Solution Section 2.1 – Graphs and Level Curves

Exercise

Find the specific values for $f(x,y,z) = \dfrac{x-y}{y^2+z^2}$

a) $f(3,-1,2)$ *b)* $f\left(1,\ \frac{1}{2},\ -\frac{1}{4}\right)$ *c)* $f\left(0,\ -\frac{1}{3},\ 0\right)$ *d)* $f(2,\ 2,\ 100)$

Solution

a) $f(3,-1,2) = \dfrac{3-(-1)}{(-1)^2+2^2}$

$$= \frac{4}{5}$$

b) $f\left(1,\ \frac{1}{2},\ -\frac{1}{4}\right) = \dfrac{1-\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^2+\left(-\frac{1}{4}\right)^2}$

$$= \frac{\frac{1}{2}}{\frac{1}{4}+\frac{1}{16}}$$

$$= \frac{\frac{1}{2}}{\frac{5}{16}}$$

$$= \frac{8}{5}$$

c) $f\left(0,\ -\frac{1}{3},\ 0\right) = \dfrac{0-\left(-\frac{1}{3}\right)}{\left(-\frac{1}{3}\right)^2+0^2}$

$$= \frac{\frac{1}{3}}{\frac{1}{9}}$$

$$= 3$$

d) $f(2,\ 2,\ 100) = \dfrac{2-(2)}{(2)^2+100^2}$

$$= 0$$

Exercise

Find the specific values for $f(x,y,z)=\sqrt{49-x^2-y^2-z^2}$

a) $f(0,\ 0,\ 0)$ *b)* $f(2,\ -3,\ 6)$ *c)* $f(-1,\ 2,\ 3)$ *d)* $f\left(\frac{4}{\sqrt{2}},\ \frac{5}{\sqrt{2}},\ \frac{6}{\sqrt{2}}\right)$

Solution

a) $f(0,\ 0,\ 0)=\sqrt{49-0^2-0^2-0^2}$

$$=7$$

b) $f(2,\ -3,\ 6)=\sqrt{49-2^2-(-3)^2-6^2}$

$$=0$$

c) $f(-1,\ 2,\ 3)=\sqrt{49-(-1)^2-2^2-3^2}$

$$=\sqrt{35}$$

d) $f\left(\frac{4}{\sqrt{2}},\ \frac{5}{\sqrt{2}},\ \frac{6}{\sqrt{2}}\right)=\sqrt{49-\left(\frac{4}{\sqrt{2}}\right)^2-\left(\frac{5}{\sqrt{2}}\right)^2-\left(\frac{6}{\sqrt{2}}\right)^2}$

$$=\sqrt{49-\frac{16}{2}-\frac{25}{2}-\frac{36}{2}}$$

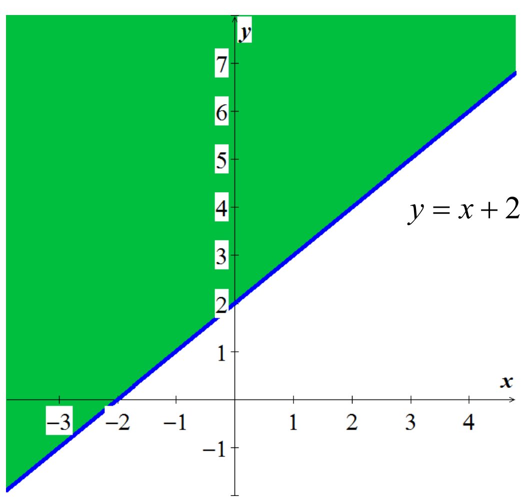
$$=\sqrt{\frac{21}{2}}$$

Exercise

Find and sketch the domain for function $f(x,y)=\sqrt{y-x-2}$

Solution

$y-x-2\ge 0 \quad \Rightarrow \quad y\ge x+2$



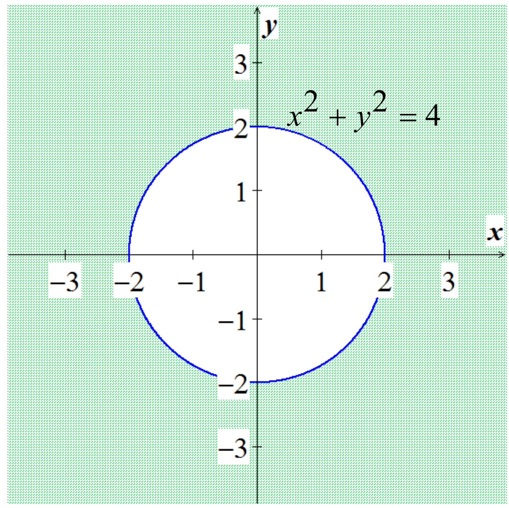
Exercise

Find and sketch the domain for function $f(x,y)=\ln\left(x^2+y^2-4\right)$

Solution

$$x^2+y^2-4>0 \quad \Rightarrow \quad x^2+y^2>4$$

Domain: All points (x,y) outside the circle



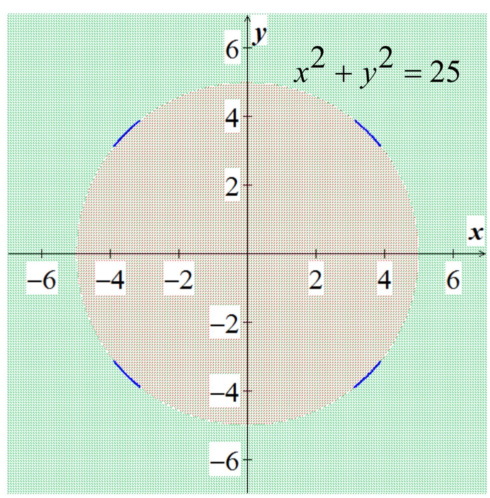
Exercise

Find and sketch the domain for function $f(x,y)=\dfrac{\sin(xy)}{x^2+y^2-25}$

Solution

$$x^2+y^2-25\neq 0 \quad \Rightarrow \quad x^2+y^2\neq 25$$

Domain: All points (x,y) not lying on the circle $x^2+y^2=25$



Exercise

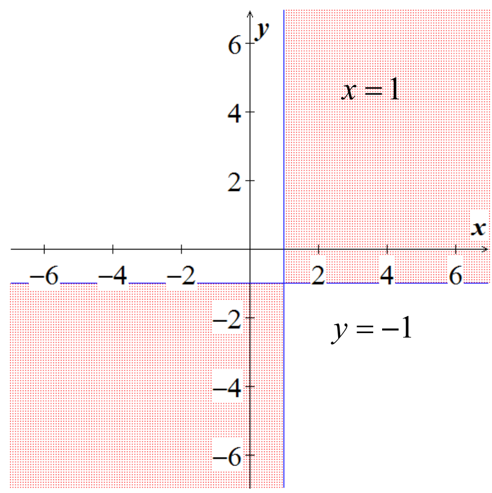
Find and sketch the domain for function $f(x,y)=\ln(xy+x-y-1)$

Solution

$$xy+x-y-1>0 \quad \Rightarrow \quad x(y+1)-(y+1)>0$$

$$(x-1)(y+1)>0$$

Domain: All points (x,y) satisfying $(x-1)(y+1)>0$



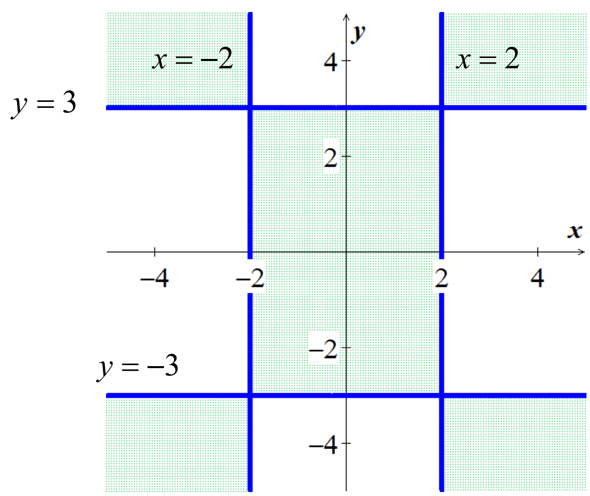
Exercise

Find and sketch the domain for function $f(x,y)=\sqrt{(x^2-4)(y^2-9)}$

Solution

$$(x^2-4)(y^2-9)\ge 0 \quad \Rightarrow \quad (x-2)(x+2)(y-3)(y+3)\ge 0$$

Domain: All points (x,y) satisfying $(x-2)(x+2)(y-3)(y+3)\ge 0$

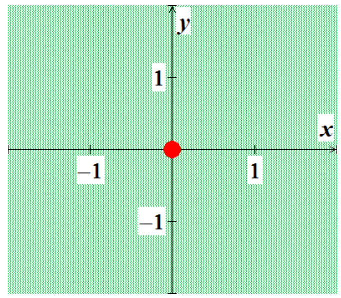


Exercise

Find and sketch the domain for function $f(x, y) = \dfrac{1}{x^2 + y^2}$

Solution

$Domain = \{ (x, y) \mid (x, y) \neq (0, 0) \}$

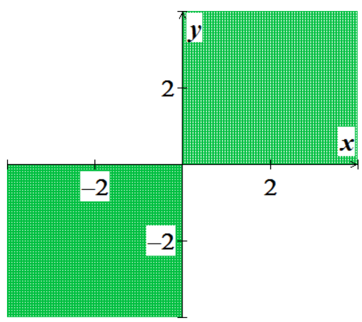


Exercise

Find and sketch the domain for function $f(x, y) = \ln xy$

Solution

$Domain = \{ (x, y) \mid xy > 0 \}$

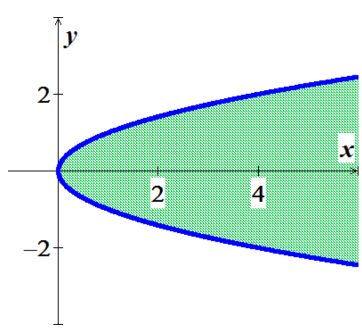


Exercise

Find and sketch the domain for function $f(x, y) = \sqrt{x - y^2}$

Solution

$Domain = \{ (x, y) \mid x > y^2 \}$

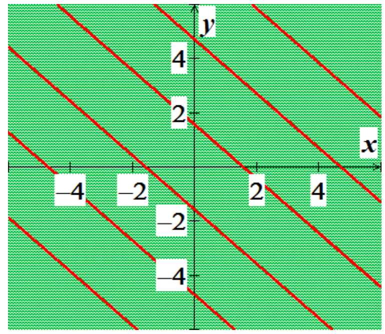


Exercise

Find and sketch the domain for function $\quad f(x, y) = \tan(x+y)$

Solution

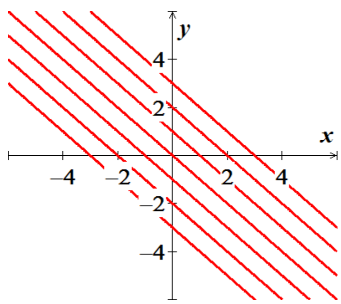
$$Domain = \left\{ (x,\ y) \mid\ x + y \neq \frac{\pi}{2} + k\pi \right\} \qquad (k \in \mathbb{Z})$$



Exercise

Find and sketch the level curves $f(x,y) = c$ on the same set of coordinate axes for the given values of c, we refer to these level curves as a contour map. $f(x,y) = x + y - 1, \quad c = -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3$

Solution

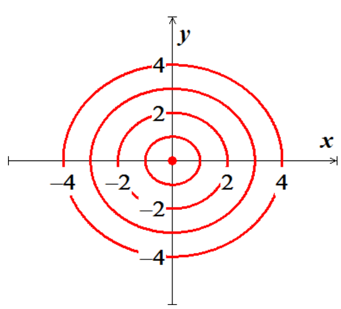


Exercise

Find and sketch the level curves $f(x,y) = c$ on the same set of coordinate axes for the given values of c, we refer to these level curves as a contour map.

$f(x,y) = x^2 + y^2, \quad c = 0,\ 1,\ 4,\ 9,\ 16,\ 25$

Solution



Exercise

For the function: $f(x,y)=4x^2+9y^2$:

a) Find the function's domain
b) Find the function's range
c) Find the function's level curves
d) Find the boundary of the function's domain
e) Determine if the domain is an open region, a closed region, or neither
f) Decide if the domain is bounded or unbounded

Solution

a) *Domain*: all points in the xy-plane

b) *Range*: $z \ge 0$

c) Level curves: For $f(x,y)=0 \quad \to \quad Origin$

For $f(x,y)=c>0 \quad \to \quad$ *ellipses* with center (0, 0) and major and minor axes along the x- and y-axes, respectively

d) No boundary points

e) Both open and closed

f) Unbounded

Exercise

For the function: $f(x,y)=xy$:

a) Find the function's domain
b) Find the function's range
c) Find the function's level curves
d) Find the boundary of the function's domain
e) Determine if the domain is an open region, a closed region, or neither
f) Decide if the domain is bounded or unbounded

Solution

a) *Domain*: all points in the xy-plane

b) *Range*: $\mathbb{R}$

c) Level curves: Hyperbolas with the x- and y-axes as asymptotes when $f(x,y)\ne 0$ and the x- and y-axes when $f(x,y)=0$

d) No boundary points

e) Both open and closed

f) Unbounded

Exercise

For the function: $f(x,y) = e^{-\left(x^2+y^2\right)}$

a) Find the function's domain
b) Find the function's range
c) Find the function's level curves
d) Find the boundary of the function's domain
e) Determine if the domain is an open region, a closed region, or neither
f) Decide if the domain is bounded or unbounded

Solution

a) *Domain*: all points in the *xy*-plane

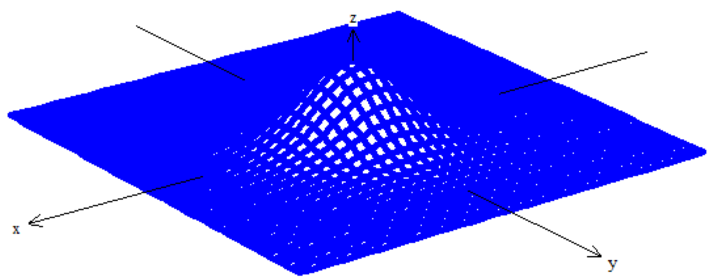
b) *Range*: $0 < z \le 1$

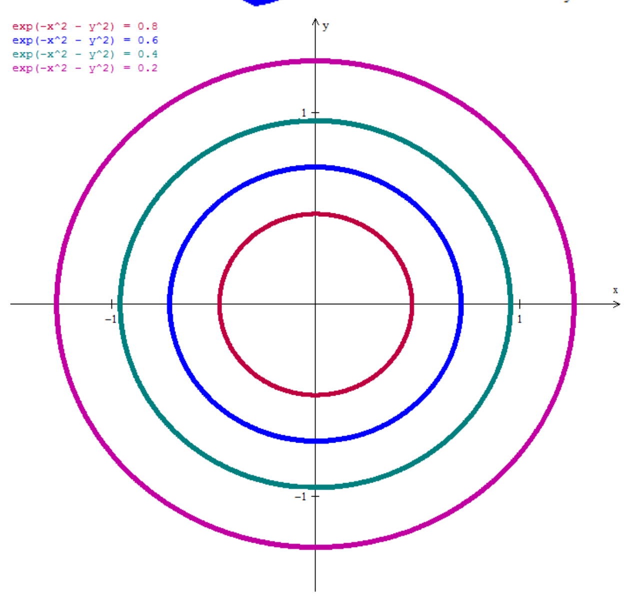
c) Level curves are the origin itself and the circles with center (0, 0) and radii $r > 0$

d) No boundary points

e) Both open and closed

f) Unbounded





Exercise

For the function: $f(x,y)=\ln\left(9-x^2-y^2\right)$

a) Find the function's domain
b) Find the function's range
c) Find the function's level curves
d) Find the boundary of the function's domain
e) Determine if the domain is an open region, a closed region, or neither
f) Decide if the domain is bounded or unbounded

Solution

$$9-x^2-y^2>0 \quad \to \quad x^2+y^2<9$$

a) *Domain*: all points inside the circle $x^2+y^2=9$

b) *Range*: $z<\ln 9$

c) Level curves are circles centered at the origin and radii $r<9$

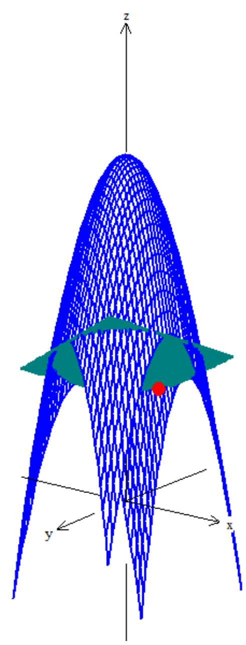
d) Boundary: the circle $x^2+y^2=9$

e) Open

f) Bounded

Exercise

Find an equation for $f(x,y)=16-x^2-y^2$ and sketch the graph of the level curve of the function $f(x,y)$ that passes through the point $\left(2\sqrt{2},\ \sqrt{2}\right)$



Solution

$$z=\left(16-x^2-y^2\right)_{\left(2\sqrt{2},\ \sqrt{2}\right)}$$

$$=16-\left(2\sqrt{2}\right)^2-\left(\sqrt{2}\right)^2$$

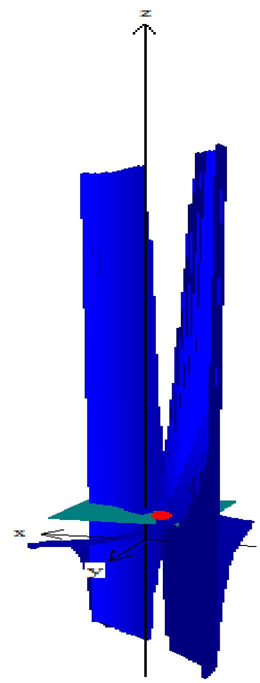
$$=6$$

$$6=16-x^2-y^2$$

$$\boxed{x^2+y^2=10}$$

Exercise

Find an equation for $f(x,y)=\dfrac{2y-x}{x+y+1}$ and sketch the graph of the level curve of the function $f(x,y)$ that passes through the point $(-1,\ 1)$



Solution

$$z=\left(\frac{2y-x}{x+y+1}\right)_{(-1,1)}$$

$$=\frac{2(1)-(-1)}{-1+1+1}$$

$$=3$$

$$3=\frac{2y-x}{x+y+1}$$

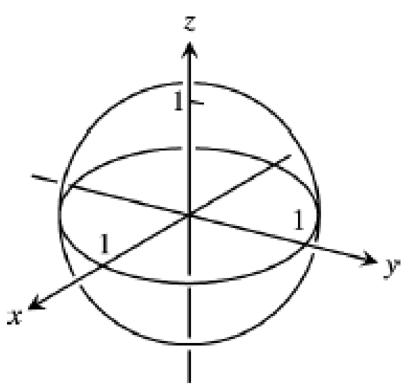
$$3x+3y+3=2y-x$$

$$\underline{y=-4x-3}$$

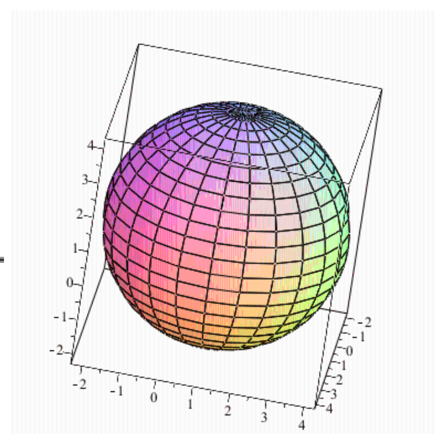
Exercise

Sketch a typical level surface for the function $f(x,y,z)=x^2+y^2+z^2$

Solution



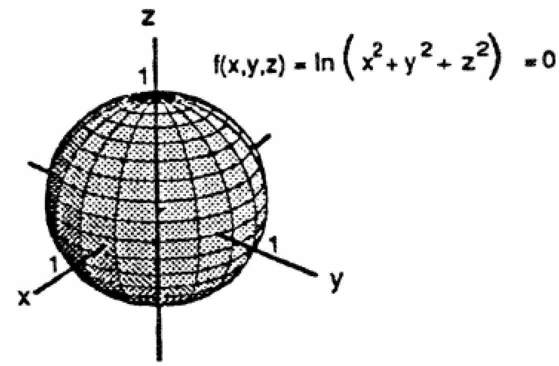
$f(x,y,z)=x^2+y^2+z^2=1$



Exercise

Sketch a typical level surface for the function $f(x,y,z)=\ln\left(x^2+y^2+z^2\right)$

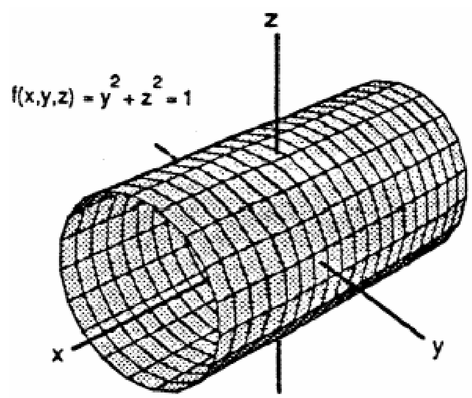
Solution



Exercise

Sketch a typical level surface for the function $f(x,y,z) = y^2 + z^2$

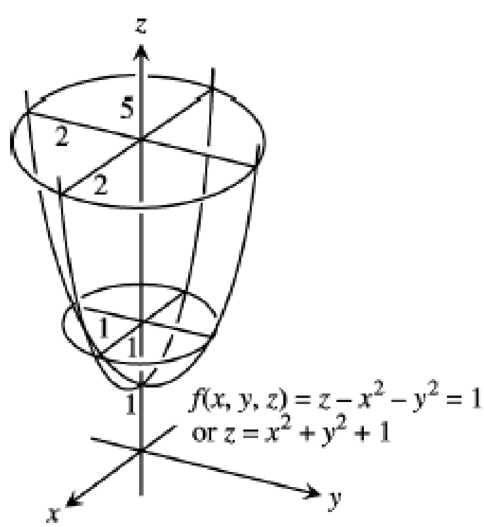
Solution



Exercise

Sketch a typical level surface for the function $f(x,y,z) = z - x^2 - y^2$

Solution



Solution *Section* 2.2 – Limits and Continuity

Exercise

Find the limits $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$

Solution

$$\lim_{(x,y)\to(0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} = \frac{3(0)^2 - (0)^2 + 5}{(0)^2 + (0)^2 + 2}$$

$$= \frac{5}{2}$$

Exercise

Find the limit $\lim\limits_{(x,y)\to(0,4)} \dfrac{x}{\sqrt{y}}$

Solution

$$\lim_{(x,y)\to(0,4)} \frac{x}{\sqrt{y}} = \frac{0}{\sqrt{4}}$$

$$= 0$$

Exercise

Find the limit $\lim\limits_{(x,y)\to(3,4)} \sqrt{x^2 + y^2 - 1}$

Solution

$$\lim_{(x,y)\to(3,4)} \sqrt{x^2 + y^2 - 1} = \sqrt{3^2 + 4^2 - 1}$$

$$= \sqrt{24}$$

$$= 2\sqrt{6}$$

Exercise

Find the limit $\lim\limits_{(x,y)\to(0,0)} \cos\dfrac{x^2 + y^3}{x + y + 1}$

Solution

$$\lim_{(x,y)\to(0,0)} \cos\frac{x^2+y^3}{x+y+1} = \cos\frac{0^2+0^3}{0+0+1}$$
$$= \cos 0$$
$$\underline{= 1}$$

Exercise

Find the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x}$

Solution

$$\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x} = e^0 \cdot \lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = 1(1)$$
$$\underline{= 1}$$

Exercise

Find the limit $\displaystyle\lim_{(x,y)\to\left(\frac{\pi}{2},0\right)} \frac{\cos y + 1}{y - \sin x}$

Solution

$$\lim_{(x,y)\to\left(\frac{\pi}{2},0\right)} \frac{\cos y+1}{y-\sin x} = \frac{\cos 0+1}{0-\sin\frac{\pi}{2}}$$
$$= \frac{1+1}{-1}$$
$$\underline{= -2}$$

Exercise

Find the limit $\displaystyle\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-2xy+y^2}{x-y}$

Solution

$$\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-2xy+y^2}{x-y} = \frac{1^2-2(1)(1)+1^2}{1-1} = \frac{0}{0}$$

$$\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-2xy+y^2}{x-y} = \lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{(x-y)^2}{x-y}$$

$$= \lim_{\substack{(x,y)\to(1,1) \\ x\neq y}} (x-y)$$

$$= 1-1$$

$$\underline{= 0}$$

Exercise

Find the limit $\displaystyle\lim_{\substack{(x,y)\to(1,1) \\ x\neq y}} \frac{x^2-y^2}{x-y}$

Solution

$$\lim_{\substack{(x,y)\to(1,1) \\ x\neq y}} \frac{x^2-y^2}{x-y} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$\lim_{\substack{(x,y)\to(1,1) \\ x\neq y}} \frac{x^2-y^2}{x-y} = \lim_{\substack{(x,y)\to(1,1) \\ x\neq y}} \frac{(x-y)(x+y)}{x-y}$$

$$= \lim_{\substack{(x,y)\to(1,1) \\ x\neq y}} (x+y)$$

$$= 1+1$$

$$\underline{= 2}$$

Exercise

Find the limit $\displaystyle\lim_{\substack{(x,y)\to(2,-4) \\ x\neq x^2,\ y\neq -4}} \frac{y+4}{x^2y-xy+4x^2-4x}$

Solution

$$\lim_{\substack{(x,y)\to(2,-4) \\ x\neq x^2,\ y\neq -4}} \frac{y+4}{x^2y-xy+4x^2-4x} = \lim_{\substack{(x,y)\to(2,-4) \\ x\neq x^2,\ y\neq -4}} \frac{y+4}{y\left(x^2-x\right)+4\left(x^2-x\right)}$$

$$= \lim_{\substack{(x,y)\to(2,-4) \\ x\neq x^2,\ y\neq -4}} \frac{y+4}{\left(x^2-x\right)(y+4)}$$

$$= \lim_{\substack{(x,y)\to(2,-4) \\ x\neq x^2,\ y\neq -4}} \frac{1}{x(x-1)}$$

$$= \frac{1}{2(2-1)}$$

$$\boxed{= \frac{1}{2}}$$

Exercise

Find the limit $\displaystyle \lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1}$

Solution

$$\lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1} = \lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{\left(\sqrt{x}-\sqrt{y+1}\right)\left(\sqrt{x}+\sqrt{y+1}\right)}$$

$$= \lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{1}{\sqrt{x}+\sqrt{y+1}}$$

$$= \frac{1}{\sqrt{4}+\sqrt{3+1}} = \frac{1}{2+2}$$

$$\boxed{= \frac{1}{4}}$$

Exercise

Find the limit $\displaystyle \lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y}$

Solution

$$\lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$\lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y} = \lim_{(x,y)\to(1,-1)} \frac{(x+y)\left(x^2-xy+y^2\right)}{x+y}$$

$$= \lim_{(x,y)\to(1,-1)} \left(x^2-xy+y^2\right)$$

$$= 1^2-(1)(-1)+(-1)^2$$

$$\boxed{= 3}$$

Exercise

Find the limit $\displaystyle\lim_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4}$

Solution

$$\lim_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4} = \lim_{(x,y)\to(2,2)} \frac{x-y}{\left(x^2-y^2\right)\left(x^2+y^2\right)}$$

$$= \lim_{(x,y)\to(2,2)} \frac{x-y}{(x-y)(x+y)\left(x^2+y^2\right)}$$

$$= \lim_{(x,y)\to(2,2)} \frac{1}{(x+y)\left(x^2+y^2\right)}$$

$$= \frac{1}{(2+2)\left(2^2+2^2\right)}$$

$$= \frac{1}{32}$$

Exercise

Find the limit $\displaystyle\lim_{P\to(1,3,4)} \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

Solution

$$\lim_{P\to(1,3,4)} \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{1}{1}+\frac{1}{3}+\frac{1}{4}$$

$$= \frac{19}{12}$$

Exercise

Find the limit $\displaystyle\lim_{P\to(1,-1,-1)} \frac{2xy+yz}{x^2+z^2}$

Solution

$$\lim_{P\to(1,-1,-1)} \frac{2xy+yz}{x^2+z^2} = \frac{2(1)(-1)+(-1)(-1)}{1^2+(-1)^2}$$

$$= -\frac{1}{2}$$

Exercise

Find the limit $\lim\limits_{P\to(\pi,0,2)} ze^{-2y}\cos 2x$

Solution

$$\lim_{P\to(\pi,0,2)} ze^{-2y}\cos 2x = 2e^{-2(0)}\cos 2\pi$$

$$= 2$$

Exercise

Find the limit $\lim\limits_{P\to(2,-3,6)} \ln\sqrt{x^2+y^2+z^2}$

Solution

$$\lim_{P\to(2,-3,6)} \ln\sqrt{x^2+y^2+z^2} = \ln\sqrt{4+9+36}$$

$$= \ln\sqrt{49}$$

$$= \ln 7$$

Exercise

Find the limit $\lim\limits_{(x,y)\to(4,-2)} (10x-5y+6xy)$

Solution

$$\lim_{(x,y)\to(4,-2)} (10x-5y+6xy) = 40+10-48$$

$$= 2$$

Exercise

Find the limit $\lim\limits_{(x,y)\to(1,1)} \dfrac{xy}{x+y}$

Solution

$$\lim_{(x,y)\to(1,1)} \frac{xy}{x+y} = \frac{1}{2}$$

Exercise

Find the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{xy}$

Solution

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{xy} = \frac{0}{0}$$

Along path $y = x$

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{xy} = \lim_{(x,y)\to(0,0)} \frac{2x}{x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{2}{x}$$

$$= \infty$$

Along path $y = -x$

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{xy} = \lim_{(x,y)\to(0,0)} \frac{0}{-x^2}$$

$$= -\infty$$

$\therefore$ Limit ***doesn't exist***

Exercise

Find the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin xy}{x^2+y^2}$

Solution

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{x^2+y^2} = \frac{0}{0}$$

Along path $y = x$

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{\sin x^2}{2x^2} = \frac{0}{0}$$

$$= \lim_{(x,y)\to(0,0)} \frac{2x\cos x^2}{4x}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\cos x^2}{2}$$

$$= \frac{1}{2}$$

Along path $y = -x$

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{-\sin x^2}{2x^2} = \frac{0}{0}$$

$$= \lim_{(x,y)\to(0,0)} \frac{-2x\cos x^2}{4x}$$

$$= \lim_{(x,y)\to(0,0)} \frac{-\cos x^2}{2}$$

$$= -\frac{1}{2}$$

$\therefore$ Limit ***doesn't exist***

Exercise

Find the limit $\displaystyle\lim_{(x,y)\to(-1,1)} \frac{x^2-y^2}{x^2-xy-2y^2}$

Solution

$$\lim_{(x,y)\to(-1,1)} \frac{x^2-y^2}{x^2-xy-2y^2} = \frac{0}{0}$$

$$= \lim_{(x,y)\to(-1,1)} \frac{(x-y)(x+y)}{(x-2y)(x+y)}$$

$$= \lim_{(x,y)\to(-1,1)} \frac{x-y}{x-2y}$$

$$= \frac{-1-1}{-1-2}$$

$$= \frac{2}{3}$$

Exercise

Find the limit $\displaystyle\lim_{(x,y)\to(1,2)} \frac{x^2y}{x^4+2y^2}$

Solution

$$\lim_{(x,y)\to(1,2)} \frac{x^2y}{x^4+2y^2} = \frac{2}{9}$$

Exercise

Find the limit $\lim\limits_{(x,y,z)\to\left(\frac{\pi}{2},0,\frac{\pi}{2}\right)} 4\cos y\sin\sqrt{xz}$

Solution

$$\lim\limits_{(x,y,z)\to\left(\frac{\pi}{2},0,\frac{\pi}{2}\right)} 4\cos y\sin\sqrt{xz} = 4(\cos 0)\sin\sqrt{\frac{\pi^2}{4}}$$

$$= 4\sin\frac{\pi}{2}$$

$$= 4$$

Exercise

Find the limit $\lim\limits_{(x,y,z)\to(5,2,-3)} \tan^{-1}\left(\frac{x+y^2}{z^2}\right)$

Solution

$$\lim\limits_{(x,y,z)\to(5,2,-3)} \tan^{-1}\left(\frac{x+y^2}{z^2}\right) = \tan^{-1}\left(\frac{9}{9}\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

Exercise

At what points (*x, y, z*) in space are the functions continuous $f(x,y,z) = x^2 + y^2 - 2z^2$

Solution

All (x,y,z)

Exercise

At what points (*x, y, z*) in space are the functions continuous $f(x,y,z) = \sqrt{x^2 + y^2 - 1}$

Solution

$x^2 + y^2 - 1 \ge 0 \quad\rightarrow\quad x^2 + y^2 \ge 1$. All (x,y,z) except the interior of the cylinder $x^2 + y^2 = 1$

Exercise

At what points (*x, y, z*) in space are the functions continuous $f(x,y,z) = \ln(xyz)$

Solution

All (x, y, z) so that $xyz > 0$

Exercise

At what points (*x, y, z*) in space are the functions continuous $f(x, y, z) = e^{x+y}\cos z$

Solution

All (x, y, z)

Exercise

At what points (*x, y, z*) in space are the functions continuous $h(x, y, z) = \dfrac{1}{|y| + |z|}$

Solution

All (x, y, z) except $(x,\ 0,\ 0)$

Exercise

At what points (*x, y, z*) in space are the functions continuous $h(x, y, z) = \dfrac{1}{z - \sqrt{x^2 + y^2}}$

Solution

All (x, y, z) except $z \neq \sqrt{x^2 + y^2}$

Solution Section 2.3 – Partial Derivatives

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x,y)=2x^2-3y-4$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(2x^2-3y-4\right)$$
$$=4x$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(2x^2-3y-4\right)$$
$$=-3$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x,y)=x^2-xy+y^2$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(x^2-xy+y^2\right)$$
$$=2x-y$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(x^2-xy+y^2\right)$$
$$=-x+2y$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x,y)=5xy-7x^2-y^2+3x-6y+2$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(5xy-7x^2-y^2+3x-6y+2\right)$$
$$=5y-14x+3$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(5xy-7x^2-y^2+3x-6y+2\right)$$
$$=5x-2y-6$$

Exercise

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ $\qquad f(x,y)=(xy-1)^2$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(xy-1)^2$$

$$=2y(xy-1)$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(xy-1)^2$$

$$=2x(xy-1)$$

Exercise

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ $\qquad f(x,y)=\left(x^3+\frac{y}{2}\right)^{2/3}$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(x^3+\frac{y}{2}\right)^{2/3}$$

$$=\frac{2}{3}\left(x^3+\frac{y}{2}\right)^{-1/3}\left(3x^2\right)$$

$$=\frac{2x^2}{\sqrt[3]{x^3+\frac{y}{2}}}$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(x^3+\frac{y}{2}\right)^{2/3}$$

$$=\frac{2}{3}\left(x^3+\frac{y}{2}\right)^{-1/3}\left(\frac{1}{2}\right)$$

$$=\frac{1}{3\sqrt[3]{x^3+\frac{y}{2}}}$$

Exercise

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ $\qquad f(x,y)=\dfrac{1}{x+y}$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{x+y}\right) \qquad\qquad \frac{\partial}{dx}\left(\frac{1}{u}\right) = -\frac{u'}{u^2}$$

$$= -\frac{1}{(x+y)^2}\frac{\partial}{\partial x}(x+y)$$

$$= -\frac{1}{(x+y)^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{x+y}\right)$$

$$= -\frac{1}{(x+y)^2}$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\qquad f(x,y) = \frac{x}{x^2+y^2}$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) \qquad\qquad \frac{\partial}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$$

$$= \frac{1\left(x^2+y^2\right) - x(2x)}{\left(x^2+y^2\right)^2}$$

$$= \frac{y^2 - x^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{(0)\left(x^2+y^2\right) - x(2y)}{\left(x^2+y^2\right)^2}$$

$$= -\frac{2xy}{\left(x^2+y^2\right)^2}$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x,y)=\tan^{-1}\frac{y}{x}$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(\tan^{-1}\frac{y}{x}\right)=\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$=-\frac{y}{1+\frac{y^2}{x^2}}\left(\frac{1}{x^2}\right)$$

$$=-\frac{y}{\frac{x^2+y^2}{x^2}}\left(\frac{1}{x^2}\right)$$

$$\boxed{=-\frac{y}{x^2+y^2}}$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(\tan^{-1}\frac{y}{x}\right)=\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$=\frac{1}{\frac{x^2+y^2}{x^2}}\left(\frac{1}{x}\right)$$

$$\boxed{=\frac{x}{x^2+y^2}}$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x,y)=e^{-x}\sin(x+y)$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(e^{-x}\sin(x+y)\right)$$

$$=\sin(x+y)\frac{\partial}{\partial x}\left(e^{-x}\right)+e^{-x}\frac{\partial}{\partial x}\left(\sin(x+y)\right)$$

$$\boxed{=-e^{-x}\sin(x+y)+e^{-x}\cos(x+y)}$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(e^{-x}\sin(x+y)\right)$$

$$\boxed{=e^{-x}\cos(x+y)}$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\qquad f(x,y) = e^{xy}\ln y$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(e^{xy}\ln y\right)$$

$$\boxed{= ye^{xy}\ln y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(e^{xy}\ln y\right)$$

$$= \ln y\frac{\partial}{\partial y}\left(e^{xy}\right) + e^{xy}\frac{\partial}{\partial y}(\ln y)$$

$$\boxed{= xe^{xy}\ln y + \frac{1}{y}e^{xy}}$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\qquad f(x,y) = \sin^2(x-3y)$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\sin^2(x-3y)\right)$$

$$= 2\sin(x-3y)\frac{\partial}{\partial x}\sin(x-3y)$$

$$= 2\sin(x-3y)\cos(x-3y)\frac{\partial}{\partial x}(x-3y)$$

$$\boxed{= 2\sin(x-3y)\cos(x-3y)}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\sin^2(x-3y)\right)$$

$$= 2\sin(x-3y)\frac{\partial}{\partial y}\sin(x-3y)$$

$$= 2\sin(x-3y)\cos(x-3y)\frac{\partial}{\partial y}(x-3y)$$

$$\boxed{= -6\sin(x-3y)\cos(x-3y)}$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\qquad f(x,y) = \cos^2\left(3x-y^2\right)$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\cos^2\left(3x - y^2\right)\right)$$

$$= 2\cos\left(3x - y^2\right)\frac{\partial}{\partial x}\left(\cos\left(3x - y^2\right)\right)$$

$$= -2\cos\left(3x - y^2\right)\sin\left(3x - y^2\right)\frac{\partial}{\partial x}\left(3x - y^2\right)$$

$$= -6\cos\left(3x - y^2\right)\sin\left(3x - y^2\right)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\cos^2\left(3x - y^2\right)\right)$$

$$= 2\cos\left(3x - y^2\right)\frac{\partial}{\partial y}\left(\cos\left(3x - y^2\right)\right)$$

$$= -2\cos\left(3x - y^2\right)\sin\left(3x - y^2\right)\frac{\partial}{\partial y}\left(3x - y^2\right)$$

$$= 4y\cos\left(3x - y^2\right)\sin\left(3x - y^2\right)$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x, y) = x^y$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^y\right)$$

$$= yx^{y-1}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^y\right)$$

$$= x^y \ln x$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x, y) = 3x^2y^5$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(3x^2y^5\right)$$

$$= 6xy^5$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(3x^2y^5\right)$$

$$= 15x^2y^4$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x, y) = x\cos y - y\sin x$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y - y\sin x)$$

$$= \cos y - y\cos x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y - y\sin x)$$

$$= -x\sin y - \sin x$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x, y) = \frac{x^2}{x^2 + y^2}$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^2}{x^2 + y^2}\right)$$

$$= \frac{2x\left(x^2 + y^2\right) - 2x^3}{\left(x^2 + y^2\right)^2}$$

$$= \frac{2xy^2}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2}{x^2 + y^2}\right)$$

$$= -\frac{2x^2y}{\left(x^2 + y^2\right)^2}$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x, y) = xye^{xy}$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(xye^{xy}\right)$$

$$= \left(y + xy^2\right)e^{xy}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xye^{xy}\right)$$

$$= \left(x + x^2 y\right)e^{xy}$$

Exercise

Find $f_x, f_y,$ *and* $f_z \quad f(x,y,z) = 1 + xy^2 - 2z^2$

Solution

$$f_x = y^2 \qquad f_y = 2xy \qquad f_z = -4z$$

Exercise

Find $f_x, f_y,$ *and* $f_z \quad f(x,y,z) = xy + yz + xz$

Solution

$$f_x = y + z \qquad f_y = x + y \qquad f_z = y + x$$

Exercise

Find $f_x, f_y,$ *and* $f_z \quad f(x,y,z) = x - \sqrt{y^2 + z^2}$

Solution

$$f_x = 1$$

$$f_y = -\frac{1}{2}\left(y^2 + z^2\right)^{-1/2} \frac{\partial}{\partial y}\left(y^2 + z^2\right)$$

$$= -\frac{1}{2}\left(y^2 + z^2\right)^{-1/2}(2y)$$

$$= -\frac{y}{\sqrt{y^2+z^2}}$$

$$f_z = -\frac{1}{2}\left(y^2+z^2\right)^{-1/2}\frac{\partial}{\partial z}\left(y^2+z^2\right)$$

$$= -\frac{1}{2}\left(y^2+z^2\right)^{-1/2}(2z)$$

$$= -\frac{z}{\sqrt{y^2+z^2}}$$

Exercise

Find $f_x, f_y,$ *and* f_z $\quad f(x,y,z)=\left(x^2+y^2+z^2\right)^{-1/2}$

Solution

$$f_x = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2x)$$

$$= -x\left(x^2+y^2+z^2\right)^{-3/2}$$

$$f_y = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2y)$$

$$= -y\left(x^2+y^2+z^2\right)^{-3/2}$$

$$f_z = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2z)$$

$$= -z\left(x^2+y^2+z^2\right)^{-3/2}$$

Exercise

Find $f_x, f_y,$ *and* f_z $\quad f(x,y,z)=\sec^{-1}(x+yz)$

Solution

$$f_x = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}\frac{\partial}{\partial x}(x+yz)$$

$$= \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}$$

$$f_y = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}\frac{\partial}{\partial y}(x+yz)$$

$$= \boxed{\frac{z}{|x+yz|\sqrt{(x+yz)^2-1}}}$$

$$f_z = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}\frac{\partial}{\partial z}(x+yz)$$

$$= \boxed{\frac{y}{|x+yz|\sqrt{(x+yz)^2-1}}}$$

Exercise

Find $f_x, f_y,$ *and* f_z $\quad f(x,y,z) = \ln(x+2y+3z)$

Solution

$$f_x = \frac{1}{x+2y+3z}\cdot\frac{\partial}{\partial x}(x+2y+3z)$$

$$= \boxed{\frac{1}{x+2y+3z}}$$

$$f_y = \frac{1}{x+2y+3z}\cdot\frac{\partial}{\partial y}(x+2y+3z)$$

$$= \boxed{\frac{2}{x+2y+3z}}$$

$$f_z = \frac{1}{x+2y+3z}\cdot\frac{\partial}{\partial z}(x+2y+3z)$$

$$= \boxed{\frac{3}{x+2y+3z}}$$

Exercise

Find $f_x, f_y,$ *and* f_z $\quad f(x,y,z) = e^{-\left(x^2+y^2+z^2\right)}$

Solution

$$f_x = e^{-\left(x^2+y^2+z^2\right)}\frac{\partial}{\partial x}\left(-\left(x^2+y^2+z^2\right)\right)$$

$$= \boxed{-2xe^{-\left(x^2+y^2+z^2\right)}}$$

$$f_y = e^{-\left(x^2+y^2+z^2\right)} \frac{\partial}{\partial y}\left(-\left(x^2+y^2+z^2\right)\right)$$

$$\boxed{= -2ye^{-\left(x^2+y^2+z^2\right)}}$$

$$f_z = e^{-\left(x^2+y^2+z^2\right)} \frac{\partial}{\partial z}\left(-\left(x^2+y^2+z^2\right)\right)$$

$$\boxed{= -2ze^{-\left(x^2+y^2+z^2\right)}}$$

Exercise

Find $f_x, f_y,$ *and* f_z $\quad f(x,y,z) = \tanh(x+2y+3z)$

Solution

$$\boxed{f_x = \operatorname{sech}^2(x+2y+3z)}$$

$$\boxed{f_y = 2\operatorname{sech}^2(x+2y+3z)}$$

$$\boxed{f_z = 3\operatorname{sech}^2(x+2y+3z)}$$

Exercise

Find $f_x, f_y,$ *and* f_z $\quad f(x,y,z) = \sinh\left(xy - z^2\right)$

Solution

$$f_x = \cosh\left(xy - z^2\right)\frac{\partial}{\partial x}\left(xy - z^2\right)$$

$$\boxed{= y\cosh\left(xy - z^2\right)}$$

$$\boxed{f_y = x\cosh\left(xy - z^2\right)}$$

$$\boxed{f_z = -2z\cosh\left(xy - z^2\right)}$$

Exercise

Find $f_x, f_y,$ *and* f_z $\qquad f(x, y, z) = 4xyz^2 - \frac{3x}{y}$

Solution

$$\boxed{f_x = 4yz^2 - \frac{3}{y}}$$

$$\boxed{f_y = 4xz^2 + \frac{3x}{y^2}}$$

$$\boxed{f_z = 8xyz}$$

Exercise

Find $f_x, f_y,$ *and* f_z $\qquad f(x, y, z) = \frac{xyz}{x+y}$

Solution

$$\boxed{f_x = \frac{y^2 z}{(x+y)^2}} \qquad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

$$\boxed{f_y = \frac{x^2 z}{(x+y)^2}}$$

$$\boxed{f_z = \frac{xy}{x+y}}$$

Exercise

Find $f_x, f_y,$ *and* f_z $\qquad f(x, y, z) = e^{x+2y+3z}$

Solution

$$\boxed{f_x = e^{x+2y+3z}}$$

$$\boxed{f_y = 2e^{x+2y+3z}}$$

$$\boxed{f_z = 3e^{x+2y+3z}}$$

Exercise

Find $f_x, f_y,$ *and* f_z $\qquad f(x, y, z) = x^2\sqrt{y+z}$

Solution

$$\boxed{f_x = 2x\sqrt{y+z}}$$

$$f_y = \frac{1}{2}\frac{x^2}{\sqrt{y+z}}$$

$$f_z = \frac{1}{2}\frac{x^2}{\sqrt{y+z}}$$

Exercise

Find partial derivatives of the function with respect to each variable $g(r,\theta) = r\cos\theta + r\sin\theta$

Solution

$$g_r = \cos\theta + \sin\theta$$

$$g_\theta = -r\sin\theta + r\cos\theta$$

Exercise

Find partial derivatives of the function with respect to each variable

$$f(x,y) = \frac{1}{2}\ln\left(x^2+y^2\right) + \tan^{-1}\frac{y}{x}$$

Solution

$$f_x = \frac{x}{x^2+y^2} - \frac{y}{x^2}\frac{1}{1+\frac{y^2}{x^2}}$$

$$= \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}$$

$$= \frac{x-y}{x^2+y^2}$$

$$f_y = \frac{y}{x^2+y^2} + \frac{1}{x}\frac{1}{1+\frac{y^2}{x^2}}$$

$$= \frac{y}{x^2+y^2} + \frac{x}{x^2+y^2}$$

$$= \frac{x+}{x^2+y^2}$$

Exercise

Find partial derivatives of the function with respect to each variable $h(x,y,z)=\sin(2\pi x+y-3z)$

Solution

$$\boxed{h_x=2\pi\cos(2\pi x+y-3z)}$$

$$\boxed{h_y=\cos(2\pi x+y-3z)}$$

$$\boxed{h(x,y,z)=-3\cos(2\pi x+y-3z)}$$

Exercise

Find partial derivatives of the function with respect to each variable $f(r,l,T,w)=\frac{1}{2rl}\sqrt{\frac{T}{\pi w}}$

Solution

$$\boxed{f_r=-\frac{1}{2r^2l}\sqrt{\frac{T}{\pi w}}}$$

$$\boxed{f_l=-\frac{1}{2rl^2}\sqrt{\frac{T}{\pi w}}}$$

$$\begin{aligned}f_T&=\frac{1}{4\pi rlw}\left(\frac{T}{\pi w}\right)^{-1/2}\\&=\frac{1}{4\pi rlw}\sqrt{\frac{\pi w}{T}}\\&=\boxed{\frac{1}{4rl}\sqrt{\frac{1}{\pi wT}}}\end{aligned}$$

$$\begin{aligned}f_w&=\frac{1}{4rl}\frac{-T}{\pi w^2}\left(\frac{T}{\pi w}\right)^{-1/2}\\&=-\frac{T}{4\pi rlw^2}\sqrt{\frac{\pi w}{T}}\\&=\boxed{-\frac{1}{4rlw}\sqrt{\frac{T}{\pi w}}}\end{aligned}$$

Exercise

Find all the second-order partial derivatives of $f(x,y)=x+y+xy$

Solution

$$\frac{\partial f}{\partial x}=1+y\quad \frac{\partial f}{\partial y}=1+x\quad \frac{\partial^2 f}{\partial x\partial y}=1$$

$$\frac{\partial^2 f}{\partial x^2} = 0 \quad \frac{\partial^2 f}{\partial y^2} = 0 \quad \frac{\partial^2 f}{\partial y \partial x} = 1$$

Exercise

Find all the second-order partial derivatives of $f(x,y) = \sin xy$

Solution

$$\boxed{\frac{\partial f}{\partial x} = y \cos xy} \qquad \boxed{\frac{\partial f}{\partial y} = x \cos xy}$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = -y^2 \sin xy} \qquad \boxed{\frac{\partial^2 f}{\partial y^2} = -x^2 \sin xy}$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \cos xy - xy \sin xy} \qquad \boxed{\frac{\partial^2 f}{\partial y \partial x} = \cos xy - xy \sin xy}$$

Exercise

Find all the second-order partial derivatives of $g(x,y) = x^2 y + \cos y + y \sin x$

Solution

$$\boxed{\frac{\partial g}{\partial x} = 2xy + y \cos x} \qquad \boxed{\frac{\partial g}{\partial y} = x^2 - \sin y + \sin x}$$

$$\boxed{\frac{\partial^2 g}{\partial x^2} = 2y - y \sin x} \qquad \boxed{\frac{\partial^2 g}{\partial y^2} = -\cos y}$$

$$\boxed{\frac{\partial^2 g}{\partial x \partial y} = 2x + \cos x} \qquad \boxed{\frac{\partial^2 g}{\partial y \partial x} = 2x + \cos x}$$

Exercise

Find all the second-order partial derivatives of $r(x,y) = \ln(x+y)$

Solution

$$\boxed{\frac{\partial r}{\partial x} = \frac{1}{x+y}} \qquad \boxed{\frac{\partial^2 r}{\partial x^2} = -\frac{1}{(x+y)^2}} \qquad \boxed{\frac{\partial^2 r}{\partial y \partial x} = -\frac{1}{(x+y)^2}}$$

$$\frac{\partial r}{\partial y} = \frac{1}{x+y} \qquad \boxed{\frac{\partial^2 r}{\partial y^2} = -\frac{1}{(x+y)^2}} \qquad \boxed{\frac{\partial^2 r}{\partial x \partial y} = -\frac{1}{(x+y)^2}}$$

Exercise

Find all the second-order partial derivatives of $\quad w = x^2 \tan(xy)$

Solution

$$\frac{\partial w}{\partial x} = 2x\tan(xy) + x^2 y\sec^2(xy)$$

$$\frac{\partial^2 w}{\partial x^2} = 2\tan(xy) + 2xy\sec^2(xy) + 2xy\sec^2(xy) + 2x^2 y\sec(xy)\frac{\partial}{\partial x}\sec(xy)$$

$$= 2\tan(xy) + 4xy\sec^2(xy) + 2x^2 y\sec(xy)\sec(xy)\tan(xy)\frac{\partial}{\partial x}(xy)$$

$$= 2\tan(xy) + 4xy\sec^2(xy) + 2x^2 y^2\sec^2(xy)\tan(xy)$$

$$\frac{\partial w}{\partial y} = x^3\sec^2(xy)$$

$$\frac{\partial^2 w}{\partial y^2} = 2x^3\sec(xy)\left[x\sec(xy)\tan(xy)\right]$$

$$= 2x^4\sec^2(xy)\tan(xy)$$

$$\frac{\partial^2 w}{\partial y\partial x} = \frac{\partial^2 w}{\partial x\partial y} = 3x^2\sec^2(xy) + x^3\left(2\sec(xy)\sec(xy)\tan(xy)\cdot y\right)$$

$$= 3x^2\sec^2(xy) + 2x^3 y\sec^2(xy)\tan(xy)$$

Exercise

Find all the second-order partial derivatives of $\quad w = ye^{x^2 - y}$

Solution

$$\frac{\partial w}{\partial x} = 2xye^{x^2 - y}$$

$$\frac{\partial^2 w}{\partial x^2} = 2ye^{x^2 - y} + 4x^2 ye^{x^2 - y}$$

$$= 2ye^{x^2 - y}\left(1 + 2x^2\right)$$

$$\frac{\partial w}{\partial y} = e^{x^2 - y} - ye^{x^2 - y}$$

$$= e^{x^2 - y}(1 - y)$$

$$\frac{\partial^2 w}{\partial y^2} = -e^{x^2 - y}(1 - y) - e^{x^2 - y}$$

$$= e^{x^2-y}(-1+y-1)$$

$$\boxed{= (y-2)e^{x^2-y}}$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}$$

$$= 2xe^{x^2-y} - 2xye^{x^2-y}$$

$$\boxed{= 2x(1-y)e^{x^2-y}}$$

Exercise

Find second-order partial derivatives of the function $g(x,y) = y + \dfrac{x}{y}$

Solution

$$\boxed{g_x = \frac{1}{y}} \qquad \boxed{g_y = 1 - \frac{x}{y^2}}$$

$$\boxed{g_{xx} = 0} \qquad \boxed{g_y = \frac{2x}{y^3}}$$

$$\boxed{g_{xy} = g_{yx} = -\frac{1}{y^2}}$$

Exercise

Find second-order partial derivatives of the function $g(x,y) = e^x + y\sin x$

Solution

$$\boxed{g_x = e^x + y\cos x} \qquad \boxed{g_y = \sin x}$$

$$\boxed{g_{xx} = e^x - y\sin x} \qquad \boxed{g_y = 0}$$

$$\boxed{g_{xy} = g_{yx} = \cos x}$$

Exercise

Find second-order partial derivatives of the function $f(x,y) = y^2 - 3xy + \cos y + 7e^y$

Solution

$$\boxed{f_x = -3y} \qquad \boxed{f_y = 2y - 3x - \sin y + 7e^y}$$

$\boxed{f_{xx} = 0}$

$\boxed{f_{xy} = f_{yx} = -3}$

$\boxed{g_y = 2 - \cos y + 7e^y}$

Exercise

Verify that the function satisfies Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$$u(x, y) = y\left(3x^2 - y^2\right)$$

Solution

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(3x^2 y - y^3\right)$$

$$= 6xy$$

$$\frac{\partial^2 u}{\partial x^2} = 6y$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(3x^2 y - y^3\right)$$

$$= 3x^2 - 3y^2$$

$$\frac{\partial^2 u}{\partial y^2} = -6y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y - 6y$$

$$\boxed{= 0} \quad \checkmark$$

$\therefore$ The given function satisfies Laplace's equation

Exercise

Verify that the function satisfies Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$$u(x, y) = \ln\left(x^2 + y^2\right)$$

Solution

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\ln\left(x^2 + y^2\right)$$

$$= \frac{2x}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right)$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right) = \frac{(ae-bd)x^2+2(af-dd)x+(bf-ce)}{\left(dx^2+ex+f\right)^2}$$

$$= \frac{-2x^2+2y^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\ln\left(x^2+y^2\right)$$

$$= \frac{2y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right)$$

$$= \frac{2x^2-2y^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = \frac{-2x^2+2y^2}{\left(x^2+y^2\right)^2}+\frac{2x^2-2y^2}{\left(x^2+y^2\right)^2}$$

$$= 0 \quad \checkmark$$

∴ The given function satisfies Laplace's equation

Exercise

Let $f(x,y) = 2x+3y-4$. Find the slope of the line tangent to this surface at the point $(2,\ -1)$ and lying in the **a**. plane $x=2$ **b**. plane $y=-1$.

Solution

a) In the plane $x=2$

$$m = f_y\big|_{(2,-1)} = 3$$

b) In the plane $y=-1$

$$m = f_z\big|_{(2,-1)} = 2$$

Exercise

Let $w = f(x, y, z)$ be a function of three independent variables and writs the formal definition of the partial derivative $\frac{\partial f}{\partial y}$ at (x_0, y_0, z_0). Use this definition to find $\frac{\partial f}{\partial y}$ at $(-1,\ 0,\ 3)$ for $f(x, y, z) = -2xy^2 + yz^2$.

Solution

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x_0, y_0 + h, z_0) - f(x_0, y_0, z_0)}{h}$$

$$\begin{aligned}
f_y(-1,0,3) &= \lim_{h \to 0} \frac{f(-1, 0+h, 3) - f(-1,0,3)}{h} \\
&= \lim_{h \to 0} \frac{-2(-1)h^2 + h(3)^2 - (0+0)}{h} \\
&= \lim_{h \to 0} \frac{2h^2 + 9h}{h} \\
&= \lim_{h \to 0} (2h + 9) \\
&= 9
\end{aligned}$$

Exercise

Find the value of $\frac{\partial x}{\partial z}$ at the point $(1, -1, -3)$ if the equation $xz + y\ln x - x^2 + 4 = 0$ defines x as a function of the two independent variables y and z and the partial derivative exists.

Solution

$$\frac{\partial x}{\partial z} z + x + y\left(\frac{1}{x}\right)\frac{\partial x}{\partial z} - 2x\frac{\partial x}{\partial z} = 0$$

$$\left(z + \frac{y}{x} - 2x\right)\frac{\partial x}{\partial z} = -x$$

$$\Rightarrow \frac{\partial x}{\partial z} = -\frac{x}{z + \frac{y}{x} - 2x}$$

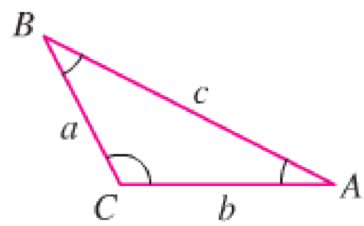
$$\left.\frac{\partial x}{\partial z}\right|_{(1,-1,-3)} = -\frac{1}{-3 + \frac{-1}{1} - 2}$$

$$= \frac{1}{6}$$

Exercise

Express A implicitly as a function of a, b, and c and calculate $\frac{\partial A}{\partial a}$ and $\frac{\partial A}{\partial b}$.

Solution



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\frac{\partial}{\partial a}\left(a^2 = b^2 + c^2 - 2bc\cos A\right)$$

$$2a = (2bc\sin A)\frac{\partial A}{\partial a} \quad \rightarrow \quad \boxed{\frac{\partial A}{\partial a} = \frac{a}{bc\sin A}}$$

$$\frac{\partial}{\partial b}\left(a^2 = b^2 + c^2 - 2bc\cos A\right)$$

$$0 = 2b - 2c\cos A + 2bc\sin A\left(\frac{\partial A}{\partial b}\right)$$

$$\left(\frac{\partial A}{\partial b}\right) = \frac{c\cos A - b}{bc\sin A}$$

Exercise

An important partial differential equation that describes the distribution of heat in a region at time t can be represented by the one-dimensional heat equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

Show that $u(x,t) = \sin(\alpha x)\cdot e^{-\beta t}$ satisfies the heat equation for constants α and β. What is the relationship between α and β for this function to be a solution?

Solution

$$u_t = -\beta\sin(\alpha x)\cdot e^{-\beta t}$$

$$u_x = \alpha\cos(\alpha x)\cdot e^{-\beta t}$$

$$u_{xx} = -\alpha^2\sin(\alpha x)\cdot e^{-\beta t}$$

For $\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \quad \rightarrow \quad u_t = u_{xx}$

$$-\beta\sin(\alpha x)\cdot e^{-\beta t} = -\alpha^2\sin(\alpha x)\cdot e^{-\beta t}$$

$$\Rightarrow \boxed{\beta = \alpha^2}$$

Exercise

Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of t.

$w = x^2 + y^2, \quad x = \cos t, \quad y = \sin t, \quad t = \pi$

Solution

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}\left(x^2 + y^2\right)\frac{d}{dt}(\cos t) + \frac{\partial f}{\partial y}\left(x^2 + y^2\right)\frac{d}{dt}(\sin t)$$

$$= 2x(-\sin t) + 2y\cos t$$

$$= -2(\cos t)\sin t + 2(\sin t)\cos t$$

$$\underline{= 0}$$

$$w = x^2 + y^2$$

$$= \cos^2 t + \sin^2 t$$

$$= 1$$

$$\frac{dw}{dt} = 0$$

$$\frac{dw}{dt}(t = \pi) \ \underline{= 0}$$

Exercise

Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of t.

$w = x^2 + y^2, \quad x = \cos t + \sin t, \quad y = \cos t - \sin t, \quad t = 0$

Solution

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}\left(x^2 + y^2\right)\frac{d}{dt}(\cos t + \sin t) + \frac{\partial f}{\partial y}\left(x^2 + y^2\right)\frac{d}{dt}(\cos t - \sin t)$$

$$= (2x)(-\sin t + \cos t) + (2y)(-\sin t - \cos t)$$

$$= 2(\cos t + \sin t)(\cos t - \sin t) - 2(\cos t - \sin t)(\sin t + \cos t)$$

$$\underline{= 0}$$

$$\frac{dw}{dt}(t = 0) \ \underline{= 0}$$

Exercise

Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of t.

