CHAPTER 3 DIFFERENTIATION

3.1 TANGENTS AND THE DERIVATIVE AT A POINT

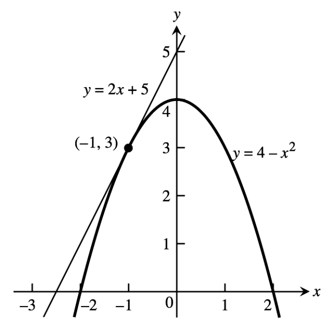
1. P_1: $m_1 = 1$, P_2: $m_2 = 5$

2. P_1: $m_1 = -2$, P_2: $m_2 = 0$

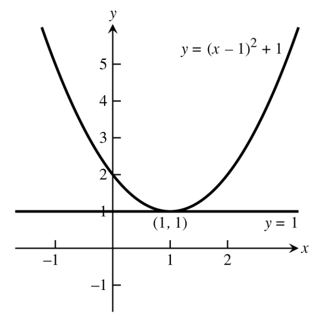
3. P_1: $m_1 = \frac{5}{2}$, P_2: $m_2 = -\frac{1}{2}$

4. P_1: $m_1 = 3$, P_2: $m_2 = -3$

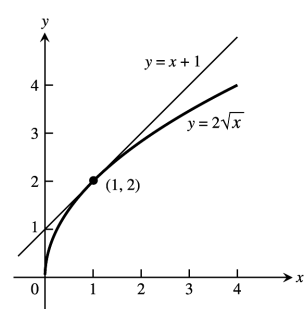
5. $m = \lim\limits_{h \to 0} \frac{[4-(-1+h)^2]-(4-(-1)^2)}{h}$
$= \lim\limits_{h \to 0} \frac{-(1-2h+h^2)+1}{h} = \lim\limits_{h \to 0} \frac{h(2-h)}{h} = 2$;
at $(-1, 3)$: $y = 3 + 2(x-(-1)) \Rightarrow y = 2x + 5$, tangent line



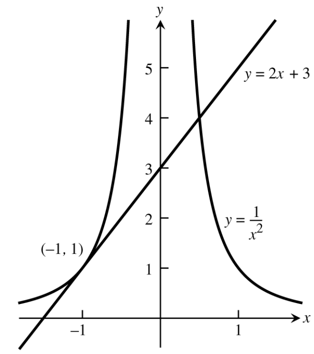
6. $m = \lim\limits_{h \to 0} \frac{[(1+h-1)^2+1]-[(1-1)^2+1]}{h} = \lim\limits_{h \to 0} \frac{h^2}{h}$
$= \lim\limits_{h \to 0} h = 0$; at $(1, 1)$: $y = 1 + 0(x-1) \Rightarrow y = 1$, tangent line



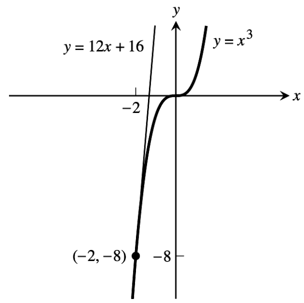
7. $m = \lim\limits_{h \to 0} \frac{2\sqrt{1+h}-2\sqrt{1}}{h} = \lim\limits_{h \to 0} \frac{2\sqrt{1+h}-2}{h} \cdot \frac{2\sqrt{1+h}+2}{2\sqrt{1+h}+2}$
$= \lim\limits_{h \to 0} \frac{4(1+h)-4}{2h\left(\sqrt{1+h}+1\right)} = \lim\limits_{h \to 0} \frac{2}{\sqrt{1+h}+1} = 1$;
at $(1, 2)$: $y = 2 + 1(x-1) \Rightarrow y = x + 1$, tangent line



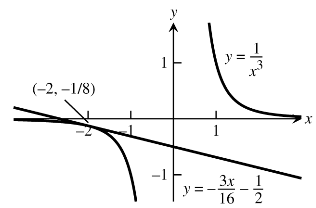
8. $m = \lim\limits_{h \to 0} \frac{\frac{1}{(-1+h)^2}-\frac{1}{(-1)^2}}{h} = \lim\limits_{h \to 0} \frac{1-(-1+h)^2}{h(-1+h)^2}$
$= \lim\limits_{h \to 0} \frac{-(-2h+h^2)}{h(-1+h)^2} = \lim\limits_{h \to 0} \frac{2-h}{(-1+h)^2} = 2$;
at $(-1, 1)$: $y = 1 + 2(x-(-1)) \Rightarrow y = 2x + 3$, tangent line



9. $m = \lim_{h \to 0} \frac{(-2+h)^3 - (-2)^3}{h} = \lim_{h \to 0} \frac{-8 + 12h - 6h^2 + h^3 + 8}{h}$
$= \lim_{h \to 0} (12 - 6h + h^2) = 12;$
at $(-2, -8)$: $y = -8 + 12(x - (-2)) \Rightarrow y = 12x + 16$, tangent line



10. $m = \lim_{h \to 0} \frac{\frac{1}{(-2+h)^3} - \frac{1}{(-2)^3}}{h} = \lim_{h \to 0} \frac{-8 - (-2+h)^3}{-8h(-2+h)^3}$
$= \lim_{h \to 0} \frac{-(12h - 6h^2 + h^3)}{-8h(-2+h)^3} = \lim_{h \to 0} \frac{12 - 6h + h^2}{8(-2+h)^3}$
$= \frac{12}{8(-8)} = -\frac{3}{16};$
at $\left(-2, -\frac{1}{8}\right)$: $y = -\frac{1}{8} - \frac{3}{16}(x - (-2))$
$\Rightarrow y = -\frac{3}{16}x - \frac{1}{2}$, tangent line



11. $m = \lim_{h \to 0} \frac{[(2+h)^2 + 1] - 5}{h} = \lim_{h \to 0} \frac{(5 + 4h + h^2) - 5}{h} = \lim_{h \to 0} \frac{h(4+h)}{h} = 4;$
at $(2, 5)$: $y - 5 = 4(x - 2)$, tangent line

12. $m = \lim_{h \to 0} \frac{[(1+h) - 2(1+h)^2] - (-1)}{h} = \lim_{h \to 0} \frac{(1 + h - 2 - 4h - 2h^2) + 1}{h} = \lim_{h \to 0} \frac{h(-3 - 2h)}{h} = -3;$
at $(1, -1)$: $y + 1 = -3(x - 1)$, tangent line

13. $m = \lim_{h \to 0} \frac{\frac{3+h}{(3+h)-2} - 3}{h} = \lim_{h \to 0} \frac{(3+h) - 3(h+1)}{h(h+1)} = \lim_{h \to 0} \frac{-2h}{h(h+1)} = -2;$
at $(3, 3)$: $y - 3 = -2(x - 3)$, tangent line

14. $m = \lim_{h \to 0} \frac{\frac{8}{(2+h)^2} - 2}{h} = \lim_{h \to 0} \frac{8 - 2(2+h)^2}{h(2+h)^2} = \lim_{h \to 0} \frac{8 - 2(4 + 4h + h^2)}{h(2+h)^2} = \lim_{h \to 0} \frac{-2h(4+h)}{h(2+h)^2} = \frac{-8}{4} = -2;$
at $(2, 2)$: $y - 2 = -2(x - 2)$

15. $m = \lim_{h \to 0} \frac{(2+h)^3 - 8}{h} = \lim_{h \to 0} \frac{(8 + 12h + 6h^2 + h^3) - 8}{h} = \lim_{h \to 0} \frac{h(12 + 6h + h^2)}{h} = 12;$
at $(2, 8)$: $y - 8 = 12(t - 2)$, tangent line

16. $m = \lim_{h \to 0} \frac{[(1+h)^3 + 3(1+h)] - 4}{h} = \lim_{h \to 0} \frac{(1 + 3h + 3h^2 + h^3 + 3 + 3h) - 4}{h} = \lim_{h \to 0} \frac{h(6 + 3h + h^2)}{h} = 6;$
at $(1, 4)$: $y - 4 = 6(t - 1)$, tangent line

17. $m = \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} = \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} = \lim_{h \to 0} \frac{(4+h) - 4}{h\left(\sqrt{4+h} + 2\right)} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{4+h} + 2\right)} = \frac{1}{\sqrt{4} + 2}$
$= \frac{1}{4}$; at $(4, 2)$: $y - 2 = \frac{1}{4}(x - 4)$, tangent line

18. $m = \lim_{h \to 0} \frac{\sqrt{(8+h) + 1} - 3}{h} = \lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h} \cdot \frac{\sqrt{9+h} + 3}{\sqrt{9+h} + 3} = \lim_{h \to 0} \frac{(9+h) - 9}{h\left(\sqrt{9+h} + 3\right)} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{9+h} + 3\right)}$
$= \frac{1}{\sqrt{9} + 3} = \frac{1}{6}$; at $(8, 3)$: $y - 3 = \frac{1}{6}(x - 8)$, tangent line

19. At $x = -1$, $y = 5 \Rightarrow m = \lim_{h \to 0} \frac{5(-1+h)^2 - 5}{h} = \lim_{h \to 0} \frac{5(1 - 2h + h^2) - 5}{h} = \lim_{h \to 0} \frac{5h(-2+h)}{h} = -10$, slope

20. At $x = 2$, $y = -3 \Rightarrow m = \lim\limits_{h \to 0} \frac{[1-(2+h)^2]-(-3)}{h} = \lim\limits_{h \to 0} \frac{(1-4-4h-h^2)+3}{h} = \lim\limits_{h \to 0} \frac{-h(4+h)}{h} = -4$, slope

21. At $x = 3$, $y = \frac{1}{2} \Rightarrow m = \lim\limits_{h \to 0} \frac{\frac{1}{(3+h)-1} - \frac{1}{2}}{h} = \lim\limits_{h \to 0} \frac{2-(2+h)}{2h(2+h)} = \lim\limits_{h \to 0} \frac{-h}{2h(2+h)} = -\frac{1}{4}$, slope

22. At $x = 0$, $y = -1 \Rightarrow m = \lim\limits_{h \to 0} \frac{\frac{h-1}{h+1} - (-1)}{h} = \lim\limits_{h \to 0} \frac{(h-1)+(h+1)}{h(h+1)} = \lim\limits_{h \to 0} \frac{2h}{h(h+1)} = 2$, slope

23. At a horizontal tangent the slope $m = 0 \Rightarrow 0 = m = \lim\limits_{h \to 0} \frac{[(x+h)^2+4(x+h)-1]-(x^2+4x-1)}{h}$
$= \lim\limits_{h \to 0} \frac{(x^2+2xh+h^2+4x+4h-1)-(x^2+4x-1)}{h} = \lim\limits_{h \to 0} \frac{(2xh+h^2+4h)}{h} = \lim\limits_{h \to 0} (2x+h+4) = 2x+4$;
$2x+4 = 0 \Rightarrow x = -2$. Then $f(-2) = 4-8-1 = -5 \Rightarrow (-2,-5)$ is the point on the graph where there is a horizontal tangent.

24. $0 = m = \lim\limits_{h \to 0} \frac{[(x+h)^3-3(x+h)]-(x^3-3x)}{h} = \lim\limits_{h \to 0} \frac{(x^3+3x^2h+3xh^2+h^3-3x-3h)-(x^3-3x)}{h}$
$= \lim\limits_{h \to 0} \frac{3x^2h+3xh^2+h^3-3h}{h} = \lim\limits_{h \to 0} (3x^2+3xh+h^2-3) = 3x^2-3$; $3x^2-3 = 0 \Rightarrow x = -1$ or $x = 1$. Then $f(-1) = 2$ and $f(1) = -2 \Rightarrow (-1,2)$ and $(1,-2)$ are the points on the graph where a horizontal tangent exists.

25. $-1 = m = \lim\limits_{h \to 0} \frac{\frac{1}{(x+h)-1} - \frac{1}{x-1}}{h} = \lim\limits_{h \to 0} \frac{(x-1)-(x+h-1)}{h(x-1)(x+h-1)} = \lim\limits_{h \to 0} \frac{-h}{h(x-1)(x+h-1)} = -\frac{1}{(x-1)^2}$
$\Rightarrow (x-1)^2 = 1 \Rightarrow x^2-2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$. If $x = 0$, then $y = -1$ and $m = -1$
$\Rightarrow y = -1-(x-0) = -(x+1)$. If $x = 2$, then $y = 1$ and $m = -1 \Rightarrow y = 1-(x-2) = -(x-3)$.

26. $\frac{1}{4} = m = \lim\limits_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim\limits_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \lim\limits_{h \to 0} \frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$
$= \lim\limits_{h \to 0} \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$. Thus, $\frac{1}{4} = \frac{1}{2\sqrt{x}} \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4 \Rightarrow y = 2$. The tangent line is
$y = 2 + \frac{1}{4}(x-4) = \frac{x}{4} + 1$.

27. $\lim\limits_{h \to 0} \frac{f(2+h)-f(2)}{h} = \lim\limits_{h \to 0} \frac{(100-4.9(2+h)^2)-(100-4.9(2)^2)}{h} = \lim\limits_{h \to 0} \frac{-4.9\,(4+4h+h^2)+4.9(4)}{h}$
$= \lim\limits_{h \to 0} (-19.6-4.9h) = -19.6$. The minus sign indicates the object is falling <u>downward</u> at a speed of 19.6 m/sec.

28. $\lim\limits_{h \to 0} \frac{f(10+h)-f(10)}{h} = \lim\limits_{h \to 0} \frac{3(10+h)^2-3(10)^2}{h} = \lim\limits_{h \to 0} \frac{3\,(20h+h^2)}{h} = 60$ ft/sec.

29. $\lim\limits_{h \to 0} \frac{f(3+h)-f(3)}{h} = \lim\limits_{h \to 0} \frac{\pi(3+h)^2-\pi(3)^2}{h} = \lim\limits_{h \to 0} \frac{\pi\,[9+6h+h^2-9]}{h} = \lim\limits_{h \to 0} \pi(6+h) = 6\pi$

30. $\lim\limits_{h \to 0} \frac{f(2+h)-f(2)}{h} = \lim\limits_{h \to 0} \frac{\frac{4\pi}{3}(2+h)^3 - \frac{4\pi}{3}(2)^3}{h} = \lim\limits_{h \to 0} \frac{\frac{4\pi}{3}[12h+6h^2+h^3]}{h} = \lim\limits_{h \to 0} \frac{4\pi}{3}[12+6h+h^2] = 16\pi$

31. At (x_0, mx_0+b) the slope of the tangent line is $\lim\limits_{h \to 0} \frac{(m(x_0+h)+b)-(m\,x_0+b)}{(x_0+h)-x_0} = \lim\limits_{h \to 0} \frac{m\,h}{h} = \lim\limits_{h \to 0} m = m$.
The equation of the tangent line is $y-(m\,x_0+b) = m(x-x_0) \Rightarrow y = mx+b$.

32. At $x = 4$, $y = \frac{1}{\sqrt{4}} = \frac{1}{2}$ and $m = \lim\limits_{h \to 0} \frac{\frac{1}{\sqrt{4+h}} - \frac{1}{2}}{h} = \lim\limits_{h \to 0} \left[\frac{\frac{1}{\sqrt{4+h}} - \frac{1}{2}}{h} \cdot \frac{2\sqrt{4+h}}{2\sqrt{4+h}}\right] = \lim\limits_{h \to 0} \left(\frac{2-\sqrt{4+h}}{2h\sqrt{4+h}}\right)$
$= \lim\limits_{h \to 0} \left[\frac{2-\sqrt{4+h}}{2h\sqrt{4+h}} \cdot \frac{2+\sqrt{4+h}}{2+\sqrt{4+h}}\right] = \lim\limits_{h \to 0} \left(\frac{4-(4+h)}{2h\sqrt{4+h}\left(2+\sqrt{4+h}\right)}\right) = \lim\limits_{h \to 0} \left(\frac{-h}{2h\sqrt{4+h}\left(2+\sqrt{4+h}\right)}\right)$

$= \lim_{h \to 0} \left(\frac{-1}{2\sqrt{4+h}\left(2+\sqrt{4+h}\right)} \right) = -\frac{1}{2\sqrt{4}\left(2+\sqrt{4}\right)} = -\frac{1}{16}$

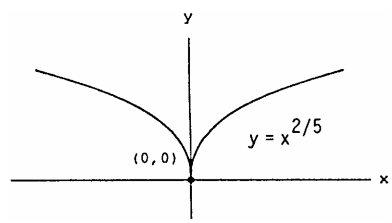
33. Slope at origin $= \lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0 \Rightarrow$ yes, f(x) does have a tangent at the origin with slope 0.

34. $\lim_{h \to 0} \frac{g(0+h)-g(0)}{h} = \lim_{h \to 0} \frac{h \sin\left(\frac{1}{h}\right)}{h} = \lim_{h \to 0} \sin \frac{1}{h}$. Since $\lim_{h \to 0} \sin \frac{1}{h}$ does not exist, f(x) has no tangent at the origin.

35. $\lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^-} \frac{-1-0}{h} = \infty$, and $\lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^+} \frac{1-0}{h} = \infty$. Therefore, $\lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \infty \Rightarrow$ yes, the graph of f has a vertical tangent at the origin.

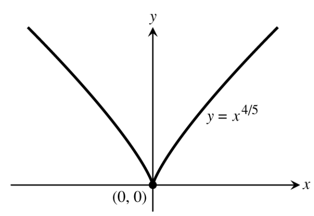
36. $\lim_{h \to 0^-} \frac{U(0+h)-U(0)}{h} = \lim_{h \to 0^-} \frac{0-1}{h} = \infty$, and $\lim_{h \to 0^+} \frac{U(0+h)-U(0)}{h} = \lim_{h \to 0^+} \frac{1-1}{h} = 0 \Rightarrow$ no, the graph of f does not have a vertical tangent at (0, 1) because the limit does not exist.

37. (a) The graph appears to have a cusp at x = 0.



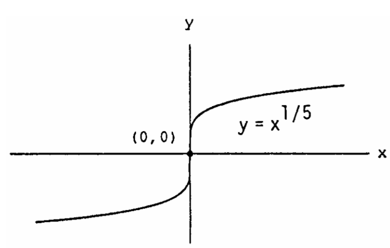
(b) $\lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^-} \frac{h^{2/5}-0}{h} = \lim_{h \to 0^-} \frac{1}{h^{3/5}} = -\infty$ and $\lim_{h \to 0^+} \frac{1}{h^{3/5}} = \infty \Rightarrow$ limit does not exist $\Rightarrow$ the graph of $y = x^{2/5}$ does not have a vertical tangent at x = 0.

38. (a) The graph appears to have a cusp at x = 0.



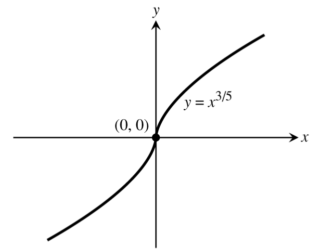
(b) $\lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^-} \frac{h^{4/5}-0}{h} = \lim_{h \to 0^-} \frac{1}{h^{1/5}} = -\infty$ and $\lim_{h \to 0^+} \frac{1}{h^{1/5}} = \infty \Rightarrow$ limit does not exist $\Rightarrow y = x^{4/5}$ does not have a vertical tangent at x = 0.

39. (a) The graph appears to have a vertical tangent at x = 0.



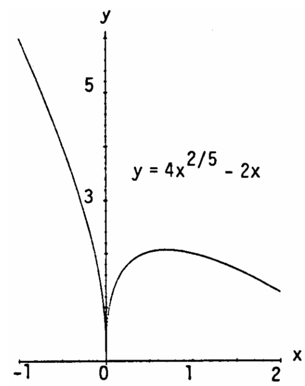
(b) $\lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0} \frac{h^{1/5}-0}{h} = \lim_{h \to 0} \frac{1}{h^{4/5}} = \infty \Rightarrow y = x^{1/5}$ has a vertical tangent at x = 0.

40. (a) The graph appears to have a vertical tangent at $x = 0$.



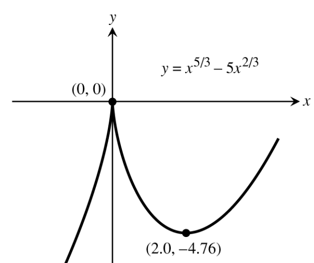
(b) $\lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0} \frac{h^{3/5}-0}{h} = \lim_{h \to 0} \frac{1}{h^{2/5}} = \infty \Rightarrow$ the graph of $y = x^{3/5}$ has a vertical tangent at $x = 0$.

41. (a) The graph appears to have a cusp at $x = 0$.



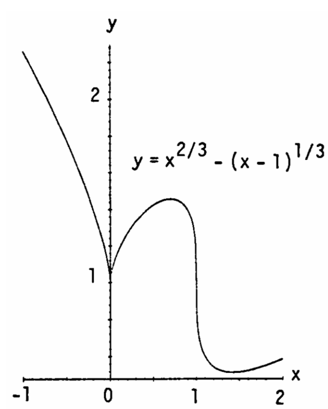
(b) $\lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^-} \frac{4h^{2/5}-2h}{h} = \lim_{h \to 0^-} \frac{4}{h^{3/5}} - 2 = -\infty$ and $\lim_{h \to 0^+} \frac{4}{h^{3/5}} - 2 = \infty$
$\Rightarrow$ limit does not exist $\Rightarrow$ the graph of $y = 4x^{2/5} - 2x$ does not have a vertical tangent at $x = 0$.

42. (a) The graph appears to have a cusp at $x = 0$.



(b) $\lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0} \frac{h^{5/3}-5h^{2/3}}{h} = \lim_{h \to 0} h^{2/3} - \frac{5}{h^{1/3}} = 0 - \lim_{h \to 0} \frac{5}{h^{1/3}}$ does not exist $\Rightarrow$ the graph of $y = x^{5/3} - 5x^{2/3}$ does not have a vertical tangent at $x = 0$.

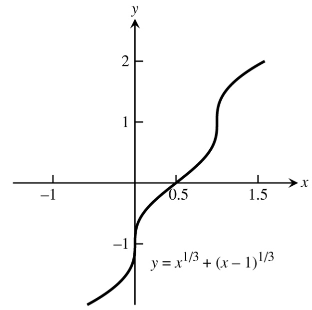
43. (a) The graph appears to have a vertical tangent at $x = 1$ and a cusp at $x = 0$.



(b) $x = 1$: $\lim_{h \to 0} \frac{(1+h)^{2/3}-(1+h-1)^{1/3}-1}{h} = \lim_{h \to 0} \frac{(1+h)^{2/3}-h^{1/3}-1}{h} = -\infty$
$\Rightarrow y = x^{2/3} - (x-1)^{1/3}$ has a vertical tangent at $x = 1$;

$x = 0$: $\lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \frac{h^{2/3} - (h-1)^{1/3} - (-1)^{1/3}}{h} = \lim\limits_{h \to 0} \left[\frac{1}{h^{1/3}} - \frac{(h-1)^{1/3}}{h} + \frac{1}{h}\right]$ does not exist $\Rightarrow$ $y = x^{2/3} - (x-1)^{1/3}$ does not have a vertical tangent at $x = 0$.

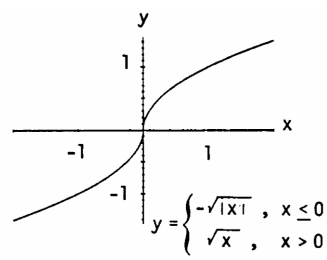
44. (a) The graph appears to have vertical tangents at $x = 0$ and $x = 1$.



(b) $x = 0$: $\lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \frac{h^{1/3} + (h-1)^{1/3} - (-1)^{1/3}}{h} = \infty \Rightarrow y = x^{1/3} + (x-1)^{1/3}$ has a vertical tangent at $x = 0$;

$x = 1$: $\lim\limits_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim\limits_{h \to 0} \frac{(1+h)^{1/3} + (1+h-1)^{1/3} - 1}{h} = \infty \Rightarrow y = x^{1/3} + (x-1)^{1/3}$ has a vertical tangent at $x = 1$.

45. (a) The graph appears to have a vertical tangent at $x = 0$.

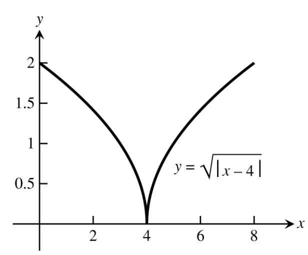


(b) $\lim\limits_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim\limits_{x \to 0^+} \frac{\sqrt{h} - 0}{h} = \lim\limits_{h \to 0} \frac{1}{\sqrt{h}} = \infty$;

$\lim\limits_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^-} \frac{-\sqrt{|h|} - 0}{h} = \lim\limits_{h \to 0^-} \frac{-\sqrt{|h|}}{-|h|} = \lim\limits_{h \to 0^-} \frac{1}{\sqrt{|h|}} = \infty$

$\Rightarrow$ y has a vertical tangent at $x = 0$.

46. (a) The graph appears to have a cusp at $x = 4$.



(b) $\lim\limits_{h \to 0^+} \frac{f(4+h) - f(4)}{h} = \lim\limits_{h \to 0^+} \frac{\sqrt{|4-(4+h)|} - 0}{h} = \lim\limits_{h \to 0^+} \frac{\sqrt{|h|}}{h} = \lim\limits_{h \to 0^+} \frac{1}{\sqrt{h}} = \infty$;

$\lim\limits_{h \to 0^-} \frac{f(4+h) - f(4)}{h} = \lim\limits_{h \to 0^-} \frac{\sqrt{|4-(4+h)|}}{h} = \lim\limits_{h \to 0^-} \frac{\sqrt{|h|}}{-|h|} = \lim\limits_{h \to 0^-} \frac{-1}{\sqrt{|h|}} = -\infty$

$\Rightarrow$ $y = \sqrt{4 - x}$ does not have a vertical tangent at $x = 4$.

47-50. Example CAS commands:

<u>Maple</u>:

```
f := x -> x^3 + 2*x;x0 := 0;
plot( f(x), x=x0-1/2..x0+3, color=black,                 # part (a)
      title="Section 3.1, #47(a)" );
q := unapply( (f(x0+h)-f(x0))/h, h );                    # part (b)
```

```
L := limit( q(h), h=0 );                                # part (c)
sec_lines := seq( f(x0)+q(h)*(x-x0), h=1..3 );          # part (d)
tan_line := f(x0) + L*(x-x0);
plot( [f(x),tan_line,sec_lines], x=x0-1/2..x0+3, color=black,
      linestyle=[1,2,5,6,7], title="Section 3.1, #47(d)",
      legend=["y=f(x)","Tangent line at x=0","Secant line (h=1)",
              "Secant line (h=2)","Secant line (h=3)"] );
```

Mathematica: (function and value for x0 may change)

```
Clear[f, m, x, h]
x0 = p;
f[x_]: = Cos[x] + 4Sin[2x]
Plot[f[x], {x, x0 − 1, x0 + 3}]
dq[h_]: = (f[x0+h] − f[x0])/h
m = Limit[dq[h], h → 0]
ytan: = f[x0] + m(x − x0)
y1: = f[x0] + dq[1](x − x0)
y2: = f[x0] + dq[2](x − x0)
y3: = f[x0] + dq[3](x − x0)
Plot[{f[x], ytan, y1, y2, y3}, {x, x0 − 1, x0 + 3}]
```

3.2 THE DERIVATIVE AS A FUNCTION

1. Step 1: $f(x) = 4 - x^2$ and $f(x+h) = 4 - (x+h)^2$

   Step 2: $\frac{f(x+h)-f(x)}{h} = \frac{[4-(x+h)^2]-(4-x^2)}{h} = \frac{(4-x^2-2xh-h^2)-4+x^2}{h} = \frac{-2xh-h^2}{h} = \frac{h(-2x-h)}{h}$
   $= -2x - h$

   Step 3: $f'(x) = \lim\limits_{h\to 0}(-2x-h) = -2x$; $f'(-3) = 6$, $f'(0) = 0$, $f'(1) = -2$

2. $F(x) = (x-1)^2 + 1$ and $F(x+h) = (x+h-1)^2 + 1 \Rightarrow F'(x) = \lim\limits_{h\to 0} \frac{[(x+h-1)^2+1]-[(x-1)^2+1]}{h}$
   $= \lim\limits_{h\to 0} \frac{(x^2+2xh+h^2-2x-2h+1+1)-(x^2-2x+1+1)}{h} = \lim\limits_{h\to 0} \frac{2xh+h^2-2h}{h} = \lim\limits_{h\to 0}(2x+h-2)$
   $= 2(x-1)$; $F'(-1) = -4$, $F'(0) = -2$, $F'(2) = 2$

3. Step 1: $g(t) = \frac{1}{t^2}$ and $g(t+h) = \frac{1}{(t+h)^2}$

   Step 2: $\frac{g(t+h)-g(t)}{h} = \frac{\frac{1}{(t+h)^2}-\frac{1}{t^2}}{h} = \frac{\left(\frac{t^2-(t+h)^2}{(t+h)^2\cdot t^2}\right)}{h} = \frac{t^2-(t^2+2th+h^2)}{(t+h)^2\cdot t^2\cdot h} = \frac{-2th-h^2}{(t+h)^2\,t^2 h}$
   $= \frac{h(-2t-h)}{(t+h)^2\,t^2 h} = \frac{-2t-h}{(t+h)^2\,t^2}$

   Step 3: $g'(t) = \lim\limits_{h\to 0} \frac{-2t-h}{(t+h)^2\,t^2} = \frac{-2t}{t^2\cdot t^2} = \frac{-2}{t^3}$; $g'(-1) = 2$, $g'(2) = -\frac{1}{4}$, $g'\left(\sqrt{3}\right) = -\frac{2}{3\sqrt{3}}$

4. $k(z) = \frac{1-z}{2z}$ and $k(z+h) = \frac{1-(z+h)}{2(z+h)} \Rightarrow k'(z) = \lim\limits_{h\to 0} \frac{\left(\frac{1-(z+h)}{2(z+h)} - \frac{1-z}{2z}\right)}{h}$
   $= \lim\limits_{h\to 0} \frac{(1-z-h)z-(1-z)(z+h)}{2(z+h)zh} = \lim\limits_{h\to 0} \frac{z-z^2-zh-z-h+z^2+zh}{2(z+h)zh} = \lim\limits_{h\to 0} \frac{-h}{2(z+h)zh} = \lim\limits_{h\to 0} \frac{-1}{2(z+h)z}$
   $= \frac{-1}{2z^2}$; $k'(-1) = -\frac{1}{2}$, $k'(1) = -\frac{1}{2}$, $k'\left(\sqrt{2}\right) = -\frac{1}{4}$

5. Step 1: $p(\theta) = \sqrt{3\theta}$ and $p(\theta+h) = \sqrt{3(\theta+h)}$

Step 2: $\frac{p(\theta+h)-p(\theta)}{h} = \frac{\sqrt{3(\theta+h)}-\sqrt{3\theta}}{h} = \frac{\left(\sqrt{3\theta+3h}-\sqrt{3\theta}\right)}{h} \cdot \frac{\left(\sqrt{3\theta+3h}+\sqrt{3\theta}\right)}{\left(\sqrt{3\theta+3h}+\sqrt{3\theta}\right)} = \frac{(3\theta+3h)-3\theta}{h\left(\sqrt{3\theta+3h}+\sqrt{3\theta}\right)}$

$= \frac{3h}{h\left(\sqrt{3\theta+3h}+\sqrt{3\theta}\right)} = \frac{3}{\sqrt{3\theta+3h}+\sqrt{3\theta}}$

Step 3: $p'(\theta) = \lim\limits_{h\to 0} \frac{3}{\sqrt{3\theta+3h}+\sqrt{3\theta}} = \frac{3}{\sqrt{3\theta}+\sqrt{3\theta}} = \frac{3}{2\sqrt{3\theta}}$; $p'(1) = \frac{3}{2\sqrt{3}}$, $p'(3) = \frac{1}{2}$, $p'\left(\frac{2}{3}\right) = \frac{3}{2\sqrt{2}}$

6. $r(s) = \sqrt{2s+1}$ and $r(s+h) = \sqrt{2(s+h)+1} \Rightarrow r'(s) = \lim\limits_{h\to 0} \frac{\sqrt{2s+2h+1}-\sqrt{2s+1}}{h}$

$= \lim\limits_{h\to 0} \frac{\left(\sqrt{2s+h+1}-\sqrt{2s+1}\right)}{h} \cdot \frac{\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)}{\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)} = \lim\limits_{h\to 0} \frac{(2s+2h+1)-(2s+1)}{h\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)}$

$= \lim\limits_{h\to 0} \frac{2h}{h\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)} = \lim\limits_{h\to 0} \frac{2}{\sqrt{2s+2h+1}+\sqrt{2s+1}} = \frac{2}{\sqrt{2s+1}+\sqrt{2s+1}} = \frac{2}{2\sqrt{2s+1}}$

$= \frac{1}{\sqrt{2s+1}}$; $r'(0) = 1$, $r'(1) = \frac{1}{\sqrt{3}}$, $r'\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}$

7. $y = f(x) = 2x^3$ and $f(x+h) = 2(x+h)^3 \Rightarrow \frac{dy}{dx} = \lim\limits_{h\to 0} \frac{2(x+h)^3-2x^3}{h} = \lim\limits_{h\to 0} \frac{2\left(x^3+3x^2h+3xh^2+h^3\right)-2x^3}{h}$

$= \lim\limits_{h\to 0} \frac{6x^2h+6xh^2+2h^3}{h} = \lim\limits_{h\to 0} \frac{h\left(6x^2+6xh+2h^2\right)}{h} = \lim\limits_{h\to 0} \left(6x^2+6xh+2h^2\right) = 6x^2$

8. $r = s^3-2s^2+3 \Rightarrow \frac{dr}{ds} = \lim\limits_{h\to 0} \frac{\left((s+h)^3-2(s+h)^2+3\right)-\left(s^3-2s^2+3\right)}{h} = \lim\limits_{h\to 0} \frac{s^3+3s^2h+3sh^2+h^3-2s^2-4sh-h^2+3-s^3+2s^2-3}{h}$

$= \lim\limits_{h\to 0} \frac{3s^2h+3sh^2+h^3-4sh-h^2}{h} = \lim\limits_{h\to 0} \frac{h\left(3s^2+3sh+h^2-4s-h\right)}{h} = \lim\limits_{h\to 0} \left(3s^2+3sh+h^2-4s-h\right) = 3s^2-2s$

9. $s = r(t) = \frac{t}{2t+1}$ and $r(t+h) = \frac{t+h}{2(t+h)+1} \Rightarrow \frac{ds}{dt} = \lim\limits_{h\to 0} \frac{\left(\frac{t+h}{2(t+h)+1}\right)-\left(\frac{t}{2t+1}\right)}{h}$

$= \lim\limits_{h\to 0} \frac{\left(\frac{(t+h)(2t+1)-t(2t+2h+1)}{(2t+2h+1)(2t+1)}\right)}{h} = \lim\limits_{h\to 0} \frac{(t+h)(2t+1)-t(2t+2h+1)}{(2t+2h+1)(2t+1)h}$

$= \lim\limits_{h\to 0} \frac{2t^2+t+2ht+h-2t^2-2ht-t}{(2t+2h+1)(2t+1)h} = \lim\limits_{h\to 0} \frac{h}{(2t+2h+1)(2t+1)h} = \lim\limits_{h\to 0} \frac{1}{(2t+2h+1)(2t+1)}$

$= \frac{1}{(2t+1)(2t+1)} = \frac{1}{(2t+1)^2}$

10. $\frac{dv}{dt} = \lim\limits_{h\to 0} \frac{\left[(t+h)-\frac{1}{t+h}\right]-\left(t-\frac{1}{t}\right)}{h} = \lim\limits_{h\to 0} \frac{h-\frac{1}{t+h}+\frac{1}{t}}{h} = \lim\limits_{h\to 0} \frac{\left(\frac{h(t+h)t-t+(t+h)}{(t+h)t}\right)}{h}$

$= \lim\limits_{h\to 0} \frac{ht^2+h^2t+h}{h(t+h)t} = \lim\limits_{h\to 0} \frac{t^2+ht+1}{(t+h)t} = \frac{t^2+1}{t^2} = 1+\frac{1}{t^2}$

11. $p = f(q) = \frac{1}{\sqrt{q+1}}$ and $f(q+h) = \frac{1}{\sqrt{(q+h)+1}} \Rightarrow \frac{dp}{dq} = \lim\limits_{h\to 0} \frac{\left(\frac{1}{\sqrt{(q+h)+1}}\right)-\left(\frac{1}{\sqrt{q+1}}\right)}{h}$

$= \lim\limits_{h\to 0} \frac{\left(\frac{\sqrt{q+1}-\sqrt{q+h+1}}{\sqrt{q+h+1}\,\sqrt{q+1}}\right)}{h} = \lim\limits_{h\to 0} \frac{\sqrt{q+1}-\sqrt{q+h+1}}{h\sqrt{q+h+1}\,\sqrt{q+1}}$

$= \lim\limits_{h\to 0} \frac{\left(\sqrt{q+1}-\sqrt{q+h+1}\right)}{h\sqrt{q+h+1}\,\sqrt{q+1}} \cdot \frac{\left(\sqrt{q+1}+\sqrt{q+h+1}\right)}{\left(\sqrt{q+1}+\sqrt{q+h+1}\right)} = \lim\limits_{h\to 0} \frac{(q+1)-(q+h+1)}{h\sqrt{q+h+1}\,\sqrt{q+1}\left(\sqrt{q+1}+\sqrt{q+h+1}\right)}$

$= \lim\limits_{h\to 0} \frac{-h}{h\sqrt{q+h+1}\,\sqrt{q+1}\left(\sqrt{q+1}+\sqrt{q+h+1}\right)} = \lim\limits_{h\to 0} \frac{-1}{\sqrt{q+h+1}\,\sqrt{q+1}\left(\sqrt{q+1}+\sqrt{q+h+1}\right)}$

$= \frac{-1}{\sqrt{q+1}\,\sqrt{q+1}\left(\sqrt{q+1}+\sqrt{q+1}\right)} = \frac{-1}{2(q+1)\sqrt{q+1}}$

12. $\frac{dz}{dw} = \lim\limits_{h\to 0} \frac{\left(\frac{1}{\sqrt{3(w+h)-2}}-\frac{1}{\sqrt{3w-2}}\right)}{h} = \lim\limits_{h\to 0} \frac{\sqrt{3w-2}-\sqrt{3w+3h-2}}{h\sqrt{3w+3h-2}\,\sqrt{3w-2}}$

$= \lim\limits_{h\to 0} \frac{\left(\sqrt{3w-2}-\sqrt{3w+3h-2}\right)}{h\sqrt{3w+3h-2}\,\sqrt{3w-2}} \cdot \frac{\left(\sqrt{3w-2}+\sqrt{3w+3h-2}\right)}{\left(\sqrt{3w-2}+\sqrt{3w+3h-2}\right)} = \lim\limits_{h\to 0} \frac{(3w-2)-(3w+3h-2)}{h\sqrt{3w+3h-2}\,\sqrt{3w-2}\left(\sqrt{3w-2}+\sqrt{3w+3h-2}\right)}$

$= \lim\limits_{h\to 0} \frac{-3}{\sqrt{3w+3h-2}\,\sqrt{3w-2}\left(\sqrt{3w-2}+\sqrt{3w+3h-2}\right)} = \frac{-3}{\sqrt{3w-2}\,\sqrt{3w-2}\left(\sqrt{3w-2}+\sqrt{3w-2}\right)}$

$= \frac{-3}{2(3w-2)\sqrt{3w-2}}$

13. $f(x) = x + \frac{9}{x}$ and $f(x+h) = (x+h) + \frac{9}{(x+h)} \Rightarrow \frac{f(x+h)-f(x)}{h} = \frac{\left[(x+h)+\frac{9}{(x+h)}\right] - \left[x+\frac{9}{x}\right]}{h}$
$= \frac{x(x+h)^2 + 9x - x^2(x+h) - 9(x+h)}{x(x+h)h} = \frac{x^3 + 2x^2h + xh^2 + 9x - x^3 - x^2h - 9x - 9h}{x(x+h)h} = \frac{x^2h + xh^2 - 9h}{x(x+h)h}$
$= \frac{h(x^2+xh-9)}{x(x+h)h} = \frac{x^2+xh-9}{x(x+h)}$; $f'(x) = \lim\limits_{h\to 0} \frac{x^2+xh-9}{x(x+h)} = \frac{x^2-9}{x^2} = 1 - \frac{9}{x^2}$; $m = f'(-3) = 0$

14. $k(x) = \frac{1}{2+x}$ and $k(x+h) = \frac{1}{2+(x+h)} \Rightarrow k'(x) = \lim\limits_{h\to 0} \frac{k(x+h)-k(x)}{h} = \lim\limits_{h\to 0} \frac{\left(\frac{1}{2+x+h} - \frac{1}{2+x}\right)}{h}$
$= \lim\limits_{h\to 0} \frac{(2+x)-(2+x+h)}{h(2+x)(2+x+h)} = \lim\limits_{h\to 0} \frac{-h}{h(2+x)(2+x+h)} = \lim\limits_{h\to 0} \frac{-1}{(2+x)(2+x+h)} = \frac{-1}{(2+x)^2}$;
$k'(2) = -\frac{1}{16}$

15. $\frac{ds}{dt} = \lim\limits_{h\to 0} \frac{[(t+h)^3 - (t+h)^2] - (t^3 - t^2)}{h} = \lim\limits_{h\to 0} \frac{(t^3 + 3t^2h + 3th^2 + h^3) - (t^2 + 2th + h^2) - t^3 + t^2}{h}$
$= \lim\limits_{h\to 0} \frac{3t^2h + 3th^2 + h^3 - 2th - h^2}{h} = \lim\limits_{h\to 0} \frac{h(3t^2 + 3th + h^2 - 2t - h)}{h} = \lim\limits_{h\to 0} (3t^2 + 3th + h^2 - 2t - h)$
$= 3t^2 - 2t$; $m = \frac{ds}{dt}\Big|_{t=-1} = 5$

16. $\frac{dy}{dx} = \lim\limits_{h\to 0} \frac{\frac{(x+h)+3}{1-(x+h)} - \frac{x+3}{1-x}}{h} = \lim\limits_{h\to 0} \frac{\frac{(x+h+3)(1-x)-(x+3)(1-x-h)}{(1-x-h)(1-x)}}{h} = \lim\limits_{h\to 0} \frac{x+h+3-x^2-xh-3x-x-3+x^2+3x+xh+3h}{h(1-x-h)(1-x)}$
$= \lim\limits_{h\to 0} \frac{4h}{h(1-x-h)(1-x)} = \lim\limits_{h\to 0} \frac{4}{(1-x-h)(1-x)} = \frac{4}{(1-x)^2}$; $\frac{dy}{dx}\Big|_{x=-2} = \frac{4}{(3)^2} = \frac{4}{9}$

17. $f(x) = \frac{8}{\sqrt{x-2}}$ and $f(x+h) = \frac{8}{\sqrt{(x+h)-2}} \Rightarrow \frac{f(x+h)-f(x)}{h} = \frac{\frac{8}{\sqrt{(x+h)-2}} - \frac{8}{\sqrt{x-2}}}{h}$
$= \frac{8\left(\sqrt{x-2} - \sqrt{x+h-2}\right)}{h\sqrt{x+h-2}\sqrt{x-2}} \cdot \frac{\left(\sqrt{x-2}+\sqrt{x+h-2}\right)}{\left(\sqrt{x-2}+\sqrt{x+h-2}\right)} = \frac{8[(x-2)-(x+h-2)]}{h\sqrt{x+h-2}\sqrt{x-2}\left(\sqrt{x-2}+\sqrt{x+h-2}\right)}$
$= \frac{-8h}{h\sqrt{x+h-2}\sqrt{x-2}\left(\sqrt{x-2}+\sqrt{x+h-2}\right)} \Rightarrow f'(x) = \lim\limits_{h\to 0} \frac{-8}{\sqrt{x+h-2}\sqrt{x-2}\left(\sqrt{x-2}+\sqrt{x+h-2}\right)}$
$= \frac{-8}{\sqrt{x-2}\sqrt{x-2}\left(\sqrt{x-2}+\sqrt{x-2}\right)} = \frac{-4}{(x-2)\sqrt{x-2}}$; $m = f'(6) = \frac{-4}{4\sqrt{4}} = -\frac{1}{2} \Rightarrow$ the equation of the tangent line at $(6,4)$ is $y - 4 = -\frac{1}{2}(x-6) \Rightarrow y = -\frac{1}{2}x + 3 + 4 \Rightarrow y = -\frac{1}{2}x + 7$.

18. $g'(z) = \lim\limits_{h\to 0} \frac{\left(1+\sqrt{4-(z+h)}\right) - \left(1+\sqrt{4-z}\right)}{h} = \lim\limits_{h\to 0} \frac{\left(\sqrt{4-z-h} - \sqrt{4-z}\right)}{h} \cdot \frac{\left(\sqrt{4-z-h}+\sqrt{4-z}\right)}{\left(\sqrt{4-z-h}+\sqrt{4-z}\right)}$
$= \lim\limits_{h\to 0} \frac{(4-z-h)-(4-z)}{h\left(\sqrt{4-z-h}+\sqrt{4-z}\right)} = \lim\limits_{h\to 0} \frac{-h}{h\left(\sqrt{4-z-h}+\sqrt{4-z}\right)} = \lim\limits_{h\to 0} \frac{-1}{\left(\sqrt{4-z-h}+\sqrt{4-z}\right)} = \frac{-1}{2\sqrt{4-z}}$;
$m = g'(3) = \frac{-1}{2\sqrt{4-3}} = -\frac{1}{2} \Rightarrow$ the equation of the tangent line at $(3,2)$ is $w - 2 = -\frac{1}{2}(z-3)$
$\Rightarrow w = -\frac{1}{2}z + \frac{3}{2} + 2 \Rightarrow w = -\frac{1}{2}z + \frac{7}{2}$.

19. $s = f(t) = 1 - 3t^2$ and $f(t+h) = 1 - 3(t+h)^2 = 1 - 3t^2 - 6th - 3h^2 \Rightarrow \frac{ds}{dt} = \lim\limits_{h\to 0} \frac{f(t+h)-f(t)}{h}$
$= \lim\limits_{h\to 0} \frac{(1-3t^2-6th-3h^2)-(1-3t^2)}{h} = \lim\limits_{h\to 0} (-6t-3h) = -6t \Rightarrow \frac{ds}{dt}\Big|_{t=-1} = 6$

20. $y = f(x) = 1 - \frac{1}{x}$ and $f(x+h) = 1 - \frac{1}{x+h} \Rightarrow \frac{dy}{dx} = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \frac{\left(1-\frac{1}{x+h}\right)-\left(1-\frac{1}{x}\right)}{h}$
$= \lim\limits_{h\to 0} \frac{\frac{1}{x}-\frac{1}{x+h}}{h} = \lim\limits_{h\to 0} \frac{h}{x(x+h)h} = \lim\limits_{h\to 0} \frac{1}{x(x+h)} = \frac{1}{x^2} \Rightarrow \frac{dy}{dx}\Big|_{x=\sqrt{3}} = \frac{1}{3}$

21. $r = f(\theta) = \frac{2}{\sqrt{4-\theta}}$ and $f(\theta+h) = \frac{2}{\sqrt{4-(\theta+h)}} \Rightarrow \frac{dr}{d\theta} = \lim\limits_{h\to 0} \frac{f(\theta+h)-f(\theta)}{h} = \lim\limits_{h\to 0} \frac{\frac{2}{\sqrt{4-\theta-h}} - \frac{2}{\sqrt{4-\theta}}}{h}$
$= \lim\limits_{h\to 0} \frac{2\sqrt{4-\theta}-2\sqrt{4-\theta-h}}{h\sqrt{4-\theta}\sqrt{4-\theta-h}} = \lim\limits_{h\to 0} \frac{2\sqrt{4-\theta}-2\sqrt{4-\theta-h}}{h\sqrt{4-\theta}\sqrt{4-\theta-h}} \cdot \frac{\left(2\sqrt{4-\theta}+2\sqrt{4-\theta-h}\right)}{\left(2\sqrt{4-\theta}+2\sqrt{4-\theta-h}\right)}$

$= \lim\limits_{h \to 0} \frac{4(4-\theta) - 4(4-\theta-h)}{2h\sqrt{4-\theta}\sqrt{4-\theta-h}\left(\sqrt{4-\theta}+\sqrt{4-\theta-h}\right)} = \lim\limits_{h \to 0} \frac{2}{\sqrt{4-\theta}\sqrt{4-\theta-h}\left(\sqrt{4-\theta}+\sqrt{4-\theta-h}\right)}$

$= \frac{2}{(4-\theta)\left(2\sqrt{4-\theta}\right)} = \frac{1}{(4-\theta)\sqrt{4-\theta}} \Rightarrow \frac{dr}{d\theta}\Big|_{\theta=0} = \frac{1}{8}$

22. $w = f(z) = z + \sqrt{z}$ and $f(z+h) = (z+h) + \sqrt{z+h} \Rightarrow \frac{dw}{dz} = \lim\limits_{h \to 0} \frac{f(z+h) - f(z)}{h}$

$= \lim\limits_{h \to 0} \frac{\left(z+h+\sqrt{z+h}\right) - \left(z+\sqrt{z}\right)}{h} = \lim\limits_{h \to 0} \frac{h+\sqrt{z+h}-\sqrt{z}}{h} = \lim\limits_{h \to 0} \left[1 + \frac{\sqrt{z+h}-\sqrt{z}}{h} \cdot \frac{\left(\sqrt{z+h}+\sqrt{z}\right)}{\left(\sqrt{z+h}+\sqrt{z}\right)}\right]$

$= 1 + \lim\limits_{h \to 0} \frac{(z+h) - z}{h\left(\sqrt{z+h}+\sqrt{z}\right)} = 1 + \lim\limits_{h \to 0} \frac{1}{\sqrt{z+h}+\sqrt{z}} = 1 + \frac{1}{2\sqrt{z}} \Rightarrow \frac{dw}{dz}\Big|_{z=4} = \frac{5}{4}$

23. $f'(x) = \lim\limits_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim\limits_{z \to x} \frac{\frac{1}{z+2} - \frac{1}{x+2}}{z - x} = \lim\limits_{z \to x} \frac{(x+2) - (z+2)}{(z-x)(z+2)(x+2)} = \lim\limits_{z \to x} \frac{x - z}{(z-x)(z+2)(x+2)} = \lim\limits_{z \to x} \frac{-1}{(z+2)(x+2)} = \frac{-1}{(x+2)^2}$

24. $f'(x) = \lim\limits_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim\limits_{z \to x} \frac{\left(z^2 - 3z + 4\right) - \left(x^2 - 3x + 4\right)}{z - x} = \lim\limits_{z \to x} \frac{z^2 - 3z - x^2 + 3x}{z - x} = \lim\limits_{z \to x} \frac{z^2 - x^2 - 3z + 3x}{z - x}$

$= \lim\limits_{z \to x} \frac{(z-x)(z+x) - 3(z-x)}{z - x} = \lim\limits_{z \to x} \frac{(z-x)\left[(z+x) - 3\right]}{z - x} = \lim\limits_{z \to x} \left[(z + x) - 3\right] = 2x - 3$

25. $g'(x) = \lim\limits_{z \to x} \frac{g(z) - g(x)}{z - x} = \lim\limits_{z \to x} \frac{\frac{z}{z-1} - \frac{x}{x-1}}{z - x} = \lim\limits_{z \to x} \frac{z(x-1) - x(z-1)}{(z-x)(z-1)(x-1)} = \lim\limits_{z \to x} \frac{-z + x}{(z-x)(z-1)(x-1)} = \lim\limits_{z \to x} \frac{-1}{(z-1)(x-1)} = \frac{-1}{(x-1)^2}$

26. $g'(x) = \lim\limits_{z \to x} \frac{g(z) - g(x)}{z - x} = \lim\limits_{z \to x} \frac{(1+\sqrt{z}) - (1+\sqrt{x})}{z - x} = \lim\limits_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \cdot \frac{\sqrt{z} + \sqrt{x}}{\sqrt{z} + \sqrt{x}} = \lim\limits_{z \to x} \frac{z - x}{(z-x)\left(\sqrt{z}+\sqrt{x}\right)} = \lim\limits_{z \to x} \frac{1}{\sqrt{z}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$

27. Note that as x increases, the slope of the tangent line to the curve is first negative, then zero (when $x = 0$), then positive $\Rightarrow$ the slope is always increasing which matches (b).

