

Solution

Section 2.5 – Subspaces, Span and Null Space

Exercise

Suppose S and T are two subspaces of a vector space V .

- a) The sum $S + T$ contains all sums $s + t$ of a vector s in S and a vector t in T . Show that $S + T$ satisfies the requirements (addition and scalar multiplication) for a vector space.
- b) If S and T are lines in \mathbb{R}^m , what is the difference between $S + T$ and $S \cup T$? That union contains all vectors from S and T or both. Explain this statement: The span of $S \cup T$ is $S + T$.

Solution

- a) Let s, s' be vectors in S , Let t, t' be vectors in T , and let c be a scalar. Then

$$(s + t) + (s' + t') = (s + s') + (t + t') \text{ and}$$

$$c(s + t) = cs + ct$$

Thus $S + T$ is closed under addition and scalar multiplication, it satisfies the two requirements for a vector space.

- b) If S and T are distinct lines, then S and T is a plane, whereas $S \cup T$ is not even closed under addition. The span of $S \cup T$ is the set of all combinations of vectors in this union. In particular, it contains all sums $s + t$ of a vector s in S and a vector t in T , and these sums form $S + T$. $S + T$ contains both S and T ; so, it contains $S \cup T$. $S + T$ is a vector space.
- c) So, it contains all combinations of vectors in itself; in particular, it contains the span of $S \cup T$. Thus, the span of $S \cup T$ is $S + T$.

Exercise

Determine which of the following are subspaces of \mathbb{R}^3 ?

- a) All vectors of the form $(a, 0, 0)$
- b) All vectors of the form $(a, 1, 1)$
- c) All vectors of the form (a, b, c) , where $b = a + c$
- d) All vectors of the form (a, b, c) , where $b = a + c + 1$
- e) All vectors of the form $(a, b, 0)$

Solution

- a) $(a_1, 0, 0) + (a_2, 0, 0) = (a_1 + a_2, 0, 0)$

$$k(a, 0, 0) = (ka, 0, 0)$$

This is a subspace of \mathbb{R}^3

b) $(a_1, 1, 1) + (a_2, 1, 1) = (a_1 + a_2, 2, 2)$ which is not in the set.

Therefore, this is not a subspace of \mathbb{R}^3

$$\begin{aligned} c) \quad (a_1, b_1, c_1) + (a_2, b_2, c_2) &= (a_1 + a_2, b_1 + b_2, c_1 + c_2) \\ &= (a_1 + a_2, a_1 + c_1 + a_2 + c_2, c_1 + c_2) \\ &= (a_1 + a_2, (a_1 + a_2) + (c_1 + c_2), c_1 + c_2) \\ &= (a_1, a_1 + c_1, c_1) + (a_2, a_2 + c_2, c_2) \end{aligned}$$

$$\begin{aligned} k(a, b, c) &= (ka, kb, kc) \\ &= (ka, k(a + c), kc) \\ &= k(a, (a + c), c) \end{aligned}$$

This is a subspace of \mathbb{R}^3

d) $k(a + c + 1) \neq ka + kc + 1$ so $k(a, b, c)$ is not in the set.

Therefore, this is not a subspace of \mathbb{R}^3

$$\begin{aligned} e) \quad (a_1, b_1, 0) + (a_2, b_2, 0) &= (a_1 + a_2, b_1 + b_2, 0) \\ k(a, b, 0) &= (ka, kb, 0) \end{aligned}$$

This is a subspace of \mathbb{R}^3

Exercise

Determine which of the following are subspaces of \mathbb{R}^∞ ?

- a) All sequences \vec{v} in \mathbb{R}^∞ of the form $\vec{v} = (v, 0, v, 0, \dots)$
- b) All sequences \vec{v} in \mathbb{R}^∞ of the form $\vec{v} = (v, 1, v, 1, \dots)$
- c) All sequences \vec{v} in \mathbb{R}^∞ of the form $\vec{v} = (v, 2v, 4v, 8v, 16v, \dots)$

Solution

$$\begin{aligned} a) \quad \text{Let } \vec{v}_1 &= (v_1, 0, v_1, 0, \dots) \quad \vec{v}_2 = (v_2, 0, v_2, 0, \dots) \\ \vec{v}_1 + \vec{v}_2 &= (v_1, 0, v_1, 0, \dots) + (v_2, 0, v_2, 0, \dots) \\ &= (v_1 + v_2, 0, v_1 + v_2, 0, \dots) \quad w = v_1 + v_2 \\ &= (w, 0, w, 0, \dots) \end{aligned}$$

$$\begin{aligned}
k\vec{v} &= k(v, 0, v, 0, \dots) \\
&= (kv, 0, kv, 0, \dots) & w = kv \\
&= (w, 0, w, 0, \dots)
\end{aligned}$$

This is a **subspace** of \mathbb{R}^∞

b) Let $\vec{v} = (v, 1, v, 1, \dots)$

$$\begin{aligned}
k\vec{v} &= k(v, 1, v, 1, \dots) \\
&= (kv, \textcolor{red}{k}, kv, \textcolor{red}{k}, \dots) \\
&\neq (kv, 1, kv, 1, \dots)
\end{aligned}$$

$k\vec{v}$ is not in the set

Since $k \neq 1$, then is **not** a **subspace** of \mathbb{R}^∞

c) Let $\vec{v}_1 = (v_1, 2v_1, 4v_1, 8v_1, \dots)$ $\vec{v}_2 = (v_2, 2v_2, 4v_2, 8v_2, \dots)$

$$\begin{aligned}
\vec{v}_1 + \vec{v}_2 &= (v_1, 2v_1, 4v_1, 8v_1, \dots) + (v_2, 2v_2, 4v_2, 8v_2, \dots) \\
&= (v_1 + v_2, 2v_1 + 2v_2, 4v_1 + 4v_2, 8v_1 + 8v_2, \dots) \\
&= (v_1 + v_2, 2(v_1 + v_2), 4(v_1 + v_2), 8(v_1 + v_2), \dots) & w = v_1 + v_2 \\
&= (w, 2w, 4w, 8w, \dots)
\end{aligned}$$

$$\begin{aligned}
k\vec{v} &= k(v, 2v, 4v, 8v, 16v, \dots) \\
&= (kv, 2kv, 4kv, 8kv, 16kv, \dots) \\
&= (w, 2w, 4w, 8w, 16w, \dots) & w = kv
\end{aligned}$$

This is a **subspace** of \mathbb{R}^∞

Exercise

Which of the following are linear combinations of $\vec{u} = (0, -2, 2)$ and $\vec{v} = (1, 3, -1)$?

- a) $(2, 2, 2)$ b) $(3, 1, 5)$ c) $(0, 4, 5)$ d) $(0, 0, 0)$

Solution

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \\ 2 & -1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$a) \quad b = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 0 & 1 & 2 \\ -2 & 3 & 2 \\ 2 & -1 & 2 \end{array} \right] \quad \text{Switch } R_1 \text{ \& } R_2$$

$$\left[\begin{array}{cc|c} -2 & 3 & 2 \\ 0 & 1 & 2 \\ 2 & -1 & 2 \end{array} \right] \quad R_3 + R_1$$

$$\left[\begin{array}{cc|c} -2 & 3 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{array} \right] \quad -\frac{1}{2}R_1$$

$$\left[\begin{array}{cc|c} 1 & -\frac{3}{2} & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{array} \right] \quad \begin{array}{l} R_1 + \frac{3}{2}R_2 \\ R_3 - 2R_2 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right]$$

$(2, 2, 2) = 2\vec{u} + 2\vec{v}$ is a linear combination of \vec{u} and \vec{v} .

$$b) \quad b = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 0 & 1 & 3 \\ -2 & 3 & 1 \\ 2 & -1 & 5 \end{array} \right] \quad \text{Switch } R_1 \text{ \& } R_2$$

$$\left[\begin{array}{cc|c} -2 & 3 & 1 \\ 0 & 1 & 3 \\ 2 & -1 & 5 \end{array} \right] \quad R_3 + R_1$$

$$\left[\begin{array}{cc|c} -2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{array} \right] \quad \begin{array}{l} R_1 - 3R_2 \\ R_3 - 2R_2 \end{array}$$

$$\left[\begin{array}{cc|c} -2 & 0 & -8 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array} \right] \quad -\frac{1}{2}R_1$$

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array} \right]$$

$(3, 1, 5) = 4\vec{u} + 3\vec{v}$ is a linear combination of \vec{u} and \vec{v} .

