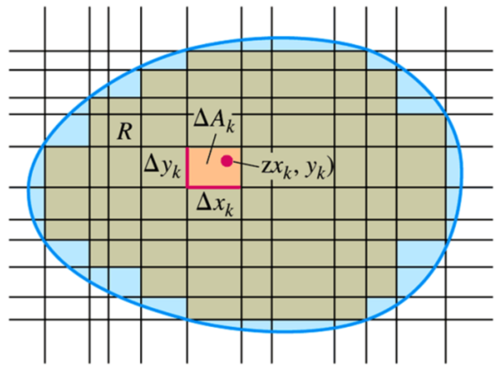
Section 3.2 – Double Integrals over General Regions



Volumes

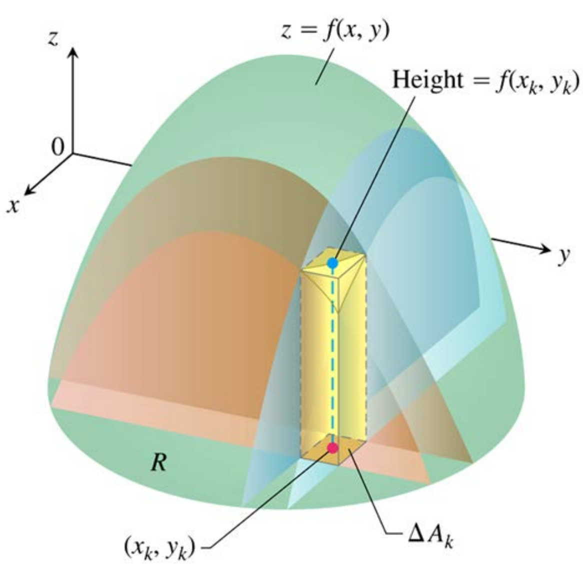
If $f(x,y)$ is positive and continuous over *R*, we define the volume of the solid region between *R* and the surface $z = f(x,y)$ to be $\iint\limits_R f(x,y)\,dA$.

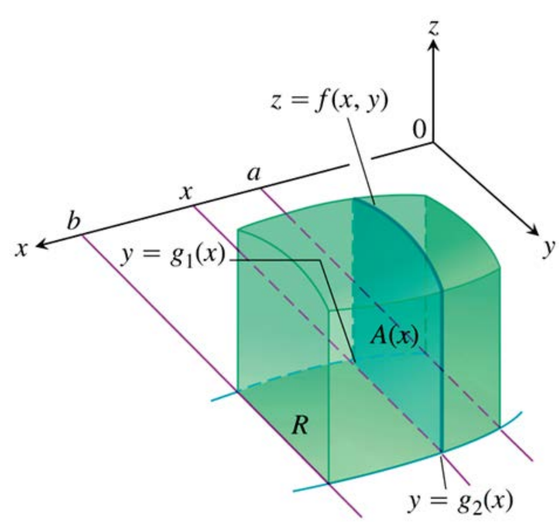
If *R* is a region in the *xy*-plane, bounded ***above*** and ***below*** by the curves $y = g_1(x)$ and $y = g_2(x)$ and on the sides by the lines $x = a,\ x = b$. Calculate the cross-sectional area

$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy$$

Then integrate $A(x)$ from $x = a\ to\ x = b$ to get the volume as an iterated integral

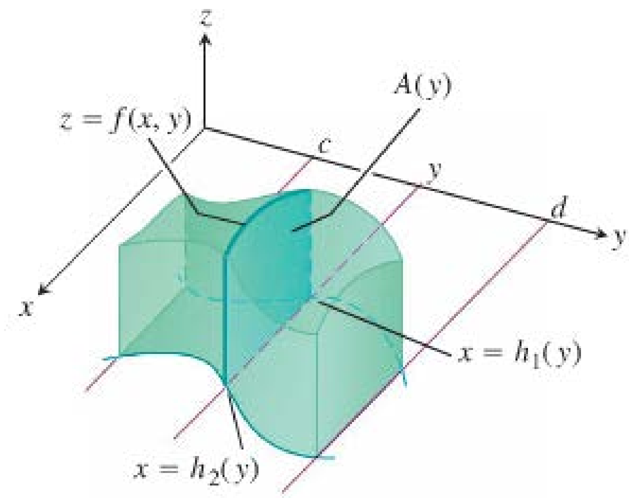
$$V = \int_a^b A(x)\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx$$





