Section 2.3 – Product and Quotient Rules

Product Rule

The derivative of the product of two differentiable functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$(f.g)' = f'.g + g'.f$$

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'gh + fg'h + fgh'$$

Example

Find the derivative of $f(x) = (2x+3)\left(3x^2\right)$

Solution

$$f(x) = 6x^3 + 9x^2$$

$$\begin{aligned} f' &= (2x+3)\left(3x^2\right)' + (2x+3)'\left(3x^2\right) \\ &= (2x+3)(6x) + (2)\left(3x^2\right) \\ &= 12x^2 + 18x + 6x^2 \\ &= 18x^2 + 18x \\ &= 18x(x+1) \end{aligned}$$

Proof of the Derivative Product Rule

$$\begin{aligned} \frac{d}{dx}(uv) &= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\ &= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \\ &= \lim_{h\to 0} \left[\frac{u(x+h)v(x+h) - u(x+h)v(x)}{h} + \frac{u(x+h)v(x) - u(x)v(x)}{h} \right] \\ &= \lim_{h\to 0} \left[u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h} \right] \\ &= \lim_{h\to 0} u(x+h) \cdot \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} + v(x) \cdot \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \\ &= u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \end{aligned}$$

Example

Find the derivative of $y=\left(3x^2+1\right)\left(x^3+3\right)$

Solution

$u=3x^2+1 \quad v=x^3+3$

$u'=6x \qquad v'=3x^2$

$y'=(6x)\left(x^3+3\right)+\left(3x^2\right)\left(3x^2+1\right)$

$=6x^4+18x+9x^4+3x^2$

$\boxed{=15x^4+3x^2+18x}$

$y=3x^5+9x^2+x^3+3$

$y'=15x^4+18x+3x^2$

Example

Find the derivative of $y=\left(3x^3+2x+5\right)\left(x^2-2x+4\right)$

Solution

$y'=\underbrace{\left(9x^2+2\right)}_{u'}\underbrace{\left(x^2-2x+4\right)}_{v}+\underbrace{(2x-2)}_{v'}\underbrace{\left(3x^3+2x+5\right)}_{u}$

$=9x^4-18x^3+36x^2+2x^2-4x+8+6x^4+4x^2+10x-6x^3-4x-10$

$\boxed{=15x^4-24x^3+42x^2+2x-2}$

Quotient Rule

$$\frac{d}{dx}\left\lfloor \frac{f(x)}{g(x)} \right\rfloor = \frac{f'(x)g(x)-g'(x)f(x)}{[g(x)]^2} = \frac{f'g-g'f}{g^2}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

$$\left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{(cx+d)^2}$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix}x^2 + 2\begin{vmatrix} a & c \\ d & f \end{vmatrix}x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

Example

Find $f'(x)$ if $f(x) = \frac{2x-1}{4x+3}$

Solution

$$f' = \frac{(2x-1)'(4x+3)-(2x-1)(4x+3)'}{(4x+3)^2} \qquad \begin{array}{ll} u = 2x-1 & v = 4x+3 \\ u' = 2 & v' = 4 \end{array}$$

$$= \frac{(2)(4x+3)-(2x-1)(4)}{(4x+3)^2}$$

$$= \frac{8x+6-8x+4}{(4x+3)^2}$$

$$= \boxed{\frac{10}{(4x+3)^2}}$$

$$f'(x) = \frac{2(3)-(-1)(4)}{4x+3} \qquad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

$$= \boxed{\frac{10}{(4x+3)^2}}$$

Example

Find the derivative of $y = \dfrac{(x-1)\left(x^2-2x\right)}{x^4}$

Solution

$$y = \frac{x^3-2x^2-x^2+2x}{x^4}$$

$$= \frac{x^3-3x^2+2x}{x^4}$$

$$= \frac{x^3}{x^4} - \frac{3x^2}{x^4} + \frac{2x}{x^4}$$

$$= x^{-1} - 3x^{-2} + 2x^{-3}$$

$$y' = -x^{-2} + 6x^{-3} - 6x^{-4}$$

$$= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}$$

Combining the product and Quotient Rules

Example

Find the derivative of $y = \dfrac{(1+x)(2x-1)}{x-1}$

Solution

$$y = \frac{(1+x)(2x-1)}{x-1}$$

$$= \frac{2x^2+x-1}{x-1}$$

$$y' = \frac{\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} x^2 + 2\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} x + \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}}{x-1}$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2\begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

$$= \frac{2x^2-4x}{(x-1)^2}$$

Or

$$y' = \frac{(x-1)\frac{d}{dx}\left[(1+x)(2x-1)\right] - (1+x)(2x-1)\frac{d}{dx}[x-1]}{(x-1)^2}$$

$$= \frac{(x-1)\left[(1)(2x-1)+2(1+x)\right]-(1+x)(2x-1)(1)}{\left(x-1\right)^2}$$

