Lecture Eight – Identities and Solving Trigonometric

Section **8.1 - Proving Identities**

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Ratio Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$$

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Example

Write $\sec\theta \ \tan\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify.

Solution

$$\sec\theta \tan\theta = \frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\cos^2\theta}$$

Example

Add $\dfrac{1}{\sin\theta} + \dfrac{1}{\cos\theta}$

Solution

$$\frac{1}{\sin\theta} + \frac{1}{\cos\theta} = \frac{\cos\theta + \sin\theta}{\sin\theta\cos\theta} \qquad\qquad \frac{1}{\sin\theta}\frac{\cos\theta}{\cos\theta} + \frac{1}{\cos\theta}\frac{\sin\theta}{\sin\theta}$$

Example

Write: $\tan\alpha + \cot\alpha$ in terms of $\sin\alpha$ *and* $\cos\alpha$

Solution

$$\tan\alpha + \cot\alpha = \frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha}$$

$$= \frac{\sin\alpha}{\cos\alpha}\frac{\sin\alpha}{\sin\alpha} + \frac{\cos\alpha}{\sin\alpha}\frac{\cos\alpha}{\cos\alpha}$$

$$= \frac{\sin^2\alpha + \cos^2\alpha}{\cos\alpha\sin\alpha}$$

$$= \frac{1}{\cos\alpha\sin\alpha}$$

Example

Prove: $\tan x + \cos x = \sin x(\sec x + \cot x)$

Solution

$$\tan x + \cos x = \frac{\sin x}{\cos x} + \cos x$$

$$= \sin x\frac{1}{\cos x} + \cos x\frac{\sin x}{\sin x}$$

$$= \sin x\sec x + \sin x\frac{\cos x}{\sin x}$$

$$= \sin x(\sec x + \cot x) \quad \checkmark$$

$\boxed{or}$

$$\sin x(\sec x + \cot x) = \sin x\left(\frac{1}{\cos x} + \frac{\cos x}{\sin x}\right)$$

$$= \frac{\sin x}{\cos x} + \sin x\frac{\cos x}{\sin x}$$

$$= \tan x + \cos x$$

Example

Prove: $\cot\alpha + 1 = \csc\alpha(\cos\alpha + \sin\alpha)$

<u>*Solution*</u>

$$\csc\alpha(\cos\alpha + \sin\alpha) = \frac{1}{\sin\alpha}(\cos\alpha + \sin\alpha)$$

$$= \frac{1}{\sin\alpha}\cos\alpha + \frac{1}{\sin\alpha}\sin\alpha$$

$$= \cot\alpha + 1 \mid \qquad \checkmark$$

Guidelines for Proving Identities

1. Work on the complicated side first (more trigonometry functions)

2. Look for trigonometry substitutions.

3. Look for algebraic operations

4. If not always change everything to sines and cosines

5. Keep an eye on the side you are not working.

Example

Prove $\qquad \dfrac{\cos^4 t - \sin^4 t}{\cos^2 t} = 1 - \tan^2 t$

<u>*Solution*</u>

$$\frac{\cos^4 t - \sin^4 t}{\cos^2 t} = \frac{\left(\cos^2 t - \sin^2 t\right)\left(\cos^2 t + \sin^2 t\right)}{\cos^2 t} \qquad\qquad a^2 - b^2 = (a-b)(a+b)$$

$$= \frac{\left(\cos^2 t - \sin^2 t\right)(1)}{\cos^2 t} \qquad\qquad \cos^2 t + \sin^2 t = 1$$

$$= \frac{\cos^2 t - \sin^2 t}{\cos^2 t}$$

$$= \frac{\cos^2 t}{\cos^2 t} - \frac{\sin^2 t}{\cos^2 t}$$

$$= 1 - \tan^2 t \mid \qquad \checkmark$$

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Example

Prove: $1 + \cos\theta = \dfrac{\sin^2\theta}{1 - \cos\theta}$

<u>*Solution*</u>

$$\dfrac{\sin^2\theta}{1 - \cos\theta} = \dfrac{1 - \cos^2\theta}{1 - \cos\theta} \qquad\qquad\qquad \sin^2\theta = 1 - \cos^2\theta$$

