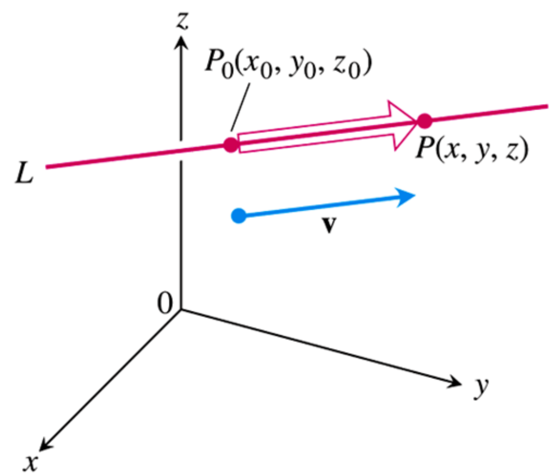
Section 1.4 – Lines and Curves in Space

Lines and Line Segments in Space



The expanded form of the equation $\overrightarrow{P_0 P} = t\vec{v}$ is

$$\left(x - x_0\right)\hat{i} + \left(y - y_0\right)\hat{j} + \left(z - z_0\right)\hat{k} = t\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)$$

Vector Equation for a Line

A **vector equation for the line *L*** through $P_0\left(x_0, y_0, z_0\right)$ parallel to ***v*** is

$$\vec{r}\left(t\right) = \vec{r}_0 + t\vec{v}, \quad -\infty < t < \infty$$

Where *r* is the position vector of a point $P\left(x, y, z\right)$ on *L* and r_0 is the position vector of $P_0\left(x_0, y_0, z_0\right)$.

Parametric Equations for a Line

A **standard parametrization** of the line through $P_0\left(x_0, y_0, z_0\right)$ parallel to $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ is

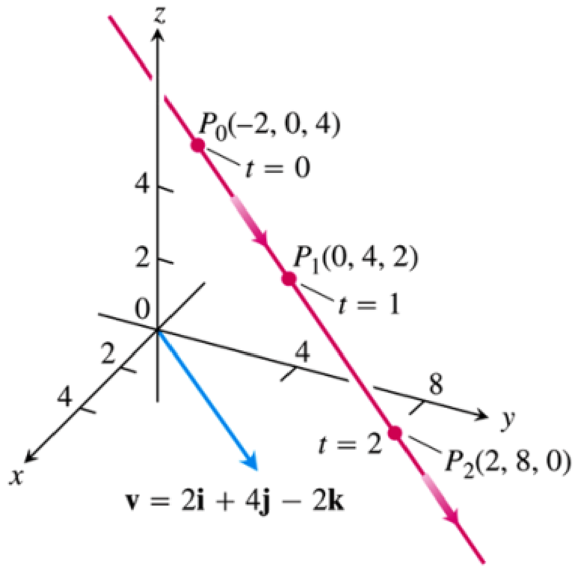
$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty$$

Example

Find the parametric equations for the line through (–2, 0, 4) parallel to $\vec{v} = 2\hat{i} + 4\hat{j} - 2\hat{k}$

Solution

$x = -2 + 2t, \quad y = 4t, \quad z = 4 - 2t$



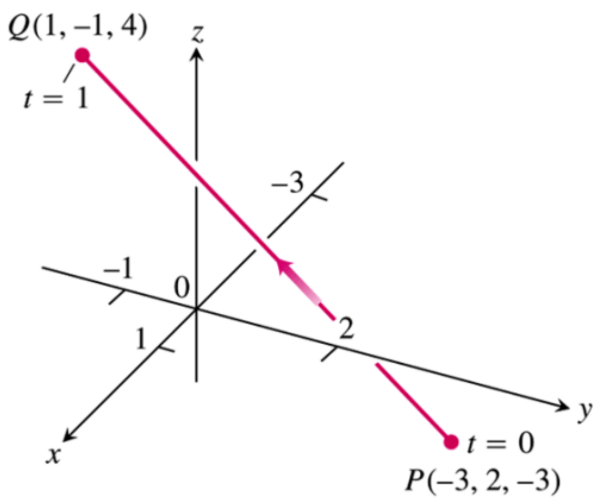
Example

Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$

Solution

$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t$

The point $(x, y, z) = (-3 + 4t,\ 2 - 3t,\ -3 + 7t)$



On the line passes through P at $t = 0$ and Q at $t = 1$.

