2.8 Lagrange Multipliers.

Ex Pt? $P(x,y,z)$ on $2x+y-z=5$ that is closest to Origin. (1)

$|\vec{OP}| = \sqrt{x^2+y^2+z^2}$

$f(x,y,z) = x^2+y^2+z^2$

$z = 2x+y-5$

$h(x,y) = x^2+y^2+(2x+y-5)^2$

$h_x = 2x+4(2x+y-5)$ $\quad h_y = 2y+2(2x+y-5)$

$= 10x+4y-20=0$ $\quad = 4x+4y-10=0$

$$\begin{cases} 10x+4y=20 \\ 4x+4y=10 \end{cases}$$

$6x = 10 \Rightarrow x = \frac{5}{3}$

$y = \left(10 - \frac{20}{3}\right)\frac{1}{4}$

$= \frac{5}{6}$

$z = \frac{10}{3} + \frac{5}{6} - 5$ $\quad \frac{25}{6} - 5$

$= -\frac{5}{6}$

$|\vec{OP}| = \sqrt{\frac{25}{9} + \frac{25}{36} + \frac{25}{36}}$

$= 5\sqrt{\frac{4+1+1}{36}}$

$= \frac{5\sqrt{6}}{6}$

Ex Pts? $x^2 - z^2 - 1 = 0$ closest to Origin

$f(x,y,z) = x^2 + y^2 + z^2$ subject to
s.t.

$x^2 - z^2 - 1 = 0$ (1)

$z^2 = x^2 - 1$

$f(x, y, \sqrt{x^2-1}) = h(x,y)$
$= x^2 + y^2 + x^2 - 1$

$h(x,y) = 2x^2 + y^2 - 1$

$h_x = 4x = 0 \Rightarrow x = 0$ $\qquad h_y = 2y = 0$
$y = 0$

c.p. $(0,0)$?

(1) $x^2 = z^2 + 1$

$h(y,z) = z^2 + 1 + y^2 + z^2$
$= y^2 + 2z^2 + 1$

$h_y = 2y = 0$ $\qquad h_z = 4z = 0$

$y = 0 = z$

$x^2 \big|_{z=0} = 1 \Rightarrow x = \pm 1$

$(\pm 1, 0, 0)$

$d = \sqrt{1+0+0} = 1$

$f(x,y,z) = x^2+y^2+z^2-a^2 = 0$

st: $x^2 - z^2 - 1 = 0 = g(x,y,z)$

$\nabla f = \lambda \nabla g$ $\qquad \nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$

$2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda(2x\hat{i} - 2z\hat{k})$

$= 2\lambda x\hat{i} - 2\lambda z\hat{k}$

$2x = 2\lambda x \qquad 2y = 0 \qquad 2z = -2\lambda z$

$\boxed{y=0}$

$x = \lambda x$

$x \neq 0 \Rightarrow \underline{\lambda = 1}$

For $\lambda = 1 \Rightarrow z = -\lambda z$

$z = -z \Rightarrow \underline{z = 0}$

$x^2 = z^2 + 1 \big|_{z=0}$

$x^2 = 1 \Rightarrow x = \pm 1.$

$(\pm 1, 0, 0)$

$\nabla f = \lambda \nabla g$

$\downarrow$

Lagrange Multiplier

Ex Greatest & Smallest

$f(x,y) = xy \quad s.t: \frac{x^2}{8} + \frac{y^2}{2} = 1$

Soln

$g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$

$\nabla f = \lambda \nabla g.$

$y\hat{i} + x\hat{j} = \frac{1}{4}x\lambda\hat{i} + \lambda y\hat{j}$

$y = \frac{1}{4}\lambda x \longrightarrow x = \lambda y. \quad (1)$

$y = \frac{1}{4}\lambda(\lambda y)$

$y = \frac{1}{4}\lambda^2 y \quad \begin{cases} y = 0 \\ \lambda^2 = 4 \\ \boxed{\lambda = \pm 2} \end{cases} \quad y \neq 0 \quad 1 = \frac{\lambda^2}{4}$

