Solution Section 3.1 – Proving Identities

Exercise

Prove the identity $\cos\theta\cot\theta+\sin\theta=\csc\theta$

Solution

$$\begin{aligned}\cos\theta\cot\theta+\sin\theta &= \cos\theta\frac{\cos\theta}{\sin\theta}+\sin\theta\\ &= \frac{\cos^2\theta}{\sin\theta}+\sin\theta\\ &= \frac{\cos^2\theta}{\sin\theta}+\sin\theta\frac{\sin\theta}{\sin\theta}\\ &= \frac{\cos^2\theta}{\sin\theta}+\frac{\sin^2\theta}{\sin\theta}\\ &= \frac{\cos^2\theta+\sin^2\theta}{\sin\theta}\\ &= \frac{1}{\sin\theta}\\ &= \csc\theta \quad \checkmark\end{aligned}$$

Exercise

Prove the identity $\sec\theta\cot\theta-\sin\theta=\dfrac{\cos^2\theta}{\sin\theta}$

Solution

$$\begin{aligned}\sec\theta\cot\theta-\sin\theta &= \frac{1}{\cos\theta}\frac{\cos\theta}{\sin\theta}-\sin\theta\\ &= \frac{1}{\sin\theta}-\sin\theta\\ &= \frac{1-\sin^2\theta}{\sin\theta}\\ &= \frac{\cos^2\theta}{\sin\theta} \quad \checkmark\end{aligned}$$

Exercise

Prove the identity $\dfrac{\csc\theta\tan\theta}{\sec\theta}=1$

Solution

$$\frac{\csc\theta\tan\theta}{\sec\theta}=\csc\theta\tan\theta\frac{1}{\sec\theta}$$

$$= \frac{1}{\sin\theta} \frac{\sin\theta}{\cos\theta} \cos\theta$$

$$\underline{= 1} \quad \checkmark$$

Exercise

Prove the identity $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

Solution

$$(\sin\theta + \cos\theta)^2 = \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$$

$$\underline{= 1 + 2\sin\theta\cos\theta} \quad \checkmark$$

Exercise

Prove the identity $\sin\theta(\sec\theta + \cot\theta) = \tan\theta + \cos\theta$

Solution

$$\begin{aligned} \sin\theta(\sec\theta + \cot\theta) &= \sin\theta\left(\frac{1}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right) \\ &= \sin\theta\frac{1}{\cos\theta} + \sin\theta\frac{\cos\theta}{\sin\theta} \\ &= \frac{\sin\theta}{\cos\theta} + \cos\theta \\ &\underline{= \tan\theta + \cos\theta} \quad \checkmark \end{aligned}$$

Exercise

Prove the identity $\cos\theta(\csc\theta + \tan\theta) = \cot\theta + \sin\theta$

Solution

$$\begin{aligned} \cos\theta(\csc\theta + \tan\theta) &= \cos\theta\frac{1}{\sin\theta} + \cos\theta\frac{\sin\theta}{\cos\theta} \\ &\underline{= \cot\theta + \sin\theta} \quad \checkmark \end{aligned}$$

Exercise

Prove the identity $\cos\theta(\sec\theta - \cos\theta) = \sin^2\theta$

Solution

$$\begin{aligned} \cos\theta(\sec\theta - \cos\theta) &= \cos\theta\frac{1}{\cos\theta} - \cos^2\theta \\ &= 1 - \cos^2\theta \\ &\underline{= \sin^2\theta} \quad \checkmark \end{aligned}$$

Exercise

Prove the identity $\cot\theta + \tan\theta = \csc\theta\sec\theta$

Solution

$$\begin{aligned}\cot\theta + \tan\theta &= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \\ &= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} \\ &= \frac{1}{\sin\theta\cos\theta} \\ &= \frac{1}{\sin\theta}\frac{1}{\cos\theta} \\ &= \csc\theta\sec\theta \quad \checkmark\end{aligned}$$

Exercise

Prove $\tan x(\cos x + \cot x) = \sin x + 1$

Solution

$$\begin{aligned}\tan x(\cos x + \cot x) &= \frac{\sin x}{\cos x}\left(\cos x + \frac{\cos x}{\sin x}\right) \\ &= \cos x\frac{\sin x}{\cos x} + \frac{\sin x}{\cos x}\frac{\cos x}{\sin x} \\ &= \sin x + 1 \quad \checkmark\end{aligned}$$

Exercise

Prove $\dfrac{1-\cos^4\theta}{1+\cos^2\theta} = \sin^2\theta$

Solution

$$\begin{aligned}\frac{1-\cos^4\theta}{1+\cos^2\theta} &= \frac{(1+\cos^2\theta)(1-\cos^2\theta)}{1+\cos^2\theta} \\ &= 1-\cos^2\theta \\ &= \sin^2\theta \quad \checkmark\end{aligned}$$

