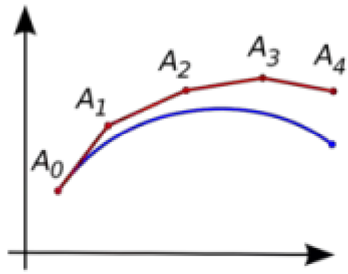
Section 1.8 – Numerical Methods

Euler's method named after ***Leonhard Euler*** is an example of a ***fixed-step*** solver.

Euler's method is a first-order numerical procedure for solving ordinary differential equations (***ODE***s) with a given initial value. It is the most basic kind of explicit method for numerical integration of ordinary differential equations.



$y' = f(x,y) \qquad y(x_0) = y_0$

The setting size: $h = \dfrac{b-a}{k} > 0$; $\quad k \in \mathbb{N}$

$$\begin{aligned}
\text{Then,} \quad & x_0 = a \\
& x_1 = x_0 + h = a + h \\
& x_k = x_{k-1} + h = a + kh \\
\textit{Last point} \quad & x_k = a + kh = b
\end{aligned}$$

By the definition of the derivative:

$$y'\left(x_k\right) \approx \frac{y\left(x_{k+1}\right) - y\left(x_k\right)}{h}$$

$$y'\left(x_k\right) \approx \frac{y_{k+1} - y_k}{h} = f\left(x_k, y_k\right) : \textit{slope}$$

The tangent line at the point $\left(x_0, y(x_0)\right)$ is:

$$y_{k+1} = y_k + h.f\left(x_k, y_k\right)$$

$$y_{k+1} = y_k + \Delta x_{step} . f\left(x_k, y_k\right)$$

This method is known as *Euler's Method* with step size h.

Example

Compute the first four step in the Euler's method approximation to the solution of $y' = y - x$ with $y(1) = 1$, using the step size $h = 0.1$. Compare the result with the actual solution to the initial value problem.

Solution

$y(1) = 1 \Rightarrow x_0 = 1$ *and* $y_0 = 1$

The *first* step:

$$y_1 = y_0 + h\left(y_0 - x_0\right)$$
$$= 1 + 0.1(1 - 1)$$
$$= 1$$

$$x_1 = x_0 + h = 1 + 0.1 = 1.1$$

The *second* step:

$$y_2 = y_1 + h\left(y_1 - x_1\right)$$
$$= 1 + 0.1(1 - 1.1)$$
$$= 0.99$$

$$x_2 = x_1 + h = 1.1 + 0.1 = 1.2$$

The *third* step:

$$y_3 = y_2 + h\left(y_2 - x_2\right)$$
$$= 0.99 + 0.1(0.99 - 1.2)$$
$$= 0.969$$

$$x_3 = x_2 + h = 1.2 + 0.1 = 1.3$$

The *fourth* step:

$$y_4 = y_3 + h\left(y_3 - x_3\right)$$
$$= 0.969 + 0.1(0.969 - 1.3)$$
$$= 0.9359$$

$$x_3 = x_2 + h = 1.3 + 0.1 = 1.4$$

The exact solution to $y' = y - x$ is $y(x) = 1 + x - e^{x-1}$

x_k	y_k : Euler's	y_k - exact	*Error*
1.0	1.0	1.0	0
1.1	1.0	0.9948	-0.0052
1.2	0.990	0.9786	-0.0114
1.3	0.969	0.9501	-0.0189
1.4	0.9359	0.9082	-0.0277

Runge-Kutta Methods

Like Euler's method, the Runge-Kutta methods are fixed-step solvers.

The second-Order Runge-Kutta Method

The second-Order Runge-Kutta method is also known as the improved Euler's method.

Starting from the initial value point $\left(x_0 , y_0\right)$, we compute two slopes:

$$s_1 = f\left(t_0 , y_0\right)$$

$$s_2 = f\left(t_0 + h , y_0 + hs_1\right)$$

$$y_1 = y_0 + h\frac{s_1 + s_2}{2}$$

But an analysis using Taylor's theorem reveals that there is an improvement in the estimate for the truncation error.
For the second-Order Runge-Kutta method, we have

$$\left|y(t_1) - y_1\right| \le Mh^3$$

The constant M depends on the function $f(t, y)$.
The second-Order Runge-Kutta method is controlled by the cube of the step size instead of the square.

