

Solution **Section 1.4 – Exact Differential Equations**

Exercise

Solve the differential equation $(2x + y)dx + (x - 6y)dy = 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = 2x + y \Rightarrow M_y = 1$$

$$\frac{\partial \psi}{\partial y} = N = x - 6y \Rightarrow N_x = 1$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = 2x + y \Rightarrow \psi = \int (2x + y)dx = x^2 + xy + h(y)$$

$$\psi_y = x + h'(y) = x - 6y \Rightarrow h'(y) = -6y$$

$$h(y) = \int -6y dy = -3y^2$$

$$\psi(x, y) = \underline{x^2 + xy - 3y^2 = C}$$

Exercise

Solve the differential equation $(2x + 3)dx + (2y - 2)dy = 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = 2x + 3 \Rightarrow M_y = 2$$

$$\frac{\partial \psi}{\partial y} = N = 2y - 2 \Rightarrow N_x = 2$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = 2x + 3 \Rightarrow \psi = \int (2x + 3)dx = x^2 + 3x + h(y)$$

$$\begin{aligned} \psi_y = h'(y) = 2y - 2 &\Rightarrow h(y) = \int (2y - 2)dy \\ &= y^2 - 2y + C \end{aligned}$$

$$\psi(x, y) = \underline{x^2 + 3x + y^2 - 2y = C}$$

Exercise

Solve the differential equation $(1 - y \sin x) + (\cos x) y' = 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = 1 - y \sin x \Rightarrow M_y = -\sin x$$

$$\frac{\partial \psi}{\partial y} = N = \cos x \Rightarrow N_x = -\sin x$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = 1 - y \sin x \Rightarrow \psi = \int (1 - y \sin x) dx = x + y \cos x + h(y)$$

$$\psi_y = \cos x + h'(y) = \cos x \Rightarrow h'(y) = 0$$

$$h(y) = C$$

$$\psi(x, y) = \underline{x + y \cos x = C}$$

Exercise

Solve the differential equation $\frac{dy}{dx} = -\frac{ax + by}{bx + cy}$

Solution

$$(bx + cy) dy = -(ax + by) dx$$

$$(ax + by) dx + (bx + cy) dy = 0$$

$$\frac{\partial \psi}{\partial x} = M = ax + by \Rightarrow M_y = b$$

$$\frac{\partial \psi}{\partial y} = N = bx + cy \Rightarrow N_x = b$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = ax + by \Rightarrow \psi = \int (ax + by) dx = \frac{1}{2} ax^2 + bxy + h(y)$$

$$\psi_y = bx + h'(y) = bx + cy \Rightarrow h'(y) = cy$$

$$h(y) = \int cy dy = \frac{1}{2} cy^2$$

$$\psi(x, y) = \frac{1}{2} ax^2 + bxy + \frac{1}{2} cy^2 = D$$

$$\boxed{ax^2 + 2bxy + cy^2 = E} \quad (E=2D)$$

Exercise

Solve the differential equation $\frac{dy}{dx} = \frac{3x^2 + y}{3y^2 - x}$

Solution

$$(3x^2 + y)dx - (3y^2 - x)dy = 0$$

$$\frac{\partial \psi}{\partial x} = M = 3x^2 + y \Rightarrow M_y = 1$$

$$\frac{\partial \psi}{\partial y} = N = -3y^2 + x \Rightarrow N_x = 1 \Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = 3x^2 + y \Rightarrow \psi = \int (3x^2 + y)dx = x^3 + xy + h(y)$$

$$\psi_y = x + h'(y) = -3y^2 + x \Rightarrow h'(y) = -3y^2$$

$$h(y) = \int -3y^2 dy = -y^3$$

$$\psi(x, y) = \underline{x^3 + xy - y^3 = C}$$

Exercise

Solve the differential equation $2xydx + (x^2 - 1)dy = 0$

Solution

$$M(x, y) = 2xy \quad N(x, y) = x^2 - 1$$

$$\frac{\partial M}{\partial y} = 2x \quad \frac{\partial N}{\partial x} = 2x \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int 2xy dx = x^2 y + h(y)$$

$$\psi_y = x^2 + h'(y) = x^2 - 1$$

$$h'(y) = -1 \Rightarrow h(y) = -y + C$$

$$\underline{x^2 y - y = C}$$

Exercise

Find the general solution $y' = \frac{x^2 + y^2}{2xy}$

Solution

$$\text{Let } y = xv \Rightarrow y' = v + xv'$$

$$v + xv' = \frac{x^2 + x^2 v^2}{2x^2 v}$$

$$xv' = \frac{1+v^2}{2v} - v$$

$$x \frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\frac{2v}{1-v^2} dv = \frac{dx}{x}$$

$$-\int \frac{1}{1-v^2} d(1-v^2) = \int \frac{dx}{x}$$

$$-\ln|1-v^2| = \ln|x| + \ln C$$

$$\ln \frac{1}{|1-v^2|} = \ln|Cx|$$

$$\frac{1}{1-\left(\frac{y}{x}\right)^2} = Cx$$

$$\frac{x^2}{x^2-y^2} = Cx$$

$$\frac{x}{C} = x^2 - y^2$$

$$\underline{y^2 = x^2 - C_1 x}$$

Exercise

Find the general solution $2xyy' = x^2 + 2y^2$

Solution

$$\text{Let } y = xv \Rightarrow y' = v + xv'$$

$$2x(xv)(v + xv') = x^2 + 2x^2v^2$$

Divide both side by x^2

$$2v^2 + 2xvv' = 1 + 2v^2$$

$$2xv \frac{dv}{dx} = 1$$

$$\int 2v dv = \int \frac{dx}{x}$$

$$v^2 = \ln x + C \quad \left(v = \frac{y}{x} \right)$$

$$\frac{y^2}{x^2} = \ln x + C$$

$$\underline{y^2 = x^2 (\ln x + C)}$$

Exercise

Find the general solution $xy' = y + 2\sqrt{xy}$

Solution

$$\text{Let } y = vx \Rightarrow y' = v + xv'$$

$$x(v + xv') = vx + 2\sqrt{x^2v}$$

$$x(v + xv') = vx + 2x\sqrt{v}$$

Divide both side by x

$$v + xv' = v + 2\sqrt{v}$$

$$x \frac{dv}{dx} = 2\sqrt{v}$$

$$\int \frac{dv}{2\sqrt{v}} = \int \frac{dx}{x}$$

$$\sqrt{v} = \ln x + C \quad \left(v = \frac{y}{x} \right)$$

$$\sqrt{\frac{y}{x}} = \ln x + C$$

$$\frac{y}{x} = (\ln x + C)^2$$

$$\underline{y = x(\ln x + C)^2}$$

Exercise

Find the general solution $xy^2y' = x^3 + y^3$

Solution

$$\text{Let } y = vx \Rightarrow y' = v + xv'$$

$$x^3v^2(v + xv') = x^3 + x^3v^3$$

Divide both side by x^3

$$v^2(v + xv') = 1 + v^3$$

$$v^3 + xv^2v' = 1 + v^3$$

$$xv^2 \frac{dv}{dx} = 1$$

$$\int v^2 dv = \int \frac{dx}{x}$$

$$\frac{1}{3}v^3 = \ln x + C \quad \left(v = \frac{y}{x} \right)$$

$$\frac{y^3}{x^3} = 3(\ln x + C)$$

$$\underline{y^3 = 3x^3(\ln x + C)}$$

Exercise

Find the general solution $x^2 y' = xy + x^2 e^{y/x}$

Solution

$$\text{Let } y = vx \Rightarrow y' = v + xv'$$

$$x^2(v + xv') = x^2 v + x^2 e^{vx/x} \quad \text{Divide both side by } x^2$$

$$v + xv' = v + e^v$$

$$x \frac{dv}{dx} = e^v$$

$$\int e^{-v} dv = \int \frac{dx}{x}$$

$$e^{-v} = \ln x + \ln C \quad \left(v = \frac{y}{x} \right)$$

$$e^{-\frac{y}{x}} = \ln C x$$

$$-\frac{y}{x} = \ln(\ln C x)$$

$$\underline{y = -x \ln(\ln C x)}$$

Exercise

Find the general solution $x^2 y' = xy + y^2$

Solution

$$\text{Let } y = vx \Rightarrow y' = v + xv'$$

$$x^2(v + xv') = x^2 v + x^2 v^2 \quad \text{Divide both side by } x^2$$

$$v + xv' = v + v^2$$

$$x \frac{dv}{dx} = v^2$$

$$\int v^{-2} dv = \int \frac{dx}{x}$$

$$-v^{-1} = \ln x + \ln C \quad \left(v = \frac{y}{x} \right)$$

$$\frac{x}{y} = -\ln C x$$

$$\frac{x}{y} = \ln \frac{1}{Cx}$$

$$\underline{y(x) = \frac{x}{\ln \frac{1}{Cx}}}$$

Exercise

Find the general solution $xyy' = x^2 + 3y^2$

Solution

$$\text{Let } y = vx \Rightarrow y' = v + xv'$$

$$vx^2(v + xv') = x^2 + 3x^2v^2 \quad \text{Divide both side by } x^2$$

$$v^2 + xv' = 1 + 3v^2$$

$$xv \frac{dv}{dx} = 1 + 2v^2$$

$$\frac{v}{1 + 2v^2} dv = \frac{dx}{x}$$

$$\frac{1}{4} \int \frac{1}{1 + 2v^2} d(1 + 2v^2) = \int \frac{dx}{x}$$

$$\frac{1}{4} \ln(1 + 2v^2) = \ln x + \ln C$$

$$\ln(1 + 2v^2) = 4 \ln C x$$

$$\ln\left(1 + 2 \frac{y^2}{x^2}\right) = \ln C x^4$$

$$\frac{x^2 + 2y^2}{x^2} = C x^4$$

$$\underline{x^2 + 2y^2 = C x^6}$$

Exercise

Find the general solution $(x^2 - y^2)y' = 2xy$

Solution

$$\text{Let } y = vx \Rightarrow y' = v + xv'$$

$$(x^2 - v^2x^2)(v + xv') = 2x^2v \quad \text{Divide both side by } x^2$$

$$(1 - v^2)(v + xv') = 2v$$

$$v + xv' = \frac{2v}{1 - v^2}$$

$$x \frac{dv}{dx} = \frac{2v}{1 - v^2} - v$$

$$x \frac{dv}{dx} = \frac{v^3 + v}{1 - v^2}$$

$$\int \frac{1-v^2}{v^3+v} dv = \int \frac{dx}{x}$$

$$\frac{1-v^2}{v(v^2+1)} = \frac{A}{v} + \frac{Bv+C}{v^2+1} = \frac{(A+B)v^2 + Cv + A}{v(v^2+1)}$$

$$\begin{cases} A+B = -1 \\ A = 1 \end{cases} \rightarrow B = -2$$

$$\int \left(\frac{1}{v} - \frac{2v}{v^2+1} \right) dv = \int \frac{dx}{x}$$

$$\int \frac{1}{v} dv - \int \frac{1}{v^2+1} d(v^2+1) = \int \frac{dx}{x}$$

$$\ln|v| - \ln(v^2+1) = \ln|x| + \ln C$$

$$\ln \frac{|v|}{v^2+1} = \ln Cx$$

$$\frac{y/x}{\frac{y^2}{x^2}+1} = Cx$$

$$\frac{y}{x} \frac{x^2}{y^2+x^2} = Cx$$

$$\underline{y(x) = C(y^2 + x^2)}$$

Exercise

Find the general solution $xyy' = y^2 + x\sqrt{4x^2 + y^2}$

Solution

$$\text{Let } y = vx \Rightarrow y' = v + xv'$$

$$vx^2(v + xv') = x^2v^2 + x\sqrt{4x^2 + v^2x^2}$$

$$vx^2(v + xv') = x^2v^2 + x^2\sqrt{4 + v^2}$$

Divide both side by x^2

$$v^2 + xvv' = v^2 + \sqrt{4 + v^2}$$

$$xv \frac{dv}{dx} = \sqrt{4 + v^2}$$

$$\int \frac{v}{\sqrt{4 + v^2}} dv = \int \frac{dx}{x}$$

$$\frac{1}{2} \int \frac{1}{\sqrt{4+v^2}} d(4+v^2) = \int \frac{dx}{x}$$

$$\sqrt{4+v^2} = \ln x + C$$

$$\sqrt{4 + \frac{y^2}{x^2}} = \ln x + C$$

$$\frac{4x^2 + y^2}{x^2} = (\ln x + C)^2$$

$$\underline{4x^2 + y^2 = x^2 (\ln x + C)^2}$$

Exercise

Find the general solution $xy' = y + \sqrt{x^2 + y^2}$

Solution

$$\text{Let } y = vx \Rightarrow y' = v + xv'$$

$$x(v + xv') = xv + \sqrt{x^2 + v^2 x^2}$$

$$x(v + xv') = xv + x\sqrt{1 + v^2}$$

Divide both side by x

$$v + xv' = v + \sqrt{1 + v^2}$$

$$x \frac{dv}{dx} = \sqrt{1 + v^2}$$

$$\int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x}$$

$$\ln \left| v + \sqrt{1 + v^2} \right| = \ln x + \ln C$$

$$\ln \left| \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} \right| = \ln C x$$

$$\frac{y}{x} + \frac{1}{x} \sqrt{x^2 + y^2} = C x$$

$$\underline{y + \sqrt{x^2 + y^2} = C x^2}$$

Exercise

Find the general solution $y^2 y' + 2xy^3 = 6x$

Solution

$$y' + 2xy = 6xy^{-2}$$

$$\text{Let } u = y^{1+2} = y^3 \Rightarrow y = u^{1/3}$$

$$\frac{du}{dx} = 3y^2 \frac{dy}{dx} \Rightarrow y' = \frac{1}{3} y^{-2} u' = \frac{1}{3} u^{-2/3} u'$$

$$\frac{1}{3} u^{-2/3} u' + 2xu^{1/3} = 6xu^{-2/3} \quad \text{Multiply both sides by } 3u^{2/3}$$

$$u' + 6xu = 18x$$

$$e^{\int 6x dx} = e^{3x^2}$$

$$\int 18xe^{3x^2} dx = 3 \int e^{3x^2} d(3x^2) = 3e^{3x^2}$$

$$u = e^{-3x^2} \left(3e^{3x^2} + C \right)$$

$$\underline{y^3 = 3 + Ce^{-3x^2}}$$

Exercise

Find the general solution $x^2 y' + 2xy = 5y^4$

Solution

$$y' + 2\frac{1}{x}y = \frac{5}{x^2}y^4 \quad \text{Divide by } x^2$$

$$\text{Let } u = y^{1-4} = y^{-3} \Rightarrow y = u^{-1/3}$$

$$\frac{du}{dx} = -3y^{-4} \frac{dy}{dx} \Rightarrow y' = -\frac{1}{3} y^4 u' = -\frac{1}{3} u^{-4/3} u'$$

$$-\frac{1}{3} u^{-4/3} u' + \frac{2}{x} u^{-1/3} = \frac{5}{x^2} u^{-4/3} \quad \text{Multiply both sides by } -3u^{4/3} \quad u = \frac{1}{x^{-6}} \left(\frac{15}{7} x^{-7} + C \right)$$

$$u' - \frac{6}{x}u = -\frac{15}{x^2}$$

$$e^{\int -\frac{6}{x} dx} = e^{-6 \ln x} = e^{\ln x^{-6}} = x^{-6}$$

$$\int x^{-6} \left(-\frac{15}{x^2} \right) dx = -15 \int x^{-8} dx = \frac{15}{7} x^{-7}$$

$$y^{-3} = \frac{15 + 7Cx^7}{7x}$$

$$\underline{y^3 = \frac{7x}{15 + 7Cx^7}}$$

Exercise

Find the general solution $2xy' + y^3 e^{-2x} = 2xy$

Solution

$$2xy' - 2xy = -e^{-2x} y^3$$

$$y' - y = -\frac{e^{-2x}}{2x} y^3 \quad \text{Divide by } 2x$$

$$\text{Let } u = y^{1-3} = y^{-2} \Rightarrow y = u^{-1/2}$$

$$\frac{du}{dx} = -2y^{-3} \frac{dy}{dx} \Rightarrow y' = -\frac{1}{2} y^3 u' = -\frac{1}{2} u^{-3/2} u'$$

$$-\frac{1}{2} u^{-3/2} u' - u^{-1/2} = -\frac{e^{-2x}}{2x} u^{-3/2} \quad \text{Multiply both sides by } -2u^{3/2}$$

$$u' + 2u = \frac{e^{-2x}}{x}$$

$$e^{\int 2dx} = e^{2x}$$

$$\int \frac{e^{-2x}}{x} e^{2x} dx = \int \frac{dx}{x} = \ln x$$

$$u = \frac{1}{e^{-2x}} (\ln x + C)$$

$$\frac{1}{y^2} = \frac{\ln x + C}{e^{2x}}$$

$$\underline{y^2 = \frac{e^{2x}}{\ln x + C}}$$

Exercise

Find the general solution $y^2(xy' + y)(1 + x^4)^{1/2} = x$

Solution

$$y^2 xy' + y^3 = x(1 + x^4)^{-1/2}$$

$$y' + \frac{1}{x} y = (1 + x^4)^{-1/2} y^{-2} \quad \text{Divide both sides by } xy^2$$

$$\text{Let } u = y^{1+2} = y^3 \Rightarrow y = u^{1/3}$$

$$\frac{du}{dx} = 3y^2 \frac{dy}{dx} \Rightarrow y' = \frac{1}{3} y^{-2} u' = \frac{1}{3} u^{-2/3} u'$$

$$\frac{1}{3}u^{-2/3}u' + \frac{1}{x}u^{1/3} = \left(1+x^4\right)^{-1/2} u^{-2/3} \quad \text{Multiply both sides by } 3u^{2/3}$$

$$u' + \frac{3}{x}u = 3\left(1+x^4\right)^{-1/2}$$

$$e^{\int \frac{3}{x} dx} = e^{3 \ln x} = e^{\ln x^3} = x^3$$

$$\int 3\left(1+x^4\right)^{-1/2} x^3 dx = \frac{3}{4} \int \left(1+x^4\right)^{-1/2} d\left(1+x^4\right) = \frac{3}{2} \sqrt{1+x^4}$$