Exercise

Prove $\dfrac{1-\sec x}{1+\sec x} = \dfrac{\cos x - 1}{\cos x + 1}$

Solution

$$\frac{1-\sec x}{1+\sec x}=\frac{1-\frac{1}{\cos x}}{1+\frac{1}{\cos x}}$$

$$=\frac{\frac{\cos x-1}{\cos x}}{\frac{\cos x+1}{\cos x}}$$

$$\boxed{=\frac{\cos x-1}{\cos x+1}} \quad \checkmark$$

Exercise

Prove $\quad \dfrac{\cos x}{1+\sin x}-\dfrac{1-\sin x}{\cos x}=0$

Solution

$$\frac{\cos x}{1+\sin x}-\frac{1-\sin x}{\cos x}=\frac{\cos x}{\cos x}\frac{\cos x}{1-\sin x}-\frac{1+\sin x}{1+\sin x}\frac{1-\sin x}{\cos x}$$

$$=\frac{\cos^2 x-(1-\sin^2 x)}{\cos x(1+\sin x)}$$

$$=\frac{\cos^2 x-1+\sin^2 x)}{\cos x(1+\sin x)}$$

$$=\frac{1-1}{\cos x(1+\sin x)}$$

$$=\frac{0}{\cos x(1+\sin x)}$$

$$\boxed{=0} \quad \checkmark$$

Exercise

Prove $\quad \dfrac{1+\cot^3 t}{1+\cot t}=\csc^2 t-\cot t$

Solution

$$\frac{1+\cot^3 t}{1+\cot t}=\frac{1+\frac{\cos^3 t}{\sin^3 t}}{1+\frac{\cos t}{\sin t}}$$

$$=\frac{\frac{\sin^3 t+\cos^3 t}{\sin^3 t}}{\frac{\sin t+\cos t}{\sin t}}$$

$$=\frac{\sin^3 t+\cos^3 t}{\sin^3 t}\cdot\frac{\sin t}{\sin t+\cos t}$$

$$= \frac{(\sin t+\cos t)(\sin^2 t-\sin t\cos t+\cos^2 t)}{\sin^2 t}\cdot\frac{1}{\sin t+\cos t}$$

$$= \frac{1-\sin t\cos t}{\sin^2 t}$$

$$= \frac{1}{\sin^2 t} - \frac{\sin t\cos t}{\sin^2 t}$$

$$= \csc^2 t - \frac{\cos t}{\sin t}$$

$$= \csc^2 t - \cot t \quad \checkmark$$

Exercise

Prove: $\tan x + \cot x = \sec x \csc x$

Solution

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x}$$

$$= \frac{1}{\cos x \sin x}$$

$$= \frac{1}{\cos x}\frac{1}{\sin x}$$

$$= \sec x \ \csc x \quad \checkmark$$

Exercise

Prove: $\dfrac{\tan x - \cot x}{\sin x \cos x} = \sec^2 x - \csc^2 x$

Solution

$$\frac{\tan x - \cot x}{\sin x\cos x} = \frac{\tan x}{\sin x\cos x} - \frac{\cot x}{\sin x\cos x}$$

$$= \tan x\frac{1}{\sin x\cos x} - \cot x\frac{1}{\sin x\cos x}$$

$$= \frac{\sin x}{\cos x}\frac{1}{\sin x\cos x} - \frac{\cos x}{\sin x}\frac{1}{\sin x\cos x}$$

$$= \frac{1}{\cos^2 x} - \frac{1}{\sin^2 x}$$

$$= \sec^2 x - \csc^2 x \quad \checkmark$$

Exercise

Prove: $\dfrac{\sec x+\tan x}{\sec x-\tan x}=\dfrac{1+2\sin x+\sin^2 x}{\cos^2 x}$

Solution

$$\frac{\sec x+\tan x}{\sec x-\tan x}=\frac{\frac{1}{\cos x}+\frac{\sin x}{\cos x}}{\frac{1}{\cos x}-\frac{\sin x}{\cos x}}\frac{\cos x}{\cos x}$$

$$=\frac{\frac{1}{\cos x}\cos x+\frac{\sin x}{\cos x}\cos x}{\frac{1}{\cos x}\cos x-\frac{\sin x}{\cos x}\cos x}$$

$$=\frac{1+\sin x}{1-\sin x}$$

$$=\frac{1+\sin x}{1-\sin x}\frac{1+\sin x}{1+\sin x}$$

$$=\frac{(1+\sin x)^2}{1-\sin^2 x}$$

$$\boxed{=\frac{1+2\sin x+\sin^2 x}{\cos^2 x}}\ \checkmark$$

Exercise

Prove the identity: $\sin^2 x-\cos^2 x=2\sin^2 x-1$

Solution

$$\sin^2 x-\cos^2 x=\sin^2 x-\left(1-\sin^2 x\right)$$

$$=\sin^2 x-1+\sin^2 x$$

$$\boxed{=2\sin^2 x-1}\ \checkmark$$

Exercise

Prove the identity: $\sin^4 x-\cos^4 x=\sin^2 x-\cos^2 x$

Solution

$$\sin^4 x-\cos^4 x=\left(\sin^2 x+\cos^2 x\right)\left(\sin^2 x-\cos^2 x\right)$$

$$=(1)\left(\sin^2 x-\cos^2 x\right)$$

$$\boxed{=\sin^2 x-\cos^2 x}\ \checkmark$$

Exercise

Prove the identity: $\dfrac{\cos\alpha}{1+\sin\alpha} = \sec\alpha - \tan\alpha$

Solution

$$\frac{\cos\alpha}{1+\sin\alpha} = \frac{\cos\alpha}{1+\sin\alpha}\frac{1-\sin\alpha}{1-\sin\alpha}$$

$$= \frac{\cos\alpha - \cos\alpha\sin\alpha}{1-\sin^2\alpha}$$

$$= \frac{\cos\alpha - \cos\alpha\sin\alpha}{\cos^2\alpha}$$

$$= \frac{\cos\alpha}{\cos^2\alpha} - \frac{\cos\alpha\sin\alpha}{\cos^2\alpha}$$

$$= \frac{1}{\cos\alpha} - \frac{\sin\alpha}{\cos\alpha}$$

$$= \sec\alpha - \tan\alpha \quad \checkmark$$

Exercise

Prove the identity: $\dfrac{\sin\alpha}{1-\sin\alpha} - \dfrac{\cos\alpha}{1-\sin\alpha} = \dfrac{1-\cot\alpha}{\csc\alpha - 1}$

