Section 3.4 – L'Hôpital's Rule

John Bernoulli discovered a rule using derivatives to calculate limits of fractions whose numerator and denominators both approach zero or $\pm\infty$. The rule is known today as ***L'Hôpital's Rule***, after Guillaume de L'Hôpital.

Indeterminate form 0/0

Theorem – L'Hôpital's Rule

Suppose that $f(a)=g(a)=0$, that f and g are differentiable on an open interval I containing a, and that $g'(x)\neq 0$ on I if $x\neq a$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Assuming the limit on the right side of this equation exists.

Example

- $\displaystyle\lim_{x\to 0}\frac{3x-\sin x}{x}=\frac{0}{0}=\lim_{x\to 0}\frac{3-\cos x}{1}=\left.\frac{3-\cos x}{1}\right|_{x=0}=\boxed{2}$

- $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\frac{0}{0}=\lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1}=\left.\frac{1}{2\sqrt{1+x}}\right|_{x=0}=\boxed{\frac{1}{2}}$

- $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2}=\frac{0}{0}=\lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}-\frac{1}{2}}{2x}=\frac{0}{0}$

  $\displaystyle=\lim_{x\to 0}\frac{-\frac{1}{4}(1+x)^{-3/2}}{2}=\left.\frac{-\frac{1}{4}(1+x)^{-3/2}}{2}\right|_{x=0}=\boxed{-\frac{1}{8}}$

- $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{x^3}=\frac{0}{0}=\lim_{x\to 0}\frac{1-\cos x}{3x^2}=\frac{0}{0}$

  $\displaystyle=\lim_{x\to 0}\frac{\sin x}{6x}=\frac{0}{0}$

  $\displaystyle=\lim_{x\to 0}\frac{\cos x}{6}$

  $\displaystyle\boxed{=\frac{1}{6}}$

Example

Use l'Hôpital Rule to find $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x+x^2}$

Solution

$$\lim_{x\to 0} \frac{1-\cos x}{x+x^2} = \frac{0}{0} = \lim_{x\to 0} \frac{\sin x}{1+2x}$$

$$= \left.\frac{\sin x}{1+2x}\right|_{x=0}$$

$$= \frac{0}{1}$$

$$= \boxed{0}$$

Example

Use l'Hôpital Rule to find $\lim\limits_{x\to 0} \dfrac{\sin x}{x^2}$

Solution

$$\lim_{x\to 0} \frac{\sin x}{x^2} = \frac{0}{0} = \lim_{x\to 0} \frac{\cos x}{2x}$$

$$= \frac{1}{0}$$

$$\boxed{= \infty}$$

Example

Use l'Hôpital Rule to find $\lim\limits_{x\to 0^-} \dfrac{\sin x}{x^2}$

Solution

$$\lim_{x\to 0^-} \frac{\sin x}{x^2} = \frac{0}{0} = \lim_{x\to 0^-} \frac{\cos x}{2x} = \boxed{-\infty}$$

Indeterminate form ∞ / ∞, $\infty - 0$, $\infty - \infty$

L'Hôpital Rule applies to the indeterminate form ∞/∞, 0/0. If

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Example

Find the limits of these ∞ / ∞ forms:

a) $\lim\limits_{x\to\pi/2} \dfrac{\sec x}{1+\tan x}$

b) $\lim\limits_{x\to\infty} \dfrac{\ln x}{2\sqrt{x}}$

c) $\lim\limits_{x\to\infty} \dfrac{e^x}{x^2}$

Solution

a)
$$\lim_{x\to\pi/2} \frac{\sec x}{1+\tan x} = \frac{\infty}{\infty} = \lim_{x\to\pi/2} \frac{\sec x \tan x}{\sec^2 x}$$
$$= \lim_{x\to\pi/2} \frac{\tan x}{\sec x}$$
$$= \lim_{x\to\pi/2} \frac{\sin x}{\cos x} \cos x$$
$$= \lim_{x\to\pi/2} \sin x$$
$$\underline{=1}$$

b)
$$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}}$$
$$= \lim_{x\to\infty} \frac{\sqrt{x}}{x}$$
$$= \lim_{x\to\infty} \frac{1}{\sqrt{x}}$$
$$\underline{=0}$$

c)
$$\lim_{x\to\infty} \frac{e^x}{x^2} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{e^x}{2x}$$
$$= \lim_{x\to\infty} \frac{e^x}{2}$$
$$\underline{=\infty}$$

