Solution Section 1.1 – Propositional Logic

Exercise

Which of these sentences are propositions? What are truth values of those that are propositions?

a) Boston is the capital of Massachusetts.
b) Miami is the capital of Florida
c) $2+3=5$
d) $5+7=10$
e) $x+2=11$
f) Answer this question
g) Do not pass go
h) What time is it?
i) The moon is made of green cheese
j) $2^n \ge 100$

Solution

a) This is a true proposition

b) This is a false proposition, Tallahassee is the capital

c) This is a true proposition

d) This is a false proposition since $5+7=12 \ne 10$

e) This is not a proposition. (The truth value depends on the value assigned to x.)

f) This is not a proposition, it is a command

g) This is not a proposition; it is a command

h) This is not a proposition; it's a question

i) This is a proposition that is false

j) This is not a proposition; its truth value depends on the value of n.

Exercise

What is the negation if each of these propositions?

a) Mei has an MP3 player
b) There is no pollution in Texas
c) $2+1=3$
d) There are 13 items in a baker's dozen,
e) 121 is a perfect square

Solution

a) Mei does not have an MP3 player

b) There is pollution in Texas

c) $2+1 \neq 3$

d) There are not 13 items in a baker's dozen.

e) 121 is not a perfect square

Exercise

Suppose the Smartphone *A* has 256 MB RAM and 32 GB ROM, and the resolution of its camera is 8 MP; Smartphone *B* has 288 MB RAM and 64 GB ROM, and the resolution of its camera is 4 MP; Smartphone *C* has 128 MB RAM and 32 GB ROM, and the resolution of its camera is 5 MP. Determine the truth value of each of these propositions.

a) Smartphone *B* has the most RAM of these three smartphones

b) Smartphone *C* has more ROM or higher resolution camera than Smartphone *B*.

c) Smartphone *B* has more RAM, more ROM, and a higher resolution camera than Smartphone *A*.

d) If Smartphone *B* has more RAM and more ROM than Smartphone *C*, then it also has a higher resolution camera.

e) Smartphone *A* has more RAM than Smartphone *B* if and only if Smartphone *B* has more RAM than Smartphone *A*.

Solution

a) This is a true proposition because 288 > 256 and 288 > 128

b) This is a true proposition, because the resolution is higher, C has 5 MP resolution compared to B's 4 MP.

c) This is not a true proposition. The resolution is not higher that *A*.

d) This is not a true proposition. Not necessary, the resolution in *B* is not higher that *C*.

e) This is not a true proposition

Exercise

Let *p* and *q* be the proposition

p: I bought a lottery ticket this week

q: I won the million dollars jackpot

$a)\ \neg p$	$b)\ p \vee q$	$c)\ p \to q$	$d)\ p \wedge q$
$e)\ p \leftrightarrow q$	$f)\ \neg p \to \neg q$	$g)\ \neg p \wedge \neg q$	$h)\ \neg p \vee (p \wedge q)$

Solution

a) I did not buy a lottery ticket this week.

b) I bought a lottery ticket this week or I won the million dollars jackpot.

c) I bought a lottery ticket this week then I won the million dollars jackpot.

d) I bought a lottery ticket this week and I won the million dollars jackpot.

e) I bought a lottery ticket this week if and only if I won the million dollars jackpot.

f) I did not buy a lottery ticket this week then I did not win the million dollars jackpot.

g) I did not buy a lottery ticket this week and I did not win the million dollars jackpot.

h) I did not buy a lottery ticket this week or either I bought a lottery ticket this week and I won the million dollars jackpot.

Exercise

Let p and q be the proposition

p: Swimming at the New Jersey shore is allowed

q: Sharks have been spotted new the shore

$a)\ \ \neg q \qquad b)\ \ p\wedge q \qquad c)\ \ \neg p\vee q \qquad d)\ \ p\to\neg q$

$e)\ \ \neg q\to p \qquad f)\ \ \neg p\to\neg q \qquad g)\ \ p\leftrightarrow\neg q \qquad h)\ \ \neg p\wedge(p\vee\neg q)$

Solution

a) Sharks have not been spotted new the shore.

b) Swimming at the New Jersey shore is allowed, and Sharks have been spotted new the shore.

c) Swimming at the New Jersey shore is not allowed, or Sharks have been spotted new the shore.

d) Swimming at the New Jersey shore is allowed then Sharks have not been spotted new the shore.

e) Sharks have not been spotted new the shore then Swimming at the New Jersey shore is allowed.

f) Swimming at the New Jersey shore is not allowed then Sharks have not been spotted new the shore.

g) Swimming at the New Jersey shore is allowed if and only if Sharks have not been spotted new the shore.

h) Swimming at the New Jersey shore is not allowed and either Swimming at the New Jersey shore is allowed, or Sharks have not been spotted new the shore

Exercise

Let p, q and r be the proposition

p: You have the flu

q: You miss the final examination

r: You pass the course

Express each of these propositions as an English sentence

$a)\ \ p\to q \qquad b)\ \ \neg q\leftrightarrow r \qquad c)\ \ q\to\neg r \qquad d)\ \ p\vee q\vee r$

$e)\ \ (p\to\neg r)\vee(q\to\neg r) \qquad f)\ \ (p\wedge q)\vee(\neg q\wedge r)$

Solution

a) You have the flu then you miss the final examination.

b) You don't miss the final examination if and only if you pass the course.

c) You miss the final examination then you don't pass the course.

d) You have the flu or you miss the final examination or you pass the course.

e) You have the flu then you don't pass the course or you miss the final examination then you don't pass the course.

f) You have the flu and you miss the final examination or both you don't miss the final examination and you pass the course.

Exercise

Determine whether each of these conditional statements is true or false.

a) If $1+1=2$, then $2 + 2 = 5$
b) If $1 + 1 = 3$, then $2 + 2 = 4$
c) If $1 + 1 = 3$, then $2 + 2 = 5$
d) If monkeys can fly, then $1 + 1 = 3$
e) If $1 + 1 = 3$, then unicorns exist
f) If $1 + 1 = 3$, then dogs can fly
g) If $1 + 1 = 2$, the dogs can fly
h) If $2 + 2 = 4$, then $1 + 2 = 3$

Solution

a) Since the hypothesis is true and the conclusion is false, this implication is false.

b) Since the hypothesis is false and the conclusion is true, this implication is true.

c) Since the hypothesis is false and the conclusion is false, this implication is true.

d) Since the hypothesis is false and the conclusion is false, this implication is true.

e) Since the hypothesis is false and the conclusion is false, this implication is true.

f) Since the hypothesis is false and the conclusion is false, this implication is true.

g) Since the hypothesis is true and the conclusion is false, this implication is false.

h) Since the hypothesis is true and the conclusion is true, this implication is true.

Exercise

Write each of these propositions in the form "p if and only if q" in English

a) If it is hot outside you buy an ice cream cone, and if you buy an ice cream cone it is hot outside.
b) For you to win the contest it is necessary and sufficient that you have only winning ticket.
c) You get promoted only if you have connections, and you have connections only if you get promoted.
d) If you watch television your mind will decay, and conversely.
e) The trains run late on exactly those days when I take it.
f) For you to get an *A* in this course, it is necessary and sufficient that you learn how to solve discrete mathematics problems.

g) If you read the newspaper every day, you will be informed, and conversely.

h) It rains if it is a weekend day, and it is a weekend day if it rains.

i) You can see the wizard only if the wizard is not in, and the wizard is not in only if you can see him

Solution

a) You buy an ice cream cone if and only if it is hot outside.

b) You win the contest if and only if you hold the only winning ticket.

c) You get promoted if and only if you have connection.

d) Your mind will decay if and only if you watch television.

e) The trains run late if and only if you watch television.

f) For you to get an *A* in this course if and only if you learn how to solve discrete mathematics problems.

Exercise

Construct a truth table for each of these compound propositions.

a) $p \wedge \neg p$

b) $p \vee \neg p$

c) $p \to \neg p$

d) $p \leftrightarrow \neg p$

e) $p \to \neg q$

f) $\neg p \leftrightarrow q$

g) $(p \vee \neg q) \to q$

h) $(p \vee q) \to (p \wedge q)$

i) $(p \to q) \leftrightarrow (\neg q \to \neg p)$

j) $(p \to q) \to (q \to p)$

k) $p \oplus (p \vee q)$

l) $(p \wedge q) \to (p \vee q)$

m) $(q \to \neg p) \leftrightarrow (p \leftrightarrow q)$

n) $(p \to q) \vee (\neg p \to q)$

o) $(p \to q) \wedge (\neg p \to q)$

p) $(p \vee q) \vee r$

q) $(p \vee q) \wedge r$

r) $(p \wedge q) \vee r$

s) $(p \wedge q) \wedge r$

t) $(p \vee q) \wedge \neg r$

Solution

a)

p	$\neg p$	$p \wedge \neg p$
T	F	***F***
F	T	***F***

b)

p	$\neg p$	$p \vee \neg p$
T	F	***T***
F	T	***T***

c)

p	$\neg p$	$p \to \neg p$
T	F	***F***
F	T	***T***

d)

p	$\neg p$	$p \leftrightarrow \neg p$
T	F	**F**
F	T	**F**

e)

p	q	$\neg q$	$p \to \neg q$
T	T	F	**F**
T	F	T	**T**
F	T	F	**T**
F	F	T	**T**

f)

p	q	$\neg q$	$\neg p \leftrightarrow q$
T	T	F	***F***
T	F	T	***T***
F	T	F	***T***
F	F	T	***F***

g)

p	q	$\neg q$	$p \vee \neg q$	$(p \vee \neg q) \to q$
T	T	F	T	***T***
T	F	T	T	***F***
F	T	F	F	***T***
F	F	T	T	***F***

h)

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \to (p \wedge q)$
T	T	T	T	***T***
T	F	T	F	***F***
F	T	T	F	***F***
F	F	F	F	***T***

i)

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$	$(p \to q) \leftrightarrow (\neg q \wedge \neg p)$
T	T	**T**	F	F	**T**	***T***
T	F	**F**	T	F	**F**	***T***
F	T	**T**	F	T	**F**	***T***
F	F	**T**	T	T	**F**	***T***

j)

p	q	$p \to q$	$q \to p$	$(p \to q) \to (q \to p)$
T	T	**T**	**T**	***T***
T	F	**F**	**T**	***T***
F	T	**T**	**F**	***F***
F	F	**T**	**T**	***T***

k)

p	q	$p \vee q$	$p \oplus (p \vee q)$
T	T	**T**	***F***
T	F	**T**	***F***
F	T	**T**	***T***
F	F	**F**	***F***

l)

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \to (p \vee q)$
T	T	**T**	**T**	***T***
T	F	**F**	**T**	***T***
F	T	**F**	**T**	***T***
F	F	**F**	**F**	***T***

m)

p	q	$\neg p$	$q \to \neg p$	$p \leftrightarrow q$	$(q \to \neg p) \leftrightarrow (p \leftrightarrow q)$
T	T	F	**F**	**T**	***F***
T	F	F	**T**	**F**	***F***
F	T	T	**T**	**F**	***F***
F	F	T	**T**	**T**	***T***

n)

p	q	$\neg p$	$p \to q$	$\neg p \to q$	$(p \to q) \vee (\neg p \to q)$
T	T	F	**T**	**T**	***T***
T	F	F	**F**	**T**	***T***
F	T	T	**T**	**T**	***T***
F	F	T	**T**	**F**	***T***

o)

p	q	$\neg p$	$p \to q$	$\neg p \to q$	$(p \to q) \wedge (\neg p \to q)$
T	T	F	**T**	**T**	***T***
T	F	F	**F**	**T**	***F***
F	T	T	**T**	**T**	***T***
F	F	T	**T**	**F**	***F***

p)

p	q	r	$p \vee q$	$(p \vee q) \vee r$
T	T	T	**T**	***T***
T	T	F	**T**	***T***
T	F	T	**T**	***T***
T	F	F	**T**	***T***
F	T	T	**T**	***T***
F	T	F	**T**	***T***
F	F	T	**F**	***T***
F	F	F	**F**	***F***

q)

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
T	T	T	**T**	***T***
T	T	F	**T**	***F***
T	F	T	**T**	***T***
T	F	F	**T**	***F***
F	T	T	**T**	***T***
F	T	F	**T**	***F***
F	F	T	**F**	***F***
F	F	F	**F**	***F***

r)

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$
T	T	T	**T**	***T***
T	T	F	**T**	***T***
T	F	T	**F**	***T***
T	F	F	**F**	***F***
F	T	T	**F**	***T***
F	T	F	**F**	***F***
F	F	T	**F**	***T***
F	F	F	**F**	***F***

s)

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$
T	T	T	**T**	***T***
T	T	F	**T**	***F***
T	F	T	**F**	***F***
T	F	F	**F**	***F***
F	T	T	**F**	***F***
F	T	F	**F**	***F***
F	F	T	**F**	***F***
F	F	F	**F**	***F***

t)

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\neg \boldsymbol{r}$	$\boldsymbol{p} \vee \boldsymbol{q}$	$(p \vee q) \wedge \neg r$
T	T	T	**F**	**T**	***F***
T	T	F	**T**	**T**	***T***
T	F	T	**F**	**T**	***F***
T	F	F	**T**	**T**	***T***
F	T	T	**F**	**T**	***F***
F	T	F	**T**	**T**	***T***
F	F	T	**F**	**F**	***F***
F	F	F	**T**	**F**	***F***

Solution Section 1.2 – Propositional Equivalences

Exercise

Use the truth table to verify these equivalences

a) $p \wedge \mathbf{T} \equiv p$ *b)* $p \vee \mathbf{F} \equiv p$ *c)* $p \wedge \mathbf{F} \equiv \mathbf{F}$

d) $p \vee \mathbf{T} \equiv \mathbf{T}$ *e)* $p \vee p \equiv p$ *f)* $p \wedge p \equiv p$

Solution

p	$p \wedge \mathbf{T}$	$p \vee \mathbf{F}$	$p \wedge \mathbf{F}$	$p \vee \mathbf{T}$	$p \vee p$	$p \wedge p$
T	T	T	F	T	T	T
F	F	F	F	T	F	F
	hold	***hold***	***hold***	***hold***	***hold***	***hold***

Exercise

Show that $\neg(\neg p)$ and p are logically equivalent

Solution

p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

Therefore, $\neg(\neg p)$ and p are logically equivalent

Exercise

Use the truth table to verify the commutative laws

a) $p \vee q \equiv q \vee p$

b) $p \wedge q \equiv q \wedge p$

Solution

p	q	$p \vee q$	$q \vee p$
T	T	***T***	***T***
T	F	***T***	***T***
F	T	***T***	***T***
F	F	***F***	***F***

Identical

p	q	$p \wedge q$	$q \wedge p$
T	T	***T***	***T***
T	F	***F***	***F***
F	T	***F***	***F***
F	F	***F***	***F***

Identical

Exercise

Use the truth table to verify the associative laws

a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Solution

a)

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$
T	T	T	**T**	***T***	**T**	***T***
T	T	F	**T**	***T***	**T**	***T***
T	F	T	**T**	***T***	**T**	***T***
T	F	F	**T**	***T***	**F**	***T***
F	T	T	**T**	***T***	**T**	***T***
F	T	F	**T**	***T***	**T**	***T***
F	F	T	**F**	***T***	**T**	***T***
F	F	F	**F**	***F***	**F**	***F***

$(p \vee q) \vee r \equiv p \vee (q \vee r)$ is true

b)

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	T	**T**	***T***	**T**	***T***
T	T	F	**T**	***F***	**F**	***F***
T	F	T	**F**	***F***	**F**	***F***
T	F	F	**F**	***F***	**F**	***F***
F	T	T	**F**	***F***	**T**	***F***
F	T	F	**F**	***F***	**F**	***F***
F	F	T	**F**	***F***	**F**	***F***
F	F	F	**F**	***F***	**F**	***F***

$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ is true

Exercise

Show that each of these conditional statements is a tautology by using truth result tables.

a) $(p \wedge q) \to p$

b) $p \to (p \vee q)$

c) $\neg p \to (p \to q)$

d) $(p \wedge q) \to (p \to q)$

e) $\neg(p \to q) \to p$

f) $\left[\neg p \wedge (p \vee q)\right] \to q$

g) $\left[(p \to q) \wedge (q \to r)\right] \to (p \to r)$

h) $\left[p \wedge (p \to q)\right] \to q$

Solution

a)

p	q	$p \vee q$	$p \to (p \vee q)$
T	T	***T***	***T***
T	F	***T***	***T***
F	T	***T***	***T***
F	F	***F***	***T***

b)

p	q	$p \wedge q$	$(p \wedge q) \to p$
T	T	***T***	***T***
T	F	***F***	***T***
F	T	***F***	***T***
F	F	***F***	***T***

c)

p	q	$\neg p$	$p \to q$	$\neg p \to (p \to q)$
T	T	***F***	***T***	***T***
T	F	***F***	***F***	***T***
F	T	***T***	***T***	***T***
F	F	***T***	***T***	***T***

d)

p	q	$p \wedge q$	$p \to q$	$(p \wedge q) \to (p \to q)$
T	T	***T***	***T***	***T***
T	F	***F***	***F***	***T***
F	T	***F***	***T***	***T***
F	F	***F***	***T***	***T***

e)

p	q	$p \to q$	$\neg(p \to q)$	$\neg(p \to q) \to p$
T	T	***T***	***F***	***T***
T	F	***F***	***T***	***T***
F	T	***T***	***F***	***T***
F	F	***T***	***F***	***T***

f)

p	q	$p \to q$	$\neg(p \to q)$	$\left[\neg p \wedge (p \vee q)\right] \to q$
T	T	***T***	***F***	***T***
T	F	***F***	***T***	***T***
F	T	***T***	***F***	***T***
F	F	***T***	***F***	***T***

g)

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	**T**	**T**	**T**	**T**	***T***
T	T	F	**T**	**F**	**F**	**F**	***T***
T	F	T	**F**	**T**	**F**	**T**	***T***
T	F	F	**F**	**T**	**F**	**F**	***T***
F	T	T	**T**	**T**	**T**	**T**	***T***
F	T	F	**T**	**F**	**F**	**T**	***T***
F	F	T	**T**	**T**	**T**	**T**	***T***
F	F	F	**T**	**T**	**T**	**T**	***T***

h)

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	***T***	***T***	***T***
T	F	***F***	***F***	***T***
F	T	***T***	***F***	***T***
F	F	***T***	***F***	***T***

Exercise

Show that $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent

Solution

The proposition $p \leftrightarrow q$ is true when p and q have the same true or false value. Since p and q are truth, then $p \wedge q$ only true. When p and q are false, then the negation $\neg p$ and $\neg q$ are true, then $\neg p \wedge \neg q$ is true. Therefore $(p \wedge q) \vee (\neg p \wedge \neg q)$ is true only when both are true. Therefore these two expressions are logically equivalent.

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$	$p \leftrightarrow q$
T	T	***T***	F	F	***F***	***T***	***T***
T	F	***F***	F	T	***F***	***F***	***F***
F	T	***F***	T	F	***F***	***F***	***F***
F	F	***F***	T	T	***T***	***T***	***T***

Exercise

Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent

Solution

The proposition $\neg(p \leftrightarrow q)$ is true when $p \leftrightarrow q$ is false. Since $p \leftrightarrow q$ is true when p and q have the same truth value, it is false when p and q have different truth values (either p is true and q is false, or vice versa). These are precisely the cases in which $p \leftrightarrow \neg q$ is true. Therefore these two expressions are logically equivalent.

p	q	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$\neg q$	$p \leftrightarrow \neg q$
T	T	***T***	***F***	F	***F***
T	F	***F***	***T***	T	***T***
F	T	***F***	***T***	F	***T***
F	F	***T***	***F***	T	***F***

Exercise

Show that $p \to q$ and $\neg q \to \neg p$ are logically equivalent

Solution

It is easy to see from the definitions of conditional statement and negation of these propositions is false in the case which p is true and q is false the proposition is false, and true in the other three cases. Therefore these two expressions are logically equivalent.

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$
T	T	***T***	F	F	***T***
T	F	***F***	T	F	***F***
F	T	***T***	F	T	***T***
F	F	***T***	T	T	***T***

Exercise

Show that $\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$ are logically equivalent

Solution

The proposition $\neg p \leftrightarrow q$ is true when $\neg p$ and q have the same truth values, which means that p and q have different truth values (either p is true and q is false, or vice versa). By the same reasoning, these are exactly the cases in which $p \leftrightarrow \neg q$ is true. Therefore these two expressions are logically equivalent.

Exercise

Show that $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$ are logically equivalent

Solution

$(p \to q) \vee (p \to r)$ will be true when either of the conditional statements is true. The conditional statement will be true if p is false, or if q in one case or r in the other case is true, when $q \vee r$ is true, which is precisely $p \to (q \vee r)$ is true. Since the two propositions are true in exactly the same situation, they are logically equivalent.

Exercise

Show that $(p \to r) \vee (q \to r)$ and $(p \wedge q) \to r$ are logically equivalent

Solution

In order for $(p \to r) \vee (q \to r)$ to be false, we must have both of the two implications false, which happens exactly when r is false and both p and q are true. But this precisely the case in which $p \wedge q$ is true and r is false, which is $(p \wedge q) \to r$ is false. Therefore these two expressions are logically equivalent.

Exercise

Show that $(p \to q) \wedge (q \to r) \to (p \to r)$ is a tautology

Solution

Given that p and $p \to q$ are both true, we conclude that q is true; from that and $q \to r$ we conclude that r is true.

Exercise

Show that $(p \vee q) \vee (\neg p \vee r) \to (q \vee r)$ is a tautology

Solution

The conclusion $q \vee r$ will be true in every case except when q and r are both false. But if q and r are both false, then one of $p \vee q$ or $\neg p \vee r$ is false, because one of p or $\neg p$ is false. Thus in this case $(p \vee q) \wedge (\neg p \vee r)$ is false. An conditional statement in which the conclusion is true or the hypothesis is false.

Exercise

Show that | (NAND) is functionally complete

Solution

Equivalence of NOT:

$p|p \equiv \neg p$

$\neg(p \wedge p) \equiv \neg p$ ***Equivalence of*** NAND

$\neg(p) \equiv \neg p$ ***Idempotent law***

Equivalence of AND:

$p \wedge q \equiv \neg(p|q)$ ***Definition of*** NAND

$p|p$

$(p|q)|(p|p)q$ ***Negation of*** $(p|q)$

Equivalence of OR:

$p \vee q \equiv \neg(\neg p \wedge \neg q)$ **DeMorgan's *equivalence of*** OR

We can do AND and OR with NANDs, also do ORs with NANDs

Thus, NAND is functionally complete.

Solution *Section* 1.3 – Predicates and Quantifiers

Exercise

Let $P(x)$ denote the statement "$x \le 4$". What are these truth values?

$a)\ P(0) \qquad b)\ P(4) \qquad c)\ P(6)$

Solution

a) True, since $0 \le 4$

b) True, since $4 \le 4$

c) False, since $7 \not\le 4$

Exercise

Let $P(x)$ be the statement "*the word* ***x*** *contains the letter* ***a***". What are these truth values?

$a)\ P(orange) \qquad b)\ P(lemon) \qquad c)\ P(true) \qquad d)\ P(false)$

Solution

a) True, since there is an *a* in *orange*.

b) False, since there is no *a* in *lemon*.

c) False, since there is no *a* in *true*.

d) True, since there is an *a* in *false*.

Exercise

State the value of x after the statement **if** $P(x)$ **then** $x := 1$ is executed, where $P(x)$ is the statement "$x > 1$", if the value of x when the statement is reached is

$a)\ x = 0 \qquad b)\ x = 1 \qquad c)\ x = 2$

Solution

a) x is still equal to 0, since the condition is false.

b) x is still equal to 0, since the condition is false.

c) x is still equal to 1 at the end, since the condition is true, so the statement $x := 1$ is executed.

Exercise

Let $P(x)$ be the statement "***x** spends more than five hours every weekday in class,*" where the domain for x consists of all students. Express each of these quantifications in English.

$a)\ \exists x\ P(x) \qquad b)\ \forall x\ P(x) \qquad c)\ \exists x\ \neg P(x) \qquad d)\ \forall x\ \neg P(x)$

Solution

a) There is a student who spends more than five hours every weekday in class.

b) Every student spends more than five hours every weekday in class.

c) There is a student who does not spend more than five hours every weekday in class.

d) No student spends more than five hours every weekday in class.

Exercise

Let $N(x)$ be the statement "***x** has visited North Dakota,*" where the domain consists of the students in your class. Express each of these quantifications in English.

$a)\ \exists x\ N(x) \qquad b)\ \forall x\ N(x) \qquad c)\ \neg\exists x\ N(x) \qquad d)\ \exists x\ \neg N(x)$

$e)\ \neg\forall x\ N(x) \qquad f)\ \forall x\ \neg N(x)$

Solution

a) Some student in the school has visited North Dakota.
Or, there exists a student in the school who has visited North Dakota

b) Every student in the school has visited North Dakota
Or, all students in the school have visited North Dakota

c) No student in the school has visited North Dakota.
Or, there does not exist a student in the school who has visited North Dakota

d) Some student in the school has not visited North Dakota.
Or, there exists a student in the school who has not visited North Dakota

e) It is not true that every student in the school has visited North Dakota
Or, not all students in the school have visited North Dakota

f) All students in the school has not visited North Dakota.
Or, no student has visited North Dakota

Exercise

Let $C(x)$ be the statement "***x** has a cat,*" let $D(x)$ be the statement "***x** has a dog,*", and let $F(x)$ be the statement "***x** has a ferret.*" Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

a) A student in your class has a cat, a dog, and a ferret.
b) All students in your class have a cat, a dog, or a ferret.
c) Some student in your class has a cat and a ferret, but not a dog.
d) No student in your class has a cat, a dog, and a ferret.
e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

Solution

a) $\exists x\left(C(x)\wedge D(x)\wedge F(x)\right)$

b) $\forall x\left(C(x)\vee D(x)\vee F(x)\right)$

c) $\exists x\left(C(x)\wedge F(x)\wedge\neg D(x)\right)$

d) $\neg\exists x\left(C(x)\wedge D(x)\wedge F(x)\right)$

e) $\left(\exists xC(x)\right)\wedge\left(\exists xD(x)\right)\wedge\left(\exists xF(x)\right)$

Exercise

Let $Q(x)$ be the statement "$x+1>2x$". If the domain consists of all integers, what are these truth values?

a) $Q(0)$ *b)* $Q(-1)$ *c)* $Q(1)$ *d)* $\exists x\ Q(x)$

e) $\forall x\ Q(x)$ *f)* $\exists x\ \neg Q(x)$ *g)* $\forall x\ \neg Q(x)$

Solution

a) Since $0+1>2\cdot 0$, that implies to $Q(0)$ is ***true***.

b) Since $(-1)+1>2\cdot(-1)$, that implies to $Q(-1)$ is ***true***.

c) Since $1+1\not>2\cdot 1$, that implies to $Q(1)$ is ***false***.

d) We showed that $Q(0)$ is *true*, therefore there is at least one x that makes $Q(x)$ *true*, so $\exists x\ Q(x)$ is ***true***.

e) We showed that $Q(1)$ is *false*, therefore there is at least one x that makes $Q(x)$ *false*, so $\forall x\ Q(x)$ is ***false***.

f) We showed that $Q(1)$ is *false*, therefore there is at least one x that makes $Q(x)$ *false*, so $\exists x\ \neg Q(x)$ is ***true***.

g) We showed that $Q(0)$ is *true*, therefore there is at least one x that makes $Q(x)$ *true*, so $\forall x\ \neg Q(x)$ is ***false***.

Exercise

Determine the truth value of each of these statements if the domain consists of all integers

$a)\ \forall n(n+1>n)$ $\qquad b)\ \exists n(2n=3n)$ $\qquad c)\ \exists n(n=-n)$ $\qquad d)\ \forall n(3n\le 4n)$

Solution

a) True, since adding 1 to a number makes it larger.

b) True, since $2\cdot 0=3\cdot 0$

c) True, since $0=-0$

d) True for all integers, $3n\le 4n \ \Rightarrow\ 3\le 4$

Exercise

Determine the truth value of each of these statements if the domain consists of all real numbers

$a)\ \exists x\left(x^3=-1\right)$ $\qquad b)\ \exists x\left(x^4<x^2\right)$ $\qquad c)\ \forall x\left((-x)^2=x^2\right)$ $\qquad d)\ \forall x(2x>x)$

Solution

a) Since $(-1)^3=-1$, the statement $\exists x\left(x^3=-1\right)$ is ***true***.

b) Since $\left(\frac{1}{2}\right)^4<\left(\frac{1}{2}\right)^2$, the statement $\exists x\left(x^4<x^2\right)$ is ***true***.

c) Since $(-x)^2=(-1)^2x^2=x^2$, the statement $\forall x\left((-x)^2=x^2\right)$ is ***true***.

d) Since $2(-1)\not> -1$, the statement $\forall x(2x>x)$ is ***false***.

Exercise

Suppose that the domain of the propositional function $P(x)$ consists of the integers 1, 2, 3, 4, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

$a)\ \exists x\ P(x)$ $\qquad b)\ \forall x\ P(x)$ $\qquad c)\ \neg\exists x\ P(x)$ $\qquad d)\ \neg\forall x\ P(x)$

$e)\ \forall x\ \left((x\ne 3)\to P(x)\right)\ \vee\ \exists x\ \neg P(x)$

Solution

a) The statement to be true, so either $P(1)$ is true or $P(2)$ is true or $P(3)$ is true or $P(4)$ is true or $P(5)$ is true. Thus, $P(1)\vee P(2)\vee P(3)\vee P(4)\vee P(5)$

b) $P(1)\wedge P(2)\wedge P(3)\wedge P(4)\wedge P(5)$

c) $\neg\left(P(1)\vee P(2)\vee P(3)\vee P(4)\vee P(5)\right)$

d) $\neg\left(P(1)\wedge P(2)\wedge P(3)\wedge P(4)\wedge P(5)\right)$

e) $\left(\left((1\neq 3)\to P(1)\wedge\left((2\neq 3)\to P(2)\right)\wedge\left((3\neq 3)\to P(3)\right)\wedge\left((4\neq 3)\to P(4)\right)\wedge\left((5\neq 3)\to P(5)\right)\right)\right.$

$\left.\vee\left(\neg\left(P(1)\vee\neg P(2)\vee\neg P(3)\vee\neg P(4)\vee\neg P(5)\right)\right)\right)$

Since the hypothesis $x\neq 3$ is false when $x=3$ and true when x is anything other than 3, we have $\left(P(1)\wedge P(2)\wedge P(3)\wedge P(4)\wedge P(5)\right)\vee\left(\neg P(1)\vee\neg P(2)\vee\neg P(3)\vee\neg P(4)\vee\neg P(5)\right)$

This statement is always true, since the first part is not true, then the second part must be true.

Exercise

For each of these statements find a domain for which the statement is true and a domain for which the statement is false.

a) Everyone is studying discrete mathematics.
b) Everyone is older than 21 years.
c) Every two people have the same mother.
d) No Two different people have the same grandmother.

Solution

Let $A(x)$ *be "x everyone at the school"*

a) Let $B(x)$ *be "x is studying discrete mathematics"*. Then we have $\forall x\ B(x)$, or $\forall x\ \left(A(x)\to B(x)\right)$

b) Let $C(x)$ *be "x is older than 21 years"*. Then we have $\forall x\ C(x)$, or $\forall x\ \left(A(x)\to C(x)\right)$

c) Let $D(x)$ *be "x has the same mother,"* $E(x)$ *two people*. $\forall x\ \left(E(x)\to D(x)\right)$

d) Let $D(x)$ *be "x has the same grandmother,"* $E(x)$ *two people*. $\neg\forall x\ \left(E(x)\to D(x)\right)$

Exercise

Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.
b) Not everyone is perfect.
c) All your friends are perfect.
d) At least one of your friends is perfect.
e) Everyone is your friend and is perfect.
f) Not everybody is your friend or someone is not perfect.

Solution

Let $X(x)$ *be "x is perfect"*; let $Y(x)$ *be "x is your friend"*

a) $\forall x\ \neg X(x)$. Alternatively, we can rewrite $\neg\exists x\ X(x)$

b) $\forall x\ \neg X(x)$

c) $\forall x\ \left(Y(x) \to X(x)\right)$

d) $\exists x\ \left(Y(x) \land X(x)\right)$

e) $\forall x\ \left(Y(x) \land X(x)\right)$

f) This is a disjunction. The expression is $\left(\forall x\ \neg Y(x)\right) \lor \left(\exists x\ \neg X(x)\right)$

Exercise

Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) Something is not in the correct place.
b) All tools are in the correct place and are in excellent condition.
c) Everything is in the correct place and in excellent condition.
d) Nothing is in the correct place and is in excellent condition.
e) One of your tools is not in the correct place, but it is in excellent condition.

Solution

Let $A(x)$ *be "x in the correct place"*; let $B(x)$ *be "x is in excellent condition"*; let $C(x)$ *be "x is a tool"*

a) $\exists x\ \neg A(x)$ *There exists something is not in the correct place.*

b) $\forall x\left(C(x) \to \left(A(x) \land B(x)\right)\right)$

c) $\forall x\left(A(x) \land B(x)\right)$

d) $\forall x \neg\left(A(x) \land B(x)\right)$

e) $\exists x\left(C(x) \land \neg A(x) \land B(x)\right)$

Solution Section 1.4 – Nested Quantifiers

Exercise

Translate these statements into English, where the domain for each variable consists of all real numbers

a) $\forall x \exists y (x < y)$

b) $\exists x \forall y (xy = y)$

c) $\forall x \forall y \left(\left((x \geq 0) \wedge (y < 0) \right) \rightarrow (x - y > 0) \right)$

d) $\forall x \forall y \left(\left((x \geq 0) \wedge (y \geq 0) \right) \rightarrow (xy \geq 0) \right)$

e) $\forall x \forall y \exists z (xy = z)$

f) $\forall x \forall y \exists z (x = y + z)$

Solution

a) For every real number x there exists a real number y such that x is less than y. Basically, there is a larger number.

b) There exists real number x such that for every a real number y, $xy = y$. This is asserting the existence of a multiplication identity for the real numbers, and the statement is true, since we can take $x = 1$.

c) For every real number x and real number y, if x is nonnegative and y is negative, then the difference $x - y$ is positive. More simply, a nonnegative number minus a negative number is positive which is true.

d) For every real number x and real number y, if x is positive and y is positive, then the product xy is positive. More simply, a product of 2 positive numbers is positive.

e) For every real number x and real number y, there exists a real number z such that the product $xy = z$. More simply, the real numbers are closed under multiplication.

f) For every real number x and real number y, there exists a real number z such that the product $x = y + z$. This is a true statement, since we can take $z = x - y$ in each case.

Exercise

Let $Q(x, y)$ be the statement "x has sent an e-mail message to y," where the domain for both x and y consists of all students in your class. Express each of these quantifications in English

a) $\exists x \exists y Q(x, y)$

b) $\exists x \forall y Q(x, y)$

c) $\forall x \exists y Q(x, y)$

d) $\exists y \forall x Q(x, y)$

e) $\forall y \exists x Q(x, y)$

f) $\forall x \forall y Q(x, y)$

Solution

a) There exist students x and y such that x has sent a message to y.
In other words, there is some student in your class who has sent a message to some student in your class.

b) There exists a student x for every student y such that x has sent a message to every y.
In other words, there is a student in your class who has sent a message to every student in your class.

c) For every student x in your class there exists a student y such that x has sent a message to y.
In other words, every student in your class has sent a message to at least one student in your class.

d) There exists a student y for every student x such that y has sent a message to every x.
In other words, there is a student in your class who has sent a message to every student in your class.

e) For every student y in your class there exists a student x such that y has sent a message to x.
In other words, every student in your class has sent a message to at least one student in your class.

f) Every student in your class has sent a message to every student in your class.

Exercise

Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators where the domain consists of all integers.

a) The product of two negative integers is positive.

b) The average of two positive integers is positive.

c) The difference of two negative integers is not necessarily negative.

d) The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers.

Solution

a) $\forall x \forall y \big((x<0) \wedge (y<0) \rightarrow (xy>0)\big)$

b) $\forall x \forall y \left((x>0) \wedge (y>0) \ \rightarrow \ \dfrac{x+y}{2} > 0 \right)$

c) $\exists x \exists y\left((x<0) \wedge (y<0) \wedge (x-y \geq 0)\right)$

d) $\forall x \forall y\left(|x+y| \leq |x|+|y|\right)$

Exercise

Rewrite these statements so that the negations only appear within the predicates

a) $\neg \exists y \forall x P(x, y)$

b) $\neg \forall x \exists y P(x, y)$

c) $\neg \exists y\left(Q(y) \wedge \forall x \neg R(x, y)\right)$

Solution

a) $\neg \exists y \forall x P(x, y) \equiv \forall y \neg \forall x P(x, y)$

$$\equiv \forall y \exists x \neg P(x, y)$$

b) $\neg \forall x \exists y P(x, y) \equiv \exists x \neg \exists y P(x, y)$

$$\equiv \exists x \forall y \neg P(x, y)$$

c) $\neg \exists y\left(Q(y) \wedge \forall x \neg R(x, y)\right) \equiv \forall y \neg\left(Q(y) \wedge \forall x \neg R(x, y)\right)$

$$\equiv \forall y\left(\neg Q(y) \vee \neg\left(\forall x \neg R(x, y)\right)\right)$$

$$\equiv \forall y\left(\neg Q(y) \vee \exists x R(x, y)\right)$$

Exercise

Express the negations of each of these statements so that all negation symbols immediately precede predicates.

a) $\forall x \exists y \forall z T(x, y, z)$

b) $\forall x \exists y P(x, y) \vee \forall x \exists y Q(x, y)$

Solution

a) $\neg\left(\forall x \exists y \forall z T(x, y, z)\right) \equiv \neg \forall x \exists y \forall z T(x, y, z)$

$$\equiv \exists x \neg \exists y \forall z T(x, y, z)$$

$$\equiv \exists x \forall y \neg \forall z T(x, y, z)$$

$$\equiv \exists x \forall y \exists z \neg T(x, y, z)$$

b) $\neg\left(\forall x \exists y P(x, y) \vee \forall x \exists y Q(x, y)\right) \equiv \neg\left(\forall x \exists y P(x, y)\right) \wedge \neg\left(\forall x \exists y Q(x, y)\right)$

$$\equiv \exists x \neg\left(\exists y P(x, y)\right) \wedge \exists x \neg\left(\exists y Q(x, y)\right)$$

$$\equiv \exists x \forall y \neg P(x, y) \wedge \exists x \forall y \neg Q(x, y)$$

Exercise

Let $T(x, y)$ mean that student *x* likes cuisine *y*, where the domain for *x* consists of all students at your school and the domain for *y* consists of all cuisines. Express each of these statements by a simple English sentence.

a) $\neg T(A, J)$

b) $\exists x T(x, Korean) \wedge \forall x Tx(x, Mexican)$

c) $\exists y(T(Monique, y) \vee T(Jay, y))$

d) $\forall x \forall z \exists y((x \neq z) \rightarrow \neg(T(x,y) \wedge T(z,y)))$

e) $\exists x \exists z \forall y(T(x,y) \leftrightarrow T(z,y))$

f) $\forall x \forall z \exists y(T(x,y) \leftrightarrow T(z,y))$

Solution

a) *A* does not like *J* cuisine

b) Some student at your school likes Korean cuisine, and everyone at your school likes Mexican cuisine.

c) There is some cuisine that Monique and Jay likes.

d) For every pair of distinct students at your school, there is some cuisine that at least one them does not like.

e) There are two students at your school who have exactly the same cuisines (tastes).

f) For every pair of students at your school, there is some cuisine about which they have the same opinion (either they both like it or they both do not like it).

Exercise

Let $L(x, y)$ be the statement " *x* loves *y*", where the domain for both *x* and *y* consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody loves Jerry.

b) Everybody loves somebody.

c) There is somebody whom everybody loves.

d) Nobody loves everybody.

e) There is somebody whom Lois does not love.

f) There is somebody whom no one loves.

g) There is exactly one person whom everybody loves.

h) There are exactly two people whom ***L*** loves.

i) Everyone loves himself or herself.

j) There is someone who loves no one besides himself or herself.

Solution

a) $\forall x\ L(x, Jerry)$

b) $\forall x \exists y\, L(x, y)$

c) $\exists y \forall x\, L(x, y)$

d) $\neg \exists x \forall y\, L(x, y)$

e) $\exists x\, \neg L(Lois, x)$

f) $\exists x \forall y\, \neg L(x, y)$

g) $\exists x \left(\forall y\, L(y, x) \wedge \forall z \left(\left(\forall w\, L(w, z) \right) \rightarrow z = x \right) \right)$

h) $\exists x \exists y \left(x \neq y \wedge L(\mathbf{L}, x) \wedge L(\mathbf{L}, y) \wedge \forall z \left(L(\mathbf{L}, z) \rightarrow (z = x \vee z = y) \right) \right)$

i) $\forall x\, L(x, x)$

j) $\exists x \forall y\, \left(L(x, y) \leftrightarrow x = y \right)$

Exercise

Let $S(x)$ be the predicate "*x* is a student," $F(x)$ the predicate "*x* is a faculty member," $A(x, y)$ the predicate "*x* has asked *y* a question," where the domain consists of all people associated with your school. Use quantifiers to express each of these statements.

a) Lois asked Professor Fred a question.
b) Every student has asked Professor Fred a question.
c) Every faculty member has either asked Professor Fred a question or been asked a question by Professor Miller.
d) Some student has not asked any faculty member a question.
e) There is a faculty member who has never been asked a question by a student.
f) Some student has asked every faculty member a question.
g) There is a faculty member who has asked every other faculty member a question.
h) Some student has never been asked a question by a faculty member.

Solution

a) $A(Lois,\ \mathrm{Pr}\,of.\,Fred)$

b) $\forall x \left(S(x) \rightarrow A(x,\ \mathrm{Pr}\,of.\,Fred) \right)$

c) $\forall x \left(F(x) \rightarrow \left(A(x,\ \mathrm{Pr}\,of.\,Fred) \vee A(\mathrm{Pr}\,of.\,Miller,\ x) \right) \right)$

d) $\exists x \left(S(x) \wedge \forall y \left(F(y) \rightarrow \neg A(x, y) \right) \right)$ ***or*** $\exists x \left(S(x) \wedge \neg \exists y \left(F(y) \rightarrow A(x, y) \right) \right)$

e) $\exists x \left(F(x) \wedge \forall y \left(S(y) \rightarrow \neg A(y, x) \right) \right)$

f) $\exists x \left(S(x) \wedge \forall y \left(F(y) \rightarrow A(x, y) \right) \right)$

g) $\exists x\left(F(x)\wedge\forall y\left(\left(F(y)\wedge y\neq x\right)\to A(x,\ y)\right)\right)$

h) $\exists x\left(S(x)\wedge\forall y\left(F(y)\to\neg A(y,\ x)\right)\right)$

Exercise

Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.

a) Every user has access to exactly one mailbox.

b) There is a process that continues to run during all error conditions only if the kernel is working correctly.

c) All users on the campus network can access all websites whose url has a .edu extension.

Solution

a) $\forall y\exists m\left(A(u,\ m)\wedge\forall n\left(n\neq m\to\neg A(u,\ n)\right)\right)$, where $A(u,\ m)$ means that user *u* has access to mailbox *m*.

b) $\exists p\forall e\left(H(e)\to S(p,\ running)\right)\to S(kernel,\ working\ correctly)$, where $H(e)$ means that error condition *e* is in effect and $S(x,\ y)$ means that the status of *x* is *y*.

c) $\forall u\forall s\left(E(s,\ edu)\to A(u,\ s)\right)$, where $E(s,x)$ means that website *s* has extension *x*, and $A(u,\ s)$ means that user *u* can access website *s*.

Exercise

Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers

a) $\exists x\forall y(x+y=y)$

b) $\forall x\forall y\left(\left((x\geq 0)\wedge(y<0)\right)\to(x-y>0)\right)$

c) $\exists x\exists y\left(\left((x\leq 0)\wedge(y\leq 0)\right)\wedge(x-y>0)\right)$

d) $\forall x\forall y\left(\left((x\neq 0)\wedge(y\neq 0)\right)\leftrightarrow(xy\neq 0)\right)$

Solution

a) There exists an additive identity for all real numbers

b) A non-negative number minus a negative number is greater than zero.

c) The difference between two non-positive numbers is not necessarily non-positive (i.e. can be positive)

d) The product of two non-zero numbers is non-zero if and only if both factors are non-zero

Exercise

Determine the truth value of each of these statements if the domain for all variables consists of all integers

a) $\forall n \exists m \ \left(n^2 < m\right)$

b) $\exists n \forall m \ \left(n < m^2\right)$

c) $\forall n \exists m \ (n + m = 0)$

d) $\exists n \forall m \ (nm = m)$

e) $\exists n \exists m \ \left(n^2 + m^2 = 5\right)$

f) $\exists n \exists m \ \left(n^2 + m^2 = 6\right)$

g) $\exists n \exists m \ (n + m = 4 \ \wedge \ n - m = 1)$

h) $\exists n \exists m \ (n + m = 4 \ \wedge \ n - m = 2)$

i) $\forall n \forall m \exists p \ \left(p = \frac{m+n}{2}\right)$

Solution

a) No matter how large *n* might be, we can always find an integer *m* bigger than n^2. This is certainly true, i.e. $m = n^2 + 1$.

b) There is an *n* that is smaller than the square of every integer. This statement is true since we could take $n = -1$, and then *n* would be less than every square, since squares are always greater than or equal to 0.

c) The order of quantifiers: *m* is allowed to depend on *n*. since we can take $m = -n$, this statement is true.

d) Clearly $n = 1$, so the statement is true.

e) $n^2 + m^2 = 5$ has a solution over the integers. This is true statement, since $n = \pm 1, \ m = \pm 2$ and vice versa (8 solutions).

f) $n^2 + m^2 = 6$ there is no integer solution. Therefore; this statement is false.

g) There is a unique solution for the statement $\{n + m = 4, \ n - m = 1\}$, namely $n = \frac{5}{2}$ *and* $m = \frac{3}{2}$.

Since there do not exist integers that make the equations true, the statement is false.

h) There is a unique solution for the statement $\{n + m = 4, \ n - m = 2\}$, namely $n = 3$ *and* $m = 1$.

Therefore; the statement is true.

i) If we take $n = 1$ *and* $m = 2$ then $p = \frac{3}{2}$ which is not an integer. Therefore; the statement is false.

Solution *Section* 1.5 – Introduction to Proofs

Exercise

Show that the square of an even number is an even number

Solution

We can rewrite the statement as: if n is even, then n^2 is even

Assume n is even, thus $n = 2k$ for some k.

$n^2 = (2k)^2 = 4k^2 = 2\left(2k^2\right)$

As n^2 is 2 times an integer, n^2 is thus even

Exercise

Prove that if n is an integer and $n^3 + 5$ is odd, then n is even

Solution

By indirect proof:

Using the contrapositive: If n is odd, then $n^3 + 5$ is even

Assume n is odd, let show that $n^3 + 5$ is even

$n = 2k + 1$ for some integer k (definition of odd numbers)

$n^3 + 5 = (2k+1)^3 + 5 = 8k^3 + 12k^2 + 6k + 6 = 2\left(4k^3 + 6k^2 + 3k + 3\right)$

As $n^3 + 5$ is 2 times an integer, it is even

Assume that $n^3 + 5$ is odd, let show that n is odd, and Assume p is true and q is false

$n = 2k + 1$ for some integer k (definition of odd numbers)

$n^3 + 5 = (2k+1)^3 + 5 = 8k^3 + 12k^2 + 6k + 6 = 2\left(4k^3 + 6k^2 + 3k + 3\right)$

As $n^3 + 5$ is 2 times an integer, it must be even. ***Contradiction***!

The indirect proof proved that the contrapositive: $\neg q \rightarrow \neg p$

If n is odd, then $n^3 + 5$ is even

The proof by contradiction assumed that the implication was false, and showed a contradiction

- If we assume p and $\neg q$, we can show that implies q
- The contradiction is q and $\neg q$
- Note that both used similar steps, but are different means of proving the implication

Exercise

Show that $m^2 = n^2$ if and only if $m = n \quad or \quad m = -n$

Solution

Rephrased: $m^2 = n^2 \leftrightarrow \left[(m=n) \vee (m=-n)\right]$. Proof by cases!

$Case\ 1$: $(m=n) \to \left(m^2 = n^2\right)$

$(m)^2 = m^2 \quad and \quad (n)^2 = n^2$, this case is proven.

$Case\ 1$: $(m=-n) \to \left(m^2 = n^2\right)$

$(m)^2 = m^2 \quad and \quad (-n)^2 = n^2$, this case is proven.

$$m^2 = n^2 \leftrightarrow \left[(m=n) \vee (m=-n)\right]$$

$$m^2 - n^2 = n^2 - n^2$$

$$m^2 - n^2 = 0 \quad \Rightarrow \quad (m-n)(m+n) = 0$$

$$m - n = 0 \quad or \quad m + n = 0$$

$$m = n \quad or \quad m = -n$$

Exercise

Use a direct proof to show that the sum of two odd integers is even.

Solution

Let m and n be two odd integers. Then there exists a and b such that $n = 2a+1$ and $m = 2b+1$.

$$\begin{aligned} n+m &= 2a+1+2b+1 \\ &= 2a+2b+2 \\ &= 2(a+b+1) \end{aligned}$$

Since this represents $n+m$ as 2 times $a+b+1$, we conclude that $n+m$ is even, as desired.

Exercise

Use a direct proof to show that the sum of two even integers is even.

Solution

Let m and n be two even integers. Then there exists a and b such that $n = 2a$ and $m = 2b$.

$$\begin{aligned} n+m &= 2a+2b \\ &= 2(a+b) \end{aligned}$$

Since this represents $n+m$ as 2 times $a+b$, we conclude that $n+m$ is even, as desired.

Exercise

Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

Solution

Let r is a rational number an s is irrational number then $t = r + s$ is an irrational.

Suppose that t is rational, then if $t = \frac{a}{b}$ and $r = \frac{c}{d}$ where *a, b, c*, and *d* are integers with $b \neq 0$ and $d \neq 0$. Then, $t + (-r) = \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$ which is rational.

$t + (-r) = r + s - r = s$, forcing that s is rational. This contradicts the hypothesis that s is irrational.

Therefore the assumption that t was rational was incorrect, and we conclude that t is irrational.

Exercise

Prove or disprove that the product of two irrational numbers is irrational.

Solution

Let $\sqrt{2}$ be the irrational number,. If we take the product of the irrational number $\sqrt{2}$ and the irrational number $\sqrt{2}$, then we obtain the rational number 2. This counterexample refutes the proposition.

Exercise

Prove that if x is irrational, then $\frac{1}{x}$ is irrational.

Solution

The contrapositive is: if $\frac{1}{x}$ is rational, then x is rational.

Since $\frac{1}{x}$ exists, then $x \neq 0$. If $\frac{1}{x}$ is rational then by definition $\frac{1}{x} = \frac{q}{p}$ and $p \neq 0$. Since $\frac{1}{x}$ can't be zero, then we would have the contradiction $1 = x \cdot 0$.

Exercise

Prove that if x is rational and $x \neq 0$, then $\frac{1}{x}$ is rational.

Solution

if x is rational and $x \neq 0$, then by definition we can write $x = \frac{p}{q}$, where p and q are nonzero integers.

Since $\frac{1}{x} = \frac{q}{p}$ and $p \neq 0$, we can conclude that $\frac{1}{x}$ is rational.

Exercise

Prove the proposition $P(0)$, where $P(n)$ is the proposition "If *n* is a positive integer greater than 1, then $n^2 > n$." What kind of proof did you use?

Solution

The proposition that we are trying to prove is If 0 is a positive integer gr2ater than 1, then $0^2 = 0$. Our proof is a vacuous one.
Since the hypothesis is false, the implication is automatically true.

Exercise

Let $P(n)$ be the proposition "If *a* and *b* are positive real numbers, then $(a+b)^n \ge a^n + b^n$." Prove that $P(1)$ is true. What kind of proof did you use?

Solution

Our proof is a direct one. By the definition of exponential, any real number to the power 1 is itself. Hence $(a+b)^1 = a+b = a^1 + b^1$. Finally, by the addition rule, we can conclude from $(a+b)^1 = a^1 + b^1$ that $(a+b)^1 \ge a^1 + b^1$.

Exercise

Show that these statements about the integer *x* are equivalent:

$\boldsymbol{i})\ 3x+2 \ \ is\ even \quad \boldsymbol{ii})\ x+5 \ \ is\ odd \quad \boldsymbol{iii})\ x^2 \ \ is\ even$

Solution

If *x* is even, then $x = 2k$ for some integer *k*.

$3x+2 = 3\cdot 2k+2 = 6k+2 = 2(3k+1)$ which is even.

$x+5 = 2k+4+1 = 2(k+2)+1$, so $x+5$ is odd

$x^2 = (2k)^2 = 4k^2 = 2\left(2k^2\right)$, so x^2 is odd

If *x* is odd, then $x = 2k+1$ for some integer *k*.

$3x+2 = 3\cdot(2k+1)+2 = 6k+3+2 = 6k+4+1 = 2(3k+2)+1$ which is odd *not* even.

$x+5 = 2k+1+5 = 2k+6 = 2(k+3)$, so $x+5$ is even not odd.

