# Solution  *Section* 3.4 – Triple Integrals

### *Exercise*

Evaluate the integral $\int_0^1\int_0^1\int_0^1 \left(x^2+y^2+z^2\right)dzdydx$

***Solution***

$$\begin{aligned}
\int_0^1\int_0^1\int_0^1 \left(x^2+y^2+z^2\right)dzdydx &= \int_0^1\int_0^1 \left[x^2z+y^2z+\frac{1}{3}z^3\right]_0^1 dydx \\
&= \int_0^1\int_0^1 \left[x^2+y^2+\frac{1}{3}\right]dydx \\
&= \int_0^1 \left[x^2y+\frac{1}{3}y^3+\frac{1}{3}y\right]_0^1 dx \\
&= \int_0^1 \left[x^2+\frac{1}{3}+\frac{1}{3}\right]dx \\
&= \left[\frac{1}{3}x^3+\frac{2}{3}x\right]_0^1 \\
&= \frac{1}{3}+\frac{2}{3} \\
&= 1
\end{aligned}$$

### *Exercise*

Evaluate the integral $\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dzdxdy$

***Solution***

$$\begin{aligned}
\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dzdxdy &= \int_0^{\sqrt{2}}\int_0^{3y}\left[8-x^2-y^2-\left(x^2+3y^2\right)\right]dxdy \\
&= \int_0^{\sqrt{2}}\int_0^{3y}\left(8-2x^2-4y^2\right)dxdy \\
&= \int_0^{\sqrt{2}}\left[8x-\frac{2}{3}x^3-4y^2x\right]_0^{3y} dy
\end{aligned}$$

$$= \int_0^{\sqrt{2}} \left(24y - 18y^3 - 12y^3\right) dy$$

$$= \int_0^{\sqrt{2}} \left(24y - 30y^3\right) dy$$

$$= \left[12y^2 - \frac{15}{2}y^4\right]_0^{\sqrt{2}}$$

$$= 24 - 30$$

$$\underline{= -6}$$

### ***Exercise***

Evaluate the integral $\int_0^{\pi/6}\int_0^1\int_{-2}^3 y\sin z\ dxdydz$

***Solution***

$$\int_0^{\pi/6}\int_0^1\int_{-2}^3 y\sin z\ dxdydz = \int_0^{\pi/6}\int_0^1 y\sin z\ \left[x\right]_{-2}^3\ dydz$$

$$= 5\int_0^{\pi/6}\int_0^1 y\sin z\ dydz$$

$$= 5\int_0^{\pi/6} \sin z\ \left[\frac{1}{2}y^2\right]_0^1 dz$$

$$= \frac{5}{2}\int_0^{\pi/6} \sin z\ dz$$

$$= -\frac{5}{2}\left[\cos z\right]_0^{\pi/6}$$

$$= -\frac{5}{2}\left(\frac{\sqrt{3}}{2} - 1\right)$$

$$\underline{= \frac{5}{4}\left(2 - \sqrt{3}\right)}$$

### Exercise

Evaluate the integral $\int_{-1}^{1}\int_{0}^{1}\int_{0}^{2}(x+y+z)\,dydxdz$

**Solution**

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{1}\int_{0}^{2}(x+y+z)\,dydxdz &= \int_{-1}^{1}\int_{0}^{1}\left[xy+\frac{1}{2}y^2+zy\right]_0^2 dxdz\\ &= \int_{-1}^{1}\int_{0}^{1}(2x+2+2z)\,dxdz\\ &= \int_{-1}^{1}\left[x^2+(2+2z)x\right]_0^1 dz\\ &= \int_{-1}^{1}(1+2+2z)\,dz\\ &= \int_{-1}^{1}(3+2z)\,dz\\ &= \left[3z+z^2\right]_{-1}^{1}\\ &= (3+1)-(-3+1)\\ &= 6\end{aligned}$$

### Exercise

Evaluate the integral $\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2}} dzdydx$

**Solution**

$$\begin{aligned}\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2}} dzdydx &= \int_{0}^{3}\int_{0}^{\sqrt{9-x^2}}\sqrt{9-x^2}\,dydx\\ &= \int_{0}^{3}\sqrt{9-x^2}\,\left[y\right]_0^{\sqrt{9-x^2}}\,dx\\ &= \int_{0}^{3}\left(9-x^2\right)dx\\ &= \left[9x-\frac{1}{3}x^3\right]_0^3\\ &= 18\end{aligned}$$

***Exercise***

Evaluate the integral $\displaystyle\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} xdzdydx$

***Solution***

$$\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} xdzdydx = \int_0^1 \int_0^{1-x^2} \left[xz\right]_3^{4-x^2-y} dydx$$

$$= \int_0^1 \int_0^{1-x^2} x\left(4 - x^2 - y - 3\right) dydx$$

$$= \int_0^1 \int_0^{1-x^2} \left(x - x^3 - xy\right) dydx$$

$$= \int_0^1 \left[\left(x - x^3\right) y - \frac{1}{2} xy^2\right]_0^{1-x^2} dx$$

$$= \int_0^1 \left[x\left(1-x^2\right)\left(1-x^2\right) - \frac{1}{2} x\left(1-x^2\right)^2\right] dx$$

$$= \int_0^1 \left(1-x^2\right)^2 \left(\frac{1}{2} x\right) dx \qquad d\left(1-x^2\right) = -2xdx$$

$$= -\frac{1}{4} \int_0^1 \left(1-x^2\right)^2 d\left(1-x^2\right)$$

$$= -\frac{1}{12}\left[\left(1-x^2\right)^3\right]_0^1$$

$$= -\frac{1}{12}(0-1)$$

$$\boxed{= \frac{1}{12}}$$

### *Exercise*

Evaluate the integral $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\cos(u+v+w)\,dudvdw$

***Solution***

$$\begin{aligned}
\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\cos(u+v+w)\,dudvdw &= \int_0^{\pi}\int_0^{\pi}\Big[\sin(u+v+w)\Big]_0^{\pi}\,dvdw \\
&= \int_0^{\pi}\int_0^{\pi}\Big[\sin(v+w+\pi)-\sin(v+w)\Big]dvdw \\
&= \int_0^{\pi}\Big[-\cos(v+w+\pi)+\cos(v+w)\Big]_0^{\pi}\,dw \\
&= \int_0^{\pi}\Big[-\cos(w+2\pi)+\cos(w+\pi)+\cos(w+\pi)-\cos(w)\Big]dw \\
&= \int_0^{\pi}\Big[-\cos(w+2\pi)+2\cos(w+\pi)-\cos(w)\Big]dw \\
&= \Big[-\sin(w+2\pi)+2\sin(w+\pi)-\sin(w)\Big]_0^{\pi} \\
&= -\sin(3\pi)+2\sin(2\pi)-\sin\pi-\big(-\sin(2\pi)+2\sin(\pi)-\sin 0\big) \\
&= 0
\end{aligned}$$

### *Exercise*

Evaluate the integral $\displaystyle\int_0^{\pi/4}\int_0^{\ln\sec v}\int_{-\infty}^{2t} e^x\,dxdtdv$

***Solution***

$$\begin{aligned}
\int_0^{\pi/4}\int_0^{\ln\sec v}\int_{-\infty}^{2t} e^x\,dxdtdv &= \int_0^{\pi/4}\int_0^{\ln\sec v}\Big[e^x\Big]_{-\infty}^{2t}\,dtdv \\
&= \int_0^{\pi/4}\int_0^{\ln\sec v}\left(e^{2t}-e^{-\infty}\right)dtdv \\
&= \int_0^{\pi/4}\int_0^{\ln\sec v}\left(e^{2t}-e^{-\infty}\right)dtdv \qquad\qquad e^{-\infty}=0 \\
&= \int_0^{\pi/4}\int_0^{\ln\sec v} e^{2t}\,dtdv
\end{aligned}$$

$$=\frac{1}{2}\int_0^{\pi/4}\left[e^{2t}\right]_0^{\ln\sec v}dv$$

$$=\frac{1}{2}\int_0^{\pi/4}\left(e^{2\ln\sec v}-1\right)dv \qquad e^{2\ln\sec v}=e^{\ln\sec^2 v}=\sec^2 v$$

$$=\frac{1}{2}\int_0^{\pi/4}\left(\sec^2 v-1\right)dv$$

$$=\frac{1}{2}\left[\tan v-v\right]_0^{\pi/4}$$

$$=\frac{1}{2}\left(1-\frac{\pi}{4}\right)$$

$$=\frac{1}{2}-\frac{\pi}{8}$$

## *Exercise*

Evaluate the integral $\displaystyle\int_0^1\int_{-z}^{z}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dydxdz$

***Solution***

$$\int_0^1\int_{-z}^{z}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dydxdz=\int_0^1\int_{-z}^{z}y\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dxdz$$

$$=2\int_0^1\int_{-z}^{z}\sqrt{1-x^2}\,dxdz \qquad \int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$=2\int_0^1\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\Big|_{-z}^{z}dz$$

$$=2\int_0^1\left(\frac{z}{2}\sqrt{1-z^2}+\frac{1}{2}\sin^{-1}z+\frac{z}{2}\sqrt{1-z^2}+\frac{1}{2}\sin^{-1}z\right)dz$$

$$=2\int_0^1\left(z\sqrt{1-z^2}+\sin^{-1}z\right)dz \qquad \int\sin^{-1}xdx=x\sin^{-1}x+\sqrt{1-x^2}$$

$$=-\int_0^1\left(1-z^2\right)^{1/2}d\left(1-z^2\right)+2\int_0^1\left(\sin^{-1}z\right)dz$$

$$= -\frac{2}{3}\left(1-z^2\right)^{3/2} + 2\left(z\sin^{-1} z + \sqrt{1-z^2}\right)\Bigg|_0^1$$

$$= 2\sin^{-1}1 + \frac{2}{3} - 2$$

$$= \pi - \frac{4}{3}$$

### *Exercise*

Evaluate the integral $\displaystyle\int_0^{\pi}\int_0^{y}\int_0^{\sin x} dzdxdy$

***Solution***

$$\int_0^{\pi}\int_0^{y}\int_0^{\sin x} dzdxdy = \int_0^{\pi}\int_0^{y} z\Big|_0^{\sin x} dxdy$$

$$= \int_0^{\pi}\int_0^{y} \sin x\, dxdy$$

$$= -\int_0^{\pi} \cos x\Big|_0^{y} dy$$

$$= -\int_0^{\pi} (\cos y - 1)\, dy$$

$$= -(\sin y - y)\Big|_0^{\pi}$$

$$= \pi$$

### *Exercise*

Evaluate the integral $\displaystyle\int_0^{9}\int_0^{1}\int_{2y}^{2} \frac{4\sin x^2}{\sqrt{z}} dxdydz$

***Solution***

$$\begin{cases} 2y \le x \le y & \rightarrow \quad 0 \le x \le 2 \\ 0 \le y \le 1 & \rightarrow \quad 0 \le y \le \frac{x}{2} \end{cases}$$

$$\int_0^9\int_0^1\int_{2y}^2 \frac{4\sin x^2}{\sqrt{z}}\,dxdydz = \int_0^9 z^{-1/2}dz\int_0^2\int_0^{x/2} 4\sin x^2 dydx$$

$$= 8z^{1/2}\Big|_0^9 \int_0^2 \sin x^2 \left[\,y\Big|_0^{x/2}\right] dx$$

$$= 4(3)\int_0^2 x\sin x^2\ dx$$

$$= 6\int_0^2 \sin x^2\ d\left(x^2\right)$$

$$= -6\cos x^2\Big|_0^2$$

$$= -6(\cos 4 - 1)$$

$$\underline{= 6 - 6\cos 4\,|}$$

## *Exercise*

Evaluate the integral $\displaystyle\int_0^\pi\int_0^\pi\int_0^\pi \cos(x+y+z)\,dxdydz$

***Solution***

$$\int_0^\pi\int_0^\pi\int_0^\pi \cos(x+y+z)\,dxdydz = \int_0^\pi\int_0^\pi \sin(x+y+z)\Big|_0^\pi\ dydz$$

$$= \int_0^\pi\int_0^\pi \big(\sin(\pi+y+z) - \sin(y+z)\big)\,dydz$$

$$= \int_0^\pi \big(-\cos(2\pi+z) + \cos(\pi+z) + \cos(\pi+z) - \cos(z)\big)\,dz$$

$\cos(2\pi+z) = \cos z \qquad \cos(\pi+z) = -\cos z$

$$= -4\int_0^\pi \cos z\ dz$$

$$= -4\sin z\Big|_0^\pi$$

$$\underline{= 0\,|}$$

### Exercise

Evaluate the integral $\displaystyle\int_1^e \int_1^x \int_0^z \frac{2y}{z^3}\, dydzdx$

**Solution**

$$\begin{aligned}
\int_1^e \int_1^x \int_0^z \frac{2y}{z^3}\, dydzdx &= \int_1^e \int_1^x \frac{1}{z^3} y^2 \Big|_0^z\, dzdx \\
&= \int_1^e \int_1^x \frac{1}{z}\, dzdx \\
&= \int_1^e \ln z \Big|_1^x\, dx \\
&= \int_1^e \ln x\, dx
\end{aligned}$$

