Solution Section 3.1 – Inner Products

Exercise

Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (1, 1)$, $\vec{v} = (3, 2)$, $\vec{w} = (0, -1)$, and $k = 3$. Compute the following.

a) $\langle \vec{u}, \vec{v} \rangle$
b) $\langle k\vec{v}, \vec{w} \rangle$
c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle$
d) $\|\vec{v}\|$
e) $d(\vec{u}, \vec{v})$
f) $\|\vec{u} - k\vec{v}\|$

Solution

a) $\langle \vec{u}, \vec{v} \rangle = 1(3) + 1(2)$

$= 5$

b) $\langle k\vec{v}, \vec{w} \rangle = \langle 3v, w \rangle$

$= 9 \cdot 0 + 6 \cdot (-1)$

$= -6$

c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

$= 1 \cdot 0 + 1 \cdot (-1) + 3 \cdot 0 + 2 \cdot (-1)$

$= -3$

d) $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

$= \sqrt{3^2 + 2^2}$

$= \sqrt{13}$

e) $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$

$= \|(-2, -1)\|$

$= \sqrt{(-2)^2 + (-1)^2}$

$= \sqrt{5}$

f) $\|\vec{u} - k\vec{v}\| = \|(1, 1) - 3(3, 2)\|$

$= \|(-8, -5)\|$

$= \sqrt{(-8)^2 + (-5)^2}$

$= \sqrt{89}$

Exercise

Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (1, 1)$, $\vec{v} = (3, 2)$, $\vec{w} = (0, -1)$ and $k = 3$.

Compute the following for the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 2u_1 v_1 + 3u_2 v_2$.

a) $\langle \vec{u}, \vec{v} \rangle$ *c)* $\langle \vec{u} + \vec{v}, \vec{w} \rangle$ *e)* $d(\vec{u}, \vec{v})$

b) $\langle k\vec{v}, \vec{w} \rangle$ *d)* $\|\vec{v}\|$ *f)* $\|\vec{u} - k\vec{v}\|$

Solution

a)
$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= 2(1)(3) + 3(1)(2) \\ &= 12 \end{aligned}$$

b)
$$\begin{aligned} \langle k\vec{v}, \vec{w} \rangle &= 2(3\cdot 3)(0) + 3(3\cdot 2)(-1) \\ &= -18 \end{aligned}$$

c)
$$\begin{aligned} \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \\ &= 1\cdot 0 + 1\cdot(-1) + 3\cdot 0 + 2\cdot(-1) \\ &= -3 \end{aligned}$$

d)
$$\begin{aligned} \|\vec{v}\| &= \sqrt{\langle \vec{v}, \vec{v} \rangle} \\ &= \sqrt{2(3)(3) + 3(2)(2)} \\ &= \sqrt{30} \end{aligned}$$

e)
$$\begin{aligned} d(\vec{u}, \vec{v}) &= \|\vec{u} - \vec{v}\| \\ &= \|\langle(-2, -1)\rangle\| \\ &= \sqrt{2(-2)(-2) + 3(-1)(-1)} \\ &= \sqrt{11} \end{aligned}$$

f)
$$\begin{aligned} \|\vec{u} - k\vec{v}\| &= \|(1, 1) - 3(3, 2)\| \\ &= \|\langle(-8, -5)\rangle\| \\ &= \sqrt{2(-8)^2 + 3(-5)^2} \\ &= \sqrt{203} \end{aligned}$$

Exercise

Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (3, -2)$, $\vec{v} = (4, 5)$, $\vec{w} = (-1, 6)$, and $k = -4$. Verify the following.

a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

d) $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$

e) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$

Solution

a) $\langle \vec{u}, \vec{v} \rangle = 3 \cdot 4 + (-2) \cdot (5)$

$= 2$

$\langle \vec{v}, \vec{u} \rangle = 4 \cdot 3 + (5) \cdot (-2)$

$= 2$

$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ ✓

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle (7,3), (-1,6) \rangle$

$= 7(-1) + 3(6)$

$= 11$

$\langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle = (3)(-1) + (-2)(6) + (4)(-1) + (5)(6)$

$= 11$

$\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ ✓

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle (3, -2), (3, 11) \rangle$

$= 3(3) + (-2)(11)$

$= -13$

$\langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle = (3)(4) + (-2)(5) + (3)(-1) + (-2)(6)$

$= -13$

$\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$ ✓

d) $\langle k\vec{u}, \vec{v} \rangle = (-4 \cdot 3) \cdot 4 + ((-4)(-2)) \cdot (5)$

