Section 4.3 – Multiplicative Inverses of Matrices

Identity Matrix

The $n \times n$ identity matrix with 1's on the main diagonal and 0's elsewhere and is denoted by

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2x2} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3x3} \qquad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{I} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

The Multiplicative Identity Matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then $AI = IA = A$

Example

$$A = \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution

$$AI = \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4(1) - 7(0) & 4(0) - 7(1) \\ -3(1) + 2(0) & -3(0) + 2(1) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix} = A$$

$$= A$$

Multiplicative inverse of a matrix

Multiplicative inverse of a matrix $A_{n\times n}$ and $A^{-1}_{n\times n}$ if exists, then:

$$A\cdot A^{-1}=A^{-1}\cdot A=I$$

Example

Show that B is Multiplicative inverse of A.

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $\qquad$ $B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

Solution

$$\boldsymbol{A.B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2(1)-1(1) & 2(-1)+1(2) \\ 1(1)+1(-1) & 1(-1)+1(2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

$$BA = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

$\therefore$ B is multiplicative inverse of a matrix A: $B = A^{-1}$

Finding Inverse matrix

To find inverse matrix using Gauss-Jordan method:

$$\left[A\middle|I\right] \;\rightarrow\; \left[I\middle|A^{-1}\right] \qquad \textit{where } A^{-1} \textit{ read as "}A \textit{ inverse"}$$

For 2 by 2 matrices (***only***)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

If $ad-bc=0$, then A^{-1} doesn't exist

Example

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix} \;\Rightarrow\; A^{-1} = ?$$

Solution

$$A^{-1} = \frac{1}{(-1)(4)-(-2)(3)}\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \Rightarrow A^{-1} = ?$$

Solution

$$A^{-1} = \frac{1}{(3)(1)-(-2)(-1)} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

To find inverse matrix using Gauss-Jordan method:

$$\left[A|I\right] \rightarrow \left[I|A^{-1}\right] \qquad \textit{where } A^{-1} \textit{ read as "A inverse"}$$

Example

$$A=\begin{bmatrix}1 & 0 & 2\\ -1 & 2 & 3\\ 1 & -1 & 0\end{bmatrix} \qquad \text{Find } A^{-1}$$

Solution

$$\left[\begin{array}{ccc|ccc}1 & 0 & 2 & 1 & 0 & 0\\ -1 & 2 & 3 & 0 & 1 & 0\\ 1 & -1 & 0 & 0 & 0 & 1\end{array}\right] \begin{matrix} \\ R_2+R_1\\ R_3-R_1\end{matrix}$$

$$\begin{array}{cccccc}-1 & 2 & 3 & 0 & 1 & 0\\ 1 & 0 & 2 & 1 & 0 & 0\\ \hline 0 & 2 & 5 & 1 & 1 & 0\end{array} \qquad \begin{array}{cccccc}1 & -1 & 0 & 0 & 0 & 1\\ -1 & 0 & -2 & -1 & 0 & 0\\ \hline 0 & -1 & -2 & -1 & 0 & 1\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 2 & 1 & 0 & 0\\ 0 & 2 & 5 & 1 & 1 & 0\\ 0 & -1 & -2 & -1 & 0 & 1\end{array}\right] \tfrac{1}{2}R_2$$

$$0 \quad 1 \quad \tfrac{5}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2} \quad 0$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 2 & 1 & 0 & 0\\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0\\ 0 & -1 & -2 & -1 & 0 & 1\end{array}\right] R_3+R_2$$

$$\begin{array}{cccccc}0 & -1 & -2 & -1 & 0 & 1\\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0\\ \hline 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 2 & 1 & 0 & 0\\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1\end{array}\right] 2R_3$$

$$0 \quad 0 \quad 1 \quad -1 \quad 1 \quad 2$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 2 & 1 & 0 & 0\\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 1 & -1 & 1 & 2\end{array}\right] \begin{matrix} R_1-2R_3\\ R_2-\frac{5}{2}R_3\\ \ \end{matrix}$$

