

Solution **Section 1.1 – Idea of Limits**

Exercise

Find the average rate of change of the function $f(x) = x^3 + 1$ over the interval $[2, 3]$

Solution

$$\begin{aligned}\frac{\Delta f}{\Delta x} &= \frac{f(3) - f(2)}{3 - 2} \\ &= \frac{3^3 + 1 - (2^3 + 1)}{1} \\ &= 27 + 1 - (8 + 1) \\ &= 19\end{aligned}$$

Exercise

Find the average rate of change of the function $f(x) = x^2$ over the interval $[-1, 1]$

Solution

$$\begin{aligned}\frac{\Delta f}{\Delta x} &= \frac{f(1) - f(-1)}{1 - (-1)} \\ &= \frac{1^2 - (-1)^2}{2} \\ &= \frac{0}{2} \\ &= 0\end{aligned}$$

Exercise

Find the average rate of change of the function $f(t) = 2 + \cos t$ over the interval $[-\pi, \pi]$

Solution

$$\begin{aligned}\frac{\Delta f}{\Delta x} &= \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)} \\ &= \frac{2 + \cos \pi - (2 + \cos(-\pi))}{2\pi} \\ &= \frac{2 - 1 - (2 - 1)}{2} \\ &= 0\end{aligned}$$

Exercise

Find the slope of $y = x^2 - 3$ at the point $P(2, 1)$ and an equation of the tangent line at this P .

Solution

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(2+h) - f(2)}{h} & \frac{\Delta y}{\Delta x} &= \frac{f(x_1 + h) - f(x_1)}{h} \\ &= \frac{(2+h)^2 - 3 - (2^2 - 3)}{h} \\ &= \frac{4 + 4h + h^2 - 3 - (4 - 3)}{h} \\ &= \frac{4h + h^2}{h} \\ &= 4 + h\end{aligned}$$

As h approaches 0. Then the secant slope $h + 4 \rightarrow 4 = \text{slope}$

$$y = 4(x - 2) + 1$$

$$y - 1 + 1 = 4x - 8 + 1$$

$$\underline{y = 4x - 7}$$

$$y = m(x - x_1) + y_1$$

Exercise

Find the slope of $y = x^2 - 2x - 3$ at the point $P(2, -3)$ and an equation of the tangent line at this P .

Solution

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(2+h) - f(2)}{h} \\ &= \frac{(2+h)^2 - 2(2+h) - 3 - (2^2 - 2(2) - 3)}{h} \\ &= \frac{4 + 4h + h^2 - 4 - 2h - 3 - (-3)}{h} \\ &= \frac{2h + h^2}{h} \\ &= 2 + h\end{aligned}$$

As h approaches 0. Then the secant slope $2 + h \rightarrow 2 = \text{slope}$

$$y + 3 = 2(x - 2)$$

$$y = 2x - 4 - 3$$

$$\underline{y = 2x - 7}$$

$$y = m(x - x_1) + y_1$$

Exercise

Find the slope of $y = x^3$ at the point $P(2, 8)$ and an equation of the tangent line at this P .

Solution

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(2+h) - f(2)}{h} \\ &= \frac{(2+h)^3 - 2^3}{h} \\ &= \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\ &= \underline{12 + 6h + h^2} \quad \text{As } h \text{ approaches } 0. \text{ Then } \text{slope} = 12\end{aligned}$$

$$y - 8 = 12(x - 2)$$

$$y = 12x - 24 + 8$$

$$\underline{y = 12x - 16}$$

$$y = m(x - x_1) + y_1$$

Exercise

Make a table of values for the function $f(x) = \frac{x+2}{x-2}$ at the points

$$x = 1.2, \quad x = \frac{11}{10}, \quad x = \frac{101}{100}, \quad x = \frac{1001}{1000}, \quad x = \frac{10001}{10000}, \quad \text{and } x = 1$$

- a) Find the average rate of change of $f(x)$ over the intervals $[1, x]$ for each $x \neq 1$ in the table
b) Extending the table if necessary, try to determine the rate of change of $f(x)$ at $x = 1$.

Solution

a)

x	1.2	1.1	1.01	1.001	1.0001	1
$f(x)$	-4.0	-3.4	-3.04	-3.004	-3.0004	-3

$$\frac{\Delta y}{\Delta x} = \frac{-4 - (-3)}{1.2 - 1} = -5.0$$

$$\frac{\Delta y}{\Delta x} = \frac{-3.4 - (-3)}{1.1 - 1} = -4.4$$

$$\frac{\Delta y}{\Delta x} = \frac{-3.04 - (-3)}{1.01 - 1} = -4.04$$

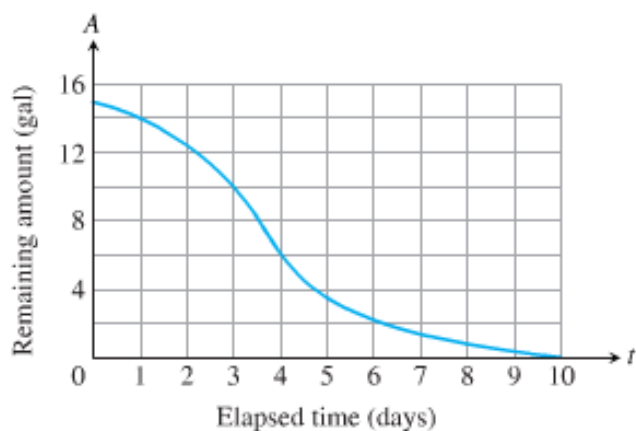
$$\frac{\Delta y}{\Delta x} = \frac{-3.004 - (-3)}{1.001 - 1} = -4.004$$

$$\frac{\Delta y}{\Delta x} = \frac{-3.0004 - (-3)}{1.0001 - 1} = -4.0004$$

b) The rate of change of $f(x)$ at $x = 1$ is -4

Exercise

The accompanying graph shows the total amount of gasoline A in the gas tank of an automobile after being driven for t days.



- a) Estimate the average rate of gasoline consumption over the time intervals $[0, 3]$, $[0, 5]$, and $[7, 10]$
- b) Estimate the instantaneous rate of gasoline consumption over the time $t = 1$, $t = 4$, and $t = 8$

Solution

- a) Average rate of gasoline consumption over the time intervals:

$[0, 3]$

$$\begin{aligned} \text{Average Rate} &= \frac{10 - 15}{3 - 0} \\ &= -\frac{5}{3} \\ &\approx -1.67 \text{ gal / day} \end{aligned}$$

$[0, 5]$

$$\begin{aligned} \text{Average Rate} &= \frac{3.9 - 15}{5 - 0} \\ &\approx -2.2 \text{ gal / day} \end{aligned}$$

$[7, 10]$

$$\begin{aligned} \text{Average Rate} &= \frac{0 - 1.4}{10 - 7} \\ &= -\frac{1.4}{3} \\ &= -\frac{7}{15} \\ &\approx -0.5 \text{ gal / day} \end{aligned}$$

- b) At $t = 1 \rightarrow P(1, 14)$
At $t = 4 \rightarrow P(4, 6)$
At $t = 8 \rightarrow P(8, 1)$