Solution Section 1.5 – Length of Curves

Exercise

Find the length of the curve $y = \frac{1}{3}\left(x^2+2\right)^{3/2}$ from $x = 0$ to $x = 3$.

Solution

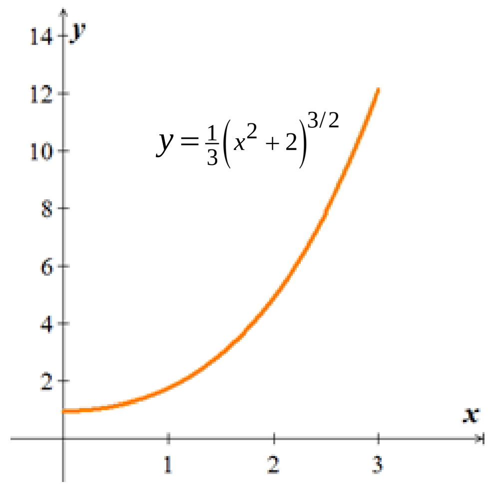
$$\frac{dy}{dx} = \frac{1}{3}\frac{3}{2}\left(x^2+2\right)^{1/2}(2x) = x\left(x^2+2\right)^{1/2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+x^2\left(x^2+2\right)}$$

$$= \sqrt{1+x^4+2x^2}$$

$$= \sqrt{\left(x^2+1\right)^2}$$

$$= x^2+1$$



$$L = \int_0^3 \left(x^2+1\right)\,dx$$

$$= \left[\frac{1}{3}x^3 + x\right]_0^3$$

$$= \frac{1}{3}(3)^3 + (3) - 0$$

$$\underline{= 12 \ \ unit}$$

Exercise

Find the length of the curve $y = (x)^{3/2}$ from $x = 0$ to $x = 4$.

Solution

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{9}{4}x}$$

$$= \sqrt{\frac{4+9x}{4}}$$

$$= \frac{1}{2}\sqrt{4+9x}$$

$$L = \int_0^4 \frac{1}{2}(4+9x)^{1/2}\,dx \qquad u = 4+9x \ \Rightarrow du = 9dx \ \rightarrow \frac{1}{9}du = dx \quad \begin{cases} x=4 & \rightarrow u = 40 \\ x = 0 & \rightarrow u = 4 \end{cases}$$

$$= \frac{1}{2}\int_4^{40} \frac{1}{9}u^{1/2}du$$

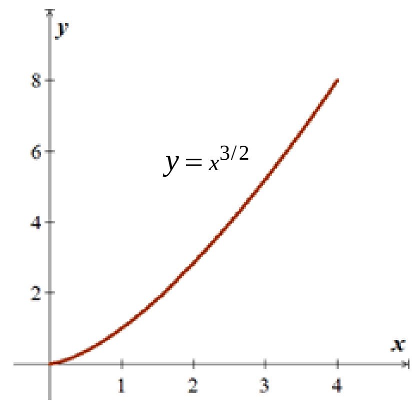
$$= \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_4^{40}$$

$$= \frac{1}{27}\left(40^{3/2} - 4^{3/2}\right)$$

$$= \frac{1}{27}\left(\sqrt{40^3} - \sqrt{4^3}\right)$$

$$= \frac{1}{27}\left(80\sqrt{10} - 8\right)$$

$$= \frac{8}{27}\left(10\sqrt{10} - 1\right) \ \ unit$$



Exercise

Find the length of the curve $x = \frac{y^{3/2}}{3} - y^{1/2}$ from $y = 1$ to $y = 9$.

Solution

$$\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} = \frac{1}{2}\left(y^{1/2} - \frac{1}{y^{1/2}}\right)$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\frac{1}{4}\left(y^{1/2} - \frac{1}{y^{1/2}}\right)^2}$$

$$= \sqrt{1+\frac{1}{4}\left(y - 2 + \frac{1}{y}\right)}$$

$$= \sqrt{1+\frac{1}{4}y - \frac{1}{2} + \frac{1}{4y}}$$

$$= \sqrt{\frac{1}{4}y + \frac{1}{2} + \frac{1}{4y}}$$

$$= \sqrt{\frac{1}{4}\left(y + 2 + \frac{1}{y}\right)}$$

$$= \frac{1}{2}\sqrt{\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2}$$

$$= \frac{1}{2}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)$$

$$L = \frac{1}{2}\int_1^9 \left(y^{1/2} + y^{-1/2}\right)dy$$

$$a = \frac{1}{3}, \quad m = \frac{3}{2}, \quad b = -1, \quad n = \frac{1}{2}$$

1. $m + n = \frac{3}{2} + \frac{1}{2} = 2$ ✓

2. $abmn = \frac{1}{3}(-1)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(\frac{1}{3}y^{3/2} + y^{1/2}\right)_1^9$$

$$= 9 + 3 - \frac{4}{3}$$

$$= \frac{32}{3} \ \ unit$$

$$= \frac{1}{2}\left[\frac{2}{3}y^{3/2} + 2y^{1/2}\right]_1^9$$

$$= \left[\frac{1}{3}y^{3/2} + y^{1/2}\right]_1^9$$

$$= \left[\frac{1}{3}9^{3/2} + 9^{1/2} - \left(\frac{1}{3}1^{3/2} + 1^{1/2}\right)\right]$$

$$= \frac{1}{3}3^3 + 3 - \left(\frac{1}{3} + 1\right)$$

$$= 9 + 3 - \frac{4}{3}$$

$$= \frac{32}{3} \ \ \textit{unit}$$

Exercise

Find the length of the curve $x = \dfrac{y^3}{6} + \dfrac{1}{2y}$ from $y = 2$ to $y = 3$.