Case 1 $y = 0 \Rightarrow x = 0 \Rightarrow (0,0)$ is not on ellipse

Case 2 if $y \neq 0 \Rightarrow \lambda = \pm 2$

$(1) \rightarrow x = \pm 2y.$

$g(x,y) = \frac{1}{8}(\pm 2y)^2 + \frac{1}{2}y^2 = 1$

$\frac{1}{2}y^2 + \frac{1}{2}y^2 = 1$

$y^2 = 1 \Rightarrow \boxed{y = \pm 1}$

$x = \pm 2y = \pm 2$

Points $(\pm 2, \pm 1)$

$f(x,y) = xy$

$f(2,1) = 2$

$f(-2,-1) = 2$

$f(-2,1) = -2$

$f(2,-1) = -2$

Ex Max / Min values $x^2+y^2=1$

$f(x,y) = 3x+4y$ $g(x,y) = x^2+y^2-1=0$

Soln. $\nabla f = \lambda \nabla g$

$3\hat{i} + 4\hat{j} = 2\lambda x\,\hat{i} + 2\lambda y\,\hat{j}$

$3 = 2\lambda x$ $4 = 2\lambda y$

$x = \frac{3}{2\lambda}$ $y = \frac{2}{\lambda}$

$x^2+y^2=1$

$\boxed{\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 = 1}$

$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1$

$\frac{9+16}{4\lambda^2} = 1$

$4\lambda^2 = 25 \rightarrow \lambda^2 = \frac{25}{4}$

$\lambda = \pm\frac{5}{2}$

$x = \frac{3}{2\lambda} = \pm\frac{3}{5}$

$y = \frac{2}{\lambda} = \pm\frac{4}{5}$

Points are $\left(\pm\frac{3}{5}, \pm\frac{4}{5}\right)$

$f(x,y) = 3x+4y$

$f\left(\frac{3}{5}, \frac{4}{5}\right) = \frac{9}{5} + \frac{16}{5} = 5$ ++ Max

$f\left(-\frac{3}{5}, -\frac{4}{5}\right) = \frac{-9}{5} - \frac{16}{5} = -5$ −− Min

$f\left(\frac{3}{5}, -\frac{4}{5}\right) = \frac{9}{5} - \frac{16}{5} = -\frac{7}{5}$ ## +−

$f\left(-\frac{3}{5}, \frac{4}{5}\right) = \frac{-9}{5} + \frac{16}{5} = \frac{7}{5}$ −+

#24 $f(x,y) = x^2y^2$ s.t. $2x^2+y^2=1$

$g(x,y) = 2x^2+y^2-1=0$

Sol^n

$$\nabla f = \lambda \nabla g$$

$$2xy^2\,\hat{i} + 2x^2y\,\hat{j} = 4\lambda x\,\hat{i} + 2\lambda y\,\hat{j}$$

$2xy^2 = 4\lambda x$ $\qquad$ $2x^2y = 2\lambda y$

$xy^2 = 2\lambda x$ $\qquad$ $x^2y = \lambda y$

if $x=0 \Rightarrow 0 = \lambda y \Rightarrow y=0$ $\quad (0,0)$ #

If $x \neq 0 \Rightarrow y^2 = 2\lambda \Rightarrow y \neq 0 \Rightarrow x^2 = \lambda$

$\boxed{\text{or } \lambda = \frac{1}{2}y^2 \to x^2 = \lambda = \frac{1}{2}y^2 \quad (y \neq 0)}$

$2x^2+y^2=1$

$2\lambda + 2\lambda = 1$

$\lambda = \frac{1}{4}$

$\begin{cases} x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2} \\ y^2 = 2\frac{1}{4} \Rightarrow y = \pm\frac{1}{\sqrt{2}} \end{cases}$

$2x^2+y^2=1$

$2(\frac{1}{2}y^2)+y^2=1$

$2y^2=1$

$y = \pm\frac{1}{\sqrt{2}}$

$x^2 = \frac{1}{2}\,\frac{1}{2} = \frac{1}{4}$

Points $\left(\pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right)$

$f(x,y) = x^2y^2$

$f\left(\pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right) = \frac{1}{4}\,\frac{1}{2} = \frac{1}{8}$

