

Solution **Section 2.6 – Improper Integrals**

Exercise

Evaluate the integral $\int_0^{\infty} \frac{dx}{x^2 + 1}$

Solution

$$\begin{aligned}\int_0^{\infty} \frac{dx}{x^2 + 1} &= \tan^{-1} x \Big|_0^b \\ &= \tan^{-1} \infty - \tan^{-1} 0 \\ &= \frac{\pi}{2} - 0 \\ &= \frac{\pi}{2} \Big| \end{aligned}$$

Exercise

Evaluate the integral $\int_0^4 \frac{dx}{\sqrt{4-x}}$

Solution

$$\begin{aligned}\int_0^4 \frac{dx}{\sqrt{4-x}} &= \int_0^4 (4-x)^{-1/2} dx \\ &= - \int_0^4 (4-x)^{-1/2} d(4-x) \\ &= -2 (4-x)^{1/2} \Big|_0^4 \\ &= -2(0-2) \\ &= 4 \Big| \end{aligned}$$

Exercise

Evaluate the integral $\int_{-\infty}^2 \frac{2dx}{x^2 + 4}$

Solution

$$\begin{aligned}\int_{-\infty}^2 \frac{2dx}{x^2 + 4} &= 2 \int_{-\infty}^2 \frac{dx}{x^2 + 2^2} \\ &= 2 \frac{1}{2} \left(\tan^{-1} \frac{x}{2} \right) \Big|_{-\infty}^2 \\ &= \tan^{-1} 1 - \tan^{-1}(-\infty) \\ &= \frac{\pi}{4} - \left(-\frac{\pi}{2} \right) \\ &= \frac{3\pi}{4}\end{aligned}$$

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} \frac{xdx}{(x^2 + 4)^{3/2}}$

Solution

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{xdx}{(x^2 + 4)^{3/2}} &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{d(x^2 + 4)}{(x^2 + 4)^{3/2}} \\ &= \frac{1}{2} \left(-2(x^2 + 4)^{-1/2} \right) \Big|_{-\infty}^{\infty} \\ &= - \frac{1}{\sqrt{x^2 + 4}} \Big|_{-\infty}^{\infty} \\ &= -(0 - 0) \\ &= 0\end{aligned}$$

$$u = x^2 + 4 \rightarrow du = 2xdx$$

Exercise

Evaluate the integral $\int_1^{\infty} \frac{dx}{x \sqrt{x^2 - 1}}$

Solution

$$\begin{aligned}
\int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}} &= \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^{\infty} \frac{dx}{x\sqrt{x^2-1}} \\
&= \lim_{b \rightarrow 1^+} \int_b^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{c \rightarrow \infty} \int_2^c \frac{dx}{x\sqrt{x^2-1}} \\
&= \lim_{b \rightarrow 1^+} \left(\sec^{-1}|x| \right) \Big|_b^2 + \lim_{c \rightarrow \infty} \left(\sec^{-1}|x| \right) \Big|_2^c \\
&= \lim_{b \rightarrow 1^+} \left(\sec^{-1} 2 - \sec^{-1} b \right) + \lim_{c \rightarrow \infty} \left(\sec^{-1} c - \sec^{-1} 2 \right) \\
&= \left(\frac{\pi}{3} - 0 \right) + \left(\frac{\pi}{2} - \frac{\pi}{3} \right) \\
&= \frac{\pi}{2}
\end{aligned}$$

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} 2xe^{-x^2} dx$

Solution

$$\begin{aligned}
\int_{-\infty}^{\infty} 2xe^{-x^2} dx &= \int_{-\infty}^0 2xe^{-x^2} dx + \int_0^{\infty} 2xe^{-x^2} dx & d(-x^2) &= -2xdx \\
&= - \lim_{b \rightarrow -\infty} \int_b^0 e^{-x^2} d(-x^2) - \lim_{c \rightarrow \infty} \int_0^c e^{-x^2} d(-x^2) \\
&= - \lim_{b \rightarrow -\infty} \left(e^{-x^2} \right) \Big|_b^0 - \lim_{c \rightarrow \infty} \left(e^{-x^2} \right) \Big|_0^c \\
&= - \lim_{b \rightarrow -\infty} \left(1 - e^{-b^2} \right) - \lim_{c \rightarrow \infty} \left(e^{-c^2} - 1 \right) \\
&= -(1-0) - (0-1) \\
&= 0
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 (-\ln x) dx$

Solution

$$u = \ln x \quad v = \int dx = x$$

$$du = \frac{1}{x} dx$$

$$\begin{aligned} \int_0^1 (-\ln x) dx &= -x \ln x + \int_0^1 x \frac{1}{x} dx \\ &= -x \ln x + \int_0^1 dx \\ &= -x \ln x + x \Big|_0^1 \\ &= 0 + 1 + 0 - 0 \\ &= \underline{1} \end{aligned}$$

Exercise

Evaluate the integral $\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$

Solution

$$\begin{aligned} \int_{-1}^4 \frac{dx}{\sqrt{|x|}} &= \int_{-1}^{0^-} \frac{dx}{\sqrt{-x}} + \int_{0^+}^4 \frac{dx}{\sqrt{x}} \\ &= \left(-2\sqrt{-x} \right) \Big|_{-1}^{0^-} + \left(2\sqrt{x} \right) \Big|_{0^+}^4 \\ &= 2 + 4 \\ &= \underline{6} \end{aligned}$$

Exercise

Evaluate the integral $\int_0^\infty e^{-3x} dx$

Solution

$$\begin{aligned}
 \int_0^{\infty} e^{-3x} dx &= -\frac{1}{3} e^{-3x} \Big|_0^{\infty} \\
 &= -\frac{1}{3} (e^{-\infty} - 1) \\
 &= \underline{\underline{\frac{1}{3}}}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Solution

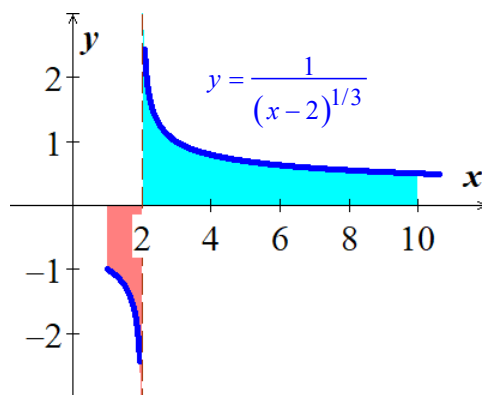
$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \tan^{-1} x \Big|_{-\infty}^{\infty} \\
 &= \tan^{-1} \infty - \tan^{-1}(-\infty) \\
 &= \frac{\pi}{2} + \frac{\pi}{2} \\
 &= \underline{\underline{\pi}}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_1^{10} \frac{dx}{(x-2)^{1/3}}$

Solution

$$\begin{aligned}
 \int_1^{10} (x-2)^{-1/3} dx &= \frac{3}{2} (x-2)^{2/3} \Big|_1^{10} \\
 &= \frac{3}{2} (8^{2/3} - (-1)^{2/3}) \\
 &= \frac{3}{2} (4 - 1) \\
 &= \underline{\underline{\frac{9}{2}}}
 \end{aligned}$$



$$\begin{aligned}
 \int_1^{10} (x-2)^{-1/3} dx &= \int_1^2 (x-2)^{-1/3} dx + \int_2^{10} (x-2)^{-1/3} dx \\
 &= \frac{3}{2} (x-2)^{2/3} \Big|_1^2 + (x-2)^{2/3} \Big|_2^{10}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{3}{2} \left(0 - (-1)^{2/3} \right) + \frac{3}{2} \left(8^{2/3} - 0 \right) \\
 &= \frac{3}{2} (-1 + 4) \\
 &= \frac{9}{2}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_1^{\infty} \frac{dx}{x^2}$

Solution

$$\begin{aligned}
 \int_1^{\infty} \frac{dx}{x^2} &= -\frac{1}{x} \Big|_1^{\infty} \\
 &= -\left(\frac{1}{\infty} - 1 \right) \\
 &= -(0 - 1) \\
 &= 1
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\infty} \frac{dx}{(x+1)^3}$

Solution

$$\begin{aligned}
 \int_0^{\infty} (x+1)^{-3} dx &= -\frac{1}{2} \frac{1}{(x+1)^2} \Big|_0^{\infty} \\
 &= -\frac{1}{2} \left(\frac{1}{\infty} - 1 \right) \\
 &= -\frac{1}{2} (0 - 1) \\
 &= \frac{1}{2}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_{-\infty}^0 e^x dx$

Solution

$$\begin{aligned}\int_{-\infty}^0 e^x dx &= e^x \Big|_{-\infty}^0 \\ &= \left(1 - e^{-\infty}\right) \\ &= 1\end{aligned}\quad e^{-\infty} \rightarrow 0$$

Exercise

Evaluate the integral $\int_1^{\infty} 2^{-x} dx$

Solution

$$\begin{aligned}\int_1^{\infty} 2^{-x} dx &= -\int_1^{\infty} 2^{-x} d(-x) \\ &= -\frac{2^{-x}}{\ln 2} \Big|_1^{\infty} \\ &= -\frac{1}{\ln 2} \left(0 - \frac{1}{2}\right) \\ &= \frac{1}{2 \ln 2}\end{aligned}\quad \int a^x dx = \frac{a^x}{\ln a}$$

Exercise

Evaluate the integral $\int_{-\infty}^0 \frac{dx}{\sqrt[3]{2-x}}$

Solution

$$\begin{aligned}\int_{-\infty}^0 \frac{dx}{\sqrt[3]{2-x}} &= -\int_{-\infty}^0 (2-x)^{-1/3} d(2-x) \\ &= -\frac{3}{2} (2-x)^{2/3} \Big|_{-\infty}^0 \\ &= -\frac{3}{2} (2^{2/3} - \infty) \\ &= \infty\end{aligned}\quad \text{diverges}$$

Exercise

Evaluate the integral $\int_{4/\pi}^{\infty} \frac{1}{x^2} \sec^2\left(\frac{1}{x}\right) dx$

Solution

$$\begin{aligned}\int_{4/\pi}^{\infty} \frac{1}{x^2} \sec^2\left(\frac{1}{x}\right) dx &= -\int_{4/\pi}^{\infty} \sec^2\left(\frac{1}{x}\right) d\left(\frac{1}{x}\right) & d\left(\frac{1}{x}\right) &= -\frac{1}{x^2} dx \\ &= -\tan\left(\frac{1}{x}\right) \Big|_{4/\pi}^{\infty} \\ &= -\left(\tan 0 - \tan \frac{\pi}{4}\right) \\ &= 1\end{aligned}$$

Exercise

Evaluate the integral $\int_{e^2}^{\infty} \frac{dx}{x \ln^p x} \quad p > 1$

Solution

$$\begin{aligned}\int_{e^2}^{\infty} \frac{dx}{x \ln^p x} &= \int_{e^2}^{\infty} (\ln x)^{-p} d(\ln x) \\ &= \frac{1}{1-p} (\ln x)^{1-p} \Big|_{e^2}^{\infty} \\ &= \frac{1}{1-p} \left((\ln x)^{-\infty} - (\ln e^2)^{1-p} \right) \\ &= \frac{-1}{1-p} 2^{1-p} \\ &= \frac{1}{(p-1)2^{p-1}}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\infty} \frac{p}{\sqrt[5]{p^2+1}} dp$

Solution

$$\begin{aligned}\int_0^{\infty} \frac{p}{\sqrt[5]{p^2+1}} dp &= \frac{1}{2} \int_0^{\infty} (p^2+1)^{-1/5} d(p^2+1) & d(p^2+1) &= 2p dp \\ &= \frac{5}{8} (p^2+1)^{4/5} \Big|_0^{\infty} \\ &= \infty \quad \text{diverges}\end{aligned}$$