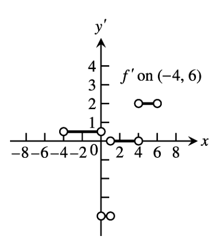
28. Note that the slope of the tangent line is never negative. For x negative, $f_2'(x)$ is positive but decreasing as x increases. When $x = 0$, the slope of the tangent line to x is 0. For $x > 0$, $f_2'(x)$ is positive and increasing. This graph matches (a).

29. $f_3(x)$ is an oscillating function like the cosine. Everywhere that the graph of f_3 has a horizontal tangent we expect f_3' to be zero, and (d) matches this condition.

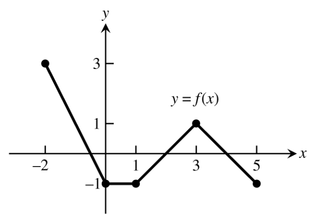
30. The graph matches with (c).

31. (a) f' is not defined at $x = 0, 1, 4$. At these points, the left-hand and right-hand derivatives do not agree. For example, $\lim\limits_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$ = slope of line joining $(-4, 0)$ and $(0, 2) = \frac{1}{2}$ but $\lim\limits_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$ = slope of line joining $(0, 2)$ and $(1, -2) = -4$. Since these values are not equal, $f'(0) = \lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ does not exist.

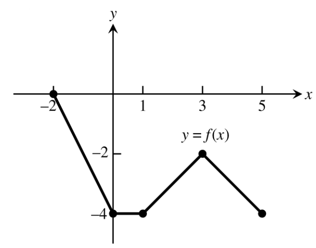
(b)



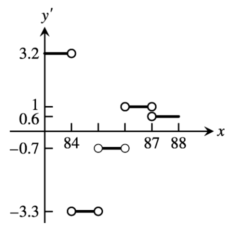
32. (a)



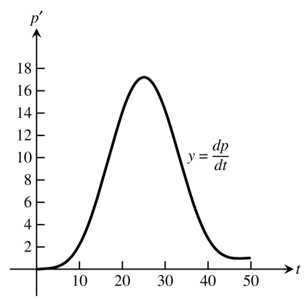
(b) Shift the graph in (a) down 3 units



33.



34. (a)



(b) The fastest is between the 20th and 30th days; slowest is between the 40th and 50th days.

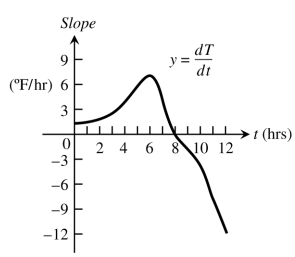
35. Answers may vary. In each case, draw a tangent line and estimate its slope.

(a) i) slope $\approx 1.54 \Rightarrow \frac{dT}{dt} \approx 1.54° \frac{F}{hr}$ ii) slope $\approx 2.86 \Rightarrow \frac{dT}{dt} \approx 2.86° \frac{F}{hr}$

iii) slope $\approx 0 \Rightarrow \frac{dT}{dt} \approx 0° \frac{F}{hr}$ iv) slope $\approx -3.75 \Rightarrow \frac{dT}{dt} \approx -3.75° \frac{F}{hr}$

(b) The tangent with the steepest positive slope appears to occur at $t = 6 \Rightarrow$ 12 p.m. and slope $\approx 7.27 \Rightarrow \frac{dT}{dt} \approx 7.27° \frac{F}{hr}$. The tangent with the steepest negative slope appears to occur at $t = 12 \Rightarrow$ 6 p.m. and slope $\approx -8.00 \Rightarrow \frac{dT}{dt} \approx -8.00° \frac{F}{hr}$

(c)



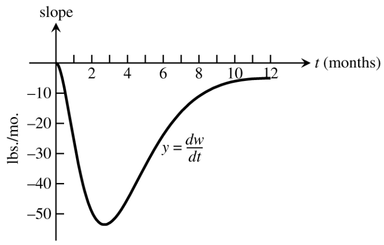
36. Answers may vary. In each case, draw a tangent line and estimate the slope.

(a) i) slope $\approx -20.83 \Rightarrow \frac{dW}{dt} \approx -20.83 \frac{lb}{month}$ ii) slope $\approx -35.00 \Rightarrow \frac{dW}{dt} \approx -35.00 \frac{lb}{month}$

iii) slope $\approx -6.25 \Rightarrow \frac{dW}{dt} \approx -6.25 \frac{lb}{month}$

(b) The tangentwith the steepest positive slope appears to occur at $t = 2.7$ months. and slope ≈ 7.27 $\Rightarrow \frac{dW}{dt} \approx -53.13 \frac{lb}{month}$

(c)



37. Left-hand derivative: For $h < 0$, $f(0+h) = f(h) = h^2$ (using $y = x^2$ curve) $\Rightarrow \lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h}$
$= \lim_{h \to 0^-} \frac{h^2 - 0}{h} = \lim_{h \to 0^-} h = 0$;
Right-hand derivative: For $h > 0$, $f(0+h) = f(h) = h$ (using $y = x$ curve) $\Rightarrow \lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h}$
$= \lim_{h \to 0^+} \frac{h-0}{h} = \lim_{h \to 0^+} 1 = 1$;
Then $\lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h} \neq \lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h} \Rightarrow$ the derivative $f'(0)$ does not exist.

38. Left-hand derivative: When $h < 0$, $1 + h < 1 \Rightarrow f(1+h) = 2 \Rightarrow \lim_{h \to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0^-} \frac{2-2}{h}$
$= \lim_{h \to 0^-} 0 = 0$;
Right-hand derivative: When $h > 0$, $1 + h > 1 \Rightarrow f(1+h) = 2(1+h) = 2 + 2h \Rightarrow \lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h}$
$= \lim_{h \to 0^+} \frac{(2+2h)-2}{h} = \lim_{h \to 0^+} \frac{2h}{h} = \lim_{h \to 0^+} 2 = 2$;
Then $\lim_{h \to 0^-} \frac{f(1+h)-f(1)}{h} \neq \lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h} \Rightarrow$ the derivative $f'(1)$ does not exist.

39. Left-hand derivative: When $h < 0$, $1 + h < 1 \Rightarrow f(1+h) = \sqrt{1+h} \Rightarrow \lim_{h \to 0^-} \frac{f(1+h)-f(1)}{h}$
$= \lim_{h \to 0^-} \frac{\sqrt{1+h}-1}{h} = \lim_{h \to 0^-} \frac{\left(\sqrt{1+h}-1\right)}{h} \cdot \frac{\left(\sqrt{1+h}+1\right)}{\left(\sqrt{1+h}+1\right)} = \lim_{h \to 0^-} \frac{(1+h)-1}{h\left(\sqrt{1+h}+1\right)} = \lim_{h \to 0^-} \frac{1}{\sqrt{1+h}+1} = \frac{1}{2}$;
Right-hand derivative: When $h > 0$, $1 + h > 1 \Rightarrow f(1+h) = 2(1+h) - 1 = 2h + 1 \Rightarrow \lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h}$
$= \lim_{h \to 0^+} \frac{(2h+1)-1}{h} = \lim_{h \to 0^+} 2 = 2$;
Then $\lim_{h \to 0^-} \frac{f(1+h)-f(1)}{h} \neq \lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h} \Rightarrow$ the derivative $f'(1)$ does not exist.

40. Left-hand derivative: $\lim_{h \to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0^-} \frac{(1+h)-1}{h} = \lim_{h \to 0^-} 1 = 1$;
Right-hand derivative: $\lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0^+} \frac{\left(\frac{1}{1+h}-1\right)}{h} = \lim_{h \to 0^+} \frac{\left(\frac{1-(1+h)}{1+h}\right)}{h}$
$= \lim_{h \to 0^+} \frac{-h}{h(1+h)} = \lim_{h \to 0^+} \frac{-1}{1+h} = -1$;
Then $\lim_{h \to 0^-} \frac{f(1+h)-f(1)}{h} \neq \lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h} \Rightarrow$ the derivative $f'(1)$ does not exist.

41. f is not continuous at $x = 0$ since $\lim_{x \to 0} f(x) =$ does not exist and $f(0) = -1$

42. Left-hand derivative: $\lim_{h \to 0^-} \frac{g(h)-g(0)}{h} = \lim_{h \to 0^-} \frac{h^{1/3}-0}{h} = \lim_{h \to 0^-} \frac{1}{h^{2/3}} = +\infty$;
Right-hand derivative: $\lim_{h \to 0^+} \frac{g(h)-g(0)}{h} = \lim_{h \to 0^+} \frac{h^{2/3}-0}{h} = \lim_{h \to 0^+} \frac{1}{h^{1/3}} = +\infty$;
Then $\lim_{h \to 0^-} \frac{g(h)-g(0)}{h} = \lim_{h \to 0^+} \frac{g(h)-g(0)}{h} = +\infty \Rightarrow$ the derivative $g'(0)$ does not exist.

43. (a) The function is differentiable on its domain $-3 \le x \le 2$ (it is smooth)
(b) none
(c) none

44. (a) The function is differentiable on its domain $-2 \le x \le 3$ (it is smooth)
(b) none
(c) none

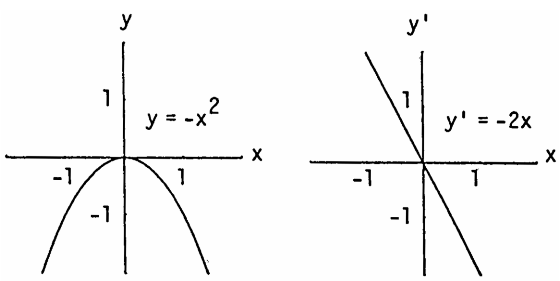
45. (a) The function is differentiable on $-3 \le x < 0$ and $0 < x \le 3$
(b) none
(c) The function is neither continuous nor differentiable at $x = 0$ since $\lim\limits_{x \to 0^-} f(x) \ne \lim\limits_{x \to 0^+} f(x)$

46. (a) f is differentiable on $-2 \le x < -1$, $-1 < x < 0$, $0 < x < 2$, and $2 < x \le 3$
(b) f is continuous but not differentiable at $x = -1$: $\lim\limits_{x \to -1} f(x) = 0$ exists but there is a corner at $x = -1$ since $\lim\limits_{h \to 0^-} \frac{f(-1+h) - f(-1)}{h} = -3$ and $\lim\limits_{h \to 0^+} \frac{f(-1+h) - f(-1)}{h} = 3 \Rightarrow f'(-1)$ does not exist
(c) f is neither continuous nor differentiable at $x = 0$ and $x = 2$:
at $x = 0$, $\lim\limits_{x \to 0^-} f(x) = 3$ but $\lim\limits_{x \to 0^+} f(x) = 0 \Rightarrow \lim\limits_{x \to 0} f(x)$ does not exist;
at $x = 2$, $\lim\limits_{x \to 2} f(x)$ exists but $\lim\limits_{x \to 2} f(x) \ne f(2)$

47. (a) f is differentiable on $-1 \le x < 0$ and $0 < x \le 2$
(b) f is continuous but not differentiable at $x = 0$: $\lim\limits_{x \to 0} f(x) = 0$ exists but there is a cusp at $x = 0$, so $f'(0) = \lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist
(c) none

48. (a) f is differentiable on $-3 \le x < -2$, $-2 < x < 2$, and $2 < x \le 3$
(b) f is continuous but not differentiable at $x = -2$ and $x = 2$: there are corners at those points
(c) none

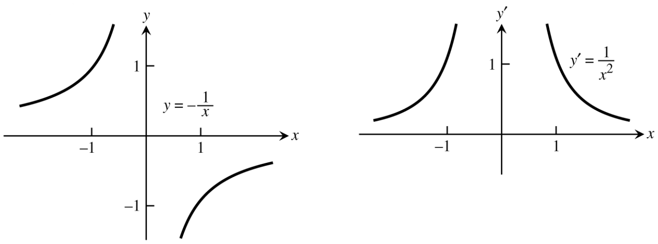
49. (a) $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \frac{-(x+h)^2 - (-x^2)}{h} = \lim\limits_{h \to 0} \frac{-x^2 - 2xh - h^2 + x^2}{h} = \lim\limits_{h \to 0} (-2x - h) = -2x$
(b)



(c) $y' = -2x$ is positive for $x < 0$, y' is zero when $x = 0$, y' is negative when $x > 0$
(d) $y = -x^2$ is increasing for $-\infty < x < 0$ and decreasing for $0 < x < \infty$; the function is increasing on intervals where $y' > 0$ and decreasing on intervals where $y' < 0$

50. (a) $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \frac{\left(\frac{-1}{x+h} - \frac{-1}{x}\right)}{h} = \lim\limits_{h \to 0} \frac{-x + (x+h)}{x(x+h)h} = \lim\limits_{h \to 0} \frac{1}{x(x+h)} = \frac{1}{x^2}$

(b)

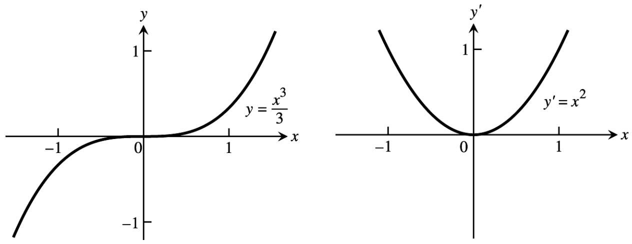


(c) y' is positive for all $x \neq 0$, y' is never 0, y' is never negative

(d) $y = -\frac{1}{x}$ is increasing for $-\infty < x < 0$ and $0 < x < \infty$

51. (a) Using the alternate formula for calculating derivatives: $f'(x) = \lim\limits_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim\limits_{z \to x} \frac{\left(\frac{z^3}{3} - \frac{x^3}{3}\right)}{z - x}$
$= \lim\limits_{z \to x} \frac{z^3 - x^3}{3(z - x)} = \lim\limits_{z \to x} \frac{(z - x)(z^2 + zx + x^2)}{3(z - x)} = \lim\limits_{z \to x} \frac{z^2 + zx + x^2}{3} = x^2 \Rightarrow f'(x) = x^2$

(b)

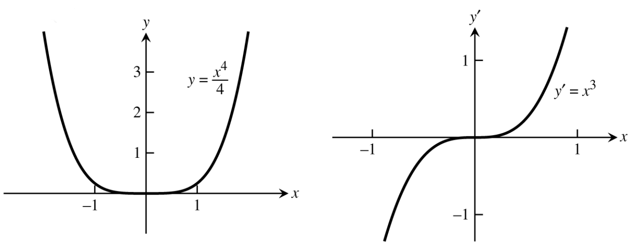


(c) y' is positive for all $x \neq 0$, and $y' = 0$ when $x = 0$; y' is never negative

(d) $y = \frac{x^3}{3}$ is increasing for all $x \neq 0$ (the graph is horizontal at $x = 0$) because y is increasing where $y' > 0$; y is never decreasing

52. (a) Using the alternate form for calculating derivatives: $f'(x) = \lim\limits_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim\limits_{z \to x} \frac{\left(\frac{z^4}{4} - \frac{x^4}{4}\right)}{z - x}$
$= \lim\limits_{z \to x} \frac{z^4 - x^4}{4(z - x)} = \lim\limits_{z \to x} \frac{(z - x)(z^3 + xz^2 + x^2 z + x^3)}{4(z - x)} = \lim\limits_{z \to x} \frac{z^3 + xz^2 + x^2 z + x^3}{4} = x^3 \Rightarrow f'(x) = x^3$

(b)



(c) y' is positive for $x > 0$, y' is zero for $x = 0$, y' is negative for $x < 0$

(d) $y = \frac{x^4}{4}$ is increasing on $0 < x < \infty$ and decreasing on $-\infty < x < 0$

53. $y' = \lim\limits_{h \to 0} \frac{(2(x + h)^2 - 13(x + h) + 5) - (2x^2 - 13x + 5)}{h} = \lim\limits_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 13x - 13h + 5 - 2x^2 + 13x - 5}{h}$
$= \lim\limits_{h \to 0} \frac{4xh + 2h^2 - 13h}{h} = \lim\limits_{h \to 0} (4x + 2h - 13) = 4x - 13$, slope at x. The slope is -1 when $4x - 13 = -1$
$\Rightarrow 4x = 12 \Rightarrow x = 3 \Rightarrow y = 2 \cdot 3^2 - 13 \cdot 3 + 5 = -16$. Thus the tangent line is $y + 16 = (-1)(x - 3)$
$\Rightarrow y = -x - 13$ and the point of tangency is $(3, -16)$.

54. For the curve $y = \sqrt{x}$, we have $y' = \lim\limits_{h \to 0} \frac{\left(\sqrt{x + h} - \sqrt{x}\right)}{h} \cdot \frac{\left(\sqrt{x + h} + \sqrt{x}\right)}{\left(\sqrt{x + h} + \sqrt{x}\right)} = \lim\limits_{h \to 0} \frac{(x + h) - x}{\left(\sqrt{x + h} + \sqrt{x}\right) h}$
$= \lim\limits_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$. Suppose $\left(a, \sqrt{a}\right)$ is the point of tangency of such a line and $(-1, 0)$ is the point on the line where it crosses the x-axis. Then the slope of the line is $\frac{\sqrt{a} - 0}{a - (-1)} = \frac{\sqrt{a}}{a + 1}$ which must also equal

$\frac{1}{2\sqrt{a}}$; using the derivative formula at $x = a \Rightarrow \frac{\sqrt{a}}{a+1} = \frac{1}{2\sqrt{a}} \Rightarrow 2a = a + 1 \Rightarrow a = 1$. Thus such a line does exist: its point of tangency is $(1, 1)$, its slope is $\frac{1}{2\sqrt{a}} = \frac{1}{2}$; and an equation of the line is $y - 1 = \frac{1}{2}(x - 1)$ $\Rightarrow y = \frac{1}{2}x + \frac{1}{2}$.

55. Yes; the derivative of $-f$ is $-f'$ so that $f'(x_0)$ exists $\Rightarrow -f'(x_0)$ exists as well.

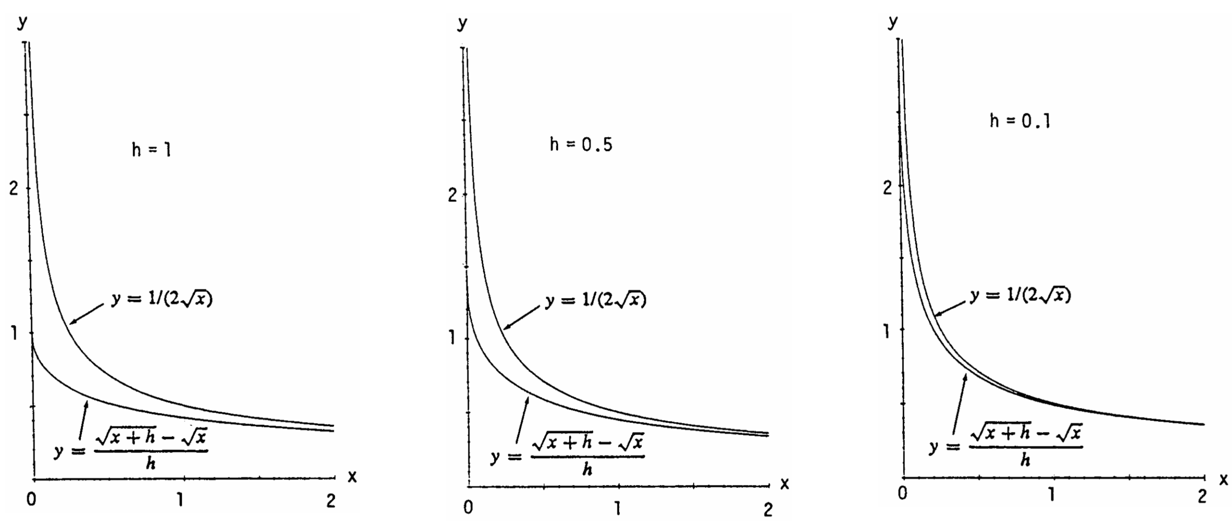
56. Yes; the derivative of $3g$ is $3g'$ so that $g'(7)$ exists $\Rightarrow 3g'(7)$ exists as well.

57. Yes, $\lim_{t \to 0} \frac{g(t)}{h(t)}$ can exist but it need not equal zero. For example, let $g(t) = mt$ and $h(t) = t$. Then $g(0) = h(0) = 0$, but $\lim_{t \to 0} \frac{g(t)}{h(t)} = \lim_{t \to 0} \frac{mt}{t} = \lim_{t \to 0} m = m$, which need not be zero.

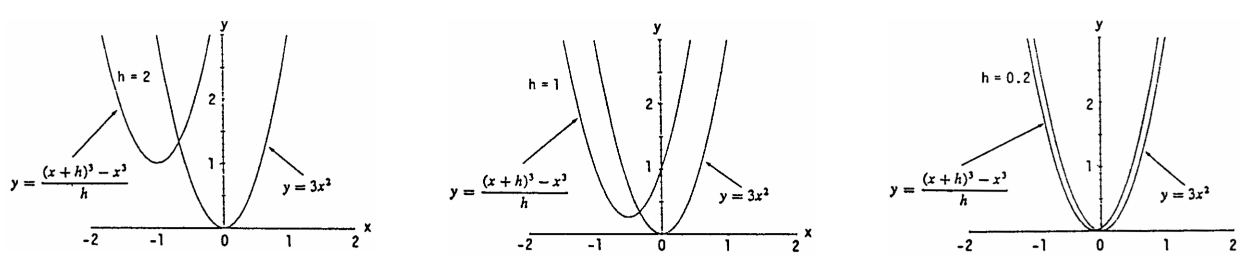
58. (a) Suppose $|f(x)| \le x^2$ for $-1 \le x \le 1$. Then $|f(0)| \le 0^2 \Rightarrow f(0) = 0$. Then $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ $= \lim_{h \to 0} \frac{f(h) - 0}{h} = \lim_{h \to 0} \frac{f(h)}{h}$. For $|h| \le 1$, $-h^2 \le f(h) \le h^2 \Rightarrow -h \le \frac{f(h)}{h} \le h \Rightarrow f'(0) = \lim_{h \to 0} \frac{f(h)}{h} = 0$ by the Sandwich Theorem for limits.

(b) Note that for $x \ne 0$, $|f(x)| = \left|x^2 \sin \frac{1}{x}\right| = |x^2|\,|\sin x| \le |x^2| \cdot 1 = x^2$ (since $-1 \le \sin x \le 1$). By part (a), f is differentiable at $x = 0$ and $f'(0) = 0$.

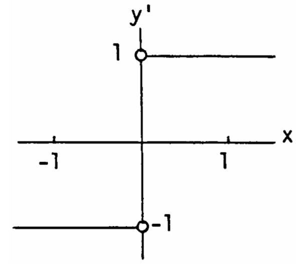
59. The graphs are shown below for $h = 1, 0.5, 0.1$. The function $y = \frac{1}{2\sqrt{x}}$ is the derivative of the function $y = \sqrt{x}$ so that $\frac{1}{2\sqrt{x}} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$. The graphs reveal that $y = \frac{\sqrt{x+h} - \sqrt{x}}{h}$ gets closer to $y = \frac{1}{2\sqrt{x}}$ as h gets smaller and smaller.



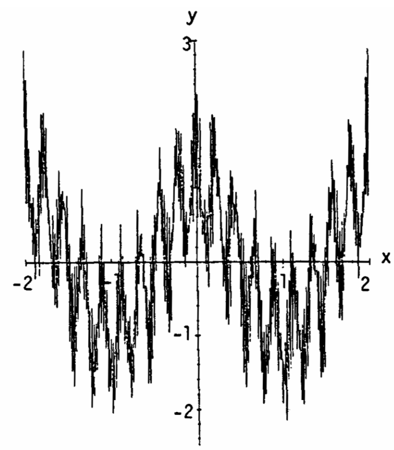
60. The graphs are shown below for $h = 2, 1, 0.5$. The function $y = 3x^2$ is the derivative of the function $y = x^3$ so that $3x^2 = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$. The graphs reveal that $y = \frac{(x+h)^3 - x^3}{h}$ gets closer to $y = 3x^2$ as h gets smaller and smaller.



61. The graphs are the same. So we know that for $f(x) = |x|$, we have $f'(x) = \frac{|x|}{x}$.



62. Weierstrass's nowhere differentiable continuous function.



$$g(x) = \cos(\pi x) + \left(\frac{2}{3}\right)^1 \cos(9\pi x) + \left(\frac{2}{3}\right)^2 \cos(9^2\pi x) + \left(\frac{2}{3}\right)^3 \cos(9^3\pi x) + \cdots + \left(\frac{2}{3}\right)^7 \cos(9^7\pi x)$$

63-68. Example CAS commands:

Maple:

```
f := x -> x^3 + x^2 - x;
x0 := 1;
plot( f(x), x=x0-5..x0+2, color=black,
        title="Section 3.2, #63(a)" );
q := unapply( (f(x+h)-f(x))/h, (x,h) );                    # (b)
L := limit( q(x,h), h=0 );                                  # (c)
m := eval( L, x=x0 );
tan_line := f(x0) + m*(x-x0);
plot( [f(x),tan_line], x=x0-2..x0+3, color=black,
       linestyle=[1,7], title="Section 3.2 #63(d)",
       legend=["y=f(x)","Tangent line at x=1"] );
Xvals := sort( [ x0+2^(-k) $ k=0..5, x0-2^(-k) $ k=0..5 ] ):    # (e)
Yvals := map( f, Xvals ):
evalf[4](< convert(Xvals,Matrix) , convert(Yvals,Matrix) >);
plot( L, x=x0-5..x0+3, color=black, title="Section 3.2 #63(f)" );
```

Mathematica: (functions and x0 may vary) (see section 2.5 re. RealOnly):

```
<<Miscellaneous`RealOnly`
Clear[f, m, x, y, h]
x0= π /4;
f[x_]:=x^2 Cos[x]
Plot[f[x], {x, x0 − 3, x0 + 3}]
```

```
q[x_, h_]:=(f[x + h] − f[x])/h
m[x_]:=Limit[q[x, h], h → 0]
ytan:=f[x0] + m[x0] (x − x0)
Plot[{f[x], ytan},{x, x0 − 3, x0 + 3}]
m[x0 − 1]//N
m[x0 + 1]//N
Plot[{f[x], m[x]},{x, x0 − 3, x0 + 3}]
```

3.3 DIFFERENTIATION RULES

1. $y = -x^2 + 3 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(-x^2) + \frac{d}{dx}(3) = -2x + 0 = -2x \Rightarrow \frac{d^2y}{dx^2} = -2$

2. $y = x^2 + x + 8 \Rightarrow \frac{dy}{dx} = 2x + 1 + 0 = 2x + 1 \Rightarrow \frac{d^2y}{dx^2} = 2$

3. $s = 5t^3 - 3t^5 \Rightarrow \frac{ds}{dt} = \frac{d}{dt}(5t^3) - \frac{d}{dt}(3t^5) = 15t^2 - 15t^4 \Rightarrow \frac{d^2s}{dt^2} = \frac{d}{dt}(15t^2) - \frac{d}{dt}(15t^4) = 30t - 60t^3$

4. $w = 3z^7 - 7z^3 + 21z^2 \Rightarrow \frac{dw}{dz} = 21z^6 - 21z^2 + 42z \Rightarrow \frac{d^2w}{dz^2} = 126z^5 - 42z + 42$

5. $y = \frac{4}{3}x^3 - x \Rightarrow \frac{dy}{dx} = 4x^2 - 1 \Rightarrow \frac{d^2y}{dx^2} = 8x$

6. $y = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4} \Rightarrow \frac{dy}{dx} = x^2 + x + \frac{1}{4} \Rightarrow \frac{d^2y}{dx^2} = 2x + 1 + 0 = 2x + 1$

7. $w = 3z^{-2} - z^{-1} \Rightarrow \frac{dw}{dz} = -6z^{-3} + z^{-2} = \frac{-6}{z^3} + \frac{1}{z^2} \Rightarrow \frac{d^2w}{dz^2} = 18z^{-4} - 2z^{-3} = \frac{18}{z^4} - \frac{2}{z^3}$

8. $s = -2t^{-1} + 4t^{-2} \Rightarrow \frac{ds}{dt} = 2t^{-2} - 8t^{-3} = \frac{2}{t^2} - \frac{8}{t^3} \Rightarrow \frac{d^2s}{dt^2} = -4t^{-3} + 24t^{-4} = \frac{-4}{t^3} + \frac{24}{t^4}$

9. $y = 6x^2 - 10x - 5x^{-2} \Rightarrow \frac{dy}{dx} = 12x - 10 + 10x^{-3} = 12x - 10 + \frac{10}{x^3} \Rightarrow \frac{d^2y}{dx^2} = 12 - 0 - 30x^{-4} = 12 - \frac{30}{x^4}$

10. $y = 4 - 2x - x^{-3} \Rightarrow \frac{dy}{dx} = -2 + 3x^{-4} = -2 + \frac{3}{x^4} \Rightarrow \frac{d^2y}{dx^2} = 0 - 12x^{-5} = \frac{-12}{x^5}$

11. $r = \frac{1}{3}s^{-2} - \frac{5}{2}s^{-1} \Rightarrow \frac{dr}{ds} = -\frac{2}{3}s^{-3} + \frac{5}{2}s^{-2} = \frac{-2}{3s^3} + \frac{5}{2s^2} \Rightarrow \frac{d^2r}{ds^2} = 2s^{-4} - 5s^{-3} = \frac{2}{s^4} - \frac{5}{s^3}$

12. $r = 12\theta^{-1} - 4\theta^{-3} + \theta^{-4} \Rightarrow \frac{dr}{d\theta} = -12\theta^{-2} + 12\theta^{-4} - 4\theta^{-5} = \frac{-12}{\theta^2} + \frac{12}{\theta^4} - \frac{4}{\theta^5} \Rightarrow \frac{d^2r}{d\theta^2} = 24\theta^{-3} - 48\theta^{-5} + 20\theta^{-6}$
$= \frac{24}{\theta^3} - \frac{48}{\theta^5} + \frac{20}{\theta^6}$

13. (a) $y = (3 - x^2)(x^3 - x + 1) \Rightarrow y' = (3 - x^2) \cdot \frac{d}{dx}(x^3 - x + 1) + (x^3 - x + 1) \cdot \frac{d}{dx}(3 - x^2)$
$= (3 - x^2)(3x^2 - 1) + (x^3 - x + 1)(-2x) = -5x^4 + 12x^2 - 2x - 3$
(b) $y = -x^5 + 4x^3 - x^2 - 3x + 3 \Rightarrow y' = -5x^4 + 12x^2 - 2x - 3$

14. (a) $y = (2x + 3)(5x^2 - 4x) \Rightarrow y' = (2x + 3)(10x - 4) + (5x^2 - 4x)(2) = 30x^2 + 14x - 12$
(b) $y = (2x + 3)(5x^2 - 4x) = 10x^3 + 7x^2 - 12x \Rightarrow y' = 30x^2 + 14x - 12$

15. (a) $y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right) \Rightarrow y' = (x^2 + 1) \cdot \frac{d}{dx}\left(x + 5 + \frac{1}{x}\right) + \left(x + 5 + \frac{1}{x}\right) \cdot \frac{d}{dx}(x^2 + 1)$
$= (x^2 + 1)(1 - x^{-2}) + (x + 5 + x^{-1})(2x) = (x^2 - 1 + 1 - x^{-2}) + (2x^2 + 10x + 2) = 3x^2 + 10x + 2 - \frac{1}{x^2}$
(b) $y = x^3 + 5x^2 + 2x + 5 + \frac{1}{x} \Rightarrow y' = 3x^2 + 10x + 2 - \frac{1}{x^2}$

16. $y = (1 + x^2)\left(x^{3/4} - x^{-3}\right)$

(a) $y' = (1 + x^2) \cdot \left(\frac{3}{4}x^{-1/4} + 3x^{-4}\right) + \left(x^{3/4} - x^{-3}\right)(2x) = \frac{3}{4x^{1/4}} + \frac{3}{x^4} + \frac{11}{4}x^{7/4} + \frac{1}{x^2}$

(b) $y = x^{3/4} - x^{-3} + x^{11/4} - x^{-1} \Rightarrow y' = \frac{3}{4x^{1/4}} + \frac{3}{x^4} + \frac{11}{4}x^{7/4} + \frac{1}{x^2}$

17. $y = \frac{2x+5}{3x-2}$; use the quotient rule: $u = 2x + 5$ and $v = 3x - 2 \Rightarrow u' = 2$ and $v' = 3 \Rightarrow y' = \frac{vu' - uv'}{v^2}$
$= \frac{(3x-2)(2)-(2x+5)(3)}{(3x-2)^2} = \frac{6x-4-6x-15}{(3x-2)^2} = \frac{-19}{(3x-2)^2}$

18. $y = \frac{4-3x}{3x^2+x}$;use the quotient rule: $u = 4 - 3x$ and $v = 3x^2 + x \Rightarrow u' = -3$ and $v' = 6x + 1 \Rightarrow y' = \frac{vu' - uv'}{v^2}$
$= \frac{(3x^2+x)(-3)-(4-3x)(6x+1)}{(3x^2+x)^2} = \frac{-9x^2-3x+18x^2-21x-4}{(3x^2+x)^2} = \frac{9x^2-24x-4}{(3x^2+x)^2}$

19. $g(x) = \frac{x^2-4}{x+0.5}$; use the quotient rule: $u = x^2 - 4$ and $v = x + 0.5 \Rightarrow u' = 2x$ and $v' = 1 \Rightarrow g'(x) = \frac{vu' - uv'}{v^2}$
$= \frac{(x+0.5)(2x)-(x^2-4)(1)}{(x+0.5)^2} = \frac{2x^2+x-x^2+4}{(x+0.5)^2} = \frac{x^2+x+4}{(x+0.5)^2}$

20. $f(t) = \frac{t^2-1}{t^2+t-2} = \frac{(t-1)(t+1)}{(t+2)(t-1)} = \frac{t+1}{t+2}, t \neq 1 \Rightarrow f'(t) = \frac{(t+2)(1)-(t+1)(1)}{(t+2)^2} = \frac{t+2-t-1}{(t+2)^2} = \frac{1}{(t+2)^2}$

21. $v = (1 - t)(1 + t^2)^{-1} = \frac{1-t}{1+t^2} \Rightarrow \frac{dv}{dt} = \frac{(1+t^2)(-1)-(1-t)(2t)}{(1+t^2)^2} = \frac{-1-t^2-2t+2t^2}{(1+t^2)^2} = \frac{t^2-2t-1}{(1+t^2)^2}$

22. $w = \frac{x+5}{2x-7} \Rightarrow w' = \frac{(2x-7)(1)-(x+5)(2)}{(2x-7)^2} = \frac{2x-7-2x-10}{(2x-7)^2} = \frac{-17}{(2x-7)^2}$

23. $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1} \Rightarrow f'(s) = \frac{(\sqrt{s}+1)\left(\frac{1}{2\sqrt{s}}\right)-(\sqrt{s}-1)\left(\frac{1}{2\sqrt{s}}\right)}{(\sqrt{s}+1)^2} = \frac{(\sqrt{s}+1)-(\sqrt{s}-1)}{2\sqrt{s}\left(\sqrt{s}+1\right)^2} = \frac{1}{\sqrt{s}\left(\sqrt{s}+1\right)^2}$

NOTE: $\frac{d}{ds}\left(\sqrt{s}\right) = \frac{1}{2\sqrt{s}}$ from Example 2 in Section 3.2

24. $u = \frac{5x+1}{2\sqrt{x}} \Rightarrow \frac{du}{dx} = \frac{(2\sqrt{x})(5)-(5x+1)\left(\frac{1}{\sqrt{x}}\right)}{4x} = \frac{5x-1}{4x^{3/2}}$

25. $v = \frac{1+x-4\sqrt{x}}{x} \Rightarrow v' = \frac{x\left(1-\frac{2}{\sqrt{x}}\right)-(1+x-4\sqrt{x})}{x^2} = \frac{2\sqrt{x}-1}{x^2}$

26. $r = 2\left(\frac{1}{\sqrt{\theta}} + \sqrt{\theta}\right) \Rightarrow r' = 2\left(\frac{\sqrt{\theta}(0)-1\left(\frac{1}{2\sqrt{\theta}}\right)}{\theta} + \frac{1}{2\sqrt{\theta}}\right) = -\frac{1}{\theta^{3/2}} + \frac{1}{\theta^{1/2}}$

27. $y = \frac{1}{(x^2-1)(x^2+x+1)}$; use the quotient rule: $u = 1$ and $v = (x^2 - 1)(x^2 + x + 1) \Rightarrow u' = 0$ and
$v' = (x^2 - 1)(2x + 1) + (x^2 + x + 1)(2x) = 2x^3 + x^2 - 2x - 1 + 2x^3 + 2x^2 + 2x = 4x^3 + 3x^2 - 1$
$\Rightarrow \frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{0-1(4x^3+3x^2-1)}{(x^2-1)^2(x^2+x+1)^2} = \frac{-4x^3-3x^2+1}{(x^2-1)^2(x^2+x+1)^2}$

28. $y = \frac{(x+1)(x+2)}{(x-1)(x-2)} = \frac{x^2+3x+2}{x^2-3x+2} \Rightarrow y' = \frac{(x^2-3x+2)(2x+3)-(x^2+3x+2)(2x-3)}{(x-1)^2(x-2)^2} = \frac{-6x^2+12}{(x-1)^2(x-2)^2} = \frac{-6(x^2-2)}{(x-1)^2(x-2)^2}$

29. $y = \frac{1}{2}x^4 - \frac{3}{2}x^2 - x \Rightarrow y' = 2x^3 - 3x - 1 \Rightarrow y'' = 6x^2 - 3 \Rightarrow y''' = 12x \Rightarrow y^{(4)} = 12 \Rightarrow y^{(n)} = 0$ for all $n \geq 5$

30. $y = \frac{1}{120}x^5 \Rightarrow y' = \frac{1}{24}x^4 \Rightarrow y'' = \frac{1}{6}x^3 \Rightarrow y''' = \frac{1}{2}x^2 \Rightarrow y^{(4)} = x \Rightarrow y^{(5)} = 1 \Rightarrow y^{(n)} = 0$ for all $n \geq 6$

31. $y = (x - 1)(x^2 + 3x - 5) = x^3 + 2x^2 - 8x + 5 \Rightarrow y' = 3x^2 + 4x - 8 \Rightarrow y'' = 6x + 4 \Rightarrow y''' = 6 \Rightarrow y^{(n)} = 0$ for all $n \geq 4$

32. $y = (4x^3 + 3x)(2 - x) = -4x^4 + 8x^3 - 3x^2 + 6x \Rightarrow y' = -16x^3 + 24x^2 - 6x + 6 \Rightarrow y'' = -48x^2 + 48x - 6$
$\Rightarrow y''' = -96x + 48 \Rightarrow y^{(4)} = -96 \Rightarrow y^{(n)} = 0$ for all $n \geq 5$

33. $y = \frac{x^3+7}{x} = x^2 + 7x^{-1} \Rightarrow \frac{dy}{dx} = 2x - 7x^{-2} = 2x - \frac{7}{x^2} \Rightarrow \frac{d^2y}{dx^2} = 2 + 14x^{-3} = 2 + \frac{14}{x^3}$

34. $s = \frac{t^2+5t-1}{t^2} = 1 + \frac{5}{t} - \frac{1}{t^2} = 1 + 5t^{-1} - t^{-2} \Rightarrow \frac{ds}{dt} = 0 - 5t^{-2} + 2t^{-3} = -5t^{-2} + 2t^{-3} = \frac{-5}{t^2} + \frac{2}{t^3}$
$\Rightarrow \frac{d^2s}{dt^2} = 10t^{-3} - 6t^{-4} = \frac{10}{t^3} - \frac{6}{t^4}$

35. $r = \frac{(\theta-1)(\theta^2+\theta+1)}{\theta^3} = \frac{\theta^3-1}{\theta^3} = 1 - \frac{1}{\theta^3} = 1 - \theta^{-3} \Rightarrow \frac{dr}{d\theta} = 0 + 3\theta^{-4} = 3\theta^{-4} = \frac{3}{\theta^4} \Rightarrow \frac{d^2r}{d\theta^2} = -12\theta^{-5} = \frac{-12}{\theta^5}$

36. $u = \frac{(x^2+x)(x^2-x+1)}{x^4} = \frac{x(x+1)(x^2-x+1)}{x^4} = \frac{x(x^3+1)}{x^4} = \frac{x^4+x}{x^4} = 1 + \frac{x}{x^4} = 1 + x^{-3}$
$\Rightarrow \frac{du}{dx} = 0 - 3x^{-4} = -3x^{-4} = \frac{-3}{x^4} \Rightarrow \frac{d^2u}{dx^2} = 12x^{-5} = \frac{12}{x^5}$

37. $w = \left(\frac{1+3z}{3z}\right)(3 - z) = \left(\frac{1}{3}z^{-1} + 1\right)(3 - z) = z^{-1} - \frac{1}{3} + 3 - z = z^{-1} + \frac{8}{3} - z \Rightarrow \frac{dw}{dz} = -z^{-2} + 0 - 1 = -z^{-2} - 1$
$= \frac{-1}{z^2} - 1 \Rightarrow \frac{d^2w}{dz^2} = 2z^{-3} - 0 = 2z^{-3} = \frac{2}{z^3}$

38. $w = (z + 1)(z - 1)(z^2 + 1) = (z^2 - 1)(z^2 + 1) = z^4 - 1 \Rightarrow \frac{dw}{dz} = 4z^3 - 0 = 4z^3 \Rightarrow \frac{d^2w}{dz^2} = 12z^2$

39. $p = \left(\frac{q^2+3}{12q}\right)\left(\frac{q^4-1}{q^3}\right) = \frac{q^6-q^2+3q^4-3}{12q^4} = \frac{1}{12}q^2 - \frac{1}{12}q^{-2} + \frac{1}{4} - \frac{1}{4}q^{-4} \Rightarrow \frac{dp}{dq} = \frac{1}{6}q + \frac{1}{6}q^{-3} + q^{-5} = \frac{1}{6}q + \frac{1}{6q^3} + \frac{1}{q^5}$
$\Rightarrow \frac{d^2p}{dq^2} = \frac{1}{6} - \frac{1}{2}q^{-4} - 5q^{-6} = \frac{1}{6} - \frac{1}{2q^4} - \frac{5}{q^6}$

40. $p = \frac{q^2+3}{(q-1)^3+(q+1)^3} = \frac{q^2+3}{(q^3-3q^2+3q-1)+(q^3+3q^2+3q+1)} = \frac{q^2+3}{2q^3+6q} = \frac{q^2+3}{2q(q^2+3)} = \frac{1}{2q} = \frac{1}{2}q^{-1}$
$\Rightarrow \frac{dp}{dq} = -\frac{1}{2}q^{-2} = -\frac{1}{2q^2} \Rightarrow \frac{d^2p}{dq^2} = q^{-3} = \frac{1}{q^3}$

41. $u(0) = 5, u'(0) = -3, v(0) = -1, v'(0) = 2$
(a) $\frac{d}{dx}(uv) = uv' + vu' \Rightarrow \frac{d}{dx}(uv)\big|_{x=0} = u(0)v'(0) + v(0)u'(0) = 5 \cdot 2 + (-1)(-3) = 13$
(b) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu'-uv'}{v^2} \Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right)\big|_{x=0} = \frac{v(0)u'(0)-u(0)v'(0)}{(v(0))^2} = \frac{(-1)(-3)-(5)(2)}{(-1)^2} = -7$
(c) $\frac{d}{dx}\left(\frac{v}{u}\right) = \frac{uv'-vu'}{u^2} \Rightarrow \frac{d}{dx}\left(\frac{v}{u}\right)\big|_{x=0} = \frac{u(0)v'(0)-v(0)u'(0)}{(u(0))^2} = \frac{(5)(2)-(-1)(-3)}{(5)^2} = \frac{7}{25}$
(d) $\frac{d}{dx}(7v - 2u) = 7v' - 2u' \Rightarrow \frac{d}{dx}(7v - 2u)\big|_{x=0} = 7v'(0) - 2u'(0) = 7 \cdot 2 - 2(-3) = 20$

42. $u(1) = 2, u'(1) = 0, v(1) = 5, v'(1) = -1$
(a) $\frac{d}{dx}(uv)\big|_{x=1} = u(1)v'(1) + v(1)u'(1) = 2 \cdot (-1) + 5 \cdot 0 = -2$
(b) $\frac{d}{dx}\left(\frac{u}{v}\right)\big|_{x=1} = \frac{v(1)u'(1)-u(1)v'(1)}{(v(1))^2} = \frac{5\cdot0-2\cdot(-1)}{(5)^2} = \frac{2}{25}$
(c) $\frac{d}{dx}\left(\frac{v}{u}\right)\big|_{x=1} = \frac{u(1)v'(1)-v(1)u'(1)}{(u(1))^2} = \frac{2\cdot(-1)-5\cdot0}{(2)^2} = -\frac{1}{2}$
(d) $\frac{d}{dx}(7v - 2u)\big|_{x=1} = 7v'(1) - 2u'(1) = 7 \cdot (-1) - 2 \cdot 0 = -7$

43. $y = x^3 - 4x + 1$. Note that $(2, 1)$ is on the curve: $1 = 2^3 - 4(2) + 1$
(a) Slope of the tangent at (x, y) is $y' = 3x^2 - 4 \Rightarrow$ slope of the tangent at $(2, 1)$ is $y'(2) = 3(2)^2 - 4 = 8$. Thus the slope of the line perpendicular to the tangent at $(2, 1)$ is $-\frac{1}{8} \Rightarrow$ the equation of the line perpendicular to the tangent line at $(2, 1)$ is $y - 1 = -\frac{1}{8}(x - 2)$ or $y = -\frac{x}{8} + \frac{5}{4}$.
(b) The slope of the curve at x is $m = 3x^2 - 4$ and the smallest value for m is -4 when $x = 0$ and $y = 1$.

(c) We want the slope of the curve to be 8 $\Rightarrow$ $y' = 8 \Rightarrow 3x^2 - 4 = 8 \Rightarrow 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$. When $x = 2$, $y = 1$ and the tangent line has equation $y - 1 = 8(x - 2)$ or $y = 8x - 15$; when $x = -2$, $y = (-2)^3 - 4(-2) + 1 = 1$, and the tangent line has equation $y - 1 = 8(x + 2)$ or $y = 8x + 17$.

44. (a) $y = x^3 - 3x - 2 \Rightarrow y' = 3x^2 - 3$. For the tangent to be horizontal, we need $m = y' = 0 \Rightarrow 0 = 3x^2 - 3 \Rightarrow 3x^2 = 3 \Rightarrow x = \pm 1$. When $x = -1$, $y = 0 \Rightarrow$ the tangent line has equation $y = 0$. The line perpendicular to this line at $(-1, 0)$ is $x = -1$. When $x = 1$, $y = -4$ $\Rightarrow$ the tangent line has equation $y = -4$. The line perpendicular to this line at $(1, -4)$ is $x = 1$.

(b) The smallest value of y' is -3, and this occurs when $x = 0$ and $y = -2$. The tangent to the curve at $(0, -2)$ has slope -3 $\Rightarrow$ the line perpendicular to the tangent at $(0, -2)$ has slope $\frac{1}{3}$ $\Rightarrow$ $y + 2 = \frac{1}{3}(x - 0)$ or $y = \frac{1}{3}x - 2$ is an equation of the perpendicular line.

45. $y = \frac{4x}{x^2+1} \Rightarrow \frac{dy}{dx} = \frac{(x^2+1)(4) - (4x)(2x)}{(x^2+1)^2} = \frac{4x^2+4-8x^2}{(x^2+1)^2} = \frac{4(-x^2+1)}{(x^2+1)^2}$. When $x = 0$, $y = 0$ and $y' = \frac{4(0+1)}{1} = 4$, so the tangent to the curve at $(0, 0)$ is the line $y = 4x$. When $x = 1$, $y = 2 \Rightarrow y' = 0$, so the tangent to the curve at $(1, 2)$ is the line $y = 2$.

46. $y = \frac{8}{x^2+4}$ $\Rightarrow$ $y' = \frac{(x^2+4)(0) - 8(2x)}{(x^2+4)^2} = \frac{-16x}{(x^2+4)^2}$. When $x = 2$, $y = 1$ and $y' = \frac{-16(2)}{(2^2+4)^2} = -\frac{1}{2}$, so the tangent line to the curve at $(2, 1)$ has the equation $y - 1 = -\frac{1}{2}(x - 2)$, or $y = -\frac{x}{2} + 2$.

47. $y = ax^2 + bx + c$ passes through $(0, 0)$ $\Rightarrow$ $0 = a(0) + b(0) + c$ $\Rightarrow$ $c = 0$; $y = ax^2 + bx$ passes through $(1, 2)$ $\Rightarrow$ $2 = a + b$; $y' = 2ax + b$ and since the curve is tangent to $y = x$ at the origin, its slope is 1 at $x = 0$ $\Rightarrow$ $y' = 1$ when $x = 0$ $\Rightarrow$ $1 = 2a(0) + b$ $\Rightarrow$ $b = 1$. Then $a + b = 2$ $\Rightarrow$ $a = 1$. In summary $a = b = 1$ and $c = 0$ so the curve is $y = x^2 + x$.

48. $y = cx - x^2$ passes through $(1, 0)$ $\Rightarrow$ $0 = c(1) - 1$ $\Rightarrow$ $c = 1$ $\Rightarrow$ the curve is $y = x - x^2$. For this curve, $y' = 1 - 2x$ and $x = 1$ $\Rightarrow$ $y' = -1$. Since $y = x - x^2$ and $y = x^2 + ax + b$ have common tangents at $x = 0$, $y = x^2 + ax + b$ must also have slope -1 at $x = 1$. Thus $y' = 2x + a$ $\Rightarrow$ $-1 = 2 \cdot 1 + a$ $\Rightarrow$ $a = -3$ $\Rightarrow$ $y = x^2 - 3x + b$. Since this last curve passes through $(1, 0)$, we have $0 = 1 - 3 + b$ $\Rightarrow$ $b = 2$. In summary, $a = -3$, $b = 2$ and $c = 1$ so the curves are $y = x^2 - 3x + 2$ and $y = x - x^2$.

