# *Solution* — *Section* 3.4 – Half-Angle Formulas

### *Exercise*

Use half-angle formulas to find the exact value of $\sin 105°$

***Solution***

$$\begin{aligned}
\sin 105° &= \sin \frac{210°}{2} \\
&= \sqrt{\frac{1-\cos 210°}{2}} \qquad\qquad \textit{reference}: 210° - 180° = 30° \\
&= \sqrt{\frac{1+\cos 30°}{2}} \\
&= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} \\
&= \sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}} \\
&= \sqrt{\frac{2+\sqrt{3}}{4}} \\
&= \frac{\sqrt{2+\sqrt{3}}}{2}
\end{aligned}$$

### *Exercise*

Find the exact of $\tan 22.5°$

***Solution***

$$\begin{aligned}
\tan 22.5° &= \tan \frac{45°}{2} \\
&= \frac{1-\cos 45°}{\sin 45°} \\
&= \frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\
&= \frac{\frac{2-\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}
\end{aligned}$$

$$=\frac{2-\sqrt{2}}{\sqrt{2}}$$

$$=\frac{2}{\sqrt{2}}-\frac{\sqrt{2}}{\sqrt{2}}$$

$$=\frac{2\sqrt{2}}{2}-1$$

$$=\sqrt{2}-1$$

***Exercise***

Given: $\cos x=\frac{2}{3},\ \ \frac{3\pi}{2}<x<2\pi$, find $\cos\frac{x}{2},\ \ \sin\frac{x}{2}$, and $\tan\frac{x}{2}$

**Solution**

$$\frac{3\pi}{2}<x<2\pi\Rightarrow\frac{3\pi}{4}<\frac{x}{2}<\pi\Rightarrow x\in QII$$

| | | |
|---|---|---|
| $\cos\frac{x}{2}=-\sqrt{\frac{1+\cos x}{2}}$<br>$=-\sqrt{\frac{1+\frac{2}{3}}{2}}$<br>$=-\sqrt{\frac{1}{2}\frac{3+2}{3}}$<br>$=-\sqrt{\frac{5}{6}}$<br>$=-\frac{\sqrt{5}}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}}$<br>$=-\frac{\sqrt{30}}{6}$ | $\sin\frac{x}{2}=\sqrt{\frac{1-\cos x}{2}}$<br>$=\sqrt{\frac{1-\frac{2}{3}}{2}}$<br>$=\sqrt{\frac{1}{2}\frac{3-2}{3}}$<br>$=\sqrt{\frac{1}{6}}$<br>$=\frac{1}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}}$<br>$=\frac{\sqrt{6}}{6}$ | $\tan\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$<br>$=\frac{\frac{\sqrt{6}}{6}}{-\frac{\sqrt{30}}{6}}$<br>$=-\frac{\sqrt{6}}{\sqrt{30}}\frac{\sqrt{30}}{\sqrt{30}}$<br>$=-\frac{6\sqrt{5}}{30}$<br>$=-\frac{\sqrt{5}}{5}$ |

### *Exercise*

Prove the identity $\quad 2\csc x\cos^2\frac{x}{2}=\frac{\sin x}{1-\cos x}$

***Solution***

$$2\csc x\cos^2\frac{x}{2}=2\frac{1}{\sin x}\frac{1+\cos x}{2}$$

$$=\frac{1+\cos x}{\sin x}\frac{1-\cos x}{1-\cos x} \qquad (a-b)(a+b)=a^2-b^2$$

$$=\frac{1-\cos^2 x}{\sin x(1-\cos x)}$$

$$=\frac{\sin^2 x}{\sin x(1-\cos x)}$$

$$=\frac{\sin x}{1-\cos x}$$

### *Exercise*

Prove the identity $\quad \tan\frac{\alpha}{2}=\sin\alpha+\cos\alpha\cot\alpha-\cot\alpha$

***Solution***

$$\tan\frac{\alpha}{2}=\frac{1-\cos\alpha}{\sin\alpha} \qquad 1=\sin^2\alpha+\cos^2\alpha$$

$$=\frac{\sin^2\alpha+\cos^2\alpha-\cos\alpha}{\sin\alpha}$$

$$=\frac{\sin^2\alpha}{\sin\alpha}+\frac{\cos^2\alpha}{\sin\alpha}-\frac{\cos\alpha}{\sin\alpha}$$

$$=\sin\alpha+\cos\alpha\frac{\cos\alpha}{\sin\alpha}-\cot\alpha$$

$$=\sin\alpha+\cos\alpha\cot\alpha-\cot\alpha$$

### *Exercise*

Prove the following equation is an identity: $\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)=\frac{\sin^2 x}{4}$

***Solution***

$$\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)=\frac{1-\cos x}{2}\cdot\frac{1+\cos x}{2}$$

$(a-b)(a+b)=a^2+b^2$

$$=\frac{1-\cos^2 x}{4}$$

$\cos^2\alpha+\sin^2\alpha=1$

$$=\frac{\sin^2 x}{4}$$

### *Exercise*

Prove the following equation is an identity: $\tan\frac{x}{2}+\cot\frac{x}{2}=2\csc x$

