Lecture R – Calculus I – Review

*Section R.*1 – Derivative

Constant Rule

$$\frac{d}{dx}[c]=0 \qquad c \text{ is constant}$$

Example

Find the derivative:

a) $f(x)=-2$ $\qquad f'(x)=0$

b) $y=\pi$ $\qquad y'=0$

c) $g(w)=\sqrt{5}$ $\qquad g'(w)=0$

d) $s(t)=320.5$ $\qquad s'(t)=0$

Power Rule

$$\frac{d}{dx}[x^n]=nx^{n-1} \qquad n \text{ is any real number}$$

Constant Times a Function

$$\frac{d}{dx}\left[cf(x)\right]=cf'(x)$$

Example

Find the derivative each function

a. $y=\dfrac{9}{4x^2}$

Solution

$$y=\frac{9}{4}x^{-2}$$

$$\to y'=\frac{9}{4}(-2)x^{-3}=-\frac{9}{2x^3}$$

b. $y=\sqrt[3]{x}$

Solution

$$y=x^{1/3}$$

$$\to y'=\frac{1}{3}x^{(1/3)-1}=\frac{1}{3}x^{-2/3}=\frac{1}{3\sqrt[3]{x^2}}$$

The Product Rule

The derivative of the product of two differentiable functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$(f.g)' = f.g' + f'.g$$

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'gh + fg'h + fgh'$$

Example

Find the derivative of $y = \left(4x + 3x^2\right)(6 - 3x)$

Solution

$$y = 24x + 6x^2 - 9x^3$$

$$\begin{aligned} y' &= \left(4x + 3x^2\right)\frac{d}{dx}(6 - 3x) + (6 - 3x)\frac{d}{dx}\left(4x + 3x^2\right) \\ &= \left(4x + 3x^2\right)(-3) + (6 - 3x)(4 + 6x) \\ &= -12x - 9x^2 + 24 + 36x - 12x - 18x^2 \\ &\boxed{= -27x^2 + 12x + 24} \end{aligned}$$

Example

Find the derivative of $y = \left(\frac{1}{x} + 1\right)(2x + 1)$

Solution

$$\begin{aligned} y' &= \left(x^{-1} + 1\right)\frac{d}{dx}(2x + 1) + (2x + 1)\frac{d}{dx}\left(x^{-1} + 1\right) \\ &= \left(x^{-1} + 1\right)(2) + (2x + 1)\left(-x^{-2}\right) \\ &= \frac{2}{x} + 2 + (2x + 1)\left(-\frac{1}{x^2}\right) \\ &= \frac{2}{x} + 2 - \frac{2x}{x^2} - \frac{1}{x^2} \\ &= \frac{2}{x} + 2 - \frac{2}{x} - \frac{1}{x^2} \\ &= 2 - \frac{1}{x^2} \\ &\boxed{= \frac{2x^2 - 1}{x^2}} \end{aligned}$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}=\frac{gf'-fg'}{g^2}$$

Example

Find the derivative of $y=\frac{x+4}{5x-2}$

Solution

$$y'=\frac{(5x-2)\frac{d}{dx}[(x+4)]-(x+4)\frac{d}{dx}[(5x-2)]}{(5x-2)^2}$$

$$=\frac{(5x-2)(1)-(x+4)(5)}{(5x-2)^2}$$

$$=\frac{5x-2-5x-20}{(5x-2)^2}$$

$$=-\frac{22}{(5x-2)^2}$$

Example

Find the derivative of $y=\frac{3-\frac{2}{x}}{x+4}$

Solution

$$y=\frac{\frac{3x-2}{x}}{x+4}=\frac{3x-2}{x}\cdot\frac{1}{x+4}=\frac{3x-2}{x^2+4x}$$

$$y'=\frac{\left(x^2+4x\right)(3)-(3x-2)(2x+4)}{\left[x(x+4)\right]^2}$$

$$=\frac{3x^2+12x-6x^2-12x+4x+8}{x^2(x+4)^2}$$

$$\boxed{=\frac{-3x^2+4x+8}{x^2(x+4)^2}}$$

Chain Rule

The General Power Rule

$$\frac{dy}{dx} = \frac{d}{dx}\left[u(x)^n\right]$$

$$= n\, u^{n\text{-}1}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[u^n\right] = n\, u^{n\text{-}1}u'$$

