Lecture Four – Applications of Trigonometry

Section 4.1 – Law of Sines

Oblique Triangle

A triangle that is not a right triangle, either acute or obtuse.
The measures of the three sides and the three angles of a triangle can be found if at least one side and any other two measures are known.

The Law of Sines

There are many relationships that exist between the sides and angles in a triangle.

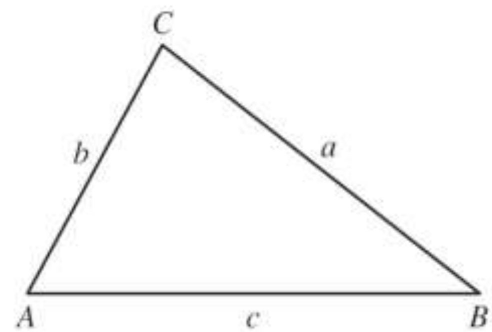
One such relation is called the law of sines.

Given triangle ABC

$$\boxed{\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}}$$

or, equivalently

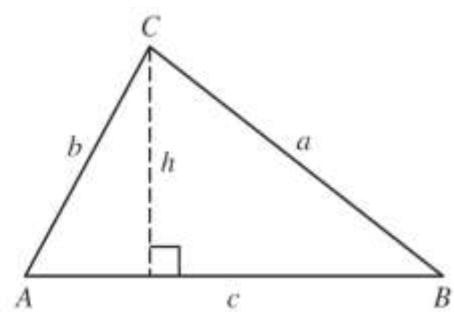
$$\boxed{\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}}$$



Proof

$$\sin A=\frac{h}{b}\ \Rightarrow h=b\sin A \quad (1)$$

$$\sin B=\frac{h}{a}\ \Rightarrow h=a\sin B \quad (2)$$



From (1) & (2)

$$h=h$$

$$b\sin A=a\sin B$$

$$\frac{b\sin A}{ab}=\frac{a\sin B}{ab}$$

$$\frac{\sin A}{a}=\frac{\sin B}{b}$$

Angle – Side - Angle (*ASA* or *AAS*)

If two angles and the included side of one triangle are equal, respectively, to two angles and the included side of a second triangle, then the triangles are congruent.

Example

In triangle *ABC*, $A = 30°$, $B = 70°$, and $a = 8.0\, cm$. Find the length of side c.

Solution

$$\begin{aligned} C &= 180° - (A + B) \\ &= 180° - (30° + 70°) \\ &= 180° - 100° \\ &= 80° \end{aligned}$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\begin{aligned} c &= \frac{a}{\sin A}\sin C \\ &= \frac{8}{\sin 30°}\sin 80° \\ &= 16\, cm \end{aligned}$$

Example

Find the missing parts of triangle *ABC* if $A = 32°$, $C = 81.8°$, , and $a = 42.9\, cm$.

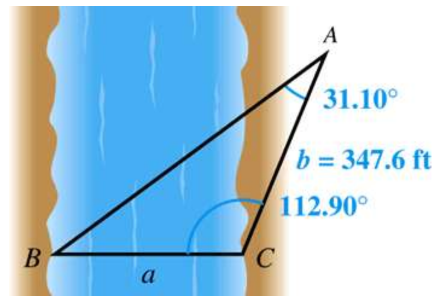
Solution

$$\begin{aligned} B &= 180° - (B + C) \\ &= 180° - (32° + 81.8°) \\ &= 66.2° \end{aligned}$$

$\frac{a}{\sin A} = \frac{b}{\sin B}$ $b = \frac{a\sin B}{\sin A}$ $= \frac{42.9\sin 66.2°}{\sin 32°}$ $\approx 74.1\, cm$	$\frac{c}{\sin C} = \frac{a}{\sin A}$ $c = \frac{a\sin C}{\sin A}$ $= \frac{42.9\sin 81.8°}{\sin 32°}$ $\approx 80.1\, cm$

Example

You wish to measure the distance across a River. You determine that $C = 112.90°$, $A = 31.10°$, and $b = 347.6\ ft$. Find the distance a across the river.



