Math 2318 – Linear Algebra **Exam 2** **Review**

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1. Find the components of the vector $\overrightarrow{P_1P_2}$ with initial point $P_1(2,\ -1,\ 4)$ and terminal point $P_2(7,\ 5,\ -8)$

2. Find $\boldsymbol{u} \times \boldsymbol{v}$, where $\boldsymbol{u} = (1,\ 2,\ -2)$ *and* $\boldsymbol{v} = (3,\ 0,\ 1)$ and show that $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ and to $\boldsymbol{v}$.

3. Calculate the scalar triple product $u \cdot (v \times w)$ of the vectors:

a) $\boldsymbol{u} = 3\boldsymbol{i} - 2\boldsymbol{j} - 5\boldsymbol{k} \quad \boldsymbol{v} = \boldsymbol{i} + 4\boldsymbol{j} - 4\boldsymbol{k} \quad \boldsymbol{w} = 3\boldsymbol{j} + 2\boldsymbol{k}$

b) $\boldsymbol{u} = (-2, 0, 6) \quad \boldsymbol{v} = (1, -3, 1) \quad \boldsymbol{w} = (-5, -1, 1)$

4. Given $\boldsymbol{u} = (3,\ 2,\ -1)$, $\boldsymbol{v} = (0,\ 2,\ -3)$, *and* $\boldsymbol{w} = (2,\ 6,\ 7)$ Compute the vectors

a) $\boldsymbol{u} \times \boldsymbol{v}$ *b)* $\boldsymbol{v} \times \boldsymbol{w}$ *c)* $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w})$

d) $(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w}$ *e)* $\boldsymbol{u} \times (\boldsymbol{v} - 2\boldsymbol{w})$ *f)* $\|\boldsymbol{u}\|$

g) Unit vector of $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ *h)* Angle between $\boldsymbol{v}$, and $\boldsymbol{w}$ *i)* $\|3\boldsymbol{u} - 5\boldsymbol{v} + \boldsymbol{w}\|$

j) $\boldsymbol{u} \cdot \boldsymbol{v}$ *k)* $\boldsymbol{u} \cdot \boldsymbol{w}$

5. Determine whether the vectors form an orthogonal set

a) $\boldsymbol{v}_1 = (2,\ 3)$, $\boldsymbol{v}_2 = (-3,\ 2)$

b) $\boldsymbol{v}_1 = (-3,\ 4,\ -1)$, $\boldsymbol{v}_2 = (1,\ 2,\ 5)$, $\boldsymbol{v}_3 = (4,\ -3,\ 0)$

c) $\boldsymbol{v}_1 = (2, -2, 1)$, $\boldsymbol{v}_2 = (2, 1, -2)$, $\boldsymbol{v}_3 = (1, 2, 2)$

6. Find the vector component $\left(proj_{\boldsymbol{a}}\, \boldsymbol{u} = \dfrac{\boldsymbol{u} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} \right)$ of $\boldsymbol{u}$ along $\boldsymbol{a}$ and the vector component of $\boldsymbol{u}$ orthogonal to $\boldsymbol{a}$.

a) $\boldsymbol{u} = (-1,\ -2)$, $\boldsymbol{a} = (-2,\ 3)$ *b)* $v = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

c) $\boldsymbol{u} = (1,\ 1,\ 1)$, $\boldsymbol{a} = (0,\ 2,\ -1)$ *d)* $\boldsymbol{u} = (2,\ 0,\ 1)$, $\boldsymbol{a} = (1,\ 2,\ 3)$

7. Find the area of the parallelogram determined by the given vectors $\boldsymbol{u} = (1,\ 1,\ 1)$, $\boldsymbol{v} = (3,\ 2,\ -5)$

8. Use the cross product to find a vector that is orthogonal to both $\boldsymbol{u} = (3,\ 3,\ 1)$, $\boldsymbol{v} = (0,\ 4,\ 2)$

9. Find the area of the triangle with the given vertices:

a) $A(2,0) \quad B(3,4) \quad C(-1,2)$ *b)* $A(2,6,-1) \quad B(1,1,1) \quad C(4,6,2)$

10. Find the volume of the parallelepiped with sides $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

$\boldsymbol{u} = (2,-6,2), \quad \boldsymbol{v} = (0,4,-2), \quad \boldsymbol{w} = (2,2,-4)$

11. Which of the following are linear combinations?

a) $(2,1,4)\ \ (1,-1,3)\ \ (3,2,5)\ \ w=(5,9,5)$

b) $(1,-1,3)\ \ (2,4,0)\ \ w=(4,2,6)$

c) $(1,-1,3)\ \ (2,4,0)\ \ w=(1,5,6)$

d) $(2,1,4)\ \ (1,-1,3)\ \ (3,2,5)\ \ w=(2,2,3)$

12. Show that the vector $\boldsymbol{w}$ is a subspace of $\boldsymbol{R}^3$?

a) All vectors of the form $\boldsymbol{w} = (a, 0, 0)$

b) $\boldsymbol{w} = (a, b, c)$, where $a + c + b = 0$, $\quad a, b, c$ are real numbers

c) $\boldsymbol{w} = (a, b, c)$, where $b = a + c$, $\quad a, b, c$ are real numbers

