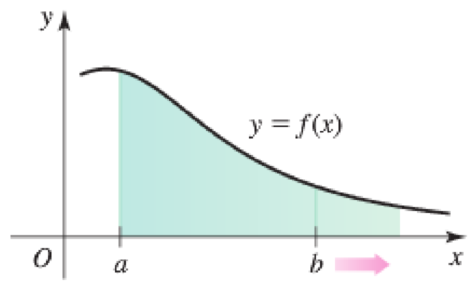
Section 2.6 – Improper Integrals

Definition

Integrals with infinite limits of integration are ***improper integrals***.

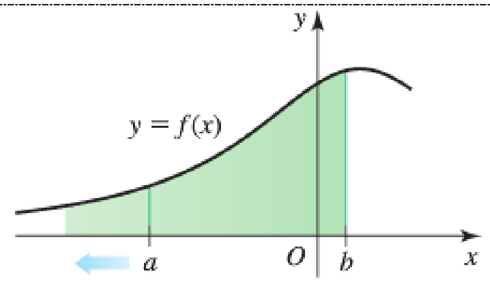
1. If $f(x)$ is continuous on $[a,\ \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$



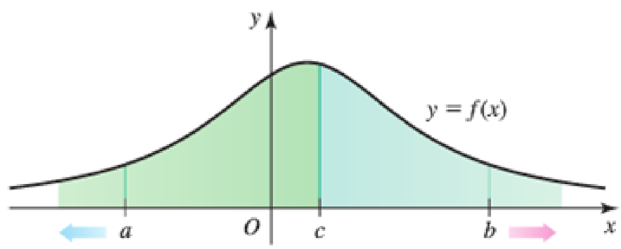
2. If $f(x)$ is continuous on $(-\infty,\ b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$



3. If $f(x)$ is continuous on $(-\infty,\ \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$



In each case, if the limit is finite we say that the improper integral ***converges*** and that the limit is the ***value*** of the improper integral. If the limit fails to exist, the improper integral ***diverges***.

Example

Is the area under the curve $y=\frac{\ln x}{x^2}$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?

Solution

$$\begin{aligned}\int_1^b \frac{\ln x}{x^2}\,dx &= -\frac{1}{x}\ln x\Big|_1^b - \int_1^b\left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx\\
&= -\left(\frac{1}{b}\ln b - \ln 1\right) + \int_1^b \frac{1}{x^2}\,dx\\
&= -\frac{1}{b}\ln b + \left[-\frac{1}{x}\right]_1^b\\
&= -\frac{1}{b}\ln b - \left(\frac{1}{b}-1\right)\\
&= -\frac{1}{b}\ln b - \frac{1}{b} + 1\end{aligned}$$

$u=\ln x \qquad dv=\frac{dx}{x^2}$

$du=\frac{1}{x}dx \qquad v=-\frac{1}{x}$

$$\int_1^{\infty} \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\int_1^{b} \frac{\ln x}{x^2}dx$$

$$= \lim_{b\to\infty}\left(-\frac{1}{b}\ln b - \frac{1}{b} + 1\right)$$

$$= -\lim_{b\to\infty}\left(\frac{1/b}{1}\right) - 0 + 1 \qquad \textit{L'Hôpital Rule}$$

$$= -0 + 1$$

$$\underline{= 1}$$

Example

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Solution

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

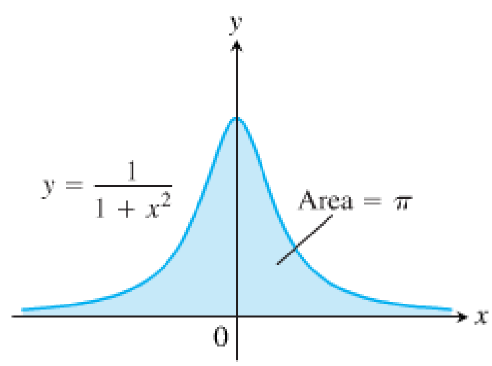
$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty}\int_{a}^{0} \frac{dx}{1+x^2}$$

$$= \lim_{a\to-\infty} \tan^{-1}x\Big|_a^0$$

$$= \lim_{a\to-\infty}\left(\tan^{-1}0 - \tan^{-1}a\right)$$

$$= 0 - \left(-\frac{\pi}{2}\right)$$

$$\underline{= \frac{\pi}{2}}$$



$$\int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_{0}^{b} \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty} \tan^{-1}x\Big|_0^b$$

$$= \lim_{b\to\infty}\left(\tan^{-1}b - \tan^{-1}0\right)$$

$$= \frac{\pi}{2} - 0$$

$$\underline{= \frac{\pi}{2}}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

Example

For what value of p does the integral $\int_1^\infty \frac{dx}{x^p}$ converge? When the integral does converge, what is its value?