Example

Find the derivative of $y = \left(x^2 + 3x\right)^4$

Solution

$$u = x^2 + 3x$$

$$y' = n \quad (u)^{n-1} \quad \frac{d}{dx}[u]$$

$$= 4\left(x^2 + 3x\right)^3 \frac{d}{dx}[x^2 + 3x]$$

$$= 4\left(x^2 + 3x\right)^3 (2x + 3)$$

Formula $\left(U^m V^n W^p\right)' = U^{m-1}V^{n-1}W^{p-1}\left(mU'VW + nUV'W + pUVW'\right)$

Proof

$$\left(U^m V^n W^p\right)' = \left(U^m\right)' V^n W^p + U^m\left(V^n\right)' W^p + U^m V^n\left(W^p\right)'$$

$$= mU^{m-1}U'V^nW^p + nU^mV^{n-1}V'W^p + pU^mV^nW^{p-1}W' \qquad \textit{factor} \;\; U^{m-1}V^{n-1}W^{p-1}$$

$$= U^{m-1}V^{n-1}W^{p-1}\left(mU'VW + nUV'W + pUVW'\right)$$

Derivatives of Trigonometric Functions

$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$	$(\tan x)' = \sec^2 x$
$(\csc x)' = -\csc x \cot x$	$(\sec x)' = \sec x \tan x$	$(\cot x)' = -\csc^2 x$

Example

Find the derivatives

a) $y = \sin x \cos x$

$$
\begin{aligned}
y' &= \sin x(\cos x)' + \cos x(\sin x)' \\
&= \sin x(-\sin x) + \cos x(\cos x) \\
&= \cos^2 x - \sin^2 x
\end{aligned}
$$

b) $y = \dfrac{\cos x}{1-\sin x}$

$$
\begin{aligned}
y' &= \frac{(1-\sin x)(\cos x)' - \cos x(1-\sin x)'}{(1-\sin x)^2} \\
&= \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2} \\
&= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \\
&= \frac{1-\sin x}{(1-\sin x)^2} \\
&= \frac{1}{1-\sin x}
\end{aligned}
$$

$\sin^2 x + \cos^2 x = 1$

Derivatives of Logarithmic

Derivative of $y = \ln x$ $\boxed{\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}}$ $x \neq 0$

The chain rule extends: $\boxed{\dfrac{d}{dx}\ln u = \dfrac{1}{u}\cdot\dfrac{du}{dx} = \dfrac{u'}{u}}$ $u > 0$

Example

Find $\dfrac{d}{dx}\ln 2x$

Solution

$$\frac{d}{dx}\ln 2x = \frac{(2x)'}{2x}$$

$$= \frac{2}{2x}$$

$$\boxed{= \frac{1}{x}}$$

Example

Find the derivative of $\ln\left(x^2+3\right)$

Solution

$$\frac{d}{dx}\ln\left(x^2+3\right) = \frac{2x}{x^2+3}$$

Derivative $\boxed{\dfrac{d}{dx}\left(\log_a u\right) = \dfrac{1}{u}\cdot\dfrac{1}{\ln a}\dfrac{du}{dx}}$

Example

- $\dfrac{d}{dx}\log_{10}(3x+1) = \dfrac{1}{(3x+1)\cdot\ln 10}\dfrac{d}{dx}(3x+1) = \boxed{\dfrac{3}{(3x+1)\cdot\ln 10}}$

Derivatives of Exponential Functions

If u is any differentiable function of x, then $\boxed{\frac{d}{dx}e^u = e^u \frac{du}{dx}}$ $\boxed{\left(e^u\right)' = u'e^u}$

Example

Find the derivative of $\frac{d}{dx}\left(5e^x\right)$

Solution

$$\frac{d}{dx}\left(5e^x\right) = 5\frac{d}{dx}e^x = \underline{5e^x}$$

Example

Find the derivative of $\frac{d}{dx}\left(e^{\sin x}\right)$

Solution

$$\frac{d}{dx}\left(e^{\sin x}\right) = e^{\sin x}\frac{d}{dx}(\sin x) \; \underline{= e^{\sin x} \cdot \cos x}$$

Example

Find the derivative of $\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right)$

Solution

$$\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right) = e^{\sqrt{3x+1}} \cdot \tfrac{1}{2}(3x+1)^{-1/2} \cdot 3$$

$$= \underline{\frac{3}{2\sqrt{3x+1}}e^{\sqrt{3x+1}}}$$

Definition

If $a > 0$ and u is a differentiable of x, then a^u is a differentiable function of x and

$$\boxed{\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}}$$

Example

- $\frac{d}{dx}3^x = 3^x \ln 3$
- $\frac{d}{dx}3^{-x} = 3^{-x}\ln 3\frac{d}{dx}(-x) = -3^{-x}\ln 3$
- $\frac{d}{dx}3^{\sin x} = 3^{\sin x}\ln 3\frac{d}{dx}(\sin x) = 3^{\sin x}\ln 3(\cos x)$