$w = \ln\left(x^2 + y^2 + z^2\right), \quad x = \cos t, \quad y = \sin t, \quad z = 4\sqrt{t}, \quad t = 3$

Solution

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \frac{2x}{x^2+y^2+z^2}(-\sin t) + \frac{2y}{x^2+y^2+z^2}(\cos t) + \frac{2z}{x^2+y^2+z^2}\left(2\frac{1}{\sqrt{t}}\right)$$

$$= \frac{-2\cos t \sin t + 2\sin t \cos t + 4\left(4\sqrt{t}\right)\left(t^{-1/2}\right)}{\cos^2 t + \sin^2 t + 16t}$$

$$= \boxed{\frac{16}{1+16t}}$$

$$w = \ln\left(x^2 + y^2 + z^2\right)$$

$$= \ln\left(\cos^2 t + \sin^2 t + 16t\right)$$

$$= \ln(1+16t)$$

$$\frac{dw}{dt} = \frac{16}{1+16t}$$

$$\frac{dw}{dt}(3) = \frac{16}{1+16(3)}$$

$$= \boxed{\frac{16}{49}}$$

Exercise

Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of t.

$w = z - \sin xy, \quad x = t, \quad y = \ln t, \quad z = e^{t-1}, \quad t = 1$

Solution

$$\frac{\partial w}{\partial t} = (-y\cos xy)(1) + (-x\cos xy)\left(\frac{1}{t}\right) + (1)\left(e^{t-1}\right) \qquad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= -(\ln t)\cos(t\ln t) - \cos(t\ln t) + e^{t-1}$$

$$= \boxed{-(\ln t + 1)\cos(t\ln t) + e^{t-1}}$$

$$w = z - \sin xy$$

$$= e^{t-1} - \sin(t\ln t)$$

$$\frac{\partial w}{\partial t} = e^{t-1} - \cos(t\ln t)\left[\ln t + t\left(\frac{1}{t}\right)\right]$$

$$= e^{t-1} - (\ln t + 1)\cos(t\ln t)$$

$$\frac{\partial w}{\partial t}(1) = -(\ln 1 + 1)\cos(1\ln 1) + e^{1-1}$$

$$= -1\cos 0 + 1$$

$$\underline{= 0}$$

Exercise

Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of t.

$$w = \sin(xy + \pi), \quad x = e^t, \quad y = \ln(t+1) \quad t = 0$$

Solution

$$\frac{\partial w}{\partial t} = y\cos(xy + \pi)e^t + x\cos(xy + \pi)\frac{1}{t+1} \qquad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= e^t\ln(t+1)\cos\left(e^t\ln(t+1) + \pi\right) + e^t\cos\left(e^t\ln(t+1) + \pi\right)\frac{1}{t+1}$$

$$= e^t\left(\ln(t+1) + \frac{1}{t+1}\right)\cos\left(e^t\ln(t+1) + \pi\right)$$

$$\left.\frac{\partial w}{\partial t}\right|_{t=0} = \cos\pi$$

$$\underline{= -1}$$

Exercise

Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of t.

$$w = xe^y + y\sin z - \cos z, \quad x = 2\sqrt{t}, \quad y = t - 1 + \ln t, \quad z = \pi t, \quad t = 1$$

Solution

$$\frac{\partial w}{\partial t} = e^y\frac{1}{\sqrt{t}} + \left(xe^y + \sin z\right)\left(1 + \frac{1}{t}\right) + (y\cos z + \sin z)(\pi) \qquad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= \frac{1}{\sqrt{t}}e^{t-1+\ln t} + \left(2\sqrt{t}e^{t-1+\ln t} + \sin(\pi t)\right)\left(1 + \frac{1}{t}\right) + \pi\left(e(t - 1 + \ln t)\cos(\pi t) + \sin(\pi t)\right)$$

$$\left.\frac{\partial w}{\partial t}\right|_{t=1} = 1 + (2 + \sin\pi)(2) + \pi\sin(\pi) \qquad = 1 + 4 + 0$$

$$\underline{= 5}$$

Exercise

Express $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as functions of u and v if $z = 4e^x \ln y, \quad x = \ln(u\cos v), \quad y = u\sin v$, then evaluate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at the point $(u,v) = \left(2,\ \frac{\pi}{4}\right)$.

Solution

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{dx}{du} + \frac{\partial z}{\partial y}\frac{dy}{du}$$

$$= \left(4e^x \ln y\right)\left(\frac{\cos v}{u\cos v}\right) + \left(4\frac{e^x}{y}\right)(\sin v)$$

$$= 4e^x\left(\frac{\ln y}{u} + \frac{\sin v}{y}\right)$$

$$= 4e^{\ln(u\cos v)}\left(\frac{\ln(u\sin v)}{u} + \frac{\sin v}{u\sin v}\right)$$

$$= 4(u\cos v)\left(\frac{\ln(u\sin v)}{u} + \frac{1}{u}\right)$$

$$= 4\cos v\ln(u\sin v) + 4\cos v$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{dx}{dv} + \frac{\partial z}{\partial y}\frac{dy}{dv}$$

$$= \left(4e^x \ln y\right)\left(\frac{-u\sin v}{u\cos v}\right) + \left(4\frac{e^x}{y}\right)(u\cos v)$$

$$= 4e^{\ln(u\cos v)}\left[\frac{-\ln(u\sin v)(u\sin v)}{u\cos v} + \frac{u\cos v}{u\sin v}\right]$$

$$= 4u\cos v\left(\frac{-u\sin^2 v\cdot\ln(u\sin v) + u\cos^2 v}{u\cos v\sin v}\right)$$

$$= 4\left(\frac{-u\sin^2 v\cdot\ln(u\sin v) + u\cos^2 v}{\sin v}\right)$$

$$= -4u\sin v\cdot\ln(u\sin v) + 4u\frac{\cos^2 v}{\sin u}$$

$$\frac{\partial z}{\partial u}\left(2,\ \frac{\pi}{4}\right) = 4\cos\frac{\pi}{4}\ln\left(2\sin\frac{\pi}{4}\right) + 4\cos\frac{\pi}{4}$$

$$= 2\sqrt{2}\ln\sqrt{2} + 2\sqrt{2}$$

$$= 2\sqrt{2}\left(\tfrac{1}{2}\ln 2 + 1\right)$$

$$= \sqrt{2}\,(\ln 2 + 2)$$

$$\frac{\partial z}{\partial v}\left(2,\ \frac{\pi}{4}\right) = -8\sin\left(\frac{\pi}{4}\right)\cdot\ln\left(2\sin\left(\frac{\pi}{4}\right)\right) + 8\frac{\cos^2\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)}$$

$$= -4\sqrt{2}\ln\left(\sqrt{2}\right) + 8\cdot\frac{1}{2}\cdot\sqrt{2}$$

$$\boxed{= -2\sqrt{2}\ln 2 + 4\sqrt{2}}$$

Exercise

Express $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ as functions of u and v if $w = xy + yz + xz, \quad x = u + v, \quad y = u - v, \quad z = uv$, then evaluate $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ at the point $(u, v) = \left(\frac{1}{2},\ 1\right)$.

Solution

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial y}\frac{dy}{du} + \frac{\partial w}{\partial z}\frac{dz}{du}$$

$$= (y+z)(1) + (x+z)(1) + (y+x)(v)$$

$$= y + z + x + z + (y+x)(v)$$

$$= y + x + 2z + yv + xv$$

$$= u - v + u + v + 2uv + uv - v^2 + uv + v^2$$

$$\boxed{= 2u + 4uv}$$

$$\frac{\partial w}{\partial u}\left(\frac{1}{2},\ 1\right) = 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right)(1)$$

$$\boxed{= 3}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{dx}{dv} + \frac{\partial w}{\partial y}\frac{dy}{dv} + \frac{\partial w}{\partial z}\frac{dz}{dv}$$

$$= (y+z)(1) + (x+z)(-1) + (y+x)(u)$$

$$= y + z - x - z + yu + xu$$

$$= y - x + yu + xu$$

$$= u - v - u - v + u^2 - uv + u^2 + uv$$

$$\boxed{= -2v + 2u^2}$$

$$\frac{\partial w}{\partial v}\left(\frac{1}{2},\ 1\right) = -2(1) + 2\left(\frac{1}{2}\right)^2$$

$$\boxed{= -\frac{3}{2}}$$

Exercise

Express $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ as functions of *x, y* and *z* if $u = e^{qr}\sin^{-1}p, \quad p = \sin x, \quad q = z^2 \ln y, \quad r = \frac{1}{z},$ then evaluate $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ at the point $(x, y, z) = \left(\frac{\pi}{4},\ \frac{1}{2},\ -\frac{1}{2}\right)$.

Solution

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{dp}{dx} + \frac{\partial u}{\partial q}\frac{dq}{dx} + \frac{\partial u}{\partial r}\frac{dr}{dx}$$

$$= \left(\frac{e^{qr}}{\sqrt{1-p^2}}\right)(\cos x) + \left(re^{qr}\sin^{-1}p\right)(0) + \left(qe^{qr}\sin^{-1}p\right)(0)$$

$$= \frac{e^{z\ln y}\cos x}{\sqrt{1-\sin^2 x}}$$

$$= e^{\ln y^z}\frac{\cos x}{|\cos x|}$$

$$\boxed{= y^z} \quad if \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\frac{\partial u}{\partial x}\left(\frac{\pi}{4},\ \frac{1}{2},\ -\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{-1/2}$$

$$= 2^{1/2}$$

$$\boxed{= \sqrt{2}}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{dp}{dy} + \frac{\partial u}{\partial q}\frac{dq}{dy} + \frac{\partial u}{\partial r}\frac{dr}{dy}$$

$$= \left(\frac{e^{qr}}{\sqrt{1-p^2}}\right)(0) + \left(re^{qr}\sin^{-1}p\right)\left(\frac{z^2}{y}\right) + \left(qe^{qr}\sin^{-1}p\right)(0)$$

$$= \frac{z^2}{y}\frac{1}{z}e^{z\ln y}\sin^{-1}(\sin x)$$

$$= \frac{z}{y}e^{\ln y^z}(x)$$

$$= \frac{xz}{y}y^z$$

$$\boxed{= xzy^{z-1}}$$

$$\frac{\partial u}{\partial y}\left(\frac{\pi}{4},\ \frac{1}{2},\ -\frac{1}{2}\right) = \left(\frac{\pi}{4}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)^{-1/2-1}$$

$$= -\left(\frac{\pi}{8}\right)2^{3/2}$$

$$= -\frac{\pi\sqrt{2}}{4}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{dp}{dz} + \frac{\partial u}{\partial q}\frac{dq}{dz} + \frac{\partial u}{\partial r}\frac{dr}{dz}$$

$$= \left(\frac{e^{qr}}{\sqrt{1-p^2}}\right)(0) + \left(re^{qr}\sin^{-1}p\right)(2z\ln y) + \left(qe^{qr}\sin^{-1}p\right)\left(-\frac{1}{z^2}\right)$$

$$= 2z\ln y\left(\frac{1}{z}y^z\sin^{-1}(\sin x)\right) - \frac{1}{z^2}\left(z^2(\ln y)y^z\sin^{-1}(\sin x)\right)$$

$$= 2xy^z\ln y - xy^z\ln y$$

$$= xy^z\ln y$$

$$\frac{\partial u}{\partial z}\left(\frac{\pi}{4},\ \frac{1}{2},\ -\frac{1}{2}\right) = \left(\frac{\pi}{4}\right)\left(\frac{1}{2}\right)^{-1/2}\ln\left(\frac{1}{2}\right)$$

$$= \left(\frac{\pi}{4}\right)\left(\sqrt{2}\right)(-\ln 2)$$

$$= -\frac{\pi\sqrt{2}}{4}\ln 2$$

Exercise

Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z^3 - xy + yz + y^3 - 2 = 0$ at the point $(1,\ 1,\ 1)$

Solution

$$F(x,y,z) = z^3 - xy + yz + y^3 - 2$$

$$F_x = -y, \quad F_y = -x + z + 3y^2, \quad F_z = 3z^2 + y$$

$$\frac{\partial z}{\partial x} = -\frac{-y}{3z^2 + y} \qquad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$= \frac{y}{3z^2 + y}$$

$$\frac{\partial z}{\partial x}(1,\ 1,\ 1) = \frac{1}{3(1)^2 + 1}$$

$$= \frac{1}{4}$$

$$\frac{\partial z}{\partial y} = -\frac{-x + z + 3y^2}{3z^2 + y} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

$$= \boxed{\frac{x - z - 3y^2}{3z^2 + y}}$$

$$\frac{\partial z}{\partial y}(1,\ 1,\ 1) = \frac{1 - 1 - 3(1)^2}{3(1)^2 + 1}$$

$$= \boxed{-\frac{3}{4}}$$

Exercise

Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $\sin(x+y) + \sin(y+z) + \sin(x+z) = 0$ at the point $(\pi,\ \pi,\ \pi)$

Solution

$$F(x,y,z) = \sin(x+y) + \sin(y+z) + \sin(x+z)$$

$$F_x = \cos(x+y) + \cos(x+z)$$

$$F_y = \cos(x+y) + \cos(y+z)$$

$$F_z = \cos(y+z) + \cos(x+z)$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\cos(x+y) + \cos(x+z)}{\cos(y+z) + \cos(x+z)}$$

$$\frac{\partial z}{\partial x}(\pi,\ \pi,\ \pi) = -\frac{\cos(2\pi) + \cos(2\pi)}{\cos(2\pi) + \cos(2\pi)}$$

$$= \boxed{-1}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\cos(x+y) + \cos(y+z)}{\cos(y+z) + \cos(x+z)}$$

$$\frac{\partial z}{\partial x}(\pi,\ \pi,\ \pi) = -\frac{\cos(2\pi) + \cos(2\pi)}{\cos(2\pi) + \cos(2\pi)}$$

$$= \boxed{-1}$$

Exercise

Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0$ at the point $(1,\ \ln 2,\ \ln 3)$

Solution

$$F(x,y,z) = xe^y + ye^z + 2\ln x - 2 - 3\ln 2$$

$$F_x = e^y + \frac{2}{x} \qquad F_y = xe^y + e^z \qquad F_z = ye^z$$

$$\frac{\partial z}{\partial x} = -\frac{e^y + \frac{2}{x}}{ye^z} \qquad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$= -\frac{xe^y + 2}{xye^z}$$

$$\frac{\partial z}{\partial x}(1, \ln 2, \ln 3) = -\frac{(1)e^{\ln 2} + 2}{\ln 2 e^{\ln 3}}$$

$$= -\frac{2+2}{3\ln 2}$$

$$= -\frac{4}{3\ln 2}$$

$$\frac{\partial z}{\partial y} = -\frac{xe^y + e^z}{ye^z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

$$\frac{\partial z}{\partial x}(1, \ln 2, \ln 3) = -\frac{e^{\ln 2} + e^{\ln 3}}{\ln 2 e^{\ln 3}}$$

$$= -\frac{2+3}{3\ln 2}$$

$$= -\frac{5}{3\ln 2}$$

Exercise

Find $\frac{\partial w}{\partial r}$ when $r = 1,\ s = -1$ if $w = (x+y+z)^2, \quad x = r - s, \quad y = \cos(r+s), \quad z = \sin(r+s)$

Solution

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{dx}{dr} + \frac{\partial w}{\partial y}\frac{dy}{dr} + \frac{\partial w}{\partial z}\frac{dz}{dr}$$

$$= 2(x+y+z)(1) + 2(x+y+z)(-\sin(r+s)) + 2(x+y+z)(\cos(r+s))$$

$$= 2(x+y+z)\left[1 - \sin(r+s) + \cos(r+s)\right]$$

$$= 2(r - s + \cos(r+s) + \sin(r+s))(1 - \sin(r+s) + \cos(r+s))$$

$$\frac{\partial w}{\partial r}(1,-1) = 2(1-(-1) + \cos(1-1) + \sin(1-1))(1 - \sin(1-1) + \cos(1-1))$$

$$= 2(1+1+1+0)(1-0+1)$$

$$= 2(3)(2)$$

$$\underline{= 12}$$

Exercise

Find $\frac{\partial z}{\partial u}$ when $u = 0,\ v = 1$ if $z = \sin xy + x\sin y, \quad x = u^2 + v^2, \quad y = uv$

Solution

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{dx}{du} + \frac{\partial z}{\partial y}\frac{dy}{du}$$

$$= (y\cos x + \sin y)(2u) + (x\cos xy + x\cos y)(v)$$

$$= 2u\left(uv\cos\left(u^2+v^2\right) + \sin uv\right) + v\left(\left(u^2+v^2\right)\cos\left(u^3v+uv^3\right) + \left(u^2+v^2\right)\cos uv\right)$$

$$= 2u\left(uv\cos\left(u^2+v^2\right) + \sin uv\right) + v\left(u^2+v^2\right)\left(\cos\left(u^3v+uv^3\right) + \cos uv\right)$$

$$\left.\frac{\partial z}{\partial u}\right|_{u=0,\ v=1} = 2(0)(0\cos(1) + \sin 0) + 1(1)(\cos(0) + \cos 0)$$

$$= 0 + 1(1+1)$$

$$\underline{= 2}$$

Exercise

Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ when $u = \ln 2,\ v = 1$ if $z = 5\tan^{-1} x, \quad x = e^u + \ln v$

Solution

$$\frac{\partial z}{\partial u} = \frac{dz}{dx}\frac{\partial x}{\partial u} = \left(\frac{5}{1+x^2}\right)e^u = \left(\frac{5}{1+\left(e^u+\ln v\right)^2}\right)e^u$$

$$\left.\frac{\partial z}{\partial u}\right|_{u=\ln 2,\ v=1} = \left(\frac{5}{1+\left(e^{\ln 2}+\ln 1\right)^2}\right)e^{\ln 2}$$

$$= \left(\frac{5}{1+(2+0)^2}\right)(2)$$

$$= 2\left(\frac{5}{5}\right)$$

$$\underline{= 2}$$

Exercise

Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ when $u=1,\ v=-2$ if $z=\ln q,\quad q=\sqrt{v+3}\ \tan^{-1}u$

Solution

$$\frac{\partial z}{\partial u}=\frac{dz}{dq}\frac{\partial q}{\partial u}$$

$$=\left(\frac{1}{q}\right)\left(\sqrt{v+3}\frac{1}{1+u^2}\right)$$

$$=\frac{1}{\sqrt{v+3}\ \tan^{-1}u}\cdot\frac{\sqrt{v+3}}{1+u^2}$$

$$=\boxed{\frac{1}{\left(1+u^2\right)\tan^{-1}u}}$$

$$\left.\frac{\partial z}{\partial u}\right|_{u=1,\ v=-2}=\frac{1}{\left(1+1^2\right)\tan^{-1}1}$$

$$=\frac{1}{2\cdot\frac{\pi}{4}}$$

$$=\boxed{\frac{2}{\pi}}$$

$$\frac{\partial z}{\partial v}=\frac{dz}{dq}\frac{\partial q}{\partial v}$$

$$=\left(\frac{1}{q}\right)\left(\frac{1}{2\sqrt{v+3}}\tan^{-1}u\right)$$

$$=\left(\frac{1}{\sqrt{v+3}\ \tan^{-1}u}\right)\left(\frac{\tan^{-1}u}{2\sqrt{v+3}}\right)$$

$$=\boxed{\frac{1}{2(v+3)}}$$

$$\left.\frac{\partial z}{\partial v}\right|_{u=1,\ v=-2}=\frac{1}{2(-2+3)}$$

$$=\boxed{\frac{1}{2}}$$

Exercise

Find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$ when $r=\pi$ and $s=0$ if $w=\sin(2x-y)$, $x=r+\sin s$, $y=rs$

Solution

$$\frac{\partial w}{\partial x}=2\cos(2x-y)$$

$$=2\cos(2r+2\sin s-rs)\Big|_{r=\pi \quad s=0}$$

$$=2\cos(2\pi)$$

$$\underline{=2}$$

$$\frac{\partial w}{\partial y}=-\cos(2x-y)$$

$$=-\cos(2r+2\sin s-rs)\Big|_{r=\pi \quad s=0}$$

$$=-\cos(2\pi)$$

$$\underline{=-1}$$

$$\frac{dx}{dr}=1$$

$$\frac{dy}{dr}=s\Big|_{r=\pi \quad s=0} \quad \underline{=0}$$

$$\frac{\partial w}{\partial r}=2(1)+(-1)(0) \qquad \frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

$$\underline{=2}$$

$$\frac{dx}{ds}=\cos s\Big|_{s=0} \quad \underline{=1}$$

$$\frac{dy}{ds}=r\Big|_{r=\pi} \quad \underline{=\pi}$$

$$\frac{\partial w}{\partial s}=2(1)+(-1)(\pi) \qquad \frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

$$\underline{=2-\pi}$$

Exercise

Assume that $w=f\left(s^3+t^2\right)$ and $f'(x)=e^x$. Find $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial s}$

Solution

$$w=f\left(s^3+t^2\right)=f(x) \quad \to \quad x=s^3+t^2$$

$$\frac{\partial w}{\partial t} = f'(x) \cdot 2t \qquad\qquad \frac{\partial w}{\partial t} = \frac{dw}{dx}\frac{\partial x}{\partial t}$$

$$= 2te^{x}$$

$$\boxed{= 2te^{s^3+t^2}}$$

$$\frac{\partial w}{\partial s} = \left(e^{x}\right)\left(3s^2\right) \qquad\qquad \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$

$$\boxed{= 3s^2 e^{s^3+t^2}}$$

Exercise

Evaluate the derivatives $w'(t)$, where $w = xy\sin z, \quad x = t^2, \quad y = 4t^3, \quad and \quad z = t+1$

Solution

$$\frac{dw}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= y\sin z(2t) + x\sin z\left(12t^2\right) + xy\cos z$$

$$= 8t^4\sin(t+1) + 12t^4\sin(t+1) + 4t^5\cos(t+1)$$

$$\boxed{= 20t^4\sin(t+1) + 4t^5\cos(t+1)}$$

Or

$$w(t) = 4t^5\sin(t+1)$$

$$\boxed{w' = 20t^4\sin(t+1) + 4t^5\cos(t+1)}$$

Exercise

Evaluate the derivatives $w'(t)$, where $w = \sqrt{x^2+y^2+z^2}, \quad x = \sin t, \quad y = \cos t, \quad and \quad z = \cos t$

Solution

$$\frac{dw}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}\cos t - \frac{y}{\sqrt{x^2+y^2+z^2}}\sin t - \frac{z}{\sqrt{x^2+y^2+z^2}}\sin t$$

$$= \frac{\sin t\cos t - \cos t\sin t - \cos t\sin t}{\sqrt{\sin^2 t + \cos^2 t + \sin^2 t}}$$

$$\boxed{= -\frac{\cos t\sin t}{\sqrt{1+\sin^2 t}}}$$

Exercise

Evaluate the derivatives w_s and w_t, where $w = xyz, \quad x = 2st, \quad y = st^2, \quad and \quad z = s^2 t$

Solution

$$w_s = w_x x_s + w_y y_s + w_z z_s$$
$$= yz(2t) + xzt^2 + xy(2st)$$
$$= 2s^3t^4 + 2s^3t^4 + 4s^3t^4$$
$$= 8s^3t^4$$

$$w_t = w_x x_t + w_y y_t + w_z z_t$$
$$= yz(2s) + xz(2st) + xy\left(s^2\right)$$
$$= 2s^4t^3 + 4s^4t^3 + 2s^4t^3$$
$$= 8s^4t^3$$

Or

$$w = (2st)\left(st^2\right)\left(s^2t\right)$$
$$= 2s^4t^4$$
$$w_s = 8s^3t^4$$
$$w_t = 8s^4t^3$$

Exercise

Evaluate the derivatives w_r, w_s, and w_t, where $w = \ln\left(xy^2\right), \quad x = rst, \quad and \quad y = r+s$

Solution

$$w_r = w_x x_r + w_y y_r$$
$$= \frac{1}{y^2}(st) + \frac{2}{xy}$$
$$= \frac{st}{(r+s)^2} + \frac{2}{rst(r+s)}$$
$$= \frac{rs^2t^2 + 2r + 2s}{rst(r+s)^2}$$

$$w_s = w_x x_s + w_y y_s$$

$$= \frac{rt}{y^2} + \frac{2}{xy}$$

$$= \frac{rt}{(r+s)^2} + \frac{2}{rst(r+s)}$$

$$\boxed{= \frac{r^2st^2 + 2r + 2s}{rst(r+s)^2}}$$

$$w_t = w_x x_t + w_y y_t$$

$$= \frac{rs}{y^2} + \frac{2}{xy}(0)$$

$$\boxed{= \frac{rs}{(r+s)^2}}$$

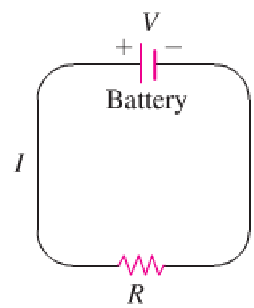
Or $\quad w = \ln\left(rst(r+s)^2\right)$

Exercise

The voltage V in a circuit that satisfies the law $V = IR$ is slowly dropping as the battery wears out. At the same time, the resistance R is increasing as the resistor heats up. Use the equation

$$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt}$$

To find how the current is changing at the instant when $R = 600\ \Omega$, $I = 0.04A$, $\frac{dR}{dt} = 0.5\ ohm/\sec$, and $\frac{dV}{dt} = -0.01\ volt/\sec$



Solution

$$V = IR \quad \rightarrow \quad \frac{\partial V}{\partial I} = R, \quad \frac{\partial V}{\partial R} = I$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt}$$

$$-0.01 = (600)\frac{dI}{dt} + (0.04)(0.5)$$

$$-0.02 - 0.01 = 600\frac{dI}{dt}$$

$$\frac{dI}{dt} = \boxed{-0.00005\ amps/\sec}$$

Exercise

The lengths *a, b*, and *c* of the edges of a rectangular box are changing with time. At the instant in question, $a = 1\ m, \quad b = 2\ m, \quad c = 3\ m$, $\frac{da}{dt} = \frac{db}{dt} = 1\ m/\sec$, and $\frac{dc}{dt} = -3\ m/\sec$. At what rates the box's volume *V* and surface area *S* changing at that instant? Are the box's interior diagonals increasing in length or decreasing?

Solution

$$V = abc \quad \Rightarrow \quad \frac{\partial V}{\partial t} = \frac{\partial V}{\partial a}\frac{da}{dt} + \frac{\partial V}{\partial b}\frac{db}{dt} + \frac{\partial V}{\partial c}\frac{dc}{dt}$$

$$\frac{\partial V}{\partial t} = (bc)\frac{da}{dt} + (ac)\frac{db}{dt} + (ab)\frac{dc}{dt}$$

$$= (2m)(3m)(1\ m/\sec) + (1m)(3m)(1\ m/\sec) + (1m)(2m)(-3\ m/\sec)$$

$$\underline{= 3\ m^3/\sec}$$

Exercise

Let $T = f(x, y)$ be the temperature at the point (x, y) on the circle $x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$ and suppose that

$$\frac{\partial T}{\partial x} = 8x - 4y, \quad \frac{\partial T}{\partial y} = 8y - 4x$$

a) Find where the maximum and minimum temperatures on the circle occur by examining the derivatives $\frac{dT}{dt}$ and $\frac{d^2T}{dt^2}$.

b) Suppose that $T = 4x^2 - 4xy + 4y^2$. Find the maximum and minimum values of *T* on the circle.

Solution

a) $\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$

$$= (8x - 4y)(-\sin t) + (8y - 4x)(\cos t)$$

$$= (8\cos t - 4\sin t)(-\sin t) + (8\sin t - 4\cos t)(\cos t)$$

$$= -8\cos t \sin t + 4\sin^2 t + 8\cos t \sin t - 4\cos^2 t$$

$$\underline{= 4\sin^2 t - 4\cos^2 t}$$

$$\frac{dT}{dt} = 0 \quad \Rightarrow \quad 4\sin^2 t - 4\cos^2 t = 0$$

$$\sin^2 t = \cos^2 t$$

$$\sin t = \pm \cos t$$

$$\underline{t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}} \quad \text{on the interval } 0 \le t \le 2\pi$$

$$\frac{d^2T}{dt^2} = 8\sin t\cos t + 8\cos t\sin t$$

$$\underline{= 16\sin t\cos t}$$

$$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{\pi}{4}} = 16\sin\frac{\pi}{4}\cos\frac{\pi}{4} > 0 \quad \Rightarrow \text{T has a minimum at } (x, y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{3\pi}{4}} = 16\sin\frac{3\pi}{4}\cos\frac{3\pi}{4} < 0 \Rightarrow \text{T has a maximum at } (x, y) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{5\pi}{4}} = 16\sin\frac{5\pi}{4}\cos\frac{5\pi}{4} > 0 \Rightarrow \text{T has a minimum at } (x, y) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

$$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{7\pi}{4}} = 16\sin\frac{7\pi}{4}\cos\frac{7\pi}{4} < 0 \quad \Rightarrow \text{T has a maximum at } (x, y) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

b) $T = 4x^2 - 4xy + 4y^2$

$$T\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = 4\left(\frac{\sqrt{2}}{2}\right)^2 - 4\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 4\left(\frac{\sqrt{2}}{2}\right)^2 = 2 - 2 + 2 = \underline{2}$$

$$T\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = 4\left(-\frac{\sqrt{2}}{2}\right)^2 - 4\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 4\left(\frac{\sqrt{2}}{2}\right)^2 = 2 + 2 + 2 = \underline{6}$$

$$T\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 4\left(-\frac{\sqrt{2}}{2}\right)^2 - 4\left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + 4\left(-\frac{\sqrt{2}}{2}\right)^2 = 2 - 2 + 2 = \underline{2}$$

$$T\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 4\left(\frac{\sqrt{2}}{2}\right)^2 - 4\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + 4\left(-\frac{\sqrt{2}}{2}\right)^2 = 2 + 2 + 2 = \underline{6}$$

The maximum value is 6 at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

The minimum value is 2 at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

Exercise

Evaluate $\frac{dy}{dx}$: $\quad x^2 - 2y^2 - 1 = 0$

Solution

$$F(x, y) = x^2 - 2y^2 - 1$$

$$\frac{dy}{dx} = -\frac{2x}{-4y} \qquad\qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= \frac{x}{2y}$$

Exercise

Evaluate $\frac{dy}{dx}$: $\quad x^3 + 3xy^2 - y^5 = 0$

Solution

$$F(x, y) = x^3 + 3xy^2 - y^5$$

$$\frac{dy}{dx} = -\frac{3x^2 + 3y^2}{6xy - 5y^4} \qquad\qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

Exercise

Evaluate $\frac{dy}{dx}$: $\quad 2\sin xy = 1$

Solution

$$F(x, y) = 2\sin xy - 1$$

$$\frac{dy}{dx} = -\frac{2y\cos xy}{2x\cos xy} \qquad\qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{y}{x}$$

Exercise

Evaluate $\frac{dy}{dx}$: $\quad ye^{xy} - 2 = 0$

Solution

$$F(x, y) = ye^{xy} - 2$$

$$\frac{dy}{dx} = -\frac{y^2 e^{xy}}{e^{xy} + xye^{xy}} \qquad\qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{y^2}{1+xy}$$

Exercise

Evaluate $\frac{dy}{dx}$: $\qquad \sqrt{x^2 + 2xy + y^4} = 3$

Solution

$$F(x, y) = \sqrt{x^2 + 2xy + y^4} - 3$$

$$\frac{dy}{dx} = -\frac{\frac{1}{2}(2x+2y)\left(x^2 + 2xy + y^4\right)^{-1/2}}{\frac{1}{2}\left(2x+4y^3\right)\left(x^2 + 2xy + y^4\right)^{-1/2}} \qquad\qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{x+y}{x+2y^3}$$

Exercise

Evaluate $\frac{dy}{dx}$: $\qquad y\ln\left(x^2 + y^2 + 4\right) = 3$

Solution

$$F(x, y) = y\ln\left(x^2 + y^2 + 4\right) - 3$$

$$\frac{dy}{dx} = -\frac{\frac{2xy}{x^2 + y^2 + 4}}{\ln\left(x^2 + y^2 + 4\right) + \frac{2y^2}{x^2 + y^2 + 4}} \qquad\qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{2xy}{2y^2 + \left(x^2 + y^2 + 4\right)\ln\left(x^2 + y^2 + 4\right)}$$

Exercise

Evaluate $\frac{dy}{dx}$: $\qquad y\ln\left(x^2 + y^2\right) = 4$

Solution

$$F(x, y) = y\ln\left(x^2 + y^2\right) - 4$$

$$\frac{dy}{dx} = -\frac{\frac{2xy}{x^2 + y^2}}{\ln\left(x^2 + y^2\right) + \frac{2y^2}{x^2 + y^2}} \qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{2xy}{\left(x^2 + y^2\right)\ln\left(x^2 + y^2\right) + 2y^2}$$

Exercise

Evaluate $\frac{dy}{dx}$: $\quad 2x^2 + 3xy - 3y^4 = 2$

Solution

$$F(x, y) = 2x^2 + 3xy - 3y^4 - 2$$

$$\frac{dy}{dx} = -\frac{4x + 3y}{3x - 12y^3} \qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

Exercise

Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at the given point. $\quad z^3 - xy + yz + y^3 - 2 = 0; \quad (1, 1, 1)$

Solution

$$F(x, y, z) = z^3 - xy + yz + y^3 - 2$$

$$F_x = -y, \quad F_y = -x + z + 3y^2, \quad and \quad F_z = 3z^2 + y\Big|_{(1,1,1)} = 4 \neq 0$$

$$\frac{dz}{dx} = -\frac{F_x}{F_z} = -\frac{-y}{3z^2 + y} \qquad \frac{dz}{dy} = -\frac{F_y}{F_z} = -\frac{e^{xz} - z\sin y}{2z + xye^{xz} + \cos y}$$

$$\frac{dz}{dx}\Big|_{(1,1,1)} = -\frac{-1}{4} = \frac{1}{4} \qquad \frac{dz}{dz}\Big|_{(1,1,1)} = -\frac{3}{4}$$

Exercise

Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at the given point. $\quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0; \quad (2, 3, 6)$

Solution

$$F(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1$$

$$F_x = -\frac{1}{x^2}\Big|_{(2,3,6)} = -\frac{1}{4}$$

$$F_y = -\frac{1}{y^2}\Big|_{(2,3,6)} = -\frac{1}{9}$$

$$F_z = -\frac{1}{z^2}\Big|_{(2,3,6)} = -\frac{1}{36} \neq 0$$

$$\frac{dz}{dx}\Big|_{(2,3,6)} = -\frac{F_x}{F_z}\Big|_{(2,3,6)}$$

$$= -\frac{-\frac{1}{4}}{-\frac{1}{36}}$$

$$\boxed{= -9}$$

$$\frac{dz}{dy}\Big|_{(2,3,6)} = -\frac{F_y}{F_z}\Big|_{(2,3,6)}$$

$$= -\frac{-\frac{1}{9}}{-\frac{1}{36}}$$

$$\boxed{= -4}$$

Exercise

Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at the given point. $\sin(x+y) + \sin(y+z) + \sin(x+z) = 0; \quad (\pi, \pi, \pi)$

Solution

$$F(x, y, z) = \sin(x+y) + \sin(y+z) + \sin(x+z)$$

$$F_x = \cos(x+y) + \cos(x+z)\Big|_{(\pi,\pi,\pi)}$$

$$= \cos 2\pi + \cos 2\pi$$

$$\boxed{= 2}$$

$$F_y = \cos(x+y) + \cos(y+z)\Big|_{(\pi,\pi,\pi)}$$

$$= \cos 2\pi + \cos 2\pi$$

$$\boxed{= 2}$$

$$F_z = \cos(y+z) + \cos(x+z)\Big|_{(\pi,\pi,\pi)}$$

$$= \cos 2\pi + \cos 2\pi$$
$$\underline{= 2} \quad \color{red}{\neq 0}$$

$$\left.\frac{dz}{dx}\right|_{(\pi,\pi,\pi)} = -\left.\frac{F_x}{F_z}\right|_{(\pi,\pi,\pi)}$$
$$= -\frac{2}{2}$$
$$\underline{\color{blue}{= -1}}$$

$$\left.\frac{dz}{dy}\right|_{(\pi,\pi,\pi)} = -\left.\frac{F_y}{F_z}\right|_{(\pi,\pi,\pi)}$$
$$= -\frac{2}{2}$$
$$\underline{\color{blue}{= -1}}$$

Exercise

Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at the given point. $\quad xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0; \quad (1,\ \ln 2,\ \ln 3)$

Solution

$$F(x, y, z) = xe^y + ye^z + 2\ln x - 2 - 3\ln 2$$

$$F_x = e^y + \left.\frac{2}{x}\right|_{(1,\ln 2,\ln 3)} = 2 + 2 \underline{= 4}$$

$$F_y = xe^y + e^z \Big|_{(1,\ln 2,\ln 3)}$$
$$= e^{\ln 2} + e^{\ln 3}$$
$$= 2 + 3$$
$$\underline{= 5}$$

$$F_z = ye^z \Big|_{(1,\ln 2,\ln 3)} = \ln 2 e^{\ln 3} \underline{= 3\ln 2} \color{red}{\neq 0}$$

$$\left.\frac{dz}{dx}\right|_{(1,\ln 2,\ln 3)} = -\left.\frac{F_x}{F_z}\right|_{(1,\ln 2,\ln 3)}$$
$$\underline{\color{blue}{= -\frac{4}{3\ln 2}}}$$

$$\left.\frac{dz}{dy}\right|_{(1,\ln 2,\ln 3)} = -\left.\frac{F_y}{F_z}\right|_{(1,\ln 2,\ln 3)}$$
$$\underline{\color{blue}{= -\frac{5}{3\ln 2}}}$$

Exercise

Consider the surface and parameterized curves C in the xy-plane

$$z = 4x^2 + y^2 - 2; \quad C: \ x = \cos t, \quad y = \sin t, \quad \text{for} \ \ 0 \le t \le 2\pi$$

a) Find $z'(t)$ on C.

b) Imagine that you are walking on the surface directly above C consistent with the positive orientation of C. Find the values of ***t*** for which you are walking uphill.

Solution

a) $z'(t) = z_x x_t + z_y y_t$

$$= 8x(-\sin t) + 2y\cos t$$

$$= -8\cos t \sin t + 2\sin t \cos t$$

$$= -6\cos t \sin t$$

$$\boxed{= -3\sin 2t}$$

b) Walking uphill $\rightarrow z'(t) > 0$

$-3\sin 2t > 0 \quad \rightarrow \quad \sin 2t < 0$

$\sin 2t = 0 \quad \rightarrow \quad 2t = 0, \ \pi, \ 2\pi, 3\pi, \ 4\pi$

$t = 0, \ \frac{\pi}{2}, \ \pi, \ \frac{3\pi}{2}, \ 2\pi$

$$\boxed{\frac{\pi}{2} \le t \le \pi \quad \& \quad \frac{3\pi}{2} \le t \le 2\pi}$$

Exercise

Consider the surface and parameterized curves C in the xy-plane

$$z = 4x^2 - 2y^2 + 4; \quad C: \ x = 2\cos t, \quad y = 2\sin t, \quad \text{for} \ \ 0 \le t \le 2\pi$$

a) Find $z'(t)$ on C.

b) Imagine that you are walking on the surface directly above C consistent with the positive orientation of C. Find the values of ***t*** for which you are walking uphill.

Solution

a) $z'(t) = z_x x_t + z_y y_t$

$$= 8x(-2\sin t) - 4y(2\cos t)$$

$$= -16\cos t \sin t - 16\sin t \cos t$$

$$= -32\cos t \sin t$$

$$\boxed{= -16\sin 2t}$$

b) Walking uphill $\rightarrow z'(t) > 0$

$-16\sin 2t > 0 \quad \rightarrow \quad \sin 2t < 0$

$\sin 2t = 0 \quad \rightarrow \quad 2t = 0, \ \pi, \ 2\pi, 3\pi, \ 4\pi$

$t = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi$

$$\boxed{\frac{\pi}{2} \le t \le \pi \quad \& \quad \frac{3\pi}{2} \le t \le 2\pi}$$

Exercise

Find the value of the derivative of $f(x,y,z) = xy + yz + xz$ with respect to t on the curve $x = \cos t, \quad y = \sin t, \quad z = \cos 2t$ at $t = 1$

Solution

$$f_x = y + z = \sin t + \cos 2t$$

$$f_y = x + z = \cos t + \cos 2t$$

$$f_x = y + z = \cos t + \sin t$$

$$\frac{dx}{dt} = -\sin t \quad \frac{dy}{dt} = \cos t \quad \frac{dz}{dt} = -2\sin 2t$$

$$\begin{aligned}\frac{df}{dt} &= f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}\\
&= (\sin t + \cos 2t)(-\sin t) + (\cos t + \cos 2t)(\cos t) + (\cos t + \sin t)(-2\sin 2t)\\
&= -\sin^2 t - \sin t \cos 2t + \cos^2 t + \cos t \cos 2t - 2\sin 2t \cos t - 2\sin 2t \sin t\\
&= \cos^2 t - \sin^2 t + (\cos t - \sin t)\cos 2t - 2\sin 2t(\cos t + \sin t)\end{aligned}$$

$$\boxed{\left.\frac{df}{dt}\right|_{t=1} = \cos 2 + (\cos 1 - \sin 1)\cos 2 - 2(\cos 1 + \sin 1)\sin 2}$$

Exercise

Define y as a differentiable function of x for $2xy + e^{x+y} - 2 = 0$, find the values of $\frac{dy}{dx}$ at point $P(0,\ \ln 2)$

Solution

$$F(x,y) = 2xy + e^{x+y} - 2$$

$$F_x = 2y + e^{x+y} \qquad\qquad F_y = 2x + e^{x+y}$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2y + e^{x+y}}{2x + e^{x+y}}$$

$$\left.\frac{dy}{dx}\right|_{(0,\ \ln 2)} = -\frac{2\ln 2 + e^{\ln 2}}{0 + e^{\ln 2}}$$

$$= -\frac{2\ln 2 + 2}{2}$$

$$\underline{= -\ln 2 - 1}$$

Solution *Section* 2.5 – Directional Derivatives and the Gradient

Exercise

Find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point $f(x,y) = y - x, \quad (2,\ 1)$

Solution

$$\frac{\partial f}{\partial x} = -1, \quad \frac{\partial f}{\partial y} = 1$$

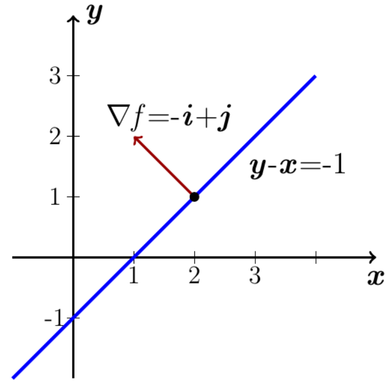
$$\nabla f = f_x \hat{i} + f_y \hat{j}$$

$$= -\hat{i} + \hat{j}$$

$$f(2,\ 1) = 1 - 2$$

$$\underline{= -1\,|}$$

$-1 = y - x$ is the level curve



Exercise

Find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point $f(x,y) = \ln\left(x^2 + y^2\right), \quad (1,\ 1)$

Solution

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1)} = \frac{2}{1+1} = 1$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,1)} = \frac{2}{1+1} = 1$$

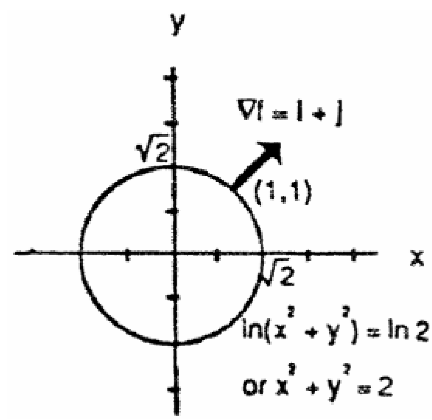
$$\nabla f = f_x \hat{i} + f_y \hat{j}$$

$$= \hat{i} + \hat{j}$$

$$\underline{f(1,\ 1) = \ln 2\,|}$$

$$\ln 2 = \ln\left(x^2 + y^2\right)$$

$\rightarrow \quad x^2 + y^2 = 2$ is the level curve



Exercise

Find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point $f(x,y)=\sqrt{2x+3y}, \quad (-1,\ 2)$

Solution

$$\frac{\partial f}{\partial x}=\frac{1}{\sqrt{2x+3y}}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(-1,2)}=\frac{1}{\sqrt{-2+6}}=\frac{1}{2}$$

$$\frac{\partial f}{\partial y}=\frac{3}{2\sqrt{2x+3y}}$$

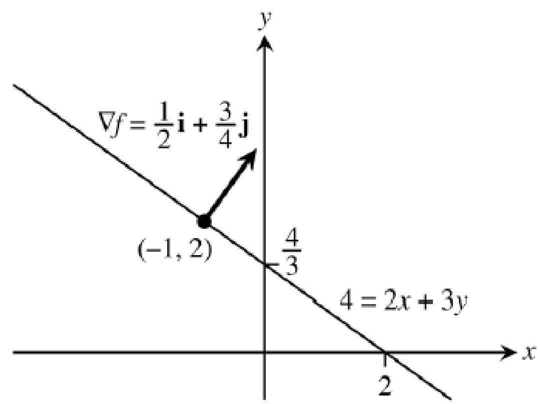
$$\left.\frac{\partial f}{\partial y}\right|_{(-1,2)}=\frac{3}{2\sqrt{-2+6}}=\frac{3}{4}$$

$$\nabla f=\frac{1}{2}\hat{i}+\frac{3}{4}\hat{j}$$

$$f(-1,\ 2)=\sqrt{2(-1)+3(2)}$$

$$\underline{=2\ |}$$

$2x+3y=4$ is the level curve



Exercise

Find ∇f at the given point $\quad f(x,y,z)=x^2+y^2-2z^2+z\ln x, \quad (1,\ 1,\ 1)$

Solution

$$\frac{\partial f}{\partial x}=2x+\frac{z}{x}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1,1)}=2+\frac{1}{1}=3$$

$$\frac{\partial f}{\partial y}=2y$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,1,1)}=2$$

$$\frac{\partial f}{\partial z}=-4z+\ln x$$

$$\left.\frac{\partial f}{\partial z}\right|_{(1,1,1)}=-4+\ln 1=-4$$

$$\underline{\nabla f=3\hat{i}+2\hat{j}-4\hat{k}\ |}$$

Exercise

Find ∇f at the given point $f(x,y,z)=2x^3-3\left(x^2+y^2\right)z+\tan^{-1}xz, \quad (1,\ 1,\ 1)$

Solution

$$\frac{\partial f}{\partial x}=6x^2-6xz+\frac{z}{1+x^2z^2}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1,1)}=6-6+\frac{1}{1+1}=\frac{1}{2}$$

$$\frac{\partial f}{\partial y}=-6yz$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,1,1)}=-6$$

$$\frac{\partial f}{\partial z}=-3\left(x^2+y^2\right)+\frac{z}{1+x^2z^2}$$

$$\left.\frac{\partial f}{\partial z}\right|_{(1,1,1)}=-3(2)+\frac{1}{2}=-\frac{11}{2}$$

$$\nabla f=\frac{1}{2}\hat{i}-6\hat{j}-\frac{11}{2}\hat{k}$$

Exercise

Find ∇f at the given point $f(x,y,z)=e^{x+y}\cos z+(y+1)\sin^{-1}x, \quad \left(0,\ 0,\ \frac{\pi}{6}\right)$

Solution

$$\frac{\partial f}{\partial x}=e^{x+y}\cos z+\frac{y+1}{\sqrt{1-x^2}}$$

$$\to \left.\frac{\partial f}{\partial x}\right|_{\left(0,0,\frac{\pi}{6}\right)}=e^0\cos\frac{\pi}{6}+\frac{0+1}{\sqrt{1-0}}$$

$$=\frac{\sqrt{3}}{2}+1$$

$$\frac{\partial f}{\partial y}=e^{x+y}\cos z+\sin^{-1}x$$

$$\to \left.\frac{\partial f}{\partial y}\right|_{\left(0,0,\frac{\pi}{6}\right)}=e^0\cos\frac{\pi}{6}+0$$

$$=\frac{\sqrt{3}}{2}$$

$$\frac{\partial f}{\partial z} = -e^{x+y}\sin z$$

$$\rightarrow \left.\frac{\partial f}{\partial z}\right|_{\left(0,0,\frac{\pi}{6}\right)} = -e^{0}\sin\frac{\pi}{6}$$

$$= -\frac{1}{2}$$

$$\nabla f = \left(\frac{\sqrt{3}}{2}+1\right)\hat{i} + \frac{\sqrt{3}}{2}\hat{j} - \frac{1}{2}\hat{k}$$

Exercise

Find the derivative of the function $f(x,y) = 2xy - 3y^2$ at $P_0(5,\ 5)$ in the direction of $\vec{v} = 4\hat{i} + 3\hat{j}$

Solution

$$\vec{u} = \frac{4\hat{i} + 3\hat{j}}{\sqrt{16+9}} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}$$

$$f_x = 2y \quad \Rightarrow \quad f_x(5,5) = 10$$

$$f_y = 2x - 6y \quad \Rightarrow \quad f_y(5,5) = 10 - 30 = -20$$

$$\nabla f = 10\hat{i} - 20\hat{j}$$

$$\left(D_{\vec{u}}f\right)_{P_0} = \left(10\hat{i} - 20\hat{j}\right)\bullet\left(\frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}\right) \qquad \left(D_{\vec{u}}f\right)_{P_0} = \nabla f \bullet \vec{u}$$

$$= 10\left(\frac{4}{5}\right) - 20\left(\frac{3}{5}\right)$$

$$= 8 - 12$$

$$= -4$$

Exercise

Find the derivative of the function $f(x,y) = \dfrac{x-y}{xy+2}$ at $P_0(1,\ -1)$ in the direction of $\vec{v} = 12\hat{i} + 5\hat{j}$

Solution

$$\vec{u} = \frac{12\hat{i} + 5\hat{j}}{\sqrt{144+25}} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{12}{13}\hat{i} + \frac{5}{13}\hat{j}$$

$$f_x = \frac{xy+2-y(x-y)}{(xy+2)^2}$$

$$= \frac{xy+2-xy+y^2}{(xy+2)^2}$$

$$= \frac{2+y^2}{(xy+2)^2}\Bigg|_{(1,-1)}$$

$$= \frac{2+1}{(-1+2)^2}$$

$$= 3$$

$$f_y = \frac{-xy-2-x(x-y)}{(xy+2)^2}$$

$$= \frac{-2-x^2}{(xy+2)^2}\Bigg|_{(1,-1)}$$

$$= \frac{-2-1}{(-1+2)^2}$$

$$= -3$$

$$\nabla f = 3\hat{i} - 3\hat{j}$$

$$\left(D_{\vec{u}} f\right)_{P_0} = \left(3\hat{i} - 3\hat{j}\right) \bullet \left(\frac{12}{13}\hat{i} + \frac{5}{13}\hat{j}\right) \qquad \left(D_{\vec{u}} f\right)_{P_0} = \nabla f \bullet \vec{u}$$

$$= \frac{36}{13} - \frac{15}{13}$$

$$= \frac{21}{13}$$

Exercise

Find the derivative of the function $h(x, y) = \tan^{-1}\left(\frac{y}{x}\right) + \sqrt{3}\sin^{-1}\left(\frac{xy}{2}\right)$ at $P_0\ (1,\ 1)$ in the direction of $\vec{v} = 3\hat{i} - 2\hat{j}$

Solution

$$\vec{u} = \frac{3\hat{i} - 2\hat{j}}{\sqrt{9+4}} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{3}{\sqrt{13}}\hat{i} - \frac{2}{\sqrt{13}}\hat{j}$$

$$h_x = \frac{-\frac{y}{x^2}}{\left(\frac{y}{x}\right)^2+1} + \sqrt{3}\frac{\frac{y}{2}}{\sqrt{1-\left(\frac{x^2y^2}{4}\right)}}$$

$$\to h_x(1,\ 1) = \frac{-1}{1+1} + \sqrt{3}\frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}}}$$

$$= -\frac{1}{2} + \sqrt{3}\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{2}$$

$$h_y = \frac{\frac{1}{x}}{\left(\frac{y}{x}\right)^2+1} + \sqrt{3}\frac{\frac{x}{2}}{\sqrt{1-\frac{x^2y^2}{4}}}$$

$$\to h_y(1,\ 1) = \frac{1}{2} + \sqrt{3}\frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}}}$$

$$= \frac{3}{2}$$

$$\nabla h = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j}$$

$$\left(D_{\vec{u}}h\right)_{P_0} = \nabla h \bullet \vec{u}$$

$$= \left(\frac{1}{2}\hat{i} + \frac{3}{2}\hat{j}\right) \bullet \left(\frac{3}{\sqrt{13}}\hat{i} - \frac{2}{\sqrt{13}}\hat{j}\right)$$

$$= \frac{3}{2\sqrt{13}} - \frac{3}{\sqrt{13}}$$

$$= -\frac{3}{2\sqrt{13}}$$

Exercise

Find the derivative of the function $f(x,y,z) = xy + yz + zx$ at $P_0(1,\ -1,\ 2)$ in the direction of $\vec{v} = 3\hat{i} + 6\hat{j} - 2\hat{k}$

Solution

$$\vec{u} = \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{9+36+4}} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$=\frac{3}{7}\hat{i}+\frac{6}{7}\hat{j}-\frac{2}{7}\hat{k}$$

$$f_x = y+z \quad \Rightarrow \quad f_x(1,-1,2) = -1+2 = 1$$

$$f_y = x+z \quad \Rightarrow \quad f_y(1,-1,2) = 1+2 = 3$$

$$f_z = y+x \quad \Rightarrow \quad f_z(1,-1,2) = -1+1 = 0$$

$$\nabla f = \hat{i}+3\hat{j}$$

$$\left(D_{\vec{u}}f\right)_{P_0} = \left(\hat{i}+3\hat{j}\right)\bullet\left(\frac{3}{7}\hat{i}+\frac{6}{7}\hat{j}-\frac{2}{7}\hat{k}\right) \qquad \left(D_{\vec{u}}f\right)_{P_0} = \nabla f\bullet\vec{u}$$

$$=\frac{3}{7}+\frac{18}{7}$$

$$=3$$

Exercise

Find the derivative of the function $g(x,y,z) = 3e^{x}\cos yz$ at $P_0\ (0,\ 0,\ 0)$ in the direction of $\vec{v} = 2\hat{i}+\hat{j}-2\hat{k}$

Solution

$$\vec{u} = \frac{2\hat{i}+\hat{j}-2\hat{k}}{\sqrt{4+1+4}} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$=\frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}-\frac{2}{3}\hat{k}$$

$$g_y = -3ze^{x}\sin yz\Big|_{(0,0,0)}$$

$$=-3(0)e^{0}\sin 0$$

$$=0$$

$$g_x = 3e^{x}\cos yz\Big|_{(0,0,0)}$$

$$=3e^{0}\cos(0)$$

$$=3$$

$$g_z = -3ye^{x}\sin yz\Big|_{(0,0,0)}$$

$$=-3(0)e^{0}\sin 0$$

$$=0$$

$$\nabla g = 3\hat{i}$$

$$\left(D_{\vec{u}}\,g\right)_{P_0} = \nabla g \bullet \vec{u}$$

$$= \left(3\hat{i}\right) \bullet \left(\tfrac{2}{3}\hat{i} + \tfrac{1}{3}\hat{j} - \tfrac{2}{3}\hat{k}\right)$$

$$= 2$$

Exercise

Find the derivative of the function $h(x, y, z) = \cos xy + e^{yz} + \ln zx$ at $P_0\left(1,\ 0,\ \frac{1}{2}\right)$ in the direction of $\vec{v} = \hat{i} + 2\hat{j} + 2\hat{k}$

Solution

$$\vec{u} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

$$h_x = -y \sin xy + \frac{1}{x}\Big|_{\left(1,0,\frac{1}{2}\right)}$$

$$= -(0)\sin(0) + \frac{1}{1}$$

$$= 1$$

$$h_y = -x \sin xy + z e^{yz}\Big|_{\left(1,0,\frac{1}{2}\right)}$$

$$= -(1)\sin 0 + \frac{1}{2}e^{0}$$

$$= \frac{1}{2}$$

$$h_z = y e^{yz} + \frac{1}{z}\Big|_{\left(1,0,\frac{1}{2}\right)}$$

$$= 0e^{0} + \frac{1}{\frac{1}{2}}$$

$$= 2$$

$$\nabla h = \hat{i} + \frac{1}{2}\hat{j} + 2\hat{k}$$

$$\left(D_{\vec{u}}\,h\right)_{P_0} = \nabla h \bullet \vec{u}$$

$$= \left(\hat{i} + \frac{1}{2}\hat{j} + 2\hat{k}\right) \bullet \left(\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$$

$$= \frac{1}{3} + \frac{1}{3} + \frac{4}{3}$$

$$\underline{= 2}$$

Exercise

Find the directions in which the function $f(x, y) = x^2 + xy + y^2$ increase and decrease most rapidly at $P_0(-1,\ 1)$. Then find the derivatives of the function in these directions.

Solution

$$\left.\begin{aligned} f_x &= 2x + y \ \Rightarrow\ f_x(-1,1) = 2(-1) + 1 = -1 \\ f_y &= x + 2y \ \Rightarrow\ f_y(-1,1) = (-1) + 2(1) = 1 \end{aligned}\right\} \rightarrow\ \nabla f = -\hat{i} + \hat{j}$$

$$\vec{u} = \frac{\nabla f}{|\nabla f|} = \frac{-\hat{i} + \hat{j}}{\sqrt{1+1}}$$

$$\underline{= -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}}$$

f *increases* most rapidly in the direction $\vec{u} = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

f *decreases* most rapidly in the direction $-\vec{u} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

$$\left(D_{\vec{u}} f\right)_{P_0} = \nabla f \bullet \vec{u}$$

$$= \left(-\hat{i} + \hat{j}\right) \bullet \left(-\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$\underline{= \sqrt{2}}$$

$$\underline{\left(D_{-\vec{u}} f\right)_{P_0} = -\sqrt{2}}$$

Exercise

Find the directions in which the function $f(x, y) = x^2 y + e^{xy} \sin y$ increase and decrease most rapidly at $P_0(1,\ 0)$. Then find the derivatives of the function in these directions.

Solution

$$f_x = 2xy + ye^{xy}\sin y\Big|_{(1,\ 0)}$$
$$= 2(1)(0) + 0e^0$$
$$\underline{= 0|}$$
$$f_y = x^2 + xe^{xy}\sin y + e^{xy}\cos y\Big|_{(1,\ 0)}$$
$$= 1^2 + 0 + 1$$
$$\underline{= 2|}$$
$$\nabla f = 2\hat{j}$$
$$\vec{u} = \frac{\nabla f}{|\nabla f|} = \hat{j}$$

f *increases* most rapidly in the direction $\vec{u} = \hat{j}$

f *decreases* most rapidly in the direction $-\vec{u} = -\hat{j}$

$$\left(D_{\vec{u}} f\right)_{P_0} = \nabla f \bullet \vec{u}$$
$$\underline{= 2\ |}$$
$$\left(D_{-\vec{u}} f\right)_{P_0} = -2$$

Exercise

Find the directions in which the function $g(x, y, z) = xe^y + z^2$ increase and decrease most rapidly at $P_0\left(1,\ \ln 2,\ \frac{1}{2}\right)$. Then find the derivatives of the function in these directions.

Solution

$$g_x = e^y \quad \Rightarrow \quad g_x\left(1, \ln 2, \frac{1}{2}\right) = e^{\ln 2} = 2$$
$$g_y = xe^y \quad \Rightarrow \quad g_y\left(1, \ln 2, \frac{1}{2}\right) = e^{\ln 2} = 2 \quad \rightarrow \quad \nabla g = 2\hat{i} + 2\hat{j} + \hat{k}$$
$$g_z = 2z \quad \Rightarrow \quad g_z\left(1, \ln 2, \frac{1}{2}\right) = 2\left(\frac{1}{2}\right) = 1$$

$$g_x = e^y\Big|_{\left(1, \ln 2, \frac{1}{2}\right)}$$
$$= e^{\ln 2}$$
$$\underline{= 2|}$$

$$g_y = xe^y\Big|_{\left(1, \ln 2, \frac{1}{2}\right)}$$

$$= e^{\ln 2}$$
$$\underline{= 2}$$

$$g_z = 2z\Big|_{\left(1, \ln 2, \frac{1}{2}\right)}$$
$$= 2\left(\frac{1}{2}\right)$$
$$\underline{= 1}$$

$$\nabla g = 2\hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{u} = \frac{\nabla g}{|\nabla g|} = \frac{2\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{4+4+1}}$$
$$\underline{= \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}}$$

g *increases* most rapidly in the direction $\vec{u} = \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$

g *decreases* most rapidly in the direction $-\vec{u} = -\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$

$$\left(D_{\vec{u}}\, g\right)_{P_0} = \nabla g \bullet \vec{u}$$
$$= \left(2\hat{i} + 2\hat{j} + \hat{k}\right) \bullet \left(\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}\right)$$
$$= \frac{4}{3} + \frac{4}{3} + \frac{1}{3}$$
$$\underline{= 3}$$

$$\underline{\left(D_{-\vec{u}}\, g\right)_{P_0} = -3}$$

Exercise

Find the directions in which the function $h(x, y, z) = \ln\left(x^2 + y^2 - 1\right) + y + 6z$ increase and decrease most rapidly at $P_0\ (1,\ 1,\ 0)$. Then find the derivatives of the function in these directions.