$$u = \frac{1}{x^3} \left(\frac{3}{2} \sqrt{1+x^4} + C \right)$$

$$\underline{y^3 = \frac{1}{x^3} \left(\frac{3}{2} \sqrt{1+x^4} + C \right)}$$

Exercise

Find the general solution $3y^2y' + y^3 = e^{-x}$

Solution

$$3y' + y = e^{-x}y^{-2} \quad \text{Divide both sides by } y^2$$

$$\text{Let } u = y^{1+2} = y^3 \Rightarrow y = u^{1/3}$$

$$\frac{du}{dx} = 3y^2 \frac{dy}{dx} \Rightarrow y' = \frac{1}{3}y^{-2}u' = \frac{1}{3}u^{-2/3}u'$$

$$u^{-2/3}u' + u^{1/3} = e^{-x}u^{-2/3} \quad \text{Multiply both sides by } u^{2/3}$$

$$u' + u = e^{-x}$$

$$e^{\int dx} = e^x$$

$$\int e^{-x}e^x dx = \int dx = x$$

$$u = \frac{1}{e^x}(x + C)$$

$$\underline{y^3 = e^{-x}(x + C)}$$

Exercise

Find the general solution $3xy^2y' = 3x^4 + y^3$

Solution

$$3xy^2y' - y^3 = 3x^4 \quad \text{Divide both sides by } xy^2$$

$$3y' - \frac{1}{x}y = 3x^3y^{-2}$$

$$\text{Let } u = y^{1+2} = y^3 \Rightarrow y = u^{1/3} \Rightarrow y' = \frac{1}{3}u^{-2/3}u'$$

$$u^{-2/3}u' - \frac{1}{x}u^{1/3} = 3x^3u^{-2/3} \quad \text{Multiply both sides by } u^{2/3}$$

$$u' - \frac{1}{x}u = 3x^3$$

$$e^{\int \frac{-1}{x} dx} = e^{-\ln x} = x^{-1}$$

$$\int 3x^3x^{-1}dx = \int 3x^2dx = x^3$$

$$u = x(x^3 + C)$$

$$y^3 = x^4 + Cx$$

$$\underline{y = \sqrt[3]{x^4 + Cx}}$$

Exercise

Find the general solution $xe^y y' = 2(e^y + x^3e^{2x})$

Solution

$$\text{Let } u = e^y \Rightarrow y = \ln u \Rightarrow y' = \frac{u'}{u}$$

$$xu \frac{1}{u}u' = 2u + 2x^3e^{2x}$$

$$u' - \frac{2}{x}u = 2x^2e^{2x}$$

$$e^{\int \frac{-2}{x} dx} = e^{-2\ln x} = x^{-2}$$

$$\int 2x^2e^{2x}x^{-2}dx = 2 \int e^{2x}dx = e^{2x}$$

$$u = x^2(e^{2x} + C)$$

$$e^y = x^2e^{2x} + Cx^2$$

$$\underline{y = \ln(x^2e^{2x} + Cx^2)}$$

Exercise

Find the general solution $(2x \sin y \cos y) y' = 4x^2 + \sin^2 y$

Solution

$$\text{Let } u = \sin y \Rightarrow u' = (\cos y) y'$$

$$2xuu' = 4x^2 + u^2$$

$$u' = 2x \frac{1}{u} + \frac{1}{2x} u$$

$$u' - \frac{1}{2x} u = 2xu^{-1}$$

$$\text{Let } v = u^{1+1} = u^2 \Rightarrow u = v^{1/2}$$

$$v' = 2uu' \Rightarrow u' = \frac{1}{2} u^{-1} v' = \frac{1}{2} v^{-1/2} v'$$

$$\frac{1}{2} v^{-1/2} v' - \frac{1}{2x} v^{1/2} = 2xv^{-1/2}$$

Multiply both sides by $2v^{1/2}$

$$v' - \frac{1}{x} v = 4x$$

$$e^{\int -\frac{1}{x} dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$\int x^{-1} (4x) dx = \int 4 dx = 4x$$

$$v = x(4x + C)$$

$$u^2 = 4x^2 + Cx$$

$$\underline{\sin^2 y = 4x^2 + Cx}$$

Exercise

Find the general solution $(x + e^y) y' = xe^{-y} - 1$

Solution

$$\text{Let } u = e^y \Rightarrow y = \ln u \Rightarrow y' = \frac{u'}{u}$$

$$(x + u) \frac{u'}{u} = xu^{-1} - 1$$

$$(x + u) u' = x - u$$

$$\text{Let } u = vx \Rightarrow u' = v + xv'$$

$$(x + vx)(v + xv') = x - vx$$

$$x(1 + v)(v + xv') = x(1 - v)$$

$$(1 + v)(v + xv') = 1 - v$$

$$v + v^2 + x(1 + v)v' = 1 - v$$

$$x(1+v)\frac{dv}{dx} = 1 - 2v - v^2$$

$$\int \frac{1+v}{1-2v-v^2} dv = \int \frac{dx}{x}$$

$$-\frac{1}{2} \ln |1-2v-v^2| = \ln x + \ln C$$

$$\ln |1-2v-v^2| = -2 \ln C x$$

$$v = \frac{u}{x} = \frac{e^y}{x}$$

$$\ln \left| 1 - 2\frac{e^y}{x} - \frac{e^{2y}}{x^2} \right| = \ln (Cx)^{-2}$$

$$\frac{x^2 - 2xe^y - e^{2y}}{x^2} = \frac{1}{(Cx)^2}$$

$$\underline{x^2 - 2xe^y - e^{2y} = C_1}$$

Exercise

Find the general solution $(x^2 + y^2)dx + (x^2 - xy)dy = 0$

Solution

$$M(x, y) = x^2 + y^2 \quad N(x, y) = x^2 - xy$$

$$\frac{\partial M}{\partial y} = 2y \quad \frac{\partial N}{\partial x} = 2x - y \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{M_y - N_x}{N} = \frac{2y - 2x + y}{x^2 + xy} \times$$

$$\text{Let } y = ux \Rightarrow dy = udx + xdu$$

$$-\frac{x^2 + y^2}{x^2 - xy} dx = udx + xdu$$

$$(x^2 + y^2)dx + (x^2 - xy)(udx + xdu) = 0$$

$$(x^2 + y^2)dx + ux(x - y)dx + x^2(x - y)du = 0$$

$$(x^2 + u^2x^2)dx + ux(x - ux)dx + x^2(x - ux)du = 0$$

$$x^2(1+u)dx + x^3(1-u)du = 0$$

$$(1+u)dx = x(u-1)du$$

$$\frac{dx}{x} = \frac{u-1}{u+1} du$$

$$\int \frac{dx}{x} = \int \left(1 - \frac{2}{u+1}\right) du$$

$$\ln|x| = u - 2\ln|u+1| + \ln C$$

$$\ln|x| + \ln\left(\frac{y}{x} + 1\right)^2 - \ln C = \frac{y}{x}$$

$$\ln \frac{x}{C} \left(\frac{(x+y)^2}{x^2} \right) = \frac{y}{x}$$

$$\frac{(x+y)^2}{Cx} = e^{\frac{y}{x}}$$

$$\boxed{(x+y)^2 = Cxe^{\frac{y}{x}}}$$

Exercise

Find the general solution $x \frac{dy}{dx} + y = x^2 y^2$

Solution

$$y' + \frac{1}{x}y = xy^2$$

$$\text{Let } u = y^{1-2} = y^{-1} \Rightarrow y = \frac{1}{u}$$

$$\frac{du}{dx} = -\frac{1}{y^2} \frac{dy}{dx} \Rightarrow y' = -y^2 u' = -\frac{1}{u^2} u'$$

$$-\frac{1}{u^2} u' + \frac{1}{x} \frac{1}{u} = x \frac{1}{u^2}$$

$$u' - \frac{1}{x}u = -x \quad \left(\times -u^2 \right)$$

$$e^{\int -\frac{1}{x} dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$\int -xx^{-1} dx = -x$$

$$u = x(-x + C)$$

$$\frac{1}{y} = -x^2 + Cx$$

$$\boxed{y(x) = \frac{1}{-x^2 + Cx}}$$

Exercise

Solve the differential equation $(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = 3x^2 - 2xy + 2 \Rightarrow M_y = -2x$$

$$\frac{\partial \psi}{\partial y} = N = 6y^2 - x^2 + 3 \Rightarrow N_x = -2x$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = 3x^2 - 2xy + 2$$

$$\psi = \int (3x^2 - 2xy + 2) dx = x^3 - x^2y + 2x + h(y)$$

$$\psi_y = -x^2 + h'(y) = 6y^2 - x^2 + 3 \Rightarrow h'(y) = 6y^2 + 3$$

$$h(y) = \int (6y^2 + 3) dy = 2y^3 + 3y$$

$$\underline{x^3 - x^2y + 2x + 2y^3 + 3y = C}$$

Exercise

Solve the differential equation $(e^x \sin y - 2y \sin x) dx + (e^x \cos y + 2 \cos x) dy = 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = e^x \sin y - 2y \sin x \Rightarrow M_y = e^x \cos y - 2 \sin x$$

$$\frac{\partial \psi}{\partial y} = N = e^x \cos y + 2 \cos x \Rightarrow N_x = e^x \cos y - 2 \sin x$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = e^x \sin y - 2y \sin x \Rightarrow \psi = \int (e^x \sin y - 2y \sin x) dx = e^x \sin y + 2y \cos x + h(y)$$

$$\psi_y = e^x \cos y + 2 \cos x + h'(y) = e^x \cos y + 2 \cos x \Rightarrow h'(y) = 0 \rightarrow \underline{h(y) = C}$$

$$\underline{e^x \sin y + 2y \cos x = C}$$

Exercise

Solve the differential equation $\left(\frac{y}{x} + 6x\right) dx + (\ln x - 2) dy = 0, \quad x > 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = \frac{y}{x} + 6x \Rightarrow M_y = \frac{1}{x}$$

$$\frac{\partial \psi}{\partial y} = N = \ln x - 2 \Rightarrow N_x = \frac{1}{x} \Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = e^x \sin y - 2y \sin x \Rightarrow \psi = \int \left(\frac{y}{x} + 6x \right) dx = y \ln x + 3x^2 + h(y)$$

$$\psi_y = \ln x + h'(y) = \ln x - 2 \Rightarrow h'(y) = -2$$

$$h(y) = \int -2dy = -2y$$

$$\underline{y \ln x + 3x^2 - 2y = C}$$

Exercise

Solve the differential equation $\frac{xdx}{(x^2 + y^2)^{3/2}} + \frac{ydy}{(x^2 + y^2)^{3/2}} = 0$

Solution

Multiply both side by $(x^2 + y^2)^{3/2}$ since $x^2 + y^2 \neq 0 \Rightarrow xdx + ydy = 0$

$$\frac{\partial \psi}{\partial x} = M = x \Rightarrow M_y = 0 \quad \frac{\partial \psi}{\partial y} = N = y \Rightarrow N_x = 0 \Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = x \Rightarrow \psi = \int xdx = \frac{1}{2}x^2 + h(y)$$

$$\psi_y = h'(y) = y \Rightarrow h(y) = \int ydy = \frac{1}{2}y^2$$

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 = C_1$$

$$\underline{x^2 + y^2 = C}$$

Exercise

Solve the differential equation $(e^{2y} - y \cos xy)dx + (2xe^{2y} - x \cos xy + 2y)dy = 0$

Solution

$$M(x, y) = e^{2y} - y \cos xy \quad N(x, y) = 2xe^{2y} - x \cos xy + 2y$$

$$\frac{\partial M}{\partial y} = 2e^{2y} - \cos x + xy \sin xy \quad \frac{\partial N}{\partial x} = 2e^{2y} - \cos x + xy \sin xy \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (e^{2y} - y \cos xy) dx = xe^{2y} - \sin xy + h(y)$$

$$\psi_y = 2xe^{2y} - x \cos xy + h'(y) = 2xe^{2y} - x \cos xy + 2y$$

$$h'(y) = 2y \Rightarrow h(y) = y^2 + C$$

$$\psi = \underline{xe^{2y} - \sin xy + y^2 = C}$$

Exercise

Solve the differential equation $(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = 3x^2 - 2xy + 2 \Rightarrow M_y = -2x$$

$$\frac{\partial \psi}{\partial y} = N = 6y^2 - x^2 + 3 \Rightarrow N_x = -2x$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = 3x^2 - 2xy + 2 \Rightarrow \psi = \int (3x^2 - 2xy + 2) dx = x^3 - x^2 y + 2x + h(y)$$

$$\psi_y = -x^2 + h'(y) = 6y^2 - x^2 + 3 \Rightarrow h'(y) = 6y^2 + 3$$

$$h(y) = \int (6y^2 + 3) dy = 2y^3 + 3y$$

$$\underline{x^3 - x^2 y + 2x + 2y^3 + 3y = C}$$

Exercise

Solve the differential equation $(e^x \sin y - 2y \sin x) dx + (e^x \cos y + 2 \cos x) dy = 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = e^x \sin y - 2y \sin x \Rightarrow M_y = e^x \cos y - 2 \sin x$$

$$\frac{\partial \psi}{\partial y} = N = e^x \cos y + 2 \cos x \Rightarrow N_x = e^x \cos y - 2 \sin x$$

$$\Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = e^x \sin y - 2y \sin x$$

$$\psi = \int (e^x \sin y - 2y \sin x) dx$$

$$= e^x \sin y + 2y \cos x + h(y)$$

$$\psi_y = e^x \cos y + 2 \cos x + h'(y) = e^x \cos y + 2 \cos x$$

$$h'(y) = 0 \rightarrow \underline{h(y) = C}$$

$$\underline{e^x \sin y + 2y \cos x = C}$$

Exercise

Solve the differential equation $\left(\frac{y}{x} + 6x\right)dx + (\ln x - 2)dy = 0, \quad x > 0$

Solution

$$\frac{\partial \psi}{\partial x} = M = \frac{y}{x} + 6x \Rightarrow M_y = \frac{1}{x} \quad \frac{\partial \psi}{\partial y} = N = \ln x - 2 \Rightarrow N_x = \frac{1}{x} \Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = e^x \sin y - 2y \sin x \Rightarrow \psi = \int \left(\frac{y}{x} + 6x\right)dx = y \ln x + 3x^2 + h(y)$$

$$\psi_y = \ln x + h'(y) = \ln x - 2 \Rightarrow h'(y) = -2$$

$$h(y) = \int -2dy = -2y$$

$$\underline{y \ln x + 3x^2 - 2y = C}$$

Exercise

Solve the differential equation $\frac{xdx}{(x^2 + y^2)^{3/2}} + \frac{ydy}{(x^2 + y^2)^{3/2}} = 0$

Solution

Multiply both side by $(x^2 + y^2)^{3/2}$ since $x^2 + y^2 \neq 0 \Rightarrow xdx + ydy = 0$

$$\frac{\partial \psi}{\partial x} = M = x \Rightarrow M_y = 0$$

$$\frac{\partial \psi}{\partial y} = N = y \Rightarrow N_x = 0 \Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = x \Rightarrow \psi = \int xdx = \frac{1}{2}x^2 + h(y)$$

$$\psi_y = h'(y) = y \Rightarrow h(y) = \int ydy = \frac{1}{2}y^2$$

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 = C_1$$

$$\underline{x^2 + y^2 = C}$$

Exercise

Find the general solution $(e^{2y} - y \cos xy)dx + (2xe^{2y} - x \cos xy + 2y)dy = 0$

Solution

$$M(x, y) = e^{2y} - y \cos xy \quad N(x, y) = 2xe^{2y} - x \cos xy + 2y$$

$$\begin{aligned}\frac{\partial M}{\partial y} &= 2e^{2y} - \cos x + xy \sin xy \\ \frac{\partial N}{\partial x} &= 2e^{2y} - \cos x + xy \sin xy\end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (e^{2y} - y \cos xy) dx = xe^{2y} - \sin xy + h(y)$$

$$\psi_y = 2xe^{2y} - x \cos xy + h'(y) = 2xe^{2y} - x \cos xy + 2y$$

$$h'(y) = 2y \Rightarrow h(y) = y^2$$

$$\underline{xe^{2y} - \sin xy + y^2 = C}$$

Exercise

Find the general solution $(2x-1)dx + (3y+7)dy = 0$

Solution

$$M(x, y) = 2x - 1 \quad N(x, y) = 3y + 7$$

$$\begin{aligned}\frac{\partial M}{\partial y} &= 0 \\ \frac{\partial N}{\partial x} &= 0\end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (2x-1) dx = x^2 - x + h(y)$$

$$\psi_y = h'(y) = 3y + 7 \Rightarrow h(y) = \frac{3}{2}y^2 + 7$$

$$\underline{x^2 - x + \frac{3}{2}y^2 + 7 = C}$$

Exercise

Find the general solution $(5x+4y)dx + (4x-8y^3)dy = 0$

Solution

$$M(x, y) = 5x + 4y \quad N(x, y) = 4x - 8y^3$$

$$\begin{aligned}\frac{\partial M}{\partial y} &= 4 \\ \frac{\partial N}{\partial x} &= 4\end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (5x+4y) dx = \frac{5}{2}x^2 + 4xy + h(y)$$

$$\psi_y = 4x + h'(y) = 4x - 8y^3 \rightarrow h'(y) = -8y^3 \Rightarrow h(y) = -2y^4$$

$$\underline{\frac{5}{2}x^2 + 4xy - 2y^4 = C}$$

Exercise

Find the general solution $(\sin y - y \sin x)dx + (\cos x + x \cos y - y)dy = 0$

Solution

$$M(x, y) = \sin y - y \sin x \quad N(x, y) = \cos x + x \cos y - y$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= \cos y - \sin x \\ \frac{\partial N}{\partial x} &= -\sin x + \cos y \end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (\sin y - y \sin x) dx = x \sin y + y \cos x + h(y)$$

$$\psi_y = x \cos y + \cos x + h'(y) = x \cos y + \cos x - y$$

$$\rightarrow h'(y) = -y \Rightarrow h(y) = -\frac{1}{2}y^2$$

$$\underline{x \sin y + y \cos x - \frac{1}{2}y^2 = C}$$

Exercise

Find the general solution $(2xy^2 - 3)dx + (2x^2y + 4)dy = 0$

Solution

$$M(x, y) = 2xy^2 - 3 \quad N(x, y) = 2x^2y + 4$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= 4xy \\ \frac{\partial N}{\partial x} &= 4xy \end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (2xy^2 - 3) dx = x^2y^2 - 3x + h(y)$$

$$\psi_y = 2x^2y + h'(y) = 2x^2y + 4$$

$$\rightarrow h'(y) = 4 \Rightarrow h(y) = 4y$$

$$\underline{x^2y^2 - 3x + 4y = C}$$

Exercise

Find the general solution $\left(1 + \ln x + \frac{y}{x}\right)dx - (1 - \ln x)dy = 0$

