Section 4.4 – Second-Order System & Mechanical Applications

Second-Order Homogeneous Linear systems

Theorem

Let matrix A $(n\times n)$, If A has distinct negative eigenvalues $-\omega_1^2,\ -\omega_2^2,\ \ldots,\ -\omega_n^2$ with associated real eigenvalues $\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n$, then a general solution of

$$\vec{x}'' = A\vec{x}$$

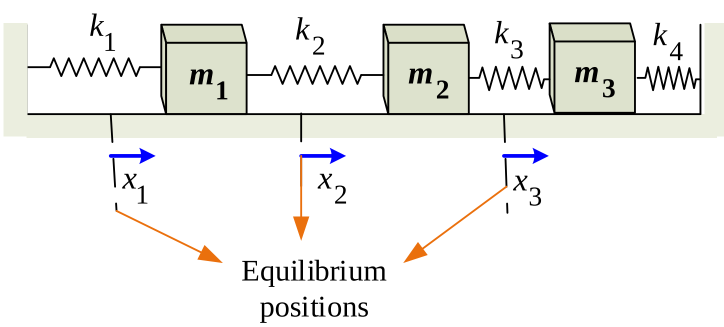
Is given by
$$\vec{x}(t) = \sum_{i=1}^{n}\left(a_i\cos\omega_i t + b_i\sin\omega_i t\right)\vec{v}_i$$

With a_i and b_i arbitrary constants.

In the special case of a nonrepeated zero eigenvalue λ_0 with associated eigenvector $\vec{v}_0$

$$\vec{x}_0(t) = \left(a_0 + b_0 t\right)\vec{v}_0$$

Example



Consider the mass-and–spring systems, as shown above. Three masses connected to each other and to two walls by 4 indicated springs. Assume the masses slide without friction and each spring obeys Hooke's law $(F = -kx)$.

By applying Newton's law $F = ma$ to the 3-masses:

$$m_1x_1'' = -k_1x_1 \qquad\qquad + k_2\left(x_2 - x_1\right)$$
$$m_2x_2'' = -k_2\left(x_2 - x_1\right) + k_3\left(x_3 - x_2\right)$$
$$m_3x_3'' = -k_3\left(x_3 - x_2\right) - k_4x_3$$

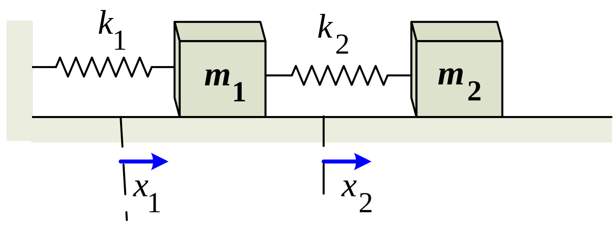
The displacement vector: $\qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

The mass matrix $$M=\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}$$

The stiffness matrix $$K=\begin{pmatrix} -k_1-k_2 & k_2 & 0 \\ k_2 & -k_2-k_3 & k_3 \\ 0 & k_3 & -k_3-k_4 \end{pmatrix}$$

Example

Consider the mass-and-spring system.



Where $m_1=2,\quad m_2=1,\quad k_1=100,\quad k_2=50$ and $M\vec{x}''=K\vec{x}$

Solution

$$\begin{cases} m_1x_1''=-k_1x_1+k_2\left(x_2-x_1\right) \\ m_2x_2''=-k_2\left(x_2-x_1\right) \end{cases} \rightarrow \begin{cases} m_1x_1''=\left(-k_1-k_2\right)x_1+k_2x_2 \\ m_2x_2''=k_2x_1-k_2x_2 \end{cases}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}x''=\begin{pmatrix} -150 & 50 \\ 50 & -50 \end{pmatrix}\vec{x} \qquad M=\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow M^{-1}=\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$x''=\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -150 & 50 \\ 50 & -50 \end{pmatrix}\vec{x} \qquad M^{-1}M\vec{x}''=M^{-1}K\vec{x}$$

$$=\begin{pmatrix} -75 & 25 \\ 50 & -50 \end{pmatrix}\vec{x} \qquad \vec{x}''=\boldsymbol{A}\vec{x}$$

$$A=\begin{pmatrix} -75 & 25 \\ 50 & -50 \end{pmatrix}$$

$$\begin{aligned} \left|A-\lambda I\right| &= \begin{vmatrix} -75-\lambda & 25 \\ 50 & -50-\lambda \end{vmatrix} \\ &=\left(-75-\lambda\right)\left(-50-\lambda\right)-1250 \\ &=\lambda^2+125\lambda+2500=0 \end{aligned}$$

