Lecture Four – Series

Section 4.1 – Introduction and Review of Power Series

Definition

A ***power series*** *about the point* x_0 is a series of the form

$$\sum_{n=0}^{\infty} a_n\left(x-x_0\right)^n = a_0 + a_1\left(x-x_0\right) + a_2\left(x-x_0\right)^2 + a_3\left(x-x_0\right)^3 + \cdots$$

The series is said to converge at x if the sequence of partial sums

$$\begin{aligned} S_N(x) &= \sum_{n=0}^{N} a_n\left(x-x_0\right)^n \\ &= a_0 + a_1\left(x-x_0\right) + a_2\left(x-x_0\right)^2 + \cdots + a_N\left(x-x_0\right)^N \end{aligned}$$

Converges as $N \to \infty$. The sum of the series at the point x is defined to be the limit at the partial sums,

$$\sum_{n=0}^{N} a_n\left(x-x_0\right)^n = \lim_{N\to\infty} S_N(x)$$

Example

Show that the geometric series $\sum_{n=0}^{\infty} x^n$ converges for $|x| < 1$ and that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \textit{for } |x| < 1$$

Show that the series diverges for $|x| \ge 1$.

Solution

The partial sums $S_N(x) = \sum_{n=0}^{N} x^n$ can be evaluates as follows.

$$\begin{aligned} (1-x)S_N(x) &= (1-x)\left(1 + x + x^2 + \cdots + x^N\right) \\ &= \left(1 + x + x^2 + \cdots + x^N\right) - \left(x + x^2 + \cdots + x^N + x^{N+1}\right) \\ &= 1 - x^{N+1} \end{aligned}$$

$$S_N(x) = \sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x} \qquad x \neq 1$$

If $|x| < 1$, then $x^{N+1} \to 0 \quad as\ N \to \infty \Rightarrow S_N(x) \to \frac{1}{1-x}$

If $|x| > 1$, then x^{N+1} diverges and therefore the $S_N(x)$ diverges

If $|x| = 1$, then $S_N(1) = N+1$

Interval of convergence

Theorem

For any power series $\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n$ there is an ***R***, either a nonnegative number or ∞, such that the series converges if $\left|x - x_0\right| < R$ and diverges if $\left|x - x_0\right| > R$

The ratio Test

Theorem

Suppose the terms of the series $\sum_{n=0}^{\infty} A_n$ have the property that

$$\lim_{n\to\infty} \frac{\left|A_{n+1}\right|}{\left|A_n\right|} = L$$

exists. If $L < 1$ the series converges, while if $L > 1$ the series diverges

Example

Find the radius of convergence for the series. $\sum_{n=0}^{\infty} \frac{2^n x^{2n}}{2n(n+1)}$

Solution

$$\frac{\left|A_{n+1}\right|}{\left|A_n\right|} = \frac{\frac{2^{n+1} x^{2(n+1)}}{2(n+1)(n+2)}}{\frac{2^n x^{2n}}{2n(n+1)}}$$

$$= \frac{2^{n+1} x^{2(n+1)}}{2(n+1)(n+2)} \frac{2n(n+1)}{2^n x^{2n}}$$

$$= \frac{2n}{(n+2)} x^2$$

$$\lim_{n\to\infty} \frac{\left|A_{n+1}\right|}{\left|A_n\right|} = \lim_{n\to\infty} \frac{2n}{n+2} x^2$$

$$\to 2x^2$$

By the ratio test, the series converges if $2x^2 < 1$, so the radius of convergence is $R = \frac{1}{\sqrt{2}}$

$$x^2 < \frac{1}{2} \qquad -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

Algebraic Operations on Series

The ***sum*** and ***difference*** of two series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots$$

$$\sum_{n=0}^{\infty} a_n x^n \pm \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} \left(a_n \pm b_n\right) x^n$$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{m=0}^{\infty} b_m x^m\right) = \sum_{p=0}^{\infty} c_p x^p \qquad c_p = \sum_{k=0}^{p} a_{p-k} b_k$$

Differentiating Power Series

Theorem

The function $f(x)=\sum_{n=0}^{\infty} a_n\left(x-x_0\right)^n = a_0 + a_1\left(x-x_0\right)+a_2\left(x-x_0\right)^2+a_3\left(x-x_0\right)^3+\cdots$

Can be differentiating the series by terms

$$f'(x)=\frac{d}{dx}\left[a_0 + a_1\left(x-x_0\right)+a_2\left(x-x_0\right)^2+a_3\left(x-x_0\right)^3+\cdots\right]$$

$$=a_1+2a_2\left(x-x_0\right)+3a_3\left(x-x_0\right)^2+\cdots$$

$$=\sum_{n=0}^{\infty} na_n\left(x-x_0\right)^{n-1}$$

$$f''(x)=\sum_{n=0}^{\infty} n(n-1)a_n\left(x-x_0\right)^{n-2}$$

Identity Theorem

Suppose that the series $f(x)=\sum_{n=0}^{\infty} a_n\left(x-x_0\right)^n$ has a positive radius of convergence. Then

$$a_n=\frac{f^{(n)}\left(x_0\right)}{n!}$$

The coefficients of a power series are determined by the values of the sum $f(x)$.

