Lecture One – First Order Equations

Section 1.1 – Differential Equations & Solutions

Ordinary Differential Equations

Involve an unknown function of a single variable with one or more of its derivatives.

$$\frac{dy}{dt} = y - t$$

y: $y(t)$ is unknown function

t: independent variable

Some other example:

$$y' = y^2 - t$$
$$ty' = y$$
$$y' + 4y = e^{-3t}$$
$$yy'' + t^2 y = \cos t$$
$$y' = \cos(ty)$$

$\therefore$ The order of a differential equation is the order of the highest derivative that occurs in the equation.

y'' : *second order*

$$\frac{\partial^2 \omega}{\partial t^2} = c^2 \frac{\partial^2 \omega}{\partial x^2}$$

is not an ODE (ω is dependent on x and t)

This equation is called a ***partial differential equation***.

Definition

A first-order differential equation of the form $\frac{dy}{dt} = y' = f(t, y)$ is said to be in normal form.

$y^{(n)} = f\left(t, y, y', \ldots, y^{(n-1)}\right)$ is said to be in normal form.

f: is a given function of 2 variables t & y (***rate function***)

Solutions

A solution of the first-order, ordinary differential equation $f(t, y, y') = 0$ is a differentiable function $y(t)$ such that $f(t, y(t), y'(t)) = 0$ for all t in the interval where $y(t)$ is defined.

1. Can be found in explicit and implicit form by applying manipulation (integration)
2. No real solution.

Example

Show that $y(t) = Ce^{-t^2}$ is a solution of the 1st order equation $y' = -2ty$

Solution

$$y(t) = Ce^{-t^2} \Rightarrow y' = -2tCe^{-t^2}$$

$$y' = -2tCe^{-t^2}$$

$$y' = -2t\, y(t) \qquad \text{True; it is a solution}$$

$y(t)$ is called the ***general solution***.

The solutions from the graph are called ***solution curves***.

Example

Is the function $y(t) = \cos t$ a solution to the differential equation $y' = 1 + y^2$

Solution

$$y' = -\sin t$$

$$y' = 1 + y^2 = -\sin t$$

$$1 + \cos^2 t \overset{?}{=} -\sin t \qquad \text{False; it is not a solution.}$$

Exercises *Section* 1.1 – Differential Equations & Solutions

1. Show that $y(t)=Ce^{-(1/2)t^2}$ is a solution of the 1st order equation $y'=-ty$ *for* $-3\le C\le 3$

2. Show that $y(t)=\dfrac{4}{1+Ce^{-4t}}$ is a solution of the 1st order equation $y'=y(4-y)$

3. A general solution may fail to produce all solutions of a differential equation $y(t)=\dfrac{4}{1+Ce^{-4t}}$.

Show that $y=0$ is a solution of the differential equation, but no value of C in the given general solution will produce this solution.

4. Use the given general solution to find a solution of the differential equation having the given initial condition. $ty'+y=t^2, \quad y(t)=\frac{1}{3}t^2+\frac{C}{t}, \quad y(1)=2$

5. Show that $y(t)=2t-2+Ce^{-t}$ is a solution of the 1st order equation $y'+y=2t$ *for* $-3\le C\le 3$

6. Use the given general solution to find a solution of the differential equation having the given initial condition. $y'+4y=\cos t, \quad y(t)=\frac{4}{17}\cos t+\frac{1}{17}\sin t+Ce^{-4t}, \quad y(0)=-1$

7. Use the given general solution to find a solution of the differential equation having the given initial condition. $ty'+(t+1)y=2te^{-t}, \quad y(t)=e^{-t}\left(t+\frac{C}{t}\right), \quad y(1)=\frac{1}{e}$

8. Use the given general solution to find a solution of the differential equation having the given initial condition. $y'=y(2+y), \quad y(t)=\dfrac{2}{-1+Ce^{-2t}}, \quad y(0)=-3$

9. Find the values of ***m*** so that the function $y=e^{mx}$ is a solution of the given differential equation

a) $y'+2y=0$

b) $5y'-2y=0$

c) $y''-5y'+6y=0$

d) $2y''+7y'-4y=0$

10. Let $x=c_1\cos t+c_2\sin t$ is 2-parameter family solutions of the second order differential equation of $x''+x=0$. Find a solution of the second-order consisting of this differential equation and the given initial conditions.

a) $x(0)=-1, \quad x'(0)=8$

b) $x\left(\frac{\pi}{2}\right)=0, \quad x'\left(\frac{\pi}{2}\right)=1$

c) $x\left(\frac{\pi}{6}\right)=\frac{1}{2}, \quad x'\left(\frac{\pi}{6}\right)=0$

d) $x\left(\frac{\pi}{4}\right)=\sqrt{2}, \quad x'\left(\frac{\pi}{4}\right)=2\sqrt{2}$

Section 1.2 – Solutions to Separable Equations

Separable Equation

Separable equation is an equation that can be written with its variables separated and then easily solved.

If f is independent of $y \Rightarrow f(x,y) = g(x) = \dfrac{dy}{dx}$

$$g(x)dx = dy$$

$$y = \int g(x)dx$$

Definition

A 1st order differential equation of the form $\dfrac{dy}{dx} = g(x)h(y)$ is said to be separable or to have separable variables.

$$\frac{dy}{h(y)} = g(x)dx$$

$$\frac{dy}{dx} = y^2xe^{3x+4y} = \left(xe^{3x}\right)\left(y^2e^{4y}\right)$$

$$\frac{dy}{dx} = y + \sin x \qquad \textit{not separable}$$

Example

At time t the sample contains $N(t)$ radioactive nuclei and is given by the differential equation:

$$N' = -\lambda N$$

This is called the ***exponential equation***.

$$N' = -\lambda N$$

$$\frac{dN}{dt} = -\lambda N \qquad \textit{Separable equation}$$

$$\frac{dN}{N} = -\lambda dt$$

$$\int \frac{dN}{N} = -\int \lambda dt$$

$$\ln|N| = -\lambda t + C$$

$$|N(t)| = e^{-\lambda t + C}$$

$$= e^{C}e^{-\lambda t}$$

$$N(t) = \begin{cases} e^{C}e^{-\lambda t} & \text{if } N > 0 \\ -e^{C}e^{-\lambda t} & \text{if } N < 0 \end{cases}$$

$$N(t) = Ae^{-\lambda t} \qquad A = \begin{cases} e^{C} & \text{if } N > 0 \\ -e^{C} & \text{if } N < 0 \end{cases}$$

Example

Solve the differential equation $y' = ty^2$

Solution

$$\frac{dy}{dt} = ty^2$$

$$\frac{dy}{y^2} = tdt$$

$$\int y^{-2}dy = \int tdt$$

$$-y^{-1} = \frac{1}{2}t^2 + C$$

$$-\frac{1}{y} = \frac{t^2+2C}{2} \qquad \textit{Cross multiplication}$$

$$-\frac{2}{t^2+2C} = y$$

$$y(t) = \boxed{-\frac{2}{t^2+2C}}$$

General Method

1. Separate the variables
2. Integrate both sides
3. Solve for the solution $y(t)$, if possible

Using definite Integration

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of change of an object's temperature (***T***) is proportional to the difference between its temperature and the ambient temperature (***A***) (i.e. the temperature of its surroundings).

$$\frac{dT}{dt} = -k(T - A)$$

Example

A can of beer at 40° F is placed into a room when the temperature is 70° F. After 10 minutes the temperature of the beer is 50° F. What is the temperature of the beer as a function of time? What is the temperature of the beer 30 minutes after the beer was placed into the room?

Solution

By Newton's law of cooling: The rate of change of an object's temperature (***T***) is proportional to the difference between its temperature and the ambient temperature (***A***).

$$\frac{dT}{dt} = -k(T - A)$$

$$\frac{dT}{T - A} = -kdt$$

$$\int_{T_0}^{T} \frac{dT}{T - A} = -\int_0^t kdt$$

$$\ln|T - A| \Big|_{T_0}^{T} = -kt \Big|_0^t$$

$$\ln|T - A| - \ln|T_0 - A| = -kt$$

$$\ln\frac{|T - A|}{|T_0 - A|} = -kt \qquad \textit{Quotient Rule}$$

$$\frac{T - A}{T_0 - A} = e^{-kt} \qquad \Rightarrow \; T - A = (T_0 - A)e^{-kt}$$

$$T(t) = A + (T_0 - A)e^{-kt}$$

<u>*Given*</u>: $T_0 = 40°F \quad \& \; A = 70°F$

$$T(t) = 70 + (40 - 70)e^{-kt} = \underline{70 - 30e^{-kt}}$$

$$T(t = 10) = 70 - 30e^{-10k} = 50$$

$$-30e^{-10k} = 50 - 70$$

$$-30e^{-10k} = -20$$

$$e^{-10k} = \frac{2}{3}$$

$$-10k = \ln\frac{2}{3}$$

$$k = \frac{\ln\frac{2}{3}}{-10} = 0.0405$$

$$T(t) = 70 - 30e^{-0.0405t}$$

$$T(t = 30) = 70 - 30e^{-0.0405(30)}$$

$$\underline{= 61.1\ \ ^\circ F|}$$

Losing a solution

When we use separate variables, the variable divisors could be zero at a point.

Example

Find a general solution to $\frac{dy}{dx} = y^2 - 4$

Solution

$$\frac{dy}{y^2-4} = dx$$

$$\left(\frac{1/4}{y-2} - \frac{1/4}{y+2}\right)dy = dx \qquad\qquad y = \pm 2 \ \text{C}\textit{ritical points}$$

$$\frac{1}{4}\left(\int\frac{dy}{y-2} - \int\frac{dy}{y+2}\right) = \int dx$$

$$\frac{1}{4}\Big[\ln|y-2| - \ln|y+2|\Big] = x + c_1$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x + c_2$$

$$\left|\frac{y-2}{y+2}\right| = e^{4x+c_2}$$

$$\frac{y-2}{y+2} = \pm e^{c_2}e^{4x}$$

$$y - 2 = Ce^{4x}(y+2)$$

$$y - Ce^{4x}y = 2Ce^{4x} + 2$$

$$\left(1 - Ce^{4x}\right)y = 2\left(Ce^{4x} + 1\right)$$

$$\underline{y = 2\frac{1+Ce^{4x}}{1-Ce^{4x}}|}$$

If $y = -2 \Rightarrow -2 = 2\frac{1+Ce^{4x}}{1-Ce^{4x}}$

$$-1 = \frac{1+Ce^{4x}}{1-Ce^{4x}}$$

$$-1 + Ce^{4x} = 1 + Ce^{4x} \quad \Rightarrow \quad -1 = 1 \quad \textit{impossible}$$

If $y = 2$ $\quad \Rightarrow 2 = 2\frac{1+Ce^{4x}}{1-Ce^{4x}}$

$$1 - Ce^{4x} = 1 + Ce^{4x}$$

$$-Ce^{4x} = Ce^{4x} \quad \Rightarrow \quad -C = C$$

$$y = 2 \Rightarrow C = 0$$

Implicitly Defined Solutions

Example

Find the solutions of the equation $y' = \frac{e^x}{1+y}$, having initial conditions $y(0) = 1$ and $y(0) = -4$

Solution

$$\frac{dy}{dx} = \frac{e^x}{1+y}$$

$$(1+y)\,dy = e^x dx$$

$$\int (1+y)\,dy = \int e^x dx$$

$$y + \tfrac{1}{2}y^2 = e^x + c$$

$$y^2 + 2y - 2\left(e^x + c\right) = 0$$

$$y(x) = \tfrac{1}{2}\left(-2 \pm \sqrt{4 + 8\left(e^x + c\right)}\right) \qquad \textit{Quadratic Formula}$$

$$= -1 \pm \sqrt{1 + 2\left(e^x + c\right)} \qquad \textit{Implicit}$$

$$y(0) = -1 + \sqrt{1 + 2\left(e^0 + c\right)} = 1$$

$$\sqrt{1 + 2(1+c)} = 2$$

$$1 + 2 + 2c = 4$$

$$2c = 1$$

$$\boxed{c = \frac{1}{2}}$$

$$y(0) = -1 - \sqrt{1 + 2\left(e^0 + c\right)} = -4$$

$$-\sqrt{1 + 2 + 2c} = -3$$

$$1 + 2 + 2c = 9$$

$$2c = 6$$

$$\boxed{c = 3}$$

$$\begin{cases} y(t) = -1 + \sqrt{2 + 2e^x} \\ y(t) = -1 - \sqrt{7 + 2e^x} \end{cases}$$

$\therefore y \neq -1$ *from* y', but it never it will be.

Explicit Solutions: $y = -1 + \sqrt{\quad}$

Implicit solutions: $y^2 + by + c$

Example

Find the solutions to the differential equation $x' = \frac{2tx}{1+x}$, having $x(0) = 1, \ -2, \ 0$

Solution

$$\frac{dx}{dt} = \frac{2tx}{1+x}$$

$$\frac{1+x}{x}dx = 2tdt$$

$$\left(\frac{1}{x} + 1\right)dx = 2tdt$$

$$\int \left(\frac{1}{x} + 1\right)dx = \int 2tdt$$

$$\ln|x| + x = t^2 + c$$

For $x(0) = 1$

$$1 = 0^2 + c$$

$$c = 1$$

$$\ln|x| + x = t^2 + c \qquad x > 0$$

We can't solve for $x(t)$

$\Rightarrow$ This solution is defined as implicit.

For $x(0) = -2$

$$\ln|-2| + (-2) = 0^2 + c$$

$$c = -2 + \ln 2$$

$$\ln|x| + x = t^2 - 2 + \ln 2$$

Since the initial condition < 0, then:

$$x + \ln(-x) = t^2 - 2 + \ln 2$$

For $x(0) = 0$

$0 = 0^2 + c$ *True statement*

$y' = 0 \ \Rightarrow \ x(t) = 0$ is a solution

Exercises *Section* 1.2 – Solutions to Separable Equations

Find the general solution of the differential equation. If possible, find an explicit solution.

