Solution Section 4.5 – Working with Integrals

Exercise

If *f* is an odd function, why is $\int_{-a}^{a} f(x)\,dx = 0$?

Solution

If $f(x)$ is an odd function then it is symmetric about the origin, which the region between $-a$ and a, there is as much area above the axis and under *f* as there is below the axis and above *f*.
Therefore, the net area must be 0.

Exercise

If *f* is an even function, why is $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

Solution

If *f* is an even function then it is symmetric about the *y*-axis, which the region that between $-a$ and 0 has the same net area as the region between 0 and a.

So $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

$$= 2\int_{0}^{a} f(x)\,dx$$

Exercise

Is x^{12} an even or odd function? Is $\sin\left(x^2\right)$ an even or odd function?

Solution

$f(x) = x^{12}$

$$f(-x) = (-x)^{12}$$
$$= x^{12}$$
$$= x^{12} = f(x)$$

Therefore; $f(x)$ is an *even* function.

$g(x) = \sin\left(x^2\right)$

$$g(-x) = \sin\left((-x)^2\right)$$
$$= \sin\left(x^2\right)$$
$$= g(x)$$

Therefore; $g(x)$ is also an even function.

Exercise

Use symmetry to evaluate the following integrals $\displaystyle\int_{-2}^{2} x^9\, dx$

Solution

Because x^9 is an *odd* function, then

$$\boxed{\int_{-2}^{2} x^9 dx = 0}$$

Exercise

Use symmetry to evaluate the following integrals $\displaystyle\int_{-200}^{200} 2x^5\, dx$

Solution

Because $2x^5$ is an *odd* function, then

$$\boxed{\int_{-200}^{200} 2x^5\, dx = 0}$$

Exercise

Use symmetry to evaluate the following integrals $\displaystyle\int_{-\pi/4}^{\pi/4} \cos x\, dx$

Solution

Because $\cos x$ is an even function, then

$$\int_{-\pi/4}^{\pi/4} \cos x\, dx = 2\int_{0}^{\pi/4} \cos x\, dx$$
$$= 2\sin x \Big|_{0}^{\pi/4}$$

$$= 2\left(\frac{\sqrt{2}}{2}\right)$$

$$\underline{= \sqrt{2}}$$

Exercise

Use symmetry to evaluate the following integrals $\displaystyle\int_{-2}^{2}\left(x^9 - 3x^5 + 2x^2 - 10\right)dx$

Solution

$$\int_{-2}^{2}\left(x^9 - 3x^5 + 2x^2 - 10\right)dx = \overbrace{\int_{-2}^{2}\left(x^9 - 3x^5\right)dx}^{Odd} + \overbrace{\int_{-2}^{2}\left(2x^2 - 10\right)dx}^{Even}$$

$$= 0 + 2\int_{0}^{2}\left(2x^2 - 10\right)dx$$

$$= 2\left(\frac{2}{3}x^3 - 10x\right)\Big|_0^2$$

$$= 2\left(\frac{16}{3} - 20\right)$$

$$\underline{= -\frac{88}{3}}$$

Exercise

Use symmetry to evaluate the following integrals $\displaystyle\int_{-\pi/2}^{\pi/2}\left(\cos 2x + \cos x \sin x - 3\sin x^5\right)\,dx$

Solution

$$\int_{-\pi/2}^{\pi/2}\left(\cos 2x + \cos x \sin x - 3\sin x^5\right)\,dx = \overbrace{\int_{-\pi/2}^{\pi/2}\cos 2x\,dx}^{Even} + \overbrace{\int_{-\pi/2}^{\pi/2}\left(\cos x \sin x - 3\sin x^5\right)dx}^{Odd}$$

$$= 2\int_{0}^{\pi/2}\cos 2x\,dx$$

$$= \sin 2x\Big|_0^{\pi/2}$$

$$\underline{= 0}$$

Exercise

Find the average value of the following functions on the given interval. $f(x) = x^3 \quad on \ [-1,\ 1]$

Solution

$$\text{Average value } = \frac{1}{1-(-1)}\int_{-1}^{1} x^3\, dx$$

$$= \frac{1}{2}\frac{1}{4}x^4 \Big|_{-1}^{1}$$

$$\boxed{= 0}$$

Exercise

Find the average value of the following functions on the given interval. $f(x) = \dfrac{1}{x^2+1} \quad on \ [-1,\ 1]$

