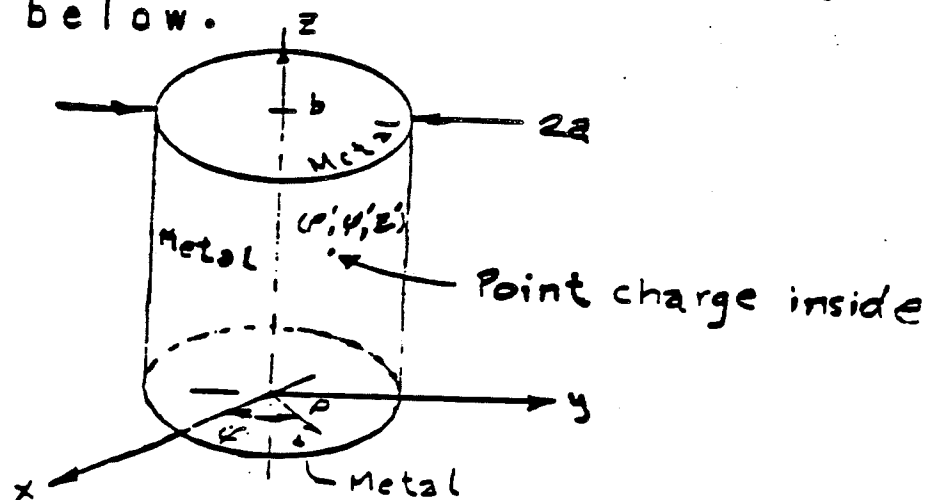


12. Potential due to a point source in a right, circular, metal cylinder — separation of variables.

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The cavity which we will analyze to provide another example of the method of separation of variables is shown below.



This problem is, in principle, no different from the rectangular cavity problem that we just solved.

In the previous problem, the elementary transcendental functions, \sin , \cos , \sinh , \cosh , etc. arose naturally from the ORDINARY differential equations obtained by SEPARATING the original PARTIAL differential equation.

In connection with the current problem, other, "higher" transcendental functions (Bessel functions) will come out the separated, ordinary differential equations.

In order to solve these differential equations, we will need to review (or learn!) the "method of Frobenius."

- a) The differential equation in cylindrical coordinates and isolation of the source point.

The differential equation in cylindrical coordinates is

$$\nabla^2 G = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\rho \frac{\partial G}{\partial \rho} \right] + \frac{1}{\rho^2} \frac{\partial^2 G}{\partial \varphi^2} + \frac{\partial^2 G}{\partial z^2} = - \frac{\delta(\rho - \rho') \delta(\varphi - \varphi') \delta(z - z')}{\epsilon \cdot \rho}$$

(Go back and review the section on curvilinear coordinates!!!!)

There are several ways to "isolate" the source point.

We saw that in the case of a rectangular metal box, the source point could be isolated by dividing the box into regions separated by a CONSTANT COORDINATE SURFACE.

Of course, this is precisely the same way we will isolate the source in this problem.

In the rectangular cavity problem, we had three different ways to divide the cavity to isolate the source point:

By a constant

x surface,
y surface, or
z surface.

Similarly, for this problem there are also three ways to isolate the source point.

b) Separation of the differential equation

Poisson's differential equation
which the potential,

$$G(\rho, \varphi, z | \rho', \varphi', z')$$

must satisfy is

$$\nabla^2 G(\rho, \varphi, z | \rho', \varphi', z') = -\frac{1}{\epsilon} \frac{\delta(\rho - \rho')}{\rho} \delta(\varphi - \varphi') \delta(z - z')$$

In the source-free regions, the potential, G , satisfies Laplace's equation:

$$\nabla^2 G(\rho, \varphi, z | \rho', \varphi', z') = 0$$

Assuming a solution of the form

$$G = R(\rho) T(\varphi) Z(z),$$

then

$$\nabla^2 G = T \cdot Z \cdot \frac{1}{\rho} [\rho R'(\rho)]' + \frac{1}{\rho^2} R \cdot Z \cdot T''(\varphi) + R \cdot T \cdot Z''(z) = 0 \Rightarrow$$

$$\frac{1}{R} \cdot \frac{1}{\rho} [\rho R']' + \frac{1}{\rho^2} \frac{T''}{T} + \frac{Z''}{Z} = 0$$

Thus, our partial differential equation is reduced to three ORDINARY differential equations:

$$\frac{Z''}{Z} = \gamma_z^2 \Rightarrow \frac{1}{R} \cdot \frac{1}{\rho} [\rho R']' + \frac{1}{\rho^2} \frac{T''}{T} = -\gamma_z^2 \quad \text{or,}$$

$$\frac{1}{R} \rho [\rho R']' + \gamma_z^2 \rho^2 + \frac{T''}{T} = 0$$

$$\Rightarrow \frac{T''}{T} = -\alpha^2 \quad \text{and} \quad \rho [\rho R']' + [\gamma_z^2 \rho^2 - \alpha^2] R = 0$$

c) Separation of the boundary conditions

The boundary conditions which will be applied to the solutions of these ordinary differential equations depend, of course, on how we isolate the source.

These imply, respectively that

$$R_{II}(\rho) T_{II}(\varphi) Z_{II}(b) = 0,$$

$$R_I(\rho) T_I(\varphi) Z_I(0) = 0,$$

$$R_k(a) T_k(\varphi) Z_k(z) = 0,$$

or

$$Z_{II}(b) = 0, \quad Z_I(0) = 0,$$

$$R_I(a) = 0, \text{ and } R_{II}(a) = 0.$$

What about a condition on

$$G_k(\rho, \varphi, z | \rho', \varphi', z') ?$$

The line $\rho = 0$ has some SINGULAR properties in cylindrical coordinates.

Usually, to define a CURVE, we must specify TWO independent conditions.

For example,

$$\rho = 1, \varphi = 0.$$

is a straight line parallel to the z axis passing through the x axis at $x = 1$.

However, the straight line $\rho = 0$ (which coincides with the z axis) does not depend on what φ is!

Thus, $G(0, \varphi, z | \rho', \varphi', z')$ is the potential along this line which should be UNIQUE (i.e., independent of φ).

Under our assumption of a separated solution, what kind of requirements does this place on $R(\rho)$, $T(\varphi)$, and $Z(z)$, in region k?

We have that

$$R_k(0) T_k(\varphi) Z_k(\bar{z}) \text{ is}$$

INDEPENDENT OF φ .

The ONLY way that this can happen
is if

$$T_k(\varphi) \text{ is constant — or —}$$

$$R_k(0) = 0.$$

This condition, then, becomes the
"boundary condition" for region k
solutions, at $\rho=0$.

(3) Source-free regions separated
by a constant angle surface

<<<<<<<<<<<<<<<>>>>>>>>>>>>>>

In this case, there is only one region and the boundary conditions are

$$\begin{aligned} G(a, \varphi, z \mid f', \varphi', z') &= 0, \\ G(\rho, \varphi, 0 \mid \rho', \varphi', z') &= 0, \quad \text{and} \\ G(\rho, \varphi, b \mid \rho', \varphi', z') &= 0. \end{aligned}$$

which implies that

$$\begin{aligned} R(a) &= 0, \\ Z(0) &= 0, \text{ and} \\ Z(b) &= 0. \end{aligned}$$

Again, the same conditions at $\rho = 0$ apply to the radial and angular solutions to ensure uniqueness of the solution.

(1) Differential equations in z.
 <<<<<<<<<<<<<<<>>>>>>>>>>>>>

The differential equation for the z variable is a simple second order ordinary differential equation with CONSTANT coefficients which we have solved before in connection with the rectangular cavity problem.

In all cases, we will want this solution to vanish at $z = 0$ and/or $z = b$.

Thus, the most convenient forms of the solution which have these properties are

$$\sin(k_z z), \text{ and } \sin[k_z(b-z)], \text{ or}$$

$$\sinh(\gamma_z z) \text{ and } \sinh[\gamma_z(b-z)]$$

where $K_z = j\gamma_z$.

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Abstract

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Well, it will be shown below that a POWER SERIES expansion for $y(x) = \sin(x)$ exists, is convergent for all x to $\sin(x)$, and is given by

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

If we were stuck out on a desert isle without or calculator and had a burning desire to compute the $\sin(.378)$, we COULD do it using such a series.

