SOLUTION *Section* 3.4 – Comparison Tests

Exercise

Use the Comparison Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+30}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent *p*-series, since $p = 2 > 1$.

Both series have nonnegative terms for $n \ge 1$

$$n^2 \le n^2 + 30 \quad \Rightarrow \quad \frac{1}{n^2} \ge \frac{1}{n^2+30}.$$

Then, by Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+30}$ ***converges***.

Exercise

Use the Comparison Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$, which is a convergent *p*-series, since $p = 3 > 1$.

Both series have nonnegative terms for $n \ge 1$

$$n^4 \le n^4 + 2 \quad \Rightarrow \quad \frac{1}{n^4} \ge \frac{1}{n^4+2}$$

$$\frac{n}{n^4} \ge \frac{n}{n^4+2} \ge \frac{n-1}{n^4+2}$$

$$\frac{1}{n^3} \ge \frac{n}{n^4+2} \ge \frac{n-1}{n^4+2}$$

Then, by Comparison Test, the given series ***converges***.

Exercise

Use the Comparison Test to determine if the series converges or diverges. $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$

Solution

Comparing with $\sum_{n=1}^{\infty} \frac{1}{n}$, which is a divergent p-series, since $p = 1 \le 1$.

Both series have nonnegative terms for $n \ge 2$

$$n^2 - n \le n^2 \quad \Rightarrow \quad \frac{1}{n^2-n} \ge \frac{1}{n^2}$$

$$\frac{n}{n^2-n} \ge \frac{n}{n^2} = \frac{1}{n}$$

Then, by Comparison Test, the given series ***diverges***.

Exercise

Use the Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$

Solution

Comparing with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which is a convergent p-series, since $p = \frac{3}{2} > 1$.

Both series have nonnegative terms for $n \ge 1$

$$0 \le \cos^2 n \le 1 \quad \Rightarrow \quad 0 \le \frac{\cos^2 n}{n^{3/2}} \le \frac{1}{n^{3/2}}$$

Then, by Comparison Test, the given series ***converges***.

Exercise

Use the Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$

Solution

Comparing with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} < 1$.

Both series have nonnegative terms for $n \ge 1$

$$\sqrt{n} \ge 1 \quad \Rightarrow \quad 2\sqrt{n} \ge 2$$

$$2\sqrt{n} + 1 \ge 3$$

Then, by Comparison Test, the given series ***diverges***.

Exercise

Use the Comparison Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$, which is a convergent series.

Then $0 < \dfrac{1}{2^n + 1} < \dfrac{1}{2^n}$

Therefore, the given series ***converges*** by comparison.

Exercise

Use the Comparison Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{3n+1}{n^3 + 1}$

Solution

$$\frac{3n+1}{n^3+1} \xrightarrow[n\to\infty]{} \frac{3}{n^2}$$

$$\begin{aligned}\frac{3n+1}{n^3+1} &= \frac{3n}{n^3+1} + \frac{1}{n^3+1} \\ &< \frac{3n}{n^3} + \frac{1}{n^3} \\ &< \frac{3}{n^2} + \frac{1}{n^2} \\ &= \frac{4}{n^2}\end{aligned}$$

Therefore, the given series ***converges*** by comparison.

Exercise

Use the Comparison Test to determine if the series converges or diverges. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

Solution

$$0 < \ln n < n \quad \Rightarrow \quad \frac{1}{\ln n} > \frac{1}{n}$$

Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ ***diverges*** to infinity (it is a harmonic series), so does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ by comparison.

Exercise

Use the Comparison Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

Solution

$$2n - 1 < 2n \quad \Rightarrow \quad \frac{1}{2n-1} > \frac{1}{2n} \quad \textit{for } n \ge 1$$

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} > \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$$

By the ***p**-series* the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $p = 1$

The given series ***diverges*** by comparison test.

Exercise

Use the Comparison Test to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2 + 2}$

Solution

$$3n^2 + 2 > 3n^2 \quad \Rightarrow \quad \frac{1}{3n^2 + 2} < \frac{1}{3n^2}$$

By the ***p**-series* the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $p > 1$

The given series ***converges*** by comparison test.