Solution

$$\text{Average value } = \frac{1}{1-(-1)}\int_{-1}^{1} \frac{1}{x^2+1}\, dx$$

$$= \frac{1}{2}\tan^{-1} x \Big|_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{\pi}{4}+\frac{\pi}{4}\right)$$

$$\boxed{= \frac{\pi}{4}}$$

Exercise

Find the average value of the following functions on the given interval. $f(x) = \dfrac{1}{x} \quad on \ [1,\ e]$

Solution

$$\text{Average value } = \frac{1}{e-1}\int_{1}^{2} \frac{1}{x}\, dx$$

$$= \frac{1}{e-1}\left(\ln|x| \,\Big|_{1}^{e}\right.$$

$$= \frac{1}{e-1}(\ln e - 0)$$

$$\boxed{= \frac{1}{e-1}}$$

Exercise

Find the average value of the following functions on the given interval. $f(x) = e^{2x}$ *on* $[0,\ \ln 2]$

Solution

$$\text{Average value } = \frac{1}{\ln 2}\int_0^{\ln 2} e^{2x}\,dx$$

$$= \frac{1}{2}\,\frac{1}{\ln 2}e^{2x}\,\Big|_0^{\ln 2}$$

$$= \frac{1}{2\ln 2}\left(e^{2\ln 2} - 1\right)$$

$$= \frac{1}{2\ln 2}(4-1)$$

$$= \frac{3}{2\ln 2}$$

Exercise

Suppose that $\int_0^4 f(x)\,dx = 10$ and $\int_0^4 g(x)\,dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_{-4}^4 f(x)\,dx$$

Solution

f is an even function.

$$\int_{-4}^4 f(x)\,dx = 2\int_0^4 f(x)\,dx$$

$$= 2(10)$$

$$= 20$$

Exercise

Suppose that $\int_0^4 f(x)dx = 10$ and $\int_0^4 g(x)dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_{-4}^{4} 3g(x)dx$$

Solution

g is an odd function

$$\int_0^4 g(x)dx = -\int_{-4}^{0} g(x)dx$$

$$\begin{aligned}\int_{-4}^{4} 3g(x)dx &= 3\int_{-4}^{0} g(x)dx + 3\int_0^4 g(x)dx \\ &= 3\int_{-4}^{0} g(x)dx - 3\int_{-4}^{0} g(x)dx \\ &= 0\end{aligned}$$

Exercise

Suppose that $\int_0^4 f(x)dx = 10$ and $\int_0^4 g(x)dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_0^1 8xf\left(4x^2\right)dx$$

Solution

$$\int_0^1 8xf\left(4x^2\right)dx = \int_0^1 f\left(4x^2\right)d\left(4x^2\right) \qquad d\left(4x^2\right) = 8xdx$$

$$\begin{cases} x = 1 & \rightarrow \ 4x^2 = 4 \\ x = 0 & \rightarrow \ 4x^2 = 0 \end{cases}$$

$$\begin{aligned}\int_0^1 8xf\left(4x^2\right)dx &= \int_0^4 f\left(4x^2\right)d\left(4x^2\right) \\ &= 10\end{aligned}$$

Exercise

Suppose that $\int_0^4 f(x)\,dx = 10$ and $\int_0^4 g(x)\,dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_{-2}^{2} 3x f(x)\,dx$$

Solution

f is an even function, that implies $xf(x)$ is an odd function.

$$\int_{-2}^{2} 3x f(x)\,dx = 0$$

Exercise

Suppose that $\int_0^4 f(x)\,dx = 10$ and $\int_0^4 g(x)\,dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_{-4}^{4} \left(4f(x) - 3g(x)\right)dx$$

Solution

f is an even function and g is an odd function.

$$\begin{aligned}\int_{-4}^{4} \left(4f(x) - 3g(x)\right)dx &= 4\int_{-4}^{4} f(x)\,dx - 3\int_{-4}^{4} g(x)\,dx \\ &= 8\int_{0}^{4} f(x)\,dx - 3(0) \\ &= 80\end{aligned}$$

Exercise

Suppose that f is an even function with $\int_0^8 f(x)\,dx = 9$. Evaluate the integral $\int_{-1}^{1} x f\left(x^2\right)dx$

Solution

f is an even function, that implies $xf(x)$ is an odd function.

$$\boxed{\int_{-1}^{1} x\, f\left(x^2\right)dx = 0}$$

Exercise

Suppose that f is an even function with $\int_0^8 f(x)dx = 9$. Evaluate the integral $\int_{-2}^{2} x^2 f\left(x^3\right)dx$

