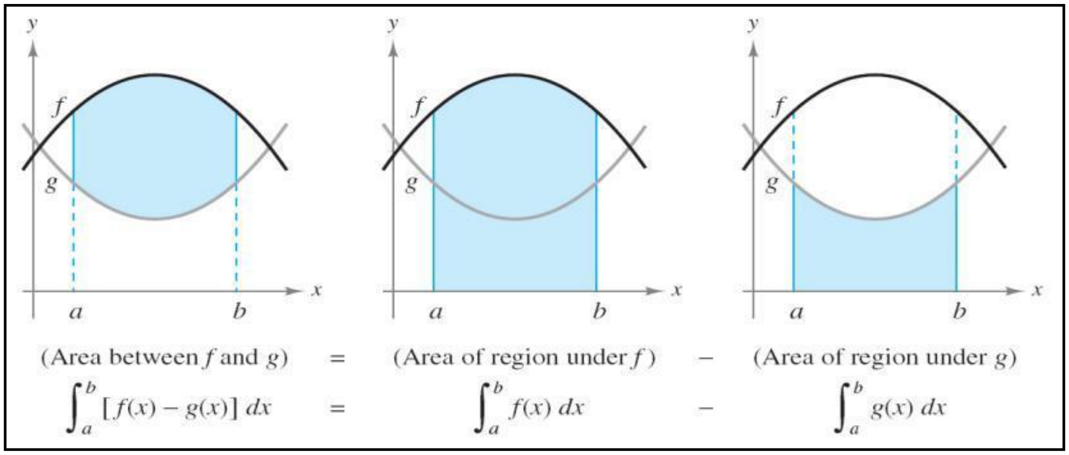
Section 4.5 – Area between Two Curves

Area of a Region Bounded by Two Graphs

If *f* and *g* are continuous on [*a*, *b*] and $g(x) \le f(x)$ fro all *x* in the interval, then the area of the region bounded by the graphs of *f*, *g*, *x* = *a*, and *x* = *b* is given by

$$A = \int_a^b [f(x) - g(x)]dx$$



Example

Find the area of the region bounded by the graphs of $y = x^2 + 1$ and $y = x$ for $0 \le x \le 2$

<u>*Solution*</u>

$$A = \int_0^2 [(x^2 + 1) - x]dx$$

$$A = \int_0^2 [x^2 - x + 1]dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + 1x \Big|_0^2$$

$$= \frac{2^3}{3} - \frac{2^2}{2} + 1(2) - 0$$

$$= \frac{8}{3}$$

Example

Find the area of the region bounded by the graphs of $y = 3 - x^2$ and $y = 2x$

<u>*Solution*</u>

Determine the intersection between two functions: $y = 3 - x^2 = 2x$

$$3 - x^2 - 2x = 0$$

$$x^2 + 2x - 3 = 0$$

$$\rightarrow \boxed{x = 1, -3}$$

$$A = \int_{-3}^{1} [(3 - x^2) - 2x]dx$$

$$A = \int_{-3}^{1} [-x^2 - 2x + 3]dx$$

$$= -\frac{x^3}{3} - 2\frac{x^2}{2} + 3x \Big|_{-3}^{1}$$

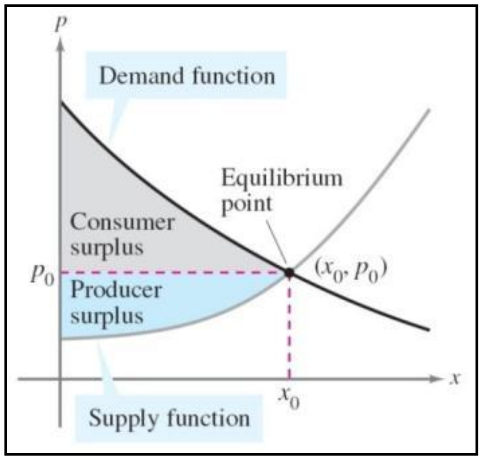
$$= -\frac{1^3}{3} - 1^2 + 3(1) - \left[-\frac{(-3)^3}{3} - (-3)^2 + 3(-3) \right]$$

$$= -\frac{1}{3} - 1 + 3 - [9 - 9 - 9]$$

$$= 11 - \frac{1}{3}$$

$$= \frac{32}{3}$$

Consumer Surplus and Producer Surplus



Demand Function: D(*x*)

Supply Function: S(*x*)

$$Consumer = \int_0^{x_0} (D - P_0)dx$$

$$Producer = \int_0^{x_0} (P_0 - S)dx$$

Example

The Demand and supply functions for a product are modeled by

Demand: $p = -0.2x + 8$ and *Supply*: $p = 0.1x + 2$

Where x is the number of units (in millions). Find the consumer and producer surpluses for this product.

