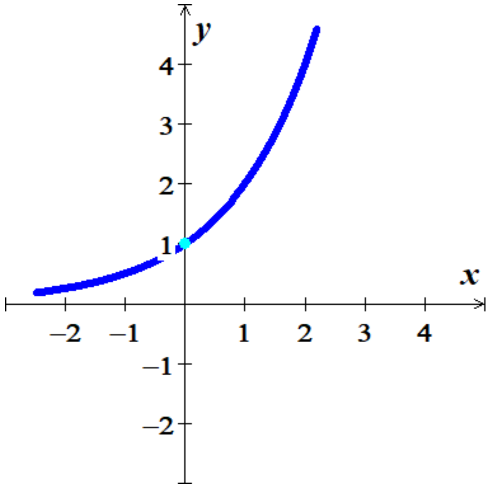
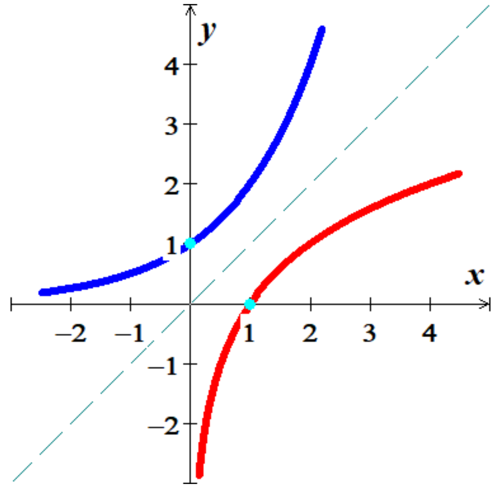
Section 3.3 – Logarithmic Functions

Graph: $x = 2^y$





Find the inverse function of $f(x) = 2^x$

$y = 2^x$

$x = 2^y$ *Solve for y?*

Logarithmic Function (*Definition*)

For $x > 0$ and $b > 0,\ b \neq 1$

$$y = \log_b x \quad \text{is equivalent to} \quad x = b^y$$

$$y = \log_b x \Leftrightarrow x = b^y$$

Base

The function $f(x) = \log_b x$ is the logarithmic function with base b.

$\log_b x$: ***read*** log base b of x

$\log x$ ***means*** $\log_{10} x$

Example

Write each equation in its equivalent exponential form:

a) $3 = \log_7 x \qquad \Rightarrow x = 7^3$

b) $2 = \log_b 25 \qquad \Rightarrow 25 = b^2$

Example

Write each equation in its equivalent logarithmic form:

a) $2^5 = x \qquad \Rightarrow 5 = \log_2 x$

b) $27 = b^3 \qquad \Rightarrow 3 = \log_b 27$

Basic Logarithmic Properties

$$log_b\, b = 1 \quad \to \quad b = b^1$$

$$\log_b 1 = 0 \quad \to \quad 1 = b^0$$

Inverse Properties

$$\log_b b^x = x \qquad\qquad \log_7 7^8 = 8$$

$$b^{\log_b x} = x \qquad\qquad 3^{\log_3 17} = 17$$

Example

Evaluate each expression without using a calculator:

a) $\log_5 \frac{1}{125}$ *b)* $\log_3 \sqrt[7]{3}$

Solution

a) $\log_5 \frac{1}{125} = \log_5 \frac{1}{5^3}$

$$= \log_5 5^{-3}$$

$$= -3$$

b) $\log_3 \sqrt[7]{3} = \log_3 3^{1/7}$

$$= \frac{1}{7}$$

Natural Logarithms

Definition

$$f(x) = \log_e x = \ln x$$

The logarithmic function with base $\boldsymbol{e}$ is called natural logarithmic function.

$\ln x$ *read* "el en of x"

$\log(-1) =$ *doesn't exist*	$\ln(-1) =$ *doesn't exist*
$\log 0 =$ *doesn't exist*	$\ln 0 =$ *doesn't exist*
$\log 0.5 \approx -0.3010$	$\ln 0.5 \approx -0.6931$
$\log 1 = 0$	$\ln 1 = 0$
$\log 2 \approx 0.3010$	$\ln 2 \approx 0.6931$
$\log 10 = 1$	$\ln e = 1$

Change-of-Base Logarithmic

$$\log_b M = \frac{\log_a M}{\log_a b} \qquad \log_b M = \frac{\log M}{\log b} \quad \textbf{or} \quad \log_b M = \frac{\ln M}{\ln b}$$

