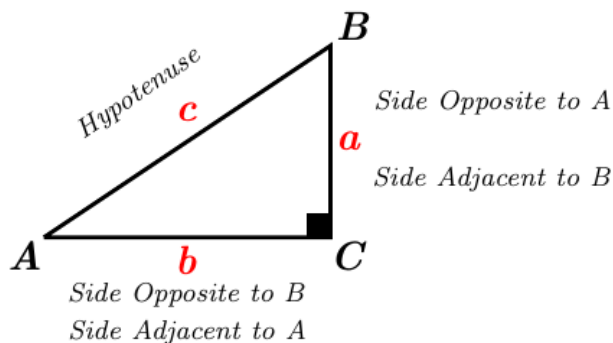
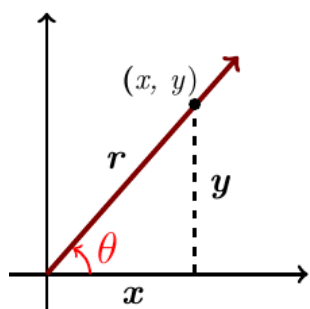


Section 3.3 – Trigonometric Functions

Let (x, y) be a point on the terminal side of an angle θ in standard position

The distance from the point to the origin is given by: $r = \sqrt{x^2 + y^2}$

Six Trigonometry Functions



$$\sin \theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}$$

$$\csc \theta = \frac{\text{hyp}}{\text{opp}} = \frac{1}{\sin \theta} = \frac{r}{y}$$

$$\cos \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$$

$$\sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\cos \theta} = \frac{r}{x}$$

$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{\sin \theta}{\cos \theta} = \frac{y}{x}$$

$$\cot \theta = \frac{\text{adj}}{\text{opp}} = \frac{\cos \theta}{\sin \theta} = \frac{1}{\tan \theta} = \frac{x}{y}$$

$$\sin A = \frac{a}{c} = \cos B$$

$$\csc A = \frac{c}{a} = \sec B$$

$$\cos A = \frac{b}{c} = \sin B$$

$$\sec A = \frac{c}{b} = \csc B$$

$$\tan A = \frac{a}{b} = \cot B$$

$$\cot A = \frac{b}{a} = \tan B$$

Example

Find the six trigonometry functions of θ if θ is in the standard position and the point $(8, 15)$ is on the terminal side of θ .

Solution

$$r = \sqrt{8^2 + 15^2} = 17$$

$$\sin \theta = \frac{y}{r} = \frac{15}{17}$$

$$\cos \theta = \frac{x}{r} = \frac{8}{17}$$

$$\tan \theta = \frac{y}{x} = \frac{15}{8}$$

$$\csc \theta = \frac{r}{y} = \frac{17}{15}$$

$$\sec \theta = \frac{r}{x} = \frac{17}{8}$$

$$\cot \theta = \frac{x}{y} = \frac{8}{15}$$

Example

Which will be greater, $\tan 30^\circ$ or $\tan 40^\circ$? How large could $\tan \theta$ be?

Solution

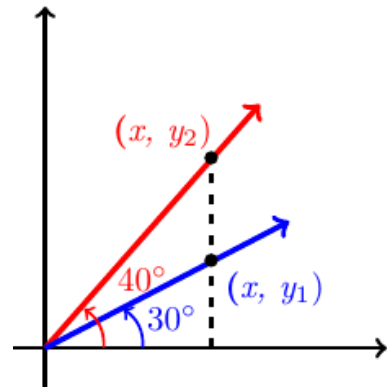
$$\tan 30^\circ = \frac{y_1}{x}$$

$$\tan 40^\circ = \frac{y_2}{x}$$

$$\text{Ratio: } \frac{y_2}{x} > \frac{y_1}{x}$$

$$\rightarrow \tan 40^\circ > \tan 30^\circ$$

No limit as to how large $\tan \theta$ can be.



Example

If $\cos \theta = \frac{\sqrt{3}}{2}$, and θ is **QIV**, find $\sin \theta$ and $\tan \theta$.

