

Exercise 1

Ox and Oy are bisector to 2 adjacent acute angles, \widehat{AOB} and \widehat{BOC} where the difference is 36° and $\widehat{AOC} = 90^\circ$. Oz is the bisector of the angle \widehat{xOy} . Determine the angle \widehat{BOz}

Solution

$$\widehat{BOC} - \widehat{AOB} = 36^\circ$$

$$\widehat{BOC} + \widehat{AOB} = 90^\circ$$

$$2 \widehat{BOC} = 126^\circ$$

$$\widehat{BOC} = 63^\circ$$

$$\widehat{AOB} = 27^\circ$$

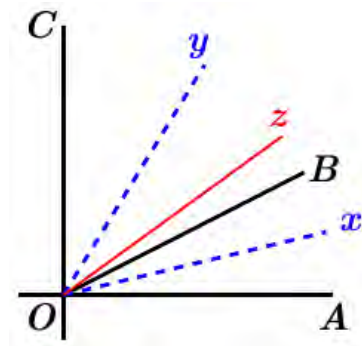
$$\begin{aligned}\widehat{xOB} &= \frac{1}{2} \widehat{AOB} \\ &= \frac{27^\circ}{2}\end{aligned}$$

$$\widehat{BOy} = \frac{63^\circ}{2}$$

$$\begin{aligned}\widehat{xOy} &= \widehat{xOB} + \widehat{BOy} \\ &= \frac{1}{2}(63^\circ + 27^\circ) \\ &= 45^\circ\end{aligned}$$

$$\widehat{xOz} = \frac{45^\circ}{2}$$

$$\begin{aligned}\widehat{BOz} &= \widehat{xOz} - \widehat{xOB} \\ &= \frac{1}{2}(45^\circ - 27^\circ) \\ &= 9^\circ\end{aligned}$$



Exercise 2

Ox and Oy are bisector to 2 adjacent acute angles, \widehat{AOB} and \widehat{BOC} where the difference is 36° . Oz is the bisector of the angle \widehat{xOy} . Determine the angle \widehat{BOz} .

Solution

$$Ox \text{ is the bisector } \widehat{AOB} \quad (1)$$

$$Oy \text{ is the bisector } \widehat{BOC} \quad (2)$$

$$Om \text{ is the bisector } \widehat{AOC} \quad (3)$$

$$Oz \text{ is the bisector } \widehat{xOy} \quad (4)$$

$$Oy \text{ is the bisector } \widehat{BOC} \quad (5)$$

$$\widehat{BOC} - \widehat{AOB} = 36^\circ$$

$$\widehat{BOC} - \widehat{BOD} = 36^\circ$$

$$\widehat{DOC} = 36^\circ$$

$$\begin{aligned} (3) \rightarrow \widehat{AOM} &= \frac{1}{2} \widehat{AOC} \\ &= \frac{1}{2} (2\widehat{AOB} + \widehat{DOC}) \\ &= \frac{1}{2} (2\widehat{AOB} + 36^\circ) \\ &= \widehat{AOB} + 18^\circ \end{aligned}$$

$$\begin{aligned} \widehat{BOM} &= \widehat{AOM} - \widehat{AOB} \\ &= \widehat{AOB} + 18^\circ - \widehat{AOB} \\ &= 18^\circ \end{aligned}$$

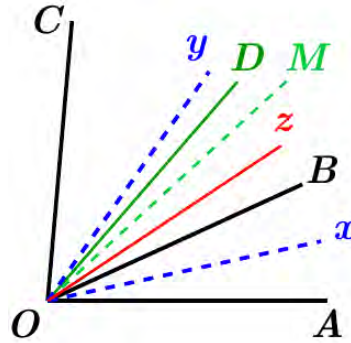
$$(1) \rightarrow \widehat{BOx} = \frac{1}{2} \widehat{AOB}$$

$$(4) \rightarrow \widehat{BOy} = \frac{1}{2} \widehat{BOC}$$

$$(1) + (4) \rightarrow \widehat{xOy} = \frac{1}{2} \widehat{AOC}$$

$$(3) \rightarrow \widehat{AOM} = \frac{1}{2} \widehat{AOC} = \widehat{xOy}$$

$$\begin{aligned} \widehat{BOz} &= \widehat{xOz} - \widehat{xOB} \\ &= \frac{1}{2} (\widehat{xOy} - \widehat{AOB}) \\ &= \frac{1}{2} (\widehat{AOM} - \widehat{AOB}) \\ &= \frac{1}{2} \widehat{BOM} \\ &= 9^\circ \end{aligned}$$



Exercise 3

Four consecutive half-lines (segments): OA , OB , OC , and OD formed angles such as

$$\widehat{DOA} = \widehat{COB} = 2\widehat{AOB} \quad \text{and} \quad \widehat{COD} = 3\widehat{AOB}$$

Calculate the angles to demonstrate that the bisectors of \widehat{AOB} and \widehat{COD} are in a straight line.

Solution

$$\widehat{AOB} + \widehat{BOC} + \widehat{COD} + \widehat{DOA} = 360^\circ$$

$$\widehat{AOB} + 2\widehat{AOB} + 3\widehat{AOB} + 2\widehat{AOB} = 360^\circ$$

$$8\widehat{AOB} = 360^\circ$$

$$\widehat{AOB} = 45^\circ$$

$$\widehat{DOA} = \widehat{COB} = 90^\circ$$

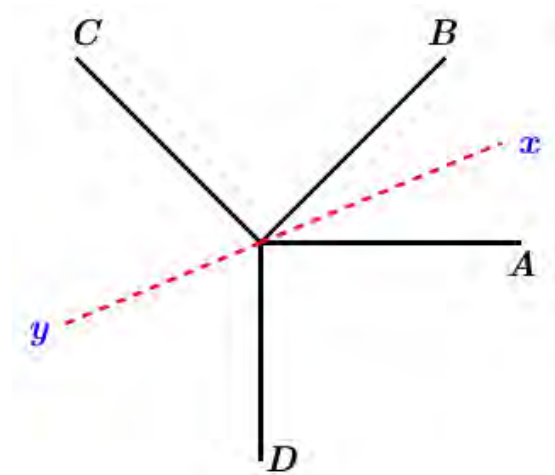
$$\widehat{COD} = 135^\circ$$

Let:

Ox is the bisector \widehat{AOB}

Oy is the bisector \widehat{COD}

$$\begin{aligned} \widehat{xOy} &= \widehat{xOB} + \widehat{BOC} + \widehat{COy} \\ &= \frac{1}{2}\widehat{AOB} + 90^\circ + \frac{1}{2}\widehat{COD} \\ &= \frac{1}{2}(45^\circ + 135^\circ) + 90^\circ \\ &= 180^\circ \end{aligned}$$



Therefore; the bisectors of \widehat{AOB} and \widehat{COD} are in a straight line

Exercise 4

The segments OA and OB formed with OX the angles α and β .

- a) Demonstrate that the bisector OC of the angle \widehat{AOB} made with OX an angle $\frac{\alpha + \beta}{2}$.
- b) Examine the cases where
 - i. $\alpha + \beta = 90^\circ$
 - ii. $\alpha + \beta = 180^\circ$

Solution

Given:

$$\widehat{XOA} = \alpha \quad \& \quad \widehat{XOB} = \beta$$

$$\begin{aligned} \widehat{AOC} &= \frac{1}{2} \widehat{AOB} \\ &= \frac{\beta - \alpha}{2} \end{aligned}$$

$$\begin{aligned} \text{a) } \widehat{XOC} &= \widehat{XOA} + \widehat{AOC} \\ &= \alpha + \frac{\beta - \alpha}{2} \\ &= \frac{\alpha + \beta}{2} \end{aligned}$$

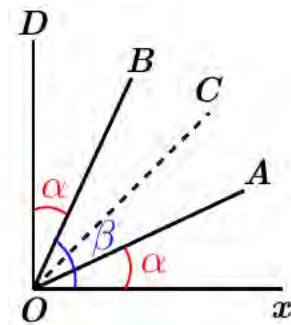
- b) **i.** If $\alpha + \beta = 90^\circ$, then

$$\widehat{XOC} = 45^\circ$$

Let: $\widehat{XOD} = 90^\circ$ that implies OC is the bisector of \widehat{XOD}

Since OC is the bisector of \widehat{AOB} , then

$$\begin{aligned} \widehat{BOD} &= 90^\circ - \beta & \beta &= 90^\circ - \alpha \\ &= 90^\circ - 90^\circ + \alpha \\ &= \alpha \end{aligned}$$



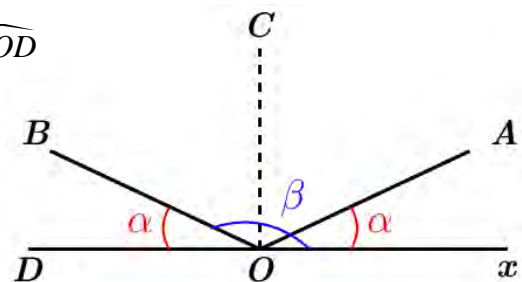
- ii.** If $\alpha + \beta = 180^\circ$, then

$$\widehat{XOC} = 90^\circ$$

Let: $\widehat{XOD} = 180^\circ$ that implies OC is the bisector of \widehat{XOD}

Since OC is the bisector of \widehat{AOB} , then

$$\begin{aligned} \widehat{BOD} &= 180^\circ - \beta & \beta &= 180^\circ - \alpha \\ &= 180^\circ - 180^\circ + \alpha \\ &= \alpha \end{aligned}$$



Exercise 5

A point O takes on an infinite right $x'Ox$ be conducted the same side half-lines OA and OB , as well as the bisectors of angles \widehat{xOA} , \widehat{AOB} , and $\widehat{BOx'}$.

Calculate the angles of the figure such that the bisector of the angle \widehat{AOB} is perpendicular to $x'Ox$ and the bisectors of the extreme angles formed an angle of 100° .

Solution

Given: $\widehat{zOz'} = 100^\circ$

$$\widehat{xOC} = 90^\circ$$

OC is the bisector \widehat{AOB}

$$\widehat{AOC} = \widehat{COB}$$

Oz is the bisector \widehat{xOA}

$$\widehat{xOz} = \widehat{zOA}$$

Oz' is the bisector $\widehat{x'OB}$

$$\widehat{x'Oz'} = \widehat{z'OB}$$

$$\widehat{xOz} = \frac{180^\circ - 100^\circ}{2}$$

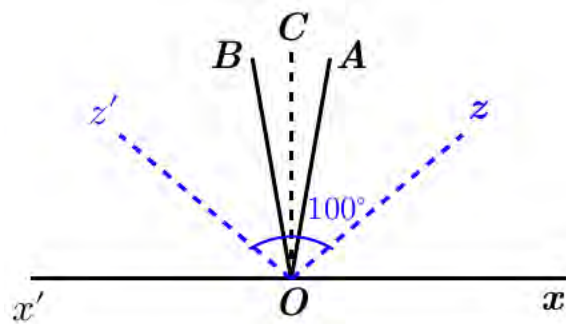
$$= 40^\circ$$

$$\widehat{AOB} = 2\widehat{AOC}$$

$$= 2(90^\circ - 2\widehat{xOz})$$

$$= 2(90^\circ - 80^\circ)$$

$$= 20^\circ$$



Exercise 6

Four consecutive half-lines OA , OB , OC , and OD formed four adjacent consecutive angles which are between them like 1, 2, 3, 4.