$x^2 = (2k+1)^2 = 4k^2+4k+1 = 2\left(2k^2+2k\right)+1$, so x^2 is odd

Exercise

Show that these statements about the real number x are equivalent:

$\boldsymbol{i})\ x\ \ is\ irrational\quad \boldsymbol{ii})\ 3x+2\ \ is\ irrational\quad \boldsymbol{iii})\ \frac{x}{2}\ \ is\ irrational$

Solution

The simplest way is to approach in indirect proof.

$\boldsymbol{i}) \to \boldsymbol{ii})$

Suppose that $3x+2$ is rational, that $3x+2=\frac{p}{q}$ for some integers p and q with $q\neq 0$. Then

$3x=\frac{p}{q}-2=\frac{p-2q}{q} \quad \Rightarrow \quad x=\frac{p-2q}{3q}$ where $3q\neq 0$. This shows that x is rational.

Suppose that x is rational, that $x=\frac{p}{q}$ for some integers p and q with $q\neq 0$. Then

$3x+2=3\frac{p}{q}-2=\frac{3p-2q}{q}$ where $q\neq 0$. This shows that $3x+2$ is rational.

$\boldsymbol{i}) \to \boldsymbol{iii})$

Suppose that $\frac{x}{2}$ is rational, that $\frac{x}{2}=\frac{p}{q}$ for some integers p and q with $q\neq 0$. Then $x=\frac{2p}{q}$ where $q\neq 0$. This shows that x is rational.

Suppose that x is rational, that $x=\frac{p}{q}$ for some integers p and q with $q\neq 0$. Then $\frac{x}{2}=\frac{p}{2q}$ where $2q\neq 0$. This shows that $\frac{x}{2}$ is rational.

Exercise

Prove that at least one of the real numbers $a_1,\ a_2,\ \ldots,\ a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?

Solution

Using proof of contradiction, then suppose all the number $a_1,\ a_2,\ \ldots,\ a_n$ are less than their average.

$a_1+a_2+\ldots+a_n<nA$

By definition: $A=\frac{a_1+a_2+\ldots+a_n}{n}$

The two displayed formulas clearly contradict each other, however: they imply that $nA<nA$. Thus our assumption must have been incorrect, and we conclude that at least one of the numbers a_1 is greater than or equal to their average.

Solution Section 1.6 – Proof Methods and Strategy

Exercise

Prove that $n^2+1 \ge 2^n$ when n is a positive integer with $1 \le n \le 4$

Solution

$n=1 \quad \to \quad 1^2+1 \ge 2^1 \Rightarrow 2 \ge 2 \ \surd$

$n=2 \quad \to \quad 2^2+1 \ge 2^2 \Rightarrow 5 \ge 4 \ \surd$

$n=3 \quad \to \quad 3^2+1 \ge 2^3 \Rightarrow 10 \ge 8 \ \surd$

$n=4 \quad \to \quad 4^2+1 \ge 2^4 \Rightarrow 17 \ge 16 \ \surd$

Exercise

Prove that there are no positive perfect cubes less than 1000 that are the sum of the cubes of two positive integers.

Solution

The cubes are: 1, 8, 27, 64, 125, 216, 343, 512, and 729.

1 + 8 = 9, 1 + 27 = 28, 1 + 64, 1 + 125, …

8 + 8, 8 + 27, 8 + 64, 8 + 125, …

27 + 27, 27 + 64, 27 + 125, …

64 + 64, 64 + 125, 64 + 216, …

125 + 125, 125 + 216, …

216 + 216, 216 + 343, …

343 + 343, 343 + 512, 343 + 729

512 + 512, 512 + 729

729 + 729

None of them works.

We can conclude the no cube less than 1000 is the sum of two cubes.

Exercise

Prove that if x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$. (*Hint*: Use a proof by cases, with the two cases corresponding to $x \ge y$ and $x < y$, respectively.)

Solution

Suppose that $x \ge y$, then by definition $max(x,\ y) = x$ and $min(x,\ y) = y$. Therefore; in this case $\max(x, y) + \min(x, y) = x + y$.

In the second case $x < y$, then by definition $max(x,\ y) = y$ and $min(x,\ y) = x$. Therefore; in this case, $\max(x, y) + \min(x, y) = y + x = x + y$.

Hence in all cases, the equality holds.

Exercise

Prove the triangle inequality, which states that if x and y are real numbers, then $|x| + |y| \ge |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \ge 0$ and equals $-x$ if $x < 0$

Solution

If x and y are both nonnegative, then $|x| + |y| = x + y = |x + y|$.

If x and y are both negative, then $|x| + |y| = (-x) + (-y) = -(x + y) = |x + y|$.

If $x \ge 0$ and $y < 0$, then there are two subcases to consider for x and $-y$:

Case 1: Suppose that $x \ge -y$, then $x + y \ge 0$. Therefore $x + y = |x + y|$, as desired.

$|x| + |y| = x + |y|$ is a positive number greater than x. Therefore $|x + y| < x < |x| + |y|$

Case 2: Suppose that $x < -y$, then $x + y < 0$. Therefore $|x + y| = -(x + y) = (-x) + (-y)$.

is a positive number less than or equal to $-y$. Therefore $|x + y| \le -y \le |x| + |y|$, as desired.

Exercise

Prove that either $2 \cdot 10^{500} + 15$ or $2 \cdot 10^{500} + 16$ is not a perfect square

Solution

A perfect square is a square of an integer

Rephrased: Show that a non-perfect square exists in the set $\left\{2 \cdot 10^{500} + 15,\ \ 2 \cdot 10^{500} + 16\right\}$

Proof: The only two perfect squares that differ by 1 are 0 and 1

Thus, any other numbers that differ by 1 cannot both be perfect squares

Thus, a non-perfect square must exist in any set that contains two numbers that differ by 1

Note that we didn't specify which one it was!

Exercise

Prove that there exists a pair of consecutive integers such that one of these integers is a perfect square and the other is a perfect cube.

Solution

$8 = 2^3 \qquad 9 = 3^2$

Exercise

Suppose that a and b are odd integers with $a \neq b$. Show there is a unique integer c such that $|a-c| = |b-c|$

Solution

The equation $|a-c| = |b-c|$ is equivalent to the disjunction of two equations:

$a-c=b-c$ *or* $a-c=-b+c$

Case: $a-c=b-c$ is equivalent to $a=b$, which contradicts the assumption $a \neq b$, so the original equation is equivalent to $a-c=-b+c$. By adding $b+c$ to both sides and dividing by 2, we see that this equation is equivalent to $c=\frac{a+b}{2}$. Thus there is a unique solution. Furthermore, this c is an integer, because the sum of the odd integers a and b is even.

Solution   Section 1.7 – Sets

Exercise

List the members of these sets

a) $\left\{x \middle| x \text{ is a real number such that } x^2 = 1\right\}$

b) $\left\{x \middle| x \text{ is a positive integer less than } 12\right\}$

c) $\left\{x \middle| x \text{ is the square of an integer and } x < 100\right\}$

d) $\left\{x \middle| x \text{ is an integer such that } x^2 = 2\right\}$

Solution

a) $\{-1,\ 1\}$

b) $\{1,2,3,4,5,6,7,8,9,10,11,12\}$

c) $\{0,1,4,9,16,25,36,49,64,81\}$

d) $\varnothing$ $\left\{\sqrt{2} \text{ is not an integer}\right\}$

Exercise

Determine whether each these pairs of sets are equal.

a) $\{1,3,3,3,5,5,5,5,5\}$, $\{5,3,1\}$

b) $\{\{1\}\}$, $\{1,\ \{1\}\}$

c) $\varnothing$, $\{\varnothing\}$

Solution

a) Yes; order and repetition do not matter.

b) No; the first set has one element, and the second has two elements.

c) No; the first set has no elements, and the second has one element (namely the empty set).

Exercise

For each of the following sets, determine whether 2 is an element of that set.

a) $\left\{x \in \mathbb{R} \middle| x \text{ is an integer greater than } 1\right\}$

b) $\left\{x \in \mathbb{R} \middle| x \text{ is the square of an integer}\right\}$

c) $\{2,\ \{2\}\}$

d) $\{\{2\},\ \{\{2\}\}\}$

e) $\{\{2\},\ \{2,\ \{2\}\}\}$

f) $\{\{\{2\}\}\}$

Solution

a) Since 2 is an integer greater than 1, 2 is an element of this set.

b) Since 2 is not a perfect square, 2 is not an element of this set.

c) This set has two elements, and clearly one of those elements is 2.

d) This set has two elements, and clearly neither of those elements is 2. Both of the elements of this set are seats; 2 is a number, not a set.

e) This set has two elements, and clearly neither of those elements is 2. Both of the elements of this set are seats; 2 is a number, not a set.

f) This set has just one element, namely the set $\{\{2\}\}$. So 2 is not an element of this set. Note $\{2\} \neq \{\{2\}\}$

Exercise

Determine whether each of these statements is true or false

a) $0 \in \varnothing$

b) $\varnothing \in \{0\}$

c) $\{0\} \subset \varnothing$

d) $\varnothing \subset \{0\}$

e) $\{0\} \in \{0\}$

f) $\{0\} \subset \{0\}$

g) $\{\varnothing\} \subseteq \{\varnothing\}$

h) $x \in \{x\}$

i) $\{x\} \subseteq \{x\}$

j) $\{x\} \in \{x\}$

k) $\{x\} \in \{\{x\}\}$

l) $\varnothing \subseteq \{x\}$

m) $\varnothing \in \{x\}$

Solution

a) *False*, since the empty set has no elements.

b) *False*, the set on the right has only one element, namely 0, not the empty set.

c) *False*, the empty set has no proper subsets.

d) *True*, every element of the set on the left, is vacuously, an element of the set on the right; and the set on the right contains an element, namely 0, that is not the set on the left.

e) *False*, the set on the right has only one element, namely 0, not the set containing the number 0.

f) *False*, for one set to be a proper subset of another, the two sets cannot be equal.

g) *True*, every set is a subset of itself.

h) *True*, *x* is the only element.

i) *True*, every set is a subset of itself.

j) *False,* the only element of $\{x\}$ is a letter, not a set.

k) *True*, $\{x\}$ is the only element

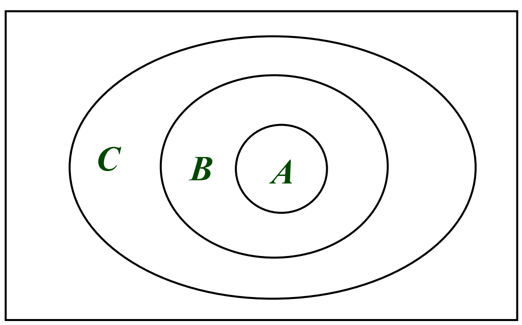
l) *True*, the empty set is a subset of every set.

m) *False*, the only element of $\{x\}$ is a letter, not a set.

Exercise

Use a Venn Diagram to illustrate the relationships $A \subset B$ and $B \subset C$.

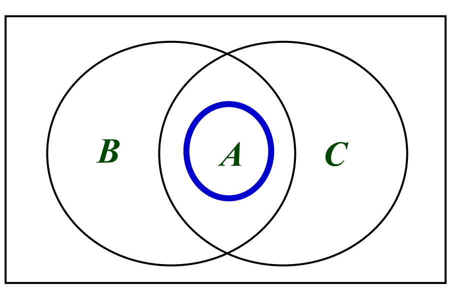
Solution



Exercise

Use a Venn Diagram to illustrate the relationships $A \subset B$ and $A \subset C$.

Solution



Since no information about the relationship between *B* and *C*, then *B* and *C* can be overlap. The set *A* must be a subset of each of them.

Exercise

Suppose that A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$

Solution

Let $x \in A$, then since $A \subseteq B$, we can conclude that $x \in B$. Furthermore, since $B \subseteq C$, the fact that $x \in B$ implies that $x \in C$. Therefore, $A \subseteq C$

Exercise

What is the cardinality of each of these sets?

a) $\{a\}$

b) $\{\{a\}\}$

c) $\{a, \{a\}\}$

d) $\{a, \{a\}, \{a, \{a\}\}\}$

Solution

a) 1

b) 1

c) 2

d) 3

Exercise

How many elements does each of these sets have where *a* and *b* are distinct elements?

a) $\mathcal{P}(\{a, b, \{a, b\}\})$

b) $\mathcal{P}(\{\varnothing, a, \{a\}, \{\{a\}\}\})$

c) $\mathcal{P}(\mathcal{P}(\varnothing))$

Solution

a) Since the set has 3 elements, the power of the set has $2^3 = 8$ elements

b) Since the set has 4 elements, the power of the set has $2^4 = 16$ elements

c) Since the set has 0 elements, the power of the set has $2^0 = 1$ elements. The power of this set therefore has $2^1 = 2$ elements.

Exercise

What is the Cartesian product $A \times B \times C$, where A is the set of all airlines and B and C are both the set of all cities in the United States? Give an example of how this Cartesian product can be used.

Solution

This is the set of triples $(a,\ b,\ c)$, where *a* is an airline and *b* and *c* are cities. A useful subset of this set of triples $(a,\ b,\ c)$ for which *a* flies between *b* and *c*. For example, Continental, Houston, Chicago) is in this subset.

Exercise

What is the Cartesian product $A \times B$, where A is the set of all courses offered by the mathematics department and B is the set of mathematics professors at this university? Give an example of how this Cartesian product can be used.

Solution

By definition it is the set of all ordered pairs $(c,\ p)$ such that *c* is a course and *p* is a professor. The elements of this set are the possible teaching assignments for the mathematics department.

Exercise

Let A be a set. Show that $\varnothing \times A = A \times \varnothing = \varnothing$

Solution

By definition, $\varnothing \times A$ consists of al pairs $(x,\ a)$ such that $x \in \varnothing$ and $a \in A$. Since there are no elements $x \in \varnothing$. There are no such pairs, so $\varnothing \times A = \varnothing$. Similar reasoning $A \times \varnothing = \varnothing$.

Solution *Section* 1.8 – Set Operations

Exercise

Let A be the set of students who live within one mile of school and let B be the set of students who walk to classes. Describe the students in each of these sets

a) $A \cap B$
b) $A \cup B$
c) $A - B$
d) $B - A$

Solution

a) The set of students who live one mile of school and walk to classes.

b) The set of students who live one mile of school or walk to classes.

c) The set of students who live one mile of school but not walk to class.

d) The set of students who live more than one mile from school but nevertheless walk to class.

Exercise

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$

a) $A \cup B$
b) $A \cap B$
c) $A - B$
d) $B - A$

Solution

a) $\{0, 1, 2, 3, 4, 5, 6\}$

b) $\{3\}$

c) $(1, 2, 4, 5\}$

d) $\{0, 6\}$

Exercise

Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$

a) $A \cup B$
b) $A \cap B$
c) $A - B$
d) $B - A$

Solution

a) $\{a, b, c, d, e, f, g, h\} = B$

b) $\{a, b, c, d, e\} = A$

c) $\varnothing$, since there are no elements in A that are not in B.

d) $\{f, g, h\}$

Exercise

Prove the domination laws by showing that

a) $A \bigcup U = U$

b) $A \bigcap U = A$

c) $A \bigcup \varnothing = A$

d) $A \bigcap \varnothing = \varnothing$

Solution

a)
$$\begin{aligned} A \bigcup U &= \left\{ x \middle| x \in A \lor x \in U \right\} \\ &= \left\{ x \middle| x \in A \lor \boldsymbol{T} \right\} \\ &= \left\{ x \middle| \boldsymbol{T} \right\} \\ &= U \end{aligned}$$

b)
$$\begin{aligned} A \bigcap U &= \left\{ x \middle| x \in A \land x \in U \right\} \\ &= \left\{ x \middle| x \in A \land \boldsymbol{T} \right\} \\ &= \left\{ x \middle| x \in A \right\} \\ &= A \end{aligned}$$

c)
$$\begin{aligned} A \bigcup \varnothing &= \left\{ x \middle| x \in A \lor x \in \varnothing \right\} \\ &= \left\{ x \middle| x \in A \lor \boldsymbol{F} \right\} \\ &= \left\{ x \middle| x \in A \right\} \\ &= A \end{aligned}$$

d)
$$\begin{aligned} A \bigcap \varnothing &= \left\{ x \middle| x \in A \land x \in \varnothing \right\} \\ &= \left\{ x \middle| x \in A \land \boldsymbol{F} \right\} \\ &= \left\{ x \mid \boldsymbol{F} \right\} \\ &= \varnothing \end{aligned}$$

Exercise

Prove the complement laws by showing that

a) $A \bigcup \overline{A} = U$

b) $A \bigcap \overline{A} = \varnothing$

Solution

a) $A \cup \overline{A} = \{x | x \in A \vee x \in \overline{A}\}$

$= \{x | x \in A \vee x \notin A\}$

$= \{x | \boldsymbol{T}\}$

$= U$

b) $A \cap \overline{A} = \{x | x \in A \wedge x \in \overline{A}\}$

$= \{x | x \in A \wedge x \notin A\}$

$= \{x | \boldsymbol{F}\}$

$= \varnothing$

Exercise

Show that

a) $A - \varnothing = A$

b) $\varnothing - A = \varnothing$

Solution

a) $A - \varnothing = \{x | x \in A \wedge x \notin \varnothing\}$

$= \{x | x \in A \wedge \boldsymbol{T}\}$

$= \{x | x \in A\}$

$= A$

b) $\varnothing - A = \{x | x \in \varnothing \wedge x \notin A\}$

$= \{x | \boldsymbol{F} \wedge x \notin A\}$

$= \{x | \boldsymbol{F}\}$

$= \varnothing$

Exercise

Prove the absorption law by showing that if *A* and *B* are sets, then

a) $A \cap (A \cup B) = A$

b) $A \cup (A \cap B) = A$

Solution

a) Suppose $x \in A \cap (A \cup B)$, then $x \in A$ and $x \in A \cup B$ by the definition of intersection. We have $x \in A$ and in the latter case $x \in A$ *or* $x \in B$ by the definition of union. Since both of these are true, $x \in A \cup B$ by the definition of intersection, and we have shown that the right-hand side is a subset of the left-hand side.

b) Suppose $x \in A \cup (A \cap B) \quad \Rightarrow \quad x \in A$ *or* $x \in (A \cap B)$ by definition of union.

$x \in A$ *or* $(x \in A$ *and* $x \in B)$

By the definition of the intersection, in any event, $x \in A$. Therefore, $x \in A \cup (A \cap B)$ as well.

That proves that the right-hand side is a subset of the left-hand side.

Exercise

Show that if *A*, *B*, and *C* are sets, then $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

Solution

Suppose $x \in \overline{A \cap B \cap C}$, then $x \notin A \cap B \cap C$, which means that *x* fails to be in at least one of these three sets. In other words, $x \notin A$ *or* $x \notin B$ *or* $x \notin C$. This is equivalent to saying that $x \in \overline{A}$ *or* $x \in \overline{B}$ *or* $x \in \overline{C}$. Therefore $x \in \overline{A} \cup \overline{B} \cup \overline{C}$.

Conversely, if $x \in \overline{A} \cup \overline{B} \cup \overline{C}$, then $x \in \overline{A}$ *or* $x \in \overline{B}$ *or* $x \in \overline{C}$. This means $x \notin A$ *or* $x \notin B$ *or* $x \notin C$, so *x* cannot be in the intersection of *A, B*, and *C*. Since $x \notin A \cap B \cap C$, we conclude that $x \in \overline{A \cap B \cap C}$, as desired.

Or

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
1	1	1	**1**	**0**	**0**	**0**	**0**	**0**
1	1	0	**0**	**1**	**0**	**0**	**1**	**1**
1	0	1	**0**	**1**	**0**	**1**	**0**	**1**
1	0	0	**0**	**1**	**0**	**1**	**1**	**1**
0	1	1	**0**	**1**	**1**	**0**	**0**	**1**
0	1	0	**0**	**1**	**1**	**0**	**1**	**1**
0	0	1	**0**	**1**	**1**	**1**	**0**	**1**
0	0	0	**0**	**1**	**1**	**1**	**1**	**1**

Exercise

Let A and B be sets. Show that

a) $(A \cap B) \subseteq A$

b) $A \subseteq (A \cup B)$

c) $(A - B) \subseteq A$

d) $A \cap (B - A) = \varnothing$

e) $A \cup (B - A) = A \cup B$

Solution

a) If x is in $A \cap B$, then, by definition of intersection, it is in A.

b) If x is in A, then perforce, by definition of union, it is in $A \cup B$.

c) If x is in $A - B$, then perforce, by definition of difference, it is in A.

d) Is $x \in A$ then $x \notin B - A$. Therefore there can be no elements in $A \cap (B - A)$, so $A \cap (B - A) = \varnothing$.

e) The left-hand side consists of elements of either A or B or both. This is precisely the definition of the right-hand side.

Exercise

Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

a) $A \cap (B - C)$

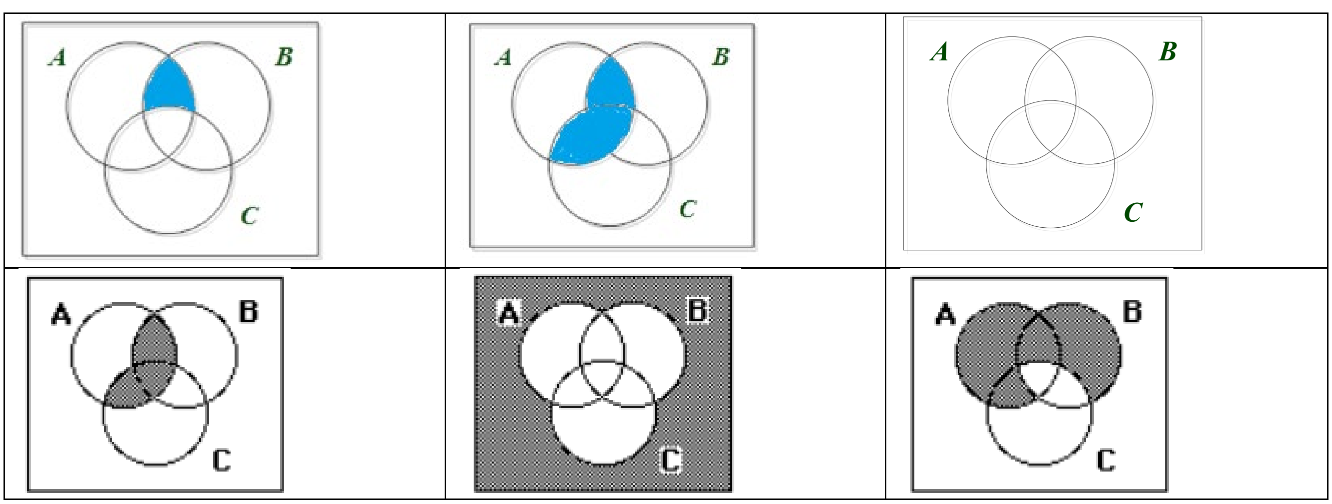
b) $(A \cap B) \cup (A \cap C)$

c) $(A \cap \overline{B}) \cup (A \cap \overline{C})$

d) $\overline{A} \cap \overline{B} \cap \overline{C}$

e) $(A - B) \cup (A - C) \cup (B - C)$

Solution



Exercise

Show that $A \oplus B = (A \cup B) - (A \cap B)$

Solution

This is just a restatement of the definition. An element is in $(A \cup B) - (A \cap B)$ if it in the union that in either *A* or *B*, but not in the intersection (i.e., not in both *A* and *B*)

Exercise

Show that $A \oplus B = (A - B) \cup (B - A)$

Solution

There are two ways that an item can be in either *A* or *B* but not both. It can be in *A* but not *B* (which is equivalent to saying that it is in $A - B$), or it can be in *B* but not *A* (which is equivalent to saying that it is in $B - A$).

Thus an element is in $A \oplus B$ if and only if it is in $(A - B) \cup (B - A)$.

Solution   *Section* 1.9 – Functions

Exercise

Why is f not a function from $\mathbb{R}$ to $\mathbb{R}$ if

a) $f(x) = \frac{1}{x}$?

b) $f(x) = \sqrt{x}$?

c) $f(x) = \pm\sqrt{x^2 + 1}$?

Solution

a) Because for $x = 0$ the value of $f(x)$ is not defined by the given rule.

b) Because for $x < 0$ the value of $f(x)$ is not defined in $\mathbb{R}$

c) Because for $f(1) = \sqrt{2} \quad or \quad f(1) = -\sqrt{2}$

Exercise

Determine whether f is a function from $\mathbb{Z}$ to $\mathbb{R}$ if

a) $f(x) = \pm x$?

b) $f(x) = \sqrt{x^2 + 1}$?

c) $f(x) = \frac{1}{x^2 - 4}$?

Solution

a) This is not a function because $f(1) = 1 \quad or \quad f(1) = -1$,

b) This is a function for all integers x.

c) This is not a function since for $x = \pm 2$ the value of $f(x)$ is not defined by the given rule.

Exercise

Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.

a) The function that assigns to each bit string the number of ones in the string minus the number of zeros in the string.

b) The function that assigns to each bit string twice the number of zeros in that string.

c) The function that assigns the number of bits over when a bit string is split into bytes (which are blocks of 8 bits).

Solution

a) The domain is the set of all bit strings. Depending on how we read the word "difference" the output values can be either integers or natural numbers. For example. If the input is 1010000, then one might read the rule as stating that the function value is 3, since there are three more 0s than 1s; but most people would probably consider the function value to be –3, obtained by subtracting in the order stated: $2-5=-3$. Then the range is $\mathbb{Z}$.

b) The domain is the set of all bit strings. Since there can be any natural number of 0s in a bit string, the value of the function can be 0, 2, 4, …. Therefore the range is the set of even natural numbers.

c) The domain is the set of all bit strings. Since the number of leftover bits can be any whole number from 0 to 7 (if it were more, then we could form another byte), the range is (0, 1, 2, 3, 4, 5, 6, 7).

Exercise

Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one and onto.

a) $f(a)=b,\ f(b)=a,\ f(c)=c,\ f(d)=d$

b) $f(a)=b,\ f(b)=b,\ f(c)=d,\ f(d)=c$

c) $f(a)=d,\ f(b)=b,\ f(c)=c,\ f(d)=d$

Solution

a) This is one–to–one.

b) This is not one–to–one, since b is the image of both a and b.

c) This is not one–to–one, since d is the image of both a and d.

Exercise

Determine whether the function $f:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ is onto if

a) $f(m,\ n)=m+n$

b) $f(m,\ n)=m^2+n^2$

c) $f(m,\ n)=m$

d) $f(m,\ n)=|n|$

e) $f(m,\ n)=m-n$

f) $f(m,\ n)=2m-n$

g) $f(m,\ n)=m^2-n^2$

h) $f(m,\ n)=m+n+1$

i) $f(m,\ n)=|m|-|n|$

j) $f(m,\ n)=m^2-4$

Solution

a) Given any integer *n*, we have $f(0,\ n)=n$, so the function is onto.

b) The range contains no negative integers, so the function is not onto.

c) Given any integer *m*, we have $f(m,\ 4)=m$, so the function is onto.

d) The range contains no negative integers, so the function is not onto.

e) Given any integer *m*, we have $f(m,\ 0)=m$, so the function is onto.

f) For any integer *n*, we have $f(0,\ -n)=n$, so the function is onto.

g) $m^2-n^2=(m-n)(m+n)\neq 2$, since 2 is not in the range, so the function is not onto.

h) For any integer *n*, we have $f(0,\ n-1)=n$, so the function is onto.

i) This is onto. To achieve negative values we set $m=0$, and to achieve nonnegative value we set $n=0$.

j) $m^2-4=(m-2)(m+2)\neq 2$ since 2 is not in the range, so the function is not onto.

Exercise

Determine whether each of these functions is a bijection from $\mathbb{R}\to\mathbb{R}$

a) $f(x)=2x+1$

b) $f(x)=x^2+1$

c) $f(x)=x^3$

d) $f(x)=\dfrac{x^2+1}{x^2+2}$

e) $f(x)=x^5+1$

Solution

a) $f(x)=2x+1 \quad\Rightarrow\quad f^{-1}(x)=\frac{x-1}{2}$. Therefore the function is a bijection.

b) It is not one-to-one since $f(1)=f(-1)=2$ and it is also not onto since the range is the interval $[1,\ \infty)$.

c) $f(x)=x^3 \quad\Rightarrow\quad f^{-1}(x)=\sqrt[3]{x}$. Therefore the function is a bijection.

d) It is not one-to-one since $f(1)=f(-1)=\frac{2}{3}$ and it is also not onto since the range is the interval $\left[\frac{1}{2},\ \infty\right)$.

e) $f(x)=x^5+1 \quad\Rightarrow\quad f^{-1}(x)=\sqrt[5]{x-1}$. Therefore the function is a bijection.

Exercise

Suppose that g is a function from A to B and f is a function from B to C.

a) Show that if both f and g are one–to–one functions, then $f \circ g$ is also one–to–one.

b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

Solution

a) Assume that both f and g are one–to–one. We need to show that $f \circ g$ is also one–to–one. If x and y are two distinct elements of A, then $f\left(g(x)\right) \neq f\left(g(y)\right)$. First, since g is one–to–one, by definition $g(x) \neq g(y)$. Second, since now $g(x)$ and $g(y)$ are distinct elements of B, and since f is one–to–one, we conclude that $f\left(g(x)\right) \neq f\left(g(y)\right)$ as desired.

b) Assume that both f and g are onto. We need to show that $f \circ g$ is onto. If z is any element of C, then there is some element $x \in A$ such that $f\left(g(x)\right) = z$. First, since f is onto, we can conclude that there is an element $y \in B$ such that $f(y) = z$. Second, since g is onto and $y \in B$, we can conclude that there is an element $x \in A$ such that $g(x) = y$. therefore $z = f(y) = f\left(g(x)\right)$ as desired.

Solution Section 2.1 – Sequences and Summations

Exercise

Find these terms of the sequence $\{a_n\}$, where $a_n = 2\cdot(-3)^n + 5^n$

$a)\ a_0 \quad b)\ a_1 \quad c)\ a_4 \quad d)\ a_5$

Solution

a) $a_0 = 2\cdot(-3)^0 + 5^0$

$\underline{= 3}$

b) $a_1 = 2\cdot(-3)^1 + 5^1$

$= -6 + 5$

$\underline{= -1}$

c) $a_4 = 2\cdot(-3)^4 + 5^4$

$= 162 + 625$

$\underline{= 787}$

d) $a_4 = 2\cdot(-3)^5 + 5^5$

$= -486 + 3125$

$\underline{= 2639}$

Exercise

What is the term a_8 of the sequence $\{a_n\}$, if a_n equals

$a)\ 2^{n-1} \quad b)\ 7 \quad c)\ 1+(-1)^n \quad d)\ -(2)^n$

Solution

a) $a_8 = 2^{8-1}$

$\underline{= 128}$

b) $\underline{a_8 = 7}$

c) $a_8 = 1+(-1)^8$

$\underline{= 2}$

d) $a_8 = -(2)^8$

$\underline{= -256}$

Exercise

What are the terms $a_0, a_1, a_2,$ *and* a_3 of the sequence $\{a_n\}$, if a_n equals

$a)\ 2^n+1 \qquad b)\ (n+1)^{n+1} \qquad c)\ \frac{n}{2} \qquad d)\ \frac{n}{2}+\frac{n}{2}$

$e)\ (-2)^n \qquad f)\ 3 \qquad g)\ 7+4^n \qquad h)\ 2^n+(-2)^n$

Solution

a) $a_0 = 2^0 + 1 = 2$

$a_1 = 2^1 + 1 = 3$

$a_2 = 2^2 + 1 = 5$

$a_3 = 2^3 + 1 = 9$

b) $a_0 = (0+1)^{0+1} = 1$

$a_1 = (1+1)^{1+1} = 4$

$a_2 = (2+1)^{2+1} = 27$

$a_3 = (3+1)^{3+1} = 256$

c) $a_0 = \frac{0}{2} = 0$

$a_1 = \frac{1}{2}$

$a_2 = \frac{2}{2} = 1$

$a_3 = \frac{3}{2}$

d) $a_0 = \frac{0}{2}+\frac{0}{2} = 0$

$a_1 = \frac{1}{2}+\frac{1}{2} = 1$

$a_2 = \frac{2}{2}+\frac{2}{2} = 1$

$a_3 = \frac{3}{2}+\frac{3}{2} = 3$

e) $a_0 = (-2)^0 = 1$

$a_1 = (-2)^1 = -2$

$a_2 = (-2)^2 = 4$

$a_3 = (-2)^3 = -8$

f) $\underline{a_0 = 3}|$

$\underline{a_1 = 3}|$

$\underline{a_2 = 3}|$

$\underline{a_3 = 3}|$

g) $a_0 = 7 + 4^0 \ \underline{= 8}|$

$a_1 = 7 + 4^1 \ \underline{= 11}|$

$a_2 = 7 + 4^2 \ \underline{= 23}|$

$a_3 = 7 + 4^3 \ \underline{= 71}|$

h) $a_0 = 2^0 + (-2)^0 \ \underline{= 2}|$

$a_1 = 2^1 + (-2)^1 \ \underline{= 0}|$

$a_2 = 2^2 + (-2)^2 \ \underline{= 0}|$

$a_3 = 2^3 + (-2)^3 \ \underline{= 0}|$

Exercise

Find at least three different sequences beginning with the terms 1, 2, 4 whose terms are generated by a simple formula or rule.

Solution

1. $2^{n-1} \quad \rightarrow \quad 1,\ 2,\ 4,\ 8,\ 16,\ \ldots$
2. The second pattern, $2-1=1 \quad 4-2=2$, as we see the difference to the previous increasing by value of 1.
   So, the next term $4+3=7 \quad 7+4=11$.
   Therefore; the sequence is 1, 2, 4, 7, 11, 16, …
3. 1, 2, 4, 1, 2, 4, … Repeating the terms

Exercise

Find at least three different sequences beginning with the terms 3, 5, 7 whose terms are generated by a simple formula or rule.

Solution

One rule should be that each term is greater than the previous term by 2; the sequence would be 3, 5, 7, 9, 11, 13, . . .

Another rule could be that the n^{th} old prime.

The sequence would be 3, 5, 7, 11, 13, 17, …

The sequence: 3, 5, 7, 12, 23, 43, 75, 122, 187, 273 from an equation $\frac{1}{2}\left(x^3 - 6x^2 + 15x - 4\right)$

Exercise

Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = 6a_{n-1}, \quad a_0 = 2$

b) $a_n = a_{n-1}^2, \quad a_1 = 2$

c) $a_n = a_{n-1} + 3a_{n-2}, \quad a_0 = 1, \ a_1 = 2$

d) $a_n = na_{n-1} + n^2 a_{n-2}, \quad a_0 = 1, \ a_1 = 1$

e) $a_n = a_{n-1} + a_{n-3}, \quad a_0 = 1, \ a_1 = 2, \ a_2 = 0$

Solution

a) $a_n = 6a_{n-1}, \quad a_0 = 2$

$a_1 = 6a_0 = 6(2) \ \underline{= 12}$

$a_2 = 6a_1 = 6(12) \ \underline{= 72}$

$a_3 = 6a_2 = 6(72) \ \underline{= 432}$

$a_4 = 6a_3 = 6(432) \ \underline{= 2592}$

b) $a_n = a_{n-1}^2, \quad a_1 = 2$

$a_2 = a_1^2 = 2^2 \ \underline{= 4}$

$a_3 = a_2^2 = 4^2 \ \underline{= 16}$

$a_4 = a_3^2 = 16^2 \ \underline{= 256}$

$a_5 = a_4^2 = 256^2 \ \underline{= 65536}$

c) $a_n = a_{n-1} + 3a_{n-2}, \quad a_0 = 1, \ a_1 = 2$

$a_2 = a_1 + 3a_0 = 2 + 3(1) \ \underline{= 5}$

$a_3 = a_2 + 3a_1 = 5 + 3(2) \ \underline{= 11}$

$a_4 = a_3 + 3a_2 = 11 + 3(5) \ \underline{= 26}$

$a_5 = a_4 + 3a_3 = 26 + 3(11) \ \underline{= 59}$

d) $a_n = na_{n-1} + n^2 a_{n-2}, \quad a_0 = 1,\ a_1 = 1$

$a_2 = 2a_1 + 2^2 a_0 = 2(1) + 4(1) \underline{= 6}$

$a_3 = 3a_2 + 3^2 a_1 = 3(6) + 9(1) \underline{= 27}$

$a_4 = 4a_3 + 4^2 a_2 = 4(27) + 16(6) \underline{= 204}$

$a_5 = 5a_4 + 5^2 a_3 = 5(204) + 25(27) \underline{= 1695}$

e) $a_n = a_{n-1} + a_{n-3}, \quad a_0 = 1,\ a_1 = 2,\ a_2 = 0$

$a_3 = a_2 + a_0 = 0 + 1 \underline{= 1}$

$a_4 = a_3 + a_1 = 1 + 2 \underline{= 3}$

$a_5 = a_4 + a_2 = 3 + 0 \underline{= 3}$

$a_6 = a_5 + a_3 = 3 + 3 \underline{= 6}$

Exercise

Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = -2a_{n-1}, \quad a_0 = -1$

b) $a_n = a_{n-1} - a_{n-2}, \quad a_0 = 2,\ a_1 = -1$

c) $a_n = 3a_{n-1}^2, \quad a_0 = 1$

d) $a_n = na_{n-1} + n^2 a_{n-2}, \quad a_0 = -1,\ a_1 = 0$

e) $a_n = a_{n-1} - a_{n-2} + a_{n-3}, \quad a_0 = 1,\ a_1 = 2,\ a_2 = 2$

Solution

a) $a_0 = -1$

$a_1 = -2a_0 \underline{= 2}$

$a_2 = -2a_1 \underline{= -4}$

$a_3 = -2a_2 \underline{= 8}$

$a_4 = -2a_3 \underline{= -16}$

$a_5 = -2a_4 \underline{= 32}$

b) $a_0 = 2, \quad a_1 = -1$

$a_2 = a_1 - a_0 \underline{= -3}$

$a_3 = a_2 - a_1 \underline{= -2}$

$a_4 = a_3 - a_2 \ \underline{= 1}$

$a_5 = a_4 - a_3 \ \underline{= 3}$

c) $a_0 = 1$

$a_1 = 3a_0^2 \ \underline{= 3}$

$a_2 = 3a_1^2 = 27 \ \underline{= 3^3}$

$a_3 = 3a_2^2 = 2187 \ \underline{= 3^7}$

$a_4 = 3a_3^2 = 14348907 \ \underline{= 3^{15}}$

$a_5 = 3a_4^2 \ \underline{= 3^{31}}$

d) $a_0 = -1, \quad a_1 = 0$

$a_2 = 2a_1 + a_0^2 \ \underline{= 1}$

$a_3 = 3a_2 + a_1^2 \ \underline{= 3}$

$a_4 = 4a_3 + a_2^2 \ \underline{= 13}$

$a_5 = 5a_4 + a_3^2 \ \underline{= 74}$

e) $a_0 = 1, \quad a_1 = 1, \quad a_2 = 2$

$a_3 = a_2 - a_1 + a_0 \ \underline{= 2}$

$a_4 = a_3 - a_2 + a_1 \ \underline{= 1}$

$a_5 = a_4 - a_3 + a_2 \ \underline{= 1}$

Exercise

Let $a_n = 2^n + 5 \cdot 3^n \quad for \ \ n = 0, 1, 2, \ldots$

a) Find $a_0, a_1, a_2, a_3, and\ a_4$

b) Show that $a_2 = 5a_1 - 6a_0$, $a_3 = 5a_2 - 6a_1$, and $a_4 = 5a_3 - 6a_2$

c) Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers n with $n \ge 2$

Solution

a) $a_0 = 2^0 + 5 \cdot 3^0 = 1 + 5 \ \underline{= 6}$

$a_1 = 2^1 + 5 \cdot 3^1 = 2 + 15 \ \underline{= 17}$

$$a_2 = 2^2 + 5\cdot 3^2 = 4 + 45 \ \underline{= 49}$$

$$a_3 = 2^3 + 5\cdot 3^3 = 8 + 5(27) \ \underline{= 143}$$

$$a_4 = 2^4 + 5\cdot 3^4 = 16 + 5(81) \ \underline{= 421}$$

b)

$$a_2 = 5a_1 - 6a_0$$

$$49 \overset{?}{=} 5(17) - 6(6)$$

$$49 = 49$$

Or

$$\begin{aligned} 5a_1 - 6a_0 &= 5\left(2^1 + 5\cdot 3^1\right) - 6\left(2^0 + 5\cdot 3^0\right) \\ &= 5\cdot 2 + 5\cdot 3 - 2\cdot 3 - 2\cdot 3\cdot 5 \\ &= (5\cdot 2 - 2\cdot 3) + 5\cdot 5\cdot 3 - 5\cdot 2\cdot 3 \\ &= 2(5-3) + 5\cdot 3(5-2) \\ &= 2\cdot 2 + 5\cdot 3\cdot 3 \\ &= 2^2 + 5\cdot 3^2 \end{aligned}$$

$$a_3 = 5a_2 - 6a_1$$

$$143 \overset{?}{=} 5(49) - 6(17)$$

$$143 = 143$$

Or

$$\begin{aligned} 5a_2 - 6a_1 &= 5\left(2^2 + 5\cdot 3^2\right) - 6\left(2^1 + 5\cdot 3^1\right) \\ &= 5\cdot 4 + 5\cdot 9 - 2\cdot 3\cdot 2 - 2\cdot 3\cdot 5\cdot 3 \\ &= (5\cdot 2 - 4\cdot 3) + 5\cdot 9 - 5\cdot 2\cdot 9 \\ &= 2(5-3) + 5\cdot 3(5-2) \\ &= 2\cdot 2 + 5\cdot 3\cdot 3 \\ &= 2^2 + 5\cdot 3^2 \end{aligned}$$

$$a_4 = 5a_3 - 6a_2$$

$$421 \overset{?}{=} 5(143) - 6(49)$$

$$421 = 421$$

Or

$$5a_3 - 6a_2 = 5\left(2^3 + 5\cdot 3^3\right) - 6\left(2^2 + 5\cdot 3^2\right)$$
$$= 5\cdot 2^3 + 5^2\cdot 3^3 - 3\cdot 2^3 - 2\cdot 3^3\cdot 5$$
$$= 5\cdot 2^3 - 3\cdot 2^3 + 5^2\cdot 3^3 - 2\cdot 3^3\cdot 5$$
$$= 2^3(5-3) + 5\cdot 3^3(5-2)$$
$$= 2^4 + 5\cdot 3^4 \quad \checkmark$$

Exercise

Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ if

a) $a_n = 0$?

b) $a_n = 1$?

c) $a_n = 2^n$?

d) $a_n = 4^n$?

e) $a_n = n4^n$?

f) $a_n = 2\cdot 4^n + 3n4^n$?

g) $a_n = (-4)^n$?

h) $a_n = n^2 4^n$?

Solution

a) Let $a_n = 8a_{n-1} - 16a_{n-2} = 0$ We get $0 = 0$ which is a true statement.

$\therefore\ a_n = 0$ is a solution of the recurrence relation.

b) Let $a_n = 8a_{n-1} - 16a_{n-2} = 1$

We get $1 = 8\cdot 1 - 16\cdot 1 = -8$ which is a false statement.

$\therefore\ a_n = 1$ is not a solution.

c) Let $a_n = 8a_{n-1} - 16a_{n-2} = 2^n$

We get $2^n = 8\cdot 2^{n-1} - 16\cdot 2^{n-2} = 2^{n-2}(8\cdot 2 - 16) = 0$

which is a false statement.

$\therefore\ a_n = 1$ is not a solution.

d) Let $a_n = 8a_{n-1} - 16a_{n-2} = 4^n$ We get

$$4^n = 8\cdot 4^{n-1} - 16\cdot 4^{n-2}$$

$$= 4^{n-2}(8\cdot 4 - 16)$$

$$= 4^{n-2}\cdot(16)$$

$$= 4^{n-2}\cdot 4^2$$

$= 4^n$ which is a true statement

$\therefore\ a_n = 4^n$ is a solution of the recurrence relation.

e) Let $a_n = 8a_{n-1} - 16a_{n-2} = n4^n$ We get

$$n4^n = 8\cdot n4^{n-1} - 16\cdot n4^{n-2}$$

$$= n4^{n-2}(8\cdot 4 - 16)$$

$$= n4^{n-2}\cdot\left(4^2\right)$$

$= n4^n$ which is a true statement

$\therefore\ a_n = n4^n$ is a solution of the recurrence relation.

f) Let $a_n = 8a_{n-1} - 16a_{n-2} = 2\cdot 4^n + 3n4^n$ We get

$$2\cdot 4^n + 3n4^n = 8\cdot\left(2\cdot 4^{n-1} + 3(n-1)4^{n-1}\right) - 16\cdot\left(2\cdot 4^{n-2} + 3(n-2)4^{n-2}\right)$$

$$= 8\cdot 4^{n-2}\left(2\cdot 4 + 3\cdot 4(n-1) - 2\cdot\left(2 + 3(n-2)\right)\right)$$

$$= 8\cdot 4^{n-2}(8 + 12n - 12 - 4 - 6n + 12)$$

$$= 8\cdot 4^{n-2}(4 + 6n)$$

$$= 4^2 4^{n-2}(2 + 3n)$$

$= 2\cdot 4^n + 3n\cdot 4^n$ which is a true statement

$\therefore\ a_n = 2\cdot 4^n + 3n\cdot 4^n$ is a solution of the recurrence relation.

g) Let $a_n = 8a_{n-1} - 16a_{n-2} = (-4)^n$ We get

$$(-4)^n = 8\cdot(-4)^{n-1} - 16\cdot(-4)^{n-2}$$

$$= (-4)^{n-2}(8\cdot(-4) - 16)$$

$$= (-4)^{n-2}(-48)$$

$$= (-4)^{n-2}(-16\cdot 3)$$

$$= (-4)^{n-2}(-4)^2\cdot 3$$

$= (-4)^n\cdot 3$ which is a false statement

$\therefore\ a_n = 4^n$ is not a solution.

h) Let $a_n = 8a_{n-1} - 16a_{n-2} = n^2 4^n$ We get

$$n^2 4^n = 8\cdot(n-1)^2 4^{n-1} - 16\cdot(n-2)^2 4^{n-2}$$

$$= 8\cdot 4^{n-2}\left(\left(n^2-2n+1\right)\cdot 4 - 2\cdot\left(n^2-4n+4\right)\right)$$

$$= 16\cdot 4^{n-2}\left(2n^2-4n+2-n^2+4n-4\right)$$

$$= 4^n\left(n^2-2\right)$$

$= n4^n$ which is a false statement

$\therefore\ a_n = n^2 4^n$ is a solution of the recurrence relation.

Exercise

Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if

a) $a_n = -n + 2$

b) $a_n = 5(-1)^n - n + 2$

c) $a_n = 3(-1)^n + 2^n - n + 2$

d) $a_n = 7\cdot 2^n - n + 2$

Solution

a) $a_{n-1} + 2a_{n-2} + 2n - 9 = -(n-1) + 2 + 2\left[-(n-2)+2\right] + 2n - 9$

$$= -n + 1 + 2 - 2n + 4 + 4 + 2n - 9$$

$$= -n + 2$$

$$= a_n$$

b) $a_{n-1} + 2a_{n-2} + 2n - 9 = 5(-1)^{n-1} - (n-1) + 2 + 2\left[5(-1)^{n-2} - (n-2) + 2\right] + 2n - 9$

$$= 5(-1)^{n-1} - n + 3 + 2\left[5(-1)^{n-2} - n + 4\right] + 2n - 9$$

$$= 5(-1)^{n-1} - n + 3 + 10(-1)^{n-2} - 2n + 8 + 2n - 9$$

$$= 5(-1)^{n-1} + 10(-1)^{n-1}(-1)^{-1} - n + 2$$

$$= 5(-1)^{n-1} - 10(-1)^{n-1} - n + 2$$

$$= -5(-1)^{n-1} - n + 2$$

$$= (-1)^1 5(-1)^{n-1} - n + 2$$

$$= 5(-1)^n - n + 2$$
$$= a_n$$

c) $a_{n-1} + 2a_{n-2} + 2n - 9 = 3(-1)^{n-1} + 2^{n-1} - (n-1) + 2$
$$+2\left[3(-1)^{n-2} + 2^{n-2} - (n-2) + 2\right] + 2n - 9$$
$$= 3(-1)^{n-1} + 2^{n-1} + 6(-1)^{n-2} + 2^{n-1} - 2n + 8 + n - 6$$
$$= 3(-1)^{n-1} - 6(-1)^{n-1} + 2^n - n + 2$$
$$= -3(-1)^{n-1} + 2^n - n + 2$$
$$= 3(-1)^n + 2^n - n + 2$$
$$= a_n$$

d) $a_{n-1} + 2a_{n-2} + 2n - 9 = 7\cdot 2^{n-1} - (n-1) + 2 + 2\left[7\cdot 2^{n-2} - (n-2) + 2\right] + 2n - 9$
$$= 7\cdot 2^{n-1} + 7\cdot 2^{n-1} - 2n + 8 + n - 6$$
$$= 2\cdot 7\cdot 2^{n-1} - n + 2$$
$$= 7\cdot 2^n - n + 2$$
$$= a_n$$

Exercise

A person deposits \$1,000.00 in an account that yields 9% interest compounded annually.

a) Set up a recurrence relation for the amount in the account at the end of *n* years.

b) Find an explicit formula for the amount in the account at the end of *n* years.

c) How much money will the account contain after 100 years?

Solution

a) The amount after $n-1$ years multiplied by 1.09 to give the amount after n years, since 9% of the value must be added to account for the interest. Therefore, we have $a_n = 1.09a_{n-1}$. The initial condition is $a_0 = 1000$.

b) Since multiplying by 1.09 for each year, the solution is $a_n = 1000(1.09)^n$.

c) $a_{100} = 1000(1.09)^{100}$
$$\approx \$5,529,041$$

Exercise

Suppose that the number of bacteria in a colony triples every hour.

a) Set up a recurrence relation for the number of bacteria after n hours have elapsed.

b) If 100 bacteria are used to begin new colony, how many bacteria will be in the colony in 10 hours?

Solution

a) Since the number of bacteria triples every hour, the recurrence relation should say that the number of bacteria after n hours is 3 times the number of bacteria after $n - 1$ hours.

Let a_n denote the number of bacteria after n hours, this statement translates into the recurrence relation $a_n = 3a_{n-1}$

b) The initial condition is $a_0 = 100$.

$$
\begin{aligned}
a_n &= 3 \cdot a_{n-1} \\
&= 3^2 \cdot a_{n-2} \\
&\ \vdots \quad \vdots \\
&= 3^n \cdot a_0
\end{aligned}
$$

$n = 10$

$$
\begin{aligned}
a_{10} &= 100 \cdot 3^{10} \\
&= \underline{5,904,900}
\end{aligned}
$$

Exercise

A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with n cars made in the nth month.

a) Set up a recurrence relation for the number of cars produced in the first n months by this factory.

b) How many cars are produced in the first year?

c) Find an explicit formula for the number of cars produced in the first n months by this factory

Solution

a) Let c_n be the number of cars produced in the first n months.

The initial condition is $c_0 = 0$.

Since n cars are made in the nth month then $c_n = c_{n-1} + n$, where c_{n-1} is the first n–1 months

b) The number of cars produced in the first year is c_{12}.

Plug in $n = 12$, we get

$$
\begin{aligned}
c_n &= n + c_{n-1} \\
&= n + (n-1) + c_{n-2}
\end{aligned}
$$

$$= n + (n-1) + (n-2) + c_{n-3}$$

$$\vdots \quad \vdots$$

$$= n + (n-1) + (n-2) + \cdots + 1 + c_0 \qquad\qquad n + (n-1) + (n-2) + \cdots + 1 = \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2} + 0$$

$$= \boxed{\frac{n^2+n}{2}}$$

$$c_{12} = \frac{12^2+12}{2}$$

$$= \boxed{78}$$

c) $\boxed{c_n = \frac{n^2+n}{2}}$

Exercise

For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

a) 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, …

b) 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, …

c) 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, …

d) 3, 6, 12, 24, 48, 96, 192, …

e) 15, 8, 1, –6, –13, –20, –27, …

f) 3, 5, 8, 12, 17, 23, 30, 38, 47, …

g) 2, 16, 54, 128, 250, 432, 686, …

h) 2, 3, 7, 25, 121, 721, 5041, 40321, …

i) 3, 6, 11, 18, 27, 38, 51, 66, 83, 102, …

j) 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, …

k) 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, …

Solution

a) We have one 1 and one 0, then two 1 and two 0, then three of each, and so on increasing the repetition by one each time. Since we have only one at the end, then we need three 1 and four 0 to continue the sequence.

b) A pattern is that the positive integers are increasing order, with odd number showing once and each even number repeated.
Thus, the next terms are 9, 10, 10.

c) The terms in the odd locations are the successive terms in the geometric sequence that starts with 1 and has ratio 2, and the terms in the even locations are all 0. The *n*th term is 0 if *n* even and is $2^{(n-1)/2}$ if *n* is odd.

Thus, the next three terms are 32, 0, 64.

d) The first term is 3 and each successive term is twice the predecessor. The *n*th term is $3\cdot 2^{n-1} \quad n>0$.

Thus, the next three terms are 384, 768, 1536.

e) The first term is 15 and each successive term is 7 less than its predecessor. The *n*th term is $15-7(n-1)=22-7n$.

Thus, the next three terms are –34, –41, –48.

f) The first term is 3 and each successive term by adding *n* to its predecessor.

$$3,\ 3+2,\ 5+3,\ 8+4,\ 12+5 \quad nth \quad 3+2+3+4+5+\cdots+n=2+1+2+3+4+5+\cdots+n$$
$$=2+\frac{n^2+n}{2}$$

The *n*th term is $\frac{1}{2}\left(n^2+n+4\right)$.