Solution

$$\cos \theta = \frac{\sqrt{3}}{2} = \frac{x}{r} \rightarrow x = \sqrt{3}, r = 2$$

$$\begin{aligned} y &= \sqrt{r^2 - x^2} \\ &= \sqrt{2^2 - (\sqrt{3})^2} \\ &= \sqrt{4 - 3} \\ &= 1 \end{aligned}$$

Since θ is **Q IV** $\Rightarrow y = -1$

$$\sin \theta = -\frac{1}{2} \quad \sin \theta = \frac{y}{r}$$

$$\tan \theta = \frac{-1}{\sqrt{3}} \quad \tan \theta = \frac{y}{x}$$

$$= -\frac{1}{\sqrt{3}} \frac{\sqrt{3}}{\sqrt{3}}$$

$$= -\frac{\sqrt{3}}{3}$$

Reciprocal Identities

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\sin \theta = \frac{1}{\csc \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cos \theta = \frac{1}{\sec \theta}$$

$$\tan \theta = \frac{1}{\cot \theta}$$

Ratio Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

Pythagorean Identities

$$x^2 + y^2 = r^2$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$(\cos \theta)^2 + (\sin \theta)^2 = 1 \Rightarrow \boxed{\cos^2 \theta + \sin^2 \theta = 1}$$

Solving for $\cos \theta$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

$$\cos \theta = \pm \sqrt{1 - \sin^2 \theta}$$

Solving for $\sin \theta$

$$\sin^2 \theta = 1 - \cos^2 \theta$$

$$\boxed{\sin \theta = \pm \sqrt{1 - \cos^2 \theta}}$$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

$$\frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

$$\left(\frac{\cos \theta}{\cos \theta}\right)^2 + \left(\frac{\sin \theta}{\cos \theta}\right)^2 = \left(\frac{1}{\cos \theta}\right)^2$$

$$\underline{1 + \tan^2 \theta = \sec^2 \theta}$$

$\cos^2 \theta + \sin^2 \theta = 1$
$\cos \theta = \pm \sqrt{1 - \sin^2 \theta}$
$\sin \theta = \pm \sqrt{1 - \cos^2 \theta}$
$1 + \tan^2 \theta = \sec^2 \theta$
$1 + \cot^2 \theta = \csc^2 \theta$

Example

Write $\tan \theta$ in terms of $\sin \theta$.

Solution

$$\begin{aligned} \tan \theta &= \frac{\sin \theta}{\cos \theta} \\ &= \frac{\sin \theta}{\pm \sqrt{1 - \sin^2 \theta}} \\ &= \pm \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} \end{aligned}$$

Example

Simplify the expression $\sqrt{x^2 + 9}$ as much as possible after substituting $3 \tan \theta$ for x

Solution

$$\begin{aligned} x &= 3 \tan \theta \\ \sqrt{x^2 + 9} &= \sqrt{(3 \tan \theta)^2 + 9} \\ &= \sqrt{9 \tan^2 \theta + 9} \\ &= \sqrt{9(\tan^2 \theta + 1)} \\ &= 3\sqrt{\sec^2 \theta} \\ &= \underline{3 \sec \theta} \end{aligned}$$

Example

Triangle ABC is a right triangle with $C = 90^\circ$. If $a = 6$ and $c = 10$, find the six trigonometric functions of A .

Solution

$$b = \sqrt{c^2 - a^2} = \sqrt{10^2 - 6^2} = 8$$

$$6, b \rightarrow 10 \rightarrow 2(3, b \rightarrow 5)$$

$$b = 4$$

$\sin A = \frac{a}{c} = \frac{3}{5}$	$\cos A = \frac{b}{c} = \frac{8}{10} = \frac{4}{5}$	$\tan A = \frac{a}{b} = \frac{6}{8} = \frac{3}{4}$
$\csc A = \frac{5}{3}$	$\sec A = \frac{5}{4}$	$\cot A = \frac{4}{3}$

$$\text{if } A + B = 90^\circ \Rightarrow \begin{cases} \sin A = \cos B \\ \sec A = \csc B \\ \tan A = \cot B \end{cases}$$

Cofunction Theorem

A trigonometric function of an angle is always equal to the cofunction of the complement of the angle.