That implies the restriction $0 \le t \le 1$ to parameterize the segment

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t, \quad 0 \le t \le 1$$

The position of a particle at time t is written:

$$\begin{aligned} r(t) &= r_0 + tv \\ &= \underbrace{r_0}_{\text{Initial position}} + \underbrace{t}_{\text{Time}} \underbrace{|v|}_{\text{Speed}} \underbrace{\frac{v}{|v|}}_{\text{Direction}} \end{aligned}$$

Example

A helicopter is to fly directly from a helipad at the origin in the direction of the point (1, 1, 1) at a speed of 60 *ft/sec*. What is the position of the helicopter after 10 *sec*.?

Solution

The unit vector: $= \dfrac{(1,\ 1,\ 1)}{\sqrt{1^2 + 1^2 + 1^2}}$

$$= \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

Therefore; the position of the helicopter at any time t is

$$\begin{aligned} \vec{r}(t) &= r_0 + t\vec{u} \\ &= 0 + t(60)\left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right) \\ &= 20\sqrt{3}\ t\left(\hat{i} + \hat{j} + \hat{k}\right) \end{aligned}$$

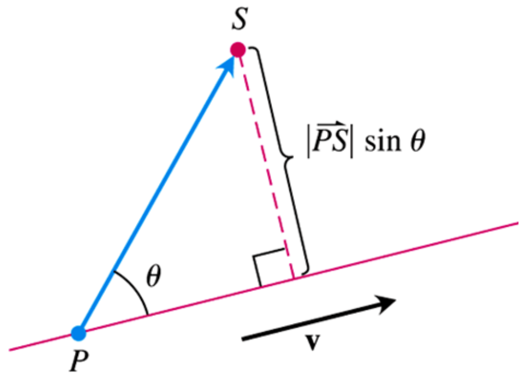
The position after 10 *sec*:

$$\begin{aligned} r(10) &= 20\sqrt{3}\ (10)\left(\hat{i} + \hat{j} + \hat{k}\right) \\ &= 200\sqrt{3}\left(\hat{i} + \hat{j} + \hat{k}\right) \end{aligned}$$

The distance is traveled:

$$\begin{aligned} |r(10)| &= 200\sqrt{3}\sqrt{1^2 + 1^2 + 1^2} \\ &= 600\ ft \end{aligned}$$

Distance from a Point *S* to a Line through *P* parallel to *v*



$$d = \frac{\left|\overrightarrow{PS}\times\vec{v}\right|}{|\vec{v}|}$$

Example

Find the distance from the point $S(1, 1, 5)$ to the line $L: \ x=1+t, \quad y=3-t, \quad z=2t$

Solution

At $t=0$, the equations for L passes through $P(1, 3, 0)$ parallel to $\vec{v}=\hat{i}-\hat{j}+2\hat{k}$

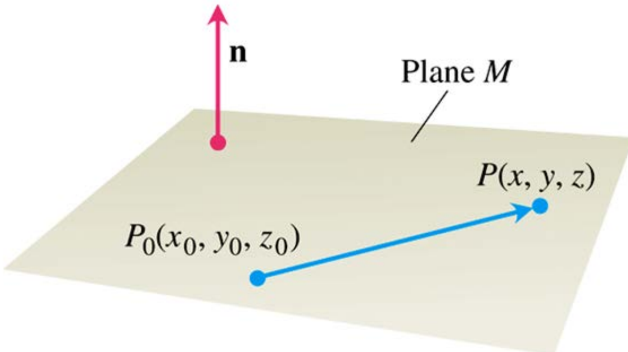
$$\begin{aligned}\overrightarrow{PS} &= (1-1)\hat{i}+(1-3)\hat{j}+(5-0)\hat{k}\\ &= -2\hat{j}+5\hat{k}\end{aligned}$$

$$\begin{aligned}\overrightarrow{PS}\times\vec{v} &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & -2 & 5\\ 1 & -1 & 2\end{vmatrix}\\ &= \hat{i}+5\hat{j}+2\hat{k}\end{aligned}$$

$$\begin{aligned}d &= \frac{\left|\overrightarrow{PS}\times\vec{v}\right|}{|\vec{v}|}\\ &= \frac{\sqrt{1+25+4}}{\sqrt{1+1+4}}\\ &= \frac{\sqrt{30}}{\sqrt{6}}\\ &= \sqrt{5}\end{aligned}$$

An Equation for a Plane in Space

A plane in space is determined by knowing a point on the plane and its “tilt” or orientation. This “tilt” is defined by specifying a vector that is perpendicular or normal to the plane.



