Section 2.4 – Law of Sines and Cosines

Oblique Triangle

A triangle, that is not a right triangle, is either acute or obtuse.
The measures of the three sides and the three angles of a triangle can be found if at least one side and any other two measures are known.

The Law of *Sines*

There are many relationships that exist between the sides and angles in a triangle.

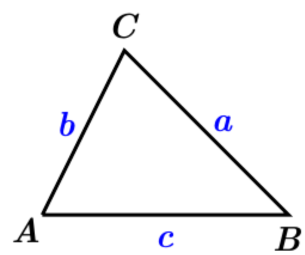
One such relation is called the law of sines.

Given triangle ***ABC***

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$$

or, equivalently

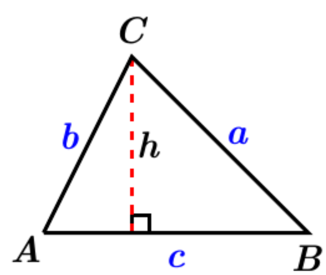
$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$$



Proof

$$\sin A = \frac{h}{b} \Rightarrow h = b\sin A \quad (1)$$

$$\sin B = \frac{h}{a} \Rightarrow h = a\sin B \quad (2)$$



From (1) & (2)

$$h = h$$

$$b\sin A = a\sin B$$

$$\frac{b\sin A}{ab} = \frac{a\sin B}{ab}$$

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

Angle – Side - Angle (*ASA* or *AAS*)

If two angles and the included side of one triangle are equal, respectively, to two angles and the included side of a second triangle, then the triangles are congruent.

Example

In triangle *ABC*, $A = 30°$, $B = 70°$, and $a = 8.0\ cm$. Find the length of side *c*.

Solution

$$\begin{aligned} C &= 180° - (A + B) \\ &= 180° - (30° + 70°) \\ &= 180° - 100° \\ &\underline{= 80°} \end{aligned}$$

$$\begin{aligned} c &= \frac{a}{\sin A}\sin C \\ &= \frac{8}{\sin 30°}\sin 80° \\ &\underline{\approx 16\ cm} \end{aligned} \qquad \frac{c}{\sin C} = \frac{a}{\sin A}$$

Example

Find the missing parts of triangle *ABC* if $A = 32°$, $C = 81.8°$, and $a = 42.9\ cm$.

Solution

$$\begin{aligned} B &= 180° - (A + C) \\ &= 180° - (32° + 81.8°) \\ &\underline{= 66.2°} \end{aligned}$$

$$\begin{aligned} b &= \frac{a \sin B}{\sin A} \\ &= \frac{42.9 \sin 66.2°}{\sin 32°} \\ &\underline{\approx 74.1\ cm} \end{aligned} \qquad \frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\begin{aligned} c &= \frac{a \sin C}{\sin A} \\ &= \frac{42.9 \sin 81.8°}{\sin 32°} \\ &\underline{\approx 80.1\ cm} \end{aligned} \qquad \frac{c}{\sin C} = \frac{a}{\sin A}$$

Example

You wish to measure the distance across a River. You determine that $C = 112.90°$, $A = 31.10°$, and $b = 347.6\ ft$. Find the distance a across the river.

Solution

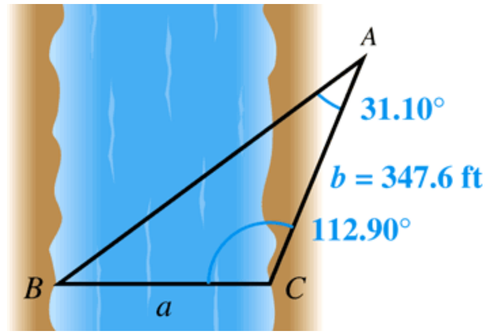
$$B = 180° - A - C$$
$$= 180° - 31.10° - 112.90°$$
$$\underline{= 36°|}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 31.1°} = \frac{347.6}{\sin 36°}$$

$$a = \frac{347.6}{\sin 36°} \sin 31.1°$$

$$\underline{a \approx 305.5 \quad ft\ |}$$



Example

Find distance x if $a = 562\ ft.$, $B = 5.7°$ and $A = 85.3°$

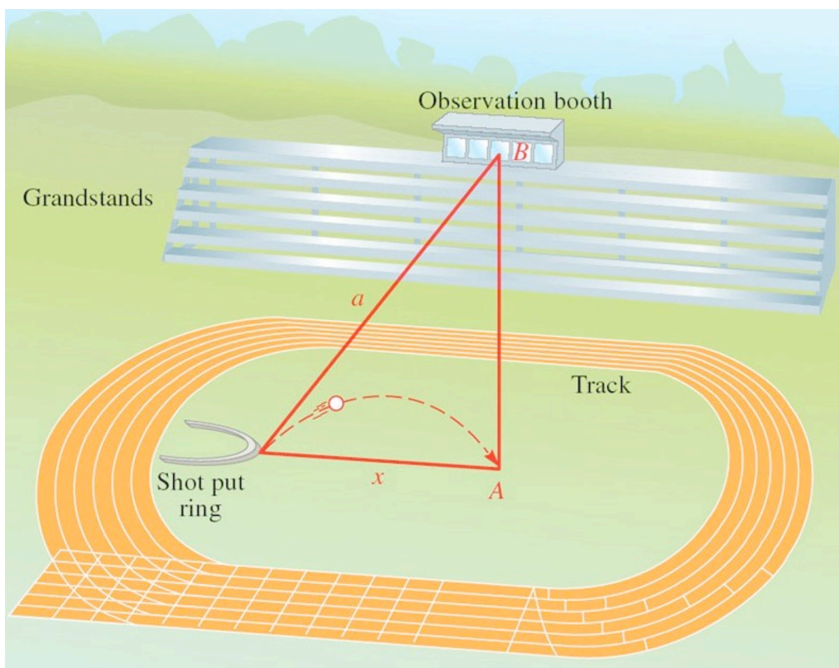
Solution

$$\frac{x}{\sin B} = \frac{a}{\sin A}$$

$$x = \frac{a \sin B}{\sin A}$$

$$= \frac{562 \sin 5.7°}{\sin 85.3°}$$

$$\underline{\approx 56.0\ \ ft\ |}$$



Ambiguous Case

Side – Angle – Side (*SAS*)

If two sides and the included angle of one triangle are equal, respectively, to two sides and the included angle of a second triangle, then the triangles are congruent.

- $0 \le \sin\theta \le 1$
- Sine (positive) in *QI* & *QII*

Example

Find angle B in triangle $\boldsymbol{ABC}$ if $a = 2$, $b = 6$, and $A = 30°$

Solution

$$\sin B = \frac{b\sin A}{a} \qquad \frac{\sin B}{b} = \frac{\sin A}{a}$$

$$= \frac{6\sin 30°}{2}$$

$$= 1.5 \quad > 1 \qquad 0 \le \sin\alpha \le 1$$

Since $sinB > 1$ is impossible, no such triangle exists.

Example

Find the missing parts in triangle $\boldsymbol{ABC}$ if $C = 35.4°$, $a = 205$ *ft*., and $c = 314$ *ft*.

Solution

$$\sin A = \frac{a\sin C}{c}$$

$$= \frac{205\sin 35.4°}{314}$$

$$A = \sin^{-1}\left(\frac{205\sin 35.4°}{314}\right)$$

$$A \approx 22.2° \qquad A' = 180° - 22.2° = 157.8°$$

$$C + A' = 35.4° + 157.8°$$

$$= 193.2° \quad > 180°$$

$$B = 180° - (22.2° + 35.4°)$$

$$\approx 122.4°$$

$$b = \frac{c\sin B}{\sin C}$$

$$= \frac{314\sin 122.4°}{\sin 35.4°}$$

$$\approx 458 \ ft$$

Example

Find the missing parts in triangle ***ABC*** if $a = 54\ cm$, $b = 62\ cm$, and $A = 40°$.

Solution

$$\sin B = \frac{b \sin A}{a}$$
$$= \frac{62 \sin 40°}{54}$$

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$B = \sin^{-1}\left(\frac{62 \sin 40°}{54}\right)$$
$$\approx 48°$$

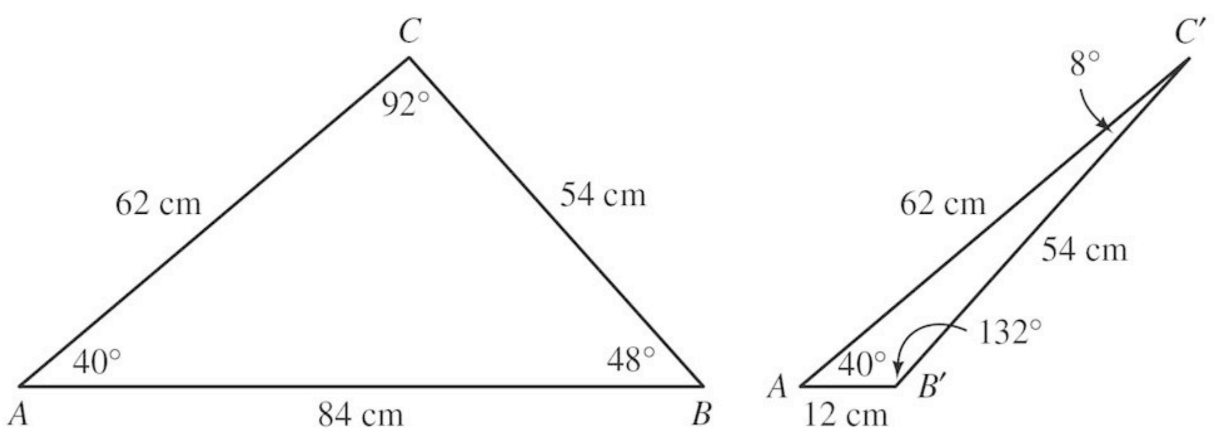
$$B' = 180° - 48°$$
$$\approx 132°$$

$$C = 180° - (40° + 48°)$$
$$\approx 92°$$

$$C' = 180° - (40° + 132°)$$
$$\approx 8°$$

$$c = \frac{a \sin C}{\sin A}$$
$$= \frac{54 \sin 92°}{\sin 40°}$$
$$\approx 84\ \ cm$$

$$c' = \frac{a \sin C'}{\sin A}$$
$$= \frac{54 \sin 8°}{\sin 40°}$$
$$\approx 12\ \ cm$$



Area of a Triangle (SAS)

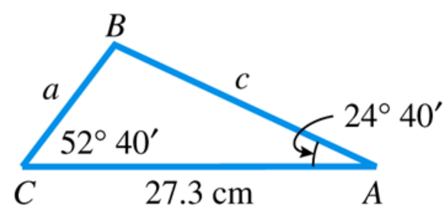
In any triangle *ABC*, the area K is given by the following formulas:

$$K = \frac{1}{2}bc\sin A \qquad K = \frac{1}{2}ac\sin B \qquad K = \frac{1}{2}ab\sin C$$

Example

Find the area of triangle *ABC* if $A = 24°40',\ b = 27.3\ cm,\ \ and\ \ C = 52°40'$

Solution



$$B = 180° - 24°40' - 52°40'$$

$$= 180° - \left(24° + \frac{40°}{60}\right) - \left(52° + \frac{40°}{60}\right)$$

$$\approx 102.667°$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin(24°40')} = \frac{27.3}{\sin(102°40')}$$

$$a = \frac{27.3\sin(24°40')}{\sin(102°40')}$$

$$\approx 11.7\ cm$$

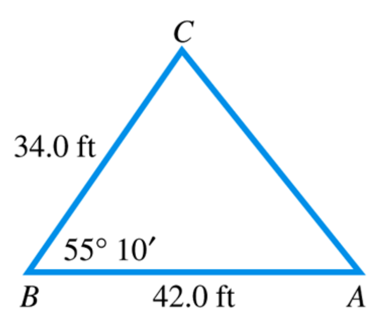
$$K = \frac{1}{2}ac\sin B$$

$$= \frac{1}{2}(11.7)(27.3)\sin(52°40')$$

$$\approx 127\ cm^2$$

Example

Find the area of triangle *ABC*.



Solution

$$K = \frac{1}{2}ac\sin B$$

$$= \frac{1}{2}(34.0)(42.0)\sin(55°10')$$

$$\approx 586\ ft^2$$

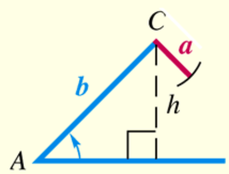
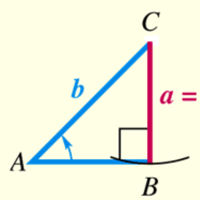
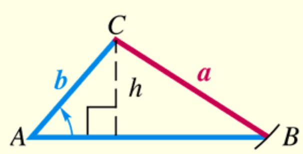
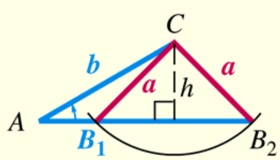
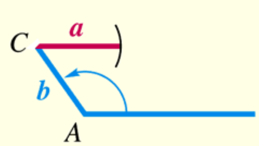
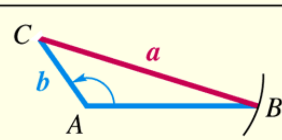
Number of Triangles Satisfying the Ambiguous Case (SSA)

Let sides a and b and angle A be given in triangle ABC. (The law of sines can be used to calculate the value of $\sin B$.)

1. If applying the law of sines results in an equation having $\sin B > 1$, then *no triangle* satisfies the given conditions.
2. If $\sin B = 1$, then *one triangle* satisfies the given conditions and $B = 90°$.
3. If $0 < \sin B < 1$, then either *one or two triangles* satisfy the given conditions.

   a) If $\sin B = k$, then let $B_1 = \sin^{-1} k$ and use B_1 for B in the first triangle.

   b) Let $B_2 = 180° - B_1$. If $A + B_2 < 180°$, then a second triangle exists. In this case, use B_2 for B in the second triangle.

Number of Triangles	Sketch	Applying Law of Sines Leads to
0		$\sin B > 1$, $a < h < b$
1		$\sin B = 1$, $a = h$ and $h < b$
1		$0 < \sin B < 1$, $a \geq b$
2		$0 < \sin B_2 < 1$, $h < a < b$
0		$\sin B \geq 1$, $a \leq b$
1		$0 < \sin B < 1$, $a > b$

Law of Cosines (*SAS*)

$$a^2 = b^2 + c^2 - 2bc\cos A \quad \rightarrow \quad a = \sqrt{b^2 + c^2 - 2bc\cos A}$$

$$b^2 = a^2 + c^2 - 2ac\cos B \quad \rightarrow \quad b = \sqrt{a^2 + c^2 - 2ac\cos B}$$

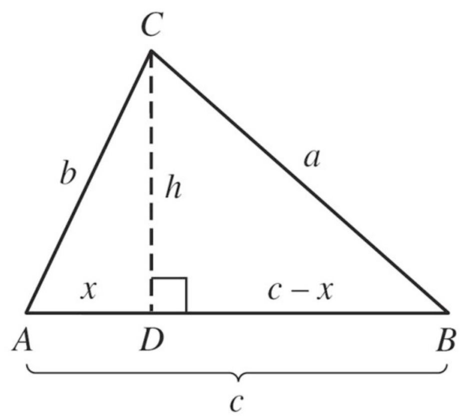
$$c^2 = a^2 + b^2 - 2ab\cos C \quad \rightarrow \quad c = \sqrt{a^2 + b^2 - 2ab\cos C}$$

Derivation

$$a^2 = (c-x)^2 + h^2$$

$$= c^2 - 2cx + x^2 + h^2 \qquad \textbf{(1)}$$

$$b^2 = x^2 + h^2 \qquad \textbf{(2)}$$



From **(2)**:

(1) $a^2 = c^2 - 2cx + b^2$

$$a^2 = c^2 + b^2 - 2cx \qquad \textbf{(3)}$$

$$\cos A = \frac{x}{b}$$

$$b\cos A = x$$

(3) $\Rightarrow$ $a^2 = c^2 + b^2 - 2cb\cos A$

Example

Find the missing parts in triangle *ABC* if $A = 60°$, $b = 20$ *in*, and $c = 30$ *in*.

