

## Section 3.5 – The Ratio and Root Tests

### **Theorem – The Ratio Test**

Let  $\sum a_n$  be a series with positive terms and suppose that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \rho$$

Then

- a) the series *converges* if  $\rho < 1$ ,
- b) the series *diverges* if  $\rho > 1$ , or  $\rho$  is infinite
- c) the test is *inconclusive* if  $\rho = 1$ ,

The value  $\rho$  doesn't mean the sum of the series.

### **Example**

Investigate the convergence of the series  $\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$

### **Solution**

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1} + 5}{3^{n+1}}}{\frac{2^n + 5}{3^n}}$$

$$= \frac{1}{3} \frac{2^{n+1} + 5}{2^n + 5}$$

$$= \frac{1}{3} \cdot \frac{2 + 5 \cdot 2^{-n}}{1 + 5 \cdot 2^{-n}}$$

$$\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{3} \cdot \frac{2 + 5 \cdot 2^{-n}}{1 + 5 \cdot 2^{-n}}$$

$$= \frac{1}{3} \cdot \frac{2}{1}$$

$$= \frac{2}{3} < 1$$

The series *converges* since  $\rho < 1$ .

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} &= \sum_{n=0}^{\infty} \frac{2^n}{3^n} + \sum_{n=0}^{\infty} \frac{5}{3^n} \\
&= \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty} \frac{5}{3^n} \\
&= \frac{1}{1 - \frac{2}{3}} + \frac{5}{1 - \frac{1}{3}} \\
&= \frac{21}{2}
\end{aligned}$$

### Example

Investigate the convergence of the series  $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

### Solution

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\frac{(2(n+1))!}{(n+1)!(n+1)!}}{\frac{(2n)!}{n!n!}} \\
&= \frac{1}{(n+1)(n+1)} \frac{(2n+2)!}{(2n)!} \\
&= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \\
&= \frac{2(n+1)(2n+1)}{(n+1)(n+1)} \\
&= \frac{4n+1}{n+1}
\end{aligned}$$

$$\begin{aligned}
\rho &= \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n \rightarrow \infty} \frac{4n+1}{n+1} \\
&= 4 > 1
\end{aligned}$$

The series *diverges* since  $\rho > 1$ .

### Example

Investigate the convergence of the series  $\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$

### Solution

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{4^{n+1} (n+1)! (n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{4^n n! n!} \\ &= \frac{4(n+1)(n+1)}{(2n+2)(2n+1)} \\ &= \frac{4(n+1)}{2(2n+1)} \\ &= \frac{2(n+1)}{2n+1} \Big| \end{aligned}$$

$$\begin{aligned}\rho &= \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \\ &= \lim_{n \rightarrow \infty} \frac{2n+2}{2n+1} \\ &= 1 \Big| \end{aligned}$$

Because the limit is  $\rho = 1$ , we can't decide from the *Ratio Test* whether the series converges.

However, since  $a_{n+1}$  *always*  $>$   $a_n$ , then the series *diverges*.

$$\begin{aligned}a_{n+1} & \stackrel{?}{>} a_n \\ \frac{4^{n+1} (n+1)! (n+1)!}{(2(n+1))!} & \stackrel{?}{>} \frac{4^n n! n!}{(2n)!} \\ \frac{4^{n+1} (n+1)! (n+1)!}{4^n n! n!} & \stackrel{?}{>} \frac{(2(n+1))!}{(2n)!} \\ 4(n+1)(n+1) & \stackrel{?}{>} (2n+1)(2n+2) \\ 4(n+1)(n+1) & \stackrel{?}{>} 2(n+1)(2n+1) \\ 2n+2 & \stackrel{?}{>} 2n+1 \\ a_{n+1} & \stackrel{?}{>} a_n \end{aligned}$$

### ***Theorem – The Root Test***

Let  $\sum a_n$  be a series with  $a_n \geq 0$  for  $n \geq N$ , and suppose that

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \rho$$

Then

- a) the series *converges* if  $\rho < 1$ ,
- b) the series *diverges* if  $\rho > 1$ , or  $\rho$  is infinite
- c) the test is *inconclusive* if  $\rho = 1$ ,

### ***Example***

Determine if the series  $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$  converges or diverges using the Root Test

#### **Solution**

$$\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}}$$

$$= \frac{\left(\sqrt[n]{n}\right)^2}{2}$$

$$\rho = \lim_{n \rightarrow \infty} \sqrt[n]{a_n}$$

$$= \lim_{n \rightarrow \infty} \frac{n^{2/n}}{2}$$

$$= \frac{\infty^0}{2}$$

$$= \frac{1}{2} < 1$$

The series *converges* by the *Root Test*.

