Section 1.8 – Applications

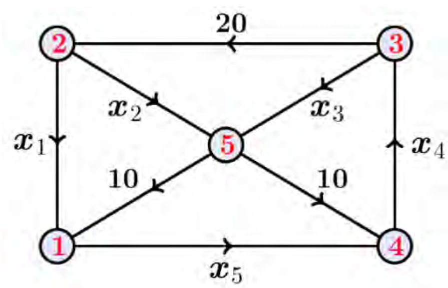
Network Analysis

Networks composed of branches and junctions are used as models in such fields as economics, traffic analysis, and electrical engineering.

In a network model, you assume that the total flow into a junction is equal to the total flow out of the junction

Example

Set up a system of linear equations to represent the network shown below. Then solve the system for $x_i, \quad i = 1,\ 2,\ 3,\ 4,\ 5.$



Solution

$1 \rightarrow \quad x_1 + 10 = x_5 \quad \Rightarrow \quad x_1 - x_5 = -10$

$2 \rightarrow \quad x_1 + x_2 = 20$

$3 \rightarrow \quad x_4 = x_3 + 20 \quad \Rightarrow \quad -x_3 + x_4 = 20$

$4 \rightarrow \quad x_4 = x_5 + 10 \quad \Rightarrow \quad x_4 - x_5 = 10$

$5 \rightarrow \quad x_2 + x_3 = 10 + 10 = 20$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 1 & 1 & 0 & 0 & 20 \end{array}\right) \quad \begin{array}{l} \\ R_2 - R_1 \\ \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 1 & 1 & 0 & 0 & 20 \end{array}\right) \quad \begin{array}{l} \\ \\ \\ \\ R_5 - R_2 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 1 & 0 & -1 & -10 \end{array}\right) \quad R_5 + R_3$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 10 \end{array}\right) \quad R_5 - R_4$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{ll} \to x_1 - x_5 = -10 & \Rightarrow \boxed{x_1 = x_5 - 10} \\ \to x_2 + x_5 = 30 & \Rightarrow \boxed{x_2 = 30 - x_5} \\ \to -x_3 + x_4 = 20 & \Rightarrow \boxed{x_3 = x_5 - 10} \\ \to x_4 - x_5 = 10 & \Rightarrow \boxed{x_4 = x_5 + 10} \end{array}$$

Solution: $\boxed{\left(x_5 - 10,\ \ 30 - x_5,\ \ x_5 - 10,\ \ 10 + x_5,\ \ x_5\right)}$

2nd Method

$$\begin{vmatrix} 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{vmatrix} - 1 \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= -1 + 1$$

$$\boxed{= 0}$$

Infinite solution:

$1 \to \ \boxed{x_1 = x_5 - 10}$

$2 \to \ x_2 = 20 - x_1 \ \boxed{= 30 - x_5}$

$4 \to \ \boxed{x_4 = x_5 + 10}$

$3 \to \ x_3 = x_4 - 20 \ \boxed{= x_5 - 10}$

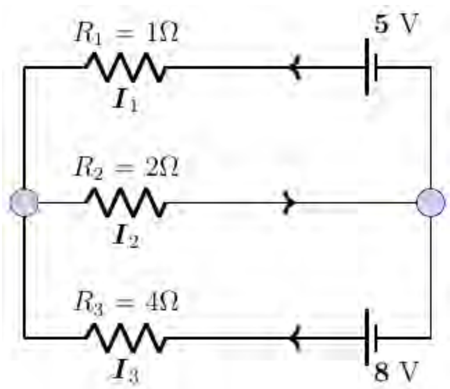
Electrical network

An electrical network is another type of network where analysis is commonly applied. An analysis of such a system uses two properties of electrical networks known as **Kirchhoff's Laws.**

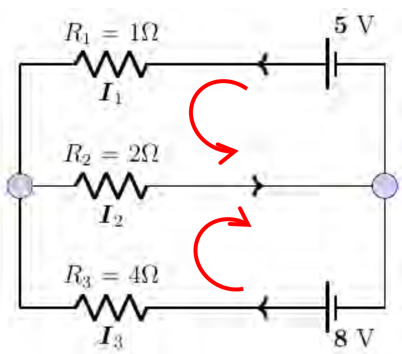
- All the current flowing into a junction must flow out of it.
- The sum of the products *IR* (*I* is current and *R* is resistance) around a closed path is equal to the total voltage in the path.