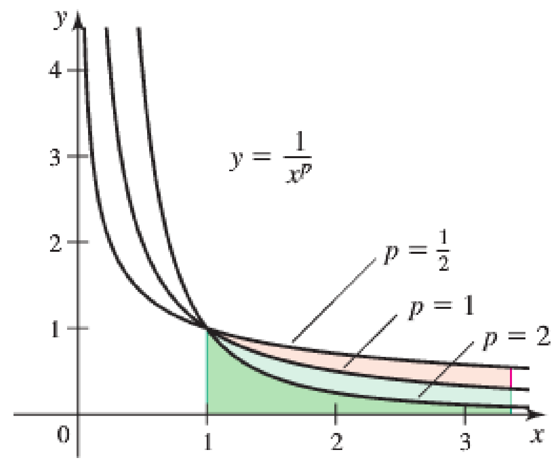
Solution

If $p \neq 1$ $\quad \int_1^b \frac{dx}{x^p} = \left.\frac{x^{-p+1}}{-p+1}\right|_1^b = \frac{1}{1-p}\left(b^{1-p} - 1\right)$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^p}$$

$$= \lim_{b\to\infty}\left[\frac{1}{1-p}\left(b^{1-p}-1\right)\right]$$

$$= \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}$$



$\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1}$, if $p > 1$.

If $p = 1$ $\quad \int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{dx}{x}$

$$= \lim_{b\to\infty} \int_1^b \frac{dx}{x}$$

$$= \lim_{b\to\infty} \left[\ln x\right]_1^b$$

$$= \lim_{b\to\infty} \left(\ln b - \ln 1\right)$$

$$= \infty$$

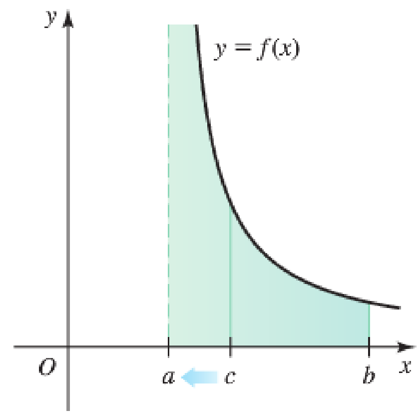
Integrands with Vertical Asymptotes

Definition

Integrals of functions that become infinite at a point within the interval of integration are ***improper integrals***. If the limit is finite we say that the improper integral ***converges*** and that the limit is the ***value*** of the improper integral. If the limit does not exist, the integral ***diverges***.

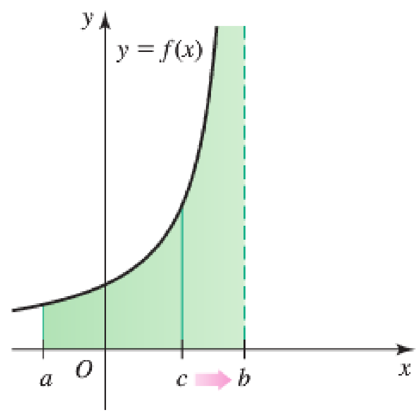
1. If $f(x)$ is continuous on $(a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$$



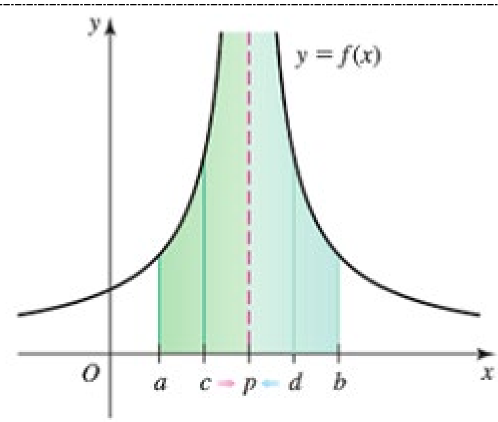
2. If $f(x)$ is continuous on $[a, b)$, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx$$



3. If $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{c\to p^-}\int_a^c f(x)\,dx + \lim_{d\to p^+}\int_d^b f(x)\,dx$$



Example

Investigate the convergence of $\displaystyle\int_0^1 \frac{1}{1-x}\,dx$

Solution

$$\int_0^1 \frac{1}{1-x}\,dx = \lim_{b\to 1^-} \int_0^b \frac{1}{1-x}\,dx$$

$$= \lim_{b\to 1^-} \Big[-\ln|1-x|\Big]_0^b$$

$$= \lim_{b\to 1^-} \Big[-\ln|1-b| + 0\Big]$$

$$= \infty$$

The limit is infinite, so the integral diverges.