Calculate the angles and the adjacent consecutive angles formed by their bisectors.

Solution

$$\widehat{AOB} + \widehat{BOC} + \widehat{COD} + \widehat{DOA} = 360^\circ$$

$$\widehat{AOB} + 2\widehat{AOB} + 3\widehat{AOB} + 4\widehat{AOB} = 360^\circ$$

$$10\widehat{AOB} = 360^\circ$$

$$\widehat{AOB} = 36^\circ$$

$$\widehat{BOC} = 72^\circ$$

$$\widehat{COD} = 108^\circ$$

$$\widehat{DOA} = 144^\circ$$

$$\widehat{xOy} = \frac{1}{2}\widehat{AOB} + \frac{1}{2}\widehat{BOC}$$

$$= \frac{1}{2}36^\circ + \frac{1}{2}72^\circ$$

$$= 18^\circ + 36^\circ$$

$$= 54^\circ$$

$$\widehat{yOz} = \frac{1}{2}\widehat{BOC} + \frac{1}{2}\widehat{COD}$$

$$= \frac{1}{2}72^\circ + \frac{1}{2}108^\circ$$

$$= 36^\circ + 54^\circ$$

$$= 90^\circ$$

$$\widehat{zOw} = \frac{1}{2}\widehat{COD} + \frac{1}{2}\widehat{DOA}$$

$$= \frac{1}{2}108^\circ + \frac{1}{2}144^\circ$$

$$= 54^\circ + 72^\circ$$

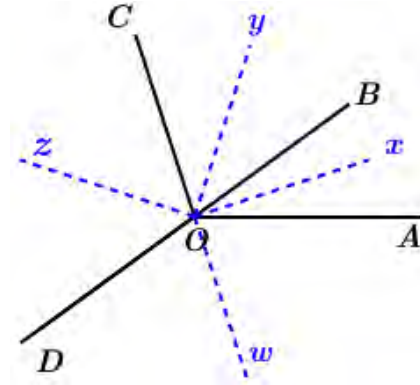
$$= 126^\circ$$

$$\widehat{wOx} = \frac{1}{2}\widehat{DOA} + \frac{1}{2}\widehat{AOB}$$

$$= \frac{1}{2}144^\circ + \frac{1}{2}36^\circ$$

$$= 72^\circ + 18^\circ$$

$$= 90^\circ$$



Exercise 7

A point P is on the base BC of an isosceles triangle ABC . The two points M and N are the middle points of the segments BP and PC , respectively, which lead the perpendicular to the base BC ; these perpendiculars meet AB in E , AC in F .

Demonstrate that the angle EPF is equal to A .

Solution

$$\widehat{BAC} = 180^\circ - \widehat{ABC} - \widehat{ACB}$$

M is the middle of the segment BP and $EM \perp$ to BP , therefore

$$EB = EP \quad \& \quad \widehat{EBP} = \widehat{EPB}$$

N is the middle of the segment CP and $FN \perp$ to CP , therefore

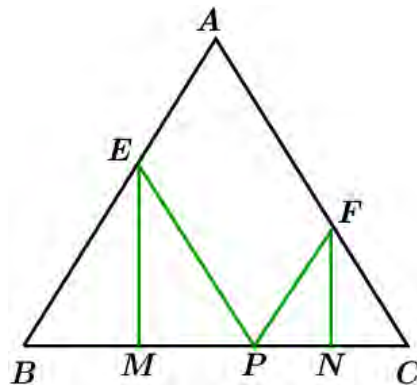
$$FP = FN \quad \& \quad \widehat{FPC} = \widehat{FCP}$$

$$\widehat{EPF} = 180^\circ - \widehat{CPF} - \widehat{BPE}$$

$$= 180^\circ - \widehat{PFC} - \widehat{PBE}$$

$$= 180^\circ - \widehat{ABC} - \widehat{ACB}$$

$$= \widehat{A} \quad \checkmark$$



Exercise 8

Given the triangle ABC and the bisectors BO and CO of the angles of the base, where the point O is the intersection of the 2 bisectors. A line DOE passes through the point O parallel to base BC .

Prove that $DE = DB + CE$

Solution

CO is the bisector of $\widehat{BCE} \Rightarrow \widehat{BCO} = \widehat{OCE}$

$OE \parallel BC \Rightarrow \widehat{COE} = \widehat{BOC}$

$\therefore \widehat{EOC} = \widehat{OCE} \rightarrow \underline{OE = EC}$

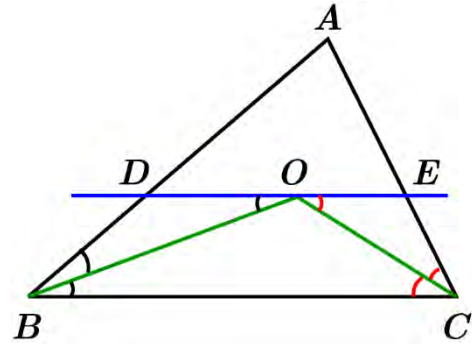
Similar; BO is the bisector of $\widehat{DBC} \Rightarrow \widehat{DBO} = \widehat{OBC}$

$DO \parallel BC \Rightarrow \widehat{DOB} = \widehat{OBC}$

$\therefore \widehat{DOB} = \widehat{OBC} \rightarrow \underline{DO = DB}$

$DE = DO + OE$

$\underline{= DB + CE}$



Exercise 9

A right triangle ABC at A with a height AH . We drop perpendiculars HE and HD from H to sides AB and AC respectively.

- Prove that $DE = AH$
- Prove that AM is perpendicular to DE , where M is the middle point of BC .
- Prove that MN (N is the middle point of AB) and the segment Bx (parallel to DE) are intersect on AH .
- Prove that AM and HD are intersect on Bx .

Solution

- The triangles AEH and ADH are right triangles and angle A is right angle.
Then $AEHD$ is a rectangle.
Therefore, $DE = AH$

- A middle point of a hypotenuse of a right triangle is the center of the circle of that triangle.
Therefore, $MC = MA = MB$

That implies to: $\widehat{MAC} = \widehat{MCA}$

From the rectangle $ADHE$: $\widehat{EAH} = \widehat{EDH}$

$$\widehat{EAH} + \widehat{HAM} + \widehat{MAC} = 90^\circ$$

$$\widehat{HAM} + \widehat{MAC} = \widehat{HAC}$$

$$\widehat{EAH} + \widehat{HAC} = 90^\circ$$

$$\widehat{EAH} + 90^\circ - \widehat{MCA} = 90^\circ$$

$$\widehat{EAH} = \widehat{MCA} = \widehat{EDH} = \widehat{MAC}$$

$$\widehat{ADE} + \widehat{EDH} = 90^\circ$$

$$\widehat{ADE} + \widehat{MAD} = 90^\circ$$

Therefore, AM is perpendicular to DE .

- N is the middle point of $AB \Rightarrow NA = NB$

$$Bx \text{ parallel to } DE \Rightarrow \widehat{ABx} = \widehat{AED} = \widehat{EDH} = \widehat{EAH}$$

Let point P the intersection of Bx and AH . Since $\widehat{ABP} = \widehat{BAP}$, then the triangle BPA is isosceles.
 PN is the perpendicular to AB as well MN . Which gives us that points M, P, N are on the same line.

Therefore, segment MN and AH intersect at point P .

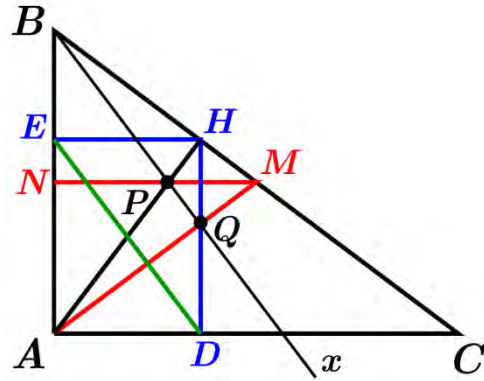
- Let Point Q be the intersection of AM and Bx .

$$\widehat{ABQ} = \widehat{BAH} \quad \& \quad \widehat{BAQ} = \widehat{ABH}$$

Then, the triangles BHA and BQA are equivalent, therefore $AQ \perp BQ$ with hypotenuse AB .

$HQ \parallel AB$, line HQ has to be perpendicular to AC .

AM and HD are intersecting on Bx at Q .



Exercise 10

Given an isosceles triangle ABC with a peak at A . Extend base BC the length $CD = AB$, then extend AB of a length $BE = \frac{1}{2}BC$, at the end draw a line EHF , H is the middle point of BC and F is located on AD .

- Prove that $\widehat{ADB} = \frac{1}{2}\widehat{ABC}$
- Prove that $EA = HD$
- Prove that $FA = FD = FH$
- Calculate the value of the angles \widehat{AFH} and \widehat{ADB} where $\widehat{BAC} = 58^\circ$.

Solution

- Triangle ABC is isosceles, then $\widehat{ABC} = \widehat{ACB}$

Since $CD = AB = AC$, then $\widehat{CAD} = \widehat{ADC}$

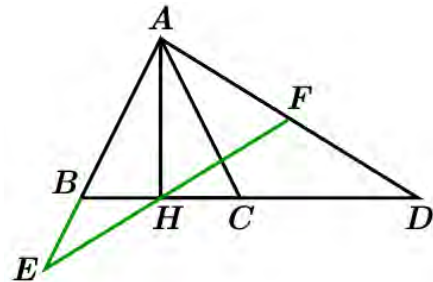
$$2\widehat{ADC} = 180^\circ - \widehat{ACD}$$

$$2\widehat{ADC} = 180^\circ - (180^\circ - \widehat{ACB})$$

$$2\widehat{ADC} = \widehat{ACB}$$

$$\widehat{ADB} = \frac{1}{2}\widehat{ACB}$$

$$= \frac{1}{2}\widehat{ABC}$$



- $BE = \frac{1}{2}BC$ H the middle point of BC
 $= HC$

$$CD = AB$$

$$HC + CD = BE + AB$$

$$EA = HD \quad \checkmark$$

- $\widehat{ADH} = \frac{1}{2}\widehat{ABD}$
 $= \frac{1}{2}(180^\circ - \widehat{HBE})$
 $= \frac{1}{2}(180^\circ - 180^\circ + 2\widehat{BHE})$
 $= \widehat{BHE}$

$$\Rightarrow \underline{FD = FH}$$

$$\widehat{AHF} = 90^\circ - \widehat{FHD}$$

$$= 90^\circ - \widehat{ADH} \quad (\triangle HDA)$$

$$= 90^\circ - (90^\circ - \widehat{HAF})$$

$$= \widehat{HAF}$$

$$\Rightarrow \underline{FA = FH}$$

$$FA = FD = FH \quad \checkmark$$

$$d) \widehat{BAC} = 58^\circ$$

$$\begin{aligned} \widehat{ADB} &= \frac{1}{2} \widehat{ACB} \\ &= \frac{1}{2} \left(\frac{1}{2} (180^\circ - \widehat{BAC}) \right) \\ &= \frac{1}{4} (180^\circ - 58^\circ) \\ &= \frac{122^\circ}{4} \\ &= \frac{61^\circ}{2} \quad \Bigg| \quad \underline{= 30.5^\circ} \end{aligned}$$

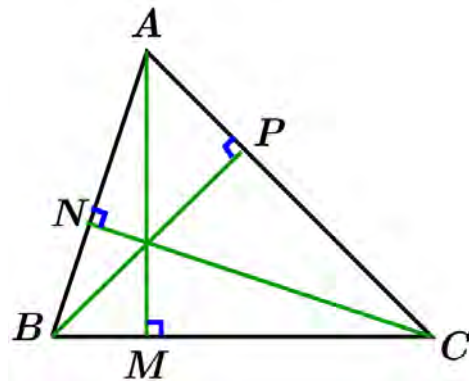
Triangle AFH is isosceles then,

$$\begin{aligned} \widehat{AFH} &= 180^\circ - \widehat{HFD} \\ &= 180^\circ - (180^\circ - 2\widehat{FDH}) \\ &= 2\widehat{FDH} \\ &= 2 \frac{61^\circ}{2} \\ &= 61^\circ \quad \Bigg| \end{aligned}$$

Exercise 11

Demonstrate that the heights of a triangle share the angles of triangle that equal to each other.

