Solution Section 3.3 – The Basics of Counting

Exercise

There are 18 mathematics majors and 325 computer science majors at a college

a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?

b) In how many ways can one representative be picked who either a mathematics major or a computer science major?

Solution

a) $18 \cdot 325 \; \underline{= 5850 \;\; \text{ways}}$

b) $18 + 325 \; \underline{= 343 \;\; \text{ways}}$

Exercise

An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

Solution

Using the product rule: there are $27 \cdot 37 \; \underline{= 999 \;\; \text{offices}}$

Exercise

A multiple-choice test contains 10 questions. There are four possible answers for each question

a) In how many ways can a student answer the questions on the test if the student answers every question?

b) In how many ways can a student answer the questions on the test if the student can leave answers blank?

Solution

a) $4 \cdot 4 \cdot 4 \cdots 4 = 4^{10} \; \underline{= 1,048,576 \; \text{ways}}$

b) There are 5 ways to answer each question 0 give any if the 4 answers or give no answer at all

$\underline{5^{10} = 9,765,625 \;\;\; \text{ways}}$

Exercise

A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of the shirts are made?

Solution

$\boxed{12 \cdot 2 \cdot 3 = 72}$ different types of shirt.

Exercise

How many different three-letter initials can people have?

Solution

$\boxed{26 \cdot 26 \cdot 26 = 17,576}$ different initials.

Exercise

How many different three-letter initials with none of the letters repeated can people have?

Solution

$\boxed{26 \cdot 25 \cdot 24 = 15,600 \;\; ways}$

Exercise

How many different three-letter initials are there that begin with an A?

Solution

$\boxed{1 \cdot 26 \cdot 26 = 676 \;\; ways}$

Exercise

How many bit strings are there of length eight?

Solution

$\boxed{2^8 = 256 \;\; bit\ strings}$

Exercise

How many bit strings of length ten both begin and end with a 1?

Solution

$\boxed{1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 2^8 = 256 \;\; bit\ strings}$

Exercise

How many bit strings of length n, where n is a positive integer, start and end with 1s?

Solution

$$\underline{1 \cdot 2^{n-2} \cdot 1 = 2^{n-2} \quad \textit{bit strings}} \Big| \qquad\qquad 1 \cdot 2 \cdot 2 \cdots 2 \cdot 2 \cdot 1$$

Exercise

How many strings are there of lowercase letters of length four or less, not counting the empty string?

Solution

The number of strings of length 4 or less by counting the number of the strings of length $0 \le i \le 4$

There are 26 letters to choose from, and a string of length i is specified by choosing its characters, one after another.

The product rules there are 26^i

$$\sum_{i=0}^{4} 26^i = 1 + 26 + 26^2 + 26^3 + 26^4$$

$$= 475,255$$

Exercise

How many strings are there of four lowercase letters that have the letter x in them?

Solution

Number of strings of length of 4 lowercase: 26^4

Number of strings of length of 4 lowercase other than ***x***: 25^4

$$26^4 - 25^4 = 66,351 \quad \textit{strings}$$

Exercise

How many positive integers between 50 and 100

a) Are divisible by 7? Which integers are these

b) Are divisible by 11? Which integers are these

c) Are divisible by 7 and 11? Which integers are these

Solution

a) Neither 50 nor 100 is divisible by 7

There are $\frac{50}{7} = 7$ integers less than 50 that are divisible by 7

There are $\frac{100}{7} = 14$ integers less than 100 that are divisible by 7

This leaves $14 - 7 = 7$ numbers between 50 and 100 that are divisible by 7.
They are 56, 63, 70, 77, 84, 91, and 98.

b) Neither 50 nor 100 is divisible by 11

There are $\frac{50}{11}=4$ integers less than 50 that are divisible by 11

There are $\frac{100}{11}=9$ integers less than 100 that are divisible by 1

This leaves $9-4=5$ numbers between 50 and 100 that are divisible by 11

They are 55, 66, 77, 88, and 99.

c) A number is divisible by 7 and 11 which is 77. There is only one such number between 50 and 100, namely 77.

