# Solution *Section* 4.2 – Series Solutions near Ordinary Points

## *Exercise*

Find a power series solution. $y' = 3y$

**Solution**

The equation $y' = 3y$ is separable with solution

$$\frac{dy}{dx} = 3y \;\Rightarrow\; \frac{dy}{y} = 3dx \quad \boxed{y = Ce^{3x}}$$

$$\ln(y) = 3x + C$$

The solution form is: $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y' - 3y = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 3\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[ (n+1) a_{n+1} - 3a_n \right] x^n = 0$$

$$(n+1) a_{n+1} - 3a_n = 0 \quad \Rightarrow \quad a_{n+1} = \frac{3a_n}{n+1}; \quad n \ge 0$$

With $y(0) = 3a_0$

$$a_1 = 3a_0$$

$$a_2 = \frac{3}{2} a_1 = \frac{3 \cdot 3}{2} a_0$$

$$a_3 = \frac{3}{3} a_2 = \frac{3 \cdot 3 \cdot 3}{2 \cdot 3} a_0$$

$$a_4 = \frac{3}{4} a_3 = \frac{3 \cdot 3 \cdot 3 \cdot 3}{2 \cdot 3 \cdot 4} a_0$$

$$\boxed{a_n = \frac{3^n}{n!} a_0}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{n!} a_0 x^n$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$

$$= a_0 e^{3x} \quad \checkmark$$

### *Exercise*

Find a power series solution. $(1+x)y' - y = 0$

***Solution***

$$(1+x)\frac{dy}{dx} = y$$

$$\frac{dy}{y} = \frac{dx}{1+x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\ln(y) = \ln(x+1) + C$$

$$y = C(x+1)$$

With $y(0) = 3a_0$

$$y(0) = C(0+1)$$

$$a_0 = C$$

$$y = a_0(x+1)$$

The solution form is: $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$(1+x)y' - y = 0$$

$$(1+x)\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} na_n x^n = 0 + \sum_{n=1}^{\infty} na_n x^n = \sum_{n=0}^{\infty} na_n x^n$$

$$\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + \sum_{n=0}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left( (n+1)a_{n+1} x^n + na_n x^n - a_n x^n \right) = 0$$

$$\left[ (n+1)a_{n+1} + na_n - a_n \right] x^n = 0$$

$$(n+1)a_{n+1} + (n-1)a_n = 0$$

$$a_{n+1} = \frac{1-n}{n+1} a_n; \quad n \ge 0$$

$$a_1 = a_0$$

$$a_2 = 0 a_1 = 0$$

$$a_3 = \frac{-1}{3} a_2 = 0$$

$$a_n = 0 \quad for \quad n \ge 2$$

$$y(x) = a_0 + a_1 x$$

$$= a_0 + a_0 x$$

$$\boxed{= a_0 (1+x)} \quad \checkmark$$

### *Exercise*

Find a power series solution. $(2-x)y' + 2y = 0$

***Solution***

$$(2-x)\frac{dy}{dx} + 2y = 0$$

$$(2-x)\frac{dy}{dx} = -2y$$

$$\frac{dy}{y} = -\frac{2dx}{2-x}$$

$$\int \frac{dy}{y} = \int \frac{2d(2-x)}{2-x}$$

$$\ln y = 2\ln(2-x) + C_1$$

$$\ln y = \ln(2-x)^2 + C_1$$

$$\ln y = \ln C(2-x)^2$$

$$y = C(2-x)^2$$

$$y(0) = C(2-0)^2$$

$$a_0 = 4C$$

$$\boxed{C = \frac{a_0}{4}}$$

$$\boxed{y = \frac{1}{4}a_0(2-x)^2}$$

The solution form is: $y(x) = \sum\limits_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$(2-x)y' + 2y = 0$$

$$(2-x)\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$2\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^n = 0 + \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n$$

$$2\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[2(n+1)a_{n+1}-na_n+2a_n\right]x^n=0$$

$$2(n+1)a_{n+1}-(n-2)a_n=0$$

$$2(n+1)a_{n+1}=(n-2)a_n$$

$$a_{n+1}=\frac{n-2}{2(n+1)}a_n, \qquad n\geq 0$$

$$a_1=\frac{-2}{2}a_0=-a_0$$

$$a_2=\frac{-1}{4}a_1=\frac{1}{4}a_0$$

$$a_3=\frac{0}{6}a_0=0$$

$$\vdots$$

$$a_n=0$$

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$=a_0-a_0x+\frac{1}{4}a_0x^2$$

$$=a_0\left(1-x+\frac{1}{4}x^2\right)$$

$$=\frac{1}{4}a_0\left(4-4x+x^2\right)$$

$$\boxed{=\frac{1}{4}a_0(2-x)^2} \quad \checkmark$$

### *Exercise*

Find a power series solution. $y' = x^2 y$

***Solution***

$$\frac{dy}{dx} = x^2 y$$

$$\frac{dy}{y} = x^2 dx$$

$$\int \frac{dy}{y} = \int x^2 dx$$

$$\ln y = \frac{1}{3}x^3 + C_1$$

$$y = e^{\frac{1}{3}x^3 + C_1}$$

$$y = Ce^{\frac{1}{3}x^3}$$

$$y(0) = C(1)$$

$$\boxed{a_0 = C}$$

$$\boxed{y = a_0 e^{x^3/3}}$$

The solution form is: $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y' - x^2 y = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{k=-2}^{\infty} (k+3) a_{k+3} x^{k+2}$$