<u>*Solution*</u>

$$\begin{aligned} B &= 180° - A - C \\ &= 180° - 31.10° - 112.90° \\ &= 36° \end{aligned}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 31.1°} = \frac{347.6}{\sin 36°}$$

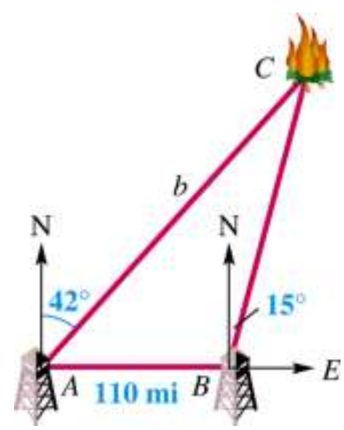
$$a = \frac{347.6}{\sin 36°} \sin 31.1°$$

$$\boxed{a = 305.5\ ft}$$

Example

Two ranger stations are on an east-west line 110 mi apart. A forest fire is located on a bearing N 42° E from the western station at A and a bearing of N 15° E from the eastern station at B. How far is the fire from the western station?

<u>*Solution*</u>



$$\angle BAC = 90° - 42° = 48°$$

$$\angle ABC = 90° + 15° = 105°$$

$$\angle C = 180° - 105° - 48° = 27°$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{b}{\sin 105°} = \frac{110}{\sin 27°}$$

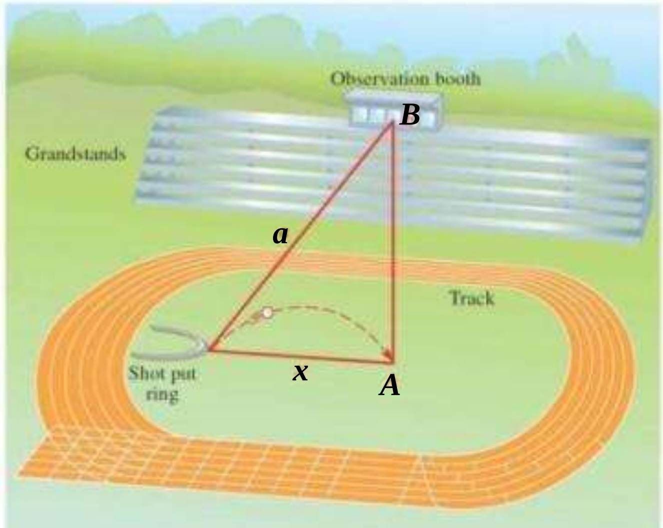
$$b = \frac{110 \sin 105°}{\sin 27°}$$

$$\boxed{b \approx 234\ mi}$$

The fire is about 234 miles from the western station.

