Lecture Three – Exponential and Logarithmic Functions

Section 3.1 – Inverse Functions

Inverse Relations

Interchanging the first and second coordinates of each ordered pair in a relation produces the inverse relation.

If a relation is defined by an equation, interchanging the variables produces an equation of the inverse relation

Given the relation: $\{(Zambia,\ 4.2),\ (Columbia,\ 4.5),\ (Poland,\ 3.3),\ (Italy,\ 3.3),\ (US,\ 2.5)\}$

Inverse Relation: $\{(4.2,\ Zambia),\ (4.5,\ Columbia),\ (3.3,\ Poland),\ (3.3,\ Italy),\ (2.5,\ US)\}$

Example

Consider the relation g given by: $G = \{(2,\ 4),\ (-1,\ 3),\ (-2,\ 0)\}$

Solution

The inverse relation: $G = \{(4,\ 2),\ (3,\ -1),\ (0,\ -2)\}$

Example

Consider the relation given by: $F = \{(-2,\ 2),\ (-1,\ 1),\ (0,\ 0),\ (1,\ 3),\ (2,\ 5)\}$

Solution

The inverse relation: $G = \{(2,\ -2),\ (1,\ -1),\ (0,\ 0),\ (3,\ 1),\ (5,\ 2)\}$

One-to-One Functions

A function *f* is one-to-one (1 – 1) if different inputs have different outputs that is,

$$if \ \ a \neq b, \quad then \quad f(a) \neq f(b)$$

A function *f* is one-to-one (1 – 1) if different outputs the same, the inputs are the same – that is,

$$if \ \ f(a) = f(b), \quad then \ \ a = b$$

Example

Given the function *f* described by $f(x) = 2x - 3$, prove that *f* is one-to-one.

Solution

$f(a) = f(b)$

$2a - 3 = 2b - 3$ *Add 3 on both sides*

$2a = 2b$ *Divide by 2*

$a = b$

$\therefore f$ is one-to-one

Example

Given the function *f* described by $f(x) = -4x + 12$, prove that *f* is one-to-one.

Solution

$f(a) = f(b)$

$-4a + 12 = -4b + 12$ *Subtract 12 from both sides*

$-4a = -4b$ *Divide by -4*

$a = b$

$\therefore f$ is one-to-one

Example

Given the function *f* described by $f(x) = x^2$, prove that *f* is one-to-one.

Solution

$-1 \neq 1$

$$\begin{cases} f(-1) = 1 \\ f(1) = 1 \end{cases} \Rightarrow f(-1) = f(1)$$

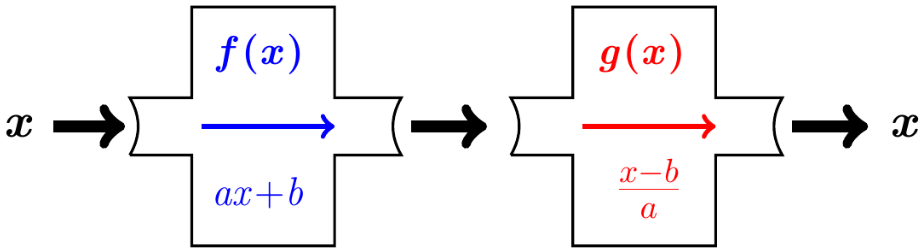
$\therefore f$ is ***not*** one-to-one

Definition of the Inverse of a Function

Let f and g be two functions such that

$$f(g(x)) = x \qquad and \qquad g(f(x)) = x$$

$$x \underset{g=f^{-1}}{\overset{f}{\rightleftarrows}} f(x) \qquad\qquad g(f(x)) = f^{-1}(f(x)) = x$$



If the inverse of a function f is also a function, it is named f^{-1} read "f – inverse"

The -1 in f^{-1} is not an exponent! And is not equal to $\cancel{\frac{1}{f(x)}}$

Domain **and** ***Range*** **of** f *and* f^{-1}

$$domain\ of\ f^{-1} = range\ of\ f$$

$$range\ of\ f^{-1} = domain\ of\ f$$

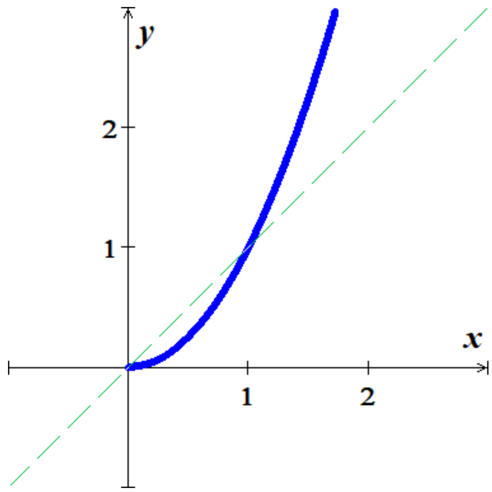
If a function f is one-to-one, then f^{-1} is the unique function such that each of the following holds.

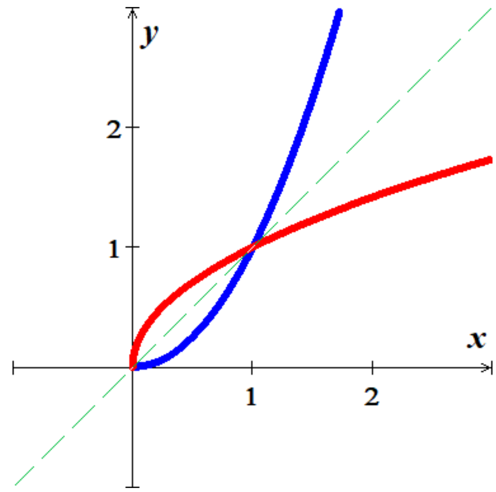
$\boxed{\left(f^{-1} \circ f\right)(x) = f^{-1}(f(x)) = x}$ for each x in the *domain* of f, and

$\boxed{\left(f \circ f^{-1}\right)(x) = f(f^{-1}(x)) = x}$ for each x in the *domain* of f^{-1}

The condition that f is one-to-one in the definition of inverse function is important; otherwise, g will not define a function

Graphing





Example

Let $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$, is g the inverse function of f?

Solution

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f\left(\sqrt[3]{x+1}\right) \\
&= \left(\sqrt[3]{x+1}\right)^3 - 1 \\
&= x + 1 - 1 \\
&= x
\end{aligned}
$$

$$
\begin{aligned}
(g \circ f)(x) &= g\left(f(x)\right) \\
&= g\left(x^3 - 1\right) \\
&= \sqrt[3]{x^3 - 1 + 1} \\
&= \sqrt[3]{x^3} \\
&= x
\end{aligned}
$$

g is the inverse function of f

Example

Show that each function is the inverse of the other: $f(x) = 4x - 7$ *and* $g(x) = \dfrac{x+7}{4}$

Solution

$$
\begin{aligned}
f(g(x)) &= f\left(\frac{x+7}{4}\right) \\
&= 4\left(\frac{x+7}{4}\right) - 7 \\
&= x + 7 - 7 \\
&= x
\end{aligned}
$$

$$
\begin{aligned}
g(f(x)) &= g(4x - 7) \\
&= \frac{4x - 7 + 7}{4} \\
&= \frac{4x}{4} \\
&= x
\end{aligned}
$$

Finding the *Inverse Function*

	Example
Finding an Inverse Function	$f(x)=2x+7$
1. Replace $f(x)$ with y	$y=2x+7$
2. Interchange x and y	$x=2y+7$
3. Solve for y	$x-7=2y$ $\frac{x-7}{2}=y$
4. Replace y with $f^{-1}(x)$	$f^{-1}(x)=\frac{x-7}{2}$

Example

Find the inverse of $f(x)=4x^3-1$

Solution

$$y=4x^3-1$$

$$x=4y^3-1$$

$$x+1=4y^3$$

$$\frac{x+1}{4}=y^3$$

$$y=\left(\frac{x+1}{4}\right)^{1/3}$$

$$\boxed{f^{-1}(x)=\sqrt[3]{\frac{x+1}{4}}}$$

Example

Find a formula for the inverse $f(x)=\frac{5x-3}{2x+1}$

Solution

$$y=\frac{5x-3}{2x+1}$$

$$x=\frac{5y-3}{2y+1}$$

$$x(2y+1)=5y-3$$

$$2xy+x=5y-3$$

$$2xy - 5y = -x - 3$$

$$y(2x - 5) = -x - 3$$

$$y = \frac{-x - 3}{2x - 5}$$

$$\boxed{f^{-1}(x) = -\frac{x + 3}{2x - 5}}$$

Exercise Section 3.1 – Inverse Functions

(1 – 9) Find the inverse relation of the given sets:

1. $A = \{(-2,\ 2),\ (1,\ -1),\ (0,\ 4),\ (1,\ 3)\}$

2. $B = \{(1,\ -1),\ (2,\ -2),\ (3,\ -3),\ (4,\ -4)\}$

3. $C = \{(a,\ -a),\ (b,\ -b),\ (c,\ -c)\}$

4. $D = \{(0,\ 0),\ (1,\ 1),\ (2,\ 2),\ (3,\ 3),\ (4,\ 4)\}$

5. $E = \{(-a,\ a),\ (-b,\ b),\ (-c,\ c),\ (-d,\ d)\}$

(6 – 14) Determine whether the function is one-to-one

6. $f(x) = 3x - 7$

7. $f(x) = x^2 - 9$

8. $f(x) = \sqrt{x}$

9. $f(x) = \sqrt[3]{x}$

10. $f(x) = |x|$

11. $f(x) = \frac{2}{x+3}$

12. $f(x) = (x-2)^3$

13. $y = x^2 + 2$

14. $f(x) = \frac{x+1}{x-3}$

15. Given that $f(x) = 5x + 8$, use composition of functions to show that $f^{-1}(x) = \frac{x-8}{5}$

16. Given the function $f(x) = (x+8)^3$

a) Find $f^{-1}(x)$

b) Graph f and f^{-1} in the same rectangular coordinate system

c) Find the domain and the range of f and f^{-1}

(17 – 32) Prove that $f(x)$ *and* $g(x)$ are inverse functions of each other.