Example

Find the limits of these $\infty \cdot 0$ forms:

a) $\lim\limits_{x\to\infty}\left(x\sin\frac{1}{x}\right)$

b) $\lim\limits_{x\to 0^+}\sqrt{x}\ln x$

Solution

a) $\lim\limits_{x\to\infty}\left(x\sin\frac{1}{x}\right)=\infty\cdot 0=\lim\limits_{h\to 0^+}\left(\frac{1}{h}\sin h\right)$ *Let* $h=\frac{1}{x}$

$$=\lim_{h\to 0^+}\left(\frac{\sin h}{h}\right)$$

$$\boxed{=1}$$

b) $\lim\limits_{x\to 0^+}\sqrt{x}\ln x=\lim\limits_{x\to 0^+}\dfrac{\ln x}{1/\sqrt{x}}$

$$=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{2x^{3/2}}}$$

$$=\lim_{x\to 0^+}\left(-2\sqrt{x}\right)$$

$$\boxed{=0}$$

Example

Find the limits of these $\infty-\infty$ form: $\lim\limits_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

Solution

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\infty-\infty=\lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right)$$

$$=\lim_{x\to 0}\left(\frac{1-\cos x}{\sin x+x\cos x}\right)=\frac{0}{0}$$

$$=\lim_{x\to 0}\left(\frac{\sin x}{\cos x+\cos x-x\sin x}\right)$$

$$=\lim_{x\to 0}\left(\frac{\sin x}{2\cos x-x\sin x}\right)$$

$$=\frac{0}{2}$$

$$\boxed{=0}$$

Indeterminate Powers

If $\lim\limits_{x\to a} \ln f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L$$

Example

Apply l'Hôpital Rule to show that $\lim\limits_{x\to 0^+} (1+x)^{1/x} = e$

Solution

$$\lim_{x\to 0^+} (1+x)^{1/x} = 1^\infty$$

$$\ln f(x) = \ln(1+x)^{1/x} = \frac{1}{x}\ln(1+x)$$

$$\begin{aligned}\lim_{x\to 0^+} \ln f(x) &= \lim_{x\to 0^+} \frac{\ln(1+x)}{x} = \frac{0}{0} \\ &= \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} \\ &= \frac{1}{1} \\ &= \underline{1}\end{aligned}$$

$$\lim_{x\to 0^+} (1+x)^{1/x} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^1 = \underline{e}$$

Example

Find $\lim\limits_{x\to\infty} x^{1/x}$

Solution

$$\lim_{x\to\infty} x^{1/x} = \infty^0$$

$$\ln f(x) = \ln x^{1/x} = \frac{\ln x}{x}$$

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \frac{0}{1} = \underline{0}$$

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = \underline{1}$$

Exercises *Section* 3.4 – L'Hôpital's Rule

Apply l'Hôpital Rule to evaluate

1. $\lim\limits_{x\to -2} \dfrac{x+2}{x^2-4}$

2. $\lim\limits_{x\to 1} \dfrac{x^3-1}{4x^3-x-3}$

3. $\lim\limits_{x\to -5} \dfrac{x^2-25}{x+5}$

4. $\lim\limits_{t\to 0} \dfrac{\sin 5t}{2t}$

5. $\lim\limits_{\theta\to -\pi/3} \dfrac{3\theta+\pi}{\sin\left(\theta+\frac{\pi}{3}\right)}$