Similarly, if R is a region bounded by the curves $x = h_1(y)$ and $x = h_2(y)$ and the lines $y = c,\ y = d$, then the volume calculated by slicing is given by the iterated integral .

$$V = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy$$



$$\int_c^d A(y)\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy$$

$$Volume = \lim \sum f(x_k, y_k)\Delta A_k = \iint_R f(x,y)\,dA$$

Theorem – Fubini's Theorem

Let $f(x,y)$ is continuous on a region R,

1. If R is defined by : $a \le x \le b,\ g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on [a, b], then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx$$

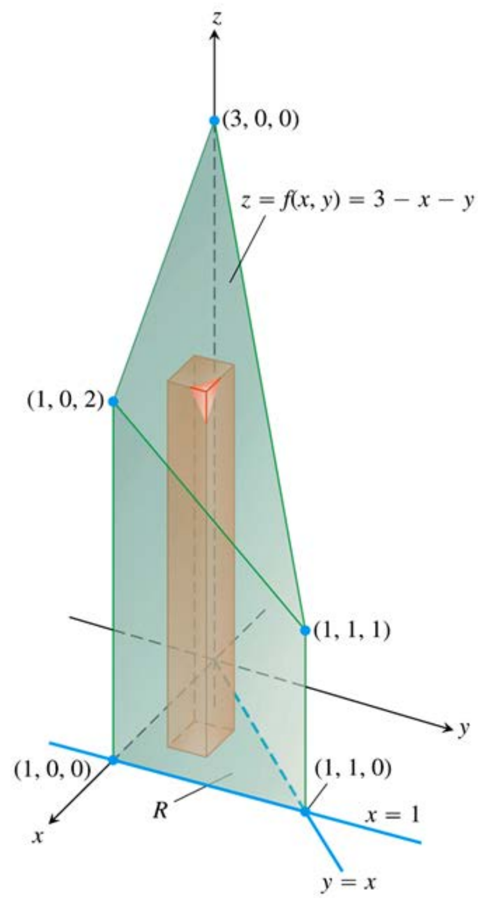
2. If R is defined by : $c \le y \le d,\ h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on [c, d], then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy$$

Example

Find the volume of the prism whose base is the triangle in the *xy*-plane bounded by the *x*-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = f(x, y) = 3 - x - y$

Solution



$$0 \le x \le 1, \quad 0 \le y \le x$$

$$V = \int_0^1 \int_0^x (3 - x - y)\,dydx$$

$$= \int_0^1 \left[3y - xy - \tfrac{1}{2}y^2\right]_0^x dx$$

$$= \int_0^1 \left(3x - x^2 - \tfrac{1}{2}x^2\right) dx$$

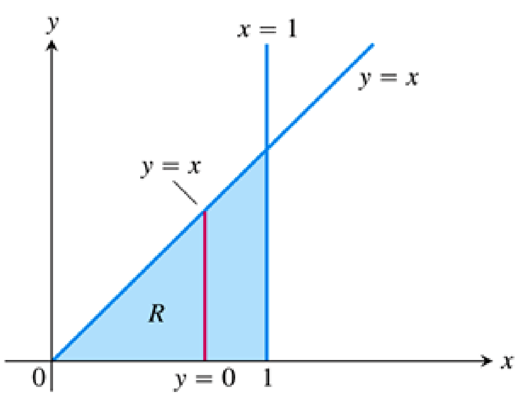
$$= \int_0^1 \left(3x - \tfrac{3}{2}x^2\right) dx$$