The eigenvalues are: $\lambda_1=-100,\quad \lambda_2=-25$

By the theorem, the natural frequencies: $\omega_1 = 10 \quad and \quad \omega_2 = 5$

For $\lambda_1 = -100 \quad \Rightarrow \quad (A + 100I)V_1 = 0$

$$\begin{pmatrix} 25 & 25 \\ 50 & 50 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = -b \qquad \to V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = -25 \quad \Rightarrow \quad (A + 25I)V_2 = 0$

$$\begin{pmatrix} -50 & 25 \\ 50 & -25 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 2a = b \qquad \to V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The free oscillation of the mass-and-spring system, follows by:

$$\vec{x}(t) = \left(a_1 \cos 10t + b_1 \sin 10t\right)V_1 + \left(a_2 \cos 5t + b_2 \sin 5t\right)V_2$$

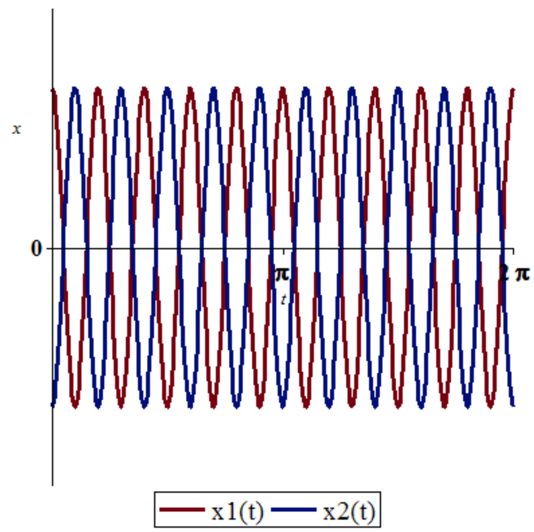
The natural mode:

$$\vec{x}_1(t) = \left(a_1 \cos 10t + b_1 \sin 10t\right)V_1$$
$$= c_1 \cos\left(10t - \alpha_1\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Where $c_1 = \sqrt{a_1^2 + b_1^2}$; $\cos\alpha_1 = \dfrac{a_1}{c_1} \quad \sin\alpha_1 = \dfrac{b_1}{c_1}$

Which has the scalar equations:

$$\begin{cases} x_1(t) = c_1 \cos\left(10t - \alpha_1\right) \\ x_2(t) = -c_1 \cos\left(10t - \alpha_1\right) \end{cases}$$



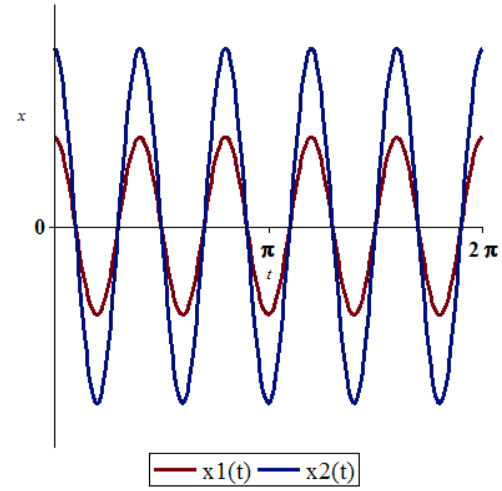
The second part:

$$\vec{x}_2(t) = \left(a_2 \cos 5t + b_2 \sin 5t\right)V_2$$
$$= c_2 \cos\left(5t - \alpha_2\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Where $c_2 = \sqrt{a_2^2 + b_2^2}$; $\cos\alpha_2 = \dfrac{a_2}{c_2} \quad \sin\alpha_2 = \dfrac{b_2}{c_2}$

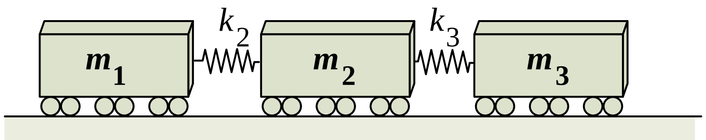
Which has the scalar equations:

$$\begin{cases} x_1(t) = c_2 \cos\left(5t - \alpha_2\right) \\ x_2(t) = 2c_2 \cos\left(5t - \alpha_2\right) \end{cases}$$



Example

Three railway cars are connected by buffer springs that react when compressed, but disengage instead of stretching.



