Linear 10/1

$$\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}$$

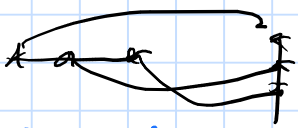
$$d = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$$d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$$
$$= \sqrt{(u_1-v_1)^2 + \cdots + (u_n-v_n)^2}$$

Dot Product – Euclidean Inner Product

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta \qquad \langle \vec{u},\vec{v}\rangle$$
$$= u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

summation of product of entries.



$v\,(0,2,2)$

Ex $\vec{u} = (0,0,1) \qquad \vec{v} = (0,2,2)$

$\theta = 45^\circ$

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$
$$= 1\sqrt{4+4}\cos 45^\circ$$
$$= 2\sqrt{2}\,\frac{\sqrt{2}}{2}$$
$$= \boxed{2}$$

Ex $\vec{u} = (4,2) \qquad \vec{w} = (-1,2)$

$$\vec{u}\cdot\vec{w} = 4(-1) + 2(2)$$
$$= -4 + 4$$
$$= \boxed{0}$$

2 vectors are perpendicular (Orthogonal) $\perp$

the dot products $= 0$.

Theorem:

$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

$\vec{u} \cdot (\vec{v} \pm \vec{w}) = \vec{u} \cdot \vec{v} \pm \vec{u} \cdot \vec{w}$

$(\vec{v} \pm \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} \pm \vec{w} \cdot \vec{u}$

$k(\vec{u} \cdot \vec{w}) = (k\vec{u}) \cdot \vec{w}$

$= \vec{u} \cdot (k\vec{w})$

$\vec{v} \cdot \vec{v} \geq 0 \quad \vec{v} \cdot \vec{v} = 0$ iff $\vec{v} = \vec{0}$

$\vec{u} \cdot \vec{0} = \vec{0} \cdot \vec{u} = 0.$

Proof $\perp$ vectors $\|\vec{v}\|^2 + \|\vec{u}\|^2 = \|\vec{v} - \vec{u}\|^2$

let $\vec{u} = (u_1, u_2) \quad \vec{v} = (v_1, v_2)$

$\|\vec{v} - \vec{u}\|^2 = (v_1 - u_1)^2 + (v_2 - u_2)^2$

$= v_1^2 - 2v_1 u_1 + u_1^2 + v_2^2 - 2v_2 u_2 + u_2^2$

Since $\vec{u} \perp \vec{v} \Rightarrow \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 = 0$

$= v_1^2 - 2(v_1 u_1 + v_2 u_2) + u_1^2 + v_2^2 + u_2^2$

$= v_1^2 + v_2^2 + u_1^2 + u_2^2$

$= \|\vec{v}\|^2 + \|\vec{u}\|^2$ ✓

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})(\vec{u}-\vec{v})$$

$$= \vec{u}\,\vec{u} - \underline{\vec{u}\,\vec{v}} - \underline{\vec{v}\,\vec{u}} + \vec{v}\,\vec{v} \qquad \|\vec{v}\| = \vec{v}\cdot\vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Schwarz Inequality

$$\|\vec{v}\cdot\vec{w}\| \leq \|\vec{v}\|\,\|\vec{w}\|$$

$$\|\vec{v}\cdot\vec{w}\| = \big|\,\|\vec{v}\|\,\|\vec{w}\|\cos\theta\,\big|$$

$$0 \leq |\cos\theta| \leq 1$$

$$\leq \|\vec{v}\|\,\|\vec{w}\|$$

$$\vec{v} = (a,b) \qquad \vec{w} = (b,a) \qquad \vec{v}\cdot\vec{w} = 2ab$$

$$\|\vec{v}\| = \|\vec{w}\| = \sqrt{a^2+b^2}$$

$$\|\vec{v}\cdot\vec{w}\| \leq \|\vec{v}\|\,\|\vec{w}\|$$

$$2ab \leq a^2+b^2 \quad ?$$

$$(a-b)^2 \geq 0$$

$$a^2 - 2ab + b^2 \geq 0$$

$$a^2+b^2 \geq 2ab$$

$$2ab \leq a^2+b^2 \quad \checkmark$$

$$\|\vec{u}+\vec{v}\|^2+\|\vec{u}-\vec{v}\|^2=2\left(\|\vec{u}\|^2+\|\vec{v}\|^2\right)$$

Proof

$$\|\vec{u}+\vec{v}\|^2+\|\vec{u}-\vec{v}\|^2=(\vec{u}+\vec{v})(\vec{u}+\vec{v})+(\vec{u}-\vec{v})(\vec{u}-\vec{v})$$
$$=\vec{u}^2+\vec{u}\vec{v}+\vec{v}\vec{u}+\vec{v}^2+\vec{u}\vec{u}-\vec{u}\vec{v}-\vec{v}\vec{u}+\vec{v}\cdot\vec{v}$$
$$=2\|\vec{u}\|^2+2\|\vec{v}\|^2$$
$$=2\left(\|\vec{u}\|^2+\|\vec{v}\|^2\right) \checkmark$$

$$\vec{u}\cdot\vec{v}=\frac{1}{4}\|\vec{u}+\vec{v}\|^2-\frac{1}{4}\|\vec{u}-\vec{v}\|^2$$

