

Section 3.5 – Exponential and logarithmic Equations

Exponential Equations

$$b^{\textcolor{red}{M}} = b^{\textcolor{blue}{N}} \leftrightarrow \textcolor{red}{M} = \textcolor{blue}{N} \text{ for any } b > 0, \neq 1$$

Example

Solve $5^{3x-6} = 125$

Solution

$$\textcolor{red}{5}^{3x-6} = \textcolor{red}{5}^3$$

$$3\textcolor{blue}{x} - 6 = 3$$

$$3\textcolor{blue}{x} = 9$$

$$\underline{\textcolor{blue}{x} = 3}$$

Example

Solve $8^{x+2} = 4^{x-3}$

Solution

$$\left(\textcolor{red}{2}^3\right)^{x+2} = \left(\textcolor{red}{2}^2\right)^{x-3}$$

$$2^{3(x+2)} = 2^{2(x-3)}$$

$$3(x+2) = 2(x-3)$$

$$3x + 6 = 2x - 6$$

$$3x - 2x = -6 - 6$$

$$\underline{\textcolor{blue}{x} = -12}$$

Using *Natural Logarithms*

1. Isolate the exponential expression
2. Take the natural logarithm on both sides of the equation
3. Simplify using one of the following properties: $\ln b^x = x \ln b$ or $\ln e^x = x$
4. Solve for the variable

Example

Solve: $7e^{2x} - 5 = 58$

Solution

$$7e^{2x} - 5 = 58$$

Isolate the exponential expression

$$7e^{2x} = 63$$

Divide by 7 both sides

$$e^{2x} = 9$$

Natural logarithm on both sides

$$\ln e^{2x} = \ln 9$$

Use inverse Property

$$2x = \ln 9$$

$$x = \frac{\ln 9}{2} \approx 1.0986$$

Example

Solve the equation $3^x = 21$

Solution

1 st method	2 nd method
$3^x = 21$	$3^x = 21 \Rightarrow x = \log_3 21$
<i>ln both sides</i>	<i>Convert to log form</i>
$\ln 3^x = \ln 21$	
$x \ln 3 = \ln 21$	$x = \frac{\ln 21}{\ln 3}$
	<i>Change of base</i>
$x = \frac{\ln 21}{\ln 3}$	

Example

Solve: $3^{2x-1} = 7^{x+1}$

Solution

$$\ln 3^{2x-1} = \ln 7^{x+1}$$

Natural logarithm on both sides

$$(2x-1)\ln 3 = (x+1)\ln 7$$

Power Rule

$$2x\ln 3 - \ln 3 = x\ln 7 + \ln 7$$

$$2x\ln 3 - x\ln 7 = \ln 3 + \ln 7$$

$$x(2\ln 3 - \ln 7) = \ln 3 + \ln 7$$

$$x = \frac{\ln 3 + \ln 7}{2\ln 3 - \ln 7} \approx 12.1143$$

Example

Solve the equation $\frac{5^x - 5^{-x}}{2} = 3$

Solution

$$5^x - 5^{-x} = 6$$

Multiply by 2 both sides

$$5^x 5^x - 5^{-x} 5^x = 6 5^x$$

Multiply by 5^x both sides

$$(5^x)^2 - 1 = 6(5^x)$$

$$(5^x)^2 - 6(5^x) - 1 = 0$$

$$5^x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-1)}}{2(1)} = \frac{6 \pm \sqrt{40}}{2} = \frac{6 \pm 2\sqrt{10}}{2} = \begin{cases} 3 + \sqrt{10} \\ 3 - \sqrt{10} < 0 \end{cases}$$

$$5^x = 3 + \sqrt{10}$$

$$\ln 5^x = \ln(3 + \sqrt{10})$$

$$x \ln 5 = \ln(3 + \sqrt{10})$$

$$x = \frac{\ln(3 + \sqrt{10})}{\ln 5} \approx 1.13$$

Logarithmic Equations

1. Express the equation in the form $\log_b M = c$
2. Use the definition of a logarithm to rewrite the equation in exponential form:

$$\log_b M = c \Rightarrow b^c = M$$

3. Solve for the variable
4. Check proposed solution in the original equation. Include only the set for $M > 0$

Example

Solve: $\log(x) + \log(x-3) = 1$

Solution

$$\log(x(x-3)) = 1$$

Product Rule

$$x(x-3) = 10^1$$

Convert to exponential form

$$x^2 - 3x = 10$$

$$x^2 - 3x - 10 = 0$$

Solve for x

$$x = -2, 5$$

Check: $x = -2 \Rightarrow \log(-2) + \log(-2-3) = 1$

$$x = 5 \Rightarrow \log(5) + \log(5-3) = 1$$

\therefore **Solution:** $x = 5$

Example

Solve: $\log_6(3x+2) + \log_6(x-1) = 1$

Solution

$$\log_6[(3x+2)(x-1)] = 1$$

Product Rule

$$(3x+2)(x-1) = 6^1$$

Convert to exponential form

$$3x^2 - x - 2 = 6$$

$$3x^2 - x - 8 = 0$$

Solve for x

$$x = \frac{1-\sqrt{97}}{6} < 0 \quad x = \frac{1+\sqrt{97}}{6} > 1$$

\therefore **Solution:** $x = \frac{1+\sqrt{97}}{6}$

Property of Logarithmic Equality

For any $M > 0, N > 0, b > 0, \neq 1$

$$\log_b M = \log_b N \Rightarrow M = N$$

Example

Solve: $\ln(x-3) = \ln(7x-23) - \ln(x+1)$

Solution

$$\ln(x-3) = \ln\left(\frac{7x-23}{x+1}\right) \quad \text{Quotient Rule}$$

$$x-3 = \frac{7x-23}{x+1}$$

$$(x-3)(x+1) = 7x-23$$

$$x^2 - 2x - 3 = 7x - 23$$

$$x^2 - 9x + 20 = 0$$

$$\underline{x = 4, 5}$$

$$\text{Check: } x = 4 \Rightarrow \ln(4-3) = \ln(7(4)-23) - \ln(4+1)$$

$$x = 5 \Rightarrow \ln(5-3) = \ln(7(5)-23) - \ln(5+1)$$

$$\therefore \text{Solution: } \underline{x = 4, 5}$$

Example

Solve: $\log(x+6) - \log(x+2) = \log x$

Solution

$$\log \frac{x+6}{x+2} = \log x \quad \text{Quotient Rule}$$

$$\frac{x+6}{x+2} = x \quad \text{Multiply by } x+2$$

$$x+6 = x(x+2)$$

$$x+6 = x^2 + 2x$$

$$x^2 + x - 6 = 0 \quad \text{Solve for } x$$

$$\underline{x = -3, 2}$$

$$\text{Check: } x = -3 \rightarrow \log(-3+6) - \log(-3+2) = \log(-3)$$

$$x = 2 \rightarrow \log(2+6) - \log(2+2) = \log(2)$$

Or Domain

$$\therefore \text{Solution: } \underline{x = 2}$$

Exercises **Section 3.5 – Exponential and logarithmic Equations**

(1 – 114) Solve the equations

1. $2^x = 128$

2. $3^x = 243$

3. $5^x = 70$

4. $6^x = 50$

5. $5^x = 134$

6. $7^x = 12$

7. $9^x = \frac{1}{\sqrt[3]{3}}$

8. $49^x = \frac{1}{343}$

9. $2^{5x+3} = \frac{1}{16}$

10. $\left(\frac{2}{5}\right)^x = \frac{8}{125}$

11. $2^{3x-7} = 32$

12. $4^{2x-1} = 64$

13. $3^{1-x} = \frac{1}{27}$

14. $2^{-x^2} = 5$

15. $2^{-x} = 8$

16. $\left(\frac{1}{3}\right)^x = 81$

17. $3^{-x} = 120$

18. $27 = 3^{5x} 9^{x^2}$

19. $4^{x+3} = 3^{-x}$

20. $2^{x+4} = 8^{x-6}$

21. $8^{x+2} = 4^{x-3}$

22. $7^x = 12$

23. $5^{x+4} = 4^{x+5}$

24. $5^{x+2} = 4^{1-x}$

25. $3^{2x-1} = 0.4^{x+2}$

26. $4^{3x-5} = 16$

27. $4^{x+3} = 3^{-x}$

28. $7^{2x+1} = 3^{x+2}$

29. $3^{x-1} = 7^{2x+5}$

30. $4^{x-2} = 2^{3x+3}$

31. $3^{5x-8} = 9^{x+2}$

32. $3^{x+4} = 2^{1-3x}$

33. $3^{2-3x} = 4^{2x+1}$

34. $4^{x+3} = 3^{-x}$

35. $7^{x+6} = 7^{3x-4}$

36. $5^{2x+1} = 6^{x-2}$

37. $2^{-100x} = (0.5)^{x-4}$

38. $4^x \left(\frac{1}{2}\right)^{3-2x} = 8 \cdot (2^x)^2$

39. $5^x + 125(5^{-x}) = 30$

40. $4^x - 3(4^{-x}) = 8$