$u = \ln x \to du = \dfrac{dx}{x} \qquad v = \displaystyle\int dx = x$

$\displaystyle\int \ln x\, dx = x\ln x - \int dx$

$$\begin{aligned}
&= x\ln x - x \Big|_1^e \\
&= e - e + 1 \\
&= 1
\end{aligned}$$

### Exercise

Evaluate the integral $\displaystyle\int_{\ln 6}^{\ln 7} \int_0^{\ln 2} \int_{\ln 4}^{\ln 5} e^{(x+y+z)}\, dzdydx$

**Solution**

$$\begin{aligned}
\int_{\ln 6}^{\ln 7} \int_0^{\ln 2} \int_{\ln 4}^{\ln 5} e^{(x+y+z)}\, dzdydx &= \int_{\ln 6}^{\ln 7} \int_0^{\ln 2} \int_{\ln 4}^{\ln 5} e^x e^y e^z\, dzdydx \\
&= \int_{\ln 6}^{\ln 7} e^x dx \int_0^{\ln 2} e^y dy \int_{\ln 4}^{\ln 5} e^z dz \\
&= e^x \Big|_{\ln 6}^{\ln 7}\ e^y \Big|_0^{\ln 2}\ e^z \Big|_{\ln 4}^{\ln 5} \\
&= (7-6)(2-1)(5-4) \qquad\qquad e^{\ln u} = u \\
&= 1
\end{aligned}$$

## *Exercise*

Evaluate the integral $\displaystyle\int_0^1\int_0^{x^2}\int_0^{x+y}(2x-y-z)\,dzdydx$

***Solution***

$$\begin{aligned}
\int_0^1\int_0^{x^2}\int_0^{x+y}(2x-y-z)\,dzdydx &= \int_0^1\int_0^{x^2}\left((2x-y)z-\frac{1}{2}z^2\right)\Big|_0^{x+y}\,dydx\\
&= \int_0^1\int_0^{x^2}\left((2x-y)(x+y)-\frac{1}{2}(x+y)^2\right)dydx\\
&= \int_0^1\int_0^{x^2}\left(2x^2+xy-y^2-\frac{1}{2}x^2-xy-\frac{1}{2}y^2\right)dydx\\
&= \int_0^1\int_0^{x^2}\left(\frac{3}{2}x^2-\frac{3}{2}y^2\right)dydx\\
&= \int_0^1\left(\frac{3}{2}x^2y-\frac{1}{2}y^3\right)\Big|_0^{x^2}\,dx\\
&= \int_0^1\left(\frac{3}{2}x^4-\frac{1}{2}x^6\right)dx\\
&= \frac{3}{10}x^5-\frac{1}{14}x^7\Big|_0^1\\
&= \frac{3}{10}-\frac{1}{14}\\
&= \frac{32}{140}\\
&= \boxed{\frac{8}{35}}
\end{aligned}$$

## *Exercise*

Evaluate the integral $\displaystyle\int_{-2}^2\int_3^6\int_0^2 dxdydz$

***Solution***

$$\begin{aligned}
\int_{-2}^2\int_3^6\int_0^2 dxdydz &= \int_{-2}^2 dz\int_3^6 dy\int_0^2 dx\\
&= z\Big|_{-2}^2\ y\Big|_3^6\ x\Big|_0^2
\end{aligned}$$

$$=(2+2)(6-3)(2-0)$$

$$\underline{=24}$$

### *Exercise*

Evaluate the integral $\displaystyle\int_{-1}^{1}\int_{-1}^{2}\int_{0}^{1} 6xyz\ dydxdz$

***Solution***

$$\int_{-1}^{1}\int_{-1}^{2}\int_{0}^{1} 6xyz\ dydxdz = 6\int_{-1}^{1} z\ dz\ \int_{-1}^{2} x\ dx\ \int_{0}^{1} y\ dy$$

$$=6\left(\frac{1}{2}z^2\right)\Big|_{-1}^{1}\ \ \frac{1}{2}x^2\Big|_{-1}^{2}\ \ \frac{1}{2}y^2\Big|_{0}^{1}$$

$$=\frac{3}{4}(1-1)(4-1)(1-0)$$

$$\underline{=0}$$

### *Exercise*

Evaluate the integral $\displaystyle\int_{-2}^{2}\int_{1}^{2}\int_{1}^{e} \frac{xy^2}{z} dzdxdy$

***Solution***

$$\int_{-2}^{2}\int_{1}^{2}\int_{1}^{e} \frac{xy^2}{z} dzdxdy = \int_{-2}^{2} y^2 dy\ \int_{1}^{2} xdx\ \int_{1}^{e} \frac{dz}{z}$$

$$=\frac{1}{3}y^3\Big|_{-2}^{2}\ \ \frac{1}{2}x^2\Big|_{1}^{2}\ \ \ln z\Big|_{1}^{e}$$

$$=\frac{1}{6}(8+8)(4-1)(1-0)$$

$$\underline{=8}$$

### *Exercise*

Evaluate the integral $\displaystyle\int_{0}^{\ln 4}\int_{0}^{\ln 3}\int_{0}^{\ln 2} e^{-x+y+z} dxdydz$

***Solution***

$$\int_{0}^{\ln 4}\int_{0}^{\ln 3}\int_{0}^{\ln 2} e^{-x+y+z} dxdydz = \int_{0}^{\ln 4} e^z dz\ \int_{0}^{\ln 3} e^y dy\ \int_{0}^{\ln 2} e^{-x} dx$$

$$= e^z \Big|_0^{\ln 4} e^y \Big|_0^{\ln 3} \left(-e^{-x}\right)\Big|_0^{\ln 2}$$

$$= -\left(e^{\ln 4} - e^0\right)\left(e^{\ln 3} - e^0\right)\left(e^{\ln 2} - e^0\right)$$

$$= -(4-1)(3-1)\left(\frac{1}{2}-1\right)$$

$$\underline{= 3}$$

### Exercise

Evaluate the integral $\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\frac{\pi}{2}} \sin \pi x \cos y \sin 2z \; dydxdz$

**Solution**

$$\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\frac{\pi}{2}} \sin \pi x \cos y \sin 2z \; dydxdz = \int_0^{\frac{\pi}{2}} \sin 2z \; dz \int_0^1 \sin \pi x \;\; cdx \int_0^{\frac{\pi}{2}} \cos y \; dy$$

$$= -\frac{1}{2}\cos 2z \Big|_0^{\frac{\pi}{2}} \quad \left[-\frac{1}{\pi}\cos \pi x\right]\Big|_0^1 \quad \sin y \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2\pi}(-1-1)(-1-1)(1-0)$$

$$\underline{= \frac{2}{\pi}}$$

### Exercise

Evaluate the integral $\int_0^2 \int_1^2 \int_0^1 yze^x dxdzdy$

**Solution**

$$\int_0^2 \int_1^2 \int_0^1 yze^x dxdzdy = \int_0^2 ydy \int_1^2 zdz \int_0^1 e^x dx$$

$$= \frac{1}{2}y^2 \Big|_0^2 \quad \frac{1}{2}z^2 \Big|_1^2 \quad e^x \Big|_0^1$$

$$= \frac{1}{4}(4)(4-1)(e-1)$$

$$\underline{= 3(e-1)}$$

## *Exercise*

Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dzdydx$

***Solution***

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dzdydx = \int_0^1 \int_0^{\sqrt{1-x^2}} z \Big|_0^{\sqrt{1-x^2}} dydx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\, dydx$$

$$= \int_0^1 \sqrt{1-x^2}\, y \Big|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \left(1-x^2\right) dx$$

$$= \left(x - \frac{1}{3}x^3\right)\Big|_0^1$$

$$= \frac{2}{3}$$

## *Exercise*

Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 2xz\ dzdydx$

***Solution***

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 2xz\ dzdydx = \int_0^1 \int_0^{\sqrt{1-x^2}} xz^2 \Big|_0^{\sqrt{1-x^2-y^2}} dydx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} x\left(1-x^2-y^2\right) dydx$$

$$= \int_0^1 \left(xy - x^3y - \frac{1}{3}xy^3\right)\Big|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \left(x\left(1-x^2\right)^{1/2} - x^3\left(1-x^2\right)^{1/2} - \frac{1}{3}x\left(1-x^2\right)^{3/2}\right) dx$$

*Switching dydx to dxdy*

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} \left(x - x^3 - xy^2\right) dxdy$$

$$= \int_0^1 \left(\frac{1}{2}x^2 - \frac{1}{4}x^4 - \frac{1}{2}x^2y^2\right)\Big|_0^{\sqrt{1-y^2}} \, dy$$

$$= \frac{1}{4}\int_0^1 \left(2\left(1-y^2\right) - \left(1-y^2\right)^2 - 2\left(1-y^2\right)y^2\right) dy$$

$$= \frac{1}{4}\int_0^1 \left(2 - 2y^2 - 1 + 2y^2 - y^4 - 2y^2 + 2y^4\right) dy$$

$$= \frac{1}{4}\int_0^1 \left(y^4 - 2y^2 + 1\right) dy$$

$$= \frac{1}{4}\left(\frac{1}{5}y^5 - \frac{2}{3}y^3 + y\right)\Big|_0^1$$

$$= \frac{1}{4}\left(\frac{1}{5} - \frac{2}{3} + 1\right)$$

$$= \frac{1}{4}\left(\frac{8}{15}\right)$$

$$= \frac{2}{15}$$

### Exercise

Evaluate the integral $\displaystyle\int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \int_0^{16-\frac{1}{4}x^2-y^2} dzdxdy$

**Solution**

$$\int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \int_0^{16-\frac{1}{4}x^2-y^2} dzdxdy = \int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} z\Big|_0^{16-\frac{1}{4}x^2-y^2} \, dxdy$$

$$= \int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \left(16 - \frac{1}{4}x^2 - y^2\right) dxdy$$

$$= \int_0^4 \left(16x - \frac{1}{12}x^3 - xy^2\right)\Big|_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \, dy$$

$$= 2\int_0^4 \left(16x - \frac{1}{12}x^3 - xy^2\right)\Big|_0^{2\sqrt{16-y^2}} dy$$

$$= 2\int_0^4 \left(32\sqrt{16-y^2} - \frac{2}{3}\left(16-y^2\right)^{3/2} - 2y^2\sqrt{16-y^2}\right) dy$$

$y = 4\sin\theta \quad \to \quad dy = 4\cos\theta d\theta \qquad \sqrt{16-y^2} = 4\cos\theta$

$$\int \sqrt{16-y^2}\; dy = 16\int \cos^2\theta\; d\theta$$

$$= 8\int (1+\cos 2\theta)\; d\theta$$

$$= 8\left(\theta + \frac{1}{2}\sin 2\theta\right)$$

$$\boxed{= 8\sin^{-1}\left(\frac{y}{4}\right) + \frac{1}{2}y\sqrt{16-y^2}}$$

$$\int \left(16-y^2\right)^{3/2} dy = \int \left(16\cos^2\theta\right)^{3/2}\; 4\cos\theta\; d\theta$$

$$= 256\int \cos^4\theta\; d\theta$$

$$= 64\int (1+\cos 2\theta)^2\; d\theta$$

$$= 64\int \left(1 + 2\cos 2\theta + \cos^2 2\theta\right)\; d\theta$$

$$= 64\int \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right)\; d\theta$$

$$= 64\left(\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right)$$

$$= 64\left(\frac{3}{2}\theta + 2\sin\theta\cos\theta + \frac{1}{4}\sin 2\theta\cos 2\theta\right)$$

$$= 64\left(\frac{3}{2}\theta + 2\sin\theta\cos\theta + \frac{1}{2}\sin\theta\cos\theta\left(1 - 2\sin^2\theta\right)\right)$$

$$= 64\left(\frac{3}{2}\sin^{-1}\left(\frac{y}{4}\right) + 2\frac{y}{4}\frac{\sqrt{16-y^2}}{4} + \frac{1}{2}\frac{y}{4}\frac{\sqrt{16-y^2}}{4}\left(1 - \frac{y^2}{8}\right)\right)$$

$$= 64\left(\frac{3}{2}\sin^{-1}\left(\frac{y}{4}\right) + \frac{1}{8}y\sqrt{16-y^2} + \frac{1}{256}y\left(8-y^2\right)\sqrt{16-y^2}\right)$$

$$= 64\left(\frac{3}{2}\sin^{-1}\left(\frac{y}{4}\right) + \frac{1}{8}y\sqrt{16-y^2} + \frac{1}{32}y\sqrt{16-y^2} - \frac{1}{256}y^3\sqrt{16-y^2}\right)$$

$$= \left(96\sin^{-1}\left(\frac{y}{4}\right) + 10y\sqrt{16-y^2} - \frac{1}{4}y^3\sqrt{16-y^2}\right)$$

$$\int y^2\sqrt{16-y^2}\ dy = 4^4\int \sin^2\theta\cos^2\theta\ d\theta$$

$$= 64\int (1-\cos 2\theta)(1+\cos 2\theta)\ d\theta$$

$$= 64\int \left(1-\cos^2 2\theta\right)\ d\theta$$

$$= 64\int \left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right)\ d\theta$$

