Solution Section 4.1 – System of linear Equations

Exercise

Use the Gauss-Jordan method to solve the system

$$\begin{aligned} x - y + 5z &= -6 \\ 3x + 3y - z &= 10 \\ x + 3y + 2z &= 5 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array} \qquad \begin{array}{cccc} 3 & 3 & -1 & 10 \\ -3 & 3 & -15 & 18 \\ \hline 0 & 6 & -16 & 28 \end{array} \qquad \begin{array}{cccc} 1 & 3 & 2 & 5 \\ -1 & 1 & -5 & 6 \\ \hline 0 & 4 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 4 & -3 & 11 \end{array}\right] \ \frac{1}{6}R_2 \qquad 0 \quad 1 \quad -\frac{8}{3} \quad \frac{14}{3}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 4 & -3 & 11 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ \\ R_3 - 4R_2 \end{array} \qquad \begin{array}{cccc} 0 & 4 & -3 & 11 \\ 0 & -4 & \frac{32}{3} & -\frac{56}{3} \\ \hline 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array} \qquad \begin{array}{cccc} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ \hline 1 & 0 & \frac{7}{3} & -\frac{4}{3} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array}\right] \begin{array}{l} \\ \\ \frac{3}{23}R_3 \end{array} \qquad 0 \quad 0 \quad 1 \quad -1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 - \frac{7}{3}R_3 \\ R_2 + \frac{8}{3}R_3 \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 0 & -\frac{7}{3} & \frac{7}{3} \\ \hline 1 & 0 & 0 & 1 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & \frac{8}{3} & -\frac{8}{3} \\ \hline 0 & 1 & 0 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$\Rightarrow$ Solution: **(1, 2, -1)**

Exercise

Use the Gauss-Jordan method to solve the system

$$\begin{cases} 2x - y + 4z = -3 \\ x - 2y - 10z = -6 \\ 3x \qquad + 4z = 7 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & -3 \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array}\right] \quad \tfrac{1}{2}R_1 \qquad 1 \quad -\tfrac{1}{2} \quad 2 \quad -\tfrac{3}{2}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 3R_1 \end{array} \qquad \begin{array}{cccc} 1 & -2 & -10 & -6 \\ -1 & \frac{1}{2} & -2 & \frac{3}{2} \\ \hline 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \end{array} \qquad \begin{array}{cccc} 3 & 0 & 4 & 7 \\ -3 & \frac{3}{2} & -6 & \frac{9}{2} \\ \hline 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}\right] \quad -\tfrac{2}{3}R_2 \qquad 0 \quad 1 \quad 8 \quad 3$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & 1 & 8 & 3 \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}\right] \begin{array}{l} R_1 + \frac{1}{2}R_2 \\ \\ R_3 - \frac{3}{2}R_2 \end{array} \qquad \begin{array}{cccc} 0 & \frac{3}{2} & -2 & \frac{23}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ \hline 0 & 0 & -14 & 7 \end{array} \qquad \begin{array}{cccc} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & \frac{1}{2} & 4 & \frac{3}{2} \\ \hline 1 & 0 & 6 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & -14 & 7 \end{array}\right] \quad -\tfrac{1}{14}R_3 \qquad 0 \quad 0 \quad 1 \quad -\tfrac{1}{2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \begin{array}{l} R_1 - 6R_3 \\ R_2 - 8R_3 \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & 0 & 6 & 0 \\ 0 & 0 & -6 & 3 \\ \hline 1 & 0 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 1 & 8 & 3 \\ 0 & 0 & -8 & 4 \\ \hline 0 & 1 & 0 & 7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \qquad \Rightarrow \underline{\textit{Solution:}} \quad \left(3,\ 7,\ -\tfrac{1}{2}\right)$$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 4x+3y-5z=-29 \\ 3x-7y-z=-19 \\ 2x+5y+2z=-10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 4 & 3 & -5 & -29 \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array}\right] \quad \frac{1}{4}R_1$$

$$1 \quad \frac{3}{4} \quad -\frac{5}{4} \quad -\frac{29}{4}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3-2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & -7 & -1 & -19 \\ -3 & -\frac{9}{4} & \frac{15}{4} & \frac{87}{4} \\ \hline 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \end{array} \qquad \begin{array}{cccc} 2 & 5 & 2 & -10 \\ -2 & -\frac{3}{2} & \frac{5}{2} & \frac{29}{2} \\ \hline 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}\right] \begin{array}{l} \\ -\frac{4}{37}R_2 \\ \\ \end{array}$$

$$0 \quad 1 \quad -\frac{11}{37} \quad -\frac{11}{37}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}\right] \begin{array}{l} R_1-\frac{3}{4}R_2 \\ \\ R_3-\frac{7}{2}R_2 \end{array}$$

$$\begin{array}{cccc} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{3}{4} & \frac{33}{148} & \frac{33}{148} \\ \hline 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \end{array} \qquad \begin{array}{cccc} 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \\ 0 & -\frac{7}{2} & \frac{77}{72} & \frac{77}{72} \\ \hline 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array}\right] \begin{array}{l} \\ \\ \frac{72}{401}R_3 \end{array}$$

$$0 \quad 0 \quad 1 \quad 1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1+\frac{38}{37}R_3 \\ R_2+\frac{11}{37}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 0 & \frac{38}{37} & \frac{38}{37} \\ \hline 1 & 0 & 0 & -6 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{11}{37} & \frac{11}{37} \\ \hline 0 & 1 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Solution: **(-6, 0, 1)**