$f(x,y,z)$ $\begin{cases} g_1(x,y,z)=0 \\ g_2(x,y,z)=0 \end{cases}$

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

Ex plane $\begin{cases} x+y+z=1 \\ x^2+y^2=1 \end{cases}$

$f(x,y,z) = x^2+y^2+z^2$

$g_1(x,y,z) = x+y+z-1=0$

$g_2(x,y,z) = x^2+y^2-1=0$

sol. $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$

$2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda\hat{i} + \lambda\hat{j} + \lambda\hat{k} + 2\mu x\hat{i} + 2\mu y\hat{j}$

$= (\lambda+2\mu x)\hat{i} + (\lambda+2\mu y)\hat{j} + \lambda\hat{k}$

$2x = \lambda + 2\mu x$ (1) $\quad 2y = \lambda + 2\mu y$ (2) $\quad 2z = \lambda$ (3)

$2(1-\mu)x = \lambda \quad 2(1-\mu)y = \lambda$

$2(1-\mu)x = 2z \quad 2(1-\mu)y = 2z$

$(1-\mu)x = z \quad (1-\mu)y = z$

$z = (1-\mu)x = (1-\mu)y$.

if $\mu=1 \Rightarrow z=0$

$\begin{cases} x+y=1 \to y=1-x \\ x^2+y^2=1 \end{cases}$

$x^2+(1-x)^2=1$

$2x^2-2x=0 \to x=0, \; x=1$

$x=0 \Rightarrow y=1-x=1$

$x=1 \Rightarrow y=0$

$(0,1,0)$ & $(1,0,0)$

若 $\mu \neq 1$ $\quad z = (1-\mu)x = (1-\mu)y$.

$$\underbrace{x = y}_{} = \frac{z}{1-\mu}$$

$x^2+y^2=1$ $\qquad\qquad$ $x+y+z=1$

$2x^2=1$ $\qquad\qquad$ $z = 1-x-y$

$\qquad\qquad\qquad\qquad = 1-2x$

$$\boxed{x = \pm\frac{1}{\sqrt{2}} = y}$$

$x = -\frac{\sqrt{2}}{2} \Rightarrow z = 1+\sqrt{2}$

$x = \frac{\sqrt{2}}{2} \Rightarrow z = 1-\sqrt{2}$

$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1+\sqrt{2}\right)$

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1-\sqrt{2}\right)$

#64 $f(x,y,z)=xyz$ $g_1(x,y,z)=x^2+y^2-4=0$

$g_2(x,y,z)=x+y+z-1=0$

soln

$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$

$yz\hat{i} + xz\hat{j} + xy\hat{k} = 2\lambda x\hat{i} + 2\lambda y\hat{j} + \mu\hat{i} + \mu\hat{j} + \mu\hat{k}$

$\hat{i}$ $yz = 2\lambda x + \mu = 2\lambda x + xy$

$\hat{j}$ $xz = 2\lambda y + \mu = 2\lambda y + xy$

$\hat{k}$ $xy = \mu$

$yz = (2\lambda + y)x$

$xz = (2\lambda + x)y$

(1) $\mu = xy = yz - 2\lambda x = xz - 2\lambda y$

$yz - xz = 2\lambda x - 2\lambda y$

$(y-x)z = 2(x-y)\lambda$

$-(x-y)z = 2(x-y)\lambda$

$z = -2\lambda$ or $x = y$

if $z = -2\lambda$

(1) $\rightarrow$ $\mu = xy = -2\lambda y - 2\lambda x = -2\lambda x - 2\lambda y$

$= -2\lambda(x+y)$

$= z(x+y)$

$z = \dfrac{xy}{x+y}$

$\begin{cases} x^2+y^2=4 \\ x+y+\dfrac{xy}{x+y}=1 \end{cases}$

$\begin{cases} x^2+y^2=4 \\ (x+y)^2+xy=x+y \end{cases}$

$$\begin{cases} \boxed{x^2+y^2=4} \\ x^2+2xy+y^2+xy-x-y=0 \end{cases} \Rightarrow y^2=4-x^2 \quad \Rightarrow$$