What we need to do to solve the problem at hand is to find a function $R(x)$ that satisfies the differential equation,

$$x[xR(x)]' + [(kx)^2 - p^2] R(x) = 0.$$

Can we find a series solution to THIS equation as can be done for the differential equation that $\sin(x)$ satisfies?

This question was answered in a theorem due to Fuchs.

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Suppose that

$$y'' + P(x)y' + Q(x)y = 0.$$

Then the point $x = u$ is called an **ORDINARY POINT** of the differential equation if and only if both $P(x)$ and $Q(x)$ possess Taylor's series about $x = u$.

For example, in the equation,

$$x[xy']' + [(kx)^2 - p^2]y = 0$$

----->

$$x^2 y'' + xy' + [(kx)^2 - p^2]y = 0$$

----->

$$y'' + y'/x + [(kx)^2 - p^2]y/x^2 = 0.$$

the point $x = 1$ is an ordinary point since

$$1/x \text{ and } [(kx)^2 - p^2]/x^2$$

both have Taylor series about $x=1$.

If $x = u$ is NOT an ordinary point,
then it is a SINGULAR POINT.

In the previous example, is $x = 0$
an ordinary or a singular point?

A singular point, $x = u$, is said
to be a REGULAR SINGULAR POINT if
and only if

$$(x - u)P(x) \text{ and } (x - u)^2 Q(x)$$

both have Taylor series expansions
about $x = u$.

A singular point which is not
regular is called an IRREGULAR
SINGULAR POINT.

In the example above, is the point
 $x = 0$ a regular, or an irregular
singular point?

We are now ready to state Fuchs' theorem:

For the differential equation,

$$y'' + P(x)y' + Q(x)y = 0.$$

If $x = u$ is an ordinary point, then a convergent series solution to the differential equation exists and is of the form

$$y = \sum_{n=0}^{\infty} a_n (x-u)^n$$

If $x = u$ is a regular singular point, then a convergent series solution to the differential equation exists and is of the form

$$y = \sum_{n=0}^{\infty} a_n (x-u)^{n+k}$$

Note that if $x = u$ is an irregular singular point, in general, a simple series solution like those above may not exist in general.

Since the point $x = 0$ (as well as any other point) is an ordinary point of the differential equation, we know from Fuchs' theorem that a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

exists.

Substituting this back into the differential equation,

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

$$y'' + y = \underbrace{\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}}_{\text{replace } n \text{ by } n+2} + \sum_{n=0}^{\infty} a_n x^n$$

replace n by $n+2$

$$= \sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} [(n+1)(n+2) a_{n+2} + a_n] x^n = 0$$

Taylor series are unique \Rightarrow

$$[(n+1)(n+2) a_{n+2} + a_n] = 0 \text{ for } n=0, 1, \dots$$

We can now use this expression, called a RECURRENCE FORMULA, or DIFFERENCE EQUATION, to generate two independent tables —

-1- relating a_{2n} to a_0

and

-2- relating a_{2n+1} to a_1 .

$$a_2 = -1/(2 \cdot 1) a_0$$

$$a_4 = -1/(4 \cdot 3) a_2 = (-1)^2 / 4! a_0$$

⋮

$$a_{2n} = (-1)^n / (2n)! a_0$$

$$a_3 = -1/(3 \cdot 2) a_1$$

$$a_5 = -1/(5 \cdot 4) a_3 = (-1)^2 / 5! a_1$$

⋮

$$a_{2n+1} = (-1)^n / (2n+1)! a_1$$

Therefore, the solution to differential equation is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}_{\cos(x)} + a_1 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}_{\sin(x)}$$

The coefficients,

$$a_0 \quad \text{and} \quad a_1$$

represent two ARBITRARY coefficients and the solution given above is the GENERAL SOLUTION to the differential equation.

The two infinite series are IDENTIFIED as $\sin(x)$ and $\cos(x)$.

These series could be taken to represent the DEFINITION of these two functions.

This series converges absolutely for all x (real or complex).

This can be seen by an application of the ratio test:

$$\frac{|a_{2n+2} x^{2n+2}|}{|a_{2n} x^{2n}|} = \frac{x^2 \cdot 1/(2n+2)!}{1/2n!}$$

$$= x^2 / (2n+2)^{(2n+1)} \rightarrow 0, n \rightarrow \infty.$$

Similarly $|a_{2n+3} x^{2n+3}| / |a_{2n+1} x^{2n+1}| \rightarrow 0$
as $n \rightarrow \infty$.

Although this series converges everywhere, it is not the most practical way to compute $\sin(x)$ and $\cos(x)$.

We will see other ways that are somewhat more practical in the computation of transcendental functions.

Now that we have covered a familiar function and differential equation, let's try to solve Bessel's differential equation:

$$x[xy']' + (x^2 - p^2)y = 0$$



$$x^2 y'' + xy' + (x^2 - p^2)y = 0.$$

Since the point $x = 0$ is a regular singular point, according to Fuchs' theorem, we must assume a solution of the form:

$$y = x^k \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+k}.$$

Substituting,

$$\begin{aligned}
 & \sum_{n=0}^{\infty} a_n \{ (n+k)(n+k-1) x^{n+k} + (n+k) x^{n+k} \\
 & \quad + x^{n+2+k} - p^2 x^{n+k} \} \\
 &= \sum_{n=0}^{\infty} \{ [(n+k)^2 - p^2] x^{n+k} \} a_n + \sum_{n=0}^{\infty} x^{n+2+k} a_n \\
 &= \sum_{n=0}^{\infty} \{ [(n+k)^2 - p^2] x^{n+k} \} a_n + \sum_{n=2}^{\infty} x^{n+k} a_{n-2} \\
 &= x^k \{ [k^2 - p^2] a_0 + [(k+1)^2 - p^2] a_1 x + \sum_{n=2}^{\infty} [(k+n)^2 - p^2] a_n + a_{n-2} \} x^n \\
 &= 0
 \end{aligned}$$

Equating coefficients to zero,

$$[k^2 - p^2] a_0 = 0$$

$$[(k+1)^2 - p^2] a_1 = 0$$

$$[(k+n)^2 - p^2] a_n + a_{n-2} = 0 \quad \text{for } n=2, 3, \dots$$

$$\rightarrow a_n = - \frac{1}{[(k+n)^2 - p^2]} a_{n-2} \quad n=2, 3, \dots$$

If

$$a_0 \neq 0$$

then

$$\underbrace{k^2 - p^2 = 0} \rightarrow k = \pm p$$

The "indicial equation"

and

$$a_1 = 0.$$

Or, If

$$a_1 \neq 0$$

then

$$(k+1)^2 - p^2 = 0 \rightarrow k = -1 \pm p \text{ and } a_0 = 0.$$

Let's first take the case

$$a_0 \neq 0, a_1 = 0, k = \pm p$$

Then the recurrence formula is

$$a_n = - \frac{1}{[(n \pm p)^2 - p^2]} a_{n-2}$$

$$= - \frac{1}{[(n \pm p) + p][(n \pm p) - p]} a_{n-2}$$

Specifically, let's take $k=+p$.
Then

$$a_n = - \frac{1}{n(n+2p)} a_{n-2}$$

$$a_2 = - \frac{1}{2 \cdot (2+2p)} a_0$$

$$a_4 = - \frac{1}{4(4+2p)} a_2 = (-1)^2 \frac{a_0}{4 \cdot 2 \cdot (4+2p)(2+2p)}$$

$$= (-1)^2 \frac{1}{2^2 \cdot [2 \cdot 1] \cdot 2^2 [(2+p)(1+p)]} a_0$$

$$\vdots$$

$$a_{2n} = (-1)^n \frac{a_0}{2^{2n} n! [(n+p)(n-1+p) \cdots (1+p)]}$$

Products of the form

$$[(n+p)(n-1+p) \cdots (1+p)]$$

can be conveniently written in terms of the GAMMA function:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

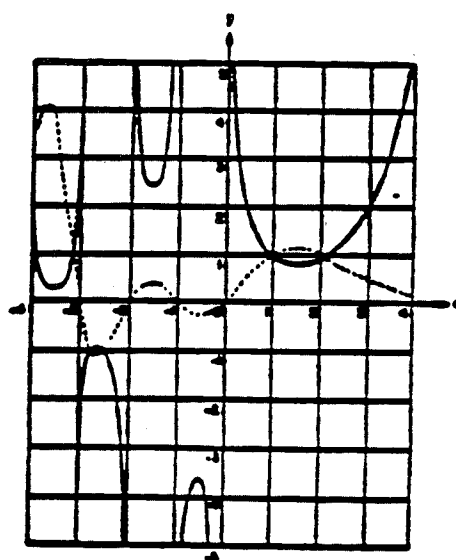


FIGURE 6.1. Gamma function. •

—, $y = \Gamma(x)$. - - - , $y = 1/\Gamma(x)$

12.3'

The gamma function has the property that

$$\Gamma(x+1) = x \Gamma(x)$$

which allows us to define $\Gamma(x)$ for $x < 0$.