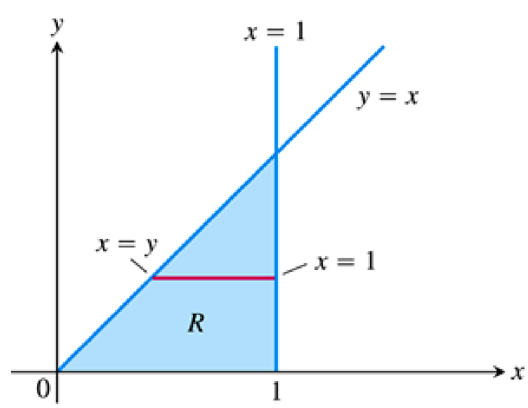
$$= \left[\tfrac{3}{2}x^2 - \tfrac{1}{2}x^3\right]_0^1$$

$$= \frac{3}{2} - \frac{1}{2}$$

$$= 1 \quad unit^3$$

$$V = \int_0^1 \int_y^1 (3 - x - y)\,dxdy = 1$$

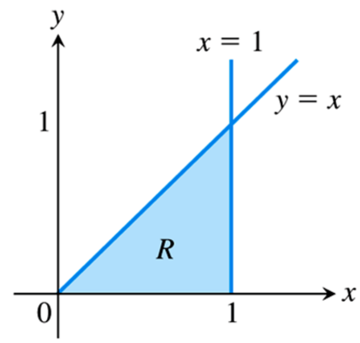




Example

Calculate $\iint\limits_R \frac{\sin x}{x}\,dA$ where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.

Solution



$$\int_0^1 \int_0^x \left(\frac{\sin x}{x}\right) dy\ dx = \int_0^1 \left(\frac{\sin x}{x} y\right)_0^x dx$$

$$= \int_0^1 \sin x dx$$

$$= -\cos x \Big|_0^1$$

$$= -\cos(1) + 1$$

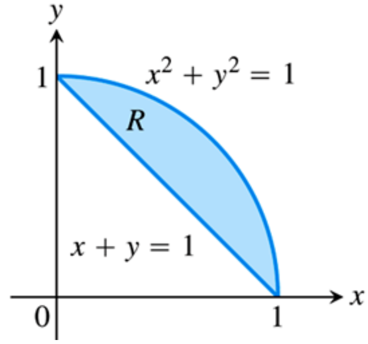
$$= 1 - \cos 1 \quad \approx 0.46$$

$\int_0^1 \int_y^1 \left(\frac{\sin x}{x}\right) dx\ dy$, we run into a problem because $\int \frac{\sin x}{x} dx$ cannot be expressed in terms of elementary functions.

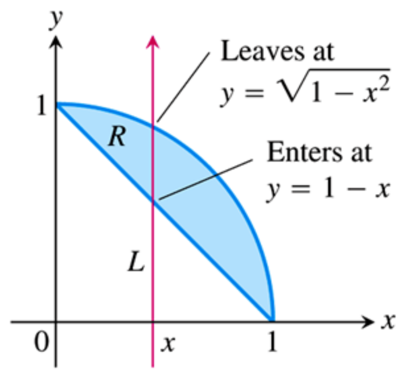
Finding Limits on Intergration

Using Vertical Cross-sections

1. Sketch the region of Integration and label the bounding curves

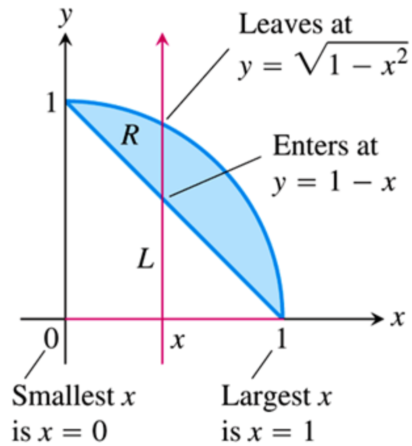


2. *Find the y-limits of integration*. Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants).



3. *Find the x-limits of integration*. Choose x-limits that include all the vertical lines through R. The integral is

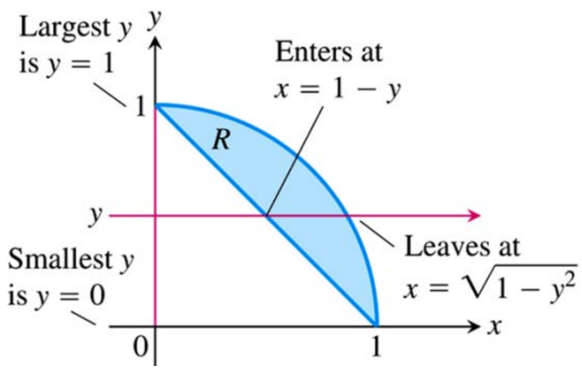
$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=1}\int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx$$



Using Horizontal Cross-sections

To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines.

