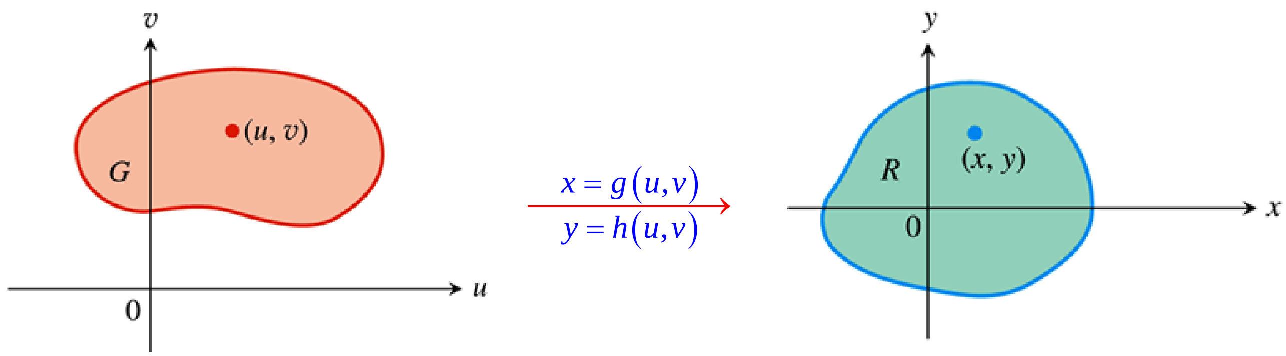
Section 3.7 – Change of variables in Multiple Integrals

Substitution in Double Integrals

Suppose that a region G in the uv-plane is transformed one-to-one into the region R in the xy-plane by equations of the form

$$x = g(u,v), \quad y = h(u,v)$$



Cartesian uv-plane ***Cartesian*** xy - plane

R is the image of G under the transformation, and G the ***preimage*** of R.

$$\iint_R f(x,y)\,dxdy = \iint_G f\big(g(u,v),h(u,v)\big)\,\left|J(u,v)\right|\,dudv$$

Definition

The ***Jacobian determinant*** or ***Jacobian*** of the coordinate transformation $x = g(u,v), \quad y = h(u,v)$ is

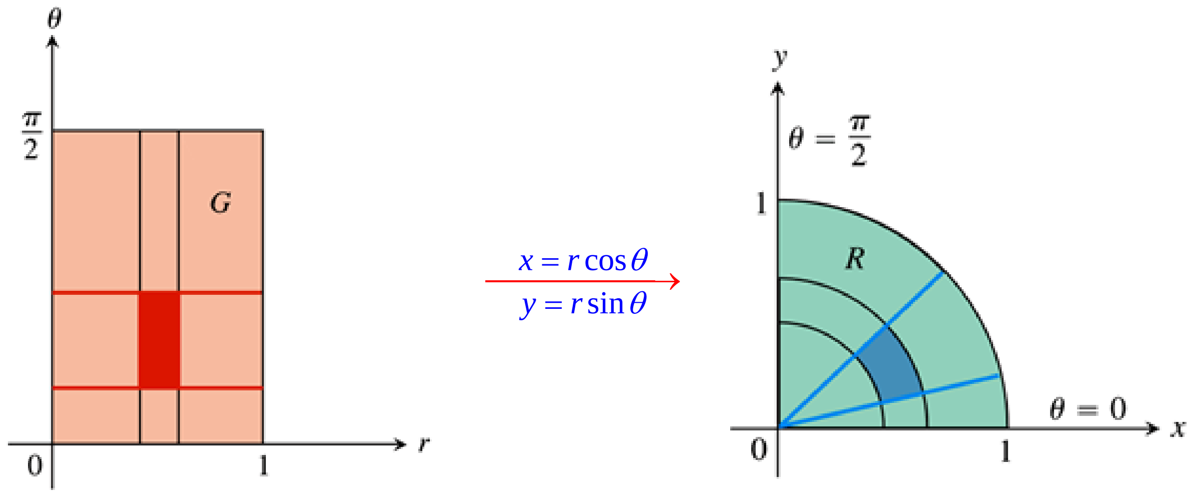
$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Example

Find the Jacobian for the polar coordinate transformation $x = r\cos\theta, \quad y = r\sin\theta$, write the Cartesian integral $\iint\limits_R f(x, y)\,dxdy$ as a polar integral.