$$= \frac{(x-1)(2x-1+2+2x)-(2x-1+2x^2-x)}{\left(x-1\right)^2}$$

$$= \frac{(x-1)(4x+1)-2x+1-2x^2+x}{\left(x-1\right)^2}$$

$$= \frac{4x^2+x-4x-1-2x+1-2x^2+x}{\left(x-1\right)^2}$$

$$= \frac{2x^2-4x}{\left(x-1\right)^2}$$

Exercises Section 2.3 – Product and Quotient Rules

Find the derivative of each function

1. $y=(x+1)\left(\sqrt{x}+2\right)$
2. $y=\left(4x+3x^2\right)(6-3x)$
3. $y=\left(\frac{1}{x}+1\right)(2x+1)$
4. $y=\dfrac{3-\frac{2}{x}}{x+4}$
5. $g(x)=\dfrac{x^2-4x+2}{x^2+3}$
6. $f(x)=\dfrac{(3-4x)(5x+1)}{7x-9}$
7. $f(x)=x\left(1-\frac{2}{x+1}\right)$
8. $f(x)=\left(\sqrt{x}+3\right)\left(x^2-5x\right)$
9. $y=(2x+3)\left(5x^2-4x\right)$
10. $y=\left(x^2+1\right)\left(x+5+\frac{1}{x}\right)$
11. $y=\dfrac{x+4}{5x-2}$
12. $z=\dfrac{4-3x}{3x^2+x}$
13. $y=(2x-7)^{-1}(x+5)$
14. $f(x)=\dfrac{\sqrt{x}-1}{\sqrt{x}+1}$
15. $y=\dfrac{1}{\left(x^2-1\right)\left(x^2+x+1\right)}$
16. $f(x)=\dfrac{x^{3/2}\left(x^2+1\right)}{x+1}$
17. $f(x)=\dfrac{x^3-4x^2+x}{x-2}$
18. $g(x)=\dfrac{x(3-x)}{2x^2}$
19. $y=\dfrac{2x^2}{3x+1}$
20. $f(x)=\dfrac{x^9+x^8+4x^5-7x}{x^4-3x^2+2x+1}$
21. $f(x)=\dfrac{x}{1+x^2}$
22. $y=\dfrac{x^2-2ax+a^2}{x-a}$
23. $f(x)=\dfrac{x^2+4x^{1/2}}{x^2}$
24. $f(x)=(2x+1)\left(3x^2+2\right)$
25. $f(x)=\dfrac{x^2-1}{x^2+1}$
26. $y=\dfrac{4x^3+3x+1}{2x^5}$
27. $y=\dfrac{4}{3-x}$
28. $y=\dfrac{2}{1-x^2}$
29. $f(x)=\dfrac{\pi}{2-\pi x}$
30. $y=\dfrac{x-4}{5x-2}$
31. $y=\dfrac{3x-4}{2x-1}$
32. $y=\dfrac{3x+4}{2x+1}$
33. $y=\dfrac{-3x+4}{2x+1}$
34. $y=\dfrac{-3x-4}{2x-1}$
35. $y=\dfrac{2x-3}{x+1}$
36. $y=\dfrac{3x}{3x-2}$
37. $y=\dfrac{x-3}{2x+5}$
38. $y=\dfrac{5x-3}{2x+5}$
39. $y=\dfrac{6x-8}{2x-3}$

40. $y=\dfrac{x^2-4}{5x^2-2}$

41. $y=\dfrac{3x^2-4}{2x^2-1}$

42. $y=\dfrac{3x^2+4}{2x^2+1}$

43. $y=\dfrac{2x^2-3}{x^2+1}$

44. $y=\dfrac{3x^2}{3x^2-2}$

45. $y=\dfrac{5x^2-3}{2x^2+5}$

46. $y=\dfrac{6x^2-8}{2x^2+1}$

47. $y=\dfrac{6x^3+8}{2x^3+1}$

48. $y=\dfrac{5x^3-3}{2x^3+5}$

49. $y=\dfrac{x^3}{3x^3-2}$

50. $y=\dfrac{2x^3-3}{2x^3+1}$

51. $y=\dfrac{2x^4-3}{2x^4+1}$

52. $y=\dfrac{x^2-4x+1}{5x^2-2x-1}$

53. $y=\dfrac{3x^2-4x+2}{2x^2+x-1}$

54. $y=\dfrac{3x^2+x-4}{2x^2+1}$

55. $y=\dfrac{2x^2-3}{x^2+5x+1}$

56. $y=\dfrac{3x^2}{3x^2+6x-8}$

57. $y=\dfrac{x^2+2x}{2x^2+x-5}$

58. $y=\dfrac{x^2+5x+1}{x^2}$

59. $y=\dfrac{x^2-3x+1}{x^2-8x+5}$

60. Find the first and second derivative $y=\dfrac{x^2+5x-1}{x^2}$

61. Find an equation of the tangent line to the graph of $y=\dfrac{x^2-4}{2x+5}$ when $x=0$

62. For what value(s) of x is the line tangent to the curve $y=x\sqrt{6-x}$ horizontal? Vertical?

63. Find y', y'', y''': $\quad y=(x-3)\sqrt{x+2}$