

Lecture 1 – Functions, Exponential & Logarithms

Section 1.1 – Functions

A **set** is a collection of objects of some type, and the objects are called **elements** of the set.

| Notation or Terminology | Meaning | Example |
|--------------------------------|--|---------------------------------|
| $a \in S$ | a is an element of S | $3 \in \mathbb{Z}$ |
| $a \notin S$ | a is not an element of S | $\frac{3}{2} \notin \mathbb{Z}$ |
| $S \subset T$ | S is a subset of T Every element of S is an element of T | $\mathbb{Z} \subset \mathbb{R}$ |
| Constant | A letter or symbol that represents a specific element of a set. | 5, $\sqrt{2}$, π |
| Variable | A letter or symbol that represents any element of a set. | Let x denote any \mathbb{R} |

Definition of a Function

A **function** is a relation between two variables such that to matches each element of a first set (called **domain**) to an element of a second set (called **range**) in such way that no element in the first set is assigned to two different elements in the second set.

The **domain** of the function is the set of all values of the independent variable for which the function is defined.

The **range** of the function is the set of all values taken on by the dependent variable.

The **Domain** of a Function

1. Rational function: $\frac{f(x)}{h(x)}$ \Rightarrow **Domain:** $h(x) \neq 0$

Example: $f(x) = \frac{1}{x-3}$ **Domain:** $x \neq 3$

2. Irrational function: $\sqrt{g(x)}$ \Rightarrow **Domain:** $g(x) \geq 0$

Example: $g(x) = \sqrt{3-x} + 5$ **Domain:** $x \leq 3$

3. Otherwise: **Domain** all real numbers

Example: $f(x) = x^3 + |x|$ **Domain:** All real numbers, \mathbb{R} , or $(-\infty, \infty)$

(1) & (2) → Find the domain: $f(x) = \frac{x+1}{\sqrt{x-3}} \Rightarrow \text{Domain: } x > 3$

$$\boxed{\begin{aligned} ax^2 + bx + c \geq 0 &\rightarrow \text{if } a > 0 \Rightarrow x \leq x_1, x \geq x_2 \\ ax^2 + bx + c \leq 0 &\rightarrow \text{if } a > 0 \Rightarrow x_1 \leq x \leq x_2 \end{aligned}}$$

Example

Let $g(x) = \frac{\sqrt{4+x}}{1-x}$. Find the domain of g .

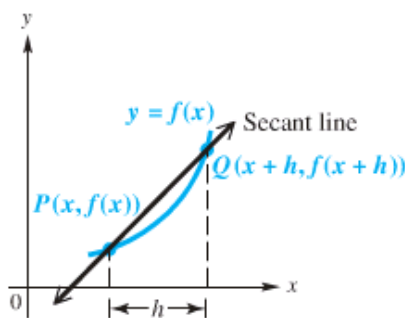
Solution

$$\begin{cases} 4+x \geq 0 \Rightarrow x \geq -4 \\ 1-x \neq 0 \Rightarrow x \neq 1 \end{cases} \rightarrow [-4, 1) \cup (1, \infty)$$

Difference Quotients

$$\frac{f(x+h)-f(x)}{(x+h)-x}$$

The difference quotient is given by: $\frac{f(x+h)-f(x)}{h}$



Example

For the function f given by $f(x) = 2x^2 - 3x$, find the difference quotient $\frac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{\overbrace{2(x+h)^2 - 3(x+h)}^{f(x+h)} - \underbrace{(2x^2 - 3x)}_{f(x)}}{h} \\ &= \frac{2x^2 + 4xh + 2h^2 - 3x - 3h - 2x^2 + 3x}{h} \\ &= \frac{4xh + 2h^2 - 3h}{h} \\ &= \frac{4xh}{h} + \frac{2h^2}{h} - \frac{3h}{h} \\ &= \underline{4x + 2h - 3} \end{aligned}$$