Solution

$$a = \sqrt{20^2 + 30^2 - 2(20)(30)\cos 60°} \qquad a = \sqrt{b^2 + c^2 - 2bc\cos A}$$

$$\approx 26 \ \ in$$

$$B = \sin^{-1}\left(\frac{20\sin 60°}{26}\right) \qquad B = \sin^{-1}\left(\frac{b\sin A}{a}\right)$$

$$\approx 42°$$

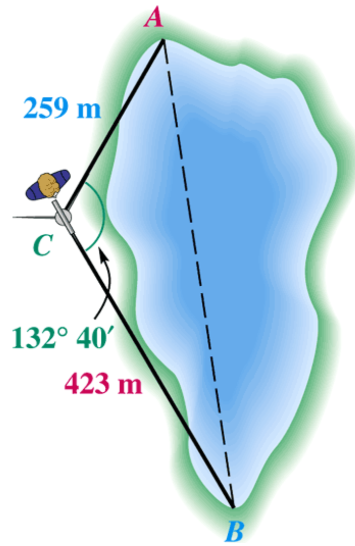
$$C = 180° - A - B$$

$$= 180° - 60° - 42°$$

$$\approx 78°$$

Example

A surveyor wishes to find the distance between two inaccessible points A and B on opposite sides of a lake. While standing at point C, she finds that $AC = 259\ m$, $BC = 423\ m$, and angle $ACB = 132°40'$. Find the distance AB.



Solution

$$AB = \sqrt{AC^2 + BC^2 - 2(AC)(BC)\cos C}$$

$$= \sqrt{259^2 + 423^2 - 2(259)(423)\cos(132°40')}$$

$$\approx 628\ m$$

Law of Cosines (*SSS*) - Three Sides

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \rightarrow \quad A = \cos^{-1}\frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \quad \rightarrow \quad B = \cos^{-1}\frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad \rightarrow \quad C = \cos^{-1}\frac{a^2 + b^2 - c^2}{2ab}$$

Example

Solve triangle *ABC* if *a* = 34 *km*, *b* = 20 *km*, and c = 18 *km*

Solution

$$A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right)$$

$$= \cos^{-1}\frac{20^2 + 18^2 - 34^2}{2(20)(18)}$$

$$\approx 127^\circ$$

$$C = \cos^{-1}\frac{a^2 + b^2 - c^2}{2ab}$$

$$= \cos^{-1}\frac{34^2 + 20^2 - 18^2}{2(34)(20)}$$

$$\approx 25^\circ$$

OR

$$\sin C = \frac{c \sin A}{a}$$

$$= \frac{18 \sin 127^\circ}{34}$$

$$C = \sin^{-1}\left(\frac{18 \sin 127^\circ}{34}\right)$$

$$\approx 25^\circ$$

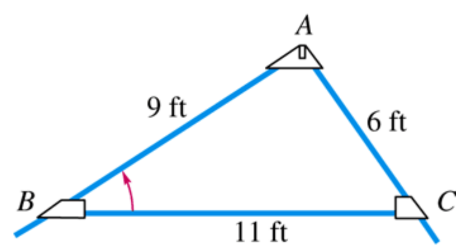
$$B = 180^\circ - A - C$$

$$= 180^\circ - 127^\circ - 25^\circ$$

$$\approx 28^\circ$$

Example

Find the measure of angle *B* in the figure of a roof truss.



Solution

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{11^2 + 9^2 - 6^2}{2(11)(9)}$$

$$B = \cos^{-1}\left(\frac{11^2 + 9^2 - 6^2}{2(11)(9)}\right)$$

$$\approx 33^\circ$$

Heron's Area Formula (*SSS*)

If a triangle has sides of lengths a, b, and c, with semi-perimeter

$$s = \frac{1}{2}(a+b+c)$$

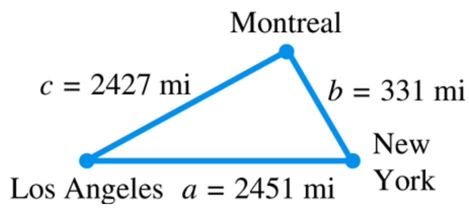
Then the area of the triangle is:

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

Example

The distance "as the crow flies" from Los Angeles to New York is 2451 *miles*, from New York to Montreal is 331 *miles*, and from Montreal to Los Angeles is 2427 *miles*. What is the area of the triangular region having these three cities as vertices? (Ignore the curvature of Earth.)

Solution



The semi-perimeter s is:

$$s = \frac{1}{2}(a+b+c)$$

$$= \frac{1}{2}(2451+331+2427)$$

$$\underline{= 2,604.5|}$$

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{2604.5(2604.5-2451)(2604.5-331)(2604.5-2427)}$$

$$\underline{\approx 401,700\ mi^2|}$$

Exercises *Section* 2.4 – Law of Sines and Cosines

1. In triangle ABC, $B = 110°$, $C = 40°$ and $b = 18\ in$. Find the length of side c.

(**2 – 10**) Find all the missing parts

2. $A = 110.4°$, $C = 21.8°$ and $c = 246\ in$

3. $B = 34°$, $C = 82°$, and $a = 5.6\ cm$

4. $B = 55°40'$, $b = 8.94\ m$, and $a = 25.1\ m$.

5. $A = 55.3°$, $a = 22.8\ ft.$, and $b = 24.9$ ft.

6. $A = 43.5°$, $a = 10.7\ in.$, and $c = 7.2\ in.$

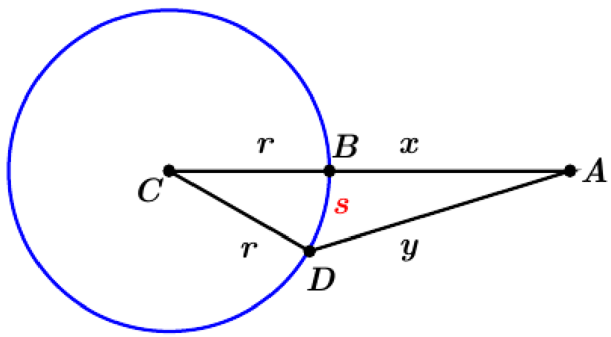
7. $b = 63.4\ km$, and $c = 75.2\ km$, $A = 124°\ 40'$

8. $A = 42.3°$, $b = 12.9m$, and $c = 15.4m$

9. $a = 832\ ft.$, $b = 623\ ft.$, and $c = 345\ ft.$

10. $a = 9.47\ ft$, $b = 15.9\ ft$, and $c = 21.1\ ft$

11. If $A = 26°$, $s = 22$, and $r = 19$, find x

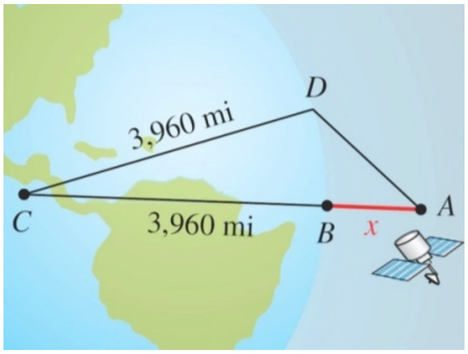


12. If $a = 13\ yd.$, $b = 14\ yd.$, and $c = 15\ yd.$, find the largest angle.

13. The diagonals of a parallelogram are 24.2 *cm* and 35.4 *cm* and intersect at an angle of 65.5°. Find the length of the shorter side of the parallelogram

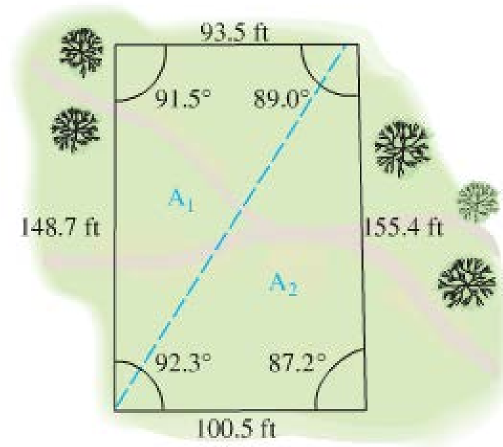
14. A man flying in a hot-air balloon in a straight line at a constant rate of 5 *feet per second*, while keeping it at a constant altitude. As he approaches the parking lot of a market, he notices that the angle of depression from his balloon to a friend's car in the parking lot is 35°. A minute and a half later, after flying directly over this friend's car, he looks back to see his friend getting into the car and observes the angle of depression to be 36°. At that time, what is the distance between him and his friend?

15. A satellite is circling above the earth. When the satellite is directly above point B, angle A is 75.4°. If the distance between points B and D on the circumference of the earth is 910 *miles* and the radius of the earth is 3,960 *miles*, how far above the earth is the satellite?

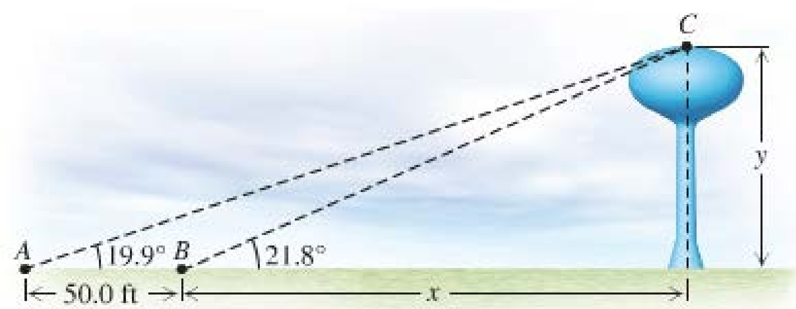


16. A pilot left Fairbanks in a light plane and flew 100 *miles* toward Fort in still air on a course with bearing of 18°. She then flew due east (bearing 90°) for some time drop supplies to a snowbound family. After the drop, her course to return to Fairbanks had bearing of 225°. What was her maximum distance from Fairbanks?

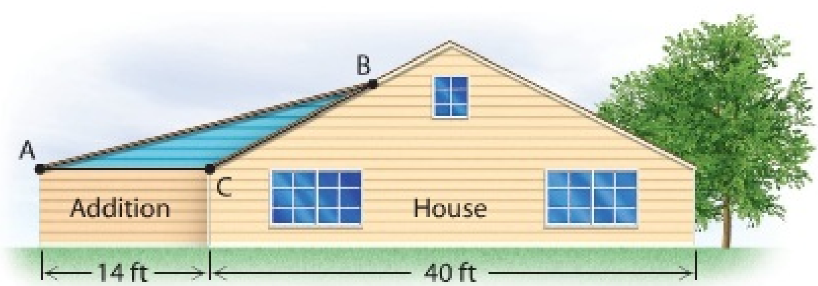
17. The dimensions of a land are given in the figure. Find the area of the property in square feet.



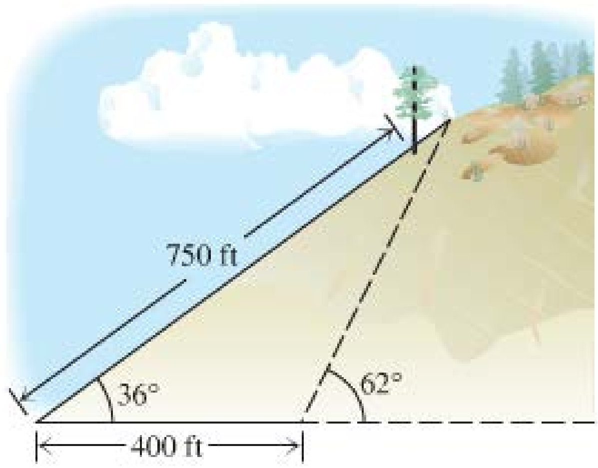
18. The angle of elevation of the top of a water tower from point A on the ground is 19.9°. From point B, 50.0 *feet* closer to the tower, the angle of elevation is 21.8°. What is the height of the tower?



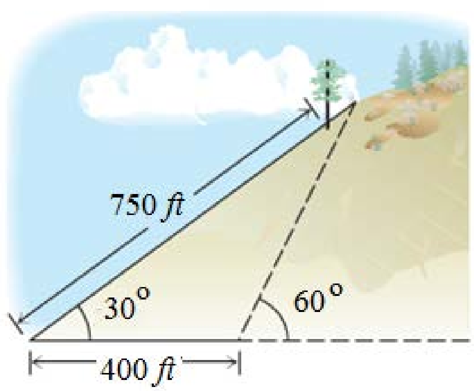
19. A 40-*feet* wide house has a roof with a 6-12 pitch (the roof rises 6 *feet* for a run of 12 *feet*). The owner plans a 14-*feet* wide addition that will have a 3-12 pitch to its roof. Find the lengths of $\overline{AB}$ and $\overline{BC}$



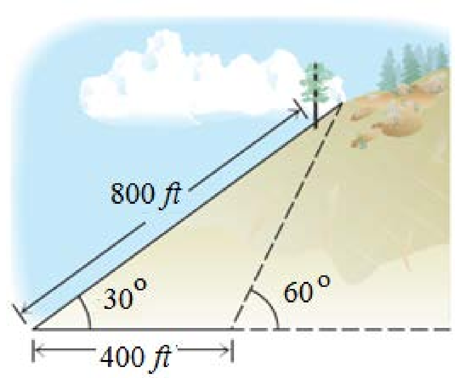
20. A hill has an angle of inclination of 36°. A study completed by a state's highway commission showed that the placement of a highway requires that 400 *feet* of the hill, measured horizontally, be removed. The engineers plan to leave a slope alongside the highway with an angle of inclination of 62°. Located 750 *feet* up the hill measured from the base is a tree containing the nest of an endangered hawk. Will this tree be removed in the excavation?



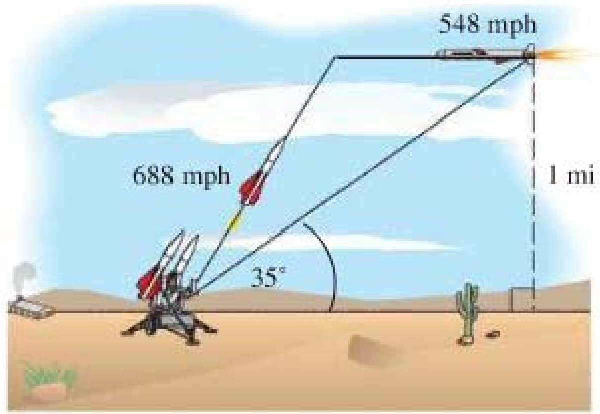
21. A hill has an angle of inclination of 30°. A study completed by a state's highway commission showed that the placement of a highway requires that 400 *feet* of the hill, measured horizontally, be removed. The engineers plan to leave a slope alongside the highway with an angle of inclination of 60°. Located 750 *feet* up the hill measured from the base is a tree containing the nest of an endangered hawk. Will this tree be removed in the excavation?



