=\frac{5}{x-2}+\frac{2x+1}{x^2-2x+3}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{1}{x^2-5x+6}$

Solution

$$\frac{1}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3}$$

$$Ax-3A+Bx-2B=1$$

$$\rightarrow \begin{cases} A+B=0 \\ -3A-2B=1 \end{cases} \rightarrow \quad A=-1 \quad B=1$$

$$\frac{1}{x^2-5x+6}=\frac{-1}{x-2}+\frac{1}{x-3}$$

Exercise

Write the partial fraction decomposition of each rational expression $\frac{1}{x^2-5x+5}$

Solution

$$\frac{1}{x^2-5x+5}=\frac{A}{x-\frac{5+\sqrt{5}}{2}}+\frac{B}{x-\frac{5-\sqrt{5}}{2}}$$

$$Ax-\left(\frac{5-\sqrt{5}}{2}\right)A+Bx-\left(\frac{5+\sqrt{5}}{2}\right)B=1$$

$$\begin{cases} A+B=0 \\ -\frac{5-\sqrt{5}}{2}A-\frac{5+\sqrt{5}}{2}B=1 \end{cases} \rightarrow \begin{matrix} \frac{5-\sqrt{5}}{2}A+\frac{5-\sqrt{5}}{2}B=0 \\ -\frac{5-\sqrt{5}}{2}A-\frac{5+\sqrt{5}}{2}B=1 \end{matrix}$$

$$-\sqrt{5}B=1 \rightarrow B=-\frac{1}{\sqrt{5}} \quad \Rightarrow \quad A=\frac{1}{\sqrt{5}}$$

$$\frac{1}{x^2-5x+5}=\frac{\sqrt{5}}{5}\frac{2}{2x-5-\sqrt{5}}-\frac{\sqrt{5}}{5}\frac{2}{2x-5+\sqrt{5}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{5x^2+20x+6}{x^3+2x^2+x}$

Solution

$$\frac{5x^2+20x+6}{x^3+2x^2+x}=\frac{5x^2+20x+6}{x(x+1)^2}$$

$$=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$Ax^2+2Ax+A+Bx^2+Bx+Cx=5x^2+20x+6$$

$$\begin{cases} A+B=5 \\ 2A+B+C=20 \quad \rightarrow \quad \underline{B=-1|} \quad \underline{C=9|} \\ \underline{A=6|} \end{cases}$$

$$\boxed{\frac{5x^2+20x+6}{x^3+2x^2+x}=\frac{6}{x}-\frac{1}{x+1}+\frac{9}{(x+1)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}$

Solution

$$\frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}=\frac{2x^3-4x-8}{x(x-1)\left(x^2+4\right)}=\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+4}$$

$$Ax^3-Ax^2+4Ax-4A+Bx^3+4Bx+Cx^3-Cx^2+Dx^2-Dx=2x^3-4x-8$$

$$\begin{cases} x^3 & A+B+C=2 \\ x^2 & -A-C+D=0 \\ x^1 & 4A+4B-D=-4 \\ x^0 & -4A=-8 \end{cases} \rightarrow \begin{cases} B+C=0 \\ -C+D=2 \\ 4B-D=-12 \\ \underline{A=2|} \end{cases}$$

$$\Rightarrow \begin{cases} B+D=2 \\ 4B-D=-12 \end{cases}$$

$$\underline{A=2 \quad B=-2 \quad C=2 \quad D=4|}$$

$$\boxed{\frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}=\frac{2}{x}-\frac{2}{x-1}+\frac{2x}{x^2+4}+\frac{4}{x^2+4}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{8x^3+13x}{\left(x^2+2\right)^2}$

Solution

$$\frac{8x^3+13x}{\left(x^2+2\right)^2}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{\left(x^2+2\right)^2}$$

$$Ax^3+2Ax+Bx^2+2B+Cx+D=8x^3+13x$$

$$\begin{cases} x^3 & A=8 \\ x^2 & B=0 \\ x^1 & 2A+C=13 \\ x^0 & D=0 \end{cases} \rightarrow \boxed{C=-3}$$

$$\boxed{\frac{8x^3+13x}{\left(x^2+2\right)^2}=\frac{8x}{x^2+2}-\frac{3x}{\left(x^2+2\right)^2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{1}{x^2-9}$

Solution

$$\frac{1}{x^2-9}=\frac{A}{x-3}+\frac{B}{x+3}$$

$$Ax+3A+Bx-3B=1$$

$$\Rightarrow \begin{cases} A+B=0 \\ 3A-3B=1 \end{cases} \rightarrow \boxed{A=\frac{1}{6} \quad B=-\frac{1}{6}}$$

$$\boxed{\frac{1}{x^2-9}=\frac{1}{6}\frac{1}{x-3}-\frac{1}{6}\frac{1}{x+3}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{2}{9x^2-1}$

Solution

$$\frac{2}{9x^2-1}=\frac{A}{3x-1}+\frac{B}{3x+1}$$

$$3Ax+A+3Bx-B=2$$

$$\Rightarrow \begin{cases} 3A+3B=0 \\ A-B=2 \end{cases} \rightarrow \underline{A=1 \quad B=-1}$$

$$\boxed{\frac{2}{9x^2-1}=\frac{1}{3x-1}-\frac{1}{3x+1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{5}{x^2+3x-4}$

Solution

$$\frac{5}{x^2+3x-4}=\frac{A}{x-1}+\frac{B}{x+4}$$

$$Ax+4A+Bx-B=5$$

$$\Rightarrow \begin{cases} A+B=0 \\ 4A-B=5 \end{cases} \rightarrow \underline{A=1 \quad B=-1}$$

$$\boxed{\frac{5}{x^2+3x-4}=\frac{1}{x-1}-\frac{1}{x+4}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{3-x}{3x^2-2x-1}$

Solution

$$\frac{3-x}{3x^2-2x-1}=\frac{A}{x-1}+\frac{B}{3x+1}$$

$$3Ax+A+Bx-B=3-x$$

$$\Rightarrow \begin{cases} 3A+B=-1 \\ A-B=3 \end{cases} \rightarrow \underline{A=\frac{1}{2} \quad B=-\frac{5}{2}}$$

$$\boxed{\frac{3-x}{3x^2-2x-1}=\frac{1}{2}\frac{1}{x-1}-\frac{5}{2}\frac{1}{3x+1}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{x^2+12x+12}{x^3-4x}$

Solution

$$\frac{x^2+12x+12}{x^3-4x}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2}$$

$$Ax^2-4A+Bx^2+2Bx+Cx^2-2Cx=x^2+12x+12$$

$$\begin{cases} x^2 & A+B+C=1 \\ x^1 & 2B-2C=12 \\ x^0 & \quad -4A=12 \end{cases} \rightarrow A=-3 \quad B=5 \quad C=-1$$

$$\boxed{\frac{x^2+12x+12}{x^3-4x}=-\frac{3}{x}+\frac{5}{x-2}-\frac{1}{x+2}}$$

Exercise

Write the partial fraction decomposition of each rational expression $\dfrac{5x-2}{(x-2)^2}$

Solution

$$\frac{5x-2}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}$$

$$Ax-2A+B=5x-2$$

$$\Rightarrow \begin{cases} \boxed{A=5} \\ -2A+B=-2 \end{cases} \rightarrow \boxed{B=8}$$

$$\boxed{\frac{5x-2}{(x-2)^2}=\frac{5}{x-2}+\frac{8}{(x-2)^2}}$$

Solution *Section* 4.6 – Infinite Sequences and Summation Notation

Exercise

Find the first four terms and the eight term of the sequence: $\{12-3n\}$

Solution

$$a_n = 12-3n$$

$$a_1 = 12-3(1) \ \underline{= 9}$$

$$a_2 = 12-3(2) = 6$$

$$a_3 = 12-3(3) \ \underline{= 3}$$

$$a_4 = 12-3(4) \ \underline{= 0}$$

$$a_8 = 12-3(8) \ \underline{= -12}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\dfrac{3n-2}{n^2+1}\right\}$

Solution

$$a_n = \frac{3n-2}{n^2+1}$$

$$a_1 = \frac{3-2}{1^2+1} \ \underline{= \frac{1}{2}}$$

$$a_2 = \frac{3(2)-2}{2^2+1} \ \underline{= \frac{4}{5}}$$

$$a_3 = \frac{3(3)-2}{3^2+1} \ \underline{= \frac{7}{10}}$$

$$a_4 = \frac{3(4)-2}{4^2+1} \ \underline{= \frac{10}{17}}$$

$$a_8 = \frac{3(8)-2}{8^2+1} \ \underline{= \frac{22}{65}}$$

Exercise

Find the first four terms and the eight term of the sequence: $\{9\}$

Solution

$$\underline{a_1 = 9}$$

$$\underline{a_2 = 9}$$

$$\underline{a_3 = 9}$$

$$\underline{a_4 = 9}$$

$$\underline{a_8 = 9}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{(-1)^{n-1}\dfrac{n+7}{2n}\right\}$

Solution

$$a_1 = (-1)^{1-1}\frac{1+7}{2(1)} \underline{= 4}$$

$$a_2 = (-1)^{2-1}\frac{2+7}{2(2)} \underline{= -\frac{9}{4}}$$

$$a_3 = (-1)^{3-1}\frac{3+7}{2(3)} \underline{= \frac{5}{3}}$$

$$a_4 = (-1)^{4-1}\frac{4+7}{2(4)} \underline{= -\frac{11}{8}}$$

$$a_8 = (-1)^{8-1}\frac{8+7}{2(8)} \underline{= -\frac{15}{16}}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\dfrac{2^n}{n^2+2}\right\}$

Solution

$$a_1 = \frac{2^1}{1^2+2} \underline{= \frac{2}{3}}$$

$$a_2 = \frac{2^2}{2^2+2} \underline{= \frac{2}{3}}$$

$$a_3 = \frac{2^3}{3^2+2} = \boxed{\frac{8}{11}}$$

$$a_4 = \frac{2^4}{4^2+2} = \boxed{\frac{8}{9}}$$

$$a_8 = \frac{2^8}{8^2+2} = \boxed{\frac{128}{33}}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{(-1)^{n-1}\frac{n}{2n-1}\right\}$

Solution

$$a_1 = (-1)^0\frac{1}{2-1} = \boxed{1}$$

$$a_2 = (-1)^1\frac{2}{4-1} = \boxed{-\frac{2}{3}}$$

$$a_3 = (-1)^2\frac{3}{6-1} = \boxed{\frac{3}{5}}$$

$$a_4 = (-1)^3\frac{4}{8-1} = \boxed{-\frac{4}{7}}$$

$$a_8 = (-1)^7\frac{8}{16-1} = \boxed{-\frac{8}{15}}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\frac{2^n}{3^n+1}\right\}$

Solution

$$a_1 = \frac{2^1}{3^1+1} = \frac{2}{4} = \boxed{\frac{1}{2}}$$

$$a_2 = \frac{2^2}{3^2+1} = \frac{4}{10} = \boxed{\frac{2}{5}}$$

$$a_3 = \frac{2^3}{3^3+1} = \frac{8}{28} = \boxed{\frac{2}{7}}$$

$$a_4 = \frac{2^4}{3^4+1} = \frac{16}{82} = \boxed{\frac{8}{41}}$$

$$a_8 = \frac{2^8}{3^8+1} = \frac{256}{6562} = \boxed{\frac{128}{3281}}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\frac{n^2}{2^n}\right\}$

Solution

$$a_1 = \frac{1^2}{2^1} = \frac{1}{2}$$

$$a_2 = \frac{2^2}{2^2} = 1$$

$$a_3 = \frac{3^2}{2^3} = \frac{9}{8}$$

$$a_4 = \frac{4^2}{2^4} = \frac{16}{16} = 1$$

$$a_8 = \frac{8^2}{2^8} = \frac{64}{256} = \frac{1}{4}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\frac{n}{e^n}\right\}$

Solution

$$a_1 = \frac{1}{e^1} = \frac{1}{e}$$

$$a_2 = \frac{2}{e^2}$$

$$a_3 = \frac{3}{e^3}$$

$$a_4 = \frac{4}{e^4}$$

$$a_8 = \frac{8}{e^8}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{c_n\right\} = \left\{(-1)^{n+1} n^2\right\}$

Solution

$$c_1 = (-1)^2 1^2 \ \underline{= 1}$$

$$c_2 = (-1)^3 2^2 \ \underline{= -4}$$

$$c_3 = (-1)^4 3^2 \ \underline{= 9}$$

$$c_4 = (-1)^5 4^2 \ \underline{= -16}$$

$$c_8 = (-1)^9 8^2 \ \underline{= -64}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{c_n\right\} = \left\{\dfrac{(-1)^n}{(n+1)(n+2)}\right\}$

Solution

$$c_1 = \frac{(-1)^1}{2\cdot 3} \ \underline{= -\frac{1}{6}}$$

$$c_2 = \frac{(-1)^2}{3\cdot 4} \ \underline{= \frac{1}{12}}$$

$$c_3 = \frac{(-1)^3}{4\cdot 5} \ \underline{= -\frac{1}{20}}$$

$$c_4 = \frac{(-1)^4}{5\cdot 6} \ \underline{= \frac{1}{30}}$$

$$c_8 = \frac{(-1)^8}{9\cdot 10} \ \underline{= \frac{1}{90}}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{c_n\right\} = \left\{\left(\frac{4}{3}\right)^n\right\}$

Solution

$$c_1 = \left(\frac{4}{3}\right)^1 = \frac{4}{3}$$

$$c_2 = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$$

$$c_3 = \left(\frac{4}{3}\right)^3 = \frac{64}{27}$$

$$c_4 = \left(\frac{4}{3}\right)^4 = \frac{256}{81}$$

$$c_8 = \left(\frac{4}{3}\right)^8 = \frac{65,536}{6,561}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{b_n\right\} = \left\{\frac{3^n}{n}\right\}$

Solution

$$b_1 = \frac{3^1}{1} = 3$$

$$b_2 = \frac{3^2}{2} = \frac{9}{2}$$

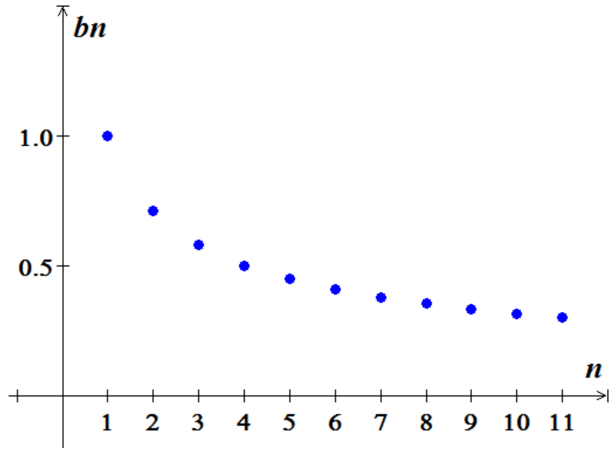
$$b_3 = \frac{3^3}{3} = 9$$

$$b_4 = \frac{3^4}{4} = \frac{81}{4}$$

$$c_8 = \frac{3^8}{8} = \frac{6,561}{8}$$

Exercise

Graph the sequence $\left\{\frac{1}{\sqrt{n}}\right\}$



Solution

$$\left\{\frac{1}{\sqrt{n}}\right\} = \left\{\frac{1}{\sqrt{1}},\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{4}},\ \cdots\right\}$$

$$\approx \{1,\ 0.71,\ 0.58,\ 0.5,\ 0.45\}$$

Exercise

Find the first four terms of the sequence of partial sums for the given sequence. $\left\{3+\frac{1}{2}n\right\}$

Solution

$$\begin{aligned} S_1 &= a_1 \\ &= 3+\tfrac{1}{2}(1) \\ &= \frac{7}{2} \end{aligned}$$

$$\begin{aligned} S_2 &= S_1 + a_2 \\ &= \frac{7}{2}+3+\tfrac{1}{2}(2) \\ &= \frac{15}{2} \end{aligned}$$

$$\begin{aligned} S_3 &= S_2 + a_3 \\ &= \frac{15}{2}+3+\tfrac{1}{2}(3) \\ &= 12 \end{aligned}$$

$$\begin{aligned} S_4 &= S_3 + a_4 \\ &= 12+3+\tfrac{1}{2}(4) \\ &= 17 \end{aligned}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_{k+1} = 3a_k - 5$

Solution

$$k = 1 \to \quad a_2 = 3a_1 - 5$$
$$= 3(2) - 5$$
$$\underline{= 1}$$

$$k = 2 \to \quad a_3 = 3a_2 - 5$$
$$= 3(1) - 5$$
$$\underline{= -2}$$

$$k = 3 \to \quad a_4 = 3a_3 - 5$$
$$= 3(-2) - 5$$
$$\underline{= -11}$$

$$k = 4 \to \quad a_5 = 3a_4 - 5$$
$$= 3(-11) - 5$$
$$\underline{= -38}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = -3, \quad a_{k+1} = a_k^2$