1. $y' = xy$

2. $xy' = 2y$

3. $y' = e^{x-y}$

4. $y' = \left(1 + y^2\right)e^x$

5. $y' = xy + y$

6. $y' = ye^x - 2e^x + y - 2$

7. $y' = \dfrac{x}{y+2}$

8. $y' = \dfrac{xy}{x-1}$

9. $y' = \dfrac{y^2 + ty + t^2}{t^2}$

10. $\dfrac{dy}{dx} = \dfrac{4x - x^3}{4 + y^3}$

11. $y' = \dfrac{2xy + 2x}{x^2 - 1}$

Find the exact solution of the initial value problem. Indicate the interval of existence.

12. $y' = \dfrac{y}{x}, \quad y(1) = -2$

13. $y' = -\dfrac{2t\left(1 + y^2\right)}{y}, \quad y(0) = 1$

14. $y' = \dfrac{\sin x}{y}, \quad y\left(\frac{\pi}{2}\right) = 1$

15. $4t\,dy = \left(y^2 + ty^2\right)dt, \quad y(1) = 1$

16. $y' = \dfrac{1 - 2t}{y}, \quad y(1) = -2$

17. $y' = y^2 - 4, \quad y(0) = 0$

18. $\dfrac{dy}{dx} = \dfrac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1$

19. $y' = \dfrac{x}{1 + 2y}, \quad y(-1) = 0$

20. A murder victim is discovered at midnight and the temperature of the body is recorded at 31°C. One hour later, the temperature of the body is 29°C. Assume that the surrounding air temperature remains constant at 21°C. Use Newton's law of cooling to calculate the victim's time of death. *Note*: The normal temperature of a living human being is approximately 37°C

21. Suppose a cold beer at 40°F is placed into a warn room at 70°F. suppose 10 minutes later, the temperature of the beer is 48°F. Use Newton's law of cooling to find the temperature 25 minutes after the beer was placed into the room.

Section 1.3 – Models of Motions

In mathematics, the rate at which a quantity changes is the derivative of that quantity.
The 2nd way of computing the rate of change comes from the application itself and is different from on application to another.

Mechanics

Law of mechanics – Newton's 2nd Law (1665-1671)

The force acting on a mass is equal to the rate of change of momentum with respect to time.
Momentum is defined as the product of mass and velocity (m.v).
The force is equal to the derivative of the momentum

$$F = \frac{d}{dt}mv = m\frac{dv}{dt} = ma$$

Position: $x(t) = -\frac{1}{2}gt^2 + v_0 t + x_0$

Universal Law of gravitation

Any body with mass M attacks any other body with mass *m* directly toward the mass M, with a magnitude proportional to the product of the 2 masses and inversely proportional to the square of the distance separating them.

$$F = \frac{GMm}{r^2}$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$F = -mg$$

$$g = \frac{GM}{r^2}$$

Motion ball: $a = -g = \frac{d^2x}{dt^2}$

$g = 32\ ft/\sec^2 = 9.8\ m/s^2$

Air Resistance

$$R(x,v) = -r(x,v).v$$

R: resistance force (*has sign opposite of the velocity*)
r: is a function that is always nonnegative

- *when a ball is falling from a high altitude, the density of the air has to be taken into account.*

$$F = -mg + R(v)$$

$$m\frac{dv}{dt} = -mg - rv$$

$$\frac{dv}{dt} = -g - \frac{r}{m}v$$

$$dv = \left(-g - \frac{r}{m}v\right)dt$$

$$\frac{dv}{g+\frac{r}{m}v} = -dt$$

$$\int \frac{dv}{g+\frac{r}{m}v} = -\int dt$$

$$\frac{m}{r}\ln\left(g + \frac{r}{m}v\right) = -t + C_1$$

$$\ln\left(g + \frac{r}{m}v\right) = -\frac{r}{m}t + C_2$$

$$g + \frac{r}{m}v = e^{-\frac{r}{m}t+C_2}$$

$$v(t) = Ce^{-rt/m} - \frac{mg}{r}$$

When $t \to \infty \quad \Rightarrow \quad v = -\frac{mg}{r}$ (*Terminal Velocity*)

$x(t) = -\frac{mC}{r}e^{-rt/m} - \frac{mg}{r}t + A$ (A: is a constant)

Example

Suppose you drop a brick from the top of a building that is 250 *m* high. The brick has a mass of 2 *kg*, and the resistance force is given by $R = -4v$. How long will it take the brick to reach the ground? what will be its velocity at that time?

Solution

$$v(t) = Ce^{-rt/m} - \frac{mg}{r}$$

$$v(0) = 0 = C - \frac{mg}{r}$$

$$\Rightarrow C = \frac{mg}{r}$$

$$= \frac{2(9.8)}{4}$$

$$= 4.9$$

$$\frac{dx}{dt} = v(t)$$

$$= 4.9\left(e^{-2t} - 1\right)$$

$$\int dx = \int 4.9\left(e^{-2t} - 1\right)dt$$

$$x(t) = 4.9\left(-\tfrac{1}{2}e^{-2t} - t\right) + A$$

$$x(0) = 250 = 4.9\left(-\tfrac{1}{2}e^{-2(0)} - (0)\right) + A$$

$$250 = 4.9\left(-\tfrac{1}{2}\right) + A$$

$$250 = -2.45 + A$$

$$A = 252.45$$

$$x(t) = 4.9\left(-\tfrac{1}{2}e^{-2t} - t\right) + 252.45$$

$x(t) = 0 \quad \Rightarrow \underline{t = 51.52\ sec}$ (Using software to solve it)

$$v(t) = 4.9\left(e^{-2t} - 1\right)$$

$$\underline{v(t = 51.52) \approx -4.9\ m/s}$$

Finding the displacement

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}.v$$

$$F = -mg + R(v) \qquad R = -k|v|.v$$

$$m\frac{dv}{dt} = -mg - k|v|v$$

$$\frac{dv}{dt} = -g - \frac{k}{m}v^2 = \boxed{\frac{dv}{dx}.v}$$

$$v\frac{dv}{dx} = -g - \frac{k}{m}v^2 \quad \boxed{= -\frac{mg + kv^2}{m}}$$

Example

A ball of mass $m = 0.2\ kg$ is protected from the surface of the earth, with velocity $v_0 = 50\ m/s$. Assume that the force of air resistance is given by $R = -k|v|.v$, where $k = 0.02$. What is the maximum height reached by the ball?

<u>Solution</u>

$$v\frac{dv}{dx} = -\frac{mg + kv^2}{m}$$

$$\frac{vdv}{mg + kv^2} = -\frac{dx}{m}$$

$$\int_{v_0}^{0} \frac{vdv}{mg + kv^2} = -\int_0^{x_{\max}} \frac{dx}{m}$$

$$d\left(mg + kv^2\right) = 2kvdv \quad \Rightarrow \quad \frac{d\left(mg+kv^2\right)}{2k} = vdv$$

$$\frac{1}{2k}\int_{v_0}^{0} \frac{d\left(mg + kv^2\right)}{mg + kv^2} = -\int_0^{x_{\max}} \frac{dx}{m}$$

$$\frac{1}{2k}\ln\left|mg + kv^2\right|_{50}^{0} = -\frac{x}{m}\Big|_0^{x_{max}}$$

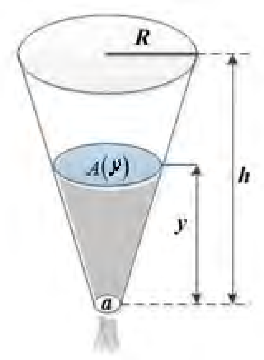
$$\frac{1}{2k}\left[\ln(mg) - \ln\left(mg + k\left(50^2\right)\right)\right] = -\frac{x_{max}}{m}$$

$$x_{max} = \frac{m}{2k}\left[\ln\left(mg + k\left(50^2\right)\right) - \ln(mg)\right]$$

$$= \frac{0.2}{2(.02)}\left[\ln\left(\frac{0.2(9.8)+(.02)(50)^2}{0.2(9.8)}\right)\right]$$

$$\boxed{= 16.4\ m}$$

Torricelli's Law



Suppose that a water tank has a ***hole with area a*** at its bottom, from which water is leaking. Denote by $y(t)$ the depth of water in the tank at time *t*, and by $V(t)$ the volume of water in the tank then. It is plausible – and true, under ideal conditions – that the velocity of water exiting through the hole is

$$v = \sqrt{2gy}$$

Which is the velocity a drop of water would acquire in falling freely from the surface of the water to the hole. One can derive this formula beginning with the assumption that the sum of the kinetic and potential energy of the system remains constant. Under real conditions, taking into account the construction of a water jet from an orifice, $v = c\sqrt{2gy}$, where *c* is an empirical constant between 0 and 1 (usually about 0.6 for a small continuous stream of water). For simplicity we take $c = 1$ in the following discussion.

$$\frac{dV}{dt} = -av = -a\sqrt{2gy}$$

$$\frac{dV}{dt} = -k\sqrt{y} \qquad \text{where} \quad k = a\sqrt{2g}$$

This is a statement of *Torricelli's* law for a draining tank.

Let $A(y)$ denote the horizontal cross-sectional area of the tank at height *y*. Then, applied to a thin horizontal slice of water at height $\bar{y}$ with area $A(\bar{y})$ and thickness $d\bar{y}$, the integral method of cross sections gives

$$V(y) = \int_0^y A(\bar{y})\,d\bar{y}$$

The fundamental theorem of calculus therefore implies that $\frac{dV}{dy} = A(y)$ and hence that

$$\frac{dV}{dt} = \frac{dV}{dy} \cdot \frac{dy}{dt} = A(y)\frac{dy}{dt}$$

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy}$$

$$= -k\sqrt{y} \qquad \text{(An alternative form of } \textit{Torricelli's} \text{ law)}$$

Example

A tank is shaped like a vertical cylinder; it initially contains water to a depth of 9 *ft*, and a bottom plug is removed at time $t = 0$ (hours). After 1 *hr*. the depth of the water has dropped to 4 *ft*. how long does it take for all the water to drain from the tank?

Solution

$$\frac{dy}{dt} = k\sqrt{y}$$

$$\frac{dy}{y^{1/2}} = kdt$$

$$\int y^{-1/2}dy = \int kdt$$

$$2y^{1/2} = kt + C$$

With initial condition $y(0) = 9$

$$2\sqrt{9} = k(0) + C$$

$$\underline{C = 6}$$

$$2\sqrt{y} = kt + 6$$

$$y(1) = 4$$

$$2\sqrt{4} = k(1) + 6$$

$$k = 6 - 4 = -2$$

$$2\sqrt{y} = -2t + 6$$

$$\sqrt{y} = 3 - t$$

$$y(t) = (3 - t)^2$$

It will take 3 hours for the tank to empty.

Exercises *Section* 1.3 – Models of Motions

1. A stone is released from rest and dropped into a deep well. Eight seconds later, the sound of the stone splashing into the water at the bottom of the well returns to the ear of the person who released the stone. How long does it take the stone to drop to the bottom of the well? How deep is the well? Ignore air resistance. The speed of sound is 340 *m/s*.

2. A rocket is fired vertically and ascends with constant acceleration $a = 100\ m/s^2$ for 1.0 *min*. At that point, the rocket motor shuts off and the rocket continues upward under the influence of gravity. Find the maximum altitude acquired by the rocket and the total time elapsed from the take-off until the rocket returns to the earth. *Ignore air resistance*.

3. A ball having mass $m = 0.1\ kg$ falls from rest under the influence of gravity in a medium that provides a resistance that is proportional to its velocity. For a velocity of $0.2\ m/s$ the force due to the resistance of the medium is −1 N. Find the terminal velocity of the ball.

1 N is the force required to accelerate a 1 kg mass at a rate of $1\ m/s^2$: $1N = 1\ \text{kg} \cdot m/s^2$

4. A ball is projected vertically upward with initial velocity v_0 from ground level. Ignore air resistance.

a) What is the maximum height acquired by the ball?

b) How long does it take the ball to reach its maximum height? How long does it take the ball to return to the ground? Are these times identical?

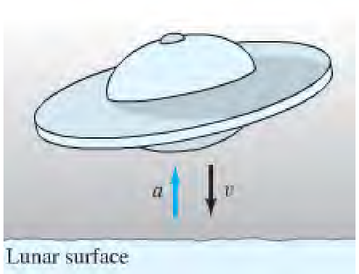
c) What is the speed of the ball when it impacts with the ground on its return?

5. An object having mass 70 kg falls from rest under the influence of gravity. The terminal velocity of the object is $-20\ m/s$. Assume that the air resistance is proportional to the velocity.

a) Find the velocity and distance traveled at the end of 2 seconds.

b) How long does it take the object to reach 80% of its terminal velocity?

6. A lunar lander is falling freely toward the surface of the moon at a speed of 450 m/s. Its retrorockets, when fired, provide a constant deceleration of 2.5 m/s^2 (the gravitational acceleration produced by the moon is assumed to be included in the given acceleration). At What height above the lunar surface should the retrorockets be activated to ensure a "soft touchdown? (v = 0 at impact)?

