Section 1.7 – Physical Applications

Density and Mass

Density is the concentration of mass in an object and is usually measured in units of mass per volume. An object with uniform density satisfies the basic relationship

$$mass = density \cdot volume$$

When density of an object varies, this formula no longer holds, and we must appeal to calculus.

Definition

Suppose a thin bar or wire can be represented as a line segment on the interval $a \le x \le b$ with a density function ρ (with units of mass per length). The mass of the object is

$$m = \int_a^b \rho(x)\,dx$$

Example

A thin 2-*m* bar, represented by the interval $0 \le x \le 2$, is made of any alloy whose density in units of *kg/m* is given by $\rho(x) = 1 + x^2$. What is the mass of the bar?

Solution

$$\begin{aligned} m &= \int_0^2 \left(1+x^2\right)dx \\ &= x + \frac{1}{3}x^3 \Big|_0^2 \\ &= 2 + \frac{8}{3} \\ &= \frac{14}{3}\ kg \end{aligned}$$

Work Done By a Constant Force

When a body moves a distance d along a straight line as a result of being acted on by a force of constant magnitude F in the direction of motion, we define the ***work*** W done by the force on the body with the formula

$$Work = force \cdot distance$$

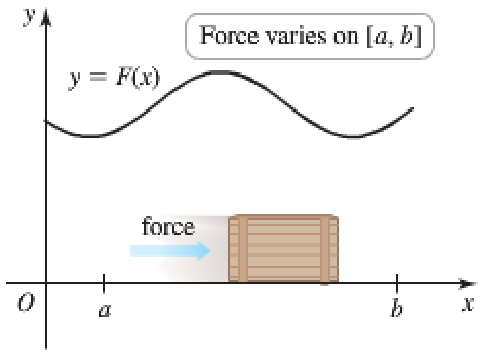
$$W = Fd \quad \textbf{(}\textit{Constant-force formula for work}\textbf{)}$$

The unit of work is a newton-meter (***N.m***), also called ***joule***.

Definition

The work done by a variable force $F(x)$ in the direction of motion along the *x*-axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\,dx$$

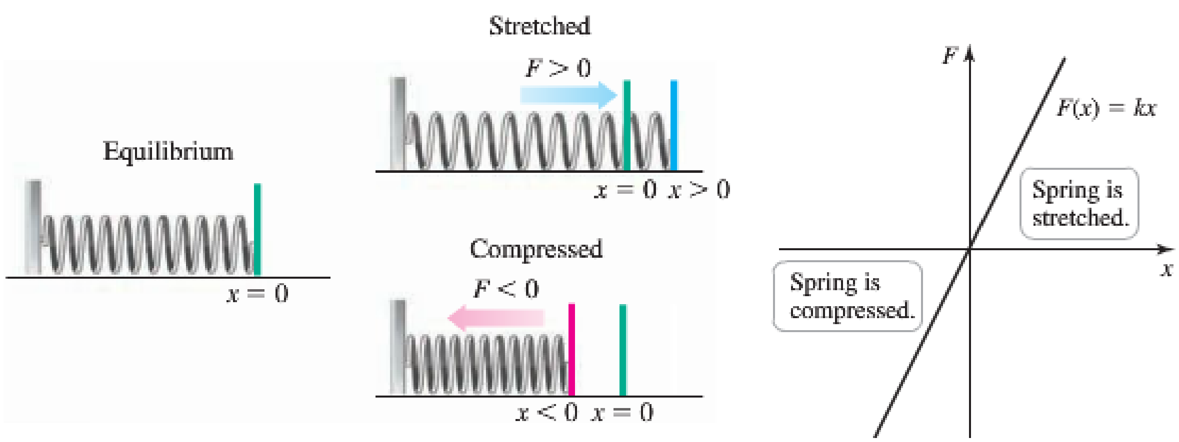


Hooke's Law for Springs: $F = kx$

Hooke's Law says that the force required to hold a stretched or compressed spring *x* units from its natural (unstressed) length is proportional to *x*. In symbols

$$F = kx$$

The constant *k*, measured in force units per unit length, is a characteristic of the spring, called ***the force constant*** (or ***spring constant***) of the spring.



- To stretch the spring to a position $x > 0$, a force $F > 0$ (in the positive direction) is required.
- To compress the spring to a position $x < 0$, a force $F < 0$ (in the negative direction) is required.

Example

Find the work required to compress a spring from its natural length of 1 *ft* to a length of 0.75 *ft* if the force constant is $k = 16$ lb./ft.

Solution

$$F = kx = 16x \qquad F(0) = 16 \cdot 0 = 0\ lb$$

$$F(0.25) = 16 \cdot (0.25) = 4\ lb$$

$$W = \int_a^b F(x)\,dx = \int_0^{0.25} 16x\,dx$$

$$= 8x^2 \Big|_0^{0.25}$$

$$= 8\left(0.25^2 - 0\right)$$

$$= 0.5\ \ ft - lb$$

Example

A spring has a natural length of 1 *m*. A force of 24 *N* holds the spring stretched to a total length of 1.8 *m*.

a) Find the force constant *k*.

b) How much work will it take to stretch the spring 2 *m* beyond its natural length?

c) How far will a 45-*N* force stretch the spring?