$$= \sum_{n=-2}^{\infty} (n+3) a_{n+3} x^{n+2}$$

$$\sum_{n=-2}^{\infty}(n+3)a_{n+3}x^{n+2}-\sum_{n=0}^{\infty}a_n x^{n+2}=0$$

$$a_1+2a_2x+\sum_{n=-2}^{\infty}(n+3)a_{n+3}x^{n+2}-\sum_{n=0}^{\infty}a_n x^{n+2}=0$$

$$a_1+2a_2x+\sum_{n=-2}^{\infty}\left[(n+3)a_{n+3}-a_n\right]x^{n+2}=0$$

If we set $a_1=a_2=0$, then

$$(n+3)a_{n+3}-a_n=0 \Rightarrow \quad a_{n+3}=\frac{a_n}{n+3}, \quad n\ge 0$$

$$a_3=\frac{1}{3}a_0$$

$$a_4=\frac{1}{4}a_1=0$$

$$a_5=\frac{1}{5}a_2=0$$

$$a_6=\frac{1}{6}a_3=\frac{1}{3\cdot 6}a_0$$

$$a_7=\frac{1}{7}a_4=0$$

$$a_9=\frac{1}{9}a_6=\frac{1}{3\cdot 6\cdot 9}a_0=\frac{1}{3^3(1\cdot 2\cdot 3)}a_0$$

$$a_{12}=\frac{1}{12}a_9=\frac{1}{3\cdot 6\cdot 9\cdot 12}a_0=\frac{1}{3^4(1\cdot 2\cdot 3\cdot 4)}a_0$$

$$a_{3n}=\frac{1}{3^n\cdot n!}a_0$$

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$=\sum_{n=0}^{\infty}a_{3n}x^{3n}$$

$$=\sum_{n=0}^{\infty}\frac{1}{3^n\cdot n!}a_0x^{3n}$$

$$=a_0\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{x^3}{3}\right)^n$$

$$\boxed{=a_0e^{x^3/3}} \quad \checkmark$$

### *Exercise*

Find a power series solution. $(x-4)y' + y = 0$

***Solution***

$$(x-4)\frac{dy}{dx} = -y$$

$$\frac{dy}{y} = -\frac{dx}{x-4}$$

$$\ln y = -\ln(x-4) + C_1$$

$$\ln y = \ln\frac{C}{x-4}$$

$$y = \frac{C}{x-4}$$

$$y(0) = \frac{C}{0-4}$$

$$a_0 = \frac{C}{-4} \quad \Rightarrow \quad C = -4a_0$$

$$\boxed{y = -\frac{4a_0}{x-4}}$$

The solution form is: $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$$

$$(x-4)y' + y = 0$$

$$(x-4)\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} na_n x^n - 4\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0 \qquad \sum_{n=1}^{\infty} na_n x^n = \sum_{n=0}^{\infty} na_n x^n$$

$$\sum_{n=0}^{\infty} na_n x^n - 4\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)a_n x^n - 4\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n = 0 \quad a_1 = \frac{1}{4}a_0$$

$$\sum_{n=0}^{\infty} \left[(n+1)a_n - 4(n+1)a_{n+1}\right]x^n = 0$$

$$(n+1)a_n - 4(n+1)a_{n+1} = 0$$

$$4(n+1)a_{n+1} = (n+1)a_n$$

$$a_{n+1} = \frac{1}{4}a_n$$

$$a_2 = \frac{1}{4}a_1 = \frac{1}{4^2}a_0$$

$$a_3 = \frac{1}{4}a_2 = \frac{1}{4^3}a_0$$

$$a_n = \frac{1}{4^n}a_0$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{4^n} a_0 x^n$$

$$= a_0 \sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$$

$$= a_0 \left(\frac{1}{1-\frac{x}{4}}\right)$$

$$= a_0 \left(\frac{4}{4-x}\right)$$

$$\boxed{= \frac{-4a_0}{x-4}} \quad \surd$$

### *Exercise*

Find a power series solution. $y'' = 9y$

***Solution***

The equation $y'' = 9y$ has a characteristic equation $\lambda^2 - 9 = 0 \Rightarrow \boxed{\lambda = \pm 3}$

$\therefore$ The general solution: $y(x) = C_1 e^{3x} + C_2 e^{-3x}$

With $y(0) = a_0$ *and* $y'(0) = a_1$

$$y(0) = C_1 e^{3(0)} + C_2 e^{-3(0)} \rightarrow C_1 + C_2 = a_0$$

$$y'(x) = 3C_1 e^{3x} - 3C_2 e^{-3x}$$

$$y(0) = 3C_1 e^{3(0)} - 3C_2 e^{-3(0)} \rightarrow 3C_1 - 3C_2 = a_1$$

$$\begin{cases} C_1 + C_2 = a_0 \\ 3C_1 - 3C_2 = a_1 \end{cases} \rightarrow \begin{cases} 3C_1 + 3C_2 = 3a_0 \\ 3C_1 - 3C_2 = a_1 \end{cases}$$

$$6C_1 = 3a_0 + a_1 \rightarrow \boxed{C_1 = \frac{3a_0 + a_1}{6}}$$

$$C_2 = a_0 - C_1 \rightarrow \boxed{C_2 = a_0 - \frac{3a_0 + a_1}{6} = \frac{3a_0 - a_1}{6}}$$

$$y(x) = \frac{3a_0 + a_1}{6} e^{3x} + \frac{3a_0 - a_1}{6} e^{-3x}$$

The solution form is: $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$y'' - 9y = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 9 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 9 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[ (n+2)(n+1) a_{n+2} - 9a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} - 9a_n = 0$$