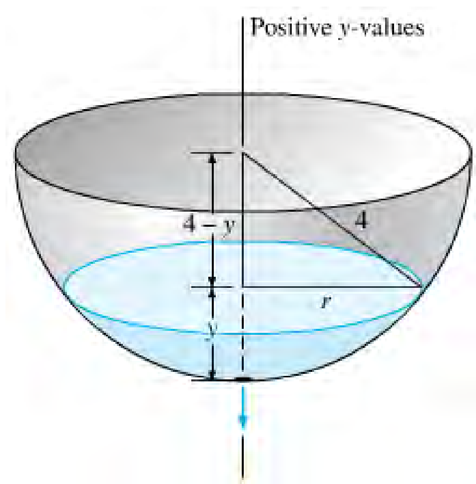


7. A body falling in a relatively dense fluid, oil for example, is acted on by three forces: a resistance force *R,* a buoyant force *B,* and its weight *w* due to gravity. The buoyant force is equal to the weight of the fluid displaced by the object. For a slowly moving spherical body of radius a, the resistive force is given by Stokes's law $R = 6\pi \frac{\mu a}{v}$, where v is the velocity of the body, and μ is the coefficient of viscosity of the surrounding fluid?

a) Find the limiting velocity of a solid sphere of radius *a* and density ρ falling freely in a medium of density ρ' and coefficient of viscosity μ.

b) In 1910 R. A. Millikan studied the motion of tiny droplets of oil falling in an electric field. A field of strength *E* exerts a force E_e on a droplet with charge *e*. Assume that *E* has been adjusted so the droplet is held stationary $(v = 0)$ and that *w* and *B* are as given. Find an expression for *e*.

8. A hemispherical bowl has top radius of 4 *ft*. and at time *t* = 0 is full of water. At that moment a circular hole with diameter 1 *in*. is opened in the bottom of the tank. How long will it take for all the water to drain from the tank?

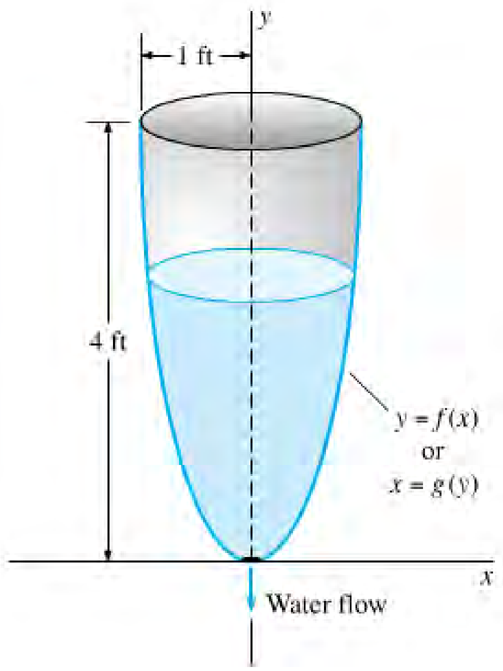


9. Suppose that the tank has a radius of 3 ft. and that its bottom hole is circular with radius 1 in. How long will it take the water (initially 9 ft. deep) to drain completely?

10. At time $t = 0$ the bottom plug (at the vertex) of a full conical water tank 16 ft. high is removed. After 1 hr the water in the tank is 9 ft. deep. When will the tank be empty?

11. Suppose that a cylindrical tank initially containing V_0 gallons of water drains (through a bottom hole) in T minutes. Use Torricelli's law to show that the volume of water in the tank after $t \le T$ minutes is $V = V_0\left(1 - \frac{t}{T}\right)^2$

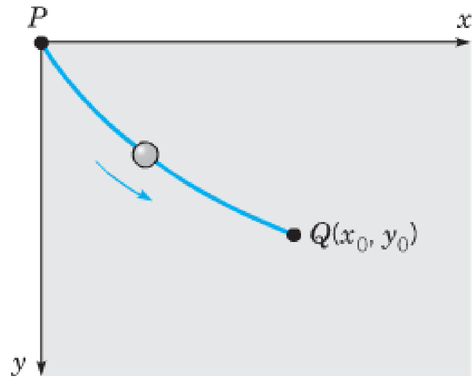
12. The clepsydra, or water clock – A 12-*hr* water clock is to be designed with the dimensions, shaped like the surface obtained by revolving the curve $y = f(x)$ around the y-axis. What should be this curve, and what should be the radius of the circular bottom hole, in order that the water level will fall at the constant rate of 4 inches per hour?



13. One of the famous problems in the history of mathematics is the brachistochrone problem: to find the curve along which a particle will slide without friction in the minimum time from one given point P to another point Q, the second point being lower than the first but not directly beneath it. This problem was posed by Johann Bernoulli in 1696 as a challenge problem to the mathematicians of his day. Correct solutions were found by Johann Bernoulli and his brother Jakob Bernoulli and by Isaac Newton, Gottfried Leibniz, and the Marquis de L'Hospital. The brachistochrone problem is important in the development of mathematics as one of the forerunners of the calculus of variations.
In solving this problem, it is convenient to take the origin as the upper point P and to orient the axes as shown. The lower point Q has coordinates (x_0, y_0). It is then possible to show that the curve of minimum time is given by a function $y = \phi(x)$ that satisfies the differential equation

$$\left(1 + y'^2\right)y = k^2 \qquad (eq.\ i)$$

Where k^2 is a certain positive constant to be determined later



a) Solve the equation $(eq.\ i)$ for y' . Why is it necessary to choose the positive square root?

b) Introduce the new variable t by the relation

$$y = k^2 \sin^2 t \qquad (eq.\ ii)$$

Show that the equation found in part (a) then takes the form

$$k^2 \sin^2 t \ dt = dx \qquad (eq.\ iii)$$

c) Letting $\theta = 2t$, show that the solution of $(eq.\ iii)$ for which $x = 0$ when $y = 0$ is given by

$$x = k^2 \frac{\theta - \sin\theta}{2}, \quad y = k^2 \frac{1 - \cos\theta}{2} \qquad (eq.\ iv)$$

Equations (*iv*) are parametric equations of the solution of (eq. *i*) that passes through (0, 0). The graph of Eqs. (*iv*) is called a cycloid.

d) If we make a proper choice of the constant *k*, then the cycloid also passes through the point (x_0, y_0) and is the solution of the brachistochrone problem. Find *k* if $x_0 = 1$ and $y_0 = 2$

Section 1.4 – Linear Equations

A first order linear equation is given by the form:

$$y' + p(x)y = f(x)$$

If $f(x) = 0 \to y' = p(x)y$. This linear equation is said to be ***homogeneous***. (Otherwise it is ***nonhomogeneous or inhomogeneous***).

$p(x)$ & $f(x)$ are called the coefficients

Linear	***Non-linear***
$x' = \sin(t)x$	$x' = t\sin x$
$y' = e^{2t}y + \cos t$	$y' = 1 - y^2$
$x' = (3t+2)x + t^2 - 1$	

Solution of the homogenous equation

$$\frac{dx}{dt} = a(t)x \quad \Rightarrow \quad \frac{dx}{x} = a(t)dt$$

$$\int \frac{dx}{x} = \int a(t)dt$$

$$\ln|x| = \int a(t)dt + C$$

Convert to exponential form

$$|x| = e^{\int a(t)dt + C} = e^C e^{\int a(t)dt}$$

Let $A = e^C$

$$\boxed{x(t) = A.e^{\int a(t)dt}}$$

Example

Solve: $x' = \sin(t)\, x$

Solution

$$\frac{dx}{dt} = \sin(t)\, x$$

$$\frac{dx}{x} = \sin(t)\, dt$$

$$\int \frac{dx}{x} = \int \sin(t)\, dt$$

$$\ln|x| = \int \sin(t)dt + C$$

$$\ln|x| = -\cos(t) + C$$

$$x = e^{-\cos(t) + C}$$

$$x(t) = A.e^{\int \sin(t)dt}$$

$$\boxed{= A.e^{-\cos t}}$$

Solving a linear first-order Equation (*Properties*)

1. Put a linear equation into a standard form $y' + p(x)y = f(x)$
2. Identify $p(x)$ then find $y_h = e^{-\int p dx}$
3. Multiply the standard form by y_h
4. Integrate both sides

Solution of the Inhomogeneous Equation $u(t) = e^{-\int a(t)dt}$

$$x' = a(t)x + f(t)$$

$$x' - ax = f$$

$$(ux)' = u\left(x' - ax\right) = uf$$

$$u(t)x(t) = \int u(t)f(t)dt + C$$

Example

Find the general solution to: $x' = x + e^{-t}$

Solution

$$x' - x = e^{-t}$$

$$e^{-\int 1dt} = e^{-t}$$

$$e^{-t}\left(x' - x\right) = e^{-t}e^{-t}$$

$$\left(e^{-t}x\right)' = e^{-2t}$$

$$e^{-t}x(t) = \int e^{-2t}dt$$

$$e^{-t}x(t) = -\frac{1}{2}e^{-2t} + C$$

$$\boxed{x(t) = -\frac{1}{2}e^{-t} + Ce^{t}}$$

Solution of the Nonhomogeneous Equation $\quad y' + p(x)y = f(x)$

Let assume: $\quad y = y_h + y_p \quad$ *where* $\begin{cases} y_h & \text{Homogeneous Solution} \\ y_p & \text{Paticular Solution} \end{cases}$

The homogeneous equation is given by $\quad y'_h + p(x)y_h = 0$

$$y'_h = -p(x)y_h$$

$$\boxed{y_h = e^{-\int p\,dx}}$$

$$y_p = u(x)y_h = u.e^{-\int p\,dx}$$

$$y'_p + p(x)y_p = f(x)$$

$$\left(uy_h\right)' + puy_h = f$$

$$u'y_h + uy'_h + puy_h = f$$

$$u'y_h + u\left(y'_h + py_h\right) = f \qquad \textit{Since } y'_h + py_h = 0 \qquad \textit{homogeneous}$$

$$u'y_h = f$$

$$\frac{du}{dx} = \frac{f}{y_h}$$

$$du = \frac{f}{e^{-\int p\,dx}}dx$$

$$= f.e^{\int p\,dx}dx$$

$$u = \int f.e^{\int p\,dx}dx$$

$$y_p = u.e^{-\int p\,dx}$$

$$u = \left(\int f.e^{\int p\,dx}dx\right)e^{-\int p\,dx}$$

$$\boxed{y_p = e^{-\int p\,dx}\int f.e^{\int p\,dx}dx}$$

$$y = y_h + y_p$$

$$\boxed{y = Ce^{-\int p\,dx} + e^{-\int p\,dx}\int f.e^{\int p\,dx}dx} \qquad \boxed{y = e^{-\int p\,dx}\left(C + \int f.e^{\int p\,dx}dx\right)}$$

Example

Find the general solution of $x' = x\sin t + 2te^{-\cos t}$ and the particular solution that satisfies $x(0) = 1$.

Solution

$$x' - x\sin t = 2te^{-\cos t} \qquad P(t) = \sin t, \quad Q(t) = 2te^{-\cos t}$$

$$x_h = e^{-\int \sin t\,dt} = e^{\cos t}$$

$$\int Q(t)x_h\,dt = \int 2te^{-\cos t}e^{\cos t}\,dt = \int 2t\,dt = t^2$$

$$x(t) = e^{-\cos t}\left(t^2 + C\right) \qquad x = \frac{1}{e^{\int P\,dt}}\left(\int Q.e^{\int P\,dt}\,dt + C\right)$$

$$x(0) = \left((0)^2 + C\right)e^{-\cos 0} = 1$$

$$Ce^{-1} = 1$$

$$C = e$$

$$\boxed{x(t) = \left(t^2 + e\right)e^{-\cos t}}$$

Example

Find the general solution of $x' = x\tan t + \sin t$ and the particular solution that satisfies $x(0) = 2$..

Solution

$$x' - (\tan t)x = \sin t \qquad P(t) = -\tan t, \quad Q(t) = \sin t$$

$$e^{-\int \tan t\,dt} = e^{\ln(\cos t)} \underline{= \cos t|}$$

$$\int (\sin t)(\cos t)\;dt = -\int \cos t\;d(\cos t) = -\frac{1}{2}\cos^2 t$$

$$x(t) = \frac{1}{\cos t}\left(-\frac{1}{2}\cos^2 t + C\right) = -\frac{1}{2}\cos t + \frac{1}{\cos t}C$$

$$\boxed{= -\frac{1}{2}\cos t + \frac{1}{\cos t}C}$$

$$x(0) = -\frac{1}{2}\cos(0) + \frac{C}{\cos(0)} = 2$$

$$-\frac{1}{2} + C = 2 \quad \Rightarrow \quad C = \frac{5}{2}$$

$$\boxed{x(t) = -\frac{1}{2}\cos t + \frac{5}{2\cos t}}$$

Notes

1. Integrating an expression that is not the differential of any elementary function is called non-elementary.

$$\int e^{x^2}\,dx \qquad \int x\tan x\,dx \qquad \int \frac{e^{-x}}{x}\,dx$$

$$\int \sin x^2\,dx \qquad \int \cos x^2\,dx \qquad \int \frac{\sin x}{x}\,dx \qquad \int \frac{\cos x}{x}\,dx$$

2. In math some important functions are defined in terms of non-elementary integrals. Two such functions are the error function and the complementary error function.

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \qquad erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt$$

Exercises *Section* 1.4 – Linear Equations

Find the general solution of the first-order, linear equation.