Given that $k_2 = k_3 = k = 3000\ lb/ft$ and $m_1 = m_3 = 750\ lbs$ *and* $m_2 = 500\ lbs$

Suppose that the leftmost car is moving to the right with velocity v_0 and at time $t=0$ strikes the other 2 cars. The corresponding initial conditions are:

$$x_1(0) = x_2(0) = x_3(0) = 0$$
$$x_1'(0) = v_0 \quad x_2'(0) = x_3'(0) = 0$$

Solution

$$m_1 x_1'' = k_2(x_2 - x_1)$$
$$m_2 x_2'' = -k_2(x_2 - x_1) + k_3(x_3 - x_2)$$
$$m_3 x_3'' = -k_3(x_3 - x_2)$$

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \vec{x}'' = \begin{pmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{pmatrix} \vec{x}$$

$$\begin{pmatrix} 750 & 0 & 0 \\ 0 & 500 & 0 \\ 0 & 0 & 750 \end{pmatrix} \vec{x}'' = \begin{pmatrix} -3000 & 3000 & 0 \\ 3000 & -6000 & 3000 \\ 0 & 3000 & -3000 \end{pmatrix} \vec{x} \qquad \begin{pmatrix} 750 & 0 & 0 \\ 0 & 500 & 0 \\ 0 & 0 & 750 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{750} & 0 & 0 \\ 0 & \frac{1}{500} & 0 \\ 0 & 0 & \frac{1}{750} \end{pmatrix}$$

$$\vec{x}'' = \begin{pmatrix} \frac{1}{750} & 0 & 0 \\ 0 & \frac{1}{500} & 0 \\ 0 & 0 & \frac{1}{750} \end{pmatrix} \begin{pmatrix} -3000 & 3000 & 0 \\ 3000 & -6000 & 3000 \\ 0 & 3000 & -3000 \end{pmatrix} \vec{x}$$

$$= \begin{pmatrix} -4 & 4 & 0 \\ 6 & -12 & 6 \\ 0 & 4 & -4 \end{pmatrix} \vec{x} \qquad A = \begin{pmatrix} -4 & 4 & 0 \\ 6 & -12 & 6 \\ 0 & 4 & -4 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -4-\lambda & 4 & 0 \\ 6 & -12-\lambda & 6 \\ 0 & 4 & -4-\lambda \end{vmatrix}$$

$$= (-4-\lambda)^2(-12-\lambda) - 24(-4-\lambda) - 24(-4-\lambda)$$

$$=(-4-\lambda)\left[48+16\lambda+\lambda^2-48\right]$$
$$=\lambda(-4-\lambda)(\lambda+16)=0$$

The eigenvalues are: $\lambda_1=0\to\omega_1=0,\quad \lambda_2=-4\to\omega_2=2,\quad \lambda_3=-16\to\omega_3=4$

For $\lambda_1=0\ (\omega_1=0)\ \Rightarrow\ (A-0I)V_1=0$

$$\begin{pmatrix}-4&4&0\\6&-12&6\\0&4&-4\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\ \Rightarrow\ \begin{matrix}a=b\\ \\ b=c\end{matrix}\ \to\ V_1=\begin{pmatrix}1\\1\\1\end{pmatrix}\ \Rightarrow \vec{x}_1(t)=\left(a_1+b_1t\right)\begin{pmatrix}1\\1\\1\end{pmatrix}$$

For $\lambda_2=-4\ (\omega_2=2)\ \Rightarrow\ (A+4I)V_2=0$

$$\begin{pmatrix}0&4&0\\6&-8&6\\0&4&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\ \Rightarrow\ \begin{matrix}b=0\\ a=-c\end{matrix}\ \to\ V_2=\begin{pmatrix}1\\0\\-1\end{pmatrix}\ \Rightarrow \vec{x}_2(t)=\left(a_2\cos 2t+b_2\sin 2t\right)\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