Solution

$$k = 1 \to \quad a_2 = a_1^2$$
$$= (-3)^2$$
$$\underline{= 9}$$

$$k = 2 \to \quad a_3 = a_2^2$$
$$= (9)^2$$
$$\underline{= 81}$$

$$k = 3 \to \quad a_4 = a_3^2$$
$$= \left(3^4\right)^2$$
$$\underline{= 3^8}$$

$k = 4 \rightarrow \quad a_5 = a_4^2$

$= \left(3^8\right)^2$

$\boxed{= 3^{16}}$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 5, \quad a_{k+1} = ka_k$

Solution

$k = 1 \rightarrow \quad a_2 = 1a_1$

$\boxed{= 5}$

$k = 2 \rightarrow \quad a_3 = 2a_2$

$= 2(5)$

$\boxed{= 10}$

$k = 3 \rightarrow \quad a_4 = 3a_3$

$= 3(10)$

$\boxed{= 30}$

$k = 4 \rightarrow \quad a_5 = 4a_4$

$= 4(30)$

$\boxed{= 120}$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_n = 3 + a_{n-1}$

Solution

$a_2 = 3 + a_1 = 3 + 2 \boxed{= 5}$

$a_3 = 3 + a_2 = 3 + 5 \boxed{= 8}$

$a_4 = 3 + a_3 = 3 + 8 \boxed{= 11}$

$a_5 = 3 + a_4 = 3 + 11 \boxed{= 14}$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 5, \quad a_n = 2a_{n-1}$

Solution

$$a_2 = 2a_1$$
$$= 2(5)$$
$$\underline{= 10}$$

$$a_3 = 2a_2$$
$$= 2(10)$$
$$\underline{= 20}$$

$$a_4 = 2a_3$$
$$= 2(20)$$
$$\underline{= 40}$$

$$a_5 = 2a_4$$
$$= 2(40)$$
$$\underline{= 80}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = \sqrt{2}, \quad a_n = \sqrt{2 + a_{n-1}}$

Solution

$$a_2 = \sqrt{2 + a_1}$$
$$\underline{= \sqrt{2 + \sqrt{2}}}$$

$$a_3 = \sqrt{2 + a_2}$$
$$\underline{= \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$

$$a_4 = \sqrt{2 + a_3}$$
$$\underline{= \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}$$

$$a_5 = \sqrt{2 + a_4}$$
$$\underline{= \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_{n+1} = 7 - 2a_n$

Solution

$$\begin{aligned} a_2 &= 7 - 2a_1 \\ &= 7 - 4 \\ &= 3 \end{aligned}$$

$$\begin{aligned} a_3 &= 7 - 2a_2 \\ &= 7 - 6 \\ &= 1 \end{aligned}$$

$$\begin{aligned} a_4 &= 7 - 2a_3 \\ &= 7 - 2 \\ &= 5 \end{aligned}$$

$$\begin{aligned} a_5 &= 7 - 2a_4 \\ &= 7 - 10 \\ &= -5 \end{aligned}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 128, \quad a_{n+1} = \frac{1}{4}a_n$

Solution

$$\begin{aligned} a_2 &= \frac{1}{4}a_1 \\ &= \frac{1}{4}128 \\ &= 32 \end{aligned}$$

$$\begin{aligned} a_3 &= \frac{1}{4}a_2 \\ &= \frac{32}{4} \\ &= 8 \end{aligned}$$

$$\begin{aligned} a_4 &= \frac{1}{4}a_3 \\ &= 2 \end{aligned}$$

$$\begin{aligned} a_5 &= \frac{1}{4}a_4 \\ &= \frac{1}{2} \end{aligned}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_{n+1} = \left(a_n\right)^n$

Solution

$$a_2 = \left(a_1\right)^1$$
$$\underline{= 2}$$

$$a_3 = \left(a_2\right)^2$$
$$= 2^2$$
$$\underline{= 4}$$

$$a_4 = \left(a_3\right)^3$$
$$= 4^3$$
$$\underline{= 64}$$

$$a_5 = \left(a_4\right)^4$$
$$\underline{= 64^4}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = A, \quad a_n = a_{n-1} + d$

Solution

$$a_2 = a_1 + d$$
$$\underline{= A + d}$$

$$a_3 = a_2 + d$$
$$= A + d + d$$
$$\underline{= A + 2d}$$

$$a_4 = a_3 + d$$
$$\underline{= A + 3d}$$

$$a_5 = a_4 + d$$
$$\underline{= A + 4d}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = A, \quad a_n = ra_{n-1}, \quad r \neq 0$

Solution

$$a_2 = ra_1$$
$$\boxed{= rA}$$

$$a_3 = ra_2$$
$$\boxed{= Ar^2}$$

$$a_4 = ra_3$$
$$\boxed{= Ar^3}$$

$$a_5 = ra_4$$
$$\boxed{= Ar^4}$$

Exercise

Find the first 5 terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_2 = 2; \quad a_n = a_{n-1} \bullet a_{n-2}$

Solution

$$a_3 = a_2 \cdot a_1$$
$$= 2 \cdot 2$$
$$\boxed{= 4}$$

$$a_4 = a_3 \cdot a_2$$
$$= 4 \cdot 2$$
$$\boxed{= 8}$$

$$a_5 = a_4 \cdot a_3$$
$$= 8 \cdot 4$$
$$\boxed{= 32}$$

$$a_6 = a_5 \cdot a_4$$
$$= 32 \cdot 8$$
$$\boxed{= 256}$$

Exercise

Express each sum using summation notation $1+2+3+\ldots+20$

Solution

$$1+2+3+4+\cdots+20=\sum_{k=1}^{20} k$$

Exercise

Express each sum using summation notation $1+2+3+\ldots+40$

Solution

$$1+2+3+\ldots+40=\sum_{k=1}^{40} k$$

Exercise

Express each sum using summation notation $1^3+2^3+3^3+\ldots+8^3$

Solution

$$1^3+2^3+3^3+\ldots+8^3=\sum_{k=1}^{8} k^3$$

Exercise

Express each sum using summation notation $1^2+2^2+3^2+\ldots+15^2$

Solution

$$1^2+2^2+3^2+\ldots+15^2=\sum_{k=1}^{15} k^2$$

Exercise

Express each sum using summation notation $2^2+2^3+2^4+\ldots+2^{11}$

Solution

$$2^2+2^3+2^4+\ldots+2^{11}=\sum_{k=2}^{11} 2^k$$

Exercise

Express each sum using summation notation $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{13}{14}$

Solution

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{13}{14}=\sum_{k=1}^{13}\frac{k}{k+1}$$

Exercise

Express each sum using summation notation $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\cdots+(-1)^6\frac{1}{3^6}$

Solution

$$1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\cdots+(-1)^6\frac{1}{3^6}=\sum_{k=0}^{6}(-1)^k\frac{1}{3^k}$$

Exercise

Express each sum using summation notation $\frac{2}{3}-\frac{4}{9}+\frac{8}{27}-\cdots+(-1)^{12}\left(\frac{2}{3}\right)^{11}$

Solution

$$\frac{2}{3}-\frac{4}{9}+\frac{8}{27}-\cdots+(-1)^{12}\left(\frac{2}{3}\right)^{11}=\sum_{k=1}^{11}(-1)^{k+1}\left(\frac{2}{3}\right)^k$$

Exercise

Express each sum using summation notation $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{14}{14+1}$

Solution

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{14}{14+1}=\sum_{k=1}^{14}\frac{k}{k+1}$$

Exercise

Express each sum using summation notation $\frac{1}{e}+\frac{2}{e^2}+\frac{3}{e^3}+\cdots+\frac{n}{e^n}$

Solution

$$\frac{1}{e}+\frac{2}{e^2}+\frac{3}{e^3}+\cdots+\frac{n}{e^n}=\sum_{k=1}^{n}\frac{k}{e^k}$$

Exercise

Find the sum: $\sum_{k=1}^{5}(2k-7)$

Solution

$$\sum_{k=1}^{5}(2k-7)=(-5)+(-3)+(-1)+1+3$$

$$\underline{=-5}$$

Exercise

Find the sum: $\sum_{k=0}^{5}k(k-2)$

Solution

$$\sum_{k=0}^{5}k(k-2)=0+(-1)+0+3+8+15$$

$$\underline{=25}$$

Exercise

Find the sum: $\sum_{k=1}^{5}(-3)^{k-1}$

Solution

$$\sum_{k=1}^{5}(-3)^{k-1}=1+(-3)+9+(-27)+81$$

$$\underline{=61}$$

Exercise

Find the sum: $\sum_{k=253}^{571}\left(\frac{1}{3}\right)$

Solution

$$\sum_{k=253}^{571}\left(\frac{1}{3}\right)=(571-253+1)\left(\frac{1}{3}\right)$$

$$\sum_{k=m}^{n} c=(n-m+1)c$$

$$=\frac{319}{3}$$

Exercise

Find the sum: $\sum_{k=1}^{50} 8$

Solution

$$\sum_{k=1}^{50} 8=(50-1+1)8$$

$$\sum_{k=m}^{n} c=(n-m+1)c$$

$$=400$$

Exercise

Find the sum: $\sum_{k=1}^{40} k$

Solution

$$\sum_{k=1}^{40} k=\frac{40(41)}{2}$$

$$\sum_{k=1}^{n} k=\frac{n(n+1)}{2}$$

$$=820$$

Exercise

Find the sum: $\sum_{k=1}^{5}(3k)$

Solution

$$\sum_{k=1}^{5} 3k=3(1)+3(2)+3(3)+3(4)+3(5)$$

$$=45$$

Exercise

Find the sum: $\sum_{k=1}^{10}\left(k^3+1\right)$

Solution

$$\sum_{k=1}^{n}k^3=\frac{n^2(n+1)^2}{4}$$

$$\begin{aligned}\sum_{k=1}^{10}\left(k^3+1\right)&=\sum_{k=1}^{10}k^3+\sum_{k=1}^{10}1\\&=\frac{10^2(10+1)^2}{4}+10(1)\\&=\frac{12100}{4}+10\\&=3025+10\\&=3035\end{aligned}$$

Exercise

Find the sum: $\sum_{k=1}^{24}\left(k^2-7k+2\right)$

Solution

$$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$\begin{aligned}\sum_{k=1}^{24}\left(k^2-7k+2\right)&=\frac{24(24+1)(2\cdot 24+1)}{6}-7\frac{24(24+1)}{2}+2(24)\\&=2848\end{aligned}$$

Exercise

Find the sum: $\sum_{k=6}^{20}\left(4k^2\right)$

Solution

$$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$\begin{aligned}\sum_{k=6}^{20}\left(4k^2\right)&=4\left(\sum_{k=1}^{20}k^2-\sum_{k=1}^{5}k^2\right)\\&=4\left(\frac{20(20+1)(2\cdot 20+1)}{6}-\frac{5(5+1)(2\cdot 5+1)}{6}\right)\end{aligned}$$

$$= 4\left(\frac{20(21)(41)}{6} - \frac{5(6)(11)}{6}\right)$$

$$= 4(2870 - 55)$$

$$\underline{= 11,260 \,|}$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{16}\left(k^2 - 4\right)$

Solution

$$\sum_{k=1}^{16}\left(k^2 - 4\right) = \sum_{k=1}^{16} k^2 - \sum_{k=1}^{16} 4$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{16(16+1)(2\cdot 16+1)}{6} - 4(16)$$

$$= 1496 - 64$$

$$\underline{= 1432 \,|}$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{6}(10 - 3k)$

Solution

$$\sum_{k=1}^{6}(10 - 3k) = 7 + 4 + 1 - 2 - 5 - 8$$

$$\underline{= -3 \,|}$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{10}\left[1 + (-1)^k\right]$

Solution

$$\sum_{k=1}^{10}\left[1+(-1)^k\right]=0+2+0+2+0+2+0+2+0+2$$

$$\underline{=10}$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{6}\frac{3}{k+1}$

Solution

$$\sum_{k=1}^{6}\frac{3}{k+1}=\frac{3}{2}+1+\frac{3}{4}+\frac{3}{5}+2+\frac{3}{7}$$

$$\underline{=\frac{879}{140}}$$

Exercise

Find the sum: $\displaystyle\sum_{k=137}^{428}2.1$

Solution

$$\sum_{k=137}^{428}2.1=(428-137+1)2.1=(292)2.1$$

$$\sum_{k=m}^{n}c=(n-m+1)c$$

$$\underline{=613.2}$$

Exercise

Write out each sum $\displaystyle\sum_{k=1}^{n}(k+2)$

Solution

$$\sum_{k=1}^{n}(k+2)\ \underline{=3+5+7+9+\cdots+(n+2)}$$

Exercise

Write out each sum $\displaystyle\sum_{k=1}^{n} k^2$

Solution

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2$$

$$\boxed{= 1 + 4 + 9 + 16 + \cdots + n^2}$$

Exercise

Write out each sum $\displaystyle\sum_{k=2}^{n} (-1)^k \ln k$

Solution

$$\sum_{k=2}^{n} (-1)^k \ln k = (-1)^2 \ln 2 + (-1)^3 \ln 3 + (-1)^4 \ln 4 + (-1)^5 \ln 5 + \cdots + (-1)^n \ln n$$

$$\boxed{= \ln 2 - \ln 3 + \ln 4 - \ln 5 + \cdots + (-1)^n \ln n}$$

Exercise

Write out each sum $\displaystyle\sum_{k=3}^{n} (-1)^{k+1} 2^k$

Solution

$$\sum_{k=3}^{n} (-1)^{k+1}\ 2^k = (-1)^4 2^3 + (-1)^5 2^4 + (-1)^6 2^5 + (-1)^7 2^6 + \cdots + (-1)^{n+1} 2^n$$

$$\boxed{= 8 - 16 + 32 - 64 + \cdots + (-1)^{n+1} 2^n}$$

Exercise

Write out each sum $\displaystyle\sum_{k=0}^{n} \frac{1}{3^k}$

Solution

$$\sum_{k=0}^{n} \frac{1}{3^k} \underline{= 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^n} \Big|}$$

Exercise

Fred has a balance of \$3,000 on his card which charges 1% interest per month on any unpaid balance. Fred can afford to pay \$100 toward the balance each *month*. His balance each month after making a \$100 payment is given by the recursively defined sequence

$$B_0 = \$3,000 \qquad B_n = 1.01B_{n-1} - 100$$

Determine Fred's balance after making the first payment. That is, determine B_1

Solution

$$B_1 = 1.01B_0 - 100$$
$$= 1.01(3,000) - 100$$
$$\underline{= \$2,930 \Big|}$$

Fred's balance is \$2,930 after making the first payment.

Exercise

A pond currently has 2,000 trout in it. A fish hatchery decides to add an additional 20 trout each month. Is it also known that the trout population is grwoing at a rate of 3% per *month*. The size of the population after *n* months is given bu the recursively defined sequence

$$P_0 = 2,000 \qquad P_n = 1.03P_{n-1} + 20$$

How many trout are in the pond after 2 months? That is, what is P_2?

Solution

$$P_1 = 1.03P_0 + 20$$
$$= 1.03(2,000) + 20$$
$$\underline{= 2,080 \Big|}$$

$$P_2 = 1.03P_1 + 20$$
$$= 1.03(2,080) + 20$$
$$\underline{= 2,162.4 \Big|}$$

There are approximately 2162 ***trout*** in the pond after 2 *months*.

Exercise

Fred bought a car by taking out a loan for $18,500 at 0.5% interest per month. Fred's normal monthly payment is $434.47 per month, but he decides that he can afford to pay $100 extra toward the balance each month. His balance each month is given by the recursively defined sequence

$$B_0 = \$18{,}500 \quad B_n = 1.005B_{n-1} - 534.47$$

Determine Fred's balance after making the first payment. That is, determine B_1

Solution

$$\begin{aligned} B_1 &= 1.005B_0 - 534.47 \\ &= 1.005(18{,}500) - 534.47 \\ &= \$18{,}058.03 \end{aligned}$$

Fred's balance is $18.058.03 after making the first payment.

Exercise

The Environmental Protection Agency (EPA) determines that Maple Lake has 250 *tons* of pollutant as a result of industrial waste and that 10% of the pollutant present is neuttralized by solar oxidation every year. The EPA imposes new pollution control laws that result in 15 *tons* of new pollutant entering the lake each year. The amount of pollutant in the lake after *n* years is given by the recursively defined sequence

$$P_0 = 250 \quad P_n = 0.9P_{n-1} + 15$$

Determine the amount of pollutant in the lake after 2 years? That is, what is P_2?

Solution

$$\begin{aligned} P_1 &= 0.9P_0 + 15 \\ &= 0.9(250) + 15 \\ &= 240 \end{aligned}$$

$$\begin{aligned} P_2 &= 0.9P_1 + 15 \\ &= 0.9(240) + 15 \\ &= 231 \end{aligned}$$

There are 231 ***tons*** of pollutants after 2 *years*.

Exercise

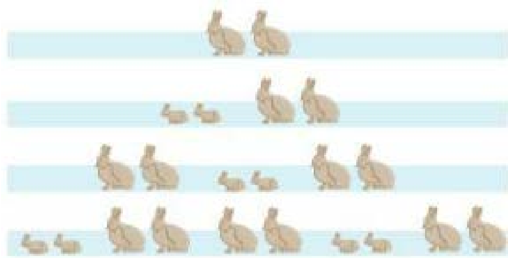
A colony of rabbits begins with one pair of mature rabbits, which will produce a pair of offspring (one male, one female) each month. Assume that all rabbits mature in 1 month and produce a pair of offspring (one male, one female) after 2 months. If no rabbits ever die, how many pairs of mature rabbits are there after 7 months?