Solution

$$M(x, y) = 1 + \ln x + \frac{y}{x} \quad N(x, y) = -1 + \ln x$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{1}{x} \\ \frac{\partial N}{\partial x} &= \frac{1}{x} \end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\begin{aligned} \psi &= \int \left(1 + \ln x + \frac{y}{x}\right) dx \\ &= x + x \ln x - x + y \ln x + h(y) \\ &= x \ln x + y \ln x + h(y) \end{aligned}$$

$$\begin{aligned} \psi_y &= \ln x + h'(y) = -1 + \ln x \\ \rightarrow h'(y) &= -1 \Rightarrow h(y) = -y \end{aligned}$$

$$\underline{x \ln x + y \ln x - y = C}$$

Exercise

Find the general solution $(x - y^3 + y^2 \sin x)dx - (3xy^2 + 2y \cos x)dy = 0$

Solution

$$M(x, y) = x - y^3 + y^2 \sin x \quad N(x, y) = -3xy^2 - 2y \cos x$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= -3y^2 + 2y \sin x \\ \frac{\partial N}{\partial x} &= -3y^2 + 2y \sin x \end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (x - y^3 + y^2 \sin x) dx = \frac{1}{2}x^2 - xy^3 - y^2 \cos x + h(y)$$

$$\begin{aligned} \psi_y &= -3xy^2 - 2y \cos x + h'(y) = -3xy^2 - 2y \cos x \\ \rightarrow h'(y) &= 0 \Rightarrow h(y) = C \end{aligned}$$

$$\underline{\frac{1}{2}x^2 - xy^3 - y^2 \cos x = C}$$

Exercise

Find the general solution $(x^3 + y^3)dx + 3xy^2dy = 0$

Solution

$$M(x, y) = x^3 + y^3 \quad N(x, y) = 3xy^2$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= 3y^2 \\ \frac{\partial N}{\partial x} &= 3y^2 \end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (x^3 + y^3) dx = \frac{1}{4}x^4 + xy^3 + h(y)$$

$$\psi_y = 3xy^2 + h'(y) = 3xy^2$$

$$\rightarrow h'(y) = 0 \Rightarrow h(y) = C$$

$$\boxed{\frac{1}{4}x^4 + xy^3 = C}$$

Exercise

Find the general solution $(3x^2y + e^y)dx + (x^3 + xe^y - 2y)dy = 0$

Solution

$$M(x, y) = 3x^2y + e^y \quad N(x, y) = x^3 + xe^y - 2y$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= 3x^2 + e^y \\ \frac{\partial N}{\partial x} &= 3x^2 + e^y \end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (3x^2y + e^y) dx = x^3y + xe^y + h(y)$$

$$\psi_y = x^3 + xe^y + h'(y) = x^3 + xe^y - 2y$$

$$\rightarrow h'(y) = -2y \Rightarrow h(y) = -y^2$$

$$\boxed{x^3y + xe^y - y^2 = C}$$

Exercise

Find the general solution $xdy + (y - 2xe^x - 6x^2)dx = 0$

Solution

$$M(x, y) = y - 2xe^x - 6x^2 \quad N(x, y) = x$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= 1 \\ \frac{\partial N}{\partial x} &= 1 \end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\begin{aligned} \psi &= \int (y - 2xe^x - 6x^2) dx \\ &= xy - (2x - 2)e^x - 2x^3 + h(y) \end{aligned}$$

$$\psi_y = x + h'(y) = x$$

$$\rightarrow h'(y) = 0 \Rightarrow h(y) = C$$

$$\underline{xy - 2xe^x + 2e^x - 2x^3 = C}$$

Exercise

Find the general solution $\left(1 - \frac{3}{y} + x\right)dy + \left(y - \frac{3}{x} + 1\right)dx = 0$

Solution

$$M(x, y) = y - \frac{3}{x} + 1 \quad N(x, y) = 1 - \frac{3}{y} + x$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= 1 \\ \frac{\partial N}{\partial x} &= 1 \end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(y - \frac{3}{x} + 1\right) dx = xy - 3\ln|x| + x + h(y)$$

$$\psi_y = x + h'(y) = 1 - \frac{3}{y} + x$$

$$\rightarrow h'(y) = 1 - \frac{3}{y} \Rightarrow h(y) = y - 3\ln|y|$$

$$xy - 3\ln|x| + x + y - 3\ln|y| = C$$

$$\underline{xy + x + y - 3\ln|xy| = C}$$

Exercise

Find the general solution $\left(x^2y^3 - \frac{1}{1+9x^2}\right)\frac{dx}{dy} + x^3y^2 = 0$

Solution

$$\left(x^2y^3 - \frac{1}{1+9x^2}\right)dx + (x^3y^2)dy = 0$$

$$M(x, y) = x^2 y^3 - \frac{1}{1+9x^2} \quad N(x, y) = x^3 y^2$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= 3x^2 y^2 \\ \frac{\partial N}{\partial x} &= 3x^2 y^2 \end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\begin{aligned} \psi &= \int \left(x^2 y^3 - \frac{1}{1+9x^2} \right) dx \\ &= \frac{1}{3} x^3 y^3 - \frac{1}{3} \arctan(3x) + h(y) \end{aligned}$$

$$\psi_y = x^3 y^2 + h'(y) = x^3 y^2$$

$$\rightarrow h'(y) = 0 \Rightarrow h(y) = C$$

$$\frac{1}{3} x^3 y^3 - \frac{1}{3} \arctan(3x) = C_1$$

$$\underline{x^3 y^3 - \arctan(3x) = C}$$

Exercise

Find the general solution $(5y - 2x)y' - 2y = 0$

Solution

$$M(x, y) = -2y \quad N(x, y) = 5y - 2x$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= -2 \\ \frac{\partial N}{\partial x} &= -2 \end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (-2y) dx = -2xy + h(y)$$

$$\psi_y = -2x + h'(y) = 5y - 2x$$

$$\rightarrow h'(y) = 5y \Rightarrow h(y) = \frac{5}{2} y^2$$

$$\underline{-2xy + \frac{5}{2} y^2 = C}$$

Exercise

Find the general solution $(x - y)dx - xdy = 0$

Solution

$$M(x, y) = x - y \quad N(x, y) = -x$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= -1 \\ \frac{\partial N}{\partial x} &= -1 \end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\begin{aligned} \psi &= \int (x - y) \, dx \\ &= \frac{1}{2}x^2 - xy + h(y) \end{aligned}$$

$$\psi_y = -x + h'(y) = -x$$

$$\rightarrow h'(y) = 0 \Rightarrow h(y) = C$$

$$\boxed{\frac{1}{2}x^2 - xy = C}$$

Exercise

Find the general solution $(x + y)dx + xdy = 0$

Solution

$$M(x, y) = x + y \quad N(x, y) = x$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= 1 \\ \frac{\partial N}{\partial x} &= 1 \end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\begin{aligned} \psi &= \int (x + y) \, dx \\ &= \frac{1}{2}x^2 + xy + h(y) \end{aligned}$$

$$\psi_y = x + h'(y) = x$$

$$\rightarrow h'(y) = 0 \Rightarrow h(y) = C$$

$$\boxed{\frac{1}{2}x^2 + xy = C}$$

Exercise

Find the general solution $\frac{dy}{dx} = -\frac{2xy^2 + 1}{2x^2y}$

Solution

$$2x^2 y dy = -(2xy^2 + 1) dx$$

$$(2xy^2 + 1) dx + 2x^2 y dy = 0$$

$$M(x, y) = 2xy^2 + 1 \quad N(x, y) = 2x^2 y$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= 4xy \\ \frac{\partial N}{\partial x} &= 4xy \end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\begin{aligned} \psi &= \int (2xy^2 + 1) dx \\ &= x^2 y^2 + x + h(y) \end{aligned}$$

$$\psi_y = 2x^2 y + h'(y) = 2x^2 y$$

$$\rightarrow h'(y) = 0 \Rightarrow h(y) = C$$

$$\boxed{x^2 y^2 + x = C}$$

Exercise

Find the general solution $(1 + e^x y + x e^x y) dx + (x e^x + 2) dy = 0$

Solution

$$M(x, y) = 1 + e^x y + x e^x y \quad N(x, y) = x e^x + 2$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= e^x + x e^x \\ \frac{\partial N}{\partial x} &= e^x + x e^x \end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\begin{aligned} \psi &= \int (1 + e^x y + x e^x y) dx \\ &= x + e^x y + (x-1)e^x y + h(y) \\ &= x + x e^x y + h(y) \end{aligned}$$

$$\psi_y = x e^x + h'(y) = x e^x + 2$$

$$\rightarrow h'(y) = 2 \Rightarrow h(y) = 2y + C$$

$$\boxed{x + x e^x y + 2y + C = 0}$$

Exercise

Find the general solution $(2xy^3 + 1)dx + \left(3x^2y^2 - \frac{1}{y}\right)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}(2xy^3 + 1) = 6xy^2$$

$$N_x = \frac{\partial}{\partial x}\left(3x^2y^2 - \frac{1}{y}\right) = 6xy^2 \Rightarrow M_y = N_x$$

$$\psi = \int (2xy^3 + 1) dx$$

$$= x^2y^3 + x + h(y)$$

$$\psi_y = 3x^2y^2 + h'(y)$$

$$= 3x^2y^2 - \frac{1}{y}$$

$$\rightarrow h'(y) = -\frac{1}{y} \Rightarrow h(y) = -\ln|y| + C$$

$$\underline{x^2y^3 + x - \ln|y| = C}$$

Exercise

Find the general solution $(2x + y)dx + (x - 2y)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}(2x + y) = 1$$

$$\Rightarrow M_y = N_x$$

$$N_x = \frac{\partial}{\partial x}(x - 2y) = 1$$

$$\psi = \int (2x + y) dx$$

$$= x^2 + xy + h(y)$$

$$\psi_y = x + h'(y) = x - 2y$$

$$\rightarrow h'(y) = -2y \Rightarrow h(y) = -y^2$$

$$\underline{x^2 + xy - y^2 = C}$$

Exercise

Find the general solution $e^x(y-x)dx + (1+e^x)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}(e^x(y-x)) = e^x$$
$$N_x = \frac{\partial}{\partial x}(1+e^x) = e^x \quad \Rightarrow M_y = N_x$$

$$\psi = \int (ye^x - xe^x) dx$$
$$= ye^x - (x-1)e^x + h(y)$$

$$\psi_y = e^x + h'(y) = 1 + e^x$$
$$\rightarrow h'(y) = 1 \Rightarrow h(y) = y$$

$$\underline{ye^x - (x-1)e^x + y = C}$$

$$\underline{y(x) = \frac{(x-1)e^x + C}{1+e^x}}$$

Exercise

Find the general solution $\left(ye^{xy} - \frac{1}{y}\right)dx + \left(xe^{xy} + \frac{x}{y^2}\right)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(ye^{xy} - \frac{1}{y}\right) = e^{xy} + xye^{xy} + \frac{1}{y^2}$$
$$N_x = \frac{\partial}{\partial x}\left(xe^{xy} + \frac{x}{y^2}\right) = e^{xy} + xye^{xy} + \frac{1}{y^2} \quad \Rightarrow M_y = N_x$$

$$\psi = \int \left(ye^{xy} - \frac{1}{y}\right) dx = e^{xy} - \frac{x}{y} + h(y)$$

$$\psi_y = xe^{xy} + \frac{x}{y^2} + h'(y) = xe^{xy} + \frac{x}{y^2}$$
$$\rightarrow h'(y) = 0 \Rightarrow h(y) = C$$

$$\underline{e^{xy} - \frac{x}{y} = C}$$

Exercise

Find the general solution $(\tan x - \sin x \sin y)dx + (\cos x \cos y)dy = 0$

Solution

$$M(x, y) = \tan x - \sin x \sin y \quad N(x, y) = \cos x \cos y$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= -\sin x \cos y \\ \frac{\partial N}{\partial x} &= -\sin x \cos y \end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (\tan x - \sin x \sin y) dx = \ln|\sec x| + \cos x \sin y + h(y)$$

$$\psi_y = \cos x \cos y + h'(y) = \cos x \cos y$$

$$\rightarrow h'(y) = 0 \Rightarrow h(y) = C$$

$$\boxed{\ln|\sec x| + \cos x \sin y = C}$$

Exercise

Find the general solution $(2x^3 - xy^2 - 2y + 3)dx - (x^2y + 2x)dy = 0$

Solution

$$M(x, y) = 2x^3 - xy^2 - 2y + 3 \quad N(x, y) = -x^2y - 2x$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= -2xy - 2 \\ \frac{\partial N}{\partial x} &= -2xy - 2 \end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (2x^3 - xy^2 - 2y + 3) dx = \frac{1}{2}x^4 - \frac{1}{2}x^2y^2 - 2xy + 3x + h(y)$$

$$\psi_y = -x^2y - 2x + h'(y) = -x^2y - 2x$$

$$\rightarrow h'(y) = 0 \Rightarrow h(y) = C$$

$$\boxed{\frac{1}{2}x^4 - \frac{1}{2}x^2y^2 - 2xy + 3x = C}$$

Exercise

Find the general solution $(x + \sin y)dx + (x \cos y - 2y)dy = 0$

Solution

$$M(x, y) = x + \sin y \quad N(x, y) = x \cos y - 2y$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= \cos y \\ \frac{\partial N}{\partial x} &= \cos y \end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int (x + \sin y) dx = \frac{1}{2}x^2 + x \sin y + h(y)$$

$$\psi_y = x \cos y + h'(y) = x \cos y - 2y$$

$$\rightarrow h'(y) = -2y \Rightarrow h(y) = -y^2$$

$$\underline{\frac{1}{2}x^2 + x \sin y - y^2 = C}$$

Exercise

Find the general solution $\left(x + \frac{1}{\sqrt{y^2 - x^2}}\right)dx + \left(1 - \frac{x}{y\sqrt{y^2 - x^2}}\right)dy = 0$

Solution

$$M(x, y) = x + \frac{1}{\sqrt{y^2 - x^2}} \quad N(x, y) = 1 - \frac{x}{y\sqrt{y^2 - x^2}}$$

$$\frac{\partial M}{\partial y} = -\frac{1}{2}(2y) \frac{1}{\sqrt{y^2 - x^2}} = -\frac{y}{\sqrt{y^2 - x^2}}$$

$$\frac{\partial N}{\partial x} = -\frac{1}{y} \frac{1}{\sqrt{y^2 - x^2}} (y^2 - x^2 + x^2) = -\frac{y}{\sqrt{y^2 - x^2}} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\psi = \int \left(x + \frac{1}{\sqrt{y^2 - x^2}}\right) dx = \frac{1}{2}x^2 + \sin^{-1} \frac{x}{y} + h(y)$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

$$\psi_y = -\frac{1}{y^2 \sqrt{1 - \left(\frac{x}{y}\right)^2}} + h'(y) = -\frac{1}{y\sqrt{y^2 - x^2}} + h'(y) = 1 - \frac{1}{y\sqrt{y^2 - x^2}}$$

$$\frac{d}{dx}(\arcsin u) = \frac{u'}{\sqrt{1 - u^2}}$$

$$\rightarrow h'(y) = 1 \Rightarrow h(y) = y$$

$$\underline{\frac{1}{2}x^2 + \sin^{-1} \frac{x}{y} + y = C}$$

Exercise

Find the general solution $(2x + y^2 - \cos(x + y))dx + (2xy - \cos(x + y) - e^y)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}(2x + y^2 - \cos(x + y)) = 2y + \sin(x + y) \quad \Rightarrow \quad M_y = N_x$$

$$N_x = \frac{\partial}{\partial x}(2xy - \cos(x + y) - e^y) = 2y + \sin(x + y)$$

$$\psi = \int (2x + y^2 - \cos(x + y)) dx = x^2 + xy^2 - \sin(x + y) + h(y)$$

$$\psi_y = 2xy - \cos(x + y) + h'(y) = 2xy - \cos(x + y) - e^y$$

$$\rightarrow h'(y) = -e^y \quad \Rightarrow \quad h(y) = -e^y$$

$$\underline{x^2 + xy^2 - \sin(x + y) - e^y = C}$$

Exercise

Find the general solution $\left(\frac{2}{\sqrt{1-x^2}} + y \cos(xy)\right)dx + (x \cos(xy) - y^{-1/3})dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}\left(\frac{2}{\sqrt{1-x^2}} + y \cos(xy)\right) = \cos(xy) - xy \sin(xy) \quad \Rightarrow \quad M_y = N_x$$

$$N_x = \frac{\partial}{\partial x}(x \cos(xy) - y^{-1/3}) = \cos(xy) - xy \sin(xy)$$

$$\psi = \int (x \cos(xy) - y^{-1/3}) dy = \sin(xy) - \frac{3}{2}y^{2/3} + h(x)$$

$$\psi_x = y \cos(xy) + h'(x) = \frac{2}{\sqrt{1-x^2}} + y \cos(xy)$$

$$\rightarrow h'(x) = \frac{2}{\sqrt{1-x^2}} \quad \Rightarrow \quad h(x) = 2 \arcsin x$$

$$\underline{\sin(xy) - \frac{3}{2}y^{2/3} + 2 \arcsin x = C}$$

Exercise

Find the general solution $(2x + y \cos(xy))dx + (x \cos(xy) - 2y)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}(2x + y \cos(xy)) = \cos(xy) - xy \sin(xy)$$

$$N_x = \frac{\partial}{\partial x}(x \cos(xy) - 2y) = \cos(xy) - xy \sin(xy) \Rightarrow M_y = N_x$$

$$\psi = \int (2x + y \cos(xy)) dy = x^2 + \sin(xy) + h(y)$$

$$\psi_y = x \cos(xy) + h'(y) = x \cos(xy) - 2y$$

$$\rightarrow h'(y) = -2y \Rightarrow h(y) = -y^2$$

$$\underline{x^2 + \sin(xy) - y^2 = C}$$

Exercise

Find the general solution $(e^x \sin y - 3x^2)dx + (e^x \cos y + \frac{1}{3}y^{-2/3})dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}(e^x \sin y - 3x^2) = e^x \cos y$$

$$N_x = \frac{\partial}{\partial x}(e^x \cos y + \frac{1}{3}y^{-2/3}) = e^x \cos y \Rightarrow M_y = N_x$$

$$\psi = \int (e^x \sin y - 3x^2) dx = e^x \sin y - x^3 + h(y)$$

$$\psi_y = e^x \cos y + h'(y) = e^x \cos y + \frac{1}{3}y^{-2/3}$$

$$\rightarrow h'(y) = \frac{1}{3}y^{-2/3} \Rightarrow h(y) = y^{1/3}$$

$$\underline{e^x \sin y - x^3 + y^{1/3} = C}$$

Exercise

Find the general solution $(2y \sin x \cos x - y + 2y^2 e^{xy^2})dx = (x - \sin^2 x - 4xy e^{xy^2})dy$

