Length

***Length* of a curve** $y = f(x)$ is given by the formula:

$$L = \int_c^d \sqrt{1+\left[f'(x)\right]^2}\ dx = \int_c^d \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

If $f(x) = ax^m + bx^n$, then

$$\boldsymbol{L} = \int_c^d \sqrt{1+\left(\frac{df}{dx}\right)^2}\ dx = \left[ax^m - bx^n\right]_c^d$$

Iff $f(x)$ satisfies these 2 conditions:

1. $m + n = 2$
2. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = max^{m-1} + nbx^{n-1}$$

$$1 + (f')^2 = 1 + \left(max^{m-1} + nbx^{n-1}\right)^2$$

$$= 1 + m^2a^2x^{2m-2} + 2abmnx^{m+n-2} + n^2b^2x^{2n-2}$$

We need to combined to a perfect square $\quad a^2 - 2ab + b^2 = (a-b)^2$

- If $x^{m+n-2} = 1 = x^0 \quad \rightarrow \quad \boxed{m+n=2}$

$$= m^2a^2x^{2m-2} + (1 + 2abmn) + n^2b^2x^{2n-2} \qquad a^2 - 2ab + b^2 = (a-b)^2$$

- Let $1 + 2abmn = -2abmn \quad \rightarrow \quad \boxed{abmn = -\frac{1}{4}}$

$$= m^2a^2x^{2m-2} - 2abmn + n^2b^2x^{2n-2} \qquad x^{2(m+n-2)} = 1$$

$$= \left(max^{m-1} - nbx^{n-1}\right)^2$$

$$\boldsymbol{L} = \int_c^d \sqrt{\left(max^{m-1} - nbx^{n-1}\right)^2}\ dx$$

$$= \int_c^d \left(max^{m-1} - nbx^{n-1}\right)\ dx$$

$$= \left[ax^m - bx^n\right]_c^d \qquad \checkmark$$

Example

Find the length of the graph of $f(x)=\frac{x^3}{12}+\frac{1}{x}, \quad 1\le x\le 4$

Solution

$$f'(x)=\frac{x^2}{4}-\frac{1}{x^2}$$

$$\begin{aligned}1+\left[f'(x)\right]^2 &=1+\left(\frac{x^2}{4}-\frac{1}{x^2}\right)^2\\ &=1+\frac{x^4}{16}-\frac{1}{2}+\frac{1}{x^4}\\ &=\frac{x^4}{16}+\frac{1}{2}+\frac{1}{x^4}\\ &=\left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2\end{aligned}$$

$$\begin{aligned}L &=\int_1^4\sqrt{1+\left(f'(x)\right)^2}\,dx\\ &=\int_1^4\sqrt{\left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2}\,dx\\ &=\int_1^4\left(\frac{x^2}{4}+\frac{1}{x^2}\right)dx\\ &=\left(\frac{x^3}{12}-\frac{1}{x}\right)\Big|_1^4\\ &=\left(\frac{4^3}{12}-\frac{1}{4}\right)-\left(\frac{1}{12}-\frac{1}{1}\right)\\ &=\frac{72}{12}\\ &=6\ \text{unit}\end{aligned}$$

$a=\frac{1}{12}, \quad m=3, \quad b=1, \quad n=-1$

1. $m+n=3-1=2$ ✓
2. $abmn=\frac{1}{12}(1)(3)(-1)=-\frac{1}{4}$ ✓

$$L=\left(\frac{x^3}{12}-\frac{1}{x}\right)\Big|_1^4$$

Example

$f(x)=\frac{1}{3}x^{3/2}-x^{1/2} \quad\rightarrow\quad L=\frac{1}{3}x^{3/2}+x^{1/2}+C$

$f(x)=\frac{2}{3}x^{3/2}-\frac{1}{2}x^{1/2} \quad\rightarrow\quad L=\frac{2}{3}x^{3/2}+\frac{1}{2}x^{1/2}$

$f(x)=\frac{1}{6}x^3+\frac{1}{2x} \quad\rightarrow\quad L=\frac{1}{6}x^3-\frac{1}{2x}+C$

$f(x)=x^3+\frac{1}{12x} \quad\rightarrow\quad L=x^3-\frac{1}{12x}+C$

$f(x)=\frac{1}{6}x^3+\frac{1}{2x} \quad\rightarrow\quad L=\frac{1}{6}x^3-\frac{1}{2x}$

$f(x)=\frac{1}{8}x^4+\frac{1}{4x^2} \quad\rightarrow\quad L=\frac{1}{8}x^4-\frac{1}{4x^2}$

$f(x)=\frac{1}{4}x^4+\frac{1}{8x^2} \quad\rightarrow\quad L=\frac{1}{4}x^4-\frac{1}{8x^2}$

$f(x)=\frac{1}{10}x^5+\frac{1}{6x^3} \quad\rightarrow\quad L=\frac{1}{10}x^5-\frac{1}{6x^3}$

$f(x)=\frac{3}{10}x^{1/3}-\frac{3}{2}x^{5/3} \quad\rightarrow\quad L=\frac{3}{10}x^{1/3}+\frac{3}{2}x^{5/3}$

$f(x)=x^{1/2}-\frac{1}{3}x^{3/2} \quad\rightarrow\quad L=x^{1/2}+\frac{1}{3}x^{3/2}$

If $f(x)=ae^{mx}+be^{nx}$, then

$$\boldsymbol{L}=\int_c^d \sqrt{1+\left(\frac{df}{dx}\right)^2}\,dx=\left[ae^{mx}-be^{nx}\right]_c^d$$

Iff $f(x)$ satisfies these 2 conditions:

1. $m=-n$
2. $abmn=-\frac{1}{4}$

Proof

$$f'(x)=ame^{mx}+bne^{nx}$$

$$1+(f')^2=1+\left(ame^{mx}+bne^{nx}\right)^2$$

$$=1+m^2a^2e^{2mx}+2abmne^{(m+n)x}+n^2b^2e^{2nx}$$

- If $e^{(m+n)x}=1=e^{(x=0)} \rightarrow \boxed{m-n}$

$$=m^2a^2x^{2m-2}+(1+2abmn)+n^2b^2x^{2n-2}$$

$a^2-2ab+b^2=(a-b)^2$

- Let $1+2abmn=-2abmn \rightarrow \boxed{abmn=-\frac{1}{4}}$

$$=m^2a^2e^{2mx}-2abmne^{(m+n)x}+n^2b^2e^{2nx}$$

$x^{2(m+n-2)}=1$

$$=\left(ame^{mx}-bne^{nx}\right)^2$$

$$\left(ame^{mx}-bne^{nx}\right)^2=m^2a^2e^{2mx}-2abmne^{(m+n)x}+n^2b^2e^{2nx}$$

Example

$f(x)=2e^x+\frac{1}{8}e^{-x} \rightarrow L=2e^x-\frac{1}{8}e^{-x}$

$f(x)=2e^{\sqrt{2}x}+\frac{1}{16}e^{-\sqrt{2}x} \rightarrow L=2e^{\sqrt{2}x}-\frac{1}{16}e^{-\sqrt{2}x}$