Solution



Cartesian rθ −plane

Cartesian xy −plane

$x = r\cos\theta, \quad y = r\sin\theta$ transform the rectangle G: $0 \le r \le 1, \ 0 \le \theta \le 2\pi$, into the quarter circle R bounded by $x^2 + y^2 = 1$ in QI.

$$J(r,\ \theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r\left(\cos^2\theta + \sin^2\theta\right)$$

$$\underline{= r \,|}$$

Example

Evaluate $\displaystyle\int_0^4 \int_{\frac{1}{2}y}^{\frac{1}{2}y+1} \frac{2x-y}{2}\, dxdy$ by applying the transformation $u=\dfrac{2x-y}{2}$, $v=\dfrac{y}{2}$ and integrating over an appropriate region in the *uv*-plane.

Solution

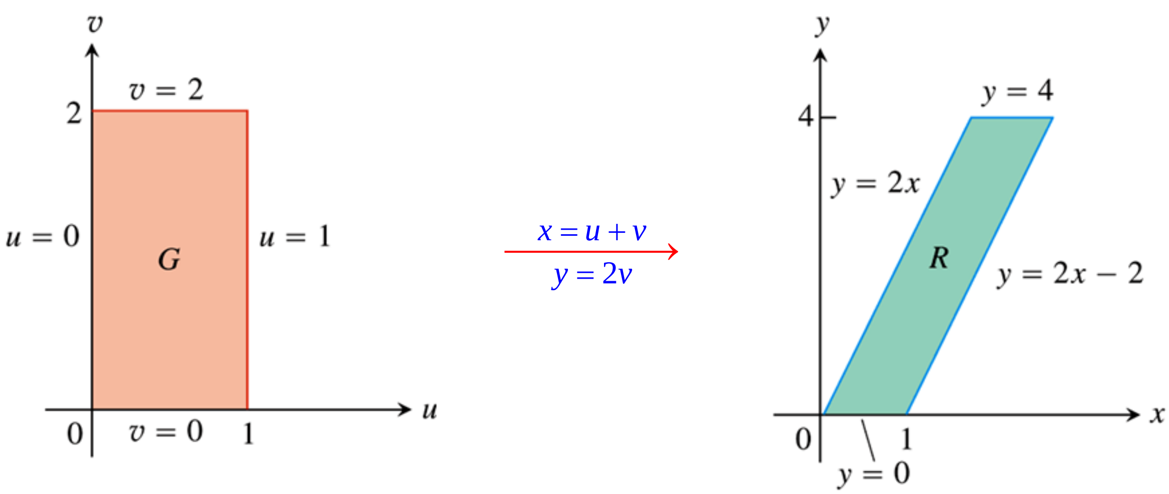
$\underline{y=2v}$

$2u=2x-y$

$x=\dfrac{2u+y}{2}$

$\quad=\dfrac{2u+2v}{2}$

$\quad=\underline{u+v}$



xy-eqns for the boundary of *R*	Corresponding *uv*-eqns. for the boundary of *G*	Simplified *uv-eqns*.
$x=\dfrac{y}{2}$	$u+v=\dfrac{2v}{2}=v$	$u=0$
$x=\dfrac{y}{2}+1$	$u+v=\dfrac{2v}{2}+1=v+1$	$u=1$
$y=0$	$2v=0$	$v=0$
$y=4$	$2v=4$	$v=2$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial u}(u+v) & \frac{\partial}{\partial v}(u+v) \\ \frac{\partial}{\partial u}(2v) & \frac{\partial}{\partial v}(2v) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}$$

$$\underline{= 2}$$

$$\int_0^4 \int_{\frac{1}{2}y}^{\frac{1}{2}y+1} \frac{2x-y}{2}\, dxdy = \int_0^{v=2} \int_{u=0}^{u=1} u \left|J(u,v)\right|\, dudv$$