The dot product $\vec{n}\bullet\overrightarrow{P_0P}=0$, since $\overrightarrow{P_0P}$ is orthogonal to $\vec{n}=A\hat{i}+B\hat{j}+C\hat{k}$.

$$\vec{n}\bullet\overrightarrow{P_0P}=0 \quad\Leftrightarrow\quad \left(A\hat{i}+B\hat{j}+C\hat{k}\right)\cdot\left(\left(x-x_0\right)\hat{i}+\left(y-y_0\right)\hat{j}+\left(z-z_0\right)\hat{k}\right)=0$$

$$A\left(x-x_0\right)+B\left(y-y_0\right)+C\left(z-z_0\right)=0$$

Equation for a Plane

The plane through $P_0\left(x_0, y_0, z_0\right)$ normal to $\vec{n}=A\hat{i}+B\hat{j}+C\hat{k}$ has

Vector equation: $\vec{n}\bullet\overrightarrow{P_0P}=0$

Component equation: $A\left(x-x_0\right)+B\left(y-y_0\right)+C\left(z-z_0\right)=0$

Component equation simplified: $Ax+By+Cz=D \quad \textit{where} \quad D=Ax_0+By_0+Cz_0$

Example

Find an equation for the plane through $P_0\left(-3,\ 0,\ 7\right)$ perpendicular to $\vec{n}=5\hat{i}+2\hat{j}-\hat{k}$

Solution

The component equation is

$$5\left(x-(-3)\right)+2\left(y-0\right)+\left(-1\right)\left(z-7\right)=0$$

$$5\left(x+3\right)+2y-z+7=0$$

$$5x+15+2y-z+7=0$$

$$\boxed{5x+2y-z=-22}$$

Example

Find an equation for the plane through $A(0,\ 0,\ 1),\quad B(2,\ 0,\ 0),\quad C(0,\ 3,\ 0)$.

Solution

The cross product

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 0 & -1\\ 0 & 3 & -1\end{vmatrix}$$

$$=3\hat{i}+2\hat{j}+6\hat{k}$$

Normal to the plane.

We substitute the components of this vector and the coordinates of $A(0, 0, 1)$ into the component form of the equation to obtain

$$3(x-0)+2(y-0)+6(z-1)=0$$

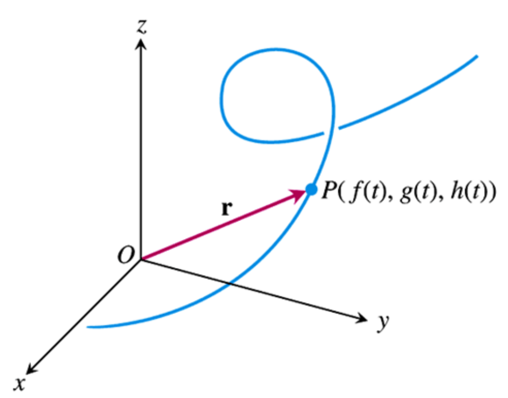
$\boxed{3x+2y+6z-6=0}$ *or* $\boxed{3x+2y+6z=6}$

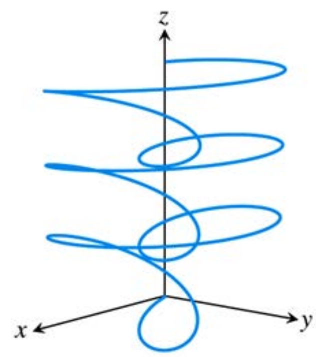
Curves

The coordinates for a particle moving through space during a time interval I, are defined as function on I:

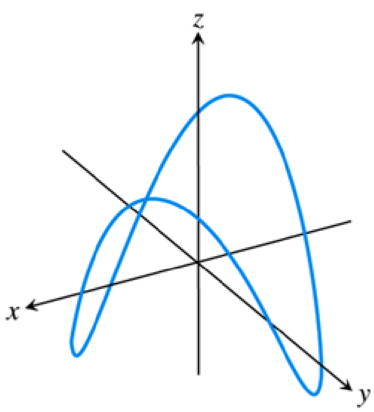
$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I.$$

The points $(x,\ y,\ z) = \left(f(t),\ g(t),\ h(t)\right), \quad t \in I$, make up the curve in space that we call the particle's path.