We also note that $\Gamma(1) = 1$.

From these equations, we can see that if x is the integer, m , then

$$\Gamma(m+1) = m!$$

The graph of $\Gamma(x)$ is shown below.

Now, how do we use the gamma function to more succinctly express the product

$$[(n+p)(n-1+p) \dots (1+p)]?$$

Using the recurrence relation of the gamma function, we see that

$$\begin{aligned} \Gamma(n+p+1) &= (n+p) \Gamma(n+p) \\ &= (n+p) \cdot (n+p-1) \cdot \Gamma(n+p-1) \\ &\vdots \\ &= (n+p)(n-1+p) \dots (1+p) \Gamma(1+p). \end{aligned}$$

Therefore,

$$[(n+p)(n-1+p)\cdots(1+p)] = \frac{\Gamma(n+p+1)}{\Gamma(1+p)}$$

Therefore, we can write

$$a_{2n} = (-1)^n \frac{\Gamma(p+1)}{2^{2n} n! \Gamma(n+p+1)} a_0$$

We can now expand one solution to Bessel's differential equation:

$$y = x^p \sum_{n=0}^{\infty} a_n x^n = x^p \sum_{n=0}^{\infty} a_{2n} x^{2n}$$

$$= x^p \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(p+1) a_0}{2^{2n} n! \Gamma(n+p+1)} x^{2n}$$

$$= [2^p \Gamma(p+1) a_0] \cdot \underbrace{\left(\frac{x}{2}\right)^p \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(n+p+1)}}_{J_p(x)}$$

The function $J_p(x)$ is called the BESSEL FUNCTION of the FIRST KIND of ORDER, p .

Remember, this solution was generated by taking $k=+p$.

We have another solution that corresponds to $k=-p$.

This solution can be found simply by replacing p by $-p$:

$$J_{-p}(x) = \left(\frac{x}{2}\right)^{-p} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(n-p+1)}.$$

We also found that we could choose

$$k = -1 \pm p, \quad a_0 = 0, \quad a_1 \neq 0$$

In this case, the recurrence formula is

$$\begin{aligned} a_n &= \frac{-1}{[(n-1 \pm p)^2 - p^2]} a_{n-2} \\ &= \frac{-1}{[n-1 \pm p - p][n-1 \pm p + p]} a_{n-2}. \end{aligned}$$

For $k = -1 + p$,

$$a_3 = -\frac{1}{2 \cdot (2+2p)} a_1$$

$$a_5 = -\frac{1}{4 \cdot (4+2p)} a_3 = (-1)^2 \frac{1}{4 \cdot 2 \cdot (4+2p)(2+2p)} a_1$$

\vdots

$$a_{2n+1} = (-1)^n \frac{\Gamma(p+1) a_1}{2^{2n} n! \Gamma(n+p+1)}$$

Thus, THIS solution is

$$x^{-1+p} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(p+1) a_1}{2^{2n} n! \Gamma(n+p+1)} x^{2n+1}$$

$$= [2^p \Gamma(p+1) a_1] \left(\frac{x}{2}\right)^p \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(n+p+1)}$$

Therefore, choosing $k = -1+p$ yields the SAME result as the choice $k = p$!

Similarly, $k = -1-p$ yields the SAME result as the choice $k = -p$.

Thus, it appears that there are at MOST two linearly independent solutions to the differential equation (as expected).

The two solutions are

$$J_{-p}(x) \text{ and } J_p(x).$$

But are $J_p(x)$ and $J_{-p}(x)$ really linearly independent?

If we look at the behavior of the solutions for x nearly zero, we can see right away that if p is NOT AN INTEGER, the two solutions are independent.

For example, for $p=0.5$,

$$J_{.5}(x) = \sqrt{\frac{x}{2}} \cdot \left[\frac{1}{\Gamma(1.5)} - \frac{1}{\Gamma(2.5)} \left(\frac{x}{2}\right)^2 + \dots \right]$$

$$J_{-.5}(x) = \sqrt{\frac{2}{x}} \left[\frac{1}{\Gamma(.5)} - \frac{1}{\Gamma(1.5)} \left(\frac{x}{2}\right)^2 + \dots \right]$$

However, if p is an integer, for example, $p=3$, then

$$J_{-3}(x) = \left(\frac{x}{2}\right)^{-3} \left\{ \frac{1}{\Gamma(-2)} - \frac{1}{1!\Gamma(-1)} \left(\frac{x}{2}\right)^2 + \frac{1}{2!\Gamma(0)} \left(\frac{x}{2}\right)^4 - \frac{1}{3!\Gamma(1)} \left(\frac{x}{2}\right)^6 + \dots \right\}$$

But, from the graph of $\Gamma(x)$, we see that $\Gamma(x)$ goes to infinity at $x =$ negative integer, or, $1/\Gamma(x)$ goes to zero at a negative integer.

$$\begin{aligned} \text{Thus,} \\ J_{-3}(x) &= \left(\frac{x}{2}\right)^{-3} \left\{ -\frac{1}{\Gamma(1)} \cdot \frac{1}{3!} \left(\frac{x}{2}\right)^6 + \frac{1}{\Gamma(2)} \cdot \frac{1}{4!} \left(\frac{x}{2}\right)^8 - \dots \right\} \\ &= -\left(\frac{x}{2}\right)^3 \left\{ \frac{1}{0!} \cdot \frac{1}{\Gamma(4)} - \frac{1}{1!} \cdot \frac{1}{\Gamma(5)} \left(\frac{x}{2}\right)^2 + \dots \right\} \end{aligned}$$

which is just $(-1)^3 J_3(x)$.

In general, we can show that

$$J_{-n}(x) = (-1)^n J_n(x).$$

It would appear from this, that only one solution to Bessel's differential equation can be found when p is an integer.

This situation is analogous to that which arises in the differential equation,

$$y'' + 2y' + y = 0$$

If we assume a solution of the form

$$y = e^{st}$$

into the equation, we obtain

$$s^2 + 2s + 1 = (s+1)^2 = 0 \rightarrow s = -1 \pm 0.$$

which has a DOUBLE root at $s = -1$.

For this differential equation, however, we know that there IS another independent solution of the form

$$y = te^{-t}$$

It is similarly also true that Bessel's differential equation also has TWO linearly independent solutions EVEN when p is an integer.

The way this other solution can be obtained is the following.

First define the BESSEL FUNCTION of the SECOND KIND of ORDER p as

$$Y_p(x) = \frac{J_p(x) \cos(p\pi) - J_{-p}(x)}{\sin(p\pi)}.$$

Because of the linearity of Bessel's equation, this also satisfies the differential equation.

If we set $p=m$, an integer, then the expression for $Y_m(x)$ becomes

$$\begin{aligned} Y_m(x) &= \frac{J_m(x) \cos(m\pi) - J_{-m}(x)}{\sin(m\pi)} \\ &= \frac{J_m(x) (-1)^m - (-1)^m J_m(x)}{\sin(m\pi)} = \frac{0}{0}. \end{aligned}$$

Clearly, we must evaluate this expression by taking its LIMIT as $p \rightarrow m$.

$$Y_m(x) = \lim_{p \rightarrow m} \left\{ \frac{J_p(x) \cos(p\pi) - J_{-p}(x)}{\sin(p\pi)} \right\}$$

Since we have an indeterminate form, we can apply L'Hospital's rule to obtain

$$Y_m(x) = \frac{\frac{\partial}{\partial p} [J_p(x) \cos(p\pi) - J_{-p}(x)]}{\frac{\partial}{\partial p} [\sin(p\pi)]} \Big|_{p=m}$$

$$= \frac{\frac{\partial J_p}{\partial p} \Big|_{p=m} (-1)^m - \frac{\partial J_{-p}}{\partial p} \Big|_{p=m}}{\pi (-1)^m}.$$

We need only differentiate (term-wise) the series for $J_{\pm p}(x)$ to obtain the expression for Y .