Solution

$$\frac{dx}{dy} = \frac{1}{2}y^2 - \frac{1}{2y^2} = \frac{1}{2}\left(y^2 - y^{-2}\right)$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \frac{1}{4}\left(y^2 - y^{-2}\right)^2}$$

$$= \frac{1}{2}\sqrt{4 + \left(y^4 - 2 + y^{-4}\right)}$$

$$= \frac{1}{2}\sqrt{y^4 + 2 + y^{-4}}$$

$$= \frac{1}{2}\sqrt{\left(y^2 + y^{-2}\right)^2}$$

$$= \frac{1}{2}\left(y^2 + y^{-2}\right)$$

$$L = \frac{1}{2}\int_2^3 \left(y^2 + y^{-2}\right)dy$$

$$= \frac{1}{2}\left[\frac{1}{3}y^3 - y^{-1}\right]_2^3$$

$$= \frac{1}{2}\left[\left(\frac{1}{3}3^3 - 3^{-1}\right) - \left(\frac{1}{3}2^3 - 2^{-1}\right)\right]$$

$$= \frac{1}{2}\left[9 - \frac{1}{3} - \left(\frac{8}{3} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left(\frac{26}{3} - \frac{13}{6}\right)$$

$$= \frac{13}{4} \ \ \textit{unit}$$

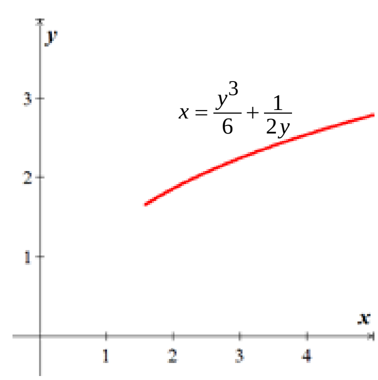
$a = \frac{1}{6}, \quad m = 3, \quad b = \frac{1}{2}, \quad n = -1$

1. $m + n = 3 - 1 = 2$ ✓
2. $abmn = \frac{1}{6}\left(\frac{1}{2}\right)(3)(-1) = -\frac{1}{4}$ ✓

$$L = \left(\frac{y^3}{6} - \frac{1}{2y}\right)_1^9$$

$$= \frac{1}{2}\left[9 - \frac{1}{3} - \left(\frac{8}{3} - \frac{1}{2}\right)\right]$$

$$= \frac{13}{4} \ \ \textit{unit}$$



Exercise

Find the length of the curve $f(x)=x^3+\frac{1}{12x}$ *for* $\frac{1}{2}\le x\le 2$

Solution

$$a=1,\quad m=3,\quad b=\frac{1}{12},\quad n=-1$$

1. $m+n=2$ ✓ **2.** $abmn=-\frac{1}{4}$ ✓

$$L=\left(x^3-\frac{1}{12x}\right)\Big|_{1/2}^{2}$$

$$=8-\frac{1}{24}-\frac{1}{8}+\frac{1}{6}$$

$$=8\ \textit{unit}$$

Exercise

Find the length of the curve of $f(x)=\frac{1}{5}x^5+\frac{1}{12x^3}$ $1\le x\le 2$

Solution

$$a=\frac{1}{5},\quad m=5,\quad b=\frac{1}{12},\quad n=-3$$

1. $m+n=5-3=2$ ✓ **2.** $abmn=\frac{1}{5}\left(\frac{1}{12}\right)(5)(-3)=-\frac{1}{4}$ ✓

$$L=\frac{1}{5}x^5-\frac{1}{12x^3}\Big|_{1}^{2}$$

$$=\frac{32}{5}-\frac{1}{96}-\frac{1}{5}+\frac{1}{12}$$

$$=\frac{31}{5}+\frac{7}{96}$$

$$=\frac{3011}{480}$$

Exercise

Find the length of the curve of $y=\frac{1}{3}x^{1/2}-x^{3/2},\quad 0\le x\le\frac{1}{3}$

Solution

$$a=\frac{1}{3},\quad m=\frac{1}{2},\quad b=-1,\quad n=\frac{3}{2}$$

3. $m+n=\frac{1}{2}+\frac{3}{2}=2$ ✓ **4.** $abmn=\frac{1}{3}(-1)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)=-\frac{1}{4}$ ✓

$$L=\frac{1}{3}x^{1/2}+x^{3/2}\Big|_{0}^{1/3}$$

$$= \frac{1}{3\sqrt{3}} + \frac{1}{3\sqrt{3}}$$

$$= \frac{2}{3\sqrt{3}} \qquad\qquad = \frac{2\sqrt{3}}{9}$$

Exercise

Find the length of the curve of $\quad y = \frac{1}{3}x^3 + \frac{1}{4x}, \quad 1 \le x \le 2$

Solution

$a = \frac{1}{3}, \quad m = 3, \quad b = \frac{1}{4}, \quad n = -1$

5. $m + n = 3 - 1 = 2$ ✓ **6.** $abmn = \frac{1}{3}\left(\frac{1}{4}\right)(3)(-1) = -\frac{1}{4}$ ✓

$$L = \frac{1}{3}x^3 - \frac{1}{4x}\Big|_1^2$$

$$= \frac{8}{3} - \frac{1}{8} - \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{3} + \frac{1}{8}$$

$$= \frac{59}{24}$$

Exercise

Find the length of the curve of $\quad y = 2e^x + \frac{1}{8}e^{-x} \quad 0 \le x \le \ln 2$

Solution

$a = 2, \quad m = 1, \quad b = \frac{1}{8}, \quad n = -1$

7. $m = -n = 1$ ✓ **8.** $abmn = 2\left(\frac{1}{8}\right)(1)(-1) = -\frac{1}{4}$ ✓

$$L = 2e^x - \frac{1}{8}e^{-x}\Big|_0^{\ln 2}$$

$$= 2e^{\ln 2} - \frac{1}{8}e^{-\ln 2} - 2 + \frac{1}{8}$$

$$= 4 - \frac{1}{16} - \frac{15}{8}$$

$$= \frac{33}{16}$$

Exercise

Find the length of the curve of $y=e^{2x}+\frac{1}{16}e^{-2x}, \quad 0\le x\le \ln 3$

Solution

$$a=1, \quad m=2, \quad b=\frac{1}{16}, \quad n=-2$$

9. $m=-n=2$ ✓ **10.** $abmn=1\left(\frac{1}{16}\right)(2)(-2)=-\frac{1}{4}$ ✓

$$L=e^{2x}-\frac{1}{16}e^{-2x}\Big|_0^{\ln 3}$$

$$=e^{2\ln 3}-\frac{1}{16}e^{-2\ln 3}-1+\frac{1}{16}$$

$$=9-\frac{1}{16}\left(\frac{1}{9}\right)-\frac{15}{16}$$

$$=\frac{1,160}{144}$$

$$\boxed{=\frac{145}{18}}$$

Exercise

Find the length of the curve $y=\ln(\cos x) \quad 0\le x\le \frac{\pi}{4}$

Solution

$$\frac{dy}{dx}=\frac{-\sin x}{\cos x}=-\tan x$$

$$L=\int_0^{\pi/4}\sqrt{1+\tan^2 x}\ dx$$

$$=\int_0^{\pi/4}\sqrt{\sec^2 x}\ dx$$

$$=\int_0^{\pi/4}\sec x\ dx$$

$$=\left[\ln|\sec x+\tan x|\right]_0^{\pi/4}$$

$$=\ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right|-\ln|\sec 0+\tan 0|$$

$$=\ln\left|\sqrt{2}+1\right|-\ln|1+0|$$

$$=\ln\left|\sqrt{2}+1\right|-0$$

$$\boxed{=\ln\left(\sqrt{2}+1\right)\ \ unit}$$

Exercise

Find the length of the curve $f(y)=2e^{\sqrt{2}y}+\frac{1}{16}e^{-\sqrt{2}y}$ *for* $0\le y\le \frac{\ln 2}{\sqrt{2}}$