$$3xy-x-y+4=0$$

$$(3x-1)\cdot y = x-4$$

$$(3x-1)^2\, y^2 = (x-4)^2$$

$$(9x^2-6x+1)(4-x^2) = x^2-8x+16$$

$$-9x^4+6x^3+35x^2-24x+4 = x^2-8x+16$$

$$-9x^4+6x^3+34x^2-16x-12=0$$

if $x=y$ $\quad \begin{cases} x^2+y^2=4 \\ 2x^2=4 \end{cases} \rightarrow x=\pm\sqrt{2}=y$

$$z=1-x-y=1-2x$$

$$x=-\sqrt{2} \rightarrow z=1+2\sqrt{2}$$

$$x=\sqrt{2} \rightarrow z=1-2\sqrt{2}$$

$$(-\sqrt{2},-\sqrt{2},1+2\sqrt{2}) \quad (\sqrt{2},\sqrt{2},1-2\sqrt{2})$$

#45 $f(x,y,z) = x + 2y - z$

$g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$

soln

$\nabla f = \lambda \nabla g$

$\hat{i} + 2\hat{j} - \hat{k} = 2\lambda x \hat{i} + 2\lambda y \hat{j} + 2\lambda z \hat{k}$

$$\begin{cases} 2\lambda x = 1 & \longrightarrow x = \frac{1}{2\lambda} \\ 2\lambda y = 2 \to \lambda y = 1 & \to y = \frac{1}{\lambda} \\ 2\lambda z = -1 & \to z = -\frac{1}{2\lambda} \end{cases}$$

$x^2 + y^2 + z^2 = 1$

$\frac{1}{4\lambda^2} + \frac{4}{4\lambda^2} + \frac{1}{4\lambda^2} = 1$

$\frac{6}{4\lambda^2} = 1 \to \frac{6}{4} = \lambda^2$

$\lambda = \pm \frac{\sqrt{6}}{2}$

$x = \pm \frac{1}{\sqrt{6}}$, $y = \pm \frac{2}{\sqrt{6}}$, $z = \mp \frac{1}{\sqrt{6}}$

$f\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) = \frac{1}{\sqrt{6}} + \frac{4}{\sqrt{6}} + \frac{1}{\sqrt{6}} = \frac{6}{\sqrt{6}} = \sqrt{6}$

$f\left(-\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = -\frac{1}{\sqrt{6}} - \frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} = -\sqrt{6}$

has a Max value @ $\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$ w/ $\sqrt{6}$

Min " @ $\left(-\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ w/ $-\sqrt{6}$

~~#46~~ $f(x,y,z) = x^2y^2z$

$g(x,y,z) = 2x^2 + 2y^2 + z^2 - 25 = 0$

$\nabla f = \lambda \nabla g$

$2xy^2z\,\hat{i} + 2x^2yz\,\hat{j} + x^2y^2\,\hat{k}$

$= 4\lambda x\,\hat{i} + 4\lambda y\,\hat{j} + 2\lambda z\,\hat{k}$

$$\begin{cases} 2xy^2z = 4\lambda x \rightarrow x = 0 \text{ or } \lambda = \frac{1}{2}y^2z \quad (1) \\ 2x^2yz = 4\lambda y \rightarrow y = 0 \text{ or } \lambda = \frac{1}{2}x^2z \quad (2) \\ x^2y^2 = 2\lambda z \rightarrow z = 0 \rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \\ \qquad \lambda = \frac{1}{2}\frac{x^2y^2}{z} \quad (3) \end{cases}$$

$(0,0,0)$ #

$g(x,y,z):\ 2x^2 + 2y^2 + z^2 = 25$

$\lambda = \underbrace{\frac{1}{2}y^2z = \frac{1}{2}x^2z}_{x^2 = y^2} = \frac{1}{2}\frac{x^2y^2}{z}$

$z = \frac{y^2}{z}$

$z^2 = y^2$

$2y^2 + 2y^2 + y^2 = 25$

$y^2 = 5 \Rightarrow y = \pm\sqrt{5}$

$x = \pm\sqrt{5} = z$

$(\pm\sqrt{5}, \pm\sqrt{5}, \pm\sqrt{5})$

+	+	+
+	+	−
+	−	+
+	−	−
−		