Example

Evaluate $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$

Solution

The integrand has a vertical asymptote at $x = 1$ and is continuous on $[0, 1)$ and $(1, 3]$.

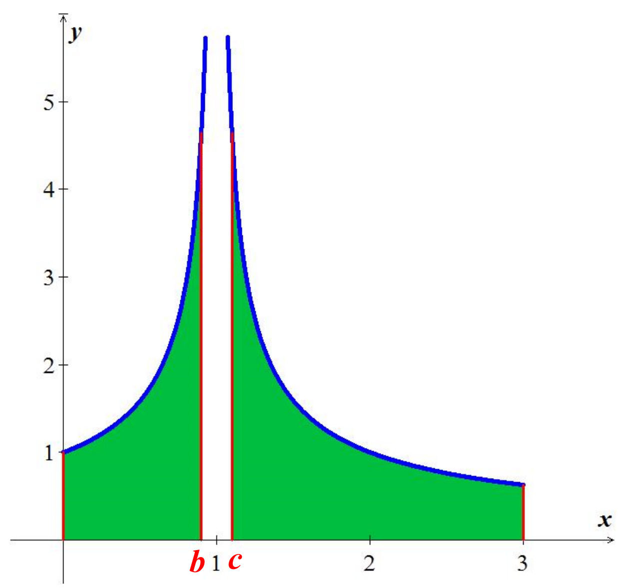
$$\int \frac{dx}{(x-1)^{2/3}} = \int (x-1)^{-2/3}\,d(x-1) = 3(x-1)^{1/3}$$

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \Big[3(x-1)^{1/3}\Big]_0^{1^-} + \Big[3(x-1)^{1/3}\Big]_{1^+}^3$$

$$= 3(0+1) + 3\left(\sqrt[3]{2} - 0\right)$$

$$= 3 + 3\sqrt[3]{2}$$



Example

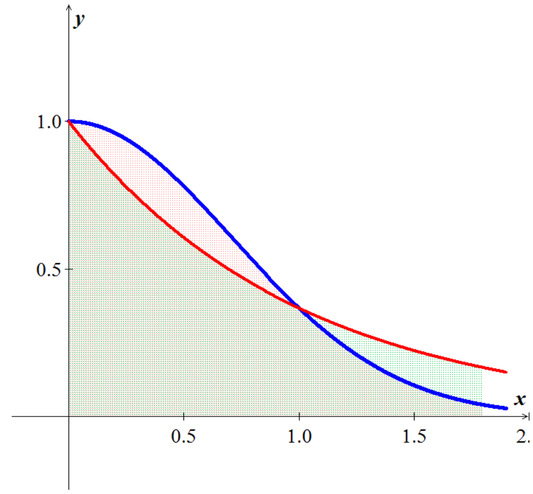
Does the integral $\displaystyle\int_1^{\infty} e^{-x^2}\,dx$ converge?

Solution

$$\int_1^{\infty} e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^{b} e^{-x^2}\,dx$$

$$\int_1^{b} e^{-x^2}\,dx \le \int_1^{b} e^{-x}\,dx = -e^{-b} + e^{-1} < e^{-1} \approx 0.36788$$

The integral converges



Theorem – Direct Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. $\displaystyle\int_a^{\infty} f(x)\,dx$ *converges if* $\displaystyle\int_a^{\infty} g(x)\,dx$ *converges*

2. $\displaystyle\int_a^{\infty} g(x)\,dx$ *diverges if* $\displaystyle\int_a^{\infty} f(x)\,dx$ *diverges*

Theorem – Limit Comparison Test

If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$$

Then

$$\int_a^{\infty} f(x)\,dx \quad \text{and} \quad \int_a^{\infty} g(x)\,dx$$

Both converge or both diverge

Example

Show that $\int_1^\infty \frac{dx}{1+x^2}$ converges by comparison with $\int_1^\infty \frac{dx}{x^2}$. Find and compare the two integral values.

