Section 3.2 – Angle and Orthogonality in Inner Product Spaces

Cosine Formula

If $\vec{u}$ and $\vec{v}$ are nonzero vectors that implies

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$\theta = \cos^{-1}\left(\frac{\langle \vec{u},\ \vec{v} \rangle}{\|\vec{u}\|\ \|\vec{v}\|}\right)$$

$$-1 \le \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\ \|\vec{v}\|} \le 1$$

Example

Let $\mathbb{R}^4$ have the Euclidean inner product. Find the cosine angle θ between the vectors $\vec{u} = (4,\ 3,\ 1,\ -2)$ and $\vec{v} = (-2,\ 1,\ 2,\ 3)$.

Solution

$$\|\vec{u}\| = \sqrt{4^2 + 3^2 + 1^2 + (-2)^2}$$

$$= \sqrt{30}$$

$$\|\vec{v}\| = \sqrt{4 + 1 + 4 + 9}$$

$$= \sqrt{18}$$

$$= 3\sqrt{2}$$

$$\langle \vec{u},\ \vec{v} \rangle = 4(-2) + 3(1) + 1(2) - 2(3)$$

$$= -9$$

$$\cos\theta = -\frac{9}{3\sqrt{30}\sqrt{2}} \qquad \cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\ \|\vec{v}\|}$$

$$= -\frac{3}{\sqrt{60}}$$

$$= -\frac{3}{2\sqrt{15}}$$

Theorem – Cauchy-Schwarz Inequality

If $\vec{v}$ and $\vec{w}$ are vectors in a real inner product space V, then

$$\left\|\langle \vec{u},\ \vec{v}\rangle\right\| \le \|\vec{u}\|\ \|\vec{v}\|$$

Proof

If either $\vec{u}$ or $\vec{v}$ is equal to zero, then both sides equal to zero
Inequality holds.

Suppose that $\vec{u},\ \vec{v} \ne 0$ and if $\vec{w}$ any vector

$\|\vec{w}\| = \vec{w}\ \vec{w} \ge 0$

Let $\vec{w} = \vec{u} - t\vec{v}$, then:

$$\begin{aligned}
0 &\le \vec{w}\vec{w} \\
&= (\vec{u} - t\vec{v})(\vec{u} - t\vec{v}) \\
&= \vec{u} \bullet \vec{u} - t(\vec{u} \bullet \vec{v}) - t(\vec{v} \bullet \vec{u}) + t^2(\vec{v} \bullet \vec{v}) \\
&= \vec{u} \bullet \vec{u} - 2t\ (\vec{u} \bullet \vec{v}) + t^2(\vec{v} \bullet \vec{v}) \qquad \text{Let } t = \frac{\vec{u} \bullet \vec{v}}{\vec{v} \bullet \vec{v}} \\
&= \vec{u} \bullet \vec{u} - 2\left(\frac{\vec{u} \bullet \vec{v}}{\vec{v} \bullet \vec{v}}\right)(\vec{u} \bullet \vec{v}) + \left(\frac{\vec{u} \bullet \vec{v}}{\vec{v} \bullet \vec{v}}\right)^2(\vec{v} \bullet \vec{v}) \\
&= \vec{u} \bullet \vec{u} - 2\frac{(\vec{u} \bullet \vec{v})^2}{\vec{v} \bullet \vec{v}} + \frac{(\vec{u} \bullet \vec{v})^2}{\vec{v} \bullet \vec{v}} \\
&= \vec{u} \bullet \vec{u} - \frac{(\vec{u} \bullet \vec{v})^2}{\vec{v} \bullet \vec{v}} \\
&= \frac{(\vec{u} \bullet \vec{u})(\vec{v} \bullet \vec{v}) - (\vec{u} \bullet \vec{v})^2}{\vec{v} \bullet \vec{v}} \qquad \text{Since } \vec{v} \bullet \vec{v} > 0 \\
&\le (\vec{u} \bullet \vec{u})(\vec{v} \bullet \vec{v}) - (\vec{u} \bullet \vec{v})^2
\end{aligned}$$

$(\vec{u} \bullet \vec{v})^2 \le (\vec{u} \bullet \vec{u})(\vec{v} \bullet \vec{v})$

