Solution *Section* 1.8 – Exponential Models

Exercise

Find the derivative of $y=\ln\left(\dfrac{\sqrt{\sin\theta\cos\theta}}{1+2\ln\theta}\right)$

Solution

$$y=\ln(\sin\theta\cos\theta)^{1/2}-\ln(1+2\ln\theta)$$
$$=\frac{1}{2}\left(\ln(\sin\theta)+\ln(\cos\theta)\right)-\ln(1+2\ln\theta)$$
$$y'=\frac{1}{2}\left(\frac{(\sin\theta)'}{\sin\theta}+\frac{(\cos\theta)'}{\cos\theta}\right)-\frac{(1+2\ln\theta)'}{1+2\ln\theta}=\frac{1}{2}\left(\frac{\cos\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta}\right)-\frac{\frac{2}{\theta}}{1+2\ln\theta}$$
$$=\frac{1}{2}(\cot\theta-\tan\theta)-\frac{2}{\theta(1+2\ln\theta)}$$

Exercise

Find the derivative of $f(x)=e^{\left(4\sqrt{x}+x^2\right)}$

Solution

$$\frac{d}{dx}e^{\left(4\sqrt{x}+x^2\right)}=e^{\left(4\sqrt{x}+x^2\right)}\frac{d}{dx}\left(4\sqrt{x}+x^2\right)=\left(\frac{2}{\sqrt{x}}+2x\right)e^{\left(4\sqrt{x}+x^2\right)}$$

Exercise

Find the derivative of $f(t)=\ln\left(3te^{-t}\right)$

Solution

$$\frac{d}{dt}\ln\left(3te^{-t}\right)=\frac{\left(3te^{-t}\right)'}{3te^{-t}}$$
$$=3\frac{e^{-t}-te^{-t}}{3te^{-t}}$$
$$\frac{e^{-t}(1-t)}{te^{-t}}$$
$$=\frac{1-t}{t}$$

$$\ln\left(3te^{-t}\right)=\ln 3+\ln t+\ln e^{-t}$$
$$=\ln 3+\ln t-t$$
$$\left(\ln\left(3te^{-t}\right)\right)'=\frac{1}{t}-1$$
$$=\frac{1-t}{t}$$

Exercise

Find the Derivative of $f(x)=\dfrac{e^{\sqrt{x}}}{\ln\left(\sqrt{x}+1\right)}$

Solution

$$f=e^{\sqrt{x}} \qquad U=\sqrt{x}=x^{1/2}\Rightarrow U'=\tfrac{1}{2}x^{-1/2} \qquad f'=\tfrac{1}{2}x^{-1/2}e^{\sqrt{x}}=\frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

$$g=\ln\left(\sqrt{x}+1\right) \qquad U=x^{1/2}+1\Rightarrow U'=\tfrac{1}{2}x^{-1/2} \qquad g'=\frac{\frac{1}{2}x^{-1/2}}{\sqrt{x}+1}=\frac{1}{2x^{1/2}\left(\sqrt{x}+1\right)}$$

$$f'(x)=\frac{\frac{e^{\sqrt{x}}}{2\sqrt{x}}\ln\left(\sqrt{x}+1\right)-\frac{1}{2\sqrt{x}\left(\sqrt{x}+1\right)}e^{\sqrt{x}}}{\left(\ln\left(\sqrt{x}+1\right)\right)^2}$$

$$=\frac{\frac{\left(\sqrt{x}+1\right)e^{\sqrt{x}}\ln\left(\sqrt{x}+1\right)-e^{\sqrt{x}}}{2\sqrt{x}\left(\sqrt{x}+1\right)}}{\left(\ln\left(\sqrt{x}+1\right)\right)^2}$$

$$\boxed{=\frac{e^{\sqrt{x}}\left[\left(\sqrt{x}+1\right)\ln\left(\sqrt{x}+1\right)-1\right]}{2\sqrt{x}\left(\sqrt{x}+1\right)\left(\ln\left(\sqrt{x}+1\right)\right)^2}}$$

Exercise

Find the Derivative of $y=\sqrt{\dfrac{(x+1)^{10}}{(2x+1)^5}}$

Solution

$$y=\left(\frac{(x+1)^{10}}{(2x+1)^5}\right)^{1/2}$$

$$\ln y=\ln\left(\frac{(x+1)^{10}}{(2x+1)^5}\right)^{1/2}$$

$$\ln y=\frac{1}{2}\ln\left(\frac{(x+1)^{10}}{(2x+1)^5}\right)$$

$$=\frac{1}{2}\left(\ln(x+1)^{10}-\ln(2x+1)^5\right)$$

$$=\frac{1}{2}\left(10\ln(x+1)-5\ln(2x+1)\right)$$

$$= 5\ln(x+1) - \frac{5}{2}\ln(2x+1)$$

$$\frac{y'}{y} = 5\frac{1}{x+1} - \frac{5}{2}\frac{2}{2x+1}$$

$$\frac{y'}{y} = \frac{5}{x+1} - \frac{5}{2x+1}$$

$$y' = y\left(\frac{5}{x+1} - \frac{5}{2x+1}\right)$$

$$\boxed{y' = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}\left(\frac{5}{x+1} - \frac{5}{2x+1}\right)}$$