1 mature pair

1 mature pair

2 mature pairs

3 mature pairs



Solution

$a_1 = 1$

$a_2 = 1$

$a_3 = 2$

$a_4 = 3$

$a_5 = 5$

$a_6 = 8$

$a_7 = 13$

$a_8 = 21$

$\vdots \quad \vdots$

$a_n = a_{n-1} + a_{n-2}$

After 7 *months* there are 21 *mature* pairs of rabbits.

Exercise

Let $\quad u_n = \dfrac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}$

Define the *n*th term of a sequence

a) Show that $u_1 = 1$ and $u_2 = 1$

b) Show that $u_{n+2} = u_{n+1} + u_n$

c) Draw the conclusion that $\{u_n\}$ is a Fibonacci sequence

d) Find the first ten terms of the sequence from part (*c*)

Solution

a) $$u_1=\frac{\left(1+\sqrt{5}\right)^1-\left(1-\sqrt{5}\right)^1}{2^1\sqrt{5}}$$

$$=\frac{2\sqrt{5}}{2\sqrt{5}}$$

$$\underline{=1}$$

$$u_2=\frac{\left(1+\sqrt{5}\right)^2-\left(1-\sqrt{5}\right)^2}{2^2\sqrt{5}} \qquad a^2-b^2=(a-b)(a+b)$$

$$=\frac{\left(1+\sqrt{5}-1+\sqrt{5}\right)-\left(1-\sqrt{5}+1-\sqrt{5}\right)}{2^2\sqrt{5}}$$

$$=\frac{4\sqrt{5}}{4\sqrt{5}}$$

$$\underline{=1}$$

b) $$u_{n+1}+u_n=\frac{\left(1+\sqrt{5}\right)^{n+1}-\left(1-\sqrt{5}\right)^{n+1}}{2^{n+1}\sqrt{5}}+\frac{\left(1+\sqrt{5}\right)^{n}-\left(1-\sqrt{5}\right)^{n}}{2^{n}\sqrt{5}}$$

$$=\frac{\left(1+\sqrt{5}\right)^{n+1}-\left(1-\sqrt{5}\right)^{n+1}+2\left(1+\sqrt{5}\right)^{n}-2\left(1-\sqrt{5}\right)^{n}}{2^{n+1}\sqrt{5}}$$

$$=\frac{\left(1+\sqrt{5}\right)^{n}\left(1+\sqrt{5}+2\right)-\left(1-\sqrt{5}\right)^{n}\left(1-\sqrt{5}+2\right)}{2^{n+1}\sqrt{5}}$$

$$=\frac{\left(1+\sqrt{5}\right)^{n}\left(3+\sqrt{5}\right)-\left(1-\sqrt{5}\right)^{n}\left(3-\sqrt{5}\right)}{2^{n+1}\sqrt{5}}$$

$$=\frac{\left(1+\sqrt{5}\right)^{n+2}\dfrac{3+\sqrt{5}}{\left(1+\sqrt{5}\right)^2}-\left(1-\sqrt{5}\right)^{n+2}\dfrac{3-\sqrt{5}}{\left(1-\sqrt{5}\right)^2}}{2^{n+1}\sqrt{5}}$$

$$=\frac{\left(1+\sqrt{5}\right)^{n+2}\dfrac{3+\sqrt{5}}{1+2\sqrt{5}+5}-\left(1-\sqrt{5}\right)^{n+2}\dfrac{3-\sqrt{5}}{1-2\sqrt{5}+5}}{2^{n+1}\sqrt{5}}$$

$$=\frac{\frac{1}{2}\left(1+\sqrt{5}\right)^{n+2}\dfrac{3+\sqrt{5}}{3+\sqrt{5}}-\frac{1}{2}\left(1-\sqrt{5}\right)^{n+2}\dfrac{3-\sqrt{5}}{3-\sqrt{5}}}{2^{n+1}\sqrt{5}}$$

$$= \frac{1}{2}\frac{\left(1+\sqrt{5}\right)^{n+2} - \left(1-\sqrt{5}\right)^{n+2}}{2^{n+1}\sqrt{5}}$$

$$= \frac{\left(1+\sqrt{5}\right)^{n+2} - \left(1-\sqrt{5}\right)^{n+2}}{2^{n+2}\sqrt{5}}$$

$$= u_{n+2}$$

c) Since $u_1 = 1$ and $u_2 = 1$ and $u_{n+2} = u_{n+1} + u_n$

$\therefore \left\{u_n\right\}$ is a Fibonacci sequence

e) $u_1 = 1$

$u_2 = 1$

$u_3 = u_2 + u_1 = 1+1 = 2$

$u_4 = u_3 + u_2 = 2+1 = 3$

$u_5 = u_4 + u_3 = 3+2 = 5$

$u_6 = u_5 + u_4 = 5+3 = 8$

$u_7 = u_6 + u_5 = 8+5 = 13$

$u_8 = u_7 + u_5 = 13+8 = 21$

$u_9 = u_8 + u_7 = 21+13 = 34$

$u_{10} = u_9 + u_8 = 34+21 = 55$

Solution Section 4.7 – Arithmetic and Geometric Sequences

Exercise

Show that the sequence $-6,\ -2,\ 2,\ \ldots,\ 4n-10,\ \ldots$ is arithmetic, and find the common difference.

Solution

We to show that $a_{k+1} - a_k$ equals to a constant.

$$\begin{aligned} a_{k+1} - a_k &= 4(k+1) - 10 - (4k-10) \\ &= 4k + 4 - 10 - 4k + 10 \\ &= 4 \end{aligned}$$

Exercise

Find the *n*th term, and the tenth term of the arithmetic sequence: $2,\ 6,\ 10,\ 14,\ \ldots$

Solution

$$\begin{aligned} d &= 6 - 2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} a_n &= 2 + (n-1)4 \\ &= 2 + 4n - 4 \\ &= 4n - 2 \end{aligned} \qquad a_n = a_1 + (n-1)d$$

$$\begin{aligned} a_{10} &= 4(10) - 2 \\ &= 38 \end{aligned}$$

Exercise

Find the *n*th term, and the tenth term of the arithmetic sequence: $3,\ 2.7,\ 2.4,\ 2.1,\ \ldots$

Solution

$$\begin{aligned} d &= 2.7 - 3 = -0.3 \\ &= -0.3 \end{aligned}$$

$$\begin{aligned} a_n &= 3 + (n-1)(-0.3) \\ &= 3 - 0.3n + 0.3 \\ &= 3.3 - 0.3n \end{aligned} \qquad a_n = a_1 + (n-1)d$$

$$\begin{aligned} a_{10} &= 3.3 - 0.3(10) \\ &= 0.3 \end{aligned}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $-6, \ -4.5, \ -3, \ -1.5, \ \ldots$

Solution

$$d = -4.5 - (-6)$$
$$= 1.5$$

$$a_n = -6 + (n-1)(1.5) \qquad\qquad a_n = a_1 + (n-1)d$$
$$= -6 + 1.5n - 1.5$$
$$= 1.5n - 7.5$$

$$a_{10} = 1.5(10) - 7.5$$
$$= 7.5$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $\ln 3, \ \ln 9, \ \ln 27, \ \ln 81, \ \ldots$

Solution

$$\ln 3, \ \ln 3^2, \ \ln 3^3, \ \ln 3^4, \ \ldots$$
$$\ln 3, \ 2\ln 3, \ 3\ln 3, \ 4\ln 3, \ \ldots$$

$$d = 2\ln 3 - \ln 3$$
$$= \ln 3$$

$$a_n = \ln 3 + (n-1)\ln 3 \qquad\qquad a_n = a_1 + (n-1)d$$
$$= \ln 3 + n\ln 3 - \ln 3$$
$$= n\ln 3$$

$$a_{10} = 10\ln 3$$
$$= \ln 3^{10}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = 2, \quad d = 3$

Solution

$$a_n = 2 + 3(n-1) \qquad\qquad a_n = a_1 + (n-1)d$$
$$= 2 + 3n - 3$$
$$= 3n - 1$$

$$a_{10} = 29$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = 5, \quad d = -3$

Solution

$$a_n = 5 + (n-1)(-3) \qquad\qquad a_n = a_1 + (n-1)d$$
$$= 5 - 3n + 3$$
$$\underline{= 8 - 3n}$$

$$a_{10} = 8 - 30$$
$$\underline{= -22}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = 1, \quad d = -\frac{1}{2}$

Solution

$$a_n = 1 + (n-1)\left(-\frac{1}{2}\right) \qquad\qquad a_n = a_1 + (n-1)d$$
$$= 1 - \frac{1}{2}n + \frac{1}{2}$$
$$\underline{= \frac{3}{2} - \frac{1}{2}n}$$

$$a_{10} = \frac{3}{2} - \frac{1}{2}(10)$$
$$\underline{= -\frac{7}{2}}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = -2, \quad d = 4$

Solution

$$a_n = -2 + (n-1)(4) \qquad\qquad a_n = a_1 + (n-1)d$$
$$= -2 + 4n - 4$$
$$\underline{= 4n - 6}$$

$$a_{10} = 4(10) - 6$$
$$\underline{= 36}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = \sqrt{2}, \quad d = \sqrt{2}$

Solution

$$a_n = \sqrt{2} + (n-1)\sqrt{2} \qquad\qquad a_n = a_1 + (n-1)d$$

$$= \sqrt{2} + \sqrt{2}n - \sqrt{2}$$

$$= \sqrt{2}\, n$$

$$a_{10} = 10\sqrt{2}$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = 0, \quad d = \pi$

Solution

$$a_n = 0 + (n-1)(\pi) \qquad\qquad a_n = a_1 + (n-1)d$$

$$= \pi n - \pi$$

$$a_{10} = 10\pi - \pi$$

$$= 9\pi$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = 13, \quad d = 4$

Solution

$$a_n = 13 + (n-1)(4) \qquad\qquad a_n = a_1 + (n-1)d$$

$$= 4n + 9$$

$$a_{10} = 4(10) + 9$$

$$= 49$$

Exercise

Find the nth term, and the tenth term of the arithmetic sequence: $a_1 = -40, \quad d = 5$

Solution

$$a_n = -40 + (n-1)(5) \qquad\qquad a_n = a_1 + (n-1)d$$

$$= 5n - 45$$

$$a_{10} = 4(10) - 45$$
$$\underline{= -5}$$

Exercise

Find the *n*th term, and the tenth term of the arithmetic sequence: $a_1 = -32, \quad d = 4$

Solution

$$a_n = -32 + (n-1)(4) \qquad\qquad a_n = a_1 + (n-1)d$$
$$\underline{= 4n - 36}$$
$$a_{10} = 4(10) - 36$$
$$\underline{= 4}$$

Exercise

Find the common difference for the arithmetic sequence with the specified terms: $a_4 = 14, \quad a_{11} = 35$

Solution

$$a_n = a_1 + (n-1)d$$
$$a_{11} = a_1 + 10d \quad \rightarrow \quad 35 = a_1 + 10d$$
$$a_4 = a_1 + 3d \quad \rightarrow \quad \underline{14 = a_1 + 3d}$$
$$21 = 7d$$
$$\underline{d = 3}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: $a_{12}; \quad a_1 = 9.1, \quad a_2 = 7.5$

Solution

$$d = a_2 - a_1$$
$$= 7.5 - 9.1$$
$$\underline{= -1.6}$$
$$a_n = a_1 + (n-1)d$$
$$a_{12} = 9.1 + (11)(-1.6)$$
$$\underline{= -8.5}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_1; $a_8 = 47$, $a_9 = 53$

Solution

$$d = a_9 - a_8$$
$$= 53 - 4$$
$$\underline{= 6}$$

$$a_8 = a_1 + (7)(6) \qquad\qquad a_n = a_1 + (n-1)d$$

$$a_1 = 47 - 42$$
$$\underline{= 5}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_{10}; $a_2 = 1$, $a_{18} = 49$

Solution

$$a_2 = a_1 + d$$
$$a_1 = a_2 - d$$

$$a_{18} = a_1 + (17)d$$
$$= a_2 - d + 17d$$
$$= a_2 + 16d$$

$$49 = 1 + 16d$$
$$16d = 48$$
$$d = \frac{48}{16} \ \underline{= 3}$$

$$a_1 = a_2 - d$$
$$= 1 - 3$$
$$\underline{= -2}$$

$$a_{10} = -2 + 9(3)$$
$$\underline{= 25}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_{10}; $a_8 = 8$, $a_{20} = 44$

Solution

$$d = \frac{44-8}{20-8} \qquad d = \frac{a_y - a_x}{y-x}$$

$$= \frac{36}{12}$$

$$\underline{= 3}$$

$$a_8 = a_1 + (8-1)(3) \qquad a_n = a_1 + (n-1)d$$

$$8 = a_1 + 21$$

$$\underline{a_1 = -13}$$

$$a_{10} = -13 + 9(3)$$

$$\underline{= 14}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_{12}; $a_8 = 4$, $a_{18} = -96$

Solution

$$d = \frac{-96-4}{18-8} \qquad d = \frac{a_y - a_x}{y-x}$$

$$= \frac{-100}{10}$$

$$\underline{= -10}$$

$$a_8 = a_1 + (8-1)(-10) \qquad a_n = a_1 + (n-1)d$$

$$4 = a_1 - 70$$

$$\underline{a_1 = 74}$$

$$a_{12} = 74 + (11)(-10)$$

$$\underline{= -36}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_8; $a_{15} = 0$, $a_{40} = -50$

Solution

$$d = \frac{-50-0}{40-15} \qquad d = \frac{a_y - a_x}{y-x}$$

$$= \frac{-50}{25}$$

$$\underline{= -2}$$

$$a_{15} = a_1 + (15-1)(-2) \qquad\qquad a_n = a_1 + (n-1)d$$

$$0 = a_1 - 28$$

$$\boxed{a_1 = 28}$$

$$a_8 = 28 + (7)(-2)$$

$$\boxed{= 14}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: $a_{20};\quad a_9 = -5,\quad a_{15} = 31$

Solution

$$d = \frac{31+5}{15-9} \qquad\qquad d = \frac{a_y - a_x}{y - x}$$

$$= \frac{36}{6}$$

$$\boxed{= 6}$$

$$a_9 = a_1 + (9-1)(6) \qquad\qquad a_n = a_1 + (n-1)d$$

$$-5 = a_1 + 42$$

$$\boxed{a_1 = -47}$$

$$a_{20} = -47 + (19)(6)$$

$$\boxed{= 67}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: $a_n;\quad a_8 = 8,\quad a_{20} = 44$

Solution

$$d = \frac{44-8}{20-8} \qquad\qquad d = \frac{a_y - a_x}{y - x}$$

$$= \frac{36}{12}$$

$$\boxed{= 3}$$

$$a_8 = a_1 + 3(8-1) \qquad\qquad a_n = a_1 + (n-1)d$$

$$8 = a_1 + 21$$

$$\boxed{a_1 = -13}$$

$$a_n = -13 + 3(n-1)$$

$$\underline{= 3n - 16}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_n; $a_8 = 4$, $a_{18} = -96$

Solution

$$d = \frac{-96-4}{18-8} \qquad d = \frac{a_y - a_x}{y - x}$$

$$\underline{= -10}$$

$$a_8 = a_1 - 10(8-1) \qquad a_n = a_1 + (n-1)d$$

$$4 = a_1 - 70$$

$$\underline{a_1 = 74}$$

$$a_n = 74 - 10(n-1)$$

$$\underline{= -10n + 84}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_n; $a_{14} = -1$, $a_{15} = 31$

Solution

$$d = \frac{31+1}{15-14} \qquad d = \frac{a_y - a_x}{y - x}$$

$$\underline{= 32}$$

$$a_{14} = a_1 + 32(14-1) \qquad a_n = a_1 + (n-1)d$$

$$-1 = a_1 + 416$$

$$\underline{a_1 = -417}$$

$$a_n = -417 + 32(n-1)$$

$$\underline{= 32n + 449}$$

Exercise

Find the specified term of the arithmetic sequence that has two given terms: a_n; $a_9 = -5$, $a_{15} = 31$

Solution

$d = \frac{31+5}{15-9}$ $\qquad d = \frac{a_y - a_x}{y - x}$

$= 6$

$a_9 = a_1 + 6(9-1)$ $\qquad a_n = a_1 + (n-1)d$

$-5 = a_1 + 48$

$a_1 = -53$

$a_n = -53 + 6(n-1)$

$= 6n - 59$

Exercise

Find the sum S_n of the arithmetic sequence that satisfies the conditions: $a_1 = 40$, $d = -3$, $n = 30$

Solution

$S_n = \frac{30}{2}\left[2(40) + (30-1)(-3)\right]$ $\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

$= -105$

Exercise

Find the sum S_n of the arithmetic sequence that satisfies the conditions: $a_7 = \frac{7}{3}$, $d = -\frac{2}{3}$, $n = 15$

Solution

$a_7 = a_1 + (6)\left(-\frac{2}{3}\right) = \frac{7}{3}$

$\frac{7}{3} = a_1 - 4$

$a_1 = \frac{7}{3} + 4$

$= \frac{19}{3}$

$S_n = \frac{15}{2}\left[2\left(\frac{19}{3}\right) + (15-1)\left(-\frac{2}{3}\right)\right]$ $\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

$= 25$

Exercise

Find the number of integers between 32 and 390 that are divisible by 6, find their sum

Solution

Number of terms: $n = \frac{390-36}{6} + 1 = 60$

$S_n = \frac{60}{2}(36+390)$ $\qquad\qquad\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$

$\underline{= 12780}$

Exercise

Find the number of terms in the arithmetic sequence with the given conditions:

$a_1 = -2, \quad d = \frac{1}{4}, \quad S = 21$

Solution

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$21 = \frac{n}{2}\left[2(-2) + (n-1)\frac{1}{4}\right]$$

$$21 = -2n + \frac{1}{8}n(n-1)$$

$$(8)21 = -2n(8) + \frac{1}{8}n(n-1)(8)$$

$$168 = -16n + \left(n^2 - n\right)$$

$$0 = n^2 - 17n - 168$$

$$\boxed{n = 24} \qquad n = -7$$

Exercise

Express the sum in terms of summation notation and find the sum $2 + 11 + 20 + \ldots + 16,058$.