$$= \int_0^2 dv \int_{u=0}^1 (u)(2)\, du$$

$$= v \Big|_0^2 \quad u^2 \Big|_0^1$$

$$= (2)(1)$$

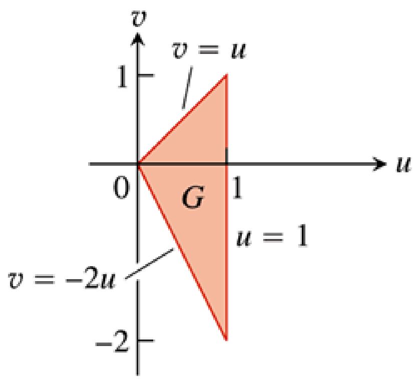
$$\underline{= 2}$$

Example

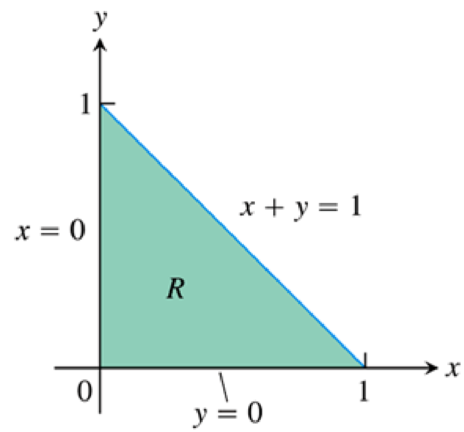
Evaluate $\displaystyle\int_0^1\int_0^{1-x}\sqrt{x+y}\,(y-2x)^2\,dydx$

Solution

$$\begin{cases} u=x+y \\ v=y-2x \end{cases} \rightarrow \begin{cases} x=\dfrac{u}{3}-\dfrac{v}{3} \\ y=\dfrac{2u}{3}+\dfrac{v}{3} \end{cases}$$



$$\xrightarrow[y=\frac{2u}{3}+\frac{v}{3}]{x=\frac{u}{3}-\frac{v}{3}}$$



xy*-eqns for the boundary of *R	**Corresponding *uv*-eqns. for the boundary of *G***	**Simplified *uv*-eqns.**
$x+y=1$	$\left(\frac{u}{3}-\frac{v}{3}\right)+\left(\frac{2u}{3}+\frac{v}{3}\right)=1$	$u=1$
$y=0$	$\frac{2u}{3}+\frac{v}{3}=1$	$v=-2u$
$x=0$	$\frac{u}{3}-\frac{v}{3}=0$	$v=u$
$x=1$	$u=3+v$	$y=2+v\big\vert_{v=0}=2>1$

$$J(u,v)=\begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} \qquad J(u,v)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$=\frac{1}{3}$$

$$\int_0^1\int_0^{1-x}\sqrt{x+y}\,(y-2x)^2\,dydx=\int_{u=0}^{u=1}\int_{v=-2u}^{v=u}u^{1/2}v^2\,|J(u,v)|\,dvdu$$

$$= \int_0^1 \int_{-2u}^{u} u^{1/2} v^2 \left(\frac{1}{3}\right) dv du$$

$$= \int_0^1 u^{1/2} \left(\frac{1}{9} v^3 \Big|_{-2u}^{u}\right) du$$

$$= \frac{1}{9} \int_0^1 u^{1/2} \left(u^3 + 8u^3\right) du$$

$$= \int_0^1 u^{7/2} du$$

$$= \frac{2}{9} u^{9/2} \Big|_0^1$$

$$\underline{= \frac{2}{9}}$$

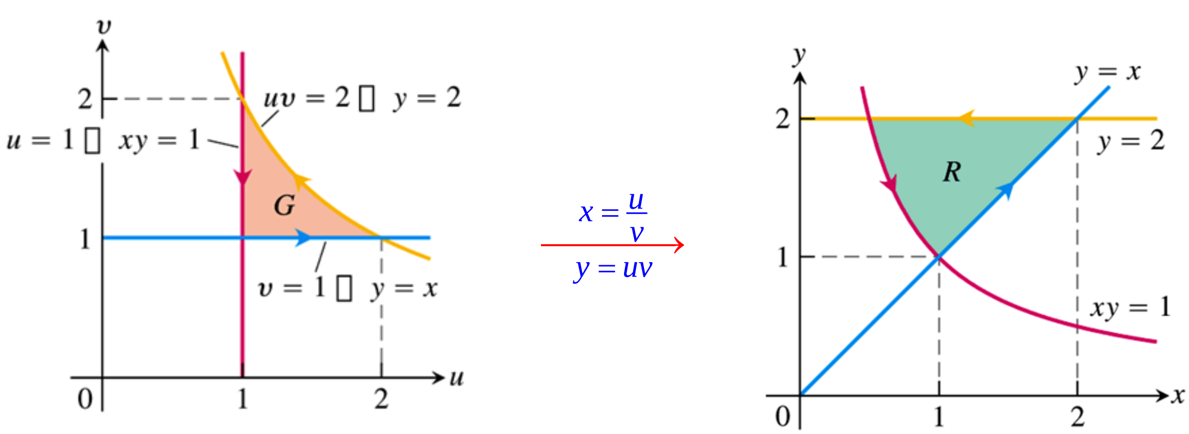
Example

Evaluate $\displaystyle\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}} dxdy$

