

Section 2.6 – Exponential & Logarithmic Functions

Exponential

Definition

The exponential function f with base b is defined by

$$f(x) = b^x \quad \text{or} \quad y = b^x$$

Where $b > 0$, $b \neq 1$ and x is any real number.

Example: $f(x) = 2^x$ $f(x) = \left(\frac{1}{2}\right)^{2x+1}$ $f(x) = 3^{-x}$ ~~$f(x) = (-2)^x$~~

Exponential Equations

$$b^x = b^y \quad \leftrightarrow \quad x = y \quad \text{for any } b > 0, \neq 1$$

Example

Solve $9^x = 27$

Solution

$$(3^2)^x = 3^3$$

$$3^{2x} = 3^3$$

$$2x = 3$$

$$x = \frac{3}{2}$$

Example

Solve $32^{2x-1} = 128^{x+3}$

Solution

$$(2^5)^{2x-1} = (2^7)^{x+3}$$

$$2^{10x-5} = 2^{7x+21}$$

$$10x - 5 = 7x + 21$$

$$3x = 26$$

$$x = \frac{26}{3}$$

Natural Base e

The irrational number e is called natural base

$f(x) = e^x$ is called natural exponential function

$$e^0 = 1$$

$$e \approx 2.7183$$

$$e^2 \approx 7.389$$

$$e^{-1} \approx 0.3679$$

Example

Biologists studying salmon have found that the oxygen consumption of yearling salmon (in appropriate units) increases exponentially with the speed of swimming according to the function defined by

$$f(x) = 100e^{0.6x}$$

where x is the speed in feet per second. Find the following

a) The oxygen consumption when the fish are still

$$\begin{aligned} f(x=0) &= 100e^{0.6(0)} \\ &= 100 \end{aligned}$$

b) The oxygen consumption at a speed of 2 ft per second

$$\begin{aligned} f(x=2) &= 100e^{0.6(2)} \\ &\approx 332 \end{aligned}$$

Logarithmic Function (Definition)

For $x > 0$ and $b > 0, b \neq 0$

$y = \log_b x$ is equivalent to $x = b^y$

$$\textcolor{red}{y} = \log_{\textcolor{blue}{b}} x \Leftrightarrow x = \textcolor{blue}{b}^{\textcolor{red}{y}}$$

Base

The function $f(x) = \log_b x$ is the logarithmic function with base b .

$\log_b x$: read log base b of x $\log x$ *means* $\log_{10} x$

Logarithmic Form	Exponential Form
$\log_2 8 = \textcolor{red}{3}$	$2^{\textcolor{red}{3}} = 8$
$\log_{1/2} 16 = \textcolor{red}{-4}$	$\left(\frac{1}{2}\right)^{\textcolor{red}{-4}} = 16$
$\log_3 \frac{1}{81} = -4$	$3^{-4} = \frac{1}{81}$
$\log_{3/4} 1 = 0$	$\left(\frac{3}{4}\right)^0 = 1$

Natural Logarithms

Definition

$$f(x) = \log_e x = \ln x$$

The logarithmic function with base e is called natural logarithmic function.

$\ln x$ read "el en of x "

$$\log(-1) = \textit{doesn't exist}$$

$$\ln(-1) = \textit{doesn't exist}$$

$$\log 0 = \textit{doesn't exist}$$

$$\ln 0 = \textit{doesn't exist}$$

$$\log 0.5 \approx \textcolor{blue}{-0.3010}$$

$$\ln 0.5 \approx \textcolor{blue}{-0.6931}$$

$$\log 1 = 0$$

$$\ln 1 = 0$$

$$\log 2 \approx \textcolor{blue}{0.3010}$$

$$\ln 2 \approx \textcolor{blue}{0.6931}$$

$$\log 10 = 1$$

$$\ln e = 1$$

Domain

The domain of a logarithmic function of the form $f(x) = \log_b x$ is the set of all positive real numbers.
(Inside the log has to be > 0)

Find the domain of $f(x) = \log_4(x-5)$

$$x-5 > 0 \Rightarrow x > 5$$

$$\text{Domain: } (5, \infty)$$

Properties of Logarithmic Functions

Product Rule

$$\log_b MN = \log_b M + \log_b N$$

Power Rule

$$\log_b M^p = p \log_b M$$

Quotient Rule

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

$$\log_b 1 = 0$$

$$\log_b b = 1$$

$$\log_b b^x = x$$

Example

Use the properties of logarithms to rewrite $\log_a \left(\frac{mnq}{p^2 r^4} \right)$