Thus, the next three terms are 57, 68, 80.

g) Since all numbers are even, then if we divide by 2 the sequence becomes: 1, 8, 27, 64, 125, 216, 343, …. This sequence appears to be n^3, therefore the *n*th term is $2n^3$.

Thus, the next three terms are 1024, 1458, 2000.

h) The *n*th term appears to be $n!\ +1$.

Thus, the next three terms are 362881, 3628801, 39916801.

i) The first term is 3 then by adding 3 to the predecessor, then 5, then 7, and so on.

$$3,\ 3+3,\ 6+5,\ 11+7,\ \ldots \to\ 1+2,\ 4+2,\ 9+2,\ 16+2,\ \ldots$$

Then the *n*th term is n^2+2.

Thus, the next three terms are 123, 146, 171.

j) This an arithmetic sequence whose difference is 4. Thus, the *n*th term is $7+4(n-1)=4n+3$.

Thus, the next three terms are 47, 51, 55.

k) This is a binary expansion of *n*. Thus, the next three terms are 1100, 1101, 1110.

Solution *Section* 2.2 – Algorithms

Exercise

List all the steps used by the Algorithm 1 to find the maximum of the list

1, 8, 12, 9, 11, 2, 14, 5, 10, 4.

Solution

The ***for*** loop then begins, with i set equal from 2 to $n = 10$ (number of the sequence).

The statement of the loop is executed since 2 < 10. This is an ***if*** … ***then*** statement.

max**:=** 1

for $i := 2$ to 10

if $\boldsymbol{max} < a_i$ **then** max $:= a_i$

$a_i = a_2 = 8$, since 1 < 8, then $\boldsymbol{max} := 8$

$a_i = a_3 = 12$, since 8 < 12, then $\boldsymbol{max} := 12$

$a_i = a_4 = 9$, since 12 < 9 ***is not true***, then $\boldsymbol{max} := 12$

$a_i = a_5 = 11$, since 12 < 11 ***is not true***, then $\boldsymbol{max} := 12$

$a_i = a_6 = 2$, since 12 < 2 ***is not true***, then $\boldsymbol{max} := 12$

$a_i = a_7 = 14$, since 12 < 14, then $\boldsymbol{max} := 14$

$a_i = a_8 = 5$, since 14 < 5 ***is not true***, then $\boldsymbol{max} := 14$

$a_i = a_9 = 10$, since 14 < 10 ***is not true***, then $\boldsymbol{max} := 14$

$a_i = a_{10} = 4$, since 14 < 4 ***is not true***, then $\boldsymbol{max} := 14$

Therefore ***max*** has the value $\boxed{14}$

Exercise

Devise an algorithm that finds the sum of all the integers in a list.

Solution

Procedure *sum* $\{a_1, a_2, \ldots, a_n : integers\}$

sum:= a_1

for $i := 2$ to n

$\quad sum := sum + a_i$

return *sum*{ is the sum of all the elements in the list}

Exercise

Describe an algorithm that takes as an input a list of n integers and produces as output the largest difference obtained by subtracting an integer in the list from the one following it.

Solution

For i going from 1 through $n - 1$, compute the value of the $(i+1)^{st}$ element in the list minus the i^{st} element in the list. If this is larger than the answer, reset the answer to be this value.

Exercise

Describe an algorithm that takes as an input a list of n integers in non-decreasing order and produces the list of all values that occur more than once.

Solution

Procedure *negatives* $\left\{a_1,\ a_2,\ \ldots,\ a_n : integers\right\}$

$k := 0$

for $i := 1$ to n

  if $a_i < 0$ ***then*** $k := k+1$

return k { the number of negative integers in the list}

Exercise

Describe an algorithm that takes as an input a list of n integers and finds the location of the last even integer in the list or returns 0 if there are no even integers in the list.

Solution

Procedure *last even loction* $\left\{a_1,\ a_2,\ \ldots,\ a_n : integers\right\}$

$k := 0$

for $i := 1$ to n

  if a_i *is even* ***then*** $k := i$

return k { *is the desired location* (*or* 0 *if there are no evens*)}

Exercise

Describe an algorithm that interchanges the values of the variables x and y, using only assignments. What is the minimum number of assignment statements needed to do this?

Solution

We cannot simply write $x := y$ followed by $y := x$.

$temp := x$
$x := y$
$y := temp$

Exercise

List all the steps used to search for 9 in the sequence 1, 3, 4, 5, 6, 7, 9, 11 using

a) a linear search *b*) a binary search

Solution

a) Note that $n = 8$ and $x = 9$.

procedure linear_search (x: integer; $a_1, a_2, \ldots, a_n$: integers)

$i := 1$

while ($i \le 8$ and $\left(i \le 8 \;\; and \;\; 9 \ne a_i\right)$

$i := i + 1$

The ***while*** loop is executed as long as $i \le 8$ and the i^{st} element is not equal to 9.

$i = 1, \quad a_1 = 1; \quad 9 \ne 1$

$i = 2, \quad a_2 = 3; \quad 9 \ne 3$

$i = 3, \quad a_3 = 4; \quad 9 \ne 4$

$i = 4, \quad a_4 = 5; \quad 9 \ne 5$

$i = 5, \quad a_5 = 6; \quad 9 \ne 6$

$i = 6, \quad a_6 = 7; \quad 9 \ne 7$

$i = 7, \quad a_7 = 7; \quad 9 = 9$

Therefore the body of the loop is not executed (so i is still equal to 7), and control passes beyond the loop.

if $i \le n$ ***then*** $location := i$

else $location := 0$

The else clause is not executed. This completes the procedure, so location has the correct value, namely 7, which indicates the location of the element x in the list: 9 is the seventh element.

b) ***procedure*** linear_search (x: integer; $a_1, a_2, \ldots, a_n$: increasing integers)

$i := 1$

$j := 8$

while $i < j$

The while step is executed, first $m = \dfrac{1+8}{2} = 4$

Then since x (= 9) is greater than $a_4\left(=5\right)$, the statement $i := m+1$ is executed, so i has the value 5.

$i = 4 + 1 = 5, \quad m = \frac{5+8}{2} = 6 \quad x(=9) > a_6\ (=6)$

$i = 6 + 1 = 7, \quad m = \frac{7+8}{2} = 7 \quad x(=9) > a_7\ (=9)$ fails thus $j := m$, so $j := 7$

At this point $i \not< j$, the condition $x = a_i$ is true, location is set to 7, as it should be, and the algorithm is finished.

Exercise

Describe an algorithm that inserts an integer *x* in the appropriate position into the list $a_1,\ a_2,\ \ldots,\ a_n$ of integers that are in increasing order.

Solution

procedure insert $\left(x, a_1,\ a_2,\ \ldots,\ a_n : \ integers\right)$

$a_{n+1} := x + 1$

$i := 1$

while $x > a_i$

  $i := i + 1$ {***The loop ends when i is the index for x***}

for $j := 0$ ***to*** $n - i$ {***Shove the rest of the list to the right***}

  $a_{n-j+1} := a_{n-j}$

$a_i := x$

{*x* has been inserted into the correct spot in the list, now of length *n* + 1}

Solution *Section* 2.3 – Divisibility and Modular Arithmetic

Exercise

Does 17 divide each of these numbers?

a) 68 ***b***) 84 ***c***) 35 ***d***) 1001

Solution

a) $68 = 17 \cdot 4$ *Yes*

b) $84 = 17 \cdot 4 + 16$ *No*, remainder 16

c) $357 = 17 \cdot 21$ *Yes*

d) $1001 = 17 \cdot 58 + 15$ *No*, remainder 15

Exercise

Prove that if a is an integer other than 0, then

a) 1 *divides* a ***b***) a *divides* 0

Solution

a) $1|a \quad \sin ce \quad a = 1 \cdot a$

b) $a|0 \quad \sin ce \quad 0 = a \cdot 0$

Exercise

Show that if $a \mid b$ and $b \mid a$, where a and b are integers, then $a = b$ or $a = -b$.

Solution

Let s and t are integers such that $a = bs$ and $b = at$.

$a = bs = ats$. Since $a \neq 0$, we conclude that $st = 1$.

The only way for this to happen, since s and t are integers, is for $s = t = 1 \quad or \quad s = t = -1$.

Therefore, either $a = b$ or $a = -b$.

Exercise

Show that if a, b, and c are integers, where $a \neq 0$ and $c \neq 0$, such that $ac \mid bc$, then $a \mid b$

Solution

Since $ac \mid bc \ \Rightarrow bc = (ac)t$ for some integers t

Since $c \neq 0$, divide both sides by c to obtain $b = at$ and this result to $a \mid b$ $\surd$

Exercise

What are the quotient and remainder when

a) 19 is divided by 7?

b) -111 is divided by 11?

c) 789 is divided by 23?

d) 1001 is divided by 13?

e) 0 is divided by 19?

f) 3 is divided by 5?

g) -1 is divided by 3?

h) 4 is divided by 1?

Solution

a) $19 = 7 \cdot 2 + 5 \qquad q = 2 \quad and \quad r = 5$

b) $-111 = 11 \cdot (-11) + 10 \qquad q = -11 \quad and \quad r = 10$

c) $789 = 23 \cdot 34 + 7 \qquad q = 34 \quad and \quad r = 7$

d) $1001 = 13 \cdot 77 + 0 \qquad q = 77 \quad and \quad r = 0$

e) $0 = 19 \cdot 0 + 0 \qquad q = 0 \quad and \quad r = 0$

f) $3 = 5 \cdot 0 + 3 \qquad q = 0 \quad and \quad r = 3$

g) $-1 = 3 \cdot (-1) + 2 \qquad q = -1 \quad and \quad r = 2$

h) $4 = 1 \cdot 4 + 0 \qquad q = 4 \quad and \quad r = 0$

Exercise

What time does a 12-hour clock read

a) 80 hours after it reads 11:00?

b) 40 hours before it reads 12:00?

c) 100 hours after it reads 6:00?

Solution

a) $11 - 80 \ \boldsymbol{mod}\ 12 = 11 - 8 = 7$, the clock reads 7:00.

b) $12 - 40 \ \boldsymbol{mod}\ 12 = -28 \ \boldsymbol{mod}\ 12$ $\qquad (12 - 40 = -28)$

$$= -28 + 36 \ \boldsymbol{mod}\ 12$$

$$= 8$$

The clock reads 8:00.

c) $6 + 100 \ \boldsymbol{mod}\ 12 = 6 + 4 = 10$, the clock reads 10:00.

Exercise

What time does a 24-hour clock read

a) 100 hours after it reads 2:00?

b) 45 hours before it reads 12:00?

c) 168 hours after it reads 19:00?

Solution

a) $2+100\ \boldsymbol{mod}\ 24 = 2+4 = 6$, the clock reads 6:00

b) $12-45\ \boldsymbol{mod}\ 24 = -33\ \boldsymbol{mod}\ 24 = -33+48\ \boldsymbol{mod}\ 24 = 15$, the clock reads 15:00

c) $168\ \boldsymbol{mod}\ 24 = 0$, the clock reads 19:00

Exercise

Suppose a and b are integers, $a \equiv 4 \pmod{13}$, and $b \equiv 9 \pmod{13}$. Find the integer c with $0 \le c \le 12$ such that

a) $c \equiv 9a \pmod{13}$

b) $c \equiv 11b \pmod{13}$

c) $c \equiv a+b \pmod{13}$

d) $c \equiv 2a+3b \pmod{13}$

e) $c \equiv a^2+b^2 \pmod{13}$

f) $c \equiv a^3-b^3 \pmod{13}$

Solution

a) $c = 9 \cdot 4\ \boldsymbol{mod}\ 13 = 36\ \boldsymbol{mod}\ 13 = 10$

b) $c = 11 \cdot 9\ \boldsymbol{mod}\ 13 = 99\ \boldsymbol{mod}\ 13 = 8$

c) $c = 4+9\ \boldsymbol{mod}\ 13 = 13\ \boldsymbol{mod}\ 13 = 0$

d) $c = 2(4)+3(9)\ \boldsymbol{mod}\ 13 = 35\ \boldsymbol{mod}\ 13 = 9$

e) $c = 4^2+9^2\ \boldsymbol{mod}\ 13 = 97\ \boldsymbol{mod}\ 13 = 6$

f) $c = 4^3-9^3\ \boldsymbol{mod}\ 13 = -665\ \boldsymbol{mod}\ 13 = 11$ $\qquad (-665 = -52 \times 13 + 11)$

Exercise

Suppose a and b are integers, $a \equiv 11 (\boldsymbol{mod}\ 19)$, and $b \equiv 3 (\boldsymbol{mod}\ 19)$. Find the integer c with $0 \le c \le 10$ such that

a) $c \equiv a - b (\boldsymbol{mod}\ 19)$

b) $c \equiv 7a + 3b (\boldsymbol{mod}\ 19)$

c) $c \equiv 2a^2 + 3b^2 (\boldsymbol{mod}\ 19)$

d) $c \equiv a^3 + 4b^3 (\boldsymbol{mod}\ 19)$

Solution

a) $c = 11 - 3\ \boldsymbol{mod}\ 19 = \underline{8}|$

b) $c = 7(11) + 3(3)\ \boldsymbol{mod}\ 19 = 86\ \boldsymbol{mod}\ 19 = \underline{10}|$ $\qquad 7(11) + 3(3) = 86 \equiv 10 (\boldsymbol{mod}\ 19)$

c) $2(11)^2 + 3(3)^2 = 263 \equiv 3\ (\boldsymbol{mod}\ 19)$

d) $(11)^3 + (3)^3 = 1439 \equiv 14\ (\boldsymbol{mod}\ 19)$

Exercise

Let m be a positive integer. Show that $a\ \boldsymbol{mod}\ m \equiv b\ \boldsymbol{mod}\ m$ if $a \equiv b\ \boldsymbol{mod}\ m$

Solution

Given $a\ \boldsymbol{mod}\ m \equiv b\ \boldsymbol{mod}\ m$ means that a and b have the same remainder $a = q_1 m + r$ and $b = q_2 m + r$ for some integer q_1, q_2 and r.

$$a - b = q_1 m + r - q_2 m - r$$
$$= (q_1 - q_2)m$$

Which says that m divides (is a factor). This precisely the definition of $a \equiv b\ \boldsymbol{mod}\ m$

Exercise

Let m be a positive integer. Show that $a \equiv b(\text{mod } m)$ if a **mod** $m = b$ **mod** m

Solution

Assume that $a \equiv b\ (\boldsymbol{mod}\ m)$. This means that $m | a - b$, $a - b = mc \Rightarrow a = b + mc$.

Computing $a\ \boldsymbol{mod}\ m$, we know that $b = qm + r$ for some nonnegative r less than m (namely, $r \equiv b\ (\boldsymbol{mod}\ m)$). Therefore $a = qm + r + mc = (q + c)m + r$. By definition this means that r must also equal $a\ \boldsymbol{mod}\ m$ $\checkmark$

Exercise

Show that if n and k are positive integers, then $\lceil n/k \rceil = \left\lfloor \dfrac{n-1}{k} \right\rfloor + 1$

Solution

The quotient $\frac{n}{k}$ lies between 2 consecutive integers, let say $b-1$ *and* b possibly equal to b. There exists a positive integer b such that $b-1 < \frac{n}{k} \le b$. In particular $\frac{n}{k} = b$. Also since $\frac{n}{k} > b-1$ we have

$n > k(b-1) \Rightarrow n-1 \ge k(b-1)$

$\left\lfloor \dfrac{n-1}{k} \right\rfloor \le \dfrac{n-1}{k} < \dfrac{n}{k} \le b$ so $\left\lfloor \dfrac{n-1}{k} \right\rfloor < b$, therefore $\left\lfloor \dfrac{n-1}{k} \right\rfloor = b-1$

Exercise

Evaluate these quantities

a) -17 ***mod*** 2
b) 144 ***mod*** 7
c) -101 ***mod*** 13
d) 199 ***mod*** 19
e) 13 ***mod*** 3
f) -97 ***mod*** 11

Solution

a) $-17 = 2\cdot(-9)+1$, the remainder is 1. That is, -17 ***mod*** $2 = 1$.
Note that we do not write $-17 = 2\cdot(-8)-1$ so -17 ***mod*** $2 = -1$

b) $144 = 7\cdot 20+4$, the remainder is 4. That is, 144 ***mod*** $7 = 4$

c) $-101 = 13\cdot(-8)+3$, the remainder is 3. That is, -101 ***mod*** $13 = 3$

d) $199 = 19\cdot 10+9$, the remainder is 9. That is, 199 ***mod*** $19 = 9$

e) $13 = 3\cdot 4+1$, the remainder is 1. That is, 13 ***mod*** $3 = 1$

f) $-97 = 11\cdot(-9)+2$, the remainder is 2. That is, -97 ***mod*** $11 = 2$

Exercise

Find a **div** m and a **mod** m when

a) $a = 228,\ m = 119$
b) $a = 9009,\ m = 223$
c) $a = -10101,\ m = 333$
d) $a = -765432,\ m = 38271$

Solution

a) $228 = 2 \cdot 119 + 109$

$228 \ \boldsymbol{div}\ 119 = 1 \quad \textit{and} \quad 228 \ \boldsymbol{mod}\ 119 = 109$.

b) $9009 = 40 \cdot 223 + 89$

$9009 \ \boldsymbol{div}\ 223 = 40 \quad \textit{and} \quad 9009 \ \boldsymbol{mod}\ 223 = 89$.

c) $-10101 = -31 \cdot 333 + 222$

$-10101 \ \boldsymbol{div}\ 333 = -31 \quad \textit{and} \quad -10101 \ \boldsymbol{mod}\ 333 = 222$.

d) $-765432 = -21 \cdot 38271 + 38259 \ \Rightarrow$
$-765432 \ \boldsymbol{div}\ 38271 = -11 \quad \textit{and} \quad -765432 \ \boldsymbol{mod}\ 38271 = 38259$.

Exercise

Find the integer a such that

a) $a \equiv -15 (\boldsymbol{mod}\ 27) \ \textit{and} \ -26 \le a \le 0$
b) $a \equiv 24 (\boldsymbol{mod}\ 31) \ \textit{and} \ -15 \le a \le 15$
c) $a \equiv 99 (\boldsymbol{mod}\ 41) \ \textit{and} \ 100 \le a \le 140$
d) $a \equiv 43 (\boldsymbol{mod}\ 23) \ \textit{and} \ -22 \le a \le 0$
e) $a \equiv 17 (\boldsymbol{mod}\ 29) \ \textit{and} \ -14 \le a \le 14$

Solution

a) -15 already satisfies the inequality, the answer $a = -15$

b) 24 is too large to satisfy the inequality, we subtract 31 and obtain $a = -7$

c) 24 is too small to satisfy the inequality, we add 41 and obtain $a = 140$

d) $a = 43 - 2 \cdot (23) = 43 - 46 = -3$

e) $a = 17 - 29 = -12$

Exercise

Decide whether each of these integers is congruent to 5 modulo 17.

$a)\ 37 \qquad b)\ 66 \qquad c) -17 \qquad d) -67$

Solution

a) $37 - 3 \ \boldsymbol{mod}\ 7 = 34 \ \boldsymbol{mod}\ 7 = 6 \ne 0$, so $37 \not\equiv 3 (\boldsymbol{mod}\ 7)$

b) $66 - 3 \ \boldsymbol{mod}\ 7 = 63 \ \boldsymbol{mod}\ 7 = 0$, so $37 \equiv 3 (\boldsymbol{mod}\ 7)$

c) $-17 - 3 \ \boldsymbol{mod}\ 7 = -20 \ \boldsymbol{mod}\ 7 = 1 \ne 0$, so $-17 \not\equiv 3 (\boldsymbol{mod}\ 7)$

d) $-67 - 3 \ \boldsymbol{mod}\ 7 = -70 \ \boldsymbol{mod}\ 7 = 0$, so $-67 \equiv 3 (\boldsymbol{mod}\ 7)$

Exercise

Find each of these values.

a) $(-133 \textbf{ mod } 23 + 261 \textbf{ mod } 23) \textbf{ mod } 23$

b) $(457 \textbf{ mod } 23 \cdot 182 \textbf{ mod } 23) \textbf{ mod } 23$

c) $(177 \textbf{ mod } 31 + 270 \textbf{ mod } 31) \textbf{ mod } 31$

d) $\left(19^2 \textbf{ mod } 41\right) \textbf{ mod } 9$

e) $\left(32^3 \textbf{ mod } 13\right)^2 \textbf{ mod } 11$

f) $\left(99^2 \textbf{ mod } 32\right)^3 \textbf{ mod } 15$

g) $\left(3^4 \textbf{ mod } 17\right)^2 \textbf{ mod } 11$

h) $\left(19^3 \textbf{ mod } 23\right)^2 \textbf{ mod } 31$

i) $\left(89^3 \textbf{ mod } 79\right)^4 \textbf{ mod } 26$

Solution

a) $-133 + 261 = 128 \equiv 13$

$-133 + 261 \textbf{ mod } 23 = 128 \textbf{ mod } 23 = 13$ $\quad 128 = 23 \cdot (5) + 13$

b) $457 \cdot 182 \textbf{ mod } 23 = 83174 \textbf{ mod } 23 = 6$ $\quad 83174 = 23 \cdot (3616) + 6$

c) $177 + 271 \textbf{ mod } 31 = 448 \textbf{ mod } 31 = 14$ $\quad 448 = 31 \cdot (14) + 14$

d)
$$\begin{aligned}\left(19^2 \textbf{ mod } 41\right) \textbf{ mod } 9 &= (361 \textbf{ mod } 41) \textbf{ mod } 9 \\ &= 33 \textbf{ mod } 9 \\ &= 6\end{aligned}$$

e)
$$\begin{aligned}\left(32^3 \textbf{ mod } 13\right)^2 \textbf{ mod } 11 &= (32768 \textbf{ mod } 13)^2 \textbf{ mod } 11 \\ &= 8^2 \textbf{ mod } 11 \\ &= 64 \textbf{ mod } 11 \\ &= 9\end{aligned}$$

f)
$$\begin{aligned}\left(99^2 \textbf{ mod } 32\right)^3 \textbf{ mod } 15 &= (9801 \textbf{ mod } 32)^3 \textbf{ mod } 15 \\ &= 9^3 \textbf{ mod } 15 \\ &= 729 \textbf{ mod } 15 \\ &= 9\end{aligned}$$

g)
$$\begin{aligned}\left(3^4 \textbf{ mod } 17\right)^2 \textbf{ mod } 11 &= (81 \textbf{ mod } 17)^2 \textbf{ mod } 11 \\ &= 13^2 \textbf{ mod } 11\end{aligned}$$

$= 169\ \boldsymbol{mod}\ 11$

$\underline{= 4}\,|$

h) $\left(19^3\ \boldsymbol{mod}\ 23\right)^2\ \boldsymbol{mod}\ 31 = (6859\ \boldsymbol{mod}\ 23)^2\ \boldsymbol{mod}\ 31$

$= 5^2\ \boldsymbol{mod}\ 31$

$= 25\ \boldsymbol{mod}\ 31$

$\underline{= 25}\,|$

i) $\left(89^3\ \boldsymbol{mod}\ 79\right)^4\ \boldsymbol{mod}\ 26 = (704969\ \boldsymbol{mod}\ 79)^4\ \boldsymbol{mod}\ 26$

$= 52^4\ \boldsymbol{mod}\ 26$

$= 7311616\ \boldsymbol{mod}\ 26$

$\underline{= 0}\,|$

Solution *Section* 2.4 – Integer Representations and Algorithms

Exercise

Convert the decimal expansion of each of these integers to a binary expansion

a) 321 *b)* 1023 *c)* 100632 *d)* 231 *e)* 4532

Solution

a)

321	160	80	40	20	10	5	2	1	
1	0	0	0	0	0	1	0	1	←

$321 = \left(1\ 0100\ 0001\right)_2$

b) $1023 = 1024 - 1 = 2^{10} - 1$ $\quad$ 1 *less than* $\left(100\ \ 0000\ \ 0000\right)_2$

1023	511	255	127	63	31	15	7	3	1	
1	1	1	1	1	1	1	1	1	1	←

$1023 = \left(11\ \ 1111\ \ 1111\right)_2$

c)

100632	50316	25158	12579	636289	3144	1572	786	393	196	98	49	24
	0	0	1	1	0	0	0	1	0	0	1	0

12	6	3	1	
0	0	1	1	←

$100632 = \left(1\ \ 1000\ \ 1001\ \ 0001\ \ 1000\right)_2$

d)

231	115	57	28	14	7	3	1	
1	1	1	0	0	1	1	1	←

$231 = \left(1110\ \ 0111\right)_2$

e)

4532	2266	1133	566	283	141	70	35	17	8	4	2	1	
0	0	1	0	1	1	0	1	1	0	0	0	1	←

$4532 = \left(1\ \ 0001\ \ 1011\ \ 0100\right)_2$

Exercise

Convert binary the expansion of each of these integers to a decimal expansion

a) $(1\,1011)_2$ *b)* $(10\,1011\,0101)_2$ *c)* $(11\,1011\,1110)_2$

d) $(111\,1100\,0001\,1111)_2$ *e)* $(1\,1111)_2$ *f)* $(10\,0000\,0001)_2$

g) $(10\,0101\,0101)_2$ *h)* $(110\,1001\,0001\,0000)_2$

Solution

Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Binary	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111

a) $(1\,1011)_2 = 1 + 2^1 + 2^3 + 2^4$

$= 1 + 2 + 8 + 16$

$\underline{= 27}$

b) $(10\,1011\,0101)_2 = 1 + 2^2 + 2^4 + 2^5 + 2^7 + 2^9$

$= 1 + 4 + 16 + 32 + 128 + 512$

$\underline{= 693}$

c) $(11\,1011\,1110)_2 = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^7 + 2^8 + 2^9$

$\underline{= 958}$

d) $(111\,1100\,0001\,1111)_2 = 1 + 2^1 + 2^2 + 2^3 + 2^4 + 2^{10} + 2^{11} + 2^{12} + 2^{13} + 2^{14}$

$\underline{= 31775}$

e) $(1\,1111)_2 = 1 + 2^1 + 2^2 + 2^3 + 2^4$

$= 1 + 2 + 8 + 16$

$\underline{= 31}$

f) $(10\,0000\,0001)_2 = 1 + 2^9$

$= 1 + 512$

$\underline{= 513}$

g) $(10\,0101\,0101)_2 = 2^9 + 2^6 + 2^4 + 2^2 + 1 \underline{= 597}$

h) $(110\,1001\,0001\,0000)_2 = 2^{14} + 2^{13} + 2^{11} + 2^8 + 2^4 \underline{= 26896}$

Exercise

Convert the binary expansion of each of these integers to an octal expansion *a)* $(1111\ 0111)_2$

b) $(1010\ 1010\ 1010)_2$

c) $(111\ 0111\ 0111\ 0111)_2$

d) $(101\ 0101\ 0101\ 0101)_2$

Solution

a) $(1111\ 0111)_2 = (11\ 110\ 111)_2 \ \underline{= (367)_8}|$

b) $(1010\ 1010\ 1010)_2 = (101\ 010\ 101\ 010)_2 = \underline{(5252)_8}|$

c) $(111\ 0111\ 0111\ 0111)_2 = (111\ 011\ 101\ 110\ 111)_2 = \underline{(73567)_8}|$

d) $(101\ 0101\ 0101\ 0101)_2 = (101\ 010\ 101\ 010\ 101)_2 = \underline{(52525)_8}|$

Exercise

Convert the octal expansion of each of these integers to a binary expansion

a) $(572)_8$ *b)* $(1604)_8$ *c)* $(423)_8$ *d)* $(2417)_8$ *e)* $(73567)_8$ *f)* $(52525)_8$

Solution

Octal	0	1	2	3	4	5	6	7	10	11	12	13	14	15	16	17
Binary	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111

a) $\begin{array}{c:c:c} 5_8 & 7_8 & 2_8 \\ \hline 101_2 & 111_2 & 010_2 \end{array} \quad \Rightarrow (572)_8 = (1\ 0111\ 1010)_2$

b) $\begin{array}{c:c:c:c} 1_8 & 6_8 & 0_8 & 4_8 \\ \hline 1_2 & 110_2 & 000_2 & 100_2 \end{array} \quad \Rightarrow (1604)_8 = (11\ 1000\ 0100)_2$

c) $\begin{array}{c:c:c} 4_8 & 2_8 & 3_8 \\ \hline 100_2 & 010_2 & 011_2 \end{array} \quad \Rightarrow (423)_8 = (1\ 0001\ 0011)_2$

d) $\begin{array}{c:c:c:c:c} 7_8 & 3_8 & 5_8 & 6_8 & 7_8 \\ \hline 111_2 & 011_2 & 101_2 & 110_2 & 111_2 \end{array} \quad \Rightarrow (73567)_8 = (111\ 0111\ 0111\ 0111)_2$

e) $\begin{array}{c:c:c:c:c} 5_8 & 2_8 & 5_8 & 2_8 & 5_8 \\ \hline 101_2 & 010_2 & 101_2 & 010_2 & 101_2 \end{array} \quad \Rightarrow (52525)_8 = (101\ 0101\ 0101\ 0101)_2$

Exercise

Convert the hexadecimal expansion of each of these integers to a binary expansion

a) $(80E)_{16}$ *b)* $(135AB)_{16}$ *c)* $(ABBA)_{16}$

d) $(DEFACED)_{16}$ *e)* $(BADFACED)_{16}$ *f)* $(ABCDEF)_{16}$

Solution

Hexadecimal	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Binary	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111

a) $\begin{array}{c:c:c} 8_{16} & 0_{16} & E_{16} \\ \hline 1000_2 & 0000_2 & 1110_2 \end{array} \quad \Rightarrow (80E)_{16} = (1000\ 0000\ 1110)_2$

b) $\begin{array}{c:c:c:c:c} 1_{16} & 3_{16} & 5_{16} & A_{16} & B_{16} \\ \hline 0001_2 & 0011_2 & 0101_2 & 1010_2 & 1011_2 \end{array}$

$$\Rightarrow (135AB)_{16} = (0001\ 0011\ 0101\ 1010\ 1011)_2$$

c) $\begin{array}{c|c|c|c} A_{16} & B_{16} & B_{16} & A_{16} \\ \hline 1010_2 & 1011_2 & 1011_2 & 1010_2 \end{array} \quad \Rightarrow (ABBA)_{16} = \underline{(1010\ 1011\ 1011\ 1010)_2}$

d) $\begin{array}{c|c|c|c|c|c|c} D_{16} & E_{16} & F_{16} & A_{16} & C_{16} & E_{16} & D_{16} \\ \hline 1101_2 & 1110_2 & 1111_2 & 1010_2 & 1100_2 & 1110_2 & 1101_2 \end{array}$

$\Rightarrow (DEFACED)_{16} = \underline{(1101\ 1110\ 1111\ 1010\ 1100\ 1110\ 1101)_2}$

e) $\begin{array}{c|c|c|c|c|c|c|c} B_{16} & A_{16} & D_{16} & F_{16} & A_{16} & C_{16} & E_{16} & D_{16} \\ \hline 1011_2 & 1010_2 & 1101_2 & 1111_2 & 1010_2 & 1100_2 & 1110_2 & 1101_2 \end{array}$

$\Rightarrow (BADFACED)_{16} = \underline{(1011\ 1010\ 1101\ 1111\ 1010\ 1100\ 1110\ 1101)_2}$

f) $\begin{array}{|c|c|c|c|c|c|} A_{16} & B_{16} & C_{16} & D_{16} & E_{16} & F_{16} \\ \hline 1010_2 & 1011_2 & 1100_2 & 1101_2 & 1110_2 & 1111_2 \end{array}$

$\Rightarrow (ABCDEF)_{16} = \underline{(1010\ 1011\ 1100\ 1101\ 1110\ 1111)_2}$

Exercise

Show that the binary expansion of a positive integer can be obtained from its hexadecimal expansion by translating each hexadecimal digit into a block of four binary digits.

Solution

Let $(\ldots h_2 h_1 h_0)_{16}$ be the hexadecimal expansion of a positive integer. The value of that integer is

$h_0 + h_1 \cdot 16 + h_2 \cdot 16^2 + \cdots = h_0 + h_1 \cdot 2^4 + h_2 \cdot 2^8 + \cdots$

If we replace each hexadecimal digit h_i by its binary expansion $(b_{i3} b_{i2} b_{i1} b_{i0})_2$, then

$h_i = b_{i0} + 2b_{i1} + 4b_{i2} + 8b_{i3}$

Therefore the value of the entire number is

$$\begin{aligned} &b_{00} + 2b_{01} + 4b_{02} + 8b_{03} + (b_{10} + 2b_{11} + 4b_{12} + 8b_{13}) \cdot 2^4 \\ &\qquad + (b_{20} + 2b_{21} + 4b_{22} + 8b_{23}) \cdot 2^8 + \cdots \\ &= b_{00} + 2b_{01} + 4b_{02} + 8b_{03} + 2^4 b_{10} + 2^5 b_{11} + 2^6 b_{12} + 2^7 b_{13} \\ &\qquad + 2^8 b_{20} + 2^9 b_{21} + 2^{10} b_{22} + 2^{11} b_{23} + \cdots \end{aligned}$$

Which is the value of the binary expansion $(\cdots b_{23} b_{22} b_{21} b_{20} b_{13} b_{12} b_{11} b_{10} b_{03} b_{02} b_{01} b_{00})_2$

Exercise

Show that the binary expansion of a positive integer can be obtained from its octal expansion by translating each octal digit into a block of three binary digits.

Solution

Let $(\ldots d_2 d_1 d_0)_8$ be the octal expansion of a positive integer. The value of that integer is

$d_0 + d_1 \cdot 8 + d_2 \cdot 8^2 + \cdots = d_0 + d_1 \cdot 2^2 + d_2 \cdot 2^6 + \cdots$

If we replace each octal digit d_i by its binary expansion $(b_{i2} b_{i1} b_{i0})_2$, then

$d_i = b_{i0} + 2b_{i1} + 4b_{i2}$

Therefore the value of the entire number is

$$b_{00} + 2b_{01} + 4b_{02} + (b_{10} + 2b_{11} + 4b_{12}) \cdot 2^3 + (b_{20} + 2b_{21} + 4b_{22}) \cdot 2^6 + \cdots$$

$$= b_{00} + 2b_{01} + 4b_{02} + 2^3 b_{10} + 2^4 b_{11} + 2^5 b_{12} + 2^6 b_{20} + 2^7 b_{21} + 2^8 b_{22} + \cdots$$

Which is the value of the binary expansion $(\cdots b_{22} b_{21} b_{20} b_{12} b_{11} b_{10} b_{02} b_{01} b_{00})_2$

Exercise

Explain how to convert from binary to base 64 expansions and from base 64 expansions to binary expansions and from octal to base 64 expansions and from base 64 expansions to octal expansions

Solution

$64 = 2^8 = 8^2$, in base 64 we need 64 symbols, from 0 to up to something representing 63. Corresponding to each such symbol would be a binary string of 6 digits, from 000000 for 0 to 001010 for ***a***, 100011 for ***z***, 100100 for ***A***, 111101 for ***Z***, for 111110 for @, and 111111 for $.

To translate from binary to base 64, we group the binary digits from the right in groups of 6 and use the list of correspondences to replace each 6 bits by one base-64 digits.

To convert from base 64 to binary, we just replace each base-64 digit by its corresponding 6 bits.

For conversion between octal and base 64, we change the binary strings in the table to octal strings, replacing each 6-bit string by its 2-digit octal equivalent, and then follow the same procedures as above, interchanging base-64 digits and 2-digits strings of octal digits.

Exercise

Find the sum and product of each of these pairs of numbers. Express your answers as a base 3 expansions

a) $(112)_3$, $(210)_3$ *b)* $(2112)_3$, $(12021)_3$

c) $(20001)_3$, $(1111)_3$ *d)* $(120021)_3$, $(2002)_3$

Solution

a)

$$\begin{array}{rcccc} & & 1 & 1 & 2 \\ + & & 2 & 1 & 0 \\ \hline & 1 & 0 & 2 & 2 \end{array} \qquad \begin{array}{rcccccc} & & & & 1 & 1 & 2 \\ \times & & & & 2 & 1 & 0 \\ \hline & & & & 0 & 0 & 0 \\ & & & 1 & 1 & 2 & \\ & 1 & 0 & 0 & 1 & & \\ \hline & 1 & 0 & 1 & 2 & 2 & 0 \end{array} \qquad \begin{array}{rcccc} & & 1 & 1 & \\ & & 1 & 1 & 2 \\ \times & & & & 2 \\ \hline & 1 & 0 & 0 & 1 \end{array}$$

b)

$$\begin{array}{rccccc} & & 2 & 1 & 1 & 2 \\ + & 1 & 2 & 0 & 2 & 1 \\ \hline & 2 & 1 & 2 & 1 & 0 \end{array} \qquad \begin{array}{rccccccccc} & & & & & & 2 & 1 & 1 & 2 \\ \times & & & & & 1 & 2 & 0 & 2 & 1 \\ \hline & & & & & & 2 & 1 & 1 & 2 \\ & & & & 1 & 2 & 0 & 0 & 1 & \\ & & & & & & & 0 & & \\ & & 1 & 2 & 0 & 0 & 1 & & & \\ & & 2 & 1 & 1 & 2 & & & & \\ \hline & 1 & 1 & 1 & 0 & 2 & 0 & 1 & 2 & 2 \end{array} \qquad \begin{array}{rccccc} & 1 & 1 & 1 & 1 & \\ & & 2 & 1 & 1 & 2 \\ \times & & & & 0 & 2 \\ \hline & 1 & 2 & 0 & 0 & 1 \end{array}$$

c)

$$\begin{array}{rccccc} & 2 & 0 & 0 & 0 & 1 \\ + & & 1 & 1 & 1 & 1 \\ \hline & 2 & 1 & 1 & 1 & 2 \end{array} \qquad \begin{array}{rcccccccc} & & & & 2 & 0 & 0 & 0 & 1 \\ \times & & & & & 1 & 1 & 1 & 1 \\ \hline & & & & 2 & 0 & 0 & 0 & 1 \\ & & & 2 & 0 & 0 & 0 & 1 & \\ & & 2 & 0 & 0 & 0 & 1 & & \\ & 2 & 0 & 0 & 0 & 1 & & & \\ \hline & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

d)

$$\begin{array}{rcccccc} & & & & 1 & 1 & \\ & 1 & 2 & 0 & 0 & 2 & 1 \\ + & & & 2 & 0 & 0 & 2 \\ \hline & 1 & 2 & 2 & 1 & 0 & 0 \end{array} \qquad \begin{array}{rcccccccccc} & & & & & 1 & 2 & 0 & 0 & 2 & 1 \\ \times & & & & & & & 2 & 0 & 0 & 2 \\ \hline & & & & 1 & 0 & 1 & 0 & 1 & 1 & 2 \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & 0 & 0 & 0 & 0 & 0 & 0 & & \\ & 1 & 0 & 1 & 0 & 1 & 1 & 2 & & & \\ \hline & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 \end{array}$$

Exercise

Find the sum and product of each of these pairs of numbers. Express your answers as an octal expansion.

a) $(763)_8,\ (147)_8$ *b)* $(6001)_8,\ (272)_8$

c) $(1111)_8,\ (777)_8$ *d)* $(54321)_8,\ (3456)_8$

Solution

a)

$$\begin{array}{rrrrr} & & 1 & 1 & \\ & & 7 & 6 & 3 \\ + & & 1 & 4 & 7 \\ \hline & 1 & 1 & 3 & 2 \end{array}$$

$$\begin{array}{rrrrrr} & & & 7 & 6 & 3 \\ \times & & & 1 & 4 & 7 \\ \hline & & 5 & 4 & 4 & 5 \\ & 3 & 7 & 1 & 4 & \\ & 7 & 6 & 3 & & \\ \hline 1 & 4 & 4 & 3 & 0 & 5 \end{array}$$

b)

$$\begin{array}{rrrrr} & 6 & 0 & 0 & 1 \\ + & & 2 & 7 & 2 \\ \hline & 6 & 2 & 7 & 3 \end{array}$$

$$\boxed{(6001)_8 + (272)_8 = 6273}$$

$6001 = 6 \cdot 8^3 + 1 = 3073$

$272 = 2 \cdot 8^2 + 7 \cdot 8 + 2 = 186$

$6001 \cdot 272 = 3073 \cdot 186 = 571,578$

$$\begin{aligned} 571,578 &= 8 \times 71447 + 2 \\ 71447 &= 8 \times 8930 + 7 \\ 8930 &= 8 \times 1116 + 2 \\ 1116 &= 8 \times 139 + 4 \\ 139 &= 8 \times 17 + 3 \\ 17 &= 8 \times 2 + 1 \\ 2 & \end{aligned}$$

$$\boxed{(6001)_8 \cdot (272)_8 = 2,134,272}$$

c)

$$\begin{array}{rrrrr} & 1 & 1 & 1 & \\ & 1 & 1 & 1 & 1 \\ + & & 7 & 7 & 7 \\ \hline & 2 & 1 & 1 & 0 \end{array}$$

$$\boxed{(1111)_8 + (777)_8 = 2110}$$

$(1111)_8 = 1 \cdot 8^3 + 1 \cdot 8^2 + 1 \cdot 8 + 1 = 585$

$(777)_8 = 7 \cdot 8^2 + 7 \cdot 8 + 7 = 511$

$(1111)_8 \cdot (777)_8 = (585)(511) = 298,935$

$$\begin{aligned} 298935 &= 8 \times 37366 + 7 \\ 37366 &= 8 \times 4670 + 6 \\ 4670 &= 8 \times 583 + 6 \\ 583 &= 8 \times 72 + 7 \\ 72 &= 8 \times 9 + 0 \\ 9 &= 8 \times 1 + 1 \\ 1 & \end{aligned}$$

$$\boxed{(1111)_8 \cdot (777)_8 = 1,107,667}$$

d)

$$\begin{array}{r} 5\ 4\ 3\ 2\ 1 \\ +\quad 3\ 4\ 5\ 6 \\ \hline 5\ 7\ 7\ 7\ 7 \end{array}$$

$$\boxed{(54321)_8 + (3456)_8 = 57,777}$$

$$(54321)_8 = 5 \cdot 8^4 + 4 \cdot 8^3 + 3 \cdot 8^2 + 2 \cdot 8 + 1 = 22,737$$

$$(3456)_8 = 3 \cdot 8^3 + 4 \cdot 8^2 + 5 \cdot 8 + 6 = 1838$$

$$(54321)_8 \cdot (3456)_8 = (22,737)(1838) = 41,790,606$$

$$41790606 = 8 \times 5223825 + 6$$
$$5223825 = 8 \times 652978 + 1$$
$$652978 = 8 \times 81622 + 2$$
$$81622 = 8 \times 10202 + 6$$
$$10202 = 8 \times 1275 + 2$$
$$1275 = 8 \times 159 + 3$$
$$159 = 8 \times 19 + 7$$
$$19 = 8 \times 2 + 3$$
$$2$$

$$\boxed{(54321)_8 \cdot (3456)_8 = 237,326,216}$$

Exercise

Find the sum and product of each of these pairs of numbers. Express your answers as a hexadecimal expansion.

a) $(1AB)_{16}$, $(BBC)_{16}$ *b)* $(20CBA)_{16}$, $(A01)_{16}$

c) $(ABCDE)_{16}$, $(1111)_{16}$ *d)* $(E0000E)_{16}$, $(BAAA)_{16}$

Solution

Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Hexadecimal	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

a) $1AB = 1*16^2 + 10*16 + 11 = 427$

$BBC = 11*16^2 + 11*16 + 12 = 3004$

$$1AB + BBC = 427 + 3004$$
$$= 3431$$

$$3431 = 16 \times 214 + 7$$
$$214 = 16 \times 14 + 6$$
$$14 \qquad\qquad 14 = D$$

$$\boxed{1AB + BBC = D67}$$

$$(1AB) \times (BBC) = (427)(3004)$$
$$= 1,282,708$$

$$1282708 = 16 \times 80169 + 4$$
$$80169 = 16 \times 5010 + 9$$
$$5010 = 16 \times 313 + 2$$
$$313 = 16 \times 19 + 9$$
$$19 = 16 \times 1 + 3$$
$$1$$

$$\boxed{(1AB) \times (BBC) = 139,294}$$

b) $(20CBA)_{16} = 2*16^4 + 0 + 12*16^2 + 11*16 + 10 = 134{,}330$

$(A01)_{16} = 10*16^2 + 0*16 + 12 = 2{,}561$

$$\begin{aligned}(20CBA)_{16} + (A01)_{16} &= 134{,}330 + 2{,}561\\ &= 136{,}891\end{aligned}$$

$136891 = 16\times 8555 + 11 \qquad 11 = B$
$8555 = 16\times 534 + 11 \qquad 11 = B$
$534 = 16\times 33 + 6$
$33 = 16\times 2 + 1$
2

$(20CBA)_{16} + (A01)_{16} = 21{,}6BB$

$$\begin{aligned}(20CBA)_{16} \times (A01)_{16} &= (134{,}330)(2{,}561)\\ &= 344{,}019{,}130\end{aligned}$$

$344019130 = 16\times 21501195 + 10 \qquad 10 = A$
$21501195 = 16\times 1343824 + 11 \qquad 11 = B$
$1343824 = 16\times 83989 + 0$
$83989 = 16\times 5249 + 5$
$5249 = 16\times 328 + 1$
$328 = 16\times 20 + 8$
$20 = 16\times 1 + 4$
1

$(20CBA)_{16} \times (A01)_{16} = 14{,}815{,}0BA$

c) $(ABCDE)_{16} = 10*16^4 + 11*16^4 + 12*16^2 + 13*16 + 14 = 703{,}710$

$(1111)_{16} = 1*16^3 + 1*16^2 + 1*16 + 1 = 4369$

$$\begin{aligned}(ABCDE)_{16} + (1111)_{16} &= 703{,}710 + 4369\\ &= 708{,}079\end{aligned}$$

$708079 = 16\times 44254 + 15 \qquad 15 = F$
$44254 = 16\times 2765 + 14 \qquad 14 = E$
$2765 = 16\times 172 + 13 \qquad 13 = D$
$172 = 16\times 10 + 12 \qquad 12 = C$
$10 \qquad 10 = A$

$(ABCDE)_{16} + (1111)_{16} = AC{,}DEF$

$$(ABCDE)_{16} \times (1111)_{16} = (703{,}710)(4369)$$
$$= 3{,}074{,}508{,}990$$

$3074508990 = 16 \times 192156811 + 14 \quad 14 = E$
$192156811 = 16 \times 12009800 + 11 \quad 11 = B$
$12009800 = 16 \times 750612 + 8$
$750612 = 16 \times 46913 + 4$
$46913 = 16 \times 2932 + 1$
$2932 = 16 \times 183 + 4$
$183 = 16 \times 11 + 7$
$11 \quad 11 = B$

$$\boxed{(ABCDE)_{16} \times (1111)_{16} = B7{,}414{,}8BE}$$

d) $(E0000E)_{16} = 14 * 16^5 + 14 = 14{,}680{,}078$

$(BAAA)_{16} = 11 * 16^3 + 10 * 16^2 + 10 * 16 + 10 = 47{,}786$

$$(E0000E)_{16} + (BAAA)_{16} = 14{,}680{,}078 + 47{,}786$$
$$= 14{,}727{,}864$$

$14727864 = 16 \times 920491 + 8$
$920491 = 16 \times 57530 + 11 \quad 11 = B$
$57530 = 16 \times 3595 + 10 \quad 10 = A$
$3595 = 16 \times 224 + 11 \quad 11 = B$
$224 = 16 \times 14 + 0$
$14 \quad 14 = E$

$$\boxed{(E0000E)_{16} + (BAAA)_{16} = E0B, AB8}$$

$$(E0000E)_{16} (BAAA)_{16} = (14{,}680{,}078)(47{,}786)$$
$$= 701{,}502{,}207{,}308$$

$701502207308 = 16 \times 43843887956 + 12 \quad 12 = C$
$43843887956 = 16 \times 2740242997 + 4$
$2740242997 = 16 \times 171265187 + 5$
$171265187 = 16 \times 10704074 + 3$
$10704074 = 16 \times 669004 + 10 \quad 10 = A$
$669004 = 16 \times 41812 + 12 \quad 12 = C$
$41812 = 16 \times 2613 + 4$
$2613 = 16 \times 163 + 5$
$163 = 16 \times 10 + 3$
$10 \quad 10 = A$

$$(E0000E)_{16} \times (BAAA)_{16} = \boxed{A{,}354{,}CA3{,}54C}$$

Solution Section 2.5 – Primes and Greatest Common Divisors

Exercise

Determine whether each of these integers is prime.

a) 21   *b)* 29   *c)* 71   *d)* 97   *e)* 111
f) 143   *g)* 19   *h)* 27   *i)* 93   *j)* 101
k) 107   *l)* 113

Solution

The numbers: 29, 71, 97, 19, 101, 107, and 113 are primes.

Not Prime: $21 = 3 \cdot 7$ $111 = 3 \cdot 37$ $143 = 13 \cdot 11$ $27 = 3^3$ $93 = 3 \cdot 31$

Exercise

Find the prime factorization of each these integers.

a) 88   *b)* 126   *c)* 729   *d)* 1001   *e)* 1111
f) 909,090   *g)* 39   *h)* 81   *i)* 101   *j)* 143
k) 289   *l)* 899

Solution

a) $88 = 2^3 \cdot 11$

b) $126 = 2 \cdot 3^2 \cdot 7$

c) $729 = 3^6$

d) $1001 = 11 \cdot 91$

e) $1111 = 11 \cdot 101$

f) $909090 = 2 \cdot 5 \cdot 9 \cdot 91 \cdot 111$

g) $39 = 3 \cdot 13$

h) $81 = 3^4$

i) $101 = 101$ (***Prime***)

j) $143 = 11 \cdot 13$

k) $289 = 17^2$

l) $899 = 29 \cdot 31$

Exercise

Find the prime factorization of 10!

Solution

$10! = 3628800$

$10! = (2 \cdot 5)\,!$

Exercise

Show that if $a^m + 1$ is composite if *a* and *m* are integers greater than 1 and *m* is odd. [*Hint*: Show that *x* + 1 is a factor of the polynomial $a^m + 1$ if *m* is odd]

Solution

Since *m* is odd, then we can factor $a^m + 1 = (a+1)\left(a^{m-1} - a^{m-2} + a^{m-3} - \cdots - 1\right)$

Because *a* and *m* are both greater than 1, we know that $1 < a+1 < a^m + 1$. This provides a factoring of $a^m + 1$ into proper factors, so $a^m + 1$ is composite.

Exercise

Show that if $2^m + 1$ is an odd prime, then $m = 2^n$ for some nonnegative integer *n*. [*Hint*: First show the polynomial identity $x^m + 1 = \left(x^k + 1\right)\left(x^{k(t-1)} - x^{k(t-2)} + \cdots - x^k + 1\right)$ holds, where $m = kt$ and *t* is odd]

Solution

Assume $y = x^k$, then the claimed identity is

$$\left(y^t + 1\right) = (y+1)\left(y^{t-1} - y^{t-2} + y^{t-3} - \cdots - y + 1\right)$$

By multiplying out the right-hand side and noticing the "telescoping" that occurs.
Let show that *m* is a power of 2 that is only prime factor is 2.

Suppose to the contrary that *m* has an odd prime factor *t* and $m = kt$, where *k* is a positive integer. Letting $x = 2$ in the identity given in the hint, we have $2^m + 1 = \left(2^k + 1\right)(\cdots)$. Because $2^k + 1 > 1$ and the prime $2^m + 1$ can have no proper factor greater than 1, we must have $2^m + 1 = 2^k + 1$, so $m = k$ and $t = 1$ contradicting the fact that *t* is prime. This completes the proof by contradiction.

Exercise

Which positive integers less than 12 are relatively prime to 12?

Solution

By inspection with mental arithmetic, the greatest common divisors of the numbers from 1 to 11 with 12 whose ***gcd*** is 1, are 1, 5, 7, and 11. These are so few since 12 had many factors – in particular, both 2 and 3.

Exercise

Which positive integers less than 30 are relatively prime to 30?

Solution

The prime factors of 30 are 2, 3, and 5.

Thus we are looking for positive integers less than 30 that have none of these prime factors. Since the smallest prime number other than these is 7, and 7^2 is already greater than 30, in fact only primes (and the number 1) will satisfy this condition.

Therefore the answer is 1, 7, 11, 13, 17, 18, 23, and 29.

