# *Section* 3.6 – Solving Linear Recurrence Relations

## *Definition*

A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

Where $c_1, c_2, \cdots, c_k$ are real numbers, and $c_k \neq 0$

✓ It is *linear* because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of *n*.

✓ It is *homogeneous* because no terms occur that are not multiples of the $a_j s$. Each coefficient is a constant.

✓ The *degree* is $k$ because $a_n$ is expressed in terms of the previous *k* terms of the sequence.

### Solving Linear Homogeneous Recurrence Relations

The basic approach is to look for solutions of the form $a_n = r^n$, where *r* is a constant.

Note that $a_n = r^n$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$.

Algebraic manipulation yields the *characteristic equation*:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r^{n-k} = 0$$

The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if *r* is a solution to the characteristic equation.

The solutions to the characteristic equation are called the *characteristic roots* of the recurrence relation. The roots are used to give an explicit formula for all the solutions of the recurrence relation.

## *Theorem*

Let $c_1$ and $c_2$ be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots $r_1$ and $r_2$. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if for $n = 0$, 1, 2, … , where $\alpha_1$ and $\alpha_2$ are constants.

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

This shows that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation.

***Example***

What is the solution to the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

***Solution***

The characteristic equation is $r^2 - r - 2 = 0$.

Its roots are $r = 2$ and $r = -1$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$, for some constants $\alpha_1$ and $\alpha_2$.

To find the constants $\alpha_1$ and $\alpha_2$, note that

$$a_0 = \alpha_1 + \alpha_2 = 2$$

$$a_1 = 2\alpha_1 - \alpha_2 = 7.$$

Solving these equations, we find that $\alpha_1 = 3$ and $\alpha_2 = -1$.

Hence, the solution is the sequence $\{a_n\}$ with $a_n = 3 \cdot 2^n - (-1)^n$

## An Explicit Formula for the Fibonacci Numbers

### *Example*

We can use Theorem to find an explicit formula for the Fibonacci numbers. The sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with the initial conditions: $f_0 = 0$ and $f_1 = 1$.

***Solution***

The roots of the characteristic equation $r^2 - r - 1 = 0$ are $r_{1,2} = \frac{1 \pm \sqrt{5}}{2}$

Therefore, from the theorem it follows that the Fibonacci numbers are given by

$$f_n = \alpha_1 \left( \frac{1+\sqrt{5}}{2} \right)^n + \alpha_2 \left( \frac{1-\sqrt{5}}{2} \right)^n$$

Using the initial conditions $f_0 = 0$ and $f_1 = 1$, we have

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \left( \frac{1+\sqrt{5}}{2} \right) + \alpha_2 \left( \frac{1-\sqrt{5}}{2} \right) = 1$$

The solution to these simultaneous equations for $\alpha_1$ and $\alpha_2$ is $\alpha_1 = \frac{1}{\sqrt{5}}$ and $\alpha_2 = -\frac{1}{\sqrt{5}}$

Consequently, the Fibonacci numbers are given by

$$f_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n$$

***Theorem***

Let $c_1$ and $c_2$ be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has one repeated root $r_0$. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ *iff* $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0,1,2,\ldots$ , where $\alpha_1$ and $\alpha_2$ are constants.

### Example

What is the solution to the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

**Solution**

The characteristic equation is $r^2 - 6r + 9 = 0$.

The only root is $r = 3$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if

$a_n = \alpha_1 3^n + \alpha_2 n(3)^n$ where $\alpha_1$ and $\alpha_2$ are constants.

$$\begin{cases} a_0 = 1 = \alpha_1 \\ a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3 \end{cases} \rightarrow \quad \alpha_1 = 1 \ \text{ and } \ \alpha_2 = 1$$

Hence, $a_n = 3^n + n(3)^n \ \boxed{= (n+1)3^n}$.

### Theorem

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

has $k$ distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

$$\text{iff} \qquad a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

### Example

Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$?

**Solution**

The characteristic equation is $r^3 - 6r^2 + 11r - 6 = 0$.

The characteristic roots are $r = 1, 2, 3$.