$$\begin{array}{cccccc}0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & -\frac{5}{2} & \frac{5}{2} & -\frac{5}{2} & -5\\ \hline 0 & 1 & 0 & 3 & -2 & -5\end{array} \qquad \begin{array}{cccccc}1 & 0 & 2 & 1 & 0 & 0\\ 0 & 0 & -2 & 2 & -2 & -4\\ \hline 1 & 0 & 0 & 3 & -2 & -4\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 3 & -2 & -4\\ 0 & 1 & 0 & 3 & -2 & -5\\ 0 & 0 & 1 & -1 & 1 & 2\end{array}\right]$$

$$\Rightarrow \qquad A^{-1}=\begin{bmatrix}3 & -2 & -4\\ 3 & -2 & -5\\ -1 & 1 & 2\end{bmatrix}$$

Solving a System Using A^{-1}

To solve the matrix equation $AX = B$.

- **X**: matrix of the variables
- **A**: Coefficient matrix
- **B**: Constant matrix

$$AX = B$$

$$A^{-1}(AX) = A^{-1}B \qquad \textit{Multiply both side by } A^{-1}$$

$$\left(A^{-1}A\right)X = A^{-1}B \qquad \textit{Associate property}$$

$$IX = A^{-1}B \qquad \textit{Multiplicative inverse property}$$

$$X = A^{-1}B \qquad \textit{Identity property}$$

Example

Solve the system using A^{-1}

$$\begin{aligned} x \quad\quad + 2z &= 6 \\ -x + 2y + 3z &= -5 \\ x - y \quad\quad &= 6 \end{aligned} \qquad \text{Given } A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

Solution

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} \underbrace{=}_{=} \underbrace{\begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix}}_{B}$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} 3(6)-2(-5)-4(6) \\ 3(6)-2(-5)-5(6) \\ -1(6)+1(-5)+2(6) \end{bmatrix} = \begin{bmatrix} 18+10-24 \\ 18+10-30 \\ -6-5+12 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

Solution: $\{(4, -2, 1)\}$

Example

Use the inverse of the coefficient matrix to solve the linear system

$$\begin{aligned} 2x - 3y &= 4 \\ x + 5y &= 2 \end{aligned}$$

Solution

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The solution of the system is $(2, 0)$

Exercise Section 4.3 – Multiplicative Inverses of Matrices

Show that B is Multiplicative inverse of A

1. $A=\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$ $B=\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$

2. $A=\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ & $B=\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$

Find the inverse, if exists, of

3. $A=\begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

4. $A=\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$

5. $A=\begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$

6. $A=\begin{bmatrix} a & b \\ 3 & 3 \end{bmatrix}$

7. $A=\begin{bmatrix} -2 & a \\ 4 & a \end{bmatrix}$

8. $A=\begin{bmatrix} 4 & 4 \\ b & a \end{bmatrix}$

9. $A=\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$

10. $A=\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

11. $A=\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

12. $A=\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

13. $A=\begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$

14. $A=\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$

15. $A=\begin{pmatrix} -6 & 9 \\ 2 & -3 \end{pmatrix}$

16. $A=\begin{pmatrix} -2 & 7 \\ 0 & 2 \end{pmatrix}$

17. $A=\begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$

18. $A=\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

19. $A=\begin{pmatrix} 2 & 1 \\ a & a \end{pmatrix}$

20. $A=\begin{pmatrix} b & 3 \\ b & 2 \end{pmatrix}$

21. $A=\begin{pmatrix} 1 & a \\ 3 & a \end{pmatrix}$

22. $A=\begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$

23. $A=\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$

24. $A=\begin{pmatrix} -3 & \frac{1}{2} \\ 6 & -1 \end{pmatrix}$

25. $A=\begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

26. $A=\begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

27. $A=\begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

28. $A=\begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$

29. $A=\begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}$

30. $A=\begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$

31. $A=\begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

32. $A=\begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 1 & 2 \end{pmatrix}$