The result of this is

$$Y_m(x) = \frac{2}{\pi} J_{-m}(x) \left[\ln\left(\frac{x}{2}\right) + \gamma \right]$$

$$- \frac{1}{\pi} \sum_{n=0}^{m-1} \frac{(x/2)^{2n-m} (m-n-1)!}{n!}$$

$$- \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+m} [\psi(m+n) + \psi(n)]}{n! (m+n)!}$$

where

$$\gamma = 0.577$$

is Euler's constant and

$$\psi(n) = \sum_{k=1}^n \frac{1}{k} \text{ with } \psi(0) = 0.$$

Thus, the general solution to Bessel's differential equation is

$$y(x) = AJ_p(x) + BY_p(x)$$

This is analogous to the general solution to

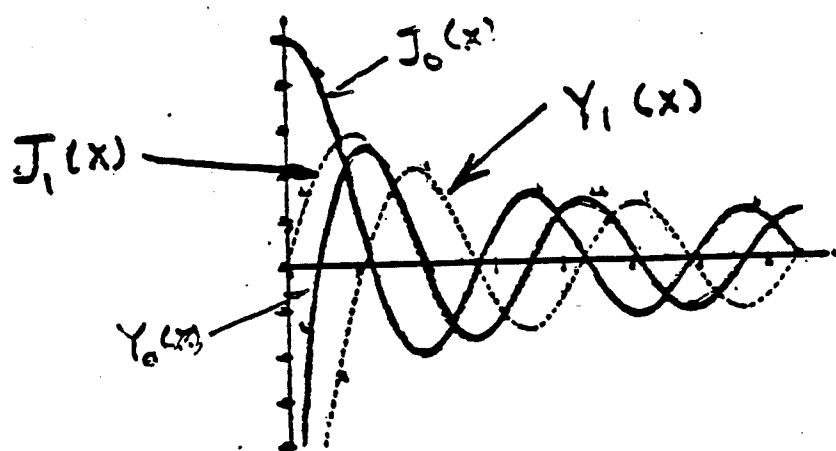
$$y'' + y = 0$$

which is

$$A\cos(x) + B\sin(x).$$

The graphs of $J_p(x)$ and $Y_p(x)$ for various values of p are shown below.

12.41



So we don't loose sight of our mission, let's see what we have got so far:

-1- We defined the problem, and wrote the differential equation for the potential due to a point source.

-2- We isolated the source by separating the "can" into source-free regions bounded by a constant z plane, or a constant radius surface, or a constant angle plane.

-3- We separated the partial differential equation into three ordinary differential equations, one in the z variable, one in the angle variable, and one in the radius variable.

-4- We found what boundary conditions the Z , radial, and angular separated functions must satisfy for the three possible methods of isolating the source point.

-5- Finally, we solved the differential equations for the z , angular, and the radial variables.

We found that a solution to this problem could only be found for DISCRETE values of k_y :

$$Y(y) = \sin[(n\pi/b)y].$$

This is an example of a STURM-LIOUVILLE PROBLEM.

The general Sturm-Liouville problem is to find solutions to

$$\frac{d}{dx} [r(x)y'] + [q(x) + p(x)]y = 0$$

subject to

$$r(x) > 0 \text{ and } w(x) > 0,$$

$$Ay'(a) + By(a) = 0$$

$$Ay'(b) + By(b) = 0$$

There are at most, two linearly independent solutions to this second order differential equation, say

$$y(x) = Cu(x;p) + Dv(x;p)$$

Then, applying the boundary conditions,

$$A[u'(2;p) \cdot C + v'(2;p) \cdot D] + B[u(2;p)C + v(2;p)D] = C$$

$$A[u'(b;p) \cdot C + v'(b;p) \cdot D] + B[u(b;p)C + v(b;p)D] = 0$$

$$\Rightarrow \begin{bmatrix} [Au'(2;p) + Bu(2;p)] & [Av'(2;p) + Bv(2;p)] \\ [Au'(b;p) + Bu(b;p)] & [Av'(b;p) + Bv(b;p)] \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} C \\ 0 \end{bmatrix}$$

If we wish to have a non-trivial solution, then the determinant of coefficients,

$$\det \begin{bmatrix} [1] & [1] \\ [1] & [1] \end{bmatrix}$$

must vanish.

In general, this will vanish ONLY for discrete values of p , say for p in the set $\{p_m\}$.

These discrete values of p are called EIGENVALUES.

There associated functions,

$$y_n(x) = C_n u_n(x;p_n) + D_n v_n(x;p_n)$$

are called EIGENFUNCTIONS.

One of the coefficients of $y_n(x)$ can be taken arbitrarily.

However, the other must be taken so that boundary conditions are satisfied:

$$[Au'(a; p_n) + Bu(a; p_n)]C_n + [Av'(a; p_n) + Bv(a; p_n)]E_n = 0$$

$$C_n = - \frac{Av'(a; p_n) + Bv(a; p_n)}{Au'(a; p_n) + Bu(a; p_n)} \cdot D_n.$$

Thus, the solution to the Sturm-Liouville problem can be taken to be

$$y_n(x) = D_n \cdot \left\{ v(x; p_n) - \frac{Av'(a; p_n) + Bv(a; p_n)}{Au'(a; p_n) + Bu(a; p_n)} u(x; p_n) \right\}$$

The set of solutions, $\{y_n(x)\}$, have a very interesting property: THEY ARE ORTHOGONAL over the interval $[a, b]$ with respect to a WEIGHT FUNCTION, $w(x)$!

This result is easy to prove. Take

$$[r(x)y_n']' + [q(x) + P_n w(x)]y_n = 0$$

$$[r(x)y_m']' + [q(x) + P_m w(x)]y_m = 0$$

Then

$$y_m[r y_n']' + q \cdot y_n y_m + P_n w \cdot y_n y_m = 0$$

$$y_n[r y_m']' + q y_m y_n + P_m w \cdot y_m y_n = 0$$

$$\{y_m[r y_n']' - y_n[r y_m']'\} + w \cdot y_n y_m (P_n - P_m) = 0$$

Notice that the bracketed term of this equation is just

$$\{y_m[r y_n']' - y_n[r y_m']'\} = \{r \cdot [y_n' y_m - y_n y_m']\}$$

Therefore,

$$(P_n - P_m) w \cdot y_n y_m = -\frac{d}{dx} \{r [y_n' y_m - y_n y_m']\}$$

Integrating this result from a to b ,

$$\begin{aligned} (P_n - P_m) \int_a^b w \cdot y_n y_m dx &= - \int_a^b \frac{d}{dx} \{ \} dx \\ &= \{ \}_a^b = r(a) [y_n'(a) y_m(a) - y_n(a) y_m'(a)] \\ &\quad - r(b) [y_n'(b) y_m(b) - y_n(b) y_m'(b)] \end{aligned}$$

But $[y'_n(z)y_m(z) - y_n(z)y'_m(z)] =$

$$\det \begin{bmatrix} y'_n(z) & y_n(z) \\ y'_m(z) & y_m(z) \end{bmatrix} = 0 \quad \text{since}$$

$A y'_n(z) + B y_n(z) = 0$ A similar result holds for $y = t$.

$$A y'_m(z) + B y_m(z) = 0$$

Therefore, $\langle y_n, y_m \rangle = 0$ for $m \neq n$.

The one arbitrary coefficient can be adjusted so that the norms of the y 's are unity.

It can also be shown that the set $\{y\}$, like the set of trigonometric functions (which are also solutions to a Sturm-Liouville problem) is COMPLETE set.

- e) Application of the boundary conditions and continuity conditions - determination of separation constants and expansion coefficients
-

Now that we have the solutions to the three ordinary differential equations,

$$Z''(z) - \gamma_z^2 Z(z) = 0$$

$$T''(\varphi) + \alpha^2 T(\varphi) = 0$$

$$\rho [\rho R'(\rho)]' + [\gamma_z^2 \rho^2 - \alpha^2] R(\rho) = 0$$

We are ready to APPLY the boundary conditions.

Application of the boundary conditions will determine the (as yet undetermined) separation constants, γ_z and α .

(1) Source-free regions separated
by constant z surface

The general solution to $T(\varphi)$'s differential equation is

$$T(\varphi) = A \cos(\alpha\varphi) + B \sin(\alpha\varphi).$$

or, what is more convenient in view of the obvious symmetry of the problem,

$$T(\varphi) = A \cos[\alpha(\varphi - \varphi')] + B \sin[\alpha(\varphi - \varphi')].$$

We had NO boundary conditions for $T(\varphi)$.