c) $b = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & 3 & 4 \\ 2 & -1 & 5 \end{array} \right] \quad \text{Switch } R_1 \text{ \& } R_2$$

$$\left[\begin{array}{cc|c} -2 & 3 & 4 \\ 0 & 1 & 0 \\ 2 & -1 & 5 \end{array} \right] \quad R_3 + R_1$$

$$\left[\begin{array}{cc|c} -2 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 2 & 9 \end{array} \right] \quad \begin{array}{l} R_1 - 3R_2 \\ R_3 - 2R_2 \end{array}$$

$$\left[\begin{array}{cc|c} -2 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{array} \right] \quad \begin{array}{l} -\frac{1}{2}R_1 \\ \frac{1}{9}R_3 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \quad R_1 + 2R_3$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

$(0, 4, 5)$ is not a linear combination of \vec{u} and \vec{v} .

d) $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & 3 & 0 \\ 2 & -1 & 0 \end{array} \right] \quad \text{Switch } R_1 \text{ \& } R_2$$

$$\left[\begin{array}{cc|c} -2 & 3 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 0 \end{array} \right] \quad R_3 + R_1$$

$$\left[\begin{array}{cc|c} -2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{array} \right] \quad \begin{array}{l} R_1 - 3R_2 \\ R_3 - 2R_2 \end{array}$$

$$\left[\begin{array}{cc|c} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad -\frac{1}{2}R_1$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$(0, 0, 0) = 0\vec{u} + 0\vec{v}$ is a linear combination of \vec{u} and \vec{v} .

Exercise

Which of the following are linear combinations of $\vec{u} = (2, 1, 4)$, $\vec{v} = (1, -1, 3)$ and $\vec{w} = (3, 2, 5)$?

a) $(-9, -7, -15)$

b) $(6, 11, 6)$

c) $(0, 0, 0)$

Solution

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{pmatrix}$$

$$a) \left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array} \right] \quad \begin{array}{l} 2R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 0 & -3 & 1 & -5 \\ 0 & 1 & -1 & 3 \end{array} \right] \quad \begin{array}{l} 3R_1 + R_2 \\ 3R_3 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 6 & 0 & 10 & -32 \\ 0 & -3 & 1 & -5 \\ 0 & 0 & -2 & 4 \end{array} \right] \quad \begin{array}{l} R_1 + 5R_3 \\ 2R_2 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 6 & 0 & 0 & -12 \\ 0 & -6 & 0 & -6 \\ 0 & 0 & -2 & 4 \end{array} \right] \quad \begin{array}{l} \frac{1}{6}R_1 \\ -\frac{1}{6}R_2 \\ -\frac{1}{2}R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

Therefore, $(-9, -7, -15) = -2\vec{u} + 1\vec{v} - 2\vec{w}$

$$b) \left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array} \right] \quad \begin{array}{l} 2R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 0 & -3 & 1 & 16 \\ 0 & 1 & -1 & -6 \end{array} \right] \quad \begin{array}{l} 3R_1 + R_2 \\ 3R_3 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 6 & 0 & 10 & 34 \\ 0 & -3 & 1 & 16 \\ 0 & 0 & -2 & -2 \end{array} \right] \quad \begin{array}{l} R_1 + 5R_3 \\ 2R_2 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 6 & 0 & 0 & 24 \\ 0 & -6 & 0 & 30 \\ 0 & 0 & -2 & -2 \end{array} \right] \quad \begin{array}{l} \frac{1}{6}R_1 \\ -\frac{1}{6}R_2 \\ -\frac{1}{2}R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Therefore, $(6, 11, 6) = 4\vec{u} - 5\vec{v} + 1\vec{w}$

$$c) \left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array} \right] \quad \begin{array}{l} 2R_2 - R_1 \\ R_2 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right] \quad \begin{array}{l} 2R_2 - R_1 \\ R_2 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 6 & 0 & 10 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array} \right] \quad \begin{array}{l} R_1 + 5R_3 \\ 2R_2 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 6 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array} \right] \quad \begin{array}{l} \frac{1}{6}R_1 \\ -\frac{1}{6}R_2 \\ -\frac{1}{2}R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Therefore, $(0, 0, 0) = 0\vec{u} + 0\vec{v} + 0\vec{w}$

Exercise

Determine whether the given vectors span \mathbb{R}^3

a) $\vec{v}_1 = (2, 2, 2), \vec{v}_2 = (0, 0, 3), \vec{v}_3 = (0, 1, 1)$

b) $\vec{v}_1 = (2, -1, 3), \vec{v}_2 = (4, 1, 2), \vec{v}_3 = (8, -1, 8)$

c) $\vec{v}_1 = (3, 1, 4), \vec{v}_2 = (2, -3, 5), \vec{v}_3 = (5, -2, 9), \vec{v}_4 = (1, 4, -1)$

Solution

a) $\det \begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{pmatrix} = -6 \neq 0$

The system is consistent for all values so the given vectors span \mathbf{R}^3 .

b) $\det \begin{pmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{pmatrix} = 0$

The system is not consistent for all values so the given vectors do not span \mathbf{R}^3 .

$$\begin{aligned}
c) \quad & \left[\begin{array}{cccc|c} 3 & 2 & 5 & 1 & b_1 \\ 1 & -3 & -2 & 4 & b_2 \\ 4 & 5 & 9 & -1 & b_3 \end{array} \right] \quad \begin{array}{l} 3R_2 - R_1 \\ 3R_3 - 4R_1 \end{array} \\
& \left[\begin{array}{cccc|c} 3 & 2 & 5 & 1 & b_1 \\ 0 & -11 & -11 & 11 & 3b_2 - b_1 \\ 0 & 7 & 7 & -7 & 3b_3 - 4b_1 \end{array} \right] \quad \begin{array}{l} 11R_1 + 2R_2 \\ 11R_3 + 7R_2 \end{array} \\
& \left[\begin{array}{cccc|c} 33 & 0 & 33 & 33 & 9b_1 + 6b_2 \\ 0 & -11 & -11 & 11 & 3b_2 - b_1 \\ 0 & 0 & 0 & 0 & 33b_3 - 51b_1 + 21b_2 \end{array} \right] \quad \begin{array}{l} \frac{1}{33}R_1 \\ -\frac{1}{11}R_2 \\ \frac{1}{33}R_2 \end{array} \\
& \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & \frac{3}{11}b_1 + \frac{2}{11}b_2 \\ 0 & 1 & 1 & -1 & \frac{1}{11}b_1 - \frac{3}{11}b_2 \\ 0 & 0 & 0 & 0 & -\frac{17}{11}b_1 + \frac{7}{11}b_2 + b_3 \end{array} \right]
\end{aligned}$$

The system has a solution only if $-\frac{17}{11}b_1 + \frac{7}{11}b_2 + b_3 = 0$. But since this is a restriction that the given vectors don't span on all of \mathbb{R}^3 . So the given vectors do not span \mathbb{R}^3 .

Exercise

Which of the following are linear combinations of $A = \begin{pmatrix} 4 & 0 \\ -2 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$

$$a) \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} \quad b) \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad c) \begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$$

Solution

$$\begin{aligned}
& \begin{pmatrix} 4 & 1 & 0 \\ 0 & -1 & 2 \\ -2 & 2 & 1 \\ -2 & 3 & 4 \end{pmatrix} \\
a) \quad & \left[\begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -8 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \end{array} \right] \quad \begin{array}{l} 2R_3 + R_1 \\ 2R_4 + R_1 \end{array}
\end{aligned}$$

$$\left[\begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -8 \\ 0 & 5 & 2 & 4 \\ 0 & 7 & 8 & -10 \end{array} \right] \quad \begin{array}{l} R_1 + R_2 \\ \\ R_3 + 5R_2 \\ R_4 + 7R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & -2 \\ 0 & -1 & 2 & -8 \\ 0 & 0 & 12 & -36 \\ 0 & 0 & 22 & -66 \end{array} \right] \quad -R_2$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & -2 \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 12 & -36 \\ 0 & 0 & 22 & -66 \end{array} \right] \quad \frac{1}{12}R_3$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & -2 \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 22 & -66 \end{array} \right] \quad \begin{array}{l} R_1 - 2R_3 \\ R_3 + 2R_3 \\ \\ R_4 - 22R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \frac{1}{4}R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{cc} 6 & -8 \\ -1 & -8 \end{array} \right] = 1A + 2B - 3C \text{ is a linear combinations of } A, B, \text{ and } C.$$