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x+2y-3z=-15 \\ 2x-3y+4z=18 \\ -3x+y+z=1 \end{cases}$

Solution

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3+3R_1 \end{array}$$

$$\begin{array}{rrrr} -2 & -4 & 6 & 30 \\ 2 & -3 & 4 & 18 \\ \hline 0 & -7 & 10 & 48 \end{array} \qquad \begin{array}{rrrr} 3 & 6 & -9 & -45 \\ -3 & 1 & 1 & 1 \\ \hline 0 & 7 & -8 & -44 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & -15 \\ 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \end{array}\right] \quad -\frac{1}{7}R_2$$

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & -15 \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 7 & -8 & -44 \end{array}\right] \begin{array}{l} R_1-2R_2 \\ \\ R_3-7R_2 \end{array}$$

$$\begin{array}{rrrr} 1 & 2 & -3 & -15 \\ 0 & -2 & \frac{20}{7} & \frac{96}{7} \\ \hline 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \end{array} \qquad \begin{array}{rrrr} 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \\ \hline 0 & 0 & 2 & 4 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 2 & 4 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{2}R_3 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1+\frac{1}{7}R_3 \\ R_2+\frac{10}{7}R_3 \\ \\ \end{array}$$

$$\begin{array}{rrrr} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 0 & \frac{1}{7} & \frac{2}{7} \\ \hline 1 & 0 & 0 & -1 \end{array} \qquad \begin{array}{rrrr} 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & \frac{10}{7} & \frac{20}{7} \\ \hline 0 & 1 & 0 & -4 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$\Rightarrow$ ***Solution***: **(-1, -4, 2)**

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x+2y+3z=10 \\ 4x+5y+6z=11 \\ 7x+8y+9z=12 \end{cases}$

Solution

$$\begin{bmatrix} 1 & 2 & 3 & | & 10 \\ 4 & 5 & 6 & | & 11 \\ 7 & 8 & 9 & | & 12 \end{bmatrix} \begin{matrix} \\ R_2-4R_1 \\ R_3-7R_1 \end{matrix}$$

-4	-8	-12	-40
4	5	6	11
0	-3	-6	-29

-7	-14	-21	-70
7	8	9	12
0	-6	-12	-58

$$\begin{bmatrix} 1 & 2 & 3 & | & 10 \\ 0 & -3 & -6 & | & -29 \\ 0 & -6 & -12 & | & -58 \end{bmatrix} \begin{matrix} \\ \frac{1}{3}R_2 \\ \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & | & 10 \\ 0 & 1 & 2 & | & \frac{29}{3} \\ 0 & -6 & -12 & | & -58 \end{bmatrix} \begin{matrix} \\ \\ R_3+6R_2 \end{matrix}$$

0	−6	−12	−58
0	6	12	58
0	0	0	0

$$\begin{bmatrix} 1 & 2 & 3 & | & 10 \\ 0 & 1 & 2 & | & \frac{29}{3} \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$\Rightarrow$ let z be the variable

From Row 1 $\Rightarrow y + 2z = \frac{29}{3}$

$\Rightarrow y = \frac{29}{3} - 2z$

From Row 1 $\Rightarrow x+2y+3z=10$

$\Rightarrow x = 10-2y-3z$

$\Rightarrow x = 10-2\left(\frac{29}{3}-2z\right)-3z$

$\Rightarrow x = 10-\frac{58}{3}+4z-3z$

$\Rightarrow x = z-\frac{28}{3}$

Solution: $\left(z-\frac{28}{3},\ \frac{29}{3}-2z,\ z\right)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 2x+y+2z=4 \\ 2x+2y \quad =5 \\ 2x-y+6z=2 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \qquad \frac{1}{2}R_1 \quad 1 \quad \frac{1}{2} \quad 1 \quad 2$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \qquad R_2 - 2R_1 \begin{array}{cccc} 2 & 2 & 0 & 5 \\ -2 & -1 & -2 & -4 \\ \hline 0 & 1 & -2 & 1 \end{array} \qquad R_3 - 2R_1 \begin{array}{cccc} 2 & -1 & 6 & 2 \\ -2 & -1 & -2 & -4 \\ \hline 0 & -2 & 4 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & -2 \end{array}\right] \qquad R_1 - \frac{1}{2}R_2 \begin{array}{cccc} 1 & \frac{1}{2} & 1 & 2 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \hline 1 & 0 & 2 & \frac{3}{2} \end{array} \qquad R_3 + 2R_2 \begin{array}{cccc} 0 & -2 & 4 & -2 \\ 0 & 2 & -4 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & \frac{3}{2} \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From Row 3: 0 = 0 is a true statement. Let ***z*** be the variable.

From Row 2: $y - 2z = 1 \Rightarrow y = 1 + 2z$

From Row 1: $x + 2z = \frac{3}{2} \Rightarrow x = -2z + \frac{3}{2}$

<u>*Solution*</u>: $\left(-2z+\frac{3}{2},\ 2z+1,\ z\right)$

Exercise

Use the Gauss-Jordan method to solve the system

$$2x-5y+3z=1$$
$$x-2y-2z=8$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 2 & -5 & 3 & 1 \end{array}\right] \quad R_2 - 2R_1 \qquad \begin{array}{cccc} 2 & -5 & 3 & 1 \\ -2 & 4 & 4 & -16 \\ \hline 0 & -1 & 7 & -15 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & -1 & 7 & -15 \end{array}\right] \quad -R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & 1 & -7 & 15 \end{array}\right] \quad R_1 + 2R_2$$

1	−2	−2	8
0	2	−14	30
1	0	−16	38

$$\left[\begin{array}{ccc|c} 1 & 0 & -16 & 38 \\ 0 & 1 & -7 & 15 \end{array}\right] \begin{array}{l} \rightarrow \; x-16z=38 \\ \rightarrow \; y-7z=15 \end{array}$$

$$\begin{cases} x = 16z+38 \\ y = 7z+15 \end{cases}$$

Solution: $\{16z+38,\ 7z+15,\ z\}$

Exercise

At SnackMix, caramel corn worth \$2.50 per pound is mixed with honey roasted missed nuts worth \$7.50 per pound in order to get 20 lb of a mixture worth \$4.50 per pound. How much of each snack is used?

