Section 1.7 – Cramer's Rule

Cramer's Rule

Theorem

If $AX = B$ is a system of a linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(B_1)}{\det(A)}$$

$$x_2 = \frac{\det(B_2)}{\det(A)}$$

$$\vdots \qquad \vdots$$

$$x_n = \frac{\det(B_n)}{\det(A)}$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

$$\det(B_1) = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & & & \\ \vdots & & & \\ b_n & a_{n2} & & a_{nn} \end{vmatrix}$$

Example

Use Cramer's rule to solve

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ -2x_1 + x_2 \phantom{{}+x_3} &= 0 \\ -4x_1 \phantom{{}+x_2} + x_3 &= 0 \end{aligned}$$

Solution

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 7$$

$$|B_1| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$|B_2| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 2$$

$$|B_3| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 0 \end{vmatrix} = 4$$

$$x_1 = \frac{|B_1|}{|A|} = \frac{1}{7} \qquad\qquad x_1 = \frac{\det(A_1)}{\det(A)}$$

$$x_2 = \frac{|B_2|}{|A|} = \frac{2}{7}$$

$$x_3 = \frac{|B_3|}{|A|} = \frac{4}{7}$$

Solution: $\left(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right)$

Example

Use Cramer's Rule to solve.

$$\begin{aligned} x_1 + \quad\quad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

Solution

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \Rightarrow \det(A) = 44$$

$$\det(A_1) = \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix} = -40$$

$$\det(A_2) = \begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix} = 72$$

$$\det(A_3) = \begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix} = 152$$

$$x_1 = \frac{-40}{44} \qquad\qquad x_1 = \frac{\det(A_1)}{\det(A)}$$

$$\boxed{= -\frac{10}{11}}$$

$$x_2 = \frac{72}{44} \qquad\qquad x_2 = \frac{\det(A_2)}{\det(A)}$$

$$\boxed{= \frac{18}{11}}$$

$$x_3 = \frac{152}{44} \qquad\qquad x_3 = \frac{\det(A_3)}{\det(A)}$$

$$\boxed{= \frac{38}{11}}$$

Solution: $\boxed{\left(-\frac{10}{11}, \frac{18}{11}, \frac{38}{11}\right)}$

A Formula for A^{-1}

Theorem: ***Inverse of a matrix using its Adjoint***

The i, j entry of A^{-1} is the cofactor C_{ji} $\left(not\ C_{ij}\right)$ divided by det(***A***):

$$\textbf{\textit{Formula for }} A^{-1}: \quad \left(A^{-1}\right)_{ij} = \frac{C_{ji}}{|A|} \quad and \quad A^{-1} = \frac{C^T}{|A|}$$

$$\boxed{A^{-1} = \frac{1}{\det(A)} adj(A)}$$

Example

Find the inverse matrix of $A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ using its adjoint.

Solution

$$C_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1; \qquad C_{12} = -\begin{vmatrix} -2 & 0 \\ -4 & 1 \end{vmatrix} = 2; \qquad C_{13} = \begin{vmatrix} -2 & 1 \\ -4 & 0 \end{vmatrix} = 4$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1; \qquad C_{22} = \begin{vmatrix} 1 & 1 \\ -4 & 1 \end{vmatrix} = 5; \qquad C_{23} = -\begin{vmatrix} 1 & 1 \\ -4 & 0 \end{vmatrix} = -4$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1; \qquad C_{32} = -\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = -2; \qquad C_{33} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3$$

$$C = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 5 & -2 \\ 4 & -4 & 3 \end{bmatrix}$$

$$\det(A) = \frac{1}{7}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{7}\begin{bmatrix} 1 & -1 & -1 \\ 2 & 5 & -2 \\ 4 & -4 & 3 \end{bmatrix}$$

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent

a) A is invertible

b) $A\boldsymbol{x} = 0$ has only the trivial solution

c) The reduced row echelon form of A is I_n

d) A can be expressed as a product of elementary matrices

e) $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $n \times 1$ matrix $\boldsymbol{b}$

f) $\det(A) \neq 0$

Exercises   *Section* 1.7 – Cramer's Rule

1. Use Cramer's Rule with ratios $\dfrac{\det B_j}{\det A}$ to solve $A\boldsymbol{x} = b$. Also find the inverse matrix $A^{-1} = \dfrac{C^T}{\det A}$. Why is the solution $\boldsymbol{x}$ is the first part the same as column 3 of A^{-1}? Which cofactors are involved in computing that column $\boldsymbol{x}$?

$$Ax = b \quad is \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2. Verify that $\det(AB) = \det(BA)$ and determine whether the equality $\det(A+B) = \det(A) + \det(B)$ holds