Solution

$$\begin{cases} u = \sqrt{xy} \\ v = \sqrt{\dfrac{y}{x}} \end{cases} \rightarrow \begin{cases} u^2 = xy \\ v^2 = \dfrac{y}{x} \end{cases}$$

$$\rightarrow \underline{x = \frac{u}{v}, \quad y = uv}$$



$$J(u,v)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$=\begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix}$$

$$\boxed{=\frac{2u}{v}}$$

xy*-eqns for the boundary of *R	**Corresponding *uv*-eqns. for the boundary of *G***	**Simplified *uv*-eqns.**
$x=y$	$\frac{u}{v}=uv$	$v=1$
$x=\frac{1}{y}$	$\frac{u}{v}=\frac{1}{uv}$	$u=1$
$y=1$	$uv=1$	
$y=2$	$uv=2$	$u=2 \quad v=\frac{2}{u}$

$$\int_1^2\int_{1/y}^{y}\sqrt{\frac{y}{x}}\,e^{\sqrt{xy}}\,dxdy=\int_1^2\int_1^{2/u} 2u\ e^u dvdu$$

$$=2\int_1^2 u\ e^u\ v\,\Big|_0^{2/u}\ du$$

$$=2\int_1^2 u\ e^u\left(\frac{2}{u}-1\right)du$$

$$=2\int_1^2 (2-u)e^u du$$

		e^u
$(+)$	$2-u$	e^u
$(-)$	-1	e^u
	0	

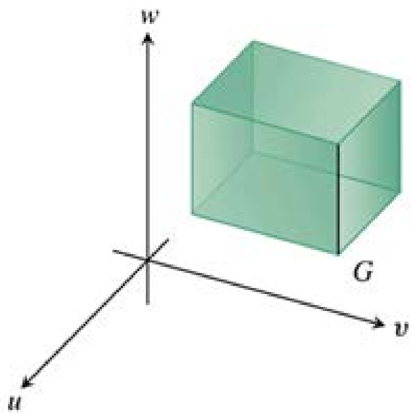
$$=2\left((2-u+1)e^u\ \Big|_1^2\right.$$

$$=2\left[(1)e^2-2e\right]$$

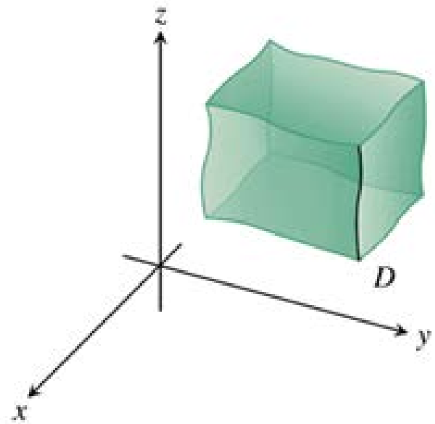
$$\boxed{=2e(e-2)}$$

Substitutions in Triple Integrals

$$x = g(u,v,w), \quad y = h(u,v,w), \quad z = k(u,v,w)$$



$x=g(u,v,w)$
$y=h(u,v,w)$
$z=k(u,v,w)$



Cartesian *uvw* - plane

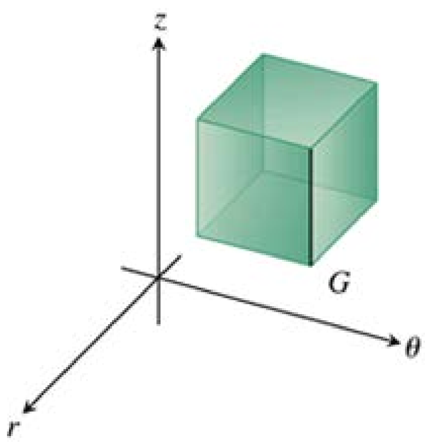
Cartesian *xyz* - plane

$$\iiint_R f(x,y,z)\; dxdydz = \iiint_R H(u,v,w)\; \left|J(u,v,w)\right|\; dudvdw$$