Solution

$x + y = 20 \qquad (1)$

$2.50x + 7.50y = 90 \qquad (2)$

(1) y = 20 - x

(2) $2.5x + 7.5\,(20 - x) = 90$

$2.5x + 150 - 7.5x = 90$

$-5x = 90 - 150$

$-5x = -60$

$x = \frac{-60}{-5} = 12$

y = 20 – x = 20 – 12 = 8

The mixture consists of 12 lb of caramel and 8 lb of nuts

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$$

Solution

$$\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix} \quad \Rightarrow \begin{bmatrix} w = 9 & x = 17 \\ 8 = y & -12 = z \end{bmatrix}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$$

Solution

$$\begin{cases} x = 12 \\ y + 3 = 5 \rightarrow y = 2 \\ 2z = 6 \rightarrow z = 3 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$$

Solution

$$\begin{bmatrix} 5 = y+3 & x-4 = 2 & 9 = 9 \\ 2 = z+4 & -3 = -3 & 8 = 8 \\ 6 = 6 & 0 = 0 & 5 = w \end{bmatrix} \qquad \rightarrow \begin{cases} y = 2 \\ x = 6 \\ z = -2 \\ 5 = w \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

Solution

$$\begin{bmatrix} a-5 & 5b & 4c+6 \\ -2d & 7f-6 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

$$a-5=15 \to a=20$$

$$5b=25 \to b=5$$

$$4c+6=6 \to 4c=0 \to c=0$$

$$-2d=-8 \to d=4$$

$$7f-6=1 \to 7f=7 \to f=1$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+11 & 12z+1 & 5m \\ 11k & 3 & 1 \end{bmatrix} + \begin{bmatrix} 9a & 9z & 4m \\ 12k & 5 & 3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

Solution

$$\begin{bmatrix} a+11+9a & 12z+1+9z & 5m+4m \\ 11k+12k & 3+5 & 1+3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 10a+11 & 21z+1 & 9m \\ 23k & 8 & 4 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

$$10a+11=41 \quad \to \quad 10a=30 \quad \Rightarrow \boxed{a=3}$$

$$21z+1=-62 \quad \to \quad 21z=-63 \quad \Rightarrow \boxed{z=-3}$$

$$9m=72 \quad \to \quad \boxed{m=8}$$

$$23k=92 \quad \to \quad \boxed{k=\frac{92}{23}=4}$$

Exercise

$$A=\begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix} \qquad B=\begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix}$$

Solution

$$A-B=\begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix}-\begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 4 & -5 \\ 2 & 1 & 9 \end{bmatrix}$$

$$3A+2B=3\begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix}+2\begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix}$$

$$=\begin{bmatrix} 9 & 3 & 3 \\ -3 & 6 & 15 \end{bmatrix}+\begin{bmatrix} 4 & -6 & 12 \\ -6 & 2 & -8 \end{bmatrix}$$

$$=\begin{bmatrix} 13 & -3 & 15 \\ -9 & 8 & 7 \end{bmatrix}$$

Exercise

$$\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}+\begin{bmatrix} -6 \\ 3 \\ 12 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}+\begin{bmatrix} -6 \\ 3 \\ 12 \end{bmatrix}=\begin{bmatrix} -4 \\ 8 \\ 20 \end{bmatrix}$$

Exercise

$$\begin{bmatrix} 5 & 8 \\ 6 & 2 \end{bmatrix}+\begin{bmatrix} 3 & 9 & 1 \\ 4 & 2 & 5 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 5 & 8 \\ 6 & 2 \end{bmatrix}+\begin{bmatrix} 3 & 9 & 1 \\ 4 & 2 & 5 \end{bmatrix} \qquad \textit{It is impossible}$$

Exercise

Find $\begin{bmatrix} -5 & 0 \\ 4 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix}$

Solution

$$\begin{bmatrix} -5 & 0 \\ 4 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -5+6 & 0+(-3) \\ 4+2 & \frac{1}{2}+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 \\ 6 & \frac{7}{2} \end{bmatrix}$$

Exercise

$\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix}$

Solution

$$\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} 5-4 & -6+6 \\ 8+8 & 9-3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 16 & 6 \end{bmatrix}$$

Exercise

Given $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ $\quad F = \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}$ $\quad$ Find $\quad 3F + 2A$

Solution

$$3F + 2A = 3\begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3(3) & 3(3) \\ 3(-1) & 3(-1) \end{bmatrix} + \begin{bmatrix} 2(1) & 2(2) \\ 2(4) & 2(3) \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 9 \\ -3 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 8 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 9+2 & 9+4 \\ -3+8 & -3+6 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 13 \\ 5 & 3 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} -5 & 6 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 5 & -8 \end{bmatrix}$

Solution

$$\begin{bmatrix} -5 & 6 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 5 & -8 \end{bmatrix} = \begin{bmatrix} -5-(-3) & 6-2 \\ 2-5 & 4-(-8) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 4 \\ -3 & 12 \end{bmatrix}$$

Exercise

Evaluate $\begin{bmatrix} 8 & 6 & -4 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -8 \end{bmatrix}$

Solution

$$\begin{bmatrix} 8 & 6 & -4 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -8 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 4 \end{bmatrix}$$

Exercise

Given $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}$. Find AB and BA.