Solution

$$h_x = \frac{2x}{x^2 + y^2 - 1} \quad \Rightarrow \quad h_x\ (1,1,0) = \frac{2}{1+1-1} = 2$$

$$h_y = \frac{2y}{x^2 + y^2 - 1} + 1 \quad \Rightarrow \quad h_y\ (1,1,0) = \frac{2}{1+1-1} + 1 = 3$$

$$h_z = 6 \quad \Rightarrow \quad h_z\ (1,1,0) = 6$$

$$h_x = \left.\frac{2x}{x^2+y^2-1}\right|_{(1,1,0)}$$

$$= \frac{2}{1+1-1}$$

$$\underline{= 2}$$

$$h_y = \left.\frac{2y}{x^2+y^2-1}+1\right|_{(1,1,0)}$$

$$= \frac{2}{1+1-1}+1$$

$$\underline{= 3}$$

$$\underline{h_z = 6}$$

$$\nabla h = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

$$\vec{u} = \frac{\nabla h}{|\nabla h|} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{4+9+36}}$$

$$\underline{= \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}}$$

h *increases* most rapidly in the direction $\vec{u} = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$

h *decreases* most rapidly in the direction $-\vec{u} = -\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}$

$$\left(D_{\vec{u}} h\right)_{P_0} = \nabla h \bullet \vec{u}$$

$$= \left(2\hat{i}+3\hat{j}+6\hat{k}\right) \bullet \left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$$

$$= \frac{4}{7} + \frac{9}{7} + \frac{36}{7}$$

$$\underline{= 7}$$

$$\left(D_{-\boldsymbol{u}} h\right)_{P_0} \underline{= -7}$$

Exercise

Sketch the curve $x^2 + y^2 = 4;$ $\left(f(x,y) = c\right)$ together with ∇f and the tangent line at the point $\left(\sqrt{2}, \sqrt{2}\right)$. Then write an equation for the tangent line.

Solution

$$f_x = 2x \quad \Rightarrow \quad f_x\left(\sqrt{2}, \sqrt{2}\right) = 2\sqrt{2}$$

$$f_y = 2y \quad \Rightarrow \quad f_y\left(\sqrt{2}, \sqrt{2}\right) = 2\sqrt{2}$$

$\nabla f = 2\sqrt{2}\hat{i} + 2\sqrt{2}\hat{j}$

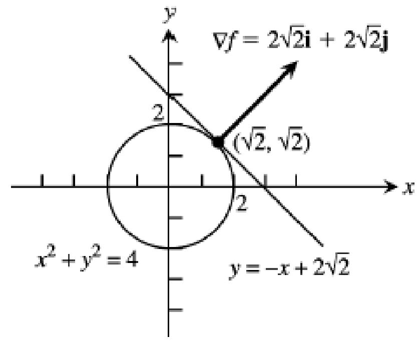
Tangent line:

$$2\sqrt{2}\left(x-\sqrt{2}\right)+2\sqrt{2}\left(y-\sqrt{2}\right)=0$$

$$2\sqrt{2}x-4+2\sqrt{2}y-4=0$$

$$2\sqrt{2}x+2\sqrt{2}y=8$$

$$\boxed{\sqrt{2}x+\sqrt{2}y=4}$$



Exercise

Sketch the curve $x^2 - y = 1$; $\left(f(x,y)=c\right)$ together with ∇f and the tangent line at the point $\left(\sqrt{2},\ 1\right)$. Then write an equation for the tangent line.

Solution

$$f_x = 2x \quad\Rightarrow\quad f_x\left(\sqrt{2},\ 1\right)=2\sqrt{2}$$

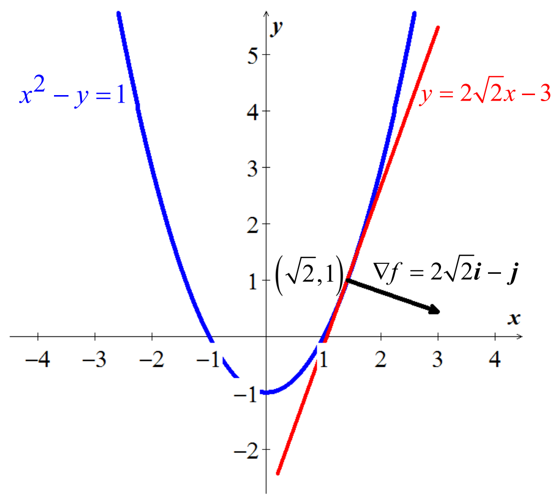
$$f_y = -1 \quad\Rightarrow\quad f_y\left(\sqrt{2},\ 1\right)=-1$$

$$\nabla f = 2\sqrt{2}\hat{i} - \hat{j}$$

Tangent line: $2\sqrt{2}\left(x-\sqrt{2}\right)-\left(y-1\right)=0$

$$2\sqrt{2}x-4-y+1=0$$

$$\boxed{y=2\sqrt{2}x-3}$$



Exercise

Sketch the curve $x^2 - xy + y^2 = 7$; $\left(f(x,y)=c\right)$ together with ∇f and the tangent line at the point $\left(-1,\ 2\right)$. Then write an equation for the tangent line.

Solution

$$f_x = 2x - y \quad\Rightarrow\quad f_x\left(-1,\ 2\right)=-4$$

$$f_y = -x + 2y \quad\Rightarrow\quad f_y\left(-1,\ 2\right)=5$$

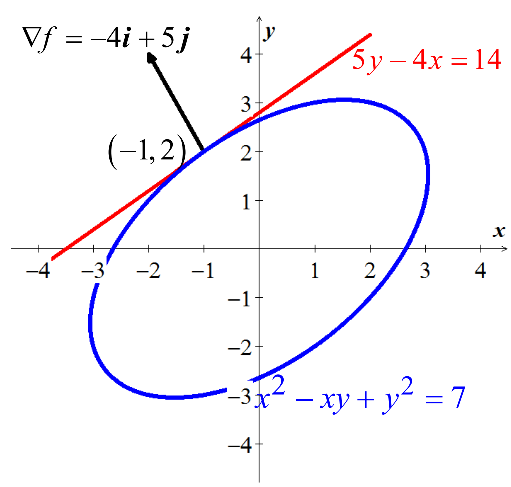
$$\rightarrow \nabla f = -4\hat{i} + 5\hat{j}$$

Tangent line:

$$-4\left(x+1\right)+5\left(y-2\right)=0$$

$$-4x+5y-14=0$$

$$\boxed{5y-4x=14}$$



Exercise

In what direction is the derivative of $f(x,y) = xy + y^2$ at $P(3,\ 2)$ equal to zero?

Solution

$$\begin{aligned} f_x &= y \\ f_y &= x + 2y \end{aligned} \quad \rightarrow \quad \nabla f = x\hat{i} + (x + 2y)\hat{j}$$

$$\nabla f(3,\ 2) = 2\hat{i} + 7\hat{j}$$

A vector is orthogonal to ∇f is $\vec{v} = 7\hat{i} - 2\hat{j}$

$$\vec{u} = \frac{7\hat{i} - 2\hat{j}}{\sqrt{49 + 4}} \qquad\qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{7}{\sqrt{53}}\hat{i} - \frac{2}{\sqrt{53}}\hat{j}$$

$$-\vec{u} = -\frac{7}{\sqrt{53}}\hat{i} + \frac{2}{\sqrt{53}}\hat{j}$$

$\vec{u}$ and $-\vec{u}$ are the directions where the derivatives are zero.

Exercise

Compute the gradient of the function, evaluate it at the given point *P*, and evaluate the directional derivative at that point in the given direction

$$f(x,\ y) = x^2; \quad P = (1,\ 2); \quad \vec{u} = \left\langle \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}} \right\rangle$$

Solution

$$\nabla f = \langle 2x,\ 0\rangle \qquad\qquad \nabla f = \left\langle f_x, f_y \right\rangle$$

$$\nabla f(1,\ 2) = \langle 2,\ 0\rangle$$

$$\begin{aligned} \left(D_{\vec{u}} f\right)_{(1,\ 2)} &= \nabla f \Big|_{(1,\ 2)} \bullet \vec{u} \\ &= \langle 2,\ 0\rangle \bullet \left\langle \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}} \right\rangle \\ &= \frac{2}{\sqrt{2}} \\ &= \sqrt{2} \end{aligned}$$

Exercise

Compute the gradient of the function, evaluate it at the given point P, and evaluate the directional derivative at that point in the given direction

$$f(x, y) = x^2 y^3; \quad P = (-1, 1); \quad \vec{u} = \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$$

Solution

$$\nabla f = \left\langle 2xy^3, 3x^2y^2 \right\rangle \qquad\qquad \nabla f = \left\langle f_x, f_y \right\rangle$$

$$\nabla f(-1, 1) = \langle -2, 3 \rangle$$

$$\begin{aligned}
\left(D_{\vec{u}} f\right)_{(-1, 1)} &= \nabla f \Big|_{(-1, 1)} \bullet \vec{u} \\
&= \langle -2, 3 \rangle \bullet \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle \\
&= -\frac{10}{13} + \frac{36}{13} \\
&= \frac{26}{3} \\
&= 2
\end{aligned}$$

Exercise

Compute the gradient of the function, evaluate it at the given point P, and evaluate the directional derivative at that point in the given direction

$$f(x, y) = \frac{x}{y^2}; \quad P = (0, 3); \quad \vec{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

Solution

$$\nabla f = \left\langle \frac{1}{y^2}, -\frac{2x}{y^3} \right\rangle \qquad\qquad \nabla f = \left\langle f_x, f_y \right\rangle$$

$$\nabla f(0, 3) = \left\langle \frac{1}{9}, 0 \right\rangle$$

$$\begin{aligned}
\left(D_{\vec{u}} f\right)_{(0, 3)} &= \nabla f \Big|_{(0, 3)} \bullet \vec{u} \\
&= \left\langle \frac{1}{9}, 0 \right\rangle \bullet \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle \\
&= \frac{\sqrt{13}}{18}
\end{aligned}$$

Exercise

Compute the gradient of the function, evaluate it at the given point P, and evaluate the directional derivative at that point in the given direction

$$f(x, y) = \sqrt{2 + x^2 + 2y^2}; \quad P = (2, 1); \quad \vec{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

Solution

$$\nabla f = \left\langle \frac{x}{\sqrt{2 + x^2 + 2y^2}}, \frac{2y}{\sqrt{2 + x^2 + 2y^2}} \right\rangle \qquad \nabla f = \left\langle f_x, f_y \right\rangle$$

$$\nabla f(2, 1) = \left\langle \frac{2}{\sqrt{8}}, \frac{2}{\sqrt{8}} \right\rangle$$

$$= \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$\left(D_{\vec{u}} f\right)_{(2, 1)} = \nabla f \Big|_{(2, 1)} \bullet \vec{u}$$

$$= \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \bullet \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

$$= \frac{3}{5\sqrt{2}} + \frac{4}{5\sqrt{2}}$$

$$= \frac{7}{5\sqrt{2}}$$

$$= \boxed{\frac{7\sqrt{2}}{10}}$$

Exercise

Compute the gradient of the function, evaluate it at the given point P, and evaluate the directional derivative at that point in the given direction

$$f(x, y, z) = xy + yz + xz + 4; \quad P = (2, -2, 1); \quad \vec{u} = \left\langle 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$

Solution

$$\nabla f = \left\langle y + z, x + z, y + x \right\rangle \qquad \nabla f = \left\langle f_x, f_y, f_z \right\rangle$$

$$\nabla f(2, -2, 1) = \left\langle -1, 3, 0 \right\rangle$$

$$\left(D_{\vec{u}} f\right)_{(2, -2, 1)} = \nabla f \Big|_{(2, -2, 1)} \bullet \vec{u}$$

$$= \left\langle -1, 3, 0 \right\rangle \bullet \left\langle 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$

$$= -\frac{3}{\sqrt{2}}$$

Exercise

Compute the gradient of the function, evaluate it at the given point *P*, and evaluate the directional derivative at that point in the given direction

$$f(x, y, z) = 1 + \sin(x + 2y - z); \quad P = \left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right); \quad \vec{u} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$$

Solution

$$\nabla f = \left\langle \cos(x + 2y - z), \ 2\cos(x + 2y - z), \ -\cos(x + 2y - z) \right\rangle \qquad \nabla f = \left\langle f_x, f_y, f_z \right\rangle$$

$$\nabla f\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right) = \left\langle \cos\frac{2\pi}{3}, \ 2\cos\frac{2\pi}{3}, \ -\cos\frac{2\pi}{3} \right\rangle$$

$$= \left\langle -\frac{1}{2}, \ -1, \ \frac{1}{2} \right\rangle$$

$$\left(D_{\vec{u}} f\right)_{\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right)} = \nabla f\Big|_{\left(\frac{\pi}{6}, \frac{\pi}{6}, -\frac{\pi}{6}\right)} \bullet \vec{u}$$

$$= \left\langle -\frac{1}{2}, \ -1, \ \frac{1}{2} \right\rangle \bullet \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$$

$$= -\frac{1}{6} - \frac{2}{3} + \frac{1}{3}$$

$$= -\frac{1}{2}$$

Exercise

Find the direction in which f increases and decreases most rapidly at P_0 and find the derivative of f in each direction. Also, find the derivative of f at P_0 in the direction of the vector ***v***.

$$f(x, y) = \cos x \cos y, \quad P_0\left(\frac{\pi}{4}, \frac{\pi}{4}\right), \quad \vec{v} = 3\hat{i} + 4\hat{j}$$

Solution

$$\nabla f = -\sin x \cos y \hat{i} - \cos x \sin y \hat{j} \qquad \nabla f = f_x \hat{i} + f_y \hat{j}$$

$$\nabla f\Big|_{\left(\frac{\pi}{4}, \frac{\pi}{4}\right)} = -\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\hat{j}$$

$$= -\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j}$$

$$|\nabla f| = \sqrt{\frac{1}{4} + \frac{1}{4}}$$

$$=\boxed{\frac{\sqrt{2}}{2}}$$

$$\vec{u}=\frac{\nabla f}{|\nabla f|}=\sqrt{2}\left(-\frac{1}{2}\hat{i}-\frac{1}{2}\hat{j}\right)$$

$$=-\frac{\sqrt{2}}{2}\hat{i}-\frac{\sqrt{2}}{2}\hat{j}$$

$\rightarrow f$ increases most rapidly in the direction of $\vec{u}=-\frac{\sqrt{2}}{2}\hat{i}-\frac{\sqrt{2}}{2}\hat{j}$

$\rightarrow f$ decreases most rapidly in the direction of $-\vec{u}=\frac{\sqrt{2}}{2}\hat{i}+\frac{\sqrt{2}}{2}\hat{j}$

$$\vec{u}_1=\frac{3\hat{i}+4\hat{j}}{\sqrt{9+16}} \qquad \vec{u}=\frac{\vec{v}}{|\vec{v}|}$$

$$=\frac{3}{5}\hat{i}+\frac{4}{5}\hat{j}$$

$$\left(D_{\vec{u}}f\right)_{\left(\frac{\pi}{4},\,\frac{\pi}{4}\right)}=\nabla f\Big|_{\left(\frac{\pi}{4},\,\frac{\pi}{4}\right)}\bullet\vec{u}_1$$

$$=\left(-\frac{1}{2}\hat{i}-\frac{1}{2}\hat{j}\right)\bullet\left(\frac{3}{5}\hat{i}+\frac{4}{5}\hat{j}\right)$$

$$=-\frac{3}{10}-\frac{4}{10}$$

$$=\boxed{-\frac{7}{10}}$$

Exercise

Find the direction in which f increases and decreases most rapidly at P_0 and find the derivative of f in each direction. Also, find the derivative of f at P_0 in the direction of the vector ***v***.

$$f(x,y)=x^2e^{-2y}, \quad P_0\,(1,\ 0), \quad \vec{v}=\hat{i}+\hat{j}$$

Solution

$$\nabla f=2xe^{-2y}\hat{i}-2x^2e^{-2y}\hat{j} \qquad \nabla f=f_x\hat{i}+f_y\hat{j}$$

$$\nabla f\Big|_{(1,\ 0)}=2\hat{i}-2\hat{j}$$

$$|\nabla f|=\sqrt{4+4}$$

$$=\boxed{2\sqrt{2}}$$

$$\vec{u}=\frac{\nabla f}{|\nabla f|}=\frac{1}{2\sqrt{2}}\left(2\hat{i}-2\hat{j}\right)$$

$$= \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$$

$\to f$ increases most rapidly in the direction of $\vec{u} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

$\to f$ decreases most rapidly in the direction of $-\vec{u} = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

$$\vec{u}_1 = \frac{\hat{i} + \hat{j}}{\sqrt{1+1}} \qquad\qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

$$\left(D_{\vec{u}} f\right)_{(1,\ 0)} = \nabla f \Big|_{(1,\ 0)} \bullet \vec{u}_1$$

$$= \left(2\hat{i} - 2\hat{j}\right) \bullet \left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)$$

$$= \sqrt{2} - \sqrt{2}$$

$$= 0$$

Exercise

Find the direction in which f increases and decreases most rapidly at P_0 and find the derivative of f in each direction. Also, find the derivative of f at P_0 in the direction of the vector $\boldsymbol{v}$.

$$f(x, y, z) = \ln(2x + 3y + 6z), \quad P_0(-1,\ -1,\ 1), \quad \vec{v} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

Solution

$$\nabla f = \frac{2}{2x+3y+6z}\hat{i} + \frac{3}{2x+3y+6z}\hat{j} + \frac{6}{2x+3y+6z}\hat{k} \qquad\qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$\nabla f \Big|_{(-1,\ -1,\ 1)} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

$$|\nabla f| = \sqrt{4+9+36}$$

$$= 7$$

$$\vec{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)$$

$$= \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

$\to f$ increases most rapidly in the direction of $\vec{u} = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$

$\to f$ decreases most rapidly in the direction of $-\vec{u} = -\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}$

$$\vec{u}_1 = \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{4+9+36}} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

$$\left(D_{\vec{u}} f\right)_{(-1,\ -1,\ 1)} = \nabla f\Big|_{(-1,\ -1,\ 1)} \bullet \vec{u}_1$$

$$= \left(2\hat{i} + 3\hat{j} + 6\hat{k}\right) \bullet \left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$$

$$= \frac{4}{7} + \frac{9}{7} + \frac{36}{7}$$

$$= 7$$

Exercise

Find the direction in which f increases and decreases most rapidly at P_0 and find the derivative of f in each direction. Also, find the derivative of f at P_0 in the direction of the vector ***v***.

$$f(x,y,z) = x^2 + 3xy - z^2 + 2y + z + 4, \quad P_0\,(0,\ 0,\ 0), \quad \vec{v} = \hat{i} + \hat{j} + \hat{k}$$

Solution

$$\nabla f = (2x+3y)\hat{i} + (3x+2)\hat{j} + (-2z+1)\hat{k} \qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$\nabla f\Big|_{(0,\ 0,\ 0)} = 2\hat{j} + \hat{k}$$

$$|\nabla f| = \sqrt{4+1}$$

$$= \sqrt{5}$$

$$\vec{u} = \frac{\nabla f}{|\nabla f|} = \frac{2}{\sqrt{5}}\hat{j} + \frac{1}{\sqrt{5}}\hat{k}$$

$\rightarrow$ f increases most rapidly in the direction of $\vec{u} = \frac{2}{\sqrt{5}}\hat{j} + \frac{1}{\sqrt{5}}\hat{k}$

$\rightarrow$ f decreases most rapidly in the direction of $-\vec{u} = -\frac{2}{\sqrt{5}}\hat{j} - \frac{1}{\sqrt{5}}\hat{k}$

$$\vec{u}_1 = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{1+1+1}} \qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

$$\left(D_{\vec{u}} f\right)_{(0,\ 0,\ 0)} = \nabla f\Big|_{(0,\ 0,\ 0)} \bullet \vec{u}_1$$

$$= \left(2\hat{j} + \hat{k}\right) \bullet \left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right)$$

$$= \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}$$

$$= \frac{3}{\sqrt{3}}$$

$$\boxed{= \sqrt{3}}$$

Exercise

Let $f(x, y) = \ln(1 + xy); \quad P = (2,\ 3)$

a) Find the unit vectors that give the direction of steepest ascent and steepest descent at *P*.

b) Find a unit vector that points in ta direction of no change.

Solution

a) $\nabla f = \frac{y}{1+xy}\hat{i} + \frac{x}{1+xy}\hat{j} \qquad \nabla f = f_x\hat{i} + f_y\hat{j}$

$$\nabla f(2,\ 3) = \frac{3}{7}\hat{i} + \frac{2}{7}\hat{j}$$

$$= \frac{1}{7}\left(3\hat{i} + 2\hat{j}\right)$$

$$\vec{u} = \frac{3\hat{i} + 2\hat{j}}{\sqrt{9+4}}$$

$$\boxed{= \frac{3}{\sqrt{13}}\hat{i} + \frac{2}{\sqrt{13}}\hat{j}}$$

The direction of steepest ascent is the unit vector in the direction of $\vec{u} = \frac{3}{\sqrt{13}}\hat{i} + \frac{2}{\sqrt{13}}\hat{j}$

The direction of steepest descent is the unit vector in the direction of $-\vec{u} = -\frac{3}{\sqrt{13}}\hat{i} - \frac{2}{\sqrt{13}}\hat{j}$

b) The unit vectors that the point in the direction of no change are $\vec{v} = \pm\left(\frac{3}{\sqrt{13}}\hat{i} - \frac{2}{\sqrt{13}}\hat{j}\right)$

Since $\vec{u} \bullet \vec{v} = 0$

Exercise

Let $f(x, y) = \sqrt{4 - x^2 - y^2}; \quad P = (-1,\ 1)$

a) Find the unit vectors that give the direction of steepest ascent and steepest descent at *P*.

b) Find a unit vector that points in ta direction of no change.

Solution

a) $\nabla f = -\frac{x}{\sqrt{4 - x^2 - y^2}}\hat{i} - \frac{y}{\sqrt{4 - x^2 - y^2}}\hat{j} \qquad \nabla f = f_x\hat{i} + f_y\hat{j}$

$$\nabla f(-1,\ 1) = \frac{1}{\sqrt{2}}\left(\hat{i} - \hat{j}\right)$$

$$\vec{u} = \frac{\hat{i} - \hat{j}}{\sqrt{1+1}}$$

$$= \frac{1}{\sqrt{2}}\left(\hat{i} - \hat{j}\right)$$

The direction of steepest ascent is the unit vector in the direction of $\vec{u} = \frac{\sqrt{2}}{2}\hat{i} - \frac{\sqrt{2}}{2}\hat{j}$

The direction of steepest descent is the unit vector in the direction of $-\vec{u} = -\frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j}$

b) The unit vectors that the point in the direction of no change are $\vec{v} = \pm\left(\frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j}\right)$

Since $\vec{u} \bullet \vec{v} = 0$

Exercise

Let $f(x,\ y) = 8 - 2x^2 - y^2$, for the level curves $f(x,\ y) = C$ and points $(a,\ b)$, compute the slope of the line tangent to the level curve at $(a,\ b)$ and verify that the tangent line is orthogonal to the gradient at that point.

$$f(x,\ y) = 5; \quad (a,\ b) = (1,\ 1)$$

Solution

$$\frac{dy}{dx} = -\frac{4x}{-2y} \qquad\qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= \frac{2x}{y}\Big|_{(1,\ 1)}$$

$$= -2$$

$m = -2 \quad f(x,\ y) = 5$

Tangent line has direction: $\quad \hat{i} - 2\hat{j}$

$$\nabla f = -4x\hat{i} - 2y\hat{j} \qquad\qquad \nabla f = f_x\hat{i} + f_y\hat{j}$$

$$\nabla f(1,\ 1) = -4\hat{i} - 2\hat{j}$$

$$\left(\hat{i} - 2\hat{j}\right) \bullet \left(-4\hat{i} - 2\hat{j}\right) = -4 + 4$$

$$= 0 \quad \checkmark$$

The gradient is orthogonal to the tangent direction.

Exercise

Let $f(x, y) = 8 - 2x^2 - y^2$, for the level curves $f(x, y) = C$ and points (a, b), compute the slope of the line tangent to the level curve at (a, b) and verify that the tangent line is orthogonal to the gradient at that point.

$$f(x, y) = 0; \quad (a, b) = (2, 0)$$

Solution

$$\frac{dy}{dx} = -\frac{4x}{-2y} \qquad\qquad \frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= \frac{2x}{y}\bigg|_{(2,\ 0)}$$

$$= \infty$$

Slope: $m = 0$

Tangent line has direction: $\hat{j}$

$$\nabla f = -4x\hat{i} - 2y\hat{j} \qquad\qquad \nabla f = f_x\hat{i} + f_y\hat{j}$$

$$\nabla f(2,\ 0) = -8\hat{i}$$

$$(\hat{j}) \cdot (-8\hat{i}) = 0 \quad ✓$$

The gradient is orthogonal to the tangent direction.

Exercise

Find the direction in which the function $f(x, y) = 4x^2 - y^2$ has zero change at the point $(1, 1, 3)$. Express the directions in terms of unit vectors.

Solution

$$\nabla f = 8x\hat{i} - 2y\hat{j} \qquad\qquad \nabla f = f_x\hat{i} + f_y\hat{j}$$

$$\nabla f(1,\ 1,\ 3) = 8\hat{i} - 2\hat{j}$$

The unit vectors in the direction of no change are

$$\vec{u} = \pm\frac{8\hat{i} - 2\hat{j}}{\sqrt{64 + 4}}$$

$$= \pm\frac{2(4\hat{i} - \hat{j})}{2\sqrt{17}}$$

$$= \pm\frac{1}{\sqrt{17}}(4\hat{i} - \hat{j})$$

Exercise

An infinitely long charged cylinder of radius R with its axis along z-axis has an electric potential $V = k\ln\left(\frac{R}{r}\right)$, where r is the distance between a variable point $P(x,\ y)$ and the axis of the cylinder $\left(r^2 = x^2 + y^2\right)$ and k is a physical constant. The electric field at a point $(x,\ y)$ in the xy-plane is given by $\vec{E} = -\nabla V$, where ∇V is the two-dimensional gradient. Compute the electric field at a point $(x,\ y)$ with $r > R$.

Solution

$$V = k(\ln R - \ln r)$$

$$= k\left(\ln R - \ln\sqrt{x^2 + y^2}\right)$$

$$= k\left(\ln R - \frac{1}{2}\ln\left(x^2 + y^2\right)\right)$$

$$= \frac{1}{2}k\left(2\ln R - \ln\left(x^2 + y^2\right)\right)$$

$$= \frac{1}{2}k\left(\ln R^2 - \ln\left(x^2 + y^2\right)\right)$$

$$\vec{E} = -\nabla V$$

$$= -\frac{1}{2}k\left(-\frac{2x}{x^2 + y^2}\hat{i} - \frac{2y}{x^2 + y^2}\hat{j}\right)$$

$$= \frac{kx}{x^2 + y^2}\hat{i} + \frac{ky}{x^2 + y^2}\hat{j}$$

Exercise

Find the tangent plane and normal line of the surface $x^2 + y^2 + z^2 = 3$ at the point $P_0\left(1,\ 1,\ 1\right)$

Solution

$$f\left(x, y, z\right) = x^2 + y^2 + z^2$$

$$f_x = 2x, \quad f_y = 2y, \quad f_z = 2z$$

$$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\nabla f\left(1,\ 1,\ 1\right) = 2\hat{i} + 2\hat{j} + 2\hat{k}$$

Tangent Line: $f_x\left(P_0\right)\left(x - x_0\right) + f_y\left(P_0\right)\left(y - y_0\right) + f_z\left(P_0\right)\left(z - z_0\right) = 0$

$$2\left(x-1\right) + 2\left(y-1\right) + 2\left(z-1\right) = 0$$

$$2x + 2y + 2z = 6$$

$$\boxed{x + y + z = 3}$$

Normal Line: $x = x_0 + f_x\left(P_0\right)t, \quad y = y_0 + f_y\left(P_0\right)t, \quad z = z_0 + f_z\left(P_0\right)t$

$$\boxed{x = 1 + 2t, \quad y = 1 + 2t, \quad z = 1 + 2t}$$

Exercise

Find the tangent plane and normal line of the surface $x^2 + 2xy - y^2 + z^2 = 7$ at the point $P_0\left(1,\ -1,\ 3\right)$

Solution

$$f\left(x, y, z\right) = x^2 + 2xy - y^2 + z^2$$

$$\rightarrow \quad f_x = 2x + 2y, \quad f_y = 2x - 2y, \quad f_z = 2z$$

$$\nabla f = \left(2x + 2y\right)\hat{i} + \left(2x - 2y\right)\hat{j} + 2z\hat{k}$$

$$\nabla f\left(1,\ -1,\ 3\right) = 4\hat{j} + 6\hat{k}$$

Tangent Line: $f_x\left(P_0\right)\left(x - x_0\right) + f_y\left(P_0\right)\left(y - y_0\right) + f_z\left(P_0\right)\left(z - z_0\right) = 0$

$$0\left(x-1\right) + 4\left(y+1\right) + 6\left(z-3\right) = 0$$

$$4y + 6z = 14$$

$$\boxed{2y + 3z = 7}$$

Normal Line: $x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$

$$\underline{x = 1, \quad y = -1 + 4t, \quad z = 3 + 6t}$$

Exercise

Find the tangent plane and normal line of the surface $\cos \pi x - x^2 y + e^{xz} + yz = 4$ at the point $P_0(0, 1, 2)$

Solution

$f(x,y,z) = \cos \pi x - x^2 y + e^{xz} + yz$

$\rightarrow \quad f_x = -\pi \sin \pi x - 2xy + ze^{xz}, \quad f_y = -x^2 + z, \quad f_z = xe^{xz} + y$

$\nabla f = \left(-\pi \sin \pi x - 2xy + ze^{xz}\right)\hat{i} + \left(z - x^2\right)\hat{j} + \left(xe^{xz} + y\right)\hat{k}$

$\nabla f(0, 1, 2) = 2\hat{i} + 2\hat{j} + \hat{k}$

Tangent Line: $f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$

$$2(x - 0) + 2(y - 1) + (z - 2) = 0$$

$$\underline{2x + 2y + z - 4 = 0}$$

Normal Line: $x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$

$$\underline{x = 2t, \quad y = 1 + 2t, \quad z = 2 + t}$$

Exercise

Find the tangent plane and normal line of the surface $x^2 - xy - y^2 - z = 0$ at the point $P_0(1, 1, -1)$

Solution

$f(x,y,z) = x^2 - xy - y^2 - z$

$\rightarrow \quad f_x = 2x - y, \quad f_y = -x - 2y, \quad f_z = -1$

$\nabla f = (2x - y)\hat{i} - (x + 2y)\hat{j} - \hat{k}$

$\nabla f(1, 1, -1) = \hat{i} - 3\hat{j} - \hat{k}$

Tangent Line: $f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$

$$(x - 1) - 3(y - 1) - (z + 1) = 0$$

$$\boxed{x-3y-z+1=0}$$

Normal Line: $x=x_0+f_x(P_0)t, \quad y=y_0+f_y(P_0)t, \quad z=z_0+f_z(P_0)t$

$$\boxed{x=1+t, \quad y=1-3t, \quad z=-1-t}$$

Exercise

Find the tangent plane and normal line of the surface $x^2+y^2-2xy-x+3y-z=-4$ at the point $P_0(2,\ -3,\ 18)$

Solution

$$f(x,y,z)=x^2+y^2-2xy-x+3y-z$$

$$\rightarrow \quad f_x=2x-2y-1, \quad f_y=2y-2x+3, \quad f_z=-1$$

$$\nabla f=(2x-2y-1)\hat{i}-(2y-2x+3)\hat{j}-\hat{k}$$

$$\nabla f(2,\ -3,\ 18)=9\hat{i}-7\hat{j}-\hat{k}$$

Tangent Line: $f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)+f_z(P_0)(z-z_0)=0$

$$9(x-2)-7(y+3)-(z-18)=0$$

$$\boxed{9x-7y-z=21}$$

Normal Line: $x=x_0+f_x(P_0)t, \quad y=y_0+f_y(P_0)t, \quad z=z_0+f_z(P_0)t$

$$\boxed{x=2+9t, \quad y=-3-7t, \quad z=18-t}$$

Exercise

Find an equation for the plane that is tangent to the surface $z=\ln(x^2+y^2)$ at the point $(1,\ 0,\ 0)$

Solution

$$z=f(x,y)=\ln(x^2+y^2)$$

$$f_x=\frac{2x}{x^2+y^2} \quad \rightarrow f_x(1,0)=2$$

$$f_y=\frac{2x}{x^2+y^2} \quad \rightarrow f_y(1,0)=0$$

Tangent Line: $2(x-1)-(y-0)-z=0 \qquad f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)-(z-z_0)=0$

$$\boxed{2x-z-2=0}$$

Exercise

Find an equation for the plane that is tangent to the surface $z=e^{-x^2-y^2}$ at the point $(0,\ 0,\ 1)$

Solution

$$z=f(x,y)=e^{-x^2-y^2}$$

$$f_x=-2xe^{-x^2-y^2} \quad \to f_x(0,0)=0$$

$$f_y=-2ye^{-x^2-y^2} \quad \to f_y(0,0)=0$$

Tangent Line: $-(z-1)=0$ $\qquad f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)-(z-z_0)=0$

$$\boxed{z=1}$$

Exercise

Find an equation for the plane that is tangent to the surface $z=\sqrt{y-x}$ at the point $(1,\ 2,\ 1)$

Solution

$$z=f(x,y)=\sqrt{y-x}$$

$$f_x=-\frac{1}{2}(y-x)^{-1/2} \quad \to f_x(1,2)=-\frac{1}{2}$$

$$f_y=\frac{1}{2}(y-x)^{-1/2} \quad \to f_y(1,2)=\frac{1}{2}$$

Tangent Line: $-\frac{1}{2}(x-1)+\frac{1}{2}(y-2)-(z-1)=0$

$$-\frac{1}{2}x+\frac{1}{2}y-z+\frac{1}{2}=0$$

$$\boxed{x-y+2z-1=0}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$z=2x^2+y^2;\quad (1,\ 1,\ 3)\ \ and\ \ (0,\ 2,\ 4)$$

Solution

$$f(x,y,z)=2x^2+y^2-z$$

$$\nabla f=4x\hat{i}+2y\hat{j}-\hat{k} \qquad \nabla f=f_x\hat{i}+f_y\hat{j}+f_z\hat{k}$$

$$\nabla f(1,\ 1,\ 3)=4\hat{i}+2\hat{j}-\hat{k}$$

The equation of the tangent plane:

$$4(x-1)+2(y-1)-(z-3)=0$$

$$\boxed{4x+2y-z=3}$$

$$\nabla f(0,\ 2,\ 3)=4\hat{j}-\hat{k}$$

The equation of the tangent plane:

$$4(y-2)-(z-4)=0$$

$$\boxed{4y-z=4}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$x^2+\frac{1}{4}y^2-\frac{1}{9}z^2=1;\quad (0,\ 2,\ 0)\ \ and\ \ \left(1,\ 1,\ \frac{3}{2}\right)$$

Solution

$$f(x,y,z)=x^2+\frac{1}{4}y^2-\frac{1}{9}z^2-1$$

$$\nabla f=2x\hat{i}+\frac{1}{2}y\hat{j}-\frac{2}{9}\hat{k} \qquad \nabla f=f_x\hat{i}+f_y\hat{j}+f_z\hat{k}$$

$$\nabla f(0,\ 2,\ 0)=\hat{j}$$

The equation of the tangent plane:

$$y-2=0$$

$$\boxed{y=2}$$

$$\nabla f\left(1,\ 1,\ \frac{3}{2}\right)=2\hat{i}+\frac{1}{2}\hat{j}-\frac{1}{3}\hat{k}$$

The equation of the tangent plane:

$$2(x-1)+\frac{1}{2}(y-1)-\frac{1}{3}\left(z-\frac{3}{2}\right)=0$$

$$2x-2+\frac{1}{2}y-\frac{1}{2}-\frac{1}{3}z+\frac{1}{2}=0$$

$$2x+\frac{1}{2}y-\frac{1}{3}z=2$$

$$\boxed{12x+3y-2z=12}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$xy\sin z-1=0;\quad \left(1,\ 2,\ \frac{\pi}{6}\right)\ \ and\ \ \left(-2,\ -1,\ \frac{5\pi}{6}\right)$$

Solution

$$f(x,y,z) = xy\sin z - 1$$

$$\nabla f = y\sin z\hat{i} + x\sin z\hat{j} + xy\cos z\hat{k} \qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$\nabla f\left(1,\ 2,\ \frac{\pi}{6}\right) = 2\sin\frac{\pi}{6}\hat{i} + \sin\frac{\pi}{6}\hat{j} + 2\cos\frac{\pi}{6}\hat{k}$$

$$= \hat{i} + \frac{1}{2}\hat{j} + \sqrt{3}\hat{k}$$

The equation of the tangent plane:

$$(x-1) + \frac{1}{2}(y-2) + \sqrt{3}\left(z - \frac{\pi}{6}\right) = 0$$

$$x + \frac{1}{2}y + \sqrt{3}z = 2 + \frac{\pi}{6}\sqrt{3}$$

$$\underline{6x + 3y + 6\sqrt{3}z = 12 + \pi\sqrt{3}}$$

$$\nabla f\left(-2,\ -1,\ \frac{5\pi}{6}\right) = -2\sin\frac{5\pi}{6}\hat{i} - \sin\frac{5\pi}{6}\hat{j} + 2\cos\frac{5\pi}{6}\hat{k}$$

$$= -\frac{1}{2}\hat{i} - \hat{j} - \sqrt{3}\hat{k}$$

The equation of the tangent plane:

$$-\frac{1}{2}(x+2) - (y+1) - \sqrt{3}\left(z - \frac{5\pi}{6}\right) = 0$$

$$-\frac{1}{2}x + -y - \sqrt{3}z = 2 - \frac{5\pi}{6}\sqrt{3}$$

$$\underline{3x + 6y + 6\sqrt{3}z = 5\pi\sqrt{3} - 12}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$yze^{xz} - 8 = 0; \quad (0,\ 2,\ 4) \ \ and \ \ (0,\ -8,\ -1)$$

Solution

$$f(x,y,z) = yze^{xz} - 8$$

$$\nabla f = yz^2e^{xz}\hat{i} + ze^{xz}\hat{j} + (y + xyz)e^{xz}\hat{k} \qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$\nabla f(0,\ 2,\ 4) = 32\hat{i} + 4\hat{j} + 2\hat{k}$$

The equation of the tangent plane:

$$32(x-0) + 4(y-2) + 2(z-4) = 0$$

$$\underline{32x + 4y + 2z = 16}$$

$$\nabla f(0,\ -8,\ -1) = -8\hat{i} - \hat{j} - 8\hat{k}$$

The equation of the tangent plane:

$$-8(x-0)-(y+8)-8(z+1)=0$$

$$\boxed{8x+y+8z=16}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$z=x^2e^{x-y};\quad (2,\ 2,\ 4)\ \textit{and}\ (-1,\ -1,\ 1)$$

Solution

$$f(x,y,z)=x^2e^{x-y}-z$$

$$\nabla f=\left(2x+x^2\right)e^{x-y}\hat{i}-x^2e^{x-y}\hat{j}-\hat{k} \qquad \nabla f=f_x\hat{i}+f_y\hat{j}+f_z\hat{k}$$

$$\nabla f(2,\ 2,\ 4)=8\hat{i}-4\hat{j}-\hat{k}$$

The equation of the tangent plane:

$$8(x-2)-4(y-2)-(z-4)=0$$

$$\boxed{8x-4y-z=4}$$

$$\nabla f(-1,\ -1,\ 1)=-\hat{i}-\hat{j}-\hat{k}$$

The equation of the tangent plane:

$$-(x+1)-(y+1)-(z-1)=0$$

$$\boxed{x+y+z=-1}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$z=\ln(1+xy);\quad (1,\ 2,\ \ln 3)\ \textit{and}\ (-2,\ -1,\ \ln 3)$$

Solution

$$f(x,y)=\ln(1+xy)$$

$$\nabla f=\frac{y}{1+xy}\hat{i}+\frac{x}{1+xy}\hat{j} \qquad \nabla f=f_x\hat{i}+f_y\hat{j}$$

$$\nabla f(1,\ 2)=\frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}$$

The equation of the tangent plane:

$$z=f(1,\ 2)+f_x(1,\ 2)(x-1)+f_y(1,\ 2)(y-2)$$

$$=\ln 3+\frac{2}{3}(x-1)+\frac{1}{3}(y-2)$$

$$= \ln 3 + \frac{2}{3}x - \frac{2}{3} + \frac{1}{3}y - \frac{2}{3}$$

$$\boxed{= \frac{2}{3}x + \frac{1}{3}y - \frac{4}{3} + \ln 3}$$

$$\nabla f(-2, \ -1,) = -\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j}$$

The equation of the tangent plane:

$$z = f(-2, \ -1) + f_x(-2, \ -1)(x+2) + f_y(-2, \ -1)(y+1)$$

$$= \ln 3 - \frac{1}{3}(x+2) - \frac{2}{3}(y+1)$$

$$= \ln 3 - \frac{1}{3}x - \frac{2}{3} - \frac{2}{3}y - \frac{2}{3}$$

$$\boxed{= -\frac{2}{3}x - \frac{1}{3}y - \frac{4}{3} + \ln 3}$$

Exercise

Find an equation of the plane tangent to the surface at the given point

$$z = f(x, y) = \frac{1}{x^2 + y^2} \quad \text{at the point } \left(1, \ 1, \ \frac{1}{2}\right)$$

Solution

$$f_x = -\frac{2x}{\left(x^2 + y^2\right)^2}\Bigg|_{\left(1, \ 1, \ \frac{1}{2}\right)}$$

$$= -\frac{2}{(1+1)^2}$$

$$\boxed{= -\frac{1}{2}}$$

$$f_y = -\frac{2y}{\left(x^2 + y^2\right)^2}\Bigg|_{\left(1, \ 1, \ \frac{1}{2}\right)}$$

$$= -\frac{2}{4}$$

$$\boxed{= -\frac{1}{2}}$$

$$f(x, y, z) = \frac{1}{x^2 + y^2} - 1$$

$$\boxed{f_z = -1}$$

Tangent plane:

$$-\frac{1}{2}(x-1)-\frac{1}{2}(y-1)-\left(z-\frac{1}{2}\right)=0$$

$$-x+1-y+1-2z+1=0$$

$$\underline{x+y+2z=3}\,\rvert$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$x^2+y+z=3; \quad (1,\ 1,\ 1) \ \textit{and} \ (2,\ 0,\ -1)$$

Solution

$$f(x,\ y,\ z)=x^2+y+z-3$$

$$\nabla f=\langle 2x,\ 1,\ 1\rangle$$

At $(1,\ 1,\ 1)$:

$$\nabla f=\langle 2,\ 1,\ 1\rangle$$

Tangent plane:

$$2(x-1)+(y-1)+(z-1)=0$$

$$\underline{2x+y+z=4}\,\rvert$$

At $(2,\ 0,\ -1)$:

$$\nabla f=\langle 4,\ 1,\ 1\rangle$$

Tangent plane:

$$4(x-2)+y+(z+1)=0$$

$$\underline{4x+y+z=7}\,\rvert$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$x^2+y^3+z^4=2; \quad (1,\ 0,\ 1) \ \textit{and} \ (-1,\ 0,\ 1)$$

Solution

$$f(x,\ y,\ z)=x^2+y^3+z^4-2$$

$$\nabla f=\left\langle 2x,\ 3y^2,\ 4z^3\right\rangle$$

At $(1,\ 0,\ 1)$:

$$\nabla f=\langle 2,\ 0,\ 4\rangle$$

Tangent plane:

$$2(x-1)+4(z-1)=0$$

$$2x+4z=6$$

$$\underline{x+2z=3}$$

At $(-1,\ 0,\ 1)$:

$$\nabla f=\langle -2,\ 0,\ 4\rangle$$

Tangent plane:

$$-2(x-1)+4(z-1)=0$$

$$\underline{x-2z=-3}$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$xy+xz+yz=12;\quad (2,\ 2,\ 2)\ and\ (2,\ 0,\ 6)$$

Solution

$$f(x,\ y,\ z)=xy+xz+yz-12$$

$$\nabla f=\langle y+z,\ \ x+z,\ \ x+y\rangle$$

At $(2,\ 2,\ 2)$:

$$\nabla f=\langle 4,\ 4,\ 4\rangle$$

Tangent plane:

$$4(x-2)+4(y-2)+4(z-2)=0$$

$$\underline{x+y+z=6}$$

At $(2,\ 0,\ 6)$:

$$\nabla f=\langle 6,\ 8,\ 2\rangle$$

Tangent plane:

$$6(x-2)+8y+2(z-6)=0$$

$$\underline{3x+4y+z=12}$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$x^2+y^2-z^2=0;\quad (3,\ 4,\ 5)\ and\ (-4,\ -3,\ 5)$$

Solution

$$f(x,\ y,\ z)=x^2+y^2-z^2$$

$$\nabla f=\langle 2x,\ \ 2y,\ \ -2z\rangle$$

At $(3,\ 4,\ 5)$:

$$\nabla f = \langle 6,\ 8,\ -10\rangle$$

Tangent plane:

$$6(x-3)+8(y-4)-10(z-5)=0$$

$$\boxed{3x+4y-5z=0}$$

At $(-4,\ -3,\ 5)$:

$$\nabla f = \langle -8,\ -6,\ -10\rangle$$

Tangent plane:

$$-8(x+4)-6(y+3)-10(z-5)=0$$

$$\boxed{4x+3y+5z=0}$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$xy\sin z = 1; \quad \left(1,\ 2,\ \frac{\pi}{6}\right) \ \textit{and} \ \left(-2,\ -1,\ \frac{5\pi}{6}\right)$$

Solution

$$f(x,\ y,\ z) = xy\sin z - 1$$

$$\nabla f = \langle y\sin z,\ \ x\sin z,\ \ xy\cos z\rangle$$

At $\left(1,\ 2,\ \frac{\pi}{6}\right)$:

$$\nabla f = \left\langle 1,\ \frac{1}{2},\ \sqrt{3}\right\rangle$$

Tangent plane:

$$(x-1)+\frac{1}{2}(y-2)+\sqrt{3}\left(z-\frac{\pi}{6}\right)=0$$

$$\boxed{x+\frac{1}{2}y+\sqrt{3}z = 2+\frac{\pi\sqrt{3}}{6}}$$

At $\left(-2,\ -1,\ \frac{5\pi}{6}\right)$:

$$\nabla f = \left\langle -\frac{1}{2},\ -1,\ -\sqrt{3}\right\rangle$$

Tangent plane:

$$-\frac{1}{2}\left(x+\frac{1}{2}\right)-(y+1)-\sqrt{3}\left(z-\frac{5\pi}{6}\right)=0$$

$$\boxed{\frac{1}{2}x+y+\sqrt{3}z = \frac{5\pi\sqrt{3}}{6}-2}$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$yze^{xz} = 8; \quad (0,\ 2,\ 4)\ \ and\ \ (0,\ -8,\ -1)$$

Solution

$$f(x,\ y,\ z) = yze^{xz} - 8$$

$$\nabla f = \left\langle yz^2 e^{xz},\ ze^{xz},\ e^{xz}(y + xyz) \right\rangle$$

At $(0,\ 2,\ 4)$:

$$\nabla f = \langle 32,\ 4,\ 8\rangle$$

Tangent plane:

$$32x + 4(y-2) + 8(z-4) = 0$$

$$\underline{8x + y + 2z = 10}$$

At $(0,\ -8,\ -1)$:

$$\nabla f = \langle -8,\ -1,\ 8\rangle$$

Tangent plane:

$$-8x - (y+8) + 8(z+1) = 0$$

$$\underline{8x + y - 8z = 0}$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$z^2 - \frac{x^2}{16} - \frac{y^2}{9} = 1; \quad \left(4,\ 3,\ -\sqrt{3}\right)\ \ and\ \ \left(-8,\ 9,\ \sqrt{14}\right)$$

Solution

$$f(x,\ y,\ z) = z^2 - \frac{x^2}{16} - \frac{y^2}{9} - 1$$

$$\nabla f = \left\langle -\frac{1}{8}x,\ \ -\frac{2}{9}y,\ 2z \right\rangle$$

At $\left(4,\ 3,\ -\sqrt{3}\right)$:

$$\nabla f = \left\langle -\frac{1}{2},\ -\frac{2}{3},\ -2\sqrt{3} \right\rangle$$

Tangent plane:

$$-\frac{1}{2}(x-4) - \frac{2}{3}(y-3) - 2\sqrt{3}\left(z + 2\sqrt{3}\right) = 0$$

$$\underline{\frac{1}{2}x + \frac{2}{3}y + 2\sqrt{3}z = -2}$$

At $\left(-8,\ 9,\ \sqrt{14}\right)$:

$\nabla f = \langle 1, \ -2, \ 2\sqrt{14} \rangle$

Tangent plane:

$$(x+8) - 2(y-9) + 2\sqrt{14}\left(z - \sqrt{14}\right) = 0$$

$$\underline{x - 2y + 2\sqrt{14}z = 2}$$

Exercise

Find an equation of the plane tangent to the surface at the given points

$$2x + y^2 - z^2 = 0; \quad (0, \ 1, \ 1) \ \textit{and} \ (4, \ 1, \ -3)$$

Solution

$f(x, y, z) = 2x + y^2 - z^2$

$\nabla f = \langle 2, \ 2y, \ -2z \rangle$

At $(0, \ 1, \ 1)$:

$\nabla f = \langle 2, \ 2, \ -2 \rangle$

Tangent plane:

$$2x + 2(y-1) - 2(z-1) = 0$$

$$\underline{x + y - z = 0}$$

At $(4, \ 1, \ -3)$:

$\nabla f = \langle 2, \ 2, \ 6 \rangle$

Tangent plane:

$$2(x-4) + 2(y-1) + 6(z+3) = 0$$

$$\underline{x - + y + 3z = -4}$$

Exercise

Find parametric equation for the line tangent to the curve of intersection of the surfaces

$$x + y^2 + 2z = 4, \quad x = 1 \quad \text{at the point } (1, \ 1, \ 1)$$

Solution

$f_x = 1, \quad f_y = 2y, \quad f_z = 2$

$\nabla f = \hat{i} + 2y\hat{j} + 2\hat{k}\Big|_{(1, \ 1, \ 1)}$

$= \hat{i} + 2\hat{j} + 2\hat{k}$

$\nabla g = \hat{i}$

$$\vec{v} = \nabla f \times \nabla g$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 2\hat{j} - 2\hat{k}$$

Tangent Line: $x = 1, \quad y = 1 + 2t, \quad z = 1 - 2t$

Exercise

Find parametric equation for the line tangent to the curve of intersection of the surfaces $xyz = 1, \quad x^2 + 2y^2 + 3z^2 = 6$ at the point $(1, 1, 1)$

Solution

$$f_x = yz, \quad f_y = xz, \quad f_z = xy$$

$$\nabla f = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$\nabla f(1, 1, 1) = \hat{i} + \hat{j} + \hat{k}$$

$$g_x = 2x, \quad g_y = 4y, \quad g_z = 6z$$

$$\nabla g = 2x\hat{i} + 4y\hat{j} + 6y\hat{k}$$

$$\nabla g(1, 1, 1) = 2\hat{i} + 4\hat{j} + 6\hat{k}$$

$$v = \nabla f \times \nabla g$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 4 & 6 \end{vmatrix}$$

$$= 2\hat{i} - 4\hat{j} + 2\hat{k}$$

Tangent Line: $x = 1 + 2t, \quad y = 1 - 4t, \quad z = 1 + 2t$

Exercise

Find parametric equation for the line tangent to the curve of intersection of the surfaces $x^3 + 3x^2y^2 + y^3 + 4xy - z^2 = 0, \quad x^2 + y^2 + z^2 = 11$ at the point $(1, 1, 3)$

Solution

$$f_x = 3x^2 + 6xy^2 + 4y \Big|_{(1, 1, 3)}$$

$$= 13$$

$f_y = 6x^2y + 3y^2 + 4x \Big|_{(1,\ 1,\ 3)}$

$\quad \underline{= 13|}$

$f_z = -2z \Big|_{(1,\ 1,\ 3)}$

$\quad \underline{= -6|}$

$\nabla f\left(1,\ 1,\ 3\right) = 13\hat{i} + 13\hat{j} - 6\hat{k}$

$g_x = 2x, \quad g_y = 2y, \quad g_z = 2z$

$\nabla g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\nabla g\left(1,\ 1,\ 3\right) = 2\hat{i} + 2\hat{j} + 6\hat{k}$

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 13 & 13 & -6 \\ 2 & 2 & 6 \end{vmatrix} \qquad \vec{v} = \nabla f \times \nabla g$$

$\quad \underline{= 90\hat{i} - 90\hat{j}|}$

Tangent Line: $\quad \underline{x = 1 + 90t, \quad y = 1 - 90t, \quad z = 3\ |}$

Exercise

Find parametric equation for the line tangent to the curve of intersection of the surfaces $x^2 + y^2 = 4, \quad x^2 + y^2 - z = 0$ at the point $\left(\sqrt{2},\ \sqrt{2},\ 4\right)$

Solution

$f_x = 2x, \quad f_y = 2u, \quad f_z = 0$

$\nabla f = 2x\hat{i} + 2y\hat{j}$

$\nabla f\left(\sqrt{2},\ \sqrt{2},\ 4\right) = 2\sqrt{2}\hat{i} + 2\sqrt{2}\hat{j}$

$g_x = 2x, \quad g_y = 2y, \quad g_z = -1$

$\nabla g = 2x\hat{i} + 2y\hat{j} - \hat{k} \quad \Rightarrow \quad \nabla g\left(\sqrt{2},\ \sqrt{2},\ 4\right) = 2\sqrt{2}\hat{i} + 2\sqrt{2}\hat{j} - \hat{k}$

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\sqrt{2} & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 2\sqrt{2} & -1 \end{vmatrix} \qquad \vec{v} = \nabla f \times \nabla g$$

$\quad \underline{= -2\sqrt{2}\hat{i} + 2\sqrt{2}\hat{j}|}$

Tangent Line: $\quad \underline{x = \sqrt{2} - 2\sqrt{2}t, \quad y = \sqrt{2} + 2\sqrt{2}t, \quad z = 4\ |}$

Exercise

Find an equation for the plane tangent to the level surface $f(x,y,z)=x^2-y-5z$ at the point $P_0(2,\ -1,\ 1)$. Also, find parametric equations for the line is normal to the surface at P_0.

Solution

$$\nabla f = 2x\hat{i}-\hat{j}-5\hat{k}\Big|_{(2,\ -1,\ 1)}$$

$$\underline{=4\hat{i}-\hat{j}-5\hat{k}}$$

Tangent Plane:

$$4(x-2)-(y+1)-5(z-1)=0$$

$$4x-8-y-1-5z+5=0$$

$$\underline{4x-y-5z=4}$$

Normal Line:

$$\begin{cases} x=2+4t \\ y=-1-t \\ z=1-5t \end{cases}$$

Exercise

By about how much will $f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}$ change if the point $P(x,y,z)$ moves from $P_0(3,\ 4,\ 12)$ a distance of $ds=0.1$ unit in the direction of $3\boldsymbol{i}+6\boldsymbol{j}-2\boldsymbol{k}$?

Solution

$$f(x,y,z)=\ln\sqrt{x^2+y^2+z^2} \qquad\qquad (\ln u)'=\frac{u'}{u}$$

$$=\frac{1}{2}\ln\left(x^2+y^2+z^2\right)$$

$$f_x=\frac{x}{x^2+y^2+z^2}\Big|_{(3,4,12)}$$

$$=\frac{3}{9+16+144}$$

$$=\frac{3}{169}$$

$$f_y=\frac{y}{x^2+y^2+z^2}\Big|_{(3,4,12)}$$

$$=\frac{4}{9+16+144}$$

$$=\frac{4}{169}$$

$$f_z = \left.\frac{z}{x^2+y^2+z^2}\right|_{(3,4,12)}$$

$$= \frac{12}{9+16+144}$$

$$= \frac{12}{169}$$

$$\nabla f = \frac{3}{169}\hat{i} + \frac{4}{169}\hat{j} + \frac{12}{169}\hat{k}$$

$$\vec{u} = \frac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{9+36+4}} \qquad\qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}$$

$$\nabla f \bullet \vec{u} = \left(\frac{3}{169}\hat{i} + \frac{4}{169}\hat{j} + \frac{12}{169}\hat{k}\right) \bullet \left(\frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}\right)$$

$$= \frac{9}{1183}$$

$$df = (\nabla f \bullet \vec{u})\, ds$$

$$= \frac{9}{1183}(0.1)$$

$$= \frac{9}{11830}$$

$$\approx 0.0008$$

Exercise

By about how much will $f(x,y,z) = e^x \cos yz$ change if the point $P(x,y,z)$ moves from origin a distance of $ds = 0.1$ unit in the direction of $2\hat{i} + 2\hat{j} - 2\hat{k}$?

