Lecture R – Introduction to Differential Equation

Solution *Section* R.1 – Derivative

Exercise

Find the derivative of $f(t) = -3t^2 + 2t - 4$

Solution

$$f'(t) = \underline{-6t + 2}$$

Exercise

Find the derivative of $g(x) = 4\sqrt[3]{x} + 2$

Solution

$$g(x) = 4x^{1/3} + 2$$

$$g'(x) = \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3x^{2/3}}$$

$$= \underline{\frac{4}{3\sqrt[3]{x^2}}}$$

Exercise

Find the derivative of $f(x) = x\left(x^2 + 1\right)$

Solution

$$f(x) = x^3 + x$$

$$f'(x) = \underline{3x^2 + 1}$$

Exercise

Find the derivative of $f(x) = \frac{2x^2 - 3x + 1}{x}$

Solution

$$f(x) = \frac{2x^2}{x} - \frac{3x}{x} + \frac{1}{x}$$

$$= 2x - 3 + \frac{1}{x}$$

$$f'(x) = \underline{2 - \frac{1}{x^2}} \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

Exercise

Find the derivative of $f(x)=\frac{4x^3-3x^2+2x+5}{x^2}$

Solution

$$f(x)=4x-3+\frac{2}{x}+5x^{-2} \qquad \left(\frac{1}{x}\right)'=-\frac{1}{x^2}$$

$$f'(x)=4-\frac{2}{x^2}-10x^{-3}$$

$$=\underline{\left|4-\frac{2}{x^2}-\frac{10}{x^3}\right|}$$

Exercise

Find the derivative of $f(x)=\frac{-6x^3+3x^2-2x+1}{x}$

Solution

$$f(x)=-6x^2+3x-2+\frac{1}{x} \qquad \left(\frac{1}{x}\right)'=-\frac{1}{x^2}$$

$$f'(x)=\underline{\left|-12x+3-\frac{1}{x^2}\right|}$$

Exercise

Find the derivative of $f(x)=x\left(1-\frac{2}{x+1}\right)$

Solution

$$f(x)=x-\frac{2x}{x+1}$$

$$\left(\frac{2x}{x+1}\right)'\Rightarrow \qquad \begin{matrix} f=2x & f'=2 \\ g=x+1 & g'=1 \end{matrix}$$

$$f'(x)=1-\frac{2(x+1)-2x}{(x+1)^2}$$

$$=1-\frac{2x+2-2x}{(x+1)^2}$$

$$=\underline{\left|1-\frac{2}{(x+1)^2}\right|}$$

Exercise

Find the derivative of $g(s)=\dfrac{s^2-2s+5}{\sqrt{s}}$

Solution

$$g(s)=\frac{s^2}{s^{1/2}}-2\frac{s}{s^{1/2}}+\frac{5}{s^{1/2}}$$
$$=s^{3/2}-2s^{1/2}+5s^{-1/2}$$

$$g'(s)=\frac{3}{2}s^{1/2}-2\frac{1}{2}s^{-1/2}+5\left(-\frac{1}{2}\right)s^{-3/2}$$
$$=\frac{3}{2}s^{1/2}-s^{-1/2}-\frac{5}{2}s^{-3/2}$$
$$=\frac{3}{2}\sqrt{s}-\frac{1}{\sqrt{s}}-\frac{5}{2s^{3/2}}$$
$$=\underline{\left|\frac{3}{2}\sqrt{s}-\frac{1}{\sqrt{s}}-\frac{5}{2s\sqrt{s}}\right|}$$

Exercise

Find the derivative of $f(x)=\dfrac{x+1}{\sqrt{x}}$

Solution

$$f(x)=\frac{x}{x^{1/2}}+\frac{1}{x^{1/2}}$$
$$=x^{1/2}+x^{-1/2}$$

$$f'(x)=\frac{1}{2}x^{-1/2}-\frac{1}{2}x^{-3/2}$$
$$=\underline{\left|\frac{1}{2x^{1/2}}-\frac{1}{2x^{3/2}}\right|}$$