Solution

$a=2,\quad m=\sqrt{2},\quad b=\frac{1}{16},\quad n=-\sqrt{2}$

1. $m=-n$ ✓ **2.** $abmn=2(\sqrt{2})\left(\frac{1}{16}\right)(-\sqrt{2})=-\frac{1}{4}$ ✓

$$L=\left(2e^{\sqrt{2}y}+\frac{1}{16}e^{-\sqrt{2}y}\right)\Big|_0^{\ln 2/\sqrt{2}}$$

$$=2e^{\ln 2}+\frac{1}{16}e^{-\ln 2}-2-\frac{1}{16}$$

$$=4+\frac{1}{32}-\frac{33}{16}$$

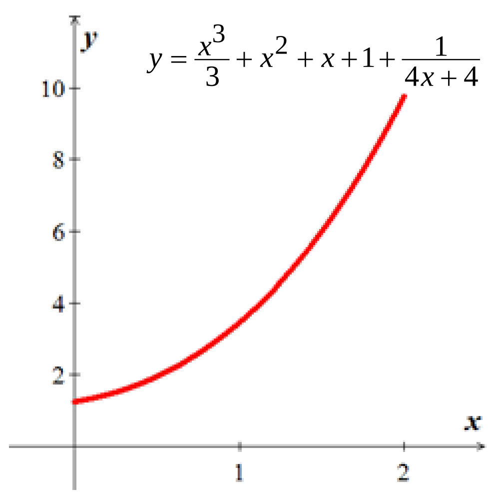
$$=\frac{63}{32}\ unit$$

Exercise

Find the length of the curve $y=\frac{x^3}{3}+x^2+x+1+\frac{1}{4x+4}\quad 0\le x\le 2$

Solution

$$\frac{dy}{dx}=x^2+2x+1-\frac{1}{4}\frac{1}{(x+1)^2}=(x+1)^2-\frac{1}{4}\frac{1}{(x+1)^2}$$



$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\left((x+1)^2-\frac{1}{4}\frac{1}{(x+1)^2}\right)^2}$$

$$=\sqrt{1+(x+1)^4-\frac{1}{2}-\frac{1}{16}\frac{1}{(x+1)^4}}$$

$$=\sqrt{(x+1)^4+\frac{1}{2}-\frac{1}{16}\frac{1}{(x+1)^4}}$$

$$=\sqrt{\left((x+1)^2+\frac{1}{4}\frac{1}{(x+1)^2}\right)^2}$$

$$=(x+1)^2+\frac{1}{4}(x+1)^{-2}$$

$$L=\int_0^2\left((x+1)^2+\frac{1}{4}(x+1)^{-2}\right)dx$$

$u=x+1\quad\Rightarrow du=dx\quad \begin{cases}x=2 & \rightarrow u=3\\ x=0 & \rightarrow u=1\end{cases}$

$$= \int_1^3 \left(u^2 + \frac{1}{4}u^{-2}\right) du$$

$$= \left[\frac{1}{3}u^3 - \frac{1}{4}u^{-1}\right]_1^3$$

$$= 9 - \frac{1}{12} - \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{53}{6} \ \ unit$$

Exercise

Find the length of the curve $y = \frac{x^3}{3} + x^2 + x + 1 + \frac{1}{4x+4} \quad 0 \le x \le 4$

Solution

$$\frac{dy}{dx} = x^2 + 2x + 1 - \frac{1}{4}\frac{1}{(x+1)^2}$$

$$= (x+1)^2 - \frac{1}{4}\frac{1}{(x+1)^2}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left((x+1)^2 - \frac{1}{4}\frac{1}{(x+1)^2}\right)^2}$$

$$= \sqrt{1 + (x+1)^4 - \frac{1}{2} - \frac{1}{16}\frac{1}{(x+1)^4}}$$

$$= \sqrt{(x+1)^4 + \frac{1}{2} - \frac{1}{16}\frac{1}{(x+1)^4}}$$

$$= \sqrt{\left((x+1)^2 + \frac{1}{4}\frac{1}{(x+1)^2}\right)^2}$$

$$= (x+1)^2 + \frac{1}{4}(x+1)^{-2}$$

$$L = \int_0^4 \left((x+1)^2 + \frac{1}{4}(x+1)^{-2}\right) dx$$

$$= \int_0^4 \left((x+1)^2 + \frac{1}{4}(x+1)^{-2}\right) d(x+1)$$

$$= \left(\frac{1}{3}(x+1)^3 - \frac{1}{4}(x+1)^{-1}\right|_0^4$$

$$= \frac{125}{3} - \frac{1}{20} - \frac{1}{3} + \frac{1}{4}$$

$$=\frac{124}{3}+\frac{1}{5}$$

$$\boxed{=\frac{623}{15}\ \ unit}$$

Exercise

Find the length of the curve $y=\ln\left(e^{x}-1\right)-\ln\left(e^{x}+1\right) \quad \ln 2\le x\le \ln 3$