The ***Jacobian determinant*** is

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

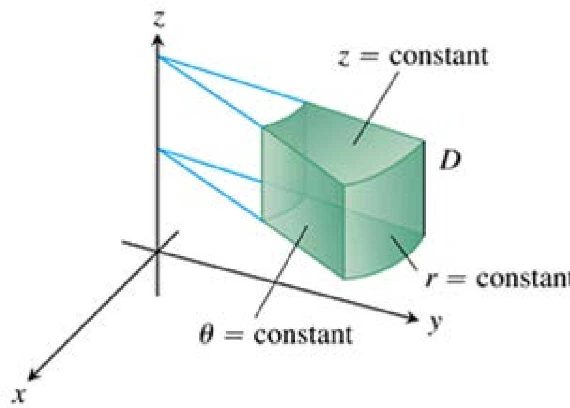
Cube with sides parallel to the axes



$x=r\cos\theta$
$y=r\sin\theta$
$z=z$

Cartesian $r\theta z$ - plane

Cube with sides parallel to the axes



Cartesian *xyz* - plane

$$J(r,\theta,z)=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z}\end{vmatrix}$$

$$=\begin{vmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}$$

$$=r\cos^2\theta+r\sin^2\theta$$

$$\underline{=r}$$

For spherical coordinates, ρ, ϕ, and θ take the place of u, v, and w. The transformation from Cartesian $\rho\phi\theta$-space to Cartesian xyz –space is given by

$$x=\rho\sin\phi\cos\theta,\quad y=\rho\sin\phi\sin\theta,\quad z=\rho\cos\phi$$

The Jacobian of the transformation

$$J(\rho,\phi,\theta)=\begin{vmatrix}\frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta}\\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta}\end{vmatrix}$$

$$=\begin{vmatrix}\sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta\\ \cos\phi & -\rho\sin\phi & 0\end{vmatrix}$$

$$=\rho^2\cos^2\phi\sin\phi\cos^2\theta+\rho^2\sin^3\phi\sin^2\theta+\rho^2\sin\phi\cos^2\phi\sin^2\theta+\rho^2\sin^3\phi\cos^2\theta$$

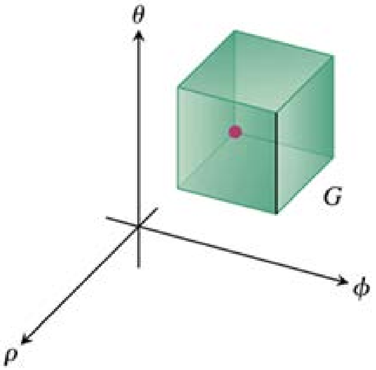
$$=\rho^2\cos^2\phi\sin\phi\left(\cos^2\theta+\sin^2\theta\right)+\rho^2\sin^3\phi\left(\sin^2\theta+\cos^2\theta\right)$$

$$=\rho^2\cos^2\phi\sin\phi+\rho^2\sin^3\phi$$

$$=\rho^2\sin\phi\left(\cos^2\phi+\sin^2\phi\right)$$

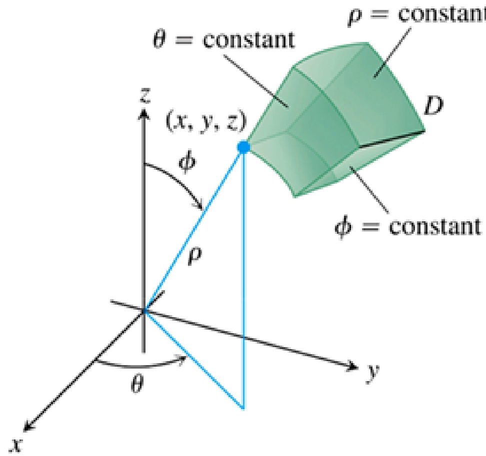
$$\underline{=\rho^2\sin\phi}$$

$$\iiint_D F(x,y,z)\,dxdydz=\iiint_G H(\rho,\phi,\theta)\left|\rho^2\sin\phi\right|\,d\rho\, d\phi\, d\theta$$



Cartesian $r\theta z$ - plane

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$



Cartesian xyz - plane

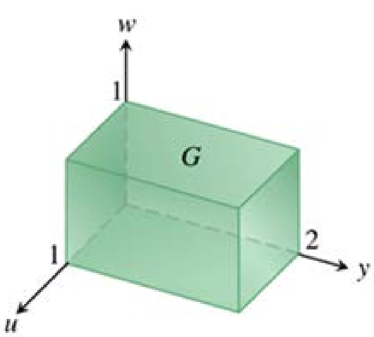
Example

Evaluate $\int_0^3 \int_0^4 \int_{\frac{1}{2}y}^{\frac{1}{2}y+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dxdydz$ by applying the transformation