Solution

a) $F = kx \quad \rightarrow \quad 24 = k(1.8 - 1)$

$$24 = k(0.8) \quad \Rightarrow \quad k = \frac{24}{0.8} = 30\ N/m$$

b) $F(x) = 30x$

$$W = \int_0^2 30x\,dx$$

$$= 15x^2 \Big|_0^2$$

$$= 15\left(2^2 - 0\right)$$

$$= 60\ J$$

c) $45 = 30x$

$$x = \frac{45}{30} = 1.5\ m$$

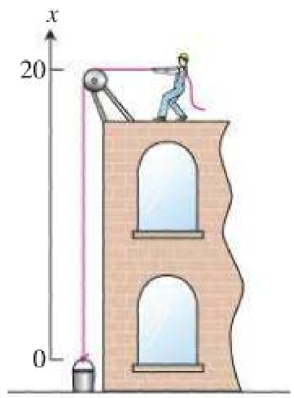
Example

A 5-*lb* bucket is lifted from the ground into the air by pulling in 20 *ft* of rope at a constant speed. The rope weighs 0.08 *lb/ft*. How much work was spent lifting the bucket and rope?

Solution

Work done on lifting the bucket only is: *weight* x *distance* = 5 (20) = 100 *ft-lb*

Work on the rope:



$$W = \int_0^{20} 0.08(20 - x)\,dx$$

$$= 0.08\left(20x - \frac{x^2}{2}\right)\Big|_0^{20}$$

$$= 0.08\left[\left(20(20) - \frac{20^2}{2}\right) - 0\right]$$

$$\underline{= 16\ \ ft - lb|}$$

The total work for the bucket and the rope combined is: 100 + 16 $\underline{= 116\ \ ft - lb|}$

Lifting

Another common work problem arises when the motion is vertical and the force is the gravitational force. The gravitational force exerted on an object with a mass of m is $F = mg$, where

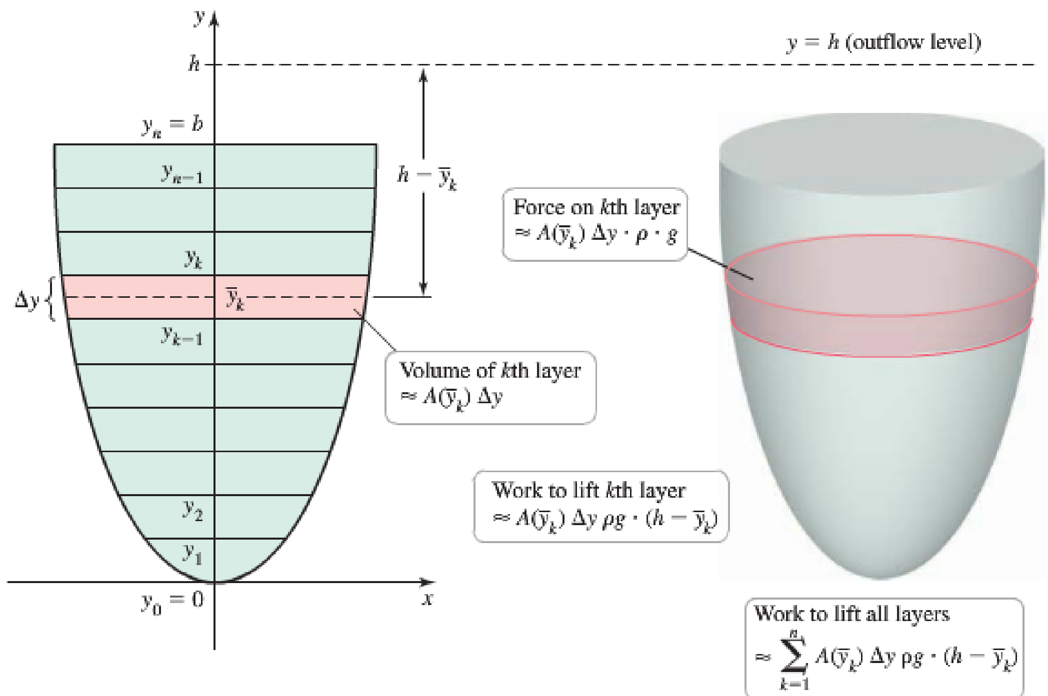
$g \approx 9.8\ m/s^2 \approx 32.2\ ft/s^2$ is the acceleration due the gravity near the surface of Earth.