$\left\|\langle \vec{u},\ \vec{v}\rangle\right\| \le \|\vec{u}\|\ \|\vec{v}\|$

The following two alternative forms of the Cauchy-Schwarz inequality are useful to know:

$$\langle \vec{u},\ \vec{v}\rangle^2 \le \langle \vec{u},\ \vec{u}\rangle\langle \vec{v},\ \vec{v}\rangle$$

$$\langle \vec{u},\ \vec{v}\rangle^2 \le \|\vec{u}\|^2\ \|\vec{v}\|^2$$

Theorem

If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in a real inner product space *V*, and if *k* is any scalar, then

a) $\|\vec{u}+\vec{v}\| \le \|\vec{u}\|+\|\vec{v}\|$ **(*Triangle inequality for vectors*)**

b) $d(\vec{u},\ \vec{v}) \le d(\vec{u},\ \vec{w})+d(\vec{w},\ \vec{v})$ **(*Triangle inequality for distances*)**

Proof **(*a*)**

$$\begin{aligned}\|\vec{u}+\vec{v}\|^2 &= \langle \vec{u}+\vec{v},\ \vec{u}+\vec{v}\rangle \\ &= \langle \vec{u},\ \vec{u}\rangle + 2\langle \vec{u},\ \vec{v}\rangle + \langle \vec{v},\ \vec{v}\rangle \\ &\le \langle \vec{u},\ \vec{u}\rangle + 2\left|\langle \vec{u},\ \vec{v}\rangle\right| + \langle \vec{v},\ \vec{v}\rangle \\ &\le \langle \vec{u},\ \vec{u}\rangle + 2\|\vec{u}\|\ \|\vec{v}\| + \langle \vec{v},\ \vec{v}\rangle \\ &= \|\vec{u}\|^2 + 2\|\vec{u}\|\ \|\vec{v}\| + \|\vec{v}\|^2 \\ &= \left(\|\vec{u}\|+\|\vec{v}\|\right)^2\end{aligned}$$

$$\|\vec{u}+\vec{v}\|^2 \le \left(\|\vec{u}\|+\|\vec{v}\|\right)^2$$

$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\|+\|\vec{v}\|$$

Definition

Two vectors $\vec{u}$ and $\vec{v}$ in an inner product space are called orthogonal if $\langle \vec{u},\ \vec{v}\rangle = 0$

Example

The vectors $\vec{u}=(1,\ 1)$ and $\vec{v}=(1,\ -1)$ are orthogonal with respect to the Euclidean inner product on $\mathbb{R}^2$, since

$$\begin{aligned}\vec{u}\bullet\vec{v} &= 1(1)+1(-1) \\ &\underline{=0\ |}\end{aligned}$$

They are not orthogonal with the respect to the weighted Euclidean inner product $\langle \vec{u},\ \vec{v}\rangle = 3u_1v_1 + 2u_2v_2$, since

$$\begin{aligned}\langle \vec{u},\ \vec{v}\rangle &= 3(1)(1)+\ 2(1)(-1) \\ &\underline{=1\ne 0\ |}\end{aligned}$$

Example

$U=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ *and* $V=\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}$ are orthogonal, since

$$U \bullet V = 1(0)+0(2)+1(0)+1(0)$$
$$\underline{=0\,|}$$

Definition

If W is a subspace of an inner product space V, then the set of all vectors are orthogonal to every vector in W is called the ***orthogonal complement*** of W and is denoted by the symbol $W^{\perp}$

Theorem

If W is a subspace of an inner product space V, then:

a) $W^{\perp}$ is a subspace of V.

b) $W \cap W^{\perp} = \{0\}$

Proof

a) Let set $W^{\perp}$ contains at least the zero vector, since $\langle \vec{0},\ \vec{w}\rangle = 0$ for every vector $\vec{w}$ in W. We need to show that $W^{\perp}$ is closed under addition and scalar multiplication.