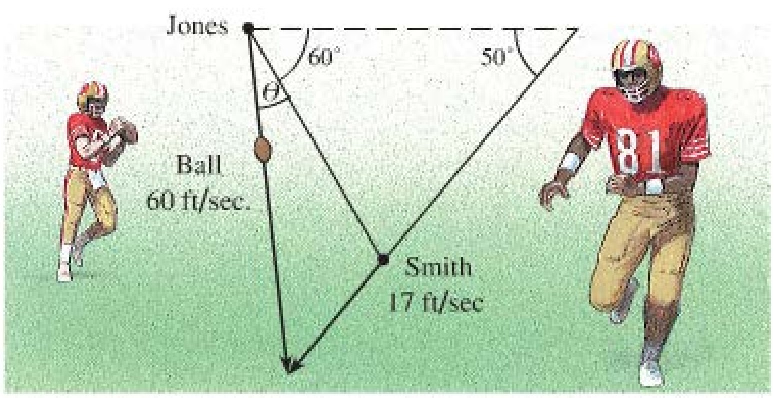
22. A hill has an angle of inclination of 30°. A study completed by a state's highway commission showed that the placement of a highway requires that 400 *feet* of the hill, measured horizontally, be removed. The engineers plan to leave a slope alongside the highway with an angle of inclination of 60°. Located 8000 *feet* up the hill measured from the base is a tree containing the nest of an endangered hawk. Will this tree be removed in the excavation?



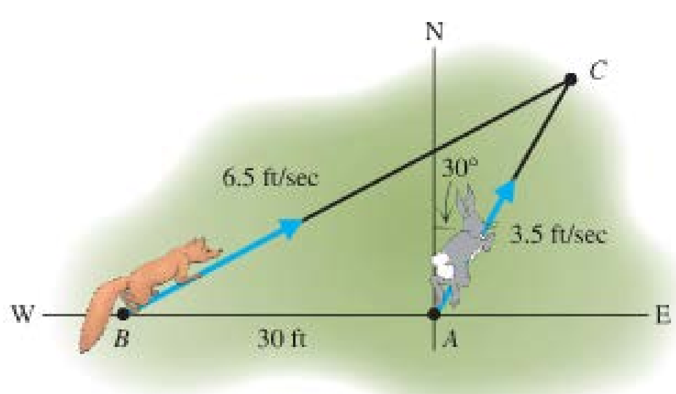
23. A cruise missile is traveling straight across the desert at 548 *mph* at an altitude of 1 *mile*. A gunner spots the missile coming in his direction and fires a projectile at the missile when the angle of elevation of the missile is 35°. If the speed of the projectile is 688 mph, then for what angle of elevation of the gun will the projectile hit the missile?



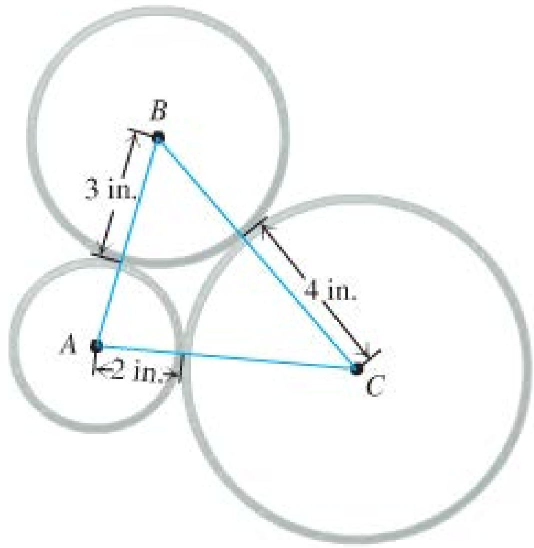
24. When the ball is snapped, Smith starts running at a 50° angle to the line of scrimmage. At the moment when Smith is at a 60° angle from Jones, Smith is running at 17 *ft/sec* and Jones passes the ball at 60 *ft/sec* to Smith. However, to complete the pass, Jones must lead Smith by the angle θ. Find θ (find θ in his head. Note that θ can be found without knowing any distances.)



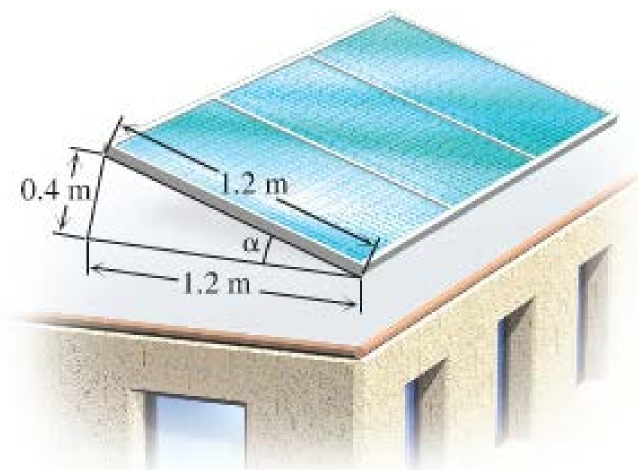
25. A rabbit starts running from point *A* in a straight line in the direction 30° from the north at 3.5 *ft/sec*. At the same time a fox starts running in a straight line from a position 30 *ft* to the west of the rabbit 6.5 *ft/sec*. The fox chooses his path so that he will catch the rabbit at point C. In how many seconds will the fox catch the rabbit?



26. An engineer wants to position three pipes at the vertices of a triangle. If the pipes A, B, and C have radii 2 *in*, 3 *in*, and 4 *in*, respectively, then what are the measures of the angles of the triangle ABC?

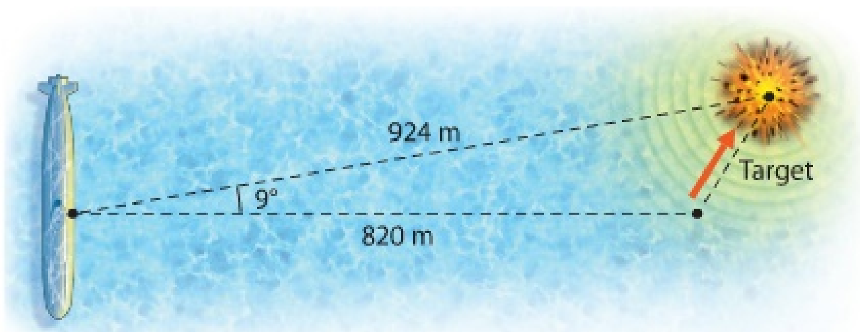


27. A solar panel with a width of 1.2 *m* is positioned on a flat roof. What is the angle of elevation α of the solar panel?

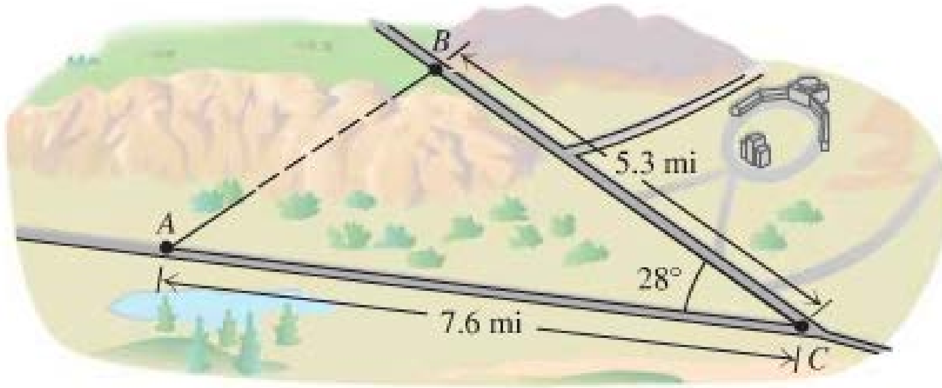


28. Andrea and Steve left the airport at the same time. Andrea flew at 180 *mph* on a course with bearing 80°, and Steve flew at 240 *mph* on a course with bearing 210°. How far apart were they after 3 *hr*.?

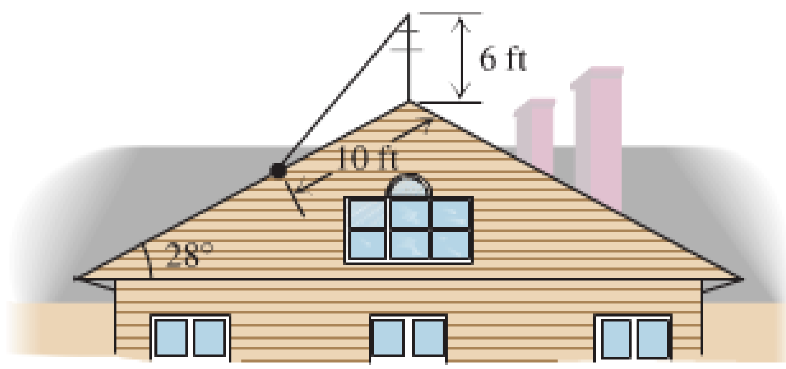
29. A submarine sights a moving target at a distance of 820 *m*. A torpedo is fired 9° ahead of the target, and travels 924 *m* in a straight line to hit the target. How far has the target moved from the time the torpedo is fired to the time of the hit?



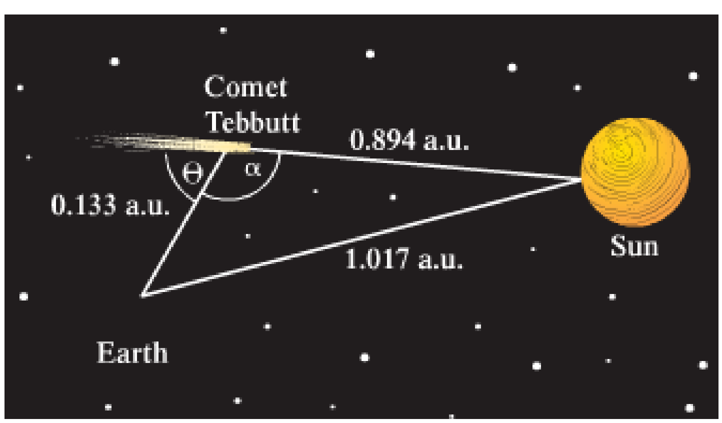
30. A tunnel is planned through a mountain to connect points A and B on two existing roads. If the angle between the roads at point C is 28°, what is the distance from point A to B? Find $\angle CBA$ and $\angle CAB$ to the nearest tenth of a degree.



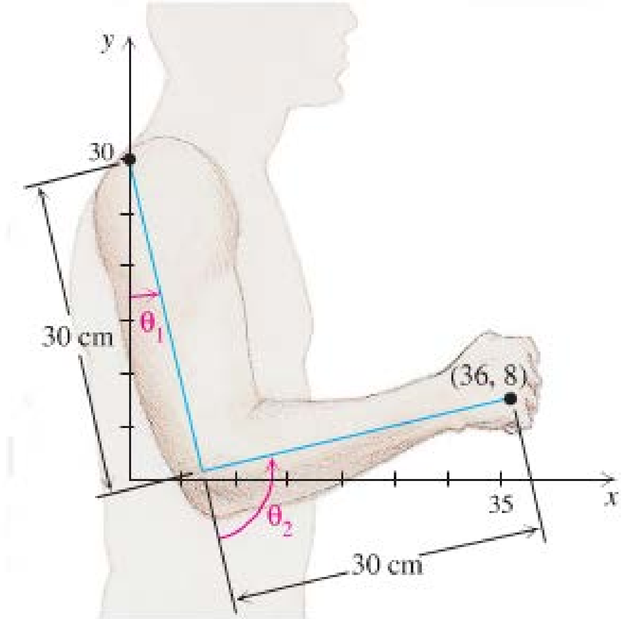
31. A 6- *feet* antenna is installed at the top of a roof. A guy wire is to be attached to the top of the antenna and to a point 10 *feet* down the roof. If the angle of elevation of the roof is 28°, then what length guy wire is needed?



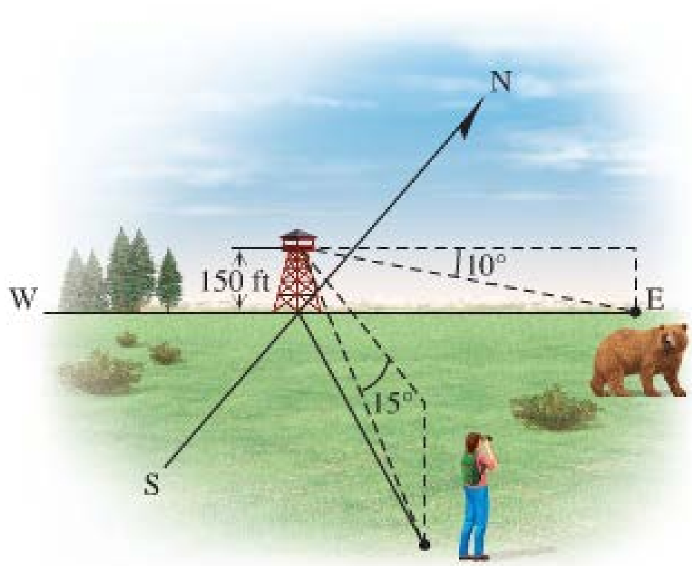
32. On June 30, 1861, Comet Tebutt, one of the greatest comets, was visible even before sunset. One of the factors that causes a comet to be extra bright is a small scattering angle θ. When Comet Tebutt was at its brightest, it was 0.133 *a.u.* from the earth, 0.894 *a.u.* from the sun, and the earth was 1.017 *a.u.* from the sun. Find the phase angle α and the scattering angle θ for Comet Tebutt on June 30, 1861. (One astronomical unit (*a.u*) is the average distance between the earth and the sub.)



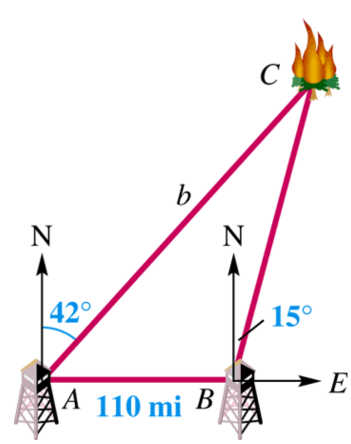
33. A human arm consists of an upper arm of 30 *cm* and a lower arm of 30 *cm*. To move the hand to the point (36, 8), the human brain chooses angle θ_1 *and* θ_2 to the nearest tenth of a degree.