Exercise

Use the Comparison Test to determine if the series converges or diverges. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

Solution

$$\sqrt{n}-1<\sqrt{n} \quad \Rightarrow \quad \frac{1}{\sqrt{n}-1}>\frac{1}{\sqrt{n}} \quad \textit{for } n\ge 2$$

By the ***p**-series* the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges $p=\frac{1}{2}<1$

The given series ***diverges*** by comparison test.

Exercise

Use the Comparison Test to determine if the series converges or diverges. $\sum_{n=0}^{\infty} \frac{4^n}{5^n+3}$

Solution

$$\frac{4^n}{5^n+3}<\frac{4^n}{5^n}=\left(\frac{4}{5}\right)^n$$

By the geometric series: $r=\frac{4}{5}<1$ converges

The given series ***converges*** by comparison test.

Exercise

Use the Comparison Test to determine if the series converges or diverges. $\sum_{n=2}^{\infty} \frac{\ln n}{n+1}$

Solution

$$\frac{\ln n}{n+1}>\frac{1}{n+1} \qquad \text{(and by integral test)}$$

The given series ***converges*** by comparison test with the divergent series.

Exercise

Use the Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

Solution

$$\frac{1}{\sqrt{n^3+1}}<\frac{1}{\sqrt{n^3}}$$

By the ***p**-series* the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges $p = \frac{3}{2} > 1$

The given series ***converges*** by comparison test.

Exercise

Use the Comparison Test to determine if the series converges or diverges. $\sum_{n=0}^{\infty} \frac{1}{n!}$

Solution

$\frac{1}{n^2} > \frac{1}{n!}$ For $n > 3$

By the ***p**-series* the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $p = 2 > 1$

The given series ***converges*** by comparison test using ***p**-series*

Exercise

Use the Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n} - 1}$

Solution

$\frac{1}{4\sqrt[3]{n} - 1} > \frac{1}{4\sqrt[3]{n}}$

By the ***p**-series* the series $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ diverges $p = \frac{1}{3} < 1$

The given series ***diverges*** by comparison test using ***p**-series*

Exercise

Use the Comparison Test to determine if the series converges or diverges. $\sum_{n=0}^{\infty} e^{-n^2}$

Solution

$\frac{1}{e^{n^2}} \le \frac{1}{e^n}$

Geometric series: $r = \frac{1}{e} < 1$ converges

The given series ***converges*** by comparison test.

Exercise

Use the Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$

Solution

$$\frac{3^n}{2^n - 1} > \left(\frac{3}{2}\right)^n$$

Geometric series: $r = \frac{3}{2} > 1$ diverges

The given series ***diverges*** by comparison test.

Exercise

Use the Limit Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n-2}{n^3 - n^2 + 3}$

Solution

Comparing with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series, since $p = 2 > 1$.

$$a_n = \frac{n-2}{n^3 - n^2 + 3} \quad \Rightarrow \quad b_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n-2}{n^3 - n^2 + 3} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{n^3 - 2n^2}{n^3 - n^2 + 3} \qquad \text{or } \textbf{\textit{L'Hopital Rule}}$$

$$= 1 > 0$$

Then, by Comparison Test, $\sum_{n=1}^{\infty} \frac{n-2}{n^3 - n^2 + 3}$ ***converges***.

Exercise

Use the Limit Comparison Test to determine if the series converges or diverges. $\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)}$

Solution

Comparing with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent p-series, since $p = 1 \le 1$.

$$a_n = \frac{n(n+1)}{(n^2+1)(n-1)} \quad \Rightarrow \quad b_n = \frac{n^2}{n^3} = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n(n+1)}{(n^2+1)(n-1)} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{n^3+n^2}{n^3-n^2+n-1}$$

$$= 1 > 0$$

Then, by Comparison Test, $\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)}$ ***diverges***.

Exercise

Use the Limit Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$

Solution

Comparing with $\sum_{n=1}^{\infty} \frac{1}{2^n}$, which is a convergent geometric, since $|r| = \frac{1}{2} < 1$.

$$a_n = \frac{2^n}{3+4^n} \quad \Rightarrow \quad b_n = \frac{1}{2^n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{3+4^n} \cdot \frac{2^n}{1} = \lim_{n\to\infty} \frac{2^{2n}}{3+4^n} = \lim_{n\to\infty} \frac{4^n}{3+4^n} = \lim_{n\to\infty} \frac{4^n \ln 4}{4^n \ln 4} = 1 > 0$$

Then, by Comparison Test, $\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$ ***converges***.