Piecewise-Defined Functions

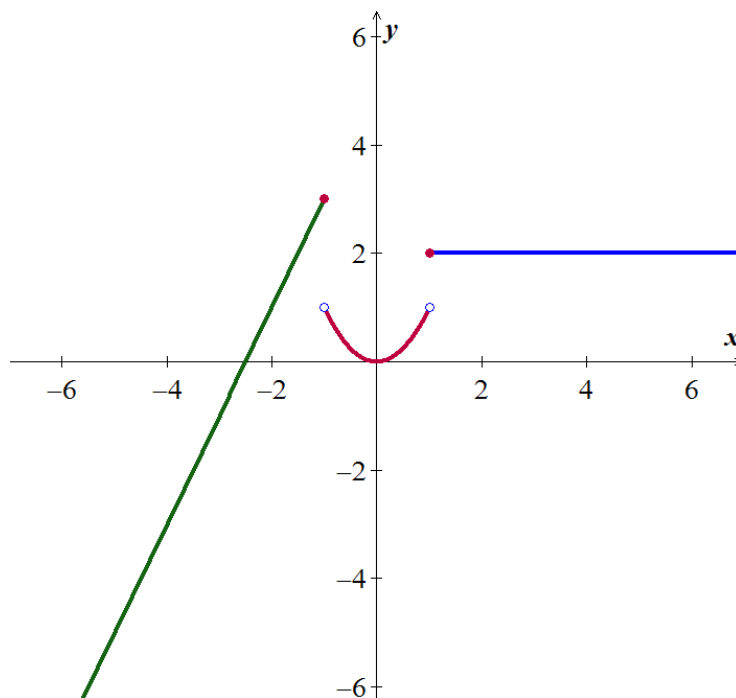
Function are sometimes described by more than one expression, we call such functions *piecewise-defined functions*.

Example

Graph each function

$$f(x) = \begin{cases} 2x + 5 & \text{if } x \leq -1 \\ x^2 & \text{if } |x| < 1 \\ 2 & \text{if } x \geq 1 \end{cases}$$

Solution

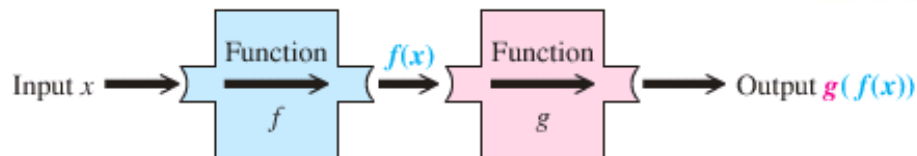
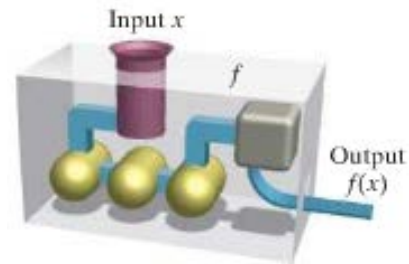


Composition of Functions

The composite function $f \circ g$, the composite of f and g , is defined as

$$(f \circ g)(x) = f(g(x))$$

Where x is in the domain of g
and $g(x)$ is in the domain of f



Example

Let $f(x) = x^2 - 1$ and $g(x) = 3x + 5$

- Find $(f \circ g)(x)$ and the domain of $f \circ g$
- Find $(g \circ f)(x)$ and the domain of $g \circ f$
- Find $(f(g))(2)$ in two different ways: first using the functions f and g separately and second using the composite function $f \circ g$.

