Section 5.7 – Mathematical Induction

If n is a positive integer and we let P_n denote the mathematical statement $(xy)^n = x^n y^n$, we obtained the following ***infinite sequence*** of statements:

$$\begin{array}{ll}
\textit{Statement } P_1: & (xy)^1 = x^1 y^1 \\
\textit{Statement } P_2: & (xy)^2 = x^2 y^2 \\
\textit{Statement } P_3: & (xy)^3 = x^3 y^3 \\
\quad\vdots & \quad\vdots \\
\textit{Statement } P_n: & (xy)^n = x^n y^n \\
\quad\vdots & \quad\vdots
\end{array}$$

Principle of Mathematical Induction

If with each positive integer n there is associated a statement P_n then all the statements P_n are true, provided the following two conditions are satisfied.

1) P_1 is true.

2) Whenever k is a positive integer such that P_k is true, then P_{k+1} is also true.

Steps in Applying the Principle of Mathematical Induction

1) Show that P_1 is true.

2) Assume that P_k is true, and then prove that P_{k+1} is true.

Example

Use the mathematical induction to prove that for every positive integer n, the sum of the first n positive integers is:

$$\frac{n(n+1)}{2}$$

Solution

(1) For $n = 1 \Rightarrow \frac{1(1+1)}{2} = 1$

$1 = 1$ √

Hence P_1 is true.

(2) Assume that P_k is true.

Thus, the induction hypothesis is: $1+2+3+\ldots+k = \frac{k(k+1)}{2}$

For $k + 1$: $1+2+3+\ldots+k+(k+1) \stackrel{?}{=} \frac{(k+1)((k+1)+1)}{2}$

$$\begin{aligned} 1+2+3+\ldots+k+(k+1) &= (1+2+3+\ldots+k)+(k+1) \\ &= \frac{k(k+1)}{2}+(k+1) && \textit{Induction hypothesis} \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} && \textit{Factor out } k+1 \\ &= \frac{(k+1)((k+1)+1)}{2} \quad \surd && \textit{Change form of } k+2 \end{aligned}$$

This shows that P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed

Example

Prove that for every positive integer n,

$$1^2+3^2+\ldots+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

Solution

(1) For $\boldsymbol{n}$ **= 1** $\Rightarrow 1^2=\frac{1(2(1)-1)(2(1)+1)}{3}$

$$1=\frac{3}{3}$$

$1=1$ ✓ hence P_1 is true.

(2) $1^2+3^2+\ldots+(2k-1)^2=\frac{k(2k-1)(2k+1)}{3}$

For k + 1:

$$1^2+3^2+\ldots+(2k-1)^2+(2(k+1)-1)^2 \overset{?}{=} \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

$$1^2+3^2+\ldots+(2k-1)^2+(2(k+1)-1)^2 \overset{?}{=} \frac{(k+1)(2k+1)(2k+3)}{3}$$

$$1^2+3^2+\ldots+(2k-1)^2+[2k+2-1]^2=\frac{k(2k-1)(2k+1)}{3}+(2k+1)^2$$

$$=\frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}$$

$$=\frac{(2k+1)[k(2k-1)+3(2k+1)]}{3}$$

$$=\frac{(2k+1)\left(2k^2-k+6k+3\right)}{3}$$

$$=\frac{(2k+1)\left(2k^2+5k+3\right)}{3}$$

$$=\frac{(2k+1)(k+1)(2k+3)}{3} \quad \surd$$

This shows that P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed

Example

Prove that 2 is a factor of $n^2 + 5n$ for every positive integer n,

Solution

(1) For $\boldsymbol{n = 1} \Rightarrow n^2 + 5n = 1^2 + 5(1)$

$$= 6$$

$$= 2.3 \quad \surd$$

Thus, 2 is a factor of $n^2 + 5n$ for $n = 1$; hence P_1 is true.

(2) 2 is a factor of $k^2 + 5k \Leftrightarrow k^2 + 5k = 2p$

is 2 a factor of $(k+1)^2 + 5(k+1)$?

$$(k+1)^2 + 5(k+1) = k^2 + 2k + 1 + 5k + 5$$

$$= k^2 + 5k + 2k + 6$$

$$= \left(k^2 + 5k\right) + 2(k+3)$$

$$= 2p + 2(k+3)$$

$$= 2.(p + k + 3) \quad \surd$$

Thus, 2 is a factor of the last expression; hence P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed

Steps in Applying the Extended Principle of Mathematical Induction

1. Show that P_1 is true.
2. Assume that P_k is true with $k \geq j$, and then prove that P_{k+1} is true.

Example

Let a be a nonzero real number such that $a > -1$. Prove that $(1+a)^n > 1+na$ for every integer $n \geq 2$.

Solution

For $\boldsymbol{n=1} \Rightarrow (1+a)^1 > 1+(1)a \Rightarrow P_1$ is false.

***Step* 1**. For $\boldsymbol{n=2} \Rightarrow (1+a)^2 \overset{?}{>} 1+(2)a$

$$1+2a+a^2 > 1+a \quad \surd$$

$\Rightarrow P_2$ is true.