Derivatives of Inverse Trigonometric Functions

$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad \lvert u\rvert < 1$	$\left(\sin^{-1}u\right)' = \frac{u'}{\sqrt{1-u^2}}$
$\frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad \lvert u\rvert < 1$	$\left(\cos^{-1}u\right)' = -\frac{u'}{\sqrt{1-u^2}}$
$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$	$\left(\tan^{-1}u\right)' = \frac{u'}{1+u^2}$
$\frac{d}{dx}\cot^{-1}u = -\frac{1}{1+u^2}\frac{du}{dx}$	$\left(\cot^{-1}u\right)' = -\frac{u'}{1+u^2}$
$\frac{d}{dx}\sec^{-1}u = \frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}, \quad \lvert u\rvert > 1$	$\left(\sec^{-1}u\right)' = \frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$
$\frac{d}{dx}\csc^{-1}u = -\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}, \quad \lvert u\rvert > 1$	$\left(\csc^{-1}u\right)' = -\frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$

Example

Find the derivative of $\frac{d}{dx}\left(\sin^{-1}x^2\right)$

Solution

$$\frac{d}{dx}\left(\sin^{-1}x^2\right) = \frac{2x}{\sqrt{1-x^4}}$$

Example

Find the derivative of $\frac{d}{dx}\left(\sec^{-1}5x^4\right)$

Solution

$$\frac{d}{dx}\left(\sec^{-1}5x^4\right) = \frac{\left(5x^4\right)'}{5x^4\sqrt{\left(5x^4\right)^2-1}} \qquad 5x^4-1>0$$

$$= \frac{20x^3}{5x^4\sqrt{25x^8-1}}$$

$$= \frac{4}{x\sqrt{25x^8-1}}$$

Exercises Section *R*.1 – Derivative

Find the derivative to the following functions

1. $f(t) = -3t^2 + 2t - 4$
2. $g(x) = 4\sqrt[3]{x} + 2$
3. $f(x) = x\left(x^2 + 1\right)$
4. $f(x) = \frac{2x^2 - 3x + 1}{x}$
5. $f(x) = \frac{4x^3 - 3x^2 + 2x + 5}{x^2}$
6. $f(x) = \frac{-6x^3 + 3x^2 - 2x + 1}{x}$
7. $f(x) = x\left(1 - \frac{2}{x+1}\right)$
8. $g(s) = \frac{s^2 - 2s + 5}{\sqrt{s}}$
9. $f(x) = \frac{x+1}{\sqrt{x}}$
10. $f(x) = 3x\left(2x^2 + 5x\right)$
11. $y = 3\left(2x^2 + 5x\right)$
12. $y = \frac{x^2 + 4x}{5}$
13. $y = \frac{3x^4}{5}$
14. $y = \frac{x^2 - 4}{2x + 5}$
15. $y = \frac{(1+x)(2x-1)}{x-1}$
16. $y = \frac{4}{2x+1}$
17. $y = \frac{2}{(x-1)^3}$
18. $f(x) = \sqrt{2t^2 + 5t + 2}$
19. $f(x) = \frac{1}{\left(x^2 - 3x\right)^2}$
20. $y = t^2\sqrt{t-2}$
21. $y = \left(\frac{6 - 5x}{x^2 - 1}\right)^2$
22. $y = x^2\sqrt{x^2 + 1}$
23. $y = \left(\frac{x+1}{x-5}\right)^2$
24. $y = \sqrt[3]{(x+4)^2}$
25. $y = x^2 \sin x$
26. $y = \frac{\sin x}{x}$
27. $y = \frac{\cot x}{1 + \cot x}$
28. $y = x^2 \sin x + 2x\cos x - 2\sin x$
29. $y = x^3 \sin x \cos x$
30. $y = \frac{4}{\cos x} + \frac{1}{\tan x}$
31. $f(x) = \frac{\left(x^2 - 6x\right)^5}{\left(3x^2 + 5x - 2\right)^4}$
32. $y = \ln\sqrt{x+5}$
33. $y = (3x+7)\ln(2x-1)$
34. $f(x) = \ln\sqrt[3]{x+1}$
35. $f(x) = \ln\left[x^2\sqrt{x^2+1}\right]$
36. $y = \ln\frac{x^2}{x^2+1}$

