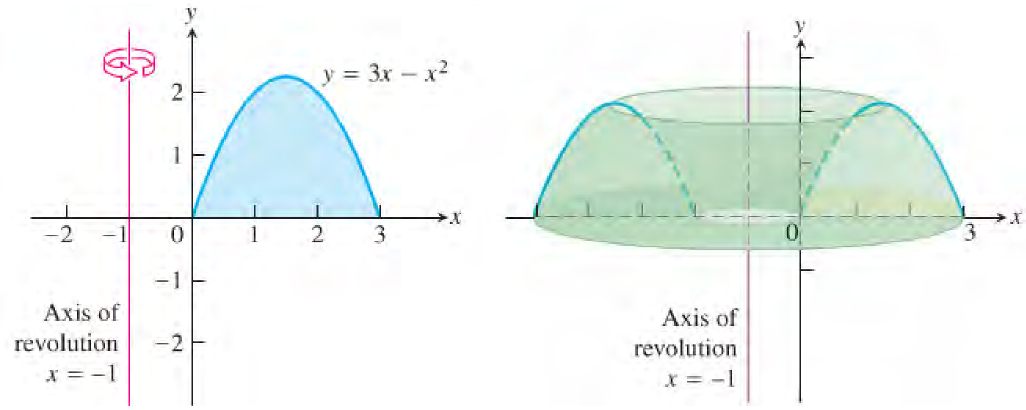
Section 1.4 – Volume by Shells

Slicing with Cylinders

Example

The region enclosed by the *x*-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate a solid. Find the volume of the solid

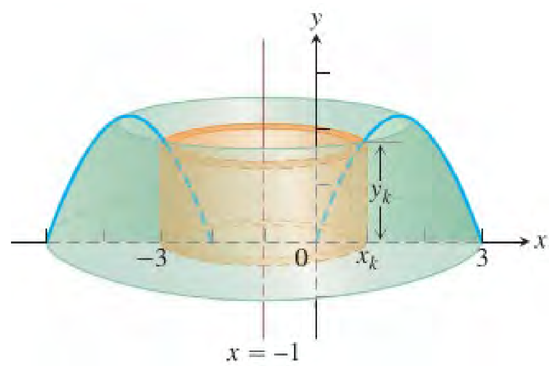
Solution

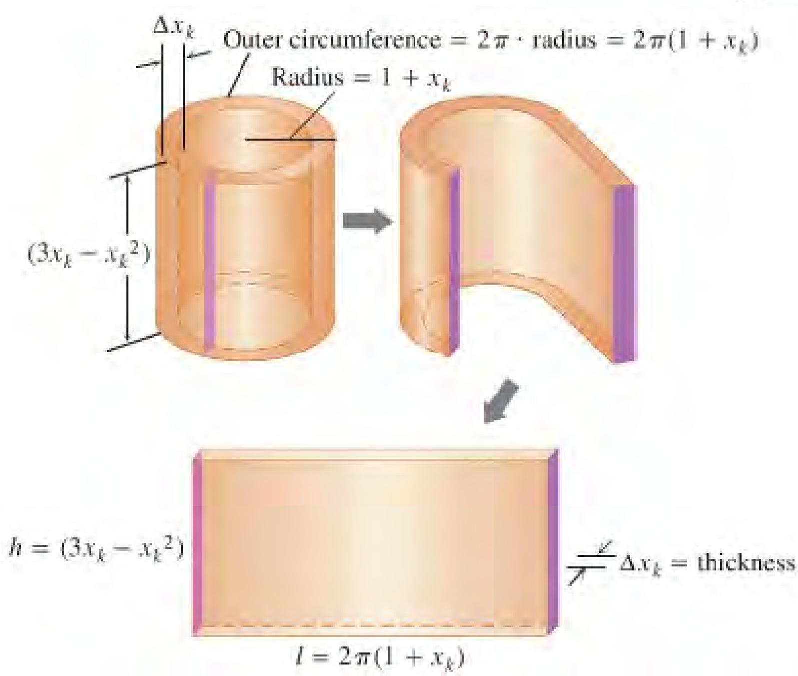


If we rotate a vertical strip of thickness Δx, this rotation produces a cylindrical shell of height y_k above a point x_k within the base of the vertical strip.

$$\Delta V_k = circumference \times height \times thickness$$

$$= 2\pi\left(1 + x_k\right) \cdot \left(3x_k - x_k^2\right) \cdot \Delta x_k$$





The Riemann sum:

$$\sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} 2\pi\left(1+x_k\right)\cdot\left(3x_k - x_k^2\right)\cdot \Delta x_k$$

Taking the limit as the thickness $\Delta x_k \to 0$ and $n \to \infty$ gives the volume integral

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi\left(1+x_k\right)\cdot\left(3x_k - x_k^2\right)\cdot \Delta x_k$$

$$= \int_0^3 2\pi\left(x+1\right)\left(3x - x^2\right)dx$$

$$= 2\pi \int_0^3 \left(3x^2 + 3x - x^2 - x^3\right)dx$$

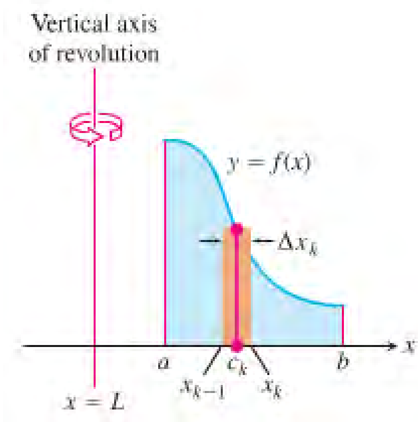
$$= 2\pi \int_0^3 \left(2x^2 + 3x - x^3\right)dx$$