$$b) \left[\begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ -2 & 2 & 1 & 0 \\ -2 & 3 & 4 & 0 \end{array} \right] \quad \begin{array}{l} \\ 2R_3 + R_1 \\ 2R_4 + R_1 \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 7 & 8 & 0 \end{array} \right] \quad \begin{array}{l} R_1 + R_2 \\ \\ R_3 + 5R_2 \\ R_4 + 7R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 22 & 0 \end{array} \right] \quad -R_2$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 22 & 0 \end{array} \right] \quad \begin{array}{l} \frac{1}{12}R_3 \\ \frac{1}{22}R_4 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \quad \begin{array}{l} R_1 - 2R_3 \\ R_3 + 2R_3 \\ R_4 - R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \frac{1}{4}R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] = 0A + 0B + 0C \text{ is a linear combinations of } A, B, \text{ and } C.$$

$$c) \left[\begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & 0 \\ -2 & 2 & 1 & 3 \\ -2 & 3 & 4 & 8 \end{array} \right] \quad \begin{array}{l} 2R_3 + R_1 \\ 2R_4 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 5 & 2 & 12 \\ 0 & 7 & 8 & 22 \end{array} \right] \quad \begin{array}{l} R_1 + R_2 \\ R_3 + 5R_2 \\ R_4 + 7R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 12 & 12 \\ 0 & 0 & 22 & 22 \end{array} \right] \quad \begin{array}{l} -R_2 \\ \frac{1}{12}R_3 \\ \frac{1}{22}R_4 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & 6 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right] \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 + 2R_3 \\ R_4 - R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \frac{1}{4}R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix} = 1A + 2B + 1C \text{ is a linear combination of } A, B, \text{ and } C.$$

Exercise

Suppose that $\vec{v}_1 = (2, 1, 0, 3)$, $\vec{v}_2 = (3, -1, 5, 2)$, $\vec{v}_3 = (-1, 0, 2, 1)$. Which of the following vectors are in $\text{span} \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$

- a) $(2, 3, -7, 3)$ b) $(0, 0, 0, 0)$ c) $(1, 1, 1, 1)$ d) $(-4, 6, -13, 4)$

Solution

In order to be $\text{span} \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$, there must exist scalars a, b, c that $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{w}$

$$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

a) $(2, 3, -7, 3)$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{array} \right] \quad \begin{array}{l} 2R_2 - R_1 \\ 2R_4 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 0 & -5 & 1 & 4 \\ 0 & 5 & 2 & -7 \\ 0 & -5 & 5 & 0 \end{array} \right] \quad \begin{array}{l} 5R_1 + 3R_2 \\ R_3 + R_2 \\ R_4 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & -2 & 22 \\ 0 & -5 & 1 & 4 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 4 & -4 \end{array} \right] \quad \begin{array}{l} \frac{1}{3}R_3 \\ \frac{1}{4}R_4 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & -2 & 22 \\ 0 & -5 & 1 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array} \right] \quad \begin{array}{l} R_1 + 2R_3 \\ R_2 - R_3 \\ R_4 - R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & 0 & 20 \\ 0 & -5 & 0 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} \frac{1}{10}R_1 \\ -\frac{1}{5}R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This system is consistent, it has only solution is $a = 2, b = -1, c = -1$

$$2\vec{v}_1 - 1\vec{v}_2 - 1\vec{v}_3 = (2, 3, -7, 3)$$

Therefore, $(2, 3, -7, 3)$ is in $\text{span} \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$

b) The vector $(0, 0, 0, 0)$ is obviously in $\text{span} \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$

$$\text{Since } 0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = (0, 0, 0, 0)$$

c) For the vector $b = (1, 1, 1, 1)$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 5 & 2 & 1 \\ 3 & 2 & 1 & 1 \end{array} \right] \quad \begin{array}{l} 2R_2 - R_1 \\ \\ 2R_4 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 0 & -5 & 1 & 1 \\ 0 & 5 & 2 & 1 \\ 0 & -5 & 5 & -1 \end{array} \right] \quad \begin{array}{l} 5R_1 + 3R_2 \\ \\ R_3 + R_2 \\ R_4 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & -2 & 8 \\ 0 & -5 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 4 & -2 \end{array} \right] \quad \begin{array}{l} 3R_1 + 2R_3 \\ 3R_2 - R_3 \\ \\ 3R_4 - 4R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & 0 & 28 \\ 0 & -5 & 0 & 1 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & -14 \end{array} \right] \quad \begin{array}{l} \\ \frac{1}{3}R_3 \\ -\frac{1}{14}R_4 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & 0 & 28 \\ 0 & -5 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad \begin{array}{l} R_1 - 28R_4 \\ R_2 - R_4 \\ R_3 + R_4 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad \begin{array}{l} \frac{1}{10}R_1 \\ -\frac{1}{5}R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

This system is inconsistent, therefore $(1, 1, 1, 1)$ is *not* in $\text{span} \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$

d) For the vector $b = (-4, 6, -13, 4)$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & -4 \\ 1 & -1 & 0 & 6 \\ 0 & 5 & 2 & -13 \\ 3 & 2 & 1 & 4 \end{array} \right] \quad \begin{array}{l} 2R_2 - R_1 \\ 2R_4 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & -4 \\ 0 & -5 & 1 & 16 \\ 0 & 5 & 2 & -13 \\ 0 & -5 & 5 & 20 \end{array} \right] \quad \begin{array}{l} 5R_1 + 3R_2 \\ R_3 + R_2 \\ R_4 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & -2 & 28 \\ 0 & -5 & 1 & 16 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 4 & 4 \end{array} \right] \quad \begin{array}{l} \frac{1}{3}R_3 \\ \frac{1}{4}R_4 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & -2 & 28 \\ 0 & -5 & 1 & 16 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right] \quad \begin{array}{l} R_1 + 2R_3 \\ R_2 - R_3 \\ R_4 - R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & 0 & 30 \\ 0 & -5 & 0 & 15 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} \frac{1}{10}R_1 \\ -\frac{1}{5}R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This system is consistent, it has only solution is $a = 3, b = -3, c = 1$

$$3\vec{v}_1 - 3\vec{v}_2 + 1\vec{v}_3 = (-4, 6, -13, 4)$$

Therefore, $(-4, 6, -13, 4)$ is in $\text{span} \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$

Exercise

Let $f = \cos^2 x$ and $g = \sin^2 x$. Which of the following lie in the space spanned by f and g

- a) $\cos 2x$ b) $3 + x^2$ c) $\sin x$ d) 0

Solution

a) $\cos 2x = \cos^2 x - \sin^2 x$, therefore $\cos 2x$ is in $\text{span} \{f, g\}$

b) In order for $3 + x^2$ to be in $\text{span} \{f, g\}$, there must exist scalars a and b such that

$$a \cos^2 x + b \sin^2 x = 3 + x^2$$

$$\text{When } \left. \begin{array}{l} x = 0 \Rightarrow a = 3 \\ x = \pi \Rightarrow a = 3 + \pi^2 \end{array} \right\} \rightarrow \text{contradiction}$$

Therefore $3 + x^2$ is *not* in $\text{span} \{f, g\}$

c) In order for $\sin x$ to be in $\text{span} \{f, g\}$, there must exist scalars a and b such that

$$a \cos^2 x + b \sin^2 x = \sin x$$

$$\text{When } \left. \begin{array}{l} x = \frac{\pi}{2} \Rightarrow b = 1 \\ x = -\frac{\pi}{2} \Rightarrow b = -1 \end{array} \right\} \rightarrow \text{contradiction}$$

Therefore $\sin x$ is *not* in $\text{span} \{f, g\}$

d) In order for 0 to be in $\text{span} \{f, g\}$, there must exist scalars a and b such that

$$0 \cos^2 x + 0 \sin^2 x = 0$$

Therefore 0 is in $\text{span} \{f, g\}$

Exercise

Let $S = \left\{ (x, y) \mid x^2 + y^2 = 0; x, y \in \mathbb{R} \right\}$, Determine:

- a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of \mathbb{R}^2 ?

Solution

$$x^2 + y^2 = 0 \rightarrow x = y = 0 \quad (x, y \in \mathbb{R})$$

a) Let $\vec{u} = (x_1, y_1) \Rightarrow x_1^2 + y_1^2 = 0$ & $x_1 = y_1 = 0$, and

$$\vec{v} = (x_2, y_2) \quad \ni \quad x_2^2 + y_2^2 = 0 \quad \& \quad x_2 = y_2 = 0$$

$$\vec{u} + \vec{v} = (x_1 + x_2, y_1 + y_2)$$

$$\begin{aligned} (x_1 + x_2)^2 + (y_1 + y_2)^2 &= x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2 \\ &= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2(x_1x_2 + y_1y_2) \\ &= (0) + (0) + 2(0+0) \quad x_i = y_i = 0 \\ &= 0 \end{aligned}$$

S is closed under addition

$$\begin{aligned} b) \quad k\vec{u} &= k(x_1, y_1) \\ &= (kx_1, ky_1) \end{aligned}$$

$$\begin{aligned} (kx_1)^2 + (ky_1)^2 &= k^2x_1^2 + k^2y_1^2 \\ &= k^2(x_1^2 + y_1^2) \\ &= k^2(0) \\ &= 0 \end{aligned}$$

S is closed under scalar multiplication

c) Since S is closed under addition and scalar multiplication, then S is a subspace of \mathbb{R}^2 .

