Section 2.2 – Limits and Continuity

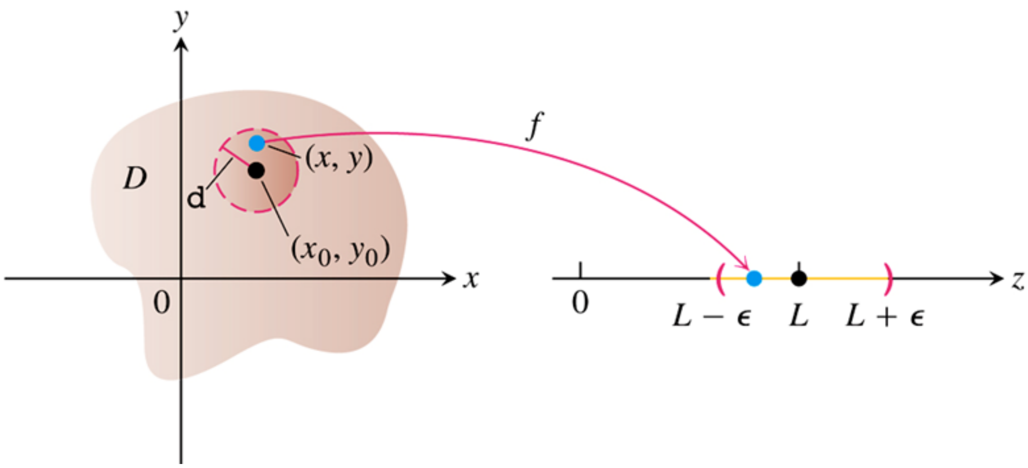
Definition

We say that a function $f(x, y)$ approaches the limit *L*, as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L$$

If, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of *f*.

$$|f(x, y) - L| < \varepsilon \quad \textit{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$



Theorem

The following rules hold if *L, M, K* are real numbers and

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L \quad \textit{and} \quad \lim_{(x, y)\to(x_0, y_0)} g(x, y) = M$$

Sum Rule: $\lim_{(x, y)\to(x_0, y_0)} \left(f(x, y) + g(x, y) \right) = L + M$

Difference Rule: $\lim_{(x, y)\to(x_0, y_0)} \left(f(x, y) - g(x, y) \right) = L - M$

Constant Multiple Rule: $\lim_{(x, y)\to(x_0, y_0)} kf(x, y) = kL$

Product Rule: $\lim_{(x, y)\to(x_0, y_0)} \left(f(x, y) \cdot g(x, y) \right) = L \cdot M$

Quotient Rule: $$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$$

Power Rule: $$\lim_{(x,y)\to(x_0,y_0)} \left[f(x,y)\right]^n = L^n \quad (\boldsymbol{n} \textit{ a positive integer})$$

Root Rule: $$\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n} \quad (\boldsymbol{n} \textit{ a positive integer})$$

Example

Find $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3}$

Solution

$$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3}$$

$$= -3$$

Example

Find $\displaystyle\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2}$

Solution

$$\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2}$$

$$= 5$$

Example

Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$

Solution

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \frac{0}{0}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x(x-y)\left(\sqrt{x} + \sqrt{y}\right)}{x - y}$$

$$= \lim_{(x,\,y)\to(0,0)} x\left(\sqrt{x}+\sqrt{y}\right)$$

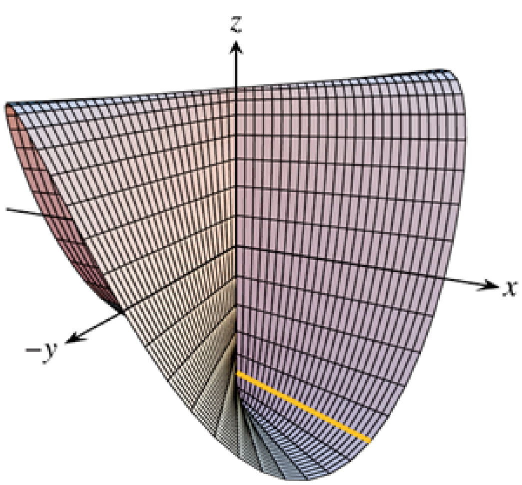
$$= 0$$

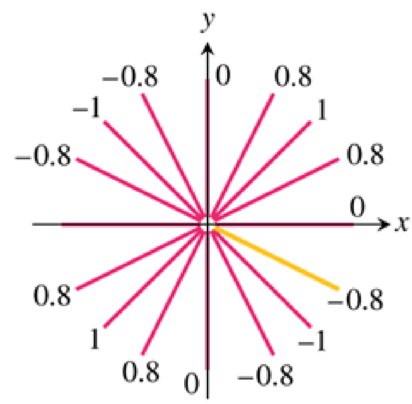
Definition

A function $f(x, y)$ is ***continuous at the point*** (x_0, y_0) if

1. f is defined at (x_0, y_0)
2. $\lim\limits_{(x,\,y)\to(x_0,\,y_0)} f(x, y)$ exists
3. $\lim\limits_{(x,\,y)\to(x_0,\,y_0)} f(x, y) = f(x_0, y_0)$

A function is ***continuous*** if it is continuous at every point of its domain.





