Lecture Two

Section 2.1 – Sequences and Summations

Sequences

Definition

A sequence is a function from a subset of the set of integers (usually either the set {0, 1, 2, …} or the set {1, 2, 3, …} to a set *S*. We use the notation a_n to denote the image of the integer *n*. We call a_n a term of the sequence.

The sequence $\{a_n\}$, where $a_n = \frac{1}{n}$

The list of the terms of this sequence: $1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots$

Definition

A ***geometric progression*** is a sequence of the form $a,\ ar,\ ar^2,\ \ldots,\ ar^n,\ \ldots$ where the *initial* term *a* and the ***common ratio*** *r* are real numbers.

The common ratio for: $6,\ -12,\ 24,\ -48,\ \ldots,\ (-2)^{n-1}(6),\ \ldots$ is $= \frac{-12}{6} = -2$

Definition

An ***arithmetic progression*** is a sequence of the form $a,\ a+d,\ a+2d,\ \ldots,\ a+nd,\ \ldots$ where the *initial* term *a* and the ***common difference*** *d* are real numbers.

Recurrence Relations

Definition

A ***recurrence relation*** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0,\ a_1,\ a_2,\ \ldots,\ a_{n-1},\ \ldots$, for all integers *n* with $n \ge n_0$, where n_0 is a nonnegative integer. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation. (A recurrence relation is said to *recursively define* a sequence.)

Example

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,\ 2,\ 3,\ \ldots$ and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

Solution

$$a_1 = a_0 + 3 = 2 + 3 = 5$$
$$a_2 = a_1 + 3 = 5 + 3 = 8$$
$$a_3 = a_2 + 3 = 8 + 3 = 11$$

Example

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2,\ 3,\ 4,\ \ldots$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2, and a_3?

Solution

$$a_2 = a_1 + a_0 = 5 - 3 = 2$$
$$a_3 = a_2 + a_1 = 2 - 5 = -3$$

Definition

The Fibonacci sequence, $f_0, f_1, f_2,\ \ldots,$ is defined by the initial conditions $f_0 = 0,\ f_1 = 1,$ and the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \quad \mathit{for} \quad n = 2,\ 3,\ 4,\ \ldots$$

Example

Find the Fibonacci number $f_2, f_3, f_4, f_5,$ *and* f_6

Solution

$$f_2 = f_1 + f_0 = 1 + 0 = 1$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5$$
$$f_6 = f_5 + f_4 5 + 3 = 8$$

Example

Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer *n*, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for *n* = 2, 3, 4, ... Answer the same question whenere $a_n = 2^n$ and where $a_n = 5$

Solution

Suppose that $a_n = 3n$. Then, for $n \ge 2$,

$$\begin{aligned} 2a_{n-1} - a_{n-2} &= 2\big(3(n-1)\big) - 3(n-2) \\ &= 6n - 2 - 3n + 6 \\ &= 3n = a_n \qquad \text{Is a solution of the recurrence relation} \end{aligned}$$

Suppose that $a_n = 2^n$. Then, for $n \ge 2$,

$$\begin{aligned} 2a_{n-1} - a_{n-2} &= 2 \cdot 2^{n-1} - 2^{n-2} \\ &= 2^n \left(2 \cdot 2^{-1} - 2^{-2}\right) \\ &= 2^n \left(1 - \tfrac{1}{4}\right) \\ &= 2^n \left(\tfrac{3}{4}\right) \\ &= 3 \cdot 2^{n-2} \\ &\ne 2^n = a_n \end{aligned}$$

Is ***not*** a solution of the recurrence relation

or $a_0 = 1, \quad a_1 = 2, \quad a_2 = 4$

$2a_1 - a_0 = 2 \cdot 2 - 1 = 3 \ne a_2$

Suppose that $a_n = 5$. Then, for $n \ge 2$,

$$2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5 = a_n$$

Is a solution of the recurrence relation

Example

Find the formalc for the sequences with the following first five terms:

a) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

b) 1, 3, 5, 7, 9

c) 1, −1, 1, −1, 1

Solution

a) The sequence with $a_n = \frac{1}{2^n}$, *n* = 0, 1, 2, …, This proposed sequence is a geometric progression with $a = 1$ and $r = \frac{1}{2}$.

b) Each term is obtained by adding 2 to the previous term, The sequence with $a_n = 2n+1$, $n = 0, 1, 2, \ldots$, This proposed sequence is an arithmetic progression with $a = 1$ and $d = 2$.

c) The terms alternate between 1 and −1, The sequence with $a_n = (-1)^n$, $n = 0, 1, 2, \ldots$, This proposed sequence is an geometric progression with $a = 1$ and $r = -1$.