Solution



Consider the 2 right triangles APB and ANC , which they have the same angle A .

Therefore, $\widehat{ABP} = \widehat{ACN}$.

Similar, consider the 2 right triangles BPC and AMC , which they have the same angle C .

Therefore, $\widehat{MAC} = \widehat{CBP}$.

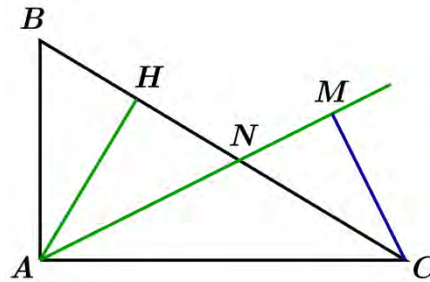
Similar, consider the 2 right triangles BNC and AMB , which they have the same angle B .

Therefore, $\widehat{BCN} = \widehat{BAM}$.

Exercise 12

A right triangle ABC at A where $AB < AC$, drop a perpendicular AH from A to the hypotenuse BC where $HN = HB$. From C drops a perpendicular CM at AN . Demonstrate that BC is the bisector of the angle \widehat{ACM} .

Solution



Consider the 2 right triangles ABC and ABH with a common angle B , then

$$\widehat{BAH} = \widehat{ACB}$$

Given: $HN = HB$, then $\widehat{HAN} = \widehat{BAH} = \widehat{ACB}$

$$\begin{aligned}\widehat{NAC} &= 90^\circ - \widehat{HAB} - \widehat{HAN} \\ &= 90^\circ - 2\widehat{HCA}\end{aligned}$$

Consider the 2 right triangles AHN and CMC , where $\widehat{HNA} = \widehat{MNC}$

Therefore, $\widehat{HAN} = \widehat{NCM}$

Since $\widehat{HAN} = \widehat{ACB}$

Then $\widehat{ACB} = \widehat{MCB}$

Therefore, BC is the bisector of the angle \widehat{ACM}

Exercise 13

On the sides of an angle that it takes the length OA and OB , so that $OA + OB = \ell$ (is given) and construct a parallelogram $OABC$. What is the place of the summit C of parallelogram?

Solution

Let segment OE extension of segment OA such that $OE = \ell$

Let segment OF extension of segment OB such that $OF = \ell$

Then, the triangle OEF is isosceles.

$$\widehat{OEF} = \widehat{OFE} = 90^\circ - \frac{1}{2}\widehat{EOF}$$

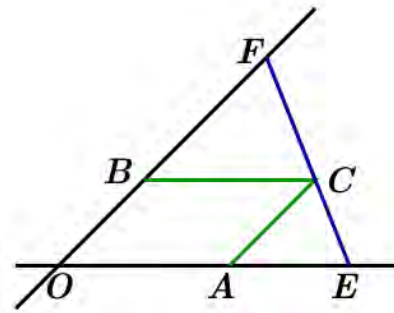
$$OA + OB = \ell$$

$$\begin{cases} OA + AC = \ell \\ OA + AE = \ell \end{cases} \Rightarrow AC = AE$$

$$\begin{cases} OB + BC = \ell \\ OB + BF = \ell \end{cases} \Rightarrow BC = BF$$

$$\widehat{OEF} = \widehat{OFE} = \widehat{FCB} = \widehat{ACE}$$

Therefore, the point C , E , and F are aligned.



Exercise 14

Demonstrate that the sum of distances from a point M on the base BC of an isosceles triangle ABC to the sides equal a constant.

Solution

The shortest distance from a point to a line is the perpendicular from that point to the line.

Therefore, let:

$$MP \perp AB$$

$$MQ \perp AC$$

Let $BH \perp AC$ (Shortest distance from B to side AC .)

Let D be the point of intersection ME and BH .

Let $ME \parallel AC$

Where the point E is the intersection of the lines MD and AB .

Since $MD \parallel AC$ then $\widehat{DMB} = \widehat{ACB}$

Since triangle ABC is an isosceles triangle.

$$\widehat{DMB} = \widehat{ACB} = \widehat{PBM}$$

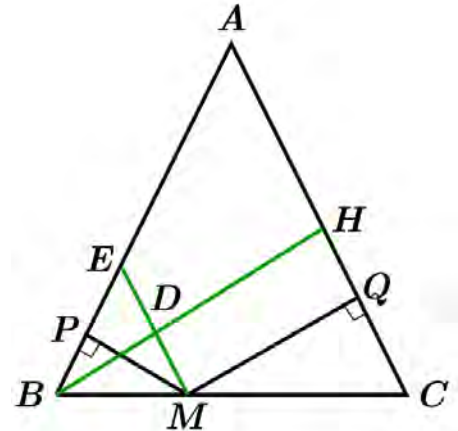
The right triangles BPM and BDM at P & D and have the same hypotenuse, then

$$\Rightarrow |MP| = |BD|$$

$MD \parallel HQ$ and $DH \perp HQ$

$$\Rightarrow |MQ| = |DH|$$

$$\begin{aligned} |MP| + |MQ| &= |BD| + |DH| \\ &= |BH| \\ &= \text{constant} \end{aligned}$$



Exercise 15

Demonstrate that the difference of distances from a point M taken on the extension of the base BC of an isosceles triangle ABC to the sides equal a constant.

Solution

The shortest distance from a point to a line is the perpendicular from that point to the line.

Therefore, let:

$$MP \perp AB$$

$$MQ \perp AC$$

Let $BH \perp AC$ (Shortest distance from B to side AC .)

Let D be the point of intersection ME and BH .

Let $ME \parallel AC$

Where the point E is the intersection of the extensions of the lines MD and AB .

Since $MD \parallel AC$ then $\widehat{DMB} = \widehat{ACB}$

Since triangle ABC is an isosceles triangle.

$$\widehat{DMB} = \widehat{ACB} = \widehat{PBM}$$

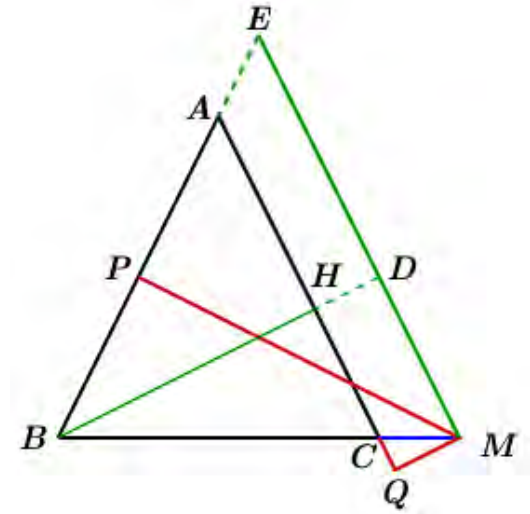
The right triangles BPM and BDM at P & D and have the same hypotenuse, then

$$\Rightarrow |MP| = |BD|$$

$$MD \parallel HQ \text{ and } DH \text{ \& } MQ \perp HQ$$

$$\Rightarrow |MQ| = |DH|$$

$$\begin{aligned} |MP| - |MQ| &= |BD| - |DH| \\ &= |BH| \\ &= \text{constant} \end{aligned}$$

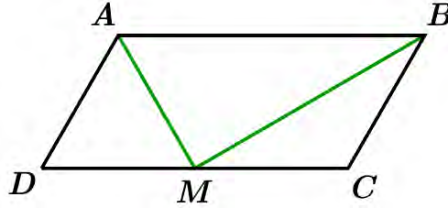


Therefore, the difference of distances from a point M taken on the extension of the base BC of an isosceles triangle ABC to the sides equal a constant.

Exercise 16

Consider a parallelogram $ABCD$ in which $CD = 2AD$. In the joint A and B the middle M of BC .
Prove that the angle \widehat{AMB} is a right angle.

Solution



Since the point M is the middle of side BC , then

$$MD = MC = \frac{1}{2} CD$$

$$\Rightarrow MD = AD = BC$$

Therefore, the triangles ADM and BCM are isosceles at D and C respectively.

Which implies that $MA = MB$

Let O be the middle point of the side AB , and $OA = OB = AD$

O and M are middle of the parallelogram $ABCD$, that implies

$$OM = BC = AD$$

$$\Rightarrow OA = OB = OM$$

The triangle MAB inscribed in a circle with center at O and diameter AB , that will imply that is a right triangle at the point M .

Or

$$\widehat{AMD} = \frac{1}{2} (180^\circ - \widehat{MDA})$$

$$\widehat{BMC} = \frac{1}{2} (180^\circ - \widehat{MCB})$$

$$\widehat{ADM} + \widehat{MCB} = 180^\circ$$

$$\widehat{DMA} + \widehat{AMB} + \widehat{BMC} = 180^\circ$$

$$\widehat{AMB} = 180^\circ - (\widehat{BMC} + \widehat{DMA})$$

$$= 180^\circ - \left(90^\circ - \frac{1}{2} \widehat{MDA} + 90^\circ - \frac{1}{2} \widehat{MCB} \right)$$

$$= \frac{1}{2} (\widehat{MDA} + \widehat{MCB})$$

$$= \frac{1}{2} (180^\circ)$$

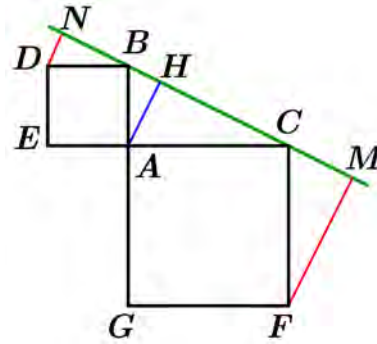
$$= 90^\circ \quad |$$

Exercise 17

From the sides AB and AC of a right triangle ABC at A , draw two squares $ABDE$ and $ACFG$. Then lead DN and FM perpendicular to the line BC .

