Solution Section 3.4 – Permutations and Combinations

Exercise

Decide whether the situation involves ***permutations*** or ***combinations***

a) A batting order for 9 players for a baseball game
b) An arrangement of 8 people for a picture
c) A committee of 7 delegates chosen from a class of 30 students to bring a petition to the administration
d) A selection of a chairman and a secretary from a committee of 14 people
e) A sample of 5 items taken from 71 items on an assembly line
f) A blend of 3 spices taken from 7 spices on a spice rack
g) From the 7 male and 10 female sales representatives for an insurance company, team of 8 will be selected to attend a national conference on insurance fraud.
h) Marbles are being drawn without replacement from a bag containing 15 marbles.
i) The new university president named 3 new officers a vice-president of finance, a vice-president of academic affairs, and a vice-president of student affairs.
j) A student checked out 4 novels from the library to read over the holiday.
k) A father ordered an ice cream cone (chocolate, vanilla, or strawberry) for each of his 4 children.

Solution

a) Permutation

b) Permutation

c) Combination

d) Permutation

e) Combination

f) Combination

g) Combination

h) Combination

i) Permutation

j) Combination

k) Neither

Exercise

How many different permutations are the of the set $\{a, b, c, d, e, f, g\}$?

Solution

$P(7,\ 7) = 5040$

Exercise

How many permutations of $\{a, b, c, d, e, f, g\}$ end with a?

Solution

To find the permutation to with a, then we may forget about the a, and leave us $\{b, c, d, e, f, g\}$

$P(6,\ 6) \ \underline{= 720 \ |}$

Exercise

Find the number of 5-permutations of a set with nine elements

Solution

$P(9,\ 5) \ \underline{= 15{,}120 \ |}$ ***by Theorem***

Exercise

In how many different orders can five runners finish a race if no ties are allowed?

Solution

$P(5,\ 5) \ \underline{= 120 \ |}$

Exercise

A coin flipped eight times where each flip comes up either heads or tails. How many possible outcomes

a) Are there in total?

b) Contain exactly three heads?

c) Contain at least three heads?

d) Contain the same number of heads and tails?

Solution

a) Each flip can be either heads or tails: There are $2^8 \underline{= 256 \ \ possible\ coutcomes \ |}$

b) $C(8,\ 3) \ \underline{= 56 \ \ outcomes \ |}$

c) At least three heads means: 3, 4, 5, 6, 7, 8 heads.

$C(8,\ 3) + C(8,\ 4) + C(8,\ 5) + C(8,\ 6) + C(8,\ 7) + C(8,\ 8) \ \underline{= 219 \ \ outcomes \ |}$

OR

$256 - C(8,\ 0) - C(8,\ 1) - C(8,\ 2) = 256 - 28 - 8 - 1 \ \underline{= 219 \ \ outcomes \ |}$

d) To have an equal number of heads and tails means 4 heads and 4 tails.

Therefore; $C(8,\ 4) \ \underline{= 70 \ \ outcomes \ |}$

Exercise

A coin flipped 10 times where each flip comes up either heads or tails. How many possible outcomes

a) Are there in total?

b) Contain exactly two heads?

c) Contain at most three tails?

d) Contain the same number of heads and tails?

Solution

a) Each flip can be either heads or tails: There are $2^{10} = 1024 \ \ \textit{possible coutcomes}$

b) $C_{10,2} = 45 \ \ \textit{outcomes}$

c) At most three tails means: 3, 2, 1, 0 tails.

$C_{10,3} + C_{10,2} + C_{10,1} + C_{10,0} = 176 \ \ \textit{outcomes}$

d) To have an equal number of heads and tails means 5 heads and 5 tails.

Therefore; $C_{10,5} = 252 \ \ \textit{outcomes}$

Exercise

How many bit strings of length 12 contain?

a) Exactly three 1s?

b) At most three 1s?

c) At least three 1s?

d) An equal number of 0s and 1s?

Solution

a) We need to choose the 3 positions that contains the 1's

$C_{12,\ 3} = 220 \ \ \textit{ways}$

b) At most three 1's means to contains 3, 2, 1, 0 –1's:

$C_{12,3} + C_{12,2} + C_{12,1} + C_{12,0} = 299 \ \ \textit{strings}$

c) At least three 1's means to contains 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 –1's:

$C_{12,3} + C_{12,4} + C_{12,5} + C_{12,6} + C_{12,7} + C_{12,8} + C_{12,9} + C_{12,10} + C_{12,11} + C_{12,12} = 4017 \ \ \textit{strings}$

OR

$2^{12} - C_{12,2} - C_{12,1} - C_{12,0} = 4096 - 66 - 12 - 1 = 4017 \ \ \textit{strings}$

d) To have an equal number of 0's and 1's means 6 1's.