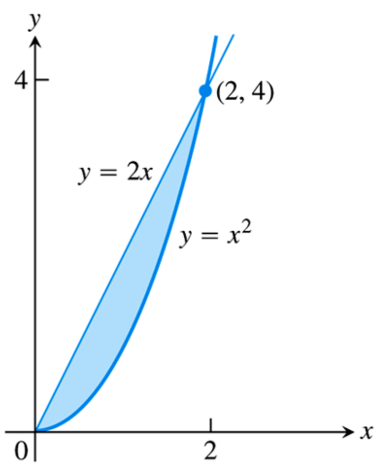
$$\iint_R f(x,y)\,dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x,y)\ dxdy$$

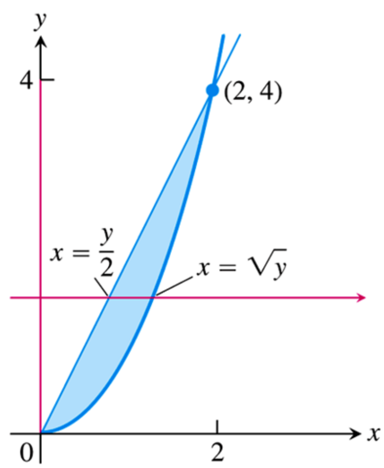


Example

Sketch the region of integration for the integral $\int_0^2 \int_{x^2}^{2x} (4x+2)\,dydx$ and write an equivalent integral with the order of integration reversed.

Solution





The given inequalities are: $x^2 \le y \le 2x$ *and* $0 \le x \le 2$

$$\to \begin{cases} y = x^2 & x = \sqrt{y} \\ y = 2x & x = \dfrac{y}{2} \end{cases} \qquad \to \begin{cases} x = 0 & y = 0 \\ x = 2 & y = 4 \end{cases}$$

The integral is $\int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,dx\,dy$

- If $f(x,y)$ and $g(x,y)$ are continuous on the bounded region R, then the following properties hold

1. *Constant Multiple*: $\iint\limits_R cf(x,y)\,dA = c\iint\limits_R f(x,y)\,dA$

2. *Sum and Difference*: $\iint\limits_R \left(f(x,y) \pm g(x,y)\right)dA = \iint\limits_R f(x,y)\,dA \pm \iint\limits_R g(x,y)\,dA$

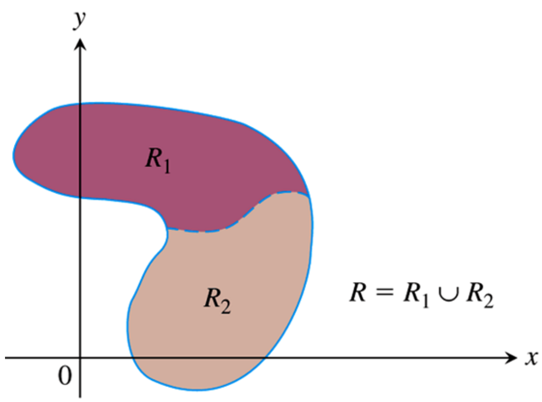
3. *Domination*:

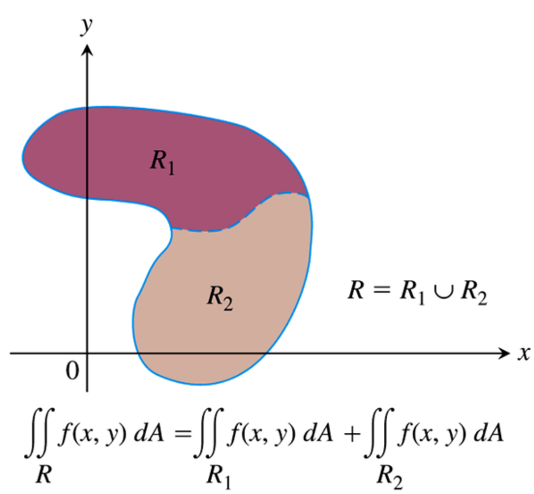
a) $\iint\limits_R f(x,y)\,dA \ge 0 \quad \text{if} \quad f(x,y) \ge 0 \ \text{on } R$

b) $\iint\limits_R f(x,y)\,dA \ge \iint\limits_R g(x,y)\,dA \quad \text{if} \quad f(x,y) \ge g(x,y) \ \text{on } R$

4. *Additivity*: $\iint\limits_R f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA + \iint\limits_{R_2} f(x,y)\,dA$

If R is the union of two non-overlapping regions R_1 and R_2.