17. $f(x) = 4x;\quad g(x) = \frac{x}{4}$

18. $f(x) = 2x;\quad g(x) = \frac{1}{2x}$

19. $f(x) = 4x - 1;\quad g(x) = \frac{x+1}{4}$

20. $f(x) = \frac{1}{2}x - \frac{3}{2};\quad g(x) = 2x + 3$

21. $f(x) = -\frac{1}{2}x - \frac{1}{2};\quad g(x) = -2x + 1$

22. $f(x) = 3x + 2;\quad g(x) = \frac{1}{3}(x-2)$

23. $f(x) = \frac{5}{x+3};\quad g(x) = \frac{5}{x} - 3$

24. $f(x) = \frac{2x}{x+1};\quad g(x) = \frac{-x}{x-2}$

25. $f(x) = \frac{3x}{x-1};\quad g(x) = \frac{x}{x-3}$

26. $f(x) = x^3 + 2;\quad g(x) = \sqrt[3]{x-2}$

27. $f(x) = x^3 - 1;\quad g(x) = \sqrt[3]{x+1}$

28. $f(x) = (x+4)^3;\quad g(x) = \sqrt[3]{x} - 4$

29. $f(x) = x^3 - 1\quad g(x) = \sqrt[3]{x+1}$

30. $f(x) = 3x - 2\quad g(x) = \frac{x+2}{3}$

31. $f(x) = x^2 + 5, x \le 0\quad g(x) = -\sqrt{x-5},\ x \ge 5$

32. $f(x) = x^3 - 4;\quad g(x) = \sqrt[3]{x+4}$

(33 – 35) Find the inverse of

33. $f(x)=(x-2)^3$

34. $f(x)=\dfrac{x+1}{x-3}$

35. $f(x)=\dfrac{2x+1}{x-3}$

(36 – 38) Determine the domain and range of f^{-1} (Hint: first find the domain and range of f)

36. $f(x)=-\dfrac{2}{x-1}$

37. $f(x)=\dfrac{5}{x+3}$

38. $f(x)=\dfrac{4x+5}{3x-8}$

(39 – 66) For the given functions

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

39. $f(x)=\dfrac{2x}{x-1}$

40. $f(x)=\dfrac{x}{x-2}$

41. $f(x)=\dfrac{x+1}{x-1}$

42. $f(x)=\dfrac{2x+1}{x+3}$

43. $f(x)=\dfrac{3x-1}{x-2}$

44. $f(x)=\dfrac{2x}{x-1}$

45. $f(x)=\dfrac{x}{x-2}$

46. $f(x)=\dfrac{x+1}{x-1}$

47. $f(x)=\dfrac{2x+1}{x+3}$

48. $f(x)=\dfrac{3x-1}{x-2}$

49. $f(x)=\dfrac{3x-2}{x+4}$

50. $f(x)=\dfrac{-3x-2}{x+4}$

51. $f(x)=\sqrt{x-1} \quad x\ge 1$

52. $f(x)=\sqrt{2-x} \quad x\le 2$

53. $f(x)=x^2+4x \quad x\ge -2$

54. $f(x)=3x+5$

55. $f(x)=\dfrac{1}{3x-2}$

56. $f(x)=\dfrac{3x+2}{2x-5}$

57. $f(x)=\dfrac{4x}{x-2}$

58. $f(x)=2-3x^2; \quad x\le 0$

59. $f(x)=2x^3-5$

60. $f(x)=\sqrt{3-x}$

61. $f(x)=\sqrt[3]{x}+1$

62. $f(x)=\left(x^3+1\right)^5$

63. $f(x)=x^2-6x; \quad x\ge 3$

64. $f(x)=(x-2)^3$

65. $f(x)=\dfrac{x+1}{x-3}$

66. $f(x)=\dfrac{2x+1}{x-3}$

67. The function $w(x)=2x+24$ can be used to convert a U.S. women's shoe size into an Italian women's shoe size. Determine the function $w^{-1}(x)$ that can use to convert an Italian women's shoe size to its equivalent U.S. shoe size.

68. The function $m(x)=1.3x-4.7$ can be used to convert a U.S. men's shoe size into an U.K. women's shoe size. Determine the function $m^{-1}(x)$ that can used to convert an U.K. men's shoe size to its equivalent U.S. shoe size.

69. A catering service use the function $c(x)=\dfrac{300+12x}{x}$ to determine the amount, in *dollars*, it charges per person for a sit-down dinner, where *x* is the number of people in attendance.

a) Find $c(30)$ and explain what it represents

b) Find $c^{-1}(x)$

c) Use $c^{-1}(x)$ to determine how many people attended a dinner for which the cost per person was \$15.00

70. A landscaping service use the function $c(x)=\dfrac{600+140x}{x}$ to determine the amount, in *dollars*, it charges per tree to deliver, where *x* is the number of trees.

a) Find $c(5)$ and explain what it represents

b) Find $c^{-1}(x)$

c) Use $c^{-1}(x)$ to determine how many trees were delivered for which the cost per tree was \$160.00

Section 3.2 – Exponential Functions

Definition

The exponential function f with base $\boldsymbol{b}$ is defined by

$$f(x)=b^x \qquad \text{or} \qquad y=b^x$$

Base

where $b>0$, $b\neq 1$ and $\boldsymbol{x}$ is any real number.

$$f(x)=2^x \qquad f(x)=\left(\tfrac{1}{2}\right)^{2x+1} \qquad f(x)=3^{-x} \qquad \cancel{f(x)=(-2)^x}$$

Example

Given: $f(x)=13.49\ (0.967)^x-1$, find $f(60)$

Solution

$$\begin{aligned} f(\mathbf{60}) &= 13.49\ (0.967)^{\mathbf{60}}-1 \\ &= 0.8014 \end{aligned}$$

Example

If $f(x)=2^x$, find each of the following. $f(-1)$, $f(3)$, $f\left(\frac{5}{2}\right)$

Solution

a) $f(-1)=2^{-1}$

$$=\frac{1}{2}$$

b) $f(3)=2^3$

$$=8$$

c) $f\left(\frac{5}{2}\right)=2^{\frac{5}{2}}$

$$=4\sqrt{2}$$

$$=5.6569$$

Graphing Exponential

1. Define the Horizontal Asymptote $f(x) = b^x \pm d$

$y = 0 \pm \text{d}$

The exponential function always equals to 0

$x \to \infty \ \ or \ \ x \to -\infty \Rightarrow f(x) \to 0$

2. Define/Make a table

(Force your exponential to = 0, then solve for x)

x	$f(x)$
$x-2$	
$x-1$	
x	
$x+1$	
$x+2$	

x	$f(x)$
-2	1/9
-1	1/3
0	1
1	3
2	9

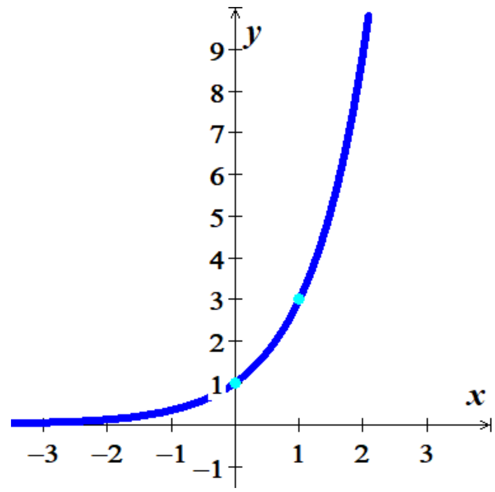
Domain: $(-\infty,\ \infty)$

Range: (d, ∞)

Example

$f(x) = 3^x$

Asymptote: $y = 0$



Example

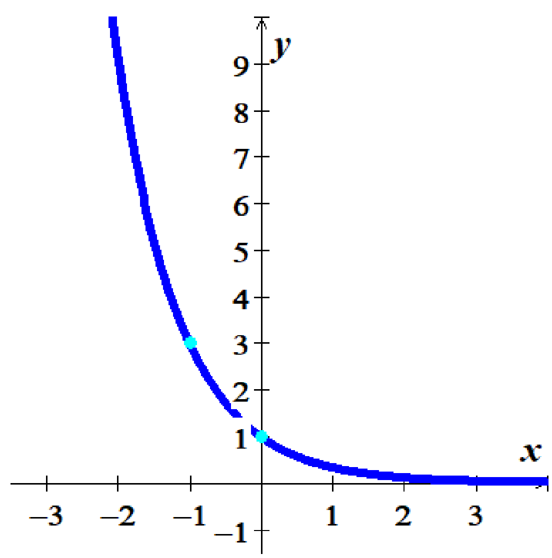
$$f(x) = \left(\tfrac{1}{3}\right)^x$$
$$= \left(3^{-1}\right)^x$$
$$= 3^{-x}$$

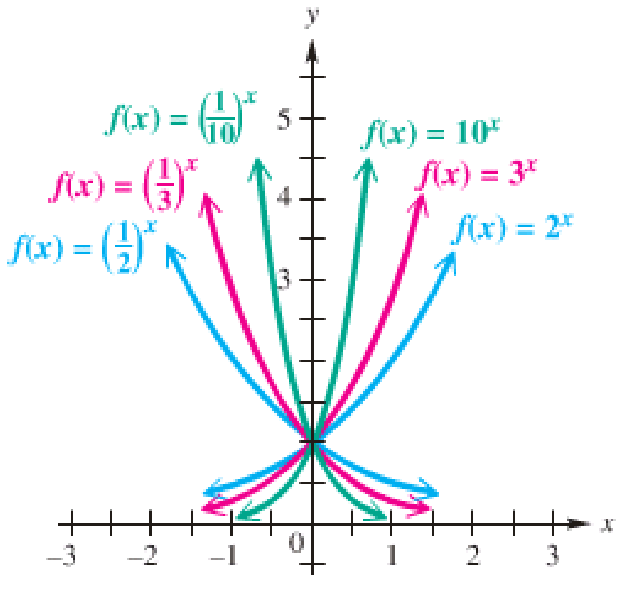
Reflected across y-axis

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$





$f(x) = a^x$

Example

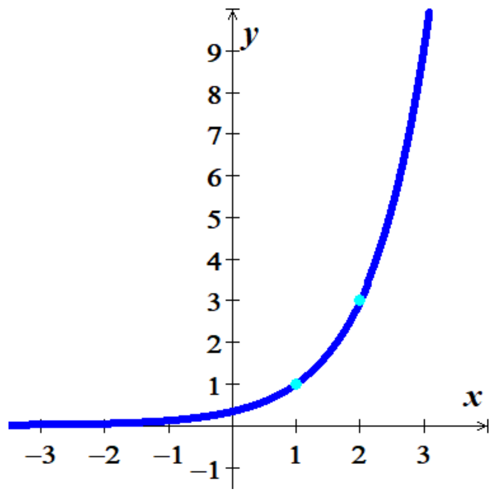
$f(x) = 3^{x-1}$

Shift right 1 unit

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$



Example

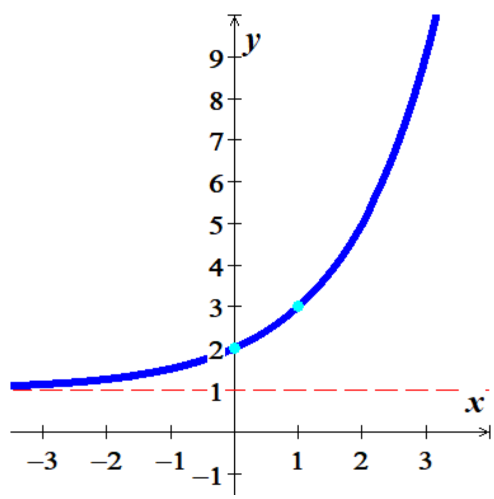
$f(x) = 2^x + 1$

Shift up 1 unit

Asymptote: $y = 1$

Domain: $(-\infty, \infty)$

Range: $(1, \infty)$



Example

$f(x) = 5 - 2^{-x}$

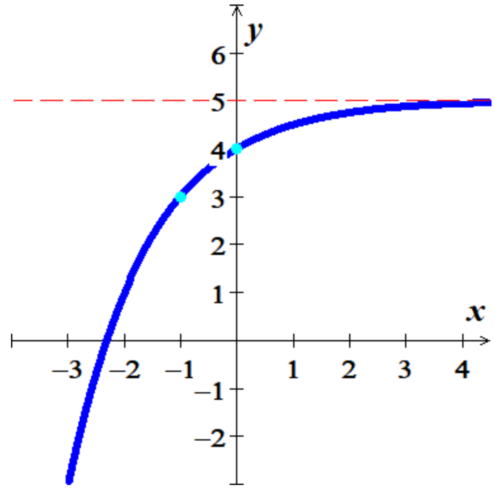
Shifted up 5 units

Reflected across x-axis and y-axis

Asymptote: $y = 5$

Domain: $(-\infty,\ \infty)$

Range: $(-\infty,\ 5)$



Example

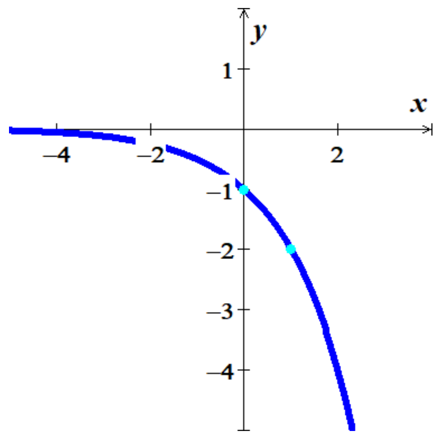
Give the *asymptote*, *domain* and *range*.