Solution

$$d\left(x^3\right) = 3x^2 dx$$

$$\begin{cases} x = 2 & \to x^3 = 8 \\ x = -2 & \to x^3 = -8 \end{cases}$$

$$\begin{aligned}
\int_{-2}^{2} x^2 f\left(x^3\right)dx &= \frac{1}{3}\int_{-8}^{8} f\left(x^3\right)d\left(x^3\right) \\
&= \frac{1}{3}\int_{-8}^{0} f\left(x^3\right)d\left(x^3\right) + \frac{1}{3}\int_{0}^{8} f\left(x^3\right)d\left(x^3\right) && f \text{ is an } \textit{even} \text{ function} \\
&= \frac{2}{3}\int_{0}^{8} f\left(x^3\right)d\left(x^3\right) \\
&= \frac{2}{3}\cdot 9 \\
&= \boxed{6}
\end{aligned}$$

Exercise

Suppose that p is a nonzero real number and f is an odd integrable function with $\int_0^1 f(x)dx = \pi$.

Evaluate the integral $\int_0^{\frac{\pi}{2p}} (\cos px)\, f(\sin px)dx$

Solution

$$\begin{cases} x = \frac{\pi}{2p} & \to \sin px = 1 \\ x = 0 & \to \sin px = 0 \end{cases}$$

$$\int_0^{\frac{\pi}{2p}} (\cos px)\, f(\sin px)dx = \frac{1}{p}\int_0^1 f(\sin px)\, d(\sin px) \qquad d(\sin px) = p\cos px\, dx$$

$$= \frac{\pi}{p}$$

Exercise

Suppose that p is a nonzero real number and f is an odd integrable function with $\int_0^1 f(x)\,dx = \pi$.

Evaluate the integral $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)\, f(\sin x)\,dx$

Solution

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos px)\, f(\sin px)\,dx = \frac{1}{p}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\sin px)\; d(\sin px) \qquad f \text{ is an odd function}$$

$$= 0$$

Exercise

The Gateway Arch in St. Louis is 630 *feet* high and has a 630-*ft* base. Its shape can be modeled by the parabola

$$y = 630\left(1 - \left(\frac{x}{315}\right)^2\right)$$

Find the average height of the arch above the ground.

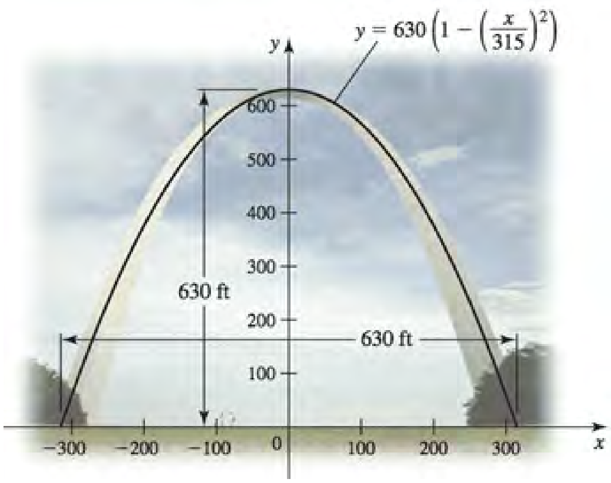
Solution

$$\text{Average height } = \frac{1}{630}\int_{-315}^{315} 630\left(1 - \frac{1}{315^2}x^2\right)dx$$

$$= x - \frac{1}{315^2}\frac{x^3}{3}\Bigg|_{-315}^{315}$$

$$= 315 - 105 + 315 - 105$$

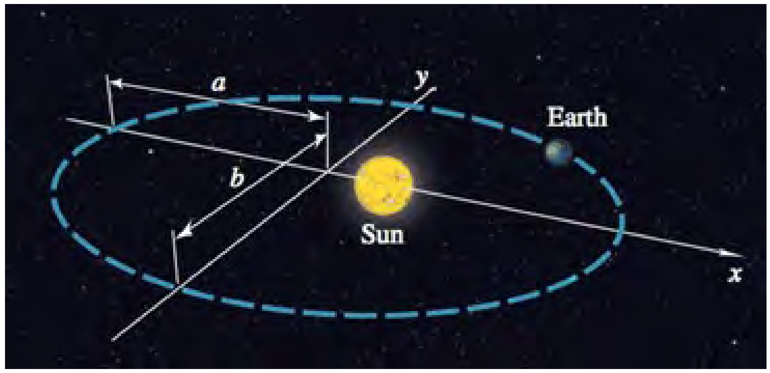
$$= 420\ \ ft$$



Exercise

The planets orbit the Sun in elliptical orbits with the Sun at one focus. The equation of an ellipse whose dimensions are 2 a in the x-direction and 2 b in the y-direction is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



a) Let d^2 denote the square of the distance from a planet to the center of the ellipse at (0, 0). Integrate over the interval $[-a,\ a]$ to show that the average value of d^2 is $\frac{a^2+2b^2}{3}$

b) Show that in the case of a circle ($a = b = R$), the average value in part (a) is R^2.