- Prove that $DN + FM = BC$
- Prove that the points D, A, F on a straight line.
- Prove that the lines DE and FG contribute on the extension of the height AH .

Solution



- a) Let consider the 2 right triangles DNB & BHA at points N & H respectively, with $DB = AB$. Then

$$\begin{aligned}\widehat{HAB} &= 90^\circ - \widehat{ABH} \\ &= 90^\circ - (90^\circ - \widehat{NBD}) \\ &= \widehat{NBD} \\ \Rightarrow \widehat{BDN} &= \widehat{ABH}\end{aligned}$$

\therefore The 2 triangles are equals, which implies that $\underline{DN = BH}$

Similar, for the 2 right triangles CMF & AHC at points M & H respectively, with $AC = CF$. Then

$$\begin{aligned}\widehat{HAC} &= 90^\circ - \widehat{ACH} \\ &= 90^\circ - (90^\circ - \widehat{MCF}) \\ &= \widehat{MCF} \\ \Rightarrow \widehat{ACH} &= \widehat{CFM}\end{aligned}$$

\therefore The 2 triangles are equals, which implies that $\underline{FM = HC}$

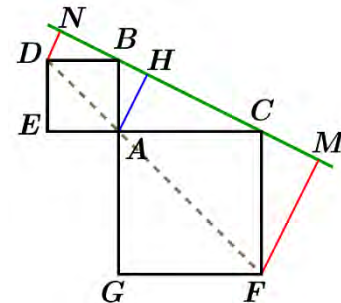
$$\begin{aligned}DN + FM &= BH + HC \\ &= \underline{BC} \quad \checkmark\end{aligned}$$

- b) Since $ABDE$ is a square, then $\widehat{BAD} = 45^\circ$

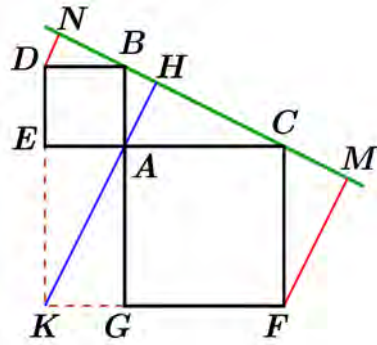
And $ACFG$ is a square, then $\widehat{CAF} = 45^\circ$

$$\begin{aligned}\widehat{DAF} &= \widehat{DAB} + \widehat{BAC} + \widehat{CAF} \\ &= 45^\circ + 90^\circ + 45^\circ \\ &= \underline{180^\circ}\end{aligned}$$

\therefore The points $D, A,$ & F are on a straight line.



- c) Let the point K be the intersection of the extension of the sides DE and FG .
Which will result of $GKEA$ is a rectangle with $AE = GK$ & $EK = AG$



Consider the 2 right triangles BAC & KGA at points A & G respectively with $AE = AB = GK$
 $\widehat{ACB} = \widehat{GAK} = \widehat{ACH}$

From the right triangle AHC :

$$\widehat{HAC} + \widehat{ACH} = 90^\circ$$

$$\rightarrow \widehat{HAC} + \widehat{KAG} = 90^\circ$$

$$\begin{aligned} \widehat{HAC} + \widehat{CAG} + \widehat{KAG} &= (\widehat{HAC} + \widehat{KAG}) + \widehat{CAG} \\ &= 90^\circ + 90^\circ \\ &= 180^\circ \end{aligned}$$

\therefore The points K , A , & H are on a straight line.

Exercise 18

Given a diamond $ABCD$; the peak B and D , the same the perpendiculars BM, BN, DP, DQ on opposite sides. These perpendiculars are intersected at E and F .

Demonstrate that the angles of the quadrilateral $BFDE$ are equals to the diamond and which is a diamond itself.

Solution

From the right triangles BPD & BMD , that implies $\widehat{MBD} = \widehat{PDB}$

$$\Rightarrow \widehat{EBD} = \widehat{EDB}$$

Similar, from the right triangles BND & BQD , that implies $\widehat{NBD} = \widehat{QDB}$

$$\Rightarrow \widehat{FBD} = \widehat{FDB}$$

$$\widehat{EBD} + \widehat{DBF} = \widehat{EDB} + \widehat{BDF}$$

$$\widehat{EBF} = \widehat{EDF}$$

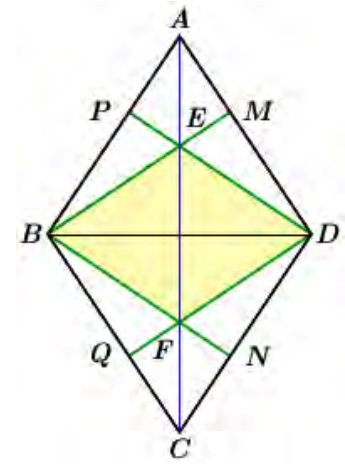
Since, $AC \perp BD$, then $EF \perp BD$

The 2 triangles EBF & EDF have EF as a common side and $\widehat{EBF} = \widehat{EDF}$, then

$$\widehat{BEF} = \widehat{DEF} = \widehat{BFE} = \widehat{DFE}$$

$$\widehat{BED} = \widehat{BFD}$$

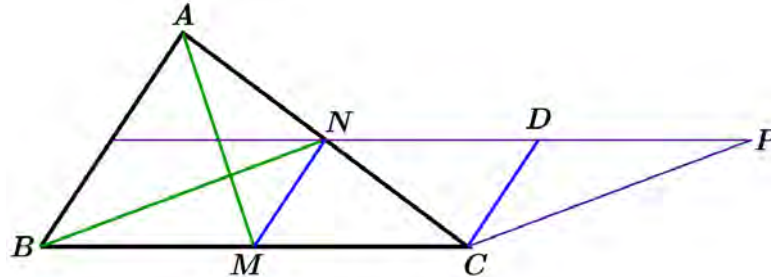
Therefore, the angles of the quadrilateral $BFDE$ are equals to the diamond and which is a diamond itself.



Exercise 19

In a triangle ABC , we trace the median AM and BN and from N a parallel to BC , from C a parallel to BN ; that the two sides intersect at a point P . Let D be the middle point of the segment PN . Prove that CD is parallel to MN .

Solution



Since the points M & N are middle of the sides BC & AC of the triangle ABC , then
 $MN \parallel AB$

Given: $NP \parallel MC$

$BN \parallel CP$

Since M & D are the middle points of the segments BC and NP respectively, then
 $BN \parallel CP \parallel MD$

Therefore, $BNPC$ is a parallelogram, and $MC = ND$.

Since $MC = ND$ & $MN = CD$

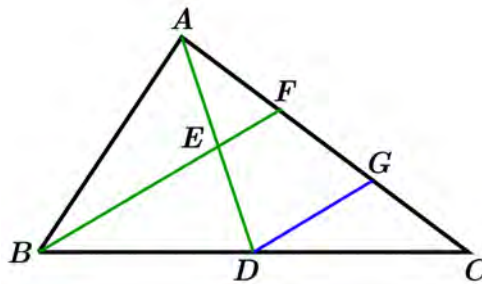
Therefore, $MCDN$ is a parallelogram which implies to CD parallel to MN

Exercise 20

The median AD of a given triangle ABC to the side BC . The same the median BE to the side AC which intersect AC at a point F .

Prove that where $AF = \frac{1}{3} AC$

Solution



Let DG be parallel to segment BEF .

Given: E is the middle point of the segment $AD \Rightarrow AE = ED$

D is the middle point of the segment $BC \Rightarrow BD = DC$

Since $EF \parallel DG$, and $AE = ED$, that implies $AF = FG$

Consider the triangles CDG and CBF :

$EF \parallel DG$, and $CD = DB$, that implies $GC = FG$

That will imply to: $AF = FG = GC$

$$AC = AF + FG + GC$$

$$= 3AF$$

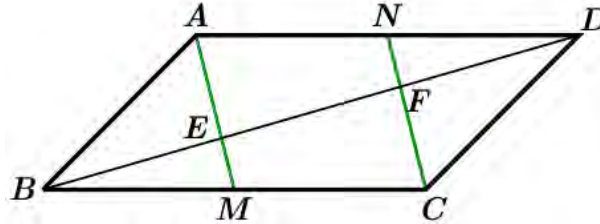
Therefore; $AF = \frac{1}{3} AC$

Exercise 21

In a parallelogram $ABCD$, from the points peak A and C joint the middle of opposite sides at M and N respectively.

Prove that the diagonal BD is divided in three equal parts.

Solution



M is the middle point of the segment BC , then $BM = CM$

N is the middle point of the segment AD , then $NA = ND$

From these, implies that $AM \parallel CN$.

From the triangles BEM & BCF , and since $ME \parallel CF$

It will give us that $BE = EF$

From the triangles DFN & DEA , and since $AE \parallel FN$

It will give us that $\Rightarrow DF = EF$

Therefore, $BE = EF = DF$

$$BD = BE + EF + FD$$

$$= 3BE$$

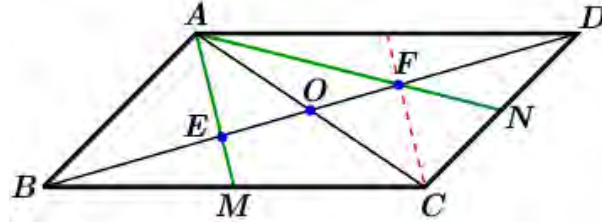
Therefore, the diagonal BD is divided in three equal parts

Exercise 22

In a parallelogram $ABCD$, from the point peak A , extend to the middle of sides BC and DC at M and N respectively.

Prove that the diagonal BD is divided in three equal parts.

Solution



Let a point E be the intersection of the segments AM & BD .

Let a point F be the intersection of the segments AN & BD .

Let O be the intersection of both diagonal AC & BD .

From the triangles BEM & BCF , and since $ME \parallel CF$

$$\Rightarrow BE = EF$$

Similar, $\Rightarrow DF = EF$

$$BO = OF \rightarrow OE = OF$$

$$BO = BE + EO$$

$$= BE + \frac{1}{2}BE$$

$$= \frac{3}{2}BE$$

$$BE = \frac{2}{3}BO$$

$$= \frac{2}{3}\left(\frac{1}{2}BD\right)$$

$$= \frac{1}{3}BD$$

$$DF = \frac{2}{3}DO$$

$$= \frac{2}{3}\left(\frac{1}{2}BD\right)$$

$$= \frac{1}{3}BD$$

Therefore, the diagonal BD is divided in three equal parts

Exercise 23

Consider a triangle ABC with a bisector AF of the angle A . by F , we lead FE parallel to AB , and by E we lead ED parallel to BC .