$$= 32\left(\theta - \frac{1}{4}\sin 4\theta\right)$$

$$= 32\left(\theta - \frac{1}{2}\sin 2\theta\cos 2\theta\right)$$

$$= 32\left(\theta - \sin\theta\cos\theta\left(1-2\sin^2\theta\right)\right)$$

$$= 32\left(\sin^{-1}\left(\frac{y}{4}\right) - \frac{y}{4}\frac{\sqrt{16-y^2}}{4}\left(1-\frac{y^2}{8}\right)\right)$$

$$= 32\left(\sin^{-1}\left(\frac{y}{4}\right) - \frac{1}{128}y\left(8-y^2\right)\sqrt{16-y^2}\right)$$

$$= 32\sin^{-1}\left(\frac{y}{4}\right) - 2y\sqrt{16-y^2} + \frac{1}{4}y^3\sqrt{16-y^2}$$

$$\int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \int_0^{16-\frac{1}{4}x^2-y^2} dzdxdy = \int_0^4 \left(64\sqrt{16-y^2} - \frac{4}{3}\left(16-y^2\right)^{3/2} - 4y^2\sqrt{16-y^2}\right)dy$$

$$= 64\left(8\sin^{-1}\left(\frac{y}{4}\right) + \frac{1}{2}y\sqrt{16-y^2}\right)$$

$$-\frac{4}{3}\left(96\sin^{-1}\left(\frac{y}{4}\right) + 10y\sqrt{16-y^2} - \frac{1}{4}y^3\sqrt{16-y^2}\right)$$

$$-4\left(32\sin^{-1}\left(\frac{y}{4}\right) - 2y\sqrt{16-y^2} + \frac{1}{4}y^3\sqrt{16-y^2}\right)$$

$$= 512\sin^{-1}\left(\frac{y}{4}\right) + 32y\sqrt{16-y^2}$$

$$-128\sin^{-1}\left(\frac{y}{4}\right) - \frac{40}{3}y\sqrt{16-y^2} + \frac{1}{3}y^3\sqrt{16-y^2}$$

$$-128\sin^{-1}\left(\frac{y}{4}\right) + 8y\sqrt{16-y^2} - y^3\sqrt{16-y^2}$$

$$= 256\sin^{-1}\left(\frac{y}{4}\right) + \frac{80}{3}y\sqrt{16-y^2} - \frac{2}{3}y^3\sqrt{16-y^2}\Big|_0^4$$

$$= 256\left(\frac{\pi}{2}\right)$$

$$= 128\pi$$

## *Exercise*

Evaluate the integral $\displaystyle\int_1^6 \int_0^{4-\frac{2}{3}y} \int_0^{12-2y-3z} \frac{1}{y}\,dxdzdy$

***Solution***

$$\int_1^6 \int_0^{4-\frac{2}{3}y} \int_0^{12-2y-3z} \frac{1}{y}\,dxdzdy = \int_1^6 \int_0^{4-\frac{2}{3}y} \frac{1}{y}x\Big|_0^{12-2y-3z}\,dzdy$$

$$= \int_1^6 \int_0^{4-\frac{2}{3}y} \frac{1}{y}(12-2y-3z)\,dzdy$$

$$= \int_1^6 \left(12\frac{1}{y}z - 2z - \frac{3}{2}\frac{1}{y}z^2\right)\Big|_0^{4-\frac{2}{3}y}\,dy$$

$$= \int_1^6 \left(12\frac{1}{y}\left(4-\frac{2}{3}y\right) - 2\left(4-\frac{2}{3}y\right) - \frac{3}{2}\frac{1}{y}\left(4-\frac{2}{3}y\right)^2\right)dy$$

$$= \int_1^6 \left(\frac{48}{y} - 16 + \frac{4}{3}y - \frac{3}{2}\frac{1}{y}\left(16 - \frac{16}{3}y + \frac{4}{9}y^2\right)\right)dy$$

$$= \int_1^6 \left(\frac{48}{y} - 16 + \frac{4}{3}y - \frac{24}{y} + 8 - \frac{2}{3}y\right)dy$$

$$= \int_1^6 \left(\frac{24}{y} - 8 + \frac{2}{3}y\right)dy$$

$$= 24\ln y - 8y + \frac{1}{3}y^2\Big|_1^6$$

$$= 24\ln 6 - 48 + 12 + 8 - \frac{1}{3}$$

$$\boxed{= 24\ln 6 - \frac{85}{3}}$$

### ***Exercise***

Evaluate the integral $\displaystyle\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{1+x^2+z^2}} dydxdz$

***Solution***

$$\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{1+x^2+z^2}} dydxdz = \int_0^3 \int_0^{\sqrt{9-z^2}} y \Big|_0^{\sqrt{1+x^2+z^2}} dxdz$$

$$= \int_0^3 \int_0^{\sqrt{9-z^2}} \sqrt{1+x^2+z^2}\ dxdz \qquad \text{Let } x^2+z^2=r^2$$

$$= \int_0^{\frac{\pi}{2}} \int_0^3 \sqrt{1+r^2}\ rdrd\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \int_0^3 \left(1+r^2\right)^{1/2} d\left(1+r^2\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)\frac{2}{3}\left(1+r^2\right)^{3/2} \Big|_0^3$$

$$\boxed{= \frac{\pi}{6}\left(10\sqrt{10}-1\right)}$$

### ***Exercise***

Evaluate the integral $\displaystyle\int_0^{\pi} \int_0^{\pi} \int_0^{\sin x} \sin y\ dzdxdy$

***Solution***

$$\int_0^{\pi} \int_0^{\pi} \int_0^{\sin x} \sin y\ dzdxdy = \int_0^{\pi} \int_0^{\pi} (\sin y)\, z \Big|_0^{\sin x} dxdy$$

$$= \int_0^{\pi} \int_0^{\pi} (\sin y \sin x)\, dxdy$$

$$= -\int_0^{\pi} \sin y \cos x \Big|_0^{\pi} dy$$

$$= 2\int_0^{\pi} \sin y \, dy$$

$$= -2\cos y \Big|_0^{\pi}$$

$$\boxed{= 4}$$

### *Exercise*

Evaluate the integral $\displaystyle\int_0^{\ln 8}\int_1^{\sqrt{z}}\int_{\ln y}^{\ln 2y} e^{x+y^2-z}\,dxdydz$

***Solution***

$$\int_0^{\ln 8}\int_1^{\sqrt{z}}\int_{\ln y}^{\ln 2y} e^{x+y^2-z}\,dxdydz = \int_0^{\ln 8}\int_1^{\sqrt{z}} e^{y^2}e^{-z}e^{x}\Big|_{\ln y}^{\ln 2y}\,dydz$$

$$= \int_0^{\ln 8}\int_1^{\sqrt{z}} e^{y^2}e^{-z}(y)\,dydz$$

$$= \frac{1}{2}\int_0^{\ln 8}\int_1^{\sqrt{z}} e^{-z}e^{y^2}\,d\left(y^2\right)dz$$

$$= \frac{1}{2}\int_0^{\ln 8} e^{-z}e^{y^2}\Big|_1^{\sqrt{z}}\,dz$$

$$= \frac{1}{2}\int_0^{\ln 8} e^{-z}\left(e^{z}-e\right)dz$$

$$= \frac{1}{2}\int_0^{\ln 8}\left(1-e^{1-z}\right)dz$$

$$= \frac{1}{2}\left(z+e^{1-z}\right)\Big|_0^{\ln 8}$$

$$= \frac{1}{2}\left(\ln 8+e^{1-\ln 8}-e\right)$$

$$= \frac{1}{2}\left(\ln 8+e\left(e^{\ln 8^{-1}}\right)-e\right)$$

$$= \frac{1}{2}\left(\ln 8+\frac{1}{8}e-e\right)$$

$$\boxed{= \frac{1}{2}\ln 8-\frac{7}{16}e}$$

### Exercise

Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{2-x} 4yz\ dzdydx$

***Solution***

$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{2-x} 4yz\ dzdydx &= \int_0^1 \int_0^{\sqrt{1-x^2}} 2yz^2 \Big|_0^{2-x}\ dydx \\
&= \int_0^1 \int_0^{\sqrt{1-x^2}} 2y(2-x)^2\ dydx \\
&= \int_0^1 \left(4-4x+x^2\right) y^2 \Big|_0^{\sqrt{1-x^2}}\ dx \\
&= \int_0^1 \left(4-4x+x^2\right)\left(1-x^2\right) dx \\
&= \int_0^1 \left(4-4x-3x^2+4x^3-x^4\right) dx \\
&= \left(4x-2x^2-x^3+x^4-\frac{1}{5}x^5\right)\Big|_0^1 \\
&= \underline{\underline{\frac{9}{5}}}
\end{aligned}$$

### Exercise

Evaluate the integral $\int_0^2 \int_0^4 \int_{y^2}^4 \sqrt{x}\ dzdxdy$

***Solution***

$$\begin{aligned}
\int_0^2 \int_0^4 \int_{y^2}^4 \sqrt{x}\ dzdxdy &= \int_0^2 \int_0^4 \sqrt{x}\ z\Big|_{y^2}^4\ dxdy \\
&= \int_0^2 \left(4-y^2\right) dy \int_0^4 x^{1/2}\ dx \\
&= \left(4y-\frac{1}{3}y^3\right)\Big|_0^2 \quad \frac{2}{3}x^{3/2}\Big|_0^4 \\
&= \frac{2}{3}\left(8-\frac{8}{3}\right)(8) \\
&= \underline{\underline{\frac{256}{9}}}
\end{aligned}$$

### Exercise

Evaluate the integral $\displaystyle\int_0^1\int_y^{2-y}\int_0^{2-x-y} xy\ dzdxdy$

**Solution**

$$\int_0^1\int_y^{2-y}\int_0^{2-x-y} xy\ dzdxdy = \int_0^1\int_y^{2-y} xyz\Big|_0^{2-x-y}\ dxdy$$

$$= \int_0^1\int_y^{2-y}\left(2xy - x^2y - xy^2\right)dxdy$$

$$= \int_0^1\left(x^2y - \frac{1}{3}x^3y - \frac{1}{2}x^2y^2\right)\Big|_y^{2-y}\ dy$$

$$= \int_0^1\left((2-y)^2 y - \frac{1}{3}(2-y)^3 y - \frac{1}{2}(2-y)^2 y^2 - y^3 + \frac{5}{6}y^4\right)dy$$

$$= \int_0^1\left(\left(4-4y+y^2\right)\left(y - \frac{2}{3}y + \frac{1}{3}y^2 - \frac{1}{2}y^2\right) - y^3 + \frac{5}{6}y^4\right)dy$$

$$= \int_0^1\left(\left(4-4y+y^2\right)\left(\frac{1}{3}y - \frac{1}{6}y^2\right) - y^3 + \frac{5}{6}y^4\right)dy$$

$$= \int_0^1\left(\frac{4}{3}y - 2y^2 + y^3 - \frac{1}{6}y^4 - y^3 + \frac{5}{6}y^4\right)dy$$

$$= \int_0^1\left(\frac{4}{3}y - 2y^2 + \frac{2}{3}y^4\right)dy$$

$$= \left(\frac{2}{3}y^2 - \frac{2}{3}y^3 + \frac{2}{15}y^5\right)\Big|_0^1$$

$$= \frac{2}{15}$$

### Exercise

Here is the region of integration of the integral

$$\int_{-1}^1\int_{x^2}^1\int_0^{1-y} dzdydx$$

***a***) *dydzdx* ***b***) *dydxdz* ***c***) *dxdydz* ***d***) *dxdzdy* ***e***) *dzdxdy*

**Solution**

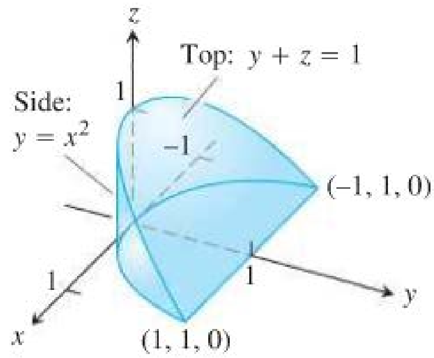
**a)** $\displaystyle\int_{-1}^{1}\int_{0}^{1-x^2}\int_{x^2}^{1-x} dydzdx$

**b)** $\displaystyle\int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-x} dydxdz$

**c)** $\displaystyle\int_{0}^{1}\int_{0}^{1-x}\int_{-\sqrt{y}}^{\sqrt{y}} dxdydz$

**d)** $\displaystyle\int_{0}^{1}\int_{0}^{1-y}\int_{-\sqrt{y}}^{\sqrt{y}} dxdzdy$

**e)** $\displaystyle\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}}\int_{0}^{1-y} dzdxdy$




## Exercise

Use another order to evaluate $\displaystyle\int_{0}^{5}\int_{-1}^{0}\int_{0}^{4x+4} dydxdz$