Exercise

Determine whether the integers in each of these sets are pairwise relatively prime.

a) 21, 34, 55 *b)* 14, 17, 85 *c)* 25, 41, 49, 64 *d)* 17, 18, 19, 23

e) 11, 15, 19 *f)* 14, 15, 21 *g)* 12, 17, 31, 37 *h)* 7, 8, 9, 11

Solution

a) $21 = 3\cdot 7, \quad 34 = 2\cdot 17, \quad 55 = 5\cdot 11$ These are pairwise relatively prime

b) $85 = 5\cdot 17$ These are not pairwise relatively prime

c) $25 = 5^2, \quad 41\, is\ prime, \quad 49 = 7^2, \quad 64 = 2^6$ These are pairwise relatively prime

d) $17,\ 19,\ and\ \ 23\ \ are\ prime \quad 18 = 2\cdot 3^2$ These are pairwise relatively prime

e) $11\ \ and\ \ 19\ \ are\ prime \quad 15 = 3\cdot 5$ These are pairwise relatively prime

f) $14 = 2\cdot 7 \quad and \quad 21 = 3\cdot 7$ These are not pairwise relatively prime

g) $17,\ 31,\ and\ \ 37\ \ are\ prime \quad 12 = 2^2\cdot 3$ These are pairwise relatively prime

h) $7\ \ and\ \ 11\ \ are\ prime \quad 8 = 2^3 \quad 9 = 3^2$ These are pairwise relatively prime

Exercise

We call a positive integer ***perfect*** if it equals the sum of its positive divisors other than itself

a) Show that 6 and 28 are perfect.

b) Show that $2^{p-1}\left(2^p-1\right)$ is a perfect number when 2^p-1 is prime

Solution

a) Since $6=1+2+3$, and these three summands are the only proper divisors of 6, we conclude that 6 is perfect.
$28=1+2+4+7+14$ are also the only proper divisors of 28

b) We need to find all proper divisors of $2^{p-1}\left(2^p-1\right)$. Certainly all the numbers

$1,\ 2,\ 4,\ 8,\ \ldots,\ 2^{p\text{-}1}$ are proper divisors, and their sum is 2^p-1 (geometric series). Also each of these divisors times 2^p-1 is also a divisor, and all but the last is proper. Again adding up this geometric series we find a sum of $2^{p-1}\left(2^p-1\right)$. There are no other proper divisors. Therefore the sum of all the divisors is

$$\begin{aligned}\left(2^p-1\right)+\left(2^p-1\right)\left(2^{p-1}-1\right)&=\left(2^p-1\right)\left(1+2^{p-1}-1\right)\\&=\left(2^p-1\right)2^{p-1}\end{aligned}$$

Which is our original number. Therefore this number is perfect.

Exercise

Show that if 2^n-1 is prime, then *n* is prime. *Hint*: Use the identity
$2^{ab}-1=\left(2^a-1\right)\cdot\left(2^{a(b-1)}+2^{a(b-2)}+\cdots+2^a+1\right)$

Solution

We will prove the assertion by proving its contrapositive.
Suppose that *n* is not prime. Then by definition $n=ab$ for some integers *a* and *b* each greater than 1.
Since $a>1$, 2^a-1, the first factor in the suggested identity, is greater than 1. The second factor is also greater than 1.
Thus $2^n-1=2^b-1$ is the product of 2 integers each greater than 1, so it is not prime.

Exercise

Determine whether each of these integers is prime, verifying some of Mersenne's claims

a) 2^7-1 *b)* 2^9-1 *c)* $2^{11}-1$ *d)* $2^{13}-1$

Solution

a) $2^7-1=127$. 2, 3, 5, 7, 11 are not factors of 127, since $\sqrt{127}<13$, therefore 127 is prime.

b) $2^9-1=511=7\cdot 73$ So this number is not prime.

c) $2^{11}-1=2047=23\cdot 89$ So this number is not prime.

d) $2^{13}-1=8191$.

Since $\sqrt{8191}<97$

then 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, and 89 are not factors of 8191, therefore 8191 is prime.

Exercise

What are the greatest common divisors of these pairs of integers?

a) $2^2\cdot 3^3\cdot 5^5,\ \ 2^5\cdot 3^3\cdot 5^2$

b) $2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13,\ \ 2^{11}\cdot 3^9\cdot 11\cdot 17^{14}$

c) $17,\ \ 17^{17}$

d) $2^2\cdot 7,\ \ 5^3\cdot 13$

e) $0,\ \ 5$

f) $2\cdot 3\cdot 5\cdot 7,\ \ 2\cdot 3\cdot 5\cdot 7$

g) $3^7\cdot 5^3\cdot 7^3,\ \ 2^{11}\cdot 3^5\cdot 5^9$

h) $11\cdot 13\cdot 17,\ \ 2^9\cdot 3^7\cdot 5^5\cdot 7^3$

i) $23^{31},\ \ 23^{17}$

j) $41\cdot 43\cdot 53,\ \ 41\cdot 43\cdot 53$

k) $1111,\ \ 0$

Solution

a) $2^2\cdot 3^3\cdot 5^2$

b) $2\cdot 3\cdot 5$

c) 17

d) 1

e) 5

f) $2\cdot 3\cdot 5\cdot 7$

g) $3^5 \cdot 5^3$

h) 1

i) 23^{17}

j) $41 \cdot 43 \cdot 53$

k) 1111

Exercise

What is the least common multiple of each pair

a) $2^2 \cdot 3^3 \cdot 5^5,\ \ 2^5 \cdot 3^3 \cdot 5^2$

b) $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13,\ \ 2^{11} \cdot 3^9 \cdot 11 \cdot 17^{14}$

c) $17,\ \ 17^{17}$

d) $2^2 \cdot 7,\ \ 5^3 \cdot 13$

e) $0,\ \ 5$

f) $2 \cdot 3 \cdot 5 \cdot 7,\ \ 2 \cdot 3 \cdot 5 \cdot 7$

g) $3^7 \cdot 5^3 \cdot 7^3,\ \ 2^{11} \cdot 3^5 \cdot 5^9$

h) $11 \cdot 13 \cdot 17,\ \ 2^9 \cdot 3^7 \cdot 5^5 \cdot 7^3$

i) $23^{31},\ \ 23^{17}$

j) $41 \cdot 43 \cdot 53,\ \ 41 \cdot 43 \cdot 53$

k) $1111,\ \ 0$

Solution

a) $2^5 \cdot 3^3 \cdot 5^5$

b) $2^{11} \cdot 3^9 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17^{14}$

c) 17^{17}

d) $2^2 \cdot 5^3 \cdot 7 \cdot 13$

e) *Undefined*

f) $2 \cdot 3 \cdot 5 \cdot 7$

g) $2^{11} \cdot 3^5 \cdot 5^9 \cdot 7^3$

h) $2^9 \cdot 3^7 \cdot 5^5 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17$

i) 23^{31}

j) $41 \cdot 43 \cdot 53$

k) *Undefined*

Exercise

Find gcd(1000, 625) and lcm(1000, 625) and verify that $\gcd(100,\ 625) \cdot lcm(100,\ 625) = 1000 \cdot 625$

Solution

$1000 = 2^3 \cdot 5^3$

$625 = 5^5$

$\gcd(1000, 625) = 5^3 = 125$

$lcm(1000, 625) = 2^3 \cdot 5^4 = 5000$

Therefore, $125 \cdot 5000 = 625000 = 1000 \cdot 625$

Exercise

Find *gcd*(92928, 123552) and lcm(92928, 123552) and verify that

$gcd(92928,\ 123552) \cdot lcm(92928,\ 123552) = 92928 \bullet 123552$

Solution

$92928 = 2^8 \cdot 3 \cdot 11^2$

$123552 = 2^5 \cdot 3^3 \cdot 11 \cdot 13$

$\gcd(92928,\ 123552) = 2^5 \cdot 3 \cdot 11 = 1056$

$lcm(92928,\ 123552) = 2^8 \cdot 3^3 \cdot 11^2 \cdot 13 = 10{,}872{,}576$

$$\begin{aligned}\gcd(92928,\ 123552) \cdot lcm(92928,\ 123552) &= \left(2^5 \cdot 3 \cdot 11\right)\left(2^8 \cdot 3^3 \cdot 11^2 \cdot 13\right)\\ &= 2^{13} \cdot 3^4 \cdot 11^3 \cdot 13\end{aligned}$$

$(92928)(123552) = \left(2^8 \cdot 3 \cdot 11^2\right)\left(2^5 \cdot 3^3 \cdot 11 \cdot 13\right) = 2^{13} \cdot 3^4 \cdot 11^3 \cdot 13$

$\gcd(92928,\ 123552) \cdot lcm(92928,\ 123552) = 92928 \bullet 123552 \ \underline{= 11{,}481{,}440{,}256}$

Exercise

Use the Euclidean algorithm to find

a) $\gcd(1,\ 5)$ *b)* $\gcd(100,\ 101)$ *c)* $\gcd(123,\ 277)$ *d)* $\gcd(1529,\ 14039)$

e) $\gcd(1529,\ 14038)$ *f)* $\gcd(12,\ 18)$ *g)* $\gcd(111,\ 201)$ *h)* $\gcd(1001,\ 1331)$

i) $\gcd(12345,\ 54321)$ *j)* $\gcd(1000,\ 5040)$ *k)* $\gcd(9888,\ 6060)$

Solution

a) $5 = 1 \cdot 5 + 0$

$\gcd(1, 5) = \gcd(1, 0) = 1$

b) $101 = 100 \cdot 1 + 1$

$1 = 1 \cdot 1 + 0$

$\gcd(100, 101) = \gcd(100, 1) = \gcd(1, 0) = 1$

c) $277 = 123 \cdot 2 + 31$

$123 = 31 \cdot 3 + 30$

$31 = 30 \cdot 1 + 1$

$30 = 1 \cdot 30 + 0$

$\gcd(123, 277) = \gcd(123, 31) = \gcd(31, 30) = \gcd(30, 1) = \gcd(1, 0) = 1$

d) $14039 = 1529 \cdot 9 + 278$

$1529 = 278 \cdot 5 + 139$

$278 = 139 \cdot 2 + 0$

$\gcd(1529, 14039) = \gcd(1529, 278) = \gcd(278, 139) = \gcd(139, 0) = 139$

e) $14038 = 1529 \cdot 9 + 277$

$1529 = 277 \cdot 5 + 144$

$277 = 144 \cdot 1 + 133$

$144 = 133 \cdot 1 + 11$

$133 = 11 \cdot 12 + 1$

$11 = 1 \cdot 11 + 0$

$\gcd(1529, 14038) = \gcd(1529, 277) = \gcd(277, 144) = \gcd(144, 133) = \gcd(133, 11)$

$= \gcd(11, 1) = \gcd(1, 0) = 1$

f) $18 = 12 \cdot 1 + 6$

$12 = 6 \cdot 2 + 0$

$\gcd(12, 18) = \gcd(12, 6) = 6$

g) $201 = 111 \cdot 1 + 90$

$111 = 90 \cdot 1 + 21$

$90 = 21 \cdot 4 + 6$

$21 = 6 \cdot 3 + 3$

$6 = 3 \cdot 2 + 0$

$\gcd(111, 201) = \gcd(111, 90) = \gcd(90, 21) = \gcd(21, 6) = \gcd(6, 3) = \gcd(3, 0) = 3$

h) $1331 = 1001 \cdot 1 + 330$

$1001 = 330 \cdot 3 + 11$

$330 = 11 \cdot 30 + 0$

$\gcd(1001, 1331) = \gcd(1001, 330) = \gcd(330, 11) = \gcd(11, 0) = 11$

i) $54321 = 12345 \cdot 4 + 4941$

$12345 = 4941 \cdot 2 + 2463$

$4941 = 2463 \cdot 2 + 15$

$2463 = 15 \cdot 164 + 3$

$15 = 3 \cdot 5 + 0$

$\gcd(12345, 54321) = \gcd(12345, 4941) = \gcd(4941, 2463) = \gcd(2463, 15)$

$= \gcd(15, 3) = \gcd(3, 0) = 3$

j) $5040 = 1000 \cdot 5 + 40$

$1000 = 40 \cdot 25 + 0$

$\gcd(1000, 5040) = \gcd(1000, 40) = \gcd(40, 0) = 40$

k) $9888 = 6060 \cdot 1 + 3828$

$6060 = 3828 \cdot 1 + 2232$

$3828 = 2232 \cdot 1 + 1596$

$2232 = 1596 \cdot 1 + 636$
$1596 = 636 \cdot 2 + 324$
$636 = 324 \cdot 1 + 312$
$324 = 312 \cdot 1 + 12$
$312 = 12 \cdot 26 + 0$

$$\gcd(9888,\ 6060) = \gcd(6060,\ 3828) = \gcd(3828,\ 2232) = \gcd(2232,\ 1596) = \gcd(1596,\ 636)$$
$$= \gcd(636,\ 324) = \gcd(324,\ 312) = \gcd(312,\ 12) = \gcd(12,\ 0) = 12$$

Exercise

Prove that the product of any three consecutive integers is divisible by 6.

Solution

Consider the product $n(n+1)(n+2)$ for some integer n.

Since every second integer is even (divisible by 2), then this product is divisible by 2.
Since every third integer is divisible by 3, then this product is divisible by 3.
Therefore, this product has both 2 and 3 in its prime factorization and is therefore divisible by $2 \cdot 3 = 6$

Exercise

Show that if *a, b*, and *m* are integers such that $m \ge 2$ and $a \equiv b(\textbf{mod}\ m)$, then $\gcd(a,\ m) = \gcd(b,\ m)$

Solution

From $a \equiv b(\textbf{mod}\ m)$ we know that $b = a + sm$ for some integer *s*. If *d* is a common divisor of *a* and *m*, then it divides the right-hand side of this equation, so it also divides *b*. We can rewrite the equation as $a = b - sm$, and they by similar reasoning, we see that every common divisor of *b* and *m* is also a divisor of *a*.

This shows that the set of common divisors of *a* and *m* is equal to the set of common divisors of *b* and *m*, so certainly $\gcd(a, m) = \gcd(b, m)$

Exercise

Prove or disprove that $n^2 - 79n + 1601$ is prime whenever *n* is a positive integer.

Solution

Using calculator or spread sheet because it is hard to get started:

$n^2 - 79n + 1601$	
$n = 1$	1523
$n = 2$	1447
$n = 3$	1373
$n = 4$	1301
$n = 5$	1231
$n = 6$	1163

All the values are prime. This may lead us to believe that the propositions is true, but it gives no clue as to how to prove it.
If we let $n = 1601$, then

$$1601^2 - 79(1601) + 1601 = 1601(1601 - 79 + 1) = 1601 \cdot 1523.$$

So we got a counterexample and the proposition is false.

The smallest n for which this expression is not prime is $n = 80$; this gives the value $1681 = 41 \cdot 41$

Solution *Section* 2.6 – Applications of Congruences

Exercise

Find the memory locations assigned by the hashing function $h(k) = k$ **mod** 97 to the records of customers with Social Security numbers?

a) 034567981 *b)* 183211232 *c)* 220195744 *d)* 987255335
e) 104578690 *f)* 432222187 *g)* 372201919 *h)* 501338753

Solution

a) $034567981 \ \boldsymbol{mod}\ 97 = 91$

b) $183211232 \ \boldsymbol{mod}\ 97 = 57$

c) $220195744 \ \boldsymbol{mod}\ 97 = 21$

d) $987255335 \ \boldsymbol{mod}\ 97 = 5$

e) $104578690 \ \boldsymbol{mod}\ 97 = 80$

f) $432222187 \ \boldsymbol{mod}\ 97 = 81$

g) $372201919 \ \boldsymbol{mod}\ 97 = 18$

h) $501338753 \ \boldsymbol{mod}\ 97 = 73$

Exercise

A parking lot has 31 visitor spaces, numbered from 0 to 30. Visitors are assigned parking spaces using the hashing function $h(k) = k$ **mod** 31, where k is the number formed from the first three digits on a visitor's license plate.

a) Which spaces are assigned by the hashing function to cars that have these first three digits on their license plates: 317, 918, 007, 100, 111, 310

b) Describe a procedure visitors should follow to find a free parking space, when the space they are assigned is occupied.

Solution

a) $317 \ \boldsymbol{mod}\ 31 = 7$

$918 \ \boldsymbol{mod}\ 31 = 19$

$007 \ \boldsymbol{mod}\ 31 = 7$

$100 \ \boldsymbol{mod}\ 31 = 7$

$111 \ \boldsymbol{mod}\ 31 = 18$

$310 \ \boldsymbol{mod}\ 31 = 0$

b) Take the next available space, where the next space is computed by adding 1 to the space number and pretending that 30 + 1 = 0.

Exercise

Find the sequence of pseudorandom numbers generated by the linear congruential generator

a) $x_{n+1} = (3x_n + 2)$ **mod** 13 with seed $x_0 = 1$.

b) $x_{n+1} = (4x_n + 1)$ **mod** 7 with seed $x_0 = 3$.

Solution

a) Given $x_0 = 1$, the $x_1 = (3x_0 + 2)\,\boldsymbol{mod}\ 13 = (3\cdot 1 + 2)\,\boldsymbol{mod}\ 13 = 5\ \boldsymbol{mod}\ 13 = 5$

$x_2 = (3\cdot 5 + 2)\,\boldsymbol{mod}\ 13 = 17\ \boldsymbol{mod}\ 13 = 4$

$x_3 = (3\cdot 4 + 2)\,\boldsymbol{mod}\ 13 = 14\ \boldsymbol{mod}\ 13 = 1$

The sequence keep continue to repeat 1, 5, 4, 1, 5, 4, …

b) Given $x_0 = 3$, the $x_1 = (4x_0 + 1)\,\boldsymbol{mod}\ 7 = (4\cdot 3 + 1)\,\boldsymbol{mod}\ 7 = 13\ \boldsymbol{mod}\ 7 = 6$

$x_2 = (4\cdot 6 + 1)\,\boldsymbol{mod}\ 7 = 25\ \boldsymbol{mod}\ 7 = 4$

$x_3 = (4\cdot 4 + 1)\,\boldsymbol{mod}\ 7 = 17\ \boldsymbol{mod}\ 7 = 3$

The sequence keep continue to repeat 3, 6, 4, 3, 6, 4, …

Exercise

Find the sequence of pseudorandom numbers generated by using the pure multiplicative generator $x_{n+1} = 3x_n$ **mod** 11 with seed $x_0 = 2$.

Solution

$x_1 = 3x_0\ \textbf{mod}\ 11 = 3\cdot 2\ \textbf{mod}\ 11 = 6$

$x_2 = 3x_1\ \textbf{mod}\ 11 = 3\cdot 6\ \textbf{mod}\ 11 = 18\ \textbf{mod}\ 11 = 7$

$x_3 = 3x_2\ \textbf{mod}\ 11 = 3\cdot 7\ \textbf{mod}\ 11 = 21\ \textbf{mod}\ 11 = 10$

$x_4 = 3x_3\ \textbf{mod}\ 11 = 3\cdot 10\ \textbf{mod}\ 11 = 30\ \textbf{mod}\ 11 = 8$

$x_5 = 3x_4\ \textbf{mod}\ 11 = 3\cdot 8\ \textbf{mod}\ 11 = 24\ \textbf{mod}\ 11 = 2$

Since $x_5 = x_0$, the sequence repeats forever: 2, 6, 7, 10, 8, 2, 6, 7, 10, 8, …

Exercise

The first nine digits of the ISBN-10 of the European version of the fifth edition of this book are $0-07-119881$. What is the check digit for that book?

Solution

Let *d* be the check digit.

$1\cdot 0 + 2\cdot 0 + 3\cdot 7 + 4\cdot 1 + 5\cdot 1 + 6\cdot 9 + 7\cdot 8 + 8\cdot 8 + 9\cdot 1 + 10\cdot d \equiv 0\,(\boldsymbol{mod}\ 11)$

$213 + 10 \cdot d \equiv 0 (\textbf{\textit{mod}}\ 11)$

So $213 \equiv 4 (\textbf{\textit{mod}}\ 11)$ *and* $10 \equiv -1 (\textbf{\textit{mod}}\ 11)$

This is equivalent to: $4 - d \equiv 0 (\textbf{\textit{mod}}\ 11)$ or $\underline{d = 4}$

Exercise

The ISBN-10 of the sixth edition of Elementary Number Theory and Its Applications is $0-321-500Q1-8$, where Q is a digit. Find the value of Q.

Solution

$1 \cdot 0 + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 1 + 5 \cdot 5 + 6 \cdot 0 + 7 \cdot 0 + 8 \cdot Q + 9 \cdot 1 + 10 \cdot 8 \equiv 0 (\textbf{\textit{mod}}\ 11)$

$130 + 8Q \equiv 0 (\textbf{\textit{mod}}\ 11)$

$8Q \equiv -130 (\textbf{\textit{mod}}\ 11) \equiv 2 (\textbf{\textit{mod}}\ 11)$ $\quad -130 \equiv (-12 \cdot 11 + 2)(\textbf{\textit{mod}}\ 11)$

$8Q \equiv 2 (\textbf{\textit{mod}}\ 11)$ $\quad$ Since $24 \equiv 2 (\textbf{\textit{mod}}\ 11)$

Therefore $8Q = 24$

This is equivalent to: $\underline{Q = 3}$

Exercise

The USPS sells money orders identified by 11-digit number $x_1, x_2, \ldots, x_{11}$. The first ten digits identify the money order: x_{11} is a check digit that satisfies $x_{11} = x_1 + x_2 + \cdots + x_{10}$ **mod** 9. Find the check digit for the USPS money orders that have identification number that start with these ten digits

a) 7555618873 *b)* 6966133421 *c)* 8018927435 *d)* 3289744134
e) 74051489623 *f)* 88382013445 *g)* 56152240784 *h)* 66606631178

Solution

a) $(7+5+5+5+6+1+8+8+7+3)\ \textbf{\textit{mod}}\ 9 = 55\ \textbf{\textit{mod}}\ 9 = 1$

b) $(6+9+6+6+1+3+3+4+2+1)\ \textbf{\textit{mod}}\ 9 = 41\ \textbf{\textit{mod}}\ 9 = 5$

c) $(8+0+1+8+9+2+7+4+3+5)\ \textbf{\textit{mod}}\ 9 = 47\ \textbf{\textit{mod}}\ 9 = 2$

d) $(3+2+8+9+7+4+4+1+3+4)\ \textbf{\textit{mod}}\ 9 = 45\ \textbf{\textit{mod}}\ 9 = 0$

e) $(7+4+0+5+1+4+8+9+6+2+3)\ \textbf{\textit{mod}}\ 9 = 49\ \textbf{\textit{mod}}\ 9 = 4$

f) $(8+8+3+8+2+0+1+3+4+4+5)\ \textbf{\textit{mod}}\ 9 = 46\ \textbf{\textit{mod}}\ 9 = 1$

g) $(5+6+1+5+2+2+4+0+7+8+4)\ \textbf{\textit{mod}}\ 9 = 44\ \textbf{\textit{mod}}\ 9 = 8$

h) $(6+6+6+0+6+6+3+1+1+7+8)\ \textbf{\textit{mod}}\ 9 = 50\ \textbf{\textit{mod}}\ 9 = 5$

Exercise

Determine which single digit errors are detected by the USPS money order code.

Solution

If one digit change to a value not congruent to it modulo 9, then the modular equivalence implied by the equation in the preamble will no longer hold. Therefore all single digit errors are detected except for the substitution of a 9 for a 0 or vice versa.

Exercise

Determine which transposition errors are detected by the USPS money order code.

Solution

Because the first ten digits are added, any transposition error involving them will go undetected. The sum of the first ten digits will be the same for the transposed number as it is for the correct number.

Suppose that the last digit is transposed with another digit; without loss of generality; we can assume it's the tenth digit and that $x_{10} \neq x_{11}$.

Then the correct equation will be

$$x_{11} \equiv x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} \ (\boldsymbol{mod}\ 9) \qquad (1)$$

But the equation resulting from the error will read

$$x_{10} \equiv x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{11} \ (\boldsymbol{mod}\ 9) \qquad (2)$$

Subtract equations (2) & (1)

$$x_{11} - x_{10} \equiv x_{10} - x_{11} \ (\boldsymbol{mod}\ 9)$$

$$2x_{11} \equiv 2x_{10} \ (\boldsymbol{mod}\ 9) \qquad \textit{Divide by 2 both sides since 2 is prime}$$

$$x_{11} \equiv x_{10} \ (\boldsymbol{mod}\ 9) \quad \text{Which is false}$$

The check equation will fail.

Therefore, we conclude that transposition errors involving the eleventh digits are detected.

Solution *Section* 3.1 – Mathematical Induction

Exercise

Prove that $1^2+3^2+5^2+\cdots+(2n+1)^2=\frac{1}{3}(n+1)(2n+1)(2n+3)$ whenever *n* is a nonnegative integer.

Solution

Since *n* is a nonnegative integer that implies to $n \geq 0$

(1) For ***n* = 0** $\Rightarrow$ $1^2=\frac{1}{3}(0+1)(0+1)(0+3)$

$1=\frac{1}{3}(1)(2)(3)=1$; hence P_1 is true.

(1) Assume that $1^2+3^2+\cdots+(2k+1)^2=\frac{1}{3}(k+1)(2k+1)(2k+3)$ is true

$$1^2+3^2+\cdots+(2k+1)^2+(2(k+1)+1)^2=\frac{1}{3}((k+1)+1)(2(k+1)+1)(2(k+1)+3)$$

$$1^2+3^2+\cdots+(2k+1)^2+(2k+3)^2=\frac{1}{3}(k+2)(2k+3)(2k+5)$$

$$\begin{aligned}1^2+3^2+\cdots+(2k+1)^2+(2k+3)^2&=\frac{1}{3}(k+1)(2k+1)(2k+3)+(2k+3)^2\\&=\frac{1}{3}(2k+3)\left[(k+1)(2k+1)+3(2k+3)\right]\\&=\frac{1}{3}(2k+3)\left(2k^2+k+2k+1+6k+9\right)\\&=\frac{1}{3}(2k+3)\left(2k^2+9k+10\right)\\&=\frac{1}{3}(2k+3)(k+2)(2k+5)\ \checkmark\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ The statement $1^2+3^2+5^2+\cdots+(2n+1)^2=\frac{1}{3}(n+1)(2n+1)(2n+3)$ is true

Exercise

Prove that $1\cdot 1!+2\cdot 2!+\cdots+n\cdot n!=(n+1)!-1$ whenever *n* is a positive integer.

Solution

Since *n* is a positive integer that implies to $n \geq 1$

(2) For ***n* = 1** $\Rightarrow$ $1\cdot 1!=(1+1)!-1$

$1=1$; hence P_1 is true.

(3) Assume that $1\cdot 1!+2\cdot 2!+\cdots+k\cdot k!=(k+1)!-1$ is true

$$1\cdot 1!+2\cdot 2!+\cdots+k\cdot k!+(k+1)\cdot(k+1)!=((k+1)+1)!-1=(k+2)!-1$$

$$1\cdot 1!+2\cdot 2!+\cdots+k\cdot k!+(k+1)\cdot(k+1)!=(k+1)!-1+(k+1)\cdot(k+1)!$$
$$=(k+1)\cdot(k+1)!+(k+1)!-1$$
$$=(k+1)!(k+1+1)-1$$
$$=(k+1)!(k+2)-1$$
$$=(k+2)!-1 \quad \checkmark$$

Hence P_{k+1} is true.

$\therefore$ The statement $1\cdot 1!+2\cdot 2!+\cdots+n\cdot n!=(n+1)!-1$ is true

Exercise

Prove that $3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^n=\frac{3}{4}\left(5^{n+1}-1\right)$ whenever n is a nonnegative integer.

Solution

(1) For $\boldsymbol{n=0} \Rightarrow 3=\frac{3}{4}(5-1)$

$3=3$; hence P_1 is true.

(4) Assume that $3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^k=\frac{3}{4}\left(5^{k+1}-1\right)$ is true

$$3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^k+3\cdot 5^{k+1}=\frac{3}{4}\left(5^{k+2}-1\right)$$

$$3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^k+3\cdot 5^{k+1}=\frac{3}{4}\left(5^{k+1}-1\right)+3\cdot 5^{k+1}$$
$$=\frac{3}{4}\left[5^{k+1}-1+4\cdot 5^{k+1}\right]$$
$$=\frac{3}{4}\left(5\cdot 5^{k+1}-1\right)$$
$$=\frac{3}{4}\left(5^{k+2}-1\right) \quad \checkmark$$

Hence P_{k+1} is true.

$\therefore$ The statement $3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^n=\frac{3}{4}\left(5^{n+1}-1\right)$ is true

Exercise

Prove that $2-2\cdot 7+2\cdot 7^2 - \cdots + 2\cdot(-7)^n = \dfrac{1-(-7)^{n+1}}{4}$ whenever n is a nonnegative integer.

Solution

(1) For $\boldsymbol{n = 0} \Rightarrow\ 2 = \dfrac{1-(-7)^1}{4}$

$2 = \frac{8}{4} = 2$; hence P_1 is true.

(2) Assume that $2-2\cdot 7+2\cdot 7^2 - \cdots + 2\cdot(-7)^k = \dfrac{1-(-7)^{k+1}}{4}$ is true

We need to prove that P_{k+1} is also true

$$2-2\cdot 7+2\cdot 7^2 - \cdots + 2\cdot(-7)^k + 2\cdot(-7)^{k+1} = \frac{1-(-7)^{(k+1)+1}}{4}$$

$$2-2\cdot 7+2\cdot 7^2 - \cdots + 2\cdot(-7)^k + 2\cdot(-7)^{k+1} = \frac{1-(-7)^{k+2}}{4}$$

$$\begin{aligned}2-2\cdot 7+2\cdot 7^2 - \cdots + 2\cdot(-7)^k + 2\cdot(-7)^{k+1} &= \frac{1-(-7)^{k+1}}{4} + 2\cdot(-7)^{k+1}\\ &= \frac{1-(-7)^{k+1}+8\cdot(-7)^{k+1}}{4}\\ &= \frac{1-(-7)^{k+1}(1-8)}{4}\\ &= \frac{1-(-7)^{k+1}(-7)}{4}\\ &= \frac{1-(-7)^{k+2}}{4}\ \checkmark\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ The statement $2-2\cdot 7+2\cdot 7^2 - \cdots + 2\cdot(-7)^n = \dfrac{1-(-7)^{n+1}}{4}$ is true

Exercise

Find a formula for the sum of the first *n* even positive integers. Prove the formula.

Solution

$$\begin{array}{c} 1+2+\cdots+(n-1)+n \\ n+(n-1)+\cdots+2+1 \\ \hline (n+1)+(n+1)+\cdots+(n+1) \end{array}$$

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

(1) For $\boldsymbol{n=1} \Rightarrow 1=\frac{1(2)}{2} \Rightarrow 1=1$; hence P_1 is true.

(2) Assume that $1+2+\cdots+k=\frac{k(k+1)}{2}$ is true

We need to prove that P_{k+1} is also true $1+2+\cdots+k+(k+1)=\frac{(k+1)((k+1)+1)}{2}=\frac{(k+1)(k+2)}{2}$

$$\begin{aligned} 1+2+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ The statement $1+2+3+\cdots+n=\frac{n(n+1)}{2}$ is true

Exercise

a) Find a formula for $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}$ by examining the values of this expression for values of this expression for small values of *n*.

b) Prove the formula.

Solution

a) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

b) For $n=1 \quad \Rightarrow \quad \frac{1}{1\cdot 2}=\frac{1}{1+1}$

$\frac{1}{2}=\frac{1}{2} \Rightarrow$ Hence P_1 is true.

Assume that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$ is true

We need to prove that P_{k+1} is also true

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{(k+1)+1}=\frac{k+1}{k+2}$$

$$\begin{aligned}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}&=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}\\&=\frac{k(k+2)+1}{(k+1)(k+2)}\\&=\frac{k^2+2k+1}{(k+1)(k+2)}\\&=\frac{(k+1)^2}{(k+1)(k+2)}\\&=\frac{k+1}{k+2}\quad\checkmark\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ The statement $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}$ is true

Exercise

Prove that $1^2-2^2+3^2-\cdots+(-1)^{n-1}n^2=(-1)^{n-1}\frac{n(n+1)}{2}$ whenever *n* is a positive integer.

Solution

(1) For $\boldsymbol{n=1}$ $\Rightarrow$ $1^2=(-1)^0\frac{1(2)}{2}$ $\Rightarrow 1=1$; hence P_1 is true.

(2) Assume that $1^2-2^2+3^2-\cdots+(-1)^{k-1}k^2=(-1)^{k-1}\frac{k(k+1)}{2}$ is true

We need to prove that $1^2-2^2+3^2-\cdots+(-1)^{k-1}k^2+(-1)^k(k+1)^2=(-1)^k\frac{(k+1)(k+2)}{2}$ is also true

$$\begin{aligned}1^2-2^2+3^2-\cdots+(-1)^{k-1}k^2+(-1)^k(k+1)^2&=(-1)^{k-1}\frac{k(k+1)}{2}+(-1)^k(k+1)^2\\&=(-1)^k(k+1)\left[(-1)^{-1}\tfrac{1}{2}k+(k+1)\right]\\&=(-1)^k(k+1)\left(-\tfrac{k}{2}+k+1\right)\\&=(-1)^k(k+1)\left(\tfrac{k}{2}+1\right)\\&=(-1)^k(k+1)\left(\tfrac{k+2}{2}\right)\quad\checkmark\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ The statement $1^2-2^2+3^2-\cdots+(-1)^{n-1}n^2=(-1)^{n-1}\frac{n(n+1)}{2}$ is true

Exercise

Prove that for very positive integer n $\qquad \sum_{k=1}^{n} k2^{k} = (n-1)2^{n+1} + 2$

Solution

For $\boldsymbol{n = 1} \Rightarrow 1 \cdot 2^{1} = (1-1)^{0} 2^{2} + 2$

$2 = 2$; Hence P_1 is true

Assume that $\sum_{k=1}^{n} k \cdot 2^{k} = (n-1)2^{n+1} + 2$ is true

We need to prove that $\sum_{k=1}^{n+1} k \cdot 2^{k} = n \cdot 2^{n+2} + 2$ is also true

$$\begin{aligned}
\sum_{k=1}^{n+1} k \cdot 2^{k} &= \sum_{k=1}^{n} k \cdot 2^{k} + (n+1) \cdot 2^{n+1} \\
&= (n-1) \cdot 2^{n+1} + 2 + (n+1) \cdot 2^{n+1} \\
&= (n-1+n+1) \cdot 2^{n+1} + 2 \\
&= 2n \cdot 2^{n+1} + 2 \\
&= n \cdot 2^{n+2} + 2 \quad \checkmark
\end{aligned}$$

$\therefore$ The statement $\sum_{k=1}^{n} k \cdot 2^{k} = (n-1)2^{n+1} + 2$ is true

Exercise

Prove that for very positive integer n $\quad 1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{1}{3} n(n+1)(n+2)$.

Solution

For $\boldsymbol{n = 1} \Rightarrow 1 \cdot 2 = \frac{1}{3} 1(1+1)(1+2)$

$2 = \frac{1}{3}(2)(3) = 2$; Hence P_1 is true

Assume that $1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) = \frac{1}{3} k(k+1)(k+2)$ is true

We need to prove that $1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) = \frac{1}{3}(k+1)(k+2)(k+3)$ is also true

$1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) = \frac{1}{3} k(k+1)(k+2) + (k+1)(k+2)$

$$= (k+1)(k+2)\left(\frac{1}{3}k+1\right)$$

$$= (k+1)(k+2)\left(\frac{k+3}{3}\right)$$

$$= \frac{1}{3}(k+1)(k+2)(k+3) \checkmark$$

$\therefore$ The statement $1\cdot 2+2\cdot 3+\cdots+n(n+1)=\frac{1}{3}n(n+1)(n+2)$ is true

Exercise

Prove that for very positive integer n $\quad 1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+n(n+1)(n+2)=\frac{1}{4}n(n+1)(n+2)(n+3)$

Solution

For $\boldsymbol{n=1} \Rightarrow 1\cdot 2\cdot 3=\frac{1}{4}1(1+1)(1+2)(1+3)$

$2=\frac{1}{3}(2)(3)=2$; Hence P_1 is true

Assume that $1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+k(k+1)(k+2)=\frac{1}{4}k(k+1)(k+2)(k+3)$ is true

$$1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+k(k+1)(k+2)+(k+1)(k+2)(k+3)=\frac{1}{4}(k+1)(k+2)(k+3)(k+4)$$

$$1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+k(k+1)(k+2)+(k+1)(k+2)(k+3)$$

$$=\frac{1}{4}k(k+1)(k+2)(k+3)+(k+1)(k+2)(k+3)$$

$$=\frac{1}{4}(k+1)(k+2)(k+3)[k+4] \checkmark$$

$\therefore$ The statement $1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+n(n+1)(n+2)=\frac{1}{4}n(n+1)(n+2)(n+3)$ is true

Exercise

Let $P(n)$ be the statement that $1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}<2-\frac{1}{n}$ where n is an integer greater than 1.

a) Show is the statement $P(2)$?

b) Show that $P(2)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step.

f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

Solution

a) $P(2)$: $\quad 1+\frac{1}{4}<2-\frac{1}{2}$

b) $1+\frac{1}{4}<2-\frac{1}{2}$

$\frac{5}{4} < \frac{3}{2}$

$10 < 12$ $\checkmark$

Exercise

Prove that $3^n < n!$ if n is an integer greater than 6.

Solution

For $\boldsymbol{n = 7} \Rightarrow 3^7 < 7! \quad \Rightarrow \quad 2187 < 5040$; Hence P_7 is true

Assume that $3^k < k!$ is true, we need to prove that $3^{k+1} < (k+1)!$

$$\begin{aligned} 3^{k+1} &= 3^k 3 \\ &< k! \cdot 3 \qquad \text{Since } k > 6 \ \Rightarrow \ 6 < k \ \rightarrow \ 3 < k+1 \\ &< k! \cdot (k+1) \\ &= (k+1)! \ \checkmark \end{aligned}$$

$\therefore$ The statement $3^n < n!$ is true

Exercise

Prove that for every positive integer n: $\quad 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1} - 1\right)$

Solution

For $\boldsymbol{n = 1} \Rightarrow 1 > 2\left(\sqrt{1+1} - 1\right) \quad \Rightarrow \quad 1 > 2\left(\sqrt{2} - 1\right) \approx 0.828$; Hence P_1 is true

Assume that $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} > 2\left(\sqrt{k+1} - 1\right)$ is true.

We need to prove that $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > 2\left(\sqrt{(k+1)+1} - 1\right) = 2\left(\sqrt{k+2} - 1\right)$

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > 2\left(\sqrt{k+1} - 1\right) + \frac{1}{\sqrt{k+1}}$$

$$2\left(\sqrt{k+1} - 1\right) + \frac{1}{\sqrt{k+1}} > 2\left(\sqrt{k+2} - 1\right)$$

$$2\sqrt{k+1} - 2 + \frac{1}{\sqrt{k+1}} > 2\sqrt{k+2} - 2$$

$$2\sqrt{k+1} + \frac{1}{\sqrt{k+1}} > 2\sqrt{k+2}$$

$$\frac{1}{\sqrt{k+1}} > 2\sqrt{k+2} - 2\sqrt{k+1}$$

$$\frac{1}{\sqrt{k+1}} > 2\left(\sqrt{k+2} - \sqrt{k+1}\right)$$

$$\left(\sqrt{k+2}+\sqrt{k+1}\right)\frac{1}{\sqrt{k+1}} > 2\left(\sqrt{k+2}-\sqrt{k+1}\right)\left(\sqrt{k+2}+\sqrt{k+1}\right)$$

$$\frac{\sqrt{k+2}}{\sqrt{k+1}}+1 > 2(k+2-k-1)$$

$$\frac{\sqrt{k+2}}{\sqrt{k+1}}+1 > 2$$

Which is clearly true since $\frac{\sqrt{k+2}}{\sqrt{k+1}} > 1$

Exercise

Use mathematical induction to prove that 2 divides n^2+n whenever n is a positive integer.

Solution

For $\boldsymbol{n=1} \Rightarrow 1^2+1=2$ since 2 divides 2; Hence P_1 is true

Assume that 2 divides k^2+k is true, we need to prove that 2 divides $(k+1)^2+(k+1)$ is true

$$\begin{aligned}(k+1)^2+(k+1) &= k^2+2k+1+k+1\\ &= k^2+k+2k+2\\ &= k^2+k+2(k+1) \quad \checkmark\end{aligned}$$

2 divides k^2+k and certainly 2 divides $2(k+1)$, so 2 divides their sum.

$\therefore$ The statement 2 divides n^2+n is true

Exercise

Use mathematical induction to prove that 3 divides n^3+2n whenever n is a positive integer.

Solution

For $\boldsymbol{n=1} \Rightarrow 1^3+2(1)=3$ since 3 divides 3; Hence P_1 is true

Assume that 3 divides k^3+2k is true.

We need to prove that 3 divides $(k+1)^3+2(k+1)$ is also true

$$\begin{aligned}(k+1)^3+2(k+1) &= k^3+3k^2+3k+1+2k+2\\ &= k^3+2k+3k^2+3k+3\\ &= k^3+2k+3\left(k^2+k+1\right) \quad \checkmark\end{aligned}$$

By the inductive hypothesis, 3 divides k^3+2k and certainly 3 divides $3\left(k^2+k+1\right)$, so 3 divides their sum.

$\therefore$ The statement 3 divides n^3+2n is true

Exercise

Use mathematical induction to prove that 5 divides $n^5 - n$ whenever n is a positive integer.

Solution

For $\boldsymbol{n = 1} \Rightarrow 1^5 - 1 = 0$, which is divisible by 5; Hence P_1 is true

Assume that 5 divides $k^5 - k$ is true.

We need to prove that 5 divides $(k+1)^5 - (k+1)$ is also true

$$\begin{aligned}(k+1)^5 - (k+1) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 \\ &= k^5 - k + 5k^4 + 10k^3 + 10k^2 + 5k \\ &= k^5 - k + 5\left(k^4 + 2k^3 + 2k^2 + k\right) \qquad \checkmark\end{aligned}$$

By the inductive hypothesis, 5 divides $k^5 - k$ and certainly 5 divides $5\left(k^4 + 2k^3 + 2k^2 + k\right)$, so 5 divides their sum.

$\therefore$ The statement 5 divides $n^5 - n$ is true

Exercise

Use mathematical induction to prove that $n^2 - 1$ is divisible by 8 whenever n is an odd positive integer.

Solution

For $\boldsymbol{n = 1} \Rightarrow 1^2 - 1 = 0$, which is divisible by 8; Hence P_1 is true

Assume that 8 divides $k^2 - 1$ is true, than implies to $k^2 - 1 = 8p$.

We need to prove that 8 divides $(k+1)^2 - 1$ is also true

$$\begin{aligned}(k+1)^2 - 1 &= k^2 + 2k + 1 - 1 \\ &= \left(k^2 - 1\right) + 2k + 1\end{aligned}$$

By the inductive hypothesis, 8 divides $k^2 - 1$ and certainly 8 divides $2k + 1$, so 8 divides their sum.

$\therefore$ The statement 8 divides $n^2 - 1$ is true

Exercise

Use mathematical induction to prove that 21 divides $4^{n+1}+5^{2n-1}$ whenever *n* is a positive integer.

Solution

For $\boldsymbol{n=1} \Rightarrow 4^2+5^1=21$, which is divisible by 21; Hence P_1 is true.

Assume that 21 divides $4^{k+1}+5^{2k-1}$ is true.

We need to prove that 21 divides $4^{(k+1)+1}+5^{2(k+1)-1}$ is also true

$$\begin{aligned}
4^{(k+1)+1}+5^{2(k+1)-1} &= 4\cdot 4^{(k+1)}+5^{2k+2-1} \\
&= 4\cdot 4^{(k+1)}+5^2\cdot 5^{2k-1} \\
&= 4\cdot 4^{(k+1)}+25\cdot 5^{2k-1} \\
&= 4\cdot 4^{(k+1)}+(4+21)\cdot 5^{2k-1} \\
&= 4\cdot 4^{(k+1)}+4\cdot 5^{2k-1}+21\cdot 5^{2k-1} \\
&= 4\cdot\left(4^{(k+1)}+5^{2k-1}\right)+21\cdot 5^{2k-1}
\end{aligned}$$

By the inductive hypothesis, 21 divides $4^{k+1}+5^{2k-1}$ and certainly 21 divides 5^{2k-1}, so 21 divides their sum.

$\therefore$ The statement 21 divides $4^{n+1}+5^{2n-1}$ is true

Exercise

Prove that the statement is true for every positive integer *n*. $1+2.2+3.2^2+\ldots+n.2^{n-1}=1+(n-1).2^n$

Solution

(2) For $\boldsymbol{n=1} \Rightarrow 1\overset{?}{=}1+(1-1)2^1=1-0=1$; hence P_1 is true.

(3) $1+2.2+3.2^2+\ldots+k.2^{k-1}=1+(k-1).2^k$ is true

$$1+2.2+3.2^2+\ldots+k.2^{k-1}+(k+1).2^{(k+1)-1}=1+((k+1)-1).2^{k+1}\ ?$$

$$\begin{aligned}
1+2.2+3.2^2+\ldots+k.2^{k-1}+(k+1).2^{(k+1)-1} &= 1+(k-1).2^k+(k+1).2^{k+1-1} \\
&= 1+k.2^k-1.2^k+(k+1).2^k \\
&= 1+k.2^k-1.2^k+k.2^k+1.2^k \\
&= 1+2^1k.2^k \\
&= 1+(k+0).2^{k+1} \\
&= 1+((k+1)-1).2^{k+1} \quad \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

The statement $1+2.2+3.2^2+\ldots+n.2^{n-1}=1+(n-1).2^n$ is true

Exercise

Prove that the statement is true for every positive integer n. $1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow 1^2 \stackrel{?}{=} \frac{1(1+1)(2(1)+1)}{6}=\frac{1(2)(3)}{6}=\frac{6}{6}=1\ \surd$; hence P_1 is true.

(2) $1^2+2^2+3^2+\ldots+k^2=\frac{k(k+1)(2k+1)}{6}$ is true

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}?$$

$$\begin{aligned}
1^2+2^2+3^2+\ldots+k^2+(k+1)^2 &= \frac{k(k+1)(2k+1)}{6}+(k+1)^2 \\
&= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} \\
&= \frac{(k+1)[k(2k+1)+6(k+1)]}{6} \\
&= \frac{(k+1)\left[2k^2+k+6k+6\right]}{6} \\
&= \frac{(k+1)\left[2k^2+7k+6\right]}{6} \\
&= \frac{(k+1)((k+2)(2k+3))}{6} \\
&= \frac{(k+1)((k+1+1)(2k+2+1))}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}\ \surd
\end{aligned}$$

Hence P_{k+1} is true.

The statement $1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$ is true

Exercise

Prove that the statement is true for every positive integer n. $\frac{1}{1\bullet 2}+\frac{1}{2\bullet 3}+\frac{1}{3\bullet 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow \frac{1}{1.2}\overset{?}{=}\frac{1}{1+1}=\frac{1}{2}=\frac{1}{1.2}$ $\surd$; hence P_1 is true.

(2) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$ is true

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{(k+1)+1}?$$

$$\begin{aligned}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)} &= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2)+1}{(k+1)(k+2)} \\ &= \frac{k^2+2k+1}{(k+1)(k+2)} \\ &= \frac{(k+1)(k+1)}{(k+1)(k+2)} \\ &= \frac{k+1}{(k+1+1)} \\ &= \frac{k+1}{(k+1)+1} \quad \surd\end{aligned}$$

Hence P_{k+1} is true.

The statement $\frac{1}{1\bullet 2}+\frac{1}{2\bullet 3}+\frac{1}{3\bullet 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$ is true

Exercise

Prove that the statement is true: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow \frac{1}{2}\overset{?}{=}1-\frac{1}{2}=\frac{1}{2}$ $\surd$; P_1 is true.

(2) $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}=1-\frac{1}{2^k}$ is true

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^{k+1}}?$$

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^k}+\frac{1}{2^{k+1}}$$

$$=1-\frac{1}{2^k}+\frac{1}{2^k\cdot 2}$$

$$=\frac{2^{k+1}-2+1}{2^{k+1}}$$

$$=\frac{2^{k+1}-1}{2^{k+1}}$$

$$=\frac{2^{k+1}}{2^{k+1}}-\frac{1}{2^{k+1}}$$

$$=1-\frac{1}{2^{k+1}} \quad \surd$$

Hence P_{k+1} is true.

The statement $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$ is true

Exercise

Prove that the statement is true: $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n-2)\cdot(3n+1)}=\frac{n}{3n+1}$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow \frac{1}{1\cdot 4}\overset{?}{=}\frac{1}{3(1)+1}=\frac{1}{4}$ $\surd$; P_1 is true.

(2) $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}$ is true

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)}\overset{?}{=}\frac{k+1}{3(k+1)+1}$$

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)}=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$=\frac{(3k+1)(k+1)}{(3k+1)(3k+3+1)}$$

$$=\frac{k+1}{3(k+1)+1} \quad \surd$$

Hence P_{k+1} is true.

The statement $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n-2)\cdot(3n+1)}=\frac{n}{3n+1}$ is true

Exercise

Prove that the statement is true: $\frac{4}{5}+\frac{4}{5^2}+\frac{4}{5^3}+\cdots+\frac{4}{5^n}=1-\frac{1}{5^n}$

Solution

(1) For $\boldsymbol{n}=\mathbf{1} \Rightarrow \frac{4}{5}\overset{?}{=}1-\frac{1}{5}=\frac{4}{5}$ $\checkmark$; P_1 is true.

(2) $\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}=1-\frac{1}{5^k}$ is true

$$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}+\frac{4}{5^{k+1}}\overset{?}{=}1-\frac{1}{5^{k+1}}$$

$$\begin{aligned}\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}+\frac{4}{5^{k+1}} &= 1-\frac{1}{5^k}+\frac{4}{5^{k+1}}\\ &= 1-\left(\frac{1}{5^k}-\frac{4}{5^{k+1}}\right)\\ &= 1-\frac{5-4}{5^{k+1}}\\ &= 1-\frac{1}{5^{k+1}} \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

The statement $\frac{4}{5}+\frac{4}{5^2}+\frac{4}{5^3}+\cdots+\frac{4}{5^n}=1-\frac{1}{5^n}$ is true

Exercise

Prove that the statement is true: $1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$

Solution

(1) For $\boldsymbol{n}=\mathbf{1} \Rightarrow 1^3\overset{?}{=}\frac{1^2(1+1)^2}{4}=\frac{4}{4}=1$ $\checkmark$; P_1 is true.

(2) $\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}=1-\frac{1}{5^k}$ is true

$$1^3+2^3+\cdots+k^3+(k+1)^3\overset{?}{=}\frac{(k+1)^2\left((k+1)+1\right)^2}{4}$$

$$\begin{aligned}1^3+2^3+\cdots+k^3+(k+1)^3 &= \frac{k^2(k+1)^2}{4}+(k+1)^3\\ &= \frac{k^2(k+1)^2+4(k+1)^3}{4}\\ &= \frac{(k+1)^2\left[k^2+4(k+1)\right]}{4}\end{aligned}$$

$$=\frac{(k+1)^2\left(k^2+4k+4\right)}{4}$$

$$=\frac{(k+1)^2(k+2)^2}{4}$$

$$=\frac{(k+1)^2((k+1)+1)^2}{4} \quad \checkmark$$

Hence P_{k+1} is true.

The statement $1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$ is true

Exercise

Prove that the statement is true for every positive integer n. $3+3^2+3^3+\ldots+3^n=\frac{3}{2}\left(3^n-1\right)$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 3\overset{?}{=}\frac{3}{2}\left(3^1-1\right)=\frac{3}{2}2=3 \quad \checkmark$; P_1 is true.

(2) $3+3^2+\cdots+3^k=\frac{3}{2}\left(3^k-1\right)$ is true

$$3+3^2+\cdots+3^k+3^{k+1}\overset{?}{=}\frac{3}{2}\left(3^{k+1}-1\right)$$

$$3+3^2+\cdots+3^k+3^{k+1}=\frac{3}{2}\left(3^k-1\right)+3^{k+1}$$

$$=\frac{1}{2}3^{k+1}-\frac{3}{2}+3^{k+1}$$

$$=\frac{3}{2}3^{k+1}-\frac{3}{2}$$

$$=\frac{3}{2}\left(3^{k+1}-1\right) \quad \checkmark$$

Hence P_{k+1} is true.

The statement $3+3^2+3^3+\ldots+3^n=\frac{3}{2}\left(3^n-1\right)$ is true

Exercise

Prove that the statement is true: $\qquad x^{2n}+x^{2n-1}y+\cdots+xy^{2n-1}+y^{2n}=\dfrac{x^{2n+1}-y^{2n+1}}{x-y}$

Solution

(1) For $\boldsymbol{n}=\mathbf{1} \Rightarrow\ x^2+xy+y^2\overset{?}{=}\dfrac{x^3-y^3}{x-y}$

$$=\frac{(x-y)\left(x^2+xy+y^2\right)}{x-y}$$

$$=x^2+xy+y^2 \qquad \checkmark$$

; P_1 is true.

(2) $x^{2k}+x^{2k-1}y+\cdots+xy^{2k-1}+y^{2k}=\dfrac{x^{2k+1}-y^{2k+1}}{x-y}$ is true

$$x^{2(k+1)}+x^{2(k+1)-1}y+\cdots+xy^{2(k+1)-1}+y^{2(k+1)}\overset{?}{=}\frac{x^{2(k+1)+1}-y^{2(k+1)+1}}{x-y}$$

$$x^{2k+2}+x^{2k+1}y+\cdots+xy^{2k+1}+y^{2k+2}=x^2\left(x^{2k}+x^{2k-1}y+\cdots+y^{2k}\right)+xy^{2k+1}+y^{2k+2}$$

$$=x^2\left(\frac{x^{2k+1}-y^{2k+1}}{x-y}\right)+xy^{2k+1}+y^{2k+2}$$

$$=\frac{x^{2k+3}-x^2y^{2k+1}+x^2y^{2k+1}+xy^{2k+2}-xy^{2k+2}-y^{2(k+1)+1}}{x-y}$$

$$=\frac{x^{2(k+1)+1}-y^{2(k+1)+1}}{x-y} \qquad \checkmark$$

Hence P_{k+1} is true.