Exercise

Let $S = \{(x, y) \mid x^2 + y^2 = 0; \ x, y \in \mathbb{C}\}$, Determine:

- Is S closed under addition?
- Is S closed under scalar multiplication?
- Is S a subspace of \mathbb{C}^2 ?

Solution

$$x^2 + y^2 = 0 \rightarrow x = \pm iy \quad (x, y \in \mathbb{C})$$

$$a) \quad \text{Let } \vec{u} = (x_1, y_1) \quad \ni \quad x_1^2 + y_1^2 = 0 \rightarrow x_1 = i y_1, \text{ and}$$

$$\vec{v} = (x_2, y_2) \quad \ni \quad x_2^2 + y_2^2 = 0 \rightarrow x_2 = -i y_2$$

$$\vec{u} + \vec{v} = (x_1 + x_2, y_1 + y_2)$$

$$\begin{aligned}
(x_1 + x_2)^2 + (y_1 + y_2)^2 &= x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2 \\
&= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2(x_1x_2 + y_1y_2) \\
&= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2(iy_1(-iy_2) + y_1y_2) \\
&= 0 + 0 + 2(-i^2y_1y_2 + y_1y_2) \\
&= 2(y_1y_2 + y_1y_2) \\
&= 4y_1y_2 \\
&\neq 0
\end{aligned}$$

S is not closed under addition

$$\begin{aligned}
b) \quad k\vec{u} &= k(x_1, y_1) \\
&= (kx_1, ky_1)
\end{aligned}$$

$$\begin{aligned}
(kx_1)^2 + (ky_1)^2 &= k^2x_1^2 + k^2y_1^2 \\
&= k^2(x_1^2 + y_1^2) \\
&= k^2(0) \\
&= 0
\end{aligned}$$

S is closed under scalar multiplication

c) Since S is *not* closed under addition, then S is **not** a subspace of \mathbb{C}^2 .

Exercise

Let $S = \{(x, y) \mid x^2 - y^2 = 0; x, y \in \mathbb{R}\}$, Determine:

- Is S closed under addition?
- Is S closed under scalar multiplication?
- Is S a subspace of \mathbb{R}^2 ?

Solution

$$x^2 - y^2 = 0 \rightarrow x = \pm y \quad (x, y \in \mathbb{R})$$

$$\begin{aligned}
a) \quad \text{Let } \vec{u} &= (x_1, y_1) \quad \ni \quad x_1^2 - y_1^2 = 0 \rightarrow x_1 = y_1, \text{ and} \\
\vec{v} &= (x_2, y_2) \quad \ni \quad x_2^2 - y_2^2 = 0 \rightarrow x_2 = -y_2
\end{aligned}$$

$$\begin{aligned}
\vec{u} + \vec{v} &= (x_1 + x_2, y_1 + y_2) \\
(x_1 + x_2)^2 - (y_1 + y_2)^2 &= x_1^2 + x_2^2 + 2x_1x_2 - y_1^2 - y_2^2 - 2y_1y_2 \\
&= (x_1^2 - y_1^2) + (x_2^2 - y_2^2) + 2(x_1x_2 - y_1y_2) \\
&= (0) + (0) + 2(y_1(-y_2) - y_1y_2) \\
&= 2(-y_1y_2 - y_1y_2) \\
&= -4y_1y_2 \\
&\neq 0
\end{aligned}$$

S is not closed under addition

$$\begin{aligned}
b) \quad k\vec{u} &= k(x_1, y_1) \\
&= (kx_1, ky_1) \\
(kx_1)^2 - (ky_1)^2 &= k^2x_1^2 - k^2y_1^2 \\
&= k^2(x_1^2 - y_1^2) \\
&= k^2(0) \\
&= 0
\end{aligned}$$

S is closed under scalar multiplication

c) Since S is *not* closed under addition, then S is **not** a subspace of \mathbb{R}^2 .

Exercise

Let $S = \{(x, y) \mid x - y = 0; x, y \in \mathbb{R}\}$, Determine:

- Is S closed under addition?
- Is S closed under scalar multiplication?
- Is S a subspace of \mathbb{R}^2 ?

Solution

$$x - y = 0 \rightarrow x = y \quad (x, y \in \mathbb{R})$$

a) Let $\vec{u} = (x_1, y_1) \ni x_1 - y_1 = 0$, and

$$\vec{v} = (x_2, y_2) \ni x_2 - y_2 = 0$$

$$\begin{aligned}\vec{u} + \vec{v} &= (x_1 + x_2, y_1 + y_2) \\ (x_1 + x_2) - (y_1 + y_2) &= x_1 + x_2 - y_1 - y_2 \\ &= (x_1 - y_1) + (x_2 - y_2) \\ &= 0\end{aligned}$$

S is closed under addition

$$\begin{aligned}b) \quad k\vec{u} &= k(x_1, y_1) \\ &= (kx_1, ky_1)\end{aligned}$$

$$\begin{aligned}kx_1 - ky_1 &= k(x_1 - y_1) \\ &= k(0) \\ &= 0\end{aligned}$$

S is closed under scalar multiplication

c) Since S is closed under addition and scalar multiplication, then S is a subspace of \mathbb{R}^2 .

Exercise

Let $S = \{(x, y) \mid x - y = 1; x, y \in \mathbb{R}\}$, Determine:

- a) Is S closed under addition?
- b) Is S closed under scalar multiplication?
- c) Is S a subspace of \mathbb{R}^2 ?

Solution

$$x - y = 0 \rightarrow x = y \quad (x, y \in \mathbb{R})$$

$$a) \quad \text{Let } \vec{u} = (x_1, y_1) \ni x_1 - y_1 = 1, \text{ and}$$

$$\vec{v} = (x_2, y_2) \ni x_2 - y_2 = 1$$

$$\begin{aligned}\vec{u} + \vec{v} &= (x_1 + x_2, y_1 + y_2) \\ (x_1 + x_2) - (y_1 + y_2) &= x_1 + x_2 - y_1 - y_2 \\ &= (x_1 - y_1) + (x_2 - y_2) \\ &= 1 + 1 \\ &= 2 \neq 1\end{aligned}$$

S is not closed under addition

$$\begin{aligned} b) \quad k\vec{u} &= k(x_1, y_1) \\ &= (kx_1, ky_1) \end{aligned}$$

$$\begin{aligned} kx_1 - ky_1 &= k(x_1 - y_1) \\ &= k(1) \\ &= \underline{k \neq 1} \end{aligned}$$

S is not closed under scalar multiplication

- c) Since S is not closed under addition and not closed scalar multiplication, then S is **not** a subspace of \mathbb{R}^2 .

Exercise

$V = \mathbb{R}^3$, $S = \{(0, s, t) \mid s, t \text{ are real numbers}\}$ where V is a vector space and S is subset of V

- a) Is S closed under addition?
- b) Is S closed under scalar multiplication?
- c) Is S a subspace of V ?

Solution

$$\begin{aligned} d) \quad \text{Let } \vec{u} &= (0, s_1, t_1) \quad \text{and} \quad \vec{v} = (0, s_2, t_2) \\ \vec{u} + \vec{v} &= (0, s_1 + s_2, t_1 + t_2) \\ &= (0, s, t) \end{aligned}$$

Yes, S is closed under addition

$$\begin{aligned} e) \quad k\vec{u} &= (0, ks_1, kt_1) \\ &= (0, s, t) \end{aligned}$$

Yes, S is closed under scalar multiplication

- f) Since S is closed under addition and scalar multiplication, then S is a subspace of V .

Exercise

$V = \mathbb{R}^3$, $S = \{(x, y, z) \mid x, y, z \geq 0\}$ where V is a vector space and S is subset of V

- a) Is S closed under addition?
- b) Is S closed under scalar multiplication?
- c) Is S a subspace of V ?