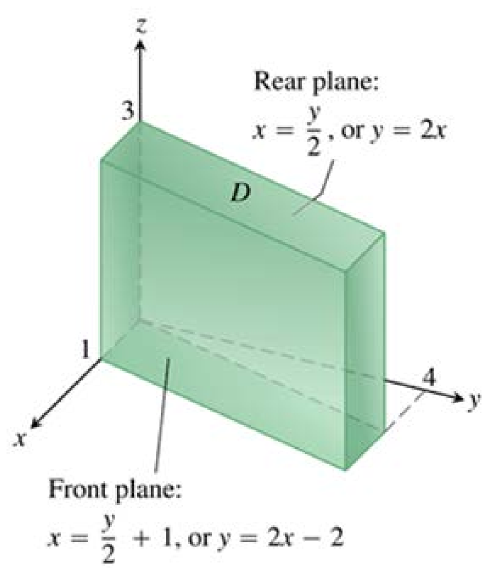
$u = \frac{2x-y}{2}, \quad v = \frac{y}{2}, \quad w = \frac{z}{3}$ and integrating over an appropriate region in the *uvw*-plane.

Solution

$$\rightarrow \begin{cases} u = \frac{2x-y}{2} \rightarrow x = u + \frac{y}{2} = u + v \\ \qquad v = \frac{y}{2} \rightarrow y = 2v \\ \qquad w = \frac{z}{3} \rightarrow z = 3w \end{cases}$$



$$x = u + v$$
$$y = 2v$$
$$z = 3w$$



xyz-eqns* for the boundary of *D	**Corresponding *uvw- eqns*. for the boundary of *G***	**Simplified *uvw- eqns*.**
$x=\frac{y}{2}$	$u+v=v$	$u=0$
$x=\frac{y}{2}+1$	$u+v=v+1$	$u=1$
$y=0$	$2v=0$	$v=0$
$y=4$	$2v=4$	$v=2$
$z=0$	$3w=0$	$w=0$
$z=3$	$3w=3$	$w=1$

$$J(u,v,w)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$$=6$$

$$\int_0^3\int_0^4\int_{\frac{1}{2}y}^{\frac{1}{2}y+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dxdydz=\int_0^1\int_0^2\int_0^1(u+w)\left|J(u,v,w)\right|\,dudvdw$$

$$=6\int_0^1\int_0^2\int_0^1(u+w)\,dudvdw$$

$$=6\int_0^1\int_0^2\left(\frac{u^2}{2}+wu\,\Big|_0^1\right.\,dvdw$$

$$=6\int_0^1\int_0^2\left(\frac{1}{2}+w\right)\,dvdw$$

$$=6\int_0^1\left(\frac{1}{2}v+wv\,\Big|_0^2\right.\,dw$$

$$= 6\int_0^1 (1+2w)\,dw$$

$$= 6\left(w + w^2\right. \Big|_0^1$$

$$= 6(1+1)$$

$$\underline{= 12}$$

Example

Evaluate $\iiint_D xz\ dV$: D is bounded by the planes: $y-x=0,\ \ y=2+x,\ \ z-y=0,\ \ z-y=2,\ \ z=0,$ and $z=3$

Solution

$$\begin{cases} y-x=0 \\ y-x=2 \end{cases} \quad let \quad \underline{u=y-x}$$

$$\Rightarrow \quad \underline{0 \le u \le 2}$$

$$\begin{cases} z-y=0 \\ z-y=2 \end{cases} \quad let \quad \underline{v=z-y}$$

$$\Rightarrow \quad \underline{0 \le v \le 2}$$

$$\begin{cases} z=0 \\ z=3 \end{cases} \quad let \quad \underline{w=z}$$

$$\Rightarrow \quad \underline{0 \le w \le 3}$$

$$\begin{cases} \underline{z=w} \\ y-x=u \qquad\qquad \rightarrow \qquad \underline{x=-u-v+w} \\ z-y=v \quad \rightarrow \quad \underline{y=w-v} \end{cases}$$

$$J(u,\ v,\ w) = \begin{vmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \qquad\qquad J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\underline{=1}$$

$$\iiint_D xz\,dV = \int_0^3 \int_0^2 \int_0^2 (w-u-v)(w)\ du\,dv\,dw$$

$$= \int_0^3 \int_0^2 \int_0^2 \left(w^2 - uw - vw\right)\ du\,dv\,dw$$

$$= \int_0^3 \int_0^2 \left(w^2 u - \frac{1}{2} wu^2 - vwu\ \Big|_0^2\ dv\,dw\right.$$