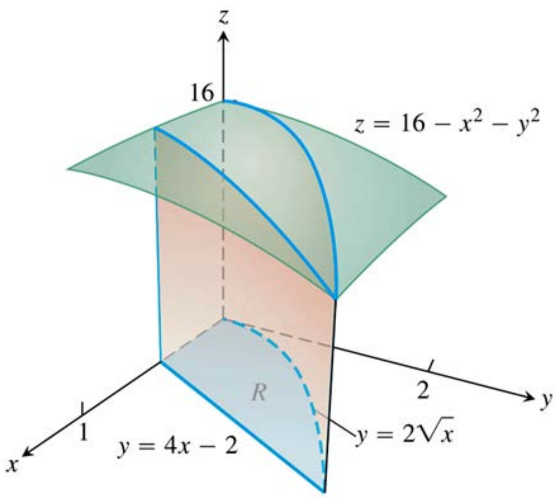


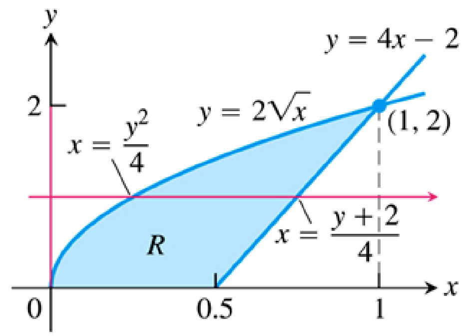


Example

Find the volume of the wedge like solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.

Solution





$$y = 2\sqrt{x} \quad \rightarrow x = \frac{y^2}{4}$$

$$y = 4x - 2 \quad \rightarrow x = \frac{y+2}{4}$$

$$y = 4\frac{y^2}{4} - 2 = y^2 - 2 \quad \rightarrow \quad y^2 - y - 2 = 0 \Rightarrow \underline{y = -1,\ 2}$$

$$Volume = \int_0^2 \int_{y^2/4}^{(y+2)/4} \left(16 - x^2 - y^2\right) dxdy$$

$$= \int_0^2 \left(16x - \frac{1}{3}x^3 - y^2 x \right|_{y^2/4}^{(y+2)/4} dy$$

$$= \int_0^2 \left[\left(16\frac{y+2}{4} - \frac{1}{3}\left(\frac{y+2}{4}\right)^3 - y^2\frac{y+2}{4}\right) - \left(16\frac{y^2}{4} - \frac{1}{3}\frac{y^6}{64} - \frac{y^4}{4}\right)\right] dy$$

$$= \int_0^2 \left[4y + 8 - \frac{1}{192}\left(y^3 + 6y^2 + 12y + 8\right) - \frac{1}{4}y^3 - \frac{1}{2}y^2 - 4y^2 + \frac{1}{192}y^6 + \frac{1}{4}y^4\right] dy$$

$$= \int_0^2 \left(4y + 8 - \frac{1}{192}y^3 - \frac{1}{32}y^2 - \frac{1}{16}y - \frac{1}{24} - \frac{1}{4}y^3 - \frac{9}{2}y^2 + \frac{1}{192}y^6 + \frac{1}{4}y^4\right) dy$$

$$= \int_0^2 \left(\frac{1}{192}y^6 + \frac{1}{4}y^4 - \frac{49}{192}y^3 - \frac{145}{32}y^2 + \frac{63}{16}y + \frac{191}{24}\right) dy$$

$$= \frac{1}{1344}y^7 + \frac{1}{20}y^5 - \frac{49}{768}y^4 - \frac{145}{96}y^3 + \frac{63}{32}y^2 + \frac{191}{24}y \Big|_0^2$$

$$= \frac{2}{21} + \frac{8}{5} - \frac{49}{48} - \frac{145}{12} + \frac{63}{8} + \frac{191}{12}$$

$$= \frac{178}{105} + \frac{513}{48}$$

$$\boxed{= \frac{62,409}{5,040}\ unit^3} \qquad \underline{\approx 12.4\ \ unit^3}$$