Solution

$$y=\ln\left(e^{x}-1\right)-\ln\left(e^{x}+1\right) \quad \Rightarrow \frac{dy}{dx}=\frac{e^{x}}{e^{x}-1}-\frac{e^{x}}{e^{x}+1}$$

$$=\frac{e^{2x}+e^{x}-e^{2x}+e^{x}}{e^{2x}-1}$$

$$=\frac{2e^{x}}{e^{2x}-1}$$

$$L=\int_{\ln 2}^{\ln 3}\sqrt{1+\left(\frac{2e^{x}}{e^{2x}-1}\right)^{2}}\ dx$$

$$=\int_{\ln 2}^{\ln 3}\sqrt{1+\frac{4e^{2x}}{e^{4x}-2e^{2x}+1}}\ dx$$

$$=\int_{\ln 2}^{\ln 3}\sqrt{\frac{e^{4x}-2e^{2x}+1+4e^{2x}}{\left(e^{2x}-1\right)^{2}}}\ dx$$

$$=\int_{\ln 2}^{\ln 3}\sqrt{\frac{e^{4x}+2e^{2x}+1}{\left(e^{2x}-1\right)^{2}}}\ dx$$

$$=\int_{\ln 2}^{\ln 3}\sqrt{\frac{\left(e^{2x}+1\right)^{2}}{\left(e^{2x}-1\right)^{2}}}\ dx$$

$$=\int_{\ln 2}^{\ln 3}\frac{e^{2x}+1}{e^{2x}-1}\ dx$$

$$=\int_{\ln 2}^{\ln 3}\frac{\frac{e^{2x}+1}{e^{x}}}{\frac{e^{2x}-1}{e^{x}}}\ dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\frac{e^{2x}}{e^x} + \frac{1}{e^x}}{\frac{e^{2x}}{e^x} - \frac{1}{e^x}} dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^x - e^{-x}} dx \qquad \textit{or} \;\; \text{Let } u = e^x - e^{-x} \;\; \Rightarrow du = \left(e^x + e^{-x}\right)dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{1}{e^x - e^{-x}} d\left(e^x - e^{-x}\right) \qquad d\left(e^x - e^{-x}\right) = \left(e^x + e^{-x}\right)dx$$

$$= \left[\ln\left|e^x - e^{-x}\right|\right]_{\ln 2}^{\ln 3}$$

$$= \ln\left(3 - \frac{1}{3}\right) - \ln\left(2 - \frac{1}{2}\right)$$

$$= \ln\left(\frac{8}{3}\right) - \ln\left(\frac{3}{2}\right)$$

$$= \ln\left(\frac{16}{9}\right) \;\; unit$$

Exercise

Find the length of the curve $f(x) = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2} \quad 1 \le x \le 4$

Solution

$$a = \frac{2}{3}, \quad m = \frac{3}{2}, \quad b = -\frac{1}{2}, \quad n = \frac{1}{2}$$

1. $m + n = \frac{3}{2} + \frac{1}{2} = 2$ ✓ **2.** $abmn = \frac{2}{3}\left(\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(\frac{2}{3}x^{3/2} + \frac{1}{2}x^{1/2}\right)\Big|_1^4$$

$$= \frac{2}{3}4^{3/2} + 1 - \frac{2}{3} - \frac{1}{2}$$

$$= \frac{16}{3} - \frac{2}{3} + \frac{1}{2}$$

$$= \frac{31}{6} \;\; unit$$

Exercise

Find the length of the curve $f(x) = x^3 + \frac{1}{12x} \quad 1 \le x \le 4$

Solution

$$a = 1, \quad m = 3, \quad b = \frac{1}{12}, \quad n = -1$$

1. $m + n = 3 - 1 = 2$ ✓ **2.** $abmn = (1)\left(\frac{1}{12}\right)(3)(-1) = -\frac{1}{4}$ ✓

$$L = \left(x^3 - \frac{1}{12x}\right)\Big|_1^4$$

$$= 4^3 - \frac{1}{48} - 1 + \frac{1}{12}$$

$$= 63 + \frac{3}{48}$$

$$= \frac{3,027}{48} \ \ unit$$

Exercise

Find the length of the curve $f(x) = \frac{1}{8}x^4 + \frac{1}{4x^2} \quad 1 \le x \le 10$

Solution

$$a = \frac{1}{8}, \quad m = 4, \quad b = \frac{1}{4}, \quad n = -2$$

1. $m + n = 4 - 2 = 2$ ✓ **2.** $abmn = \left(\frac{1}{8}\right)\left(\frac{1}{4}\right)(4)(-2) = -\frac{1}{4}$ ✓

$$L = \left(\frac{1}{8}x^4 - \frac{1}{4x^2}\right)\Big|_1^{10}$$

$$= \frac{10^4}{8} - \frac{1}{400} - \frac{1}{8} + \frac{1}{4}$$

$$= \frac{9,999}{8} + \frac{99}{400}$$

$$= \frac{9}{8}\left(1111 + \frac{11}{50}\right)$$

$$= \frac{9}{8}\left(\frac{55,561}{50}\right)$$

$$= \frac{500,049}{400} \ \ unit$$

Exercise

Find the length of the curve $f(x)=\frac{1}{4}x^4+\frac{1}{8x^2} \quad 3\le x\le 8$

Solution

$$a=\frac{1}{4}, \quad m=4, \quad b=\frac{1}{8}, \quad n=-2$$

1. $m+n=4-2=2$ ✓ **2.** $abmn=\left(\frac{1}{4}\right)\left(\frac{1}{8}\right)(4)(-2)=-\frac{1}{4}$ ✓

$$L=\left(\frac{1}{4}x^4-\frac{1}{8x^2}\right)\Big|_3^8$$

$$=\frac{8^4}{4}-\frac{1}{8^3}-\frac{81}{4}+\frac{1}{72}$$

$$=\frac{4,015}{4}-\frac{1}{512}+\frac{1}{72}$$

$$=\frac{1}{4}\left(4,015-\frac{1}{128}+\frac{1}{18}\right)$$

$$=\frac{1}{4}\left(4,015+\frac{55}{1152}\right)$$

$$=\frac{4,625,335}{4,608} \text{ unit}$$

Exercise

Find the length of the curve $f(x)=\frac{1}{10}x^5+\frac{1}{6x^3} \quad 1\le x\le 7$

Solution

$$a=\frac{1}{10}, \quad m=5, \quad b=\frac{1}{6}, \quad n=-3$$

1. $m+n=5-3=2$ ✓ **2.** $abmn=\left(\frac{1}{10}\right)\left(\frac{1}{6}\right)(5)(-3)=-\frac{1}{4}$ ✓

$$L=\left(\frac{1}{10}x^5-\frac{1}{6x^3}\right)\Big|_1^7$$

$$=\frac{7^5}{10}-\frac{1}{2058}-\frac{1}{10}+\frac{1}{6}$$

$$=\frac{8403}{5}+\frac{57}{343}$$

$$=\frac{2,882,514}{1,715} \text{ unit}$$

Exercise

Find the length of the curve $f(x)=\frac{3}{10}x^{1/3}-\frac{3}{2}x^{5/3} \quad 0\le x\le 12\,\text{b}$