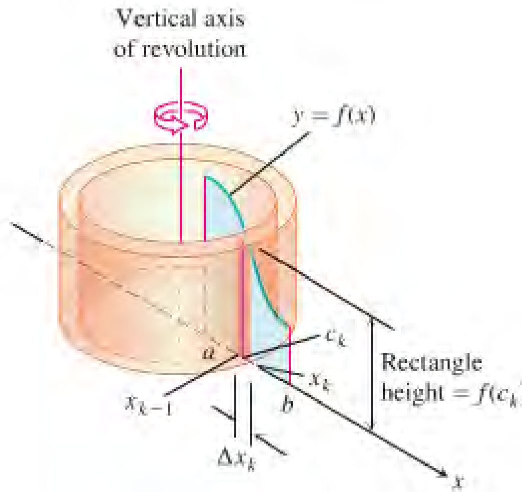
$$= 2\pi \left(\tfrac{2}{3}x^3 + \tfrac{3}{2}x^2 - \tfrac{1}{4}x^4 \right|_0^3$$

$$= 2\pi \left(\tfrac{2}{3}(3)^3 + \tfrac{3}{2}(3)^2 - \tfrac{1}{4}(3)^4\right)$$

$$= \frac{45\pi}{2} \quad unit^3$$

Shell Method



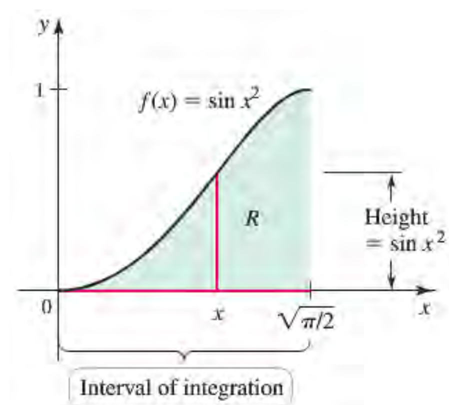


$$V = 2\pi \int_a^b \begin{pmatrix} shell \\ radius \end{pmatrix} \begin{pmatrix} shell \\ height \end{pmatrix} dx$$

Example

Let R be the region bounded by the graph of $f(x) = \sin x^2$, the x-axis, and the vertical line $x = \sqrt{\frac{\pi}{2}}$. Find the volume of the solid generated when R is revolved about the y-axis.

Solution



$$V = 2\pi \int_a^b \begin{pmatrix} shell \\ radius \end{pmatrix} \begin{pmatrix} shell \\ height \end{pmatrix} dx$$

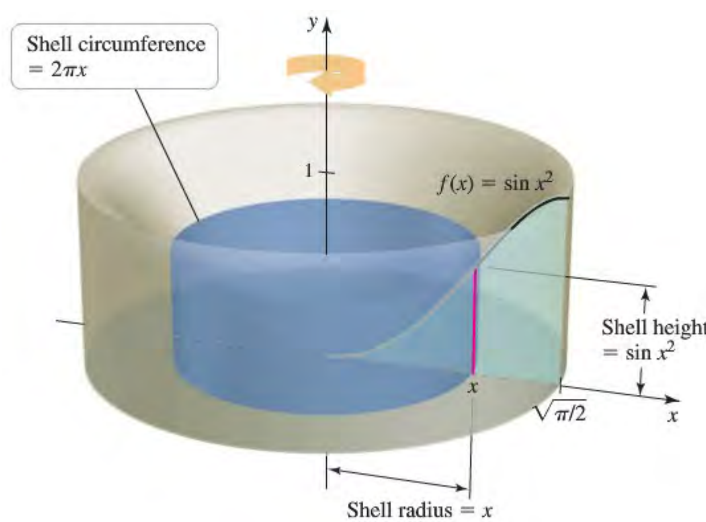
$$= 2\pi \int_0^{\sqrt{\pi/2}} x \sin x^2 \; dx$$

$$= \pi \int_0^{\sqrt{\pi/2}} \sin x^2 \; d\left(x^2\right)$$

$$= -\pi \cos\left(x^2\right) \Big|_0^{\sqrt{\pi/2}}$$

$$= -\pi\left(\cos\left(\frac{\pi}{2}\right) - \cos 0\right)$$

$$= \pi \quad unit^3$$



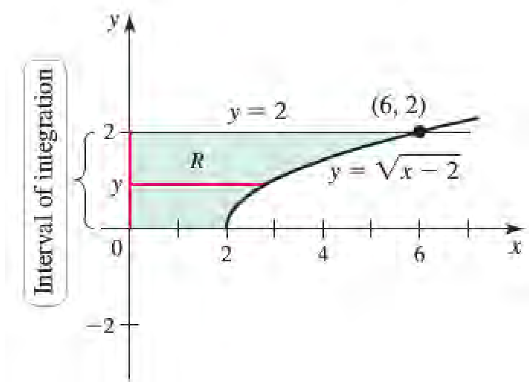
Example

Let R be the region in the first region bounded by the graph $y=\sqrt{x-2}$ and the line $y=2$.

a) Find the volume of the solid generated when R is revolved about the x-axis.

b) Find the volume of the solid generated when R is revolved about the line $y=-2$.