Solution

Difference in terms:

$d = 11 - 2 = 9$

Number of terms:

$n = \frac{16058-2}{9} + 1 \underline{= 1785}$

$a_n = 2 + (n-1)(9)$ $\qquad\qquad\qquad a_n = a_1 + (n-1)d$

$= 2 + 9n - 9$

$\underline{= 9n - 7}$

Hence the nth term is: $\sum_{n=1}^{1785}(9n-7)$

$S_{1785} = \frac{1789}{2}(2+16058)$ $\qquad$ $S_n = \frac{n}{2}(a_1 + a_n)$

$\underline{= 14,333,550}$

Exercise

Express the sum in terms of summation notation and find the sum $60+64+68+72+\cdots+120$.

Solution

Difference in terms:

$d = 64 - 60 = 4$

Number of terms:

$n = \frac{120-60}{4}+1 \underline{=16}$ $\qquad$ $n = \frac{a_n - a_1}{d}+1$

$a_n = 60+(n-1)(4)$

$\underline{= 4n-54}$

Hence the nth term is: $\sum_{n=1}^{16}(4n-54)$

$S = \frac{16}{2}(60+120)$ $\qquad$ $S_n = \frac{n}{2}(a_1 + a_n)$

$\underline{= 1440}$

Exercise

Find each arithmetic sum $\quad 1+3+5+\cdots+(2n-1)$

Solution

Difference in terms:

$d = 3 - 1 = 2$

Number of terms:

$n = \frac{(2n-1)-1}{2}+1$ $\qquad$ $n = \frac{a_n - a_1}{d}+1$

$= \frac{2n-2}{2}+1$

$= n-1+1$

$\underline{= n}$

$$S = \frac{n}{2}\left(1 + (2n - 1)\right) \qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$= \frac{n}{2}(2n)$$

$$\underline{= n^2}$$

Exercise

Find each arithmetic sum $2 + 4 + 6 + \cdots + 2n$

Solution

Difference in terms: $d = 4 - 2 = 2$

Number of terms:

$$n = \frac{2n - 2}{2} + 1 \qquad n = \frac{a_n - a_1}{d} + 1$$

$$= n - 1 + 1$$

$$\underline{= n}$$

$$S = \frac{n}{2}(2 + 2n) \qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$= n(n + 1)$$

$$\underline{= n^2 + n}$$

Exercise

Find each arithmetic sum $2 + 5 + 8 + \cdots + 41$

Solution

Difference in terms:

$d = 5 - 2 = 3$

Number of terms:

$$n = \frac{41 - 2}{3} + 1 \qquad n = \frac{a_n - a_1}{d} + 1$$

$$\underline{= 14}$$

$$S = \frac{14}{2}(2 + 41) \qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$\underline{= 301}$$

Exercise

Find each arithmetic sum $7+12+17+\cdots+(2+5n)$

Solution

Difference in terms:

$d = 12 - 7 = 5$

Number of terms:

$$n = \frac{2+5n-7}{5}+1 \qquad n = \frac{a_n - a_1}{d}+1$$

$$= \frac{5n-5}{5}+1$$

$$= \frac{5n}{5} - \frac{5}{5}+1$$

$$\underline{= n}$$

$$S = \frac{n}{2}(7+2+5n) \qquad S_n = \frac{n}{2}(a_1 + a_n)$$

$$\underline{= \frac{n}{2}(9+5n)}$$

Exercise

Find each arithmetic sum $73+78+83+88+\cdots+558$

Solution

Difference in terms:

$d = 78 - 73 = 5$

Number of terms:

$$n = \frac{558-73}{5}+1 \qquad n = \frac{a_n - a_1}{d}+1$$

$$\underline{= 98}$$

$$S = \frac{98}{2}(73+558) \qquad S_n = \frac{n}{2}(a_1 + a_n)$$

$$\underline{= 30,919}$$

Exercise

Find each arithmetic sum $7+1-5-11-\cdots-299$

Solution

Difference in terms:

$d = 1 - 7 = -6$

Number of terms:

$$n=\frac{-299-7}{-6}+1 \qquad n=\frac{a_n-a_1}{d}+1$$

$$=52$$

$$S=\frac{52}{2}(7-299) \qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$=-7592$$

Exercise

Find each arithmetic sum $\quad -1+2+7+\cdots+(4n-5)$

Solution

$$S=\frac{n}{2}(-1+4n-5) \qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$=n(2n-3)$$

Exercise

Find each arithmetic sum $\quad 5+9+13+\cdots+49$

Solution

Difference in terms: $d=9-5=4$

Number of terms: $n=\frac{49-5+4}{4}=12 \qquad n=\frac{a_n-a_1+d}{d}$

$$S=\frac{12}{2}(5+49) \qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$=324$$

Exercise

Find each arithmetic sum $\quad 2+4+6+\cdots+70$

Solution

Difference in terms: $\quad d=4-2=2$

Number of terms:

$$n=\frac{70-2+2}{2} \qquad n=\frac{a_n-a_1+d}{d}$$

$$=35$$

$$S=\frac{35}{2}(70+2) \qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$=1,260$$

Exercise

Find each arithmetic sum $\quad 1+3+5+\cdots+59$

Solution

Difference in terms: $d=3-1=2$

Number of terms:

$$n=\frac{59-1+2}{2} \qquad\qquad n=\frac{a_n-a_1+d}{d}$$

$$=30$$

$$S=\frac{30}{2}(59+1) \qquad\qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$=900$$

Exercise

Find each arithmetic sum $\quad 4+4.5+5+5.5+\cdots+100$

Solution

Difference in terms: $d=4.5-4=0.5$

Number of terms:

$$n=\frac{100-4+0.5}{0.5} \qquad\qquad n=\frac{a_n-a_1+d}{d}$$

$$=193$$

$$S=\frac{193}{2}(4+100) \qquad\qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$=10,036$$

Exercise

Find each arithmetic sum $\quad 8+8\frac{1}{4}+8\frac{1}{2}+8\frac{3}{4}+9+\cdots+50$

Solution

Difference in terms: $d=8\frac{1}{4}-8=\frac{1}{4}$

Number of terms:

$$n=\frac{50-8+0.25}{0.25} \qquad\qquad n=\frac{a_n-a_1+d}{d}$$

$$=169$$

$$S=\frac{169}{2}(8+50) \qquad\qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$=4,901$$

Exercise

Show that the given sequence is geometric, and find the common ratio

$$5,\ -\frac{5}{4},\ \frac{5}{16},\ \dots,\ 5\left(-\frac{1}{4}\right)^{n-1},\ \dots$$

Solution

To be geometric, we must show that $\frac{a_{k+1}}{a_k}=r$ is equal to some constant, which is the common ratio.

The common ratio:

$$r=\frac{a_2}{a_1} \qquad\qquad r=\frac{a_{k+1}}{a_k}$$

$$=\frac{-\frac{5}{4}}{5}$$

$$\boxed{=-\frac{1}{4}}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $8,\ 4,\ 2,\ 1,\ \dots$

Solution

Given: $a_1=8,\quad r=\frac{4}{8}=\frac{1}{2}$

$$a_n=a_1r^{n-1}=8\left(\frac{1}{2}\right)^{n-1}$$

$$=2^3\left(2^{-1}\right)^{n-1}$$

$$=2^3 2^{-n+1}$$

$$\boxed{=2^{4-n}}$$

$$a_5=2^{4-5}$$

$$=2^{-1}$$

$$\boxed{=\frac{1}{2}}$$

$$a_8=2^{4-8}$$

$$=2^{-4}$$

$$\boxed{=\frac{1}{16}}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence

$$300,\ -30,\ 3,\ -0.3,\ \ldots$$

Solution

Given: $a_1 = 300, \quad r = \frac{-30}{300} = -0.1$

$a_n = a_1 r^{n-1} = 300(-0.1)^{n-1}$

$\quad = 3\left(10^2\right)\left(-10^{-1}\right)^{n-1} = 3(10)^2(-10)^{-n+1} = 3(-10)^{-n+3}$

$a_5 = 300(-0.1)^{5-1}$

$\quad = 300\left(-10^{-1}\right)^4$

$\quad \underline{= 0.03}$

$a_8 = 3(-10)^{-8+3}$

$\quad = 3(-10)^{-5}$

$\quad \underline{= -.00003}$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $1,\ -\sqrt{3},\ 3,\ -3\sqrt{3},\ \ldots$

Solution

Given: $a_1 = 1, \quad r = \frac{a_2}{a_1} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$

$a_n = 1\left(-\sqrt{3}\right)^{n-1}$ $\qquad a_n = a_1 r^{n-1}$

$\quad = a_1 r^{n-1}$

$a_5 = 1\left(-\sqrt{3}\right)^{5-1}$

$\quad \underline{= 9}$

$a_8 = 1\left(-\sqrt{3}\right)^{8-1}$

$\quad = \left(-\sqrt{3}\right)^7$

$\quad \underline{= -27\sqrt{3}}$

Exercise

Find the **n**th term, the *fifth* term, and the *eighth* term of the geometric sequence $4, \ -6, \ 9, \ -13.5, \ \ldots$

Solution

Given: $a_1 = 4, \quad r = \dfrac{a_2}{a_1} = \dfrac{-6}{4} = -\dfrac{3}{2}$

$a_n = 4\left(-\dfrac{3}{2}\right)^{n-1}$ $\qquad a_n = a_1 r^{n-1}$

$$a_5 = 4\left(-\dfrac{3}{2}\right)^{5-1}$$

$$= 4\left(-\dfrac{3}{2}\right)^{4}$$

$$= 4\left(\dfrac{3^4}{2^4}\right)$$

$$\boxed{= \dfrac{81}{4}}$$

$$a_8 = 4\left(-\dfrac{3}{2}\right)^{7}$$

$$= -4\left(\dfrac{3^7}{2^7}\right)$$

$$\boxed{= -\dfrac{2187}{32}}$$

Exercise

Find the **n**th term, the *fifth* term, and the *eighth* term of the geometric sequence $1, \ -x^2, \ x^4, \ -x^6, \ \ldots$

Solution

Given: $a_1 = 1, \quad r = \dfrac{a_2}{a_1} = \dfrac{-x^2}{1} = -x^2$

$a_n = \left(-x^2\right)^{n-1}$ $\qquad a_n = a_1 r^{n-1}$

$$a_5 = \left(-x^2\right)^{4}$$

$$\boxed{= x^8}$$

$$a_8 = \left(-x^2\right)^{7}$$

$$\boxed{= -x^{14}}$$

Exercise

Find the $\boldsymbol{n}$th term, the *fifth* term, and the *eighth* term of the geometric sequence

$$10,\ 10^{2x-1},\ 10^{4x-3},\ 10^{6x-5},\ \ldots$$

Solution

Given: $a_1 = 10$

$$r = \frac{a_2}{a_1} = \frac{10^{2x-1}}{10} = 10^{2x-1-1} = 10^{2x-2}$$

$$a_n = 10\left(10^{2x-2}\right)^{n-1} \qquad\qquad a_n = a_1 r^{n-1}$$

$$= 10\left(10^{(2x-2)(n-1)}\right)$$

$$= 10\left(10^{(2x-2)n-2x+2}\right)$$

$$= 10^{2nx-2n-2x+2+1}$$

$$\boxed{= 10^{2(n-1)x-2n+3}}$$

$$a_5 = 10^{2(5-1)x-2(5)+3}$$

$$\boxed{= 10^{8x-7}}$$

$$a_8 = 10^{2(8-1)x-2(8)+3}$$

$$\boxed{= 10^{14x-13}}$$

Exercise

Find the $\boldsymbol{n}$th term, the *fifth* term, and the *eighth* term of the geometric sequence $a_1 = 2, \quad r = 3$

Solution

Given: $a_1 = 10, \quad r = 3$

$$\boxed{a_n = 2 \cdot 3^{n-1}} \qquad\qquad a_n = a_1 r^{n-1}$$

$$a_5 = 2 \cdot 3^4$$

$$\boxed{= 162}$$

$$a_8 = 2 \cdot 3^7$$

$$\boxed{= 4374}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $a_1 = 1, \quad r = -\frac{1}{2}$

Solution

Given: $\quad a_1 = 1, \quad r = -\frac{1}{2}$

$$\boxed{a_n = \left(-\frac{1}{2}\right)^{n-1}} \qquad\qquad a_n = a_1 r^{n-1}$$

$$a_5 = \left(-\frac{1}{2}\right)^4$$

$$\boxed{= \frac{1}{16}}$$

$$a_8 = \left(-\frac{1}{2}\right)^7$$

$$\boxed{= -\frac{1}{128}}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $a_1 = -2, \quad r = 4$

Solution

Given: $\quad a_1 = -2, \quad r = 4$

$$\boxed{a_n = -2 \bullet (4)^{n-1}} \qquad\qquad a_n = a_1 r^{n-1}$$

$$a_5 = \left(-\frac{1}{2}\right)^4$$

$$\boxed{= \frac{1}{16}}$$

$$a_8 = \left(-\frac{1}{2}\right)^7$$

$$\boxed{= -\frac{1}{128}}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $a_1 = \sqrt{2}, \quad r = \sqrt{2}$

Solution

Given: $\quad a_1 = \sqrt{2}, \quad r = \sqrt{2}$

$$a_n = \sqrt{2}\left(\sqrt{2}\right)^{n-1} \qquad\qquad a_n = a_1 r^{n-1}$$

$$\boxed{= \left(\sqrt{2}\right)^n}$$

$$a_5 = \left(\sqrt{2}\right)^5$$

$$\boxed{= 4\sqrt{2}}$$

$$a_8 = \left(\sqrt{2}\right)^8$$

$$\boxed{= 16}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $a_1 = 0, \quad r = \pi$

Solution

Given: $a_1 = 0, \quad r = \pi$

$$a_n = 0(\pi)^{n-1} \qquad\qquad a_n = a_1 r^{n-1}$$

$$\boxed{= 0}$$

$$a_5 = 0^5$$

$$\boxed{= 0}$$

$$a_8 = 0^8$$

$$\boxed{= 0}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $\left\{s_n\right\} = \left\{3^n\right\}$

Solution

$$\boxed{a_n = 3^n}$$

$$\boxed{a_5 = 3^5}$$

$$\boxed{a_8 = 3^8}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $\left\{s_n\right\}=\left\{(-5)^n\right\}$

Solution

$$\boxed{a_n = 3^n}$$

$$a_5 = (-5)^5$$
$$\boxed{= -5^5}$$

$$a_8 = (-5)^8$$
$$\boxed{= 5^8}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $\left\{s_n\right\}=\left\{-3\left(\frac{1}{2}\right)^n\right\}$

Solution

$$\boxed{a_n = -3\left(\frac{1}{2}\right)^n}$$

$$a_5 = -3\left(\frac{1}{2}\right)^5$$
$$\boxed{= -\frac{3}{32}}$$

$$a_8 = -3\left(\frac{1}{2}\right)^8$$
$$\boxed{= -\frac{3}{256}}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $\left\{u_n\right\}=\left\{\frac{3^{n-1}}{2^n}\right\}$

Solution

$$\boxed{a_n = \frac{3^{n-1}}{2^n}}$$

$$a_5 = \frac{3^4}{2^5}$$

$$=\frac{81}{32}$$

$$a_8=\frac{3^7}{2^8}$$

$$=\frac{3^7}{256}$$

Exercise

Find the ***n***th term, the *fifth* term, and the *eighth* term of the geometric sequence $\left\{u_n\right\}=\left\{\frac{2^n}{3^{n-1}}\right\}$

Solution

$$a_n=\frac{2^n}{3^{n-1}}$$

$$a_5=\frac{2^5}{3^4}$$

$$=\frac{32}{81}$$

$$a_8=\frac{2^8}{3^7}$$

$$=\frac{256}{3^7}$$

Exercise

Find all possible values of ***r*** for a geometric sequence with the two given terms $a_4=3, \quad a_6=9$