Exercise

Evaluate the integral $\int_{-1}^1 \ln y^2 \, dy$

Solution

$$\int_{-1}^1 \ln y^2 \, dy = 2 \int_0^1 \ln y^2 \, dy$$

$$\int \ln x^2 \, dx = 2 \int \ln x \, dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x} dx$$

$$v = \int dx = x$$

$$= 2 \left[x \ln x - \int dx \right]$$

$$= 2(x \ln x - x) \Big|$$

$$\int_{-1}^1 \ln y^2 \, dy = 4 \left(y \ln y - y \right) \Big|_0^1$$

$$= 4(-1 - 0)$$

$$= \underline{-4}$$

Exercise

Evaluate the integral $\int_{-2}^6 \frac{dx}{\sqrt{|x-2|}}$

Solution

$$\int_{-2}^6 \frac{dx}{\sqrt{|x-2|}} = \int_{-2}^2 \frac{dx}{\sqrt{2-x}} + \int_2^6 \frac{dx}{\sqrt{x-2}}$$

$$= - \int_{-2}^2 (2-x)^{-1/2} d(2-x) + \int_2^6 (x-2)^{-1/2} d(x-2)$$

$$= -2 \left(\sqrt{2-x} \right) \Big|_{-2}^2 + 2 \left(\sqrt{x-2} \right) \Big|_2^6$$

$$= -2(0 - 2) + 2(2 - 0)$$

$$= \underline{8}$$

Exercise

Evaluate $\int_0^{\infty} x e^{-x} dx$

Solution

		$\int e^{-x} dx$
+	x	$-e^{-x}$
-	1	e^{-x}

$$\begin{aligned}\int_0^{\infty} x e^{-x} dx &= -x e^{-x} - e^{-x} \Big|_0^{\infty} \\ &= 0 - (-1) \\ &= 1\end{aligned}$$

Exercise

Evaluate $\int_0^1 x \ln x dx$

Solution

$$u = \ln x \quad dv = x dx$$

$$du = \frac{dx}{x} \quad v = \frac{1}{2} x^2$$

$$\begin{aligned}\int x \ln x dx &= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x dx \\ &= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2\end{aligned}$$

$$\begin{aligned}\int_0^1 x \ln x dx &= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \Big|_0^1 \\ &= -\frac{1}{4}\end{aligned}$$

Exercise

Evaluate $\int_1^{\infty} \frac{\ln x}{x^2} dx$

Solution

$$u = \ln x \quad dv = -\frac{1}{x^2} dx$$

$$du = \frac{dx}{x} \quad v = -\frac{1}{x}$$

$$\begin{aligned} \int \frac{\ln x}{x^2} dx &= -\frac{1}{x} \ln x + \int \frac{1}{x^2} dx \\ &= -\frac{1}{x} \ln x - \frac{1}{x} \end{aligned}$$

$$\begin{aligned} \int_1^{\infty} \frac{\ln x}{x^2} dx &= -\frac{1}{x} (\ln x + 1) \Big|_1^{\infty} \\ &= \underline{\underline{1}} \end{aligned}$$

Exercise

Evaluate $\int_1^{\infty} (1-x)e^{-x} dx$

Solution

$$\begin{aligned} \int_1^{\infty} (1-x)e^{-x} dx &= \left(-e^{-x} - (-x-1)e^{-x} \right) \Big|_1^{\infty} \\ &= xe^{-x} \Big|_1^{\infty} \quad e^{-\infty} \rightarrow 0 \\ &= 0 - e^{-1} \\ &= \underline{\underline{-\frac{1}{e}}} \end{aligned}$$

Exercise

Evaluate $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx$

Solution

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx &= \int_{-\infty}^{\infty} \frac{d(e^x)}{1+(e^x)^2} \\ &= \arctan e^x \Big|_{-\infty}^{\infty} \\ &= \arctan \infty - \arctan 0 \\ &= \underline{\underline{\frac{\pi}{2}}} \end{aligned}$$

Exercise

Evaluate $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

Solution

$$\int_0^1 x^{-1/3} dx = \frac{3}{2} x^{2/3} \Big|_0^1$$

$$= \frac{3}{2} \Big|$$

Exercise

Evaluate $\int_1^\infty 4x^{-1/4} dx$

Solution

$$\int_1^\infty 4x^{-1/4} dx = \frac{16}{3} x^{3/4} \Big|_1^\infty$$

$$= \infty \quad \text{Diverges}$$

Exercise

Evaluate $\int_0^2 \frac{dx}{x^3}$

Solution

$$\int_0^2 \frac{dx}{x^3} = -\frac{1}{2x^2} \Big|_0^2$$

$$= -\frac{1}{8} + \infty$$

$$= \infty \quad \text{Diverges}$$

Exercise

Evaluate $\int_1^\infty \frac{dx}{x^3}$

Solution

$$\int_1^\infty \frac{dx}{x^3} = -\frac{1}{2x^2} \Big|_1^\infty$$

$$= -\frac{1}{\infty} + \frac{1}{2}$$

$$\underline{= \frac{1}{2}}$$

Exercise

Evaluate $\int_1^{\infty} 6x^{-4} dx$

Solution

$$\int_1^{\infty} 6x^{-4} dx = -2 \frac{1}{x^3} \Big|_1^{\infty}$$

$$\underline{= 2}$$

Exercise

Evaluate $\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$

Solution

$$u = \sqrt{x} \rightarrow u^2 = x$$

$$dx = 2u du$$

$$\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)} = \int_0^{\infty} \frac{2u}{u(u^2+1)} du$$

$$= 2 \int_0^{\infty} \frac{1}{u^2+1} du$$

$$= 2 \arctan \sqrt{x} \Big|_0^{\infty}$$

$$= 2 \left(\frac{\pi}{2} - 0 \right)$$

$$\underline{= \pi}$$

Exercise

Evaluate $\int_{-\infty}^0 xe^{-4x} dx$

Solution

$$\int_{-\infty}^0 x e^{-4x} dx = \left(-\frac{x}{4} - \frac{1}{16} \right) e^{-4x} \Big|_{-\infty}^0$$

$$= -\frac{1}{16} - \infty$$

$$= -\infty \quad \text{Diverges}$$

$$\int x^n e^{ax} dx = e^{ax} \sum_{k=0}^n \frac{(-1)^k n!}{a^{k+1} (n-k)!} x^{n-k}$$

Exercise

Evaluate $\int_0^{\infty} x e^{-x/3} dx$

Solution

$$\int_0^{\infty} x e^{-x/3} dx = (-3x - 9) e^{-x/3} \Big|_0^{\infty}$$

$$= 9$$

$$\int x^n e^{ax} dx = e^{ax} \sum_{k=0}^n \frac{(-1)^k n!}{a^{k+1} (n-k)!} x^{n-k}$$

Exercise

Evaluate $\int_0^{\infty} x^2 e^{-x} dx$

Solution

$$\int_0^{\infty} x^2 e^{-x} dx = (-x^2 - 2x - 2) e^{-x} \Big|_0^{\infty}$$

$$= 2$$

$$\int x^n e^{ax} dx = e^{ax} \sum_{k=0}^n \frac{(-1)^k n!}{a^{k+1} (n-k)!} x^{n-k}$$

Exercise

Evaluate $\int_0^{\infty} e^{-x} \cos x dx$

Solution

		$\int \cos x dx$
+	e^{-x}	$\sin x$
-	$-e^{-x}$	$-\cos x$
+	e^{-x}	$-\int \cos x dx$

$$\int e^{-x} \cos x \, dx = e^{-x} (\sin x - \cos x) - \int e^{-x} \cos x \, dx$$

$$2 \int e^{-x} \cos x \, dx = e^{-x} (\sin x - \cos x)$$

$$\int_0^{\infty} e^{-x} \cos x \, dx = \frac{1}{2} e^{-x} (\sin x - \cos x) \Big|_0^{\infty}$$

$$\underline{= \frac{1}{2}}$$

Exercise

Evaluate $\int_4^{\infty} \frac{1}{x(\ln x)^3} \, dx$

Solution

$$\int_4^{\infty} \frac{1}{x(\ln x)^3} \, dx = \int_4^{\infty} (\ln x)^{-3} d(\ln x)$$

$$= -\frac{1}{2} \frac{1}{(\ln x)^2} \Big|_4^{\infty}$$

$$= \frac{1}{2} \left(0 - \frac{1}{(\ln 4)^2} \right)$$

$$\underline{= -\frac{1}{2(\ln 4)^2}}$$

Exercise

Evaluate $\int_1^{\infty} \frac{\ln x}{x} \, dx$

Solution

$$\int_1^{\infty} \frac{\ln x}{x} \, dx = \int_1^{\infty} \ln x \, d(\ln x)$$

$$= \frac{1}{2} (\ln x)^2 \Big|_1^{\infty}$$

$$\underline{= \infty} \quad \text{diverges}$$

Exercise

Evaluate $\int_{-\infty}^{\infty} \frac{4}{16+x^2} dx$

Solution

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{4}{16+x^2} dx &= \arctan\left(\frac{x}{4}\right) \Big|_{-\infty}^{\infty} \\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \\ &= \pi\end{aligned}$$

Exercise

Evaluate $\int_0^{\infty} \frac{x^3}{(x^2+1)^2} dx$

Solution

$$\begin{aligned}\frac{x^3}{(x^2+1)^2} &= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \\ x^3 &= Ax^3 + Ax + Bx^2 + B + Cx + D\end{aligned}$$

$$\begin{cases} x^3 & \underline{A=1} \\ x^2 & \underline{B=0} \\ x & A+C=0 \rightarrow \underline{C=-1} \\ x^0 & B+D=0 \rightarrow \underline{D=0} \end{cases}$$

$$\begin{aligned}\int_0^{\infty} \frac{x^3}{(x^2+1)^2} dx &= \int_0^{\infty} \frac{x}{x^2+1} dx - \int_0^{\infty} \frac{x}{(x^2+1)^2} dx \\ &= \frac{1}{2} \int_0^{\infty} \frac{1}{x^2+1} d(x^2+1) - \frac{1}{2} \int_0^{\infty} \frac{1}{(x^2+1)^2} d(x^2+1) \\ &= \frac{1}{2} \ln(x^2+1) + \frac{1}{2} \frac{1}{x^2+1} \Big|_0^{\infty} \\ &= \infty \quad \text{diverges}\end{aligned}$$

Exercise

Evaluate $\int_0^{\infty} \frac{1}{e^x + e^{-x}} dx$

Solution

$$\begin{aligned}\int_0^{\infty} \frac{1}{e^x + e^{-x}} dx &= \int_0^{\infty} \frac{1}{e^x + e^{-x}} \frac{e^x}{e^x} dx \\&= \int_0^{\infty} \frac{e^x}{e^{2x} + 1} dx \\&= \int_0^{\infty} \frac{1}{(e^x)^2 + 1} d(e^x) \\&= \arctan e^x \Big|_0^{\infty} \\&= \arctan(\infty) - \arctan(1) \\&= \frac{\pi}{2} - \frac{\pi}{4} \\&= \frac{\pi}{4}\end{aligned}$$

Exercise

Evaluate $\int_0^{\infty} \frac{e^x}{1 + e^x} dx$

Solution

$$\begin{aligned}\int_0^{\infty} \frac{e^x}{1 + e^x} dx &= \int_0^{\infty} \frac{1}{1 + e^x} d(e^x) \\&= \ln(1 + e^x) \Big|_0^{\infty} \\&= \infty \quad \text{diverges}\end{aligned}$$

Exercise

Evaluate $\int_0^{\infty} \cos \pi x dx$

Solution

$$\int_0^{\infty} \cos \pi x \, dx = \frac{1}{\pi} \sin \pi x \Big|_0^{\infty}$$

$$\underline{= \infty} \quad \text{diverges}$$

Exercise

Evaluate $\int_0^{\infty} \sin \frac{x}{2} \, dx$

Solution

$$\int_0^{\infty} \sin \frac{x}{2} \, dx = -2 \cos \frac{x}{2} \Big|_0^{\infty}$$

$$\underline{= \infty} \quad \text{diverges}$$

Exercise

Evaluate the integral $\int_1^{\infty} \frac{dx}{(x+1)^9}$

Solution

$$\int_1^{\infty} \frac{dx}{(x+1)^9} = \int_1^{\infty} (x+1)^{-9} d(x+1)$$

$$= -\frac{1}{8} (x+1)^{-8} \Big|_1^{\infty}$$

$$= -\frac{1}{8} \left(0 - \frac{1}{2^8} \right)$$

$$\underline{= \frac{1}{2,048}}$$

Exercise

Evaluate the integral $\int_1^{\infty} \frac{3x-1}{4x^3-x^2} \, dx$

Solution

$$\frac{3x-1}{4x^3-x^2} = \frac{3x-1}{x^2(4x-1)}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{4x-1}$$

$$3x-1 = 4Ax^2 - Ax + 4Bx - B + Cx^2$$

$$x^2 \quad 4A + C = 0 \quad \rightarrow \quad \underline{C = -4}$$

$$x^1 \quad -A + 4B = 3 \quad \rightarrow \quad \underline{A = 1}$$

$$x^0 \quad -B = -1 \quad \rightarrow \quad \underline{B = 1}$$

$$\begin{aligned} \int_1^{\infty} \frac{3x-1}{4x^3-x^2} dx &= \int_1^{\infty} \left(\frac{1}{x} + \frac{1}{x^2} - \frac{4}{4x-1} \right) dx \\ &= \ln x - \frac{1}{x} - \int_1^{\infty} \frac{1}{4x-1} d(4x-1) \\ &= \ln x - \frac{1}{x} - \ln(4x-1) \Big|_1^{\infty} \\ &= \ln \frac{x}{4x-1} - \frac{1}{x} \Big|_1^{\infty} \\ &= \ln \frac{1}{4} - 0 - \ln \frac{1}{3} - 1 \\ &= -\ln 4 + \ln 3 - 1 \\ &= \underline{\ln \frac{3}{4} - 1} \end{aligned}$$