a) $f(x) = -2^x$

Asymptote: $y = 0$

Domain: $(-\infty,\ \infty)$

Range: $(-\infty,\ 0)$

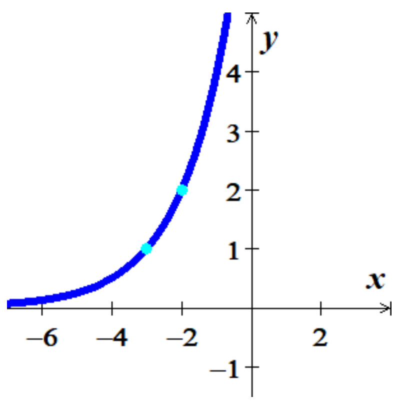


b) $f(x) = 2^{x+3}$

Asymptote: $y = 0$

Domain: $(-\infty,\ \infty)$

Range: $(0, \infty)$

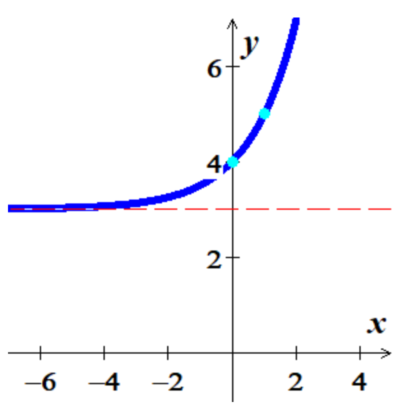


c) $f(x) = 2^x + 3$

Asymptote: $y = 3$

Domain: $(-\infty, \infty)$

Range: $(3, \infty)$



Natural Base *e*

The irrational number e is called natural base

$f(x)=e^x$ is called natural exponential function

$e^0=1$ $\qquad e\approx 2.7183$ $\qquad e^2\approx 7.3891$ $\qquad e^{-1}\approx 0.3679$

Example

The exponential function $f(x)=1066e^{0.042x}$ models the gray wolf population of the Western Great Lakes, $f(x)$, in billions, x *years* after 1978. Project the gray population in the recovery area in 2012.

Solution

$x = 2012 - 1978 = 34$

$$f(x=34)=1066e^{0.042(34)}$$
$$=4445.6$$
$$\approx 4446$$

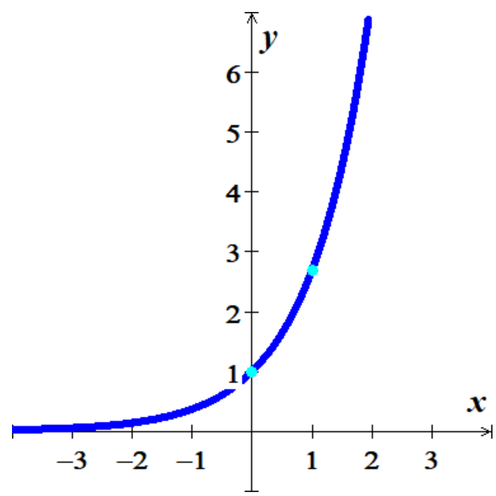
Example

Graph $\qquad f(x)=e^x$

Solution

Asymptote: $y=0$

x	$f(x)$
-1	.4
0	1
1	2.7



Domain: $(-\infty,\ \infty)$

Range: $(0,\infty)$

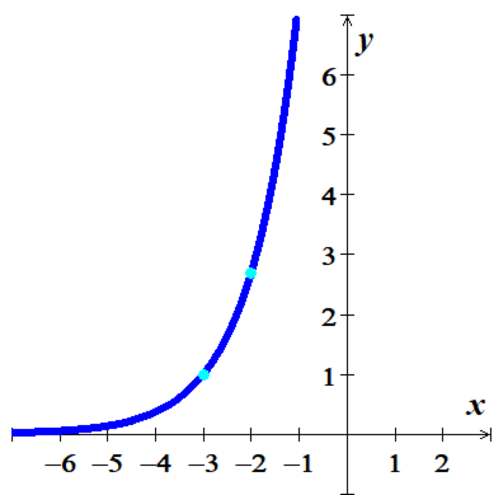
Example

$f(x)=e^{x+3}$

Solution

Shifted left 3 units

Asymptote: $y=0$



Exercises Section 3.2 – Exponential Functions

(**1 – 8**) Evaluate to four decimal places using a calculator

1. $2^{3.4}$

2. $5^{\sqrt{3}}$

3. $6^{-1.2}$

4. $e^{-0.75}$

5. $e^{2.3}$

6. $e^{-0.95}$

7. $\pi^{\sqrt{\pi}}$

8. $e^{\sqrt{2}}$

(**9 – 20**) Find the ***asymptote***, ***domain***, and ***range*** of the given functions. Then, sketch the graph

9. $f(x)=2^x+3$

10. $f(x)=2^{3-x}$

11. $f(x)=\left(\frac{2}{5}\right)^{-x}$

12. $f(x)=-\left(\frac{1}{2}\right)^x+4$

13. $f(x)=4^x$

14. $f(x)=2-4^x$

15. $f(x)=-3+4^{x-1}$

16. $f(x)=1+\left(\frac{1}{4}\right)^{x+1}$

17. $f(x)=e^{x-2}$

18. $f(x)=3-e^{x-2}$

19. $f(x)=e^{x+4}$

20. $f(x)=2+e^{x-1}$

21. The exponential function $f(x)=1066e^{0.042x}$ models the gray wolf population of the Western Great Lakes, $f(x)$, in *billions*, *x years* after 1978. Project the gray population in the recovery area in 2012.

22. The function $f(x)=6.4e^{0.0123x}$ describes world population, $f(x)$, in billions, x years after 2004 subject to a growth rate of 1.23% *annually*. Use the function to predict world population in 2050.

23. A cup of coffee is heated to $160°F$ and placed in a room that maintains a temperature of $70°F$. The temperature T of the coffee, in *degree Fahrenheit*, after t minutes is given by

$$T(t)=70+90e^{-0.0485t}$$

a) Find the temperature of the coffee 20 *minutes* after it is placed in the room
b) Determine when the temperature of the coffee will reach $90°F$

24. A cup of coffee is heated to $180°F$ and placed in a room that maintains a temperature of $65°F$. The temperature T of the coffee, in *degree Fahrenheit*, after t minutes is given by

$$T(t)=65+115e^{-0.042t}$$

a) Find the temperature of the coffee 10 *minutes* after it is placed in the room
b) Determine when the temperature of the coffee will reach $100°F$

25. The percent $I(x)$ of the original intensity of light striking the surface of a lake that is available x *feet* below the surface of the lake is given by the equation

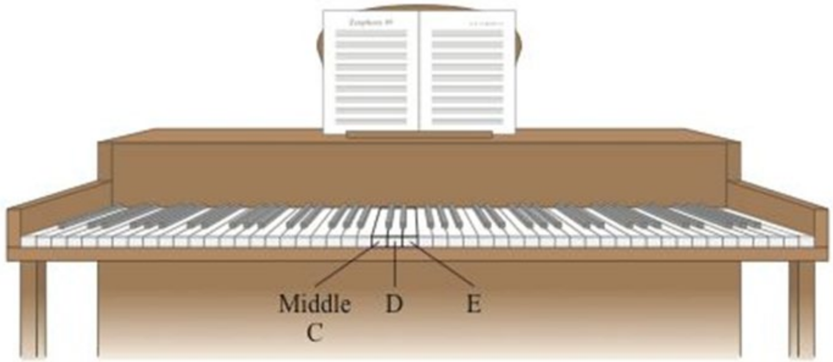
$$I(x)=100e^{-.95x}$$

a) What percentage of the light is available 2 *feet* below the surface of the lake?

b) At what depth is the intensity of the light one-half the intensity at the surface?

26. Starting on the left side of a standard 88–*key* piano, the frequency, in *vibrations* per *second*, of the *n*th note is given by

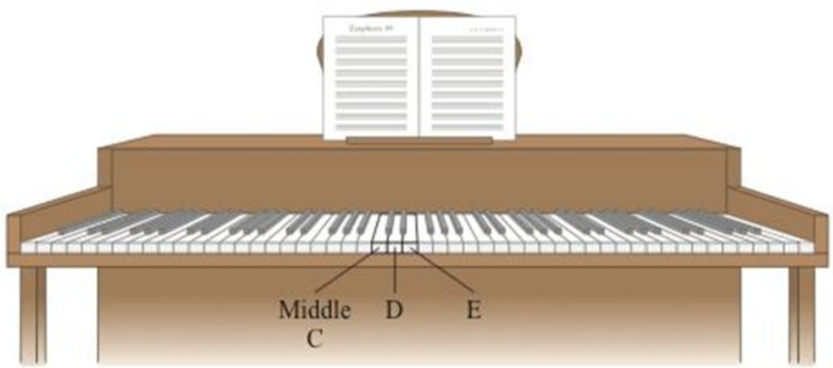
$$f(n) = (2.75)\ 2^{\frac{n-1}{12}}$$



a) Determine the frequency of middle *C*, key number 40 on an 88–*key* piano.
b) Is the difference in frequency between middle *C* (key number 40) and *D* (key number 42) the same as the difference in frequency between *D* (key number 42) and *E* (key number 44)?

27. Starting on the left side of a standard 88–*key* piano, the frequency, in *vibrations* per *second*, of the *n*th note is given by

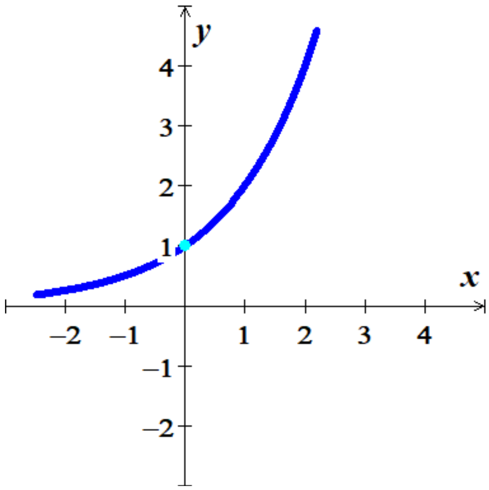
$$f(n) = (27.5)\ 2^{\frac{n-1}{12}}$$

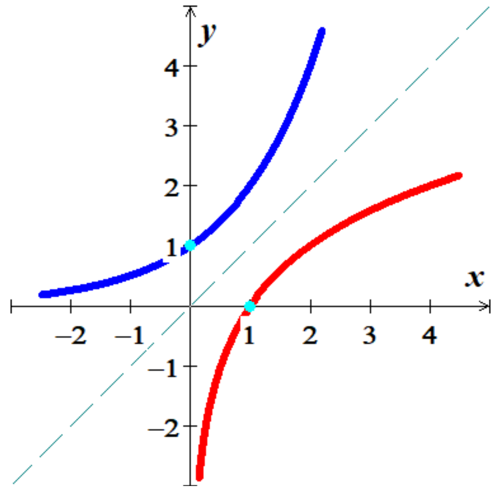


a) Determine the frequency of middle *C*, key number 40 on an 88–*key* piano.
b) Is the difference in frequency between middle *C* (key number 40) and *D* (key number 42) the same as the difference in frequency between *D* (key number 42) and *E* (key number 44)?