The statement $x^{2n}+x^{2n-1}y+\cdots+xy^{2n-1}+y^{2n}=\dfrac{x^{2n+1}-y^{2n+1}}{x-y}$ is true

Exercise

Prove that the statement is true: $5\cdot 6+5\cdot 6^2+5\cdot 6^3+\cdots+5\cdot 6^n=6\left(6^n-1\right)$

Solution

(1) For $\boldsymbol{n}=\mathbf{1} \Rightarrow 5\cdot 6\overset{?}{=}6\left(6^1-1\right)=6(5)$ $\checkmark$; P_1 is true.

(2) $5\cdot 6+5\cdot 6^2+\cdots+5\cdot 6^k=6\left(6^k-1\right)$ is true

$$5\cdot 6+5\cdot 6^2+\cdots+5\cdot 6^k+5\cdot 6^{k+1}\overset{?}{=}6\left(6^{k+1}-1\right)$$

$$\begin{aligned}5\cdot 6+5\cdot 6^2+\cdots+5\cdot 6^k+5\cdot 6^{k+1}&=6\left(6^k-1\right)+5\cdot 6^{k+1}\\&=6^{k+1}-6+5\cdot 6^{k+1}\\&=6^{k+1}(1+5)-6\\&=6\cdot 6^{k+1}-6\\&=6\left(6^{k+1}-1\right)\quad\checkmark\end{aligned}$$

Hence P_{k+1} is true.

The statement $5\cdot 6+5\cdot 6^2+5\cdot 6^3+\cdots+5\cdot 6^n=6\left(6^n-1\right)$ is true

Exercise

Prove that the statement is true: $7\cdot 8+7\cdot 8^2+7\cdot 8^3+\cdots+7\cdot 8^n=8\left(8^n-1\right)$

Solution

(1) For $\boldsymbol{n}=\mathbf{1} \Rightarrow 7\cdot 8\overset{?}{=}8\left(8^1-1\right)=8(7)$ $\checkmark$; P_1 is true.

(2) $7\cdot 8+7\cdot 8^2+\cdots+7\cdot 8^k=8\left(8^k-1\right)$ is true

$$7\cdot 8+7\cdot 8^2+\cdots+7\cdot 8^k+7\cdot 8^{k+1}\overset{?}{=}8\left(8^{k+1}-1\right)$$

$$\begin{aligned}7\cdot 8+7\cdot 8^2+\cdots+7\cdot 8^k+7\cdot 8^{k+1}&=8\left(8^k-1\right)+7\cdot 8^{k+1}\\&=8^{k+1}-8+7\cdot 8^{k+1}\\&=8^{k+1}(1+7)-8\\&=8\left(8^{k+1}-1\right)\quad\checkmark\end{aligned}$$

Hence P_{k+1} is true.

The statement $7\cdot 8+7\cdot 8^2+7\cdot 8^3+\cdots+7\cdot 8^n=8\left(8^n-1\right)$ is true.

Exercise

Prove that the statement is true: $3+6+9+\cdots+3n=\dfrac{3n(n+1)}{2}$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 3 \overset{?}{=} \dfrac{3(1)(1+1)}{2}=3$ √; P_1 is true.

(2) $3+6+9+\cdots+3k=\dfrac{3k(k+1)}{2}$ is true

$$3+6+9+\cdots+3k+3(k+1) \overset{?}{=} \frac{3(k+1)(k+2)}{2}$$

$$\begin{aligned} 3+6+9+\cdots+3k+3(k+1) &= \frac{3k(k+1)}{2}+3(k+1) \\ &= \frac{3k(k+1)+6(k+1)}{2} \\ &= \frac{(k+1)(3k+6)}{2} \\ &= \frac{3(k+1)(k+2)}{2} \quad \surd \end{aligned}$$

Hence P_{k+1} is true.

The statement $3+6+9+\cdots+3n=\dfrac{3n(n+1)}{2}$ is true.

Exercise

Prove that the statement is true: $5+10+15+\cdots+5n=\dfrac{5n(n+1)}{2}$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 5 \overset{?}{=} \dfrac{5(1)(1+1)}{2}=5$ √; P_1 is true.

(2) $5+10+15+\cdots+5k=\dfrac{5k(k+1)}{2}$ is true

$$5+10+15+\cdots+5k+5(k+1) \overset{?}{=} \frac{5(k+1)(k+2)}{2}$$

$$\begin{aligned} 5+10+15+\cdots+5k+5(k+1) &= \frac{5k(k+1)}{2}+5(k+1) \\ &= \frac{5k(k+1)+10(k+1)}{2} \\ &= \frac{(k+1)(5k+10)}{2} \end{aligned}$$

$$=\frac{5(k+1)(k+2)}{2} \quad \checkmark$$

Hence P_{k+1} is true.

The statement $5+10+15+\cdots+5n=\frac{5n(n+1)}{2}$ is true.

Exercise

Prove that the statement is true: $\quad 1+3+5+\cdots+(2n-1)=n^2$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 1 \stackrel{?}{=} 1^2=1 \ \checkmark$; P_1 is true.

(2) $1+3+5+\cdots+(2k-1)=k^2$ is true

$$1+3+5+\cdots+(2k-1)+(2(k+1)-1) \stackrel{?}{=} (k+1)^2$$

$$\begin{aligned}1+3+5+\cdots+(2k-1)+(2(k+1)-1)&=k^2+2k+2-1\\&=k^2+2k+1\\&=(k+1)^2 \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

The statement $1+3+5+\cdots+(2n-1)=n^2$ is true.

Exercise

Prove that the statement is true: $\quad 4+7+10+\cdots+(3n+1)=\frac{n(3n+5)}{2}$

Solution

(1) For $\boldsymbol{n=1} \Rightarrow 4 \stackrel{?}{=} \frac{1(3+5)}{2}=4 \ \checkmark$; P_1 is true.

(2) $4+7+10+\cdots+(3k+1)=\frac{k(3k+5)}{2}$ is true

$$4+7+10+\cdots+(3k+1)+(3(k+1)+1) \stackrel{?}{=} \frac{(k+1)(3(k+1)+5)}{2}=\frac{(k+1)(3k+8)}{2}$$

$$\begin{aligned}4+7+10+\cdots+(3k+1)+(3k+4)&=\frac{k(3k+5)}{2}+3k+4\\&=\frac{3k^2+5k+6k+8}{2}\end{aligned}$$

$$= \frac{3k^2 + 5k + 3k + 3k + 8}{2}$$

$$= \frac{k(3k+8) + (3k+8)}{2}$$

$$= \frac{(3k+8)(k+1)}{2} \quad \surd$$

Hence P_{k+1} is true.

The statement $4 + 7 + 10 + \cdots + (3n+1) = \dfrac{n(3n+5)}{2}$ is true.

Exercise

Prove that the statement by mathematical induction: $\left(a^m\right)^n = a^{mn}$ (a and m are constant)

Solution

- For $\boldsymbol{n = 1} \Rightarrow \left(a^m\right)^1 \overset{?}{=} a^{m(1)} \rightarrow a^m = a^m \ \surd$; P_1 is true.

- $\left(a^m\right)^k = a^{mk}$ is true

  $$\left(a^m\right)^{(k+1)} \overset{?}{=} a^{m(k+1)}$$

  $$\begin{aligned}\left(a^m\right)^{(k+1)} &= \left(a^m\right)^k a^m \\ &= a^{km} a^m \\ &= a^{km+m} \\ &= a^{m(k+1)} \quad \surd\end{aligned}$$

  Hence P_{k+1} is true.

The statement $\left(a^m\right)^n = a^{mn}$ is true.

Exercise

Prove that the statement is true for every positive integer n. $n < 2^n$

Solution

Step **1**. For $\boldsymbol{n = 1} \Rightarrow 1 \overset{?}{<} 2^1$ ✓ $\Rightarrow P_1$ is true.

Step **2**. Assume that P_k is true $k < 2^k$

We need to prove that P_{k+1} is true, that is $k+1 < 2^{k+1}$

$$\begin{aligned} k+1 &< k+k = 2k \\ &< 2 \cdot 2^k \\ &= 2^{k+1} \quad ✓ \end{aligned}$$

Thus, P_{k+1} is true.

The statement $n < 2^n$ is true.

Exercise

Prove that the statement is true for every positive integer n. 3 is a factor of $n^3 - n + 3$

Solution

- For $\boldsymbol{n = 1} \Rightarrow 1^3 - 1 + 3 = 3 = 3(1)$ ✓ $\Rightarrow P_1$ is true.
- Assume that P_k is true 3 is a factor of $k^3 - k + 3$

We need to prove that P_{k+1} is true, that is $(k+1)^3 - (k+1) + 3$

$$\begin{aligned} (k+1)^3 - (k+1) + 3 &= k^3 + 3k^2 + 3k + 1 - k - 1 + 3 \\ &= \left(k^3 - k + 3\right) + 3k^2 + 3k \qquad\qquad k^3 - k + 3 = 3K \\ &= 3K + 3k^2 + 3k \\ &= 3\left(K + k^2 + k\right) \quad ✓ \end{aligned}$$

Thus, P_{k+1} is true.

The statement $n^3 - n + 3$ is true.

Exercise

Prove that the statement is true for every positive integer n. 4 is a factor of $5^n - 1$

Solution

- For $\boldsymbol{n=1} \Rightarrow 5^1 - 1 = 4 = 4(1)$ $\checkmark$ $\Rightarrow P_1$ is true.
- Assume that P_k is true 4 is a factor of $5^k - 1$

  We need to prove that P_{k+1} is true, that is $5^{k+1} - 1$

$$\begin{aligned} 5^{k+1} - 1 &= 5^k 5^1 - 5 + 4 \\ &= 5\left(5^k - 1\right) + 4 \\ &= 5\left(5^k - 1\right) + 4 \end{aligned}$$

By the induction hypothesis, 4 is a factor of $5^k - 1$ and 4 is a factor of 4, so 4 is a factor of the $(k+1)$ term. $\checkmark$

Thus, P_{k+1} is true.

The statement $5^n - 1$ is true.

Exercise

Prove that the statement by mathematical induction: $2^n > 2n \quad if \quad n \ge 3$

Solution

- For $\boldsymbol{n=3} \Rightarrow 2^3 \ge 2(3) \quad \Rightarrow \quad 8 \ge 6$ $\checkmark$ $\Rightarrow P_3$ is true.
- Assume that P_k is true: $2^k > 2k$; we need to prove that P_{k+1} : $2^{k+1} > 2(k+1)$ is true

$$\begin{aligned} 2^k &> 2k \\ 2^k \cdot 2 &> 2k \cdot 2 \\ 2^{k+1} &> 4k = 2k + 2k \qquad k \ge 3 \\ &> 2k + 2 \\ &= 2(k+1) \checkmark \end{aligned}$$

Thus, P_{k+1} is true.

The statement $2^n > 2n \quad if \quad n \ge 3$ is true.

Exercise

Prove that the statement by mathematical induction: If $0 < a < 1$, then $a^n < a^{n-1}$

Solution

- For $\boldsymbol{n} = \mathbf{1}$

$$a^1 < a^{1-1} \quad \Rightarrow \quad a < 1 \checkmark$$

since $0 < a < 1 \Rightarrow P_1$ is true.

- Assume that P_k is true: $a^k < a^{k-1}$; we need to prove that $P_{k+1}: a^{k+1} < a^k$ is true

$$a^k < a^{k-1} \quad \rightarrow \quad a^k \cdot a < a^{k-1} \cdot a$$

$$a^{k+1} < a^k \quad \checkmark$$

Thus, P_{k+1} is true.

The statement $a^n < a^{n-1}$ is true.

Exercise

Prove that the statement by mathematical induction: If $n \ge 4$, then $n! > 2^n$

Solution

- For $\boldsymbol{n} = \mathbf{4}$

$$4! > 2^4 \quad \Rightarrow \quad 24 > 16 \checkmark$$

$\Rightarrow P_4$ is true.

- Assume that P_k is true: $k! > 2^k$; we need to prove that $P_{k+1}: (k+1)! > 2^{k+1}$ is true

$$\begin{aligned}(k+1)! &= k! \cdot (k+1) \\ &> 2^k (k+1) \qquad \qquad \textit{Since} \quad k \ge 4 \Rightarrow k+1 > 2 \\ &> 2^k \cdot 2 \\ &= 2^{k+1} \quad \checkmark\end{aligned}$$

Thus, P_{k+1} is true.

The statement $n! > 2^n$ is true.

Exercise

Prove that the statement by mathematical induction: $3^n > 2n+1 \quad \textit{if} \quad n \ge 2$

Solution

- For $\boldsymbol{n} = \mathbf{2}$

$$3^2 > 2(2)+1 \quad \Rightarrow \quad 9 > 5 \checkmark$$

$\Rightarrow P_2$ is true.

- Assume that P_k is true: $3^k > 2k+1$;

  We need to prove that $P_{k+1}: 3^{k+1} > 2(k+1)+1$ is true

  $$\begin{aligned} 3^k > 2k+1 \quad \Rightarrow \quad 3^k \cdot 3 &> (2k+1)\cdot 3 \\ 3^{k+1} &> 6k+3 \\ &> 2k+2+1 \\ &= 2(k+1)+1 \ \checkmark \end{aligned}$$

  Thus, P_{k+1} is true.

The statement $3^n > 2n+1 \quad if \quad n \ge 2$ is true.

Exercise

Prove that the statement by mathematical induction: $2^n > n^2 \quad for \quad n > 4$

Solution

- For $\boldsymbol{n} = 5$

  $2^5 > 5^2 \quad \Rightarrow \quad 32 > 25 \ \checkmark$

  $\Rightarrow P_5$ is true.

- Assume that P_k is true: $2^k > k^2$

  Wwe need to prove that $P_{k+1}: 2^{k+1} > (k+1)^2$ is true

  $$\begin{aligned} 2^k > k^2 \quad \Rightarrow \quad 2^k \cdot 2 &> k^2 \cdot 2 \\ 2^{k+1} &> 2k^2 \\ &= k^2 + k^2 \qquad\qquad k < k+1 \ \Rightarrow \ k\cdot k > k+k+1 \Rightarrow \ k^2 > 2k+1 \\ &> k^2 + 2k+1 \\ &= (k+1)^2 \ \checkmark \end{aligned}$$

  Thus, P_{k+1} is true

The statement $2^n > n^2 \quad for \quad n > 4$ is true.

Exercise

Prove that the statement by mathematical induction: $4^n > n^4 \quad for \quad n \ge 5$

Solution

- For $\boldsymbol{n} = 5$

  $4^5 > 5^4 \quad \Rightarrow \quad 1024 > 625 \ \checkmark$

$\Rightarrow P_5$ is true.

- Assume that P_k is true: $4^k > k^4$

We need to prove that $P_{k+1}: 4^{k+1} > (k+1)^4$ is true

$4^k > k^4 \quad \Rightarrow \quad 4^k \cdot 4 > k^4 \cdot 4$

$4^{k+1} > 4k^4 \qquad\qquad k < k+1 \quad \Rightarrow \quad 4k > k+1 \ \rightarrow\ 4k^4 > (k+1)^4$

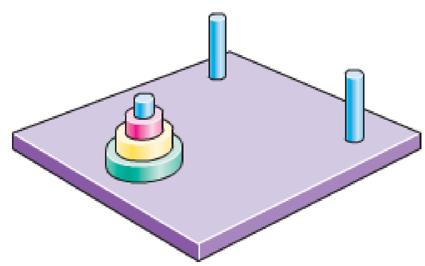
$> (k+1)^4 \ \checkmark \qquad$ Thus, P_{k+1} is true

The statement $4^n > n^4 \quad for \quad n \ge 5$ is true.

Exercise

A pile of *n* rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must be moved, one at a time, to a different peg with no ring ever placed on top of a smaller ring. Find the least number of moves that would be required. Prove your result by mathematical induction.

Solution



With 1 ring, 1 move is required.

With 2 rings, 3 moves are required $\Rightarrow \quad 3 = 2+1$

With 3 rings, 7 moves are required $\Rightarrow \quad 7 = 2^2+2+1$

With *n* rings, $2^{n-1}+\cdots+2^2+2^1+2^0 = 2^n-1$ moves are required

- For $\boldsymbol{n=1} \Rightarrow 2^0 = 2^1-1 = 1 \ \checkmark \quad \Rightarrow P_1$ is true.
- Assume that P_k is true: $2^{k-1}+\cdots+2^2+2^1+2^0 = 2^k-1$

$$2^k+2^{k-1}+\cdots+2^2+2^1+1 \overset{?}{=} 2^{k+1}-1$$

$$\begin{aligned} 2^k+2^{k-1}+\cdots+2^2+2^1+1 &= 2^k+2^k-1 \\ &= 2\cdot 2^k-1 \\ &= 2^{k+1}-1 \quad \checkmark \end{aligned}$$

Exercise

Find $f(1), f(2), f(3)$, and $f(4)$ if $f(n)$ is defined recursively by $f(0)=1$ and for $n=0,\ 1,\ 2,\ldots$

a) $f(n+1)=f(n)+2$ *b)* $f(n+1)=3f(n)$

c) $f(n+1)=2^{f(n)}$ *d)* $f(n+1)=f(n)^2+f(n)+1$

Solution

a) $f(1)=f(0)+2$

$=1+2$

$=3$

$f(2)=f(1)+2$

$=3+2$

$=5$

$f(3)=f(2)+2$

$=5+2$

$=7$

$f(4)=f(3)+2$

$=7+2$

$=9$

b) $f(n+1)=3f(n)$

$f(1)=3\cdot f(0)$

$=3(1)$

$=3$

$f(2)=3\cdot f(1)$

$=3(3)$

$=9$

$f(3)=3\cdot f(2)$

$=3(9)$

$=27$

$f(4)=3\cdot f(3)$

$=3(27)$

$=81$

c) $f(n+1) = 2^{f(n)}$

$f(1) = 2^{f(0)}$

$= 2^1$

$\underline{= 2}$

$f(2) = 2^{f(1)}$

$= 2^2$

$\underline{= 4}$

$f(3) = 2^{f(2)}$

$= 2^4$

$\underline{= 16}$

$f(4) = 2^{f(3)}$

$= 2^{16}$

$\underline{= 65536}$

d) $f(n+1) = f(n)^2 + f(n) + 1$

$f(1) = f(0)^2 + f(0) + 1$

$= 1^2 + 1 + 1$

$\underline{= 3}$

$f(2) = f(1)^2 + f(1) + 1$

$= 3^2 + 3 + 1$

$\underline{= 13}$

$f(3) = f(2)^2 + f(2) + 1$

$= 13^2 + 13 + 1$

$\underline{= 183}$

$f(4) = f(3)^2 + f(3) + 1$

$= 183^2 + 183 + 1$

$\underline{= 33673}$

Exercise

Find $f(1), f(2), f(3)$, $f(4)$ and $f(5)$ if $f(n)$ is defined recursively by $f(0)=3$ and for $n=0,\ 1,\ 2,\ldots$

a) $f(n+1)=-2f(n)$

b) $f(n+1)=3f(n)+7$

c) $f(n+1)=3^{f(n)/3}$

d) $f(n+1)=f(n)^2-2f(n)-2$

Solution

a) $f(n+1)=-2f(n)$

$$f(1)=-2f(0)$$
$$=-2(3)$$
$$=-6$$

$$f(2)=-2f(1)$$
$$=-2(-6)$$
$$=12$$

$$f(3)=-2f(2)$$
$$=-2(12)$$
$$=-24$$

$$f(4)=-2f(3)$$
$$=-2(-24)$$
$$=48$$

$$f(5)=-2f(4)$$
$$=-2(48)$$
$$=-96$$

b) $f(1)=3\cdot f(0)+7$

$$=3(3)+7$$
$$=16$$

$$f(2)=3\cdot f(1)+7$$
$$=3(16)+7$$
$$=55$$

$$f(3)=3\cdot f(2)+7$$
$$=3(55)+7$$
$$=172$$

$f(4) = 3 \cdot f(3) + 7$
$= 3(172) + 7$
$\underline{= 523}$

$f(5) = 3 \cdot f(4) + 7$
$= 3(523) + 7$
$\underline{= 1576}$

c) $f(1) = 3^{f(0)/3}$
$= 3^{3/3}$
$= 3^1$
$\underline{= 3}$

$f(2) = 3^{f(1)/3}$
$= 3^{3/3}$
$= 3^1$
$\underline{= 3}$

$f(3) = 3^{f(2)/3}$
$= 3^{3/3}$
$= 3^1$
$\underline{= 3}$

$f(4) = 3^{f(3)/3}$
$= 3^{3/3}$
$= 3^1$
$\underline{= 3}$

$f(5) = 3^{f(3)/3}$
$= 3^{3/3}$
$= 3^1$
$\underline{= 3}$

d) $f(1) = f(0)^2 - 2f(0) - 2$
$= 3^2 - 2(3) - 2$
$\underline{= 1}$

$f(2) = f(1)^2 - 2f(1) - 2$

$$=1^2-2(1)-2$$
$$\underline{=-3}$$

$$f(3)=f(2)^2-2f(2)-2$$
$$=(-3)^2-2(-3)-2$$
$$\underline{=13}$$

$$f(4)=f(3)^2-2f(3)-2$$
$$=(13)^2-2(13)-2$$
$$\underline{=141}$$

$$f(5)=f(4)^2-2f(4)-2$$
$$=(141)^2-2(141)-2$$
$$\underline{=19,597}$$

Exercise

Find $f(2), f(3), f(4)$ and $f(5)$ if $f(n)$ is defined recursively by $f(0)=f(1)=1$ and for $n=1, 2, \dots$

a) $f(n+1)=f(n)-f(n-1)$

b) $f(n+1)=f(n)f(n-1)$

c) $f(n+1)=f(n)^2+f(n-1)^3$

d) $f(n+1)=f(n)/f(n-1)$

Solution

a) $f(2)=f(1)-f(0)$
$$=1-1$$
$$\underline{=0}$$

$$f(3)=f(2)-f(1)$$
$$=0-1$$
$$\underline{=-1}$$

$$f(4)=f(3)-f(2)$$
$$=-1-0$$
$$\underline{=-1}$$

$$f(5)=f(4)-f(3)$$
$$=-1-(-1)$$
$$\underline{=0}$$

b) $f(2) = f(1)f(0)$

$= (1)(1)$

$\underline{= 1}$

$f(3) = f(2)f(1)$

$= (1)(1)$

$\underline{= 1}$

$f(4) = f(3)f(2)$

$= (1)(1)$

$\underline{= 1}$

$f(5) = f(4)f(3)$

$= (1)(1)$

$\underline{= 1}$

c) $f(2) = f(1)^2 + f(0)^3$

$= 1^2 + 1^3$

$\underline{= 2}$

$f(3) = f(2)^2 + f(1)^3$

$= 2^2 + 1^3$

$\underline{= 5}$

$f(4) = f(3)^2 + f(2)^3$

$= 5^2 + 2^3$

$\underline{= 33}$

$f(5) = f(4)^2 + f(3)^3$

$= 33^2 + 5^3$

$\underline{= 1,214}$

d) $f(2) = \dfrac{f(1)}{f(0)}$

$= \dfrac{1}{1}$

$\underline{= 1}$

$f(3) = \dfrac{f(2)}{f(1)}$

$= \dfrac{1}{1}$

$\underline{=1}$

$$f(4)=\frac{f(3)}{f(2)}$$
$$=\frac{1}{1}$$
$$\underline{=1}$$

$$f(5)=\frac{f(4)}{f(3)}$$
$$=\frac{1}{1}$$
$$\underline{=1}$$

Exercise

Determine whether each of these proposed definitions is a valid recursive definition of a function *f* from the set of nonnegative integers to the set of integers. If *f* is well defined, fund a formula for $f(n)$ when *n* is nonnegative integer and prove that your formula is valid.

a) $f(0)=0, \quad f(n)=2f(n-2) \quad for \quad n\ge 1$

b) $f(0)=1, \quad f(n)=-f(n-1) \quad for \quad n\ge 1$

c) $f(0)=1, \quad f(n)=f(n-1)-1 \quad for \quad n\ge 1$

d) $f(0)=2, \quad f(1)=3, \quad f(n)=f(n-1)-1 \quad for \quad n\ge 2$

e) $f(0)=1, \quad f(1)=2, \quad f(n)=2f(n-2) \quad for \quad n\ge 2$

f) $f(0)=1, \quad f(1)=0, \quad f(2)=2, \quad f(n)=2f(n-3) \quad for \quad n\ge 3$

g) $f(0)=0, \quad f(1)=1, \quad f(n)=2f(n+1) \quad for \quad n\ge 2$

h) $f(0)=0, \quad f(1)=1, \quad f(n)=2f(n-1) \quad for \quad n\ge 2$

i) $f(0)=2, \quad f(n)=f(n-1)$ if *n* is odd and $n\ge 1$ and $f(n)=2f(n-2)$ if *n* is even and $n\ge 2$

j) $f(0)=1, \quad f(n)=3f(n-1)$ if *n* is odd and $n\ge 1$ and $f(n)=9f(n-2)$ if *n* is even and $n\ge 2$

Solution

a) This is invalid, since $f(1)=2f(1-2)=\cancel{2f(-1)} \quad for \quad n\ge 1, \ f(-1)$ is not defined.

b) $f(1)=-f(0)=-1$, this is a valid, since $n=0$ is provided and each subsequent value is determined by the previous one. $f(n)=(-1)^n$, this is true for $n=0 \quad since \quad (-1)^0=1$.

Assume it is true for $n=k$, then

$$f(k+1)=-f(k+1-1)=-f(k)=-(-1)^k=(-1)^{k+1}$$

c) $f(1)=f(0)-1=1-1=0$, this is a valid.

$f(2) = f(1) - 1 = 0 - 1 = -1$

The sequence: $1,\ 0,\ -1,\ -2,\ -3,\ \ldots \Rightarrow f(n) = 1 - n$

By induction:

The basis step: $f(0) = 1 - 0 = 1$

If $\quad f(k) = 1 - k$

Then $\quad f(k+1) = f(k) - 1 = 1 - k - 1 = 1 - (k+1)$

d) $f(2) = f(1) - 1 = 3 - 1 = 2$

$f(3) = f(2) - 1 = 2 - 1 = 1$

Given: $f(0) = 2,\ \ f(1) = 3$

Then the sequence: 2, 3, 2, 1, 0, … $\Rightarrow f(n) = 4 - n$

By induction: Basis step: $f(0) = 2$ and $f(1) = 4 - 1 = 3$

If $\quad f(k) = 4 - k$

Then $\quad f(k+1) = f(k) - 1 = 4 - k - 1 = 4 - (k+1)$

e) $f(2) = 2f(0) = 2 \qquad f(1) = 2$

$f(3) = 2f(1) = 2(2) = 4 \qquad f(4) = 2f(2) = 2(2) = 4$

$f(5) = 2f(3) = 2(4) = 8 \qquad f(6) = 2f(4) = 2(4) = 8$

Then the sequence: 1, 2, 2, 4, 4, 8, 8, … $\Rightarrow f(n) = 2^{(n+1)/2}$

By induction: Basis step: $f(0) = 2^{(0+1)/2} = 1 \quad and \quad f(1) = 2^{(1+1)/2} = 2$ and

If $\quad f(k) = 2^{(k+1)/2}$

Then

$f(k+1) = 2f(k-1) = 2 \cdot 2^{(k-1+1)/2} = 2 \cdot 2^{k/2} = 2^{(k/2)+1} = 2^{(k+2)/2} = 2^{((k+1)+1)/2}$

f) $f(3) = 2f(0) = 2(1) = 2 \qquad f(4) = 2f(1) = 2(0) = 0 \qquad f(5) = 2f(2) = 2(2) = 4$

$f(6) = 2f(3) = 2(2) = 4 \qquad f(7) = 2f(4) = 2(0) = 0 \qquad f(8) = 2f(5) = 2(4) = 8$

This is valid, since the values $n = 0, 1, 2$ are given. The sequence: 1, 0, 2, 2, 0, 4, 4, 0, 8, …

We conjecture the formula	***Prove***
$f(n) = 2^{n/3}$ when $n \equiv 0(\boldsymbol{mod}\,3)$	$f(0) = 2^{0/3} = 1$
$f(n) = 0$ when $n \equiv 1(\boldsymbol{mod}\,3)$	$f(1) = 0$
$f(n) = 2^{(n+1)/3}$ when $n \equiv 2(\boldsymbol{mod}\,3)$	$f(2) = 2^{(2+1)/3} = 2^1 = 2$

Assume the inductive hypothesis that the formula is valid for smaller inputs. Then

For $n \equiv 0(\boldsymbol{mod}\,3)$ we have $f(n) = 2f(n) = 2 \cdot 2^{(n-3)/3} = 2 \cdot 2^{n/3} \cdot 2^{-1} = 2^{n/3}$ as desired

For $n \equiv 1(\boldsymbol{mod}\,3)$ we have $f(n) = 2f(n-3) = 2 \cdot 0 = 0$ as desired

For $n \equiv 2(\boldsymbol{mod}\,3)$ we have $f(n) = 2f(n-3) = 2 \cdot 2^{(n-3+1)/3} = 2 \cdot 2^{(n+1)/3} \cdot 2^{-1} = 2^{(n+1)/3}$ as desired

g) $f(2) = 2f(3)$ This is not valid, since $f(3)$ has not been defined

h) $f(2) = 2 \cdot f(1) = 2(1) = 2 \qquad f(3) = 2f(2) = 2(2) = 4$

This is ***invalid***, because the value at $n = 1$ is defined in 2 conflicting ways, first as $f(1) = 1$ and then as $f(1) = 2f(1-1) = 2f(0) = 2(0) = 0$

i) $f(1) = f(0) = 2 \qquad f(2) = 2f(0) = 2(2) = 4$

$f(3) = f(2) = 4 \qquad f(4) = 2f(2) = 2(4) = 8$

This is ***invalid***, since we have a conflict for odd $n \ge 3$.

On one hand $f(3) = f(2)$, but the other hand $f(3) = 2f(1)$.

However, $f(1) = f(0) = 2$ and $f(2) = 2f(0) = 4$, so these apparently conflicting rules tell us that $f(3) = 2 \cdot 2 = 4$ on the other hand. We got the same answer either way.

The sequence: 2, 2, 4, 4, 8, 8, …

j) $f(1) = 3f(0) = 3(1) = 3 \qquad f(2) = 9f(0) = 9(1) = 9$

$f(3) = 3f(2) = 3(9) = 27 \qquad f(4) = 9f(2) = 9(9) = 81$

The sequence: 1, 3, 9, 27, 81, …

This is a valid, since we conjecture the formula $f(n) = 3^n$

By induction: Basis step: $f(0) = 3^0 = 1$

If $\quad f(k) = 3^k$

Then $f(k+1) = 3f(k) = 3 \cdot 3^k \underline{= 3^{k+1}}|$

Exercise

Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$ if

a) $a_n = 6n$ *b)* $a_n = 2n+1$ *c)* $a_n = 10^n$ *d)* $a_n = 5$

e) $a_n = 4n-2$ *f)* $a_n = 1+(-1)^n$ *g)* $a_n = n(n+1)$ *h)* $a_n = n^2$

Solution

a) $a_1 = 6$

$a_2 = 12 = 6 + 6$

$a_3 = 18 = 12 + 6$

$\vdots \qquad \vdots$

$\rightarrow \boxed{a_{n+1} = a_n + 6}$ *with* $a_1 = 6$ *for all* $n \ge 1$

b) $a_1 = 3$

$a_2 = 5 = 3 + 2$

$a_3 = 7 = 5 + 2$

$\vdots \quad \vdots$

$\rightarrow \boxed{a_{n+1} = a_n + 2}$ *with* $a_1 = 3$ *for all* $n \ge 1$

c) $a_1 = 10$

$a_2 = 10^2 = 10 \cdot 10$

$a_3 = 10^3 = 10 \cdot 10^2$

$\vdots \quad \vdots$

$\boxed{a_{n+1} = 10a_n}$ *with* $a_1 = 10$ *for all* $n \ge 1$

d) $a_1 = 5$

$a_2 = 5$

$a_3 = 5$

$\vdots \quad \vdots$

$\boxed{a_{n+1} = a_1}$ *with* $a_1 = 5,$ *for all* $n \ge 1$

e) $a_1 = 2$

$a_2 = 6 = 2 + 4$

$a_3 = 10 = 6 + 4$

$\vdots \quad \vdots$

$\boxed{a_{n+1} = a_n + 4}$ *with* $a_1 = 2,$ *for all* $n \ge 1$

f) $a_1 = 1 - 1 = 0,$

$a_2 = 1 + 1 = 2$

$a_3 = 1 - 1 = 0$

$\vdots \quad \vdots$

The sequence alternate: 0,2, 0, 2, …

$\boxed{a_n = a_{n-2}}$ *with* $a_1 = 0,\ a_2 = 2,$ *for all* $n \ge 3$

g) $a_1 = 1(2) = 2$

$a_2 = 2(3) = 6$

$a_3 = 12$

$\vdots \quad \vdots$

The sequence alternate: 2,6, 12, 20, 30, …

The difference between successive terms are 4, 6, 8, 10, ….

$\underline{a_n = a_{n-1} + 2n}|$ *with* $a_1 = 2,$ *for all* $n \ge 2$

h) $a_1 = 1^2 = 1$

$a_2 = 2^2 = 4$

$a_3 = 3^2 = 9$

$\vdots \quad \vdots$

The sequence alternate: 1, 4, 9, 16, 25, …

The difference between successive terms are 3, 5, 7, 9, ….

$\underline{a_n = a_{n-1} + 2n - 1}|$ *with* $a_1 = 1,$ *for all* $n \ge 2$

Exercise

Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ when n is a positive integer and f_n is the nth Fibonacci number.

Solution

For $n = 1$: $f_1^2 = f_1 f_2 = 1 \cdot 1 = 1$ is true since both values are 1

Assume the inductive hypothesis. Then

$$\begin{aligned} f_1^2 + f_2^2 + \cdots + f_n^2 + f_{n+1}^2 &= f_n f_{n+1} + f_{n+1}^2 \\ &= f_{n+1}\left(f_n + f_{n+1}\right) \\ &= f_{n+1} f_{n+2} \end{aligned}$$

Exercise

Prove that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ when n is a positive integer and f_n is the nth Fibonacci number.

Solution

Using the principle of mathematical induction

For $n=1$: $f_1 = f_2$ is true since both values are 1

Let assume that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$

We need to prove that $f_1 + f_3 + \cdots + f_{2n-1} + f_{2n+1} = f_{2(n+1)}$

$$\begin{aligned} f_1 + f_3 + \cdots + f_{2n-1} + f_{2n+1} &= f_{2n} + f_{2n+1} \\ &= f_{2n+2} \end{aligned}$$

(by the *definition* of the Fibonacci numbers)

Exercise

Give a recursive definition of

a) The set of odd positive integers
b) The set of positive integers powers of 3
c) The set of polynomial with integer coefficients
d) The set of even integers
e) The set of positive integers congruent to 2 modulo 3.
f) The set of positive integers not divisible by 5

Solution

a) Off integers are obtained from other odd integers by adding 2.
Thus, we can define this set S as follows $1 \in S$; and if $n \in S,\ then\ \ n+2 \in S$.

b) Powers of 3 are obtained from other powers of 3 by multiplying by 3.
Thus, we can define this set S as follows $3 \in S$; and if $n \in S,\ then\ \ 3n \in S$.

c) There are several ways to do this. One that is suggested by Horner's method is as follows. We assume that the variable for these polynomials is the letter *x*. All integers are in *S*; if $p(x) \in S$ and *n* is any integer, then $xp(x)+n$ is in *S*.
Another method constructs the polynomials term by term. Its base case is to let 0 be in *S*; and its inductive step is to say that if $p(x) \in S$, *c* is an integer, and *n* is a nonnegative integer, then $p(x)+cx^n$ is in *S*.

d) Off integers are obtained from other even integers by adding 2.
Thus, we can define this set S as follows $2 \in S$; and if $n \in S,\ then\ \ n-2 \in S\ \ and\ \ n+2 \in S$.

e) The smallest positive integer congruent to 2 modulo 3 is 2, so $2 \in S$. All the others can be obtained by adding multiples of 3, so the inductive step is that $n \in S,\ then\ \ n+3 \in S$

f) The positive integers no divisible by 5 are the ones congruent to 1, 2, 3, or 4 *modulo* 5.
Thus, we can define this set S as follows $1 \in S,\ 2 \in S,\ 3 \in S,\ and\ 4 \in S$; and if $n \in S,\ then\ \ n+5 \in S$

Exercise

Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: (0, 0) ∈ S.

Recursive step: If $(a, b) \in S$, then $(a+2,\ b+3) \in S$ *and* $(a+3,\ b+2) \in S$

a) List the elements of S produced by the first five applications of the recursive definition.

b) Use strong induction on the number of applications of the recursive step of the definition to show that $5 \mid a+b$ when $(a, b) \in S$.

c) Use structural induction to show that $5 \mid a+b$ when $(a, b) \in S$.

Solution

a) Apply each recursive step rules to the only element given in the basis step, we see that (2, 3) and (3, 2) are in S.

If we apply the recursive step to these we add (4, 6), (5, 5) and (6, 4).

The next round gives us (6, 9), (7, 8), and (9, 6). Add (8, 12), (9, 11), (10, 10), (11, 9), and (12, 8); and a fifth set of applications adds (10, 15), (11, 4), (12, 13), (13, 12), (14, 1), and (15, 10).

b) Let $P(n)$ be the statement that $5 \mid a+b$ when $(a, b) \in S$ is obtained by n applications to the recursive step.

For $n=0$, $P(0)$ is true, since the only element of S obtained with no applications of the recursive step is (0, 0), and $5 \mid 0+0$ $\checkmark$

Assume the inductive hypothesis that $5 \mid a+b$ whenever $(a, b) \in S$ is obtained by k or fewer applications of the recursive step, and consider an element obtained with $k+1$ applications of the recursive step. Since the final application of the recursive step to an element $(a,\ b)$ must applied to an element, that $5 \mid a+b$.

We need to check that this inequality implies $5 \mid a+2+b+3$ and $5 \mid a+3+b+2$.

This is clear, since each is equivalent to $5 \mid a+b+5$ and 5 divides both $a+b$ and 5.

c) This holds for the basis step, since $5 \mid 0+0$

If this holds for $(a,\ b)$, then it also holds for the elements obtained from $(a,\ b)$ in the recursive step by the same argument as in part (b).

Exercise

Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0, 0) \in S$.

Recursive step: If $(a, b) \in S$, then $(a,\ b+1) \in S$, $(a+1,\ b+1) \in S$ *and* $(a+2,\ b+1) \in S$

a) List the elements of S produced by the first five applications of the recursive definition.

b) Use strong induction on the number of applications of the recursive step of the definition to show that $a \le 2b$ whenever $(a, b) \in S$.

c) Use structural induction to show that $a \le 2b$ whenever $(a, b) \in S$.

Solution

a) Apply each recursive step rules to the only element given in the basis step, we see that (0, 1), (1, 1) and (2, 1) are in S.

2nd step: (0, 2), (1, 2), (2, 2), (3, 2) and (4, 2).

3rd step: (0,3), (1, 3), (2, 3), (3, 3), (4, 3), (5, 3) and (6, 3).

4th step: (0, 4), (1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4), (7, 4) and (8,4)

5th step: (0, 5), (1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (6, 5), (7, 5), (8, 5), (9, 5), and (10, 5)

b) Let $P(n)$ be the statement that $a \le 2b$ whenever $(a, b) \in S$ is obtained with no applications of the recursive step.

For the basis step, the only element of S obtained with no applications of the recursive step is $(0,\ 0)$, then $0 \le 2 \cdot 0$ is true. Therefore $P(0)$ is true.

Assume that $a \le 2b$ whenever $(a, b) \in S$ is obtained by k or fewer applications of the recursive step. Consider an element obtained with $k + 1$ applications of the recursive step.

We know that $a \le 2b$, we need to check this inequality implies $a \le 2(b+1)$, $a+1 \le 2(b+1)$, and $a+2 \le 2(b+1)$.

Thus is clear that $0 \le 2,\ 1 \le 2$ *and* $2 \le 2$, respectively, to $a \le 2b$ to obtain these inequalities.

c) This holds for the basis step, since $0 \le 0$.

If this holds for $(a,\ b)$, then it also holds for the elements obtained from $(a,\ b)$ in the recursive step, since adding $0 \le 2,\ 1 \le 2$ *and* $2 \le 2$, respectively, to $a \le 2b$ yiekds $a \le 2(b+1)$, $a+1 \le 2(b+1)$, and $a+2 \le 2(b+1)$.

Solution *Section* 3.3 – The Basics of Counting

Exercise

There are 18 mathematics majors and 325 computer science majors at a college

a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?

b) In how many ways can one representative be picked who either a mathematics major or a computer science major?

Solution

a) $18 \cdot 325 \ \boxed{= 5850 \ \ ways}$

b) $18 + 325 \ \boxed{= 343 \ \ ways}$

Exercise

An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

Solution

Using the product rule: there are $27 \cdot 37 \ \boxed{= 999 \ \ offices}$

Exercise

A multiple-choice test contains 10 questions. There are four possible answers for each question

a) In how many ways can a student answer the questions on the test if the student answers every question?

b) In how many ways can a student answer the questions on the test if the student can leave answers blank?

Solution

a) $4 \cdot 4 \cdot 4 \cdots 4 = 4^{10} \ \boxed{= 1,048,576 \ ways}$

b) There are 5 ways to answer each question 0 give any if the 4 answers or give no answer at all

$\boxed{5^{10} = 9,765,625 \quad ways}$

Exercise

A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of the shirts are made?

Solution

$\boxed{12 \cdot 2 \cdot 3 = 72}$ different types of shirt.

Exercise

How many different three-letter initials can people have?

Solution

$\boxed{26 \cdot 26 \cdot 26 = 17,576}$ different initials.

Exercise

How many different three-letter initials with none of the letters repeated can people have?

Solution

$\boxed{26 \cdot 25 \cdot 24 = 15,600 \ \ ways}$

Exercise

How many different three-letter initials are there that begin with an A?

Solution

$\boxed{1 \cdot 26 \cdot 26 = 676 \ \ ways}$

Exercise

How many bit strings are there of length eight?

Solution

$\boxed{2^8 = 256 \ \ bit\ strings}$

Exercise

How many bit strings of length ten both begin and end with a 1?

Solution

$\boxed{1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 2^8 = 256 \ \ bit\ strings}$

Exercise

How many bit strings of length n, where n is a positive integer, start and end with 1s?

Solution

$\boxed{1 \cdot 2^{n-2} \cdot 1 = 2^{n-2} \quad \textit{bit strings}}$ $\qquad 1 \cdot 2 \cdot 2 \cdots 2 \cdot 2 \cdot 1$

Exercise

How many strings are there of lowercase letters of length four or less, not counting the empty string?

Solution

The number of strings of length 4 or less by counting the number of the strings of length $0 \le i \le 4$
There are 26 letters to choose from, and a string of length i is specified by choosing its characters, one after another.

The product rules there are 26^i

$$\sum_{i=0}^{4} 26^i = 1 + 26 + 26^2 + 26^3 + 26^4$$

$$\boxed{= 475,255}$$

Exercise

How many strings are there of four lowercase letters that have the letter x in them?

Solution

Number of strings of length of 4 lowercase: 26^4

Number of strings of length of 4 lowercase other than ***x***: 25^4

$\boxed{26^4 - 25^4 = 66,351 \quad \textit{strings}}$

Exercise

How many positive integers between 50 and 100

a) Are divisible by 7? Which integers are these
b) Are divisible by 11? Which integers are these
c) Are divisible by 7 and 11? Which integers are these

Solution

a) Neither 50 nor 100 is divisible by 7

There are $\frac{50}{7} = 7$ integers less than 50 that are divisible by 7

There are $\frac{100}{7} = 14$ integers less than 100 that are divisible by 7

This leaves $14 - 7 = 7$ numbers between 50 and 100 that are divisible by 7.
They are 56, 63, 70, 77, 84, 91, and 98.

b) Neither 50 nor 100 is divisible by 11

There are $\frac{50}{11} = 4$ integers less than 50 that are divisible by 11

There are $\frac{100}{11} = 9$ integers less than 100 that are divisible by 1

This leaves $9 - 4 = 5$ numbers between 50 and 100 that are divisible by 11

They are 55, 66, 77, 88, and 99.

c) A number is divisible by 7 and 11 which is 77. There is only one such number between 50 and 100, namely 77.

Exercise

How many positive integers less than 100

a) Are divisible by 7?
b) Are divisible by 7 but not by 11?
c) Are divisible by both 7 and 11?
d) Are divisible by either 7 or 11?
e) Are divisible by exactly one of 7 and 11?
f) Are divisible by neither 7 nor 11?

Solution

7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98 *are divisible by* 7
11, 22, 33, 44, 55, 66, 77, 88, 99 *are divisible by* 11

a) Every 7^{th} number is divisible by 7. Therefore, $\frac{99}{7} \approx 14$ such numbers. The k^{th} multiple of 7 does not occur until the number $7k$ has been reached.

b) There are 13 such numbers since 77 is the only one divisible by 11.

c) There is only **1** number (77) divisible by both 7 and 11

d) $\frac{99}{11} \approx 9$ such numbers and 14 such numbers divisible by 7 and only 1 is divisible by 11. Therefore, there are $14 + 9 - 1 \underline{= 22}$ divisible by either 7 or 11

e) The number of numbers divisible of them: $22 - 1 \underline{= 21}$ $\quad$ $(\textit{subtract } (d) \textit{ from } (c))$

f) Subtract part (d) from the total number of positive integers less than 100.
$99 - 22 \underline{= 77}$

Exercise

How many positive integers less than 1000

a) Are divisible by 7?
b) Are divisible by 7 but not by 11?
c) Are divisible by both 7 and 11?
d) Are divisible by either 7 or 11?
e) Are divisible by exactly one of 7 and 11?
f) Are divisible by neither 7 nor 11?
g) Have distinct digits?
h) Have distinct digits and are even?

Solution

a) Every 7^{th} number is divisible by 7. Therefore, $\frac{999}{7} \approx 142$ such numbers. The k^{th} multiple of 7 does not occur until the number $7k$ has been reached.

b) Every 11^{th} number is divisible by 11. Therefore, $\frac{999}{11} \approx 90$ numbers.

Since 77 is the first number that is divisible by 7 and 11, and there are $\frac{999}{77} \approx 12$ numbers divisible by 77.

There are $142 - 12 \underline{=130}|$ numbers divisible by 7 but not by 11.

c) There are **12** numbers divisible by both 7 and 11 (from part ***b***)

d) There are $142 + 90 - 12 \underline{=220}|$ divisible by either 7 or 11

e) The number of numbers divisible of them: $220 - 12 \underline{=208}|$ (*subtract* (***d***) *from* (***c***))

f) Subtract part (**d**) from the total number of positive integers less than 1000.
$999 - 220 \underline{=779}|$

g) If we assume that numbers are written without leading 0's, then we can break down this part in three cases: one-digit numbers, two-digit numbers and three-digit numbers.
There are 9 one-digit numbers, and each of them has distinct digits.
There are 90 two-digit numbers (10 – 99), and all but 9 of them have distinct digits, so there are 81 two-digit numbers with distinct digits. Or the first digit 1 through 9 (9 choices), using the product rule: $9 \cdot 9 = 81$ choices in all.
For three-digit numbers there are $9 \cdot 9 \cdot 8 = 648$ distinct digits
Therefore $9 + 81 + 648 = \underline{738}|$ total distinct digits.

h) If we use to count the odd numbers with distinct digits and subtract from part (***g***), we can get the numbers distinct digits and are even.
There are 5 odd one-digit numbers.
For two-digit numbers; first the ones digits (5 choices), then the tens digit (8 choices) – neither the ones digit value nor 0 is available, therefore there are 40 such two-digit numbers (half of 81).
For three-digit numbers, first the ones digits (5 choices), the hundreds digit (8 choices), then the tens digit (8 choices). There are $5 \cdot 8 \cdot 8 = 320$ distinct digits

So $5+40+320=365$ total odd numbers with distinct digits.

Therefore $738-365 \underline{=373}$ total distinct digits.

Exercise

A committee is formed consisting of one representative from each of the 50 states in the United States, where the representative from a state is either the governor or one of the two senators from that state. How many ways are there to form this committee?

Solution

There are 50 choices to make each of which can be done in 3 ways, namely by choosing the governor, choosing the senior senator, or choosing the junior senator.

$\underline{3^{50} \approx 7.2\times 10^{23}}$

Exercise

How many license plates can be made using either three digits followed by three uppercase English letters or three uppercase English letters followed by three digits?

Solution

$10^3 \cdot 26^3 + 26^3 \cdot 10^3 \underline{= 35{,}152{,}000 \quad \textit{license plates}}$

Exercise

How many license plates can be made using either two uppercase English letters followed by four digits or two digits followed by four uppercase English letters?

Solution

Letters		***Digits***			
L	L	D	D	D	D
26	26	10	10	10	10

$26\cdot 26\cdot 10\cdot 10\cdot 10\cdot 10 = 6,760,000$

Digits		***Letters***			
D	D	L	L	L	L
10	10	26	26	26	26

$10\cdot 10\cdot 26\cdot 26\cdot 26\cdot 26 = 45,697,600$

Therefore: $6,760,000 + 45,697,600 \underline{= 52,457,600 \quad \textit{license plates}}$

Exercise

How many license plates can be made using either three uppercase English letters followed by three digits or four uppercase English letters followed by two digits?

Solution

$$26^3 \cdot 10^3 + 26^4 \cdot 10^2 \underline{= 63,273,600 \quad \textit{license plates}}$$

Exercise

How many strings of eight English letter are there.

a) That contains no vowels, if letters can be repeated?
b) That contains no vowels, if letters cannot be repeated?
c) That starts with a vowel, if letters can be repeated?
d) That starts with a vowel, if letters cannot be repeated?
e) That contains at least one vowel, if letters can be repeated?
f) That contains at least one vowel, if letters cannot be repeated?

Solution

	1	2	3	4	5	6	7	8
	NV	NV	NV	NV	NV	NV	NV	NV
a	21	21	21	21	21	21	21	21
b	21	20	19	18	17	16	15	14
	V	L	L	L	L	L	L	L
c	5	26	26	26	26	26	26	26
d	5	25	24	23	22	21	20	19

a) There are 8 slots which can be filled with $26 - 5 = 21$ non-vowels.

By the product rule: $21^8 \underline{= 37,822,859,361 \quad \textit{strings}}$

b) $21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \underline{= 8,204,716,800 \quad \textit{strings}}$

c) $5 \cdot 26^7 \underline{= 40,159,050,880 \quad \textit{strings}}$

d) $5 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \underline{= 12,113,640,000 \quad \textit{strings}}$

e) By the product rule: $26^8 - 21^8 \underline{= 171,004,205,215 \quad \textit{strings}}$

f) $8 \cdot 5 \cdot 21^7 \underline{= 72,043,541,640 \quad \textit{strings}}$

Exercise

How many ways are there to seat four of a group of ten people around a circular table where two seatings are considered the same when everyone has the same immediate left and immediate right neighbor?

Solution

The count ordered arrangements of length 4 from the 10 people, then we get $10 \cdot 9 \cdot 8 \cdot 7 = 5040$ arrangements.

However, we can rotate the people around the table in 4 ways and get the same seating arrangement, so the overcounts by a factor of 4.

Therefore, there are $\frac{5040}{4} \underline{= 1260 \quad ways}$

Exercise

In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and the groom are among these 10 people, if

a) The bride must be in the picture?

b) Both the bride and groom must be in the picture?

c) Exactly one of the bride and the groom is in the picture?

Solution

a) The bride is in any of the 6 positions.

1	2	3	4	5	6
B	***P***	***P***	***P***	***P***	***P***
1	9	8	7	6	5

Then, it will leave us with 5 remaining positions.

This can be done in $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15120$ ways.

Therefore $6 \cdot 15120 \underline{= 90,720 \quad ways}$

b) The bride is in any of the 6 positions.

1	2	3	4	5	6
B	***G***	***P***	***P***	***P***	***P***
1	1	8	7	6	5

Then place the groom in any of the 5 remaining positions.

Then, it will leave us with 4 remaining positions in the picture.

This can be done in $8 \cdot 7 \cdot 6 \cdot 5 = 1680$ ways.

Therefore $6 \cdot 5 \cdot 1680 \underline{= 50,400 \quad ways}$

c) For the just the bride to be in the picture: $90720 - 50400 = 40,320$ ways.

There are 40,320 ways for just the groom to be in the picture.

Therefore, $40320 + 40320 \underline{= 80,640 \quad ways}$

Exercise

How many different types of homes are available if a builder offers a choice of 6 basic plans, 3 roof styles, and 2 exterior finishes?

Solution

$6.3.2 = \underline{36\ |}$ *different homes types*

Exercise

A menu offers a choice of 3 salads, 8 main dishes, and 7 desserts. How many different meals consisting of one salad, one main dish, and one dessert are possible?

Solution

$3.8.7 = \underline{168\ |}$ ***different meals***

Exercise

A couple has narrowed down the choice of a name for their new baby to 4 first names and 5 middle names. How many different first- and middle-name arrangements are possible?

Solution

$4.5 = \underline{20\ |}$ *possible arrangements*

Exercise

An automobile manufacturer produces 8 models, each available in 7 different exterior colors, with 4 different upholstery fabrics and 5 interior colors. How many varieties of automobile are available?

Solution

$8.7.4.5 = \underline{1120\ |}$

Exercise

A biologist is attempting to classify 52,000 species of insects by assigning 3 initials to each species. Is it possible to classify all the species in this way? If not, how many initials should be used?