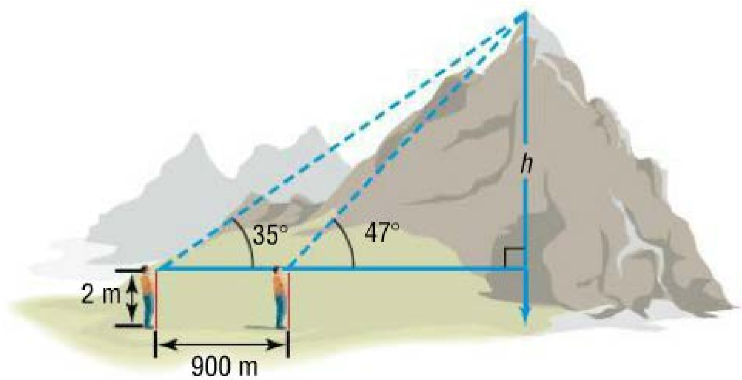
34. A forest ranger is 150 *feet* above the ground in a fire tower when she spots an angry grizzly bear east of the tower with an angle of depression of 10°. Southeast of the tower she spots a hiker with an angle of depression of 15°. Find the distance between the hiker and the angry bear.



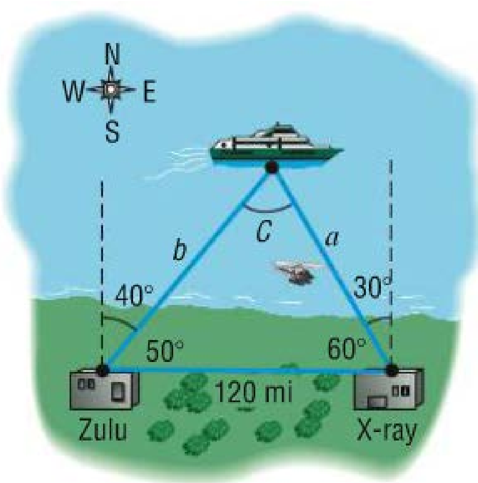
35. Two ranger stations are on an east-west line 110 *mi* apart. A forest fire is located on a bearing N 42° E from the western station at *A* and a bearing of N 15° E from the eastern station at *B*. How far is the fire from the western station?



36. To measure the height of a mountain, a surveyor takes two sightings of the peak at a distance 900 *meters* apart on a direct line to the mountain. The first observation results in an angle of elevation of 47°, and the second results in an angle of elevation of 35°. If the transit is 2 *meters* high, what is the height h of the mountain?



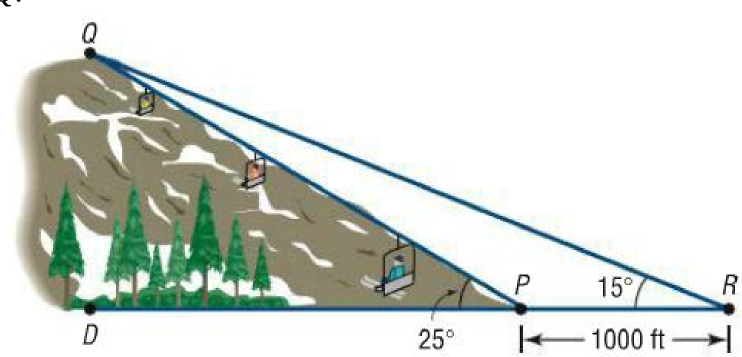
37. A Station Zulu is located 120 *miles* due west of Station X-ray. A ship at sea sends an SOS call that is received by each station. The call to Station Zulu indicates that the bearing of the ship from Zulu is $N\ 40°\ E$. The call to Station X-ray indicates that the bearing of the ship from X-ray is $N\ 30°\ W$.



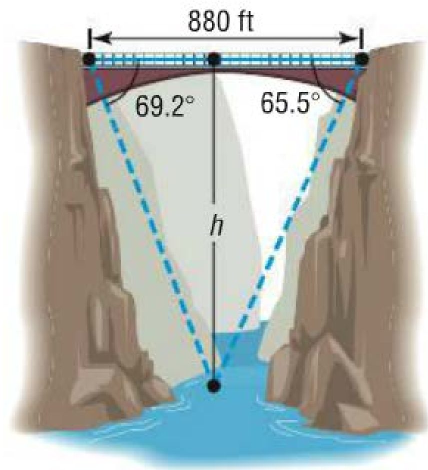
a) How far is each station from the ship?

b) If a helicopter capable of flying 200 *miles* per *hour* is dispatched from the nearest station to the ship, how long will it take to reach the ship?

38. To find the length of the span of a proposed ski lift from *P* to *Q*, a surveyor measures $\angle DPQ$ to be $25°$ and then walks back a distance of 1000 *feet* to *R* and measures $\angle DRQ$ to be $15°$. What is the distance from *P* to *Q*.



39. The highest bridge in the world is the bridge over the Royal Gorge of the Arkansas River in Colorado, sightings to the same point at water level directly under the bridge are taken from each side of the 880–foot–long bridge.



(**40 – 53**) Find the area of the triangle

40. $b=1, \quad c=3, \quad A=80°$

41. $b=4, \quad c=1, \quad A=120°$

42. $a=2, \quad c=1, \quad B=10°$

43. $a=3, \quad c=2, \quad B=110°$

44. $a=8, \quad b=6, \quad C=30°$

45. $a=3, \quad b=4, \quad C=60°$

46. $a=6, \quad b=4, \quad C=60°$

47. $a=4, \quad b=5, \quad c=7$

48. $a=12, \quad b=13, \quad c=5$

49. $a=3, \quad b=3, \quad c=2$

50. $a=4, \quad b=5, \quad c=3$

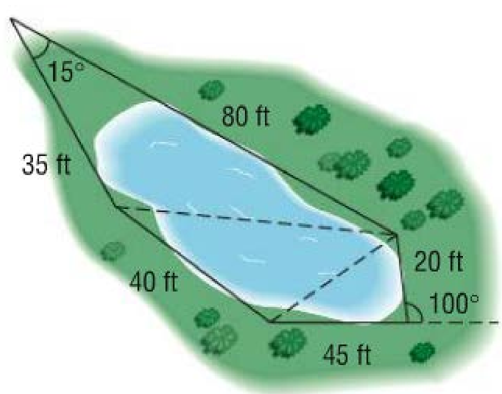
51. $a=5, \quad b=8, \quad c=9$

52. $a=2, \quad b=2, \quad c=2$

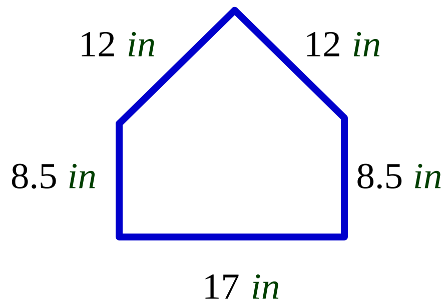
53. $a=4, \quad b=3, \quad c=6$

54. The dimensions of a triangular lot are 100 *feet* by 50 *feet* by 75 *feet*. If the price of such land is \$3 per square foot, how much does the lot cost?

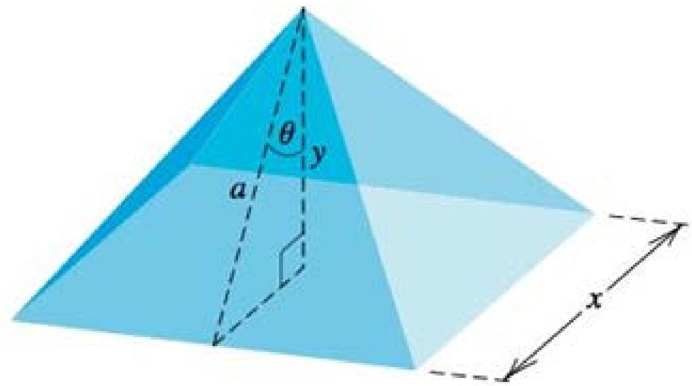
55. To approximate the area of a lake, a surveyor walks around the perimeter of the lake. What is the approximate area of the lake?



56. The dimensions of home plate at any major league baseball stadium are shown. Find the area of home plate



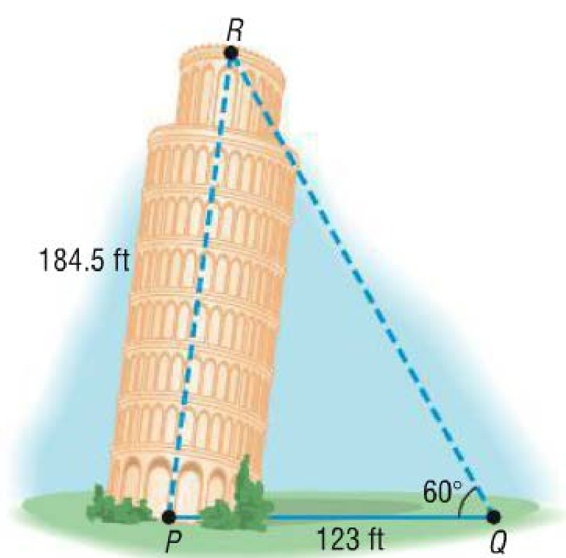
57. A pyramid has a square base and congruent triangular faces. Let θ be the angle that the altitude a of a triangular face makes with the altitude y of the pyramid, and let x be the length of a side.



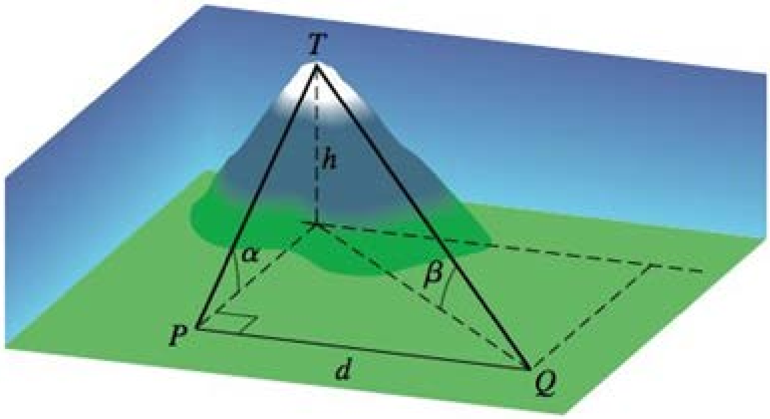
a) Express the total surface area S of the four faces in terms of a and θ.

b) The volume V of the pyramid equals one-third the area of the base times the altitude. Express V in terms of a and θ.

58. The famous Leaning Tower of Pisa was originally 184.5 *feet* high. At a distance of 123 *feet* from the base if the tower, the angle of elevation to the top of the tower is found to be 60°. Find the $\angle RPQ$ indicated in the figure. Also, find the perpendicular distance from R to PQ.



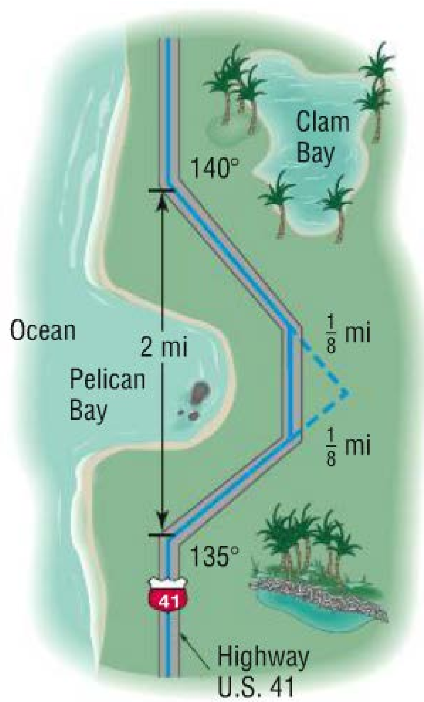
59. If a mountaintop is viewed from a point P due south of the mountain, the angle of elevation is α. If viewed from a point Q that is d miles cast of P, the angle of elevation is β.



a) Show that the height h of the mountain is given by $h = \dfrac{d \sin\alpha \sin\beta}{\sqrt{\sin^2\alpha - \sin^2\beta}}$

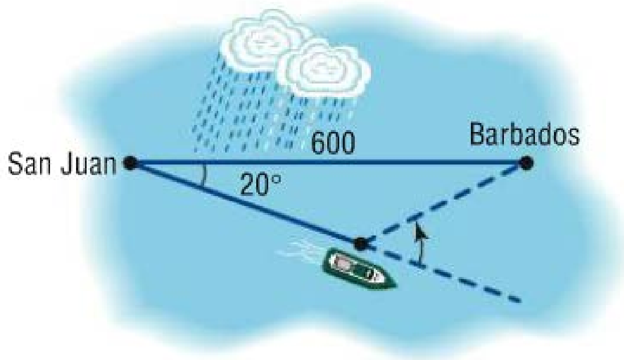
b) If $\alpha = 30°, \quad \beta = 20°, \quad and \quad d = 10\ mi$, approximate h.

60. A highway whose primary directions are north–south, is being constructed along the west coast of Florida. Near Naples, a bay obstructs the straight path of the road. Since the cost of a bridge is prohibitive, engineers decide to go around the bay. The path that they decide on and the measurements taken as shown in the picture. What is the length of highway needed to go around the bay?



61. Derive the Mollweide's formula: $\dfrac{a-b}{c} = \dfrac{\sin\left[\frac{1}{2}(A-B)\right]}{\cos\left(\frac{1}{2}C\right)}$

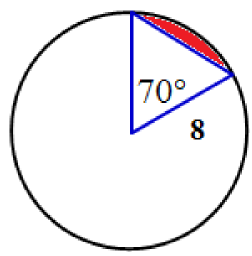
62. A cruise ship maintains an average speed of 15 *knots* in going from San Juan, Puerto Rico, to Barbados, West Indies, a distance of 600 nautical miles. To avoid a tropical storm, the captain heads out to San Juan in a direction of 20° off a direct heading to Barbados. The captain maintains the 15–knots speed for 10 *hours*, after which time the path to Barbados becomes clear of storms.



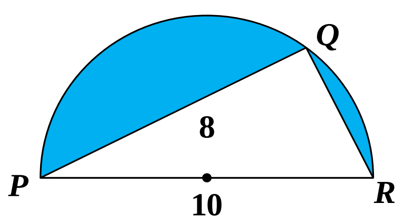
a) Through what angle should the captain turn to head directly to Barbados?

b) Once the turn is made, how long will it be before the ship reaches Barbados if the same 15–knot speed is maintained?

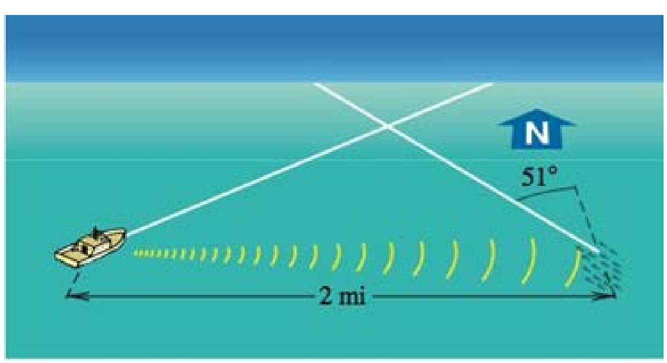
63. Find the area of the segment (shaded in blue in the figure) of a circle whose radius is 8 *feet*, formed by a cebtral angle of $70°$



64. Find the area of the shaded region enclosed in a semicircle of diameter 10 *inches*. The length of the chord *PQ* is 8 *inches*.