What properties should our solution have?

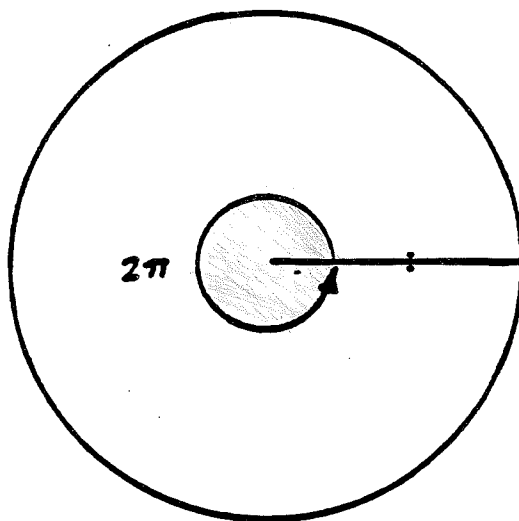
First of all, the symmetry of the problem demands EVEN SYMMETRY about

$$\varphi = \varphi'$$

Therefore, we can eliminate the \sin term leaving

$$T(\varphi) = A \cos[\alpha(\varphi - \varphi')]$$

Furthermore, if $\varphi' = 0$, then how should $T(0)$ and $T(2\pi)$ be related?



The potential is continuous!

Thus,

$$\begin{aligned} T(2\pi) &= T(0) \rightarrow \cos(\alpha \cdot 2\pi) = \cos(0) = 1 \\ &\rightarrow \alpha \cdot 2\pi = n \cdot 2\pi \rightarrow \alpha = n, \quad n = 0, 1, 2, \dots \end{aligned}$$

Therefore,

$$T_n(\varphi) = \cos[n(\varphi - \varphi')]$$

satisfies the differential equation and the PERIODICITY condition.

We note that these functions form an orthogonal set:

$$\begin{aligned} \int_0^{2\pi} T_n(\varphi) T_k(\varphi) d\varphi &= \int_0^{2\pi} \cos[n(\varphi - \varphi')] \cos[k(\varphi - \varphi')] d\varphi \\ &= \frac{2\pi}{E_n} \delta_{nk} \quad \text{where } E_n = \begin{cases} 1 & n=0 \\ 2 & n \neq 0 \end{cases} \end{aligned}$$

Next, we turn to the differential equation in ρ .

We found that the solution to Bessel's differential equation,

$$x [x y'(x)]' + [x^2 - p^2] y(x) = 0$$

was

$$y(x) = A J_p(x) + B Y_p(x).$$

But we need to solve

$$\rho [\rho R'(\rho)]' + [\gamma_z^2 \rho^2 - \underset{\uparrow n^2}{\alpha^2}] R(\rho) = 0$$

Note that if $x = \gamma_z \rho$,

$$x \frac{d}{dx} = x \frac{d\rho}{dx} \cdot \frac{d}{d\rho} = \gamma_z \rho \frac{1}{\gamma_z} \frac{d}{d\rho} = \rho \frac{d}{d\rho}$$

so that

$$R(\rho) = y(\gamma_z \rho) = A J_n(\gamma_z \rho) + B Y_n(\gamma_z \rho).$$

Now remember that a "boundary condition" on $R(\rho)$ was that $R(0) = 0$ if $T(\varphi)$ was not a constant.

We note that $Y_n(0)$ is infinite for every n and that $J_n(0) = 0$ for $n = 1, 2, 3, \dots$.

Thus, we must choose $B=0$ for $n=1, 2, 3, \dots$.

(Note that for $n=0$, $T_0(\varphi) = \cos(0) = 1$ is a constant. In this case, however, the physics demands that $B=0$ so that the potential be finite along $\rho=0$.)

Furthermore, $R(a) = 0$ in both regions I and II and therefore, we must find a γ_z such that

$$J_n(\gamma_z a) = J_n(\xi_{mn}) = 0$$

where $\xi_{mn} = \gamma_z a$ are the roots or zeros of the Bessel function, J_n .

We can determine these roots from the graph of the Bessel function. It is clear from that graph that there are an infinite number of such roots.

(Compare this to the situation we had in the metal box problem where we had to solve $\sin(ka)=0$ and found an infinite number of roots, $k = m\pi/a$).

A more practical way to determine these Bessel function zeros is to look them up in a table, such as the one below.

Thus, the radial solutions are

$$R_{mn}(\rho) = J_n\left(\frac{\xi_{mn}}{a} \cdot \rho\right).$$

Some of these solutions are graphed below.

From our consideration of the Sturm-Liouville problem, we know that $\{J_n(\xi_{mn}\rho/a)\}$ forms an orthogonal set:

$$\int_0^a \rho J_n(\xi_{mn}\rho/a) J_n(\xi_{kn}\rho/a) d\rho = \delta_{km} \cdot \frac{a^2}{2} [J_n'(\xi_{mn})]^2$$

It can be shown that

$$\|J_n(\xi_{mn}\rho/a)\|^2 = \frac{a^2}{2} [J_n'(\xi_{mn})]^2.$$

$$\int_0^a \rho J_n(\xi_{mn}\rho/a) d\rho = 0$$

BESSEL FUNCTIONS OF INTEGER ORDER

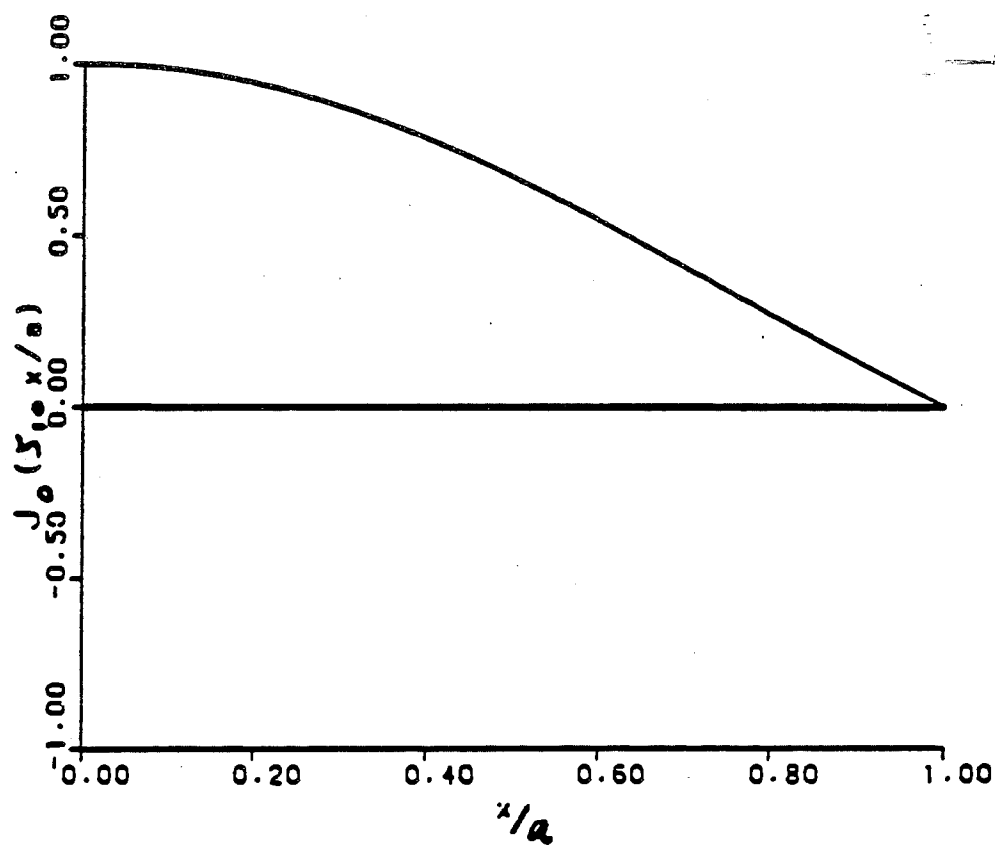
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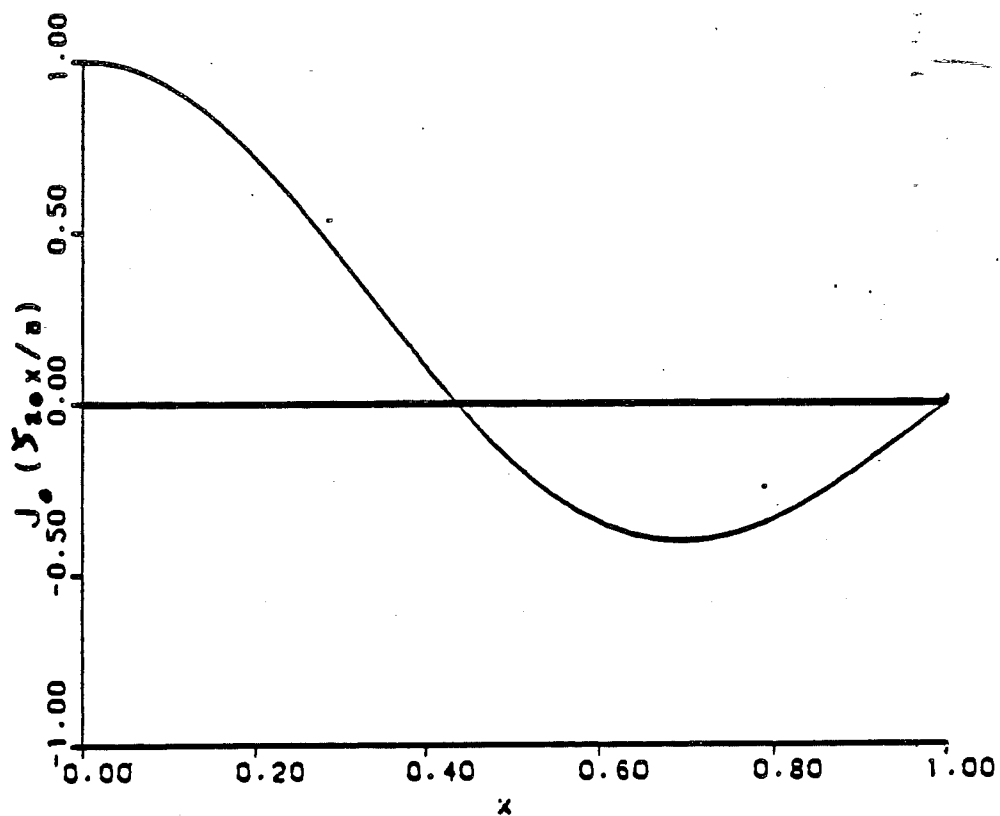
Table 9.5
ZEROS AND ASSOCIATED VALUES OF BESSEL FUNCTIONS AND THEIR DERIVATIVES