***Solution***

$-1 \le x \le 0 \quad 0 \le y \le 4x+4 \quad 0 \le z \le 5$

$$\begin{cases} x=-1 & \to y=0 \\ x=0 & \to y=4 \end{cases}$$

$$y = 4x+4 \quad \to \quad x = \frac{y-4}{4}$$

$$\int_{0}^{5}\int_{-1}^{0}\int_{0}^{4x+4} dydxdz = \int_{0}^{4}\int_{\frac{y-4}{4}}^{0}\int_{0}^{5} dzdxdy$$

$$= \int_{0}^{4} x\Big|_{\frac{y-4}{4}}^{0} dy \quad z\Big|_{0}^{5}$$

$$= \frac{5}{4}\int_{0}^{4}(4-y)\,dy$$

$$= \frac{5}{4}\left(4y - \frac{1}{2}y^2\right)\Big|_{0}^{4}$$

$$= 10$$

### Exercise

Use another order to evaluate $\displaystyle\int_0^1\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} dzdydx$

***Solution***

$0 \le x \le 1 \quad -2 \le y \le 2 \quad 0 \le z \le \sqrt{4-y^2}$

$0 \le z \le 2$

$z = \sqrt{4-y^2} \quad \to \quad y = \pm\sqrt{4-z^2}$

$$\begin{aligned}
\int_0^1\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} dzdydx &= \int_0^1\int_0^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dydzdx \\
&= \int_0^1 dx \int_0^2 y\Big|_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \ dz \\
&= 2\int_0^2 \sqrt{4-z^2}\ dz \\
&= 2\left(\frac{z}{2}\sqrt{4-z^2} + 2\sin^{-1}\frac{z}{2}\right)\Big|_0^2 \\
&= 2\left(2\frac{\pi}{2}\right) \\
&= 2\pi
\end{aligned}$$

### Exercise

Use another order to evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dydzdx$

***Solution***

$$\begin{aligned}
\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dydzdx &= \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dzdydx \\
&= \int_0^1\int_0^{\sqrt{1-x^2}} z\Big|_0^{\sqrt{1-x^2}} \ dydx \\
&= \int_0^1 \sqrt{1-x^2}\ y\Big|_0^{\sqrt{1-x^2}} \ dx
\end{aligned}$$

$$= \int_0^1 \left(1 - x^2\right) dx$$

$$= \left(x - \frac{1}{3}x^3\right)\Big|_0^1$$

$$= \frac{2}{3}$$

### Exercise

Use another order to evaluate $\displaystyle \int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-z^2}} dydzdx$

**Solution**

$0 \le x \le 4 \quad 0 \le y \le \sqrt{16 - x^2 - z^2} \quad 0 \le z \le \sqrt{16 - x^2}$

$x^2 + y^2 + z^2 = 16$

$0 \le x \le \sqrt{16 - y^2 - z^2} \quad 0 \le y \le \sqrt{16 - z^2} \quad 0 \le z \le 4$

$$\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-z^2}} dydzdx = \int_0^4 \int_0^{\sqrt{16-z^2}} \int_0^{\sqrt{16-y^2-z^2}} dxdydz$$

$$= \int_0^4 \int_0^{\sqrt{16-z^2}} x\Big|_0^{\sqrt{16-y^2-z^2}} dydz$$

$$= \int_0^4 \int_0^{\sqrt{16-z^2}} \sqrt{16 - y^2 - z^2}\ dydz \qquad \text{Let } y^2 + z^2 = r^2$$

$$= \int_0^{\frac{\pi}{2}} \int_0^4 \sqrt{16 - r^2}\ r\ drd\theta$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta \int_0^4 \left(16 - r^2\right)^{1/2} d\left(16 - r^2\right)$$

$$= -\frac{1}{2}\left(\frac{\pi}{2}\right)\left(\frac{2}{3}\right)\left(16 - r^2\right)^{3/2}\Big|_0^4$$

$$= -\frac{\pi}{6}(-64)$$

$$= \frac{32\pi}{3}$$

## *Exercise*

Use another order to evaluate $\displaystyle\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin\sqrt{yz}}{x^{3/2}}\, dydxdz$

***Solution***

$$\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin\sqrt{yz}}{x^{3/2}}\, dydxdz = \int_0^{\pi^2} \int_1^4 \int_z^{4z} x^{-3/2} \sin\sqrt{yz}\; dxdzdy$$

$$= -2\int_0^{\pi^2} \int_1^4 \sin\sqrt{yz}\left(x^{-1/2}\right)\Big|_z^{4z}\; dzdy$$

$$= -2\int_0^{\pi^2} \int_1^4 \sin\sqrt{yz}\left(\frac{1}{2\sqrt{z}} - \frac{1}{\sqrt{z}}\right) dzdy$$

$$= \int_0^{\pi^2} \int_1^4 \frac{\sin\sqrt{yz}}{\sqrt{z}}\; dzdy$$

$$= 2\int_0^{\pi^2} \int_1^4 \frac{1}{\sqrt{y}} \sin\sqrt{yz}\; d\left(\sqrt{yz}\right) dy \qquad d\left(\sqrt{yz}\right) = \frac{1}{2}\frac{y}{\sqrt{yz}} dz$$

$$= -2\int_0^{\pi^2} \frac{1}{\sqrt{y}} \cos\sqrt{yz}\,\Big|_1^4\; dy$$

$$= -2\int_0^{\pi^2} \frac{1}{\sqrt{y}}\left(\cos\left(2\sqrt{y}\right) - \cos\sqrt{y}\right) dy$$

$$= -4\int_0^{\pi^2} \left(\cos\left(2\sqrt{y}\right) - \cos\sqrt{y}\right) d\left(\sqrt{y}\right)$$

$$= -4\left(\frac{1}{2}\sin\left(2\sqrt{y}\right) - \sin\sqrt{y}\right)\Big|_0^{\pi^2}$$

$$= 0$$

## *Exercise*

Evaluate $\displaystyle\iiint_D (xy + xz + yz)\, dV$ ; $D = \{(x,\ y,\ z): \ -1 \le x \le 1, \ \ -2 \le y \le 2, \ \ -3 \le z \le 3\}$

***Solution***

$$\iiint_D (xy + xz + yz)\,dV = \int_{-3}^{3}\int_{-2}^{2}\int_{-1}^{1} (xy + xz + yz)\,dxdydz$$

$$= \int_{-3}^{3}\int_{-2}^{2} \left(\frac{1}{2}x^2 y + \frac{1}{2}x^2 z + xyz\right)\Big|_{-1}^{1}\, dydz$$

$$= \int_{-3}^{3}\int_{-2}^{2} \left(\frac{1}{2}y + \frac{1}{2}z + yz - \frac{1}{2}y - \frac{1}{2}z + yz\right) dydz$$

$$= \int_{-3}^{3}\int_{-2}^{2} 2yz\ dydz$$

$$= \int_{-3}^{3} zy^2\Big|_{-2}^{2}\ dz$$

$$\underline{= 0}$$

## Exercise

Evaluate $\displaystyle\iiint_D xyze^{-x^2-y^2}\,dV$; $\quad D = \left\{(x,\ y,\ z):\ \ 0 \le x \le \sqrt{\ln 2},\ \ 0 \le y \le \sqrt{\ln 4},\ \ 0 \le z \le 1\right\}$

***Solution***

$$\iiint_D xyze^{-x^2-y^2}\,dV = \int_0^1\int_0^{\sqrt{\ln 4}}\int_0^{\sqrt{\ln 2}} xyze^{-x^2-y^2}\,dxdydz$$

$$= \int_0^1 z\ dz\ \int_0^{\sqrt{\ln 4}} ye^{-y^2}\,dy\ \int_0^{\sqrt{\ln 2}} xe^{-x^2}\,dx$$

$$= \frac{1}{2}z^2\Big|_0^1\ \int_0^{\sqrt{\ln 4}} \left(-\frac{1}{2}\right)e^{-y^2}\,d\left(-y^2\right)\ \int_0^{\sqrt{\ln 2}} \left(-\frac{1}{2}\right)e^{-x^2}\,d\left(-x^2\right)$$

$$= \frac{1}{8}\ e^{-y^2}\Big|_0^{\sqrt{\ln 4}}\ \ e^{-x^2}\Big|_0^{\sqrt{\ln 2}}$$

$$= \frac{1}{8}\left(\frac{1}{4} - 1\right)\left(\frac{1}{2} - 1\right)$$

$$\underline{= \frac{3}{64}}$$

### *Exercise*

Let $D = \left\{(x, y, z):\ 0 \le x \le y^2,\ 0 \le y \le z^3,\ 0 \le z \le 2\right\}$

*a)* Use a triple integral to find the volume of *D*.

*b)* In theory, how many other possible orderings of the variables (besides the one used in part (*a*)) can be used to find the volume of *D*? Verify the result of part (*a*) using one of these other ordering.

*c)* What is the volume of the region $D = \left\{(x, y, z):\ 0 \le x \le y^p,\ 0 \le y \le z^q,\ 0 \le z \le 2\right\}$, where *p* and *q* are positive real numbers?

***Solution***

***a)*** $V = \int_0^2 \int_0^{z^3} \int_0^{y^2} dxdydz$

$$= \int_0^2 \int_0^{z^3} x \Big|_0^{y^2} dydz$$

$$= \int_0^2 \int_0^{z^3} y^2\ dydz$$

$$= \frac{1}{3}\int_0^2 y^3 \Big|_0^{z^3} dz$$

$$= \frac{1}{3}\int_0^2 z^9\ dz$$

$$= \frac{1}{30} z^{10} \Big|_0^2$$

$$= \frac{512}{15}\ unit^3$$

***b)*** There are total of 6: *dxdydz*, *dxdzdy*, *dydxdz*, *dydzdx*, *dzdxdy*, *dzdydx*

$0 \le x \le y^2$

$z = 2 \to y = 2^3 = 8 \qquad 0 \le y \le 8$

$y = z^3 \to z = \sqrt[3]{y} \qquad \sqrt[3]{y} \le z \le 2$

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \int_0^{y^2} dxdzdy = \int_0^8 \int_{\sqrt[3]{y}}^2 x \Big|_0^{y^2} dzdy$$

$$= \int_0^8 \int_{\sqrt[3]{y}}^2 y^2 dzdy$$

$$= \int_0^8 y^2 z \Big|_{\sqrt[3]{y}}^{2} dy$$

$$= \int_0^8 y^2 \left(2 - y^{1/3}\right) dy$$

$$= \int_0^8 \left(2y^2 - y^{7/3}\right) dy$$

$$= \left(\frac{2}{3} y^3 - \frac{3}{10} y^{10/3}\right)\Big|_0^{2^3}$$

$$= \frac{2^{10}}{3} - \frac{3}{5} 2^9$$

$$= 2^9 \left(\frac{2}{3} - \frac{3}{5}\right)$$

$$= \frac{2^9}{15} \qquad\qquad = \frac{512}{15}$$

***c)*** $D = \left\{(x, y, z):\ 0 \le x \le y^p,\ 0 \le y \le z^q,\ 0 \le z \le 2\right\},\ (p,\ q \in \mathbb{R})$

$$V = \int_0^2 \int_0^{z^q} \int_0^{y^p} dxdydz$$

$$= \int_0^2 \int_0^{z^q} x \Big|_0^{y^p} dydz$$

$$= \int_0^2 \int_0^{z^q} y^p \ dydz$$

$$= \frac{1}{p+1} \int_0^2 y^{p+1} \Big|_0^{z^q} dz$$

$$= \frac{1}{p+1} \int_0^2 z^{q(p+1)} \ dz$$

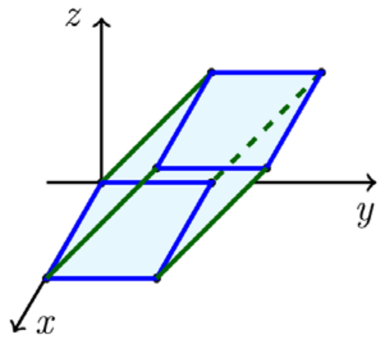
$$= \frac{1}{(p+1)(q(p+1)+1)} z^{q(p+1)+1} \Big|_0^2$$

$$= \frac{2^{q(p+1)+1}}{(p+1)(q(p+1)+1)}$$

### *Exercise*

Find the volume the parallelepiped (slanted box) with vertices $(0,\ 0,\ 0)$, $(1,\ 0,\ 0)$, $(0,\ 1,\ 0)$, $(1,\ 1,\ 0)$, $(0,\ 1,\ 1)$, $(1,\ 1,\ 1)$, $(0,\ 2,\ 1)$, $(1,\ 2,\ 1)$

***Solution***




$$V=\int_0^1\int_z^{z+1}\int_0^1 dxdydz$$

$$=\int_0^1\int_z^{z+1} x\Big|_0^1\ dydz$$

$$=\int_0^1\int_z^{z+1} dydz$$

$$=\int_0^1 y\Big|_z^{z+1}\ dz$$

$$=\int_0^1 dz$$

$$=z\Big|_0^1$$

$$\underline{=1}$$

### *Exercise*

Find the volume the larger of two solids formed when the parallelepiped with vertices $(0,\ 0,\ 0)$, $(2,\ 0,\ 0)$, $(0,\ 2,\ 0)$, $(2,\ 2,\ 0)$, $(0,\ 1,\ 1)$, $(2,\ 1,\ 1)$, $(0,\ 3,\ 1)$, $(2,\ 3,\ 1)$ is sliced by the plane $y=2$.