Therefore; $C(12,\ 6) = 924 \ \ \textit{strings}$

Exercise

A group contains *n* men and *n* women. How many ways are there to arrange these people in a row if the men and women alternate?

Solution

Consider the order in which the men appear relative to each other. There are *n* men $P(n,n)=n!$ arrangements is allowed.

Consider the order in which the women appear relative to each other. There are *n* women $P(n,n)=n!$ arrangements is allowed.

Men and women must alternate, and there are the same number of men and women; therefore there are exactly 2 possibilities: either the row with a man ends with a woman ***or*** it starts with a woman ends with a man.

By the product rule there are $n!\ n!\ 2 \ \underline{= 2(n!)^2 \quad ways}$

Exercise

In how many ways can a set of two positive integers less than 100 be chosen?

Solution

$$C_{99,\ 2} \ \underline{= 4851 \quad ways}$$

Exercise

In how many ways can a set of five letters be selected from the English alphabet?

Solution

$$C_{26,\ 5} \ \underline{= 65,780 \quad ways}$$

Exercise

How many subsets with an odd number of elements does a set with 10 elements have?

Solution

$$C_{10,1} + C_{10,3} + C_{10,5} + C_{10,7} + C_{10,9} \ \underline{= 512 \quad subsets}$$

Exercise

How many subsets with more than two elements does a set with 100 elements have?

Solution

There are 2^{100} subsets of a set with 100 elements. All of them have more than 2 subsets except the empty set, the 100 subsets consisting of one element each, and $C_{100,\ 2} = 4950$ *subsets* with 2 elements.

Therefore; $2^{100} - 4950 = 1.26\times 10^{30}$

Exercise

How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other?

Solution

First position the men relative to each other. Since there are 8 men, there are $P(8,8)$ ways to do this.

This creates 9 slots where a woman may stand: in front of the first man, between the first and second men, …, between the 7^{th} and 8 men, and behind the 8^{th} man.
We need to choose 5 of these positions, in order, for the first through 5^{th} woman to occupy.

Therefore, $P(8,8)\cdot P(9,5) = 609,638,400 \ \ ways$

Exercise

How many ways are there for six men and 10 women to stand in a line so that no two men stand next to each other?

Solution

16	**15**	**14**	**13**	**12**	**11**	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**
W	W	W	W	W	M	W	M	W	M	W	M	W	M	W	M

Since there are 10 women, there are $P(10,10) = 3,628,800$

This creates 11 slots where a man may stand.
This can be done is $P(11,6) = 332,640$

Therefore $P(10,10)\cdot P(11,6) = 1,207,084,032,000 \ \ ways$

Exercise

A professor writes 40 discrete mathematics true/false questions of the statements in these questions. 17 are true. If the questions can be positioned in any order, how many different answer keys are possible?

Solution

$$C_{40,17} \underline{= 8.9\times 10^{10} \quad answers}$$

Exercise

Thirteen people on a softball team show up for a game.

a) How many ways are there to choose 10 players to take the field?

b) How many ways are there to assign the 10 positions by selecting players from the 13 people who show up?

c) Of the 13 people who show up, there are three women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?

Solution

a) $C_{13,10} \underline{= 286 \quad ways}$

b) The order in which the choices are made: $P_{13,10} \underline{= 1,037,836,800 \quad ways}$

c) There is only one way to choose the 10 players without choosing a woman, since there are exactly 10 men.

Therefore, there are $286-1 \underline{= 285 \quad ways}$ to choose the players if at least one of them must be a woman.

Exercise

A club has 25 members

a) How many ways are there to choose four members of the club to serve on an executive committee?

b) How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office?

Solution

a) Since the order of choosing the members is not relevant, we need to use a combination

$C(25,4) \underline{= 12,650 \quad ways}$

b) Since the order of choosing the members is matter, we need to use a permutation.

$P(25,4) \underline{= 303,600 \quad ways}$

Exercise

How many 4-permutations of the positive integers not exceeding 100 contain three consecutive integers, k, $k+1$, $k+2$, in the order

a) Where these consecutive integers can perhaps be separated by other integers in the permutation?

b) Where they are in consecutive positions in the permutation?