Section 3.3 – Logarithmic Functions

Graph: $x = 2^y$





Find the inverse function of $f(x) = 2^x$

$y = 2^x$

$x = 2^y$ *Solve for y?*

Logarithmic Function (*Definition)*

For $\boldsymbol{x > 0}$ and $b > 0,\ b \neq 1$

$$y = \log_b x \quad \text{is equivalent to} \quad x = b^y$$

$$y = \log_b x \iff x = b^y$$

Base

The function $f(x) = \log_b x$ is the logarithmic function with base b.

$\log_b x$: ***read*** log base b of x

$\log x$ ***means*** $\log_{10} x$

Example

Write each equation in its equivalent exponential form:

a) $3 = \log_7 x \qquad \Rightarrow x = 7^3$

b) $2 = \log_b 25 \qquad \Rightarrow 25 = b^2$

Example

Write each equation in its equivalent logarithmic form:

a) $2^5 = x \qquad \Rightarrow 5 = \log_2 x$

b) $27 = b^3 \qquad \Rightarrow 3 = \log_b 27$

Basic Logarithmic Properties

$$\log_b b = 1 \quad \to \quad b = b^1$$

$$\log_b 1 = 0 \quad \to \quad 1 = b^0$$

Inverse Properties

$$\log_b b^x = x \qquad\qquad \log_7 7^8 = 8$$

$$b^{\log_b x} = x \qquad\qquad 3^{\log_3 17} = 17$$

Example

Evaluate each expression without using a calculator:

a) $\log_5 \frac{1}{125}$ *b)* $\log_3 \sqrt[7]{3}$

Solution

a) $\log_5 \frac{1}{125} = \log_5 \frac{1}{5^3}$

$= \log_5 5^{-3}$

$\underline{= -3}$

b) $\log_3 \sqrt[7]{3} = \log_3 3^{1/7}$

$\underline{= \frac{1}{7}}$

Natural Logarithms

Definition

$$f(x) = \log_e x = \ln x$$

The logarithmic function with base ***e*** is called natural logarithmic function.

$\ln x$ *read* "el en of x"

$\log(-1) =$ *doesn't exist*	$\ln(-1) =$ *doesn't exist*
$\log 0 =$ *doesn't exist*	$\ln 0 =$ *doesn't exist*
$\log 0.5 \approx -0.3010$	$\ln 0.5 \approx -0.6931$
$\log 1 = 0$	$\ln 1 = 0$
$\log 2 \approx 0.3010$	$\ln 2 \approx 0.6931$
$\log 10 = 1$	$\ln e = 1$

Change-of-Base Logarithmic

$$\log_b M = \frac{\log_a M}{\log_a b} \qquad \log_b M = \frac{\log M}{\log b} \quad \textit{or} \quad \log_b M = \frac{\ln M}{\ln b}$$

Evaluate

$$\log_7 2506 = \frac{\log 2506}{\log 7}$$ log(2506) / log(7)

$$\approx 4.02$$

Or

$$\log_7 2506 = \frac{\ln 2506}{\ln 7} \approx 4.02$$ ln(2506) / ln(7)

$$\log_5 17 = \frac{\ln 17}{\ln 5} \approx 1.7604$$

$$\log_2 0.1 = \frac{\ln 0.1}{\ln 2} \approx -3.3219$$

Domain

The domain of a logarithmic function of the form $f(x)=\log_b x$ is the set of all positive real numbers.
(*Inside* the log has to be > 0)

Range: $(-\infty,\ \infty)$

Example

Find the ***domain*** of

a) $f(x)=\log_4(x-5)$

$$x-5>0 \quad \Rightarrow \; x>5$$

Domain: $\boxed{(5,\ \infty)}$

b) $f(x)=\ln(4-x)$

$$4-x>0$$
$$-x>-4$$
$$x<4$$

Domain: $\boxed{(-\infty,\ 4)}$

c) $h(x)=\ln\left(x^2\right)$

$x^2>0 \Rightarrow$ all real numbers except 0.

Domain: $\{x|\, x\neq 0\}$

or $\boxed{(-\infty,\ 0)\cup(0,\ \infty)}$

or $\boxed{\mathbb{R}-\{0\}}$

Graphs of *Logarithmic* Functions

Example

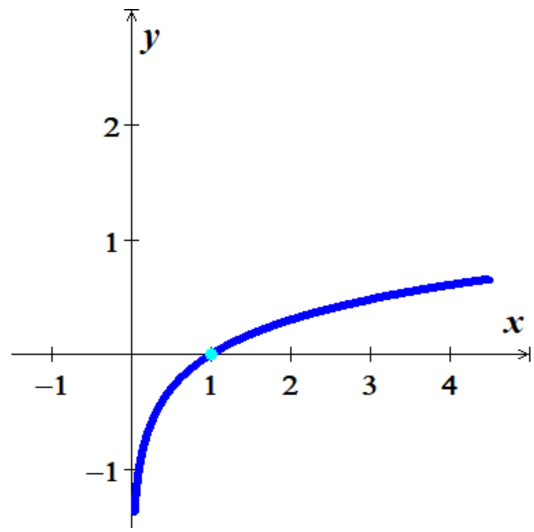
Graph $g(x) = \log x$

Solution

Asymptote: $x = 0$
(*Force inside log to be equal to zero, then solve for* x)