Solution

$$(2y \sin x \cos x - y + 2y^2 e^{xy^2})dx - (x - \sin^2 x - 4xy e^{xy^2})dy = 0$$

$$M(x, y) = 2y \sin x \cos x - y + 2y^2 e^{xy^2} \quad N(x, y) = -x + \sin^2 x + 4xy e^{xy^2}$$

$$\frac{\partial M}{\partial y} = 2 \sin x \cos x - 1 + 4ye^{xy^2} + 4xy^3 e^{xy^2}$$

$$\frac{\partial N}{\partial x} = -1 + 2 \sin x \cos x + 4ye^{xy^2} + 4xy^3 e^{xy^2} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\begin{aligned}\psi &= \int \left(2y \sin x \cos x - y + 2y^2 e^{xy^2} \right) dx \\ &= \int (2y \sin x) d(\sin x) - xy + 2e^{xy^2} \\ &= y \sin^2 x - xy + 2e^{xy^2}\end{aligned}$$

$$\begin{aligned}\psi_y &= \sin^2 x - x + 4xye^{xy^2} + h'(y) = -x + \sin^2 x + 4xye^{xy^2} \\ &\rightarrow h'(y) = 0 \Rightarrow h(y) = C\end{aligned}$$

$$\underline{y \sin^2 x - xy + 2e^{xy^2} = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$x^2 y^3 + x(1 + y^2)y' = 0, \quad \mu(x, y) = \frac{1}{xy^3}$$

Solution

$$\begin{aligned}M_y &= \frac{\partial}{\partial y}(x^2 y^3) = 3x^2 y^2 \\ N_x &= \frac{\partial}{\partial x}(x + xy^2) = 1 + y^2\end{aligned} \Rightarrow M_y \neq N_x$$

$$x^2 y^3 \left(\frac{1}{xy^3} \right) + x(1 + y^2) \left(\frac{1}{xy^3} \right) y' = 0$$

$$x + \left(\frac{1 + y^2}{y^3} \right) \frac{dy}{dx} = 0 \Rightarrow \left(\frac{1 + y^2}{y^3} \right) dy = -x dx$$

$$\begin{aligned}\int \left(y^{-3} + \frac{1}{y} \right) dy &= - \int x dx \\ -\frac{1}{2} y^{-2} + \ln|y| &= -\frac{1}{2} x^2 + C_0\end{aligned}$$

$$\underline{x^2 - y^{-2} + \ln|y| = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$y^2 - xy + (x^2)y' = 0, \quad \mu(x, y) = \frac{1}{xy^2}$$

Solution

$$M_y = \frac{\partial}{\partial y}(y^2 - xy) = 2y - x \quad N_x = \frac{\partial}{\partial x}(x^2) = 2x \quad \Rightarrow M_y \neq N_x$$

$$(y^2 - xy)\left(\frac{1}{xy^2}\right) + (x^2)\left(\frac{1}{xy^2}\right)y' = 0$$

$$\left(\frac{1}{x} - \frac{1}{y}\right) + \left(\frac{x}{y^2}\right)y' = 0$$

$$M_y = \frac{\partial}{\partial y}\left(\frac{1}{x} - \frac{1}{y}\right) = \frac{1}{y^2}$$

$$N_x = \frac{\partial}{\partial x}\left(\frac{x}{y^2}\right) = \frac{1}{y^2} \quad \Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = \frac{1}{x} - \frac{1}{y} \quad \Rightarrow \quad \psi = \int \left(\frac{1}{x} - \frac{1}{y}\right) dx = \ln|x| - \frac{x}{y} + h(y)$$

$$\psi_y = \frac{x}{y^2} + h'(y) = \frac{x}{y^2}$$

$$\Rightarrow h'(y) = 0 \rightarrow h(y) = C$$

$$\boxed{\ln|x| - \frac{x}{y} = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$x^2y^3 - y + x(1 + x^2y^2)y' = 0, \quad \mu(x, y) = \frac{1}{xy}$$

Solution

$$M_y = \frac{\partial}{\partial y}(x^2y^3 - y) = 3y^2 - 1 \quad N_x = \frac{\partial}{\partial x}(x + x^3y^2) = 1 + 3x^2y^2 \quad \Rightarrow M_y \neq N_x$$

$$(x^2y^3 - y)\left(\frac{1}{xy}\right) + x(1 + x^2y^2)\left(\frac{1}{xy}\right)y' = 0$$

$$\left(xy^2 - \frac{1}{x}\right) + \left(\frac{1}{y} + x^2y\right)y' = 0$$

$$M_y = \frac{\partial}{\partial y} \left(xy^2 - \frac{1}{x} \right) = 2xy$$

$$N_x = \frac{\partial}{\partial x} \left(\frac{1}{y} + x^2 y \right) = 2xy \Rightarrow M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = xy^2 - \frac{1}{x} \Rightarrow \psi = \int \left(xy^2 - \frac{1}{x} \right) dx = \frac{1}{2} x^2 y^2 - \ln|x| + h(y)$$

$$\psi_y = x^2 y + h'(y) = \frac{1}{y} + x^2 y$$

$$\Rightarrow h'(y) = \frac{1}{y} \rightarrow h(y) = \ln|y|$$

$$\underline{\frac{1}{2} x^2 y^2 - \ln x + \ln y = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$\left(\frac{\sin y}{y} - 2e^{-x} \sin x \right) dx + \left(\frac{\cos y + 2e^{-x} \cos x}{y} \right) dy = 0, \quad \mu(x, y) = ye^x$$

Solution

$$M_y = \frac{\partial}{\partial y} \left(\frac{\sin y}{y} - 2e^{-x} \sin x \right) = \frac{y \cos y - \sin y}{y^2}$$

$$N_x = \frac{\partial}{\partial x} \left(\frac{\cos y + 2e^{-x} \cos x}{y} \right) = \frac{1}{y} \left(-2e^{-x} \cos x - 2e^{-x} \sin x \right)$$

$$\Rightarrow M_y \neq N_x$$

$$\left(ye^x \right) \left(\frac{\sin y}{y} - 2e^{-x} \sin x \right) dx + \left(ye^x \right) \left(\frac{\cos y + 2e^{-x} \cos x}{y} \right) dy = 0$$

$$\left(e^x \sin y - 2y \sin x \right) dx + \left(e^x \cos y + 2 \cos x \right) dy = 0$$

$$M_y = \frac{\partial}{\partial y} \left(e^x \sin y - 2y \sin x \right) = e^x \cos y - 2 \sin x$$

$$N_x = \frac{\partial}{\partial x} \left(e^x \cos y + 2 \cos x \right) = e^x \cos y - 2 \sin x \Rightarrow M_y = N_x$$

$$\psi = \int \left(e^x \sin y - 2y \sin x \right) dx$$

$$= e^x \sin y + 2y \cos x + h(y)$$

$$\psi_y = e^x \cos y + 2 \cos x + h'(y)$$

$$= e^x \cos y + 2 \cos x$$

$$\Rightarrow h'(y) = 0 \rightarrow h(y) = C$$

$$\underline{e^x \sin y + 2y \cos x = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$(x+2)\sin y dx + x \cos y dy = 0, \quad \mu(x, y) = xe^x$$

Solution

$$M_y = \frac{\partial}{\partial y}((x+2)\sin y) = (x+2)\cos y$$

$$N_x = \frac{\partial}{\partial x}(x \cos y) = \cos y \Rightarrow M_y \neq N_x$$

$$(xe^x)(x+2)\sin y dx + (xe^x)x \cos y dy = 0$$

$$(x^2 + 2x)e^x \sin y dx + x^2 e^x \cos y dy = 0$$

$$M_y = \frac{\partial}{\partial y}((x^2 + 2x)e^x \sin y) = (x^2 + 2x)e^x \cos y$$

$$N_x = \frac{\partial}{\partial x}(x^2 e^x \cos y) = (2xe^x + x^2)e^x \cos y \Rightarrow M_y = N_x$$

$$\psi = \int (x^2 e^x \cos y) dy = x^2 e^x \sin y + h(x)$$

$$\psi_x = (x^2 + 2x)e^x \sin y + h'(x) = (x^2 + 2x)e^x \sin y$$

$$\Rightarrow h'(x) = 0 \rightarrow h(x) = C$$

$$\underline{x^2 e^x \sin y = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

$$(x^2 + y^2 - x)dx - y dy = 0, \quad \mu(x, y) = \frac{1}{x^2 + y^2}$$

Solution

$$M_y = \frac{\partial}{\partial y}(x^2 + y^2 - x) = 2y$$

$$N_x = \frac{\partial}{\partial x}(-y) = 0 \quad \Rightarrow M_y \neq N_x$$

$$\frac{1}{x^2 + y^2}(x^2 + y^2 - x)dx - \frac{y}{x^2 + y^2}dy = 0$$

$$\left(1 - \frac{x}{x^2 + y^2}\right)dx - \frac{y}{x^2 + y^2}dy = 0$$

$$M_y = \left(1 - \frac{x}{x^2 + y^2}\right) = \frac{2xy}{(x^2 + y^2)^2}$$

$$N_x = \left(\frac{-y}{x^2 + y^2}\right) = \frac{2xy}{(x^2 + y^2)^2} \quad \Rightarrow M_y = N_x$$

$$\frac{d\psi}{dx} = 1 - \frac{x}{x^2 + y^2}$$

$$\psi = \int \left(1 - \frac{x}{x^2 + y^2}\right) dx$$

$$= \int dx - \frac{1}{2} \int \frac{1}{x^2 + y^2} d(x^2 + y^2)$$

$$= x - \frac{1}{2} \ln(x^2 + y^2) + h(y)$$

$$\psi_y = -\frac{y}{x^2 + y^2} + h'(y)$$

$$= -\frac{y}{x^2 + y^2}$$

$$h'(y) = 0 \rightarrow h(y) = C$$

$$\underline{x - \frac{1}{2} \ln(x^2 + y^2) = C}$$

Exercise

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation $(2y - 6x)dx + (3x - 4x^2y^{-1})dy = 0$, $\mu(x, y) = xy^2$

Solution

$$M_y = \frac{\partial}{\partial y}(2y - 6x) = 2$$

$$N_x = \frac{\partial}{\partial x}(3x - 4x^2y^{-1}) = 3 - \frac{8x}{y} \Rightarrow M_y \neq N_x$$

$$xy^2(2y - 6x)dx + xy^2(3x - 4x^2y^{-1})dy = 0$$

$$(2xy^3 - 6x^2y^2)dx + (3x^2y^2 - 4x^3y)dy = 0$$

$$M_y = \frac{\partial}{\partial y}(2xy^3 - 6x^2y^2) = 6xy^2 - 12x^2y$$

$$N_x = \frac{\partial}{\partial x}(3x^2y^2 - 4x^3y) = 6xy^2 - 12x^2y \Rightarrow M_y = N_x$$

$$\psi = \int (2xy^3 - 6x^2y^2) dx$$

$$= x^2y^3 - 2x^3y^2 + h(y)$$

$$\psi_y = 3x^2y^2 - 4x^3y + h'(y)$$

$$= 3x^2y^2 - 4x^3y$$

$$h'(y) = 0 \rightarrow h(y) = C$$

$$\underline{x^2y^3 - 2x^3y^2 = C}$$

Exercise

Find the general solution of the homogenous equation $(x^2 + y^2)dx - 2xydy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}(x^2 + y^2) = 2y \Rightarrow M_y \neq N_x$$

$$N_x = \frac{\partial}{\partial x}(-2xy) = -2y$$

$$\frac{M_y - N_x}{N} = \frac{2y + 2y}{-2xy} = -\frac{4y}{2xy} = -\frac{2}{x}$$

$$\frac{d\mu}{dx} = -\mu \frac{2}{x} \Rightarrow \int \frac{d\mu}{\mu} = -2 \int \frac{dx}{x}$$

$$\ln \mu = -2 \ln x$$

$$\ln \mu = \ln x^{-2} \Rightarrow \underline{\mu = \frac{1}{x^2}}$$

$$\frac{1}{x^2}(x^2 + y^2)dx - \frac{1}{x^2}2xydy = 0 \quad \Rightarrow \quad \left(1 + \frac{y^2}{x^2}\right)dx - \frac{2y}{x}dy = 0$$

$$M_y = \frac{\partial}{\partial y}\left(1 + \frac{y^2}{x^2}\right) = \frac{2y}{x^2} \quad \Rightarrow \quad M_y = N_x$$

$$N_x = \frac{\partial}{\partial x}\left(-\frac{2y}{x}\right) = \frac{2y}{x^2}$$

$$\psi = \int \left(1 + \frac{y^2}{x^2}\right) dx$$

$$= x - \frac{y^2}{x} + h(y)$$

$$\psi_y = -\frac{2y}{x} + h'(y) = -\frac{2y}{x}$$

$$h'(y) = 0 \quad \Rightarrow \quad h(y) = C$$

$$x - \frac{y^2}{x} = C \quad \text{multiply by } x$$

$$\boxed{x^2 - y^2 = Cx}$$

Exercise

Find the general solution of the homogenous equation $(x + y)dx + (y - x)dy = 0$

Solution

$$(x + y)dx = -(y - x)dy$$

$$\frac{dy}{dx} = \frac{x + y}{x - y}$$

$$\begin{aligned} & \frac{x + y}{x - y} \\ &= \frac{\frac{x + y}{x}}{\frac{x - y}{x}} \end{aligned}$$

$$\begin{aligned} &= \frac{1 + \frac{y}{x}}{1 - \frac{y}{x}} \end{aligned}$$

$$\frac{dy}{dx} = \frac{1 + v}{1 - v} = x \frac{dv}{dx} + v$$

$$x \frac{dv}{dx} = \frac{1 + v}{1 - v} - v = \frac{1 + v^2}{1 - v}$$

$$\frac{1 - v}{1 + v^2} dv = \frac{dx}{x}$$

$$\int \frac{dx}{x} = \int \frac{1}{1+v^2} dv - \int \frac{v}{1+v^2} dv$$

$$\ln x = \arctan v - \frac{1}{2} \int \frac{1}{1+v^2} d(1+v^2)$$

$$\ln x + C = \arctan v - \frac{1}{2} \ln(1+v^2)$$

$$\ln x + C = \arctan \frac{y}{x} - \frac{1}{2} \ln \left(1 + \frac{y^2}{x^2} \right)$$

$$\boxed{\arctan \frac{y}{x} - \frac{1}{2} \ln \left(1 + \frac{y^2}{x^2} \right) - \ln x = C}$$

$$\int \frac{1}{a^2 + x^2} dx = \arctan \frac{x}{a}$$

Exercise

Find the general solution of the homogenous equation

$$\frac{dy}{dx} = \frac{y(x^2 + y^2)}{xy^2 - 2x^3}$$

Solution

$$\begin{aligned} \frac{dy}{dx} &= \frac{y}{x} \frac{\frac{x^2 + y^2}{x^2}}{\frac{y^2 - 2x^2}{x^2}} \\ &= \frac{y}{x} \frac{1 + \left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^2 - 2} \\ &= v \frac{1 + v^2}{v^2 - 2} = x \frac{dv}{dx} + v \end{aligned}$$

$$\frac{v + v^3}{v^2 - 2} - v = x \frac{dv}{dx}$$

$$x \frac{dv}{dx} = \frac{v + v^3 - v^3 + 2v}{v^2 - 2} = \frac{3v}{v^2 - 2}$$

$$\int \frac{dx}{x} = \int \frac{v^2 - 2}{3v} dv = \frac{1}{3} \int \left(v - \frac{2}{v} \right) dv$$

$$3 \ln x + C = \frac{1}{2} v^2 - 2 \ln v$$

$$3 \ln x + C = \frac{1}{2} \frac{y^2}{x^2} - 2 \ln \frac{y}{x}$$

$$3 \ln x + C = \frac{1}{2} \frac{y^2}{x^2} - 2(\ln y - \ln x)$$

$$6 \ln x + C = \frac{y^2}{x^2} - 4 \ln y + 4 \ln x$$

$$\boxed{\frac{y^2}{x^2} - 4 \ln y - 2 \ln x = C}$$

Exercise

Find an integrating factor and solve the given equation $(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}(3x^2y + 2xy + y^3) = 3x^2 + 2x + 3y^2$$

$$N_x = \frac{\partial}{\partial x}(x^2 + y^2) = 2x$$

$$\Rightarrow M_y \neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2} = 3$$

$$\frac{d\mu}{dx} = 3\mu \Rightarrow \int \frac{d\mu}{\mu} = 3 \int dx$$

$$\ln \mu = 3x \Rightarrow \mu = e^{3x}$$

$$e^{3x}(3x^2y + 2xy + y^3)dx + e^{3x}(x^2 + y^2)dy = 0$$

$$M_y = \frac{\partial}{\partial y} \left[e^{3x}(3x^2y + 2xy + y^3) \right] = e^{3x}(3x^2 + 2x + 3y^2)$$

$$N_x = \frac{\partial}{\partial x} e^{3x}(x^2 + y^2) = 3e^{3x}(x^2 + y^2) + 2xe^{3x} = e^{3x}(3x^2 + 3y^2 + 2x)$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int \left(e^{3x}(3x^2y + 2xy + y^3) \right) dx = +h(y)$$

$$= e^{3x} \left(x^2y + \frac{2}{3}xy + \frac{1}{3}y^3 - \frac{2}{3}xy - \frac{2}{9}y + \frac{2}{9}y \right) + h(y)$$

$$= e^{3x} \left(x^2y + \frac{1}{3}y^3 \right) + h(y)$$

$$\psi_y = e^{3x}(x^2 + y^2) + h'(y) = e^{3x}(x^2 + y^2)$$

$$h'(y) = 0 \Rightarrow h(y) = C$$

	$\int e^{3x}$
$3x^2y + 2xy + y^3$	$\frac{1}{3}e^{3x}$
$6xy + 2y$	$\frac{1}{9}e^{3x}$
$6y$	$\frac{1}{27}e^{3x}$

$$\underline{e^{3x} \left(x^2 y + \frac{1}{3} y^3 \right) = C}$$

Exercise

Find an integrating factor and solve the given equation $dx + \left(\frac{x}{y} - \sin y \right) dy = 0$