Solution

The functions $f(x)=\frac{1}{x^2}$ and $g(x)=\frac{1}{1+x^2}$ are positive and continuous on $[1, \infty)$. Also,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{1}{1+x^2}}$$

$$= \lim_{x\to\infty} \frac{1+x^2}{x^2}$$

$$\boxed{=1}$$

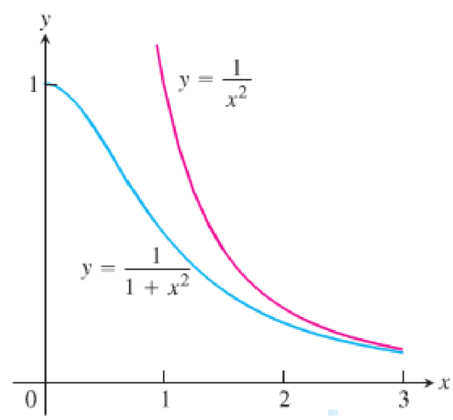
Therefore, $\int_1^\infty \frac{dx}{1+x^2}$ converges because $\int_1^\infty \frac{dx}{x^2}$ converges.

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 1\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$\boxed{= \frac{\pi}{4}}$$



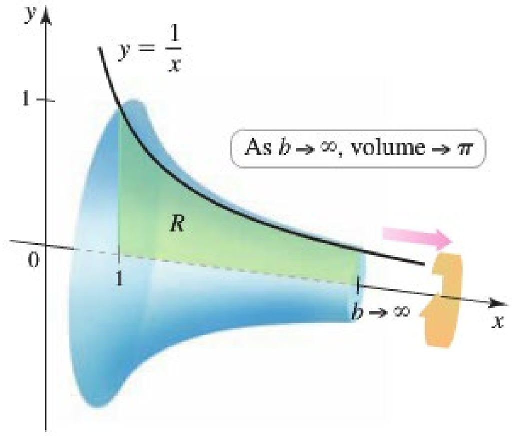
Example

Let R be the region bounded by the graph of $y = x^{-1}$ and the x-$axis$, for $x \ge 1$.

a) What is the volume of the solid generated when R is revolved about the x-$axis$?

b) What is the surface area of the solid generated when R is revolved about the x-$axis$?

c) What is the volume of the solid generated when R is revolved about the y-$axis$?



Solution

a) $V = \pi \int_1^{\infty} \frac{1}{x^2} dx \qquad\qquad V = \pi \int_a^b \left(f(x)\right)^2 dx$

$$= -\pi \frac{1}{x}\Big|_1^{\infty}$$

$$= -\pi(0-1)$$

$$= \pi \ \ unit^3$$

b) $S = 2\pi \int_1^{\infty} \frac{1}{x}\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\, dx \qquad\qquad S = 2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\, dx$

$$= 2\pi \int_1^{\infty} \frac{1}{x^3}\sqrt{x^4+1}\, dx$$

$$> 2\pi \int_1^{\infty} \frac{x^2}{x^3}\, dx \qquad\qquad \sqrt{x^4+1} > \sqrt{x^4} = x^2$$

$$= 2\pi \int_1^{\infty} \frac{1}{x}\, dx$$

$$= 2\pi(\ln x)\Big|_1^{\infty}$$

$$= \infty \ \ unit^2$$

c) $V = 2\pi \int_1^{\infty} x\frac{1}{x} dx \qquad\qquad V = 2\pi \int_a^b x \cdot f(x)\, dx \quad (\text{Shell method})$