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$



x	$g(x)$
~~0~~	
0.5	$-.3$
1	0
2	.3
3	.5

Example

$f(x) = \log_5 x$

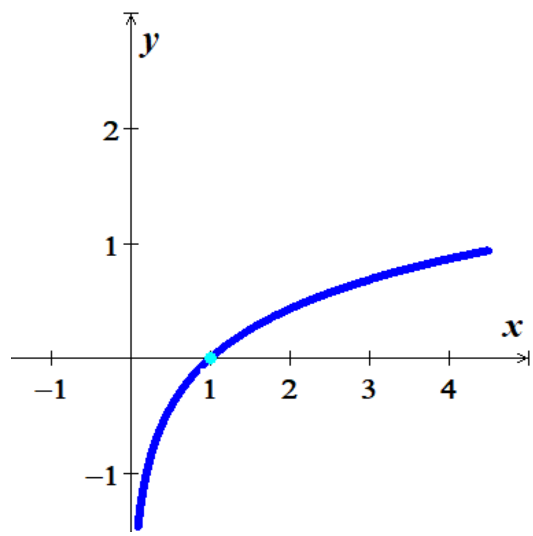
Solution

$$f(x) = \frac{\log x}{\log 5}$$

Asymptote: $x = 0$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$



x	$f(x)$
~~0~~	
$\frac{1}{5}$	-1
1	0
5	1

Example

Graph: $f(x) = \log_{1/2} x$

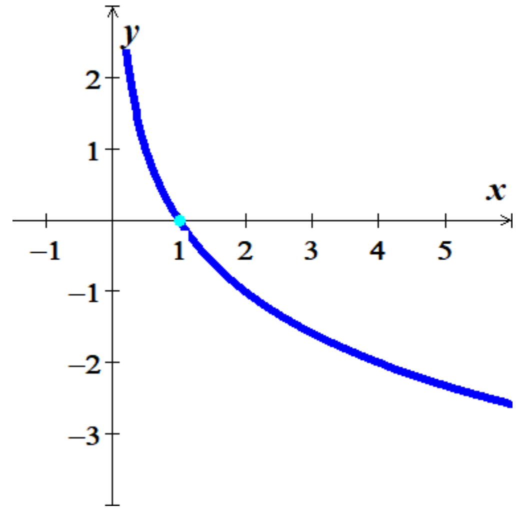
Solution

Asymptote: $x = 0$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

x	$f(x)$
0	
2	-1
1	0
$\frac{1}{2}$	1



Example

Graph: $f(x) = \log_2 (x-1)$

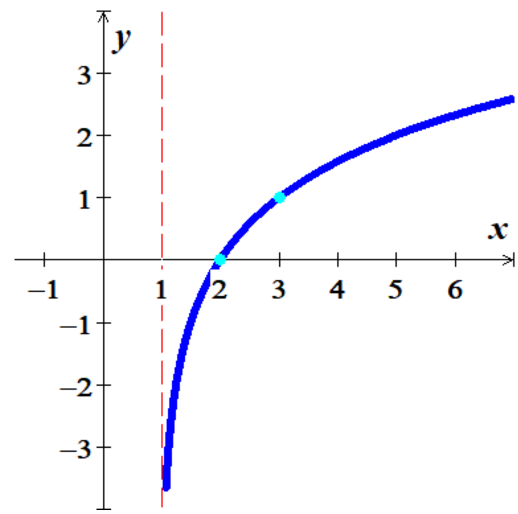
Solution

Asymptote: $x = 1$

Domain: $(1, \infty)$

Range: $(-\infty, \infty)$

x	$f(x)$
1	
$1\frac{1}{3}$	-1
2	0
3	1



Example

$f(x) = |\ln(x-1)|$

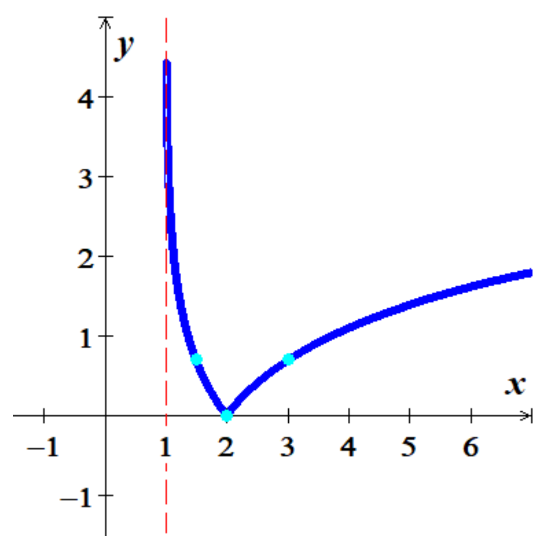
Solution

Asymptote: $x = 1$

Domain: $(1, \infty)$

Range: $[0, \infty)$

x	$f(x)$
1	
$\frac{3}{2}$	-0.7
2	0
3	0.7



Exercises *Section* 3.3 – Logarithmic Functions

(**1 – 12**) Write the equation in its equivalent logarithmic form

1. $2^6 = 64$

2. $5^4 = 625$

3. $5^{-3} = \frac{1}{125}$

4. $\sqrt[3]{64} = 4$

5. $b^3 = 343$

6. $8^y = 300$

7. $\sqrt[n]{x} = y$

8. $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$

9. $\left(\frac{1}{2}\right)^{-5} = 32$

10. $e^{x-2} = 2y$

11. $e = 3x$

12. $\sqrt[3]{e^{2x}} = y$

(**13 – 24**) Write the equation in its equivalent exponential form

13. $\log_5 125 = y$

14. $\log_4 16 = x$

15. $\log_5 \frac{1}{5} = x$

16. $\log_2 \frac{1}{8} = x$

17. $\log_6 \sqrt{6} = x$

18. $\log_3 \frac{1}{\sqrt{3}} = x$

19. $6 = \log_2 64$

20. $2 = \log_9 x$

21. $\log_{\sqrt{3}} 81 = 8$

22. $\log_4 \frac{1}{64} = -3$

23. $\log_4 26 = y$

24. $\ln M = c$

(**25 – 31**) Evaluate the expression without using a calculator

25. $\log_4 16$

26. $\log_2 \frac{1}{8}$

27. $\log_6 \sqrt{6}$

28. $\log_3 \frac{1}{\sqrt{3}}$

29. $\log_3 \sqrt[7]{3}$

30. $\log_3 \sqrt{9}$

31. $\log_{\frac{1}{2}} \sqrt{\frac{1}{2}}$

(**32 – 40**) Simplify

32. $\log_5 1$

33. $\log_7 7^2$

34. $3^{\log_3 8}$

35. $10^{\log 3}$

36. $e^{2+\ln 3}$

37. $\ln e^{-3}$

38. $\ln e^{x-5}$

39. $\log_b b^n$

40. $\ln e^{x^2+3x}$

(**41 – 64**) Find the domain of

41. $f(x) = \log_5 (x+4)$

42. $f(x) = \log_5 (x+6)$

43. $f(x) = \log(2-x)$

44. $f(x) = \log(7-x)$

45. $f(x) = \ln(x-2)^2$

46. $f(x) = \ln(x-7)^2$

47. $f(x) = \log\left(x^2 - 4x - 12\right)$

48. $f(x)=\log\left(\frac{x-2}{x+5}\right)$

49. $f(x)=\log\left(\frac{3-x}{x-2}\right)$

50. $f(x)=\ln\left(\frac{x^2}{x-4}\right)$

51. $f(x)=\log_3\left(x^3-x\right)$

52. $f(x)=\log\sqrt{2x-5}$

53. $f(x)=3\ln(5x-6)$

54. $f(x)=\log\left(\frac{x}{x-2}\right)$

55. $f(x)=\ln\left(x^2+4\right)$

56. $f(x)=\ln|4x-8|$

57. $f(x)=\ln\left(x^2-9\right)$

58. $f(x)=\ln|5-x|$

59. $f(x)=\ln(x-4)^2$

60. $f(x)=\ln\left(x^2-4\right)$

61. $f(x)=\ln\left(x^2-4x+3\right)$

62. $f(x)=\ln\left(2x^2-5x+3\right)$

63. $f(x)=\log\left(x^2+4x+3\right)$

64. $f(x)=\ln\left(x^4-x^2\right)$

(65 – 73) Find the ***asymptote***, ***domain***, and ***range*** of the given functions. Then, sketch the graph

65. $f(x)=\log_4(x-2)$

66. $f(x)=\log_4|x|$

67. $f(x)=\left(\log_4 x\right)-2$

68. $f(x)=\log(3-x)$

69. $f(x)=2-\log(x+2)$

70. $f(x)=\ln(x-2)$

71. $f(x)=\ln(3-x)$

72. $f(x)=2+\ln(x+1)$

73. $f(x)=1-\ln(x-2)$

74. On a study by psychologists Bornstein and Bornstein, it was found that the average walking speed w, in feet per second, of a person living in a city of population P, in ***thousands***, is given by the function:

$$w(P)=0.37\ln P+0.05$$

a) The population is 124,848. Find the average walking speed of people living in Hartford.
b) The population is 1,236,249. Find the average walking speed of people living in San Antonio.

75. The loudness of sounds is measured in a unit called a decibel. To measure with this unit, we first assign an intensity of I_0 to a very faint sound, called the threshold sound. If a particular sound has intensity I, then the decibel rating of this louder sound is

$$d=10\log\frac{I}{I_0}$$

Find the exact decibel rating of a sound with intensity $10{,}000I_0$

76. Students in an accounting class took a final exam and then took equivalent forms of the exam at monthly intervals thereafter. The average score $S(t)$, as a percent, after t months was found to be given by the function

$$S(t) = 78 - 15\ \log(t+1); \quad t \ge 0$$

a) What was the average score when the students initially took the test, $t = 0$?

b) What was the average score after 4 *months*? 24 *months*?

77. A model for advertising response is given by the function

$$N(a) = 1,000 + 200\ \ln a, \quad a \ge 1$$

Where $N(a)$ is the number of units sold when $\boldsymbol{a}$ is the amount spent on advertising, in thousands of dollars.

a) $N(1)$

b) $N(5)$

Section 3.4 – Properties of Logarithms

Product Rule

$$\log_b MN = \log_b M + \log_b N$$ ***For*** ***M*** **> 0** ***and*** ***N*** **> 0**

$$\begin{cases} \log_b M = x \quad \Rightarrow \quad M = b^x \\ \log_b N = y \quad \Rightarrow \quad N = b^y \end{cases} \qquad \Rightarrow \text{MN} = b^x\, b^y = b^{x+y}$$

Convert back to logarithmic form: $\log_b MN = x + y$

$$\log_b MN = \log_b M + \log_b N$$

Example

Use the product rule to expand the logarithmic expression

$$\log(100x) = \log 100 + \log x$$

Power Rule

$$\log_b M^{p} = p \log_b M$$

Example

Use the power rule to expand each logarithmic expression

$$\ln \sqrt[3]{x} = \ln(x)^{1/3} = \tfrac{1}{3}\ln x$$

Quotient Rule

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

Example

Use the quotient rule to expand the logarithmic expression

$$\ln\left(\frac{e^5}{11}\right) = \ln e^5 - \ln 11$$

$$= 5 - \ln 11$$

Example

Express each of the following in terms of sums and differences of logarithm: $\log_6 (7 \times 9)$

Solution

$$\log_6 (7 \times 9) = \log_6 7 + \log_6 9 \qquad \textit{Product Rule}$$

Example

Express each of the following in terms of sums and differences of logarithm: $\log_9 \left(\frac{15}{7}\right)$

Solution

$$\log_9 \left(\frac{15}{7}\right) = \log_9 15 - \log_9 7 \qquad \textit{Quotient Rule}$$

Example

Express each of the following in terms of sums and differences of logarithm: $\log_5 \sqrt{8}$

Solution

$$\log_5 \sqrt{8} = \log_5 \left(2^3\right)^{1/2}$$

$$= \log_5 2^{3/2} \qquad \textit{Power Rule}$$

$$\boxed{= \frac{3}{2} \log_5 2}$$

Example

Express each of the following in terms of sums and differences of logarithm: $\log_b \left(x^4 \sqrt[3]{y}\right)$

Solution

$$\log_b \left(x^4 \sqrt[3]{y}\right) = \log_b \left(x^4\right) + \log_b \left(\sqrt[3]{y}\right) \qquad \textit{Product Rule}$$

$$= \log_b \left(x^4\right) + \log_b \left(y^{1/3}\right) \qquad \textit{Power Rule}$$

$$\boxed{= 4 \log_b x + \frac{1}{3} \log_b y}$$

Example

Express each of the following in terms of sums and differences of logarithm: $\log_a\left(\dfrac{mnq}{p^2r^4}\right)$

Solution

$$\log_a\left(\frac{mnq}{p^2r^4}\right)=\log_a(mnq)-\log_a\left(p^2r^4\right) \qquad \textit{Quotient Rule}$$

$$=\log_a m+\log_a n+\log_a q-\left(\log_a p^2+\log_a r^4\right) \qquad \textit{Product Rule}$$

$$=\log_a m+\log_a n+\log_a q-\log_a p^2-\log_a r^4$$

$$\underline{=\log_a m+\log_a n+\log_a q-2\log_a p-4\log_a r} \qquad \textit{Power Rule}$$

Example

Express each of the following in terms of sums and differences of logarithm: $\log_5\left(\dfrac{\sqrt{x}}{25y^3}\right)$

Solution

$$\log_5\left(\frac{\sqrt{x}}{25y^3}\right)=\log_5\left(x^{1/2}\right)-\log_5\left(25y^3\right) \qquad \textit{Quotient Rule}$$

$$=\log_5\left(x^{1/2}\right)-\left[\log_5\left(5^2\right)+\log_5\left(y^3\right)\right] \qquad \textit{Product Rule}$$

$$=\log_5\left(x^{1/2}\right)-\log_5\left(5^2\right)-\log_5\left(y^3\right) \qquad \log_5\left(5^2\right)=2$$

$$\underline{=\frac{1}{2}\log_5 x-2-3\log_5 y}$$

Example

Write as a single logarithmic $\log(7x+6)-\log x$

Solution

$$\log(7x+6)-\log x=\log\frac{7x+6}{x}$$ *Quotient Rule*

Example

Write as a single logarithmic $\log_3(x+2)+\log_3 x-\log_3 2$

Solution

$$\log_3(x+2)+\log_3 x-\log_3 2=\log_3 x(x+2)-\log_3 2$$ *Product Rule*

$$=\boxed{\log_3\frac{x(x+2)}{2}}$$ *Quotient Rule*

Example

Write as a single logarithmic $2\ln x+\frac{1}{3}\ln(x+5)$

Solution

$$2\ln x+\tfrac{1}{3}\ln(x+5)=\ln x^2+\ln(x+5)^{1/3}$$ *Power Rule*

$$=\ln x^2\,(x+5)^{1/3}$$ *Product Rule*

$$=\boxed{\ln\left(x^2\,\sqrt[3]{x+5}\right)}$$

Example

Write as a single logarithmic $2\log(x-3)-\log x$

Solution

$$2\log(x-3)-\log x=\log(x-3)^2-\log x$$ *Power Rule*

$$=\boxed{\log\frac{(x-3)^2}{x}}$$ *Quotient Rule*

Exercises *Section* 3.4 – Properties of Logarithms

(1 – 31) Express the following in terms of sums and differences of logarithms

1. $\log_3(ab)$

2. $\log_7(7x)$

3. $\log\frac{x}{1000}$

4. $\log_5\left(\frac{125}{y}\right)$

5. $\log_b x^7$

6. $\ln\sqrt[7]{x}$

7. $\log_a \frac{x^2 y}{z^4}$

8. $\log_b \frac{x^2 y}{b^3}$

9. $\log_b\left(\frac{x^3 y}{z^2}\right)$

10. $\log_b\left(\frac{\sqrt[3]{x}y^4}{z^5}\right)$

11. $\log\left(\frac{100x^3\sqrt[3]{5-x}}{3(x+7)^2}\right)$

12. $\log_a \sqrt[4]{\frac{m^8 n^{12}}{a^3 b^5}}$

13. $\log_p \sqrt[3]{\frac{m^5 n^4}{t^2}}$

14. $\log_b \sqrt[n]{\frac{x^3 y^5}{z^m}}$

15. $\log_a \sqrt[3]{\frac{a^2 b}{c^5}}$

16. $\log_b\left(x^4\sqrt[3]{y}\right)$

17. $\log_5\left(\frac{\sqrt{x}}{25y^3}\right)$

18. $\log_a \frac{x^3 w}{y^2 z^4}$

19. $\log_a \frac{\sqrt{y}}{x^4\sqrt[3]{z}}$

20. $\ln\sqrt[4]{\frac{x^7}{y^5 z}}$

21. $\ln x\sqrt[3]{\frac{y^4}{z^5}}$

22. $\log_b \sqrt[5]{\frac{m^4 n^5}{x^2 a b^{10}}}$

23. $\log_b \frac{a^5 b^{10}}{c^2\sqrt[4]{d^3}}$

24. $\ln\left(x^2\sqrt{x^2+1}\right)$

25. $\ln\frac{x^2}{x^2+1}$

26. $\ln\left(\frac{x^2(x+1)^3}{(x+3)^{1/2}}\right)$

27. $\ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$

28. $\ln\frac{\left(x^2+1\right)^5}{\sqrt{1-x}}$

29. $\ln\left(\sqrt[3]{\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}}\right)$