Prove that $AE = BD$

Solution

Given: $\widehat{EAF} = \widehat{FAB}$

Since $FE \parallel AB$, then

$$\widehat{FEC} = \widehat{BAE} = 2\widehat{EAF}$$

$$\begin{aligned}\widehat{AEF} &= 180^\circ - \widehat{FEC} \\ &= 180^\circ - 2\widehat{EAF}\end{aligned}$$

Consider the triangle AEF :

$$\begin{aligned}\widehat{EAF} + \widehat{EFA} + \widehat{AEF} &= 180^\circ \\ \widehat{EAF} + \widehat{EFA} + 180^\circ - 2\widehat{EAF} &= 180^\circ \\ \widehat{EFA} - \widehat{EAF} &= 0^\circ \\ \widehat{EFA} &= \widehat{EAF}\end{aligned}$$

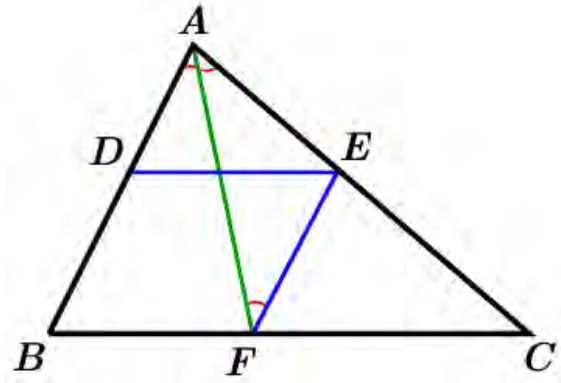
\therefore Triangle AEF is isosceles

$$\Rightarrow \underline{AE = EF}$$

Given $DE \parallel BF$ & $FE \parallel DB$

$FEDB$ is a parallelogram.

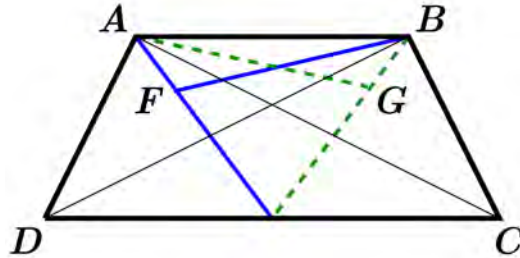
Then, $\underline{EF = DB = AE}$



Exercise 24

Given an isosceles trapezoid $ABCD$ ($AD = BC$) with diagonals AC and BD . The bisector of angles \widehat{DAB} and \widehat{DBA} intersect in F , and the bisector of angles \widehat{CBA} and \widehat{CAB} intersect in G . Demonstrate that FG is parallel to AB .

Solution



Consider the 2 triangles ABD & ABC :

- Both has the AB as common
- $AD = BC$

That implies to: $\widehat{ABD} = \widehat{CAB}$

Since BF is the bisector of the angle \widehat{ABD}

$$\widehat{ABF} = \widehat{FBD}$$

$$\begin{aligned} \Rightarrow \widehat{ABF} &= \frac{1}{2} \widehat{ABD} \\ &= \frac{1}{2} \widehat{CAB} \\ &= \frac{1}{2} (2\widehat{BAG}) \\ &= \widehat{BAG} \end{aligned}$$

$$\widehat{ABF} = \widehat{BAG} \quad |$$

From the 2 triangles AFB & AGB

- Both has the AB as common
- $\widehat{ABF} = \widehat{BAG}$

$$\underline{FG \parallel AB} \quad |$$

Exercise 25

Let M and N be the middle points of the bases AB and CD of a trapezoid $ABCD$.

Let P and Q be the middle points of the diagonals AC and BD respectively.

Demonstrate that the angles \widehat{M} and \widehat{N} of quadrilateral $MNPQ$ are equals to the angle formed by extending the sides not parallel to BC and AD , where intersect at point E .

Solution

Since N is the mid-point of the side DC , and
 P is the mid-point of the side AC , then

$$\Rightarrow NP \parallel AD$$

Since M is the mid-point of the side AB , and
 Q is the mid-point of the side DB , then

$$\Rightarrow QM \parallel AD$$

$$\therefore NP \parallel QM \parallel AE$$

Since N is the mid-point of the side DC , and
 Q is the mid-point of the side DB , then

$$\Rightarrow NQ \parallel CB$$

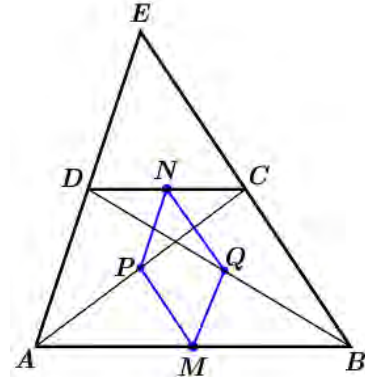
Since M is the mid-point of the side AB , and
 P is the mid-point of the side AC , then

$$\Rightarrow MP \parallel CB$$

$$\therefore NQ \parallel MP \parallel BE$$

$$\rightarrow \begin{cases} NP \parallel QM \\ NQ \parallel PM \end{cases}$$

$$\therefore \underline{\widehat{PMQ} = \widehat{PNQ}}$$



Exercise 26

In a triangle ABC , the medians segment BM and CN intersect in right angles and the measurement are 3 and 6 units respectively.

1. Construct a geometrical to the triangle ABC .
2. In the trace of third median AP which leads MN extension such the distance $MD = MN$, which lead to the segments AD and PD . Calculate AD and DP .
3. What is the nature of the triangle APD ?

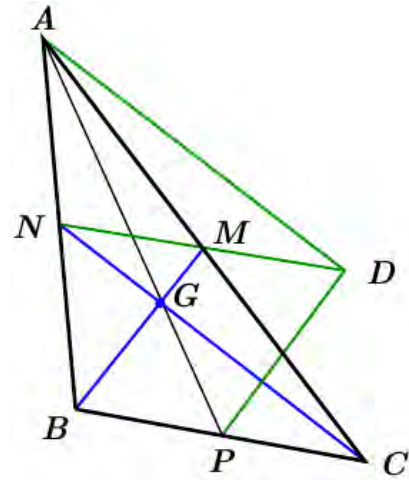
Solution

1. Since M and N are the middle point of the side's AC & AB , then

$$\begin{aligned} BG &= \frac{2}{3} BM \\ &= \frac{2}{3}(3) \\ &= 2 \text{ units} \end{aligned}$$

Similar,

$$\begin{aligned} CG &= \frac{2}{3} CN \\ &= \frac{2}{3}(6) \\ &= 4 \text{ units} \end{aligned}$$



Wish, we lead to: $GM = 1$ & $GN = 2$

We can construct 2 perpendicular lines intersect at a point G , then we use to measure the distance from the point G to get the points B , C , M , & N .

By extending the segment BN and CM with equal distance and which it will intersect at point A .

2. Since $ND \parallel BC$ & $MD = MN$

The parallelogram $BPDM$, $BP = MD = MN$

Then $PD = MB = 3 \text{ units}$

$AD \parallel CN$ and M is the intersection of the diagonals of the parallelogram $ADCN$, then

$AD = CN = 6 \text{ units}$

3. $PD \parallel BN$ & $MB \perp CN$, then $PD \perp CN$

$AD \parallel CN$ & $PD \perp CN$, then $AD \perp PD$

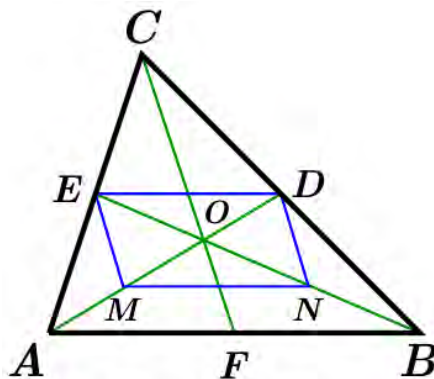
Therefore, the triangle ADP is right triangle at point D .

Exercise 27

Inside the triangle ABC , the median AD , BE , and CF intersect at a point O . We take M the middle point of the segment OA , N the middle point of segment OB .

Show that $DEMN$ is a parallelogram.

Solution



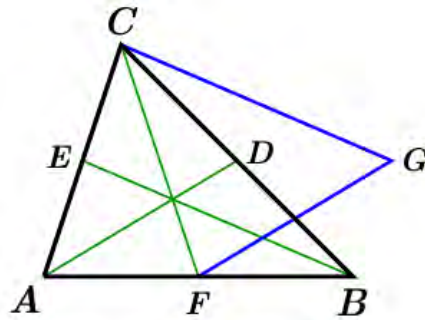
DE & MN are parallel to AB and equals to $\frac{1}{2}|AB|$

That implies to $ME \parallel DN$.

Therefore, $DEMN$ is a parallelogram

Exercise 28

Inside the triangle ABC , the median AD , BE , and CF intersect at a point O . From the point F , draw FG parallel to AD and are equals, then joint A to G .



Show that $CG = BE$.

Solution

Given: $FG \parallel AD$ & $FG = AD$

Then, the quadrilateral $AFGD$ is a parallelogram which it results to $DG \parallel AF$ & $DG = AF$.

$\rightarrow DG \parallel BF$

Since F is the mid-point of the side AB , then $AF = DG = FB$.

Then, the quadrilateral $BFDG$ is a parallelogram which it results to $FD \parallel BG$ & $DF = GB$.

So, $FD \parallel BG \parallel CE$

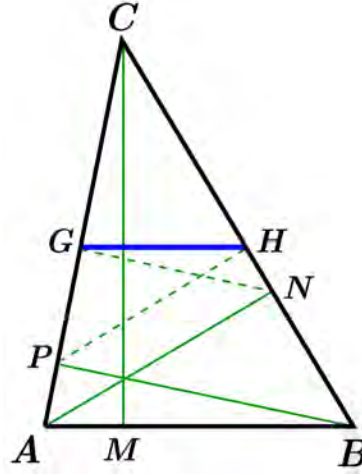
Given that D & F are midpoints, then $DF = \frac{1}{2}AC = CE$

And $CE \parallel BG$ & $DF = CE$, then $BGCE$ is a parallelogram.

Therefore, $CG = BE$

Exercise 29

The height of a triangle ABC (each of the three lines passing through a vertex of the triangle and perpendicular to the opposite side to this vertex) are AN , BP , CM .



From P , let PH perpendicular to BC , same from N , let NG perpendicular to AC . Show that GH is parallel to AB .

Solution

Let the point O be the middle of the segment AB .

Then O is the center of the 2 triangles ANB & APB .