Integrating Power Series

Theorem

Suppose the power series $f(x)=\sum_{n=0}^{\infty} a_n\left(x-x_0\right)^n$ converges for $\left|x-x_0\right|<R, \quad R>0$

$$\int f(x)dx=C+\sum_{n=0}^{\infty} a_n\frac{\left(x-x_0\right)^{n+1}}{n+1}$$

Section 4.2 – Series Solutions near Ordinary Points

Example of a First-Order Equation

Find the series solution for the differential equation $y' - 2xy = 0$

Solution

We look for a solution of the form: $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$= \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y' - 2xy = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - 2x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 2 a_n x^{n+1} = 0 \qquad \text{let} \quad n-1 = p+1$$

$$\sum_{p=-1}^{\infty} (p+2) a_{p+2} x^{p+1} - \sum_{n=0}^{\infty} 2 a_n x^{n+1} = 0$$

$$a_1 + \sum_{p=0}^{\infty} (p+2) a_{p+2} x^{p+1} - \sum_{n=0}^{\infty} 2 a_n x^{n+1} = 0$$

$$a_1 + \sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1} - \sum_{n=0}^{\infty} 2 a_n x^{n+1} = 0$$

$$a_1 + \sum_{n=0}^{\infty} \left[(n+2) a_{n+2} - 2a_n \right] x^{n+1} = 0$$

$$(n+2) a_{n+2} - 2a_n = 0$$

$$a_{n+2} = \frac{2a_n}{n+2}$$

By the identity theorem: $a_n = \dfrac{f^{(n)}\left(x_0\right)}{n!}$

$\Rightarrow a_0 = y(0)$ $\qquad a_1 = 0$

$a_2 = \dfrac{2a_0}{2} = y(0)$ $\qquad a_3 = \dfrac{2a_1}{3} = 0$

$a_4 = \dfrac{2a_2}{4} = \frac{1}{2} y(0)$ $\qquad a_5 = \dfrac{2a_3}{5} = 0$

$a_6 = \dfrac{2a_4}{6} = \frac{1}{6} y(0)$

$a_8 = \dfrac{2a_6}{8} = \frac{1}{2.3.4} y(0)$

$$y(x) = \sum_{k=0}^{\infty} a_{2k} x^{2k}$$

$$= y(0) \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$$

Example

Find the general series solution to the equation

$$y'' + xy' + y = 0$$

Find the particular solution with $y(0) = 0$ and $y'(0) = 2$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + xy' + y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + n a_n + a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + (n+1) a_n = 0$$

$$(n+2)(n+1) a_{n+2} = -(n+1) a_n$$

$$a_{n+2} = -\frac{1}{n+2} a_n$$

$$a_0 = y(0) = 0 \qquad a_1 = y'(0) = 2$$

$$a_2 = -\tfrac{1}{2} a_0 \qquad a_3 = -\tfrac{1}{3} a_1$$

$$a_4 = -\tfrac{1}{4} a_2 = \frac{1}{2 \cdot 4} a_0 \qquad a_5 = -\tfrac{1}{5} a_3 = \frac{1}{3 \cdot 5} a_1$$

$$a_6 = -\tfrac{1}{6} a_4 = -\frac{1}{2 \cdot 4 \cdot 6} a_0 \qquad a_7 = -\tfrac{1}{7} a_7 = -\frac{1}{3 \cdot 5 \cdot 7} a_1$$

The general solution can be written as:

$$y(x) = a_0\left[1 - \frac{1}{2}x^2 + \frac{1}{2\cdot 4}x^4 - \frac{1}{2\cdot 4\cdot 6}x^6 + \cdots\right]$$
$$+ a_1\left[x - \frac{1}{3}x^3 + \frac{1}{3\cdot 5}x^5 - \frac{1}{3\cdot 5\cdot 7}x^7 + \cdots\right]$$

For the given initial $y(0) = 0$ and $y'(0) = 2$, the solution is:

$$y(x) = 2\left[x - \frac{1}{3}x^3 + \frac{1}{3\cdot 5}x^5 - \frac{1}{3\cdot 5\cdot 7}x^7 + \cdots\right]$$

Exercises *Section* 4.2 – Series Solutions near Ordinary Points

Find a power series solution.

1. $y' = 3y$
2. $(1+x)y' - y = 0$
3. $(2-x)y' + 2y = 0$
4. $y' = x^2 y$
5. $(x-4)y' + y = 0$
6. $y'' = 9y$
7. $y'' + y = 0$

Find the series solution to the initial value problems

8. $y'' + (x-1)y' + y = 0 \quad y(1) = 2 \quad y'(1) = 0$

9. $y'' + xy' + y = 0 \quad y(0) = 1 \quad y'(0) = 0$

10. $y'' - xy' - y = 0 \quad y(0) = 2 \quad y'(0) = 1$

11. $\left(2 + x^2\right)y'' - xy' + 4y = 0 \quad y(0) = -1 \quad y'(0) = 3$

Section 4.3 – Legendre's Equation

Solving a homogeneous linear differential equation with constant coefficients can be reduced to the algebraic problem of finding the roots of its characteristic equation.

The Legendre's equation of order n is important in many applications. It has the form

$$\left(1-x^2\right)y''-2xy'+n(n+1)y=0$$

$$y''-\frac{2x}{1-x^2}y'+\frac{n(n+1)}{1-x^2}y=0$$

$$y''+P(x)y'+Q(x)y=0$$

Any solution of that equation is called a Legendre function.

Note that: $P(x)=\frac{2x}{1-x^2}$ and $Q(x)=\frac{n(n+1)}{1-x^2}$ are analytic at $x=0$. P are $x=\pm 1$.