$$a_{n+2} = \frac{9}{(n+2)(n+1)} a_n, \quad n \ge 0$$

$$a_2 = \frac{9}{(2)(1)} a_0 = \frac{9}{2} a_0 \qquad a_3 = \frac{9}{(3)(2)} a_1 = \frac{9}{2\cdot 3} a_1$$

$$a_4 = \frac{3^2}{(4)(3)} a_2 = \frac{9\cdot 9}{2\cdot 3\cdot 4} a_0 = \frac{3^4}{2\cdot 3\cdot 4} a_0 \qquad a_5 = \frac{9}{(5)(4)} a_3 = \frac{3^4}{2\cdot 3\cdot 4\cdot 5} a_1$$

$$a_6 = \frac{3^2}{(6)(5)} a_4 = \frac{3^6}{6!} a_0 \qquad a_7 = \frac{9}{(7)(6)} a_5 = \frac{3^6}{7!} a_1$$

$$a_{2n} = \frac{3^{2n}}{(2n)!} a_0 \qquad a_{2n+1} = \frac{3^{2n}}{(2n+1)!} a_1$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 \left[ 1 + \frac{3^2}{2!} x^2 + \frac{3^4}{4!} x^4 + \frac{3^6}{6!} x^6 + \cdots \right] + a_1 \left[ x + \frac{3^2}{3!} x^3 + \frac{3^4}{5!} x^5 + \frac{3^6}{7!} x^7 + \cdots \right]$$

$$y(x) = \frac{3a_0 + a_1}{6} e^{3x} + \frac{3a_0 - a_1}{6} e^{-3x}$$

$$= \frac{3a_0 + a_1}{6} \left[ 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \cdots \right] + \frac{3a_0 - a_1}{6} \left[ 1 - 3x + \frac{(-3x)^2}{2!} + \frac{(-3x)^3}{3!} + \cdots \right]$$

$$= \frac{3a_0}{6} \left[ 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \cdots \right] + \frac{a_1}{6} \left[ 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \cdots \right]$$

$$+ \frac{3a_0}{6} \left[ 1 - 3x + \frac{(3x)^2}{2!} - \frac{(3x)^3}{3!} + \cdots \right] - \frac{a_1}{6} \left[ 1 - 3x + \frac{(3x)^2}{2!} - \frac{(3x)^3}{3!} + \cdots \right]$$

$$= \frac{1}{2} a_0 \left[ 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \cdots + 1 - 3x + \frac{(3x)^2}{2!} - \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \cdots \right]$$

$$+ \frac{a_1}{6} \left[ 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \cdots - 1 + 3x - \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} - \frac{(3x)^4}{4!} + \cdots \right]$$

$$= \frac{1}{2} a_0 \left[ 2 + 2\frac{(3x)^2}{2!} + 2\frac{(3x)^4}{4!} + \cdots \right] + \frac{a_1}{6} \left[ 6x + 2\frac{(3x)^3}{3!} + 2\frac{(3x)^5}{5!} + \cdots \right]$$

$$= a_0 \left[ 1 + \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} + \cdots \right] + a_1 \left[ x + \frac{3^2 x^3}{3!} + \frac{3^4 x^5}{5!} + \cdots \right]$$

Which are identical.

### ***Exercise***

Find a power series solution. $y'' + y = 0$

***Solution***

The equation $y'' + y = 0$ has a characteristic equation $\lambda^2 + 1 = 0 \quad \Rightarrow \quad \boxed{\lambda = \pm i}$

$\therefore$ The general solution: $y(x) = C_1 \sin x + C_2 \cos x$

With $y(0) = a_0 \quad$ *and* $\quad y'(0) = a_1$

$$y(0) = C_1 \sin(0) + C_2 \cos(0) \quad \rightarrow \quad C_2 = a_0$$

$$y'(x) = C_1 \cos x - C_2 \sin x$$

$$y(0) = C_1 \cos(0) - C_2 \sin(0) \quad \rightarrow \quad C_1 = a_1$$

$$y(x) = a_1 \sin x + a_0 \cos x$$

$$= a_0 \cos x + a_1 \sin x$$

The solution form is: $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad\qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$y'' + y = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[ (n+2)(n+1) a_{n+2} + a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + a_n = 0$$

$$a_{n+2} = \frac{-1}{(n+2)(n+1)} a_n, \quad n \ge 0$$

$$a_2 = \frac{1}{(2)(1)} a_0 = -\frac{1}{2} a_0 \qquad\qquad a_3 = \frac{1}{(3)(2)} a_1 = -\frac{1}{2 \cdot 3} a_1$$

$$a_4 = -\frac{1}{(4)(3)} a_2 = \frac{1}{2\cdot 3\cdot 4} a_0 \qquad a_5 = -\frac{1}{(5)(4)} a_3 = \frac{1}{2\cdot 3\cdot 4\cdot 5} a_1$$

$$a_6 = -\frac{1}{(6)(5)} a_4 = -\frac{1}{6!} a_0 \qquad a_7 = -\frac{1}{(7)(6)} a_5 = -\frac{1}{7!} a_1$$

$$a_{2n} = \frac{(-1)^n}{(2n)!} a_0 \qquad a_{2n+1} = \frac{(-1)^n}{(2n+1)!} a_1$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 \left[ 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \cdots + \frac{(-1)^n}{(2n)!} x^{2n} + \cdots \right]$$

$$+ a_1 \left[ x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \cdots \right]$$

$$\boxed{= a_0 \cos x + a_1 \sin x} \quad \surd$$

### *Exercise*

Find the series solution to the initial value problem $y'' + (x-1)y' + y = 0 \qquad y(1) = 2 \qquad y'(1) = 0$