49. $y = 8x + 5 \Rightarrow m = 8$; $f(x) = 3x^2 - 4x \Rightarrow f'(x) = 6x - 4$; $6x - 4 = 8 \Rightarrow x = 2 \Rightarrow f(2) = 3(2)^2 - 4(2) = 4 \Rightarrow (2, 4)$

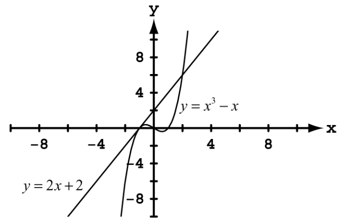
50. $8x - 2y = 1 \Rightarrow y = 4x - \frac{1}{2} \Rightarrow m = 4$; $g(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 1 \Rightarrow g'(x) = x^2 - 3x$; $x^2 - 3x = 4 \Rightarrow x = 4$ or $x = -1$ $\Rightarrow g(4) = \frac{1}{3}(4)^3 - \frac{3}{2}(4)^2 + 1 = -\frac{5}{3}$, $g(-1) = \frac{1}{3}(-1)^3 - \frac{3}{2}(-1)^2 + 1 = -\frac{5}{6} \Rightarrow \left(4, -\frac{5}{3}\right)$ or $\left(-1, -\frac{5}{6}\right)$

51. $y = 2x + 3 \Rightarrow m = 2 \Rightarrow m_{\perp} = -\frac{1}{2}$; $y = \frac{x}{x-2} \Rightarrow y' = \frac{(x-2)(1) - x(1)}{(x-2)^2} = \frac{-2}{(x-2)^2}$; $\frac{-2}{(x-2)^2} = -\frac{1}{2} \Rightarrow 4 = (x-2)^2$ $\Rightarrow \pm 2 = x - 2 \Rightarrow x = 4$ or $x = 0 \Rightarrow$ if $x = 4$, $y = \frac{4}{4-2} = 2$, and if $x = 0$, $y = \frac{0}{0-2} = 0 \Rightarrow (4, 2)$ or $(0, 0)$.

52. $m = \frac{y-8}{x-3}$; $f(x) = x^2 \Rightarrow f'(x) = 2x$; $m = f'(x) \Rightarrow \frac{y-8}{x-3} = 2x \Rightarrow \frac{x^2-8}{x-3} = 2x \Rightarrow x^2 - 8 = 2x^2 - 6x \Rightarrow x^2 - 6x + 8 = 0$ $\Rightarrow x = 4$ or $x = 2 \Rightarrow f(4) = 4^2 = 16$, $f(2) = 2^2 = 4 \Rightarrow (4, 16)$ or $(2, 4)$.

53. (a) $y = x^3 - x$ $\Rightarrow$ $y' = 3x^2 - 1$. When $x = -1$, $y = 0$ and $y' = 2$ $\Rightarrow$ the tangent line to the curve at $(-1, 0)$ is $y = 2(x + 1)$ or $y = 2x + 2$.

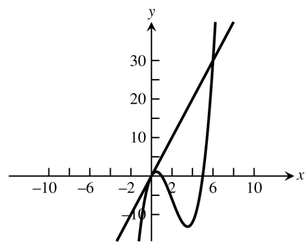
(b)



(c) $\left.\begin{array}{l} y = x^3 - x \\ y = 2x + 2 \end{array}\right\} \Rightarrow x^3 - x = 2x + 2 \Rightarrow x^3 - 3x - 2 = (x-2)(x+1)^2 = 0 \Rightarrow x = 2$ or $x = -1$. Since $y = 2(2) + 2 = 6$; the other intersection point is $(2, 6)$

54. (a) $y = x^3 - 6x^2 + 5x \Rightarrow y' = 3x^2 - 12x + 5$. When $x = 0$, $y = 0$ and $y' = 5 \Rightarrow$ the tangent line to the curve at $(0, 0)$ is $y = 5x$.

(b)



(c) $\left.\begin{array}{l} y = x^3 - 6x^2 + 5x \\ y = 5x \end{array}\right\} \Rightarrow x^3 - 6x^2 + 5x = 5x \Rightarrow x^3 - 6x^2 = 0 \Rightarrow x^2(x-6) = 0 \Rightarrow x = 0$ or $x = 6$. Since $y = 5(6) = 30$, the other intersection point is $(6, 30)$.

55. $\lim\limits_{x \to 1} \frac{x^{50} - 1}{x - 1} = 50\,x^{49}\Big|_{x=1} = 50\,(1)^{49} = 50$

56. $\lim\limits_{x \to -1} \frac{x^{2/9} - 1}{x + 1} = \frac{2}{9}x^{-7/9}\Big|_{x=-1} = \frac{2}{9(-1)^{7/9}} = -\frac{2}{9}$

57. $g'(x) = \begin{cases} 2x - 3 & x > 0 \\ a & x < 0 \end{cases}$, since g is differentiable at $x = 0 \Rightarrow \lim\limits_{x \to 0^+} (2x - 3) = -3$ and $\lim\limits_{x \to 0^-} a = a \Rightarrow a = -3$

58. $f'(x) = \begin{cases} a & x > -1 \\ 2bx & x < -1 \end{cases}$, since f is differentiable at $x = -1 \Rightarrow \lim\limits_{x \to -1^+} a = a$ and $\lim\limits_{x \to -1^-} (2bx) = -2b \Rightarrow a = -2b$, and since f is continuous at $x = -1 \Rightarrow \lim\limits_{x \to -1^+} (ax + b) = -a + b$ and $\lim\limits_{x \to -1^-} (bx^2 - 3) = b - 3 \Rightarrow -a + b = b - 3$ $\Rightarrow a = 3 \Rightarrow 3 = -2b \Rightarrow b = -\frac{3}{2}$.

59. $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 \Rightarrow P'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1$

60. $R = M^2\left(\frac{C}{2} - \frac{M}{3}\right) = \frac{C}{2}M^2 - \frac{1}{3}M^3$, where C is a constant $\Rightarrow \frac{dR}{dM} = CM - M^2$

61. Let c be a constant $\Rightarrow \frac{dc}{dx} = 0 \Rightarrow \frac{d}{dx}(u \cdot c) = u \cdot \frac{dc}{dx} + c \cdot \frac{du}{dx} = u \cdot 0 + c\frac{du}{dx} = c\frac{du}{dx}$. Thus when one of the functions is a constant, the Product Rule is just the Constant Multiple Rule $\Rightarrow$ the Constant Multiple Rule is a special case of the Product Rule.

62. (a) We use the Quotient rule to derive the Reciprocal Rule (with $u = 1$): $\frac{d}{dx}\left(\frac{1}{v}\right) = \frac{v \cdot 0 - 1 \cdot \frac{dv}{dx}}{v^2} = \frac{-1 \cdot \frac{dv}{dx}}{v^2} = -\frac{1}{v^2} \cdot \frac{dv}{dx}$.

(b) Now, using the Reciprocal Rule and the Product Rule, we'll derive the Quotient Rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{d}{dx}\left(u \cdot \frac{1}{v}\right)$
$= u \cdot \frac{d}{dx}\left(\frac{1}{v}\right) + \frac{1}{v} \cdot \frac{du}{dx}$ (Product Rule) $= u \cdot \left(\frac{-1}{v^2}\right)\frac{dv}{dx} + \frac{1}{v}\frac{du}{dx}$ (Reciprocal Rule) $\Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{-u\frac{dv}{dx} + v\frac{du}{dx}}{v^2}$
$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, the Quotient Rule.

63. (a) $\frac{d}{dx}(uvw) = \frac{d}{dx}((uv) \cdot w) = (uv)\frac{dw}{dx} + w \cdot \frac{d}{dx}(uv) = uv\frac{dw}{dx} + w\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) = uv\frac{dw}{dx} + wu\frac{dv}{dx} + wv\frac{du}{dx}$
$= uvw' + uv'w + u'vw$
(b) $\frac{d}{dx}(u_1u_2u_3u_4) = \frac{d}{dx}((u_1u_2u_3)u_4) = (u_1u_2u_3)\frac{du_4}{dx} + u_4\frac{d}{dx}(u_1u_2u_3) \Rightarrow \frac{d}{dx}(u_1u_2u_3u_4)$
$= u_1u_2u_3\frac{du_4}{dx} + u_4\left(u_1u_2\frac{du_3}{dx} + u_3u_1\frac{du_2}{dx} + u_3u_2\frac{du_1}{dx}\right)$ (using (a) above)
$\Rightarrow \frac{d}{dx}(u_1u_2u_3u_4) = u_1u_2u_3\frac{du_4}{dx} + u_1u_2u_4\frac{du_3}{dx} + u_1u_3u_4\frac{du_2}{dx} + u_2u_3u_4\frac{du_1}{dx}$
$= u_1u_2u_3u_4' + u_1u_2u_3'u_4 + u_1u_2'u_3u_4 + u_1'u_2u_3u_4$
(c) Generalizing (a) and (b) above, $\frac{d}{dx}(u_1 \cdots u_n) = u_1u_2 \cdots u_{n-1}u_n' + u_1u_2 \cdots u_{n-2}u_{n-1}'u_n + \ldots + u_1'u_2 \cdots u_n$

64. $\frac{d}{dx}(x^{-m}) = \frac{d}{dx}\left(\frac{1}{x^m}\right) = \frac{x^m \cdot 0 - 1(m \cdot x^{m-1})}{(x^m)^2} = \frac{-m \cdot x^{m-1}}{x^{2m}} = -m \cdot x^{m-1-2m} = -m \cdot x^{-m-1}$

65. $P = \frac{nRT}{V-nb} - \frac{an^2}{V^2}$. We are holding T constant, and a, b, n, R are also constant so their derivatives are zero
$\Rightarrow \frac{dP}{dV} = \frac{(V-nb)\cdot 0 - (nRT)(1)}{(V-nb)^2} - \frac{V^2(0) - (an^2)(2V)}{(V^2)^2} = \frac{-nRT}{(V-nb)^2} + \frac{2an^2}{V^3}$

66. $A(q) = \frac{km}{q} + cm + \frac{hq}{2} = (km)q^{-1} + cm + \left(\frac{h}{2}\right)q \Rightarrow \frac{dA}{dq} = -(km)q^{-2} + \left(\frac{h}{2}\right) = -\frac{km}{q^2} + \frac{h}{2} \Rightarrow \frac{d^2A}{dt^2} = 2(km)q^{-3} = \frac{2km}{q^3}$

3.4 THE DERIVATIVE AS A RATE OF CHANGE

1. $s = t^2 - 3t + 2,\ 0 \le t \le 2$
(a) displacement $= \Delta s = s(2) - s(0) = 0\text{m} - 2\text{m} = -2$ m, $v_{av} = \frac{\Delta s}{\Delta t} = \frac{-2}{2} = -1$ m/sec
(b) $v = \frac{ds}{dt} = 2t - 3 \Rightarrow |v(0)| = |-3| = 3$ m/sec and $|v(2)| = 1$ m/sec;
$a = \frac{d^2s}{dt^2} = 2 \Rightarrow a(0) = 2$ m/sec^2 and $a(2) = 2$ m/sec^2
(c) $v = 0 \Rightarrow 2t - 3 = 0 \Rightarrow t = \frac{3}{2}$. v is negative in the interval $0 < t < \frac{3}{2}$ and v is positive when $\frac{3}{2} < t < 2 \Rightarrow$ the body changes direction at $t = \frac{3}{2}$.

2. $s = 6t - t^2,\ 0 \le t \le 6$
(a) displacement $= \Delta s = s(6) - s(0) = 0$ m, $v_{av} = \frac{\Delta s}{\Delta t} = \frac{0}{6} = 0$ m/sec
(b) $v = \frac{ds}{dt} = 6 - 2t \Rightarrow |v(0)| = |6| = 6$ m/sec and $|v(6)| = |-6| = 6$ m/sec;
$a = \frac{d^2s}{dt^2} = -2 \Rightarrow a(0) = -2$ m/sec^2 and $a(6) = -2$ m/sec^2
(c) $v = 0 \Rightarrow 6 - 2t = 0 \Rightarrow t = 3$. v is positive in the interval $0 < t < 3$ and v is negative when $3 < t < 6 \Rightarrow$ the body changes direction at $t = 3$.

3. $s = -t^3 + 3t^2 - 3t,\ 0 \le t \le 3$
(a) displacement $= \Delta s = s(3) - s(0) = -9$ m, $v_{av} = \frac{\Delta s}{\Delta t} = \frac{-9}{3} = -3$ m/sec
(b) $v = \frac{ds}{dt} = -3t^2 + 6t - 3 \Rightarrow |v(0)| = |-3| = 3$ m/sec and $|v(3)| = |-12| = 12$ m/sec; $a = \frac{d^2s}{dt^2} = -6t + 6$
$\Rightarrow a(0) = 6$ m/sec^2 and $a(3) = -12$ m/sec^2
(c) $v = 0 \Rightarrow -3t^2 + 6t - 3 = 0 \Rightarrow t^2 - 2t + 1 = 0 \Rightarrow (t-1)^2 = 0 \Rightarrow t = 1$. For all other values of t in the interval the velocity v is negative (the graph of $v = -3t^2 + 6t - 3$ is a parabola with vertex at $t = 1$ which opens downward $\Rightarrow$ the body never changes direction).

4. $s = \frac{t^4}{4} - t^3 + t^2,\ 0 \le t \le 3$
   (a) $\Delta s = s(3) - s(0) = \frac{9}{4}$ m, $v_{av} = \frac{\Delta s}{\Delta t} = \frac{\frac{9}{4}}{3} = \frac{3}{4}$ m/sec
   (b) $v = t^3 - 3t^2 + 2t \Rightarrow |v(0)| = 0$ m/sec and $|v(3)| = 6$ m/sec; $a = 3t^2 - 6t + 2 \Rightarrow a(0) = 2$ m/sec^2 and $a(3) = 11$ m/sec^2
   (c) $v = 0 \Rightarrow t^3 - 3t^2 + 2t = 0 \Rightarrow t(t-2)(t-1) = 0 \Rightarrow t = 0, 1, 2 \Rightarrow v = t(t-2)(t-1)$ is positive in the interval for $0 < t < 1$ and v is negative for $1 < t < 2$ and v is positive for $2 < t < 3 \Rightarrow$ the body changes direction at $t = 1$ and at $t = 2$.

5. $s = \frac{25}{t^2} - \frac{5}{t},\ 1 \le t \le 5$
   (a) $\Delta s = s(5) - s(1) = -20$ m, $v_{av} = \frac{-20}{4} = -5$ m/sec
   (b) $v = \frac{-50}{t^3} + \frac{5}{t^2} \Rightarrow |v(1)| = 45$ m/sec and $|v(5)| = \frac{1}{5}$ m/sec; $a = \frac{150}{t^4} - \frac{10}{t^3} \Rightarrow a(1) = 140$ m/sec^2 and $a(5) = \frac{4}{25}$ m/sec^2
   (c) $v = 0 \Rightarrow \frac{-50+5t}{t^3} = 0 \Rightarrow -50 + 5t = 0 \Rightarrow t = 10 \Rightarrow$ the body does not change direction in the interval

6. $s = \frac{25}{t+5},\ -4 \le t \le 0$
   (a) $\Delta s = s(0) - s(-4) = -20$ m, $v_{av} = -\frac{20}{4} = -5$ m/sec
   (b) $v = \frac{-25}{(t+5)^2} \Rightarrow |v(-4)| = 25$ m/sec and $|v(0)| = 1$ m/sec; $a = \frac{50}{(t+5)^3} \Rightarrow a(-4) = 50$ m/sec^2 and $a(0) = \frac{2}{5}$ m/sec^2
   (c) $v = 0 \Rightarrow \frac{-25}{(t+5)^2} = 0 \Rightarrow$ v is never 0 $\Rightarrow$ the body never changes direction

7. $s = t^3 - 6t^2 + 9t$ and let the positive direction be to the right on the s-axis.
   (a) $v = 3t^2 - 12t + 9$ so that $v = 0 \Rightarrow t^2 - 4t + 3 = (t-3)(t-1) = 0 \Rightarrow t = 1$ or 3; $a = 6t - 12 \Rightarrow a(1) = -6$ m/sec^2 and $a(3) = 6$ m/sec^2. Thus the body is motionless but being accelerated left when $t = 1$, and motionless but being accelerated right when $t = 3$.
   (b) $a = 0 \Rightarrow 6t - 12 = 0 \Rightarrow t = 2$ with speed $|v(2)| = |12 - 24 + 9| = 3$ m/sec
   (c) The body moves to the right or forward on $0 \le t < 1$, and to the left or backward on $1 < t < 2$. The positions are $s(0) = 0$, $s(1) = 4$ and $s(2) = 2 \Rightarrow$ total distance $= |s(1) - s(0)| + |s(2) - s(1)| = |4| + |-2| = 6$ m.

8. $v = t^2 - 4t + 3 \Rightarrow a = 2t - 4$
   (a) $v = 0 \Rightarrow t^2 - 4t + 3 = 0 \Rightarrow t = 1$ or $3 \Rightarrow a(1) = -2$ m/sec^2 and $a(3) = 2$ m/sec^2
   (b) $v > 0 \Rightarrow (t-3)(t-1) > 0 \Rightarrow 0 \le t < 1$ or $t > 3$ and the body is moving forward; $v < 0 \Rightarrow (t-3)(t-1) < 0 \Rightarrow 1 < t < 3$ and the body is moving backward
   (c) velocity increasing $\Rightarrow a > 0 \Rightarrow 2t - 4 > 0 \Rightarrow t > 2$; velocity decreasing $\Rightarrow a < 0 \Rightarrow 2t - 4 < 0 \Rightarrow 0 \le t < 2$

9. $s_m = 1.86t^2 \Rightarrow v_m = 3.72t$ and solving $3.72t = 27.8 \Rightarrow t \approx 7.5$ sec on Mars; $s_j = 11.44t^2 \Rightarrow v_j = 22.88t$ and solving $22.88t = 27.8 \Rightarrow t \approx 1.2$ sec on Jupiter.

10. (a) $v(t) = s'(t) = 24 - 1.6t$ m/sec, and $a(t) = v'(t) = s''(t) = -1.6$ m/sec^2
   (b) Solve $v(t) = 0 \Rightarrow 24 - 1.6t = 0 \Rightarrow t = 15$ sec
   (c) $s(15) = 24(15) - .8(15)^2 = 180$ m
   (d) Solve $s(t) = 90 \Rightarrow 24t - .8t^2 = 90 \Rightarrow t = \frac{30 \pm 15\sqrt{2}}{2} \approx 4.39$ sec going up and 25.6 sec going down
   (e) Twice the time it took to reach its highest point or 30 sec

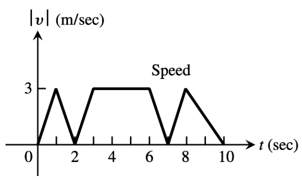
11. $s = 15t - \frac{1}{2}g_s t^2 \Rightarrow v = 15 - g_s t$ so that $v = 0 \Rightarrow 15 - g_s t = 0 \Rightarrow g_s = \frac{15}{t}$. Therefore $g_s = \frac{15}{20} = \frac{3}{4} = 0.75$ m/sec^2

12. Solving $s_m = 832t - 2.6t^2 = 0 \Rightarrow t(832 - 2.6t) = 0 \Rightarrow t = 0$ or $320 \Rightarrow$ 320 sec on the moon; solving $s_e = 832t - 16t^2 = 0 \Rightarrow t(832 - 16t) = 0 \Rightarrow t = 0$ or $52 \Rightarrow$ 52 sec on the earth. Also, $v_m = 832 - 5.2t = 0$ $\Rightarrow t = 160$ and $s_m(160) = 66{,}560$ ft, the height it reaches above the moon's surface; $v_e = 832 - 32t = 0$ $\Rightarrow t = 26$ and $s_e(26) = 10{,}816$ ft, the height it reaches above the earth's surface.

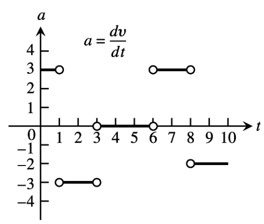
13. (a) $s = 179 - 16t^2 \Rightarrow v = -32t \Rightarrow \text{speed} = |v| = 32t$ ft/sec and $a = -32$ ft/sec^2
(b) $s = 0 \Rightarrow 179 - 16t^2 = 0 \Rightarrow t = \sqrt{\frac{179}{16}} \approx 3.3$ sec
(c) When $t = \sqrt{\frac{179}{16}}$, $v = -32\sqrt{\frac{179}{16}} = -8\sqrt{179} \approx -107.0$ ft/sec

14. (a) $\lim_{\theta \to \frac{\pi}{2}} v = \lim_{\theta \to \frac{\pi}{2}} 9.8(\sin\theta)t = 9.8t$ so we expect $v = 9.8t$ m/sec in free fall
(b) $a = \frac{dv}{dt} = 9.8$ m/sec^2

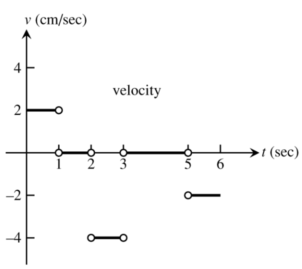
15. (a) at 2 and 7 seconds
(b) between 3 and 6 seconds: $3 \le t \le 6$
(c)

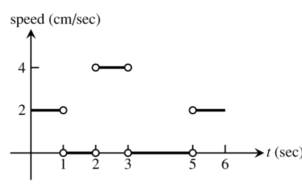


(d)



16. (a) P is moving to the left when $2 < t < 3$ or $5 < t < 6$; P is moving to the right when $0 < t < 1$; P is standing still when $1 < t < 2$ or $3 < t < 5$
(b)





17. (a) 190 ft/sec
(b) 2 sec
(c) at 8 sec, 0 ft/sec
(d) 10.8 sec, 90 ft/sec
(e) From $t = 8$ until $t = 10.8$ sec, a total of 2.8 sec
(f) Greatest acceleration happens 2 sec after launch
(g) From $t = 2$ to $t = 10.8$ sec; during this period, $a = \frac{v(10.8) - v(2)}{10.8 - 2} \approx -32$ ft/sec^2

18. (a) Forward: $0 \le t < 1$ and $5 < t < 7$; Backward: $1 < t < 5$; Speeds up: $1 < t < 2$ and $5 < t < 6$; Slows down: $0 \le t < 1$, $3 < t < 5$, and $6 < t < 7$
(b) Positive: $3 < t < 6$; negative: $0 \le t < 2$ and $6 < t < 7$; zero: $2 < t < 3$ and $7 < t < 9$
(c) $t = 0$ and $2 \le t \le 3$
(d) $7 \le t \le 9$

19. $s = 490t^2 \Rightarrow v = 980t \Rightarrow a = 980$
(a) Solving $160 = 490t^2 \Rightarrow t = \frac{4}{7}$ sec. The average velocity was $\frac{s(4/7) - s(0)}{4/7} = 280$ cm/sec.

(b) At the 160 cm mark the balls are falling at $v(4/7) = 560$ cm/sec. The acceleration at the 160 cm mark was 980 cm/sec^2.

(c) The light was flashing at a rate of $\frac{17}{4/7} = 29.75$ flashes per second.

20. (a)

(b)

21. $C =$ position, $A =$ velocity, and $B =$ acceleration. Neither A nor C can be the derivative of B because B's derivative is constant. Graph C cannot be the derivative of A either, because A has some negative slopes while C has only positive values. So, C (being the derivative of neither A nor B) must be the graph of position. Curve C has both positive and negative slopes, so its derivative, the velocity, must be A and not B. That leaves B for acceleration.

22. $C =$ position, $B =$ velocity, and $A =$ acceleration. Curve C cannot be the derivative of either A or B because C has only negative values while both A and B have some positive slopes. So, C represents position. Curve C has no positive slopes, so its derivative, the velocity, must be B. That leaves A for acceleration. Indeed, A is negative where B has negative slopes and positive where B has positive slopes.

23. (a) $c(100) = 11{,}000 \Rightarrow c_{av} = \frac{11{,}000}{100} = \$110$

(b) $c(x) = 2000 + 100x - .1x^2 \Rightarrow c'(x) = 100 - .2x$. Marginal cost $= c'(x) \Rightarrow$ the marginal cost of producing 100 machines is $c'(100) = \$80$

(c) The cost of producing the 101st machine is $c(101) - c(100) = 100 - \frac{201}{10} = \$79.90$

24. (a) $r(x) = 20000\left(1 - \frac{1}{x}\right) \Rightarrow r'(x) = \frac{20000}{x^2}$, which is marginal revenue. $r'(100) = \frac{20000}{100^2} = \$2$.

(b) $r'(101) = \$1.96$.

(c) $\lim_{x \to \infty} r'(x) = \lim_{x \to \infty} \frac{20000}{x^2} = 0$. The increase in revenue as the number of items increases without bound will approach zero.

25. $b(t) = 10^6 + 10^4 t - 10^3 t^2 \Rightarrow b'(t) = 10^4 - (2)(10^3 t) = 10^3(10 - 2t)$

(a) $b'(0) = 10^4$ bacteria/hr

(b) $b'(5) = 0$ bacteria/hr

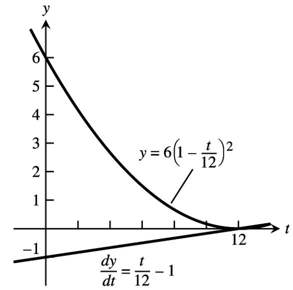
(c) $b'(10) = -10^4$ bacteria/hr

26. $Q(t) = 200(30 - t)^2 = 200\left(900 - 60t + t^2\right) \Rightarrow Q'(t) = 200(-60 + 2t) \Rightarrow Q'(10) = -8{,}000$ gallons/min is the rate the water is running at the end of 10 min. Then $\frac{Q(10) - Q(0)}{10} = -10{,}000$ gallons/min is the average rate the water flows during the first 10 min. The negative signs indicate water is <u>leaving</u> the tank.

27. (a) $y = 6\left(1 - \frac{t}{12}\right)^2 = 6\left(1 - \frac{t}{6} + \frac{t^2}{144}\right) \Rightarrow \frac{dy}{dt} = \frac{t}{12} - 1$

(b) The largest value of $\frac{dy}{dt}$ is 0 m/h when $t = 12$ and the fluid level is falling the slowest at that time. The smallest value of $\frac{dy}{dt}$ is -1 m/h, when $t = 0$, and the fluid level is falling the fastest at that time.

(c) In this situation, $\frac{dy}{dt} \le 0 \Rightarrow$ the graph of y is always decreasing. As $\frac{dy}{dt}$ increases in value, the slope of the graph of y increases from -1 to 0 over the interval $0 \le t \le 12$.



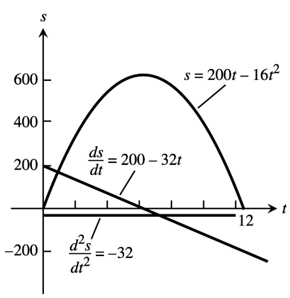
28. (a) $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2 \Rightarrow \left.\frac{dV}{dr}\right|_{r=2} = 4\pi(2)^2 = 16\pi$ ft^3/ft

(b) When $r = 2$, $\frac{dV}{dr} = 16\pi$ so that when r changes by 1 unit, we expect V to change by approximately 16π. Therefore when r changes by 0.2 units V changes by approximately $(16\pi)(0.2) = 3.2\pi \approx 10.05$ ft^3. Note that $V(2.2) - V(2) \approx 11.09$ ft^3.

29. 200 km/hr $= 55\frac{5}{9}$m/sec $= \frac{500}{9}$ m/sec, and $D = \frac{10}{9}t^2 \Rightarrow V = \frac{20}{9}t$. Thus $V = \frac{500}{9} \Rightarrow \frac{20}{9}t = \frac{500}{9} \Rightarrow t = 25$ sec. When $t = 25$, $D = \frac{10}{9}(25)^2 = \frac{6250}{9}$ m

30. $s = v_0 t - 16t^2 \Rightarrow v = v_0 - 32t$; $v = 0 \Rightarrow t = \frac{v_0}{32}$; $1900 = v_0 t - 16t^2$ so that $t = \frac{v_0}{32} \Rightarrow 1900 = \frac{v_0^2}{32} - \frac{v_0^2}{64}$
$\Rightarrow v_0 = \sqrt{(64)(1900)} = 80\sqrt{19}$ ft/sec and, finally, $\frac{80\sqrt{19}\text{ ft}}{\text{sec}} \cdot \frac{60\text{ sec}}{1\text{ min}} \cdot \frac{60\text{ min}}{1\text{ hr}} \cdot \frac{1\text{ mi}}{5280\text{ ft}} \approx 238$ mph.

31.



(a) $v = 0$ when $t = 6.25$ sec

(b) $v > 0$ when $0 \le t < 6.25 \Rightarrow$ body moves right (up); $v < 0$ when $6.25 < t \le 12.5 \Rightarrow$ body moves left (down)

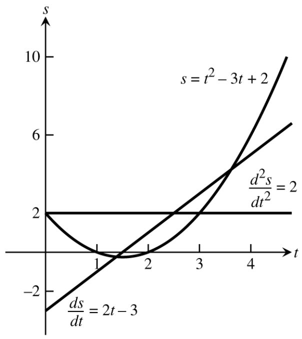
(c) body changes direction at $t = 6.25$ sec

(d) body speeds up on $(6.25, 12.5]$ and slows down on $[0, 6.25)$

(e) The body is moving fastest at the endpoints $t = 0$ and $t = 12.5$ when it is traveling 200 ft/sec. It's moving slowest at $t = 6.25$ when the speed is 0.

(f) When $t = 6.25$ the body is $s = 625$ m from the origin and farthest away.

32.



(a) $\mathrm{v} = 0$ when $\mathrm{t} = \frac{3}{2}$ sec

(b) $\mathrm{v} < 0$ when $0 \le \mathrm{t} < 1.5 \Rightarrow$ body moves left (down); $\mathrm{v} > 0$ when $1.5 < \mathrm{t} \le 5 \Rightarrow$ body moves right (up)

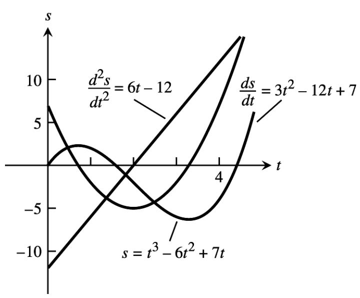
(c) body changes direction at $\mathrm{t} = \frac{3}{2}$ sec

(d) body speeds up on $\left(\frac{3}{2}, 5\right]$ and slows down on $\left[0, \frac{3}{2}\right)$

(e) body is moving fastest at $\mathrm{t} = 5$ when the speed $= |\mathrm{v}(5)| = 7$ units/sec; it is moving slowest at $\mathrm{t} = \frac{3}{2}$ when the speed is 0

(f) When $\mathrm{t} = 5$ the body is $\mathrm{s} = 12$ units from the origin and farthest away.

33.



(a) $\mathrm{v} = 0$ when $\mathrm{t} = \frac{6 \pm \sqrt{15}}{3}$ sec

(b) $\mathrm{v} < 0$ when $\frac{6-\sqrt{15}}{3} < \mathrm{t} < \frac{6+\sqrt{15}}{3} \Rightarrow$ body moves left (down); $\mathrm{v} > 0$ when $0 \le \mathrm{t} < \frac{6-\sqrt{15}}{3}$ or $\frac{6+\sqrt{15}}{3} < \mathrm{t} \le 4$ $\Rightarrow$ body moves right (up)

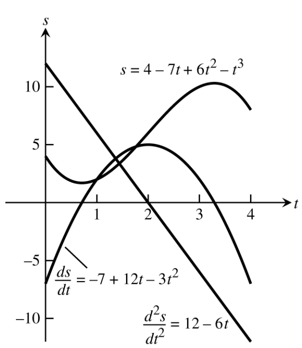
(c) body changes direction at $\mathrm{t} = \frac{6 \pm \sqrt{15}}{3}$ sec

(d) body speeds up on $\left(\frac{6-\sqrt{15}}{3}, 2\right) \cup \left(\frac{6+\sqrt{15}}{3}, 4\right]$ and slows down on $\left[0, \frac{6-\sqrt{15}}{3}\right) \cup \left(2, \frac{6+\sqrt{15}}{3}\right)$.

(e) The body is moving fastest at $\mathrm{t} = 0$ and $\mathrm{t} = 4$ when it is moving 7 units/sec and slowest at $\mathrm{t} = \frac{6\pm\sqrt{15}}{3}$ sec

(f) When $\mathrm{t} = \frac{6+\sqrt{15}}{3}$ the body is at position $\mathrm{s} \approx -6.303$ units and farthest from the origin.

34.



(a) $\mathrm{v} = 0$ when $\mathrm{t} = \frac{6 \pm \sqrt{15}}{3}$

(b) $v < 0$ when $0 \le t < \frac{6-\sqrt{15}}{3}$ or $\frac{6+\sqrt{15}}{3} < t \le 4 \Rightarrow$ body is moving left (down); $v > 0$ when $\frac{6-\sqrt{15}}{3} < t < \frac{6+\sqrt{15}}{3} \Rightarrow$ body is moving right (up)

(c) body changes direction at $t = \frac{6 \pm \sqrt{15}}{3}$ sec

(d) body speeds up on $\left(\frac{6-\sqrt{15}}{3}, 2\right) \cup \left(\frac{6+\sqrt{15}}{3}, 4\right]$ and slows down on $\left[0, \frac{6-\sqrt{15}}{3}\right) \cup \left(2, \frac{6+\sqrt{15}}{3}\right)$

(e) The body is moving fastest at 7 units/sec when $t = 0$ and $t = 4$; it is moving slowest and stationary at $t = \frac{6 \pm \sqrt{15}}{3}$

(f) When $t = \frac{6+\sqrt{15}}{3}$ the position is $s \approx 10.303$ units and the body is farthest from the origin.

3.5 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

1. $y = -10x + 3\cos x \Rightarrow \frac{dy}{dx} = -10 + 3\frac{d}{dx}(\cos x) = -10 - 3\sin x$

2. $y = \frac{3}{x} + 5\sin x \Rightarrow \frac{dy}{dx} = \frac{-3}{x^2} + 5\frac{d}{dx}(\sin x) = \frac{-3}{x^2} + 5\cos x$

3. $y = x^2\cos x \Rightarrow \frac{dy}{dx} = x^2(-\sin x) + 2x\cos x = -x^2\sin x + 2x\cos x$

4. $y = \sqrt{x}\sec x + 3 \Rightarrow \frac{dy}{dx} = \sqrt{x}\sec x\tan x + \frac{\sec x}{2\sqrt{x}} + 0 = \sqrt{x}\sec x\tan x + \frac{\sec x}{2\sqrt{x}}$

5. $y = \csc x - 4\sqrt{x} + 7 \Rightarrow \frac{dy}{dx} = -\csc x\cot x - \frac{4}{2\sqrt{x}} + 0 = -\csc x\cot x - \frac{2}{\sqrt{x}}$

6. $y = x^2\cot x - \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = x^2\frac{d}{dx}(\cot x) + \cot x \cdot \frac{d}{dx}(x^2) + \frac{2}{x^3} = -x^2\csc^2 x + (\cot x)(2x) + \frac{2}{x^3}$
$= -x^2\csc^2 x + 2x\cot x + \frac{2}{x^3}$

7. $f(x) = \sin x\tan x \Rightarrow f'(x) = \sin x\sec^2 x + \cos x\tan x = \sin x\sec^2 x + \cos x\frac{\sin x}{\cos x} = \sin x(\sec^2 x + 1)$

8. $g(x) = \csc x\cot x \Rightarrow g'(x) = \csc x(-\csc^2 x) + (-\csc x\cot x)\cot x = -\csc^3 x - \csc x\cot^2 x = -\csc x(\csc^2 x + \cot^2 x)$

9. $y = (\sec x + \tan x)(\sec x - \tan x) \Rightarrow \frac{dy}{dx} = (\sec x + \tan x)\frac{d}{dx}(\sec x - \tan x) + (\sec x - \tan x)\frac{d}{dx}(\sec x + \tan x)$
$= (\sec x + \tan x)(\sec x\tan x - \sec^2 x) + (\sec x - \tan x)(\sec x\tan x + \sec^2 x)$
$= (\sec^2 x\tan x + \sec x\tan^2 x - \sec^3 x - \sec^2 x\tan x) + (\sec^2 x\tan x - \sec x\tan^2 x + \sec^3 x - \tan x\sec^2 x) = 0.$
$\left(\text{Note also that } y = \sec^2 x - \tan^2 x = (\tan^2 x + 1) - \tan^2 x = 1 \Rightarrow \frac{dy}{dx} = 0.\right)$

10. $y = (\sin x + \cos x)\sec x \Rightarrow \frac{dy}{dx} = (\sin x + \cos x)\frac{d}{dx}(\sec x) + \sec x\frac{d}{dx}(\sin x + \cos x)$
$= (\sin x + \cos x)(\sec x\tan x) + (\sec x)(\cos x - \sin x) = \frac{(\sin x + \cos x)\sin x}{\cos^2 x} + \frac{\cos x - \sin x}{\cos x}$
$= \frac{\sin^2 x + \cos x\sin x + \cos^2 x - \cos x\sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$
$\left(\text{Note also that } y = \sin x\sec x + \cos x\sec x = \tan x + 1 \Rightarrow \frac{dy}{dx} = \sec^2 x.\right)$

11. $y = \frac{\cot x}{1+\cot x} \Rightarrow \frac{dy}{dx} = \frac{(1+\cot x)\frac{d}{dx}(\cot x) - (\cot x)\frac{d}{dx}(1+\cot x)}{(1+\cot x)^2} = \frac{(1+\cot x)(-\csc^2 x) - (\cot x)(-\csc^2 x)}{(1+\cot x)^2}$
$= \frac{-\csc^2 x - \csc^2 x\cot x + \csc^2 x\cot x}{(1+\cot x)^2} = \frac{-\csc^2 x}{(1+\cot x)^2}$

12. $y = \frac{\cos x}{1+\sin x} \Rightarrow \frac{dy}{dx} = \frac{(1+\sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2} = \frac{(1+\sin x)(-\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2}$
$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} = \frac{-\sin x - 1}{(1+\sin x)^2} = \frac{-(1+\sin x)}{(1+\sin x)^2} = \frac{-1}{1+\sin x}$

13. $y = \frac{4}{\cos x} + \frac{1}{\tan x} = 4 \sec x + \cot x \Rightarrow \frac{dy}{dx} = 4 \sec x \tan x - \csc^2 x$

14. $y = \frac{\cos x}{x} + \frac{x}{\cos x} \Rightarrow \frac{dy}{dx} = \frac{x(-\sin x) - (\cos x)(1)}{x^2} + \frac{(\cos x)(1) - x(-\sin x)}{\cos^2 x} = \frac{-x \sin x - \cos x}{x^2} + \frac{\cos x + x \sin x}{\cos^2 x}$

15. $y = x^2 \sin x + 2x \cos x - 2 \sin x \Rightarrow \frac{dy}{dx} = (x^2 \cos x + (\sin x)(2x)) + ((2x)(-\sin x) + (\cos x)(2)) - 2 \cos x$
$= x^2 \cos x + 2x \sin x - 2x \sin x + 2 \cos x - 2 \cos x = x^2 \cos x$

16. $y = x^2 \cos x - 2x \sin x - 2 \cos x \Rightarrow \frac{dy}{dx} = (x^2(-\sin x) + (\cos x)(2x)) - (2x \cos x + (\sin x)(2)) - 2(-\sin x)$
$= -x^2 \sin x + 2x \cos x - 2x \cos x - 2 \sin x + 2 \sin x = -x^2 \sin x$

17. $f(x) = x^3 \sin x \cos x \Rightarrow f'(x) = x^3 \sin x(-\sin x) + x^3 \cos x(\cos x) + 3x^2 \sin x \cos x = -x^3 \sin^2 x + x^3 \cos^2 x + 3x^2 \sin x \cos x$

18. $g(x) = (2-x)\tan^2 x \Rightarrow g'(x) = (2-x)(2 \tan x \sec^2 x) + (-1)\tan^2 x = 2(2-x)\tan x \sec^2 x - \tan^2 x$
$= 2(2-x)\tan x(\sec^2 x - \tan x)$

19. $s = \tan t - t \Rightarrow \frac{ds}{dt} = \sec^2 t - 1$

20. $s = t^2 - \sec t + 1 \Rightarrow \frac{ds}{dt} = 2t - \sec t \tan t$

21. $s = \frac{1 + \csc t}{1 - \csc t} \Rightarrow \frac{ds}{dt} = \frac{(1 - \csc t)(-\csc t \cot t) - (1 + \csc t)(\csc t \cot t)}{(1 - \csc t)^2}$
$= \frac{-\csc t \cot t + \csc^2 t \cot t - \csc t \cot t - \csc^2 t \cot t}{(1 - \csc t)^2} = \frac{-2 \csc t \cot t}{(1 - \csc t)^2}$

22. $s = \frac{\sin t}{1 - \cos t} \Rightarrow \frac{ds}{dt} = \frac{(1 - \cos t)(\cos t) - (\sin t)(\sin t)}{(1 - \cos t)^2} = \frac{\cos t - \cos^2 t - \sin^2 t}{(1 - \cos t)^2} = \frac{\cos t - 1}{(1 - \cos t)^2} = -\frac{1}{1 - \cos t} = \frac{1}{\cos t - 1}$

23. $r = 4 - \theta^2 \sin \theta \Rightarrow \frac{dr}{d\theta} = -\left(\theta^2 \frac{d}{d\theta}(\sin \theta) + (\sin \theta)(2\theta)\right) = -(\theta^2 \cos \theta + 2\theta \sin \theta) = -\theta(\theta \cos \theta + 2 \sin \theta)$

24. $r = \theta \sin \theta + \cos \theta \Rightarrow \frac{dr}{d\theta} = (\theta \cos \theta + (\sin \theta)(1)) - \sin \theta = \theta \cos \theta$

25. $r = \sec \theta \csc \theta \Rightarrow \frac{dr}{d\theta} = (\sec \theta)(-\csc \theta \cot \theta) + (\csc \theta)(\sec \theta \tan \theta)$
$= \left(\frac{-1}{\cos \theta}\right)\left(\frac{1}{\sin \theta}\right)\left(\frac{\cos \theta}{\sin \theta}\right) + \left(\frac{1}{\sin \theta}\right)\left(\frac{1}{\cos \theta}\right)\left(\frac{\sin \theta}{\cos \theta}\right) = \frac{-1}{\sin^2 \theta} + \frac{1}{\cos^2 \theta} = \sec^2 \theta - \csc^2 \theta$

26. $r = (1 + \sec \theta) \sin \theta \Rightarrow \frac{dr}{d\theta} = (1 + \sec \theta) \cos \theta + (\sin \theta)(\sec \theta \tan \theta) = (\cos \theta + 1) + \tan^2 \theta = \cos \theta + \sec^2 \theta$

27. $p = 5 + \frac{1}{\cot q} = 5 + \tan q \Rightarrow \frac{dp}{dq} = \sec^2 q$

28. $p = (1 + \csc q) \cos q \Rightarrow \frac{dp}{dq} = (1 + \csc q)(-\sin q) + (\cos q)(-\csc q \cot q) = (-\sin q - 1) - \cot^2 q = -\sin q - \csc^2 q$

29. $p = \frac{\sin q + \cos q}{\cos q} \Rightarrow \frac{dp}{dq} = \frac{(\cos q)(\cos q - \sin q) - (\sin q + \cos q)(-\sin q)}{\cos^2 q}$
$= \frac{\cos^2 q - \cos q \sin q + \sin^2 q + \cos q \sin q}{\cos^2 q} = \frac{1}{\cos^2 q} = \sec^2 q$

30. $p = \frac{\tan q}{1 + \tan q} \Rightarrow \frac{dp}{dq} = \frac{(1 + \tan q)(\sec^2 q) - (\tan q)(\sec^2 q)}{(1 + \tan q)^2} = \frac{\sec^2 q + \tan q \sec^2 q - \tan q \sec^2 q}{(1 + \tan q)^2} = \frac{\sec^2 q}{(1 + \tan q)^2}$

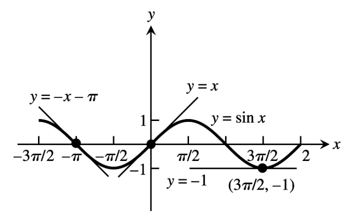
31. $p = \frac{q \sin q}{q^2 - 1} \Rightarrow \frac{dp}{dq} = \frac{(q^2 - 1)(q \cos q + \sin q(1)) - (q \sin q)(2q)}{(q^2 - 1)^2} = \frac{q^3 \cos q + q^2 \sin q - q \cos q - \sin q - 2q^2 \sin q}{(q^2 - 1)^2}$
$= \frac{q^3 \cos q - q^2 \sin q - q \cos q - \sin q}{(q^2 - 1)^2}$

32. $p = \frac{3q + \tan q}{q \sec q} \Rightarrow \frac{dp}{dq} = \frac{(q \sec q)(3 + \sec^2 q) - (3q + \tan q)(q \sec q \tan q + \sec q(1))}{(q \sec q)^2}$
$= \frac{3q \sec q + q \sec^3 q - (3q^2 \sec q \tan q + 3q \sec q + q \sec q \tan^2 q + \sec q \tan q)}{(q \sec q)^2} = \frac{q \sec^3 q - 3q^2 \sec q \tan q - q \sec q \tan^2 q - \sec q \tan q}{(q \sec q)^2}$

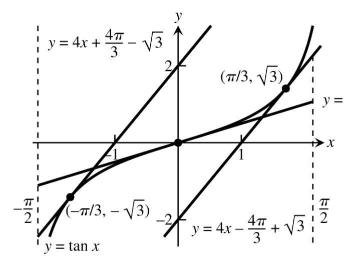
33. (a) $y = \csc x \Rightarrow y' = -\csc x \cot x \Rightarrow y'' = -\left((\csc x)(-\csc^2 x) + (\cot x)(-\csc x \cot x)\right) = \csc^3 x + \csc x \cot^2 x$
$= (\csc x)(\csc^2 x + \cot^2 x) = (\csc x)(\csc^2 x + \csc^2 x - 1) = 2\csc^3 x - \csc x$
(b) $y = \sec x \Rightarrow y' = \sec x \tan x \Rightarrow y'' = (\sec x)(\sec^2 x) + (\tan x)(\sec x \tan x) = \sec^3 x + \sec x \tan^2 x$
$= (\sec x)(\sec^2 x + \tan^2 x) = (\sec x)(\sec^2 x + \sec^2 x - 1) = 2\sec^3 x - \sec x$

34. (a) $y = -2\sin x \Rightarrow y' = -2\cos x \Rightarrow y'' = -2(-\sin x) = 2\sin x \Rightarrow y''' = 2\cos x \Rightarrow y^{(4)} = -2\sin x$
(b) $y = 9\cos x \Rightarrow y' = -9\sin x \Rightarrow y'' = -9\cos x \Rightarrow y''' = -9(-\sin x) = 9\sin x \Rightarrow y^{(4)} = 9\cos x$

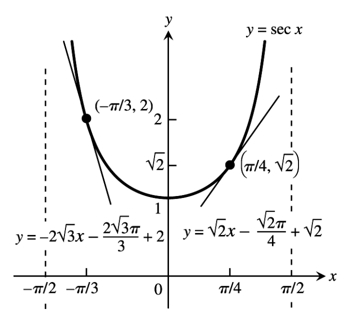
35. $y = \sin x \Rightarrow y' = \cos x \Rightarrow$ slope of tangent at $x = -\pi$ is $y'(-\pi) = \cos(-\pi) = -1$; slope of tangent at $x = 0$ is $y'(0) = \cos(0) = 1$; and slope of tangent at $x = \frac{3\pi}{2}$ is $y'\left(\frac{3\pi}{2}\right) = \cos\frac{3\pi}{2} = 0$. The tangent at $(-\pi, 0)$ is $y - 0 = -1(x + \pi)$, or $y = -x - \pi$; the tangent at $(0, 0)$ is $y - 0 = 1(x - 0)$, or $y = x$; and the tangent at $\left(\frac{3\pi}{2}, -1\right)$ is $y = -1$.



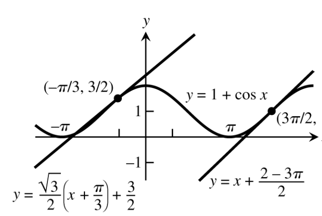
36. $y = \tan x \Rightarrow y' = \sec^2 x \Rightarrow$ slope of tangent at $x = -\frac{\pi}{3}$ is $\sec^2\left(-\frac{\pi}{3}\right) = 4$; slope of tangent at $x = 0$ is $\sec^2(0) = 1$; and slope of tangent at $x = \frac{\pi}{3}$ is $\sec^2\left(\frac{\pi}{3}\right) = 4$. The tangent at $\left(-\frac{\pi}{3}, \tan\left(-\frac{\pi}{3}\right)\right) = \left(-\frac{\pi}{3}, -\sqrt{3}\right)$ is $y + \sqrt{3} = 4\left(x + \frac{\pi}{3}\right)$; the tangent at $(0, 0)$ is $y = x$; and the tangent at $\left(\frac{\pi}{3}, \tan\left(\frac{\pi}{3}\right)\right) = \left(\frac{\pi}{3}, \sqrt{3}\right)$ is $y - \sqrt{3} = 4\left(x - \frac{\pi}{3}\right)$.



37. $y = \sec x \Rightarrow y' = \sec x \tan x \Rightarrow$ slope of tangent at $x = -\frac{\pi}{3}$ is $\sec\left(-\frac{\pi}{3}\right)\tan\left(-\frac{\pi}{3}\right) = -2\sqrt{3}$; slope of tangent at $x = \frac{\pi}{4}$ is $\sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = \sqrt{2}$. The tangent at the point $\left(-\frac{\pi}{3}, \sec\left(-\frac{\pi}{3}\right)\right) = \left(-\frac{\pi}{3}, 2\right)$ is $y - 2 = -2\sqrt{3}\left(x + \frac{\pi}{3}\right)$; the tangent at the point $\left(\frac{\pi}{4}, \sec\left(\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{4}, \sqrt{2}\right)$ is $y - \sqrt{2} = \sqrt{2}\left(x - \frac{\pi}{4}\right)$.



38. $y = 1 + \cos x \Rightarrow y' = -\sin x \Rightarrow$ slope of tangent at $x = -\frac{\pi}{3}$ is $-\sin\left(-\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$; slope of tangent at $x = \frac{3\pi}{2}$ is $-\sin\left(\frac{3\pi}{2}\right) = 1$. The tangent at the point $\left(-\frac{\pi}{3}, 1 + \cos\left(-\frac{\pi}{3}\right)\right) = \left(-\frac{\pi}{3}, \frac{3}{2}\right)$ is $y - \frac{3}{2} = \frac{\sqrt{3}}{2}\left(x + \frac{\pi}{3}\right)$; the tangent at the point $\left(\frac{3\pi}{2}, 1 + \cos\left(\frac{3\pi}{2}\right)\right) = \left(\frac{3\pi}{2}, 1\right)$ is $y - 1 = x - \frac{3\pi}{2}$



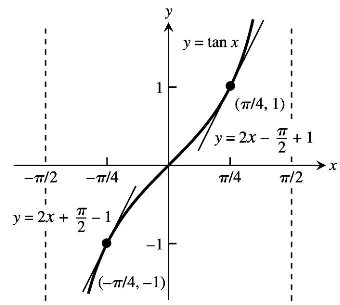
39. Yes, $y = x + \sin x \Rightarrow y' = 1 + \cos x$; horizontal tangent occurs where $1 + \cos x = 0 \Rightarrow \cos x = -1 \Rightarrow x = \pi$

40. No, $y = 2x + \sin x \Rightarrow y' = 2 + \cos x$; horizontal tangent occurs where $2 + \cos x = 0 \Rightarrow \cos x = -2$. But there are no x-values for which $\cos x = -2$.

