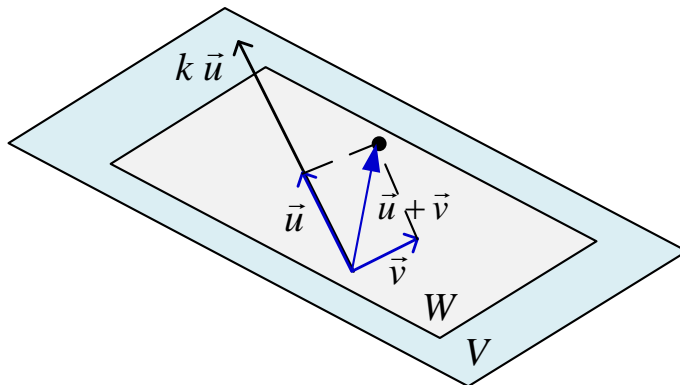


Section 2.5 – Subspaces, Span and Null Spaces

Subspaces

Definition

A subset W of a vector space V is called a **subspace** of V if W itself a vector space under the addition and scalar multiplication defined in V .



Theorem

If W is a set of one or more vectors in a vector space V , then W is a subspace of V iff the following conditions holds

1. If \vec{u} and \vec{v} are vectors in W , then $\vec{u} + \vec{v}$ is in W .
2. If k is any scalar and \vec{v} is any vector in W , the $k\vec{v}$ is in the subspace in W .

- The most fundamental ideas in linear algebra are that the plane is a subspace of the full vector space \mathbb{R}^n .
- Every subspace contains the zero vector. The plane vector in \mathbb{R}^3 has to go through $(0, 0, 0)$. From rule (2), if we choose $k = 0$ and the rule requires $0\vec{v}$ to be in the subspace.

The **axioms** that are **not** inherited by W are

Axiom 1 – Closure of W under addition

Axiom 4 – Existence of a zero vector in W

Axiom 5 – Existence of a negative in W for every vector in W

Axiom 6 – Closure of W under scalar multiplication

Example

Keep only the vectors (x, y) whose components are positive or zero (first quadrant “*quarter-plane*”). The vector $(2, 3)$ is included but $(-2, -3)$ is not. So, rule (2) is violated when we try $k = -1$. **The quarter-plane is not a subspace.**

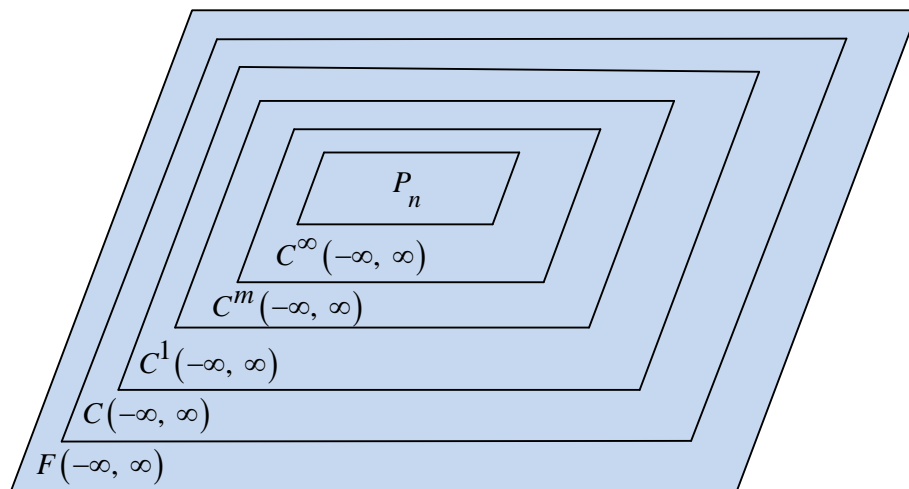
Example

Include also the vectors whose components are both negative. Now we have two quarter-planes. Rule (ii) satisfies when we multiply by any c . But rule (i) fails. The sum of $v = (2, 3)$ and $w = (-3, -2)$ is $(-1, 1)$ which is outside the quarter-plane. **Two quarter-planes don't make a subspace.**

Example

The Subspace $C(-\infty, \infty)$

There is a theorem in calculus which states that a sum of continuous functions is continuous and then a constant times a continuous function is continuous. In vector word, the set of continuous functions on $(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$. We denote this subspace by $C(-\infty, \infty)$



Theorem

If W_1, W_2, \dots, W_n are subspaces of a vector space V , then intersection of these subspaces is also a subspace of V .