Exercise

Find the derivative of $f(x) = (2x)^{4x}$

Solution

$$\ln f(x) = 4x\ln(2x)$$

$$\frac{f'}{f} = 4\left(\ln 2x + x\frac{2}{2x}\right)$$

$$\boxed{f'(x) = 4(\ln 2x + 1)(2x)^{4x}}$$

Exercise

Find the derivative of $f(x) = 2^{x^2}$

Solution

$$\boxed{f'(x) = 2x \cdot 2^{x^2} \ln 2}$$

Exercise

Find the derivative of $h(y) = y^{\sin y}$

Solution

$$\ln h = \ln y^{\sin y} = \sin y \ln y$$

$$\frac{h'}{h} = \cos y \ln y + \frac{\sin y}{y}$$

$$\boxed{h'(y) = y^{\sin y}\left(\cos y \ln y + \frac{\sin y}{y}\right)}$$

Exercise

Find the derivative of $f(x) = x^{\pi}$

Solution

$$\ln f = \pi \ln x$$

$$\frac{f'}{f} = \frac{\pi}{x}$$

$$\boxed{f'(x) = \pi x^{\pi - 1}}$$

Exercise

Find the derivative of $h(t) = (\sin t)^{\sqrt{t}}$

Solution

$$\ln h = \ln(\sin t)^{\sqrt{t}} = \sqrt{t}\ln(\sin t)$$

$$\frac{h'}{h} = \frac{1}{2\sqrt{t}}\ln \sin t + \sqrt{t}\frac{\cos t}{\sin t}$$

$$\boxed{h'(t) = \frac{1}{2\sqrt{t}}(\ln \sin t + 2t \cot t)(\sin t)^{\sqrt{t}}}$$

Exercise

Find the derivative of $p(x) = x^{-\ln x}$

Solution

$$\ln p(x) = \ln x^{-\ln x} = -(\ln x)^2$$

$$\frac{p'}{p} = -\frac{2\ln x}{x}$$

$$p'(x) = -\frac{2\ln x}{x}x^{-\ln x} \boxed{= -\frac{2\ln x}{x^{1+\ln x}}}$$

Exercise

Find the derivative of $f(x) = x^{2x}$

Solution

$$\ln f = \ln x^{2x} = 2x \ln x$$

$$\frac{f'}{f} = 2\ln x + 2\frac{x}{x}$$

$$\boxed{f'(x) = 2(1 + \ln x)x^{2x}}$$

Exercise

Find the derivative of $f(x) = x^{\tan x}$

Solution

$$\ln f(x) = \ln x^{\tan x} = \tan x \ln x$$

$$\frac{f'}{f} = \sec^2 x \ln x + \frac{\tan x}{x}$$

$$\boxed{f'(x) = \left(\sec^2 x \ln x + \frac{\tan x}{x}\right) x^{\tan x}}$$

Exercise

Find the derivative of $f(x) = x^e + e^x$

Solution

$$\boxed{f'(x) = ex^{e-1} + e^x}$$

Exercise

Find the derivative of $f(x) = x^{x^{10}}$

Solution

$$\ln f = x^{10} \ln x$$

$$\frac{f'}{f} = 10x^9 \ln x + \frac{x^{10}}{x}$$

$$f'(x) = x^{x^{10}} \left(10x^9 \ln x + x^9\right)$$

$$\boxed{= x^{9+x^{10}} (10 \ln x + 1)}$$

Exercise

Find the derivative of $f(x) = \left(1 + \frac{4}{x}\right)^x$

Solution

$$\ln f = x \ln\left(1 + \frac{4}{x}\right)$$

$$\frac{f'}{f} = \ln\left(1 + \frac{4}{x}\right) + x \frac{-\frac{4}{x^2}}{1 + \frac{4}{x}}$$

$$\boxed{f'(x) = \left(1 + \frac{4}{x}\right)^x \left(\ln\left(1 + \frac{4}{x}\right) - \frac{4}{x+4}\right)}$$

Exercise

Find the derivative of $f(x)=\cos\left(x^{2\sin x}\right)$

Solution

$$f'=-\left(x^{2\sin x}\right)'\sin\left(x^{2\sin x}\right)$$

$$Let\ y=x^{2\sin x}\quad\to\quad \ln y=(2\sin x)\ln x$$

$$\frac{y'}{y}=2\cos x\ln x+\frac{2\sin x}{x}$$

$$\boxed{f'=-x^{2\sin x}\left(2\cos x\ln x+\frac{2\sin x}{x}\right)\sin\left(x^{2\sin x}\right)}$$