***Solution***

The initial value is $x = 1$ $\quad y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n$$

$$y'' + (x-1)y' + y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + (x-1)\sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + \sum_{n=1}^{\infty} n a_n (x-1)^{n} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} \left[ (n+2)(n+1) a_{n+2} + n a_n + a_n \right] x^{n-1} = 0$$

$$(n+2)(n+1) a_{n+2} + (n+1) a_n = 0$$

$$(n+2) a_{n+2} + a_n = 0$$

$$a_{n+2} = -\frac{1}{n+2} a_n$$

$$a_0 = y(1) = 2 \qquad\qquad a_1 = y'(1) = 0$$

$$a_2 = -\frac{1}{2} a_0 = -1 \qquad\qquad a_3 = -\frac{1}{3} a_1 = 0$$

$$a_4 = -\frac{1}{4} a_2 = \frac{1}{2 \bullet 4} a_0 = \frac{1}{4} \qquad\qquad a_5 = -\frac{1}{5} a_3 = 0$$

$$a_6 = -\frac{1}{6} a_4 = -\frac{1}{24} \qquad\qquad a_7 = -\frac{1}{7} a_5 = 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n = a_0 + a_1 (x-1) + a_2 (x-1) + a_3 (x-1)^3 + a_4 (x-1)^4 + \cdots$$

$$= 2 - (x-1)^2 + \frac{1}{4}(x-1)^4 - \frac{1}{24}(x-1)^6 + \cdots \qquad \boxed{= \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{2n}}{n!\, 2^n}}$$

### *Exercise*

Find the series solution to the initial value problem $y'' + xy' + y = 0 \qquad y(0) = 1 \qquad y'(0) = 0$

***Solution***

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + xy' + y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[ (n+2)(n+1) a_{n+2} + n a_n + a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + (n+1) a_n = 0$$

$$(n+2)(n+1) a_{n+2} = -(n+1) a_n$$

$$a_{n+2} = -\frac{1}{n+2} a_n$$

$$a_0 = y(0) = 1 \qquad\qquad a_1 = y'(0) = 0$$

$$a_2 = -\frac{1}{2} a_0 = -\frac{1}{2} \qquad\qquad a_3 = -\frac{1}{3} a_1 = 0$$

$$a_4 = -\frac{1}{4} a_2 = \frac{1}{2 \cdot 4} = \frac{1}{2^2 \cdot 1 \cdot 2} \qquad\qquad a_5 = -\frac{1}{5} a_3 = 0$$

$$a_6 = -\frac{1}{6} a_4 = -\frac{1}{2^3 \cdot 1 \cdot 2 \cdot 3} \qquad\qquad a_7 = -\frac{1}{7} a_7 = 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= 1 - \frac{1}{2} x^2 + \frac{1}{2^2 2!} x^4 - \frac{1}{2^3 3!} x^5 + \cdots$$

### *Exercise*

Find the series solution to the initial value problem $y'' - xy' - y = 0 \quad y(0) = 2 \quad y'(0) = 1$

**Solution**

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' - y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[ (n+2)(n+1) a_{n+2} - n a_n - a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} - (n+1) a_n = 0$$

$$(n+2)(n+1) a_{n+2} = -(n+1) a_n$$

$$a_{n+2} = \frac{1}{n+2} a_n$$

$$a_0 = y(0) = 2 \qquad a_1 = y'(0) = 1$$

$$a_2 = \frac{1}{2} a_0 = 1 \qquad a_3 = \frac{1}{3} a_1 = \frac{1}{3}$$

$$a_4 = \frac{1}{4} a_2 = \frac{1}{4} \qquad a_5 = \frac{1}{5} a_3 = \frac{1}{3 \cdot 5}$$

$$a_6 = \frac{1}{6} a_4 = \frac{1}{4} \frac{1}{6} = \frac{1}{24} \qquad a_7 = \frac{1}{7} a_7 = \frac{1}{3 \cdot 5 \cdot 7}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$y(x) \; \underline{= 2 + x + x^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4 + \frac{1}{15} x^5 + \frac{1}{24} x^6 + \cdots}$$

### Exercise

Find the series solution to the initial value problem $\left(2+x^2\right)y'' - xy' + 4y = 0 \qquad y(0) = -1 \qquad y'(0) = 3$

***Solution***

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\left(2+x^2\right)y'' - xy' + 4y = 0$$

$$\left(2+x^2\right)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} na_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} na_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$2(n+2)(n+1)a_{n+2} + n(n-1)a_n - na_n + 4a_n = 0$$

$$2(n+2)(n+1)a_{n+2} + \left(n^2 - 2n + 4\right)a_n = 0$$

$$a_{n+2} = -\frac{n^2-2n+4}{2(n+2)(n+1)}a_n$$

$a_0 = y(0) = -1 \qquad\qquad a_1 = y'(0) = 3$

$n = 0 \rightarrow \quad a_2 = -\frac{4}{4}a_0 = 1 \qquad\qquad n = 1 \rightarrow \quad a_3 = -\frac{3}{12}a_1 = -\frac{1}{4}(3) = -\frac{3}{4}$

$n = 2 \rightarrow \quad a_4 = -\frac{4}{24}a_2 = -\frac{1}{6} \qquad\qquad n = 3 \rightarrow \quad a_5 = -\frac{7}{40}a_3 = -\frac{7}{40}\left(-\frac{3}{4}\right) = \frac{21}{160}$

$$y(x) = \boxed{-1 + 3x + x^2 - \frac{3}{4}x^3 - \frac{1}{6}x^4 + \frac{21}{160}x^5 + \cdots}$$