Exercise

Find the derivative to the following functions $y=3x\left(2x^2+5x\right)$

Solution

$$y=6x^3+15x^2$$
$$\Rightarrow \underline{\left|y'=18x^2+30x\right|}$$

Exercise

Find the derivative to the following functions $y = 3\left(2x^2 + 5x\right)$

Solution

$$y = 6x^2 + 15x$$

$$\Rightarrow \boxed{y' = 12x + 15}$$

Exercise

Find the derivative to the following functions $y = \frac{x^2 + 4x}{5}$

Solution

$$\underline{\left| y' = \frac{1}{5}(2x + 4) \right|}$$

Exercise

Find the derivative to the following functions $y = \frac{3x^4}{5}$

Solution

$$\underline{\left| y' = \frac{12}{5}x^3 \right|}$$

Exercise

Find the derivative to the following functions $y = \frac{x^2 - 4}{2x + 5}$

Solution

$$y' = \frac{(2x + 5)(2x) - (x^2 - 4)(2)}{(2x + 5)^2}$$

$$= \frac{4x^2 + 10x - 2x^2 + 8}{(2x + 5)^2}$$

$$\underline{\left| = \frac{2x^2 + 10x + 8}{(2x + 5)^2} \right|}$$

Exercise

Find the derivative to the following functions $y = \frac{(1 + x)(2x - 1)}{x - 1}$

Solution

$$y' = \frac{(x-1)\frac{d}{dx}\left[(1+x)(2x-1)\right]-(1+x)(2x-1)\frac{d}{dx}[x-1]}{(x-1)^2}$$

$$= \frac{(x-1)\left[(1)(2x-1)+2(1+x)\right]-(1+x)(2x-1)(1)}{(x-1)^2}$$

$$= \frac{(x-1)(2x-1+2+2x)-(2x-1+2x^2-x)}{(x-1)^2}$$

$$= \frac{(x-1)(4x+1)-2x+1-2x^2+x}{(x-1)^2}$$

$$= \frac{4x^2+x-4x-1-2x+1-2x^2+x}{(x-1)^2}$$

$$= \boxed{\frac{2x^2-4x}{(x-1)^2}}$$

Or

$$y = \frac{(1+x)(2x-1)}{x-1}$$

$$= \frac{2x-1+2x^2-x}{x-1}$$

$$= \frac{2x^2+x-1}{x-1}$$

$$y' = \frac{(x-1)(4x+1)-(2x^2+x-1)(1)}{(x-1)^2}$$

$$= \frac{4x^2+x-4x-1-2x^2-x+1}{(x-1)^2}$$

$$= \boxed{\frac{2x^2-4x}{(x-1)^2}}$$

Exercise

Find the derivative to the following functions $y = \frac{4}{2x+1}$

Solution

$$y = 4(2x+1)^{-1}$$

$$y' = -4(2x+1)^{-2}(2)$$

$$= -8(2x+1)^{-2}$$

$$= \boxed{-\frac{8}{(2x+1)^2}}$$

Exercise

Find the derivative to the following functions $y = \dfrac{2}{(x-1)^3} = 2(x-1)^{-3}$

Solution

$$y = 2(x-1)^{-3}$$

$$y' = 2(-3)(x-1)^{-4}(1)$$

$$= \boxed{-\frac{6}{(x-1)^4}}$$

Exercise

Find the derivative to the following functions $y = \sqrt[3]{(x+4)^2}$

Solution

$$y = (x+4)^{2/3}$$

$$y' = \frac{2}{3}(x+4)^{-1/3}$$

$$= \frac{2}{3}\frac{1}{(x+4)^{1/3}}$$

$$= \boxed{\frac{2}{3\sqrt[3]{x+4}}}$$

Exercise

Find the derivative of $f(x) = \sqrt{2t^2+5t+2}$

Solution

$$f(t) = \left(2t^2+5t+2\right)^{1/2} \qquad U = 2t^2+5t+2 \;\rightarrow\; U' = 4t+5$$

$$f'(t) = \frac{1}{2}(4t+5)\left(2t^2+5t+2\right)^{-1/2} \qquad \left(U^n\right)' = nU'U^{n-1}$$