Solution

$$AB = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 13 \\ 4 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} -16 & 29 \\ -4 & 10 \end{bmatrix} \quad Note: \ AB \neq BA$$

Exercise

Find $\begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix}\begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix}$

Solution

$$\begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix}\begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -3(-6)+4(2)+2(3) & -3(4)+4(3)+2(-2) \\ 5(-6)+0(2)+4(3) & 5(4)+0(3)+4(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 32 & -4 \\ -18 & 12 \end{bmatrix}$$

Exercise

Find $AB = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix}$

Solution

$$\Rightarrow AB = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1(4)+3(1) & 1(6)+3(0) \\ 2(4)+5(1) & 2(6)+5(0) \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 13 & 12 \end{bmatrix}$$

Exercise

Find $\begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}.\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}.\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1(1)-1(1)+4(1) & 1(1)-1(2)+4(-1) & 1(0)-1(4)+4(3) \\ 4(1)-1(1)+3(1) & 4(1)-1(2)+3(-1) & 4(0)-1(4)+3(3) \\ 2(1)+0(1)-2(1) & 2(1)+0(2)-2(-1) & 2(0)+0(4)-2(3) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -5 & 8 \\ 6 & -1 & 5 \\ 0 & 4 & -6 \end{bmatrix}$$

Exercise

Find $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix}.\begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & -1 & 3 \end{bmatrix}.\begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1(1)+1(4)+0(2) & 1(-1)+1(-1)+0(0) & 1(4)+1(3)+0(-2) \\ 1(1)+2(4)+4(2) & 1(-1)+2(-1)+4(0) & 1(4)+2(3)+4(-2) \\ 1(1)-1(4)+3(2) & 1(-1)-1(-1)+3(0) & 1(4)-1(3)+3(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2 & 7 \\ 17 & -3 & 2 \\ 3 & 0 & -5 \end{bmatrix}$$

Exercise

Find $\begin{bmatrix} -2 & -3 & -4 \\ 2 & -1 & 0 \\ 4 & -2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$

Solution

$$\begin{bmatrix} -2 & -3 & -4 \\ 2 & -1 & 0 \\ 4 & -2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix} = \begin{pmatrix} -15 & -16 & 3 \\ -1 & 0 & 9 \\ 7 & 6 & 12 \end{pmatrix}$$

Exercise

Find $\begin{bmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{18} \\ \sqrt{3} & \sqrt{27} & 0 \end{bmatrix}\begin{bmatrix} 8 & -10 \\ 9 & 12 \\ 0 & 2 \end{bmatrix}$

Solution

$$\begin{bmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{18} \\ \sqrt{3} & \sqrt{27} & 0 \end{bmatrix}\begin{bmatrix} 8 & -10 \\ 9 & 12 \\ 0 & 2 \end{bmatrix} = \begin{pmatrix} 17\sqrt{2} & -4\sqrt{2} \\ 35\sqrt{3} & 26\sqrt{3} \end{pmatrix}$$

Exercise

A contractor builds three kinds of houses, models *A*, *B*, and *C*, with a choice of two styles, Spanish and contemporary. Matrix *P* shows the number of each kind of house planned for a new 100-home subdivision. The amounts for each of the exterior materials depend primarily on the style of the house. These amounts are shown in matrix *Q*. (concrete is in cubic yards, lumber in units of 1000 board feet, brick in 1000s, and shingles in units of 100 ft^2.) Matrix *R* gives the cost in dollars for each kind of material.

a) What is the total cost of these materials for each model?

b) How much of each of four kinds of material must be ordered

c) What is the total cost for exterior materials?

Solution

$$\begin{array}{c} \\ Model\ A \\ Model\ B \\ Model\ C \end{array} \begin{array}{c} \begin{array}{cc} Spanish & Contemporary \end{array} \\ \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} \end{array} = P$$

$$\begin{array}{c} \\ Spanish \\ Contemporary \end{array} \begin{array}{c} \begin{array}{cccc} Concrete & Lumber & Brick & Shingles \end{array} \\ \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix} \end{array} = Q$$

$$\begin{array}{l} \\ \textit{Concrete} \\ \textit{Lumber} \\ \textit{Brick} \\ \textit{Shingles} \end{array} \begin{array}{c} \textit{Cost per unit} \\ \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} = R \end{array}$$

a) What is the total cost of these materials for each model?

$$PQ = \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix}$$

$$\begin{array}{cc} & \begin{array}{cccc} \textit{Concrete} & \textit{Lumber} & \textit{Brick} & \textit{Shingles} \end{array} \\ = & \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \end{array} \begin{array}{l} \\ \textit{Model A} \\ \textit{Model B} \\ \textit{Model C} \end{array}$$

$$(PQ)R = \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} 72,900 \\ 54,700 \\ 60,800 \end{bmatrix} \begin{array}{l} \textit{Model A} \\ \textit{Model B} \\ \textit{Model C} \end{array}$$