13. Determine whether the given vectors span $\boldsymbol{R}^3$

a) $v_1 = (1,1,1), \quad v_2 = (2,2,0), \quad v_3 = (3,0,0)$

b) $v_1 = (1,3,3), \quad v_2 = (1,3,4), \quad v_3 = (1,4,3), \quad v_4 = (6,2,1)$

14. Determine whether the vectors are linearly independent or linearly dependent

a) (1, 1, −1), (2, −3, 1), (8, −7, 1)

b) (1, −2, −3), (2, 3, −1), (3, 2, 1)

c) (1, −2, 1), (1, 2, −1), (7, −4, 1)

d) (1, −3, 7), (2, 0, −6), (3, −1, −1), (2, 4, −5)

e) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

15. Find the coordinate vector of $\boldsymbol{w}$ relative to the basis $S = \{u_1, u_2\}$ for $\boldsymbol{R}^2$

a) $u_1 = (1,-1), \quad u_2 = (1,1), \quad w = (1,0)$

b) $u_1 = (2,-4), \quad u_2 = (3,8), \quad w = (1,1)$

16. Find the coordinate vector of **v** relative to the basis $S = \{v_1, v_2, v_3\}$

a) $v = (2,-1,1), \quad v_1 = (2,1,3), \quad v_2 = (1,0,1), \quad v_3 = (1,1,1)$

b) $v = (2,1,0), \quad v_1 = (1,2,1), \quad v_2 = (-1,1,2), \quad v_3 = (1,2,3)$

17. Given the matrix A and b:

a) Reduce A to row-reduced echelon form.
b) What is the dimension of A?
c) What is the rank of A?
d) What are the pivots?
e) What are the free variables?
f) Find the special (homogeneous) solutions.
g) What is the nullspace $N(A)$?
h) Find the particular solution to $Ax = b$
i) Give the complete solution.

i. $$A = \begin{pmatrix} -1 & 2 & 5 & 0 \\ 2 & 1 & 0 & 0 \\ 6 & -1 & -8 & -1 \\ 0 & 2 & 4 & 3 \end{pmatrix} \qquad b = \begin{pmatrix} 5 \\ -15 \\ -47 \\ 16 \end{pmatrix}$$

ii. $$A = \begin{pmatrix} 1 & -2 & 1 & 2 \\ 2 & -4 & 2 & 4 \\ -1 & 2 & -1 & -2 \\ 3 & -6 & 3 & 6 \end{pmatrix} \qquad b = \begin{pmatrix} -1 \\ -2 \\ 1 \\ -3 \end{pmatrix}$$

iii. $$A = \begin{pmatrix} 1 & 2 & -3 & 1 \\ -2 & 1 & 2 & 1 \\ -1 & 3 & -1 & 2 \\ 4 & -7 & 0 & -5 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ -1 \\ 3 \\ -5 \end{pmatrix}$$

Solution

1. $(5, 6, -12)$

2. $(2, -7, -6)$, $\boldsymbol{u} \times \boldsymbol{v}$ is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$.

3. *a*) 49 *b*) –92

4. *a*) $(-4,\ 9,\ 6)$ *b*) $(32,\ -6,\ -4)$ *c*) $(-14,\ -20,\ -82)$

d) $(27,\ 40,\ -42)$ *e*) $(-44,\ 55,\ -22)$ *e*) $\sqrt{14}$

g) $\left(\frac{3}{\sqrt{14}},\ \frac{2}{\sqrt{14}},\ -\frac{1}{\sqrt{14}}\right)$, $\left(0,\ \frac{2}{\sqrt{13}},\ -\frac{3}{\sqrt{13}}\right)$, $\left(\frac{2}{\sqrt{89}},\ \frac{6}{\sqrt{89}},\ \frac{7}{\sqrt{89}}\right)$

h) 105.343° *i*) 22.045 *j*) 7 *k*) 11

5. *a*) *Yes* *b*) *No* *c*) *Yes*

6. *a*) $\left(\frac{8}{13},\ -\frac{12}{13}\right)$ $\left(-\frac{21}{13},\ -\frac{14}{13}\right)$ *b*) $(\cos\theta,\ 0)$ $(0,\ \sin\theta)$

c) $\left(0,\ \frac{2}{5},\ \frac{-1}{5}\right)$ $\left(1,\ \frac{3}{5},\ \frac{6}{5}\right)$ *d*) $\left(\frac{5}{14},\ \frac{5}{7},\ \frac{15}{14}\right)$ $\left(\frac{23}{14},\ -\frac{5}{7},\ -\frac{1}{14}\right)$