2.3 Orthogonality.

Defn

2 nonzero vectors $\vec{u}$ & $\vec{v}$ in $\mathbb{R}^n$ are said to be orthogonal ($\perp$) if their dot product is zero ($\vec{u}\cdot\vec{v}=0$)

A set of orthogonal vectors : orthogonal set.
An orthogonal set of unit vectors is :
orthonormal set.

Ex $\vec{u}=(-2,3,1,4)$ $\vec{v}=(1,2,0,-1)$ $\mathbb{R}^4$

$$\vec{u}\cdot\vec{v}=-2+6+0-4$$
$$=0$$

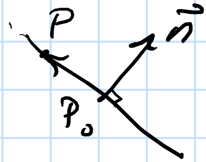
$\therefore$ $\vec{u}$ & $\vec{v}$ are orthogonal in $\mathbb{R}^4$

$$\hat{i}\cdot\hat{j}=\hat{i}\cdot\hat{k}=\hat{j}\cdot\hat{k}=0$$
$$\hat{i}\cdot\hat{j}=(1,0,0)\cdot(0,1,0)$$
$$=0+0+0$$
$$=0 \checkmark$$

$$\vec{N} = (N_1, N_2, N_3)$$
$$= N_1 \vec{i} + N_2 \vec{j} + N_3 \vec{k}$$

Normal is vector $\perp$ to line or plane

$$\vec{n} = (a, b) = a\vec{i} + b\vec{j}$$
$$\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$$



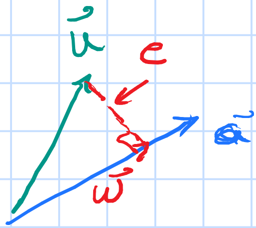
$$\vec{n} \cdot \vec{PP_0} = 0$$

2-dim: $(a, b) \cdot (x_0 - x, y_0 - y) = 0$

$$a(x_0 - x) + b(y_0 - y) = 0$$
$$ax_0 - ax + by_0 - by = 0$$
$$ax + by = ax_0 + by_0$$
$$ax + by = C$$

$$\vec{n} = (a, b)$$

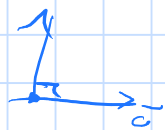
$$ax + by + cz = ax_0 + by_0 + cy_0$$
$$= D$$
$$Ax + By + Cz = D$$
$$\vec{n} = (a, b, c)$$



Projections

$$\text{proj}_{\vec{a}} \vec{u} = \frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} \vec{a}$$

$$\vec{u} \cdot \vec{a} = \mathbb{R}$$
$$\|\vec{a}\|^2 = \mathbb{R}$$



$$\vec{e} = \vec{u} - \text{proj}_{\vec{a}} \vec{u}$$

if $\frac{\vec{u} \cdot \vec{a}}{\|\vec{a}\|^2} = 1 \Rightarrow \vec{a}$ $\qquad$ $\vec{a}$ onto $\vec{a}$

Ex $\vec{u} = (2,-1,3)$ $\vec{a} = (4,-1,2)$

$$\text{proj}_{\vec{a}}\vec{u} = \frac{\vec{u}\cdot\vec{a}}{\|a\|^2}\vec{a}$$

$$= \frac{(2,-1,3)\cdot(4,-1,2)}{16+1+4}(4,-1,2)$$

$$= \frac{8+1+6}{21}(4,-1,2)$$

$$= \frac{5}{7}(4,-1,2)$$

$$= \boxed{\left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right)}$$

vector component of $\vec{u}$ orth. to $\vec{a}$

$$\vec{u} - \text{proj}_{\vec{a}}\vec{u} = (2,-1,3) - \left(\frac{20}{7}, \frac{-5}{7}, \frac{10}{7}\right)$$

$$= \boxed{\left(-\frac{6}{7}, -\frac{2}{7}, \frac{11}{7}\right)}$$

Theorem Pythagoras $\mathbb{R}^2$: $\vec{u}$ & $\vec{v}$ orthogonal

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

$\vec{u}\cdot\vec{v} = \vec{v}\vec{u} = 0$

$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})(\vec{u}+\vec{v})$$

$$= \vec{u}\vec{u} + \vec{u}\vec{v} + \vec{v}\vec{u} + \vec{v}\vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2$$

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}$$

$ax + by + c = 0$

$$d = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$$

$ax + by + cz + d = 0$