Section 2.6 – Exponential & Logarithmic Functions

Exponential

Definition

The exponential function *f* with base *b* is defined by

$$f(x) = b^x \quad \text{or} \quad y = b^x$$

Where $b > 0$, $b \neq 1$ and x is any real number.

Example: $f(x) = 2^x \qquad f(x) = \left(\frac{1}{2}\right)^{2x+1} \qquad f(x) = 3^{-x} \qquad \cancel{f(x) = (-2)^x}$

Exponential Equations

$$b^x = b^y \quad \leftrightarrow \quad x = y \qquad \text{for any b} > 0, \neq 1$$

Example

Solve $9^x = 27$

<u>Solution</u>

$$\left(3^2\right)^x = 3^3$$

$$3^{2x} = 3^3$$

$$2x = 3$$

$$x = \frac{3}{2}$$

Example

Solve $32^{2x-1} = 128^{x+3}$

<u>Solution</u>

$$\left(2^5\right)^{2x-1} = \left(2^7\right)^{x+3}$$

$$2^{10x-5} = 2^{7x+21}$$

$$10x - 5 = 7x + 21$$

$$3x = 26$$

$$x = \frac{26}{3}$$

Natural Base *e*

The irrational number e is called natural base

$f(x) = e^x$ is called natural exponential function

$e^0 = 1$ $\qquad$ $e \approx 2.7183$ $\qquad$ $e^2 \approx 7.389$ $\qquad$ $e^{-1} \approx 0.3679$

Example

Biologists studying salmon have found that the oxygen consumption of yearling salmon (in appropriate units) increases exponentially with the speed of swimming according to the function defined by

$$f(x) = 100e^{0.6x}$$

where x is the speed in feet per second. Find the following

a) The oxygen consumption when the fish are still

$$f(x=0) = 100e^{0.6(0)}$$

$$= 100$$

b) The oxygen consumption at a speed of 2 *ft* per second

$$f(x=2) = 100e^{0.6(2)}$$

$$\approx 332$$

Logarithmic Function (*Definition)*

For $x > 0$ and $b > 0,\ b \neq 0$

$$y = \log_b x \quad \text{is equivalent to} \quad x = b^y$$

$$y = \log_b x \Leftrightarrow x = b^y$$

Base

The function $f(x) = \log_b x$ is the logarithmic function with base b.

$\log_b x$: <u>*read*</u> log base b of x *log x* ***means*** $\log_{10} x$

Logarithmic Form	*Exponential Form*
$\log_2 8 = 3$	$2^3 = 8$
$\log_{1/2} 16 = -4$	$\left(\frac{1}{2}\right)^{-4} = 16$
$\log_3 \frac{1}{81} = -4$	$3^{-4} = \frac{1}{81}$
$\log_{3/4} 1 = 0$	$\left(\frac{3}{4}\right)^0 = 1$

Natural Logarithms

Definition

$$f(x) = \log_e x = \ln x$$

The logarithmic function with base ***e*** is called natural logarithmic function.
$\ln x$ *read* "el en of x"

$\log(-1) =$ *doesn't exist* $\ln(-1) =$ *doesn't exist*

$\log 0 =$ *doesn't exist* $\ln 0 =$ *doesn't exist*

$\log 0.5 \approx -0.3010$ $\ln 0.5 \approx -0.6931$

$\log 1 = 0$ $\ln 1 = 0$

$\log 2 \approx 0.3010$ $\ln 2 \approx 0.6931$

$\log 10 = 1$ $\ln e = 1$

Domain

The domain of a logarithmic function of the form $f(x) = \log_b x$ is the set of all positive real numbers. (Inside the log has to be > 0)

Find the domain of $f(x) = \log_4 (x-5)$

$x - 5 > 0 \Rightarrow x > 5$ *Domain:* $(5, \infty)$

Properties of Logarithmic Functions

Product Rule

$$\log_b MN = \log_b M + \log_b N$$

Power Rule

$$\log_b M^p = p \log_b M$$

Quotient Rule

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

$\log_b 1 = 0$ $\log_b b = 1$ $\log_b b^x = x$

Example

Use the properties of logarithms to rewrite $\log_a \left(\frac{mnq}{p^2 r^4} \right)$