33. $A=\begin{pmatrix}3 & 3 & 1\\ 1 & 2 & 1\\ 2 & -1 & 1\end{pmatrix}$

34. $A=\begin{pmatrix}-3 & 1 & -1\\ 1 & -4 & -7\\ 1 & 2 & 5\end{pmatrix}$

35. $A=\begin{pmatrix}1 & 1 & -3\\ 2 & -4 & 1\\ -5 & 7 & 1\end{pmatrix}$

36. $A=\begin{pmatrix}1 & 2 & -1\\ 3 & 5 & 3\\ 2 & 4 & 3\end{pmatrix}$

37. $A=\begin{bmatrix}-2 & -3 & 4 & 1\\ 0 & 1 & 1 & 0\\ 0 & 4 & -6 & 1\\ -2 & -2 & 5 & 1\end{bmatrix}$

38. $A=\begin{bmatrix}1 & -14 & 7 & 38\\ -1 & 2 & 1 & -2\\ 1 & 2 & -1 & -6\\ 1 & -2 & 3 & 6\end{bmatrix}$

39. $A=\begin{bmatrix}10 & 20 & -30 & 15\\ 3 & -7 & 14 & -8\\ -7 & -2 & -1 & 2\\ 4 & 4 & -3 & 1\end{bmatrix}$

State the conditions under which A^{-1} exists. Then find a formula for A^{-1}

40. $A=[x]$

41. $A=\begin{bmatrix}x & 0\\ 0 & y\end{bmatrix}$

42. $A=\begin{bmatrix}0 & 0 & x\\ 0 & y & 0\\ z & 0 & 0\end{bmatrix}$

43. $A=\begin{bmatrix}x & 1 & 1 & 1\\ 0 & y & 0 & 0\\ 0 & 0 & z & 0\\ 0 & 0 & 0 & w\end{bmatrix}$

44. Solve the system using A^{-1} $\begin{cases}x \qquad + 2z = 6\\ -x+2y+3z=-5\\ x-y \qquad = 6\end{cases}$ Given $A^{-1}=\begin{bmatrix}3 & -2 & -4\\ 3 & -2 & -5\\ -1 & 1 & 2\end{bmatrix}$

45. Solve the system using A^{-1}

$$\begin{cases}x+2y+5z=2\\ 2x+3y+8z=3\\ -x+y+2z=3\end{cases}$$

a) Write the linear system as a matrix equation in the form *AX*= *B*
b) Solve the system using the inverse that is given for the coefficient matrix

the inverse of $\begin{bmatrix}1 & 2 & 5\\ 2 & 3 & 8\\ -1 & 1 & 2\end{bmatrix}$ *is* $\begin{bmatrix}2 & -1 & -1\\ 12 & -7 & -2\\ -5 & 3 & 1\end{bmatrix}$

46. Solve the system using A^{-1}

$$\begin{cases} x - y + z = 8 \\ 2y - z = -7 \\ 2x + 3y = 1 \end{cases}$$

a) Write the linear system as a matrix equation in the form $AX = B$

b) Solve the system using the inverse that is given for the coefficient matrix

$$\textit{the inverse is } \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

(47– 75) Use the ***inverse*** of the coefficient matrix to solve the linear system

47. $\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$

48. $\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$

49. $\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$

50. $\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$

51. $\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$

52. $\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$

53. $\begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$

54. $\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$

55. $\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$

56. $\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$

57. $\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$

58. $\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$

59. $\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$

60. $\begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$

61. $\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$

62. $\begin{cases} -2x + 3y = 4 \\ -3x + 4y = 5 \end{cases}$

63. $\begin{cases} x - 2y = 6 \\ 4x + 3y = 2 \end{cases}$

64. $\begin{cases} 2x - 3y = 7 \\ 4x + y = -7 \end{cases}$

65. $\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$

66. $\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$

67. $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

68. $\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$

69. $\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$

70. $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

71. $\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$

72. $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

73. $\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$