Solution

$$\frac{\sin\alpha}{1-\sin\alpha} - \frac{\cos\alpha}{1-\sin\alpha} = \frac{\sin\alpha - \cos\alpha}{1-\sin\alpha}$$

$$= \frac{\frac{\sin\alpha - \cos\alpha}{\sin\alpha}}{\frac{1-\sin\alpha}{\sin\alpha}}$$

$$= \frac{\frac{\sin\alpha}{\sin\alpha} - \frac{\cos\alpha}{\sin\alpha}}{\frac{1}{\sin\alpha} - \frac{\sin\alpha}{\sin\alpha}}$$

$$= \frac{1-\cot\alpha}{\csc\alpha - 1} \quad \checkmark$$

Exercise

Prove the identity: $\dfrac{\frac{1}{\tan x} + \cot x}{\frac{1}{\tan x} + \tan x} = \dfrac{2}{\sec^2 x}$

Solution

$$\frac{\frac{1}{\tan x}+\cot x}{\frac{1}{\tan x}+\tan x}=\frac{\frac{1}{\tan x}+\frac{1}{\tan x}}{\frac{1+\tan^2 x}{\tan x}}$$

$$=\frac{\frac{2}{\tan x}}{\frac{\sec^2 x}{\tan x}}$$

$$=\frac{2}{\sec^2 x}\ \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cot^2\theta+3\cot\theta-4}{\cot\theta+4}=\cot\theta-1$

Solution

$$\frac{\cot^2\theta+3\cot\theta-4}{\cot\theta+4}=\frac{(\cot\theta+4)(\cot\theta-1)}{\cot\theta+4}$$

$$=\cot\theta-1\ \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin\theta}{1+\cos\theta}=\dfrac{1-\cos\theta}{\sin\theta}$

Solution

$$\frac{\sin\theta}{1+\cos\theta}=\frac{\sin\theta}{1+\cos\theta}\frac{1-\cos\theta}{1-\cos\theta}$$

$$=\frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta}$$

$$=\frac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$$

$$=\frac{1-\cos\theta}{\sin\theta}\ \checkmark$$

Exercise

Prove the following equation is an identity: $\tan x(\csc x-\sin x)=\cos x$

Solution

$$\tan x(\csc x-\sin x)=\frac{\sin x}{\cos x}\left(\frac{1}{\sin x}-\sin x\right)$$

$$= \frac{\sin x}{\cos x}\left(\frac{1-\sin^2 x}{\sin x}\right)$$

$$= \frac{1}{\cos x}\left(\frac{\cos^2 x}{1}\right)$$

$$= \cos x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\sin x(\tan x \cos x - \cot x \cos x) = 1 - 2\cos^2 x$

Solution

$$\begin{aligned}
\sin x(\tan x \cos x - \cot x \cos x) &= \sin x \cos x\left(\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x}\right) \\
&= \sin x \cos x\left(\frac{\sin^2 x - \cos^2 x}{\cos x \sin x}\right) \\
&= 1 - \cos^2 x - \cos^2 x \\
&= 1 - 2\cos^2 x \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $(1+\tan x)^2 + (\tan x - 1)^2 = 2\sec^2 x$

Solution

$$\begin{aligned}
(1+\tan x)^2 + (\tan x - 1)^2 &= 1 + 2\tan x + \tan^2 x + 1 - 2\tan x + \tan^2 x \\
&= 2 + 2\tan^2 x \\
&= 2\left(1 + \tan^2 x\right) \\
&= 2\sec^2 x \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\sec x + \tan x = \frac{\cos x}{1 - \sin x}$

Solution

$$\begin{aligned}
\sec x + \tan x &= \frac{1}{\cos x} + \frac{\sin x}{\cos x} \\
&= \frac{1+\sin x}{\cos x}\frac{1-\sin x}{1-\sin x}
\end{aligned}$$

$$=\frac{1-\sin^2 x}{\cos x(1-\sin x)}$$

$$=\frac{\cos^2 x}{\cos x(1-\sin x)}$$

$$=\frac{\cos x}{1-\sin x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\tan x-1}{\tan x+1}=\frac{1-\cot x}{1+\cot x}$

Solution

$$\frac{\tan x-1}{\tan x+1}=\frac{\frac{1}{\cot x}-1}{\frac{1}{\cot x}+1}$$

$$=\frac{\frac{1-\cot x}{\cot x}}{\frac{1+\cot x}{\cot x}}$$

$$=\frac{1-\cot x}{1+\cot x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $7\csc^2 x-5\cot^2 x=2\csc^2 x+5$

Solution

$$7\csc^2 x-5\cot^2 x=7\csc^2 x-5\left(\csc^2 x-1\right)$$

$$=7\csc^2 x-5\csc^2 x+5$$

$$=2\csc^2 x+5 \quad \checkmark$$

Exercise

Prove the following equation is an identity: $1-\frac{\cos^2 x}{1-\sin x}=-\sin x$