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} \frac{4}{x^2+16} dx$

Solution

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{4}{x^2+16} dx &= 2 \int_0^{\infty} \frac{4}{x^2+4^2} dx \\ &= 2 \arctan \frac{x}{4} \Big|_0^{\infty} \\ &= 2 \arctan \infty - 2 \arctan 0 \\ &= 2 \left(\frac{\pi}{2} - 0 \right) \\ &= \underline{\pi} \end{aligned}$$

Exercise

Evaluate the integral $\int_{-\infty}^{-1} \frac{dx}{(x-1)^4}$

Solution

$$\begin{aligned}\int_{-\infty}^{-1} \frac{dx}{(x-1)^4} &= \int_{-\infty}^{-1} (x-1)^{-4} d(x-1) \\ &= -\frac{1}{3} \frac{1}{(x-1)^3} \Big|_{-\infty}^{-1} \\ &= -\frac{1}{3} \left(-\frac{1}{8} - 0 \right) \\ &= \frac{1}{24}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\infty} x e^{-x} dx$

Solution

$$\begin{aligned}\int_0^{\infty} x e^{-x} dx &= -e^{-x} (x+1) \Big|_0^{\infty} \\ &= -0 + 1 \\ &= 1\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\infty} \frac{6x}{1+x^6} dx$

Solution

$$\begin{aligned}\text{Let } u &= x^2 \\ du &= 2x dx\end{aligned}$$

$$\int_0^{\infty} \frac{6x}{1+x^6} dx = \int_0^{\infty} \frac{3 du}{1+u^3}$$

$$\frac{3}{1+u^3} = \frac{A}{1+u} + \frac{Bu+C}{1-u+u^2}$$

$$3 = A - Au + Au^2 + Bu + Bu^2 + C + Cu$$

$$\begin{cases} u^2 & A+B=0 \\ u^1 & -A+B+C=0 \\ u^0 & A+C=3 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 3 \quad \Delta_A = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} = 3$$

$$A = \frac{3}{3} \equiv 1 \mid$$

$$B = -A \equiv -1 \mid$$

$$C = 3 - A \equiv 2 \mid$$

$$\begin{aligned} \int_0^\infty \frac{6x}{1+x^6} dx &= \int_0^\infty \frac{1}{1+u} du + \int_0^\infty \frac{-u+2}{1-u+u^2} du \\ &= \int_0^\infty \frac{1}{1+u} d(u+1) - \frac{1}{2} \int_0^\infty \frac{2u-1+3}{1-u+u^2} du \\ &= \ln(u+1) \Big|_0^\infty - \frac{1}{2} \int_0^\infty \frac{2u-1}{1-u+u^2} du - \frac{3}{2} \int_0^\infty \frac{1}{\left(u-\frac{1}{2}\right)^2 + \frac{3}{4}} du \\ &= \ln(x^2+1) \Big|_0^\infty - \frac{1}{2} \int_0^\infty \frac{1}{1-u+u^2} d(1-u+u^2) - \frac{3}{2} \int_0^\infty \frac{1}{\left(u-\frac{1}{2}\right)^2 + \frac{3}{4}} d\left(u-\frac{1}{2}\right) \\ &= \ln(x^2+1) - \frac{1}{2} \ln|1-u+u^2| - \frac{3}{2} \frac{2}{\sqrt{3}} \tan\left(\left(u-\frac{1}{2}\right) \frac{2}{\sqrt{3}}\right) \Big|_0^\infty \\ &= \ln(x^2+1) - \frac{1}{2} \ln|1-x^2+x^4| - \sqrt{3} \tan\left(\frac{2x^2-1}{\sqrt{3}}\right) \Big|_0^\infty \\ &= \ln(x^2+1) - \ln\sqrt{1-x^2+x^4} - \sqrt{3} \tan\left(\frac{2x^2-1}{\sqrt{3}}\right) \Big|_0^\infty \\ &= \ln \frac{x^2+1}{\sqrt{1-x^2+x^4}} - \sqrt{3} \tan\left(\frac{2x^2-1}{\sqrt{3}}\right) \Big|_0^\infty \end{aligned}$$

$$\begin{aligned}
&= \ln 1 - \sqrt{3} \tan(\infty) - \ln 1 + \sqrt{3} \tan\left(-\frac{1}{\sqrt{3}}\right) \\
&= -\sqrt{3} \frac{\pi}{2} - \sqrt{3} \frac{\pi}{6} \\
&= -\frac{2\pi}{3} \sqrt{3}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^2 \frac{dx}{\sqrt[3]{|x-1|}}$

Solution

$$\begin{aligned}
\int_0^2 \frac{dx}{\sqrt[3]{|x-1|}} &= \int_0^1 \frac{dx}{\sqrt[3]{|x-1|}} + \int_1^2 \frac{dx}{\sqrt[3]{|x-1|}} \\
&= -\int_0^1 (1-x)^{-1/3} d(1-x) + \int_1^2 (x-1)^{-1/3} d(x-1) \\
&= -\frac{3}{2}(1-x)^{2/3} \Big|_0^1 + \frac{3}{2}(x-1)^{2/3} \Big|_1^2 \\
&= -\frac{3}{2}(0-1) + \frac{3}{2}(1-0) \\
&= \frac{3}{2} + \frac{3}{2} \\
&= 3
\end{aligned}$$

$$\begin{aligned}
\int_0^2 \frac{dx}{\sqrt[3]{|x-1|}} &= \int_0^2 (x-1)^{-1/3} d(x-1) \\
&= \frac{3}{2}(x-1)^{2/3} \Big|_0^2 \\
&= \frac{3}{2}(1+1) \\
&= 3
\end{aligned}$$

Exercise

Evaluate the integral $\int_{-1}^1 \frac{dx}{x^2 + 2x + 5}$

Solution

$$\begin{aligned}\int_{-1}^1 \frac{dx}{x^2 + 2x + 5} &= \int_{-1}^1 \frac{dx}{(x+1)^2 + 4} \\ &= \frac{1}{2} \arctan \frac{x+1}{2} \Big|_{-1}^1 \\ &= \frac{1}{2} (\arctan 1 - \arctan 0) \\ &= \frac{\pi}{8}\end{aligned}$$

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 5}$

Solution

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 5} &= \int_{-\infty}^{\infty} \frac{dx}{(x+1)^2 + 4} \\ &= \frac{1}{2} \arctan \frac{x+1}{2} \Big|_{-\infty}^{\infty} \\ &= \frac{1}{2} (\arctan \infty - \arctan (-\infty)) \\ &= \frac{1}{2} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) \\ &= \frac{\pi}{2}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\infty} \cos x \, dx$

Solution

$$\begin{aligned}
 \int_0^{\infty} \cos x \, dx &= \sin x \Big|_0^{\infty} \\
 &= \sin \infty - \sin 0 \\
 &= \cancel{\infty} \\
 &= \text{doesn't exist} \quad \rightarrow \text{diverges}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_2^{\infty} \frac{\cos\left(\frac{\pi}{x}\right)}{x^2} dx$

Solution

$$\begin{aligned}
 \int_2^{\infty} \frac{\cos\left(\frac{\pi}{x}\right)}{x^2} dx &= -\frac{1}{\pi} \int_2^{\infty} \cos\left(\frac{\pi}{x}\right) d\left(\frac{\pi}{x}\right) & d\left(\frac{\pi}{x}\right) &= -\frac{\pi}{x^2} dx \\
 &= -\frac{1}{\pi} \sin\left(\frac{\pi}{x}\right) \Big|_2^{\infty} \\
 &= -\frac{1}{\pi} \left(\sin 0 - \sin \frac{\pi}{2} \right) \\
 &= -\frac{1}{\pi} (0 - 1) \\
 &= \frac{1}{\pi}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_{-\infty}^a \sqrt{e^x} \, dx$

Solution

$$\begin{aligned}
 \int_{-\infty}^a \sqrt{e^x} \, dx &= \int_{-\infty}^a e^{x/2} \, dx \\
 &= 2e^{x/2} \Big|_{-\infty}^a \\
 &= 2\left(e^{a/2} - e^{-\infty}\right) \\
 &= 2e^{a/2}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\infty} \frac{e^x}{e^{2x} + 1} dx$

Solution

$$\begin{aligned}\int_0^{\infty} \frac{e^x}{e^{2x} + 1} dx &= \int_0^{\infty} \frac{d(e^x)}{(e^x)^2 + 1} \\&= \tan^{-1} e^x \Big|_0^{\infty} \\&= \tan^{-1} \infty - \tan^{-1} 1 \\&= \frac{\pi}{2} - \frac{\pi}{4} \\&= \frac{\pi}{4}\end{aligned}$$

Exercise

Evaluate the integral $\int_1^{\infty} \frac{dx}{x(x+1)}$

Solution

$$\begin{aligned}\int_1^{\infty} \frac{dx}{x(x+1)} &= \int_1^{\infty} \left(\frac{1}{x} - \frac{1}{x+1} \right) dx \\&= \ln|x| - \ln|x+1| \Big|_1^{\infty} \\&= \ln\left(\frac{x}{x+1}\right) \Big|_1^{\infty} \\&= \ln 1 - \ln \frac{1}{2} \\&= \ln 2\end{aligned}$$

Exercise

Evaluate the integral $\int_1^{\infty} \frac{dx}{x^2(x+1)}$

Solution

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$1 = Ax^2 + Ax + Bx + B + Cx^2$$

$$\begin{cases} x^2 & A + C = 0 \rightarrow \underline{C = 1} \\ x^1 & A + B = 0 \rightarrow \underline{A = -1} \\ x^0 & B = 1 \end{cases}$$

$$\begin{aligned} \int_1^{\infty} \frac{dx}{x^2(x+1)} &= \int_1^{\infty} \left(-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1} \right) dx \\ &= -\ln x - \frac{1}{x} + \ln(x+1) \Big|_1^{\infty} \\ &= \ln \frac{x+1}{x} - \frac{1}{x} \Big|_1^{\infty} \\ &= \ln 1 - 0 - (\ln 2 - 1) \\ &= \underline{1 - \ln 2} \end{aligned}$$

Exercise

Evaluate the integral $\int_1^{\infty} \frac{3x^2+1}{x^3+x} dx$

Solution

$$\begin{aligned} \int_1^{\infty} \frac{3x^2+1}{x^3+x} dx &= \int_1^{\infty} \frac{1}{x^3+x} d(x^3+x) \\ &= \ln(x^3+x) \Big|_1^{\infty} \\ &= \ln \infty - \ln 2 \\ &= \underline{\infty} \quad \text{diverges} \end{aligned}$$

Exercise

Evaluate the integral $\int_1^{\infty} \frac{1}{x^2} \sin \frac{\pi}{x} dx$

Solution

$$\begin{aligned}
 \int_1^{\infty} \frac{1}{x^2} \sin \frac{\pi}{x} dx &= -\frac{1}{\pi} \int_1^{\infty} \sin \frac{\pi}{x} d\left(\frac{\pi}{x}\right) & d\left(\frac{\pi}{x}\right) &= -\frac{\pi}{x^2} dx \\
 &= \frac{1}{\pi} \cos \frac{\pi}{x} \Big|_1^{\infty} \\
 &= \frac{1}{\pi} (\cos 0 - \cos \pi) \\
 &= \frac{1}{\pi} (1 + 1) \\
 &= \frac{2}{\pi}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_2^{\infty} \frac{dx}{(x+2)^2}$

Solution

$$\begin{aligned}
 \int_2^{\infty} \frac{dx}{(x+2)^2} &= \int_2^{\infty} \frac{d(x+2)}{(x+2)^2} \\
 &= -\frac{1}{x+2} \Big|_2^{\infty} \\
 &= -\left(0 - \frac{1}{4}\right) \\
 &= \frac{1}{4}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_1^{\infty} \frac{\tan^{-1} x}{x^2 + 1} dx$