Solution



a) $y=\sqrt{x-2} \;\to\; y^2=x-2 \;\Rightarrow\; x=y^2+2$

$0\le y\le 2$

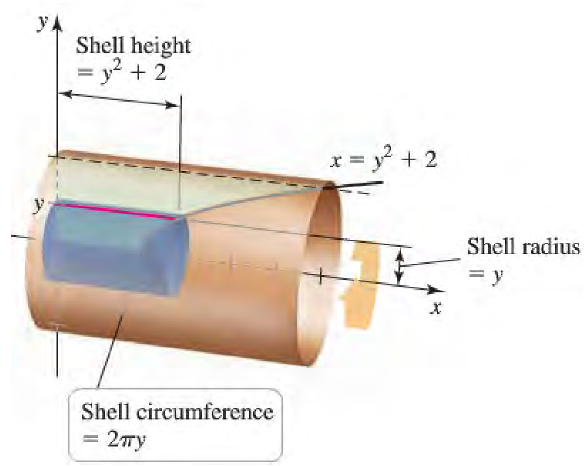
$$V=2\pi\int_c^d \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix}dy$$

$$=2\pi\int_0^2 y\left(y^2+2\right)dy$$

$$=2\pi\int_0^2 \left(y^3+2y\right)dy$$

$$=2\pi\left(\frac{y^4}{4}+y^2\right.\Bigg|_0^2$$

$$=16\pi \quad unit^3$$



b) Revolved R about the line $y=-2$.

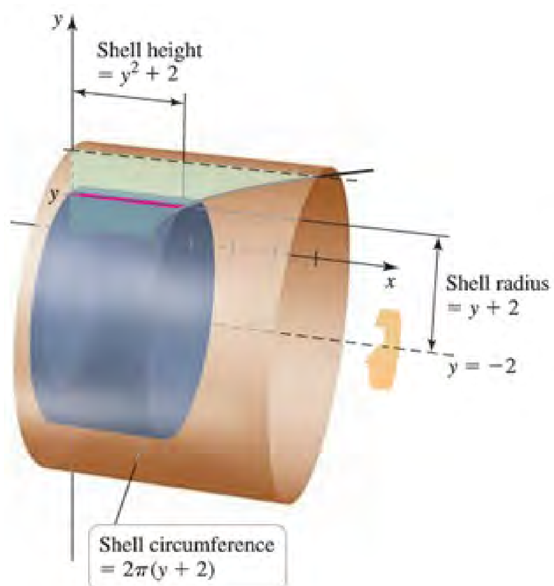
$$V=2\pi\int_c^d \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix}dy$$

$$=2\pi\int_0^2 (y+2)\left(y^2+2\right)dy$$

$$=2\pi\left(\frac{1}{4}y^4+\frac{2}{3}y^3+y^2+4y\right.\Bigg|_0^2$$

$$=2\pi\left(4+\frac{16}{3}+4+8\right)$$

$$=\frac{128\pi}{3} \quad unit^3$$



Example

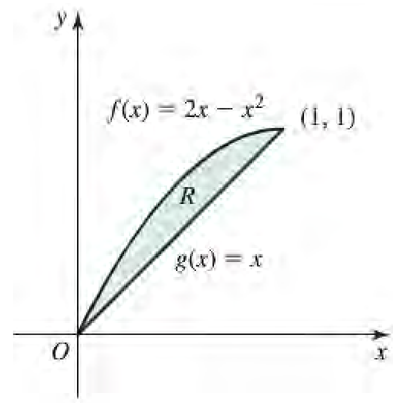
The region R is bounded by the graphs of $f(x) = 2x - x^2$ and $g(x) = x$ on the interval $[0, 1]$.
Use the washer method and the shell method to find the volume of the solid formed when R is revolved about the x-axis.

Solution

$$f(x) = g(x)$$

$$2x - x^2 = x$$

$$x^2 - x = 0 \Rightarrow \boxed{x = 0, 1}$$



Washer Method:

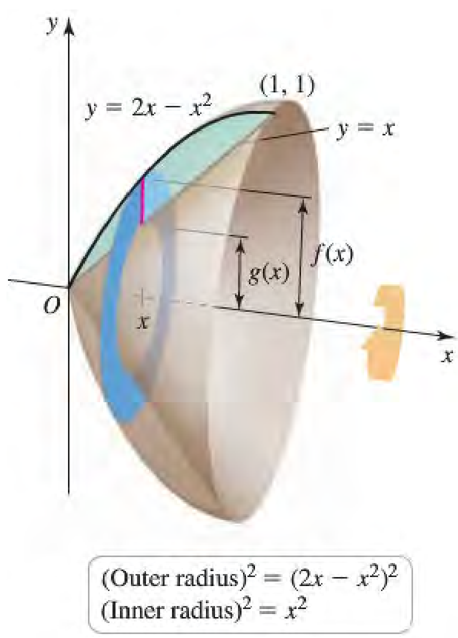
$$V = \pi \int_0^1 \left[\left(2x - x^2\right)^2 - x^2\right] dx$$