1. $y' - 3y = 5$
2. $y' + 2ty = 5t$
3. $x' - 2\frac{x}{t+1} = (t+1)^2$
4. $(1+x)y' + y = \cos x$
5. $y' - y = 3e^t$
6. $y' + y = \sin t$
7. $y' + y = \frac{1}{1+e^t}$
8. $y' + 3t^2 y = t^2$
9. $(\cos t)y' + (\sin t)y = 1$
10. $y^2 + (y')^2 = 1$
11. $y' + \frac{3}{t}y = \frac{\sin t}{t^3}, \quad (t \neq 0)$
12. $\left(1+e^t\right)y' + e^t y = 0$
13. $\left(t^2+9\right)y' + ty = 0$
14. $y' + y = \frac{1}{1+e^t}$
15. $\frac{dy}{dt} - 2y = 4 - t$
16. $\left(1+x^3\right)y' = 3x^2 y + x^2 + x^5$
17. $y' = \cos x - y\sec x$

Solve the differential equations

18. $x\frac{dy}{dx} + y = e^x, \quad x > 0$
19. $y' + (\tan x)y = \cos^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$
20. $x\frac{dy}{dx} + 2y = 1 - \frac{1}{x}, \quad x > 0$
21. $(1+x)y' + y = \sqrt{x}$
22. $e^{2x}y' + 2e^{2x}y = 2x$
23. $(t+1)\frac{ds}{dt} + 2s = 3(t+1) + \frac{1}{(t+1)^2}, \quad t > -1$
24. $\tan\theta\frac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \frac{\pi}{2}$

Find the solution of the initial value problem

25. $y' - 3y = 4\ ; \quad y(0) = 2$
26. $y' = y + 2xe^{2x}; \qquad y(0) = 3$
27. $\left(x^2+1\right)y' + 3xy = 6x; \qquad y(0) = -1$
28. $t\frac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(2) = 1$
29. $\theta\frac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0, \quad y\left(\frac{\pi}{2}\right) = 1$
30. $\frac{dy}{dx} + xy = x, \quad y(0) = -6$
31. $ty' + 2y = 4t^2, \quad y(1) = 2$
32. $\left(1+t^2\right)y' + 4ty = \left(1+t^2\right)^{-2}, \quad y(1) = 0$

Find the solution of the initial value problem. Discuss the interval of existence and provide a sketch of your solution

33. $xy' + 2y = \sin x; \quad y\left(\frac{\pi}{2}\right) = 0$
34. $(2x+3)y' = y + (2x+3)^{1/2}; \qquad y(-1) = 0$

Find the general solution of the given differential equation. Then find the particular solution satisfying the given initial condition of

35. $y' - 3y = 4, \quad y(0) = 2$

36. $y' + \frac{1}{2}y = t, \quad y(0) = 1$

37. $y' + y = e^t, \quad y(0) = 1$

Section 1.5 – Mixing Problems

The physical representation of the rate of change:

$$\frac{dx}{dt} = \text{rate of change} = \text{rate in - rate out}$$

This is referred to as a ***balance law***.

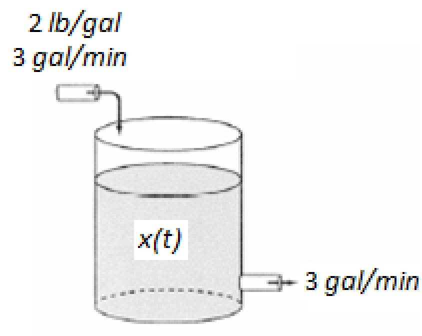
Rate = Volume Rate (*gal/min*) *x* Concentration (*lb/gal*)

Example

The tank initially holds 100 gal of pure water. At time $t = 0$, a solution containing 2 *lb* of salt per *gallon* begins to enter the tank at the rate of 3 *gallons* per *minute*. At the same time a drain is opened at the bottom of the tank so that the volume of solution in the tank remains constant.
How much salt is in the tank after 60 *min*?
What will be the eventual salt content in the tank?

Solution

$x(t)$: number of pounds of salt in the tank after t min.



Volume: $V(t) = 100 + (3-3)t = 100$

Concentration at time t: $c(t) = \frac{x(t)}{V(t)} = \frac{x(t)}{100} \ \ lb\,/\,gal$

Rate in = Volume Rate *x* Concentration

$$= 3\frac{gal}{min} \times 2\frac{lb}{gal}$$

$$= 6 \ \ lb\,/\,\text{min}$$

Rate out = Volume Rate *x* Concentration

$$= 3\frac{gal}{\text{min}} \times \frac{x(t)}{100}\frac{lb}{gal}$$

$$= \frac{3x(t)}{100} \ \ lb\,/\,\text{min}$$

$$\frac{dx}{dt} = \text{rate of change}$$

$$= \text{rate in} - \text{rate out}$$

$$= 6 - \frac{3x}{100}$$

$$\frac{dx}{dt} + \frac{3}{100}x = 6$$

$$u(t) = e^{\int\left(\frac{3}{100}\right)dt} = e^{0.03t}$$

$$\int 6e^{0.03t}\,dt = \frac{6}{0.03}e^{0.03t} = 200e^{0.03t}$$

$$x(t) = e^{-0.03t}\left(200e^{0.03t} + C\right)$$

$$\boxed{x(t) = 200 + Ce^{-0.03t}}$$

Since there was no salt present in the tank initially, the initial condition is $x(0) = 0$

$$x(t = 0) = 200 + Ce^{-0.03(0)} = 0$$

$$200 + C = 0$$

$$C = -200$$

$$\boxed{x(t) = 200 - 200e^{-0.03t}}$$

After 60 min:

$$x(60) = 200 - 200e^{-0.03(60)}$$

$$\boxed{\approx 167\ lb}$$

As $t \to \infty$ then $x(t) = \lim_{t\to\infty}\left(200 - 200e^{-0.03t}\right)$

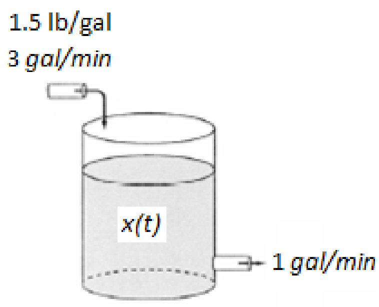
$$= 200 - 200\lim_{t\to\infty}\left(e^{-0.03t}\right) \qquad \lim_{t\to\infty}\left(e^{-0.03t}\right) = e^{-\infty} = 0$$

$$\boxed{= 200\ \ lb}$$

Example

The 600-*gal* tank is filled with 300 *gal* of pure water. A spigot is opened above the tank and a salt solution containing 1.5 *lb*. of salt per gallon of solution begins flowing into the tank at the rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave tank at a rate of 1 *gal/min*. What will be the salt content in the tank at the precise moment that the volume of solution in the tank is equal to the tank's capacity (600 *gal*)?

Solution



$$\begin{aligned} V(t) &= 300 + (3-1)t \\ &= 300 + 2t \end{aligned}$$

$$c(t) = \frac{x(t)}{300+2t}$$

$$\begin{aligned} \text{Rate in } &= 3\frac{gal}{min} \times 1.5\frac{lb}{gal} \\ &= 4.5 \ \ lb\,/\,min \end{aligned}$$

$$\begin{aligned} \text{Rate out } &= 1 \times \frac{x}{300+2t} \\ &= \frac{x}{300+2t} \ \ lb\,/\,\text{min} \end{aligned}$$

$$\frac{dx}{dt} = 4.5 - \frac{x}{300+2t}$$

$$\frac{dx}{dt} + \frac{1}{300+2t}x = 4.5$$

$$\begin{aligned} u(t) &= e^{\int \frac{1}{300+2t}dt} \qquad\qquad d(300+2t) = 2dt \\ &= e^{\frac{1}{2}\int \frac{1}{300+2t}d(300+2t)} \\ &= e^{\frac{1}{2}\ln(300+2t)} \\ &= e^{\ln(300+2t)^{1/2}} \\ &= \sqrt{300+2t} \end{aligned}$$

$$\int 4.5\sqrt{300+2t}\ dt = 4.5\frac{1}{2}\frac{2}{3}(300+2t)^{2/3}$$

$$\begin{aligned} x(t) &= \frac{1}{\sqrt{300+2t}}\left(1.5(300+2t)^{3/2} + C\right) \\ &= 1.5(300+2t) + \frac{C}{\sqrt{300+2t}} \\ &= 450 + 3t + \frac{C}{\sqrt{300+2t}} \end{aligned}$$

$$x(0) = 450 + 3(0) + \frac{C}{\sqrt{300+2(0)}} = 0$$

$$450+\frac{C}{\sqrt{300}}=0$$

$$C=-450\sqrt{300}$$

$$=-4500\sqrt{3}$$

$$x(t)=450+3t-\frac{4500\sqrt{3}}{\sqrt{300+2t}}$$

$$V=300+2t=600$$

$$t=150\ min$$

$$x(t=150)=450+3(150)-\frac{4500\sqrt{3}}{\sqrt{300+2(150)}}$$

$$\underline{\approx 582\ lb}$$

Exercises *Section* 1.5 – Mixing Problems

1. Consider two tanks, label tank A and tank B for reference. Tank A contains 100 *gal* of solution in which is dissolved 20 *lb* of salt. Tank B contains 200 *gal* of solution which is dissolved 40 *lb* of salt. Pure water flows into the tank A at rate of 5 *gal/s*. There is a drain at the bottom of tank A. The solution leaves tank A via the drain at a rate of 5 *gal/s* and flows immediately into tank B at the same rate. A drain at the bottom of tank B allows the solution to leave tank B at a rate of 2.5 *gal/s*. What is the salt content in tank B at the precise moment that tank B contains 250 *gal* of solution?

2. A tank contains 100 *gal* of pure water. At time zero, a sugar-water solution containing 0.2 *lb* of sugar per gal enters the tank at a rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the sugar solution to leave the tank at 3 *gal/min*. Assume that the solution in the tank is kept perfectly mixed at all times.

a) What will be the sugar content in the tank after 20 minutes?

b) How long will it take the sugar content in the tank to reach 15 *lb*?

c) What will be the eventual sugar content in the tank?

3. A tank initially contains 50 *gal* of sugar water having a concentration of 2 *lb*. of sugar for each gal of water. At time zero, pure water begins pouring into the tank at a rate of 2 *gal* per *minute*. Simultaneously, a drain is opened at the bottom of the tank so that the volume of sugar-water solution in the tank remains constant.

a) How much sugar is in the tank after 10 minutes?

b) How long will it take the sugar content in the tank to dip below 20 *lb*.?

c) What will be the eventual sugar content in the tank?

4. A 50-*gal* tank initially contains 20 *gal* of pure water. Salt-water solution containing 0.5 *lb.* of salt for each gallon of water begins entering the tank at a rate of 4 *gal/min.* simultaneously; a drain is opened at the bottom of the tank, allowing the salt-water solution to leave the tank at a rate of 2 *gal/min*. What is the salt content (*lb*) in the tank at the precise moment that the tank is full of salt-water solution?

5. A tank contains 500 *gal* of a salt-water solution containing 0.05 *lb* of salt per gallon of water. Pure water is poured into the tank and a drain at the bottom of the tank is adjusted so as to keep the volume of solution in the tank constant. At what rate (gal/min) should the water be poured into the tank to lower the salt concentration to 0.01 lb/gal of water in less than one hour?

6. A tank contains 100 *gal* of fresh water. A solution containing 1 *lb./gal* of soluble lawn fertilizer runs into the tank at the rate of 1 gal/min, and the mixture is pumped out of the tank at a rate of 3 *gal/min*. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.

7. A 200-*gal* tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 *lb./gal* of concentrate enters the tank at the rate of 5 *gal/min*, and the well-stirred mixture is withdrawn at the rate of 3 *gal/min*.

a) At what time will the tank be full?

b) At the time the tank is full, how many pounds of concentrate will it contain?

8. Suppose that an Iowa class battleship has mass 51,000 metric tons (51,000,000 *kg*) and $k \approx 59{,}000\ kg/\sec$. Assume that the ship loses power when it is moving at a speed of 9 *m/sec*.

a) About how far will the ship coast before it is dead in the water?

b) About how long will it take the ship's speed to drop to 1 m/sec?

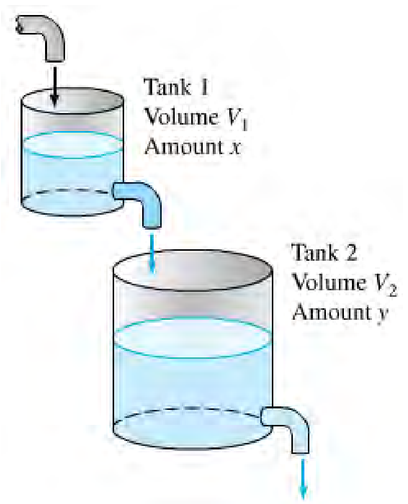
9. A 66-*kg* cyclist on a 7-*kg* bicycle starts coasting on level ground at 9 m/sec. The $k \approx 3.9\ kg/\sec$

a) About how far will the cyclist coast before reaching a complete stop?

b) How long will it take the cyclist's speed to drop to 1 *m/sec*?

10. An Executive conference room of a corporation contains 4500 ft^3 of air initially free of carbon monoxide. Starting at time $t = 0$, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 $ft^3/\min$. A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 $ft^3/\min$. Find the time when the concentration of carbon monoxide in the room reaches 0.01%.

11. Consider the cascade of 2 tanks with $V_1 = 100\ gal$ and $V_2 = 200\ gal$ the volumes of brine in the 2 tanks. Each tank also initially contains 50 lb. of salt. The three flow rates indicated in the figure are each 5 gal/min, with pure water flowing into tank.



a) Find the amount $x(t)$ of salt in tank 1 at time *t*.

b) Suppose that $y(t)$ is the amount of salt in tank 2 at time *t*. Show first that

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200}$$

And then solve for $y(t)$, using the function $x(t)$ found in part (*a*).

c) Finally, find the maximum amount of salt ever in tank 2.

12. Suppose that in the cascade tank 1 initially 100 *gal* of pure ethanol and tank 2 initially contains 100 *gal* of pure water. Pure water flows into tank 1 at 10 *gal/min*, and the other two flow rates are also 10 *gal/min*.

a) Find the amounts $x(t)$ and $y(t)$ of ethanol in the two tanks at time $t \geq 0$.

b) Find the maximum amount of ethanol ever in tank 2.