***Solution***

$$V=\int_0^1\int_z^{2}\int_0^2 dxdydz$$

$$=\int_0^1\int_z^{2} x\Big|_0^2\ dydz$$

$$=2\int_0^1\int_z^{2} dydz$$

$$=2\int_0^1 y\Big|_z^{2}\ dz$$

$$= 2\int_0^1 (2-z)\,dz$$

$$= 2\left(2z - \frac{1}{2}z^2\right)\Big|_0^1$$

$$= 2\left(2-\frac{1}{2}\right)$$

$$\underline{= 3 \,|}$$

### *Exercise*

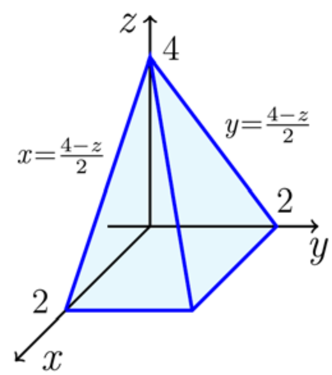
Find the volume of the pyramid with vertices $(0,\ 0,\ 0)$, $(2,\ 0,\ 0)$, $(2,\ 2,\ 0)$, $(0,\ 2,\ 0)$, $(0,\ 0,\ 4)$

***Solution***

$$(2,\ 0)\ \ \&\ \ (0,\ 4) \to z = \frac{4}{-2}(x-2) = -2x+4$$

$$\underline{x = \frac{4-z}{2}\,|}$$

$$(2,\ 0)\ \ \&\ \ (0,\ 4)\ \to\ \underline{y = \frac{4-z}{2}\,|}$$




$$V = \int_0^4 \int_0^{\frac{4-z}{2}} \int_0^{\frac{4-z}{2}} dxdydz$$

$$= \int_0^4 \int_0^{\frac{4-z}{2}} x\Big|_0^{\frac{4-z}{2}}\ dydz$$

$$= \frac{1}{2}\int_0^4 \int_0^{\frac{4-z}{2}} (4-z)\,dydz$$

$$= \frac{1}{2}\int_0^4 (4-z)\,y\Big|_0^{\frac{4-z}{2}}\ dz$$

$$= -\frac{1}{4}\int_0^4 (4-z)^2\, d(4-z)$$

$$= -\frac{1}{12}(4-z)^3\Big|_0^4$$

$$= -\frac{1}{12}(-64)$$

$$\underline{= \frac{16}{3}\,|}$$

### *Exercise*

Two different tetrahedrons fill the region in the first octant bounded by the coordinate planes and the plane $x+y+z=4$. Both solids have densities that vary in the $z$–direction between $\rho=4$ and $\rho=8$, according to the functions $\rho_1=8-z$ and $\rho_2=4+z$. Find the mass of each solid

***Solution***

$$m_1=\int_0^4\int_0^{4-x}\int_0^{4-x-y}(8-z)\,dzdydx$$

$$=\int_0^4\int_0^{4-x}\left(8z-\frac{1}{2}z^2\right)\Big|_0^{4-x-y}\,dydx$$

$$=\int_0^4\int_0^{4-x}\left(32-8x-8y-\frac{1}{2}(4-x-y)^2\right)dydx$$

$$=\int_0^4\int_0^{4-x}\left(24-4x-\frac{1}{2}x^2-4y-xy-\frac{1}{2}y^2\right)dydx$$

$$=\int_0^4\left(24y-4xy-\frac{1}{2}x^2y-2y^2-\frac{1}{2}xy^2-\frac{1}{6}y^3\right)\Big|_0^{4-x}\,dx$$

$$=\int_0^4\left(\frac{160}{3}-24x+2x^2+\frac{1}{6}x^3\right)dx$$

$$=\left(\frac{160}{3}x-12x^2+\frac{2}{3}x^3+\frac{1}{24}x^4\right)\Big|_0^4$$

$$=\frac{224}{3}$$

$$m_2=\int_0^4\int_0^{4-x}\int_0^{4-x-y}(4+z)\,dzdydx$$

$$=\int_0^4\int_0^{4-x}\left(4z+\frac{1}{2}z^2\right)\Big|_0^{4-x-y}\,dydx$$

$$=\int_0^4\int_0^{4-x}\left(24-8x-\frac{1}{2}x^2-8y+xy-\frac{1}{2}y^2\right)dydx$$

$$=\int_0^4\left(24y-8xy-\frac{1}{2}x^2y-4y^2+\frac{1}{2}xy^2-\frac{1}{6}y^3\right)\Big|_0^{4-x}\,dx$$

$$=\int_0^4\left(\frac{128}{3}-24x+4x^2-\frac{1}{6}x^3\right)dx$$

$$= \left( \frac{128}{3}x - 12x^2 + \frac{4}{3}x^3 - \frac{1}{24}x^4 \right) \Big|_0^4$$

$$\boxed{= \frac{160}{3}}$$

Solid 1 has greater mass.

### *Exercise*

Suppose a wedge of cheese fills the region in the first octant bounded by the planes $y = z$, $y = 4$ and $x = 4$. You could divide the wedge into two equal pieces (by volume) if you sliced the wedge with the plane $x = 2$. Instead find *a* with $0 < a < 1$ such that slicing the wedge with the plane $y = a$ divides the wedge into two pieces of equal volume

***Solution***

$$V = \int_0^4 \int_0^4 \int_0^y dzdydx$$

$$= \int_0^4 dx \int_0^4 z \Big|_0^y \, dy$$

$$= x \Big|_0^4 \quad \int_0^4 y \, dy$$

$$= 2y^2 \Big|_0^4$$

$$\boxed{= 32}$$

$$V = \frac{1}{2}(32) = \int_0^4 \int_0^a \int_0^y dzdydx = 16$$

$$\int_0^4 \int_0^a \int_0^y dzdydx = \int_0^4 dx \int_0^a z \Big|_0^y \, dy$$

$$= x \Big|_0^4 \quad \int_0^a y \, dy$$

$$= 2y^2 \Big|_0^a$$

$$= 2a^2 = 16$$

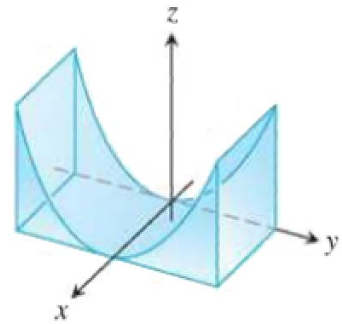
$$\boxed{a = 2\sqrt{2}}$$

### *Exercise*

Find the volumes of the region between the cylinder $z = y^2$ and the *xy*-plane that is bounded by the planes $x = 0, \quad x = 1, \quad y = -1, \quad y = 1$

***Solution***

$$V = \int_0^1 \int_{-1}^1 \int_0^{y^2} dzdydx$$




$$= \int_0^1 \int_{-1}^1 [z]_0^{y^2} \; dydx$$

$$= \int_0^1 \int_{-1}^1 y^2 dydx$$

$$= \frac{1}{3} \int_0^1 \left[ y^3 \right]_{-1}^1 dx$$

$$= \frac{2}{3} \int_0^1 dx$$

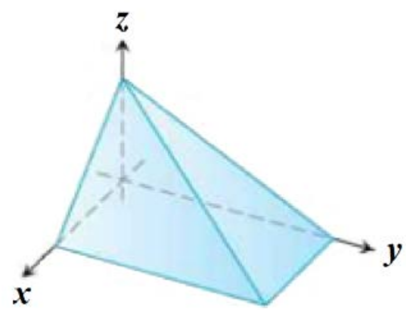
$$= \frac{2}{3}$$

### *Exercise*

Find the volumes of the region in the first octant bounded by the coordinate planes and the planes $x + z = 1, \quad y + 2z = 2$

***Solution***

$$V = \int_0^1 \int_0^{1-x} \int_0^{2-2z} dydzdx$$




$$= \int_0^1 \int_0^{1-x} (2 - 2z) \, dzdx$$

$$= \int_0^1 \left[ 2z - z^2 \right]_0^{1-x} dx$$

$$= \int_0^1 \left[ 2(1-x) - (1-x)^2 \right] dx$$

$$= \int_0^1 (1-x)(2 - 1 + x) \, dx$$

$$= \int_0^1 (1-x)(1+x)\,dx$$

$$= \int_0^1 \left(1-x^2\right) dx$$

$$= \left[x - \frac{1}{3}x^3\right]_0^1$$

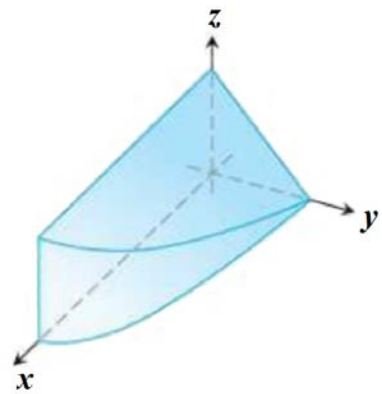
$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$

***Exercise***

Find the volumes of the region in the first octant bounded by the coordinate planes and the plane $y + z = 2$, and the cylinder $x = 4 - y^2$

***Solution***




$$V = \int_0^4 \int_0^{\sqrt{4-x}} \int_0^{2-y} dzdydx$$

$$= \int_0^4 \int_0^{\sqrt{4-x}} (2-y)\,dydx$$

$$= \int_0^4 \left[2y - \frac{1}{2}y^2\right]_0^{\sqrt{4-x}} dydx$$

$$= \int_0^4 \left[2\sqrt{4-x} - \frac{1}{2}(4-x)\right] dx$$

$$= -\int_0^4 \left[2(4-x)^{1/2} - \frac{1}{2}(4-x)\right] d(4-x)$$

$$= -\left[\frac{4}{3}(4-x)^{3/2} - \frac{1}{4}(4-x)^2\right]_0^4$$

$$= -\left[0 - \left(\frac{4}{3}4^{3/2} - \frac{1}{4}4^2\right)\right]$$

$$= \frac{20}{3}$$

### *Exercise*

Find the volumes of the wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y, \quad z = 0$

***Solution***

$$V = 2\int_0^1 \int_{-\sqrt{1-x^2}}^0 \int_0^{-y} dzdydx$$

$$= -2\int_0^1 \int_{-\sqrt{1-x^2}}^0 ydydx$$

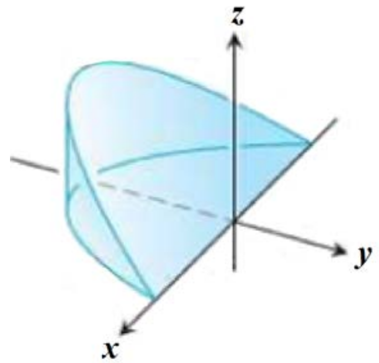
$$= -2\int_0^1 \left[\frac{1}{2}y^2\right]_{-\sqrt{1-x^2}}^0 dx$$

$$= \int_0^1 \left(1 - x^2\right) dx$$

$$= \left[x - \frac{1}{3}x^3\right]_0^1$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$




### *Exercise*

Find the volumes of the region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$

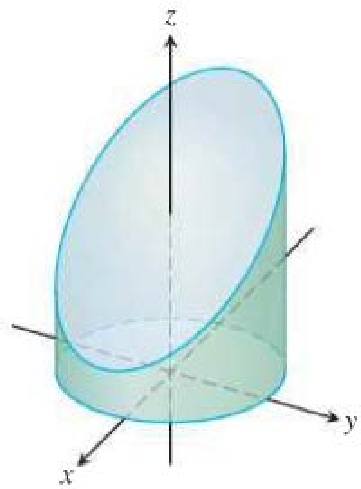
***Solution***

$$V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{3-x} dzdydx$$

$$= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3 - x)\, dydx$$

$$= 2\int_{-2}^2 (3 - x)\sqrt{4 - x^2}\, dx$$

$$= 6\int_{-2}^2 \sqrt{4 - x^2}\, dx - 2\int_{-2}^2 x\sqrt{4 - x^2}\, dx \qquad d\left(4 - x^2\right) = -2xdx$$



$$= 6\int_{-2}^{2}\sqrt{4-x^2}\,dx + \int_{-2}^{2}\left(4-x^2\right)^{1/2} d\left(4-x^2\right)$$

$$= 3\left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_{-2}^{2} + \frac{2}{3}\left[\left(4-x^2\right)^{3/2}\right]_{-2}^{2}$$

$$= 3\left[4\sin^{-1}1 - 4\sin^{-1}(-1)\right] + \frac{2}{3}(0)$$

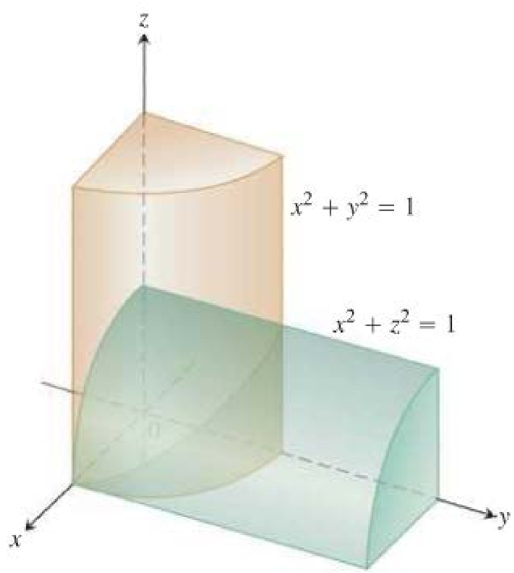
$$= 12\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= 12\pi$$