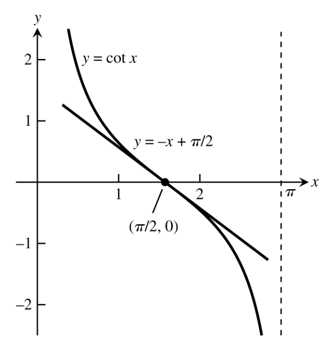
41. No, $y = x - \cot x \Rightarrow y' = 1 + \csc^2 x$; horizontal tangent occurs where $1 + \csc^2 x = 0 \Rightarrow \csc^2 x = -1$. But there are no x-values for which $\csc^2 x = -1$.

42. Yes, $y = x + 2\cos x \Rightarrow y' = 1 - 2\sin x$; horizontal tangent occurs where $1 - 2\sin x = 0 \Rightarrow 1 = 2\sin x$ $\Rightarrow \frac{1}{2} = \sin x \Rightarrow x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$

43. We want all points on the curve where the tangent line has slope 2. Thus, $y = \tan x \Rightarrow y' = \sec^2 x$ so that $y' = 2 \Rightarrow \sec^2 x = 2 \Rightarrow \sec x = \pm\sqrt{2}$ $\Rightarrow x = \pm\frac{\pi}{4}$. Then the tangent line at $\left(\frac{\pi}{4}, 1\right)$ has equation $y - 1 = 2\left(x - \frac{\pi}{4}\right)$; the tangent line at $\left(-\frac{\pi}{4}, -1\right)$ has equation $y + 1 = 2\left(x + \frac{\pi}{4}\right)$.



44. We want all points on the curve $y = \cot x$ where the tangent line has slope -1. Thus $y = \cot x$ $\Rightarrow y' = -\csc^2 x$ so that $y' = -1 \Rightarrow -\csc^2 x = -1$ $\Rightarrow \csc^2 x = 1 \Rightarrow \csc x = \pm 1 \Rightarrow x = \frac{\pi}{2}$. The tangent line at $\left(\frac{\pi}{2}, 0\right)$ is $y = -x + \frac{\pi}{2}$.



45. $y = 4 + \cot x - 2\csc x \Rightarrow y' = -\csc^2 x + 2\csc x \cot x = -\left(\frac{1}{\sin x}\right)\left(\frac{1 - 2\cos x}{\sin x}\right)$
    (a) When $x = \frac{\pi}{2}$, then $y' = -1$; the tangent line is $y = -x + \frac{\pi}{2} + 2$.
    (b) To find the location of the horizontal tangent set $y' = 0 \Rightarrow 1 - 2\cos x = 0 \Rightarrow x = \frac{\pi}{3}$ radians. When $x = \frac{\pi}{3}$, then $y = 4 - \sqrt{3}$ is the horizontal tangent.

46. $y = 1 + \sqrt{2}\csc x + \cot x \Rightarrow y' = -\sqrt{2}\csc x \cot x - \csc^2 x = -\left(\frac{1}{\sin x}\right)\left(\frac{\sqrt{2}\cos x + 1}{\sin x}\right)$
    (a) If $x = \frac{\pi}{4}$, then $y' = -4$; the tangent line is $y = -4x + \pi + 4$.
    (b) To find the location of the horizontal tangent set $y' = 0 \Rightarrow \sqrt{2}\cos x + 1 = 0 \Rightarrow x = \frac{3\pi}{4}$ radians. When $x = \frac{3\pi}{4}$, then $y = 2$ is the horizontal tangent.

47. $\lim_{x \to 2} \sin\left(\frac{1}{x} - \frac{1}{2}\right) = \sin\left(\frac{1}{2} - \frac{1}{2}\right) = \sin 0 = 0$

48. $\lim_{x \to -\frac{\pi}{6}} \sqrt{1 + \cos(\pi \csc x)} = \sqrt{1 + \cos\left(\pi \csc\left(-\frac{\pi}{6}\right)\right)} = \sqrt{1 + \cos(\pi \cdot (-2))} = \sqrt{2}$

49. $\lim_{\theta \to \frac{\pi}{6}} \frac{\sin\theta - \frac{1}{2}}{\theta - \frac{\pi}{6}} = \frac{d}{d\theta}(\sin\theta)\Big|_{\theta = \frac{\pi}{6}} = \cos\theta\Big|_{\theta = \frac{\pi}{6}} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

50. $\lim\limits_{\theta \to \frac{\pi}{4}} \frac{\tan\theta - 1}{\theta - \frac{\pi}{4}} = \frac{d}{d\theta}(\tan\theta)\Big|_{\theta=\frac{\pi}{4}} = \sec^2\theta\Big|_{\theta=\frac{\pi}{4}} = \sec^2\left(\frac{\pi}{4}\right) = 2$

51. $\lim\limits_{x \to 0} \sec\left[\cos x + \pi \tan\left(\frac{\pi}{4 \sec x}\right) - 1\right] = \sec\left[1 + \pi \tan\left(\frac{\pi}{4 \sec 0}\right) - 1\right] = \; = \sec\left[\pi \tan\left(\frac{\pi}{4}\right)\right] = \sec\pi = -1$

52. $\lim\limits_{x \to 0} \sin\left(\frac{\pi + \tan x}{\tan x - 2\sec x}\right) = \sin\left(\frac{\pi + \tan 0}{\tan 0 - 2\sec 0}\right) = \sin\left(-\frac{\pi}{2}\right) = -1$

53. $\lim\limits_{t \to 0} \tan\left(1 - \frac{\sin t}{t}\right) = \tan\left(1 - \lim\limits_{t \to 0} \frac{\sin t}{t}\right) = \tan(1 - 1) = 0$

54. $\lim\limits_{\theta \to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right) = \cos\left(\pi \lim\limits_{\theta \to 0} \frac{\theta}{\sin\theta}\right) = \cos\left(\pi \cdot \frac{1}{\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta}}\right) = \cos\left(\pi \cdot \frac{1}{1}\right) = -1$

55. $s = 2 - 2\sin t \Rightarrow v = \frac{ds}{dt} = -2\cos t \Rightarrow a = \frac{dv}{dt} = 2\sin t \Rightarrow j = \frac{da}{dt} = 2\cos t$. Therefore, velocity $= v\left(\frac{\pi}{4}\right)$ $= -\sqrt{2}$ m/sec; speed $= \left|v\left(\frac{\pi}{4}\right)\right| = \sqrt{2}$ m/sec; acceleration $= a\left(\frac{\pi}{4}\right) = \sqrt{2}$ m/sec^2; jerk $= j\left(\frac{\pi}{4}\right) = \sqrt{2}$ m/sec^3.

56. $s = \sin t + \cos t \Rightarrow v = \frac{ds}{dt} = \cos t - \sin t \Rightarrow a = \frac{dv}{dt} = -\sin t - \cos t \Rightarrow j = \frac{da}{dt} = -\cos t + \sin t$. Therefore velocity $= v\left(\frac{\pi}{4}\right) = 0$ m/sec; speed $= \left|v\left(\frac{\pi}{4}\right)\right| = 0$ m/sec; acceleration $= a\left(\frac{\pi}{4}\right) = -\sqrt{2}$ m/sec^2; jerk $= j\left(\frac{\pi}{4}\right) = 0$ m/sec^3.

57. $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \frac{\sin^2 3x}{x^2} = \lim\limits_{x \to 0} 9\left(\frac{\sin 3x}{3x}\right)\left(\frac{\sin 3x}{3x}\right) = 9$ so that f is continuous at $x = 0 \Rightarrow \lim\limits_{x \to 0} f(x) = f(0) \Rightarrow 9 = c$.

58. $\lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^-} (x + b) = b$ and $\lim\limits_{x \to 0^+} g(x) = \lim\limits_{x \to 0^+} \cos x = 1$ so that g is continuous at $x = 0 \Rightarrow \lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^+} g(x) \Rightarrow b = 1$. Now g is not differentiable at $x = 0$: At $x = 0$, the left-hand derivative is $\frac{d}{dx}(x + b)\big|_{x=0} = 1$, but the right-hand derivative is $\frac{d}{dx}(\cos x)\big|_{x=0} = -\sin 0 = 0$. The left- and right-hand derivatives can never agree at $x = 0$, so g is not differentiable at $x = 0$ for any value of b (including $b = 1$).

59. $\frac{d^{999}}{dx^{999}}(\cos x) = \sin x$ because $\frac{d^4}{dx^4}(\cos x) = \cos x \Rightarrow$ the derivative of cos x any number of times that is a multiple of 4 is cos x. Thus, dividing 999 by 4 gives $999 = 249 \cdot 4 + 3 \Rightarrow \frac{d^{999}}{dx^{999}}(\cos x)$ $= \frac{d^3}{dx^3}\left[\frac{d^{249\cdot 4}}{dx^{249\cdot 4}}(\cos x)\right] = \frac{d^3}{dx^3}(\cos x) = \sin x$.

60. (a) $y = \sec x = \frac{1}{\cos x} \Rightarrow \frac{dy}{dx} = \frac{(\cos x)(0) - (1)(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos^2 x} = \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) = \sec x \tan x$
$\Rightarrow \frac{d}{dx}(\sec x) = \sec x \tan x$

(b) $y = \csc x = \frac{1}{\sin x} \Rightarrow \frac{dy}{dx} = \frac{(\sin x)(0) - (1)(\cos x)}{(\sin x)^2} = \frac{-\cos x}{\sin^2 x} = \left(\frac{-1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) = -\csc x \cot x$
$\Rightarrow \frac{d}{dx}(\csc x) = -\csc x \cot x$

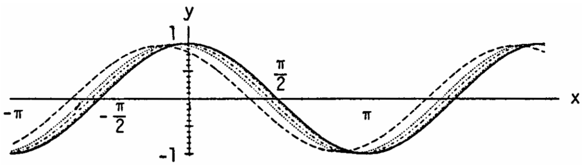
(c) $y = \cot x = \frac{\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$
$\Rightarrow \frac{d}{dx}(\cot x) = -\csc^2 x$

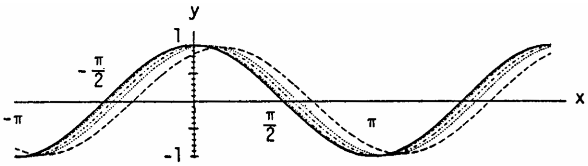
61. (a) $t = 0 \to x = 10\cos(0) = 10\,\text{cm}$; $t = \frac{\pi}{3} \to x = 10\cos\left(\frac{\pi}{3}\right) = 5\,\text{cm}$; $t = \frac{3\pi}{4} \to x = 10\cos\left(\frac{3\pi}{4}\right) = -5\sqrt{2}\,\text{cm}$

(b) $t = 0 \to v = -10\sin(0) = 0\,\frac{\text{cm}}{\text{sec}}$; $t = \frac{\pi}{3} \to v = -10\sin\left(\frac{\pi}{3}\right) = -5\sqrt{3}\,\frac{\text{cm}}{\text{sec}}$; $t = \frac{3\pi}{4} \to v = -10\sin\left(\frac{3\pi}{4}\right) = -5\sqrt{2}\,\frac{\text{cm}}{\text{sec}}$

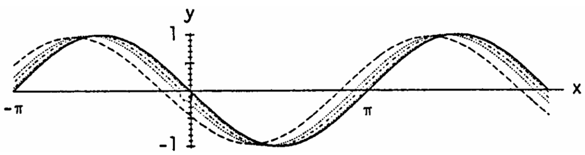
62. (a) $t = 0 \to x = 3\cos(0) + 4\sin(0) = 3$ ft; $t = \frac{\pi}{2} \to x = 3\cos\left(\frac{\pi}{2}\right) + 4\sin\left(\frac{\pi}{2}\right) = 4$ ft;
$t = \pi \to x = 3\cos(\pi) + 4\sin(\pi) = -3$ ft

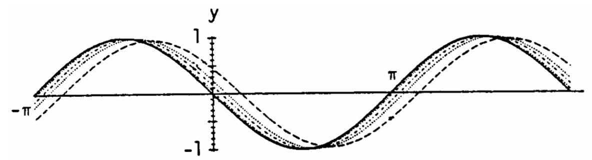
(b) $t = 0 \to v = -3\sin(0) + 4\cos(0) = 4\,\frac{\text{ft}}{\text{sec}}$; $t = \frac{\pi}{2} \to v = -3\sin\left(\frac{\pi}{2}\right) + 4\cos\left(\frac{\pi}{2}\right) = -3\,\frac{\text{ft}}{\text{sec}}$;
$t = \pi \to v = -3\sin(\pi) + 4\cos(\pi) = -4\,\frac{\text{ft}}{\text{sec}}$

63. 



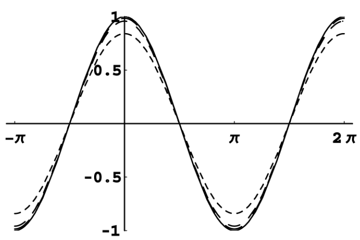
As h takes on the values of 1, 0.5, 0.3 and 0.1 the corresponding dashed curves of $y = \frac{\sin(x+h) - \sin x}{h}$ get closer and closer to the black curve $y = \cos x$ because $\frac{d}{dx}(\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \cos x$. The same is true as h takes on the values of -1, -0.5, -0.3 and -0.1.

64. 



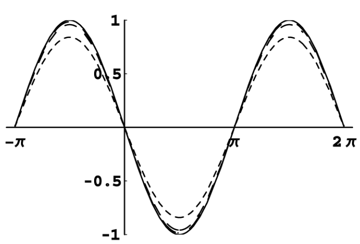
As h takes on the values of 1, 0.5, 0.3, and 0.1 the corresponding dashed curves of $y = \frac{\cos(x+h) - \cos x}{h}$ get closer and closer to the black curve $y = -\sin x$ because $\frac{d}{dx}(\cos x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = -\sin x$. The same is true as h takes on the values of -1, -0.5, -0.3, and -0.1.

65. (a)



The dashed curves of $y = \frac{\sin(x+h) - \sin(x-h)}{2h}$ are closer to the black curve $y = \cos x$ than the corresponding dashed curves in Exercise 63 illustrating that the centered difference quotient is a better approximation of the derivative of this function.

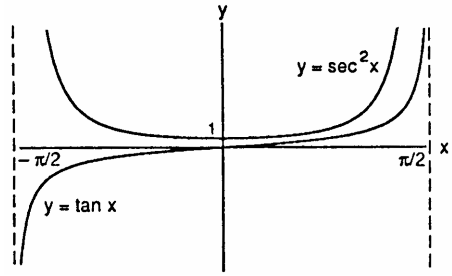
(b)



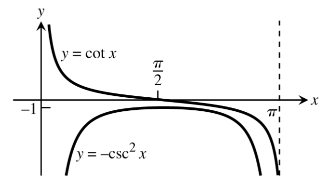
The dashed curves of $y = \frac{\cos(x+h) - \cos(x-h)}{2h}$ are closer to the black curve $y = -\sin x$ than the corresponding dashed curves in Exercise 64 illustrating that the centered difference quotient is a better approximation of the derivative of this function.

66. $\lim_{h \to 0} \frac{|0+h| - |0-h|}{2h} = \lim_{x \to 0} \frac{|h| - |h|}{2h} = \lim_{h \to 0} 0 = 0 \Rightarrow$ the limits of the centered difference quotient exists even though the derivative of $f(x) = |x|$ does not exist at $x = 0$.

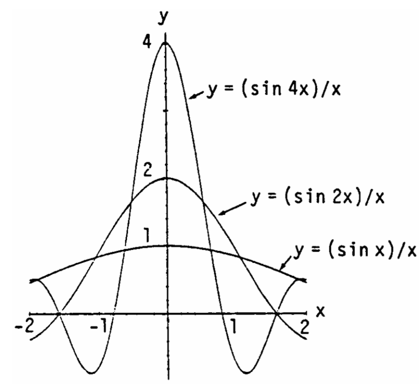
67. $y = \tan x \Rightarrow y' = \sec^2 x$, so the smallest value $y' = \sec^2 x$ takes on is $y' = 1$ when $x = 0$; y' has no maximum value since $\sec^2 x$ has no largest value on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; y' is never negative since $\sec^2 x \geq 1$.



68. $y = \cot x \Rightarrow y' = -\csc^2 x$ so y' has no smallest value since $-\csc^2 x$ has no minimum value on $(0, \pi)$; the largest value of y' is -1, when $x = \frac{\pi}{2}$; the slope is never positive since the largest value $y' = -\csc^2 x$ takes on is -1.



69. $y = \frac{\sin x}{x}$ appears to cross the y-axis at $y = 1$, since $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$; $y = \frac{\sin 2x}{x}$ appears to cross the y-axis at $y = 2$, since $\lim\limits_{x \to 0} \frac{\sin 2x}{x} = 2$; $y = \frac{\sin 4x}{x}$ appears to cross the y-axis at $y = 4$, since $\lim\limits_{x \to 0} \frac{\sin 4x}{x} = 4$. However, none of these graphs actually cross the y-axis since $x = 0$ is not in the domain of the functions. Also, $\lim\limits_{x \to 0} \frac{\sin 5x}{x} = 5$, $\lim\limits_{x \to 0} \frac{\sin(-3x)}{x} = -3$, and $\lim\limits_{x \to 0} \frac{\sin kx}{x} = k \Rightarrow$ the graphs of $y = \frac{\sin 5x}{x}$, $y = \frac{\sin(-3x)}{x}$, and $y = \frac{\sin kx}{x}$ approach 5, -3, and k, respectively, as $x \to 0$. However, the graphs do not actually cross the y-axis.



70. (a)

h	$\frac{\sin h}{h}$	$\left(\frac{\sin h}{h}\right)\left(\frac{180}{\pi}\right)$
1	.017452406	.99994923
0.01	.017453292	1
0.001	.017453292	1
0.0001	.017453292	1

$$\lim_{h \to 0} \frac{\sin h^\circ}{h} = \lim_{x \to 0} \frac{\sin\left(h \cdot \frac{\pi}{180}\right)}{h} = \lim_{h \to 0} \frac{\frac{\pi}{180} \sin\left(h \cdot \frac{\pi}{180}\right)}{\frac{\pi}{180} \cdot h} = \lim_{\theta \to 0} \frac{\frac{\pi}{180} \sin \theta}{\theta} = \frac{\pi}{180} \qquad \left(\theta = h \cdot \frac{\pi}{180}\right)$$

(converting to radians)

(b)

h	$\frac{\cos h - 1}{h}$
1	-0.0001523
0.01	-0.0000015
0.001	-0.0000001
0.0001	0

$\lim\limits_{h \to 0} \frac{\cos h - 1}{h} = 0$, whether h is measured in degrees or radians.

(c) In degrees, $\frac{d}{dx}(\sin x) = \lim\limits_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim\limits_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}$

$$= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h}\right) = (\sin x) \cdot \lim_{h \to 0} \left(\frac{\cos h - 1}{h}\right) + (\cos x) \cdot \lim_{h \to 0} \left(\frac{\sin h}{h}\right)$$

$$= (\sin x)(0) + (\cos x)\left(\frac{\pi}{180}\right) = \frac{\pi}{180} \cos x$$

(d) In degrees, $\frac{d}{dx}(\cos x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h}$
$= \lim_{h \to 0} \frac{(\cos x)(\cos h - 1) - \sin x \sin h}{h} = \lim_{h \to 0} \left(\cos x \cdot \frac{\cos h - 1}{h}\right) - \lim_{h \to 0} \left(\sin x \cdot \frac{\sin h}{h}\right)$
$= (\cos x) \lim_{h \to 0} \left(\frac{\cos h - 1}{h}\right) - (\sin x) \lim_{h \to 0} \left(\frac{\sin h}{h}\right) = (\cos x)(0) - (\sin x)\left(\frac{\pi}{180}\right) = -\frac{\pi}{180} \sin x$

(e) $\frac{d^2}{dx^2}(\sin x) = \frac{d}{dx}\left(\frac{\pi}{180}\cos x\right) = -\left(\frac{\pi}{180}\right)^2 \sin x$; $\frac{d^3}{dx^3}(\sin x) = \frac{d}{dx}\left(-\left(\frac{\pi}{180}\right)^2 \sin x\right) = -\left(\frac{\pi}{180}\right)^3 \cos x$;
$\frac{d^2}{dx^2}(\cos x) = \frac{d}{dx}\left(-\frac{\pi}{180}\sin x\right) = -\left(\frac{\pi}{180}\right)^2 \cos x$; $\frac{d^3}{dx^3}(\cos x) = \frac{d}{dx}\left(-\left(\frac{\pi}{180}\right)^2 \cos x\right) = \left(\frac{\pi}{180}\right)^3 \sin x$

3.6 THE CHAIN RULE

1. $f(u) = 6u - 9 \Rightarrow f'(u) = 6 \Rightarrow f'(g(x)) = 6$; $g(x) = \frac{1}{2}x^4 \Rightarrow g'(x) = 2x^3$; therefore $\frac{dy}{dx} = f'(g(x))g'(x) = 6 \cdot 2x^3 = 12x^3$

2. $f(u) = 2u^3 \Rightarrow f'(u) = 6u^2 \Rightarrow f'(g(x)) = 6(8x-1)^2$; $g(x) = 8x - 1 \Rightarrow g'(x) = 8$; therefore $\frac{dy}{dx} = f'(g(x))g'(x)$
$= 6(8x-1)^2 \cdot 8 = 48(8x-1)^2$

3. $f(u) = \sin u \Rightarrow f'(u) = \cos u \Rightarrow f'(g(x)) = \cos(3x+1)$; $g(x) = 3x + 1 \Rightarrow g'(x) = 3$; therefore $\frac{dy}{dx} = f'(g(x))g'(x)$
$= (\cos(3x+1))(3) = 3\cos(3x+1)$

4. $f(u) = \cos u \Rightarrow f'(u) = -\sin u \Rightarrow f'(g(x)) = -\sin\left(\frac{-x}{3}\right)$; $g(x) = \frac{-x}{3} \Rightarrow g'(x) = -\frac{1}{3}$; therefore $\frac{dy}{dx} = f'(g(x))g'(x)$
$= -\sin\left(\frac{-x}{3}\right) \cdot \left(\frac{-1}{3}\right) = \frac{1}{3}\sin\left(\frac{-x}{3}\right)$

5. $f(u) = \cos u \Rightarrow f'(u) = -\sin u \Rightarrow f'(g(x)) = -\sin(\sin x)$; $g(x) = \sin x \Rightarrow g'(x) = \cos x$; therefore
$\frac{dy}{dx} = f'(g(x))g'(x) = -(\sin(\sin x))\cos x$

6. $f(u) = \sin u \Rightarrow f'(u) = \cos u \Rightarrow f'(g(x)) = \cos(x - \cos x)$; $g(x) = x - \cos x \Rightarrow g'(x) = 1 + \sin x$; therefore
$\frac{dy}{dx} = f'(g(x))g'(x) = (\cos(x - \cos x))(1 + \sin x)$

7. $f(u) = \tan u \Rightarrow f'(u) = \sec^2 u \Rightarrow f'(g(x)) = \sec^2(10x - 5)$; $g(x) = 10x - 5 \Rightarrow g'(x) = 10$; therefore
$\frac{dy}{dx} = f'(g(x))g'(x) = (\sec^2(10x-5))(10) = 10\sec^2(10x-5)$

8. $f(u) = -\sec u \Rightarrow f'(u) = -\sec u \tan u \Rightarrow f'(g(x)) = -\sec(x^2+7x)\tan(x^2+7x)$; $g(x) = x^2 + 7x$
$\Rightarrow g'(x) = 2x + 7$; therefore $\frac{dy}{dx} = f'(g(x))g'(x) = -(2x+7)\sec(x^2+7x)\tan(x^2+7x)$

9. With $u = (2x+1)$, $y = u^5$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 5u^4 \cdot 2 = 10(2x+1)^4$

10. With $u = (4-3x)$, $y = u^9$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 9u^8 \cdot (-3) = -27(4-3x)^8$

11. With $u = \left(1 - \frac{x}{7}\right)$, $y = u^{-7}$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -7u^{-8} \cdot \left(-\frac{1}{7}\right) = \left(1 - \frac{x}{7}\right)^{-8}$

12. With $u = \left(\frac{x}{2} - 1\right)$, $y = u^{-10}$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -10u^{-11} \cdot \left(\frac{1}{2}\right) = -5\left(\frac{x}{2} - 1\right)^{-11}$

13. With $u = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)$, $y = u^4$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 4u^3 \cdot \left(\frac{x}{4} + 1 + \frac{1}{x^2}\right) = 4\left(\frac{x^2}{8} + x - \frac{1}{x}\right)^3\left(\frac{x}{4} + 1 + \frac{1}{x^2}\right)$

14. With $u = 3x^2 - 4x + 6$, $y = u^{1/2}$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2}u^{-1/2} \cdot (6x - 4) = \frac{3x-2}{\sqrt{3x^2-4x+6}}$

15. With $u = \tan x$, $y = \sec u$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\sec u \tan u)(\sec^2 x) = (\sec(\tan x)\tan(\tan x))\sec^2 x$

16. With $u = \pi - \frac{1}{x}$, $y = \cot u$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (-\csc^2 u)\left(\frac{1}{x^2}\right) = -\frac{1}{x^2}\csc^2\left(\pi - \frac{1}{x}\right)$

17. With $u = \sin x$, $y = u^3$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2\cos x = 3(\sin^2 x)(\cos x)$

18. With $u = \cos x$, $y = 5u^{-4}$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (-20u^{-5})(-\sin x) = 20(\cos^{-5} x)(\sin x)$

19. $p = \sqrt{3-t} = (3-t)^{1/2} \Rightarrow \frac{dp}{dt} = \frac{1}{2}(3-t)^{-1/2} \cdot \frac{d}{dt}(3-t) = -\frac{1}{2}(3-t)^{-1/2} = \frac{-1}{2\sqrt{3-t}}$

20. $q = \sqrt[3]{2r - r^2} = (2r - r^2)^{1/3} \Rightarrow \frac{dq}{dr} = \frac{1}{3}(2r - r^2)^{-2/3} \cdot \frac{d}{dr}(2r - r^2) = \frac{1}{3}(2r - r^2)^{-2/3}(2 - 2r) = \frac{2-2r}{3(2r-r^2)^{2/3}}$

21. $s = \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\cos 5t \Rightarrow \frac{ds}{dt} = \frac{4}{3\pi}\cos 3t \cdot \frac{d}{dt}(3t) + \frac{4}{5\pi}(-\sin 5t) \cdot \frac{d}{dt}(5t) = \frac{4}{\pi}\cos 3t - \frac{4}{\pi}\sin 5t$
$= \frac{4}{\pi}(\cos 3t - \sin 5t)$

22. $s = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right) \Rightarrow \frac{ds}{dt} = \cos\left(\frac{3\pi t}{2}\right) \cdot \frac{d}{dt}\left(\frac{3\pi t}{2}\right) - \sin\left(\frac{3\pi t}{2}\right) \cdot \frac{d}{dt}\left(\frac{3\pi t}{2}\right) = \frac{3\pi}{2}\cos\left(\frac{3\pi t}{2}\right) - \frac{3\pi}{2}\sin\left(\frac{3\pi t}{2}\right)$
$= \frac{3\pi}{2}\left(\cos\frac{3\pi t}{2} - \sin\frac{3\pi t}{2}\right)$

23. $r = (\csc\theta + \cot\theta)^{-1} \Rightarrow \frac{dr}{d\theta} = -(\csc\theta + \cot\theta)^{-2}\frac{d}{d\theta}(\csc\theta + \cot\theta) = \frac{\csc\theta\cot\theta + \csc^2\theta}{(\csc\theta + \cot\theta)^2} = \frac{\csc\theta(\cot\theta + \csc\theta)}{(\csc\theta + \cot\theta)^2} = \frac{\csc\theta}{\csc\theta + \cot\theta}$

24. $r = 6(\sec\theta - \tan\theta)^{3/2} \Rightarrow \frac{dr}{d\theta} = 6 \cdot \frac{3}{2}(\sec\theta - \tan\theta)^{1/2}\frac{d}{d\theta}(\sec\theta - \tan\theta) = 9\sqrt{\sec\theta - \tan\theta}(\sec\theta\tan\theta - \sec^2\theta)$

25. $y = x^2\sin^4 x + x\cos^{-2} x \Rightarrow \frac{dy}{dx} = x^2\frac{d}{dx}(\sin^4 x) + \sin^4 x \cdot \frac{d}{dx}(x^2) + x\frac{d}{dx}(\cos^{-2} x) + \cos^{-2} x \cdot \frac{d}{dx}(x)$
$= x^2\left(4\sin^3 x\frac{d}{dx}(\sin x)\right) + 2x\sin^4 x + x\left(-2\cos^{-3} x \cdot \frac{d}{dx}(\cos x)\right) + \cos^{-2} x$
$= x^2(4\sin^3 x\cos x) + 2x\sin^4 x + x((-2\cos^{-3} x)(-\sin x)) + \cos^{-2} x$
$= 4x^2\sin^3 x\cos x + 2x\sin^4 x + 2x\sin x\cos^{-3} x + \cos^{-2} x$

26. $y = \frac{1}{x}\sin^{-5} x - \frac{x}{3}\cos^3 x \Rightarrow y' = \frac{1}{x}\frac{d}{dx}(\sin^{-5} x) + \sin^{-5} x \cdot \frac{d}{dx}\left(\frac{1}{x}\right) - \frac{x}{3}\frac{d}{dx}(\cos^3 x) - \cos^3 x \cdot \frac{d}{dx}\left(\frac{x}{3}\right)$
$= \frac{1}{x}(-5\sin^{-6} x\cos x) + (\sin^{-5} x)\left(-\frac{1}{x^2}\right) - \frac{x}{3}((3\cos^2 x)(-\sin x)) - (\cos^3 x)\left(\frac{1}{3}\right)$
$= -\frac{5}{x}\sin^{-6} x\cos x - \frac{1}{x^2}\sin^{-5} x + x\cos^2 x\sin x - \frac{1}{3}\cos^3 x$

27. $y = \frac{1}{21}(3x-2)^7 + \left(4 - \frac{1}{2x^2}\right)^{-1} \Rightarrow \frac{dy}{dx} = \frac{7}{21}(3x-2)^6 \cdot \frac{d}{dx}(3x-2) + (-1)\left(4 - \frac{1}{2x^2}\right)^{-2} \cdot \frac{d}{dx}\left(4 - \frac{1}{2x^2}\right)$
$= \frac{7}{21}(3x-2)^6 \cdot 3 + (-1)\left(4 - \frac{1}{2x^2}\right)^{-2}\left(\frac{1}{x^3}\right) = (3x-2)^6 - \frac{1}{x^3\left(4 - \frac{1}{2x^2}\right)^2}$

28. $y = (5-2x)^{-3} + \frac{1}{8}\left(\frac{2}{x} + 1\right)^4 \Rightarrow \frac{dy}{dx} = -3(5-2x)^{-4}(-2) + \frac{4}{8}\left(\frac{2}{x} + 1\right)^3\left(-\frac{2}{x^2}\right) = 6(5-2x)^{-4} - \left(\frac{1}{x^2}\right)\left(\frac{2}{x} + 1\right)^3$
$= \frac{6}{(5-2x)^4} - \frac{\left(\frac{2}{x}+1\right)^3}{x^2}$

29. $y = (4x+3)^4(x+1)^{-3} \Rightarrow \frac{dy}{dx} = (4x+3)^4(-3)(x+1)^{-4} \cdot \frac{d}{dx}(x+1) + (x+1)^{-3}(4)(4x+3)^3 \cdot \frac{d}{dx}(4x+3)$
$= (4x+3)^4(-3)(x+1)^{-4}(1) + (x+1)^{-3}(4)(4x+3)^3(4) = -3(4x+3)^4(x+1)^{-4} + 16(4x+3)^3(x+1)^{-3}$
$= \frac{(4x+3)^3}{(x+1)^4}[-3(4x+3) + 16(x+1)] = \frac{(4x+3)^3(4x+7)}{(x+1)^4}$

30. $y = (2x-5)^{-1}\left(x^2-5x\right)^6 \Rightarrow \frac{dy}{dx} = (2x-5)^{-1}(6)\left(x^2-5x\right)^5(2x-5) + \left(x^2-5x\right)^6(-1)(2x-5)^{-2}(2)$
$= 6\left(x^2-5x\right)^5 - \frac{2\left(x^2-5x\right)^6}{(2x-5)^2}$

31. $h(x) = x\tan\left(2\sqrt{x}\right) + 7 \Rightarrow h'(x) = x\frac{d}{dx}\left(\tan\left(2x^{1/2}\right)\right) + \tan\left(2x^{1/2}\right)\cdot\frac{d}{dx}(x) + 0$
$= x\sec^2\left(2x^{1/2}\right)\cdot\frac{d}{dx}\left(2x^{1/2}\right) + \tan\left(2x^{1/2}\right) = x\sec^2\left(2\sqrt{x}\right)\cdot\frac{1}{\sqrt{x}} + \tan\left(2\sqrt{x}\right) = \sqrt{x}\sec^2\left(2\sqrt{x}\right) + \tan\left(2\sqrt{x}\right)$

32. $k(x) = x^2\sec\left(\frac{1}{x}\right) \Rightarrow k'(x) = x^2\frac{d}{dx}\left(\sec\frac{1}{x}\right) + \sec\left(\frac{1}{x}\right)\cdot\frac{d}{dx}\left(x^2\right) = x^2\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)\cdot\frac{d}{dx}\left(\frac{1}{x}\right) + 2x\sec\left(\frac{1}{x}\right)$
$= x^2\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right) + 2x\sec\left(\frac{1}{x}\right) = 2x\sec\left(\frac{1}{x}\right) - \sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)$

33. $f(x) = \sqrt{7 + x\sec x} \Rightarrow f'(x) = \frac{1}{2}(7 + x\sec x)^{-1/2}(x\cdot(\sec x\tan x) + (\sec x)\cdot 1) = \frac{x\sec x\tan x + \sec x}{2\sqrt{7+x\sec x}}$

34. $g(x) = \frac{\tan 3x}{(x+7)^4} \Rightarrow g'(x) = \frac{(x+7)^4\left(\sec^2 3x\cdot 3\right) - (\tan 3x)4(x+7)^3\cdot 1}{\left[(x+7)^4\right]^2} = \frac{(x+7)^3\left(3(x+7)\sec^2 3x - 4\tan 3x\right)}{(x+7)^8}$
$= \frac{\left(3(x+7)\sec^2 3x - 4\tan 3x\right)}{(x+7)^5}$

35. $f(\theta) = \left(\frac{\sin\theta}{1+\cos\theta}\right)^2 \Rightarrow f'(\theta) = 2\left(\frac{\sin\theta}{1+\cos\theta}\right)\cdot\frac{d}{d\theta}\left(\frac{\sin\theta}{1+\cos\theta}\right) = \frac{2\sin\theta}{1+\cos\theta}\cdot\frac{(1+\cos\theta)(\cos\theta) - (\sin\theta)(-\sin\theta)}{(1+\cos\theta)^2}$
$= \frac{(2\sin\theta)\left(\cos\theta + \cos^2\theta + \sin^2\theta\right)}{(1+\cos\theta)^3} = \frac{(2\sin\theta)(\cos\theta + 1)}{(1+\cos\theta)^3} = \frac{2\sin\theta}{(1+\cos\theta)^2}$

36. $g(t) = \left(\frac{1+\sin 3t}{3-2t}\right)^{-1} = \frac{3-2t}{1+\sin 3t} \Rightarrow g'(t) = \frac{(1+\sin 3t)(-2) - (3-2t)(3\cos 3t)}{(1+\sin 3t)^2} = \frac{-2 - 2\sin 3t - 9\cos 3t + 6t\cos 3t}{(1+\sin 3t)^2}$

37. $r = \sin\left(\theta^2\right)\cos(2\theta) \Rightarrow \frac{dr}{d\theta} = \sin\left(\theta^2\right)(-\sin 2\theta)\frac{d}{d\theta}(2\theta) + \cos(2\theta)\left(\cos\left(\theta^2\right)\right)\cdot\frac{d}{d\theta}\left(\theta^2\right)$
$= \sin\left(\theta^2\right)(-\sin 2\theta)(2) + (\cos 2\theta)\left(\cos\left(\theta^2\right)\right)(2\theta) = -2\sin\left(\theta^2\right)\sin(2\theta) + 2\theta\cos(2\theta)\cos\left(\theta^2\right)$

38. $r = \left(\sec\sqrt{\theta}\right)\tan\left(\frac{1}{\theta}\right) \Rightarrow \frac{dr}{d\theta} = \left(\sec\sqrt{\theta}\right)\left(\sec^2\frac{1}{\theta}\right)\left(-\frac{1}{\theta^2}\right) + \tan\left(\frac{1}{\theta}\right)\left(\sec\sqrt{\theta}\tan\sqrt{\theta}\right)\left(\frac{1}{2\sqrt{\theta}}\right)$
$= -\frac{1}{\theta^2}\sec\sqrt{\theta}\sec^2\left(\frac{1}{\theta}\right) + \frac{1}{2\sqrt{\theta}}\tan\left(\frac{1}{\theta}\right)\sec\sqrt{\theta}\tan\sqrt{\theta} = \left(\sec\sqrt{\theta}\right)\left[\frac{\tan\sqrt{\theta}\tan\left(\frac{1}{\theta}\right)}{2\sqrt{\theta}} - \frac{\sec^2\left(\frac{1}{\theta}\right)}{\theta^2}\right]$

39. $q = \sin\left(\frac{t}{\sqrt{t+1}}\right) \Rightarrow \frac{dq}{dt} = \cos\left(\frac{t}{\sqrt{t+1}}\right)\cdot\frac{d}{dt}\left(\frac{t}{\sqrt{t+1}}\right) = \cos\left(\frac{t}{\sqrt{t+1}}\right)\cdot\frac{\sqrt{t+1}\,(1) - t\cdot\frac{d}{dt}\left(\sqrt{t+1}\right)}{\left(\sqrt{t+1}\right)^2}$
$= \cos\left(\frac{t}{\sqrt{t+1}}\right)\cdot\frac{\sqrt{t+1} - \frac{t}{2\sqrt{t+1}}}{t+1} = \cos\left(\frac{t}{\sqrt{t+1}}\right)\left(\frac{2(t+1)-t}{2(t+1)^{3/2}}\right) = \left(\frac{t+2}{2(t+1)^{3/2}}\right)\cos\left(\frac{t}{\sqrt{t+1}}\right)$

40. $q = \cot\left(\frac{\sin t}{t}\right) \Rightarrow \frac{dq}{dt} = -\csc^2\left(\frac{\sin t}{t}\right)\cdot\frac{d}{dt}\left(\frac{\sin t}{t}\right) = \left(-\csc^2\left(\frac{\sin t}{t}\right)\right)\left(\frac{t\cos t - \sin t}{t^2}\right)$

41. $y = \sin^2(\pi t - 2) \Rightarrow \frac{dy}{dt} = 2\sin(\pi t - 2)\cdot\frac{d}{dt}\sin(\pi t - 2) = 2\sin(\pi t - 2)\cdot\cos(\pi t - 2)\cdot\frac{d}{dt}(\pi t - 2)$
$= 2\pi\sin(\pi t - 2)\cos(\pi t - 2)$

42. $y = \sec^2\pi t \Rightarrow \frac{dy}{dt} = (2\sec\pi t)\cdot\frac{d}{dt}(\sec\pi t) = (2\sec\pi t)(\sec\pi t\tan\pi t)\cdot\frac{d}{dt}(\pi t) = 2\pi\sec^2\pi t\tan\pi t$

43. $y = (1+\cos 2t)^{-4} \Rightarrow \frac{dy}{dt} = -4(1+\cos 2t)^{-5}\cdot\frac{d}{dt}(1+\cos 2t) = -4(1+\cos 2t)^{-5}(-\sin 2t)\cdot\frac{d}{dt}(2t) = \frac{8\sin 2t}{(1+\cos 2t)^5}$

44. $y = \left(1+\cot\left(\frac{t}{2}\right)\right)^{-2} \Rightarrow \frac{dy}{dt} = -2\left(1+\cot\left(\frac{t}{2}\right)\right)^{-3}\cdot\frac{d}{dt}\left(1+\cot\left(\frac{t}{2}\right)\right) = -2\left(1+\cot\left(\frac{t}{2}\right)\right)^{-3}\cdot\left(-\csc^2\left(\frac{t}{2}\right)\right)\cdot\frac{d}{dt}\left(\frac{t}{2}\right)$
$= \frac{\csc^2\left(\frac{t}{2}\right)}{\left(1+\cot\left(\frac{t}{2}\right)\right)^3}$

45. $y = (t\tan t)^{10} \Rightarrow \frac{dy}{dt} = 10(t\tan t)^9(t\cdot\sec^2 t + 1\cdot\tan t) = 10t^9\tan^9 t(t\sec^2 t + \tan t) = 10t^{10}\tan^9 t\sec^2 t + 10t^9\tan^{10} t$

46. $y = \left(t^{-3/4}\sin t\right)^{4/3} = t^{-1}(\sin t)^{4/3} \Rightarrow \frac{dy}{dt} = t^{-1}\left(\frac{4}{3}\right)(\sin t)^{1/3}\cos t - t^{-2}(\sin t)^{4/3} = \frac{4(\sin t)^{1/3}\cos t}{3t} - \frac{(\sin t)^{4/3}}{t^2}$
$= \frac{(\sin t)^{1/3}(4t\cos t - 3\cos t)}{3t^2}$

47. $y = \left(\frac{t^2}{t^3-4t}\right)^3 \Rightarrow \frac{dy}{dt} = 3\left(\frac{t^2}{t^3-4t}\right)^2\cdot\frac{(t^3-4t)(2t)-t^2(3t^2-4)}{(t^3-4t)^2} = \frac{3t^4}{(t^3-4t)^2}\cdot\frac{2t^4-8t^2-3t^4+4t^2}{(t^3-4t)^2} = \frac{3t^4(-t^4-4t^2)}{t^4(t^2-4)^4} = \frac{-3t^2(t^2+4)}{(t^2-4)^4}$

48. $y = \left(\frac{3t-4}{5t+2}\right)^{-5} \Rightarrow \frac{dy}{dt} = -5\left(\frac{3t-4}{5t+2}\right)^{-6}\cdot\frac{(5t+2)\cdot 3-(3t-4)\cdot 5}{(5t+2)^2} = -5\left(\frac{5t+2}{3t-4}\right)^6\cdot\frac{15t+6-15t+20}{(5t+2)^2} = -5\frac{(5t+2)^6}{(3t-4)^6}\cdot\frac{26}{(5t+2)^2}$
$= \frac{-130(5t+2)^4}{(3t-4)^6}$

49. $y = \sin(\cos(2t-5)) \Rightarrow \frac{dy}{dt} = \cos(\cos(2t-5))\cdot\frac{d}{dt}\cos(2t-5) = \cos(\cos(2t-5))\cdot(-\sin(2t-5))\cdot\frac{d}{dt}(2t-5)$
$= -2\cos(\cos(2t-5))(\sin(2t-5))$

50. $y = \cos\left(5\sin\left(\frac{t}{3}\right)\right) \Rightarrow \frac{dy}{dt} = -\sin\left(5\sin\left(\frac{t}{3}\right)\right)\cdot\frac{d}{dt}\left(5\sin\left(\frac{t}{3}\right)\right) = -\sin\left(5\sin\left(\frac{t}{3}\right)\right)\left(5\cos\left(\frac{t}{3}\right)\right)\cdot\frac{d}{dt}\left(\frac{t}{3}\right)$
$= -\frac{5}{3}\sin\left(5\sin\left(\frac{t}{3}\right)\right)\left(\cos\left(\frac{t}{3}\right)\right)$

51. $y = \left[1+\tan^4\left(\frac{t}{12}\right)\right]^3 \Rightarrow \frac{dy}{dt} = 3\left[1+\tan^4\left(\frac{t}{12}\right)\right]^2\cdot\frac{d}{dt}\left[1+\tan^4\left(\frac{t}{12}\right)\right] = 3\left[1+\tan^4\left(\frac{t}{12}\right)\right]^2\left[4\tan^3\left(\frac{t}{12}\right)\cdot\frac{d}{dt}\tan\left(\frac{t}{12}\right)\right]$
$= 12\left[1+\tan^4\left(\frac{t}{12}\right)\right]^2\left[\tan^3\left(\frac{t}{12}\right)\sec^2\left(\frac{t}{12}\right)\cdot\frac{1}{12}\right] = \left[1+\tan^4\left(\frac{t}{12}\right)\right]^2\left[\tan^3\left(\frac{t}{12}\right)\sec^2\left(\frac{t}{12}\right)\right]$

52. $y = \frac{1}{6}\left[1+\cos^2(7t)\right]^3 \Rightarrow \frac{dy}{dt} = \frac{3}{6}\left[1+\cos^2(7t)\right]^2\cdot 2\cos(7t)(-\sin(7t))(7) = -7\left[1+\cos^2(7t)\right]^2(\cos(7t)\sin(7t))$

53. $y = (1+\cos(t^2))^{1/2} \Rightarrow \frac{dy}{dt} = \frac{1}{2}(1+\cos(t^2))^{-1/2}\cdot\frac{d}{dt}(1+\cos(t^2)) = \frac{1}{2}(1+\cos(t^2))^{-1/2}\left(-\sin(t^2)\cdot\frac{d}{dt}(t^2)\right)$
$= -\frac{1}{2}(1+\cos(t^2))^{-1/2}(\sin(t^2))\cdot 2t = -\frac{t\sin(t^2)}{\sqrt{1+\cos(t^2)}}$

54. $y = 4\sin\left(\sqrt{1+\sqrt{t}}\right) \Rightarrow \frac{dy}{dt} = 4\cos\left(\sqrt{1+\sqrt{t}}\right)\cdot\frac{d}{dt}\left(\sqrt{1+\sqrt{t}}\right) = 4\cos\left(\sqrt{1+\sqrt{t}}\right)\cdot\frac{1}{2\sqrt{1+\sqrt{t}}}\cdot\frac{d}{dt}\left(1+\sqrt{t}\right)$
$= \frac{2\cos\left(\sqrt{1+\sqrt{t}}\right)}{\sqrt{1+\sqrt{t}}\cdot 2\sqrt{t}} = \frac{\cos\left(\sqrt{1+\sqrt{t}}\right)}{\sqrt{t+t\sqrt{t}}}$

55. $y = \tan^2(\sin^3 t) \Rightarrow \frac{dy}{dt} = 2\tan(\sin^3 t)\cdot\sec^2(\sin^3 t)\cdot(3\sin^2 t\cdot(\cos t)) = 6\tan(\sin^3 t)\sec^2(\sin^3 t)\sin^2 t\cos t$

56. $y = \cos^4(\sec^2 3t) \Rightarrow \frac{dy}{dt} = 4\cos^3(\sec^2(3t))(-\sin(\sec^2(3t))\cdot 2(\sec(3t))(\sec(3t)\tan(3t)\cdot 3))$
$= -24\cos^3(\sec^2(3t))\sin(\sec^2(3t))\sec^2(3t)\tan(3t)$

57. $y = 3t(2t^2-5)^4 \Rightarrow \frac{dy}{dt} = 3t\cdot 4(2t^2-5)^3(4t) + 3\cdot(2t^2-5)^4 = 3(2t^2-5)^3\left[16t^2+2t^2-5\right] = 3(2t^2-5)^3(18t^2-5)$

58. $y = \sqrt{3t+\sqrt{2+\sqrt{1-t}}} \Rightarrow \frac{dy}{dt} = \frac{1}{2}\left(3t+\sqrt{2+\sqrt{1-t}}\right)^{-1/2}\left(3+\frac{1}{2}\left(2+\sqrt{1-t}\right)^{-1/2}\frac{1}{2}(1-t)^{-1/2}(-1)\right)$
$= \frac{1}{2\sqrt{3t+\sqrt{2+\sqrt{1-t}}}}\left(3+\frac{1}{2\sqrt{2+\sqrt{1-t}}}\cdot\frac{-1}{2\sqrt{1-t}}\right) = \frac{1}{2\sqrt{3t+\sqrt{2+\sqrt{1-t}}}}\left(\frac{12\sqrt{1-t}\sqrt{2+\sqrt{1-t}}-1}{4\sqrt{1-t}\sqrt{2+\sqrt{1-t}}}\right) = \frac{12\sqrt{1-t}\sqrt{2+\sqrt{1-t}}-1}{8\sqrt{1-t}\sqrt{2+\sqrt{1-t}}\sqrt{3t+\sqrt{2+\sqrt{1-t}}}}$

59. $y = \left(1+\frac{1}{x}\right)^3 \Rightarrow y' = 3\left(1+\frac{1}{x}\right)^2\left(-\frac{1}{x^2}\right) = -\frac{3}{x^2}\left(1+\frac{1}{x}\right)^2 \Rightarrow y'' = \left(-\frac{3}{x^2}\right)\cdot\frac{d}{dx}\left(1+\frac{1}{x}\right)^2 - \left(1+\frac{1}{x}\right)^2\cdot\frac{d}{dx}\left(\frac{3}{x^2}\right)$
$= \left(-\frac{3}{x^2}\right)\left(2\left(1+\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)\right) + \left(\frac{6}{x^3}\right)\left(1+\frac{1}{x}\right)^2 = \frac{6}{x^4}\left(1+\frac{1}{x}\right) + \frac{6}{x^3}\left(1+\frac{1}{x}\right)^2 = \frac{6}{x^3}\left(1+\frac{1}{x}\right)\left(\frac{1}{x}+1+\frac{1}{x}\right)$
$= \frac{6}{x^3}\left(1+\frac{1}{x}\right)\left(1+\frac{2}{x}\right)$

60. $y = \left(1-\sqrt{x}\right)^{-1} \Rightarrow y' = -\left(1-\sqrt{x}\right)^{-2}\left(-\frac{1}{2}x^{-1/2}\right) = \frac{1}{2}\left(1-\sqrt{x}\right)^{-2}x^{-1/2}$
$\Rightarrow y'' = \frac{1}{2}\left[\left(1-\sqrt{x}\right)^{-2}\left(-\frac{1}{2}x^{-3/2}\right) + x^{-1/2}(-2)\left(1-\sqrt{x}\right)^{-3}\left(-\frac{1}{2}x^{-1/2}\right)\right]$
$= \frac{1}{2}\left[\frac{-1}{2}x^{-3/2}\left(1-\sqrt{x}\right)^{-2} + x^{-1}\left(1-\sqrt{x}\right)^{-3}\right] = \frac{1}{2}x^{-1}\left(1-\sqrt{x}\right)^{-3}\left[-\frac{1}{2}x^{-1/2}\left(1-\sqrt{x}\right)+1\right]$
$= \frac{1}{2x}\left(1-\sqrt{x}\right)^{-3}\left(-\frac{1}{2\sqrt{x}}+\frac{1}{2}+1\right) = \frac{1}{2x}\left(1-\sqrt{x}\right)^{-3}\left(\frac{3}{2}-\frac{1}{2\sqrt{x}}\right)$