30. $\ln\left(\sqrt{\frac{1}{x(x+1)}}\right)$

31. $\ln\left(\sqrt{\left(x^2+1\right)(x-1)^2}\right)$

(32 – 55) Write the expression as a single logarithm and simplify if necessary

32. $\log(x+5)+2\log x$

33. $3\log_b x-\frac{1}{3}\log_b y+4\log_b z$

34. $\frac{1}{2}\log_b(x+5)-5\log_b y$

35. $\ln\left(x^2-y^2\right)-\ln(x-y)$

36. $\ln(xz)-\ln\left(x\sqrt{y}\right)+2\ln\frac{y}{z}$

37. $\log\left(x^2 y\right)-\log z$

38. $\log\left(z^2\sqrt{y}\right)-\log z^{1/2}$

39. $2\log_a x+\frac{1}{3}\log_a(x-2)-5\log_a(2x+3)$

40. $5\log_a x-\frac{1}{2}\log_a(3x-4)-3\log_a(5x+1)$

41. $\log\left(x^3y^2\right)-2\log\left(x\sqrt[3]{y}\right)-3\log\left(\frac{x}{y}\right)$

42. $\ln y^3+\frac{1}{3}\ln\left(x^3y^6\right)-5\ln y$

43. $2\ln x-4\ln\left(\frac{1}{y}\right)-3\ln(xy)$

44. $4\ln x+7\ln y-3\ln z$

45. $\frac{1}{3}\left[5\ln(x+6)-\ln x-\ln(x^2-25)\right]$

46. $\frac{2}{3}\left[\ln\left(x^2-4\right)-\ln(x+2)\right]+\ln(x+y)$

47. $\frac{1}{2}\log_b m+\frac{3}{2}\log_b 2n-\log_b m^2n$

48. $\frac{1}{2}\log_y p^3q^4-\frac{2}{3}\log_y p^4q^3$

49. $\frac{1}{2}\log_a x+4\log_a y-3\log_a x$

50. $\frac{2}{3}\left[\ln\left(x^2-9\right)-\ln(x+3)\right]+\ln(x+y)$

51. $\frac{1}{4}\log_b x-2\log_b 5-10\log_b y$

52. $2\ln(x+4)-\ln x-\ln\left(x^2-3\right)$

53. $\ln x+\ln(y+3)+\ln(y+2)-\ln\left(y^2+5y+6\right)$

54. $\ln x+\ln(x+4)+\ln(x+1)-\ln\left(x^2+5x+4\right)$

55. $\ln\left(x^2-25\right)-2\ln(x+5)+\ln(x-5)$

56. Assume that $\log_{10} 2=.3010$. Find each logarithm $\log_{10} 4$, $\log_{10} 5$

57. Given that: $\log_a 2\approx 0.301$, $\log_a 7\approx 0.845$, and $\log_a 11\approx 1.041$ find each of the following:

a) $\log_a \frac{2}{11}$

b) $\log_a 14$

c) $\log_a 98$

d) $\log_a \frac{1}{7}$

e) $\log_a 9$

f) $\log_a \frac{77}{8}$

Section 3.5 – Exponential and logarithmic Equations

Exponential Equations

$$b^M = b^N \quad \leftrightarrow \quad M = N \quad \text{for any b > 0, } \neq 1$$

Example

Solve $5^{3x-6} = 125$

Solution

$5^{3x-6} = 5^3$

$3x - 6 = 3$

$3x = 9$

$\underline{x = 3}$

Example

Solve $8^{x+2} = 4^{x-3}$

Solution

$\left(2^3\right)^{x+2} = \left(2^2\right)^{x-3}$

$2^{3(x+2)} = 2^{2(x-3)}$

$3(x+2) = 2(x-3)$

$3x + 6 = 2x - 6$

$3x - 2x = -6 - 6$

$\underline{x = -12}$

Using *Natural Logarithms*

1. Isolate the exponential expression
2. Take the natural logarithm on both sides of the equation
3. Simplify using one of the following properties: $\ln b^{x} = x\ln b$ *or* $\ln e^{x} = x$
4. Solve for the variable

Example

Solve: $7e^{2x} - 5 = 58$

Solution

$7e^{2x} - 5 = 58$ ***Isolate the exponential expression***

$7e^{2x} = 63$ ***Divide by 7 both sides***

$e^{2x} = 9$ ***Natural logarithm on both sides***

$\ln e^{2x} = \ln 9$ ***Use inverse Property***

$2x = \ln 9$

$x = \dfrac{\ln 9}{2} \approx 1.0986$

Example

Solve: $3^{2x-1} = 7^{x+1}$

Solution

$\ln 3^{2x-1} = \ln 7^{x+1}$ ***Natural logarithm on both sides***

$(2x-1)\ln 3 = (x+1)\ln 7$ ***Power Rule***

$2x\ln 3 - \ln 3 = x\ln 7 + \ln 7$

$2x\ln 3 - x\ln 7 = \ln 3 + \ln 7$

$x\left(2\ln 3 - \ln 7\right) = \ln 3 + \ln 7$

$x = \dfrac{\ln 3 + \ln 7}{2\ln 3 - \ln 7}$ ≈ 12.1143

Logarithmic Equations

1. Express the equation in the form $\log_b M = c$
2. Use the definition of a logarithm to rewrite the equation in exponential form:

$$\log_b M = c \quad \Rightarrow \quad b^c = M$$

3. Solve for the variable
4. Check proposed solution in the original equation. Include only the set for M > 0

Example

Solve: $\log(x) + \log(x-3) = 1$

Solution

$\log(x(x-3)) = 1$ ***Product Rule***

$x(x-3) = 10^1$ ***Convert to exponential form***

$x^2 - 3x = 10$

$x^2 - 3x - 10 = 0$ ***Solve for x***

$x = -2,\ 5$

Check: $x = -2 \Rightarrow \cancel{\log(-2) + \log(x-3) = 1}$

$x = 5 \Rightarrow \log(5) + \log(5-3) = 1$

$\therefore$ ***Solution***: $\boxed{x = 5}$

Example

Solve: $\log_6(3x+2) + \log_6(x-1) = 1$

Solution

$\log_6[(3x+2)(x-1)] = 1$ ***Product Rule***

$(3x+2)(x-1) = 6^1$ ***Convert to exponential form***

$3x^2 - x - 2 = 6$

$3x^2 - x - 8 = 0$ ***Solve for x***

$x = \frac{1-\sqrt{97}}{6} < 0 \qquad x = \frac{1+\sqrt{97}}{6} > 1$

$\therefore$ ***Solution***: $\boxed{x = \frac{1+\sqrt{97}}{6}}$

Property of Logarithmic Equality

For any $M > 0, N > 0, b > 0, \neq 1$

$$log_b M = log_b N \quad \Rightarrow \quad M = N$$

Example

Solve: $\ln(x-3) = \ln(7x-23) - \ln(x+1)$

Solution

$\ln(x-3) = \ln\left(\frac{7x-23}{x+1}\right)$ ***Quotient Rule***

$x - 3 = \frac{7x-23}{x+1}$

$(x-3)(x+1) = 7x - 23$

$x^2 - 2x - 3 = 7x - 23$

$x^2 - 9x + 20 = 0$

$\underline{x = 4,\ 5|}$

Check: $x = 4 \Rightarrow \ln(4-3) = \ln(7(4)-23) - \ln(4+1)$

$x = 5 \Rightarrow \ln(5-3) = \ln(7(5)-23) - \ln(5+1)$

$\therefore$ ***Solution***: $\underline{x = 4,\ 5|}$

Example

Solve: $\log(x+6) - \log(x+2) = \log x$

Solution

$\log\frac{x+6}{x+2} = \log x$ ***Quotient Rule***

$\frac{x+6}{x+2} = x$ ***Multiply by x + 2***

$x + 6 = x(x+2)$

$x + 6 = x^2 + 2x$

$x^2 + x - 6 = 0$ ***Solve for x***

$\underline{x = -3,\ 2|}$

Check: $x = -3 \rightarrow \log(-3+6) - \log(-3+2) = \log(-3)$ Or ***Domain***

$x = 2 \rightarrow \log(2+6) - \log(2+2) = \log(2)$

$\therefore$ ***Solution***: $\underline{x = 2|}$

Exercises *Section* 3.5 – Exponential and logarithmic Equations

(**1 – 105**) Solve the equations

1. $2^x = 128$
2. $3^x = 243$
3. $5^x = 70$
4. $6^x = 50$
5. $5^x = 134$
6. $7^x = 12$
7. $9^x = \frac{1}{\sqrt[3]{3}}$
8. $49^x = \frac{1}{343}$
9. $2^{5x+3} = \frac{1}{16}$
10. $\left(\frac{2}{5}\right)^x = \frac{8}{125}$
11. $2^{3x-7} = 32$
12. $4^{2x-1} = 64$
13. $3^{1-x} = \frac{1}{27}$
14. $2^{-x^2} = 5$
15. $2^{-x} = 8$
16. $\left(\frac{1}{3}\right)^x = 81$
17. $3^{-x} = 120$
18. $27 = 3^{5x}\ 9^{x^2}$
19. $4^{x+3} = 3^{-x}$
20. $2^{x+4} = 8^{x-6}$
21. $8^{x+2} = 4^{x-3}$
22. $7^x = 12$
23. $5^{x+4} = 4^{x+5}$
24. $5^{x+2} = 4^{1-x}$
25. $3^{2x-1} = 0.4^{x+2}$
26. $4^{3x-5} = 16$
27. $4^{x+3} = 3^{-x}$
28. $7^{2x+1} = 3^{x+2}$
29. $3^{x-1} = 7^{2x+5}$
30. $4^{x-2} = 2^{3x+3}$
31. $3^{5x-8} = 9^{x+2}$
32. $3^{x+4} = 2^{1-3x}$
33. $3^{2-3x} = 4^{2x+1}$
34. $4^{x+3} = 3^{-x}$
35. $7^{x+6} = 7^{3x-4}$
36. $2^{-100x} = (0.5)^{x-4}$
37. $4^x\left(\frac{1}{2}\right)^{3-2x} = 8.\left(2^x\right)^2$
38. $5^x + 125\left(5^{-x}\right) = 30$
39. $4^x - 3\left(4^{-x}\right) = 8$
40. $5^{3x-6} = 125$
41. $e^x = 15$
42. $e^{x+1} = 20$
43. $9e^x = 107$
44. $e^{x\ln 3} = 27$
45. $e^{x^2} = e^{7x-12}$