Solution

$$\begin{aligned}
 \int_1^{\infty} \frac{\tan^{-1} x}{x^2 + 1} dx &= \int_1^{\infty} \tan^{-1} x d(\tan^{-1} x) \\
 &= \frac{1}{2} (\tan^{-1} x)^2 \Big|_1^{\infty} \\
 &= \frac{1}{2} \left((\tan^{-1} \infty)^2 - (\tan^{-1} 1)^2 \right)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left(\left(\frac{\pi}{2} \right)^2 - \left(\frac{\pi}{4} \right)^2 \right) \\
&= \frac{\pi^2}{2} \left(\frac{1}{4} - \frac{1}{16} \right) \\
&= \frac{\pi^2}{2} \left(\frac{3}{16} \right) \\
&= \frac{3\pi^2}{32} \Big|
\end{aligned}$$

Exercise

Evaluate the integral $\int_{-3}^1 \frac{dx}{(2x+6)^{2/3}}$

Solution

$$\begin{aligned}
\int_{-3}^1 \frac{dx}{(2x+6)^{2/3}} &= \frac{1}{2} \int_{-3}^1 (2x+6)^{-2/3} d(2x+6) \\
&= \frac{3}{2} (2x+6)^{1/3} \Big|_{-3}^1 \\
&= \frac{3}{2} (\sqrt[3]{8} - 0) \\
&= 3 \Big|
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

Solution

$$\begin{aligned}
\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx &= 2 \int_0^1 e^{\sqrt{x}} d(\sqrt{x}) \\
&= 2e^{\sqrt{x}} \Big|_0^1 \\
&= 2(e-1) \Big|
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\ln 3} \frac{e^x}{(e^x - 1)^{2/3}} dx$

Solution

$$\begin{aligned}\int_0^{\ln 3} \frac{e^x}{(e^x - 1)^{2/3}} dx &= \int_0^{\ln 3} (e^x - 1)^{-2/3} d(e^x - 1) \\ &= 3(e^x - 1)^{1/3} \Big|_0^{\ln 3} \\ &= 3\left((e^{\ln 3} - 1)^{1/3} - 0\right) \\ &= 3(3 - 1)^{1/3} \\ &= 3\sqrt[3]{2}\end{aligned}$$

Exercise

Evaluate the integral $\int_1^2 \frac{dx}{\sqrt{x-1}}$

Solution

$$\begin{aligned}\int_1^2 \frac{dx}{\sqrt{x-1}} &= \int_1^2 (x-1)^{-1/2} d(x-1) \\ &= 2(x-1)^{1/2} \Big|_1^2 \\ &= 2(1-0) \\ &= 2\end{aligned}$$

Exercise

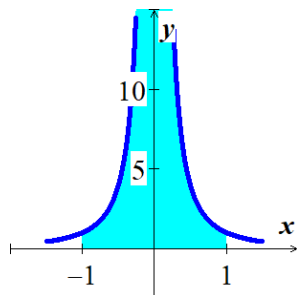
Evaluate the integral $\int_{-1}^1 \frac{dx}{x^2}$

Solution

$$\int_{-1}^1 \frac{dx}{x^2} = \int_{-1}^0 \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2}$$

$$= -\frac{1}{x} \Big|_{-1}^0 - \frac{1}{x} \Big|_1^0$$

$$= -\infty \quad \text{diverges}$$



$$\int_{-1}^1 \frac{dx}{x^2} = -\frac{1}{x} \Big|_{-1}^1$$

$$= -1 - 1$$

$$= -2$$

Exercise

Evaluate the integral $\int_0^2 \frac{dx}{(x-1)^2}$

Solution

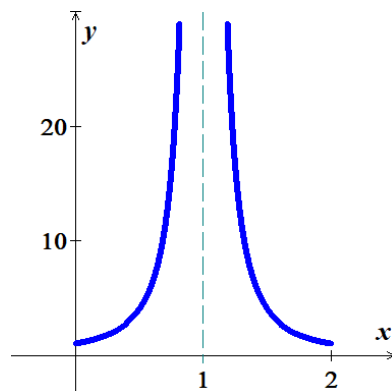
$$\int_0^2 \frac{dx}{(x-1)^2} = \int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2}$$

$$= \int_0^1 \frac{d(x-1)}{(x-1)^2} + \int_1^2 \frac{d(x-1)}{(x-1)^2}$$

$$= -\frac{1}{x-1} \Big|_0^1 - \frac{1}{x-1} \Big|_1^2$$

$$= -\infty - 1 - 1 + \infty$$

$$= \infty \quad \text{diverges}$$



$$\int_0^2 \frac{dx}{(x-1)^2} = -\frac{1}{x-1} \Big|_0^2$$

$$= -(1+1)$$

$$= -2$$

Exercise

Evaluate the integral $\int_{-1}^2 \frac{dx}{(x-1)^2}$

Solution

$$\begin{aligned}
\int_{-1}^2 \frac{dx}{(x-1)^2} &= \int_{-1}^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2} \\
&= \int_{-1}^1 \frac{d(x-1)}{(x-1)^2} + \int_1^2 \frac{d(x-1)}{(x-1)^2} \\
&= -\frac{1}{x-1} \bigg|_{-1}^1 - \frac{1}{x-1} \bigg|_1^2 \\
&= -\infty - \frac{1}{2} - 1 + \infty \\
&= \underline{\infty} \quad \text{diverges}
\end{aligned}$$

Exercise

Evaluate the integral $\int_1^{\infty} \frac{dx}{x \sqrt{x^2 - 1}}$

Solution

$$\begin{aligned}
\int_1^{\infty} \frac{dx}{x \sqrt{x^2 - 1}} &= \sec^{-1} x \bigg|_1^{\infty} \\
&= \sec^{-1} \infty - \sec^{-1} 1 \\
&= \frac{\pi}{2} - 0 \\
&= \underline{\frac{\pi}{2}}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\infty} x e^{-x^2} dx$

Solution

$$\begin{aligned}
\int_0^{\infty} x e^{-x^2} dx &= -\frac{1}{2} \int_0^{\infty} e^{-x^2} d(-x^2) \\
&= -\frac{1}{2} e^{-x^2} \bigg|_0^{\infty} \\
&= -\frac{1}{2} (0 - 1) \\
&= \underline{\frac{1}{2}}
\end{aligned}$$

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} x e^{-x^2} dx$

Solution

$$\begin{aligned}\int_{-\infty}^{\infty} x e^{-x^2} dx &= -\frac{1}{2} \int_{-\infty}^0 e^{-x^2} d(-x^2) - \frac{1}{2} \int_0^{\infty} e^{-x^2} d(-x^2) \\ &= -\frac{1}{2} e^{-x^2} \Big|_{-\infty}^0 - \frac{1}{2} e^{-x^2} \Big|_0^{\infty} \\ &= -\frac{1}{2}(1-0) - \frac{1}{2}(0-1) \\ &= -\frac{1}{2} + \frac{1}{2} \\ &= 0\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{\infty} x e^{-x^2} dx &= -\frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} d(-x^2) \\ &= -\frac{1}{2} e^{-x^2} \Big|_{-\infty}^{\infty} \\ &= -\frac{1}{2}(0-0) \\ &= 0\end{aligned}$$

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} \frac{x}{x^2+1} dx$

Solution

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x}{x^2+1} dx &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{x^2+1} d(x^2+1) \\ &= \frac{1}{2} \ln(x^2+1) \Big|_{-\infty}^{\infty} \\ &= \frac{1}{2}(\infty - \infty) \\ &= \infty \quad \text{diverges}\end{aligned}$$

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2}$

Solution

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2} &= \int_{-\infty}^{\infty} \frac{dx}{(x+1)^2 + 1} \\&= \int_{-\infty}^{\infty} \frac{d(x+1)}{(x+1)^2 + 1} \\&= \tan^{-1}(x+1) \Big|_{-\infty}^{\infty} \\&= \tan^{-1} \infty - \tan^{-1}(-\infty) \\&= \frac{\pi}{2} + \frac{\pi}{2} \\&= \pi\end{aligned}$$

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 6x + 12}$

Solution

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{x^2 + 6x + 12} &= \int_{-\infty}^{\infty} \frac{dx}{(x+3)^2 + 3} \\&= \int_{-\infty}^{\infty} \frac{d(x+3)}{(x+3)^2 + 3} \\&= \frac{1}{\sqrt{3}} \tan^{-1} \frac{x+3}{\sqrt{3}} \Big|_{-\infty}^{\infty} \\&= \frac{1}{\sqrt{3}} \left(\tan^{-1} \infty - \tan^{-1}(-\infty) \right) \\&= \frac{1}{\sqrt{3}} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) \\&= \frac{\pi}{\sqrt{3}}\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{2 - \sqrt{3x}}$

Solution

$$\text{Let } u = 2 - \sqrt{3x} \rightarrow \sqrt{3x} = 2 - u$$

$$du = -\frac{3}{2}(3x)^{-1/2} dx$$

$$dx = -\frac{2}{3}\sqrt{3x} du$$

$$= -\frac{2}{3}(2 - u) du$$

$$\begin{aligned} \int \frac{dx}{2 - \sqrt{3x}} &= -\frac{2}{3} \int \frac{1}{u}(2 - u) du \\ &= -\frac{2}{3} \int \left(\frac{2}{u} - 1 \right) du \\ &= -\frac{2}{3} (2 \ln|u| - u) + C \\ &= -\frac{2}{3} (2 \ln|2 - \sqrt{3x}| - 2 - \sqrt{3x}) + C \\ &= -\frac{2}{3} \left(\ln(2 - \sqrt{3x})^2 - 2 - \sqrt{3x} \right) + C \end{aligned}$$

Exercise

Evaluate the integral $\int \theta \cos(2\theta + 1) d\theta$

Solution

		$\int \cos(2\theta + 1) d\theta$
+	θ	$\frac{1}{2} \sin(2\theta + 1)$
-	1	$-\frac{1}{4} \cos(2\theta + 1)$

$$\int \theta \cos(2\theta + 1) d\theta = \frac{1}{2} \theta \sin(2\theta + 1) + \frac{1}{4} \cos(2\theta + 1) + C$$

Exercise

Evaluate the integral $\int \sqrt{x} \sqrt{1+\sqrt{x}} \, dx$

Solution

Let $u = \sqrt{x} \Rightarrow u^2 = x$
 $2u \, du = dx$

$$\begin{aligned} \int \sqrt{x} \sqrt{1+\sqrt{x}} \, dx &= \int u \sqrt{1+u} (2u \, du) \\ &= \int 2u^2 (1+u)^{1/2} \, du \end{aligned}$$

		$\int (1+u)^{1/2}$
+	$2u^2$	$\frac{2}{3}(1+u)^{3/2}$
-	$4u$	$\frac{4}{15}(1+u)^{5/2}$
+	4	$\frac{8}{105}(1+u)^{7/2}$

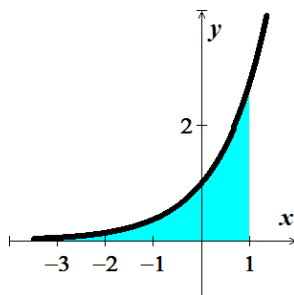
$$\begin{aligned} \int \sqrt{x} \sqrt{1+\sqrt{x}} \, dx &= \frac{4}{3}u^2 (1+u)^{3/2} - \frac{16}{15}u(1+u)^{5/2} + \frac{32}{105}(1+u)^{7/2} + C \\ &= \frac{4}{3}x(1+\sqrt{x})^{3/2} - \frac{16}{15}\sqrt{x}(1+\sqrt{x})^{5/2} + \frac{32}{105}(1+\sqrt{x})^{7/2} + C \end{aligned}$$

Exercise

Find the area of the unbounded shaded region $y = e^x, \quad -\infty < x \leq 1$

Solution

$$\begin{aligned} A &= \int_{-\infty}^1 e^x \, dx \\ &= e^x \Big|_{-\infty}^1 \\ &= e - 0 \\ &= e \text{ unit}^2 \end{aligned}$$



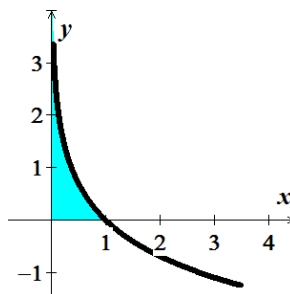
Exercise

Find the area of the unbounded shaded region

$$y = -\ln x$$

Solution

$$\begin{aligned} A &= -\int_0^1 \ln x \, dx \\ &= -(x \ln x - x) \Big|_0^1 \\ &= \underline{1 \text{ unit}^2} \end{aligned}$$