Solution

a) Let $\vec{u} = (x_1, y_1, z_1)$ and $\vec{v} = (x_2, y_2, z_2)$

$$\begin{aligned}\vec{u} + \vec{v} &= (x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (x, y, z)\end{aligned}$$

where $x = x_1 + x_2, y = y_1 + y_2, z = z_1 + z_2$

Yes, S is closed under addition

b) $(-1)\vec{u} = (-x_1, -y_1, -z_1)$

S is **not** closed under scalar multiplication since $x_1 \geq 0 \Rightarrow -x_1 \leq 0$

c) Since S is closed under addition but it is not closed scalar multiplication, then S is **not** a subspace of V .

Exercise

$V = \mathbb{R}^3$, $S = \{(x, y, z) \mid z = x + y + 1\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of V ?

Solution

a) Let $\vec{u} = (0, 1, 2)$ and $\vec{v} = (1, 2, 4)$

$$\vec{u} + \vec{v} = (1, 3, 6)$$

$$\neq (1, 3, 1+3+1)$$

S is **not** closed under addition

b) $k\vec{u} = (kx_1, ky_1, kz_1)$

$$= (kx_1, ky_1, k(x_1 + y_1 + 1))$$

$$= (kx_1, ky_1, kx_1 + ky_1 + k)$$

$$= (x, y, z)$$

Where $x = kx_1, y = ky_1, z = k(x_1 + y_1 + 1)$

S is closed under scalar multiplication

c) Since S is not closed under addition and closed scalar multiplication, then S is **not** a subspace of V .

Exercise

Let $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2 \right\}$, Determine:

- a) Is S closed under addition?
- b) Is S closed under scalar multiplication?
- c) Is S a subspace of \mathbb{R}^3 ?

Solution

a) Let $\vec{u} = (a_1, a_2, a_3)$ and $\vec{v} = (b_1, b_2, b_3)$

$$\begin{aligned}\vec{u} + \vec{v} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\ &= (3a_2, a_2, -a_2) + (3b_2, b_2, -b_2) \\ &= (3a_2 + 3b_2, a_2 + b_2, -a_2 - b_2) \\ &= (3(a_2 + b_2), a_2 + b_2, -(a_2 + b_2)) \quad c_2 = a_2 + b_2 \\ &= (3c_2, c_2, -c_2) \\ &= (c_1, c_2, c_3) : c_1 = 3c_2 \quad c_3 = -c_2\end{aligned}$$

S is closed under addition

b) $k\vec{u} = k(a_1, a_2, a_3)$

$$\begin{aligned}&= k(3a_2, a_2, -a_2) \\ &= (3ka_2, ka_2, -ka_2) \quad c_2 = ka_2 \\ &= (3c_2, c_2, -c_2) \\ &= (c_1, c_2, c_3) : c_1 = 3c_2 \quad c_3 = -c_2\end{aligned}$$

S is closed under scalar multiplication.

- c) Since S is closed under addition and scalar multiplication, then S is a subspace of V .

Exercise

Let $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2 \right\}$, Determine:

- a) Is S closed under addition?
- b) Is S closed under scalar multiplication?
- c) Is S a subspace of \mathbb{R}^3 ?

Solution

a) Let $\vec{u} = (2, 1, 0)$ and $\vec{v} = (3, 0, 1)$ $a_1 = a_3 + 2$

$$\vec{u} + \vec{v} = (2, 1, 0) + (3, 0, 1)$$

$$= (5, 1, 1) \quad \begin{matrix} ? \\ 5 = 1 + 2 \end{matrix}$$

$$\neq (3, 1, 1)$$

S is *not* closed under addition

b) $k\vec{u} = k(a_1, a_2, a_3)$

$$= k(a_3 + 2, a_2, a_3)$$

$$= (ka_3 + 2k, ka_2, ka_3)$$

$$a_1 = a_3 + 2 \rightarrow ka_3 + 2k = a_3 + 2$$

$$2k \neq 2 \quad (\forall k)$$

S is *not* closed under scalar multiplication.

- c) Since S is not closed under addition and not closed scalar multiplication, then S is **not** a subspace of V .

Exercise

Let $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0 \right\}$, Determine:

- a) Is S closed under addition?
- b) Is S closed under scalar multiplication?
- c) Is S a subspace of \mathbb{R}^3 ?

Solution

a) Let $\vec{u} = (a_1, a_2, a_3)$ and $\vec{v} = (b_1, b_2, b_3)$

$$2a_1 - 7a_2 + a_3 = 0 \rightarrow a_3 = 7a_2 - 2a_1$$

$$\begin{aligned}
\vec{u} + \vec{v} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\
&= (a_1, a_2, 7a_2 - 2a_1) + (b_1, b_2, 7b_2 - 2b_1) \\
&= (a_1 + b_1, a_2 + b_2, 7a_2 - 2a_1 + 7b_2 - 2b_1) \\
&= (a_1 + b_1, a_2 + b_2, 7(a_2 + b_2) - 2(a_1 + b_1)) \quad \text{Let } c_1 = a_1 + b_1 \quad c_2 = a_2 + b_2 \\
&= (c_1, c_2, 7c_2 - 2c_1) \quad c_3 = 7c_2 - 2c_1 \rightarrow 2c_1 - 7c_2 + c_3 = 0 \\
&= (c_1, c_2, c_3)
\end{aligned}$$

S is closed under addition

$$\begin{aligned}
b) \quad k\vec{u} &= k(a_1, a_2, a_3) \\
&= k(a_1, a_2, 7a_2 - 2a_1) \\
&= (ka_1, ka_2, 7ka_2 - 2ka_1) \quad \text{Let } c_1 = ka_1 \quad c_2 = ka_2 \\
&= (c_1, c_2, 7c_2 - 2c_1) \quad c_3 = 7c_2 - 2c_1 \rightarrow 2c_1 - 7c_2 + c_3 = 0 \\
&= (c_1, c_2, c_3)
\end{aligned}$$

S is closed under scalar multiplication.

c) Since S is closed under addition and scalar multiplication, then S is a subspace of V .

Exercise

Let $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0 \right\}$, Determine:

- a) Is S closed under addition?
- b) Is S closed under scalar multiplication?
- c) Is S a subspace of \mathbb{R}^3 ?

Solution

$$a_1 - 4a_2 - a_3 = 0 \rightarrow a_1 = 4a_2 + a_3$$

$$\begin{aligned}
a) \quad \text{Let } \vec{u} &= (a_1, a_2, a_3) \quad \text{and} \quad \vec{v} = (b_1, b_2, b_3) \\
\vec{u} + \vec{v} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\
&= (4a_2 + a_3, a_2, a_3) + (4b_2 + b_3, b_2, b_3) \\
&= (4a_2 + a_3 + 4b_2 + b_3, a_2 + b_2, a_3 + b_3)
\end{aligned}$$

$$\begin{aligned}
&= \left(4(a_2 + b_2) + (a_3 + b_3), a_2 + b_2, a_3 + b_3 \right) \quad \text{Let } c_2 = a_2 + b_2 \quad c_3 = a_3 + b_3 \\
&= \left(4c_2 + c_3, c_2, c_3 \right) \quad c_1 - 4c_2 - c_3 = 0 \rightarrow c_1 = 4c_2 + c_3 \\
&= \left(c_1, c_2, c_3 \right)
\end{aligned}$$

S is closed under addition

$$\begin{aligned}
b) \quad k\vec{u} &= k(a_1, a_2, a_3) \\
&= k(4a_2 + a_3, a_2, a_3) \\
&= (4ka_2 + ka_3, ka_2, ka_3) \quad \text{Let } c_2 = ka_2 \quad c_3 = ka_3 \\
&= (4c_2 + c_3, c_2, c_3) \quad c_1 = 4c_2 + c_3 \rightarrow c_1 - 4c_2 - c_3 = 0 \\
&= (c_1, c_2, c_3)
\end{aligned}$$

S is closed under scalar multiplication.

c) Since S is closed under addition and scalar multiplication, then S is a subspace of V .

Exercise

Let $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 0 \right\}$, Determine:

- a) Is S closed under addition?
- b) Is S closed under scalar multiplication?
- c) Is S a subspace of \mathbb{R}^3 ?

