Solution Section 2.4 – The Chain Rule

Exercise

Find the derivative of $y=\left(3x^4+1\right)^4\left(x^3+4\right)$

Solution

$$y'=4\left(12x^3\right)\left(3x^4+1\right)^3\left(x^3+4\right)+3x^2\left(3x^4+1\right)^4$$

$$=48x^3\left(3x^4+1\right)^3\left(x^3+4\right)+3x^2\left(3x^4+1\right)^4$$

$$=3x^2\left(3x^4+1\right)^3\left[16x\left(x^3+4\right)+3x^4+1\right]$$

$$=3x^2\left(3x^4+1\right)^3\left[16x^4+64x+3x^4+1\right]$$

$$=3x^2\left(3x^4+1\right)^3\left[19x^4+64x+1\right]$$

Exercise

Find the derivative of $y=\left(x^3+1\right)^2$

Solution

$$u=x^3+1 \quad \rightarrow y=u^2$$

$$\frac{d}{dx}y=\frac{dy}{du}\frac{du}{dx}=2u\left(3x^2\right)$$

$$y'=2\left(x^3+1\right)\left(3x^2\right)$$

$$=6x^2\left(x^3+1\right)$$

Exercise

Find the derivative of $y=\left(x^2+3x\right)^4$

Solution

$$u=x^2+3x$$

$$y' = n \quad (u)^{n-1} \quad \frac{d}{dx}[u]$$

$$= 4\left(x^2 + 3x\right)^3 \frac{d}{dx}[x^2 + 3x]$$

$$= 4\left(x^2 + 3x\right)^3 (2x+3)$$

Exercise

Find the derivative of $y = \dfrac{4}{2x+1}$

Solution

$$y = 4(2x+1)^{-1}$$

$$y' = -4(2x+1)^{-2}(2)$$

$$= -8(2x+1)^{-2}$$

$$= -\frac{8}{(2x+1)^2}$$

Exercise

Find the derivative of $y = \dfrac{2}{(x-1)^3}$

Solution

$$y = 2(x-1)^{-3}$$

$$y' = 2(-3)(x-1)^{-4}(1)$$

$$= -\frac{6}{(x-1)^4}$$

Exercise

Find and simplify the derivative of $y = x^2\sqrt{x^2+1}$

Solution

$$y = x^2\left(x^2+1\right)^{1/2}$$

$$y' = x^2 \frac{d}{dx}\left[(x^2+1)^{1/2}\right] + (x^2+1)^{1/2} \frac{d}{dx}\left[x^2\right]$$

$$= x^2\left[\frac{1}{2}(x^2+1)^{-1/2}(2x)\right] + (x^2+1)^{1/2}[2x]$$

$$= x^3(x^2+1)^{-1/2} + 2x(x^2+1)^{1/2}$$

$$= \frac{(x^2+1)^{1/2}}{(x^2+1)^{1/2}}\left[x^3(x^2+1)^{-1/2} + 2x(x^2+1)^{1/2}\right]$$

$$= \frac{x^3(x^2+1)^{-1/2}(x^2+1)^{1/2}+2x(x^2+1)^{1/2}(x^2+1)^{1/2}}{(x^2+1)^{1/2}}$$

$$= \frac{x^3+2x(x^2+1)}{(x^2+1)^{1/2}}$$

$$= \frac{x^3+2x^3+2x}{\sqrt{x^2+1}}$$

$$= \frac{3x^3+2x}{\sqrt{x^2+1}}$$

$$= \frac{x(3x^2+2)}{\sqrt{x^2+1}}$$

Exercise

Find and simplify the derivative of $y = \left(\frac{x+1}{x-5}\right)^2$

Solution

$$y' = 2\left(\frac{x+1}{x-5}\right)\frac{d}{dx}\left[\frac{x+1}{x-5}\right]$$

$$= 2\left(\frac{x+1}{x-5}\right)\left[\frac{(1)(x-5)-(1)(x+1)}{(x-5)^2}\right]$$