For $\lambda_3=-16\ (\omega_3=4)\ \Rightarrow\ (A+16I)V_3=0$

$$\begin{pmatrix}12&4&0\\6&4&6\\0&4&12\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\ \Rightarrow\ \begin{matrix}3a=-b\\ \\ b=-3c\end{matrix}\ \to\ V_3=\begin{pmatrix}1\\-3\\1\end{pmatrix}\ \Rightarrow \vec{x}_3(t)=\left(a_3\cos 4t+b_3\sin 4t\right)\begin{pmatrix}1\\-3\\1\end{pmatrix}$$

$$\vec{x}(t)=a_1\begin{pmatrix}1\\1\\1\end{pmatrix}+b_1t\begin{pmatrix}1\\1\\1\end{pmatrix}+a_2\begin{pmatrix}1\\0\\-1\end{pmatrix}\cos 2t+b_2\begin{pmatrix}1\\0\\-1\end{pmatrix}\sin 2t+a_3\begin{pmatrix}1\\-3\\1\end{pmatrix}\cos 4t+b_3\begin{pmatrix}1\\-3\\1\end{pmatrix}\sin 4t$$

$$\begin{cases}\vec{x}_1(t)=a_1+b_1t+a_2\cos 2t+b_2\sin 2t+a_3\cos 4t+b_3\sin 4t\\ \vec{x}_2(t)=a_1+b_1t-3a_3\cos 4t-3b_3\sin 4t\\ \vec{x}_3(t)=a_1+b_1t-a_2\cos 2t-b_2\sin 2t+a_3\cos 4t+b_3\sin 4t\end{cases}$$

Applying the initial values

$$\vec{x}_1(0)=a_1+a_2+a_3=0$$
$$\vec{x}_2(0)=a_1-3a_3=0 \qquad a_1=3a_3 \qquad \Rightarrow\ \underline{a_1=a_2=a_3=0}$$
$$\vec{x}_3(0)=a_1-a_2+a_3=0\quad (1)\,\&\,(3)\to 2a_1+2a_3=0$$

$$\begin{cases}\vec{x}_1(t)=b_1t+b_2\sin 2t+b_3\sin 4t\\ \vec{x}_2(t)=b_1t-3b_3\sin 4t\\ \vec{x}_3(t)=b_1t-b_2\sin 2t+b_3\sin 4t\end{cases}$$

$$\begin{cases}\vec{x}_1'(t)=b_1+2b_2\cos 2t+4b_3\cos 4t\\ \vec{x}_2'(t)=b_1-12b_3\cos 4t\\ \vec{x}_3'(t)=b_1-2b_2\cos 2t+4b_3\cos 4t\end{cases}$$

$$\begin{cases} \vec{x}_1'(0) = b_1 + 2b_2 + 4b_3 = v_0 \\ \vec{x}_2'(0) = b_1 \qquad -12b_3 = 0 \\ \vec{x}_3'(0) = b_1 - 2b_2 + 4b_3 = 0 \end{cases} \to \begin{matrix} b_1 = 12b_3 \\ 2b_2 = 16b_3 \end{matrix} \to 12b_3 + 16b_3 + 4b_3 = v_0 \Rightarrow \begin{matrix} b_3 = \frac{1}{32}v_0 \\ b_1 = \frac{3}{8}v_0 \\ b_2 = \frac{1}{4}v_0 \end{matrix}$$

$$\begin{cases} \vec{x}_1(t) = \frac{1}{32}v_0(12t + 8\sin 2t + \sin 4t) \\ \vec{x}_2(t) = \frac{1}{32}v_0(12t \qquad -3\sin 4t) \\ \vec{x}_3(t) = \frac{1}{32}v_0(12t - 8\sin 2t + \sin 4t) \end{cases} \qquad \begin{cases} \vec{x}_1'(t) = \frac{1}{32}v_0(12 + 16\cos 2t + 4\cos 4t) \\ \vec{x}_2'(t) = \frac{1}{32}v_0(12 - 12\cos 4t) \\ \vec{x}_3'(t) = \frac{1}{32}v_0(12 - 16\cos 2t + 4\cos 4t) \end{cases}$$