Solution

$$1-\frac{\cos^2 x}{1-\sin x}=1-\frac{1-\sin^2 x}{1-\sin x}$$

$$=1-\frac{(1-\sin x)(1+\sin x)}{1-\sin x}$$

$$=1-(1+\sin x)$$

$$=1-1-\sin x$$

$$\boxed{=-\sin x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{1-\cos x}{1+\cos x}=\dfrac{\sec x-1}{\sec x+1}$

Solution

$$\frac{1-\cos x}{1+\cos x}=\frac{\frac{1}{\cos x}-\frac{\cos x}{\cos x}}{\frac{1}{\cos x}+\frac{\cos x}{\cos x}}$$

$$\boxed{=\frac{\sec x-1}{\sec x+1}} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\sec x-1}{\tan x}=\dfrac{\tan x}{\sec x+1}$

Solution

$$\frac{\sec x-1}{\tan x}=\frac{\sec x-1}{\tan x}\frac{\sec x+1}{\sec x+1}$$

$$=\frac{\sec^2 x-1}{\tan x(\sec x+1)}$$

$$=\frac{\tan^2 x}{\tan x(\sec x+1)}$$

$$\boxed{=\frac{\tan x}{\sec x+1}} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos x}{\cos x-\sin x}=\dfrac{1}{1-\tan x}$

Solution

$$\frac{\cos x}{\cos x-\sin x}=\frac{\frac{\cos x}{\cos x}}{\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}}$$

$$\boxed{=\frac{1}{1-\tan x}} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $(\sec x+\tan x)^2=\frac{1+\sin x}{1-\sin x}$

Solution

$$\begin{aligned}(\sec x+\tan x)^2&=\left(\frac{1}{\cos x}+\frac{\sin x}{\cos x}\right)^2\\&=\left(\frac{1+\sin x}{\cos x}\right)^2\\&=\frac{(1+\sin x)^2}{\cos^2 x}\\&=\frac{(1+\sin x)^2}{1-\sin^2 x}\\&=\frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)}\\&=\frac{1+\sin x}{1-\sin x}\end{aligned}$$

√

Exercise

Prove the following equation is an identity: $\frac{\cos x}{1+\tan x}-\frac{\sin x}{1+\cot x}=\cos x-\sin x$

Solution

$$\begin{aligned}\frac{\cos x}{1+\tan x}-\frac{\sin x}{1+\cot x}&=\frac{\cos x}{1+\frac{\sin x}{\cos x}}-\frac{\sin x}{1+\frac{\cos x}{\sin x}}\\&=\frac{\cos x}{\frac{\cos x+\sin x}{\cos x}}-\frac{\sin x}{\frac{\sin x+\cos x}{\sin x}}\\&=\frac{\cos^2 x}{\cos x+\sin x}-\frac{\sin^2 x}{\sin x+\cos x}\\&=\frac{\cos^2 x-\sin^2 x}{\cos x+\sin x}\\&=\frac{(\cos x+\sin x)(\cos x-\sin x)}{\cos x+\sin x}\\&=\cos x-\sin x\end{aligned}$$

√

Exercise

Prove the following equation is an identity: $\dfrac{\cot x+\csc x-1}{\cot x-\csc x+1}=\csc x+\cot x$

Solution

$$\begin{aligned}\frac{\cot x+\csc x-1}{\cot x-\csc x+1}&=\frac{\cot x+\csc x-\left(\csc^2 x-\cot^2 x\right)}{\cot x-\csc x+1}\\&=\frac{\cot x+\csc x-(\csc x-\cot x)(\csc x+\cot x)}{\cot x-\csc x+1}\\&=\frac{(\csc x+\cot x)(1-(\csc x-\cot x))}{\cot x-\csc x+1}\\&=\frac{(\csc x+\cot x)(1-\csc x+\cot x)}{\cot x-\csc x+1}\\&=\csc x+\cot x\end{aligned}$$

√

Exercise

Prove the following equation is an identity: $\dfrac{\tan x+\cot x}{\tan x-\cot x}=\dfrac{1}{\sin^2 x-\cos^2 x}$

Solution

$$\begin{aligned}\frac{\tan x+\cot x}{\tan x-\cot x}&=\frac{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}}{\frac{\sin x}{\cos x}-\frac{\cos x}{\sin x}}\\&=\frac{\frac{\sin^2 x+\cos^2 x}{\cos x\sin x}}{\frac{\sin^2 x-\cos^2 x}{\cos x\sin x}}\\&=\frac{1}{\sin^2 x-\cos^2 x}\end{aligned}$$

√

Exercise

Prove the following equation is an identity: $\dfrac{1-\cot^2 x}{1+\cot^2 x}+1=2\sin^2 x$

Solution

$$\begin{aligned}\frac{1-\cot^2 x}{1+\cot^2 x}+1&=\frac{1-\cot^2 x+1+\cot^2 x}{1+\cot^2 x}\\&=\frac{2}{\csc^2 x}\\&=2\sin^2 x\end{aligned}$$

√

Exercise

Prove the following equation is an identity: $\frac{1+\cos x}{1-\cos x}-\frac{1-\cos x}{1+\cos x}=4\cot x\csc x$