# Solution    Section 4.3 – Legendre's Equation

### *Exercise*

Establish the recursion formula using the following two steps

*a)* Differentiate both sides of equation

$$g(t,\ x)=\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n \text{ with respect to } t \text{ to show that}$$

$$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\left(1-2xt+t^2\right)\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

*b)* Equate the coefficients of $t^n$ in this equation to show that

$$P_1(x)=xP_0(x) \quad \textit{and} \quad (n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x) \quad \textit{for } n\ge 1$$

***Solution***

***a)*** Let: $\left(1-2xt+t^2\right)^{-1/2}=\displaystyle\sum_{n=0}^{\infty}P_n(x)t^n$

Differentiate both sides with respect to *t*: $\left(\left(1-2xt+t^2\right)^{-1/2}\right)'=\left(\displaystyle\sum_{n=0}^{\infty}P_n(x)t^n\right)'$

$$-\tfrac{1}{2}(-2x+2t)\left(1-2xt+t^2\right)^{-3/2}=\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

$$(x-t)\left(1-2xt+t^2\right)^{-1/2}=\left(1-2xt+t^2\right)\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$ ***Multiply both sides by:*** $1-2xt+t^2$

$$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\left(1-2xt+t^2\right)\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

***b)*** $$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\sum_{n=0}^{\infty}xP_n(x)t^n-\underbrace{\sum_{n=0}^{\infty}P_n(x)t^{n+1}}_{n=n+1}$$

$$=\sum_{n=0}^{\infty}xP_n(x)t^n-\sum_{n=1}^{\infty}P_{n-1}(x)t^n$$

$$\left(1-2xt+t^2\right)\sum_{n=1}^{\infty} nP_n(x)t^{n-1} = \sum_{n=1}^{\infty} nP_n(x)t^{n-1} - \sum_{n=1}^{\infty} 2xnP_n(x)t^n + \sum_{n=1}^{\infty} nP_n(x)t^{n+1}$$

$$= \sum_{n=0}^{\infty}(n+1)P_{n+1}(x)t^n - \sum_{n=1}^{\infty} 2nxP_n(x)t^n + \sum_{n=2}^{\infty}(n-1)P_{n-1}(x)t^n$$

Thus,

$$\sum_{n=0}^{\infty} xP_n(x)t^n - \sum_{n=1}^{\infty} P_{n-1}(x)t^n = \sum_{n=0}^{\infty}(n+1)P_{n+1}(x)t^n - \sum_{n=1}^{\infty} 2nxP_n(x)t^n + \sum_{n=2}^{\infty}(n-1)P_{n-1}(x)t^n$$

Therefore;

$$0 = \left[xP_0(x) - P_1(x)\right]t^0 + \left[xP_1(x) - P_0(x) - 2P_2(x) + 2xP_1(x)\right]t^1$$
$$+\sum_{n=0}^{\infty}\left[xP_n(x) - P_{n-1}(x) - (n+1)P_{n+1}(x) + 2nxP_n(x) - (n-1)P_{n-1}(x)\right]t^n$$

$$0 = \left[xP_0(x) - P_1(x)\right]t^0 + \left[3xP_1(x) - P_0(x) - 2P_2(x)\right]t^1$$
$$+\sum_{n=0}^{\infty}\left[(2n+1)xP_n(x) - nP_{n-1}(x) - (n+1)P_{n+1}(x)\right]t^n$$

That implies:

$$xP_0(x) - P_1(x) = 0 \quad \Rightarrow \quad P_1(x) = xP_0(x)$$

$$3xP_1(x) - P_0(x) - 2P_2(x) = 0 \quad \Rightarrow \quad 2P_2(x) = P_0(x) - 3xP_1(x)$$

$$(2n+1)xP_n(x) - nP_{n-1}(x) - (n+1)P_{n+1}(x) = 0$$
$$\Rightarrow \quad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

If $n$ = 1 then: $2P_2(x) = 3xP_1(x) - P_0(x)$ √

### Exercise

Show that $P_{2n+1}(0)=0$ *and* $P_{2n}(0)=(-1)^n \dfrac{2n!}{2^{2n}(n!)^2}$

**Solution**

Given the formula:

$$nP_n(x)=(2n-1)xP_{n-1}(x)-(n-1)P_{n-2}(x) \qquad \textit{for } n\ge 2$$

By letting $x=0$, then the formula can be:

$$nP_n(0)=-(n-1)P_{n-2}(0)$$

Replacing $n$ with $2n$, then

$$2nP_{2n}(0)=-(2n-1)P_{2n-2}(0)$$

$$P_{2n}(0)=\frac{1-2n}{2n}P_{2n-2}(0)$$

$$P_2(0)=\frac{1-2}{2}P_0(0)=-\frac{1}{2}P_0(0)$$

$$P_4(0)=\frac{1-4}{4}P_2(0)=\frac{1-2}{2}\cdot\frac{1-4}{4}P_0(0)=\frac{1\cdot 3}{2^2\cdot 1\cdot 2}P_0(0)$$

$$P_6(0)=\frac{1-6}{6}P_4(0)=\frac{1-2}{2}\cdot\frac{1-4}{4}\cdot\frac{1-6}{6}P_0(0)=-\frac{1\cdot 3\cdot 5}{2^3\cdot 1\cdot 2\cdot 3}P_0(0)$$