For these equations to hold, only when the 2 buffer springs remain compressed; that is, while both

$$x_2 - x_1 < 0 \quad and \quad x_3 - x_2 < 0$$

$$\begin{aligned} x_2(t) - x_1(t) &= \frac{1}{32}v_0(12t - 3\sin 4t) - \frac{1}{32}v_0(12t + 8\sin 2t + \sin 4t) \\ &= \frac{1}{32}v_0(-8\sin 2t - 4\sin 4t) \\ &= -\frac{1}{8}v_0(2\sin 2t + 2\sin 2t\cos 2t) \\ &= -\frac{1}{4}v_0 \sin 2t(1 + \cos 2t) < 0 \end{aligned}$$

$$\sin 2t = 0 \Rightarrow (2t = 0, \pi) \to \underline{t = 0,\ \tfrac{\pi}{2}} \qquad \cos 2t = -1 \to (2t = \pi) \to \underline{t = \tfrac{\pi}{2}}$$

$$x_2 - x_1 < 0 \Rightarrow t \in \left(0,\ \frac{\pi}{2}\right)$$

$$\begin{aligned} x_3(t) - x_2(t) &= \frac{1}{32}v_0(12t - 8\sin 2t + \sin 4t) - \frac{1}{32}v_0(12t - 3\sin 4t) \\ &= \frac{1}{32}v_0(-8\sin 2t + 4\sin 4t) \\ &= -\frac{1}{8}v_0(2\sin 2t - 2\sin 2t\cos 2t) \\ &= -\frac{1}{4}v_0(\sin 2t)(1 - \cos 2t) < 0 \end{aligned}$$

$$\sin 2t = 0 \Rightarrow (2t = 0, \pi) \to \underline{t = 0,\ \tfrac{\pi}{2}} \qquad \cos 2t = 1 \to (2t = 0) \to \underline{t = 0}$$

$$x_3 - x_2 < 0 \Rightarrow t \in \left(0,\ \frac{\pi}{2}\right)$$

$x_2 - x_1 < 0 \quad and \quad x_3 - x_2 < 0$ until $\underline{t = \frac{\pi}{2} \approx 1.57\ sec}$

$$x_1\left(\frac{\pi}{2}\right) = x_2\left(\frac{\pi}{2}\right) = x_3\left(\frac{\pi}{2}\right) = \frac{1}{32}v_0\left(12\frac{\pi}{2}\right) \underline{= \frac{3\pi}{16}v_0}$$

$$x_1'\left(\frac{\pi}{2}\right) = x_2'\left(\frac{\pi}{2}\right) = 0 \quad x_3'\left(\frac{\pi}{2}\right) = \frac{1}{32}v_0(32) \underline{= v_0}$$

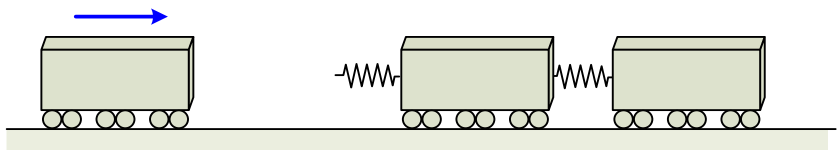
We conclude that the 3 railway cars remain engaged and moving to the right until disengagement occurs at time $t = \frac{\pi}{2}$.

At $t > \frac{\pi}{2}$

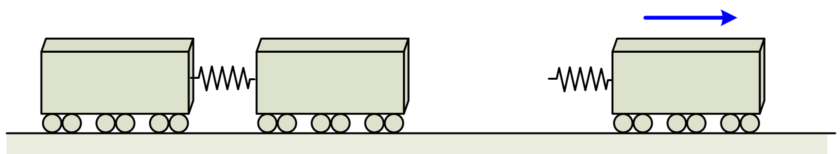
$$x_1(t) = x_2(t) = \frac{3\pi}{16} v_0$$

$$\frac{3\pi}{16} v_0 = v_0\left(\frac{\pi}{2} - \beta\right) \quad \rightarrow \quad \beta = \frac{\pi}{2} - \frac{3\pi}{16} = \frac{5\pi}{16}$$