Example

Write each function in terms of its cofunction

a) $\cos 52^\circ$

Solution

$$\cos 52^\circ = \sin(90^\circ - 52^\circ) = \sin 38^\circ$$

b) $\tan 71^\circ$

Solution

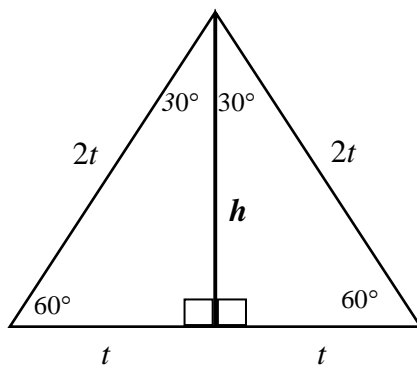
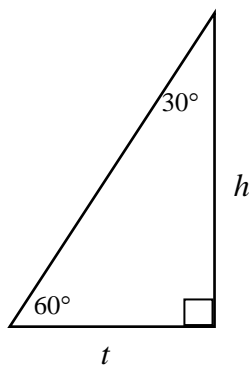
$$\tan 71^\circ = \cot(90^\circ - 71^\circ) = \cot 19^\circ$$

c) $\sec 24^\circ$

Solution

$$\sec 24^\circ = \csc(90^\circ - 24^\circ) = \csc 66^\circ$$

30° - 60° - 90° Triangle



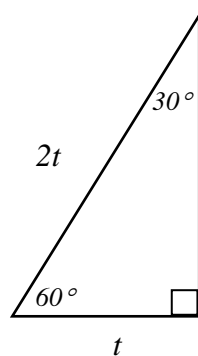
$$t^2 + h^2 = (2t)^2$$

$$t^2 + h^2 = 4t^2$$

$$h^2 = 4t^2 - t^2$$

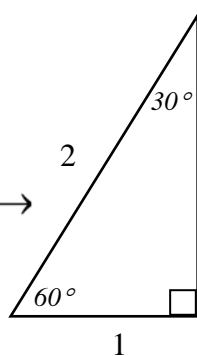
$$h^2 = 3t^2$$

$$h = t\sqrt{3}$$



$t\sqrt{3}$

\rightarrow

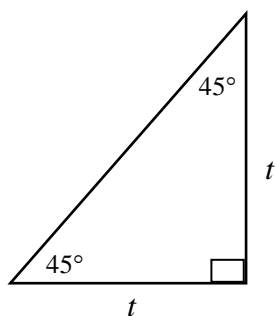


$\sqrt{3}$

\Rightarrow

$$\sin 60^\circ = \frac{\sqrt{3}}{2}$$

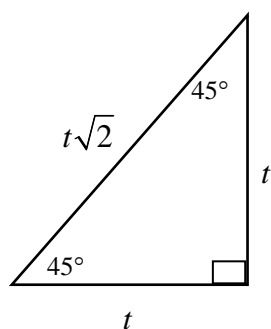
45° - 45° - 90° Triangle



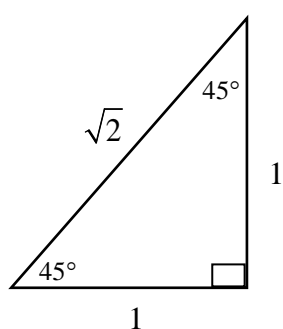
$$\text{hypotenuse}^2 = t^2 + t^2$$

$$\text{hypotenuse} = \sqrt{2t^2}$$

$$\text{hypotenuse} = t\sqrt{2}$$



t



1

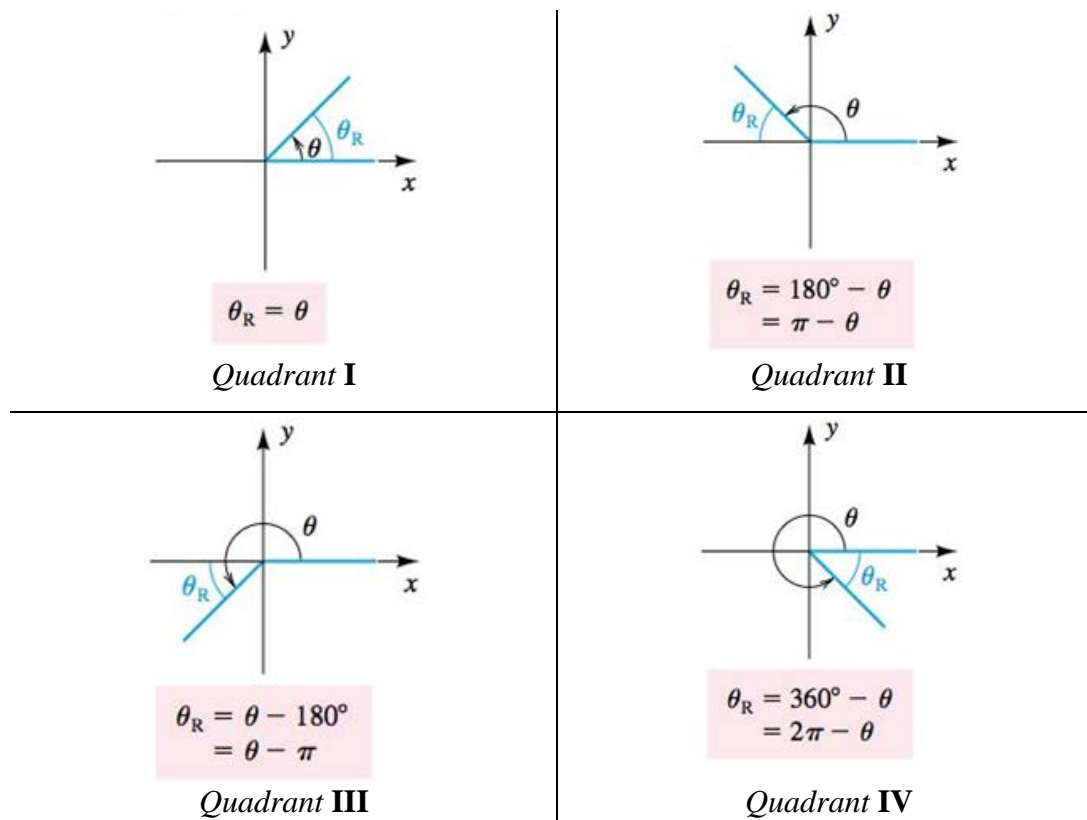
\Rightarrow

$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$

Reference Angle

Definition

The reference angle or related angle for any angle θ in standard position is the positive acute angle between the terminal side of θ and the x -axis, and it is denoted $\hat{\theta}$



