3.2

Geometric Series: $\sum_{n=1}^{\infty} a r^{n-1}$

$S = \frac{a}{1-r}$ if $|r| < 1$ → converges.

$S = \infty$ if $|r| \geq 1$ → diverges

Ex $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = \sum_{n=0}^{\infty} 5\left(-\frac{1}{4}\right)^n$

$|r| = \frac{1}{4} < 1$; $r = -\frac{1}{4}$

$S = \frac{5}{1-\left(-\frac{1}{4}\right)}$

$= \frac{5}{1+\frac{1}{4}}$

$= 4$

∴ By the Geometric Series, the given series converges w/ sum of 4

Divergent Series

$\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$

$\lim_{n\to\infty} a_n = 0$

a) $\sum_{n=1}^{\infty} \frac{n+1}{n}$, $\frac{n+1}{n} \longrightarrow 1 \neq 0$

By the divergent series, the given series diverges

Ex $\sum_{n=1}^{\infty} n^2$ $\quad n^2 \to \infty$

$\therefore$ By the divergent series, the given series diverges

Ex: $\sum_{n=1}^{\infty} (-1)^{n+1}$; diverges because limit given series doesn't exist

Ex $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ $\quad \frac{-n}{2n+5} \to -\frac{1}{2} \neq 0$.

By the divergent series, the given series diverges

Ex $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{3^{n-1}}{6^{n-1}} - \frac{1}{6^{n-1}} \right)$

$$= \sum_{n=1}^{\infty} \left(\frac{3}{6}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{n-1}$$

$|r| = \frac{1}{2} < 1$

$|r| = \frac{1}{6} < 1$

$$S = \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{6}}$$

$$= 2 - \frac{6}{5}$$

$$= \frac{4}{5}$$

$\therefore$ By the Geometric series, the given series converges w/ sum $= \frac{4}{5}$

3.3 Integral Test.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges?

$$\int_1^{\infty} \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_1^{\infty}$$
$$= -(0-1)$$
$$= 1$$

P- series. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\begin{cases} \text{if } p \leq 1 \to \text{diverges.} \\ \text{if } p > 1 \to \text{converges.} \end{cases}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$; $p = 2 > 1$

$\therefore$ By p-series, the given series converges

EX

$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$$\int_1^{\infty} \frac{dx}{x^2+1} = \tan^{-1} x \Big|_1^{\infty}$$
$$= \tan^{-1}\infty - \tan^{-1} 1$$
$$= \frac{\pi}{2} - \frac{\pi}{4}$$
$$= \frac{\pi}{4}$$

$\therefore$ By the integral Test, the given series converges

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#1 $\sum_{n=1}^{\infty} \frac{1}{n^{0.2}}$ $\quad p = 0.2 \leq 1$

$\therefore$ By p-series, the given series diverges

#17. $\sum_{n=1}^{\infty} \frac{1}{n^5}$ $\quad p = 5 > 1$

$\therefore$ By the p-series, the given series converges

4. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \int_2^{\infty} \frac{d(\ln x)}{(\ln x)^2}$$

$$= -\frac{1}{\ln x}\Big|_2^{\infty}$$

$$= -\left(\frac{1}{\ln \infty} - \frac{1}{\ln 2}\right)$$

$$= \frac{1}{\ln 2}$$

By the integral Test, the given series converges

#5. $\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$

$$\int_1^{\infty} x^2 e^{-x/3} dx = e^{-x/3}(-3x^2 - 18x - 54)\Big|_1^{\infty}$$

	$\int e^{-x/3}$
$+x^2$	$-3e^{-x/3}$
$-2x$	$9e^{-x/3}$
$+2$	$-27e^{-x/3}$

$$= 0 - e^{-1/3}(-3 - 18 - 54)$$

$$= \frac{75}{e^{1/3}}$$

$\therefore$ By the integral Test, the given series converges

3.4 Comparison Test

$\sum a_n$, $\sum c_n$ $\sum d_n$

$d_n \leq a_n \leq c_n$

If $\sum c_n$ converges $\Rightarrow$ $\sum a_n$ converges

If $\sum d_n$ diverges $\Rightarrow$ $\sum a_n$ diverges

Ex $\sum \frac{5}{5n-1}$

$$5n > 5n-1$$

$$\frac{1}{5n} < \frac{1}{5n-1}$$

$$\frac{5}{5n-1} > \frac{5}{5n}$$

$$= \frac{1}{n}$$; $p = 1 \leq 1$ diverges by p-series

$\therefore$ By the Comparison Test, the given series diverges

Limit Comparison Test.