6. $\lim\limits_{x\to 0} \dfrac{x^2}{\ln(\sec x)}$

7. $\lim\limits_{\theta\to 0} \dfrac{3^{\sin\theta}-1}{\theta}$

8. $\lim\limits_{x\to 0} \dfrac{3^x-1}{2^x-1}$

9. $\lim\limits_{x\to 0^+} (\ln x-\ln\sin x)$

10. $\lim\limits_{x\to 0} \dfrac{\left(e^x-1\right)^2}{x\sin x}$

11. $\lim\limits_{x\to \pi/2^-} \dfrac{1+\tan x}{\sec x}$

12. $\lim\limits_{x\to \infty} \dfrac{4x^3-6x^2+1}{2x^3-10x+3}$

13. $\lim\limits_{x\to 0} \dfrac{3\sin 4x}{5x}$

14. $\lim\limits_{x\to 2\pi} \dfrac{x\sin x+x^2-4\pi^2}{x-2\pi}$

15. $\lim\limits_{x\to 0} \dfrac{\tan 4x}{\tan 7x}$

16. $\lim\limits_{x\to 0} \dfrac{\sin^2 3x}{x^2}$

17. $\lim\limits_{x\to -1} \dfrac{x^3-x^2-5x-3}{x^4+2x^3-x^2-4x-2}$

18. $\lim\limits_{x\to 1} \dfrac{x^n-1}{x-1} \quad (n>0)$

19. $\lim\limits_{x\to 1^-} (1-x)\tan\left(\frac{\pi x}{2}\right)$

20. $\lim\limits_{x\to \infty} \dfrac{3}{x}\csc\dfrac{5}{x}$

21. $\lim\limits_{x\to \pi/4} \dfrac{\tan x-\cot x}{x-\frac{\pi}{4}}$

22. $\lim\limits_{x\to 0} \dfrac{1-\cos 3x}{8x^2}$

23. $\lim\limits_{x\to 3} \dfrac{x-1-\sqrt{x^2-5}}{x-3}$

24. $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{\sqrt{8-x^2}-x}$

25. $\lim\limits_{h\to 0} \dfrac{\sin(x+h)-\sin x}{h}$ *x is a real number*

26. $\lim\limits_{x\to 2} \dfrac{\sqrt[3]{3x+2}-2}{x-2}$

27. $\lim\limits_{x\to \infty} \dfrac{3x^4-x^2}{6x^4+12}$

28. $\lim\limits_{x\to \infty} \dfrac{4x^3-2x^2+6}{\pi x^3+4}$

29. $\lim\limits_{x\to \infty} \dfrac{8-4x^2}{3x^3+x-1}$

30. $\lim\limits_{x\to \pi/2} \dfrac{2\tan x}{\sec^2 x}$

31. $\lim\limits_{x\to 0} \dfrac{e^x-x-1}{5x^2}$

32. $\lim\limits_{x\to 0} \dfrac{e^x-\sin x-1}{x^4+8x^3+12x^2}$

33. $\lim\limits_{x\to \infty} \dfrac{e^{1/x}-1}{1/x}$

34. $\lim\limits_{x\to \infty} \dfrac{e^{3x}}{3e^{3x}+5}$

35. $\lim_{x\to\infty} \dfrac{\ln(3x+5)}{\ln(7x+3)+1}$

36. $\lim_{x\to\infty} \dfrac{\ln\left(3x+e^x\right)}{\ln\left(7x+3e^{2x}\right)}$

37. $\lim_{x\to\infty} \dfrac{x^2-\ln\left(\frac{2}{x}\right)}{3x^2+2x}$

38. $\lim_{x\to 1^+} x^{1/(x-1)}$

39. $\lim_{x\to e^+} (\ln x)^{1/(x-e)}$

40. $\lim_{x\to\infty} (1+2x)^{1/(2\ln x)}$

41. $\lim_{x\to\infty} \left(\dfrac{x^2+1}{x+2}\right)^{1/x}$

42. $\lim_{t\to 2} \dfrac{t^3-t^2-2t}{t^2-4}$

43. $\lim_{x\to 0} \dfrac{1-\cos 6x}{2x}$

44. $\lim_{x\to\infty} \dfrac{5x^2+2x-5}{\sqrt{x^4-1}}$

45. $\lim_{\theta\to 0} \dfrac{3\sin^2 2\theta}{\theta^2}$

46. $\lim_{x\to\infty} \left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)$