➤ A subspace containing \vec{v} and \vec{w} must contain all linear combination $c\vec{v} + d\vec{w}$.

Example

Inside the vector space M of all 2 by 2 matrices, given two subspaces:

U all upper triangular matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

D all diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$

Solution

If we add 2 matrices in **U**: $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 0 & 2d \end{bmatrix}$ is in **U**.

If we add 2 matrices in **D**: $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 0 & 2d \end{bmatrix}$ is in **D**.

In this case **D** is also a subspace of **U**!. The zero matrix is in these subspaces, when a , b , and d all equal zero.

Span

Definition

The subspace of a vector space V that is formed from all possible linear combinations of the vectors in a nonempty set S is called the **span of S** , and we say that the vectors in S *span* that subspace. If

$S = \{w_1, w_2, \dots, w_r\}$, then we denoted the span of S by

$$\text{span}\{w_1, w_2, \dots, w_r\} \quad \text{or} \quad \text{span}(S)$$

Theorem

Let $\vec{v}_1, \dots, \vec{v}_n$ be vectors in vector space V and S be their span. Then,

a) S is a subspace of V .

Proof: $\forall \vec{u}, \vec{v} \in S$, $\vec{u} = a_1 \vec{v}_1 + \dots + a_n \vec{v}_n$ and $\vec{v} = b_1 \vec{v}_1 + \dots + b_n \vec{v}_n$

$$\vec{u} + \vec{v} = (a_1 + b_1) \vec{v}_1 + \dots + (a_n + b_n) \vec{v}_n \in S$$

$$k\vec{u} = ka_1 \vec{v}_1 + \dots + ka_n \vec{v}_n \in S$$

b) S is the smallest subspace of V that contains $\vec{v}_1, \dots, \vec{v}_k$. i.e. any other subspace \vec{w} containing

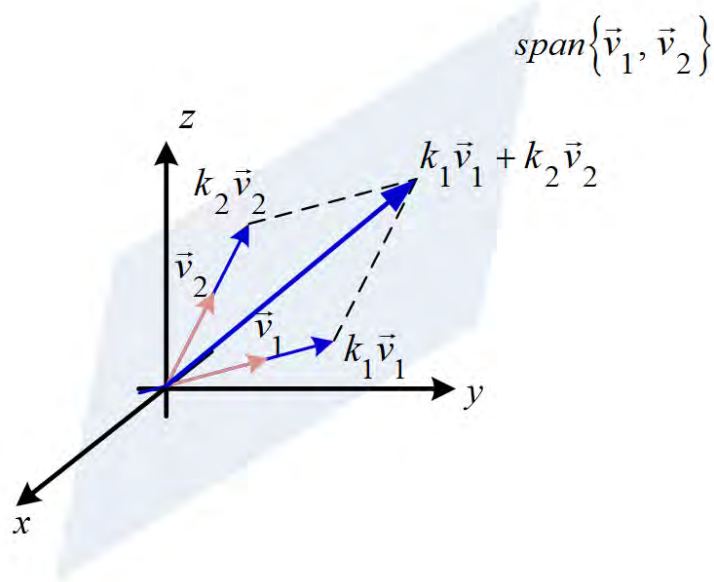
$\vec{v}_1, \dots, \vec{v}_n$ also contains S .

Proof: let $\vec{u} \in S$, $\vec{u} = a_1 \vec{v}_1 + \dots + a_n \vec{v}_n$

But $a_1 \vec{v}_1, \dots, a_n \vec{v}_n \in \vec{w} \therefore \vec{w}$ closed under scalar multiplication.

$a_1 \vec{v}_1, \dots, a_n \vec{v}_n \in \vec{w} \therefore \vec{w}$ closed under addition.

$\therefore \vec{u} \in \vec{w}$



Example

a) $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ span the full two-dimensional space \mathbb{R}^2 .

b) $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\vec{v}_3 = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$ span the full space \mathbb{R}^2 .

c) $\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{w}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ only span a line in \mathbb{R}^2 .