Evaluate

$$\log_7 2506 = \frac{\log 2506}{\log 7}$$

log(2506) / log(7)

$$\approx 4.02$$

Or

$$\log_7 2506 = \frac{\ln 2506}{\ln 7} \approx 4.02$$

ln(2506) / ln(7)

$$\log_5 17 = \frac{\ln 17}{\ln 5} \approx 1.7604$$

$$\log_2 0.1 = \frac{\ln 0.1}{\ln 2} \approx -3.3219$$

Domain

The domain of a logarithmic function of the form $f(x) = \log_b x$ is the set of all positive real numbers.
(*Inside* the log has to be > 0)

Range: $(-\infty, \infty)$

Example

Find the ***domain*** of

a) $f(x) = \log_4 (x-5)$

$$x - 5 > 0 \quad \Rightarrow \quad x > 5$$

Domain: $\boxed{(5, \infty)}$

b) $f(x) = \ln(4 - x)$

$$4 - x > 0$$
$$-x > -4$$
$$x < 4$$

Domain: $\boxed{(-\infty, 4)}$

c) $h(x) = \ln\left(x^2\right)$

$x^2 > 0 \Rightarrow$ all real numbers except 0.

Domain: $\{x | x \neq 0\}$

or $\boxed{(-\infty, 0) \cup (0, \infty)}$

or $\boxed{\mathbb{R} - \{0\}}$

Graphs of *Logarithmic* Functions

Example

Graph $g(x) = \log x$

Solution

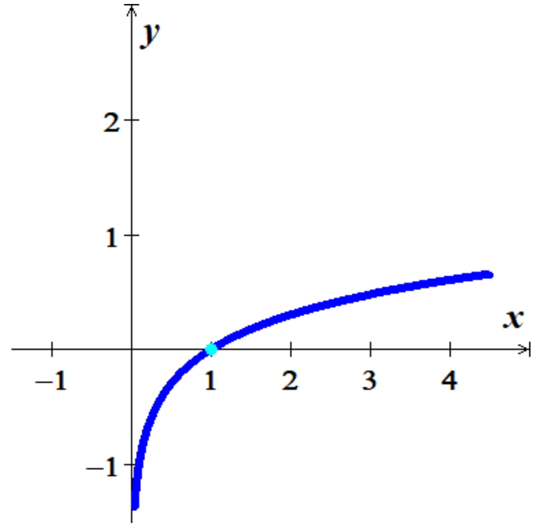
Asymptote: $x = 0$

(*Force inside log to be equal to zero, then solve for* x)