Solution

$$a_1 + 2a_2 - 3a_3 = 0 \rightarrow a_1 = -2a_2 + 3a_3$$

$$\begin{aligned}
a) \quad \text{Let } \vec{u} &= (a_1, a_2, a_3) \quad \text{and} \quad \vec{v} = (b_1, b_2, b_3) \\
\vec{u} + \vec{v} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\
&= (-2a_2 + 3a_3, a_2, a_3) + (-2b_2 + 3b_3, b_2, b_3) \\
&= (-2a_2 + 3a_3 - 2b_2 + 3b_3, a_2 + b_2, a_3 + b_3) \\
&= \left(-2(a_2 + b_2) + 3(a_3 + b_3), a_2 + b_2, a_3 + b_3 \right) \quad \text{Let } c_2 = a_2 + b_2 \quad c_3 = a_3 + b_3 \\
&= (-2c_2 + 3c_3, c_2, c_3) \quad c_1 + 2c_2 - 3c_3 = 0 \rightarrow c_1 = -2c_2 + 3c_3 \\
&= (c_1, c_2, c_3)
\end{aligned}$$

S is closed under addition

$$\begin{aligned}
 b) \quad k\vec{u} &= k(a_1, a_2, a_3) \\
 &= k(4a_2 + a_3, a_2, a_3) \\
 &= (-2ka_2 + 3ka_3, ka_2, ka_3) \quad \text{Let } c_2 = ka_2 \quad c_3 = ka_3 \\
 &= (-2c_2 + 3c_3, c_2, c_3) \quad c_1 = -2c_2 + 3c_3 \rightarrow c_1 - 2c_2 + 3c_3 = 0 \\
 &= (c_1, c_2, c_3)
 \end{aligned}$$

S is closed under scalar multiplication.

c) Since S is closed under addition and scalar multiplication, then S is a subspace of V .

Exercise

Let $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1 \right\}$, Determine:

- a) Is S closed under addition?
- b) Is S closed under scalar multiplication?
- c) Is S a subspace of \mathbb{R}^3 ?

Solution

$$a_1 + 2a_2 - 3a_3 = 1 \rightarrow a_1 = 1 - 2a_2 + 3a_3$$

$$\begin{aligned}
 a) \quad \text{Let } \vec{u} &= (a_1, a_2, a_3) \quad \text{and} \quad \vec{v} = (b_1, b_2, b_3) \\
 \vec{u} + \vec{v} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\
 &= (1 - 2a_2 + 3a_3, a_2, a_3) + (1 - 2b_2 + 3b_3, b_2, b_3) \\
 &= (1 - 2a_2 + 3a_3 + 1 - 2b_2 + 3b_3, a_2 + b_2, a_3 + b_3) \\
 &= (2 - 2(a_2 + b_2) + 3(a_3 + b_3), a_2 + b_2, a_3 + b_3) \quad \text{Let } c_2 = a_2 + b_2 \quad c_3 = a_3 + b_3 \\
 &= (2 - 2c_2 + 3c_3, c_2, c_3) \quad c_1 + 2c_2 - 3c_3 = 1 \rightarrow c_1 = 1 - 2c_2 + 3c_3 \\
 &\neq (1 - 2c_2 + 3c_3, c_2, c_3)
 \end{aligned}$$

S is *not* closed under addition

$$\begin{aligned}
 b) \quad \vec{u} &= (2, 1, 1) \\
 k\vec{u} &= k(2, 1, 1)
 \end{aligned}$$

$$= (2k, k, k) \quad a_1 + 2a_2 - 3a_3 = 1 \rightarrow 2k + 2k - 3k = 1 \quad ?$$

$$k \neq 1 \quad (\forall k)$$

S is *not* closed under scalar multiplication.

- c) Since S is not closed under addition and not closed scalar multiplication, then S is **not** a subspace of V .

Exercise

Let $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0 \right\}$, Determine:

- a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of \mathbb{R}^3 ?

Solution

$$5a_1^2 - 3a_2^2 + 6a_3^2 = 0 \rightarrow a_2^2 = \frac{5}{3}a_1^2 + 2a_3^2$$

a) Let $\vec{u} = (0, \sqrt{2}, 1)$ and $\vec{v} = (3, \sqrt{17}, 1)$

$$\begin{aligned} \vec{u} + \vec{v} &= (0, \sqrt{2}, 1) + (3, \sqrt{17}, 1) \\ &= (3, \sqrt{2} + \sqrt{17}, 2) \end{aligned}$$

$$a_2^2 = \frac{5}{3}a_1^2 + 2a_3^2 \rightarrow (\sqrt{2} + \sqrt{17})^2 \neq 15 + 8$$

S is *not* closed under addition

b) $k\vec{u} = k(a_1, a_2, a_3)$

$$= (ka_1, ka_2, ka_3)$$

$$5(ka_1)^2 - 3(ka_2)^2 + 6(ka_3)^2 = 0$$

$$5k^2a_1^2 - 3k^2a_2^2 + 6k^2a_3^2 = 0$$

$$5a_1^2 - 3a_2^2 + 6a_3^2 = 0$$

S is closed under scalar multiplication.

- c) Since S is *not* closed under addition and is closed scalar multiplication, then S is **not** a subspace of V .

Exercise

Let $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_3 = a_1 + a_2 \right\}$, Determine:

- a) Is S closed under addition?
- b) Is S closed under scalar multiplication?
- c) Is S a subspace of \mathbb{R}^3 ?

Solution

a) Let $\vec{u} = (a_1, a_2, a_3)$ and $\vec{v} = (b_1, b_2, b_3)$

$$\begin{aligned}\vec{u} + \vec{v} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\ &= (a_1, a_2, a_1 + a_2) + (b_1, b_2, b_1 + b_2) \\ &= (a_1 + b_1, a_2 + b_2, a_1 + a_2 + b_1 + b_2) \quad \text{Let } c_1 = a_1 + b_1 \quad c_2 = a_2 + b_2 \\ &= (c_1, c_2, c_1 + c_2) \quad \text{Then, } c_3 = c_1 + c_2 \\ &= (c_1, c_2, c_3)\end{aligned}$$

S is closed under addition

b) $k\vec{u} = k(a_1, a_2, a_3)$

$$\begin{aligned}&= k(a_1, a_2, a_1 + a_2) \\ &= (ka_1, ka_2, k(a_1 + a_2)) \\ &= (ka_1, ka_2, ka_1 + ka_2) \quad \text{Where } c_1 = ka_1, \quad c_2 = ka_2, \quad c_3 = ka_1 + ka_2 \\ &= (c_1, c_2, c_3) \quad c_3 = ka_1 + ka_2 = c_1 + c_2\end{aligned}$$

S is closed under scalar multiplication.

- c) Since S is closed under addition and scalar multiplication, then S is a subspace of V .

Exercise

Let $S = \left\{ (a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + a_2 + a_3 = 0 \right\}$, Determine:

- a) Is S closed under addition?
- b) Is S closed under scalar multiplication?
- c) Is S a subspace of \mathbb{R}^3 ?

Solution

a) Let $\vec{u} = (a_1, a_2, a_3) \ni a_1 + a_2 + a_3 = 0$

$\vec{v} = (b_1, b_2, b_3) \ni b_1 + b_2 + b_3 = 0$

$$\begin{aligned}\vec{u} + \vec{v} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\ &= (a_1 + b_1, a_2 + b_2, a_3 + b_3)\end{aligned}$$

Since $a_1 + a_2 + a_3 = 0$ & $b_1 + b_2 + b_3 = 0$

Then, $\rightarrow (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) = 0$

S is closed under addition

b) $k\vec{u} = k(a_1, a_2, a_3)$
 $= (ka_1, ka_2, ka_3)$

$ka_1 + ka_2 + ka_3 = k(a_1 + a_2 + a_3) = k(0) = 0$

S is closed under scalar multiplication.

c) Since S is closed under addition and scalar multiplication, then S is a subspace of V .

Exercise

Let $S = \{(x_1, x_2, 1) : x_1 \text{ and } x_2 \text{ are real numbers}\}$, Determine:

- a) Is S closed under addition?
- b) Is S closed under scalar multiplication?
- c) Is S a subspace of \mathbb{R}^3 ?

Solution

a) Let $\vec{u} = (x_1, x_2, 1)$ & $\vec{v} = (y_1, y_2, 1)$

$$\begin{aligned}\vec{u} + \vec{v} &= (x_1, x_2, 1) + (y_1, y_2, 1) \\ &= (x_1 + y_1, x_2 + y_2, 2) \quad \text{If we let } z_1 = x_1 + y_1 \quad z_2 = x_2 + y_2 \\ &= (z_1, z_2, 2) \\ &\neq (z_1, z_2, 1)\end{aligned}$$

S is **not** closed under addition

b) $k\vec{u} = k(x_1, x_2, 1)$

$$= (kx_1, kx_2, k) \quad \text{If we let } z_1 = kx_1 \quad z_2 = kx_2$$

$$\neq (z_1, z_2, 1) \quad k \neq 1 \quad (\forall k)$$

S is **not** closed under scalar multiplication.

- c) Since S is *not* closed under addition and is *not* closed scalar multiplication, then S is **not** a subspace of V .