Definition

The area of a closed, bounded plane region R is $\qquad A = \iint\limits_R dA$

Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution

$$y = x = x^2 \quad \to \quad x = 0,\ 1$$

$$A = \int_0^1 \int_{x^2}^{x} dydx$$

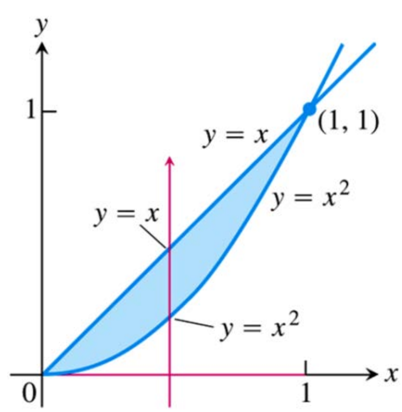
$$= \int_0^1 y \Big|_{x^2}^{x} dx$$

$$= \int_0^1 \left(x - x^2\right) dx$$

$$= \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

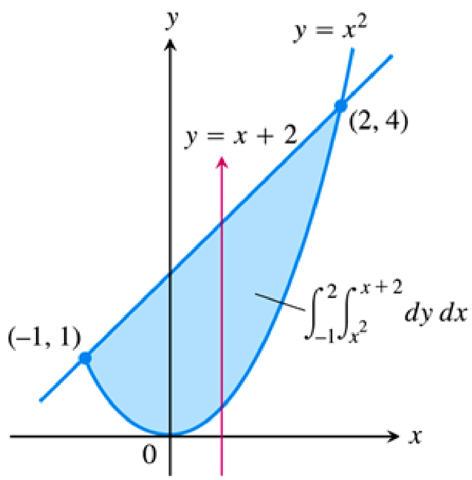
$$\boxed{= \frac{1}{6}\ \ unit^2}$$

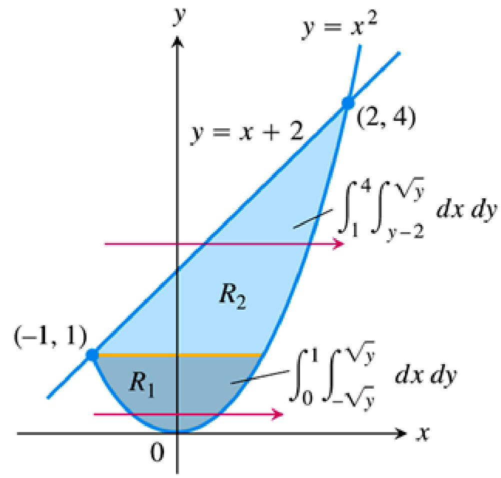


Example

Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Solution





$y = x^2 = x + 2$

$x^2 - x - 2 = 0$

$\underline{x = -1,\ 2}$

$$A = \int_{-1}^{2}\int_{x^2}^{x+2} dydx$$

$$= \int_{-1}^{2} y\Big|_{x^2}^{x+2} dx$$

$$= \int_{-1}^{2} \left(x + 2 - x^2\right) dx$$

$$= \frac{1}{2}x^2 + 2x - \frac{1}{3}x^3 \Big|_{-1}^{2}$$

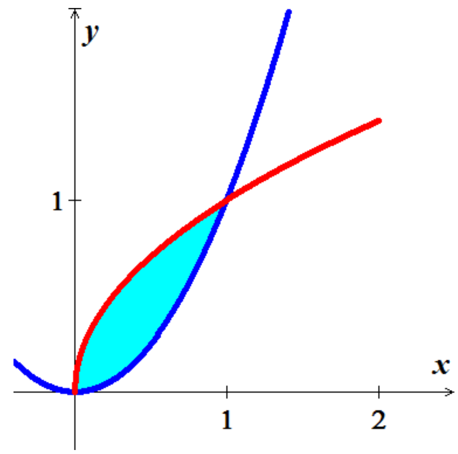
$$= \frac{1}{2}(4) + 2(4) - \frac{1}{3}(8) - \left(\frac{1}{2}(-1)^2 - 2 + \frac{1}{3}\right)$$

$$\boxed{= \frac{9}{2}\quad unit^2}$$

Example

Find the area of the region R between $y = x^2$ and $y^2 = x$.