Solution

a) The consecutive numbers are 5, 6, 7, since can be separate by other integers the permutation can be written as 5, 6, 32, 7.

In order to specify such 4-permutation, we need to choose 3 consecutive integers. They can be {1, 2, 3} to (98, 99, 100}; thus, there are 98 possibilities. There are 4 possibilities, we need to decide which 97 other positive integers not exceeding 100 is to fill this slot, and there are 97 choices.

In fact, every 4-permutation consisting of 4 consecutive numbers, in order, has been double counted.

Therefore, we need to subtract the number of such 4-permutations. Clearly there are 97 of them. Further thought shows that every other 4-permutation in our collection arises in a unique way.

Therefore $98 \cdot 4 \cdot 97 - 97 = 37,927$

b) The consecutive numbers be consecutive in the 4-permutation.

There are only 2 places to put the fourth number in slot 1 and slot 4.

Therefore, $98 \cdot 2 \cdot 97 - 97 = 18,915$

Exercise

The English alphabet contains 21 constants and five vowels. How many strings of six lowercase letters of the English alphabet contain?

a) Exactly one vowel?

b) Exactly two vowels?

c) At least one vowel?

d) At least two vowels?

Solution

a) This can be done 6 ways. We need to choose the vowel and this can be done in 5 ways. Each other 5 positions can contain any of the 21 consonants, so there are 21^5 ways to fill the rest of the string.

Therefore, the answer is $6 \cdot 5 \cdot 21^5 = 122,533,030 \ \ ways$

b) The position of the vowels can be done in $C(6,\ 2) = 15 \ ways$. We need to choose the 2 vowels in 5^2 ways. Each other 4 positions can contain any of the 21 consonants, so there are 21^4 ways to fill the rest of the string.

Therefore, the answer is $15 \cdot 5^2 \cdot 21^4 = 72,930,375 \ \ ways$

c) Count the number of strings with no vowels and subtract this from the total number of stings.

$26^6 - 21^6 = 223,149,655 \ \ ways$

d) Subtracting the total number of strings from the number of strings with no vowels and the number of strings with one vowel. Answer: $26^6 - 21^6 - 6 \cdot 5 \cdot 21^5 \ \underline{= 100,626,625 \quad ways}|$

Exercise

Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have

a) The same number of men and women?

b) More women than men?

Solution

a) $C_{10,3} \cdot C_{15,3} \ \underline{= 54,600 \ \ ways}|$

b) There are $C_{15,6}$ *ways* to choose the committee to be composed only of women

$C_{15,5} \cdot C_{10,1}$ *ways* if there are to be five women and one man, and $C_{15,4} \cdot C_{10,2}$ *ways* if there are to be four women and two men.

Therefore, $C_{15,6} + C_{15,5} \cdot C_{10,1} + C_{15,4} \cdot C_{10,2} \ \underline{= 96,460 \quad ways}|$

Exercise

How many bit strings contain exactly eight 0s and 10 1s if every 0 must be immediately followed by a 1?

Solution

0 1	0 1	0 1	0 1	0 1	0 1	0 1	0 1	1	1

There are 8 blocks consisting of the string 01

$C_{10,2} \ \underline{= 45 \quad ways}|$

Exercise

How many bit strings contain exactly five 0s and 14 1s if every 0 must be immediately followed by two 1s?

Solution

Glue 2 1's to the right of each 0, giving a collection of 9 tokens: five 001's and four 1's.

$C_{9,4} \ \underline{= 126 \quad ways}|$

Exercise

A concert to raise money for an economics prize is to consist of 5 works; 2 overtures, 2 sonatas, and a piano concerto.

a) In how many ways can the program be arranged?
b) In how many ways can the program be arranged if an overture must come first?

Solution

a) $P(5,5) \underline{= 120 \quad ways}|$

b) $P(2,1)\cdot P(4,4) \underline{= 48 \quad ways}|$

Exercise

A zydeco band from Louisiana will play 5 traditional and 3 original Cajun compositions at a concert. In how many ways can they arrange the program if

a) The begin with a traditional piece?
b) An original piece will be played last?

Solution

a) $P(5,1)\cdot P(7,7) \underline{= 25,200 \quad ways}|$

b) $P(7,7)\cdot P(3,1) \underline{= 15,120 \quad ways}|$

Exercise

In an election with 3 candidates for one office and 6 candidates for another office, how many different ballots may be printed?