Solution

$$\frac{a_6}{a_4}=\frac{a_1r^5}{a_1r^3}$$

$$\frac{9}{3}=r^2$$

$$r^2=3$$

$$r=\pm\sqrt{3}$$

Exercise

Find the *sixth* term of the geometric sequence whose first two terms are 4 and 6

Solution

Given: $a_1 = 4, \quad a_2 = 6$

$$r = \frac{a_2}{a_1}$$

$$= \frac{6}{4}$$

$$= \frac{3}{2}$$

$$a_6 = a_1 r^{n-1}$$

$$= 4\left(\frac{3}{2}\right)^5$$

$$= \frac{243}{8}$$

Exercise

Given a geometric sequence with $a_4 = 4, \quad a_7 = 12$, find r and a_{10}

Solution

$$r = \left(\frac{12}{4}\right)^{1/(7-4)} \qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$= 3^{1/3}$$

$$= \sqrt[3]{3}$$

$$a_1 = \frac{a_4}{r^3} \qquad a_4 = a_1 r^{n-1}$$

$$= \frac{4}{3}$$

$$a_{10} = \frac{4}{3}\left(\sqrt[3]{3}\right)^9 \qquad a_{10} = a_1 r^{n-1}$$

$$= 36$$

Exercise

Find the specified term of the geometric sequence $a_6;\ \ a_1 = 4,\ \ a_2 = 6$

Solution

$$r = \left(\frac{6}{4}\right)^{1/(2-1)} \qquad\qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$= \frac{3}{2}$$

$$a_6 = 4\left(\frac{3}{2}\right)^5 \qquad\qquad a_n = a_1 r^{n-1}$$

$$= \frac{3^5}{8}$$

Exercise

Find the specified term of the geometric sequence $a_7;\ \ a_2 = 3,\ \ a_3 = -\sqrt{3}$

Solution

$$r = \left(\frac{-\sqrt{3}}{3}\right)^{1/(3-2)} \qquad\qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$= -\frac{\sqrt{3}}{3}$$

$$a_2 = a_1\left(-\frac{\sqrt{3}}{3}\right)^1 \qquad\qquad a_n = a_1 r^{n-1}$$

$$3 = -\frac{\sqrt{3}}{3}a_1$$

$$a_1 = -\frac{9}{\sqrt{3}}$$

$$= -3\sqrt{3}$$

$$a_7 = -3\sqrt{3}\left(-\frac{\sqrt{3}}{3}\right)^6$$

$$= -3\sqrt{3}\,\frac{3^3}{3^6}$$

$$= -\frac{\sqrt{3}}{9}$$

Exercise

Find the specified term of the geometric sequence a_6; $a_2 = 3$, $a_3 = -\sqrt{2}$

Solution

$$r = \left(\frac{-\sqrt{2}}{3}\right)^{1/(3-2)} \qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$= -\frac{\sqrt{2}}{3}$$

$$a_2 = a_1\left(-\frac{\sqrt{2}}{3}\right)^1 \qquad a_n = a_1 r^{n-1}$$

$$3 = -\frac{\sqrt{2}}{3}a_1$$

$$a_1 = -\frac{9}{\sqrt{2}}$$

$$a_6 = -\frac{9}{\sqrt{2}}\left(-\frac{\sqrt{2}}{3}\right)^5$$

$$= 9\frac{\sqrt{2}^4}{3^5}$$

$$= \frac{4}{27}$$

Exercise

Find the specified term of the geometric sequence a_5; $a_1 = 4$, $a_2 = 7$

Solution

$$r = \frac{7}{4} \qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$a_5 = 4\left(\frac{7}{4}\right)^4 \qquad a_n = a_1 r^{n-1}$$

$$= \frac{7^4}{64}$$

Exercise

Find the specified term of the geometric sequence $a_9;\quad a_2 = 3,\quad a_5 = -81$

Solution

$$r = \left(\frac{-81}{3}\right)^{1/(5-2)} \qquad\qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$= (-27)^{1/3}$$

$$\underline{= -3}$$

$$a_2 = a_1(-3)^3 \qquad\qquad a_n = a_1 r^{n-1}$$

$$3 = -27a_1$$

$$\underline{a_1 = -\frac{1}{9}}$$

$$a_9 = -\frac{1}{9}(-3)^8$$

$$\underline{= -3^6}$$

Exercise

Find the specified term of the geometric sequence $a_7;\quad a_1 = -4,\quad a_3 = -1$

Solution

$$r = \left(\frac{-1}{-4}\right)^{1/(3-1)} \qquad\qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$= \left(\frac{1}{4}\right)^{1/2}$$

$$\underline{= \frac{1}{2}}$$

$$a_7 = -4\left(\frac{1}{2}\right)^6 \qquad\qquad a_n = a_1 r^{n-1}$$

$$\underline{= -\frac{1}{16}}$$

Exercise

Find the specified term of the geometric sequence $a_8;\quad a_2 = 3,\quad a_4 = 6$

Solution

$$r = \left(\frac{-81}{3}\right)^{1/(5-2)} \qquad\qquad r = \left(\frac{a_y}{a_x}\right)^{1/(y-x)}$$

$$= (-27)^{1/3}$$

$$\underline{= -3}$$

$$a_2 = a_1(-3)^3 \qquad\qquad a_n = a_1 r^{n-1}$$

$$3 = -81a_1$$

$$\underline{a_1 = -\frac{1}{9}}$$

$$a_8 = -\frac{1}{9}(-3)^7$$

$$\underline{= 3^5}$$

Exercise

Express the sum in terms of summation notation: $4+11+18+25+32$. (Answers are not unique)

Solution

$n = 5$

$d = 11 - 4 = 7$

$$a_n = 4 + (n-1)7 \qquad\qquad a_n = a_1 + (n-1)d$$

$$= 4 + 7n - 7$$

$$\underline{= 7n - 3}$$

$$4+11+18+25+32 = \sum_{n=1}^{5}(7n-3)$$

Exercise

Express the sum in terms of summation notation: $4+11+18+\ldots+466$. (Answers are not unique)

Solution

Difference in terms: $d = 11 - 4 = 7$

Number of terms: $n = \dfrac{466-4}{7} + 1 \underline{= 67|}$

$a_n = 4 + (n-1)7$ $\qquad\qquad$ $a_n = a_1 + (n-1)d$

$\quad = 4 + 7n - 7$

$\quad \underline{= 7n - 3|}$

$$4 + 11 + 18 + \ldots + 466 = \sum_{n=1}^{67} (7n-3)$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*) $2+4+8+16+32+64+128$

Solution

$$2+4+8+16+32+64+128 = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7$$

$$= \sum_{n=1}^{7} 2^n$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*) $2-4+8-16+32-64$

Solution

$r = \dfrac{-4}{2} = -2$ $\qquad\qquad$ $r = \dfrac{a_2}{a_1}$

$a_n = 2(-2)^{n-1}$ $\qquad\qquad$ $a_n = a_1 r^{n-1}$

$\quad = (-1)^{n-1} 2^n$

$$2-4+8-16+32-64 = \sum_{n=1}^{6} (-1)^{n-1} 2^n$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*) $3+8+13+18+23$

Solution

$d = 8-3$ $\qquad\qquad$ $d = a_2 - a_1$

$\quad \underline{= 5|}$

$a_n = 3 + 5(n-1)$ $\qquad\qquad$ $a_n = a_1 + (n-1)d$

$$=5n-2$$

$$3+8+13+18+23=\sum_{n=1}^{5}(5n-2)$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*) $256+192+144+108+\cdots$

Solution

$$r=\frac{192}{256} \qquad\qquad r=\frac{a_2}{a_1}$$

$$=\frac{3}{4}$$

$$a_n=256\left(\frac{3}{4}\right)^{n-1} \qquad\qquad a_n=a_1r^{n-1}$$

$$256+192+144+108+\cdots=\sum_{n=1}^{\infty}256\left(\frac{3}{4}\right)^{n-1}$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*): $\frac{5}{13}+\frac{10}{11}+\frac{15}{9}+\frac{20}{7}$

Solution

Number of terms: $n = 4$

Numerators : $5,10,15,20$ *common difference* 5

Denominators : $13,11,9,7$ *common difference* -2

Numerator:

$$a_n=5+(n-1)5 \qquad\qquad a_n=a_1+(n-1)d$$

$$=5+5n-5$$

$$=5n$$

Denominator:

$$a_n=13+(n-1)(-2)$$

$$=13-2n+2$$

$$=15-2n$$

Hence the nth term is: $\frac{5}{13}+\frac{10}{11}+\frac{15}{9}+\frac{20}{7}=\sum_{n=1}^{4}\frac{5n}{15-2n}$

Exercise

Express the sum in terms of summation notation (Answers are not unique.) $\frac{1}{4}-\frac{1}{12}+\frac{1}{36}-\frac{1}{108}$

Solution

$$\frac{1}{4}-\frac{1}{12}+\frac{1}{36}-\frac{1}{108}=\frac{1}{4}-\frac{1}{4}\frac{1}{3^1}+\frac{1}{4}\frac{1}{3^2}-\frac{1}{4}\frac{1}{3^3}$$

$$=\sum_{n=1}^{4}(-1)^{n+1}\frac{1}{4}\left(\frac{1}{3}\right)^{n-1}$$

Exercise

Express the sum in terms of summation notation (Answers are not unique.) $3+\frac{3}{5}+\frac{3}{25}+\frac{3}{125}+\frac{3}{625}$

Solution

$$3+\frac{3}{5}+\frac{3}{25}+\frac{3}{125}+\frac{3}{625}=\frac{3}{5^0}+\frac{3}{5^1}+\frac{3}{5^2}+\frac{3}{5^3}+\frac{3}{5^4}$$

$$=\sum_{n=0}^{4}\frac{3}{5^n}$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*): $\frac{3}{7}+\frac{6}{11}+\frac{9}{15}+\frac{12}{19}+\frac{15}{23}+\frac{18}{27}$

Solution

Numerators : 3, 6, 9, 12, 15,18 *common difference* 3

Denominators : 7, 11,15, 19, 13, 27 *common difference* 4

Numerator:

$$a_n=3+3(n-1) \qquad a_n=a_1+(n-1)d$$

$$\underline{=3n}$$

***Denominator*:**

$$a_n=7+4(n-1)$$

$$\underline{=4n+3}$$

$$\frac{3}{7}+\frac{6}{11}+\frac{9}{15}+\frac{12}{19}+\frac{15}{23}+\frac{18}{27}=\sum_{n=1}^{6}\frac{3n}{4n+3}$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*.) $\frac{x}{3}+\frac{x^2}{9}+\frac{x^3}{27}+\cdots, \quad |x|<3$

Solution

$$\frac{x}{3}+\frac{x^2}{9}+\frac{x^3}{27}+\cdots = \frac{x}{3}+\frac{x^2}{3^2}+\frac{x^3}{3^3}+\cdots$$

$$= \sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^n$$

Exercise

Express the sum in terms of summation notation (*Answers are not unique*.) $2x+4x^2+8x^3+\cdots, \quad |x|<\frac{1}{2}$

Solution

$$2x+4x^2+8x^3+\cdots = 2x+(2x)^2+(2x)^3+\cdots$$

$$= \sum_{n=1}^{\infty}(2x)^n$$

Exercise

Find the sum of the infinite geometric series if it exists: $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$

Solution

$$a_1=1, \quad r=-\frac{1}{2}$$

$$S=\frac{1}{1+\frac{1}{2}} \qquad\qquad S=\frac{a_1}{1-r}$$

$$=\frac{1}{\frac{3}{2}}$$

$$=\frac{2}{3}$$

Exercise

Find the sum of the infinite geometric series if it exists: $1.5+0.015+0.00015+\ldots$

Solution

$$a_1=0.015$$

$a_2 = .00015$

$r = \dfrac{.00015}{.015}$

$\underline{= .01}|$

$S = 1.5 + \dfrac{a_1}{1-r}$

$= 1.5 + \dfrac{.015}{1-.01}$

$= \dfrac{15}{10} + \dfrac{.015}{.99}$

$= \dfrac{15}{10} + \dfrac{15}{990}$

$= \dfrac{15}{10} + \dfrac{15}{990}$

$= \dfrac{1500}{990}$

$\underline{= \dfrac{50}{33}}|$

Exercise

Find the sum of the infinite geometric series if it exists: $\sqrt{2} - 2 + \sqrt{8} - 4 + \ldots$

Solution

$r = \dfrac{-2}{\sqrt{2}}$ $\qquad r = \dfrac{a_2}{a_1}$

$\underline{= -\sqrt{2}}|$

$|r| = \sqrt{2} > 1 \Rightarrow$ The sum ***doesn't exist***.

Exercise

Find the sum of the infinite geometric series if it exists: $256 + 192 + 144 + 108 + \ldots$

Solution

$r = \dfrac{192}{256}$ $\qquad r = \dfrac{a_2}{a_1}$

$\underline{= \dfrac{3}{4}}|$

$S = \dfrac{256}{1 - \frac{3}{4}}$ $\qquad S = \dfrac{a_1}{1-r}$

$\underline{= 1024}|$

Exercise

Find the sum of the infinite geometric series if it exists: $\frac{1}{4}+\frac{2}{4}+\frac{2^2}{4}+\frac{2^3}{4}+\ldots+\frac{2^{n-1}}{4}$

Solution

$$r=\frac{\frac{2}{4}}{\frac{1}{4}} \qquad\qquad r=\frac{a_2}{a_1}$$

$$=2$$

$$S_n=\frac{1}{4}\left(\frac{1-2^n}{1-2}\right) \qquad\qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$=-\frac{1}{4}\left(1-2^n\right)$$

Exercise

Find the sum of the infinite geometric series if it exists: $\frac{3}{9}+\frac{3^2}{9}+\frac{3^3}{9}+\ldots+\frac{3^n}{9}$

Solution

$$r=\frac{\frac{3^2}{9}}{\frac{3}{9}} \qquad\qquad r=\frac{a_2}{a_1}$$

$$=3$$

$$S_n=\frac{3}{9}\left(\frac{1-3^n}{1-3}\right) \qquad\qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$=-\frac{1}{6}\left(1-3^n\right)$$

Exercise

Find the sum of the infinite geometric series if it exists: $-1-2-4-8-\cdots-2^{n-1}$

Solution

$$r=\frac{-2}{-1} \qquad\qquad r=\frac{a_2}{a_1}$$

$$=2$$

$$S_n=-1\left(\frac{1-2^n}{1-2}\right) \qquad\qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$=1-2^n$$

Exercise

Find the sum of the infinite geometric series if it exists: $2+\frac{6}{5}+\frac{18}{25}+\ldots+2\left(\frac{3}{5}\right)^{n-1}$

Solution

$$r=\frac{\frac{6}{5}}{2} \qquad r=\frac{a_2}{a_1}$$

$$=\frac{3}{5}<1$$

$$S_n=2\cdot\frac{1-\left(\frac{3}{5}\right)^n}{1-\frac{3}{5}} \qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$=2\cdot\frac{1-\left(\frac{3}{5}\right)^n}{\frac{2}{5}}$$

$$=5\left(1-\left(\frac{3}{5}\right)^n\right)$$

Exercise

Find the sum of the infinite geometric series if it exists: $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$

Solution

$$r=\frac{1}{3}<1 \qquad r=\frac{a_2}{a_1}$$

$$S=\frac{1}{1-\frac{1}{3}} \qquad S=\frac{a_1}{1-r}$$

$$=\frac{3}{2}$$ The series ***converges***

Exercise

Find the sum of the infinite geometric series if it exists: $2+\frac{4}{3}+\frac{8}{9}+\frac{16}{27}+\cdots$

Solution

$$r=\frac{\frac{4}{3}}{2} \qquad r=\frac{a_2}{a_1}$$

$= \frac{2}{3} < 1$

$S = \dfrac{2}{1-\frac{2}{3}}$ $\qquad S = \dfrac{a_1}{1-r}$

$\boxed{= 6}$ The series ***converges***

Exercise

Find the sum of the infinite geometric series if it exists: $2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \cdots$

Solution

$a_1 = 2$

$|r| = \left|-\frac{1}{4}\right| < 1$ $\qquad r = \dfrac{a_2}{a_1}$

$S = \dfrac{2}{1+\frac{1}{4}}$ $\qquad S = \dfrac{a_1}{1-r}$

$\boxed{= \frac{8}{5}}$ The series ***converges***

Exercise

Find the sum of the infinite geometric series if it exists: $1 - \frac{3}{4} + \frac{9}{16} - \frac{27}{64} + \cdots$

Solution

$a_1 = 1$

$|r| = \left|-\frac{3}{4}\right| < 1$ $\qquad r = \dfrac{a_2}{a_1}$

$S = \dfrac{1}{1+\frac{3}{4}}$ $\qquad S = \dfrac{a_1}{1-r}$

$\boxed{= \frac{4}{7}}$ The series ***converges***

Exercise

Find the sum of the infinite geometric series if it exists: $9+12+16+\frac{64}{3}+\cdots$

