Section 4.2 – Inferences About Two Means: Dependent

Definitions

Two samples are ***independent*** if the sample values selected from one population are not related to or somehow paired or matched with the sample values from the other population.

Two samples are ***dependent*** if the sample values are *paired*. (That is, each pair of sample values consists of two measurements from the same subject (such as before/after data), or each pair of sample values consists of matched pairs (such as husband/wife data), where the matching is based on some inherent relationship.)

Objectives

Test a claim about the mean of the differences from dependent samples or construct a confidence interval estimate of the mean of the differences from dependent samples.

Statistical inference methods on matched-pairs data use the same methods as inference on a single population mean, except that the ***differences*** are analyzed.

Testing Hypotheses Regarding the Difference of Two Means Using a Matched-Pairs Design

To test hypotheses regarding the mean difference of matched-pairs data, the following must be satisfied:

- ✓ The sample is obtained using simple random sampling,
- ✓ The sample data are matched pairs,
- ✓ The differences are normally distributed with no outliers or the sample size, n, is large ($n \geq 30$),
- ✓ The sampled values are independent

$$t = \frac{\bar{d} - \mu_d}{\frac{s_d}{\sqrt{n}}}$$

Notation

d : Individual difference between the two values in a single matched pair

μ_d : Mean value of the differences d for the *population* of all pairs of data

$\bar{d}$: Mean value of the differences d for the paired *sample* data

s_d : Standard deviation of the differences d for the paired *sample* data

n : Number of all *pairs* of data

Confidence Intervals for Dependent Samples

$$\bar{d} - E < \mu_d < \bar{d} + E \qquad \textit{where} \quad E = t_{\alpha/2}\frac{s_d}{\sqrt{n}}$$

Critical values of $t_{\alpha/2}$: Use Table with $n - 1$ degrees of freedom.

***Step* 1:** Determine the null and alternative hypotheses. The hypotheses can be structured in one of three ways, where μ_d is the population mean difference of the matched-pairs data.

Two–Tailed	***Left–Tailed***	***Right–Tailed***
$H_0 : \mu_d = 0$	$H_0 : \mu_d = 0$	$H_0 : \mu_d = 0$
$H_1 : \mu_d \neq 0$	$H_1 : \mu_d < 0$	$H_1 : \mu_d > 0$

***Step* 2:** Select a level of significance, α, based on the seriousness of making a Type I error.

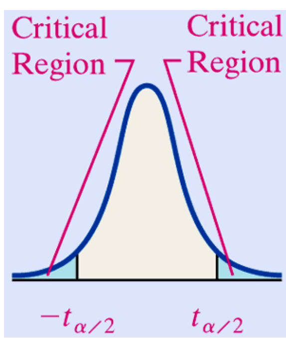
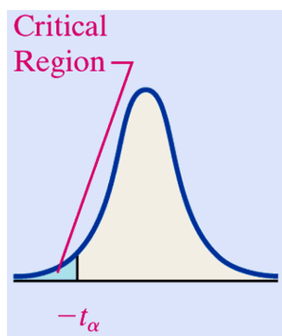
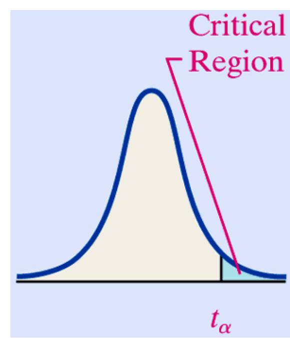
***Step* 3:** Compute the test statistic

