Solution *Section* 2.1 – Functions and Graphs

Exercise

$$f(x)=\begin{cases} 2+x & \text{if } x<-4 \\ -x & \text{if } -4\le x\le 2 \\ 3x & \text{if } x>2 \end{cases}$$

Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

Solution

a) $f(-5)=2-5=-3$

b) $f(-1)=-(-1)=1$

c) $f(0)=-0=0$

d) $f(3)=3(3)=9$

Exercise

$$f(x)=\begin{cases} -2x & \text{if } x<-3 \\ 3x-1 & \text{if } -3\le x\le 2 \\ -4x & \text{if } x>2 \end{cases}$$

Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

Solution

a) $f(-5)=-2(-5)=10$

b) $f(-1)=3(-1)-1=-4$

c) $f(0)=3(0)-1=-1$

d) $f(3)=-4(3)=-12$

Exercise

$$f(x)=\begin{cases} x^3+3 & \text{if } -2\le x\le 0 \\ x+3 & \text{if } 0<x<1 \\ 4+x-x^2 & \text{if } 1\le x\le 3 \end{cases}$$

Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

Solution

a) $f(-5)=$ *doesn't exist*

b) $f(-1)=(-1)^3+3$

$\quad\quad\underline{=2}$

c) $f(0)=(0)^3+3$

$= 3$

d) $f(3) = 4 + (3) - (3)^2$

$= -2$

Exercise

$$h(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & if \quad x \neq 3 \\ 6 & if \quad x = 3 \end{cases}$$

Find: $h(5)$, $h(0)$, *and* $h(3)$

Solution

a) $h(5) = \dfrac{5^2 - 9}{5 - 3}$

$= 8$

b) $h(0) = \dfrac{0^2 - 9}{0 - 3}$

$= 3$

c) $h(3) = 6$

Exercise

$$f(x) = \begin{cases} 3x + 5 & if \quad x < 0 \\ 4x + 7 & if \quad x \geq 0 \end{cases}$$

Find

a) $f(0)$ *b)* $f(-2)$ *c)* $f(1)$ *d)* $f(3) + f(-3)$ *e)* Graph $f(x)$

Solution

a) $f(0) = 4(0) + 7$

$= 7$

b) $f(-2) = 3(-2) + 5$

$= -1$

c) $f(1) = 4(1) + 7$

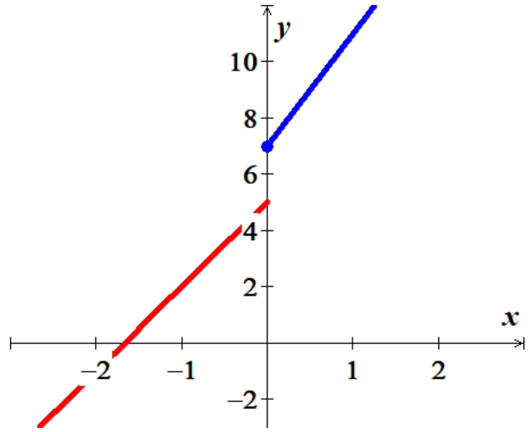
$= 11$

d) $f(3) + f(-3) = 4(3) + 7 + 3(-3) + 5$

$= 12 + 7 - 9 + 5$

$= 15$

e)



Exercise

$$f(x)=\begin{cases} 6x-1 & \text{if} \quad x<0 \\ 7x+3 & \text{if} \quad x\ge 0 \end{cases} \quad \text{Find}$$

a) $f(0)$ *b)* $f(-1)$ *c)* $f(4)$ *d)* $f(2)+f(-2)$ *e)* Graph $f(x)$

Solution

a) $f(0)=7(0)+3$

$\underline{=3}$

b) $f(-2)=6(-1)-1$

$\underline{=-7}$

c) $f(4)=7(4)+3$

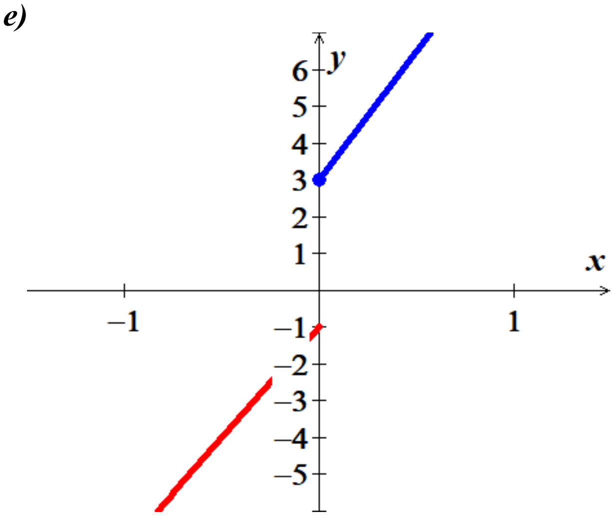
$\underline{=31}$

d) $f(2)+f(-2)=7(2)+3+6(-2)-1$

$=14+3-12-1$

$\underline{=4}$

e)