## *Exercise*

Find the volumes of the region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown below

***Solution***




$$V = 8\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dzdydx$$

$$= 8\int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dydx$$

$$= 8\int_0^1 \sqrt{1-x^2}\,\left[y\right]_0^{\sqrt{1-x^2}}\,dx$$

$$= 8\int_0^1 \left(1-x^2\right)dx$$

$$= 8\left[x - \frac{1}{3}x^3\right]_0^1$$

$$= \frac{16}{3}$$

## *Exercise*

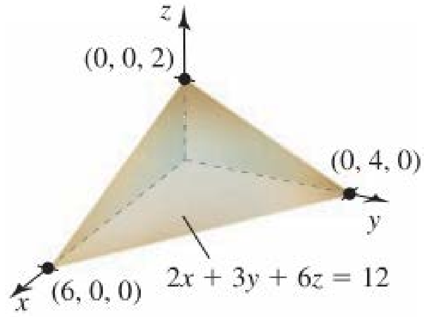
Find the volume of the solid in the first octant bounded by the plane $2x + 3y + 6z = 12$ and the coordinate planes

***Solution***

$$z = \frac{12 - 2x - 3y}{6} = 2 - \frac{x}{3} - \frac{y}{2} \quad z = 0 \rightarrow \quad 2x + 3y = 12 \rightarrow y = 4 - \frac{2x}{3}$$

$$0 \le z \le 2 - \frac{x}{3} - \frac{y}{2}; \quad 0 \le y \le 4 - \frac{2x}{3}; \quad 0 \le x \le 6$$

$$V = \int_0^6 \int_0^{4-\frac{2x}{3}} \int_0^{2-\frac{x}{3}-\frac{y}{2}} 1\, dzdydx$$




$$= \int_0^6 \int_0^{4-\frac{2x}{3}} z \Big|_0^{2-\frac{x}{3}-\frac{y}{2}} dydx$$

$$= \int_0^6 \int_0^{4-\frac{2x}{3}} \left(2 - \frac{x}{3} - \frac{y}{2}\right) dydx$$

$$= \int_0^6 \left(2y - \frac{x}{3}y - \frac{1}{4}y^2\right) \Big|_0^{4-\frac{2x}{3}} dx$$

$$= \int_0^6 \left(8 - \frac{4}{3}x - \frac{4}{3}x + \frac{2}{9}x^2 - \frac{1}{4}\left(16 - \frac{16}{3}x + \frac{4}{9}x^2\right)\right) dx$$

$$= \int_0^6 \left(4 - \frac{4}{3}x + \frac{1}{9}x^2\right) dx$$

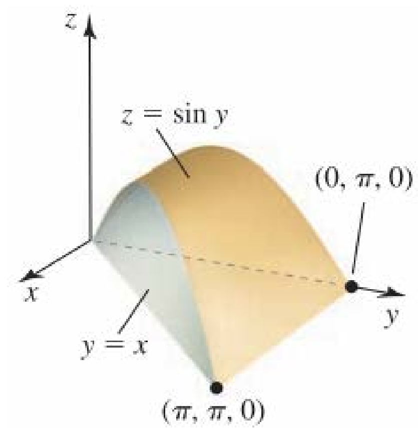
$$= 4x - \frac{2}{3}x^2 + \frac{1}{27}x^3 \Big|_0^6$$

$$\underline{= 8} \quad unit^3$$

## Exercise

Find the volume of the solid in the first octant formed when the cylinder $z = \sin y$, for $0 \le y \le \pi$, is sliced by the planes $y = x$ and $x = 0$

***Solution***




$$V = \int_0^\pi \int_x^\pi \int_0^{\sin y} 1\, dzdydx$$

$$= \int_0^\pi \int_x^\pi z \Big|_0^{\sin y} dydx$$

$$= \int_0^\pi \int_x^\pi \sin y\, dydx$$

$$= -\int_0^\pi \cos y \Big|_x^\pi dx$$

$$= -\int_0^{\pi} (-1 - \cos x)\, dx$$

$$= (x + \sin x)\Big|_0^{\pi}$$

$$= \pi \quad unit^3$$

### *Exercise*

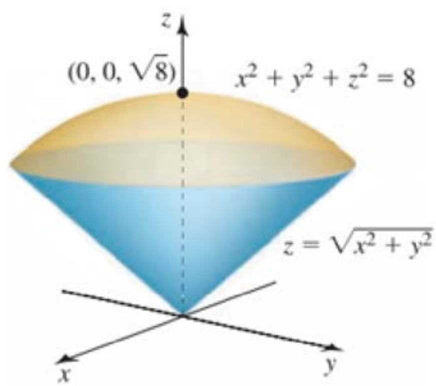
Find the volume of the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and bounded above the sphere $x^2 + y^2 + z^2 = 8$

**Solution**

$$z = \sqrt{x^2 + y^2} \qquad z = \sqrt{8 - x^2 - y^2}$$

$$x^2 + y^2 + \left(\sqrt{x^2 + y^2}\right)^2 = 8 \;\Rightarrow\; x^2 + y^2 = 4 \;\rightarrow\; y = \pm\sqrt{4 - x^2}$$

$$(y = 0) \rightarrow x^2 = 4 \quad x = \pm 2$$




$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} 1\, dzdydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} z \Big|_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} dydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(\sqrt{8 - x^2 - y^2} - \sqrt{x^2 + y^2}\right) dydx$$

*Convert to **Polar coordinates***

$$= \int_0^{2\pi} \int_0^{2} \left(\sqrt{8 - r^2} - r\right) r\, drd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{2} \left(r\sqrt{8 - r^2} - r^2\right) dr$$

$$= (2\pi)\left(\int_0^{2} \frac{-1}{2}\left(8 - r^2\right)^{1/2} d\left(8 - r^2\right) - \left(\frac{1}{3}r^3\right)\Big|_0^2\right)$$

$$= (\pi)\left(-\frac{2}{3}\left(8 - r^2\right)^{3/2}\Big|_0^2 - \frac{8}{3}\right)$$

$$= \pi\left(-\frac{2}{3}\left(8-16\sqrt{2}\right)-\frac{8}{3}\right)$$

$$= \frac{32\pi}{3}\left(\sqrt{2}-1\right) \quad unit^3$$

### *Exercise*

The solid between the sphere $x^2+y^2+z^2=19$ and the hyperboloid $z^2-x^2-y^2=1$, for $z>0$

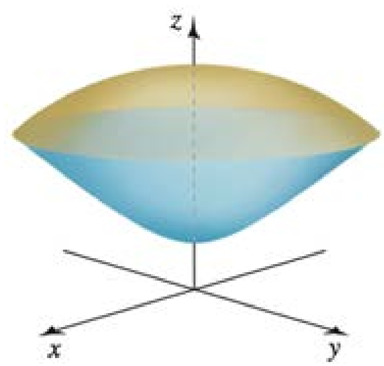
***Solution***

$z^2 = 1+x^2+y^2$

The intersection of the sphere and hyperboloid:

$x^2+y^2+z^2=19 \quad \to \quad x^2+y^2+1+x^2+y^2=19$

$2x^2+2y^2=18 \quad \to \quad x^2+y^2=9$




$$V = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{\sqrt{1+x^2+y^2}}^{\sqrt{19-x^2-y^2}} dzdydx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z\Big|_{\sqrt{1+x^2+y^2}}^{\sqrt{19-x^2-y^2}} dydx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\left(\sqrt{19-x^2-y^2}-\sqrt{1+x^2+y^2}\right)dydx$$

$$= \int_{0}^{2\pi}\int_{0}^{3}\left(\sqrt{19-r^2}-\sqrt{1+r^2}\right) rdrd\theta$$

$$= \int_{0}^{2\pi} d\theta\left(-\frac{1}{2}\int_{0}^{3}\left(19-r^2\right)^{1/2} d\left(19-r^2\right)-\frac{1}{2}\int_{0}^{3}\left(1+r^2\right)^{1/2} d\left(1+r^2\right)\right)$$

$$= -\frac{2\pi}{3}\left(\left(19-r^2\right)^{3/2}+\left(1+r^2\right)^{3/2}\right)\Big|_{0}^{3}$$

$$= -\frac{2\pi}{3}\left(10\sqrt{10}+10\sqrt{10}-19\sqrt{19}-1\right)$$

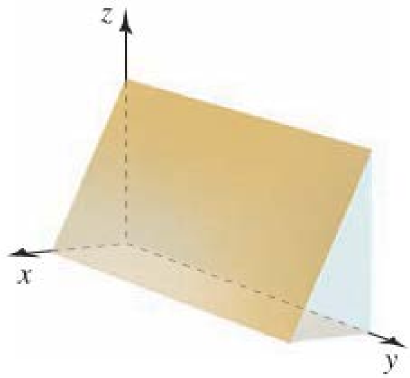
$$= \frac{2\pi}{3}\left(1-20\sqrt{10}+19\sqrt{19}\right)$$

### *Exercise*

Find the volume of the prism in the first octant bounded below by $z = 2 - 4x$ and $y = 8$

***Solution***

$$z = 2 - 4x = 0 \quad \Rightarrow \quad x = \frac{1}{2}$$



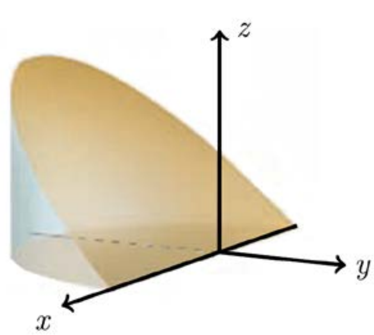

$$\begin{aligned}
V &= \int_0^{1/2} \int_0^8 \int_0^{2-4x} 1\, dzdydx \\
&= \int_0^{1/2} \int_0^8 (2-4x)\, dydx \\
&= \int_0^{1/2} (2-4x)\, y \Big|_0^8\, dx \\
&= 16 \int_0^{1/2} (1-2x)\, dx \\
&= 16\left(x - x^2\right) \Big|_0^{1/2} \\
&= 16\left(\frac{1}{2} - \frac{1}{4}\right) \\
&= 4 \quad unit^3
\end{aligned}$$

### *Exercise*

Find the volume of the wedge above the *xy*-plane formed when the cylinder $x^2 + y^2 = 4$ is cut by the planes $z = 0$ and $y = -z$

***Solution***

$$0 \le z \le -y \;\; (y<0); \quad -\sqrt{4-x^2} \le y \le 0; \quad y = 0 \to x^2 = 4 \;\Rightarrow\; -2 \le x \le 2$$




$$\begin{aligned}
V &= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{0} \int_0^{-y} 1\, dzdydx \\
&= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{0} (-y)\, dydx \\
&= -\frac{1}{2} \int_{-2}^{2} \left(y^2\right) \Big|_{-\sqrt{4-x^2}}^{0}\, dx
\end{aligned}$$

$$=\frac{1}{2}\int_{-2}^{2}\left(4-x^2\right)\,dx$$

$$=\frac{1}{2}\left(4x-\frac{1}{3}x^3\right)\Big|_{-2}^{2}$$

$$=8-\frac{8}{3}$$

$$=\frac{16}{3}\quad unit^3$$

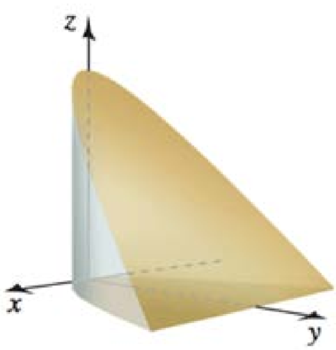
## Exercise

The wedge bounded by the parabolic cylinder $y=x^2$ and the planes $z=3-y$ and $z=0$

**Solution**

$$z=3-y=0\quad\to\quad \underline{y=3}$$

$$y=x^2=3\quad\to\quad \underline{x=\pm\sqrt{3}}$$




$$V=\int_{-\sqrt{3}}^{\sqrt{3}}\int_{x^2}^{3}\int_{0}^{3-y}dzdydx$$

$$=\int_{-\sqrt{3}}^{\sqrt{3}}\int_{x^2}^{3}z\Big|_{0}^{3-y}\,dydx$$

$$=\int_{-\sqrt{3}}^{\sqrt{3}}\int_{x^2}^{3}(3-y)\,dydx$$

$$=\int_{-\sqrt{3}}^{\sqrt{3}}\left(3y-\frac{1}{2}y^2\right)\Big|_{x^2}^{3}\,dx$$

$$=\int_{-\sqrt{3}}^{\sqrt{3}}\left(\frac{9}{2}-3x^2+\frac{1}{2}x^4\right)dx$$

$$=\left(\frac{9}{2}x-x^3+\frac{1}{10}x^5\right)\Big|_{-\sqrt{3}}^{\sqrt{3}}$$

$$=2\left(\frac{9}{2}\sqrt{3}-3\sqrt{3}+\frac{9}{10}\sqrt{3}\right)$$

$$=\frac{24\sqrt{3}}{5}\quad unit^3$$

### Exercise

Find the volume of the solid bounded by the surfaces $z = e^y$ and $z = 1$ over the rectangle $\{(x, y):\ 0 \le x \le 1,\ 0 \le y \le \ln 2\}$