Hence Legendre's equation has power series solutions of the form $y=\sum_{m=0}^{\infty} a_m x^m$

$$y'=\sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$y''=\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}$$

$$\left(1-x^2\right)y''-2xy'+n(n+1)y=0$$

$$\left(1-x^2\right)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}-2x\sum_{m=1}^{\infty} m a_m x^{m-1}+n(n+1)\sum_{m=0}^{\infty} a_m x^m=0$$

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}-\sum_{m=2}^{\infty} m(m-1)a_m x^m-\sum_{m=1}^{\infty} 2m a_m x^m+\sum_{m=0}^{\infty} n(n+1)a_m x^m=0$$

To obtain the same general power x^k, then we must set $m-2=k \quad\Rightarrow\quad m=k+2$

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k-\sum_{k=2}^{\infty}k(k-1)a_k x^k-\sum_{k=1}^{\infty}2k a_k x^k+\sum_{k=0}^{\infty}n(n+1)a_k x^k=0$$

$k=0$	$2\cdot 1\cdot a_2+n(n+1)a_0$

$k=1$	$3\cdot 2\cdot a_3+\left[-2+n(n+1)\right]a_1$
$k=2$	$4\cdot 3\cdot a_4+\left[-2-4+n(n+1)\right]a_2$
k	$(k+2)(k+1)a_{k+2}+\left[-k(k-1)-2k+n(n+1)\right]a_k$

$$(k+2)(k+1)a_{k+2}+\left[-k^2-k+n(n+1)\right]a_k=0$$

$$\begin{aligned} a_{k+2} &= -\frac{-k^2-k+n^2+n}{(k+2)(k+1)}a_k \\ &= -\frac{(n-k)(n+k+1)}{(k+2)(k+1)}a_k \end{aligned}$$

This is called a ***recurrence relation*** or ***recursion formula***.

$$\begin{aligned} a_2 &= -\frac{n(n+1)}{2!}a_0 \\ a_4 &= -\frac{(n-2)(n+3)}{4\cdot 3}a_2 \\ &= \frac{(n-2)n(n+1)(n+3)}{4!}a_0 \\ &\vdots \quad \vdots \end{aligned}$$

$$\begin{aligned} a_3 &= -\frac{(n-1)(n+2)}{3!}a_1 \\ a_5 &= -\frac{(n-3)(n+4)}{5\cdot 4}a_3 \\ &= \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1 \\ &\vdots \quad \vdots \end{aligned}$$

The general Legendre equation solution is: $y(x)=a_0y_1(x)+a_1y_2(x)$

Where

$$\begin{cases} y_1(x)=1-\dfrac{n(n+1)}{2!}x^2+\dfrac{(n-2)n(n+1)(n+3)}{4!}x^4-\cdots \\ y_2(x)=x-\dfrac{(n-1)(n+2)}{3!}x^3+\dfrac{(n-3)(n-1)(n+2)(n+4)}{5!}x^4-\cdots \end{cases}$$

Legendre Polynomials $P_n(x)$

For Legendre's equation $\left(1-x^2\right)y''-2xy'+n(n+1)y=0$ will happen when the parameter ***n*** is nonnegative integer. Otherwise, when *n* is even, $y_1(x)$ reduces to a polynomial of degree *n*. If *n* is odd, $y_2(x)$ reduces (the same) to a polynomial of degree *n*.

$$a_n=\frac{(2n)!}{2^n(n!)^2}=\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\qquad n\in\mathbb{Z}^+$$

If $n=0 \quad\Rightarrow\quad a_n=1$

$$a_k=-\frac{(k+2)(k+1)}{(n-k)(n+k+1)}a_{k+2}\qquad (k\le n-2)$$

If $k=n-2$

$$\begin{aligned}
a_{n-2}&=-\frac{n(n-1)}{2(2n-1)}a_n\\
&=-\frac{n(n-1)(2n)!}{2(2n-1)2^n(n!)^2}\\
&=-\frac{n(n-1)(2n)(2n-1)(2n-2)!}{2(2n-1)2^n\left[n(n-1)!\right]\left[n(n-1)(n-2)!\right]}\\
&=-\frac{(2n-2)!}{2^n(n-1)!(n-2)!}
\end{aligned}$$

$$\begin{aligned}
a_{n-4}&=-\frac{(n-2)(n-3)}{4(2n-3)}a_{n-2}\\
&=\frac{(n-2)(n-3)}{4(2n-3)}\cdot\frac{(2n-2)!}{2^n(n-1)!(n-2)!}\\
&=\frac{(n-2)(n-3)}{4(2n-3)}\cdot\frac{(2n-2)(2n-3)(2n-4)!}{2^n(n-1)(n-2)(n-3)(n-4)!(n-2)!}\\
&=\frac{2(n-1)(2n-4)!}{4\cdot 2^n(n-1)(n-4)!(n-2)!}\\
&=\frac{(2n-4)!}{2^n 2!(n-2)!(n-4)!}
\end{aligned}$$

In general; $a_{n-2k}=(-1)^k\dfrac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!}$

The resulting solution of Legendre's differential equation is called the Legendre polynomial of degree ***n*** and is denoted by $P_n(x)$.

$$P_n(x) = \sum_{k=0}^{K} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}$$

$$= \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!} x^{n-2} + \cdots$$

$P_0(x) = 1$

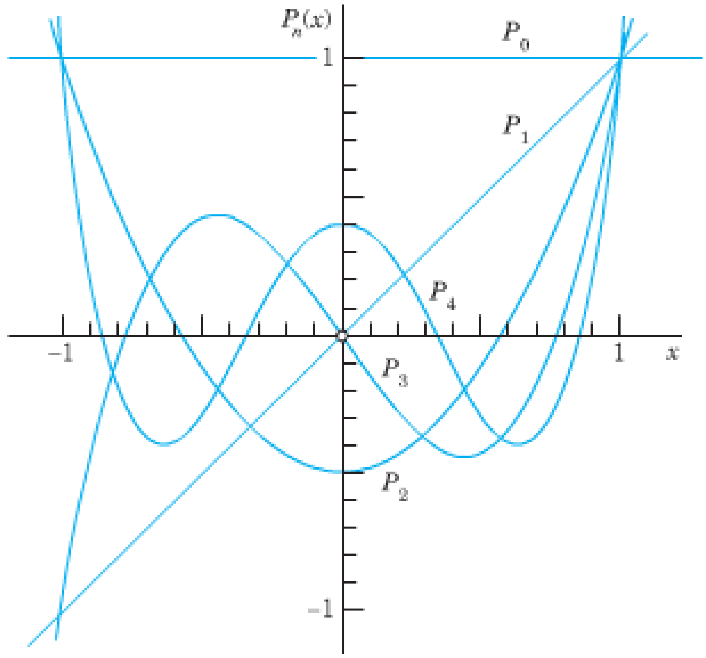
$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$