Exercise

$$f(x)=\begin{cases} 2x+1 & if \quad x\le 1 \\ 3x-2 & if \quad x>1 \end{cases}$$ Find

a) $f(0)$ *b)* $f(2)$ *c)* $f(-2)$ *d)* $f(1)+f(-1)$ *e)* Graph $f(x)$

Solution

a) $f(0)=2(0)+1$

$\quad =1$

b) $f(2)=3(2)-2$

$\quad =4$

c) $f(-2)=2(-2)+1$

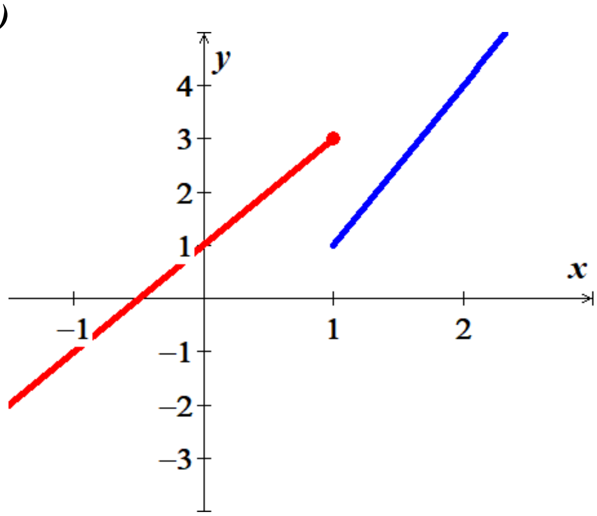
$\quad =-3$

d) $f(1)+f(-1)=(2(1)+1)+(2(-1)+1)$

$\quad =2+1-2+1$

$\quad =2$

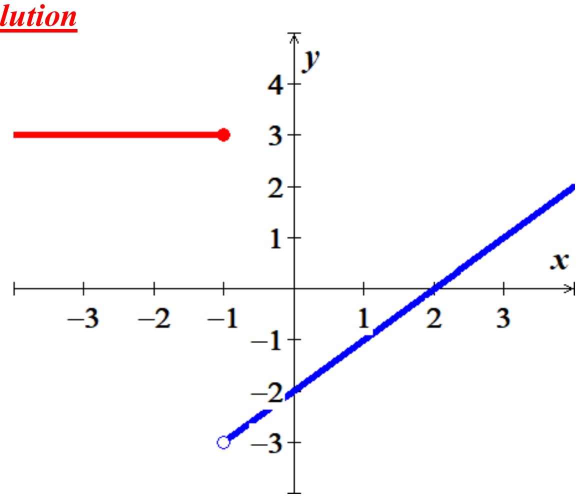
e)



Exercise

Graph the piecewise function defined by $f(x)=\begin{cases} 3 & if \quad x \le -1 \\ x-2 & if \quad x > -1 \end{cases}$

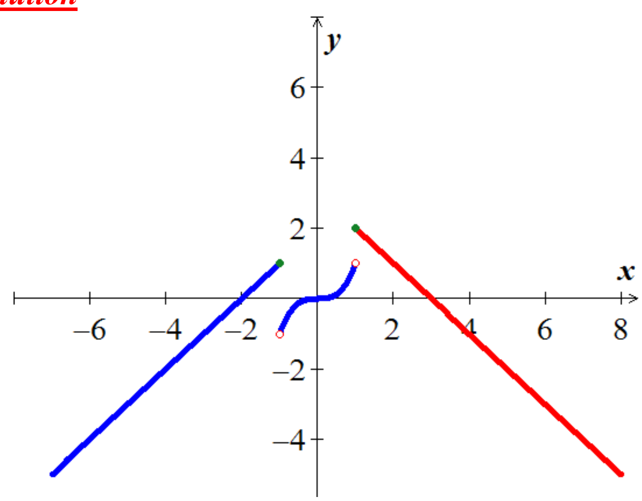
Solution



Exercise

Sketch the graph $f(x)=\begin{cases} x+2 & if & x \le -1 \\ x^3 & if & -1 < x < 1 \\ -x+3 & if & x \ge 1 \end{cases}$