Exercise

How many positive integers less than 100

a) Are divisible by 7?
b) Are divisible by 7 but not by 11?
c) Are divisible by both 7 and 11?
d) Are divisible by either 7 or 11?
e) Are divisible by exactly one of 7 and 11?
f) Are divisible by neither 7 nor 11?

Solution

7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98 *are divisible by* 7
11, 22, 33, 44, 55, 66, 77, 88, 99 *are divisible by* 11

a) Every 7^{th} number is divisible by 7. Therefore, $\frac{99}{7}\approx 14$ such numbers. The k^{th} multiple of 7 does not occur until the number $7k$ has been reached.

b) There are 13 such numbers since 77 is the only one divisible by 11.

c) There is only **1** number (77) divisible by both 7 and 11

d) $\frac{99}{11}\approx 9$ such numbers and 14 such numbers divisible by 7 and only 1 is divisible by 11. Therefore, there are $14+9-1 \underline{=22}$ divisible by either 7 or 11

e) The number of numbers divisible of them: $22-1 \underline{=21}$ $\quad$ $(subtract\ (d)\ from\ (c))$

f) Subtract part (d) from the total number of positive integers less than 100.
$99-22 \underline{=77}$

Exercise

How many positive integers less than 1000

a) Are divisible by 7?
b) Are divisible by 7 but not by 11?
c) Are divisible by both 7 and 11?
d) Are divisible by either 7 or 11?
e) Are divisible by exactly one of 7 and 11?
f) Are divisible by neither 7 nor 11?
g) Have distinct digits?
h) Have distinct digits and are even?

Solution

a) Every 7th number is divisible by 7. Therefore, $\frac{999}{7} \approx 142$ such numbers. The k^{th} multiple of 7 does not occur until the number $7k$ has been reached.

b) Every 11th number is divisible by 11. Therefore, $\frac{999}{11} \approx 90$ numbers.

Since 77 is the first number that is divisible by 7 and 11, and there are $\frac{999}{77} \approx 12$ numbers divisible by 77.

There are $142-12 \ \underline{=130}\ |$ numbers divisible by 7 but not by 11.

c) There are **12** numbers divisible by both 7 and 11 (from part ***b***)

d) There are $142+90-12 \ \underline{=220}\ |$ divisible by either 7 or 11

e) The number of numbers divisible of them: $220-12 \ \underline{=208}\ |$ (*subtract* (***d***) *from* (***c***))

f) Subtract part (**d**) from the total number of positive integers less than 1000.

$999-220 \ \underline{=779}\ |$

g) If we assume that numbers are written without leading 0's, then we can break down this part in three cases: one-digit numbers, two-digit numbers and three-digit numbers.

There are 9 one-digit numbers, and each of them has distinct digits.

There are 90 two-digit numbers (10 – 99), and all but 9 of them have distinct digits, so there are 81 two-digit numbers with distinct digits. Or the first digit 1 through 9 (9 choices), using the product rule: $9 \cdot 9 = 81$ choices in all.

For three-digit numbers there are $9 \cdot 9 \cdot 8 = 648$ distinct digits

Therefore $9+81+648 = \underline{738}\ |$ total distinct digits.

h) If we use to count the odd numbers with distinct digits and subtract from part (***g***), we can get the numbers distinct digits and are even.

There are 5 odd one-digit numbers.

For two-digit numbers; first the ones digits (5 choices), then the tens digit (8 choices) – neither the ones digit value nor 0 is available, therefore there are 40 such two-digit numbers (half of 81).

For three-digit numbers, first the ones digits (5 choices), the hundreds digit (8 choices), then the tens digit (8 choices). There are $5 \cdot 8 \cdot 8 = 320$ distinct digits

So $5+40+320=365$ total odd numbers with distinct digits.

Therefore $738-365 \underline{=373}$ total distinct digits.