Exercise *Section* 4.3 – Legendre's Equation

1. Establish the recursion formula using the following two steps

a) Differentiate both sides of equation

$$g(t,\ x)=\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n \text{ with respect to } t \text{ to show that}$$

$$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\left(1-2xt+t^2\right)\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

b) Equate the coefficients of t^n in this equation to show that

$P_1(x)=xP_0(x)$ and

$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)$ *for* $n\ge 1$

2. Show that $P_{2n+1}(0)=0$ *and* $P_{2n}(0)=(-1)^n\dfrac{2n!}{2^{2n}(n!)^2}$

3. Show that $P_n'(0)=(-1)^{n+1}P_n'(-1)=\dfrac{n(n+1)}{2}$

Hint: Use Legendre's equation $\left(1-x^2\right)y''-2xy'+n(n+1)y=0$

4. The differential equation $y''+xy=0$ is called ***Airy's equation***, and its solutions are called ***Airy functions***. Find the series for the solutions y_1 and y_2 where $y_1(0)=1$ and $y_1'(0)=0$, while $y_2(0)=0$ and $y_2'(0)=1$. What is the radius of convergence for these two series?

Section 4.4 – Solution about Singular Points

Solution about Singular Points

The Standard form $\qquad y'' + P(x)y' + Q(x)y = 0$

Definition (*Regular and Irregular Singular Points*)

A singular point x_0 is said to be a ***regular singular point*** of the differential equation if the functions

$$p(x) = \left(x - x_0\right)P(x) \qquad q(x) = \left(x - x_0\right)^2 Q(x)$$

Are both analytic at x_0.

If it isn't regular $\Rightarrow$ irregular singular point of the equation

Example

Determine the singular points for $\left(x^2 - 4\right)^2 y'' + 3(x-2)y' + 5y = 0$

Solution

$$(x-2)^2(x+2)^2 y'' + 3(x-2)y' + 5y = 0$$

$$y'' + 3\frac{x-2}{(x-2)^2(x+2)^2}y' + \frac{5}{(x-2)^2(x+2)^2}y = 0$$

$$P(x) = \frac{3}{(x-2)(x+2)^2} \qquad Q(x) = \frac{5}{(x-2)^2(x+2)^2}$$

The points are: $x = -2,\ 2$

At $x = -2$

$$p(x) = (x+2)\frac{3}{(x-2)(x+2)^2} = \frac{3}{(x-2)(x+2)}$$

$\boxed{x = -2,\ \ 2}$ $\Rightarrow$ is not an analytic at $x = -2$

$$q(x) = (x+2)^2\frac{5}{(x-2)^2(x+2)^2} = \frac{5}{(x-2)^2}$$

$\boxed{x = 2}$ $\Rightarrow$ It is an analytic at $x = 2$

At $x = 2$

$$p(x) = (x-2)\frac{3}{(x-2)(x+2)^2} = \frac{3}{(x+2)}$$

$\boxed{x = -2}$ $\Rightarrow$ It is an analytic at $x = -2$

$$q(x) = (x-2)^2\frac{5}{(x-2)^2(x+2)^2} = \frac{5}{(x+2)^2}$$

$\boxed{x = -2}$ $\Rightarrow$ It is an analytic at $x = -2$

Frobenius *Theorem*

If $x = x_0$ is a regular singular point of the differential equation. There exists at least one solution of the form

$$y = \left(x - x_0\right)^r \sum_{n=0}^{\infty} c_n \left(x - x_0\right)^n = \sum_{n=0}^{\infty} c_n \left(x - x_0\right)^{n+r}$$

r: constant to be determined.

The series will converge at least on some interval $0 < x - x_0 < R$

The model of Frobenius

The simplest equation, of a second-order linear differential equation near the regular singular point $x = 0$, is the constant-coefficient ***equidimensional*** equation

$$x^2 y'' + p_0 x y' + q_0 y = 0$$

If r is a root of the quadratic equation

$$r(r-1) + p_0 r + q_0 = 0$$

Example

Find the exponents in the possible Frobenius series solutions of the equation

$$2x^2(1+x)y'' + 3x(1+x)^3 y' - \left(1 - x^2\right)y = 0$$

Solution

$$y'' + \frac{3x(1+x)^3}{2x^2(1+x)} y' - \frac{(1-x)(1+x)}{2x^2(1+x)} y = 0$$

$$y'' + \frac{3}{2}\frac{(1+x)^2}{x} y' - \frac{1}{2}\frac{1-x}{x^2} y = 0$$

Therefore; $p_0 = \frac{3}{2}, \quad q_0 = -\frac{1}{2}$

The indicial equation is $r(r-1) + \frac{3}{2}r - \frac{1}{2} = r^2 + \frac{1}{2}r - \frac{1}{2} = (r+1)\left(r - \frac{1}{2}\right) = 0$

With roots $r_1 = \frac{1}{2}$ *and* $r_2 = -1$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2} \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{-1} \sum_{n=0}^{\infty} b_n x^n$$

Theorem – Frobenius Series Solutions

Suppose that $x=0$ is a regular point of the equation $\quad x^2y''+p_0xy'+q_0y=0$

Let $\rho>0$ denote the minimum of the radii of convergence of the power series

$$p(x)=\sum_{n=0}^{\infty}p_nx^n \quad \text{and} \quad q(x)=\sum_{n=0}^{\infty}q_nx^n$$

Let r_1 and r_2 be the (real) roots, with $r_1\ge r_2$, of the ***indicial equation*** $I(x)=r(r-1)+p_0r+q_0=0$.