Solution

$26^3 = 17,576$ *This would not be enough.*

$26^4 = 456,976$ *Which is more than enough*

Exercise

How many 4-letter code words are possible using the first 10 letters of the alphabet under:

a) No letter can be repeated

b) Letters can be repeated

c) Adjacent can't be alike

Solution

a) $10.9.8.7 = 5040$

b) $10.10.10.10 = 10,000$

c) $10.9.9.9 = 7290$

Exercise

How many 3 letters license plate without repeats

Solution

$26.25.24 = 15600$ possible

Exercise

How many ways can 2 coins turn up heads, H, or tails, T – if the combined outcome (H, T) is to be distinguished from the outcome (T, H)?

Solution

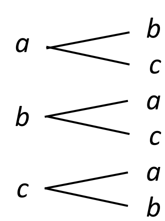
$2 \times 2 = 4 \quad \textit{outcomes}$

Exercise

How many 2-letter code words can be formed from the first 3 letters of the alphabet if no letter can be used more than once?

Solution

$3 \times 2 = 6 \quad \textit{outcomes}$



Exercise

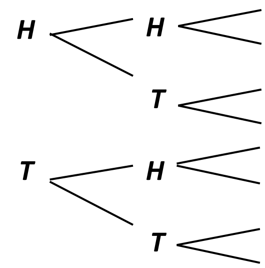
A coin is tossed with possible outcomes heads, H, or tails, T. Then a single die is tossed with possible outcomes 1, 2, 3, 4, 5, or 6. How many combined outcomes are there?

Solution

$2 \times 6 = 12 \quad \textit{outcomes}$

Exercise

In how many ways can 3 coins turn up heads, H, or tails, T – if combined outcomes such as (H,T,H), (H, H, T), and (T, H, H) are to be considered different?



Solution

$\underline{2\times2\times2=8 \quad \textit{outcomes}}$

Exercise

An entertainment guide recommends 6 restaurants and 3 plays that appeal to a couple.

a) If the couple goes to dinner or to a play, how many selections are possible?

b) If the couple goes to dinner and then to a play, how many combined selections are possible?

Solution

a) 3 + 6 = 9

b) 6.3 = 18

Solution *Section* 3.4 – Permutations and Combinations

Exercise

Decide whether the situation involves ***permutations*** or ***combinations***

a) A batting order for 9 players for a baseball game
b) An arrangement of 8 people for a picture
c) A committee of 7 delegates chosen from a class of 30 students to bring a petition to the administration
d) A selection of a chairman and a secretary from a committee of 14 people
e) A sample of 5 items taken from 71 items on an assembly line
f) A blend of 3 spices taken from 7 spices on a spice rack
g) From the 7 male and 10 female sales representatives for an insurance company, team of 8 will be selected to attend a national conference on insurance fraud.
h) Marbles are being drawn without replacement from a bag containing 15 marbles.
i) The new university president named 3 new officers a vice-president of finance, a vice-president of academic affairs, and a vice-president of student affairs.
j) A student checked out 4 novels from the library to read over the holiday.
k) A father ordered an ice cream cone (chocolate, vanilla, or strawberry) for each of his 4 children.

Solution

a) Permutation

b) Permutation

c) Combination

d) Permutation

e) Combination

f) Combination

g) Combination

h) Combination

i) Permutation

j) Combination

k) Neither

Exercise

How many different permutations are the of the set $\{a, b, c, d, e, f, g\}$?

Solution

$P(7,\ 7) = 5040$

Exercise

How many permutations of $\{a, b, c, d, e, f, g\}$ end with a?

Solution

To find the permutation to with a, then we may forget about the a, and leave us $\{b, c, d, e, f, g\}$

$P(6,\ 6)\ \underline{=720\ |}$

Exercise

Find the number of 5-permutations of a set with nine elements

Solution

$P(9,\ 5)\ \underline{=15{,}120\ |}$ ***by Theorem***

Exercise

In how many different orders can five runners finish a race if no ties are allowed?

Solution

$P(5,\ 5)\ \underline{=120\ |}$

Exercise

A coin flipped eight times where each flip comes up either heads or tails. How many possible outcomes

a) Are there in total?

b) Contain exactly three heads?

c) Contain at least three heads?

d) Contain the same number of heads and tails?

Solution

a) Each flip can be either heads or tails: There are $2^8 \underline{= 256\ \ possible\ coutcomes\ |}$

b) $C(8,\ 3)\ \underline{=56\ \ outcomes\ |}$

c) At least three heads means: 3, 4, 5, 6, 7, 8 heads.

$C(8,\ 3)+C(8,\ 4)+C(8,\ 5)+C(8,\ 6)+C(8,\ 7)+C(8,\ 8)\ \underline{=219\ \ outcomes\ |}$

OR

$256-C(8,\ 0)-C(8,\ 1)-C(8,\ 2)=256-28-8-1\ \underline{=219\ \ outcomes\ |}$

d) To have an equal number of heads and tails means 4 heads and 4 tails.

Therefore; $C(8,\ 4)\ \underline{=70\ \ outcomes\ |}$

Exercise

A coin flipped 10 times where each flip comes up either heads or tails. How many possible outcomes

a) Are there in total?

b) Contain exactly two heads?

c) Contain at most three tails?

d) Contain the same number of heads and tails?

Solution

a) Each flip can be either heads or tails: There are $2^{10} = 1024$ *possible coutcomes*

b) $C_{10,2} = 45$ *outcomes*

c) At most three tails means: 3, 2, 1, 0 tails.

$C_{10,3} + C_{10,2} + C_{10,1} + C_{10,0} = 176$ *outcomes*

d) To have an equal number of heads and tails means 5 heads and 5 tails.

Therefore; $C_{10,5} = 252$ *outcomes*

Exercise

How many bit strings of length 12 contain?

a) Exactly three 1s?

b) At most three 1s?

c) At least three 1s?

d) An equal number of 0s and 1s?

Solution

a) We need to choose the 3 positions that contains the 1's

$C_{12,\ 3} = 220$ *ways*

b) At most three 1's means to contains 3, 2, 1, 0 –1's:

$C_{12,3} + C_{12,2} + C_{12,1} + C_{12,0} = 299$ *strings*

c) At least three 1's means to contains 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 –1's:

$$C_{12,3} + C_{12,4} + C_{12,5} + C_{12,6} + C_{12,7} + C_{12,8} + C_{12,9} + C_{12,10} + C_{12,11} + C_{12,12} = 4017 \text{ strings}$$

OR

$$2^{12} - C_{12,2} - C_{12,1} - C_{12,0} = 4096 - 66 - 12 - 1 = 4017 \text{ strings}$$

d) To have an equal number of 0's and 1's means 6 1's.

Therefore; $C(12,\ 6) = 924$ *strings*

Exercise

A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate?

Solution

Consider the order in which the men appear relative to each other. There are n men $P(n,n) = n!$ arrangements is allowed.

Consider the order in which the women appear relative to each other. There are n women $P(n,n) = n!$ arrangements is allowed.

Men and women must alternate, and there are the same number of men and women; therefore there are exactly 2 possibilities: either the row with a man ends with a woman ***or*** it starts with a woman ends with a man.

By the product rule there are $n!\ n!\ 2 \underline{= 2(n!)^2 \quad ways}$

Exercise

In how many ways can a set of two positive integers less than 100 be chosen?

Solution

$$C_{99,\ 2} \underline{= 4851 \quad ways}$$

Exercise

In how many ways can a set of five letters be selected from the English alphabet?

Solution

$$C_{26,\ 5} \underline{= 65,780 \quad ways}$$

Exercise

How many subsets with an odd number of elements does a set with 10 elements have?

Solution

$$C_{10,1} + C_{10,3} + C_{10,5} + C_{10,7} + C_{10,9} \underline{= 512 \quad subsets}$$

Exercise

How many subsets with more than two elements does a set with 100 elements have?

Solution

There are 2^{100} subsets of a set with 100 elements. All of them have more than 2 subsets except the empty set, the 100 subsets consisting of one element each, and $C_{100,\ 2} = 4950$ *subsets* with 2 elements.

Therefore; $2^{100} - 4950 \underline{=1.26\times 10^{30}}$

Exercise

How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other?

Solution

First position the men relative to each other. Since there are 8 men, there are $P(8,8)$ ways to do this.

This creates 9 slots where a woman may stand: in front of the first man, between the first and second men, …, between the 7^{th} and 8 men, and behind the 8^{th} man.
We need to choose 5 of these positions, in order, for the first through 5^{th} woman to occupy.

Therefore, $P(8,8)\cdot P(9,5) \underline{=609,638,400 \ \ ways}$

Exercise

How many ways are there for six men and 10 women to stand in a line so that no two men stand next to each other?

Solution

16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
W	W	W	W	W	M	W	M	W	M	W	M	W	M	W	M

Since there are 10 women, there are $P(10,10) = 3,628,800$

This creates 11 slots where a man may stand.
This can be done is $P(11,6) = 332,640$

Therefore $P(10,10)\cdot P(11,6) \underline{=1,207,084,032,000 \ \ ways}$

Exercise

A professor writes 40 discrete mathematics true/false questions of the statements in these questions. 17 are true. If the questions can be positioned in any order, how many different answer keys are possible?

Solution

$$C_{40,17} \underline{= 8.9\times 10^{10} \ \ answers}|$$

Exercise

Thirteen people on a softball team show up for a game.

a) How many ways are there to choose 10 players to take the field?

b) How many ways are there to assign the 10 positions by selecting players from the 13 people who show up?

c) Of the 13 people who show up, there are three women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?

Solution

a) $C_{13,10} \underline{= 286 \ \ ways}|$

b) The order in which the choices are made: $P_{13,10} \underline{= 1,037,836,800 \ \ ways}|$

c) There is only one way to choose the 10 players without choosing a woman, since there are exactly 10 men.

Therefore, there are $286-1 \underline{= 285 \ \ ways}|$ to choose the players if at least one of them must be a woman.

Exercise

A club has 25 members

a) How many ways are there to choose four members of the club to serve on an executive committee?

b) How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office?

Solution

a) Since the order of choosing the members is not relevant, we need to use a combination

$C(25,4) \underline{= 12,650 \ \ ways}|$

b) Since the order of choosing the members is matter, we need to use a permutation.

$P(25,4) \underline{= 303,600 \ \ ways}|$

Exercise

How many 4-permutations of the positive integers not exceeding 100 contain three consecutive integers, k, $k+1$, $k+2$, in the order

a) Where these consecutive integers can perhaps be separated by other integers in the permutation?

b) Where they are in consecutive positions in the permutation?

Solution

a) The consecutive numbers are 5, 6, 7, since can be separate by other integers the permutation can be written as 5, 6, 32, 7.

In order to specify such 4-permutation, we need to choose 3 consecutive integers. They can be {1, 2, 3} to (98, 99, 100}; thus, there are 98 possibilities. There are 4 possibilities, we need to decide which 97 other positive integers not exceeding 100 is to fill this slot, and there are 97 choices.

In fact, every 4-permutation consisting of 4 consecutive numbers, in order, has been double counted.

Therefore, we need to subtract the number of such 4-permutations. Clearly there are 97 of them.

Further thought shows that every other 4-permutation in our collection arises in a unique way.

Therefore $98 \cdot 4 \cdot 97 - 97 \underline{= 37,927}$

b) The consecutive numbers be consecutive in the 4-permutation.

There are only 2 places to put the fourth number in slot 1 and slot 4.

Therefore, $98 \cdot 2 \cdot 97 - 97 \underline{= 18,915}$

Exercise

The English alphabet contains 21 constants and five vowels. How many strings of six lowercase letters of the English alphabet contain?

a) Exactly one vowel?

b) Exactly two vowels?

c) At least one vowel?

d) At least two vowels?

Solution

a) This can be done 6 ways. We need to choose the vowel and this can be done in 5 ways. Each other 5 positions can contain any of the 21 consonants, so there are 21^5 ways to fill the rest of the string.

Therefore, the answer is $6 \cdot 5 \cdot 21^5 \underline{= 122,533,030 \ \ ways}$

b) The position of the vowels can be done in $C(6,\ 2) = 15\ ways$. We need to choose the 2 vowels in 5^2 ways. Each other 4 positions can contain any of the 21 consonants, so there are 21^4 ways to fill the rest of the string.

Therefore, the answer is $15 \cdot 5^2 \cdot 21^4 \underline{= 72,930,375 \ \ ways}$

c) Count the number of strings with no vowels and subtract this from the total number of stings.

$26^6 - 21^6 \underline{= 223,149,655 \ \ ways}$

d) Subtracting the total number of strings from the number of strings with no vowels and the number of strings with one vowel. Answer: $26^6 - 21^6 - 6 \cdot 5 \cdot 21^5 \underline{= 100,626,625 \quad ways}$

Exercise

Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have

a) The same number of men and women?

b) More women than men?

Solution

a) $C_{10,3} \cdot C_{15,3} \underline{= 54,600 \ \ ways}$

b) There are $C_{15,6}$ *ways* to choose the committee to be composed only of women

$C_{15,5} \cdot C_{10,1}$ *ways* if there are to be five women and one man, and $C_{15,4} \cdot C_{10,2}$ *ways* if there are to be four women and two men.

Therefore, $C_{15,6} + C_{15,5} \cdot C_{10,1} + C_{15,4} \cdot C_{10,2} \underline{= 96,460 \quad ways}$

Exercise

How many bit strings contain exactly eight 0s and 10 1s if every 0 must be immediately followed by a 1?

Solution

0 1	0 1	0 1	0 1	0 1	0 1	0 1	0 1	1	1

There are 8 blocks consisting of the string 01

$C_{10,2} \underline{= 45 \quad ways}$

Exercise

How many bit strings contain exactly five 0s and 14 1s if every 0 must be immediately followed by two 1s?

Solution

Glue 2 1's to the right of each 0, giving a collection of 9 tokens: five 001's and four 1's.

$C_{9,4} \underline{= 126 \quad ways}$

Exercise

A concert to raise money for an economics prize is to consist of 5 works; 2 overtures, 2 sonatas, and a piano concerto.

a) In how many ways can the program be arranged?
b) In how many ways can the program be arranged if an overture must come first?

Solution

a) $P(5,5)\ \underline{=120 \quad ways}|$

b) $P(2,1)\cdot P(4,4)\ \underline{=48 \quad ways}|$

Exercise

A zydeco band from Louisiana will play 5 traditional and 3 original Cajun compositions at a concert. In how many ways can they arrange the program if

a) The begin with a traditional piece?
b) An original piece will be played last?

Solution

a) $P(5,1)\cdot P(7,7)\underline{=25,200 \quad ways}|$

b) $P(7,7)\cdot P(3,1)\ \underline{=15,120 \quad ways}|$

Exercise

In an election with 3 candidates for one office and 6 candidates for another office, how many different ballots may be printed?

Solution

Office **1**: $P(3,3)$

Office **2**: $P(6,6)$

Multiplication principle: $2\cdot P(3,3)P(6,6)\ \underline{=8640}|$

Exercise

A business school gives courses in typing, shorthand, transcription, business English, technical writing, and accounting. In how many ways can a student arrange a schedule if 3 courses are taken? assume that the order in which courses are schedules matters.

Solution

$P(6,3)\ \underline{=120 \quad ways}|$

Exercise

If your college offers 400 courses, 25 of which are in mathematics, and your counselor arranges your schedule of 4 courses by random selection, how many schedules are possible that do not include a math course? Assume that the order in which courses are scheduled matters.

Solution

$$P(nonmath) = P(375,4) = 1.946\times 10^{10}$$

Exercise

A baseball team has 19 players. How many 9-player batting orders are possible?

Solution

$$P(19,9) = 3.352\times 10^{10}$$

Exercise

A chapter of union Local 715 has 35 members. In how many different ways can the chapter select a president, a vice-president, a treasurer, and a secretary?

Solution

$$P(35,4) = 1,256,640$$

Exercise

An economics club has 31 members.

a) If a committee of 4 is to be selected, in how many ways can the selection be made?
b) In how many ways can a committee of at least 1 and at most 3 be selected?

Solution

a) $C_{31,4} = 31,465$

b)
$$\begin{aligned} P(\text{at least } 1 \text{ and at most } 3 \text{ be selected}) &= C_{31,1} + C_{31,2} + C_{31,3} \\ &= 31 + 465 + 4495 \\ &= 4991 \end{aligned}$$

Exercise

In a club with 9 male and 11 female members, how many 5-member committees can be chosen that have

a) All men?
b) All women?
c) 3 men and 2 women?

Solution

a) $C(9,5) \underline{=126}|$

b) $C(11,5) \underline{=462}|$

c) $C(9,3)\cdot C(11,2)=(84)(55) \underline{=4,620}|$

Exercise

In a club with 9 male and 11 female members, how many 5-member committees can be selected that have

a) At least 4 women?
b) No more than 2 men?

Solution

a) $C(11,4)\cdot C(9,1)+C(11,5)\cdot C(9,0) \underline{=3,432}|$

b) $C(9,0)\cdot C(11,5)+C(9,1)\cdot C(11,4)+C(9,2)\cdot C(11,3) \underline{=9,372}|$

Exercise

In a game of musical chairs, 12 children will sit in 11 chairs arranged in a row (one will be left out). In how many ways can this happen, if we count rearrangements of the children in the chairs as different outcomes?

Solution

$P(12,11) \underline{=479,001,600 \quad \textit{different outcomes}}|$

Exercise

A group of 3 students is to be selected from a group of 14 students to take part in a class in cell biology.

a) In how many ways can this be done?
b) In how many ways can the group who will not take part be chosen?

Solution

a) $\binom{14}{3} \underline{=364 \quad \textit{ways}}|$

b) $\binom{14}{11} \underline{=364 \quad \textit{ways}}|$

Exercise

Marbles are being drawn without replacement from a bag containing 16 marbles.

a) How many samples of 2 marbles can be drawn?
b) How many samples of 2 marbles can be drawn?
c) If the bag contains 3 yellow, 4 white, and 9 blue marbles, how many samples of 2 marbles can be drawn in which both marbles are blue?

Solution

a) $C(16,2) \ \underline{= 120 \ \ samples}$

b) $C(16,4) \ \underline{= 1820 \ \ samples}$

c) $C(9,2) \ \underline{= 36 \ \ samples}$

Exercise

A bag contains 5 black, 1 red, and 3 yellow jelly beans; you take 3 at random. How many samples are possible in which the jelly beans are

a) All black?
b) All red?
c) All yellow?
d) 2 black and 1 red?
e) 2 black and 1 yellow?
f) 2 yellow and 1 black?
g) 2 red and 1 yellow?

Solution

a) $C_{5,3} \ \underline{= 10}$

b) No 3 read. $C_{1,3} \ \underline{= 0}$

c) $C_{3,3} \ \underline{= 1}$

d) $C_{5,2}C_{1,1} \ \underline{= 10}$

e) $C_{5,2}C_{3,1} \ \underline{= 30}$

f) $C_{3,2}C_{5,1} \ \underline{= 15}$

g) *There is only* **1** ***red***.

Solution Section 3.5 – Applications of Recurrence Relations

Exercise

a) Find a recurrence relation for the number of permutation of a set with *n* elements

b) Use the recurrence relation to find the number of permutations of a set with *n* elements using iteration.

Solution

a) A permutation of a set with *n* elements of a choice of a first element, followed by a permutation of a set of $n-1$ elements. Therefore $P_n = nP_{n-1} \quad with \quad P_0 = 1$

b)
$$\begin{aligned} P_n &= nP_{n-1} \\ &= n(n-1)P_{n-2} \\ &= n(n-1)\cdots 2\cdot 1\cdot P_0 \\ &= n! \end{aligned}$$

Exercise

A vending machine dispensing books of stamps accepts only one-dollar coins, \$1 bills, and \$5 bills.

a) Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the coins and bills are deposited matter.

b) What are the initial conditions?

c) How many ways are there to deposit \$10 for a book of stamps?

Solution

a) Let a_n be the number of ways to deposit *n* dollars in the vending machine. We must express a_n in terms of earlier terms in the sequence. If we want to deposit *n* dollars, we may start with a dollar coin and then deposit *n* – 1 dollars. This gives us a_{n-1} ways to deposit *n* dollars.

We can start with a dollar bill and then deposit *n* – 1 dollars. This gives us a_{n-1} more ways to deposit *n* dollars.

Finally, we can deposit a five-dollar bill and follow that with *n* – 5 dollars; there are a_{n-5} ways to do this, Therefore the recurrence relation is $a_n = 2a_{n-1} + a_{n-5} \quad for \quad n \ge 5$

b) We need initial conditions for all *n* from 0 to 4. Clearly, $a_0 = 1$ (deposit nothing) and $a_1 = 2$ (deposit either the dollar coin or the dollar bill)

$a_2 = 2^2 = 4; \quad a_3 = 2^3 = 8 \quad and \quad a_4 = 2^4 = 16$

c)
$$a_5 = 2a_4 + a_0 = 2(16) + 1 = 33$$
$$a_6 = 2a_5 + a_1 = 2(33) + 2 = 68$$
$$a_7 = 2a_6 + a_2 = 2(68) + 4 = 140$$

$a_8 = 2a_7 + a_3 = 2 \cdot 140 + 8 = 288$

$a_9 = 2a_8 + a_4 = 2 \cdot 288 + 16 = 592$

$a_{10} = 2a_8 + a_5 = 2 \cdot 592 + 33 = 1217$

Therefore, there are 1217 ways to deposit \$10.

Exercise

a) Find a recurrence relation for the number of bit strings of length *n* that contain three consecutive 0s.
b) What are the initial conditions?
c) How many bit strings of length seven contain three consecutive 0s?

Solution

a) Let a_n be the number of bit strings of length *n* containing three consecutive 0's. In order to construct a bit string of length *n* containing three consecutive 0's we could start with 1 and follow with a string of length *n* – 1 three consecutive 0's, or we could start with a 01 and follow with a string of length *n* – 2 three consecutive 0's, or we could start with a 001 and follow with a string of length *n* – 3 three consecutive 0's, or we could start with a 000 and follow with a string of length *n* – 3.

These 4 cases are mutually exclusive and exhaust the possibilities for how the string might start. We can write down the recurrence relation, valid for all $n \geq 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$

b) There are no bit strings of length 0, 1, or 2 containing 3 consecutive 0's, so the initial conditions are $a_0 = a_1 = a_2 = 0$

c)
$$\begin{aligned} a_3 &= a_2 + a_1 + a_0 + 2^0 \\ &= 0 + 0 + 0 + 1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} a_4 &= a_3 + a_2 + a_1 + 2^1 \\ &= 1 + 0 + 0 + 2 \\ &= 3 \end{aligned}$$

$$\begin{aligned} a_5 &= a_4 + a_3 + a_2 + 2^2 \\ &= 3 + 1 + 0 + 4 \\ &= 8 \end{aligned}$$

$$\begin{aligned} a_6 &= a_5 + a_4 + a_3 + 2^3 \\ &= 8 + 3 + 1 + 8 \\ &= 20 \end{aligned}$$

$$a_7 = a_6 + a_5 + a_4 + 2^4$$

$= 20 + 8 + 3 + 16$

$\underline{= 47}$

Therefore, there are 47 bits of length 7 containing three consecutive 0's.

Exercise

a) Find a recurrence relation for the number of bit strings of length *n* that do not contain three consecutive 0s.

b) What are the initial conditions?

c) How many bit strings of length seven do not contain three consecutive 0s?

Solution

a) Let a_n be the number of bit strings of length *n* do not contain three consecutive 0's. In order to construct a bit string of length *n* of this type we could start with 1 and follow with a string of length *n* – 1 not containing three consecutive 0's, or we could start with 01 and follow with a string of length *n* – 2 not containing three consecutive 0's, or we could start with a 001 and follow with a string of length *n* – 3 not containing three consecutive 0's.

These 3 cases are mutually exclusive and exhaust the possibilities for how the string might start since it cannot start 000.

We can write down the recurrence relation, valid for all $n \geq 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3}$

b) There are no bit strings of length 0, 1, or 2 containing 3 consecutive 0's, so the initial conditions are $a_0 = 1;\quad a_1 = 2 \quad and \quad a_2 = 4$

c) $a_3 = a_2 + a_1 + a_0$

$= 4 + 2 + 1$

$\underline{= 7}$

$a_4 = a_3 + a_2 + a_1$

$= 7 + 4 + 2$

$\underline{= 13}$

$a_5 = a_4 + a_3 + a_2$

$= 13 + 7 + 4$

$\underline{= 24}$

$a_6 = a_5 + a_4 + a_3$

$= 24 + 13 + 7$

$\underline{= 44}$

$a_7 = a_6 + a_5 + a_4$

$= 44 + 24 + 13$

$\underline{= 81}$

Therefore, there are 81 bits of length 7 that do not contain three consecutive 0's.

Exercise

a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.

b) What are the initial conditions?

c) In how many can this person climb a flight of eight stairs

Solution

a) Let a_n be the number of ways to climb n stairs. In order to climb n stairs, a person must either start with a step of one stair and climb $n - 1$ stairs $\left(a_{n-1}\right)$ or else start with a step of two stairs and then climb $n - 2$ stairs $\left(a_{n-2}\right)$ or else start with a step of two stairs and then climb $n - 3$ stairs $\left(a_{n-3}\right)$.
From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3}$

b) The initial conditions are $a_0 = 1, \quad a_1 = 1 \quad and \quad a_2 = 2$, since there is one way to climb no stairs (do nothing), clearly only one way to climb one stair, and two ways to climb stairs (one step twice or two steps at once).

c) Each term in our sequence $\left\{a_n\right\}$ is the sum of the previous three terms, so the sequence begins

$a_0 = 1,\ a_1 = 1,\ a_2 = 2,\ a_3 = 4,\ a_4 = 7,\ a_5 = 13,\ a_6 = 24,\ a_7 = 44,\ a_8 = 81$

Thus, a person can climb a flight of 8 stairs in 81 ways under the restrictions in this problem.

Solution *Section* 3.6 – Solving Linear Recurrence Relations

Exercise

Determine which of these are linear and homogeneous recurrence relations with constant coefficients. Also find the degree of those that are

a) $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$

b) $a_n = 2na_{n-1} + a_{n-2}$

c) $a_n = a_{n-1} + a_{n-4}$

d) $a_n = a_{n-1} + 2$

e) $a_n = a_{n-1}^2 + a_{n-2}$

f) $a_n = a_{n-2}$

g) $a_n = a_{n-1} + n$

h) $a_n = 3a_{n-2}$

i) $a_n = 3$

j) $a_n = a_{n-1}^2$

k) $a_n = a_{n-1} + 2a_{n-3}$

l) $a_n = \dfrac{a_{n-1}}{n}$

Solution

a) Linear (terms a_i all to the first power), has constant coefficients (3, 4 and 5), and is homogeneous (no terms are functions of just n); has degree 3

b) Linear (terms a_i all to the first power), doesn't have constant coefficients ($2n$), and is homogeneous

c) Linear, homogeneous, with constant coefficients; degree 4

d) Linear with constant coefficients, not homogeneous because of 2

e) Not linear since a_{n-1}^2

f) Linear, homogeneous, with constant coefficients; degree 2

g) Linear but not homogeneous because of the n.

h) Linear, homogeneous, with constant coefficients; degree 2

i) Linear with constant coefficients, not homogeneous because of 3

j) Not linear since a_{n-1}^2

k) Linear, homogeneous, with constant coefficients; degree 3

l) Linear with constant coefficients, not homogeneous

Exercise

Solve these recurrence relations together with the initial conditions given

a) $a_n = 2a_{n-1} \quad for \quad n \ge 1, \ a_0 = 3$

b) $a_n = 5a_{n-1} - 6a_{n-2} \quad for \quad n \ge 2, \ a_0 = 1, \ a_1 = 0$

c) $a_n = 4a_{n-1} - 4a_{n-2} \quad for \quad n \ge 2, \ a_0 = 6, \ a_1 = 8$

d) $a_n = 4a_{n-2} \quad for \quad n \ge 2, \ a_0 = 0, \ a_1 = 4$

e) $a_n = \dfrac{a_{n-2}}{4} \quad for \quad n \ge 2, \ a_0 = 1, \ a_1 = 0$

f) $a_n = a_{n-1} + 6a_{n-2} \quad for \quad n \ge 2, \ a_0 = 3, \ a_1 = 6$

g) $a_n = 7a_{n-1} - 10a_{n-2} \quad for \quad n \ge 2, \ a_0 = 2, \ a_1 = 1$

h) $a_n = -6a_{n-1} - 9a_{n-2} \quad for \quad n \ge 2, \ a_0 = 3, \ a_1 = -3$

i) $a_{n+2} = -4a_{n-1} + 5a_n \quad for \quad n \ge 0, \ a_0 = 2, \ a_1 = 8$

Solution

a) The characteristic polynomial is $r - 2 = 0 \quad \Rightarrow \quad r = 2$

The general solution: $a_n = \alpha_1 2^n$

$$3 = \alpha_1 2^0 \quad \to \quad \alpha_1 = 3$$

Therefore, the solution is $\boxed{a_n = 3 \cdot 2^n}$

b) The characteristic polynomial is $r^2 - 5r + 6 = 0 \quad \Rightarrow \quad r = 2, \ 3$

The general solution: $a_n = \alpha_1 2^n + \alpha_2 3^n$

$$\begin{aligned} 1 = \alpha_1 2^0 + \alpha_2 3^0 \quad &\to \quad 1 = \alpha_1 + \alpha_2 \\ 0 = \alpha_1 2^1 + \alpha_2 3^1 \quad &\to \quad 0 = 2\alpha_1 + 3\alpha_2 \end{aligned} \quad \Rightarrow \quad \alpha_1 = 3, \quad \alpha_2 = -2$$

Therefore, the solution is $\boxed{a_n = 3 \cdot 2^n - 2 \cdot 3^n}$

c) The characteristic polynomial is $r^2 - 4r + 4 = 0 \quad \Rightarrow \quad r = 2, \ 2$

The general solution: $a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n$

$$\left.\begin{array}{l} 6=\alpha_1 2^0+\alpha_2 (0)2^0 \quad \rightarrow \quad 6=\alpha_1 \\ 8=\alpha_1 2^1+\alpha_2 (1)2^1 \quad \rightarrow \quad 8=2\alpha_1+2\alpha_2 \end{array}\right. \Rightarrow \quad \alpha_1=6, \quad \alpha_2=-2$$

Therefore, the solution is $a_n=6\cdot 2^n-2n\cdot 2^n \ \boxed{=(6-2n)2^n}$

d) The characteristic polynomial is $r^2-4=0 \quad \Rightarrow \quad r=\pm 2$

The general solution: $a_n=\alpha_1(-2)^n+\alpha_2 2^n$

$$\left.\begin{array}{l} 0=\alpha_1(-2)^0+\alpha_2 2^0 \quad \rightarrow \quad 0=\alpha_1+\alpha_2 \\ 4=\alpha_1(-2)^1+\alpha_2 2^1 \quad \rightarrow \quad 4=-2\alpha_1+2\alpha_2 \end{array}\right. \Rightarrow \quad \alpha_1=-1, \quad \alpha_2=1$$

Therefore, the solution is $\boxed{a_n=2^n-(-2)^n}$

e) The characteristic polynomial is $r^2-\frac{1}{4}=0 \quad \Rightarrow \quad r=\pm\frac{1}{2}$

The general solution: $a_n=\alpha_1\left(-\frac{1}{2}\right)^n+\alpha_2\left(\frac{1}{2}\right)^n=\alpha_1(-2)^{-n}+\alpha_2(2)^{-n}$

$$\left.\begin{array}{l} 1=\alpha_1\left(-\frac{1}{2}\right)^0+\alpha_2\left(\frac{1}{2}\right)^0 \quad \rightarrow \quad 1=\alpha_1+\alpha_2 \\ 0=\alpha_1\left(-\frac{1}{2}\right)^1+\alpha_2\left(\frac{1}{2}\right)^1 \quad \rightarrow \quad 0=-\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2 \end{array}\right. \Rightarrow \quad \alpha_1=\frac{1}{2}, \quad \alpha_2=\frac{1}{2}$$

Therefore, the solution is $a_n=\frac{1}{2}\left(-\frac{1}{2}\right)^n+\frac{1}{2}\left(\frac{1}{2}\right)^n$

$$\boxed{=\left(\frac{1}{2}\right)^{n+1}-\left(-\frac{1}{2}\right)^{n+1}}$$

f) The characteristic polynomial is $r^2-r-6=0 \quad \Rightarrow \quad r=-2,\ 3$

The general solution: $a_n=\alpha_1(-2)^n+\alpha_2 3^n$

$$\left.\begin{array}{l} 3=\alpha_1(-2)^0+\alpha_2 3^0 \quad \rightarrow \quad 3=\alpha_1+\alpha_2 \\ 6=\alpha_1(-2)^1+\alpha_2 3^1 \quad \rightarrow \quad 6=-2\alpha_1+3\alpha_2 \end{array}\right. \Rightarrow \quad \alpha_1=\frac{3}{5}, \quad \alpha_2=\frac{12}{5}$$

Therefore, the solution is $\boxed{a_n=\frac{3}{5}(-2)^n+\frac{12}{5}3^n}$

g) The characteristic polynomial is $r^2-7r+10=0 \quad \Rightarrow \quad r=2,\ 5$

The general solution: $a_n=\alpha_1 2^n+\alpha_2 5^n$

$$\left.\begin{array}{l} 2=\alpha_1 2^0+\alpha_2 5^0 \quad \rightarrow \quad 2=\alpha_1+\alpha_2 \\ 1=\alpha_1 2^1+\alpha_2 5^1 \quad \rightarrow \quad 1=2\alpha_1+5\alpha_2 \end{array}\right. \Rightarrow \quad \alpha_1=3, \quad \alpha_2=-1$$

Therefore, the solution is $\boxed{a_n=3\cdot 2^n-5^n}$

h) The characteristic polynomial is $r^2+6r+9=0 \quad \Rightarrow \quad r=-3,\ -3$

The general solution: $a_n=\alpha_1(-3)^n+\alpha_2 n(-3)^n$

$$\begin{aligned} 3&=\alpha_1(-3)^0+\alpha_2(0)(-3)^0 \quad \rightarrow \quad 3=\alpha_1 \\ -3&=\alpha_1(-3)^1+\alpha_2(1)(-3)^1 \quad \rightarrow \quad -3=-3\alpha_1+-3\alpha_2 \end{aligned} \quad \Rightarrow \quad \alpha_1=3, \quad \alpha_2=-2$$

Therefore, the solution is $\boxed{a_n=3\cdot(-3)^n-2n(-3)^n=(3-2n)(-3)^n}$

i) The characteristic polynomial is $r^2+4r-5=0 \quad \Rightarrow \quad r=-5,\ 1$

The general solution: $a_n=\alpha_1(-5)^n+\alpha_2 1^n=\alpha_1(-5)^n+\alpha_2$

$$\begin{aligned} 2&=\alpha_1(-5)^0+\alpha_2 \quad \rightarrow \quad 2=\alpha_1+\alpha_2 \\ 8&=\alpha_1(-5)^1+\alpha_2 \quad \rightarrow \quad 8=-5\alpha_1+\alpha_2 \end{aligned} \quad \Rightarrow \quad \alpha_1=-1, \quad \alpha_2=3$$

Therefore, the solution is $\boxed{a_n=-(-5)^n+3}$

Exercise

How many different messages can be transmitted in n microseconds using three different signals if one signal requires 1 microsecond for transmittal, the other two signals require 2 microseconds each for transmittal, and a signal in a message is followed immediately by the next signal?

Solution

The model is the recurrence relation $a_n=a_{n-1}+a_{n-2}+a_{n-2}=a_{n-1}+2a_{n-2}$ with $a_0=a_1=1$

The characteristic polynomial is $r^2-r-2=0$

So, the roots are -1, and 2

The general solution: $\quad a_n=\alpha_1(-1)^n+\alpha_2 2^n$

Plugging in initial conditions gives

$$\begin{aligned} 1&=\alpha_1(-1)^0+\alpha_2 2^0 \quad \rightarrow \quad 1=\alpha_1+\alpha_2 \\ 1&=\alpha_1(-1)^1+\alpha_2 2^1 \quad \rightarrow \quad 1=-\alpha_1+2\alpha_2 \end{aligned} \quad \Rightarrow \quad \alpha_1=\tfrac{1}{3}, \quad \alpha_2=\tfrac{2}{3}$$

Therefore, the solution is in n microseconds $\boxed{a_n=\frac{1}{3}(-1)^n+\frac{2}{3}2^n}$ messages can be transmitted.

Exercise

In how many ways can a $2\times n$ rectangular checkerboard be tiled using 1×2 and 2×2 pieces?

Solution

Let t_n be the number of ways like to tile a $2\times n$ board with 1×2 and 2×2 pieces. To obtain the recurrence relation, imagine what tiles are placed at the left-hand end of the board. We can place a 2×2 tile there, leaving a $2\times(n-2)$ board to be tiled, which of course can be done in t_{n-2} ways.

We can place a 1×2 tile at the edge, oriented vertically, leaving $2\times(n-1)$ board to be tiled, which of course can be done in t_{n-1} ways.

Finally, we can place two 1×2 tiles horizontally, one above the other, leaving a $2\times(n-2)$ board to be tiled, which of course can be done in t_{n-2} ways. These 3 possibilities are disjoint.

Therefore, our recurrence relation is $t_n = t_{n-1} + 2t_{n-2}$

The initial conditions are $t_0 = t_1 = 1$, since there is only one way to tile as 2×0 board and 2×1 board.

This recurrence relation has characteristic roots -1 and 2.

So, the general solution is $t_n = \alpha_1(-1)^n + \alpha_2 2^n$

Plugging in initial conditions gives

$$\left.\begin{array}{l} 1=\alpha_1(-1)^0+\alpha_2 2^0 \quad \rightarrow \quad 1=\alpha_1+\alpha_2 \\ 1=\alpha_1(-1)^1+\alpha_2 2^1 \quad \rightarrow \quad 1=-\alpha_1+2\alpha_2 \end{array}\right. \Rightarrow \quad \alpha_1=\frac{1}{3}, \quad \alpha_2=\frac{2}{3}$$

Therefore, the solution is $a_n = \frac{1}{3}(-1)^n + \frac{2}{3}\cdot 2^n$

$$= \frac{(-1)^n}{3} + \frac{2^{n+1}}{3}$$

Exercise

Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3} \quad for \quad n\geq 3,\ a_0 = 3,\ a_1 = 6\ and\ a_2 = 0$

Solution

$$a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$$

The characteristic polynomial is $r^3 - 2r^2 - r + 2 = 0$

That implies to: $r^2(r-2) - (r-2) = (r-2)\left(r^2-1\right) = 0$

So, the roots are $1, -1$, and 2

The general solution:

$$a_n = \alpha_1 1^n + \alpha_2(-1)^n + \alpha_3 2^n$$
$$= \alpha_1 + \alpha_2(-1)^n + \alpha_3 2^n$$

Plugging in initial conditions gives

$$3=\alpha_1+\alpha_2(-1)^0+\alpha_3 2^0 \quad \rightarrow \quad 3=\alpha_1+\alpha_2+\alpha_3$$

$$6 = \alpha_1 + \alpha_2 (-1)^1 + \alpha_3 2^1 \quad \rightarrow \quad 6 = \alpha_1 - \alpha_2 + 2\alpha_3$$

$$\Rightarrow \ \alpha_1 = 6, \quad \alpha_2 = -2, \quad \alpha_3 = -1$$

$$0 = \alpha_1 + \alpha_2 (-1)^2 + \alpha_3 2^2 \quad \rightarrow \quad 0 = \alpha_1 + \alpha_2 + 4\alpha_3$$

Therefore, the solution is $a_n = 6 - 2(-1)^n - 2^n$

Exercise

Find the solution to $a_n = 7a_{n-2} + 6a_{n-3} \quad \text{with} \quad a_0 = 9,\ a_1 = 10 \text{ and } a_2 = 32$

Solution

This is a third-degree recurrence relation.

The characteristic polynomial is $r^3 - 7r - 6 = 0$

By the rational root test, the possible rational roots are $\pm\left\{\frac{6}{1}\right\} = \pm\{1,\ 2,\ 3,\ 6\}$

We find that $r = -1$ (using calculator).

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array} \qquad Q(x) = r^2 - r - 6 = (r+2)(r-3)$$

$$r^3 - 6r^2 + 12r - 8 = (r+1)(r+2)(r-3) = 0$$

So, the roots are $-2, -1$, and 3.

The general solution:

$$a_n = \alpha_1 (-2)^n + \alpha_2 (-1)^n + \alpha_3 3^n$$

Plugging in initial conditions gives

$$a_0 = 9 = \alpha_1 (-2)^0 + \alpha_2 (-1)^0 + \alpha_3 3^0 \quad \rightarrow \quad 9 = \alpha_1 + \alpha_2 + \alpha_3$$

$$a_1 = 10 = \alpha_1 (-2)^1 + \alpha_2 (-1)^1 + \alpha_3 3^1 \quad \rightarrow \quad 10 = -2\alpha_1 - \alpha_2 + 3\alpha_3$$

$$a_2 = 32 = \alpha_1 (-2)^2 + \alpha_2 (-1)^2 + \alpha_3 3^2 \quad \rightarrow \quad 32 = 4\alpha_1 + \alpha_2 + 9\alpha_3$$

The solution to the system of equations is $\alpha_1 = -3, \ \alpha_2 = 8 \ \text{ and } \ \alpha_3 = 4$

Therefore, the specific solution is $a_n = -3(-2)^n + 8(-1)^n + 4 \cdot 3^n$

Exercise

Find the solution to $a_n = 5a_{n-2} - 4a_{n-4}$ *with* $a_0 = 3,\ a_1 = 2,\ a_2 = 6\ and\ a_3 = 8$

Solution

This is a fourth-degree recurrence relation.

The characteristic polynomial is $r^4 - 5r^2 - 4 = 0$

That implies to: $\left(r^2 - 1\right)\left(r^2 - 4\right) = (r-1)(r+1)(r-2)(r+2) = 0$

So, the roots are $1, -1, 2, -2$

The general solution: $a_n = \alpha_1 + \alpha_2 (-1)^n + \alpha_3 2^n + \alpha_4 (-2)^n$

Plugging in initial conditions gives

$$3 = \alpha_1 + \alpha_2 (-1)^0 + \alpha_3 2^0 + \alpha_4 (-2)^0 \quad \rightarrow \quad 3 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

$$10 = \alpha_1 + \alpha_2 (-1)^1 + \alpha_3 2^1 + \alpha_4 (-2)^1 \quad \rightarrow \quad 10 = \alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4$$

$$6 = \alpha_1 + \alpha_2 (-1)^2 + \alpha_3 2^2 + \alpha_4 (-2)^2 \quad \rightarrow \quad 6 = \alpha_1 + \alpha_2 + 4\alpha_3 + 4\alpha_4$$

$$8 = \alpha_1 + \alpha_2 (-1)^3 + \alpha_3 2^3 + \alpha_4 (-2)^3 \quad \rightarrow \quad 8 = \alpha_1 - \alpha_2 + 8\alpha_3 - 8\alpha_4$$

The solution to the system of equations is $\alpha_1 = \alpha_2 = \alpha_3 = 1$ *and* $\alpha_4 = 0$

Therefore, the solution is $\boxed{a_n = 1 + (-1)^n + 2^n}$

Exercise

Find the recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ *with* $a_0 = -5,\ a_1 = 4\ and\ a_2 = 88$

Solution

This is a third-degree recurrence relation.

The characteristic polynomial is $r^3 - 6r^2 + 12r - 8 = 0$

By the rational root test, the possible rational roots are $\pm 1,\ \pm 2,\ \pm 4,\ \pm 8$

We find that $r = 2$ (using calculator).

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 12 & -8 \\ & & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array} \qquad Q(x) = r^2 - 4r + 4 = (r-2)^2$$

$$r^3 - 6r^2 + 12r - 8 = (r-2)^3 = 0$$

Hence the only root is 2, with multiplicity 3.

The general solution: $\boxed{a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n + \alpha_3 n^2 \cdot (-2)^n}$

Plugging in initial conditions gives

$$\boxed{-5 = a_0 = \alpha_1}$$

$$4 = a_1 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 \qquad \alpha_1 + \alpha_2 + \alpha_3 = 2 \rightarrow \alpha_2 + \alpha_3 = 7$$

$$88 = a_2 = 4\alpha_1 + 8\alpha_2 + 16\alpha_3 \qquad \alpha_1 + 2\alpha_2 + 4\alpha_3 = 22 \rightarrow 2\alpha_2 + 4\alpha_3 = 27$$

$$\rightarrow \begin{cases} \alpha_2 + \alpha_3 = 7 \\ 2\alpha_2 + 4\alpha_3 = 27 \end{cases} \Rightarrow \quad \boxed{\alpha_2 = \frac{1}{2}} \quad \boxed{\alpha_3 = \frac{13}{2}}$$

Therefore, the solution: $\quad a_n = -5 \cdot 2^n + \frac{1}{2} n \cdot 2^n + \frac{13}{2} n^2 \cdot (-2)^n$

$$\boxed{= -5 \cdot 2^n + n \cdot 2^{n-1} + 13n^2 \cdot (-2)^{n-1}}$$

Exercise

Find the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ *with* $a_0 = 5,\ a_1 = -9$ *and* $a_2 = 15$

Solution

This is a third-degree recurrence relation.

The characteristic polynomial is $r^3 + 3r^2 + 3r + 1 = 0$

$r^3 + 3r^2 + 3r + 1 = 0 = (r+1)^3 = 0$

Hence the only root is -1, with multiplicity 3.

The general solution: $\quad \boxed{a_n = \alpha_1 (-1)^n + \alpha_2 n \cdot (-1)^n + \alpha_3 n^2 \cdot (-1)^n}$

Plugging in initial conditions gives

$$\boxed{5 = a_0 = \alpha_1}$$

$$a_1 = -9 = -\alpha_1 - \alpha_2 - \alpha_3 \quad \rightarrow \quad \alpha_2 + \alpha_3 = 9 - \alpha_1 = 4$$

$$a_2 = 15 = \alpha_1 + 2\alpha_2 + 4\alpha_3 \quad \rightarrow \quad 2\alpha_2 + 4\alpha_3 = 15 - \alpha_1 = 10$$

$$\rightarrow \begin{cases} \alpha_2 + \alpha_3 = 4 \\ 2\alpha_2 + 4\alpha_3 = 10 \end{cases} \Rightarrow \quad \boxed{\alpha_2 = 3} \quad \boxed{\alpha_3 = 1}$$

Therefore, the specific solution is $\quad a_n = 5(-1)^n + 3n \cdot (-1)^n + n^2 \cdot (-1)^n$

$$\boxed{= \left(n^2 + 3n + 5\right)(-1)^n}$$

Exercise

Find the general form of the solutions of the recurrence relation $a_n = 8a_{n-2} - 16a_{n-4}$

Solution

This is a fourth-degree recurrence relation.

The characteristic polynomial is $r^4 - 8r^2 + 16 = \left(r^2 - 4\right)^2$

$\left(r^2 - 4\right)^2 = (r-2)^2 (r+2)^2 = 0$

The roots are –2 and 2, each with multiplicity 2.

The general solution: $a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n + \alpha_3 (-2)^n + \alpha_4 n \cdot (-2)^n$

Exercise

What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots 1, 1, 1, 1, –2, –2, –2, 3, 3, –4?

Solution

There are 4 distinct roots, so $t = 4$. The multiplicities are 4, 3, 2, and 1.

The general solution:

$$a_n = \left(\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2 + \alpha_{1,3} n^3\right) + \left(\alpha_{2,0} + \alpha_{2,1} n + \alpha_{2,2} n^2\right)(-2)^n + \left(\alpha_{3,0} + \alpha_{3,1} n\right)3^n + \alpha_{4,0} (-4)^n$$

Exercise

What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots $-1, -1, -1, 2, 2, 5, 5, 7$?

Solution

There are 4 distinct roots, so $t = 4$. The multiplicities are 3, 2, 2, and 1.

The general solution:

$$a_n = \left(\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2\right)(-1)^n + \left(\alpha_{2,0} + \alpha_{2,1} n\right)2^n + \left(\alpha_{3,0} + \alpha_{3,1} n\right)5^n + \alpha_{4,0} 7^n$$

SOLUTION *Section* 4.1 – Relations and Their Properties

Exercise

List the ordered pairs in the relation R from A = {0, 1, 2, 3, 4} to B = {0, 1, 2, 3} where $(a,\ b) \in R$ if and only if

a) $a = b$ ***b***) $a + b = 4$ ***c***) $a > b$

d) $a \mid b$ ***e***) $gcd(a,b) = 1$ ***f***) $lcm(a,b) = 2$

Solution

a) {(0, 0), (1, 1), (2, 2), (3, 3)}

b) {(4, 0), (1, 3), (3, 1), (2, 2)}

c) {(1, 0), (2, 0), (2,1), (3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2), (4, 3)}

d) {(1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 0) , (3, 3) , 4, 0)} **(**means b is multiple of $a \neq 0$**)**

e) {(1, 0), (0, 1), (1,1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (4, 1), (4, 3)} **(**means *relatively prime***)**

f) {(1, 2), (2, 1), (2, 2)} **(**Mean *least common multiple* is 2**)**.

Exercise

a) List all the ordered pairs in the relation $R = \{(a,\ b) \mid a\ divides\ \ b\}$ on the set {1, 2, 3, 4, 5, 6}

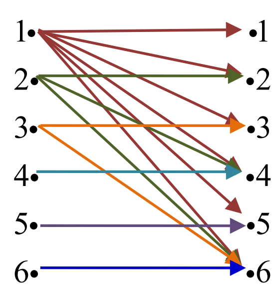
b) Display this relation graphically.

c) Display this relation in tabular form.

Solution

a) {(1, 1), (1, 2), (1,3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)}

b)



c)

R	1	2	3	4	5	6
1	×	×	×	×	×	×
2		×		×		×
3			×			×
4				×		
5					×	
6						×

Exercise

For each of these relations on the set {1, 2, 3, 4}, decide whether it is reflexive, symmetric, antisymmetric and transitive

a) {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}
b) {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}
c) {(2, 4), (4, 2)}
d) {(1, 2), (2, 3), (3, 4)}
e) {(1, 1), (2, 2), (3, 3), (4, 4)}
f) {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}

Solution

a) This relation is not reflexive, since (1, 1) is not included
It is not symmetric, since (2, 4) is included but not (4, 2)
It is not antisymmetric, since it includes (2, 3) and (3, 2) but $2 \neq 3$

$$\begin{matrix} (2,\ 3) \ \& \ (3,\ 4) \ \to \ (2,\ 4) \\ (2,\ 3) \ \& \ (3,\ 2) \ \to \ (2,\ 2) \end{matrix} \ \& \ \begin{matrix} (3,\ 2) \ \& \ (2,\ 4) \ \to \ (3,\ 4) \\ (3,\ 2) \ \& \ (2,\ 3) \ \to \ (3,\ 3) \end{matrix}$$ It is transitive.

b) This relation is reflexive, since (1, 1), (2, 2), (3, 3), and (4, 4)} are included
It is symmetric, since (2, 1) and (1, 2) are included
It is not antisymmetric, since it includes (2, 1) and (1, 2) but $2 \neq 1$

$$\begin{matrix} (2,\ 1) \ \& \ (1,\ 2) \ \to \ (2,\ 2) \\ (1,\ 2) \ \& \ (2,\ 1) \ \to \ (1,\ 1) \end{matrix}$$ It is transitive.

c) This relation is not reflexive, since (1, 1) is not included
It is symmetric, since (2, 4) and (4, 2) are included
It is not antisymmetric, since it includes (2, 4) and (4, 2) but $2 \neq 4$
It is not transitive, since it includes (2, 4) and (4, 2) but not (2, 2)

d) This relation is not reflexive, since (1, 1) is not included
It is not symmetric, since (1, 2) is included but not (2, 1)
It is antisymmetric, since no cases of (*a, b*) and (*b, a*) both being in the relation
It is not transitive, since it includes (1, 2) and (2, 3) but not (1, 3)

e) This relation is reflexive, since (1, 1), (2, 2), (3, 3), and (4, 4)} are included and it is *symmetric*
It is antisymmetric, since no cases of (*a, b*) and (*b, a*) both being in the relation
It is transitive, since the only time the hypothesis $(a,\ b) \in R \wedge (b,\ c) \in R$ is met is when $a \equiv b \equiv c$

f) This relation is not reflexive, since (1, 1) is not included
It is not symmetric, since (1, 4) is included but not (4, 1)
It is not antisymmetric, since it includes (1, 3) and (3, 1)
It is not transitive, since it includes (2, 3) and (3, 1) but not (2, 1)

Exercise

Determine whether the relation R on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a,\ b) \in R$ if and only if

a) a is taller than b.
b) a and b were born on the same day
c) a has the same first name as b.
d) a and b have a common grandparent.