Exercise

A committee is formed consisting of one representative from each of the 50 states in the United States, where the representative from a state is either the governor or one of the two senators from that state. How many ways are there to form this committee?

Solution

There are 50 choices to make each of which can be done in 3 ways, namely by choosing the governor, choosing the senior senator, or choosing the junior senator.

$\underline{3^{50} \approx 7.2\times 10^{23}}$

Exercise

How many license plates can be made using either three digits followed by three uppercase English letters or three uppercase English letters followed by three digits?

Solution

$10^3\cdot 26^3+26^3\cdot 10^3 \underline{=35{,}152{,}000 \ \ \textit{license plates}}$

Exercise

How many license plates can be made using either two uppercase English letters followed by four digits or two digits followed by four uppercase English letters?

Solution

Letters		***Digits***			
L	L	D	D	D	D
26	26	10	10	10	10

$26\cdot 26\cdot 10\cdot 10\cdot 10\cdot 10=6{,}760{,}000$

Digits		***Letters***			
D	D	L	L	L	L
10	10	26	26	26	26

$10\cdot 10\cdot 26\cdot 26\cdot 26\cdot 26=45{,}697{,}600$

Therefore: $6{,}760{,}000+45{,}697{,}600 \underline{=52{,}457{,}600 \ \ \textit{license plates}}$

Exercise

How many license plates can be made using either three uppercase English letters followed by three digits or four uppercase English letters followed by two digits?

Solution

$$26^3 \cdot 10^3 + 26^4 \cdot 10^2 \underline{= 63,273,600 \quad \textit{license plates}}$$

Exercise

How many strings of eight English letter are there.

a) That contains no vowels, if letters can be repeated?
b) That contains no vowels, if letters cannot be repeated?
c) That starts with a vowel, if letters can be repeated?
d) That starts with a vowel, if letters cannot be repeated?
e) That contains at least one vowel, if letters can be repeated?
f) That contains at least one vowel, if letters cannot be repeated?

Solution

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
	NV	**NV**	**NV**	**NV**	**NV**	**NV**	**NV**	**NV**
a	21	21	21	21	21	21	21	21
b	21	20	19	18	17	16	15	14
	V	**L**	**L**	**L**	**L**	**L**	**L**	**L**
c	5	26	26	26	26	26	26	26
d	5	25	24	23	22	21	20	19

a) There are 8 slots which can be filled with $26-5=21$ non-vowels.

By the product rule: $\underline{21^8 = 37,822,859,361 \quad \textit{strings}}$

b) $21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \underline{= 8,204,716,800 \quad \textit{strings}}$

c) $5 \cdot 26^7 \underline{= 40,159,050,880 \quad \textit{strings}}$

d) $5 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \underline{= 12,113,640,000 \quad \textit{strings}}$

e) By the product rule: $26^8 - 21^8 \underline{= 171,004,205,215 \quad \textit{strings}}$

f) $8 \cdot 5 \cdot 21^7 \underline{= 72,043,541,640 \quad \textit{strings}}$

Exercise

How many ways are there to seat four of a group of ten people around a circular table where two seatings are considered the same when everyone has the same immediate left and immediate right neighbor?

Solution

The count ordered arrangements of length 4 from the 10 people, then we get $10 \cdot 9 \cdot 8 \cdot 7 = 5040$ arrangements.

However, we can rotate the people around the table in 4 ways and get the same seating arrangement, so the overcounts by a factor of 4.

Therefore, there are $\frac{5040}{4} \underline{= 1260 \quad ways}$

Exercise

In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and the groom are among these 10 people, if

a) The bride must be in the picture?

b) Both the bride and groom must be in the picture?

c) Exactly one of the bride and the groom is in the picture?

Solution

a) The bride is in any of the 6 positions.

1	2	3	4	5	6
B	***P***	***P***	***P***	***P***	***P***
1	9	8	7	6	5

Then, it will leave us with 5 remaining positions.

This can be done in $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15120$ ways.