<u>*Solution*</u>

$$-0.2x + 8 = 0.1x + 2$$

$$\Rightarrow -0.2x - 0.1x = 2 - 8$$

$$\Rightarrow -0.3x = -6$$

$$\Rightarrow x = 20$$

$$\text{Consumer} = \int_0^{20} \left[(-0.2x + 8) - 4\right] dx$$

$$= \int_0^{20} (-0.2x + 4) dx$$

$$= -0.2\frac{x^2}{2} + 4x \Big|_0^{20}$$

$$= -0.1(20)^2 + 4(20) - 0$$

$$= 40$$

$$\text{Producer} = \int_0^{20} \left[4 - (0.1x + 2)\right] dx$$

$$= \int_0^{20} \left[2 - 0.1x\right] dx$$

$$= 2x - 0.1\frac{x^2}{2} \Big|_0^{20}$$

$$= \left(2(20) - 0.1\frac{20^2}{2}\right) - 0$$

$$= 20$$

Example

The projected fuel cost C (in millions dollars per year) for a trucking company from 2008 through 2020 is $C_1 = 5.6 + 2.21t, \;\; 8 \le t \le 20$, where t = 8 corresponds to 2008. If the company purchases more efficient truck engines, fuel cost is expected to decrease and to follow the model $C_2 = 4.7 + 2.04t, \;\; 8 \le t \le 20$. How much can the company save with the more efficient engines?

<u>*Solution*</u>

$$\text{Petroleum saved} = \int_8^{20} (C_1 - C_2)dt$$

$$= \int_8^{20} \left[5.6 - 2.21t - (4.7 + 2.04t)\right]dt$$

$$= \int_8^{20} \left[5.6 - 2.21t - 4.7 - 2.04t\right]dt$$

$$= \int_8^{20} (0.17t + 0.9)dt$$

$$= 0.17\frac{t^2}{2} + 0.9t \Big|_8^{20}$$

$$= \left(0.17\frac{20^2}{2} + 0.9(20)\right) - \left(0.17\frac{8^2}{2} + 0.9(8)\right)$$

$= \$\ 39.36$ *millions*

Exercises *Section* 4.5 – Area between Two Curves

1. Find the area of the region bounded by the graphs of $y = x^2 - x - 2$ and x-axis

2. Find the area of the region bounded by the graphs of $f(x) = x^3 + 2x^2 - 3x$ and $g(x) = x^2 + 3x$

3. Find the area bounded by $f(x) = -x^2 + 1$, $g(x) = 2x + 4$, $x = -1$, and $x = 2$

4. Find the area between the curves $y = x^{1/2}$ *and* $y = x^3$

5. Find the area of the region bounded by the graphs of $y = x^2 - 2x$ and $y = x$ on [0, 4].

6. Find the area between the curves $x = 1,\quad x = 2,\quad y = x^3 + 2,\quad y = 0$

7. Find the area between the curves $y = x^2 - 18,\quad y = x - 6$

8. Find the area between the curves $x = -1,\quad x = 2,\quad y = e^{-x},\quad y = e^x$

9. Find the area between the curves $y = \sqrt{x},\quad y = x\sqrt{x}$

10. A company is considering a new manufacturing process in one of its plants. The new process provides substantial initial savings, with the savings declining with time *t* (in years) according to the rate-of-savings function

$$S'(t) = 100 - t^2$$

where $S'(t)$ is in thousands of dollars per year. At the same time, the cost of operating the new process increases with time *t* (in years), according to the rate-of-cost function (in thousands of dollars per year)

$$C'(t) = t^2 + \frac{14}{3}t$$

a) For how many years will the company realize savings?
b) What will be the net total savings during this period?

11. Find the producers' surplus if the supply function for pork bellies is given by

$$S(x) = x^{5/2} + 2x^{3/2} + 50$$

Assume supply and demand are in equilibrium at $x = 16$.

12. Suppose the supply function for concrete is given by

$$S(q) = 100 + 3q^{3/2} + q^{5/2}$$

And that supply and demand are in equilibrium at $q = 9$. Find the producers' surplus.

14. Find the consumers' surplus if the demand function for grass seed is given by

$$D(x) = \frac{200}{(3x+1)^2}$$

Assuming supply and demand are in equilibrium at $x = 3$.

15. Find the consumers' surplus if the demand function for olive oil is given by

$$D(x) = \frac{32{,}000}{(2x+8)^3}$$

And if supply and demand are in equilibrium at $x = 6$.