Solution

$$\begin{aligned}\frac{1+\cos x}{1-\cos x}-\frac{1-\cos x}{1+\cos x}&=\frac{(1+\cos x)^2-(1-\cos x)^2}{1-\cos^2 x}\\&=\frac{(1+\cos x+1-\cos x)(1+\cos x-1+\cos x)}{\sin^2 x} \qquad a^2-b^2=(a-b)(a+b)\\&=\frac{(2)(2\cos x)}{\sin^2 x}\\&=4\frac{\cos x}{\sin x}\frac{1}{\sin x}\\&=4\cot x\csc x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin^3 x-\cos^3 x}{\sin x-\cos x}=1+\sin x\cos x$

Solution

$$\begin{aligned}\frac{\sin^3 x-\cos^3 x}{\sin x-\cos x}&=\frac{(\sin x-\cos x)\left(\sin^2 x+\sin x\cos x+\cos^2 x\right)}{\sin x-\cos x} \qquad a^3-b^3=(a-b)\left(a^2+ab+b^2\right)\\&=1+\sin x\cos x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $1+\sec^2 x\sin^2 x=\sec^2 x$

Solution

$$\begin{aligned}1+\sec^2 x\sin^2 x&=1+\frac{1}{\cos^2 x}\sin^2 x\\&=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}\\&=\frac{1}{\cos^2 x}\\&=\sec^2 x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{1+\csc x}{\sec x} = \cos x + \cot x$

Solution

$$\begin{aligned}\frac{1+\csc x}{\sec x} &= \frac{1}{\sec x} + \frac{\csc x}{\sec x}\\ &= \cos x + \frac{\frac{1}{\sin x}}{\frac{1}{\cos x}}\\ &= \cos x + \frac{\cos x}{\sin x}\\ &= \cos x + \cot x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\tan^2 x = \sec^2 x - \sin^2 x - \cos^2 x$

Solution

$$\begin{aligned}\sec^2 x - \sin^2 x - \cos^2 x &= \frac{1}{\cos^2 x} - \left(\sin^2 x + \cos^2 x\right)\\ &= \frac{1}{\cos^2 x} - 1\\ &= \frac{1-\cos^2 x}{\cos^2 x}\\ &= \frac{\sin^2 x}{\cos^2 x}\\ &= \tan^2 x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin x}{1-\cos x} + \frac{\sin x}{1+\cos x} = 2\csc x$

Solution

$$\begin{aligned}\frac{\sin x}{1-\cos x} + \frac{\sin x}{1+\cos x} &= \sin x\left(\frac{1}{1-\cos x} + \frac{1}{1+\cos x}\right)\\ &= \sin x\left(\frac{1+\cos x + 1 - \cos x}{1-\cos^2 x}\right)\\ &= \sin x\left(\frac{2}{\sin^2 x}\right)\end{aligned}$$

$$= \frac{2}{\sin x}$$

$$\underline{= 2\csc x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\cos^2(\alpha - \beta) - \cos^2(\alpha + \beta) = \sin^2(\alpha + \beta) - \sin^2(\alpha - \beta)$

Solution

$$\cos^2(\alpha - \beta) - \cos^2(\alpha + \beta) = 1 - \sin^2(\alpha - \beta) - \left[1 - \sin^2(\alpha + \beta)\right]$$

$$= 1 - \sin^2(\alpha - \beta) - 1 + \sin^2(\alpha + \beta)$$

$$\underline{= \sin^2(\alpha + \beta) - \sin^2(\alpha - \beta)} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\tan x \csc x - \sec^2 x \cos x = 0$

Solution

$$\tan x \csc x - \sec^2 x \cos x = \frac{\sin x}{\cos x}\frac{1}{\sin x} - \frac{1}{\cos^2 x}\cos x$$

$$= \frac{1}{\cos x} - \frac{1}{\cos x}$$

$$\underline{= 0} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $(1 + \tan x)^2 - 2\tan x = \frac{1}{(1 - \sin x)(1 + \sin x)}$

Solution

$$(1 + \tan x)^2 - 2\tan x = 1 + 2\tan x + \tan^2 x - 2\tan x$$

$$= 1 + \tan^2 x$$

$$= \sec^2 x$$

$$= \frac{1}{\cos^2 x}$$

$$= \frac{1}{1 - \sin^2 x}$$

$$\underline{= \frac{1}{(1 - \sin x)(1 + \sin x)}} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{3\csc^2 x - 5\csc x - 28}{\csc x - 4} = \dfrac{3}{\sin x} + 7$

Solution

$$\begin{aligned}\frac{3\csc^2 x - 5\csc x - 28}{\csc x - 4} &= \frac{(3\csc x + 7)(\csc x - 4)}{\csc x - 4} \\ &= 3\csc x + 7 \\ &= \frac{3}{\sin x} + 7 \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\left(\sec^2 x - 1\right)\left(\sec^2 x + 1\right) = \tan^4 x + 2\tan^2 x$

Solution

$$\begin{aligned}\left(\sec^2 x - 1\right)\left(\sec^2 x + 1\right) &= \sec^4 x - 1 \qquad\qquad (a-b)(a+b) = a^2 - b^2 \quad a = \sec^2 x \\ &= \left(\sec^2 x\right)^2 - 1 \\ &= \left(1 + \tan^2 x\right)^2 - 1 \\ &= 1 + 2\tan^2 x + \tan^4 x - 1 \\ &= \tan^4 x + 2\tan^2 x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\csc x}{\cot x} - \dfrac{\cot x}{\csc x} = \dfrac{\sin x}{\cot x}$

