

Section 3.8 – Taylor and Maclaurin Series

The sum of a power series:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

$$= a_0 + a_1 (x-a) + a_2 (x-a)^2 + \cdots + a_n (x-a)^n + \cdots$$

$$f'(x) = a_1 + 2a_2 (x-a) + 3a_3 (x-a)^2 + \cdots + na_n (x-a)^{n-1} + \cdots$$

$$f''(x) = 2a_2 + 2 \cdot 3a_3 (x-a) + 3 \cdot 4a_4 (x-a)^2 + \cdots + (n-1) \cdot na_n (x-a)^{n-2} + \cdots$$

$$f'''(x) = 1 \cdot 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4 (x-a) + 3 \cdot 4 \cdot 5a_5 (x-a)^2 + \cdots + (n-2) \cdot (n-1) \cdot na_n (x-a)^{n-3} + \cdots$$

$$f^{(n)}(x) = n!a_n + \text{a sum of terms with } (x-a) \text{ as a factor}$$

In general: $\boxed{f^{(n)}(x) = n!a_n} \Rightarrow a_n = \frac{f^{(n)}(a)}{n!}$

If f has a series representation, then the series must be

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Taylor and Maclaurin Series

Definitions

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point.

Then the **Taylor series generated by f** at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The **Maclaurin series generated by f** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

The Taylor series generated by f at $x = 0$.

Example

Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$. Where, if anywhere, does the series converge to $\frac{1}{x}$.

Solution

$$f(x) = x^{-1}$$

$$f(2) = 2^{-1} = \frac{1}{2}$$

$$f'(x) = -x^{-2}$$

$$f'(2) = -\frac{1}{2^2}$$

$$f''(x) = 2!x^{-3}$$

$$\frac{f''(2)}{2!} = 2^{-3} = \frac{(-1)^3}{2^3}$$

$$f'''(x) = -3!x^{-4}$$

$$\frac{f^{(3)}(2)}{3!} = 2^{-4} = \frac{(-1)^4}{2^4}$$

$$\vdots \quad \vdots$$

$$\vdots \quad \vdots$$

$$f^{(n)}(x) = (-1)^n n! x^{-(n+1)}$$

$$\frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}$$

The Taylor series is:

$$\begin{aligned} f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n \\ = \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} + \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots \end{aligned}$$

Taylor Polynomials

Definition

Let f be a function with derivatives of order k for $k = 1, 2, \dots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N , the Taylor polynomial of order n generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Example

Find the Taylor series and the Taylor polynomials generated by $f(x) = e^x$ at $x = 0$

Solution

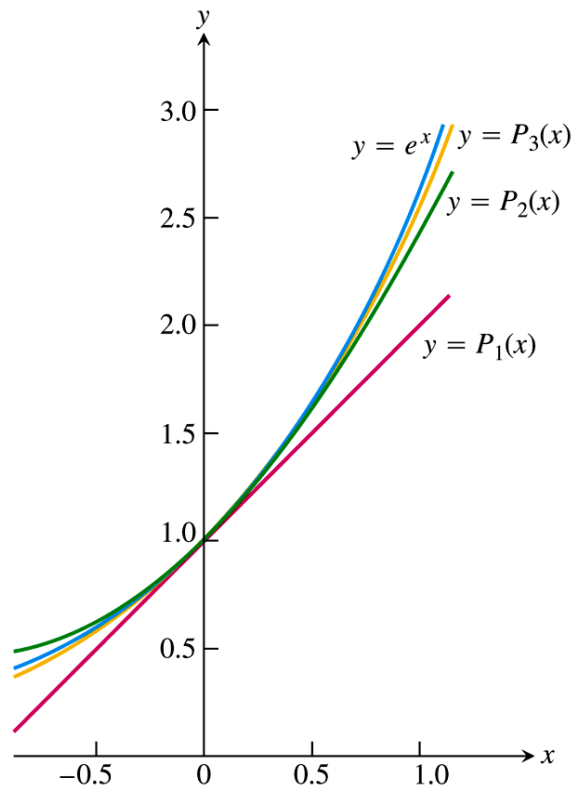
$$f^{(n)}(x) = e^x \rightarrow f^{(n)}(0) = 1$$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$

This is also the Maclaurin series of e^x



The Taylor polynomial of order n at $x = 0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$$

Example

Find the Taylor series and the Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$