Solution

$$y dx + (x - y \sin y) dy = 0 \quad \text{Multiply by } y \text{ both sides}$$

$$M_y = \frac{\partial}{\partial y}(y) = 1; \quad N_x = \frac{\partial}{\partial x}(x - y \sin y) = 1; \quad M_y = N_x$$

$$\frac{\partial \psi}{\partial x} = 2x + y^2 \Rightarrow \psi = \int y dx = xy + h(y)$$

$$\psi_y = x + h'(y) = x - y \sin y \Rightarrow h'(y) = -y \sin y$$

$$h(y) = - \int y \sin y dy = y \cos y - \sin y$$

$$\underline{xy + y \cos y - \sin y = C}$$

	$\int -\sin y$
y	$\cos y$
1	$\sin y$

Exercise

Find an integrating factor and solve the given equation $e^x dx + (e^x \cot y + 2y \csc y) dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}(e^x) = 0$$

$$N_x = \frac{\partial}{\partial x}(e^x \cot y + 2y \csc y) = e^x \Rightarrow M_y \neq N_x$$

$$e^x dx + \left(e^x \frac{\cos y}{\sin y} + 2y \frac{1}{\sin y} \right) dy = 0 \quad \text{Multiply by } \sin y \text{ both sides}$$

$$(\sin y) e^x dx + (\sin y) \left(e^x \frac{\cos y}{\sin y} + 2y \frac{1}{\sin y} \right) dy = 0$$

$$e^x \sin y dx + (e^x \cos y + 2y) dy = 0$$

$$M_y = \frac{\partial}{\partial y}(e^x \sin y) = e^x \cos y$$

$$N_x = \frac{\partial}{\partial x}(e^x \cos y + 2y) = e^x \cos y \Rightarrow M_y = N_x$$

$$\psi = \int (e^x \sin y) dx = e^x \sin y + h(y)$$

$$\psi_y = e^x \cos y + h'(y) = e^x \cos y + 2y$$

$$\rightarrow h'(y) = 2y \Rightarrow h(y) = y^2$$

$$\psi(x, y) = e^x \sin y + y^2 = C$$

$$\underline{e^x \sin y + y^2 = C}$$

Exercise

Find an integrating factor and solve the given equation $\left(3x + \frac{6}{y}\right)dx + \left(\frac{x^2}{y} + 3\frac{y}{x}\right)dy = 0$

Solution

$$xy\left(3x + \frac{6}{y}\right)dx + xy\left(\frac{x^2}{y} + 3\frac{y}{x}\right)dy = 0$$

$$(3x^2y + 6x)dx + (x^3 + 3y^2)dy = 0$$

$$M_y = \frac{\partial}{\partial y}(3x^2y + 6x) = 3x^2$$

$$N_x = \frac{\partial}{\partial x}(x^3 + 3y^2) = 3x^2 \Rightarrow M_y = N_x$$

$$\psi = \int (3x^2y + 6x)dx = x^3y + 3x^2 + h(y)$$

$$\psi_y = x^3 + h'(y) = x^3 + 3y^2$$

$$h'(y) = 3y^2 \Rightarrow h(y) = y^3$$

$$\underline{x^3y + 3x^2 + y^3 = C}$$

Exercise

Find an integrating factor and solve the given equation $(x + 3x^3 \sin y)dx + (x^4 \cos y)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}(x + 3x^3 \sin y) = 3x^3 \cos y$$

$$\Rightarrow M_y \neq N_x$$

$$N_x = \frac{\partial}{\partial x}(x^4 \cos y) = 4x^3 \cos y$$

$$\frac{M_y - N_x}{N} = \frac{3x^3 \cos y - 4x^3 \cos y}{x^4 \cos y} = -\frac{1}{x}$$

$$\mu = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$$

$$\mu = e^{\int \frac{M_y - N_x}{N} dx}$$

$$\frac{1}{x} \left(x + 3x^3 \sin y \right) dx + \frac{1}{x} \left(x^4 \cos y \right) dy = 0$$

$$\left(1 + 3x^2 \sin y \right) dx + \left(x^3 \cos y \right) dy = 0$$

$$M_y = \frac{\partial}{\partial y} \left(1 + 3x^2 \sin y \right) = 3x^2 \cos y$$

$$\Rightarrow M_y = N_x$$

$$N_x = \frac{\partial}{\partial x} \left(x^3 \cos y \right) = 3x^2 \cos y$$

$$\psi = \int \left(1 + 3x^2 \sin y \right) dx = x + x^3 \sin y + h(y)$$

$$\psi_y = -x^3 \cos y + h'(y) = -x^3 \cos y$$

$$h'(y) = 0 \Rightarrow h(y) = C$$

$$\underline{x + x^3 \sin y = C}$$

Exercise

Find an integrating factor and solve the given equation $(2x^2 + y)dx + (x^2y - x)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y} (2x^2 + y) = 1$$

$$\Rightarrow M_y \neq N_x$$

$$N_x = \frac{\partial}{\partial x} (x^2y - x) = 2xy - 1$$

$$\frac{M_y - N_x}{N} = \frac{1 - 2xy + 1}{x^2y - x} = \frac{2(1 - xy)}{x(xy - 1)} = -\frac{2}{x}$$

$$\mu = e^{-\int \frac{2}{x} dx} = e^{-2 \ln x} = x^{-2}$$

$$\mu = e^{\int \frac{M_y - N_x}{N} dx}$$

$$x^{-2} (2x^2 + y) dx + x^{-2} (x^2y - x) dy = 0$$

$$\left(2 + \frac{y}{x^2} \right) dx + \left(y - \frac{1}{x} \right) dy = 0$$

$$M_y = \frac{\partial}{\partial y} \left(2 + \frac{y}{x^2} \right) = \frac{1}{x^2}$$

$$\Rightarrow M_y = N_x$$

$$N_x = \frac{\partial}{\partial x} \left(y - \frac{1}{x} \right) = \frac{1}{x^2}$$

$$\psi = \int \left(2 + \frac{y}{x^2} \right) dx$$

$$= 2x - \frac{y}{x} + h(y)$$

$$\psi_y = -\frac{1}{x} + h'(y) = y - \frac{1}{x}$$

$$h'(y) = y \Rightarrow h(y) = \frac{1}{2}y^2$$

$$\underline{2x - \frac{y}{x} + \frac{1}{2}y^2 = C}$$

Exercise

Find an integrating factor and solve the given equation

$$(3x^2 + y)dx + (x^2y - x)dy = 0$$

Solution

$$M_y = \frac{\partial}{\partial y}(3x^2 + y) = 1$$

$$N_x = \frac{\partial}{\partial x}(x^2y - x) = 2xy - 1$$

$$\Rightarrow M_y \neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{1 - 2xy + 1}{x^2y - x}$$

$$= \frac{2(1 - xy)}{x(xy - 1)}$$

$$= -\frac{2}{x}$$

$$e^{-\int \frac{2}{x} dx} = e^{-2 \ln x} = \underline{x^{-2}}$$

$$x^{-2}(3x^2 + y)dx + x^{-2}(x^2y - x)dy = 0$$

$$\left(3 + \frac{y}{x^2} \right) dx + \left(y - \frac{1}{x} \right) dy = 0$$

$$M_y = \frac{\partial}{\partial y} \left(3 + \frac{y}{x^2} \right) = \frac{1}{x^2}$$

$$\Rightarrow M_y = N_x$$

$$N_x = \frac{\partial}{\partial x} \left(y - \frac{1}{x} \right) = \frac{1}{x^2}$$

$$\psi = \int \left(3 + \frac{y}{x^2} \right) dx$$

$$= 3x - \frac{y}{x} + h(y)$$

$$\psi_y = -\frac{1}{x} + h'(y)$$

$$= y - \frac{1}{x}$$

$$h'(y) = y \Rightarrow h(y) = \frac{1}{2}y^2$$

$$\underline{3x - \frac{y}{x} + \frac{1}{2}y^2 = C}$$

Exercise

Find an integrating factor and solve the given equation $(2y^2 + 2y + 4x^2)dx + (2xy + x)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}(2y^2 + 2y + 4x^2) = 4y + 2 \Rightarrow M_y \neq N_x$$

$$N_x = \frac{\partial}{\partial x}(2xy + x) = 2y + 1$$

$$\frac{M_y - N_x}{N} = \frac{4y + 2 - 2y - 1}{x(2y + 1)} = \frac{2y + 1}{x(2y + 1)} = \frac{1}{x}$$

$$\mu = e^{\int \frac{1}{x} dx} = e^{\ln x} = x \quad \mu = e^{\int \frac{M_y - N_x}{N} dx}$$

$$x(2y^2 + 2y + 4x^2)dx + x(2xy + x)dy = 0$$

$$(2xy^2 + 2xy + 4x^3)dx + (2x^2y + x^2)dy = 0$$

$$M_y = \frac{\partial}{\partial y}(2xy^2 + 2xy + 4x^3) = 4xy + 2x \Rightarrow M_y = N_x$$

$$N_x = \frac{\partial}{\partial x}(2x^2y + x^2) = 4xy + 2x$$

$$\psi = \int (2xy^2 + 2xy + 4x^3)dx$$

$$= x^2y^2 + x^2y + x^4 + h(y)$$

$$\psi_y = 2x^2y + x^2 + h'(y) = 2x^2y + x^2$$

$$h'(y) = 0 \Rightarrow h(y) = C$$

$$\underline{x^2y^2 + x^2y + x^4 = C}$$

Exercise

Find an integrating factor and solve the given equation $(x^4 - x + y)dx - xdy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}(x^4 - x + y) = 1 \quad \Rightarrow \quad M_y \neq N_x$$

$$N_x = \frac{\partial}{\partial x}(-x) = -1$$

$$\frac{M_y - N_x}{N} = -\frac{2}{x}$$

$$\mu = e^{-\int \frac{2}{x} dx} = e^{-2 \ln x} = x^{-2}$$

$$\mu = e^{\int \frac{M_y - N_x}{N} dx}$$

$$x^{-2}(x^4 - x + y)dx - x^{-2}xdy = 0$$

$$(x^2 - x^{-1} + yx^{-2})dx - x^{-1}dy = 0$$

$$M_y = \frac{\partial}{\partial y}(x^2 - x^{-1} + yx^{-2}) = x^{-2} \quad \Rightarrow \quad M_y = N_x$$

$$N_x = \frac{\partial}{\partial x}(-x^{-1}) = x^{-2}$$

$$\psi = \int (x^2 - x^{-1} + yx^{-2})dx$$

$$= \frac{1}{3}x^3 - \ln|x| - \frac{y}{x} + h(y)$$

$$\psi_y = -\frac{1}{x} + h'(y)$$

$$= -\frac{1}{x}$$

$$h'(y) = 0 \Rightarrow h(y) = C$$

$$\underline{\frac{1}{3}x^3 - \ln|x| - \frac{y}{x} = C}$$

Exercise

Find an integrating factor and solve the given equation $(2xy)dx + (y^2 - 3x^2)dy = 0$

Solution

$$M_y = \frac{\partial}{\partial y}(2xy) = 2x \quad \Rightarrow \quad M_y \neq N_x$$

$$N_x = \frac{\partial}{\partial x}(y^2 - 3x^2) = -6x$$

$$\frac{N_x - M_y}{M} = \frac{-8x}{2xy} = -\frac{4}{y}$$

$$\mu = e^{-\int \frac{4}{y} dy} = e^{-4 \ln y} = \underline{y^{-4}}$$

$$\mu = e^{\int \frac{N_x - M_y}{M} dy}$$

$$y^{-4}(2xy)dx + y^{-4}(y^2 - 3x^2)dy = 0$$

$$(2xy^{-3})dx + (y^{-2} - 3x^2y^{-4})dy = 0$$

$$M_y = \frac{\partial}{\partial y}(2xy^{-3}) = -6xy^{-4}$$

$$N_x = \frac{\partial}{\partial x}(y^{-2} - 3x^2y^{-4}) = -6xy^{-4} \Rightarrow M_y = N_x$$

$$\psi = \int (2xy^{-3})dx = x^2y^{-3} + h(y)$$

$$\psi_y = -3x^2y^{-4} + h'(y) = y^{-2} - 3x^2y^{-4}$$

$$h'(y) = \frac{1}{y^2} \Rightarrow h(y) = -\frac{1}{y}$$

$$\underline{x^2y^{-3} - \frac{1}{y} = C}$$

Exercise

Solve the given initial-value problem $\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}, \quad y(0) = 2$

Solution

$$(xy^2 - \cos x \sin x)dx - y(1-x^2)dy = 0$$

$$M_y = \frac{\partial}{\partial y}(xy^2 - \cos x \sin x) = 2xy \quad N_x = \frac{\partial}{\partial x}(-y + yx^2) = 2xy \Rightarrow \underline{M_y = N_x}$$

$$\psi = \int (xy^2 - \cos x \sin x)dx$$

$$= \int \left(xy^2 - \frac{1}{2} \sin 2x\right)dx$$

$$= \frac{1}{2}x^2y^2 + \frac{1}{4}\cos 2x + h(y)$$

$$\psi_y = x^2y + h'(y) = -y + yx^2$$

$$h'(y) = -y \Rightarrow h(y) = -\frac{1}{2}y^2$$

$$\psi = \frac{1}{2}x^2y^2 + \frac{1}{4}\cos 2x - \frac{1}{2}y^2 = C$$

$$y(0) = 2 \Rightarrow \frac{1}{4} - \frac{1}{2}(4) = C \rightarrow \boxed{C = -\frac{7}{4}}$$

$$\frac{1}{2}x^2y^2 + \frac{1}{4}\cos 2x - \frac{1}{2}y^2 = -\frac{7}{4}$$

$$\boxed{2x^2y^2 + \cos 2x - 2y^2 = -7}$$

$$\underline{x^2y^2 + \cos^2 x - y^2 = -3}$$

$$2x^2y^2 + 2\cos^2 x - 1 - 2y^2 = -7$$

Exercise

Solve the given initial-value problem

$$(x+y)^2 dx + (2xy + x^2 - 1)dy, \quad y(1) = 1$$

Solution

$$M_y = \frac{\partial}{\partial y}(x+y)^2 = 2(x+y)$$

$$N_x = \frac{\partial}{\partial x}(2xy + x^2 - 1) = 2y + 2x$$

$$\Rightarrow \underline{M_y = N_x}$$

$$\psi = \int (x^2 + 2xy + y^2) dx$$

$$= \frac{1}{3}x^3 + x^2y + xy^2 + h(y)$$

$$\psi_y = x^2 + 2xy + h'(y)$$

$$= 2xy + x^2 - 1$$

$$h'(y) = -1 \Rightarrow h(y) = -y$$

$$\psi = \frac{1}{3}x^3 + x^2y + xy^2 - y = C$$

$$y(1) = 1 \Rightarrow \frac{1}{3} + 1 + 1 - 1 = C \rightarrow \underline{C = \frac{4}{3}}$$

$$\underline{\frac{1}{3}x^3 + x^2y + xy^2 - y = \frac{4}{3}}$$

Exercise

Solve the given initial-value problem

$$(e^x + y)dx + (2 + x + ye^y)dy, \quad y(0) = 1$$

Solution

$$M_y = \frac{\partial}{\partial y}(e^x + y) = 1$$

$$N_x = \frac{\partial}{\partial x}(2 + x + ye^y) = 1$$

$$\Rightarrow \underline{M_y = N_x}$$

$$\begin{aligned}\psi &= \int (e^x + y) dx \\ &= e^x + xy + h(y)\end{aligned}$$

$$\begin{aligned}\psi_y &= x + h'(y) \\ &= 2 + x + ye^y\end{aligned}$$

$$h'(y) = 2 + ye^y$$

$$h(y) = 2y + e^y(y-1)$$

$$e^x + xy + 2y + e^y(y-1) = C$$

$$y(0) = 1 \Rightarrow 1 + 2 = C \rightarrow \underline{C = 3}$$

$$\underline{e^x + xy + 2y + e^y(y-1) = 3}$$

Exercise

Solve the given initial-value problem $(2x - y)dx + (2y - x)dy, \quad y(1) = 3$

Solution

$$\begin{cases} M_y = \frac{\partial}{\partial y}(2x - y) = -1 \\ N_x = \frac{\partial}{\partial x}(2y - x) = -1 \end{cases} \Rightarrow \underline{M_y = N_x}$$

$$\begin{aligned}\psi &= \int (2x - y) dx \\ &= x^2 - xy + h(y)\end{aligned}$$

$$\psi_y = -x + h'(y) = 2y - x$$

$$h'(y) = 2y \rightarrow h(y) = y^2$$

$$x^2 - xy + y^2 = C$$

$$y(1) = 3 \Rightarrow 1 - 3 + 9 = C \rightarrow \underline{C = 7}$$

$$\underline{x^2 - xy + y^2 = 7}$$

$$y^2 - xy + x^2 - 7 = 0 \rightarrow y = \frac{x \pm \sqrt{x^2 - 4x^2 + 28}}{2} = \frac{x \pm \sqrt{-3x^2 + 28}}{2}$$

$$\text{since } y(1) = 3 \rightarrow y = \frac{1}{2} \left(1 \pm \sqrt{-3(1)^2 + 28} \right) = \frac{1}{2} (1 \pm 5) \left\{ \begin{array}{l} \frac{1+5}{2} = 3 \\ \frac{1-5}{2} = -2 \end{array} \right.$$

$$\underline{y(x) = \frac{x + \sqrt{-3x^2 + 28}}{2}} \quad |x| < \sqrt{\frac{28}{3}}$$