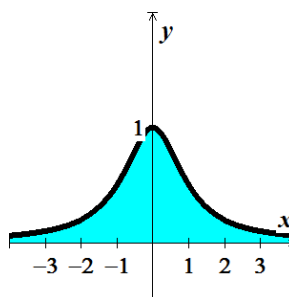
Exercise

Find the area of the unbounded shaded region

$$y = \frac{1}{x^2 + 1}$$

Solution

$$\begin{aligned} A &= \int_{-\infty}^{\infty} \frac{1}{x^2 + 1} \, dx \\ &= \arctan x \Big|_{-\infty}^{\infty} \\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \\ &= \underline{\pi \text{ unit}^2} \end{aligned}$$



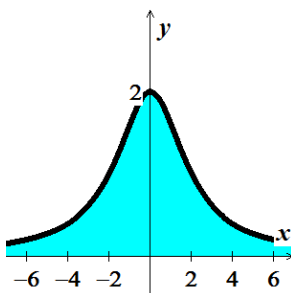
Exercise

Find the area of the unbounded shaded region

$$y = \frac{8}{x^2 + 4}$$

Solution

$$\begin{aligned} A &= \int_{-\infty}^{\infty} \frac{8}{x^2 + 4} \, dx \\ &= 4 \arctan \frac{x}{2} \Big|_{-\infty}^{\infty} \\ &= 4 \left(\frac{\pi}{2} + \frac{\pi}{2} \right) \\ &= \underline{4\pi \text{ unit}^2} \end{aligned}$$



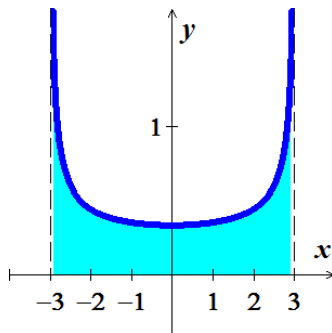
Exercise

Find the area of the region R between the graph of $f(x) = \frac{1}{\sqrt{9-x^2}}$ and the x -axis on the interval $(-3, 3)$

(if it exists)

Solution

$$\begin{aligned} A &= \int_{-3}^3 \frac{dx}{\sqrt{9-x^2}} \\ &= 2 \int_0^3 \frac{dx}{\sqrt{9-x^2}} \\ &= 2 \sin^{-1} \frac{x}{3} \Big|_0^3 \\ &= 2 \left(\sin^{-1} 1 - \sin^{-1} 0 \right) \\ &= \pi \text{ unit}^2 \end{aligned}$$



Exercise

Find the volume of the region bounded by $f(x) = (x^2 + 1)^{-1/2}$ and the x -axis on the interval $[2, \infty)$ is revolved about the x -axis.

Solution

$$\begin{aligned} V &= \pi \int_2^{\infty} \frac{1}{x^2 + 1} dx & V &= \pi \int_a^b (f(x))^2 dx \\ &= \pi \tan^{-1} x \Big|_2^{\infty} \\ &= \pi \left(\tan^{-1} \infty - \tan^{-1} 2 \right) \\ &= \pi \left(\frac{\pi}{2} - \tan^{-1} 2 \right) \text{ unit}^3 \end{aligned}$$

Exercise

Find the volume of the region bounded by $f(x) = \sqrt{\frac{x+1}{x^3}}$ and the x -axis on the interval $[1, \infty)$ is revolved about the x -axis.

Solution

$$\begin{aligned}
 V &= \pi \int_1^{\infty} \frac{x+1}{x^3} dx & V &= \pi \int_a^b (f(x))^2 dx \\
 &= \pi \int_1^{\infty} \left(\frac{1}{x^2} + x^{-3} \right) dx \\
 &= \pi \left(-\frac{1}{x} - \frac{1}{2} \frac{1}{x^2} \right) \Big|_1^{\infty} \\
 &= \pi \left(1 + \frac{1}{2} \right) \\
 &= \frac{3\pi}{2} \text{ unit}^3
 \end{aligned}$$

Exercise

Find the volume of the region bounded by $f(x) = (x+1)^{-3}$ and the x -axis on the interval $[0, \infty)$ is revolved about the y -axis.

Solution

$$\begin{aligned}
 V &= 2\pi \int_0^{\infty} x \frac{1}{(x+1)^3} dx & V &= 2\pi \int_a^b x \cdot f(x) dx \quad (\text{Shell method}) \\
 &= 2\pi \int_0^{\infty} \left(\frac{1}{(x+1)^2} - \frac{1}{(x+1)^3} \right) d(x+1) \\
 &\quad \frac{x}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} \\
 &\quad x = Ax^2 + 2Ax + A + Bx + B + C \\
 &\quad \begin{cases} A=0 \\ 2A+B=1 \rightarrow B=1 \\ B+C=0 \end{cases} \\
 &= 2\pi \left(\frac{-1}{x+1} + \frac{1}{2} \frac{1}{(x+1)^2} \right) \Big|_0^{\infty} \\
 &= 2\pi \left(1 - \frac{1}{2} \right) \\
 &= \pi \text{ unit}^3
 \end{aligned}$$

Exercise

Find the volume of the region bounded by $f(x) = \frac{1}{\sqrt{x} \ln x}$ and the x -axis on the interval $[2, \infty)$ is revolved about the x -axis.

Solution

$$\begin{aligned} V &= \pi \int_2^{\infty} \frac{1}{x \ln^2 x} dx & V &= \pi \int_a^b (f(x))^2 dx \\ &= \pi \int_2^{\infty} \frac{1}{\ln^2 x} d(\ln x) \\ &= \pi \left(-\frac{1}{\ln x} \right) \Big|_2^{\infty} \\ &= \pi \left(-0 + \frac{1}{\ln 2} \right) \\ &= \frac{\pi}{\ln 2} \text{ unit}^3 \end{aligned}$$

Exercise

Find the volume of the region bounded by $f(x) = \frac{\sqrt{x}}{\sqrt[3]{x^2 + 1}}$ and the x -axis on the interval $[0, \infty)$ is revolved about the x -axis.

Solution

$$\begin{aligned} V &= \pi \int_0^{\infty} \frac{x}{(x^2 + 1)^{2/3}} dx & V &= \pi \int_a^b (f(x))^2 dx \\ &= \frac{\pi}{2} \int_0^{\infty} (x^2 + 1)^{-2/3} d(x^2 + 1) \\ &= \frac{3\pi}{2} (x^2 + 1)^{1/3} \Big|_0^{\infty} \\ &= \frac{3\pi}{2} (\infty - 1) \\ &= \infty \text{ diverges} \end{aligned}$$

Therefore, the volume doesn't exist

Exercise

Find the volume of the region bounded by $f(x) = (x^2 - 1)^{-1/4}$ and the x -axis on the interval $(1, 2]$ is revolved about the y -axis.

Solution

$$\begin{aligned} V &= 2\pi \int_1^2 x (x^2 - 1)^{-1/4} dx & V &= 2\pi \int_a^b x \cdot f(x) dx \quad (\text{Shell method}) \\ &= \pi \int_1^2 (x^2 - 1)^{-1/4} d(x^2 - 1) \\ &= \frac{4\pi}{3} (x^2 - 1)^{3/4} \Big|_1^2 \\ &= \frac{4\pi}{3} (3)^{3/4} \\ &= \frac{4\pi}{3^{1/4}} \text{ unit}^3 \end{aligned}$$

Exercise

Find the volume of the region bounded by $f(x) = \tan x$ and the x -axis on the interval $\left[0, \frac{\pi}{2}\right)$ is revolved about the x -axis.

Solution

$$\begin{aligned} V &= \pi \int_0^{\pi/2} \tan^2 x dx & V &= \pi \int_a^b (f(x))^2 dx \\ &= \pi \int_0^{\pi/2} (\sec^2 x - 1) dx \\ &= \pi (\tan x - x) \Big|_0^{\pi/2} & \left(\tan \frac{\pi}{2} = \infty \right) \\ &= \infty \text{ diverges} \end{aligned}$$

Therefore, the volume doesn't exist

Exercise

Find the volume of the region bounded by $f(x) = -\ln x$ and the x -axis on the interval $(0, 1]$ is revolved about the x -axis.

Solution

$$V = \pi \int_0^1 \ln^2 x \, dx \qquad V = \pi \int_a^b (f(x))^2 \, dx$$

$$u = \ln x \quad dv = \ln x \, dx$$

$$du = \frac{dx}{x} \quad v = x \ln x - x$$

$$u = \ln x \quad dv = dx$$

$$du = \frac{dx}{x} \quad v = x$$

$$\begin{aligned} \int \ln x \, dx &= x \ln x - \int dx \\ &= x \ln x - x \end{aligned}$$

$$\begin{aligned} \int \ln^2 x \, dx &= \ln x (x \ln x - x) - \int (\ln x - 1) \, dx \\ &= x \ln^2 x - x \ln x - (x \ln x - x - x) \\ &= x \ln^2 x - 2x \ln x + 2x \end{aligned}$$

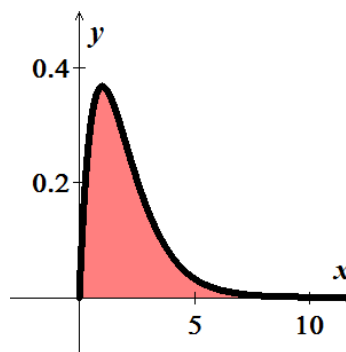
$$\begin{aligned} V &= \pi \left(x \ln^2 x - 2x \ln x + 2x \right) \Big|_0^1 \\ &= \underline{2\pi \text{ unit}^3} \end{aligned}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by the graphs of $y = xe^{-x}$, $y = 0$, and $x = 0$ about the x -axis.

Solution

$$\begin{aligned} V &= \pi \int_0^{\infty} (xe^{-x})^2 \, dx \\ &= \pi \int_0^{\infty} x^2 e^{-2x} \, dx \\ &= \pi e^{-2x} \left(-\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4} \right) \Big|_0^{\infty} \\ &= \pi \left(0 + \frac{1}{4} \right) \\ &= \underline{\frac{\pi}{4} \text{ unit}^3} \end{aligned}$$



Exercise

The region between the x -axis and the curve

$$f(x) = \begin{cases} 0, & x = 0 \\ x \ln x, & 0 < x \leq 2 \end{cases}$$

is revolved about the x -axis to generate the solid.

Find the volume of the solid.

Solution

$$V = \pi \int_0^2 y^2 dx$$

$$= \pi \int_0^2 (x \ln x)^2 dx$$

$$u = (\ln x)^2 \quad dv = x^2 dx$$

$$du = 2 \frac{\ln x}{x} dx \quad v = \frac{1}{3} x^3$$

$$= \frac{\pi}{3} x^3 (\ln x)^2 \Big|_0^2 - \frac{2\pi}{3} \int_0^2 x^2 \ln x dx$$

$$u = \ln x \quad dv = x^2 dx$$

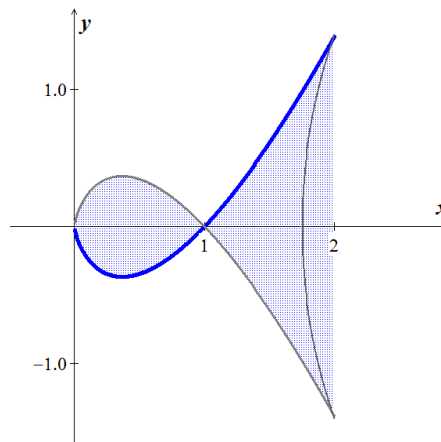
$$du = \frac{1}{x} dx \quad v = \frac{1}{3} x^3$$

$$= \frac{\pi}{3} \left(8(\ln 2)^2 - 0 \right) - \frac{2\pi}{9} x^3 \ln x \Big|_0^2 + \frac{2\pi}{9} \int_0^2 x^2 dx$$

$$= \frac{8\pi}{3} (\ln 2)^2 - \frac{2\pi}{9} (8 \ln 2 - 0) + \frac{2\pi}{27} x^3 \Big|_0^2$$

$$= \frac{8\pi}{3} (\ln 2)^2 - \frac{16\pi}{9} \ln 2 + \frac{16\pi}{27}$$

$$= \frac{8\pi}{3} \left((\ln 2)^2 - \frac{2}{3} \ln 2 + \frac{2}{9} \right) \text{ unit}^3$$



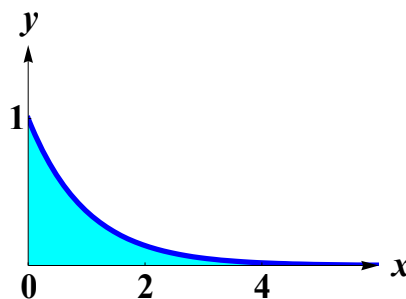
Exercise

Consider the region satisfying the inequalities $y \leq e^{-x}$, $y \geq 0$, $x \geq 0$

- Find the area of the region
- Find the volume of the solid generated by revolving the region about the x -axis.
- Find the volume of the solid generated by revolving the region about the y -axis.