$$= 2\left(\frac{x+1}{x-5}\right)\left(\frac{x-5-x-1}{(x-5)^2}\right)$$

$$= 2\left(\frac{x+1}{x-5}\right)\left(\frac{-6}{(x-5)^2}\right)$$

$$= -\frac{12(x+1)}{(x-5)^3}$$

Exercise

Find the derivative of $p(t)=\dfrac{(2t+3)^3}{4t^2-1}$

Solution

$$P'(x)=\frac{2(3)(2t+3)^2\left(4t^2-1\right)-8t(2t+3)^3}{\left(4t^2-1\right)^2}$$

$$=\frac{(2t+3)^2\left[6\left(4t^2-1\right)-8t(2t+3)\right]}{\left(4t^2-1\right)^2}$$

$$=\frac{(2t+3)^2\left[24t^2-6-16t^2-24t\right]}{\left(4t^2-1\right)^2}$$

$$=\frac{(2t+3)^2\left(8t^2-24t-6\right)}{\left(4t^2-1\right)^2}$$

$$=\frac{2(2t+3)^2\left(4t^2-12t-3\right)}{\left(4t^2-1\right)^2}$$

Exercise

Find the derivative of $s(t)=\sqrt{2t^2+5t+2}$

Solution

$$s(t)=\left(2t^2+5t+2\right)^{1/2} \qquad U=2t^2+5t+2 \ \rightarrow\ U'=4t+5$$

$$s'(t)=\frac{1}{2}(4t+5)\left(2t^2+5t+2\right)^{-1/2} \qquad \left(U^n\right)'=nU'U^{n-1}$$

$$=\frac{1}{2}\frac{4t+5}{\sqrt{2t^2+5t+2}}$$

Exercise

Find the derivative of $f(x)=\dfrac{1}{\left(x^2-3x\right)^2}$

Solution

$$f(x)=\left(x^2-3x\right)^{-2}$$

$$f'(x)=-2(2x-3)\left(x^2-3x\right)^{-3}$$

$$\boxed{=-\dfrac{2(2x-3)}{\left(x^2-3x\right)^3}}$$

Exercise

Find the derivative of $y=t^2\sqrt{t-2}$

Solution

$$f=t^2 \qquad g=(t-2)^{1/2}$$

$$f'=2t \qquad g'=\tfrac{1}{2}(t-2)^{-1/2}$$

$$y'=2t\sqrt{t-2}+t^2\,\tfrac{1}{2}(t-2)^{-1/2}$$

$$=\left[2t(t-2)^{1/2}+t^2\,\tfrac{1}{2}(t-2)^{-1/2}\right]\frac{2(t-2)^{1/2}}{2(t-2)^{1/2}}$$

$$=\frac{4t(t-2)+t^2}{2(t-2)^{1/2}}$$

$$=\frac{4t^2-8t+t^2}{2\sqrt{t-2}}$$

$$\boxed{=\frac{5t^2-4t}{2\sqrt{t-2}}}$$

Exercise

Differentiate the function: $y = \dfrac{1}{(9x-4)^8}$

Solution

$$y = (9x-4)^{-8}$$

$U = 9x-4 \qquad (U^n)' = nU^{n-1}U'$

$U' = 9$

$$\frac{dy}{dx} = -8(9x-4)^{-9}(9)$$

$$\boxed{= -\frac{72}{(9x-4)^9}}$$

$$y = \frac{u}{v} \quad \to \quad y' = \frac{u'v - v'u}{v^2}$$

$u = 1 \qquad v = (9x-4)^8$

$u' = 0 \quad v' = 8(9x-4)^7(9) = 72(9x-4)^7$

$$\frac{dy}{dx} = \frac{0 - 72(9x-4)^7}{(9x-4)^{16}}$$

$$\boxed{= -\frac{72}{(9x-4)^9}}$$

Exercise

Find the derivative of $y = \left(\dfrac{6-5x}{x^2-1}\right)^2$

Solution

$f = 6-5x \quad g = x^2-1$

$f' = -5 \qquad g' = 2x$

$$y = 2\frac{-5\left(x^2-1\right) - 2x(6-5x)}{\left(x^2-1\right)^2}\left(\frac{6-5x}{x^2-1}\right) \qquad (U^n)' = nU'U^{n-1}$$