Exercise

Solve the given initial-value problem $(9x^2 + y - 1)dx - (4y - x)dy, \quad y(1) = 0$

Solution

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}(9x^2 + y - 1) = 1 \\ N_x &= \frac{\partial}{\partial x}(x - 4y) = 1 \end{aligned} \Rightarrow \underline{M_y = N_x}$$

$$\begin{aligned} \psi &= \int (9x^2 + y - 1)dx \\ &= 3x^3 + xy - x + h(y) \end{aligned}$$

$$\psi_y = x + h'(y) = x - 4y$$

$$h'(y) = -4y \rightarrow h(y) = -2y^2$$

$$3x^3 + xy - x - 2y^2 = C$$

$$y(1) = 0 \Rightarrow 3 - 1 = C \rightarrow \underline{C = 2}$$

$$\underline{-2y^2 + xy + 3x^3 - x = 2}$$

$$-2y^2 + xy + 3x^3 - x - 2 = 0 \rightarrow y = \frac{-x \pm \sqrt{x^2 + 24x^3 - 8x - 16}}{-4}$$

$$\text{since } y(1) = 0 \rightarrow y = -\frac{1}{4}(-1 \pm \sqrt{1 + 24 - 8 - 16}) = -\frac{1}{4}(-1 \pm 1) \left\{ \begin{array}{l} -\frac{-1+1}{4} = 0 \\ \cancel{-\frac{-1-1}{4} = \frac{1}{2}} \end{array} \right.$$

$$\underline{y(x) = \frac{x - \sqrt{x^2 + 24x^3 - 8x - 16}}{4}}$$

Exercise

Solve $\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1 - x^2)}, \quad y(0) = 2$

Solution

$$(xy^2 - \cos x \sin x)dx - y(1 - x^2)dy = 0$$

$$(xy^2 - \cos x \sin x)dx + y(x^2 - 1)dy = 0$$

$$M(x, y) = xy^2 - \cos x \sin x \quad N(x, y) = y(x^2 - 1)$$

$$\begin{aligned}\frac{\partial M}{\partial y} &= 2xy \\ \frac{\partial N}{\partial x} &= 2xy\end{aligned} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\begin{aligned}\psi &= \int (xy^2 - \cos x \sin x) dx \\ &= \int xy^2 dx - \int \sin x d(\sin x) \\ &= \frac{1}{2}x^2y^2 - \frac{1}{2}\sin^2 x + h(y)\end{aligned}$$

$$\psi_y = x^2y + h'(y) = x^2y - y$$

$$h'(y) = -y \Rightarrow h(y) = -\frac{1}{2}y^2$$

$$\frac{1}{2}x^2y^2 - \frac{1}{2}\sin^2 x - \frac{1}{2}y^2 = C$$

$$y(0) = 2 \rightarrow -2 = C$$

$$\frac{1}{2}x^2y^2 - \frac{1}{2}\sin^2 x - \frac{1}{2}y^2 = -2$$

Exercise

Solve the given initial-value problem $\frac{dy}{dx} = (-2x + y)^2 - 7, \quad y(0) = 0$

Solution

$$\text{Let } u = -2x + y \Rightarrow \frac{du}{dx} = -2 + \frac{dy}{dx}$$

$$\frac{du}{dx} + 2 = u^2 - 7$$

$$\frac{du}{dx} = u^2 - 9$$

$$\int \frac{du}{u^2 - 9} = \int dx \quad \frac{1}{u^2 - 9} = \frac{A}{u - 3} + \frac{B}{u + 3}$$

$$Au + 3A + Bu - 3B = 0 \quad \begin{cases} A + B = 0 \\ 3A - 3B = 1 \end{cases} \rightarrow A = \frac{1}{6}, \quad B = -\frac{1}{6}$$

$$\frac{1}{6} \int \left(\frac{1}{u - 3} - \frac{1}{u + 3} \right) du = \int dx$$

$$\frac{1}{6} (\ln|u - 3| - \ln|u + 3|) = x + C$$

$$\ln \left| \frac{u - 3}{u + 3} \right| = 6x + C$$

$$\frac{u - 3}{u + 3} = e^{6x + C} = Ae^{6x}$$

$$u - 3 = Aue^{6x} + 3Ae^{6x}$$

$$u = \frac{3 + 3Ae^{6x}}{1 - Ae^{6x}} = -2x + y$$

$$y = 2x + \frac{3 + 3Ae^{6x}}{1 - Ae^{6x}} \quad y(0) = 0$$

$$0 = \frac{3 + 3A}{1 - A} \rightarrow \underline{A = -1}$$

$$\underline{y(x) = 2x + \frac{3(1 - e^{6x})}{1 + e^{6x}}}$$

Exercise

Solve the given initial-value problem $(2y - x)y' - y + 2x = 0, \quad y(1) = 0$

Solution

$$M_y = \frac{\partial}{\partial y}(2y - x) = 2 \quad \Rightarrow \quad M_y = N_x$$

$$N_x = \frac{\partial}{\partial x}(-y + 2x) = 2$$

$$\psi = \int (-y + 2x) dx = x^2 - xy + h(y)$$

$$\psi_y = -x + h'(y) = 2y - x \Rightarrow h'(y) = 2y \rightarrow h(y) = y^2$$

$$x^2 - xy + y^2 = C$$

$$y(1) = 0 \rightarrow 1 = C$$

$$x^2 - xy + y^2 = 1$$

$$y^2 - xy + x^2 - 1 = 0$$

$$y = \frac{x \pm \sqrt{x^2 - 4x^2 + 4}}{2} = \frac{x \pm \sqrt{4 - 3x^2}}{2}$$

$$\text{Since } y(1) = 0 \rightarrow y(x) = \underline{\frac{x - \sqrt{4 - 3x^2}}{2}}$$

Exercise

Solve the given initial-value problem $(x + y^3)y' + y + x^3 = 0, \quad y(0) = -2$

Solution

$$M_y = \frac{\partial}{\partial y}(y + x^3) = 1$$

$$N_x = \frac{\partial}{\partial x}(x + y^3) = 1 \quad \Rightarrow \quad \underline{M_y = N_x}$$

$$\psi = \int (y + x^3) dx = xy + \frac{1}{4}x^4 + h(y)$$

$$\psi_y = x + h'(y) = x + y^3$$

$$h'(y) = y^3 \rightarrow h(y) = \frac{1}{4}y^4$$

$$xy + \frac{1}{4}x^4 + \frac{1}{4}y^4 = C$$

$$y(0) = -2 \rightarrow \underline{4 = C}$$

$$\underline{xy + \frac{1}{4}x^4 + \frac{1}{4}y^4 = 4}$$

Exercise

Solve the given initial-value problem $y' = (3x^2 + 1)(y^2 + 1)$, $y(0) = 1$

Solution

$$\frac{1}{y^2 + 1} y' - (3x^2 + 1) = 0$$

$$M_y = \frac{\partial}{\partial y}(-3x^2 - 1) = 0$$

$$N_x = \frac{\partial}{\partial x}\left(\frac{1}{y^2 + 1}\right) = 0 \quad \Rightarrow \quad M_y = N_x$$

$$\psi = \int (-3x^2 - 1) dx = -x^3 - x + h(y)$$

$$\psi_y = h'(y) = \frac{1}{y^2 + 1} \rightarrow h(y) = \tan^{-1} y$$

$$-x^3 - x + \tan^{-1} y = C$$

$$y(0) = 1 \rightarrow \tan^{-1} 1 = C \Rightarrow \underline{C = \frac{\pi}{4}}$$

$$-x^3 - x + \tan^{-1} y = \frac{\pi}{4}$$

$$\tan^{-1} y = x^3 + x + \frac{\pi}{4}$$

$$\underline{y(x) = \tan\left(x^3 + x + \frac{\pi}{4}\right)}$$

Exercise

Solve the given initial-value problem $(y^3 + \cos t)y' = 2 + y \sin t, \quad y(0) = -1$

Solution

$$(y^3 + \cos t)y' - (2 + y \sin t) = 0$$

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}(-2 - y \sin t) = -\sin t \\ N_t &= \frac{\partial}{\partial t}(y^3 + \cos t) = -\sin t \end{aligned} \Rightarrow \underline{M_y = N_t}$$

$$\begin{aligned} \psi &= \int (-2 - y \sin t) dt \\ &= -2t + y \cos t + h(y) \end{aligned}$$

$$\psi_y = \cos t + h'(y) = y^3 + \cos t$$

$$h'(y) = y^3 \rightarrow h(y) = \frac{1}{4}y^4$$

$$-2t + y \cos t + \frac{1}{4}y^4 = C$$

$$y(0) = -1 \rightarrow -1 + \frac{1}{4} = C \Rightarrow \underline{C = -\frac{3}{4}}$$

$$\underline{-2t + y \cos t + \frac{1}{4}y^4 = -\frac{3}{4}}$$

Exercise

Solve the given initial-value problem $(y^3 - t^3)y' = 3t^2y + 1, \quad y(-2) = -1$

Solution

$$(y^3 - t^3)y' - (3t^2y + 1) = 0$$

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}(-3t^2y - 1) = -3t^2 \\ N_t &= \frac{\partial}{\partial t}(y^3 - t^3) = -3t^2 \end{aligned} \Rightarrow \underline{M_y = N_t}$$

$$\begin{aligned} \psi &= \int (-3t^2y - 1) dt \\ &= -t^3y - t + h(y) \end{aligned}$$

$$\begin{aligned} \psi_y &= -t^3 + h'(y) \\ &= y^3 - t^3 \end{aligned}$$

$$h'(y) = y^3 \rightarrow h(y) = \frac{1}{4}y^4$$

$$-t^3y - t + \frac{1}{4}y^4 = C$$

$$y(-2) = -1 \quad -8 + 2 + \frac{1}{4} = C \Rightarrow C = -\frac{23}{4}$$

$$\underline{-t^3y - t + \frac{1}{4}y^4 = -\frac{23}{4}}$$

Exercise

Solve the given initial-value problem $(e^{2y} + t^2y)y' + ty^2 + \cos t = 0, \quad y\left(\frac{\pi}{2}\right) = 0$

Solution

$$M_y = \frac{\partial}{\partial y}(ty^2 + \cos t) = 2yt$$

$$N_t = \frac{\partial}{\partial t}(e^{2y} + t^2y) = 2ty \Rightarrow \underline{M_y = N_t}$$

$$\begin{aligned} \psi &= \int (ty^2 + \cos t) dt \\ &= \frac{1}{2}t^2y^2 + \sin t + h(y) \end{aligned}$$

$$\begin{aligned} \psi_y &= t^2y + h'(y) \\ &= t^2y + e^{2y} \end{aligned}$$

$$h'(y) = e^{2y} \rightarrow h(y) = \frac{1}{2}e^{2y}$$

$$\frac{1}{2}t^2y^2 + \sin t + \frac{1}{2}e^{2y} = C$$

$$y\left(\frac{\pi}{2}\right) = 0 \quad 1 + \frac{1}{2} = C \Rightarrow C = \frac{3}{2}$$

$$\underline{\frac{1}{2}t^2y^2 + \sin t + \frac{1}{2}e^{2y} = \frac{3}{2}}$$

Exercise

Solve the given initial-value problem $y' = -\frac{y \cos(ty) + 1}{t \cos(ty) + 2ye^{y^2}}, \quad y(\pi) = 0$

Solution

$$\left(t \cos(ty) + 2ye^{y^2}\right)y' + (y \cos(ty) + 1) = 0$$

$$M_y = \frac{\partial}{\partial y}(y \cos ty + 1) = \cos ty - ty \sin ty$$

$$N_t = \frac{\partial}{\partial t}(t \cos ty + 2ye^{y^2}) = \cos ty - ty \sin ty \quad \Rightarrow \quad \underline{M_y = N_t}$$

$$\psi = \int (y \cos ty + 1) dt$$

$$= \sin ty + t + h(y)$$

$$\psi_y = t \cos ty + h'(y)$$

$$= t \cos ty + 2ye^{y^2}$$

$$h'(y) = 2ye^{y^2} \rightarrow h(y) = e^{y^2}$$

$$\sin ty + t + e^{y^2} = C$$

$$\textcolor{red}{y(\pi) = 0} \Rightarrow \underline{C = -\pi - 1}$$

$$\underline{\textcolor{blue}{\sin ty + t + e^{y^2} = \pi + 1}}$$

Exercise

Solve the given initial-value problem $\left(2ty + \frac{1}{y}\right)y' + y^2 = 1, \quad y(1) = 1$

Solution

$$\left(2ty + \frac{1}{y}\right)y' + y^2 - 1 = 0$$

$$M_y = \frac{\partial}{\partial y}(y^2 - 1) = 2y$$

$$N_t = \frac{\partial}{\partial t}\left(2ty + \frac{1}{y}\right) = 2y \quad \Rightarrow \quad \underline{M_y = N_t}$$

$$\psi = \int (y^2 - 1) dt$$

$$= ty^2 - t + h(y)$$

$$\psi_y = t \cos ty + h'(y)$$

$$= 2ty + h'(y)$$

$$= 2ty + \frac{1}{y}$$

$$h'(y) = \frac{1}{y} \rightarrow h(y) = \ln y$$

$$ty^2 - t + \ln|y| = C$$

$$y(1) = 1 \Rightarrow \underline{C = 0}$$

$$\underline{ty^2 - t + \ln|y| = 0}$$

Exercise

Solve the given initial-value problem $(ye^x + 1)dx + (e^x - 1)dy = 0$ $y(1) = 1$

Solution

$$M_y = \frac{\partial}{\partial y}(ye^x + 1) = e^x$$

$$N_x = \frac{\partial}{\partial x}(e^x - 1) = e^x \Rightarrow M_y = N_x$$

$$\psi = \int (ye^x + 1)dx$$

$$= ye^x + x + h(y)$$

$$\psi_y = e^x + h'(y) = e^x - 1$$

$$\rightarrow h'(y) = -1 \Rightarrow h(y) = -y$$

$$\underline{ye^x + x - y = C}$$

$$y(1) = 1 \rightarrow \underline{C = e}$$

$$\underline{ye^x + x - y = e}$$

Exercise

Solve the given initial-value problem $2xy^2 + 4 = 2(3 - x^2y)y'$ $y(-1) = 8$

Solution

$$2xy^2 + 4 - 2(3 - x^2y)y' = 0$$

$$M = 2xy^2 + 4 \Rightarrow M_y = 4xy$$

$$N = -6 + 2x^2y \Rightarrow N_x = 4xy \rightarrow M_y = N_x$$

$$\psi = \int (2xy^2 + 4) dx$$

$$= x^2y^2 + 4x + h(y)$$

$$\psi = \int M dx$$

$$\psi_y = 2x^2y + h'(y)$$

$$= 2x^2y - 6$$

$$h'(y) = -6 \rightarrow h(y) = -6y$$

$$\underline{x^2y^2 + 4x - 6y = C}$$

$$\textcolor{red}{y(-1)=8} \rightarrow 64 - 4 - 48 = C \quad \underline{C=12}$$

$$\underline{\textcolor{blue}{x^2y^2 + 4x - 6y = 12}}$$

Exercise

Solve the given initial-value problem $y' + \frac{4}{x}y = x^3y^2 \quad y(2) = -1$

Solution

$$\text{Let } u = y^{1-2} = y^{-1} \Rightarrow y = \frac{1}{u}$$

$$\frac{du}{dx} = -\frac{1}{y^2} \frac{dy}{dx} \Rightarrow u' = -\frac{1}{y^2} y'$$

$$y' = -y^2 u' = -\frac{1}{u^2} u'$$

$$y' + \frac{4}{x}y = x^3y^2$$

$$-\frac{1}{u^2} u' + \frac{4}{x} \frac{1}{u} = x^3 \frac{1}{u^2}$$

$$u' - \frac{4}{x}u = -x^3$$

$$e^{\int -\frac{4}{x} dx} = e^{-4 \ln x} = x^{-4}$$

$$\int -x^3 x^{-4} dx = -\int \frac{1}{x} dx = -\ln x$$

$$u = x^4(-\ln x + C)$$

$$y = \frac{1}{x^4(C - \ln x)} \quad y = \frac{1}{u}$$

$$\textcolor{red}{y(2)=-1} \rightarrow -1 = \frac{1}{16(C - \ln 2)} \quad \underline{C = \ln 2 - \frac{1}{16}}$$

$$\underline{y(x) = \frac{1}{x^4 \left(\ln 2 - \frac{1}{16} - \ln x \right)}}$$

Exercise

Solve the given initial-value problem $y' = 5y + e^{-2x}y^{-2}$ $y(0) = 2$

Solution

$$y^2 y' - 5y^3 = e^{-2x}$$

$$\text{Let } u = y^3 \Rightarrow y = u^{1/3}$$

$$u' = 3y^2 y' \Rightarrow y' = \frac{1}{3}u^{-2/3}u'$$

$$\frac{1}{3}u' - 5u = e^{-2x}$$

$$u' - 15u = 3e^{-2x}$$

$$e^{\int -15dx} = e^{-15x}$$

$$\int 3e^{-2x}e^{-15x}dx = 3 \int e^{-17x}dx = -\frac{3}{17}e^{-17x}$$

$$u = e^{15x} \left(-\frac{3}{17}e^{-17x} + C \right) \quad u = y^3$$

$$y^3 = e^{15x}C - \frac{3}{17}e^{-2x}$$

$$y(0) = 2 \rightarrow 8 = C - \frac{3}{17} \quad C = \frac{139}{17}$$

$$y(x) = \left(\frac{139e^{15x} - 3e^{-2x}}{17} \right)^{1/3}$$

Exercise

Solve the given initial-value problem $6y' - 2y = xy^4$ $y(0) = -2$

Solution

$$6y^{-4}y' - 2y^{-3} = x$$

$$\text{Let } u = y^{-3} \Rightarrow y = u^{-1/3}$$

$$y' = -\frac{1}{3}u^{-4/3}u'$$

$$6u^{4/3} \left(-\frac{1}{3}u^{-4/3} \right) u' - 2u = x$$

$$-2u' - 2u = x$$

$$u' + u = -\frac{1}{2}x$$

$$e^{\int dx} = e^x$$

$$-\int \frac{1}{2} x e^x dx = -\frac{1}{2} (x-1) e^x$$

$$u = \frac{1}{e^x} \left(-\frac{1}{2} (x-1) e^x + C \right) \quad u = y^{-3}$$

$$y^{-3} = \frac{C}{e^x} - \frac{1}{2} (x-1)$$

$$y(0) = -2 \rightarrow -\frac{1}{8} = C + \frac{1}{2} \quad \underline{C = -\frac{5}{8}}$$

$$\frac{1}{y^3} = -\frac{5}{8} e^{-x} - \frac{1}{2} (x-1)$$

$$= -\frac{5e^{-x} - 4x + 4}{8}$$

$$y(x) = -\frac{2}{\left(5e^{-x} - 4x + 4\right)^{1/3}}$$

Exercise

Solve the given initial-value problem $y' + \frac{y}{x} - \sqrt{y} = 0 \quad y(1) = 0$