$$= 2\pi x\Big|_1^{\infty}$$

$$= \infty \ \ unit^3$$

Exercises Section 2.6 – Improper Integrals

(1 – 81) Evaluate the integrals

1. $\displaystyle\int_0^{\infty} \frac{dx}{x^2+1}$

2. $\displaystyle\int_0^{4} \frac{dx}{\sqrt{4-x}}$

3. $\displaystyle\int_{-\infty}^{2} \frac{2dx}{x^2+4}$

4. $\displaystyle\int_{-\infty}^{\infty} \frac{xdx}{\left(x^2+4\right)^{3/2}}$

5. $\displaystyle\int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}}$

6. $\displaystyle\int_{-\infty}^{\infty} 2xe^{-x^2}dx$

7. $\displaystyle\int_0^{1} (-\ln x)dx$

8. $\displaystyle\int_{-1}^{4} \frac{dx}{\sqrt{|x|}}$

9. $\displaystyle\int_0^{\infty} e^{-3x}\ dx$

10. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

11. $\displaystyle\int_1^{10} \frac{dx}{(x-2)^{1/3}}$

12. $\displaystyle\int_1^{\infty} \frac{dx}{x^2}$

13. $\displaystyle\int_0^{\infty} \frac{dx}{(x+1)^3}$

14. $\displaystyle\int_{-\infty}^{0} e^x\ dx$

15. $\displaystyle\int_1^{\infty} 2^{-x}\ dx$

16. $\displaystyle\int_{-\infty}^{0} \frac{dx}{\sqrt[3]{2-x}}$

17. $\displaystyle\int_{4/\pi}^{\infty} \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)dx$