Solution

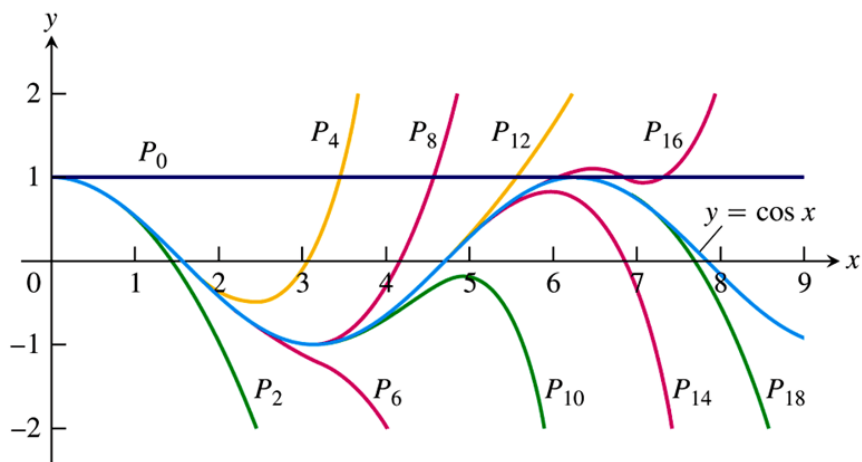
$$\begin{aligned} f(x) &= \cos x, & f'(x) &= -\sin x, \\ f''(x) &= -\cos x, & f''(x) &= \sin x, \\ &\vdots & &\vdots \\ f^{(2n)}(x) &= (-1)^n \cos x, & f^{(2n+1)}(x) &= (-1)^{n+1} \sin x \end{aligned}$$

$$f^{(2n)}(0) = (-1)^n, \quad f^{(2n+1)}(0) = 0$$

The Taylor series generated by f at $x = 0$ is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ = 1 + 0 \cdot x - \frac{x^2}{2!} + 0 \cdot x^3 + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\ = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\ = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \end{aligned}$$

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$$



Example

Find the Taylor series for $\cos x$ about $\frac{\pi}{3}$. Where is the series valid?

Solution

$$\begin{aligned}\cos x &= \cos\left(x - \frac{\pi}{3} + \frac{\pi}{3}\right) \\&= \cos\left(x - \frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(x - \frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) \\&= \frac{1}{2}\left[1 - \frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{3}\right)^4 - \dots\right] - \frac{\sqrt{3}}{2}\left[\left(x - \frac{\pi}{3}\right) - \frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{5!}\left(x - \frac{\pi}{3}\right)^5 - \dots\right] \\&= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{2} \frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2 - \frac{\sqrt{3}}{2} \frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{2} \frac{1}{4!}\left(x - \frac{\pi}{3}\right)^4 + \frac{1}{5!} \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)^5 - \dots\end{aligned}$$

This series representation is valid for all x .

Example

Find the Taylor series for $\ln x$ in powers of $x - 2$. Where does the series converge to $\ln x$?

Solution

Let $t = \frac{x-2}{2}$, then

$$\begin{aligned}\ln x &= \ln(2 + (x-2)) \\ &= \ln\left[2\left(1 + \frac{x-2}{2}\right)\right] \\ &= \ln 2 + \ln(1+t)\end{aligned}$$

$$f(t) = \ln(1+t) \qquad f(0) = \ln(1) = 0$$

$$f'(t) = \frac{1}{1+t} \qquad f'(0) = 1$$

$$f''(t) = \frac{-1}{(1+t)^2} \qquad f''(0) = -1$$

$$f'''(t) = \frac{2}{(1+t)^3} \qquad f'''(0) = 2$$

$$f^{(4)}(t) = \frac{-6}{(1+t)^4} \qquad f^{(4)}(0) = -6$$

$$f^{(n)}(t) = (-1)^{n+1} \frac{(n-1)!}{(1+t)^n}$$

$$\begin{aligned}\ln(1+t) &= f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \frac{f'''(0)}{3!}t^3 + \frac{f^{(IV)}(0)}{4!}t^4 + \dots \\ &= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots\end{aligned}$$

$$\begin{aligned}\ln x &= \ln 2 + \ln(1+t) \\ &= \ln 2 + t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots \\ &= \ln 2 + \frac{x-2}{2} - \frac{(x-2)^2}{2 \times 2^2} + \frac{(x-2)^3}{3 \times 2^3} - \frac{(x-2)^4}{4 \times 2^4} + \dots \\ &= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \times 2^n} (x-2)^n\end{aligned}$$