Solution

$$\begin{aligned}\frac{\csc x}{\cot x} - \frac{\cot x}{\csc x} &= \frac{\csc^2 x - \cot^2 x}{\cot x \csc x} \\ &= \frac{\csc^2 x - \left(\csc^2 x - 1\right)}{\cot x \csc x} \\ &= \frac{\csc^2 x - \csc^2 x + 1}{\cot x \csc x} \\ &= \frac{1}{\cot x \frac{1}{\sin x}} \\ &= \frac{\sin x}{\cot x} \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{1-\cos^2 x}{1+\cos x}=\dfrac{\sec x-1}{\sec x}$

Solution

$$\frac{1-\cos^2 x}{1+\cos x}=\frac{(1-\cos x)(1+\cos x)}{1+\cos x}$$

$$=1-\cos x$$

$$=1-\frac{1}{\sec x}$$

$$=\frac{\sec x-1}{\sec x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos x}{1+\cos x}=\dfrac{\sec x-1}{\tan^2 x}$

Solution

$$\frac{\cos x}{1+\cos x}=\frac{\cos x}{1+\cos x}\frac{1-\cos x}{1-\cos x}$$

$$=\frac{\cos x-\cos^2 x}{\cos^2 x-1}$$

$$=\frac{\cos x-\cos^2 x}{\sin^2 x}\frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}}$$

$$=\frac{\frac{1}{\cos x}-1}{\frac{\sin^2 x}{\cos^2 x}}$$

$$=\frac{\sec x-1}{\tan^2 x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{1-2\sin^2 x}{1+2\sin x\cos x}=\dfrac{\cos x-\sin x}{\cos x+\sin x}$

Solution

$$\frac{1-2\sin^2 x}{1+2\sin x\cos x}=\frac{\cos^2 x+\sin^2 x-2\sin^2 x}{\cos^2 x+\sin^2 x+2\sin x\cos x}$$

$$=\frac{\cos^2 x-\sin^2 x}{(\cos x+\sin x)^2}$$

$$= \frac{(\cos x - \sin x)(\cos x + \sin x)}{(\cos x + \sin x)^2}$$

$$= \frac{\cos x - \sin x}{\cos x + \sin x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $(\cos x - \sin x)^2 + (\cos x + \sin x)^2 = 2$

Solution

$$\begin{aligned}(\cos x - \sin x)^2 + (\cos x + \sin x)^2 &= \cos^2 x - 2\sin x \cos x + \sin^2 x + \cos^2 x + 2\sin x \cos x + \sin^2 x \\ &= \cos^2 x + \sin^2 x + \cos^2 x + \sin^2 x \\ &= 1 + 1 \\ &= 2 \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} = 2\csc x$

Solution

$$\begin{aligned}\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} &= \frac{\sin x \sin x + (1+\cos x)(1+\cos x)}{(1+\cos x)\sin x} \\ &= \frac{\sin^2 x + 1 + 2\cos x + \cos^2 x}{(1+\cos x)\sin x} \\ &= \frac{1 + 1 + 2\cos x}{(1+\cos x)\sin x} \\ &= \frac{2 + 2\cos x}{(1+\cos x)\sin x} \\ &= \frac{2(1+\cos x)}{(1+\cos x)\sin x} \\ &= \frac{2}{\sin x} \\ &= 2\csc x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sin x+\tan x}{\cot x+\csc x}=\sin x\tan x$

Solution

$$\frac{\sin x+\tan x}{\cot x+\csc x}=\frac{\sin x+\tan x}{\frac{1}{\tan x}+\frac{1}{\sin x}}$$

$$=\frac{\sin x+\tan x}{\frac{\sin x+\tan x}{\tan x\sin x}}$$

$$=\left(\sin x+\tan x\right)\frac{\tan x\sin x}{\sin x+\tan x}$$

$$=\tan x\sin x \checkmark$$

Exercise

Prove the following equation is an identity: $\csc^2 x\sec^2 x=\sec^2 x+\csc^2 x$

Solution

$$\csc^2 x\sec^2 x=\frac{1}{\sin^2 x}\frac{1}{\cos^2 x}$$

$$=\frac{1}{\sin^2 x\cos^2 x}$$

$$=\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}$$

$$=\frac{\sin^2 x}{\sin^2 x\cos^2 x}+\frac{\cos^2 x}{\sin^2 x\cos^2 x}$$

$$=\frac{1}{\cos^2 x}+\frac{1}{\sin^2 x}$$

$$=\sec^2 x+\csc^2 x \checkmark$$

Exercise

Prove the following equation is an identity: $\cos^2 x+1=2\cos^2 x+\sin^2 x$

Solution

$$\cos^2 x+1=\cos^2 x+\cos^2 x+\sin^2 x$$

$$=2\cos^2 x+\sin^2 x \checkmark$$

Exercise

Prove the following equation is an identity: $1-\dfrac{\cos^2 x}{1+\sin x}=\sin x$