The triangle OBN is isosceles, implies to $\widehat{ONB} = \widehat{OBN}$

The triangle OPA is isosceles, implies to $\widehat{OPA} = \widehat{OAP}$

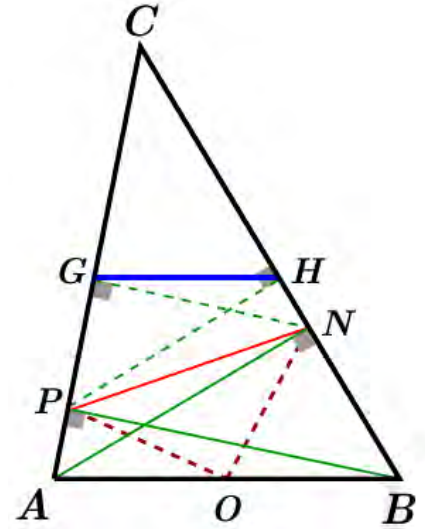
$$\begin{aligned}\widehat{PON} &= 180^\circ - (\widehat{NOB} + \widehat{POA}) \\ &= 180^\circ - (180^\circ - 2\widehat{NBO} + 180^\circ - 2\widehat{OAP}) \\ &= 2\widehat{B} + 2\widehat{A} - 180^\circ\end{aligned}$$

Consider the triangle PON with $OP = ON$, then

$$\begin{aligned}\widehat{OPN} &= \widehat{ONP} \\ \widehat{OPN} &= \frac{1}{2}(180^\circ - \widehat{PON}) \\ &= \frac{1}{2}(180^\circ - 2\widehat{B} - 2\widehat{A} + 180^\circ) \\ &= 180^\circ - \widehat{B} - \widehat{A}\end{aligned}$$

$$\begin{aligned}\widehat{APN} &= \widehat{APO} + \widehat{OPN} \\ &= \widehat{A} + 180^\circ - \widehat{B} - \widehat{A} \\ &= 180^\circ - \widehat{B}\end{aligned}$$

$$\begin{aligned}\widehat{CPN} &= 180^\circ - \widehat{APN} \\ &= 180^\circ - 180^\circ + \widehat{B} \\ &= \widehat{B}\end{aligned}$$



From the 2 right triangles CHP & CGN

$$\widehat{HPC} = \widehat{GNC}$$

$$\begin{aligned}\widehat{GHN} &= 180^\circ - \widehat{HGN} - \widehat{HNG} \\ &= 180^\circ - \widehat{HGN} - \widehat{CPH}\end{aligned}$$

$$180^\circ - \widehat{GHC} = 180^\circ - \widehat{HGN} - \widehat{CPH}$$

$$\widehat{GHC} = \widehat{HGN} + \widehat{CPH} \quad |$$

Let Q be the middle point of the segment PN .

Since PGN & PHN are right triangle with the same hypothesis.

Then, the triangles HQN & GQN are isosceles.

$$\widehat{H} = \widehat{N} \quad \& \quad \widehat{G} = \widehat{P}$$

$$\begin{aligned}\widehat{GQH} &= 180^\circ - (180^\circ - 2\widehat{P} + 180^\circ - 2\widehat{N}) \\ &= 2\widehat{P} + 2\widehat{N} - 180^\circ\end{aligned}$$

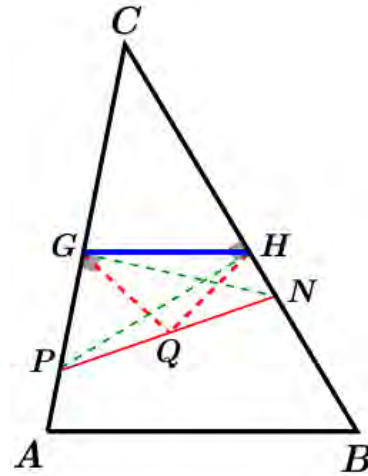
$$\text{Since } QG = QH \Rightarrow \widehat{QGH} = \widehat{QHG}$$

$$\begin{aligned}\widehat{QGH} &= \frac{1}{2}(180^\circ - \widehat{GQH}) \\ &= \frac{1}{2}(180^\circ - 2\widehat{P} - 2\widehat{N} + 180^\circ) \\ &= 180^\circ - \widehat{P} - \widehat{N}\end{aligned}$$

$$\begin{aligned}\widehat{HGN} &= \widehat{QGH} - \widehat{QGN} \\ &= 180^\circ - \widehat{P} - \widehat{N} - 90^\circ + \widehat{QGP} \\ &= 90^\circ - \widehat{P} - \widehat{N} + \widehat{P} \\ &= 90^\circ - \widehat{N} \\ &= \widehat{NPH}\end{aligned}$$

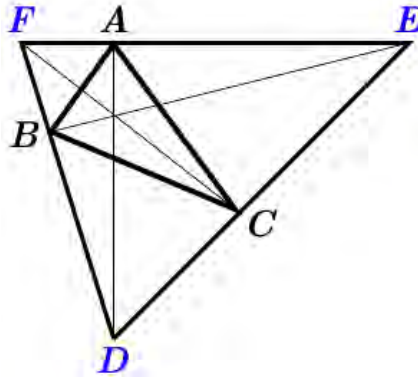
$$\begin{aligned}\widehat{CHG} &= \widehat{HGN} + \widehat{CPH} \\ &= \widehat{NPH} + \widehat{CPH} \\ &= \widehat{NPC} \\ &= \widehat{B} \quad | \end{aligned}$$

Therefore, $GH \parallel AB$



Exercise 30

From the top of a triangle, we lead the external bisectors of angles such that formed an outside triangle such that the top of the first are the feet of the second heights.



Solution

Let the triangle DEF where DA , BE , and FC are heights (perpendicular to sides).

Let the point M be the middle points of the same hypotenuse of the 2 right triangles FAD & FCD . Then, the 2 triangles inscribed the same circle with the center at point M .

$$MF = MA = MC = MD$$

$$\widehat{MFA} = \widehat{MAF} \quad \& \quad \widehat{MCD} = \widehat{MDC}$$

Therefore, the triangle AMC is isosceles.

$$MA = MC \quad \& \quad \widehat{MAC} = \widehat{ACM}$$

$$\begin{aligned} \widehat{AMC} &= 180^\circ - (\widehat{FMA} + \widehat{CMD}) \\ &= 180^\circ - (180^\circ - 2\widehat{F} + 180^\circ - 2\widehat{D}) \\ &= \underline{2\widehat{F} + 2\widehat{D} - 180^\circ} \end{aligned}$$

$$\begin{aligned} \widehat{ACM} &= \frac{1}{2}(180^\circ - \widehat{AMC}) \\ &= \frac{1}{2}(180^\circ + 180^\circ - 2\widehat{F} - 2\widehat{D}) \\ &= \underline{180^\circ - \widehat{F} - \widehat{D}} \end{aligned}$$

$$\begin{aligned} \widehat{DCA} &= \widehat{DCM} + \widehat{MCA} \\ &= \widehat{D} + 180^\circ - \widehat{F} - \widehat{D} \end{aligned}$$

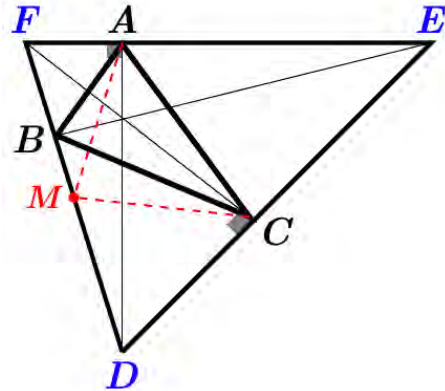
$$180^\circ - \widehat{ACE} = 180^\circ - \widehat{F}$$

$$\underline{\widehat{ACE} = \widehat{F}}$$

Similar,

Let the point N be the middle points of the same hypotenuse of the 2 right triangles FCE & FBE . Then, the 2 triangles inscribed the same circle with the center at point N .

$$NF = NB = NC = NE$$



$$\widehat{NBF} = \widehat{BFN} \quad \& \quad \widehat{NEC} = \widehat{NCE}$$

Therefore, the triangle NBC is isosceles.

$$MA = MC \quad \& \quad \widehat{MAC} = \widehat{ACM}$$

$$\begin{aligned} \widehat{BNC} &= 180^\circ - (\widehat{FNB} + \widehat{CNE}) \\ &= 180^\circ - (180^\circ - 2\hat{F} + 180^\circ - 2\hat{E}) \\ &= \underline{2\hat{F} + 2\hat{E} - 180^\circ} \end{aligned}$$

$$\begin{aligned} \widehat{BCN} &= \frac{1}{2}(180^\circ - \widehat{BNC}) \\ &= \frac{1}{2}(180^\circ + 180^\circ - 2\hat{F} - 2\hat{E}) \\ &= \underline{180^\circ - \hat{F} - \hat{E}} \end{aligned}$$

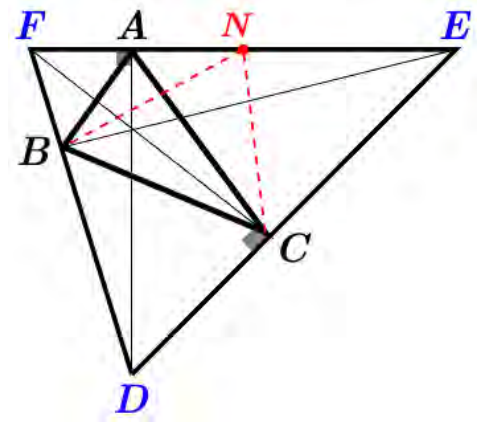
$$\begin{aligned} \widehat{BCE} &= \widehat{BCN} + \widehat{NCE} \\ &= \hat{E} + 180^\circ - \hat{F} - \hat{E} \end{aligned}$$

$$180^\circ - \widehat{BCD} = 180^\circ - \hat{F}$$

$$\underline{\widehat{BCD} = \hat{F}}$$

Then, $\widehat{ACE} = \hat{F} = \widehat{BCD}$

Therefore, CF is the interior bisector of \widehat{BCA} and DCE is the exterior bisector.



$$\begin{aligned} \widehat{MAC} &= \frac{1}{2}(180^\circ - \widehat{AMC}) \\ &= \frac{1}{2}(180^\circ + 180^\circ - 2\hat{F} - 2\hat{D}) \\ &= \underline{180^\circ - \hat{F} - \hat{D}} \end{aligned}$$

$$\begin{aligned} \widehat{FAC} &= \widehat{FAM} + \widehat{MAC} \\ &= \hat{F} + 180^\circ - \hat{F} - \hat{D} \end{aligned}$$

$$180^\circ - \widehat{CAE} = 180^\circ - \hat{D}$$

$$\underline{\widehat{CAE} = \hat{D}}$$

Let the point P be the middle points of the same hypotenuse of the 2 right triangles DAE & BDE . Then, the 2 triangles inscribed the same circle with the center at point P .

$$PE = PA = PB = PD$$

$$\widehat{PAE} = \widehat{PAE} \quad \& \quad \widehat{PBD} = \widehat{PDB}$$

Therefore, the triangle APB is isosceles.

$$PA = PB \quad \& \quad \widehat{PAB} = \widehat{PBA}$$

$$\begin{aligned}
\widehat{APB} &= 180^\circ - (\widehat{DPB} + \widehat{APE}) \\
&= 180^\circ - (180^\circ - 2\widehat{D} + 180^\circ - 2\widehat{E}) \\
&= \underline{2\widehat{D} + 2\widehat{E} - 180^\circ}
\end{aligned}$$

$$\begin{aligned}
\widehat{PAB} &= \frac{1}{2}(180^\circ - \widehat{APB}) \\
&= \frac{1}{2}(180^\circ + 180^\circ - 2\widehat{D} - 2\widehat{E}) \\
&= \underline{180^\circ - \widehat{D} - \widehat{E}}
\end{aligned}$$