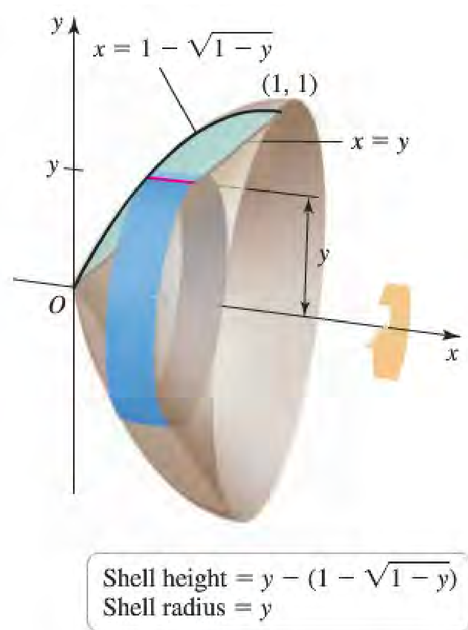
$$= \pi \int_0^1 \left(3x^2 - 4x^3 + x^4\right) dx$$

$$= \pi \left(x^3 - x^4 + \frac{x^5}{5} \right|_0^1$$

$$= \pi\left(1 - 1 + \frac{1}{5}\right)$$

$$\boxed{= \frac{\pi}{5} \quad unit^3}$$





Shell Method:

$x = y$

$y = 2x - x^2$

$x^2 - 2x + y = 0$

$$x = \frac{2 \pm \sqrt{4-4y}}{2}$$

$$= \begin{cases} 1-\sqrt{1-y} \\ \cancel{1+\sqrt{1-y}} \end{cases}$$

$x = 0 \quad \to y = 0$

$x = 1 \quad \to y = 1$

$$V = 2\pi \int_0^1 y \left[y - \left(1-\sqrt{1-y}\right) \right] dy$$

$$= 2\pi \int_0^1 y \left(y - 1 + \sqrt{1-y} \right) dy$$

$$= 2\pi \int_0^1 \left(y^2 - y + y(1-y)^{1/2} \right) dy$$

$$= 2\pi \left(\frac{1}{3}y^3 - \frac{1}{2}y^2 + \frac{2}{5}(1-y)^{5/2} - \frac{2}{3}(1-y)^{3/2} \right|_0^1$$

$$= 2\pi \left(\frac{1}{3} - \frac{1}{2} - \left(\frac{2}{5} - \frac{2}{3} \right) \right)$$

$$= 2\pi \left(-\frac{1}{6} + \frac{4}{15} \right)$$

$$= 2\pi \left(\frac{9}{90} \right)$$

$$= \frac{\pi}{5} \quad unit^3$$

Let $u = 1-y \quad \to \quad y = 1-u$

$dy = -du$

$$\int y(1-y)^{1/2} dy = -\int (1-u)u^{1/2} du$$

$$= -\int \left(u^{1/2} - u^{3/2} \right) du$$

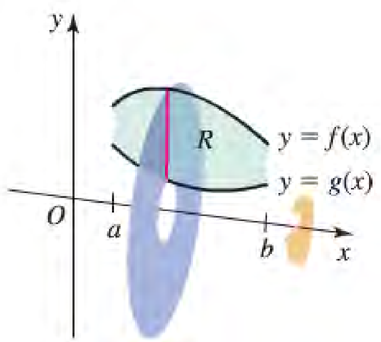
$$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}$$

$$= \frac{2}{5}(1-y)^{5/2} - \frac{2}{3}(1-y)^{3/2}$$

Summary of the Shell Method

1. Draw the region and sketch a line segment across it parallel to the axis of revolution. Label the segment's height or length (*shell height*) and distance from the axis of revolution (*shell radius*)
2. Find the limits of integration for the thickness variable.
3. Integrate the product 2π (*shell radius*) (*shell height*) with respect to the thickness variable (*x* or *y*) to find the volume