Exercise

Let $S = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 = x_1 + 2x_3 \right\}$, Determine:

- a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of \mathbb{R}^3 ?

Solution

a) Let $\vec{u} = (x_1, x_2, x_3)$ & $\vec{v} = (y_1, y_2, y_3)$

$$\vec{u} + \vec{v} = (x_1, x_2, x_3) + (y_1, y_2, y_3)$$

$$= (x_1 + y_1, x_2 + y_2, x_3 + y_3) \quad S = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 = x_1 + 2x_3 \right\}$$

$$x_2 + y_2 = x_1 + 2x_3 + y_1 + 2y_3$$

$$= x_1 + y_1 + 2(x_3 + y_3)$$

S is closed under addition

b) $k\vec{u} = k(x_1, x_2, x_3)$

$$= (kx_1, kx_2, kx_3) \quad \text{If we let } z_1 = kx_1 \quad z_2 = kx_2$$

$$kx_2 = kx_1 + 2kx_3$$

$$kx_2 = k(x_1 + 2x_3)$$

$$x_2 = x_1 + 2x_3$$

S is closed under scalar multiplication.

- c) Since S is closed under addition and scalar multiplication, then S is a subspace of V .

Exercise

Let $S = \left\{ \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \in M_{2 \times 2} \mid a, c, d \in \mathbb{R} \right\}$ and $V = M_{2,2}$, Determine:

- a) Is S closed under addition?
- b) Is S closed under scalar multiplication?
- c) Is S a subspace of V ?

Solution

a) Let $A = \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$ & $B = \begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix}$

$$\begin{aligned} A + B &= \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + a_2 & 2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix} && \text{If we let } a = a_1 + a_2 \quad c = c_1 + c_2 \quad d = d_1 + d_2 \\ &= \begin{pmatrix} a & 2 \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \end{aligned}$$

S is **not** closed under addition

b) $kA = k \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$

$$\begin{aligned} &= \begin{pmatrix} ka_1 & k \\ kc_1 & kd_1 \end{pmatrix} && \text{If we let } a = ka_1 \quad c = kc_1 \quad d = kd_1 \\ &= \begin{pmatrix} a & k \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} && k \neq 1 \quad (\forall k) \end{aligned}$$

S is **not** closed under scalar multiplication.

- c) Since S is **not** closed under addition and is **not** closed scalar multiplication, then S is **not** a subspace of V .

Exercise

Let $S = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in M_{2 \times 2} \mid a, c, d \in \mathbb{R} \right\}$ and $V = M_{2,2}$, Determine:

- a) Is S closed under addition?
- b) Is S closed under scalar multiplication?
- c) Is S a subspace of V ?

Solution

a) Let $A = \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$ & $B = \begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix}$

$$A + B = \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + a_2 & 2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

$$= \begin{pmatrix} a & 2 \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$$

If we let $a = a_1 + a_2$ $c = c_1 + c_2$ $d = d_1 + d_2$

S is **not** closed under addition

b) $kA = k \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$

$$= \begin{pmatrix} ka_1 & k \\ kc_1 & kd_1 \end{pmatrix}$$

If we let $a = ka_1$ $c = kc_1$ $d = kd_1$

$$= \begin{pmatrix} a & k \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$$

$k \neq 1 \quad (\forall k)$

S is **not** closed under scalar multiplication.

- c) Since S is **not** closed under addition and is **not** closed scalar multiplication, then S is **not** a subspace of V .

Exercise

Let $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in M_{2 \times 2} \mid a, d \in \mathbb{R} \text{ \& } ad \geq 0 \right\}$ and $V = M_{2,2}$, Determine:

- a) Is S closed under addition?
- b) Is S closed under scalar multiplication?
- c) Is S a subspace of V ?

Solution

a) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \rightarrow 1(2) > 0$ & $B = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow (-2)(-1) > 0$

$$A + B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \\ = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$ad \geq 0 \rightarrow (-1)(1) = -1 < 0$$

S is **not** closed under addition

b) $kA = k \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \\ = \begin{pmatrix} ka & 0 \\ 0 & kd \end{pmatrix}$

$$(ka)(kd) = k^2(ad)$$

$$\text{Since, } ad \geq 0 \text{ \& } k^2 \geq 0$$

$$k^2(ad) \geq 0$$

S is closed under scalar multiplication.

c) Since S is **not** closed under addition and is closed scalar multiplication, then S is **not** a subspace of V .

Exercise

$V = M_{33}$, $S = \{A \mid A \text{ is invertible}\}$ where V is a vector space and S is subset of V

- a) Is S closed under addition?
- b) Is S closed under scalar multiplication?
- c) Is S a subspace of V ?

Solution

a) Let assume: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ are invertible

$$\text{But } A + B = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \text{ is not invertible.}$$

S is **not** closed under addition

b) S is **not** closed under scalar multiplication if $k = 0$

c) Since S is **not** closed under addition and is **not** closed scalar multiplication, then S is **not** a subspace of V .

Exercise

Let $S = \{p(t) = a + 2at + 3at^3 \mid a \in \mathbb{R} \text{ \& } p(t) \in P_2\}$ and $V = P_2$, Determine:

- a) Is S closed under addition?
- b) Is S closed under scalar multiplication?
- c) Is S a subspace of V ?

Solution

a) Let $p_1(t) = a + 2at + 3at^3$ \& $p_2(t) = b + 2bt + 3bt^3$

$$\begin{aligned} p_1(t) + p_2(t) &= a + 2at + 3at^3 + b + 2bt + 3bt^3 \\ &= (a+b) + 2(a+b)t + 3(a+b)t^3 \quad \text{Let } c = a+b \in \mathbb{R} \\ &= c + 2ct + 3ct^3 \end{aligned}$$

S is closed under addition

b) $kp_1(t) = k(a + 2at + 3at^3)$

$$\begin{aligned} &= ka + 2kat + 3kat^3 \quad \text{Let } c = ka \in \mathbb{R} \\ &= c + 2ct + 3ct^3 \end{aligned}$$

S is closed under scalar multiplication.

- c) Since S is closed under addition and is closed scalar multiplication, then S is a subspace of V .

Exercise

Let $S = \{p(t) \mid p(t) \in P[t] \text{ has degree } 3\}$, Determine:

- a) Is S closed under addition?
- b) Is S closed under scalar multiplication?
- c) Is S a subspace of $P[t]$?

Solution

a) Let $p_1(t) = at^3 + b_1t^2 + c_1t + d_1$ \& $p_2(t) = -at^3 + b_2t^2 + c_2t + d_2$

$$\begin{aligned} p_1(t) + p_2(t) &= at^3 + b_1t^2 + c_1t + d_1 - at^3 + b_2t^2 + c_2t + d_2 \\ &= (b_1 + b_2)t^2 + (c_1 + c_2)t + (d_1 + d_2) \\ &= bt^2 + ct + d \end{aligned}$$

Has no 3rd degree polynomial.

S is *not* closed under addition

$$\begin{aligned}b) \quad kp_1(t) &= k(at^3 + b_1t^2 + c_1t + d_1) \\&= kat^3 + kb_1t^2 + kc_1t + kd_1 \\&= k_1t^3 + k_2t^2 + k_3t + k_4\end{aligned}$$

It is 3rd degree polynomial.

S is closed under scalar multiplication.

c) Since S is *not* closed under addition and is closed scalar multiplication, then S is ***not*** a subspace of V .

Exercise

Let $S = \{p(t) \mid p(0) = 0, p(t) \in \mathbf{P}[t]\}$, Determine:

- d) Is S closed under addition?
- e) Is S closed under scalar multiplication?
- f) Is S a subspace of $\mathbf{P}[t]$?

Solution

$$\begin{aligned}a) \quad \text{Let } p_1(t) &= a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t \Rightarrow p_1(0) = 0 \\p_2(t) &= b_n t^n + b_{n-1} t^{n-1} + \dots + b_1 t \Rightarrow p_2(0) = 0 \\p_1(t) + p_2(t) &= a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + b_n t^n + b_{n-1} t^{n-1} + \dots + b_1 t \\&= (a_n + b_n) t^n + (a_{n-1} + b_{n-1}) t^{n-1} + \dots + (a_1 + b_1) t\end{aligned}$$

$$p_1(0) + p_2(0) = 0$$

S is closed under addition

$$\begin{aligned}b) \quad kp_1(t) &= k(a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t) \\&= ka_n t^n + ka_{n-1} t^{n-1} + \dots + ka_1 t \\&= k_1 t^3 + k_2 t^2 + k_3 t + k_4\end{aligned}$$

$$kp_1(0) = 0$$

S is closed under scalar multiplication.

c) Since S is closed under addition and is closed scalar multiplication, then S is a subspace of V .