61. $y = \frac{1}{9}\cot(3x-1) \Rightarrow y' = -\frac{1}{9}\csc^2(3x-1)(3) = -\frac{1}{3}\csc^2(3x-1) \Rightarrow y'' = \left(-\frac{2}{3}\right)\left(\csc(3x-1)\cdot\frac{d}{dx}\csc(3x-1)\right)$
$= -\frac{2}{3}\csc(3x-1)\left(-\csc(3x-1)\cot(3x-1)\cdot\frac{d}{dx}(3x-1)\right) = 2\csc^2(3x-1)\cot(3x-1)$

62. $y = 9\tan\left(\frac{x}{3}\right) \Rightarrow y' = 9\left(\sec^2\left(\frac{x}{3}\right)\right)\left(\frac{1}{3}\right) = 3\sec^2\left(\frac{x}{3}\right) \Rightarrow y'' = 3\cdot 2\sec\left(\frac{x}{3}\right)\left(\sec\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)\right)\left(\frac{1}{3}\right) = 2\sec^2\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)$

63. $y = x(2x+1)^4 \Rightarrow y' = x\cdot 4(2x+1)^3(2) + 1\cdot(2x+1)^4 = (2x+1)^3(8x+(2x+1)) = (2x+1)^3(10x+1)$
$\Rightarrow y'' = (2x+1)^3(10) + 3(2x+1)^2(2)(10x+1) = 2(2x+1)^2(5(2x+1)+3(10x+1)) = 2(2x+1)^2(40x+8)$
$= 16(2x+1)^2(5x+1)$

64. $y = x^2(x^3-1)^5 \Rightarrow y' = x^2\cdot 5(x^3-1)^4(3x^2) + 2x(x^3-1)^5 = x(x^3-1)^4\left[15x^3+2(x^3-1)\right] = (x^3-1)^4(17x^4-2x)$
$\Rightarrow y'' = (x^3-1)^4(68x^3-2) + 4(x^3-1)^3(3x^2)(17x^4-2x) = 2(x^3-1)^3\left[(x^3-1)(34x^3-1)+6x^2(17x^4-2x)\right]$
$= 2(x^3-1)^3(136x^6-47x^3+1)$

65. $g(x) = \sqrt{x} \Rightarrow g'(x) = \frac{1}{2\sqrt{x}} \Rightarrow g(1) = 1$ and $g'(1) = \frac{1}{2}$; $f(u) = u^5+1 \Rightarrow f'(u) = 5u^4 \Rightarrow f'(g(1)) = f'(1) = 5$;
therefore, $(f\circ g)'(1) = f'(g(1))\cdot g'(1) = 5\cdot\frac{1}{2} = \frac{5}{2}$

66. $g(x) = (1-x)^{-1} \Rightarrow g'(x) = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2} \Rightarrow g(-1) = \frac{1}{2}$ and $g'(-1) = \frac{1}{4}$; $f(u) = 1-\frac{1}{u}$
$\Rightarrow f'(u) = \frac{1}{u^2} \Rightarrow f'(g(-1)) = f'\left(\frac{1}{2}\right) = 4$; therefore, $(f\circ g)'(-1) = f'(g(-1))g'(-1) = 4\cdot\frac{1}{4} = 1$

67. $g(x) = 5\sqrt{x} \Rightarrow g'(x) = \frac{5}{2\sqrt{x}} \Rightarrow g(1) = 5$ and $g'(1) = \frac{5}{2}$; $f(u) = \cot\left(\frac{\pi u}{10}\right) \Rightarrow f'(u) = -\csc^2\left(\frac{\pi u}{10}\right)\left(\frac{\pi}{10}\right) = \frac{-\pi}{10}\csc^2\left(\frac{\pi u}{10}\right)$
$\Rightarrow f'(g(1)) = f'(5) = -\frac{\pi}{10}\csc^2\left(\frac{\pi}{2}\right) = -\frac{\pi}{10}$; therefore, $(f\circ g)'(1) = f'(g(1))g'(1) = -\frac{\pi}{10}\cdot\frac{5}{2} = -\frac{\pi}{4}$

68. $g(x) = \pi x \Rightarrow g'(x) = \pi \Rightarrow g\left(\frac{1}{4}\right) = \frac{\pi}{4}$ and $g'\left(\frac{1}{4}\right) = \pi$; $f(u) = u + \sec^2 u \Rightarrow f'(u) = 1 + 2\sec u\cdot\sec u\tan u$
$= 1 + 2\sec^2 u\tan u \Rightarrow f'\left(g\left(\frac{1}{4}\right)\right) = f'\left(\frac{\pi}{4}\right) = 1 + 2\sec^2\frac{\pi}{4}\tan\frac{\pi}{4} = 5$; therefore, $(f\circ g)'\left(\frac{1}{4}\right) = f'\left(g\left(\frac{1}{4}\right)\right)g'\left(\frac{1}{4}\right) = 5\pi$

69. $g(x) = 10x^2+x+1 \Rightarrow g'(x) = 20x+1 \Rightarrow g(0) = 1$ and $g'(0) = 1$; $f(u) = \frac{2u}{u^2+1} \Rightarrow f'(u) = \frac{(u^2+1)(2)-(2u)(2u)}{(u^2+1)^2}$
$= \frac{-2u^2+2}{(u^2+1)^2} \Rightarrow f'(g(0)) = f'(1) = 0$; therefore, $(f\circ g)'(0) = f'(g(0))g'(0) = 0\cdot 1 = 0$

70. $g(x) = \frac{1}{x^2}-1 \Rightarrow g'(x) = -\frac{2}{x^3} \Rightarrow g(-1) = 0$ and $g'(-1) = 2$; $f(u) = \left(\frac{u-1}{u+1}\right)^2 \Rightarrow f'(u) = 2\left(\frac{u-1}{u+1}\right)\frac{d}{du}\left(\frac{u-1}{u+1}\right)$
$= 2\left(\frac{u-1}{u+1}\right)\cdot\frac{(u+1)(1)-(u-1)(1)}{(u+1)^2} = \frac{2(u-1)(2)}{(u+1)^3} = \frac{4(u-1)}{(u+1)^3} \Rightarrow f'(g(-1)) = f'(0) = -4$; therefore,
$(f\circ g)'(-1) = f'(g(-1))g'(-1) = (-4)(2) = -8$

71. $y = f(g(x)), f'(3) = -1, g'(2) = 5, g(2) = 3 \Rightarrow y' = f'(g(x))g'(x) \Rightarrow y'\big|_{x=2} = f'(g(2))g'(2) = f'(3) \cdot 5$
$= (-1) \cdot 5 = -5$

72. $r = \sin(f(t)), f(0) = \frac{\pi}{3}, f'(0) = 4 \Rightarrow \frac{dr}{dt} = \cos(f(t)) \cdot f'(t) \Rightarrow \frac{dr}{dt}\big|_{t=0} = \cos(f(0)) \cdot f'(0) = \cos\left(\frac{\pi}{3}\right) \cdot 4 = \left(\frac{1}{2}\right) \cdot 4 = 2$

73. (a) $y = 2f(x) \Rightarrow \frac{dy}{dx} = 2f'(x) \Rightarrow \frac{dy}{dx}\big|_{x=2} = 2f'(2) = 2\left(\frac{1}{3}\right) = \frac{2}{3}$

(b) $y = f(x) + g(x) \Rightarrow \frac{dy}{dx} = f'(x) + g'(x) \Rightarrow \frac{dy}{dx}\big|_{x=3} = f'(3) + g'(3) = 2\pi + 5$

(c) $y = f(x) \cdot g(x) \Rightarrow \frac{dy}{dx} = f(x)g'(x) + g(x)f'(x) \Rightarrow \frac{dy}{dx}\big|_{x=3} = f(3)g'(3) + g(3)f'(3) = 3 \cdot 5 + (-4)(2\pi) = 15 - 8\pi$

(d) $y = \frac{f(x)}{g(x)} \Rightarrow \frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \Rightarrow \frac{dy}{dx}\big|_{x=2} = \frac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2} = \frac{(2)\left(\frac{1}{3}\right) - (8)(-3)}{2^2} = \frac{37}{6}$

(e) $y = f(g(x)) \Rightarrow \frac{dy}{dx} = f'(g(x))g'(x) \Rightarrow \frac{dy}{dx}\big|_{x=2} = f'(g(2))g'(2) = f'(2)(-3) = \frac{1}{3}(-3) = -1$

(f) $y = (f(x))^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(f(x))^{-1/2} \cdot f'(x) = \frac{f'(x)}{2\sqrt{f(x)}} \Rightarrow \frac{dy}{dx}\big|_{x=2} = \frac{f'(2)}{2\sqrt{f(2)}} = \frac{\left(\frac{1}{3}\right)}{2\sqrt{8}} = \frac{1}{6\sqrt{8}} = \frac{1}{12\sqrt{2}} = \frac{\sqrt{2}}{24}$

(g) $y = (g(x))^{-2} \Rightarrow \frac{dy}{dx} = -2(g(x))^{-3} \cdot g'(x) \Rightarrow \frac{dy}{dx}\big|_{x=3} = -2(g(3))^{-3}g'(3) = -2(-4)^{-3} \cdot 5 = \frac{5}{32}$

(h) $y = \left((f(x))^2 + (g(x))^2\right)^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left((f(x))^2 + (g(x))^2\right)^{-1/2}(2f(x) \cdot f'(x) + 2g(x) \cdot g'(x))$
$\Rightarrow \frac{dy}{dx}\big|_{x=2} = \frac{1}{2}\left((f(2))^2 + (g(2))^2\right)^{-1/2}(2f(2)f'(2) + 2g(2)g'(2)) = \frac{1}{2}\left(8^2 + 2^2\right)^{-1/2}\left(2 \cdot 8 \cdot \frac{1}{3} + 2 \cdot 2 \cdot (-3)\right) = -\frac{5}{3\sqrt{17}}$

74. (a) $y = 5f(x) - g(x) \Rightarrow \frac{dy}{dx} = 5f'(x) - g'(x) \Rightarrow \frac{dy}{dx}\big|_{x=1} = 5f'(1) - g'(1) = 5\left(-\frac{1}{3}\right) - \left(\frac{-8}{3}\right) = 1$

(b) $y = f(x)(g(x))^3 \Rightarrow \frac{dy}{dx} = f(x)\left(3(g(x))^2g'(x)\right) + (g(x))^3f'(x) \Rightarrow \frac{dy}{dx}\big|_{x=0} = 3f(0)(g(0))^2g'(0) + (g(0))^3f'(0)$
$= 3(1)(1)^2\left(\frac{1}{3}\right) + (1)^3(5) = 6$

(c) $y = \frac{f(x)}{g(x)+1} \Rightarrow \frac{dy}{dx} = \frac{(g(x)+1)f'(x) - f(x)g'(x)}{(g(x)+1)^2} \Rightarrow \frac{dy}{dx}\big|_{x=1} = \frac{(g(1)+1)f'(1) - f(1)g'(1)}{(g(1)+1)^2}$
$= \frac{(-4+1)\left(-\frac{1}{3}\right) - (3)\left(-\frac{8}{3}\right)}{(-4+1)^2} = 1$

(d) $y = f(g(x)) \Rightarrow \frac{dy}{dx} = f'(g(x))g'(x) \Rightarrow \frac{dy}{dx}\big|_{x=0} = f'(g(0))g'(0) = f'(1)\left(\frac{1}{3}\right) = \left(-\frac{1}{3}\right)\left(\frac{1}{3}\right) = -\frac{1}{9}$

(e) $y = g(f(x)) \Rightarrow \frac{dy}{dx} = g'(f(x))f'(x) \Rightarrow \frac{dy}{dx}\big|_{x=0} = g'(f(0))f'(0) = g'(1)(5) = \left(-\frac{8}{3}\right)(5) = -\frac{40}{3}$

(f) $y = \left(x^{11} + f(x)\right)^{-2} \Rightarrow \frac{dy}{dx} = -2\left(x^{11} + f(x)\right)^{-3}\left(11x^{10} + f'(x)\right) \Rightarrow \frac{dy}{dx}\big|_{x=1} = -2(1 + f(1))^{-3}\left(11 + f'(1)\right)$
$= -2(1+3)^{-3}\left(11 - \frac{1}{3}\right) = \left(-\frac{2}{4^3}\right)\left(\frac{32}{3}\right) = -\frac{1}{3}$

(g) $y = f(x + g(x)) \Rightarrow \frac{dy}{dx} = f'(x + g(x))\left(1 + g'(x)\right) \Rightarrow \frac{dy}{dx}\big|_{x=0} = f'(0 + g(0))\left(1 + g'(0)\right) = f'(1)\left(1 + \frac{1}{3}\right)$
$= \left(-\frac{1}{3}\right)\left(\frac{4}{3}\right) = -\frac{4}{9}$

75. $\frac{ds}{dt} = \frac{ds}{d\theta} \cdot \frac{d\theta}{dt}$: $s = \cos\theta \Rightarrow \frac{ds}{d\theta} = -\sin\theta \Rightarrow \frac{ds}{d\theta}\big|_{\theta = \frac{3\pi}{2}} = -\sin\left(\frac{3\pi}{2}\right) = 1$ so that $\frac{ds}{dt} = \frac{ds}{d\theta} \cdot \frac{d\theta}{dt} = 1 \cdot 5 = 5$

76. $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$: $y = x^2 + 7x - 5 \Rightarrow \frac{dy}{dx} = 2x + 7 \Rightarrow \frac{dy}{dx}\big|_{x=1} = 9$ so that $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = 9 \cdot \frac{1}{3} = 3$

77. With $y = x$, we should get $\frac{dy}{dx} = 1$ for both (a) and (b):

(a) $y = \frac{u}{5} + 7 \Rightarrow \frac{dy}{du} = \frac{1}{5}$; $u = 5x - 35 \Rightarrow \frac{du}{dx} = 5$; therefore, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{5} \cdot 5 = 1$, as expected

(b) $y = 1 + \frac{1}{u} \Rightarrow \frac{dy}{du} = -\frac{1}{u^2}$; $u = (x-1)^{-1} \Rightarrow \frac{du}{dx} = -(x-1)^{-2}(1) = \frac{-1}{(x-1)^2}$; therefore $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
$= \frac{-1}{u^2} \cdot \frac{-1}{(x-1)^2} = \frac{-1}{((x-1)^{-1})^2} \cdot \frac{-1}{(x-1)^2} = (x-1)^2 \cdot \frac{1}{(x-1)^2} = 1$, again as expected

78. With $y = x^{3/2}$, we should get $\frac{dy}{dx} = \frac{3}{2}x^{1/2}$ for both (a) and (b):

(a) $y = u^3 \Rightarrow \frac{dy}{du} = 3u^2$; $u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}}$; therefore, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \cdot \frac{1}{2\sqrt{x}} = 3\left(\sqrt{x}\right)^2 \cdot \frac{1}{2\sqrt{x}} = \frac{3}{2}\sqrt{x}$, as expected.

(b) $y = \sqrt{u} \Rightarrow \frac{dy}{du} = \frac{1}{2\sqrt{u}}$; $u = x^3 \Rightarrow \frac{du}{dx} = 3x^2$; therefore, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot 3x^2 = \frac{1}{2\sqrt{x^3}} \cdot 3x^2 = \frac{3}{2}x^{1/2}$, again as expected.

79. $y = \left(\frac{x-1}{x+1}\right)^2$ and $x = 0 \Rightarrow y = \left(\frac{0-1}{0+1}\right)^2 = (-1)^2 = 1$. $y' = 2\left(\frac{x-1}{x+1}\right) \cdot \frac{(x+1)\cdot 1 - (x-1)\cdot 1}{(x+1)^2} = 2\frac{(x-1)}{(x+1)}\frac{2}{(x+1)^2} = \frac{4(x-1)}{(x+1)^3}$

$y'\Big|_{x=0} = \frac{4(0-1)}{(0+1)^3} = \frac{-4}{1^3} = -4 \Rightarrow y - 1 = -4(x-0) \Rightarrow y = -4x + 1$

80. $y = \sqrt{x^2 - x + 7}$ and $x = 2 \Rightarrow y = \sqrt{(2)^2 - (2) + 7} = \sqrt{9} = 3$. $y' = \frac{1}{2}(x^2 - x + 7)^{-1/2}(2x - 1) = \frac{2x-1}{2\sqrt{x^2-x+7}}$

$y'\Big|_{x=2} = \frac{2(2)-1}{2\sqrt{(2)^2-(2)+7}} = \frac{3}{6} = \frac{1}{2} \Rightarrow y - 3 = \frac{1}{2}(x-2) \Rightarrow y = \frac{1}{2}x + 2$

81. $y = 2\tan\left(\frac{\pi x}{4}\right) \Rightarrow \frac{dy}{dx} = \left(2\sec^2\frac{\pi x}{4}\right)\left(\frac{\pi}{4}\right) = \frac{\pi}{2}\sec^2\frac{\pi x}{4}$

(a) $\frac{dy}{dx}\Big|_{x=1} = \frac{\pi}{2}\sec^2\left(\frac{\pi}{4}\right) = \pi \Rightarrow$ slope of tangent is 2; thus, $y(1) = 2\tan\left(\frac{\pi}{4}\right) = 2$ and $y'(1) = \pi \Rightarrow$ tangent line is given by $y - 2 = \pi(x-1) \Rightarrow y = \pi x + 2 - \pi$

(b) $y' = \frac{\pi}{2}\sec^2\left(\frac{\pi x}{4}\right)$ and the smallest value the secant function can have in $-2 < x < 2$ is $1 \Rightarrow$ the minimum value of y' is $\frac{\pi}{2}$ and that occurs when $\frac{\pi}{2} = \frac{\pi}{2}\sec^2\left(\frac{\pi x}{4}\right) \Rightarrow 1 = \sec^2\left(\frac{\pi x}{4}\right) \Rightarrow \pm 1 = \sec\left(\frac{\pi x}{4}\right) \Rightarrow x = 0$.

82. (a) $y = \sin 2x \Rightarrow y' = 2\cos 2x \Rightarrow y'(0) = 2\cos(0) = 2 \Rightarrow$ tangent to $y = \sin 2x$ at the origin is $y = 2x$; $y = -\sin\left(\frac{x}{2}\right) \Rightarrow y' = -\frac{1}{2}\cos\left(\frac{x}{2}\right) \Rightarrow y'(0) = -\frac{1}{2}\cos 0 = -\frac{1}{2} \Rightarrow$ tangent to $y = -\sin\left(\frac{x}{2}\right)$ at the origin is $y = -\frac{1}{2}x$. The tangents are perpendicular to each other at the origin since the product of their slopes is -1.

(b) $y = \sin(mx) \Rightarrow y' = m\cos(mx) \Rightarrow y'(0) = m\cos 0 = m$; $y = -\sin\left(\frac{x}{m}\right) \Rightarrow y' = -\frac{1}{m}\cos\left(\frac{x}{m}\right) \Rightarrow y'(0) = -\frac{1}{m}\cos(0) = -\frac{1}{m}$. Since $m \cdot \left(-\frac{1}{m}\right) = -1$, the tangent lines are perpendicular at the origin.

(c) $y = \sin(mx) \Rightarrow y' = m\cos(mx)$. The largest value $\cos(mx)$ can attain is 1 at $x = 0 \Rightarrow$ the largest value y' can attain is $|m|$ because $|y'| = |m\cos(mx)| = |m|\,|\cos mx| \le |m| \cdot 1 = |m|$. Also, $y = -\sin\left(\frac{x}{m}\right) \Rightarrow y' = -\frac{1}{m}\cos\left(\frac{x}{m}\right) \Rightarrow |y'| = \left|\frac{-1}{m}\cos\left(\frac{x}{m}\right)\right| \le \left|\frac{1}{m}\right|\left|\cos\left(\frac{x}{m}\right)\right| \le \frac{1}{|m|} \Rightarrow$ the largest value y' can attain is $\left|\frac{1}{m}\right|$.

(d) $y = \sin(mx) \Rightarrow y' = m\cos(mx) \Rightarrow y'(0) = m \Rightarrow$ slope of curve at the origin is m. Also, $\sin(mx)$ completes m periods on $[0, 2\pi]$. Therefore the slope of the curve $y = \sin(mx)$ at the origin is the same as the number of periods it completes on $[0, 2\pi]$. In particular, for large m, we can think of "compressing" the graph of $y = \sin x$ horizontally which gives more periods completed on $[0, 2\pi]$, but also increases the slope of the graph at the origin.

83. $s = A\cos(2\pi bt) \Rightarrow v = \frac{ds}{dt} = -A\sin(2\pi bt)(2\pi b) = -2\pi bA\sin(2\pi bt)$. If we replace b with 2b to double the frequency, the velocity formula gives $v = -4\pi bA\sin(4\pi bt) \Rightarrow$ doubling the frequency causes the velocity to double. Also $v = -2\pi bA\sin(2\pi bt) \Rightarrow a = \frac{dv}{dt} = -4\pi^2b^2A\cos(2\pi bt)$. If we replace b with 2b in the acceleration formula, we get $a = -16\pi^2b^2A\cos(4\pi bt) \Rightarrow$ doubling the frequency causes the acceleration to quadruple. Finally, $a = -4\pi^2b^2A\cos(2\pi bt) \Rightarrow j = \frac{da}{dt} = 8\pi^3b^3A\sin(2\pi bt)$. If we replace b with 2b in the jerk formula, we get $j = 64\pi^3b^3A\sin(4\pi bt) \Rightarrow$ doubling the frequency multiplies the jerk by a factor of 8.

84. (a) $y = 37\sin\left[\frac{2\pi}{365}(x-101)\right] + 25 \Rightarrow y' = 37\cos\left[\frac{2\pi}{365}(x-101)\right]\left(\frac{2\pi}{365}\right) = \frac{74\pi}{365}\cos\left[\frac{2\pi}{365}(x-101)\right]$.
The temperature is increasing the fastest when y' is as large as possible. The largest value of $\cos\left[\frac{2\pi}{365}(x-101)\right]$ is 1 and occurs when $\frac{2\pi}{365}(x-101) = 0 \Rightarrow x = 101 \Rightarrow$ on day 101 of the year ($\sim$ April 11), the temperature is increasing the fastest.

(b) $y'(101) = \frac{74\pi}{365}\cos\left[\frac{2\pi}{365}(101-101)\right] = \frac{74\pi}{365}\cos(0) = \frac{74\pi}{365} \approx 0.64$ °F/day

85. $s = (1+4t)^{1/2} \Rightarrow v = \frac{ds}{dt} = \frac{1}{2}(1+4t)^{-1/2}(4) = 2(1+4t)^{-1/2} \Rightarrow v(6) = 2(1+4\cdot 6)^{-1/2} = \frac{2}{5}$ m/sec;
$v = 2(1+4t)^{-1/2} \Rightarrow a = \frac{dv}{dt} = -\frac{1}{2}\cdot 2(1+4t)^{-3/2}(4) = -4(1+4t)^{-3/2} \Rightarrow a(6) = -4(1+4\cdot 6)^{-3/2} = -\frac{4}{125}$ m/sec^2

86. We need to show $a = \frac{dv}{dt}$ is constant: $a = \frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt}$ and $\frac{dv}{ds} = \frac{d}{ds}\left(k\sqrt{s}\right) = \frac{k}{2\sqrt{s}} \Rightarrow a = \frac{dv}{ds}\cdot\frac{ds}{dt} = \frac{dv}{ds}\cdot v$
$= \frac{k}{2\sqrt{s}}\cdot k\sqrt{s} = \frac{k^2}{2}$ which is a constant.

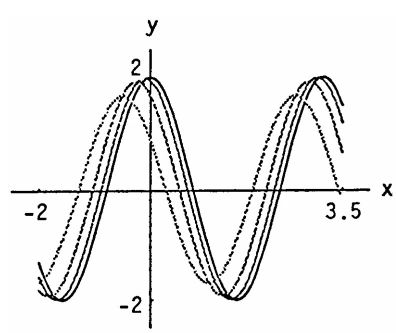
87. v proportional to $\frac{1}{\sqrt{s}} \Rightarrow v = \frac{k}{\sqrt{s}}$ for some constant k $\Rightarrow \frac{dv}{ds} = -\frac{k}{2s^{3/2}}$. Thus, $a = \frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt} = \frac{dv}{ds}\cdot v$
$= -\frac{k}{2s^{3/2}}\cdot\frac{k}{\sqrt{s}} = -\frac{k^2}{2}\left(\frac{1}{s^2}\right) \Rightarrow$ acceleration is a constant times $\frac{1}{s^2}$ so a is inversely proportional to s^2.

88. Let $\frac{dx}{dt} = f(x)$. Then, $a = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = \frac{dv}{dx}\cdot f(x) = \frac{d}{dx}\left(\frac{dx}{dt}\right)\cdot f(x) = \frac{d}{dx}(f(x))\cdot f(x) = f'(x)f(x)$, as required.

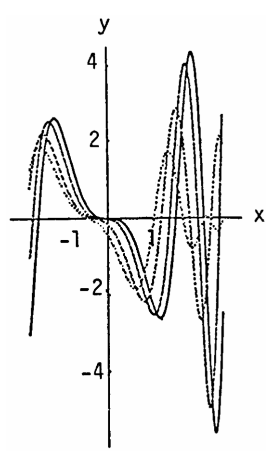
89. $T = 2\pi\sqrt{\frac{L}{g}} \Rightarrow \frac{dT}{dL} = 2\pi\cdot\frac{1}{2\sqrt{\frac{L}{g}}}\cdot\frac{1}{g} = \frac{\pi}{g\sqrt{\frac{L}{g}}} = \frac{\pi}{\sqrt{gL}}$. Therefore, $\frac{dT}{du} = \frac{dT}{dL}\cdot\frac{dL}{du} = \frac{\pi}{\sqrt{gL}}\cdot kL = \frac{\pi k\sqrt{L}}{\sqrt{g}} = \frac{1}{2}\cdot 2\pi k\sqrt{\frac{L}{g}}$
$= \frac{kT}{2}$, as required.

90. No. The chain rule says that when g is differentiable at 0 and f is differentiable at g(0), then $f \circ g$ is differentiable at 0. But the chain rule says nothing about what happens when g is not differentiable at 0 so there is no contradiction.

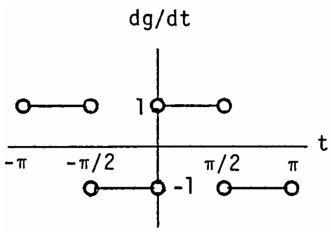
91. As $h \to 0$, the graph of $y = \frac{\sin 2(x+h) - \sin 2x}{h}$ approaches the graph of $y = 2\cos 2x$ because $\lim\limits_{h\to 0}\frac{\sin 2(x+h)-\sin 2x}{h} = \frac{d}{dx}(\sin 2x) = 2\cos 2x$.



92. As $h \to 0$, the graph of $y = \frac{\cos[(x+h)^2]-\cos(x^2)}{h}$ approaches the graph of $y = -2x\sin(x^2)$ because $\lim\limits_{h\to 0}\frac{\cos[(x+h)^2]-\cos(x^2)}{h} = \frac{d}{dx}[\cos(x^2)] = -2x\sin(x^2)$.

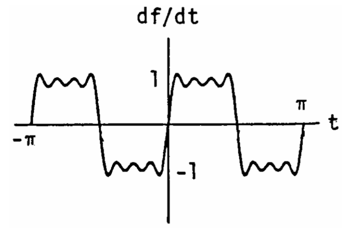


93. (a)

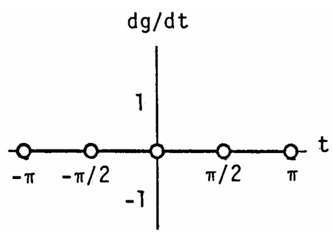


(b) $\frac{df}{dt} = 1.27324 \sin 2t + 0.42444 \sin 6t + 0.2546 \sin 10t + 0.18186 \sin 14t$

(c) The curve of $y = \frac{df}{dt}$ approximates $y = \frac{dg}{dt}$ the best when t is not $-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}$, nor π.

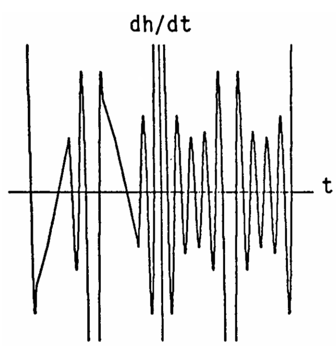


94. (a)



(b) $\frac{dh}{dt} = 2.5464 \cos(2t) + 2.5464 \cos(6t) + 2.5465 \cos(10t) + 2.54646 \cos(14t) + 2.54646 \cos(18t)$

(c)



3.7 IMPLICIT DIFFERENTIATION

1. $x^2y + xy^2 = 6$:

Step 1: $\left(x^2 \frac{dy}{dx} + y \cdot 2x\right) + \left(x \cdot 2y \frac{dy}{dx} + y^2 \cdot 1\right) = 0$

Step 2: $x^2 \frac{dy}{dx} + 2xy \frac{dy}{dx} = -2xy - y^2$

Step 3: $\frac{dy}{dx}(x^2 + 2xy) = -2xy - y^2$

Step 4: $\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}$

2. $x^3 + y^3 = 18xy \Rightarrow 3x^2 + 3y^2 \frac{dy}{dx} = 18y + 18x \frac{dy}{dx} \Rightarrow (3y^2 - 18x) \frac{dy}{dx} = 18y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{6y - x^2}{y^2 - 6x}$

3. $2xy + y^2 = x + y$:

Step 1: $\left(2x \frac{dy}{dx} + 2y\right) + 2y \frac{dy}{dx} = 1 + \frac{dy}{dx}$

Step 2: $2x \frac{dy}{dx} + 2y \frac{dy}{dx} - \frac{dy}{dx} = 1 - 2y$

Step 3: $\frac{dy}{dx}(2x + 2y - 1) = 1 - 2y$

Step 4: $\frac{dy}{dx} = \frac{1-2y}{2x+2y-1}$

4. $x^3 - xy + y^3 = 1 \Rightarrow 3x^2 - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0 \Rightarrow (3y^2 - x)\frac{dy}{dx} = y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{y-3x^2}{3y^2-x}$

5. $x^2(x-y)^2 = x^2 - y^2$:

Step 1: $x^2\left[2(x-y)\left(1 - \frac{dy}{dx}\right)\right] + (x-y)^2(2x) = 2x - 2y\frac{dy}{dx}$

Step 2: $-2x^2(x-y)\frac{dy}{dx} + 2y\frac{dy}{dx} = 2x - 2x^2(x-y) - 2x(x-y)^2$

Step 3: $\frac{dy}{dx}\left[-2x^2(x-y) + 2y\right] = 2x\left[1 - x(x-y) - (x-y)^2\right]$

Step 4: $\frac{dy}{dx} = \frac{2x\left[1-x(x-y)-(x-y)^2\right]}{-2x^2(x-y)+2y} = \frac{x\left[1-x(x-y)-(x-y)^2\right]}{y-x^2(x-y)} = \frac{x\left(1-x^2+xy-x^2+2xy-y^2\right)}{x^2y-x^3+y}$

$= \frac{x-2x^3+3x^2y-xy^2}{x^2y-x^3+y}$

6. $(3xy+7)^2 = 6y \Rightarrow 2(3xy+7)\cdot\left(3x\frac{dy}{dx} + 3y\right) = 6\frac{dy}{dx} \Rightarrow 2(3xy+7)(3x)\frac{dy}{dx} - 6\frac{dy}{dx} = -6y(3xy+7)$

$\Rightarrow \frac{dy}{dx}\left[6x(3xy+7) - 6\right] = -6y(3xy+7) \Rightarrow \frac{dy}{dx} = -\frac{y(3xy+7)}{x(3xy+7)-1} = \frac{3xy^2+7y}{1-3x^2y-7x}$

7. $y^2 = \frac{x-1}{x+1} \Rightarrow 2y\frac{dy}{dx} = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2} \Rightarrow \frac{dy}{dx} = \frac{1}{y(x+1)^2}$

8. $x^3 = \frac{2x-y}{x+3y} \Rightarrow x^4 + 3x^3y = 2x - y \Rightarrow 4x^3 + 9x^2y + 3x^3y' = 2 - y' \Rightarrow (3x^3+1)y' = 2 - 4x^3 - 9x^2y$

$\Rightarrow y' = \frac{2-4x^3-9x^2y}{3x^3+1}$

9. $x = \tan y \Rightarrow 1 = (\sec^2 y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$

10. $xy = \cot(xy) \Rightarrow x\frac{dy}{dx} + y = -\csc^2(xy)\left(x\frac{dy}{dx} + y\right) \Rightarrow x\frac{dy}{dx} + x\csc^2(xy)\frac{dy}{dx} = -y\csc^2(xy) - y$

$\Rightarrow \frac{dy}{dx}\left[x + x\csc^2(xy)\right] = -y\left[\csc^2(xy) + 1\right] \Rightarrow \frac{dy}{dx} = \frac{-y\left[\csc^2(xy)+1\right]}{x\left[1+\csc^2(xy)\right]} = -\frac{y}{x}$

11. $x + \tan(xy) = 0 \Rightarrow 1 + \left[\sec^2(xy)\right]\left(y + x\frac{dy}{dx}\right) = 0 \Rightarrow x\sec^2(xy)\frac{dy}{dx} = -1 - y\sec^2(xy) \Rightarrow \frac{dy}{dx} = \frac{-1-y\sec^2(xy)}{x\sec^2(xy)}$

$= \frac{-1}{x\sec^2(xy)} - \frac{y}{x} = \frac{-\cos^2(xy)}{x} - \frac{y}{x} = \frac{-\cos^2(xy)-y}{x}$

12. $x^4 + \sin y = x^3y^2 \Rightarrow 4x^3 + (\cos y)\frac{dy}{dx} = 3x^2y^2 + x^3\cdot 2y\frac{dy}{dx} \Rightarrow (\cos y - 2x^3y)\frac{dy}{dx} = 3x^2y^2 - 4x^3 \Rightarrow \frac{dy}{dx} = \frac{3x^2y^2-4x^3}{\cos y - 2x^3y}$

13. $y\sin\left(\frac{1}{y}\right) = 1 - xy \Rightarrow y\left[\cos\left(\frac{1}{y}\right)\cdot(-1)\frac{1}{y^2}\cdot\frac{dy}{dx}\right] + \sin\left(\frac{1}{y}\right)\cdot\frac{dy}{dx} = -x\frac{dy}{dx} - y \Rightarrow$

$\frac{dy}{dx}\left[-\frac{1}{y}\cos\left(\frac{1}{y}\right) + \sin\left(\frac{1}{y}\right) + x\right] = -y \Rightarrow \frac{dy}{dx} = \frac{-y}{-\frac{1}{y}\cos\left(\frac{1}{y}\right)+\sin\left(\frac{1}{y}\right)+x} = \frac{-y^2}{y\sin\left(\frac{1}{y}\right)-\cos\left(\frac{1}{y}\right)+xy}$

14. $x\cos(2x+3y) = y\sin x \Rightarrow -x\sin(2x+3y)(2+3y') + \cos(2x+3y) = y\cos x + y'\sin x$

$\Rightarrow -2x\sin(2x+3y) - 3xy'\sin(2x+3y) + \cos(2x+3y) = y\cos x + y'\sin x$

$\Rightarrow \cos(2x+3y) - 2x\sin(2x+3y) - y\cos x = (\sin x + 3x\sin(2x+3y))y' \Rightarrow y' = \frac{\cos(2x+3y)-2x\sin(2x+3y)-y\cos x}{\sin x+3x\sin(2x+3y)}$

15. $\theta^{1/2} + r^{1/2} = 1 \Rightarrow \frac{1}{2}\theta^{-1/2} + \frac{1}{2}r^{-1/2}\cdot\frac{dr}{d\theta} = 0 \Rightarrow \frac{dr}{d\theta}\left[\frac{1}{2\sqrt{r}}\right] = \frac{-1}{2\sqrt{\theta}} \Rightarrow \frac{dr}{d\theta} = -\frac{2\sqrt{r}}{2\sqrt{\theta}} = -\frac{\sqrt{r}}{\sqrt{\theta}}$

16. $r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4} \Rightarrow \frac{dr}{d\theta} - \theta^{-1/2} = \theta^{-1/3} + \theta^{-1/4} \Rightarrow \frac{dr}{d\theta} = \theta^{-1/2} + \theta^{-1/3} + \theta^{-1/4}$

17. $\sin(r\theta) = \frac{1}{2} \Rightarrow [\cos(r\theta)]\left(r + \theta\,\frac{dr}{d\theta}\right) = 0 \Rightarrow \frac{dr}{d\theta}\,[\theta\cos(r\theta)] = -r\cos(r\theta) \Rightarrow \frac{dr}{d\theta} = \frac{-r\cos(r\theta)}{\theta\cos(r\theta)} = -\frac{r}{\theta}$, $\cos(r\theta) \neq 0$

18. $\cos r + \cot\theta = r\theta \Rightarrow (-\sin r)\,\frac{dr}{d\theta} - \csc^2\theta = r + \theta\,\frac{dr}{d\theta} \Rightarrow \frac{dr}{d\theta}\,[-\sin r - \theta] = r + \csc^2\theta \Rightarrow \frac{dr}{d\theta} = -\frac{r + \csc^2\theta}{\sin r + \theta}$

19. $x^2 + y^2 = 1 \;\Rightarrow\; 2x + 2yy' = 0 \;\Rightarrow\; 2yy' = -2x \;\Rightarrow\; \frac{dy}{dx} = y' = -\frac{x}{y}$; now to find $\frac{d^2y}{dx^2}$, $\frac{d}{dx}(y') = \frac{d}{dx}\left(-\frac{x}{y}\right)$

$\Rightarrow\; y'' = \frac{y(-1) + xy'}{y^2} = \frac{-y + x\left(-\frac{x}{y}\right)}{y^2}$ since $y' = -\frac{x}{y} \;\Rightarrow\; \frac{d^2y}{dx^2} = y'' = \frac{-y^2 - x^2}{y^3} = \frac{-y^2 - (1 - y^2)}{y^3} = \frac{-1}{y^3}$

20. $x^{2/3} + y^{2/3} = 1 \;\Rightarrow\; \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\,\frac{dy}{dx} = 0 \;\Rightarrow\; \frac{dy}{dx}\left[\frac{2}{3}y^{-1/3}\right] = -\frac{2}{3}x^{-1/3} \;\Rightarrow\; y' = \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3}$;

Differentiating again, $y'' = \frac{x^{1/3}\cdot\left(-\frac{1}{3}y^{-2/3}\right)y' + y^{1/3}\left(\frac{1}{3}x^{-2/3}\right)}{x^{2/3}} = \frac{x^{1/3}\cdot\left(-\frac{1}{3}y^{-2/3}\right)\left(-\frac{y^{1/3}}{x^{1/3}}\right) + y^{1/3}\left(\frac{1}{3}x^{-2/3}\right)}{x^{2/3}}$

$\Rightarrow\; \frac{d^2y}{dx^2} = \frac{1}{3}x^{-2/3}y^{-1/3} + \frac{1}{3}y^{1/3}x^{-4/3} = \frac{y^{1/3}}{3x^{4/3}} + \frac{1}{3y^{1/3}x^{2/3}}$

21. $y^2 = x^2 + 2x \;\Rightarrow\; 2yy' = 2x + 2 \;\Rightarrow\; y' = \frac{2x+2}{2y} = \frac{x+1}{y}$; then $y'' = \frac{y - (x+1)y'}{y^2} = \frac{y - (x+1)\left(\frac{x+1}{y}\right)}{y^2}$

$\Rightarrow\; \frac{d^2y}{dx^2} = y'' = \frac{y^2 - (x+1)^2}{y^3}$

22. $y^2 - 2x = 1 - 2y \;\Rightarrow\; 2y\cdot y' - 2 = -2y' \;\Rightarrow\; y'(2y+2) = 2 \;\Rightarrow\; y' = \frac{1}{y+1} = (y+1)^{-1}$; then $y'' = -(y+1)^{-2}\cdot y'$

$= -(y+1)^{-2}(y+1)^{-1} \;\Rightarrow\; \frac{d^2y}{dx^2} = y'' = \frac{-1}{(y+1)^3}$

23. $2\sqrt{y} = x - y \;\Rightarrow\; y^{-1/2}y' = 1 - y' \;\Rightarrow\; y'\left(y^{-1/2} + 1\right) = 1 \;\Rightarrow\; \frac{dy}{dx} = y' = \frac{1}{y^{-1/2}+1} = \frac{\sqrt{y}}{\sqrt{y}+1}$; we can differentiate the equation $y'\left(y^{-1/2} + 1\right) = 1$ again to find y'': $y'\left(-\frac{1}{2}y^{-3/2}y'\right) + \left(y^{-1/2} + 1\right)y'' = 0$

$\Rightarrow\; \left(y^{-1/2} + 1\right)y'' = \frac{1}{2}[y']^2y^{-3/2} \;\Rightarrow\; \frac{d^2y}{dx^2} = y'' = \frac{\frac{1}{2}\left(\frac{1}{y^{-1/2}+1}\right)^2 y^{-3/2}}{\left(y^{-1/2}+1\right)} = \frac{1}{2y^{3/2}\left(y^{-1/2}+1\right)^3} = \frac{1}{2\left(1+\sqrt{y}\right)^3}$

24. $xy + y^2 = 1 \;\Rightarrow\; xy' + y + 2yy' = 0 \;\Rightarrow\; xy' + 2yy' = -y \;\Rightarrow\; y'(x+2y) = -y \;\Rightarrow\; y' = \frac{-y}{(x+2y)}$; $\frac{d^2y}{dx^2} = y''$

$= \frac{-(x+2y)y' + y(1+2y')}{(x+2y)^2} = \frac{-(x+2y)\left[\frac{-y}{(x+2y)}\right] + y\left[1 + 2\left(\frac{-y}{(x+2y)}\right)\right]}{(x+2y)^2} = \frac{\frac{1}{(x+2y)}\left[y(x+2y) + y(x+2y) - 2y^2\right]}{(x+2y)^2}$

$= \frac{2y(x+2y) - 2y^2}{(x+2y)^3} = \frac{2y^2 + 2xy}{(x+2y)^3} = \frac{2y(x+y)}{(x+2y)^3}$

25. $x^3 + y^3 = 16 \;\Rightarrow\; 3x^2 + 3y^2y' = 0 \;\Rightarrow\; 3y^2y' = -3x^2 \;\Rightarrow\; y' = -\frac{x^2}{y^2}$; we differentiate $y^2y' = -x^2$ to find y'':

$y^2y'' + y'[2y\cdot y'] = -2x \;\Rightarrow\; y^2y'' = -2x - 2y[y']^2 \;\Rightarrow\; y'' = \frac{-2x - 2y\left(-\frac{x^2}{y^2}\right)^2}{y^2} = \frac{-2x - \frac{2x^4}{y^3}}{y^2}$

$= \frac{-2xy^3 - 2x^4}{y^5} \;\Rightarrow\; \left.\frac{d^2y}{dx^2}\right|_{(2,2)} = \frac{-32-32}{32} = -2$

26. $xy + y^2 = 1 \;\Rightarrow\; xy' + y + 2yy' = 0 \;\Rightarrow\; y'(x+2y) = -y \;\Rightarrow\; y' = \frac{-y}{(x+2y)} \;\Rightarrow\; y'' = \frac{(x+2y)(-y') - (-y)(1+2y')}{(x+2y)^2}$;

since $y'|_{(0,-1)} = -\frac{1}{2}$ we obtain $y''|_{(0,-1)} = \frac{(-2)\left(\frac{1}{2}\right) - (1)(0)}{4} = -\frac{1}{4}$

27. $y^2 + x^2 = y^4 - 2x$ at $(-2,1)$ and $(-2,-1) \;\Rightarrow\; 2y\,\frac{dy}{dx} + 2x = 4y^3\,\frac{dy}{dx} - 2 \;\Rightarrow\; 2y\,\frac{dy}{dx} - 4y^3\,\frac{dy}{dx} = -2 - 2x$

$\Rightarrow\; \frac{dy}{dx}\left(2y - 4y^3\right) = -2 - 2x \;\Rightarrow\; \frac{dy}{dx} = \frac{x+1}{2y^3-y} \;\Rightarrow\; \left.\frac{dy}{dx}\right|_{(-2,1)} = -1$ and $\left.\frac{dy}{dx}\right|_{(-2,-1)} = 1$

28. $(x^2+y^2)^2 = (x-y)^2$ at $(1,0)$ and $(1,-1)$ $\Rightarrow$ $2(x^2+y^2)\left(2x+2y\,\frac{dy}{dx}\right) = 2(x-y)\left(1-\frac{dy}{dx}\right)$
$\Rightarrow$ $\frac{dy}{dx}\left[2y(x^2+y^2)+(x-y)\right] = -2x(x^2+y^2)+(x-y)$ $\Rightarrow$ $\frac{dy}{dx} = \frac{-2x(x^2+y^2)+(x-y)}{2y(x^2+y^2)+(x-y)}$ $\Rightarrow$ $\left.\frac{dy}{dx}\right|_{(1,0)} = -1$
and $\left.\frac{dy}{dx}\right|_{(1,-1)} = 1$

29. $x^2+xy-y^2=1$ $\Rightarrow$ $2x+y+xy'-2yy'=0$ $\Rightarrow$ $(x-2y)y' = -2x-y$ $\Rightarrow$ $y' = \frac{2x+y}{2y-x}$;
(a) the slope of the tangent line $m = y'|_{(2,3)} = \frac{7}{4}$ $\Rightarrow$ the tangent line is $y-3=\frac{7}{4}(x-2)$ $\Rightarrow$ $y=\frac{7}{4}x-\frac{1}{2}$
(b) the normal line is $y-3=-\frac{4}{7}(x-2)$ $\Rightarrow$ $y=-\frac{4}{7}x+\frac{29}{7}$

30. $x^2+y^2=25$ $\Rightarrow$ $2x+2yy'=0$ $\Rightarrow$ $y'=-\frac{x}{y}$;
(a) the slope of the tangent line $m = y'|_{(3,-4)} = -\left.\frac{x}{y}\right|_{(3,-4)} = \frac{3}{4}$ $\Rightarrow$ the tangent line is $y+4=\frac{3}{4}(x-3)$ $\Rightarrow$ $y=\frac{3}{4}x-\frac{25}{4}$
(b) the normal line is $y+4=-\frac{4}{3}(x-3)$ $\Rightarrow$ $y=-\frac{4}{3}x$

31. $x^2y^2=9$ $\Rightarrow$ $2xy^2+2x^2yy'=0$ $\Rightarrow$ $x^2yy'=-xy^2$ $\Rightarrow$ $y'=-\frac{y}{x}$;
(a) the slope of the tangent line $m = y'|_{(-1,3)} = -\left.\frac{y}{x}\right|_{(-1,3)} = 3$ $\Rightarrow$ the tangent line is $y-3=3(x+1)$ $\Rightarrow$ $y=3x+6$
(b) the normal line is $y-3=-\frac{1}{3}(x+1)$ $\Rightarrow$ $y=-\frac{1}{3}x+\frac{8}{3}$

32. $y^2-2x-4y-1=0$ $\Rightarrow$ $2yy'-2-4y'=0$ $\Rightarrow$ $2(y-2)y'=2$ $\Rightarrow$ $y'=\frac{1}{y-2}$;
(a) the slope of the tangent line $m = y'|_{(-2,1)} = -1$ $\Rightarrow$ the tangent line is $y-1=-1(x+2)$ $\Rightarrow$ $y=-x-1$
(b) the normal line is $y-1=1(x+2)$ $\Rightarrow$ $y=x+3$

33. $6x^2+3xy+2y^2+17y-6=0$ $\Rightarrow$ $12x+3y+3xy'+4yy'+17y'=0$ $\Rightarrow$ $y'(3x+4y+17)=-12x-3y$
$\Rightarrow$ $y'=\frac{-12x-3y}{3x+4y+17}$;
(a) the slope of the tangent line $m = y'|_{(-1,0)} = \left.\frac{-12x-3y}{3x+4y+17}\right|_{(-1,0)} = \frac{6}{7}$ $\Rightarrow$ the tangent line is $y-0=\frac{6}{7}(x+1)$
$\Rightarrow$ $y=\frac{6}{7}x+\frac{6}{7}$
(b) the normal line is $y-0=-\frac{7}{6}(x+1)$ $\Rightarrow$ $y=-\frac{7}{6}x-\frac{7}{6}$

34. $x^2-\sqrt{3}xy+2y^2=5$ $\Rightarrow$ $2x-\sqrt{3}xy'-\sqrt{3}y+4yy'=0$ $\Rightarrow$ $y'\left(4y-\sqrt{3}x\right)=\sqrt{3}y-2x$ $\Rightarrow$ $y'=\frac{\sqrt{3}y-2x}{4y-\sqrt{3}x}$;
(a) the slope of the tangent line $m = y'|_{(\sqrt{3},2)} = \left.\frac{\sqrt{3}y-2x}{4y-\sqrt{3}x}\right|_{(\sqrt{3},2)} = 0$ $\Rightarrow$ the tangent line is $y=2$
(b) the normal line is $x=\sqrt{3}$

35. $2xy+\pi\sin y=2\pi$ $\Rightarrow$ $2xy'+2y+\pi(\cos y)y'=0$ $\Rightarrow$ $y'(2x+\pi\cos y)=-2y$ $\Rightarrow$ $y'=\frac{-2y}{2x+\pi\cos y}$;
(a) the slope of the tangent line $m = y'|_{(1,\frac{\pi}{2})} = \left.\frac{-2y}{2x+\pi\cos y}\right|_{(1,\frac{\pi}{2})} = -\frac{\pi}{2}$ $\Rightarrow$ the tangent line is
$y-\frac{\pi}{2}=-\frac{\pi}{2}(x-1)$ $\Rightarrow$ $y=-\frac{\pi}{2}x+\pi$
(b) the normal line is $y-\frac{\pi}{2}=\frac{2}{\pi}(x-1)$ $\Rightarrow$ $y=\frac{2}{\pi}x-\frac{2}{\pi}+\frac{\pi}{2}$

36. $x\sin 2y=y\cos 2x$ $\Rightarrow$ $x(\cos 2y)2y'+\sin 2y=-2y\sin 2x+y'\cos 2x$ $\Rightarrow$ $y'(2x\cos 2y-\cos 2x)$
$=-\sin 2y-2y\sin 2x$ $\Rightarrow$ $y'=\frac{\sin 2y+2y\sin 2x}{\cos 2x-2x\cos 2y}$;
(a) the slope of the tangent line $m = y'|_{(\frac{\pi}{4},\frac{\pi}{2})} = \left.\frac{\sin 2y+2y\sin 2x}{\cos 2x-2x\cos 2y}\right|_{(\frac{\pi}{4},\frac{\pi}{2})} = \frac{\pi}{\frac{\pi}{2}} = 2$ $\Rightarrow$ the tangent line is
$y-\frac{\pi}{2}=2\left(x-\frac{\pi}{4}\right)$ $\Rightarrow$ $y=2x$
(b) the normal line is $y-\frac{\pi}{2}=-\frac{1}{2}\left(x-\frac{\pi}{4}\right)$ $\Rightarrow$ $y=-\frac{1}{2}x+\frac{5\pi}{8}$

37. $y = 2\sin(\pi x - y) \Rightarrow y' = 2[\cos(\pi x - y)] \cdot (\pi - y') \Rightarrow y'[1 + 2\cos(\pi x - y)] = 2\pi\cos(\pi x - y) \Rightarrow y' = \frac{2\pi\cos(\pi x - y)}{1 + 2\cos(\pi x - y)}$;

(a) the slope of the tangent line $m = y'|_{(1,0)} = \left.\frac{2\pi\cos(\pi x - y)}{1 + 2\cos(\pi x - y)}\right|_{(1,0)} = 2\pi \Rightarrow$ the tangent line is $y - 0 = 2\pi(x - 1) \Rightarrow y = 2\pi x - 2\pi$

(b) the normal line is $y - 0 = -\frac{1}{2\pi}(x - 1) \Rightarrow y = -\frac{x}{2\pi} + \frac{1}{2\pi}$

38. $x^2\cos^2 y - \sin y = 0 \Rightarrow x^2(2\cos y)(-\sin y)y' + 2x\cos^2 y - y'\cos y = 0 \Rightarrow y'[-2x^2\cos y\sin y - \cos y] = -2x\cos^2 y \Rightarrow y' = \frac{2x\cos^2 y}{2x^2\cos y\sin y + \cos y}$;

(a) the slope of the tangent line $m = y'|_{(0,\pi)} = \left.\frac{2x\cos^2 y}{2x^2\cos y\sin y + \cos y}\right|_{(0,\pi)} = 0 \Rightarrow$ the tangent line is $y = \pi$

(b) the normal line is $x = 0$

39. Solving $x^2 + xy + y^2 = 7$ and $y = 0 \Rightarrow x^2 = 7 \Rightarrow x = \pm\sqrt{7} \Rightarrow \left(-\sqrt{7}, 0\right)$ and $\left(\sqrt{7}, 0\right)$ are the points where the curve crosses the x-axis. Now $x^2 + xy + y^2 = 7 \Rightarrow 2x + y + xy' + 2yy' = 0 \Rightarrow (x + 2y)y' = -2x - y \Rightarrow y' = -\frac{2x + y}{x + 2y} \Rightarrow m = -\frac{2x + y}{x + 2y} \Rightarrow$ the slope at $\left(-\sqrt{7}, 0\right)$ is $m = -\frac{-2\sqrt{7}}{-\sqrt{7}} = -2$ and the slope at $\left(\sqrt{7}, 0\right)$ is $m = -\frac{2\sqrt{7}}{\sqrt{7}} = -2$. Since the slope is -2 in each case, the corresponding tangents must be parallel.