Solution

$a_1 = 9$

$|r| = \left|\frac{4}{3}\right| > 1$ The series ***diverges***

Exercise

Find the sum of the infinite geometric series if it exists: $8+12+18+27+\cdots$

Solution

$a_1 = 8$

$r = \frac{12}{8}$ $\qquad r = \frac{a_2}{a_1}$

$= \frac{3}{2} > 1$ The series ***diverges***

Exercise

Find the sum of the infinite geometric series if it exists: $6+2+\frac{2}{3}+\frac{2}{9}+\cdots$

Solution

$a_1 = 6$

$|r| = \frac{2}{6}$ $\qquad r = \frac{a_2}{a_1}$

$= \frac{1}{3} < 1$

$S = \frac{6}{1-\frac{1}{3}}$ $\qquad S = \frac{a_1}{1-r}$

$= \frac{6}{\frac{2}{3}}$

$\underline{= 9}$ The series ***converges***

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{20}(3k-5)$

Solution

$a_1 = 3(1)-5 \underline{=-2|}$

$a_{20} = 3(20)-5 \underline{=55|}$

$$\sum_{k=1}^{20}(3k-5) = \frac{20}{2}(-2+55) \qquad\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$\underline{=530\ |}$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{18}\left(\frac{1}{2}k+7\right)$

Solution

$a_1 = \frac{1}{2}(1)+7 \underline{=\frac{15}{2}|}$

$a_{18} = \frac{1}{2}(18)+7 \underline{=16|}$

$$\sum_{k=1}^{18}\left(\frac{1}{2}k+7\right) = \frac{18}{2}\left(\frac{15}{2}+16\right) \qquad\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$\underline{=\frac{423}{2}\ |}$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{80}(2k-5)$

Solution

$a_1 = 2(1)-5 \underline{=-3|}$

$a_{80} = 2(80)-5 \underline{=155|}$

$$\sum_{k=1}^{80} (2k-5) = \frac{80}{2}(-3+155) \qquad\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$= 40(152)$$

$$= 6080$$

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{90} (3-2n)$

Solution

$$a_1 = 3-2(1) = 1$$

$$a_{90} = 3-2(90) = -177$$

$$\sum_{n=1}^{80} (3-2n) = \frac{90}{2}(1-177) \qquad\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$= 45(-176)$$

$$= -7920$$

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{100} \left(6-\frac{1}{2}n\right)$

Solution

$$a_1 = 6-\frac{1}{2}(1) = \frac{11}{2}$$

$$a_{100} = 6-\frac{1}{2}(100) = -44$$

$$\sum_{n=1}^{100} \left(6-\frac{1}{2}n\right) = \frac{100}{2}\left(\frac{11}{2}-44\right) \qquad\qquad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

$$= 50\left(-\frac{77}{2}\right)$$

$$= -1925$$

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{80}\left(\frac{1}{3}n+\frac{1}{2}\right)$

Solution

$$a_1=\frac{1}{3}(1)+\frac{1}{2}=\frac{5}{6}$$

$$a_{80}=\frac{1}{3}(80)+\frac{1}{2}=\frac{163}{6}$$

$$\sum_{n=1}^{80}\left(\frac{1}{3}n+\frac{1}{2}\right)=\frac{80}{2}\left(\frac{5}{6}+\frac{163}{6}\right) \qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$=40\left(\frac{168}{6}\right)$$

$$=1,120$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{10}3^k$

Solution

$$\sum_{k=1}^{10}3^k=3\frac{1-3^{10}}{1-3} \qquad S_n=a_1\frac{1-r^n}{1-r}$$

$$=3\frac{-59048}{-2}$$

$$=88,572$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{9}\left(-\sqrt{5}\right)^k$

Solution

$$\begin{cases} a_1=-\sqrt{5} \\ a_2=\left(-\sqrt{5}\right)^2=5 \end{cases}$$

$$r = \frac{5}{-\sqrt{5}} \qquad\qquad r = \frac{a_2}{a_1}$$

$$= -\sqrt{5}$$

$$\sum_{k=1}^{9} \left(-\sqrt{5}\right)^k = \left(-\sqrt{5}\right)\frac{1-\left(-\sqrt{5}\right)^9}{1-\left(-\sqrt{5}\right)} \qquad\qquad S_n = a_1\frac{1-r^n}{1-r}$$

$$= \frac{\left(-\sqrt{5}\right)\left(1+625\sqrt{5}\right)}{1+\sqrt{5}} \cdot \frac{1-\sqrt{5}}{1-\sqrt{5}}$$

$$= \frac{3124\sqrt{5}-3120}{-4}$$

$$= 780 - 781\sqrt{5}$$

Exercise

Find the sum: $\displaystyle\sum_{k=0}^{9} \left(-\frac{1}{2}\right)^{k+1}$

Solution

$$\sum_{k=0}^{9} \left(-\frac{1}{2}\right)^{k+1} = \left(-\frac{1}{2}\right)\frac{1-\left(-\frac{1}{2}\right)^{10}}{1+\frac{1}{2}} \qquad\qquad S_n = a_1\frac{1-r^n}{1-r}$$

$$= -\frac{1}{2}\frac{1-\frac{1}{2^{10}}}{\frac{3}{2}}$$

$$= -\frac{\frac{1024-1}{1024}}{3}$$

$$= -\frac{1023}{3072}$$

$$= -\frac{341}{1024}$$

Exercise

Find the sum : $\displaystyle\sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1}$

Solution

$$|r| = \frac{2}{3} \ < 1$$

$$\sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1} = \frac{2}{1-\frac{2}{3}} \qquad S = \frac{a_1}{1-r}$$

$$= \frac{2}{\frac{1}{3}}$$

$$\underline{= 6 \ |}, \qquad \text{the series } \textbf{\textit{converges}}$$

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n}$

Solution

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n} = \sum_{n=1}^{\infty} \frac{2}{3}\left(\frac{2}{3}\right)^{n-1}$$

$$|r| = \frac{2}{3} \ < 1$$

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n} = \frac{2}{3}\frac{1}{1-\frac{2}{3}} \qquad S = \frac{a_1}{1-r}$$

$$= \frac{2}{3}(3)$$

$$\underline{= 2 \ |} \qquad \text{The series } \textbf{\textit{converges}}$$

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{\infty} 3\left(\frac{3}{2}\right)^{n}$

Solution

Since $|r| = \frac{3}{2} > 1$, the series ***diverges***

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{\infty} 5\left(\frac{1}{4}\right)^{n-1}$

Solution

$$|r| = \frac{1}{4} < 1$$

$$a_1 = 5$$

$$\sum_{n=1}^{\infty} 5\left(\frac{1}{4}\right)^{n-1} = \frac{5}{1-\frac{1}{4}} \qquad S = \frac{a_1}{1-r}$$

$$= \frac{20}{3}$$

The series ***converges***

Exercise

Find the sum: $\displaystyle\sum_{n=1}^{\infty} 8\left(\frac{1}{3}\right)^{n-1}$

Solution

$$|r| = \frac{1}{3} < 1$$

$$a_1 = 8$$

$$\sum_{n=1}^{\infty} 8\left(\frac{1}{3}\right)^{n-1} = \frac{8}{1-\frac{1}{3}} \qquad S = \frac{a_1}{1-r}$$

$$= 12$$

The series ***converges***

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2}\cdot 3^{k-1}$

Solution

Since $|r| = 3 > 1$, the series ***diverges***

Exercise

Find the sum: $\sum_{k=1}^{\infty} 6\left(-\frac{2}{3}\right)^{k-1}$

Solution

$|r| = \frac{2}{3} \ < 1$

$a_1 = 6$

$$\sum_{k=1}^{\infty} 6\left(-\frac{2}{3}\right)^{k-1} = \frac{6}{1+\frac{2}{3}} \qquad S = \frac{a_1}{1-r}$$

$$= \frac{18}{5}$$

The series ***converges***

Exercise

Find the sum: $\sum_{k=1}^{\infty} 4\left(-\frac{1}{2}\right)^{k-1}$

Solution

$|r| = \frac{1}{2} \ < 1$

$a_1 = 4$

$$\sum_{k=1}^{\infty} 4\left(-\frac{1}{2}\right)^{k-1} = \frac{4}{1+\frac{1}{2}} \qquad S = \frac{a_1}{1-r}$$

$$= \frac{8}{3}$$

The series ***converges***

Exercise

Find the sum: $\sum_{k=8}^{14} \left(3^{k-7} + 2j^2\right)$

Solution

$a_n = 3^{n-7} \ \rightarrow \ a_1 = 3^{-6}$

$r = 3$

$n = 14 - 8 + 1 = 7$

$$\sum_{k=8}^{14}\left(3^{k-7}+2j^2\right)=\sum_{k=8}^{14}3^{k-7}+2\sum_{k=8}^{14}j^2$$

$$=3^{-6}\cdot\frac{1-3^7}{1-3}+2(7)j^2$$

$$=-\frac{1}{2}\left(\frac{1-3^7}{3^6}\right)+14j^2$$

$$=-\frac{1}{2}\left(\frac{-2,186}{729}\right)+14j^2$$

$$=\frac{1,093}{729}+14j^2$$

Exercise

Find the sum of the first 120 terms of: $\quad 14,\ 16,\ 18,\ 20,\ \cdots$

Solution

$n=120$

$a_1=14$

$d=16-14=2$

$$S_{120}=\frac{120}{2}\left[2(14)+2(120-1)\right] \qquad S_n=\frac{n}{2}\left[2a_1+(n-1)d\right]$$

$$=60(48+238)$$

$$=17,160$$

Exercise

Find the sum of the first 46 terms of $\quad 2,\ -1,\ -4,\ -7,\ \cdots$

Solution

$n=46$

$a_1=2$

$d=-1-2=-3$

$$S_{46}=\frac{46}{2}\left[2(2)-3(46-1)\right] \qquad S_n=\frac{n}{2}\left[2a_1+(n-1)d\right]$$

$$=23(4-135)$$

$$=-3,013$$

Exercise

Find the rational number represented by the repeating decimal $0.\overline{23}$

Solution

$$0.\overline{23} = 0.23 + 0.0023 + .000023 + \dots$$

$$a_1 = 0.23$$

$$r = \frac{.0023}{.23} = 0.01$$

$$S = \frac{0.23}{1-0.01} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= \frac{0.23}{0.99}$$

$$\boxed{= \frac{23}{99}}$$

Exercise

Find the rational number represented by the repeating decimal $0.0\overline{71}$

Solution

$$0.0\overline{71} = 0.071 + 0.00071 + .0000071 + \dots$$

$$a_1 = 0.071$$

$$r = \frac{.00071}{.071} = 0.01$$

$$S = \frac{0.071}{1-0.01} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= \frac{0.071}{0.990}$$

$$\boxed{= \frac{71}{990}}$$

Exercise

Find the rational number represented by the repeating decimal $2.4\overline{17}$

Solution

$$2.4\overline{17} = 2.4 + 0.017 + 0.00017 + .0000017 + \dots$$

$$a_1 = 0.017$$

$$r = \frac{.00017}{.017} = 0.01$$

$$S = 2.4 + \frac{0.017}{1-0.01} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= \frac{24}{10} + \frac{0.017}{0.990}$$

$$= \frac{24}{10} + \frac{17}{990}$$

$$= \frac{240+17}{990}$$

$$\boxed{= \frac{2,393}{990}}$$

Exercise

Find the rational number represented by the repeating decimal $10.\overline{5}$

Solution

$$10.\overline{5} = 10 + 0.5 + 0.05 + .005 + \ldots$$

$$a_1 = 0.5$$

$$r = \frac{0.05}{0.5} = 0.1$$

$$S = 10 + \frac{0.5}{1-0.1} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= 10 + \frac{0.5}{0.9}$$

$$= 10 + \frac{5}{9}$$

$$\boxed{= \frac{95}{9}}$$

Exercise

Find the rational number represented by the repeating decimal $5.\overline{146}$

Solution

$$5.\overline{146} = 5 + 0.146 + 0.000146 + .000000146 + \ldots$$

$$a_1 = 0.146$$

$$r = \frac{0.000146}{0.146} = 0.001$$

$$S = 5 + \frac{0.146}{1-0.001} \qquad\qquad S = \frac{a_1}{1-r}$$

$$= 5 + \frac{0.146}{0.999}$$

$$= 5 + \frac{146}{999}$$

$$\boxed{= \frac{5,141}{999}}$$

Exercise

Find the rational number represented by the repeating decimal $3.2\overline{394}$

Solution

$$3.2\overline{394} = 3.2 + 0.0394 + 0.0000394 + \dots$$

$$a_1 = 0.0394$$

$$r = \frac{0.0000394}{0.0394} = 0.001$$

$$S = 3.2 + \frac{0.0394}{1 - 0.001} \qquad S = \frac{a_1}{1-r}$$

$$= \frac{32}{10} + \frac{0.0394}{0.9990}$$

$$= \frac{32}{10} + \frac{394}{9990}$$

$$= \frac{31968 + 394}{9990}$$

$$= \frac{32,362}{9,990}$$

$$\boxed{= \frac{16,181}{4,995}}$$

Exercise

Find the rational number represented by the repeating decimal $1.\overline{6124}$

Solution

$$1.\overline{6124} = 1 + 0.6124 + 0.00006124 + \dots$$

$$a_1 = 0.6124$$

$$r = \frac{0.00006124}{0.6124} = 0.0001$$

$$S = 1 + \frac{0.6124}{1 - 0.0001} \qquad S = \frac{a_1}{1-r}$$

$$= 1 + \frac{0.6124}{0.9999}$$

$$= 1 + \frac{6124}{9999}$$

$$= \boxed{\frac{16,123}{9,999}}$$

Exercise

Find x so that $x+3$, $2x+1$, and $5x+2$ are consecutive terms of an arithmetic sequence.

Solution

$$d = 2x+1-(x+3)$$
$$= x-2$$
$$d = 5x+2-(2x+1)$$
$$= 3x+1$$
$$d = 3x+1 = x-2$$
$$2x = -3$$
$$\boxed{x = -\frac{3}{2}}$$

Exercise

Find x so that $2x$, $3x+2$, and $5x+3$ are consecutive terms of an arithmetic sequence.

Solution

$$d = 3x+2-2x$$
$$= x+2$$
$$d = 5x+3-(3x+2)$$
$$= 2x+1$$
$$d = 2x+1 = x+2$$
$$\boxed{x = 1}$$

Exercise

Find x so that x, $x+2$, and $x+3$ are consecutive terms of a geometric sequence.

Solution

$$r = \frac{x+2}{x}$$
$$r = \frac{x+3}{x+2}$$

$$r = \frac{x+2}{x} = \frac{x+3}{x+2}$$

$$(x+2)^2 = x^2 + 3x$$

$$x^2 + 4x + 4 - x^2 - 3x = 0$$

$$x + 4 = 0$$

$$\underline{x = -4}$$

Exercise

Find x so that $x-1$, x and $x+2$ are consecutive terms of a geometric sequence.

Solution

$$r = \frac{x}{x-1} = \frac{x+2}{x}$$

$$x^2 = x^2 + x - 2$$

$$x - 2 = 0$$

$$\underline{x = 2}$$

Exercise

How many terms must be added in an arithmetic sequence whose first term is 11 and whose common difference is 3 to obtain a sum of 1092?

Solution

Given: $a_1 = 11;\quad d = 3;\quad S = 1092$

$$1092 = \frac{n}{2}\left(22 + 3(n-1)\right) \qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$n(3n+19) = 2184$$

$$3n^2 + 19n - 2184 = 0$$

$$n = \frac{-19 \pm \sqrt{361 + 26208}}{6}$$

$$= \frac{-19 \pm 163}{6}$$

$\underline{n = 24}$ & $\cancel{n = -\frac{91}{3}}$

Exercise

How many terms must be added in an arithmetic sequence whose first term is 78 and whose common difference is –4 to obtain a sum of 702?

Solution

Given: $a_1 = 78;\quad d = -4;\quad S = 702$

$$702 = \frac{n}{2}\left(2(78) - 4(n-1)\right) \qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$n(160 - 4n) = 1404$$

$$-4n^2 + 160n - 1404 = 0$$

$$n = \frac{-160 \pm \sqrt{25,600 - 22464}}{-8}$$

$$= \frac{160 \pm 56}{8}$$

$$\underline{n = 13} \quad \& \quad \underline{n = 27}$$

Exercise

The first ten rows of seating in a certain section of a stadium have 30 seats, 32 seats, 34 seats, and so on. The eleventh through the twentieth rows each contain 50 seats. Find the total number of seats in the section.

Solution

Given: $a_1 = 30;\quad d = 2$

$$S = S_{10} + 50(20 - 11 + 1) \qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

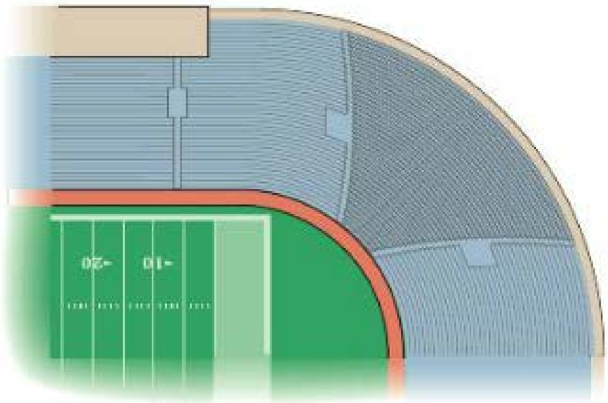
$$= \frac{10}{2}\left(2(30) + 2(9)\right) + 50(10)$$

$$= 5(78) + 500$$

$$\underline{= 890 \ \ seats}$$

Exercise

The corner section of a football stadium has 15 seats in the first row and 40 rows in all. Each successive row contains two additional seats. How many seats are in this section?