Example

Find distance x if a = 562 *ft.*, $B = 5.7°$ and $A = 85.3°$



<u>*Solution*</u>

$$\frac{x}{\sin B} = \frac{a}{\sin A}$$

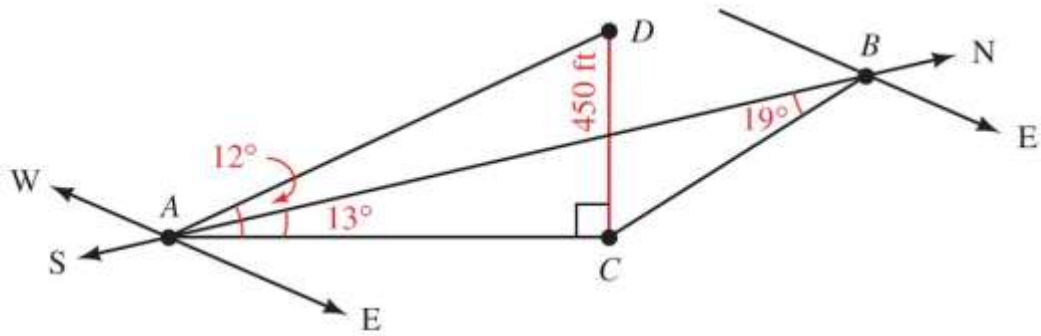
$$x = \frac{a \sin B}{\sin A}$$

$$= \frac{562 \sin 5.7°}{\sin 85.3°}$$

$$= 56.0 \ ft$$

Example

A hot-air balloon is flying over a dry lake when the wind stops blowing. The balloon comes to a stop 450 feet above the ground at point *D*. A jeep following the balloon runs out of gas at point *A*. The nearest service station is due north of the jeep at point *B*. The bearing of the balloon from the jeep at *A* is N 13° E, while the bearing of the balloon from the service station at *B* is S 19° E. If the angle of elevation of the balloon from *A* is 12°, how far will the people in the jeep have to walk to reach the service station at point *B*?



<u>*Solution*</u>

$$\tan 12° = \frac{DC}{AC}$$

$$AC = \frac{DC}{\tan 12°}$$

$$= \frac{450}{\tan 12°}$$

$$= 2{,}117\ ft$$

$$\angle ACB = 180° - (13° + 19°)$$

$$= 148°$$

Using triangle *ABC*

$$\frac{AB}{\sin 148°} = \frac{2117}{\sin 19°}$$

$$AB = \frac{2117 \sin 148°}{\sin 19°}$$

$$\underline{= 3{,}400\ ft}$$

Ambiguous Case

Side – Angle – Side (*SAS*)

If two sides and the included angle of one triangle are equal, respectively, to two sides and the included angle of a second triangle, then the triangles are congruent.

Example

Find angle B in triangle ABC if $a = 2$, $b = 6$, and $A = 30°$

Solution

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\begin{aligned} \sin B &= \frac{b \sin A}{a} \\ &= \frac{6 \sin 30°}{2} \\ &= 1.5 \qquad\qquad -1 \le \sin\alpha \le 1 \end{aligned}$$

Since $sinB > 1$ is impossible, no such triangle exists.

Example

Find the missing parts in triangle ABC if $C = 35.4°$, $a = 205$ ft., and $c = 314$ ft.

Solution

$$\begin{aligned} \sin A &= \frac{a \sin C}{c} \\ &= \frac{205 \sin 35.4°}{314} \\ &= 0.3782 \end{aligned}$$

$$A = \sin^{-1}(0.3782)$$

$$A = 22.2° \qquad\qquad A' = 180° - 22.2° = 157.8°$$

$$\begin{aligned} C + A' &= 35.4° + 157.8° \\ &= 193.2° \quad > 180° \end{aligned}$$

$$B = 180° - (22.2° + 35.4°) = 122.4°$$

$$\begin{aligned} b &= \frac{c \sin B}{\sin C} \\ &= \frac{314 \sin 122.4°}{\sin 35.4°} \\ &= 458 \ ft \end{aligned}$$

Example

Find the missing parts in triangle ABC if a = 54 cm, b = 62 cm, and A = 40°.

<u>*Solution*</u>

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\sin B = \frac{b \sin A}{a}$$