$$= \boxed{\frac{1}{2}\frac{4t+5}{\sqrt{2t^2+5t+2}}}$$

Exercise

Find the derivative of $f(x)=\dfrac{1}{\left(x^2-3x\right)^2}$

Solution

$$f(x)=\left(x^2-3x\right)^{-2}$$

$$f'(x)=-2(2x-3)\left(x^2-3x\right)^{-3}$$

$$=\boxed{-\frac{2(2x-3)}{\left(x^2-3x\right)^3}}$$

Exercise

Find the derivative of $y=t^2\sqrt{t-2}$

Solution

$$y'=2t\sqrt{t-2}+t^2\tfrac{1}{2}(t-2)^{-1/2}$$

$$f=t^2 \qquad f'=2t$$

$$g=(t-2)^{1/2} \qquad g'=\tfrac{1}{2}(t-2)^{-1/2}$$

$$=\left[2t(t-2)^{1/2}+t^2\tfrac{1}{2}(t-2)^{-1/2}\right]\frac{2(t-2)^{1/2}}{2(t-2)^{1/2}}$$

$$=\frac{4t(t-2)+t^2}{2(t-2)^{1/2}}$$

$$=\frac{4t^2-8t+t^2}{2\sqrt{t-2}}$$

$$=\boxed{\frac{5t^2-4t}{2\sqrt{t-2}}}$$

Exercise

Find the derivative of $y=\left(\dfrac{6-5x}{x^2-1}\right)^2$

Solution

$$f=6-5x \qquad f'=-5$$

$$g=x^2-1 \qquad g'=2x$$

$$y' = 2\frac{-5\left(x^2-1\right)-2x(6-5x)}{\left(x^2-1\right)^2}\left(\frac{6-5x}{x^2-1}\right) \qquad \left(U^n\right)' = nU'U^{n-1}$$

$$= 2\frac{-5x^2+5-12x+10x^2}{\left(x^2-1\right)^3}(6-5x)$$

$$= \boxed{\frac{2\left(5x^2-12x+5\right)(6-5x)}{\left(x^2-1\right)^3}}$$

Exercise

Find the derivative to the following functions $y = x^2\sqrt{x^2+1}$

Solution

$$y = x^2\left(x^2+1\right)^{1/2}$$

$$y' = x^2\frac{d}{dx}\left[(x^2+1)^{1/2}\right] + (x^2+1)^{1/2}\frac{d}{dx}\left[x^2\right]$$

$$= x^2\left[\frac{1}{2}(x^2+1)^{-1/2}(2x)\right] + (x^2+1)^{1/2}\left[2x\right]$$

$$= x^3(x^2+1)^{-1/2} + 2x(x^2+1)^{1/2}$$

$$= \frac{(x^2+1)^{1/2}}{(x^2+1)^{1/2}}\left[x^3(x^2+1)^{-1/2} + 2x(x^2+1)^{1/2}\right]$$

$$= \frac{x^3(x^2+1)^{-1/2}(x^2+1)^{1/2} + 2x(x^2+1)^{1/2}(x^2+1)^{1/2}}{(x^2+1)^{1/2}}$$

$$= \frac{x^3+2x(x^2+1)}{(x^2+1)^{1/2}}$$

$$= \frac{x^3+2x^3+2x}{\sqrt{x^2+1}}$$

$$= \frac{3x^3+2x}{\sqrt{x^2+1}}$$

$$= \boxed{\frac{x(3x^2+2)}{\sqrt{x^2+1}}}$$

Exercise

Find the derivative to the following functions $y=\left(\frac{x+1}{x-5}\right)^2$