Solution

$$a=\frac{1}{10},\quad m=5,\quad b=\frac{1}{6},\quad n=-3$$

1. $m+n=5-3=2$ ✓ **2.** $abmn=\left(\frac{1}{10}\right)\left(\frac{1}{6}\right)(5)(-3)=-\frac{1}{4}$ ✓

$$L=\left(\frac{3}{10}x^{1/3}+\frac{3}{2}x^{5/3}\right)\Big|_0^{12}$$

$$=\frac{3}{10}\sqrt[3]{12}+\frac{3}{2}12\sqrt[3]{144}$$

$$=\frac{3}{10}\sqrt[3]{12}+18\sqrt[3]{144}\ \ unit \qquad =\frac{3}{10}\sqrt[3]{12}\left(1+600\sqrt[3]{12}\right)$$

Exercise

Find the length of the curve $f(x)=x^{1/2}-\frac{1}{3}x^{3/2} \quad 2\le x\le 9$

Solution

$$a=1,\quad m=\frac{1}{2},\quad b=-\frac{1}{3},\quad n=\frac{3}{2}$$

1. $m+n=\frac{1}{2}+\frac{3}{2}=2$ ✓ **2.** $abmn=(1)\left(-\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)=-\frac{1}{4}$ ✓

$$L=\left(x^{1/2}+\frac{1}{3}x^{3/2}\right)\Big|_2^{9}$$

$$=3+9-\sqrt{2}-\frac{2\sqrt{2}}{3}$$

$$=\frac{1}{3}\left(36-5\sqrt{2}\right)\ \ unit$$

Exercise

Find the length of the curve $y=x^{1/2}-\frac{1}{3}x^{3/2} \quad 1\le x\le 4$

Solution

$$a=1,\quad m=\frac{1}{2},\quad b=-\frac{1}{3},\quad n=\frac{3}{2}$$

1. $m+n=\frac{1}{2}+\frac{3}{2}=2$ ✓ **2.** $abmn=(1)\left(-\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)=-\frac{1}{4}$ ✓

$$L=\left(x^{1/2}+\frac{1}{3}x^{3/2}\right)\Big|_1^{4}$$

$$= 2 + \frac{8}{3} - 1 - \frac{1}{3}$$

$$= 1 + \frac{7}{3}$$

$$\boxed{= \frac{10}{3} \ \ unit}$$

Exercise

Find the length of the curve $x = y^{2/3}, \quad 1 \le y \le 8$

Solution

$$x' = \frac{2}{3} y^{-1/3}$$

$$\left(x'\right)^2 = \frac{4}{9} y^{-2/3}$$

$$L = \int_1^8 \sqrt{1 + \frac{4}{9y^{2/3}}} \ dy$$

$$= \int_1^8 \frac{1}{3y^{1/3}} \sqrt{9y^{2/3} + 4} \ dy$$

$$= \frac{1}{3} \int_1^8 y^{-1/3} \sqrt{9y^{2/3} + 4} \ dy$$

$$= \frac{1}{18} \int_1^8 \left(9y^{2/3} + 4\right)^{1/2} \ d\left(9y^{2/3} + 4\right)$$

$$= \frac{1}{27} \left(9y^{2/3} + 4\right)^{3/2} \Big|_1^8$$

$$= \frac{1}{27} \left(\left(9\left(2^3\right)^{2/3} + 4 \right)^{3/2} - 13^{3/2} \right)$$

$$\boxed{= \frac{1}{27} \left(40^{3/2} - 13^{3/2}\right) \ \ unit}$$

Exercise

Find the length of the curve $y = 2x + 4 \quad -2 \le x \le 2$

Solution

$$\sqrt{1 + \left(y'\right)^2} = \sqrt{1 + 2^2}$$

$$= \sqrt{5}$$

$$L = \int_{-2}^{2} \sqrt{5}\ dx$$

$$= \sqrt{5}\ x \Big|_{-2}^{2}$$

$$= 4\sqrt{5}\ \ unit$$

Exercise

Find the length of the curve $y = \frac{x^3}{6} + \frac{1}{2x} \quad x \in [1,\ 2]$

Solution

$$a = \frac{1}{6}, \quad m = 3, \quad b = \frac{1}{2}, \quad n = -1$$

1. $m + n = 3 - 1 = 2$ ✓ **2.** $abmn = \left(\frac{1}{6}\right)(3)\left(\frac{1}{2}\right)(-1) = -\frac{1}{4}$ ✓

$$L = \frac{x^3}{6} - \frac{1}{2x}\Big|_{1}^{2}$$

$$= \frac{4}{3} - \frac{1}{4} - \frac{1}{6} + \frac{1}{2}$$

$$= \frac{7}{12}\ \ unit$$

Exercise

Find the length of the curve $f(x) = x^{1/2} - \frac{1}{3}x^{3/2} \quad 1 \le x \le 3$

Solution

$$a = 1, \quad m = \frac{1}{2}, \quad b = -\frac{1}{3}, \quad n = \frac{3}{2}$$

1. $m + n = \frac{1}{2} + \frac{3}{2} = 2$ ✓ **2.** $abmn = (1)\left(-\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(x^{1/2} + \frac{1}{3}x^{3/2}\right)\Big|_{1}^{3}$$

$$= \sqrt{3} + \sqrt{3} - 1 - \frac{1}{3}$$

$$= 2\sqrt{3} - \frac{4}{3}\ \ unit$$

Exercise

Find the length of the curve $y=\frac{3}{4}x^{4/3}-\frac{3}{8}x^{2/3}+5, \quad 1\le x\le 8$

Solution

$$a=\frac{3}{4}, \quad m=\frac{4}{3}, \quad b=-\frac{3}{8}, \quad n=\frac{2}{3}$$