Exercise

Evaluate the integral $\displaystyle\int\frac{2y\,dy}{y^2-25}$

Solution

$$\int\frac{2y\,dy}{y^2-25}=\int\frac{d\left(y^2-25\right)}{y^2-25}\qquad d\left(y^2-25\right)=2y\,dy$$

$$\boxed{=\ln\left|y^2-25\right|+C}$$

Exercise

Evaluate the integral $\displaystyle\int\frac{\sec y\tan y}{2+\sec y}dy$

Solution

$$\int\frac{\sec y\tan y}{2+\sec y}dy=\int\frac{d(2+\sec y)}{2+\sec y}\qquad d(2+\sec y)=\sec y\tan y\,dy$$

$$\boxed{=\ln|2+\sec y|+C}$$

Exercise

Find the integral $\displaystyle\int\frac{5}{e^{-5x}+7}dx$

Solution

$$\int\frac{5}{e^{-5x}+7}\frac{e^{5x}}{e^{5x}}dx=\int\frac{5e^{5x}}{1+7e^{5x}}dx\qquad d\left(1+7e^{5x}\right)=35e^{5x}dx$$

$$= \frac{1}{7}\int \frac{d\left(1+7e^{5x}\right)}{1+7e^{5x}}$$

$$= \frac{1}{7}\ln\left|1+7e^{5x}\right| + C$$

Exercise

Find the integral $\displaystyle\int \frac{e^{2x}}{4+e^{2x}}dx$

Solution

$$\int \frac{e^{2x}}{4+e^{2x}}dx = \frac{1}{2}\int \frac{d\left(4+e^{2x}\right)}{4+e^{2x}} \qquad d\left(4+e^{2x}\right) = 2e^{2x}dx$$

$$= \frac{1}{2}\ln\left(4+e^{2x}\right) + C$$

Exercise

Find the integral $\displaystyle\int \frac{dx}{x\ln x\ln(\ln x)}$

Solution

$$\int \frac{dx}{x\ln x\ln(\ln x)} = \int \frac{d(\ln(\ln x))}{\ln(\ln x)} \qquad d(\ln(\ln x)) = \frac{1/x}{\ln x} = \frac{1}{x\ln x}$$

$$= \ln\ln(\ln x) + C$$

Exercise

Find the integral $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

Solution

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\int e^{\sqrt{x}}d\left(\sqrt{x}\right) \qquad d\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}dx$$

$$= 2e^{\sqrt{x}} + C$$

Exercise

Find the integral $\displaystyle\int \frac{e^{\sin x}}{\sec x}dx$

Solution

$$\int \frac{e^{\sin x}}{\sec x}dx = \int e^{\sin x} d(\sin x) \qquad d(\sin x) = \cos x dx = \frac{dx}{\sec x}$$

$$= e^{\sin x} + C$$

Exercise

Find the integral $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}dx$

Solution

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}dx = \int \frac{1}{e^x - e^{-x}} d\left(e^x - e^{-x}\right) \qquad d\left(e^x - e^{-x}\right) = \left(e^x + e^{-x}\right)dx$$

$$= \ln\left(e^x - e^{-x}\right) + C$$

Exercise

Find the integral $\int \frac{4^{\cot x}}{\sin^2 x}dx$

Solution

$$\int \frac{4^{\cot x}}{\sin^2 x}dx = -\int 4^{\cot x} d(\cot x) \qquad d(\cot x) = -\csc^2 x dx = -\frac{dx}{\sin^2 x}$$

$$= \frac{4^{\cot x}}{\ln 4} + C \qquad \int a^x dx = \frac{a^x}{\ln a} + C$$

Exercise

Find the integral $\int \frac{4x^2 + 2x + 4}{x+1}dx$

Solution

$$\int \frac{4x^2 + 2x + 4}{x+1}dx = \int \left(4x + 2 + \frac{6}{x+1}\right)dx$$

$$= \int (4x-2)dx + \int \frac{6}{x+1}dx$$

$$= \int (4x-2)dx + 6\int \frac{d(x+1)}{x+1} \qquad \int \frac{d(U)}{U} = \ln|U|$$