Solution

Office **1**: $P(3,3)$

Office **2**: $P(6,6)$

Multiplication principle: $2\cdot P(3,3)P(6,6) \underline{= 8640}|$

Exercise

A business school gives courses in typing, shorthand, transcription, business English, technical writing, and accounting. In how many ways can a student arrange a schedule if 3 courses are taken? assume that the order in which courses are schedules matters.

Solution

$P(6,3) \underline{= 120 \quad ways}|$

Exercise

If your college offers 400 courses, 25 of which are in mathematics, and your counselor arranges your schedule of 4 courses by random selection, how many schedules are possible that do not include a math course? Assume that the order in which courses are scheduled matters.

Solution

$$P(nonmath) = P(375, 4) \ \underline{= 1.946 \times 10^{10}}$$

Exercise

A baseball team has 19 players. How many 9-player batting orders are possible?

Solution

$$P(19, 9) \ \underline{= 3.352 \times 10^{10}}$$

Exercise

A chapter of union Local 715 has 35 members. In how many different ways can the chapter select a president, a vice-president, a treasurer, and a secretary?

Solution

$$P(35, 4) \ \underline{= 1,256,640}$$

Exercise

An economics club has 31 members.

a) If a committee of 4 is to be selected, in how many ways can the selection be made?
b) In how many ways can a committee of at least 1 and at most 3 be selected?

Solution

a) $\underline{C_{31,4} = 31,465}$

b)
$$\begin{aligned} P(\text{at least } 1 \text{ and at most } 3 \text{ be selected}) &= C_{31,1} + C_{31,2} + C_{31,3} \\ &= 31 + 465 + 4495 \\ &\underline{= 4991} \end{aligned}$$

Exercise

In a club with 9 male and 11 female members, how many 5-member committees can be chosen that have

a) All men?
b) All women?
c) 3 men and 2 women?

Solution

a) $C(9,5) \underline{=126}$

b) $C(11,5) \underline{=462}$

c) $C(9,3)\cdot C(11,2)=(84)(55) \underline{=4,620}$

Exercise

In a club with 9 male and 11 female members, how many 5-member committees can be selected that have

a) At least 4 women?
b) No more than 2 men?

Solution

a) $C(11,4)\cdot C(9,1)+C(11,5)\cdot C(9,0) \underline{=3,432}$

b) $C(9,0)\cdot C(11,5)+C(9,1)\cdot C(11,4)+C(9,2)\cdot C(11,3) \underline{=9,372}$

Exercise

In a game of musical chairs, 12 children will sit in 11 chairs arranged in a row (one will be left out). In how many ways can this happen, if we count rearrangements of the children in the chairs as different outcomes?

Solution

$P(12,11) \underline{=479,001,600 \quad \textit{different outcomes}}$

Exercise

A group of 3 students is to be selected from a group of 14 students to take part in a class in cell biology.

a) In how many ways can this be done?
b) In how many ways can the group who will not take part be chosen?

Solution

a) $\binom{14}{3} \underline{=364 \quad \textit{ways}}$

b) $\binom{14}{11} \underline{=364 \quad \textit{ways}}$

Exercise

Marbles are being drawn without replacement from a bag containing 16 marbles.

a) How many samples of 2 marbles can be drawn?
b) How many samples of 2 marbles can be drawn?
c) If the bag contains 3 yellow, 4 white, and 9 blue marbles, how many samples of 2 marbles can be drawn in which both marbles are blue?

Solution

a) $C(16,2) = 120 \ \ samples$

b) $C(16,4) = 1820 \ \ samples$

c) $C(9,2) = 36 \ \ samples$

Exercise

A bag contains 5 black, 1 red, and 3 yellow jelly beans; you take 3 at random. How many samples are possible in which the jelly beans are

a) All black?
b) All red?
c) All yellow?
d) 2 black and 1 red?
e) 2 black and 1 yellow?
f) 2 yellow and 1 black?
g) 2 red and 1 yellow?

Solution

a) $C_{5,3} = 10$

b) No 3 read. $C_{1,3} = 0$

c) $C_{3,3} = 1$

d) $C_{5,2}C_{1,1} = 10$

e) $C_{5,2}C_{3,1} = 30$

f) $C_{3,2}C_{5,1} = 15$

g) *There is only* **1** ***red***.