n	$J_n(x)$	$J'_n(x)$	$J_n(x)$	$J'_n(x)$	$J_n(x)$	$J'_n(x)$
1	2.40482 55577	-0.51914 78973	3.83171	-0.40276	5.13562	-0.33967
2	5.52007 81103	-0.34026 48065	7.01559	-0.30012	8.41774	-0.27138
3	8.65372 78129	-0.27145 22004	10.17347	-0.24970	11.61984	-0.23264
4	11.79153 44191	-0.23245 98314	13.32349	-0.21856	14.79505	-0.20454
5	14.93091 77086	-0.20454 64331	16.47063	-0.19447	17.95982	-0.18773
6	18.07106 39679	-0.18772 88030	19.61586	-0.18006	21.11700	-0.17326
7	21.21163 66799	-0.17326 58042	22.76008	-0.16718	24.27511	-0.16170
8	24.35247 15308	-0.16170 15507	25.90367	-0.15672	27.42577	-0.15218
9	27.49347 91327	-0.15218 12138	29.04663	-0.14801	30.56970	-0.14417
10	30.63460 64686	-0.14416 59777	32.18968	-0.14061	33.71652	-0.13738
11	33.77582 02136	-0.13729 68434	35.33231	-0.13421	36.86786	-0.13152
12	36.91709 87537	-0.13132 46267	38.47477	-0.12862	40.00845	-0.12607
13	40.05842 57666	-0.12606 98771	41.61709	-0.12367	43.15345	-0.12140
14	43.19979 17132	-0.12139 86748	44.75952	-0.11975	46.29800	-0.11721
15	46.34118 8717	-0.11721 11789	47.90166	-0.11527	49.44216	-0.11343
16	49.48260 98974	-0.11342 91926	51.04354	-0.11167	52.58602	-0.10999
17	52.62405 18412	-0.10999 11850	54.18555	-0.10839	55.72943	-0.10605
18	55.76541 87543	-0.10606 78887	57.32753	-0.10537	58.87302	-0.10396
19	58.90685 39281	-0.10395 95729	60.46966	-0.10260	62.01622	-0.10129
20	62.04826 91902	-0.10129 34989	63.61136	-0.10004	65.15927	-0.99882
n	$J_n(x)$	$J'_n(x)$	$J_n(x)$	$J'_n(x)$	$J_n(x)$	$J'_n(x)$
1	6.38016	-0.29627	7.58834	-0.26836	8.77148	-0.24543
2	9.76102	-0.24042	11.06471	-0.23180	12.33860	-0.21743
3	13.01570	-0.21828	14.37254	-0.20636	15.70212	-0.19615
4	16.22347	-0.19644	17.61597	-0.18766	18.98013	-0.17993
5	19.40942	-0.18005	20.82693	-0.17323	22.21780	-0.16712
6	22.58273	-0.16718	24.01902	-0.16168	25.49034	-0.15669
7	25.74817	-0.15672	27.19809	-0.15217	28.62662	-0.14799
8	28.90855	-0.14801	30.37101	-0.14616	31.61172	-0.14059
9	32.06485	-0.14060	33.53714	-0.13729	34.96878	-0.13420
10	35.21867	-0.13421	36.69900	-0.13132	38.15987	-0.12861
11	38.37067	-0.12862	39.85763	-0.12607	41.32638	-0.12366
12	41.52077	-0.12367	43.01374	-0.12140	44.48932	-0.11925
13	44.66974	-0.11925	46.16785	-0.11721	47.64960	-0.11527
14	47.81779	-0.11527	49.32036	-0.11343	50.80717	-0.11167
15	50.96503	-0.11167	52.47155	-0.10999	53.96303	-0.10838
16	54.11162	-0.10999	55.62165	-0.10605	57.11730	-0.10537
17	57.25765	-0.10537	58.77084	-0.10396	60.27025	-0.10260
18	60.40372	-0.10260	61.91974	-0.10129	63.42205	-0.10004
19	63.54960	-0.10004	65.06705	-0.09882	66.57289	-0.09765
20	66.69324	-0.09765	68.21417	-0.09657	69.72289	-0.09543
n	$J_n(x)$	$J'_n(x)$	$J_n(x)$	$J'_n(x)$	$J_n(x)$	$J'_n(x)$
1	9.93611	-0.22713	11.06637	-0.21209	12.22509	-0.19944
2	13.50929	-0.20575	14.82127	-0.19479	16.03777	-0.18569
3	17.00382	-0.18726	18.78758	-0.17942	19.45454	-0.17764
4	20.37079	-0.17375	22.64154	-0.16666	22.64517	-0.16130
5	23.50658	-0.16159	26.43493	-0.15657	26.26681	-0.15196
6	26.82015	-0.15212	28.19119	-0.14792	29.54566	-0.14404
7	30.03372	-0.14613	31.42279	-0.14095	32.79580	-0.13722
8	33.23304	-0.13727	34.65709	-0.13418	36.02562	-0.13127
9	36.42202	-0.13131	37.89872	-0.12859	39.24045	-0.12603
10	39.60324	-0.12606	41.07077	-0.12365	42.44589	-0.12137
11	42.77848	-0.12139	44.21541	-0.11924	45.63844	-0.11719
12	45.94907	-0.11721	47.34417	-0.11526	48.82594	-0.11342
13	49.11577	-0.11343	50.50618	-0.11166	52.00769	-0.10998
14	52.27944	-0.10999	53.73843	-0.10838	55.18475	-0.10605
15	55.44059	-0.10605	56.90525	-0.10537	58.35789	-0.10396
16	58.59961	-0.10396	60.06648	-0.10260	61.52774	-0.10129
17	61.74682	-0.10129	63.23142	-0.10003	64.69478	-0.09882
18	64.91251	-0.09882	66.39141	-0.09765	67.85443	-0.09652
19	68.06649	-0.09652	69.54971	-0.09643	71.02255	-0.09538
20	71.22313	-0.09538	72.70655	-0.09536	74.18277	-0.09237