Two-Path Test for Nonexistence of a Limit

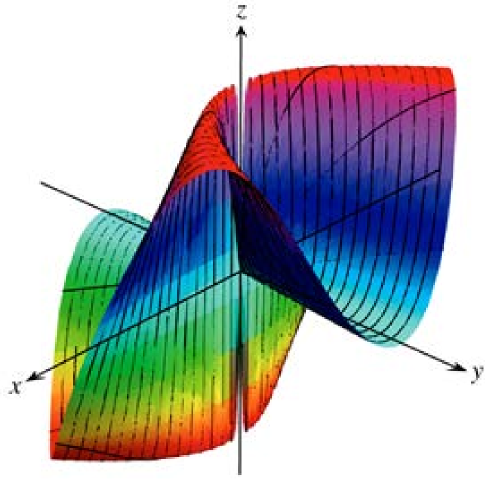
If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim\limits_{(x,\,y)\to(x_0,\,y_0)} f(x, y)$ does not exist.

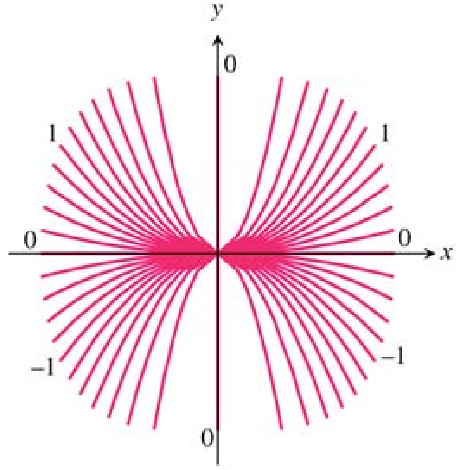
Example

Show that the function $f(x,y)=\dfrac{2x^2y}{x^4+y^2}$ has no limit as $(x,\ y)$ approaches $(0,\ 0)$

Solution

$$\lim_{(x,\,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}=\frac{0}{0}$$





We examine the curve $y=kx^2, \quad x\neq 0$

$$\left.\frac{2x^2y}{x^4+y^2}\right|_{y=kx^2}=\frac{2x^2\left(kx^2\right)}{x^4+\left(kx^2\right)^2}$$

$$=\frac{2kx^4}{x^4+k^2x^4}$$

$$=\frac{2kx^4}{x^4\left(1+k^2\right)}$$

$$=\frac{2k}{1+k^2}$$

$$\lim_{\substack{(x,\,y)\to(0,0)\\ along\ y=kx^2}} f(x,y)=\frac{2k}{1+k^2}$$

This limit varies with the path of approach. If $(x,\ y)$ approaches (0, 0) along the parabola $y=x^2$.

Continuity of Composites

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0).

Functions of More Than Two Variables

The definitions of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, products, quotients, powers, and composites all extend to functions of three or more variables. Functions like

$$\ln(x + y + z) \quad \textit{and} \quad \frac{y \sin z}{x - 1}$$

Exercises *Section* 2.2 – Limits and Continuity

(**1 – 24**) Find the limit

1. $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$

2. $\lim\limits_{(x,y)\to(0,4)} \dfrac{x}{\sqrt{y}}$

3. $\lim\limits_{(x,y)\to(3,4)} \sqrt{x^2 + y^2 - 1}$

4. $\lim\limits_{(x,y)\to(0,0)} \cos\dfrac{x^2 + y^3}{x + y + 1}$

5. $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^y \sin x}{x}$

6. $\lim\limits_{(x,y)\to\left(\frac{\pi}{2},0\right)} \dfrac{\cos y + 1}{y - \sin x}$

7. $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq y}} \dfrac{x^2 - 2xy + y^2}{x - y}$

8. $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq y}} \dfrac{x^2 - y^2}{x - y}$

9. $\lim\limits_{\substack{(x,y)\to(2,-4)\\ x\neq x^2,\ y\neq -4}} \dfrac{y + 4}{x^2 y - xy + 4x^2 - 4x}$

10. $\lim\limits_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \dfrac{\sqrt{x} - \sqrt{y+1}}{x - y - 1}$

11. $\lim\limits_{(x,y)\to(1,-1)} \dfrac{x^3 + y^3}{x + y}$

12. $\lim\limits_{(x,y)\to(2,2)} \dfrac{x - y}{x^4 - y^4}$

13. $\lim\limits_{P\to(1,3,4)} \left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right)$

14. $\lim\limits_{P\to(1,-1,-1)} \dfrac{2xy + yz}{x^2 + z^2}$

15. $\lim\limits_{P\to(\pi,0,2)} ze^{-2y}\cos 2x$

16. $\lim\limits_{P\to(2,-3,6)} \ln\sqrt{x^2 + y^2 + z^2}$

17. $\lim\limits_{(x,y)\to(4,-2)} (10x - 5y + 6xy)$

18. $\lim\limits_{(x,y)\to(1,1)} \dfrac{xy}{x + y}$

19. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x + y}{xy}$

20. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin xy}{x^2 + y^2}$

21. $\lim\limits_{(x,y)\to(-1,1)} \dfrac{x^2 - y^2}{x^2 - xy - 2y^2}$

22. $\lim\limits_{(x,y)\to(1,2)} \dfrac{x^2 y}{x^4 + 2y^2}$

23. $\lim\limits_{(x,y,z)\to\left(\frac{\pi}{2},0,\frac{\pi}{2}\right)} 4\cos y \sin\sqrt{xz}$

24. $\lim\limits_{(x,y,z)\to(5,2,-3)} \tan^{-1}\left(\dfrac{x + y^2}{z^2}\right)$

(25 – 30) At what points (x, y, z) in space are the functions continuous

25. $f(x,y,z) = x^2 + y^2 - 2z^2$

26. $f(x,y,z) = \sqrt{x^2 + y^2 - 1}$

27. $f(x,y,z) = \ln(xyz)$

28. $f(x,y,z) = e^{x+y}\cos z$

29. $h(x,y,z) = \dfrac{1}{|y| + |z|}$

30. $h(x,y,z) = \dfrac{1}{z - \sqrt{x^2 + y^2}}$