Domain: $(0,\ \infty)$

Range: $(-\infty,\ \infty)$

x	$g(x)$
0	
0.5	−.3
1	0
2	.3
3	.5



Example

$f(x) = \log_5 x$

Solution

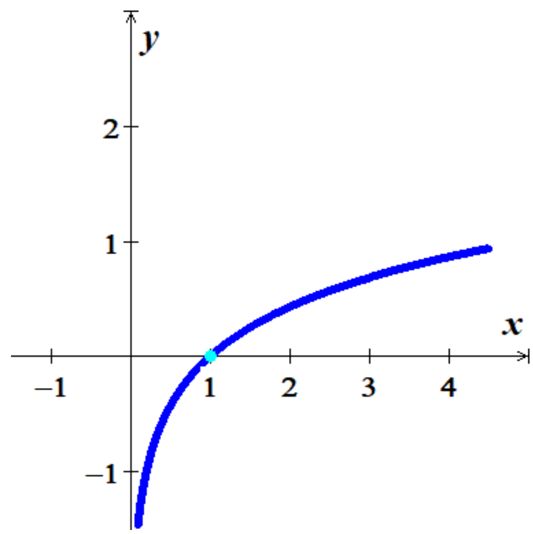
$$f(x) = \frac{\log x}{\log 5}$$

Asymptote: $x = 0$

Domain: $(0,\ \infty)$

Range: $(-\infty,\ \infty)$

x	$f(x)$
0	
$\frac{1}{5}$	−1
1	0
5	1



Example

Graph: $f(x) = \log_{1/2} x$

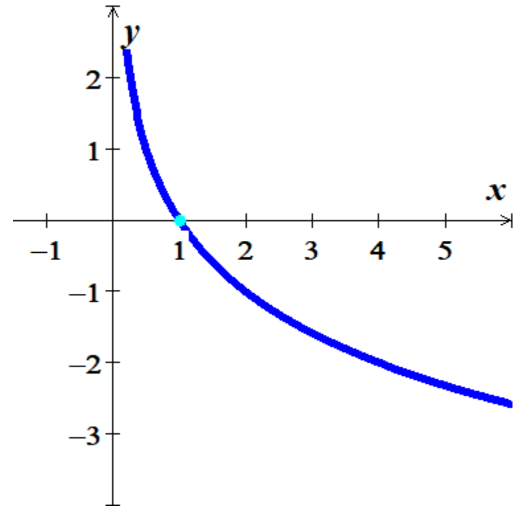
Solution

Asymptote: $x = 0$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

x	$f(x)$
0	
2	−1
1	0
$\frac{1}{2}$	1



Example

Graph: $f(x) = \log_2 (x-1)$

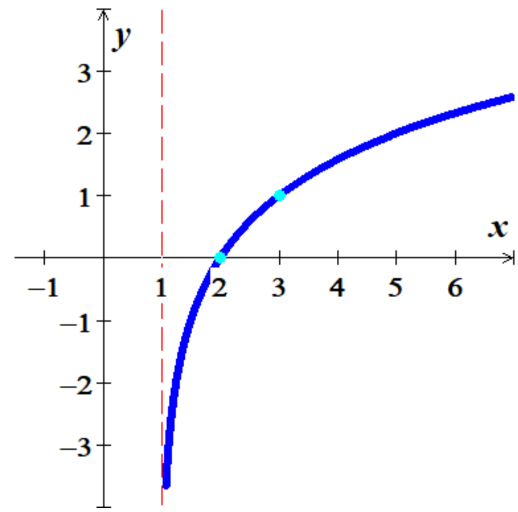
Solution

Asymptote: $x = 1$

Domain: $(1, \infty)$

Range: $(-\infty, \infty)$

x	$f(x)$
1	
$\frac{1}{3}$	−1
2	0
3	1



Example

$f(x) = |\ln(x-1)|$

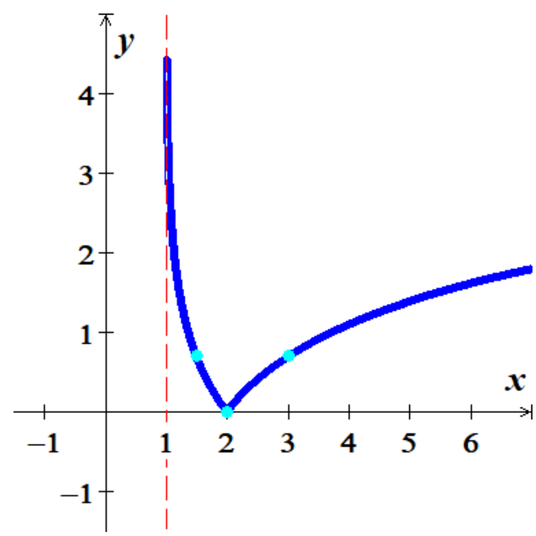
Solution

Asymptote: $x = 1$

Domain: $(1, \infty)$

Range: $[0, \infty)$

x	$f(x)$
1	
$\frac{3}{2}$	−0.7
2	0
3	0.7



Exercises *Section* 3.3 – Logarithmic Functions

(**1 – 12**) Write the equation in its equivalent logarithmic form

1. $2^6 = 64$

2. $5^4 = 625$

3. $5^{-3} = \frac{1}{125}$

4. $\sqrt[3]{64} = 4$

5. $b^3 = 343$

6. $8^y = 300$

7. $\sqrt[n]{x} = y$

8. $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$

9. $\left(\frac{1}{2}\right)^{-5} = 32$

10. $e^{x-2} = 2y$

11. $e = 3x$

12. $\sqrt[3]{e^{2x}} = y$

(**13 – 24**) Write the equation in its equivalent exponential form

13. $\log_5 125 = y$

14. $\log_4 16 = x$

15. $\log_5 \frac{1}{5} = x$

16. $\log_2 \frac{1}{8} = x$

17. $\log_6 \sqrt{6} = x$

18. $\log_3 \frac{1}{\sqrt{3}} = x$

19. $6 = \log_2 64$

20. $2 = \log_9 x$

21. $\log_{\sqrt{3}} 81 = 8$

22. $\log_4 \frac{1}{64} = -3$

23. $\log_4 26 = y$

24. $\ln M = c$

(**25 – 31**) Evaluate the expression without using a calculator

25. $\log_4 16$

26. $\log_2 \frac{1}{8}$

27. $\log_6 \sqrt{6}$

28. $\log_3 \frac{1}{\sqrt{3}}$

29. $\log_3 \sqrt[7]{3}$

30. $\log_3 \sqrt{9}$

31. $\log_{\frac{1}{2}} \sqrt{\frac{1}{2}}$

(**32 – 40**) Simplify

32. $\log_5 1$

33. $\log_7 7^2$

34. $3^{\log_3 8}$

35. $10^{\log 3}$

36. $e^{2+\ln 3}$

37. $\ln e^{-3}$

38. $\ln e^{x-5}$

39. $\log_b b^n$

40. $\ln e^{x^2+3x}$

(**41 – 64**) Find the domain of

41. $f(x) = \log_5 (x+4)$

42. $f(x) = \log_5 (x+6)$

43. $f(x) = \log(2-x)$

45. $f(x) = \ln(x-2)^2$

46. $f(x) = \ln(x-7)^2$

47. $f(x) = \log\left(x^2 - 4x - 12\right)$

44. $f(x) = \log(7 - x)$

48. $f(x) = \log\left(\frac{x-2}{x+5}\right)$