Solution

$$f_x = e^x \cos yz \quad \Rightarrow \quad f_x(0,0,0) = 1$$

$$f_y = -ze^x \sin yz \quad \Rightarrow \quad f_y(0,0,0) = 0$$

$$f_z = -ze^x \sin yz \quad \Rightarrow \quad f_z(0,0,0) = 0$$

$$\nabla f = \hat{i}$$

$$\vec{u} = \frac{2\hat{i}+2\hat{j}-2\hat{k}}{\sqrt{4+4+4}} \qquad\qquad \vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{2}{2\sqrt{3}}\hat{i} + \frac{2}{2\sqrt{3}}\hat{j} - \frac{2}{2\sqrt{3}}\hat{k}$$

$$= \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

$$\nabla f \bullet \vec{u} = \left(\hat{i}\right) \bullet \left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}\right)$$

$$= \frac{1}{\sqrt{3}}$$

$$df = \left(\nabla f \bullet \vec{u}\right) ds$$

$$= \frac{1}{\sqrt{3}}(0.1)$$

$$\boxed{= \frac{1}{10\sqrt{3}}} \qquad \boxed{\approx 0.0577}$$

Exercise

Find the linearization $L(x, y)$ of $f(x, y) = x^2 + y^2 + 1$ at the point (0, 0) and (1, 1)

Solution

$$f(0,0) = 1$$

$$f_x = 2x \quad \Rightarrow \quad f_x(0,0) = 0$$

$$f_y = 2y \quad \Rightarrow \quad f_y(0,0) = 0$$

$$L(x, y) = f\left(x_0, y_0\right) + f_x\left(x_0, y_0\right)\left(x - x_0\right) + f_y\left(x_0, y_0\right)\left(y - y_0\right)$$

$$L(x, y) = 1 + 0(x - 0) + 0(y - 0) \ \boxed{= 1}$$

$$f(1,1) = 3$$

$$f_x = 2x \quad \Rightarrow \quad f_x(1,1) = 2$$

$$f_y = 2y \quad \Rightarrow \quad f_y(1,1) = 2$$

$$L(x, y) = f\left(x_0, y_0\right) + f_x\left(x_0, y_0\right)\left(x - x_0\right) + f_y\left(x_0, y_0\right)\left(y - y_0\right)$$

$$L(x, y) = 3 + 2(x - 1) + 2(y - 1)$$

$$\boxed{= 2x + 2y - 1}$$

Exercise

Find the linearization $L(x, y)$ of $f(x, y) = (x + y + 2)^2$ at the point (0, 0) and (1, 2)

Solution

$$f(0,0) = 4$$

$f_x = 2(x+y+2) \quad \Rightarrow \quad f_x(0,0) = 4$

$f_y = 2(x+y+2) \quad \Rightarrow \quad f_y(0,0) = 4$

$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

$L(x,y) = 4 + 4(x-0) + 4(y-0)$

$\quad = 4 + 4x + 4y$

$f(1,2) = (1+2+2)^2 = 25$

$f_x = 2(x+y+2) \quad \Rightarrow \quad f_x(1,2) = 10$

$f_y = 2(x+y+2) \quad \Rightarrow \quad f_y(1,2) = 10$

$L(x,y) = 25 + 10(x-1) + 10(y-2)$

$\quad = 10x + 10y - 5$

Exercise

Find the linearization $L(x,y)$ of $f(x,y) = x^3y^4$ at the point (1, 1) and (0, 0)

Solution

$f(1,\ 1) = 1$

$f_x = 3x^2y^4 \quad \Rightarrow \quad f_x(1,1) = 3$

$f_y = 4x^2y^3 \quad \Rightarrow \quad f_y(1,1) = 4$

$L(x,y) = 1 + 3(x-1) + 4(y-1)$

$\quad = 3x + 4y - 6$

$f(0,0) = 0$

$f_x = 3x^2y^4 \quad \Rightarrow \quad f_x(0,0) = 0$

$f_y = 4x^2y^3 \quad \Rightarrow \quad f_y(0,0) = 0$

$L(x,\ y) = 0 + 0(x-0) + 0(y-0)$

$\quad = 0$

Exercise

Find the linearization $L(x,y)$ of $f(x,y) = e^{2y-x}$ at the point (0, 0) and (1, 2)

Solution

$$f(0,\ 0) = e^0 = 1$$

$$f_x = -e^{2y-x} \quad \Rightarrow \quad f_x(0,0) = -1$$

$$f_y = 2e^{2y-x} \quad \Rightarrow \quad f_y(0,0) = 2$$

$$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$L(x,\ y) = 1 - 1(x - 0) + 2(y - 0)$$

$$\underline{= 1 - x + 2y \,\big|}$$

$$f(1,\ 2) = e^3$$

$$f_x = -e^{2y-x} \quad \Rightarrow \quad f_x(1,2) = -e^3$$

$$f_y = 2e^{2y-x} \quad \Rightarrow \quad f_y(0,0) = 2e^3$$

$$L(x,\ y) = e^3 - e^3(x - 1) + 2e^3(y - 2)$$

$$\underline{= -e^3 x + 2e^3 y - 2e^3 \,\big|}$$

Exercise

Find the linearization $L(x,y,z)$ of $f(x,y,z) = x^2 + y^2 + z^2$ at the point (1, 1, 1)

Solution

$$f(1,\ 1,\ 1) = 3$$

$$f_x = 2x \quad \Rightarrow \quad f_x(1,1,1) = 2$$

$$f_y = 2y \quad \Rightarrow \quad f_y(1,1,1) = 2$$

$$f_z = 2z \quad \Rightarrow \quad f_z(1,1,1) = 2$$

$$L(x,y,z) = f(x_0, y_0, z_0) + f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0)$$

$$L(x,y,z) = 3 + 2(x - 1) + 2(y - 1) + 2(z - 1)$$

$$\underline{= 2x + 2y + 2z - 3 \,\big|}$$

Exercise

Find the linearization $L(x,y,z)$ of $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$ at the point (1, 2, 2)

Solution

$$f(1,1,1) = \sqrt{1 + 4 + 4} = 3$$

$$f_x = \frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}(2x)$$

$$= \left.\frac{x}{\sqrt{x^2+y^2+z^2}}\right|_{(1,2,2)}$$

$$= \frac{1}{3}$$

$$f_y = \frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}(2y)$$

$$= \left.\frac{y}{\sqrt{x^2+y^2+z^2}}\right|_{(1,2,2)}$$

$$= \frac{2}{3}$$

$$f_z = \frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}(2z)$$

$$= \left.\frac{z}{\sqrt{x^2+y^2+z^2}}\right|_{(1,2,2)}$$

$$= \frac{2}{3}$$

$$L(x,y,z) = f\left(x_0,y_0,z_0\right) + f_x\left(x-x_0\right) + f_y\left(y-y_0\right) + f_z\left(z-z_0\right)$$

$$L(x,y,z) = 3 + \frac{1}{3}(x-1) + \frac{2}{3}(y-2) + \frac{2}{3}(z-2)$$

$$= \frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z$$

Exercise

Find the linearization $L(x,y,z)$ of $f(x,y,z) = \dfrac{\sin xy}{z}$ at the point $\left(\frac{\pi}{2},\ 1,\ 1\right)$

Solution

$$f\left(\frac{\pi}{2},\ 1,\ 1\right) = \frac{\sin\frac{\pi}{2}}{1} = 1$$

$$f_x = \left.\frac{y\cos xy}{z}\right|_{\left(\frac{\pi}{2},1,1\right)}$$

$$= 0$$

$$f_y = \left.\frac{x\cos xy}{z}\right|_{\left(\frac{\pi}{2},1,1\right)}$$

$$= 0$$

$$f_z = -\frac{\sin xy}{z^2}\Big|_{\left(\frac{\pi}{2},1,1\right)}$$

$$\underline{= -1}$$

$$L(x,y,z) = f\left(x_0, y_0, z_0\right) + f_x\left(x - x_0\right) + f_y\left(y - y_0\right) + f_z\left(z - z_0\right)$$

$$L(x,y,z) = 1 + 0\left(x - \frac{\pi}{2}\right) + 0(y-1) - (z-1)$$

$$\underline{= 2 - z}$$

Exercise

Find the linearization $L(x,y,z)$ of $f(x,y,z) = e^x + \cos(y+z)$ at the point $\left(0,\ \frac{\pi}{4},\ \frac{\pi}{4}\right)$

Solution

$$f\left(0,\ \frac{\pi}{4},\ \frac{\pi}{4}\right) = 1$$

$$f_x = e^x\Big|_{\left(0,\ \frac{\pi}{4},\ \frac{\pi}{4}\right)}$$

$$\underline{= 1}$$

$$f_y = -\sin(y+z)\Big|_{\left(0,\ \frac{\pi}{4},\ \frac{\pi}{4}\right)}$$

$$\underline{= -1}$$

$$f_z = -\sin(y+z)\Big|_{\left(0,\ \frac{\pi}{4},\ \frac{\pi}{4}\right)}$$

$$\underline{= -1}$$

$$L(x,y,z) = f\left(x_0, y_0, z_0\right) + f_x\left(x - x_0\right) + f_y\left(y - y_0\right) + f_z\left(z - z_0\right)$$

$$L(x,y,z) = 1 + x - \left(y - \frac{\pi}{4}\right) - \left(z - \frac{\pi}{4}\right)$$

$$\underline{= x - y - z + 1 + \frac{\pi}{2}}$$

Exercise

Find the linear approximation to the function *f* at the point $(a,\ b)$ and estimate the given function value

$f(x,\ y) = 4\cos(2x - y)$; $\quad (a,\ b) = \left(\frac{\pi}{4},\ \frac{\pi}{4}\right)$; estimate $f(0.8,\ 0.8)$

Solution

$$f\left(\frac{\pi}{4},\ \frac{\pi}{4}\right) = 4\cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$
$$= 4\cos\left(\frac{\pi}{4}\right)$$
$$= 2\sqrt{2}$$

$$f_x(x,\ y) = -8\sin(2x - y)$$
$$f_x\left(\frac{\pi}{4},\ \frac{\pi}{4}\right) = -8\sin\left(\frac{\pi}{4}\right)$$
$$= -4\sqrt{2}$$

$$f_y(x,\ y) = 4\sin(2x - y)$$
$$f_x\left(\frac{\pi}{4},\ \frac{\pi}{4}\right) = 4\sin\left(\frac{\pi}{4}\right)$$
$$= 2\sqrt{2}$$

$$L(x, y) = 2\sqrt{2} - 4\sqrt{2}\left(x - \frac{\pi}{4}\right) + 2\sqrt{2}\left(y - \frac{\pi}{4}\right) \qquad L(x,y) = f\left(x_0, y_0\right) + f_x\left(x - x_0\right) + f_y\left(y - y_0\right)$$
$$= 2\sqrt{2} - 4\sqrt{2}x + \pi\sqrt{2} + 2\sqrt{2}y - \frac{\pi}{2}\sqrt{2}$$
$$= -4\sqrt{2}x + 2\sqrt{2}y + 2\sqrt{2} + \frac{\pi}{2}\sqrt{2}$$

$$f(0.8,\ 0.8) = -4\sqrt{2}\,\frac{4}{5} + 2\sqrt{2}\,\frac{4}{5} + 2\sqrt{2} + \frac{\pi}{2}\sqrt{2}$$
$$= -\frac{8}{5}\sqrt{2} + 2\sqrt{2} + \frac{\pi}{2}\sqrt{2}$$
$$= \left(\frac{2}{5} + \frac{\pi}{2}\right)\sqrt{2}$$
$$= (4 + 5\pi)\frac{\sqrt{2}}{10}$$
$$\approx 2.787$$

Exercise

Find the linear approximation to the function *f* at the point $(a,\ b)$ and estimate the given function value

$f(x,\ y) = (x + y)e^{xy}$; $(a,\ b) = (2,\ 0)$; estimate $f(1.95,\ 0.05)$

Solution

$$f(2,\ 0) = 2e^0$$
$$= 2$$

$$f_x(x,\ y) = \left(1 + xy + y^2\right)e^{xy}$$

$\underline{f_x(2,\ 0) = 1}$

$f_y(x,\ y) = \left(1 + x^2 + xy\right)e^{xy}$

$\underline{f_y(2,\ 0) = 5}$

$L(x, y) = 2 + (x-2) + 5(y-0)$ $\qquad L(x, y) = f\left(x_0, y_0\right) + f_x\left(x - x_0\right) + f_y\left(y - y_0\right)$

$= x + 5y$

$f(1.95,\ 0.05) = \frac{39}{20} + 5\left(\frac{1}{20}\right)$

$\underline{= \frac{41}{20}}$

$\underline{\approx 2.205}$

Exercise

Find the linear approximation to the function f at the point $(a,\ b)$ and estimate the given function value

$f(x,\ y) = xy + x - y; \quad (a,\ b) = (2,\ 3)$; estimate $f(2.1,\ 2.99)$

Solution

$f(2,\ 3) = 6 + 2 - 3$

$\underline{= 5}$

$f_x = y + 1\Big|_{(2,\ 3)}$

$\underline{= 4}$

$f_y = x - 1\Big|_{(2,\ 3)}$

$\underline{= 1}$

$L(x, y) = 5 + 4(x-2) + (y-3)$ $\qquad L(x, y) = f\left(x_0, y_0\right) + f_x\left(x - x_0\right) + f_y\left(y - y_0\right)$

$= 4x + y - 6$

$f(2.1,\ 2.99) = 4(2.1) + 2.99 - 6$

$= 4\frac{21}{10} + \frac{299}{100} - 6$

$= \frac{840 + 299 - 600}{100}$

$= \frac{539}{100}$

$\underline{= 5.39}$

Exercise

Find the linear approximation to the function f at the point $(a,\ b)$ and estimate the given function value

$$f(x,\ y) = 12 - 4x^2 - 8y^2; \quad (a,\ b) = (-1,\ 4);\ \text{estimate } f(-1.05,\ 3.95)$$

Solution

$$f(-1,\ 4) = 12 - 4 - 128$$

$$\underline{= -120}$$

$$f_x = -8x\Big|_{(-1,\ 4)}$$

$$\underline{= 8}$$

$$f_y = -16y\Big|_{(-1,\ 4)}$$

$$\underline{= -64}$$

$$L(x, y) = -120 + 8(x+1) - 64(y-4) \qquad L(x,y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$$

$$\underline{= 8x - 64y + 144}$$

$$f(-1.05,\ 3.95) = 8(-1.05) - 64(3.95) + 144$$

$$\underline{= -117.2}$$

Exercise

Find the linear approximation to the function f at the point $(a,\ b)$ and estimate the given function value

$$f(x,\ y) = -x^2 + 2y^2; \quad (a,\ b) = (3,\ -1);\ \text{estimate } f(3.1,\ -1.04)$$

Solution

$$f(3,\ -1) = -9 + 2$$

$$\underline{= -7}$$

$$f_x = -2x\Big|_{(3,\ -1)}$$

$$\underline{= -6}$$

$$f_y = 4y\Big|_{(3,\ -1)}$$

$$\underline{= -4}$$

$$L(x, y) = -7 - 6(x-3) - 4(y+1) \qquad L(x,y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$$

$$\underline{= -6x - 4y + 7}$$

$$f(3.1,\ -1.04) = -6(3.1) - 4(-1.04) + 7$$
$$= -7.44$$

Exercise

Find the linear approximation to the function *f* at the point $(a,\ b)$ and estimate the given function value

$$f(x,\ y) = \sqrt{x^2 + y^2};\quad (a,\ b) = (3,\ -4);\ \text{estimate } f(3.06,\ -3.92)$$

Solution

$$f(3,\ -4) = \sqrt{9+16}$$
$$= 5$$

$$f_x = \left.\frac{x}{\sqrt{x^2+y^2}}\right|_{(3,\ -4)}$$
$$= \frac{3}{5}$$

$$f_y = \left.\frac{y}{\sqrt{x^2+y^2}}\right|_{(3,\ -4)}$$
$$= -\frac{4}{5}$$

$$L(x,y) = 5 + \frac{3}{5}(x-3) - \frac{4}{5}(y+4) \qquad L(x,y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$$
$$= \frac{3}{5}x - \frac{4}{5}y$$

$$f(3.06,\ -3.92) = \frac{3}{5}\left(\frac{306}{100}\right) - \frac{4}{5}\left(-\frac{392}{100}\right)$$
$$= \frac{918+1568}{500}$$
$$= \frac{1,243}{250}$$
$$= 4.972$$

Exercise

Find the linear approximation to the function *f* at the point $(a,\ b)$ and estimate the given function value

$$f(x,\ y) = \ln(1+x+y);\quad (a,\ b) = (0,\ 0);\ \text{estimate } f(0.1,\ -0.2)$$

Solution

$$f(0,\ 0) = 0$$

$$f_x = \left.\frac{1}{1+x+y}\right|_{(0,\ 0)}$$
$$\underline{=1}$$

$$f_y = \left.\frac{1}{1+x+y}\right|_{(0,\ 0)}$$
$$\underline{=1}$$

$L(x,y) = x + y$ $\qquad L(x,y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$

$$f(0.1,\ -0.2) = 0.1 - 0.2$$
$$\underline{= -0.1}$$

Exercise

Find the linear approximation to the function f at the point $(a,\ b)$ and estimate the given function value

$$f(x,\ y) = \frac{x+y}{x-y};\quad (a,\ b) = (3,\ 2);\ \text{estimate}\ \ f(2.95,\ 2.05)$$

Solution

$$\underline{f(3,\ 2) = 5}$$

$$f_x = \left.\frac{-2y}{(x-y)^2}\right|_{(3,\ 2)}$$
$$\underline{= -4}$$

$$f_y = \left.\frac{2x}{(x-y)^2}\right|_{(3,\ 2)}$$
$$\underline{= 6}$$

$L(x,y) = 5 - 4(x-3) + 6(y-2)$ $\qquad L(x,y) = f(x_0, y_0) + f_x(x - x_0) + f_y(y - y_0)$

$$\underline{= -4x + 6y + 5}$$

$$f(2.95,\ 2.05) = -4(2.95) + 6(2.05) + 5$$
$$\underline{= 5.5}$$

Exercise

Estimate the change in the function $f(x,\ y) = -2y^2 + 3x^2 + xy$ when $(x,\ y)$ changes from $(1,\ -2)$ to $(1.05,\ -1.9)$.

Solution

$f_x = 6x + y$

$\boxed{f_x(1, \ -2) = 4}$

$f_y = -4y + x$

$\boxed{f_y(1, \ -2) = 9}$

$$\begin{aligned}\Delta f &\approx f_x(1, \ -2)\Delta x + f_y(1, \ -2)\Delta y \\ &= 4(1.05 - 1) + 9(-1.9 + 2) \\ &= .2 + .9 \\ &= 1.1\end{aligned}$$

Exercise

What is the largest value that the directional derivative of $f(x, y, z) = xyz$ can have at the point $(1, \ 1, \ 1)$?

Solution

$$\begin{aligned}\nabla f &= yz\hat{i} + xz\hat{j} + xy\hat{k}\Big|_{(1, \ 1, \ 1)} \\ &= \hat{i} + \hat{j} + \hat{k}\end{aligned}$$

The ***maximum value***: $\quad |\nabla f| = \sqrt{1+1+1}$

$$= \sqrt{3}$$

Exercise

You plan to calculate the volume inside a stretch of pipeline that is about 36 *in*. in diameter and 1 *mile* long. With which measurement should you be more careful, the length or the diameter? Why?

Solution

$1 \ mile = 5280 \ ft$

$r = \frac{36}{2} in \frac{1 \ ft}{12 \ in} \quad = \frac{3}{2} \ ft$

$V = \pi r^2 h$

$$\begin{aligned}dV &= 2\pi r h dr + \pi r^2 dh \\ &= 2\pi\left(\frac{3}{2}\right)(5280)\,dr + \pi\left(\frac{3}{2}\right)^2 dh \\ &= 15,840\pi dr + \frac{9\pi}{4}\pi dh\end{aligned}$$

We have to be more careful with the diameter, since it has a greater effect on dV.

Exercise

The volume of a cylinder with radius r and height h is $V = \pi r^2 h$. Find the approximate percentage change in the volume when the radius decreases by 3% and the height increases by 2%.

Solution

$$\Delta V = 2\pi rh\Delta r + \pi r^2 \Delta h$$

$$\begin{aligned}\frac{dV}{V} &= \frac{2\pi rh}{\pi r^2 h}dr + \frac{\pi r^2}{\pi r^2 h}dh \\ &= 2\frac{dr}{r} + \frac{dh}{h} \\ &= 2(-3\%) + 2\% \\ &= 4\%\end{aligned}$$

Approximate change volume.

Exercise

The volume of an ellipsoid with axes of length $2a$, $2b$, and $2c$ is $V = \pi abc$. Find the percentage change in the volume when a increases by 2%, b increases by 1.5%, and c decreases by 2.5%.

Solution

$$dV = \pi(bc\Delta a + ac\Delta b + ab\Delta c)$$

$$\begin{aligned}\frac{dV}{V} &= \frac{da}{a} + \frac{db}{b} + \frac{dc}{c} \\ &= 2\% + 1.5\% - 2.5\% \\ &= 1\%\end{aligned}$$

Approximate change volume.

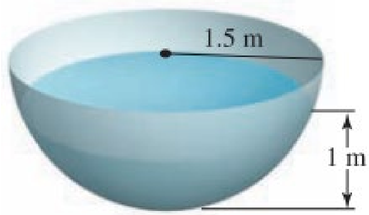
Exercise

A hemispherical tank with a radius of 1.50 m is filled with water to a depth of 1.00 m. Water level drops by 0.05 m (from 1.00 m to 0.95 m)

a) Approximate the change in the volume of water in the tank. The volume of a spherical cap is $V = \frac{1}{3}\pi h^2(3r - h)$, where r is the radius of the sphere and h is the thickness of the cap (in this case, the depth of the water).

b) Approximate the change in the surface area of the water in the tank.

Solution



a) $V = \frac{1}{3}\pi h^2(3r - h)$

$$= \frac{1}{3}\pi\left(3rh^2 - h^3\right)$$

$$dV = \frac{1}{3}\pi\left(6rh - 3h^2\right)dh$$

$$= \pi\left(2rh - h^2\right)dh$$

$$= \pi\left(2(1.5)(1) - 1^2\right)(-.05)$$

$$\boxed{= -0.1\pi \ \ m^3}$$

b) $S = \pi\left(2rh - h^2\right)$

$$dS = \pi(2r - 2h)dh$$

$$= 2\pi(1.5 - 1)(-.05)$$

$$\boxed{= -0.05\pi \ \ m^2}$$

Exercise

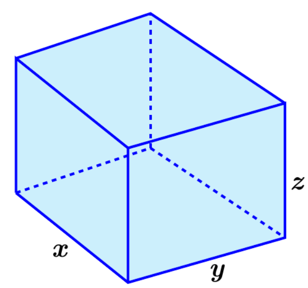
Find the linearization $L(x, y, z)$ of $f(x, y, z) = e^x + \cos(y + z)$ at the point $\left(0, \ \frac{\pi}{4}, \ \frac{\pi}{4}\right)$

Consider a closed rectangular box with a square base. If x is measured with error at most 2% and y is measured with error at most 3% use a differential to estimate the corresponding percentage error in computing the box's

a) Surface area

b) Volume

Solution



Given: $\frac{dx}{x} \le 0.02, \quad \frac{dy}{y} \le 0.03$

a) $S = 2(xx + xy + xy) = 2x^2 + 4xy$

$$dS = (4x + 4y)dx + 4xdy$$

$$= (4x + 4y)\left(x\frac{dx}{x}\right) + 4xy\frac{dy}{y}$$

$$= \left(4x^2 + 4xy\right)\frac{dx}{x} + 4xy\frac{dy}{y}$$

$$\le \left(4x^2 + 4xy\right)(0.02) + 4xy(0.03)$$

$$= 0.02\left(4x^2\right) + 0.02(4xy) + 0.03(4xy)$$

$$= 0.04\left(2x^2\right) + 0.05(4xy)$$

$$\le 0.05\left(2x^2\right) + 0.05(4xy)$$

$$= 0.05\left(2x^2 + 4xy\right)$$

$$\boxed{= 0.05\,S}$$

b) $V = x^2 y$

$$dV = 2xydx + x^2 dy$$

$$= 2x^2 y \frac{dx}{x} + x^2 y \frac{dy}{y}$$

$$\leq 2x^2 y(0.02) + x^2 y(.03)$$

$$= .07\left(x^2 y\right)$$

$$\underline{= .07\ V}$$

Exercise

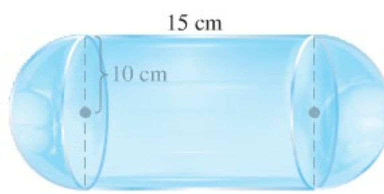
Consider a closed container in the shape of a cylinder of radius 10 *cm* and height 15 *cm* with a hemisphere on each end.

The container is coated with a layer of ice $\frac{1}{2}$ *cm* thick. Use a differential to estimate the total volume of ice.

(*Hint*: assume *r* is radius with $dr = \frac{1}{2}$ and *h* is height with $dh = 0$)

Solution

$$V = \frac{4\pi}{3} r^3 + \pi r^2 h$$



$$dV = 4\pi r^2 dr + 2\pi rhdr + \pi r^2 dh$$

$$= \left(4\pi r^2 + 2\pi rh\right)dr + \pi r^2 dh$$

$$= \left(4\pi(10)^2 + 2\pi(10)(15)\right)\left(\tfrac{1}{2}\right) + \pi(10)^2(0)$$

$$\underline{= 350\pi\ cm^3}$$

Exercise

A standard 12-*fl-oz* can of soda is essentially a cylinder of radius $r = 1\ in$ and height $h = 5\ in$.

a) At these dimensions, how sensitive is the can's volume to a small change in radius versus a small change in height?

b) Could you design a soda can that appears to hold more soda but in fact holds the same 12-fl-oz? What might its dimensions be? (There is more than one correct answer.)

Solution

Given: $r = 1\ in \quad h = 5\ in.$

a) $V = \pi r^2 h \quad \Rightarrow \quad dV = 2\pi rhdr + \pi r^2 dh$

$$dV = 10\pi dr + \pi dh$$

$= \pi(10dr + dh)$

The volume is about 10 times more sensitive to a change in *r*.

b) $dV = 0 \quad \Rightarrow \quad 2\pi rhdr + \pi r^2 dh = 0$

$2hdr + rdh = 0$

$10dr + dh = 0 \quad \Rightarrow \quad dr = -\frac{1}{10}dh$

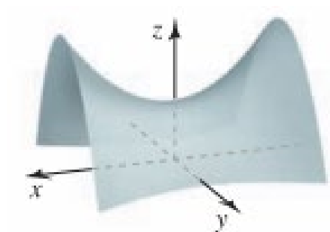
Assume $dh = 1.5$, then $dr = -.15$

$2h(-0.15) + r(1.5) = 0$

$r = 0.85\ in \quad h = 6.5\ in.$ is one solution for $\Delta V \approx dV = 0$

Exercise

Consider the function $f(x, y) = 2x^2 - 4y^2 + 10$, whose graph is shown



a) Fill in the table showing the value of the directional derivative at points (a, b) in the direction given by the unit vectors ***u***, ***v***, *and* ***w***

b) Interpret each of the directional derivatives computed in part(a) at the point $(2, 0)$

Solution

a) $f_x = 4x \quad f_y = -8y$

$\nabla f \bullet \vec{u} = (4a\hat{i} - 8b\hat{j}) \bullet (u_x \hat{i} + u_y \hat{j})$

$= 4au_x - 8bu_y$

	$(0, 0)$	$(a, b) = (2, 0)$	$(a, b) = (1, 1)$
$\vec{u} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	0	$4(2)\frac{\sqrt{2}}{2} - 0 = 4\sqrt{2}$	$4(1)\left(\frac{\sqrt{2}}{2}\right) - 8(1)\left(\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$
$\vec{v} = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	0	$4(2)\left(-\frac{\sqrt{2}}{2}\right) - 0 = -4\sqrt{2}$	$4(1)\left(-\frac{\sqrt{2}}{2}\right) - 8(1)\left(\frac{\sqrt{2}}{2}\right) = -6\sqrt{2}$
$\vec{w} = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	0	$4(2)\left(-\frac{\sqrt{2}}{2}\right) - 0 = -4\sqrt{2}$	$4(1)\left(-\frac{\sqrt{2}}{2}\right) - 8(1)\left(-\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$

b) The function is ***increasing*** @ $(2, 0)$ in the direction of $\vec{u}$

The function is ***decreasing*** @ $(2, 0)$ in the direction of $\vec{v} + \vec{w}$

Exercise

Two spheres have the same center and radii r and R, where $0 < r < R$. The volume of the region between the sphere is $V(r,\ R) = \frac{4\pi}{3}\left(R^3 - r^3\right)$.

a) First use your intuition. If r is held fixed, how does V change as R increases? What is the sign of V_R? If R is held fixed, how does V change as r increases (up to the value of R)? What is the sign of V_r?

b) Compute V_r and V_R. Are the results consistent with part (a)?

c) Consider spheres with $R = 3$ and $r = 1$. Does the volume change more if R is increased by $\Delta R = 0.1$ (with r fixed) or if r is decreased by $\Delta r = 0.1$ (with R fixed)?

Solution

a) r is fixed, then $V_R = 4\pi R^2 > 0$

$\therefore$ If R increases then V increases.

R is fixed, then $V_r = -4\pi r^2 < 0$

$\therefore$ If r increases then V decreases.

b) Yes, $V_r = -4\pi r^2 < 0 \quad V_R = 4\pi R^2 > 0$

c) If $R = 3$, $r = 1$, $\Delta r = 0.1$, and $\Delta R = 0.1$

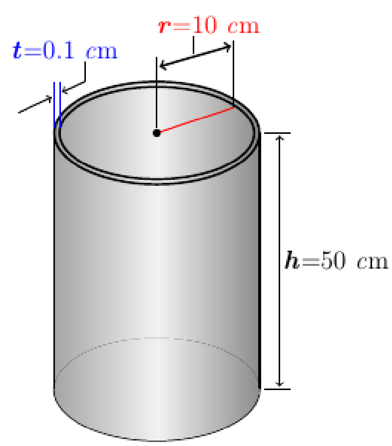
$\Delta R = 0.1 \quad \Rightarrow \quad \Delta V = 4\pi(3)^2(0.1) \ \underline{= 3.6\pi}$

If r is decreased by 0.1

$\Delta V = -4\pi(1)^2(-.1) \ \underline{= 0.4\pi}$

$\therefore$ Volume changes more if R is increased.

Exercise



A company manufactures cylindrical aluminum tubes to rigid specifications. The tubes are designed to have an outside radius of $r = 10\ cm$, a height of $h = 50\ cm$, and a thickness of $t = 0.1\ cm$. The manufacturing process produces tubes with a maximum error of $\pm 0.05\ cm$ in the radius and height and a maximum error of $\pm 0.0005\ cm$ in the thickness. The volume of the material used to construct a cylindrical tube is $V(r,h,t) = \pi ht(2r - t)$. Estimate maximum error in the volume of the tube.

Solution

$V(r,h,t) = 2\pi rht - \pi ht^2$

$$dV = 2\pi htdr + \left(2\pi rt - \pi t^2\right)dh + 2\pi h(r-t)dt$$

$$= 2\pi(50)(0.1)(.05) + \left(2\pi(10)(0.1) - \pi(0.1)^2\right)(.05) + 2\pi(50)(10-0.1)(.0005)$$

$$= \pi\left(0.5 + 1.99(.05) + 990(.0005)\right)$$

$$\approx 3.4385$$

The maximum error in the volume is approximately $3.4385\ cm^3$.
The volume is far more sensitive to errors in the thickness, since for the thickness 990π is more than for the radius (10π) and height (1.99π)

Exercise

The volume of a right circular cone with radius r and height h is $V = \frac{1}{3}\pi hr^2$

a) Approximate the change in the volume of the cone when the radius changes from $r = 6.5$ to $r = 6.6$ and the height changes from $h = 4.20$ to $h = 4.15$

b) Approximate the change in the volume of the cone when the radius changes from $r = 5.4$ to $r = 5.37$ and the height changes from $h = 12.0$ to $h = 11.96$

Solution

$$V = \frac{1}{3}\pi hr^2$$

$$dV = \frac{1}{3}\pi\left(2rhdr + r^2dh\right)$$

a) $dV = \frac{\pi}{3}\left(2(6.5)(4.2)(6.6-6.5) + (6.5)^2(4.15-4.2)\right)$

$$= \frac{\pi}{3}\left(54.6(0.5) + 42.25(-.05)\right)$$

$$\approx 3.505$$

b) $dV = \frac{\pi}{3}\left(2(5.4)(12)(-.03) + (5.4)^2(-.04)\right)$

$$\approx -5.293$$

Exercise

The area of an ellipse with axes of length $2a$ and $2b$ is $A = \pi ab$. Approximate the percent change in the area when a increases by 2% and b increases by 1.5%.

Solution

$$dA = \pi(b\,da + a\,db)$$

$$\frac{dA}{A} = \frac{\pi}{\pi ab}(b\,da + a\,db)$$

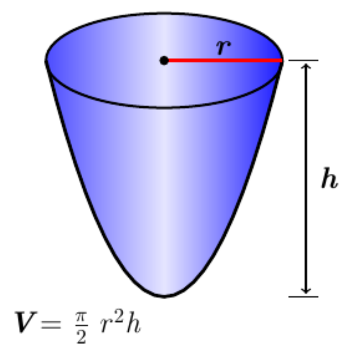
$$\frac{dA}{A} = \frac{da}{a} + \frac{db}{b}$$
$$= 2\% + 1.5\%$$
$$\underline{= 3.5\%}$$

Exercise

The Volume of a segment of a circular paraboloid with radius r and height h is $V = \frac{1}{2}\pi hr^2$.

Approximate the percent change in the volume when the radius decreases by 1.5% and the height increases by 2.2%

Solution



$$dV = \frac{\pi}{2}\left(2rhdr + r^2 dh\right)$$

$$\frac{1}{V}dV = \frac{1}{\frac{1}{2}\pi hr^2}\frac{\pi}{2}\left(2rhdr + r^2 dh\right)$$

$$\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h}$$
$$= 2(-1.5\%) + 2.2\%$$
$$\underline{= -0.8\%}$$

Exercise

Batting averages in baseball are defined by $A = \frac{x}{y}$, where $x \ge 0$ is the total number of hits and $y > 0$ is the total number of at-bats. Treat x and y as positive real numbers and note that $0 \le A \le 1$.

a) Estimate the change in the batting average if the number of hits increases from 60 to 62 and the number of at-bats increases from 175 to 180.

b) If a batter currently has a batting average of $A = 0.35$, does the average decrease if the batter fails to get a hit more than it increases if the batter gets a hit?

c) Does the answer in part (*b*) depend on the current batting average? Explain.

Solution

a) $dA = \frac{1}{y}dx - \frac{x}{y^2}dy$

$$= \frac{1}{175}(62-60) - \frac{60}{175^2}(180-175)$$
$$= \frac{2}{175} - \frac{300}{175^2}$$
$$= \frac{50}{30,625}$$

$$= \boxed{\frac{2}{1,225}}$$

$$\boxed{\approx 0.001633}$$

b) If the batter fails to get a hit, the average decreases by

$$\frac{x}{y} - \frac{x}{y+1} = \frac{x}{y(y+1)}$$

$$= \boxed{\frac{A}{y+1}}$$

If the batter gets a hit, the average increases by

$$\frac{x+1}{y+1} - \frac{x}{y} = \frac{y-x}{y(y+1)}$$

$$= \frac{1-\frac{x}{y}}{y+1}$$

$$= \boxed{\frac{1-A}{y+1}}$$

If $A = 0.35$, the second of these quantities is larger, therefore the answer is no; the batting average changes more if the batter gets a hit than if he fails to get a hit.

c) The answer depends on whether A is less than or greater than 0.50.

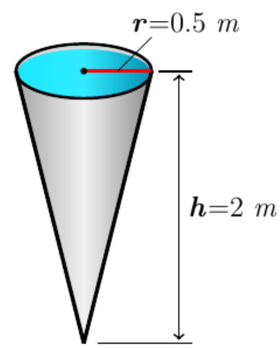
Exercise

A conical tank with radius 0.50 *m* and height 2.0 *m* is filled with water. Water released from the tank, and the water level drops by 0.05 *m* (from 2.0 *m* to 1.95 *m*).

Approximate the change in volume of water in the tank.

(***Hint***: When the water level drops, both the radius and height of the cone of water change).

Solution



$$\frac{x}{r} = \frac{y}{h}$$

$$\frac{x}{.5} = \frac{1.95}{2}$$

$$x = .4875$$

$$dr = 0.4875 - 0.5$$

$$\boxed{= -0.0125}$$

$$V = \frac{1}{3}\pi h r^2$$

$$dV = \frac{1}{3}\pi\left(2rhdr + r^2 dh\right)$$

$$\frac{1}{V}dV = \frac{1}{\frac{1}{3}\pi h r^2}\frac{\pi}{3}\left(2rhdr + r^2 dh\right)$$

$$\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h}$$

$$= -2\frac{0.0125}{0.5} - \frac{.05}{2}$$

$$= -0.05 - \frac{.05}{2}$$

$$= -0.075$$

$$V = \frac{1}{3}\pi(2)(0.5)^2$$

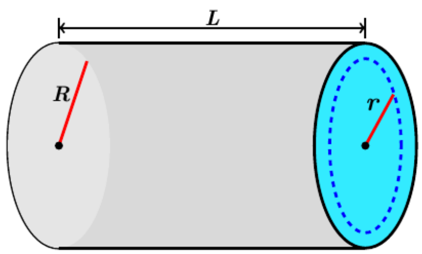
$$\approx 0.5236$$

$$dV = (-.075)(.5236)$$

$$\approx 0.03927 \ m^3$$

Exercise

Poiseuille's law is a fundamental law of fluid dynamics that describes the flow velocity of a viscous incompressible fluid in a cylinder (it is used to model blood flow through veins and arteries). It says that in a cylinder of radius *R* and length *L*, the velocity of the fluid $r \le R$ units from the centerline of the cylinder is $V = \frac{P}{4L\upsilon}\left(R^2 - r^2\right)$, where *P* is the difference in the pressure between the ends of the cylinder and υ is the viscosity of the fluid. Assuming that *P* and υ are constant, the velocity *V* along the centerline of the cylinder $(r = 0)$ is $V = \frac{kR^2}{L}$, where *k* is a constant that we will take to be $k = 1$.



a) Estimate the change in the centerline velocity $(r = 0)$ if the radius of the flow cylinder increases from $R = 3\ cm$ to $R = 3.05\ cm$ and the length increases from $L = 50\ cm$ to $L = 50.5\ cm$.

b) Estimate the percent change in the centerline velocity if the radius of the flow cylinder *R* decreases by 1% and the length increases by 2%.

Solution

$$k = 1 \quad \to \quad V = \frac{R^2}{L}$$

a) $dV = \frac{2R}{L}dR - \frac{R^2}{L^2}dL$

$$= \frac{2(3)}{50}(3.05 - 3) - \frac{3^2}{50^2}(50.5 - 50)$$

$$= \frac{3}{25}(0.05) - \frac{9}{2500}(0.5)$$

$$= \frac{3}{500} - \frac{9}{5000}$$

$$= \frac{21}{5,000}$$

$$= 0.0042\ cm^3$$

b) $\frac{1}{V} dV = \frac{L}{R^2}\left(\frac{2R}{L} dR - \frac{R^2}{L^2} dL\right)$

$$\frac{dV}{V} = 2\frac{dR}{R} - \frac{dL}{L}$$

R decreases by 1% and the length increases by 2%.

$$= 2(-1\%) - (2\%)$$

$$= -4\%$$

V will decrease by approximately 4%.

Exercise

Suppose that in a large group of people a fraction $0 \le r \le 1$ of the people have flu. The probability that in *n* random encounters, you will meet at least one person with flu is $P = f(n, r) = 1 - (1-r)^n$. Although *n* is a positive integer, regard it as a positive real number.

a) Compute f_r and f_n.

b) How sensitive is the probability *P* to the flu rate *r*? Suppose you meet $n = 20$ people. Approximately how much does the probability *P* increase if the flu rate increases from $r = 0.1$ to $r = 0.11$ (with *n* fixed)?

c) Approximately how much does the probability *P* increase the flu rate increases from $r = 0.9$ to $r = 0.91$

d) Interpret the results of parts (*b*) and (*c*).

Solution

a) $f_r = n(1-r)^{n-1}$

$$f = 1 - (1-r)^n$$

$$f_n = -\frac{\partial}{\partial n}(1-r)^n$$

$$\ln y = \ln (1-r)^n$$

$$\ln y = n \ln (1-r)$$

$$\frac{y_n}{y} = \ln (1-r)$$

$$y_n = (1-r)^n \ln (1-r)$$

$$\boxed{f_n = -(1-r)^n \ln(1-r)}$$

b) $n = 20 \quad r = 0.1$

$$\Delta P \approx f_r(20,\ 0.1)(0.11 - 0.1) \qquad\qquad f_r = n(1-r)^{n-1}$$

$$= 20(1-0.1)^{19}(0.01)$$

$$\boxed{\approx 0.027}$$

c) $n = 20 \quad r = 0.9$

$$\Delta P \approx f_r(20,\ 0.9)(.01) \qquad\qquad f_r = n(1-r)^{n-1}$$

$$= 20(1-0.9)^{19}(0.01)$$

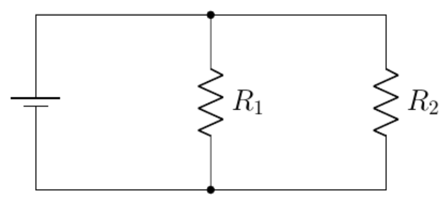
$$\boxed{\approx 2 \times 10^{-20}}$$

d) Small changes in the flu rate have a greater effect on the probability of catching the flu when the flu rate is small compared to when the flu rate is large.

Exercise

When two electrical resistors with resistance $R_1 > 0$ and $R_2 > 0$ are wired in parallel in a circuit, the combined resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$



a) Estimate the change in R if R_1 increases from 2 Ω to 2.05 Ω and R_2 decreases from 3 Ω to 2.95 Ω.

b) Is it true that if $R_1 = R_2$ and R_1 increases by the same small amount as R_2 decreases, then R is approximately unchanged? Explain.

c) Is it true that if R_1 and R_2 increase, then R increases? Explain.

d) Suppose $R_1 > R_2$ and R_1 increases by the same small amount as R_2 decreases. Does R increase or decrease?

Solution

a) $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}.$

$$\frac{1}{R} = \frac{R_2 + R_1}{R_1 R_2}$$

$$R = \frac{R_1 R_2}{R_2 + R_1}$$

$$dR = \frac{R_2^2}{\left(R_2 + R_1\right)^2} dR_1 + \frac{R_1^2}{\left(R_2 + R_1\right)^2} dR_2$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

$$= \frac{R_2^2}{\left(R_2 + R_1\right)^2} \frac{R_1^2}{R_1^2} dR_1 + \frac{R_1^2}{\left(R_2 + R_1\right)^2} \frac{R_2^2}{R_2^2} dR_2$$

$$= \left(\frac{R_1 R_2}{R_2 + R_1}\right)^2 \frac{dR_1}{R_1^2} + \left(\frac{R_1 R_2}{R_2 + R_1}\right)^2 \frac{dR_2}{R_2^2}$$

$$= R^2 \left(\frac{dR_1}{R_1^2} + \frac{dR_2}{R_2^2}\right)$$

$$= \left(\frac{6}{5}\right)^2 \left(\frac{2.05-2}{4} + \frac{2.95-3}{9}\right)$$

$$= \frac{36}{25}\left(\frac{.05}{4} - \frac{.05}{9}\right)$$

$$= \frac{36}{500}\left(\frac{5}{36}\right)$$

$$= \frac{1}{100}$$

$$\underline{= 0.01\ \ \Omega}$$

b) If $R_1 = R_2$

R_1 increases by the same small amount as R_2 decreases.

$dR_1 = -dR_2$

$$dR = R^2 \left(\frac{dR_1}{R_1^2} + \frac{dR_2}{R_2^2}\right)$$

$$= R^2 \left(-\frac{dR_2}{R_2^2} + \frac{dR_2}{R_2^2}\right)$$

$$\underline{= 0}$$

c) If R_1 and R_2 increase

$$dR = R^2 \left(\frac{dR_1}{R_1^2} + \frac{dR_2}{R_2^2}\right) > 0$$

Therefore, R increases

d) *Given*: $R_1 > R_2$

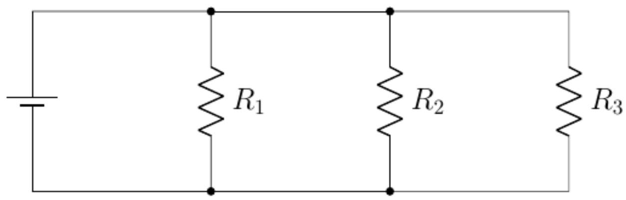
$$R = \frac{R_1 R_2}{R_1 + R_2}$$

R is more sensitive to changes in R_2, so if R_1 increases by the same small amount as R_2 decreases, then R will decrease.

Exercise

When three electrical resistors with resistance $R_1 > 0$, $R_2 > 0$ and $R_3 > 0$ are wired in parallel in a circuit, the combined resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$



Estimate the change in R if R_1 increases from 2 Ω to 2.05 Ω, R_2 decreases from 3 Ω to 2.95 Ω, and R_3 increases from 1.5 Ω to 1.55 Ω

Solution

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

$$\frac{1}{R} = \frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_1 R_2 R_3}$$

$$R = \frac{R_1 R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

$$-\frac{1}{R^2} dR = -\frac{1}{R_1^2} dR_1 - \frac{1}{R_2^2} dR_2 - \frac{1}{R_3^2} dR_3$$

$$dR = R^2 \left(\frac{1}{R_1^2} dR_1 + \frac{1}{R_2^2} dR_2 + \frac{1}{R_3^2} dR_3 \right)$$

$$= \left(\frac{2(3)(1.5)}{6+3+4.5} \right)^2 \left(\tfrac{1}{4}(2.05-2) + \tfrac{1}{9}(2.95-3) + \tfrac{100}{225}(1.55-1.5) \right)$$

$$= \left(\frac{9}{13.5}\right)^2 \left(\frac{1}{4}(.05) - \frac{1}{9}(.05) + \frac{100}{225}(.05)\right)$$

$$= \left(\frac{90}{135}\right)^2 \left(\frac{5}{400} - \frac{5}{900} + \frac{1}{45}\right)$$

$$= \left(\frac{2}{3}\right)^2 \left(\frac{1}{80} - \frac{1}{180} + \frac{1}{45}\right)$$

$$= \frac{4}{9}\left(\frac{9 - 4 + 16}{720}\right)$$

$$= \frac{1}{9}\left(\frac{21}{180}\right)$$

$$= \frac{7}{540}\ \Omega$$

$$\approx 0.013\ \Omega$$

Exercise

Consider the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and the plane P given by $Ax + By + Cz + 1 = 0$. Let

$h = \frac{1}{\sqrt{A^2 + B^2 + C^2}}$ and $m = \sqrt{a^2A^2 + b^2B^2 + c^2C^2}$

a) Find the equation of the plane tangent to the ellipsoid at the point $(p,\ q,\ r)$.

b) Find the two points on the ellipsoid at which the tangent plane parallel to P and find equations of the tangent planes.

c) Show that the distance between the origin and the plane P is h.

d) Show that the distance between the origin and the tangent planes is hm.

e) Find a condition that guarantees the plane P does not intersect the ellipsoid.

Solution

a) $f(x,\ y,\ z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

$$\nabla f = \left\langle \frac{2x}{a^2},\ \frac{2y}{b^2},\ \frac{2z}{c^2} \right\rangle \Bigg|_{(p,\ q,\ r)}$$

$$= \left\langle \frac{2p}{a^2},\ \frac{2q}{b^2},\ \frac{2r}{c^2} \right\rangle$$

Equation of the ***plane tangent*** to the ellipsoid at the point $(p,\ q,\ r)$ is:

$$\frac{2p}{a^2}(x - p) + \frac{2q}{b^2}(y - q) + \frac{2r}{c^2}(z - r) = 0$$

$$\frac{p}{a^2}x + \frac{q}{b^2}y + \frac{r}{c^2}z = \frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2}$$

$$\left.\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\right|_{(p,\ q,\ r)}$$

$$\frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}=1$$

$$\therefore\ \boxed{\frac{p}{a^2}x+\frac{q}{b^2}y+\frac{r}{c^2}z=1}$$

b) ***Given***: $Ax+By+Cz+1=0$ → The vector will be $\langle A,\ B,\ C\rangle$ and $\left\langle \frac{p}{a^2},\ \frac{q}{b^2},\ \frac{r}{c^2}\right\rangle$ must be proportional.

$$\left\langle \frac{p}{a^2},\ \frac{q}{b^2},\ \frac{r}{c^2}\right\rangle=\lambda\langle A,\ B,\ C\rangle$$

$$\begin{cases}\frac{p}{a^2}=\lambda A & \to & p=\lambda Aa^2\\ \frac{q}{b^2}=\lambda B & \to & q=\lambda Bb^2\\ \frac{r}{c^2}=\lambda C & \to & r=\lambda Cc^2\end{cases}$$

$$\langle p,\ q,\ r\rangle=\lambda\left\langle Aa^2,\ Bb^2,\ Cc^2\right\rangle$$

$$\frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}=1$$

$$\frac{\lambda^2A^2a^4}{a^2}+\frac{\lambda^2B^2b^4}{b^2}+\frac{\lambda^2C^2c^4}{c^2}=1$$

$$\lambda^2\left(A^2a^2+B^2b^2+C^2c^2\right)=1 \qquad\qquad m=\sqrt{a^2A^2+b^2B^2+c^2C^2}$$

$$\lambda^2m^2=1$$

$$\lambda=\pm\frac{1}{m}$$

$$\langle p,\ q,\ r\rangle=\pm\frac{1}{m}\left\langle Aa^2,\ Bb^2,\ Cc^2\right\rangle$$

Equations of the tangent planes: $\boxed{(p,\ q,\ r)=\pm\left(Aa^2,\ Bb^2,\ Cc^2\right)}$

c) The distance between the plane $Ax+By+Cz+1=0$ to the origin:

Let $S=(x,\ y,\ z)$ be the point on the plane, then

$$\overrightarrow{OS}=\langle x,\ y,\ z\rangle$$

$$\vec{n}=\langle A,\ B,\ C\rangle$$

$$|\vec{n}|=\sqrt{A^2+B^2+C^2}$$

$$d = \left| \frac{\langle x, y, z \rangle \cdot \langle A, B, C \rangle}{\sqrt{A^2+B^2+C^2}} \right|$$

Distance from a Point to a Plane: $d = \left| \frac{\overrightarrow{OS} \cdot \vec{n}}{|\vec{n}|} \right|$

$$= \left| \frac{Ax+By+Cz}{\sqrt{A^2+B^2+C^2}} \right|$$

$Ax + By + Cz + 1 = 0$

$$= \left| \frac{-1}{\sqrt{A^2+B^2+C^2}} \right|$$

$$= \frac{1}{\sqrt{A^2+B^2+C^2}}$$

$$= h \quad \checkmark$$

d) The tangent plane at $Q(p, q, r) = \pm\left(Aa^2, Bb^2, Cc^2\right)$ has an equation $Ax + By + Cz = \pm m$

$$\overline{OQ} = \left\langle Aa^2, Bb^2, Cc^2 \right\rangle$$

$$\vec{n} = \langle A, B, C \rangle$$

$$|\vec{n}| = \sqrt{A^2 + B^2 + C^2}$$

$$d = \left| \frac{\left\langle Aa^2, Bb^2, Cc^2 \right\rangle \cdot \langle A, B, C \rangle}{\sqrt{A^2+B^2+C^2}} \right|$$

$d = \left| \frac{\overrightarrow{OQ} \cdot \vec{n}}{|\vec{n}|} \right|$

$$= \frac{a^2A^2+b^2B^2+c^2C^2}{\sqrt{A^2+B^2+C^2}}$$

$m = \sqrt{a^2A^2 + b^2B^2 + c^2C^2}$ $\quad h = \frac{1}{\sqrt{A^2 + B^2 + C^2}}$

$$\underline{= hm} \quad \checkmark$$

e) For the plane *P* does not intersect the ellipsoid if and only if the 2 tangent planes parallel to *P* are closer to the origin than *P*; this is equivalent to the condition $m < 1$.

Solution Section 2.7 – Maximum/Minimum Problems

Exercise

Find all the local maxima, local minima, and saddle points of the function

$f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$

Solution

$$f_x = 2x + y + 3 = 0 \qquad f_y = x + 2y - 3 = 0$$

$$\begin{cases} 2x + y = -3 \\ x + 2y = 3 \end{cases} \rightarrow \quad x = -3 \quad y = 3$$

Therefore, the critical point is $(-3,\ 3)$

$$f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = 1$$

$$f_{xx} f_{yy} - f_{xy}^2 = (2)(2) - 1^2 = 3 > 0 \quad \textit{and} \quad f_{xx} = 2 > 0$$

The function *f* has a ***local minimum*** at $(-3,\ 3)$ and the value is

$$f(-3,\ 3) = (-3)^2 + (-3)(3) + 3^2 + 3(-3) - 3(3) + 4$$

$$= -5$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x,y) = x^2 - xy + y^2 + 2x + 2y - 4$$

Solution

$$f_x = 2x - y + 2 = 0 \qquad f_y = -x + 2y + 2 = 0$$

$$\begin{cases} 2x - y = -2 \\ x - 2y = 2 \end{cases} \qquad \Delta = \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} = -3 \quad \Delta_x = \begin{vmatrix} -2 & -1 \\ 2 & -2 \end{vmatrix} = 6$$

$$x = \frac{6}{-3} = -2 \quad y = -4 + 2 = -2$$

Therefore, the critical point is $(-2,\ -2)$

$$f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = -1$$

At $(-2,\ -2)$: $f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = -1$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - 1 = 3 > 0 \quad \textit{and} \quad f_{xx} = 2 > 0$$

The function *f* has a ***local minimum*** at $(-2,\ -2)$ and the value is

$$f(-2,\ -2) = 4 - 4 + 4 - 4 - 4 - 4$$

$$\underline{= -8}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x,y) = x^3 + y^3 - 3xy + 15$$

Solution

$$f_x = 3x^2 - 3y = 0 \qquad f_y = 3y^2 - 3x = 0$$

$$\begin{cases} x^2 = y \\ y^2 = x \end{cases} \quad \left(x^2\right)^2 = x \quad \rightarrow \quad x^4 = x$$

$$x\left(x^3 - 1\right) = 0 \quad \rightarrow \quad \underline{x = 0,\ \ 1}$$

$$\begin{cases} x = 0 & y = 0 \\ x = 1 & y = 1 \end{cases}$$

Therefore, the critical point is $(0,\ 0)$ & $(1,\ 1)$

$$f_{xx} = 6x \quad f_{yy} = 6y \quad f_{xy} = -3$$

At $(0,\ 0)$

$$f_{xx} = 0 \quad f_{yy} = 0 \quad f_{xy} = -3$$

$$f_{xx}f_{yy} - f_{xy}^2 = 0 - 9 = -9 < 0$$

The function *f* has a ***saddle point*** at $(0,\ 0)$ and the value is $\underline{f(0,\ 0) = 15}$

At $(1,\ 1)$

$$f_{xx} = 6 \quad f_{yy} = 6 \quad f_{xy} = -3$$

$$f_{xx}f_{yy} - f_{xy}^2 = 36 - 9 = 27 > 0 \quad \textit{and} \quad f_{xx} = 6 > 0$$

The function *f* has a ***local minimum*** at $(1,\ 1)$ and the value is $\underline{f(1,\ 1) = 14}$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x,y)=x^4-8x^2+3y^2-6y$$

Solution

$$f_x=4x^3-16x=0 \qquad f_y=6y-6=0$$

$$\begin{cases} 4x\left(x^2-4\right)=0 & \rightarrow \ \underline{x=0,\ \pm 2} \\ \quad y=1 \end{cases}$$

Therefore, the critical point is $(0,\ 1)$ & $(\pm 2,\ 1)$

$$f_{xx}=12x^2-16 \quad f_{yy}=6 \quad f_{xy}=0$$

At $(0,\ 1)$

$$f_{xx}=-16 \quad f_{yy}=6 \quad f_{xy}=0$$

$$f_{xx}f_{yy}-f_{xy}^2=-96<0$$

The function f has a ***saddle point*** at $(0,\ 1)$ and the value is $\underline{f(0,\ 1)=-3}$

At $(2,\ 1)$

$$f_{xx}=32 \quad f_{yy}=6 \quad f_{xy}=0$$

$$f_{xx}f_{yy}-f_{xy}^2=192>0 \quad \textit{and} \quad f_{xx}=32>0$$

The function f has a ***local minimum*** at $(2,\ 1)$ and the value is $\underline{f(2,\ 1)=-19}$

At $(-2,\ 1)$

$$f_{xx}=32 \quad f_{yy}=6 \quad f_{xy}=0$$

$$f_{xx}f_{yy}-f_{xy}^2=192>0 \quad \textit{and} \quad f_{xx}=32>0$$

The function f has a ***local minimum*** at $(-2,\ 1)$ and the value is $\underline{f(-2,\ 1)=-19}$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$$f(x,y)=2xy-5x^2-2y^2+4x+4y-4$$

Solution

$$f_x=2y-10x+4=0 \qquad f_y=2x-4y+4=0$$

$$\begin{cases} -5x + y = -2 \\ x - 2y = -2 \end{cases} \rightarrow \quad x = \frac{2}{3} \quad y = \frac{4}{3}$$

Therefore, the critical point is $\left(\frac{2}{3}, \frac{4}{3}\right)$

$$f_{xx}\Big|_{\left(\frac{2}{3}, \frac{4}{3}\right)} = -10 \quad f_{yy}\Big|_{\left(\frac{2}{3}, \frac{4}{3}\right)} = -4 \quad f_{xy}\Big|_{\left(\frac{2}{3}, \frac{4}{3}\right)} = 2$$

$$f_{xx}f_{yy} - f_{xy}^2 = (-10)(-4) - 2^2 = 36 > 0 \quad \text{and} \quad f_{xx} = -10 < 0$$

The function f has a local maximum at $\left(\frac{2}{3}, \frac{4}{3}\right)$ and the value is

$$f\left(\frac{2}{3}, \frac{4}{3}\right) = 2\left(\frac{2}{3}\right)\left(\frac{4}{3}\right) - 5\left(\frac{2}{3}\right)^2 - 2\left(\frac{4}{3}\right)^2 + 4\left(\frac{2}{3}\right) + 4\left(\frac{4}{3}\right) - 4$$

$$= 0$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$f(x, y) = x^2 - 4xy + y^2 + 6y + 2$

Solution

$$f_x = 2x - 4y = 0 \qquad f_y = -4x + 2y + 6 = 0$$

$$\begin{cases} x - 2y = 0 \\ -2x + y = -3 \end{cases} \rightarrow \quad x = 2 \quad y = 1$$

Therefore, the critical point is $(2, 1)$

$$f_{xx}\Big|_{(2,1)} = 2, \quad f_{yy}\Big|_{(2,1)} = 2, \quad f_{xy}\Big|_{(2,1)} = -4$$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - (-4)^2 = -12 < 0 \quad \Rightarrow \quad \textit{Saddle point}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$

Solution

$$f_x = 4x + 3y - 5 = 0 \qquad f_y = 3x + 8y + 2 = 0$$

$$\begin{cases} 4x+3y=5 \\ 3x+8y=-2 \end{cases} \rightarrow \quad x=2 \quad y=-1$$

Therefore, the critical point is $(2,\ -1)$

$$f_{xx}\Big|_{(2,-1)}=4, \quad f_{yy}\Big|_{(2,-1)}=8, \quad f_{xy}\Big|_{(2,-1)}=3$$

$$f_{xx}f_{yy}-f_{xy}^2=(4)(8)-3^2=23>0 \quad \text{and} \quad f_{xx}=4>0$$

The function f has a local minimum at $(2,\ -1)$ and the value is

$$f(2,\ -1)=2(2)^2+3(2)(-1)+4(-1)^2-5(2)+2(-1)$$

$$\underline{=-6}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$f(x,y)=x^2-y^2-2x+4y+6$

Solution

$$f_x=2x-2=0 \qquad f_y=-2y+4=0$$

$$\begin{cases} 2x=2 \\ 2y=4 \end{cases} \rightarrow \quad x=1 \quad y=2$$

Therefore, the critical point is $(1,\ 2)$

$$f_{xx}\Big|_{(1,2)}=2, \quad f_{yy}\Big|_{(1,2)}=-2, \quad f_{xy}\Big|_{(1,2)}=0$$

$$f_{xx}f_{yy}-f_{xy}^2=(2)(-2)-0^2=-4<0 \quad \rightarrow \quad \textit{Saddle Point}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$f(x,y)=\sqrt{56x^2-8y^2-16x-31}+1-8x$

Solution

$$f_x=\frac{1}{2}\frac{112x-16}{\sqrt{56x^2-8y^2-16x-31}}-8=0 \qquad f_y=\frac{1}{2}\frac{-16y}{\sqrt{56x^2-8y^2-16x-31}}=0$$

$$\begin{cases} 56x - 8 = 8\sqrt{56x^2 - 8y^2 - 16x - 31} \\ -8y = 0 \end{cases} \rightarrow \begin{array}{l} x = \frac{16}{7} \quad \cancel{x = -2} \\ y = 0 \end{array}$$

Therefore, the critical point is $\left(\frac{16}{7},\ 0\right)$

$$f_{xx}\Big|_{\left(\frac{16}{7},\ 0\right)} = \frac{56\sqrt{56x^2 - 8y^2 - 16x - 31} - (56x-8)(56x-8)\left(56x^2 - 8y^2 - 16x - 31\right)^{-1/2}}{56x^2 - 8y^2 - 16x - 31}$$

$$= -\frac{8}{15}$$

$$f_{yy}\Big|_{\left(\frac{16}{7},\ 0\right)} = \frac{-8\sqrt{56x^2 - 8y^2 - 16x - 31} - (-8y)\left(56x^2 - 8y^2 - 16x - 31\right)^{-1/2}(-8y)}{56x^2 - 8y^2 - 16x - 31}$$

$$= -\frac{8}{15}$$

$$f_{xy}\Big|_{\left(\frac{16}{7},\ 0\right)} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = \left(-\frac{8}{15}\right)\left(-\frac{8}{15}\right) - 0 = \frac{34}{225} > 0 \quad \textit{and} \quad f_{xx} = -\frac{8}{15} < 0$$

The function f has a local maximum at $\left(\frac{16}{7},\ 0\right)$ and the value is

$$f\left(\frac{16}{7},\ 0\right) = \sqrt{56\left(\frac{16}{7}\right)^2 - 8(0)^2 - 16\left(\frac{16}{7}\right) - 31} + 1 - 8\left(\frac{16}{7}\right)$$

$$= -\frac{16}{7}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y) = 1 - \sqrt[3]{x^2 + y^2}$

Solution

$$f_x = -\frac{1}{3}2x\left(x^2 + y^2\right)^{-2/3}$$

$$= \frac{-2x}{3\left(x^2 + y^2\right)^{2/3}} = 0$$

$$f_y = -\frac{1}{3}2y\left(x^2 + y^2\right)^{-2/3}$$

$$= \frac{-2y}{3\left(x^2+y^2\right)^{2/3}} = 0$$

There are no solutions to the system $f_x(x,y)=0$ *and* $f_y(x,y)=0$, however, this occurs when $x=0 \quad y=0$. The critical point is $(0,\ 0)$

We cannot use the second derivative test, but this is the only possible local maximum, local minimum, or saddle point. $f(x,y)$ has a local maximum of $f(0,0)=1$ since

$$f(x,y)=1-\sqrt[3]{x^2+y^2} \le 1 \quad \forall (x,y)-\{(0,0)\}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function

$f(x,y)=x^3+y^3+3x^2-3y^2-8$

Solution

$$f_x = 3x^2+6x=0 \qquad f_y = 3y^2-6y=0$$

$$\begin{cases} 3x(x+2)=0 \\ 3y(y-2)=0 \end{cases} \rightarrow \begin{array}{l} x=0,\ -2 \\ y=0,\ 2 \end{array}$$

Therefore, the critical point is $(0,0)$, $(0,2)$, $(-2,0)$, and $(-2,2)$

$$f_{xx}=6x+6, \quad f_{yy}=6y-6, \quad f_{xy}=0$$

For $(0,0)$ $\quad f_{xx}\big|_{(0,0)}=6, \quad f_{yy}\big|_{(0,0)}=-6, \quad f_{xy}\big|_{(0,0)}=0$

$$f_{xx}f_{yy}-f_{xy}^2=(6)(-6)-0^2=-36<0 \quad \Rightarrow \quad \textit{Saddle Point}$$

For $(0,2)$ $\quad f_{xx}\big|_{(0,2)}=6, \quad f_{yy}\big|_{(0,2)}=6, \quad f_{xy}\big|_{(0,2)}=0$

$$f_{xx}f_{yy}-f_{xy}^2=(6)(6)-0^2=36>0 \quad \textit{and} \quad f_{xx}>0$$

The function f has a local minimum at $(0,2)$ and the value is $f(0,2)\ \underline{=-12}$

For $(-2,0)$ $\quad f_{xx}\big|_{(-2,0)}=-6, \quad f_{yy}\big|_{(-2,0)}=-6, \quad f_{xy}\big|_{(-2,0)}=0$

$$f_{xx}f_{yy}-f_{xy}^2=(-6)(-6)-0^2=36>0 \quad \textit{and} \quad f_{xx}<0$$

The function f has a local maximum at $(-2,0)$ and the value is $f(-2,0)\ \underline{=-4}$

For $(-2,2)$ $\quad f_{xx}\Big|_{(-2,2)} = 6, \quad f_{yy}\Big|_{(-2,2)} = 6, \quad f_{xy}\Big|_{(-2,2)} = 0$

$$f_{xx}f_{yy} - f_{xy}^2 = (-6)(6) - 0^2 = -36 < 0 \quad \Rightarrow \quad \textit{Saddle Point}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y) = 4xy - x^4 - y^4$

Solution

$$f_x = 4y - 4x^3 = 0 \qquad f_y = 4x - 4y^3 = 0$$

$$\begin{cases} y - x^3 = 0 \\ x - y^3 = 0 \end{cases} \Rightarrow x = y \to x - x^3 = 0 \to x\left(1 - x^2\right) = 0 \quad \to x = 0, \ \pm 1$$

Therefore, the critical point is $(0,0)$, $(1,1)$, and $(-1,-1)$

$$f_{xx} = -12x^2, \quad f_{yy} = -12y^2, \quad f_{xy} = 4$$

For $(0,0)$ $\quad f_{xx}\Big|_{(0,0)} = 0, \quad f_{yy}\Big|_{(0,0)} = 0, \quad f_{xy}\Big|_{(0,0)} = 4$

$$f_{xx}f_{yy} - f_{xy}^2 = 0 - 4^2 = -16 < 0 \quad \Rightarrow \quad \textit{Saddle Point}$$

For $(1,1)$ $\quad f_{xx}\Big|_{(1,1)} = -12, \quad f_{yy}\Big|_{(1,1)} = -12, \quad f_{xy}\Big|_{(1,1)} = 4$

$$f_{xx}f_{yy} - f_{xy}^2 = (-12)(-12) - 4^2 = 128 > 0 \quad \textit{and} \quad f_{xx} < 0$$

The function has a local maximum at $(1,1)$ and the value is $f(1,1) \underline{= 2}$

For $(-1,-1)$ $\quad f_{xx}\Big|_{(-1,-1)} = -12, \quad f_{yy}\Big|_{(-1,-1)} = -12, \quad f_{xy}\Big|_{(-1,-1)} = 0$

$$f_{xx}f_{yy} - f_{xy}^2 = (-12)(-12) - 4^2 = 128 > 0 \quad \textit{and} \quad f_{xx} < 0 \ f$$

The function f has a local maximum at $(-1,-1)$ and the value is $f(-1,-1) \underline{= 2}$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y) = \dfrac{1}{x^2 + y^2 - 1}$

Solution

$$f_x = \frac{-2x}{\left(x^2 + y^2 - 1\right)^2} = 0 \qquad f_y = \frac{-2y}{\left(x^2 + y^2 - 1\right)^2} = 0$$

$\Rightarrow x = y = 0$ Therefore, the critical point is $(0,0)$

$$f_{xx} = \frac{-2\left(x^2+y^2-1\right)^2 - (-2x)(4x)\left(x^2+y^2-1\right)}{\left(x^2+y^2-1\right)^4}$$

$$= \frac{-2x^2-2y^2+2+8x^2}{\left(x^2+y^2-1\right)^3}$$

$$= \frac{6x^2-2y^2+2}{\left(x^2+y^2-1\right)^3}$$

$$f_{yy} = \frac{-2\left(x^2+y^2-1\right)^2 - (-2y)(4y)\left(x^2+y^2-1\right)}{\left(x^2+y^2-1\right)^4}$$

$$= \frac{-2x^2-2y^2+2+8y^2}{\left(x^2+y^2-1\right)^3}$$

$$= \frac{-2x^2+6y^2+2}{\left(x^2+y^2-1\right)^3}$$

$$f_{xy} = \frac{-2x(4y)\left(x^2+y^2-1\right)}{\left(x^2+y^2-1\right)^4}$$

$$= \frac{-8xy}{\left(x^2+y^2-1\right)^3}$$

$$f_{xx}\Big|_{(0,0)} = -2, \quad f_{yy}\Big|_{(0,0)} = -2, \quad f_{xy}\Big|_{(0,0)} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - 0^2 = 4 > 0 \quad \textit{and} \quad f_{xx} < 0$$

The function f has a local maximum at $(0,0)$ and the value is $f(0,0) = -1$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y) = \frac{1}{x} + xy + \frac{1}{y}$

Solution

$$f_x = -\frac{1}{x^2} + y = 0 \qquad f_y = x - \frac{1}{y^2} = 0$$

$$\to \begin{cases} y = \frac{1}{x^2} & (x \neq 0) \\ x = \frac{1}{y^2} & (y \neq 0) \end{cases} \quad x = x^4 \quad \Rightarrow x = 1 = y$$

Therefore, the critical point is $(1,1)$

$$f_{xx}\Big|_{(1,1)} = \left(\frac{2}{x^3}\right)\Big|_{(1,1)} = 2, \quad f_{yy}\Big|_{(1,1)} = \left(\frac{2}{y^3}\right)\Big|_{(1,1)} = -2, \quad f_{xy}\Big|_{(1,1)} = (1)\Big|_{(1,1)} = 1$$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - 1^2 = 3 > 0 \quad \text{and} \quad f_{xx} > 0$$

The function f has a local minimum at $(1,1)$ and the value is $\underline{f(1,\ 1) = 3}$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y) = y\sin x$

Solution

$$f_x = y\cos x = 0 \qquad f_y = \sin x = 0$$

$$\to \begin{cases} y\cos x = 0 \\ \sin x = 0 \end{cases} \quad x = n\pi \quad y = 0 \quad \text{Therefore, the critical point is } (n\pi, 0)$$

$$f_{xx}\Big|_{(n\pi,0)} = -y\sin x\Big|_{(n\pi,0)} = 0$$

$$f_{yy}\Big|_{(n\pi,0)} = 0$$

$$f_{xy}\Big|_{(n\pi,0)} = \cos x\Big|_{(n\pi,0)} = \pm 1$$

If n is even: $\quad f_{xx}f_{yy} - f_{xy}^2 = 0 - 1^2 = -1 < 0 \quad \Rightarrow \quad$ *Saddle Point*

If n is odd: $\quad f_{xx}f_{yy} - f_{xy}^2 = 0 - (-1)^2 = -1 < 0 \quad \Rightarrow \quad$ *Saddle Point*

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y) = e^{2x}\cos y$

Solution

$$f_x = 2e^{2x}\cos y = 0 \qquad f_y = -e^{2x}\sin y = 0$$

Since $e^{2x} \neq 0 \quad \forall x$, the functions $\cos y$ and $\sin y$ cannot equal to zero for the same y.