37. $f(x) = e^{-2x^3}$

38. $f(x) = 4e^{x^2}$

39. $f(x) = 2x^3 e^x$

40. $f(x) = \dfrac{3e^x}{1+e^x}$

41. $f(x) = 5e^x + 3x + 1$

42. $f(x) = x^2 e^x$

43. $f(x) = \dfrac{e^x + e^{-x}}{2}$

44. $f(x) = \dfrac{e^x}{x^2}$

45. $f(x) = x^2 e^x - e^x$

46. $f(x) = (1+2x)e^{4x}$

47. $y = x^2 e^{5x}$

48. $y = x^2 e^{-2x}$

49. $f(x) = \dfrac{e^x}{x^2+1}$

50. $f(x) = \dfrac{1-e^x}{1+e^x}$

51. $y = \dfrac{\ln x}{e^{2x}}$

52. $f(x) = e^{2x} \ln\left(xe^x + 1\right)$

53. $f(x) = \dfrac{xe^x}{\ln\left(x^2+1\right)}$

54. $y = \cos^{-1}\left(\frac{1}{x}\right)$

55. $y = \sin^{-1}\sqrt{2}t$

56. $y = \sec^{-1}(5s)$

57. $y = \cot^{-1}\sqrt{t-1}$

58. $y = \ln\left(\tan^{-1} x\right)$

59. $y = \tan^{-1}(\ln x)$

Section R.2 – Integration

Definition of Antiderivative

A Function F is an Antiderivative of a function f if for every x in the domain of f, it follows that $F'(x) = f(x)$

Notation for Antiderivatives and indefinite integrals

The notation

$$\int f(x)dx = F(x) + C$$

where C is an arbitrary constant, means that F is an Antiderivative of f.
That is $F'(x) = f(x)$ for all x in the domain of f.

$\int f(x)dx$ Indefinite integral

$$\text{Integral sign} \longrightarrow \int \overbrace{f(x)}^{\text{Integrand}} \underbrace{dx}_{\text{Differential}} = \overbrace{F(x)}^{\text{Antiderivative}} + C$$

Basic Integration Rules

$$\int kdx = kx + C$$

$$\int kf(x)dx = k\int f(x)dx$$

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

The General Power Rule

The Simple Power Rule is given by:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

$$\int \overbrace{\left(x^2+1\right)^3}^{u^3} \underbrace{2x dx}_{du} = \int u^3 du$$

$$= \frac{u^4}{4} + C$$

General Power Rule for Integration

If u is a differentiable function of x, then

$$\int u^n \frac{du}{dx} dx = \int u^n du = \frac{u^{n+1}}{n+1} + C \qquad n \neq -1$$

Example

Find each indefinite integral.

$$\int 5x dx = \int 5x^1 dx$$

$$= 5\frac{x^{1+1}}{1+1} + C$$

$$\boxed{= \frac{5}{2}x^2 + C}$$

$$\int \sqrt[3]{x} dx = \int x^{1/3} dx$$

$$= \frac{x^{1/3+1}}{1/3+1} + C$$

$$= \frac{x^{4/3}}{4/3} + C$$

$$\boxed{= \frac{3}{4}x^{4/3} + C} \qquad or \qquad = \frac{3}{4}x\sqrt[3]{x} + C$$

Example

Find the integral $\quad \int x^2 \sin\left(x^3\right) dx$

Solution

$$\int x^2 \sin\left(x^3\right) dx = \frac{1}{3}\int \sin x^3 \cdot d\left(x^3\right) \qquad d\left(x^3\right) = 3x^2 dx$$

$$\boxed{= -\frac{1}{3}\cos\left(x^3\right) + C}$$

Example

Find the integral $\quad \int x\sqrt{2x+1}\, dx$

Solution

Let: $u = 2x+1 \quad \Rightarrow \quad du = 2dx$

$$dx = \frac{1}{2}du$$

$$u = 2x+1 \to 2x = u-1 \quad \Rightarrow \quad x = \frac{u-1}{2}$$

$$\int x\sqrt{2x+1}\, dx = \int \frac{1}{2}(u-1)\sqrt{u}\ \frac{1}{2}du$$

$$= \frac{1}{4}\int (u-1)u^{1/2}\, du$$

$$= \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right) du$$

$$= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$$

$$\boxed{= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C}$$

Theorem – The Fundamental Theorem of Calculus, P-2

If f is continuous at every point in $[a, b]$, then F is any antiderivative of f on $[a, b]$, then