13. A multiple cascade is shown in the figure. At time t = 0, tank 0 contains 1 *gal* of ethanol and 1 *gal* of water; all the remaining tanks contain 2 *gal* of pure water each. Pure water is pumped into tank 0 at 1 *gal/.min*, and the varying mixture in each tank is pumped into the one below it at the same rate. Assume, as usual, that the mixtures are kept perfectly uniform by stirring. Let $x_n(t)$ denote the amount of ethanol in tank n at time t.

a) Show that $x_0(t) = e^{-t/2}$

b) Show that the maximum value of $x_n(t)$ for $n > 0$ is $M_n = x_n(2n) = \dfrac{n^n e^{-n}}{n!}$

14. Assume that Lake Erie has a volume of 480 km^3 and that its rate of inflow (from Lake Huron) and outflow (to Lake Ontario) are both 350 km^3 per year. Suppose that at the time t = 0 (years), the pollutant concentration of Lake Erie – caused by past industrial pollution that has now been ordered to cease – is 5 times that of Lake Huron. If the outflow henceforth is perfectly mixed lake water, how long will it take to reduce the pollution concentration in Lake Erie to twice that of Lake Huron?

15. A 120 *gal* tank initially contains 90 *lb*. of salt dissolved in 90 *gal* of water. Brine containing 2 *lb./gal* of salt flows into the tank at rate of 4 *gal/min*, and the well-stirred mixture flows out the tank at the rate of 3 *gal/min*. How much salt does the tank contain when it is full?

Section 1.6 – Exact Differential Equations

A class of equations known as exact equations for which there is also a well-defined method of solution

Theorem

Let the function M, N, M_y *and* N_x, where M_y *and* N_x are partial derivatives, be continuous in the rectangular region $R: \alpha < x < \beta, \ \gamma < y < \delta$ then

$$M(x, y) + N(x, y)y' = 0$$

Is an exact differential equation in R, *iff* $\quad M_y(x, y) = N_x(x, y)$

At each point in R. That is, there exists a function ψ satisfying

$$\psi_x(x, y) = M(x, y) \quad \text{and} \quad \psi_y(x, y) = N(x, y) \qquad \underline{\text{Iff}}\ M_y(x, y) = N_x(x, y)$$

$$\psi(x, y) = \int M(x, y)\,dx$$

Example

Solve the differential equation: $\quad 2x + y^2 + 2xyy' = 0$

Solution

$$\frac{\partial\psi}{\partial x} = M = 2x + y^2 \quad \Rightarrow \quad M_y = 2y$$

$$\frac{\partial\psi}{\partial y} = N = 2xy \quad \Rightarrow \quad N_x = 2y$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial\psi}{\partial x} = 2x + y^2 \quad \Rightarrow \quad \psi = \int\left(2x + y^2\right)dx = x^2 + xy^2 + h(y)$$

$$\psi_y = 2xy + h'(y) = 2xy \quad \Rightarrow \quad h'(y) = 0$$

$$\Rightarrow \ h(y) = C$$

$$\psi(x, y) = x^2 + xy^2 + C$$

$$\boxed{x^2 + xy^2 = C}$$

Example

Solve the differential equation: $y\cos x + 2xe^{y} + \left(\sin x + x^2 e^{y} - 1\right)y' = 0$

Solution

$$M = y\cos x + 2xe^{y} = \frac{\partial\psi}{\partial x} \quad \Rightarrow \quad M_y = \cos x + 2xe^{y}$$

$$\frac{\partial\psi}{\partial y} = N = \sin x + x^2 e^{y} - 1 \quad \Rightarrow \quad N_x = \cos x + 2xe^{y}$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int\left(y\cos x + 2xe^{y}\right)dx = y\sin x + x^2 e^{y} + h(y)$$

$$\psi_y = \sin x + x^2 e^{y} + h'(y) = \sin x + x^2 e^{y} - 1 \quad \Rightarrow \quad h'(y) = -1$$

$$\Rightarrow h(y) = -y$$

$$\psi(x, y) = y\sin x + x^2 e^{y} - y = C$$

$$\boxed{y\sin x + x^2 e^{y} - y = C}$$

Example

Solve the differential equation: $3xy + y^2 + \left(x^2 + xy\right)y' = 0$

Solution

$$M = 3xy + y^2 = \frac{\partial\psi}{\partial x} \quad \Rightarrow \quad M_y = 3x + 2y$$

$$N = x^2 + xy = \frac{\partial\psi}{\partial y} \quad \Rightarrow \quad N_x = 2x + y$$

$$\Rightarrow M_y \neq N_x$$

Can be solved by this procedure.

Integrating Factors

It is sometimes possible to convert a differential equation that is not exact equation by multiplying the equation by a suitable integrating factor.

Definition

An integrating factor for the differential equation $\omega = Mdx + Ndy = 0$ is a function $\mu(x, y)$ such that the form $\mu\omega = \mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy$ is exact.

$$(\mu M)_y = (\mu N)_x$$

$$M\mu_y - N\mu_x + \left(M_y - N_x\right)\mu = 0$$

Assuming that μ is a function of x only, we have

$$(\mu M)_y = \mu M_y \quad \& \quad (\mu N)_x = \mu N_x + N\frac{d\mu}{dx}$$

$$\Rightarrow \quad \mu M_y = \mu N_x + N\frac{d\mu}{dx}$$

$$\boxed{\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu}$$

Example

Find an integrating factor for the equation $\left(3xy + y^2\right) + \left(x^2 + xy\right)y' = 0$

And then solve the equation.

Solution

$$M_y = \frac{\partial}{\partial y}\left(3xy + y^2\right) = 3x + 2y \qquad N_x = \frac{\partial}{\partial x}\left(x^2 + xy\right) = 2x + y$$

$$\Rightarrow M_y \neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{3x + 2y - 2x - y}{x^2 + xy} = \frac{x + y}{x(x + y)} = \frac{1}{x}$$

$$\frac{d\mu}{dx} = \frac{\mu}{x} \quad \Rightarrow \quad \int \frac{d\mu}{\mu} = \int \frac{dx}{x}$$

$$\ln \mu = \ln x$$

$$\mu = x$$

$$x\left(3xy+y^2\right)+x\left(x^2+xy\right)y'=0$$

$$M_y=\frac{\partial}{\partial y}\left(3x^2y+xy^2\right)=3x^2+2xy \qquad N_x=\frac{\partial}{\partial x}\left(x^3+x^2y\right)=3x^2+2xy$$

$$\Rightarrow M_y=N_x$$

$$\psi=\int\left(3x^2y+xy^2\right)dx=x^3y+\tfrac{1}{2}x^2y^2+h(y)$$

$$\psi_y=x^3+x^2y+h'(y)=x^3+x^2y \quad\Rightarrow\quad h'(y)=0$$

$$\Rightarrow h(y)=C$$

$$\psi(x,y)=x^3y+\tfrac{1}{2}x^2y^2=C$$

$$\boxed{x^3y+\tfrac{1}{2}x^2y^2=C}$$

Bernoulli Equations

An equation of the form $y'+P(x)y=Q(x)y^n, \quad n\neq 0,\,1$ is called a ***Bernoulli equation***.

If $n=0 \quad\Rightarrow\quad y'+Py=Q$ First–order linear differential equation

If $n=1 \quad\Rightarrow\quad y'+Py=Qy \ \rightarrow\ y'+(P-Q)y=0$ Separable equation.

For $n\neq 0,\,1$, the Bernoulli equation can be written as $y^{-n}\frac{dy}{dx}+Py^{1-n}=Q \qquad (1)$

Let $u=y^{1-n} \quad\Rightarrow\quad \frac{du}{dx}=(1-n)y^{-n}\frac{dy}{dx}$

$$y^{-n}\frac{dy}{dx}=\frac{1}{1-n}\frac{du}{dx}$$

$$(1)\quad\Rightarrow\quad \frac{1}{1-n}\frac{du}{dx}+Pu=Q$$

$\underline{u'+(1-n)Pu=(1-n)Q}$ Which is 1^{st}–order linear differential equation.

Example

Find the general solution $y'-4y=2e^x\sqrt{y}$

Solution

$$\sqrt{y}=y^{1/2} \quad\Rightarrow\quad n=\frac{1}{2}$$

Let $u = y^{1-\frac{1}{2}} = y^{1/2} \quad \Rightarrow \quad y = u^2$

$$\frac{du}{dx} = \frac{1}{2} y^{-1/2} \frac{dy}{dx} \quad \Rightarrow \quad 2y^{1/2} \frac{du}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx} - 4y = 2e^x u$$

$$2u\frac{du}{dx} - 4u^2 = 2ue^x \qquad \textit{Divide by } 2u$$

$$u' - 2u = e^x$$

$$e^{\int -2dx} = e^{-2x}$$

$$\int e^x e^{-2x} dx = \int e^{-x} dx = -e^{-x}$$

$$u = \frac{1}{e^{-2x}}\left(-e^{-x} + C\right)$$

$$y^{1/2} = -e^x + Ce^{2x}$$

$$\boxed{y = \left(Ce^{2x} - e^x\right)^2}$$

Example

Find the general solution $xy' + y = 3x^3y^2$

Solution

$$y' + \frac{1}{x}y = 3x^2y^2$$

Let $u = y^{1-2} = y^{-1} \quad \Rightarrow \quad y = \frac{1}{u}$

$$\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \quad \Rightarrow \quad y' = -y^2u' = -\frac{1}{u^2}u'$$

$$-\frac{1}{u^2}u' + \frac{1}{x}\frac{1}{u} = 3x^2\frac{1}{u^2} \qquad \textit{Multiply both sides by } -u^2$$

$$u' - \frac{1}{x}u' = -3x^2$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$\int -3x^2x^{-1}dx = -3\int xdx = -\frac{3}{2}x^2$$

$$u = x\left(-\frac{3}{2}x^2 + C_1\right)$$

$$\frac{1}{y} = \frac{-3x^3 + 2C_1x}{2}$$

$$\boxed{y = \frac{2}{Cx - 3x^3}}$$

Homogeneous Equations $\frac{dy}{dx}=f(x,y)$

The form of a homogeneous equation suggests that it may be simplified by using a variable denoted by 'v', to represent the ratio of y to x. This

$$y=xv \quad \Rightarrow \quad \frac{dy}{dx}=F(v)$$

Let assume that v is a function of x, then

$$\frac{dy}{dx}=x\frac{dv}{dx}+v \quad \Rightarrow \quad F(v)=x\frac{dv}{dx}+v$$

The most significant fact about this equation is that the variables x & v can always be separated, regardless of the form of the function F.

$$\frac{dx}{x}=\frac{dv}{F(v)-v}$$

Solving this equation and then replacing v by $\frac{y}{x}$ gives the solution of the original equation.

Example

Solve the differential equation $\qquad \frac{dy}{dx}=\frac{y^2+2xy}{x^2}$

Solution

$$\frac{dy}{dx}=\left(\frac{y}{x}\right)^2+2\frac{y}{x}=v^2+2v$$

$$x\frac{dv}{dx}+v=v^2+2v \qquad \Rightarrow x\frac{dv}{dx}=v^2+v$$

$$xdv=v(v+1)dx$$

$$\int\frac{dx}{x}=\int\frac{dv}{v(v+1)}$$

$$\int\frac{dx}{x}=\int\left(\frac{1}{v}-\frac{1}{v+1}\right)dv$$

$$\ln x+\ln C=\ln v-\ln(v+1)$$

$$\ln(\mathrm{C}x)=\ln\frac{v}{v+1}$$

$$\mathrm{C}x=\frac{v}{v+1}=\frac{\frac{y}{x}}{\frac{y}{x}+1}=\frac{y}{y+x} \qquad \Rightarrow Cxy+Cx^2=y$$

$$Cx^2=y-Cxy$$

$$\boxed{y=\frac{Cx^2}{1-Cx}}$$

Example

Find the general solution $y' = \dfrac{x^2 e^{y/x} + y^2}{xy}$

Solution

Let $y = xv \quad \Rightarrow \quad y' = v + xv'$

$$v + xv' = \frac{x^2 e^{xv/x} + (xv)^2}{x(xv)}$$

$$xv' = \frac{x^2 e^{v} + x^2 v^2}{x^2 v} - v$$

$$x\frac{dv}{dx} = \frac{e^{v} + v^2}{v} - v$$

$$x\frac{dv}{dx} = \frac{e^{v}}{v}$$

$$\int \frac{v}{e^{v}} dv = \int \frac{dx}{x}$$

$$-ve^{-v} - e^{-v} = \ln x + C$$

$$-e^{-v}(v+1) = \ln x + C$$

$$-e^{-y/x}\left(\frac{y}{x} + 1\right) = \ln x + C$$

$$\underline{y + x = -xe^{y/x}(\ln x + C)}$$

Exercises *Section* 1.6 – Exact Differential Equations

Solve the differential equation

1. $(2x+y)dx+(x-6y)dy=0$
2. $(2x+3)dx+(2y-2)dy=0$
3. $(1-y\sin x)+(\cos x)y'=0$
4. $\dfrac{dy}{dx}=-\dfrac{ax+by}{bx+cy}$
5. $\dfrac{dy}{dx}=\dfrac{3x^2+y}{3y^2-x}$
6. $\left(3x^2-2xy+2\right)+\left(6y^2-x^2+3\right)y'=0$
7. $\left(e^x\sin y-2y\sin x\right)dx+\left(e^x\cos y+2\cos x\right)dy=0$
8. $\left(\dfrac{y}{x}+6x\right)dx+(\ln x-2)dy=0, \qquad x>0$
9. $\dfrac{xdx}{\left(x^2+y^2\right)^{3/2}}+\dfrac{ydy}{\left(x^2+y^2\right)^{3/2}}=0$