The solutions to the recurrence relation are of the form $a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_2 \cdot 3^n$ where $\alpha_1$, $\alpha_2$ and $\alpha_3$ are constants.

$$\begin{cases} a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3 \\ a_1 = 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3 \\ a_2 = 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9 \end{cases} \rightarrow \alpha_1 = 1 \quad \alpha_2 = -1 \quad and \quad \alpha_3 = 2$$

$$\boxed{a_n = 1 - 2^n + 2 \cdot 3^n}$$

## *Theorem*

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

has $t$ distinct roots $r_1, r_2, \ldots, r_k$ with multiplicities $m_1, m_2, \ldots, m_k$ respectively so that $m_i \geq 1$ for $i$ = 0, 1, 2, …, $t$ and $m_1 + m_2 + \ldots + m_t = k$ Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

$$\begin{aligned} \textit{iff} \qquad a_n = & \left( \alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} \cdot n^{m_1-1} \right) r_1^n \\ & + \left( \alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} \cdot n^{m_2-1} \right) r_2^n \\ & + \cdots + \left( \alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} \cdot n^{m_t-1} \right) r_t^n \end{aligned}$$

for $n$ = 0, 1, 2, …, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_{i-1}$.

## *Example*

Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 2, 2, 2, 5, 5, and 9 (that is, there are three roots, the root 2 with multiplicity three, the root 5 with multiplicity two, and the root 9 with multiplicity one). What is the form of the general solution?

***Solution***

The general form of the solution is:

$$\left( \alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} \cdot n^2 \right) 2^n + \left( \alpha_{2,0} + \alpha_{2,1} n \right) 5^n + \alpha_{3,0} 9^n$$

### *Example*

Find the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$?

***Solution***

The characteristic equation is $r^3 + 3r^2 + 3r + 1 = 0$.

The characteristic root is a single root $r = -1$ of multiplicity three.

The solutions to the recurrence relation are of the form

$a_n = \left(\alpha_{1,0} + \alpha_{1,1} \cdot n + \alpha_{1,2} \cdot n^2\right) \cdot (-1)^n$ where $\alpha_1$, $\alpha_2$ and $\alpha_3$ are constants.

$$\begin{cases} a_0 = 1 = \alpha_{1,0} \\ a_1 = -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2} \\ a_2 = -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2} \end{cases} \rightarrow \quad \alpha_{1,0} = 1 \;\; \alpha_{1,1} = 3 \;\; \textit{and} \;\; \alpha_{1,2} = -2$$

$$a_n = \left(1 + 3n - 2n^2\right) \cdot (-1)^n$$

## Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

### *Definition*

A *linear nonhomogeneous recurrence relation with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $F(n)$ is a function not identically zero depending only on $n$.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is called the ***associated homogeneous recurrence relation***.

- The following are linear nonhomogeneous recurrence relations with constant coefficients:

$$a_n = a_{n-1} + 2^n$$

$$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$$

$$a_n = 3a_{n-1} + n3^n$$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$$

where the following are the associated linear homogeneous recurrence relations, respectively:

$$a_n = a_{n-1}$$

$$a_n = a_{n-1} + a_{n-2}$$
$$a_n = 3a_{n-1}$$
$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

### *Theorem*

If $\left\{a_n^{(p)}\right\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

then every solution is of the form $\left\{a_n^{(p)} + a_n^{(k)}\right\}$, where $\left\{a_n^{(k)}\right\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

### *Example*

Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

***Solution***

The associated linear homogeneous equation is $a_n = 3a_{n-1}$.

Its solutions are $a_n^{(k)} = \alpha 3^n$, where α is a constant.

Because $F(n)= 2n$ is a polynomial in $n$ of degree one.
Let the linear function $p_n = cn + d$ be such a solution

Then $a_n = 3a_{n-1} + 2n$ becomes $cn + d = 3(c(n-1) + d) + 2n$.

$\Rightarrow (2 + 2c)n + (2d - 3c) = 0.$

It follows that $cn + d$ is a solution if and only if $2 + 2c = 0$ and $2d - 3c = 0$.
Therefore, $cn + d$ is a solution if and only if $c = -1$ and $d = -3/2$.

Consequently, $a_n^{(p)} = -n - \frac{3}{2}$ is a particular solution.

By Theorem, all solutions are of the form $a_n = a_n^{(p)} + a_n^{(k)} = -n - \frac{3}{2} + \alpha 3^n$, where α is a constant.

$a_1 = 3$, let $n = 1$. Then $3 = -1 - \frac{3}{2} + 3\alpha \quad \to \quad 3\alpha = 3 + \frac{5}{2} \Rightarrow \boxed{\alpha = \frac{11}{6}}$.

Hence, the solution is $\underline{a_n = -n - \frac{3}{2} + \frac{11}{6}3^n}$.