1. $m+n=\frac{4}{3}+\frac{2}{3}=2$ ✓ **2.** $abmn=\left(\frac{3}{4}\right)\left(\frac{4}{3}\right)\left(-\frac{3}{8}\right)\left(\frac{2}{3}\right)=-\frac{1}{4}$ ✓

$$L=\left(\frac{3}{4}x^{4/3}+\frac{3}{8}x^{2/3}\right)\Big|_1^8$$

$$=\frac{3}{4}\left(2^3\right)^{4/3}+\frac{3}{8}\left(2^3\right)^{2/3}-\frac{3}{4}-\frac{3}{8}$$

$$=12+\frac{3}{2}-\frac{3}{4}-\frac{3}{8}$$

$$=\frac{96+12-6-3}{8}$$

$$=\frac{99}{8} \ \ unit$$

Exercise

Find the length of the curve $y=\ln x-\frac{1}{8}x^2; \quad 1\le x\le 2$

Solution

$$\sqrt{1+(y')^2}=\sqrt{1+\left(\frac{1}{x}-\frac{1}{4}x\right)^2}$$

$$=\sqrt{1+\frac{1}{x^2}-\frac{1}{2}+\frac{1}{16}x^2}$$

$$=\sqrt{\frac{1}{x^2}+\frac{1}{2}+\frac{1}{16}x^2}$$

$$=\sqrt{\left(\frac{1}{x}+\frac{1}{4}x\right)^2}$$

$$=\frac{1}{x}+\frac{1}{4}x$$

$$L=\int_1^2\left(\frac{1}{x}+\frac{1}{4}x\right)\,dx$$

$$=\ln x+\frac{1}{8}x^2\Big|_1^2$$

$$= \ln 2 + \frac{1}{2} - \frac{1}{8}$$

$$= \ln 2 + \frac{3}{8} \ \ unit$$

Exercise

Find the length of the curve $\quad y = \frac{1}{2}x^2 - \frac{1}{4}\ln x; \quad 1 \le x \le 3$

Solution

$$\sqrt{1+(y')^2} = \sqrt{1+\left(x-\frac{1}{4x}\right)^2}$$

$$= \sqrt{1+x^2-\frac{1}{2}+\frac{1}{16x^2}}$$

$$= \sqrt{x^2+\frac{1}{2}+\frac{1}{16x^2}}$$

$$= \sqrt{\left(x+\frac{1}{4x}\right)^2}$$

$$= x + \frac{1}{4x}$$

$$L = \int_1^3 \left(x+\frac{1}{4x}\right) dx$$

$$= \frac{1}{2}x^2 + \frac{1}{4}\ln x \ \Big|_1^3$$

$$= \frac{9}{2} + \frac{1}{4}\ln 3 - \frac{1}{2}$$

$$= 4 + \frac{1}{4}\ln 2 \ \ unit$$

Exercise

Find the length of the curve $\quad y = \int_{-2}^{x} \sqrt{2t^4-1}\, dt \quad -2 \le x \le -1$

Solution

$$\frac{dy}{dt} = \sqrt{2t^4-1}$$

$$\sqrt{1+\left(\frac{dy}{dt}\right)^2} = \sqrt{1+2t^4-1}$$

$$= \sqrt{2t^4}$$

$$= \sqrt{2}\ t^2$$

$$L = \sqrt{2}\int_{-2}^{-1} t^2\ dt$$

$$= \frac{\sqrt{2}}{3} t^3 \Big|_{-2}^{-1}$$

$$= \frac{\sqrt{2}}{3}(-1+8)$$

$$= \frac{7\sqrt{2}}{3}\ \ unit$$

Exercise

Find the length of the curve $x = \int_0^y \sqrt{\sec^4 t - 1}\ dt \quad -\frac{\pi}{4} \le y \le \frac{\pi}{4}$

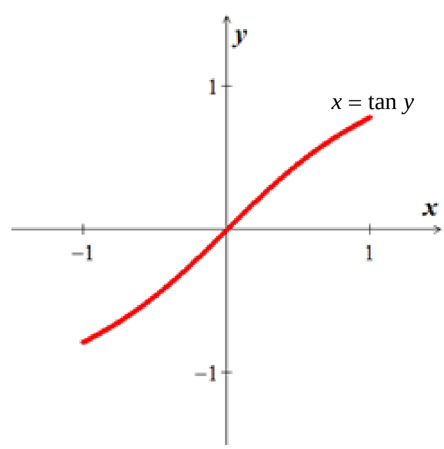
Solution

$$\frac{dx}{dy} = \sqrt{\sec^4 y - 1}$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\sec^4 y - 1}$$

$$= \sqrt{\sec^4 y}$$

$$= \sec^2 y$$



$$L = \int_{-\pi/4}^{\pi/4} \sec^2 y\ \ dy$$

$$= \tan y \Big|_{-\pi/4}^{\pi/4}$$

$$= 1-(-1)$$

$$= 2\ \ unit$$

Exercise

Find the length of the curve $y = 3-2x \quad 0 \le x \le 2$. Check your answer by finding the length of the segment as the hypotenuse of a right triangle.

Solution

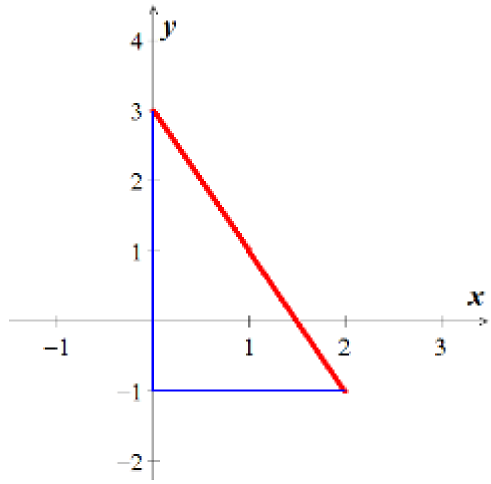
$$\frac{dy}{dx} = -2$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+4} = \sqrt{5}$$

$$L = \int_0^2 \sqrt{5}\ dx$$

$$= \sqrt{5}x \Big|_0^2$$

$$\underline{= 2\sqrt{5}\ \ unit}$$



$$\begin{cases} x = 0 & \rightarrow y = 3 \\ x = 2 & \rightarrow y = -1 \end{cases}$$

$$d = \sqrt{(2-0)^2 + (3+1)^2}$$

$$= \sqrt{4+16}$$

$$= \sqrt{20}$$

$$\underline{= 2\sqrt{5}\ \ unit}$$

Exercise

Find a curve through the origin in the xy-plane whose length from $x = 0$ to $x = 1$ is $L = \int_0^1 \sqrt{1+\frac{1}{4}e^x}\ dx$