Then

- ✓ For $x>0$, there exist a solution of the form

$$y_1(x)=x^{r_1}\sum_{n=0}^{\infty}a_nx^n \quad (a_0\neq 0)$$ corresponding to the larger root r_1.

$$y_2(x)=y_1(x)\ln x+x^{r_1+1}\sum_{n=0}^{\infty}b_nx^n$$

- ✓ If $r_1-r_2=N$, a positive integer, then the equation has two solutions y_1 and y_2 of the form

$$y_1(x)=x^{r_1}\sum_{n=0}^{\infty}a_nx^n \quad \text{and} \quad y_2(x)=Cy_1(x)+x^{r_2}\sum_{n=0}^{\infty}b_nx^n \quad (a_0,b_0\neq 0)$$

The radii of convergence of the power series of this theorem are all at least ρ. The coefficients in these series (and the constant C) may be determined by direct substitution of the series.

Example

Find the general solution to the equation $2xy'' + y' - 4y = 0$ near the point $x_0 = 0$

Solution

$$\left(\frac{x}{2}\right)2xy'' + \left(\frac{x}{2}\right)y' - \left(\frac{x}{2}\right)4y = 0$$

$$x^2y'' + \frac{1}{2}xy' - 2xy = 0$$

That implies to $p(x) = \frac{1}{2}$ and $q(x) = -2x$, both are analytic. Hence, $x_0 = 0$ is a regular point

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} \qquad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}$$

$$2x^2y'' + xy' - 4xy = 0$$

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - 4x\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - 4x\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[2(n+r)(n+r-1) + (n+r)\right]a_n x^{n+r} - 4x\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-2) + (n+r)\right]a_n x^{n+r} - 4\sum_{n=0}^{\infty}a_n x^{n+r+1} = 0$$

$$x^r\left(\sum_{n=0}^{\infty}\left[(n+r)(2n+2r-1)\right]a_n x^{n} - 4\sum_{n=0}^{\infty}a_n x^{n+1}\right) = 0$$

$$x^r\left(r(2r-1)a_0 + \underbrace{\sum_{n=1}^{\infty}(n+r)(2n+2r-1)a_n x^{n-1}}_{k=n} - \underbrace{4\sum_{n=0}^{\infty}a_n x^{n+1}}_{k=n+1}\right) = 0$$

$$x^r\left(r(2r-1)a_0 + \sum_{k=1}^{\infty}(k+r)(2k+2r-1)a_k x^{k} - 4\sum_{k=1}^{\infty}a_{k-1} x^{k}\right) = 0$$

$$x^r\left(r(2r-1)a_0 + \sum_{k=1}^{\infty}\left[(k+r)(2k+2r-1)a_k - 4a_{k-1}\right]x^{k}\right) = 0$$

$$\begin{cases} r(2r-1)a_0 = 0 & \Rightarrow \boxed{r=0} \quad \boxed{r=\frac{1}{2}} \\ (k+r)(2k+2r-1)a_k - 4a_{k-1} = 0 & \Rightarrow \boxed{a_k = \dfrac{4}{(k+r)(2k+2r-1)}a_{k-1}} \end{cases}$$

$r = 0$

$$a_k = \frac{4}{k(2k-1)}a_{k-1}$$

$$a_1 = \frac{4}{1}a_0$$

$$a_2 = \frac{4}{2\cdot 3}a_1 = \frac{4^2}{1\cdot 2\cdot 3}a_0$$

$$a_3 = \frac{4}{3\cdot 5}a_2 = \frac{4^3}{1\cdot 2\cdot 3\cdot 3\cdot 5}a_0$$

$$a_4 = \frac{4}{4\cdot 7}a_3 = \frac{4^3}{4!(1\cdot 3\cdot 5\cdot 7)}a_0$$

$$a_n = \frac{4^n}{n!\ 1\cdot 3\cdot 5\cdots(2n-1)}a_0$$

$r = \frac{1}{2}$

$$a_k = \frac{4}{\left(k+\frac{1}{2}\right)\left(2k+2\frac{1}{2}-1\right)}a_{k-1} = \frac{4}{k(2k+1)}a_{k-1}$$

$$a_1 = \frac{4}{1\cdot 3}a_0$$

$$a_2 = \frac{4}{2\cdot 5}a_1 = \frac{4^2}{1\cdot 2\cdot 3\cdot 5}a_0$$

$$a_3 = \frac{4}{3\cdot 7}a_2 = \frac{4^3}{3!(3\cdot 5\cdot 7)}a_0$$

$$a_4 = \frac{4}{4\cdot 9}a_3 = \frac{4^3}{4!(3\cdot 5\cdot 7\cdot 9)}a_0$$

$$a_n = \frac{4^n}{n!\ 3\cdot 5\cdots(2n+1)}a_0$$

$$y_1(x) = x^0\left(a_0 + \sum_{n=0}^{\infty}\frac{4}{n!\ 1\cdot 3\cdot 5\cdots(2n-1)}a_0x^n\right) = a_0\left(1+\sum_{n=0}^{\infty}\frac{4}{n!\ 1\cdot 3\cdot 5\cdots(2n-1)}x^n\right)$$

$$y_2(x) = x^{1/2}\left(a_0 + \sum_{n=0}^{\infty}\frac{4^n}{n!\ 3\cdot 5\cdots(2n+1)}a_0x^n\right) = a_0x^{1/2}\left(1+\sum_{n=0}^{\infty}\frac{4^n}{n!\ 3\cdot 5\cdots(2n+1)}x^n\right)$$

$$y(x) = C_1y_1(x) + C_2y_2(x)$$

$$= C_1\left(1+\sum_{n=0}^{\infty}\frac{4}{n!\ 1\cdot 3\cdot 5\cdots(2n-1)}x^n\right) + C_2x^{1/2}\left(1+\sum_{n=0}^{\infty}\frac{4^n}{n!\ 3\cdot 5\cdots(2n+1)}x^n\right)$$

Example

Find the general solution to the equation $3xy'' + y' - y = 0$

Solution

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \qquad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