Example

Determine the currents I_1, I_2, and I_3 for the electrical network



Solution



$$I_2 = I_1 + I_3$$
$$I_1 + 2I_2 = 5$$
$$2I_2 + 4I_3 = 8$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ I_1 + 2I_2 \qquad = 5 \\ \qquad I_2 + 2I_3 = 4 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 7 \qquad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 5 & 2 & 0 \\ 4 & 1 & 2 \end{vmatrix} = 7 \qquad D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 5 & 0 \\ 0 & 4 & 2 \end{vmatrix} = 14 \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 5 \\ 0 & 1 & 4 \end{vmatrix} = 7$$

$\underline{I_1 = 1\ A}$ $\underline{I_2 = 2\ A}$ $\underline{I_3 = 1\ A}$

Cryptography

A ***cryptogram*** is a message written according to a secret code (the Greek word ***kryptos*** means "*hidden*"). One method of using matrix multiplication to ***encode*** and ***decode*** messages.

Let assign a number to each letter in the alphabet (with 0 assigned to a blank space), as shown

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

Example

Consider the invertible matrix: $A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix}$

The message: ***MEET ME MONDAY***

a) Write the uncoded row matrices 1×3 for the message.

b) Use the matrix A to encode the message.

c) Decode a message from part *b*) given the matrix *A*.

Solution

a)

$$\begin{array}{ccccc} M\ E\ E & T\ \_\ M & E\ \_\ M & O\ N\ D & A\ Y\ \_ \\ \begin{bmatrix}13 & 5 & 5\end{bmatrix} & \begin{bmatrix}20 & 0 & 13\end{bmatrix} & \begin{bmatrix}5 & 0 & 13\end{bmatrix} & \begin{bmatrix}15 & 14 & 4\end{bmatrix} & \begin{bmatrix}1 & 25 & 0\end{bmatrix} \end{array}$$

b) Let encode the message ***MEET ME MONDAY***

$$\begin{bmatrix}13 & 5 & 5\end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix}13 & -26 & 21\end{bmatrix}$$

$$\begin{bmatrix}20 & 0 & 13\end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix}33 & -53 & -12\end{bmatrix}$$

$$\begin{bmatrix}5 & 0 & 13\end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix}18 & -23 & -42\end{bmatrix}$$

$$\begin{bmatrix}15 & 14 & 4\end{bmatrix}\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}=\begin{bmatrix}5 & -20 & 56\end{bmatrix}$$

$$\begin{bmatrix}1 & 25 & 0\end{bmatrix}\begin{bmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{bmatrix}=\begin{bmatrix}-24 & 23 & 77\end{bmatrix}$$

The sequence of coded row matrices is

$$[13 \quad -26 \quad -21] \quad [33 \quad -53 \quad -12] \quad [18 \quad -23 \quad -42] \quad [5 \quad -20 \quad 56] \quad [-24 \quad 23 \quad 77]$$

The cryptogram:

$$13 \quad -26 \quad -21 \quad 33 \quad -53 \quad -12 \quad 18 \quad -23 \quad -42 \quad 5 \quad -20 \quad 56 \quad -24 \quad 23 \quad 77$$

c) To decode a message given the matrix A.

$$|A|=\begin{vmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{vmatrix}=1$$

$$A^{-1}=\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}$$

With the cryptogram:

$$[13 \quad -26 \quad -21] \quad [33 \quad -53 \quad -12] \quad [18 \quad -23 \quad -42] \quad [5 \quad -20 \quad 56] \quad [-24 \quad 23 \quad 77]$$

$$\begin{bmatrix}13 & -26 & 21\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}13 & 5 & 5\end{bmatrix}$$

$$\begin{bmatrix}33 & -53 & -1\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}20 & 0 & 13\end{bmatrix}$$

$$\begin{bmatrix}18 & -23 & -42\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}5 & 0 & 13\end{bmatrix}$$

$$\begin{bmatrix}5 & -20 & 56\end{bmatrix}\begin{bmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}15 & 14 & 4\end{bmatrix}$$