$$= 2\frac{-5x^2+5-12x+10x^2}{\left(x^2-1\right)^3}(6-5x)$$

$$\boxed{= \frac{2\left(5x^2-12x+5\right)(6-5x)}{\left(x^2-1\right)^3}}$$

Exercise

Find the derivative $f(x)=(3x+1)^4$

Solution

$U=3x+1 \quad \rightarrow \quad U'=3$

$f'(x)=4(3x+1)^3(3)$ $\qquad$ $\left(U^n\right)'=nU^{n-1}U'$

$\boxed{=12(3x+1)^3}$

Exercise

Find the derivative $f(x)=\dfrac{1}{\left(x^2+x+4\right)^3}$

Solution

$U=x^2+x+4 \quad \rightarrow \quad U'=2x+1$

$f'(x)=-\dfrac{3(2x+1)}{\left(x^2+x+4\right)^4}$ $\qquad$ $\left(\dfrac{1}{U^n}\right)'=-\dfrac{nU'}{U^{n+1}}$

$\boxed{=-\dfrac{6x+3}{\left(x^2+x+4\right)^4}}$

Exercise

Find the derivative $f(x)=\sqrt{3-x}$

Solution

$U=3-x \quad \rightarrow \quad U'=-1$

$f'(x)\ \boxed{=\dfrac{1}{2\sqrt{3-x}}}$ $\qquad$ $\left(\sqrt{U}\right)'=\dfrac{U'}{2\sqrt{U}}$

Exercise

Find the derivative $f(x)=\left(3x^2+5\right)^5$

Solution

$U=3x^2+5 \quad \rightarrow \quad U'=6x$

$$f'(x) = 5\left(3x^2+5\right)^4 (6x) \qquad\qquad \left(U^n\right)' = nU^{n-1}U'$$

$$\boxed{= 30x\left(3x^2+5\right)^4}$$

Exercise

Find the derivative $f(x) = \left(5x^2-3\right)^6$

Solution

$$U = 5x^2 - 3 \quad \rightarrow \quad U' = 10x$$

$$f'(x) = 6\left(5x^2-3\right)^5 (10x) \qquad\qquad \left(U^n\right)' = nU^{n-1}U'$$

$$\boxed{= 60x\left(5x^2-3\right)^5}$$

Exercise

Find the derivative $f(x) = \left(x^4+1\right)^{-2}$

Solution

$$U = x^4 + 1 \quad \rightarrow \quad U' = 4x^3$$

$$f'(x) = -2\left(x^4+1\right)^{-3}\left(4x^3\right) \qquad\qquad \left(U^n\right)' = nU^{n-1}U'$$

$$\boxed{= -8x^3\left(x^4+1\right)^{-3}}$$

Exercise

Find the derivative $f(x) = (4x+3)^{1/2}$

Solution

$$U = 4x + 3 \quad \rightarrow \quad U' = 4$$

$$f'(x) = \frac{1}{2}(4x+3)^{-1/2}(4)$$

$$\boxed{= \frac{2}{\sqrt{4x+3}}}$$

Exercise

Suppose a demand function is given by $\qquad q = D(p) = 30\left(5 - \dfrac{p}{\sqrt{p^2+1}}\right)$

Where q is the demand for a product and p is the price per item in dollars. Find the rate of change in the demand for the product per unit change in price (i.e. find dq/dp)

Solution

$$\frac{dq}{dp} = 30\frac{1\sqrt{p^2+1} - \frac{1}{2}(2p)\left(p^2+1\right)^{-1/2}\,p}{\left(\sqrt{p^2+1}\right)^2}$$

$$= 30\frac{\left(p^2+1\right)^{1/2} - p^2\left(p^2+1\right)^{-1/2}}{p^2+1}\frac{\left(p^2+1\right)^{1/2}}{\left(p^2+1\right)^{1/2}}$$

$$= 30\frac{p^2+1-p^2}{\left(p^2+1\right)^{3/2}}$$

$$= \frac{30}{\left(p^2+1\right)^{3/2}}$$

Exercise

Find the tangent line to the graph of $y=\sqrt[3]{(x+4)^2}$ when x = 4.