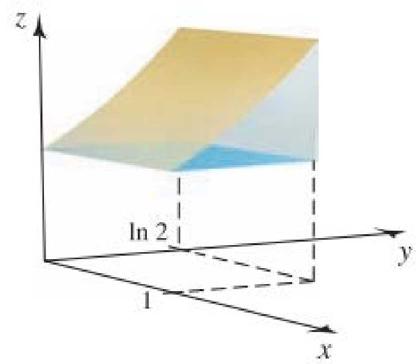
**Solution**

$$V = \int_0^1 \int_0^{\ln 2} \int_1^{e^y} 1\, dzdydx$$

$$= \int_0^1 dx \int_0^{\ln 2} \left(e^y - 1\right) dy$$

$$= x\Big|_0^1 \left(e^y - y\right)\Big|_0^{\ln 2}$$

$$= 1 - \ln 2 \quad unit^3$$




### Exercise

Find the volume of the wedge of the cylinder $x^2 + 4y^2 = 4$ created by the planes $z = 3 - x$ and $z = x - 3$

**Solution**

$$y^2 = \frac{1}{4}\left(4 - x^2\right) \quad \rightarrow \quad y = \pm\frac{1}{2}\sqrt{4 - x^2} \text{ v}$$

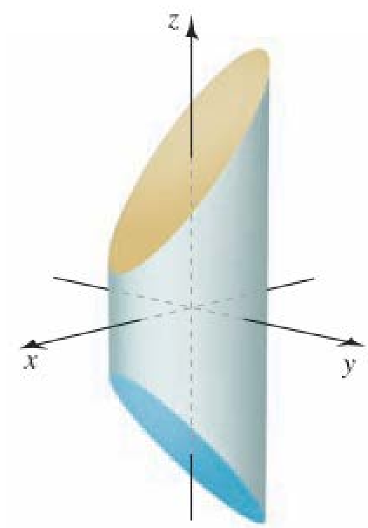
$$x^2 = 4 \quad \rightarrow \quad -2 \le x \le 2$$

$$V = \int_{-2}^{2} \int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} \int_{x-3}^{3-x} 1\, dzdydx$$

$$= \int_{-2}^{2} \int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} (6 - 2x)\, dydx$$

$$= \int_{-2}^{2} (6 - 2x)\, y \Big|_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} (6 - 2x)\sqrt{4 - x^2}\, dx$$




$$= \int_{-2}^{2} 6\sqrt{4 - x^2}\, dx + \int_{-2}^{2} \sqrt{4 - x^2}\, d\left(4 - x^2\right)$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$= 3x\sqrt{4-x^2} + 12\sin^{-1}\frac{x}{2} + \frac{2}{3}\sqrt{4-x^2}\Big|_{-2}^{2}$$

$$= 12\frac{\pi}{2} + 12\frac{\pi}{2}$$

$$= 12\pi \quad unit^3$$

## *Exercise*

Find the volume of the solid in the first octant bounded by the cone $z = 1 - \sqrt{x^2 + y^2}$ and the plane $x + y + z = 1$

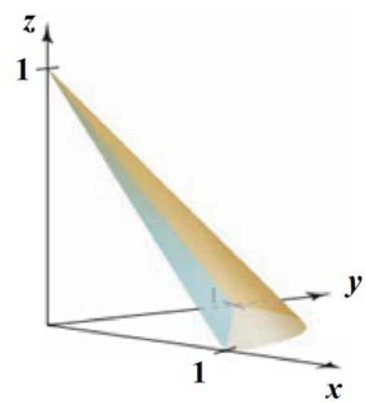
***Solution***

$$0 \le z \le 1$$

$$z = 1 - \sqrt{x^2+y^2} \quad \to x^2 + y^2 = (1-z)^2 \Rightarrow x = \sqrt{(1-z)^2 - y^2}$$

$$1 - y - z \le x \le \sqrt{(1-z)^2 - y^2}$$

$$0 \le y \le 1 - z$$




$$V = \int_0^1 \int_1^{1-z} \int_{1-y-z}^{\sqrt{(1-z)^2-y^2}} 1\, dxdydz$$

$$= \int_0^1 \int_1^{1-z} x \Big|_{1-y-z}^{\sqrt{(1-z)^2-y^2}} dydz$$

$$= \int_0^1 \int_1^{1-z} \left(\sqrt{(1-z)^2 - y^2} - 1 + y + z\right) dydz \qquad \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$= \int_0^1 \frac{y}{2}\sqrt{(1-z)^2 - y^2} + \frac{1}{2}(1-z)^2 \sin^{-1}\left(\frac{y}{1-z}\right) - y + \frac{1}{2}y^2 + zy \Big|_0^{1-z} dz$$

$$= \int_0^1 \left(\frac{1}{2}(1-z)^2 \sin^{-1}(1) + \frac{1}{2}(1-z)^2 - (1-z)^2\right) dz$$

$$= \int_0^1 \left(\frac{\pi}{4}(1-z)^2 - \frac{1}{2}(1-z)^2\right) dz$$

$$= \frac{\pi - 2}{4}\int_0^1 (1-z)^2 d(1-z)$$

$$= \frac{\pi - 2}{12}(1-z)^3 \Big|_0^1$$

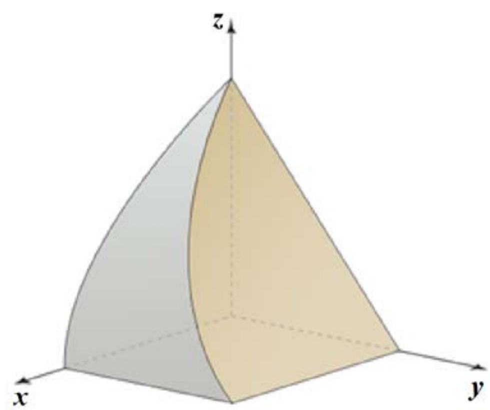
$$= \frac{\pi - 2}{12} \quad unit^3$$

### *Exercise*

Find the volume of the solid bounded by $x = 0$, $x = 1 - z^2$, $y = 0$, $z = 0$, and $z = 1 - y$

***Solution***

$$V = \int_0^1 \int_0^{1-z^2} \int_0^{1-z} 1 \, dydxdz$$




$$= \int_0^1 \int_0^{1-z^2} (1-z)\, dxdz$$

$$= \int_0^1 (1-z)\, x \Big|_0^{1-z^2} dz$$

$$= \int_0^1 (1-z)\left(1-z^2\right) dz$$

$$= \int_0^1 \left(1 - z^2 - z + z^3\right) dz$$

$$= z - \frac{1}{3}z^3 - \frac{1}{2}z^2 + \frac{1}{4}z^4 \Big|_0^1$$

$$= 1 - \frac{1}{3} - \frac{1}{2} + \frac{1}{4}$$

$$= \frac{5}{12} \quad unit^3$$

### *Exercise*

Find the volume of the solid bounded by $x = 0$, $x = 2$, $y = 0$, $y = e^{-z}$, $z = 0$, and $z = 1$

***Solution***

$$V = \int_0^2 \int_0^1 \int_0^{e^{-z}} 1 \, dydzdx$$

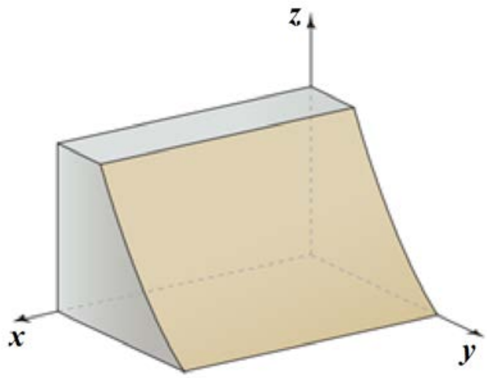
$$= \int_0^2 dx \int_0^1 y\Big|_0^{e^{-z}} \, dz$$

$$= 2\int_0^1 e^{-z}\, dz$$

$$= -2e^{-z}\Big|_0^1$$

$$= -2\left(e^{-1} - 1\right)$$

$$= 2 - \frac{2}{e} \quad unit^3$$




### Exercise

Find the volume of the solid bounded by $x = 0$, $x = 2$, $y = z$, $y = z + 1$, $z = 0$, and $z = 4$

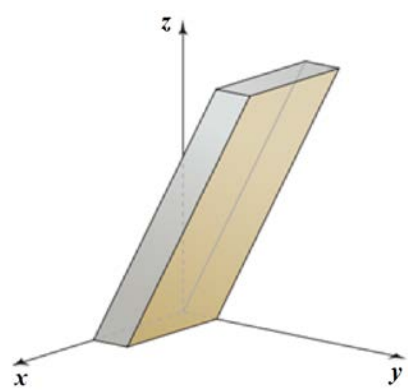
***Solution***

$$V = \int_0^2 \int_0^4 \int_z^{z+1} 1\, dydzdx$$

$$= \int_0^2 \int_0^4 y\Big|_z^{z+1} dzdx$$

$$= \int_0^2 dx \int_0^4 dz \quad = (2)(4)$$

$$= 8 \quad unit^3$$




### Exercise

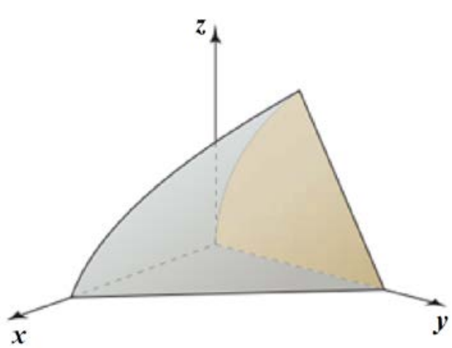
Find the volume of the solid bounded by $x = 0$, $y = z^2$, $z = 0$, and $z = 2 - x - y$

***Solution***

$$y = 2 - x - z; \quad x = 2 - z - y = 2 - z - z^2$$

$$V = \int_0^1 \int_0^{2-z-z^2} \int_{z^2}^{2-x-z} 1\, dydxdz$$

$$= \int_0^1 \int_0^{2-z-z^2} \left(2 - x - z - z^2\right) dxdz$$



$$= \int_0^1 \left( \left(2 - z - z^2\right) x - \frac{1}{2} x^2 \right) \Big|_0^{2-z-z^2} \; dz$$

$$= \frac{1}{2} \int_0^1 \left(2 - z - z^2\right)^2 dz$$

$$= \frac{1}{2} \int_0^1 \left(4 - 4z - 3z^2 + 2z^3 + z^4\right) dz$$

$$= \frac{1}{2} \left(4z - 2z^2 - z^3 + \frac{1}{2} z^4 + \frac{1}{5} z^5\right) \Big|_0^1$$

$$= \frac{1}{2} \left(4 - 2 - 1 + \frac{1}{2} + \frac{1}{5}\right)$$

$$= \frac{17}{20} \quad unit^3$$

## *Exercise*

Find the volume of the solid common to the cylinders $z = \sin x$ and $z = \sin y$ over the square

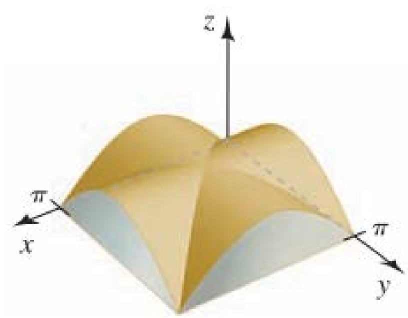
$$R = \left\{ (x, y): \; 0 \le x \le \pi, \; 0 \le y \le \pi \right\}$$

***Solution***

$$z = \sin x = \sin y \quad \to \quad x = y \;\; or \;\; y = \pi - x$$

$$V = 4 \int_0^{\pi/2} \int_x^{\pi - x} \int_0^{\sin y} 1 dz dy dx \qquad \textbf{4:} \; by\ symmetry,\ volume - \textbf{4}\ times$$

$$= 4 \int_0^{\pi/2} \int_x^{\pi - x} \sin y \; dy dx$$




$$= -4 \int_0^{\pi/2} \cos y \Big|_x^{\pi - x} \; dx$$

$$= -4 \int_0^{\pi/2} \left( \cos(\pi - x) - \cos x \right) \; dx$$

$$= -4 \int_0^{\pi/2} \left( -2\cos x \right) \; dx$$

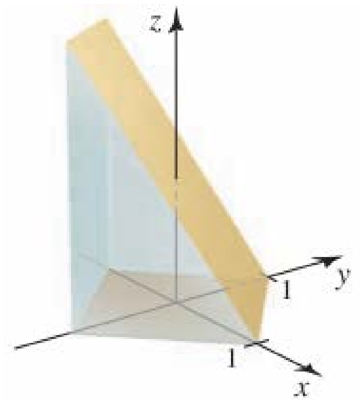
$$= 8 \sin x \Big|_0^{\pi/2}$$

$$= 8 \quad unit^3$$

## *Exercise*

Find the volume of the wedge of the square column $|x|+|y|=1$ created by the planes $z=0$ and $x+y+z=1$

***Solution***




$$0 \le z \le 1-x-y$$

$$|x|+|y|=1 \quad \to \quad \begin{cases} x+y=1 \Rightarrow y=1-x \\ -x+y=1 \Rightarrow y=1+x \\ x-y=1 \Rightarrow y=x-1 \\ -x-y=1 \Rightarrow y=-x-1 \end{cases}$$

$$\begin{cases} y=-x-1 \\ y=x+1 \end{cases} \Rightarrow -1 \le x \le 0 \qquad \begin{cases} y=x-1 \\ y=-x+1 \end{cases} \Rightarrow 0 \le x \le 1$$

$$V=\int_{-1}^{0}\int_{-x-1}^{x+1}\int_{0}^{1-x-y} 1dzdydx+\int_{0}^{1}\int_{x-1}^{-x+1}\int_{0}^{1-x-y} 1dzdydx$$

$$=\int_{-1}^{0}\int_{-x-1}^{x+1}(1-x-y)\ dydx+\int_{0}^{1}\int_{x-1}^{-x+1}(1-x-y)\ dydx$$

$$=\int_{-1}^{0}\left((1-x)y-\frac{1}{2}y^2\right)\Big|_{-x-1}^{x+1}\ dx+\int_{0}^{1}\left((1-x)y-\frac{1}{2}y^2\right)\Big|_{x-1}^{-x+1}\ dx$$

$$=\int_{-1}^{0} 2(1-x)(x+1)\ dx+\int_{0}^{1} 2(1-x)^2\ dx$$

$$=\int_{-1}^{0} 2\left(1-x^2\right)\ dx+2\int_{0}^{1}\left(1-2x+x^2\right)\ dx$$

$$=2\left(x-\frac{1}{3}x^3\right)\Big|_{-1}^{0}+2\left(x-x^2+\frac{1}{3}x^3\right)\Big|_{0}^{1}$$

$$=2\left(1-\frac{1}{3}\right)+\frac{2}{3}$$

$$=2 \quad unit^3$$

## *Exercise*

Find the volume of a right circular cone with height $h$ and base radius $r$.