Input t_0 *and* y_0

For k = 1 to N

$$s_1 = f\left(t_{k-1} , y_{k-1}\right)$$

$$s_2 = f\left(t_{k-1} + h , y_{k-1} + hs_1\right)$$

$$y_k = y_{k-1} + h\frac{s_1 + s_2}{2}$$

$$t_k = t_{k-1} + h$$

Example

Compute the first four step in the second-Order Runge-Kutta method approximation to the solution of $y' = y - t$ with $y(1) = 1$, using the step size $h = 0.1$. Compare the result with the actual solution to the initial value problem.

Solution

$\Rightarrow t_0 = 1$ *and* $y_0 = 1$

The *first* step:

$$\begin{aligned} s_1 &= f\left(t_0, y_0\right) \\ &= y_0 - t_0 \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} s_2 &= f\left(t_0 + h, y_0 + hs_1\right) \\ &= \left(y_0 + hs_1\right) - \left(t_0 + h\right) \\ &= (1 + .1(0)) - (1 + .1) \\ &= 1 - 1.1 \\ &= -0.1 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + h\frac{s_1 + s_2}{2} \\ &= 1 + 0.1\left(\frac{0 - 0.1}{2}\right) \\ &= 0.995 \end{aligned}$$

$$\begin{aligned} t_1 &= t_0 + h \\ &= 1 + 0.1 \\ &= 1.1 \end{aligned}$$

The *second* step:

$$s_1 = y_1 - t_1 = 0.995 - 1.1 = -0.105$$

$$s_2 = \left(y_1 + hs_1\right) - \left(t_1 + h\right) = (0.995 + .1(-0.105)) - (1.1 + .1) = -.2155$$

$$y_2 = y_1 + h\frac{s_1 + s_2}{2} = .995 + 0.1\left(\frac{-.105 - .2155}{2}\right) = .978975$$

$$t_2 = t_1 + h = 1.1 + .1 = 1.2$$

The *third* step:

$$s_1 = y_2 - t_2 = 0.978975 - 1.2 = -0.221025$$

$$s_2 = \left(y_2 + hs_1\right) - \left(t_2 + h\right) = \left(0.978975 + .1(-0.221025)\right) - \left(1.2 + .1\right) = -.3431275$$

$$y_3 = y_2 + h\frac{s_1 + s_2}{2} = .978975 + 0.1\left(\frac{-.221025 - .3431275}{2}\right) = 0.9507673$$

$$t_3 = t_2 + h = 1.3$$

The *fourth* step:

$$s_1 = y_3 - t_3 = 0.9507673 - 1.3 = -0.3492327$$

$$s_2 = \left(y_3 + hs_1\right) - \left(t_3 + h\right) = \left(0.9507673 + .1(-0.3492327)\right) - \left(1.3 + .1\right) = -.48415597$$

$$y_4 = y_3 + h\frac{s_1 + s_2}{2} = .9507673 + 0.1\left(\frac{-.3492327 - .48415597}{2}\right) = 0.9090979$$

$$t_4 = t_3 + h = 1.4$$

t_k	y_k : Runge-Kutta	y_k - Exact	***Runge-Kutta Error***	***Euler's Error***
1.0	1.0	1.0	**0**	**0**
1.1	0.9950000	0.994829081	**-0.000170918**	**-0.0052**
1.2	0.9789750	0.978597241	**-0.000377758**	**-0.0114**
1.3	0.9507673	0.950141192	**-0.000626182**	**-0.0189**
1.4	0.9090979	0.908175302	**-0.000922647**	**-0.0277**

Fourth-Order Runge-Kutta Method

This method is the most commonly used solution algorithm. For most equations and systems it is suitably fast and accurate.