Solution

$$L = \int_0^1 \sqrt{1+\frac{1}{4}e^x}\ dx$$

$$\frac{dy}{dx} = \frac{e^{x/2}}{2} \qquad \rightarrow\ dy = \frac{e^{x/2}}{2}dx$$

$$y = \int \frac{e^{x/2}}{2}dx = e^{x/2} + C$$

$$0 = e^{0/2} + C$$

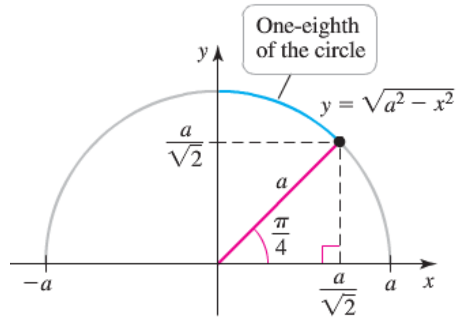
$$0 = 1 + C \qquad \Rightarrow\ \underline{C = -1}$$

$$\underline{y = e^{x/2} - 1}$$

Exercise

Confirm that the circumference of a circle of radius a is $2\pi a$.

Solution



$$f(x) = \sqrt{a^2 - x^2} \quad for \quad -a \le x \le a$$

$$f'(x) = -\frac{x}{\sqrt{a^2 - x^2}} \qquad \text{but } \underline{x \ne \pm a}$$

$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2}}$$

$$= \frac{a}{\sqrt{a^2 - x^2}}$$

Let's compute the length of $\frac{1}{8}$ of the circle on $\left[0, \ \frac{a}{\sqrt{2}}\right]$

$$L = 8a \int_0^{a/\sqrt{2}} \frac{dx}{\sqrt{a^2 - x^2}}$$

$$= 8a \sin^{-1}\left(\frac{x}{a}\right)\Big|_0^{a/\sqrt{2}}$$

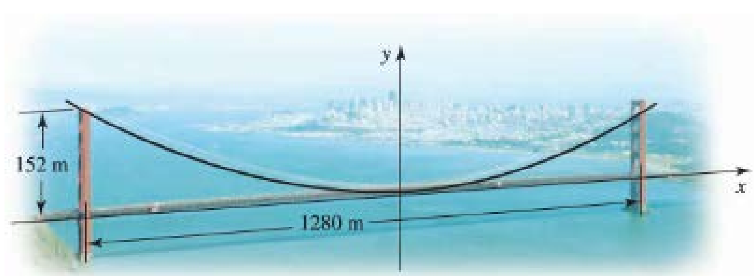
$$= 8a \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= 8a\left(\frac{\pi}{4}\right)$$

$$\underline{= 2\pi a \ \ unit}$$

Exercise

The profile of the cables on a suspension bridge may be modeled by a parabola. The central span of the Golden Gate Bridge is 1280 *m* long and 152 *m* high. The parabola $y = 0.00037x^2$ gives a good fit to the shape of the cables, where $|x| \le 640$, and x and y are measured in *meters*. Approximate the length of the cables that stretch between the tops of the two towers.



Solution

$$y' = 0.00074x$$

$$L = \int_{-640}^{640} \sqrt{1+(.00074x)^2}\ dx \qquad\qquad \int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|$$

$$= \left(\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\left|x+\sqrt{1+x^2}\right|\right)\Bigg|_{-640}^{640}$$

$$= 320\sqrt{1+640^2} + \frac{1}{2}\ln\left|640+\sqrt{1+640^2}\right| + 320\sqrt{1+x^2} - \frac{1}{2}\ln\left|-640+\sqrt{1+640^2}\right|$$

$$\approx 1326.4\ \ m$$

Exercise

Electrical wires suspended between two towers form a caternary modeled by the equation

$$y = 20\cosh\frac{x}{20}, \quad -20 \le x \le 20$$

Where x and y are measured in *meters*. The towers are 40 *meters* apart. Find the length of the suspended cable.

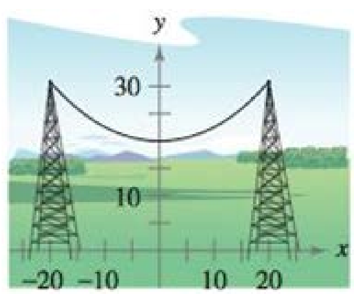
Solution

$$y = 20\cosh\frac{x}{20} \quad \to y' = \sinh\frac{x}{20}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\sinh^2\frac{x}{20}}$$

$$= \sqrt{\cosh^2\frac{x}{20}}$$

$$= \cosh\frac{x}{20}$$



$$L = \int_{-20}^{20} \cosh\frac{x}{20}\,dx$$

$$= 2(20)\sinh\frac{x}{20}\Big|_{0}^{20}$$

$$= 40(\sinh 1 - \sinh 0)$$

$$= 40\sinh 1\ \ unit$$

Exercise

A barn is 100 *feet* long and 40 *feet* wide. A cross section of the roof is the inverted caternary $y = 31 - 10\left(e^{x/20} + e^{-x/20}\right)$. Find the number of ***square feet*** of roofing on the barn.

Solution

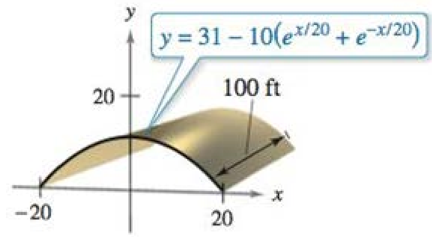
$$a = 10, \quad m = \frac{1}{20}, \quad b = 10, \quad n = -\frac{1}{20}$$

1. $m = -n$ ✓ 2. $abmn = 10(10)\left(\frac{1}{20}\right)\left(-\frac{1}{20}\right) = -\frac{1}{4}$ ✓

$$L = 10\left(e^{x/20} - e^{-x/20}\right)\Big|_{-20}^{20}$$

$$= 10\left(e - \frac{1}{e} - \frac{1}{e} + e\right)$$

$$= 20\left(e - \frac{1}{e}\right) \approx 47\ ft$$



$\therefore$ There are $100(47) = 4,700\ ft^2$ of roofing on the barn

Exercise

A cable for a suspension bridge has the shape of a parabola with equation $y = kx^2$. Let h represent the height of the cable from it lowest point to its highest point and let $2w$ represent the total span of the bridge.