Solution

a) I am *not* taller than myself, therefore *being taller* is not reflexive
It is *not* symmetric, since I am taller than my kid but my kid is not
It is antisymmetric since we never have a taller than b and b taller than a even if $a = b$
It is transitive since if a taller than b and b taller than c that implies that A taller then c

b) The relation is reflexive since a is born on the same day
It is symmetric, since a and b were born on the same day
It is *not* antisymmetric since a and b were born on the same day but $a \neq b$
It is transitive since if a and b were born on the same day and b and c were born on the same day that implies that a and c were born on the same day

c) The relation is reflexive since a has the same first name as a
It is symmetric, since a has the same first name as b than b has the same first name as a
It is *not* antisymmetric since a has the same first name as b but $a \neq b$
It is transitive since if a has the same first name as b and c has the same first name as c that implies that a has the same first name as c

d) The relation is reflexive since a and a have a common grandparent
It is symmetric, since a and b have a common grandparent than b and a have a common grandparent
It is *not* antisymmetric since a and b have a common grandparent but $a \neq b$
It is transitive since if a and b have a common grandparent and b and c have a common grandparent that implies that a and c have a common grandparent

Exercise

Determine whether the relation $\boldsymbol{R}$ on the set of all ***real numbers*** is reflexive, symmetric, antisymmetric, and/or transitive, where $(x,\ y) \in \boldsymbol{R}$ if and only if

a) $x + y = 0$ ***b)*** $x = \pm y$ ***c)*** $x - y$ *is a rational number* ***d)*** $x = 2y$
e) $xy \geq 0$ ***f)*** $xy = 0$ ***g)*** $x = 1$ ***h)*** $x = 1$ *or* $y = 1$

Solution

a) The relation is *not* reflexive since $1 + 1 \neq 0$
It is symmetric, since $x + y = 0$ then $y + x = 0$ because $x + y = y + x$
It is *not* antisymmetric since $(1, -1) \in \boldsymbol{R}$ and $(-1, 1) \in \boldsymbol{R}$ but $1 \neq -1$

It is *not* transitive since (1, −1) and (−1, 1) $\in \boldsymbol{R}$ but $(1,\ 1) \notin \boldsymbol{R}$

b) The relation is reflexive since $x = \pm x$

It is symmetric, since $x = \pm y$ *iff* $y = \pm x$

It is *not* antisymmetric since (1, −1) $\in \boldsymbol{R}$ and (−1, 1) $\in \boldsymbol{R}$ but $1 \neq -1$

It is transitive since the product 1's and −1's is ± 1

c) The relation is reflexive since $x - x = 0$ is a rational number

It is symmetric, since $x - y$ is rational, then $-(x-y) = y - x$

It is *not* antisymmetric since (1, −1) $\in \boldsymbol{R}$ but (−1, 1) $\in \boldsymbol{R}$ but $1 \neq -1$

It is transitive since $(x, y) \in \boldsymbol{R}$ then $x - y$ is a rational number $(y, z) \in \boldsymbol{R}$ then $x - y$ is a rational number, therefore $x - z$ is rational that means that $(x, z) \in \boldsymbol{R}$

d) The relation is *not* reflexive since $1 \neq 2 \cdot 1$

It is *not* symmetric, since $(2,\ 1) \in \boldsymbol{R}$ then $2 = 2 \cdot 1$ but $1 \neq 2 \cdot 2$ therefore $(1,\ 2) \notin \boldsymbol{R}$

It is antisymmetric since $x = 2y$ and $y = 2x$ that implies to $y = 2(2y) = 4y$ which $y = 0$

It is *not* transitive since $2 = 2 \cdot 1$ and $4 = 2 \cdot 2 \;\Rightarrow\; 4 \neq 2 \cdot 1$ so $(4,\ 1) \notin \boldsymbol{R}$

e) The relation is reflexive since $x^2 \geq 0$ always positive

It is symmetric, since $xy \geq 0 \quad \Rightarrow \quad yx \geq 0$

It is *not* antisymmetric since (2, 3) $\in \boldsymbol{R}$ and (3, 2) $\in \boldsymbol{R}$ but $2 \neq 3$

It is *not* transitive since $(1,\ 0) \in \boldsymbol{R} \Rightarrow 1 \cdot 0 \geq 0 \quad (0,\ -1) \in \boldsymbol{R} \Rightarrow 0 \cdot (-1) \geq 0$ but $1 \cdot (-1) \not\geq 0 \Rightarrow (1,\ -1) \notin \boldsymbol{R}$

f) $\boxed{xy = 0}$ The relation is *not* reflexive since $(1,\ 1) \notin \boldsymbol{R}$

It is symmetric, since $xy = 0 \quad \to \quad yx = 0$

It is antisymmetric since (2, 0) $\in \boldsymbol{R}$ and (0, 2) $\in \boldsymbol{R}$ but $2 \neq 0$

It is *not* transitive since $2 \cdot 0 = 0$ (2, 0) $\in \boldsymbol{R}$ and $0 \cdot (-2) = 0$ (0, −2) $\in \boldsymbol{R} \Rightarrow$ $2 \cdot (-2) \neq 0$ so $(2,\ -2) \notin \boldsymbol{R}$

g) The relation is *not* reflexive since $(2,\ 2) \notin \boldsymbol{R}$

It is *not* symmetric, since $(1,\ 2) \in \boldsymbol{R}$ but $(2,\ 1) \notin \boldsymbol{R}$

It is antisymmetric since $(x, y) \in \boldsymbol{R}$ and $(y, x) \in \boldsymbol{R}$ then $x = 1$ and $y = 1$, so $x = y$

It is transitive since $(x, y) \in \boldsymbol{R}$ and $(y, z) \in \boldsymbol{R}$ then $x = 1$ and $y = 1$, so $(x, z) \in \boldsymbol{R}$

h) The relation is *not* reflexive since $(2,\ 2) \notin \boldsymbol{R}$

It is symmetric, since $(1,\ 2) \in \boldsymbol{R}$ and $(2,\ 1) \in \boldsymbol{R}$

It is *not* antisymmetric since $(1,\ 2) \in \boldsymbol{R}$ and $(2,\ 1) \in \boldsymbol{R}$ but $1 \neq 2$

It is *not* transitive since $(2,\ 1) \in \boldsymbol{R}$ and $(1,\ 3) \in \boldsymbol{R}$ but $(2,\ 3) \notin \boldsymbol{R}$

Exercise

Determine whether the relation R on the set of all ***integers numbers*** is reflexive, symmetric, antisymmetric, and/or transitive, where $(x,\ y) \in R$ if and only if

$\boldsymbol{a)}\ x \neq y$ $\quad\boldsymbol{b)}\ xy \geq 1$ $\quad\boldsymbol{c)}\ x = y+1\ \ or\ \ x = y-1$ $\quad\boldsymbol{d)}\ x \equiv y(\boldsymbol{mod}\ 7)$

$\boldsymbol{e)}\ x\ is\ a\ multiple\ of\ y$ $\quad\boldsymbol{f)}\ x = y^2$ $\quad\boldsymbol{g)}\ x \geq y^2$

Solution

a) This relation is not reflexive, since $1 \neq 1$ for instance

It is symmetric, if $x \neq y \Rightarrow y \neq x$

It is *not* antisymmetric since $1 \neq 2 \Rightarrow 2 \neq 1$

It is *not* transitive since $1 \neq 2\ and\ 2 \neq 1 \Rightarrow 1 \neq 1$

b) This relation is not reflexive, since (0, 0) is not included $(0 \not\geq 1)$

It is symmetric, because $xy = yx$ (commutative property of multiplication)

It is *not* antisymmetric since (2, 3) and (3, 2) are both included

It is transitive holds between x and y if and only if either x and y are both positive or x and y are both negative

c) This relation is *not* reflexive, since (1, 1) is not included $(1 \neq 1+1)$

It is symmetric, because $x = y-1$ is equivalent to $y = x+1$

It is *not* antisymmetric since (1, 2) and (2, 1) are in the relation

It is *not* transitive since (1, 2) and (2, 1) are in the relation but (1, 1) is not

d) $x \equiv y(\boldsymbol{mod}\ 7)$ means that $x - y = 7t$ for some t.

This relation is reflexive since $x - x = 7 \cdot 0$

It is symmetric since is $x \equiv y(\boldsymbol{mod}\ 7)$ then $x - y = 7t$, therefore $y - x = 7(-t)$ so $y \equiv x(\boldsymbol{mod}\ 7)$

It is *not* antisymmetric since $1 \equiv 8(\boldsymbol{mod}\ 7)$ and $8 \equiv 1(\boldsymbol{mod}\ 7)$

It is transitive since $x \equiv y(\boldsymbol{mod}\ 7)$ means $x - y = 7t$ and $y \equiv z(\boldsymbol{mod}\ 7)$ means $y - z = 7s$

$x - y = x - y + y - z = 7t + 7s = 7(t+s)$; therefore $x \equiv z(\boldsymbol{mod}\ 7)$

e) $x\ is\ a\ multiple\ of\ y$ means that $x = ty$ for some t.

This relation is reflexive since $x = x \cdot 1$

It is *not* symmetric since is $6 = 3 \cdot 2$ but $2 \neq 3 \cdot 6$

It is *not* antisymmetric since 2 is multiple of -2 but $2 \neq -2$

It is transitive since $x = ty$ and $y = sz \Rightarrow x = ty = tsz = (ts)z$ therefore x is a multiply of z.

f) This relation is *not* reflexive, since $3 \neq 3^2$

It is *not* symmetric since is $9 = 3^2$ but $3 \neq 9^2$

It is antisymmetric since $x = y^2$ and $y = x^2$

$$\Rightarrow x = y^2 = x^4$$

$$x - x^4 = 0$$

$$x\left(1 - x^3\right) = 0$$

$$x(1-x)\left(1+x+x^2\right)=0 \qquad \rightarrow x=0,\ 1$$

$x=y^2$ and $y=x^2$ when $x=y$

It is *not* transitive since $81=9^2$ and $9=3^2$ but $81\neq 3^2$

g) This relation is *not* reflexive, since $3 \not\geq 3^2$

It is *not* symmetric since is $9\geq 3^2$ but $3 \not\geq 9^2$

It is antisymmetric since $x\geq y^2$ and $y\geq x^2$, only when $x=0,\ 1$.

It is transitive since $x\geq y^2$ and $y\geq z^2$

$$\begin{aligned} x &\geq y^2 \\ &\geq \left(z^2\right)^2 \\ &= z^4 \\ &\geq z^2 \end{aligned}$$

Exercise

Show that the relation $R=\varnothing$ on nonempty set S is symmetric and transitive, but not reflexive.

Solution

If $R=\varnothing$, then the hypothesis of the conditional statements in the definitions of symmetric and transitive are never true, so those statements are always true by definition.

$S\neq\varnothing$ the statement $(a,a)\in R$ is false for an element of *S*, so $\forall a\ (a,a)\in R$ is not true, thus *R* is not reflexive.

Exercise

Show that the relation $R=\varnothing$ on nonempty set $S=\varnothing$ is reflexive, symmetric and transitive.

Solution

Since the domain is empty, then the relation is vacuously reflexive, symmetric and transitive s

Exercise

Give an example of a relation on a set that is

a) both symmetric and antisymmetric

b) neither symmetric nor antisymmetric

Solution

a) The empty set on $\{a\}$ – vacuously symmetric and antisymmetric

b) $\{(a, b), (b, a), (a, c)\}$ on $\{a, b, c\}$

Exercise

A relation R is called ***asymmetric*** if $(a,\ b) \in R$ implies that $(b,\ a) \notin R$. Explore the notion of an asymmetric relation to the following

a) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
b) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
c) $\{(2, 4), (4, 2)\}$
d) $\{(1, 2), (2, 3), (3, 4)\}$
e) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
f) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$
g) a is taller than b.
h) a and b were born on the same day
i) a has the same first name as b.
j) a and b have a common grandparent.

Solution

The relations (***a***), (***b***), and (***c***) are not *asymmetric* since they contain pairs of the form (x, x)

The relation (***f***) is not *asymmetric* since both (1, 3) and (3, 1) are in the relation

The relation (***d***) is not *asymmetric*

The relation (***g***) is *asymmetric* since if a taller than b, then b can't be taller than a.

The relation (***h***) is not *asymmetric* since a and b were born on the same day but $a \ne b$

The relation (***i***) is not *asymmetric* since a has the same first name as b but $a \ne b$

The relation (***j***) is not *asymmetric* since a and b have a common grandparent but $a \ne b$

Exercise

Let R be the relation $R = \{(a,\ b) \mid a < b\}$ on the set of integers. Find

a) R^{-1} ***b)*** $\overline{R}$

Solution

a) $R^{-1} = \{(b,a) \mid (a,b) \in R\} = \{(b,a) \mid a < b\} = \{(a,b) \mid a > b\}$

b) $\overline{R} = \{(b,a) \mid (a,b) \notin R\} = \{(b,a) \mid a \not< b\} = \{(a,b) \mid a \ge b\}$

Exercise

Let R be the relation $R = \{(a,\ b) \mid a \text{ divides } b\}$ on the set of positive integers. Find

a) R^{-1} ***b)*** $\overline{R}$

Solution

a) $R^{-1} = \{(a,\ b) \mid b \text{ divides } a\}$

b) $\overline{R} = \{(a,\ b) \mid a \text{ does not divide } b\}$

Exercise

Let *R* be the relation on the set of all states in the U.S. consisting of pairs (*a, b*) where state *a* borders state *b*. Find

a) R^{-1} ***b)*** $\overline{R}$

Solution

a) Since this relation is symmetric, $R^{-1} = R$

b) This relation consists of all pairs (*a, b*) in which state *a* does not border state *b*.

Exercise

Let $R_1 = \{(1,\ 2),\ (2,\ 3),\ (3,\ 4)\}$ and

$R_2 = \{(1,\ 1),\ (1,\ 2),\ (2,\ 1),\ (2,\ 2),\ (2,\ 3),\ (3,\ 1),\ (3,\ 2),\ (3,\ 3),\ (3,\ 4)\}$ be relation from {1, 2, 3} to {1, 2, 3, 4}.

Find

a) $R_1 \cup R_2$ ***b***) $R_1 \cap R_2$ ***c***) $R_1 - R_2$ ***d***) $R_2 - R_1$

Solution

a) $R_1 \cup R_2 = \{(1,\ 1),\ (1,\ 2),\ (2,\ 1),\ (2,\ 2),\ (2,\ 3),\ (3,\ 1),\ (3,\ 2),\ (3,\ 3),\ (3,\ 4)\} = R_2$

b) $R_1 \cap R_2 = \{(1,\ 2),\ (2,\ 3),\ (3,\ 4)\} = R_1$

c) $R_1 - R_2 = \varnothing$

d) $R_2 - R_1 = \{(1,\ 1),\ (2,\ 1),\ (2,\ 2),\ (3,\ 1),\ (3,\ 2),\ (3,\ 3)\}$

Exercise

Let the relation R = {(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)} and the relation S = {(2, 1), (3, 1), (3, 2), (4, 2)}
Find $S \circ R$

Solution

$(1,\ 2) \in R \text{ and } (2,\ 1) \in S \Rightarrow (1,\ 1) \in S \circ R$

$(1,\ 3) \in R \text{ and } (3,\ 2) \in S \Rightarrow (1,\ 2) \in S \circ R$

$(2,\ 3) \in R \text{ and } (3,\ 1) \in S \Rightarrow (2,\ 1) \in S \circ R$

$(2,\ 4) \in R \text{ and } (4,\ 2) \in S \Rightarrow (2,\ 2) \in S \circ R$

$$S \circ R = \{(1,\ 1),\ (1,\ 2),\ (2,\ 1),\ (2,\ 2)\}$$

Exercise

$$R_1 = \{(a,b) \in \boldsymbol{R}^2 | a > b\} \qquad R_3 = \{(a,b) \in \boldsymbol{R}^2 | a < b\} \qquad R_5 = \{(a,b) \in \boldsymbol{R}^2 | a = b\}$$

$$R_2 = \{(a,b) \in \boldsymbol{R}^2 | a \ge b\} \qquad R_4 = \{(a,b) \in \boldsymbol{R}^2 | a \le b\} \qquad R_6 = \{(a,b) \in \boldsymbol{R}^2 | a \ne b\}$$

Find the following:

a) $R_1 \cup R_3$ ***b***) $R_1 \cup R_5$ ***c***) $R_2 \cap R_4$ ***d***) $R_3 \cap R_5$ ***e***) $R_1 - R_2$

f) $R_2 - R_1$ ***g***) $R_1 \oplus R_3$ ***h***) $R_2 \oplus R_4$ ***i***) $R_1 \circ R_1$ ***j***) $R_1 \circ R_2$

k) $R_1 \circ R_3$ ***l***) $R_1 \circ R_4$ ***m***) $R_1 \circ R_5$ ***n***) $R_1 \circ R_6$ ***o***) $R_2 \circ R_3$

Solution

a) $R_1 \cup R_3 = \{(a,b) \in \boldsymbol{R}^2 \mid a > b \text{ or } a < b\}$

$= \{(a,b) \in \boldsymbol{R}^2 \mid a \ne b\}$

$= R_6$

b) $R_1 \cup R_5 = \{(a,b) \in \boldsymbol{R}^2 \mid a > b \text{ or } a = b\}$

$= \{(a,b) \in \boldsymbol{R}^2 \mid a \le b\}$

$= R_2$

c) $R_2 \cap R_4 = \{(a,b) \in \boldsymbol{R}^2 \mid a \ge b \text{ and } a \le b\}$

$= \{(a,b) \in \boldsymbol{R}^2 \mid a = b\}$

$= R_5$

d) $R_3 \cap R_5 = \{(a,b) \in \boldsymbol{R}^2 \mid a < b \text{ and } a = b\}$

$= \varnothing$

e) $R_1 - R_2 = R_1 \cap \overline{R}_2$

$= \{(a,b) \in \boldsymbol{R}^2 \mid a > b \text{ and } a < b\}$

$= \varnothing$

f) $R_2 - R_1 = R_2 \cap \overline{R}_1$

$= \{(a,b) \in \boldsymbol{R}^2 \mid a \ge b \text{ and } a \le b\}$

$= \{(a,b) \in \boldsymbol{R}^2 \mid a = b\}$

$= R_5$

g) $R_1 \oplus R_3 = \left(R_1 \cap \bar{R}_3\right) \cup \left(R_3 \cap \bar{R}_1\right)$

$= \left\{(a,b) \in \boldsymbol{R}^2 \mid a > b \text{ and } a \ge b\right\} \cup \left\{(a,b) \in \boldsymbol{R}^2 \mid a < b \text{ and } a \le b\right\}$

$= R_1 \cup R_3$ **(From part *a*)**

$= R_6$

h) $R_2 \oplus R_4 = \left(R_2 \cap \bar{R}_4\right) \cup \left(R_4 \cap \bar{R}_2\right)$

$= \left\{(a,b) \in \boldsymbol{R}^2 \mid a \ge b \text{ and } a > b\right\} \cup \left\{(a,b) \in \boldsymbol{R}^2 \mid a \le b \text{ and } a < b\right\}$

$= R_1 \cup R_3$ **(From part *a*)**

$= R_6$

i) $R_1 \circ R_1 = \left\{(a,b) \in R_1 \text{ and } (b,c) \in R_1\right\}$

$a > b$ *and* $b > c \Rightarrow a > c$ (*clearly*) that means $(a,c) \in R_1$ (*Transitive*).

Therefore, $R_1 \circ R_1 = R_1$

j) $R_1 \circ R_2 = \left\{(a,b) \in R_2 \text{ and } (b,c) \in R_1\right\}$

$a \ge b$ *and* $b > c \Rightarrow a > c$ (*clearly*) that means $(a,c) \in R_1$. Therefore, $R_1 \circ R_2 = R_1$

k) $R_1 \circ R_3 = \left\{(a,b) \in R_3 \text{ and } (b,c) \in R_1\right\}$

$a < b$ *and* $b > c$. Therefore, $R_1 \circ R_3 = \boldsymbol{R}^2$

l) $R_1 \circ R_4 = \left\{(a,b) \in R_4 \text{ and } (b,c) \in R_1\right\}$

$a \le b$ *and* $b > c$. Clearly this can always be done simply by choosing b to be large enough.

Therefore, $R_1 \circ R_4 = \boldsymbol{R}^2$

m) $R_1 \circ R_5 = \left\{(a,b) \in R_5 \text{ and } (b,c) \in R_1\right\}$

$a = b$ *and* $b > c$ *iff* $a > c$. Therefore, $R_1 \circ R_5 = R_1$

n) $R_1 \circ R_6 = \left\{(a,b) \in R_6 \text{ and } (b,c) \in R_1\right\}$

$a \ne b$ *and* $b > c$. Clearly this can always be done simply by choosing b to be large enough.

Therefore, $R_1 \circ R_6 = \boldsymbol{R}^2$

o) $R_2 \circ R_3 = \left\{(a,b) \in R_3 \text{ and } (b,c) \in R_2\right\}$

$a < b$ *and* $b \ge c$. Clearly this can always be done simply by choosing b to be large enough.

Therefore, $R_2 \circ R_3 = \boldsymbol{R}^2$

Exercise

Let R_1 *and* R_2 be the "*divides*" and "*is a multiple of*" relations on the set of all positive integers, respectively.

That is $R_1 = \{(a,b) \mid a\ divides\ b\}$ and $R_2 = \{(a,b) \mid a\ is\ a\ multiple\ of\ b\}$

Find the following:

a) $R_1 \cup R_2$ ***b***) $R_1 \cap R_2$ ***c***) $R_1 - R_2$ ***d***) $R_2 - R_1$ ***e***) $R_1 \oplus R_2$

Solution

a) $(a,\ b) \in R_1 \cup R_2$ if and only if $a|b$ or $b|a$

b) $(a,\ b) \in R_1 \cup R_2$ if and only if $a|b$ and $b|a$ with $a = \pm b$ and $a \neq 0$

c) $R_1 - R_2 = R_1 \cap \overline{R}_2$ this relation holds between 2 integers if R_1 holds and R_2 does not hold.

$(a,\ b) \in R_1 \cap R_2$ if and only if $a|b$ and $b \nmid a$ $(a \neq \pm b)$

d) $R_2 - R_1 = R_2 \cap \overline{R}_1$ this relation holds between 2 integers if R_2 holds and R_1 does not hold.

$(a,\ b) \in R_1 \cap R_2$ if and only if $b|a$ and $a \nmid b$ $(a \neq \pm b)$

e) $R_1 \oplus R_2 = (R_1 - R_2) \cup (R_2 - R_1)$ this relation holds between 2 integers if R_2 holds and R_1 does not hold and R_2 holds and R_1 does not hold. if and only if $a|b$ or $b|a$ $(a \neq \pm b)$

SOLUTION *Section* 4.2 – Representing Relations

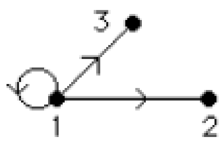
Exercise

Represent each of these relations on {1, 2, 3} with a matrix (with the elements of this set listed in increasing order). Then draw the directed graphs representing each relation

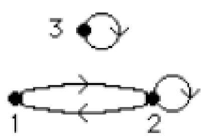
a) {(1, 1), (1, 2), (1, 3)}
b) {(1, 2), (2, 1), (2, 2), (3, 3)}
c) {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}
d) {(1, 3), (3, 1)}

Solution

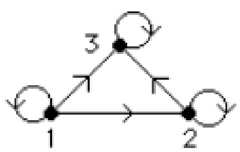
a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$



b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



c) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$



d) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

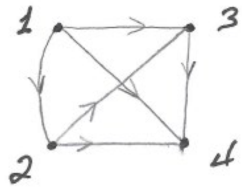
Exercise

Represent each of these relations on {1, 2, 3, 4} with a matrix (with the elements of this set listed in increasing order). Then draw the directed graphs representing each relation

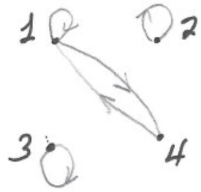
a) {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}
b) {(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)}
c) { (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)}
d) {(2, 4), (3, 1), (3, 2), (3, 4)}

Solution

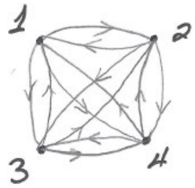
a) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$



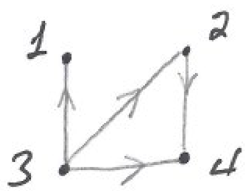
b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$



c) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$



d) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$



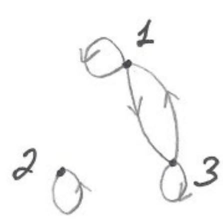
Exercise

List the ordered pairs in the relations on {1, 2, 3} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order). Then draw the directed graphs representing each relation

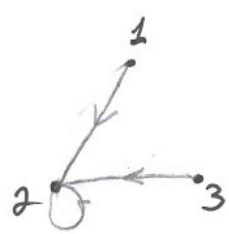
$a)$ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $b)$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $c)$ $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Solution

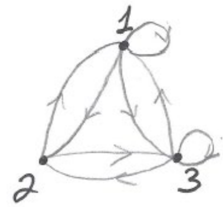
a) $\{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$



b) $\{(1, 2), (2, 2), (3, 2)\}$



c) $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (3, 3)\}$



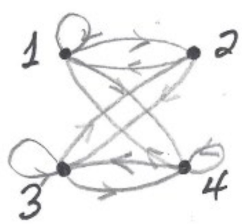
Exercise

List the ordered pairs in the relations on $\{1, 2, 3, 4\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order). Then draw the directed graphs representing each relation

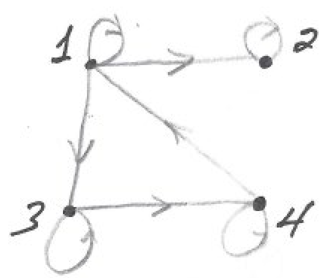
$a)\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad b)\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad c)\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

Solution

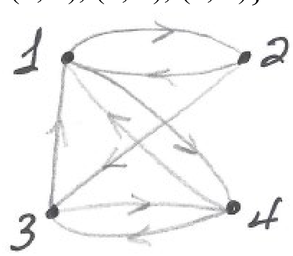
a) $\{(1, 1), (1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4)\}$



b) $\{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3), (3, 4), (4, 1), (4, 4)\}$



c) $\{(1, 2), (1, 4), (2, 1), (2, 3), (3, 1), (3, 4), (4, 1), (4, 3)\}$



Exercise

Let R be the relation represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find: $a)\ R^2$ $\quad b)\ R^3$ $\quad c)\ R^4$

Solution

a) $M_{R^2} = M^2_R = M_R \odot M_R$

$$R^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

b) $M_{R^3} = M^3_R = M_R \odot M^2_R$

$$R^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

c) $M_{R^4} = M^{(4)}_R = M_R \odot M^{(3)}_R$

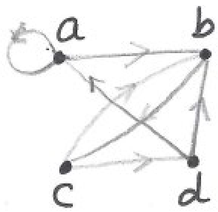
$$R^4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Exercise

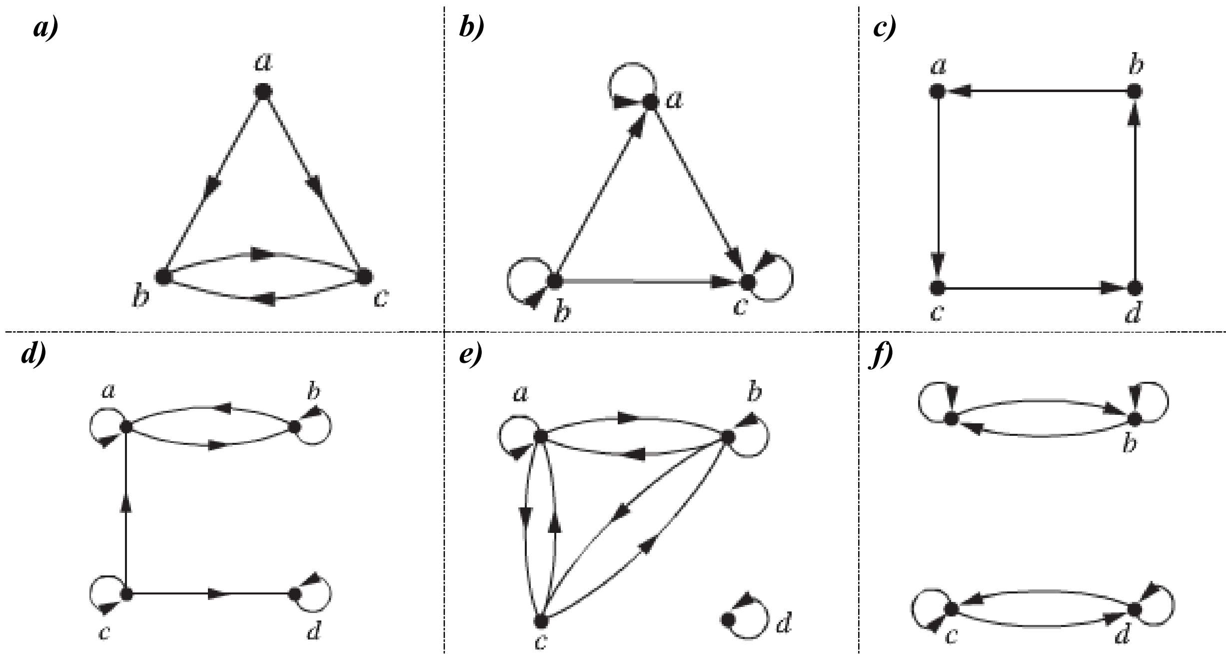
Draw the directed graph that represents the relation $\{(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)\}$

Solution



Exercise

Determine whether the relations represented by the directed graphs are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive



Solution

a) $\{(a,\ b),\ (a,\ c),\ (b,\ c),\ (c,\ b)\}$
It is not reflexive since $(a,\ a)$ doesn't exist
It is not symmetric
It is transitive since $(a,\ b),\ (b,\ c) \Rightarrow (a,\ c)$

b) $\{(a,\ a),\ (a,\ c),\ (b,\ a),\ (b,\ b),\ (b,\ c),\ (c,\ c)\}$
It is reflexive
It is not symmetric
It is transitive since $(b,\ a),\ (a,\ c) \Rightarrow (b,\ c)$

c) $\{(a,\ c),\ (b,\ a),\ (c,\ d),\ (d,\ b)\}$
It is not reflexive; it is not symmetric, and not transitive since

d) $\{(a,\ a),\ (a,\ b),\ (b,\ a),\ (b,\ b),\ (c,\ a),\ (c,\ c),\ (c,\ d),\ (d,\ d)\}$
It is reflexive, not symmetric (no $(a,\ c)$), and not transitive $(c,\ a),\ (a,\ b)$ but no $(c,\ b)$

e) $\{(a,\ a),\ (a,\ b),\ (a,\ c),\ (b,\ a),\ (b,\ b),\ (b,\ c),\ (c,\ a),\ (c,\ b),\ (d,\ d)\}$
It is not reflexive; it is symmetric and transitive

f) $\{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, d), (d, c)\}$
It is reflexive; it is symmetric and transitive

SOLUTION *Section* 4.3 – Closures of Relations

Exercise

Let R be the relation on the set {0, 1, 2, 3} containing the ordered pairs (0, 1), (1, 1), (1, 2), (2, 0), (2, 2), and (3, 0). Find the

a) Reflexive closure of R.

b) Symmetric closure of R.

Solution

a) The reflexive closure of R is R with all (a, a). In this case the closure of R is
{(0, 0), (0, 1), (1, 1), (1, 2), (2,0), (2, 2), (3, 0), (3, 3)}

b) The symmetric closure of R is R with (b, a) for which (a, b) is in R.
In this case the symmetric of R is
{(0, 1), (0, 2), (0, 3), (1, 0) (1, 1), (1, 2), (2,0), (2, 1) (2, 2), (3, 0)}

Exercise

Let R be the relation $\{(a,\ b) \mid a \neq b\}$ on the set of integers. What is the reflexive closure of R?

Solution

When we add all the pairs (x, x) to the given relation we have all of $\mathbb{Z}\times\mathbb{Z}$, which the relation will always holds.

Exercise

Let R be the relation $\{(a,\ b) \mid a \text{ divides } b\}$ on the set of integers. What is the symmetric closure of R?

Solution

To form the symmetric closure, we need to add all the pairs (b, a) such that (a, b) is in R.
We need to include pairs (b, a) such that a divides b, which is equivalent to saying that we need to include all the pairs (a, b) such that b divides a.
Thus the closure is $\{(a, b) \mid a$ divides b **or** b divides $a\}$

Exercise

How can the directed graph representing the reflexive closure of a relation on a finite set be constructed from the directed graph of the relation?

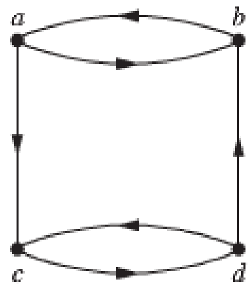
Solution

To form a reflexive closure, we simply need to add a loop at each vertex that does not already have one.

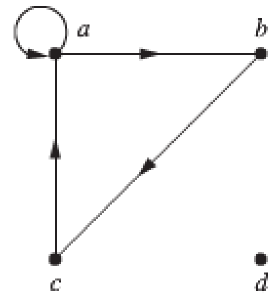
Exercise

Draw the directed graph of the *reflexive*, *symmetric*, and *both reflexive and symmetric* closure of the relations with the directed graph shown

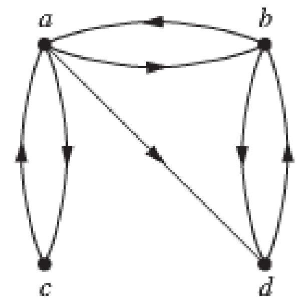
a)



b)

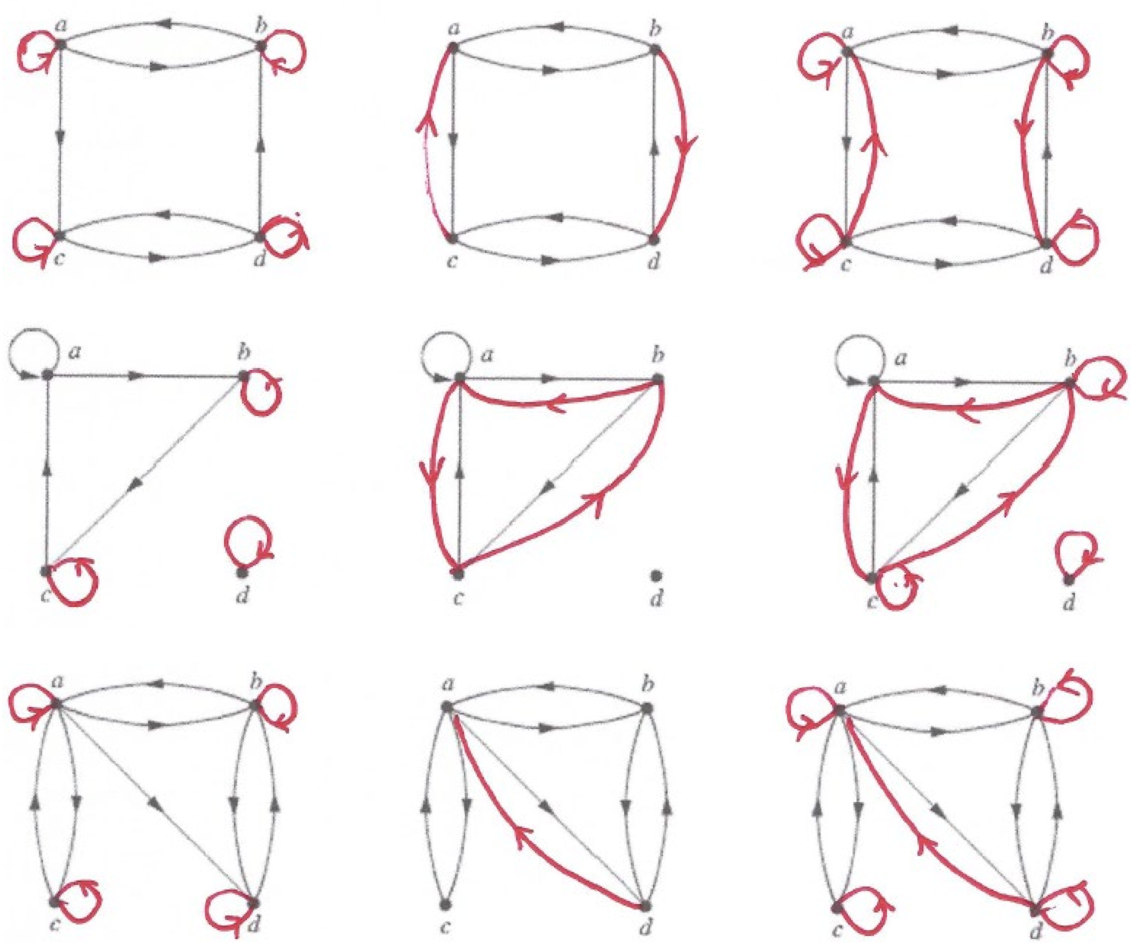


c)



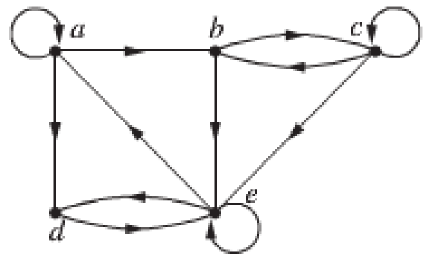
Solution

Reflexive | ***Symmetric*** | ***Reflexive** and **Symmetric***



Exercise

1. Determine whether these sequences of vertices are paths in this directed graph



a) *a, b, c, e*
b) *b, e, c, b, e*
c) *a, a, b, e, d, e*
d) *b, c, e, d, a, a, b*
e) *b, c, c, b, e, d, e, d*
f) *a, a, b, b, c, ,c, b, e, d*

2. Find all circuits of length three in the directed graph

Solution

a) This is a path
b) This is not a path (no edge from *e* to *c*)
c) This is a path
d) This is not a path (no edge from *d* to *a*)
e) This is a path
f) This is not a path (no loop at *b*)

2. A circuit of length 3 can be written as a sequence of 4 vertices.

Start @ ***b***: *bccb* and *beab*

Start @ ***c***: *ccbc* and *cbcc*

Start @ ***d***: *deed*, *eede* and *edee*

eabe, dead, eade, abea, adea, aaaa, cccc, and eeee

Exercise

Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs (1, 3), (2, 4), (3, 1), (3, 5), (4, 3), (5, 1), and (5, 2). Find

a) R^2 ***b***) R^3 ***c***) R^4 ***d***) R^5 ***e***) R^6 ***f***) R^*

Solution

$$M_R = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

a) $M_{R^2} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$

b) $M_{R^3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$

c) $M_{R^4} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

d) $M_{R^5} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

e) $M_{R^6} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

f) $M_{R^*} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

Exercise

Let R be the relation on the pair (a, b) if a and b are cities such that there is a direct non-stop airline flight from a to b. When is (a, b) in

a) R^2 ***b***) R^3 ***c***) R^*

Solution

a) The pair (a, b) is in R^2 precisely when there is a city c such that there is a direct flight from a to c and a direct flight from c to b – when it is possible to fly from a to b with a scheduled stop in some intermediate city.

b) The pair (a, b) is in R^3 precisely when there are cities c and d such that there is a direct flight from a to c, a direct flight from c to d, and a direct flight from d to b – when it is possible to fly from a to b with two scheduled stops in some intermediate cities.

c) The pair (a, b) is in R^* precisely when it is possible to fly from a to b.

Exercise

Let R be the relation on the set of all students containing the ordered pair (a, b) if a and b are in at least one common class and $a \neq b$. When is (a, b) in

a) R^2 ***b***) R^3 ***c***) R^*

Solution

a) The pair $(a,\ b) \in R^2$ if there is a person c other than a or b who is in a class with a and a class with b. $(a,\ a) \in R^2$ as long a is taking a class that has at least one other person in it, that person serves as the "c".

b) The pair $(a,\ b) \in R^3$ if there are persons c different from a and d different from b ***and*** c such that c is in a class with a, c is in class with d, and d is in class with b.

c) The pair $(a,\ b) \in R^*$ if there is a sequence of persons $c_0,\ c_1,\ c_2,\ \ldots,\ c_n$, with $n \geq 1$ such that $c_0 = a,\ c_n = b$, and for each i from 1 to n, $c_{i-1} \neq c_i$ and c_{i-1} is at least one class with c_i

Exercise

Suppose that the relation R is reflexive. Show that R^* is reflexive.

Solution

Since $R \subseteq R^*$, clearly if $\Delta \subseteq R$, then $\Delta \subseteq R^*$

Exercise

Suppose that the relation R is symmetric. Show that R^* is symmetric.

Solution

Suppose $(a,\ b) \in R^*$, then there is a path from a to b in R. Given such a path, if R is symmetric, then the reverse of every edge in the path is also in R; therefore there is a path from b to a in R.
This means that $(b,\ a) \in R^*$ whenever (a, b) is.

Exercise

Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive.

Solution

It is certainly possibly for R^2 to contain some pairs (a, a).
For example: $R = \{(1, 2), (2, 1)\}$

SOLUTION Section 4.4 – Equivalence Relations

Exercise

Which of these relations on {0, 1, 2, 3} are equivalence relations?
Determine the properties of an equivalence relation that the others lack.
What are the equivalence classes of the equivalence relations?

a) {(0, 0), (1, 1), (2, 2), (3, 3)}
b) {(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}
c) {(0, 0), (1, 1), (1, 2), (2, 1), (3, 2), (3, 3)}
d) {(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}
e) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}

Solution

a) This is an equivalence relation. It is reflexive, symmetric and transitive.
The equivalence classes all have just one element.
Each element is in an equivalence class by itself.

b) This is not reflexive since pair (1, 1) is missing. It is symmetric and it is not transitive since the pairs (0, 2) and (2, 3) are there, but not (0, 3).
This is not an equivalence relation.

c) This is an equivalence relation. It is reflexive, symmetric and transitive.
The elements 1 and 2 are in one equivalence class, and 0 and 3 are each in their own equivalence class.

d) This is reflexive and symmetric and it is not transitive since the pairs (1, 3) and (3, 2) are there, but not (1, 2).
This is not an equivalence relation.

e) This is reflexive, it is not symmetric since (2, 1) is missing and it is not transitive since the pairs (2, 0) and (0, 1) are there, but not (2, 1).
This is not an equivalence relation.

Exercise

Which of these relations on the set of all people are equivalence relations?
Determine the properties of an equivalence relation that the others lack.
What are the equivalence classes of the equivalence relations?

a) $\left\{(a,\ b) \mid a \text{ and } b \text{ are the same age}\right\}$
b) $\left\{(a,\ b) \mid a \text{ and } b \text{ have the same parents}\right\}$
c) $\left\{(a,\ b) \mid a \text{ and } b \text{ share a common parent}\right\}$
d) $\left\{(a,\ b) \mid a \text{ and } b \text{ have met}\right\}$
e) $\left\{(a,\ b) \mid a \text{ and } b \text{ speak a common language}\right\}$

Solution

a) This relation is reflexive, since *a* is the same person (same age).
If *a* is the same age as *b*, then *b* has to be the same age as *a*. this relation is symmetric.
If *a* is the same age as *b* and *b* is the same age as *c*, then *a* has to be the same age as *c*. this relation is transitive.
An equivalence class is the set of all people who are the same age. To really identify the equivalence class and the equivalence relation itself, one would need to specify exactly what ine meant by the "same age". For example, we could define two people to be the same age if their official dates of birth were identical.

b) For each pair (*m, w*) of a man and a woman, the set of offspring of their union, if nonempty, is an equivalence class. In many cases, then, an equivalence class consists of all the children in a nuclear family with children.

c) Let assume the relation is biological parentage. It is possible that *a* to be the the child of *W* and *X*, *b* is the child of *X* and *Y*, and *c* is the child of *Y* and *Z*. Then *a* is related to *b*, and *b* is related to *c*, but *a* is not related to *c*. This is not an equivalence relation, since it is not transitive. Therefore, this is not an equivalence relation.

d) If *a* met *b* and *b* met *c*, then it is not neccesary that *a* met *c*. This is not an equivalence relation, since it is not transitive. Therefore, this is not an equivalence relation.

e) If *a* speaks the same language (english) as *b* and *b* speaks the same language (spanish) as *c*, then it is not neccesary that *a* can speak spanish as *c*. This is not an equivalence relation, since it is not transitive.

Exercise

Which of these relations on the set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$ are equivalence relations?
Determine the properties of an equivalence relation that the others lack.
What are the equivalence classes of the equivalence relations?

a) $\{(f,\ g) \mid f(1) = g(1)\}$

b) $\{(f,\ g) \mid f(0) = g(0)\ or\ f(1) = g(1)\}$

c) $\{(f,\ g) \mid f(x) - g(x) = 1\ for\ all\ x \in \mathbb{Z}\}$

d) $\{(f,\ g) \mid for\ some\ C \in \mathbb{Z},\ for\ all\ x \in \mathbb{Z},\ f(x) - g(x) = C\}$

e) $\{(f,\ g) \mid f(0) = g(1)\ or\ f(1) = g(0)\}$

Solution

a) This is an equivalence relation, one of the general form that 2 things are considered equivalent if they have the same "something" (is 1).
$\{(f, f) \mid f(1) = f(1)\}$ This relation is reflexive.
If $\{(f,\ g) \mid f(1) = g(1)\}$ then $\{(g, f) \mid g(1) = f(1)\}$ $\therefore$ this relation is symmetric
If $\{(f,\ g) \mid f(1) = g(1)\}$ and $\{(g,\ h) \mid g(1) = h(1)\}$, then $\{(f,\ h) \mid f(1) = h(1)\}$ $\therefore$ this relation is transitive.

There is one equivalence class for each $n \in \mathbb{Z}$ and it contains all those functions whose value at 1 is n.

b) Let $f(x)=0, \quad g(x)=x, \quad and \quad h(x)=1$ for all $x \in \mathbb{Z}$. Then *f* is related to *g* since $f(0)=g(0)$ and *g* is related to *h* since $g(1)=h(1)$, but $f(0) \neq h(1)$, therefore *f* is related to *h* since that have no values in common.
Hence, this is not an equivalence relation because it is not transitive.

c) It is not reflexive relation since $f(x)-f(x)=0 \neq 1$.
It is not symmetric since if $f(x)-g(x)=1$, then $g(x)-f(x)=-1 \neq 1$
It is not transitive since $f(x)-g(x)=1$ and $g(x)-h(x)=1 \quad \Rightarrow \quad f(x)-h(x)=2 \neq 1$,
This is not an equivalence relation.

d) This relation is reflexive, $f(x)-f(x)=0 \in \mathbb{Z}$
$f(x)-g(x)=C \quad \Rightarrow \quad g(x)-f(x)=-C \in \mathbb{Z}$, this relation is symmetric
$f(x)-g(x)=C_1 \quad g(x)-h(x)=C_2 \quad f(x)-h(x)=C_1+C_2 \in \mathbb{Z}$, this relation is transitive.
This is an equivalence relation.
The set of equivalence classes is uncountable. For each function $f:\mathbb{Z} \to \mathbb{Z}$, there is the equivalence class consisting of all functions *g* for which there is a constant *C* such that $g(n)=f(n)+C$ for all $n \in \mathbb{Z}$.

e) It is not reflexive relation since $f(0) \neq f(1)$ since it not given.
This is not an equivalence relation.

Exercise

Define three equivalence relations on the set of students in your discrete mathematics class different from the relations discussed in the text. Determine the equivalence classes for each of these equivalence relations.

Solution

One relation is that *a* and *b* are related if they were born in the same state. Here the equivalence classes are the nonempty sets of students from each state.
Another example is for *a* to be related to *b* if *a* and *b* have lived the same number of complete decades. The equivalence classes are the set of all 10 to 19 years olds. The set of all 20 to 29 year olds, and so on.
A Third example is *a* to be related to *b* if 10 is a divisor of the difference between *a*'s age and *b*'s age, where "age" means the whole number of years since birth, as of the first day of class.
For each *i* = 0, 1, 2, …, 9, there is the equivalence class (if it is nonempty) of those students whose age ends with the digit *i*.

Exercise

Define three equivalence relations on the set of buildings on a college campus. Determine the equivalence classes for each of these equivalence relations.

Solution

Two buildings are equivalent, if they were opened during the same year; an equivalence class consists of the set of buildings opened in a given year.

For another example, we can define 2 buildings to be equivalent if they have the same number of stories; the equivalence classes are the set of 1-story buildings, the set of 2-story buildings, and so on.

The third example, partition the set of all buildings into 2 classes – those in which you do have a class this semester and those in which you don't. Every building in which you have a class is equivalent to every building in which you have a class (including itself), and every building in which you don't have a class is equivalent to every building in which you don't have a class.

Exercise

Let R be the relation on the set of all sets of real numbers such that $S\ R\ T$ if and only if S and T have the same cardinality. Show that R is an equivalence relation. What are the equivalence classes of the sets $\{0, 1, 2\}$ and $\boldsymbol{Z}$?

Solution

Two sets have the same cardinality if there is a bijection (1−1 and onto function) from one set to the other.

We need to prove that R is reflexive, symmetric, and transitive.

Every set has the same cardinality as itself because of the identity function.

If f is a bijection from S to T, then f^{-1} is a bijection from T tot S, so R is symmetric.

If f is a bijection from S to T, and g is a bijection from T to U, then $g \circ f$ is a bijection from S to U, so R is transitive.

The equivalence class $\{1, 2, 3\}$ is the set of all 3-element sets of real numbers, including such sets as $\{4, 25, 1948\}$ and $\left(e,\ \pi,\ \sqrt{2}\right)$.

Similarly, $[\boldsymbol{Z}]$ is the set of all infinite countable sets of real numbers, such as the set of natural numbers, the set of rational numbers, and the set of the prime numbers, but not including the set $\{1, 2, 3\}$ (it's too small) or the set of all real numbers (too big).

Exercise

Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) such that $f(x) = f(y)$

a) Show that R is an equivalence relation on A.

b) What are the equivalence classes of R?

Solution

a) It is reflexive since $f(x) = f(x)$ for all $x \in A$

It is symmetric since $f(x) = f(y)$, then $f(y) = f(x)$

It is transitive since $f(x) = f(y)$ and $f(y) = f(z)$ then $f(x) = f(z)$

b) The equivalence class of x is the set of all $y \in A$ such that $f(y) = f(x)$ (definition of inverse).

Thus the equivalence classes are precisely the sets $f^{-1}(b)$ for every b in the range of f.

Exercise

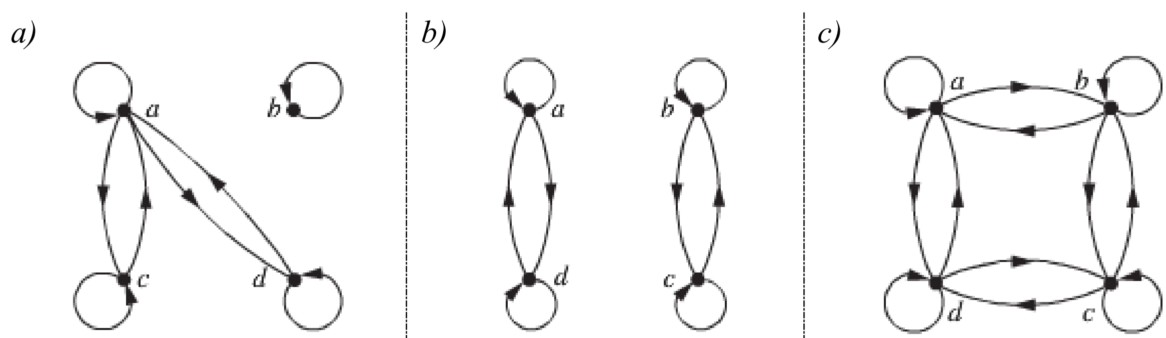
Suppose that A is a nonempty set, and R is an equivalence relation on A. Show that there is a function f with A as its domain such that $(x, y) \in R$ if and only if $f(x) = f(y)$

Solution

The function that sends each $x \in A$ to its equivalence class $[x]$ is obviously such a function.

Exercise

Determine whether the relation with the directed graph shown is an equivalence relation



Solution

a) The relation is reflexive since there is a loop at each vertex.
It is symmetric since every edge has 2 vertices and pointing in the both direction.
It is not transitive since we have {d, a} and {a, c} but not {d, a}

b) The relation is reflexive since there is a loop at each vertex.
It is symmetric since every edge has 2 vertices and pointing in the both direction.
It is transitive since paths of length 2 are accompanied by the path of length 1, edge between the same 2 vertices in the same direction.

This relation is an equivalence relation.
The equivalence classes are $\{a, d\}$ and $\{b, c\}$

c) The relation is reflexive since there is a loop at each vertex.
It is symmetric since every edge has 2 vertices and pointing in the both direction.
It is not transitive (a, b) and (b, c) but not (a, c).