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

10. $x^2y^3+x\left(1+y^2\right)y'=0, \qquad \mu(x,\ y)=\dfrac{1}{xy^3}$
11. $y^2-xy+\left(x^2\right)y'=0, \qquad \mu(x,\ y)=\dfrac{1}{xy^2}$
12. $x^2y^3-y+x\left(1+x^2y^2\right)y'=0, \qquad \mu(x,\ y)=\dfrac{1}{xy}$
13. $\left(\dfrac{\sin y}{y}-2e^{-x}\sin x\right)dx+\left(\dfrac{\cos y+2e^{-x}\cos x}{y}\right)dy=0, \qquad \mu(x,\ y)=ye^x$
14. $(x+2)\sin y\,dx+x\cos y\,dy=0, \qquad \mu(x,\ y)=xe^x$
15. $\left(x^2+y^2-x\right)dx-ydy=0, \qquad \mu(x,\ y)=\dfrac{1}{x^2+y^2}$

Find the general solution of each homogenous equation

16. $\left(x^2+y^2\right)dx-2xydy=0$
17. $(x+y)dx+(y-x)dy=0$
18. $\dfrac{dy}{dx}=\dfrac{y\left(x^2+y^2\right)}{xy^2-2x^3}$

Find an integrating factor and solve the given equation

19. $\left(3x^2y+2xy+y^3\right)dx+\left(x^2+y^2\right)dy=0$
20. $dx+\left(\dfrac{x}{y}-\sin y\right)dy=0$
21. $e^xdx+\left(e^x\cot y+2y\csc y\right)dy=0$
22. $\left(3x+\dfrac{6}{y}\right)dx+\left(\dfrac{x^2}{y}+3\dfrac{y}{x}\right)dy=0$

Solve the given initial-value problem

23. $\dfrac{dy}{dx}=\dfrac{xy^2-\cos x\sin x}{y\left(1-x^2\right)},\qquad y(0)=2$

24. $(x+y)^2\,dx+\left(2xy+x^2-1\right)dy,\qquad y(1)=1$

25. $(4y+2x-5)\,dx+(6y+4x-1)\,dy,\qquad y(-1)=2$

26. $\left(e^x+y\right)dx+\left(2+x+ye^y\right)dy,\qquad y(0)=1$

27. $(2x-y)\,dx+(2y-x)\,dy,\qquad y(1)=3$

28. $\left(9x^2+y-1\right)dx-(4y-x)\,dy,\qquad y(1)=0$

Find the general solution

29. $y'=\dfrac{x^2+y^2}{2xy}$

30. $2xyy'=x^2+2y^2$

31. $xy'=y+2\sqrt{xy}$

32. $xy^2y'=x^3+y^3$

33. $x^2y'=xy+x^2e^{y/x}$

34. $x^2y'=xy+y^2$

35. $xyy'=x^2+3y^2$

36. $\left(x^2-y^2\right)y'=2xy$

37. $xyy'=y^2+x\sqrt{4x^2+y^2}$

38. $xy'=y+\sqrt{x^2+y^2}$

39. $y^2y'+2xy^3=6x$

40. $x^2y'+2xy=5y^4$

41. $2xy'+y^3e^{-2x}=2xy$

42. $y^2(xy'+y)\left(1+x^4\right)^{1/2}=x$

43. $3y^2y'+y^3=e^{-x}$

44. $3xy^2y'=3x^4+y^3$

45. $xe^yy'=2\left(e^y+x^3e^{2x}\right)$

46. $(2x\sin y\cos y)\,y'=4x^2+\sin^2 y$

47. $\left(x+e^y\right)y'=xe^{-y}-1$

Section 1.7 - Existence and Uniqueness of Solutions

The questions of existence and uniqueness

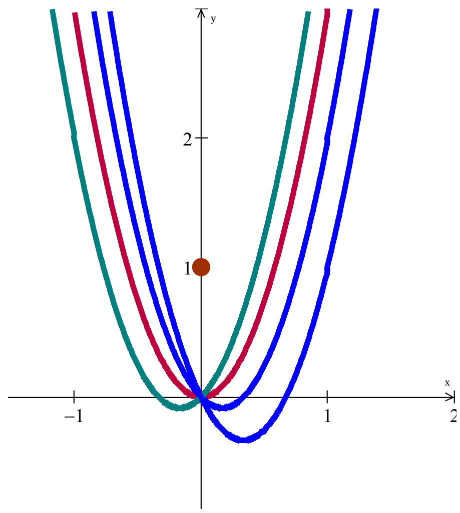
- When can we be sure that a solution exists**?**
- How many different solutions are there

Existence of Solutions

Example

Consider the initial value problem: $tx' = x + 3t^2$ with $x(0) = 1$

Solution



$$x' = \frac{1}{t}x + 3t$$

$$x' = \frac{1}{t}x + 3t \qquad t \neq 0$$

There is ***no solution*** to the given initial value

$$\begin{aligned} u(t) &= e^{-\int \frac{1}{t}dt} \\ &= e^{-\ln t} \\ &= \frac{1}{t} \end{aligned}$$

$$\left[\frac{x}{t}\right]' = 3$$

$$\begin{aligned} \frac{x}{t} &= \int 3dt \\ &= 3t + C \end{aligned}$$

$$\Rightarrow x(t) = 3t^2 + Ct$$

Theorem: Existence of Solutions

Suppose the function $f(t,x)$ is defined and continuous on the rectangle $\boldsymbol{R}$ in the *tx*-plane. Then given any point $\left(t_0,\, x_0\right) \in \boldsymbol{R}$, the initial value problem

$$x' = f(t,x) \quad \textit{and} \quad x(t_0) = x_0$$

has a solution $x(t)$ defined in an interval containing x_0. Furthermore, the solution will be defined at least until the solution curve $t \to \left(t,\, x(t)\right)$ leaves the rectangle $\boldsymbol{R}$.

Interval of Existence of a Solution

Example

Consider the initial value problem $x' = 1 + x^2$ with $x(0) = 0$. Find the solution and its interval of existence.

Solution

The right-hand side is $f(t,x) = 1 + x^2$ which is continuous on the entire *tx*-plane.
The solution to the initial value problem is:

$$\frac{dx}{dt} = 1 + x^2$$

$$\frac{dx}{1+x^2} = dt$$

$$\int \frac{dx}{1+x^2} = \int dt$$

$$\tan^{-1} x = t$$

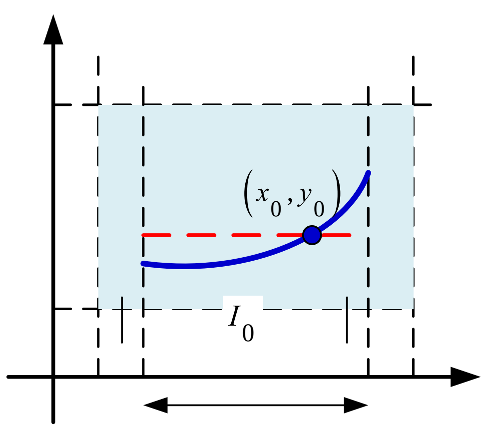
$$x(t) = \tan t$$

$x(t)$ is discontinuous at $t = \pm\frac{\pi}{2}$. Hence the solution to the initial value problem is defined only for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

The interval: $\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$

Theorem: Existence of a Unique Solution

Let R be a rectangular region in the *xy*-plane defined by $a \le x \le b,\ \ c \le y \le d$ that contains the point $\left(x_0, y_0\right)$ in its interior. If $f\left(x, y\right)$ and $\frac{\partial f}{\partial y}$ are continuous on R, then there exists some interval $I_0 : \left(x_0 - h,\ x_0 + h\right),\ \ h > 0$, contained in [$a$, b], and a unique function $y\left(x\right)$, defined on I_0 that is a solution of the initial-value problem (IVP)



Mathematics & Theorems

Any theorem is a logical statement which has hypotheses (when it's true) and conclusions (true)

The Hypotheses of the Uniqueness of Solutions Theorem

1. The equation is in normal form $y' = f(t, y)$
2. The right-hand side $f(t, y)$ and its derivative $\frac{\partial f}{\partial y}$ are both continuous in the rectangle ***R***.
3. The initial point $\left(t_0, y_0\right)$ is in the rectangle ***R***.

For the uniqueness theorem the conclusions are as follows:

1- There is one and only one solution to the initial value problem.

2- The solution exists until the solution curve $t \to \left(t, y(t)\right)$ leaves the rectangle ***R***.

Example

Consider the initial value problem $tx' = x + 3t^2$. Is there a solution to this equation with initial condition $x(1) = 2$? If so, is the solution unique?

Solution

$$x' = \frac{x}{t} + 3t$$

The right-hand side: $f(t, x) = \frac{x}{t} + 3t$ is continuous except where $t = 0$.

We can take ***R*** to be any rectangle which contains the point $(1,\ 2)$ to avoid $t = 0$, we can choose $\frac{1}{2} < t < 2$ and $0 < x < 4$

Then *f* is continuous everywhere in ***R*** $\Rightarrow$ hypotheses of the existence theorem are satisfied.

Since $\frac{\partial f}{\partial x} = \frac{1}{t}$ is also continuous in ***R***.

There is only one solution.

Exercises   *Section* 1.7 - Existence and Uniqueness of Solutions

Which of the initial value problems are guaranteed a unique solution

1. $y' = 4 + y^2, \quad y(0) = 1$

2. $y' = \sqrt{y}, \quad y(4) = 0$

3. $y' = t\tan^{-1} y, \quad y(0) = 2$

4. $\omega' = \omega \sin \omega + s, \quad \omega(0) = -1$

5. $x' = \dfrac{t}{x+1}, \quad x(0) = 0$

6. $y' = \dfrac{1}{x}\text{y} + 2, \quad y(0) = 1$

7. Show that $y(t) = 0$ and $y(t) = t^3$ are both solutions of the initial value problem $y' = 3y^{2/3}$, where $y(0) = 0$. Explain why this fact doesn't contradict Theorem

8. Use a numerical solver to sketch the solution of the given initial value problem

$$\frac{dy}{dt} = \frac{t}{y+1}, \qquad y(2) = 0$$

a) Where does your solver experience difficulty? why? Use the image of your solution to estimate the interval of existence.

b) Find an explicit solution; then use your formula to determine the interval of existence. How does it compare with the approximation found in part (*a*).

Section 1.8 - Autonomous Equations and Stability

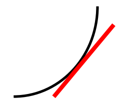
A first-order autonomous equation is an equation of the form

$$x' = f(x)$$

$$\frac{dy}{dx} = f(x, y)$$

Definition

The value $f(x, y)$ where the function f assigns to the point represent the slope of a line (line segment) call ***a lineal element***.

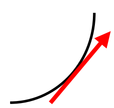


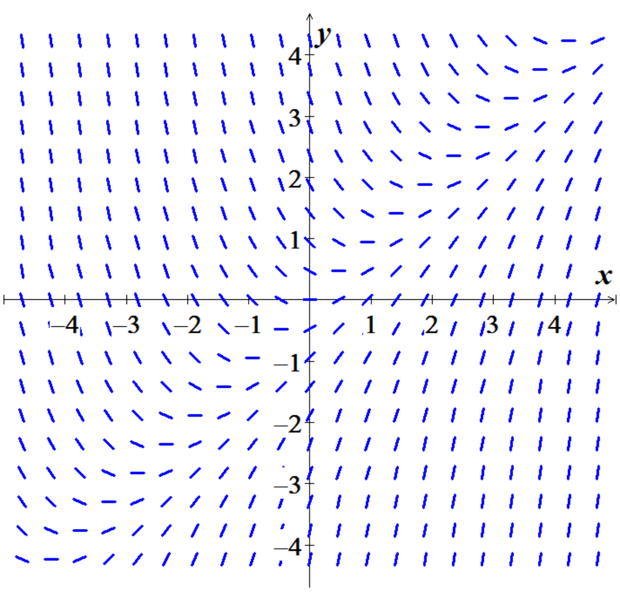
Example: Given $\frac{dy}{dx} = 0.2xy$ and consider the point $(2,\ 3)$

The slope of the lineal element is $\frac{dy}{dx} = 0.2xy = 0.2(2)(3) = 1.2$ $\quad$ (*positive sign*)

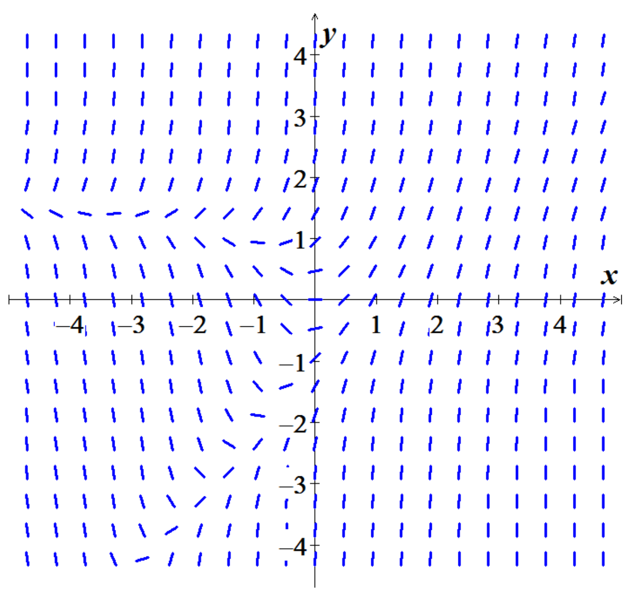
The Direction Fields

What we draw a lineal element at each point (x, y) with slope $f(x, y)$ then the collection of these lineal elements is called a ***direction field*** or a ***slope field*** of the differential equation $\frac{dy}{dx} = f(x, y)$.