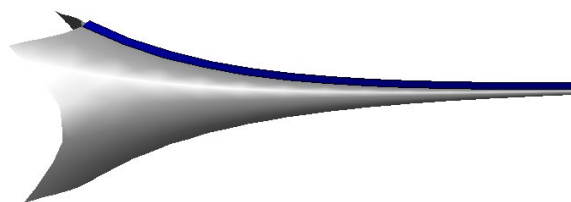
Solution

$$\begin{aligned}
 a) \quad A &= \int_0^{\infty} e^{-x} dx \\
 &= -e^{-x} \Big|_0^{\infty} \\
 &= -(0-1) \\
 &= \underline{1 \text{ unit}^2}
 \end{aligned}$$



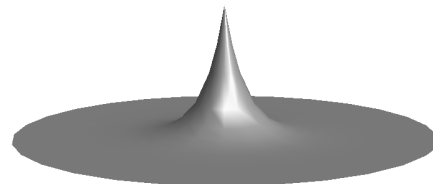
$$\begin{aligned}
 b) \quad V &= \pi \int_0^{\infty} (e^{-x})^2 dx \\
 &= \pi \int_0^{\infty} e^{-2x} dx \\
 &= -\frac{\pi}{2} e^{-2x} \Big|_0^{\infty} \\
 &= -\frac{\pi}{2} (0-1) \\
 &= \underline{\frac{\pi}{2} \text{ unit}^3}
 \end{aligned}$$

$$V = \pi \int_a^b (R(x)^2 - r(x)^2) dx \quad (\text{Disk Method})$$



$$\begin{aligned}
 c) \quad V &= 2\pi \int_0^{\infty} x e^{-x} dx \\
 &= -2\pi (e^{-x}(x+1)) \Big|_0^{\infty} \\
 &= -2\pi (0-1) \\
 &= \underline{2\pi \text{ unit}^3}
 \end{aligned}$$

$$V = 2\pi \int_a^b x f(x) dx \quad (\text{Shell Method})$$



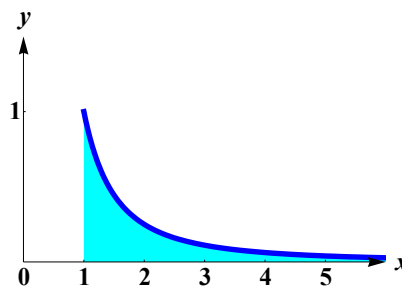
Exercise

Consider the region satisfying the inequalities $y \leq \frac{1}{x^2}$, $y \geq 0$, $x \geq 1$

- Find the area of the region
- Find the volume of the solid generated by revolving the region about the x -axis.
- Find the volume of the solid generated by revolving the region about the y -axis.

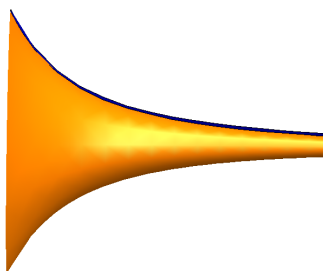
Solution

$$\begin{aligned} a) \quad A &= \int_1^{\infty} \frac{1}{x^2} dx \\ &= -\frac{1}{x} \Big|_1^{\infty} \\ &= -(0-1) \\ &= \underline{1 \text{ unit}^2} \end{aligned}$$



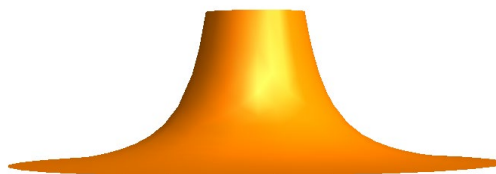
$$\begin{aligned} b) \quad V &= \pi \int_0^{\infty} \left(\frac{1}{x^2} \right)^2 dx \\ &= \pi \int_0^{\infty} x^{-4} dx \\ &= -\frac{\pi}{3x^3} \Big|_1^{\infty} \\ &= -\frac{\pi}{3}(0-1) \\ &= \underline{\frac{\pi}{3} \text{ unit}^3} \end{aligned}$$

$$V = \pi \int_a^b (R(x)^2 - r(x)^2) dx \quad (\text{Disk Method})$$



$$\begin{aligned} c) \quad V &= 2\pi \int_0^{\infty} x \left(\frac{1}{x^2} \right) dx \\ &= 2\pi \int_0^{\infty} \frac{1}{x} dx \\ &= 2\pi \ln x \Big|_1^{\infty} \\ &= \underline{\infty} \quad \text{Diverges} \end{aligned}$$

$$V = 2\pi \int_a^b x f(x) dx \quad (\text{Shell Method})$$



Exercise

Find the perimeter of the hypocycloid of four cusps $x^{2/3} + y^{2/3} = 4$

Solution

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}$$

$$\begin{aligned}
 \sqrt{1+(y')^2} &= \sqrt{1+\frac{y^{2/3}}{x^{2/3}}} \\
 &= \frac{\sqrt{x^{2/3}+y^{2/3}}}{x^{1/3}} \\
 &= \frac{\sqrt{4}}{x^{1/3}} \\
 &= 2x^{-1/3}
 \end{aligned}$$

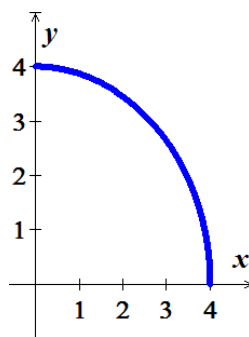
$$\begin{aligned}
 S &= 4 \int_0^8 2x^{-1/3} dx \\
 &= 12x^{2/3} \Big|_0^8 \\
 &= 12(4-0) \\
 &= \underline{48 \text{ unit}^2}
 \end{aligned}$$

Exercise

Find the arc length of the graph $y = \sqrt{16-x^2}$ over the interval $[0, 4]$

Solution

$$\begin{aligned}
 y' &= -\frac{x}{\sqrt{16-x^2}} \\
 \sqrt{1+(y')^2} &= \sqrt{1+\frac{x^2}{16-x^2}} \\
 &= \frac{4}{\sqrt{16-x^2}} \\
 L &= \int_0^4 \frac{4}{\sqrt{16-x^2}} dx \\
 &= 4 \arcsin \frac{x}{4} \Big|_0^4 \\
 &= 4(\arcsin 1 - \arcsin 0) \\
 &= 4\left(\frac{\pi}{2}\right) \\
 &= \underline{2\pi \text{ unit}}
 \end{aligned}$$



Exercise

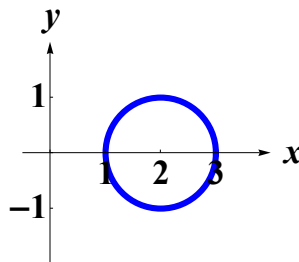
The region bounded by $(x-2)^2 + y^2 = 1$ is revolved about the y -axis to form a torus. Find the surface area of the torus.

Solution

$$2(x-2) + 2yy' = 0$$

$$y' = -\frac{x-2}{y}$$

$$\begin{aligned}\sqrt{1+(y')^2} &= \sqrt{1+\frac{(x-2)^2}{y^2}} \\ &= \sqrt{\frac{y^2+(x-2)^2}{y^2}} \quad (x-2)^2 + y^2 = 1 \\ &= \frac{1}{y} \\ &= \frac{1}{\sqrt{1-(x-2)^2}}\end{aligned}$$



$$S = 4\pi \int_1^3 \frac{x}{\sqrt{1-(x-2)^2}} dx$$

$$= 4\pi \int_1^3 \frac{x-2+2}{\sqrt{1-(x-2)^2}} dx$$

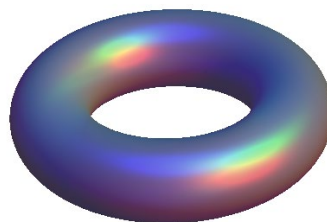
$$= 4\pi \int_1^3 \frac{x-2}{\sqrt{1-(x-2)^2}} dx + 4\pi \int_1^3 \frac{2}{\sqrt{1-(x-2)^2}} dx$$

$$= -2\pi \int_1^3 \left(1-(x-2)^2\right)^{-1/2} d\left(1-(x-2)^2\right) + 8\pi \arctan(x-2) \Big|_1^3$$

$$= -4\pi \sqrt{1-(x-2)^2} \Big|_1^3 + 8\pi (\arctan(1) - \arctan(-1))$$

$$= -4\pi(0-0) + 8\pi\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= 8\pi^2 \text{ unit}^2$$



Exercise

Find the surface area formed by revolving the graph $y = 2e^{-x}$ on the interval $[0, \infty)$ about the x -axis

Solution

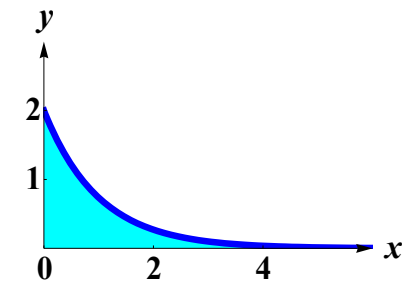
$$y' = -2e^{-x}$$

$$\sqrt{1+(y')^2} = \sqrt{1+4e^{-2x}}$$

$$S = 2\pi \int_0^{\infty} 2e^{-x} \sqrt{1+4e^{-2x}} dx$$

$$= -4\pi \int_0^{\infty} \sqrt{1+4(e^{-x})^2} d(e^{-x})$$

$$\int \sqrt{1+4u^2} du = \frac{1}{2} \int \sec^3 \theta d\theta$$



$$\begin{aligned} 2u &= \tan \theta & \sqrt{4u^2 + 1} &= \sec \theta \\ du &= \frac{1}{2} \sec^2 \theta d\theta \end{aligned}$$

$$\begin{aligned} u &= \sec \theta & dv &= \sec^2 \theta d\theta \\ du &= \sec \theta \tan \theta d\theta & v &= \tan \theta \end{aligned}$$

$$\int \sec^3 \theta d\theta = \sec \theta \tan \theta - \int \tan \theta (\sec \theta \tan \theta d\theta)$$

$$= \sec \theta \tan \theta - \int \tan^2 \theta \sec \theta d\theta$$

$$= \sec \theta \tan \theta - \int (\sec^2 \theta - 1) \sec \theta d\theta$$

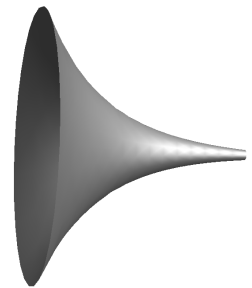
$$2 \int \sec^3 \theta d\theta = \sec \theta \tan \theta + \int \sec \theta d\theta$$

$$\int \sqrt{1+4u^2} du = \frac{1}{4} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|)$$

$$S = -\pi \left(2e^{-x} \sqrt{1+4e^{-2x}} + \ln \left| 2e^{-x} + \sqrt{1+4e^{-2x}} \right| \right) \Big|_0^{\infty}$$

$$= -\pi (-2\sqrt{5} + \ln(2 + \sqrt{5}))$$

$$= \pi (2\sqrt{5} - \ln(2 + \sqrt{5})) \quad \text{unit}^2$$



Exercise

The magnetic potential P at a point on the axis of a circular coil is given by

$$P = \frac{2\pi NI r}{k} \int_c^{\infty} \frac{1}{(r^2 + x^2)^{3/2}} dx$$

Where N , I , r , k , and c are constants. Find P .