Solution

Given: $a_1 = 15;\quad d = 2;\quad n = 40$

$$S_{40} = \frac{40}{2}\left(30 + 2(40-1)\right) \qquad\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$= 20(30 + 78)$$

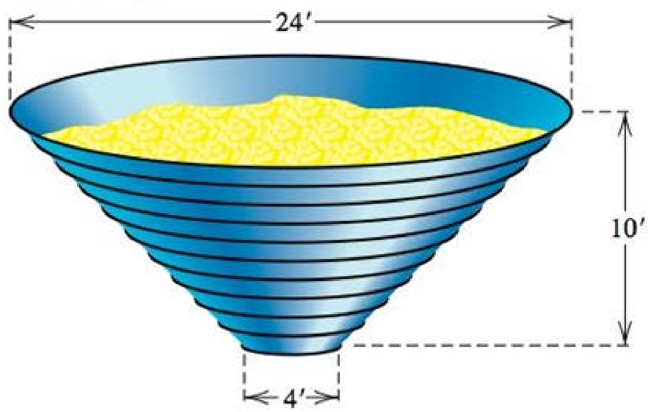
$$= 20(108)$$

$$\underline{= 2,160}$$

The corner section has 2,160 seats.

Exercise

A gain bin is to be constructed in the shape of a frustum of a cone.



The bin is to be 10 *feet* tall with 11 metal rings positioned uniformly around it, from the 4-foot opening at the bottom to the 24-foot opening at the top. Find the total length of metal needed to make the rings.

Solution

The circumference of each ring is πD.

$a_1 = 4\pi;\quad a_{11} = 24\pi$

$24 = 4 + (11-1)d$ $\qquad\qquad a_n = a_1 + (n-1)d$

$10d = 20$

$\underline{d = 2}$

$S_{11} = \frac{11}{2}(4\pi + 24\pi)$ $\qquad\qquad S_n = \frac{n}{2}(a_1 + a_n)$

$\quad = 154\pi \ \ ft$

Exercise

A bicycle rider coasts downhill, traveling 4 *feet* the first second. In each succeeding second, the rider travels 5 *feet* farther than in the preceding second. If the rider reaches the bottom of the hill in 11 *seconds*, find the total distance traveled.

Solution

Given: $a_1 = 4 \ \ ft \quad \& \quad d = 5 \ \ ft$

$S_{11} = \frac{11}{2}(8 + 5(10))$ $\qquad\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

$\quad = 319 \ \ ft$

∴ the total distance traveled 319 *feet*.

Exercise

A contest will have five each prizes totaling \$5,000, and there will be a \$100 difference between successive prices. Find the first prize.

Solution

Given: $n = 5 \quad S_5 = 5000 \quad d = -100$

$5,000 = \frac{5}{2}\left[2a_1 + 4(-100)\right]$ $\qquad\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

$2,000 = 2a_1 - 400$

$a_1 = \$1,200$

Exercise

A Company is to distribute \$46,000 in bonuses to its top ten salespeople. The tenth salesperson on the list will receive \$1,000, and the difference in bonus money between successively ranked salesperson is to be constant. Find the bonus for each salesperson.

Solution

Given: $n=10 \quad S_{10}=46,000 \quad a_{10}=1,000$

$$46,000=\frac{10}{2}\left(a_1+1000\right) \qquad\qquad S_n=\frac{n}{2}\left(a_1+a_n\right)$$

$$9,200=a_1+1000$$

$$a_1=8,200$$

$$d=\frac{1,000-8,200}{9} \qquad\qquad a_n=a_1+(n-1)d$$

$$\underline{=-800}$$

$8,200 $7,400 $6,600 $5,800 $5,000 $4,200 $3,400 $2,600 $1,800 $1,000

Exercise

Assuming air resistance is negligible, a small object that is dropped from a hot air balloon falls 16 *feet* during the first second, 48 *feet* during the second second, 80 *feet* during the third second, 112 *feet* during the fourth second, and so on. Find an expression for the distance the object falls in *n* seconds.

Solution

Given the sequence: 16, 48, 80, 112, $\cdots$

This is an arithmetic sequence with:

$$a_1=16 \quad \& \quad d=48-16=32$$

$$S_n=\frac{n}{2}\left(32+32(n-1)\right) \qquad\qquad S_n=\frac{n}{2}\left[2a_1+(n-1)d\right]$$

$$=\frac{n}{2}(32n)$$

$$\underline{=16n^2}$$

Exercise

A brick staircase has a total of 30 steps. The bottom step requires 100 bricks. Each successive step requires two fewer bricks than the prior step.

a) How many bricks are required for the top step?

b) How many bricks are required to build the staircase?

Solution

a) ***Given***: $n=30 \quad a_1=100 \quad d=-2$

$$a_n=100-2(n-1) \qquad\qquad a_n=a_1+(n-1)d$$

$$\underline{=-2n+102}$$

$$a_{30}=102-60$$

$$\underline{=42}$$

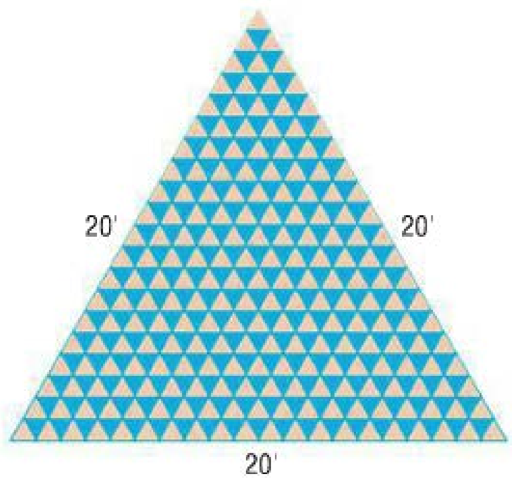
b) $S_{30} = 15(100 + 42)$ $\qquad\qquad$ $S_n = \frac{n}{2}\left(a_1 + a_n\right)$

$\quad\quad = 2{,}130$

It required 2130 *bricks* to build the staircase.

Exercise

A mosaic is designed in the shape of an equilateral triangle, 20 *feet* on each side. Each tile in the mosaic is in the shape of an equilateral triangle, 12 *inches* to a side. The tiles are to alternate in color as shown below.



How many tiles of each color will be required?

Solution

Bottom row has 20 lighter colored tiles.
Top row has 1 lighter colored tile.
The number decreases by 1 as we move up the triangle.
∴ This is an arithmetic sequence with: $a_1 = 20;\quad d = -1;\quad n = 20$

$$S_{20} = \frac{20}{2}\left(40 + (-1)(20 - 1)\right) \qquad\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$= 10(40 - 19)$$

$$= 10(21)$$

$$= 210$$

∴ There are 210 *lighter colored* tiles.

Bottom row has 19 darker colored tiles.
Top row has 1 darker colored tile.

∴ This is an arithmetic sequence with: $a_1 = 1;\quad d = -1;\quad n = 19$

$$S_{19} = \frac{19}{2}\left(2(19) + (-1)(19 - 1)\right) \qquad\qquad S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$=\frac{19}{2}(38-18)$$

$$\underline{=190}$$

∴ There are 190 *darker colored* tiles.

Solution *Section* 4.8 – Mathematical Induction

Exercise

Find all positive integers n for which the given statement is not true

$a)\quad 3^n > 6n \qquad b)\quad 3^n > 2n+1 \qquad c)\quad 2^n > n^2 \qquad d)\quad n! > 2n$

Solution

a) $n=1 \quad 3<6$

$n=2 \quad 3^2 < 18$

$n=3, \quad 27 > 18$

The statement is true for all $n \ge 3 \quad 3^n > 6n$

The statement is not true for $\boxed{n=1,\ 2}$

b) $n=1; \quad 3=3$

$n=2; \quad 9>5$

The statement is true for all $n \ge 2 \quad 3^n > 2n+1$

The statement is not true for $\boxed{n=1}$

c) $n=1; \quad 2<4$

$n=2; \quad 4=4$

$n=3; \quad 8<9$

$n=4; \quad 16=16$

$n=5; \quad 32>25$

The statement is true for all $n \ge 5; \quad 2^n > n^2$

The statement is not true for $\boxed{n=1,\ 2,\ 3,\ 4}$

d) $n=1; \quad 1<2$

$n=2; \quad 2<4$

$n=3; \quad 6=6$

$n=4; \quad 12>8$

The statement is true for all $n \ge 4; \quad n! > 2n$

The statement is not true for $\boxed{n=1,\ 2,\ 3}$

Exercise

Prove that the statement is true for every positive integer n. $2+4+6+\ldots+2n=n(n+1)$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 2=1(1+1)=2$; hence P_1 is true.

(2) Assume $2+4+6+\ldots+2k=k(k+1)$ is true

$$\Rightarrow 2+4+6+\ldots+2k+2(k+1)=(k+1)(k+1+1)?$$

$$\begin{aligned}2+4+6+\ldots+2k+2(k+1) &= 2+4+6+\ldots+2k+2(k+1)\\ &= k(k+1)+2(k+1)\\ &= (k+1)(k+2)\\ &= (k+1)(k+1+1)\ \surd\end{aligned}$$

***Factor* (k + 1)**

Hence P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1+3+5+\ldots+(2n-1)=n^2$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 1\overset{?}{=}1^2=1$; hence P_1 is true.

(2) Assume $1+3+5+\ldots+(2k-1)=k^2$ is true

$$\Rightarrow 1+3+5+\ldots+(2(k+1)-1)=(k+1)^2?$$

$$\begin{aligned}1+3+5+\ldots+(2k-1)+(2(k+1)-1) &= 1+3+5+\ldots+(2k-1)+(2k+2-1)\\ &= k^2+(2k+1)\\ &= k^2+2k+1\\ &= (k+1)^2\ \surd\end{aligned}$$

Hence P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $2+7+12+\dots+(5n-3)=\frac{1}{2}n(5n-1)$

Solution

(1) For ***n* = 1** $\Rightarrow$ $2\overset{?}{=}\frac{1}{2}(1)(5(1)-1)=\frac{1}{2}(4)=2$; hence P_1 is true.

(2) Assume $2+7+12+\dots+(5k-3)=\frac{1}{2}k(5k-1)$ is true

$$2+7+12+\dots+(5(k+1)-3)=\frac{1}{2}(k+1)(5(k+1)-1)?$$

$$\begin{aligned}2+7+12+\dots+(5k-3)+(5(k+1)-3) &= 2+7+12+\dots+(5k-3)+(5k+5-3)\\ &=\frac{1}{2}k(5k-1)+(5k+2)\frac{2}{2}\\ &=\frac{1}{2}\left[5k^2-k+10k+4\right]\\ &=\frac{1}{2}\left[5k^2-k+5k+5k+5-1\right]\\ &=\frac{1}{2}\left[k(5k-1+5)+5k+5-1\right]\\ &=\frac{1}{2}\left[(k+1)(5k+5-1)\right]\\ &=\frac{1}{2}\left[(k+1)(5(k+1)-1)\right] \quad \surd \quad P_{k+1} \text{ is also true.}\end{aligned}$$

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1+2.2+3.2^2+\dots+n.2^{n-1}=1+(n-1).2^n$

Solution

(1) For ***n* = 1** $\Rightarrow$ $1\overset{?}{=}1+(1-1)2^1=1-0=1$; hence P_1 is true.

(2) $1+2.2+3.2^2+\dots+k.2^{k-1}=1+(k-1).2^k$ is true

$$1+2.2+3.2^2+\dots+k.2^{k-1}+(k+1).2^{(k+1)-1}=1+((k+1)-1).2^{k+1}?$$

$$\begin{aligned}1+2.2+3.2^2+\dots+k.2^{k-1}+(k+1).2^{(k+1)-1} &= 1+(k-1).2^k+(k+1).2^{k+1-1}\\ &=1+k.2^k-1.2^k+(k+1).2^k\\ &=1+k.2^k-1.2^k+k.2^k+1.2^k\\ &=1+2^1k.2^k\\ &=1+(k+0).2^{k+1}\\ &=1+((k+1)-1).2^{k+1} \quad \surd \quad P_{k+1} \text{ is also true.}\end{aligned}$$

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow 1^2 \overset{?}{=} \frac{1(1+1)(2(1)+1)}{6}=\frac{1(2)(3)}{6}=\frac{6}{6}=1$ √ ; hence P_1 is true.

(2) $1^2+2^2+3^2+\ldots+k^2=\frac{k(k+1)(2k+1)}{6}$ is true

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}?$$

$$\begin{aligned}1^2+2^2+3^2+\ldots+k^2+(k+1)^2&=\frac{k(k+1)(2k+1)}{6}+(k+1)^2\\&=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}\\&=\frac{(k+1)[k(2k+1)+6(k+1)]}{6}\\&=\frac{(k+1)\left[2k^2+k+6k+6\right]}{6}\\&=\frac{(k+1)\left[2k^2+7k+6\right]}{6}\\&=\frac{(k+1)((k+2)(2k+3))}{6}\\&=\frac{(k+1)((k+1+1)(2k+2+1))}{6}\\&=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}\quad\checkmark\end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $\frac{1}{1\bullet 2}+\frac{1}{2\bullet 3}+\frac{1}{3\bullet 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow \frac{1}{1.2} \overset{?}{=} \frac{1}{1+1}=\frac{1}{2}=\frac{1}{1.2}$ √ ; hence P_1 is true.

(2) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$ is true

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{(k+1)+1}?$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)(k+1)}{(k+1)(k+2)}$$

$$=\frac{k+1}{(k+1+1)}$$

$$=\frac{k+1}{(k+1)+1} \quad \surd$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$

Solution

(1) For ***n* = 1** $\Rightarrow \frac{1}{2}\overset{?}{=}1-\frac{1}{2}=\frac{1}{2} \quad \surd$; P_1 is true.

(2) $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}=1-\frac{1}{2^k}$ is true

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^{k+1}}?$$

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^k}+\frac{1}{2^{k+1}}$$

$$=1-\frac{1}{2^k}+\frac{1}{2^k\cdot 2}$$

$$=\frac{2^{k+1}-2+1}{2^{k+1}}$$

$$=\frac{2^{k+1}-1}{2^{k+1}}$$

$$=\frac{2^{k+1}}{2^{k+1}}-\frac{1}{2^{k+1}}$$

$$=1-\frac{1}{2^{k+1}} \quad \checkmark$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n-2)\cdot(3n+1)}=\frac{n}{3n+1}$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow \frac{1}{1\cdot 4}\overset{?}{=}\frac{1}{3(1)+1}=\frac{1}{4}$ $\checkmark$; P_1 is true.

(2) $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}$ is true

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)}\overset{?}{=}\frac{k+1}{3(k+1)+1}$$

$$\begin{aligned}\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)}&=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}\\&=\frac{3k^2+4k+1}{(3k+1)(3k+4)}\\&=\frac{(3k+1)(k+1)}{(3k+1)(3k+3+1)}\\&=\frac{k+1}{3(k+1)+1} \quad \checkmark\end{aligned}$$

P_{k+1} is also true

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\frac{4}{5}+\frac{4}{5^2}+\frac{4}{5^3}+\cdots+\frac{4}{5^n}=1-\frac{1}{5^n}$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow \frac{4}{5}\overset{?}{=}1-\frac{1}{5}=\frac{4}{5}$ $\checkmark$; P_1 is true.

(2) $\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}=1-\frac{1}{5^k}$ is true

$$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}+\frac{4}{5^{k+1}}\overset{?}{=}1-\frac{1}{5^{k+1}}$$

$$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}+\frac{4}{5^{k+1}}=1-\frac{1}{5^k}+\frac{4}{5^{k+1}}$$

$$=1-\left(\frac{1}{5^k}-\frac{4}{5^{k+1}}\right)$$

$$=1-\frac{5-4}{5^{k+1}}$$

$$=1-\frac{1}{5^{k+1}} \quad \surd$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $1^3+2^3+3^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 1^3\overset{?}{=}\dfrac{1^2(1+1)^2}{4}=\dfrac{4}{4}=1$ $\surd$; P_1 is true.

(2) $\dfrac{4}{5}+\dfrac{4}{5^2}+\cdots+\dfrac{4}{5^k}=1-\dfrac{1}{5^k}$ is true

$$1^3+2^3+\cdots+k^3+(k+1)^3\overset{?}{=}\frac{(k+1)^2\left((k+1)+1\right)^2}{4}$$

$$1^3+2^3+\cdots+k^3+(k+1)^3=\frac{k^2(k+1)^2}{4}+(k+1)^3$$

$$=\frac{k^2(k+1)^2+4(k+1)^3}{4}$$

$$=\frac{(k+1)^2\left[k^2+4(k+1)\right]}{4}$$

$$=\frac{(k+1)^2\left(k^2+4k+4\right)}{4}$$

$$=\frac{(k+1)^2(k+2)^2}{4}$$

$$=\frac{(k+1)^2\left((k+1)+1\right)^2}{4} \quad \surd$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $3+3^2+3^3+\ldots+3^n=\frac{3}{2}\left(3^n-1\right)$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 3\overset{?}{=}\frac{3}{2}\left(3^1-1\right)=\frac{3}{2}2=3$ $\checkmark$; P_1 is true.