$\therefore$ No critical points $\Rightarrow$ no extrema and no saddle points.

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y)=e^{y}-ye^{x}$

Solution

$$f_x=-ye^{x}=0 \qquad f_y=e^{y}-e^{x}=0$$

$$\rightarrow \begin{cases} -ye^{x}=0 \\ e^{y}-e^{x}=0 \end{cases} \quad y=0 \quad e^{x}=e^{y}=1=e^{0} \Rightarrow x=0$$

$\therefore$ The critical point is $(0,0)$

$$f_{xx}\Big|_{(0,0)}=-ye^{x}\Big|_{(0,0)}=0$$

$$f_{yy}\Big|_{(0,0)}=e^{y}=1$$

$$f_{xy}\Big|_{(0,0)}=-e^{x}\Big|_{(0,0)}=-1$$

$$f_{xx}f_{yy}-f_{xy}^2=0(1)-(-1)^2=-1<0 \quad \Rightarrow \quad \textit{Saddle Point}$$

Exercise

Find all the local maxima, local minima, and saddle points of the function $f(x,y)=e^{-y}\left(x^2+y^2\right)$

Solution

$$f_x=2xe^{-y}=0$$

$$f_y=-e^{-y}\left(x^2+y^2\right)+2ye^{-y}=e^{-y}\left(2y-x^2-y^2\right)=0$$

$$\rightarrow \begin{cases} 2xe^{-y}=0 & \rightarrow \boxed{x=0} \\ e^{-y}\left(2y-x^2-y^2\right)=0 \end{cases} \quad 2y-x^2-y^2=0 \rightarrow y(2-y=0) \quad \boxed{y=0,\ 2}$$

$\therefore$ The critical point is $(0,0)$ and $(0,2)$

$$f_{xx}=2e^{-y}$$

$$f_{yy}=-e^{-y}\left(2y-x^2-y^2\right)+e^{-y}(2-2y)=e^{-y}\left(2-4y+x^2+y^2\right)$$

$f_{xy} = -2xye^{-y}$

For $(0,0)$ $f_{xx}\Big|_{(0,0)} = 2,\quad f_{yy}\Big|_{(0,0)} = 2,\quad f_{xy}\Big|_{(0,0)} = 0$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - 0^2 = 4 > 0 \quad \text{and} \quad f_{xx} > 0$$

The function f has a local minimum at $(0,0)$ and the value is $\underline{f(0,\ 0) = 0}$

For $(0,2)$ $f_{xx}\Big|_{(0,2)} = \frac{2}{e^2},\quad f_{yy}\Big|_{(0,2)} = -\frac{2}{e^2},\quad f_{xy}\Big|_{(0,2)} = 0$

$$f_{xx}f_{yy} - f_{xy}^2 = \frac{2}{e^2}\left(-\frac{2}{e^2}\right) - 0^2 = -\frac{4}{e^4} < 0 \quad \Rightarrow \quad \textit{Saddle Point}$$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x,y) = 2\ln x + \ln y - 4x - y$

Solution

$$f_x = \frac{2}{x} - 4 = 0 \qquad f_y = \frac{1}{y} - 1 = 0$$

$$\rightarrow \begin{cases} 2 = 4x \\ 1 = y \end{cases} \quad x = \frac{1}{2}$$

$\therefore$ The critical point is $\left(\frac{1}{2},\ 1\right)$

$$f_{xx}\Big|_{\left(\frac{1}{2},1\right)} = \left(-\frac{2}{x^2}\right)\Bigg|_{\left(\frac{1}{2},1\right)} = -8$$

$$f_{yy}\Big|_{\left(\frac{1}{2},1\right)} = \left(-\frac{1}{y^2}\right)\Bigg|_{\left(\frac{1}{2},1\right)} = -1$$

$$f_{xy}\Big|_{\left(\frac{1}{2},1\right)} = 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = (-8)(-1) - 0^2 = 8 > 0 \quad \text{and} \quad f_{xx} < 0$$

The function f has a local maximum at $\left(\frac{1}{2},\ 1\right)$ and the value is $\underline{f\left(\frac{1}{2},\ 1\right) = -3 - 2\ln 2}$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x,y)=\ln(x+y)+x^2-y$

Solution

$$f_x=\frac{1}{x+y}+2x=0$$

$$f_y=\frac{1}{x+y}-1=0$$

$$\rightarrow \begin{cases} \frac{1}{x+y}=-2x & \rightarrow -2x(x+y)=1 \\ \frac{1}{x+y}=1 & \rightarrow 1=x+y \end{cases} \Rightarrow -2x(1)=1 \rightarrow x=-\frac{1}{2} \quad y=\frac{3}{2}$$

$\therefore$ The critical point is $\left(-\frac{1}{2},\ \frac{3}{2}\right)$

$$f_{xx}=-\frac{1}{(x+y)^2}+2, \quad f_{yy}=-\frac{1}{(x+y)^2}, \quad f_{xy}=-\frac{1}{(x+y)^2}$$

$$f_{xx}\Big|_{\left(-\frac{1}{2},\frac{3}{2}\right)}=1$$

$$f_{yy}\Big|_{\left(-\frac{1}{2},\frac{3}{2}\right)}=-1$$

$$f_{xy}\Big|_{\left(-\frac{1}{2},\frac{3}{2}\right)}=-1$$

$f_{xx}f_{yy}-f_{xy}^2=(1)(-1)-(-1)^2=-2<0$ *and* *Saddle Point*

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x,\ y)=1+x^2+y^2$

Solution

$$f_x=2x=0 \quad \rightarrow \quad \underline{x=0}$$

$$f_y=2y=0 \quad \rightarrow \quad \underline{y=0}$$

$\therefore$ The critical point is $(0,\ 0)$

$$f_{xx}=2, \quad f_{yy}=2, \quad f_{xy}=0$$

$f_{xx}f_{yy}-f_{xy}^2\Big|_{(0,0)}=4>0$ *and* $f_{xx}>0$

The function f has a local minimum at $(0,\ 0)$ and the value is $\underline{f(0,\ 0)=1}$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = x^2 - 6x + y^2 + 8y$

Solution

$f_x = 2x - 6 = 0 \quad \rightarrow \quad \underline{x = 3}$

$f_y = 2y + 8 = 0 \quad \rightarrow \quad \underline{y = -4}$

$\therefore$ The critical point is $(3, \ -4)$

$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 0$

$$f_{xx} f_{yy} - f_{xy}^2 \Big|_{(3,-4)} = 4 > 0 \quad \textit{and} \quad f_{xx} > 0$$

The function f has a local minimum at $(3, \ -4)$ and the value is

$\underline{f(3, \ -4) = 9 - 18 + 16 - 32 = -25}$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = (3x - 2)^2 + (y - 4)^2$

Solution

$f_x = 6(3x - 2) = 0 \quad \rightarrow \quad \underline{x = \frac{2}{3}}$

$f_y = 2(y - 4) = 0 \quad \rightarrow \quad \underline{y = 4}$

$\therefore$ The critical point is $\left(\frac{2}{3}, \ 4\right)$

$f_{xx} = 18, \quad f_{yy} = 2, \quad f_{xy} = 0$

$$f_{xx} f_{yy} - f_{xy}^2 \Big|_{\left(\frac{2}{3}, \ 4\right)} = 36 > 0 \quad \textit{and} \quad f_{xx} > 0$$

The function f has a local minimum at $\left(\frac{2}{3}, \ 4\right)$ and the value is $\underline{f\left(\frac{2}{3}, \ 4\right) = 0}$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = 3x^2 - 4y^2$

Solution

$f_x = 6x = 0 \quad \rightarrow \quad \underline{x = 0}$

$f_y = -8y = 0 \quad \to \quad \underline{y = 0|}$

$\therefore$ The critical point is $(0,\ 0)$

$f_{xx} = 6, \quad f_{yy} = -8, \quad f_{xy} = 0$

$$f_{xx} f_{yy} - f_{xy}^2 \Big|_{(0,\ 0)} = -48 < 0 \quad \textit{and} \quad \textit{Saddle point}$$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x,\ y) = x^4 + y^4 - 16xy$

Solution

$f_x = 4x^3 - 16y = 0$

$f_y = 4y^3 - 16x = 0$

$$\begin{cases} x^3 = 4y \\ y^3 = 4x \ \to \ x = \dfrac{y^3}{4} \end{cases} \quad \to \quad \left(\frac{y^3}{4}\right)^3 = 4y$$

$y^9 = 4^4 y$

$y\left(y^8 - 2^8\right) = 0 \to y = 0, \ \pm 2$

$\therefore$ The critical point is $(0,\ 0)$, $(-2,\ -2)$, $(2,\ 2)$

$f_{xx} = 12x^2 > 0, \quad f_{yy} = 12y^2, \quad f_{xy} = -16$

$f_{xx} f_{yy} - f_{xy}^2 = 144x^2y^2 - 256$

@ $(0,\ 0)$

$f_{xx} f_{yy} - f_{xy}^2 = -256 < 0 \quad \textit{and} \quad \textit{Saddle point}$

@ $(-2,\ -2)$

$f_{xx} f_{yy} - f_{xy}^2 = 2,048 > 0 \quad \textit{and} \quad f_{xx} > 0$

The function f has a local minimum at $(-2,\ -2)$ and the value is $\underline{f(-2,\ -2) = -32\,|}$

@ $(2,\ 2)$

$f_{xx} f_{yy} - f_{xy}^2 = 2,048 > 0 \quad \textit{and} \quad f_{xx} > 0$

The function f has a local minimum at $(2,\ 2)$ and the value is $\underline{f(2,\ 2) = -32\,|}$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = \frac{1}{3}x^3 - \frac{1}{3}y^3 + 3xy$

Solution

$$f_x = x^2 + 3y = 0$$

$$f_y = -y^2 + 3x = 0$$

$$\begin{cases} x^2 = -3y \\ y^2 = 3x \ \rightarrow \ x = \frac{y^2}{3} \end{cases} \rightarrow \left(\frac{y^2}{3}\right)^2 = -3y$$

$$y^4 = -3^3 y$$

$$y\left(y^3 + 3^3\right) = 0 \rightarrow y = 0, \ -3$$

$\therefore$ The critical point is $(0, \ 0)$, $(3, \ -3)$

$$f_{xx} = 2x, \quad f_{yy} = -2y, \quad f_{xy} = 3$$

$$f_{xx}f_{yy} - f_{xy}^2 = -4xy - 9$$

@ $(0, \ 0)$

$f_{xx}f_{yy} - f_{xy}^2 = -9 < 0$ *and* *Saddle point*

@ $(3, \ -3)$

$f_{xx}f_{yy} - f_{xy}^2 = 36 - 9 = 27 > 0$ *and* $f_{xx} = 6 > 0$

The function f has a local minimum at $(3, \ -3)$ and the value is

$$f(3, \ -3) = 9 + 9 - 27 = -9$$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x, y) = x^4 - 2x^2 + y^2 - 4y + 5$

Solution

$$f_x = 4x^3 - 4x = 4x\left(x^2 - 1\right) = 0 \quad \rightarrow \quad x = 0, \ \pm 1$$

$$f_y = 2y - 4 = 0 \quad \rightarrow \quad y = 2$$

$\therefore$ The critical point is $(0, \ 2)$, $(-1, \ 2)$, $(1, \ 2)$

$f_{xx} = 12x^2 - 4, \quad f_{yy} = 2, \quad f_{xy} = 0$

$f_{xx} f_{yy} - f_{xy}^2 = 24x^2 - 8$

@ $(0,\ 2)$

$f_{xx} f_{yy} - f_{xy}^2 = -8 < 0$ *and* *Saddle point*

@ $(-1,\ 2)$

$f_{xx} f_{yy} - f_{xy}^2 = 16 > 0$ *and* $f_{xx} = 8 > 0$

The function f has a local minimum at $(-1,\ 2)$ and the value is

$\underline{f(-1,\ 2) = 1 - 2 + 4 - 8 + 5 = 0}$

@ $(1,\ 2)$

$f_{xx} f_{yy} - f_{xy}^2 = 16 > 0$ *and* $f_{xx} = 8 > 0$

The function f has a local minimum at $(1,\ 2)$ and the value is

$\underline{f(1,\ 2) = 1 - 2 + 4 - 8 + 5 = 0}$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x,\ y) = x^2 + xy - 2x - y + 1$

Solution

$f_x = 2x + y - 2 = 0$

$f_y = x - 1 = 0 \quad \rightarrow \quad \underline{x = 1}$

$y = 2 - 2x \ \underline{= 0}$

$\therefore$ The critical point is $(1,\ 0)$

$f_{xx} = 2, \quad f_{yy} = 0, \quad f_{xy} = 1$

$f_{xx} f_{yy} - f_{xy}^2 = -1 < 0$ *and* *Saddle point*

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x,\ y) = x^2 + 6x + y^2 + 8$

Solution

$f_x = 2x + 6 = 0 \quad \rightarrow \quad \underline{x = -3|}$

$f_y = 2y = 0 \quad \rightarrow \quad \underline{y = 0|}$

$\therefore$ The critical point is $(-3,\ 0)$

$f_{xx} = 2 > 0, \quad f_{yy} = 2, \quad f_{xy} = 0$

$f_{xx}f_{yy} - f_{xy}^2 \Big|_{(-3,\ 0)} = 4 > 0 \quad$ *and* $\quad f_{xx} > 0$

The function f has a local minimum at $(-3,\ 0)$ and the value is

$\underline{f(-3,\ 0) = 9 - 18 + 8 = -1\,|}$

Exercise

Find all the local maxima, minima, and saddle points of the function $f(x,\ y) = e^{x^2y^2 - 2xy^2 + y^2}$

Solution

$f_x = \left(2xy^2 - 2y^2\right)e^{x^2y^2 - 2xy^2 + y^2} = 0 \quad \rightarrow \quad 2(x-1)y^2 = 0$

$f_y = \left(2x^2y - 4xy + 2y\right)e^{x^2y^2 - 2xy^2 + y^2} = 0 \quad \rightarrow \quad 2y\left(x^2 - 2x + 1\right) = 0$

$$\begin{cases} 2(x-1)y^2 = 0 & \rightarrow y = 0,\ x = 1 \\ 2y\left(x^2 - 2x + 1\right) = 0 & \rightarrow y = 0,\ x = 1 \end{cases}$$

$\therefore$ The critical point is $(1,\ 0),\ (x,\ 0),\ (1,\ y)$

@ $(1,\ 0)$

$$\begin{aligned} f_{xx} &= \left(2y^2 + 2xy^2 - 2y^2\right)e^{x^2y^2 - 2xy^2 + y^2} \\ &= 2xy^2 e^{x^2y^2 - 2xy^2 + y^2} \Big|_{(1,0)} \\ &= 0 \end{aligned}$$

$$\begin{aligned} f_{yy} &= \left(2x^2 - 4x + 2 + 2x^2y - 4xy + 2y\right)e^{x^2y^2 - 2xy^2 + y^2} \Big|_{(1,0)} \\ &= 0 \end{aligned}$$

$$\begin{aligned} f_{xy} &= \left(4xy - 4y + 2x^2y - 4xy + 2y\right)e^{x^2y^2 - 2xy^2 + y^2} \\ &= \left(2x^2y - 2y\right)e^{x^2y^2 - 2xy^2 + y^2} \Big|_{(1,0)} \\ &= 0 \end{aligned}$$

$$f_{xx}f_{yy}-f_{xy}^2\Big|_{(1,\ 0)}=0$$

Inconclusive. No extreme values.

Exercise

Identify the critical points of the functions. Then determine whether each critical point corresponds to a local maximum, local minimum, or saddle point. State when your analysis is inconclusive.

$$f(x,\ y)=x^4+y^4-16xy$$

Solution

$$f_x=4x^3-16y=0 \quad (1)$$

$$f_y=4y^3-16x=0 \quad (2)$$

$$\begin{cases}(1)\to\ x^3=4y\\(2)\to\ y^3=4x\end{cases}$$

$$\left(\frac{y^3}{4}\right)^3=4y$$

$$y^9=4^4y$$

$$y\left(y^8-2^8\right)=0 \quad\to\quad \underline{y=0,\ \pm 2}\,|$$

C.P: $(0,\ 0)$, $(2,\ 2)$, $(-2,\ -2)$

$$f_{xx}=12x^2 \qquad f_{yy}=12y^2 \qquad f_{xy}=-16$$

$$f_{xx}f_{yy}-f_{xy}^2=144x^2y^2-256$$

@ $(0,\ 0)$

$$f_{xx}f_{yy}-f_{xy}^2=-256<0$$

$(0,\ 0)$ is a ***saddle point***.

@ $\pm(2,\ 2)$

$$f_{xx}f_{yy}-f_{xy}^2=144(4)(4)-256=2,048>0$$

f has a ***local Min*** @ $(2,\ 2)$ & $(-2,\ -2)$

Exercise

Identify the critical points of the functions. Then determine whether each critical point corresponds to a local maximum, local minimum, or saddle point. State when your analysis is inconclusive.

$$f(x, y) = \frac{1}{3}x^3 - \frac{1}{3}y^3 + 2xy$$

Solution

$$f_x = x^2 + 2y = 0 \quad (1)$$

$$f_y = -y^2 + 2x = 0 \quad (2)$$

$$(1) \to y = -\frac{1}{2}x^2$$

$$(2) \to \quad -\left(-\frac{1}{2}x^2\right)^2 + 2x = 0$$

$$-\frac{1}{4}x^4 + 2x = 0$$

$$-\frac{1}{4}x\left(x^3 - 8\right) = 0 \qquad \to \begin{cases} x = 0 & \to y = 0 \\ x = 2 & \to y = -2 \end{cases}$$

C.P: $(0,\ 0)$ & $(2,\ -2)$

$$f_{xx} = 2x \qquad f_{yy} = -2y \qquad f_{xy} = 2 > 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = -4xy - 4$$

@ $(0,\ 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = -4 < 0$$

$(0,\ 0)$ is a ***saddle point***.

@ $(2,\ -2)$

$$f_{xx}f_{yy} - f_{xy}^2 = 16 - 4 = 12 > 0$$

f has a ***local Min*** @ $(2,\ -2)$

Exercise

Identify the critical points of the functions. Then determine whether each critical point corresponds to a local maximum, local minimum, or saddle point. State when your analysis is inconclusive.

$$f(x, y) = xy(2+x)(y-3)$$

Solution

$$f(x, y) = 2xy^2 - 6xy + x^2y^2 - 3x^2y$$

$$\begin{aligned} f_x &= 2y^2 - 6y + 2xy^2 - 6xy = 0 \\ &= 2y(y-3) + 2xy(y-3) \\ &= 2y(y-3)(x+1) = 0 \quad (1) \end{aligned}$$

$$\rightarrow \underline{y = 0, \ \ y = 3, \ \ x = -1}|$$

$$\begin{aligned} f_y &= 4xy - 6x + 2x^2y - 3x^2 \\ &= 2x(2y-3) + x^2(2y-3) \\ &= x(2y-3)(x+2) = 0 \quad (2) \end{aligned}$$

$$\rightarrow \underline{x = 0, \ \ x = -2, \ \ y = \frac{2}{3}}|$$

$$y = 0 \quad (2) \rightarrow \quad x(-3)(x+2) = 0 \ \Rightarrow \underline{x = 0, -2}|$$

$$y = 3 \quad (2) \rightarrow \quad x(3)(x+2) = 0 \ \Rightarrow \underline{x = 0, -2}|$$

$$x = -1 \quad (2) \rightarrow \quad -(2y-3) = 0 \ \Rightarrow \underline{y = \frac{3}{2}}|$$

C.P: $(0,\ 0),\ (-2,\ 0),\ (0,\ 3),\ (-2,\ 3),\ \&\ \left(1,\ \frac{3}{2}\right)$

$$f_{xx} = 2y^2 - 6y$$

$$f_{xy} = 4y - 6 + 4xy - 6x$$

$$f_{yy} = 4x + 2x^2$$

$$f_{xx}f_{yy} - f_{xy}^2 = \left(2y^2 - 6y\right)\left(4x + 2x^2\right) - (4y - 6 + 4xy - 6x)^2$$

@ $(0,\ 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = -36 < 0$$

$(0,\ 0)$ is a ***saddle point***.

@ $(-2,\ 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = -36 < 0$$

$(-2,\ 0)$ is a ***saddle point***.

@ $(0,\ 3)$

$$f_{xx}f_{yy} - f_{xy}^2 = -36 < 0$$

$(0,\ 3)$ is a ***saddle point***.

@ $(-2,\ 3)$

$$f_{xx}f_{yy}-f_{xy}^2=-64<0$$

$(-2,\ 3)$ is a ***saddle point***.

@ $\left(-1,\ \frac{3}{2}\right)$

$$f_{xx}f_{yy}-f_{xy}^2=9>0 \qquad f_{xx}=-\frac{9}{2}<0$$

Function has a ***local max*** @ $\left(-1,\ \frac{3}{2}\right)$

Exercise

Identify the critical points of the functions. Then determine whether each critical point corresponds to a local maximum, local minimum, or saddle point. State when your analysis is inconclusive.

$$f(x,\ y)=10-x^3-y^3-3x^2+3y^2$$

Solution

$$f_x=-3x^2-6x=0$$

$$=-3x(x+2)=0 \quad (1)$$

$\to$ $x=0,\ \ x=-2$

$$f_y=-3y^2+6y$$

$$=-3y(y-2)=0 \quad (2)$$

$\to$ $y=0,\ \ y=2$

C.P: $(0,\ 0),\ (-2,\ 0),\ (0,\ 2),\ \&\ (-2,\ 2)$

$$f_{xx}=-6x-6$$

$$f_{xy}=0$$

$$f_{yy}=-6y+6$$

$$f_{xx}f_{yy}-f_{xy}^2=(-6x-6)(-6y+6)$$

$$=-36(-xy+x-y+1)$$

@ $(0,\ 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = -36 < 0$$

$(0,\ 0)$ is a ***saddle point***.

@ $(0,\ 2)$

$$f_{xx}f_{yy} - f_{xy}^2 = 36 > 0 \qquad f_{xx} = -6 < 0$$

Function has a ***local min*** @ $(0,\ 2)$

@ $(-2,\ 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = 36 > 0 \qquad f_{xx} = 6 > 0$$

Function has a ***local max*** @ $(-2,\ 0)$

@ $(-2,\ 2)$

$$f_{xx}f_{yy} - f_{xy}^2 = -36 < 0$$

$(-2,\ 2)$ is a ***saddle point***.

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

$f(x,\ y) = \frac{1}{3}x^3 - \frac{1}{3}y^3 + 2xy$ on the rectangle $R = \{(x,\ y):\ 0 \le x \le 3,\ -1 \le y \le 1\}$

Solution

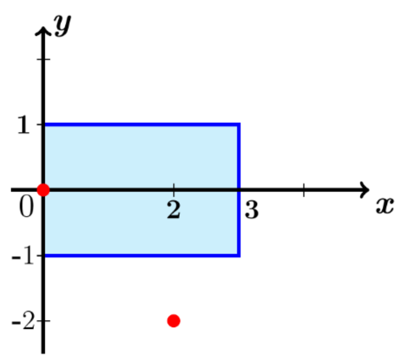
$$f_x = x^2 + 2y = 0 \quad \to \quad y = -\frac{1}{2}x^2$$

$$f_y = -y^2 + 2x = 0$$

$$-\frac{1}{4}x^4 + 2x = 0$$

$$-\frac{1}{4}x\left(x^3 - 8\right) = 0 \qquad \to \quad \underline{x = 0,\ \ x = 2}$$

$$\to \begin{cases} x = 0 & y = 0 \\ x = 2 & y = -2 \end{cases}$$



C.P: $(0,\ 0)$ & $(2,\ -2)$ neither in the interior of R.

$0 \le x \le 3 \quad \to \quad y = -1$

$$g(x) = f(x,\ -1) = \frac{1}{3}x^3 - 2x + \frac{1}{3}$$

$$f_x = x^2 - 2 = 0 \quad \to \quad \underline{x = \sqrt{2}},\ \ \cancel{x = -\sqrt{2}}$$

$$g\left(\sqrt{2}\right) = \frac{2}{3}\sqrt{2} - 2\sqrt{2} + \frac{1}{3}$$

$$= \boxed{\frac{1-4\sqrt{2}}{3}}$$

$\rightarrow\ y = 1$

$g(x) = f(x,\ 1) = \frac{1}{3}x^3 + 2x - \frac{1}{3}$

$f_x = x^2 + 2 \neq 0$

$-1 \le y \le 1 \quad \rightarrow \quad x = 0$

$h(y) = f(0,\ y) = -\frac{1}{3}y^3$

$f_y = y^2 = 0 \quad \rightarrow \quad \boxed{y = 0}$

$\rightarrow\ x = 3$

$h(y) = f(3,\ y) = -\frac{1}{3}y^3 + 6y + 9$

$f_y = -y^2 + 6 = 0 \quad \rightarrow \quad \boxed{y = \sqrt{6}}$

$h(\sqrt{6}) = f(3,\ \sqrt{6})\ \boxed{= 4\sqrt{6} + 9}$

Absolute minimum: $\boxed{f(\sqrt{2},\ 1) = \frac{1-4\sqrt{2}}{3}}$

Absolute maximum: $\boxed{f(3,\ \sqrt{6}) = 4\sqrt{6} + 9}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

$f(x,\ y) = x^4 + y^4 - 4xy + 1$ on the square $R = \{(x,\ y):\ -2 \le x \le 2,\ -2 \le y \le 2\}$

Solution

$f_x = 4x^3 - 4y = 0 \quad \rightarrow \quad y = x^3$

$f_y = 4y^3 - 4x = 0$

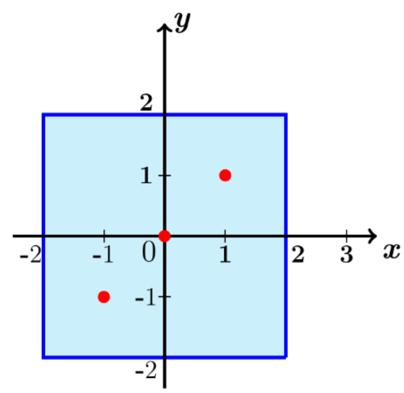
$x^9 - x = 0 \quad \rightarrow \quad \boxed{x = 0,\ \pm 1}$

$x = 0 \quad \rightarrow \quad y = 0$

$x = -1 \quad \rightarrow \quad y = -1$

$x = 1 \quad \rightarrow \quad y = 1$

C.P: $(0,\ 0),\ (1,\ 1)\ \&\ (-1,\ -1)$



	$f(x, y)$
$(0,\ 0)$	1
$(-1,\ -1)$	$1+1-4+1=-1$
$(1,\ 1)$	$1+1-4+1=-1$
$(2,\ 2)$	$16+16-16+1=17$
$(-2,\ 2)$	$16+16+16+1=49$
$(2,\ -2)$	$16+16+16+1=49$
$(-2,\ -2)$	$16+16-16+1=17$

Absolute minimum: $\boxed{f(1,\ 1)=f(-1,\ -1)=-1}$

Absolute maximum: $\boxed{f(-2,\ 2)=f(2,\ -2)=49}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

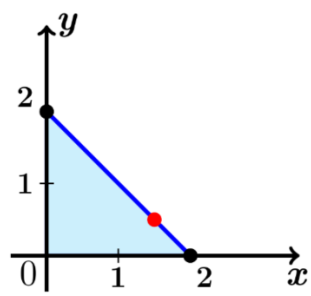
$f(x, y)=x^2y-y^3$ on the triangle $R=\{(x, y):\ 0\le x\le 2,\ 0\le y\le 2-x\}$

Solution

$f_x=2xy=0 \quad (0,\ 0)$

$f_y=x^2-3y^2=0 \quad (2)$

C.P: ***None*** inside the triangle



$y=2-x$

$$f(x, y)=x^2(2-x)-(2-x)^3$$

$$=(2-x)\left(x^2-4+4x-x^2\right)$$

$$=(2-x)(4x-4)$$

$$=12x-8-4x^2$$

$g'=12-8x=0 \quad \to \quad \boxed{x=\frac{3}{2}}$

$x=\frac{3}{2} \quad \to \quad \boxed{y=2-\frac{3}{2}=\frac{1}{2}}$

	$f(x, y)$
$(0,\ 0)$	0
$(2,\ 0)$	0
$(0,\ 2)$	-8
$\left(\frac{3}{2},\ \frac{1}{2}\right)$	$\frac{9}{8}-\frac{1}{8}=1$

Absolute minimum: $\boxed{f(0,\ 2)=-8}$

Absolute maximum: $\boxed{f\left(\frac{3}{2},\ \frac{1}{2}\right)=1}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

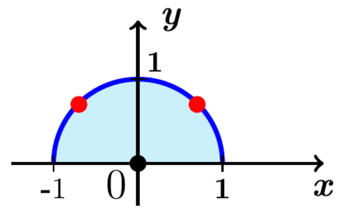
$f(x, y) = xy$ on the semicircular disk $R = \left\{(x, y): \ -1 \le x \le 1, \ 0 \le y \le \sqrt{1-x^2}\right\}$

Solution

$$f_x = y = 0$$

$$f_y = x = 0$$

C.P.: $(0, \ 0)$



$$y = \sqrt{1-x^2} \quad \to \quad y^2 + x^2 = 1$$

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$$

$$f(x, y) = g(t) = \cos t \sin t$$

$$= \frac{1}{2}\sin 2t$$

$$g' = \cos 2t = 0 \quad \to \quad 2t = \frac{\pi}{2}, \ \frac{3\pi}{2}$$

$$\underline{t = \frac{\pi}{4}, \ \frac{3\pi}{4}|}$$

$$g\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

$$g\left(\frac{3\pi}{4}\right) = -\frac{1}{2}$$

Absolute minimum: $\quad \underline{f\left(-\frac{\sqrt{2}}{2}, \ \frac{\sqrt{2}}{2}\right) = -\frac{1}{2}}|$

Absolute maximum: $\quad \underline{f\left(\frac{\sqrt{2}}{2}, \ \frac{\sqrt{2}}{2}\right) = \frac{1}{2}}|$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

$$f(x, y) = x^2 + y^2 - 2y + 1; \quad R = \left\{(x, y): \ x^2 + y^2 \le 4\right\}$$

Solution

$$f_x = 2x = 0 \quad \to \quad \underline{x = 0|}$$

$$f_y = 2y - 2 = 0 \quad \to \quad \underline{y = 1|}$$

C.P.: $(0, \ 1)$

$y^2 + x^2 = 4$

$$\begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}$$

$$f(x,\ y) = g(t) = 4\cos^2 t + 4\sin^2 t - 4\sin t + 1$$
$$= 5 - 4\sin t$$

$g' = -4\cos t = 0 \quad \rightarrow \quad \boxed{t = \frac{\pi}{2},\ \frac{3\pi}{2}}$

$g\left(\frac{\pi}{2}\right) = 1 = f(0,\ 2)$

$g\left(\frac{3\pi}{2}\right) = 9 = f(0,\ -2)$

$f(0,\ 1) = 0$

Absolute minimum: $\boxed{f(0,\ 1) = 0}$

Absolute maximum: $\boxed{f(0,\ -2) = 9}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

$$f(x,\ y) = 2x^2 + y^2; \quad R = \left\{(x,\ y):\ x^2 + y^2 \le 16\right\}$$

Solution

$f_x = 4x = 0 \quad \rightarrow \quad \boxed{x = 0}$

$f_y = 2y = 0 \quad \rightarrow \quad \boxed{y = 0}$

C.P.: $(0,\ 0)$

$y^2 + x^2 = 16 \qquad \begin{cases} x = 4\cos t \\ y = 4\sin t \end{cases}$

$$f(x,\ y) = g(t) = 32\cos^2 t + 16\sin^2 t$$
$$= 16\cos^2 t + 16$$

$$g' = -32\sin t\cos t$$
$$= -16\sin 2t = 0$$

$2t = n\pi \quad \rightarrow \quad t = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$

t	(x, y)	$f(x, y)$
	$(0,\ 0)$	0
0	$(4,\ 0)$	32
$\frac{\pi}{2}$	$(0,\ 4)$	16
π	$(-4,\ 0)$	32
$\frac{3\pi}{2}$	$(0,\ -4)$	16

Absolute minimum: $\boxed{f(0,\ 0) = 0}$

Absolute maximum: $\boxed{f(\pm 4,\ 0) = 32}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

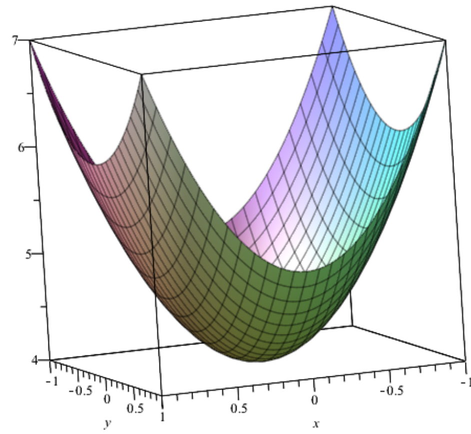
$$f(x, y) = 4 + 2x^2 + y^2; \quad R = \{(x, y): \ -1 \le x \le 1, \ -1 \le y \le 1\}$$

Solution

$f_x = 4x = 0 \quad \rightarrow \quad \underline{x = 0}$

$f_y = 2y = 0 \quad \rightarrow \quad \underline{y = 0}$

C.P.: $(0, \ 0)$



	$f(x, y)$
$(0, \ 0)$	4
$(\pm 1, \ \pm 1)$	$4 + 2 + 1 = 7$

Absolute minimum: $\underline{f(0, \ 0) = 4}$

Absolute maximum: $\underline{f(\pm 1, \ \pm 1) = 7}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

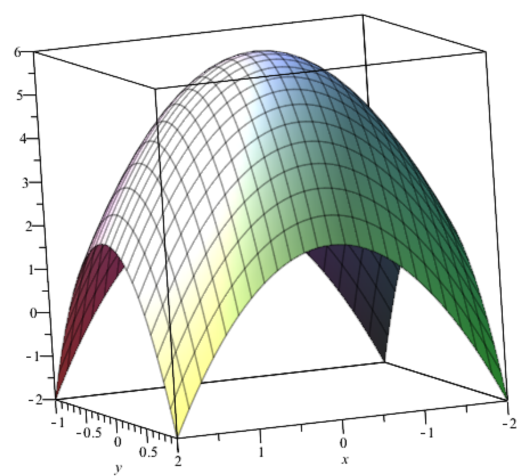
$$f(x, y) = 6 - x^2 - 4y^2; \quad R = \{(x, y): \ -2 \le x \le 2, \ -1 \le y \le 1\}$$

Solution

$f_x = -2x = 0 \quad \rightarrow \quad \underline{x = 0}$

$f_y = -8y = 0 \quad \rightarrow \quad \underline{y = 0}$

C.P.: $(0, \ 0)$



	$f(x, y)$
$(0, \ 0)$	6
$(\pm 2, \ \pm 1)$	$6 - 4 - 4 = -2$

Absolute minimum: $\underline{f(\pm 2, \ \pm 1) = -2}$

Absolute maximum: $\underline{f(0, \ 0) = 6}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

$$f(x, y) = 2x^2 - 4x + 3y^2 + 2; \quad R = \left\{(x, y):\ (x-1)^2 + y^2 \le 1\right\}$$

Solution

$f_x = 4x = 0 \quad \rightarrow \quad \underline{x = 0|}$

$f_y = 2y = 0 \quad \rightarrow \quad \underline{y = 0|}$

C.P.: $(0,\ 0)$

$$(x-1)^2 + y^2 = 1 \quad \begin{cases} x - 1 = \cos t \rightarrow x = 1 + \cos t \\ \qquad\qquad\qquad y = \sin t \end{cases}$$

$$f(x, y) = g(t) = 2(1 + \cos t)^2 - 4 - 4\cos t + 3\sin^2 t + 2$$

$$= 2\cos^2 t + 3\sin^2 t$$

$$= 2 + \sin^2 t$$

$$g' = 2\sin t \cos t$$

$$= \sin 2t = 0$$

$$2t = n\pi \quad \rightarrow \quad t = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$$

t	(x, y)	$f(x, y)$
	$(0,\ 0)$	2
0	$(2,\ 0)$	2
$\frac{\pi}{2}$	$(1,\ 1)$	3
π	$(0,\ 0)$	2
$\frac{3\pi}{2}$	$(1,\ -1)$	3

Absolute minimum: $\underline{f(0,\ 0) = f(2,\ 0) = 2\ |}$

Absolute maximum: $\underline{f(1,\ \pm 1) = 3\ |}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

$$f(x, y) = -2x^2 + 4x - 3y^2 - 6y - 1; \quad R = \left\{(x, y):\ (x-1)^2 + (y+1)^2 \le 1\right\}$$

Solution

$f_x = -4x + 4 = 0 \quad \rightarrow \quad \underline{x = 1|}$

$$f_y = -6y - 6 = 0 \quad \rightarrow \quad \underline{y = -1|}$$

C.P.: $(1, \ -1)$

$$(x-1)^2 + (y+1)^2 = 1 \qquad \begin{cases} x - 1 = \cos t \rightarrow \ x = 1 + \cos t \\ y + 1 = \sin t \rightarrow \ y = \sin t - 1 \end{cases}$$

$$\begin{aligned} f(x, y) = g(t) &= -2(1+\cos t)^2 + 4 + 4\cos t - 3(\sin t - 1)^2 - 6\sin t + 6 - 1 \\ &= 4 - 2\cos^2 t - 3\sin^2 t \\ &= 1 + \cos^2 t \end{aligned}$$

$$\begin{aligned} g' &= -2\sin t\cos t \\ &= -\sin 2t = 0 \end{aligned}$$

$$2t = n\pi \quad \rightarrow \quad t = 0, \ \frac{\pi}{2}, \ \pi, \ \frac{3\pi}{2}$$

t	(x, y)	$f(x, y)$
	$(1, \ -1)$	$-2+4-3+6-1=4$
0	$(2, \ -1)$	$-8+8-3+6-1=2$
$\frac{\pi}{2}$	$(1, \ 0)$	1
π	$(0, \ -1)$	2
$\frac{3\pi}{2}$	$(1, \ -2)$	1

Absolute ***minimum***: $\underline{f(1, \ 0) = f(1, \ -2) = 1 \ |}$

Absolute ***maximum***: $\underline{f(1, \ -1) = 4 \ |}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

$$f(x, y) = \sqrt{x^2 + y^2 - 2x + 2}; \quad R = \{(x, y): \ x^2 + y^2 \le 4, \ y \ge 0\}$$

Solution

$$g(x, y) = x^2 + y^2 - 2x + 2$$

$$g_x = 2x - 2 = 0 \quad \rightarrow \quad \underline{x = 1|}$$

$$g_y = 2y = 0 \quad \rightarrow \quad \underline{y = 0|}$$

C.P.: $(1, \ 0)$

$$y^2 + x^2 = 4 \qquad \begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}$$

$g(x,\ y) = h(t) = 4 - 4\cos t + 2$

$\qquad\qquad\qquad = 6 - \cos t$

$g' = \sin t = 0$

$t = 0,\ \pi$

t	(x, y)	$f(x, y)$
	$(1,\ 0)$	1
0	$(2,\ 0)$	$\sqrt{2}$
π	$(-2,\ 0)$	$\sqrt{10}$

Absolute ***minimum***: $\quad f(1,\ 0) = 1$

Absolute ***maximum***: $\quad f(-2,\ 0) = \sqrt{10}$

Exercise

Find the absolute maximum and minimum values of the function on the specified region *R*.

$$f(x,\ y) = \frac{-x^2 + 2y^2}{2 + 2x^2y^2};\ R \text{ is the closed region bounded by the lines } y = x,\ y = 2x, \text{ and } y = 2$$

Solution

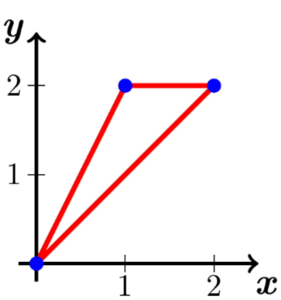
$$f_x = \frac{2\left(-2 - 4y^4\right)x}{\left(2 + 2x^2y^2\right)^2}$$

$$\left(\frac{ax^n + b}{cx^n + d}\right)' = \frac{n(ad - bc)x^{n-1}}{\left(cx^n + d\right)^2}$$

$$= -\frac{\left(2y^4 + 1\right)x}{\left(1 + x^2y^2\right)^2} = 0 \quad \to \quad x = 0$$

$$f_y = \frac{2\left(4 + 2x^4\right)y}{\left(2 + 2x^2y^2\right)^2}$$

$$= \frac{\left(2 + x^4\right)y}{\left(1 + x^2y^2\right)^2} = 0 \quad \to \quad y = 0$$



C.P.: $(0,\ 0)$

@ $y = x \qquad f(x,\ y) = \dfrac{x^2}{2 + 2x^4} = 0$

$$f' = \frac{4x - 4x^5}{\left(2 + 2x^4\right)^2}$$

$$= \frac{4x\left(1 - x^4\right)}{\left(2 + 2x^4\right)^2} = 0 \quad \rightarrow \quad x = 0, \pm 1$$

$$y = x = 0 \quad \rightarrow \quad f(0,\ 0) = 0$$

$$y = x = \pm 1 \quad \rightarrow \quad f(1,\ 1) = \frac{1}{4}$$

@ $y = 2x$ $\quad f(x,\ y) = \dfrac{7x^2}{2 + 8x^4}$

$$f' = \frac{28x - 112x^5}{\left(2 + 8x^4\right)^2}$$

$$= \frac{28x\left(1 - 8x^4\right)}{\left(2 + 8x^4\right)^2} = 0 \quad \rightarrow \quad x = 0,\ \pm\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$y = 2x = \sqrt{2} \quad \rightarrow \quad f\left(\frac{\sqrt{2}}{2},\ \sqrt{2}\right) = \frac{7}{8}$$

@ $y = 2$ $\quad f(x,\ y) = \dfrac{-x^2 + 8}{8x^2 + 2}$

$$f' = \frac{-132x}{\left(8x^2 + 2\right)^2} = 0 \quad \rightarrow \quad \underline{x = 0}|$$

$$\left(\frac{ax^n + b}{cx^n + d}\right)' = \frac{n(ad - bc)x^{n-1}}{\left(cx^n + d\right)^2}$$

$$y = x = 2 \quad \rightarrow \quad f(2,\ 2) = \frac{4}{34} = \frac{2}{17}$$

$$y = 2x = 2 \Rightarrow x = 1 \quad \rightarrow \quad f(1,\ 2) = \frac{7}{10}$$

Absolute ***minimum***: $\quad \underline{f(0,\ 0) = 0}|$

Absolute ***maximum***: $\quad \underline{f\left(\frac{\sqrt{2}}{2},\ \sqrt{2}\right) = \frac{7}{8}}|$

Exercise

Find the absolute maximum and minimum values of the function on the specified region R.

$f(x,\ y) = \sqrt{x^2 + y^2}$; R is the closed region bounded by the ellipse $\dfrac{x^2}{4} + y^2 = 1$

Solution

$$f_x = \frac{x}{\sqrt{x^2+y^2}} = 0 \quad \rightarrow \quad x = 0$$

$$f_y = \frac{y}{\sqrt{x^2+y^2}} = 0 \quad \rightarrow \quad y = 0$$

C.P.: $(0,\ 0)$

$$\frac{x^2}{4} + y^2 = 1 \qquad \begin{cases} x = 2\cos t \\ y = \sin t \end{cases}$$

$$\begin{aligned} f(x,\ y) = g(t) &= \sqrt{4\cos^2 t + \sin^2 t} \\ &= \sqrt{3\cos^2 t + 1} \end{aligned}$$

$$\begin{aligned} g' &= \frac{-3\cos t \sin t}{\sqrt{3\cos^2 t + 1}} \\ &= -\frac{3}{2}\frac{\sin 2t}{\sqrt{3\cos^2 t + 1}} = 0 \end{aligned}$$

$$2t = n\pi \quad \rightarrow \quad t = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$$

t	(x, y)	$f(x, y)$
	$(0,\ 0)$	0
0	$(2,\ 0)$	2
$\frac{\pi}{2}$	$(0,\ 1)$	1
π	$(-2,\ 0)$	2
$\frac{3\pi}{2}$	$(0,\ -1)$	1

Absolute ***minimum***: $f(0,\ 0) = 0$

Absolute ***maximum***: $f(-2,\ 0) = f(2,\ 0) = 2$

Exercise

Find the absolute maximum and minimum values of the function on the specified region *R*.

$$f(x,\ y) = x^2 + y^2 - 4; \quad R = \left\{(x,\ y):\ x^2 + y^2 < 4\right\}$$

Solution

$$f_x = 2x = 0 \quad \rightarrow \quad x = 0$$

$$f_y = 2y = 0 \quad \rightarrow \quad y = 0$$

C.P.: $(0,\ 0)$

$f(0,0) = -4$

$y^2 + x^2 = 4 \qquad \begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}$

$f(x,\ y) = g(t) = 4\cos^2 t + 4\sin^2 t - 4$

$\qquad\qquad = 0$ No extreme points.

$f(x,\ y) \ge -4$

Absolute ***minimum***: $\underline{f(0,\ 0) = -4}|$

Exercise

Find the absolute maximum and minimum values of the function on the specified region *R*.

$$f(x,\ y) = x + 3y; \quad R = \{(x,\ y):\ |x| < 1,\ |y| < 2\}$$

Solution

$f_x = 1 \ne 0$

$f_y = 3 \ne 0$

C.P.: *None*

$-1 < x < 1 \quad -2 < y < 2$

(x, y)	$f(x, y)$
$(-1,\ -2)$	-7
$(1,\ -2)$	-5
$(1,\ 2)$	7
$(-1,\ 2)$	5

The range of the function $f(x,\ y)$ on *R* is the interval $(-7,\ 7)$.

$\therefore\ f(x,\ y)$ has ***neither*** an absolute minimum or maximum on *R*.

Exercise

Find the absolute maximum and minimum values of the function on the specified region *R*.

$$f(x,\ y) = 2e^{-x-y}; \quad R = \{(x,\ y):\ x \ge 0,\ y \ge 0\}$$

Solution

$f_x = -2e^{-x-y} \ne 0$

$f_y = -2e^{-x-y} \neq 0$

C.P.: *None*

$R = \{(x,\ y):\ x \ge 0,\ y \ge 0\}$

$f(0,\ 0) = 2$

$(x,\ y) \to \infty \quad \Rightarrow \quad f(x,\ y) \to 0$

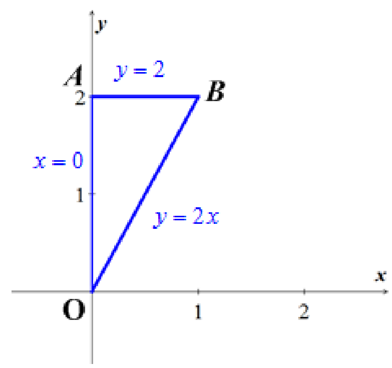
Absolute ***minimum***: *None*

Absolute ***maximum***: $\underline{f(0,\ 0) = 2}|$

Exercise

Find the absolute maxima and minima of the function $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0, \quad y = 2, \quad y = 2x$ in the first quadrant.

Solution



$f_x = 4x - 4 = 0 \quad f_y = 2y - 4 = 0$

$x = 1 \quad y = 2$

The critical point is $(1,\ 2)$ and the value is $f(1,\ 2) \underline{= -5}|$

i. On the segment OA. The function $f(0,\ y) = y^2 - 4y + 1$

This function is defined on the closed interval $0 \le y \le 2$.

$$f'(0,\ y) = 2y - 4 = 0 \quad \to \quad y = 2$$

$$\begin{cases} y = 0 & \to f(0,\ 0) = \underline{1}| \\ y = 2 & \to f(0,\ 2) = \underline{-3}| \end{cases}$$

ii. On the segment OB

$$f(x,\ 2x) = 2x^2 - 4x + (2x)^2 - 4(2x) + 1 = 6x^2 - 12x + 1 \qquad 0 \le x \le 1$$

$$f'(x, 2x) = 12x - 12 = 0 \quad \to \quad x = 1$$

$$\begin{cases} x = 0 & \to f(0,\ 0) = \underline{1}| \\ x = 1 & \to f(1,\ 2) = \underline{-5}| \end{cases} \qquad \therefore\ (1,\ 2) \text{ is not interior point of } OB$$

iii. On the segment AB

$$f(x,\ 2) = 2x^2 - 4x + (2)^2 - 4(2) + 1 = 2x^2 - 4x - 3 \qquad 0 \le x \le 1$$

$$f'(x, 2) = 4x - 4 = 0 \quad \to \quad x = 1$$

$$\begin{cases} x=0 & \to f(0,\ 2)=\underline{-3}| \\ x=1 & \to f(1,\ 2)=\underline{-5}| \end{cases}$$

$\Rightarrow (1,\ 2)$ is not interior point of triangular region.

Therefore; the absolute ***maximum*** is 1 at (0, 0) and the absolute ***minimum*** is −5 at $(1,\ 2)$

Exercise

Find the absolute maxima and minima of the function $D(x,y)=x^2-xy+y^2+1$ on the closed triangular plate bounded by the lines $x=0,\quad y=4,\quad y=x$ in the first quadrant.

Solution

$D_x=2x-y=0,\quad D_y=-x+2y=0,\quad \Rightarrow x=y=0$

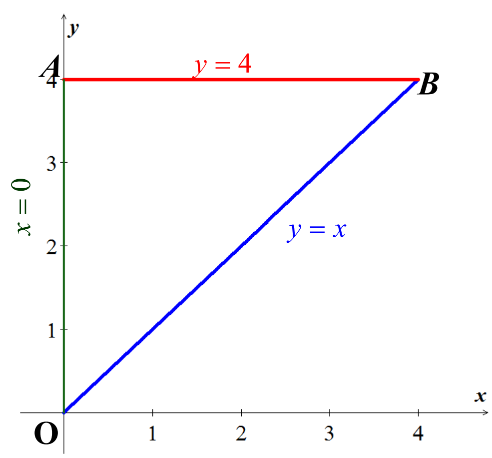
The critical point is $(0,\ 0)$ and the value is $D(0,\ 0)\ \underline{=1}|$

i. On the segment OA.

$D(0,\ y)=y^2+1,\quad 0\le y\le 4$

$D'(0,\ y)=2y=0\quad \to\quad y=0$

$$\begin{cases} y=0 & \to D(0,\ 0)=\underline{1}| \\ y=4 & \to D(0,\ 4)=\underline{17}| \end{cases}$$



ii. On the segment OB

$D(x,\ x)=x^2+1\qquad 0\le x\le 4$

$D'(x,x)=2x=0\quad \to\quad x=0$

$x=0\quad \to D(0,\ 0)=\underline{1}|$

iii. On the segment AB

$D(x,4)=x^2-4x+17\qquad 0\le x\le 4$

$D'(x,2)=2x-4=0\quad \to\quad x=2$

$$\begin{cases} x=2 & \to D(2,\ 4)=13 \\ x=4 & \to D(4,\ 4)=\underline{17}| \end{cases}$$

$\Rightarrow (0,0)$ is not interior point of triangular region.

Therefore; the absolute ***maximum*** is 11 at $(0,4)$ *and* $(4,4)$ and the absolute ***minimum*** is 1 at $(0,0)$

Exercise

Find the absolute maxima and minima of the function $T(x,y)=x^2+xy+y^2-6x+2$ on the triangular plate $0\le x\le 5,\quad -3\le y\le 0$.

Solution

$$T_x=2x+y-6=0,\quad T_y=x+2y=0$$

$$\begin{cases}2x+y=6\\ x+2y=0\end{cases}\ \rightarrow\ \boxed{x=4,\ y=-2}$$

The critical point is $(4,-2)$ and the value is $T(4,-2)\ \underline{=-10}|$

i. On the segment *OA*.

$$T(0,\ y)=y^2+2,\quad -3\le y\le 0$$

$$T'(0,\ y)=2y=0\ \rightarrow\ y=0$$

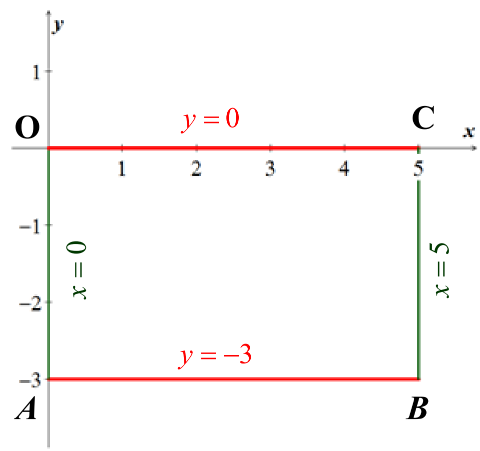
$$\begin{cases}y=0 & \rightarrow T(0,\ 0)=\underline{2}|\\ y=-3 & \rightarrow T(0,-3)=\underline{11}|\end{cases}$$

ii. On the segment *AB*

$$T(x,-3)=x^2-9x+11\qquad 0\le x\le 5$$

$$T'(x,-3)=2x-9=0\ \rightarrow\ x=\frac{9}{2}$$

$$\begin{cases}x=\frac{9}{2} & \rightarrow T\left(\frac{9}{2},-3\right)=\underline{-\frac{37}{4}}|\\ x=0 & \rightarrow T(0,-3)=\underline{11}|\end{cases}$$



iii. On the segment *BC*

$$T(5,y)=y^2+5y-3\qquad -3\le y\le 0$$

$$T'(5,y)=2y+5=0\ \rightarrow\ y=-\frac{5}{2}$$

$$\begin{cases}y=0 & \rightarrow T(5,0)=\underline{-3}|\\ y=-\frac{5}{2} & \rightarrow T\left(5,-\frac{5}{2}\right)=\underline{-\frac{37}{4}}|\\ y=-3 & \rightarrow T(5,-3)=\underline{-9}|\end{cases}$$

iv. On the segment *CO*

$$T(x,0)=x^2-6x+2\qquad 0\le x\le 5$$

$$T'(x,0)=2x-6=0\ \rightarrow\ x=3$$

$$(3,0)\ \rightarrow T(3,0)=\underline{-7}|$$

Therefore; the absolute ***maximum*** is 11 at $(0,-3)$ and the absolute ***minimum*** is -10 at $(4,-2)$

Exercise

Find the absolute maxima and minima of the function $f(x,y)=\left(4x-x^2\right)\cos y$ on the triangular plate $1\le x\le 3,\quad -\frac{\pi}{4}\le y\le \frac{\pi}{4}$.