Suppose that $\vec{u}$ and $\vec{v}$ are vectors in $W^{\perp}$, so every vector $\vec{w}$ in W we have $\langle \vec{u},\ \vec{w}\rangle = 0$ and $\langle \vec{v},\ \vec{w}\rangle = 0$

$$\langle \vec{u}+\vec{v},\ \vec{w}\rangle = \langle \vec{u},\ \vec{w}\rangle + \langle \vec{v},\ \vec{w}\rangle$$
$$= 0+0$$
$$\underline{=0\,|}$$

Closed under addition

$$\langle k\vec{u},\ \vec{w}\rangle = k\langle \vec{u},\ \vec{w}\rangle$$
$$= k(0)$$
$$\underline{=0\,|}$$

Closed under scalar multiplication

Which proves that $\vec{u}+\vec{w}$ and $k\vec{u}$ are in $W^{\perp}$

b) If $\vec{v}$ is any vector in both W and $W^{\perp}$, then $\vec{v}$ is orthogonal to itself; that is, $\langle \vec{v},\ \vec{v}\rangle = 0$. It follows from the positivity axiom for inner products that $\vec{v}=\vec{0}$

Theorem

If W is a subspace of a finite-dimensional inner product space V, then the orthogonal complement of $W^{\perp}$ is W; that is

$$\left(W^{\perp}\right)^{\perp} = W$$

Example

Let W be the subspace of $\mathbb{R}^6$ spanned by the vectors

$$\vec{w}_1 = (1,\ 3,\ -2,\ 0,\ 2,\ 0), \qquad \vec{w}_2 = (2,\ 6,\ -5,\ -2,\ 4,\ -3)$$
$$\vec{w}_3 = (0,\ 0,\ 5,\ 10,\ 0,\ 15), \qquad \vec{w}_4 = (2,\ 6,\ 0,\ 8,\ 4,\ 18)$$

Find a basis for the orthogonal complement of W.

Solution

The Space W is the same as the row space of the matrix

$$A = \begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 0 & 0 & 4 & 8 & 0 & 18 \end{pmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ R_3 + 5R_2 \\ R_4 + 4R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 \\ 0 & 0 & -1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix} \begin{matrix} \\ -R_2 \\ \\ \frac{1}{6}R_3 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} R_1 - 6R_3 \\ R_2 - 3R_3 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution

$$\left(x_1, x_2, x_3, x_4, x_5, x_6\right) = \left(-3x_2 - 4x_4 - 2x_5, x_2, -2x_4, x_4, x_5, 0\right)$$

$$= x_2\left(-3, 1, 0, 0, 0, 0\right) + x_4\left(-4, 0, -2, 1, 0, 0\right) + x_5\left(-2, 0, 0, 0, 1, 0\right)$$

$$\vec{v}_1 = \left(-3, 1, 0, 0, 0, 0\right), \quad \vec{v}_2 = \left(-4, 0, -2, 1, 0, 0\right), \quad \vec{v}_3 = \left(-2, 0, 0, 0, 1, 0\right)$$

Definition

A collection of vectors in $\mathbb{R}^n$ (or inner space) is called orthogonal if any 2 are perpendicular.

$$\vec{v}_i . \vec{v}_j = \vec{v}^T \vec{v} = \begin{cases} 0 & for\ i \neq j \quad (orthogonal\ vectors) \\ 1 & for\ i = j \quad (unit\ vectors) \end{cases}$$

Theorem

If $\vec{v}_1, \ldots, \vec{v}_m$ are nonzero orthogonal vectors, then they are linearly independent.

Definition

A vector $\vec{v}$ is called normal if $\left\|\vec{v}\right\| = 1$

A collection of vectors $\vec{v}_1, \ldots, \vec{v}_m$ is called orthonormal if they are orthogonal and each $\left\|\vec{v}_i\right\| = 1$.

An orthonormal basis is a basis made up of orthonormal vectors.

Example

Q rotates every vector in the plane through the angle θ.

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$Q^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$ $\cos^2\theta + \sin^2\theta = 1$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\boxed{= Q^T}$$

The dot product $(\cos\theta\sin\theta - \sin\theta\cos\theta = 0)$, the columns are orthogonal.

They are unit vectors because $\cos^2\theta + \sin^2\theta = 1$. Those columns give an orthonormal basis for the plane $\mathbb{R}^2$.