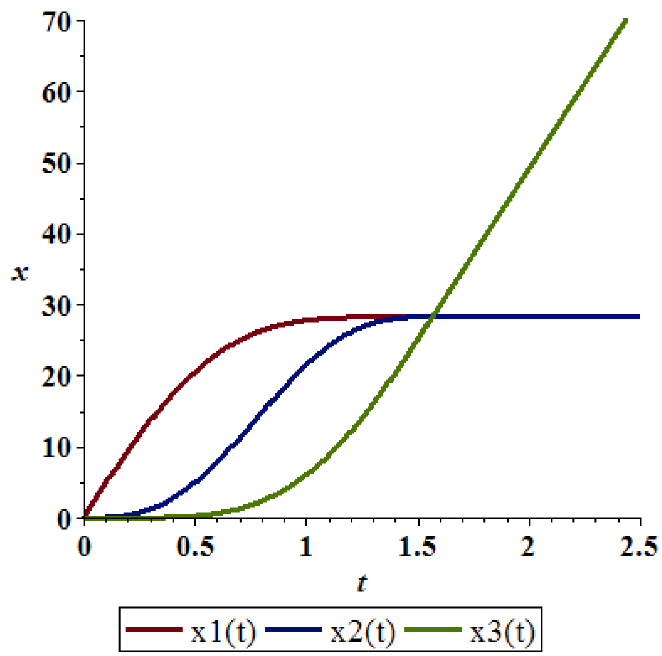
$$x_3(t) = v_0\left(t - \frac{5\pi}{16}\right) \boxed{= v_0 t - \frac{5\pi}{16} v_0}$$



At rest

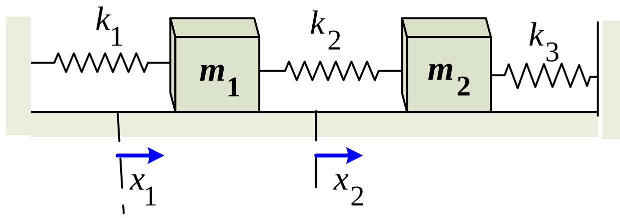


At rest



Exercises Section 4.4 – Second-Order System & Mechanical Applications

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

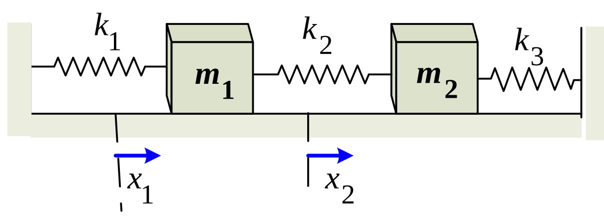
1. $m_1 = m_2 = 1; \quad k_1 = 0,\ k_2 = 2,\ k_3 = 0 \quad (no\ walls)$

2. $m_1 = m_2 = 1; \quad k_1 = 1,\ k_2 = 2,\ k_3 = 1$

3. $m_1 = m_2 = 1; \quad k_1 = 2,\ k_2 = 1,\ k_3 = 2$

4. $m_1 = 1,\ m_2 = 2; \quad k_1 = 2,\ k_2 = k_3 = 4$

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



The mass-and-spring system is set in motion from rest $x_1'(0) = x_2'(0) = 0$ in its equilibrium position $x_1(0) = x_2(0) = 0$.

Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

For the given external forces $F_1(t)$ and $F_2(t)$ acting on the masses m_1 and m_2, respectively.

Find the resulting motion of the system and describe it as a superposition of oscillations at three different frequencies.

5. $m_1 = m_2 = 1; \quad k_1 = 1,\ k_2 = 4,\ k_3 = 1 \quad F_1(t) = 96\cos 5t,\ F_2(t) = 0$

6. $m_1 = 1,\ m_2 = 2; \quad k_1 = 1,\ k_2 = k_3 = 2; \quad F_1(t) = 0,\ F_2(t) = 120\cos 3t$

7. $m_1 = m_2 = 1; \quad k_1 = 4,\ k_2 = 6,\ k_3 = 4; \quad F_1(t) = 30\cos t,\ F_2(t) = 60\cos t$

8. Consider a mass-and-spring system containing two masses $m_1 = m_2 = 1$ whose displacement functions $x(t)$ and $y(t)$ satisfy the differential equations

$$x'' = -40x + 8y$$
$$y'' = 12x - 60y$$

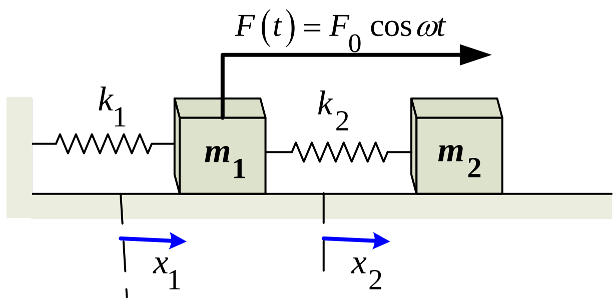
a) Describe the two fundamental modes of free oscillation of the system.

b) Assume that the two masses start in motion with the initial conditions

$$x(0)=19,\ x'(0)=12 \quad and \quad y(0)=3,\ y'(0)=6$$

And are acted on by the same force, $F_1(t)=F_2(t)=-195\cos 7t$. Describe the resulting motion as a superposition of oscillations at three different frequencies.