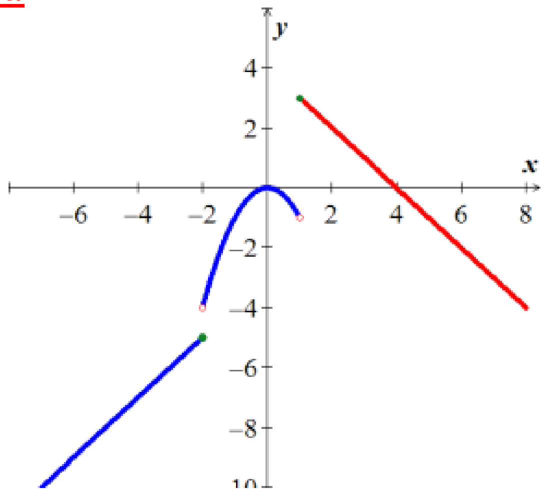
Solution



Exercise

Sketch the graph $f(x)=\begin{cases} x-3 & if \quad x\le -2 \\ -x^2 & if \quad -2<x<1 \\ -x+4 & if \quad x\ge 1 \end{cases}$

Solution



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$f(x)=x^2-2x+3$

Solution

Relative Maximum: *None*

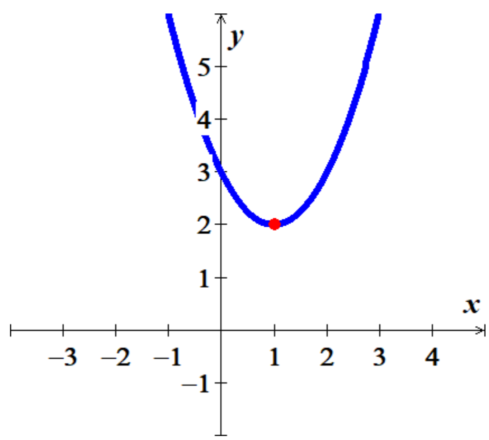
Minimum Point: $(1,\ 2)$

Increasing: $(1,\ \infty)$

Decreasing: $(-\infty,\ 1)$

Domain: $\mathbb{R}$

Range: $[2,\ \infty)$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = -x^2 - 2x + 3$$

Solution

Maximum Point: $(-1,\ 4)$

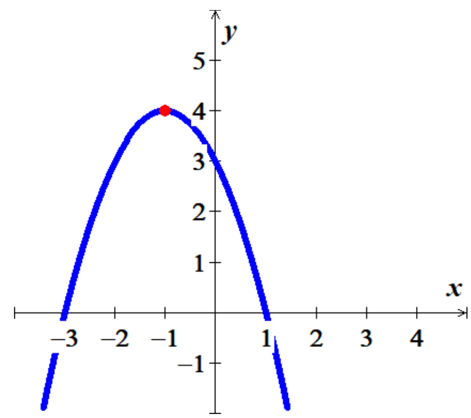
Relative Minimum: *None*

Increasing: $(-\infty,\ -1)$

Decreasing: $(-1,\ \infty)$

Domain: $\mathbb{R}$

Range: $(-\infty,\ 4]$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = -x^3 + 3x^2$$

Solution

Relative Maximum: $(2,\ 4)$

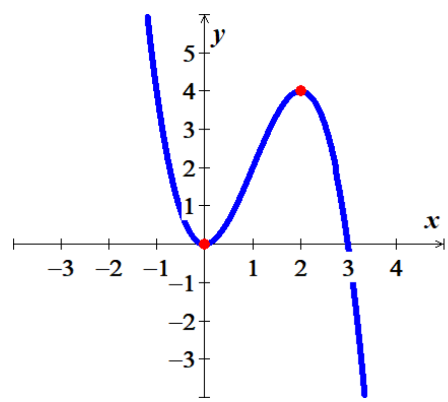
Relative Minimum: $(0,\ 0)$

Increasing: $(0,\ 2)$

Decreasing: $(-\infty,\ 0)\quad(2,\ \infty)$

Domain: $\mathbb{R}$

Range: $\mathbb{R}$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = x^3 - 3x^2$$