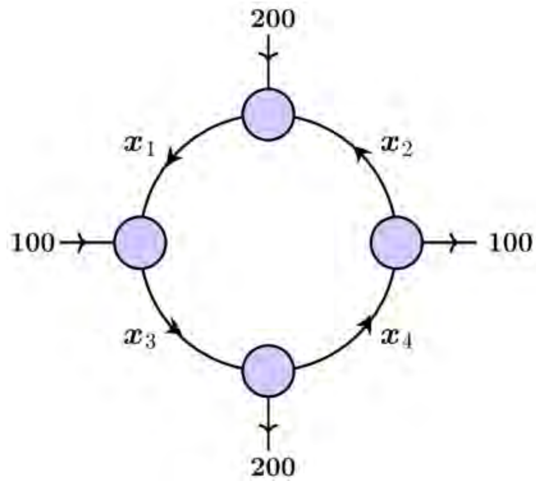
$$\begin{bmatrix} -24 & 23 & 77 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 25 & 0 \end{bmatrix}$$

The message is:

13	5	5	20	0	13	5	0	13	15	14	4	1	25	0
M	*E*	*E*	*T*	_	*M*	*E*	_	*M*	*O*	*N*	*D*	*A*	*Y*	_

Exercises *Section* 1.8 – Applications

1. The flow of traffic, in vehicles per hour, through a network of streets as is shown below



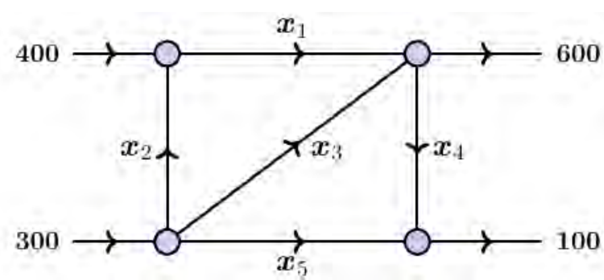
a) Solve this system for x_i, $i = 1, 2, 3, 4$.

b) Find the traffic flow when $x_4 = 0$.

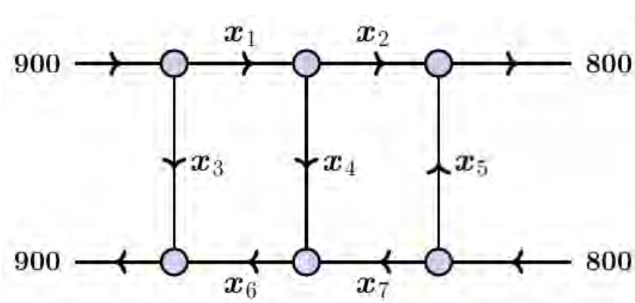
c) Find the traffic flow when $x_4 = 100$.

d) Find the traffic flow when $x_1 = 2x_2$.

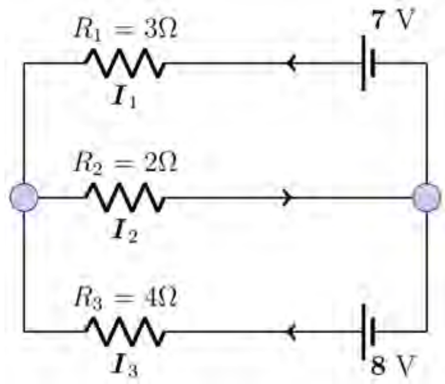
2. Through a network, Express x_n's in terms of the parameters ***s*** and ***t***.



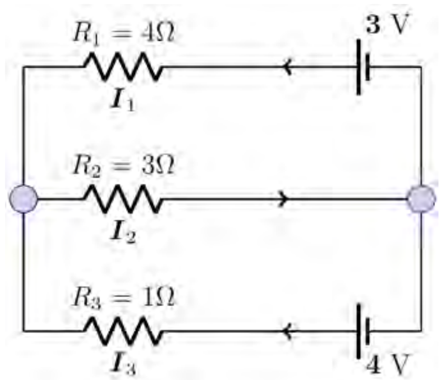
3. Water is flowing through a network of pipes. Express x_n's in terms of the parameters ***s*** and ***t***.



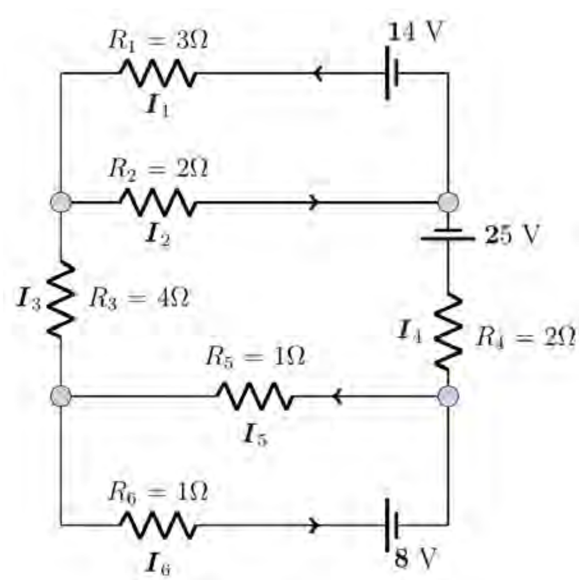
4. Determine the currents $I_1, I_2,$ *and* I_3 for the electrical network shown below