Solution

$$y=(x+4)^{2/3}$$

$$y'=\frac{2}{3}(x+4)^{-1/3}$$

$$=\frac{2}{3}\frac{1}{(x+4)^{1/3}}$$

$$=\frac{2}{3\sqrt[3]{x+4}}$$

$$x=4\to m=y'=\frac{2}{3\sqrt[3]{4+4}}$$

$$=\frac{2}{3\sqrt[3]{2^3}}$$

$$=\frac{2}{3(2)}=\frac{1}{3}$$

$$=\frac{1}{3}$$

$$x=4\to y=\sqrt[3]{(4+4)^2}=4$$

$$y-4=\frac{1}{3}(x-4)$$

$$y=\frac{1}{3}x-\frac{4}{3}+4$$

$$\boxed{y=\frac{1}{3}x+\frac{8}{3}}$$

Exercise

The revenue realized by a small city from the collection of fines from parking tickets is given by

$$R(n)=\frac{8000n}{n+2}$$

where n is the number of work-hours each day that can be devoted to parking patrol. At the outbreak of a flu epidemic, 30 work-hours are used daily in parking patrol, but during the epidemic that number is decreasing at the rate of 6 work-hours per day. How fast is revenue from parking fines decreasing at the outbreak of the epidemic?

Solution

$$\frac{dn}{dt}=-6$$

$$\frac{dR}{dt} = \frac{dR}{dn} \cdot \frac{dn}{dt}$$

$$= \frac{8000(n+2) - (1)8000n}{(n+2)^2} \bullet (-6)$$

$$= (-6)\frac{8000n + 16000 - 8000n}{(n+2)^2}$$

$$= -\frac{96000}{(n+2)^2}$$

$$\frac{dR}{dt}(30) = -\frac{96000}{(30+2)^2}$$

$$= -93.75 \approx -94$$

The revenue is being lost at the rate of \$94 per day.

Exercise

To test an individual's use of a certain mineral, a researcher injects small amount form of that material into the person's bloodstream. The mineral remaining in the bloodstream is measured each day for several days. Suppose the amount of the mineral remaining in the bloodstream (in milligrams per cubic centimeter) t days after the injection is approximated by $C(t) = \frac{1}{2}(2t+1)^{-1/2}$. Find the rate of change of the mineral level with respect to time for 4 days.

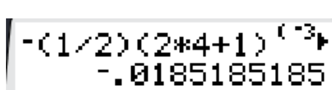
Solution

$u = 2t + 1 \quad \rightarrow \quad u' = 2$

$$C'(t) = \frac{1}{2}\underbrace{\left(-\frac{1}{2}\right)}_{n}\underbrace{(2t+1)^{-1/2-1}}_{u^{n-1}}\underbrace{(2)}_{u'} \qquad\qquad (u^n)' = uu^{n-1}u'$$

$$= -\frac{1}{2}(2t+1)^{-3/2}$$

$$C'(t=4) = -\frac{1}{2}(2(4)+1)^{-3/2} = -0.0185 \approx -0.02$$



Exercise

The total cost (in hundreds of dollars) of producing *x* cameras per week is

$$C(x)=6+\sqrt{4x+4} \quad 0\le x\le 30$$

a) Find $C'(x)$

b) Find $C'(15)$ and $C'(24)$. Interpret the results

Solution

a) $C(x)=6+(4x+4)^{1/2}$

$$C'(x)=\frac{1}{2}(4x+4)^{-1/2}(4)$$

$$=\boxed{\frac{2}{(4x+4)^{1/2}}}$$

b) $C'(15)=\dfrac{2}{(4(15)+4)^{1/2}}=0.25 \quad \text{or} \quad \boxed{\$25}$

$$C'(24)=\frac{2}{(4(24)+4)^{1/2}}=0.2 \quad \text{or} \quad \boxed{\$20}$$

At a production level of 24 cameras, total costs are increasing at the rate of \$20 per camera. Therefore, the cost of purchasing the 25th camera is approximately \$20.