Example

Find the exact value of $\sin 240^\circ$

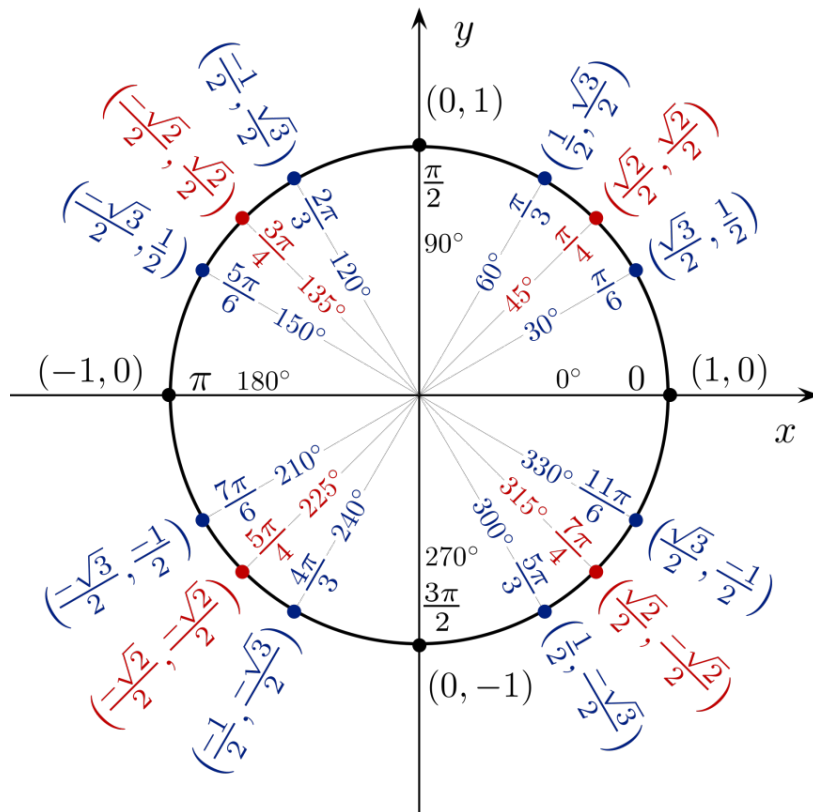
Solution

$$\hat{\theta} = 240^\circ - 180^\circ = 60^\circ$$

$$\rightarrow 240^\circ \in Q_{III}$$

$$\sin 240^\circ = -\sin 60^\circ$$

$$= -\frac{\sqrt{3}}{2}$$



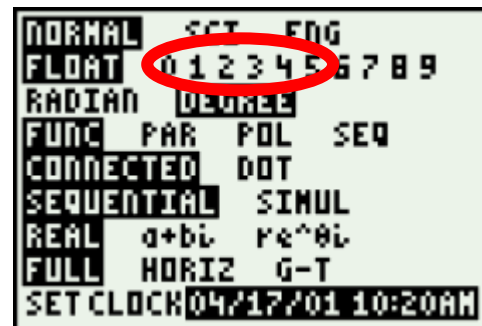
Approximation- Simply using calculator

$$\sin 250^\circ \approx -0.9397$$

$$\cos 250^\circ \approx -0.3420$$

$$\tan 250^\circ \approx 2.7475$$

$$\csc 250^\circ = \frac{1}{\sin 250^\circ} \approx -1.0642$$



To find the angle by using the inverse trigonometry functions, always enter a **positive** value.

Example

Find θ if $\sin \theta = -0.5592$ and θ terminates in QIII with $0^\circ \leq \theta < 360^\circ$.

Solution

$$\hat{\theta} = \sin^{-1} 0.5592 \approx 34^\circ$$

$$\theta \in \text{QIII}$$

$$\Rightarrow \theta = 180^\circ + 34^\circ = 214^\circ$$

Example

Find θ to the nearest degree if $\cot \theta = -1.6003$ and θ terminates in QII with $0^\circ \leq \theta < 360^\circ$.

Solution

$$\cot \theta = -1.6003 = \frac{1}{\tan \theta}$$

$$\tan \theta = \frac{1}{-1.6003}$$

$$\hat{\theta} = \tan^{-1} \frac{1}{1.6003} = \underline{32^\circ}$$

$$\theta \in QII \quad \Rightarrow \theta = 180^\circ - 32^\circ = \underline{148^\circ}$$

Exercise

Section 5.3 – Trigonometric Functions

(1 – 16) Find the *six* trigonometry functions of θ if θ is in the standard position and the given point is on the terminal side of θ .

- | | | | |
|---------------|----------------|-----------------|-----------------|
| 1. $(-2, 3)$ | 5. $(5, -12)$ | 9. $(-6, 8)$ | 13. $(7, 24)$ |