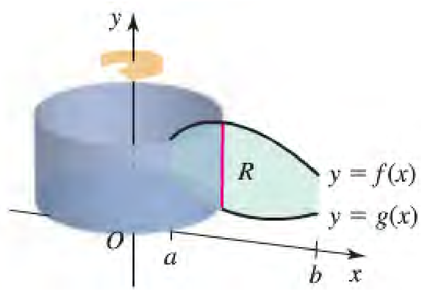
Integration With respect to x



Disk/washer method about the x-axis

Disks/washers are ***perpendicular*** to the *x*-axis

$$V = \pi \int_a^b \left(f(x)^2 - g(x)^2 \right) dx$$

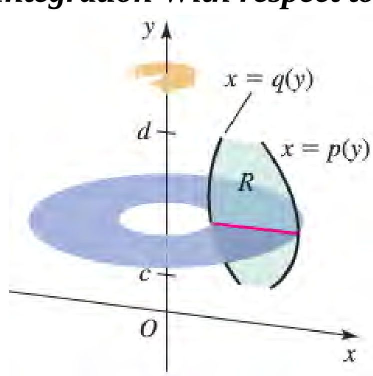


Shell method about the y-axis

Shells are ***parallel*** to the *y*-axis

$$V = 2\pi \int_a^b x \left(f(x) - g(x) \right) dx$$

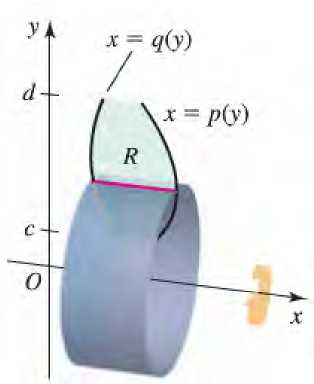
Integration With respect to y



Disk/washer method about the y-axis

Disks/washers are ***perpendicular*** to the *y*-axis

$$V = \pi \int_c^d \left(p(y)^2 - q(y)^2 \right) dy$$



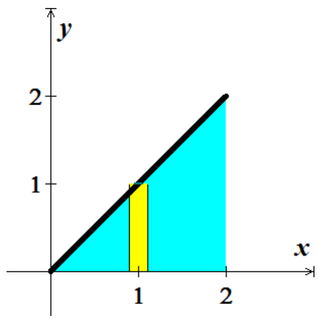
Shells are ***parallel*** to the *x*-axis

$$V = 2\pi \int_c^d y \left(p(y) - q(y) \right) dy$$

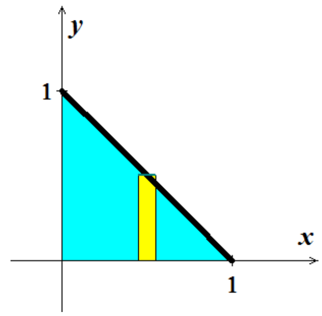
Exercises   *Section* 1.4 – Volume by Shells

(1 – 13) Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

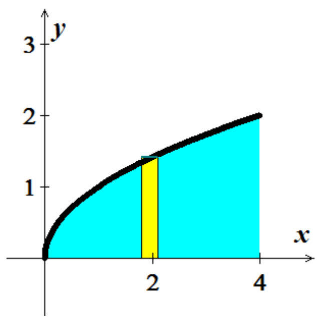
1. $y = x$



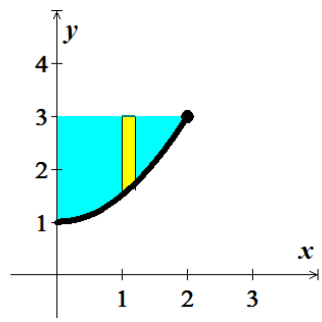
2. $y = 1 - x$



3. $y = \sqrt{x}$



4. $y = \frac{1}{2}x^2 + 1$



5. $y = \frac{1}{4}x^2, \quad y = 0, \quad x = 4$

6. $y = \frac{1}{2}x^3, \quad y = 0, \quad x = 3$

7. $y = x^2, \quad y = 4x - x^2$

8. $y = 9 - x^2, \quad y = 0$

9. $y = 4x - x^2, \quad x = 0, \quad y = 4$

10. $y = x^{3/2}, \quad y = 8, \quad x = 0$

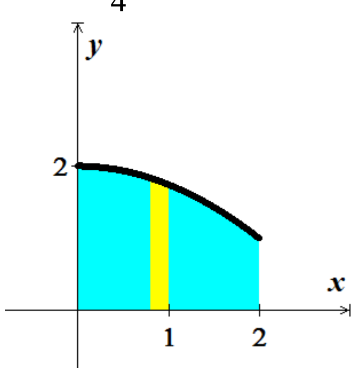
11. $y = \sqrt{x-2}, \quad y = 0, \quad x = 4$

12. $y = -x^2 + 1, \quad y = 0$

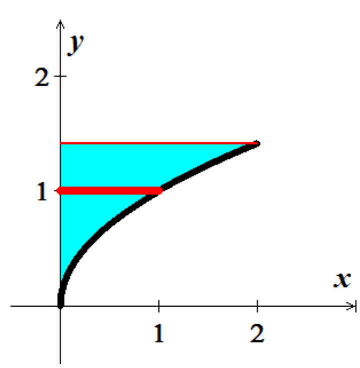
13. $y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad y = 0, \quad x = 0, \quad x = 1$

(14 – 15) Use the shell method to find the volume of the solid generated by revolving the shaded region about the indicated axis