Solution

$$1-\frac{\cos^2 x}{1+\sin x}=1-\frac{1-\sin^2 x}{1+\sin x}$$

$$=1-\frac{(1-\sin x)(1+\sin x)}{1+\sin x}$$

$$=1-(1-\sin x)$$

$$=1-1+\sin x$$

$$=\sin x \;\checkmark$$

Exercise

Prove the following equation is an identity: $\cot^2 x=(\csc x-1)(\csc x+1)$

Solution

$$\cot^2 x=\csc^2 x-1$$

$$=(\csc x-1)(\csc x+1) \;\checkmark$$

Exercise

Prove the following equation is an identity: $10\csc^2 x-6\cot^2 x=4\csc^2 x+6$

Solution

$$10\csc^2 x-6\cot^2 x=10\csc^2 x-6\left(\csc^2 x-1\right)$$

$$=10\csc^2 x-6\csc^2 x+6$$

$$=4\csc^2 x+6 \;\checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\csc x+\cot x}{\tan x+\sin x}=\csc x\cot x$

Solution

$$\frac{\csc x+\cot x}{\tan x+\sin x}=\frac{\csc x+\cot x}{\frac{1}{\cot x}+\frac{1}{\csc x}}$$

$$=\frac{\csc x+\cot x}{\frac{\csc x+\cot x}{\cot x\csc x}}$$

$$= \csc x + \cot x \frac{\cot x \csc x}{\csc x + \cot x}$$

$$= \cot x \csc x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{1-\sec x}{\tan x} + \frac{\tan x}{1-\sec x} = -2\csc x$

Solution

$$\frac{1-\sec x}{\tan x} + \frac{\tan x}{1-\sec x} = \frac{(1-\sec x)(1-\sec x) + \tan^2 x}{\tan x(1-\sec x)}$$

$$= \frac{(1-\sec x)^2 + \sec^2 x - 1}{\tan x(1-\sec x)}$$

$$= \frac{(1-\sec x)^2 + (\sec x + 1)(\sec x - 1)}{\tan x(1-\sec x)}$$

$$= \frac{(1-\sec x)^2 - (\sec x + 1)(1 - \sec x)}{\tan x(1-\sec x)}$$

$$= \frac{(1-\sec x)\left[(1-\sec x) - (\sec x + 1)\right]}{\tan x(1-\sec x)}$$

$$= \frac{1-\sec x - \sec x - 1}{\tan x}$$

$$= \frac{-2\sec x}{\tan x}$$

$$= -2\frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}}$$

$$= -2\frac{1}{\sin x}$$

$$= -2\csc x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\csc x - \sin x = \cos x \cot x$

Solution

$$\csc x - \sin x = \frac{1}{\sin x} - \sin x$$

$$= \frac{1-\sin^2 x}{\sin x}$$

$$= \frac{\cos^2 x}{\sin x}$$

$$= \cos x \frac{\cos x}{\sin x}$$

$$= \cos x \cot x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\tan x + \sec x}{\sec x} - \frac{\tan x + \sec x}{\tan x} = -\cos x \cot x$

Solution

$$\frac{\tan x + \sec x}{\sec x} - \frac{\tan x + \sec x}{\tan x} = \frac{(\tan x + \sec x)\tan x - \sec x(\tan x + \sec x)}{\sec x \tan x}$$

$$= \frac{\tan^2 x + \sec x \tan x - \sec x \tan x - \sec^2 x}{\sec x \tan x}$$

$$= \frac{\tan^2 x - \sec^2 x}{\sec x \tan x} \qquad 1 + \tan^2 \alpha = \sec^2 \alpha$$

$$= \frac{-1}{\sec x \tan x}$$

$$= -\frac{1}{\sec x}\frac{1}{\tan x}$$

$$= -\cos x \cot x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\cot^3 x = \cot x\left(\csc^2 x - 1\right)$

Solution

$$\cot^3 x = \cot x \cot^2 x$$

$$= \cot x\left(\csc^2 x - 1\right) \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\cot^2 x}{\csc x - 1} = \frac{1 + \sin x}{\sin x}$

Solution

$$\frac{\cot^2 x}{\csc x - 1} = \frac{\csc^2 x - 1}{\csc x - 1}$$

$$= \frac{(\csc x - 1)(\csc x + 1)}{\csc x - 1}$$

$$= \csc x + 1$$

$$= \frac{1}{\sin x} + 1$$

$$= \frac{1 + \sin x}{\sin x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\cot^2 x + \csc^2 x = 2\csc^2 x - 1$

Solution

$$\cot^2 x + \csc^2 x = \csc^2 x - 1 + \csc^2 x$$

$$= 2\csc^2 x - 1 \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cot^2 x}{1+\csc x} = \csc x - 1$

Solution

$$\frac{\cot^2 x}{1+\csc x} = \frac{\csc^2 x - 1}{1+\csc x}$$

$$= \frac{(\csc x - 1)(\csc x + 1)}{1+\csc x}$$

$$= \csc x - 1 \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\sec^4 x - \tan^4 x = \sec^2 x + \tan^2 x$

Solution

$$\sec^4 x - \tan^4 x = \left(\sec^2 x + \tan^2 x\right)\left(\sec^2 x - \tan^2 x\right) \qquad a^2 - b^2 = (a-b)(a+b)$$

$$= \left(\sec^2 x + \tan^2 x\right)(1)$$

$$= \sec^2 x + \tan^2 x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos x}{1+\sin x} + \dfrac{1+\sin x}{\cos x} = 2\sec x$