$$= 2x^2 - 2x + 6\ln|x+1| + C$$

Exercise

Evaluate the integral $\displaystyle\int_{\ln 4}^{\ln 9} e^{x/2}dx$

Solution

$$\int_{\ln 4}^{\ln 9} e^{x/2}dx = 2e^{x/2}\Big|_{\ln 2^2}^{\ln 3^2}$$

$$= 2\left(e^{(2\ln 3)/2} - e^{(2\ln 2)/2}\right)$$

$$= 2\left(e^{\ln 3} - e^{\ln 2}\right)$$

$$= 2(3-2)$$

$$\underline{= 2}$$

Exercise

Evaluate the integral $\displaystyle\int_0^3 \frac{2x-1}{x+1}dx$

Solution

$$\int_0^3 \frac{2x-1}{x+1}dx = \int_0^3 \left(2 - \frac{3}{x+1}\right)dx$$

$$\begin{array}{r} 2 \\ x+1\overline{)2x-1} \\ \underline{-2x-2} \\ -3 \end{array}$$

$$= \left(2x - 3\ln|x+1|\right)\Big|_0^3$$

$$\underline{= 6 - 3\ln 4}$$

Exercise

Evaluate the integral $\displaystyle\int_e^{e^2} \frac{dx}{x\ln^3 x}$

Solution

$$\int_e^{e^2} \frac{dx}{x\ln^3 x} = \int_e^{e^2} \ln^{-3} x\, d(\ln x)$$

$$= -\frac{1}{2}\ln^{-2} x\Big|_e^{e^2}$$

$$= -\frac{1}{2}(2-1)$$

$$\underline{= -\frac{1}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_{e^2}^{e^3} \frac{dx}{x\ln x \ln^2(\ln x)}$

Solution

$$\int_{e^2}^{e^3} \frac{dx}{x\ln x \ln^2(\ln x)} = \int_{e^2}^{e^3} (\ln(\ln x))^{-2}\, d(\ln(\ln x)) \qquad d(\ln(\ln x)) = \frac{1/x}{\ln x} = \frac{1}{x\ln x}$$

$$= -\frac{1}{\ln(\ln x)}\Big|_{e^2}^{e^3}$$

$$= -\frac{1}{\ln\left(\ln e^3\right)} + \frac{1}{\ln\left(\ln e^2\right)}$$

$$= -\frac{1}{\ln 3} + \frac{1}{\ln 2}$$

Exercise

Evaluate the integral $\displaystyle\int_0^1 \frac{y\ln^4\left(y^2+1\right)}{y^2+1}dy$

Solution

$$\int_0^1 \frac{y\ln^4\left(y^2+1\right)}{y^2+1}dy = \frac{1}{2}\int_0^1 \ln^4\left(y^2+1\right) d\left(\ln\left(y^2+1\right)\right) \qquad d\left(\ln\left(y^2+1\right)\right) = \frac{2y}{y^2+1}dy$$

$$= \frac{1}{10}\ln^5\left(y^2+1\right)\Big|_0^1$$

$$= \frac{1}{10}(\ln 2)^5$$

Exercise

Evaluate the integral $\displaystyle\int_{\ln 2}^{\ln 3} \frac{e^x+e^{-x}}{e^{2x}-2+e^{-2x}}dx$

Solution

$$\int_{\ln 2}^{\ln 3} \frac{e^x+e^{-x}}{e^{2x}-2+e^{-2x}}dx = \int_{\ln 2}^{\ln 3} \frac{e^x+e^{-x}}{\left(e^x-e^{-x}\right)^2}dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{1}{\left(e^x-e^{-x}\right)^2} d\left(e^x-e^{-x}\right)$$

$$= -\frac{1}{e^x - e^{-x}}\Bigg|_{\ln 2}^{\ln 3}$$

$$= -\frac{1}{e^{\ln 3} - e^{-\ln 3}} + \frac{1}{e^{\ln 2} - e^{-\ln 2}}$$

$$= \frac{1}{2-\frac{1}{2}} - \frac{1}{3-\frac{1}{3}}$$

$$= \frac{2}{3} - \frac{3}{8}$$

$$= \frac{7}{24}$$

Exercise

Evaluate the integral $\displaystyle\int_{-2}^{2} \frac{e^{z/2}}{e^{z/2}+1}dz$

Solution

$$\int_{-2}^{2} \frac{e^{z/2}}{e^{z/2}+1}dz = 2\int_{-2}^{2} \frac{1}{e^{z/2}+1}d\left(e^{z/2}+1\right)$$

$$d\left(e^{z/2}+1\right) = \frac{1}{2}e^{z/2}dz$$

$$= 2\ln\left(e^{z/2}+1\right)\Big|_{-2}^{2}$$

$$= 2\left(\ln(e+1) - \ln\left(e^{-1}+1\right)\right)$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{\pi/2} 4^{\sin x}\cos x\, dx$