Solution

$$y' + \frac{1}{x} y = y^{1/2}$$

$$y^{-1/2} y' + \frac{1}{x} y^{1/2} = 1$$

$$\text{Let } u = y^{1/2} \Rightarrow y = u^2$$

$$y' = 2u u'$$

$$u^{-1} (2u u') + \frac{1}{x} u = 1$$

$$2u' + \frac{1}{x} u = 1$$

$$u' + \frac{1}{2x} u = \frac{1}{2}$$

$$e^{\int \frac{1}{2x} dx} = e^{\frac{1}{2} \ln x} = x^{1/2}$$

$$\int \frac{1}{2} x^{1/2} dx = \frac{1}{3} x^{3/2}$$

$$u = \frac{1}{x^{1/2}} \left(\frac{1}{3} x^{3/2} + C \right)$$

$$y^{1/2} = \frac{1}{3} x + C x^{-1/2}$$

$$y(1) = 0 \rightarrow 0 = C + \frac{1}{3} \quad \underline{C = -\frac{1}{3}}$$

$$\underline{y(x) = \left(\frac{1}{3}x + \frac{1}{3}x^{-1/2} \right)^2}$$

Exercise

Solve the given initial-value problem $xyy' + 4x^2 + y^2 = 0 \quad y(2) = -7$

Solution

$$xyy' = -4x^2 - y^2$$

$$\frac{y}{x} y' = -4 - \left(\frac{y}{x} \right)^2$$

$$\text{Let } \frac{y}{x} = v \rightarrow y = xv \quad \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$v \left(v + x \frac{dv}{dx} \right) = -4 - v^2$$

$$vx \frac{dv}{dx} = -4 - 2v^2$$

$$-\frac{v}{4 + 2v^2} dv = \frac{dx}{x}$$

$$-\frac{1}{4} \int \frac{1}{4 + 2v^2} d(4 + 2v^2) = \int \frac{1}{x} dx$$

$$-\frac{1}{4} \ln(4 + 2v^2) = \ln x + \ln C_1$$

$$\ln(4 + 2v^2)^{-\frac{1}{4}} = \ln C_1 x$$

$$(4 + 2v^2)^{-\frac{1}{4}} = C_1 x$$

$$4 + 2 \left(\frac{y}{x} \right)^2 = Cx^{-4}$$

$$\frac{2y^2}{x^2} = \frac{C}{x^4} - 4$$

$$y^2 = \frac{1}{2} \frac{C - 4x^4}{x^2}$$

$$y(2) = -7 \rightarrow 49 = \frac{1}{2} \frac{C - 64}{4} \rightarrow \underline{C = 456}$$

$$y^2 = \frac{1}{2} \frac{456 - 4x^4}{x^2}$$

$$y^2 = \frac{228 - 2x^4}{x^2}$$

$$y = \pm \frac{\sqrt{228 - 2x^4}}{x}$$

Since the given initial $y(2) = -7$

$$y = -\frac{\sqrt{228 - 2x^4}}{x}$$

Exercise

Solve the given initial-value problem $xy' = y(\ln x - \ln y)$ $y(1) = 4$

Solution

$$y' = \frac{y}{x} \ln \frac{x}{y}$$

$$\text{Let } \frac{y}{x} = v \rightarrow y = xv \quad \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$v + xv' = v \ln \frac{1}{v}$$

$$x \frac{dv}{dx} = -v(1 + \ln v)$$

$$\int \frac{dv}{v(1 + \ln v)} = - \int \frac{dx}{x}$$

$$\int \frac{d(1 + \ln v)}{1 + \ln v} = - \int \frac{dx}{x}$$

$$\ln(1 + \ln v) = -\ln x + \ln C$$

$$\ln\left(1 + \ln \frac{y}{x}\right) = \ln \frac{C}{x}$$

$$1 + \ln \frac{y}{x} = \frac{C}{x}$$

$$y(1) = 4 \rightarrow \underline{C = 1 + \ln 4}$$

$$\ln \frac{y}{x} = \frac{1 + \ln 4}{x} - 1$$

$$y(x) = xe^{\frac{1 + \ln 4}{x} - 1}$$

Exercise

Solve the given initial-value problem $y' - (4x - y + 1)^2 = 0$ $y(0) = 2$

Solution

$$\text{Let } v = 4x - y \Rightarrow v' = 4 - y'$$

$$y' = 4 - v'$$

$$4 - v' - (v + 1)^2 = 0$$

$$v' = 4 - (v + 1)^2$$

$$= -v^2 - 2v + 3$$

$$\int \frac{dv}{v^2 + 2v - 3} = - \int dx$$

$$\int \frac{dv}{(v-1)(v+3)} = - \int dx$$

$$\frac{1}{4} \int \left(\frac{1}{v-1} - \frac{1}{v+3} \right) = - \int dx$$

$$\frac{1}{4} (\ln(v-1) - \ln(v+3)) = -x + C_1$$

$$\ln \frac{v-1}{v+3} = C - 4x$$

$$\ln \left| \frac{4x - y - 1}{4x - y + 3} \right| = C - 4x$$

$$y(0) = 2 \rightarrow \underline{C = \ln 3}$$

$$\frac{4x - y - 1}{4x - y + 3} = e^{\ln 3 - 4x}$$

$$4x - y - 1 = 3(4x - y + 3)e^{-4x}$$

$$(4x - y - 1)e^{4x} = 12x - 3y + 9$$

$$3y - ye^{4x} = 12x + 9 + (1 - 4x)e^{4x}$$

$$\underline{y(x) = \frac{12x + 9 + (1 - 4x)e^{4x}}{3 - e^{4x}}}$$

$$\frac{1}{(v-1)(v+3)} = \frac{A}{v-1} + \frac{B}{v+3}$$

$$Av + 3A + Bv - B = 1$$

$$\begin{cases} A + B = 0 \\ 3A - B = 1 \end{cases} \rightarrow \underline{A = \frac{1}{4}, B = -\frac{1}{4}}$$

Exercise

Solve the given initial-value problem $(e^{t+y} + 2y)y' + (e^{t+y} + 3t^2) = 0$, $y(0) = 0$

Solution

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}(e^{t+y} + 3t^2) = e^{t+y} \\ N_t &= \frac{\partial}{\partial t}(e^{t+y} + 2y) = e^{t+y} \end{aligned} \Rightarrow \underline{M_y = N_t}$$

$$\begin{aligned} \psi &= \int (e^{t+y} + 3t^2) dt \\ &= e^{t+y} + t^3 + h(y) \end{aligned}$$

$$\psi_y = e^{t+y} + h'(y) = e^{t+y} + 2y$$

$$h'(y) = 2y \rightarrow h(y) = y^2$$

$$e^{t+y} + t^3 + y^2 = C$$

$$y(0) = 0 \Rightarrow \underline{C = 1}$$

$$\underline{e^{t+y} + t^3 + y^2 = 1}$$

Exercise

Solve the given initial-value problem $(4y + 2x - 5)dx + (6y + 4x - 1)dy$, $y(-1) = 2$

Solution

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}(4y + 2x - 5) = 4 \\ N_x &= \frac{\partial}{\partial x}(6y + 4x - 1) = 4 \end{aligned} \Rightarrow \underline{M_y = N_x}$$

$$\begin{aligned} \psi &= \int (4y + 2x - 5) dx \\ &= 4xy + x^2 - 5x + h(y) \end{aligned}$$

$$\begin{aligned} \psi_y &= 4x + h'(y) \\ &= 6y + 4x - 1 \end{aligned}$$

$$\Rightarrow h'(y) = 6y - 1 \rightarrow h(y) = 3y^2 - y$$

$$\psi = 4xy + x^2 - 5x + 3y^2 - y = C$$

$$y(-1) = 2 \Rightarrow 4(-1)(2) + 1 + 5 + 12 - 2 = C \rightarrow \underline{C = 8}$$

$$\underline{4xy + x^2 - 5x + 3y^2 - y = 8}$$

Exercise

Solve the given initial-value problem $\left(ye^{xy} - \frac{1}{y} \right) dx + \left(xe^{xy} + \frac{x}{y^2} \right) dy = 0$ $y(1) = 1$

Solution

$$M_y = \frac{\partial}{\partial y} \left(ye^{xy} - \frac{1}{y} \right) = (1 + xy)e^{xy} + \frac{1}{y^2}$$
$$N_x = \frac{\partial}{\partial x} \left(xe^{xy} + \frac{x}{y^2} \right) = (1 + xy)e^{xy} + \frac{1}{y^2} \Rightarrow M_y = N_x$$

$$\psi = \int \left(ye^{xy} - \frac{1}{y} \right) dx = e^{xy} - \frac{x}{y} + h(y)$$

$$\psi_y = xe^{xy} + \frac{x}{y^2} + h'(y) = xe^{xy} + \frac{x}{y^2}$$

$$\rightarrow h'(y) = 0 \Rightarrow h(y) = C$$

$$\underline{e^{xy} - \frac{x}{y} = C}$$

$$y(1) = 1 \rightarrow \underline{C = e - 1}$$

$$\underline{e^{xy} - \frac{x}{y} = e - 1}$$

Exercise

Solve the given initial-value problem $(2y \ln t - t \sin y) y' + \frac{1}{t} y^2 + \cos y = 0$, $y(2) = 0$

Solution

$$M_y = \frac{\partial}{\partial y} \left(\frac{1}{t} y^2 + \cos y \right) = \frac{2y}{t} - \sin y$$
$$N_t = \frac{\partial}{\partial t} (2y \ln t - t \sin y) = \frac{2y}{t} - \sin y \Rightarrow \underline{M_y = N_t}$$

$$\psi = \int \left(\frac{1}{t} y^2 + \cos y \right) dt$$

$$= y^2 \ln|t| + t \cos y + h(y)$$

$$\psi_y = 2y \ln t - t \sin y + h'(y) = 2y \ln t - t \sin y$$

$$\Rightarrow h'(y) = 0 \rightarrow h(y) = C$$

$$y^2 \ln|t| + t \cos y + C = 0$$

$$y(2) = 0 \Rightarrow \underline{C = -2}$$

$$\underline{y^2 \ln|t| + t \cos y - 2 = 0}$$

Exercise

Solve the given initial-value problem $(\tan y - 2)dx + \left(x \sec^2 y + \frac{1}{y}\right)dy = 0$ $y(0) = 1$

Solution

$$M_y = \frac{\partial}{\partial y}(\tan y - 2) = \sec^2 y$$

$$N_x = \frac{\partial}{\partial x}\left(x \sec^2 y + \frac{1}{y}\right) = \sec^2 y \Rightarrow M_y = N_x$$

$$\psi = \int (\tan y - 2)dx$$

$$= x \tan y - 2x + h(y)$$

$$\psi_y = x \sec^2 y + h'(y)$$

$$= x \sec^2 y + \frac{1}{y}$$

$$\rightarrow h'(y) = \frac{1}{y} \Rightarrow h(y) = \ln|y|$$

$$\underline{x \tan y - 2x + \ln|y| = C}$$

$$y(0) = 1 \rightarrow \underline{C = 0}$$

$$\underline{x \tan y - 2x + \ln y = 0}$$

Exercise

Solve the given initial-value problem $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0$ $y(0) = -3$

Solution

$$M = 2xy - 9x^2 \Rightarrow M_y = 2x$$

$$N = 2y + x^2 + 1 \Rightarrow N_x = 2x \rightarrow M_y = N_x$$

$$\psi = \int (2xy - 9x^2) dx$$

$$= x^2 y - 3x^3 + h(y)$$

$$\psi_y = 2y + x^2 + h'(y)$$

$$= 2y + x^2 + 1$$

$$\psi = \int M dx$$

$$h'(y) = 2y + 1 \rightarrow h(y) = y^2 + y$$

$$\underline{x^2 y - 3x^3 + y^2 + y = C}$$

$$y(0) = -3 \rightarrow 9 - 3 = C \quad \underline{C = 6}$$

$$\underline{x^2 y - 3x^3 + y^2 + y = 6}$$

Exercise

Solve the given initial-value problem $\frac{2t}{t^2+1}y - 2t + \left(2 - \ln(t^2+1)\right)\frac{dy}{dt} = 0 \quad y(5) = 0$

Solution

$$M = \frac{2t}{t^2+1}y - 2t \Rightarrow M_y = \frac{2t}{t^2+1} \rightarrow M_y = N_t$$

$$N = 2 - \ln(t^2+1) \Rightarrow N_t = \frac{2t}{t^2+1}$$

$$\psi = \int \left(\frac{2t}{t^2+1}y - 2t \right) dt \quad \psi = \int M dt$$

$$= y \int \frac{1}{t^2+1} d(t^2+1) - t^2$$

$$= \ln(t^2+1)y - t^2 + h(y)$$

$$\psi_y = \ln(t^2+1) + h'(y)$$

$$= 2 - \ln(t^2+1)$$

$$h'(y) = -2 \rightarrow h(y) = -2y$$

$$\ln(t^2+1)y - t^2 - 2y = C$$

$$y(5) = 0 \rightarrow \underline{C = -25}$$

$$(\ln(t^2+1) - 2)y - t^2 = -25$$

$$\underline{y(t) = \frac{t^2 - 25}{\ln(t^2+1) - 2}}$$

Exercise

Solve the given initial-value problem $3y^3e^{3xy} - 1 + (2ye^{3xy} + 3xy^2e^{3xy})y' = 0$ $y(0) = 1$

Solution

$$M = 3y^3e^{3xy} - 1 \Rightarrow M_y = 9y^2e^{3xy} + 9xy^3e^{3xy}$$

$$N = 2ye^{3xy} + 3xy^2e^{3xy} \Rightarrow N_x = 6y^2e^{3xy} + 3y^2e^{3xy} + 9xy^3e^{3xy}$$

$$\rightarrow M_y = N_x$$

$$\begin{aligned}\psi &= \int (3y^3e^{3xy} - 1) dx \\ &= y^2e^{3xy} - x + h(y)\end{aligned}$$

$$\psi = \int M dx$$

$$\begin{aligned}\psi_y &= 2ye^{3xy} + 3xy^2e^{3xy} + h'(y) \\ &= 2ye^{3xy} + 3xy^2e^{3xy}\end{aligned}$$

$$\Rightarrow h'(y) = 0 \rightarrow h(y) = C$$

$$y^2e^{3xy} - x = C$$

$$y(0) = 1 \rightarrow \underline{C = 1}$$

$$\underline{y^2e^{3xy} - x = 1}$$

Exercise

Solve the given initial-value problem $2xydx + (1 + x^2)dy = 0$; $y(2) = -5$

Solution

$$M = 2xy \Rightarrow M_y = 2x$$

$$N = 1 + x^2 \Rightarrow N_x = 2x \rightarrow M_y = N_x$$

$$\psi = \int (2xy) dx = x^2y + h(y)$$

$$\psi_y = x^2 + h'(y) = 1 + x^2$$

$$\Rightarrow h'(y) = 1 \rightarrow h(y) = y$$

$$x^2y + y = C$$

$$y(x^2 + 1) = C$$

$$y(2) = -5 \rightarrow -20 - 5 = C \Rightarrow \underline{C = -25}$$

$$\underline{y(x) = -\frac{25}{1+x^2}}$$

Exercise

Solve the given initial-value problem $\frac{dy}{dx} = -\frac{2x \cos y + 3x^2 y}{x^3 - x^2 \sin y - y}$; $y(0) = 2$

Solution

$$(x^3 - x^2 \sin y - y)dy = -(2x \cos y + 3x^2 y)dx$$

$$(2x \cos y + 3x^2 y)dx + (x^3 - x^2 \sin y - y)dy = 0$$

$$M = 2x \cos y + 3x^2 y \Rightarrow M_y = -2x \sin y + 3x^2$$

$$N = x^3 - x^2 \sin y - y \Rightarrow N_x = 3x^2 - 2x \sin y$$

$$\rightarrow M_y = N_x$$

$$\psi = \int (2x \cos y + 3x^2 y) dx = x^2 \cos y + x^3 y + h(y)$$

$$\psi_y = -x^2 \sin y + x^3 + h'(y) = x^3 - x^2 \sin y - y$$

$$\Rightarrow h'(y) = -y \rightarrow h(y) = -\frac{1}{2}y^2$$

$$x^2 \cos y + x^3 y - \frac{1}{2}y^2 = C$$

$$y(0) = 2 \rightarrow \underline{-2 = C}$$

$$\underline{x^2 \cos y + x^3 y - \frac{1}{2}y^2 = -2}$$

Exercise

Find an integrating factor of the form $x^n y^m$ and solve the equation

$$(2y^2 - 6xy)dx + (3xy - 4x^2)dy = 0$$

Solution

$$x^n y^m (2y^2 - 6xy)dx + x^n y^m (3xy - 4x^2)dy = 0$$

$$(2x^n y^{m+2} - 6x^{n+1} y^{m+1})dx + (3x^{n+1} y^{m+1} - 4x^{n+2} y^m)dy = 0$$

$$M_y = \frac{\partial}{\partial y} (2x^n y^{m+2} - 6x^{n+1} y^{m+1}) = 2(m+2)x^n y^{m+1} - 6(m+1)x^{n+1} y^m$$

$$N_x = \frac{\partial}{\partial x} (3x^{n+1} y^{m+1} - 4x^{n+2} y^m) = 3(n+1)x^n y^{m+1} - 4(n+2)x^{n+1} y^m$$