Solution

$$\begin{aligned} \log_a \left(\frac{mnq}{p^2 r^4} \right) &= \log_a (mnq) - \log_a (p^2 r^4) && \text{Quotient Rule} \\ &= \log_a m + \log_a n + \log_a q - (\log_a p^2 + \log_a r^4) && \text{Product Rule} \\ &= \log_a m + \log_a n + \log_a q - \log_a p^2 - \log_a r^4 \\ &= \log_a m + \log_a n + \log_a q - 2\log_a p - 4\log_a r && \text{Power Rule} \end{aligned}$$

Changing Logarithmic Bases

$$\log_b M = \frac{\log_a M}{\log_a b}$$

$$\log_b M = \frac{\log M}{\log b} \quad \text{or} \quad \log_b M = \frac{\ln M}{\ln b}$$

Example

Find: $\log_5 27$

Solution

$$\log_5 27 = \frac{\ln 27}{\ln 5}$$
$$\approx 2.05$$

$$\ln(27) / \ln(5)$$

Property of Logarithmic

$$M = N \leftrightarrow \ln M = \ln N$$

1. Isolate the exponential expression
2. Take the natural logarithm on both sides of the equation
3. Simplify using one of the following properties: $\ln b^x = x \ln b$ or $\ln e^x = x$
4. Solve for the variable

Example

Solve $3^{2x} = 4^{x+1}$

Solution

$$\ln 3^{2x} = \ln 4^{x+1}$$

$$2x \ln 3 = (x+1) \ln 4$$

$$2x \ln 3 = x \ln 4 + \ln 4$$

$$2x \ln 3 - x \ln 4 = \ln 4$$

$$x(2 \ln 3 - \ln 4) = \ln 4$$

$$x = \frac{\ln 4}{2 \ln 3 - \ln 4}$$

$$\approx 1.710$$

$$\ln(4) / (2 \ln(3) - \ln(4))$$

Solving Logarithmic Equations

1. Express the equation in the form $\log_b M = c$
2. Use the definition of a logarithm to rewrite the equation in exponential form:

$$\log_b M = c \Rightarrow b^c = M$$

3. Solve for the variable
4. Check proposed solution in the original equation. Include only the set for $M > 0$

Example

Solve $\log_2 x - \log_2 (x-1) = 1$

Solution

$$\log_2 \frac{x}{x-1} = 1 \quad \text{Write in exponential form}$$

$$\frac{x}{x-1} = 2^1 = 2$$

$$x = 2(x-1)$$

$$x = 2x - 2$$

$$-x = -2$$

$$x = 2$$

$$\text{Check: } \log_2 2 - \log_2 (2-1) = 1$$

✚ For any $M > 0, N > 0, b > 0, \neq 1$

$$\log_b M = \log_b N \Leftrightarrow M = N$$

Check proposed solution in the original equation. Include only the set inside the log for > 0

Example

Solve: $\log(x+6) - \log(x+2) = \log x$

Solution

$$\log(x+6) - \log(x+2) = \log x$$

Quotient Rule

$$\log \frac{x+6}{x+2} = \log x$$

$$\frac{x+6}{x+2} = x$$

Multiply by $x+2$

$$x+6 = x(x+2)$$

$$x+6 = x^2 + 2x$$

$$0 = x^2 + 2x - x - 6$$

$$x^2 + x - 6 = 0$$

Solve for x

$$x = -3, 2$$

Check: $x = -3 \rightarrow \log(-3 + 6) - \log(-3 + 2) = \log(-3)$

or Domain

$$x = 2 \rightarrow \log(2 + 6) - \log(2 + 2) = \log(2)$$

Solution: $x = 2$

$$\log_e x = \ln x$$

$$\log_{10} x = \log x$$

Exercises **Section 2.6 – Exponential & Logarithmic Functions**

Solve

1. $4^{2x-1} = 64$

2. $3^{1-x} = \frac{1}{27}$

3. $9^x = \frac{1}{\sqrt[3]{3}}$

4. $5^{3x-6} = 125$

5. $8^{x+2} = 4^{x-3}$

Solve

6. $7e^{2x} - 5 = 58$

7. $4\ln(3x) = 8$

8. $\ln(x-3) = \ln(7x-23) - \ln(x+1)$

Use the properties of logarithms to rewrite

9. $\log_b \left(\frac{x^3 y}{z^2} \right)$

10. $\log_b \left(\frac{\sqrt[3]{xy^4}}{z^5} \right)$

11. $\log_b n \sqrt{\frac{x^3 y^5}{z^m}}$

12. $\log_p 3 \sqrt{\frac{m^5 n^4}{t^2}}$

13. $\log_a 4 \sqrt{\frac{m^8 n^{12}}{a^3 b^5}}$