Solution

$$y'=2\left(\frac{x+1}{x-5}\right)\frac{d}{dx}\left[\frac{x+1}{x-5}\right]$$

$$=2\left(\frac{x+1}{x-5}\right)\left[\frac{(1)(x-5)-(1)(x+1)}{(x-5)^2}\right]$$

$$=2\left(\frac{x+1}{x-5}\right)\left(\frac{x-5-x-1}{(x-5)^2}\right)$$

$$=2\left(\frac{x+1}{x-5}\right)\left(\frac{-6}{(x-5)^2}\right)$$

$$=\underline{\left|-\frac{12(x+1)}{(x-5)^3}\right|}$$

Exercise

Find the derivative to the following functions $y=x^2\sin x$

Solution

$$y'=\underline{\left|2x\sin x+x^2\cos x\right|}$$

$u=x^2 \quad v=\sin x$

$u'=2x \quad v'=\cos x$

Exercise

Find the derivative to the following functions $y=\frac{\sin x}{x}$

Solution

$$y'=\underline{\left|\frac{x\cos x-\sin x}{x^2}\right|}$$

$u=\sin x \quad v=x$

$u'=\cos x \quad v'=1$

Exercise

Find the derivative to the following functions $y=\frac{\cot x}{1+\cot x}$

Solution

$$y'=\frac{-\csc^2 x(1+\cot x)+\csc^2 x\cot x}{(1+\cot x)^2}$$

$u=\cot x \quad v=1+\cot x$

$u'=-\csc^2 x \quad v'=-\csc^2 x$

$$= \frac{-\csc^2 x - \csc^2 x \cot x + \csc^2 x \cot x}{(1+\cot x)^2}$$

$$= \boxed{\frac{-\csc^2 x}{(1+\cot x)^2}}$$

Exercise

Find the derivative to the following functions $y = x^2 \sin x + 2x\cos x - 2\sin x$

Solution

$$y' = 2x\sin x + x^2 \cos x + 2\cos x - 2x\sin x - 2\cos x$$

$$= \boxed{x^2 \cos x}$$

Exercise

Find the derivative to the following functions $y = x^3 \sin x \cos x$

Solution

$$y' = \left(x^3\right)' \sin x \cos x + x^3 (\sin x)' \cos x + x^3 \sin x (\cos x)'$$

$$= \boxed{3x^2 \sin x \cos x + x^3 \cos^2 x - x^3 \sin^2 x}$$

Exercise

Find the derivative to the following functions $y = \frac{4}{\cos x} + \frac{1}{\tan x}$

Solution

$$y' = \frac{-4\sin x}{\cos^2 x} - \frac{\sec^2 x}{\tan^2 x} \qquad \left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$$

$$= -4\frac{\sin x}{\cos x}\frac{1}{\cos x} - \frac{1}{\cos^2 x}\frac{\cos^2 x}{\sin^2 x}$$

$$= \boxed{-4\tan x \sec x - \csc^2 x}$$

Exercise

Find the derivative to the following functions $f(x) = x^2 e^x$

Solution

$$f'(x) = e^x \frac{d}{dx}\left(x^2\right) + x^2 \frac{d}{dx}\left(e^x\right)$$

$$= e^x(2x) + x^2e^x$$

$$\underline{\boxed{= xe^x(2+x)}}$$

Exercise

Find the derivative to the following functions $f(x) = \frac{e^x + e^{-x}}{2}$

Solution

$$f(x) = \frac{e^x + e^{-x}}{2}$$

$$= \frac{1}{2}(e^x + e^{-x})$$

$$f'(x) = \frac{1}{2}\left(\frac{d}{dx}\left[e^x\right] + \frac{d}{dx}\left[e^{-x}\right]\right)$$

$$\underline{\boxed{= \frac{1}{2}(e^x - e^{-x})}}$$

Exercise

Find the derivative to the following functions $f(x) = \frac{e^x}{x^2}$

Solution

$$f'(x) = \frac{x^2e^x - e^x(2x)}{x^4}$$

$$= \frac{x^2e^x - 2xe^x}{x^4}$$

$$= \frac{xe^x(x-2)}{x^4}$$

$$\underline{\boxed{= \frac{e^x(x-2)}{x^3}}}$$

Exercise

Find the derivative to the following functions $f(x) = x^2e^x - e^x$

Solution

$$f'(x) = e^x\frac{d}{dx}[x^2] + x^2\frac{d}{dx}[e^x] - \frac{d}{dx}[e^x]$$

$$= e^x(2x) + x^2e^x - e^x$$

$$\underline{\boxed{= e^x(x^2 + 2x - 1)}}$$

Exercise

Find the derivative to the following functions $f(x) = (1+2x)e^{4x}$

Solution

$$\begin{aligned} f'(x) &= (2)e^{4x} + (1+2x)(4e^{4x}) \\ &= 2e^{4x} + (1+2x)(4e^{4x}) \\ &= 2e^{4x}(1+2(1+2x)) \\ &= 2e^{4x}(1+2+4x) \\ &= \underline{\boxed{2e^{4x}(3+4x)}} \end{aligned}$$

Exercise

Find the derivative to the following functions $y = x^2 e^{5x}$

Solution

$$\begin{aligned} y' &= x^2\left(5e^{5x}\right) + 2x\left(e^{5x}\right) \\ &= \underline{\boxed{xe^{5x}(5x+2)}} \end{aligned}$$