41. $5^{3x-6} = 125$

42. $e^x = 15$

43. $e^{x+1} = 20$

44. $9e^x = 107$

45. $e^{x \ln 3} = 27$

46. $e^{x^2} = e^{7x-12}$
47. $f(x) = xe^x + e^x$
48. $f(x) = x^3(4e^{4x}) + 3x^2e^{4x}$
49. $e^{2x} - 2e^x - 3 = 0$
50. $e^{0.08t} = 2500$
51. $e^{x^2} = 200$
52. $e^{2x+1} \cdot e^{-4x} = 3e$
53. $e^{2x} - 8e^x + 7 = 0$
54. $e^{2x} + 2e^x - 15 = 0$
55. $e^x + e^{-x} - 6 = 0$
56. $e^{1-3x} \cdot e^{5x} = 2e$
57. $6\ln(2x) = 30$
58. $\log_5(x-7) = 2$
59. $\log_4(5+x) = 3$
60. $\log(4x-18) = 1$
61. $\log_3 x = -2$
62. $\log(x^2 + 19) = 2$
63. $\ln(x^2 - 12) = \ln x$
64. $\log(2x^2 + 3x) = \log(10x + 30)$
65. $\log_5(2x+3) = \log_5 11 + \log_5 3$
66. $\log_3 x - \log_9(x+42) = 0$
67. $\log_5 x + \log_5(4x-1) = 1$
68. $\log x - \log(x+3) = 1$
69. $\log x + \log(x-9) = 1$
70. $\log_2(x+1) + \log_2(x-1) = 3$
71. $\log_8(x+1) - \log_8 x = 2$
72. $\ln(x+8) + \ln(x-1) = 2\ln x$
73. $\ln(4x+6) - \ln(x+5) = \ln x$
74. $\ln(5+4x) - \ln(x+3) = \ln 3$
75. $\ln \sqrt[4]{x} = \sqrt{\ln x}$
76. $\sqrt{\ln x} = \ln \sqrt{x}$
77. $\log x^2 = (\log x)^2$
78. $\log x^3 = (\log x)^2$
79. $\log(\log x) = 1$
80. $\log(\log x) = 2$
81. $\ln(\ln x) = 2$
82. $\ln(e^{x^2}) = 64$
83. $e^{\ln(x-1)} = 4$
84. $10^{\log(2x+5)} = 9$
85. $\log \sqrt{x^3 - 9} = 2$
86. $\log \sqrt{x^3 - 17} = \frac{1}{2}$
87. $\log_4 x = \log_4(8-x)$
88. $\log_7(x-5) = \log_7(6x)$
89. $\ln x^2 = \ln(12-x)$
90. $\log_2(x+7) + \log_2 x = 3$
91. $\ln x = 1 - \ln(x+2)$
92. $\ln x = 1 + \ln(x+1)$

$$93. \log_6 (2x-3) = \log_6 12 - \log_6 3$$

$$98. \log_3 (x+3) + \log_3 (x+5) = 1$$

$$94. \log (3x+2) + \log (x-1) = 1$$

$$99. \ln x = \frac{1}{2} \ln \left(2x + \frac{5}{2} \right) + \frac{1}{2} \ln 2$$

$$95. \log_5 (x+2) + \log_5 (x-2) = 1$$

$$100. \ln (-4-x) + \ln 3 = \ln (2-x)$$

$$96. \log_2 x + \log_2 (x-4) = 2$$

$$101. \log_4 x + \log_4 (x-2) = \log_4 (15)$$

$$97. \log_3 x + \log_3 (x+6) = 3$$

$$102. \ln (x-5) - \ln (x+4) = \ln (x-1) - \ln (x+2)$$

$$103. \ln (4-x) = \ln (x+8) + \ln (2x+13)$$

$$104. \log (x^2 + 4) - \log (x+2) = 2 + \log (x-2)$$

$$105. \log_3 (x-2) = \log_3 27 - \log_3 (x-4) - 5^{\log_5 1}$$

$$106. \log_2 (x+3) = \log_2 (x-3) + \log_3 9 + 4^{\log_4 3}$$

$$107. \frac{10^x - 10^{-x}}{2} = 20$$

$$112. \frac{e^x - e^{-x}}{2} = 15$$

$$108. \frac{10^x + 10^{-x}}{2} = 8$$

$$113. \frac{1}{e^x - e^{-x}} = 4$$

$$109. \frac{10^x + 10^{-x}}{10^x - 10^{-x}} = 5$$

$$114. \frac{e^x + e^{-x}}{e^x - e^{-x}} = 3$$

$$110. \frac{10^x + 10^{-x}}{10^x - 10^{-x}} = 2$$

$$115. \frac{e^x - e^{-x}}{e^x + e^{-x}} = 6$$

$$111. \frac{e^x + e^{-x}}{2} = 15$$

(170 – 173) Use common logarithms to solve for x in terms of y

$$116. y = \frac{10^x + 10^{-x}}{2}$$

$$118. y = \frac{e^x - e^{-x}}{2}$$

$$117. y = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$$

$$119. y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$120. \text{Solve for } t \text{ using logarithms with base } a: 2a^{t/3} = 5$$

$$121. \text{Solve for } t \text{ using logarithms with base } a: K = H - Ca^t$$