The total cost of materials is \$72,900 for model *A*, \$54,700 for model *B*, \$60,800 for model *C*.

b) How much of each of four kinds of material must be ordered

$$\begin{array}{c} \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \\ \hline 3800 \quad 130 \quad 1400 \quad 200 \end{array} \qquad T = \begin{bmatrix} 3800 & 130 & 1400 & 200 \end{bmatrix}$$

3800 yd^3 of concrete, 130,000 board feet of lumber, 1,400,000 bricks, and 20,000 ft^2 of shingles are needed.

c) What is the total cost for exterior materials?

$$TR = \begin{bmatrix} 3800 & 130 & 1400 & 200 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} = \begin{bmatrix} 188,400 \end{bmatrix}$$

The total cost for exterior materials is \$188,400

Exercise

Mitchell Fabricators manufactures three styles of bicycle frames in its two plants. The following table shows the number of each style produced at each plant

	Mountain Bike	Racing Bike	Touring Bike
North Plant	150	120	100
South Plant	180	90	130

a) Write a 2 *x* 3 matrix ***A*** that represents the information in the table

b) The manufacturer increased production of each style by 20%. Find a Matrix ***M*** that represents the increased production figures.

c) Find the matrix ***A* + *M*** and tell what it represents

Solution

a) $A = \begin{bmatrix} 150 & 120 & 100 \\ 180 & 90 & 130 \end{bmatrix}$

b) The 20% production will represents ***A*** + 20%(***A***) $\rightarrow$ ***A*** + .2 ***A*** = 1.2***A***

$$M = (1.2)\begin{bmatrix} 150 & 120 & 100 \\ 180 & 90 & 130 \end{bmatrix}$$

$$= \begin{bmatrix} 180 & 144 & 120 \\ 216 & 108 & 156 \end{bmatrix}$$

c) $A + M = \begin{bmatrix} 150 & 120 & 100 \\ 180 & 90 & 130 \end{bmatrix} + \begin{bmatrix} 180 & 144 & 120 \\ 216 & 108 & 156 \end{bmatrix}$

$$= \begin{bmatrix} 330 & 264 & 220 \\ 396 & 198 & 286 \end{bmatrix}$$

The matrix A + M represents the total production of each style at each plant for the time period (2 months)

Exercise

Sal's Shoes and Fred's Footwear both have outlets in California and Arizona. Sal's sells shoes for $80, sandals for $40, and boots for $120. Fred's prices are $60, $30, and $150 for shoes, sandals and boots, respectively. Half of all sales in California stores are shoes, 1/4 are sandals, and 1/4 are boots. In Arizona the fractions are 1/5 shoes, 1/5 are sandals, and 3/5 are boots.

a) Write a 2 x 3 matrix called P representing prices for the two stores and three types of footwear.

b) Write a 2 x 3 matrix called F representing fraction of each type of footwear sold in each state.

c) Only one of the two products PF and FP is meaningful. Determine which one it is, calculate the product, and describe what the entries represent.

Solution

a) Write a 2 x 3 matrix called P representing prices for the two stores and three types of footwear.

$$P = \begin{array}{c} \begin{bmatrix} 80 & 40 & 120 \\ 60 & 30 & 150 \end{bmatrix} \end{array} \begin{array}{l} Sal's \\ Fred's \end{array}$$

b) Write a 2 x 3 matrix called F representing fraction of each type of footwear sold in each state.

$$F = \begin{array}{c} \begin{array}{cc} CA & AR \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{3}{5} \end{bmatrix} \end{array}$$

c) Only one of the two products PF and FP is meaningful. Determine which one it is, calculate the product, and describe what the entries represent.

$$PF = \begin{bmatrix} 80 & 40 & 120 \\ 60 & 30 & 150 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{3}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 80\frac{1}{2} + 40\frac{1}{4} + 120\frac{1}{4} & 80\frac{1}{5} + 40\frac{1}{5} + 120\frac{3}{5} \\ 60\frac{1}{2} + 30\frac{1}{4} + 150\frac{1}{4} & 60\frac{1}{5} + 30\frac{1}{5} + 150\frac{3}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 96 \\ 75 & 108 \end{bmatrix}$$

Solution   *Section* 4.3 – Multiplicative Inverses of Matrices

Exercise

Show that B is Multiplicative inverse of A

$$A=\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \qquad B=\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

Solution

$$AB=\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}=\begin{pmatrix} -6 & \\ & \end{pmatrix}$$

B is not multiplicative inverse of A

Exercise

$$\text{A} = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix} \Rightarrow A^{-1} = ?$$

Solution

$$A^{-1} = \frac{1}{-4+6}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ -\frac{1}{3} & 1 \end{bmatrix}$$

Exercise

$$A=\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix} \Rightarrow A^{-1} = ?$$