Solution

Relative Maximum: $(0,\ 0)$

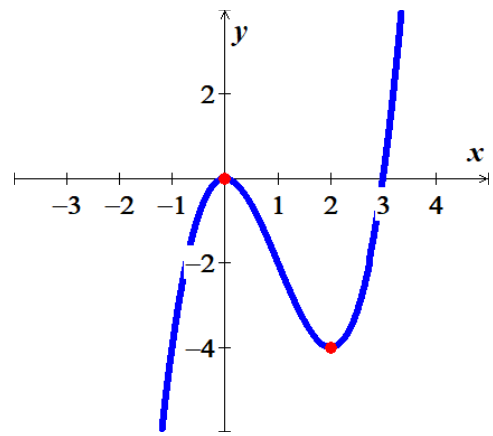
Relative Minimum: $(2,\ -4)$

Increasing: $(-\infty,\ 0)\ (2,\ \infty)$

Decreasing: $(0,\ 2)$

Domain: $\mathbb{R}$

Range: $\mathbb{R}$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x) = \frac{1}{4}x^4 - 2x^2$$

Solution

Relative Maximum: $(0,\ 0)$

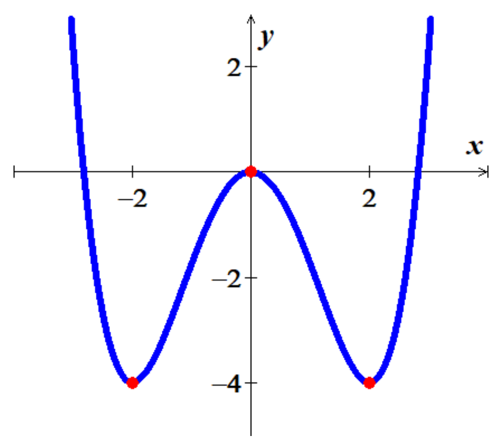
Minimum Points: $(-2,\ -4) \& (2,\ -4)$

Increasing: $(-2,\ 0) \cup (2,\ \infty)$

Decreasing: $(-\infty,\ -2) \cup (0,\ 2)$

Domain: $\mathbb{R}$

Range: $[-4,\ \infty)$



Exercise

Determine any ***relative maximum*** or ***minimum*** of the function, determine the intervals on which the function ***increasing*** or ***decreasing***, and then find the ***domain*** and the ***range***.

$$f(x)=\frac{4}{81}x^4-\frac{8}{9}x^2+4$$

Solution

Relative Maximum: $(0,\ 4)$

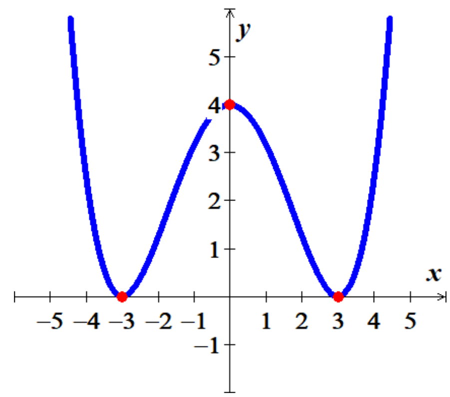
Minimum Points: $(-3,\ 0)\ \&\ (3,\ 0)$

Increasing: $(-3,\ 0)\cup(3,\ \infty)$

Decreasing: $(-\infty,\ -3)\cup(0,\ 3)$

Domain: $\mathbb{R}$

Range: $[0,\ \infty)$



Exercise

The elevation H, in *meters*, above sea level at which the boiling point of water is in t degrees Celsius is given by the function

$$H(t)=1000(100-t)+580(100-t)^2$$

At what elevation is the boiling point 99.5°.

Solution

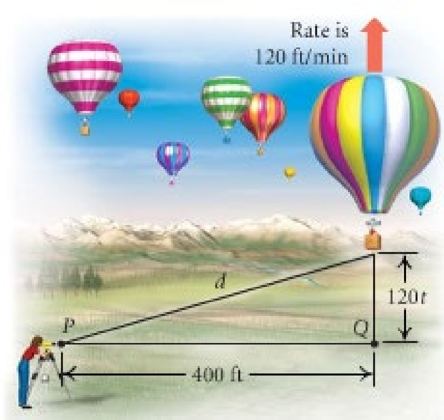
$$H(99.5)=1000(100-99.5)+580(100-99.5)^2$$

$$=645\ m$$

Exercise

A hot-air balloon rises straight up from the ground at a rate of 120 *ft./min*. The balloon is tracked from a rangefinder on the ground at point P, which is 400 *ft*. from the release point Q of the balloon. Let d = the distance from the balloon to the rangefinder and t – the time, in minutes, since the balloon was released. Express d as a function of t.

