

## ***Solution***      **Section 3.4 – Orthogonal Matrices**

### ***Exercise***

Show that the matrix is orthogonal       $A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$

### **Solution**

$$\begin{aligned} AA^T &= \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \end{aligned}$$

$$\begin{aligned} A^T A &= \begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \end{aligned}$$

$$AA^T = A^T A = I$$

∴  $A$  is an orthogonal

### ***Exercise***

Show that the matrix is orthogonal       $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

### **Solution**

$$\begin{aligned}
 AA^T &= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I
 \end{aligned}$$

$$\begin{aligned}
 A^T A &= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I
 \end{aligned}$$

$$AA^T = A^T A = I$$

$\therefore A$  is an orthogonal.

### ***Exercise***

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

### ***Solution***

$$\begin{aligned}
 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I
 \end{aligned}$$

$$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ is orthogonal with inverse } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

### Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

### Solution

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\therefore \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ is orthogonal with inverse } \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ (It is a standard matrix for a rotation of } 45^\circ \text{)}$$

### Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

### Solution

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}^T = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\therefore \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \text{ is orthogonal with inverse } \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

### Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$

### Solution

$$\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}^T = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\therefore \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \text{ is orthogonal with an inverse } \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$

### Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$$

### Solution

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 7 \\ 1 & 3 & -5 \\ -1 & 4 & 2 \end{pmatrix} \\ = \begin{pmatrix} 3 & & \\ & & \\ & & \end{pmatrix} \neq I$$

$$\therefore \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix} \text{ is not an orthogonal}$$

### Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

### Solution

$$\begin{aligned} \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}^T &= \begin{pmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{2} & & \\ & & \\ & & \end{pmatrix} \neq I \end{aligned}$$

$$\text{Or } \|r_1\| = \sqrt{0+1^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{3}{2}} \neq 1 \quad \therefore A \text{ is not orthogonal}$$

### Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

### Solution

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \text{ is orthogonal with inverse } \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

### ***Exercise***

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

### ***Solution***

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \text{ is orthogonal with inverse } \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

### Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

### Solution

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix} \text{ is orthogonal with inverse } \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

### Exercise

Determine if the matrix is orthogonal. For those that is orthogonal find the inverse

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$$

### Solution

$$\begin{aligned}
\|r_2\| &= \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{2}\right)^2} \\
&= \sqrt{\frac{1}{3} + \frac{1}{4}} \\
&= \sqrt{\frac{7}{12}} \neq 1
\end{aligned}$$

**Or**

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{5}{6} & & \\ & & & \end{pmatrix} \neq I
\end{aligned}$$

∴ The matrix is **not** an orthogonal

### Exercise

Find a last column so that the resulting matrix is orthogonal

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \dots \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \dots \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & \dots \end{bmatrix}$$

### Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}^T$$

$$\begin{aligned}
\|\vec{q}_1\| &= \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} \\
&= 1
\end{aligned}$$

$$\vec{q}_2 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}^T$$



$$\|\vec{q}_2\| = \sqrt{\frac{1}{6} + \frac{1}{6} + \frac{4}{6}} \\ = 1$$

$$\text{Let } \vec{q}_3 = [x \ y \ z]^T$$

$$\vec{q}_1 \cdot \vec{q}_3 = \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y - \frac{1}{\sqrt{3}}z = 0 \rightarrow x + y - z = 0$$

$$\vec{q}_2 \cdot \vec{q}_3 = \frac{1}{\sqrt{6}}x + \frac{1}{\sqrt{6}}y - \frac{2}{\sqrt{6}}z = 0 \rightarrow x + y - 2z = 0$$

$$\begin{cases} x + y - z = 0 \\ x + y - 2z = 0 \end{cases} \rightarrow \underline{z = 0} \quad \text{and} \quad x + y = 0 \Rightarrow \underline{x = -y}$$

$$\vec{q}_3 = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$$

### Exercise

Determine if the given matrix is orthogonal. If it is, find its inverse

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{5} & \frac{3}{7} \\ \frac{4}{9} & \frac{3}{5} & -\frac{2}{7} \\ \frac{8}{9} & -\frac{2}{5} & \frac{3}{7} \end{bmatrix}$$

### Solution

$$\vec{q}_1 = \begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{8}{9} \end{bmatrix}^T \quad \vec{q}_2 = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & -\frac{2}{5} \end{bmatrix}^T \quad \vec{q}_3 = \begin{bmatrix} \frac{3}{7} & -\frac{2}{7} & \frac{3}{7} \end{bmatrix}^T$$

$$\vec{q}_1 \cdot \vec{q}_2 = \frac{4}{45} + \frac{12}{45} - \frac{16}{45} \\ = 0$$

$$\vec{q}_1 \cdot \vec{q}_3 = \frac{3}{63} - \frac{8}{63} + \frac{24}{63} \\ = \frac{19}{63} \neq 0$$

$$\vec{q}_2 \cdot \vec{q}_3 = \frac{12}{35} - \frac{6}{35} + \frac{6}{35} \\ = \frac{12}{35} \neq 0$$

The given matrix is **not** orthogonal

### Exercise

Prove that if  $A$  is orthogonal, then  $A^T$  is orthogonal.