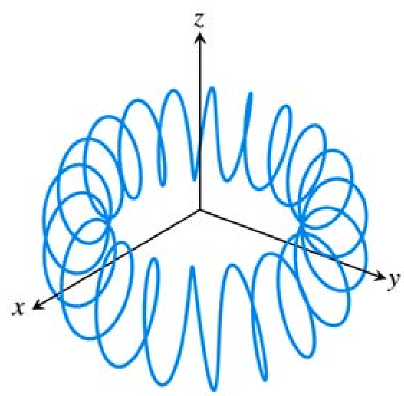




$\vec{r}(t) = (\sin 3t)(\cos t)\hat{i} + (\sin 3t)(\sin t)\hat{j} + t\hat{k}$



$\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + (\sin 2t)\hat{k}$



$\vec{r}(t) = (4 + \sin 20t)(\cos t)\hat{i} + (4 + \sin 20t)(\sin t)\hat{j} + (\cos 20t)\hat{k}$

Example

Graph the vector function $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}$

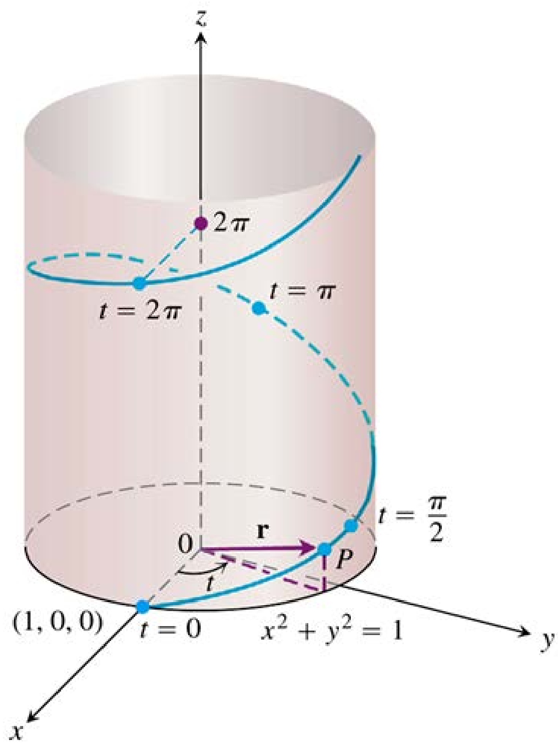
Solution

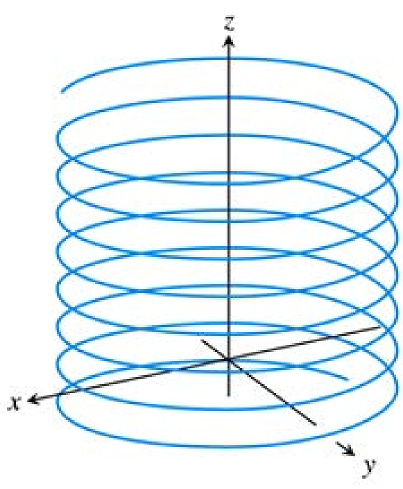
$x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1$

The curves traced by $\vec{r}(t)$ winds around a circular cylinder, satisfies the equation.

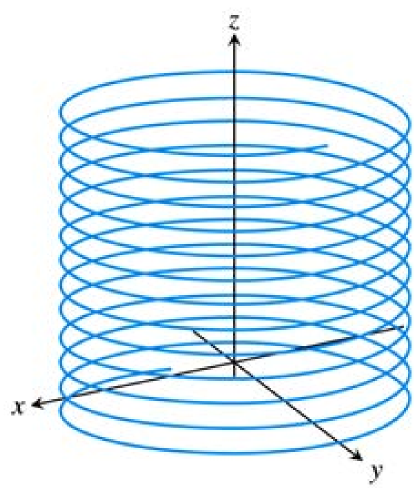
The curve rises as the ***k***-components $z = t$ increases. Each time t increases by 2π, the curve completes one turn around the cylinder. The curve is called a helix (from an old Greek word for “spiral”). The equations

$$x = \cos t, \quad y = \sin t, \quad z = t$$





$\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + 0.3t\,\hat{k}$



$\vec{r}(t) = (\cos 5t)\hat{i} + (\sin 5t)\hat{j} + t\,\hat{k}$

Limits and Continuity

Definition

Let $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ be a vector function with domain *D*, and ***L*** a vector. We say that ***r*** has limit ***L*** as *t* approaches t_0 and write

$$\lim_{t\to t_0} \vec{r}(t) = \boldsymbol{L}$$

If, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$

$$\left|\vec{r}(t) - \boldsymbol{L}\right| < \varepsilon \quad \textit{whenever} \quad 0 < \left|t - t_0\right| < \delta$$