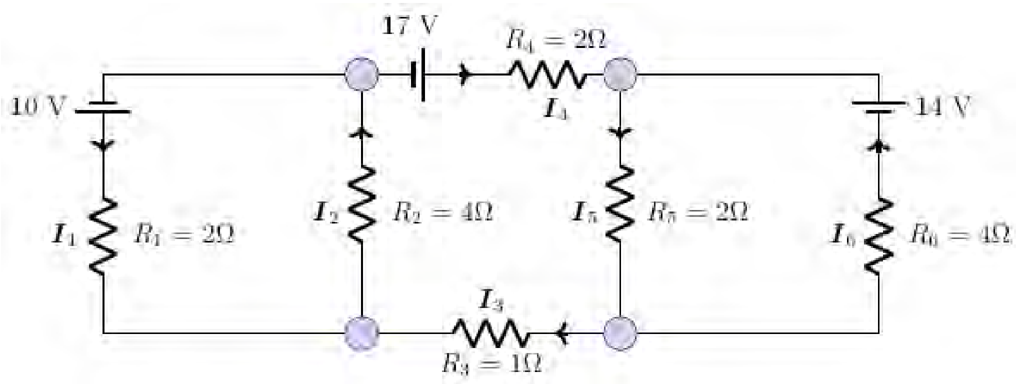
5. Determine the currents $I_1, I_2,$ *and* I_3 for the electrical network shown below



6. Determine the currents $I_1, I_2, I_3, I_4, I_5,$ *and* I_6 for the electrical network shown below



7. Determine the currents $I_1, I_2, I_3, I_4, I_5, \text{ and } I_6$ for the electrical network shown below



8. Consider the invertible matrix: $A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -2 & -2 & 7 \end{pmatrix}$

The message: ***ICEBERG DEAD AHEAD***

a) Write the uncoded row matrices 1×3 for the message.
b) Use the matrix *A* to encode the message.
c) Decode a message from part *b*) given the matrix *A*.

9. You want to send the message: ***LINEAR ALGEBRA*** with a key word ***MATH***

a) Write the matrix *A*.
b) Write the uncoded row matrices 1×2 for the message.
c) Use the matrix *A* to encode the message.
d) Decode a message from part *b*) given the matrix *A*.

10. You want to send the message: ***CRYPTOGRAPHY IS A METHOD OF PROTECTING INFORMATIONS*** with a key word ***CODE***

a) Write the matrix *A*.
b) Write the uncoded row matrices 1×2 for the message.
c) Use the matrix *A* to encode the message.
d) Decode a message from part *b*) given the matrix *A*.

11. Write the matrix *A* with a key word ***MATH,*** then decode the cryptogram

117 9 456 132 386 62 260 104 413 161 104 8

12. Write the matrix *A* with a key word ***MATH,*** then decode the cryptogram

438 150 145 37 240 96 635 191 445 157 260 104 413 161 104 8

13. Consider the invertible matrix: $A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix}$

Decode the cryptogram

1 –5 11 19 –25 –45 11 –16 –28 20 –29 –27

12 –12 –53 40 –61 –35 8 –17 7

14. Determine the key word, then decode the given cryptogram

6 18 5 4 15 13 1 20 8

102 649 238 57 324 112 128 622 207

180 613 290 102 360 259 151 580 297

Hint: First row is the key

15. Determine the key word, then decode the given cryptogram

5	17	21	1	20	9	15	14	19
259	863	783	77	378	357	301	448	565
106	266	318	325	365	485	301	522	653
326	653	738	103	566	495	115	640	555
290	791	762	115	474	507	119	332	279
305	454	513	339	645	611	226	341	426
260	338	368	406	657	830	270	649	590
110	337	418	74	318	330	261	561	469
114	426	390	160	543	372	89	535	441
323	842	783	97	344	245	84	601	444
424	851	944	175	262	339	379	698	755
226	341	426	37	454	217	156	694	536