Solution

$$\text{Let } K = \frac{2\pi NI r}{k}$$

$$P = K \int_c^{\infty} \frac{1}{(r^2 + x^2)^{3/2}} dx$$

$$= K \int_c^{\infty} \frac{r \sec^2 \theta}{r^3 \sec^3 \theta} d\theta$$

$$= \frac{K}{r^2} \int_c^{\infty} \cos \theta d\theta$$

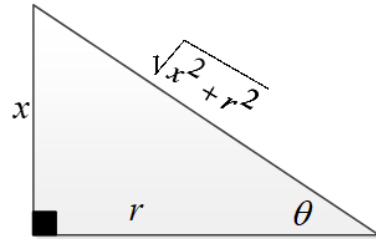
$$= \frac{K}{r^2} \sin \theta \Big|_c^{\infty}$$

$$= \frac{K}{r^2} \frac{x}{\sqrt{r^2 + x^2}} \Big|_c^{\infty}$$

$$= \frac{K}{r^2} \left(1 - \frac{c}{\sqrt{r^2 + c^2}} \right)$$

$$= \frac{2\pi NI \left(\sqrt{r^2 + c^2} - c \right)}{kr \sqrt{r^2 + c^2}}$$

$$\begin{aligned} x &= r \tan \theta & \sqrt{x^2 + r^2} &= r \sec \theta \\ dx &= r \sec^2 \theta d\theta \end{aligned}$$



Exercise

A “semi-infinite” uniform rod occupies the nonnegative x -axis. The rod has a linear density δ , which means that a segment of length dx has a mass of δdx . A particle of mass M is located at the point $(-a, 0)$. The gravitational force F that the rod exerts on the mass is given by

$$F = \int_0^{\infty} \frac{GM\delta}{(a+x)^2} dx$$

Where G is the gravitational constant. Find F .

Solution

$$\begin{aligned}
 F &= \int_0^{\infty} \frac{GM\delta}{(a+x)^2} dx \\
 &= -\frac{GM\delta}{a+x} \Bigg|_0^{\infty} \\
 &= \frac{GM\delta}{a}
 \end{aligned}$$

Exercise

Let R be the region bounded by the graph of $f(x) = x^{-p}$ and the x -axis

- a) Let S be the solid generated when R is revolved about the x -axis. For what values of p is the volume of S finite for $0 < x \leq 1$?
- b) Let S be the solid generated when R is revolved about the y -axis. For what values of p is the volume of S finite for $0 < x \leq 1$?
- c) Let S be the solid generated when R is revolved about the x -axis. For what values of p is the volume of S finite for $x \geq 1$?
- d) Let S be the solid generated when R is revolved about the y -axis. For what values of p is the volume of S finite for $x \geq 1$?

Solution

$$\begin{aligned} \text{a) } V &= \pi \int_0^1 \left(x^{-p}\right)^2 dx & V &= \pi \int_a^b f(x)^2 dx \\ &= \pi \int_0^1 x^{-2p} dx \\ &= \pi \left. \frac{x^{-2p+1}}{1-2p} \right|_0^1 \\ &= \frac{\pi}{1-2p} \left(1 - 0^{-2p+1}\right) \end{aligned}$$

The volume of S finite when $1-2p > 0 \Rightarrow \underline{p < \frac{1}{2}}$

$$\begin{aligned} \text{b) } V &= 2\pi \int_0^1 x \cdot x^{-p} dx & V &= 2\pi \int_a^b xf(x) dx \\ &= 2\pi \int_0^1 x^{1-p} dx \\ &= \frac{2\pi}{2-p} x^{2-p} \Big|_0^1 \\ &= \frac{2\pi}{2-p} \left(1 - 0^{2-p}\right) \end{aligned}$$

The volume of S finite when $2-p > 0 \Rightarrow \underline{p < 2}$

$$\begin{aligned} \text{c) } V &= \pi \int_1^\infty \left(x^{-p}\right)^2 dx & V &= \pi \int_a^b f(x)^2 dx \\ &= \pi \int_1^\infty x^{-2p} dx \end{aligned}$$

$$= \pi \frac{x^{-2p+1}}{1-2p} \Big|_1^{\infty}$$

$$= \frac{\pi}{1-2p} \left(\infty^{1-2p} - 1 \right)$$

The volume of S finite when $1-2p < 0 \Rightarrow \underline{p > \frac{1}{2}} \quad \left(\frac{1}{\infty} = 0 \right)$

$$d) \quad V = 2\pi \int_0^1 x \cdot x^{-p} dx$$

$$V = 2\pi \int_a^b xf(x) dx$$

$$= 2\pi \int_0^1 x^{1-p} dx$$

$$= \frac{2\pi}{2-p} x^{2-p} \Big|_0^1$$

$$= \frac{2\pi}{2-p} \left(1 - 0^{2-p} \right)$$

The volume of S finite when $2-p > 0 \Rightarrow \underline{p < 2}$

Exercise

The solid formed by revolving (about the x -axis) the unbounded region lying between the graph of $f(x) = \frac{1}{x}$ and the x -axis ($x \geq 1$) is called **Gabriel's Horn**.

Show that this solid has a finite volume and an infinite surface area

Solution

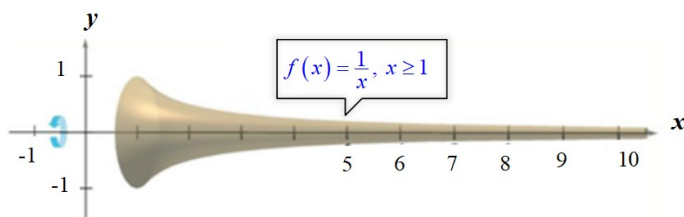
$$V = \pi \int_1^{\infty} \frac{1}{x^2} dx$$

$$V = \pi \int_a^b (f(x))^2 dx \text{ (disk method)}$$

$$= -\pi \frac{1}{x} \Big|_1^{\infty}$$

$$= -\pi (0 - 1)$$

$$= \pi \text{ unit}^3$$



$$f'(x) = -\frac{1}{x^2}$$

$$S = 2\pi \int_1^{\infty} \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx$$

$$S = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx$$

Since $1 + \frac{1}{x^4} > 1$ and $\int_1^{\infty} \frac{1}{x} dx$ diverges

Therefore the surface area is *infinite*.

Exercise

Water is drained from a 3000-gal tank at a rate that starts at 100 gal/hr. and decreases continuously by 5% /hr. If the drain left open indefinitely, how much water drains from the tank? Can a full tank be emptied at this rate?

Solution

$$\begin{aligned} \text{Rate of the drain water: } r(t) &= 100(1 - .05)^t \\ &= 100(0.95)^t \\ &= 100 e^{(\ln 0.95)t} \end{aligned}$$

Total water amount drained:

$$\begin{aligned} D &= \int_0^{\infty} 100 e^{(\ln 0.95)t} dt \\ &= \frac{100}{\ln 0.95} e^{(\ln 0.95)t} \Big|_0^{\infty} \\ &= \frac{100}{\ln 0.95} (0 - 1) \\ &\quad \ln 0.95 < 0 \xrightarrow[t \rightarrow \infty]{} e^{(\ln 0.95)t} = e^{-\infty} = 0 \\ &= -\frac{100}{\ln 0.95} \approx 1950 \text{ gal} \end{aligned}$$

Since 1950 gal < 3000 gal which it takes infinite time.

Therefore; the full 3,000-gallon tank cannot be emptied at this rate.

Exercise

Let $I(a) = \int_0^{\infty} \frac{dx}{(1+x^a)(1+x^2)}$, where a is a real number.

a) Evaluate $I(a)$ and show that its value is independent of a .

(**Hint:** split the integral into two integrals over $[0, 1]$ and $[1, \infty)$; then use a change of variables to convert the second integral into an integral over $[0, 1]$.)

b) Let f be any positive continuous function on $\left[0, \frac{\pi}{2}\right]$

Evaluate
$$\int_0^{\pi/2} \frac{f(\cos x)}{f(\cos x) + f(\sin x)} dx$$

(**Hint:** Use the identity $\cos\left(\frac{\pi}{2} - x\right) = \sin x$)

Solution

$$\begin{aligned} \text{a) } I(a) &= \int_0^{\infty} \frac{dx}{(1+x^a)(1+x^2)} \\ &= \int_0^1 \frac{dx}{(1+x^a)(1+x^2)} + \int_1^{\infty} \frac{dx}{(1+x^a)(1+x^2)} \end{aligned}$$

$$\text{Let } u = \frac{1}{x} \Rightarrow x = \frac{1}{u}$$

$$dx = -\frac{1}{u^2} du$$

$$x=1 \rightarrow u=1$$

$$x=\infty \rightarrow u=0$$

$$\begin{aligned} I(a) &= \int_0^1 \frac{dx}{(1+x^a)(1+x^2)} - \int_1^0 \frac{du}{u^2 \left(1+u^{-a}\right) \left(1+u^{-2}\right)} \\ &= \int_0^1 \frac{dx}{(1+x^a)(1+x^2)} + \int_0^1 \frac{du}{u^2 \left(1+\frac{1}{u^a}\right) \left(1+\frac{1}{u^2}\right)} \\ &= \int_0^1 \frac{dx}{(1+x^a)(1+x^2)} + \int_0^1 \frac{u^a du}{(1+u^a)(1+u^2)} \quad (x=u) \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \frac{dx}{(1+x^a)(1+x^2)} + \int_0^1 \frac{x^a du}{(1+x^a)(1+x^2)} \\
&= \int_0^1 \frac{1+x^a}{(1+x^a)(1+x^2)} dx \\
&= \int_0^1 \frac{dx}{1+x^2} \\
&= \tan^{-1} x \Big|_0^1 \\
&= \tan^{-1} 1 - \tan^{-1} 0 \\
&= \frac{\pi}{4} - 0 \\
&= \frac{\pi}{4}
\end{aligned}$$

$$b) \quad I = \int_0^{\pi/2} \frac{f(\cos x)}{f(\cos x) + f(\sin x)} dx$$

$$\begin{aligned}
\text{Let } u = \frac{\pi}{2} - x &\Rightarrow x = \frac{\pi}{2} - u \\
&dx = -du
\end{aligned}$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$x = 1 \rightarrow u = 1$$

$$x = \infty \rightarrow u = 0$$

$$\begin{aligned}
I &= - \int_{\pi/2}^0 \frac{f(\sin u)}{f(\sin u) + f(\cos u)} du \\
&= \int_0^{\pi/2} \frac{f(\sin u)}{f(\sin u) + f(\cos u)} du
\end{aligned}$$

$$\begin{aligned}
2I &= \int_0^{\pi/2} \frac{f(\cos x)}{f(\cos x) + f(\sin x)} dx + \int_0^{\pi/2} \frac{f(\sin x)}{f(\sin x) + f(\cos x)} dx \\
&= \int_0^{\pi/2} dx \\
&= x \Big|_0^{\pi/2}
\end{aligned}$$

$$= \frac{\pi}{2} \Big|$$

$$I = \frac{\pi}{4} \Big|$$

Exercise

Let R be the region bounded by $y = \ln x$, the x -axis, and the line $x = a$, where $a > 1$.

- Find the volume $V_1(a)$ of the solid generated when R is revolved about the x -axis (as a function of a).
- Find the volume $V_2(a)$ of the solid generated when R is revolved about the y -axis (as a function of a).
- Graph V_1 and V_2 . For what values of $a > 1$ is $V_1(a) > V_2(a)$?

Solution

$$a) \quad V_1(a) = \pi \int_1^a (\ln x)^2 dx$$

$$\text{Let } z = \ln x \Rightarrow x = e^z$$

$$dx = e^z dz$$

$$V_1(a) = \pi \int_1^a z^2 e^z dz$$

		$\int e^z dz$
+	z^2	e^z
-	$2z$	e^z
+	2	e^z

$$V_1(a) = \pi e^z (z^2 - 2z + 2) \Big|_1^a$$

$$= \pi x ((\ln x)^2 - 2 \ln x + 2) \Big|_1^a$$

$$= \pi \left[a \left((\ln a)^2 - 2 \ln a + 2 \right) - (\ln 1 - 2 \ln 1 + 2) \right]$$

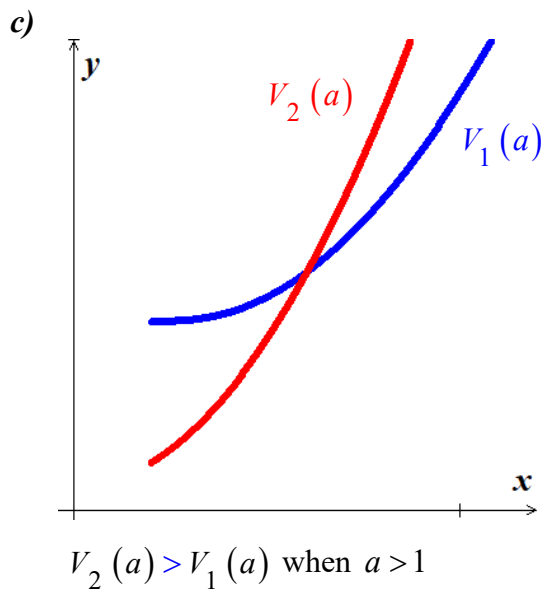
$$= \pi \left(a \ln^2 a - 2a \ln a + 2a + 2 \right) \text{ unit}^3 \Big|$$

- About y -axis

$$V_2(a) = 2\pi \int_1^a x \ln x \, dx$$

		$\int x \, dx$
+	$\ln x$	$\frac{1}{2}x^2$
-	$\frac{1}{x}$	$\int \frac{1}{2}x^2 \, dx$

$$\begin{aligned}
 V_2(a) &= 2\pi \left(\frac{1}{2}x^2 \ln x \Big|_1^a - \frac{1}{2} \int_1^a x \, dx \right) \\
 &= 2\pi \left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 \Big|_1^a \right) \\
 &= \pi \left(a^2 \ln a - \frac{1}{2}a^2 - \ln 1 + \frac{1}{2} \right) \\
 &= \frac{\pi}{2} (2a^2 \ln a - a^2 + 1) \quad \text{unit}^3
 \end{aligned}$$



Exercise

Let R be the region bounded by the graph of $f(x) = x^{-p}$ and the x -axis, for $x \geq 1$. Let V_1 and V_2 be the volumes of the solids generated when R is revolved about the x -axis and the y -axis, respectively, if they exist.

a) For what values of p (if any) is $V_1 = V_2$?

b) Repeat part (a) on the interval $(0, 1]$.