Definition

The **row space** of a matrix is the subspace of \mathbb{R}^n spanned by the rows.

Example

Determine whether $\vec{v}_1 = (1, 1, 2)$, $\vec{v}_2 = (1, 0, 1)$, and $\vec{v}_3 = (2, 1, 3)$ span the vector space \mathbb{R}^3

Solution

Let $b = (b_1, b_2, b_3)$ be the arbitrary vector in \mathbb{R}^3 can be expressed as a linear combination

$$\vec{b} = k_1 \vec{v}_1 + k_2 \vec{v}_2 + k_3 \vec{v}_3$$

$$(b_1, b_2, b_3) = k_1 (1, 1, 2) + k_2 (1, 0, 1) + k_3 (2, 1, 3)$$

$$(b_1, b_2, b_3) = (k_1 + k_2 + 2k_3, k_1 + k_3, 2k_1 + k_2 + 3k_3)$$

$$\rightarrow \begin{cases} k_1 + k_2 + 2k_3 = b_1 \\ k_1 + k_3 = b_2 \\ 2k_1 + k_2 + 3k_3 = b_3 \end{cases}$$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} \\ = 0$$

Since the determinant is zero, the \vec{v}_1 , \vec{v}_2 , and \vec{v}_3 **do not** span space \mathbb{R}^3

Solution Spaces of *Homogeneous (Null Space) Systems*

Theorem

The solution set of a homogeneous linear system $A\vec{x} = \vec{0}$ in n unknowns is a subspace of \mathbb{R}^n

Proof

Let W be the solution set for the system. The set W is not empty because it contains at least the trivial solution $\vec{x} = \vec{0}$.

To show that W is a subspace of \mathbb{R}^n , we must show that it is closed under addition and scalar multiplication.

Let \vec{x}_1 and \vec{x}_2 be vectors in W and these vectors are solution of $A\vec{x} = \vec{0}$.

$$A\vec{x}_1 = \vec{0} \quad \text{and} \quad A\vec{x}_2 = \vec{0}$$

Therefore,

$$\begin{aligned}
 A(\vec{x}_1 + \vec{x}_2) &= A\vec{x}_1 + A\vec{x}_2 \\
 &= \vec{0} + \vec{0} \\
 &= \underline{\vec{0}}
 \end{aligned}$$

So, W is closed under addition.

$$A(k\vec{x}_1) = kA\vec{x}_1 = k\vec{0} = \vec{0}$$

So, W is closed under scalar multiplication.

Null Spaces

Definition

The nullspace of A consists of all solutions to $A\vec{x} = \vec{0}$. These solution vectors \vec{x} are in \mathbb{R}^n . The Nullspace containing all solutions is denoted by $N(A)$ *or* $NS(A)$.

$$\left\{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \right\} \text{ is the nullspace of } A, NS(A)$$

(Can also be called **Kernel** of A : $Ker(A)$)

Theorem

Suppose $NS(A)$ is a subspace of \mathbb{R}^n for $A_{m \times n}$

✓ Let \vec{x} and \vec{y} are in the nullspace ($\vec{x}, \vec{y} \in NS(A)$) then

$$\begin{aligned} A(\vec{x} + \vec{y}) &= A\vec{x} + A\vec{y} \\ &= \vec{0} + \vec{0} \\ &= \underline{\vec{0}} \end{aligned}$$

✓ Let $\vec{x} \in NS(A)$ then $c\vec{x} \in NS(A)$

$$\begin{aligned} \therefore A(c\vec{x}) &= cA\vec{x} \\ &= c\vec{0} \\ &= \underline{\vec{0}} \end{aligned}$$

Since we can add and multiply without leaving the Nullspace, it is a subspace.

Example

The equation $x + 2y + 3z = 0$ comes from the 1 by 3 matrix $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. This equation produces a plane through the origin. The plane is a subspace of \mathbb{R}^3 . *It is the Nullspace of A .*

Solution

The solution to $x + 2y + 3z = 6$ also form a plane, but not a subspace.