Solution

$$\begin{aligned} a) \quad (f \circ g)(x) &= f(g(x)) \\ &= f(3x + 5) \\ &= (\underline{\quad})^2 - 1 \\ &= (3x + 5)^2 - 1 \\ &= 9x^2 + 30x + 25 - 1 \\ &= 9x^2 + 30x + 24 \end{aligned}$$

$$\text{Domain} : (3x + 5) \rightarrow \mathbb{R}$$

$$\text{Domain} : (9x^2 + 30x + 24) \rightarrow \mathbb{R}$$

Domain of $f \circ g : \mathbb{R}$

$$\begin{aligned} b) \quad (g \circ f)(x) &= g(f(x)) \\ &= g(x^2 - 1) \\ &= 3(x^2 - 1) + 5 \\ &= 3x^2 - 3 + 5 \\ &= 3x^2 + 2 \end{aligned}$$

$$\text{Domain} : (x^2 - 1) \rightarrow \mathbb{R}$$

$$\text{Domain} : (3x^2 + 2) \rightarrow \mathbb{R}$$

Domain of $g \circ f : \mathbb{R}$

$$c) \quad g(2) = 3(2) + 5 = 11$$

$$\begin{aligned}(f \circ g)(2) &= f(g(2)) \\ &= f(11) \\ &= 11^2 - 1 \\ &= 120\end{aligned}$$

$$(f \circ g)(x) = 9x^2 + 30x + 24$$

$$(f \circ g)(2) = 9(2)^2 + 30(2) + 24 = 120$$

Example

Let $f(x) = x^2 - 16$ and $g(x) = \sqrt{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

$$a) \quad (f \circ g)(x) = f(g(x))$$

$$= f(\sqrt{x})$$

$$= (\sqrt{x})^2 - 16$$

$$= x - 16$$

$$\text{Domain} : (\sqrt{x}) \rightarrow x \geq 0$$

$$\text{Domain} : (x - 16) \rightarrow \mathbb{R}$$

Domain of $f \circ g : x \geq 0$

$$b) \quad (g \circ f)(x) = g(f(x))$$

$$= g(x^2 - 16)$$

$$= \sqrt{x^2 - 16}$$

$$\text{Domain} : (x^2 - 16) \rightarrow \mathbb{R}$$

$$\text{Domain} : (\sqrt{x^2 - 16}) \rightarrow |x| \geq 4$$

Domain of $g \circ f : |x| \geq 4$ or $(-\infty, -4] \cup [4, \infty)$

Even and Odd Functions

Given the function $f(x)$ then find $f(-x)$ and simplify:

- If $f(-x) = f(x) \Rightarrow f$ is ***even***, or
- If $f(-x) = -f(x) \Rightarrow f$ is ***odd***
- ***Neither***

Example

Decide whether each function is even, odd, or neither

a) $f(x) = 8x^4 - 3x^2$

$$\begin{aligned}f(-x) &= 8(-x)^4 - 3(-x)^2 \\&= 8x^4 - 3x^2 \\&= f(x)\end{aligned}$$

Function is *Even*

b) $f(x) = 6x^3 - 9x$

$$\begin{aligned}f(-x) &= 6(-x)^3 - 9(-x) \\&= -6x^3 + 9x \\&= -(6x^3 - 9x) \\&= -f(x)\end{aligned}$$