$3xy'' + y' - y = 0$

$$3x \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$3 \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} c_n (n+r)(3n+3r-3+1) x^{n-1} x^r - \sum_{n=0}^{\infty} c_n x^n x^r = 0$$

$$x^r \left(\sum_{n=0}^{\infty} c_n (n+r)(3n+3r-2) x^{n-1} - \sum_{n=0}^{\infty} c_n x^n \right) = 0$$

$$x^r \left(c_0 r(3r-2) x^{-1} + \underbrace{\sum_{n=1}^{\infty} c_n (n+r)(3n+3r-2) x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \right) = 0$$

$$x^r \left(c_0 r(3r-2) x^{-1} + \sum_{k=0}^{\infty} c_{k+1} (k+r+1)(3k+3r+1) x^k - \sum_{k=0}^{\infty} c_k x^k \right) = 0$$

$$x^r \left(c_0 r(3r-2) x^{-1} + \sum_{k=0}^{\infty} \left[c_{k+1} (k+r+1)(3k+3r+1) - c_k \right] x^k \right) = 0$$

$$\begin{cases} c_0 r(3r-2) = 0 & \Rightarrow \boxed{r=0} \quad \boxed{r=\frac{2}{3}} \\ c_{k+1}(k+r+1)(3k+3r+1) - c_k = 0 & \Rightarrow \boxed{c_{k+1} = \dfrac{c_k}{(k+r+1)(3k+3r+1)}} \end{cases}$$

$r=0$

$$c_{k+1}=\frac{c_k}{(k+1)(3k+1)}$$

$$c_1=c_0$$

$$c_2=\frac{c_1}{2\cdot 4}=\frac{c_0}{(2)(4)}$$

$$c_3=\frac{c_2}{3\cdot 7}=\frac{c_0}{2\cdot 3(4\cdot 7)}$$

$$c_4=\frac{c_3}{4\cdot 10}=\frac{c_0}{2\cdot 3\cdot 4(4\cdot 7\cdot 10)}$$

$$c_n=\frac{c_0}{n!\ 4\cdot 7\cdot 10\cdots(3n-2)}$$

$$y_1(x)=x^0\left(c_0+\sum_{n=0}^{\infty}\frac{c_0}{n!\ 4\cdot 7\cdot 10\cdots(3n-2)}x^n\right)$$

$$=c_0\left(1+\sum_{n=0}^{\infty}\frac{1}{n!\ 4\cdot 7\cdot 10\cdots(3n-2)}x^n\right)$$

$r=\frac{2}{3}$

$$c_{k+1}=\frac{c_k}{\left(k+\frac{5}{3}\right)(3k+3)}=\frac{c_k}{(3k+5)(k+1)}$$

$$c_1=\frac{c_0}{5\cdot 1}$$

$$c_2=\frac{c_1}{8\cdot 2}=\frac{c_0}{5\cdot 8\cdot 1\cdot 2}$$

$$c_3=\frac{c_2}{11\cdot 3}=\frac{c_0}{(5\cdot 8\cdot 11)(1\cdot 2\cdot 3)}$$

$$c_4=\frac{c_3}{14\cdot 4}=\frac{c_0}{(5\cdot 8\cdot 11\cdot 14)(1\cdot 2\cdot 3\cdot 4)}$$

$$c_n=\frac{c_0}{n!\ 5\cdot 8\cdot 11\cdots(3n+2)}$$

$$y_2(x)=x^{2/3}\left(c_0+\sum_{n=0}^{\infty}\frac{c_0}{n!\ 5\cdot 8\cdots(3n+2)}x^n\right)$$

$$=c_0x^{2/3}\left(1+\sum_{n=0}^{\infty}\frac{1}{n!\ 5\cdot 8\cdots(3n+2)}x^n\right)$$

$$y(x)=C_1y_1(x)+C_2y_2(x)$$

$$=C_1\left(1+\sum_{n=0}^{\infty}\frac{1}{n!\ 4\cdot 7\cdot 10\cdots(3n-2)}x^n\right)+C_2x^{2/3}\left(1+\sum_{n=0}^{\infty}\frac{1}{n!\ 5\cdot 8\cdots(3n+2)}x^n\right)$$

OR

$$y''+\frac{1}{3x}y'-\frac{1}{3x}y=0$$

$$p(x)=\left(x-x_0\right)P(x)=x\frac{1}{3x}=\frac{1}{3} \qquad p(x)=a_0+a_1x+\cdots$$

$$q(x)=\left(x-x_0\right)^2Q(x)=x^2\left(-\frac{1}{3x}\right)=-\frac{1}{3}x \qquad q(x)=b_0+b_1x+\cdots$$

$$r(r-1)+a_0r+b_0=0$$

$$r(r-1)+\frac{1}{3}r+0=0$$

$$r^2-r+\frac{1}{3}r=0$$

$$3r^2-2r=0$$

$$r(3r-2)=0$$

Exercises *Section* 4.4 – Solution about Singular Points

1. Find the Frobenius series solutions of $2x^2y'' + 3xy' - \left(1 + x^2\right)y = 0$

2. Find the general solution to the equation $2xy'' + (1 + x)y' + y = 0$

3. Find a Frobenius solution of Bessel's equation of order zero $x^2y'' + xy' + x^2y = 0$

Section 4.5 – Bessel's Equation and Bessel Functions

In this section we consider three special cases of ***Bessel's equation***

$$x^2y'' + xy' + \left(x^2 - \upsilon^2\right)y = 0$$

Where υ is a constant, and the solutions are called ***Bessel functions***.
The indicial equation is

$$I(r) = r(r-1) + p_0 r + q_0 = r(r-1) + r - \upsilon^2 = 0$$

$$r^2 - \upsilon^2 = 0 \quad \rightarrow \quad r = \pm\upsilon$$

We will consider the three cases $\upsilon = 0$, $\upsilon = \frac{1}{2}$, and $\upsilon = 1$ for the interval $x > 0$.