$= -8$

$k\langle \vec{u}, \vec{v} \rangle = (-4)(3 \cdot 4 + (-2) \cdot (5))$

$= -8$

$\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$ ✓

e) $\left\langle \vec{0},\ \vec{v} \right\rangle = 0\cdot 4 + 0\cdot(5)$

$\underline{= 0}$

$\left\langle \vec{v},\ \vec{0} \right\rangle = 4\cdot 0 + (5)\cdot(0)$

$\underline{= 0}$

$\left\langle \vec{0},\ \vec{v} \right\rangle = \left\langle \vec{v},\ \vec{0} \right\rangle = 0 \quad \checkmark$

Exercise

Let $\langle \vec{u},\ \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (3,\ -2)$, $\vec{v} = (4,\ 5)$, $\vec{w} = (-1,\ 6)$, and $k = -4$. Verify the following for the weighted Euclidean inner product $\langle \vec{u},\ \vec{v} \rangle = 4u_1v_1 + 5u_2v_2$.

a) $\langle \vec{u},\ \vec{v} \rangle = \langle \vec{v},\ \vec{u} \rangle$

b) $\langle \vec{u}+\vec{v},\ \vec{w} \rangle = \langle \vec{u},\ \vec{w} \rangle + \langle \vec{v},\ \vec{w} \rangle$

c) $\langle \vec{u},\ \vec{v}+\vec{w} \rangle = \langle \vec{u},\ \vec{v} \rangle + \langle \vec{u},\ \vec{w} \rangle$

d) $\langle k\vec{u},\ \vec{v} \rangle = k\langle \vec{u},\ \vec{v} \rangle$

e) $\left\langle \vec{0},\ \vec{v} \right\rangle = \left\langle \vec{v},\ \vec{0} \right\rangle = 0$

Solution

a) $\langle \vec{u},\ \vec{v} \rangle = 4\cdot 3\cdot 4 + 5\cdot(-2)\cdot(5)$

$\underline{= -2}$

$\langle \vec{v},\ \vec{u} \rangle = 4\cdot 4\cdot 3 + 5\cdot(5)\cdot(-2)$

$\underline{= -2}$

$\langle \vec{u},\ \vec{v} \rangle = \langle \vec{v},\ \vec{u} \rangle \quad \checkmark$

b) $\langle \vec{u}+\vec{v},\ \vec{w} \rangle = \langle (7,3),\ (-1,6) \rangle$

$= 4\cdot 7(-1) + 5\cdot 3(6)$

$\underline{= 62}$

$\langle \vec{u},\ \vec{w} \rangle + \langle \vec{v},\ \vec{w} \rangle = 4\cdot(3)(-1) + 5\cdot(-2)(6) + 4\cdot(4)(-1) + 5\cdot(5)(6)$

$\underline{= 62}$

$\langle \vec{u}+\vec{v},\ \vec{w} \rangle = \langle \vec{u},\ \vec{w} \rangle + \langle \vec{v},\ \vec{w} \rangle \quad \checkmark$

c) $\langle \vec{u},\ \vec{v}+\vec{w} \rangle = \langle (3,\ -2),\ (3,\ 11) \rangle$

$= 4\cdot 3(3) + 5\cdot(-2)(11)$

$\underline{= -74}$

$\langle \vec{u},\ \vec{v} \rangle + \langle \vec{u},\ \vec{w} \rangle = 4\cdot(3)(4) + 5\cdot(-2)(5) + 4\cdot(3)(-1) + 5\cdot(-2)(6)$

$\underline{= -74}$

$\langle \vec{u},\ \vec{v}+\vec{w} \rangle = \langle \vec{u},\ \vec{v} \rangle + \langle \vec{u},\ \vec{w} \rangle$ ✓

d) $\langle k\vec{u},\ \vec{v} \rangle = 4\cdot(-4\cdot 3)\cdot 4 + 5\cdot\big((-4)(-2)\big)\cdot(5)$

$\underline{= 8}$

$k\langle \vec{u},\ \vec{v} \rangle = (-4)\big(4\cdot 3\cdot 4 + 5\cdot(-2)\cdot(5)\big)$

$\underline{= 8}$

$\langle k\vec{u},\ \vec{v} \rangle = k\langle \vec{u},\ \vec{v} \rangle$ ✓

e) $\langle \vec{0},\ \vec{v} \rangle = 4\cdot 0\cdot 4 + 5\cdot 0\cdot(5)$

$\underline{= 0}$

$\langle \vec{v},\ \vec{0} \rangle = 4\cdot 4\cdot 0 + 5\cdot(5)\cdot(0)$

$\underline{= 0}$

$\langle \vec{0},\ \vec{v} \rangle = \langle \vec{v},\ \vec{0} \rangle = 0$ ✓

Exercise

Let $\vec{u} = (u_1,\ u_2)$ and $\vec{v} = (v_1,\ v_2)$. Show that the following are inner product on $\mathbb{R}^2$ by verifying that the inner product axioms hold. $\langle \vec{u},\ \vec{v} \rangle = 3u_1v_1 + 5u_2v_2$