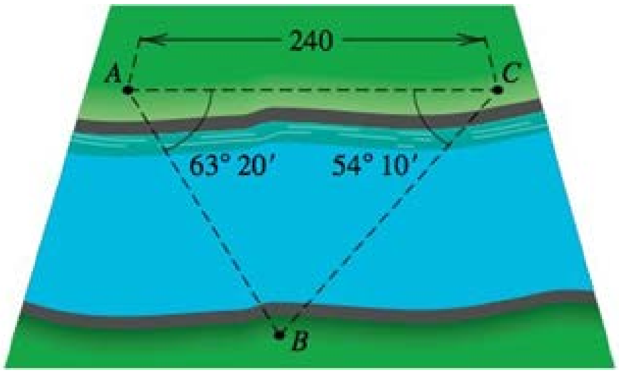
65. A commercial fishing boat uses sonar equipment to detect a school of fish 2 *miles* east of the boat and traveling in the direction of $N\ 51°\ W$ at a rate of $8\ mi/hr$



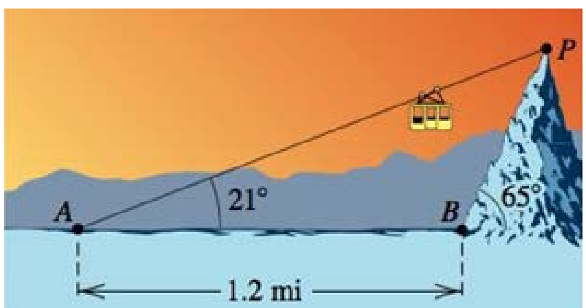
a) The boat travels at 20 mi / hr, approximate the direction it should head to intercept the school of fish.

b) Find, to the nearest minute, the time it will take the boat to reach the fish.

66. To find the distance between two points *A* and *B* that lie on opposite banks of a river, a surveyor lays off a line segment *AC* of length 240 *yards* along one bank and determines that the measures of $\angle BAC$ and $\angle ACB$ are $63^\circ\ 20'$ and $54^\circ\ 10'$, respectively.



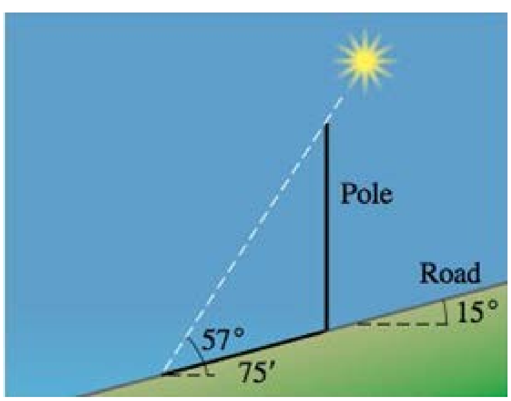
67. A cable car carries passengers from a point *A*, which is 1.2 *miles* from a point *B* at the base of a mountain, to a point *P* at the top of the mountain. The angle of elevation of *P* from *A* and *B* are 21° and 65°, respectively.



a) Approximate the distance between *A* and *P*.

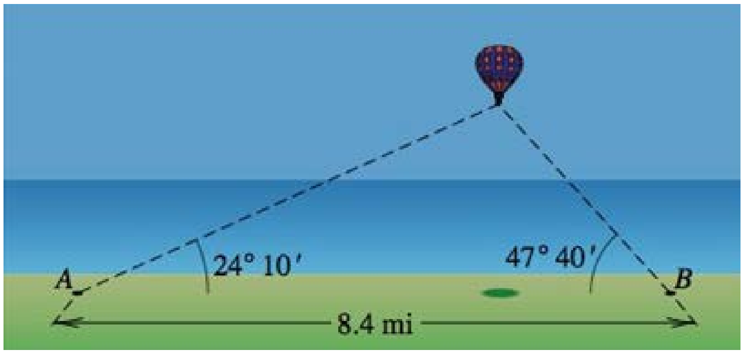
b) Approximate the height of the mountain.

68. A straight road makes an angle of 15° with the horizontal. When the angle of elevation of the sun is 57°, a vertical pole at the side of the road casts a shadow 75 *feet* long directly down the road.

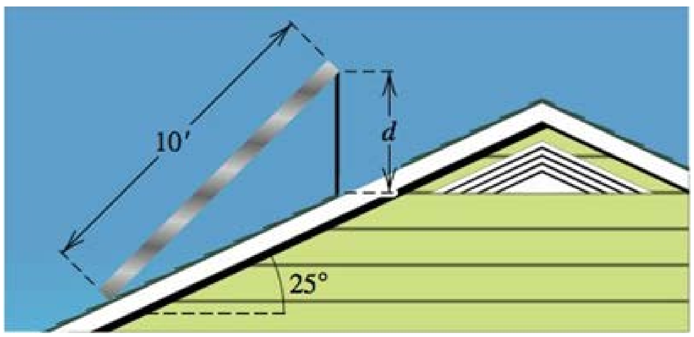


Approximate the length of the pole.

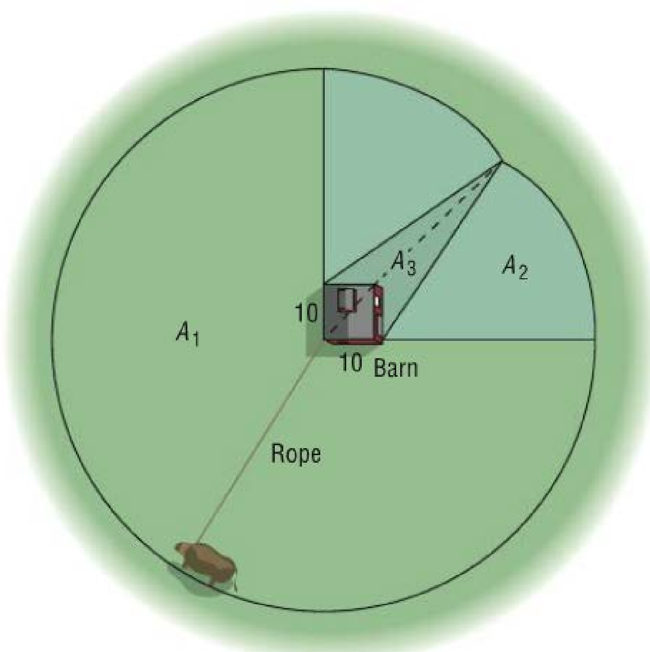
69. The angles of elevation of a balloon from two points A and B on level ground are $24° 10'$ and $47° 40'$, respectively. Points A and B are 8.4 *miles* apart, and the balloon is between the points, in the same vertical plane. Approximate the height of the balloon above the ground.



70. A solar panel 10 *feet* in width, which is to be attached to a roof that makes an angle of $25°$ with the horizontal. Approximate the length d of the brace that is needed for the panel to make an angle of $45°$ with the horizontal.



71. A cow is tethered to one corner of a square barn, 10 *feet* by 10 *feet*, with a rope 100 *feet* long.

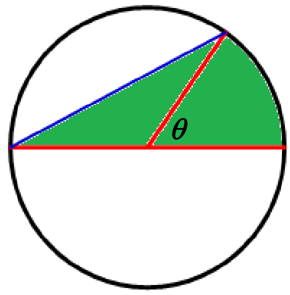


a) What is the maximum grazing area for the cow?

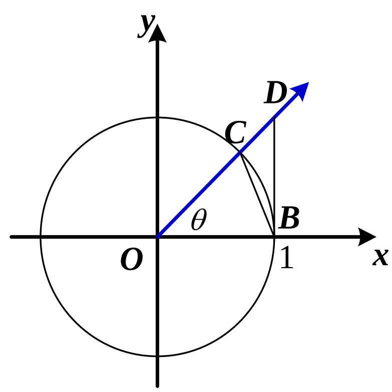
b) If the barn is rectangular, 10 *feet* by 20 *feet*, what is the maximum grazing area for the cow?

72. For any triangle, show that $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$ *where* $s=\frac{1}{2}(a+b+c)$

73. The figure shows a circle of radius r with center at O. find the area K of the shaded region as a function of the central angle θ.



74. Refer to the figure, in which a unit circle is drawn. The line segment DB is tangent to the circle and θ is acute.



a) Express the area of ΔOBC in terms of $\sin\theta$ and $\cos\theta$.

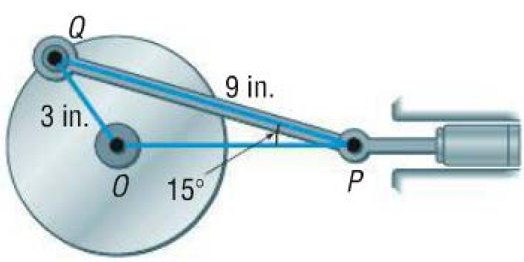
b) Express the area of ΔOBD in terms of $\sin\theta$ and $\cos\theta$.

c) The area of the sector $\overset{\frown}{OBC}$ if the circle is $\frac{1}{2}\theta$, where θ is measured in *radians*. Use the results of part (*a*) and (*b*) and the fact that

$$Area\ \Delta OBC < Area\ \overset{\frown}{OBC} < Area\ \Delta OBD$$

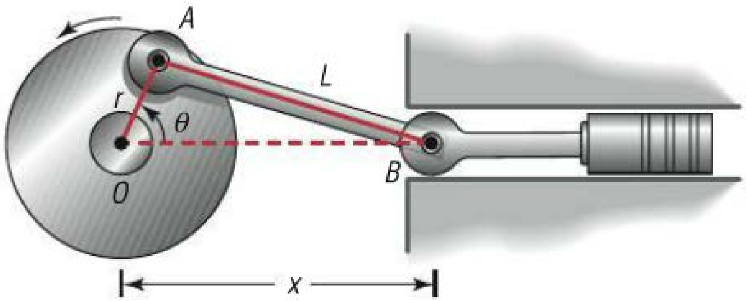
To show that $1<\frac{\theta}{\sin\theta}<\frac{1}{\cos\theta}$

75. On a certain automobile, the crankshaft is 3 *inches* long and the connecting rod is 9 *inches* long. At the time when $\angle OPQ$ is 15°, how far is the piston P from the center O of the crankshaft?



76. Rod OA rotates about the fixed point O so that point A travels on a circle of radius r. Connected to point A is another rod AB of length $L > 2r$, and point B is connected to a piston. Show that the distance x between point O and point B is given by

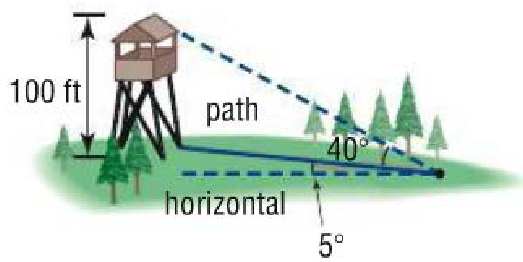
$$x = r\cos\theta + \sqrt{r^2\cos^2\theta + L^2 - r^2}$$



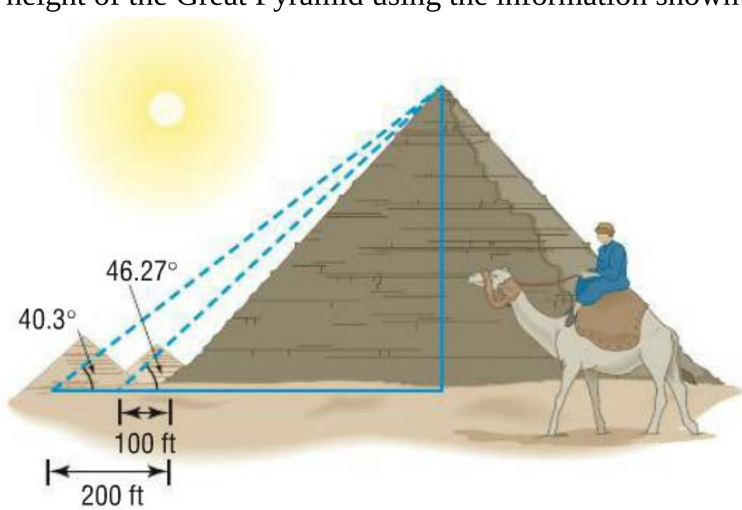
Where θ is the angle of rotation of rod OA.

77. Find the area of the segment of a circle whose radius is 5 *inches*, formed by a central angle of $40°$.

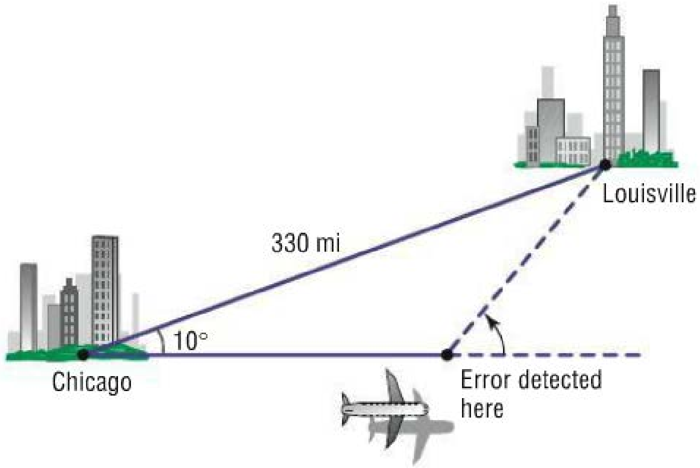
78. A forest ranger is walking on a path inclined at 5° to the horizontal directly toward a 100–foot–tall fire observation tower. The angle of elevation from the path to the top of the tower is 40°. How far is the ranger from the tower at this time?



79. One of the original Seven Wonders of the world, the Great Pyramid of Cheops was built about 2580 BC. Its original height was 480 *feet* 11 *inches*, but owing to the loss of its topmost stones, it is now shorter. Find the current height of the Great Pyramid using the information shown in the picture.



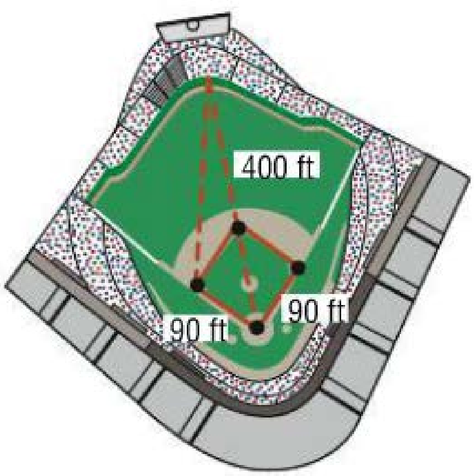
80. In attempting to fly from Chicago to Louisville, a distance of 330 *miles*, a pilot inadvertently took a course that was 10° in error.



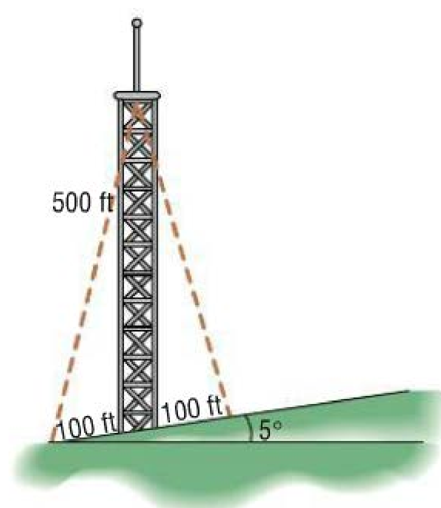
a) If the aircraft maintains an average speed of 220 *miles* per *hours*, and if the error in direction is discovered after 15 *minutes*, through what angle should the pilot turn to head toward Louisville?

b) What new average speed should the pilot maintain so that the total time of the trip is 90 *minutes*?