$$= \frac{62 \sin 40°}{54}$$

$$= 0.738$$

$$B = \sin^{-1}(0.738) = 48°$$

$$C = 180° - (40° + 48°)$$

$$= 92°$$

$$c = \frac{a \sin C}{\sin A}$$

$$= \frac{54 \sin 92°}{\sin 40°}$$

$$= 84\ cm$$

$$B = 180° - 48° = 132°$$

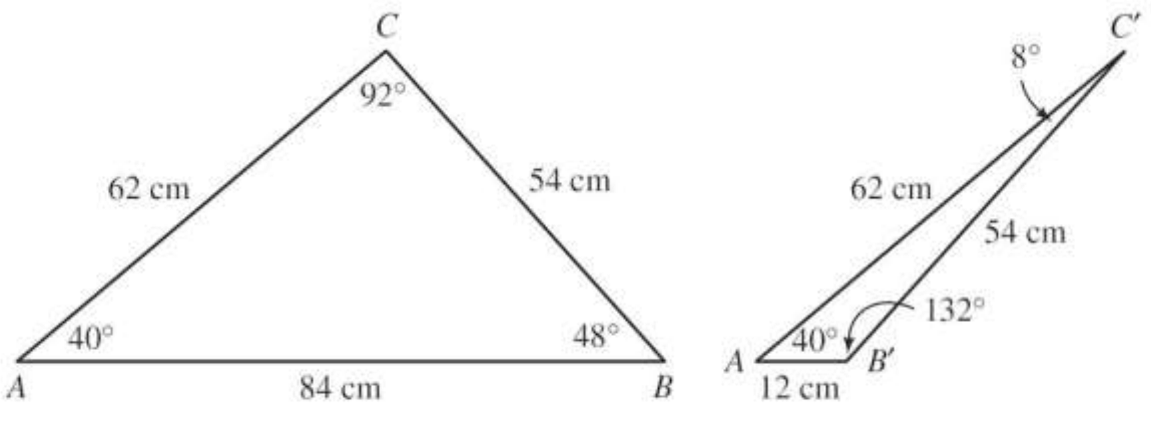
$$C' = 180° - (40° + 132°)$$

$$= 8°$$

$$c' = \frac{a \sin C'}{\sin A}$$

$$= \frac{54 \sin 8°}{\sin 40°}$$

$$= 12\ cm$$



Area of a Triangle (SAS)

In any triangle *ABC*, the area A is given by the following formulas:

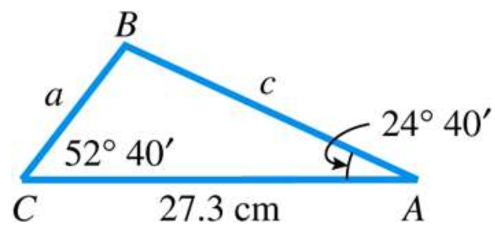
$$A = \frac{1}{2} bc \sin A \qquad A = \frac{1}{2} ac \sin B \qquad A = \frac{1}{2} ab \sin C$$

Example

Find the area of triangle ABC if $A = 24°40',\ b = 27.3\ cm,\ \ and\ C = 52°40'$

<u>*Solution*</u>

$$\begin{aligned} B &= 180° - 24°40' - 52°40' \\ &= 180° - \left(24° + \frac{40°}{60}\right) - \left(52° + \frac{40°}{60}\right) \\ &\approx 102.667° \end{aligned}$$



$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

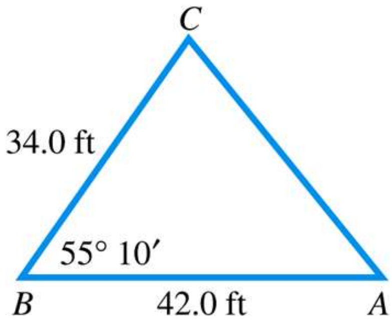
$$\frac{a}{\sin(24°40')} = \frac{27.3}{\sin(102°40')}$$

$$\begin{aligned} a &= \frac{27.3 \sin(24°40')}{\sin(102°40')} \\ &\approx 11.7\ cm \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2} ac \sin B \\ &= \frac{1}{2}(11.7)(27.3)\sin(52°40') \\ &\approx 127\ cm^2 \end{aligned}$$

Example

Find the area of triangle *ABC*.



<u>*Solution*</u>

$$A = \frac{1}{2}ac\sin B$$

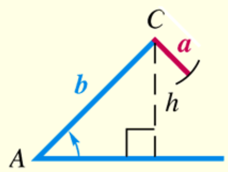
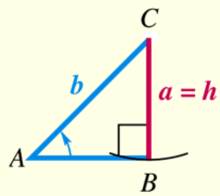
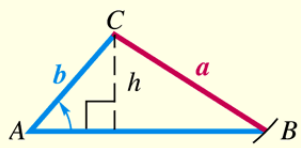
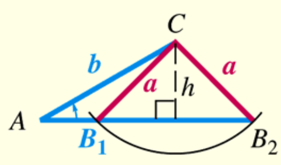
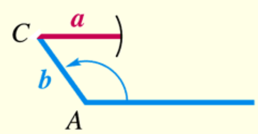
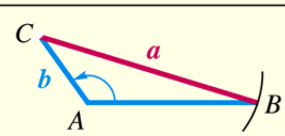
$$= \frac{1}{2}(34.0)(42.0)\sin(55°10')$$

$$\approx 586\ ft^2$$

Number of Triangles Satisfying the Ambiguous Case (SSA)

Let sides a and b and angle A be given in triangle ABC. (The law of sines can be used to calculate the value of $\sin B$.)