Solution

$$A^{-1} = \frac{1}{10-10}\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

$$= \frac{1}{0}\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

Doesn't exist

Exercise

Find the inverse of $A=\begin{bmatrix}-2 & 3\\ -3 & 4\end{bmatrix}$

Solution

$$\left[\begin{array}{cc|cc}-2 & 3 & 1 & 0\\ -3 & 4 & 0 & 1\end{array}\right] \quad -\tfrac{1}{2}R_1$$

$$\begin{array}{cccc}1 & -\frac{3}{2} & -\frac{1}{2} & 0\end{array}$$

$$\begin{array}{cccc}-3 & 4 & 0 & 1\\ 3 & -\frac{9}{2} & -\frac{3}{2} & 0\\ \hline 0 & -\frac{1}{2} & -\frac{3}{2} & 1\end{array}$$

$$\left[\begin{array}{cc|cc}1 & -\frac{3}{2} & -\frac{1}{2} & 0\\ -3 & 4 & 0 & 1\end{array}\right] \quad R_2+3R_1$$

$$\left[\begin{array}{cc|cc}1 & -\frac{3}{2} & -\frac{1}{2} & 0\\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1\end{array}\right] \quad -2R_2$$

$$\begin{array}{cccc}0 & 1 & 3 & -2\end{array}$$

$$\begin{array}{cccc}1 & -\frac{3}{2} & -\frac{1}{2} & 0\\ 0 & \frac{3}{2} & \frac{9}{2} & -3\\ \hline 1 & 0 & 4 & -3\end{array}$$

$$\left[\begin{array}{cc|cc}1 & -\frac{3}{2} & -\frac{1}{2} & 0\\ 0 & 1 & 3 & -2\end{array}\right] \quad R_1+\tfrac{3}{2}R_2$$

$$\left[\begin{array}{cc|cc}1 & 0 & 4 & -3\\ 0 & 1 & 3 & -2\end{array}\right]$$

$$\Rightarrow\ A^{-1}=\begin{bmatrix}4 & -3\\ 3 & -2\end{bmatrix}$$

Exercise

Find the inverse of $A=\begin{bmatrix}a & b\\ 3 & 3\end{bmatrix}$

Solution

$$A^{-1}=\frac{1}{3a-3b}\begin{bmatrix}3 & -b\\ -3 & a\end{bmatrix}$$

$$=\begin{bmatrix}\frac{3}{3(a-b)} & \frac{-b}{3(a-b)}\\ \frac{-3}{3(a-b)} & \frac{a}{3(a-b)}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{a-b} & \frac{-b}{3(a-b)}\\ \frac{-1}{a-b} & \frac{a}{3(a-b)}\end{bmatrix}$$

Exercise

Find the inverse of $A=\begin{bmatrix}-2 & a\\ 4 & a\end{bmatrix}$

Solution

$$A^{-1}=\frac{1}{-2a-4a}\begin{bmatrix}a & -a\\ -4 & -2\end{bmatrix}$$

$$=\begin{bmatrix}\frac{a}{-6a} & \frac{-a}{-6a}\\ \frac{-4}{-6a} & \frac{-2}{-6a}\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{1}{6} & \frac{1}{6}\\ \frac{2}{3a} & \frac{1}{3a}\end{bmatrix}$$

Exercise

Find the inverse of $A=\begin{bmatrix}4 & 4\\ b & a\end{bmatrix}$

Solution

$$A^{-1}=\frac{1}{4a-4b}\begin{bmatrix}a & -4\\ -b & 4\end{bmatrix}$$

$$=\begin{bmatrix}\frac{a}{4(a-b)} & \frac{-4}{4(a-b)}\\ \frac{-b}{4(a-b)} & \frac{4}{4(a-b)}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{a}{4(a-b)} & \frac{-1}{a-b}\\ \frac{-b}{4(a-b)} & \frac{1}{a-b}\end{bmatrix}$$

Exercise

Find A^{-1} *if* $A=\begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-3R_1 \end{array} \qquad \begin{array}{cccccc} 2 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -2 & 0 & 0 \\ \hline 0 & -2 & -3 & -2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 3 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & -3 & 0 & 0 \\ \hline 0 & 0 & -3 & -3 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \; -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \; -\tfrac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \begin{array}{l} R_1-R_3 \\ R_2-\frac{3}{2}R_3 \\ \\ \end{array} \qquad \begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ \hline 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array} \qquad \begin{array}{cccccc} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ \hline 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \qquad A^{-1}=\begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

Exercise

Find A^{-1}, where $A=\begin{bmatrix}1 & 2 & -1\\ 3 & 5 & 3\\ 2 & 4 & 3\end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 3 & 5 & 3 & 0 & 1 & 0\\ 2 & 4 & 3 & 0 & 0 & 1\end{array}\right]\quad \begin{array}{l}R_2-3R_1\\ R_3+2R_1\end{array}$$

$$\begin{array}{cccccc}3 & 5 & 3 & 0 & 1 & 0\\ -3 & -6 & 3 & -3 & 0 & 0\\ \hline 0 & -1 & 6 & -3 & 1 & 0\end{array}\qquad \begin{array}{cccccc}2 & 4 & 3 & 0 & 0 & 1\\ -2 & -4 & 2 & -2 & 0 & 0\\ \hline 0 & 0 & 5 & -2 & 0 & 1\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & -1 & 6 & -3 & 1 & 0\\ 0 & 0 & 5 & -2 & 0 & 1\end{array}\right]\quad -R_2$$