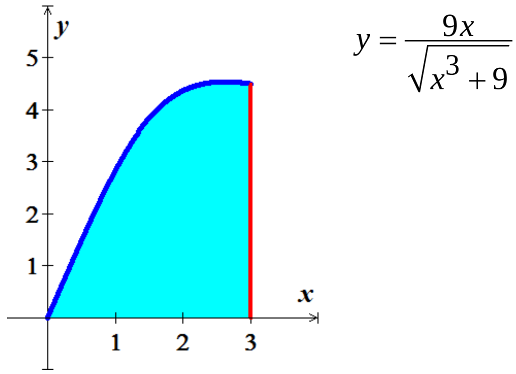
14. $y = 2 - \frac{1}{4}x^2$



15. $x = y^2$



16. Use the shell method to find the volume of the solid generated by revolving the shaded region about the *y*-axis

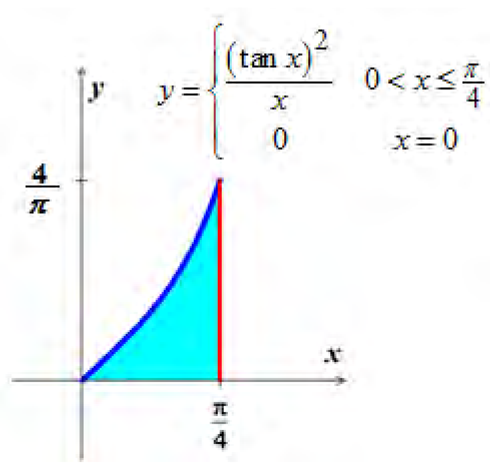


17. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y=x^2, \quad y=2-x, \quad x=0, \quad for\ x\geq 0$ about the *y*-axis.

18. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y=2-x^2, \quad y=x^2, \quad x=0$ about the *y*-axis.

19. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y=\frac{3}{2\sqrt{x}}, \quad y=0, \quad x=1, \quad x=4$ about the *y*-axis.

20. Let $g(x)=\begin{cases} \frac{(\tan x)^2}{x} & 0<x\leq\frac{\pi}{4} \\ 0 & x=0 \end{cases}$



a) Show that $x\cdot g(x)=(\tan x)^2, \quad 0\leq x\leq\frac{\pi}{4}$

b) Find the volume of the solid generated by revolving the shaded region about the *y*-axis.

21. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $x=\sqrt{y}, \quad x=-y, \quad y=2$ about the *x*-axis.

22. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $x=y^2, \quad x=-y, \quad y=2, \quad y\geq 0$ about the *x*-axis.

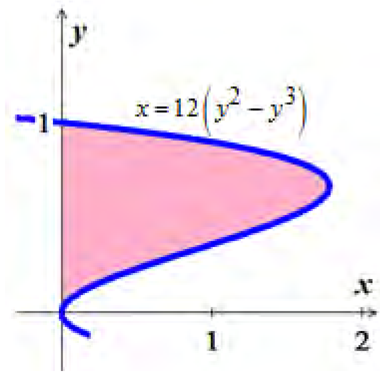
23. Compute the volume of the solid generated by revolving the region bounded by the lines $y = x$ *and* $y = x^2$ about each coordinate axis using

a) The *shell* method

b) The *washer* method

24. Use the shell method to find the volumes of the solids generated by revolving the shaded regions about the *indicated* axes.

a) The x-axis
b) The line $y = 1$
c) The line $y = \frac{8}{5}$
d) The line $y = -\frac{2}{5}$



25. Use the *washer* method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = \sqrt{x}, \quad y = 2, \quad x = 0$ about

a) the x-axis
b) the y-axis
c) the line $x = 4$
d) the line $y = 1$

26. Find the volume of the solid generated by revolving the region bounded by $y = \frac{4}{x^3}$ and the lines $x = 1,$ *and* $y = \frac{1}{2}$ about

a) the x-axis;
b) the y-axis;
c) the line $x = 2$;
d) the line $y = 4$.

27. The region in the first quadrant that is bounded by the curve $y = \frac{1}{\sqrt{x}}$, on the left by the line $x = \frac{1}{4}$, and below by the line $y = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid by

a) The *shell* method
b) The *washer* method

28. The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ to generate a solid. Find the volume of the solid.

a) revolved about the x-axis
b) revolved about the y-axis

29. Find the volume of the solid generated by revolving the region bounded by $y = \sin x$ and the lines $x = 0, \quad x = \pi,$ *and* $y = 2$ about the line $y = 2$.

30. A cylinder hole with radius r is drilled symmetrically through the center of a sphere with radius R, where $r \le R$. What is the volume of the remaining material?

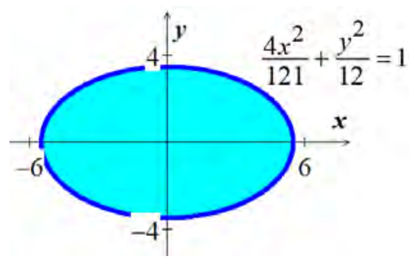
31. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y=\sqrt{x}, \quad y=2-x, \quad y=0$ about the x-axis.

32. Find the volume of the region bounded by $y=\dfrac{\ln x}{x^2}, \quad y=0, \quad x=1, \quad and\ x=3$ revolved about the *y-axis*

33. Find the volume of the region bounded by $y=\dfrac{e^x}{x}, \quad y=0, \quad x=1, \quad and\ x=2$ revolved about the *y-axis*

34. Find the volume of the region bounded by $y^2=\ln x, \quad y^2=\ln x^3, \quad and\ y=2$ revolved about the *x-axis*

35. The profile of a football resembles the ellipse. Find the football's volume to the nearest *cubic inch*.



(**36 – 38**) Find the volume using both the *disk/washer* and *shell* methods of

36. $y=(x-2)^3-2, \quad x=0, \quad y=25;$ revolved about the *y-axis*

37. $y=\sqrt{\ln x}, \quad y=\sqrt{\ln x^2}, \quad y=1;$ revolved about the *x-axis*