40. $xy + 2x - y = 0 \Rightarrow x\frac{dy}{dx} + y + 2 - \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y + 2}{1 - x}$; the slope of the line $2x + y = 0$ is -2. In order to be parallel, the normal lines must also have slope of -2. Since a normal is perpendicular to a tangent, the slope of the tangent is $\frac{1}{2}$. Therefore, $\frac{y + 2}{1 - x} = \frac{1}{2} \Rightarrow 2y + 4 = 1 - x \Rightarrow x = -3 - 2y$. Substituting in the original equation, $y(-3 - 2y) + 2(-3 - 2y) - y = 0 \Rightarrow y^2 + 4y + 3 = 0 \Rightarrow y = -3$ or $y = -1$. If $y = -3$, then $x = 3$ and $y + 3 = -2(x - 3) \Rightarrow y = -2x + 3$. If $y = -1$, then $x = -1$ and $y + 1 = -2(x + 1) \Rightarrow y = -2x - 3$.

41. $y^4 = y^2 - x^2 \Rightarrow 4y^3y' = 2yy' - 2x \Rightarrow 2(2y^3 - y)y' = -2x \Rightarrow y' = \frac{x}{y - 2y^3}$; the slope of the tangent line at $\left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)$ is $\left.\frac{x}{y - 2y^3}\right|_{\left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)} = \frac{\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2} - \frac{6\sqrt{3}}{8}} = \frac{\frac{1}{4}}{\frac{1}{2} - \frac{3}{4}} = \frac{1}{2 - 3} = -1$; the slope of the tangent line at $\left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right)$ is $\left.\frac{x}{y - 2y^3}\right|_{\left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right)} = \frac{\frac{\sqrt{3}}{4}}{\frac{1}{2} - \frac{2}{8}} = \frac{2\sqrt{3}}{4 - 2} = \sqrt{3}$

42. $y^2(2 - x) = x^3 \Rightarrow 2yy'(2 - x) + y^2(-1) = 3x^2 \Rightarrow y' = \frac{y^2 + 3x^2}{2y(2 - x)}$; the slope of the tangent line is $m = \left.\frac{y^2 + 3x^2}{2y(2 - x)}\right|_{(1,1)} = \frac{4}{2} = 2 \Rightarrow$ the tangent line is $y - 1 = 2(x - 1) \Rightarrow y = 2x - 1$; the normal line is $y - 1 = -\frac{1}{2}(x - 1) \Rightarrow y = -\frac{1}{2}x + \frac{3}{2}$

43. $y^4 - 4y^2 = x^4 - 9x^2 \Rightarrow 4y^3y' - 8yy' = 4x^3 - 18x \Rightarrow y'(4y^3 - 8y) = 4x^3 - 18x \Rightarrow y' = \frac{4x^3 - 18x}{4y^3 - 8y} = \frac{2x^3 - 9x}{2y^3 - 4y} = \frac{x(2x^2 - 9)}{y(2y^2 - 4)} = m$; $(-3, 2)$: $m = \frac{(-3)(18 - 9)}{2(8 - 4)} = -\frac{27}{8}$; $(-3, -2)$: $m = \frac{27}{8}$; $(3, 2)$: $m = \frac{27}{8}$; $(3, -2)$: $m = -\frac{27}{8}$

44. $x^3 + y^3 - 9xy = 0 \Rightarrow 3x^2 + 3y^2y' - 9xy' - 9y = 0 \Rightarrow y'(3y^2 - 9x) = 9y - 3x^2 \Rightarrow y' = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}$

(a) $y'|_{(4,2)} = \frac{5}{4}$ and $y'|_{(2,4)} = \frac{4}{5}$;

(b) $y' = 0 \Rightarrow \frac{3y - x^2}{y^2 - 3x} = 0 \Rightarrow 3y - x^2 = 0 \Rightarrow y = \frac{x^2}{3} \Rightarrow x^3 + \left(\frac{x^2}{3}\right)^3 - 9x\left(\frac{x^2}{3}\right) = 0 \Rightarrow x^6 - 54x^3 = 0 \Rightarrow x^3(x^3 - 54) = 0 \Rightarrow x = 0$ or $x = \sqrt[3]{54} = 3\sqrt[3]{2} \Rightarrow$ there is a horizontal tangent at $x = 3\sqrt[3]{2}$. To find the corresponding y-value, we will use part (c).

(c) $\frac{dx}{dy} = 0 \Rightarrow \frac{y^2 - 3x}{3y - x^2} = 0 \Rightarrow y^2 - 3x = 0 \Rightarrow y = \pm\sqrt{3x}$; $y = \sqrt{3x} \Rightarrow x^3 + \left(\sqrt{3x}\right)^3 - 9x\sqrt{3x} = 0 \Rightarrow x^3 - 6\sqrt{3}\,x^{3/2} = 0 \Rightarrow x^{3/2}\left(x^{3/2} - 6\sqrt{3}\right) = 0 \Rightarrow x^{3/2} = 0$ or $x^{3/2} = 6\sqrt{3} \Rightarrow x = 0$ or $x = \sqrt[3]{108} = 3\sqrt[3]{4}$.

Since the equation $x^3 + y^3 - 9xy = 0$ is symmetric in x and y, the graph is symmetric about the line $y = x$. That is, if (a, b) is a point on the folium, then so is (b, a). Moreover, if $y'|_{(a,b)} = m$, then $y'|_{(b,a)} = \frac{1}{m}$. Thus, if the folium has a horizontal tangent at (a, b), it has a vertical tangent at (b, a) so one might expect that with a horizontal tangent at $x = \sqrt[3]{54}$ and a vertical tangent at $x = 3\sqrt[3]{4}$, the points of tangency are $\left(\sqrt[3]{54}, 3\sqrt[3]{4}\right)$ and $\left(3\sqrt[3]{4}, \sqrt[3]{54}\right)$, respectively. One can check that these points do satisfy the equation $x^3 + y^3 - 9xy = 0$.

45. $x^2 + 2xy - 3y^2 = 0 \Rightarrow 2x + 2xy' + 2y - 6yy' = 0 \Rightarrow y'(2x - 6y) = -2x - 2y \Rightarrow y' = \frac{x+y}{3y-x} \Rightarrow$ the slope of the tangent line $m = y'|_{(1,1)} = \left.\frac{x+y}{3y-x}\right|_{(1,1)} = 1 \Rightarrow$ the equation of the normal line at (1, 1) is $y - 1 = -1(x - 1) \Rightarrow y = -x + 2$. To find where the normal line intersects the curve we substitute into its equation: $x^2 + 2x(2 - x) - 3(2 - x)^2 = 0$ $\Rightarrow x^2 + 4x - 2x^2 - 3(4 - 4x + x^2) = 0 \Rightarrow -4x^2 + 16x - 12 = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x - 3)(x - 1) = 0$ $\Rightarrow x = 3$ and $y = -x + 2 = -1$. Therefore, the normal to the curve at (1, 1) intersects the curve at the point (3, −1). Note that it also intersects the curve at (1, 1).

46. Let p and q be integers with $q > 0$ and suppose that $y = \sqrt[q]{x^p} = x^{p/q}$. Then $y^q = x^p$. Since p and q are integers and assuming y is a differentiable function of x, $\frac{d}{dx}(y^q) = \frac{d}{dx}(x^p) \Rightarrow qy^{q-1}\frac{dy}{dx} = px^{p-1} \Rightarrow \frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}}{y^{q-1}}$ $= \frac{p}{q} \cdot \frac{x^{p-1}}{(x^{p/q})^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}}{x^{p-p/q}} = \frac{p}{q} \cdot x^{p-1-(p-p/q)} = \frac{p}{q} \cdot x^{(p/q)-1}$

47. $y^2 = x \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$. If a normal is drawn from (a, 0) to (x_1, y_1) on the curve its slope satisfies $\frac{y_1 - 0}{x_1 - a} = -2y_1$ $\Rightarrow y_1 = -2y_1(x_1 - a)$ or $a = x_1 + \frac{1}{2}$. Since $x_1 \ge 0$ on the curve, we must have that $a \ge \frac{1}{2}$. By symmetry, the two points on the parabola are $(x_1, \sqrt{x_1})$ and $(x_1, -\sqrt{x_1})$. For the normal to be perpendicular, $\left(\frac{\sqrt{x_1}}{x_1 - a}\right)\left(\frac{\sqrt{x_1}}{a - x_1}\right) = -1$ $\Rightarrow \frac{x_1}{(a - x_1)^2} = 1 \Rightarrow x_1 = (a - x_1)^2 \Rightarrow x_1 = \left(x_1 + \frac{1}{2} - x_1\right)^2 \Rightarrow x_1 = \frac{1}{4}$ and $y_1 = \pm\frac{1}{2}$. Therefore, $\left(\frac{1}{4}, \pm\frac{1}{2}\right)$ and $a = \frac{3}{4}$.

48. $2x^2 + 3y^2 = 5 \Rightarrow 4x + 6yy' = 0 \Rightarrow y' = -\frac{2x}{3y} \Rightarrow y'|_{(1,1)} = -\left.\frac{2x}{3y}\right|_{(1,1)} = -\frac{2}{3}$ and $y'|_{(1,-1)} = -\left.\frac{2x}{3y}\right|_{(1,-1)} = \frac{2}{3}$; also, $y^2 = x^3 \Rightarrow 2yy' = 3x^2 \Rightarrow y' = \frac{3x^2}{2y} \Rightarrow y'|_{(1,1)} = \left.\frac{3x^2}{2y}\right|_{(1,1)} = \frac{3}{2}$ and $y'|_{(1,-1)} = \left.\frac{3x^2}{2y}\right|_{(1,-1)} = -\frac{3}{2}$. Therefore the tangents to the curves are perpendicular at (1, 1) and (1, −1) (i.e., the curves are orthogonal at these two points of intersection).

49. (a) $x^2 + y^2 = 4, x^2 = 3y^2 \Rightarrow (3y^2) + y^2 = 4 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$. If $y = 1 \Rightarrow x^2 + (1)^2 = 4 \Rightarrow x^2 = 3$ $\Rightarrow x = \pm\sqrt{3}$. If $y = -1 \Rightarrow x^2 + (-1)^2 = 4 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$.

$x^2 + y^2 = 4 \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow m_1 = \frac{dy}{dx} = -\frac{x}{y}$ and $x^2 = 3y^2 \Rightarrow 2x = 6y\frac{dy}{dx} \Rightarrow m_2 = \frac{dy}{dx} = \frac{x}{3y}$

At $\left(\sqrt{3}, 1\right)$: $m_1 = \frac{dy}{dx} = -\frac{\sqrt{3}}{1} = -\sqrt{3}$ and $m_2 = \frac{dy}{dx} = \frac{\sqrt{3}}{3(1)} = \frac{\sqrt{3}}{3} \Rightarrow m_1 \cdot m_2 = \left(-\sqrt{3}\right)\left(\frac{\sqrt{3}}{3}\right) = -1$

At $\left(\sqrt{3}, -1\right)$: $m_1 = \frac{dy}{dx} = -\frac{\sqrt{3}}{(-1)} = \sqrt{3}$ and $m_2 = \frac{dy}{dx} = \frac{\sqrt{3}}{3(-1)} = -\frac{\sqrt{3}}{3} \Rightarrow m_1 \cdot m_2 = \left(\sqrt{3}\right)\left(-\frac{\sqrt{3}}{3}\right) = -1$

At $\left(-\sqrt{3}, 1\right)$: $m_1 = \frac{dy}{dx} = -\frac{\left(-\sqrt{3}\right)}{1} = \sqrt{3}$ and $m_2 = \frac{dy}{dx} = \frac{-\sqrt{3}}{3(1)} = -\frac{\sqrt{3}}{3} \Rightarrow m_1 \cdot m_2 = \left(\sqrt{3}\right)\left(-\frac{\sqrt{3}}{3}\right) = -1$

At $\left(-\sqrt{3}, -1\right)$: $m_1 = \frac{dy}{dx} = -\frac{\left(-\sqrt{3}\right)}{(-1)} = -\sqrt{3}$ and $m_2 = \frac{dy}{dx} = \frac{\left(-\sqrt{3}\right)}{3(-1)} = \frac{\sqrt{3}}{3} \Rightarrow m_1 \cdot m_2 = \left(-\sqrt{3}\right)\left(\frac{\sqrt{3}}{3}\right) = -1$

(b) $x = 1 - y^2, x = \frac{1}{3}y^2 \Rightarrow \left(\frac{1}{3}y^2\right) = 1 - y^2 \Rightarrow y^2 = \frac{3}{4} \Rightarrow y = \pm\frac{\sqrt{3}}{2}$. If $y = \frac{\sqrt{3}}{2} \Rightarrow x = 1 - \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4}$. If $y = -\frac{\sqrt{3}}{2} \Rightarrow x = 1 - \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4}$. $x = 1 - y^2 \Rightarrow 1 = -2y\frac{dy}{dx} \Rightarrow m_1 = \frac{dy}{dx} = -\frac{1}{2y}$ and $x = \frac{1}{3}y^2$ $\Rightarrow 1 = \frac{2}{3}y\frac{dy}{dx} \Rightarrow m_2 = \frac{dy}{dx} = \frac{3}{2y}$

At $\left(\frac{1}{4}, \frac{\sqrt{3}}{2}\right)$: $m_1 = \frac{dy}{dx} = -\frac{1}{2\left(\sqrt{3}/2\right)} = -\frac{1}{\sqrt{3}}$ and $m_2 = \frac{dy}{dx} = \frac{3}{2\left(\sqrt{3}/2\right)} = \frac{3}{\sqrt{3}} \Rightarrow m_1 \cdot m_2 = \left(-\frac{1}{\sqrt{3}}\right)\left(\frac{3}{\sqrt{3}}\right) = -1$

At $\left(\frac{1}{4}, -\frac{\sqrt{3}}{2}\right)$: $m_1 = \frac{dy}{dx} = -\frac{1}{2\left(-\sqrt{3}/2\right)} = \frac{1}{\sqrt{3}}$ and $m_2 = \frac{dy}{dx} = \frac{3}{2\left(-\sqrt{3}/2\right)} = -\frac{3}{\sqrt{3}} \Rightarrow m_1 \cdot m_2 = \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{3}{\sqrt{3}}\right) = -1$

50. $y = -\frac{1}{3}x + b$, $y^2 = x^3 \Rightarrow \frac{dy}{dx} = -\frac{1}{3}$ and $2y\frac{dy}{dx} = 3x^2 \Rightarrow \frac{dy}{dx} = \frac{3x^2}{2y} \Rightarrow \left(-\frac{1}{3}\right)\left(\frac{3x^2}{2y}\right) = -1 \Rightarrow \frac{x^2}{2} = y \Rightarrow \left(\frac{x^2}{2}\right)^2 = x^3$ $\Rightarrow \frac{x^4}{4} = x^3 \Rightarrow x^4 - 4x^3 = 0 \Rightarrow x^3(x-4) = 0 \Rightarrow x = 0$ or $x = 4$. If $x = 0 \Rightarrow y = \frac{(0)^2}{2} = 0$ and $\left(-\frac{1}{3}\right)\left(\frac{3x^2}{2y}\right) = -1$ is indeterminant at $(0,0)$. If $x = 4 \Rightarrow y = \frac{(4)^2}{2} = 8$. At $(4, 8)$, $y = -\frac{1}{3}x + b \Rightarrow 8 = -\frac{1}{3}(4) + b \Rightarrow b = \frac{28}{3}$.

51. $xy^3 + x^2y = 6 \;\Rightarrow\; x\left(3y^2\,\frac{dy}{dx}\right) + y^3 + x^2\,\frac{dy}{dx} + 2xy = 0 \;\Rightarrow\; \frac{dy}{dx}\left(3xy^2 + x^2\right) = -y^3 - 2xy \;\Rightarrow\; \frac{dy}{dx} = \frac{-y^3 - 2xy}{3xy^2 + x^2}$ $= -\frac{y^3 + 2xy}{3xy^2 + x^2}$; also, $xy^3 + x^2y = 6 \;\Rightarrow\; x\left(3y^2\right) + y^3\,\frac{dx}{dy} + x^2 + y\left(2x\,\frac{dx}{dy}\right) = 0 \;\Rightarrow\; \frac{dx}{dy}\left(y^3 + 2xy\right) = -3xy^2 - x^2$ $\Rightarrow\; \frac{dx}{dy} = -\frac{3xy^2 + x^2}{y^3 + 2xy}$; thus $\frac{dx}{dy}$ appears to equal $\frac{1}{\frac{dy}{dx}}$. The two different treatments view the graphs as functions symmetric across the line $y = x$, so their slopes are reciprocals of one another at the corresponding points (a, b) and (b, a).

52. $x^3 + y^2 = \sin^2 y \;\Rightarrow\; 3x^2 + 2y\,\frac{dy}{dx} = (2\sin y)(\cos y)\,\frac{dy}{dx} \;\Rightarrow\; \frac{dy}{dx}(2y - 2\sin y\cos y) = -3x^2 \;\Rightarrow\; \frac{dy}{dx} = \frac{-3x^2}{2y - 2\sin y\cos y}$ $= \frac{3x^2}{2\sin y\cos y - 2y}$; also, $x^3 + y^2 = \sin^2 y \;\Rightarrow\; 3x^2\,\frac{dx}{dy} + 2y = 2\sin y\cos y \;\Rightarrow\; \frac{dx}{dy} = \frac{2\sin y\cos y - 2y}{3x^2}$; thus $\frac{dx}{dy}$ appears to equal $\frac{1}{\frac{dy}{dx}}$. The two different treatments view the graphs as functions symmetric across the line $y = x$ so their slopes are reciprocals of one another at the corresponding points (a, b) and (b, a).

53-60. Example CAS commands:

Maple:

```
q1 := x^3-x*y+y^3 = 7;
pt := [x=2,y=1];
p1 := implicitplot( q1, x=-3..3, y=-3..3 ):
p1;
eval( q1, pt );
q2 := implicitdiff( q1, y, x );
m := eval( q2, pt );
tan_line := y = 1 + m*(x-2);
p2 := implicitplot( tan_line, x=-5..5, y=-5..5, color=green ):
p3 := pointplot( eval([x,y],pt), color=blue ):
display( [p1,p2,p3], ="Section 3.7 #57(c)" );
```

Mathematica: (functions and x0 may vary):

Note use of double equal sign (logic statement) in definition of eqn and tanline.

```
<<Graphics`ImplicitPlot`
Clear[x, y]
{x0, y0}={1, π/4};
eqn=x + Tan[y/x]==2;
ImplicitPlot[eqn,{ x, x0 − 3, x0 + 3},{y, y0 − 3, y0 + 3}]
eqn/.{x → x0, y → y0}
eqn/.{ y → y[x]}
D[%, x]
Solve[%, y'[x]]
slope=y'[x]/.First[%]
m=slope/.{x → x0, y[x] → y0}
tanline=y==y0 + m (x − x0)
ImplicitPlot[{eqn, tanline}, {x, x0 − 3, x0 + 3},{y, y0 − 3, y0 + 3}]
```

3.8 RELATED RATES

1. $A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

2. $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

3. $y = 5x, \frac{dx}{dt} = 2 \Rightarrow \frac{dy}{dt} = 5\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = 5(2) = 10$

4. $2x + 3y = 12, \frac{dy}{dt} = -2 \Rightarrow 2\frac{dx}{dt} + 3\frac{dy}{dt} = 0 \Rightarrow 2\frac{dx}{dt} + 3(-2) = 0 \Rightarrow \frac{dx}{dt} = 3$

5. $y = x^2, \frac{dx}{dt} = 3 \Rightarrow \frac{dy}{dt} = 2x\frac{dx}{dt}$; when $x = -1 \Rightarrow \frac{dy}{dt} = 2(-1)(3) = -6$

6. $x = y^3 - y, \frac{dy}{dt} = 5 \Rightarrow \frac{dx}{dt} = 3y^2\frac{dy}{dt} - \frac{dy}{dt}$; when $y = 2 \Rightarrow \frac{dx}{dt} = 3(2)^2(5) - (5) = 55$

7. $x^2 + y^2 = 25, \frac{dx}{dt} = -2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$; when $x = 3$ and $y = -4 \Rightarrow 2(3)(-2) + 2(-4)\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{3}{2}$

8. $x^2y^3 = \frac{4}{27}, \frac{dy}{dt} = \frac{1}{2} \Rightarrow 3x^2y^2\frac{dy}{dt} + 2xy^3\frac{dx}{dt} = 0$; when $x = 2 \Rightarrow (2)^2y^3 = \frac{4}{27} \Rightarrow y = \frac{1}{3}$. Thus
$3(2)^2\left(\frac{1}{3}\right)^2\left(\frac{1}{2}\right) + 2(2)\left(\frac{1}{3}\right)^3\frac{dx}{dt} = 0 \Rightarrow \frac{dx}{dt} = -\frac{9}{2}$

9. $L = \sqrt{x^2 + y^2}, \frac{dx}{dt} = -1, \frac{dy}{dt} = 3 \Rightarrow \frac{dL}{dt} = \frac{1}{2\sqrt{x^2+y^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right) = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2+y^2}}$; when $x = 5$ and $y = 12$
$\Rightarrow \frac{dL}{dt} = \frac{(5)(-1)+(12)(3)}{\sqrt{(5)^2+(12)^2}} = \frac{31}{13}$

10. $r + s^2 + v^3 = 12, \frac{dr}{dt} = 4, \frac{ds}{dt} = -3 \Rightarrow \frac{dr}{dt} + 2s\frac{ds}{dt} + 3v^2\frac{dv}{dt} = 0$; when $r = 3$ and $s = 1 \Rightarrow (3) + (1)^2 + v^3 = 12 \Rightarrow v = 2$
$\Rightarrow 4 + 2(1)(-3) + 3(2)^2\frac{dv}{dt} = 0 \Rightarrow \frac{dv}{dt} = \frac{1}{6}$

11. (a) $S = 6x^2, \frac{dx}{dt} = -5\frac{m}{min} \Rightarrow \frac{dS}{dt} = 12x\frac{dx}{dt}$; when $x = 3 \Rightarrow \frac{dS}{dt} = 12(3)(-5) = -180\ \frac{m^2}{min}$
(b) $V = x^3, \frac{dx}{dt} = -5\frac{m}{min} \Rightarrow \frac{dV}{dt} = 3x^2\frac{dx}{dt}$; when $x = 3 \Rightarrow \frac{dV}{dt} = 3(3)^2(-5) = -135\ \frac{m^3}{min}$

12. $S = 6x^2, \frac{dS}{dt} = 72\ \frac{in^2}{sec} \Rightarrow \frac{dS}{dt} = 12x\frac{dx}{dt} \Rightarrow 72 = 12(3)\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 2\ \frac{in}{sec}$; $V = x^3 \Rightarrow \frac{dV}{dt} = 3x^2\frac{dx}{dt}$; when $x = 3$
$\Rightarrow \frac{dV}{dt} = 3(3)^2(2) = 54\ \frac{in^3}{sec}$

13. (a) $V = \pi r^2 h \Rightarrow \frac{dV}{dt} = \pi r^2\frac{dh}{dt}$
(b) $V = \pi r^2 h \Rightarrow \frac{dV}{dt} = 2\pi rh\frac{dr}{dt}$
(c) $V = \pi r^2 h \Rightarrow \frac{dV}{dt} = \pi r^2\frac{dh}{dt} + 2\pi rh\frac{dr}{dt}$

14. (a) $V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{dV}{dt} = \frac{1}{3}\pi r^2\frac{dh}{dt}$
(b) $V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{dV}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt}$
(c) $\frac{dV}{dt} = \frac{1}{3}\pi r^2\frac{dh}{dt} + \frac{2}{3}\pi rh\frac{dr}{dt}$

15. (a) $\frac{dV}{dt} = 1$ volt/sec
(b) $\frac{dI}{dt} = -\frac{1}{3}$ amp/sec
(c) $\frac{dV}{dt} = R\left(\frac{dI}{dt}\right) + I\left(\frac{dR}{dt}\right) \Rightarrow \frac{dR}{dt} = \frac{1}{I}\left(\frac{dV}{dt} - R\frac{dI}{dt}\right) \Rightarrow \frac{dR}{dt} = \frac{1}{I}\left(\frac{dV}{dt} - \frac{V}{I}\frac{dI}{dt}\right)$
(d) $\frac{dR}{dt} = \frac{1}{2}\left[1 - \frac{12}{2}\left(-\frac{1}{3}\right)\right] = \left(\frac{1}{2}\right)(3) = \frac{3}{2}$ ohms/sec, R is increasing

16. (a) $P = RI^2 \Rightarrow \frac{dP}{dt} = I^2\frac{dR}{dt} + 2RI\frac{dI}{dt}$
(b) $P = RI^2 \Rightarrow 0 = \frac{dP}{dt} = I^2\frac{dR}{dt} + 2RI\frac{dI}{dt} \Rightarrow \frac{dR}{dt} = -\frac{2RI}{I^2}\frac{dI}{dt} = -\frac{2\left(\frac{P}{I^2}\right)}{I}\frac{dI}{dt} = -\frac{2P}{I^3}\frac{dI}{dt}$

17. (a) $s = \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2} \Rightarrow \frac{ds}{dt} = \frac{x}{\sqrt{x^2+y^2}} \frac{dx}{dt}$

(b) $s = \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2} \Rightarrow \frac{ds}{dt} = \frac{x}{\sqrt{x^2+y^2}} \frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2}} \frac{dy}{dt}$

(c) $s = \sqrt{x^2 + y^2} \Rightarrow s^2 = x^2 + y^2 \Rightarrow 2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow 2s \cdot 0 = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow \frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt}$

18. (a) $s = \sqrt{x^2 + y^2 + z^2} \Rightarrow s^2 = x^2 + y^2 + z^2 \Rightarrow 2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} + 2z \frac{dz}{dt}$
$\Rightarrow \frac{ds}{dt} = \frac{x}{\sqrt{x^2+y^2+z^2}} \frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2+z^2}} \frac{dy}{dt} + \frac{z}{\sqrt{x^2+y^2+z^2}} \frac{dz}{dt}$

(b) From part (a) with $\frac{dx}{dt} = 0 \Rightarrow \frac{ds}{dt} = \frac{y}{\sqrt{x^2+y^2+z^2}} \frac{dy}{dt} + \frac{z}{\sqrt{x^2+y^2+z^2}} \frac{dz}{dt}$

(c) From part (a) with $\frac{ds}{dt} = 0 \Rightarrow 0 = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} + 2z \frac{dz}{dt} \Rightarrow \frac{dx}{dt} + \frac{y}{x} \frac{dy}{dt} + \frac{z}{x} \frac{dz}{dt} = 0$

19. (a) $A = \frac{1}{2} ab \sin\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2} ab \cos\theta \frac{d\theta}{dt}$ (b) $A = \frac{1}{2} ab \sin\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2} ab \cos\theta \frac{d\theta}{dt} + \frac{1}{2} b \sin\theta \frac{da}{dt}$

(c) $A = \frac{1}{2} ab \sin\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2} ab \cos\theta \frac{d\theta}{dt} + \frac{1}{2} b \sin\theta \frac{da}{dt} + \frac{1}{2} a \sin\theta \frac{db}{dt}$

20. Given $A = \pi r^2$, $\frac{dr}{dt} = 0.01$ cm/sec, and $r = 50$ cm. Since $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$, then $\left.\frac{dA}{dt}\right|_{r=50} = 2\pi(50)\left(\frac{1}{100}\right) = \pi$ cm^2/min.

21. Given $\frac{d\ell}{dt} = -2$ cm/sec, $\frac{dw}{dt} = 2$ cm/sec, $\ell = 12$ cm and $w = 5$ cm.

(a) $A = \ell w \Rightarrow \frac{dA}{dt} = \ell \frac{dw}{dt} + w \frac{d\ell}{dt} \Rightarrow \frac{dA}{dt} = 12(2) + 5(-2) = 14$ cm^2/sec, increasing

(b) $P = 2\ell + 2w \Rightarrow \frac{dP}{dt} = 2 \frac{d\ell}{dt} + 2 \frac{dw}{dt} = 2(-2) + 2(2) = 0$ cm/sec, constant

(c) $D = \sqrt{w^2 + \ell^2} = (w^2 + \ell^2)^{1/2} \Rightarrow \frac{dD}{dt} = \frac{1}{2}(w^2 + \ell^2)^{-1/2}\left(2w \frac{dw}{dt} + 2\ell \frac{d\ell}{dt}\right) \Rightarrow \frac{dD}{dt} = \frac{w \frac{dw}{dt} + \ell \frac{d\ell}{dt}}{\sqrt{w^2+\ell^2}}$
$= \frac{(5)(2) + (12)(-2)}{\sqrt{25+144}} = -\frac{14}{13}$ cm/sec, decreasing

22. (a) $V = xyz \Rightarrow \frac{dV}{dt} = yz \frac{dx}{dt} + xz \frac{dy}{dt} + xy \frac{dz}{dt} \Rightarrow \left.\frac{dV}{dt}\right|_{(4,3,2)} = (3)(2)(1) + (4)(2)(-2) + (4)(3)(1) = 2$ m^3/sec

(b) $S = 2xy + 2xz + 2yz \Rightarrow \frac{dS}{dt} = (2y + 2z) \frac{dx}{dt} + (2x + 2z) \frac{dy}{dt} + (2x + 2y) \frac{dz}{dt}$
$\Rightarrow \left.\frac{dS}{dt}\right|_{(4,3,2)} = (10)(1) + (12)(-2) + (14)(1) = 0$ m^2/sec

(c) $\ell = \sqrt{x^2 + y^2 + z^2} = (x^2 + y^2 + z^2)^{1/2} \Rightarrow \frac{d\ell}{dt} = \frac{x}{\sqrt{x^2+y^2+z^2}} \frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2+z^2}} \frac{dy}{dt} + \frac{z}{\sqrt{x^2+y^2+z^2}} \frac{dz}{dt}$
$\Rightarrow \left.\frac{d\ell}{dt}\right|_{(4,3,2)} = \left(\frac{4}{\sqrt{29}}\right)(1) + \left(\frac{3}{\sqrt{29}}\right)(-2) + \left(\frac{2}{\sqrt{29}}\right)(1) = 0$ m/sec

23. Given: $\frac{dx}{dt} = 5$ ft/sec, the ladder is 13 ft long, and $x = 12$, $y = 5$ at the instant of time

(a) Since $x^2 + y^2 = 169 \Rightarrow \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} = -\left(\frac{12}{5}\right)(5) = -12$ ft/sec, the ladder is sliding down the wall

(b) The area of the triangle formed by the ladder and walls is $A = \frac{1}{2} xy \Rightarrow \frac{dA}{dt} = \left(\frac{1}{2}\right)\left(x \frac{dy}{dt} + y \frac{dx}{dt}\right)$. The area is changing at $\frac{1}{2}[12(-12) + 5(5)] = -\frac{119}{2} = -59.5$ ft^2/sec.

(c) $\cos\theta = \frac{x}{13} \Rightarrow -\sin\theta \frac{d\theta}{dt} = \frac{1}{13} \cdot \frac{dx}{dt} \Rightarrow \frac{d\theta}{dt} = -\frac{1}{13 \sin\theta} \cdot \frac{dx}{dt} = -\left(\frac{1}{5}\right)(5) = -1$ rad/sec

24. $s^2 = y^2 + x^2 \Rightarrow 2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow \frac{ds}{dt} = \frac{1}{s}\left(x \frac{dx}{dt} + y \frac{dy}{dt}\right) \Rightarrow \frac{ds}{dt} = \frac{1}{\sqrt{169}}[5(-442) + 12(-481)] = -614$ knots

25. Let s represent the distance between the girl and the kite and x represents the horizontal distance between the girl and kite $\Rightarrow s^2 = (300)^2 + x^2 \Rightarrow \frac{ds}{dt} = \frac{x}{s} \frac{dx}{dt} = \frac{400(25)}{500} = 20$ ft/sec.

26. When the diameter is 3.8 in., the radius is 1.9 in. and $\frac{dr}{dt} = \frac{1}{3000}$ in/min. Also $V = 6\pi r^2 \Rightarrow \frac{dV}{dt} = 12\pi r \frac{dr}{dt}$
$\Rightarrow \frac{dV}{dt} = 12\pi(1.9)\left(\frac{1}{3000}\right) = 0.0076\pi$. The volume is changing at about 0.0239 in^3/min.

27. $V = \frac{1}{3}\pi r^2 h$, $h = \frac{3}{8}(2r) = \frac{3r}{4} \Rightarrow r = \frac{4h}{3} \Rightarrow V = \frac{1}{3}\pi\left(\frac{4h}{3}\right)^2 h = \frac{16\pi h^3}{27} \Rightarrow \frac{dV}{dt} = \frac{16\pi h^2}{9}\frac{dh}{dt}$

(a) $\left.\frac{dh}{dt}\right|_{h=4} = \left(\frac{9}{16\pi 4^2}\right)(10) = \frac{90}{256\pi} \approx 0.1119$ m/sec $= 11.19$ cm/sec

(b) $r = \frac{4h}{3} \Rightarrow \frac{dr}{dt} = \frac{4}{3}\frac{dh}{dt} = \frac{4}{3}\left(\frac{90}{256\pi}\right) = \frac{15}{32\pi} \approx 0.1492$ m/sec $= 14.92$ cm/sec

28. (a) $V = \frac{1}{3}\pi r^2 h$ and $r = \frac{15h}{2} \Rightarrow V = \frac{1}{3}\pi\left(\frac{15h}{2}\right)^2 h = \frac{75\pi h^3}{4} \Rightarrow \frac{dV}{dt} = \frac{225\pi h^2}{4}\frac{dh}{dt} \Rightarrow \left.\frac{dh}{dt}\right|_{h=5} = \frac{4(-50)}{225\pi(5)^2} = \frac{-8}{225\pi}$
≈ -0.0113 m/min $= -1.13$ cm/min

(b) $r = \frac{15h}{2} \Rightarrow \frac{dr}{dt} = \frac{15}{2}\frac{dh}{dt} \Rightarrow \left.\frac{dr}{dt}\right|_{h=5} = \left(\frac{15}{2}\right)\left(\frac{-8}{225\pi}\right) = \frac{-4}{15\pi} \approx -0.0849$ m/sec $= -8.49$ cm/sec

29. (a) $V = \frac{\pi}{3}y^2(3R - y) \Rightarrow \frac{dV}{dt} = \frac{\pi}{3}\left[2y(3R - y) + y^2(-1)\right]\frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \left[\frac{\pi}{3}\left(6Ry - 3y^2\right)\right]^{-1}\frac{dV}{dt} \Rightarrow$ at $R = 13$ and $y = 8$ we have $\frac{dy}{dt} = \frac{1}{144\pi}(-6) = \frac{-1}{24\pi}$ m/min

(b) The hemisphere is on the circle $r^2 + (13 - y)^2 = 169 \Rightarrow r = \sqrt{26y - y^2}$ m

(c) $r = \left(26y - y^2\right)^{1/2} \Rightarrow \frac{dr}{dt} = \frac{1}{2}\left(26y - y^2\right)^{-1/2}(26 - 2y)\frac{dy}{dt} \Rightarrow \frac{dr}{dt} = \frac{13 - y}{\sqrt{26y - y^2}}\frac{dy}{dt} \Rightarrow \left.\frac{dr}{dt}\right|_{y=8} = \frac{13 - 8}{\sqrt{26\cdot 8 - 64}}\left(\frac{-1}{24\pi}\right)$
$= \frac{-5}{288\pi}$ m/min

30. If $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$, and $\frac{dV}{dt} = kS = 4k\pi r^2$, then $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \Rightarrow 4k\pi r^2 = 4\pi r^2\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = k$, a constant. Therefore, the radius is increasing at a constant rate.

31. If $V = \frac{4}{3}\pi r^3$, $r = 5$, and $\frac{dV}{dt} = 100\pi$ ft^3/min, then $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = 1$ ft/min. Then $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r\frac{dr}{dt} = 8\pi(5)(1) = 40\pi$ ft^2/min, the rate at which the surface area is increasing.

32. Let s represent the length of the rope and x the horizontal distance of the boat from the dock.

(a) We have $s^2 = x^2 + 36 \Rightarrow \frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt} = \frac{s}{\sqrt{s^2 - 36}}\frac{ds}{dt}$. Therefore, the boat is approaching the dock at $\left.\frac{dx}{dt}\right|_{s=10} = \frac{10}{\sqrt{10^2 - 36}}(-2) = -2.5$ ft/sec.

(b) $\cos\theta = \frac{6}{r} \Rightarrow -\sin\theta\frac{d\theta}{dt} = -\frac{6}{r^2}\frac{dr}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{6}{r^2\sin\theta}\frac{dr}{dt}$. Thus, $r = 10$, $x = 8$, and $\sin\theta = \frac{8}{10}$
$\Rightarrow \frac{d\theta}{dt} = \frac{6}{10^2\left(\frac{8}{10}\right)}\cdot(-2) = -\frac{3}{20}$ rad/sec

33. Let s represent the distance between the bicycle and balloon, h the height of the balloon and x the horizontal distance between the balloon and the bicycle. The relationship between the variables is $s^2 = h^2 + x^2$
$\Rightarrow \frac{ds}{dt} = \frac{1}{s}\left(h\frac{dh}{dt} + x\frac{dx}{dt}\right) \Rightarrow \frac{ds}{dt} = \frac{1}{85}\left[68(1) + 51(17)\right] = 11$ ft/sec.

34. (a) Let h be the height of the coffee in the pot. Since the radius of the pot is 3, the volume of the coffee is $V = 9\pi h \Rightarrow \frac{dV}{dt} = 9\pi\frac{dh}{dt} \Rightarrow$ the rate the coffee is rising is $\frac{dh}{dt} = \frac{1}{9\pi}\frac{dV}{dt} = \frac{10}{9\pi}$ in/min.

(b) Let h be the height of the coffee in the pot. From the figure, the radius of the filter $r = \frac{h}{2} \Rightarrow V = \frac{1}{3}\pi r^2 h$
$= \frac{\pi h^3}{12}$, the volume of the filter. The rate the coffee is falling is $\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt} = \frac{4}{25\pi}(-10) = -\frac{8}{5\pi}$ in/min.

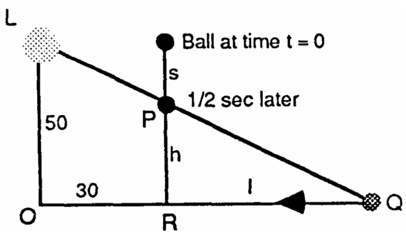
35. $y = QD^{-1} \Rightarrow \frac{dy}{dt} = D^{-1}\frac{dQ}{dt} - QD^{-2}\frac{dD}{dt} = \frac{1}{41}(0) - \frac{233}{(41)^2}(-2) = \frac{466}{1681}$ L/min $\Rightarrow$ increasing about 0.2772 L/min

36. Let $P(x, y)$ represent a point on the curve $y = x^2$ and θ the angle of inclination of a line containing P and the origin. Consequently, $\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{x^2}{x} = x \Rightarrow \sec^2\theta\frac{d\theta}{dt} = \frac{dx}{dt} \Rightarrow \frac{d\theta}{dt} = \cos^2\theta\frac{dx}{dt}$. Since $\frac{dx}{dt} = 10$ m/sec and $\left.\cos^2\theta\right|_{x=3} = \frac{x^2}{y^2 + x^2} = \frac{3^2}{9^2 + 3^2} = \frac{1}{10}$, we have $\left.\frac{d\theta}{dt}\right|_{x=3} = 1$ rad/sec.

37. The distance from the origin is $s = \sqrt{x^2 + y^2}$ and we wish to find $\left.\frac{ds}{dt}\right|_{(5,12)} = \frac{1}{2}\left(x^2 + y^2\right)^{-1/2}\left(2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt}\right)\Big|_{(5,12)}$
$= \frac{(5)(-1)+(12)(-5)}{\sqrt{25+144}} = -5$ m/sec

38. Let s = distance of car from foot of perpendicular in the textbook diagram $\Rightarrow\ \tan\theta = \frac{s}{132}\ \Rightarrow\ \sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{132}\,\frac{ds}{dt}$
$\Rightarrow\ \frac{d\theta}{dt} = \frac{\cos^2\theta}{132}\,\frac{ds}{dt}$; $\frac{ds}{dt} = -264$ and $\theta = 0\ \Rightarrow\ \frac{d\theta}{dt} = -2$ rad/sec. A half second later the car has traveled 132 ft right of the perpendicular $\Rightarrow\ |\theta| = \frac{\pi}{4}$, $\cos^2\theta = \frac{1}{2}$, and $\frac{ds}{dt} = 264$ (since s increases) $\Rightarrow\ \frac{d\theta}{dt} = \frac{\left(\frac{1}{2}\right)}{132}(264) = 1$ rad/sec.

39. Let $s = 16t^2$ represent the distance the ball has fallen, h the distance between the ball and the ground, and I the distance between the shadow and the point directly beneath the ball. Accordingly, $s + h = 50$ and since the triangle LOQ and triangle PRQ are similar we have $I = \frac{30h}{50-h} \Rightarrow h = 50 - 16t^2$ and $I = \frac{30\left(50-16t^2\right)}{50-\left(50-16t^2\right)} = \frac{1500}{16t^2} - 30\ \Rightarrow \frac{dI}{dt} = -\frac{1500}{8t^3}$
$\Rightarrow\ \left.\frac{dI}{dt}\right|_{t=\frac{1}{2}} = -1500$ ft/sec.



40. When x represents the length of the shadow, then $\tan\theta = \frac{80}{x}\ \Rightarrow\ \sec^2\theta\,\frac{d\theta}{dt} = -\frac{80}{x^2}\,\frac{dx}{dt}\ \Rightarrow\ \frac{dx}{dt} = \frac{-x^2\sec^2\theta}{80}\,\frac{d\theta}{dt}$. We are given that $\frac{d\theta}{dt} = 0.27° = \frac{3\pi}{2000}$ rad/min. At $x = 60$, $\cos\theta = \frac{3}{5} \Rightarrow \left|\frac{dx}{dt}\right| = \left|\frac{-x^2\sec^2\theta}{80}\,\frac{d\theta}{dt}\right|\Big|_{\left(\frac{d\theta}{dt} = \frac{3\pi}{2000}\text{ and }\sec\theta = \frac{5}{3}\right)} = \frac{3\pi}{16}$ ft/min
≈ 0.589 ft/min ≈ 7.1 in./min.

41. The volume of the ice is $V = \frac{4}{3}\pi r^3 - \frac{4}{3}\pi 4^3\ \Rightarrow\ \frac{dV}{dt} = 4\pi r^2\,\frac{dr}{dt}\ \Rightarrow\ \left.\frac{dr}{dt}\right|_{r=6} = \frac{-5}{72\pi}$ in./min when $\frac{dV}{dt} = -10$ in^3/min, the thickness of the ice is decreasing at $\frac{5}{72\pi}$ in/min. The surface area is $S = 4\pi r^2\ \Rightarrow\ \frac{dS}{dt} = 8\pi r\,\frac{dr}{dt}\ \Rightarrow\ \left.\frac{dS}{dt}\right|_{r=6} = 48\pi\left(\frac{-5}{72\pi}\right)$
$= -\frac{10}{3}$ in^2/min, the outer surface area of the ice is decreasing at $\frac{10}{3}$ in^2/min.

42. Let s represent the horizontal distance between the car and plane while r is the line-of-sight distance between the car and plane $\Rightarrow\ 9 + s^2 = r^2 \Rightarrow \frac{ds}{dt} = \frac{r}{\sqrt{r^2-9}}\,\frac{dr}{dt} \Rightarrow \left.\frac{ds}{dt}\right|_{r=5} = \frac{5}{\sqrt{16}}(-160) = -200$ mph $\Rightarrow$ speed of plane + speed of car
$= 200$ mph $\Rightarrow$ the speed of the car is 80 mph.

43. Let x represent distance of the player from second base and s the distance to third base. Then $\frac{dx}{dt} = -16$ ft/sec

(a) $s^2 = x^2 + 8100\ \Rightarrow\ 2s\,\frac{ds}{dt} = 2x\,\frac{dx}{dt}\ \Rightarrow\ \frac{ds}{dt} = \frac{x}{s}\,\frac{dx}{dt}$. When the player is 30 ft from first base, $x = 60$
$\Rightarrow\ s = 30\sqrt{13}$ and $\frac{ds}{dt} = \frac{60}{30\sqrt{13}}(-16) = \frac{-32}{\sqrt{13}} \approx -8.875$ ft/sec

(b) $\sin\theta_1 = \frac{90}{s}\ \Rightarrow\ \cos\theta_1\,\frac{d\theta_1}{dt} = -\frac{90}{s^2}\cdot\frac{ds}{dt}\ \Rightarrow\ \frac{d\theta_1}{dt} = -\frac{90}{s^2\cos\theta_1}\cdot\frac{ds}{dt} = -\frac{90}{s\cdot x}\cdot\frac{ds}{dt}$. Therefore, $x = 60$ and $s = 30\sqrt{13}$
$\Rightarrow\ \frac{d\theta_1}{dt} = -\frac{90}{\left(30\sqrt{13}\right)(60)}\cdot\left(\frac{-32}{\sqrt{13}}\right) = \frac{8}{65}$ rad/sec; $\cos\theta_2 = \frac{90}{s}\ \Rightarrow\ -\sin\theta_2\,\frac{d\theta_2}{dt} = -\frac{90}{s^2}\cdot\frac{ds}{dt}\ \Rightarrow\ \frac{d\theta_2}{dt} = \frac{90}{s^2\sin\theta_2}\cdot\frac{ds}{dt}$
$= \frac{90}{s\cdot x}\cdot\frac{ds}{dt}$. Therefore, $x = 60$ and $s = 30\sqrt{13}\ \Rightarrow\ \frac{d\theta_2}{dt} = \frac{90}{\left(30\sqrt{13}\right)(60)}\cdot\left(\frac{-32}{\sqrt{13}}\right) = -\frac{8}{65}$ rad/sec.

(c) $\frac{d\theta_1}{dt} = -\frac{90}{s^2\cos\theta_1}\cdot\frac{ds}{dt} = -\frac{90}{\left(s^2\cdot\frac{x}{s}\right)}\cdot\left(\frac{x}{s}\right)\cdot\left(\frac{dx}{dt}\right) = \left(-\frac{90}{s^2}\right)\left(\frac{dx}{dt}\right) = \left(-\frac{90}{x^2+8100}\right)\frac{dx}{dt}\ \Rightarrow\ \lim\limits_{x\to 0}\frac{d\theta_1}{dt}$
$= \lim\limits_{x\to 0}\left(-\frac{90}{x^2+8100}\right)(-15) = \frac{1}{6}$ rad/sec; $\frac{d\theta_2}{dt} = \frac{90}{s^2\sin\theta_2}\cdot\frac{ds}{dt} = \left(\frac{90}{s^2\cdot\frac{x}{s}}\right)\left(\frac{x}{s}\right)\left(\frac{dx}{dt}\right) = \left(\frac{90}{s^2}\right)\left(\frac{dx}{dt}\right)$
$= \left(\frac{90}{x^2+8100}\right)\frac{dx}{dt}\ \Rightarrow\ \lim\limits_{x\to 0}\frac{d\theta_2}{dt} = -\frac{1}{6}$ rad/sec

44. Let a represent the distance between point O and ship A, b the distance between point O and ship B, and D the distance between the ships. By the Law of Cosines, $D^2 = a^2 + b^2 - 2ab\cos 120°\ \Rightarrow \frac{dD}{dt} = \frac{1}{2D}\left[2a\,\frac{da}{dt} + 2b\,\frac{db}{dt} + a\,\frac{db}{dt} + b\,\frac{da}{dt}\right]$.
When $a = 5$, $\frac{da}{dt} = 14$, $b = 3$, and $\frac{db}{dt} = 21$, then $\frac{dD}{dt} = \frac{413}{2D}$ where $D = 7$. The ships are moving $\frac{dD}{dt} = 29.5$ knots apart.