$$\vdots \qquad \vdots \quad \vdots$$

$$P_{2n}(0)=\frac{1-2}{2}\cdot\frac{1-4}{4}\cdot\frac{1-6}{6}\cdots\frac{1-2n}{2n}P_0(0)$$

$$=(-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\cdot 1\cdot 2\cdot 3\cdots n}P_0(0)$$

$$1\cdot 3\cdot 5\cdots(2n-1)=\frac{1\cdot 2\cdot 3\cdot 4\cdots(2n-1)(2n)}{2\cdot 4\cdot 6\cdots(2n)}$$

$$=\frac{(2n)!}{2^n n!}$$

$$=(-1)^n\frac{(2n)!}{2^n\cdot(n!)^2}P_0(0)$$

With $P_0(0)=1$

$$\boxed{P_{2n}(0)=(-1)^n\frac{(2n)!}{2^n\cdot(n!)^2}}$$

## Exercise

Show that $P_n'(0) = (-1)^{n+1} P_n'(-1) = \frac{n(n+1)}{2}$

***Hint***: Use Legendre's equation $\left(1-x^2\right)y'' - 2xy' + n(n+1)y = 0$

## Solution

Because $P_n(x)$ is a solution of Legendre's equation, then

$$\left(1-x^2\right)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$$

Let $x = 1$, then

$$-2P_n'(1) + n(n+1)P_n(1) = 0$$

$$P_n'(1) = \frac{n(n+1)}{2}P_n(1)$$

Let $x = -1$, then

$$2P_n'(-1) + n(n+1)P_n(-1) = 0$$

$$P_n'(-1) = -\frac{n(n+1)}{2}P_n(-1)$$

However, $P_n(1) = P_n(-1) = 1$

$$(-1)^{n+1}P_n'(-1) = \frac{n(n+1)}{2}$$

## Exercise

The differential equation $y'' + xy = 0$ is called ***Airy's equation***, and its solutions are called ***Airy functions***. Find the series for the solutions $y_1$ and $y_2$ where $y_1(0) = 1$ and $y_1'(0) = 0$, while $y_2(0) = 0$ and $y_2'(0) = 1$. What is the radius of convergence for these two series?

## Solution

Let $y = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad$ *and* $\quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$

$$y'' + xy = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

***Shifting the index to get a common power of x***

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}a_{n-1}x^n=0$$

$$2a_2+\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}a_{n-1}x^n=0$$

$$2a_2+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_{n-1}\right]x^n=0$$

$2a_2=0$ *or* $(n+2)(n+1)a_{n+2}+a_{n-1}=0$

$a_2=0$ *or* $a_{n+2}=\dfrac{-a_{n-1}}{(n+2)(n+1)}$ $\quad n\geq 1$

$a_3=\dfrac{-a_0}{3\cdot 2}$ $\quad a_4=-\dfrac{a_1}{4\cdot 3}$ $\quad a_5=-\dfrac{a_2}{5\cdot 4}=0$

$a_6=-\dfrac{a_3}{6\cdot 5}=\dfrac{a_0}{6\cdot 5\cdot 3\cdot 2}$ $\quad a_7=-\dfrac{a_4}{7\cdot 6}=\dfrac{a_1}{7\cdot 6\cdot 4\cdot 3}$ $\quad a_8=-\dfrac{a_5}{8\cdot 7}=0$

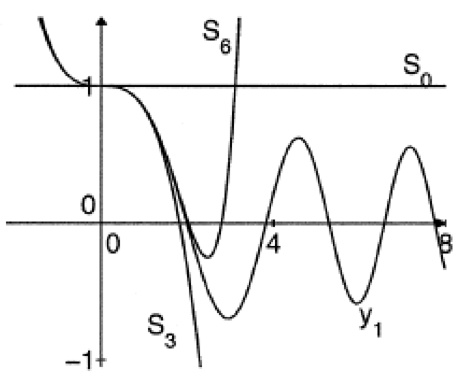
$a_9=-\dfrac{a_6}{9\cdot 8}=-\dfrac{a_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}$ $\quad a_{10}=-\dfrac{a_7}{10\cdot 9}=-\dfrac{a_1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}$ $\quad a_{11}=0$

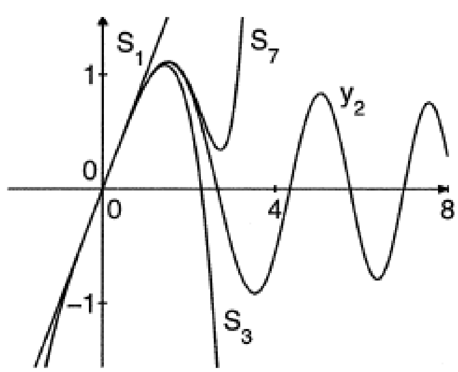
$a_{3n}=\dfrac{(-1)^n a_0}{(2\cdot 3)\cdot(5\cdot 6)\cdots(3n-1)(3n)}$ $\quad a_{3n+1}=\dfrac{(-1)^n a_1}{(3\cdot 4)\cdot(6\cdot 7)\cdots(3n)(3n+1)}$ $\quad a_{3n+2}=0$

$$y(x)=a_0\left[1-\frac{1}{2\cdot 3}x^2+\frac{1}{2\cdot 3\cdot 5\cdot 6}x^6-\cdots\right]+a_1\left[x-\frac{1}{3\cdot 4}x^4+\frac{1}{3\cdot 4\cdot 6\cdot 7}x^7-\cdots\right]$$

$$y_1(x)=1-\frac{1}{2\cdot 3}x^2+\frac{1}{2\cdot 3\cdot 5\cdot 6}x^6-\cdots=1+\sum_{n=1}^{\infty}\frac{(-1)^n x^{3n}}{(2\cdot 3)\cdot(5\cdot 6)\cdots(3n-1)(3n)}$$

$$y_2(x)=x-\frac{1}{3\cdot 4}x^4+\frac{1}{3\cdot 4\cdot 6\cdot 7}x^7-\cdots=x+\sum_{n=1}^{\infty}\frac{(-1)^n x^{3n+1}}{(3\cdot 4)\cdot(6\cdot 7)\cdots(3n)(3n+1)}$$