$a_n > 0$ $b_n > 0$

1. If $\lim_{n \to \infty} \frac{a_n}{b_n} = c > 0$ $\Rightarrow$ $\sum a_n$ & $\sum b_n$ $\Big\{$ Both converge / diverge

$\frac{a_n}{b_n} \to 0$ $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges

$\to \infty$ diverges $\Rightarrow$ diverges

Ex $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$

$a_n = \frac{2n+1}{n^2+2n+1}$

$\boxed{b_n = \frac{2n}{n^2} = \frac{2}{n}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n+1}{n^2+2n+1} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2}$$

$$= 2$$

$\sum b_n = \frac{1}{n}$: $p = 1 \leq 1$ diverges by p-series

∴ By the Limit Comparison Test, the given series diverges

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

$2^n - 1 < 2^n$

$\frac{1}{2^n-1} \to \frac{1}{2^n}$ $\quad b_n = \frac{1}{2^n} \to 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{2^n-1} \cdot \frac{2^n}{1}$$

$$= \lim_{n\to\infty} \frac{2^n}{2^n}$$

$$= 1$$

∴ By the Limit Comparison Test, the given series converges

Ex $\displaystyle\sum_{n=2}^{\infty} \frac{1+n\ln n}{n^2+5}$

$$a_n = \frac{1+n\ln n}{n^2+5}$$

$$b_n = \frac{n\ln n}{n^2} = \frac{\ln n}{n} > \frac{1}{n},$$

$p = 1 \leq 1$; $\sum b_n$ diverges by p-series

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1+n\ln n}{n^2+5} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{n^2 \ln n}{n^2}$$

$$= \lim_{n\to\infty} \ln n$$

$$= \infty$$

∴ By the Limit Comparison Test, the given series diverges

$\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^{3/2}}$

$$\int_2^{\infty} x^{-3/2} \ln x \, dx$$

$y = \ln x \to x = e^y$

$dx = e^y dy$

$$= \int y e^{-3y/2} e^y dy$$

$$= \int y e^{-y/2} dy$$

	$\int e^{-y/2}$
$+ y$	$-2e^{-y/2}$
-1	$4e^{-y/2}$

$$= (-2y-4)e^{-y/2} \Big|_2^{\infty}$$

$$= 0 - (-4-4)e^{-1}$$

$$= \frac{8}{e}$$

∴ By the Integral Test, the given series converges

$\underbrace{\cos x}(x^2+1)$ $\qquad$ $(x^2+1)\cos x$

$$\sum_{n=2}^{\infty} \frac{\ln n}{n^{3/2}}$$

$$\frac{\ln n}{n^{3/2}} > \frac{1}{n^{3/2}}$$

3.4

#1 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+30}$ $\qquad$ $\tan^{-1}\frac{1}{\sqrt{30}}$

$$n^2+30 > n^2$$

$$\frac{1}{n^2+30} < \frac{1}{n^2}$$; $p=2>1$ converges by p-series

$\therefore$ By the Comparison Test, the given series converges

#2 $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$

$$n^4+2 > n^4$$

$$\frac{1}{n^4+2} < \frac{1}{n^4}$$

$$\frac{n-1}{n^4+2} < \frac{n-1}{n^4}$$

$$\frac{n}{n^4} = \frac{1}{n^3}$$

$p=3>1 \Rightarrow$ converges by p-series

$\therefore$ By the Comparison Test, the given series converges

#60 $\sum_{k=1}^{\infty} (-1)^k k \sin \frac{1}{k}$

$-1 \leq \sin \frac{1}{k} \leq 1$

$-k \leq k\sin \frac{1}{k} \leq k$

$\lim_{k \to \infty} k = \infty$

$\therefore$ By Comparison Test, the given series diverges

#63 $\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$

$-1 \leq \cos n \leq 1$

$-\frac{1}{n^3} \leq \frac{\cos n}{n^3} \leq \frac{1}{n^3}$

$\sum \frac{1}{n^3}$; $p = 3 > 1$ converges by p-series

$\therefore$ By the Comparison Test, the given series converges.