Example

Find the limit of $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}$ as *t* approaches $\frac{\pi}{4}$

Solution

$$\lim_{t\to\pi/4} \vec{r}(t) = \left(\lim_{t\to\pi/4} \cos t\right)\hat{i} + \left(\lim_{t\to\pi/4} \sin t\right)\hat{j} + \left(\lim_{t\to\pi/4} t\right)\hat{k}$$

$$= \frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j} + \frac{\pi}{4}\hat{k}$$

Definition

A vector fumction $\vec{r}(t)$ is ***continuous at a point*** $t = t_0$ in its domain if $\lim_{t\to t_0} \vec{r}(t) = \vec{r}(t_0)$. The function is continuous if it is continuous at every point in its domain.

Lines of Intersection

Example

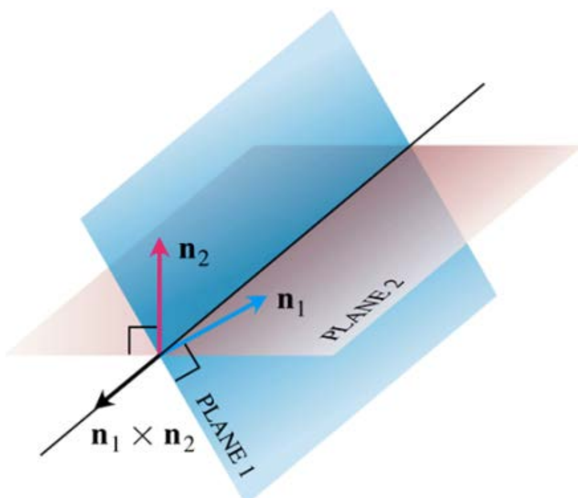
Find a vector parallel to the line of intersection of the planes $3x-6y-2z=15$ *and* $2x+y-2z=5$

Solution

The line of intersection of two planes is perpendicular to both planes' normal vectors $\vec{n}_1$ and $\vec{n}_2$ and therefore parallel to $\vec{n}_1 \times \vec{n}_2$.

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= 14\hat{i} + 2\hat{j} + 15\hat{k}$$



Example

Find the point where the line $x=\frac{8}{3}+2t, \quad y=-2t, \quad z=1+t$ intersects the plane $3x+2y+6z=6$.

Solution

The point: $\left(\frac{8}{3}+2t,\ -2t,\ 1+t\right)$

lies in the plane if its coordinates satisfy the equation of the plane, that is, if

$$3\left(\frac{8}{3}+2t\right)+2(-2t)+6(1+t)=6$$

$$8+6t-4t+6+6t=6$$

$$8t=-8$$

$$t=-1$$

The point of intersection is: $\left(\frac{8}{3}+2t,\ -2t,\ 1+t\right)\Big|_{t=-1} = \left(\frac{2}{3},\ 2,\ 0\right)$

The distance from a Point to a Plane

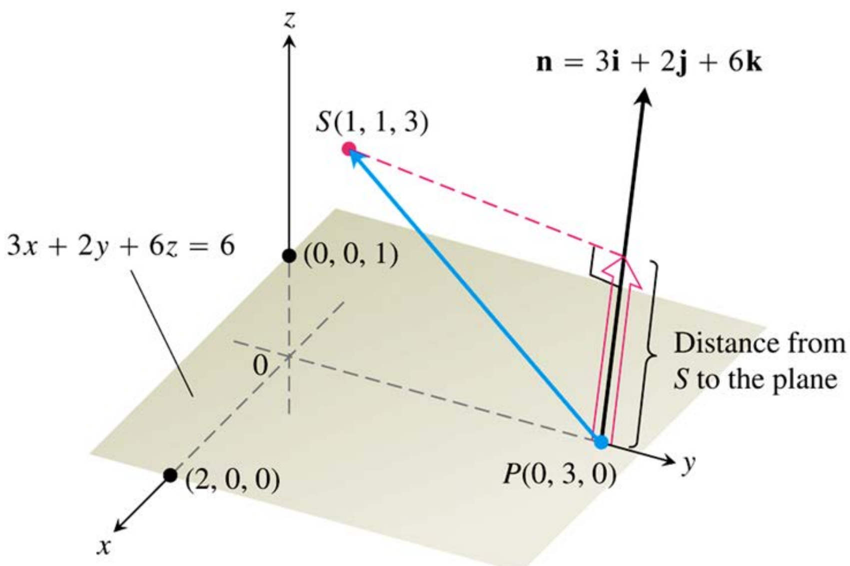
$$d = \left| \overrightarrow{PS} \bullet \frac{\vec{n}}{|\vec{n}|} \right|$$