Solution

$$f_x=(4-2x)\cos y=0,\quad f_y=\left(x^2-4x\right)\sin y=0$$

$$\begin{cases}(4-2x)\cos y=0 & \to x=2,\ y=\dfrac{(n+1)\pi}{2}\\ x(x-4)\sin y=0 & \quad\to x=0,4,\ y=n\pi\end{cases}$$

$\boxed{x=2,\ y=0}$ *because* $1\le x\le 3,\quad -\frac{\pi}{4}\le y\le \frac{\pi}{4}$

The critical point is $(2,0)$ and the value is $f(2,0)=4$

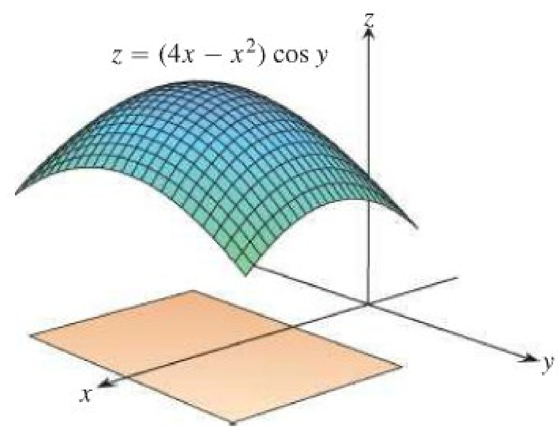
Values of all 4 corner points:

$$A\left(1,-\tfrac{\pi}{4}\right)\ \to f\left(1,-\tfrac{\pi}{4}\right)=\frac{3\sqrt{2}}{2}$$

$$B\left(1,\tfrac{\pi}{4}\right)\ \to f\left(1,\tfrac{\pi}{4}\right)=\frac{3\sqrt{2}}{2}$$

$$C\left(3,\tfrac{\pi}{4}\right)\ \to f\left(3,\tfrac{\pi}{4}\right)=\frac{3\sqrt{2}}{2}$$

$$A\left(3,-\tfrac{\pi}{4}\right)\ \to f\left(3,-\tfrac{\pi}{4}\right)=\frac{3\sqrt{2}}{2}$$

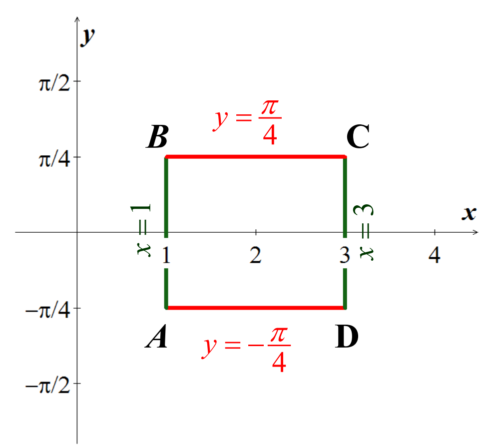


i. On the segment *AB*

$$f(1,y)=3\cos y\qquad -\frac{\pi}{4}\le y\le\frac{\pi}{4}$$

$$f'(1,y)=-3\sin y=0\ \to\ y=0$$

$$x=1\ \to f(1,0)=3$$



ii. On the segment *BC*

$$f\left(x,\tfrac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\left(4x-x^2\right)\qquad 1\le x\le 3$$

$$f'\left(x,\tfrac{\pi}{4}\right)=\frac{\sqrt{2}}{2}(4-2x)=0\ \Rightarrow\ x=2$$

$$x=2\ \to f\left(2,\tfrac{\pi}{4}\right)=2\sqrt{2}$$

iii. On the segment *CD*

$$f(3,y)=3\cos y\qquad -\frac{\pi}{4}\le y\le\frac{\pi}{4}$$

$$f'(3,y)=-3\sin y=0\ \to\ y=0$$

iv. On the segment *DA*

$$f\left(x,-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\left(4x-x^2\right) \qquad 1\le x\le 3$$

$$f'\left(x,-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}(4-2x)=0 \quad\Rightarrow\quad x=2$$

Therefore; the absolute ***maximum*** is 4 at $(2,0)$ and the absolute ***minimum*** is $\frac{3\sqrt{2}}{2}$ at $\left(1,-\frac{\pi}{4}\right)$, $\left(1,\frac{\pi}{4}\right)$, $\left(3,-\frac{\pi}{4}\right)$, *and* $\left(3,\frac{\pi}{4}\right)$

Exercise

Find the point on the graph of $z=x^2+y^2+10$ nearest the plane $x+2y-z=0$

Solution

The point on $z=x^2+y^2+10$ where the tangent plane is parallel to the plane $x+2y-z=0$.

Let $w=z-x^2-y^2-10 \quad\to\quad \nabla w=-2x\boldsymbol{i}-2y\boldsymbol{j}+\boldsymbol{k}$ is normal to $z=x^2+y^2+10$ at (x,y).

The vector ∇w is parallel to $\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$ which is normal to the plane if $x=\frac{1}{2}$ *and* $y=1$

$(-2x=-1 \text{ and } -2y=-2)$, $z=\left(\frac{1}{2}\right)^2+1^2+10=\frac{45}{4}$

Thus, the point $\left(\frac{1}{2},\ 1,\ \frac{45}{4}\right)$ is the point on the surface $z=x^2+y^2+10$ nearest the plane $x+2y-z=0$

Exercise

Find the minimum distance from the point $(2,\ -1,\ 1)$ to the plane $x+y-z=2$

Solution

$$d(x,y,z)=\sqrt{(x-2)^2+(y+1)^2+(z-1)^2}$$

$$x+y-z=2 \quad\Rightarrow\quad z=x+y-2$$

Let: $D(x,y,z)=(x-2)^2+(y+1)^2+(z-1)^2$

$$D(x,y)=(x-2)^2+(y+1)^2+(x+y-2-1)^2$$
$$=(x-2)^2+(y+1)^2+(x+y-3)^2$$

$$D_x=2(x-2)+2(x+y-3)$$
$$=4x+2y-10=0$$

$$D_y=2(y+1)+2(x+y-3)$$
$$=2x+4y-4=0$$

$$\begin{cases} 4x+2y=10 \\ 2x+4y=4 \end{cases} \Rightarrow \boxed{x=\frac{8}{3},\ \ y=-\frac{1}{3}}$$

$\therefore$ The critical point is $\left(\frac{8}{3},\ -\frac{1}{3}\right)$.

$$\underline{\left| z=\frac{8}{3}-\frac{1}{3}-2=\frac{1}{3} \right|}$$

$$D_{xx}\Big|_{\left(\frac{8}{3},-\frac{1}{3}\right)}=4, \quad D_{yy}\Big|_{\left(\frac{8}{3},-\frac{1}{3}\right)}=4, \quad D_{xy}\Big|_{\left(\frac{8}{3},-\frac{1}{3}\right)}=2$$

$$D_{xx}D_{yy}-D_{xy}^2=(4)(4)-2^2=12>0 \quad \textit{and} \quad D_{xx}>0$$

Therefore, the local ***minimum*** of the distance is

$$d\left(\frac{8}{3},-\frac{1}{3},\frac{1}{3}\right)=\sqrt{\left(\frac{8}{3}-2\right)^2+\left(-\frac{1}{3}+1\right)^2+\left(\frac{1}{3}-1\right)^2}$$

$$\underline{=\frac{2}{\sqrt{3}}\ \Big|}$$

Exercise

Find the maximum value of $s=xy+yz+xz$ where $x+y+z=6$

Solution

$$x+y+z=6 \quad\Rightarrow\quad z=6-x-y$$

$$s(x,y,z)=xy+yz+xz$$

$$\begin{aligned} s(x,y)&=xy+y(6-x-y)+x(6-x-y) \\ &=xy+6y-xy-y^2+6x-x^2-xy \\ &=-x^2-y^2+6y+6x-xy \end{aligned}$$

$$s_x=-2x+6-y=0 \qquad s_y=-2y+6-x=0$$

$$\begin{cases} 2x+y=6 \\ x+2y=6 \end{cases} \Rightarrow \boxed{x=2,\ \ y=2}$$

$\therefore$ The critical point is $(2,\ 2)$.

$$\underline{\left| z=6-2-2=2 \right|}$$

$$s_{xx}\Big|_{(2,2)}=-2, \quad s_{yy}\Big|_{(2,2)}=-2, \quad s_{xy}\Big|_{(2,2)}=-1$$

$$s_{xx}s_{yy}-s_{xy}^2=(-2)(-2)-(-1)^2=3>0 \quad \textit{and} \quad s_{xx}<0$$

Therefore, the local *maximum* of the distance is

$$s(2,\ 2,\ 2) = (2)(2) + (2)(2) + (2)(2)$$
$$= 12$$

Exercise

Among all triangles with a perimeter of 9 *units*, find the dimensions of the triangle with the maximum area. It may be easiest to use Heron's formula, which states that the area of a triangle with side length a, b, and c is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $2s$ is the perimeter of the triangle.

Solution

The semi-perimeter is: $\quad s = \dfrac{a+b+c}{2}$

$$c = 2s - a - b$$

$$\begin{aligned} A^2 &= s(s-a)(s-b)(s-2s+a+b) \\ &= s(s-a)(s-b)(a+b-s) \end{aligned}$$

$$f(a,\ b) = s(s-a)(s-b)(a+b-s)$$

$$\begin{aligned} f_a &= s(s-b)(-a-b+s) + s(s-a)(s-b) \\ &= s(s-b)(-a-b+s+s-a) \\ &= s(s-b)(2s-2a-b) = 0 \end{aligned}$$

$$\begin{aligned} f_a &= s(s-a)(-a-b+s) + s(s-a)(s-b) \\ &= s(s-a)(-a-b+s+s-b) \\ &= s(s-a)(2s-a-2b) = 0 \end{aligned}$$

$$\begin{cases} (s-b)(2s-2a-b) = 0 & \to b = s \quad 2a+b = 2s \\ (s-a)(2s-a-2b) = 0 & \to a = s \quad a+2b = 2s \end{cases}$$

$$\begin{cases} 2a+b = 2s \\ a+2b = 2s \end{cases} \qquad \Delta = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 \quad \Delta_a = \begin{vmatrix} 2s & 1 \\ 2s & 2 \end{vmatrix} = 2s \quad \Delta_b = \begin{vmatrix} 2 & 2s \\ 1 & 2s \end{vmatrix} = 2s$$

$$\to \quad a = b = \frac{2}{3}s$$

$$\begin{aligned} c &= 2s - 2\frac{2}{3}s \\ &= \frac{2}{3}s \end{aligned}$$

$a = b = c = \dfrac{2}{3}s$ $\therefore$ Equilateral triangle

The maximum area is obtained when all three sides are equal with each side length is 3 *units* (since the perimeter is 9 *units*).

Exercise

Let P be a plane tangent to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at a point in the first octane. Let T be the tetrahedron in the first octant bounded by P and the coordinate planes $x=0$, $y=0$, and $z=0$. Find the minimum volume T. (the volume of a tetrahedron in one-third the area of the base times the height.)

Solution

Let $Q\left(x_0, y_0, z_0\right)$ be a point on the ellipsoid.

The tangent plane P at the point Q has an equation:

$$\frac{x_0}{a^2}x+\frac{y_0}{b^2}y+\frac{z_0}{c^2}z=1$$

The intersection points of the plane with the axes are:

$$\left(\frac{a^2}{x_0},\ 0,\ 0\right),\ \left(0,\ \frac{b^2}{y_0},\ 0\right),\ \text{and}\ \left(0,\ 0,\ \frac{c^2}{z_0}\right)$$

$\therefore$ The tetrahedron T has base area

$$A=\frac{a^2b^2}{2x_0y_0}$$

Height: $h=\frac{c^2}{z_0}$

$$\therefore\ V=\frac{1}{3}\frac{a^2b^2c^2}{2x_0y_0z_0}$$

$$=\frac{1}{6}\frac{a^2b^2c^2}{x_0y_0z_0}$$

$$\frac{x_0}{a^2}x+\frac{y_0}{b^2}y+\frac{z_0}{c^2}z=1\ \rightarrow\ \frac{1}{a^2}x_0^2+\frac{1}{b^2}y_0^2+\frac{1}{c^2}z_0^2=1$$

$$z_0^2=c^2\left(1-\frac{1}{a^2}x_0^2-\frac{1}{b^2}y_0^2\right)$$

$$V=\frac{1}{6}\frac{a^2b^2c^2}{\left(x_0y_0c\right)\sqrt{\frac{a^2b^2-b^2x_0^2-a^2y_0^2}{a^2b^2}}}$$

$$=\frac{a^3b^3c}{6}\frac{1}{x_0y_0\sqrt{a^2b^2-b^2x_0^2-a^2y_0^2}}$$

$$=\frac{a^3b^3c}{6}\left(a^2b^2x_0^2y_0^2-b^2x_0^4y_0^2-a^2x_0^2y_0^4\right)^{-1/2}$$

$$V_{x_0} = -\frac{a^3b^3c}{12}\frac{2a^2b^2x_0y_0^2 - 4b^2x_0^3y_0^2 - 2a^2x_0y_0^4}{\left(a^2b^2x_0^2y_0^2 - b^2x_0^4y_0^2 - a^2x_0^2y_0^4\right)^{3/2}} = 0$$

$$V_{y_0} = -\frac{a^3b^3c}{12}\frac{2a^2b^2x_0^2y_0 - 2b^2x_0^4y_0 - 4a^2x_0^2y_0^3}{\left(a^2b^2x_0^2y_0^2 - b^2x_0^4y_0^2 - a^2x_0^2y_0^4\right)^{3/2}} = 0$$

$$\begin{cases} 2x_0y_0^2\left(a^2b^2 - 2b^2x_0^2 - a^2y_0^2\right) = 0 \\ 2x_0^2y_0\left(a^2b^2 - b^2x_0^2 - 2a^2y_0^2\right) = 0 \end{cases}$$

$$\begin{cases} 2b^2x_0^2 + a^2y_0^2 = a^2b^2 \\ b^2x_0^2 + 2a^2y_0^2 = a^2b^2 \end{cases}$$

$$\Delta = \begin{vmatrix} 2b^2 & a^2 \\ b^2 & 2a^2 \end{vmatrix} = 3a^2b^2 \quad \Delta_x = \begin{vmatrix} a^2b^2 & a^2 \\ a^2b^2 & 2a^2 \end{vmatrix} = a^4b^2 \quad \Delta_y = \begin{vmatrix} 2b^2 & a^2b^2 \\ b^2 & a^2b^2 \end{vmatrix} = a^2b^4$$

$$x_0^2 = \frac{a^4b^2}{3a^2b^2} = \frac{a^2}{3}, \quad y_0^2 = \frac{a^2b^4}{3a^2b^2} = \frac{b^2}{3}$$

$$\begin{aligned} z_0^2 &= c^2\left(1 - \frac{1}{a^2}\frac{a^2}{3} - \frac{1}{b^2}\frac{b^2}{3}\right) \\ &= c^2\left(1 - \frac{1}{3} - \frac{1}{3}\right) \\ &= \frac{1}{3}c^2 \end{aligned}$$

$$x_0 = \frac{a}{\sqrt{3}}, \quad y_0 = \frac{b}{\sqrt{3}}, \quad z_0 = \frac{c}{\sqrt{3}}$$

$$V = \frac{1}{6}\frac{a^2b^2c^2}{\frac{a}{\sqrt{3}}\frac{b}{\sqrt{3}}\frac{c}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}}{2}abc$$

Exercise

Given three distinct noncollinear points A, B, and C in the plane, find the point P in the plane such the sum of the distances $|AP|+|BP|+|CP|$ is a minimum. Here is how to procees with three points, assuming that the triangle formed by the three points has no angle greater than $\left(120° = \frac{2\pi}{3}\right)$

a) Assume the coordinates of the threee given points are $A\left(x_1, y_1\right)$, $B\left(x_2, y_2\right)$, and $C\left(x_3, y_3\right)$. Let $d_1(x, y)$ be the distance between $A\left(x_1, y_1\right)$ and a variable point $P(x, y)$. Compute the gradient of d_1 and show that it is a unit vector pointing along the line between the two points.

b) Define d_2 and d_3 in a similar way and show that ∇d_2 and ∇d_3 are also unit vectors in the direction of line between the two points.

c) The goal is to minimize $f(x, y)=d_1+d_2+d_3$. Show that the condition $f_x=f_y=0$ implies that $\nabla d_1+\nabla d_2+\nabla d_3=0$.

d) Explain why part (*c*) implies that the optimal point P has the property the three line segments AP, BP, and CP all intersect symmetrically in angles of $\frac{2\pi}{3}$.

e) What is the optimal solution if one of the angles in the triangle is greater than $\frac{2\pi}{3}$ (draw a picture)?

f) Estimate the Steiner point for the three points $(0,\ 0)$, $(0,\ 1)$, $(2,\ 0)$

Solution

a) $d_1(x, y)=\sqrt{\left(x-x_1\right)^2+\left(y-y_1\right)^2}$

$$\nabla d_1(x, y)=\frac{x-x_1}{\sqrt{\left(x-x_1\right)^2+\left(y-y_1\right)^2}}\hat{i}+\frac{y-y_1}{\sqrt{\left(x-x_1\right)^2+\left(y-y_1\right)^2}}\hat{j}$$

$$=\frac{x-x_1}{d_1(x, y)}\hat{i}+\frac{y-y_1}{d_1(x, y)}\hat{j}$$

$$\left|\nabla d_1(x, y)\right|=\frac{1}{d_1(x, y)}\sqrt{\left(x-x_1\right)^2+\left(y-y_1\right)^2}$$

$$=\frac{d_1(x, y)}{d_1(x, y)}$$

$$=1$$

$\therefore$ The gradient of d_1 is a unit vector

b) $d_2(x, y)=\sqrt{\left(x-x_2\right)^2+\left(y-y_2\right)^2}$

$$\nabla d_2(x, y) = \frac{x - x_2}{d_2(x, y)}\hat{i} + \frac{y - y_2}{d_2(x, y)}\hat{j}$$

$$\begin{aligned}\left|\nabla d_2(x, y)\right| &= \frac{1}{d_2(x, y)}\sqrt{(x - x_2)^2 + (y - y_2)^2} \\ &= \frac{d_2(x, y)}{d_2(x, y)} \\ &= 1\end{aligned}$$

$\therefore\ \nabla d_2$ is a unit vector

$$d_3(x, y) = \sqrt{(x - x_3)^2 + (y - y_3)^2}$$

$$\nabla d_3(x, y) = \frac{x - x_3}{d_3(x, y)}\hat{i} + \frac{y - y_3}{d_3(x, y)}\hat{j}$$

$$\begin{aligned}\left|\nabla d_3(x, y)\right| &= \frac{1}{d_3(x, y)}\sqrt{(x - x_3)^2 + (y - y_3)^2} \\ &= \frac{d_3(x, y)}{d_3(x, y)} \\ &= 1\end{aligned}$$

$\therefore\ \nabla d_3$ is a unit vector

c) $f(x, y) = d_1 + d_2 + d_3$

$\nabla f = \nabla d_1 + \nabla d_2 + \nabla d_3$

Given that $f_x = f_y = 0$

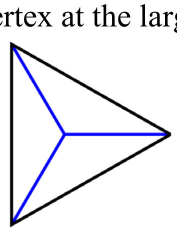
$$\begin{aligned}\nabla f &= f_x\hat{i} + f_y\hat{j} = 0 \\ &= \nabla d_1 + \nabla d_2 + \nabla d_3\end{aligned}$$

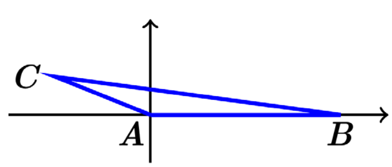
$\nabla d_1 + \nabla d_2 + \nabla d_3 = 0$ ✓

d) Since $\nabla d_1 + \nabla d_2 + \nabla d_3 = 0$ that implies all the 3 *unit* vectors add to 0.

Therefore, all three divide the unit circle into 3 equal sectors, they must make angles of $\pm\frac{2\pi}{3}$.

e) The optimal point is the vertex at the large angle.





f) Three points $(0,\ 0),\ (0,\ 1),\ (2,\ 0)$

$$f_x = f_y = 0$$

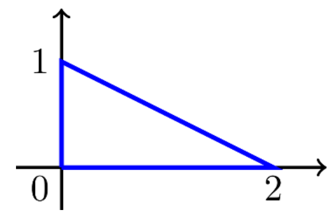
$$f(x,\ y) = d_1 + d_2 + d_3$$

$$= \sqrt{\left(x-x_1\right)^2 + \left(y-y_1\right)^2} + \sqrt{\left(x-x_2\right)^2 + \left(y-y_2\right)^2} + \sqrt{\left(x-x_3\right)^2 + \left(y-y_3\right)^2}$$

$$= \sqrt{x^2+y^2} + \sqrt{x^2+(y-1)^2} + \sqrt{(x-2)^2+y^2}$$

$$f_x = \frac{x}{\sqrt{x^2+y^2}} + \frac{x}{\sqrt{x^2+(y-1)^2}} + \frac{x-2}{\sqrt{(x-2)^2+y^2}} = 0$$

$$f_y = \frac{y}{\sqrt{x^2+y^2}} + \frac{y-1}{\sqrt{x^2+(y-1)^2}} + \frac{y}{\sqrt{(x-2)^2+y^2}} = 0$$



Using maple:

$$solvefor\left(\left\{\frac{x}{\sqrt{x^2+y^2}} + \frac{x}{\sqrt{x^2+(y-1)^2}} + \frac{x-2}{\sqrt{(x-2)^2+y^2}} = 0,\ \frac{y}{\sqrt{x^2+y^2}} + \frac{y-1}{\sqrt{x^2+(y-1)^2}} + \frac{y}{\sqrt{(x-2)^2+y^2}} = 0\right\}\right)$$

$$x = \frac{1}{13} + \frac{4\sqrt{3}}{39}\ \approx 0.25456931$$

$$y = \frac{8}{13} - \frac{7\sqrt{3}}{39}\ \approx 0.30450371$$

Exercise

Show that the following two functions have two local maxima but no other extreme points (thus no saddle or basin between the mountains).

$$f(x,\ y) = -\left(x^2-1\right)^2 - \left(x^2-e^y\right)^2$$

Solution

$$f_x = -4x\left(x^2-1\right) - 4x\left(x^2-e^y\right)$$

$$= -4x^3 + 4x - 4x^3 + 4xe^y$$

$$= -8x^3 + 4x\left(1+e^y\right) = 0$$

$$f_y = 2e^y\left(x^2-e^y\right) = 0$$

$$\begin{cases} -4x\left(2x^2-1-e^y\right)=0 & \rightarrow x=0, \quad e^y=2x^2-1 \ (1) \\ 2e^y\left(x^2-e^y\right)=0 & \qquad e^y=x^2 \ (2) \end{cases}$$

$$(1) \rightarrow e^y = 2x^2-1\Big|_{x=0} = \cancel{-1}$$

$$from\ (2) \rightarrow (1):\ x^2 = 2x^2-1$$

$$x^2 = 1 \rightarrow \underline{x=\pm 1}$$

$$e^y = x^2\Big|_{x=\pm 1} = 1 \rightarrow \underline{y=0}$$

$\therefore$ ***C.P.***: $(\pm 1,\ 0)$

$$f_{xx} = -24x^2 + 4\left(1+e^y\right)$$

$$f_{yy} = 2x^2e^y - 4e^{2y}$$

$$f_{xy} = 4xe^y$$

$$f_{xx}f_{yy} - f_{xy}^2 = \left(-24x^2+4+4e^y\right)\left(2x^2e^y-4e^{2y}\right) - 16x^2e^{2y}$$

@ $(-1,\ 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = (-24+4+4)(2-4) - 16 = 16 > 0$$

$$f_{xx} = -24+8 = -16 < 0$$

f has a ***local Max*** @ $(-1,\ 0)$

@ $(1,\ 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = (-24+4+4)(2-4) - 16 = 16 > 0$$

$$f_{xx} = -24+8 = -16 < 0$$

f has a ***local Max*** @ $(1,\ 0)$

Exercise

Show that the following two functions have two local maxima but no other extreme points (thus no saddle or basin between the mountains).

$$f(x,\ y) = 4x^2e^y - 2x^4 - e^{4y}$$

Solution

$$f_x = 8xe^y - 8x^3 = 0$$

$$f_y = 4x^2e^y - 4e^{4y} = 0$$

$$\begin{cases} 8x\left(e^y - x^2\right) = 0 & \rightarrow x = 0, \quad e^y = x^2 \quad (1) \\ 4e^y\left(x^2 - e^{3y}\right) = 0 & \qquad e^{3y} = x^2 \quad (2) \end{cases}$$

$$(1) \rightarrow \ e^y = x^2\Big|_{x=0} = \cancel{0}$$

$$\textit{from } (2) \rightarrow (1): \ e^{3y} = e^y \ \Rightarrow \ 3y = y$$

$$\underline{y = 0}\,\Big| \ \rightarrow e^y = x^2 = 1 \quad \underline{x = \pm 1}\,\Big|$$

$\therefore$ ***C.P.***: $(\pm 1, \ 0)$

$$f_{xx} = 8e^y - 24x^2$$

$$f_{yy} = 4x^2e^y - 16e^{4y}$$

$$f_{xy} = 8xe^y$$

$$f_{xx}f_{yy} - f_{xy}^2 = \left(8e^y - 24x^2\right)\left(4x^2e^y - 16e^{4y}\right) - 64x^2e^{2y}$$

@ $(-1, \ 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = (8-24)(4-16) - 64 = 128 > 0$$

$$f_{xx} = 8 - 24 = -16 < 0$$

f has a ***local Max*** @ $(-1, \ 0)$

@ $(1, \ 0)$

$$f_{xx}f_{yy} - f_{xy}^2 = (8-24)(4-16) - 64 = 128 > 0$$

$$f_{xx} = 8 - 24 = -16 < 0$$

f has a ***local Max*** @ $(1, \ 0)$

Solution Section 2.8 – Lagrange Multipliers

Exercise

Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x,y) = xy$ has its extreme values.

Solution

$$g(x,y) = x^2 + 2y^2 - 1$$

$$\nabla f = y\,\hat{i} + x\,\hat{j}, \quad \nabla g = 2x\,\hat{i} + 4y\,\hat{j}$$

$$\nabla f = \lambda \nabla g$$

$$y\,\hat{i} + x\,\hat{j} = 2\lambda x\,\hat{i} + 4\lambda y\,\hat{j}$$

$$y = 2\lambda x \qquad x = 4\lambda y$$

$$x = 8x\lambda^2$$

$$8x\lambda^2 - x = 0$$

$$x\left(8\lambda^2 - 1\right) = 0 \quad \Rightarrow \quad \begin{cases} \lambda^2 = \frac{1}{8} \to \lambda = \pm\frac{1}{2\sqrt{2}} \\ x = 0 \end{cases}$$

Case 1: If $x = 0 \;\Rightarrow y = 2\lambda x = 0$. But (0, 0) is not on the ellipse so $x \neq 0$

Case 2: If $x \neq 0$ *and* $\lambda = \pm\frac{\sqrt{2}}{4}$

$$\Rightarrow x = 4\lambda y = \pm\sqrt{2}y$$

$$\left(\pm\sqrt{2}y\right)^2 + 2y^2 = 1$$

$$2y^2 + 2y^2 = 1$$

$$y^2 = \frac{1}{4} \quad \Rightarrow \quad \boxed{y = \pm\frac{1}{2}}$$

$$x \pm \sqrt{2}y \quad \Rightarrow \quad \boxed{x = \pm\frac{\sqrt{2}}{2}}$$

$$f(x,y) = xy = \pm\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \pm\frac{\sqrt{2}}{4}$$

Therefore, f has extreme values at $\left(\pm\frac{\sqrt{2}}{2},\ \pm\frac{1}{2}\right)$

The extreme values of f on the ellipse are $\pm\frac{\sqrt{2}}{4}$

Exercise

Find the extreme values of $f(x,y) = xy$ subject to the constraint $g(x,y) = x^2 + y^2 - 10 = 0$.

Solution

$$g(x,y) = x^2 + y^2 - 10$$

$$\nabla f = y\,\hat{i} + x\,\hat{j}, \quad \nabla g = 2x\,\hat{i} + 2y\,\hat{j}$$

$$\nabla f = \lambda \nabla g$$

$$y\,\hat{i} + x\,\hat{j} = 2\lambda x\,\hat{i} + 2\lambda y\,\hat{j}$$

$$y = 2\lambda x \qquad x = 2\lambda y = 4x\lambda^2$$

$$x\left(4\lambda^2 - 1\right) = 0$$

$$\Rightarrow \begin{cases} \lambda^2 = \frac{1}{4} \to \lambda = \pm\frac{1}{2} \\ x = 0 \end{cases}$$

Case 1: If $x = 0 \ \Rightarrow y = 2\lambda x = 0$. But (0, 0) is not on the circle so $x \neq 0$

Case 2: If $x \neq 0$ *and* $\lambda = \pm\frac{1}{2}$

$$\Rightarrow x = 2\lambda y = \pm y$$

$$g(x,y) = x^2 + y^2 - 10 = 0$$

$$(\pm y)^2 + y^2 = 10$$

$$2y^2 = 10$$

$$y^2 = 5 \quad \Rightarrow \quad \underline{y = \pm\sqrt{5} = \pm x}\,\big|$$

$$f(x,y) = xy = \pm\sqrt{5}\left(\sqrt{5}\right) = \pm 5$$

Therefore, f has extreme values at $\left(\pm\sqrt{5},\ \pm\sqrt{5}\right)$

The extreme values of f on the circle are ± 5

Exercise

Find the extreme values of $f(x,y) = x^3 + y^2$ on the circle $x^2 + y^2 = 1$

Solution

$$\nabla f = 3x^2\hat{i} + 2y\hat{j} \qquad \nabla f = f_x\,\hat{i} + f_y\,\hat{j}$$

$$g(x,y) = x^2 + y^2 - 1 \quad \to \quad \nabla g = 2x\hat{i} + 2y\hat{j}$$

$$3x^2\hat{i} + 2y\hat{j} = \lambda\left(2x\hat{i} + 2y\hat{j}\right) \qquad \nabla f = \lambda \nabla g$$

$$= 2\lambda x\hat{i} + 2\lambda y\hat{j}$$

$$\rightarrow \begin{cases} 3x^2 = 2\lambda x \\ 2y = 2\lambda y \quad \rightarrow \lambda = 1 \ \ or \ \ y = 0 \end{cases}$$

For $\lambda = 1$

$$3x^2 = 2x$$

$$x(3x-2) = 0 \quad \rightarrow \quad x = 0, \ \frac{2}{3}$$

$$y = \pm\sqrt{1-x^2}$$

$$\rightarrow \begin{cases} x = 0 & \rightarrow y = \pm 1 \\ x = \frac{2}{3} & \rightarrow y = \pm\sqrt{1-\frac{4}{9}} = \pm\frac{\sqrt{5}}{3} \end{cases}$$

$$(0, \ -1), \ (0, \ 1), \ \left(\frac{2}{3}, \ -\frac{\sqrt{5}}{3}\right) \ \ \& \ \ \left(\frac{2}{3}, \ \frac{\sqrt{5}}{3}\right)$$

For $y = 0$

$$x^2 = 1 \ \Rightarrow \ x = \pm 1$$

$$(\pm 1, \ 0)$$

$$f(x,y) = x^3 + y^2$$

$$f(-1, \ 0) = -1$$

$$f(1, \ 0) = 1$$

$$f(0, \ -1) = 1$$

$$f(0, \ 1) = 1$$

$$f\left(\frac{2}{3}, \ -\frac{\sqrt{5}}{3}\right) = \frac{8}{27} + \frac{5}{9}$$

$$= \frac{23}{27}$$

$$f\left(\frac{2}{3}, \ \frac{\sqrt{5}}{3}\right) = \frac{8}{27} + \frac{5}{9}$$

$$= \frac{23}{27}$$

Absolute Max. is 1 @ $(0, \ \pm 1)$, $(1, \ 0)$

Absolute Min. is -1 @ $(-1, \ 0)$

Exercise

Find the extreme values of $f(x,y)=x^2+y^2-3x-xy$ on the circle $x^2+y^2\le 9$

Solution

$$\nabla f=(2x-3-y)\hat{i}+(2y-x)\hat{j} \qquad \nabla f=f_x\hat{i}+f_y\hat{j}$$

$$g(x,y)=x^2+y^2-9 \quad\to\quad \nabla g=2x\hat{i}+2y\hat{j}$$

$$(2x-3-y)\hat{i}+(2y-x)\hat{j}=2\lambda x\hat{i}+2\lambda y\hat{j} \qquad \nabla f=\lambda\nabla g$$

$$\to\begin{cases}2x-3-y=2x\lambda & \to\ 2x(1-\lambda)-y=3 \quad (1)\\ 2y-x=2y\lambda & \to\ 2y(1-\lambda)=x \quad (2)\end{cases}$$

$$(2)\to\quad 1-\lambda=\frac{x}{2y}$$

$$(1)\to\quad 2x\left(\frac{x}{2y}\right)-y=3$$

$$x^2-y^2=3y$$

$$x^2=y^2+3y$$

$$x^2+y^2=9$$

$$y^2+3y+y^2=9$$

$$2y^2+3y-9=0 \quad\to\quad \underline{y=-3,\ \frac{3}{2}}$$

For $y=-3$

$$x=\pm\sqrt{9-y^2}=0$$

$$\therefore (0,\ -3)$$

For $y=\frac{3}{2}$

$$x=\pm\sqrt{9-\frac{9}{4}}=\pm\frac{3\sqrt{3}}{2}$$

$$\therefore \left(\pm\frac{3\sqrt{3}}{2},\ \frac{3}{2}\right)$$

$$\begin{cases}f_x=2x-3-y=0 & 3y=3\to \underline{y=1}\\ f_y=2y-x=0 & \to x=2y\end{cases}$$

$\therefore (2,\ 1)$ ***C.P.***

$$f(0,\ -3)=9$$

$$f(2,\ 1)=-3$$

$$f\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) = 9 - \frac{27\sqrt{3}}{4}$$

$$= \frac{36 - 27\sqrt{3}}{4}$$

$$f\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) = 9 + \frac{27\sqrt{3}}{4}$$

$$= \frac{36 + 27\sqrt{3}}{4}$$

Absolute Max. is $\frac{36 + 27\sqrt{3}}{4}$ @ $\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$

Absolute Min. is -3 @ $(2, 1)$

Exercise

Find the maximum value of $f(x,y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$.

Solution

$$\nabla f = -2x\,\hat{i} - 2y\,\hat{j}, \quad \nabla g = \hat{i} + 3\,\hat{j}$$

$$\nabla f = \lambda \nabla g$$

$$-2x\,\hat{i} - 2y\,\hat{j} = \lambda\,\hat{i} + 3\lambda\,\hat{j}$$

$$-2x = \lambda \qquad -2y = 3\lambda$$

$$x = -\frac{\lambda}{2} \qquad y = -\frac{3\lambda}{2}$$

$$x + 3y = 10$$

$$-\frac{\lambda}{2} + 3\left(-\frac{3\lambda}{2}\right) = 10$$

$$-5\lambda = 10 \quad \Rightarrow \quad \boxed{\lambda = -2}$$

$$x = -\frac{\lambda}{2} = 1$$

$$y = -\frac{3\lambda}{2} = 3$$

$$f(x,y) = 49 - 1^2 - 3^2$$

$$= 39$$

Therefore, f has extreme values at $(1, 3)$.

The extreme values of f is 39

Exercise

Find the points on the curve $x^2y=2$ nearest the origin.

Solution

Let $f(x,y)=x^2+y^2$, the square of the distance to the origin subject to the constraint

$g(x,y)=x^2y-2=0$

$\nabla f=2x\,\hat{i}+2y\,\hat{j},\quad \nabla g=2xy\,\hat{i}+x^2\,\hat{j}$

$\nabla f=\lambda\nabla g$

$2x\,\hat{i}+2y\,\hat{j}=2xy\lambda\,\hat{i}+x^2\lambda\,\hat{j}$

$$2x=2xy\lambda \qquad 2y=x^2\lambda$$

$$y=\frac{1}{\lambda} \qquad x^2=\frac{2y}{\lambda}=\frac{2}{\lambda^2}$$

$x^2y-2=0$

$\left(\frac{2}{\lambda^2}\right)\left(\frac{1}{\lambda}\right)-2=0$

$\frac{2}{\lambda^3}=2 \quad\Rightarrow\quad \lambda^3=1\to\boxed{\lambda=1}$

$\boxed{y=1}\quad x^2=2\Rightarrow\boxed{x=\pm\sqrt{2}}$

$\therefore\ \left(\pm\sqrt{2},\ 1\right)$ are the points on the curve $x^2y=2$ nearest the origin.

Exercise

Use the method of Lagrange multipliers to find

a) The minimum value of $x+y$, subject to the constraints $xy=16,\quad x>0,\quad y>0$

b) The maximum value of xy, subject to the constraints $x+y=16$

Solution

a) $\nabla f=\hat{i}+\hat{j},\quad \nabla g=y\,\hat{i}+x\,\hat{j}$

$\nabla f=\lambda\nabla g$

$\hat{i}+\hat{j}=y\lambda\,\hat{i}+x\lambda\,\hat{j}$

$$1=y\lambda,\quad 1=x\lambda$$

$$y=\frac{1}{\lambda},\quad x=\frac{1}{\lambda}$$

$g(x,y)=xy-16=0$

$$\frac{1}{\lambda^2}-16=0 \quad \Rightarrow \quad \lambda^2=\frac{1}{16} \to \boxed{\lambda=\pm\frac{1}{4}}$$

For $\lambda=-\frac{1}{4} \quad \to \quad \cancel{x=y=-4} \qquad \text{since} \quad x>0, \quad y>0$

For $\lambda=\frac{1}{4} \quad \to \quad \boxed{x=y=4}$

The minimum value is $f=x+y=4+4=8$.

$xy=16, \quad x>0, \quad y>0$ is a branch of a hyperbola in the first quadrant with x- and y-axes as asymptotes.

The equations $x+y=c$ give a family of parallel lines with $m=-1$. Thus the minimum value of $\boldsymbol{c}$ occurs where $x+y=c$ is tangent to the hyperbola's branch.

b) $\nabla f=y\,\hat{i}+x\,\hat{j}, \quad \nabla g=\hat{i}+\hat{j}$

$$\nabla f=\lambda\nabla g$$

$$y\,\hat{i}+x\,\hat{j}=\lambda\,\hat{i}+\lambda\,\hat{j}$$

$$y=\lambda, \quad x=\lambda$$

$$g(x,y)=x+y-16=0$$

$$\to \quad 2\lambda=16 \quad \Rightarrow \quad \lambda=8$$

For $\lambda=8 \quad \to \quad x=y=8$

The maximum value is $f=xy=8\times 8=64$.

The equations $xy=c, \quad x>0, y>0 \quad or \quad x<0, y<0$ give a family of hyperbolas in the first and third quadrants with x- and y-axes as asymptotes. Thus the maximum value of $\boldsymbol{c}$ occurs where $xy=c$ is tangent to the line $x+y=16$.

Exercise

Find the radius and height of the open right circular cylinder of largest surface area that can be inscribed in a sphere of radius a. What is the largest surface area?

Solution

For a cylinder of radius r and height h, to maximize the surface area $S=2\pi rh$ subject to the constraint $g(r,h)=r^2+\left(\frac{h}{2}\right)^2-a^2=0$

$$\nabla S=2\pi h\,\hat{i}+2\pi r\,\hat{j} \quad and \quad \nabla g=2r\,\hat{i}+\frac{h}{2}\,\hat{j}$$

$$\nabla S=\lambda\nabla g$$

$$2\pi h\,\hat{i}+2\pi r\,\hat{j}=2r\lambda\,\hat{i}+\frac{h}{2}\lambda\,\hat{j}$$

$$2\pi h = 2r\lambda, \quad 2\pi r = \frac{h}{2}\lambda$$

$$\lambda = \frac{\pi h}{r} \quad \to 2\pi r = \frac{h}{2}\frac{\pi h}{r}$$

$$4r^2 = h^2 \quad \Rightarrow \quad h = 2r$$

$$r^2 + \left(\frac{h}{2}\right)^2 = a^2$$

$$r^2 + r^2 = a^2$$

$$2r^2 = a^2 \quad \to \quad \boxed{r = \frac{a}{\sqrt{2}}}$$

$$\boxed{h = \frac{2a}{\sqrt{2}} = a\sqrt{2}}$$

$$S = 2\pi rh$$

$$= 2\pi \frac{a}{\sqrt{2}} a\sqrt{2}$$

$$\boxed{= 2\pi a^2}$$

Exercise

Use the method of Lagrange multipliers to find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ with sides parallel to the coordinate axes.

Solution

The area of a rectangle is $A(x,y) = (2x)(2y) = 4xy$ subject to the constraint

$$g(x,y) = \frac{x^2}{16} + \frac{y^2}{9} - 1 = 0.$$

$$\nabla A = 4y\,\hat{i} + 4x\,\hat{j} \quad \text{and} \quad \nabla g = \frac{1}{8}x\,\hat{i} + \frac{2}{9}y\,\hat{j}$$

$$\nabla A = \lambda \nabla g$$

$$4y\,\hat{i} + 4x\,\hat{j} = \frac{1}{8}x\lambda\,\hat{i} + \frac{2}{9}y\lambda\,\hat{j}$$

$$4y = \frac{1}{8}x\lambda \quad \text{and} \quad 4x = \frac{2}{9}y\lambda$$

$$\lambda = \frac{32y}{x} \quad \Rightarrow \quad 4x = \frac{2y}{9}\frac{32y}{x}$$

$$\to x^2 = \frac{64y^2}{36}$$

$$\boxed{x = \pm\frac{4}{3}y}$$

$$\frac{1}{16}\frac{16y^2}{9} + \frac{1}{9}y^2 = 1$$

$\frac{2}{9}y^2 = 1 \quad \rightarrow \quad y^2 = \frac{9}{2}$

$$\boxed{y = \pm\frac{3\sqrt{2}}{2}}$$

Since x and y represents distance, then

$$y = \frac{3\sqrt{2}}{2} \quad \rightarrow \quad x = \frac{4}{3}\frac{3\sqrt{2}}{2} = 2\sqrt{2}$$

$\therefore$ The length is $2x = 4\sqrt{2}$ and the width is $2y = 3\sqrt{2}$

Exercise

Find the maximum and minimum values of $x^2 + y^2$ subject to the constraint $x^2 - 2x + y^2 - 4y = 0$

Solution

$\nabla f = 2x\,\hat{i} + 2y\,\hat{j} \quad and \quad \nabla g = (2x-2)\,\hat{i} + (2y-4)\,\hat{j}$

$\nabla f = \lambda \nabla g$

$2x\,\hat{i} + 2y\,\hat{j} = (2x-2)\lambda\,\hat{i} + (2y-4)\lambda\,\hat{j}$

$$\begin{array}{ll} 2x = 2(x-1)\lambda \quad and & 2y = 2(y-2)\lambda \\ x = x\lambda - \lambda & y = y\lambda - 2\lambda \\ x(\lambda - 1) = \lambda & y(\lambda - 1) = 2\lambda \\ x = \dfrac{\lambda}{\lambda - 1} & y = \dfrac{2\lambda}{\lambda - 1} = 2x \quad (\lambda \neq 1) \end{array}$$

$x^2 - 2x + y^2 - 4y = 0$

$x^2 - 2x + 4x^2 - 8x = 0$

$5x^2 - 10x = 0$

$5x(x-2) = 0 \quad \Rightarrow \quad x = 0,\ \ 2$

$x = 0 \quad y = 2x = 0 \rightarrow (0,\ 0)$

$x = 2 \quad y = 2x = 4 \rightarrow (2,\ 4)$

$\therefore\ f(0,0) = 0$ is the minimum value, and $f(2,4) = 20$ is the maximum value.

Exercise

The temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant on the plate walks around the circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant?

Solution

$$g(x, y) = x^2 + y^2 - 25 = 0$$

$$\nabla T = (8x - 4y)\,\hat{i} + (-4x + 2y)\,\hat{j} \quad \textit{and} \quad \nabla g = 2x\,\hat{i} + 2y\,\hat{j}$$

$$\nabla T = \lambda \nabla g$$

$$(8x - 4y)\,\hat{i} + (-4x + 2y)\,\hat{j} = 2x\lambda\,\hat{i} + 2y\lambda\,\hat{j}$$

$$8x - 4y = 2x\lambda, \quad -4x + 2y = 2y\lambda$$

$$4x - 2y = x\lambda, \quad y - y\lambda = 2x \to y = \frac{2x}{1-\lambda} \quad (\lambda \neq 1)$$

$$4x - 2\frac{2x}{1-\lambda} = x\lambda$$

$$4x - \frac{4x}{1-\lambda} - x\lambda = 0$$

$$x\left(4 - 4\lambda - 4 - \lambda + \lambda^2\right) = 0$$

$$x\left(\lambda^2 - 5\lambda\right) = 0 \quad \Rightarrow \quad \boxed{x = 0}, \quad \boxed{\lambda = 0,\ 5}$$

Case 1: $x = 0$ $\quad \left\lfloor y = \frac{2x}{1-\lambda} = 0 \right\rfloor$, but (0, 0) is not on the circle $x^2 + y^2 = 25$

Case 2: $\lambda = 0 \quad y = 2x$

$$\Rightarrow x^2 + (2x)^2 = 25 \quad \to 5x^2 = 25 \quad \boxed{x = \pm\sqrt{5}} \quad \left(\sqrt{5}, 2\sqrt{5}\right) \ \left(-\sqrt{5}, -2\sqrt{5}\right)$$

Case 3: $\lambda = 5 \quad y = -\frac{x}{2}$

$$\Rightarrow x^2 + \frac{x^2}{4} = 25 \quad \to 5x^2 = 100 \quad \boxed{x = \pm 2\sqrt{5}} \quad \left(2\sqrt{5}, -\sqrt{5}\right) \ \left(-2\sqrt{5}, \sqrt{5}\right)$$

$$T\left(\sqrt{5}, 2\sqrt{5}\right) = 4\left(\sqrt{5}\right)^2 - 4\left(\sqrt{5}\right)\left(2\sqrt{5}\right) + \left(2\sqrt{5}\right)^2 = 0°$$

$$T\left(-\sqrt{5}, -2\sqrt{5}\right) = 4\left(-\sqrt{5}\right)^2 - 4\left(-\sqrt{5}\right)\left(-2\sqrt{5}\right) + \left(-2\sqrt{5}\right)^2 = 0°$$

$$T\left(2\sqrt{5}, -\sqrt{5}\right) = 4\left(2\sqrt{5}\right)^2 - 4\left(2\sqrt{5}\right)\left(-\sqrt{5}\right) + \left(-\sqrt{5}\right)^2 = 125°$$

$$T\left(-2\sqrt{5}, \sqrt{5}\right) = 4\left(-2\sqrt{5}\right)^2 - 4\left(-2\sqrt{5}\right)\left(\sqrt{5}\right) + \left(\sqrt{5}\right)^2 = 125°$$

$\therefore$ The minimum temperature is 0° at $\left(\sqrt{5}, 2\sqrt{5}\right)$ $\left(-\sqrt{5}, -2\sqrt{5}\right)$

The maximum temperature is 125° at $\left(2\sqrt{5}, -\sqrt{5}\right)$ $\left(-2\sqrt{5}, \sqrt{5}\right)$

Exercise

Your firm has been asked to design a storage tank for liquid petroleum gas. The customer's specifications call for a cylindrical tank with hemispherical ends, and the tank is to hold $8000\ m^3$ of gas. He customer also wants to use the smallest amount of material possible in building the tank. What radius and height do you recommend for the cylindrical portion of the tank?

Solution

The surface area is: $S = 4\pi r^2 + 2\pi rh$ subject to the constraint $V = \frac{4}{3}\pi r^3 + \pi r^2 h = 8000$.

$\nabla S = (8\pi r + 2\pi h)\,\hat{i} + 2\pi r\,\hat{j} \quad and \quad \nabla V = \left(4\pi r^2 + 2\pi rh\right)\hat{i} + \pi r^2\,\hat{j}$

$\nabla S = \lambda \nabla V$

$(8\pi r + 2\pi h)\,\hat{i} + 2\pi r\,\hat{j} = \left(4\pi r^2 + 2\pi rh\right)\lambda\,\hat{i} + \pi r^2 \lambda\,\hat{j}$

$8\pi r + 2\pi h = 2\pi r(2r + h)\lambda \quad and \quad 2\pi r = \pi r^2 \lambda$

$4r + h = r(2r + h)\lambda \qquad r^2\lambda - 2r = 0 \to r(\lambda r - 2) = 0$

$r = 0 \quad and \quad \lambda = \frac{2}{r}\ (r \neq 0)$

$4r + h = r(2r + h)\frac{2}{r} \quad \to \quad 4r + h = 4r + 2h \ \Rightarrow \boxed{h = 0}$

The tank is a sphere, there is no cylindrical part, and

$\frac{4}{3}\pi r^3 + \pi r^2(0) = 8000$

$r^3 = \frac{6000}{\pi}$

$\boxed{r = 10\left(\frac{6}{\pi}\right)^{1/3} \approx 12.4}$

Exercise

A closed rectangular box is to have volume $V\ cm^3$. The cost of the material used in the box is a $a\ cents/cm^2$ for top and bottom, $b\ cents/cm^2$ for front and back, and $c\ cents/cm^2$ for the remaining sides. What dimensions minimize the total cost of materials?

Solution

The cost is given by: $f(x, y, z) = 2axy + 2bxz + 2cyz$

Subject to the constraint $xyz = V$

$\nabla f = (2ay + 2bz)\hat{i} + (2ax + 2cz)\hat{j} + (2bx + 2cy)\hat{k}$

$g(x, y, z) = xyz - V$

$\nabla g = yz\hat{i} + xz\hat{j} + xy\hat{k}$

$$(2ay+2bz)\hat{i}+(2ax+2cz)\hat{j}+(2bx+2cy)\hat{k}=\lambda\left(yz\hat{i}+xz\hat{j}+xy\hat{k}\right)$$

$$\begin{matrix} x \\ -y \\ \end{matrix}\begin{cases} 2ay+2bz=yz\lambda \\ 2ax+2cz=xz\lambda \\ 2bx+2cy=xy\lambda \end{cases} \qquad \begin{array}{r} 2axy+2xbz=xyz\lambda \\ -2axy-2cyz=-xyz\lambda \\ \hline 2bxz-2cyz=0 \end{array} \quad \to bx=cy$$

$$\boxed{y=\frac{b}{c}x}$$

$$\begin{matrix} x \\ \\ -z \end{matrix}\begin{cases} 2ay+2bz=yz\lambda \\ \\ 2bx+2cy=xy\lambda \end{cases} \qquad \begin{array}{r} 2axy+2bxz=xyz\lambda \\ -2bxz-2cyz=-xyz\lambda \\ \hline 2axy-2cyz=0 \end{array} \quad \to ax=cz$$

$$\boxed{z=\frac{a}{c}x}$$

$$\begin{aligned} V &= xyz \\ &= x\left(\frac{b}{c}x\right)\left(\frac{a}{c}x\right) \\ &= \frac{ab}{c^2}x^3 \end{aligned}$$

$$x^3=\frac{c^2}{ab}V$$

$$x=\left(\frac{c^2}{ab}V\right)^{1/3} \quad (width)$$

$$y=\frac{b}{c}\left(\frac{c^2}{ab}V\right)^{1/3}=\left(\frac{b^2}{ac}V\right)^{1/3} \quad (depth)$$

$$z=\frac{a}{c}\left(\frac{c^2}{ab}V\right)^{1/3}=\left(\frac{a^2}{bc}V\right)^{1/3} \quad (height)$$

Exercise

Find the extreme values of $f(x,y,z)=x(y+z)$ on the curve of intersection of the right circular cylinder $x^2+y^2=1$ and the hyperbolic cylinder $xz=1$.

Solution

$$g(x,y,z)=x^2+y^2-1$$

$$h(x,y,z)=xz-1$$

$$\nabla f=(y+z)\hat{i}+x\hat{j}+x\hat{k}$$

$$\nabla g=2x\hat{i}+2y\hat{j}$$

$$\nabla h=z\hat{i}+x\hat{k}$$

$$(y+z)\hat{i}+x\hat{j}+x\hat{k}=\lambda\left(2x\hat{i}+2y\hat{j}\right)+\mu\left(z\hat{i}+x\hat{k}\right) \qquad \nabla f=\lambda\nabla g+\mu\nabla h$$

$$\begin{cases} y+z=2\lambda x+\mu z \\ \quad x=2\lambda y \\ \quad x=\mu x \qquad \rightarrow x=0 \ \ or \ \ \mu=1 \end{cases}$$

For $x=0$ $\Rightarrow$ impossible since $xz=1$

For $\mu=1$

$$\begin{cases} y+z=2\lambda x+z \quad \rightarrow y=2\lambda x \\ \quad x=2\lambda y \end{cases}$$

$$y=2\lambda(2\lambda y)$$

$$=4\lambda^2 y$$

$$y\left(4\lambda^2-1\right)=0\rightarrow\begin{cases} y=0 \\ \lambda=\pm\frac{1}{2} \end{cases}$$

If $y=0$ $\rightarrow x^2=1$ $\Rightarrow x=\pm1$

$xz=1$ $\Rightarrow$ $(1,\ 0,\ 1)$ $(-1,\ 0,\ -1)$

If $\lambda=-\frac{1}{2}$ $\rightarrow y=-x$

$$x^2+y^2=2x^2=1 \rightarrow \underline{x=\pm\frac{1}{\sqrt{2}}}$$

$$y=\mp\frac{1}{\sqrt{2}}$$

$$z=\frac{1}{x}=\pm\sqrt{2}$$

$$\Rightarrow \left(-\frac{1}{\sqrt{2}},\ \pm\frac{1}{\sqrt{2}},\ -\sqrt{2}\right) \ \&\ \left(\frac{1}{\sqrt{2}},\ \pm\frac{1}{\sqrt{2}},\ \sqrt{2}\right)$$

$$f(1,\ 0,\ 1)=1$$

$$f(-1,\ 0,\ -1)=1$$

$$f\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ -\sqrt{2}\right)=\frac{1}{2}$$

$$f\left(-\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ -\sqrt{2}\right)=\frac{3}{2}$$

$$f\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ \sqrt{2}\right)=\frac{1}{2}$$

$$f\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ \sqrt{2}\right)=\frac{3}{2}$$

Abs. Max is $\frac{3}{2}$ @ $f\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ \sqrt{2}\right)$ & $f\left(-\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ -\sqrt{2}\right)$

Abs. Min is $\frac{1}{2}$ @ $f\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ \sqrt{2}\right)$ & $f\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ -\sqrt{2}\right)$

Exercise

Find the point closest to the origin on the curve of intersection of the plane $x+y+z=1$ and the cone $z^2=2x^2+2y^2$

Solution

$$f(x,y,z)=x^2+y^2+z^2 \qquad \to \nabla f=2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$g(x,y,z)=x+y+z-1 \qquad \to \nabla g=\hat{i}+\hat{j}+\hat{k}$$

$$h(x,y,z)=2x^2+2y^2-z^2 \qquad \to \nabla h=4x\hat{i}+4y\hat{j}-2z\hat{k}$$

$$2x\hat{i}+2y\hat{j}+2z\hat{k}=\lambda\left(\hat{i}+\hat{j}+\hat{k}\right)+\mu\left(4x\hat{i}+4y\hat{j}-2z\hat{k}\right) \qquad \nabla f=\lambda\nabla g+\mu\nabla h$$

$$=(\lambda+4\mu x)\hat{i}+(\lambda+4\mu y)\hat{j}+(\lambda-2\mu z)\hat{k}$$

$$\begin{cases} 2x=\lambda+4\mu x & \to \lambda=2x(1-2\mu) \\ 2y=\lambda+4\mu y & \to \lambda=2y(1-2\mu) \\ 2z=\lambda-2\mu z & \to \lambda=2z(1-\mu) \end{cases}$$

$$\begin{aligned}\lambda&=2x(1-2\mu)\\&=2y(1-2\mu)\\&=2z(1-\mu)\end{aligned}$$

$$x(1-2\mu)=y(1-2\mu) \quad \to \quad \begin{cases} x=y \\ 1-2\mu=0 \end{cases}$$

If $x=y$ $\to$ $z^2=2x^2+2x^2=4x^2$

$\underline{z=\pm 2x}$

$$x+y+z=1 \ \to \ \begin{cases} x+x+2x=1 \Rightarrow \underline{x=\frac{1}{4}} \\ x+x-2x\neq 1 \end{cases}$$

$\therefore\ \left(\frac{1}{4},\ \frac{1}{4},\ \frac{1}{2}\right)$

If $\mu=\frac{1}{2}$ $\to$ $\lambda=0=2z\left(1-\frac{1}{2}\right)$

$\underline{z=0}$

$\to\ z^2=2x^2+2y^2=0$

$$x^2+y^2=0 \ \to \ \boxed{x=y=0}$$

But $x+y+z=1$, therefore $x=y=z\neq 0$

$\therefore$ The point $\left(\frac{1}{4},\ \frac{1}{4},\ \frac{1}{2}\right)$ on the curve of intersection is closest to the origin.

Exercise

Find the point on the plane $x+2y+3z=13$ closest to the point $(1,\ 1,\ 1)$

Solution

Let $f(x,y,z)=(x-1)^2+(y-1)^2+(z-1)^2$ (be the square of the distance from $(1,1,1)$)

$\nabla f=2(x-1)\,\hat{i}+2(y-1)\,\hat{j}+2(z-1)\hat{k}$ *and* $\nabla g=\hat{i}+2\hat{j}+3\hat{k}$

$\nabla f=\lambda\nabla g$

$2(x-1)\,\hat{i}+2(y-1)\,\hat{j}+2(z-1)\hat{k}=\lambda\ \hat{i}+2\lambda\,\hat{j}+3\lambda\hat{k}$

$$\to\begin{cases}2(x-1)=\lambda\to x=\frac{\lambda}{2}+1\\ 2(y-1)=2\lambda\to y=\lambda+1\\ 2(z-1)=3\lambda\to z=\frac{3\lambda}{2}+1\end{cases}\quad\to\quad \frac{\lambda}{2}+1+2(\lambda+1)+3\left(\frac{3\lambda}{2}+1\right)=13$$

$$\frac{\lambda}{2}+1+2\lambda+2+\frac{9\lambda}{2}+3=13$$

$$7\lambda=7\quad\Rightarrow\quad \boxed{\lambda=1}$$

$$x=\frac{\lambda}{2}+1=\frac{3}{2},\quad y=\lambda+1=2,\quad z=\frac{3\lambda}{2}+1=\frac{5}{2}$$

$\therefore$ The point $\left(\frac{3}{2},\ 2,\ \frac{5}{2}\right)$ is closet.

Exercise

Find the point on the sphere $x^2+y^2+z^2=4$ farthest from the point $(1,\ -1,\ 1)$

Solution

Let $f(x,y,z)=(x-1)^2+(y+1)^2+(z-1)^2$ (be the square of the distance from $(1,-1,1)$)

$\nabla f=2(x-1)\,\hat{i}+2(y+1)\,\hat{j}+2(z-1)\hat{k}$ *and* $\nabla g=2x\,\hat{i}+2y\,\hat{j}+2z\hat{k}$

$\nabla f=\lambda\nabla g$

$2(x-1)\,\hat{i}+2(y+1)\,\hat{j}+2(z-1)\hat{k}=2x\lambda\,\hat{i}+2y\lambda\,\hat{j}+2z\lambda\hat{k}$

$$\rightarrow\begin{cases} x-1=x\lambda \rightarrow x=\dfrac{1}{1-\lambda} \\ y+1=y\lambda \rightarrow y=-\dfrac{1}{1-\lambda} \\ z-1=z\lambda \rightarrow z=\dfrac{1}{1-\lambda}\end{cases} \rightarrow \left(\frac{1}{1-\lambda}\right)^2+\left(-\frac{1}{1-\lambda}\right)^2+\left(\frac{1}{1-\lambda}\right)^2=4$$

$$3\left(\frac{1}{1-\lambda}\right)^2=4 \quad \rightarrow \left(\frac{1}{1-\lambda}\right)^2=\frac{4}{3} \quad \rightarrow \frac{1}{1-\lambda}=\pm\frac{2}{\sqrt{3}}$$

$$x=\pm\frac{2}{\sqrt{3}}, \quad y=\mp\frac{2}{\sqrt{3}}, \quad z=\pm\frac{2}{\sqrt{3}}$$

$$\left(\frac{2}{\sqrt{3}},\ -\frac{2}{\sqrt{3}},\ \frac{2}{\sqrt{3}}\right) \quad and \quad \left(-\frac{2}{\sqrt{3}},\ \frac{2}{\sqrt{3}},\ -\frac{2}{\sqrt{3}}\right)$$

$$f\left(\frac{2}{\sqrt{3}},-\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}}\right)=\left(\frac{2}{\sqrt{3}}-1\right)^2+\left(-\frac{2}{\sqrt{3}}+1\right)^2+\left(\frac{2}{\sqrt{3}}-1\right)^2\approx 0.72$$

$$f\left(-\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}},-\frac{2}{\sqrt{3}}\right)=\left(-\frac{2}{\sqrt{3}}-1\right)^2+\left(\frac{2}{\sqrt{3}}+1\right)^2+\left(-\frac{2}{\sqrt{3}}-1\right)^2\approx 13.928$$

$\therefore$ The largest value of f occurs at $\left(-\frac{2}{\sqrt{3}},\ \frac{2}{\sqrt{3}},\ -\frac{2}{\sqrt{3}}\right)$ on the sphere.

Exercise

Find the minimum distance from the surface $x^2-y^2-z^2=1$ to the origin

Solution

Let $f(x,y,z)=x^2+y^2+z^2$ (be the square of the distance from origin)

$\nabla f=2x\,\hat{i}+2y\,\hat{j}+2z\hat{k} \quad and \quad \nabla g=2x\,\hat{i}-2y\,\hat{j}-2z\hat{k}$

$\nabla f=\lambda\nabla g$

$2x\,\hat{i}+2y\,\hat{j}+2z\hat{k}=2x\lambda\,\hat{i}-2y\lambda\,\hat{j}-2z\lambda\hat{k}$

$$\rightarrow\begin{cases} 2x=2x\lambda \\ 2y=-2y\lambda \\ 2z=-2z\lambda\end{cases} \rightarrow \quad \lambda=1 \ or \ x=0$$

Case 1: $\lambda=1 \quad \rightarrow\begin{cases} 2y=-2y\lambda & \boxed{y=0} \\ 2z=-2z\lambda & \boxed{z=0}\end{cases} \quad x^2-y^2-z^2=1\Rightarrow\boxed{x=\pm1}$

Case 2: $x=0 \quad \rightarrow -y^2-z^2=1$ *No solution*

$\therefore$ The points on the unit circle $y^2+z^2=1$ are the points on the surface $x^2-y^2-z^2=1$ closest to the origin.