Solution



$$y = x^2 = \left(y^2\right)^2$$

$$y = y^4 \quad \to \quad \boxed{y = 0,\ 1}$$

$$\boxed{0 \le y \le 1}$$

$$y = x^2 \quad \to \quad x = \sqrt{y}$$

$$\boxed{y^2 \le x \le \sqrt{y}}$$

$$Area = \int_0^1 \int_{y^2}^{\sqrt{y}} dxdy$$

$$= \int_0^1 x \Big|_{y^2}^{\sqrt{y}} dy$$

$$= \int_0^1 \left(y^{1/2} - y^2\right) dy$$

$$= \frac{2}{3} y^{3/2} - \frac{1}{3} y^3 \Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{3}$$

$$\boxed{= \frac{1}{3}\ unit^2}$$

Average values of f over R $= \dfrac{1}{area\ of\ R} \iint\limits_R f dA$

❖ Average value of f over $R = \dfrac{1}{area\ of\ R} \iint\limits_R f dA \quad \boxed{= \dfrac{2}{\pi}}$

Example

Find the average value of $f(x,y) = x\cos xy$ over the rectangle $R:\ 0 \le x \le \pi, \quad 0 \le y \le 1.$

Solution

$$\int_0^{\pi}\int_0^1 x\cos xy\ dydx = \int_0^{\pi} \sin xy \Big|_0^1 dx \qquad\qquad \int x\cos xy\ dy = \sin xy + C$$

$$= \int_0^{\pi} (\sin x - 0)\ dx$$

$$= \int_0^{\pi} \sin x\ dx$$

$$= -\cos x \Big|_0^{\pi}$$

$$= 1 + 1$$

$$\boxed{= 2}$$

Exercises *Section* 3.2 – Double Integrals over General Regions

(1 – 4) Sketch the region of integration and evaluate the integral

1. $\displaystyle\int_0^{\pi}\int_0^{x} x\sin y\,dy\,dx$

2. $\displaystyle\int_0^{\pi}\int_0^{\sin x} y\,dy\,dx$

3. $\displaystyle\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dx\,dy$

4. $\displaystyle\int_1^{4}\int_0^{\sqrt{x}} \frac{3}{2}e^{y/\sqrt{x}}\,dy\,dx$

5. Integrate $f(x,y)=\frac{x}{y}$ over the region in the first quadrant bounded by the lines $y=x,\quad y=2x,\quad x=1,\quad and\quad x=2$

6. Integrate $f(x,y)=x^2+y^2$ over the triangular region with vertices $(0,0),\quad (1,0)\quad and\quad (0,1)$

7. Integrate $f(s,t)=e^{s}\ln t$ over the region in the first quadrant of the *st*-plane that lies above the curve $s=\ln t$ from $t=1$ to $t=2$.

8. Evaluate $\displaystyle\int_{-2}^{0}\int_{v}^{-v} 2\,dp\,dv$

9. Evaluate $\displaystyle\int_{-\pi/3}^{\pi/3}\int_0^{\sec t} 3\cos t\;du\,dt$

(10 – 13) Sketch the region of integration, reverse the order of integration, and evaluate the integral

10. $\displaystyle\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\,dy\,dx$

11. $\displaystyle\int_0^{2}\int_x^{2} 2y^2\sin xy\;dy\,dx$

12. $\displaystyle\int_0^{1/16}\int_{y^{1/4}}^{1/2} \cos\left(16\pi x^5\right)dx\,dy$

13. $\displaystyle\int_0^{2}\int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx$

14. Find the volume of the region bounded above the paraboloid $z=x^2+y^2$ and below by the triangle enclosed by the lines $y=x$, $x=0$, and $x+y=2$ in the *xy*-plane

15. Find the volume of the solid that is bounded above the cylinder $z=x^2$ and below by the region enclosed by the parabola $y=2-x^2$ and the line $y=x$ in the *xy*-plane

16. Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$ and the plane $z + y = 3$

17. Find the volume of the solid that is bounded on the front and back by the planes $x = 2$, and $x = 1$, on the sides by the cylinders $y = \pm\frac{1}{x}$ and above and below the planes $z = x + 1$ and $z = 0$.