49. $f(x) = \log\left(\frac{3-x}{x-2}\right)$

50. $f(x) = \ln\left(\frac{x^2}{x-4}\right)$

51. $f(x) = \log_3\left(x^3 - x\right)$

52. $f(x) = \log\sqrt{2x-5}$

53. $f(x) = 3\ln(5x-6)$

54. $f(x) = \log\left(\frac{x}{x-2}\right)$

55. $f(x) = \ln\left(x^2 + 4\right)$

56. $f(x) = \ln|4x - 8|$

57. $f(x) = \ln\left(x^2 - 9\right)$

58. $f(x) = \ln|5 - x|$

59. $f(x) = \ln(x-4)^2$

60. $f(x) = \ln\left(x^2 - 4\right)$

61. $f(x) = \ln\left(x^2 - 4x + 3\right)$

62. $f(x) = \ln\left(2x^2 - 5x + 3\right)$

63. $f(x) = \log\left(x^2 + 4x + 3\right)$

64. $f(x) = \ln\left(x^4 - x^2\right)$

(**65 – 73**) Find the ***asymptote***, ***domain***, and ***range*** of the given functions. Then, sketch the graph

65. $f(x) = \log_4(x-2)$

66. $f(x) = \log_4|x|$

67. $f(x) = \left(\log_4 x\right) - 2$

68. $f(x) = \log(3 - x)$

69. $f(x) = 2 - \log(x+2)$

70. $f(x) = \ln(x-2)$

71. $f(x) = \ln(3 - x)$

72. $f(x) = 2 + \ln(x+1)$

73. $f(x) = 1 - \ln(x-2)$

74. On a study by psychologists Bornstein and Bornstein, it was found that the average walking speed *w*, in feet per second, of a person living in a city of population *P*, in ***thousands***, is given by the function:

$$w(P) = 0.37\ln P + 0.05$$

a) The population is 124,848. Find the average walking speed of people living in Hartford.
b) The population is 1,236,249. Find the average walking speed of people living in San Antonio.

75. The loudness of sounds is measured in a unit called a decibel. To measure with this unit, we first assign an intensity of I_0 to a very faint sound, called the threshold sound. If a particular sound has intensity *I*, then the decibel rating of this louder sound is

$$d = 10\log\frac{I}{I_0}$$

Find the exact decibel rating of a sound with intensity $10{,}000 I_0$

76. Students in an accounting class took a final exam and then took equivalent forms of the exam at monthly intervals thereafter. The average score $S(t)$, as a percent, after t months was found to be given by the function

$$S(t) = 78 - 15\ \log(t+1); \quad t \ge 0$$

a) What was the average score when the students initially took the test, $t = 0$?

b) What was the average score after 4 *months*? 24 *months*?

77. A model for advertising response is given by the function

$$N(a) = 1{,}000 + 200\ \ln a, \quad a \ge 1$$

Where $N(a)$ is the number of units sold when $\boldsymbol{a}$ is the amount spent on advertising, in thousands of dollars.

a) $N(1)$

b) $N(5)$