Therefore $6 \cdot 15120 \underline{= 90,720 \quad ways}$

b) The bride is in any of the 6 positions.

1	2	3	4	5	6
B	***G***	***P***	***P***	***P***	***P***
1	1	8	7	6	5

Then place the groom in any of the 5 remaining positions.

Then, it will leave us with 4 remaining positions in the picture.

This can be done in $8 \cdot 7 \cdot 6 \cdot 5 = 1680$ ways.

Therefore $6 \cdot 5 \cdot 1680 \underline{= 50,400 \quad ways}$

c) For the just the bride to be in the picture: $90720 - 50400 = 40,320$ ways.

There are 40,320 ways for just the groom to be in the picture.

Therefore, $40320 + 40320 \underline{= 80,640 \quad ways}$

Exercise

How many different types of homes are available if a builder offers a choice of 6 basic plans, 3 roof styles, and 2 exterior finishes?

Solution

$6.3.2 = \underline{36}$ *different homes types*

Exercise

A menu offers a choice of 3 salads, 8 main dishes, and 7 desserts. How many different meals consisting of one salad, one main dish, and one dessert are possible?

Solution

$3.8.7 = \underline{168}$ ***different meals***

Exercise

A couple has narrowed down the choice of a name for their new baby to 4 first names and 5 middle names. How many different first- and middle-name arrangements are possible?

Solution

$4.5 = \underline{20}$ *possible arrangements*

Exercise

An automobile manufacturer produces 8 models, each available in 7 different exterior colors, with 4 different upholstery fabrics and 5 interior colors. How many varieties of automobile are available?

Solution

$8.7.4.5 = \underline{1120}$

Exercise

A biologist is attempting to classify 52,000 species of insects by assigning 3 initials to each species. Is it possible to classify all the species in this way? If not, how many initials should be used?

Solution

$26^3 = 17,576$ *This would not be enough.*

$26^4 = 456,976$ *Which is more than enough*

Exercise

How many 4-letter code words are possible using the first 10 letters of the alphabet under:

a) No letter can be repeated
b) Letters can be repeated
c) Adjacent can't be alike

Solution

a) $10.9.8.7 = 5040$

b) $10.10.10.10 = 10,000$

c) $10.9.9.9 = 7290$

Exercise

How many 3 letters license plate without repeats

Solution

$26.25.24 = 15600$ possible

Exercise

How many ways can 2 coins turn up heads, H, or tails, T – if the combined outcome (H, T) is to be distinguished from the outcome (T, H)?

Solution

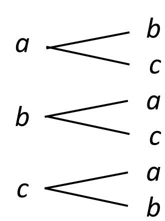
$2 \times 2 = 4 \quad \textit{outcomes}$

Exercise

How many 2-letter code words can be formed from the first 3 letters of the alphabet if no letter can be used more than once?

Solution

$3 \times 2 = 6 \quad \textit{outcomes}$



Exercise

A coin is tossed with possible outcomes heads, H, or tails, T. Then a single die is tossed with possible outcomes 1, 2, 3, 4, 5, or 6. How many combined outcomes are there?

Solution

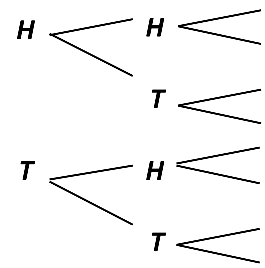
$2 \times 6 = 12 \quad \textit{outcomes}$

Exercise

In how many ways can 3 coins turn up heads, H, or tails, T – if combined outcomes such as (H,T,H), (H, H, T), and (T, H, H) are to be considered different?

Solution

$\underline{2\times 2\times 2 = 8 \quad outcomes}$



Exercise

An entertainment guide recommends 6 restaurants and 3 plays that appeal to a couple.

a) If the couple goes to dinner or to a play, how many selections are possible?
b) If the couple goes to dinner and then to a play, how many combined selections are possible?

Solution

a) 3 + 6 = 9

b) 6.3 = 18