$$= \dfrac{(1 - \cos\theta)(1 + \cos\theta)}{1 - \cos\theta} \qquad\qquad a^2 - b^2 = (a - b)(a + b)$$

$$\underline{= 1 + \cos\theta} \quad \checkmark$$

Example

Prove: $\tan^2\alpha\left(1 + \cot^2\alpha\right) = \dfrac{1}{1 - \sin^2\alpha}$

<u>*Solution*</u>

$$\tan^2\alpha\left(1 + \cot^2\alpha\right) = \tan^2\alpha + \tan^2\alpha\cot^2\alpha$$

$$= \tan^2\alpha + \tan^2\alpha\dfrac{1}{\tan^2\alpha}$$

$$= \tan^2\alpha + 1 \qquad\qquad \tan^2\alpha + 1 = \sec^2\alpha$$

$$= \sec^2\alpha$$

$$= \dfrac{1}{\cos^2\alpha} \qquad\qquad \cos^2\alpha = 1 - \sin^2\alpha$$

$$\underline{= \dfrac{1}{1 - \sin^2\alpha}} \quad \checkmark$$

Example

Prove: $\dfrac{\sin\alpha}{1 + \cos\alpha} + \dfrac{1 + \cos\alpha}{\sin\alpha} = 2\csc\alpha$

<u>*Solution*</u>

$$\dfrac{\sin\alpha}{1 + \cos\alpha} + \dfrac{1 + \cos\alpha}{\sin\alpha} = \dfrac{\sin\alpha}{\sin\alpha}\cdot\dfrac{\sin\alpha}{1 + \cos\alpha} + \dfrac{1 + \cos\alpha}{\sin\alpha}\cdot\dfrac{1 + \cos\alpha}{1 + \cos\alpha}$$

$$= \dfrac{\sin^2\alpha + (1 + \cos\alpha)^2}{\sin\alpha(1 + \cos\alpha)}$$

$$= \dfrac{\sin^2\alpha + 1 + \cos^2\alpha + 2\cos\alpha}{\sin\alpha(1 + \cos\alpha)}$$

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$$= \frac{2 + 2\cos\alpha}{\sin\alpha(1 + \cos\alpha)}$$

$$= \frac{2(1 + \cos\alpha)}{\sin\alpha(1 + \cos\alpha)}$$

$$= \frac{2}{\sin\alpha}$$

$$= 2\csc\alpha \mid \quad \checkmark$$

Example

Prove $\quad \dfrac{1 + \sin t}{\cos t} = \dfrac{\cos t}{1 - \sin t}$

Solution

$$\frac{1 + \sin t}{\cos t} = \frac{1 + \sin t}{\cos t} \cdot \frac{1 - \sin t}{1 - \sin t}$$

$$= \frac{1 - \sin^2 t}{\cos t(1 - \sin t)}$$

$$= \frac{\cos^2 t}{\cos t(1 - \sin t)}$$

$$= \frac{\cos t}{1 - \sin t} \mid \qquad \checkmark$$

Example

Show that $\cot^2\theta + \cos^2\theta = \cot^2\theta\cos^2\theta$ is not an identity by finding a counterexample

Solution

$$\cot^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} = \cot^2\frac{\pi}{4}\cos^2\frac{\pi}{4}$$

$$1^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1^2 \left(\frac{1}{\sqrt{2}}\right)^2$$