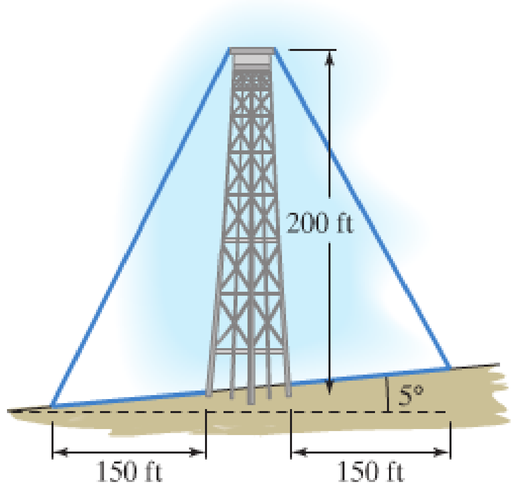
81. The distance from home plate to the fence in dead center is 400 *feet*. How far is it for the fence in dead center to third base?



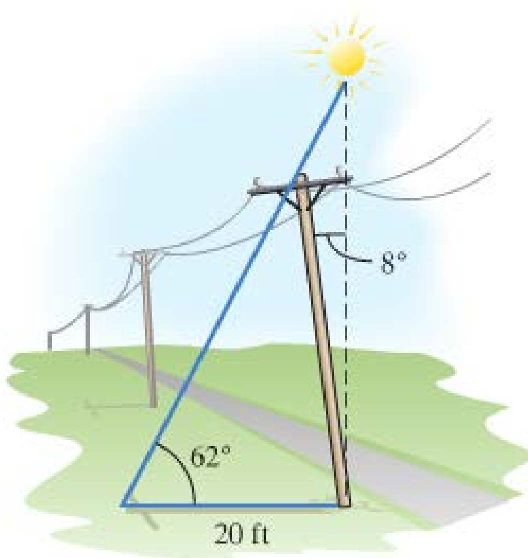
82. A radio tower 500 *feet* high is located on the side of a hill with an inclination to the horizontal of 5°. How long should two guy wires be if they are to connect to the top of the tower and be secured at two points 100 *feet* directly above and directly below the base of the tower?



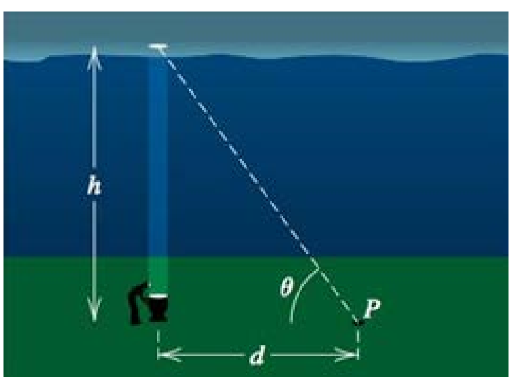
83. A 200-*foot* tower on the side of a hill that forms a 5° angle with the horizontal. Find the length of each of the two guy wires that are anchored 150 *feet* uphill and downhill from the tower's base and extend to the top of the tower.



84. When the angle of elevation of the sun is 62°, a telephone pole that is tilted at an angle of 8° directly away from the sun casts a shadow 20 *feet* long. Determine the length of the pole.



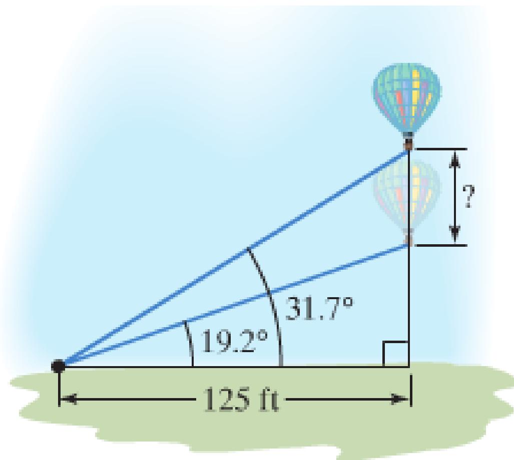
85. To measure the height h of a cloud cover, a meteorology student directs a spotlight vertically upward from the ground. From a point P on level ground that is d *meters* from the spotlight, the angle of elevation θ of the light image on the clouds is then measured.



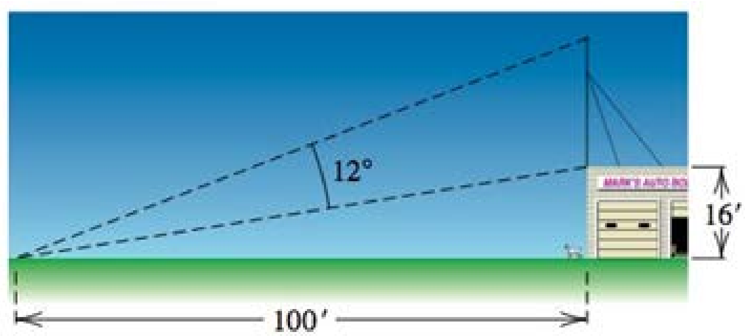
a) Express h in terms of d and θ

b) Approximate h if $d = 1000\ m$ and $\theta = 59°$

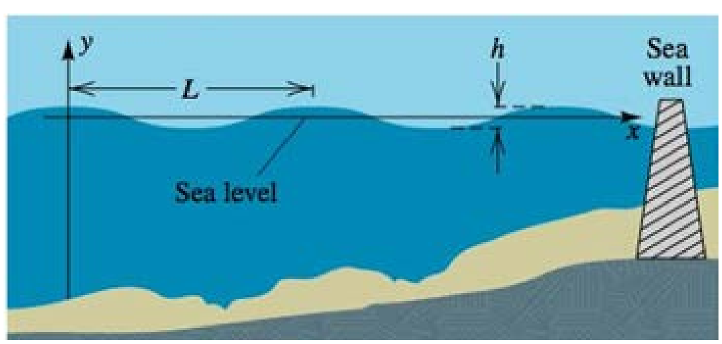
86. A hot–air balloon is rising vertically. From a point on level ground 125 *feet* from the point directly under the passenger compartment, the angle of elevation to the balloon changes from $19.2°$ to $31.7°$. How far does the balloon rise during this period?



87. A *CB* antenna is located on the top of a garage that is 16 *feet* tall. From a point on level ground that is 100 *feet* from a point directly below the antenna, the antenna subtends an angle of 12°. Approximate the length of the antenna.



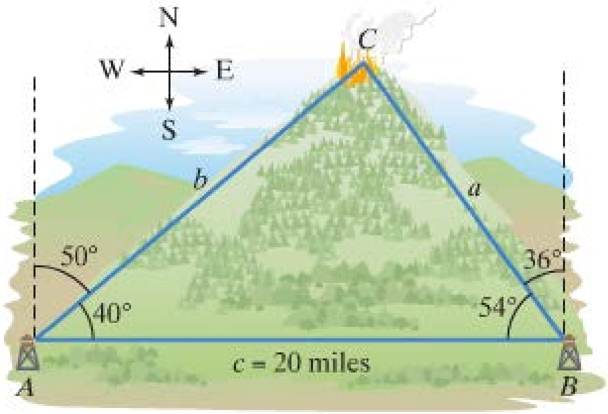
88. A tsunami is a tidal wave caused by an earthquake beneath the sea. These waves can be more than 100 *feet* in height and can travel at great speeds. Engineers sometimes represent such waves by trigonometric expressions of the form $y = a\cos bt$ and use these representations to estimate the effectiveness of sea walls. Suppose that a wave has height $h = 50\ ft$ and period time 30 *minutes* and is traveling at the rate of $180\ ft/\sec$



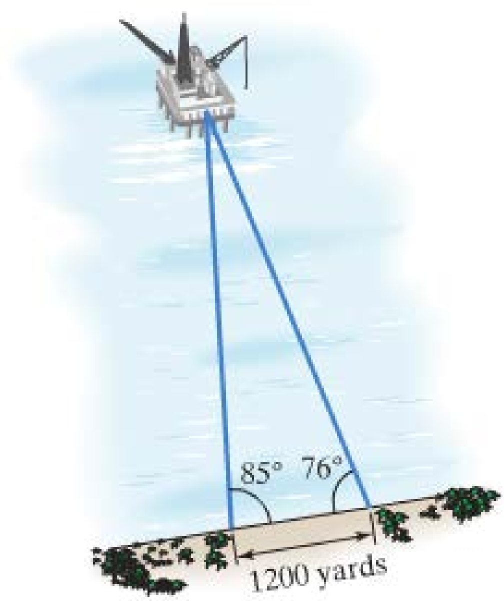
a) Let (x, y) be a point on the wave represented in the figure. Express y as a function of t if $y = 25\ ft$ when $t = 0$.

b) The wave length L is the distance between two successive crests of the wave. Approximate L in *feet*.

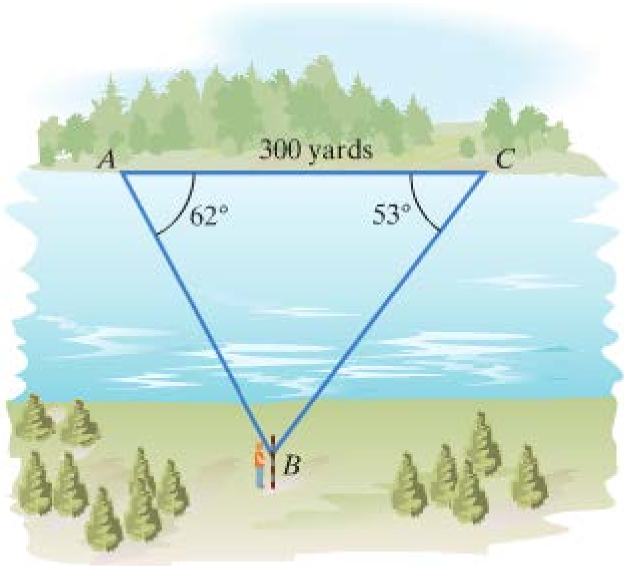
89. Two fire–lookout stations are 20 *miles* apart, with station B directly east of station A. Both stations spot fire on a mountain to the north. The bearing from station A to the fire is N50°E. The bearing from station B to the fire is N36°W. How far is the fire from station A?



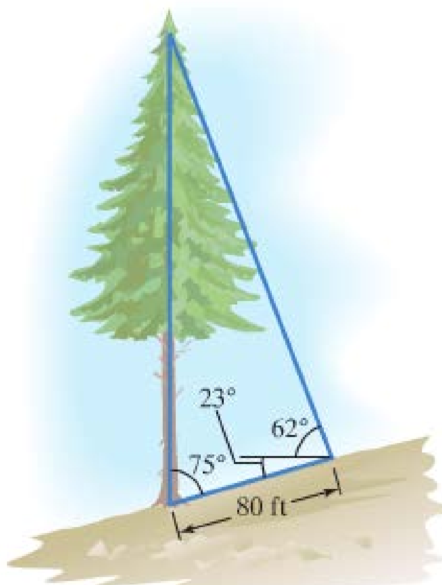
90. A 1200–yard–long sand beach and an oils platform in the ocean. The angle made with the platform from one end of the beach is 85° and from the other end is 76°. Find the distance of the oil platform from each end of the beach.



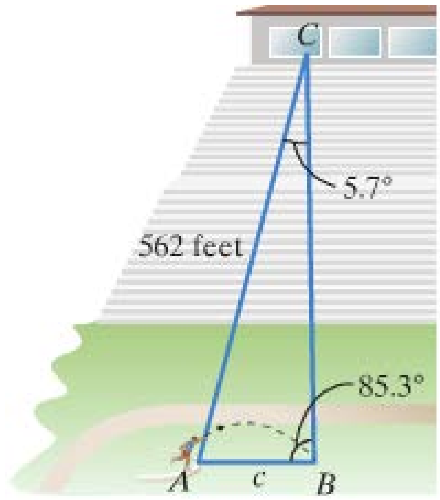
91. A surveyor needs to determine the distance between two points that lie on opposite banks of a river. 300 *yards* are measured along one bank. The angle from each end of this line segment to a point on the opposite bank are 62° and 53°. Find the distance between *A* and *B*.



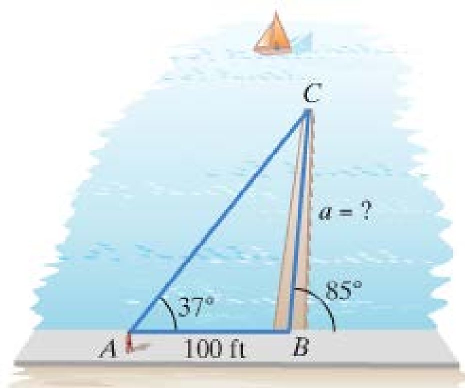
92. A pine tree growing on a hillside makes a 75° angle with the hill. From a point 80 *feet* up the hill, the angle of elevation to the top of the tree is 62° and the angle of depression to the bottom is 23°. Find the height of the tree.



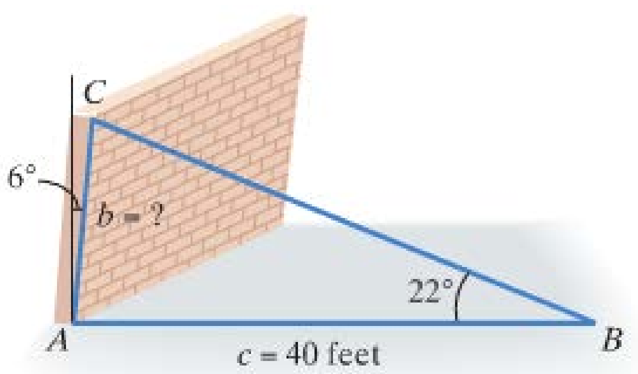
93. The shot of a hot-put ring is tossed from *A* lands at *B*. Using modern electronic equipment, the distance of the toss can be measured without the use of measuring tapes. When the shot lands at *B*, an electronic transmitter placed at *B* sends a signal to a device in the official's booth above the track. The device determines the angles *B* and *C*. At a track meet, the distance from the official' booth to the shot-ring is 562 *feet*. If $B = 85.3°$ and $C = 5.7°$, determine the length of the toss.



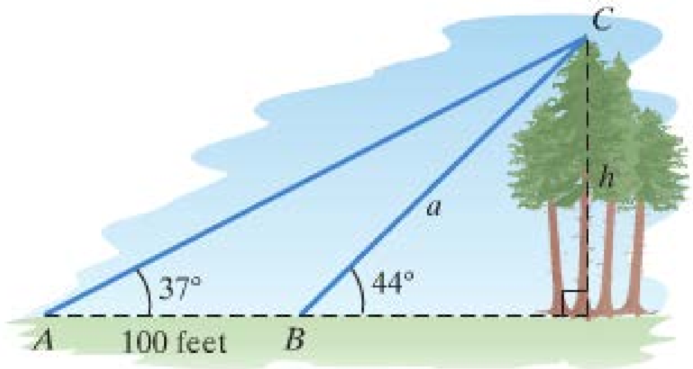
94. A pier forms an 85° angle with a straight shore. At a distance of 100 *feet* from a pier, the line of sight to the tip forms a 37° angle. Find the length of the pier.