Function is *Odd*

c) $f(x) = 3x^2 + 5x$

$$\begin{aligned}f(-x) &= 3(-x)^2 + 5(-x) \\&= 3x^2 - 5x\end{aligned}$$

Function is *Neither*

Exercises

Section 1.1 – Functions

(1 – 80) Find the Domain

1. $f(x) = 7x + 4$

2. $f(x) = |3x - 2|$

3. $f(x) = 3x + \pi$

4. $f(x) = \sqrt{7}x + \frac{1}{2}$

5. $f(x) = -2x^2 + 3x - 5$

6. $f(x) = x^3 - 2x^2 + x - 3$

7. $f(x) = x^2 - 2x - 15$

8. $f(x) = 4 - \frac{2}{x}$

9. $f(x) = \frac{1}{x^4}$

10. $g(x) = \frac{3}{x-4}$

11. $y = \frac{2}{x-3}$

12. $y = \frac{-7}{x-5}$

13. $f(x) = \frac{x+5}{2-x}$

14. $f(x) = \frac{8}{x+4}$

15. $f(x) = \frac{1}{x+4}$

16. $f(x) = \frac{1}{x-4}$

17. $f(x) = \frac{3x}{x+2}$

18. $f(x) = x - \frac{2}{x-3}$

19. $f(x) = x + \frac{3}{x-5}$

20. $f(x) = \frac{1}{2}x - \frac{8}{x+7}$

21. $f(x) = \frac{1}{x-3} - \frac{8}{x+7}$

22. $f(x) = \frac{1}{x+4} - \frac{2x}{x-4}$

23. $f(x) = \frac{3x^2}{x+3} - \frac{4x}{x-2}$

24. $f(x) = \frac{1}{x^2 - 2x + 1}$

25. $f(x) = \frac{x}{x^2 + 3x + 2}$

26. $f(x) = \frac{x^2}{x^2 - 5x + 4}$

27. $f(x) = \frac{1}{x^2 - 4x - 5}$

28. $g(x) = \frac{2}{x^2 + x - 12}$

29. $h(x) = \frac{5}{\frac{4}{x} - 1}$

30. $y = \sqrt{x}$

31. $f(x) = \sqrt{8-3x}$

32. $y = \sqrt{4x+1}$

33. $y = \sqrt{7-2x}$

34. $f(x) = \sqrt{8-x}$

35. $f(x) = \sqrt{3-2x}$

36. $f(x) = \sqrt{3+2x}$

37. $f(x) = \sqrt{5-x}$

38. $f(x) = \sqrt{x-5}$

39. $f(x) = \sqrt{6-3x}$

40. $f(x) = \sqrt{3x-6}$

41. $f(x) = \sqrt{2x+7}$

42. $f(x) = \sqrt{x^2-16}$

43. $f(x) = \sqrt{16-x^2}$

44. $f(x) = \sqrt{9-x^2}$

45. $f(x) = \sqrt{x^2-25}$

46. $f(x) = \sqrt{x^2-5x+4}$

47. $f(x) = \sqrt{x^2+5x+4}$

48. $f(x) = \sqrt{x^2+3x+2}$

49. $f(x) = \sqrt{x^2-3x+2}$

50. $f(x) = \sqrt{x-4} + \sqrt{x+1}$

51. $f(x) = \sqrt{3-x} + \sqrt{x-2}$

52. $f(x) = \sqrt{1-x} + \sqrt{4-x}$

53. $f(x) = \sqrt{1-x} - \sqrt{x-3}$

54. $f(x) = \sqrt{x+4} - \sqrt{x-1}$

55. $f(x) = \frac{\sqrt{x+1}}{x}$

56. $g(x) = \frac{\sqrt{x-3}}{x-6}$

57. $f(x) = \frac{\sqrt{x+4}}{\sqrt{x-1}}$

58. $f(x) = \frac{\sqrt{5-x}}{x}$

59. $f(x) = \frac{x}{\sqrt{5-x}}$

$$60. f(x) = \frac{1}{x\sqrt{5-x}}$$

$$61. f(x) = \frac{x+1}{x^3-4x}$$

$$62. f(x) = \frac{\sqrt{x+5}}{x}$$

$$63. f(x) = \frac{x}{\sqrt{x+5}}$$