7. $\sqrt{114}$

8. $(2,\ -6,\ 12)$

9. *a*) 7 *b*) $\frac{\sqrt{374}}{2}$

10. 16

11. *a*) $(5,9,5) = 3(2,1,4) - 4(1,-1,3) + 1(3,2,5)$

b) $(4,2,6) = 2(1,-1,3) + 1(2,4,0)$

c) *not a linear combination*

d) $(2,2,3) = \frac{1}{2}(2,1,4) - \frac{1}{2}(1,-1,3) + \frac{1}{2}(3,2,5)$

12. *a*) *Yes*

b) *Yes*

c) *Yes*

13. *a*) $\det = -6$, *Yes*

b) $\begin{pmatrix} 1 & 0 & 0 & 39 & 7b_1 - b_2 - b_3 \\ 0 & 1 & 0 & -17 & b_3 - 3b_1 \\ 0 & 0 & 1 & -16 & b_2 - 3b_1 \end{pmatrix}$, *Yes*

14. $a)$ *Lineraly dependent*

$b)$ *Lineraly independent*

$c)$ *Lineraly independent*

$d)$ *Lineraly dependent*

$e)$ *Lineraly independent*

15. $a)$ $(w)_S = \left(\frac{5}{28}, \frac{3}{14}\right)$

$b)$ $(w)_S = \left(\frac{1}{2}, \frac{1}{2}\right)$

16. $a)$ $(v)_S = (-1, 4, 0)$

$b)$ $(v)_S = \left(\frac{1}{2}, -1, \frac{1}{2}\right)$

17.

i) $A = \begin{pmatrix} -1 & 2 & 5 & 0 \\ 2 & 1 & 0 & 0 \\ 6 & -1 & -8 & -1 \\ 0 & 2 & 4 & 3 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ -15 \\ -47 \\ 16 \end{pmatrix}$

a) $\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

b) $Dim = 1$

c) $Rank = 3$

d) x_1, x_2, x_4

e) x_3

f) $s_1 = (1, -2, 1, 0)$

g) $\mathbf{x}_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

h) $\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -7 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow x_p = (-7, -1, 6, 0)$

i) $\mathbf{x} = \begin{bmatrix} -7 \\ -1 \\ 6 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

ii) $A = \begin{pmatrix} 1 & -2 & 1 & 2 \\ 2 & -4 & 2 & 4 \\ -1 & 2 & -1 & -2 \\ 3 & -6 & 3 & 6 \end{pmatrix} \quad b = \begin{pmatrix} -1 \\ -2 \\ 1 \\ -3 \end{pmatrix}$

a) $\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

b) $Dim = 3$

c) $Rank = 1$

d) x_1

e) x_2, x_3, x_4

f) $s_1 = (1,1,0,0) \quad s_2 = (-2,0,1,0) \quad s_3 = (-3,0,0,1)$

g) $\mathbf{x}_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \mathbf{x}_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \mathbf{x}_4 \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

h) $x_p = (-1,0,0,0)$

i) $\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

iii) $A = \begin{pmatrix} 1 & 2 & -3 & 1 \\ -2 & 1 & 2 & 1 \\ -1 & 3 & -1 & 2 \\ 4 & -7 & 0 & -5 \end{pmatrix} \quad b = \begin{pmatrix} 4 \\ -1 \\ 3 \\ -5 \end{pmatrix}$

a) $\left[\begin{array}{cccc|c} 1 & 0 & -\frac{7}{5} & -\frac{1}{5} & \frac{6}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

b) $Dim = 2$

c) $Rank = 2$

d) $x_1, \quad x_2$

e) $x_3, \quad x_4$

f) $s_1 = \left(\frac{7}{5}, \frac{4}{5}, 1, 0\right) \quad s_2 = \left(\frac{1}{5}, -\frac{3}{5}, 0, 1\right)$

g) $\boldsymbol{x}_3 \begin{bmatrix} \frac{7}{5} \\ \frac{4}{5} \\ 1 \\ 0 \end{bmatrix} + \boldsymbol{x}_4 \begin{bmatrix} \frac{1}{5} \\ -\frac{3}{5} \\ 0 \\ 1 \end{bmatrix}$

h) $x_p = \left(\frac{6}{5}, \frac{7}{5}, 0, 0\right)$

i) $\boldsymbol{x} = \begin{bmatrix} \frac{6}{5} \\ \frac{7}{5} \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{7}{5} \\ \frac{4}{5} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \frac{1}{5} \\ -\frac{3}{5} \\ 0 \\ 1 \end{bmatrix}$