1. If applying the law of sines results in an equation having $\sin B > 1$, then *no triangle* satisfies the given conditions.
2. If $\sin B = 1$, then *one triangle* satisfies the given conditions and $B = 90°$.
3. If $0 < \sin B < 1$, then either *one or two triangles* satisfy the given conditions.
   - ***a)*** If $\sin B = k$, then let $B_1 = \sin^{-1} k$ and use B_1 for B in the first triangle.
   - ***b)*** Let $B_2 = 180° - B_1$. If $A + B_2 < 180°$, then a second triangle exists. In this case, use B_2 for B in the second triangle.

Number of Triangles	Sketch	Applying Law of Sines Leads to
0		$\sin B > 1$, $a < h < b$
1		$\sin B = 1$, $a = h$ and $h < b$
1		$0 < \sin B < 1$, $a \geq b$
2		$0 < \sin B_2 < 1$, $h < a < b$
0		$\sin B \geq 1$, $a \leq b$
1		$0 < \sin B < 1$, $a > b$

Exercises **Section 4.1 – Law of Sines**

1. In triangle *ABC*, $B = 110°, C = 40°$ and $b = 18\,in$. Find the length of side *c*.

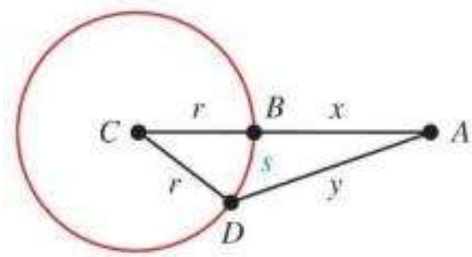
2. In triangle *ABC*, $A = 110.4°$, $C = 21.8°$ and $c = 246\,in$. Find all the missing parts.

3. Find the missing parts of triangle *ABC*, if $B = 34°$, $C = 82°$, and $a = 5.6\,cm$.

4. Solve triangle *ABC* if $B = 55°40'$, $b = 8.94$ m, and $a = 25.1$ m.

5. Solve triangle *ABC* if $A = 55.3°$, $a = 22.8$ *ft*., and $b = 24.9$ ft.

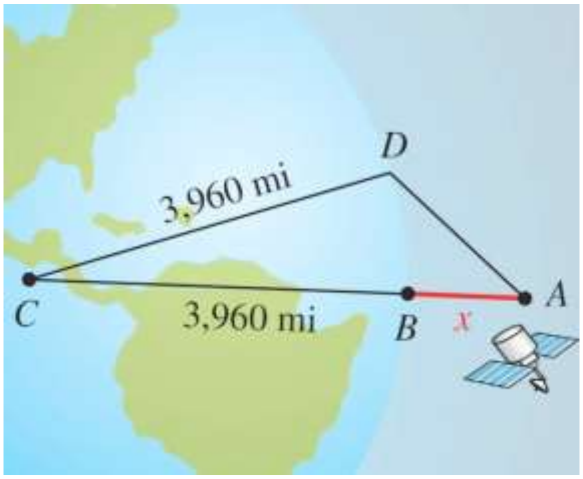
6. Solve triangle *ABC* given $A = 43.5°$, $a = 10.7$ in., and $c = 7.2$ in.