### ***Example***

Determine if the series  $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$  converges or diverges using the Root Test

#### **Solution**

$$\begin{aligned}\sqrt[n]{a_n} &= \sqrt[n]{\frac{2^n}{n^3}} \\ &= \frac{\sqrt[n]{2^n}}{\sqrt[n]{n^3}} \\ &= \frac{2}{\left(\sqrt[n]{n}\right)^3}\end{aligned}$$

$$\begin{aligned}\rho &= \lim_{n \rightarrow \infty} \frac{2}{\left(\sqrt[n]{n}\right)^3} \\ &= \frac{2}{1} \\ &= 2 > 1\end{aligned}$$

The series *diverges* by the *Root Test*.

### Example

Determine if the series  $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$  converges or diverges using the Root Test

### Solution

$$\begin{aligned}\sqrt[n]{\left(\frac{1}{1+n}\right)^n} &= \frac{1}{1+n} \\ \lim_{n \rightarrow \infty} \frac{1}{1+n} &= 0 < 1\end{aligned}$$

The series *converges* by the *Root Test*.

## Exercises      Section 3.5 – The Ratio and Root Tests

Use the **Ratio Test** to determine if the series converges or diverges.

1.  $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

2.  $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}}$

3.  $\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$

4.  $\sum_{n=1}^{\infty} \frac{n^2(n+2)!}{n!3^{2n}}$

5.  $\sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)\ln(n+1)}$

6.  $\sum_{n=1}^{\infty} \frac{99^n}{n!}$

7.  $\sum_{n=1}^{\infty} \frac{n^5}{2^n}$

8.  $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

9.  $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

10.  $\sum_{n=1}^{\infty} \frac{1}{5^n}$

11.  $\sum_{n=1}^{\infty} \frac{1}{n!}$

12.  $\sum_{n=0}^{\infty} \frac{n!}{3^n}$

13.  $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

14.  $\sum_{n=1}^{\infty} n\left(\frac{6}{5}\right)^n$

15.  $\sum_{n=1}^{\infty} n\left(\frac{7}{8}\right)^n$

16.  $\sum_{n=1}^{\infty} \frac{n}{4^n}$

17.  $\sum_{n=1}^{\infty} \frac{5^n}{n^4}$

18.  $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

19.  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

20.  $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

21.  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\frac{3}{2}\right)^n}{n^2}$

22.  $\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

23.  $\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

Use the **Root Test** to determine if the series converges or diverges.

24.  $\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$

25.  $\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$

26.  $\sum_{n=1}^{\infty} \left(\ln\left(e^2 + \frac{1}{n}\right)\right)^{n+1}$

27.  $\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$

28.  $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$

29.  $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

30.  $\sum_{n=1}^{\infty} \frac{1}{5^n}$

31.  $\sum_{n=1}^{\infty} \frac{1}{n^n}$

32.  $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

$$33. \sum_{n=1}^{\infty} \left( \frac{2n}{n+1} \right)^n$$

$$34. \sum_{n=1}^{\infty} \left( \frac{3n+2}{n+3} \right)^n$$

$$35. \sum_{n=1}^{\infty} \left( \frac{n-2}{5n+1} \right)^n$$

$$36. \sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$$

$$37. \sum_{n=1}^{\infty} \left( \frac{-3n}{2n+1} \right)^{3n}$$

$$38. \sum_{n=1}^{\infty} \left( 2\sqrt[n]{n} + 1 \right)^n$$

$$39. \sum_{n=0}^{\infty} e^{-3n}$$

$$40. \sum_{n=1}^{\infty} \frac{n}{3^n}$$

$$41. \sum_{n=1}^{\infty} \left( \frac{n}{500} \right)^n$$

$$42. \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n^2} \right)^n$$

$$43. \sum_{n=1}^{\infty} \left( \frac{\ln n}{n} \right)^n$$

$$44. \sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$$

Use any method to determine if the series converges or diverges.