46. $f(x) = xe^x + e^x$

47. $f(x) = x^3\left(4e^{4x}\right) + 3x^2 e^{4x}$

48. $e^{2x} - 2e^x - 3 = 0$

49. $e^{0.08t} = 2500$

50. $e^{x^2} = 200$

51. $e^{2x+1} \cdot e^{-4x} = 3e$

52. $e^{2x} - 8e^x + 7 = 0$

53. $e^{2x} + 2e^x - 15 = 0$

54. $e^x + e^{-x} - 6 = 0$

55. $e^{1-3x} \cdot e^{5x} = 2e$

56. $6\ln(2x) = 30$

57. $\log_5 (x-7) = 2$

58. $\log_4 (5+x) = 3$

59. $\log(4x-18) = 1$

60. $\log_3 x = -2$

61. $\log\left(x^2+19\right) = 2$

62. $\ln\left(x^2-12\right) = \ln x$

63. $\log\left(2x^2+3x\right) = \log(10x+30)$

64. $\log_5 (2x+3) = \log_5 11 + \log_5 3$

65. $\log_3 x - \log_9 (x+42) = 0$

66. $\log_5 x + \log_5 (4x-1) = 1$

67. $\log x - \log(x+3) = 1$

68. $\log x + \log(x-9) = 1$

69. $\log_2 (x+1) + \log_2 (x-1) = 3$

70. $\log_8 (x+1) - \log_8 x = 2$

71. $\ln(x+8) + \ln(x-1) = 2\ln x$

72. $\ln(4x+6) - \ln(x+5) = \ln x$

73. $\ln(5+4x) - \ln(x+3) = \ln 3$

74. $\ln \sqrt[4]{x} = \sqrt{\ln x}$

75. $\sqrt{\ln x} = \ln \sqrt{x}$

76. $\log x^2 = (\log x)^2$

77. $\log x^3 = (\log x)^2$

78. $\log(\log x) = 1$

79. $\log(\log x) = 2$

80. $\ln(\ln x) = 2$

81. $\ln\left(e^{x^2}\right) = 64$

82. $e^{\ln(x-1)} = 4$

83. $10^{\log(2x+5)} = 9$

84. $\log\sqrt{x^3-9} = 2$

85. $\log\sqrt{x^3-17} = \frac{1}{2}$

86. $\log_4 x = \log_4 (8-x)$

87. $\log_7 (x-5) = \log_7 (6x)$

88. $\ln x^2 = \ln(12-x)$

89. $\log_2 (x+7) + \log_2 x = 3$

90. $\ln x = 1 - \ln(x+2)$

91. $\ln x = 1 + \ln(x+1)$

92. $\log_6 (2x-3) = \log_6 12 - \log_6 3$

93. $\log(3x+2) + \log(x-1) = 1$

94. $\log_5 (x+2) + \log_5 (x-2) = 1$

95. $\log_2 x + \log_2 (x-4) = 2$

96. $\log_3 x + \log_3 (x+6) = 3$

97. $\log_3 (x+3) + \log_3 (x+5) = 1$

98. $\ln x = \frac{1}{2}\ln\left(2x + \frac{5}{2}\right) + \frac{1}{2}\ln 2$

99. $\ln(-4-x) + \ln 3 = \ln(2-x)$

100. $\log_4 x + \log_4 (x-2) = \log_4 (15)$

101. $\ln(x-5) - \ln(x+4) = \ln(x-1) - \ln(x+2)$

102. $\ln(4-x) = \ln(x+8) + \ln(2x+13)$

103. $\log\left(x^2+4\right) - \log(x+2) = 2 + \log(x-2)$

104. $\log_3 (x-2) = \log_3 27 - \log_3 (x-4) - 5^{\log_5 1}$

105. $\log_2 (x+3) = \log_2 (x-3) + \log_3 9 + 4^{\log_4 3}$

106. Solve for t using logarithms with base a: $2a^{t/3} = 5$

107. Solve for t using logarithms with base a: $K = H - Ca^t$

Section 3.6 – Exponential Growth and Decay

Exponential Growth and Decay

The mathematical model for exponential growth or decay is given by

$$A(t) = A_0 e^{kt}$$

$A(t)$: Exponential Function (After time *t*)

A_0 : At time zero (initial value).

t: Time

k: Exponential rate.

- If $k > 0$, the function models of a growing entity
- If $k < 0$, the function models of a decay entity.

Example

In 1990, the population of Africa was 643 *million* and by 2000 it had grown to 813 *million*

a. Use the exponential growth function $A(t) = A_0 e^{kt}$, in which *t* is the number of years after 1990, to find the exponential growth function that models data

b. By which year will Africa's population reach 2000 *million*, or two *billion*?

Solution

a. $A(t) = A_0 e^{kt}$

From 1990 to 2000, is 10 years, that implies in 10 years the population grows from 643 to 813

$813 = 643 e^{k(10)}$

$\frac{813}{643} = e^{10k}$

$\ln \frac{813}{643} = \ln e^{10k}$

$\ln \frac{813}{643} = 10k$

$\frac{1}{10} \ln \frac{813}{643} = k$

$k \approx 0.023$

$\Rightarrow A(t) = 643 e^{0.023t}$

b. $2000 = 643e^{0.023t}$

$$\frac{2000}{643} = e^{0.023t}$$

$$\ln\frac{2000}{643} = \ln e^{0.023t}$$

$$\ln\frac{2000}{643} = 0.023t$$

$$\frac{\ln\frac{2000}{643}}{0.023} = t$$

$t \approx 49$

$\underline{Year:\ 2039}$

Doubling Time

$$P(t) = P_0 e^{kt}$$

$$2P_0 = P_0 e^{kt} \qquad P(t) = 2P_0$$

$$2 = e^{kt}$$

$$\ln 2 = \ln e^{kt}$$

$$\ln 2 = kt \ln e \qquad \ln e = 1$$

$$\ln 2 = kt$$

$$t = \frac{\ln 2}{k}$$

Growth Rate and Doubling Time

$$\boxed{Tk = \ln 2} \qquad or\ \ T = \frac{\ln 2}{k} \qquad or\ \ k = \frac{\ln 2}{T}$$

Example

A country's population doubled in 45 years. What was the exponential growth rate?

Solution

$$k = \frac{\ln 2}{t}$$

$$= \frac{\ln 2}{45}$$

$$\approx 0.0154$$

Finding K or T

$$A = A_0 e^{kt} \quad \Rightarrow \quad \boxed{kT = \ln \frac{A}{A_0}}$$

Proof

$$A = A_0 e^{kt}$$

$$\frac{A}{A_0} = e^{kt}$$

$$\ln \frac{A}{A_0} = \ln e^{kt}$$

$$\boxed{\ln \frac{A}{A_0} = kt} \quad \checkmark$$

Example

According to the U.S. Census Bureau, the world population reached 6 *billion* people on July 18, 1999, and was growing exponentially. By the end of 2000, the population had grown to 6.079 *billion*. The projected world population (in *billion* of people) t years after 2000, is given by the function defined by

$$f(t) = 6.079e^{.0126t}$$

a) Based on this model, what will the world population be in 2010?

b) In what year will the world population reach 7 *billion*?

Solution

a) In 2010 $\rightarrow t = 10$

$$f(t = 10) = 6.079e^{.0126(10)}$$
$$\approx 6.895$$

b) $t = \frac{1}{.0126} \ln\left(\frac{7}{6.079}\right)$ $\qquad kT = \ln \frac{A}{A_0}$

$$\approx 11.2$$

Year : 2011

Example

Strontium-90 is a waste product from nuclear reactors. As a consequence of fallout from atmosphere nuclear tests, we all have a measurable amount of strontium-90 in our bones.

a) The half-life of Strontium-90 is 28 *years*, meaning that all after 28 *years* a given amount of the substance will have decayed to half the original amount. Find the exponential decay model for Strontium-90.

b) Suppose the nuclear accident occurs and releases 60 *grams* of Strontium-90 into the atmosphere. How long will it take for Strontium-90 to decay to a level of 10 *grams*?

Solution

a) $k = \frac{1}{28}\ln\frac{1}{2}$ $\qquad kT = \ln\frac{A}{A_0}$

≈ -0.0248

$\boxed{A(t) = A_0 e^{-0.0248t}}$

b) $A = A_0 e^{-0.0248t}$

$t = \dfrac{\ln\frac{1}{6}}{-0.0248}$ $\qquad kT = \ln\frac{A}{A_0}$

$\approx 72.25\ yrs$

Logistic Model Function

Definition

The magnitude of a population at time $t \ge 0$ is given by

$$P(t) = \frac{c}{1 + ae^{-bt}}$$

where

c is the ***carrying capacity*** (the maximum population that can be supported by available resources as $t \to \infty$)

b is positive constant called the ***growth rate constant***.

$P_0 = P(0)$ is the ***initial population***.

a is related to the initial population P_0 and the carrying capacity c by the formula

$$a = \frac{c - P_0}{P_0}$$

Example

The coyote population in a wilderness area was estimated at 200 in 2007. By the beginning of 2009, the coyote population had increased to 250. A park ranger estimates that he carrying capacity of the wilderness area is 500 *coyotes*.

a) Use the given data to determine the growth rate constant for the logistic model of this coyote population.

b) Use the logistic model (part *a*) to predict the year in which the coyote population will first reach 400.

Solution

a) ***Given***: $P(0)=200 \quad P(2)=250 \quad c=500$

$$a=\frac{500-200}{200} \qquad\qquad a=\frac{c-P_0}{P_0}$$

$$=\boxed{\frac{3}{2}}$$

$$P(2)=\frac{500}{1+\frac{3}{2}e^{-2b}}=250 \qquad\qquad P(t)=\frac{c}{1+ae^{-bt}}$$

$$1+\frac{3}{2}e^{-2b}=\frac{500}{250}$$

$$\frac{3}{2}e^{-2b}=2-1$$

$$e^{-2b}=\frac{2}{3}$$

$$-2b=\ln\frac{2}{3}$$

$$b=-\frac{1}{2}\ln\frac{2}{3} \qquad\qquad \ln\frac{1}{x}=-\ln x$$

$$=\frac{1}{2}\ln\frac{3}{2}$$

$$\boxed{=\ln\sqrt{\frac{3}{2}}}$$

$$\boxed{\approx 0.202732554}$$

$$P(t)=\frac{500}{1+\frac{3}{2}e^{-\ln\sqrt{\frac{3}{2}}\,t}}$$

$$\boxed{=\frac{1,000}{2+3e^{-\ln\sqrt{\frac{3}{2}}\,t}}}$$

b) $400=\dfrac{1,000}{2+3e^{-\ln\sqrt{\frac{3}{2}}\,t}}$

$$2+3e^{-\ln\sqrt{\frac{3}{2}}\,t}=\frac{1{,}000}{400}$$

$$3e^{-\ln\sqrt{\frac{3}{2}}\,t}=\frac{5}{2}-2$$

$$e^{-\ln\sqrt{\frac{3}{2}}\,t}=\frac{1}{6}$$

$$-\ln\sqrt{\frac{3}{2}}\ t=\ln\frac{1}{6}$$

$$-\frac{1}{2}\ln\frac{3}{2}\ t=-\ln 6$$

$$t=2\frac{\ln 6}{\ln\frac{3}{2}}$$

$$=2\log_{\frac{3}{2}}6$$

$$\approx 8.84\ \ yrs.$$

Isotope	***Half–Life***
Carbon $\left({}^{14}C\right)$	5,730 *years*
Radium $\left({}^{226}Ra\right)$	1,660 *years*
Polonium $\left({}^{210}Po\right)$	138 *days*
Phosphorus $\left({}^{32}P\right)$	14 *days*
Polonium $\left({}^{214}Po\right)$	$\frac{1}{10{,}000}$ of a *second*

Formulas for Compound Interest

1. For n compounding per year: $A = P\left(1+\frac{r}{n}\right)^{nt}$

2. For Continuous compounding: $A = Pe^{rt}$

P: Principal, initial value
n: number of period per year
t: number of years
r: interest rate

A is also called ***Future value***

P is also called ***Present value***

Example

A sum of \$10,000 is invested at an annual rate of 8%. Find the balance in the account after 5 *years* subject to quarterly compounding

Solution

Given:

$P = \$10{,}000$

$r = 8\% = 0.08$

$t = 5$

Quarterly $n = 4$

$$A = 10000\left(1+\frac{0.08}{4}\right)^{4(5)} \qquad A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$\approx \$14{,}859.47$$

Example

Suppose \$5000 is deposited in an account paying 3% interest compounded continuously for 5 *yrs*. Find the total amount on deposit at the end of 5 *yrs*.