$$t_0 = \frac{\bar{d}}{\frac{s_d}{\sqrt{n}}}$$

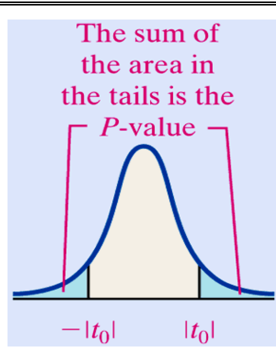
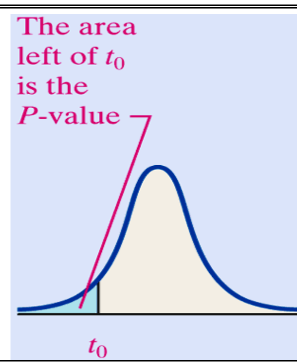
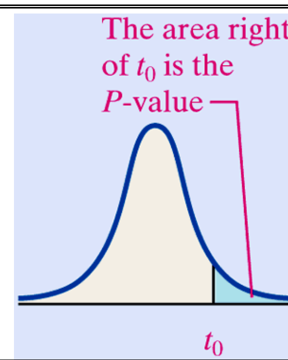
which approximately follows Student's t-distribution with $n - 1$ degrees of freedom. The values of $\bar{d}$ and s_d are the mean and standard deviation of the differenced data.

Use Table to determine the critical value using $n - 1$ degrees of freedom.

Classical Approach

Two–Tailed	***Left–Tailed***	***Right–Tailed***
$t_0 < -t_{\alpha/2}$ *or* $t_0 > t_{\alpha/2}$	$t_0 < -t_{\alpha}$	$t_0 > t_{\alpha}$
Reject the null hypothesis	*Reject the null hypothesis*	*Reject the null hypothesis*
		

***Step* 4**: Estimate the P–value

Two–Tailed	***Left–Tailed***	***Right–Tailed***
		

*If **P-value** < **α**, reject the null hypothesis*

Example

Data Set below includes measured weights of college students in September and April of their freshman year. (Here we use only a small portion of the available data so that we can better illustrate the method of hypothesis testing.) Use the sample data in Table below with a 0.05 significance level to test the claim that for the population of students, the mean change in weight from September to April is equal to 0 *kg*.

Weight (kg) Measurements of Students in Their Freshman Year

April weight	66	52	68	69	71
September weight	67	53	64	71	70
Difference *d* = (April weight) – (September weight)	–1	–1	4	–2	1

Solution

Requirements are satisfied: samples are dependent, values paired from each student; although a volunteer study, we'll proceed as if simple random sample and deal with this in the interpretation.

Weight gained = April weight – Sept. weight

μ_d denotes the mean of the "April – Sept." differences in weight; the claim is $\mu_d = 0\ kg$

Step 1: claim is $\mu_d = 0\ kg$

Step 2: If original claim is not true, we have $\mu_d \neq 0\ kg$

Step 3: $H_0 : \mu_d = 0\ kg$ original claim $H_1 : \mu_d \neq 0\ kg$

Step 4: significance level is $\alpha = 0.05$

Step 5: use the student *t* distribution

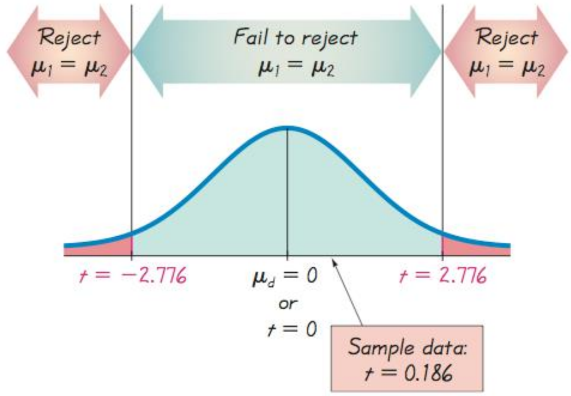
Step 6: find values of *d* and s_d differences are: –1, –1, 4, –2, 1 $d = 0.2$ and $s_d = 2.4$ now find the test statistic

$$t = \frac{\bar{d} - \mu_d}{\frac{s_d}{\sqrt{n}}} = \frac{0.2 - 0}{\frac{2.4}{\sqrt{5}}} = 0.186$$

From (*t*–Distribution Table): df = *n* – 1, area in two tails is 0.05, yields a critical value *t* = ±2.776

Degrees of Freedom	Area in Two Tails 0.01	0.02	0.05	0.10	0.20
4	**4.604**	3.747	$\boxed{2.776}$	2.132	1.533

Step 7: Because the test statistic does not fall in the critical region, we fail to reject the null *hypothesis*.