The work in joules required to lift an object of mass m a vertical distance of y meters is

$$work = force \cdot distance = mgy$$

This type of problem leads to 3 key observation to the solution:

- ✓ Water from different levels of the tank is lifted different vertical distances, requiring different amounts of work,
- ✓ Water from the same horizontal plane is lifted the same distance, requiring the same amount of work.
- ✓ A volume V of water has mass ρV, where $\rho = 1\ g/cm^3 = 1000\ kg/m^3$ is the density of water.



$$F_k = mg \approx \overbrace{\underbrace{A\left(\bar{y}_k\right)\Delta y}_{volume} \cdot \rho}^{mass} \cdot g \;\Rightarrow\; W_k = \underbrace{A\left(\bar{y}_k\right)\Delta y \rho\, g}_{force} \cdot \underbrace{\left(h - \bar{y}_i\right)}_{distance}$$

$$W \approx \sum_{k=1}^{n} W_k = \sum_{k=1}^{n} \rho\, g A\left(\bar{y}_k\right)\left(h - \bar{y}_k\right)\Delta y$$

$$W = \lim_{n\to\infty} \sum_{k=1}^{n} \rho\, g A\left(\bar{y}_k\right)\left(h - \bar{y}_k\right)\Delta y = \int_0^b \rho\, g A(y) \underbrace{(h-y)}_{D(y)} dy$$

Solving Lifting Problems

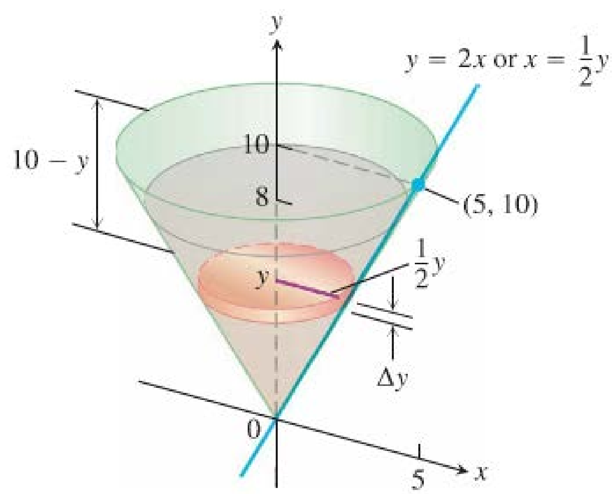
1. Draw a y-*axis* in the vertical direction (parallel to gravity) and choose a convenient origin. Assume the interval $[a,\ b]$ corresponds to the vertical extent of the fluid.
2. For $a \le y \le b$, find the cross-sectional area $A(y)$ of the horizontal slices and the distance $D(y)$ the slices must be lifted.
3. The work required to lift the water is

$$W = \int_a^b \rho\, g A(y) D(y)\, dy$$

Example

The conical tank is filled to within 2 *ft* of the top with olive oil weighing 57 lb/ft^3. How much work does it take to pump the oil to the rim of the tank?

Solution



The volume of a slab between the planes y and Δy:

$$\Delta V = \pi (radius)^2 (thickness)$$

$$= \pi\left(\tfrac{1}{2}y\right)^2 \Delta y$$

$$= \tfrac{\pi}{4} y^2 \Delta y \ \ ft^3$$

The force $F(y)$ required to lift this slab is equal to its weight

$$F(y) = 57\Delta V$$

$$= 57\tfrac{\pi}{4} y^2 \Delta y$$

Distance to lift to the level of the rim of the cone is about $(10 - y)$ *ft*, so the work done lifting the slab

$$W = \int_0^8 \frac{57\pi}{4} y^2 (10 - y)\, dy$$

$$= \frac{57\pi}{4}\int_0^8 \left(10y^2 - y^3\right) dy$$

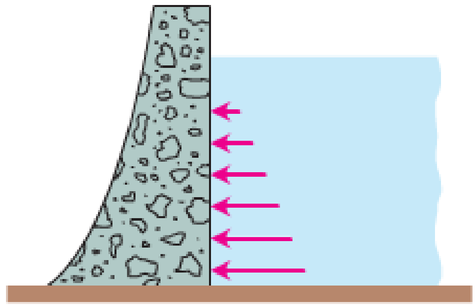
$$= \frac{57\pi}{4}\left[\tfrac{10}{3}y^3 - \tfrac{1}{4}y^4\right]_0^8$$

$$= \frac{57\pi}{4}\left[\left(\tfrac{10}{3}(8)^3 - \tfrac{1}{4}(8)^4\right) - 0\right]$$

$$\approx 30{,}561 \ \ ft-lb$$

Pressure and Force

Dams are built thicker at the bottom than at the top because the pressure against them increases with depth



Pressure is a force per unit area, measured in units such as N/m^2.