Exercise

Find the maximum and minimum values of $f(x,y,z)=x-2y+5z$ on the sphere $x^2+y^2+z^2=30$

Solution

$$\nabla f = \hat{i}-2\hat{j}+5\hat{k} \quad \text{and} \quad \nabla g = 2x\,\hat{i}+2y\,\hat{j}+2z\,\hat{k}$$

$$\nabla f = \lambda \nabla g$$

$$\hat{i}-2\hat{j}+5\hat{k} = 2x\lambda\,\hat{i}+2y\lambda\,\hat{j}+2z\lambda\hat{k}$$

$$\rightarrow \begin{cases} 2x\lambda = 1 \\ 2y\lambda = -2 \\ 2z\lambda = 5 \end{cases} \rightarrow \quad \begin{aligned} x &= \frac{1}{2\lambda} \\ y &= -\frac{1}{\lambda} \\ z &= \frac{5}{2\lambda} \end{aligned}$$

$$\left(\frac{1}{2\lambda}\right)^2+\left(-\frac{1}{\lambda}\right)^2+\left(\frac{5}{2\lambda}\right)^2=30$$

$$\frac{1}{4\lambda^2}+\frac{1}{\lambda^2}+\frac{25}{4\lambda^2}=30$$

$$\frac{30}{4\lambda^2}=30$$

$$\lambda^2=\frac{1}{4} \Rightarrow \lambda=\pm\frac{1}{2}$$

$$\lambda=\frac{1}{2} \quad \Rightarrow \quad \boxed{x=1,\ \ y=-2,\ \ z=5}$$

$$\lambda=-\frac{1}{2} \quad \Rightarrow \quad \boxed{x=-1,\ \ y=2,\ \ z=-5}$$

$$f(1,-2,5)=1+4+25=30$$

$$f(-1,2,-5)=-1-4-25=-30$$

$\therefore$ The maximum value $f(1,-2,5)=30$ and the minimum is $f(-1,2,-5)=-30$

Exercise

Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

Solution

$$f(x,y,z)=x^2+y^2+z^2 \quad \text{and} \quad g(x,y,z)=x+y+z-9=0$$

$$\nabla f = 2x\,\hat{i}+2y\,\hat{j}+2z\,\hat{k} \quad \text{and} \quad \nabla g = \hat{i}+\hat{j}+\hat{k}$$

$$\nabla f = \lambda \nabla g$$

$$2x\,\hat{i}+2y\,\hat{j}+2z\,\hat{k}=\lambda\,\hat{i}+\lambda\,\hat{j}+\lambda\,\hat{k}$$

$$\to \begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \end{cases} \quad \to \quad x = y = z = \frac{1}{2\lambda}$$

$$\frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{2\lambda} = 9$$

$$\frac{3}{2\lambda} = 9 \quad \Rightarrow \quad \lambda = \frac{1}{6}$$

$$x = y = z = \frac{1}{2\frac{1}{6}} \ \underline{= 3}$$

Exercise

A space probe in the shape of the ellipsoid $4x^2 + y^2 + 4z^2 = 16$ enters Earth's atmosphere and its surface begins to heat. After 1 hour, the temperature at the point (x, y, z) on the probe's surface is $T(x, y, z) = 8x^2 + 4yz - 16z + 600$. Find the hottest point on the probe's surface.

Solution

$$\nabla T = 16x\,\hat{i} + 4z\,\hat{j} + (4y - 16)\,\hat{k} \quad and \quad \nabla g = 8x\,\hat{i} + 2y\,\hat{j} + 8z\,\hat{k}$$

$$\nabla f = \lambda \nabla g$$

$$16x\,\hat{i} + 4z\,\hat{j} + (4y - 16)\,\hat{k} = 8x\lambda\,\hat{i} + 2y\lambda\,\hat{j} + 8z\lambda\,\hat{k}$$

$$\to \begin{cases} 16x = 8x\lambda \\ 4z = 2y\lambda \\ 4y - 16 = 8z\lambda \end{cases} \quad \to \quad \lambda = 2 \ \text{or} \ x = 0$$

$$\textbf{\textit{Case}}\ 1\text{:}\ \lambda = 2 \quad \to \begin{cases} 2z = y\lambda & \to 2z = 2y \ \Rightarrow \ z = y \\ y - 4 = 2z\lambda & \quad y - 4 = 2y(2) \end{cases}$$

$$3y = -4$$

$$\underline{y = -\frac{4}{3} = z}$$

$$4x^2 + \frac{16}{9} + 4\left(\frac{16}{9}\right) = 16 \quad \to \quad x^2 = \frac{16}{9} \quad \underline{x = \pm\frac{4}{3}}$$

$$\textbf{\textit{Case}}\ 2\text{:}\ x = 0 \quad \to \lambda = \frac{2z}{y} \quad \Rightarrow y - 4 = 2z\frac{2z}{y}$$

$$y^2 - 4y = 4z^2$$

$$4x^2 + y^2 + 4z^2 = 16 \quad \to \quad y^2 + y^2 - 4y = 16$$

$$2y^2 - 4y - 16 = 0 \quad \to \quad \underline{y = 4, \ -2}$$

$$\begin{cases} y=4 & \rightarrow 4z^2 = 4^2 - 16 = 0 \Rightarrow \boxed{z=0} \\ y=-2 & \rightarrow 4z^2 = (-2)^2 + 8 = 13 \Rightarrow \boxed{z=\pm\sqrt{3}} \end{cases}$$

$$T\left(-\frac{4}{3},-\frac{4}{3},-\frac{4}{3}\right) = 8\left(-\frac{4}{3}\right)^2 + 4\left(-\frac{4}{3}\right)\left(-\frac{4}{3}\right) - 16\left(-\frac{4}{3}\right) + 600$$

$$\approx 642.667^\circ$$

$$T\left(\frac{4}{3},-\frac{4}{3},-\frac{4}{3}\right) = 8\left(\frac{4}{3}\right)^2 + 4\left(-\frac{4}{3}\right)\left(-\frac{4}{3}\right) - 16\left(-\frac{4}{3}\right) + 600$$

$$\approx 642.667^\circ$$

$$T(0,4,0) = 0 + 0 - 0 + 600$$

$$\approx 600^\circ$$

$$T\left(0,-2,-\sqrt{3}\right) = 0 + 4(-2)\left(-\sqrt{3}\right) - 16\left(-\sqrt{3}\right) + 600$$

$$\approx 641.6^\circ$$

$$T\left(0,-2,\sqrt{3}\right) = 0 + 4(-2)\left(\sqrt{3}\right) - 16\left(\sqrt{3}\right) + 600$$

$$\approx 558.43^\circ$$

$\therefore \left(\pm\frac{4}{3},-\frac{4}{3},-\frac{4}{3}\right)$ are the hottest points on the space probe.

Exercise

Find the extreme values of $f(x,y,z) = xyz$

Subject to the constraint $\begin{cases} x+y+z=32 \\ x-y+z=0 \end{cases}$

Solution

$f(x,y,z) = xyz$	$g_1(x,y,z) = x+y+z-32 = 0$	$g_2(x,y,z) = x-y+z = 0$
$\nabla f = yz\hat{i} + xz\hat{j} + xy\hat{k}$	$\nabla g_1 = \hat{i} + \hat{j} + \hat{k}$	$\nabla g_2 = \hat{i} - \hat{j} + \hat{k}$

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

$$yz\hat{i} + xz\hat{j} + xy\hat{k} = \lambda\hat{i} + \lambda\hat{j} + \lambda\hat{k} + \mu\hat{i} + \mu\hat{j} + \mu\hat{k}$$

$$\begin{cases} yz = \lambda + \mu & (1) \\ xz = \lambda - \mu & (2) \\ xy = \lambda + \mu & (3) \end{cases} \Rightarrow \begin{cases} (1)+(2) \rightarrow & 2\lambda = yz + zx \\ (3)+(2) \rightarrow & 2\lambda = xy + zx \end{cases} \rightarrow 2\lambda = yz + zx = xy + zx$$

$$yz = xy \quad \Rightarrow \quad y = 0 \quad or \quad x = z \quad (y \neq 0)$$

Case 1:

If $y=0 \Rightarrow \begin{cases} g_1(x,y,z) = x+z-32 = 0 \\ g_2(x,y,z) = x+z = 0 \end{cases} \quad \rightarrow x = -z \qquad \cancel{-32=0}$

Case 2:

If $x=z \Rightarrow \begin{cases} g_1(x,y,z) = 2x+y-32 = 0 \\ g_2(x,y,z) = 2x-y = 0 \end{cases} \quad \rightarrow y = 2x \qquad 4x = 32 \rightarrow \boxed{x=8=z} \quad \boxed{y=16}$

$f(x,y,z) = xyz = (8)(16)(8) \; \underline{=1024}|$

The extreme point is $(8,\ 16,\ 8)$ with a value of 1024.

Exercise

Find the extreme values of $f(x,y,z) = x^2+y^2+z^2$

Subject to the constraint $\begin{cases} x+2z=6 \\ x+y=12 \end{cases}$

Solution

$f(x,y,z) = x^2+y^2+z^2$	$g_1(x,y,z) = x+2z-6=0$	$g_2(x,y,z) = x+y-12=0$
$\nabla f = 2x\hat{i}+2y\hat{j}+2z\hat{k}$	$\nabla g_1 = \hat{i}+2\hat{k}$	$\nabla g_2 = \hat{i}+\hat{j}$

$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$

$2x\hat{i}+2y\hat{j}+2z\hat{k} = \lambda\hat{i}+2\lambda\hat{k}+\mu\hat{i}+\mu\hat{j}$

$\begin{cases} 2x = \lambda+\mu & (1) \\ 2y = \mu & (2) \\ 2z = 2\lambda & (3) \end{cases} \quad \Rightarrow \; 2x = z+2y \;\Rightarrow\; z = 2x-2y$

$\begin{cases} x+2z=6 \\ x+y=12 \end{cases}$

$\begin{cases} x+4x-4y=6 \\ x+y=12 \end{cases}$

$\begin{cases} 5x-4y=6 \\ x+y=12 \end{cases}$

$x=6, \quad y=6 \;\Rightarrow\; z=0$

$f(x,y,z) = x^2+y^2+z^2$

$\underline{=72}\,|$

The extreme point is $(6,\ 6,\ 0)$ with a value of 72.

Exercise

What point on the plane $x + y + 4z = 8$ is closest to the origin? Give an argument showing you have found an absolute minimum of the distance function.

Solution

It suffices to minimize the function $f(x, y, z) = x^2 + y^2 + z^2$

$x + y + 4z = 8 \quad \to \quad x = 8 - y - 4z$

$f(y, z) = (8 - y - 4z)^2 + y^2 + z^2$

$$\begin{aligned} f_y &= -2(8 - y - 4z) + 2y \\ &= 4y + 8z - 16 = 0 \end{aligned}$$

$$\begin{aligned} f_z &= -8(8 - y - 4z) + 2z \\ &= 8y + 34z - 64 = 0 \end{aligned}$$

$$\begin{cases} y + 2z = 4 \\ 4y + 17z = 32 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 2 \\ 4 & 17 \end{vmatrix} = 9 \quad \Delta_y = \begin{vmatrix} 4 & 2 \\ 32 & 17 \end{vmatrix} = 4$$

$y = \frac{4}{9} \quad \to z = \frac{16}{9}$

$x = 8 - \frac{4}{9} - \frac{64}{9} \; \boxed{= \frac{4}{9}}$

$\therefore$ The closest point is $\left(\frac{4}{9}, \; \frac{4}{9}, \; \frac{16}{9} \right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$f(x, y) = 2x + y + 10 \quad \textit{subject to} \quad 2(x-1)^2 + 4(y-1)^2 = 1$

Solution

$f(x, y) = 2x + y + 10$

$\nabla f = 2\hat{i} + \hat{j} \qquad\qquad \nabla f = f_x \hat{i} + f_y \hat{j}$

$g(x, y) = 2(x-1)^2 + 4(y-1)^2 - 1$

$\nabla g = 4(x-1)\hat{i} + 8(y-1)\hat{j}$

$2\hat{i} + \hat{j} = \lambda\left(4(x-1)\hat{i} + 8(y-1)\hat{j}\right) \qquad \nabla f = \lambda \nabla g$

$$\begin{matrix} \hat{i} \\ \hat{j} \end{matrix} \begin{cases} 2 = 4\lambda(x-1) \\ 1 = 8\lambda(y-1) \end{cases} \to \begin{cases} x - 1 = \frac{1}{2\lambda} \\ y - 1 = \frac{1}{8\lambda} \end{cases}$$

$$2(x-1)^2 + 4(y-1)^2 = 1$$

$$2\left(\frac{1}{2\lambda}\right)^2 + 4\left(\frac{1}{8\lambda}\right)^2 = 1$$

$$\frac{1}{2\lambda^2} + \frac{1}{16\lambda^2} = 1$$

$$\frac{9}{16\lambda^2} = 1$$

$$\lambda^2 = \frac{9}{16} \quad \rightarrow \quad \lambda = \pm\frac{3}{4}$$

For $\lambda = -\frac{3}{4}$

$$\begin{cases} x = -\frac{1}{2}\frac{4}{3} + 1 = \frac{1}{3} \\ y = -\frac{1}{8}\frac{4}{3} + 1 = \frac{5}{6} \end{cases} \qquad \left(\frac{1}{3},\ \frac{5}{6}\right)$$

For $\lambda = \frac{3}{4}$

$$\begin{cases} x = \frac{1}{2}\frac{4}{3} + 1 = \frac{1}{3} \\ y = \frac{1}{8}\frac{4}{3} + 1 = \frac{5}{6} \end{cases} \qquad \left(\frac{5}{3},\ \frac{7}{6}\right)$$

$$f\left(\frac{1}{3},\ \frac{5}{6}\right) = \frac{2}{3} + \frac{5}{6} + 10 = \frac{23}{2}$$

$$f\left(\frac{5}{3},\ \frac{7}{6}\right) = \frac{10}{3} + \frac{7}{6} + 10 = \frac{29}{2}$$

Maximum is $\frac{29}{2}$ @ $\left(\frac{5}{3},\ \frac{7}{6}\right)$

Minimum is $\frac{23}{2}$ @ $\left(\frac{1}{3},\ \frac{5}{6}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x,\ y) = x^2y^2 \quad \textit{subject to} \quad 2x^2 + y^2 = 1$$

Solution

$$\nabla f = 2xy^2\hat{i} + 2x^2y\hat{j} \qquad\qquad \nabla f = f_x\hat{i} + f_y\hat{j}$$

$$g(x,\ y) = 2x^2 + y^2 - 1$$

$$\nabla g = 4x\hat{i} + 2y\hat{j}$$

$$2xy^2\hat{i} + 2x^2y\hat{j} = \lambda\left(4x\hat{i} + 2y\hat{j}\right) \qquad \nabla f = \lambda \nabla g$$

$$\begin{matrix} \hat{i} \\ \hat{j} \end{matrix}\begin{cases} 2xy^2 = 4\lambda x \\ 2x^2y = 2\lambda y \end{cases}$$

$$2x\left(y^2 - 2\lambda\right) = 0 \quad \to \quad \underline{x = 0,\ y^2 = 2\lambda|}$$

$$2y\left(x^2 - \lambda\right) = 0 \quad \to \quad \underline{y = 0,\ x^2 = \lambda|}$$

$$(0,\ 0)$$

$$2\lambda + 2\lambda = 1 \quad \to \quad \underline{\lambda = \tfrac{1}{4}|}$$

For $\lambda = \frac{1}{4}$

$$\begin{cases} x^2 = \frac{1}{4} \to x = \pm\frac{1}{2} \\ y^2 = \frac{1}{2} \to y = \pm\frac{1}{\sqrt{2}} \end{cases} \to \pm\left(\frac{1}{2},\ \frac{\sqrt{2}}{2}\right)$$

$$f\left(\pm\frac{1}{2},\ \pm\frac{\sqrt{2}}{2}\right) = \left(\pm\frac{1}{2}\right)^2\left(\pm\frac{\sqrt{2}}{2}\right)^2 \underline{= \frac{1}{8}|}$$

$$\underline{f(0,\ \pm 1) = 0|}$$

$$\underline{f\left(\pm\frac{\sqrt{2}}{2},\ 0\right) = 0|}$$

Maximum is $\frac{1}{8}$ @ $\left(\pm\frac{1}{2},\ \pm\frac{\sqrt{2}}{2}\right)$

Minimum is 0 @ $(0,\ \pm 1)$ & $\left(\pm\frac{\sqrt{2}}{2},\ 0\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x,\ y) = x + 2y \quad \textit{subject to} \quad x^2 + y^2 = 4$$

Solution

$$\nabla f = \hat{i} + 2\hat{j} \qquad \nabla f = f_x\hat{i} + f_y\hat{j}$$

$$g(x,\ y) = x^2 + y^2 - 4$$

$$\nabla g = 2x\hat{i} + 2y\hat{j}$$

$$\hat{i} + 2\hat{j} = \lambda\left(2x\hat{i} + 2y\hat{j}\right) \qquad \nabla f = \lambda \nabla g$$

$$\begin{matrix} \hat{i} \\ \hat{j} \end{matrix} \begin{cases} 1 = 2\lambda x & \rightarrow 2\lambda = \frac{1}{x} \\ 2 = 2\lambda y & \rightarrow 2\lambda = \frac{2}{y} \end{cases}$$

$$2\lambda = \frac{1}{x} = \frac{2}{y} \quad \Rightarrow \quad y = 2x$$

$$x^2 + (2x)^2 = 4$$

$$5x^2 = 4 \quad \rightarrow \quad \boxed{x = \pm\frac{2}{\sqrt{5}}} \quad \Rightarrow \quad \boxed{y = \pm\frac{4}{\sqrt{5}}}$$

The points: $\pm\left(\frac{2}{\sqrt{5}},\ \frac{4}{\sqrt{5}}\right)$

$$\begin{aligned} f\left(\frac{2}{\sqrt{5}},\ \frac{4}{\sqrt{5}}\right) &= \frac{2}{\sqrt{5}} + \frac{8}{\sqrt{5}} \\ &= \frac{10}{\sqrt{5}} \\ &= \boxed{2\sqrt{5}} \end{aligned}$$

$$\begin{aligned} f\left(-\frac{2}{\sqrt{5}},\ -\frac{4}{\sqrt{5}}\right) &= -\frac{2}{\sqrt{5}} - \frac{8}{\sqrt{5}} \\ &= -\frac{10}{\sqrt{5}} \\ &= \boxed{-2\sqrt{5}} \end{aligned}$$

Maximum is $2\sqrt{5}$ @ $\left(\frac{2}{\sqrt{5}},\ \frac{4}{\sqrt{5}}\right)$

Minimum is $-2\sqrt{5}$ @ $\left(-\frac{2}{\sqrt{5}},\ -\frac{4}{\sqrt{5}}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x, y) = xy^2 \quad \textit{subject to} \quad x^2 + y^2 = 1$$

Solution

$$\nabla f = y^2\hat{i} + 2xy\hat{j} \qquad \nabla f = f_x\hat{i} + f_y\hat{j}$$

$$g(x, y) = x^2 + y^2 - 1$$

$$\nabla g = 2x\hat{i} + 2y\hat{j}$$

$$y^2\hat{i} + 2xy\hat{j} = \lambda\left(2x\hat{i} + 2y\hat{j}\right) \qquad \nabla f = \lambda \nabla g$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \end{array} \begin{cases} y^2 = 2\lambda x & (1) \\ 2xy = 2\lambda y & \rightarrow y = 0,\ x = \lambda \end{cases}$$

For $y = 0 = \lambda$

$$x^2 + 0 = 1 \quad \rightarrow \quad \underline{x = \pm 1}$$

The points: $(\pm 1,\ 0)$

For $\lambda = x$

$$(1) \rightarrow y^2 = 2x^2$$

$$x^2 + 2x^2 = 1$$

$$3x^2 = 1 \quad \rightarrow \quad \underline{x = \pm \frac{1}{\sqrt{3}}}$$

$$y^2 = 2\left(\frac{1}{\sqrt{3}}\right)^2$$

$$y = \pm\sqrt{\frac{2}{3}} = \underline{\pm\frac{\sqrt{6}}{3}}$$

The points: $\left(\pm\frac{\sqrt{3}}{3},\ \pm\frac{\sqrt{6}}{3}\right)$

	$f(x, y) = xy^2$
$(1,\ 0)$	0
$(-1,\ 0)$	0
$\left(\frac{\sqrt{3}}{3},\ \frac{\sqrt{6}}{3}\right)$	$\frac{\sqrt{3}}{3}\frac{2}{3} = \frac{2\sqrt{3}}{9}$
$\left(\frac{\sqrt{3}}{3},\ -\frac{\sqrt{6}}{3}\right)$	$\frac{\sqrt{3}}{3}\frac{2}{3} = \frac{2\sqrt{3}}{9}$
$\left(-\frac{\sqrt{3}}{3},\ \frac{\sqrt{6}}{3}\right)$	$-\frac{\sqrt{3}}{3}\frac{2}{3} = -\frac{2\sqrt{3}}{9}$
$\left(-\frac{\sqrt{3}}{3},\ -\frac{\sqrt{6}}{3}\right)$	$-\frac{\sqrt{3}}{3}\frac{2}{3} = -\frac{2\sqrt{3}}{9}$

Maximum is $\frac{2\sqrt{3}}{9}$ @ $\left(\frac{\sqrt{3}}{3},\ \pm\frac{\sqrt{6}}{3}\right)$

Minimum is $-\frac{2\sqrt{3}}{9}$ @ $\left(\pm\frac{\sqrt{3}}{3},\ \frac{\sqrt{6}}{3}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x, y) = x + y \quad \textit{subject to} \quad x^2 - xy + y^2 = 1$$

Solution

$\nabla f = \hat{i} + \hat{j}$ $\qquad\qquad\qquad$ $\nabla f = f_x \hat{i} + f_y \hat{j}$

$g(x, y) = x^2 + y^2 - xy - 1$

$\nabla g = (2x - y)\hat{i} + (2y - x)\hat{j}$

$\hat{i} + \hat{j} = \lambda(2x - y)\hat{i} + (2y - x)\hat{j}$ $\qquad\qquad$ $\nabla f = \lambda \nabla g$

$$\begin{matrix} \hat{i} \\ \hat{j} \end{matrix} \begin{cases} 1 = \lambda(2x - y) & (1) \\ 1 = \lambda(2y - x) & (2) \end{cases}$$

$\lambda = \dfrac{1}{2x - y} = \dfrac{1}{2y - x}$

$2y - x = 2x - y \quad \rightarrow \quad \underline{y = x}$

$x^2 - xy + y^2 = 1$

$x^2 - x^2 + x^2 = 1$

$x^2 = 1 \quad \rightarrow \quad \underline{x = \pm 1 = y}$

The points: $\pm(1, 1)$

	$f(x, y) = x + y$
$(1, 1)$	2
$(-1, -1)$	-2

Maximum is 2 @ $(1, 1)$

Minimum is -2 @ $(-1, -1)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x, y) = x^2 + y^2 \quad \textit{subject to} \quad 2x^2 + 3xy + 2y^2 = 7$$

Solution

$\nabla f = 2x\hat{i} + 2y\hat{j}$ $\qquad\qquad\qquad$ $\nabla f = f_x \hat{i} + f_y \hat{j}$

$$g(x, y) = 2x^2 + 3xy + 2y^2 - 7$$

$$\nabla g = (4x + 3y)\hat{i} + (3x + 4y)\hat{j}$$

$$2x\hat{i} + 2y\hat{j} = \lambda\left((4x + 3y)\hat{i} + (3x + 4y)\hat{j}\right) \qquad \nabla f = \lambda \nabla g$$

$$\begin{matrix} \hat{i} \\ \hat{j} \end{matrix} \begin{cases} 2x = \lambda(4x + 3y) \\ 2y = \lambda(3x + 4y) \end{cases} \rightarrow \lambda = \frac{2x}{4x + 3y} = \frac{2y}{3x + 4y}$$

$$3x^2 + 4xy = 4xy + 3y^2$$

$$x^2 = y^2 \rightarrow x = \pm y$$

For $x = y$

$$2x^2 + 3xy + 2y^2 = 7$$

$$2y^2 + 3y^2 + 2y^2 = 7$$

$$7y^2 = 7$$

$$y^2 = 1 \rightarrow \underline{y = \pm 1 = x}|$$

The points: $(\pm 1, \ \pm 1)$

For $x = -y$

$$2y^2 - 3y^2 + 2y^2 = 7$$

$$y^2 = 7$$

$$\underline{y = \pm\sqrt{7} = -x}|$$

The points: $(\sqrt{7}, \ -\sqrt{7})$ & $(-\sqrt{7}, \ \sqrt{7})$

	$f(x, y) = x^2 + y^2$
$(1, \ 1)$	2
$(-1, \ -1)$	2
$(-\sqrt{7}, \ \sqrt{7})$	$7 + 7 = 14$
$(\sqrt{7}, \ -\sqrt{7})$	$7 + 7 = 14$

Maximum is 14 @ $(\sqrt{7}, \ -\sqrt{7})$ & $(-\sqrt{7}, \ \sqrt{7})$

Minimum is 2 @ $(1, \ 1), \ (-1, \ -1)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x, y) = xy \quad subject\ to \quad x^2 + y^2 - xy = 9$$

Solution

$\nabla f = y\hat{i} + x\hat{j}$ $\qquad\qquad$ $\nabla f = f_x \hat{i} + f_y \hat{j}$

$g(x, y) = x^2 + y^2 - xy - 9$

$\nabla g = (2x - y)\hat{i} + (2y - x)\hat{j}$

$y\hat{i} + x\hat{j} = \lambda\left((2x - y)\hat{i} + (2y - x)\hat{j}\right)$ $\qquad\qquad$ $\nabla f = \lambda \nabla g$

$$\begin{matrix} \hat{i} \\ \hat{j} \end{matrix} \begin{cases} y = \lambda(2x - y) \\ x = \lambda(2y - x) \end{cases} \rightarrow \lambda = \frac{y}{2x - y} = \frac{x}{2y - x}$$

$2y^2 - xy = 2x^2 - xy$

$x^2 = y^2 \rightarrow x = \pm y$

For $x = y$

$x^2 + y^2 - xy = 9$

$y^2 + y^2 - y^2 = 9$

$y^2 = 9$

$\underline{y = \pm 3 = x}|$

The points: $(3, 3)$, $(-3, -3)$

For $x = -y$

$x^2 + y^2 - xy = 9$

$y^2 + y^2 + y^2 = 9$

$y^2 = 3$

$\underline{y = \pm\sqrt{3} = -x}|$

The points: $\left(\sqrt{3}, -\sqrt{3}\right)$ & $\left(-\sqrt{3}, \sqrt{3}\right)$

	$f(x, y) = xy$
$(3, 3)$	9
$(-3, -3)$	9
$\left(\sqrt{3}, -\sqrt{3}\right)$	-3

$\left(-\sqrt{3},\ \sqrt{3}\right)$	-3

Maximum is 9 @ $(3,\ 3),\ (-3,\ -3)$

Minimum is -3 @ $\left(\sqrt{3},\ -\sqrt{3}\right)$ & $\left(-\sqrt{3},\ \sqrt{3}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x,\ y) = x - y \quad \textit{subject to} \quad x^2 - 3xy + y^2 = 20$$

Solution

$\nabla f = \hat{i} - \hat{j}$ $\qquad\qquad \nabla f = f_x \hat{i} + f_y \hat{j}$

$g(x,\ y) = x^2 - 3xy + y^2 - 20$

$\nabla g = (2x - 3y)\hat{i} + (2y - 3x)\hat{j}$

$\hat{i} - \hat{j} = \lambda\left((2x - 3y)\hat{i} + (2y - 3x)\hat{j}\right)$ $\qquad\qquad \nabla f = \lambda \nabla g$

$$\begin{matrix} \hat{i} \\ \hat{j} \end{matrix} \begin{cases} 1 = \lambda(2x - 3y) \\ -1 = \lambda(2y - 3x) \end{cases} \rightarrow \lambda = \frac{1}{2x - 3y} = -\frac{1}{2y - 3x}$$

$2y - 3x = -2x + 3y$

$-x = y$

For $x = -y$

$x^2 - 3xy + y^2 = 20$

$y^2 + 3y^2 + y^2 = 20$

$y^2 = 4$

$y = \pm 2 = -x$

The points: $(2,\ -2)$ & $(-2,\ 2)$

	$f(x,\ y) = x - y$
$(2,\ -2)$	4
$(-2,\ 2)$	-4

Maximum is 4 @ $(2,\ -2)$

Minimum is -4 @ $(-2,\ 2)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x, y) = e^{2xy} \quad \textit{subject to} \quad x^2 + y^2 = 16$$

Solution

$$\nabla f = 2ye^{2xy}\hat{i} + 2xe^{2xy}\hat{j} \qquad \nabla f = f_x\hat{i} + f_y\hat{j}$$

$$g(x, y) = x^2 + y^2 - 16$$

$$\nabla g = 2x\hat{i} + 2y\hat{j}$$

$$2ye^{2xy}\hat{i} + 2xe^{2xy}\hat{j} = \lambda\left(2x\hat{i} + 2y\hat{j}\right) \qquad \nabla f = \lambda \nabla g$$

$$\begin{matrix}\hat{i} \\ \hat{j}\end{matrix}\begin{cases} 2ye^{2xy} = 2\lambda x \\ 2xe^{2xy} = 2\lambda y \end{cases} \rightarrow \lambda = \frac{ye^{2xy}}{x} = \frac{xe^{2xy}}{y}$$

$$x^2 = y^2 \rightarrow x = \pm y$$

For $x = y$

$$x^2 + y^2 = 16$$

$$y^2 + y^2 = 16$$

$$y^2 = 8$$

$$\underline{y = \pm 2\sqrt{2} = x}$$

The points: $\left(2\sqrt{2},\ 2\sqrt{2}\right)$ & $\left(-2\sqrt{2},\ -2\sqrt{2}\right)$

For $x = -y$

$$x^2 + y^2 = 16$$

$$y^2 + y^2 = 16$$

$$y^2 = 8 \rightarrow \underline{y = \pm 2\sqrt{2} = -x}$$

The points: $\left(2\sqrt{2},\ -2\sqrt{2}\right)$ & $\left(-2\sqrt{2},\ 2\sqrt{2}\right)$

	$f(x, y) = e^{2xy}$
$\left(2\sqrt{2},\ 2\sqrt{2}\right)$	e^{16}
$\left(-2\sqrt{2},\ -2\sqrt{2}\right)$	e^{16}
$\left(2\sqrt{2},\ -2\sqrt{2}\right)$	e^{-16}
$\left(-2\sqrt{2},\ 2\sqrt{2}\right)$	e^{-16}

Maximum is e^{16} @ $\left(2\sqrt{2},\ 2\sqrt{2}\right)$ & $\left(-2\sqrt{2},\ -2\sqrt{2}\right)$

Minimum is e^{-16} @ $\left(2\sqrt{2},\ -2\sqrt{2}\right)$ & $\left(-2\sqrt{2},\ 2\sqrt{2}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x, y) = x^2 + y^2 \quad \textit{subject to} \quad x^6 + y^6 = 1$$

Solution

$$\nabla f = 2x\hat{\imath} + 2y\hat{\jmath} \qquad\qquad \nabla f = f_x\hat{\imath} + f_y\hat{\jmath}$$

$$g(x, y) = x^6 + y^6 - 1$$

$$\nabla g = 6x^5\hat{\imath} + 6y^5\hat{\jmath}$$

$$2x\hat{\imath} + 2y\hat{\jmath} = \lambda\left(6x^5\hat{\imath} + 6y^5\hat{\jmath}\right) \qquad\qquad \nabla f = \lambda\nabla g$$

$$\begin{array}{l} \hat{\imath} \\ \hat{\jmath} \end{array} \begin{cases} x = 3\lambda x^5 \\ y = 3\lambda y^5 \end{cases} \to \lambda = \frac{1}{x^4} = \frac{1}{y^4} \quad or \quad x = 0 \quad or \quad y = 0$$

$$x^4 = y^4 \to x = \pm y$$

For $x = y$

$$x^6 + y^6 = 1$$

$$y^6 = \frac{1}{2}$$

$$\boxed{y = \pm\frac{1}{\sqrt[6]{2}} = x}$$

The points: $\left(\frac{1}{\sqrt[6]{2}},\ \frac{1}{\sqrt[6]{2}}\right)$ & $\left(-\frac{1}{\sqrt[6]{2}},\ -\frac{1}{\sqrt[6]{2}}\right)$

For $x = -y$

$$y^6 = \frac{1}{2}$$

$$\boxed{y = \pm\frac{1}{\sqrt[6]{2}} = -x}$$

The points: $\left(-\frac{1}{\sqrt[6]{2}},\ \frac{1}{\sqrt[6]{2}}\right)$ & $\left(\frac{1}{\sqrt[6]{2}},\ -\frac{1}{\sqrt[6]{2}}\right)$

For $x = 0$

$y^6 = 1$

$\underline{y = \pm 1}$

The points: $(0,\ \pm 1)$

For $y = 0$

$x^6 = 1 \quad \rightarrow \quad \underline{x = \pm 1}$

The points: $(\pm 1,\ 0)$

	$f(x,\ y) = x^2 + y^2$
$\left(\frac{1}{\sqrt[6]{2}},\ \frac{1}{\sqrt[6]{2}}\right)$	$\frac{1}{2^{1/3}} + \frac{1}{2^{1/3}} = \frac{2}{2^{1/3}} = 2^{2/3}$
$\left(-\frac{1}{\sqrt[6]{2}},\ -\frac{1}{\sqrt[6]{2}}\right)$	$2^{2/3}$
$\left(-\frac{1}{\sqrt[6]{2}},\ \frac{1}{\sqrt[6]{2}}\right)$	$2^{2/3}$
$\left(\frac{1}{\sqrt[6]{2}},\ -\frac{1}{\sqrt[6]{2}}\right)$	$2^{2/3}$
$(0,\ \pm 1)$	1
$(\pm 1,\ 0)$	1

Maximum is $2^{2/3}$ @ $\left(\frac{1}{\sqrt[6]{2}},\ \frac{1}{\sqrt[6]{2}}\right), \left(-\frac{1}{\sqrt[6]{2}},\ -\frac{1}{\sqrt[6]{2}}\right), \left(-\frac{1}{\sqrt[6]{2}},\ \frac{1}{\sqrt[6]{2}}\right) \& \left(\frac{1}{\sqrt[6]{2}},\ -\frac{1}{\sqrt[6]{2}}\right)$

Minimum is 1 @ $(0,\ \pm 1)$ & $(\pm 1,\ 0)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x,\ y) = y^2 - 4x^2 \quad \textit{subject to} \quad x^2 + 2y^2 = 4$$

Solution

$\nabla f = -8x\hat{i} + 2y\hat{j} \qquad \nabla f = f_x\hat{i} + f_y\hat{j}$

$g(x,\ y) = x^2 + 2y^2 - 4$

$\nabla g = 2x\hat{i} + 4y\hat{j}$

$-8x\hat{i} + 2y\hat{j} = \lambda\left(2x\hat{i} + 4y\hat{j}\right) \qquad \nabla f = \lambda \nabla g$

$$\begin{array}{l} \hat{i} \\ \hat{j} \end{array}\begin{cases} -4x = \lambda x & \rightarrow x = 0, \quad \lambda = -4 \ (x \neq 0) \\ y = 2\lambda y & \rightarrow y = 0, \quad \lambda = \frac{1}{2} \ (y \neq 0) \end{cases}$$

$$x^2 + 2y^2 = 4$$

For $x = 0$

$$y^2 = 2 \quad \rightarrow \quad \underline{y = \pm\sqrt{2}|}$$

The points: $\left(0, \ \pm\sqrt{2}\right)$

For $y = 0$

$$x^2 = 4 \quad \rightarrow \quad \underline{x = \pm 2|}$$

The points: $(\pm 2, \ 0)$

For $\lambda = -4 = \frac{1}{2}$ contradiction

	$f(x, y) = y^2 - 4x^2$
$\left(0, \ \pm\sqrt{2}\right)$	2
$(\pm 2, \ 0)$	-16

Maximum is 2 @ $\left(0, \ \pm\sqrt{2}\right)$

Minimum is -16 @ $(\pm 2, \ 0)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x, y) = xy + x + y \quad \textit{subject to} \quad x^2y^2 = 4$$

Solution

$$\nabla f = (y+1)\hat{i} + (x+1)\hat{j} \qquad\qquad \nabla f = f_x\hat{i} + f_y\hat{j}$$

$$g(x, y) = x^2y^2 - 4$$

$$\nabla g = 2xy^2\hat{i} + 2yx^2\hat{j}$$

$$(y+1)\hat{i} + (x+1)\hat{j} = \lambda\left(2xy^2\hat{i} + 2yx^2\hat{j}\right) \qquad\qquad \nabla f = \lambda\nabla g$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \end{array}\begin{cases} y+1 = 2\lambda xy^2 \\ x+1 = 2\lambda yx^2 \end{cases} \rightarrow \lambda = \frac{y+1}{2xy^2} = \frac{x+1}{2yx^2}$$

$$2x^2y^2 + 2yx^2 = 2x^2y^2 + 2xy^2$$

$$x^2 y = xy^2 \quad \to \quad \begin{cases} x = 0 \\ y = 0 \\ x = y \end{cases}$$

$$x^2 y^2 = 4$$

For $x = 0$ & $y = 0$

$$0 = 4 \quad \to \quad \textit{Impossible}$$

For $x = y$

$$y^4 = 4 \quad \to \quad \underline{y = \pm\sqrt{2} = x|}$$

The points: $\left(\pm\sqrt{2},\ \pm\sqrt{2}\right)$

	$f(x, y) = xy + x + y$
$\left(\sqrt{2},\ \sqrt{2}\right)$	$2 + 2\sqrt{2}$
$\left(-\sqrt{2},\ -\sqrt{2}\right)$	$2 - 2\sqrt{2}$

Maximum is $2 + 2\sqrt{2}$ @ $\left(\sqrt{2},\ \sqrt{2}\right)$

Minimum is $2 - 2\sqrt{2}$ @ $\left(-\sqrt{2},\ -\sqrt{2}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x, y, z) = x + 3y - z \quad \textit{subject to} \quad x^2 + y^2 + z^2 = 4$$

<u>Solution</u>

$$\nabla f = \hat{i} + 3\hat{j} - \hat{k} \qquad\qquad \nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$

$$g(x, y) = x^2 + y^2 + z^2 - 4$$

$$\nabla g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{i} + 3\hat{j} - \hat{k} = \lambda\left(2x\hat{i} + 2y\hat{j} + 2z\hat{k}\right) \qquad\qquad \nabla f = \lambda \nabla g$$

$$\begin{matrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{matrix} \begin{cases} 1 = 2\lambda x \\ 3 = 2\lambda y \\ -1 = 2\lambda z \end{cases} \quad \to \quad \lambda = \frac{1}{2x} = \frac{3}{2y} = -\frac{1}{2z}$$

$$y = 3x, \quad y = -3z, \quad x = -z \quad x, y, z \neq 0$$

$$x^2 + y^2 + z^2 = 4$$

For $y = 3x$ & $z = -x$

$$x^2+9x^2+x^2=4$$

$$11x^2=4 \quad \to \quad \boxed{x=\pm\frac{2}{\sqrt{11}}}$$

$$y=\pm\frac{6}{\sqrt{11}} \qquad z=\mp\frac{2}{\sqrt{11}}$$

The points: $\left(\pm\frac{2}{\sqrt{11}},\ \pm\frac{6}{\sqrt{11}},\ \mp\frac{2}{\sqrt{11}}\right)$

	$f(x,\ y,\ z)=x+3y-z$
$\left(\frac{2}{\sqrt{11}},\ \frac{6}{\sqrt{11}},\ -\frac{2}{\sqrt{11}}\right)$	$\frac{2}{\sqrt{11}}+\frac{18}{\sqrt{11}}+\frac{2}{\sqrt{11}}=\frac{22}{\sqrt{11}}=2\sqrt{11}$
$\left(-\frac{2}{\sqrt{11}},\ -\frac{6}{\sqrt{11}},\ \frac{2}{\sqrt{11}}\right)$	$-\frac{2}{\sqrt{11}}-\frac{18}{\sqrt{11}}-\frac{2}{\sqrt{11}}=-2\sqrt{11}$

Maximum is $2\sqrt{11}$ @ $\left(\frac{2}{\sqrt{11}},\ \frac{6}{\sqrt{11}},\ -\frac{2}{\sqrt{11}}\right)$

Minimum is $-2\sqrt{11}$ @ $\left(-\frac{2}{\sqrt{11}},\ -\frac{6}{\sqrt{11}},\ \frac{2}{\sqrt{11}}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x,\ y,\ z)=xyz \quad \textit{subject to} \quad x^2+2y^2+4z^2=9$$

Solution

$$\nabla f=yz\hat{i}+xz\hat{j}+xy\hat{k} \qquad\qquad \nabla f=f_x\hat{i}+f_y\hat{j}+f_z\hat{k}$$

$$g(x,\ y)=x^2+2y^2+4z^2-9$$

$$\nabla g=2x\hat{i}+4y\hat{j}+8z\hat{k}$$

$$yz\hat{i}+xz\hat{j}+xy\hat{k}=\lambda\left(2x\hat{i}+4y\hat{j}+8z\hat{k}\right) \qquad\qquad \nabla f=\lambda\nabla g$$

$$\begin{matrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{matrix}\begin{cases}yz=2\lambda x\\ xz=4\lambda y\\ xy=8\lambda z\end{cases} \to \lambda=\frac{yz}{2x}=\frac{xz}{4y}=\frac{xy}{8z}$$

$$2y^2=x^2,\quad 2z^2=y^2,\quad 4z^2=x^2 \qquad x,\ y,\ z\neq 0$$

$$x^2+2y^2+4z^2=9$$

For $y^2=2z^2$ & $x^2=4z^2$

$$4z^2 + 4z^2 + 4z^2 = 9$$

$$12z^2 = 9 \quad \rightarrow \quad \underline{z = \pm\frac{3}{2\sqrt{3}} = \pm\frac{\sqrt{3}}{2}}$$

$$y^2 = \frac{9}{6} \quad \rightarrow y = \pm\frac{3}{\sqrt{6}} = \pm\frac{\sqrt{6}}{2}$$

$$x^2 = 3 \quad \rightarrow x = \pm\sqrt{3}$$

The points: $\left(\pm\frac{\sqrt{3}}{2},\ \pm\frac{\sqrt{6}}{2},\ \pm\sqrt{3}\right)$

	$f(x,\ y,\ z) = xyz$
$\left(\frac{\sqrt{3}}{2},\ \frac{\sqrt{6}}{2},\ \sqrt{3}\right)$	$\frac{\sqrt{54}}{4} = \frac{3\sqrt{6}}{4}$
$\left(-\frac{\sqrt{3}}{2},\ -\frac{\sqrt{6}}{2},\ -\sqrt{3}\right)$	$-\frac{3\sqrt{6}}{4}$

Maximum is $\frac{3\sqrt{6}}{4}$ @ $\left(\frac{\sqrt{3}}{2},\ \frac{\sqrt{6}}{2},\ \sqrt{3}\right)$

Minimum is $-\frac{3\sqrt{6}}{4}$ @ $\left(-\frac{\sqrt{3}}{2},\ -\frac{\sqrt{6}}{2},\ -\sqrt{3}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x,\ y,\ z) = x \quad \textit{subject to} \quad x^2 + y^2 + z^2 - z = 1$$

Solution

$$\nabla f = \hat{i} \qquad\qquad \nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$

$$g(x,\ y) = x^2 + y^2 + z^2 - z - 1$$

$$\nabla g = 2x\hat{i} + 2y\hat{j} + (2z-1)\hat{k}$$

$$\hat{i} = \lambda\left(2x\hat{i} + 2y\hat{j} + (2z-1)\hat{k}\right) \qquad\qquad \nabla f = \lambda \nabla g$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} 1 = 2\lambda x \\ 0 = 2\lambda y \\ 0 = \lambda(2z-1) \end{cases} \quad \rightarrow \lambda = \frac{1}{2x}, \quad y = 0 \ or \ z = \frac{1}{2}$$

$$x^2 + y^2 + z^2 - z = 1$$

For $y = 0$ & $z = \frac{1}{2}$

$$x^2 + \frac{1}{4} - \frac{1}{2} = 1$$

$$x^2 = \frac{5}{4} \quad \to \quad \underline{x = \pm\frac{\sqrt{5}}{2}}$$

The points: $\left(\pm\frac{\sqrt{5}}{2},\ 0,\ \frac{1}{2}\right)$

	$f(x, y, z) = x$
$\left(\frac{\sqrt{5}}{2},\ 0,\ \frac{1}{2}\right)$	$\frac{\sqrt{5}}{2}$
$\left(-\frac{\sqrt{5}}{2},\ 0,\ \frac{1}{2}\right)$	$-\frac{\sqrt{5}}{2}$

Maximum is $\frac{\sqrt{5}}{2}$ @ $\left(\frac{\sqrt{5}}{2},\ 0,\ \frac{1}{2}\right)$

Minimum is $-\frac{\sqrt{5}}{2}$ @ $\left(-\frac{\sqrt{5}}{2},\ 0,\ \frac{1}{2}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x, y, z) = x - z \quad \textit{subject to} \quad x^2 + y^2 + z^2 - y = 2$$

Solution

$$\nabla f = \hat{i} - \hat{k} \qquad\qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$g(x, y) = x^2 + y^2 + z^2 - y - 2$$

$$\nabla g = 2x\hat{i} + (2y-1)\hat{j} + 2z\hat{k}$$

$$\hat{i} - \hat{k} = \lambda\left(2x\hat{i} + (2y-1)\hat{j} + 2z\hat{k}\right) \qquad\qquad \nabla f = \lambda\nabla g$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array}\begin{cases} 1 = 2\lambda x \\ 0 = \lambda(2y-1) \\ -1 = 2\lambda z \end{cases} \quad \to \quad \lambda = \frac{1}{2x} = -\frac{1}{2z}, \quad y = \frac{1}{2}$$

$$\underline{z = -x}$$

$$x^2 + y^2 + z^2 - y = 2$$

For $y = \frac{1}{2}$ & $z = -x$

$$x^2 + \frac{1}{4} + x^2 - \frac{1}{2} = 2$$

$$2x^2 = \frac{9}{4} \quad \rightarrow \quad \boxed{x = \pm\frac{3}{2\sqrt{2}} = \pm\frac{3\sqrt{2}}{4} = -z}$$

The points: $\left(\pm\frac{3\sqrt{2}}{4},\ \frac{1}{2},\ \mp\frac{3\sqrt{2}}{4}\right)$

	$f(x, y, z) = x - z$
$\left(\frac{3\sqrt{2}}{4},\ \frac{1}{2},\ -\frac{3\sqrt{2}}{4}\right)$	$\frac{3\sqrt{2}}{2}$
$\left(-\frac{3\sqrt{2}}{4},\ \frac{1}{2},\ \frac{3\sqrt{2}}{4}\right)$	$-\frac{3\sqrt{2}}{2}$

Maximum is $\frac{3\sqrt{2}}{2}$ @ $\left(\frac{3\sqrt{2}}{4},\ \frac{1}{2},\ -\frac{3\sqrt{2}}{4}\right)$

Minimum is $-\frac{3\sqrt{2}}{2}$ @ $\left(-\frac{3\sqrt{2}}{4},\ \frac{1}{2},\ \frac{3\sqrt{2}}{4}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x, y, z) = x^2 + y^2 + z^2 \quad \textit{subject to} \quad x^2 + y^2 + z^2 - 4xy = 1$$

Solution

$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ $\qquad\qquad \nabla f = f_x\,\hat{i} + f_y\,\hat{j} + f_z\,\hat{k}$

$g(x, y) = x^2 + y^2 + z^2 - 4xy - 1$

$\nabla g = (2x - 4y)\hat{i} + (2y - 4x)\hat{j} + 2z\hat{k}$

$2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda\left((2x - 4y)\hat{i} + (2y - 4x)\hat{j} + 2z\hat{k}\right)$ $\qquad\qquad \nabla f = \lambda\nabla g$

$$\begin{matrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{matrix} \begin{cases} 2x = \lambda(2x - 4y) \\ 2y = \lambda(2y - 4x) \\ \quad 2z = 2\lambda z \qquad \rightarrow z = 0 \ or \ \lambda = 1 \end{cases} \quad \rightarrow \ \lambda = \frac{x}{x - 2y} = \frac{y}{y - 2x}$$

$x^2 + y^2 + z^2 - 4xy = 1$

For $z = 0$

$2z = 2z \ \rightarrow \ z = \pm 1$

$xy - 2x^2 = xy - 2y^2$

$x^2 = y^2$

$\boxed{x = \pm y}$

For $x = y$

$$x^2 + x^2 - 4x^2 = 1$$

$$-2x^2 = 1 \quad \to x^2 = -\frac{1}{2} \quad \textit{impossible}$$

For $x = -y$

$$x^2 + x^2 + 4x^2 = 1$$

$$6x^2 = 1$$

$$\underline{x = \pm\frac{1}{\sqrt{6}} = -y}$$

The points: $\left(\pm\frac{\sqrt{6}}{6},\ \mp\frac{\sqrt{6}}{6},\ 0\right)$

For $\lambda = 1$

$$\lambda = \frac{x}{x-2y} = \frac{y}{y-2x} = 1$$

$$x - 2y = x \quad \to \quad y = 0$$

$$y = y - 2x \quad \to \quad x = 0$$

$$z^2 = 1 \quad \to \quad z = \pm 1$$

The points: $(0,\ 0,\ \pm 1)$

	$f(x, y, z) = x^2 + y^2 + z^2$
$\left(\frac{\sqrt{6}}{6},\ -\frac{\sqrt{6}}{6},\ 0\right)$	$\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$
$\left(-\frac{\sqrt{6}}{6},\ \frac{\sqrt{6}}{6},\ 0\right)$	$\frac{1}{3}$
$(0,\ 0,\ \pm 1)$	1

Maximum is 1 @ $(0,\ 0,\ \pm 1)$

Minimum is $\frac{1}{3}$ @ $\left(-\frac{\sqrt{6}}{6},\ \frac{\sqrt{6}}{6},\ 0\right)$ & $\left(\frac{\sqrt{6}}{6},\ -\frac{\sqrt{6}}{6},\ 0\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x, y, z) = x + y + z \quad \textit{subject to} \quad x^2 + y^2 + z^2 - 2x - 2y = 1$$

Solution

$\nabla f = \hat{i} + \hat{j} + \hat{k}$ $\qquad\qquad$ $\nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$

$g(x, y) = x^2 + y^2 + z^2 - 2x - 2y - 1$

$\nabla g = (2x-2)\hat{i} + (2y-2)\hat{j} + 2z\hat{k}$

$\hat{i} + \hat{j} + \hat{k} = \lambda\left((2x-2)\hat{i} + (2y-2)\hat{j} + 2z\hat{k}\right)$ $\qquad\qquad$ $\nabla f = \lambda \nabla g$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} 1 = \lambda(2x-2) \\ 1 = \lambda(2y-2) \\ \quad 1 = 2\lambda z \end{cases} \rightarrow \lambda = \frac{1}{2x-2} = \frac{1}{2y-2} = \frac{1}{2z}$$

$$\frac{1}{x-1} = \frac{1}{y-1} = \frac{1}{z}$$

$x = y, \quad z = y - 1 = x - 1, \quad x, y \neq 1, \quad z \neq 0$

$x^2 + y^2 + z^2 - 2x - 2y = 1$

For $x = y = z + 1$

$(z+1)^2 + (z+1)^2 + z^2 - 2(z+1) - 2(z+1) = 1$

$2z^2 + 4z + 2 + z^2 - 4z - 4 = 1$

$3z^2 = 3 \quad \rightarrow \quad \underline{z = \pm 1}$

$z = -1 \quad \rightarrow \quad x = y = 0$

$z = 1 \quad \rightarrow \quad x = y = 2$

The points: $(0, 0, -1)$ & $(2, 2, 1)$

	$f(x, y, z) = x + y + z$
$(0, 0, -1)$	-1
$(2, 2, 1)$	5

Maximum is 5 @ $(2, 2, 1)$

Minimum is -1 @ $(0, 0, -1)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x, y, z) = 2x + z^2 \quad \textit{subject to} \quad x^2 + y^2 + 2z^2 = 25$$

Solution

$\nabla f = 2\hat{i} + 2z\hat{k}$ $\qquad\qquad\qquad\qquad$ $\nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$

$g(x, y) = x^2 + y^2 + 2z^2 - 25$

$\nabla g = 2x\hat{i} + 2y\hat{j} + 4z\hat{k}$

$2\hat{i} + 2z\hat{k} = \lambda\left(2x\hat{i} + 2y\hat{j} + 4z\hat{k}\right)$ $\qquad\qquad\qquad\qquad$ $\nabla f = \lambda \nabla g$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array}\begin{cases} 2 = 2\lambda x & \rightarrow \lambda = \dfrac{1}{x} \\ 0 = 2\lambda y & \rightarrow y = 0 \\ 2z = 4\lambda z & \rightarrow \lambda = \dfrac{1}{2} \ \textit{or } z = 0 \end{cases}$$

$x^2 + y^2 + 2z^2 = 25$

For $\lambda = \frac{1}{2}$ $\quad y = 0$

$$x = \frac{1}{\lambda} = 2$$

$$4 + 2z^2 = 25$$

$$2z^2 = 21 \quad \rightarrow \quad \underline{z = \pm\sqrt{\frac{21}{2}}}$$

The points: $\left(2,\ 0,\ \pm\sqrt{\frac{21}{2}}\right)$

For $z = 0$ $\quad y = 0$

$$x^2 = 25 \quad \rightarrow \quad \underline{x = \pm 5}$$

The points: $(\pm 5,\ 0,\ 0)$

	$f(x, y, z) = 2x + z^2$
$\left(2,\ 0,\ \pm\sqrt{\frac{21}{2}}\right)$	$4 + \frac{21}{2} = \frac{29}{2}$
$(5,\ 0,\ 0)$	10
$(-5,\ 0,\ 0)$	-10

Maximum is $\frac{29}{2}$ @ $\left(2,\ 0,\ \pm\sqrt{\frac{21}{2}}\right)$

Minimum is -10 @ $(-5,\ 0,\ 0)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$$f(x, y, z) = x^2 + y^2 - z \quad \textit{subject to} \quad z = 2x^2y^2 + 1$$

Solution

$\nabla f = 2x\hat{i} + 2y\hat{j} - \hat{k}$ $\qquad\qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$

$g(x, y) = 2x^2y^2 + 1 - z$

$\nabla g = 4xy^2\hat{i} + 4x^2y\hat{j} - \hat{k}$

$2x\hat{i} + 2y\hat{j} - \hat{k} = \lambda\left(4xy^2\hat{i} + 4x^2y\hat{j} - \hat{k}\right)$ $\qquad\qquad \nabla f = \lambda\nabla g$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} 2x = 4\lambda xy^2 & \rightarrow x = 0,\ \lambda = \dfrac{1}{2y^2} \\ 2y = 4\lambda x^2 y & \rightarrow y = 0,\ \lambda = \dfrac{1}{2x^2} \\ -1 = -\lambda & \rightarrow \lambda = 1 \end{cases}$$

$z = 2x^2y^2 + 1$

For $x = 0 \quad y = 0$

$\underline{z = 1}|$

The points: $(0,\ 0,\ 1)$

For $\lambda = 1$

$\lambda = \dfrac{1}{2y^2} = \dfrac{1}{2x^2} = 1$

$x^2 = \dfrac{1}{2} \rightarrow \underline{x = \pm\dfrac{1}{\sqrt{2}}}|$

$y^2 = \dfrac{1}{2} \rightarrow \underline{y = \pm\dfrac{1}{\sqrt{2}}}|$

$z = 2\dfrac{1}{2}\dfrac{1}{2} + 1 = \underline{\dfrac{3}{2}}|$

The points: $\left(\pm\dfrac{1}{\sqrt{2}},\ \pm\dfrac{1}{\sqrt{2}},\ \dfrac{3}{2}\right)$

	$f(x, y, z) = x^2 + y^2 - z$
$(0,\ 0,\ 1)$	-1
$\left(\pm\frac{1}{\sqrt{2}},\ \pm\frac{1}{\sqrt{2}},\ \frac{3}{2}\right)$	$\frac{1}{2} + \frac{1}{2} - \frac{3}{2} = -\frac{1}{2}$

Maximum is $-\frac{1}{2}$ @ $\left(-\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ \frac{3}{2}\right),\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ \frac{3}{2}\right),\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ \frac{3}{2}\right),\left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ \frac{3}{2}\right)$

Minimum is -1 @ $(0,\ 0,\ 1)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x,\ y,\ z)=\sqrt{xyz} \quad \textit{subject to} \quad x+y+z=1 \quad \textit{with} \ \ x\ge 0,\ y\ge 0,\ z\ge 0$$

Solution

$$\nabla f=\frac{1}{2}\frac{yz}{\sqrt{xyz}}\hat{i}+\frac{1}{2}\frac{xz}{\sqrt{xyz}}\hat{j}+\frac{1}{2}\frac{xy}{\sqrt{xyz}}\hat{k} \qquad \nabla f=f_x\hat{i}+f_y\hat{j}+f_z\hat{k}$$

$$g(x,\ y)=x+y+z-1=0$$

$$\nabla g=\hat{i}+\hat{j}+\hat{k}$$

$$\frac{1}{2}\frac{yz}{\sqrt{xyz}}\hat{i}+\frac{1}{2}\frac{xz}{\sqrt{xyz}}\hat{j}+\frac{1}{2}\frac{xy}{\sqrt{xyz}}\hat{k}=\lambda\left(\hat{i}+\hat{j}+\hat{k}\right) \qquad \nabla f=\lambda\nabla g$$

$$\begin{array}{l}\hat{i}\\ \hat{j}\\ \hat{k}\end{array}\begin{cases}\lambda=\frac{1}{2}\frac{yz}{\sqrt{xyz}}\\[2ex] \lambda=\frac{1}{2}\frac{xz}{\sqrt{xyz}}\\[2ex] \lambda=\frac{1}{2}\frac{xy}{\sqrt{xyz}}\end{cases} \rightarrow \lambda=\frac{1}{2}\frac{yz}{\sqrt{xyz}}=\frac{1}{2}\frac{xz}{\sqrt{xyz}}=\frac{1}{2}\frac{xy}{\sqrt{xyz}}$$

$$xy=xz=yz \quad x,\ y,\ z\ \neq 0$$

$$\underline{x=y=z}|$$

$$\begin{cases}xy=xz & \rightarrow x=0, \quad y=z\\ xy=yz & \rightarrow y=0, \quad x=z\\ xz=yz & \rightarrow z=0, \quad x=y\end{cases}$$

$$x+y+z=1$$

For $x=y=z$

$$3z=1 \quad \rightarrow \underline{z=\frac{1}{3}=y=x}|$$

The point: $\left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$

For $x=y=0$

$$\underline{z=1}|$$

The point: $(0,\ 0,\ 1)$

$x = z = 0 \rightarrow \quad (0,\ 1,\ 0)$

$y = z = 0 \rightarrow \quad (1,\ 0,\ 0)$

	$f(x,\ y,\ z) = \sqrt{xyz}$
$(0,\ 0,\ 1)$	0
$(0,\ 1,\ 0)$	0
$(1,\ 0,\ 0)$	0
$\left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$	$\frac{1}{3\sqrt{3}}$

Maximum is $\frac{1}{3\sqrt{3}}$ @ $\left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$

Minimum is 0 @ $(0,\ 0,\ 1),\ (0,\ 1,\ 0),\ (1,\ 0,\ 0)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$$f(x,\ y,\ z) = x^2 + y^2 + z^2 \quad \textit{subject to} \quad x^2 + y^2 + z^2 - 4xy = 1$$

Solution

$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ $\qquad\qquad$ $\nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$

$g(x,\ y) = x^2 + y^2 + z^2 - 4xy - 1$

$\nabla g = (2x - 4y)\hat{i} + (2y - 4x)\hat{j} + 2z\hat{k}$

$2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda\left((2x - 4y)\hat{i} + (2y - 4x)\hat{j} + 2z\hat{k}\right)$ $\qquad\qquad$ $\nabla f = \lambda \nabla g$

$$\begin{array}{c} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} 2x = 2\lambda(x - 2y) & \rightarrow \lambda = \dfrac{x}{x - 2y} \\ 2y = 2\lambda(y - 2x) & \rightarrow \lambda = \dfrac{y}{y - 2x} \\ \quad 2z = 2\lambda z & \rightarrow z = 0,\quad \lambda = 1 \end{cases}$$

$$\lambda = \frac{x}{x - 2y} = \frac{y}{y - 2x}$$

$$xy - 2x^2 = xy - 2y^2$$

$$x^2 = y^2 \quad \rightarrow \quad x = \pm y$$

$x^2 + y^2 + z^2 - 4xy = 1$

For $z = 0$

$x = y$

$$x^2 + x^2 - 4x^2 = 1 \quad \rightarrow \quad -2x^2 = 1 \quad (\textit{impossible})$$

$x = -y$

$$x^2 + x^2 + 4x^2 = 1$$

$$x^2 = \frac{1}{6}$$

$$\underline{x = \pm\frac{1}{\sqrt{6}} = -y}|$$

The point: $\left(\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0\right), \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ 0\right)$

For $\lambda = 1$

$$\frac{x}{x-2y} = 1 \ \rightarrow\ \underline{y = 0}|$$

$$\frac{y}{y-2x} = 1 \ \rightarrow\ \underline{x = 0}|$$

$$z^2 = 1 \ \rightarrow\ \underline{z = \pm 1}|$$

The point: $(0,\ 0,\ \pm 1)$

	$f(x, y, z) = x^2 + y^2 + z^2$
$\left(\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0\right)$	$\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$
$\left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ 0\right)$	$\frac{1}{3}$
$(0,\ 0,\ \pm 1)$	1

Maximum is 1 @ $(0,\ 0,\ \pm 1)$

Minimum is $\frac{1}{3}$ @ $\left(\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0\right), \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ 0\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of *f* (when they exist) subject to the given constraint

$f(x, y, z) = x + 2y - z \quad \textit{subject to} \quad x^2 + y^2 + z^2 = 1$

Solution

$$\nabla f = \hat{i} + 2\hat{j} - \hat{k} \qquad\qquad \nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$

$$g(x, y) = x^2 + y^2 + z^2 - 1$$

$$\nabla g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{i} + 2\hat{j} - \hat{k} = \lambda\left(2x\hat{i} + 2y\hat{j} + 2z\hat{k}\right) \qquad \nabla f = \lambda \nabla g$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} 2\lambda x = 1 & \to x = \dfrac{1}{2\lambda} \\ 2\lambda y = 2 & \to y = \dfrac{1}{\lambda} \\ 2\lambda z = -1 & \to z = -\dfrac{1}{2\lambda} \end{cases}$$

$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} = 1$$

$$\frac{6}{4\lambda^2} = 1 \;\to\; \lambda^2 = \frac{6}{4}$$

$$\boxed{\lambda = \pm\frac{\sqrt{6}}{2}}$$

For $\lambda = -\frac{\sqrt{6}}{2}$

$$\begin{cases} x = -\dfrac{\sqrt{6}}{6} \\ y = -\dfrac{\sqrt{6}}{3} \\ z = \dfrac{\sqrt{6}}{6} \end{cases} \to \left(-\frac{\sqrt{6}}{6},\ -\frac{\sqrt{6}}{3},\ \frac{\sqrt{6}}{6}\right)$$

For $\lambda = \frac{\sqrt{6}}{2}$

$$\begin{cases} x = \dfrac{\sqrt{6}}{6} \\ y = \dfrac{\sqrt{6}}{3} \\ z = -\dfrac{\sqrt{6}}{6} \end{cases} \to \left(\frac{\sqrt{6}}{6},\ \frac{\sqrt{6}}{3},\ -\frac{\sqrt{6}}{6}\right)$$

$$f\left(-\frac{\sqrt{6}}{6},\ -\frac{\sqrt{6}}{3},\ \frac{\sqrt{6}}{6}\right) = -\frac{\sqrt{6}}{6} - \frac{2\sqrt{6}}{3} - \frac{\sqrt{6}}{6} \; \boxed{= -\sqrt{6}}$$

$$f\left(\frac{\sqrt{6}}{6},\ \frac{\sqrt{6}}{3},\ -\frac{\sqrt{6}}{6}\right) = \frac{\sqrt{6}}{6} + \frac{2\sqrt{6}}{3} + \frac{\sqrt{6}}{6} \; \boxed{= \sqrt{6}}$$

Maximum is $\sqrt{6}$ @ $\left(\frac{\sqrt{6}}{6},\ \frac{\sqrt{6}}{3},\ -\frac{\sqrt{6}}{6}\right)$

Minimum is $-\sqrt{6}$ @ $\left(-\frac{\sqrt{6}}{6},\ -\frac{\sqrt{6}}{3},\ \frac{\sqrt{6}}{6}\right)$

Exercise

Use Lagrange multipliers to find the maximum and minimum values of f (when they exist) subject to the given constraint

$f(x, y, z) = x^2y^2z$ *subject to* $2x^2 + y^2 + z^2 = 25$

Solution

$$\nabla f = 2xy^2z\hat{i} + 2x^2yz\hat{j} + x^2y^2\hat{k} \qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$g(x, y) = 2x^2 + y^2 + z^2 - 25$$

$$\nabla g = 4x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$2xy^2z\hat{i} + 2x^2yz\hat{j} + x^2y^2\hat{k} = \lambda\left(4x\hat{i} + 2y\hat{j} + 2z\hat{k}\right) \qquad \nabla f = \lambda\nabla g$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} 4\lambda x = 2xy^2z & \to x = 0, \ \ \lambda = \frac{1}{2}y^2z \\ 2\lambda y = 2x^2yz & \to y = 0, \ \ \lambda = x^2z \\ 2\lambda z = x^2y^2 & \to \ \lambda = \dfrac{x^2y^2}{2z} \end{cases}$$

$$\therefore \ (0, \ 0, \ 0)$$

$$\lambda = \frac{1}{2}y^2z = x^2z \ \to \ y^2 = 2x^2$$

$$\lambda = \frac{1}{2}y^2z = \frac{x^2y^2}{2z} \ \to \ z^2 = x^2 = \frac{1}{2}y^2$$

$$2x^2 + y^2 + z^2 = 25$$

$$y^2 + y^2 + \frac{1}{2}y^2 = 25$$

$$\frac{5}{2}y^2 = 25 \ \to \ y = \pm 5\sqrt{\frac{2}{5}} = \pm\sqrt{10}$$

$$x^2 = \frac{1}{2}y^2 = 5 \ \to \ x = z = \pm\sqrt{5}$$

$$\therefore \ \left(\pm\sqrt{5}, \ \pm\sqrt{10}, \ \pm\sqrt{5}\right)$$

$$f(x, y, z) = x^2y^2z$$

$$f\left(\pm\sqrt{5}, \ \pm\sqrt{10}, \ \sqrt{5}\right) = (5)(10)\sqrt{5} = 50\sqrt{5}$$

$$f\left(\pm\sqrt{5}, \ \pm\sqrt{10}, \ -\sqrt{5}\right) = (5)(10)\left(-\sqrt{5}\right) = -50\sqrt{5}$$

Maximum is $50\sqrt{5}$ @ $\left(\pm\sqrt{5}, \ \pm\sqrt{10}, \ \sqrt{5}\right)$

Minimum is $-50\sqrt{5}$ @ $\left(\pm\sqrt{5}, \ \pm\sqrt{10}, \ -\sqrt{5}\right)$

Exercise

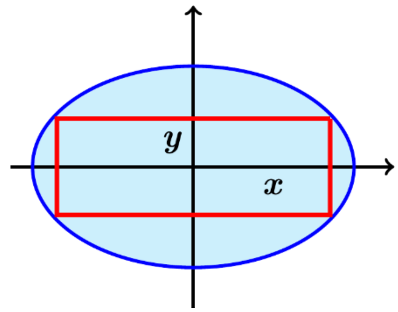
Use Lagrange multipliers to find the dimensions of the rectangle with the maximum perimeter that can be inscribed with sides parallel to the coordinate axes in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution

Let $(x,\ y)$ be the corner of the rectangle in QI.