Since the series for $\ln(1+t)$ is valid for $-1 < t \leq 1$, this series for $\ln x$ is valid for $-1 < \frac{x-2}{2} \leq 1$

$$-2 < x-2 \leq 2 \quad \rightarrow \quad \underline{0 < x \leq 4}$$

Exercises Section 3.8 – Taylor and Maclaurin Series

(1 – 23) Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a

1. $f(x) = e^{2x}, \quad a = 0$
2. $f(x) = \sin x, \quad a = 0$
3. $f(x) = \ln(1+x), \quad a = 0$
4. $f(x) = \frac{1}{x+2}, \quad a = 0$
5. $f(x) = \sqrt{1-x}, \quad a = 0$
6. $f(x) = x^3, \quad a = 1$
7. $f(x) = 8\sqrt{x}, \quad a = 1$
8. $f(x) = \sin x, \quad a = \frac{\pi}{4}$
9. $f(x) = \cos x, \quad a = \frac{\pi}{6}$
10. $f(x) = \sqrt{x}, \quad a = 9$
11. $f(x) = \sqrt[3]{x}, \quad a = 8$
12. $f(x) = \ln x, \quad a = e$
13. $f(x) = \sqrt[4]{x}, \quad a = 8$
14. $f(x) = \tan^{-1} x + x^2 + 1, \quad a = 1$
15. $f(x) = e^x, \quad a = \ln 2$
16. $f(x) = e^{3x}; \quad a = 0$
17. $f(x) = \frac{1}{x}; \quad a = 1$
18. $f(x) = \cos x; \quad a = \frac{\pi}{2}$
19. $f(x) = \frac{1}{x+1}; \quad a = 0$
20. $f(x) = \tan^{-1} 4x; \quad a = 0$
21. $f(x) = \sin 2x; \quad a = -\frac{\pi}{2}$
22. $f(x) = \cosh 3x; \quad a = 0$
23. $f(x) = \frac{1}{4+x^2}; \quad a = 0$

(25 – 35) Find the n th Maclaurin polynomial for the function

24. $f(x) = e^{4x}, \quad n = 4$
25. $f(x) = e^{-x}, \quad n = 5$
26. $f(x) = e^{-x/2}, \quad n = 4$
27. $f(x) = e^{x/3}, \quad n = 4$
28. $f(x) = \sin x, \quad n = 5$
29. $f(x) = \cos \pi x, \quad n = 4$
30. $f(x) = xe^x, \quad n = 4$
31. $f(x) = x^2 e^{-x}, \quad n = 4$
32. $f(x) = \frac{1}{x+1}, \quad n = 5$
33. $f(x) = \frac{x}{x+1}, \quad n = 4$
34. $f(x) = \sec x, \quad n = 2$
35. $f(x) = \tan x, \quad n = 3$

(36 – 39) Find out the **third** term of the Maclaurin series for the following function.

36. $f(x) = (1+x)^{1/3}$
37. $f(x) = (1+x)^{-1/2}$
38. $f(x) = \left(1 + \frac{x}{2}\right)^{-3}$
39. $f(x) = (1+2x)^{-5}$

(40 – 55) Find the Maclaurin series for

40. xe^x

46. $x^2 \sin\left(\frac{x}{3}\right)$

51. $\frac{1+x^3}{1+x^2}$

41. $5 \cos \pi x$

47. $\cos^2\left(\frac{x}{2}\right)$

52. $\ln \frac{1+x}{1-x}$

42. $\frac{x^2}{x+1}$

48. $\sin x \cos x$

53. $\frac{e^{2x^2}-1}{x^2}$

43. e^{3x+1}

49. $\tan^{-1}(5x^2)$

54. $\cosh x - \cos x$

44. $\cos(2x^3)$

50. $\ln(2+x^2)$

55. $\sinh x - \sin x$

45. $\cos(2x - \pi)$

(56 – 59) Finding Taylor and Maclaurin Series generated by f at $x = a$

56. $f(x) = x^3 - 2x + 4, \quad a = 2$

58. $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, \quad a = -1$

57. $f(x) = 2x^3 + x^2 + 3x - 8, \quad a = 1$

59. $f(x) = \cos\left(2x + \frac{\pi}{2}\right), \quad a = \frac{\pi}{4}$

(60 – 68) Find the Taylor series of the functions, where is each series representation valid?