$$45. \sum_{n=1}^{\infty} \frac{n\sqrt{2}}{2^n}$$

$$46. \sum_{n=1}^{\infty} n^2 e^{-n}$$

$$47. \sum_{n=1}^{\infty} \frac{n!}{10^n}$$

$$48. \sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$$

$$49. \sum_{n=1}^{\infty} \frac{n2^n (n+1)!}{3^n n!}$$

$$60. \sum_{n=3}^{\infty} \frac{1}{n(\ln n)\sqrt{\ln \ln n}}$$

$$61. \sum_{n=1}^{\infty} \frac{1+(-1)^n}{\sqrt{n}}$$

$$62. \sum_{n=1}^{\infty} \frac{n!}{n^2 e^n}$$

$$50. \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$$

$$51. \sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$$

$$52. \sum_{n=1}^{\infty} \left| \sin \frac{1}{n^2} \right|$$

$$53. \sum_{n=8}^{\infty} \frac{1}{\pi^n + 5}$$

$$54. \sum_{n=2}^{\infty} \frac{1}{(\ln n)^3}$$

$$63. \sum_{n=1}^{\infty} \frac{(2n)!6^n}{(3n)!}$$

$$64. \sum_{n=2}^{\infty} \frac{\sqrt{n}}{3^n \ln n}$$

$$65. \sum_{n=0}^{\infty} \frac{n^{100} 2^n}{\sqrt{n}!}$$

$$55. \sum_{n=1}^{\infty} \frac{1}{\pi^n - n^\pi}$$

$$56. \sum_{n=0}^{\infty} \frac{1+n}{2+n}$$

$$57. \sum_{n=1}^{\infty} \frac{1+n^{4/3}}{2+n^{5/3}}$$

$$58. \sum_{n=1}^{\infty} \frac{n^2}{1+n\sqrt{n}}$$

$$59. \sum_{n=2}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)^2}$$

$$66. \sum_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$$

$$67. \sum_{n=1}^{\infty} \frac{2^n}{3^n - n^3}$$

$$68. \sum_{n=1}^{\infty} \frac{n^n}{\pi^n n!}$$

69.  $\sum_{n=0}^{\infty} \frac{2^n}{n!}$
70.  $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$
71.  $\sum_{n=1}^{\infty} \frac{n^n}{n!}$
72.  $\sum_{n=1}^{\infty} \frac{100}{n}$
73.  $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$
74.  $\sum_{n=1}^{\infty} \left(\frac{2\pi}{3}\right)^n$
75.  $\sum_{n=1}^{\infty} \frac{5n}{2n-1}$
76.  $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$
77.  $\sum_{n=1}^{\infty} (-1)^n \frac{3^{n-2}}{2^n}$
78.  $\sum_{n=1}^{\infty} \frac{10}{3\sqrt[3]{n^3}}$
79.  $\sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$
80.  $\sum_{n=1}^{\infty} \frac{2^n}{4n^2-1}$
81.  $\sum_{n=1}^{\infty} \frac{\cos n}{3^n}$
82.  $\sum_{n=1}^{\infty} \frac{n!}{n 7^n}$
83.  $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$
84.  $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$
85.  $\sum_{k=1}^{\infty} \left(\frac{1}{\ln(k+1)}\right)^k$
86.  $\sum_{k=2}^{\infty} \frac{1}{k^2(\ln k)^2}$
87.  $\sum_{k=3}^{\infty} \frac{1}{\ln k}$
88.  $\sum_{k=2}^{\infty} \frac{5 \ln k}{k}$
89.  $\sum_{k=1}^{\infty} \ln\left(\frac{k+2}{k+1}\right)$
90.  $\sum_{k=2}^{\infty} \frac{1}{k^2 \ln k}$
91.  $\sum_{k=2}^{\infty} \frac{1}{k^{\ln k}}$
92.  $\sum_{n=1}^{\infty} \frac{n!}{n^n}$
93.  $\frac{1+\sqrt{2}}{2} + \frac{1+\sqrt{3}}{4} + \frac{1+\sqrt{4}}{8} + \dots$
94.  $\sum_{n=1}^{\infty} \frac{(-3)^n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$
95.  $\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{18^n (2n-1)n!}$

96. Use the integral test to show that  $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$  converges. Show that the sum  $s$  of the series is less than

$$\frac{\pi}{2}$$

97. Use the root test to show that  $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$  converges

98. Use the root test to test that  $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$  converges



99. Try to use the ratio test to determine whether  $\sum_{n=1}^{\infty} \frac{2^{2n} (n!)^2}{(2n)!}$  converges. What happens?

$$\begin{aligned} \text{Now observe that } \frac{2^{2n} (n!)^2}{(2n)!} &= \frac{[2n(2n-2)(2n-4) \cdots 6 \times 4 \times 2]^2}{2n(2n-1)(2n-2) \cdots 3 \times 2 \times 1} \\ &= \frac{2n}{2n-1} \times \frac{2n-2}{2n-3} \times \frac{4}{3} \times \frac{2}{1} \end{aligned}$$

Does the given series converge? Why or why not?

100. Suppose  $a_n > 0$  and  $\frac{a_{n+1}}{a_n} \geq \frac{n}{n+1}$  for all  $n$ . Show that  $\sum_{n=1}^{\infty} a_n$  diverges.

$$\left( a_n \geq \frac{K}{n} \text{ for some constant } K \right)$$

101. Working in the early 1600s, the mathematicians Wallis, Pascal, and Fermat were calculating the area of the region under the curve  $y = x^p$  between  $x = 0$  and  $x = 1$ , where  $p$  is the positive integer. Using arguments that predated the Fundamental Theorem of Calculus, they were able to prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left( \frac{k}{n} \right)^p = \frac{1}{p+1}$$

Use Riemann sums and integrals to verify this limit.

102. Complete the following steps to find the values of  $p > 0$  for which the series  $\sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{p^k k!}$

converges

a) Use the Ratio Test to show that  $\sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{p^k k!}$  converges for  $p > 2$ .

- b) Use Stirling's formula,  $k! = \sqrt{2\pi k} k^k e^{-k}$  for large  $k$ , to determine whether the series converges when  $p = 2$ .

$$\left( \text{Hint : } 1 \cdot 3 \cdot 5 \cdots (2k-1) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2k-1) 2k}{2 \cdot 4 \cdot 6 \cdots 2k} \right)$$