c) Assuming $0 < b <$ a, the coordinates of the Sun are $\left(\sqrt{a^2-b^2},\ 0\right)$. Let D^2 denote the square of the distance from the planet to the Sun. Integrate over the interval $[-a,\ a]$ to show that the average value of D^2 is $\frac{4a^2-b^2}{3}$.

Solution

a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2-\frac{b^2}{a^2}x^2$$

$$d^2=x^2+y^2=x^2+b^2-\frac{b^2}{a^2}x^2$$

$$=b^2+\left(1-\frac{b^2}{a^2}\right)x^2$$

The average value of $d^2=\frac{1}{2a}\int_{-a}^{a}\left(b^2+\left(1-\frac{b^2}{a^2}\right)x^2\right)dx$

$$= \frac{1}{2a}\left(b^2x + \frac{1}{3}\left(1 - \frac{b^2}{a^2}\right)x^3\right)\Bigg|_{-a}^{a}$$

$$= \frac{1}{2a}\left(ab^2 + \frac{1}{3}\left(1 - \frac{b^2}{a^2}\right)a^3 + ab^2 + \frac{1}{3}\left(1 - \frac{b^2}{a^2}\right)a^3\right)$$

$$= \frac{1}{2a}\left(2ab^2 + \frac{2}{3}a^3 - \frac{2}{3}b^2a\right)$$

$$\boxed{= \frac{2}{3}b^2 + \frac{a^2}{3}}$$

b) If $a = b = R$

The average value of $d^2 = \frac{2R^2}{3} + \frac{R^2}{3}$

$$\boxed{= R^2}$$

c) $D^2 = \left(x - \sqrt{a^2 - b^2}\right)^2 + y^2$

$$= x^2 - 2x\sqrt{a^2 - b^2} + a^2 - b^2 + b^2 - \frac{b^2}{a^2}x^2$$

$$= \left(1 - \frac{b^2}{a^2}\right)x^2 - 2x\sqrt{a^2 - b^2} + a^2$$

The average value of $D^2 = \frac{1}{2a}\int_{-a}^{a}\left(\left(1 - \frac{b^2}{a^2}\right)x^2 - 2x\sqrt{a^2 - b^2} + a^2\right)dx$

$$= \frac{1}{2a}\left(\frac{1}{3}\left(1 - \frac{b^2}{a^2}\right)x^3 - x^2\sqrt{a^2 - b^2} + a^2x\right)\Bigg|_{-a}^{a}$$

$$= \frac{1}{2a}\left(\frac{1}{3}\left(1 - \frac{b^2}{a^2}\right)a^3 - a^2\sqrt{a^2 - b^2} + a^3 + \frac{1}{3}\left(1 - \frac{b^2}{a^2}\right)a^3 + a^2\sqrt{a^2 - b^2} + a^3\right)$$

$$= \frac{1}{3}\left(1 - \frac{b^2}{a^2}\right)a^2 + a^2$$

$$= \frac{1}{3}a^2 - \frac{1}{3}b^2 + a^2$$

$$\boxed{= \frac{4}{3}a^2 - \frac{1}{3}b^2}$$

Exercise

A particle moves along a line with a velocity given by $v(t)=5\sin \pi t$ starting with an initial position $s(0)=0$. Find the displacement of the particle between $t=0$ and $t=2$, which is given by $s(t)=\int_0^2 v(t)\,dt$. Find the distance traveled by the particle during this interval, which is $\int_0^2 |v(t)|\,dt$.