Solution

$$\int_{0}^{\pi/2} 4^{\sin x}\cos x\, dx = \int_{0}^{\pi/2} 4^{\sin x}\, d(\sin x)$$

$$\int a^x dx = \frac{a^x}{\ln a}$$

$$= \frac{1}{\ln 4}4^{\sin x}\Big|_{0}^{\pi/2}$$

$$= \frac{1}{\ln 4}(4-1)$$

$$= \frac{3}{\ln 4}$$

Exercise

Evaluate the integral $\int_{1/3}^{1/2} \frac{10^{1/p}}{p^2} dp$

Solution

$$\int_{1/3}^{1/2} \frac{10^{1/p}}{p^2} dp = -\int_{1/3}^{1/2} 10^{1/p} d\left(\frac{1}{p}\right)$$

$$= -\frac{1}{\ln 10} 10^{1/p} \Big|_{1/3}^{1/2}$$

$$= -\frac{1}{\ln 10}\left(10^2 - 10^3\right)$$

$$= \frac{900}{\ln 10}$$

Exercise

Evaluate the integral $\int_1^2 (1+\ln x)x^x dx$

Solution

$$y = x^x \quad \to \quad \ln y = x \ln x$$

$$\frac{y'}{y} = 1 + \ln x \quad \Rightarrow \quad \left(x^x\right)' = x^x(1+\ln x)$$

$$\int_1^2 (1+\ln x)x^x dx = \int_1^2 d\left(x^x\right)$$

$$= x^x \Big|_1^2$$

$$= 2^2 - 1$$

$$= 3$$

Exercise

Find a curve through the origin in the xy-plane whose length from $x = 0$ to $x = 1$ is $L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x}\, dx$

Solution

$$L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x}\, dx \quad \Rightarrow \frac{dy}{dx} = \frac{e^{x/2}}{2}$$

$$dy = \frac{e^{x/2}}{2} dx$$

$$y = \int \frac{e^{x/2}}{2}\,dx = e^{x/2} + C$$

$$0 = e^{0/2} + C$$

$$0 = 1 + C \quad \to \ C = -1$$

$$\boxed{y = e^{x/2} - 1}$$

Exercise

Find the length of the curve $y = \ln\left(e^x - 1\right) - \ln\left(e^x + 1\right)$ *from* $x = \ln 2$ *to* $x = \ln 3$

Solution

$$y = \ln\left(e^x - 1\right) - \ln\left(e^x + 1\right) \quad \Rightarrow \frac{dy}{dx} = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1}$$

$$= \frac{e^{2x} + e^x - e^{2x} + e^x}{e^{2x} - 1}$$

$$= \frac{2e^x}{e^{2x} - 1}$$

$$L = \int_{\ln 2}^{\ln 3} \sqrt{1 + \left(\frac{2e^x}{e^{2x} - 1}\right)^2}\,dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{1 + \frac{4e^{2x}}{e^{4x} - 2e^{2x} + 1}}\,dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} - 2e^{2x} + 1 + 4e^{2x}}{\left(e^{2x} - 1\right)^2}}\,dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} + 2e^{2x} + 1}{\left(e^{2x} - 1\right)^2}}\,dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{\left(e^{2x} + 1\right)^2}{\left(e^{2x} - 1\right)^2}}\,dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{e^{2x} + 1}{e^{2x} - 1}\,dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\frac{e^{2x}+1}{e^x}}{\frac{e^{2x}-1}{e^x}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\frac{e^{2x}}{e^x}+\frac{1}{e^x}}{\frac{e^{2x}}{e^x}-\frac{1}{e^x}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{e^x+e^{-x}}{e^x-e^{-x}}\, dx$$

Let $u = e^x - e^{-x} \quad \Rightarrow du = \left(e^x + e^{-x}\right)dx \quad \rightarrow \begin{cases} x = \ln 2 & u = e^{\ln 2} - e^{-\ln 2} = 2 - \frac{1}{2} = \frac{3}{2} \\ x = \ln 3 & u = e^{\ln 3} - e^{-\ln 3} = 3 - \frac{1}{3} = \frac{8}{3} \end{cases}$

$$L = \int_{3/2}^{8/3} \frac{du}{u}$$

$$= \left[\ln|u|\ \right]_{3/2}^{8/3}$$

$$= \ln\frac{8}{3} - \ln\frac{3}{2}$$

$$= \ln\frac{8/3}{3/2}$$

$$= \ln\left(\frac{16}{9}\right)$$

Exercise

Find the length of the curve $y = \ln(\cos x)$ *from* $x = 0$ *to* $x = \frac{\pi}{4}$

Solution

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$$

$$L = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\ dx$$

$$= \int_0^{\pi/4} \sqrt{\sec^2 x}\ dx$$

$$= \int_0^{\pi/4} \sec x\ dx$$

$$= \left[\ln|\sec x + \tan x|\right]_0^{\pi/4}$$

$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln\left|\sec 0 + \tan 0\right|$$

$$= \ln\left|\sqrt{2} + 1\right| - \ln\left|1 + 0\right|$$

$$= \ln\left|\sqrt{2} + 1\right| - 0$$

$$= \ln\left(\sqrt{2} + 1\right)$$

Exercise

Find the area of the surface generated by revolving the curve $x = \frac{1}{2}\left(e^{y} + e^{-y}\right), \quad 0 \le y \le \ln 2,$ about *y*-axis