Show that the length C of the cable is given by $$C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\ dx$$

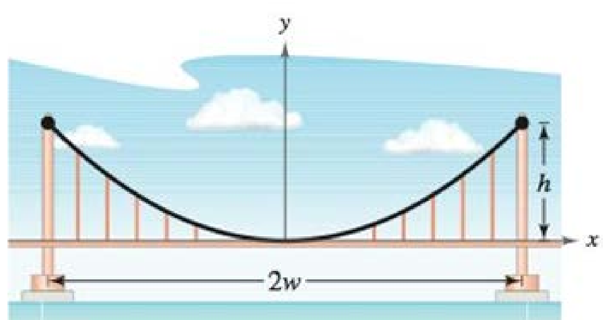
Solution

$y' = 2kx$

$\sqrt{1 + (y')^2} = \sqrt{1 + 4k^2x^2}$

At $(w, h) \rightarrow h = kw^2$

$\Rightarrow\ k = \frac{h}{w^2}$



$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{4h^2}{w^4}x^2}$

$\therefore$ By symmetry:

$$C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\ dx$$

Exercise

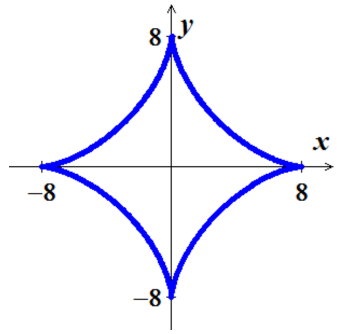
Find the total length of the graph of the astroid $x^{2/3} + y^{2/3} = 4$

Solution

$x^{2/3} + y^{2/3} = 4 \quad \Rightarrow \quad y = \left(4 - x^{2/3}\right)^{3/2}$

$$y' = \frac{3}{2}\left(-\frac{2}{3}x^{-1/3}\right)\left(4 - x^{2/3}\right)^{1/2}$$

$$= -\frac{1}{x^{1/3}}\left(4 - x^{2/3}\right)^{1/2}$$



$$1 + \left(y'\right)^2 = 1 + \frac{1}{x^{2/3}}\left(4 - x^{2/3}\right)$$

$$= \frac{4}{x^{2/3}}$$

$$y = 0 \quad \to x^{2/3} = 4 \;\Rightarrow\; \underline{x = 4^{3/2} = 8}$$

$$L = 4\int_0^8 \sqrt{\frac{4}{x^{2/3}}}\, dx$$

$$= 8\int_0^8 x^{-1/3}\, dx$$

$$= 12x^{2/3}\Big|_0^8$$

$$= 12(4 - 0)$$

$$\underline{= 48}$$

Exercise

Find the arc length from $(0,\ 3)$ clockwise to $\left(2,\ \sqrt{5}\right)$ along the circle $x^2 + y^2 = 9$

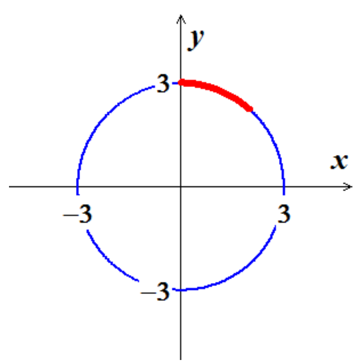
Solution

$$y = \sqrt{9 - x^2} \quad \Rightarrow \quad y' = -\frac{x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + \left(y'\right)^2} = \sqrt{1 + \frac{x^2}{9 - x^2}}$$

$$= \sqrt{\frac{9}{9 - x^2}}$$

$$= \frac{3}{\sqrt{9 - x^2}}$$



$$L = \int_0^2 \frac{3}{\sqrt{9 - x^2}}\, dx$$

$$= 3\arcsin\frac{x}{3}\Big|_0^2$$

$$= 3\arcsin\frac{2}{3} \quad \approx 2.1892$$

Exercise

Find the arc length from $(-3,\ 4)$ clockwise to $(4,\ 3)$ along the circle $x^2 + y^2 = 25$. Show that the result is one-fourth the circumference of the circle.

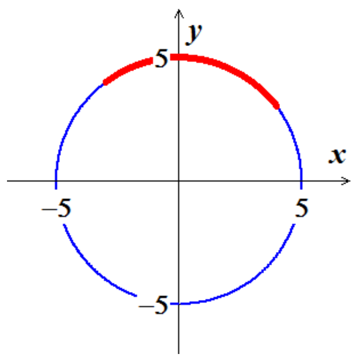
Solution

$$y = \sqrt{25 - x^2} \quad \Rightarrow \quad y' = -\frac{x}{\sqrt{25 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{25 - x^2}}$$

$$= \sqrt{\frac{25}{25 - x^2}}$$

$$= \frac{5}{\sqrt{25 - x^2}}$$



$$L = \int_{-3}^{4} \frac{5}{\sqrt{25 - x^2}}\, dx$$

$$= 5\arcsin\frac{x}{5}\Big|_{-3}^{4}$$

$$= 5\left(\arcsin\frac{4}{5} + \arcsin\frac{3}{5}\right) \quad \approx 7.854$$

Exercise

$y = \ln x$ between $x = 1$ and $x = b > 1$ that

$$\int \frac{\sqrt{x^2 + a^2}}{x} dx = \sqrt{x^2 + a^2} - a\ln\left(\frac{a + \sqrt{x^2 + a^2}}{x}\right) + C$$

Use any means to approximate the value of b for which the curve has length 2.

Solution

Given: $\quad L = 2$

$$y = \ln x \quad \to \quad y' = \frac{1}{x}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{1}{x^2}}$$

$$= \frac{\sqrt{x^2+1}}{x}$$

$$L = \int_1^b \frac{\sqrt{x^2+1}}{x}\,dx$$

$$= \sqrt{x^2+1} - \ln\left(\frac{1+\sqrt{x^2+1}}{x}\right)\Bigg|_1^b$$

$$= \sqrt{b^2+1} - \ln\left(\frac{1+\sqrt{b^2+1}}{b}\right) - \sqrt{2} + \ln\left(1+\sqrt{2}\right) \quad = 2$$

Using Mapple:

$$fsolve\left(\sqrt{b^2+1} - \ln\left(\frac{1+\sqrt{b^2+1}}{b}\right) - \sqrt{2} + \ln\left(1+\sqrt{2}\right) = 2\ ,\ \ b\right)$$

$$b = 2.714999998$$

$\underline{b \approx 2.715}$