Starting from the initial value point (t_0, y_0), we compute two slopes:

$$s_1 = f(t_0, y_0)$$

$$s_2 = f\left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}s_1\right)$$

$$s_3 = f\left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}s_2\right)$$

$$s_4 = f\left(t_0 + h, y_0 + hs_3\right)$$

$$y_1 = y_0 + h\frac{s_1 + 2s_2 + 2s_3 + s_4}{6}$$

Example

Compute the first four step in the second-Order Runge-Kutta method approximation to the solution of $y' = y - t$ with $y(1) = 1$, using the step size $h = 0.1$. Compare the result with the actual solution to the initial value problem.

Solution

$\Rightarrow t_0 = 1$ *and* $y_0 = 1$

The *first* step:

$$s_1 = f(t_0, y_0)$$
$$= 1 - 1$$
$$= 0$$

$$s_2 = f\left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}s_1\right)$$
$$= f(1.05, 1)$$
$$= 1 - 1.05$$
$$= -0.05$$

$$s_3 = f\left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}s_2\right)$$
$$= f(1.05, .9975)$$
$$= .9975 - 1.05$$
$$= -0.0525$$

$$s_4 = f\left(t_0 + h, y_0 + hs_3\right)$$

$$= f\left(1.1\,,.99475\right)$$
$$= .99475 - 1.1$$
$$= -0.10525$$

$$y_1 = y_0 + h\frac{s_1 + 2s_2 + 2s_3 + s_4}{6}$$
$$= 1 + 0.1\left(\frac{0 + 2(-.05) + 2(-.0525) + (-.10525)}{6}\right)$$
$$= 0.99482916667$$

$$t_1 = t_0 + h$$
$$= 1.1$$

t_k	y_k : Runge-Kutta	y_k - Exact	***Runge-Kutta Error***
1.0	1.0	1.0	**0**
1.1	0.994829167	0.994829081	**-0.000000086**
1.2	0.978597429	0.978597241	**0.000000295**
1.3	0.950141502	0.950141192	**-0.000000310**
1.4	0.908175759	0.908175302	**-0.000000457**

Exercises Section 1.8 – Numerical Methods

Calculate the first five iterations of Euler's method with step $h = 0.1$ of

1. $y' = ty \quad y(0) = 1$

2. $z' = x - 2z \quad z(0) = 1$

3. $z' = 5 - z \quad z(0) = 0$

4. Given: $y' + 2xy = x \quad y(0) = 8$

a) Use a computer and Euler's method to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

5. Given: $y' + 2y = 2 - e^{-4t} \quad y(0) = 1$

a) Solve the differential equation

b) Use Euler's method and Runge-Kutta methods to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$. Plot the exact solution and approximate solutions as discrete points.

6. Given: $z' - 2z = xe^{2x} \quad z(0) = 1$

a) Use a computer and Euler's method to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a third with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

7. Consider the initial value problem $y' = 12y(4 - y) \quad y(0) = 1$

Use Euler's method with step size $h = 0.04$ to sketch solution on the interval $[0,\ 2]$

8. You've seen that the error in Euler's method varies directly as the first power of the step size $\left(i.e\ E_h \approx \lambda h\right)$. This makes Euler's method an order to halve the error? How does this affect the number of required iterations?

9. Use Euler's method to provide an approximate solution over the given time interval using the given steps sizes. Provide a plot of ***v*** versus ***y*** for each step size

$$y'' + 4y = 0, \quad y(0) = 4, \quad y'(0) = 0, \quad [0,\ 2\pi]; \quad h = 0.1,\ 0.01,\ 0.001$$

10. Given $z' + z = \cos x \quad z(0) = 1$

a) Use a computer and Runge-Kutta method to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

11. Given $x' = \frac{t}{x} \quad x(0) = 1$

a) Use a computer and Runge-Kutta method to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

12. Consider the initial value problem $y' = \frac{t}{y^2} \quad y(0) = 1$

Use Runge-Kutta method with step size $h = 0.04$ to sketch solution on the interval $[0,\ 2]$

13. Consider the initial value problem $y' - y = -\frac{1}{2}e^{t/2}\sin 5t + 5e^{t/2}\cos 5t \quad y(0) = 0$

Use Runge-Kutta method with step size $h = 0.05$ to sketch solution on the interval $[0,\ 5]$