38. $y=\dfrac{6}{x+3}, \quad y=2-x;$ revolved about the *x-axis*

(**39 – 42**) Use the shell method to find the volume of the solid generated by the revolving the plane region about the given line

39. $y=2x-x^2, \quad y=0,$ *about the line* $x=4$

40. $y=\sqrt{x}, \quad y=0, \quad x=4,$ *about the line* $x=6$

41. $y=x^2, \quad y=4x-x^2,$ *about the line* $x=4$

42. $y=\frac{1}{3}x^3, \quad y=6x-x^2,$ *about the line* $x=3$

(**43 – 44**) Use the disk method or shell method to find the volume of the solid generated vy revolving the region bounded by the graph of the equations about the given lines.

43. $y=x^3, \quad y=0, \quad x=2$

a) the x-axis *b) the y-axis* *c) the line* $x=4$

44. $y=\dfrac{10}{x^2}$, $y=0$, $x=1$, $x=5$

a) the x-axis *b) the y-axis* *c) the line* $y=10$

45. Let V_1 and V_2 be the volumes of the solids that result when the plane region bounded by $y=\dfrac{1}{x}$, $y=0$, $x=\dfrac{1}{4}$, and $x=c$ $\left(\textit{where } c>\dfrac{1}{4}\right)$ is revolved about the *x-axis* and the *y-axis*, respectively. Find the value of c for which $V_1=V_2$

46. The region bounded by $y=r^2-x^2$, $y=0$, and $x=0$ is revolved about the *y-axis* to form a paraboloid. A hole, centered along the axis of revolution, is drilled through this solid. The hole has a radius k, $0<k<r$. Find the volume of the resulting ring

a) By integrating with respect to *x*.
b) By integrating with respect to *y*.

47. The region *R* in the first quadrant bounded by the parabola $y=4-x^2$ and the coordinate axes is revolved about the *y*-axis to produce a dome-shaped solid. Find the volume of the solid in the following ways.

a) Apply the disk method and integrate with respect to *y*.
b) Apply the shell method and integrate with respect to *x*.

48. The region bounded by the curves $y=1+\sqrt{x}$, $y=1-\sqrt{x}$, and the line $x=1$ is revolved about the *y*-axis. Find the volume of the resulting solid by

a) Integrating with respect to *x* and
b) Integrating with respect to *y*.

49. The region bounded by the graphs of $x=0$, $x=\sqrt{\ln y}$, and $x=\sqrt{2-\ln y}$ in the first quadrant is revolved about the *y*-axis. What is the volume of the resulting solid?

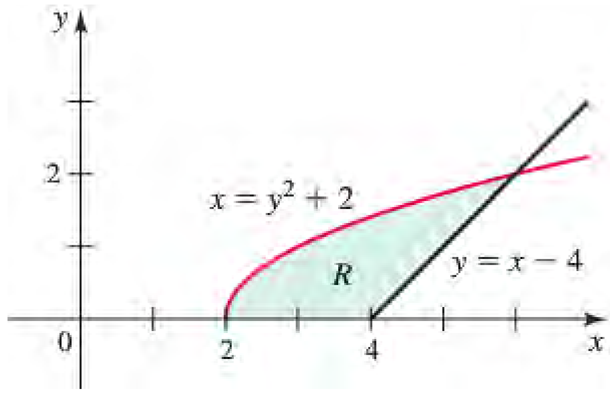
50. The region bounded by $y=\left(1-x^2\right)^{-1/2}$ and the *x*-axis over the interval $\left[0,\ \dfrac{\sqrt{3}}{2}\right]$ is revolved about the *y*-axis. What is the volume of the solid that is generated?

51. The region bounded by the graph $y=4-x^2$ and the *x*-axis over the interval $[-2,\ 2]$ is revolved about the line $x=-2$. What is the volume of the solid that is generated?

52. The region bounded by the graph $y=6x$ and $y=x^2+5$ is revolved about the line $y=-1$ and the line $x=-1$. Find the volumes of the resulting solids. Which one is greater?

53. The region bounded by the graph $y = 2x$, $y = 6 - x$ and $y = 0$ is revolved about the line $y = -2$ and the line $x = -2$. Find the volumes of the resulting solids. Which one is greater?

54. The region R is bounded by the curves $x = y^2 + 2$, $y = x - 4$, *and* $y = 0$



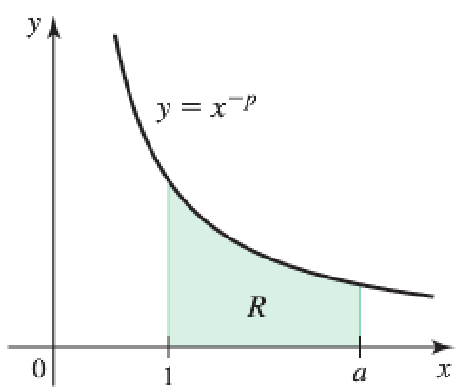
a) Write a single integral that gives the area of R.

b) Write a single integral that gives the volume of the solid generated when R is revolved about the x-axis.

c) Write a single integral that gives the volume of the solid generated when R is revolved about the y-axis.

d) Suppose S is a solid whose base is R and whose cross sections perpendicular to R and parallel to the x-axis are semicircles. Write a single integral that gives the volume of S.