$$\boxed{\int_a^b f(x)dx = F(b) - F(a)}$$

Example

a) $\displaystyle\int_0^\pi \cos x \, dx = \sin x \Big|_0^\pi$

$$= \sin \pi - \sin 0$$

$$= \underline{0}$$

b) $\displaystyle\int_{-\frac{\pi}{4}}^0 \sec x \tan x \, dx = \sec x \Big|_{-\frac{\pi}{4}}^0$

$$= \sec 0 - \sec\left(-\frac{\pi}{4}\right)$$

$$= \underline{1 - \sqrt{2}}$$

c) $\displaystyle\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4$

$$= \left((4)^{3/2} + \frac{4}{4}\right) - \left((1)^{3/2} + \frac{4}{1}\right)$$

$$= (9) - (5)$$

$$= \underline{4}$$

Other Indefinite Integrals

$$\frac{d}{dx}\left(e^{ax}\right)=ae^{ax} \quad\rightarrow\quad \int e^{ax}dx=\frac{1}{a}e^{ax}+C$$

$$\frac{d}{dx}\left(\ln|x|\right)=\frac{1}{x} \quad\rightarrow\quad \int \frac{dx}{x}=\ln|x|+C$$

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right)=\frac{1}{\sqrt{a^2-x^2}} \quad\rightarrow\quad \int \frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)+C$$

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right)=\frac{a}{x^2+a^2} \quad\rightarrow\quad \int \frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

$$\frac{d}{dx}\left(\sec^{-1}\left|\frac{x}{a}\right|\right)=\frac{a}{x\sqrt{x^2-a^2}} \quad\rightarrow\quad \int \frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|+C$$

Example

Evaluate $\int e^{-10x}dx$

Solution

$$\int e^{-10x}dx=-\frac{1}{10}e^{-10x}+C$$

Example

Evaluate $\int \frac{5}{x}dx$

Solution

$$\int \frac{5}{x}dx=5\ln|x|+C$$

Example

Evaluate $\int \frac{4}{\sqrt{9-x^2}}dx$

Solution

$$\int \frac{4}{\sqrt{9-x^2}}dx=4\sin^{-1}\left(\frac{x}{3}\right)+C \qquad a^2=9 \quad\rightarrow\quad a=3$$

Exercises Section R.2 – Integration

Find each indefinite integral.

1. $\int \frac{x+2}{\sqrt{x}} dx$
2. $\int 4y^{-3} dy$
3. $\int (x^3 - 4x + 2) dx$
4. $\int \left(\sqrt[4]{x^3} + 1\right) dx$
5. $\int \sqrt{x}(x+1) dx$
6. $\int (1+3t)t^2 dt$
7. $\int \frac{x^2 - 5}{x^2} dx$
8. $\int (-40x + 250) dx$
9. $\int \left(7 - 3x - 3x^2\right)(2x+1)\, dx$
10. $\int \left(1 + \cos 3\theta\right) d\theta$
11. $\int 2\sec^2\theta\, d\theta$
12. $\int \sec 2x \tan 2x\, dx$
13. $\int 2e^{2x} dx$
14. $\int \frac{12}{x} dx$
15. $\int \frac{dx}{\sqrt{1-x^2}}$
16. $\int \frac{dx}{x^2+1}$
17. $\int \frac{1+\tan\theta}{\sec\theta} d\theta$

Find the general solution of the differential equation

18. $y' = 2t + 3$
19. $y' = 3t^2 + 2t + 3$
20. $y' = \sin 2t + 2\cos 3t$
21. $y' = x^3(3x^4+1)^2$
22. $y' = 5x\sqrt{x^2-1}$
23. $y' = x\sqrt{x^2+4}$

Evaluate the integrals

24. $\int_{-2}^{2} \left(x^3 - 2x + 3\right) dx$
25. $\int_{0}^{1} \left(x^2 + \sqrt{x}\right) dx$
26. $\int_{0}^{\pi/3} 4\sec u \tan u\, du$
27. $\int_{\pi/4}^{3\pi/4} \csc\theta \cot\theta\, d\theta$
28. $\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt$
29. $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy$
30. $\int_{1}^{8} \frac{\left(x^{1/3}+1\right)\left(2 - x^{2/3}\right)}{x^{1/3}} dx$
31. $\int_{0}^{1} (2t+3)^3 dt$
32. $\int_{-1}^{1} r\sqrt{1-r^2}\, dr$