***Solution***

$$\tan\frac{x}{2}+\cot\frac{x}{2}=\tan\frac{x}{2}+\frac{1}{\tan\frac{x}{2}}$$

$$=\frac{\sin x}{1+\cos x}+\frac{\sin x}{1-\cos x}$$

$$=\sin x\frac{(1-\cos x)+(1+\cos x)}{1-\cos^2 x}$$

$$=\sin x\frac{2}{\sin^2 x}$$

$$=\frac{2}{\sin x}$$

$$=2\csc x$$

### *Exercise*

Prove the following equation is an identity: $2\sin^2\left(\frac{x}{2}\right)=\frac{\sin^2 x}{1+\cos x}$

***Solution***

$$2\sin^2\left(\frac{x}{2}\right)=2\frac{1-\cos x}{2}$$

$$=1-\cos x\cdot\frac{1+\cos x}{1+\cos x}$$

$$=\frac{1-\cos^2 x}{1+\cos x}$$

$$=\frac{\sin^2 x}{1+\cos x}$$

***Exercise***

Prove the following equation is an identity: $\tan^2\left(\frac{x}{2}\right)=\frac{\sec x+\cos x-2}{\sec x-\cos x}$

***Solution***

$$\tan^2\left(\frac{x}{2}\right)=\frac{1-\cos x}{1+\cos x} \qquad \tan^2\alpha=\frac{1-\cos 2\alpha}{1+\cos 2\alpha}; \quad x=2\alpha; \quad \tan^2\frac{x}{2}=\frac{1-\cos x}{1+\cos x}$$

$$=\frac{1-\cos x}{1+\cos x}\frac{1-\cos x}{1-\cos x}$$

$$=\frac{1-2\cos x+\cos^2 x}{1-\cos^2 x}\ \frac{\frac{1}{\cos x}}{\frac{1}{\cos x}}$$

$$=\frac{\frac{1-2\cos x+\cos^2 x}{\cos x}}{\frac{1-\cos^2 x}{\cos x}}$$

$$=\frac{\frac{1}{\cos x}-\frac{2\cos x}{\cos x}+\frac{\cos^2 x}{\cos x}}{\frac{1}{\cos x}-\frac{\cos^2 x}{\cos x}}$$

$$=\frac{\sec x-2+\cos x}{\sec x-\cos x}$$

$$\frac{\sec x+\cos x-2}{\sec x-\cos x}=\frac{\frac{1}{\cos x}+\cos x-2}{\frac{1}{\cos x}-\cos x}$$

$$=\frac{\frac{1+\cos^2 x-2\cos x}{\cos x}}{\frac{1-\cos^2 x}{\cos x}}$$

$$=\frac{(1-\cos x)^2}{(1-\cos x)(1+\cos x)}$$

$$=\frac{1-\cos x}{1+\cos x} \qquad \tan^2\alpha=\frac{1-\cos 2\alpha}{1+\cos 2\alpha}; \quad x=2\alpha; \quad \tan^2\frac{x}{2}=\frac{1-\cos x}{1+\cos x}$$

$$=\tan^2\left(\frac{x}{2}\right)$$

***Exercise***

Prove the following equation is an identity: $\sec^2\left(\frac{x}{2}\right) = \frac{2\sec x + 2}{\sec x + 2 + \cos x}$

***Solution***

$$\sec^2\left(\frac{x}{2}\right) = \frac{1}{\cos^2\left(\frac{x}{2}\right)} \qquad \qquad \cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}} \Rightarrow \cos^2\left(\frac{\alpha}{2}\right) = \frac{1+\cos\alpha}{2}$$

$$= \frac{1}{\frac{1+\cos x}{2}}$$

$$= \frac{2}{1+\cos x}\frac{1+\cos x}{1+\cos x}$$

$$= \frac{2+2\cos x}{1+2\cos x+\cos^2 x}$$

$$= \frac{2+2\cos x}{1+2\cos x+\cos^2 x}\frac{\frac{1}{\cos x}}{\frac{1}{\cos x}}$$

$$= \frac{\frac{2}{\cos x}+2\frac{\cos x}{\cos x}}{\frac{1}{\cos x}+\frac{2\cos x}{\cos x}+\frac{\cos^2 x}{\cos x}}$$

$$= \frac{2\sec x+2}{\sec x+2+\cos x}$$

***Exercise***

Prove the following equation is an identity: $\dfrac{1-\sin^2\left(\frac{x}{2}\right)}{1+\sin^2\left(\frac{x}{2}\right)} = \dfrac{1+\cos x}{3-\cos x}$

***Solution***

$$\frac{1-\sin^2\left(\frac{x}{2}\right)}{1+\sin^2\left(\frac{x}{2}\right)} = \frac{1-\frac{1-\cos x}{2}}{1+\frac{1-\cos x}{2}}$$

$$= \frac{\frac{2-1-\cos x}{2}}{\frac{2+1-\cos x}{2}}$$

$$= \frac{1-\cos x}{3-\cos x}$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{1-\cos^2\left(\frac{x}{2}\right)}{1-\sin^2\left(\frac{x}{2}\right)}=\dfrac{1-\cos x}{1+\cos x}$

***Solution***

$$\frac{1-\cos^2\left(\frac{x}{2}\right)}{1-\sin^2\left(\frac{x}{2}\right)}=\frac{1-\frac{1+\cos x}{2}}{1-\frac{1-\cos x}{2}}$$

$$=\frac{\frac{2-(1+\cos x)}{2}}{\frac{2-(1-\cos x)}{2}}$$

$$=\frac{\frac{2-1-\cos x}{2}}{\frac{2-1+\cos x}{2}}$$

$$=\frac{1-\cos x}{1+\cos x}$$