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

3. Verify that $\det(kA) = k^n \det(A)$

$a)\quad A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}, \quad k = 2$

$b)\quad A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}, \quad k = -2$

$c)\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}, \quad k = 3$

(4 – 58) Use Cramer's rule to solve the system

4. $\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$

5. $\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$

6. $\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$

7. $\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$

8. $\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$

9. $\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$

10. $\begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$

11. $\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$

12. $\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$

13. $\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$

14. $\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$

15. $\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$

16. $\begin{cases} x-2y=5 \\ -10x+2y=4 \end{cases}$

17. $\begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$

18. $\begin{cases} 4x-4y=-12 \\ 4x+4y=-20 \end{cases}$

19. $\begin{cases} x+y=7 \\ x-y=3 \end{cases}$

20. $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

21. $\begin{cases} 12x+3y=15 \\ 2x-3y=13 \end{cases}$

22. $\begin{cases} x-2y=5 \\ 5x-y=-2 \end{cases}$

23. $\begin{cases} 3x+2y=2 \\ 2x+2y=3 \end{cases}$

24. $\begin{cases} 4x-5y=17 \\ 2x+3y=3 \end{cases}$

25. $\begin{cases} x-3y=4 \\ 3x-4y=12 \end{cases}$

26. $\begin{cases} 2x-9y=5 \\ 3x-3y=11 \end{cases}$

27. $\begin{cases} 3x-4y=4 \\ x+y=6 \end{cases}$

28. $\begin{cases} 3x=7y+1 \\ 2x=3y-1 \end{cases}$

29. $\begin{cases} 2x=3y+2 \\ 5x=51-4y \end{cases}$

30. $\begin{cases} y=-4x+2 \\ 2x=3y-1 \end{cases}$

31. $\begin{cases} 3x=2-3y \\ 2y=3-2x \end{cases}$

32. $\begin{cases} x+2y-3=0 \\ 12=8y+4x \end{cases}$

33. $\begin{cases} 7x-2y=3 \\ 3x+y=5 \end{cases}$

34. $\begin{cases} 3x+2y-z=4 \\ 3x-2y+z=5 \\ 4x-5y-z=-1 \end{cases}$

35. $\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$

36. $\begin{cases} 2x+y+z=9 \\ -x-y+z=1 \\ 3x-y+z=9 \end{cases}$

37. $\begin{cases} 3y-z=-1 \\ x+5y-z=-4 \\ -3x+6y+2z=11 \end{cases}$

38. $\begin{cases} x+3y+4z=14 \\ 2x-3y+2z=10 \\ 3x-y+z=9 \end{cases}$

39. $\begin{cases} x+4y-z=20 \\ 3x+2y+z=8 \\ 2x-3y+2z=-16 \end{cases}$

40. $\begin{cases} -2x+6y+7z=3 \\ -4x+5y+3z=7 \\ -6x+3y+5z=-4 \end{cases}$

41. $\begin{cases} 2x-y+z=1 \\ 3x-3y+4z=5 \\ 4x-2y+3z=4 \end{cases}$

42. $\begin{cases} 3x-4y+4z=7 \\ x-y-2z=2 \\ 2x-3y+6z=5 \end{cases}$

43. $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

44. $\begin{cases} x+y+z=3 \\ -y+2z=1 \\ -x+z=0 \end{cases}$

45. $\begin{cases} 3x+y+3z=14 \\ 7x+5y+8z=37 \\ x+3y+2z=9 \end{cases}$

46. $\begin{cases} 4x-2y+z=7 \\ x+y+z=-2 \\ 4x+2y+z=3 \end{cases}$

47. $\begin{cases} 2y-z=7 \\ x+2y+z=17 \\ 2x-3y+2z=-1 \end{cases}$

48. $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

49. $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

50. $\begin{cases} 2x-y+2z=-8 \\ x+2y-3z=9 \\ 3x-y-4z=3 \end{cases}$

51. $\begin{cases} x \quad -3z=-5 \\ 2x-y+2z=16 \\ 7x-3y-5z=19 \end{cases}$

52. $\begin{cases} x+2y-z=5 \\ 2x-y+3z=0 \\ 2y+z=1 \end{cases}$

53. $\begin{cases} x+y+z=6 \\ 3x+4y-7z=1 \\ 2x-y+3z=5 \end{cases}$

54. $\begin{cases} 3x+2y+3z=3 \\ 4x-5y+7z=1 \\ 2x+3y-2z=6 \end{cases}$

55. $\begin{cases} 4x+5y \quad\quad\; =2 \\ 11x+y+2z=3 \\ x+5y+2z=1 \end{cases}$

56. $\begin{cases} x-4y+z=6 \\ 4x-y+2z=-1 \\ 2x+2y-3z=-20 \end{cases}$

57. $\begin{cases} 2x-\;\; y+\;\; z=-1 \\ 3x+4y-\;\; z=-1 \\ 4x-\;\;\; y+2z=-1 \end{cases}$

58. $\begin{cases} -x_1-4x_2+2x_3+x_4=-32 \\ 2x_1-x_2+7x_3+9x_4=14 \\ -x_1+x_2+3x_3+x_4=11 \\ -x_1-2x_2+x_3-4x_4=-4 \end{cases}$

59. Show that the matrix A is invertible for all values of θ, then find A^{-1} using $A^{-1}=\frac{1}{\det(A)}adj(A)$

$$A=\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$