Perimeter: $P=4(x+y)$

$f(x,\ y)=x+y$ subject to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



$\nabla f=\hat{i}+\hat{j}$ $\qquad\qquad$ $\nabla f=f_x\hat{i}+f_y\hat{j}$

$g(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$

$\nabla g=\frac{2x}{a^2}\hat{i}+\frac{2y}{b^2}\hat{j}$

$\hat{i}+\hat{j}=\lambda\left(\frac{2x}{a^2}\hat{i}+\frac{2y}{b^2}\hat{j}\right)$ $\qquad\qquad$ $\nabla f=\lambda\nabla g$

$$\begin{array}{l} \hat{i} \\ \hat{j} \end{array}\begin{cases} \frac{2x}{a^2}\lambda=1 & \rightarrow\ 2\lambda=\frac{a^2}{x} \\ \frac{2y}{b^2}\lambda=1 & \rightarrow\ 2\lambda=\frac{b^2}{y} \end{cases}$$

$\frac{a^2}{x}=\frac{b^2}{y}\quad\rightarrow\quad y=\frac{b^2}{a^2}x$

$\frac{x^2}{a^2}+\frac{1}{b^2}\frac{b^4}{a^4}x^2=1$

$\left(\frac{1}{a^2}+\frac{b^2}{a^4}\right)x^2=1$

$x^2=\frac{a^4}{a^2+b^2}\quad\Rightarrow\quad x=\frac{a^2}{\sqrt{a^2+b^2}}\quad\in QI$

$\rightarrow\ y=\frac{a^2}{\sqrt{a^2+b^2}}$

Dimension of rectangle with greatest perimeter are $\frac{a^2}{\sqrt{a^2+b^2}}$ *by* $\frac{b^2}{\sqrt{a^2+b^2}}$

Exercise

Use Lagrange multipliers to find the dimensions of the right circular cylinder of minimum surface area (including the circular ends) with a volume of $32\pi\ in^3$

Solution

$V = 32\pi\ in^3$

Circular cylinder: Let r: radius h: height

Surface area: $A = 2\pi r^2 + 2\pi rh$

Volume: $V = \pi r^2 h$

Suffices to Minimize $f(r,\ h) = \pi r^2 + \pi rh$ subject to $\pi r^2 h = 32$

$f(r,\ h) = \pi r^2 + \pi rh$

$\nabla f = (2r + h)\hat{i} + r\hat{j}$

$g(r,\ h) = \pi r^2 h - 32$

$\nabla g = 2rh\hat{i} + r^2\hat{j}$

$(2r + h)\hat{i} + r\hat{j} = 2rh\lambda\hat{i} + r^2\lambda\hat{j}$ $\qquad \nabla f = \lambda \nabla g$

$$\begin{cases} 2\lambda rh = 2r + h \\ \lambda r^2 = r \quad \rightarrow \boxed{r = \dfrac{1}{\lambda}} \end{cases}$$

$2\lambda \dfrac{1}{\lambda} h = 2\dfrac{1}{\lambda} + h$

$\boxed{h = \dfrac{2}{\lambda}}$

$\pi r^2 h = 32$

$\pi \dfrac{2}{\lambda^3} = 32$

$\lambda^3 = \dfrac{\pi}{16} \quad \rightarrow \quad \boxed{\lambda = \dfrac{1}{2}\sqrt[3]{\dfrac{\pi}{2}}}$

$\boxed{r = 2\sqrt[3]{\dfrac{2}{\pi}}\ in}$

$\boxed{h = 4\sqrt[3]{\dfrac{2}{\pi}}\ in}$

Exercise

Find the point(s) on the cone $z^2 - x^2 - y^2 = 0$ that are closest to the point $(1,\ 3,\ 1)$. Give an argument showing you have found an absolute minimum of the distance function.

Solution

$$f(x,y,z) = (x-1)^2 + (y-3)^2 + (z-1)^2$$

Subject to $g(x,y,z) = z^2 - x^2 - y^2$

$$\nabla f = 2(x-1)\hat{i} + 2(y-3)\hat{j} + 2(z-1)\hat{k}$$

$$\nabla g = -2x\hat{i} - 2y\hat{j} + 2z\hat{k}$$

$$\nabla f = \lambda \nabla g$$

$$2(x-1) = -2\lambda x$$

$$x - 1 = -\lambda x \quad \rightarrow \quad x = \frac{1}{1+\lambda}$$

$$2(y-3) = -2\lambda y$$

$$y - 3 = -\lambda y \quad \rightarrow \quad x = \frac{3}{1+\lambda}$$

$$2(z-1) = 2\lambda z$$

$$z - 1 = \lambda z \quad \rightarrow \quad z = \frac{1}{1-\lambda}$$

$$z^2 - x^2 - y^2 = 0$$

$$\frac{1}{(1-\lambda)^2} - \frac{1}{(1+\lambda)^2} - \frac{9}{(1+\lambda)^2} = 0$$

$$\frac{1}{(1-\lambda)^2} - \frac{10}{(1+\lambda)^2} = 0$$

$$1 + 2\lambda + \lambda^2 - 10 + 20\lambda - 10\lambda^2 = 0$$

$$-9\lambda^2 + 22\lambda - 9 = 0 \quad \rightarrow \quad \lambda = \frac{-22 \pm \sqrt{160}}{-18} = \frac{11 \mp 2\sqrt{10}}{9}$$

For $\lambda = \frac{11 - 2\sqrt{10}}{9}$

$$x = \frac{9}{9 + 11 - 2\sqrt{10}}$$

$$= \frac{1}{2}\frac{9}{10 - \sqrt{10}}\frac{10+\sqrt{10}}{10+\sqrt{10}}$$

$$= \frac{1}{20}\left(10 + \sqrt{10}\right)$$

$$y = \frac{27}{9 + 11 - 2\sqrt{10}}$$

$$= \frac{1}{2}\frac{27}{10-\sqrt{10}}\frac{10+\sqrt{10}}{10+\sqrt{10}}$$

$$\boxed{= \frac{3}{20}\left(10+\sqrt{10}\right)}$$

$$z = \frac{9}{9-11+2\sqrt{10}}$$

$$= -\frac{1}{2}\frac{9}{1-\sqrt{10}}\frac{1+\sqrt{10}}{1+\sqrt{10}}$$

$$\boxed{= \frac{1}{2}\left(1+\sqrt{10}\right)}$$

$$\therefore \left(\frac{1}{2}+\frac{\sqrt{10}}{20},\ \frac{3}{2}+\frac{3\sqrt{10}}{20},\ \frac{1}{2}+\frac{\sqrt{10}}{2}\right)$$

For $\lambda = \frac{11+2\sqrt{10}}{9}$

$$x = \frac{9}{9+11+2\sqrt{10}}$$

$$= \frac{1}{2}\frac{9}{10+\sqrt{10}}\frac{10-\sqrt{10}}{10-\sqrt{10}}$$

$$\boxed{= \frac{1}{20}\left(10-\sqrt{10}\right)}$$

$$y = \frac{27}{9+11+2\sqrt{10}}$$

$$= \frac{1}{2}\frac{27}{10+\sqrt{10}}\frac{10-\sqrt{10}}{10-\sqrt{10}}$$

$$\boxed{= \frac{3}{20}\left(10-\sqrt{10}\right)}$$

$$z = \frac{9}{9-11-2\sqrt{10}}$$

$$= \frac{1}{2}\frac{9}{1+\sqrt{10}}\frac{1-\sqrt{10}}{1-\sqrt{10}}$$

$$\boxed{= \frac{1}{20}\left(1-\sqrt{10}\right)}$$

$$\therefore \left(\frac{1}{2}-\frac{\sqrt{10}}{20},\ \frac{3}{2}-\frac{3\sqrt{10}}{20},\ \frac{1}{2}-\frac{\sqrt{10}}{2}\right)$$

Therefore, there are 3 solutions to the Lagrange conditions:

$$\left(\frac{1}{2}\pm\frac{\sqrt{10}}{20},\ \frac{3}{2}\pm\frac{3\sqrt{10}}{20},\ \frac{1}{2}\pm\frac{\sqrt{10}}{2}\right) \ \&\ (0,\ 0,\ 0)$$

$$f(x,y,z)=(x-1)^2+(y-3)^2+(z-1)^2$$

$$f(0,\ 0,\ 0)\ \underline{=11}$$

$$\begin{aligned}
f\left(\frac{1}{2}+\frac{\sqrt{10}}{20},\ \frac{3}{2}+\frac{3\sqrt{10}}{20},\ \frac{1}{2}+\frac{\sqrt{10}}{2}\right) &= \left(\frac{1}{2}+\frac{\sqrt{10}}{20}-1\right)^2+\left(\frac{3}{2}+\frac{3\sqrt{10}}{20}-3\right)^2+\left(\frac{1}{2}+\frac{\sqrt{10}}{2}-1\right)^2 \\
&= \left(\frac{\sqrt{10}}{20}-\frac{1}{2}\right)^2+\left(\frac{3\sqrt{10}}{20}-\frac{3}{2}\right)^2+\left(\frac{\sqrt{10}}{2}-\frac{1}{2}\right)^2 \\
&= \frac{1}{400}\left(\sqrt{10}-10\right)^2+\frac{9}{400}\left(\sqrt{10}-10\right)^2+\frac{1}{4}\left(\sqrt{10}-1\right)^2 \\
&= \frac{1}{40}\left(110-20\sqrt{10}\right)+\frac{1}{4}\left(11-2\sqrt{10}\right) \\
&= \frac{11}{4}-\frac{1}{2}\sqrt{10}+\frac{11}{4}-\frac{1}{2}\sqrt{10} \\
&= \underline{\frac{11}{2}-\sqrt{10}}
\end{aligned}$$

$$\begin{aligned}
f\left(\frac{1}{2}-\frac{\sqrt{10}}{20},\ \frac{3}{2}-\frac{3\sqrt{10}}{20},\ \frac{1}{2}-\frac{\sqrt{10}}{2}\right) &= \left(\frac{1}{2}-\frac{\sqrt{10}}{20}-1\right)^2+\left(\frac{3}{2}-\frac{3\sqrt{10}}{20}-3\right)^2+\left(\frac{1}{2}-\frac{\sqrt{10}}{2}-1\right)^2 \\
&= \left(-\frac{\sqrt{10}}{20}-\frac{1}{2}\right)^2+\left(-\frac{3\sqrt{10}}{20}-\frac{3}{2}\right)^2+\left(-\frac{\sqrt{10}}{2}-\frac{1}{2}\right)^2 \\
&= \frac{1}{400}\left(\sqrt{10}+10\right)^2+\frac{9}{400}\left(\sqrt{10}+10\right)^2+\frac{1}{4}\left(\sqrt{10}+1\right)^2 \\
&= \frac{1}{40}\left(110+20\sqrt{10}\right)+\frac{1}{4}\left(11+2\sqrt{10}\right) \\
&= \frac{11}{4}+\frac{1}{2}\sqrt{10}+\frac{11}{4}+\frac{1}{2}\sqrt{10} \\
&= \underline{\frac{11}{2}+\sqrt{10}}
\end{aligned}$$

The closest point is $\left(\frac{1}{2}+\frac{\sqrt{10}}{20},\ \frac{3}{2}+\frac{3\sqrt{10}}{20},\ \frac{1}{2}+\frac{\sqrt{10}}{2}\right)$

Exercise

Let $P_0(a,\ b,\ c)$ be a fixed point in $\mathbb{R}^3$ and let $d(x,\ y,\ z)$ be the distance between P_0 and a variable point $P(x,\ y,\ z)$.

a) Compute $\nabla d(x,\ y,\ z)$

b) Show that $\nabla d(x,\ y,\ z)$ points in the direction from P_0 to P and has magnitude 1 for all $(x,\ y,\ z)$.

c) Describe the level surfaces of *d* and give the direction of $\nabla d(x,\ y,\ z)$ relative to the level surfaces of *d*.

d) Discuss $\lim\limits_{P\to P_0} \nabla d(x,\ y,\ z)$

Solution

$$d(x,y,z) = \left|\overrightarrow{PP_0}\right|$$

$$= \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

a) $\nabla d(x,y,z) = \dfrac{1}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}}\langle x-a,\ \ y-b,\ \ z-c\rangle$

b) $\nabla d(x,y,z) = \dfrac{1}{d(x,y,z)}\langle x-a,\ \ y-b,\ \ z-c\rangle$

$$= \frac{1}{\left|\overrightarrow{PP_0}\right|}\overrightarrow{PP_0}$$

$$\underline{= d(x,y,z)\ \big|}$$

$\therefore\ \ \nabla d(x,y,z)$ is a unit vector.

c) The level surfaces of *d* are spheres centered at *a, b, c* and $\nabla d(x,y,z)$ is $\perp$ to these spheres, pointing outwards.

d) $\lim\limits_{P\to P_0} \nabla d(x,y,z) = \nexists$

Because of $P \to P_0$ in the direction of a unit vector $\vec{u}$.

$\nabla d(x,y,z) = \pm\vec{u}$, but the limit must be the same in all directions.

Exercise

A shipping company requires that the sum of length plus girth of rectangular boxes must not exceed 108 *in*. Find the dimensions of the box with maximum volume that meets this condition. (the girth is the perimeter of the smallest base of the box).

Solution

Let *x, y*, and *z* represent the length, width, and height.

The girth is: $= 2y + 2z(= P)$

Volume: $V = xyz$

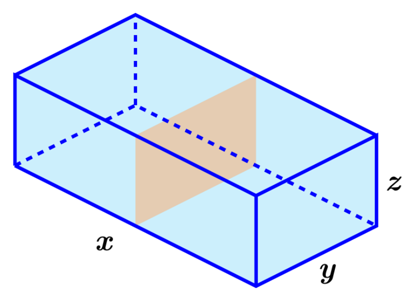
We want to maximize the volume of the box satisfying: $x + 2y + 2z = 108$

$f(x, y, z) = xyz \quad subject\ to \quad g(x, y) = x + 2y + 2z - 108$

$\nabla f = yz\hat{i} + xz\hat{j} + xy\hat{k} \qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$

$\nabla g = \hat{i} + 2\hat{j} + 2\hat{k}$

$yz\hat{i} + xz\hat{j} + xy\hat{k} = \lambda\left(\hat{i} + 2\hat{j} + 2\hat{k}\right) \qquad \nabla f = \lambda \nabla g$



$$\begin{array}{c} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \begin{cases} yz = \lambda \\ xz = 2\lambda \\ xy = 2\lambda \end{cases} \rightarrow \lambda = yz = \frac{xz}{2} = \frac{xy}{2} \quad x, y, z \neq 0$$

$$\begin{cases} yz = \frac{1}{2}xz & \rightarrow y = \frac{1}{2}x \\ yz = \frac{1}{2}xy & \rightarrow z = \frac{1}{2}x \\ xz = xy & \rightarrow y = z \end{cases}$$

$x + 2y + 2z = 108$

For $y = \frac{1}{2}x \quad z = \frac{1}{2}x$

$x + x + x = 108$

$3x = 108 \quad \rightarrow \quad \underline{x = 36|}$

$\underline{y = 18 \quad z = 18\ |}$

For $x = 0 \quad y = 0$

$2z = 108 \quad \rightarrow \quad \underline{z = 54|}$

For $x = 0 \quad z = 0$

$2y = 108 \quad \rightarrow \quad \underline{y = 54|}$

For $y = 0 \quad z = 0$

$\underline{x = 108|}$

$\therefore$ The critical points are: $(0, 0, 0), (0, 54, 0), (0, 0, 54), (108, 0, 0), (18, 18, 36)$

$$\boxed{V(0,\ 0,\ 0) = 0}$$

$$\boxed{V(0,\ 54,\ 0) = 0}$$

$$\boxed{V(0,\ 0,\ 54) = 0}$$

$$\boxed{V(108,\ 0,\ 0) = 0}$$

$$V = 36(18)(18)$$
$$\boxed{= 11,664}$$

The dimensions of the package are: $\boldsymbol{x} = 36\ in.$, $\boldsymbol{y} = 18\ in$, $\boldsymbol{z} = 18\ in$.

The maximum volume is $11,664\ \ in^3$

Exercise

Find the rectangular box with a volume of $16\ ft^3$ that has minimum surface area.

Solution

Let *x, y*, and *z* represent the length, width, and height (positive values)

Volume: $V = xyz = 16$

We want to minimum surface area: $\quad 2xy + 2yz + 2xz$

$$f(x,\ y,\ z) = 2xy + 2yz + 2xz \quad \textit{subject to} \quad g(x,\ y) = xyz - 16$$

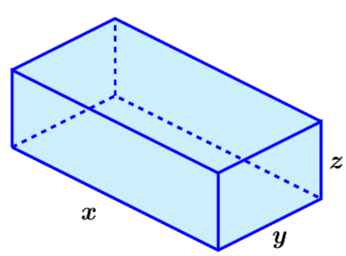
$$\nabla f = 2(y+z)\hat{i} + 2(x+z)\hat{j} + 2(x+y)\hat{k} \qquad \nabla f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

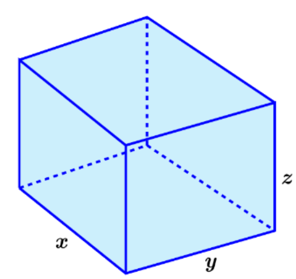
$$\nabla g = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$2(y+z)\hat{i} + 2(x+z)\hat{j} + 2(x+y)\hat{k} = \lambda\left(yz\hat{i} + xz\hat{j} + xy\hat{k}\right) \qquad \nabla f = \lambda \nabla g$$

$$\begin{matrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{matrix} \begin{cases} 2(y+z) = \lambda yz \\ 2(x+z) = \lambda xz \\ 2(x+y) = \lambda xy \end{cases} \rightarrow \lambda = \frac{2(y+z)}{yz} = \frac{2(x+z)}{xz} = \frac{2(x+y)}{xy} \quad x,\ y,\ z \neq 0$$

$$\begin{cases} xy + xz = xy + yz & \rightarrow x = y \\ xy + xz = xz + yz & \rightarrow x = z \\ xy + yz = xz + yz & \rightarrow y = z \end{cases}$$





$xyz = 16$

For $x = y = z$

$$x^3 = 16 = 2^4 \quad \rightarrow \quad \boxed{x = 2\sqrt[3]{2} = y = z}$$

The box is a cube shape box with length of $2\sqrt[3]{2}$

Exercise

Find the minimum and maximum distances between the ellipse $x^2 + xy + 2y^2 = 1$ and the origin.

Solution

$$f(x,\ y) = x^2 + y^2 \quad \textit{subject to} \quad x^2 + xy + y^2 = 1$$

$$\nabla f = 2x\hat{i} + 2y\hat{j} \qquad\qquad \nabla f = f_x \hat{i} + f_y \hat{j}$$

$$g(x,\ y) = x^2 + xy + 2y^2 - 1 = 0$$

$$\nabla g = (2x + y)\hat{i} + (4y + x)\hat{j}$$

$$2x\hat{i} + 2y\hat{j} = \lambda\left((2x + y)\hat{i} + (4y + x)\hat{j}\right) \qquad\qquad \nabla f = \lambda \nabla g$$

$$\begin{matrix} \hat{i} \\ \hat{j} \end{matrix} \begin{cases} 2x = \lambda(2x + y) \\ 2y = \lambda(4y + x) \end{cases} \rightarrow \lambda = \frac{2x}{2x + y} = \frac{2y}{4y + x} \quad x,\ y \neq 0$$

$$8xy + 2x^2 = 4xy + 2y^2$$

$$x^2 + 2xy - y^2 = 0$$

$$\begin{cases} x^2 + xy + 2y^2 = 1 \\ -x^2 - 2xy + y^2 = 0 \end{cases} \rightarrow 3y^2 - xy = 1$$

$$\Rightarrow \quad x = \frac{3y^2 - 1}{y}$$

$$\left(\frac{3y^2 - 1}{y}\right)^2 + 3y^2 - 1 + 2y^2 = 1$$

$$\frac{9y^4 - 6y^2 + 1}{y^2} + 5y^2 - 2 = 0$$

$$14y^4 - 8y^2 + 1 = 0$$

$$y^2 = \frac{8 \pm \sqrt{8}}{28} = \frac{4 \pm \sqrt{2}}{14} \quad \rightarrow \quad \begin{cases} y = \pm\sqrt{\dfrac{4 - \sqrt{2}}{14}} \approx \pm.429766 \\ y = \pm\sqrt{\dfrac{4 + \sqrt{2}}{14}} \approx \pm.621876 \end{cases}$$

$$\begin{cases} y = \sqrt{\dfrac{4 - \sqrt{2}}{14}} & \rightarrow x = \dfrac{3\dfrac{4 - \sqrt{2}}{14} - 1}{\sqrt{\dfrac{4 - \sqrt{2}}{14}}} = \dfrac{\left(-2 - 3\sqrt{2}\right)\sqrt{14}}{14\sqrt{4 - \sqrt{2}}} \approx -1.0375475 \\ y = -\sqrt{\dfrac{4 - \sqrt{2}}{14}} & \rightarrow x = \dfrac{3\dfrac{4 - \sqrt{2}}{14} - 1}{-\sqrt{\dfrac{4 - \sqrt{2}}{14}}} = \dfrac{\left(2 + 3\sqrt{2}\right)\sqrt{14}}{14\sqrt{4 - \sqrt{2}}} \approx 1.0375475 \end{cases}$$

$$\begin{cases} y=\sqrt{\dfrac{4+\sqrt{2}}{14}} & \to x=\dfrac{3\dfrac{4+\sqrt{2}}{14}-1}{\sqrt{\dfrac{4+\sqrt{2}}{14}}}=\dfrac{\left(-2+3\sqrt{2}\right)\sqrt{14}}{14\sqrt{4+\sqrt{2}}}\approx .2575894 \\ y=-\sqrt{\dfrac{4+\sqrt{2}}{14}} & \to x=\dfrac{3\dfrac{4+\sqrt{2}}{14}-1}{-\sqrt{\dfrac{4+\sqrt{2}}{14}}}=\dfrac{\left(2-3\sqrt{2}\right)\sqrt{14}}{14\sqrt{4+\sqrt{2}}}\approx -.2575894 \end{cases}$$

$$f(1.0375475,\ .429766)=(1.0375475)^2+(.429766)^2$$

$$\approx 1.2612 \qquad \textit{\textbf{Maximum}} \text{ value}$$

$$f(.2575894,\ .621876)=(.2575894)^2+(.621876)^2$$

$$\approx 0.453 \qquad \textit{\textbf{Minimum}} \text{ value}$$

Exercise

Find the dimensions of the rectangle of maximum area with sides parallel to the coordinate axes that can be inscribed in the ellipse $4x^2+16y^2=16$

Solution

$$f(x,\ y)=xy \quad \textit{subject to} \quad 4x^2+16y^2=16$$

$$\nabla f=y\hat{i}+x\hat{j}$$

$$g(x,\ y)=4x^2+16y^2-16$$

$$\nabla g=8x\hat{i}+32y\hat{j}$$

$$y\hat{i}+x\hat{j}=\lambda\left(8x\hat{i}+32y\hat{j}\right)$$

$$\begin{matrix}\hat{i} \\ \hat{j}\end{matrix}\begin{cases} y=8\lambda x \\ x=32\lambda y\end{cases} \to \lambda=\frac{y}{8x}=\frac{x}{32y} \quad x,\ y\neq 0$$

$$32y^2=8x^2 \quad \to \quad \underline{x=\pm 2y|}$$

$$4x^2+16y^2=16$$

For $x=2y$ (only positive since its length)

$$16y^2+16y^2=16$$

$$y^2=\frac{1}{2}$$

$$y=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

$\boxed{x = \sqrt{2}}$

$$f\left(\sqrt{2},\ \frac{\sqrt{2}}{2}\right) = 1$$

Maximum area of the rectangle is:

$$Area = 4 \times f\left(\sqrt{2},\ \frac{\sqrt{2}}{2}\right)$$

$$\boxed{= 4\ \ unit^2}$$

Exercise

Find the dimensions of the rectangle of maximum perimeter with sides parallel to the coordinate axes that can be inscribed in the ellipse $2x^2 + 4y^2 = 3$

Solution

$$Perimeter = 4(x + y)$$

$$f(x,\ y) = x + y \quad subject\ to \quad 2x^2 + 4y^2 = 3$$

$$\nabla f = \hat{i} + \hat{j}$$

$$g(x,\ y) = 2x^2 + 4y^2 - 3$$

$$\nabla g = 4x\hat{i} + 8y\hat{j}$$

$$\hat{i} + \hat{j} = \lambda\left(4x\hat{i} + 8y\hat{j}\right)$$

$$\begin{matrix} \hat{i} \\ \hat{j} \end{matrix} \begin{cases} 1 = 4\lambda x \\ 1 = 8\lambda y \end{cases} \rightarrow \lambda = \frac{1}{4x} = \frac{1}{8y} \quad x,\ y > 0$$

$$8y = 4x \quad \rightarrow \quad \boxed{x = 2y}$$

$$2x^2 + 4y^2 = 3$$

For $x = 2y$

$$8y^2 + 4y^2 = 3$$

$$y^2 = \frac{1}{4} \quad \rightarrow \quad \boxed{y = \frac{1}{2}}$$

$$\boxed{x = 1}$$

Dimensions of the rectangle of maximum perimeter is: 2×1

Exercise

Find the point on the plane $2x+3y+6z-10=0$ *closest to the point* $(-2,\ 5,\ 1)$

Solution

$$f(x,\ y,\ z)=(x+2)^2+(y-5)^2+(z-1)^2 \quad \textit{subject to} \quad 2x+3y+6z-10=0$$

$$\nabla f=2(x+2)\hat{i}+2(y-5)\hat{j}+2(z-1)\hat{k}$$

$$g(x,\ y,\ z)=2x+3y+6z-10$$

$$\nabla g=2\hat{i}+3\hat{j}+6\hat{k}$$

$$2(x+2)\hat{i}+2(y-5)\hat{j}+2(z-1)\hat{k}=\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$$

$$\begin{matrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{matrix}\begin{cases}2(x+2)=2\lambda\\ 2(y-5)=3\lambda\\ 2(z-1)=6\lambda\end{cases} \rightarrow \lambda=x+2=\tfrac{2}{3}(y-5)=\tfrac{1}{3}(z-1)$$

$$\begin{cases}x=\lambda-2\\ y=\dfrac{3}{2}\lambda+5\\ z=3\lambda+1\end{cases}$$

$$2x+3y+6z-10=0$$

$$2\lambda-4+\frac{9}{2}\lambda+15+18\lambda+6-10=0$$

$$\frac{49}{2}\lambda=-7 \rightarrow \underline{\lambda=-\frac{2}{7}|}$$

$$\begin{cases}x=-\dfrac{2}{7}-2=-\dfrac{16}{7}\\ y=-\dfrac{3}{7}+5=\dfrac{32}{7}\\ z=-\dfrac{6}{7}+1=\dfrac{1}{7}\end{cases}$$

The closest point is $\left(-\frac{16}{7},\ \frac{32}{7},\ \frac{1}{7}\right)$

$$\begin{aligned}f\left(-\tfrac{16}{7},\ \tfrac{32}{7},\ \tfrac{1}{7}\right)&=\left(-\tfrac{16}{7}+2\right)^2+\left(\tfrac{32}{7}-5\right)^2+\left(\tfrac{1}{7}-1\right)^2\\ &=\frac{4}{49}+\frac{9}{49}+\frac{36}{49}\\ &=1\end{aligned}$$

The distance is 1.

Exercise

Find the point on the surface $4x+y-1=0$ *closest to the point* $(1,\ 2,\ -3)$

Solution

$$f(x,\ y,\ z)=(x-1)^2+(y-2)^2+(z+3)^2 \quad \textit{subject to} \quad 4x+y-1=0$$

$$\nabla f=2(x-1)\hat{i}+2(y-2)\hat{j}+2(z+3)\hat{k}$$

$$g(x,\ y,\ z)=4x+y-1$$

$$\nabla g=4\hat{i}+\hat{j}$$

$$2(x-1)\hat{i}+2(y-2)\hat{j}+2(z+3)\hat{k}=\lambda\left(4\hat{i}+\hat{j}\right)$$

$$\begin{array}{l} \hat{i} \\ \hat{j} \\ \hat{k} \end{array}\begin{cases} 2(x-1)=4\lambda \\ 2(y-2)=\lambda & \rightarrow \lambda=\frac{1}{2}(x-1)=2(y-2) \\ 2(z+3)=0 & \qquad \underline{z=-3} \end{cases}$$

$$\begin{cases} x=2\lambda+1 \\ y=\frac{1}{2}\lambda+2 \end{cases}$$

$$4x+y-1=0$$

$$8\lambda+4+\frac{1}{2}\lambda+2-1=0$$

$$\frac{17}{2}\lambda=-5 \ \rightarrow \ \underline{\lambda=-\frac{10}{17}}$$

$$\begin{cases} x=-\frac{20}{17}+1=-\frac{3}{17} \\ y=-\frac{5}{17}+2=\frac{29}{17} \end{cases}$$

The closest point is $\left(-\frac{3}{17},\ \frac{29}{17},\ -3\right)$

Exercise

Find the points on the cone $z^2=x^2+y^2$ *closest to the point* $(1,\ 2,\ 0)$

Solution

$$f(x,\ y,\ z)=(x-1)^2+(y-2)^2+z^2 \quad \textit{subject to} \quad z^2=x^2+y^2$$

$$\nabla f=2(x-1)\hat{i}+2(y-2)\hat{j}+2z\hat{k}$$

$$g(x,\ y,\ z)=x^2+y^2-z^2$$

$$\nabla g=2x\hat{i}+2y\hat{j}-2z\hat{k}$$

$$2(x-1)\hat{i}+2(y-2)\hat{j}+2z\hat{k}=\lambda\left(2x\hat{i}+2y\hat{j}-2z\hat{k}\right)$$

$$\begin{matrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{matrix}\begin{cases}x-1=\lambda x\\ y-2=\lambda y & \rightarrow \lambda=\dfrac{x-1}{x}=\dfrac{y-2}{y}\\ z=-\lambda z & z=0,\quad \lambda=-1\end{cases}$$

$$xy-y=xy-2x \quad \rightarrow \quad y=2x$$

$$z^2=x^2+y^2$$

For $z=0$

$$x^2+y^2=0 \rightarrow \underline{x=y=0}$$

For $\lambda=-1$

$$\begin{cases}\dfrac{x-1}{x}=-1 & \rightarrow x-1=-x & \Rightarrow \underline{x=\dfrac{1}{2}}\\ \dfrac{y-2}{y}=-1 & \rightarrow y-2=-y & \Rightarrow \underline{y=1}\end{cases}$$

$$z^2=\frac{1}{4}+1=\frac{5}{4}$$

$$z=\pm\frac{\sqrt{5}}{2}$$

The closest point is $\left(\frac{1}{2},\ 1,\ \pm\frac{\sqrt{5}}{2}\right)$

Exercise

Find the minimum and maximum distances between the sphere

$$x^2+y^2+z^2=9 \quad \textit{closest to the point} \quad (2,\ 3,\ 4)$$

Solution

$$f(x,\ y,\ z)=(x-2)^2+(y-3)^2+(z-4)^2 \quad \textit{subject to} \quad x^2+y^2+z^2=9$$

$$\nabla f=2(x-2)\hat{i}+2(y-3)\hat{j}+2(z-4)\hat{k}$$

$$g(x,\ y,\ z)=x^2+y^2+z^2-9$$

$$\nabla g=2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$2(x-2)\hat{i}+2(y-3)\hat{j}+2(z-4)\hat{k}=\lambda\left(2x\hat{i}+2y\hat{j}+2z\hat{k}\right)$$

$$\begin{matrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{matrix}\begin{cases}x-2=\lambda x\\ y-3=\lambda y & \rightarrow \lambda=\dfrac{x-2}{x}=\dfrac{y-3}{y}=\dfrac{z-4}{z}\\ z-4=\lambda z\end{cases}$$

$$\begin{cases} x\lambda = x-2 & \to x = \dfrac{2}{1-\lambda} \\ y\lambda = y-3 & \to y = \dfrac{3}{1-\lambda} \\ z\lambda = z-4 & \to z = \dfrac{4}{1-\lambda} \end{cases}$$

$x^2 + y^2 + z^2 - 9 = 0$

$$\left(\frac{2}{1-\lambda}\right)^2 + \left(\frac{3}{1-\lambda}\right)^2 + \left(\frac{4}{1-\lambda}\right)^2 - 9 = 0$$

$$\frac{29}{(1-\lambda)^2} - 9 = 0$$

$$\frac{29 - 9 + 18\lambda - 9\lambda^2}{(1-\lambda)^2} = 0$$

$$9\lambda^2 - 18\lambda - 20 = 0$$

$$\lambda = \frac{18 \pm \sqrt{1044}}{18}$$

$$= \boxed{\frac{3 \pm \sqrt{29}}{3}}$$

For $\lambda = \frac{3-\sqrt{29}}{3} = 1 - \frac{\sqrt{29}}{3}$

$$\begin{cases} x = \dfrac{2}{1-1+\frac{\sqrt{29}}{3}} = \dfrac{6}{\sqrt{29}} \\ y = \dfrac{3}{1-1+\frac{\sqrt{29}}{3}} = \dfrac{9}{\sqrt{29}} \\ z = \dfrac{4}{1-1+\frac{\sqrt{29}}{3}} = \dfrac{12}{\sqrt{29}} \end{cases}$$

The points are: $\pm\left(\frac{6}{\sqrt{29}}, \frac{9}{\sqrt{29}}, \frac{12}{\sqrt{29}}\right)$

$$f\left(\frac{6}{\sqrt{29}}, \frac{9}{\sqrt{29}}, \frac{12}{\sqrt{29}}\right) = \left(\frac{6}{\sqrt{29}} - 2\right)^2 + \left(\frac{9}{\sqrt{29}} - 3\right)^2 + \left(\frac{12}{\sqrt{29}} - 4\right)^2$$

$$= \frac{36 - 24\sqrt{29} + 116}{29} + \frac{81 - 54\sqrt{29} + 261}{29} + \frac{144 - 96\sqrt{29} + 464}{29}$$

$$= \frac{1{,}102 - 174\sqrt{29}}{29}$$

$$= \boxed{38 - 6\sqrt{29}}$$

$$f\left(-\frac{6}{\sqrt{29}},\ -\frac{9}{\sqrt{29}},\ -\frac{12}{\sqrt{29}}\right)=\left(-\frac{6}{\sqrt{29}}-2\right)^2+\left(-\frac{9}{\sqrt{29}}-3\right)^2+\left(-\frac{12}{\sqrt{29}}-4\right)^2$$

$$=\frac{36+24\sqrt{29}+116}{29}+\frac{81+54\sqrt{29}+261}{29}+\frac{144+96\sqrt{29}+464}{29}$$

$$\underline{=38+6\sqrt{29}}\,\big|$$

$$\sqrt{38-6\sqrt{29}}=\sqrt{9-2\left(3\sqrt{29}\right)+29}$$

$$=\sqrt{\left(\sqrt{29}-3\right)^2}$$

$$=\sqrt{29}-3$$

$\therefore$ ***Minimum*** distance: $\sqrt{29}-3$

$\therefore$ ***Maximum*** distance: $\sqrt{29}+3$

Exercise

Find the maximum value of $x_1+x_2+x_3+x_4$ *subject to* $x_1^2+x_2^2+x_3^2+x_4^2=16$

Solution

$$f\left(x_1,x_2,x_3,x_4\right)=x_1+x_2+x_3+x_4$$

$$\nabla f=\langle 1,\ 1,\ 1,\ 1\rangle$$

$$g\left(x_1,x_2,x_3,x_4\right)=x_1^2+x_2^2+x_3^2+x_4^2-16$$

$$\nabla g=\left\langle 2x_1,\ 2x_2,\ 2x_3,\ 2x_4\right\rangle$$

$$\langle 1,\ 1,\ 1,\ 1\rangle=\lambda\left\langle 2x_1,\ 2x_2,\ 2x_3,\ 2x_4\right\rangle$$

$$\begin{cases}2\lambda x_1=1\\ 2\lambda x_2=1\\ 2\lambda x_3=1\\ 2\lambda x_4=1\end{cases}\rightarrow\ \lambda=\frac{1}{2x_1}=\frac{1}{2x_2}=\frac{1}{2x_2}=\frac{1}{2x_4}$$

$$x_1=x_2=x_3=x_4$$

$$x_1^2+x_2^2+x_3^2+x_4^2=16$$

$$4x_1^2=16$$

$$\underline{x_1=\pm 2=x_2=x_3=x_4}\,\big|$$

$f(2,\ 2,\ 2,\ 2) = 8$ ***Maximum*** value

$f(-2,\ -2,\ -2,\ -2) = -8$ ***Minimum*** value

Exercise

Find the maximum value of $x_1 + x_2 + \cdots + x_n$ *subject to* $x_1^2 + x_2^2 + \cdots + x_n^2 = c^2$

Solution

$$f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \ldots + x_n$$

$$\nabla f = \langle 1,\ 1,\ \ldots,\ 1\rangle$$

$$g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 - c^2$$

$$\nabla g = \langle 2x_1,\ 2x_2, \cdots,\ 2x_n\rangle$$

$$\langle 1,\ 1,\ \ldots,\ 1\rangle = \lambda\langle 2x_1,\ 2x_2,\ \ldots,\ 2x_n\rangle$$

$$\begin{cases} 2\lambda x_1 = 1 \\ 2\lambda x_2 = 1 \\ \vdots \quad\quad \vdots \\ 2\lambda x_n = 1 \end{cases} \rightarrow \lambda = \frac{1}{2x_1} = \frac{1}{2x_2} = \ldots = \frac{1}{2x_n}$$

$$x_1 = x_2 = \cdots = x_n$$

$$x_1^2 + x_2^2 + \cdots + x_n^2 = c^2$$

$$nx_1^2 = c^2$$

$$\boxed{x_1 = \pm\frac{c}{\sqrt{n}} = x_2 = \ldots = x_n}$$

$$f\left(\frac{c}{\sqrt{n}},\ \frac{c}{\sqrt{n}},\ \cdots,\ \frac{c}{\sqrt{n}}\right) = \frac{nc}{\sqrt{n}}$$

$$\boxed{= c\sqrt{n}}$$ ***Maximum*** value

Exercise

Find the maximum value of $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ *subject to* $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ for the given positive real numbers $a_1, a_2, \cdots, a_n$

Solution

$$f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

$$\nabla f = \langle a_1,\ a_2,\ \ldots,\ a_n \rangle$$

$$g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 - 1$$

$$\nabla g = \langle 2x_1,\ 2x_2, \cdots,\ 2x_n \rangle$$

$$\langle a_1,\ a_2,\ \ldots,\ a_n \rangle = \lambda \langle 2x_1,\ 2x_2,\ \ldots,\ 2x_n \rangle$$

$$\begin{cases} 2\lambda x_1 = a_1 \\ 2\lambda x_2 = a_2 \\ \ \vdots \qquad \vdots \\ 2\lambda x_n = a_n \end{cases} \quad \to \quad \lambda = \frac{a_1}{2x_1} = \frac{a_2}{2x_2} = \ldots = \frac{a_n}{2x_n}$$

$$\frac{a_1}{2x_1} = \frac{a_2}{2x_2} = \ldots = \frac{a_n}{2x_n}$$

$$x_2 = \frac{a_2}{a_1}x_1, \quad x_1 = \frac{a_3}{a_1}x_1, \quad \ldots, \quad x_n = \frac{a_n}{a_1}x_1$$

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1$$

$$x_1^2 + \frac{a_2^2}{a_1^2}x_1^2 + \frac{a_3^2}{a_1^2}x_1^2 + \ldots + \frac{a_n^2}{a_1^2}x_1^2 = 1$$

$$a_1^2x_1^2 + a_2^2x_1^2 + a_3^2x_1^2 + \ldots + a_n^2x_1^2 = a_1^2$$

$$\left(a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2\right)x_1^2 = a_1^2$$

$$x_1^2 = \frac{a_1^2}{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}$$

$$x_1 = \pm\frac{a_1}{\sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}}$$

$$x_2 = \pm \frac{a_2}{\sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$x_n = \pm \frac{a_n}{\sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}}$$

The negative values of x's will give us the minimum

$$f\left(x_1, x_2, \ldots, x_n\right) = \frac{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}{\sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}}$$

$$= \boxed{\sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}}$$ ***Maximum*** value

Exercise

The planes $x + 2z = 12$ and $x + y = 6$ intersect in a line L. Find the point on L nearest the origin.

Solution

$f(x, y, z) = x^2 + y^2 + z^2$ *subject to* $g(x, y, z) = x + 2z - 12$ $\quad h(x, y, z) = x + y - 6$

$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$g(x, y, z) = x + 2z - 12 \;\rightarrow\; \nabla g = \hat{i} + 2\hat{k}$

$h(x, y, z) = x + y - 6 \;\rightarrow\; \nabla h = \hat{i} + \hat{j}$

$2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda\left(\hat{i} + 2\hat{k}\right) + \mu\left(\hat{i} + \hat{j}\right) \qquad \nabla f = \lambda \nabla g + \mu \nabla h$

$$\begin{cases} 2x = \lambda + \mu \\ 2y = \mu \\ 2z = 2\lambda \quad \rightarrow z = \lambda \end{cases}$$

$2x = z + 2y \;\rightarrow\; x = \frac{1}{2}z + y$

$x + 2z = 12 \;\rightarrow\; \frac{1}{2}z + y + 2z = 12$

$2y + 5z = 24$

$x + y = 6 \;\; \& \;\; 2y + \frac{1}{2}z = 6$

$4y + z = 12$

$$\begin{cases} 2y + 5z = 24 \\ 4y + z = 12 \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix} = -18 \quad \Delta_y = \begin{vmatrix} 24 & 5 \\ 12 & 1 \end{vmatrix} = -36 \quad \Delta_z = \begin{vmatrix} 2 & 24 \\ 4 & 12 \end{vmatrix} = -72$$

$\underline{y = 2, \quad z = 4, \quad \rightarrow x = 4}$

The point $(4, \ 2, \ 4)$ is the pint on the line closet to the rogin.

Exercise

Find the maximum and minimum values of

$f(x, y, z) = xyz \quad subject\ to \quad x^2 + y^2 = 4 \ \ and \ \ x + y + z = 1$

Solution

$$\nabla f = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$g(x, y, z) = x^2 + y^2 - 4 \quad \rightarrow \quad \nabla g = 2x\hat{i} + 2y\hat{j}$$

$$h(x, y, z) = x + y + z - 1 \quad \rightarrow \quad \nabla h = \hat{i} + \hat{j} + \hat{k}$$

$$yz\hat{i} + xz\hat{j} + xy\hat{k} = \lambda\left(2x\hat{i} + 2y\hat{j}\right) + \mu\left(\hat{i} + \hat{j} + \hat{k}\right) \qquad \nabla f = \lambda \nabla g + \mu \nabla h$$

$$\begin{cases} yz = 2x\lambda + \mu & \rightarrow yz = 2x\lambda + xy \\ xz = 2\lambda y + \mu & \rightarrow xz = 2\lambda y + xy \\ \quad xy = \mu \end{cases}$$

$$xy = yz - 2x\lambda = xz - 2\lambda y$$

$$2\lambda(y - x) = z(x - y)$$

$$z = -2\lambda \quad or \quad y = x$$

For $y = x$

$$x^2 + y^2 = 4 \ \rightarrow \ 2x^2 = 4 \ \Rightarrow \ \underline{x = \pm\sqrt{2} = y}$$

For $y = x = \sqrt{2}$

$$x + y + z = 1 \ \rightarrow \ \underline{z = 1 - 2\sqrt{2}}$$

For $y = x = -\sqrt{2}$

$$\underline{z = 1 + 2\sqrt{2}}$$

$\therefore$ Points: $\left(\sqrt{2}, \ \sqrt{2}, \ 1 - 2\sqrt{2}\right) \ \left(-\sqrt{2}, \ -\sqrt{2}, \ 1 + 2\sqrt{2}\right)$

For $z = -2\lambda$

$$\mu = xy = yz + xz$$

$$z = \frac{xy}{x + y}$$

$$x + y + \frac{xy}{x+y} = 1$$

$$x^2 + y^2 + 3xy = x + y$$

$$4 + 3xy = x + y$$

$$4 + (3x-1)\,y - x = 0$$

$$\left((3x-1)\sqrt{4-x^2}\right)^2 = (x-4)^2$$

$$\left(9x^2 - 6x + 1\right)\left(4 - x^2\right) = x^2 - 8x + 16$$

$$36x^2 - 9x^4 - 24x + 6x^3 + 4 - x^2 - x^2 + 8x - 16 = 0$$

$$9x^4 - 6x^3 - 34x^2 + 16x + 12 = 0$$

Using Maple: evalf (solve ($9x^4 - 6x^3 - 34x^2 + 16x + 12, x$))

$x \approx -1.78,\ \ -0.42,\ .912,\ 1.955$

x	y	$z = 1 - x - y$
−1.78	.912	1.868
	−.912	3.692
−.42	1.955	−.535
	−1.955	3.375
.912	1.78	−1.692
	−1.78	1.868
1.955	.42	−1.375
	−.42	−.535

	$f(x, y, z) = xyz$
$\left(\sqrt{2},\ \sqrt{2},\ 1-2\sqrt{2}\right)$	$2 - 4\sqrt{2} \approx -3.657$
$\left(-\sqrt{2},\ -\sqrt{2},\ 1+2\sqrt{2}\right)$	$2 + 4\sqrt{2} \approx 7.657$
$(-1.78,\ .912,\ 1.868)$	−3.032
$(-1.78,\ -.912,\ 3.692)$	5.99
$(-.42,\ 1.955,\ -.535)$	0.439
$(-.42,\ -1.955,\ 3.375)$	2.77
$(.912,\ 1.78,\ -1.692)$	−2.747
$(.912,\ -1.78,\ 1.868)$	−3.032

$(1.955,\ .42,\ -1.375)$	-1.129
$(1.955,\ -.42,\ -.535)$	0.439

Minimum value of $2-4\sqrt{2}$ @ $\left(\sqrt{2},\ \sqrt{2},\ 1-2\sqrt{2}\right)$

Maximum value of $2+4\sqrt{2}$ @ $\left(-\sqrt{2},\ -\sqrt{2},\ 1+2\sqrt{2}\right)$

Exercise

The paraboloid $z=x^2+2y^2+1$ and the plane $x-y+2z=4$ intersect in a curve *C*. Find the points on *C* that have minimum and maximum distance from the origin.

Solution

$$f(x,\ y,\ z)=x^2+y^2+z^2 \quad \textit{subject to} \quad g(x,\ y,\ z)=x^2+2y^2-z+1 \quad h(x,\ y,\ z)=x-y+2z-4$$

$$\nabla f=2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$g(x,\ y,\ z)=x^2+2y^2-z+1 \quad \rightarrow \quad \nabla g=2x\hat{i}+4y\hat{j}-\hat{k}$$

$$h(x,\ y,\ z)=x-y+2z-4 \quad \rightarrow \quad \nabla h=\hat{i}-\hat{j}+2\hat{k}$$

$$2x\hat{i}+2y\hat{j}+2z\hat{k}=\lambda\left(2x\hat{i}+4y\hat{j}-\hat{k}\right)+\mu\left(\hat{i}-\hat{j}+2\hat{k}\right) \qquad \nabla f=\lambda\nabla g+\mu\nabla h$$

$$\begin{cases} 2x=2\lambda x+\mu & \rightarrow x=\dfrac{\mu}{2-2\lambda} & (1) \\ 2y=4\lambda y-\mu & \rightarrow y=\dfrac{\mu}{4\lambda-2} & (2) \\ 2z=-\lambda+2\mu & \rightarrow z=-\dfrac{1}{2}\lambda+\mu & (3) \end{cases}$$

$$(1)\ \&\ (2) \ \rightarrow \ \mu=2x-2\lambda x=4\lambda y-2y$$

$$(1-\lambda)x=(2\lambda-1)y$$

$$y=\frac{1-\lambda}{2\lambda-1}x$$

$$(1)\ \&\ (3) \ \rightarrow \ \mu=2x-2\lambda x=z+\frac{1}{2}\lambda$$

$$\begin{cases} 2\lambda x+\mu=2x \\ 4\lambda y-\mu=2y \end{cases} \rightarrow \ 2\lambda(x+2y)=2(x+y) \ \Rightarrow \ \boxed{\lambda=\frac{x+y}{x+2y}}$$

$$\mu=2x\left(1-\frac{x+y}{x+2y}\right)$$

$$=\boxed{\frac{2xy}{x+2y}}$$

$$(3) \ \rightarrow \ z=-\frac{1}{2}\frac{x+y}{x+2y}+\frac{2xy}{x+2y}$$

$$= \frac{4xy - x - y}{2(x+2y)}$$

$x - y + 2z = 4$

$$x - y + \frac{4xy - x - y}{x+2y} = 4$$

$x^2 + xy - 2y^2 + 4xy - x - y = 4x + 8y$

$x^2 - 2y^2 + 5xy - 5x - 9y = 0$

$x^2 + 2y^2 - z + 1 = 0$

$z = x^2 + 2y^2 + 1$

$z = 2 - \frac{1}{2}x + \frac{1}{2}y = x^2 + 2y^2 + 1$

$2x^2 + x + 4y^2 - y - 2 = 0$

$$x = \frac{-1 \pm \sqrt{-32y^2 + 8y + 17}}{4}$$

$$\begin{array}{l} -2\times \begin{cases} x^2 - 2y^2 + 5xy - 5x - 9y = 0 \\ 2x^2 + 4y^2 + x - y - 2 = 0 \end{cases} \\ \hline 8y^2 - 10xy + 11x + 17y - 2 = 0 \end{array}$$

$$x = \frac{8y^2 + 17y - 2}{10y - 11} = \frac{-1 \pm \sqrt{-32y^2 + 8y + 17}}{4}$$

$32y^2 + 68y - 8 = -10y + 11 \pm (10y - 11)\sqrt{-32y^2 + 8y + 17}$

$32y^2 + 78y - 19 = \pm(10y - 11)\sqrt{-32y^2 + 8y + 17}$

Using Maple: Ploy1:= $32y^2 + 78y - 19 = (10y - 11)\sqrt{-32y^2 + 8y + 17}$;

fsolve(Poly1)

Using Maple: Ploy2:= $32y^2 + 78y - 19 = -(10y - 11)\sqrt{-32y^2 + 8y + 17}$;

fsolve(Poly2)

$$\begin{cases} y \approx -0.3917 & \rightarrow x \approx 0.4982 & z \approx 1.9468 \\ y \approx 0.4613 & \rightarrow x \approx -1.1814 & z \approx 2.36 \end{cases}$$

The points on C are: $(0.4982,\ -0.3917,\ 1.9468)$ $(-1.1814,\ 0.4613,\ 2.36)$

Minimum distance from the origin: $\sqrt{(-.3917)^2 + .4982^2 + 1.9468^2}$ ≈ 2.04735

Maximum distance from the origin: $\sqrt{1.1814^2 + .4613^2 + 2.36^2}$ ≈ 2.3235

Exercise

Find the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + z^2$ on the curve on which the cone $z^2 = 4x^2 + 4y^2$ and the plane $2x + 4z = 5$ intersect.

Solution

$$f(x, y, z) = x^2 + y^2 + z^2 \quad \textit{subject to} \quad g(x, y, z) = 4x^2 + 4y^2 - z^2 \quad h(x, y, z) = 2x + 4z - 5$$

$$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$
$$\nabla g = 8x\hat{i} + 8y\hat{j} - 2z\hat{k}$$

$$\nabla h = 2\hat{i} + 4\hat{k}$$

$$2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda\left(8x\hat{i} + 8y\hat{j} - 2z\hat{k}\right) + \mu\left(2\hat{i} + 4\hat{k}\right) \qquad \nabla f = \lambda\nabla g + \mu\nabla h$$

$$\begin{cases} 2x = 8\lambda x + 2\mu & (1) \\ 2y = 8\lambda y & \rightarrow y = 0 \ \ or \ \ \lambda = \frac{1}{4} \\ 2z = -2\lambda z + 4\mu & (2) \end{cases}$$

For $y = 0$

$$z^2 = 4x^2 + 4y^2 \quad \rightarrow \quad z = \pm 2x$$

For $z = -2x$

$$2x - 8x = 5 \ \rightarrow \ x = -\frac{5}{6}$$

$$\rightarrow \ z = \frac{5}{3}$$

Point: $\left(-\frac{5}{6}, \ 0, \ \frac{5}{3}\right)$

For $z = 2x$

$$2x + 8x = 5 \ \rightarrow \ x = \frac{1}{2}$$

$$\rightarrow \ z = 1$$

Point: $\left(\frac{1}{2}, \ 0, \ 1\right)$

For $\lambda = \frac{1}{4}$

$$(1) \rightarrow \ x = x + \mu \quad \Rightarrow \quad \underline{\mu = 0}$$

$$(2) \ \rightarrow \ z = -\frac{1}{4}z \quad \Rightarrow \quad z = 0$$

$$z^2 = 4x^2 + 4y^2 = 0 \quad \text{(impossible)}$$

There are no solutions to the Lagrange conditions.

	$f(x, y, z) = x^2 + y^2 + z^2$
$\left(-\frac{5}{6},\ 0,\ \frac{5}{3}\right)$	$\frac{25}{36} + \frac{25}{9} = \frac{125}{36}$
$\left(\frac{1}{2},\ 0,\ 1\right)$	$\frac{5}{4}$

The maximum value of f is $\frac{125}{36}$ @ $\left(-\frac{5}{6},\ 0,\ \frac{5}{3}\right)$

The minimum value of f is $\frac{5}{4}$ @ $\left(\frac{1}{2},\ 0,\ 1\right)$

Exercise

The temperature of points on a elliptical plate $x^2 + y^2 + xy \le 1$ is given by $T(x, y) = 25\left(x^2 + y^2\right)$. Find the hottest and coldest temperatures on the edge of the elliptical plate.

Solution

$$T(x, y) = 25x^2 + 25y^2 \quad \textit{subject to} \quad g(x, y) = x^2 + y^2 + xy - 1$$

$$\nabla T = 50x\hat{i} + 50y\hat{j}$$

$$\nabla g = (2x + y)\hat{i} + (2y + x)\hat{j}$$

$$50x\hat{i} + 50y\hat{j} = \lambda\left((2x + y)\hat{i} + (2y + x)\hat{j}\right) \qquad \nabla f = \lambda \nabla g$$

$$\begin{cases} 50x = (2x + y)\lambda & \rightarrow \lambda = \dfrac{50x}{2x + y} \\ 50y = (2y + x)\lambda & \rightarrow \lambda = \dfrac{50y}{2y + x} \end{cases}$$

$$\frac{50x}{2x + y} = \frac{50y}{2y + x}$$

$$2xy + x^2 = 2xy + y^2$$

$$x = \pm y$$

$$x^2 + y^2 + xy = 1$$

For $x = -y$

$$x^2 + x^2 - x^2 = 1$$

$$x^2 = 1 \quad \rightarrow \quad x = \pm 1 = -y$$

Points: $(1,\ -1)$ $(-1,\ 1)$

For $x = y$

$$3x^2 = 1 \quad \rightarrow \quad x = \pm\frac{1}{\sqrt{3}} = y$$

Points: $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

	$T(x, y) = 25\left(x^2 + y^2\right)$
$(1, -1)$	50
$(-1, 1)$	50
$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$25\left(\frac{2}{3}\right) = \frac{50}{3}$
$\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$	$\frac{50}{3}$

The hottest temperature on the edge of the plate is 50 @ $(1, -1)$ $(-1, 1)$

The coldest temperature on the edge of the plate is $\frac{50}{3}$ @ $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$