We have: $QQ^T = I = Q^T Q$ (This type is called ***rotation***)

Exercises *Section* 3.2 – Angle and Orthogonality in Inner Product Spaces

1. Which of the following form orthonormal sets?

a) $(1, 0), (0, 2)$ in $\mathbb{R}^2$

b) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^2$

c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^2$

d) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^3$

e) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ in $\mathbb{R}^3$

f) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ in $\mathbb{R}^3$

2. Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

a) $\vec{u} = (1, -3), \quad \vec{v} = (2, 4)$

b) $\vec{u} = (-1, 0), \quad \vec{v} = (3, 8)$

c) $\vec{u} = (-1, 5, 2), \quad \vec{v} = (2, 4, -9)$

d) $\vec{u} = (4, 1, 8), \quad \vec{v} = (1, 0, -3)$

e) $\vec{u} = (1, 0, 1, 0), \quad \vec{v} = (-3, -3, -3, -3)$

f) $\vec{u} = (2, 1, 7, -1), \quad \vec{v} = (4, 0, 0, 0)$

g) $\vec{u} = (1, 3, -5, 4), \quad \vec{v} = (2, -4, 4, 1)$

h) $\vec{u} = (1, 2, 3, 4), \quad \vec{v} = (-1, -2, -3, -4)$

3. Find the cosine of the angle between ***A*** and ***B***.

a) $A = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$

b) $A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$

c) $A = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

d) $A = \begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$

4. Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

a) $\vec{u} = (-1, 3, 2), \quad \vec{v} = (4, 2, -1)$

b) $\vec{u} = (a, b), \quad \vec{v} = (-b, a)$

c) $\vec{u} = (-2, -2, -2), \quad \vec{v} = (1, 1, 1)$

d) $\vec{u} = (-4, 6, -10, 1), \quad \vec{v} = (2, 1, -2, 9)$

e) $\vec{u} = (-4, 6, -10, 1), \quad \vec{v} = (2, 1, -2, 9)$

5. Do there exist scalars *k* and *l* such that the vectors $\vec{u} = (2, k, 6), \quad \vec{v} = (l, 5, 3), \quad and \quad \vec{w} = (1, 2, 3)$ are mutually orthogonal with respect to the Euclidean inner product?

6. Let $\mathbb{R}^3$ have the Euclidean inner product. For which values of k are $\vec{u}$ and $\vec{v}$ orthogonal?

a) $\vec{u} = (2, 1, 3), \quad \vec{v} = (1, 7, k)$ *b)* $\vec{u} = (k, k, 1), \quad \vec{v} = (k, 5, 6)$

7. Let V be an inner product space. Show that if $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors in V, then $\|\vec{u} - \vec{v}\| = \sqrt{2}$

8. Let $\boldsymbol{S}$ be a subspace of $\mathbb{R}^n$. Explain what $\left(\boldsymbol{S}^{\perp}\right)^{\perp} = \boldsymbol{S}$ means and why it is true.

9. The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ – (***note*** that all six edges have length $\sqrt{2}$, so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ to the vertices?

10. Determine if the given vectors are orthogonal.

$\vec{x}_1 = (1, 0, 1, 0), \quad \vec{x}_2 = (0, 1, 0, 1), \quad \vec{x}_3 = (1, 0, -1, 0), \quad \vec{x}_4 = (1, 1, -1, -1)$

11. Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

a) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \ \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

b) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \ \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) \ \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

12. Consider vectors $\vec{u} = (2, 3, 5) \quad \vec{v} = (1, -4, 3) \quad in \quad \mathbb{R}^3$

a) $\langle \vec{u}, \vec{v} \rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine between $\vec{u}$ *and* $\vec{v}$

13. Consider vectors $\vec{u} = (1, 1, 1) \quad \vec{v} = (1, 2, -3) \quad in \quad \mathbb{R}^3$

a) $\langle \vec{u}, \vec{v} \rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine θ between $\vec{u}$ *and* $\vec{v}$

14. Consider vectors $\vec{u} = (1, 2, 5) \quad \vec{v} = (2, -3, 5) \quad \vec{w} = (4, 2, -3) \quad in \quad \mathbb{R}^3$

a) $\langle \vec{u}, \vec{v} \rangle$ *d)* $\|\vec{u}\|$ *g)* Cosine α between $\vec{u}$ *and* $\vec{v}$

b) $\langle \vec{u}, \vec{w} \rangle$ *e)* $\|\vec{v}\|$ *h)* Cosine β between $\vec{u}$ *and* $\vec{w}$