Exercise

Which of these collections of subsets are partitions of $\{1, 2, 3, 4, 5, 6\}$

a) $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}$
b) $\{1\}, \{2, 3, 6\}, \{4\}, \{5\}$
c) $\{2, 4, 6\}, \{1, 3, 5\}$
d) $\{1, 4, 5\}, \{2, 6\}$

Solution

a) This is not a partition, since the sets are not pairwise disjoint. 2 and 4 appear in 2 of the sets.

b) This is a partition

c) This is a partition

d) This is not a partition, since element 3 is missing from the sets

Exercise

Which of these collections of subsets are partitions of $\{-3, -2, -1, 0, 1, 2, 3\}$

a) $\{-3, -1, 1, 3\}, \{-2, 0, 2\}$
b) $\{-3, -2, -1, 0\}, \{0, 1, 2, 3\}$
c) $\{-3, 3\}, \{-2, 2\}, \{-1, 1\}, \{0\}$
d) $\{-3, -2, 2, 3\}, \{-1, 1\}$

Solution

a) This a partition, since it satisfies the definition

b) This is not a partition, since the subsets are not disjoint

c) This a partition, since it satisfies the definition

d) This is not a partition, since the union of the subsets leaves out 0

SOLUTION *Section* 4.5 – Partial Orderings

Exercise

Which of these relations on {0, 1, 2, 3} are partial orderings? Determine the properties of a partial ordering that the others lack.

a) {(0, 0), (1, 1), (2, 2), (3, 3)}
b) {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}
c) {(0, 0), (1, 1), (1, 2), (2, 2), (3, 3)}
d) {(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}
e) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}
f) {(0, 0), (2, 2), (3, 3)}
g) {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 3)}
h) {(0, 0), (1, 1), (1, 2), (2, 2), (3, 1), (3, 3)}
i) {(0, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (2, 3), (3, 0), (3, 3)}
j) {(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 3)}

Solution

a) This relation is reflexive because each of 0, 1, 2, 3 is related to itself.
This relation is antisymmetric because a to be related to b is for a to be equal to b
Since a is related to b and b related to c and $a = b = c$, then a is related to c. So the relation is transitive.
The equality relation on any set satisfies all three conditions, therefore is a partial ordering.

b) It is reflexive but it is not antisymmetric since we have $2R3$ and $3R2$ but $2 \neq 3$.
Therefore, this is not a partial ordering.

c) This relation is reflexive because each of 0, 1, 2, 3 is related to itself.
This relation is antisymmetric because a to be related to b is for a to be equal to b
It is transitive for the same reason and $1R1 \text{ and } 1R2 \quad \Rightarrow 1R2$
Therefore, is a partial ordering.

d) This relation is reflexive because each of 0, 1, 2, 3 is related to itself.
This relation is antisymmetric because a to be related to b is for a to be equal to b
It is transitive for the same reason and $1R1 \text{ and } 1R2 \quad \Rightarrow 1R2$, $1R3 \text{ and } 3R3 \quad \Rightarrow 1R3$, and
$2R3 \text{ and } 3R3 \quad \Rightarrow 2R3$
Therefore, is a partial ordering.

e) It is reflexive but it is not antisymmetric since we have $0R1$ and $1R0$ but $0 \neq 1$.
Therefore, this is not a partial ordering.

f) Since 1 is not related to itself, so this relation is not reflexive.
Therefore, R is not a partial ordering.

g) This relation is reflexive because each of 0, 1, 2, 3 is related to itself.
This relation is antisymmetric because a to be related to b is for a to be equal to b
It is transitive for the same reason and $2R0 \text{ and } 0R0 \quad \Rightarrow 2R0$, and $2R3 \text{ and } 3R3 \quad \Rightarrow 2R3$

Therefore, is a partial ordering.

h) Since $3R1$ *and* $1R2 \quad \Rightarrow \cancel{3R2}$, so this relation is not transitive.
Therefore, R is not a partial ordering.

i) Since $1R2$ *and* $2R0 \quad \Rightarrow \cancel{1R0}$, so this relation is not transitive.
Therefore, R is not a partial ordering.

j) Since $0R1$ *and* $1R0$ but $0 \neq 1$, so this relation is not antisymmetric and it is not transitive because $2R0$ *and* $0R1 \quad \Rightarrow \cancel{2R1}$.
Therefore, R is not a partial ordering.

Exercise

Is (*S, R*) a poset If *S* is the set of all people in the world and $(a,\ b) \in R$, where *a* and *b* are people, if

a) a is a taller than b?
b) a is not taller than b?
c) $a = b$ or a is an ancestor of b?
d) a and b have a common friend?
e) a is a shorter than b?
f) a weighs more than b?
g) $a = b$ or a is a descendant of b?
h) a and b do not have a common friend?

Solution

a) Since nobody is taller than himself, this relation is not reflexive, so (*S, R*) is not a poset.

b) To be not a taller means exactly the same height or shorter. 2 different people *x* and *y* could have the same height, in which case xRy *and* yRx *but* $x \neq y$, so *R* is not antisymmetric.
Therefore, this relation is not a poset.

c) The equality clause in the given of *R* guarantees that *R* is reflexive.
If *a* is ancestor to *b*, then *b* can't be ancestor to *a*, so the relation is vacuously antisymmetric.
If *a* is ancestor to *b* and *b* is ancestor to *c*, then *a* is ancestor to *c*, thus *R* is transitive.
Therefore, this relation is a poset.

d) Let *x* and *y* be any 2 distinct friends, xRy *and* yRx *but* $x \neq y$, so *R* is not antisymmetric.
Therefore, this relation is not a poset.

e) Let 2 people can be the same height since are not the same person, so *R* is not antisymmetric.
Therefore, this relation is not a poset.

f) Since nobody is weight more than himself, this relation is not reflexive, so this relation is not a poset.

g) Since $a = a$, then the *R* is reflexive.

Given that $a = b$ but if a is a descendant of b, then b cannot be a descendant of a. So, the relation is vacuously antisymmetric.
if a is a descendant of b and b is a descendant of c, then a is a descendant of c. So, the R is transitive.
Therefore, this relation is a poset.

h) Since anyone and himself have a common friend, then this relation is not reflexive, so this relation is not a poset.

Exercise

Which of these are posets?

a) $(\mathbb{Z}, =)$	*b)* $(Z,\ \neq)$	*c)* $(Z,\ \geq)$	*d)* $(Z,\ \nmid)$
e) $(R,\ =)$	*f)* $(R,\ <)$	*g)* $(R,\ \leq)$	*h)* $(R,\ \neq)$

Solution

a) The equality relation of any set satisfies all three conditions. Therefore, a partial order.

b) This is not a poset since the relation is not reflexive $(a \neq a)$

c) The relation is reflexive since the relation involved the equality sign.

d) This is not a poset since the relation is not reflexive $(2 \nmid 2)$

e) The equality relation of any set satisfies all three conditions. Therefore, a partial order.

f) This is not a poset since the relation is not reflexive $(2 \not< 2)$

g) The relation is reflexive since the relation involved the equality sign.

h) This is not a poset since the relation is not reflexive $(2 = 2)$

It is not antisymmetric since $1R2$ *and* $2R1$ *but* $1 \neq 2$

It is not transitive $1R2$ *and* $2R1$ *but* $1 = 1 \Rightarrow 1 \not R 1$

Exercise

Determine whether the relations represented by these zero-one matrices are partial orders

a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *b)* $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *c)* $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *d)* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ *f)* $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

Solution

a) The relation is $\{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (2,\ 1),\ (2,\ 2),\ (3,\ 3)\}$

This is not antisymmetric because $1R2 \quad and \quad 2R1 \quad but\ 1 \neq 2$.

Therefore, this matrix is not a partial order.

b) The relation is $\{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (2,\ 2),\ (3,\ 3)\}$

It is clearly reflexive.

The pairs (1, 2) and (1, 3) are in the relation that neither can be part of a counterexample to antisymmetry or transitivity.

c) The relation is $\{(1,\ 1),\ (1,\ 3),\ (2,\ 1),\ (2,\ 2),\ (3,\ 3)\}$

It is clearly reflexive. The pairs (1, 3) and (2, 1) are in the relation that neither can be part of a counterexample to antisymmetry.

It is not transitive since, (2, 1) and (1, 3) that will lead to (2, 3) which is not in the relation.

Therefore, this matrix is not a partial order.

d) The relation is $\{(1,\ 1),\ (2,\ 2),\ (3,\ 1),\ (3,\ 3)\}$

It is clearly reflexive.

The pair (3, 1) is in the relation that can't be part of a counterexample to antisymmetry or transitivity.

e) The relation is $\{(1,\ 1),\ (1,\ 2),\ (1,\ 3),\ (2,\ 2),\ (2,\ 3),\ (3,\ 3),\ (3,\ 4),\ (4,\ 1),\ (4,\ 2),\ (4,\ 4)\}$

It is not transitive since, (4, 1) and (1, 3) are in the relation but not (4, 3).

Therefore, this matrix is not a partial order.

f) The relation is $\{(1,\ 1),\ (1,\ 3),\ (2,\ 2),\ (2,\ 3),\ (3,\ 3),\ (3,\ 4),\ (4,\ 1),\ (4,\ 2),\ (4,\ 4)\}$

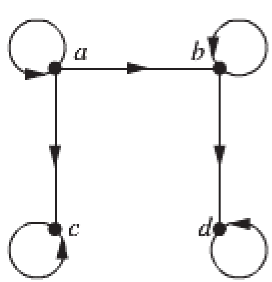
It is not transitive since, (4, 1) and (1, 3) are in the relation but not (4, 3).

Therefore, this matrix is not a partial order.

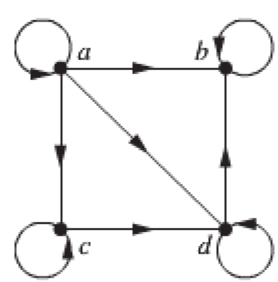
Exercise

Determine whether the relation with the directed graph shown is a partial order.

a)



b)



c)

Solution

a) This is relation is not transitive since there no relation (arrow) between *a* and *d*.

$aRb \text{ and } bRd \quad \Rightarrow \cancel{aRd}$

b) This is relation is not transitive since there no relation (arrow) from *c* and *b*.

c) This relation is reflexive since all points have an arrow to itself.
This relation is antisymmetric since no pair of arrows going in opposite directions between 2 different points.
Therefore, this relation is a partial order.

Exercise

Let $(S,\ R)$ be a poset. Show that $\left(S,\ R^{-1}\right)$ is also a poset, where R^{-1} is the inverse of *R*. The poset $\left(S,\ R^{-1}\right)$ is called the dual of $(S,\ R)$.

Solution

Since *R* is reflexive, then R^{-1} is clearly reflexive.

Suppose that $(a,b) \in R^{-1}$ and $a \neq b$. Then $(b,a) \in R$, so $(a,b) \notin R$, so $(b,a) \notin R^{-1}$

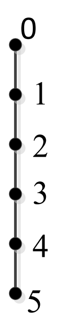
If $(a,b) \in R^{-1}$ and $(b,c) \in R^{-1}$, then $(b,a) \in R$ and $(c,b) \in R$, since *R* is transitive, so $(c,a) \in R$, therefore $(a,c) \in R^{-1}$, thus R^{-1} is transitive.

Therefore $\left(S,\ R^{-1}\right)$ is a poset

Exercise

Draw the Hasse diagram for the “greater than or equal to” relation on {0, 1, 2, 3, 4, 5}

Solution

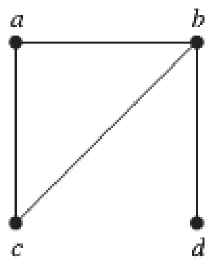


SOLUTION *Section* 4.6 – Graphs: Definitions and Basic Properties

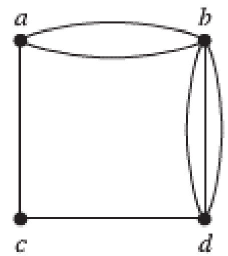
Exercise

Determine whether the graph shown has directed or undirected edges, whether it has multiple edges, and whether it has one or more loops.

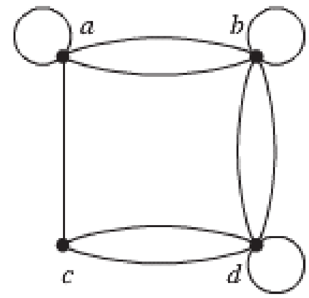
a)



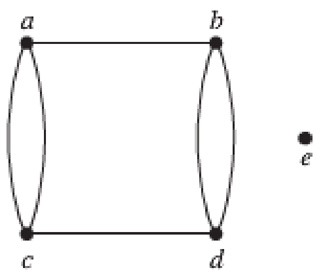
b)



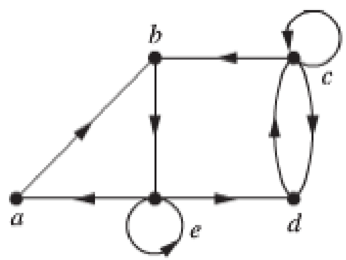
c)



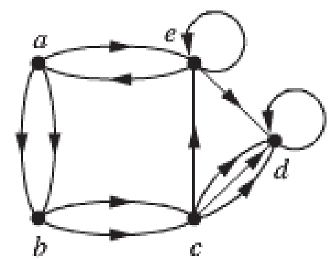
d)



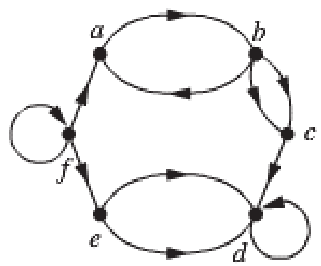
e)



f)



g)



Solution

a) This is a simple graph; the edges are undirected, and there are no parallel edges or loops.

b) This is a multigraph; the edges are undirected, and there are no loops, but there are parallel edges.

c) This is a pseudograph; the edges are undirected, and there are no parallel edges or loops.

d) This is a multigraph; the edges are undirected, and there are no loops, but there are parallel edges.

e) This is a directed graph; the edges are directed, and there are no parallel edges.

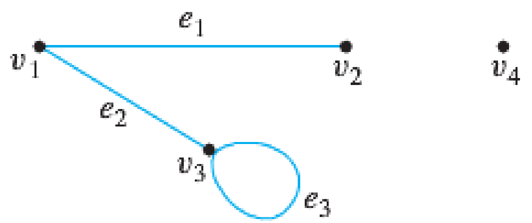
f) This is a directed multigraph; the edges are directed, and there are parallel edges.

g) This is a directed multigraph; the edges are directed, and there is a set of parallel edges.

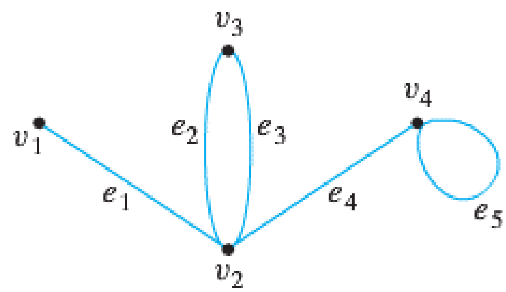
Exercise

Define each graph formally by specifying its vertex set, its edge set, and a table giving the edge-endpoint function

a)



b)



Solution

a) Vertex set $\{v_1, v_2, v_3, v_4\}$

Edge set $\{e_1, e_2, e_3\}$

Edge-endpoint function:

Edge	***Endpoints***
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_3\}$

b) Vertex set $\{v_1, v_2, v_3, v_4\}$

Edge set $\{e_1, e_2, e_3, e_4, e_5\}$

Edge-endpoint function:

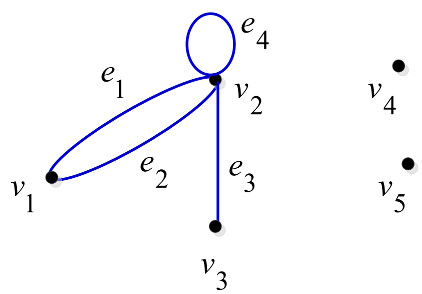
Edge	***Endpoints***
e_1	$\{v_1, v_2\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_2, v_4\}$
e_5	$\{v_4\}$

Exercise

Graph G has vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{e_1, e_2, e_3, e_4\}$, with edge-endpoint function as follow

Edge	***Endpoints***
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_2\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_2\}$

Solution

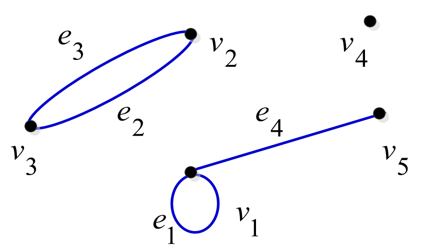


Exercise

Graph H has vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{e_1, e_2, e_3, e_4\}$, with edge-endpoint function as follow

Edge	***Endpoints***
e_1	$\{v_1\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_1, v_5\}$

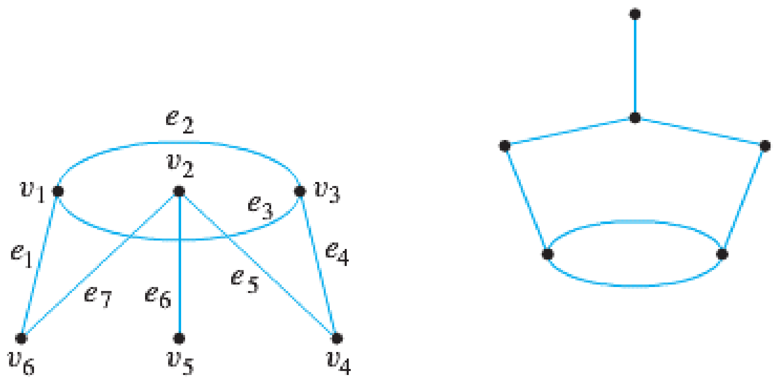
Solution



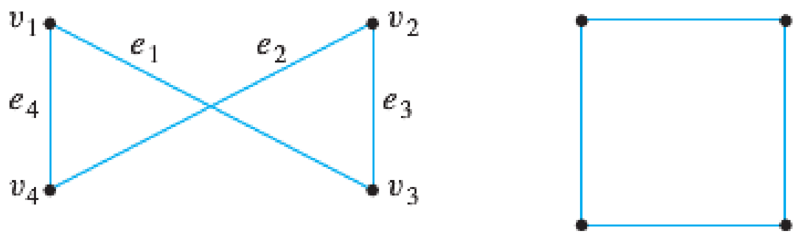
Exercise

Show that the 2 drawings represent the same graph by labeling the vertices and edges of the right-hand drawing to correspond to those of the left-hand drawing.

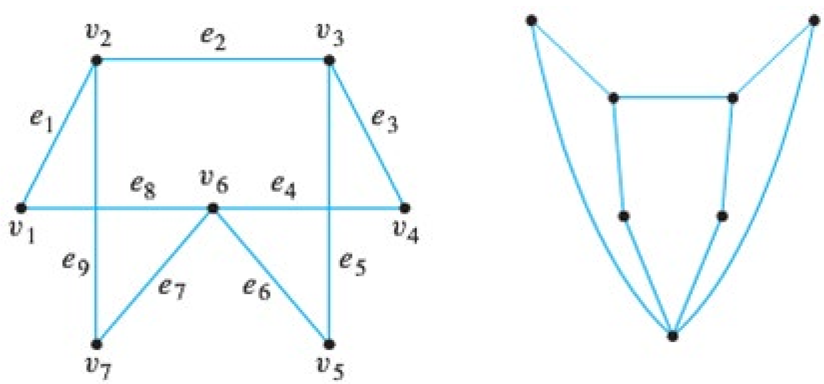
a)



b)

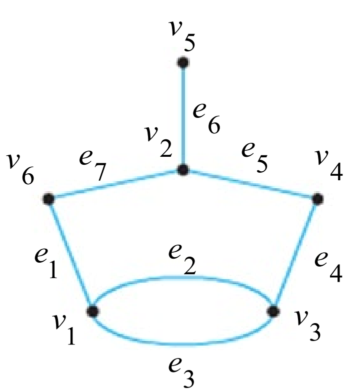


c)

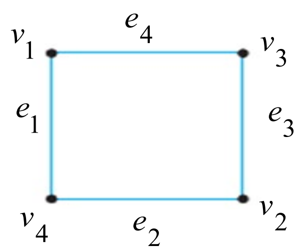


Solution

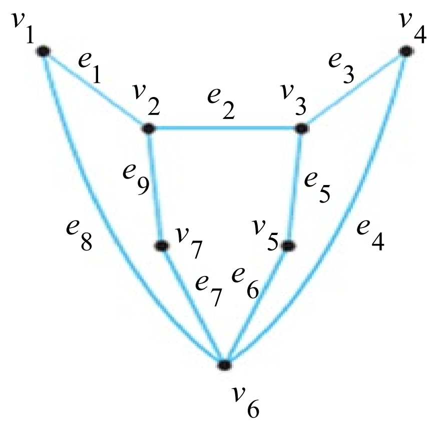
a) If you just hold the vertex v_5 turn it around to up position and stretch vertically little



b) Hold the edge e_4 and twisted as the vertices switch position.



c)



Exercise

For each of the graphs

i. Find all edges that are incident on v_1

ii. Find all vertices that are adjacent to v_3

iii. Find all edges that are adjacent to e_1

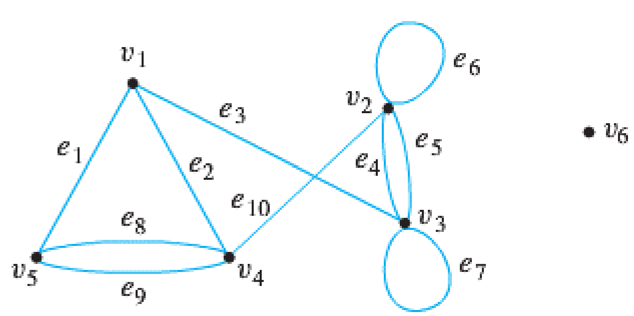
iv. Find all loops

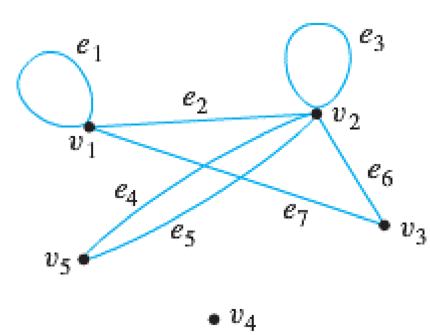
v. Find all parallel edges

vi. Find all isolated vertices

vii. Find the degree of v_3

viii. Find the total degree of the graph





Solution

a) e_1, e_2, and e_3 are incident on v_1

v_1, v_2 and v_3 are adjacent to v_3

e_2, e_3, e_8, and e_9 are adjacent to e_1

e_6 and e_7 are loops.

e_4 and e_5 are parallel; e_8 and e_9 are parallel

v_6 is an isolated vertex.

Degree of $v_3 = 5$

Total degree = 20

b) e_1, e_2, and e_7 are incident on v_1

v_1, v_2 and v_3 are adjacent to v_3

e_2 and e_7 are adjacent to e_1

e_1 and e_3 are loops.

e_4 and e_5 are parallel

Isolated vertex: none.

Degree of $v_3 = 2$

Total degree = 14

Exercise

Let *G* be a simple graph. Show that the relation *R* on the set of vertices of *G* such that uRv if and only if there is an edge associated to $\{u,\ v\}$ is a symmetric, irreflexive relation on *G*.

Solution

In a simple graph, edges are undirected.

If uRv, then there is edge associated with $\{u,\ v\}$. But $\{u,\ v\} = \{v,\ u\}$, so this edge is associated with $\{v,\ u\}$ and, therefore. So, *R* is symmetric.

A simple graph does not allow loops; that is if there is an edge associated with $\{u,\ v\}$, then $u \neq v$.

Thus uRu never holds, and so by definition *R* is irreflexive.

Exercise

Let G be an undirected graph with a loop at every vertex. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to $\{u,\ v\}$ is a symmetric, reflexive relation on G.

Solution

If uRv, then there is edge associated with $\{u,\ v\}$, and since the graph is undirected, this is also edge joining vertices $\{v,\ u\}$ and therefore. So, R is symmetric.

The relation is reflexive because the loops guarantees that uRu for each vertex u.

Exercise

Explain how graphs can be used to model electronic mail messages in a network. Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed? Describe a graph that models the electronic mail sent in a network in a particular week.

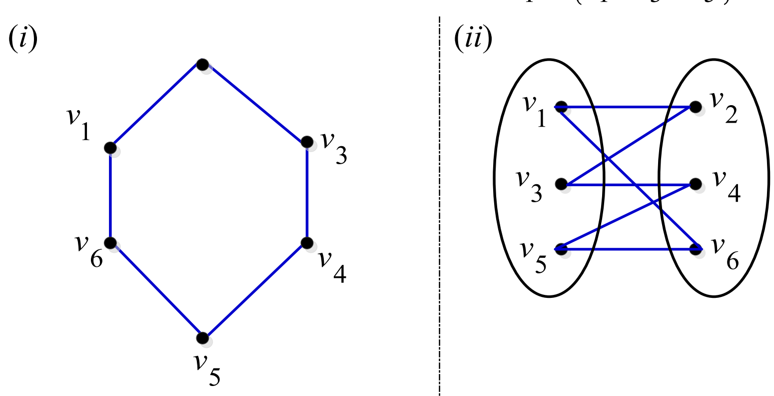
Solution

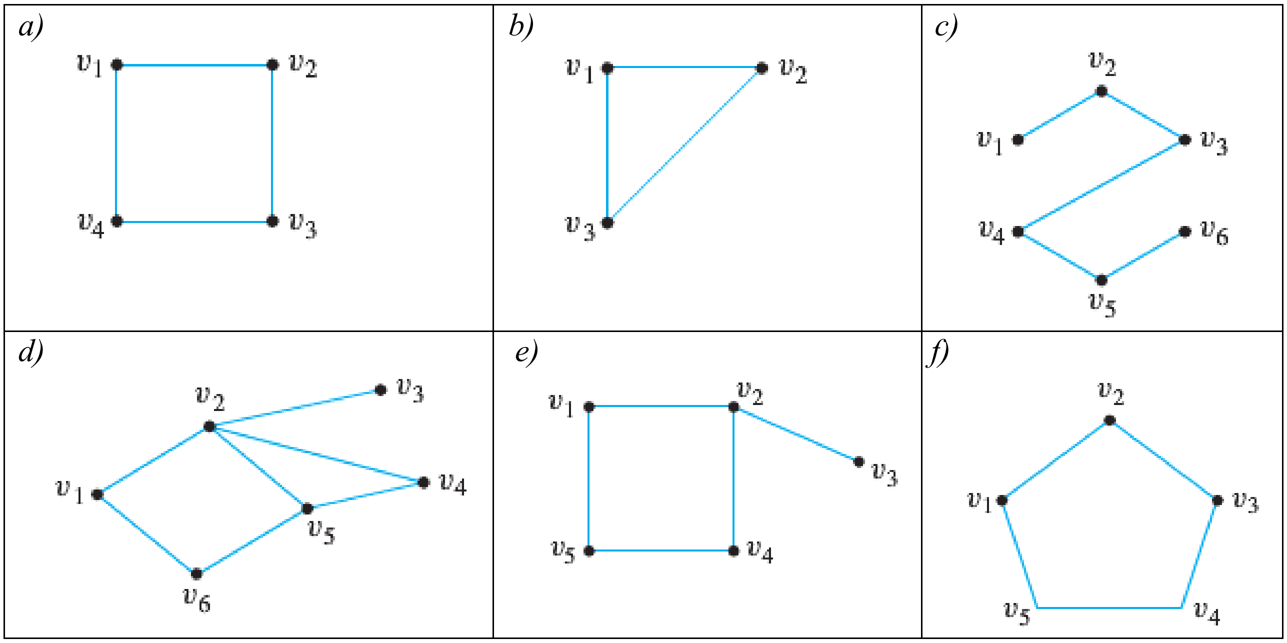
We can have a vertex for each mailbox or e-mail address in the network, with a directed edge between two vertices if a message is sent from the tail of the edge to the head.

We use directed edge for each message sent during the week.

Exercise

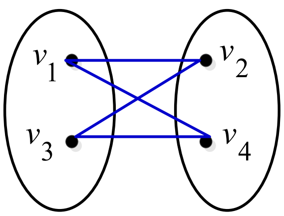
A bipartite graph G is a simple graph whose vertex set can be portioned into two disjoint nonempty subsets V_1 and V_2 such that vertices in V_1 may be connected to vertices in V_2, but no vertices in V_1 are connected to other vertices in V_1 and no vertices in V_2 are connected to other vertices in V_2. For example, the graph G illustrated in (i) can be redrawn as shown in (ii). From the drawing in (ii), you can see that G is bipartite with mutually disjoint vertex set $V_1 = \{v_1,\ v_3,\ v_5\}$ and $V_1 = \{v_2,\ v_4,\ v_6\}$





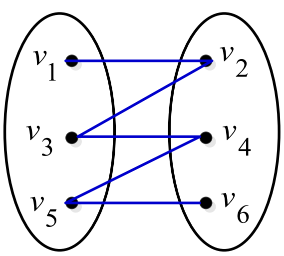
<u>***Solution***</u>

a)



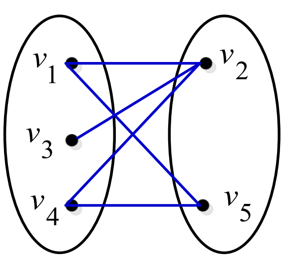
b) $\{v_1, v_2, v_3\}$ form a triangle, we can't create a bipartite graph *G*.

c)



d) $\{v_2, v_4, v_5\}$ form a triangle, therefore we can't create a bipartite graph *G*.

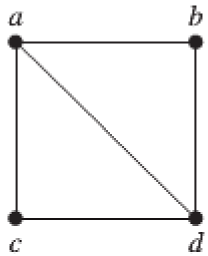
e)



SOLUTION *Section* 4.7 – Representing Graphs and Graph Isomorphism

Exercise

Use the adjacency list to represent the given graph, then represent with an adjacency matrix



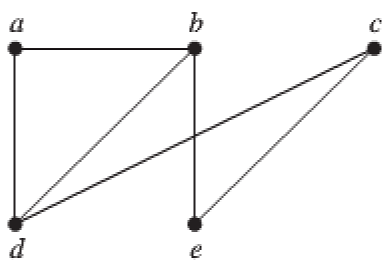
Solution

Vertex	*Adjacent Vertices*
a	*b, c*
b	*a, d*
c	*a, d*
d	*a, b, c*

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Exercise

Use the adjacency list to represent the given graph, then represent with an adjacency matrix



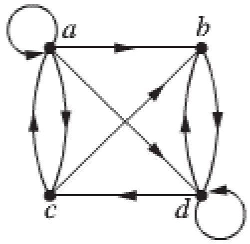
Solution

Vertex	*Adjacent Vertices*
a	*b, d*
b	*a, d, e*
c	*d, e*
d	*a, b, c*
e	*b, c*

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Exercise

Use the adjacency list to represent the given graph, then represent with an adjacency matrix



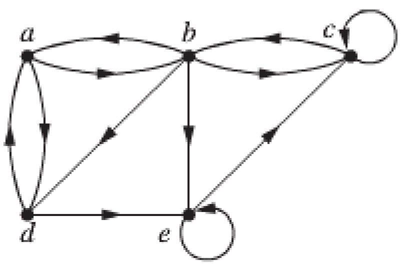
Solution

Initial Vertex	*Terminal Vertices*
a	*a, b , c, d*
b	*d*
c	*a, b*
d	*b, c, d*

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Exercise

Use the adjacency list to represent the given graph, then represent with an adjacency matrix



Solution

Initial Vertex	*Terminal Vertices*
a	*b, d*
b	*a, c, d, e*
c	*b, c*
d	*a, e*
e	*c , e*

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

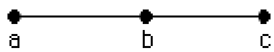
Exercise

Draw a graph with the given adjacency

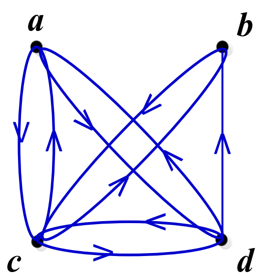
$a)\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad b)\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ $\quad c)\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

Solution

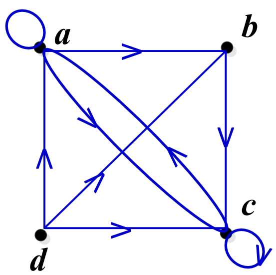
a)



b)



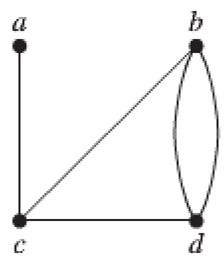
c)



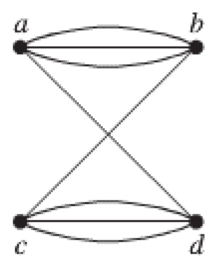
Exercise

Represent the given graph using adjacency matrix

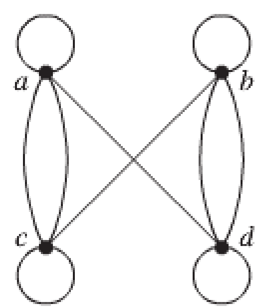
a)



b)



c)



Solution

a) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

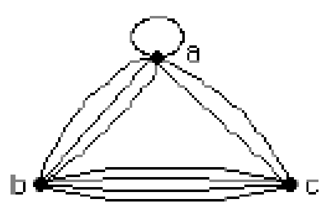
Exercise

Draw an undirected graph represented by the given adjacency

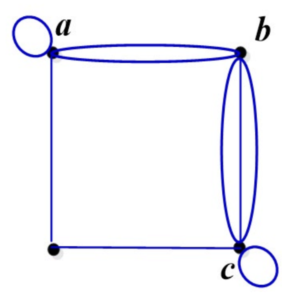
$a)$ $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$ $b)$ $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ $c)$ $\begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$

Solution

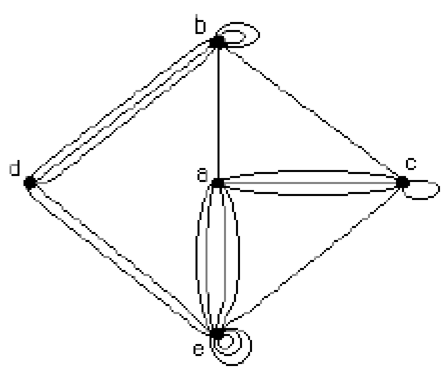
a)



b)

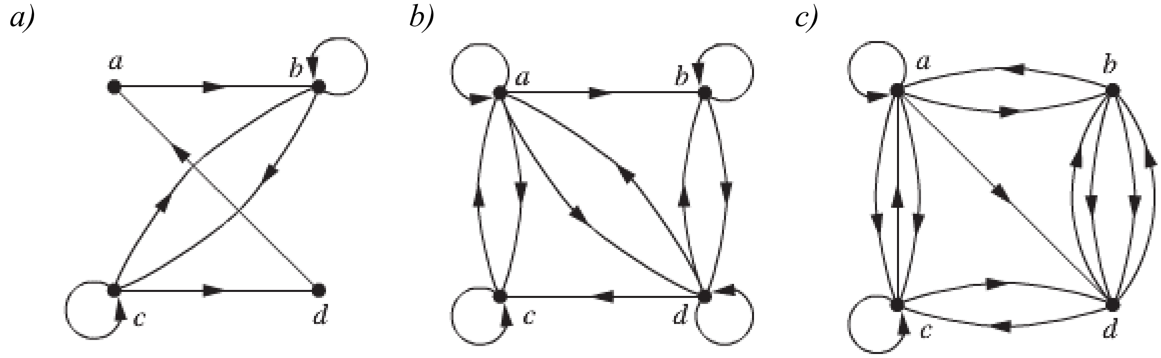


c)



Exercise

Find the adjacency matrix of the given directed multigraph with respect to the vertices listed in alphabetic order.



Solution

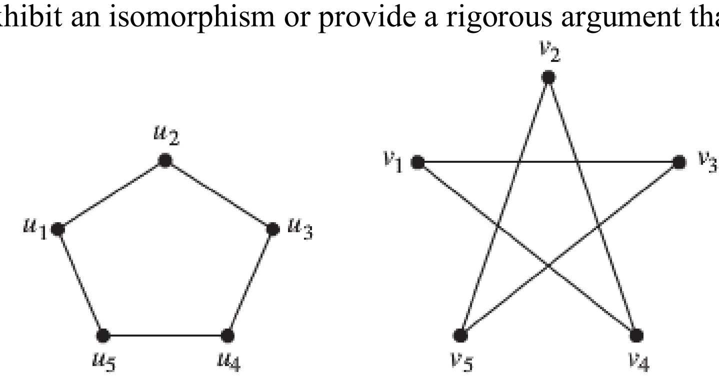
a) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$

Exercise

Determine whether the given pair of graphs is isomorphic.
Exhibit an isomorphism or provide a rigorous argument that none exists.



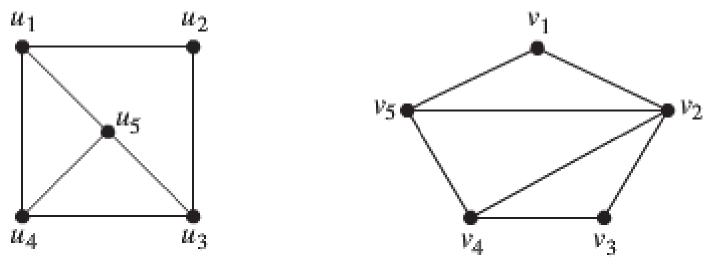
Solution

Both graphs have 5 vertices and 5 edges.

However, each vertex in the second graph has of degree 2, whereas the first does not.

Exercise

Determine whether the given pair of graphs is isomorphic.
Exhibit an isomorphism or provide a rigorous argument that none exists.

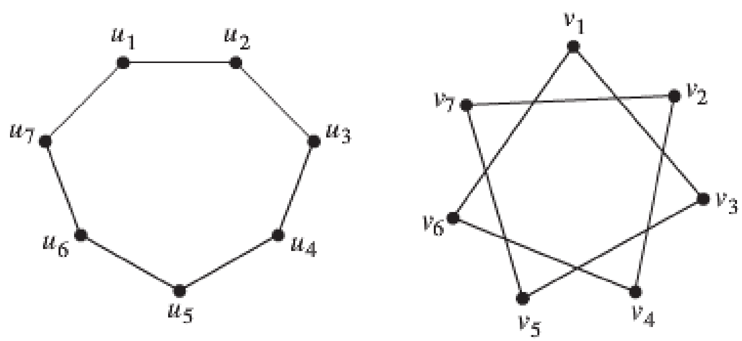


Solution

Both graphs have 5 vertices and 7 edges.
However, the second graph has a vertex of degree 4, whereas the first does not.

Exercise

Determine whether the given pair of graphs is isomorphic.
Exhibit an isomorphism or provide a rigorous argument that none exists.



Solution

Both graphs have 7 vertices and 7 edges.

$f(u_1) = v_1$

$f(u_2) = v_3$

$f(u_3) = v_5$

$f(u_4) = v_7$

$f(u_5) = v_2$

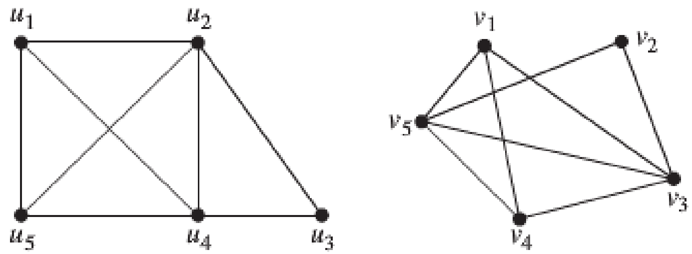
$f(u_6) = v_4$

and $f(u_7) = v_6$

$\therefore$ The graphs are isomorphic.

Exercise

Determine whether the given pair of graphs is isomorphic.
Exhibit an isomorphism or provide a rigorous argument that none exists.



Solution

Both graphs have 5 vertices and 8 edges.

$f(u_1) = v_1$

$f(u_2) = v_3$

$f(u_3) = v_2$

$f(u_4) = v_5$

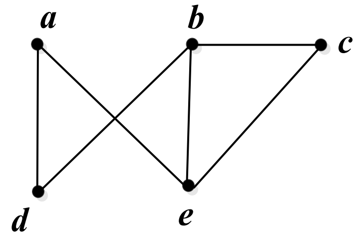
and $f(u_5) = v_4$

$\therefore$ The graphs are isomorphic.

SOLUTION **Section 4.8 – Connectivity**

Exercise

Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? Which are the lengths of those that are paths?



a) *a, e, b, c, b* ***b)*** *a, e, a, d, b, c, a* ***c)*** *e, b, a, d, b, e* ***d)*** *c, b , d, a, e, c*

Solution

a) This is a path of length 4, but it is not a circuit, since it ends at a vertex other than the one at which it began. It is not a simple, since it uses an edge more than once.

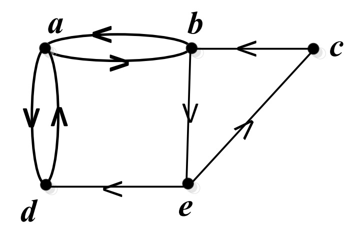
b) This is not a path, since there is no edge from *c* to *a*.

c) This is not a path, since there is no edge from *b* to *a*.

d) This is a path of length 5, which is a circuit. It is simple, since no edges are repeated.

Exercise

Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? Which are the lengths of those that are paths?



a) *a, b, e, c, b* ***b)*** *a, d, a, d, a* ***c)*** *a, d, b, e, a* ***d)*** *a, b , e, c, b, d, a*

Solution

a) This is a path of length 4, but it is not a circuit, since it ends at a vertex other than the one at which it began. It is simple, since no edges are repeated.

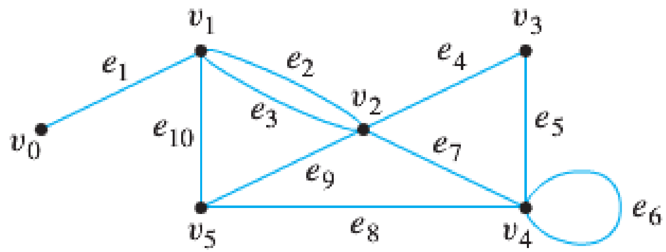
b) This is a path of length 4, which is a circuit. It is not simple, since it uses an edge more than once.

c) This is not a path, since there is no edge from *d* to *b*.

d) This is not a path, since there is no edge from *b* to *d*.

Exercise

Determine whether of the following walks are trails, paths, circuits, or simple circuits or just walk to the graph below.



a) $v_0e_1v_1e_{10}v_5e_9v_2e_2v_1$ *b)* $v_4e_7v_2e_9v_5e_{10}v_1e_3v_2e_9v_5$ *c)* v_2

d) $v_5v_2v_3v_4v_4v_5$ *e)* $v_2v_3v_4v_5v_2v_4v_3v_2$ *f)* $e_5e_8e_{10}e_3$

Solution

a) It is trail since no repeated edge.
It is not a path, repeated vertex v_1
It is not a circuit, since it ends at a vertex other than the one at which it began v_0

b) It is a walk; it is not a trail since it has a repeated edge e_9.
It is not a circuit, since it ends at a vertex other than the one at which it began v_4.

c) It is a closed walk, starts and ends at the same vertex v_2.
It is a trail since no repeated edge.
It is not a path or a circuit, since no edge.

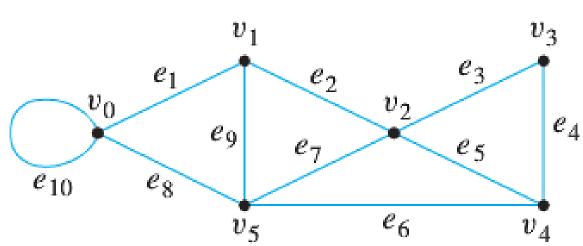
d) It is a path and it is circuit but not a simple circuit since it has a repeated vertex v_4

e) It is a closed walk, starts and ends at the same vertex v_2.
It is not a trail since it has repeated edges $\{v_2, v_3\}$ & $\{v_3, v_4\}$.

f) It is a path, it is not a circuit, since it ends at a vertex other than the one at which it began.

Exercise

Determine whether of the following walks are trails, paths, circuits, or simple circuits or just walk to the graph below.



a) $v_1 e_2 v_2 e_3 v_3 e_4 v_4 e_5 v_2 e_2 v_1 e_1 v_0$ *b)* $v_2 v_3 v_4 v_5 v_2$ *c)* $v_4 v_2 v_3 v_4 v_5 v_2 v_4$

d) $v_2 v_1 v_5 v_2 v_3 v_4 v_2$ *e)* $v_0 v_5 v_2 v_3 v_4 v_2 v_1$ *f)* $v_5 v_4 v_2 v_1$

Solution

a) It is not a trail since it has a repeated edge e_2. It is not a path, repeated vertex v_1, it is not a circuit, since it ends at a vertex other than the one at which it began v_1

b) It is a closed walk, starts and ends at the same vertex v_2. It is a trail since no repeated edge. It is a circuit.

c) It is not a trail since it has repeated edges $\{v_2, v_4\}$.

It is a circuit, but not a simple circuit.

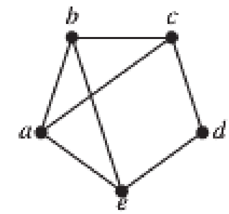
d) It is a path and it is circuit but not a simple circuit since it has a repeated vertex v_2

e) It is a trail since no repeated edge.
It is not a circuit, since it ends at a vertex other than the one at which it began.

f) It is a path, it is not a circuit, since it ends at a vertex other than the one at which it began.

Exercise

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

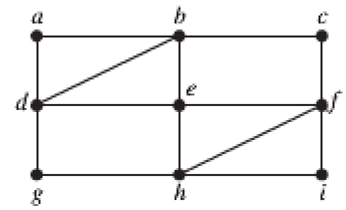


Solution

The vertices *a, b, c, e* have degree 3, therefore the graph has no Euler circuit.
It is not Euler path since there is more than 2 vertices with an odd degree.

Exercise

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

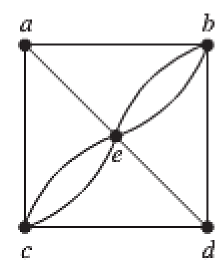


Solution

All the vertex degrees are even, so there is an Euler circuit.
Circuit form: *a, b, c, f, i, h, g, d, e, h, f, e, b, d, a*

Exercise

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

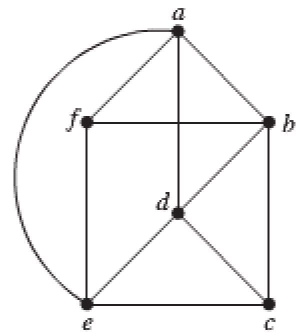


Solution

The vertices *a, d* have degree 3, therefore the graph has no Euler circuit.
It has an Euler path *a, e, c, e, b, e, d, b, a, c, d*. (it has exactly 2 vertices of odd degree)

Exercise

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

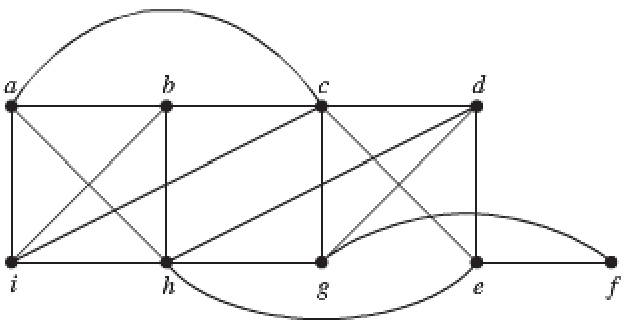


Solution

The vertices *c, f* have degrees 3, therefore the graph has no Euler circuit.
There is an Euler path between the two vertices of odd degree.
One such path is: *f, a, b, c, d, e, f, b, d, a, e, c*.

Exercise

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

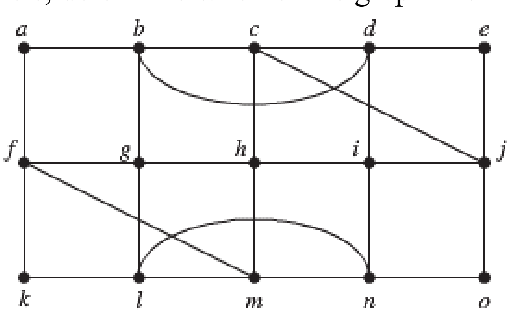


Solution

All the vertex degree are even, so there is an Euler circuit.
Form: *a, i, h, g, d, e, f, g, c, e, h, d, c, a, b, I, c, b, h, a*

Exercise

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

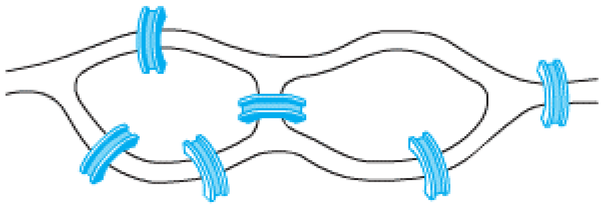


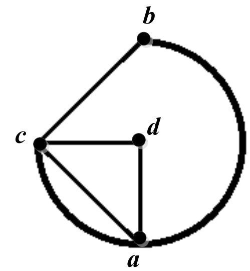
Solution

All the vertex degree are even, so there is an Euler circuit.
Circuit: *a, b, c, d, e, j, c, h, i, d, b, g, h, m, n, o, j, i, n, l, m, f, g, l, k, f, a*

Exercise

Can someone cross all the bridges shown in this map exactly once and return to the starting point?





Solution

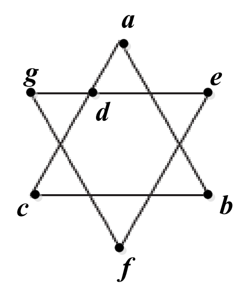
Vertices *a* and *b* are the banks of the river, and vertices *c* and *d* are the islands.

Each vertex has even degree, so the graph has an Euler circuit, such as: *a, c, b, a, d, c, a*.

Therefore, a walk of the type described is possible.

Exercise

Determine whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture

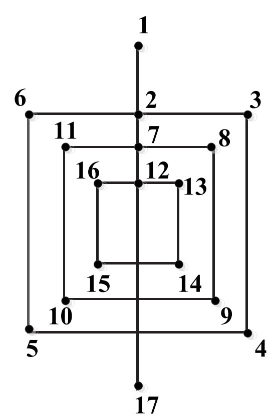


Solution

Yes, the path: *a, b, c, d, e, f, g, d, a*.

Exercise

Determine whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture

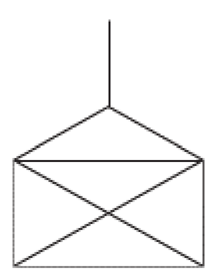


Solution

1, 2, 3, 4, 5, 6, 2, 7, 8, 9, 10, 11, 7, 12, 13, 14, 15, 16, 12, 17

Exercise

Determine whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture

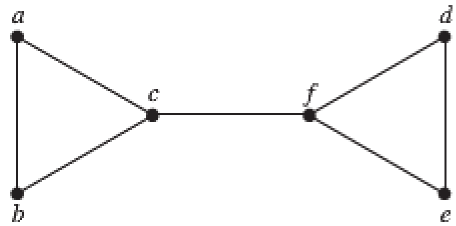


Solution

No

Exercise

Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

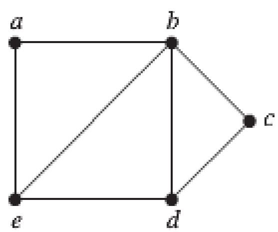


Solution

The graph is not a Hamilton circuit because of the cut edge $\{c, f\}$.
Every simple circuit must be confined to one of the 2 components obtained by deleting this edge.

Exercise

Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

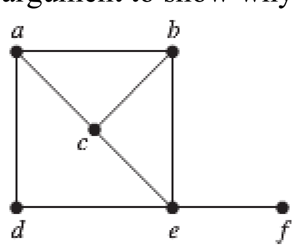


Solution

Hamilton circuit: *a, b, c, d, e, a*.

Exercise

Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

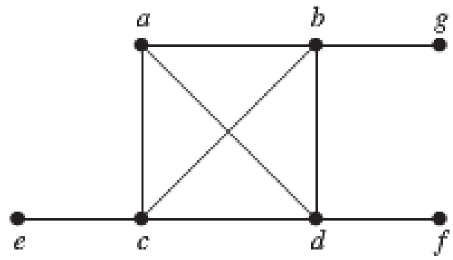


Solution

The graph is not a Hamilton circuit because of the cut edge $\{e, f\}$.

Exercise

Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

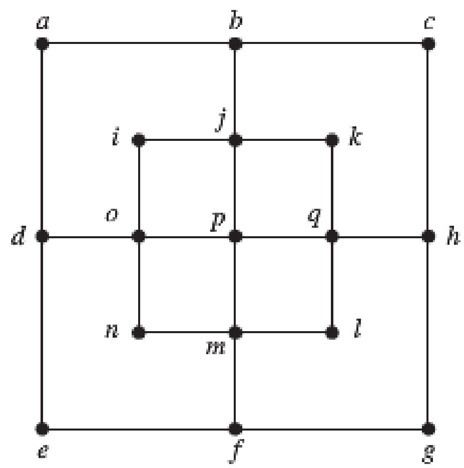


Solution

No Hamilton circuit exists, because once a purported circuit has reached *e* it would be nowhere to go.

Exercise

Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.



Solution

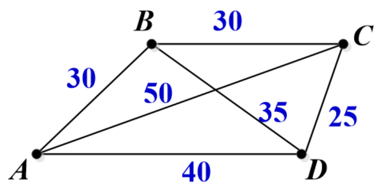
This graph has no Hamilton circuit.

If it did, then certainly the circuit would have to contain edges $\{d, a\}$ and $\{a, b\}$, since these are the only edges incident to vertex *a*. By the same reasoning, the circuit would have to contain the other six edges around the outside of the figure. These 8 edges already complete a circuit, and this circuit omits the 9 vertices on the inside.

Therefore, there is no Hamilton circuit.

Exercise

Imagine that the drawing below is a map showing 4 cities and the distances in kilometers between them. Suppose that a salesman must travel to each city exactly once, starting and ending in city A. Which route from city to city will minimize the total distance that must be traveled?



Solution

Route	***Total Distance (Km)***
ABCDA	30 + 30 + 25 + 40 = 125
ABDCA	30 + 35 + 25 + 50 = 140
ACBDA	50 + 30 + 35 + 40 = 155
ACDBA	50 + 25 + 35 + 30 = 140
ADBCA	40 + 35 + 30 + 50 = 155
ADCBA	40 + 25 + 30 + 30 = 125

Thus either route *ABCDA* or *ADCBA* fives the minimum total distance of 125 km.