| 2. $(-3, -4)$ | 6. $(9, -12)$ | 10. $(-15, 8)$ | 14. $(-7, -24)$ |
| 3. $(-3, 0)$ | 7. $(16, -12)$ | 11. $(-7, 24)$ | 15. $(-24, -7)$ |
| 4. $(12, -5)$ | 8. $(15, -8)$ | 12. $(10, -24)$ | 16. $(24, -10)$ |

17. Find the values of the six trigonometric functions for an angle of 90° .

18. Indicate the two quadrants θ could terminate in if $\cos \theta = \frac{1}{2}$

19. Indicate the two quadrants θ could terminate in if $\csc \theta = -2.45$

(20 – 38) Find the remaining trigonometric function of θ if

- | | |
|--|--|
| 20. $\sin \theta = \frac{12}{13}$ and θ terminates in QI . | 29. $\sin \theta = -\frac{3}{5}$ & $\theta \in QIV$ |
| 21. $\cot \theta = -2$ and θ terminates in QII . | 30. $\cos \theta = -\frac{12}{13}$ & $\theta \in QIII$ |
| 22. $\tan \theta = \frac{3}{4}$ and θ terminates in $QIII$. | 31. $\cos \theta = -\frac{5}{13}$ & $\theta \in QII$ |
| 23. $\cos \theta = \frac{24}{25}$ and θ terminates in QIV . | 32. $\cos \theta = \frac{12}{13}$ & $\theta \in QIV$ |
| 24. $\cos \theta = \frac{\sqrt{3}}{2}$ and θ is terminates in QIV . | 33. $\sin \theta = -\frac{8}{17}$ & $\theta \in QIII$ |
| 25. $\tan \theta = -\frac{1}{2}$ and $\cos \theta > 0$ | 34. $\cos \theta = -\frac{15}{17}$ & $\theta \in QII$ |
| 26. $\cos \theta = \frac{3}{5}$ & $\theta \in QI$ | 35. $\cos \theta = -\frac{8}{17}$ & $\theta \in QII$ |
| 27. $\cos \theta = -\frac{4}{5}$ & $\theta \in QII$ | 36. $\cos \theta = -\frac{7}{25}$ & $\theta \in QII$ |
| 28. $\sin \theta = -\frac{3}{5}$ & $\theta \in QIII$ | 37. $\sin \theta = -\frac{7}{25}$ & $\theta \in QIII$ |
| | 38. $\sin \theta = -\frac{24}{25}$ & $\theta \in QIV$ |