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & 1 & -6 & 3 & -1 & 0\\ 0 & 0 & 5 & -2 & 0 & 1\end{array}\right]\quad R_1-2R_2$$

$$\begin{array}{cccccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & -2 & 12 & -6 & 2 & 0\\ \hline 1 & 0 & 11 & -5 & 2 & 0\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 11 & -5 & 2 & 0\\ 0 & 1 & -6 & 3 & -1 & 0\\ 0 & 0 & 5 & -2 & 0 & 1\end{array}\right]\quad \tfrac{1}{5}R_3$$

$$\begin{array}{cccccc}0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5}\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 11 & -5 & 2 & 0\\ 0 & 1 & -6 & 3 & -1 & 0\\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5}\end{array}\right]\quad \begin{array}{l}R_1-11R_3\\ R_2+6R_3\end{array}$$

$$\begin{array}{cccccc}0 & 1 & -6 & 3 & -1 & 0\\ 0 & 0 & 6 & -\frac{12}{5} & 0 & \frac{6}{5}\\ \hline 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5}\end{array}\qquad \begin{array}{cccccc}1 & 0 & 11 & -5 & 2 & 0\\ 0 & 0 & -11 & \frac{22}{5} & 0 & -\frac{11}{5}\\ \hline 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5}\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5}\\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5}\\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5}\end{array}\right]\qquad A^{-1}=\begin{bmatrix}-\frac{3}{5} & 2 & -\frac{11}{5}\\ \frac{3}{5} & -1 & \frac{6}{5}\\ -\frac{2}{5} & 0 & \frac{1}{5}\end{bmatrix}$$

Exercise

Find A^{-1}, where $A=\begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2+2R_1 \\ R_3-R_1 \end{array}$$

$$\begin{array}{cccccc} -2 & 0 & 1 & 0 & 1 & 0 \\ 2 & 4 & -2 & 2 & 0 & 0 \\ \hline 0 & 4 & -1 & 2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & -2 & 1 & -1 & 0 & 0 \\ \hline 0 & -3 & 1 & -1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 4 & -1 & 2 & 1 & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array}\right] \quad \tfrac{1}{4}R_2$$

$$\begin{array}{cccccc} 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1-2R_2 \\ \\ R_3+3R_2 \end{array}$$

$$\begin{array}{cccccc} 0 & -3 & 1 & -1 & 0 & 1 \\ 0 & 3 & -\frac{3}{4} & \frac{3}{2} & \frac{3}{4} & 0 \\ \hline 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{array} \qquad \begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & \frac{1}{2} & -1 & -\frac{1}{2} & 0 \\ \hline 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{array}\right] \quad 4R_3$$

$$\begin{array}{cccccc} 0 & 0 & 1 & 2 & 3 & 4 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}\right] \quad \begin{array}{l} R_1+\frac{1}{2}R_3 \\ R_2+\frac{1}{4}R_3 \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 \\ \hline 1 & 0 & 0 & 1 & 1 & 2 \end{array} \qquad \begin{array}{cccccc} 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ \hline 0 & 1 & 0 & 1 & 1 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}\right]$$

$$A^{-1}=\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

Exercise

Find A^{-1}, where $A=\begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \qquad \frac{1}{-2}R_1 \qquad 1 \quad -\frac{5}{2} \quad -\frac{3}{2} \quad -\frac{1}{2} \quad 0 \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \qquad R_2-4R_1 \qquad \begin{array}{cccccc} 4 & -1 & 3 & 0 & 1 & 0 \\ -4 & 10 & 6 & 2 & 0 & 0 \\ \hline 0 & 9 & 9 & 2 & 1 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \qquad R_3-7R_1 \qquad \begin{array}{cccccc} 7 & -2 & 5 & 0 & 0 & 1 \\ -7 & \frac{35}{2} & \frac{21}{2} & \frac{7}{2} & 0 & 0 \\ \hline 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}\right] \qquad \frac{1}{9}R_2 \quad 0 \quad 1 \quad 1 \quad \frac{2}{9} \quad \frac{1}{9} \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}\right] \quad R_3-\frac{31}{2}R_2 \quad \begin{array}{cccccc} 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \\ 0 & -\frac{31}{2} & -\frac{31}{2} & -\frac{31}{9} & -\frac{31}{18} & 0 \\ \hline 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array}\right]$$

The inverse matrix doesn't exist

Exercise

Solve the system using A^{-1} $\begin{cases} x \qquad\quad +2z=6 \\ -x+2y+3z=-5 \\ x-y \qquad\quad =6 \end{cases}$ Given $A^{-1}=\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

Solution

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3(6)-2(-5)-4(6) \\ 3(6)-2(-5)-5(6) \\ -1(6)+1(-5)+2(6) \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

Solution: $\{(4, -2, 1)\}$

Exercise

Solve the system using A^{-1} $\begin{aligned} x+2y+5z&=2 \\ 2x+3y+8z&=3 \\ -x+\ y+2z&=3 \end{aligned}$

a. Write the linear system as a matrix equation in the form $AX= B$
b. Solve the system using the inverse that is given for the coefficient matrix

the inverse of $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix}$ *is* $\begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}$

Solution

a) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$

b) $\begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}=\begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$

Exercise

Solve the system using A^{-1} $\begin{aligned} x-y+z&=8 \\ 2y-z&=-7 \\ 2x+3y\quad &=1 \end{aligned}$

a) Write the linear system as a matrix equation in the form $AX = B$

b) Solve the system using the inverse that is given for the coefficient matrix

$$\textit{the inverse is } \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

Solution

a) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -7 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

Exercise

Use the inverse of the coefficient matrix to solve the linear system

$$\begin{aligned} x \qquad\quad +z &= -1 \\ 2x-2y-z &= 5 \\ 3x \qquad\qquad &= 6 \end{aligned}$$

Solution

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix} \quad \rightarrow \quad A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

Exercise

Evaluate $\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix}$

Solution

$$\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix} = -9 - (-6) = -3$$

Exercise

Evaluate $\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix} = -6 - (0) = -6$$