Solution

a) $p = ?$ if $V_1 = V_2$

$$\begin{aligned} V_1 &= \pi \int_1^{\infty} (x^{-p})^2 dx \\ &= \pi \int_1^{\infty} x^{-2p} dx \\ &= \frac{\pi}{1-2p} x^{-2p+1} \Big|_1^{\infty} \end{aligned}$$

$$\text{If } -2p+1 \geq 0 \rightarrow p \leq \frac{1}{2}$$

$$\underline{V_1 = \infty}$$

If $p > \frac{1}{2}$

$$\begin{aligned} V_1 &= \frac{\pi}{1-2p} (0-1) \\ &= \underline{\frac{\pi}{2p-1} \text{ unit}^3} \end{aligned}$$

$$\begin{aligned} V_2 &= 2\pi \int_1^{\infty} x^{1-p} dx \\ &= \frac{2\pi}{2-p} x^{2-p} \Big|_1^{\infty} \end{aligned}$$

$$\text{If } 2-p \geq 0 \rightarrow p \leq 2$$

$$\underline{V_2 = \infty}$$

If $p > 2$

$$\begin{aligned} V_2 &= \frac{2\pi}{2-p} (0-1) \\ &= \underline{\frac{2\pi}{p-2} \text{ unit}^3} \end{aligned}$$

$$V_1 = V_2$$

$$\frac{\pi}{2p-1} = \frac{2\pi}{2-p}$$

$$\frac{1}{2p-1} = \frac{2}{2-p}$$

$$2-p = 4p-2$$

$$5p = 4$$

$$p = \frac{4}{5} < 2$$

$\therefore V_1 = V_2$ only when both volumes are infinite.

$$\begin{aligned} b) \quad V_1 &= \pi \int_0^1 x^{-2p} dx \\ &= \frac{\pi}{1-2p} x^{-2p+1} \Big|_0^1 \\ &= \frac{\pi}{1-2p} (1-0) \\ &= \frac{\pi}{1-2p} \text{ unit}^3 \end{aligned}$$

$$\begin{aligned} V_2 &= 2\pi \int_0^1 x^{1-p} dx \\ &= \frac{2\pi}{2-p} x^{2-p} \Big|_0^1 \\ &= \frac{2\pi}{2-p} (1-0) \\ &= \frac{2\pi}{2-p} \text{ unit}^3 \end{aligned}$$

$$\frac{\pi}{1-2p} = \frac{2\pi}{2-p}$$

$$\frac{1}{1-2p} = \frac{2}{2-p}$$

$$2-p = 2-4p$$

$$p = 0$$

$\therefore V_1 \neq V_2$ for any p .

Exercise

Let R_1 be the region bounded by the graph of $y = e^{-ax}$ and the x -axis on the interval $[0, b]$ where $a > 0$ and $b > 0$. Let R_2 be the region bounded by the graph of $y = e^{-ax}$ and the x -axis on the interval $[b, \infty)$. Let V_1 and V_2 be the volumes of the solids generated when R_1 and R_2 are revolved about the x -axis. Find and graph the relationship between a and b for which $V_1 = V_2$.

Solution

Given: $R_1 : y = e^{-ax} \quad [0, b]$

$R_2 : y = e^{-ax} \quad [b, \infty]$

$$\begin{aligned} V_1 &= \pi \int_0^b \left(e^{-ax}\right)^2 dx \\ &= \pi \int_0^b e^{-2ax} dx \\ &= -\frac{\pi}{2a} e^{-2ax} \Big|_0^b \\ &= -\frac{\pi}{2a} \left(e^{-2ab} - 1\right) \text{ unit}^3 \end{aligned}$$

$$\begin{aligned} V_2 &= \pi \int_b^{\infty} \left(e^{-ax}\right)^2 dx \\ &= \pi \int_b^{\infty} e^{-2ax} dx \\ &= -\frac{\pi}{2a} e^{-2ax} \Big|_b^{\infty} \\ &= \frac{\pi}{2a} e^{-2ab} \text{ unit}^3 \end{aligned}$$

$$V_1 = V_2$$

$$-\frac{\pi}{2a} \left(e^{-2ab} - 1\right) = \frac{\pi}{2a} e^{-2ab}$$

$$1 - e^{-2ab} = e^{-2ab}$$

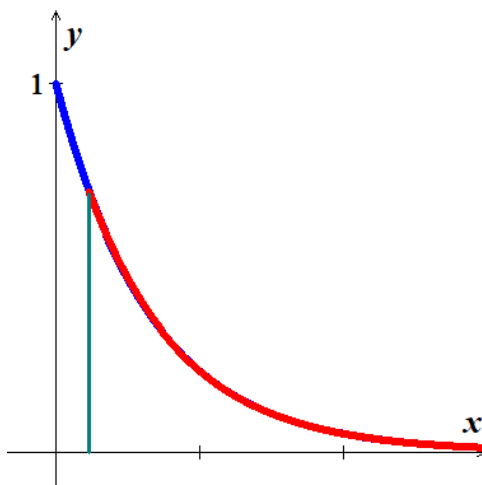
$$2e^{-2ab} = 1$$

$$\frac{1}{e^{2ab}} = \frac{1}{2}$$

$$e^{2ab} = 2$$

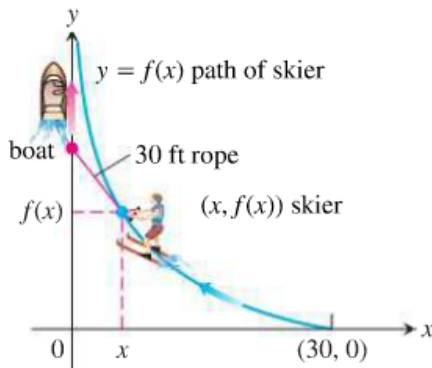
$$2ab = \ln 2$$

$$ab = \frac{1}{2} \ln 2$$



Exercise

Suppose that a boat is positioned at the origin with a water skier tethered to the boat at the point $(30, 0)$ on a rope 30 ft. long. As the boat travels along the positive y-axis, the skier is pulled behind the boat along an unknown path $y = f(x)$, as shown



a) Show that $f'(x) = \frac{-\sqrt{900-x^2}}{x}$

(Hint: Assume that the skier is always pointed directly at the boat and the rope is on line is on a line tangent to the path $y = f(x)$.)

b) Solve the equation in part (a) for $f(x)$, using $f(30) = 0$

Solution

a) From the triangle: $h^2 + x^2 = 30^2 \Rightarrow h = \sqrt{900 - x^2}$

The slope of the tangent line (line is going down) is: $m = \frac{-\sqrt{900-x^2}}{x}$

Thus, $f'(x) = \frac{-\sqrt{900-x^2}}{x}$

$$b) \quad f(x) = \int \frac{-\sqrt{900-x^2}}{x} dx$$

$$x = 30 \sin \theta \rightarrow dx = 30 \cos \theta d\theta, \quad 0 < \theta < \frac{\pi}{2}$$

$$\sqrt{900-x^2} = \sqrt{900-900 \sin^2 \theta} = 30 \cos \theta$$

$$f(x) = - \int \frac{30 \cos \theta}{30 \sin \theta} (30 \cos \theta) d\theta$$

$$= -30 \int \frac{\cos^2 \theta}{\sin \theta} d\theta$$

$$= -30 \int \frac{1 - \sin^2 \theta}{\sin \theta} d\theta$$

$$= -30 \int (\csc \theta - \sin \theta) d\theta$$

$$= -\ln |\csc \theta + \cot \theta| - 30 \cos \theta + C$$

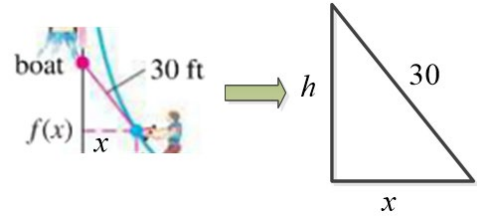
$$= -\ln \left| \frac{30}{x} + \frac{\sqrt{900-x^2}}{x} \right| - \sqrt{900-x^2} + C$$

Given (30, 0), then

$$0 = -\ln \left| \frac{30}{30} + \frac{\sqrt{900-30^2}}{30} \right| - \sqrt{900-30^2} + C$$

$$0 = -\ln |1| + C \Rightarrow \underline{C=0}$$

$$f(x) = -\ln \left| \frac{30}{x} + \frac{\sqrt{900-x^2}}{x} \right| - \sqrt{900-x^2}$$



Exercise

Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the substance B at time $t = 0$, and if x is the amount of product at time t , then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt} = k(a-x)(b-x) \quad \text{or} \quad \frac{1}{(a-x)(b-x)} \frac{dx}{dt} = k$$

Where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t .

a) If $a = b$

b) If $a \neq b$

Assume in each case that $x = 0$ when $t = 0$

Solution

$$\frac{1}{(a-x)(b-x)} dx = k dt$$

$$a) \quad a = b \Rightarrow \frac{1}{(a-x)^2} dx = k dt$$

$$\int \frac{1}{(a-x)^2} dx = \int k dt$$

$$\frac{1}{a-x} = kt + C$$

$$x(t=0) = 0 \Rightarrow \left. \frac{1}{a} = C \right|$$

$$\frac{1}{a-x} = kt + \frac{1}{a} = \frac{k at + 1}{a}$$

$$a-x = \frac{a}{k at + 1}$$

$$x = a - \frac{a}{k at + 1}$$

$$= \frac{a^2 kt}{k at + 1} \Big|$$

$$b) \quad a \neq b \Rightarrow \frac{1}{(a-x)(b-x)} dx = k dt$$

$$\int \frac{1}{(a-x)(b-x)} dx = \int k dt$$

$$\frac{1}{(a-x)(b-x)} = \frac{A}{a-x} + \frac{B}{b-x}$$

$$\begin{cases} -A - B = 0 \\ bA + aB = 1 \end{cases} \rightarrow \begin{cases} B = \frac{1}{a-b} \\ A = -\frac{1}{a-b} \end{cases}$$

$$\frac{-1}{a-b} \int \frac{1}{a-x} dx + \frac{1}{a-b} \int \frac{1}{b-x} dx = \int k dt$$

$$\frac{1}{a-b} \ln|a-x| - \frac{1}{a-b} \ln|b-x| = kt + C$$

$$\frac{1}{a-b} \ln \left| \frac{a-x}{b-x} \right| = kt + C$$

$$x(0) = 0 \Rightarrow \underline{\frac{1}{a-b} \ln \left(\frac{a}{b} \right) = C}$$

$$\frac{1}{a-b} \ln \left| \frac{a-x}{b-x} \right| = kt + \frac{1}{a-b} \ln \left(\frac{a}{b} \right)$$

$$\ln \left| \frac{a-x}{b-x} \right| = (a-b)kt + \ln \left(\frac{a}{b} \right)$$

$$\frac{a-x}{b-x} = e^{(a-b)kt + \ln \left(\frac{a}{b} \right)}$$

$$\frac{a-x}{b-x} = \frac{a}{b} e^{(a-b)kt}$$

$$a-x = b \frac{a}{b} e^{(a-b)kt} - x \frac{a}{b} e^{(a-b)kt}$$

$$x \left(\frac{a}{b} e^{(a-b)kt} - 1 \right) = a e^{(a-b)kt} - a$$

$$\underline{x = \frac{abe^{(a-b)kt} - ab}{ae^{(a-b)kt} - b}}$$