Solution

$$A = 5000e^{(.03)(5)} \qquad A = Pe^{rt}$$

$$\approx \$5{,}809.17$$

Exercises *Section* 3.6 – Exponential Growth and Decay

1. Suppose that \$10,000 is invested at interest rate of 5.4% per year, compounded continuously.

a) Find the exponential growth function
b) What will the balance be after, 1 *yr*. 10 *yrs*.?
c) After how long will the investment be double?

2. In 1990, the population of Africa was 643 *million* and by 2000 it had grown to 813 *million*

a) Use the exponential growth function $A(t) = A_0 e^{kt}$, in which t is the number of years after 1990, to find the exponential growth function that models data
b) By which year will Africa's population reach 2000 *million*, or two *billion*?

3. The radioactive element carbon-14 has a half-life of 5750 *yrs*. The percentage of carbon-14 present in the remains of organic matter can be used to determine the age of that organic matter. Archaeologists discovered that the linen wrapping from one of the Dead Sea Scrolls had lost 22.3% of its carbon-14 at the time it was found. How old was the linen wrapping?

4. Suppose that \$2000 is invested at interest rate $\boldsymbol{k}$, compounded continuously, and grows to \$2983.65 in 5 *yrs*.

a) What is the interest rate?
b) Find the exponential growth function
c) What will the balance be after 10 *yrs*?
d) After how long will the \$2000 have doubled?

5. In 2005, the population of China was about 1.306 *billion*, and the exponential growth rate was 0.6% per *year*.

a) Find the exponential growth function
b) Estimate the population in 2008
c) After how long will the population be double what it was in 2005?

6. How long will it take for the money in an account that is compounded continuously at 3% interest to double?

7. If 600 *g* of radioactive substance are present initially and 3 *yrs.* later only 300 *g* remain, how much of the substance will be present after 6 *yrs.*?

8. The population of an endangered species of bird was 4200 in 1990. Thirteen years later, in 2003, the bird population declined to 3000. The population of the birds is decreasing exponentially according to the function $A(t) = 4200e^{kt}$ where A is the bird population t years after 1990. Use this information to find the value of $\boldsymbol{k}$.

9. A city had a population of 21,400 in 2000 and a population of 23,200 in 2005.

a) Find the exponential growth function for the city.
b) Use the growth function to predict the population of the city in 2018.

10. A city had a population of 53,700 in 2002 and a population of 58,100 in 2006.

a) Find the exponential growth function for the city.
b) Use the growth function to predict the population of the city in 2013.

11. The population of Charlotte, North Carolina, is growing exponentially. The population of Charlotte was 395,934 in 1990 and 610,949 in 2005. Find the expoenential growth function that models the population of Charlote and use it to predict the popultion of Charlotte in 2017.

12. The population of Las Vegas, Nevada, is growing exponentially. The population of Las Vegas was 258,295 in 1990 and 545,147 in 2005. Find the expoenential growth function that models the population oc Las Vegas and use it to predict the popultion of Las Vegas in 2017.

13. Find the decay function for the amount of Polonium $\left({}^{210}Po\right)$ that remains in a sample after t days.

14. Estimate the percentage of polonium $\left({}^{210}Po\right)$ that remains in a sample after 2 *years*.

15. Estimate the age of a bone if it now contains 65% of its original amount of carbon-14.

16. Geologists have determined that Crater Lake inn Oregon was formed by a volcanic eruption. Chemical analysis of a wood chip assumed to be from a tree that died during the eruption has shown that it contains approximately 45% of its original carbon-14. Estimate how long ago the volcanic eruption occurred.

17. Lead shielding is used to contain radiation. The percentage of a certain radiation that can penetrate x *millimeters* of lead shielding is given by $I(x) = 100e^{-1.5x}$

a) What percentage of radiation will penetrate a lead shield that is 1 *millimeter* thick?
b) How many *millimeters* of lead shielding are required so that less than 0.02% of the radiations penetrates the shielding?

18. After a race, a runner's pulse rate R, in *beats* per *minute*, decreases according to the function

$$R(t) = 145e^{-0.092t}, \quad 0 \le t \le 15$$

Where t is measured in minutes.

a) Find the runner's pulse rate at the end of the race and 1 *minute* after the end of the race.
b) How long after the end of the race will the runner's pulse rate be 80 *beats* per *minute*?

19. A can of soda at $79°F$ is placed in a refrigerator that maintains a constant temperature of $36°F$. The temperature T of the soda t *minutes* after it is placed in the refrigerator is given by

$$T(t) = 36 + 43e^{-0.058t}$$

a) Find the temperature of the soda 10 *minutes* after it is placed in the refrigerator.

b) When will the temperature of the soda be $45°F$

20. During surgery, a patient's circulatory system requires at least 50 *milligrams* of an anesthetic. The amount of anesthetic present t *hours* after 80 *milligrams* of anesthetic is administered is given by

$$T(t) = 80(0.727)^t$$

a) How much of the anesthetic is present in the patient's circulatory system 30 *minutes* after the anesthetic is administered?

b) How long can the operation last if the patient does not receive additional anesthetic?

21. The following function models the average typing speed S, in *words* per *minute*, for a student who has been typing for t *months*.

$$S(t) = 5 + 29\ln(t+1), \quad 0 \le t \le 9$$

Use S to determine how long it takes the student to achieve an average speed of 65 *words* per *minute*.

22. The exponential function

$$S(x) = 8320(0.73)^x, \quad 10 \le x \le 20$$

models the speed of the dragster during the 10-*second* period immediately following the time when the dragster crosses the finish line. This is the deceleration period.

How long after the start of the race did the dragster attain a speed of 275 *miles* per *hour*?

23. If \$8.000 is invested at an annual interest rate of 5% and compounded annually, find the balance after

a) 4 *years*. *b)* 8 *years*.

24. If \$20.000 is invested at an annual interest rate of 4.5% and compounded annually, find the balance after

a) 3 *years*. *b)* 5 *years*.

25. If \$10.000 is invested at an annual interest rate of 3% for 5 *years*, find the balance if the interest rate is compounded

a) Annually. *c)* Quarterly *e)* Daily (365) *g)* Continuously

b) Semi-annually. *d)* Monthly *f)* Hourly

26. If \$20.000 is invested at an annual interest rate of 2% for 10 *years*, find the balance if the interest rate is compounded

a) Annually. *c)* Quarterly *e)* Daily (365) *g)* Continuously

b) Semi-annually. *d)* Monthly *f)* Hourly

27. Find the accumulated value of an investment of \$10,000 for 5 *years* at an interest rate of 5.5% if the money is

a) Compounded *semiannually*
b) Compounded *quarterly*
c) Compounded *monthly*
d) Compounded *Continuously*

28. Suppose \$1000 is deposited in an account paying 4% interest per year compounded *quarterly*.

a) Find the amount in the account after 10 *years* with no withdraws.
b) How much interest is earned over the 10 *years* period?

29. An investment of 1,000 increased to \$13,464 in 20 *years*. If the interest was compounded continuously, find the interest rate.

30. Becky must pay a lump sum of \$6000 in 5 *yrs*.

a) What amount deposited today at 3.1% compounded annually will grow to \$6000 in 5 *yrs*.?
b) If only \$5000 is available to deposit now, what annual interest rate is necessary for the money to increase to \$6000 in 5 *yrs*.?

31. Find the present value of \$4,000 if the annual interest rate is 3.5% compounded *quarterly* for 6 *years*.

32. How much money will there be in an account at the end of 8 years if \$18,000 is deposited at 3% interest compounded *semi-annually*?

33. The function defined by $P(x) = 908e^{-0.0001348x}$ approximates the atmospheric pressure (in *millibars*) at an altitude of x *meters*. Use P to predict the pressure:

a) At 0 *meters*
b) At 12,000 *meters*

34. How long, to the nearest tenth of a year, will it take \$1000 to grow to \$3600 at 8% annual interest compounded quarterly?

35. The annual revenue R, in *dollars*, of a new company can be closely modeled by the logistic function

$$R(t) = \frac{625,000}{1 + \frac{3}{10}e^{-.045t}}$$

Where the natural number t is the time, in *years*, since the company was founded.

a) According to the model, what will the company's annual revenue for its first year and its second year?
b) According to the model, what will the company's annual revenue approach in the long-term future?

36. The number of cars A sold annually by an automobile dealership can be closely modeled by the logistic function

$$A(t) = \frac{1{,}650}{1 + \frac{12}{5} e^{-.055t}}$$

a) According to the model, what number of cars will the dealership sell during its first year and its second year?
b) According to the model, what will the dealership's car sales approach in the long-term future?

37. The population of wolves in a preserve satisfies a logistic model in which $P_0 = 312$ in 2008, $c = 1{,}600$, and $P(6) = 416$.

a) Determine the logistic model for this population, where t is the number of years after 2008.
b) Use the logistic model from part (*a*) to predict the size of the groundhog population in 2014.

38. The population of walruses on an island satisfies a logistic model in which $P_0 = 800$ in 2006, $c = 5{,}500$, and $P(1) = 900$.

a) Determine the logistic model for this population, where t is the number of years after 2006.
b) Use the logistic model from part (*a*) to predict the year in which the walrus population will first exceed 2000.

39. Newton's Law of Cooling states that is an object at temperature T_0 is placed into an environment at constant temperature A, then the temperature of the object, $T(t)$ (in degrees Fahrenheit), after t *minutes* is given by $T(t) = A + (T_0 - A)e^{-kt}$, where k is a constant that depends on the object.

a) Determine the constant k for a canned soda drink that takes 5 *minutes* to cool from $75°F$ to $65°F$ after being placed in a refrigerator that maintains a constant temperature of $34°F$
b) What will be the temperature of the soda after 30 *minutes*?
c) When will the temperature of the soda drink be $36°F$?

40. According to a software company, the users of its typing tutorial can expect to type $N(t)$ words per minute after t *hours* of practice with the product, according to the function $N(t) = 100\left(1.04 + 0.99^t\right)$

a) How many words per minute can a student expect to type after 2 *hours* of practice?
b) How many words per minute can a student expect to type after 40 *hours* of practice?
c) How many hours of practice will be required before a student can expect to type 60 words per minute?

41. A lawyer has determined that the number of people $P(t)$ in a city of 1.2 *million* people who have been exposed to a news item after *t days* is given by the function

$$P(t) = 1{,}200{,}000\left(1 - e^{-0.03t}\right)$$

a) How many days after a major crime has been reported has 40% of the population heard of the crime?

b) A defense lawyer knows it will be difficult to pick an unbiased jury after 80% of the population has heard of the crime. After how many days will 80% of the population have heard of the crime?