$$1 + \frac{1}{2} = \frac{1}{2}$$

$$\frac{3}{2} \neq \frac{1}{2} \qquad \checkmark$$

Exercises   *Section* 8.1 – Proving Identities

(1–80)   Prove the identity

1. $\cos\theta\cot\theta + \sin\theta = \csc\theta$

2. $\sec\theta\cot\theta - \sin\theta = \dfrac{\cos^2\theta}{\sin\theta}$

3. $\dfrac{\csc\theta\tan\theta}{\sec\theta} = 1$

4. $\left(\sin\theta + \cos\theta\right)^2 = 1 + 2\sin\theta\cos\theta$

5. $\sin\theta(\sec\theta + \cot\theta) = \tan\theta + \cos\theta$

6. $\cos\theta\left(\csc\theta + \tan\theta\right) = \cot\theta + \sin\theta$

7. $\cot\theta + \tan\theta = \csc\theta\sec\theta$

8. $\tan x(\cos x + \cot x) = \sin x + 1$

9. $\dfrac{1 - \cos^4\theta}{1 + \cos^2\theta} = \sin^2\theta$

10. $\dfrac{1 - \sec x}{1 + \sec x} = \dfrac{\cos x - 1}{\cos x + 1}$

11. $\dfrac{\cos x}{1 + \sin x} - \dfrac{1 - \sin x}{\cos x} = 0$

12. $\dfrac{1 + \cot^3 t}{1 + \cot t} = \csc^2 t - \cot t$

13. $\tan x + \cot x = \sec x\csc x$

14. $\dfrac{\tan x - \cot x}{\sin x\cos x} = \sec^2 x - \csc^2 x$

15. $\dfrac{\sec x + \tan x}{\sec x - \tan x} = \dfrac{1 + 2\sin x + \sin^2 x}{\cos^2 x}$

16. $\sin^2 x - \cos^2 x = 2\sin^2 x - 1$

17. $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$

18. $\dfrac{\cos\alpha}{1 + \sin\alpha} = \sec\alpha - \tan\alpha$

19. $\dfrac{\sin\alpha}{1 - \sin\alpha} - \dfrac{\cos\alpha}{1 - \sin\alpha} = \dfrac{1 - \cot\alpha}{\csc\alpha - 1}$

20. $\dfrac{\dfrac{1}{\tan x} + \cot x}{\dfrac{1}{\tan x} + \tan x} = \dfrac{2}{\sec^2 x}$

21. $\dfrac{\cot^2\theta + 3\cot\theta - 4}{\cot\theta + 4} = \cot\theta - 1$

22. $\dfrac{\sin\theta}{1 + \cos\theta} = \dfrac{1 - \cos\theta}{\sin\theta}$

23. $\tan x\left(\csc x - \sin x\right) = \cos x$

24. $\sin x\left(\tan x\cos x - \cot x\cos x\right) = 1 - 2\cos^2 x$

25. $\left(1 + \tan x\right)^2 + \left(\tan x - 1\right)^2 = 2\sec^2 x$

26. $\sec x + \tan x = \dfrac{\cos x}{1 - \sin x}$

27. $\dfrac{\tan x - 1}{\tan x + 1} = \dfrac{1 - \cot x}{1 + \cot x}$

28. $7\csc^2 x - 5\cot^2 x = 2\csc^2 x + 5$

29. $1 - \dfrac{\cos^2 x}{1 - \sin x} = -\sin x$

30. $\dfrac{1 - \cos x}{1 + \cos x} = \dfrac{\sec x - 1}{\sec x + 1}$

31. $\dfrac{\sec x - 1}{\tan x} = \dfrac{\tan x}{\sec x + 1}$

32. $\dfrac{\cos x}{\cos x - \sin x} = \dfrac{1}{1 - \tan x}$

33. $\left(\sec x + \tan x\right)^2 = \dfrac{1 + \sin x}{1 - \sin x}$

34. $\dfrac{\cos x}{1 + \tan x} - \dfrac{\sin x}{1 + \cot x} = \cos x - \sin x$

35. $\dfrac{\cot x + \csc x - 1}{\cot x - \csc x + 1} = \csc x + \cot x$

36. $\dfrac{\tan x + \cot x}{\tan x - \cot x} = \dfrac{1}{\sin^2 x - \cos^2 x}$

37. $\dfrac{1 - \cot^2 x}{1 + \cot^2 x} + 1 = 2\sin^2 x$

38. $\dfrac{1 + \cos x}{1 - \cos x} - \dfrac{1 - \cos x}{1 + \cos x} = 4\cot x\csc x$

39. $\dfrac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = 1 + \sin x\cos x$

40. $1 + \sec^2 x \sin^2 x = \sec^2 x$

41. $\dfrac{1 + \csc x}{\sec x} = \cos x + \cot x$

42. $\tan^2 x = \sec^2 x - \sin^2 x - \cos^2 x$

43. $\dfrac{\sin x}{1 - \cos x} + \dfrac{\sin x}{1 + \cos x} = 2\csc x$

44. $\dfrac{\tan x + \sec x}{\sec x} - \dfrac{\tan x + \sec x}{\tan x} = -\cos x \cot x$