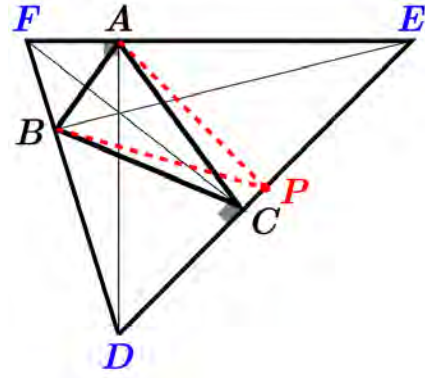
$$\begin{aligned}
\widehat{BAE} &= \widehat{PAE} + \widehat{PAB} \\
&= \widehat{E} + 180^\circ - \widehat{D} - \widehat{E}
\end{aligned}$$

$$180^\circ - \widehat{FAB} = 180^\circ - \widehat{D}$$

$$\underline{\widehat{FAB} = \widehat{D}}$$

Then, $\widehat{FAB} = \widehat{D} = \widehat{CAE}$

Therefore, AD is the interior bisector of \widehat{BAC} and FAE is the exterior bisector.



$$\begin{aligned}
\widehat{NBC} &= \frac{1}{2}(180^\circ - \widehat{BNC}) \\
&= \frac{1}{2}(180^\circ + 180^\circ - 2\widehat{F} - 2\widehat{E}) \\
&= \underline{180^\circ - \widehat{F} - \widehat{E}}
\end{aligned}$$

$$\begin{aligned}
\widehat{CBF} &= \widehat{CBN} + \widehat{NBF} \\
&= 180^\circ - \widehat{F} - \widehat{E} + \widehat{F}
\end{aligned}$$

$$180^\circ - \widehat{CBD} = 180^\circ - \widehat{E}$$

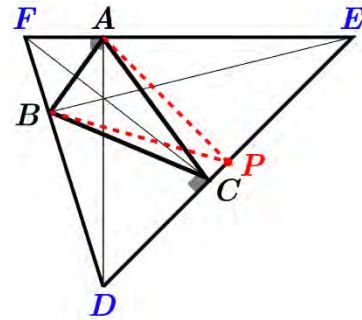
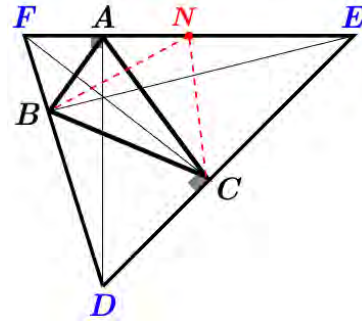
$$\underline{\widehat{CBD} = \widehat{E}}$$

$$\begin{aligned}
\widehat{PBA} &= \frac{1}{2}(180^\circ - \widehat{APB}) \\
&= \frac{1}{2}(180^\circ + 180^\circ - 2\widehat{D} - 2\widehat{E}) \\
&= \underline{180^\circ - \widehat{D} - \widehat{E}}
\end{aligned}$$

$$\begin{aligned}
\widehat{ABD} &= \widehat{DBP} + \widehat{PBA} \\
&= \widehat{D} + 180^\circ - \widehat{D} - \widehat{E}
\end{aligned}$$

$$180^\circ - \widehat{ABF} = 180^\circ - \widehat{E}$$

$$\underline{\widehat{ABF} = \widehat{E}}$$



Then, $\widehat{CBD} = \widehat{E} = \widehat{ABF}$

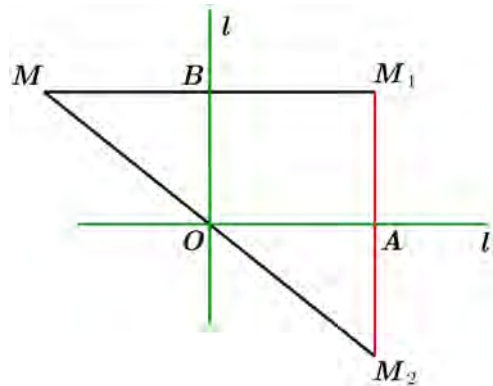
Therefore, BE is the interior bisector of \widehat{ABC} and DBF is the exterior bisector.

Exercise 31

Consider a point O on a vertical line ℓ , a point M outside the line ℓ . We take the symmetries M_1 and M_2 from M across the line ℓ and the point O , respectively.

Demonstrate that the points M_1 and M_2 are symmetries with regard to a line perpendicular to the line ℓ_1 passing through the point O .

Solution



Since M_1 is the symmetry of M across the line ℓ , let B the middle point of the segment MM_1 .

That implies to: $BM = BM_1$.

Similarly, the point O is the middle point of the segment MM_2 .

That implies to: $OM = OM_2$.

Let A be the point intersection of the segment M_1M_2 and line ℓ_1 .

Since $MM_1 \perp \ell$ and $\ell \perp \ell_1 \Rightarrow MM_1 \parallel \ell_1$

From the right triangle MM_1M_2 Since O is the middle of MM_2 and $OA \perp MM_1$.

Therefore, the point A the middle point of the segment M_1M_2

Exercise 32

In a quadrilateral $ABCD$ (Kite), the sides $AB = AD$, $\angle A = 135^\circ$ and $\angle B = \angle D = 90^\circ$.

1. Prove the symmetry in the figure.
2. Prove that the middles of the sides are the top of rectangle.
3. Prove there exists an interior of the given quadrilateral a point equidistant of 4 sides; determine these points.
4. On the same exterior bisector of angles A, B, C, D ; they formed a quadrilateral

Solution

$$\begin{aligned} 1. \quad C &= 180^\circ - 135^\circ \\ &= 45^\circ \end{aligned}$$

Since $AB = AD$, then the side AC is the bisector of \widehat{BAD} .

The 2 right triangles ABC & ADC are rights on B and D respectively, with same hypotenuse AC , therefore the figure is symmetric about the side AC .

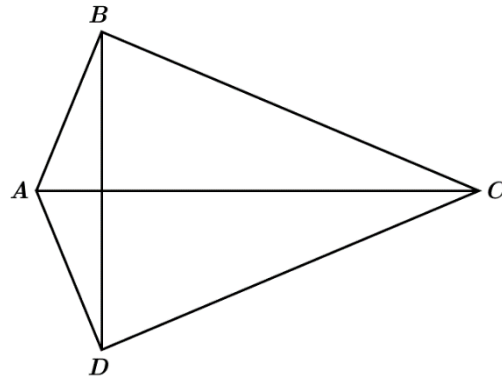
$$\begin{aligned} \widehat{BAC} &= \frac{135^\circ}{2} \\ &= 67.5^\circ \end{aligned}$$

$$\begin{aligned} \widehat{BCA} &= \frac{45^\circ}{2} \\ &= 22.5^\circ \end{aligned}$$

$$\begin{aligned} \widehat{ABD} &= 90^\circ - 67.5^\circ \\ &= 22.5^\circ \end{aligned}$$

$$\widehat{ADB} = 22.5^\circ$$

$$\begin{aligned} \widehat{DBC} &= 90^\circ - 22.5^\circ \\ &= 67.5^\circ \end{aligned}$$



2. Since $AB = AD$, then the triangle BAD is isosceles.

Let M and Q the middle points of the sides AB and AD respectively.

Therefore, the segment $MQ \parallel BD$.

Similar, the triangle BCD is isosceles, since $CB = CD$.

Let N and P the middle points of the sides CA and CD respectively.

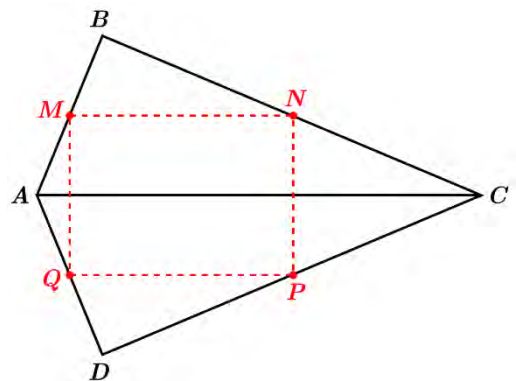
Therefore, the segment $NP \parallel BD$.

The points M and N are the middle points of the sides AB and CB respectively in the right triangle ABC .

Therefore, the segment $MN \parallel AC$.

The points P and Q are the middle points of the sides DC and DA respectively in the right triangle ADC .

Therefore, the segment $PQ \parallel AC$.



$$PQ \parallel MN \parallel AC \quad \& \quad MQ \parallel NP \parallel BD$$

That implies to $MQ \parallel NP$ and $MN \parallel PQ$, which implies that $MNPQ$ is a parallelogram.

From the quadrilateral $ABCD$, $AC \perp BD$

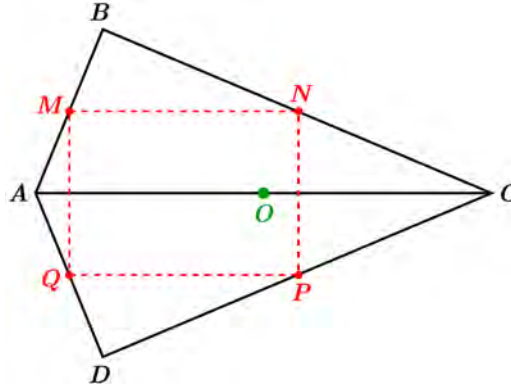
Therefore, $MNPQ$ is rectangular.

3. Let the point O be the middle point of the segment AC .

Given that the 2 right triangles ABC & ADC have the same hypothesis segment AC .

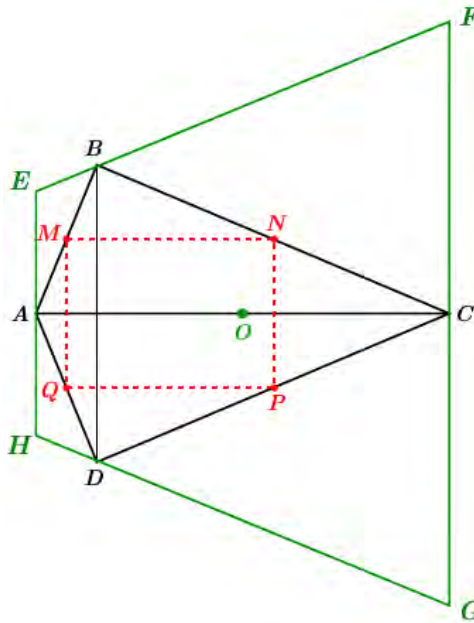
The 4 point are inscribed in a circle with diameter AC where the radius is:

$$OA = OB = OC = OD$$



Therefore, the point O is an interior of the given quadrilateral which is a point equidistant of 4 sides.