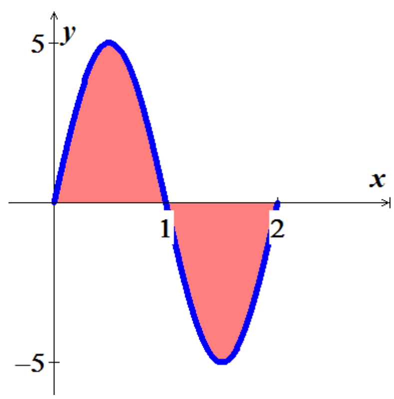
Solution

$$s(t)=\int_0^2 5\sin \pi t\ dt$$

$$=-\frac{5}{\pi}\cos \pi t\ \Big|_0^2$$

$$=-\frac{5}{\pi}(1-1)$$

$$\boxed{=0}$$



$$s(t)=\int_0^2 |5\sin \pi t|\ dt$$

$$=5\int_0^1 \sin \pi t\ dt\ +\ 5\int_1^2 (-\sin \pi t)\ dt$$

$$=-\frac{5}{\pi}\cos \pi t\ \Big|_0^1\ +\ \frac{5}{\pi}\cos \pi t\ \Big|_1^2$$

$$=-\frac{5}{\pi}(-1-1)+\frac{5}{\pi}(1+1)$$

$$=\frac{10}{\pi}+\frac{10}{\pi}$$

$$\boxed{=\frac{20}{\pi}}$$

Exercise

A baseball is launched into the outfield on a parabolic trajectory given by $y=0.01x(200-x)$. Find the average height of the baseball over the horizontal extent of its flight.

Solution

$$y=0.01x(200-x)=0\quad\to\quad x=0,\ 200$$

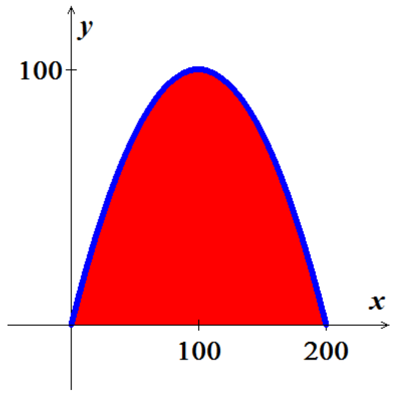
$$Avg=\frac{1}{200}\int_0^{200}\left(2x-0.01x^2\right)\ dx$$

$$= \frac{1}{200}\left(x^2 - \frac{1}{300}x^3\right)\Big|_0^{200}$$

$$= \frac{1}{200}\left(4\times 10^4 - \frac{1}{300}8\times 10^6\right)$$

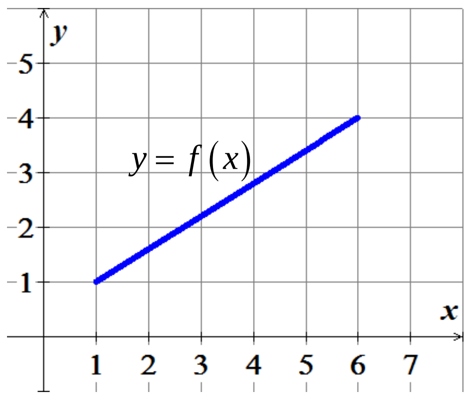
$$= \frac{4\times 10^4}{200}\left(1 - \frac{2}{3}\right)$$

$$= \frac{200}{3}$$



Exercise

Find the average value of f shown in the figure on the interval $[1,\ 6]$ and then find the point(s) c in $(1,\ 6)$ guaranteed to exist by the Mean Value Theorem for Integrals



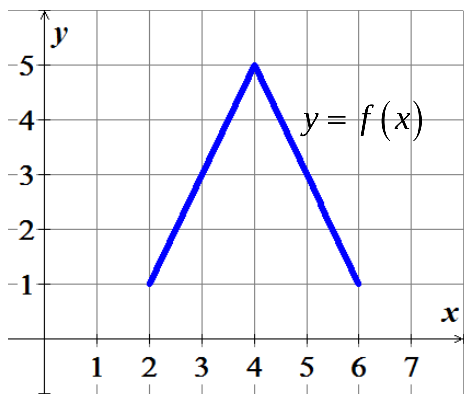
Solution

Since it is a straight line, then the average value is 2.5

The average value occurs at the midpoint of the interval which is $(3.5,\ 2.5)$

Exercise

Find the average value of f shown in the figure on the interval $[2,\ 6]$ and then find the point(s) c in $(2,\ 6)$ guaranteed to exist by the Mean Value Theorem for Integrals



Solution

Over interval $[2,\ 4]$; it is a straight line, then the average value is 3

Over interval $[4,\ 6]$; it is a straight line, then the average value is 3 .

Therefore; the overage is 3 over $[2,\ 6]$