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + x\sum_{n=1}^{\infty}(n+r)a_n x^{n+r-1} + \left(x^2 - \upsilon^2\right)\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=1}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r+2} - \upsilon^2\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$x^r\left(\sum_{n=0}^{\infty}\left[(n+r)(n+r-1) + (n+r) - \upsilon^2\right]a_n x^n + \sum_{n=0}^{\infty}a_n x^{n+2}\right) = 0$$

$$(n+r)(n+r-1) + (n+r) = (n+r)(n+r-1+1) = (n+r)^2$$

$$\left(r^2 - \upsilon^2\right)a_0 + \left((1+r)^2 - \upsilon^2\right)a_1 + \underbrace{\sum_{n=2}^{\infty}\left[(n+r)^2 - \upsilon^2\right]a_n x^n}_{k=n-2} + \underbrace{\sum_{n=0}^{\infty}a_n x^{n+2}}_{k=n} = 0$$

$$\sum_{k=0}^{\infty}\left[(k+2+r)^2 - \upsilon^2\right]a_{k+2}x^{k+2} + \sum_{k=0}^{\infty}a_k x^{k+2} = 0$$

$$\sum_{k=0}^{\infty}\left[\left((k+2+r)^2 - \upsilon^2\right)a_{k+2} + a_k\right]x^{k+2} = 0$$

$$\left((k+2+r)^2 - \upsilon^2\right)a_{k+2} + a_k = 0$$

$$a_{k+2} = \frac{-a_k}{(k+2+r)^2 - \upsilon^2}$$

$$(k+2+r)^2 - \upsilon^2 = (k+2)^2 + 2r(k+2) + r^2 - \upsilon^2$$

$$= (k+2)(k+2+2r) + r^2 - \upsilon^2 \qquad r^2 - \upsilon^2 = 0$$

$$a_{k+2} = \frac{-a_k}{(k+2)(k+2+2\upsilon)}$$

We must choose $a_1 = 0 \quad \rightarrow \quad a_3 = a_5 = \cdots = 0$

$$a_{2n} = -\frac{1}{2n(2n+2\upsilon)} a_{n-2} = -\frac{1}{2^2 n(n+\upsilon)} a_{n-2} \qquad (2n = k+2)$$

$$a_2 = -\frac{1}{2^2 \cdot 1 \cdot (1+\nu)} a_0$$

$$a_4(0) = -\frac{1}{2^2 \cdot 2(2+\nu)} a_2 = \frac{1}{2^4 \cdot 1 \cdot 2(1+\nu)(2+\nu)} a_0$$

$$a_6(0) = -\frac{1}{2^2 \cdot 3(3+\nu)} a_4 = -\frac{1}{2^6 \cdot 1 \cdot 2 \cdot 3(1+\nu)(2+\nu)(3+\nu)} a_0$$

$$\vdots \qquad \vdots \;\; \vdots$$

$$a_{2n} = \frac{(-1)^n}{2^{2n} n!(1+\upsilon)(2+\upsilon)\cdots(n+\upsilon)} a_0, \quad n = 1, 2, 3, \ldots$$

Gamma Function

$$(\upsilon+1)\cdot(\upsilon+2)\cdot\ldots\cdot(\upsilon+n) = \frac{(\upsilon+n)!}{\upsilon!}$$

The gamma function is defined by $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \quad$ *for* $x > 0$

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} = \frac{\Gamma(x+n)}{x\cdot(x+1)\cdot\ldots\cdot(x+n-1)}$$

$$x! = \Gamma(x+1)$$

$$(\upsilon+n)! = \Gamma(\upsilon+n+1)$$

$$a_{2n} = \frac{(-1)^n}{2^{2n+\upsilon} n!\Gamma(1+\upsilon+n)}, \quad n = 0, 1, 2, 3, \ldots$$

The series solution is denoted by $J_\upsilon(x)$: $\qquad J_\upsilon(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\upsilon+n)} \left(\frac{x}{2}\right)^{2n+\upsilon}$

For $r_2 = -\upsilon$, then $\qquad J_{-\upsilon}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-\upsilon+n)} \left(\frac{x}{2}\right)^{2n-\upsilon}$

The functions $J_\upsilon(x)$ and $J_\upsilon(x)$ are called the ***Bessel function of the first kind*** of order υ and $-\upsilon$.

Bessel Equation of Order Zero

In this case $\upsilon = 0$, that implies to Bessel's equation: $x^2 y'' + xy' + x^2 y = 0$

The roots of the indicial equation are equal: $r_1 = r_2 = 0$

Hence, $y_1(x) = a_0 \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} \right]$

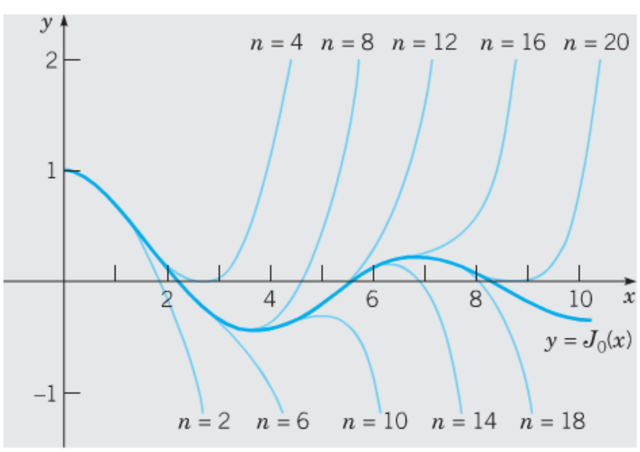
$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n)} \left(\frac{x}{2}\right)^{2n}$$