Example

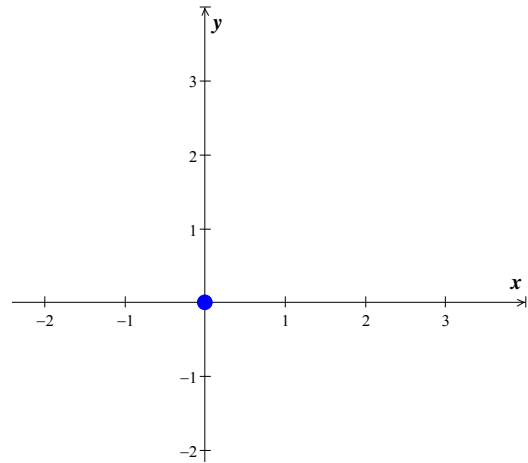
Find the null space of

$$a) A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad b) B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

Solution

$$a) \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{cases} x_1 + 2x_2 = 0 \\ 3x_2 = 0 \end{cases}$$
$$\Rightarrow x_1 = x_2 = 0$$

$$\text{So } NS(A) = \{\vec{0}\}$$



$$b) \begin{bmatrix} 1 & 2 & 0 \\ 3 & 6 & 0 \end{bmatrix} \xrightarrow{R_2 - 3R_1}$$

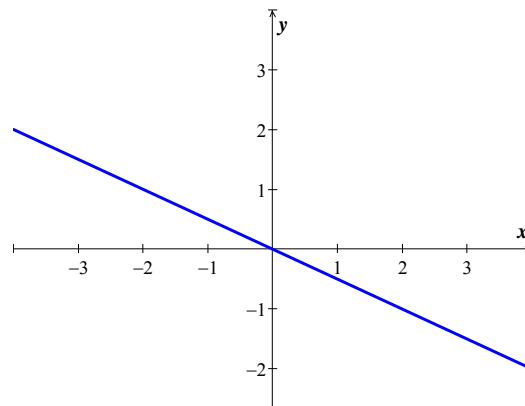
$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x_1 = -2x_2$$

If we let $x_2 = s$, then

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ is in } NS(B) \text{ if and only if}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



Example

Describe the nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

Solution

Apply the elimination to the linear equations $Ax = 0$:

$$\begin{bmatrix} x_1 + 2x_2 = 0 \\ 3x_1 + 6x_2 = 0 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} x_1 + 2x_2 = 0 \\ 0 = 0 \end{bmatrix}$$

There is only one equation ($x_1 + 2x_2 = 0$), this line is the Nullspace $N(A)$.

Example

Consider the linear system

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution

$$z = t, \quad y = s, \quad x = 2s - 3t$$

$$\Rightarrow x - 2y + 3z = 0$$

This is the equation of a plane through the origin that has $\vec{n} = (1, -2, 3)$ as a normal.