18. Find the volume under the parabolic cylinder $z = x^2$ above the region enclosed by the parabola $y = 6 - x^2$ and the line $y = x$ in the *xy*-plane

19. Find the area of the region enclosed by the line $y = 2x + 4$ and the parabola $y = 4 - x^2$ in the *xy*-plane.

20. Find the area of the region enclosed by the coordinate axes and the lin $x = 0$ e $x + y = 2$.

21. Find the area of the region enclosed by the lines , $y = 2x$, and $y = 4$

22. Find the area of the region enclosed by the parabola $x = y - y^2$ and the line $y = -x$.

23. Find the area of the region enclosed by the curve $y = e^x$ and the lines $y = 0$, $x = 0$ and $x = \ln 2$

24. Find the area of the region enclosed by the curve $y = \ln x$ and $y = 2\ln x$ and the lines $x = e$ in the first quadrant.

25. Find the area of the region enclosed by the lines $y = x$, $y = \frac{x}{3}$, and $y = 2$

26. Find the area of the region enclosed by the lines $y = x - 2$ and $y = -x$ and the curve $y = \sqrt{x}$

27. Find the area of the region enclosed by the parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$

28. Find the area of the region bounded by the lines $y = -x - 4$, $y = x$, and $y = 2x - 4$. Make a sketch of the region.

29. Find the area of the region bounded by the lines $y = |x|$ and $y = 20 - x^2$. Make a sketch of the region.

30. Find the area of the region bounded by the lines $y = x^2$ and $y = 1 + x - x^2$. Make a sketch of the region.

(31 – 34) Find the area of the region

31. $\int_0^6 \int_{y^2/3}^{2y} dxdy$

32. $\int_0^{\pi/4} \int_{\sin x}^{\cos x} dydx$

33. $\int_{-1}^2 \int_{y^2}^{y+2} dxdy$

34. $\int_0^2 \int_{x^2-4}^{0} dydx + \int_0^4 \int_0^{\sqrt{x}} dydx$

35. Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2, \quad 0 \le y \le 2$

36. Find the average height of $f(x,y) = \dfrac{1}{xy}$ over the square $\ln 2 \le x \le 2\ln 2, \quad \ln 2 \le y \le 2\ln 2$

(**37 – 40**) Evaluate the integral over the given region

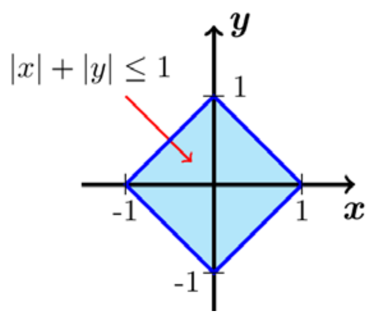
37. $\displaystyle\iint_R y\,dA \quad R = \left\{(x,\ y):\ \ 0 \le x \le \frac{\pi}{3},\quad 0 \le y \le \sec x\right\}$

38. $\displaystyle\iint_R (x+y)\,dA$ *R is the region bounded by* $y = \dfrac{1}{x}$ *and* $y = \dfrac{5}{2} - x$

39. $\displaystyle\iint_R \frac{xy}{1+x^2+y^2}\,dA \quad R = \{(x,\ y):\ \ 0 \le y \le x,\quad 0 \le x \le 2\}$

40. $\displaystyle\iint_R x\sec^2 y\ dA \quad R = \left\{(x,\ y):\ \ 0 \le y \le x^2,\quad 0 \le x \le \frac{\sqrt{\pi}}{2}\right\}$

41. Consider the region $\quad R = \{(x,\ y):\ \ |x| + |y| \le 1\}$



a) Use a double integral to show that the area of *R* is 2.

b) Find the volume of the square column whose base is *R* and whose upper surface is $z = 12 - 3x - 4y$.

c) Find the volume of the solid above *R* and beneath the cylinder $x^2 + z^2 = 1$.

d) Find the volume of the pyramid whose base is *R* and whose vertex is on the *z*-axis at $(0,\ 0,\ 6)$