***Solution***

The equation of a circle is centered at the origin with radius $r$: $x^2+y^2=r^2$

$$-\sqrt{r^2-x^2} \le y \le \sqrt{r^2-x^2} \quad \& \quad -r \le x \le r$$

$$z = a - b\sqrt{x^2+y^2} \begin{cases} z = h & \underline{h = a} \\ z = 0 & 0 = a - br = h - br \ \Rightarrow b = \frac{h}{r} \end{cases}$$

The equation of a cone with height $h$: $z = h - \frac{h}{r}\sqrt{x^2+y^2}$

$$V = \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}}\int_{0}^{h-\frac{h}{r}\sqrt{x^2+y^2}} 1dzdydx$$

$$= \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \left(h - \frac{h}{r}\sqrt{x^2+y^2}\right)dydx$$

Let $x^2 + y^2 = R^2$ (*Polar Coordinates*)

$$= \int_{0}^{2\pi}\int_{0}^{r}\left(h - \frac{h}{r}R\right) R\ dRd\theta$$

$$= \int_{0}^{2\pi} d\theta \int_{0}^{r}\left(hR - \frac{h}{r}R^2\right) dR$$

$$= 2\pi\left(\frac{1}{2}hR^2 - \frac{h}{3r}R^3\right)\Big|_0^r$$

$$= 2\pi\left(\frac{1}{2}hr^2 - \frac{1}{3}hr^2\right)$$

$$= \frac{1}{3}\pi r^2 h \quad unit^3$$

## *Exercise*

Find the volume of a tetrahedron whose vertices are located at $(0,\ 0,\ 0)$, $(a,\ 0,\ 0)$, $(0,\ b,\ 0)$, and $(0,\ 0,\ c)$

***Solution***

The equation of the plane through the vertices: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$0 \le z \le c\left(1 - \frac{x}{a} - \frac{y}{b}\right) \quad 0 \le y \le b\left(1 - \frac{x}{a}\right) \quad 0 \le x \le a$$

$$V = \int_{0}^{a}\int_{0}^{b\left(1-\frac{x}{a}\right)}\int_{0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} 1dzdydx$$

$$= \int_{0}^{a}\int_{0}^{b\left(1-\frac{x}{a}\right)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right)dydx$$

$$= c\int_0^a \left(\left(1-\frac{x}{a}\right)y - \frac{1}{2b}y^2\right)\Bigg|_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= c\int_0^a \left(b\left(1-\frac{x}{a}\right)^2 - \frac{1}{2}b\left(1-\frac{x}{a}\right)^2\right)dx$$

$$= \frac{1}{2}bc\int_0^a \left(1-\frac{2}{a}x+\frac{1}{a^2}x^2\right)dx$$

$$= \frac{1}{2}bc\left(x-\frac{1}{a}x^2+\frac{1}{3a^2}x^3\right)\Bigg|_0^a$$

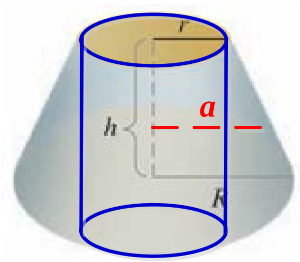
$$= \frac{1}{2}bc\left(a-a+\frac{1}{3}a\right)$$

$$\boxed{= \frac{abc}{6}}$$

## *Exercise*

Find the volume of a truncated cone of height $h$ whose ends have radii $r$ and $R$.

***Solution***




There are 2 volumes to consider:

1. Volume of the cylinder: $V_1 = \pi r^2 h$
2. Volume $V_2$ that remains when cylinder is removed.

   $V_2$ is the annulus centered at the origin with inner radius $r$ and outer radius $R$.

   Using Polar Coordinates: the equation of the frustum is: $z = \frac{h}{R-r}(R-a)$

$$V_2 = \int_0^{2\pi}\int_r^R\int_0^{\frac{h}{R-r}(R-a)} adzdad\theta$$

$$= \int_0^{2\pi}\int_r^R \frac{h}{R-r}(R-a)a\ dad\theta$$

$$= \frac{h}{R-r}\int_0^{2\pi}d\theta\int_r^R \left(Ra-a^2\right)da$$

$$= \frac{2\pi h}{R-r}\left(\frac{1}{2}Ra^2-\frac{1}{3}a^3\right)\Bigg|_r^R$$

$$= \frac{2\pi h}{R-r}\left(\frac{1}{2}R^3-\frac{1}{3}R^3-\frac{1}{2}Rr^2+\frac{1}{3}r^3\right)$$

$$= \frac{2\pi h}{R-r}\left(\frac{1}{6}R^3 - \frac{1}{2}Rr^2 + \frac{1}{3}r^3\right)$$

$$= \frac{1}{3}\frac{\pi h}{R-r}\left(R^3 - 3Rr^2 + 2r^3\right)$$

$$V_1 + V_1 = \pi r^2 h + \frac{1}{3}\frac{\pi h}{R-r}\left(R^3 - 3Rr^2 + 2r^3\right)$$

$$= \frac{1}{3}\frac{\pi h}{R-r}\left(3r^2(R-r) + R^3 - 3Rr^2 + 2r^3\right)$$

$$= \frac{1}{3}\frac{\pi h}{R-r}\left(R^3 - r^3\right)$$

$$= \frac{1}{3}\frac{\pi h}{R-r}(R-r)\left(R^3 + rR + r^2\right)$$

$$= \frac{1}{3}\pi h\left(R^3 + rR + r^2\right)$$

### *Exercise*

Find the volume of the solid that is bounded above by the cylinder $z = 4 - x^2$, on the sides by the cylinder $x^2 + y^2 = 4$, and below by the $xy$-plane.

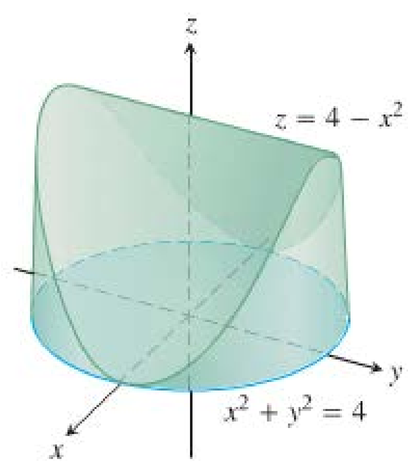
***Solution***

$z = 4 - x^2 \quad \to \quad 0 \le z \le 4 - x^2$

$x^2 + y^2 = 4 \quad \to \quad -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

Since it is symmetric, then $0 \le y \le \sqrt{4-x^2}$

$y = 0 \quad \to \quad x = \pm 2 \qquad 0 \le x \le 2$




$$V = 4\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{4-x^2} dzdydx$$

$$= 4\int_0^2 \int_0^{\sqrt{4-x^2}} z\Big|_0^{4-x^2} dydx$$

$$= 4\int_0^2 \int_0^{\sqrt{4-x^2}} \left(4 - x^2\right) dydx$$

$$= 4\int_0^2 \left(4 - x^2\right) y\Big|_0^{\sqrt{4-x^2}} dx$$

$$= 4\int_0^2 \left(4 - x^2\right)^{3/2} dx$$

$$x = 2\sin\alpha \quad \to \quad 4 - x^2 = 4\cos^2\alpha$$

$$dx = 2\cos\alpha\, d\alpha$$

$$\begin{cases} x = 2 & \rightarrow \alpha = \sin^{-1} 1 = \frac{\pi}{2} \\ x = 0 & \rightarrow \alpha = \sin^{-1} 0 = 0 \end{cases}$$

$$= 4\int_0^{\pi/2} 16\cos^4 \alpha \, d\alpha$$

$$= 64\int_0^{\pi/2} \left(\frac{1+\cos 2\alpha}{2}\right)^2 \, d\alpha$$

$$= 16\int_0^{\pi/2} \left(1 + 2\cos 2\alpha + \cos^2 2\alpha\right) \, d\alpha$$

$$= 16\int_0^{\pi/2} \left(1 + 2\cos 2\alpha + \frac{1}{2} + \frac{1}{2}\cos 4\alpha\right) \, d\alpha$$

$$= 16\left(\frac{3}{2}\alpha + \sin 2\alpha + \frac{1}{8}\sin 4\alpha\right)\Big|_0^{\pi/2}$$

$$= 16\left(\frac{3}{2}\alpha + \sin 2\alpha + \frac{1}{8}\sin 4\alpha\right)\Big|_0^{\pi/2}$$

$$= 16\left(\frac{3\pi}{4}\right)$$

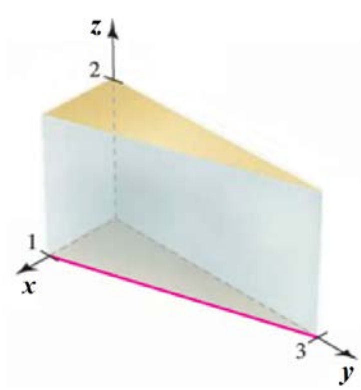
$$\underline{= 12\pi \ \ unit^3}$$

## *Exercise*

Find the volume of the prism in the first octant bounded by the planes $y = 3 - 3x$ *and* $z = 2$

***Solution***

$$V = \int_0^1 \int_0^{3-3x} \int_0^2 dz dy dx$$




$$= \int_0^1 \int_0^{3-3x} z\Big|_0^2 \, dy dx$$

$$= 2\int_0^1 \int_0^{3-3x} dy dx$$

$$= 2\int_0^1 y\Big|_0^{3-3x} \, dx$$

$$= 2\int_0^1 (3 - 3x)\, dx$$

$$= 2\left(3x - \frac{3}{2}x^2\right)\Big|_0^1$$

$$= 2\left(3 - \frac{3}{2}\right)$$

$$= 3 \ \ unit^3$$

### *Exercise*

Find the volume of the prism in the first octant bounded by the planes $x^2 + y^2 = 4$ *and* $x^2 + z^2 = 4$

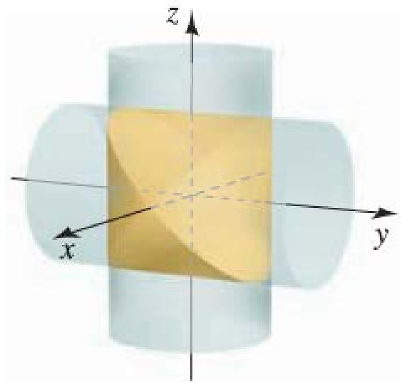
***Solution***

$$x^2 + y^2 = 4 \quad \rightarrow \quad y^2 \le 4 - x^2$$

$$-\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}$$

$$x^2 + z^2 = 4 \quad \rightarrow \quad z^2 \le 4 - x^2$$

$$-\sqrt{4 - x^2} \le z \le \sqrt{4 - x^2}$$




$$x^2 \le 4 \quad \rightarrow \quad -2 \le x \le 2$$

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dzdydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} z \Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dydx$$

$$= 2\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4 - x^2}\, dydx$$

$$= 2\int_{-2}^{2} \sqrt{4 - x^2}\ y \Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= 4\int_{-2}^{2} \left(4 - x^2\right) dx$$

$$= 4\left(4x - \frac{1}{3}x^3\right)\Big|_{-2}^{2}$$

$$= 8\left(8 - \frac{8}{3}\right)$$

$$= \frac{128}{3} \ \ unit^3$$