Exercise

Given: $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix}$

- a) Find $NS(A)$
- b) For which n is $NS(A)$ a subspace of \mathbb{R}^n
- c) Sketch $NS(A)$ in \mathbb{R}^2 or \mathbb{R}^3

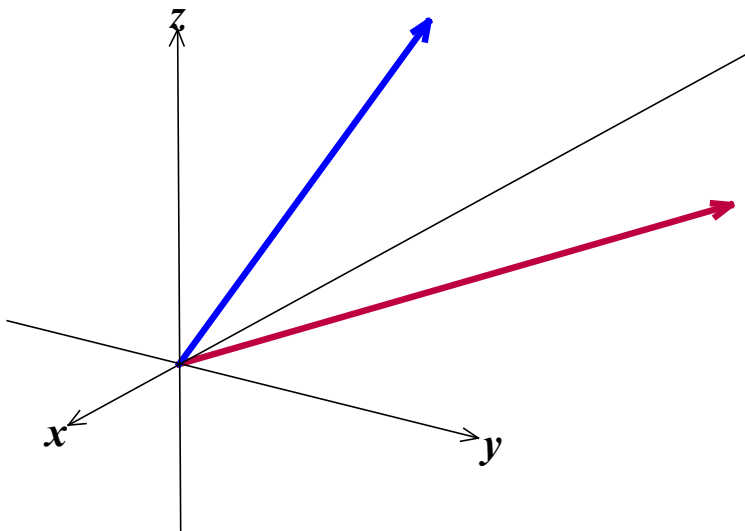
Solution

a) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad x = -3y - 2z$

$$\left\{ y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \mid y, z \in \mathbb{R} \right\}$$

b) $n = 3$

c)



Exercise

Determine which of the following are subspaces of M_{22}

- a) All 2×2 matrices with integer entries
- b) All matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a + b + c + d = 0$

Solution

a) Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are integers.

$$A + B = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix}$$

where $a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4$ are integers too.

Then, it is closed under addition.

$$\begin{aligned} \frac{1}{2}A &= \frac{1}{2} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix} \end{aligned}$$

It is not closed under multiplication if the scalar is a real number.

Therefore; it is **not** a subspace of M_{22}

b) Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ $a_1 + a_2 + a_3 + a_4 = 0$

and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ $b_1 + b_2 + b_3 + b_4 = 0$

$$A + B = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix}$$

$$a_1 + a_2 + a_3 + a_4 + b_1 + b_2 + b_3 + b_4 = 0$$

$$(a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + (a_4 + b_4) = 0$$

Then, it is closed under addition.

$$kA = \begin{bmatrix} ka_1 & ka_2 \\ ka_3 & ka_4 \end{bmatrix}$$

$$ka_1 + ka_2 + ka_3 + ka_4 = k(a_1 + a_2 + a_3 + a_4) = k(0) = 0$$

It is closed under multiplication

Therefore; it is a subspace of M_{22}

Exercise

Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1 \right\}$. Is V a vector space?

Solution

$$k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

$$\begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k^2 ad - k^2 bc$$

$$= k^2 (ad - bc)$$

$$= k^2 \neq k$$

$\therefore V$ is *not* a vector space

Exercise

Let $V = \{(x, 0, y) : x \text{ \& } y \text{ are arbitrary } \mathbb{R}\}$. Define addition and scalar multiplication as follows:

$$\begin{cases} (x_1, 0, y_1) + (x_2, 0, y_2) = (x_1 + x_2, y_1 + y_2) \\ c(x, 0, y) = (cx, cy) \end{cases}$$

Is V a vector space?

Solution

$$\text{Let } \vec{v}_1(x_1, 0, y_1) \text{ \& } \vec{v}_2(x_2, 0, y_2)$$

$$\vec{v}_1 + \vec{v}_2 = (x_1, 0, y_1) + (x_2, 0, y_2)$$

$$= (x_1 + x_2, y_1 + y_2)$$

$$\neq (x_1 + x_2, 0, y_1 + y_2)$$

$$= \vec{V}_1 + \vec{V}_2$$

$\therefore V$ is *not* a vector space

Exercise

Construct a matrix whose column space contains $(1, 1, 0)$, $(0, 1, 1)$, and whose nullspace contains $(1, 0, 1)$ and $(0, 0, 1)$

Solution

It is *not* possible.

Since a matrix (A) must be 3×3 .

Since the nullspace contains 2 independent vectors, then A can have at most $3 - 2 = 1$ pivot.

But the column space contains 2 independent vectors, A must have at least 2 pivots.

These 2 conditions can't both be met.

Exercise

How is the nullspace $N(C)$ related to the spaces $N(A)$ and $N(B)$, is $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

Solution

$$N(C) = N(A) \cap N(B)$$

$$Cx = \begin{bmatrix} Ax \\ Bx \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{Iff } Ax = 0 \quad \& \quad Bx = 0$$

Exercise

True or False (check addition or give a counterexample)

- If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V .
- The empty set is a subspace of every vector space.
- If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$.
- The intersection of any two subsets of V is a subspace of V .
- Let W be the xy -plane in \mathbb{R}^3 ; that is, $W = \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$. Then $W = \mathbb{R}^2$

Solution

a) False

W is a subset of V , but not necessary that the scalar of a vector in W is in V .

Therefore, W is *not* a subspace of V

b) False

Since not every subspace has an empty space, example \mathbb{R}

c) True

If V is a vector space in \mathbb{R}^n and W is a vector space in \mathbb{Z}^n . Then V contains a subspace W and $W \neq V$

d) False

e) False

Exercise

Let $A\vec{x} = \vec{0}$ be a homogeneous system of n linear equations in n unknowns that has only the trivial solution. Show that if k is any positive integer, then the system $A^k \vec{x} = \vec{0}$ also has only trivial solution.

Solution

Since A is a square matrix, thus A has only the trivial solution that implies A is invertible.

But A^k is also invertible so $A^k \vec{x} = \vec{0}$ has only trivial solution.

Exercise

Let $A\vec{x} = \vec{0}$ be a homogeneous system of n linear equations in n unknowns and let Q be an invertible $n \times n$ matrix. Show that $A\vec{x} = \vec{0}$ has just trivial solution if and only if $(QA)\vec{x} = \vec{0}$ has just trivial solution.

Solution

Since A is a square matrix $n \times n$. If $A\vec{x} = \vec{0}$ has just trivial solution, then A is invertible. Since Q is an invertible $n \times n$ matrix that implies QA is also invertible. Thus, $(QA)\vec{x} = \vec{0}$ has trivial solution.

On the other hand, if $(QA)\vec{x} = \vec{0}$ has trivial solution then QA is invertible.

Since Q is invertible that implies Q^{-1} is also invertible.

Thus, $A = Q^{-1}QA$ is invertible i.e. $A\vec{x} = \vec{0}$ has just trivial solution.

$A\vec{x} = \vec{0}$ has just trivial solution *iff* $(QA)\vec{x} = \vec{0}$ has just trivial solution.

Exercise

Let $A\vec{x} = \vec{b}$ be a consistent system of linear equations and let \vec{x}_1 be a fixed solution. Show that every solution to the system can be written in the form $\vec{x} = \vec{x}_1 + \vec{x}_0$ where \vec{x}_0 is a solution to $A\vec{x} = \vec{0}$. Show also that every matrix of this form is a solution.

Solution

Since \vec{x}_0 is a solution to $A\vec{x} = \vec{0}$, we have $A\vec{x}_0 = \vec{0}$.

The sum of $A\vec{x}_0 = \vec{0}$ and $A\vec{x} = \vec{b}$

$$\begin{array}{r} A\vec{x}_0 = \vec{0} \\ + \quad A\vec{x} = \vec{b} \\ \hline A\vec{x}_0 + A\vec{x} = \vec{0} + \vec{b} \end{array}$$

$$A(\vec{x} + \vec{x}_0) = \vec{b}$$

As adding an equation to the original equation does not affect the solution.

If we let \vec{x}_1 be a fixed solution, then every solution to $A\vec{x} = \vec{b}$ is $\vec{x} = \vec{x}_1 + \vec{x}_0$

Besides that

$$\begin{aligned} A(\vec{x}_1 + \vec{x}_0) &= A\vec{x}_1 + A\vec{x}_0 \\ &= \vec{b} + \vec{0} \\ &= \vec{b} \end{aligned}$$

So, every matrix (vector) in the form $\vec{x}_1 + \vec{x}_0$ is a solution to $A\vec{x} = \vec{b}$.