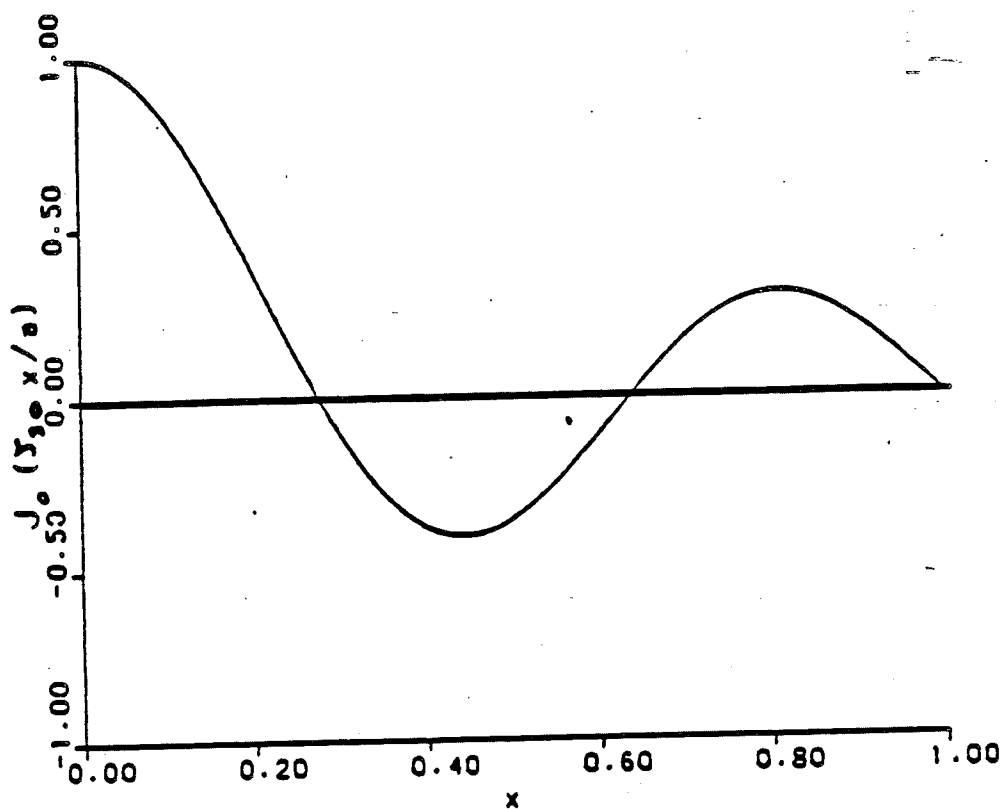
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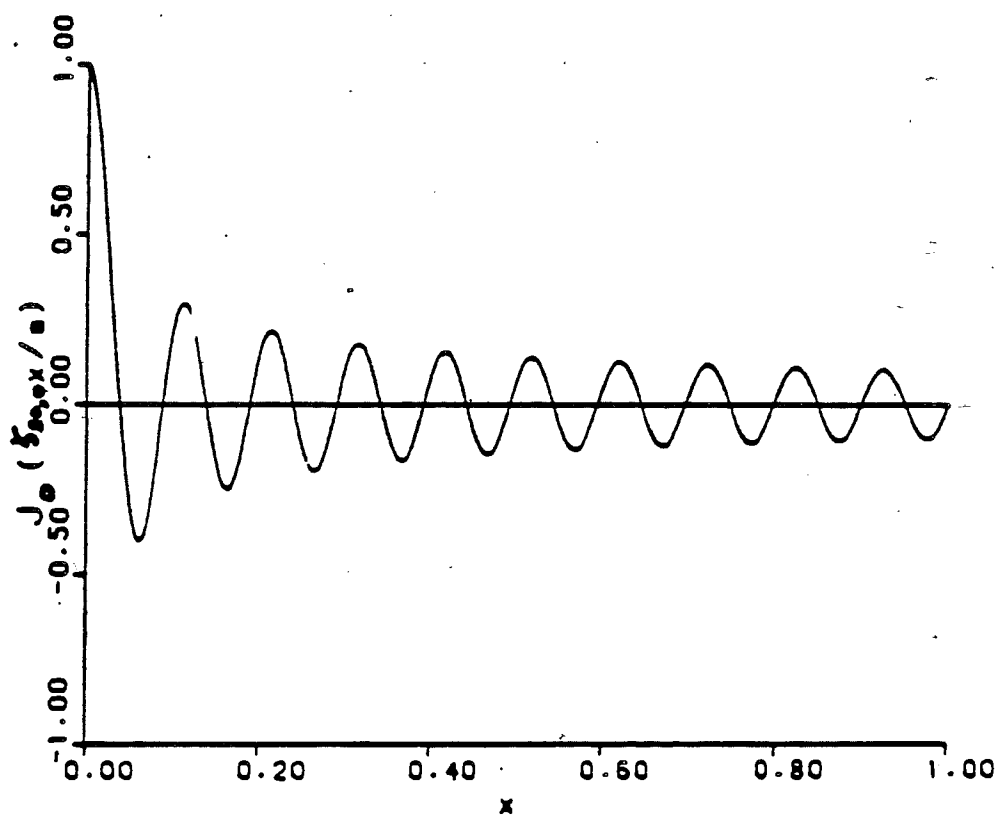
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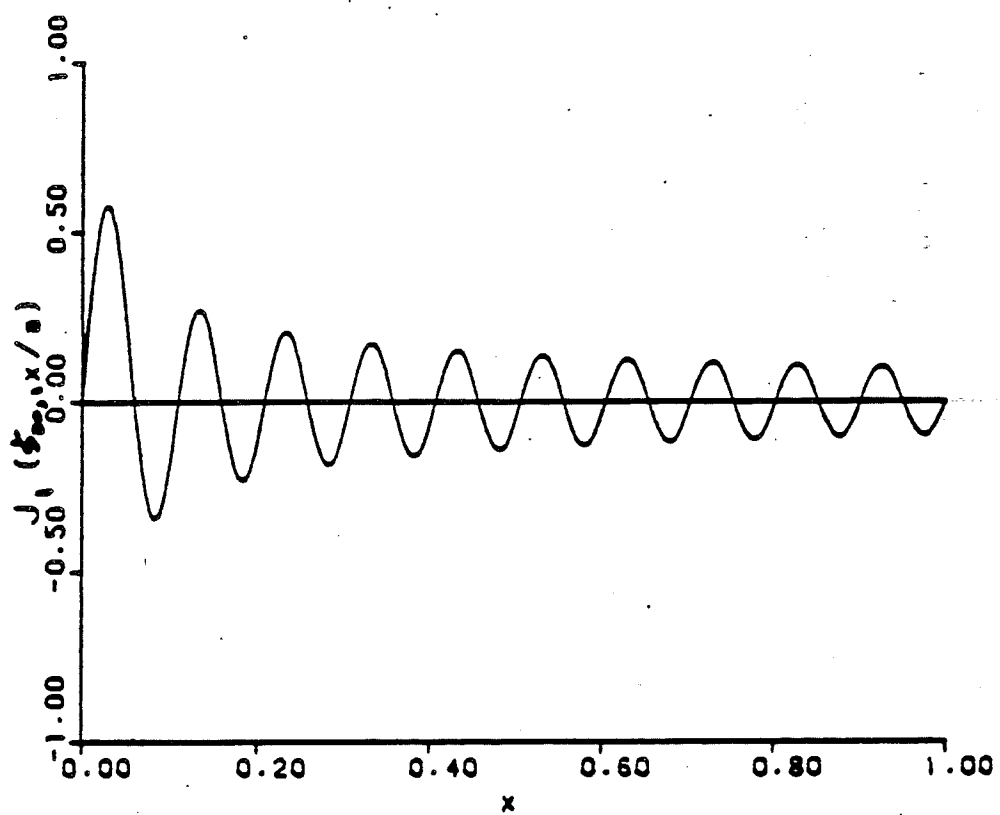
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Finally, we must find $Z_I(z)$ and $Z_{II}(z)$ for regions I and II, respectively:

$$Z''(z) - \gamma_z^2 Z(z) = 0 \rightarrow Z(z) = A \cosh(\gamma_z z) + B \sinh(\gamma_z z)$$

subject to

$$Z_I(0) = 0, \quad Z_{II}(b) = 0$$

Therefore, we take

$$Z_I(z) = \sinh(\gamma_z z) = \sinh\left(\frac{\xi_{mn}}{a} z\right).$$

Similarly,

$$Z_{II}(z) = \sinh[\gamma_z (b-z)] = \sinh\left[\frac{\xi_{mn}}{a} (b-z)\right].$$

Now we are ready to put all the solutions back together to find $\psi_{mn}(r, \varphi, z)$ that

$$G_I(r, \varphi, z | r', \varphi', z') = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left\{ A_{mn}^I \overbrace{J_n\left(\frac{\xi_{mn}}{a} r\right) \cos[n(\varphi - \varphi')] \sinh\left(\frac{\xi_{mn}}{a} z\right)} \right\}$$

satisfies Laplace's equation and the boundary conditions in region I.