7. If $A = 26°,\ s = 22,\ \ and\ \ r = 19,$ find x

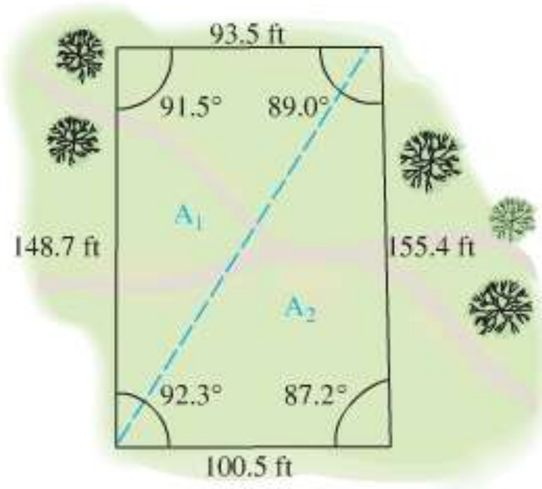
8. A man flying in a hot-air balloon in a straight line at a constant rate of 5 feet per second, while keeping it at a constant altitude. As he approaches the parking lot of a market, he notices that the angle of depression from his balloon to a friend's car in the parking lot is 35°. A minute and a half later, after flying directly over this friend's car, he looks back to see his friend getting into the car and observes the angle of depression to be 36°. At that time, what is the distance between him and his friend?

9. A satellite is circling above the earth. When the satellite is directly above point *B*, angle *A* is 75.4°. If the distance between points *B* and *D* on the circumference of the earth is 910 miles and the radius of the earth is 3,960 miles, how far above the earth is the satellite?

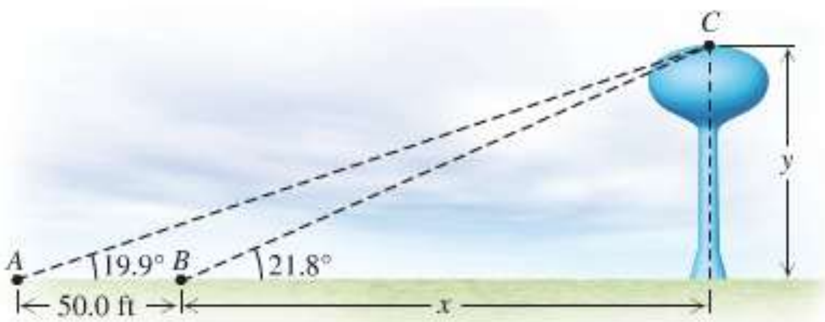


10. A pilot left Fairbanks in a light plane and flew 100 miles toward Fort in still air on a course with bearing of 18°. She then flew due east (bearing 90°) for some time drop supplies to a snowbound family. After the drop, her course to return to Fairbanks had bearing of 225°. What was her maximum distance from Fairbanks?

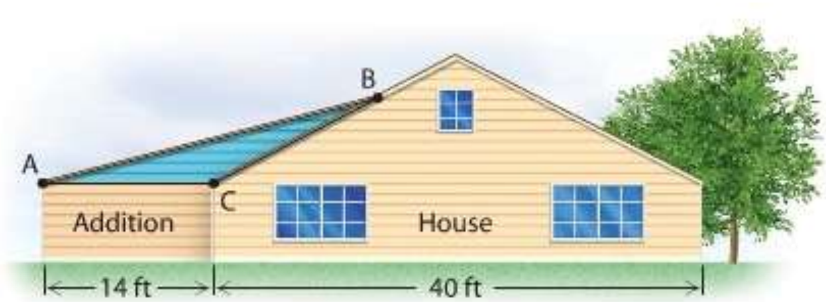
11. The dimensions of a land are given in the figure. Find the area of the property in square feet.