18. $\displaystyle\int_{e^2}^{\infty} \frac{dx}{x\ln^p x} \qquad p>1$

19. $\displaystyle\int_0^{\infty} \frac{p}{\sqrt[5]{p^2+1}}\ dp$

20. $\displaystyle\int_{-1}^{1} \ln y^2\ dy$

21. $\displaystyle\int_{-2}^{6} \frac{dx}{\sqrt{|x-2|}}$

22. $\displaystyle\int_0^{\infty} xe^{-x}dx$

23. $\displaystyle\int_0^{1} x\ln x\ dx$

24. $\displaystyle\int_0^{1} x\ln x\ dx$

25. $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}dx$

26. $\displaystyle\int_1^{\infty} (1-x)e^x\ dx$

27. $\displaystyle\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\ dx$

28. $\displaystyle\int_0^{1} \frac{dx}{\sqrt[3]{x}}$

29. $\displaystyle\int_1^{\infty} \frac{3}{\sqrt[3]{x}}\ dx$

30. $\displaystyle\int_1^{\infty} \frac{4}{\sqrt[4]{x}}\ dx$

31. $\displaystyle\int_0^{2} \frac{dx}{x^3}$

32. $\displaystyle\int_1^{\infty} \frac{dx}{x^3}$

33. $\displaystyle\int_1^{\infty} \frac{6}{x^4}\ dx$

34. $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x}\,(x+1)}$

35. $\displaystyle\int_{-\infty}^{0} xe^{-4x}dx$

36. $\displaystyle\int_0^\infty xe^{-x/3}dx$

37. $\displaystyle\int_0^\infty x^2e^{-x}dx$

38. $\displaystyle\int_0^\infty e^{-x}\cos x\,dx$

39. $\displaystyle\int_4^\infty \frac{1}{x(\ln x)^3}dx$

40. $\displaystyle\int_1^\infty \frac{\ln x}{x}dx$

41. $\displaystyle\int_{-\infty}^\infty \frac{4}{16+x^2}dx$

42. $\displaystyle\int_0^\infty \frac{x^3}{\left(x^2+1\right)^2}dx$

43. $\displaystyle\int_0^\infty \frac{1}{e^x+e^{-x}}dx$

44. $\displaystyle\int_0^\infty \frac{e^x}{1+e^x}dx$

45. $\displaystyle\int_0^\infty \cos\pi x\,dx$

46. $\displaystyle\int_0^\infty \sin\frac{x}{2}\,dx$

47. $\displaystyle\int_1^\infty \frac{dx}{(x+1)^9}$

48. $\displaystyle\int_1^\infty \frac{3x-1}{4x^3-x^2}\,dx$

49. $\displaystyle\int_{-\infty}^\infty \frac{4}{x^2+16}\,dx$

50. $\displaystyle\int_{-\infty}^{-1} \frac{dx}{(x-1)^4}$

51. $\displaystyle\int_0^\infty xe^{-x}\,dx$

52. $\displaystyle\int_0^\infty \frac{6x}{1+x^6}\,dx$

53. $\displaystyle\int_0^2 \frac{dx}{\sqrt[3]{|x-1|}}$

54. $\displaystyle\int_{-1}^1 \frac{dx}{x^2+2x+5}$

55. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+2x+5}$

56. $\displaystyle\int_0^\infty \cos x\,dx$

57. $\displaystyle\int_2^\infty \frac{\cos\left(\frac{\pi}{x}\right)}{x^2}\,dx$

58. $\displaystyle\int_{-\infty}^a \sqrt{e^x}\,dx$

59. $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx$

60. $\displaystyle\int_1^\infty \frac{dx}{x(x+1)}$

61. $\displaystyle\int_1^\infty \frac{dx}{x^2(x+1)}$

62. $\displaystyle\int_1^\infty \frac{3x^2+1}{x^3+x}\,dx$

63. $\displaystyle\int_1^\infty \frac{1}{x^2}\sin\frac{\pi}{x}\,dx$

64. $\displaystyle\int_2^\infty \frac{dx}{(x+2)^2}$

65. $\displaystyle\int_1^\infty \frac{\tan^{-1}x}{x^2+1}\,dx$

66. $\displaystyle\int_{-3}^1 \frac{dx}{(2x+6)^{2/3}}$

67. $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

68. $\displaystyle\int_0^{\ln 3} \frac{e^x}{\left(e^x-1\right)^{2/3}}\,dx$

69. $\displaystyle\int_1^2 \frac{dx}{\sqrt{x-1}}$

70. $\displaystyle\int_{-1}^1 \frac{dx}{x^2}$

71. $\displaystyle\int_0^2 \frac{dx}{(x-1)^2}$

72. $\displaystyle\int_{-1}^2 \frac{dx}{(x-1)^2}$

73. $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$

74. $\displaystyle\int_0^\infty xe^{-x^2}\,dx$

75. $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

76. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx$

77. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$

78. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+6x+12}$

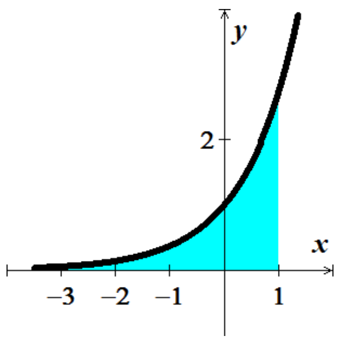
79. $\displaystyle\int \frac{dx}{2-\sqrt{3x}}$

80. $\displaystyle\int \theta\cos(2\theta+1)\,d\theta$

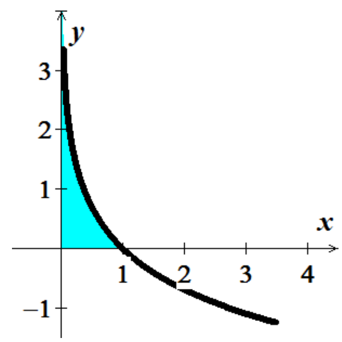
81. $\displaystyle\int \sqrt{x}\,\sqrt{1+\sqrt{x}}\,dx$

(82 – 85) Find the area of the unbounded shaded region

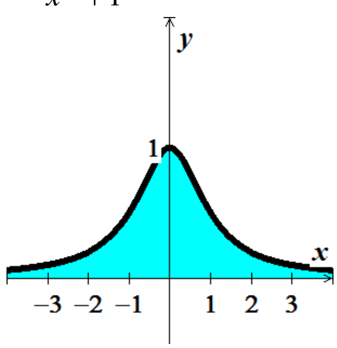
82. $y=e^x,\quad -\infty < x \le 1$



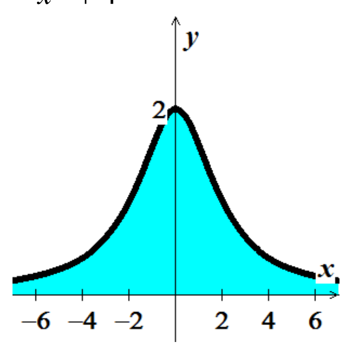
83. $y=-\ln x$



84. $y=\dfrac{1}{x^2+1}$



85. $y=\dfrac{8}{x^2+4}$



86. Find the area of the region R between the graph of $f(x)=\dfrac{1}{\sqrt{9-x^2}}$ and the x-*axis* on the interval $(-3,\ 3)$ (if it exists)

87. Find the volume of the region bounded by $f(x)=\left(x^2+1\right)^{-1/2}$ and the x-*axis* on the interval $[2,\ \infty)$ is revolved about the x-*axis*.

88. Find the volume of the region bounded by $f(x) = \sqrt{\dfrac{x+1}{x^3}}$ and the *x-axis* on the interval $[1, \infty)$ is revolved about the *x-axis*.

89. Find the volume of the region bounded by $f(x) = (x+1)^{-3}$ and the *x-axis* on the interval $[0, \infty)$ is revolved about the *y-axis*.

90. Find the volume of the region bounded by $f(x) = \dfrac{1}{\sqrt{x}\ln x}$ and the *x-axis* on the interval $[2, \infty)$ is revolved about the *x-axis*.