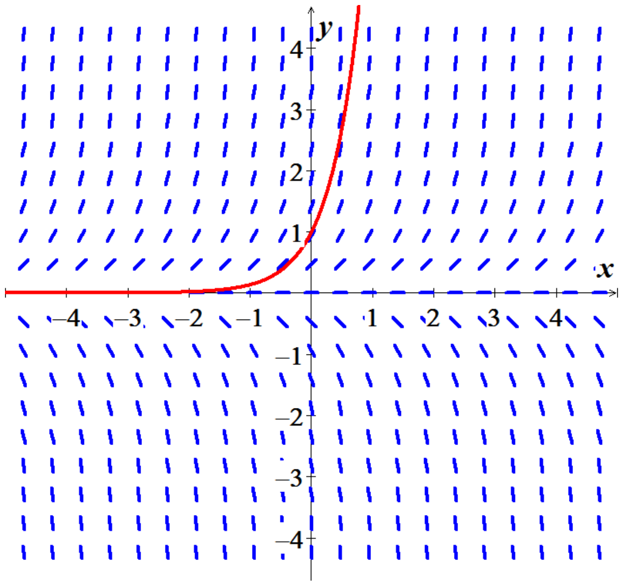
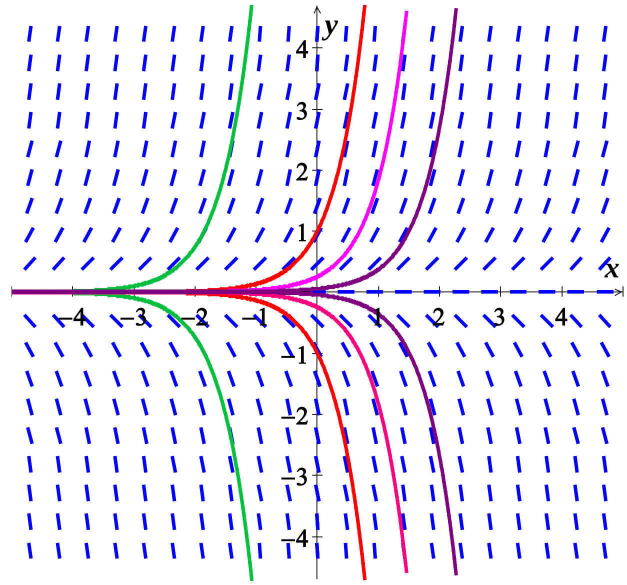




$y' = x - y$



$y' = y^2 - xy + 2x$

$y' = 2y, \quad with\ y(0) = 1 \quad \Rightarrow \quad y = e^{2x}$

Example

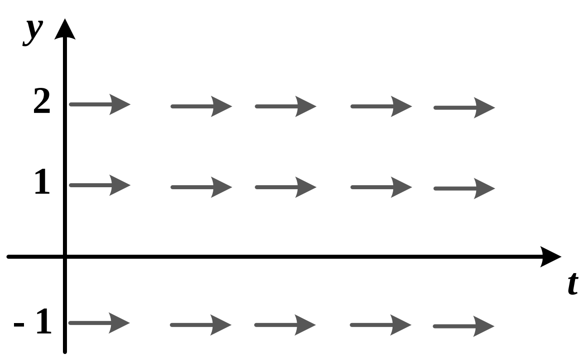
Sketch the direction field for the following differential equation. Sketch the set of integral curves for this differential equation, how the solutions behave as $t \to \infty$ and if this behavior depends on the value of $y(0)$ describe this dependency

$$y' = \left(y^2 - y - 2\right)(1 - y)^2$$

Solution

$$y' = 0 \quad \Rightarrow \quad \left(y^2 - y - 2\right)(1 - y)^2 = 0$$

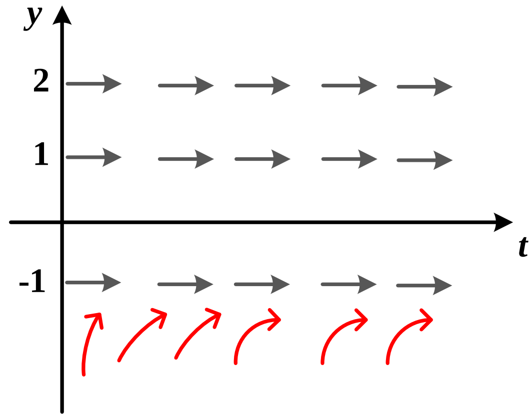
$\underline{y = \pm 1,\ 2|}$ *Slope of the tangent lines*



This divided into 4 regions.

For $y < -1$, assume $y = -2 \Rightarrow y' = \left(4^2 + 2 - 2\right)(1 + 2)^2 = 36 > \mathbf{0}\ (\nearrow)$

$y = -1$, the slopes will flatten out while staying positive

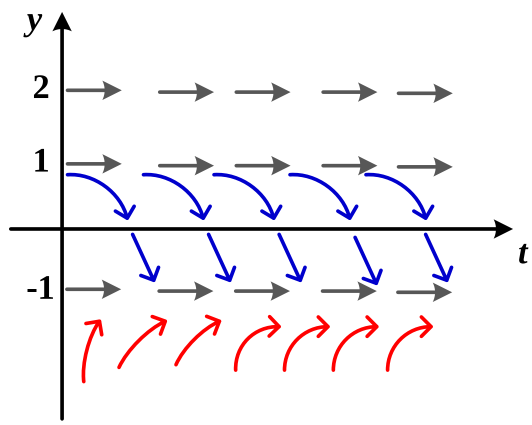


For $-1<y<1$, assume $y=0 \Rightarrow y'=(-2)(1)^2=-2<\mathbf{0}\ (\searrow)$

Therefore, tangent lines in this region will have negative slopes and apparently not very steep.

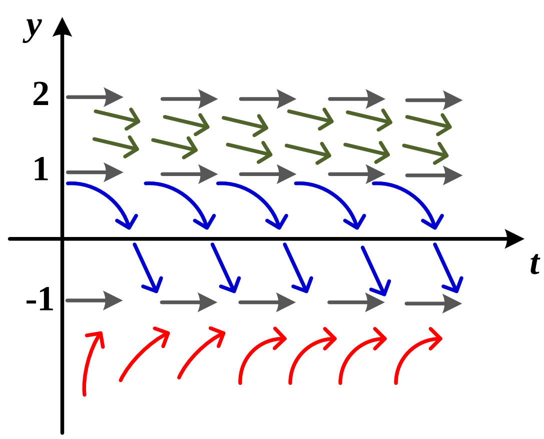
$y=.9 \quad \Rightarrow \quad y'=-.0209$

$y=-.9 \quad \Rightarrow \quad y'=-1.0469$ (Steeper)



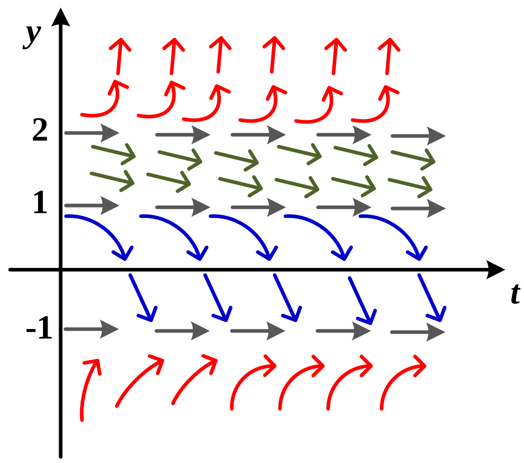
For $1<y<2$, assume $y=1.5 \Rightarrow y'=\left(1.5^2-1.5-2\right)(-.5)^2=-0.3125<\mathbf{0}\ (\searrow)$

Not to steep

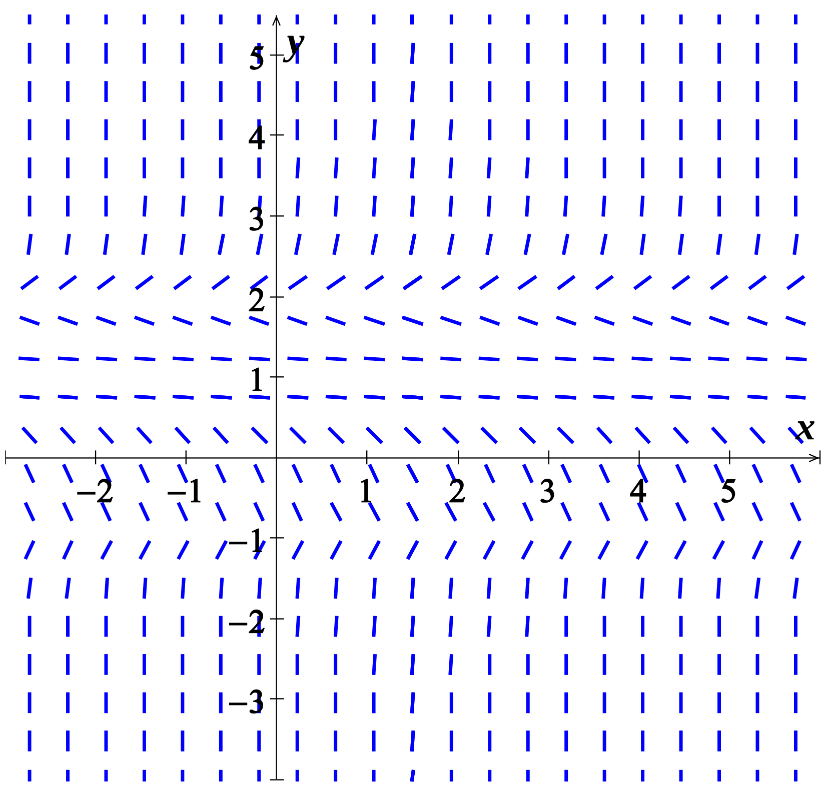


For $y > 2$, assume $y = 3 \Rightarrow y' = (4)(-2)^2 = 16 > \mathbf{0}$ $(\nearrow)$

Start out fairly flat neary = 2, then will get fairly steep.



Value of $y(0)$	$t \to \infty$
$y(0) < -1$	$y \to -1$
$-1 \le y(0) < 2$	$y \to 1$
$y(0) = 2$	$y \to 2$
$y(0) > 2$	$y \to \infty$



Autonomous 1st order DE

A system $\dot{y} = rx - y - xz = 0$, which does not explicitly contain the independent variable t is called an ***autonomous system***. Otherwise, the system is called <u>*non-autonomous system*</u>.

Autonomous	***Not- Autonomous***
$x' = \sin x$	$x' = \sin(tx)$
$y' = y^2 + 1$	$y' = y^2 + t$
$z' = e^z$	$z' = t^2$

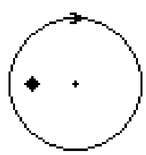
Equilibrium Points & Solutions

$x'(t) = 0 = f(x_0) \Rightarrow x_0$ *is an equilibrium point* and also called a ***critical point***.

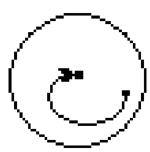
$x'(t) = x_0$ *called equilibrium solution*

From these equilibrium points, we can determine the stability of the system.

- An equilibrium point is ***stable*** if all nearby solutions stay nearby.



- An equilibrium point is ***asymptotically stable*** if all nearby solutions not only stay nearby, but also tend to the equilibrium point. An equilibrium point is stable if all nearby solutions.



1. If $f'(x_0) < 0$, then f is ***decreasing*** at x_0 and x_0 is asymptotically stable.
2. If $f'(x_0) > 0$, then f is ***increasing*** at x_0 and x_0 is unstable.
3. If $f'(x_0) = 0$, no conclusion can be drawn.

The family of all solution curves without the presence of the independent variable is called the ***phase portrait***.

When an independent variable ***t*** is interpreted as time and the solution curve $-P_+ < x < P_+$ could be thought of as the path of a particle moving in the solution space, then the system $f_\mu(x)$ is considered as a ***dynamical system***, where the solution curves are called ***trajectories*** or ***orbits***.

Example

Discover the behavior as $t \to \infty$ of all solutions to the differential equation

$$x' = f(x) = \left(x^2 - 1\right)(x - 2)$$

Solution

The equilibrium points: $f(x) = 0$

$$\left(x^2 - 1\right)(x - 2) = 0$$

$\Rightarrow x_1 = -1,\ x_2 = 1,\ x_3 = 2$ are equilibrium.

$$f' = 3x^2 - 4x - 1$$

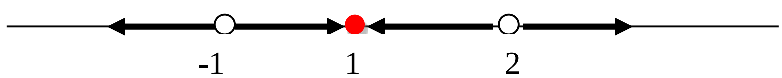
$f'(-1) = 3(-1)^2 - 4(-1) - 1 = 6 > 0$ ***unstable***

$f'(1) = 3(1)^2 - 4(1) - 1 = -2 < 0$ is ***asymptotically stable***

$f'(2) = 3(2)^2 - 4(2) - 1 = 3 > 0$ ***unstable***

$x(t) = -1,\ x(t) = 1,\ x(t) = 2$

These are constant functions, the position of the point the phase line modeled by them is also constant

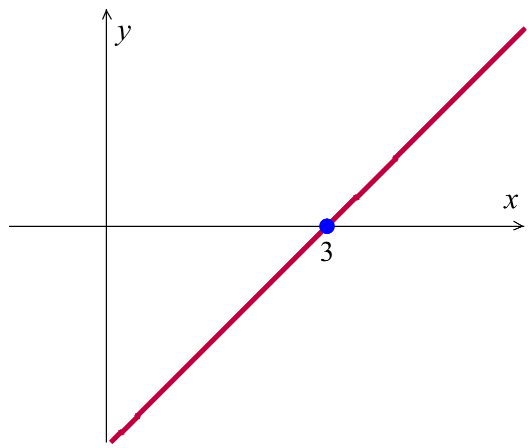


Phase Portrait

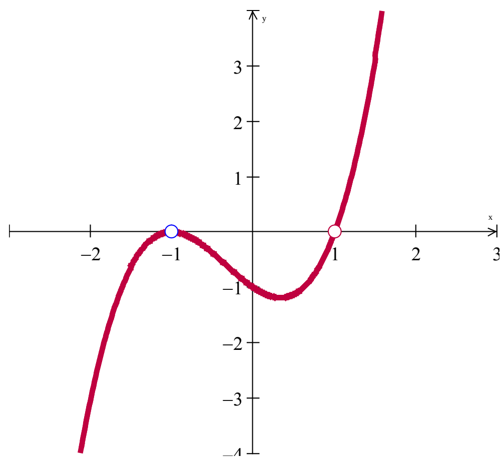
Exercises *Section* 1.8 - Autonomous Equations and Stability

The graph of the right-hand side $y' = f(y)$ is shown. Identify the equilibrium points and sketch the equilibrium solutions in the *ty*-plane. Classify each equilibrium point as either unstable or asymptotically stable.