Solution



$$d^2=(120t)^2+400^2$$

$$d=\sqrt{14400t^2+160000}$$

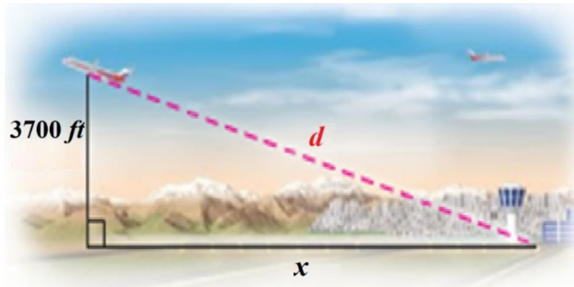
$$d=\sqrt{1600\left(9t^2+100\right)}$$

$$d(t)=40\sqrt{9t^2+100}$$

Exercise

An airplane is flying at an altitude of 3700 *feet*. The slanted distance directly to the airport is d *feet*. Express the horizontal distance x as a function of d.

Solution



$$d^2 = (3,700)^2 + x^2$$

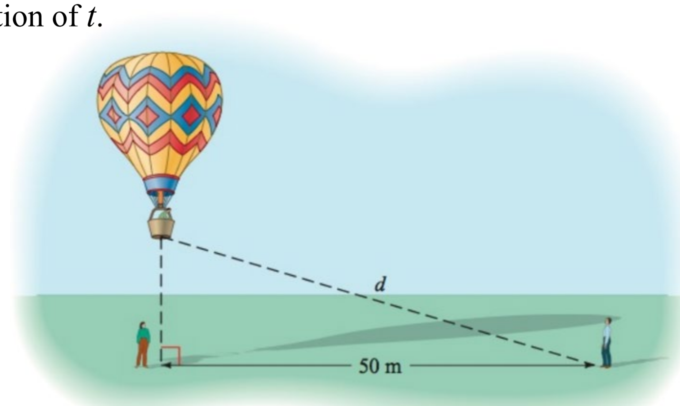
$$h^2 = d^2 - (3700)^2$$

$$\boxed{h(t) = \sqrt{d^2 - (3,700)^2}}$$

Exercise

For the first minute of flight, a hot air balloon rises vertically at a rate of 3 *m/sec*. If t is the time in *seconds* that the balloon has been airborne, write the distance d between the balloon and a point on the ground 50 *meters* from the point to lift off as a function of t.

Solution



$$\boxed{h = 3t} \qquad v = \frac{h}{t}$$

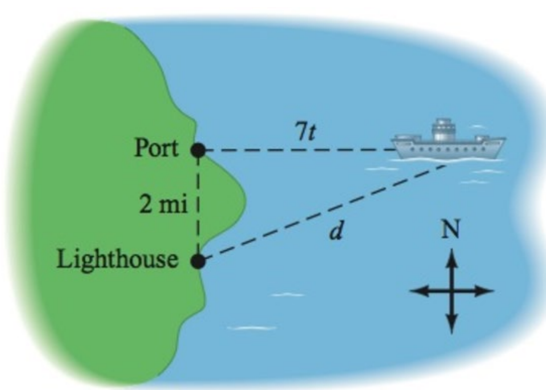
$$d^2 = h^2 + 50^2$$

$$\boxed{d(t) = \sqrt{9t^2 + 2,500}}$$

Exercise

A light house is 2 *miles* south of a port. A ship leaves port and sails east at a rate of 7 *miles* per *hour*. Express the distance d between the ship and the lighthouse as a function of time, given that the ship has been sailing for t hours.

Solution

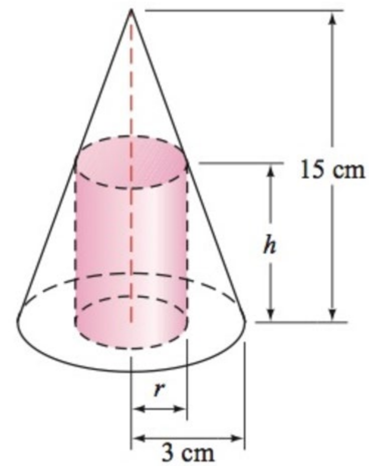


$$d^2 = 4^2 + (7t)^2$$

$$\boxed{d(t) = \sqrt{16 + 49t^2}}$$

Exercise

A cone has an altitude of 15 *cm* and a radius of 3 *cm*. A right circular cylinder of radius *r* and height *h* is inscribed in the cone. Use similar triangles to write *h* as a function of *r*.