Solution

$$S = 2\pi \int_0^{\ln 2} \frac{1}{2}\left(e^{y} + e^{-y}\right)\sqrt{1 + \left(\frac{e^{y} - e^{-y}}{2}\right)^2}\, dy$$

$$= \pi \int_0^{\ln 2} \left(e^{y} + e^{-y}\right)\sqrt{1 + \frac{e^{2y} + e^{-2y} - 2}{4}}\, dy$$

$$= \pi \int_0^{\ln 2} \left(e^{y} + e^{-y}\right)\sqrt{\frac{4 + e^{2y} + e^{-2y} - 2}{4}}\, dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^{y} + e^{-y}\right)\sqrt{e^{2y} + e^{-2y} + 2}\, dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^{y} + e^{-y}\right)\sqrt{\left(e^{y} + e^{-y}\right)^2}\, dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^{y} + e^{-y}\right)^2\, dy$$

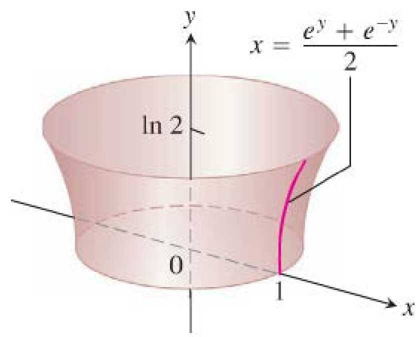
$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^{2y} + e^{-2y} + 2\right)\, dy$$

$$= \frac{\pi}{2}\left[\frac{1}{2}e^{2y} - \frac{1}{2}e^{-2y} + 2y\right]_0^{\ln 2}$$

$$= \frac{\pi}{2}\left[\left(\frac{1}{2}e^{2\ln 2} - \frac{1}{2}e^{-2\ln 2} + 2\ln 2\right) - \left(\frac{1}{2}e^{0} - \frac{1}{2}e^{0} + 0\right)\right]$$

$$= \frac{\pi}{2}\left(\frac{1}{2}\cdot 4 - \frac{1}{2}\cdot\frac{1}{4} + 2\ln 2\right)$$

$$= \frac{\pi}{2}\left(\frac{15}{8} + 2\ln 2\right)$$



Exercise

The population of a town with a 2010 population of 90,000 grows at a rate of 2.4% /yr. In what year will the population coudle its initial value (to 180,000)?

Solution

$$k = \frac{\ln \frac{1.024(90,000)}{90,000}}{1} = \ln(1.024)$$

$$T_2 = \frac{\ln 2}{\ln 1.024} \approx 29.226 \ \ yrs$$

It reaches 180,000 around the year 2039.

Exercise

How long will it take an initial deposit of \$1500 to increase in value to \$2500 in a saving account with an APY of 3.1%? Assume the interest rate reamins constant and no additional deposits or withdrawals are made.

Solution

$$y(t) = 1500\, e^{kt}$$

$$k = \frac{\ln 1.031}{1} = \ln(1.031)$$

$$T = \frac{\ln\left(\frac{2500}{1500}\right)}{\ln 1.031} \approx 16.7 \ \ yrs$$

$$kT = \ln\left(\frac{y}{y_0}\right)$$

Exercise

The number of cells in a tumor doubles every 6 weeks starting with 8 cells. After how many weeks doses the tumor have 1500 cells?

Solution

$$k = \frac{\ln 2}{T_2} = \frac{\ln 2}{6} \quad \Rightarrow \quad y(t) = 8\, e^{(t \ln 2)/6}$$

$$t = 6\frac{\ln\left(\frac{1500}{8}\right)}{\ln 2} \approx 45.3 \ \ weeks$$

$$kT = \ln\left(\frac{y}{y_0}\right)$$

Exercise

According to the 2010 census, the U.S. population was 309 million with an estimated growth rate of 0.8% /*yr*.

a) Based on these figures, find the doubling time and project the population in 2050.

b) Suppose the actual growth rate is just 0.2 percentage point lower than 0.8% /*yr* (0.6%). What are the resulting doubling time and projected 2050 population? Repeat these calculstions assuming the growth rate is 0.2 percentage point higher than 0.8% /yr.

c) Comment on th sensitivity of these projections to the growth rate.