60. $f(x) = e^{-2x}$ about -1

65. $f(x) = \sin x - \cos x$ about $\frac{\pi}{4}$

61. $f(x) = \sin x$ about $\frac{\pi}{2}$

66. $f(x) = \cos^2 x$ about $\frac{\pi}{8}$

62. $f(x) = \ln x$ in powers of $x - 3$

67. $f(x) = \frac{x}{1+x}$ in powers of $x - 1$

63. $f(x) = \ln(2+x)$ in powers of $x - 2$

68. $f(x) = xe^x$ in powers of $x + 2$

64. $f(x) = e^{2x+3}$ in powers of $x + 1$

(69 – 81) Find the n th-order Taylor polynomial centered at c for the function

69. $f(x) = \frac{2}{x}, \quad n = 3, \quad c = 1$

75. $f(x) = \sin 2x; \quad n = 3, \quad c = 0$

70. $f(x) = \frac{1}{x^2}, \quad n = 4, \quad c = 2$

76. $f(x) = \cos x^2; \quad n = 2, \quad c = 0$

71. $f(x) = \sqrt{x}, \quad n = 3, \quad c = 4$

77. $f(x) = e^{-x}; \quad n = 2, \quad c = 0$

72. $f(x) = \sqrt[3]{x}, \quad n = 3, \quad c = 8$

78. $f(x) = \cos x; \quad n = 2, \quad c = \frac{\pi}{4}$

73. $f(x) = \ln x, \quad n = 4, \quad c = 2$

79. $f(x) = \ln x; \quad n = 2, \quad c = 1$

74. $f(x) = x^2 \cos x, \quad n = 2, \quad c = \pi$

80. $f(x) = \sinh 2x; \quad n = 4, \quad c = 0$

81. $f(x) = \cosh x; \quad n = 3, \quad c = \ln 2$

(82 – 84) Find the sums of the series

$$82. \quad 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots$$

$$83. \quad 1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \frac{x^8}{9!} + \dots$$

$$84. \quad x^3 - \frac{x^9}{3! \times 4} + \frac{x^{15}}{5! \times 16} - \frac{x^{21}}{7! \times 64} + \frac{x^{27}}{9! \times 256} - \dots$$

(85 – 90) Use the geometric series $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for $|x| < 1$, to determine the Maclaurin series and the interval of convergence for the following functions.

$$85. \quad f(x) = \frac{1}{1-x^2}$$

$$88. \quad f(x) = \frac{10}{1+x}$$

$$86. \quad f(x) = \frac{1}{1+x^3}$$

$$89. \quad f(x) = \frac{1}{(1-10x)^2}$$

$$87. \quad f(x) = \frac{1}{1+5x}$$

$$90. \quad f(x) = \ln(1-4x)$$

91. The limit $\lim_{n \rightarrow \infty} \frac{n!}{\sqrt{2\pi} n^{n+1/2} e^{-n}} = 1$ that is the relative error in the approximation

$$n! \approx \sqrt{2\pi} n^{n+1/2} e^{-n}$$

Approaches zero as n increases. That is $n!$ grows at a rate comparable to $\sqrt{2\pi} n^{n+1/2} e^{-n}$. This result, known as Stirling's Formula, is often very useful in applied mathematics and statistics. Prove it by carrying out the following steps.

a) Use the identity $\ln(n!) = \sum_{j=1}^n \ln j$ and the increasing nature of \ln to show that if $n \geq 1$,

$$\int_0^n \ln x \, dx < \ln(n!) < \int_1^{n+1} \ln x \, dx$$

And hence that $n \ln n - n < \ln(n!) < (n+1) \ln(n+1) - n$

b) If $c_n = \ln(n!) - \left(n + \frac{1}{2}\right) \ln n + n$, show that

$$\begin{aligned} c_n - c_{n+1} &= \left(n + \frac{1}{2}\right) \ln\left(\frac{n+1}{n}\right) - 1 \\ &= \left(n + \frac{1}{2}\right) \ln\left(\frac{1 + \frac{1}{2n+1}}{1 - \frac{1}{2n+1}}\right) - 1 \end{aligned}$$

c) Use the Maclaurin series for $\ln \frac{1+t}{1-t}$ to show that

$$0 < c_n - c_{n+1} < \frac{1}{3} \left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^3} + \dots \right) \\ = \frac{1}{12} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

and therefore that $\{c_n\}$ is decreasing and $\{c_n - \frac{1}{12n}\}$ is increasing. Hence conclude that

$\lim_{n \rightarrow \infty} c_n = c$ exists, and that

$$\lim_{n \rightarrow \infty} \frac{n!}{n^{n+1/2} e^{-n}} = \lim_{n \rightarrow \infty} e^{c_n} = e^c$$

92. Suppose you want to approximate $\sqrt[3]{128}$ to within 10^{-4} of the exact value.

- Use a Taylor polynomial for $f(x) = (125 + x)^{1/3}$ centered at 0.
- Use a Taylor polynomial for $f(x) = x^{1/3}$ centered at 125.
- Compare the two approaches. Are they equivalent?

93. Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

- Use the definition of the derivative to show that $f'(0) = 0$
- Assume the fact that $f^k(0) = 0$ for $k = 1, 2, 3, \dots$ (prove using the definition of the derivative.)
Write the Taylor series for f centered at 0.
- Explain why the Taylor series for f does not converge to f for $x \neq 0$

94. Teams A and B go into sudden death overtime after playing to a tie. The teams alternate possession of the ball and the first team to score wins. Each team has a $\frac{1}{6}$ chance of scoring when it has the ball, with Team A having the ball first.

- The probability that Team A ultimately wins is $\sum_{k=0}^{\infty} \frac{1}{6} \left(\frac{5}{6} \right)^{2k}$. Evaluate this series.

- The expected number of rounds (possessions by either team) required for the overtime to end is

$$\frac{1}{6} \sum_{k=1}^{\infty} k \left(\frac{5}{6} \right)^{k-1}. \text{ Evaluate this series.}$$