91. Find the volume of the region bounded by $f(x) = \dfrac{\sqrt{x}}{\sqrt[3]{x^2+1}}$ and the *x-axis* on the interval $[0, \infty)$ is revolved about the *x-axis*.

92. Find the volume of the region bounded by $f(x) = \left(x^2-1\right)^{-1/4}$ and the *x-axis* on the interval $(1, 2]$ is revolved about the *y-axis*.

93. Find the volume of the region bounded by $f(x) = \tan x$ and the *x-axis* on the interval $\left[0, \frac{\pi}{2}\right)$ is revolved about the *x-axis*.

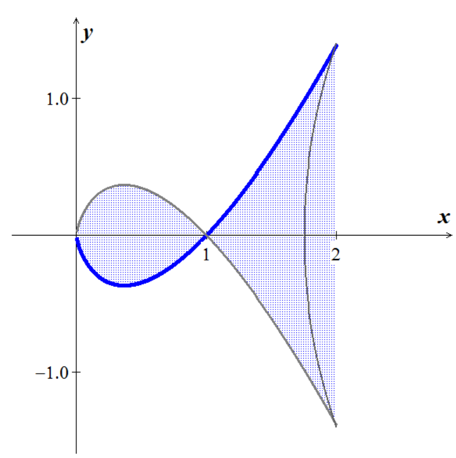
94. Find the volume of the region bounded by $f(x) = -\ln x$ and the *x-axis* on the interval $(0, 1]$ is revolved about the *x-axis*.

95. Find the volume of the solid generated by revolving the region bounded by the graphs of $y = xe^{-x}$, $y = 0$, and $x = 0$ about the *x-axis*.

96. The region between the *x*-axis and the curve

$$f(x) = \begin{cases} 0, & x = 0 \\ x\ln x, & 0 < x \le 2 \end{cases}$$

is revolved about the *x*-axis to generate the solid. Find the volume of the solid.



(97 – 98) Consider the region satisfying the inequalities

a) Find the area of the region
b) Find the volume of the solid generated by revolving the region about the *x-axis*.
c) Find the volume of the solid generated by revolving the region about the *y-axis*.

97. $y \le e^{-x}, \quad y \ge 0, \quad x \ge 0$

98. $y \le \frac{1}{x^2}, \quad y \ge 0, \quad x \ge 1$

99. Find the perimeter of the hypocycloid of four cusps $x^{2/3} + y^{2/3} = 4$

100. Find the arc length of the graph $y = \sqrt{16 - x^2}$ over the interval $[0, 4]$

101. The region bounded by $(x-2)^2 + y^2 = 1$ is revolved about the *y-axis* to form a torus. Find the surface area of the torus.

102. Find the surface area formed by revolving the graph $y = 2e^{-x}$ on the interval $[0, \infty)$ about the *x-axis*

103. The magnetic potential P at a point on the axis of a circular coil is given by

$$P = \frac{2\pi NIr}{k} \int_c^{\infty} \frac{1}{\left(r^2 + x^2\right)^{3/2}}\, dx$$

Where N, I, r, k, and c are constants. Find P.

104. A "semi-infinite" uniform rod occupies the nonnegative *x-axis*. The rod has a linear density δ, which means that a segment of length dx has a mass of δdx. A particle of mass M is located at the point $(-a, 0)$. The gravitational force F that the rod exerts on the mass is given by

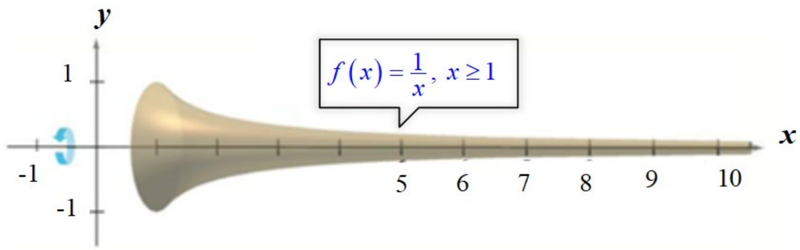
$$F = \int_0^{\infty} \frac{GM\delta}{(a+x)^2}\, dx$$

Where G is the gravitational constant. Find F.