Exercise

Use the Limit Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\,4^n}$

Solution

Comparing with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} < 1$.

$$a_n = \frac{5^n}{\sqrt{n}\,4^n} \quad \Rightarrow \quad b_n = \frac{1}{\sqrt{n}}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{5^n}{\sqrt{n}\,4^n}\cdot\frac{\sqrt{n}}{1}=\lim_{n\to\infty}\frac{5^n}{4^n}=\lim_{n\to\infty}\left(\frac{5}{4}\right)^n=\infty$$

Then, by Comparison Test, $\sum_{n=1}^{\infty}\frac{5^n}{\sqrt{n}\,4^n}$ ***diverges***.

Exercise

Use the Limit Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty}\left(\frac{2n+3}{5n+4}\right)^n$

Solution

Comparing with $\sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n$, which is a convergent geometric, since $|r|=\frac{2}{5}<1$.

$$a_n=\left(\frac{2n+3}{5n+4}\right)^n \quad\Rightarrow\quad b_n=\left(\frac{2}{5}\right)^n$$

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n}&=\lim_{n\to\infty}\left(\frac{2n+3}{5n+4}\right)^n\cdot\left(\frac{5}{2}\right)^n\\&=\lim_{n\to\infty}\left(\frac{10n+15}{10n+8}\right)^n\\&=\lim_{n\to\infty}\left(\frac{10n}{10n}\right)^n\\&=\lim_{n\to\infty}1^n=1>0\end{aligned}$$

Then, by Comparison Test, $\sum_{n=1}^{\infty}\left(\frac{2n+3}{5n+4}\right)^n$ ***converges***.

Exercise

Use the Limit Comparison Test to determine if the series converges or diverges. $\sum_{n=2}^{\infty}\frac{1}{\ln n}$

Solution

Comparing with $\sum_{n=2}^{\infty}\frac{1}{n}$, which is a divergent *p*-series, since $p=1\le 1$.

$$a_n=\frac{1}{\ln n} \quad\Rightarrow\quad b_n=\frac{1}{n}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{1}{\ln n}\cdot\frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{n}{\ln n}$$

$$= \lim_{n\to\infty} \frac{1}{1/n}$$

$$= \lim_{n\to\infty} n$$

$$\boxed{= \infty}$$

Then, by Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ ***diverges***.

Exercise

Use the Limit Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$

Solution

Let $b_n = \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{1}{1+\sqrt{n}} \bigg/ \frac{1}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}}$$

$$\boxed{=1}$$

Since the ***p**-series* diverges to infinity $\left(p = \frac{1}{2}\right)$, so the given series ***diverges*** by the limit comparison test.

Exercise

Use the Limit Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n+5}{n^3 - 2n + 3}$

Solution

Let $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{n+5}{n^3 - 2n + 3} \bigg/ \frac{1}{n^2} = \lim_{n\to\infty} \frac{n^3 + 5n^2}{n^3 - 2n + 3}$$

$$\boxed{= 1 < \infty}$$

Since the ***p**-series* converges $(p = 2)$, therefore the given series ***converges*** by the limit comparison test.

Exercise

Use the Limit Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Solution

$$b_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{n}{n^2+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+1}$$

$$\boxed{= 1 < \infty}$$

Since the ***p**-series* diverges to infinity $(p = 1)$, so the given series ***diverges*** by the limit comparison test.

Exercise

Use the Limit Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{5}{4^n+1}$

Solution

$$b_n = \frac{1}{4^n} = \left(\frac{1}{4}\right)^n$$

$$\lim_{n\to\infty} \frac{5}{4^n+1} \cdot \frac{4^n}{1} = 5$$

The given series converges by a limit comparison with the convergent geometric series $\left(r = \frac{1}{4} < 1\right)$

Exercise

Use the Limit Comparison Test to determine if the series converges or diverges. $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$

Solution

$$b_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} \cdot \frac{n}{1} = 1 < \infty$$

Since the ***p**-series* diverges to infinity $(p = 1)$, so the given series ***diverges*** by the limit comparison test.