39. If $\sin \theta = -\frac{5}{13}$, and θ is $QIII$, find $\cos \theta$ and $\tan \theta$.

40. If $\cos \theta = \frac{3}{5}$, and θ is QIV , find $\sin \theta$ and $\tan \theta$.

41. Use the reciprocal identities if $\cos \theta = \frac{\sqrt{3}}{2}$ find $\sec \theta$

42. Find $\cos \theta$, given that $\sec \theta = \frac{5}{3}$

43. Find $\sin \theta$, given that $\csc \theta = -\frac{\sqrt{12}}{2}$
44. Use a ratio identity to find $\tan \theta$ if $\sin \theta = \frac{3}{5}$ and $\cos \theta = -\frac{4}{5}$
45. If $\cos \theta = -\frac{1}{2}$ and θ terminates in QII, find $\sin \theta$
46. If $\sin \theta = \frac{3}{5}$ and θ terminated in QII, find $\cos \theta$ and $\tan \theta$.
47. Find $\tan \theta$ if $\sin \theta = \frac{1}{3}$ and θ terminates in QI
48. Find the remaining trigonometric ratios of θ , if $\sec \theta = -3$ and $\theta \in QIII$
49. Using the calculator and rounding your answer to the nearest hundredth, find the remaining trigonometric ratios of θ if $\csc \theta = -2.45$ and $\theta \in QIII$.
50. Write $\frac{\sec \theta}{\csc \theta}$ in terms of $\sin \theta$ and $\cos \theta$, and then simplify if possible.
51. Write $\cot \theta - \csc \theta$ in terms of $\sin \theta$ and $\cos \theta$, and then simplify if possible.
52. Write $\frac{\sin \theta}{\cos \theta} + \frac{1}{\sin \theta}$ in terms of $\sin \theta$ and/or $\cos \theta$, and then simplify if possible.
53. Write $\sin \theta \cot \theta + \cos \theta$ in terms of $\sin \theta$ and $\cos \theta$, and then simplify if possible.
54. Multiply $(1 - \cos \theta)(1 + \cos \theta)$
55. Multiply $(\sin \theta + 2)(\sin \theta - 5)$
56. Simplify the expression $\sqrt{25 - x^2}$ as much as possible after substituting $5 \sin \theta$ for x .
57. Simplify the expression $\sqrt{4x^2 + 16}$ as much as possible after substituting $2 \tan \theta$ for x
- (58 – 60) Simplify by using the table
58. $5 \sin^2 30^\circ$ 59. $\sin^2 60^\circ + \cos^2 60^\circ$ 60. $(\tan 45^\circ + \tan 60^\circ)^2$
61. Find θ if $\sin \theta = -\frac{1}{2}$ and θ terminates in QIII with $0^\circ \leq \theta \leq 360^\circ$.
62. Find θ to the nearest degree if $\sec \theta = 3.8637$ and θ terminates in QIV with $0^\circ \leq \theta < 360^\circ$.
- (63 – 67) Find the exact value of
63. $\cos 225^\circ$ 65. $\tan 315^\circ$ 67. $\cot 480^\circ$
64. $\csc 300^\circ$ 66. $\cos 420^\circ$

(68 – 70) Use the calculator to find the value of

68. $\csc 166.7^\circ$

69. $\sec 590.9^\circ$

70. $\tan 195^\circ 10'$

71. Use the calculator to find θ to the nearest degree if $\sin \theta = -0.3090$ with $\theta \in QIV$ with $0^\circ \leq \theta < 360^\circ$

72. Use the calculator to find θ to the nearest degree if $\cos \theta = -0.7660$ with $\theta \in QIII$ with $0^\circ \leq \theta < 360^\circ$

73. Use the calculator to find θ to the nearest degree if $\sec \theta = -3.4159$ with $\theta \in QII$ with $0^\circ \leq \theta < 360^\circ$

74. Find θ to the nearest tenth of a degree if $\tan \theta = -0.8541$ and θ terminates in QIV with $0^\circ \leq \theta < 360^\circ$

75. Use the calculator to find θ to the nearest degree if $\sin \theta = 0.49368329$ with $\theta \in QII$ with $0^\circ \leq \theta < 360^\circ$