4. Let two segments pass through the points A and C perpendicular to the segment AC , which are parallel to segment BD .



Let the segment EF passes through the point B such that:

$$\begin{aligned} \widehat{EBA} &= \widehat{FBC} \\ &= \frac{90^\circ}{2} \\ &= 45^\circ \end{aligned}$$

$$\begin{aligned}\widehat{EAB} &= 90^\circ - \widehat{BAC} \\ &= 90^\circ - 67.5^\circ \\ &= \underline{22.5^\circ}\end{aligned}$$

$$\begin{aligned}\widehat{AEB} &= 180^\circ - \widehat{EBA} - \widehat{BAE} \\ &= 180^\circ - 45^\circ - 22.5^\circ \\ &= \underline{112.5^\circ}\end{aligned}$$

$$\begin{aligned}\widehat{BCF} &= 90^\circ - \widehat{ACB} \\ &= 90^\circ - 22.5^\circ \\ &= \underline{67.5^\circ}\end{aligned}$$

$$\begin{aligned}\widehat{BFC} &= 180^\circ - \widehat{FBC} - \widehat{BCF} \\ &= 180^\circ - 45^\circ - 67.5^\circ \\ &= \underline{67.5^\circ}\end{aligned}$$

Exercise 33

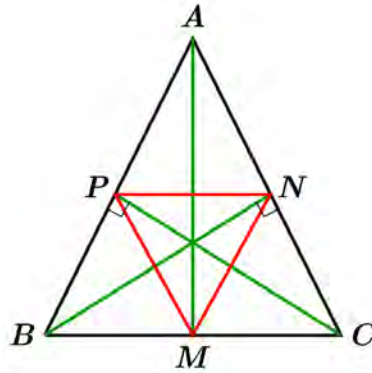
Given an isosceles triangle ABC with a peak at A with angle of 47° .

The height of a triangle ABC of each of the three lines passing through a vertex of the triangle and perpendicular to the opposite side to this vertex, are AM , BN , CP .

What is the angle of the peak of the isosceles triangle MNP ?

Solution

The triangle ABC is isosceles triangle at A then the height AM is the bisector. So, the point M is the middle point of the side BC .



Since the 2 right triangles BPC & BNC have the same hypothesis segment BC .

There are inscribed in a circle with diameter BC with the center at point M .

Therefore, the triangle MNP is isosceles with a peak at the point M .

$$C = \widehat{MNC} = B = \widehat{BPM}$$

$$\begin{cases} \widehat{BMP} = 180^\circ - 2B \\ \widehat{CMN} = 180^\circ - 2C \end{cases}$$

$$\begin{aligned} B &= \frac{180^\circ - 47^\circ}{2} \\ &= \frac{133^\circ}{2} \end{aligned}$$

$$\begin{aligned} \widehat{PMN} &= 180^\circ - \widehat{BMP} - \widehat{CMN} \\ &= 180^\circ - 180^\circ + 2B - 180^\circ + 2B \\ &= 4B - 180^\circ \\ &= 4\left(\frac{133^\circ}{2}\right) - 180^\circ \\ &= 266^\circ - 180^\circ \\ &= 86^\circ \end{aligned}$$

Exercise 34

The angles A , B , and C of a triangle are:

$$A = 68^\circ \quad B = 62^\circ \quad C = 50^\circ$$

Let the point H be the intersection point of the heights of the triangle ABC .

The bisectors inside the triangle BHC intersect at a point O .

Find the angles: \widehat{BOH} , \widehat{HOC} , and \widehat{COB}

Solution

Triangle BNC :

$$\begin{aligned}\widehat{NBC} &= 90^\circ - C \\ &= 90^\circ - 50^\circ \\ &= 40^\circ\end{aligned}$$

Triangle BPC :

$$\begin{aligned}\widehat{PCB} &= 90^\circ - B \\ &= 90^\circ - 62^\circ \\ &= 28^\circ\end{aligned}$$

Triangle BHC :

$$\begin{aligned}\widehat{BHC} &= 180^\circ - C - \widehat{BPC} \\ &= 180^\circ - 40^\circ - 28^\circ \\ &= 112^\circ\end{aligned}$$

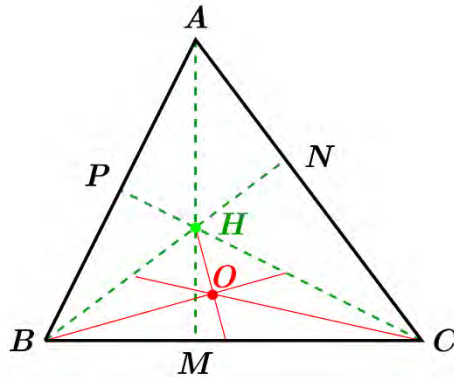
$$\begin{aligned}\widehat{OCB} &= \frac{1}{2} \widehat{BCP} \\ &= \frac{1}{2} (28^\circ) \\ &= 14^\circ\end{aligned}$$

$$\begin{aligned}\widehat{OBC} &= \frac{1}{2} \widehat{NBC} \\ &= \frac{1}{2} (40^\circ) \\ &= 20^\circ\end{aligned}$$

Triangle BOC :

$$\begin{aligned}\widehat{COB} &= 180^\circ - \widehat{OCB} - \widehat{OBC} \\ &= 180^\circ - 14^\circ - 20^\circ \\ &= 146^\circ\end{aligned}$$

$$\begin{aligned}\widehat{BHO} &= \frac{1}{2} \widehat{BHC} \\ &= \frac{1}{2} (112^\circ) \\ &= 56^\circ\end{aligned}$$



Triangle BHO :

$$\begin{aligned}\widehat{HOB} &= 180^\circ - \widehat{OBH} - \widehat{OHB} \\ &= 180^\circ - 20^\circ - 56^\circ \\ &= \underline{104^\circ} \quad | \end{aligned}$$

Exercise 35

On the sides of an angle A , where the longest of $AB = AB'$ is $AC = AC'$

- a) Prove that $BC' = B'C$
- b) Let a point I be the intersection of the sides BC' and $B'C$. Prove that $IB = IB'$ and $IC = IC'$.
- c) Prove that the point I is located on the bisector of angle A .

Solution

- a) From the 2 triangles $AB'C$ and ABC' , they have common:

- ✓ Common angle A .
- ✓ $AB = AB'$
- ✓ $AC = AC'$

The 2 triangles are equivalent.

Therefore, that implies to $BC' = B'C$.

- b) From the 2 triangles IBC and $IB'C'$.

- ✓ $BC = B'C'$
- ✓ $\widehat{BIC} = \widehat{B'IC'}$
- ✓ $C = C'$

That implies to the 2 triangles IBC and $IB'C'$ are equivalent

Therefore, $IB = IB'$ and $IC = IC'$

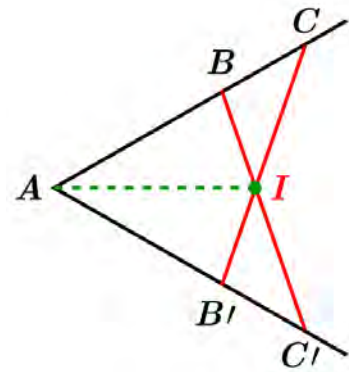
- c) Consider the 2 triangles ABI and $AB'I$

- ✓ $AB = AB'$
- ✓ $IB = IB'$
- ✓ Common side AI

That implies to

$$\widehat{IAB} = \widehat{IAB'}$$

Therefore, AI is the **bisector** of angle A .



Exercise 36

Given two adjacent angles of 60° , \widehat{AOB} and \widehat{BOC} .

A point Z is located inside the angle \widehat{BOC} , where leads the perpendicular ZP , ZT , ZS on OC , OA , and OB .

Prove that $ZT = ZS + ZP$

Solution

Let the segment DE passing through the point Z and parallel to OA .

So that,

$$\widehat{CDE} = 120^\circ$$

$$\begin{aligned}\widehat{ODE} &= 180^\circ - 120^\circ \\ &= 60^\circ\end{aligned}$$

And it is given that

$$\widehat{DOE} = 60^\circ$$

Which implies that the triangle ODE is equilateral.

Let OF perpendicular to the side OE .

And since all the heights in the equilateral ODE triangle are equals ($OO' = DF$).

Since side ZT is perpendicular to the line OA , then the points $OO'ZT$ is a rectangle with $OO' = ZT$

That implies to $DF = ZT$

Let the point G on the segment DF such that ZG perpendicular to DF and $ZS = GF$ from the rectangle $ZSFG$.

From the right triangle DGZ at G .

$$\widehat{GDZ} = 30^\circ \text{ (from the equilateral } ODE \text{ triangle).}$$

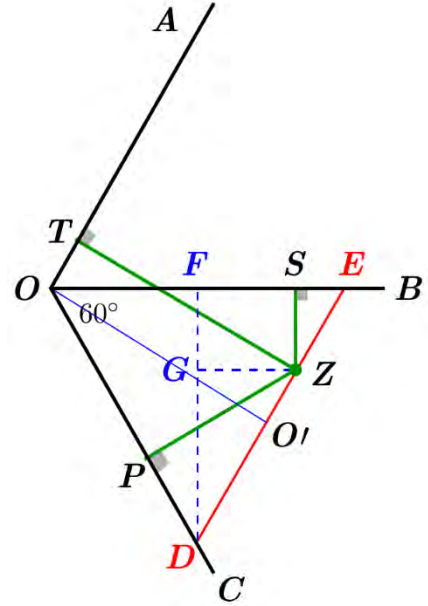
$$\Rightarrow \widehat{GZD} = 60^\circ$$

Consider the 2 right triangles DPZ and DGZ , they have

- ✓ ZD common hypotenuse
- ✓ $\widehat{P} = \widehat{G} = 90^\circ$
- ✓ $\widehat{PDZ} = \widehat{GZD} = 60^\circ$

That imply to $\underline{GD = ZP}$

$$\begin{aligned}ZT &= \overline{DF} \\ &= \overline{DG} + \overline{GF} \\ &= \underline{ZP + ZS}\end{aligned}$$



Exercise 37

Let two perpendicular lines, $x'ox$ & $y'oy$. Consider a point P in xoy plane.

Let PA be perpendicular to ox and PB be perpendicular to oy .

From the point P , a secant line PZ intersect $x'ox$ & $y'oy$, a point Q on $x'x$ and R on $y'y$ such that $QS = PR$.

Let SD and SC be perpendicular to $x'ox$ & $y'oy$ respectively.

- Prove that BD and AC are parallel
- Prove that $\overline{PS} = \overline{BD} + \overline{AC}$
- Prove that $\overline{QR} = \overline{BD} - \overline{AC}$

Solution

- Consider the 3 right triangles PBR , AOC & QDS :

- ✓ $BP \parallel DQ \parallel OA$
- ✓ $BR \parallel DS \parallel OC$
- ✓ $BP = OA$
- ✓ $DS = OC$
- ✓ $\widehat{RBP} = \widehat{QDS} = \widehat{AOC} = 90^\circ$
- ✓ $PR = QS$ (hypotenuse)

The 3 triangles are equals.

Therefore, $AC \parallel PR \parallel QS \Rightarrow AC \parallel PS$

Since, $BP = DQ$ then $BD \parallel PQ$ from the parallelogram $PBDQ$.

$BD \parallel PQ \parallel AC$

Therefore, BD and AC are *parallel*.

- $\overline{BP} = \overline{DQ}$

$$\begin{aligned}\overline{PS} &= \overline{PQ} + \overline{QS} \\ &= \overline{AC} + \overline{BD}\end{aligned}$$

- $\overline{QR} = \overline{PQ} - \overline{PR}$
 $= \overline{BD} - \overline{AC}$