Exercise

Use the Limit Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{2^n+1}{5^n+1}$

Solution

$$b_n = \left(\frac{2}{5}\right)^n$$

$$\lim_{n\to\infty} \frac{2^n+1}{5^n+1} \cdot \frac{5^n}{2^n} = 1$$

The given series converges by a limit comparison with the convergent geometric series $\left(r = \frac{2}{5} < 1\right)$

Exercise

Use the Limit Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$

Solution

Let $b_n = \frac{1}{n^3}$

$$\lim_{n\to\infty} \frac{2n^2-1}{3n^5+2n+1} \Big/ \frac{1}{n^3} = \lim_{n\to\infty} \frac{2n^5-n^3}{3n^5+2n+1}$$

$$= \frac{2}{3} < \infty$$

Since the ***p**-series* converges $(p=3)$, therefore the given series ***converges*** by the limit comparison test.

Exercise

Use the Limit Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{n^2(n+3)}$

Solution

Let $b_n = \frac{1}{n^3}$ By the ***p**-series* converges $(p=3)$

$$\lim_{n\to\infty} \frac{1}{n^2(n+3)} \Big/ \frac{1}{n^3} = \lim_{n\to\infty} \frac{n^3}{n^2(n+3)}$$

$$= 1 < \infty$$

Therefore, the given series ***converges*** by the limit comparison test.

Exercise

Use the Limit Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

Solution

Let $b_n = \frac{1}{n^2}$ By the ***p**-series* converges $(p = 2)$,

$$\lim_{n\to\infty} \frac{1}{n\sqrt{n^2+1}} \cdot \frac{n^2}{1} = 1 < \infty$$

The given series ***converges*** by the limit comparison test.

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+\sqrt[3]{n}}$

Solution

Comparing with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} < 1$.

$$a_n = \frac{1}{2\sqrt{n}+\sqrt[3]{n}} \Rightarrow b_n = \frac{1}{\sqrt{n}}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{1}{2\sqrt{n}+\sqrt[3]{n}} \cdot \frac{\sqrt{n}}{1} \\
&= \lim_{n\to\infty} \frac{\sqrt{n}}{2\sqrt{n}+\sqrt[3]{n}} \\
&= \lim_{n\to\infty} \frac{1}{2+n^{1/3-1/2}} \\
&= \lim_{n\to\infty} \frac{1}{2+n^{-1/6}} \\
&= \frac{1}{2} > 0
\end{aligned}$$

Then, by Comparison Test, the given series ***diverges***.

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$

Solution

Comparing with $\sum_{n=1}^{\infty}\frac{1}{2^n}=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$, which is a convergent geometric, since $|r|=\frac{1}{2}<1$.

By the Direct Comparison Test: $\frac{\sin^2 n}{2^n}\le\frac{1}{2^n}$

The given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty}\frac{n+1}{n^2\sqrt{n}}$

Solution

Comparing with $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$, which is a convergent p-series, since $p=\frac{3}{2}>1$.

$$a_n=\frac{n+1}{n^2\sqrt{n}} \quad\Rightarrow\quad b_n=\frac{1}{n^{3/2}}$$

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n}&=\lim_{n\to\infty}\frac{n+1}{n^2\sqrt{n}}\cdot\frac{n^{3/2}}{1}\\&=\lim_{n\to\infty}\frac{n+1}{n}\\&=1>0\end{aligned}$$

Then, by Comparison Test, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty}\frac{10n+1}{n(n+1)(n+2)}$

Solution

Comparing with $\sum_{n=1}^{\infty}\frac{1}{n^2}$, which is a convergent p-series, since $p=2>1$.

$$a_n=\frac{10n+1}{n(n+1)(n+2)} \quad\Rightarrow\quad b_n=\frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{10n+1}{n(n+1)(n+2)}\cdot\frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{10n^3 + n^2}{n\left(n^2 + 3n + 2\right)}$$

$$= \lim_{n\to\infty} \frac{10n^3 + n^2}{n^3 + 3n^2 + 2}$$

$$= 10 > 0$$

Then, by Comparison Test, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$

Solution

By the Direct Comparison Test: $\left(\frac{n}{3n+1}\right)^n < \left(\frac{n}{3n}\right)^n = \left(\frac{1}{3}\right)^n$

$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$, which is a convergent geometric, since $|r| = \frac{1}{3} < 1$.