We conclude that there is not sufficient evidence to warrant rejection of the claim that for the population of students, the mean change in weight from September to April is equal to 0 *kg*. Based on the sample results listed in Table, there does not appear to be a significant weight gain from September to April.

The P-value method:

Using technology, we can find the *P*-value of 0.8605. (Using *t*–Distribution Table: with the test statistic of $t = 0.186$ and 4 degrees of freedom, we can determine that the *P*-value is greater than 0.20.) We again fail to reject the null hypothesis, because the *P*-value is greater than the significance level of $\alpha = 0.05$.

Confidence Interval method:

Construct a 95% confidence interval estimate of μ_d, which is the mean of the "April–September" weight differences of college students in their freshman year.

$\bar{d} = 0.2, \quad s_d = 2.4 \quad n = 5, \quad t = 2.776$

The margin error: $\quad E = t_{\alpha/2} \dfrac{s_d}{\sqrt{n}} = 2.776 \cdot \dfrac{2.4}{\sqrt{5}} = 3.0$

The confidence interval:

$$\bar{d} - E < \mu_d < \bar{d} + E$$

$$0.2 - 3.0 < \mu_d < 0.2 + 3.0$$

$$-2.8 < \mu_d < 3.2$$

Conclusion:

We have 95% confidence that the limits of –2.8 *kg* and 3.2 *kg* contain the true value of the mean weight change from September to April. In the long run, 95% of such samples will lead to confidence interval limits that actually do contain the true population mean of the differences. Note that the confidence interval includes the value of 0 *kg*, so it is very possible that the mean of the weight changes is equal to 0 *kg*.

Exercises Section 4.2 – Inferences about Two Means: Dependent

1. Listed below are the time intervals (in minutes) before and after eruptions of the Old Faithful geyser. Find the values of $\bar{d}$ and s_d. In general, what does μ_d represent?

Time interval before eruption	98	92	95	87	96
Time interval after eruption	92	95	92	100	90

2. Listed below are measured fuel consumption amount (in miles/gal) from a sample of cars.

City fuel consumption	18	22	21	21
Highway fuel consumption	26	31	29	29

Assume that wou want to use a 0.05 significance level to test the claim that the paired sample data come from a population for which the mean differecne is $\mu_d = 0$. Find

a) $\bar{d}$ *b)* s_d *c)* The t test statistic *d)* The critical values.

3. Listed below are predicted high temperatures that were forecast different days.

Predicted high temperatures forecast 3 *days ahead*	79	86	79	83	80
Predicted high temperatures forecast 5 *days ahead*	80	80	79	80	79

Assume that wou want to use a 0.05 significance level to test the claim that the paired sample data come from a population for which the mean differecne is $\mu_d = 0$. Find

a) $\bar{d}$ *b)* s_d *c)* The t test statistic *d)* The critical values.

4. Listed below are body mass indices (BMI). The BMI of each student was measured in September and April of the freshman year.

a) Use a 0.05 significance level to tet the claim that the mean change in BMI for all students is equal to 0. Does BMI appear to change during freshman year?

b) Construct a 95% confidence interval estimate of the change in BMI during freshman year. Does the confidence interval include 0, and what does that suggest about BMI during freshman year?

April BMI	20.15	19.24	20.77	23.85	21.32
September BMI	20.68	19.48	19.59	24.57	20.96

5. Listed below are body temperature (in °F) of subjects measured at 8:00 AM and at 12:00 AM. Construct a 95% confidence interval estimate of the difference between the 8:00 AM temperatures and the 12:00 AM temperatures. Is body temperature basically the same at both times?