# Solution   *Section* 4.4 – Solution about Singular Points

### *Exercise*

Find the Frobenius series solutions of $2x^2y'' + 3xy' - \left(1 + x^2\right)y = 0$

***Solution***

$$y'' + \frac{3}{2}\frac{1}{x}y' - \frac{1}{2}\frac{1+x^2}{x^2}y = 0 \qquad \textit{Divide each term by } 2x^2$$

Therefore, $x = 0$ is a regular singular point, and that $p_0 = \frac{3}{2}, \quad q_0 = -\frac{1}{2}$

$p(x) \equiv \frac{3}{2}, \quad q(x) = -\frac{1}{2} - \frac{1}{2}x^2$ are polynomials.

The Frobenius series will converge for all $x > 0$. The indicial equation is

$$r(r-1) + \frac{3}{2}r - \frac{1}{2} = r^2 + \frac{1}{2}r - \frac{1}{2} = (r+1)\left(r - \frac{1}{2}\right) = 0$$

So the roots are $r_1 = \frac{1}{2}$ *and* $r_2 = -1$.

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2}\sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{-1}\sum_{n=0}^{\infty} b_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} \qquad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$2x^2y'' + 3xy' - \left(1 + x^2\right)y = 0$$

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 3x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty}a_n x^{n+r} - x^2\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(2n+2r-2)a_n x^{n+r} + \sum_{n=0}^{\infty}3(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}a_n x^{n+r} - \sum_{n=0}^{\infty}a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2) + 3(n+r) - 1\right]a_n x^{n+r} - \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r+1) - 1\right]a_n x^{n+r} - \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0$$

$$\left(2r^2+r-1\right)a_0+\left(2r^2+5r+2\right)a_1x+$$

$$\sum_{n=2}^{\infty}\left[(n+r)(2n+2r+1)-1\right]a_nx^{n+r}-\sum_{n=2}^{\infty}a_{n-2}x^{n+r}=0$$

$$\left(2r^2+r-1\right)a_0+\left(2r^2+5r+2\right)a_1x+\sum_{n=2}^{\infty}\left(\left[(n+r)(2n+2r+1)-1\right]a_n-a_{n-2}\right)x^{n+r}=0$$

For $n=0 \rightarrow \left(2r^2+r-1\right)a_0=0$

$$r=-1 \quad or \quad r=\tfrac{1}{2} \quad \surd$$

For $n=1 \rightarrow \left(2r^2+5r+2\right)a_1=0$

$$\cancel{r=-2 \quad or \quad r=-\tfrac{1}{2}} \quad \text{Therefore } a_1=0$$

$$\left[(n+r)(2n+2r+1)-1\right]a_n-a_{n-2}=0$$

$$a_n=\frac{1}{(n+r)\left(2(n+r)+1\right)-1}a_{n-2}$$

$$=\frac{1}{2(n+r)^2+(n+r)-1}a_{n-2} \qquad for \quad n\geq 2$$

$r=\frac{1}{2}$

$$a_n=\frac{1}{2\left(n+\frac{1}{2}\right)^2+\left(n+\frac{1}{2}\right)-1}a_{n-2}=\frac{1}{2n^2+3n}a_{n-2}$$

$$a_2=\frac{1}{14}a_0$$

$$a_3=\frac{1}{24}a_1=0$$

$$a_4=\frac{1}{44}a_2=\frac{1}{616}a_0$$

$$a_5=0$$

$$a_6=\frac{1}{90}a_4=\frac{1}{55440}a_0$$

$$y_1(x)=a_0x^{1/2}\left(1+\frac{x^2}{14}+\frac{x^4}{616}+\frac{x^6}{55,440}+\cdots\right)$$

$r=-1$

$$b_n=\frac{1}{2(n-1)^2+(n-1)-1}b_{n-2}$$

$$b_2=\frac{1}{2}b_0$$

$$b_3=\frac{1}{9}b_1=0$$

$$b_4=\frac{1}{20}b_2=\frac{1}{40}b_0$$

$$b_5=0$$

$$b_6=\frac{1}{54}b_4=\frac{1}{2160}b_0$$

$$y_2(x)=b_0x^{-1}\left(1+\frac{x^2}{2}+\frac{x^4}{40}+\frac{x^6}{2160}+\cdots\right)$$

$$y(x)=C_1\left(1+\sum_{n=0}^{\infty}\frac{4}{n!\;1\cdot3\cdot5\cdots(2n-1)}x^n\right)+C_2x^{1/2}\left(1+\sum_{n=0}^{\infty}\frac{4^n}{n!\;3\cdot5\cdots(2n+1)}x^n\right)$$

### ***Exercise***

Find the general solution to the equation $2xy'' + (1+x)y' + y = 0$

***Solution***

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} \qquad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$2xy'' + (1+x)y' + y = 0$$