***Step* 2**. Assume that P_k is true $(1+a)^k > 1+ka$

We need to prove that P_{k+1} is true, that is $(1+a)^{k+1} > 1+(k+1)a$

$$\begin{aligned}(1+a)^{k+1} &= (1+a)^k(1+a)^1 \\ &> (1+ka)(1+a)\end{aligned}$$

$$\begin{aligned}(1+ka)(1+a) &= 1+a+ka+ka^2 \\ &= 1+(a+ka)+ka^2 \\ &= 1+a(k+1)+ka^2 \\ &> 1+(k+1)a\end{aligned}$$

$$\begin{aligned}(1+a)^{k+1} &> (1+ka)(1+a) \\ &> 1+(k+1)a \quad \surd\end{aligned}$$

Thus, P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed

Exercises *Section* 5.7 – Mathematical Induction

1. Find all positive integers n for which the given statement is not true

a) $3^n > 6n$ *b)* $3^n > 2n+1$ *c)* $2^n > n^2$ *d)* $n! > 2n$

2. Prove that the statement is true for every positive integer n. $2+4+6+\ldots+2n = n(n+1)$

3. Prove that the statement is true for every positive integer n. $1+3+5+\ldots+(2n-1) = n^2$

4. Prove that the statement is true for every positive integer n. $2+7+12+\ldots+(5n-3) = \frac{1}{2}n(5n-1)$

(5 – 35) Prove that the statement is true by the mathematical induction

5. $1+2{\bullet}2+3{\bullet}2^2+\ldots+n{\bullet}2^{n-1} = 1+(n-1).2^n$

6. $1^2+2^2+3^2+\ldots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

7. $\dfrac{1}{1{\bullet}2}+\dfrac{1}{2{\bullet}3}+\dfrac{1}{3{\bullet}4}+\ldots+\dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

8. $\dfrac{1}{2}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+\ldots+\dfrac{1}{2^n} = 1-\dfrac{1}{2^n}$

9. $\dfrac{1}{1\cdot 4}+\dfrac{1}{4\cdot 7}+\dfrac{1}{7\cdot 10}+\cdots+\dfrac{1}{(3n-2)\cdot(3n+1)} = \dfrac{n}{3n+1}$

10. $\dfrac{4}{5}+\dfrac{4}{5^2}+\dfrac{4}{5^3}+\cdots+\dfrac{4}{5^n} = 1-\dfrac{1}{5^n}$

11. $1^3+2^3+3^3+\cdots+n^3 = \dfrac{n^2(n+1)^2}{4}$

12. $3+3^2+3^3+\ldots+3^n = \frac{3}{2}\left(3^n-1\right)$

13. $x^{2n}+x^{2n-1}y+\cdots+xy^{2n-1}+y^{2n} = \dfrac{x^{2n+1}-y^{2n+1}}{x-y}$

14. $5{\bullet}6+5{\bullet}6^2+5{\bullet}6^3+\cdots+5{\bullet}6^n = 6\left(6^n-1\right)$

15. $7{\bullet}8+7{\bullet}8^2+7{\bullet}8^3+\cdots+7{\bullet}8^n = 8\left(8^n-1\right)$

16. $3+6+9+\cdots+3n = \dfrac{3n(n+1)}{2}$

17. $5+10+15+\cdots+5n = \dfrac{5n(n+1)}{2}$

18. $1+3+5+\cdots+(2n-1) = n^2$

19. $4+7+10+\cdots+(3n+1)=\dfrac{n(3n+5)}{2}$

20. $2+4+6+\cdots+2(n-1)+2n=n(n+1)$

21. $1+(1+2)+(1+2+3)+\ \cdots\ +(1+2+\cdots+n)=\dfrac{n(n+1)(n+2)}{6}\quad =\sum_{k=1}^{n}\left(\sum_{i=1}^{k} i\right)$

22. $1+2+3+\ \cdots\ +n\ <\ \dfrac{(2n+3)^2}{7}$

23. $\dfrac{1}{2n}\ \le\ \dfrac{1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot(2n)}$

24. $\dfrac{2n+1}{2n+2}\ \le\ \dfrac{\sqrt{n+1}}{\sqrt{n+2}}$

25. $n!\ <n^n\quad \textit{for}\quad n>1$

26. For every positive integer *n*. $\quad n<2^n$

27. For every positive integer *n*. $\quad$ 3 is a factor of n^3-n+3

28. For every positive integer *n*. $\quad$ 4 is a factor of 5^n-1

29. $\left(a^m\right)^n=a^{mn}$ (*a* and *m* are constant)

30. $2^n>2n\quad \textit{if}\quad n\ge 3$

31. If $0<a<1$, then $a^n<a^{n-1}$

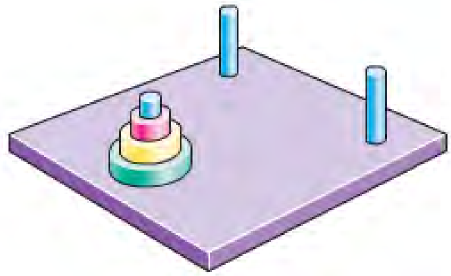
32. If $n\ge 4$, then $n!\ >2^n$

33. $3^n>2n+1\quad \textit{if}\quad n\ge 2$

34. $2^n>n^2\quad \textit{for}\quad n>4$

35. $4^n>n^4\quad \textit{for}\quad n\ge 5$

36. A pile of *n* rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must moved one at a time, to a different peg with no ring ever placed on top of a smaller ring.



Find the least number of moves that would be required.
Prove your result by mathematical induction.