Exercise

Evaluate $\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix}$

Solution

$$\begin{aligned} \begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix} &= x(8x) - 4x(2x) \\ &= 8x^2 - 8x^2 \\ &= 0 \end{aligned}$$

Exercise

Evaluate $\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix}$

Solution

$$\begin{aligned} \begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix} &= 3x - 2x(4) \\ &= 3x - 8x \\ &= -5x \end{aligned}$$

Exercise

Evaluate $\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix}$

Solution

$$\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix} = \underline{-3x^4 - 2x}$$

Exercise

Evaluate $\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix}$

Solution

$$\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix} = \underline{-8a + 5b}$$

Exercise

Evaluate $\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$

Solution

$$\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix} = 0 + 126 + 48 - (-48 + 36 + 0) = 90$$

Exercise

Evaluate $\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix} = 0 - 4 - 16 - (-14 - 16 + 0) = 10$$

Exercise

Evaluate $\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix}$

Solution

$$\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix} = -109$$

Exercise

Evaluate $\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix}$

Solution

$$\left|\begin{array}{ccc|cc} 2x & 1 & -1 & 2x & 1 \\ 0 & 4 & x & 0 & 4 \\ 3 & 0 & 2 & 3 & 0 \end{array}\right| = 2x(4)(2) + 3x + 0 - (-1)(4)(3) - 0 - 0$$

$$= 16x + 3x + 12$$

$$\underline{= 19x + 12}$$

Exercise

Evaluate $\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix}$

Solution

$$\left|\begin{array}{ccc|cc} 0 & x & x & 0 & x \\ x & x^2 & 5 & x & x^2 \\ x & 7 & -5 & x & 7 \end{array}\right| = 0 + x(5)(x) + x(x)(7) - (x)\left(x^2\right)(x) - 0 - x(x)(-5)$$

$$= 5x^2 + 7x^2 - x^4 + 5x^2$$

$$\underline{= 17x^2 - x^4}$$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}=-3-4=-7$$

$$D_x=\begin{vmatrix} -4 & 2 \\ -5 & -1 \end{vmatrix}=4-(-10)=14$$

$$D_y=\begin{vmatrix} 3 & -4 \\ 2 & -5 \end{vmatrix}=-15-(-8)=-7$$

$$x=\frac{D_x}{D}=\frac{14}{-7}=-2$$

$$y=\frac{D_y}{D}=\frac{-7}{-7}=1$$

Solution: $\boxed{(-2,\ 1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix}=-29 \qquad D_x=\begin{vmatrix} 7 & 5 \\ -3 & -2 \end{vmatrix}=1 \qquad D_y=\begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix}=-41$$

$$x=\frac{1}{-29}=-\frac{1}{29} \qquad y=\frac{41}{29}$$

Solution: $\boxed{\left(-\frac{1}{29},\ \frac{41}{29}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y-z=4 \\ 3x-2y+z=5 \\ 4x-5y-z=-1 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 2 & -1 \\ 3 & -2 & 1 \\ 4 & -5 & -1 \end{vmatrix}=42 \qquad D_x=\begin{vmatrix} 4 & 2 & -1 \\ 5 & -2 & 1 \\ -1 & -5 & -1 \end{vmatrix}=63$$

$$D_y=\begin{vmatrix} 3 & 4 & -1 \\ 3 & 5 & 1 \\ 4 & -1 & -1 \end{vmatrix}=39 \quad D_z=\begin{vmatrix} 3 & 2 & 4 \\ 3 & -2 & 5 \\ 4 & -5 & -1 \end{vmatrix}=99$$

$$x=\frac{D_x}{D}=\frac{63}{42}=\frac{3}{2}, \qquad y=\frac{D_y}{D}=\frac{39}{42}=\frac{13}{14}, \qquad z=\frac{D_z}{D}=\frac{99}{42}=\frac{33}{14}$$

Solution: $\left(\frac{3}{2}, \frac{13}{14}, \frac{33}{14}\right)$

Exercise

Find the quadratic function $f(x)=ax^2+bx+c$ for which $f(1)=-10, \quad f(-2)=-31, \quad f(2)=-19$. What is the function?

Solution

$$f(1)=a(1)^2+b(1)+c \quad \Rightarrow \quad -10=a+b+c$$

$$f(-2)=a(-2)^2+b(-2)+c \quad \Rightarrow \quad -31=4a-2b+c$$

$$f(2)=a(2)^2+b(2)+c \quad \Rightarrow \quad -19=4a+2b+c$$

$$\begin{cases} a+b+c=-10 \\ 4a-2b+c=-31 \\ 4a+2b+c=-19 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 1 \\ 4 & -2 & 1 \\ 4 & 2 & 1 \end{vmatrix}=12 \qquad D_a=\begin{vmatrix} -10 & 1 & 1 \\ -31 & -2 & 1 \\ -19 & 2 & 1 \end{vmatrix}=-48$$

$$D_b=\begin{vmatrix} 1 & -10 & 1 \\ 4 & -31 & 1 \\ 4 & -19 & 1 \end{vmatrix}=36 \qquad D_c=\begin{vmatrix} 1 & 1 & -10 \\ 4 & -2 & -31 \\ 4 & 2 & -19 \end{vmatrix}=-108$$

$$a=\frac{D_a}{D}=\frac{-48}{12}=-4, \quad b=\frac{D_b}{D}=\frac{36}{12}=3, \quad c=\frac{D_c}{D}=\frac{-108}{12}=-9$$

Solution: $f(x)=-x^2+3x-9$