47. $\lim_{\theta\to 0} 2\theta\cot 3\theta$

48. $\lim_{x\to 0} \dfrac{e^{-2x}-1+2x}{x^2}$

49. $\lim_{x\to 1} \dfrac{x^4-x^3-3x^2+5x-2}{x^3+x^2-5x+3}$

50. $\lim_{y\to 0^+} \dfrac{\ln^{10} y}{\sqrt{y}}$

51. $\lim_{\theta\to 0} \dfrac{3\sin 8\theta}{8\sin 3\theta}$

52. $\lim_{x\to\infty} \dfrac{\ln x^{100}}{\sqrt{x}}$

53. $\lim_{x\to 0} \csc x \sin^{-1} x$

54. $\lim_{x\to\infty} \dfrac{\ln^3 x}{\sqrt{x}}$

55. $\lim_{x\to\infty} \ln\left(\dfrac{x+1}{x-1}\right)$

56. $\lim_{x\to 0^+} (1+x)^{\cot x}$

57. $\lim_{x\to \frac{\pi}{2}^+} (\sin x)^{\tan x}$

58. $\lim_{x\to\infty} \left(\sqrt{x}+1\right)^{1/x}$

59. $\lim_{x\to 0^+} |\ln x|^x$

60. $\lim_{x\to\infty} x^{1/x}$

61. $\lim_{x\to\infty} \left(1-\dfrac{3}{x}\right)^x$

62. $\lim_{x\to\infty} \left(\dfrac{2}{\pi}\tan^{-1} x\right)^x$

63. $\lim_{x\to 1} (x-1)^{\sin \pi x}$

64. $\lim_{x\to\infty} \dfrac{2x^5-x+1}{5x^6+x}$

65. $\lim_{x\to\infty} \dfrac{4x^4-\sqrt{x}}{2x^4+x^{-1}}$

66. $\lim_{x\to 0} \dfrac{1-\cos x^n}{x^{2n}}$

67. $\lim_{x\to 0} \dfrac{1-\cos^n x}{x^2}$

68. $\lim_{x\to 0} \dfrac{1-\cos x^n}{x^2}$

69. $\lim_{x\to 0} \dfrac{3x}{\tan 4x}$

70. $\lim_{x\to 0} \dfrac{\sin ax}{\sin bx}$

71. $\lim_{x\to 2} \dfrac{\ln(2x-3)}{x^2-4}$

72. $\lim\limits_{x\to 0} \frac{1-\cos ax}{1-\cos bx}$

73. $\lim\limits_{x\to 0} \frac{\sin^{-1} x}{\tan^{-1} x}$

74. $\lim\limits_{x\to 1} \frac{x^{1/3}-1}{x^{2/3}-1}$

75. $\lim\limits_{x\to 0} x\cot x$

76. $\lim\limits_{x\to 0} \frac{1-\cos x}{\ln\left(1+x^2\right)}$

77. $\lim\limits_{x\to \pi} \frac{\sin^2 x}{x-\pi}$

78. $\lim\limits_{x\to 0} \frac{10^x - e^x}{x}$

79. $\lim\limits_{x\to \frac{\pi}{2}} \frac{\cos 3x}{\pi - 2x}$

80. $\lim\limits_{x\to 1} \frac{\ln(ex)-1}{\sin \pi x}$

81. $\lim\limits_{x\to \infty} x\sin\frac{1}{x}$

82. $\lim\limits_{x\to 0} \frac{x-\sin x}{x^3}$

83. $\lim\limits_{x\to 0} \frac{x-\sin x}{x-\tan x}$

84. $\lim\limits_{x\to 0} \frac{2-x^2-2\cos x}{x^4}$

85. $\lim\limits_{x\to 0^+} \frac{\sin^2 x}{\tan x - x}$

86. $\lim\limits_{x\to \frac{\pi}{2}} \frac{\ln\sin x}{\cos x}$

87. $\lim\limits_{x\to \frac{\pi}{2}} \frac{\sin x}{x}$

88. $\lim\limits_{x\to 1^-} \frac{\arccos x}{x-1}$

89. $\lim\limits_{x\to \infty} x\left(2\tan^{-1} x - \pi\right)$

90. $\lim\limits_{x\to \frac{\pi}{2}^+} x(\sec x - \tan x)$

91. $\lim\limits_{x\to 0} \left(\frac{1}{x} - \frac{1}{xe^{ax}}\right)$

92. $\lim\limits_{x\to 0^+} x^{\sqrt{x}}$

93. $\lim\limits_{x\to \pi} \frac{\cos x + 1}{(x-\pi)^2}$

94. $\lim\limits_{x\to 0} \frac{\sin x - x}{7x^3}$