105. Let R be the region bounded by the graph of $f(x) = x^{-p}$ and the *x-axis*

a) Let S be the solid generated when R is revolved about the *x-axis*. For what values of p is the volume of S finite for $0 < x \le 1$?
b) Let S be the solid generated when R is revolved about the *y-axis*. For what values of p is the volume of S finite for $0 < x \le 1$?
c) Let S be the solid generated when R is revolved about the *x-axis*. For what values of p is the volume of S finite for $x \ge 1$?
d) Let S be the solid generated when R is revolved about the *y-axis*. For what values of p is the volume of S finite for $x \ge 1$?

106. The solid formed by revolving (about the x-*axis*) the unbounded region lying between the graph of $f(x)=\frac{1}{x}$ and the x-*axis* $(x \ge 1)$ is called ***Gabriel's Horn***.



Show that this solid has a finite volume and an infinite surface area.

107. Water is drained from a 3000-*gal* tank at a rate that starts at 100 *gal/hr*. and decreases continuously by 5% /*hr*. If the drain left open indefinitely, how much water drains from the tank? Can a full tank be emptied at this rate?

108. Let $I(a)=\displaystyle\int_0^{\infty} \frac{dx}{\left(1+x^a\right)\left(1+x^2\right)}$, where a is a real number.

a) Evaluate $I(a)$ and show that its value is independent of a.

(***Hint***: split the integral into two integrals over $[0,\ 1]$ and $[1,\ \infty)$; then use a change of variables to convert the second integral into an integral over $[0,\ 1]$.)

b) Let f be any positive continuous function on $\left[0,\ \frac{\pi}{2}\right]$

Evaluate $\displaystyle\int_0^{\pi/2} \frac{f(\cos x)}{f(\cos x)+f(\sin x)}dx$

(***Hint***: Use the identity $\cos\left(\frac{\pi}{2}-x\right)=\sin x$)

109. Let R be the region bounded by $y=\ln x$, the x-*axis*, and the line $x=a$, where $a>1$.

a) Find the volume $V_1(a)$ of the solid generated when R is revolved about the x-*axis* (as a function of a).

b) Find the volume $V_2(a)$ of the solid generated when R is revolved about the y-*axis* (as a function of a).

c) Graph V_1 and V_2. For what values of $a>1$ is $V_1(a)>V_2(a)$?

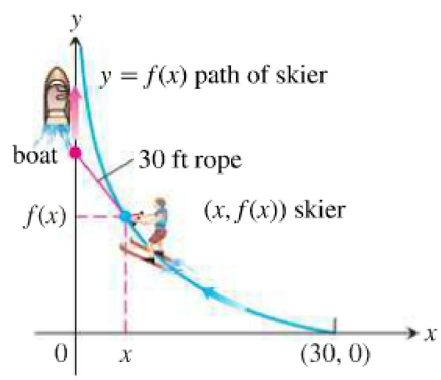
110. Let R be the region bounded by the graph of $f(x) = x^{-p}$ and the x-axis, for $x \geq 1$. Let V_1 and V_2 be the volumes of the solids generated when R is revolved about the x-axis and the y-axis, respectively, if they exist.

a) For what values of p (if any) is $V_1 = V_2$?

b) Repeat part (a) on the interval $(0,\ 1]$.

111. Let R_1 be the region bounded by the graph of $y = e^{-ax}$ and the x-axis on the interval $[0,\ b]$ where $a > 0$ and $b > 0$. Let R_2 be the region bounded by the graph of $y = e^{-ax}$ and the x-axis on the interval $[b,\ \infty)$. Let V_1 and V_2 be the volumes of the solids generated when R_1 and R_2 are revolved about the x-axis. Find and graph the relationship between a and b for which $V_1 = V_2$.

112. Suppose that a boat is positioned at the origin with a water skier tethered to the boat at the point (30, 0) on a rope 30 *feett.* long. As the boat travels along the positive y-axis, the skier is pulled behind the boat along an unknown path $y = f(x)$, as shown



a) Show that $f'(x) = \dfrac{-\sqrt{900 - x^2}}{x}$

(*Hint*: Assume that the skier is always pointed directly at the boat and the rope is on line is on a line tangent to the path $y = f(x)$.)

b) Solve the equation in part (a) for $f(x)$, using $f(30) = 0$

113. Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the substance B at time $t = 0$, and if x is the amount of product at time t, then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt} = k(a - x)(b - x) \quad \textit{or} \quad \frac{1}{(a - x)(b - x)}\frac{dx}{dt} = k$$

Where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t.

a) If $a = b$

b) If $a \neq b$

Assume in each case that $x = 0$ when $t = 0$