Section 1.5 – Other Types of Equations

The numbers of solutions to a polynomial with ***n*** degree, where ***n*** is Natural Number, are ***n*** solutions.

Solving a Polynomial Equation by factoring

Example

Solve: $4x^4 = 12x^2$

Solution

$4x^4 - 12x^2 = 0$

$4x^2(x^2 - 3) = 0$

$4x^2 = 0$ $\qquad$ $x^2 - 3 = 0$

$x^2 = 0$ $\qquad$ $x^2 = 3$

$\rightarrow \underline{x = 0, 0}$ $\qquad$ $\underline{x = \pm\sqrt{3}}$

Example

Solve: $2x^3 + 3x^2 = 8x + 12$

Solution

$2x^3 + 3x^2 - 8x - 12 = 0$

$x^2(2x + 3) - 4(2x + 3) = 0$

$(2x + 3)(x^2 - 4) = 0$

$2x + 3 = 0$ $\qquad$ $x^2 - 4 = 0$

$2x = -3$ $\qquad$ $x^2 = 4$

$\underline{x = -\frac{3}{2}}$ $\qquad$ $\underline{x = \pm\sqrt{4} \quad = \pm 2}$

Equations that Are Quadratic in Form

$$ax^2+bx+c=0$$
$$a(x)^2+b(x)^1+c=0$$
$$a(u)^2+b(u)^1+c=0$$
$$a\left(x^n\right)^2+b\left(x^n\right)^1+c=0$$
$$au^2+bu+c=0$$

Example

Solve: $x^4-5x^2+6=0$

Solution

$$\left(x^2\right)^2-5\left(x^2\right)+6=0$$
$$(U)^2-5(U)+6=0$$
$$U^2-5U+6=0$$

Solve for U

$$\Rightarrow U=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(6)}}{2(1)}$$
$$=\frac{5\pm\sqrt{25-24}}{2}$$
$$=\frac{5\pm\sqrt{1}}{2}$$
$$\rightarrow\begin{cases}U=\frac{5-1}{2}=2\\U=\frac{5+1}{2}=3\end{cases}$$

$$x^2=U \qquad \rightarrow\begin{cases}x^2=2\rightarrow \underline{x=\pm\sqrt{2}}\\x^2=3\rightarrow \underline{x=\pm\sqrt{3}}\end{cases}$$

or $\left(x^2-2\right)\left(x^2-3\right)=0$

$x^2-2=0$ $\quad$ $x^2-3=0$

$x^2=2$ $\quad$ $x^2=3$

$x=\pm\sqrt{2}$ $\quad$ $x=\pm\sqrt{3}$

Example

Solve: $(x+1)^{2/3}-(x+1)^{1/3}-2=0$

Solution

$u=(x+1)^{1/3}$

$u^2-u-2=0$

$(u-2)(u+1)=0$

$u-2=0$	$u+1=0$
$u=2$	$u=-1$
$u=(x+1)^{1/3}=2$	$u=(x+1)^{1/3}=-1$
$x+1=2^3$	$x+1=(-1)^3$
$x+1=8$	$x+1=-1$
$x=7$	$x=-2$

$\left((x+1)^{1/3}-2\right)\left((x+1)^{1/3}+1\right)=0$

Example

Solve: $3x^{2/3}-11x^{1/3}-4=0$

Solution

$3\left(x^{1/3}\right)^2-11\left(x^{1/3}\right)-4=0$

$x^{1/3}=\dfrac{11\pm\sqrt{121+48}}{2(3)}$

$x^{1/3}=\frac{11-13}{6}$	$x^{1/3}=\frac{11+13}{6}$
$=-\frac{1}{3}$	$=4$
$x=\left(-\frac{1}{3}\right)^3$	$x=4^3$
$=-\frac{1}{27}$	$=64$

Or factor

$\left(3x^{1/3}+1\right)\left(x^{1/3}-4\right)=0$

$3x^{1/3}+1=0$ $\qquad$ $x^{1/3}-4=0$

Solving a *Radical* Equation

Power Property

If P and Q are algebraic expressions, then every solution of the equation $P = Q$ is also a solution of the equation $P^n = Q^n$; for any positive integer n.