For example, the pressure of the atmosphere on the surface of Earth is about 14 lb/in^2
$\left(\approx 100 \text{ kilopascals}, \text{ or } 10^5 \; N/m^2\right)$

Another example, if you stood on the bottom of a swimming pool, you would feel pressure due to the weight (force) of the column of water above your head. If your head is flat and has surface area $A\ m^2$ and it is h meters below the surface, then the column of water above your head has volume $Ah\ m^3$. That column of water exerts a force:

$$F = mass \cdot acceleration = \underbrace{volume \cdot density}_{mass} \cdot g = Ah\rho g$$

Where ρ is the density of water
g is the acceleration due to gravity.

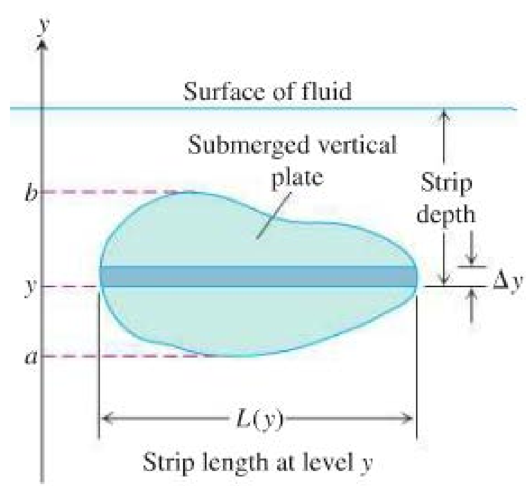
Therefore, the pressure on your head is the force divided by the surface area of your head

$$pressure = \frac{force}{A} = \frac{Ah\rho g}{A} = \rho g h$$

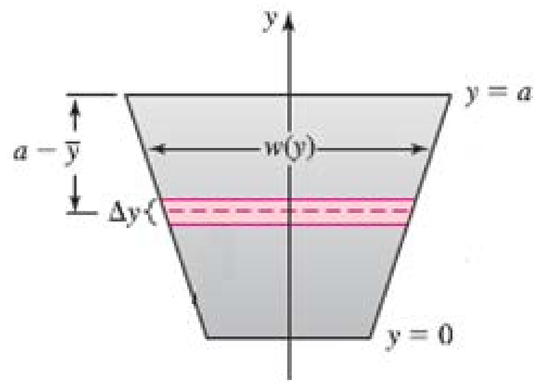
This pressure is called ***hydrostatic pressure*** (meaning the pressure of water at rest), and it has the following important property: it has the same magnitude in all directions.

Suppose that a plate submerge vertically in fluid of weight-density w runs from $y = a$ to $y = b$ on the y-axis. Let $L(y)$ be the ***length*** (or ***width***) of the horizontal strip measured from left to right along the surface of the plate at level y. Then the force exerted by the fluid against one side of the plate is

$$F = \int_a^b w \cdot (strip\ depth) \cdot L(y)\,dy$$



Solving Force / Pressure Problems



1. Draw a y-*axis* on the face of the dam in the vertical direction and choose a convenient origin (often taken to be the base of the dam).
2. Find the width function $w(y)$ for each value of y on the face of the dam
3. If the base of the dam is at $y=0$ and the top of the dam is at $y=a$, then the total force on the dam is

$$F=\int_0^a \rho g \underbrace{(a-y)}_{depth}\underbrace{w(y)}_{width}dy$$

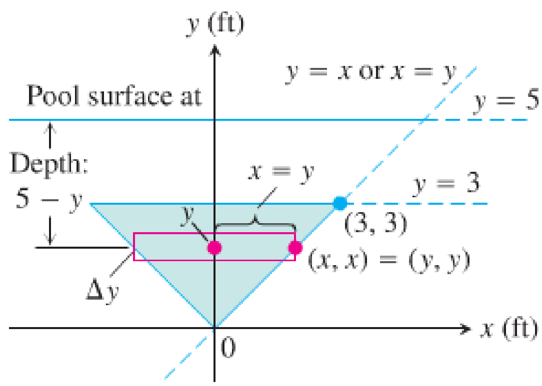
Example

A flat isosceles right-triangular plate with base 6 *ft* and height 3 *ft* is submerged vertically, base up, 2 *ft* below the surface of a swimming pool. Find the force exerted by the water against on side of the plate. (Freshwater Weight density: 62.4 lb/ft^3)