3.9 LINEARIZATION AND DIFFERENTIALS

1. $f(x) = x^3 - 2x + 3 \Rightarrow f'(x) = 3x^2 - 2 \Rightarrow L(x) = f'(2)(x-2) + f(2) = 10(x-2) + 7 \Rightarrow L(x) = 10x - 13$ at $x = 2$

2. $f(x) = \sqrt{x^2+9} = (x^2+9)^{1/2} \Rightarrow f'(x) = \left(\frac{1}{2}\right)(x^2+9)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+9}} \Rightarrow L(x) = f'(-4)(x+4) + f(-4)$
$= -\frac{4}{5}(x+4) + 5 \Rightarrow L(x) = -\frac{4}{5}x + \frac{9}{5}$ at $x = -4$

3. $f(x) = x + \frac{1}{x} \Rightarrow f'(x) = 1 - x^{-2} \Rightarrow L(x) = f(1) + f'(1)(x-1) = 2 + 0(x-1) = 2$

4. $f(x) = x^{1/3} \Rightarrow f'(x) = \frac{1}{3x^{2/3}} \Rightarrow L(x) = f'(-8)(x-(-8)) + f(-8) = \frac{1}{12}(x+8) - 2 \Rightarrow L(x) = \frac{1}{12}x - \frac{4}{3}$

5. $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \Rightarrow L(x) = f(\pi) + f'(\pi)(x-\pi) = 0 + 1(x-\pi) = x - \pi$

6. (a) $f(x) = \sin x \Rightarrow f'(x) = \cos x \Rightarrow L(x) = f(0) + f'(0)(x-0) = x \Rightarrow L(x) = x$
(b) $f(x) = \cos x \Rightarrow f'(x) = -\sin x \Rightarrow L(x) = f(0) + f'(0)(x-0) = 1 \Rightarrow L(x) = 1$
(c) $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \Rightarrow L(x) = f(0) + f'(0)(x-0) = x \Rightarrow L(x) = x$

7. $f(x) = x^2 + 2x \Rightarrow f'(x) = 2x + 2 \Rightarrow L(x) = f'(0)(x-0) + f(0) = 2(x-0) + 0 \Rightarrow L(x) = 2x$ at $x = 0$

8. $f(x) = x^{-1} \Rightarrow f'(x) = -x^{-2} \Rightarrow L(x) = f'(1)(x-1) + f(1) = (-1)(x-1) + 1 \Rightarrow L(x) = -x + 2$ at $x = 1$

9. $f(x) = 2x^2 + 4x - 3 \Rightarrow f'(x) = 4x + 4 \Rightarrow L(x) = f'(-1)(x+1) + f(-1) = 0(x+1) + (-5) \Rightarrow L(x) = -5$ at $x = -1$

10. $f(x) = 1 + x \Rightarrow f'(x) = 1 \Rightarrow L(x) = f'(8)(x-8) + f(8) = 1(x-8) + 9 \Rightarrow L(x) = x + 1$ at $x = 8$

11. $f(x) = \sqrt[3]{x} = x^{1/3} \Rightarrow f'(x) = \left(\frac{1}{3}\right)x^{-2/3} \Rightarrow L(x) = f'(8)(x-8) + f(8) = \frac{1}{12}(x-8) + 2 \Rightarrow L(x) = \frac{1}{12}x + \frac{4}{3}$ at $x = 8$

12. $f(x) = \frac{x}{x+1} \Rightarrow f'(x) = \frac{(1)(x+1)-(1)(x)}{(x+1)^2} = \frac{1}{(x+1)^2} \Rightarrow L(x) = f'(1)(x-1) + f(1) = \frac{1}{4}(x-1) + \frac{1}{2}$
$\Rightarrow L(x) = \frac{1}{4}x + \frac{1}{4}$ at $x = 1$

13. $f'(x) = k(1+x)^{k-1}$. We have $f(0) = 1$ and $f'(0) = k$. $L(x) = f(0) + f'(0)(x-0) = 1 + k(x-0) = 1 + kx$

14. (a) $f(x) = (1-x)^6 = [1+(-x)]^6 \approx 1 + 6(-x) = 1 - 6x$
(b) $f(x) = \frac{2}{1-x} = 2[1+(-x)]^{-1} \approx 2[1+(-1)(-x)] = 2 + 2x$
(c) $f(x) = (1+x)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)x = 1 - \frac{x}{2}$
(d) $f(x) = \sqrt{2+x^2} = \sqrt{2}\left(1+\frac{x^2}{2}\right)^{1/2} \approx \sqrt{2}\left(1+\frac{1}{2}\frac{x^2}{2}\right) = \sqrt{2}\left(1+\frac{x^2}{4}\right)$
(e) $f(x) = (4+3x)^{1/3} = 4^{1/3}\left(1+\frac{3x}{4}\right)^{1/3} \approx 4^{1/3}\left(1+\frac{1}{3}\frac{3x}{4}\right) = 4^{1/3}\left(1+\frac{x}{4}\right)$
(f) $f(x) = \left(1-\frac{1}{2+x}\right)^{2/3} = \left[1+\left(-\frac{1}{2+x}\right)\right]^{2/3} \approx 1 + \frac{2}{3}\left(-\frac{1}{2+x}\right) = 1 - \frac{2}{6+3x}$

15. (a) $(1.0002)^{50} = (1+0.0002)^{50} \approx 1 + 50(0.0002) = 1 + .01 = 1.01$
(b) $\sqrt[3]{1.009} = (1+0.009)^{1/3} \approx 1 + \left(\frac{1}{3}\right)(0.009) = 1 + 0.003 = 1.003$

16. $f(x) = \sqrt{x+1} + \sin x = (x+1)^{1/2} + \sin x \Rightarrow f'(x) = \left(\frac{1}{2}\right)(x+1)^{-1/2} + \cos x \Rightarrow L_f(x) = f'(0)(x-0) + f(0)$
$= \frac{3}{2}(x-0) + 1 \Rightarrow L_f(x) = \frac{3}{2}x + 1$, the linearization of $f(x)$; $g(x) = \sqrt{x+1} = (x+1)^{1/2} \Rightarrow g'(x)$

$= \left(\frac{1}{2}\right)(x+1)^{-1/2} \Rightarrow L_g(x) = g'(0)(x-0) + g(0) = \frac{1}{2}(x-0) + 1 \Rightarrow L_g(x) = \frac{1}{2}x + 1$, the linearization of g(x); $h(x) = \sin x \Rightarrow h'(x) = \cos x \Rightarrow L_h(x) = h'(0)(x-0) + h(0) = (1)(x-0) + 0 \Rightarrow L_h(x) = x$, the linearization of h(x). $L_f(x) = L_g(x) + L_h(x)$ implies that the linearization of a sum is equal to the sum of the linearizations.

17. $y = x^3 - 3\sqrt{x} = x^3 - 3x^{1/2} \Rightarrow dy = \left(3x^2 - \frac{3}{2}x^{-1/2}\right)dx \Rightarrow dy = \left(3x^2 - \frac{3}{2\sqrt{x}}\right)dx$

18. $y = x\sqrt{1-x^2} = x\left(1-x^2\right)^{1/2} \Rightarrow dy = \left[(1)\left(1-x^2\right)^{1/2} + (x)\left(\frac{1}{2}\right)\left(1-x^2\right)^{-1/2}(-2x)\right]dx$
$= \left(1-x^2\right)^{-1/2}\left[\left(1-x^2\right) - x^2\right]dx = \frac{\left(1-2x^2\right)}{\sqrt{1-x^2}}dx$

19. $y = \frac{2x}{1+x^2} \Rightarrow dy = \left(\frac{(2)\left(1+x^2\right) - (2x)(2x)}{\left(1+x^2\right)^2}\right)dx = \frac{2-2x^2}{\left(1+x^2\right)^2}dx$

20. $y = \frac{2\sqrt{x}}{3\left(1+\sqrt{x}\right)} = \frac{2x^{1/2}}{3\left(1+x^{1/2}\right)} \Rightarrow dy = \left(\frac{x^{-1/2}\left(3\left(1+x^{1/2}\right)\right) - 2x^{1/2}\left(\frac{3}{2}x^{-1/2}\right)}{9\left(1+x^{1/2}\right)^2}\right)dx = \frac{3x^{-1/2}+3-3}{9\left(1+x^{1/2}\right)^2}dx$
$\Rightarrow dy = \frac{1}{3\sqrt{x}\left(1+\sqrt{x}\right)^2}dx$

21. $2y^{3/2} + xy - x = 0 \Rightarrow 3y^{1/2}\,dy + y\,dx + x\,dy - dx = 0 \Rightarrow \left(3y^{1/2} + x\right)dy = (1-y)\,dx \Rightarrow dy = \frac{1-y}{3\sqrt{y}+x}dx$

22. $xy^2 - 4x^{3/2} - y = 0 \Rightarrow y^2\,dx + 2xy\,dy - 6x^{1/2}\,dx - dy = 0 \Rightarrow (2xy-1)\,dy = \left(6x^{1/2} - y^2\right)dx$
$\Rightarrow dy = \frac{6\sqrt{x} - y^2}{2xy-1}dx$

23. $y = \sin\left(5\sqrt{x}\right) = \sin\left(5x^{1/2}\right) \Rightarrow dy = \left(\cos\left(5x^{1/2}\right)\right)\left(\frac{5}{2}x^{-1/2}\right)dx \Rightarrow dy = \frac{5\cos\left(5\sqrt{x}\right)}{2\sqrt{x}}dx$

24. $y = \cos\left(x^2\right) \Rightarrow dy = \left[-\sin\left(x^2\right)\right](2x)\,dx = -2x\sin\left(x^2\right)dx$

25. $y = 4\tan\left(\frac{x^3}{3}\right) \Rightarrow dy = 4\left(\sec^2\left(\frac{x^3}{3}\right)\right)\left(x^2\right)dx \Rightarrow dy = 4x^2\sec^2\left(\frac{x^3}{3}\right)dx$

26. $y = \sec\left(x^2-1\right) \Rightarrow dy = \left[\sec\left(x^2-1\right)\tan\left(x^2-1\right)\right](2x)\,dx = 2x\left[\sec\left(x^2-1\right)\tan\left(x^2-1\right)\right]dx$

27. $y = 3\csc\left(1-2\sqrt{x}\right) = 3\csc\left(1-2x^{1/2}\right) \Rightarrow dy = 3\left(-\csc\left(1-2x^{1/2}\right)\right)\cot\left(1-2x^{1/2}\right)\left(-x^{-1/2}\right)dx$
$\Rightarrow dy = \frac{3}{\sqrt{x}}\csc\left(1-2\sqrt{x}\right)\cot\left(1-2\sqrt{x}\right)dx$

28. $y = 2\cot\left(\frac{1}{\sqrt{x}}\right) = 2\cot\left(x^{-1/2}\right) \Rightarrow dy = -2\csc^2\left(x^{-1/2}\right)\left(-\frac{1}{2}\right)\left(x^{-3/2}\right)dx \Rightarrow dy = \frac{1}{\sqrt{x^3}}\csc^2\left(\frac{1}{\sqrt{x}}\right)dx$

29. $f(x) = x^2 + 2x,\ x_0 = 1,\ dx = 0.1 \Rightarrow f'(x) = 2x + 2$
(a) $\Delta f = f(x_0 + dx) - f(x_0) = f(1.1) - f(1) = 3.41 - 3 = 0.41$
(b) $df = f'(x_0)\,dx = [2(1) + 2](0.1) = 0.4$
(c) $|\Delta f - df| = |0.41 - 0.4| = 0.01$

30. $f(x) = 2x^2 + 4x - 3,\ x_0 = -1,\ dx = 0.1 \Rightarrow f'(x) = 4x + 4$
(a) $\Delta f = f(x_0 + dx) - f(x_0) = f(-.9) - f(-1) = .02$
(b) $df = f'(x_0)\,dx = [4(-1) + 4](.1) = 0$
(c) $|\Delta f - df| = |.02 - 0| = .02$

31. $f(x) = x^3 - x$, $x_0 = 1$, $dx = 0.1 \Rightarrow f'(x) = 3x^2 - 1$
    (a) $\Delta f = f(x_0 + dx) - f(x_0) = f(1.1) - f(1) = .231$
    (b) $df = f'(x_0)\,dx = [3(1)^2 - 1](.1) = .2$
    (c) $|\Delta f - df| = |.231 - .2| = .031$

32. $f(x) = x^4$, $x_0 = 1$, $dx = 0.1 \Rightarrow f'(x) = 4x^3$
    (a) $\Delta f = f(x_0 + dx) - f(x_0) = f(1.1) - f(1) = .4641$
    (b) $df = f'(x_0)\,dx = 4(1)^3(.1) = .4$
    (c) $|\Delta f - df| = |.4641 - .4| = .0641$

33. $f(x) = x^{-1}$, $x_0 = 0.5$, $dx = 0.1 \Rightarrow f'(x) = -x^{-2}$
    (a) $\Delta f = f(x_0 + dx) - f(x_0) = f(.6) - f(.5) = -\frac{1}{3}$
    (b) $df = f'(x_0)\,dx = (-4)\left(\frac{1}{10}\right) = -\frac{2}{5}$
    (c) $|\Delta f - df| = \left|-\frac{1}{3} + \frac{2}{5}\right| = \frac{1}{15}$

34. $f(x) = x^3 - 2x + 3$, $x_0 = 2$, $dx = 0.1 \Rightarrow f'(x) = 3x^2 - 2$
    (a) $\Delta f = f(x_0 + dx) - f(x_0) = f(2.1) - f(2) = 1.061$
    (b) $df = f'(x_0)\,dx = (10)(0.10) = 1$
    (c) $|\Delta f - df| = |1.061 - 1| = .061$

35. $V = \frac{4}{3}\pi r^3 \Rightarrow dV = 4\pi r_0^2\,dr$

36. $V = x^3 \Rightarrow dV = 3x_0^2\,dx$

37. $S = 6x^2 \Rightarrow dS = 12x_0\,dx$

38. $S = \pi r\sqrt{r^2 + h^2} = \pi r\left(r^2 + h^2\right)^{1/2}$, h constant $\Rightarrow \frac{dS}{dr} = \pi\left(r^2 + h^2\right)^{1/2} + \pi r \cdot r\left(r^2 + h^2\right)^{-1/2}$
    $\Rightarrow \frac{dS}{dr} = \frac{\pi\left(r^2 + h^2\right) + \pi r^2}{\sqrt{r^2 + h^2}} \Rightarrow dS = \frac{\pi\left(2r_0^2 + h^2\right)}{\sqrt{r_0^2 + h^2}}\,dr$, h constant

39. $V = \pi r^2 h$, height constant $\Rightarrow dV = 2\pi r_0 h\,dr$

40. $S = 2\pi rh \Rightarrow dS = 2\pi r\,dh$

41. Given $r = 2$ m, $dr = .02$ m
    (a) $A = \pi r^2 \Rightarrow dA = 2\pi r\,dr = 2\pi(2)(.02) = .08\pi\ m^2$
    (b) $\left(\frac{.08\pi}{4\pi}\right)(100\%) = 2\%$

42. $C = 2\pi r$ and $dC = 2$ in. $\Rightarrow dC = 2\pi\,dr \Rightarrow dr = \frac{1}{\pi} \Rightarrow$ the diameter grew about $\frac{2}{\pi}$ in.; $A = \pi r^2 \Rightarrow dA = 2\pi r\,dr$
    $= 2\pi(5)\left(\frac{1}{\pi}\right) = 10\ in.^2$

43. The volume of a cylinder is $V = \pi r^2 h$. When h is held fixed, we have $\frac{dV}{dr} = 2\pi rh$, and so $dV = 2\pi rh\,dr$. For $h = 30$ in., $r = 6$ in., and $dr = 0.5$ in., the volume of the material in the shell is approximately $dV = 2\pi rh\,dr = 2\pi(6)(30)(0.5)$
    $= 180\pi \approx 565.5\ in^3$.

44. Let θ = angle of elevation and h = height of building. Then $h = 30\tan\theta$, so $dh = 30\sec^2\theta\,d\theta$. We want $|dh| < 0.04h$, which gives: $|30\sec^2\theta\,d\theta| < 0.04|30\tan\theta| \Rightarrow \frac{1}{\cos^2\theta}|d\theta| < \frac{0.04\sin\theta}{\cos\theta} \Rightarrow |d\theta| < 0.04\sin\theta\cos\theta \Rightarrow |d\theta| < 0.04\sin\frac{5\pi}{12}\cos\frac{5\pi}{12}$
    $= 0.01$ radian. The angle should be measured with an error of less than 0.01 radian (or approximatley 0.57 degrees), which is a percentage error of approximately 0.76%.

45. The percentage error in the radius is $\frac{\left(\frac{dr}{dt}\right)}{r} \times 100 \le 2\%$.

(a) Since $C = 2\pi r \Rightarrow \frac{dC}{dt} = 2\pi \frac{dr}{dt}$. The percentage error in calculating the circle's circumference is $\frac{\left(\frac{dC}{dt}\right)}{C} \times 100$ $= \frac{\left(2\pi \frac{dr}{dt}\right)}{2\pi r} \times 100 = \frac{\left(\frac{dr}{dt}\right)}{r} \times 100 \le 2\%$.

(b) Since $A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$. The percentage error in calculating the circle's area is given by $\frac{\left(\frac{dA}{dt}\right)}{A} \times 100$ $= \frac{\left(2\pi r \frac{dr}{dt}\right)}{\pi r^2} \times 100 = 2\,\frac{\left(\frac{dr}{dt}\right)}{r} \times 100 \le 2(2\%) = 4\%$.

46. The percentage error in the edge of the cube is $\frac{\left(\frac{dx}{dt}\right)}{x} \times 100 \le 0.5\%$.

(a) Since $S = 6x^2 \Rightarrow \frac{dS}{dt} = 12x\,\frac{dx}{dt}$. The percentage error in the cube's surface area is $\frac{\left(\frac{dS}{dt}\right)}{S} \times 100 = \frac{\left(12x\,\frac{dx}{dt}\right)}{6x^2} \times 100$ $= 2\frac{\left(\frac{dx}{dt}\right)}{x} \times 100 \le 2(0.5\%) = 1\%$

(b) Since $V = x^3 \Rightarrow \frac{dV}{dt} = 3x^2\frac{dx}{dt}$. The percentage error in the cube's volume is $\frac{\left(\frac{dV}{dt}\right)}{V} \times 100 = \frac{\left(3x^2\frac{dx}{dt}\right)}{x^3} \times 100$ $= 3\frac{\left(\frac{dx}{dt}\right)}{x} \times 100 \le 3(0.5\%) = 1.5\%$

47. $V = \pi h^3 \;\Rightarrow\; dV = 3\pi h^2\,dh$; recall that $\Delta V \approx dV$. Then $|\Delta V| \le (1\%)(V) = \frac{(1)(\pi h^3)}{100} \;\Rightarrow\; |dV| \le \frac{(1)(\pi h^3)}{100}$ $\Rightarrow\; |3\pi h^2\,dh| \le \frac{(1)(\pi h^3)}{100} \;\Rightarrow\; |dh| \le \frac{1}{300}\,h = \left(\frac{1}{3}\,\%\right)h$. Therefore the greatest tolerated error in the measurement of h is $\frac{1}{3}\,\%$.

48. (a) Let D_i represent the interior diameter. Then $V = \pi r^2 h = \pi\left(\frac{D_i}{2}\right)^2 h = \frac{\pi D_i^2 h}{4}$ and $h = 10 \;\Rightarrow\; V = \frac{5\pi D_i^2}{2} \;\Rightarrow$ $dV = 5\pi D_i\,dD_i$. Recall that $\Delta V \approx dV$. We want $|\Delta V| \le (1\%)(V) \;\Rightarrow\; |dV| \le \left(\frac{1}{100}\right)\left(\frac{5\pi D_i^2}{2}\right) = \frac{\pi D_i^2}{40}$ $\Rightarrow\; 5\pi D_i\,dD_i \le \frac{\pi D_i^2}{40} \;\Rightarrow\; \frac{dD_i}{D_i} \le 200$. The inside diameter must be measured to within 0.5%.

(b) Let D_e represent the exterior diameter, h the height and S the area of the painted surface. $S = \pi D_e h \Rightarrow dS = \pi h dD_e$ $\Rightarrow \frac{dS}{S} = \frac{dD_e}{D_e}$. Thus for small changes in exterior diameter, the approximate percentage change in the exterior diameter is equal to the approximate percentage change in the area painted, and to estimate the amount of paint required to within 5%, the tanks's exterior diameter must be measured to within 5%.

49. Given $D = 100$ cm, $dD = 1$ cm, $V = \frac{4}{3}\,\pi\left(\frac{D}{2}\right)^3 = \frac{\pi D^3}{6} \;\Rightarrow\; dV = \frac{\pi}{2}\,D^2\,dD = \frac{\pi}{2}\,(100)^2(1) = \frac{10^4\pi}{2}$. Then $\frac{dV}{V}\,(100\%)$ $= \left[\frac{\frac{10^4\pi}{2}}{\frac{10^6\pi}{6}}\right](10^2\%) = \left[\frac{\frac{10^6\pi}{2}}{\frac{10^6\pi}{6}}\right]\% = 3\%$

50. $V = \frac{4}{3}\,\pi r^3 = \frac{4}{3}\,\pi\left(\frac{D}{2}\right)^3 = \frac{\pi D^3}{6} \;\Rightarrow\; dV = \frac{\pi D^2}{2}\,dD$; recall that $\Delta V \approx dV$. Then $|\Delta V| \le (3\%)V = \left(\frac{3}{100}\right)\left(\frac{\pi D^3}{6}\right)$ $= \frac{\pi D^3}{200} \;\Rightarrow\; |dV| \le \frac{\pi D^3}{200} \;\Rightarrow\; \left|\frac{\pi D^2}{2}\,dD\right| \le \frac{\pi D^3}{200} \;\Rightarrow\; |dD| \le \frac{D}{100} = (1\%)\,D \;\Rightarrow$ the allowable percentage error in measuring the diameter is 1%.

51. $W = a + \frac{b}{g} = a + bg^{-1} \;\Rightarrow\; dW = -bg^{-2}\,dg = -\,\frac{b\,dg}{g^2} \;\Rightarrow\; \frac{dW_{moon}}{dW_{earth}} = \frac{\left(-\frac{b\,dg}{(5.2)^2}\right)}{\left(-\frac{b\,dg}{(32)^2}\right)} = \left(\frac{32}{5.2}\right)^2 = 37.87$, so a change of gravity on the moon has about 38 times the effect that a change of the same magnitude has on Earth.

52. (a) $T = 2\pi\left(\frac{L}{g}\right)^{1/2} \;\Rightarrow\; dT = 2\pi\sqrt{L}\left(-\,\frac{1}{2}\,g^{-3/2}\right)dg = -\pi\sqrt{L}\,g^{-3/2}\,dg$

(b) If g increases, then $dg > 0 \;\Rightarrow\; dT < 0$. The period T decreases and the clock ticks more frequently. Both the pendulum speed and clock speed increase.

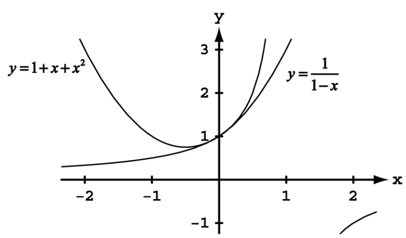
(c) $0.001 = -\pi\sqrt{100}\left(980^{-3/2}\right)dg \;\Rightarrow\; dg \approx -0.977$ cm/sec^2 $\Rightarrow$ the new $g \approx 979$ cm/sec^2

53. $E(x) = f(x) - g(x) \Rightarrow E(x) = f(x) - m(x-a) - c$. Then $E(a) = 0 \Rightarrow f(a) - m(a-a) - c = 0 \Rightarrow c = f(a)$. Next we calculate m: $\lim_{x \to a} \frac{E(x)}{x-a} = 0 \Rightarrow \lim_{x \to a} \frac{f(x) - m(x-a) - c}{x-a} = 0 \Rightarrow \lim_{x \to a} \left[\frac{f(x) - f(a)}{x-a} - m\right] = 0$ (since $c = f(a)$) $\Rightarrow f'(a) - m = 0 \Rightarrow m = f'(a)$. Therefore, $g(x) = m(x-a) + c = f'(a)(x-a) + f(a)$ is the linear approximation, as claimed.

54. (a) i. $Q(a) = f(a)$ implies that $b_0 = f(a)$.
ii. Since $Q'(x) = b_1 + 2b_2(x-a)$, $Q'(a) = f'(a)$ implies that $b_1 = f'(a)$.
iii. Since $Q''(x) = 2b_2$, $Q''(a) = f''(a)$ implies that $b_2 = \frac{f''(a)}{2}$.
In summary, $b_0 = f(a)$, $b_1 = f'(a)$, and $b_2 = \frac{f''(a)}{2}$.

(b) $f(x) = (1-x)^{-1}$; $f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2}$; $f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$
Since $f(0) = 1$, $f'(0) = 1$, and $f''(0) = 2$, the coefficients are $b_0 = 1$, $b_1 = 1$, $b_2 = \frac{2}{2} = 1$. The quadratic approximation is $Q(x) = 1 + x + x^2$.

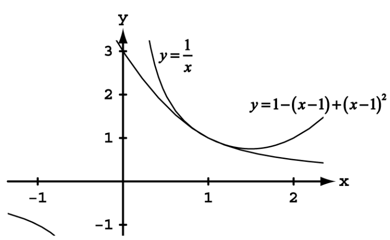
(c)



[-2.35, 2.35] by [-1.25, 3.25]

As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

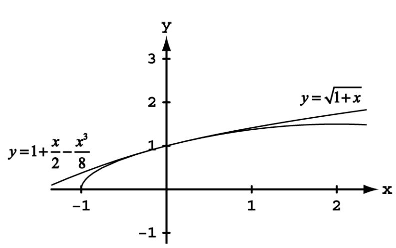
(d) $g(x) = x^{-1}$; $g'(x) = -1x^{-2}$; $g''(x) = 2x^{-3}$
Since $g(1) = 1$, $g'(1) = -1$, and $g''(1) = 2$, the coefficients are $b_0 = 1$, $b_1 = -1$, $b_2 = \frac{2}{2} = 1$. The quadratic approximation is $Q(x) = 1 - (x-1) + (x-1)^2$.



[-1.35, 3.35] by [-1.25, 3.25]

As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

(e) $h(x) = (1+x)^{1/2}$; $h'(x) = \frac{1}{2}(1+x)^{-1/2}$; $h''(x) = -\frac{1}{4}(1+x)^{-3/2}$
Since $h(0) = 1$, $h'(0) = \frac{1}{2}$, and $h''(0) = -\frac{1}{4}$, the coefficients are $b_0 = 1$, $b_1 = \frac{1}{2}$, $b_2 = \frac{-\frac{1}{4}}{2} = -\frac{1}{8}$. The quadratic approximation is $Q(x) = 1 + \frac{x}{2} - \frac{x^2}{8}$.



As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

(f) The linearization of any differentiable function $u(x)$ at $x = a$ is $L(x) = u(a) + u'(a)(x-a) = b_0 + b_1(x-a)$, where b_0 and b_1 are the coefficients of the constant and linear terms of the quadratic approximation. Thus, the linearization for $f(x)$ at $x = 0$ is $1 + x$; the linearization for $g(x)$ at $x = 1$ is $1 - (x-1)$ or $2 - x$; and the linearization for $h(x)$ at $x = 0$ is $1 + \frac{x}{2}$.

55-58. Example CAS commands:

Maple:

```
with(plots):
a:= 1: f:=x -> x ∧ 3 + x ∧ 2 − 2*x;
plot(f(x), x=−1..2);
diff(f(x),x);
fp := unapply (″,x);
L:=x -> f(a) + fp(a)*(x − a);
plot({f(x), L(x)}, x=−1..2);
err:=x -> abs(f(x) − L(x));
plot(err(x), x=−1..2, title = #absolute error function#);
err(−1);
```

Mathematica: (function, x1, x2, and a may vary):

```
Clear[f, x]
{x1, x2} = {−1, 2}; a = 1;
f[x_]:=x^3 + x^2 − 2x
Plot[f[x], {x, x1, x2}]
lin[x_]=f[a] + f'[a](x − a)
Plot[{f[x], lin[x]}, {x, x1, x2}]
err[x_]=Abs[f[x] − lin[x]]
Plot[err[x], {x, x1,x 2}]
err//N
```

After reviewing the error function, plot the error function and epsilon for differing values of epsilon (eps) and delta (del)

```
eps = 0.5; del = 0.4
Plot[{err[x], eps},{x, a − del, a + del}]
```

CHAPTER 3 PRACTICE EXERCISES

1. $y = x^5 - 0.125x^2 + 0.25x \Rightarrow \frac{dy}{dx} = 5x^4 - 0.25x + 0.25$

2. $y = 3 - 0.7x^3 + 0.3x^7 \Rightarrow \frac{dy}{dx} = -2.1x^2 + 2.1x^6$

3. $y = x^3 - 3\left(x^2 + \pi^2\right) \Rightarrow \frac{dy}{dx} = 3x^2 - 3(2x + 0) = 3x^2 - 6x = 3x(x - 2)$

4. $y = x^7 + \sqrt{7}x - \frac{1}{\pi+1} \Rightarrow \frac{dy}{dx} = 7x^6 + \sqrt{7}$

5. $y = (x + 1)^2\left(x^2 + 2x\right) \Rightarrow \frac{dy}{dx} = (x + 1)^2(2x + 2) + \left(x^2 + 2x\right)(2(x + 1)) = 2(x + 1)\left[(x + 1)^2 + x(x + 2)\right]$
$= 2(x + 1)\left(2x^2 + 4x + 1\right)$

6. $y = (2x - 5)(4 - x)^{-1} \Rightarrow \frac{dy}{dx} = (2x - 5)(-1)(4 - x)^{-2}(-1) + (4 - x)^{-1}(2) = (4 - x)^{-2}\left[(2x - 5) + 2(4 - x)\right]$
$= 3(4 - x)^{-2}$

7. $y = \left(\theta^2 + \sec\theta + 1\right)^3 \Rightarrow \frac{dy}{d\theta} = 3\left(\theta^2 + \sec\theta + 1\right)^2(2\theta + \sec\theta\tan\theta)$

8. $y = \left(-1 - \frac{\csc\theta}{2} - \frac{\theta^2}{4}\right)^2 \Rightarrow \frac{dy}{d\theta} = 2\left(-1 - \frac{\csc\theta}{2} - \frac{\theta^2}{4}\right)\left(\frac{\csc\theta\cot\theta}{2} - \frac{\theta}{2}\right) = \left(-1 - \frac{\csc\theta}{2} - \frac{\theta^2}{4}\right)(\csc\theta\cot\theta - \theta)$

9. $s = \frac{\sqrt{t}}{1+\sqrt{t}} \Rightarrow \frac{ds}{dt} = \frac{(1+\sqrt{t})\cdot\frac{1}{2\sqrt{t}} - \sqrt{t}\left(\frac{1}{2\sqrt{t}}\right)}{(1+\sqrt{t})^2} = \frac{(1+\sqrt{t})-\sqrt{t}}{2\sqrt{t}\,(1+\sqrt{t})^2} = \frac{1}{2\sqrt{t}\,(1+\sqrt{t})^2}$

10. $s = \frac{1}{\sqrt{t}-1} \Rightarrow \frac{ds}{dt} = \frac{(\sqrt{t}-1)(0) - 1\left(\frac{1}{2\sqrt{t}}\right)}{(\sqrt{t}-1)^2} = \frac{-1}{2\sqrt{t}\,(\sqrt{t}-1)^2}$

11. $y = 2\tan^2 x - \sec^2 x \Rightarrow \frac{dy}{dx} = (4\tan x)(\sec^2 x) - (2\sec x)(\sec x\tan x) = 2\sec^2 x\tan x$

12. $y = \frac{1}{\sin^2 x} - \frac{2}{\sin x} = \csc^2 x - 2\csc x \Rightarrow \frac{dy}{dx} = (2\csc x)(-\csc x\cot x) - 2(-\csc x\cot x) = (2\csc x\cot x)(1-\csc x)$

13. $s = \cos^4(1-2t) \Rightarrow \frac{ds}{dt} = 4\cos^3(1-2t)(-\sin(1-2t))(-2) = 8\cos^3(1-2t)\sin(1-2t)$

14. $s = \cot^3\left(\frac{2}{t}\right) \Rightarrow \frac{ds}{dt} = 3\cot^2\left(\frac{2}{t}\right)\left(-\csc^2\left(\frac{2}{t}\right)\right)\left(\frac{-2}{t^2}\right) = \frac{6}{t^2}\cot^2\left(\frac{2}{t}\right)\csc^2\left(\frac{2}{t}\right)$

15. $s = (\sec t + \tan t)^5 \Rightarrow \frac{ds}{dt} = 5(\sec t + \tan t)^4(\sec t\tan t + \sec^2 t) = 5(\sec t)(\sec t + \tan t)^5$

16. $s = \csc^5(1-t+3t^2) \Rightarrow \frac{ds}{dt} = 5\csc^4(1-t+3t^2)\left(-\csc(1-t+3t^2)\cot(1-t+3t^2)\right)(-1+6t)$
$= -5(6t-1)\csc^5(1-t+3t^2)\cot(1-t+3t^2)$

17. $r = \sqrt{2\theta\sin\theta} = (2\theta\sin\theta)^{1/2} \Rightarrow \frac{dr}{d\theta} = \frac{1}{2}(2\theta\sin\theta)^{-1/2}(2\theta\cos\theta + 2\sin\theta) = \frac{\theta\cos\theta+\sin\theta}{\sqrt{2\theta\sin\theta}}$

18. $r = 2\theta\sqrt{\cos\theta} = 2\theta(\cos\theta)^{1/2} \Rightarrow \frac{dr}{d\theta} = 2\theta\left(\frac{1}{2}\right)(\cos\theta)^{-1/2}(-\sin\theta) + 2(\cos\theta)^{1/2} = \frac{-\theta\sin\theta}{\sqrt{\cos\theta}} + 2\sqrt{\cos\theta}$
$= \frac{2\cos\theta - \theta\sin\theta}{\sqrt{\cos\theta}}$

19. $r = \sin\sqrt{2\theta} = \sin(2\theta)^{1/2} \Rightarrow \frac{dr}{d\theta} = \cos(2\theta)^{1/2}\left(\frac{1}{2}(2\theta)^{-1/2}(2)\right) = \frac{\cos\sqrt{2\theta}}{\sqrt{2\theta}}$

20. $r = \sin\left(\theta+\sqrt{\theta+1}\right) \Rightarrow \frac{dr}{d\theta} = \cos\left(\theta+\sqrt{\theta+1}\right)\left(1+\frac{1}{2\sqrt{\theta+1}}\right) = \frac{2\sqrt{\theta+1}+1}{2\sqrt{\theta+1}}\cos\left(\theta+\sqrt{\theta+1}\right)$

21. $y = \frac{1}{2}x^2\csc\frac{2}{x} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^2\left(-\csc\frac{2}{x}\cot\frac{2}{x}\right)\left(\frac{-2}{x^2}\right) + \left(\csc\frac{2}{x}\right)\left(\frac{1}{2}\cdot 2x\right) = \csc\frac{2}{x}\cot\frac{2}{x} + x\csc\frac{2}{x}$

22. $y = 2\sqrt{x}\sin\sqrt{x} \Rightarrow \frac{dy}{dx} = 2\sqrt{x}\left(\cos\sqrt{x}\right)\left(\frac{1}{2\sqrt{x}}\right) + \left(\sin\sqrt{x}\right)\left(\frac{2}{2\sqrt{x}}\right) = \cos\sqrt{x} + \frac{\sin\sqrt{x}}{\sqrt{x}}$

23. $y = x^{-1/2}\sec(2x)^2 \Rightarrow \frac{dy}{dx} = x^{-1/2}\sec(2x)^2\tan(2x)^2(2(2x)\cdot 2) + \sec(2x)^2\left(-\frac{1}{2}x^{-3/2}\right)$
$= 8x^{1/2}\sec(2x)^2\tan(2x)^2 - \frac{1}{2}x^{-3/2}\sec(2x)^2 = \frac{1}{2}x^{1/2}\sec(2x)^2\left[16\tan(2x)^2 - x^{-2}\right]$ or $\frac{1}{2x^{3/2}}\sec(2x)^2\left[16x^2\tan(2x)^2 - 1\right]$

24. $y = \sqrt{x}\csc(x+1)^3 = x^{1/2}\csc(x+1)^3$
$\Rightarrow \frac{dy}{dx} = x^{1/2}\left(-\csc(x+1)^3\cot(x+1)^3\right)\left(3(x+1)^2\right) + \csc(x+1)^3\left(\frac{1}{2}x^{-1/2}\right)$
$= -3\sqrt{x}(x+1)^2\csc(x+1)^3\cot(x+1)^3 + \frac{\csc(x+1)^3}{2\sqrt{x}} = \frac{1}{2}\sqrt{x}\csc(x+1)^3\left[\frac{1}{x} - 6(x+1)^2\cot(x+1)^3\right]$
or $\frac{1}{2\sqrt{x}}\csc(x+1)^3\left[1 - 6x(x+1)^2\cot(x+1)^3\right]$

25. $y = 5\cot x^2 \Rightarrow \frac{dy}{dx} = 5\left(-\csc^2 x^2\right)(2x) = -10x\csc^2(x^2)$

26. $y = x^2\cot 5x \Rightarrow \frac{dy}{dx} = x^2\left(-\csc^2 5x\right)(5) + (\cot 5x)(2x) = -5x^2\csc^2 5x + 2x\cot 5x$

27. $y = x^2 \sin^2(2x^2) \Rightarrow \frac{dy}{dx} = x^2(2\sin(2x^2))(\cos(2x^2))(4x) + \sin^2(2x^2)(2x) = 8x^3 \sin(2x^2)\cos(2x^2) + 2x\sin^2(2x^2)$

28. $y = x^{-2}\sin^2(x^3) \Rightarrow \frac{dy}{dx} = x^{-2}(2\sin(x^3))(\cos(x^3))(3x^2) + \sin^2(x^3)(-2x^{-3}) = 6\sin(x^3)\cos(x^3) - 2x^{-3}\sin^2(x^3)$

29. $s = \left(\frac{4t}{t+1}\right)^{-2} \Rightarrow \frac{ds}{dt} = -2\left(\frac{4t}{t+1}\right)^{-3}\left(\frac{(t+1)(4)-(4t)(1)}{(t+1)^2}\right) = -2\left(\frac{4t}{t+1}\right)^{-3}\frac{4}{(t+1)^2} = -\frac{(t+1)}{8t^3}$

30. $s = \frac{-1}{15(15t-1)^3} = -\frac{1}{15}(15t-1)^{-3} \Rightarrow \frac{ds}{dt} = -\frac{1}{15}(-3)(15t-1)^{-4}(15) = \frac{3}{(15t-1)^4}$

31. $y = \left(\frac{\sqrt{x}}{x+1}\right)^2 \Rightarrow \frac{dy}{dx} = 2\left(\frac{\sqrt{x}}{x+1}\right)\cdot\frac{(x+1)\left(\frac{1}{2\sqrt{x}}\right)-(\sqrt{x})(1)}{(x+1)^2} = \frac{(x+1)-2x}{(x+1)^3} = \frac{1-x}{(x+1)^3}$

32. $y = \left(\frac{2\sqrt{x}}{2\sqrt{x}+1}\right)^2 \Rightarrow \frac{dy}{dx} = 2\left(\frac{2\sqrt{x}}{2\sqrt{x}+1}\right)\left(\frac{(2\sqrt{x}+1)\left(\frac{1}{\sqrt{x}}\right)-(2\sqrt{x})\left(\frac{1}{\sqrt{x}}\right)}{(2\sqrt{x}+1)^2}\right) = \frac{4\sqrt{x}\left(\frac{1}{\sqrt{x}}\right)}{(2\sqrt{x}+1)^3} = \frac{4}{(2\sqrt{x}+1)^3}$

33. $y = \sqrt{\frac{x^2+x}{x^2}} = \left(1+\frac{1}{x}\right)^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(1+\frac{1}{x}\right)^{-1/2}\left(-\frac{1}{x^2}\right) = -\frac{1}{2x^2\sqrt{1+\frac{1}{x}}}$

34. $y = 4x\sqrt{x+\sqrt{x}} = 4x\left(x+x^{1/2}\right)^{1/2} \Rightarrow \frac{dy}{dx} = 4x\left(\frac{1}{2}\right)\left(x+x^{1/2}\right)^{-1/2}\left(1+\frac{1}{2}x^{-1/2}\right) + \left(x+x^{1/2}\right)^{1/2}(4)$
$= \left(x+\sqrt{x}\right)^{-1/2}\left[2x\left(1+\frac{1}{2\sqrt{x}}\right)+4\left(x+\sqrt{x}\right)\right] = \left(x+\sqrt{x}\right)^{-1/2}\left(2x+\sqrt{x}+4x+4\sqrt{x}\right) = \frac{6x+5\sqrt{x}}{\sqrt{x+\sqrt{x}}}$

35. $r = \left(\frac{\sin\theta}{\cos\theta-1}\right)^2 \Rightarrow \frac{dr}{d\theta} = 2\left(\frac{\sin\theta}{\cos\theta-1}\right)\left[\frac{(\cos\theta-1)(\cos\theta)-(\sin\theta)(-\sin\theta)}{(\cos\theta-1)^2}\right] = 2\left(\frac{\sin\theta}{\cos\theta-1}\right)\left(\frac{\cos^2\theta-\cos\theta+\sin^2\theta}{(\cos\theta-1)^2}\right)$
$= \frac{(2\sin\theta)(1-\cos\theta)}{(\cos\theta-1)^3} = \frac{-2\sin\theta}{(\cos\theta-1)^2}$

36. $r = \left(\frac{\sin\theta+1}{1-\cos\theta}\right)^2 \Rightarrow \frac{dr}{d\theta} = 2\left(\frac{\sin\theta+1}{1-\cos\theta}\right)\left[\frac{(1-\cos\theta)(\cos\theta)-(\sin\theta+1)(\sin\theta)}{(1-\cos\theta)^2}\right] = \frac{2(\sin\theta+1)}{(1-\cos\theta)^3}(\cos\theta-\cos^2\theta-\sin^2\theta-\sin\theta)$
$= \frac{2(\sin\theta+1)(\cos\theta-\sin\theta-1)}{(1-\cos\theta)^3}$

37. $y = (2x+1)\sqrt{2x+1} = (2x+1)^{3/2} \Rightarrow \frac{dy}{dx} = \frac{3}{2}(2x+1)^{1/2}(2) = 3\sqrt{2x+1}$

38. $y = 20(3x-4)^{1/4}(3x-4)^{-1/5} = 20(3x-4)^{1/20} \Rightarrow \frac{dy}{dx} = 20\left(\frac{1}{20}\right)(3x-4)^{-19/20}(3) = \frac{3}{(3x-4)^{19/20}}$

39. $y = 3\left(5x^2+\sin 2x\right)^{-3/2} \Rightarrow \frac{dy}{dx} = 3\left(-\frac{3}{2}\right)\left(5x^2+\sin 2x\right)^{-5/2}[10x+(\cos 2x)(2)] = \frac{-9(5x+\cos 2x)}{\left(5x^2+\sin 2x\right)^{5/2}}$

40. $y = \left(3+\cos^3 3x\right)^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}\left(3+\cos^3 3x\right)^{-4/3}\left(3\cos^2 3x\right)(-\sin 3x)(3) = \frac{3\cos^2 3x\sin 3x}{\left(3+\cos^3 3x\right)^{4/3}}$

41. $xy + 2x + 3y = 1 \Rightarrow (xy'+y) + 2 + 3y' = 0 \Rightarrow xy' + 3y' = -2 - y \Rightarrow y'(x+3) = -2 - y \Rightarrow y' = -\frac{y+2}{x+3}$

42. $x^2 + xy + y^2 - 5x = 2 \Rightarrow 2x + \left(x\frac{dy}{dx}+y\right) + 2y\frac{dy}{dx} - 5 = 0 \Rightarrow x\frac{dy}{dx} + 2y\frac{dy}{dx} = 5 - 2x - y \Rightarrow \frac{dy}{dx}(x+2y) = 5 - 2x - y$
$\Rightarrow \frac{dy}{dx} = \frac{5-2x-y}{x+2y}$

43. $x^3 + 4xy - 3y^{4/3} = 2x \Rightarrow 3x^2 + \left(4x\frac{dy}{dx}+4y\right) - 4y^{1/3}\frac{dy}{dx} = 2 \Rightarrow 4x\frac{dy}{dx} - 4y^{1/3}\frac{dy}{dx} = 2 - 3x^2 - 4y$
$\Rightarrow \frac{dy}{dx}\left(4x-4y^{1/3}\right) = 2 - 3x^2 - 4y \Rightarrow \frac{dy}{dx} = \frac{2-3x^2-4y}{4x-4y^{1/3}}$

44. $5x^{4/5} + 10y^{6/5} = 15 \Rightarrow 4x^{-1/5} + 12y^{1/5}\,\frac{dy}{dx} = 0 \Rightarrow 12y^{1/5}\,\frac{dy}{dx} = -4x^{-1/5} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}\,x^{-1/5}y^{-1/5} = -\frac{1}{3(xy)^{1/5}}$

45. $(xy)^{1/2} = 1 \Rightarrow \frac{1}{2}\,(xy)^{-1/2}\left(x\,\frac{dy}{dx} + y\right) = 0 \Rightarrow x^{1/2}y^{-1/2}\,\frac{dy}{dx} = -x^{-1/2}y^{1/2} \Rightarrow \frac{dy}{dx} = -x^{-1}y \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

46. $x^2y^2 = 1 \Rightarrow x^2\left(2y\,\frac{dy}{dx}\right) + y^2(2x) = 0 \Rightarrow 2x^2y\,\frac{dy}{dx} = -2xy^2 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

47. $y^2 = \frac{x}{x+1} \Rightarrow 2y\,\frac{dy}{dx} = \frac{(x+1)(1)-(x)(1)}{(x+1)^2} \Rightarrow \frac{dy}{dx} = \frac{1}{2y(x+1)^2}$

48. $y^2 = \left(\frac{1+x}{1-x}\right)^{1/2} \Rightarrow y^4 = \frac{1+x}{1-x} \Rightarrow 4y^3\,\frac{dy}{dx} = \frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2} \Rightarrow \frac{dy}{dx} = \frac{1}{2y^3(1-x)^2}$

49. $p^3 + 4pq - 3q^2 = 2 \Rightarrow 3p^2\,\frac{dp}{dq} + 4\left(p + q\,\frac{dp}{dq}\right) - 6q = 0 \Rightarrow 3p^2\,\frac{dp}{dq} + 4q\,\frac{dp}{dq} = 6q - 4p \Rightarrow \frac{dp}{dq}\,(3p^2+4q) = 6q - 4p$
$\Rightarrow \frac{dp}{dq} = \frac{6q-4p}{3p^2+4q}$

50. $q = (5p^2+2p)^{-3/2} \Rightarrow 1 = -\frac{3}{2}\,(5p^2+2p)^{-5/2}\left(10p\,\frac{dp}{dq} + 2\,\frac{dp}{dq}\right) \Rightarrow -\frac{2}{3}\,(5p^2+2p)^{5/2} = \frac{dp}{dq}\,(10p+2)$
$\Rightarrow \frac{dp}{dq} = -\frac{(5p^2+2p)^{5/2}}{3(5p+1)}$

51. $r\cos 2s + \sin^2 s = \pi \Rightarrow r(-\sin 2s)(2) + (\cos 2s)\left(\frac{dr}{ds}\right) + 2\sin s\cos s = 0 \Rightarrow \frac{dr}{ds}\,(\cos 2s) = 2r\sin 2s - 2\sin s\cos s$
$\Rightarrow \frac{dr}{ds} = \frac{2r\sin 2s - \sin 2s}{\cos 2s} = \frac{(2r-1)(\sin 2s)}{\cos 2s} = (2r-1)(\tan 2s)$

52. $2rs - r - s + s^2 = -3 \Rightarrow 2\left(r + s\,\frac{dr}{ds}\right) - \frac{dr}{ds} - 1 + 2s = 0 \Rightarrow \frac{dr}{ds}\,(2s-1) = 1 - 2s - 2r \Rightarrow \frac{dr}{ds} = \frac{1-2s-2r}{2s-1}$

53. (a) $x^3 + y^3 = 1 \Rightarrow 3x^2 + 3y^2\,\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^2}{y^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{y^2(-2x) - (-x^2)\left(2y\,\frac{dy}{dx}\right)}{y^4}$
$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2xy^2 + (2yx^2)\left(-\frac{x^2}{y^2}\right)}{y^4} = \frac{-2xy^2 - \frac{2x^4}{y}}{y^4} = \frac{-2xy^3 - 2x^4}{y^5}$

(b) $y^2 = 1 - \frac{2}{x} \Rightarrow 2y\,\frac{dy}{dx} = \frac{2}{x^2} \Rightarrow \frac{dy}{dx} = \frac{1}{yx^2} \Rightarrow \frac{dy}{dx} = (yx^2)^{-1} \Rightarrow \frac{d^2y}{dx^2} = -(yx^2)^{-2}\left[y(2x) + x^2\,\frac{dy}{dx}\right]$
$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2xy - x^2\left(\frac{1}{yx^2}\right)}{y^2x^4} = \frac{-2xy^2 - 1}{y^3x^4}$

54. (a) $x^2 - y^2 = 1 \Rightarrow 2x - 2y\,\frac{dy}{dx} = 0 \Rightarrow -2y\,\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

(b) $\frac{dy}{dx} = \frac{x}{y} \Rightarrow \frac{d^2y}{dx^2} = \frac{y(1) - x\,\frac{dy}{dx}}{y^2} = \frac{y - x\left(\frac{x}{y}\right)}{y^2} = \frac{y^2 - x^2}{y^3} = \frac{-1}{y^3}$ (since $y^2 - x^2 = -1$)

55. (a) Let $h(x) = 6f(x) - g(x) \Rightarrow h'(x) = 6f'(x) - g'(x) \Rightarrow h'(1) = 6f'(1) - g'(1) = 6\left(\frac{1}{2}\right) - (-4) = 7$

(b) Let $h(x) = f(x)g^2(x) \Rightarrow h'(x) = f(x)\,(2g(x))\,g'(x) + g^2(x)f'(x) \Rightarrow h'(0) = 2f(0)g(0)g'(0) + g^2(0)f'(0)$
$= 2(1)(1)\left(\frac{1}{2}\right) + (1)^2(-3) = -2$

(c) Let $h(x) = \frac{f(x)}{g(x)+1} \Rightarrow h'(x) = \frac{(g(x)+1)f'(x) - f(x)g'(x)}{(g(x)+1)^2} \Rightarrow h'(1) = \frac{(g(1)+1)f'(1) - f(1)g'(1)}{(g(1)+1)^2} = \frac{(5+1)\left(\frac{1}{2}\right) - 3\,(-4)}{(5+1)^2} = \frac{5}{12}$

(d) Let $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x))g'(x) \Rightarrow h'(0) = f'(g(0))g'(0) = f'(1)\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

(e) Let $h(x) = g(f(x)) \Rightarrow h'(x) = g'(f(x))f'(x) \Rightarrow h'(0) = g'(f(0))f'(0) = g'(1)f'(0) = (-4)\,(-3) = 12$

(f) Let $h(x) = (x + f(x))^{3/2} \Rightarrow h'(x) = \frac{3}{2}\,(x + f(x))^{1/2}\,(1 + f'(x)) \Rightarrow h'(1) = \frac{3}{2}\,(1 + f(1))^{1/2}\,(1 + f'(1))$
$= \frac{3}{2}\,(1+3)^{1/2}\left(1 + \frac{1}{2}\right) = \frac{9}{2}$

(g) Let $h(x) = f(x + g(x)) \Rightarrow h'(x) = f'(x + g(x))\,(1 + g'(x)) \Rightarrow h'(0) = f'(g(0))\,(1 + g'(0))$
$= f'(1)\left(1 + \frac{1}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = \frac{3}{4}$

56. (a) Let $h(x) = \sqrt{x}\, f(x) \Rightarrow h'(x) = \sqrt{x}\, f'(x) + f(x) \cdot \frac{1}{2\sqrt{x}} \Rightarrow h'(1) = \sqrt{1}\, f'(1) + f(1) \cdot \frac{1}{2\sqrt{1}} = \frac{1}{5} + (-3)\left(\frac{1}{2}\right) = -\frac{13}{10}$

(b) Let $h(x) = (f(x))^{1/2} \Rightarrow h'(x) = \frac{1}{2}(f(x))^{-1/2}(f'(x)) \Rightarrow h'(0) = \frac{1}{2}(f(0))^{-1/2} f'(0) = \frac{1}{2}(9)^{-1/2}(-2) = -\frac{1}{3}$