$$= \int_0^3 \int_0^2 \left(2w^2 - 2w - 2vw\right)\ dv\,dw$$

$$= \int_0^3 \left(2w^2 v - 2wv - wv^2\ \Big|_0^2\ dw\right.$$

$$= \int_0^3 \left(4w^2 - 4w - 4w\right)\ dw$$

$$= \int_0^3 \left(4w^2 - 8w\right)\ dw$$

$$= \frac{4}{3} w^3 - 4w^2\ \Big|_0^3$$

$$= 36 - 36$$

$$\underline{= 0}$$

Exercises Section 3.7 – Change of Variables in Multiple Integrals

Let $S = \{0 \le u \le 1,\ 0 \le v \le 1\}$ be a unit square in the *uv*-plane. Find the image of *S* in the *xy*-plane under the following transformations.

1. $T: \quad x = v,\ y = u$

2. $T: \quad x = -v,\ y = u$

3. $T: \quad x = \frac{u+v}{2},\ \ y = \frac{u-v}{2}$

4. $T: \quad x = u,\ \ y = 2v + 2$

5. *a*) Solve the system $u = x - y, \quad v = 2x + y$ for *x* and *y* in terms of *u* and *v*. Then find the value of the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$

b) Find the image under the transformation $u = x - y,\ \ v = 2x + y$ of the triangular region with vertices (0, 0), (1, 1), and (1, −2) in the *xy*-plane. Sketch the transformed region in the *uv*-plane.

6. Let *R* be the region in the first quadrant of the *xy*-plane bounded by the hyperbolas $xy = 1,\ xy = 9$ and the lines $y = x,\ y = 4x$. Use the transformation $x = \frac{u}{v},\ y = uv$ with $u > 0,$ *and* $v > 0$ to rewrite

$$\iint\limits_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy$$

As an integral over an appropriate region *G* in the uv-plane. Then evaluate the *uv*-integral over *G*.

7. The area πab of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be found by integrating the function $f(x,y) = 1$ over the region bounded by the ellipse in the *xy*-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x = au,\ y = bv$ and evaluate the transformed integral over the disk *G*: $u^2 + v^2 \le 1$ in the *uv*-plane. Find the area this way.

8. Use the transformation $x = u + \frac{1}{2}v,\ \ y = v$ to evaluate the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3 (2x - y) e^{(2x-y)^2}\, dxdy$$

By first writing it as an integral over a region *G* in the *uv*-plane.

9. Use the transformation $x = \frac{u}{v},\ \ y = uv$ to evaluate the integral

$$\int_1^2 \int_{1/y}^{y} \left(x^2 + y^2\right) dxdy + \int_2^4 \int_{y/4}^{4/y} \left(x^2 + y^2\right) dxdy$$

10. Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ of the transformation

a) $x = u\cos v, \quad y = u\sin v$

b) $x = u\sin v, \quad y = u\cos v$

11. Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ of the transformation

a) $x = u\cos v, \quad y = u\sin v, \quad z = w$

b) $x = 2u - 1, \quad y = 3v - 4, \quad z = \frac{1}{2}(w - 4)$

12. Evaluate the appropriate determinant to show that the Jacobian of the transformation from Cartesian $\rho\phi\theta$–space to Cartesian *xyz*–space is $\rho^2 \sin\phi$

13. How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.

14. Find the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

(***Hint***: Let $x = au,\ y = bv,\ and\ z = cw$. Then find the volume of an appropriate region in *uvw*-space)

15. Use the transformation $x = u^2 - v^2,\ y = 2uv$ to evaluate the integral

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2}\ dydx$$

(***Hint***: Show that the image of the triangular region ***G*** with vertices (0, 0), (1, 0), (1, 1) in the *uv*-plane is the region of integration ***R*** in the *xy*-plane defined by the limits of integration.)