<u>Solution</u>

$$\log_a \left(\frac{mnq}{p^2 r^4} \right) = \log_a (mnq) - \log_a \left(p^2 r^4\right) \qquad \textit{Quotient Rule}$$

$$= \log_a m + \log_a n + \log_a q - \left(\log_a p^2 + \log_a r^4\right) \qquad \textit{Product Rule}$$

$$= \log_a m + \log_a n + \log_a q - \log_a p^2 - \log_a r^4$$

$$= \log_a m + \log_a n + \log_a q - 2\log_a p - 4\log_a r \qquad \textit{Power Rule}$$

Changing Logarithmic Bases

$$\log_b M = \frac{\log_a M}{\log_a b} \qquad\qquad \log_b M = \frac{\log M}{\log b} \quad \textbf{or} \quad \log_b M = \frac{\ln M}{\ln b}$$

Example

Find: $\log_5 27$

<u>*Solution*</u>

$$\log_5 27 = \frac{\ln 27}{ln5}$$

ln(27) / ln(5)

$$\approx 2.05$$

Property of *Logarithmic*

- $M = N \quad \leftrightarrow \quad \ln M = \ln N$
  1. Isolate the exponential expression
  2. Take the natural logarithm on both sides of the equation
  3. Simplify using one of the following properties: $\ln b^x = x \ln b$ *or* $\ln e^x = x$
  4. Solve for the variable

Example

Solve $3^{2x} = 4^{x+1}$

<u>*Solution*</u>

$$\ln 3^{2x} = \ln 4^{x+1}$$

$$2x \ln 3 = (x+1) \ln 4$$

$$2x \ln 3 = x \ln 4 + \ln 4$$

$$2x \ln 3 - x \ln 4 = \ln 4$$

$$x(2\ln 3 - \ln 4) = \ln 4$$

$$x = \frac{\ln 4}{2\ln 3 - \ln 4}$$

$\ln(4) / (2\ln(3) - \ln(4))$

$$\approx 1.710$$

Solving Logarithmic Equations

1. Express the equation in the form $\log_b M = c$
2. Use the definition of a logarithm to rewrite the equation in exponential form:
$$\log_b M = c \quad \Rightarrow \quad b^c = M$$
3. Solve for the variable
4. Check proposed solution in the original equation. Include only the set for M > 0

Example

Solve $\quad \log_2 x - \log_2 (x-1) = 1$

Solution

$\log_2 \frac{x}{x-1} = 1$ *Write in exponential form*

$\frac{x}{x-1} = 2^1 = 2$

$x = 2(x-1)$

$x = 2x - 2$

$-x = -2$

$x = 2$ *Check*: $\log_2 2 - \log_2 (2-1) = 1$

For any M > 0, N > 0, b > 0, ≠ 1

$$\log_b M = \log_b N \leftrightarrow M = N$$

Check proposed solution in the original equation. Include only the set inside the log for > 0

Example

Solve: $\quad \log(x+6) - \log(x+2) = \log x$

Solution

$\log(x+6) - \log(x+2) = \log x$ *Quotient Rule*

$\log \frac{x+6}{x+2} = \log x$

$\frac{x+6}{x+2} = x$ *Multiply by x + 2*

$x + 6 = x(x+2)$

$x + 6 = x^2 + 2x$

$0 = x^2 + 2x - x - 6$

$x^2 + x - 6 = 0$ *Solve for x*

$x = -3,\ 2$

Check: $x = -3 \rightarrow \log(-3+6) - \log(-3+2) = \log(-3)$ *Or Domain*

$x = 2 \rightarrow \log(2+6) - \log(2+2) = \log(2)$

Solution: $\boxed{x = 2}$

$\log_e x = \ln x$

$log_{10}\ x = logx$

Exercises *Section* 2.6 – Exponential & Logarithmic Functions

Solve

1. $4^{2x-1} = 64$

2. $3^{1-x} = \frac{1}{27}$

3. $9^x = \frac{1}{\sqrt[3]{3}}$

4. $5^{3x-6} = 125$

5. $8^{x+2} = 4^{x-3}$

Solve

6. $7e^{2x} - 5 = 58$

7. $4\ln(3x) = 8$

8. $\ln(x-3) = \ln(7x-23) - \ln(x+1)$

Use the properties of logarithms to rewrite

9. $\log_b \left(\frac{x^3 y}{z^2} \right)$

10. $\log_b \left(\frac{\sqrt[3]{x} y^4}{z^5} \right)$

11. $\log_b \sqrt[n]{\frac{x^3 y^5}{z^m}}$

12. $\log_p \sqrt[3]{\frac{m^5 n^4}{t^2}}$

13. $log_a \sqrt[4]{\frac{m^8 n^{12}}{a^3 b^5}}$