95. $\lim\limits_{x\to \infty} \frac{\tan^{-1} x - \frac{\pi}{2}}{\frac{1}{x}}$

96. $\lim\limits_{x\to 3} \frac{x-1-\sqrt{x^2-5}}{x-3}$

97. $\lim\limits_{x\to 2} \frac{x^2+x-6}{\sqrt{8-x^2}-x}$

98. $\lim\limits_{x\to 2} \frac{x^2-4x+4}{\sin^2 \pi x}$

99. $\lim\limits_{x\to 2} \frac{(3x+2)^{1/3}-2}{x-2}$

100. $\lim\limits_{x\to \infty} \frac{3x^4 - x^2}{6x^4+12}$

101. $\lim\limits_{x\to \infty} \frac{4x^3-2x^2+6}{\pi x^3 + 4}$

102. $\lim\limits_{x\to \frac{\pi}{2}^-} \frac{1}{3}(2x-\pi)\tan x$

103. $\lim\limits_{x\to \infty} x\ln\left(1+\frac{1}{x}\right)$

104. $\lim\limits_{x\to \frac{\pi}{2}^-} \left(\frac{\pi}{2} - x\right)\sec x$

105. $\lim\limits_{x\to \infty} \frac{e^{1/x}-1}{\sin\frac{1}{x}}$

106. $\lim\limits_{x\to 0^+} \sin x \sqrt{\frac{1-x}{x}}$

107. $\lim\limits_{x\to 0} \left(\cot x - \frac{1}{x}\right)$

108. $\lim\limits_{x\to \infty} \left(x - \sqrt{x^2+1}\right)$

109. $\lim\limits_{\theta \to \frac{\pi}{2}^-} \left(\tan\theta - \sec\theta\right)$

110. $\lim\limits_{x\to 0^+} \ln x^{2x}$

111. $\lim\limits_{x\to 0} \ln\left(1+4x\right)^{3/x}$

112. $\lim\limits_{\theta \to \frac{\pi}{2}^-} \ln\left(\tan\theta\right)^{\cos\theta}$

113. $\lim\limits_{x\to 0^+} \left(1+x\right)^{\cot x}$

114. $\lim\limits_{x\to\infty} \left(1+\frac{1}{x}\right)^{\ln x}$

115. $\lim\limits_{x\to\infty} \left(1+\frac{a}{x}\right)^{x}$

116. $\lim\limits_{x\to 0} \left(e^{5x}+x\right)^{1/x}$

117. $\lim\limits_{x\to 0} \left(e^{ax}+x\right)^{1/x}$

118. $\lim\limits_{x\to 0} \left(2^{ax}+x\right)^{1/x}$

119. $\lim\limits_{x\to 0^+} \left(\tan x\right)^{x}$

120. The functions $f(x)=\left(x^x\right)^x$ *and* $g(x)=x^{\left(x^x\right)}$ are different functions. For example, $f(3)=19{,}683$ and $g(3)\approx 7.6\times 10^{12}$. Determine whether $\lim\limits_{x\to 0^+} f(x)$ and $\lim\limits_{x\to 0^+} g(x)$ are intermediate forms and evaluate the limits.

121. Consider the function $g(x)=\left(1+\frac{1}{x}\right)^{x+a}$. show that if $0\le a<\frac{1}{2}$, then $g(x)\to e$ from below as $x\to\infty$; if $\frac{1}{2}\le a<1$, then $g(x)\to e$ from above as $x\to\infty$

122. Let $f(x)=(a+x)^x$, where $a>0$

- ***a)*** What is the domain of f (in terms of a)?
- ***b)*** Describe the end behavior of f (near the left boundary of its domain and as $x\to\infty$).
- ***c)*** Compute f'.
- ***d)*** Show that f has a single local minimum at the point z that satisfies $(z+a)\ln(z+a)+z=0$
- ***e)*** Describe how $f(z)$ varies as a increases.