Similarly, the most general form of the solution in region II is

$$G_{II}(r, \varphi, z | r', \varphi', z') = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left\{ A_{mn}^{II} J_n\left(\frac{\xi_{mn}}{a} r\right) \cos[n(\varphi - \varphi')] \sinh\left[\frac{\xi_{mn}}{a} (b-z)\right] \right\}$$

We note that since, $\{\cos[n(\varphi-\varphi')]\}$ and $\{J_n(\xi_{mn}\rho/a)\}$ are orthogonal sets, then,

$$\{\Psi_{mn}(\rho, \varphi)\} = \{\cos[n(\varphi-\varphi')] J_n(\xi_{mn}\rho/a)\}$$

is an orthogonal set also:

$$\int_0^{2\pi} \int_0^a \rho \Psi_{mn}(\rho, \varphi) \Psi_{km}(\rho, \varphi) d\rho d\varphi = \delta_{km} \delta_{ln} \cdot \frac{a^2 \pi}{\epsilon_n} \cdot [J_n'(\xi_{mn})]^2.$$

Using this orthogonality property and the continuity conditions, we are now ready to obtain the final solution to the problem (for this particular way of isolating the source).

Applying the condition of continuity of the potential at $z=z'$,

$$\begin{aligned} G_I(\rho, \varphi, z') | \rho', \varphi', z') &= G_{II}(\rho, \varphi, z') | \rho', \varphi', z') \rightarrow \\ \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} A_{mn}^I \Psi_{mn}(\rho, \varphi) \sinh\left(\frac{\xi_{mn}}{a} z'\right) \\ &= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} A_{mn}^{II} \Psi_{mn}(\rho, \varphi) \sinh\left[\frac{\xi_{mn}}{a} (b-z')\right] \rightarrow \\ A_{mn}^I \sinh\left(\frac{\xi_{mn}}{a} z'\right) &= A_{mn}^{II} \sinh\left[\frac{\xi_{mn}}{a} (b-z')\right] \end{aligned}$$

Applying the condition,

$$\frac{\partial G_{II}(P, \varphi, z')}{\partial z} - \frac{\partial G_I(P, \varphi, z')}{\partial z} =$$

$$\text{we have } -\frac{1}{\epsilon} \delta(P-P') \frac{\delta(\varphi-\varphi')}{\rho},$$

$$\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left\{ A_{mn}^{II} \frac{\xi_{mn}}{2} \cosh \left[\frac{\xi_{mn}}{2} (b-z') \right] + A_{mn}^I \frac{\xi_{mn}}{2} \cosh \left(\frac{\xi_{mn}}{2} z' \right) \right\} \psi_{mn}(\rho, \varphi)$$

$$= -\frac{1}{\epsilon} \delta(P-P') \delta(\varphi-\varphi') / \rho$$

Using the orthogonality of $(\psi_{mn}(\rho, \varphi))$, we find that

$$B_{mn} = -\frac{\epsilon_n}{2^2 \pi} \frac{1}{[J_n'(\xi_{mn})]^2} \cdot$$

$$\frac{1}{\epsilon} \int_0^{2\pi} \int_0^2 \delta(P-P') \frac{\delta(\varphi-\varphi')}{\rho} \cdot \psi_{mn}(\rho, \varphi) \rho d\rho d\varphi$$

$$= -\frac{\epsilon_n}{\epsilon 2^2 \pi} \frac{1}{[J_n'(\xi_{mn})]^2} \cdot \psi_{mn}(\rho; \varphi')$$

- or -

$$A_{mn}^{II} \cosh \left[\frac{\xi_{mn}}{2} (b-z') \right] + A_{mn}^I \cosh \left(\frac{\xi_{mn}}{2} z' \right)$$

$$= +\frac{1}{\epsilon} \cdot \frac{\epsilon_n}{\pi 2} \frac{1}{\xi_{mn}} \frac{1}{[J_n'(\xi_{mn})]^2} \psi_{mn}(\rho; \varphi')$$

Thus, just as in the metal box problem, we have two equations in two unknowns (for each m and n):

$$A_{mn}^{\text{II}} \sinh \left[\frac{\gamma_{mn}}{2} (b - z') \right] - A_{mn}^{\text{I}} \sinh \left(\frac{\gamma_{mn}}{2} z' \right) = 0$$

$$A_{mn}^{\text{II}} \cosh \left[\frac{\gamma_{mn}}{2} (b - z') \right] + A_{mn}^{\text{I}} \cosh \left(\frac{\gamma_{mn}}{2} z' \right) = \\ + \frac{1}{\epsilon} \frac{\epsilon_n}{\pi a} \frac{1}{\gamma_{mn}} \frac{1}{[J_n'(\gamma_{mn})]^2} \psi_{mn}(\rho', \varphi')$$

The coefficients, therefore, are

$$A_{mn}^{\text{I}} = \frac{\begin{vmatrix} \sinh \left[\frac{\gamma_{mn}}{2} (b - z') \right] & 0 \\ \cosh \left[\frac{\gamma_{mn}}{2} (b - z') \right] & + \frac{1}{\epsilon} \frac{\epsilon_n}{\pi a} \frac{\psi_{mn}(\rho', \varphi')}{\gamma_{mn} [J_n'(\gamma_{mn})]^2} \end{vmatrix}}{\begin{vmatrix} \sinh \left[\frac{\gamma_{mn}}{2} (b - z') \right] & - \sinh \left(\frac{\gamma_{mn}}{2} z' \right) \\ \cosh \left[\frac{\gamma_{mn}}{2} (b - z') \right] & \cosh \left(\frac{\gamma_{mn}}{2} z' \right) \end{vmatrix}} \\ = + \frac{1}{\epsilon} \frac{\epsilon_n}{\pi a} \frac{1}{[J_n'(\gamma_{mn})]^2} \frac{\sinh \left[\frac{\gamma_{mn}}{2} (b - z') \right] \psi_{mn}(\rho', \varphi')}{\gamma_{mn} \sinh \left(\gamma_{mn} \frac{b}{2} \right)}$$

Similarly,

$$A_{mn}^{\text{II}} = + \frac{1}{\epsilon} \frac{\epsilon_n}{\pi a} \frac{1}{[J_n'(\gamma_{mn})]^2} \frac{\sinh \left(\frac{\gamma_{mn}}{2} z' \right) \psi_{mn}(\rho', \varphi')}{\gamma_{mn} \sinh \left(\gamma_{mn} \frac{b}{2} \right)}$$

Therefore, the final solution is

$$G(\rho, \varphi, z | \rho', \varphi', z') =$$

$$+ \frac{1}{\pi a \epsilon} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left\{ \frac{\epsilon_n}{[J'_n(\gamma_{mn})]^2} \cdot \frac{\psi_{mn}(\rho, \varphi) \psi_{mn}(\rho', \varphi')}{\gamma_{mn} \sinh(\gamma_{mn} \frac{b}{a})} \cdot \right.$$

$$\left. \begin{cases} \sinh\left[\frac{\gamma_{mn}}{a}(b-z')\right] \sinh\left(\frac{\gamma_{mn}}{a}z\right) & z < z' \\ \sinh\left(\frac{\gamma_{mn}}{a}z'\right) \sinh\left[\frac{\gamma_{mn}}{a}(b-z)\right] & z > z' \end{cases} \right.$$

where $J_n(\gamma_{mn}) = 0$,

$$\psi_{mn}(\rho, \varphi) = J_n\left(\frac{\gamma_{mn}}{a}\rho\right) \cos[n(\varphi - \varphi')].$$