Example

How can we produce the terms of a sequence if the first 10 terms are 1, 2, 2, 3, 3, 3, 4, 4, 4, 4?

Solution

In this sequence, the integer 1 appears once, the integer 2 appears twice, the integer 3 appears three times, the integer 4 appears four times. A reasonable rule for generating this sequence is that the integer *n* appears exactly *n* times.
The sequence generated this is possible match.

Example

How can we produce the terms of a sequence if the first 10 terms are 5, 11, 17, 23, 29, 35, 41, 47, 53, 59?

Solution

$d = 11 - 5 = 6$

The sequence can be obtained by adding 6 to previous term. This produce to $a_n = 5 + 6(n-1)$.

This sequence is an arithmetic progression with $a = 5$ and $d = 6$.

Example

How can we produce the terms of a sequence if the first 10 terms are 1, 3, 4, 7, 11, 18, 29, 47, 76, 123?

Solution

4 = 1 + 3
7 = 4 + 3
11 = 4 + 7
And so on. We can see that the third term is the sum of the two previous term.
The sequence is determined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$ with initial conditions $L_1 = 1$ and $L_1 = 2$.

Some Useful Sequences	
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, …
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, …
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, …
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, …
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, …
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, …
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …

Summations

To find the sum of many terms of an infinite sequence, it is easy to express using ***summation notation***.

$$\sum_{k=1}^{m} a_k = a_1 + a_2 + a_3 + \ldots + a_m$$

$$\sum_{k=m}^{n} a_k \quad or \quad \sum_{m \le k \le n} a_k$$

The index of summation runs through all integers starting with its ***lower limit*** and ending with its ***upper limit***.

The large uppercase Greek letter ***sigma***, Σ, is used to denote summation.

Example

Use the summation notation to express the sun of the first 100 terms of the sequence $\left\{a_j\right\}$, where $a_j = \frac{1}{j}$ for $j = 1, 2, 3, \ldots$

Solution

$$\sum_{j=1}^{100} \frac{1}{j}$$

Example

What is the value of $\sum_{j=1}^{5} j^2$

Solution

$$
\begin{aligned}
\sum_{j=1}^{5} j^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \\
&= 1 + 4 + 9 + 16 + 25 \\
&= 55
\end{aligned}
$$

Example

What is the value of $\sum_{k=4}^{8} (-1)^k$

Solution

$$
\begin{aligned}
\sum_{k=4}^{8} (-1)^k &= (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8 \\
&= 1 + (-1) + 1 + (-1) + 1 \\
&= 1
\end{aligned}
$$

Theorem

If a and r are real numbers and $r \neq 0$, then

$$
\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & if \quad r \neq 1 \\ (n+1)a & if \quad r11 \end{cases}
$$

Proof

Let $S_n = \sum_{j=0}^{n} ar^j$

$$
\begin{aligned}
rS_n &= r\sum_{j=0}^{n} ar^j \\
&= \sum_{j=0}^{n} ar^{j+1}
\end{aligned}
$$

$$= \sum_{k=1}^{n+1} ar^k$$ *Shifting the index of summation with* $k = j+1$

$$= \sum_{k=0}^{n} ar^k + \left(ar^{n+1} - a\right)$$

$$= S_n + \left(ar^{n+1} - a\right)$$

$$rS_n = S_n + \left(ar^{n+1} - a\right)$$

$$(r-1)S_n = ar^{n+1} - a$$

$$S_n = \frac{ar^{n+1} - a}{r-1}$$

If $r = 1$, then the $S_n = \sum_{j=0}^{n} a(1)^j = \sum_{j=0}^{n} a = (n+1)a$

Double summations

Double summations arise in many contexts (as in the analysis of nested loops in computer programs). An example of a double summation is

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij$$

To evaluate the double sum, first expand the inner summation and then continue by computing the outer summation

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i + 2i + 3i)$$

$$= \sum_{i=1}^{4} 6i$$

$$= 6 + 12 + 18 + 24$$

$$\underline{= 60}$$

Some Useful Summation Formulae	
Sum	Closed Form
$\sum_{k=0}^{n} ar^k \quad (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1}, \quad r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \quad \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \quad \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Example

What is the value of $\sum_{s\in[0,2,4]} s$

Solution

$$\sum_{s\in[0,2,4]} s = 0+2+4 \underline{=6}$$

Example

What is the value of $\displaystyle\sum_{k=50}^{100} k^2$

Solution

$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{100\cdot 101\cdot 201}{6} - \frac{49\cdot 50\cdot 99}{6} = 338,350 - 40,425 \underline{= 297,925}$$