Solution

The width of a thin strip at level y is: $L(y)=2x=2y$

The depth of the strip beneath the surface is: $(5-y)$



$$F=\int_a^b w\cdot(strip\ depth)\cdot L(y)dy$$

$$=\int_0^3 62.4(5-y)\cdot(2y)dy$$

$$=124.8\int_0^3\left(5y-y^2\right)dy$$

$$=124.8\left[\frac{5}{2}y^2-\frac{1}{3}y^3\right]_0^3$$

$$=124.8\left[\left(\frac{5}{2}(3)^2-\frac{1}{3}(3)^3\right)-0\right]$$

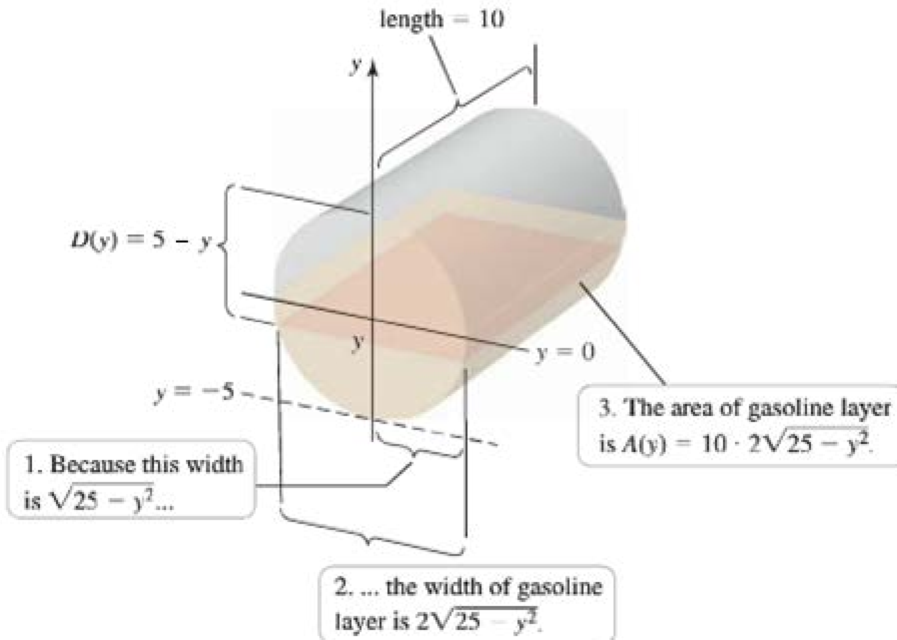
$$\underline{=1684.8\ \ lb}$$

Example

A cylindrical tank with a length of 10 *m* and radius of 5 *m* is on its side and half-full of gasoline. How much work is required to empty the tank through an outlet pipe at the top of the tank?

The density of gasoline is $\rho \approx 737\ kg / m^3$.

Solution



$$x^2 + y^2 = 5^2 \rightarrow x = \pm\sqrt{25 - y^2}$$

$$A(y) = 2(10)\sqrt{25 - x^2}$$

$$W = 737(9.8)\int_{-5}^{0} 20\sqrt{25 - y^2}\,(5 - y)\,dy$$

$$= 144,452\underbrace{\int_{-5}^{0} 5\sqrt{25 - y^2}\,dy}_{\text{area of } \frac{1}{4} \text{ circle}} - 144,452\int_{-5}^{0} y\sqrt{25 - y^2}\,dy$$

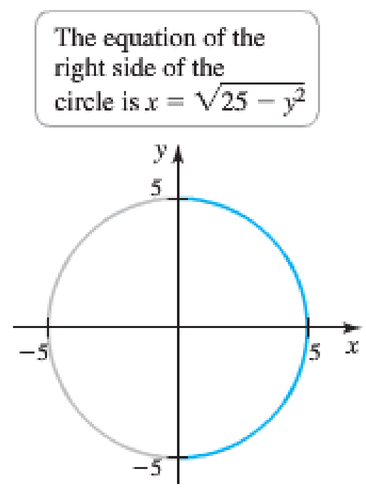
$$= 144,452\left(5 \cdot \frac{25\pi}{4} + \frac{1}{2}\int_{-5}^{0} \sqrt{25 - y^2}\ d\left(25 - y^2\right)\right)$$

$$= 144,452\left(\frac{125\pi}{4} + \frac{1}{3}\left(25 - y^2\right)^{3/2}\Big|_{-5}^{0}\right)$$

$$= 144,452\left(\frac{125\pi}{4} + \frac{125}{3}\right)$$

$$= 18,056,500\left(\frac{3\pi + 4}{12}\right)$$

$$\approx 20.2 \times 10^6\ \ joules$$



Material	***Density*** ρ	
	$\left(kg / m^3\right)$	$\left(lb / ft^3\right)$
Aluminum	2700	169
Copper	8940	558
Freshwater	1000	62.4
Gasoline	737	42 45
Gold	19320	1206
Iron	7870	491
Lead	11.34×10^3	708
Magnesium	1740	109
Mercury	13546	849
Milk	1030	64.5
Molasses	1600	100
Olive Oil	913	57
Platinum	21.45×10^3	1340
Seawater	1030	64

Exercises *Section* 1.7 – Physical Applications

Find the mass of a thin bar with the given density function

1. $\rho(x) = 1 + \sin x; \quad 0 \le x \le \pi$

2. $\rho(x) = 1 + x^3; \quad 0 \le x \le 1$

3. $\rho(x) = 2 - \frac{x}{2}; \quad 0 \le x \le 2$

4. $\rho(x) = 5e^{-2x}; \quad 0 \le x \le 4$

5. $\rho(x) = x\sqrt{2 - x^2}; \quad 0 \le x \le 1$

6. $\rho(x) = \begin{cases} 1 & \text{if } 0 \le x \le 2 \\ 2 & \text{if } 2 < x \le 3 \end{cases}$