(2) $3+3^2+\cdots+3^k=\frac{3}{2}\left(3^k-1\right)$ is true $\rightarrow$ Is $3+3^2+\cdots+3^k+3^{k+1}\overset{?}{=}\frac{3}{2}\left(3^{k+1}-1\right)$

$$\begin{aligned}3+3^2+\cdots+3^k+3^{k+1}&=\frac{3}{2}\left(3^k-1\right)+3^{k+1}\\&=\frac{1}{2}3^{k+1}-\frac{3}{2}+3^{k+1}\\&=\frac{3}{2}3^{k+1}-\frac{3}{2}\\&=\frac{3}{2}\left(3^{k+1}-1\right)\quad\checkmark\end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $x^{2n}+x^{2n-1}y+\cdots+xy^{2n-1}+y^{2n}=\dfrac{x^{2n+1}-y^{2n+1}}{x-y}$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow x^2+xy+y^2\overset{?}{=}\dfrac{x^3-y^3}{x-y}$

$$=\frac{(x-y)\left(x^2+xy+y^2\right)}{x-y}$$

$$=x^2+xy+y^2 \quad \checkmark$$; P_1 is true.

(2) $x^{2k}+x^{2k-1}y+\cdots+xy^{2k-1}+y^{2k}=\dfrac{x^{2k+1}-y^{2k+1}}{x-y}$ is true

$$x^{2(k+1)}+x^{2(k+1)-1}y+\cdots+xy^{2(k+1)-1}+y^{2(k+1)}\overset{?}{=}\frac{x^{2(k+1)+1}-y^{2(k+1)+1}}{x-y}$$

$$\begin{aligned}x^{2k+2}+x^{2k+1}y+\cdots+xy^{2k+1}+y^{2k+2}&=x^2\left(x^{2k}+x^{2k-1}y+\cdots+y^{2k}\right)+xy^{2k+1}+y^{2k+2}\\&=x^2\left(\frac{x^{2k+1}-y^{2k+1}}{x-y}\right)+xy^{2k+1}+y^{2k+2}\\&=\frac{x^{2k+3}-x^2y^{2k+1}+x^2y^{2k+1}+xy^{2k+2}-xy^{2k+2}-y^{2(k+1)+1}}{x-y}\end{aligned}$$

$$= \frac{x^{2(k+1)+1} - y^{2(k+1)+1}}{x-y} \quad \checkmark$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $5 \cdot 6 + 5 \cdot 6^2 + 5 \cdot 6^3 + \cdots + 5 \cdot 6^n = 6\left(6^n - 1\right)$

Solution

(1) For $\boldsymbol{n = 1} \Rightarrow 5 \cdot 6 \overset{?}{=} 6\left(6^1 - 1\right) = 6(5)$ $\checkmark$; P_1 is true.

(2) $5 \cdot 6 + 5 \cdot 6^2 + \cdots + 5 \cdot 6^k = 6\left(6^k - 1\right)$ is true

$$5 \cdot 6 + 5 \cdot 6^2 + \cdots + 5 \cdot 6^k + 5 \cdot 6^{k+1} \overset{?}{=} 6\left(6^{k+1} - 1\right)$$

$$\begin{aligned} 5 \cdot 6 + 5 \cdot 6^2 + \cdots + 5 \cdot 6^k + 5 \cdot 6^{k+1} &= 6\left(6^k - 1\right) + 5 \cdot 6^{k+1} \\ &= 6^{k+1} - 6 + 5 \cdot 6^{k+1} \\ &= 6^{k+1}(1+5) - 6 \\ &= 6 \cdot 6^{k+1} - 6 \\ &= 6\left(6^{k+1} - 1\right) \quad \checkmark \end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $7 \cdot 8 + 7 \cdot 8^2 + 7 \cdot 8^3 + \cdots + 7 \cdot 8^n = 8\left(8^n - 1\right)$

Solution

(1) For $\boldsymbol{n = 1} \Rightarrow 7 \cdot 8 \overset{?}{=} 8\left(8^1 - 1\right) = 8(7)$ $\checkmark$; P_1 is true.

(2) $7 \cdot 8 + 7 \cdot 8^2 + \cdots + 7 \cdot 8^k = 8\left(8^k - 1\right)$ is true

$$7 \cdot 8 + 7 \cdot 8^2 + \cdots + 7 \cdot 8^k + 7 \cdot 8^{k+1} \overset{?}{=} 8\left(8^{k+1} - 1\right)$$

$$7 \cdot 8 + 7 \cdot 8^2 + \cdots + 7 \cdot 8^k + 7 \cdot 8^{k+1} = 8\left(8^k - 1\right) + 7 \cdot 8^{k+1}$$

$$=8^{k+1}-8+7\cdot 8^{k+1}$$
$$=8^{k+1}(1+7)-8$$
$$=8\cdot 8^{k+1}-8$$
$$=8\left(8^{k+1}-1\right) \quad \surd$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $3+6+9+\cdots+3n=\dfrac{3n(n+1)}{2}$

Solution

(1) For ***n* = 1** $\Rightarrow$ $3\overset{?}{=}\dfrac{3(1)(1+1)}{2}=3$ $\surd$; P_1 is true.

(2) $3+6+9+\cdots+3k=\dfrac{3k(k+1)}{2}$ is true

$$3+6+9+\cdots+3k+3(k+1)\overset{?}{=}\frac{3(k+1)(k+2)}{2}$$

$$\begin{aligned}3+6+9+\cdots+3k+3(k+1)&=\frac{3k(k+1)}{2}+3(k+1)\\&=\frac{3k(k+1)+6(k+1)}{2}\\&=\frac{(k+1)(3k+6)}{2}\\&=\frac{3(k+1)(k+2)}{2} \quad \surd\end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $5+10+15+\cdots+5n=\dfrac{5n(n+1)}{2}$

Solution

(1) For ***n* = 1** $\Rightarrow$ $5\overset{?}{=}\dfrac{5(1)(1+1)}{2}=5$ $\surd$; P_1 is true.

(2) $5+10+15+\cdots+5k=\dfrac{5k(k+1)}{2}$ is true

$$5+10+15+\cdots+5k+5(k+1)\overset{?}{=}\frac{5(k+1)(k+2)}{2}$$

$$\begin{aligned}5+10+15+\cdots+5k+5(k+1) &= \frac{5k(k+1)}{2}+5(k+1)\\ &= \frac{5k(k+1)+10(k+1)}{2}\\ &= \frac{(k+1)(5k+10)}{2}\\ &= \frac{5(k+1)(k+2)}{2} \quad \surd\end{aligned}$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $1+3+5+\cdots+(2n-1)=n^2$

Solution

(1) For ***n* = 1** $\Rightarrow$ $1\overset{?}{=}1^2=1$ $\surd$; P_1 is true.

(2) $1+3+5+\cdots+(2k-1)=k^2$ is true

$$1+3+5+\cdots+(2k-1)+(2(k+1)-1)\overset{?}{=}(k+1)^2$$

$$\begin{aligned}1+3+5+\cdots+(2k-1)+(2(k+1)-1) &= k^2+2k+2-1\\ &= k^2+2k+1\\ &= (k+1)^2 \quad \surd\end{aligned}$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $4+7+10+\cdots+(3n+1)=\dfrac{n(3n+5)}{2}$

Solution

(1) For ***n* = 1** $\Rightarrow$ $4\overset{?}{=}\dfrac{1(3+5)}{2}=4$ $\surd$; P_1 is true.

(2) $4+7+10+\cdots+(3k+1)=\dfrac{k(3k+5)}{2}$ is true

$$4+7+10+\cdots+(3k+1)+(3(k+1)+1)\overset{?}{=}\frac{(k+1)(3(k+1)+5)}{2}=\frac{(k+1)(3k+8)}{2}$$

$$\begin{aligned}4+7+10+\cdots+(3k+1)+(3k+4)&=\frac{k(3k+5)}{2}+3k+4\\&=\frac{3k^2+5k+6k+8}{2}\\&=\frac{3k^2+5k+3k+3k+8}{2}\\&=\frac{k(3k+8)+(3k+8)}{2}\\&=\frac{(3k+8)(k+1)}{2}\quad\surd\end{aligned}$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $\left(a^m\right)^n=a^{mn}$ (a and m are constant)

Solution

- For ***n* = 1** $\Rightarrow$ $\left(a^m\right)^1\overset{?}{=}a^{m(1)}$ $\rightarrow$ $a^m=a^m$ $\surd$; P_1 is true.
- $\left(a^m\right)^k=a^{mk}$ is true

$$\left(a^m\right)^{(k+1)}\overset{?}{=}a^{m(k+1)}$$

$$\begin{aligned}\left(a^m\right)^{(k+1)}&=\left(a^m\right)^k a^m\\&=a^{km}a^m\\&=a^{km+m}\end{aligned}$$

$$= a^{m(k+1)} \quad \checkmark$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $n < 2^n$

Solution

***Step* 1**. For $\boldsymbol{n = 1} \Rightarrow 1 \overset{?}{<} 2^1 \ \checkmark \Rightarrow P_1$ is true.

***Step* 2**. Assume that P_k is true $k < 2^k$

We need to prove that P_{k+1} is true, that is $k+1 < 2^{k+1}$

$$\begin{aligned} k+1 &< k+k = 2k \\ &< 2 \cdot 2^k \\ &= 2^{k+1} \quad \checkmark \end{aligned}$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. 3 is a factor of $n^3 - n + 3$

Solution

- For $\boldsymbol{n = 1} \Rightarrow 1^3 - 1 + 3 = 3 = 3(1) \ \checkmark \Rightarrow P_1$ is true.
- Assume that P_k is true 3 is a factor of $k^3 - k + 3$

We need to prove that P_{k+1} is true, that is $(k+1)^3 - (k+1) + 3$

$$\begin{aligned} (k+1)^3 - (k+1) + 3 &= k^3 + 3k^2 + 3k + 1 - k - 1 + 3 \\ &= \left(k^3 - k + 3\right) + 3k^2 + 3k \\ &= 3K + 3k^2 + 3k \\ &= 3\left(K + k^2 + k\right) \quad \checkmark \end{aligned}$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. 4 is a factor of $5^n - 1$

Solution

- For $\boldsymbol{n = 1} \Rightarrow 5^1 - 1 = 4 = 4(1)$ $\checkmark$ $\Rightarrow P_1$ is true.
- Assume that P_k is true 4 is a factor of $5^k - 1$

  We need to prove that P_{k+1} is true, that is $5^{k+1} - 1$

$$\begin{aligned} 5^{k+1} - 1 &= 5^k 5^1 - 5 + 4 \\ &= 5\left(5^k - 1\right) + 4 \\ &= 5\left(5^k - 1\right) + 4 \end{aligned}$$

By the induction hypothesis, 4 is a factor of $5^k - 1$ and 4 is a factor of 4, so 4 is a factor of the $(k+1)$ term. $\checkmark$

Thus, P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $2^n > 2n \quad if \quad n \ge 3$

Solution

- For $\boldsymbol{n = 3} \Rightarrow 2^3 \ge 2(3) \Rightarrow 8 \ge 6$ $\checkmark$ $\Rightarrow P_1$ is true.
- Assume that P_k is true: $2^k > 2k$;

  we need to prove that P_{k+1}: $2^{k+1} > 2(k+1)$ is true

$$\begin{aligned} &2^k > 2k \\ &2^k \cdot 2 > 2k \cdot 2 \\ &2^{k+1} > 4k = 2k + 2k \qquad k \ge 3 \\ &\qquad > 2k + 2 \\ &\qquad = 2(k+1) \ \checkmark \end{aligned}$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: If $0 < a < 1$, then $a^n < a^{n-1}$

Solution

- For $\boldsymbol{n = 1} \Rightarrow a^1 < a^{1-1} \Rightarrow a < 1$ ✓ since $0 < a < 1 \Rightarrow P_1$ is true.

- Assume that P_k is true: $a^k < a^{k-1}$;

  We need to prove that P_{k+1}: $a^{k+1} < a^k$ is true

  $$a^k < a^{k-1} \rightarrow a^k \cdot a < a^{k-1} \cdot a$$
  $$a^{k+1} < a^k \quad ✓$$

  P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: If $n \ge 4$, then $n! > 2^n$

Solution

- For $\boldsymbol{n = 4} \Rightarrow 4! > 2^4 \Rightarrow 24 > 16$ ✓ $\Rightarrow P_1$ is true.
- Assume that P_k is true: $k! > 2^k$;

  We need to prove that P_{k+1}: $(k+1)! > 2^{k+1}$ is true

  $$\begin{aligned}(k+1)! &= k! \cdot (k+1) \\ &> 2^k (k+1) && k \ge 4 \Rightarrow k+1 > 2 \\ &> 2^k \cdot 2 \\ &= 2^{k+1} \quad ✓\end{aligned}$$

  Thus, P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $3^n > 2n + 1 \quad if \quad n \ge 2$

Solution

- For $\boldsymbol{n = 2} \Rightarrow 3^2 > 2(2) + 1 \Rightarrow 9 > 5$ ✓ $\Rightarrow P_1$ is true.

- Assume that P_k is true: $3^k > 2k+1$

  We need to prove that P_{k+1}: $3^{k+1} > 2(k+1)+1$ is true

  $$3^k > 2k+1 \quad \Rightarrow \quad 3^k \cdot 3 > (2k+1)\cdot 3$$
  $$3^{k+1} > 6k+3$$
  $$> 2k+2+1$$
  $$= 2(k+1)+1 \quad \checkmark$$

  Thus, P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $2^n > n^2 \quad for \quad n > 4$

Solution

- For $\boldsymbol{n = 5}$ $\Rightarrow 2^5 > 5^2 \quad \Rightarrow \quad 32 > 25 \checkmark \quad \Rightarrow P_1$ is true.

- Assume that P_k is true: $2^k > k^2$

  We need to prove that P_{k+1}: $2^{k+1} > (k+1)^2$ is true

  $$2^k > k^2 \quad \Rightarrow \quad 2^k \cdot 2 > k^2 \cdot 2$$
  $$2^{k+1} > 2k^2 \qquad\qquad k < k+1 \quad \Rightarrow \quad k^2 > 2k+1$$
  $$> (k+1)^2 \quad \checkmark$$

  Thus, P_{k+1} is also true

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $4^n > n^4 \quad for \quad n \ge 5$

Solution

- For $\boldsymbol{n = 5}$ $\Rightarrow 4^5 > 5^4 \quad \Rightarrow \quad 1024 > 625 \checkmark \quad \Rightarrow P_1$ is true.

- Assume that P_k is true: $4^k > k^4$

  We need to prove that P_{k+1}: $4^{k+1} > (k+1)^4$ is true

$$4^k > k^4 \quad \Rightarrow \quad 4^k \cdot 4 > k^4 \cdot 4$$

$$4^{k+1} > 4k^4 \qquad\qquad k < k+1 \quad \Rightarrow \quad k^2 > 2k+1$$

$$> (k+1)^4 \quad \surd$$

Thus, P_{k+1} is also true

∴ By the mathematical induction, the proof is completed.

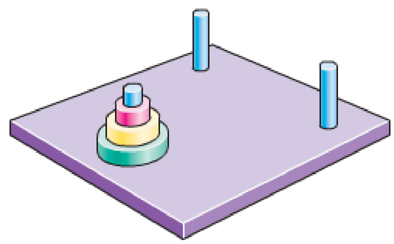
Exercise

A pile of *n* rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must moved, one at a time, to a different peg with no ring ever placed on top of a smaller ring. Find the least number of moves that would be required. Prove your result by mathematical induction.

Solution

With 1 ring, 1 move is required.

With 2 rings, 3 moves are required $\Rightarrow \quad 3 = 2+1$



With 3 rings, 7 moves are required $\Rightarrow \quad 7 = 2^2 + 2 + 1$

With *n* rings, $2^{n-1} + \cdots + 2^2 + 2^1 + 2^0 = 2^n - 1$ moves are required

- For $\boldsymbol{n = 1} \Rightarrow 2^0 = 2^1 - 1 = 1 \surd \quad \Rightarrow P_1$ is true.

- Assume that P_k is true: $2^{k-1} + \cdots + 2^2 + 2^1 + 2^0 = 2^k - 1$

$$2^k + 2^{k-1} + \cdots + 2^2 + 2^1 + 1 \overset{?}{=} 2^{k+1} - 1$$

$$\begin{aligned} 2^k + 2^{k-1} + \cdots + 2^2 + 2^1 + 1 &= 2^k + 2^k - 1 \\ &= 2 \cdot 2^k - 1 \\ &= 2^{k+1} - 1 \quad \surd \end{aligned}$$

∴ By the mathematical induction, the proof is completed.