$$64. f(x) = \frac{1}{x\sqrt{x+5}}$$

$$65. f(x) = \frac{x+3}{\sqrt{x-3}}$$

$$66. f(x) = \frac{\sqrt{x+3}}{\sqrt{x-3}}$$

$$67. f(x) = \frac{\sqrt{x-2}}{\sqrt{x+2}}$$

$$68. f(x) = \frac{\sqrt{2-x}}{\sqrt{x+2}}$$

$$69. f(x) = \frac{x-4}{\sqrt{x-2}}$$

$$70. f(x) = \frac{1}{(x-3)\sqrt{x+3}}$$

$$71. f(x) = \sqrt{x+2} + \sqrt{2-x}$$

$$72. f(x) = \sqrt{(x-2)(x-6)}$$

$$73. f(x) = \sqrt{x+3} - \sqrt{4-x}$$

$$74. f(x) = \frac{\sqrt{4x-3}}{x^2-4}$$

$$75. f(x) = \frac{4x}{6x^2+13x-5}$$

$$76. f(x) = \frac{\sqrt{2x-3}}{x^2-5x+4}$$

$$77. f(x) = \frac{x^2}{\sqrt{x^2-5x+4}}$$

$$78. f(x) = \frac{x+2}{\sqrt{x^2+5x+4}}$$

$$79. f(x) = \frac{\sqrt{x+2}}{\sqrt{x^2+3x+2}}$$

$$80. f(x) = \frac{\sqrt{2x+3}}{x^2-6x+5}$$

(81 – 97) Find and simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$ for the given function

$$81. f(x) = 9x + 5$$

$$82. f(x) = 6x + 2$$

$$83. f(x) = 4x + 11$$

$$84. f(x) = 3x - 5$$

$$85. f(x) = -2x - 3$$

$$86. f(x) = -4x + 3$$

$$87. f(x) = 3x - 6$$

$$88. f(x) = -5x - 7$$

$$89. f(x) = 2x^2$$

$$90. f(x) = 5x^2$$

$$91. f(x) = 3x^2 - 4x$$

$$92. f(x) = 2x^2 - 3x$$

$$93. f(x) = 2x^2 - x - 3$$

$$94. f(x) = x^2 - 2x + 5$$

$$95. f(x) = 3x^2 - 2x + 5$$

$$96. f(x) = -2x^2 - 3x + 7$$

$$97. f(x) = \sqrt{x-3}$$

98. Let $f(x) = 4x - 3$ and $g(x) = 5x + 7$. Find each of the following and give the domain

$$a) (f+g)(x)$$

$$b) (f-g)(x)$$

$$c) (fg)(x)$$

$$d) \left(\frac{f}{g}\right)(x)$$

99. Let $f(x) = 2x^2 + 3$ and $g(x) = 3x - 4$. Find each of the following and give the domain

$$a) (f+g)(x)$$

$$b) (f-g)(x)$$

$$c) (fg)(x)$$

$$d) \left(\frac{f}{g}\right)(x)$$

100. Let $f(x) = x^2 - 2x - 3$ and $g(x) = x^2 + 3x - 2$. Find each of the following and give the domain

$$a) (f+g)(x)$$

$$b) (f-g)(x)$$

$$c) (fg)(x)$$

$$d) \left(\frac{f}{g}\right)(x)$$

101. Let $f(x) = \sqrt{4x-1}$ and $g(x) = \frac{1}{x}$. Find each of the following and give the domain

a) $(f+g)(x)$ b) $(f-g)(x)$ c) $(fg)(x)$ d) $\left(\frac{f}{g}\right)(x)$

102. Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, and $(f/g)(x)$ and the domain of

$$f(x) = \sqrt{3-2x}, \quad g(x) = \sqrt{x+4}$$

103. Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, and $(f/g)(x)$ and the domain of