Example

Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$

Solution

The coefficients in the equation $3x + 2y + 6z = 6$ give $\vec{n} = 3\hat{i} + 2\hat{j} + 6\hat{k}$



$\overrightarrow{PS} = \hat{i} - 2\hat{j} + 3\hat{k}$

$$|\vec{n}| = \sqrt{3^2 + 2^2 + 6^2}$$
$$= 7$$

The distance from S to the plane is

$$d = \left| \overrightarrow{PS} \bullet \frac{\vec{n}}{|\vec{n}|} \right|$$
$$= \left| \left(\hat{i} - 2\hat{j} + 3\hat{k}\right) \bullet \left(\frac{3}{7}\hat{i} + \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}\right) \right|$$
$$= \left| \frac{3}{7} - \frac{4}{7} + \frac{18}{7} \right|$$
$$= \frac{17}{7}$$

Angles Between Planes

Example

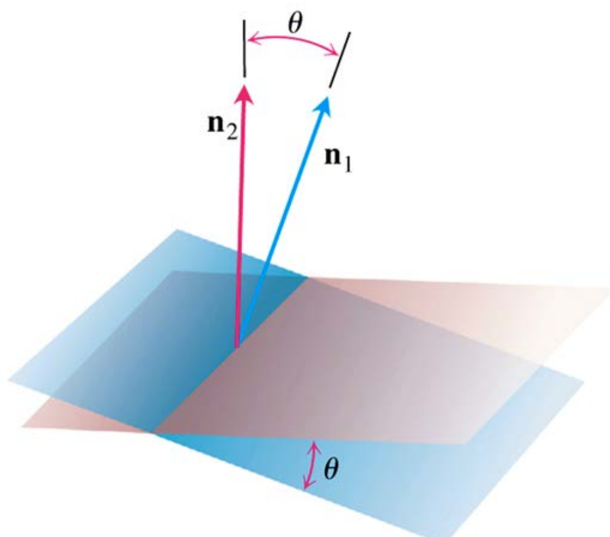
Find the angle between the planes $3x-6y-2z=15$ and $2x+y-2z=5$

Solution

The vectors: $\vec{n}_1=3\hat{i}-6\hat{j}-2\hat{k},\quad \vec{n}_2=2\hat{i}+\hat{j}-2\hat{k}$ are normal to the planes.

The angle between them is:

$$\begin{aligned}\theta&=\cos^{-1}\left(\frac{\vec{n}_1\bullet\vec{n}_2}{\left|\vec{n}_1\right|\left|\vec{n}_2\right|}\right)\\&=\cos^{-1}\left(\frac{6-6+4}{\sqrt{9+36+4}\sqrt{4+1+4}}\right)\\&=\cos^{-1}\left(\frac{4}{21}\right)\\&\approx 1.38\ rad\end{aligned}$$



Exercises *Section* 1.4 – Lines and Curves in Space

1. Find the parametric equation for the line through the point $P(3,\ -4,\ -1)$ parallel to the vector $\hat{i}+\hat{j}+\hat{k}$

2. Find the parametric equation for the line through the points $P(1,\ 2,\ -1)$ and $Q(-1,\ 0,\ 1)$

3. Find the parametric equation for the line through the points $P(-2,\ 0,\ 3)$ and $Q(3,\ 5,\ -2)$

4. Find the parametric equation for the line through the origin parallel to the vector $2\hat{j}+\hat{k}$

5. Find the parametric equation for the line through the point $P(3,\ -2,\ 1)$ parallel to the line $x=1+2t, \quad y=2-t, \quad z=3t$

6. Find the parametric equation for the line through $(2,\ 4,\ 5)$ perpendicular to the plane $3x+7y-5z=21$

7. Find the parametric equation for the line through $(2,\ 3,\ 0)$ perpendicular to the vectors $\vec{u}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{v}=3\hat{i}+4\hat{j}+5\hat{k}$

8. Find the parameterization for the line segment joining the points $(0,\ 0,\ 0)$, $\left(1,\ 1,\ \frac{3}{2}\right)$. Draw coordinate axes and sketch the segment, indicate the direction on increasing *t* for the parametrization.