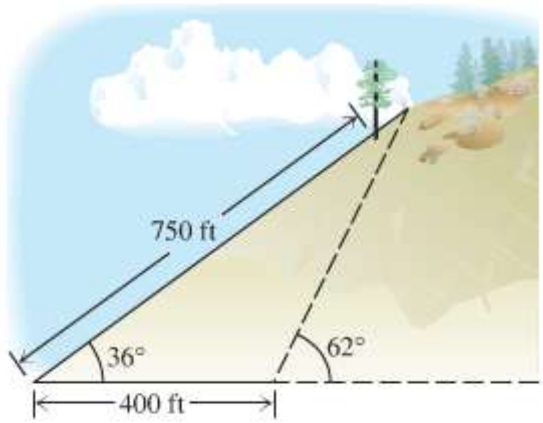
12. The angle of elevation of the top of a water tower from point A on the ground is 19.9°. From point B, 50.0 feet closer to the tower, the angle of elevation is 21.8°. What is the height of the tower?



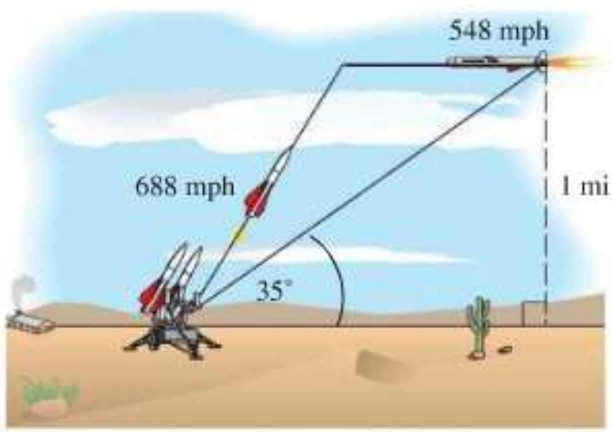
13. A 40-ft wide house has a roof with a 6-12 pitch (the roof rises 6 ft for a run of 12 ft). The owner plans a 14-ft wide addition that will have a 3-12 pitch to its roof. Find the lengths of $\overline{AB}$ and $\overline{BC}$



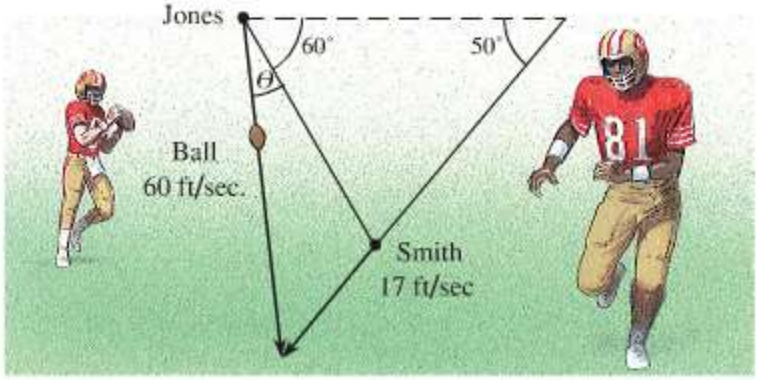
14. A hill has an angle of inclination of 36°. A study completed by a state's highway commission showed that the placement of a highway requires that 400 ft of the hill, measured horizontally, be removed. The engineers plan to leave a slope alongside the highway with an angle of inclination of 62°. Located 750 ft up the hill measured from the base is a tree containing the nest of an endangered hawk. Will this tree be removed in the excavation?



15. A cruise missile is traveling straight across the desert at 548 mph at an altitude of 1 mile. A gunner spots the missile coming in his direction and fires a projectile at the missile when the angle of elevation of the missile is 35°. If the speed of the projectile is 688 mph, then for what angle of elevation of the gun will the projectile hit the missile?



16. When the ball is snapped, Smith starts running at a 50° angle to the line of scrimmage. At the moment when Smith is at a 60° angle from Jones, Smith is running at 17 ft/sec and Jones passes the ball at 60 ft/sec to Smith. However, to complete the pass, Jones must lead Smith by the angle θ. Find θ (find θ in his head. Note that θ can be found without knowing any distances.)



17. A rabbit starts running from point *A* in a straight line in the direction 30° from the north at 3.5 *ft/sec*. At the same time a fox starts running in a straight line from a position 30 *ft* to the west of the rabbit 6.5 *ft/sec*. The fox chooses his path so that he will catch the rabbit at point C. In how many seconds will the fox catch the rabbit?