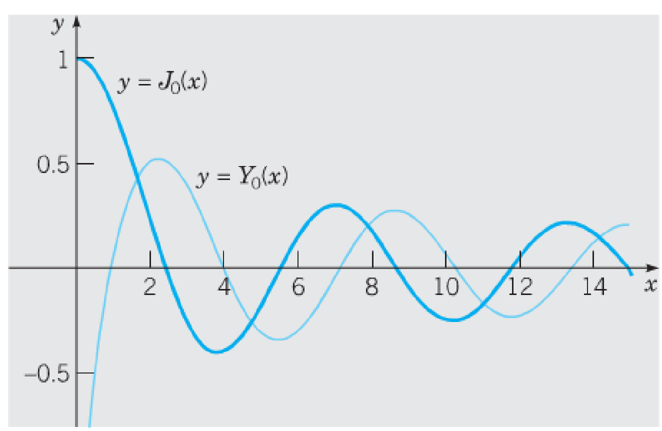
$$y_2(x) = J_0(x)\ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H(n)}{(n!)^2} \left(\frac{x}{2}\right)^{2n} \qquad H(n) = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{n}$$

$$\begin{aligned} Y_0(x) &= \frac{2}{\pi}\left[y_2(x) + (\gamma - \ln 2) J_0(x) \right] \\ &= \frac{2}{\pi} \left[\left(\ln \frac{x}{2} + \gamma \right) J_0(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} H(n)}{(n!)^2} \left(\frac{x}{2}\right)^{2n} \right] \end{aligned}$$

Where γ is ***Euler's constant***, defined by

$$\begin{aligned} \gamma &= \lim_{n\to\infty} \left[H(n) - \ln n \right] \\ &= \lim_{x\to\infty} \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n \right] \\ &= 0.5772156\cdots \end{aligned}$$





Bessel Equation of Order One-Half

In this case $\upsilon = \frac{1}{2}$, that implies to Bessel's equation: $x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$

The roots of the indicial equation are equal: $r_1 = \frac{1}{2}, \quad r_2 = -\frac{1}{2}$

$$a_{2n} = -\frac{1}{2^2 n(n+\upsilon)} a_{n-2} = -\frac{1}{2^2 n\left(n+\frac{1}{2}\right)} a_{n-2} = -\frac{1}{2n(2n+1)} a_{n-2}$$

$$a_{2n} = \frac{(-1)^n}{2^{2n} n!(1+\upsilon)(2+\upsilon)\cdots(n+\upsilon)} a_0, \quad n = 1,2,3,\ldots$$

Taking $a_0 = 1$, we obtain

$$y_1(x) = x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\right] = x^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x, \quad x > 0$$

For $r_2 = -\frac{1}{2}$, $a_{2n} = \frac{(-1)^n}{(2n)!} a_0$, $\quad a_{2n+1} = \frac{(-1)^n}{(2n+1)!} a_1$, $\quad n = 1,2,\ldots$

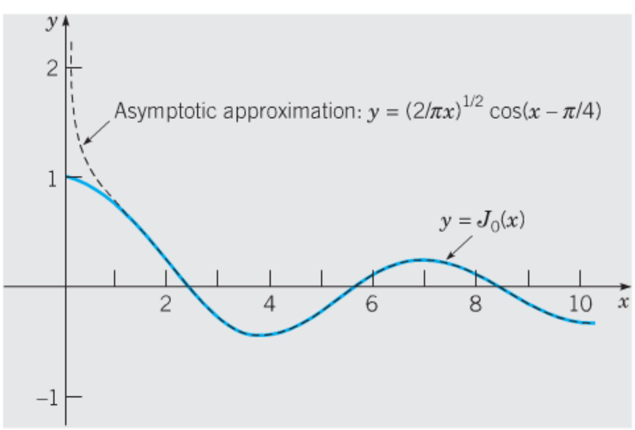
$$y_2(x) = x^{-1/2}\left[a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right]$$

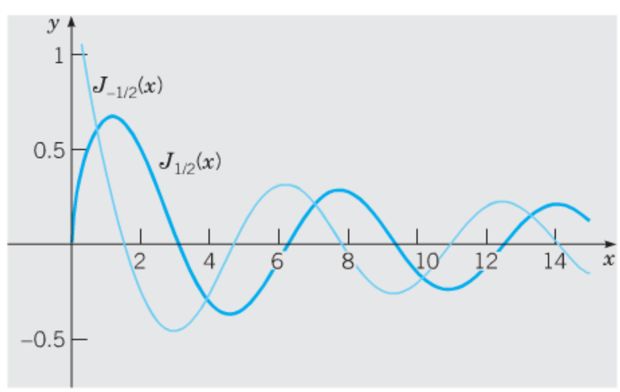
$$= a_0 \frac{\cos x}{x^{1/2}} + a_1 \frac{\sin x}{x^{1/2}}$$

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \quad x > 0$$

The general solution is:

$$y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$$





Bessel Equation of Order One

In this case $\upsilon = 1$, that implies to Bessel's equation: $x^2 y'' + xy' + \left(x^2 - 1\right) y = 0$

The roots of the indicial equation are equal: $r_1 = 1, \quad r_2 = -1$

$$a_{2n} = -\frac{1}{2^2 n(n+1)} a_{n-2}$$

$$a_{2n} = \frac{(-1)^n}{2^{2n} n!(n+1)!} a_0, \quad n = 1, 2, 3, \ldots$$

Taking $a_0 = \frac{1}{2}$, we obtain

$$J_1(x) = \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n+1)!n!}$$

$$y_2(x) = -J_1(x)\ln x + \frac{1}{x}\left[1 - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(H_n + H_{n-1}\right)}{2^{2n} n!(n-1)!} x^{2n}\right]$$

$$Y_1(x) = \frac{2}{\pi}\left[-y_2(x) + (\gamma - \ln 2) J_1(x)\right]$$

The general solution is:

$$y = c_1 J_1(x) + c_2 Y_1(x)$$