Exercises *Section* 2.1 – Sequences and Summations

1. Find these terms of the sequence $\{a_n\}$, where $a_n = 2\cdot(-3)^n + 5^n$

a) a_0 *b*) a_1 *c*) a_4 *d*) a_5

2. What is the term a_8 of the sequence $\{a_n\}$, if a_n equals

a) 2^{n-1} *b*) 7 *c*) $1+(-1)^n$ *d*) $-(2)^n$

3. What are the terms a_0, a_1, a_2, *and* a_3 of the sequence $\{a_n\}$, if a_n equals

a) 2^n+1 *b*) $(n+1)^{n+1}$ *c*) $\frac{n}{2}$ *d*) $\frac{n}{2}+\frac{n}{2}$

e) $(-2)^n$ *f*) 3 *g*) $7+4^n$ *h*) $2^n+(-2)^n$

4. Find at least three different sequences beginning with the terms 1, 2, 4 whose terms are generated by a simple formula or rule.

5. Find at least three different sequences beginning with the terms 3, 5, 7 whose terms are generated by a simple formula or rule.

6. Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = 6a_{n-1}$, $a_0 = 2$

b) $a_n = a_{n-1}^2$, $a_1 = 2$

c) $a_n = a_{n-1} + 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$

d) $a_n = na_{n-1} + n^2 a_{n-2}$, $a_0 = 1$, $a_1 = 1$

e) $a_n = a_{n-1} + a_{n-3}$, $a_0 = 1$, $a_1 = 2$, $a_2 = 0$

7. Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = -2a_{n-1}$, $a_0 = -1$

b) $a_n = a_{n-1} - a_{n-2}$, $a_0 = 2$, $a_1 = -1$

c) $a_n = 3a_{n-1}^2$, $a_0 = 1$

d) $a_n = na_{n-1} + n^2 a_{n-2}$, $a_0 = -1$, $a_1 = 0$

e) $a_n = a_{n-1} - a_{n-2} + a_{n-3}$, $a_0 = 1$, $a_1 = 2$, $a_2 = 2$

8. Let $a_n = 2^n + 5 \cdot 3^n \quad for \quad n = 0,\ 1,\ 2,\ \ldots$

a) Find $a_0,\ a_1,\ a_2,\ a_3, and\ a_4$

b) Show that $a_2 = 5a_1 - 6a_0$, $a_3 = 5a_2 - 6a_1$, and $a_4 = 5a_3 - 6a_2$

c) Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers *n* with $n \geq 2$

9. Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ if

a) $a_n = 0$?

b) $a_n = 1$?

c) $a_n = 2^n$?

d) $a_n = 4^n$?

e) $a_n = n4^n$?

f) $a_n = 2 \cdot 4^n + 3n4^n$?

g) $a_n = (-4)^n$?

h) $a_n = n^2 4^n$?

10. Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if

i) $a_n = -n + 2$

j) $a_n = 5(-1)^n - n + 2$

k) $a_n = 3(-1)^n + 2^n - n + 2$

l) $a_n = 7 \cdot 2^n - n + 2$

11. A person deposits \$1,000.00 in an account that yields 9% interest compounded annually.

a) Set up a recurrence relation for the amount in the account at the end of *n* years.

b) Find an explicit formula for the amount in the account at the end of *n* years.

c) How much money will the account contain after 100 years?

12. Suppose that the number of bacteria in a colony triples every hour.

a) Set up a recurrence relation for the number of bacteria after *n* hours have elapsed.

b) If 100 bacteria are used to begin new colony, how many bacteria will be in the colony in 10 hours?

13. A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with n cars made in the *n*th month.

a) Set up a recurrence relation for the number of cars produced in the first n months by this factory.

b) How many cars are produced in the first year?

c) Find an explicit formula for the number of cars produced in the first n months by this factory

14. For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

a) 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, …

b) 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, …

c) 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, …

d) 3, 6, 12, 24, 48, 96, 192, …

e) 15, 8, 1, −6, −13, −20, −27, …

f) 3, 5, 8, 12, 17, 23, 30, 38, 47, …

g) 2, 16, 54, 128, 250, 432, 686, …

h) 2, 3, 7, 25, 121, 721, 5041, 40321, …

i) 3, 6, 11, 18, 27, 38, 51, 66, 83, 102, …

j) 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, …

k) 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, …