95. A leaning wall is inclined 6° from the vertical. At a distance of 40 *feet* from the wall, the angle of elevation to the top is 22°.Find the height of the wall.



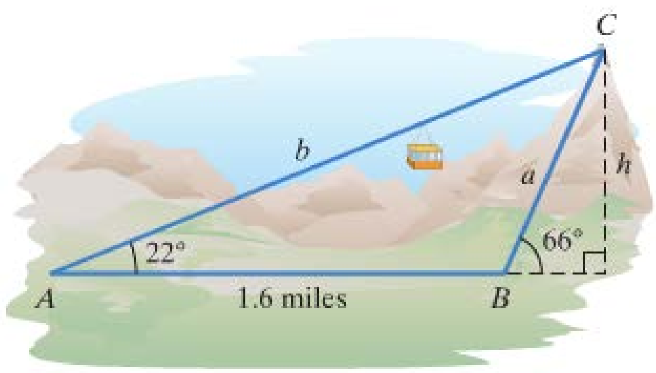
96. Redwood trees are hundreds of feet tall. The height of one of these is represented by h.



a) Find the height of the tree.

b) Find a.

97. A carry cable car that carries passengers from A to C. Point A is 1.6 *miles* from the base of the mountain. The angles of elevation from A and B to the mountain's peak are 22° and 66°, respectively.

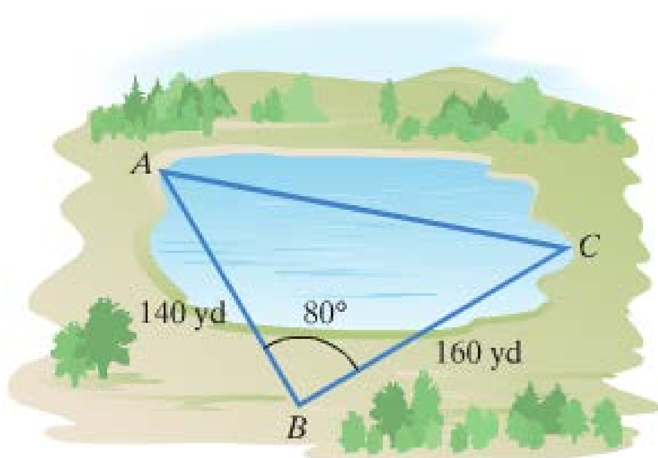


a) Find the height of the mountain.

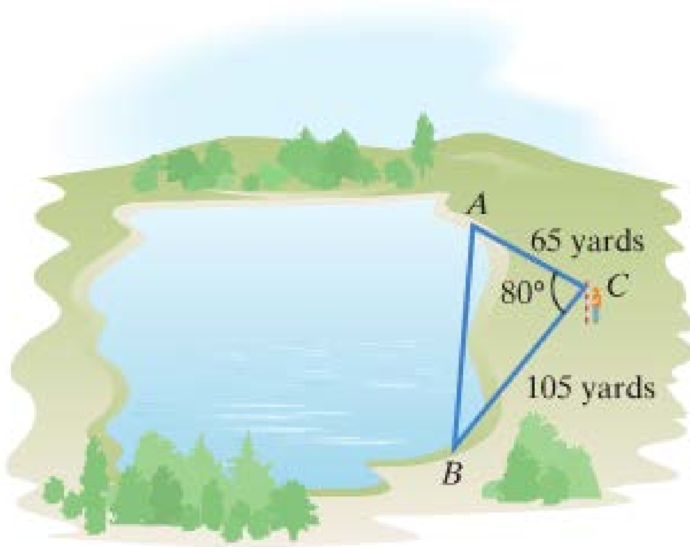
b) Determine the distance covered by the cable car.

c) Find a.

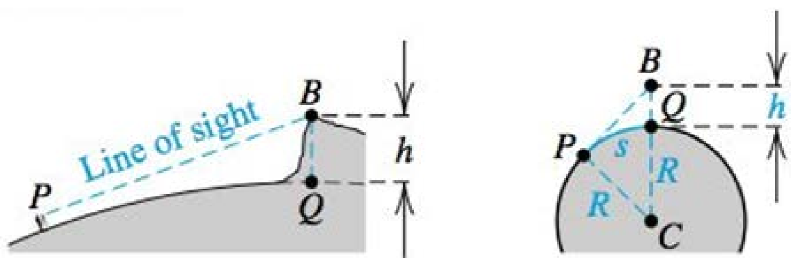
98. Find the distance across the lake from A to C.



99. To find the distance across a protected cove at a lake, a surveyor makes the measurements. Find the distance from A to B.



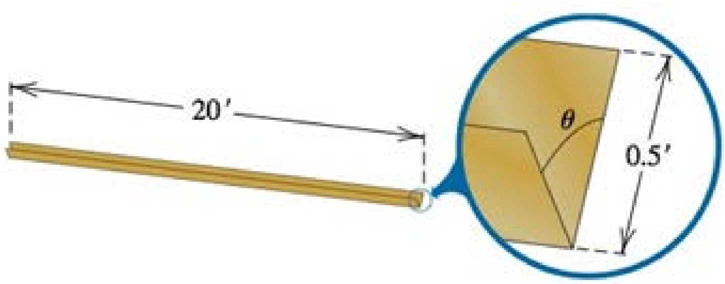
100. A surveyor using a transit, sights the edge B of a bluff, as shown in the left of the figure. Because of the curvature of Earth, the true elevation $\boldsymbol{h}$ of the bluff is larger than that measured by the surveyor. A cross-sectional schematic view of Earth is shown in the right part of the figure.



a) If s is the length of arc PQ and R is the distance from P to the center C of Earth, express h in terms of R and s.

b) If $R = 4,000\ mi$ and $s = 50\ mi$, estimate the elevation of the bluff in *feet*.

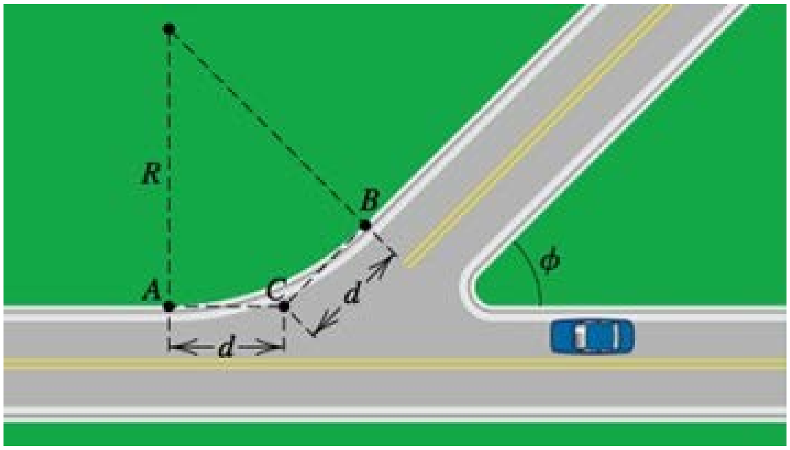
101. Shown in the figure is a design for a rain gutter.



a) Express the volume V as a function of θ.

b) Approximate the acute angle θ that results in a volume of $2\ ft^3$

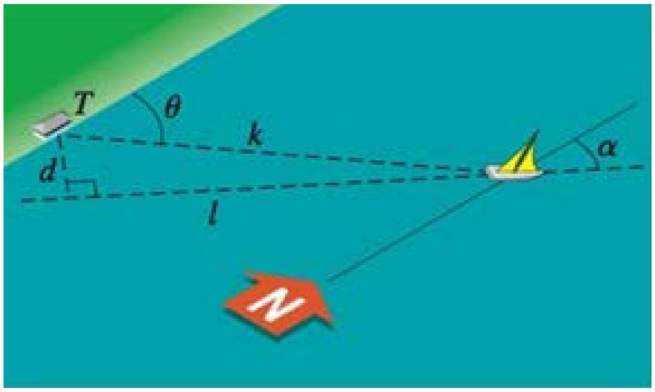
102. A highway engineer is designing curbing for a street at an intersection where two highways meet at an angle ϕ, as shown in the figure, the curbing between points *A* and *B* is to be constructed using a circle that is tangent to the highway at these two points.



a) Show that the relationship between the radius *R* of the circle and the distance *d* in the figure is given by the equation $d = R\tan\frac{\phi}{2}$.

b) If $\phi = 45°$ and $d = 20\ ft$, approximate *R* and the length of the curbing.

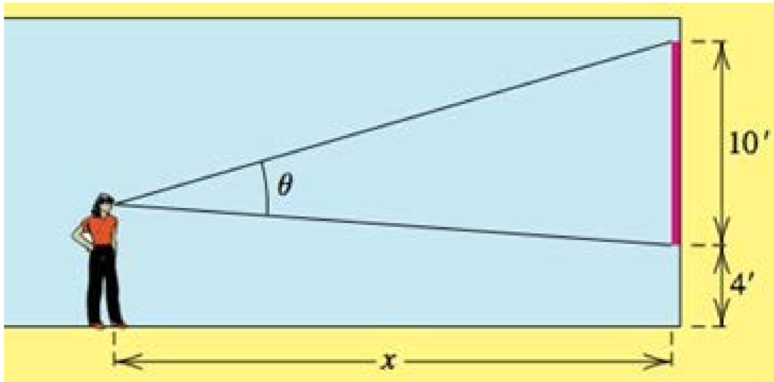
103. A sailboat is following a straight line course l. (Assume that the shoreline is parallel to the north-south line.) The shortest distance from a tracking station *T* to the course is *d* miles. As the boat sails, the tracking station records its distance *k* from *T* and its direction θ with respect to *T*. Angle α specifies the direction of the sailboat.



a) Express α in terms of *d*, *k*, and θ.

b) Estimate α to the nearest degree if $d = 50\ mi$, $k = 210\ mi$, and $\theta = 53.4°$

104. An art critic whose eye level is 6 *feet* above the floor views a painting that is 10 *feet* in height and is mounted 4 *feet* above the floor.



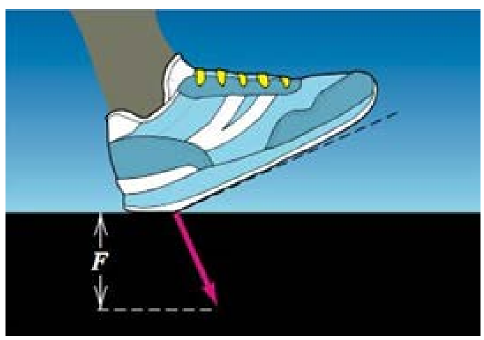
a) If the critic is standing x *feet* from the wall, express the viewing angle θ in terms of x.

b) Use the addition formula for the tangent to show that $\theta = \tan^{-1}\left(\dfrac{10x}{x^2 - 16}\right)$

c) For what value of x is $\theta = 45°$?

105. When an individual is walking, the magnitude F of the vertical force of one foot on the ground can be described by

$$F = A(\cos bt - a\cos 3bt), \quad \text{where } t \text{ is time in seconds, } A > 0,\ b > 0 \text{ and } 0 < a < 1$$



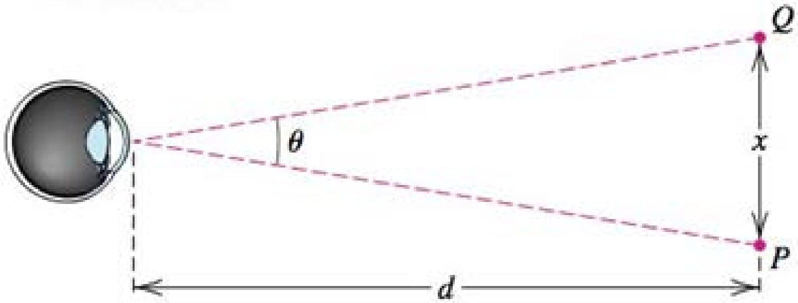
a) Show that $F = 0$, when $t = -\dfrac{\pi}{2b}$ and $t = \dfrac{\pi}{2b}$. (the time $t = -\dfrac{\pi}{2b}$ corresponds to the moment when the foot first touches the ground and the weight of the body is being supported by the other foot.)

b) The maximum force occurs when $3a\sin 3bt = \sin bt$.

If $a = \dfrac{1}{3}$, find the solutions of this equation for the interval $-\dfrac{\pi}{2b} < t < \dfrac{\pi}{2b}$.

c) If $a = \dfrac{1}{3}$, express the maximum force in terms of A.

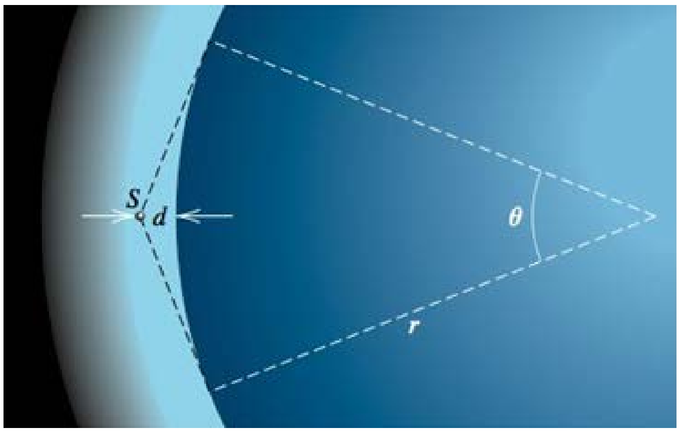
106. The human eye can distinguish between two distant points P and Q provided the angle of resolution θ is not too small. Suppose P and Q are $\underline{x}$ units apart and are d units from the eye.



a) Express x in terms of d and θ.

b) For a person with normal vision, the smallest distinguishable angle of resolution is about 0.0005 *radian*. If a pen 6 *inches* long is viewed by such an individual at a distance of d feet, for what values of d will be the end points of the pen be distinguishable?

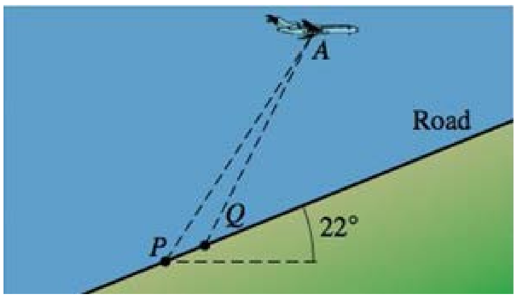
107. A satellite S circles a planet at a distance d *miles* from the planet's surface. The portion of the planet's surface that is visible from the satellite is determined by the angle θ.



a) Assuming that the planet is spherical in shape, express d in terms of θ and the radius r of the planet.