Example

Consider the linear system

$$\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution

$$x = 0, \quad y = 0, \quad z = 0$$

The solution space is $\{\vec{0}\}$

Exercises Section 2.5 – Subspaces, Span and Null Spaces

1. Suppose S and T are two subspaces of a vector space \mathbf{V} .
 - a) The sum $S + T$ contains all sums $\vec{s} + \vec{t}$ of a vector \vec{s} in S and a vector \vec{t} in T . Show that $S + T$ satisfies the requirements (addition and scalar multiplication) for a vector space.
 - b) If S and T are lines in \mathbb{R}^m , what is the difference between $S + T$ and $S \cup T$? That union contains all vectors from S and T or both. Explain this statement: The span of $S \cup T$ is $S + T$.
2. Determine which of the following are subspaces of \mathbb{R}^3 ?
 - a) All vectors of the form $(a, 0, 0)$
 - b) All vectors of the form $(a, 1, 1)$
 - c) All vectors of the form (a, b, c) , where $b = a + c$
 - d) All vectors of the form (a, b, c) , where $b = a + c + 1$
 - e) All vectors of the form $(a, b, 0)$
3. Determine which of the following are subspaces of \mathbb{R}^∞ ?
 - a) All sequences \vec{v} in \mathbb{R}^∞ of the form $\vec{v} = (v, 0, v, 0, \dots)$
 - b) All sequences \vec{v} in \mathbb{R}^∞ of the form $\vec{v} = (v, 1, v, 1, \dots)$
 - c) All sequences \vec{v} in \mathbb{R}^∞ of the form $\vec{v} = (v, 2v, 4v, 8v, 16v, \dots)$
4. Which of the following are linear combinations of $\vec{u} = (0, -2, 2)$ and $\vec{v} = (1, 3, -1)$?
 - a) $(2, 2, 2)$
 - b) $(3, 1, 5)$
 - c) $(0, 4, 5)$
 - d) $(0, 0, 0)$
5. Which of the following are linear combinations of $\vec{u} = (2, 1, 4)$, $\vec{v} = (1, -1, 3)$ and $\vec{w} = (3, 2, 5)$?
 - a) $(-9, -7, -15)$
 - b) $(6, 11, 6)$
 - c) $(0, 0, 0)$
6. Determine whether the given vectors span \mathbb{R}^3
 - a) $\vec{v}_1 = (2, 2, 2)$, $\vec{v}_2 = (0, 0, 3)$, $\vec{v}_3 = (0, 1, 1)$
 - b) $\vec{v}_1 = (2, -1, 3)$, $\vec{v}_2 = (4, 1, 2)$, $\vec{v}_3 = (8, -1, 8)$
 - c) $\vec{v}_1 = (3, 1, 4)$, $\vec{v}_2 = (2, -3, 5)$, $\vec{v}_3 = (5, -2, 9)$, $\vec{v}_4 = (1, 4, -1)$
7. Which of the following are linear combinations of $A = \begin{pmatrix} 4 & 0 \\ -2 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$
 - a) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$
 - b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
 - c) $\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$

8. Suppose that $\vec{v}_1 = (2, 1, 0, 3)$, $\vec{v}_2 = (3, -1, 5, 2)$, $\vec{v}_3 = (-1, 0, 2, 1)$. Which of the following vectors are in $\text{span} \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$
- a) $(2, 3, -7, 3)$ b) $(0, 0, 0, 0)$ c) $(1, 1, 1, 1)$ d) $(-4, 6, -13, 4)$
9. Let $f = \cos^2 x$ and $g = \sin^2 x$. Which of the following lie in the space spanned by f and g
- a) $\cos 2x$ b) $3 + x^2$ c) $\sin x$ d) 0
10. Let $S = \left\{ (x, y) \mid x^2 + y^2 = 0; x, y \in \mathbb{R} \right\}$, Determine:
- a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of \mathbb{R}^2 ?
11. Let $S = \left\{ (x, y) \mid x^2 + y^2 = 0; x, y \in \mathbb{C} \right\}$, Determine:
- a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of \mathbb{C}^2 ?
12. Let $S = \left\{ (x, y) \mid x^2 - y^2 = 0; x, y \in \mathbb{R} \right\}$, Determine:
- a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of \mathbb{R}^2 ?
13. Let $S = \left\{ (x, y) \mid x - y = 0; x, y \in \mathbb{R} \right\}$, Determine:
- a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of \mathbb{R}^2 ?
14. Let $S = \left\{ (x, y) \mid x - y = 1; x, y \in \mathbb{R} \right\}$, Determine:
- a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of \mathbb{R}^2 ?

15. $V = \mathbb{R}^3$, $S = \{(0, s, t) \mid s, t \text{ are real numbers}\}$ where V is a vector space and S is subset of V
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of V ?
16. $V = \mathbb{R}^3$, $S = \{(x, y, z) \mid x, y, z \geq 0\}$ where V is a vector space and S is subset of V
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of V ?
17. $V = \mathbb{R}^3$, $S = \{(x, y, z) \mid z = x + y + 1\}$ where V is a vector space and S is subset of V
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of V ?
18. Let $S = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$, Determine:
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of \mathbb{R}^3 ?
19. Let $S = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$, Determine:
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of \mathbb{R}^3 ?
20. Let $S = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$, Determine:
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of \mathbb{R}^3 ?
21. Let $S = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0\}$, Determine:
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of \mathbb{R}^3 ?