For the equation to be exact, then

$$2(m+2)x^n y^{m+1} - 6(m+1)x^{n+1} y^m = 3(n+1)x^n y^{m+1} - 4(n+2)x^{n+1} y^m$$

$$2(m+2)y - 6(m+1)x = 3(n+1)y - 4(n+2)x$$

$$\begin{cases} 2m+4 = 3n+3 \\ 3m+3 = 2n+4 \end{cases} \rightarrow \begin{cases} 2m-3n = -1 \\ 3m-2n = 1 \end{cases} \Rightarrow \underline{m=1, n=1}$$

$$xy(2y^2 - 6xy)dx + xy(3xy - 4x^2)dy = 0$$

$$(2xy^3 - 6x^2y^2)dx + (3x^2y^2 - 4x^3y)dy = 0$$

$$M_y = \frac{\partial}{\partial y} (2xy^3 - 6x^2y^2) = 6xy^2 - 12x^2y$$

$$N_x = \frac{\partial}{\partial x} (3x^2y^2 - 4x^3y) = 6xy^2 - 12x^2y$$

$$\Rightarrow M_y = N_x$$

$$\psi = \int (2xy^3 - 6x^2y^2)dx$$

$$= x^2y^3 - 2x^3y^2 + h(y)$$

$$\psi_y = 3x^2y^2 - 4x^3y + h'(y)$$

$$= 3x^2y^2 - 4x^3y$$

$$h'(y) = 0 \Rightarrow h(y) = C$$

$$\underline{x^2y^3 - 2x^3y^2 = C}$$

Exercise

Find an integrating factor of the form $x^n y^m$ and solve the equation

$$(12 + 5xy)dx + (6xy^{-1} + 3x^2)dy = 0$$

Solution

$$x^n y^m (12 + 5xy)dx + x^n y^m (6xy^{-1} + 3x^2)dy = 0$$

$$(12x^n y^m + 5x^{n+1} y^{m+1})dx + (6x^{n+1} y^{m-1} + 3x^{n+2} y^m)dy = 0$$

$$M_y = \frac{\partial}{\partial y} (12x^n y^m + 5x^{n+1} y^{m+1}) = 12mx^n y^{m-1} + 5(m+1)x^{n+1} y^m$$

$$N_x = \frac{\partial}{\partial x} (6x^{n+1} y^{m-1} + 3x^{n+2} y^m) = 6(n+1)x^n y^{m-1} + 3(n+2)x^{n+1} y^m$$

For the equation to be exact, then

$$12mx^n y^{m-1} + 5(m+1)x^{n+1} y^m = 6(n+1)x^n y^{m-1} + 3(n+2)x^{n+1} y^m$$

$$12m + 5(m+1)xy = 6(n+1) + 3(n+2)xy$$

$$\begin{cases} 12m = 6n + 6 \\ 5m + 5 = 3n + 6 \end{cases} \rightarrow \begin{cases} 2m - n = 1 \\ 5m - 6n = 1 \end{cases} \Rightarrow \underline{n = 3, m = 2}$$

$$x^3 y^2 (12 + 5xy) dx + x^3 y^2 (6xy^{-1} + 3x^2) dy = 0$$

$$(12x^3 y^2 + 5x^4 y^3) dx + (6x^4 y + 3x^5 y^2) dy = 0$$

$$M_y = \frac{\partial}{\partial y} (12x^3 y^2 + 5x^4 y^3) = 24x^3 y + 15x^4 y^2$$

$$N_x = \frac{\partial}{\partial x} (6x^4 y + 3x^5 y^2) = 24x^3 y + 15x^4 y^2 \Rightarrow M_y = N_x$$

$$\psi = \int (12x^3 y^2 + 5x^4 y^3) dx$$

$$= 3x^4 y^2 + x^5 y^3 + h(y)$$

$$\psi_y = 6x^4 y + 3x^5 y^2 + h'(y)$$

$$= 6x^4 y + 3x^5 y^2$$

$$h'(y) = 0 \Rightarrow h(y) = C$$

$$\underline{3x^4 y^2 + x^5 y^3 = C}$$

Exercise

Find an integrating factor of the form $x^n y^m$ and solve the equation

$$(3y + 4xy^2) dx + (2x + 3x^2 y) dy = 0$$

Solution

$$x^n y^m (3y + 4xy^2) dx + x^n y^m (2x + 3x^2 y) dy = 0$$

$$(3x^n y^{m+1} + 4x^{n+1} y^{m+2}) dx + (2x^{n+1} y^m + 3x^{n+2} y^{m+1}) dy = 0$$

$$M = 3x^n y^{m+1} + 4x^{n+1} y^{m+2} ; \quad N = 2x^{n+1} y^m + 3x^{n+2} y^{m+1}$$

$$M_y = \frac{\partial}{\partial y} (3x^n y^{m+1} + 4x^{n+1} y^{m+2}) = 3(m+1)x^n y^m + 4(m+2)x^{n+1} y^{m+1}$$

$$N_x = \frac{\partial}{\partial x} (2x^{n+1} y^m + 3x^{n+2} y^{m+1}) = 2(n+1)x^n y^m + 3(n+2)x^{n+1} y^{m+1}$$

For the equation to be exact, then

$$3(m+1)x^n y^m + 4(m+2)x^{n+1} y^{m+1} = 2(n+1)x^n y^m + 3(n+2)x^{n+1} y^{m+1}$$

$$3(m+1) + 4(m+2)xy = 2(n+1) + 3(n+2)xy$$

$$\begin{cases} 3m+3 = 2n+2 \\ 4m+8 = 3n+6 \end{cases} \rightarrow \begin{cases} 3m-2n = -1 \\ 4m-3n = -2 \end{cases}$$

$$\underline{m} = \frac{\begin{vmatrix} -1 & -2 \\ -2 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 4 & -3 \end{vmatrix}} = \frac{-1}{-1} = \underline{1} \quad \underline{n} = \frac{\begin{vmatrix} 3 & -1 \\ 4 & -2 \end{vmatrix}}{-1} = \underline{2}$$

$$x^2 y (3y + 4xy^2) dx + x^2 y (2x + 3x^2 y) dy = 0$$

$$(3x^2 y^2 + 4x^3 y^3) dx + (2x^3 y + 3x^4 y^2) dy = 0$$

$$M_y = \frac{\partial}{\partial y} (3x^2 y^2 + 4x^3 y^3) = 6x^2 y + 12x^3 y^2$$

$$N_x = \frac{\partial}{\partial x} (2x^3 y + 3x^4 y^2) = 6x^2 y + 12x^3 y^2 \Rightarrow M_y = N_x$$

$$\psi = \int (3x^2 y^2 + 4x^3 y^3) dx$$

$$= x^3 y^2 + x^4 y^3 + h(y)$$

$$\psi_y = 2x^3 y + 3x^4 y^2 + h'(y) = 2x^3 y + 3x^4 y^2$$

$$h'(y) = 0 \Rightarrow h(y) = C$$

$$\underline{x^3 y^2 + x^4 y^3 = C}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx} - 5y = -\frac{5}{2}xy^3$

Solution

$$y^3 \Rightarrow n = 3$$

$$\text{Let } u = y^{1-3} = y^{-2} = \frac{1}{y^2} \Rightarrow y = \frac{1}{\sqrt{u}}$$

$$\frac{du}{dx} = -2y^{-3} \frac{dy}{dx} \Rightarrow -\frac{1}{2}y^3 \frac{du}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx} - 5y = -\frac{5}{2}xy^3$$

$$-\frac{1}{2}u^{-3/2} \frac{du}{dx} - 5u^{-1/2} = -\frac{5}{2}xu^{-3/2} \quad \times -2u^{3/2}$$

$$\frac{du}{dx} + 10u = 5x$$

$$e^{\int 10dx} = e^{10x}$$

$$\int 5xe^{10x} dx = \left(\frac{1}{2}x - \frac{1}{20}\right)e^{10x}$$

$$u = \frac{1}{e^{10x}} \left(\left(\frac{1}{2}x - \frac{1}{20}\right)e^{10x} + C \right)$$

$$\boxed{\frac{1}{y^2} = \frac{1}{2}x - \frac{1}{20} + Ce^{-10x}}$$

		$\int e^{10x}$
+	$5x$	$\frac{1}{10}e^{10x}$
-	5	$\frac{1}{100}e^{10x}$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx} + \frac{y}{x} = x^2y^2$

Solution

$$y^2 \Rightarrow n = 2$$

$$\text{Let } u = y^{1-2} = \frac{1}{y} \Rightarrow y = \frac{1}{u}$$

$$\frac{du}{dx} = -\frac{1}{y^2} \frac{dy}{dx} \Rightarrow -\frac{1}{u^2} \frac{du}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx} + \frac{y}{x} = x^2y^2$$

$$-\frac{1}{u^2} \frac{du}{dx} + \frac{1}{xu} = x^2 \frac{1}{u^2} \quad \times -u^2$$

$$\frac{du}{dx} - \frac{1}{x}u = -x^2$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{x}$$

$$\int x^2 \frac{1}{x} dx = \frac{1}{2}x^2$$

$$u = x \left(\frac{1}{2}x^2 + C_1 \right)$$

$$\frac{1}{y} = \frac{1}{2}x^3 + C_1x$$

$$\underline{y(x) = \frac{2}{x^3 + Cx}}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx} - y = e^{2x}y^3$

Solution

$$y^3 \Rightarrow n = 3$$

$$\text{Let } u = y^{1-3} = y^{-2} = \frac{1}{y^2} \Rightarrow y = \frac{1}{\sqrt{u}} = u^{-1/2}$$

$$\frac{du}{dx} = -2y^{-3} \frac{dy}{dx} \Rightarrow -\frac{1}{2}y^3 \frac{du}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx} - y = e^{2x}y^3$$

$$-\frac{1}{2}u^{-3/2} \frac{du}{dx} - u^{-1/2} = e^{2x}u^{-3/2} \quad \times -2u^{3/2}$$

$$\frac{du}{dx} + 2u = -2e^{2x}$$

$$e^{\int 2dx} = e^{2x}$$

$$\int -2e^{2x}e^{2x}dx = -\frac{1}{2}e^{4x}$$

$$u = e^{-2x} \left(-\frac{1}{2}e^{4x} + C \right)$$

$$\underline{y^{-2} = -\frac{1}{2}e^{2x} + Ce^{-2x}} \quad y = \pm \sqrt{\frac{-2}{e^{2x} + Ce^{-2x}}}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx} + \frac{y}{x-2} = 5(x-2)y^{1/2}$

Solution

$$y^{1/2} \Rightarrow n = \frac{1}{2}$$

$$\text{Let } u = y^{1-\frac{1}{2}} = y^{1/2} \Rightarrow y = u^2 \quad \rightarrow \frac{dy}{dx} = 2u \frac{du}{dx}$$

$$\frac{dy}{dx} + \frac{1}{x-2}y = 5(x-2)y^{1/2}$$

$$2u \frac{du}{dx} + \frac{1}{x-2}u^2 = 5(x-2)u \quad \times \frac{1}{2u}$$

$$\frac{du}{dx} + \frac{1}{2(x-2)}u = \frac{5}{2}(x-2)$$

$$e^{\frac{1}{2} \int \frac{1}{x-2} dx} = e^{\frac{1}{2} \ln|x-2|} = \sqrt{x-2}$$

$$\int \frac{1}{2}(x-2)\sqrt{x-2} dx = \int \frac{1}{2}(x-2)^{3/2} dx = \frac{1}{5}(x-2)^{5/2}$$

$$u = \frac{1}{\sqrt{x-2}} \left(\frac{1}{5}(x-2)^{5/2} + C \right)$$

$$y^{1/2} = \frac{1}{5}(x-2)^{3/2} + \frac{C}{\sqrt{x-2}}$$

$$y(x) = \left(\frac{1}{5}(x-2)^{3/2} + \frac{C}{\sqrt{x-2}} \right)^2$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx} + y = e^x y^{-2}$

Solution

$$y^{-2} \Rightarrow n = -2$$

$$\text{Let } u = y^{1+2} = y^3 \Rightarrow y = u^{1/3}$$

$$\rightarrow \frac{dy}{dx} = \frac{1}{3} u^{-2/3} \frac{du}{dx}$$

$$\frac{dy}{dx} + y = e^x y^{-2}$$

$$\frac{1}{3} u^{-2/3} \frac{du}{dx} + u^{1/3} = e^x u^{-2/3} \quad \times 3u^{2/3}$$

$$\frac{du}{dx} + 3u = 3e^x$$

$$e^{\int 3dx} = e^{3x}$$

$$\int 3e^x e^{3x} dx = \frac{3}{4} e^{4x}$$

$$u = e^{-3x} \left(\frac{3}{4} e^{4x} + C \right)$$

$$y^3 = \frac{3}{4} e^x + C e^{-3x}$$

Exercise

Find the general solution by using either Bernoulli $\frac{dy}{dx} + y^3x + y = 0$

Solution

$$\frac{dy}{dx} + y = -xy^3$$

$$y^3 \Rightarrow n = 3$$

$$\text{Let } u = y^{1-3} = y^{-2} = \frac{1}{y^2} \Rightarrow y = \frac{1}{\sqrt{u}} = u^{-1/2}$$

$$\frac{dy}{dx} = -\frac{1}{2}u^{-3/2} \frac{du}{dx}$$

$$-\frac{1}{2}u^{-3/2} \frac{du}{dx} + u^{-1/2} = -xu^{-3/2} \quad \times -2u^{3/2}$$

$$\frac{du}{dx} - 2u = 2x$$

$$e^{\int -2dx} = e^{-2x}$$

$$\int 2xe^{-2x} dx = \left(-x - \frac{1}{2}\right)e^{-2x}$$

$$u = e^{2x} \left(\left(-x - \frac{1}{2}\right)e^{-2x} + C \right)$$

$$\frac{1}{y^2} = -x - \frac{1}{2} + Ce^{2x}$$

$$y(x) = \pm \frac{1}{\sqrt{Ce^{2x} - x - \frac{1}{2}}}$$

		$\int e^{-2x}$
+	$2x$	$-\frac{1}{2}e^{-2x}$
-	2	$\frac{1}{4}e^{-2x}$

Exercise

Find the general solution by using homogeneous equations. $(xy + y^2)dx - x^2dy = 0$

Solution

$$\frac{dy}{dx} = \frac{xy + y^2}{x^2} = \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

$$\text{Let } v = \frac{y}{x} \rightarrow y = xv \Rightarrow y' = v + xv'$$

$$v + xv' = v + v^2$$

$$x \frac{dv}{dx} = v^2$$

$$\int \frac{dv}{v^2} = \int \frac{dx}{x}$$

$$-\frac{1}{v} = \ln|x| + C$$

$$-\frac{x}{y} = \ln|x| + C$$

$$\underline{y(x) = -\frac{x}{\ln|x| + C}}$$

Exercise

Find the general solution by using homogeneous equations. $(x^2 + y^2)dx + 2xydy = 0$

Solution

$$\frac{dy}{dx} = -\frac{x^2 + y^2}{2xy}$$

$$= -\frac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right)$$

$$\text{Let } v = \frac{y}{x} \rightarrow y = xv \Rightarrow y' = v + xv'$$

$$v + xv' = -\frac{1}{2}v - \frac{1}{2v}$$

$$xv' = -\frac{3}{2}v - \frac{1}{2v}$$

$$x \frac{dv}{dx} = -\left(\frac{3v^2 + 1}{2v}\right)$$

$$\int \frac{2v}{3v^2 + 1} dv = -\int \frac{1}{x} dx$$

$$\frac{1}{3} \ln(3v^2 + 1) = -\ln|x| + \ln C_1$$

$$\ln(3v^2 + 1) = -3\ln|C_1 x|$$

$$3v^2 + 1 = \frac{C}{|x|^3}$$

$$3\frac{y^2}{x^2} = \frac{C}{|x|^3} - 1$$

$$3xy^2 = C - x^3$$

$$\underline{3xy^2 + x^3 = C}$$

Exercise

Find the general solution by using homogeneous equations.

$$(y^2 - xy)dx + x^2 dy = 0$$

Solution

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2}$$

$$= \frac{y}{x} - \left(\frac{y}{x}\right)^2$$

$$\text{Let } v = \frac{y}{x} \rightarrow y = xv \Rightarrow y' = v + xv'$$

$$v + xv' = v - v^2$$

$$x \frac{dv}{dx} = -v^2$$

$$-\int \frac{1}{v^2} dv = \int \frac{1}{x} dx$$

$$\frac{1}{v} = \ln|x| + C$$

$$\frac{x}{y} = \ln|x| + C$$

$$\underline{y(x) = \frac{x}{\ln|x| + C}}$$

Exercise

Find the general solution by using homogeneous equations.

$$\frac{dy}{d\theta} = \frac{1}{\theta} \left(\theta \sec\left(\frac{y}{\theta}\right) + y \right)$$

Solution

$$\frac{dy}{d\theta} = \sec\left(\frac{y}{\theta}\right) + \frac{y}{\theta}$$

$$\text{Let } v = \frac{y}{\theta} \rightarrow y = \theta v \Rightarrow y' = v + \theta v'$$

$$v + \theta v' = \sec v + v$$

$$\int \cos v \, dv = \int \frac{d\theta}{\theta}$$

$$\sin v = \ln|\theta| + C$$

$$\frac{y}{\theta} = \arcsin(\ln|\theta| + C)$$

$$\underline{y(\theta) = \theta \arcsin(\ln|\theta| + C)}$$

Exercise

Find the general solution by using homogeneous equations.

$$\frac{dy}{dx} = \frac{y(\ln y - \ln x + 1)}{x}$$

Solution

$$\frac{dy}{dx} = \frac{y}{x} \left(\ln \frac{y}{x} + 1 \right)$$

$$\text{Let } v = \frac{y}{x} \rightarrow y = xv \Rightarrow y' = v + xv'$$

$$v + xv' = v \ln |v| + v$$

$$x \frac{dv}{dx} = v \ln |v|$$

$$\frac{dv}{v \ln |v|} = \frac{dx}{x}$$

$$\int \frac{1}{\ln |v|} d(\ln v) = \int \frac{dx}{x}$$

$$\ln |\ln v| = \ln |x| + \ln C$$

$$\ln |\ln v| = \ln |Cx|$$

$$\ln v = Cx$$

$$v = e^{Cx} = \frac{y}{x}$$

$$\underline{y(x) = xe^{Cx}}$$