Solution

a) $T_2 = \dfrac{\ln 2}{\ln 1.008} \approx 87\ \ yrs$

The population in 2050: $P(50) = 309e^{40\ln 1.008} \approx 425\ \ million$

b) If the growth rate is 0.6%: $T_2 = \dfrac{\ln 2}{\ln 1.006} \approx 116\ \ yrs$

The population in 2050: $P(50) = 309e^{40\ln 1.006} \approx 392.5\ \ million$

If the growth rate is 1%: $T_2 = \dfrac{\ln 2}{\ln 1.01} \approx 69.7\ \ yrs$

The population in 2050: $P(50) = 309e^{40\ln 1.01} \approx 460.1\ \ million$

c) A growth rate of just 0.2% produces large differences in population growth.

Exercise

The homicide rate decreases at a rate of 3%/yr in a city that had 800 homicides /yr in 2010. At this rate, when will the homicide rate reach 600 homicides/yr?

Solution

The homicide rate is modeled by: $H(t) = 800e^{-kt}$

$k = \ln(1-.03) \approx -0.03$

$H(t) = 800e^{-0.03t}$

$t = \dfrac{\ln(6/8)}{-0.03} \approx 9.6\ \ yrs$ $\qquad kT = \ln\left(\dfrac{y}{y_0}\right)$

So it should achieve this rate in 2019.

Exercise

A drug is eliminated from the body at a rate of 15% /hr. after how many hours does the amount of drug reach 10% of the initial dose?

Solution

$k = \ln(1-.15) \approx -\ln(.85)$

$t = \dfrac{\ln(.1)}{\ln(.85)} \approx 14.17\ \ hrs$ $\qquad kT = \ln\left(\dfrac{y}{y_0}\right)$

Exercise

A large die-casting machine used to make automobile engine blocks is purchased for \$2.5 *million*. For tax purposes, the value of the machine can be depreciated by 6.8% of its current value each year.

a) What is the value of the machine after 10 *years*?

b) After how many years is the value of the machine 10% of its original value?

Solution

a) $V(t) = 2.5e^{-kt}$

$$k = \frac{\ln(1-.068)}{1} \approx \underline{\ln(.932)} \qquad\qquad kT = \ln\left(\frac{y}{y_0}\right)$$

$$V(t) = 2.5e^{-t\ln .932} \quad \to \quad V(10) = 2.5e^{-10\cdot\ln .932} \;\; \underline{\approx \$1.2 \ million}$$

b) $t = \dfrac{\ln(.1)}{\ln(.932)} \;\; \underline{\approx 32.7 \ \ yrs}$

Exercise

Roughly 12,000 Americans are diagmosed with thyroid cancer every year, which accounts for 1% of all cancer cases. It occurs in women three times as frequency as in men. Fortunately, thyroid cancer can be treated successfully in many cases with radioactive iodine, or I-131. This unstable form of iodine has a half-life of 8 days and is given in small doses meansured in millicuries.

a) Suppose a patient is given an initial dose of 100 millicuries. Find the function that gives the amount of I-131 in the body after $t \ge 0$ days.

b) How long does it take the amount of I-131 to reach 10% of the initial dose?

c) Finding the initial dose to give a particular patient is a critical calculation. How does the time reach 10% of the initial dose change if the initial dose is increased by 5%?

Solution

a) $\underline{k = \dfrac{\ln 2}{8}} \qquad\qquad kT = \ln(y/y_0)$

After *t* days would be: $y = 100e^{-(t\ln 2)/8}$ millicuries.

b) $t = \dfrac{-8\ln\left(\frac{10}{100}\right)}{\ln(2)} \;\; \underline{\approx 26.58 \ \ days}$

c) $t = \dfrac{-8\ln\left(\frac{10}{105}\right)}{\ln(2)} \;\; \underline{\approx 27.14 \ \ days}$

Exercise

City ***A*** has a current population of 500,000 people and grows at a rate of 3% /*yr*. City ***B*** has a cuurent population of 300,000 and grows at a rate of 5%/*yr*.

a) When will the cities have the same population?

b) Suppose City ***C*** has a current population of $y_0 < 500,000$ and a growth rate of $p > 3\% / yr$. What is the relationship between y_0 and *p* such that the Cities ***A*** and ***C*** have the same population in 10 years?

Solution

a) $500,000e^{\ln(1.03)t} = 300,000e^{\ln(1.05)t}$

$5e^{\ln(1.03)t} = 3e^{\ln(1.05)t}$

$\frac{5}{3} = e^{(\ln(1.05)-\ln 1.03)t}$

$\ln\frac{5}{3} = \left(\ln\frac{1.05}{1.03}\right)t \quad \to \quad t = \frac{\ln(5/3)}{\ln(1.05/1.03)} \approx 26.56 \; yrs$

b) $500,000e^{\ln(1.03)(10)} = y_0 e^{\ln(1+p)(10)}$

$y_0 = 500,000e^{10(\ln(1.03)-\ln(1+p))}$

$= 500,000e^{\ln\left(\frac{1.03}{1+p}\right)^{10}}$

$= 500,000\left(\frac{1.03}{1+p}\right)^{10}$

Exercise

Suppose the acceleration of an object moving along a line is given by $a(t) = -kv(t)$, where *k* is a positive constant and *v* is the object's velocity. Assume that the initial velocity and position are given by $v(0) = 10$ and $s(0) = 0$, respectively.

a) Use $a(t) = v'(t)$ to find the velocity of the object as a function of time.

b) Use $v(t) = s'(t)$ to find the position of the object as a function of time.

c) Use the fact that $\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt}$ (by the *Chain Rule*) to find the velocity as a function of position.