7. $\rho(x) = \begin{cases} 1 & \text{if } 0 \le x \le 2 \\ 1 + x & \text{if } 2 < x \le 4 \end{cases}$

8. $\rho(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 1 \\ x(2 - x) & \text{if } 1 < x \le 2 \end{cases}$

9. A heavy-duty shock absorber is compressed 2 *cm* from its equilibrium position by a mass of 500 *kg*. How much work is required tocompress the shock absorber 4 *cm* from its equilibrium position? (A mass of 500 *kg* exerts a force (in newtons) of 500 *g*)

10. A spring has a restoring force given by $F(x) = 25x$. Let $W(x)$ be the work required to stretch the spring from its equilibrium position $(x = 0)$ to a variable distance *x*. Graph the work function. Compare the work required to stretch the spring *x* units from equilibrium to the work required to compress the spring *x* units from equilibrium.

11. A swimming pool has the shape of a box with a base that measures 25 *m* by 15 *m* and a depth of 2.5 *m*. How much work is required to pump the water out of the pool when it is full?

12. It took 1800 *J* of work to stretch a spring from its natural length of 2 *m* to a length of 5 *m*. Find the spring's force constant.

13. How much work is required to move am object from $x = 1$ to $x = 5$ (measired in meters) in the presence.

14. How much work is required to move am object from $x = 0$ to $x = 3$ (measired in meters) with a force (in *N*) is given by $F(x) = \frac{2}{x^2}$ acting along the *x-axis* .

15. A force of 200 N will stretch a garage door spring 0.8-*m* beyond its unstressed length.

a) How far will a 300-N-force stretch the spring?
b) How much work does it take to stretch the spring this far?

16. A spring on a horizontal surface can be stretched and held 0.5 *m* from its equilibrium position with a force of 50 *N*.

a) How much work is done in stretching the spring 1.5 *m* from its equilibrium position?
b) How much work is done in compressing the spring 0.5 *m* from its equilibrium position?

17. Suppose a force of 10 *N* is required to stretch a spring 0.1 *m* from its equilibrium position and hold it in that position.

a) Assuming that the spring obeys Hooke's law, find the spring constant *k*.

b) How much work is needed to ***compress*** the spring 0.5 *m* from its equilibrium position?

c) How much work is needed to ***stretch*** the spring 0.25 *m* from its equilibrium position?

d) How much additional work is required to stretch the spring 0.25 *m* if it has already been stretched 0.1 *m* from its equilibrium position?

18. A spring has a natural length of 10 *in*. An 800-*lb* force stretches the spring to 14 *in*.

a) Find the force constant.

b) How much work is done in stretching the spring from 10 *in* to 12 *in*?

c) How far beyond its natural length will a 1600-*lb* force stretch the spring?

19. It takes a force of 21,714 *lb*. to compress a coil spring assembly on a Transit Authority subway car from its free height of 8 *in*. to its fully compressed height of 5 *in*.

a) What is the assembly's force constant?

b) How much work does it take to compress the assembly the first half inch? The second half inch? Answer to the nearest *in-lb*.

20. A bag of sand originally weighing 144 *lb* was lifted a constant rate. As it rose, sand also leaked out at a constant rate. The sand was half gone by the time the bag had been lifted 10 18 *ft*. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment.)

21. A mountain climber is about to haul up a 50 *m* length of hanging rope. How much work will it take if the rope weighs 0.624 *N/m*?

22. An electric elevator with a motor at the top has a multistrand cable weighing 4.5 *lb/ft*. When the car is at the first floor, 180 *ft* of cable are paid out, and effectively 0 *ft* are out when the car is at the top floor. How much work does the motor do just lifting the cable when it takes the car from the first floor to the top?

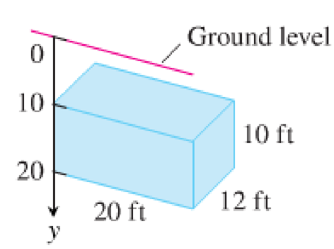
23. The rectangular cistern (storage tank for rainwater) shown has its top 10 *ft* below ground level. The cistern, currently full, is to be emptied for inspection by pumping its contents to ground level.

a) How much work will it take to empty the cistern?

b) How long will it take a 1/-*hp* pump, rated at 275 *ft-lb/sec*, to pump the tank dry?

c) How long will it take the pump in part (b) to empty the tank hallway? (It will be less than half the time required to empty the tank completely)

d) What are the answers to parts (a) through (c) in a location where water weighs 62.6 lb/ft^3? 62.59 lb/ft^3?



