Notebook 17: Integration in Vector Fields

Line Integrals

Compute the line integral of $f(x, y, z) = \sqrt{1 + x^2 + y^2}$ along $r(t) = \langle t, t^2, t^3 \rangle$ from $t = 0$ to $t = \frac{\pi}{2}$.

First, f and r are defined and the magnitude of the velocity vector is calculated.

> $with(VectorCalculus):$
> $f(x, y, z) := \sqrt{1 + x^2 + y^2}$; $r(t) := \langle t, t^2, t^3 \rangle$:'$r(t)$' $= r(t)$

$$f := (x, y, z) \rightarrow \sqrt{1 + x^2 + y^2}$$
$$r(t) = (t)e_x + (t^2)e_y + (t^3)e_z$$

> $dsdt := \sqrt{r'(t) . r'(t)}$

$$dsdt := \sqrt{1 + 4t^2 + 9t^4}$$

The integrand is then calculated by

> $integrand := f(r(t)[1], r(t)[2], r(t)[3]) \cdot dsdt$

$$integrand := \sqrt{1 + t^2 + t^4}\,\sqrt{1 + 4t^2 + 9t^4}$$

Maple does not return an exact answer for this integral, but a decimal approximation can be obtained.

> $\int_0^{\pi/2} integrand\, dt$; $evalf(\%)$

$$\int_0^{\frac{1}{2}\pi} \sqrt{1 + t^2 + t^4}\,\sqrt{1 + 4t^2 + 9t^4}\, dt$$
$$10.45184189$$

Vector Fields, Work, Circulation, and Flux

Find the work done by the force $F = \langle x \cdot z, z, y \cdot z \rangle$ over the curve $r(t) = \langle t^2, t, t^3 \rangle$ from $t = 0$ to $t = 1$.

The force and curve vectors are defined

> $F(x, y, z) := \langle x \cdot z, z, y \cdot z \rangle$:'F' $= F(x, y, z)$;
> $r(t) := \langle t^2, t, t^3 \rangle$:'$r(t)$' $= r(t)$

$$F = (xz)e_x + (z)e_y + (yz)e_z$$
$$r(t) = (t^2)e_x + (t)e_y + (t^3)e_z$$

The force along the curve is

> $F(r(t)[1], r(t)[2], r(t)[3])$

$$(t^5)e_x + (t^3)e_y + (t^4)e_z$$

The integrand is defined as

> $integrand := F(r(t)[1], r(t)[2], r(t)[3]).r'(t)$

$$integrand := 5\,t^6 + t^3$$

So, the total work is

> $\int_0^1 integrand\,\mathrm{d}t$

$$\frac{27}{28}$$

Green's Theorem in the Plane

Find the counterclockwise circulation of the field $F = \langle x - 2\,y, 3\,x + y\rangle$ around the simple closed curve $C: 4\,x^2 + y^2 = 16$.

The form of Green's Theorem to use is the following

> $m(x, y) := x - 2\,y : n(x, y) := 3\,x + y :$
> $integrand := \frac{\partial}{\partial x}\,n(x, y) - \frac{\partial}{\partial y}\,m(x, y)$

$$integrand := 5$$

> $\int_{-2}^{2}\int_{-\sqrt{16-4\,x^2}}^{\sqrt{16-4\,x^2}} integrand\,\mathrm{d}y\,\mathrm{d}x$

$$40\,\pi$$