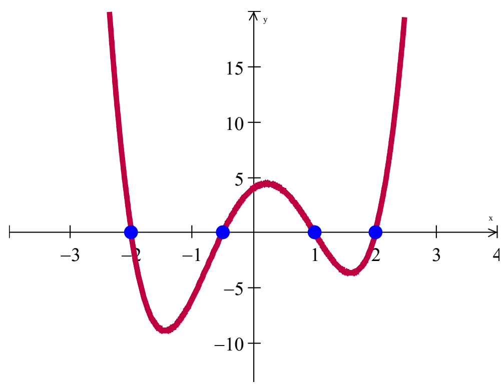
1.



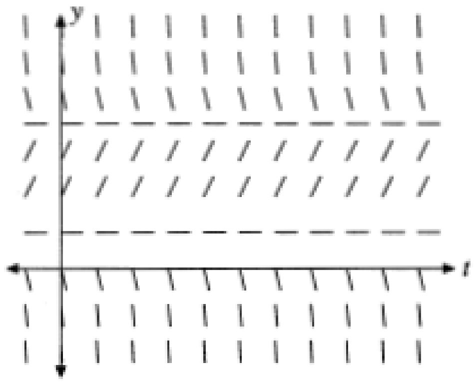
2.



3.



4. Impose the equilibrium solution(s), classifying each as either unstable or asymptotically stable



An autonomous differential equation is given. Perform each of the following to exercises 5 - 8

a) Sketch a graph of $f(y)$

b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

5. $y' = 2 - y$

6. $y' = (y+1)(y-4)$

7. $y' = 9y - y^3$

8. $y' = \sin y$

Determine the stability of the equilibrium solutions

9. $x' = 4 - x^2$

10. $x' = x(x-1)(x+2)$

11. A tank contains 100 *gal* of pure water. A salt solution with concentration 3 *lb/gal* enters the tank at a rate of 2 *gal/min*. Solution drains from the tank at a rate of 2 *gal/min*. Use the qualitative analysis to find the eventual concentration of the salt in the tank.

Section 1.9 - Modeling Population Growth

Modeling Population Growth

The mathematical model of the growth of a population is given by:

$$P' = rP$$

Where $\boldsymbol{r}$: reproductive rate.

The natural of the predictions of the model depend on the nature of the reproductive rate r.

Malthusian Method

Since r is a constant because the birth or death rates do not depend on time or on the size.
Therefore the solution to $P' = rP$ is given by:

$$\begin{aligned} P(t) &= Ce^{rt} \\ &= P_0 e^{rt} \end{aligned}$$

The population at time $t = 0$ is P_0.

Example

A biologist starts with 10 cells in a culture. Exactly 24 *hrs* later he counts 25. Assuming a Malthusian model, what the reproductive rate? What will be the number of cells of the end of 10 days?

Solution

$$P = P_0 e^{rt}$$

$$P = 10e^{rt}$$

$$25 = 10e^{r(1)}$$ **24 *hrs* = 1 *day* *P* = 25**

$$\frac{25}{10} = e^r$$

$$\ln\frac{25}{10} = \ln e^r$$

$$\begin{aligned} r &= \ln 2.5 \\ &\approx 0.9163 \end{aligned}$$

$$P(t) = 10e^{0.9163t}$$

$$\begin{aligned} P(10) &= 10e^{0.9163(10)} \\ &\approx 95367 \ \ cells \end{aligned}$$

Example

A certain radioactive material is decaying at a rate proportional to the amount present. If a sample of 50 grams of the material was present initially and after 2 hours the sample lost 10% of its mass, find:

a) An expression for the mass of the material remaining at any time.

b) The mass of the material after 4 hours.

c) How long will it take for 75% of the material to decay?

d) The half-life of the material.

Solution

Given: $A_0 = 50g \qquad A(2) = 50 - .1(50) = 45g$

a) $A(2) = 50e^{-2r}$

$$45 = 50e^{-2r}$$

$$e^{-2r} = \frac{45}{50} = \frac{9}{10}$$

Convert to logarithm

$$-2r = \ln\frac{9}{10}$$

$$r = -\frac{1}{2}\ln\frac{9}{10}$$

$$A(t) = 50e^{\frac{1}{2}\ln\left(\frac{9}{10}\right)t} = 50e^{\ln\left(\frac{9}{10}\right)^{t/2}}$$

$$\boxed{= 50\left(\frac{9}{10}\right)^{t/2}}$$

b) $A(4) = 50\left(\frac{9}{10}\right)^2 = \boxed{40.5\ g}$

c) $At\ 75\% \quad \Rightarrow \quad A(t) = 50 \times .25 = 12.5\ g$

$$12.5 = 50\left(\frac{9}{10}\right)^{t/2}$$

$$\left(\frac{9}{10}\right)^{t/2} = \frac{12.5}{50}$$

$$\frac{t}{2}\ln\left(\frac{9}{10}\right) = \ln\left(\frac{12.5}{50}\right)$$

$$t = \frac{2\ln\left(\frac{12.5}{50}\right)}{\ln\left(\frac{9}{10}\right)} \boxed{\approx 26.32\ hours}$$

d) $T = \dfrac{\ln 2}{-\frac{1}{2}\ln 0.9} \boxed{\approx 13.16\ \text{hrs}}$

Note

We can this formula to solve most of the questions $\boxed{rT = \ln\frac{A}{A_0}}$

Logistic Model of Growth

Suppose an environment is capable of sustaining no more than a fixed number K of individuals in its populations. The quantity K is called the ***carrying capacity*** of the environment. In reality this model in unrealistic because environments impose limitations to population growth.

The logistic equation is given by:

$$P' = rP\left(1-\frac{P}{K}\right) = KrP\left(K-P\right) \qquad P' = kP\left(M-P\right) \quad where \quad k=\frac{r}{K} \ \ \& \ \ M=K$$

The logistic equation can be solved by separation of variables

$$\frac{dP}{dt} = rP\left(1-\frac{P}{K}\right)$$

$$\int \frac{dP}{P\left(1-\frac{P}{K}\right)} = \int r\,dt \qquad\qquad \frac{1}{P\left(1-\frac{P}{K}\right)} = \frac{K}{P\left(K-P\right)} = \frac{1}{P} + \frac{1}{K-P}$$

$$\int \frac{dP}{P} + \int \frac{dP}{K-P} = \int r\,dt$$

$$\ln\left|P\right| - \ln\left|K-P\right| = rt + C$$

$$\ln\left|\frac{P}{K-P}\right| = rt + C$$

$$\frac{P}{K-P} = e^{rt+C}$$

$$\frac{P}{K-P} = Ae^{rt}$$

$$t=0 \quad \Rightarrow \quad A = \frac{P_0}{K-P_0}$$

$$P = KAe^{rt} - PAe^{rt}$$

$$P\left(1+Ae^{rt}\right) = KAe^{rt}$$

$$\begin{aligned} P &= \frac{KAe^{rt}}{1+Ae^{rt}} \\ &= \frac{KA}{e^{-rt}+A} \\ &= \frac{K\frac{P_0}{K-P_0}}{e^{-rt}+\frac{P_0}{K-P_0}} \\ &= \frac{KP_0}{\left(K-P_0\right)e^{-rt}+P_0} \end{aligned}$$

$$P(t) = \frac{KP_0}{P_0+\left(K-P_0\right)e^{-r(t-t_0)}} \quad or \quad P(t) = \frac{MP_0}{P_0+\left(M-P_0\right)e^{-kMt}}$$

Example

Suppose we start at time $t_0 = 0$ with a sample of 1000 cells. One day later we see that the population has doubled, and sometime later we notice that the population has stabilized at 100,000.

Solution

$$K = 100,000 = 10^5$$

$$P_0 = 1000$$

$$P(t) = \frac{10^5 10^3}{10^3 + \left(10^5 - 10^3\right)e^{-r(t-0)}}$$

$$= \frac{10^8}{10^3 + 10^3(100-1)e^{-rt}}$$

$$= \frac{10^5}{1+99e^{-rt}}$$

$$2P_0 = \frac{10^5}{1+99e^{-r(1)}}$$

$$2\times 10^3 = \frac{10^5}{1+99e^{-r}}$$

$$1+99e^{-r} = \frac{10^5}{2\times 10^3}$$

$$1+99e^{-r} = 50$$

$$99e^{-r} = 49$$

$$e^{-r} = \frac{49}{99}$$

$$-r = \ln\left(\frac{49}{99}\right)$$

$$r \approx 0.7033$$

$$P(t) = \frac{10^5}{1+99e^{-0.7033t}}$$

Pollution

Consider a lake that has a volume of $V = 100\ km^3$, it is fed by an input river, and there is another river which is fed by the lake at a rate that keeps the volume of the lake constant.

The input rate: $r(t) = 50 + 20\cos\left[2\pi\left(t - \frac{1}{4}\right)\right]$

The maximum flow into the lake occurs when $t = \frac{1}{4}$

In addition, there is a factory on the lake that introduces a pollutant into the lake at the rate of 2 km^3 / yr.

Let $x(t)$ denote the total amount of pollution in the lake at time t. If we make the assumption that the pollutant is rapidly mixed throughout the lake, then

$$\frac{dx}{dt} = 2 - \left(50 + 20\cos\left[2\pi\left(t - \frac{1}{4}\right)\right]\right) . \frac{x}{100}$$

Exercises *Section* 1.9 - Modeling Population Growth

1. The rate of growth of bacteria in a petri dish is proportional to the number of bacteria in the dish.

2. The rate of growth of a population of field mice is inversely proportional to the square root of the population.

3. A biologist starts with 100 cells in a culture. After 24 *hrs,* he counts 300. Assuming a Malthusian model, what the reproductive rate? What will be the number of cells of the end of 5 *days*?

4. A biologist prepares a culture. After 1 *day* of growth, the biologist counts 1000 cells. After 2*days,* he counts 3000. Assuming a Malthusian model, what the reproductive rate and how many cells were present initially?

5. A population of bacteria is growing according to the Malthusian model. If the population is triples in 10 *hrs*, what is the reproduction rate? How often does the population double itself?

6. Consider a lake that is stocked with walleye pike and that the population of pike is governed by the logistic equation

$$P' = 0.1P\left(1 - \frac{P}{10}\right)$$

where time is measured in days and P in thousands of fish. Suppose that fishing is started in this lake and that 100 fish are removed each day.

a) Modify the logistic model to account for the fishing.
b) Find and classify the equilibrium points for your model.
c) Use qualitative analysis to completely discuss the fate of the fish population with this model. In particular, if the initial fish population is 1000, what happens to the fish as time passes? What will happen to an initial population having 2000 fish?

7. Suppose that in 1885 the population of a certain country was 50 million and was growing at the rate of 750,000 people per year at that time. Suppose also that in 1940 its population was 100 million and was then growing at the rate of 1 million per year. Assume that this population satisfies the logistic equation. Determine both the limiting population M and the predicted population for the year 2000.

8. The time rate of change of a rabbit population P is proportional to the square root of P. At time $t = 0$ (months) the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be one year later?

9. Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to reproduce (so that the birth rate is $\beta = 0$) and the death rate δ (deaths per week per fish) is thereafter proportional to $\frac{1}{\sqrt{P}}$. If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long did it take all the fish in the lake to die?

10. Suppose that when a certain lake is stocked with fish, the birth and death rates β and δ are both inversely proportional to $\sqrt{P}$

a) Show that $P(t)=\left(\frac{1}{2}kt+\sqrt{P_0}\right)^2$, where k is a constant.

b) If $P_0=100$ and after 6 months there are 169 fish in the lake, how many will there be after 1 year?

11. The time rate of change of an alligator population P in a swamp is proportional to the square of P. The swamp contained a dozen alligators in 1988, two dozen in 1998.

a) When will there be four dozen alligators in the swamp?

b) What happens thereafter?

12. Consider a prolific breed of rabbits whose birth and death rates, β and δ, are each proportional to the rabbit population $P=P(t)$, with $\beta>\delta$

a) Show that $P(t)=\dfrac{P_0}{1-kP_0t}$, $\quad k$ *constant*

Note that $P(t)\to+\infty \quad as \quad t\to\dfrac{1}{kP_0}$. This is doomsday

b) Suppose that $P_0=6$ and that there are nine rabbits after ten months. When does doomsday occur?

c) With $\beta<\delta$, repeat part (*a*)

d) What now happens to the rabbit population in the long run?

13. Consider a population $P(t)$ satisfying the logistic equation $\frac{dP}{dt}=aP-bP^2$, where $B=aP$ is the time rate at which births occur and $D=bP^2$ is the rate at which deaths occur.

a) If the initial population is $P(0)=P_0$, and B_0 births per month and D_0 deaths per month are occurring at time $t=0$, show that the limiting population is $M=\dfrac{B_0P_0}{D_0}$.

b) If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occurring at time $t=0$, how many months does it take for $P(t)$ to reach 95% of the limiting population *M*?

c) If the initial population is 240 rabbits and there are 9 births per month and 12 deaths per month occurring at time $t=0$, how many months does it take for $P(t)$ to reach 105% of the limiting population *M*?