9. Find the parameterization for the line segment joining the points $(1,\ 0,\ -1)$, $(0,\ 3,\ 0)$. Draw coordinate axes and sketch the segment, indicate the direction on increasing *t* for the parametrization.

10. Find equation for the plane through $P_0(0,\ 2,\ -1)$ normal to $\vec{n}=3\hat{i}-2\hat{j}-\hat{k}$

11. Find equation for the plane through $(1,\ -1,\ 3)$ parallel to the plane $3x+y+z=7$

12. Find equation for the plane through $(1,\ 1,\ -1)$, $(2,\ 0,\ 2)$ and $(0,\ -2,\ 1)$

13. Find equation for the plane through $P_0(2,\ 4,\ 5)$ perpendicular to the line $x=5+t, \quad y=1+3t, \quad z=4t$

14. Find equation for the plane through $A(1,\ -2,\ 1)$ perpendicular to the vector from the origin to *A*.

15. Find the point of intersection of the lines $x=2t+1, \quad y=3t+2, \quad z=4t+3$ and $x=s+2, \quad y=2s+4, \quad z=-4s-1$, and find the plane determined by these lines.

16. Find the plane determined by the intersecting lines:

$L_1: \ x = -1+t, \quad y = 2+t, \quad z = 1-t; \quad -\infty < t < \infty$

$L_2: \ x = 1-4s, \quad y = 1+2s, \quad z = 2-2s; \quad -\infty < s < \infty$

17. Find a plane through $P_0(2,1,-1)$ and perpendicular to the line of intersection of the planes

$2x + y - z = 3, \quad x + 2y + z = 2$

(**18 – 25**) Find the distance from the point to the plane

18. $(0,\ 0,\ 12), \quad x = 4t, \quad y = -2t, \quad z = 2t$

19. $(2,\ 1,\ -1), \quad x = 2t, \quad y = 1+2t, \quad z = 2t$

20. $(3,\ -1,\ 4), \quad x = 4-t, \quad y = 3+2t, \quad z = -5+3t$

21. $(2,\ -3,\ 4), \quad x + 2y + 2z = 13$

22. $(0,\ 0,\ 0), \quad 3x + 2y + 6z = 6$

23. $(0,\ 1,\ 1), \quad 4y + 3z = -12$

24. $(6,\ 0,\ -6), \quad x - y = 4$

25. $(3,\ 0,\ 10), \quad 2x + 3y + z = 2$

(**26 – 27**) Find the distance from the point to the line

26. $(2,\ 2,\ 0); \quad x = -t, \quad y = t, \quad z = -1+t$

27. $(0,\ 4,\ 1); \quad x = 2+t, \quad y = 2+t, \quad z = t$

28. Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$

(**29 – 32**) Find the angle between the planes

29. $x + y = 1, \quad 2x + y - 2z = 2$

30. $5x + y - z = 10, \quad x - 2y + 3z = -1$