24. When a particle of mass m is at $(x, 0)$, it is attracted toward the origin with a force whose magnitude is $\frac{k}{x^2}$. If the particle starts from rest at $x = b$ and is acted on by no other forces, find the work done on it by the time reaches $x = a$, $0 < a < b$.

25. The strength of Earth's gravitation field varies with the distance r from Earth's center, and the magnitude of the gravitational force experienced by a satellite of mass m during and after launch is

$$F(r) = \frac{mMG}{r^2}$$

Here, $M = 5.975 \times 10^{24}\ kg$ is Earth's mass, $G = 6.6720 \times 10^{-11}\ N \cdot m^2 kg^{-2}$ is the universal gravitational constant, and r is measured in meters. The work it takes to lift a 1000-kg satellite from Earth's surface to a circular orbit 35,780 km above Earth's center is therefore given by the integral

$$W = \int_{6,370,000}^{35,780,000} \frac{1000MG}{r^2}\, dr \ \ \text{joules}$$

Evaluate the integral. The lower limit of integration is Earth's radius in meters at the launch site. (This calculation does not take into account energy spend lifting the launch vehicle or energy spent bringing the satellite to orbit velocity.)

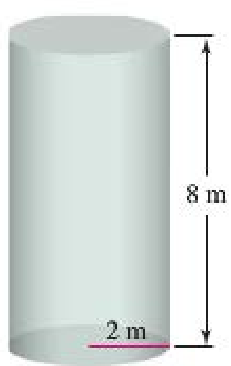
26. You drove an 800-*gal* truck from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 4750-*ft* elevation change in 50 minutes.

Assuming that the water leaked out at a steady rate, how much work was spent in carrying the water to the top? Do not count the work done in getting yourself and the truck there. Water weighs 8-*lb*./*gal*.

27. A cylindrical water tank has height 8 m and radius 2 m

a) If the tank is full of water, how much work is required to pump the water to the level of the top of the tank and out of the tank?

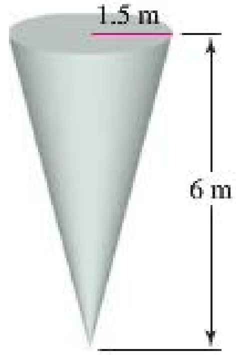
b) Is it true it takes half as much work to pump the water out of the tank when it is half full as when it is full? Explain.



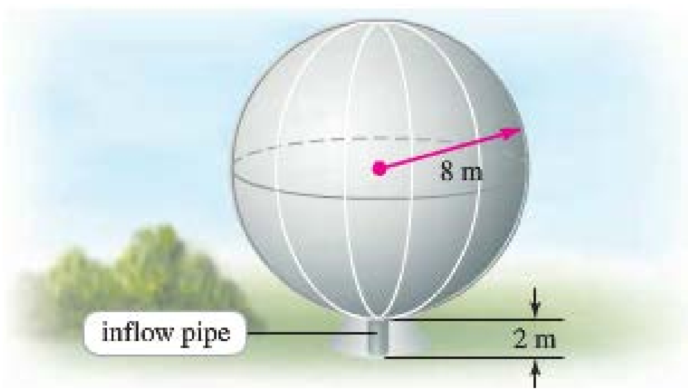
28. A water tank is shaped like an inverted cone with height 6 *m* and base radius 1.5 *m*.

a) If the tank is full of water, how much work is required to pump the water to the level of the top of the tank and out of the tank?

b) Is it true it takes half as much work to pump the water out of the tank when it is half full as when it is full? Explain



29. A spherical water tank with an inner radius of 8 *m* has its lowest point 2 *m* above the ground. It is filled by a pipe that feed the tank at its lowest point.4



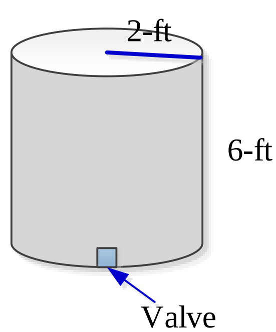
a) Neglecting the volume of the inflow pipe, how much work is required to fill the tank if it is initially empty?

b) Now assume that the inflow pipe feeds the tank at the top of the tank. Neglecting the volume of the inflow pipe, how much work is required to fill the tank if it is initially empty?

30. A large vertical dam in the shape of a symmetric trapezoid has a height of 30 *m*. a width of 20 *m*. at its base, and a width of 40 *m*. at the top. What is the total force on the face of the dam when the reservoir is full? $\left(\rho = 1000\frac{kg}{m^3},\ g = 9.8\frac{m}{s^2}\right)$

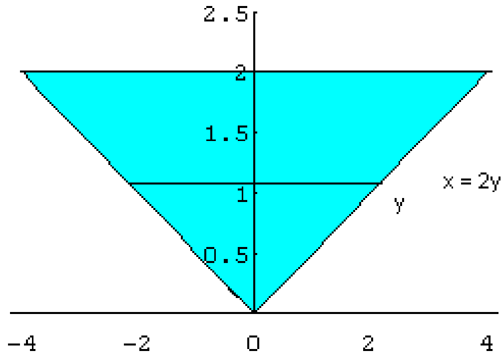
31. Pumping water from a lake 15-*ft* below the bottom of the tank can fill the cylindrical tank shown here.