### *Example*

Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.

***Solution***

The linear nonhomogeneous equation is $a_n = 5a_{n-1} - 6a_{n-2}$.

Its solutions are $a_n^{(k)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$, where $\alpha_1$ and $\alpha_2$ are constants

The trial solution is $a_n^{(p)} = C \cdot 7^n$

$$C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n$$

$$C \cdot 7^n = 7^{n-2}\left(35C - 6C + 49\right)$$

$$C \cdot 7^2 = 29C + 49$$

$$49C - 29C = 49$$

$$20C = 49$$

$$C = \frac{49}{20}$$

Hence, $a_n^{(p)} = \frac{49}{20} \cdot 7^n$

Hence, the solution is $a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + \frac{49}{20} \cdot 7^n$.

# *Exercises* *Section* 3.6 – Solving Linear Recurrence Relations

**1.** Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also find the degree of those that are

*a)* $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$

*b)* $a_n = 2na_{n-1} + a_{n-2}$

*c)* $a_n = a_{n-1} + a_{n-4}$

*d)* $a_n = a_{n-1} + 2$

*e)* $a_n = a_{n-1}^2 + a_{n-2}$

*f)* $a_n = a_{n-2}$

*g)* $a_n = a_{n-1} + n$

*h)* $a_n = 3a_{n-2}$

*i)* $a_n = 3$

*j)* $a_n = a_{n-1}^2$

*k)* $a_n = a_{n-1} + 2a_{n-3}$

*l)* $a_n = \dfrac{a_{n-1}}{n}$

**2.** Solve these recurrence relations together with the initial conditions given

*a)* $a_n = 2a_{n-1}$ *for* $n \ge 1,\ a_0 = 3$

*b)* $a_n = 5a_{n-1} - 6a_{n-2}$ *for* $n \ge 2,\ a_0 = 1,\ a_1 = 0$

*c)* $a_n = 4a_{n-1} - 4a_{n-2}$ *for* $n \ge 2,\ a_0 = 6,\ a_1 = 8$

*d)* $a_n = 4a_{n-2}$ *for* $n \ge 2,\ a_0 = 0,\ a_1 = 4$

*e)* $a_n = \dfrac{a_{n-2}}{4}$ *for* $n \ge 2,\ a_0 = 1,\ a_1 = 0$

*f)* $a_n = a_{n-1} + 6a_{n-2}$ *for* $n \ge 2,\ a_0 = 3,\ a_1 = 6$

*g)* $a_n = 7a_{n-1} - 10a_{n-2}$ *for* $n \ge 2,\ a_0 = 2,\ a_1 = 1$

*h)* $a_n = -6a_{n-1} - 9a_{n-2}$ *for* $n \ge 2,\ a_0 = 3,\ a_1 = -3$

*i)* $a_{n+2} = -4a_{n-1} + 5a_n$ *for* $n \ge 0,\ a_0 = 2,\ a_1 = 8$

**3.** How many different messages can be transmitted in $n$ microseconds using three different signals if one signal requires 1 microsecond for transmittal, the other two signals require 2 microseconds each for transmittal, and a signal in a message is followed immediately by the next signal?

**4.** In how many ways can a $2 \times n$ rectangular checkerboard be tiled using $1 \times 2$ and $2 \times 2$ pieces?

**5.** Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3} \quad for \quad n \geq 3,\ a_0 = 3,\ a_1 = 6\ and\ a_2 = 0$

**6.** Find the solution to $a_n = 7a_{n-2} + 6a_{n-3} \quad with \quad a_0 = 9,\ a_1 = 10\ and\ a_2 = 32$

**7.** Find the solution to $a_n = 5a_{n-2} - 4a_{n-4} \quad with \quad a_0 = 3,\ a_1 = 2,\ a_2 = 6\ and\ a_3 = 8$

**8.** Find the recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} \quad with \quad a_0 = -5,\ a_1 = 4\ and\ a_2 = 88$

**9.** Find the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3} \quad with \quad a_0 = 5,\ a_1 = -9\ \ and\ a_2 = 15$

**10.** Find the general form of the solutions of the recurrence relation $a_n = 8a_{n-2} - 16a_{n-4}$

**11.** What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots 1, 1, 1, 1, –2, –2, –2, 3, 3, –4?

**12.** What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots $-1, -1, -1, 2, 2, 5, 5, 7$?