55. The region R is bounded by $y = \frac{1}{x^p}$ and the x-axis on the interval $[1, a]$, where $p > 0$ and $a > 1$.



Let V_x and V_y be the volumes of the solids generated when R is revolved about the x- and y-axes, respectively.

a) With $a = 2$ and $p = 1$, which is greater, V_x or V_y?

b) With $a = 4$ and $p = 3$, which is greater, V_x or V_y?

c) Find a general expression for V_x in terms of a and p. Note that $p = \frac{1}{2}$ is a special case, what is V_x when $p = \frac{1}{2}$?

d) Find a general expression for V_y in terms of a and p. Note that $p=2$ is a special case, what is V_y when $p=2$?

e) Explain how parts (*c*) and (*d*) demonstrate that $\lim\limits_{h\to 0}\dfrac{a^h-1}{h}=\ln a$

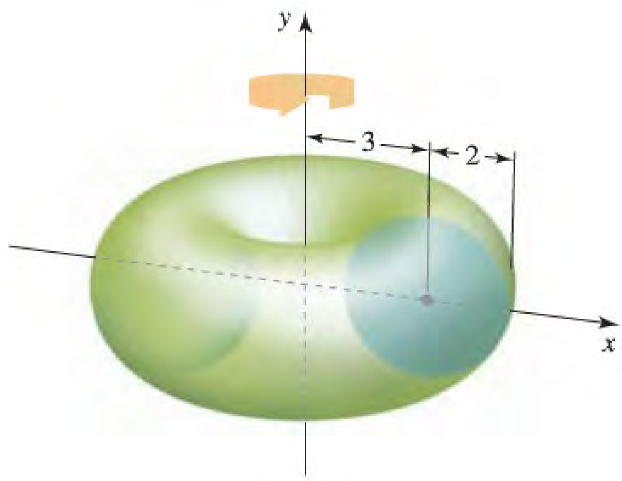
f) Find any values of a and p for which $V_x > V_y$

56. Let R be the region bounded by the graph of $f(x)=cx(1-x)$ and the x-*axis* on $[0,\ 1]$. Find the positive value of c such that the volume of the solid generated by revolving R about the x-*axis* equals the volume of the solid generated by revolving R about the y-*axis*.

57. Find the volume of the torus (doughnut formed when the circle of radius 2 centered at (3, 0) is revolved about the *y*-axis.

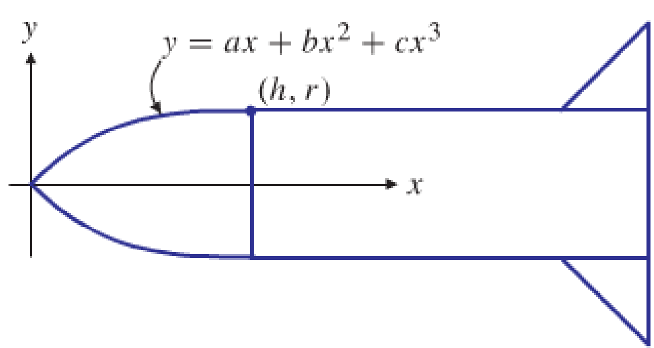
a) Use geometry to evaluate the integral

b) Use Shell method (use integral table)



58. The nose of a rocket is a solid of revolution of base radius *r* and height *h* that must join smoothly to the cylindrical body of the rocket. Taking the origin at the tip of the nose and the *x*-axis along the central axis of the rocket, various nose shapes can be obtained by revolving the cubic curve about *x*-axis.

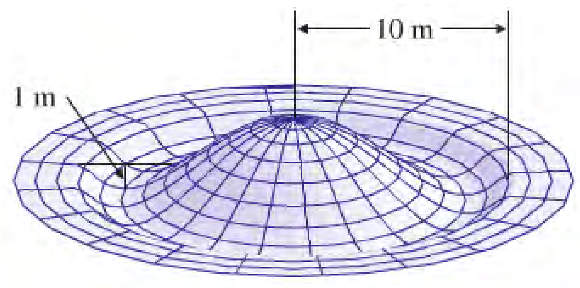
$$y=f(x)=ax+bx^2+cx^3$$



The cubic curve must have slope 0 at $x = h$, and its slope must be positive for $0 < x < h$. Find the particular cubic curve that maximizes the volume of the nose. Also show that his choice of the cubic makes the slope $\frac{dy}{dx}$ at the origin as large as possible and, hence, corresponds to the bluntest nose.

59. A landscaper wants to create on level ground a ring-shaped pool having an outside radius of 10 *m* and a maximum depth of 1 *m* surrounding a hill that will be built up using all the earth excavated from the pool. She decided to use a fourth-degree polynomial to determine the cross-sectional shape of the hill and pool bottom: at distance ***r*** *m* from the center of the development the height above or below normal ground level will be

$$h(r) = a\left(r^2 - 100\right)\left(r^2 - k^2\right) \quad m$$



For some $a > 0$, where k is the inner radius of the pool.
Find k and a so that the requirements given above are all satisfied.
How much earth must be moved from the pool to build the hill?