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + (1+x)\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)c_n x^{n-1}x^r + \sum_{n=0}^{\infty}(n+r)c_n x^{n-1}x^r + \sum_{n=0}^{\infty}(n+r)c_n x^n x^r + \sum_{n=0}^{\infty} c_n x^n x^r = 0$$

$$x^r\left(\sum_{n=0}^{\infty}(n+r)(2n+2r-2+1)c_n x^{n-1} + \sum_{n=0}^{\infty}(n+r+1)c_n x^n\right) = 0$$

$$x^r\left(\sum_{n=0}^{\infty}(n+r)(2n+2r-1)c_n x^{n-1} + \sum_{n=0}^{\infty}(n+r+1)c_n x^n\right) = 0$$

$$x^r\left(c_0 r(2r-1)x^{-1} + \underbrace{\sum_{n=1}^{\infty}c_n(n+r)(2n+2r-1)x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=0}^{\infty}(n+r+1)c_n x^n}_{k=n}\right) = 0$$

$$x^r\left(c_0 r(2r-1)x^{-1} + \sum_{k=0}^{\infty}c_{k+1}(r+k+1)(2k+2r+1)x^k + \sum_{k=0}^{\infty}c_k(r+k+1)x^k\right) = 0$$

$$x^r\left(c_0 r(2r-1)x^{-1} + \sum_{k=0}^{\infty}\left[c_{k+1}(r+k+1)(2k+2r+1) + c_k(r+k+1)\right]x^k\right) = 0$$

$$\begin{cases} c_0 r(2r-1)=0 & \Rightarrow \qquad \boxed{r=0} \quad \boxed{r=\frac{1}{2}} \\ c_{k+1}(r+k+1)(2k+2r+1)+c_k(r+k)=0 & \Rightarrow \boxed{c_{k+1}=-\frac{r+k+1}{(r+k+1)(2k+2r+1)}c_k} \end{cases}$$

$r=0$

$$c_{k+1}=-\frac{1}{2k+1}c_k$$

$$c_1=-\frac{1}{1}c_0$$

$$c_2=-\frac{1}{3}c_1=\frac{1}{3}c_0$$

$$c_3=-\frac{1}{5}c_2=-\frac{1}{1\cdot 3\cdot 5}c_0$$

$$c_4=-\frac{1}{7}c_3=\frac{1}{1\cdot 3\cdot 5\cdot 7}c_0$$

$$c_n=\frac{(-1)^n}{1\cdot 3\cdot 5\cdots(2n-1)}c_0$$

$$y_1(x)=c_0\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{1\cdot 3\cdot 5\cdots(2n-1)}x^n\right)$$

$r=\frac{1}{2}$

$$c_{k+1}=-\frac{k+\frac{3}{2}}{\left(k+\frac{3}{2}\right)(2k+2)}c_k=-\frac{1}{2(k+1)}c_k$$

$$c_1=-\frac{1}{2}c_0$$

$$c_2=-\frac{1}{2\cdot 2}c_1=\frac{1}{2\cdot 2\cdot 2}c_0$$

$$c_3=-\frac{1}{2\cdot 3}c_2=-\frac{1}{2^3(2\cdot 3)}c_0=-\frac{1}{2^3\cdot 3!}c_0$$

$$c_4=-\frac{1}{2\cdot 4}c_3=\frac{1}{2^4\cdot 4!}c_0$$

$$c_n=\frac{(-1)^n}{2^n n!}c_0$$

$$y_2(x)=c_0x^{1/2}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n n!}x^n\right)$$

$$y(x)=C_1y_1(x)+C_2y_2(x)$$

$$\boxed{=C_1\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{1\cdot 3\cdot 5\cdots(2n-1)}x^n\right)+C_2\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}x^{n+1/2}\right)}$$

### *Exercise*

Find a Frobenius solution of Bessel's equation of order zero $x^2y'' + xy' + x^2y = 0$

***Solution***

$$y'' + \frac{1}{x}y' + y = 0$$

Therefore, $x = 0$ is a regular singular point, and that $p_0 = 1,\quad q_0 = 0$ and $p(x) \equiv 1,\quad q(x) = x^2$.

The indicial equation is: $r(r-1) + r = r^2 = 0 \quad \rightarrow \quad \boxed{r = 0}$

There is only one Frobenius series solution: $y(x) = \sum\limits_{n=0}^{\infty} a_n x^n$

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$x^2y'' + xy' + x^2y = 0$$

$$x^2 \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} n a_n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} \left[n(n-1) + n\right] a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\sum_{n=0}^{\infty} n^2 a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0 \quad a_4 = -\frac{a_2}{4^2} = \frac{a_0}{2^2 \cdot 4^2}$$

$$0 + a_1 x + \sum_{n=2}^{\infty} n^2 a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$a_1 x + \sum_{n=2}^{\infty} \left(n^2 a_n + a_{n-2}\right) x^n = 0$$

$$a_1 = 0 \quad \rightarrow \quad a_{n(odd)} = 0$$

$$n^2 a_n + a_{n-2} = 0 \quad \Rightarrow \quad a_n = -\frac{a_{n-2}}{n^2} \quad (n \ge 2)$$

$$a_2 = -\frac{a_0}{4}$$

$$a_6 = -\frac{a_4}{6^2} = -\frac{a_0}{2^2 \cdot 4^2 \cdot 6^2}$$

$$a_{2n} = \frac{(-1)^n}{2^2 \cdot 4^2 \cdots (2n)^2} a_0 = \frac{(-1)^n}{2^{2n} \cdot (n!)^2} a_0$$

The choice $a_0 = 1$ gives us the Bessel function of order zero of the first kind.

$$J_0(x) = \frac{(-1)^n x^{2n}}{2^{2n} \cdot (n!)^2} \quad = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots$$