22. Let $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 0 \right\}$, Determine:
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of \mathbb{R}^3 ?
23. Let $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1 \right\}$, Determine:
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of \mathbb{R}^3 ?
24. Let $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0 \right\}$, Determine:
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of \mathbb{R}^3 ?
25. Let $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_3 = a_1 + a_2 \right\}$, Determine:
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of \mathbb{R}^3 ?
26. Let $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + a_2 + a_3 = 0 \right\}$, Determine:
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of \mathbb{R}^3 ?
27. $S = \left\{ (x_1, x_2, 1) : x_1 \text{ and } x_2 \text{ are real numbers} \right\}$, Determine:
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of \mathbb{R}^3 ?
28. $S = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 = x_1 + 2x_3 \right\}$, Determine:
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of \mathbb{R}^3 ?

29. $S = \left\{ \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \in M_{2 \times 2} \mid a, b, c \in \mathbb{R} \right\}$ and $V = M_{2,2}$, Determine:
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of V ?
30. $S = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in M_{2 \times 2} \mid a, b, c \in \mathbb{R} \right\}$ and $V = M_{2,2}$, Determine:
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of V ?
31. Let $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in M_{2 \times 2} \mid a, d \in \mathbb{R} \text{ \& } ad \geq 0 \right\}$ and $V = M_{2,2}$, Determine:
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of V ?
32. $V = M_{33}$, $S = \{A \mid A \text{ is invertible}\}$ where V is a vector space and S is subset of V
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of V ?
33. Let $S = \left\{ p(t) = a + 2at + 3at^3 \mid a \in \mathbb{R} \text{ \& } p(t) \in \mathbb{P}_2 \right\}$ and $V = \mathbb{P}_2$, Determine:
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of V ?
34. Let $S = \{p(t) \mid p(t) \in \mathbb{P}[t] \text{ has degree } 3\}$, Determine:
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of $\mathbb{P}[t]$?
35. Let $S = \{p(t) \mid p(0) = 0, p(t) \in \mathbb{P}[t]\}$, Determine:
- Is S closed under addition?
 - Is S closed under scalar multiplication?
 - Is S a subspace of $\mathbb{P}[t]$?

36. Given: $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix}$
- Find $NS(A)$
 - For which n is $NS(A)$ a subspace of \mathbb{R}^n
 - Sketch $NS(A)$ in \mathbb{R}^2 or \mathbb{R}^3
37. Determine which of the following are subspaces of M_{22}
- All 2×2 matrices with integer entries
 - All matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a + b + c + d = 0$
38. Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1 \right\}$. Is V a vector space?
39. Let $V = \{(x, 0, y) : x \text{ \& } y \text{ are arbitrary } \mathbb{R}\}$. Define addition and scalar multiplication as follows:
- $$\begin{cases} (x_1, 0, y_1) + (x_2, 0, y_2) = (x_1 + x_2, y_1 + y_2) \\ c(x, 0, y) = (cx, cy) \end{cases}$$
- Is V a vector space?
40. Construct a matrix whose column space contains $(1, 1, 0)$ and $(0, 1, 1)$ and whose nullspace contains $(1, 0, 1)$ and $(0, 0, 1)$
41. How is the nullspace $N(C)$ related to the spaces $N(A)$ and $N(B)$, is $C = \begin{bmatrix} A \\ B \end{bmatrix}$?
42. True or False (check addition or give a counterexample)
- If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V .
 - The empty set is a subspace of every vector space.
 - If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$.
 - The intersection of any two subsets of V is a subspace of V .
 - Let W be the xy -plane in \mathbb{R}^3 ; that is, $W = \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$. Then $W = \mathbb{R}^2$
43. Let $A\vec{x} = \vec{0}$ be a homogeneous system of n linear equations in n unknowns that has only the trivial solution. Show that if k is any positive integer, then the system $A^k \vec{x} = \vec{0}$ also has only trivial solution.

- 44.** Let $A\vec{x} = \vec{0}$ be a homogeneous system of n linear equations in n unknowns and let Q be an invertible $n \times n$ matrix. Show that $A\vec{x} = \vec{0}$ has just trivial solution if and only if $(QA)\vec{x} = \vec{0}$ has just trivial solution.
- 45.** Let $A\vec{x} = \vec{b}$ be a consistent system of linear equations and let \vec{x}_1 be a fixed solution. Show that every solution to the system can be written in the form $\vec{x} = \vec{x}_1 + \vec{x}_0$ where \vec{x}_0 is a solution to $A\vec{x} = \vec{0}$. Show also that every matrix of this form is a solution.