16. Evaluate $\iint\limits_R y^4 dA$; R is the region bounded by the hyperbolas $xy = 1$ and $xy = 4$ and the lines $\dfrac{y}{x} = 1$, and $\dfrac{y}{x} = 3$

17. Evaluate $\iint\limits_R \left(y^2 + xy - 2x^2\right) dA$; R is the region bounded by the lines $y = x$, $y = x - 3$, $y = -2x + 3$, and $y = -2x - 3$

18. Evaluate $\iint\limits_D x\; dV$; R is bounded by the planes $y - 2x = 0$, $y - 2x = 1$, $z - 3y = 0$, $z - 3y = 1$, $z - 4x = 0$ and $z - 4x = 3$

19. Let R be the region bounded by the lines $x + y = 1$; $x + y = 4$; $x - 2y = 0$; $x - 2y = -4$

Evaluate the integral $\iint\limits_R 3xydA$

20. Let R be the region bounded by the square with vertices (0, 1), (1, 2), (2, 1), & (1, 0).

Evaluate the integral $\iint\limits_R (x + y)^2 \sin^2 (x - y)\, dA$

21. Evaluate $\iiint_D yzdV$ D is bounded by the planes: $x + 2y = 1$, $x + 2y = 2$, $x - z = 0$, $x - z = 2$, $2y - z = 0$, and $2y - z = 3$

22. Evaluate $\iint\limits_R xy\; dA$; R is the square with vertices $(0,\ 0)$, $(1,\ 1)$, $(2,\ 0)$, and $(1,\ -1)$

23. Evaluate $\iint\limits_R x^2 y\; dA$; $R = \left\{(x,\ y):\quad 0 \le x \le 2, \quad x \le y \le x + 4\right\}$

24. Evaluate $\iint\limits_R x^2 \sqrt{x + 2y}\; dA$; $R = \left\{(x,\ y):\quad 0 \le x \le 2, \quad -\frac{x}{2} \le y \le 1 - x\right\}$

25. Evaluate $\iint\limits_R xy\; dA$; where R is bounded by the ellipse $9x^2 + 4y^2 = 36$.

26. Evaluate $\int_0^1 \int_y^{y+2} \sqrt{x - y}\; dxdy$

27. Evaluate $\iint\limits_R \sqrt{y^2 - x^2}\ dA$; where R is the diamond bounded by $y - x = 0$, $y - x = 2$, $y + x = 0$, and $y + x = 2$

28. Evaluate $\iint\limits_R \left(\dfrac{y-x}{y+2x+1}\right)^4 dA$; where R is the parallelogram bounded by $y - x = 1$, $y - x = 2$, $y + 2x = 0$, and $y + 2x = 4$

29. Evaluate $\iint\limits_R e^{xy}\ dA$; where R is the region bounded by $xy = 1$, $xy = 4$, $\dfrac{y}{x} = 1$, and $\dfrac{y}{x} = 3$

30. Evaluate $\iint\limits_R xy\ dA$; where R is the region bounded by the hyperbolas $xy = 1$, $xy = 4$, $y = 1$, and $y = 3$

31. Evaluate $\iint\limits_R (x - y)\sqrt{x - 2y}\ dA$; where R is the triangular region bounded by $y = 0$, $x - 2y = 0$, and $x - y = 1$

32. Evaluate $\iiint_D xy\ dV$: D is bounded by the planes: $y - x = 0$, $y - x = 2$, $z - y = 0$, $z - y = 2$, $z = 0$, and $z = 3$

33. Evaluate $\iiint_D dV$: D is bounded by the planes: $y - 2x = 0$, $y - 2x = 1$, $z - 3y = 0$, $z - 3y = 1$, $z - 4x = 0$, and $z - 4x = 3$

34. Evaluate $\iiint_D z\ dV$: D is bounded by the paraboloid $z = 16 - x^2 - 4y^2$ and the xy-plane.

35. Evaluate $\iiint_D dV$: D is bounded by the upper half of the ellipsoid $\dfrac{x^2}{9} + \dfrac{y^2}{4} + z^2 = 1$ and the xy-plane.

36. Evaluate $\iiint_D xz\ dV$: D is bounded by the planes: $y = x$, $y = x + 2$, $x - z = 0$, $z = x + 3$, $z = 0$, and $z = 4$

(37 – 41) Let R be the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>0$ and $b>0$ are real numbers.

37. Find the area of R.

38. Evaluates $\iint_R |xy|\,dA$

39. Find the center of mass of the upper half of R $(y \geq 0)$ assuming it has a constant density.

40. Find the average square of the distance between points of R and the origin.

41. Find the average distance between points in the upper half of R and the x-axis.

(42 – 45) Let D be the region bounded by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, where $a>0$, $b>0$ and $c>0$ are real numbers.

42. Find the Volume of D.

43. Evaluates $\iint_D |xyz|\,dV$

44. Find the center of mass of the upper half of D $(z \geq 0)$ assuming it has a constant density.

45. Find the average square of the distance between points of D and the origin.