Exercise

Find the derivative to the following functions $y = e^{x^2+1}\sqrt{5x+2}$

Solution

$$\begin{aligned} y &= (2x)e^{x^2+1}\sqrt{5x+2} + e^{x^2+1}\frac{5}{2\sqrt{5x+2}} \\ &= 2xe^{x^2+1}\sqrt{5x+2}\frac{2\sqrt{5x+2}}{2\sqrt{5x+2}} + \frac{5e^{x^2+1}}{2\sqrt{5x+2}} \\ &= \frac{4xe^{x^2+1}(5x+2)}{2\sqrt{5x+2}} + \frac{5e^{x^2+1}}{2\sqrt{5x+2}} \\ &= \frac{20x^2e^{x^2+1}+8xe^{x^2+1}+5e^{x^2+1}}{2\sqrt{5x+2}} \\ &= \underline{\boxed{\frac{e^{x^2+1}\left(20x^2+8x+5\right)}{2\sqrt{5x+2}}}} \end{aligned}$$

Exercise

Find the derivative to the following functions $f(x)=\ln\sqrt[3]{x+1}$

Solution

$$f(x)=\ln(x+1)^{1/3}$$

$$=\frac{1}{3}\ln(x+1)$$

$$u=x+1\Rightarrow\frac{du}{dx}=1$$

$$f'(x)=\frac{1}{3}\frac{1}{x+1}$$

$$=\boxed{\frac{1}{3(x+1)}}$$

Exercise

Find the derivative to the following functions $f(x)=\ln\left[x^2\sqrt{x^2+1}\right]$

Solution

$$f(x)=\ln(x^2)+\ln\sqrt{x^2+1}$$ ***Product Property***

$$f(x)=\ln(x^2)+\ln(x^2+1)^{1/2}$$

$$f(x)=2\ln x+\frac{1}{2}\ln(x^2+1)$$ ***Power Property***

$$f'(x)=2\frac{1}{x}+\frac{1}{2}\frac{2x}{x^2+1}$$ ***Differentiate***

$$=\boxed{\frac{2}{x}+\frac{x}{x^2+1}}$$

Exercise

Find the derivative to the following functions $y=\ln\frac{x^2}{x^2+1}$

Solution

$$y=\ln x^2-\ln x^2+1$$

$$y'=\frac{2x}{x^2}-\frac{2x}{x^2+1}$$

$$=\boxed{\frac{2}{x}-\frac{2x}{x^2+1}}$$

Exercise

Find the derivative to the following functions $y = \ln \dfrac{1+e^x}{1-e^x}$

Solution

$$y = \ln\left(1+e^x\right) - \ln\left(1-e^x\right)$$

$$y' = \frac{e^x}{1+e^x} - \frac{-e^x}{1-e^x}$$

$$= \frac{e^x}{1+e^x} + \frac{e^x}{1-e^x}$$

$$= \frac{e^x - e^{2x} + e^x + e^{2x}}{\left(1+e^x\right)\left(1-e^x\right)}$$

$$= \boxed{\frac{2e^x}{\left(1+e^x\right)\left(1-e^x\right)}}$$

Exercise

Find the derivative to the following functions $y = x\ 3^{x+1}$

Solution

$$y' = 3^{x+1} + x\ 3^{x+1}\ln 3$$

$$= \boxed{3^{x+1}\left(1 + x\ \ln 3\right)}$$

Exercise

Find the derivative to the following functions $f(t) = \dfrac{\log_8\left(t^{3/2}+1\right)}{t}$

Solution

$$f' = \frac{\dfrac{\frac{1}{\ln 8}\dfrac{\frac{3}{2}t^{1/2}}{t^{3/2}+1}\cdot t - \log_8\left(t^{3/2}+1\right)}{t}}{t}$$

$$= \frac{\dfrac{\frac{1}{\ln 8}\dfrac{\frac{3}{2}t^{3/2}}{t^{3/2}+1} - \log_8\left(t^{3/2}+1\right)}{t^2}}{t^2} \cdot \frac{2\ln 8\left(t^{3/2}+1\right)}{2\ln 8\left(t^{3/2}+1\right)}$$

$$= \boxed{\frac{3t^{3/2} - 2\left(t^{3/2}+1\right)(\ln 8)\log_8\left(t^{3/2}+1\right)}{t^2\left(t^{3/2}+1\right)\ln 8}}$$