Solution

Axiom 1: $\langle \vec{u},\ \vec{v} \rangle = 3u_1v_1 + 5u_2v_2$

$= 3v_1u_1 + 5v_2u_2$

$= \langle \vec{v},\ \vec{u} \rangle$ ✓

Axiom 2: $\langle \vec{u}+\vec{v},\ \vec{w} \rangle = 3(u_1+v_1)w_1 + 5(u_2+v_2)w_2$

$= 3(u_1w_1 + v_1w_1) + 5(u_2w_2 + v_2w_2)$

$= 3u_1w_1 + 3v_1w_1 + 5u_2w_2 + 5v_2w_2$

$= (3u_1w_1 + 5u_2w_2) + (3v_1w_1 + 5v_2w_2)$

$= \langle \vec{u},\ \vec{w} \rangle + \langle \vec{v},\ \vec{w} \rangle$ ✓

Axiom 3: $\langle k\vec{u},\ \vec{v} \rangle = 3(ku_1)v_1 + 5(ku_2)v_2$

$= k(3u_1v_1 + 5u_2v_2)$

$$= k\langle \vec{u}, \vec{v}\rangle \quad \checkmark$$

$$Axiom\ 4: \quad \langle \vec{v}, \vec{v}\rangle = 3v_1v_1 + 5v_2v_2$$

$$= 3v_1^2 + 5v_2^2 \ge 0$$

$$v_1 = v_2 = 0 \quad iff \quad \vec{v} = \vec{0} \quad \checkmark$$

Exercise

Show that the following identity holds for the vectors in any inner product space

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

Solution

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 &= \langle \vec{u}+\vec{v}, \vec{u}+\vec{v}\rangle + \langle \vec{u}-\vec{v}, \vec{u}-\vec{v}\rangle \\
&= \langle \vec{u}, \vec{u}+\vec{v}\rangle + \langle \vec{v}, \vec{u}+\vec{v}\rangle + \langle \vec{u}, \vec{u}-\vec{v}\rangle - \langle \vec{v}, \vec{u}-\vec{v}\rangle \\
&= \langle \vec{u}, \vec{u}\rangle + \langle \vec{u}, \vec{v}\rangle + \langle \vec{v}, \vec{u}\rangle + \langle \vec{v}, \vec{v}\rangle + \langle \vec{u}, \vec{u}\rangle - \langle \vec{u}, \vec{v}\rangle - \langle \vec{v}, \vec{u}\rangle + \langle \vec{v}, \vec{v}\rangle \\
&= 2\langle \vec{u}, \vec{u}\rangle + 2\langle \vec{v}, \vec{v}\rangle \\
&= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2 \quad \checkmark
\end{aligned}$$

Exercise

Show that the following identity holds for the vectors in any inner product space

$$\langle \vec{u}, \vec{v}\rangle = \frac{1}{4}\left(\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2\right)$$

Solution

$$\|\vec{u}+\vec{v}\|^2 = \langle \vec{u}+\vec{v}, \vec{u}+\vec{v}\rangle = \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v}\rangle + \|\vec{v}\|^2$$

$$\|\vec{u}-\vec{v}\|^2 = \langle \vec{u}-\vec{v}, \vec{u}-\vec{v}\rangle = \|\vec{u}\|^2 - 2\langle \vec{u}, \vec{v}\rangle + \|\vec{v}\|^2$$

$$\begin{array}{rl}
& \|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v}\rangle + \|\vec{v}\|^2 \\
- & \underline{\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 - 2\langle \vec{u}, \vec{v}\rangle + \|\vec{v}\|^2} \\
& \|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2 = 4\langle \vec{u}, \vec{v}\rangle
\end{array}$$

$$\langle \vec{u}, \vec{v}\rangle = \frac{1}{4}\left(\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2\right) \quad \checkmark$$

Exercise

Prove that $\|k\vec{v}\| = |k| \; \|\vec{v}\|$

Solution

$$\|k\vec{v}\|^2 = \langle k\vec{v},\ \vec{v}\rangle$$

$$= k^2 \langle \vec{v},\ \vec{v}\rangle$$

$$= k^2 \|\vec{v}\|^2$$

$$\|k\vec{v}\| = k \; \|\vec{v}\| \quad \checkmark$$