Solution

$$\frac{15-h}{15}=\frac{r}{3}$$

$$15-h=5r$$

$$\boxed{h(r)=15-5r}$$

Exercise

Water is flowing into a conical drinking cup with an altitude of 4 *inches* am a radius of 2 *inches*.

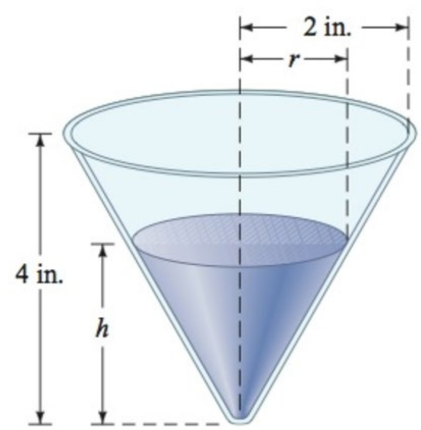
a) Write the radius *r* of the surface of the water as a function of its depth *h*.

b) Write the volume *V* of the water as a function of its depth *h*.

Solution

a) $\frac{h}{4}=\frac{r}{2}$

$$\boxed{r(h)=\frac{1}{2}h}$$



b) $Area=\pi r^2$

$$V=\frac{1}{3}\pi r^2 h$$

$$=\frac{1}{3}\pi\left(\frac{h^2}{4}\right)h$$

$$\boxed{=\frac{1}{12}\pi h^3}$$

Exercise

A water tank has the shape of a right circular cone with height 16 *feet* and radius 8 *feet*. Water is running into the tank so that the radius *r* (in *feet*) of the surface of the water is given by $r=1.5t$, where *t* is the time (in *minutes*) that the water has been running.

a) The area *A* of the surface of the water is $A=\pi r^2$. Find $A(t)$ and use it to determine the area of the surface of the water when $t=2$ *minutes*.

b) The volume *V* of the water is given by $V=\frac{1}{3}\pi r^2 h$. Find $V(t)$ and use it to determine the volume of the water when $t=3$ *minutes*

Solution

c) $Area = \pi r^2$

$$A(t) = \pi\left(\frac{3}{2}t\right)^2$$

$$\boxed{= \frac{9\pi}{4}t^2}$$

d) $\frac{h}{16} = \frac{r}{8}$

$$\boxed{h = 2r}$$

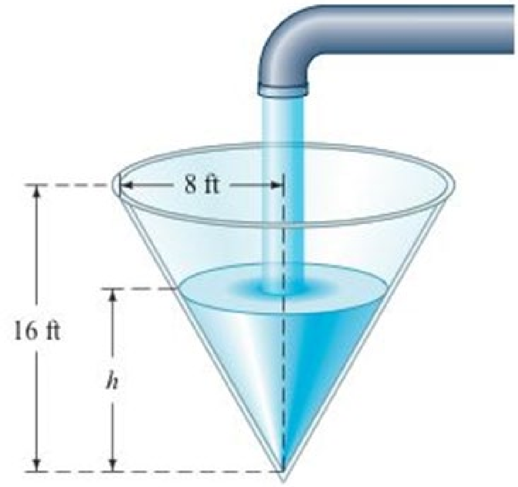
$$V(t) = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi r^2 (2r)$$

$$= \frac{2}{3}\pi r^3$$

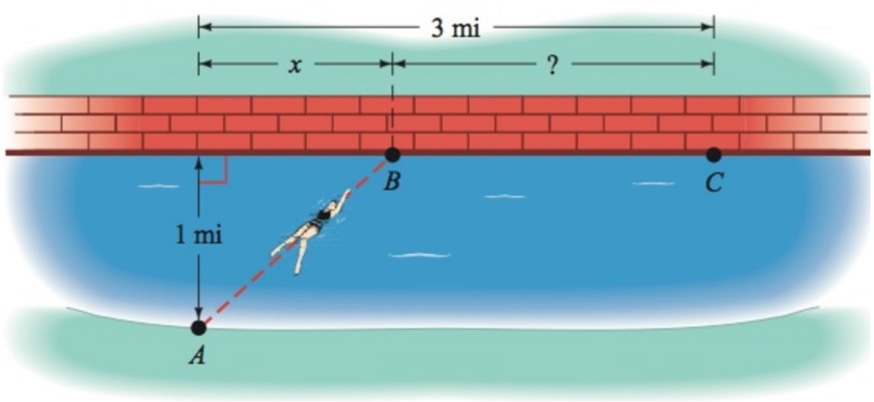
$$= \frac{2}{3}\pi\left(\frac{3}{2}t\right)^3$$

$$\boxed{= \frac{9}{4}\pi t^3}$$



Exercise

An athlete swims from point ***A*** to point ***B*** at a rate of 2 *miles* per *hour* and runs from point ***B*** to point ***C*** at a rate of 8 *miles* per *hour*. Use the dimensions in the figure to write the time t required to reach point ***C*** as a function of x.