Therefore, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$

Solution

Comparing with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent *p*-series, since $p = 2 > 1$.

$$a_n = \frac{(\ln n)^2}{n^3} \quad \Rightarrow \quad b_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(\ln n)^2}{n^3} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{(\ln n)^2}{n}$$

$$= \lim_{n\to\infty} \frac{2\ln n\left(\frac{1}{n}\right)}{1}$$

$$= 2 \lim_{n\to\infty} \frac{\ln n}{n}$$

$$\underline{= 0|}$$

Then, by Comparison Test, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin n}{n^2}$

Solution

Let $b_n = \dfrac{1}{n^2}$

$$\lim_{n\to\infty} \frac{1+\sin n}{n^2} \Big/ \frac{1}{n^2} = \lim_{n\to\infty} (1+\sin n) \text{ which does not exist.}$$

$\dfrac{1+\sin n}{n^2} \le \dfrac{2}{n^2}$, then the given series converges by comparison test

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

Solution

$$a_n = \frac{1}{2+3^n} < \frac{1}{3^n} = b_n$$

So, by the Direct Comparison Test, the series converges.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$

Solution

$$2+\sqrt{n} < n \quad \Rightarrow \quad \frac{1}{2+\sqrt{n}} \ge \frac{1}{n}$$

By the ***p****-series* the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $p = 1$

The given series ***diverges*** by Comparison Test using ***p****-series*

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{1}{an+b}$

Solution

$$an+b<n \quad \Rightarrow \quad \frac{1}{an+b} \geq \frac{1}{n} \qquad a,\ b>0$$

By the ***p****-series* the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $p=1$

The given series ***diverges*** by Comparison Test using ***p****-series.*

$$\lim_{n\to\infty} \frac{1}{an+b} \cdot \frac{n}{1} = \frac{1}{a} > 0$$

The given series ***diverges*** by Limit Comparison Test

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

Solution

Let $b_n = \frac{1}{n^{3/2}}$ By the ***p****-series* converges $\left(p = \frac{3}{2} > 1\right)$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n^2+1} \cdot \frac{n^{3/2}}{1} = 1$$

Therefore, the given series ***converges*** by the limit comparison test.

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$

Solution

$$\frac{\sqrt[3]{n}}{n} = \frac{1}{n^{2/3}}$$

By the ***p****-series* the series $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges $p = \frac{2}{3} < 1$

The given series ***diverges*** by comparison test using ***p****-series*

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=0}^{\infty} 5\left(-\frac{4}{3}\right)^n$

Solution

$$|r| = \frac{4}{3} > 1$$

The given series ***diverges*** by Geometric series

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n + 1}$

Solution

$$\frac{1}{5^n + 1} < \left(\frac{1}{5}\right)^n$$

The given series converges by a Direct Comparison with the convergent geometric series $\left(r = \frac{1}{5} < 1\right)$

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3 - 8}$

Solution

Let $b_n = \frac{1}{n^3}$ By the ***p**-series* converges $(p = 3)$

$$\lim_{n \to \infty} \frac{1}{n^3 - 8} \cdot \frac{n^3}{1} = 1$$

Therefore, the given series ***converges*** by the limit comparison test with ***p**-series*.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n - 2}$

Solution

$$\lim_{n \to \infty} \frac{2n}{3n - 2} = \frac{2}{3} \neq 0$$

The given series ***diverges*** by the Limit

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$

Solution

$$\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots$$

$$=\frac{1}{2}$$

The given series ***converges*** by telescoping series.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty}\frac{n}{\left(n^2+1\right)^2}$

Solution

$$\int_1^{\infty}\frac{x}{\left(x^2+1\right)^2}dx=\frac{1}{2}\int_1^{\infty}\frac{1}{\left(x^2+1\right)^2}d\left(x^2+1\right)$$

$$=-\frac{1}{2}\frac{1}{x^2+1}\Big|_1^{\infty}$$

$$=-\frac{1}{2}\left(0-\frac{1}{2}\right)$$

$$\boxed{=\frac{1}{4}}$$

The given series ***converges*** by the Integral test.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty}\frac{n2^n}{4n^3+1}$

Solution

$$b_n=\frac{n^2}{2^n}$$

$$\lim_{n\to\infty}\frac{n2^n}{4n^3+1}\cdot\frac{n^2}{2^n}=\frac{1}{4}$$

The given series ***diverges*** by the Limit Comparison test