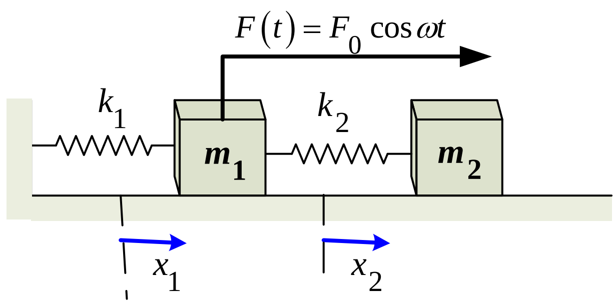
9. Consider a mass-and-spring system shown below. Assume that $m_1=1$; $k_1=50$; $F_0=5$ in mks units, and that $\omega=10$. Then find m_2 so that in the resulting steady periodic oscillations, the mass m_1 will remain at rest (!).



Thus the effect of the second mass-and-spring pair will be to neutralize the effect of the force on the first mass. This is an example of a dynamic damper. It has an electrical analogy that some cable companies use to prevent your reception of certain cable channels.

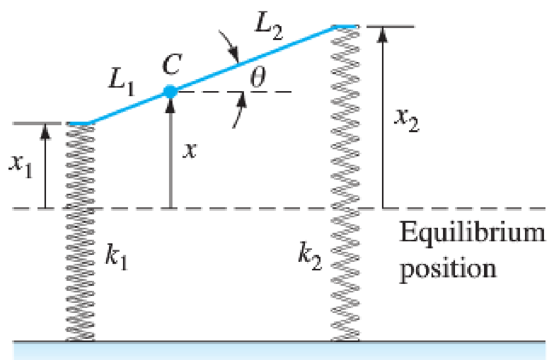
10. Consider a mass-and-spring system shown below. Assume that

$$m_1=2,\ m_2=\frac{1}{2};\quad k_1=75,\ k_2=25;\quad F_0=100 \quad and\ \omega=10\ \text{(in mks units)}.$$



Find the solution of the system $M\vec{x}''=K\vec{x}+F$ that satisfies the initial conditions $\vec{x}(0)=\vec{x}'(0)=\mathbf{0}$

11. A car with two axles and with separate front and rear suspension systems.



We assume that the car body acts as would a solid bar of mass m and length $L = L_1 + L_2$. It has moment of inertia I about its center of mass C, which is at distance L_1 from the front of the car. The car has front and back suspension springs with Hooke's constants k_1 and k_2, respectively. When the car is in motion, let $x(t)$ denote the vertical displacement of the center of mass of the car from equilibrium; let $\theta(t)$ denote its angular displacement (in radians) from the horizontal. Then Newton's laws of motion for linear and angular acceleration can be used to derive the equations.

$$mx'' = -(k_1 + k_2)x + (k_1L_1 - k_2L_2)\theta$$
$$I\theta'' = (k_1L_1 - k_2L_2)x - \left(k_1L_1^2 + k_2L_2^2\right)\theta$$

Suppose that $m = 75$ *slugs* (the car weighs 2400 *lb*), $L_1 = 7\,ft$, $L_2 = 3\,ft$ (it's a rear engine car), $k_1 = k_2 = 2000$ *lb / ft*, and $I = 1000$ *ft.lb.s*^{2}.

a) Find the two natural frequencies ω_1 and ω_2 of the car.

b) Now suppose that the car is driven at a speed of v *ft* / sec along a washboard surface shaped like a sine curve with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1$ *or* $\omega = \omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

The system is taken as a model for an undamped car with the given parameters in *fps* units.

a) Find the two natural frequencies ω_1 and ω_2 of the car (in hertz).

b) Assume that his car is driven along a sinusoidal washboard surface with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1$ *or* $\omega = \omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

12. $m = 100;\quad I = 800;\quad L_1 = L_2 = 5;\quad k_1 = k_2 = 2000$

13. $m = 100;\quad I = 1000;\quad L_1 = 6,\ L_2 = 4;\quad k_1 = k_2 = 2000$

14. $m = 100;\quad I = 800;\quad L_1 = L_2 = 5;\quad k_1 = 1000,\ k_2 = 2000$