31. $x = 7, \quad x + y + \sqrt{2}z = -3$

32. $x + y = 1, \quad y + z = 1$

33. Find the point in which the line meets the plane $x = 1-t, \quad y = 3t, \quad z = 1+t; \quad 2x - y + 3z = 6$

34. Find the point in which the line meets the plane

$x = 2, \quad y = 3+2t, \quad z = -2-2t; \quad 6x + 3y - 4z = -12$

35. Find an equation of the line through the point $(0,\ 1,\ 1)$ and parallel to the line

$\overrightarrow{R(t)} = \langle 1+2t,\ \ 3-5t,\ \ 7+6t \rangle$

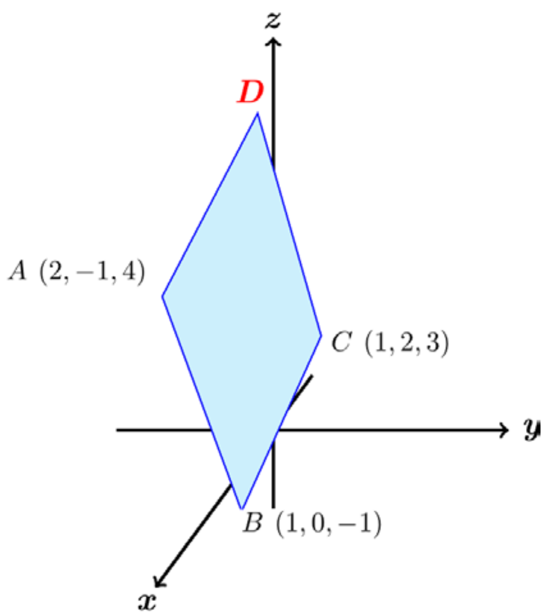
36. Find an equation of the line through the point $(0,\ 1,\ 1)$ that is orthogonal to both $\langle 0,\ -1,\ 3\rangle$ and $\langle 2,\ -1,\ 2\rangle$

37. Find an equation of the line through the point $(0,\ 1,\ 1)$ that is orthogonal to the vector $\langle -2,\ 1,\ 7\rangle$ and the *y-axis*

38. Suppose that $\vec{n}$ is normal to a plane and that $\vec{v}$ is parallel to the plane. Describe how you would find a vector $\vec{n}$ that is both perpendicular to $\vec{v}$ and parallel to the plane.

39. Given a point $\left(x_0,\ y_0,\ 0\right)$ and a vector $\boldsymbol{v} = \langle a,\ b,\ 0\rangle$ in $\mathbb{R}^3$, describe the set of points that satisfy the equation $\langle a,\ b,\ 0\rangle \times \left\langle x - x_0,\ y - y_0,\ 0\right\rangle = \mathbf{0}$. Use this result to determine an equation of a line in $\mathbb{R}^2$ passing through $\left(x_0,\ y_0\right)$ parallel to the vector $\langle a,\ b\rangle$.

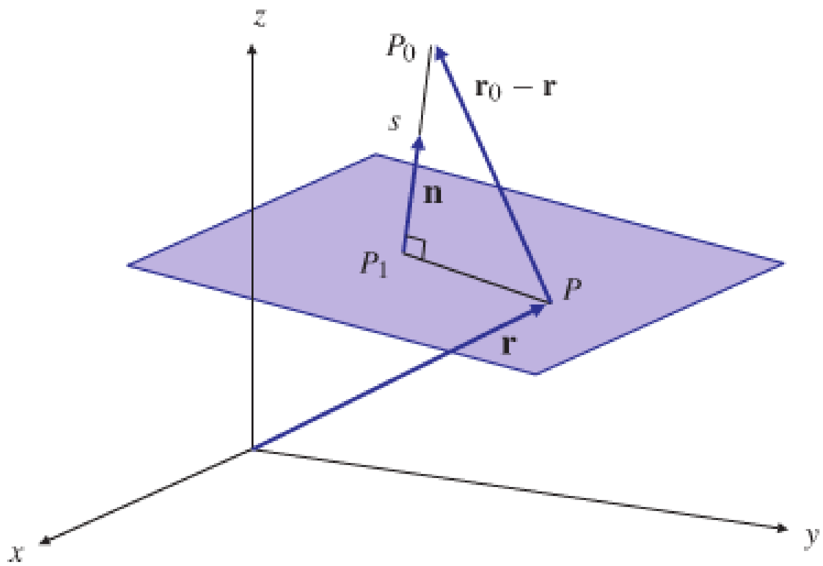
40. The parallelogram has vertices at $A(2, -1, 4)$, $B(1, 0, -1)$, $C(1, 2, 3)$ and D. Find



a) The coordinates of D,
b) The cosine of the interior angle of B
c) The vector projection of $\overrightarrow{BA}$ onto $\overrightarrow{BC}$,
d) The area of the parallelogram,
e) An equation for the plane of the parallelogram,
f) The areas of the orthogonal projection of the parallelogram on the three coordinate planes.

41. *a*) Find the distance from the point $P_0\left(x_0, y_0, z_0\right)$ to the plane *P* having equation

$$Ax + By + Cz = D$$



b) What is the distance from $(2, -1, 3)$ to the plane $2x - 2y - z = 9$?