8:00 AM	97.0	96.2	97.6	96.4	97.8	99.2
12:00 AM	98.0	98.6	98.8	98.0	98.6	97.6

6. Listed below are systolic blood pressure measuremetns (mm Hg) taking rom the right and left arms of the same woman. Use a 0.05 significance level to test for a difference the measurements from the two arms. What do you conclude?

Right arm	102	101	94	79	79
Left arm	175	169	182	146	144

7. As part of the National Health and Nutrition Examination Survey, the Department of Health and Human Services obtained self-reported heights and measured heights for males ages 12 – 16. All measurement are in inches. Listed below are sample results

Reported height	68	71	63	70	71	60	65	64	54	63	66	72
Measured height	67.9	69.9	64.9	68.3	70.3	60.6	64.5	67.0	55.6	74.2	65.0	70.8

a) Is there sufficient evidence to support the claim that there is a difference between self-reported heights and measured heights of males? Use a 0.05 significance level.

b) Construct a 95% confidence interval estimate of the man difference between reported heights and measured heights. Interpret the resulting confidence interval, and comment on the implications of whether the confidence interval limits contain 0.

8. Listed below are combined city – highway fuel consumption ratings (in miles/gal) for different cars measured under both the old rating system and a new rating system introducing in 2008. The new ratings were implemented in response to complaints that the old ratings were too high. Use a 0.01 significance level to test the claim the old ratings are higher than the new ratings.

Old rating	16	18	27	17	33	28	33	18	24	19	18	27	22	18	20	29	19	27	20	21
New rating	15	16	24	15	29	25	29	16	22	17	16	24	20	16	18	26	17	25	18	19

9. Listed below are 2 tables. Construct a 95% confidence interval estimate of the mean of the differences between weights of discarded paper and weights of discarded plastic. Which seems to weigh more: discarded paper or discarded plastic?

Paper

2.41	7.57	9.55	8.82	8.72	6.96	6.83	11.42	16.08	6.38	13.05	11.36	15.09
2.80	6.44	5.86	11.08	12.43	6.05	13.61	6.98	14.33	13.31	3.27	6.67	17.65
12.73	9.83	16.39	6.33	9.19	9.41	9.45	12.32	20.12	7.72	6.16	7.98	9.64
8.08	10.99	13.11	3.26	1.65	10.00	8.96	9.46	5.88	8.26	12.45	10.58	5.87
8.78	11.03	12.29	20.58	12.56	9.92	3.45	9.09	3.69	2.61			

Plastic

0.27	1.41	2.19	2.83	2.19	1.81	0.85	3.05	3.42	2.10	2.93	2.44	2.17
1.41	2.00	0.93	2.97	2.04	0.65	2.13	0.63	1.53	4.69	0.15	1.45	2.68
3.53	1.49	2.31	0.92	0.89	0.80	0.72	2.66	4.37	0.92	1.40	1.45	1.68
1.53	1.44	1.44	1.36	0.38	1.74	2.35	2.30	1.14	2.88	2.13	5.28	1.48
3.36	2.83	2.87	2.96	1.61	1.58	1.15	1.28	0.58	0.74			

10. Suppose you wish to test the claim that μ_d, the mean value of the differences d for a population of paired data, is different from 0. Given a sample of $n = 23$ and a significance level of $\alpha = 0.05$, what criterion would be used for rejecting the null hypothesis?

11. Assume that he paired data came from a population that is normally distributed. Using a 0.05 significance level, find $\bar{d}$, s_d, the t test statistic, and the critical values to test the claim that $\mu_d = 0$

x	14	8	4	14	3	12	4	13
y	15	8	7	13	5	11	6	15

12. Assume that he paired data came from a population that is normally distributed. Using a 0.05 significance level, find $\bar{d}$, s_d, the t test statistic, and the critical values to test the claim that $\mu_d = 0$

x	12	5	1	20	3	16	12	8
y	7	10	5	15	7	14	10	13