Solution

$$Swimming\ distance = \sqrt{x^2 + 1}$$

$$t_{swim} = \frac{\sqrt{x^2 + 1}}{2} \qquad t = \frac{d}{v}$$

$$Running\ distance = 3 - x$$

$$t_{run} = \frac{3-x}{8} \qquad\qquad t = \frac{d}{v}$$

$$\boxed{t_{total} = \frac{\sqrt{x^2+1}}{2} + \frac{3-x}{8}}$$

Exercise

A device used in golf to estimate the distance d, in *yards*, to a hole measures the size $\boldsymbol{s}$, in *inches*, that the 7-*foot* pin appears to be in a viewfinder. Express the distance d as a function of $\boldsymbol{s}$.

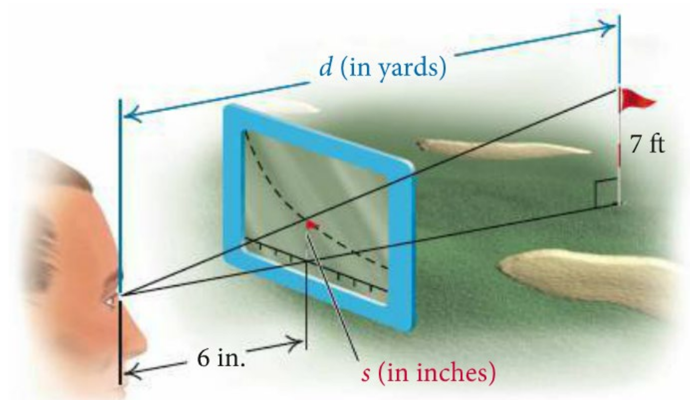
Solution

$$\frac{d}{6} = \frac{7}{s}\frac{ft}{in}$$

$$d = \frac{7}{s}\frac{ft}{in}6in$$

$$d = \frac{42}{s}ft\frac{1yd}{3ft}$$

$$\boxed{d(s) = \frac{14}{s}}$$



Exercise

A *rhombus* is inscribed in a rectangle that is w *meters* wide with a perimeter of 40 m. Each vertex of the rhombus is a midpoint of a side of the rectangle. Express the area of the *rhombus* as a function of the rectangle's width.

Solution

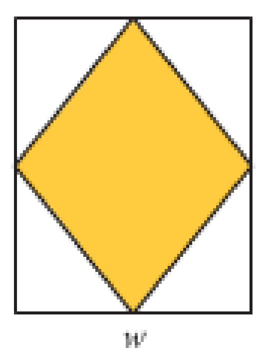
The area of the rhombus = $\frac{1}{2}$ area of the rectangle, since each vertex of the rhombus is a midpoint of a side of the rectangle.

$$\begin{aligned} Perimter: \quad & 2l + 2w = 40 \qquad \textit{Divide both sides by } \mathbf{2} \\ & l + w = 20 \\ & l = 20 - w \end{aligned}$$

$$Area\ of\ the\ rectangle = lw = (20-w)w$$

$$Area\ of\ the\ rhombus = \frac{1}{2}\left(20w - w^2\right)$$

$$\boxed{= -\frac{1}{2}w^2 + 10w}$$



Exercise

The surface area *S* of a right circular cylinder is given by the formula $S = 2\pi rh + 2\pi r^2$. if the height is twice the radius, find each of the following.

a) A function $S(r)$ for the surface area as a function of *r*.

b) A function $S(h)$ for the surface area as a function of *h*.