There are two ways to go about it. One is to pump the water through a hose attached to a valve in the bottom of the tank. The other is to attach the hose to the rim of the tank and let the water pour in. Which way will be faster? Give reason for answer

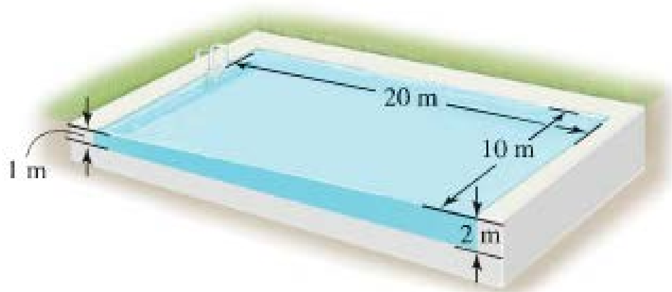


32. A tank truck hauls milk in a 6-*ft* diameter horizontal right circular cylindrical tank. How must force does the milk exert on each end of the tank when the tank is half full?

33. The vertical triangular plate shown here is the end plate of a trough full of water. What is the fluid force against the plate?



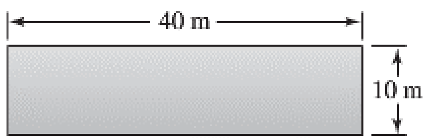
34. A swimming pool is 20 *m* leg long and 10 *m* wide, with a bottom that slopes uniformly from a depth of 1 *m* at one end to a depth of 2 m at the other end.



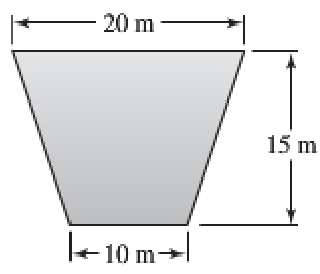
Assuming the pool is full, how much work is required to pump the water to a level 0.2 *m* above the top of the pool?

Find the total force on the face of the given dam

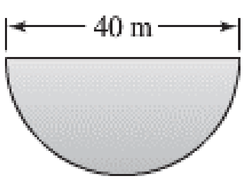
35.



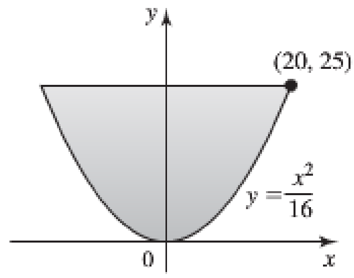
36.



37.



38.



39. A large building shaped like a box is 50 *m* high with a face that is 80 *m* wide. A strong wind blows directly at the face of the building, exerting a pressure of 150 N/m^2 at the ground and increasing with height according to $P(y) = 150 + 2y$, where *y* is the height above the ground. Calculate the total force on the building, which is a meaure of the resistance that must be included in the design of the building.

40. Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a square window, 0.5 *m* on side, with the lower edge of the window on the bottom of the pool.

41. Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a square window, 0.5 *m* on side, with the lower edge of the window 1 *m* from the bottom of the pool.

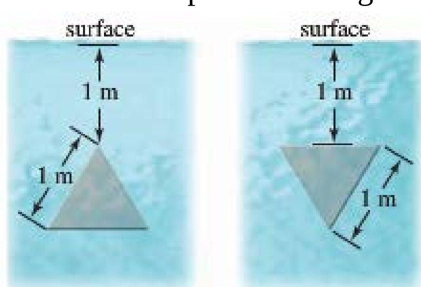
42. Adiving pool that is 4 *m* deep full of water has a viewing window on one of its vertical walls. Find the force of the a circle window, with a radius of 0.5 *m*, tangent to the bottom of the pool.

43. A rigid body with a mass of 2 *kg* moves along a line due to a force that produces a position function $x(t) = 4t^2$, where *x* is measured in *meters* and *t* is measured in *seconds*. Find the work done during the first 5 *sec*. in two ways.

a) Note that $x''(t) = 8$; then use Newton's second law, $\left(F = ma = mx''(t)\right)$ to evaluate the work integral $W = \int_{x_0}^{x_f} F(x)\,dx$, where x_0 and x_f are the initial and final positions, repectively.

b) Change variables in the work integral and integrate with respect to *t*.

44. A plate shaped like an equilateral triangle 1 *m* on a side is placed on a vetical wall 1 *m* below the surface of a pool filled with water. On which plate in the figure is the force is greater



45. A square plate 1 *m* on a side is placed on a vetical wall 1 *m* below the surface of a pool filled with water. On which plate in the figure is the force is greater