Solution

$$\frac{\cos x}{1+\sin x} + \frac{1+\sin x}{\cos x} = \frac{\cos^2 x + (1+\sin x)^2}{(1+\sin x)\cos x}$$

$$= \frac{\cos^2 x + 1 + 2\sin x + \sin^2 x}{(1+\sin x)\cos x}$$

$$= \frac{2 + 2\sin x}{(1+\sin x)\cos x}$$

$$= \frac{2(1+\sin x)}{(1+\sin x)\cos x}$$

$$= \frac{2}{\cos x}$$

$$= 2\sec x$$ √

Exercise

Prove the following equation is an identity: $\frac{\sin x+\cos x}{\sin x-\cos x} = \frac{1+2\sin x\cos x}{2\sin^2 x-1}$

Solution

$$\frac{\sin x+\cos x}{\sin x-\cos x} = \frac{\sin x+\cos x}{\sin x-\cos x}\frac{\sin x+\cos x}{\sin x+\cos x}$$

$$= \frac{\sin^2 x+2\sin x\cos x+\cos^2 x}{\sin^2 x-\cos^2 x}$$

$$= \frac{1+2\sin x\cos x}{\sin^2 x-\left(1-\sin^2 x\right)}$$

$$= \frac{1+2\sin x\cos x}{\sin^2 x-1+\sin^2 x}$$

$$= \frac{1+2\sin x\cos x}{2\sin^2 x-1}$$ √

Exercise

Prove the following equation is an identity: $\frac{\csc x-1}{\csc x+1} = \frac{\cot^2 x}{\csc^2 x+2\csc x+1}$

Solution

$$\frac{\csc x-1}{\csc x+1} = \frac{\csc x-1}{\csc x+1}\frac{\csc x+1}{\csc x+1}$$

$$= \frac{\csc^2 x-1}{\csc^2 x+2\csc x+1}$$

$$= \frac{\cot^2 x}{\csc^2 x+2\csc x+1}$$ √

Exercise

Prove the following equation is an identity: $\csc^4 x - \cot^4 x = \csc^2 x + \cot^2 x$

Solution

$$\begin{aligned}\csc^4 x - \cot^4 x &= \left(\csc^2 x + \cot^2 x\right)\left(\csc^2 x - \cot^2 x\right)\\ &= \left(\csc^2 x + \cot^2 x\right)(1)\\ &= \underline{\csc^2 x + \cot^2 x} \end{aligned}$$

$\checkmark$

Exercise

Prove the following equation is an identity: $\tan\left(\frac{\pi}{4} + x\right) = \cot\left(\frac{\pi}{4} - x\right)$

Solution

$$\begin{aligned}\tan\left(\frac{\pi}{4} + x\right) &= \cot\left[\frac{\pi}{2} - \left(\frac{\pi}{4} + x\right)\right]\\ &= \cot\left(\frac{\pi}{2} - \frac{\pi}{4} - x\right)\\ &= \underline{\cot\left(\frac{\pi}{4} - x\right)}\end{aligned}$$

$\checkmark$

Exercise

Prove the following equation is an identity: $\dfrac{\sin\theta}{1+\sin\theta} - \dfrac{\sin\theta}{1-\sin\theta} = -2\tan^2\theta$

Solution

$$\begin{aligned}\frac{\sin\theta}{1+\sin\theta} - \frac{\sin\theta}{1-\sin\theta} &= \sin\theta\left[\frac{1-\sin\theta-(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}\right]\\ &= \sin\theta\left[\frac{1-\sin\theta-1-\sin\theta}{1-\sin^2\theta}\right]\\ &= \sin\theta\left(\frac{-2\sin\theta}{\cos^2\theta}\right)\\ &= -2\frac{\sin^2\theta}{\cos^2\theta}\\ &= \underline{-2\tan^2\theta}\end{aligned}$$

$\checkmark$

Exercise

Prove the following equation is an identity: $\csc^2 x - \cos^2 x \csc^2 x = 1$

Solution

$$\csc^2 x - \cos^2 x \csc^2 x = \csc^2 x\left(1 - \cos^2 x\right)$$
$$= \frac{1}{\sin^2 x}\left(\sin^2 x\right)$$
$$= 1 \quad \checkmark$$

Exercise

Prove the following equation is an identity: $1 - 2\sin^2 x = 2\cos^2 x - 1$

Solution

$$1 - 2\sin^2 x = 1 - 2\left(1 - \cos^2 x\right)$$
$$= 1 - 2 + 2\cos^2 x$$
$$= 2\cos^2 x - 1 \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\csc^2 x - \cos x \sec x = \cot^2 x$

Solution

$$\csc^2 x - \cos x \sec x = \frac{1}{\sin^2 x} - \cos x \frac{1}{\cos x}$$
$$= \frac{1}{\sin^2 x} - 1$$
$$= \frac{1 - \sin^2 x}{\sin^2 x}$$
$$= \frac{\cos^2 x}{\sin^2 x}$$
$$= \cot^2 x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $(\sec x - \tan x)(\sec x + \tan x) = 1$

Solution

$$(\sec x - \tan x)(\sec x + \tan x) = \sec^2 x - \tan^2 x$$
$$= 1 + \tan^2 x - \tan^2 x$$
$$\underline{= 1} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\left(1 + \tan^2 x\right)\left(1 - \sin^2 x\right) = 1$

Solution

$$\left(1 + \tan^2 x\right)\left(1 - \sin^2 x\right) = \sec^2 x \cos^2 x$$
$$= \frac{1}{\cos^2 x}\cos^2 x$$
$$\underline{= 1} \quad \checkmark$$