#### Solution

Since  $A$  is orthogonal then  $A^T = A^{-1}$  and  $A = (A^T)^T$

Then  $(A^T)^T A^T = AA^T = I \Rightarrow A^T$  is orthogonal

Another word, since  $A$  is orthogonal, then both column and row vectors of  $A$  form an orthonormal set.

$A^T$  is just  $A$  with its row and column vectors are swapped.

The column vectors of  $A^T$  (which are the row vectors of  $A$ ) and row vectors of  $A^T$  (which are the column vectors of  $A$ ) form orthonormal sets, therefore  $A^T$  is orthogonal

### Exercise

Prove that if  $A$  is orthogonal, then  $A^{-1}$  is orthogonal

#### Solution

Since  $A$  is orthogonal then  $A^T = A^{-1}$  and  $A = (A^{-1})^{-1}$

$$\begin{aligned} (A^{-1})^{-1} &= (A^T)^{-1} & A^T &= A^{-1} \\ &= (A^{-1})^T \end{aligned}$$

$\therefore A^{-1}$  is orthogonal

### Exercise

Prove that if  $A$  and  $B$  are orthogonal, then  $AB$  is orthogonal

#### Solution

Since  $A$  is orthogonal then  $A^T = A^{-1}$

and  $B$  is orthogonal then  $B^T = B^{-1}$

$$\begin{aligned} (AB)^T &= B^T A^T \\ &= B^{-1} A^{-1} \\ &= (AB)^{-1} \end{aligned}$$

$\therefore AB$  is orthogonal

### Exercise

Let  $Q$  be an  $n \times n$  orthogonal matrix, and let  $A$  be an  $n \times n$  matrix.

Show that  $\det(QAQ^T) = \det(A)$

### Solution

$$\begin{aligned}\det(QAQ^T) &= \det(Q)\det(A)\det(Q^T) \\ &= \det(A)\det(QQ^T) \quad \text{Since } Q \text{ is an orthogonal matrix } \det(QQ^T) = \det(I) \\ &= \det(A)\det(I) \\ &= \det(A) \quad \checkmark\end{aligned}$$

### Exercise

$$\text{Let } A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$$

- a) Is matrix  $A$  an orthogonal matrix?
- b) Let  $B$  be the matrix obtained by normalizing each row of  $A$ , find  $B$ .
- c) Is  $B$  an orthogonal matrix?
- d) Are the columns of  $B$  orthogonal?

### Solution

$$\begin{aligned}a) \quad AA^T &= \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 7 \\ 1 & 3 & -5 \\ -1 & 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & & \\ & & \\ & & \end{pmatrix} \neq I\end{aligned}$$

$$\therefore \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix} \text{ is } \textbf{not} \text{ an orthogonal}$$

$$\begin{aligned}b) \quad \|(1, 1, -1)\| &= \sqrt{1+1+1} \\ &= \sqrt{3}\end{aligned}$$

$$\begin{aligned}\|(1, 3, 4)\| &= \sqrt{1+9+16} \\ &= \sqrt{26}\end{aligned}$$

$$\begin{aligned}\|(7, -5, 2)\| &= \sqrt{49 + 25 + 4} \\ &= \sqrt{78}\end{aligned}$$

$$B = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{26}} & \frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \\ \frac{7}{\sqrt{78}} & -\frac{5}{\sqrt{78}} & \frac{2}{\sqrt{78}} \end{pmatrix}$$

c) Yes, since the rows are orthogonal with unit vectors.

$$\begin{aligned}BB^T &= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{26}} & \frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \\ \frac{7}{\sqrt{78}} & -\frac{5}{\sqrt{78}} & \frac{2}{\sqrt{78}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{26}} & \frac{7}{\sqrt{78}} \\ \frac{1}{\sqrt{3}} & \frac{3}{\sqrt{26}} & -\frac{5}{\sqrt{78}} \\ -\frac{1}{\sqrt{3}} & \frac{4}{\sqrt{26}} & \frac{2}{\sqrt{78}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I\end{aligned}$$

d) Yes, since the rows of  $B$  form an orthonormal set of vectors. Then, the column of  $B$  must form an orthonormal set.

$$\begin{aligned}\left\|\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{26}}, \frac{7}{\sqrt{78}}\right)\right\| &= \sqrt{\frac{1}{3} + \frac{1}{26} + \frac{49}{78}} \\ &= \sqrt{\frac{26 + 3 + 49}{78}} \\ &= \sqrt{\frac{78}{78}} \\ &= 1\end{aligned}$$

$$\begin{aligned}\left\|\left(\frac{1}{\sqrt{3}}, \frac{3}{\sqrt{26}}, -\frac{5}{\sqrt{78}}\right)\right\| &= \sqrt{\frac{1}{3} + \frac{9}{26} + \frac{25}{78}} \\ &= \sqrt{\frac{26 + 27 + 25}{78}} \\ &= \sqrt{\frac{78}{78}} \\ &= 1\end{aligned}$$

$$\left\|\left(-\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{26}}, \frac{2}{\sqrt{78}}\right)\right\| = \sqrt{\frac{1}{3} + \frac{16}{26} + \frac{4}{78}}$$

$$= \sqrt{\frac{78}{78}}$$

$$= \underline{1}$$