c) $\langle \vec{v}, \vec{w} \rangle$ *f)* $\|\vec{w}\|$ *i)* Cosine θ between $\vec{v}$ *and* $\vec{w}$

j) $(\vec{u} + \vec{v}) \bullet \vec{w}$

15. Consider polynomial $f(t)=3t-5;\quad g(t)=t^2 \quad in \quad \mathbb{P}(t)$

a) $\langle f, g\rangle$ *b)* $\|f\|$ *c)* $\|g\|$ *d)* Cosine between f *and* g

16. Consider polynomial $f(t)=t+2\ ;\quad g(t)=3t-2\ ;\quad h(t)=t^2-2t-3 \quad in \quad \mathbb{P}(t)$

a) $\langle f, g\rangle$
b) $\langle f, h\rangle$
c) $\langle g, h\rangle$
d) $\|f\|$
e) $\|g\|$
f) $\|h\|$
g) Cosine α between f *and* g
h) Cosine β between f *and* h
i) Cosine θ between g *and* h

17. Suppose $\langle \vec{u}, \vec{v}\rangle = 3+2\,i$ in a complex inner product space *V*. Find:

a) $\langle (2-4\,i)\vec{u}, \vec{v}\rangle$ *b)* $\langle \vec{u}, (4+3\,i)\vec{v}\rangle$ *c)* $\langle (3-6\,i)\vec{u}, (5-2\,i)\vec{v}\rangle$ *d)* $\|\vec{u}, \vec{v}\|$

18. Find the Fourier coefficient *c* and the projection $c\vec{v}$ of $\vec{u}=(3+4\,i,\ 2-3\,i)$ along $\vec{v}=(5+i,\ 2i)$ in $\mathbb{C}^2$

19. Suppose $\vec{v}=(1,\ 3,\ 5,\ 7)$. Find the projection of $\vec{v}$ onto *W* or find $\vec{w}\in W$ that minimizes $\|\vec{v}-\vec{w}\|$, where *W* is the subspace of $\mathbb{R}^4$ spanned by:

a) $\vec{u}_1=(1,\ 1,\ 1,\ 1)$ *and* $\vec{u}_2=(1,\ -3,\ 4,\ -2)$
b) $\vec{v}_1=(1,\ 1,\ 1,\ 1)$ *and* $\vec{v}_2=(1,\ 2,\ 3,\ 2)$

20. Suppose $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ is an orthogonal set of vectors. Prove that (*Pythagoras*)

$$\|\vec{u}_1+\vec{u}_2+\ \ldots+\vec{u}_n\|^2=\|\vec{u}_1\|^2+\|\vec{u}_2\|^2+\ \ldots+\|\vec{u}_n\|^2$$

21. Suppose *A* is an orthogonal matrix. Show that $\langle \vec{u}A, \vec{v}A\rangle = \langle \vec{u}, \vec{v}\rangle$ for any $\vec{u}, \vec{v}\ \in V$

22. Suppose *A* is an orthogonal matrix. Show that $\|\vec{u}A\|=\|\vec{u}\|$ for every $\vec{u}\ \in V$

23. Let *V* be an inner product space over $\mathbb{R}$ or $\mathbb{C}$. Show that

$$\|\vec{u}+\vec{v}\|=\|\vec{u}\|+\|\vec{v}\|$$

If and only if

$$\|s\vec{u}+t\vec{v}\|=s\|\vec{u}\|+t\|\vec{v}\| \quad \textit{for all} \quad s,\ t\ge 0$$

24. Let *V* be an inner product vector space over $\mathbb{R}$.

a) If e_1, e_2, e_3 are three vectors in *V* with pairwise product negative, that is,

$$\langle e_1, e_2\rangle < 0, \quad i, j=1,\ 2,\ 3, \quad i\ne j$$

Show that e_1, e_2, e_3 are linearly independent.

b) Is it possible for three vectors on the *xy*–plane to have pairwise negative products?

c) Does part (a) remain valid when the word "negative: is replaced with positive?

d) Suppose $\vec{u},\ \vec{v},\ and\ \vec{w}$ are three unit vectors in the *xy*–plane. What are the maximum and minimum values that

$$\langle \vec{u},\ \vec{v} \rangle + \langle \vec{v},\ \vec{w} \rangle + \langle \vec{w},\ \vec{u} \rangle$$

Can attain? And when?