Solution

Given: $h = 2r$

a) $S = 2\pi rh + 2\pi r^2$

$$S(r) = 2\pi r(2r) + 2\pi r^2$$

$$= 4\pi r^2 + 2\pi r^2$$

$$\underline{= 6\pi r^2}$$

b) $r = \frac{1}{2}h$

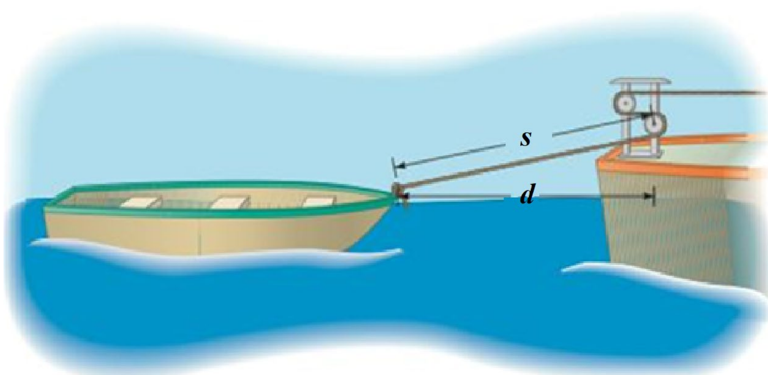
$$S(h) = 2\pi\left(\frac{1}{2}h\right)h + 2\pi\left(\frac{1}{2}h\right)^2$$

$$= \pi h^2 + \frac{1}{2}\pi h^2$$

$$\underline{= \frac{3}{2}\pi h^2}$$

Exercise

A boat is towed by a rope that runs through a pulley that is 4 *feet* above the point where the rope is tied to the boat. The length (in *feet*) of the rope from the boat to the pulley is given by $s = 48 - t$, where *t* is the time in *seconds* that the boat has been in tow. The horizontal distance from the pulley to the boat is *d*.



a) Find $d(t)$

b) Evaluate $s(35)$ and $d(35)$

Solution

a) $s^2 = d^2 + 4^2$

$d^2 = (48-t)^2 - 16$

$d(t) = \sqrt{2,304 - 96t + t^2 - 16}$

$= \sqrt{t^2 - 96t + 2,288}$

b) $s(35) = 48 - 35$

$= 13\ \ feet$

$d(35) = \sqrt{(48-35)^2 - 16}$

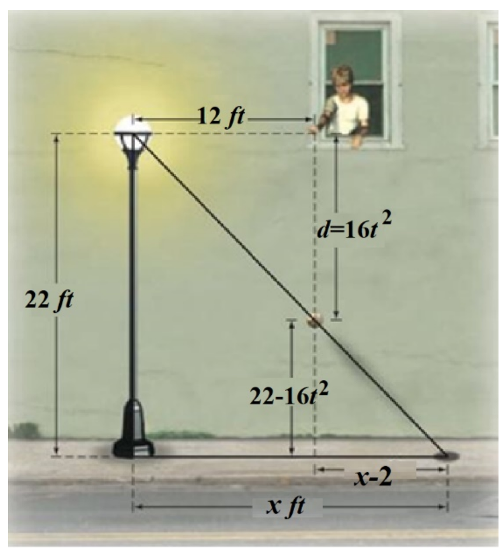
$= \sqrt{13^2 - 16}$

$= \sqrt{153}\ \ feet$

Exercise

The light from a lamppost casts a shadow from a ball that was dropped from a height of 22 *feet* above the ground. The distance *d*, in *feet*, the ball has dropped *t seconds* after it is released is given by $d(t) = 16t^2$. Find the distance *x*, in *feet*, of the shadow from the base of the lamppost as a function of time *t*.

Solution



$$\frac{22-16t^2}{22} = \frac{x-12}{x}$$

$$\left(22-16t^2\right)x = 22(x-12)$$

$$\left(22-16t^2\right)x = 22x - 264$$

$$\left(22-16t^2-22\right)x = -264$$

$$-16t^2 x = -264$$

$$x(t) = \frac{33}{2t^2}$$

Exercise

A right circular cylinder of height *h* and a radius *r* is inscribed in a right circular cone with a height of 10 *feet* and a base with radius 6 *feet*.

a) Express the height *h* of the cylinder as a function of *r*.

b) Express the volume *V* of the cylinder as a function of *r*.

c) Express the volume *V* of the cylinder as a function of *h*.

Solution

a) $\dfrac{h}{10}=\dfrac{6-r}{6}$

$\boxed{h(r)=\dfrac{5}{3}(6-r)}$

b) $V=\pi r^2 h$

$V(r)=\dfrac{5}{3}\pi r^2(6-r)$

$\boxed{=\dfrac{5}{3}\pi\left(6r^2-r^3\right)}$

c) $\dfrac{3}{5}h=6-r$

$r=6-\dfrac{3}{5}h$

$V=\pi r^2 h$

$V(h)=\pi\left(\dfrac{30-3h}{5}\right)^2 h$

$\boxed{=\dfrac{1}{25}\pi h(30-3h)^2}$