Solution

a) If $a(t) = \frac{dv}{dt} = -kv \quad \to \quad \frac{dv}{v} = -kdt$

$\int \frac{dv}{v} = -k\int dt$

$\ln v = -kt + C \qquad$ Since $v(0) = 10$

$\ln 10 = C$

$\ln v = -kt + \ln 10$

$v = e^{-kt+\ln 10} = e^{-kt}e^{\ln 10}$

$v(t) = 10e^{-kt}$

b) $v(t)=\frac{ds}{dt}=10e^{-kt}$

$$\int ds=10\int e^{-kt}dt$$

$s(t)=-\frac{10}{k}e^{-kt}+C$ Since $s(0)=0$

$0=-\frac{10}{k}+C \quad \rightarrow \quad \underline{C=\frac{10}{k}}$

$\underline{s(t)=-\frac{10}{k}e^{-kt}+\frac{10}{k}}$

c) $\frac{dv}{dt}=\frac{dv}{ds}\frac{ds}{dt}$

$-10ke^{-kt}=\frac{dv}{ds}\left(10e^{-kt}\right)$

$-k=\frac{dv}{ds}$

$$\int dv=-k\int ds$$

$v=-ks+C$ Since $v(0)=10$

$\underline{v=10-ks}$

Exercise

On the first day of the year $(t=0)$, a city uses electricity at a rate of 2000 *MW*. That rate is projected to increase at a rate of 1.3% per *year*.

a) Based on these figures, find an exponential growth function for the power (rate of electricity use) for the city.

b) Find the total energy (in MW-*yr*) used by the city over four full years beginning at $t=0$

c) Find a function that gives the total energy used (in MW-*yr*) between $t=0$ and any future time $t>0$

Solution

a) $P(t)=2000e^{kt}$

At a rate of 1.3% per year: $k=\ln(1.013)$

$\underline{P(t)=2000e^{t\ln 1.013}}$

b) $\int_0^4 P(t)\,dt=2000\int_0^4 e^{t\ln 1.013}dt$

$=\frac{2000}{\ln 1.013}e^{t\ln 1.013}\Big|_0^4$

$\underline{\approx 8210.3}$

c) $\int_0^t P(s)\,ds = 2000\int_0^t e^{s\ln 1.013}ds$

$$= \frac{2000}{\ln 1.013} e^{s\ln 1.013}\Big|_0^t$$

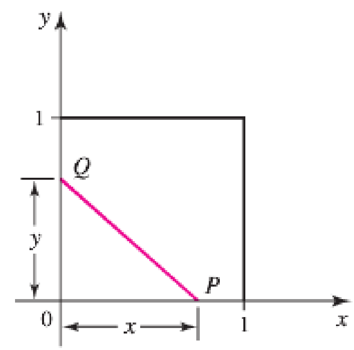
$$\boxed{= -154{,}844\left(1 + e^{t\ln(1.013)}\right)}$$

Exercise

Two points P and Q are chosen randomly, one on each of two adjacent sides of a unit square.

What is the probability that the area of the triangle formed by the sides of the square and the line segment PQ is less than one-fourth the area of the square? Begin by showing that x and y must satisfy $xy < \frac{1}{2}$ in order for the area consition to be met. Then argue that the required probability is $\frac{1}{2} + \int_{1/2}^{1} \frac{dx}{2x}$ and evaluate the integral.

Solution



The area of the triangle is $\frac{1}{2}xy$

If $xy < \frac{1}{2}$, then if we let $0 < x < \frac{1}{2}$ we have $0 < y < 1$

Because there is a probability of $\frac{1}{2}$ of choosing $0 < x < \frac{1}{2}$, the probability we seek is at least $\frac{1}{2}$.

In addtion, for $\frac{1}{2} < x < 1$, if $y < \frac{1}{2x}$,

$$\int_{1/2}^{1} \frac{dx}{2x} = \frac{1}{2}\ln x\Big|_{1/2}^{1} \; \boxed{= \frac{\ln 2}{2}}$$

$$\frac{1}{2} + \int_{1/2}^{1} \frac{dx}{2x} \; \boxed{= \frac{1}{2}(1 + \ln 2)}$$