$$f(x) = \frac{2x}{x-4}, \quad g(x) = \frac{x}{x+5}$$

104. Let $f(x) = \sqrt{4x-1}$ and $g(x) = \frac{1}{x}$. Find each of the following and give the domain

e) $(f+g)(x)$ f) $(f-g)(x)$ g) $(fg)(x)$ h) $\left(\frac{f}{g}\right)(x)$

105. Given that $f(x) = x+1$ and $g(x) = \sqrt{x+3}$

- a) Find $(f+g)(x)$
- b) Find the domain of $(f+g)(x)$
- c) Find: $(f+g)(6)$

106. Given that $f(x) = x^2 - 4$ and $g(x) = x+2$

- a) Find $(f+g)(x)$ and its domain
- b) Find $(f/g)(x)$ and its domain

107. Find $(f \circ g)(x)$, $(g \circ f)(x)$, $f(g(-2))$ and $g(f(3))$

$$f(x) = 2x^2 + 3x - 4, \quad g(x) = 2x - 1$$

108. Find $(f \circ g)(x)$, $(g \circ f)(x)$, $f(g(-2))$ and $g(f(3))$

$$f(x) = x^3 + 2x^2, \quad g(x) = 3x$$

109. Find $(f \circ g)(x)$, $(g \circ f)(x)$, $f(g(-2))$ and $g(f(3))$

$$f(x) = |x|, \quad g(x) = -7$$

(110 – 139) For the given function; find:

a) Find $(f \circ g)(x)$ and the **domain** of $f \circ g$

b) Find $(g \circ f)(x)$ and the **domain** of $g \circ f$

110. $f(x) = x - 3$ and $g(x) = x + 3$

111. $f(x) = \frac{2}{3}x$ and $g(x) = \frac{3}{2}x$

112. $f(x) = x - 1$ and $g(x) = 3x^2 - 2x - 1$

113. $f(x) = 3x - 2$ and $g(x) = x^2 - 5$

114. $f(x) = x^2 - 2$ and $g(x) = 4x - 3$

115. $f(x) = 4x^2 - x + 10$ and $g(x) = 2x - 7$

116. $f(x) = \sqrt{x}$ and $g(x) = x + 3$

117. $f(x) = \sqrt{x}$ and $g(x) = 2 - 3x$

118. $f(x) = 3x + 2$ and $g(x) = \sqrt{x}$

119. $f(x) = x^4$ and $g(x) = \sqrt[4]{x}$

120. $f(x) = x^n$ and $g(x) = \sqrt[n]{x}$

121. $f(x) = x^2 - 3x$ and $g(x) = \sqrt{x+2}$

122. $f(x) = \sqrt{x-2}$ and $g(x) = \sqrt{x+5}$

123. $f(x) = x^2 + 2$ and $g(x) = \sqrt{3-x}$

124. $f(x) = x^5 - 2$ and $g(x) = \sqrt[5]{x+2}$

125. $f(x) = 1 - x^2$ and $g(x) = \sqrt{x^2 - 25}$

126. $f(x) = 2x + 3$ and $g(x) = \frac{x-3}{2}$

127. $f(x) = 4x - 5$ and $g(x) = \frac{x+5}{4}$

128. $f(x) = \frac{4}{1-5x}$ and $g(x) = \frac{1}{x}$

129. $f(x) = \frac{1}{x-2}$ and $g(x) = \frac{x+2}{x}$

130. $f(x) = \frac{1}{1+x}$ and $g(x) = \frac{1-x}{x}$

131. $f(x) = \frac{3x+5}{2}$ and $g(x) = \frac{2x-5}{3}$

132. $f(x) = \frac{x-1}{x-2}$ and $g(x) = \frac{x-3}{x-4}$

133. $f(x) = \frac{6}{x-3}$ and $g(x) = \frac{1}{x}$

134. $f(x) = \frac{6}{x}$ and $g(x) = \frac{1}{2x+1}$

135. $f(x) = 3x - 7$ and $g(x) = \frac{x+7}{3}$

136. $f(x) = \frac{2x+3}{x-4}$ and $g(x) = \frac{4x+3}{x-2}$

137. $f(x) = \frac{2x+3}{x+4}$ and $g(x) = \frac{-4x+3}{x-2}$

138. $f(x) = x + 1$ and $g(x) = x^3 - 5x^2 + 3x + 7$

139. $f(x) = x - 1$ and $g(x) = x^3 + 2x^2 - 3x - 9$

140. Given that $f(x) = 2x - 5$ and $g(x) = x^2 - 3x + 8$, find $(f \circ g)(x)$, $(g \circ f)(x)$ and their domain then find $(f \circ g)(7)$