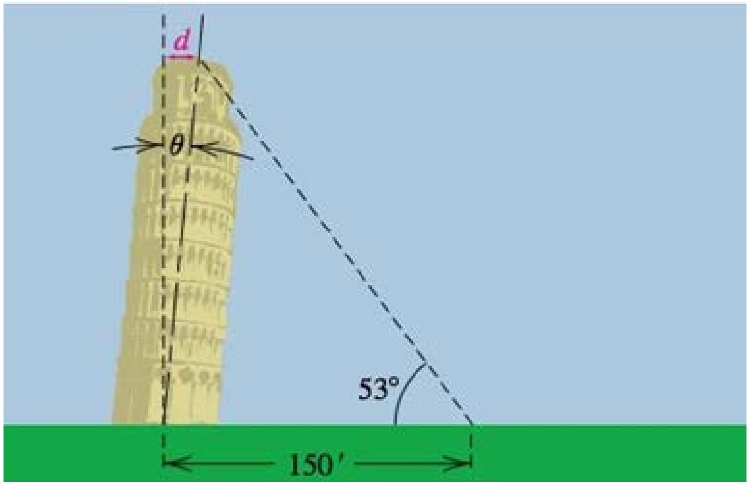
b) Approximate θ for a satellite 300 *miles* from the surface of Earth, using $r = 4,000\ mi$.

108. A straight road makes an angle of $22°$ with the horizontal. From a certain point P on the road, the angle of elevation of an airplane at point A is $57°$. At the same instant, form another point Q, 100 *meters* farther up the road, the angle of elevation is $63°$. The points P, Q, and A lie in the same vertical plane.



Approximate the distance from P to the airplane.

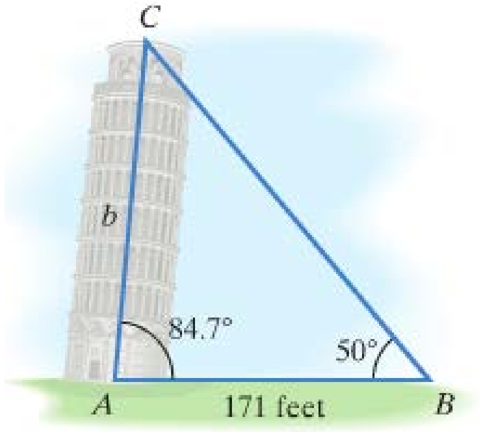
109. The leaning tower of Pisa was originally perpendicular to the ground and 179 *feet* tall. Because of sinking into the earth, it now leans at a certain angle θ from the perpendicular. When the top of the tower is viewed from a point 150 *feet* from the center of its base, the angle of elevation is $53°$.



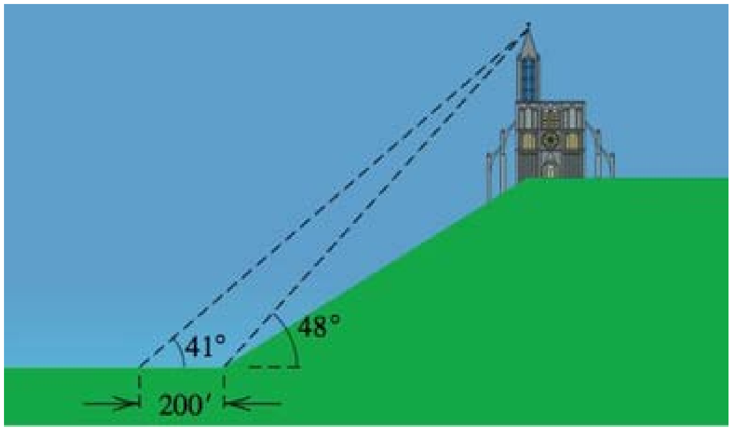
a) Approximate the angle θ.

b) Approximate the distance *d* that the center of the top the tower has moved from the perpendicular.

110. The leaning Tower of Pisa in Italy leans at an angle of about 84.7°, 171 *feet* from the base of the tower, the angle of elevation to the top is 50°. Find the distance from the base to the top of the tower.

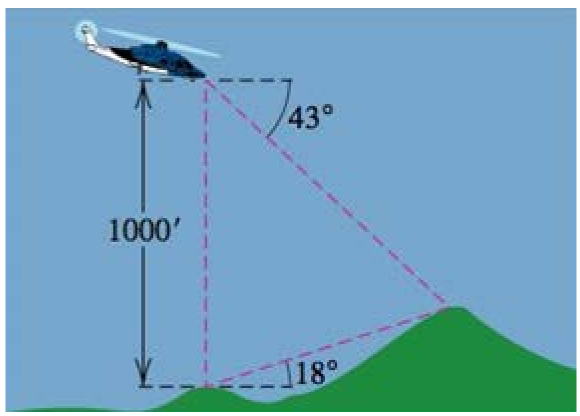


111. A cathedral is located on a hill. When the top of the spire is viewed from the base of the hill, the angle of elevation is $48°$. When it is viewed at a distance of 200 *feet* from the base of the hill, the angle is $41°$. The hill rises at an angle of $32°$.



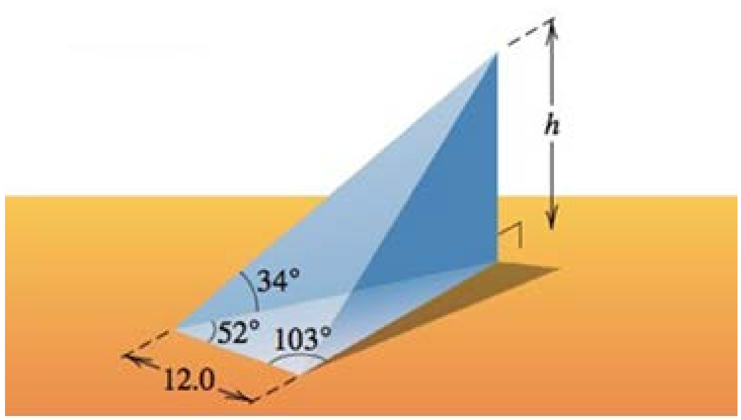
Approximate the height of the cathedral.

112. A helicopter hovers at an altitude that is 1,000 *feet* above a mountain peak of altitude 5,210 *feet*. A second, taller peak is viewed from both the mountaintop and the helicopter. From the helicopter, the angle of depression is $43°$, and from the mountaintop, the angle of elevation is $18°$.



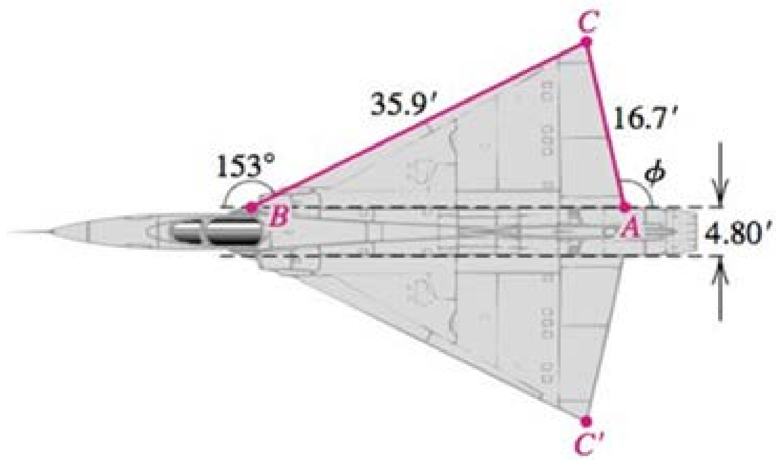
a) Approximate the distance from peak to peak.
b) Approximate the altitude of the taller peak.

113. The volume *V* of the right triangular prism shown in the figure is $\frac{1}{3}Bh$, where *B* is the area of the base and h is the height of the prism.



a) Approximate h.
b) Approximate *V*.

114. Shown in the figure is a plan for the top of a wing of a jet fighter.

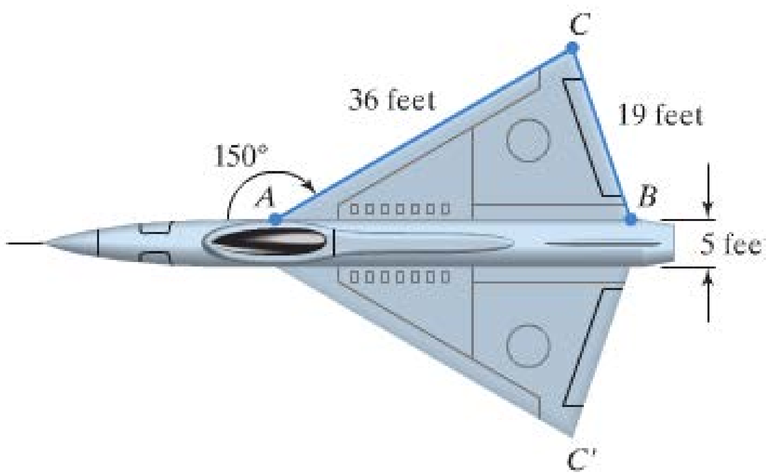


a) Approximate angle ϕ.

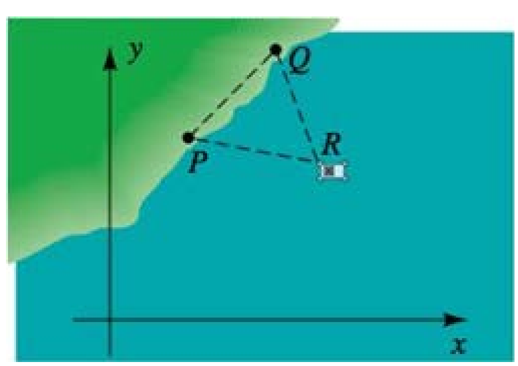
b) If the fuselage is 4.80 *feet* wide, approximate the wing span CC'.

c) Approximate the area of the triangle *ABC*.

115. Shown in the figure is a plan for the top of a wing of a jet fighter. The fuselage is 5 *feet* wide. Find the wing span CC'

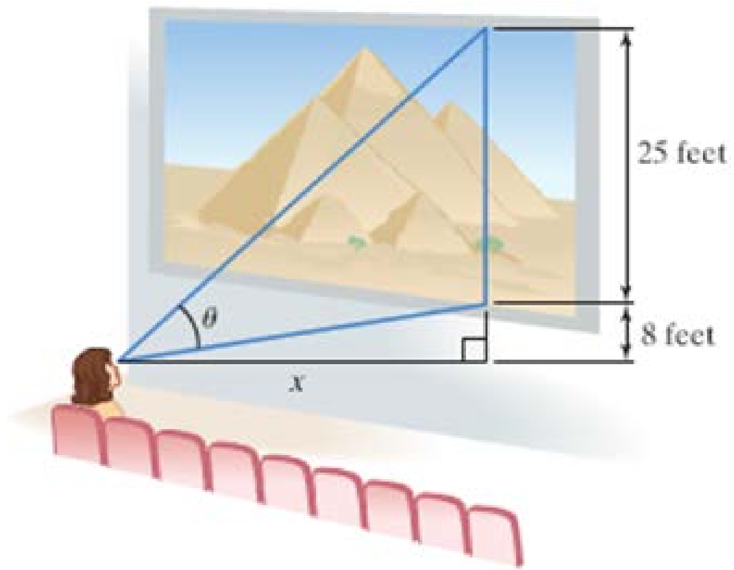


116. Computer software for surveyors makes use of coordinate systems to locate geographic positions. An offshore oil well at point *R* is viewed from points *P* and *A* and $\angle QPR$ and $\angle RQP$ are found to be $55°\ 50'$ and $65°\ 22'$, respectively. If points *P* and *Q* have coordinates $(1487.7,\ 3452.8)$ and $(3145.8,\ 5127.5)$, respectively. Approximate the coordinates of *R*.



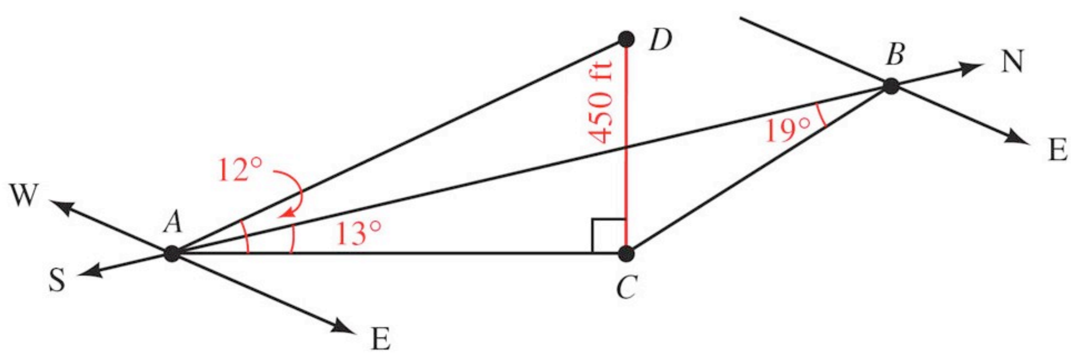
117. Your movie theater has a 25–foot–high screen located 8 *feet* above your eye level. If you sit too close to the screen, your viewing angle is too small resulting in a distorted picture. By contrast, if you sit too far back, the image is quite small, diminishing the movie's visual impact. If you sit *x feet* back from the screen, your viewing angle θ, is giving by

$$\theta = \tan^{-1}\frac{33}{x} - \tan^{-1}\frac{8}{x}$$

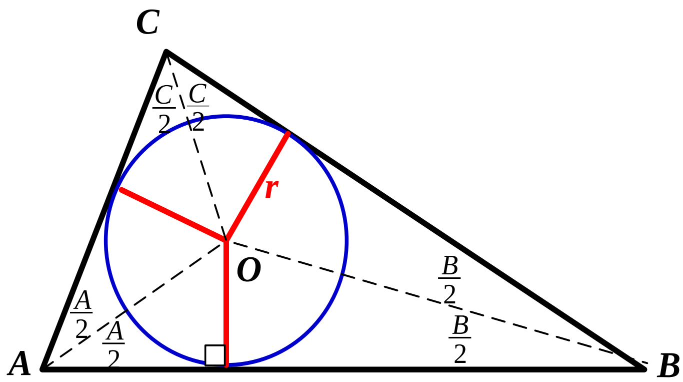


Find the viewing angle, in radians, at distances of 5 *feet*, 10 *feet*, 15 *feet*, 25 *feet*, and 25 *feet*.

118. A hot-air balloon is flying over a dry lake when the wind stops blowing. The balloon comes to a stop 450 *feet* above the ground at point *D*. A jeep following the balloon runs out of gas at point *A*. The nearest service station is due north of the jeep at point *B*. The bearing of the balloon from the jeep at *A* is N 13° E, while the bearing of the balloon from the service station at *B* is S 19° E. If the angle of elevation of the balloon from *A* is 12°, how far will the people in the jeep have to walk to reach the service station at point *B*?



119. The lines that bisect each angle of a triangle meet in a single point *O*, and perpendicular distance *r* from *O* to each side of the triangle is the same. The circle with center at *O* and radius *r* is called the inscribed circle of the triangle.



a) Show that $r = \dfrac{c\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{C}{2}}$

b) Show that $\cot\frac{C}{2} = \frac{s-c}{r}$ where $s = \frac{1}{2}(a+b+c)$

c) Show that $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{s}{r}$

d) Show that the area *K* of triangle *ABC* is $K = rs$, where $s = \frac{1}{2}(a+b+c)$.

e) Show that $r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$

120. Derive the formula: $\dfrac{a-b}{a+b} = \dfrac{\tan\left[\frac{1}{2}(A-B)\right]}{\tan\left[\frac{1}{2}(A+B)\right]}$

121. For any triangle, show that $\sin\frac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$ *where* $s = \frac{1}{2}(a+b+c)$

122. Prove the identity $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2+b^2+c^2}{2abc}$