45. $\cos^2(\alpha - \beta) - \cos^2(\alpha + \beta) = \sin^2(\alpha + \beta) - \sin^2(\alpha - \beta)$

46. $\tan x \csc x - \sec^2 x \cos x = 0$

47. $(1 + \tan x)^2 - 2\tan x = \dfrac{1}{(1 - \sin x)(1 + \sin x)}$

48. $\dfrac{3\csc^2 x - 5\csc x - 28}{\csc x - 4} = \dfrac{3}{\sin x} + 7$

49. $\left(\sec^2 x - 1\right)\left(\sec^2 x + 1\right) = \tan^4 x + 2\tan^2 x$

50. $\dfrac{\csc x}{\cot x} - \dfrac{\cot x}{\csc x} = \dfrac{\sin x}{\cot x}$

51. $\dfrac{1 - \cos^2 x}{1 + \cos x} = \dfrac{\sec x - 1}{\sec x}$

52. $\dfrac{\cos x}{1 + \cos x} = \dfrac{\sec x - 1}{\tan^2 x}$

53. $\dfrac{1 - 2\sin^2 x}{1 + 2\sin x \cos x} = \dfrac{\cos x - \sin x}{\cos x + \sin x}$

54. $(\cos x - \sin x)^2 + (\cos x + \sin x)^2 = 2$

55. $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} = 2\csc x$

56. $\dfrac{\sin x + \tan x}{\cot x + \csc x} = \sin x \tan x$

57. $\csc^2 x \sec^2 x = \sec^2 x + \csc^2 x$

58. $\cos^2 x + 1 = 2\cos^2 x + \sin^2 x$

59. $1 - \dfrac{\cos^2 x}{1 + \sin x} = \sin x$

60. $\cot^2 x = (\csc x - 1)(\csc x + 1)$

61. $\dfrac{\csc x + \cot x}{\tan x + \sin x} = \csc x \cot x$

62. $\dfrac{1 - \sec x}{\tan x} + \dfrac{\tan x}{1 - \sec x} = -2\csc x$

63. $\csc x - \sin x = \cos x \cot x$

64. $\cot^3 x = \cot x \left(\csc^2 x - 1\right)$

65. $\dfrac{\cot^2 x}{\csc x - 1} = \dfrac{1 + \sin x}{\sin x}$

66. $\cot^2 x + \csc^2 x = 2\csc^2 x - 1$

67. $\dfrac{\cot^2 x}{1 + \csc x} = \csc x - 1$

68. $\sec^4 x - \tan^4 x = \sec^2 x + \tan^2 x$

69. $\dfrac{\cos x}{1 + \sin x} + \dfrac{1 + \sin x}{\cos x} = 2\sec x$

70. $\dfrac{\sin x + \cos x}{\sin x - \cos x} = \dfrac{1 + 2\sin x \cos x}{2\sin^2 x - 1}$

71. $\dfrac{\csc x - 1}{\csc x + 1} = \dfrac{\cot^2 x}{\csc^2 x + 2\csc x + 1}$

72. $\csc^4 x - \cot^4 x = \csc^2 x + \cot^2 x$

73. $\tan\left(\dfrac{\pi}{4} + x\right) = \cot\left(\dfrac{\pi}{4} - x\right)$

74. $\dfrac{\sin\theta}{1 + \sin\theta} - \dfrac{\sin\theta}{1 - \sin\theta} = -2\tan^2\theta$

75. $\csc^2 x - \cos^2 x \csc^2 x = 1$

76. $1 - 2\sin^2 x = 2\cos^2 x - 1$

77. $\csc^2 x - \cos x \sec x = \cot^2 x$

78. $(\sec x - \tan x)(\sec x + \tan x) = 1$

79. $\left(1 + \tan^2 x\right)\left(1 - \sin^2 x\right) = 1$

80. $10\csc^2 x - 6\cot^2 x = 4\csc^2 x + 6$

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