141. Given that $f(x) = \sqrt{x}$ and $g(x) = x - 1$, find

a) $(f \circ g)(x) = f(g(x))$

b) $(g \circ f)(x) = g(f(x))$

c) $(f \circ g)(2) = f(g(2))$

142. Given that $f(x) = \frac{x}{x+5}$ and $g(x) = \frac{6}{x}$, find

a) $(f \circ g)(x) = f(g(x))$

b) $(g \circ f)(x) = g(f(x))$

c) $(f \circ g)(2) = f(g(2))$

(143 – 167) Determine whether f is even, odd, or neither

143. $f(x) = 3x^4 + 2x^2 - 5$

144. $f(x) = 8x^3 - 3x^2$

145. $f(x) = \sqrt{x^2 + 4}$

146. $f(x) = 3x^2 - 5x + 1$

147. $f(x) = \sqrt[3]{x^3 - x}$

148. $f(x) = |x| - 3$

149. $f(x) = x^3 - \frac{1}{x}$

150. $f(x) = -x^3 + 2x$

151. $f(x) = x^5 - 2x^3$

152. $f(x) = .5x^4 - 2x^2 + 6$

153. $f(x) = .75x^2 + |x| + 4$

154. $f(x) = x^3 - x + 9$

155. $f(x) = x^4 - 5x + 8$

156. $f(x) = x^3 + x$

157. $g(x) = x^2 - x$

158. $h(x) = 2x^2 + x^4$

159. $f(x) = 2x^2 + x^4 + 1$

160. $f(x) = \frac{1}{5}x^6 - 3x^2$

161. $f(x) = x\sqrt{1-x^2}$

162. $f(x) = x^2\sqrt{1-x^2}$

163. $f(x) = 5x^7 - 6x^3 - 2x$

164. $f(x) = 5x^6 - 3x^2 - 7$

165. $f(x) = x^2 + 6$

166. $f(x) = 7x^3 - x$

167. $h(x) = x^5 + 1$

168. $f(x) = \begin{cases} 2+x & \text{if } x < -4 \\ -x & \text{if } -4 \leq x \leq 2 \\ 3x & \text{if } x > 2 \end{cases}$

Find: $f(-5)$, $f(-1)$, $f(0)$, and $f(3)$

169. $f(x) = \begin{cases} -2x & \text{if } x < -3 \\ 3x-1 & \text{if } -3 \leq x \leq 2 \\ -4x & \text{if } x > 2 \end{cases}$

Find: $f(-5)$, $f(-1)$, $f(0)$, and $f(3)$

170. $f(x) = \begin{cases} x^3 + 3 & \text{if } -2 \leq x \leq 0 \\ x + 3 & \text{if } 0 < x < 1 \\ 4 + x - x^2 & \text{if } 1 \leq x \leq 3 \end{cases}$ Find: $f(-5)$, $f(-1)$, $f(0)$, and $f(3)$

171. $h(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$ Find: $h(5)$, $h(0)$, and $h(3)$

172. Graph the piecewise function defined by $f(x) = \begin{cases} 3 & \text{if } x \leq -1 \\ x - 2 & \text{if } x > -1 \end{cases}$

173. Sketch the graph $f(x) = \begin{cases} x + 2 & \text{if } x \leq -1 \\ x^3 & \text{if } -1 < x < 1 \\ -x + 3 & \text{if } x \geq 1 \end{cases}$

174. Sketch the graph $f(x) = \begin{cases} x - 3 & \text{if } x \leq -2 \\ -x^2 & \text{if } -2 < x < 1 \\ -x + 4 & \text{if } x \geq 1 \end{cases}$