Example

Solve $x - \sqrt{15 - 2x} = 0$

Solution

$$x = \sqrt{15 - 2x}$$

$$x^2 = \left(\sqrt{15 - 2x}\right)^2$$

$$x^2 = 15 - 2x$$

$$x^2 + 2x - 15 = 0$$

$$(x - 3)(x + 5) = 0$$

$x - 3 = 0$ $\qquad$ $x + 5 = 0$

$x = 3$ $\qquad$ $x = -5$

Check

$x = 3$ $\qquad$ $x = -5$

$3 - \sqrt{15 - 2(3)} = 0$ $\qquad$ $-5 - \sqrt{15 - 2(-5)} = 0$

$3 - \sqrt{9} = 0$ $\qquad$ $-5 - \sqrt{25} = 0$

$3 - 3 = 0$ $\quad$ (*true*) $\qquad$ $-5 - 5 \neq 0$ $\quad$ (*false*)

$\boxed{x = 3}$ is the only solution

Solving Radical Equations of the Form $x^{\frac{m}{n}} = k$

Assume that m and n are positive integers

If m is ***even***: $x^{\frac{m}{n}} = k \quad \Rightarrow \left(x^{\frac{m}{n}} \right)^{\frac{n}{m}} = k^{\frac{n}{m}} \quad \Rightarrow x = \pm k^{\frac{n}{m}}$

If m is ***odd***: $x^{\frac{m}{n}} = k \quad \Rightarrow \left(x^{\frac{m}{n}} \right)^{\frac{n}{m}} = k^{\frac{n}{m}} \quad \Rightarrow x = k^{\frac{n}{m}}$

Example

Solve: $5x^{3/2} - 25 = 0$

Solution

$$5x^{3/2} = 25$$
$$x^{3/2} = \frac{25}{5} = 5$$
$$x = 5^{\frac{2}{3}}$$
$$= \sqrt[3]{5^2}$$
$$= \sqrt[3]{25}$$

Example

Solve: $x^{2/3} - 8 = -4$

Solution

$$x^{2/3} = 4$$
$$x = \pm(4)^{3/2}$$
$$= \pm\left(2^2\right)^{3/2}$$
$$= \pm 2^3$$
$$= \pm 8$$

Solving an Absolute Value Equation

If c is a positive real number and X represents any algebraic expression, then $|X| = c$ is equivalent to $X = c$ or $X = -c$

$$|X| = c \quad \rightarrow \quad X = c \text{ or } X = -c$$

Properties of Absolute Value

1. For b > 0, $|a| = b$ if and only if (*iff*) $a = b$ *or* $a = -b$
2. $|a| = |b|$ *iff* $a = b$ *or* $a = -b$

For any positive number b:

3. $|a| < b$ *iff* $-b < a < b$
4. $|a| < b$ *iff* $a < -b$ *or* $a > b$

Example

Solve: $|2x - 1| = 5$

Solution

$2x - 1 = 5$ $\qquad$ $2x - 1 = -5$

$2\boldsymbol{x} = 6$ $\qquad$ $2\boldsymbol{x} = -4$

$\boldsymbol{x} = 3$ $\qquad$ $\boldsymbol{x} = -2$

Solutions: $x = -2,\ 3$

Example

Solve: $4|1 - 2x| - 20 = 0$

Solution

$4|1 - 2x| = 20$

$|1 - 2x| = 5$

$1 - 2x = 5$ $\qquad$ $1 - 2x = -5$

$-2x = 4$ $\qquad$ $-2x = -6$

$x = -2$ $\qquad$ $x = 3$

Solutions: $x = -2,\ 3$

Exercise *Section* 1.5 – Other Types of Equations

(1 – 112) Solve

1. $3x^3 + 2x^2 = 12x + 8$

2. $x^3 + x^2 - 4x - 4 = 0$

3. $x^3 + x^2 + 4x + 4 = 0$

4. $x^3 + 4x^2 - 25x - 100 = 0$

5. $x^3 - 2x^2 - x + 2 = 0$

6. $x^3 - x^2 - 25x + 25 = 0$

7. $x^3 - x^2 = 16x - 16$

8. $x^3 + x^2 + 25x + 25 = 0$

9. $x^3 + 2x^2 = 16x + 32$

10. $2x^3 + 3x^2 - 6x - 9 = 0$

11. $2x^3 + x^2 - 8x - 4 = 0$

12. $2x^3 + 16x^2 + 30x = 0$

13. $3x^3 - 9x^2 - 30x = 0$

14. $x^4 + 3x^2 = 10$

15. $5x^4 = 40x$

16. $9x^4 - 9x^2 + 2 = 0$

17. $x^4 + 720 = 89x^2$

18. $12x^4 - 11x^2 + 2 = 0$

19. $2x^4 - 7x^2 + 5 = 0$

20. $x^4 - 5x^2 + 4 = 0$

21. $x^4 + 3x^2 = 10$

22. $3x^4 - 48x^2 = 0$

23. $5x^4 - 20x^2 = 0$

24. $x^4 - 4x^3 - 4x^2 = 0$

25. $x^4 - 6x^3 + 9x^2 = 0$

26. $x^4 - 4x^3 + 3x^2 = 0$

27. $x^4 - 4x^2 + 3 = 0$

28. $x^4 + 4x^2 + 3 = 0$

29. $x^4 + 6x^2 - 7 = 0$

30. $x^4 - 6x^2 - 7 = 0$

31. $3x^4 + 4x^2 - 7 = 0$

32. $3x^4 - 4x^2 - 7 = 0$

33. $3x^4 - x^2 - 2 = 0$

34. $3x^4 + x^2 - 2 = 0$

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35. $x - 3\sqrt{x} - 4 = 0$

36. $\left(5x^2 - 6\right)^{1/4} = x$

37. $\left(x^2 + 24x\right)^{1/4} = 3$

38. $x^{5/2} = 32$

39. $\sqrt[3]{2x + 11} = 3$

40. $\sqrt[3]{6x - 3} = 3$

41. $\sqrt[3]{2x - 6} = 4$

42. $\sqrt[3]{4x - 3} - 5 = 0$

43. $(3x - 1)^{1/3} + 4 = 0$

44. $(2x + 3)^{1/3} + 4 = 6$

45. $(3x - 6)^{1/3} + 5 = 8$

46. $(3x + 1)^{1/4} + 7 = 9$

47. $(2x + 3)^{1/4} + 7 = 10$

48. $\sqrt[3]{4x^2 - 4x + 1} - \sqrt[3]{x} = 0$

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49. $\sqrt{2x + 3} = 5$

50. $\sqrt{x - 3} + 6 = 5$

51. $\sqrt{3x - 2} = 4$

52. $\sqrt{5x - 4} = 9$

53. $\sqrt{5x - 1} = 8$

54. $\sqrt{3x - 2} - 5 = 0$

55. $\sqrt{2x + 5} + 11 = 6$

56. $\sqrt{3x + 7} + 10 = 4$

57. $x = \sqrt{7x + 8}$

58. $x = \sqrt{6x + 7}$

59. $\sqrt{5x + 1} = x + 1$

60. $x = \sqrt{2x - 2} + 1$

61. $x - 2\sqrt{x - 3} = 3$

62. $x + \sqrt{26 - 11x} = 4$

63. $x - \sqrt{2x + 3} = 0$

64. $\sqrt{x + 3} + 3 = x$

65. $x - \sqrt{x + 11} = 1$

66. $\sqrt{x - 7} = 7 - \sqrt{x}$

67. $\sqrt{x - 8} = \sqrt{x} - 2$

68. $\sqrt{2x - 5} = \sqrt{x + 4}$

69. $\sqrt{6x + 2} = \sqrt{5x + 3}$

70. $\sqrt{3x + 1} - \sqrt{x + 4} = 1$

71. $\sqrt{x + 2} + \sqrt{x - 1} = 3$

72. $\sqrt{x - 4} + \sqrt{x + 4} = 4$

73. $\sqrt{2x - 3} - \sqrt{x - 2} = 1$

74. $\sqrt{x + 2} + \sqrt{3x + 7} = 1$

75. $2\sqrt{4x + 1} - 9 = x - 5$

76. $\sqrt{2x - 3} + \sqrt{x - 2} = 1$

77. $\sqrt{2x + 3} = 1 + \sqrt{x + 1}$

78. $\sqrt{x + 5} - \sqrt{x - 3} = 2$

79. $|x| = -9$

80. $|x| = 9$

81. $|x-2| = 7$

82. $|x-2| = 0$

83. $|2x-3| = 6$

84. $|2x-1| = 11$

85. $7|5x| + 2 = 16$

86. $4\left|1 - \frac{3}{4}x\right| + 7 = 10$

87. $|x+7| + 6 = 2$

88. $|5-3x| = 12$

89. $|4x+2| = 5$

90. $3|x+5| = 12$

91. $2|x-6| = 8$

92. $3|2x-1| = 21$

93. $2|3x-2| = 14$

94. $|3x-1| + 2 = 16$

95. $|6x-2| + 4 = 32$

96. $7|5x| + 2 = 16$

97. $|4x+1| + 10 = 4$

98. $|4x+1| + 4 = 10$

99. $|3x-2| + 8 = 1$

100. $|3x-2| + 1 = 8$

101. $\left|\frac{6x+1}{x-1}\right| = 3$

102. $|x+1| = |1-3x|$

103. $|3x-1| = |x+5|$

104. $|5x-8| = |3x+2|$

105. $|4x-9| = |2x+1|$

106. $|2x-4| = |x-1|$

107. $|3x-4| = |3x+4|$

108. $|3x-5| = |3x+5|$

109. $|x-3| = |5-x|$

110. $|x-3| = |6-x|$

111. $\left|\frac{2}{3}x - 2\right| = \left|\frac{1}{3}x + 3\right|$

112. $\left|\frac{1}{2}x - 2\right| = \left|x - \frac{1}{2}\right|$