(c) Let $h(x) = f\left(\sqrt{x}\right) \Rightarrow h'(x) = f'\left(\sqrt{x}\right) \cdot \frac{1}{2\sqrt{x}} \Rightarrow h'(1) = f'\left(\sqrt{1}\right) \cdot \frac{1}{2\sqrt{1}} = \frac{1}{5} \cdot \frac{1}{2} = \frac{1}{10}$

(d) Let $h(x) = f(1 - 5\tan x) \Rightarrow h'(x) = f'(1 - 5\tan x)(-5\sec^2 x) \Rightarrow h'(0) = f'(1 - 5\tan 0)(-5\sec^2 0)$
$= f'(1)(-5) = \frac{1}{5}(-5) = -1$

(e) Let $h(x) = \frac{f(x)}{2 + \cos x} \Rightarrow h'(x) = \frac{(2 + \cos x)f'(x) - f(x)(-\sin x)}{(2 + \cos x)^2} \Rightarrow h'(0) = \frac{(2 + 1)f'(0) - f(0)(0)}{(2 + 1)^2} = \frac{3(-2)}{9} = -\frac{2}{3}$

(f) Let $h(x) = 10 \sin\left(\frac{\pi x}{2}\right) f^2(x) \Rightarrow h'(x) = 10 \sin\left(\frac{\pi x}{2}\right)(2f(x)f'(x)) + f^2(x)\left(10\cos\left(\frac{\pi x}{2}\right)\right)\left(\frac{\pi}{2}\right)$
$\Rightarrow h'(1) = 10\sin\left(\frac{\pi}{2}\right)(2f(1)f'(1)) + f^2(1)\left(10\cos\left(\frac{\pi}{2}\right)\right)\left(\frac{\pi}{2}\right) = 20(-3)\left(\frac{1}{5}\right) + 0 = -12$

57. $x = t^2 + \pi \Rightarrow \frac{dx}{dt} = 2t$; $y = 3\sin 2x \Rightarrow \frac{dy}{dx} = 3(\cos 2x)(2) = 6\cos 2x = 6\cos(2t^2 + 2\pi) = 6\cos(2t^2)$; thus,
$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = 6\cos(2t^2) \cdot 2t \Rightarrow \left.\frac{dy}{dt}\right|_{t=0} = 6\cos(0) \cdot 0 = 0$

58. $t = (u^2 + 2u)^{1/3} \Rightarrow \frac{dt}{du} = \frac{1}{3}(u^2 + 2u)^{-2/3}(2u + 2) = \frac{2}{3}(u^2 + 2u)^{-2/3}(u + 1)$; $s = t^2 + 5t \Rightarrow \frac{ds}{dt} = 2t + 5$
$= 2(u^2 + 2u)^{1/3} + 5$; thus $\frac{ds}{du} = \frac{ds}{dt} \cdot \frac{dt}{du} = \left[2(u^2 + 2u)^{1/3} + 5\right]\left(\frac{2}{3}\right)(u^2 + 2u)^{-2/3}(u + 1)$
$\Rightarrow \left.\frac{ds}{du}\right|_{u=2} = \left[2(2^2 + 2(2))^{1/3} + 5\right]\left(\frac{2}{3}\right)(2^2 + 2(2))^{-2/3}(2 + 1) = 2\left(2 \cdot 8^{1/3} + 5\right)\left(8^{-2/3}\right) = 2(2 \cdot 2 + 5)\left(\frac{1}{4}\right) = \frac{9}{2}$

59. $r = 8\sin\left(s + \frac{\pi}{6}\right) \Rightarrow \frac{dr}{ds} = 8\cos\left(s + \frac{\pi}{6}\right)$; $w = \sin\left(\sqrt{r} - 2\right) \Rightarrow \frac{dw}{dr} = \cos\left(\sqrt{r} - 2\right)\left(\frac{1}{2\sqrt{r}}\right)$
$= \frac{\cos\sqrt{8\sin\left(s + \frac{\pi}{6}\right)} - 2}{2\sqrt{8\sin\left(s + \frac{\pi}{6}\right)}}$; thus, $\frac{dw}{ds} = \frac{dw}{dr} \cdot \frac{dr}{ds} = \frac{\cos\left(\sqrt{8\sin\left(s + \frac{\pi}{6}\right)} - 2\right)}{2\sqrt{8\sin\left(s + \frac{\pi}{6}\right)}} \cdot \left[8\cos\left(s + \frac{\pi}{6}\right)\right]$
$\Rightarrow \left.\frac{dw}{ds}\right|_{s=0} = \frac{\cos\left(\sqrt{8\sin\left(\frac{\pi}{6}\right)} - 2\right) \cdot 8\cos\left(\frac{\pi}{6}\right)}{2\sqrt{8\sin\left(\frac{\pi}{6}\right)}} = \frac{(\cos 0)(8)\left(\frac{\sqrt{3}}{2}\right)}{2\sqrt{4}} = \sqrt{3}$

60. $\theta^2 t + \theta = 1 \Rightarrow \left(\theta^2 + t\left(2\theta\frac{d\theta}{dt}\right)\right) + \frac{d\theta}{dt} = 0 \Rightarrow \frac{d\theta}{dt}(2\theta t + 1) = -\theta^2 \Rightarrow \frac{d\theta}{dt} = \frac{-\theta^2}{2\theta t + 1}$; $r = (\theta^2 + 7)^{1/3}$
$\Rightarrow \frac{dr}{d\theta} = \frac{1}{3}(\theta^2 + 7)^{-2/3}(2\theta) = \frac{2}{3}\theta(\theta^2 + 7)^{-2/3}$; now $t = 0$ and $\theta^2 t + \theta = 1 \Rightarrow \theta = 1$ so that $\left.\frac{d\theta}{dt}\right|_{t=0,\,\theta=1} = \frac{-1}{1} = -1$
and $\left.\frac{dr}{d\theta}\right|_{\theta=1} = \frac{2}{3}(1 + 7)^{-2/3} = \frac{1}{6} \Rightarrow \left.\frac{dr}{dt}\right|_{t=0} = \left.\frac{dr}{d\theta}\right|_{t=0} \cdot \left.\frac{d\theta}{dt}\right|_{t=0} = \left(\frac{1}{6}\right)(-1) = -\frac{1}{6}$

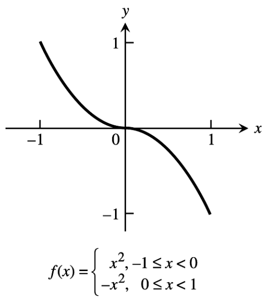
61. $y^3 + y = 2\cos x \Rightarrow 3y^2\frac{dy}{dx} + \frac{dy}{dx} = -2\sin x \Rightarrow \frac{dy}{dx}(3y^2 + 1) = -2\sin x \Rightarrow \frac{dy}{dx} = \frac{-2\sin x}{3y^2 + 1} \Rightarrow \left.\frac{dy}{dx}\right|_{(0,1)}$
$= \frac{-2\sin(0)}{3+1} = 0$; $\frac{d^2y}{dx^2} = \frac{(3y^2 + 1)(-2\cos x) - (-2\sin x)\left(6y\frac{dy}{dx}\right)}{(3y^2 + 1)^2}$
$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{(0,1)} = \frac{(3 + 1)(-2\cos 0) - (-2\sin 0)(6 \cdot 0)}{(3 + 1)^2} = -\frac{1}{2}$

62. $x^{1/3} + y^{1/3} = 4 \Rightarrow \frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y^{2/3}}{x^{2/3}} \Rightarrow \left.\frac{dy}{dx}\right|_{(8,8)} = -1$; $\frac{dy}{dx} = \frac{-y^{2/3}}{x^{2/3}}$
$\Rightarrow \frac{d^2y}{dx^2} = \frac{\left(x^{2/3}\right)\left(-\frac{2}{3}y^{-1/3}\frac{dy}{dx}\right) - \left(-y^{2/3}\right)\left(\frac{2}{3}x^{-1/3}\right)}{\left(x^{2/3}\right)^2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{(8,8)} = \frac{\left(8^{2/3}\right)\left[-\frac{2}{3} \cdot 8^{-1/3} \cdot (-1)\right] + \left(8^{2/3}\right)\left(\frac{2}{3} \cdot 8^{-1/3}\right)}{8^{4/3}}$
$= \frac{\frac{1}{3} + \frac{1}{3}}{8^{2/3}} = \frac{\frac{2}{3}}{4} = \frac{1}{6}$

63. $f(t) = \frac{1}{2t + 1}$ and $f(t + h) = \frac{1}{2(t + h) + 1} \Rightarrow \frac{f(t + h) - f(t)}{h} = \frac{\frac{1}{2(t+h)+1} - \frac{1}{2t+1}}{h} = \frac{2t + 1 - (2t + 2h + 1)}{(2t + 2h + 1)(2t + 1)h}$
$= \frac{-2h}{(2t + 2h + 1)(2t + 1)h} = \frac{-2}{(2t + 2h + 1)(2t + 1)} \Rightarrow f'(t) = \lim_{h \to 0}\frac{f(t + h) - f(t)}{h} = \lim_{h \to 0}\frac{-2}{(2t + 2h + 1)(2t + 1)}$
$= \frac{-2}{(2t + 1)^2}$

64. $g(x) = 2x^2 + 1$ and $g(x+h) = 2(x+h)^2 + 1 = 2x^2 + 4xh + 2h^2 + 1 \Rightarrow \frac{g(x+h) - g(x)}{h} = \frac{(2x^2 + 4xh + 2h^2 + 1) - (2x^2 + 1)}{h}$
$= \frac{4xh + 2h^2}{h} = 4x + 2h \Rightarrow g'(x) = \lim\limits_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim\limits_{h \to 0} (4x + 2h) = 4x$

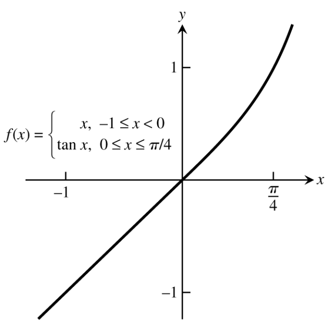
65. (a)



(b) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} x^2 = 0$ and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} -x^2 = 0 \Rightarrow \lim\limits_{x \to 0} f(x) = 0$. Since $\lim\limits_{x \to 0} f(x) = 0 = f(0)$ it follows that f is continuous at $x = 0$.

(c) $\lim\limits_{x \to 0^-} f'(x) = \lim\limits_{x \to 0^-} (2x) = 0$ and $\lim\limits_{x \to 0^+} f'(x) = \lim\limits_{x \to 0^+} (-2x) = 0 \Rightarrow \lim\limits_{x \to 0} f'(x) = 0$. Since this limit exists, it follows that f is differentiable at $x = 0$.

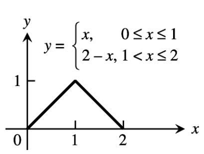
66. (a)



(b) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} x = 0$ and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \tan x = 0 \Rightarrow \lim\limits_{x \to 0} f(x) = 0$. Since $\lim\limits_{x \to 0} f(x) = 0 = f(0)$, it follows that f is continuous at $x = 0$.

(c) $\lim\limits_{x \to 0^-} f'(x) = \lim\limits_{x \to 0^-} 1 = 1$ and $\lim\limits_{x \to 0^+} f'(x) = \lim\limits_{x \to 0^+} \sec^2 x = 1 \Rightarrow \lim\limits_{x \to 0} f'(x) = 1$. Since this limit exists it follows that f is differentiable at $x = 0$.

67. (a)



(b) $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x = 1$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (2 - x) = 1 \Rightarrow \lim\limits_{x \to 1} f(x) = 1$. Since $\lim\limits_{x \to 1} f(x) = 1 = f(1)$, it follows that f is continuous at $x = 1$.

(c) $\lim\limits_{x \to 1^-} f'(x) = \lim\limits_{x \to 1^-} 1 = 1$ and $\lim\limits_{x \to 1^+} f'(x) = \lim\limits_{x \to 1^+} -1 = -1 \Rightarrow \lim\limits_{x \to 1^-} f'(x) \neq \lim\limits_{x \to 1^+} f'(x)$, so $\lim\limits_{x \to 1} f'(x)$ does not exist $\Rightarrow$ f is not differentiable at $x = 1$.

68. (a) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \sin 2x = 0$ and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} mx = 0 \Rightarrow \lim\limits_{x \to 0} f(x) = 0$, independent of m; since $f(0) = 0 = \lim\limits_{x \to 0} f(x)$ it follows that f is continuous at $x = 0$ for all values of m.

(b) $\lim\limits_{x \to 0^-} f'(x) = \lim\limits_{x \to 0^-} (\sin 2x)' = \lim\limits_{x \to 0^-} 2\cos 2x = 2$ and $\lim\limits_{x \to 0^+} f'(x) = \lim\limits_{x \to 0^+} (mx)' = \lim\limits_{x \to 0^+} m = m \Rightarrow$ f is differentiable at $x = 0$ provided that $\lim\limits_{x \to 0^-} f'(x) = \lim\limits_{x \to 0^+} f'(x) \Rightarrow m = 2$.

69. $y = \frac{x}{2} + \frac{1}{2x-4} = \frac{1}{2}x + (2x-4)^{-1} \Rightarrow \frac{dy}{dx} = \frac{1}{2} - 2(2x-4)^{-2}$; the slope of the tangent is $-\frac{3}{2} \Rightarrow -\frac{3}{2} = \frac{1}{2} - 2(2x-4)^{-2}$ $\Rightarrow -2 = -2(2x-4)^{-2} \Rightarrow 1 = \frac{1}{(2x-4)^2} \Rightarrow (2x-4)^2 = 1 \Rightarrow 4x^2 - 16x + 16 = 1 \Rightarrow 4x^2 - 16x + 15 = 0$ $\Rightarrow (2x-5)(2x-3) = 0 \Rightarrow x = \frac{5}{2}$ or $x = \frac{3}{2} \Rightarrow \left(\frac{5}{2}, \frac{9}{4}\right)$ and $\left(\frac{3}{2}, -\frac{1}{4}\right)$ are points on the curve where the slope is $-\frac{3}{2}$.

70. $y = x - \frac{1}{2x} \Rightarrow \frac{dy}{dx} = 1 + \frac{2}{(2x)^2} = 1 + \frac{1}{2x^2}$; the slope of the tangent is 3 $\Rightarrow 3 = 1 + \frac{1}{2x^2} \Rightarrow 2 = \frac{1}{2x^2} \Rightarrow x^2 = \frac{1}{4}$ $\Rightarrow x = \pm\frac{1}{2} \Rightarrow \left(\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$ are points on the curve where the slope is 3.

71. $y = 2x^3 - 3x^2 - 12x + 20 \Rightarrow \frac{dy}{dx} = 6x^2 - 6x - 12$; the tangent is parallel to the x-axis when $\frac{dy}{dx} = 0$ $\Rightarrow 6x^2 - 6x - 12 = 0 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2$ or $x = -1 \Rightarrow (2, 0)$ and $(-1, 27)$ are points on the curve where the tangent is parallel to the x-axis.

72. $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2 \Rightarrow \left.\frac{dy}{dx}\right|_{(-2,-8)} = 12$; an equation of the tangent line at $(-2, -8)$ is $y + 8 = 12(x + 2)$ $\Rightarrow y = 12x + 16$; x-intercept: $0 = 12x + 16 \Rightarrow x = -\frac{4}{3} \Rightarrow \left(-\frac{4}{3}, 0\right)$; y-intercept: $y = 12(0) + 16 = 16 \Rightarrow (0, 16)$

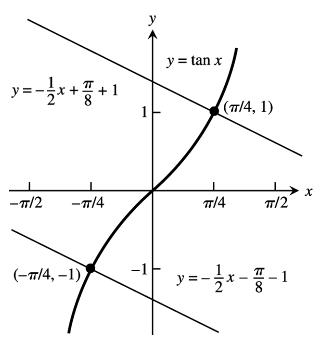
73. $y = 2x^3 - 3x^2 - 12x + 20 \Rightarrow \frac{dy}{dx} = 6x^2 - 6x - 12$

(a) The tangent is perpendicular to the line $y = 1 - \frac{x}{24}$ when $\frac{dy}{dx} = -\left(\frac{1}{-\left(\frac{1}{24}\right)}\right) = 24$; $6x^2 - 6x - 12 = 24$ $\Rightarrow x^2 - x - 2 = 4 \Rightarrow x^2 - x - 6 = 0 \Rightarrow (x-3)(x+2) = 0 \Rightarrow x = -2$ or $x = 3 \Rightarrow (-2, 16)$ and $(3, 11)$ are points where the tangent is perpendicular to $y = 1 - \frac{x}{24}$.

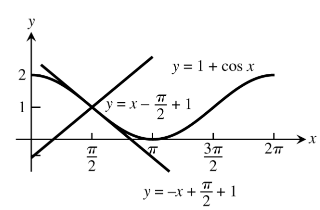
(b) The tangent is parallel to the line $y = \sqrt{2} - 12x$ when $\frac{dy}{dx} = -12 \Rightarrow 6x^2 - 6x - 12 = -12 \Rightarrow x^2 - x = 0$ $\Rightarrow x(x-1) = 0 \Rightarrow x = 0$ or $x = 1 \Rightarrow (0, 20)$ and $(1, 7)$ are points where the tangent is parallel to $y = \sqrt{2} - 12x$.

74. $y = \frac{\pi \sin x}{x} \Rightarrow \frac{dy}{dx} = \frac{x(\pi \cos x) - (\pi \sin x)(1)}{x^2} \Rightarrow m_1 = \left.\frac{dy}{dx}\right|_{x=\pi} = \frac{-\pi^2}{\pi^2} = -1$ and $m_2 = \left.\frac{dy}{dx}\right|_{x=-\pi} \frac{\pi^2}{\pi^2} = 1$. Since $m_1 = -\frac{1}{m_2}$ the tangents intersect at right angles.

75. $y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = \sec^2 x$; now the slope of $y = -\frac{x}{2}$ is $-\frac{1}{2} \Rightarrow$ the normal line is parallel to $y = -\frac{x}{2}$ when $\frac{dy}{dx} = 2$. Thus, $\sec^2 x = 2 \Rightarrow \frac{1}{\cos^2 x} = 2$ $\Rightarrow \cos^2 x = \frac{1}{2} \Rightarrow \cos x = \frac{\pm 1}{\sqrt{2}} \Rightarrow x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$ for $-\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow \left(-\frac{\pi}{4}, -1\right)$ and $\left(\frac{\pi}{4}, 1\right)$ are points where the normal is parallel to $y = -\frac{x}{2}$.



76. $y = 1 + \cos x \Rightarrow \frac{dy}{dx} = -\sin x \Rightarrow \left.\frac{dy}{dx}\right|_{\left(\frac{\pi}{2}, 1\right)} = -1$ $\Rightarrow$ the tangent at $\left(\frac{\pi}{2}, 1\right)$ is the line $y - 1 = -\left(x - \frac{\pi}{2}\right)$ $\Rightarrow y = -x + \frac{\pi}{2} + 1$; the normal at $\left(\frac{\pi}{2}, 1\right)$ is $y - 1 = (1)\left(x - \frac{\pi}{2}\right) \Rightarrow y = x - \frac{\pi}{2} + 1$



77. $y = x^2 + C \Rightarrow \frac{dy}{dx} = 2x$ and $y = x \Rightarrow \frac{dy}{dx} = 1$; the parabola is tangent to $y = x$ when $2x = 1 \Rightarrow x = \frac{1}{2} \Rightarrow y = \frac{1}{2}$; thus, $\frac{1}{2} = \left(\frac{1}{2}\right)^2 + C \Rightarrow C = \frac{1}{4}$

78. $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2 \Rightarrow \left.\frac{dy}{dx}\right|_{x=a} = 3a^2 \Rightarrow$ the tangent line at (a, a^3) is $y - a^3 = 3a^2(x - a)$. The tangent line intersects $y = x^3$ when $x^3 - a^3 = 3a^2(x - a) \Rightarrow (x - a)(x^2 + xa + a^2) = 3a^2(x - a) \Rightarrow (x - a)(x^2 + xa - 2a^2) = 0$ $\Rightarrow (x - a)^2(x + 2a) = 0 \Rightarrow x = a$ or $x = -2a$. Now $\left.\frac{dy}{dx}\right|_{x=-2a} = 3(-2a)^2 = 12a^2 = 4(3a^2)$, so the slope at $x = -2a$ is 4 times as large as the slope at (a, a^3) where $x = a$.

79. The line through $(0, 3)$ and $(5, -2)$ has slope $m = \frac{3-(-2)}{0-5} = -1 \Rightarrow$ the line through $(0, 3)$ and $(5, -2)$ is $y = -x + 3$; $y = \frac{c}{x+1} \Rightarrow \frac{dy}{dx} = \frac{-c}{(x+1)^2}$, so the curve is tangent to $y = -x + 3 \Rightarrow \frac{dy}{dx} = -1 = \frac{-c}{(x+1)^2}$ $\Rightarrow (x + 1)^2 = c,\ x \neq -1$. Moreover, $y = \frac{c}{x+1}$ intersects $y = -x + 3 \Rightarrow \frac{c}{x+1} = -x + 3,\ x \neq -1$ $\Rightarrow c = (x + 1)(-x + 3),\ x \neq -1$. Thus $c = c \Rightarrow (x + 1)^2 = (x + 1)(-x + 3) \Rightarrow (x + 1)[x + 1 - (-x + 3)]$ $= 0,\ x \neq -1 \Rightarrow (x + 1)(2x - 2) = 0 \Rightarrow x = 1$ (since $x \neq -1$) $\Rightarrow c = 4$.

80. Let $\left(b, \pm\sqrt{a^2 - b^2}\right)$ be a point on the circle $x^2 + y^2 = a^2$. Then $x^2 + y^2 = a^2 \Rightarrow 2x + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$ $\Rightarrow \left.\frac{dy}{dx}\right|_{x=b} = \frac{-b}{\pm\sqrt{a^2-b^2}} \Rightarrow$ normal line through $\left(b, \pm\sqrt{a^2 - b^2}\right)$ has slope $\frac{\pm\sqrt{a^2-b^2}}{b} \Rightarrow$ normal line is $y - \left(\pm\sqrt{a^2 - b^2}\right) = \frac{\pm\sqrt{a^2-b^2}}{b}(x - b) \Rightarrow y \mp \sqrt{a^2 - b^2} = \frac{\pm\sqrt{a^2-b^2}}{b}x \mp \sqrt{a^2 - b^2} \Rightarrow y = \pm\frac{\sqrt{a^2-b^2}}{b}x$ which passes through the origin.

81. $x^2 + 2y^2 = 9 \Rightarrow 2x + 4y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{2y} \Rightarrow \left.\frac{dy}{dx}\right|_{(1,2)} = -\frac{1}{4} \Rightarrow$ the tangent line is $y = 2 - \frac{1}{4}(x - 1)$ $= -\frac{1}{4}x + \frac{9}{4}$ and the normal line is $y = 2 + 4(x - 1) = 4x - 2$.

82. $x^3 + y^2 = 2 \Rightarrow 3x^2 + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-3x^2}{2y} \Rightarrow \left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{3}{2} \Rightarrow$ the tangent line is $y = 1 + \frac{-3}{2}(x - 1)$ $= -\frac{3}{2}x + \frac{5}{2}$ and the normal line is $y = 1 + \frac{2}{3}(x - 1) = \frac{2}{3}x + \frac{1}{3}$.

83. $xy + 2x - 5y = 2 \Rightarrow \left(x \frac{dy}{dx} + y\right) + 2 - 5\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx}(x - 5) = -y - 2 \Rightarrow \frac{dy}{dx} = \frac{-y-2}{x-5} \Rightarrow \left.\frac{dy}{dx}\right|_{(3,2)} = 2$ $\Rightarrow$ the tangent line is $y = 2 + 2(x - 3) = 2x - 4$ and the normal line is $y = 2 + \frac{-1}{2}(x - 3) = -\frac{1}{2}x + \frac{7}{2}$.

84. $(y - x)^2 = 2x + 4 \Rightarrow 2(y - x)\left(\frac{dy}{dx} - 1\right) = 2 \Rightarrow (y - x)\frac{dy}{dx} = 1 + (y - x) \Rightarrow \frac{dy}{dx} = \frac{1+y-x}{y-x} \Rightarrow \left.\frac{dy}{dx}\right|_{(6,2)} = \frac{3}{4}$ $\Rightarrow$ the tangent line is $y = 2 + \frac{3}{4}(x - 6) = \frac{3}{4}x - \frac{5}{2}$ and the normal line is $y = 2 - \frac{4}{3}(x - 6) = -\frac{4}{3}x + 10$.

85. $x + \sqrt{xy} = 6 \Rightarrow 1 + \frac{1}{2\sqrt{xy}}\left(x \frac{dy}{dx} + y\right) = 0 \Rightarrow x \frac{dy}{dx} + y = -2\sqrt{xy} \Rightarrow \frac{dy}{dx} = \frac{-2\sqrt{xy}-y}{x} \Rightarrow \left.\frac{dy}{dx}\right|_{(4,1)} = \frac{-5}{4}$ $\Rightarrow$ the tangent line is $y = 1 - \frac{5}{4}(x - 4) = -\frac{5}{4}x + 6$ and the normal line is $y = 1 + \frac{4}{5}(x - 4) = \frac{4}{5}x - \frac{11}{5}$.

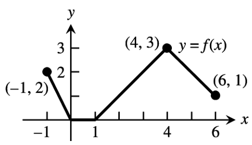
86. $x^{3/2} + 2y^{3/2} = 17 \Rightarrow \frac{3}{2}x^{1/2} + 3y^{1/2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x^{1/2}}{2y^{1/2}} \Rightarrow \left.\frac{dy}{dx}\right|_{(1,4)} = -\frac{1}{4} \Rightarrow$ the tangent line is $y = 4 - \frac{1}{4}(x - 1) = -\frac{1}{4}x + \frac{17}{4}$ and the normal line is $y = 4 + 4(x - 1) = 4x$.

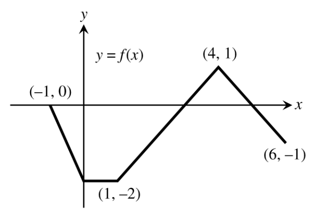
87. $x^3y^3 + y^2 = x + y \Rightarrow \left[x^3\left(3y^2\frac{dy}{dx}\right) + y^3(3x^2)\right] + 2y\frac{dy}{dx} = 1 + \frac{dy}{dx} \Rightarrow 3x^3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 1 - 3x^2y^3$ $\Rightarrow \frac{dy}{dx}(3x^3y^2 + 2y - 1) = 1 - 3x^2y^3 \Rightarrow \frac{dy}{dx} = \frac{1-3x^2y^3}{3x^3y^2+2y-1} \Rightarrow \left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{2}{4}$, but $\left.\frac{dy}{dx}\right|_{(1,-1)}$ is undefined. Therefore, the curve has slope $-\frac{1}{2}$ at $(1, 1)$ but the slope is undefined at $(1, -1)$.

88. $y = \sin(x - \sin x) \Rightarrow \frac{dy}{dx} = [\cos(x - \sin x)](1 - \cos x)$; $y = 0 \Rightarrow \sin(x - \sin x) = 0 \Rightarrow x - \sin x = k\pi$, $k = -2, -1, 0, 1, 2$ (for our interval) $\Rightarrow \cos(x - \sin x) = \cos(k\pi) = \pm 1$. Therefore, $\frac{dy}{dx} = 0$ and $y = 0$ when $1 - \cos x = 0$ and $x = k\pi$. For $-2\pi \le x \le 2\pi$, these equations hold when $k = -2, 0$, and 2 (since $\cos(-\pi) = \cos\pi = -1$). Thus the curve has horizontal tangents at the x-axis for the x-values -2π, 0, and 2π (which are even integer multiples of π) $\Rightarrow$ the curve has an infinite number of horizontal tangents.

89. B = graph of f, A = graph of f′. Curve B cannot be the derivative of A because A has only negative slopes while some of B's values are positive.

90. A = graph of f, B = graph of f′. Curve A cannot be the derivative of B because B has only negative slopes while A has positive values for $x > 0$.

91. 

92. 

93. (a) 0, 0 (b) largest 1700, smallest about 1400

94. rabbits/day and foxes/day

95. $\lim\limits_{x \to 0} \frac{\sin x}{2x^2 - x} = \lim\limits_{x \to 0} \left[\left(\frac{\sin x}{x}\right) \cdot \frac{1}{(2x - 1)}\right] = (1)\left(\frac{1}{-1}\right) = -1$

96. $\lim\limits_{x \to 0} \frac{3x - \tan 7x}{2x} = \lim\limits_{x \to 0} \left(\frac{3x}{2x} - \frac{\sin 7x}{2x \cos 7x}\right) = \frac{3}{2} - \lim\limits_{x \to 0} \left(\frac{1}{\cos 7x} \cdot \frac{\sin 7x}{7x} \cdot \frac{1}{\left(\frac{2}{7}\right)}\right) = \frac{3}{2} - \left(1 \cdot 1 \cdot \frac{7}{2}\right) = -2$

97. $\lim\limits_{r \to 0} \frac{\sin r}{\tan 2r} = \lim\limits_{r \to 0} \left(\frac{\sin r}{r} \cdot \frac{2r}{\tan 2r} \cdot \frac{1}{2}\right) = \left(\frac{1}{2}\right)(1) \lim\limits_{r \to 0} \frac{\cos 2r}{\left(\frac{\sin 2r}{2r}\right)} = \left(\frac{1}{2}\right)(1)\left(\frac{1}{1}\right) = \frac{1}{2}$

98. $\lim\limits_{\theta \to 0} \frac{\sin(\sin\theta)}{\theta} = \lim\limits_{\theta \to 0} \left(\frac{\sin(\sin\theta)}{\sin\theta}\right)\left(\frac{\sin\theta}{\theta}\right) = \lim\limits_{\theta \to 0} \frac{\sin(\sin\theta)}{\sin\theta}$. Let $x = \sin\theta$. Then $x \to 0$ as $\theta \to 0$ $\Rightarrow \lim\limits_{\theta \to 0} \frac{\sin(\sin\theta)}{\sin\theta} = \lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

99. $\lim\limits_{\theta \to \left(\frac{\pi}{2}\right)^-} \frac{4\tan^2\theta + \tan\theta + 1}{\tan^2\theta + 5} = \lim\limits_{\theta \to \left(\frac{\pi}{2}\right)^-} \frac{\left(4 + \frac{1}{\tan\theta} + \frac{1}{\tan^2\theta}\right)}{\left(1 + \frac{5}{\tan^2\theta}\right)} = \frac{(4 + 0 + 0)}{(1 + 0)} = 4$

100. $\lim\limits_{\theta \to 0^+} \frac{1 - 2\cot^2\theta}{5\cot^2\theta - 7\cot\theta - 8} = \lim\limits_{\theta \to 0^+} \frac{\left(\frac{1}{\cot^2\theta} - 2\right)}{\left(5 - \frac{7}{\cot\theta} - \frac{8}{\cot^2\theta}\right)} = \frac{(0 - 2)}{(5 - 0 - 0)} = -\frac{2}{5}$

101. $\lim\limits_{x \to 0} \frac{x \sin x}{2 - 2\cos x} = \lim\limits_{x \to 0} \frac{x \sin x}{2(1 - \cos x)} = \lim\limits_{x \to 0} \frac{x \sin x}{2\left(2\sin^2\left(\frac{x}{2}\right)\right)} = \lim\limits_{x \to 0} \left[\frac{\frac{x}{2} \cdot \frac{x}{2}}{\sin^2\left(\frac{x}{2}\right)} \cdot \frac{\sin x}{x}\right] = \lim\limits_{x \to 0} \left[\frac{\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} \cdot \frac{\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} \cdot \frac{\sin x}{x}\right]$ $= (1)(1)(1) = 1$

102. $\lim\limits_{\theta \to 0} \frac{1 - \cos\theta}{\theta^2} = \lim\limits_{\theta \to 0} \frac{2\sin^2\left(\frac{\theta}{2}\right)}{\theta^2} = \lim\limits_{\theta \to 0} \left[\frac{\sin\left(\frac{\theta}{2}\right)}{\left(\frac{\theta}{2}\right)} \cdot \frac{\sin\left(\frac{\theta}{2}\right)}{\left(\frac{\theta}{2}\right)} \cdot \frac{1}{2}\right] = (1)(1)\left(\frac{1}{2}\right) = \frac{1}{2}$

103. $\lim\limits_{x\to 0} \frac{\tan x}{x} = \lim\limits_{x\to 0} \left(\frac{1}{\cos x} \cdot \frac{\sin x}{x}\right) = 1$; let $\theta = \tan x \Rightarrow \theta \to 0$ as $x \to 0 \Rightarrow \lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} \frac{\tan(\tan x)}{\tan x}$ $= \lim\limits_{\theta\to 0} \frac{\tan\theta}{\theta} = 1$. Therefore, to make g continuous at the origin, define $g(0) = 1$.

104. $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \frac{\tan(\tan x)}{\sin(\sin x)} = \lim\limits_{x\to 0} \left[\frac{\tan(\tan x)}{\tan x} \cdot \frac{\sin x}{\sin(\sin x)} \cdot \frac{1}{\cos x}\right] = 1 \cdot \lim\limits_{x\to 0} \frac{\sin x}{\sin(\sin x)}$ (using the result of #105); let $\theta = \sin x \Rightarrow \theta \to 0$ as $x \to 0 \Rightarrow \lim\limits_{x\to 0} \frac{\sin x}{\sin(\sin x)} = \lim\limits_{\theta\to 0} \frac{\theta}{\sin\theta} = 1$. Therefore, to make f continuous at the origin, define $f(0) = 1$.

105. (a) $S = 2\pi r^2 + 2\pi rh$ and h constant $\Rightarrow \frac{dS}{dt} = 4\pi r \frac{dr}{dt} + 2\pi h \frac{dr}{dt} = (4\pi r + 2\pi h) \frac{dr}{dt}$
(b) $S = 2\pi r^2 + 2\pi rh$ and r constant $\Rightarrow \frac{dS}{dt} = 2\pi r \frac{dh}{dt}$
(c) $S = 2\pi r^2 + 2\pi rh \Rightarrow \frac{dS}{dt} = 4\pi r \frac{dr}{dt} + 2\pi \left(r \frac{dh}{dt} + h \frac{dr}{dt}\right) = (4\pi r + 2\pi h) \frac{dr}{dt} + 2\pi r \frac{dh}{dt}$
(d) S constant $\Rightarrow \frac{dS}{dt} = 0 \Rightarrow 0 = (4\pi r + 2\pi h) \frac{dr}{dt} + 2\pi r \frac{dh}{dt} \Rightarrow (2r + h) \frac{dr}{dt} = -r \frac{dh}{dt} \Rightarrow \frac{dr}{dt} = \frac{-r}{2r+h} \frac{dh}{dt}$

106. $S = \pi r\sqrt{r^2 + h^2} \Rightarrow \frac{dS}{dt} = \pi r \cdot \frac{\left(r \frac{dr}{dt} + h \frac{dh}{dt}\right)}{\sqrt{r^2+h^2}} + \pi\sqrt{r^2 + h^2} \frac{dr}{dt}$;
(a) h constant $\Rightarrow \frac{dh}{dt} = 0 \Rightarrow \frac{dS}{dt} = \frac{\pi r^2 \frac{dr}{dt}}{\sqrt{r^2+h^2}} + \pi\sqrt{r^2 + h^2} \frac{dr}{dt} = \left[\pi\sqrt{r^2 + h^2} + \frac{\pi r^2}{\sqrt{r^2+h^2}}\right] \frac{dr}{dt}$
(b) r constant $\Rightarrow \frac{dr}{dt} = 0 \Rightarrow \frac{dS}{dt} = \frac{\pi rh}{\sqrt{r^2+h^2}} \frac{dh}{dt}$
(c) In general, $\frac{dS}{dt} = \left[\pi\sqrt{r^2 + h^2} + \frac{\pi r^2}{\sqrt{r^2+h^2}}\right] \frac{dr}{dt} + \frac{\pi rh}{\sqrt{r^2+h^2}} \frac{dh}{dt}$

107. $A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$; so $r = 10$ and $\frac{dr}{dt} = -\frac{2}{\pi}$ m/sec $\Rightarrow \frac{dA}{dt} = (2\pi)(10)\left(-\frac{2}{\pi}\right) = -40$ m^2/sec

108. $V = s^3 \Rightarrow \frac{dV}{dt} = 3s^2 \cdot \frac{ds}{dt} \Rightarrow \frac{ds}{dt} = \frac{1}{3s^2} \frac{dV}{dt}$; so $s = 20$ and $\frac{dV}{dt} = 1200$ cm^3/min $\Rightarrow \frac{ds}{dt} = \frac{1}{3(20)^2}(1200) = 1$ cm/min

109. $\frac{dR_1}{dt} = -1$ ohm/sec, $\frac{dR_2}{dt} = 0.5$ ohm/sec; and $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow \frac{-1}{R^2} \frac{dR}{dt} = \frac{-1}{R_1^2} \frac{dR_1}{dt} - \frac{1}{R_2^2} \frac{dR_2}{dt}$. Also, $R_1 = 75$ ohms and $R_2 = 50$ ohms $\Rightarrow \frac{1}{R} = \frac{1}{75} + \frac{1}{50} \Rightarrow R = 30$ ohms. Therefore, from the derivative equation, $\frac{-1}{(30)^2} \frac{dR}{dt} = \frac{-1}{(75)^2}(-1) - \frac{1}{(50)^2}(0.5) = \left(\frac{1}{5625} - \frac{1}{5000}\right) \Rightarrow \frac{dR}{dt} = (-900)\left(\frac{5000-5625}{5625\cdot 5000}\right) = \frac{9(625)}{50(5625)} = \frac{1}{50} = 0.02$ ohm/sec.

110. $\frac{dR}{dt} = 3$ ohms/sec and $\frac{dX}{dt} = -2$ ohms/sec; $Z = \sqrt{R^2 + X^2} \Rightarrow \frac{dZ}{dt} = \frac{R \frac{dR}{dt} + X \frac{dX}{dt}}{\sqrt{R^2 + X^2}}$ so that $R = 10$ ohms and $X = 20$ ohms $\Rightarrow \frac{dZ}{dt} = \frac{(10)(3)+(20)(-2)}{\sqrt{10^2 + 20^2}} = \frac{-1}{\sqrt{5}} \approx -0.45$ ohm/sec.

111. Given $\frac{dx}{dt} = 10$ m/sec and $\frac{dy}{dt} = 5$ m/sec, let D be the distance from the origin $\Rightarrow D^2 = x^2 + y^2 \Rightarrow 2D \frac{dD}{dt}$ $= 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow D \frac{dD}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt}$. When $(x, y) = (3, -4)$, $D = \sqrt{3^2 + (-4)^2} = 5$ and $5 \frac{dD}{dt} = (3)(10) + (-4)(5) \Rightarrow \frac{dD}{dt} = \frac{10}{5} = 2$. Therefore, the particle is moving <u>away</u> <u>from</u> the origin at 2 m/sec (because the distance D is increasing).

112. Let D be the distance from the origin. We are given that $\frac{dD}{dt} = 11$ units/sec. Then $D^2 = x^2 + y^2 = x^2 + \left(x^{3/2}\right)^2$ $= x^2 + x^3 \Rightarrow 2D \frac{dD}{dt} = 2x \frac{dx}{dt} + 3x^2 \frac{dx}{dt} = x(2 + 3x) \frac{dx}{dt}$; $x = 3 \Rightarrow D = \sqrt{3^2 + 3^3} = 6$ and substitution in the derivative equation gives $(2)(6)(11) = (3)(2 + 9) \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 4$ units/sec.

113. (a) From the diagram we have $\frac{10}{h} = \frac{4}{r} \Rightarrow r = \frac{2}{5} h$.
(b) $V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi \left(\frac{2}{5} h\right)^2 h = \frac{4\pi h^3}{75} \Rightarrow \frac{dV}{dt} = \frac{4\pi h^2}{25} \frac{dh}{dt}$, so $\frac{dV}{dt} = -5$ and $h = 6 \Rightarrow \frac{dh}{dt} = -\frac{125}{144\pi}$ ft/min.

114. From the sketch in the text, $s = r\theta \Rightarrow \frac{ds}{dt} = r\,\frac{d\theta}{dt} + \theta\,\frac{dr}{dt}$. Also $r = 1.2$ is constant $\Rightarrow \frac{dr}{dt} = 0 \Rightarrow \frac{ds}{dt} = r\,\frac{d\theta}{dt} = (1.2)\,\frac{d\theta}{dt}$. Therefore, $\frac{ds}{dt} = 6$ ft/sec and $r = 1.2$ ft $\Rightarrow \frac{d\theta}{dt} = 5$ rad/sec

115. (a) From the sketch in the text, $\frac{d\theta}{dt} = -0.6$ rad/sec and $x = \tan\theta$. Also $x = \tan\theta \Rightarrow \frac{dx}{dt} = \sec^2\theta\,\frac{d\theta}{dt}$; at point A, $x = 0$ $\Rightarrow \theta = 0 \Rightarrow \frac{dx}{dt} = (\sec^2 0)\,(-0.6) = -0.6$. Therefore the speed of the light is $0.6 = \frac{3}{5}$ km/sec when it reaches point A.

(b) $\frac{(3/5)\text{ rad}}{\text{sec}} \cdot \frac{1\text{ rev}}{2\pi\text{ rad}} \cdot \frac{60\text{ sec}}{\text{min}} = \frac{18}{\pi}$ revs/min

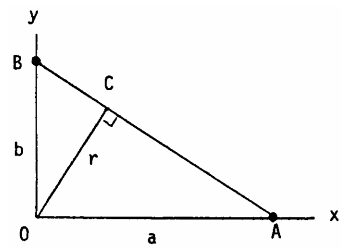
116. From the figure, $\frac{a}{r} = \frac{b}{BC} \Rightarrow \frac{a}{r} = \frac{b}{\sqrt{b^2 - r^2}}$. We are given that r is constant. Differentiation gives,

$$\frac{1}{r} \cdot \frac{da}{dt} = \frac{\left(\sqrt{b^2 - r^2}\right)\left(\frac{db}{dt}\right) - (b)\left(\frac{b}{\sqrt{b^2 - r^2}}\right)\left(\frac{db}{dt}\right)}{b^2 - r^2}.$$

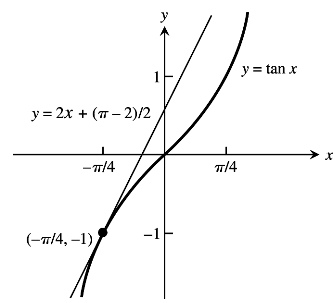
Then, $b = 2r$ and $\frac{db}{dt} = -0.3r$

$$\Rightarrow \frac{da}{dt} = r\left[\frac{\sqrt{(2r)^2 - r^2}\,(-0.3r) - (2r)\left(\frac{2r(-0.3r)}{\sqrt{(2r)^2 - r^2}}\right)}{(2r)^2 - r^2}\right]$$

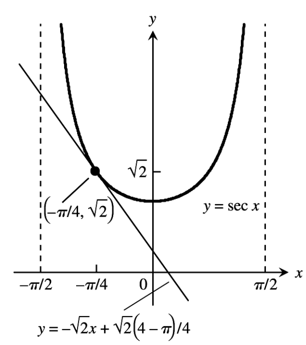
$$= \frac{\sqrt{3r^2}\,(-0.3r) + \frac{4r^2(0.3r)}{\sqrt{3r^2}}}{3r} = \frac{(3r^2)\,(-0.3r) + (4r^2)\,(0.3r)}{3\sqrt{3}\,r^2} = \frac{0.3r}{3\sqrt{3}} = \frac{r}{10\sqrt{3}}$$ m/sec. Since $\frac{da}{dt}$ is positive, the distance OA is increasing when $OB = 2r$, and B is moving toward O at the rate of $0.3r$ m/sec.



117. (a) If $f(x) = \tan x$ and $x = -\frac{\pi}{4}$, then $f'(x) = \sec^2 x$, $f\left(-\frac{\pi}{4}\right) = -1$ and $f'\left(-\frac{\pi}{4}\right) = 2$. The linearization of f(x) is $L(x) = 2\left(x + \frac{\pi}{4}\right) + (-1) = 2x + \frac{\pi - 2}{2}$.



(b) If $f(x) = \sec x$ and $x = -\frac{\pi}{4}$, then $f'(x) = \sec x \tan x$, $f\left(-\frac{\pi}{4}\right) = \sqrt{2}$ and $f'\left(-\frac{\pi}{4}\right) = -\sqrt{2}$. The linearization of f(x) is $L(x) = -\sqrt{2}\left(x + \frac{\pi}{4}\right) + \sqrt{2}$ $= -\sqrt{2}x + \frac{\sqrt{2}(4 - \pi)}{4}$.



118. $f(x) = \frac{1}{1 + \tan x} \Rightarrow f'(x) = \frac{-\sec^2 x}{(1 + \tan x)^2}$. The linearization at $x = 0$ is $L(x) = f'(0)(x - 0) + f(0) = 1 - x$.

119. $f(x) = \sqrt{x + 1} + \sin x - 0.5 = (x + 1)^{1/2} + \sin x - 0.5 \Rightarrow f'(x) = \left(\frac{1}{2}\right)(x + 1)^{-1/2} + \cos x$ $\Rightarrow L(x) = f'(0)(x - 0) + f(0) = 1.5(x - 0) + 0.5 \Rightarrow L(x) = 1.5x + 0.5$, the linearization of f(x).

120. $f(x) = \frac{2}{1-x} + \sqrt{1 + x} - 3.1 = 2(1 - x)^{-1} + (1 + x)^{1/2} - 3.1 \Rightarrow f'(x) = -2(1 - x)^{-2}(-1) + \frac{1}{2}(1 + x)^{-1/2}$ $= \frac{2}{(1 - x)^2} + \frac{1}{2\sqrt{1 + x}} \Rightarrow L(x) = f'(0)(x - 0) + f(0) = 2.5x - 0.1$, the linearization of f(x).

121. $S = \pi r\sqrt{r^2 + h^2}$, r constant $\Rightarrow dS = \pi r \cdot \frac{1}{2}(r^2 + h^2)^{-1/2} 2h\, dh = \frac{\pi r h}{\sqrt{r^2 + h^2}} dh$. Height changes from h_0 to $h_0 + dh$ $\Rightarrow dS = \frac{\pi r h_0 (dh)}{\sqrt{r^2 + h_0^2}}$

122. (a) $S = 6r^2 \Rightarrow dS = 12r\, dr$. We want $|dS| \le (2\%)\, S \Rightarrow |12r\, dr| \le \frac{12r^2}{100} \Rightarrow |dr| \le \frac{r}{100}$. The measurement of the edge r must have an error less than 1%.

(b) When $V = r^3$, then $dV = 3r^2\, dr$. The accuracy of the volume is $\left(\frac{dV}{V}\right)(100\%) = \left(\frac{3r^2\, dr}{r^3}\right)(100\%)$ $= \left(\frac{3}{r}\right)(dr)(100\%) = \left(\frac{3}{r}\right)\left(\frac{r}{100}\right)(100\%) = 3\%$

123. $C = 2\pi r \Rightarrow r = \frac{C}{2\pi}$, $S = 4\pi r^2 = \frac{C^2}{\pi}$, and $V = \frac{4}{3}\pi r^3 = \frac{C^3}{6\pi^2}$. It also follows that $dr = \frac{1}{2\pi}\, dC$, $dS = \frac{2C}{\pi}\, dC$ and $dV = \frac{C^2}{2\pi^2}\, dC$. Recall that $C = 10$ cm and $dC = 0.4$ cm.

(a) $dr = \frac{0.4}{2\pi} = \frac{0.2}{\pi}$ cm $\Rightarrow \left(\frac{dr}{r}\right)(100\%) = \left(\frac{0.2}{\pi}\right)\left(\frac{2\pi}{10}\right)(100\%) = (.04)(100\%) = 4\%$

(b) $dS = \frac{20}{\pi}(0.4) = \frac{8}{\pi}$ cm $\Rightarrow \left(\frac{dS}{S}\right)(100\%) = \left(\frac{8}{\pi}\right)\left(\frac{\pi}{100}\right)(100\%) = 8\%$

(c) $dV = \frac{10^2}{2\pi^2}(0.4) = \frac{20}{\pi^2}$ cm $\Rightarrow \left(\frac{dV}{V}\right)(100\%) = \left(\frac{20}{\pi^2}\right)\left(\frac{6\pi^2}{1000}\right)(100\%) = 12\%$

124. Similar triangles yield $\frac{35}{h} = \frac{15}{6} \Rightarrow h = 14$ ft. The same triangles imply that $\frac{20+a}{h} = \frac{a}{6} \Rightarrow h = 120a^{-1} + 6$ $\Rightarrow dh = -120a^{-2}\, da = -\frac{120}{a^2}\, da = \left(-\frac{120}{a^2}\right)\left(\pm\frac{1}{12}\right) = \left(-\frac{120}{15^2}\right)\left(\pm\frac{1}{12}\right) = \pm\frac{2}{45} \approx \pm .0444$ ft $= \pm 0.53$ inches.

CHAPTER 3 ADDITIONAL AND ADVANCED EXERCISES

1. (a) $\sin 2\theta = 2\sin\theta\cos\theta \Rightarrow \frac{d}{d\theta}(\sin 2\theta) = \frac{d}{d\theta}(2\sin\theta\cos\theta) \Rightarrow 2\cos 2\theta = 2[(\sin\theta)(-\sin\theta) + (\cos\theta)(\cos\theta)]$ $\Rightarrow \cos 2\theta = \cos^2\theta - \sin^2\theta$

(b) $\cos 2\theta = \cos^2\theta - \sin^2\theta \Rightarrow \frac{d}{d\theta}(\cos 2\theta) = \frac{d}{d\theta}(\cos^2\theta - \sin^2\theta) \Rightarrow -2\sin 2\theta = (2\cos\theta)(-\sin\theta) - (2\sin\theta)(\cos\theta)$ $\Rightarrow \sin 2\theta = \cos\theta\sin\theta + \sin\theta\cos\theta \Rightarrow \sin 2\theta = 2\sin\theta\cos\theta$

2. The derivative of $\sin(x + a) = \sin x\cos a + \cos x\sin a$ with respect to x is $\cos(x + a) = \cos x\cos a - \sin x\sin a$, which is also an identity. This principle does not apply to the equation $x^2 - 2x - 8 = 0$, since $x^2 - 2x - 8 = 0$ is not an identity: it holds for 2 values of x (-2 and 4), but not for all x.

3. (a) $f(x) = \cos x \Rightarrow f'(x) = -\sin x \Rightarrow f''(x) = -\cos x$, and $g(x) = a + bx + cx^2 \Rightarrow g'(x) = b + 2cx \Rightarrow g''(x) = 2c$; also, $f(0) = g(0) \Rightarrow \cos(0) = a \Rightarrow a = 1$; $f'(0) = g'(0) \Rightarrow -\sin(0) = b \Rightarrow b = 0$; $f''(0) = g''(0) \Rightarrow -\cos(0) = 2c$ $\Rightarrow c = -\frac{1}{2}$. Therefore, $g(x) = 1 - \frac{1}{2}x^2$.

(b) $f(x) = \sin(x + a) \Rightarrow f'(x) = \cos(x + a)$, and $g(x) = b\sin x + c\cos x \Rightarrow g'(x) = b\cos x - c\sin x$; also, $f(0) = g(0)$ $\Rightarrow \sin(a) = b\sin(0) + c\cos(0) \Rightarrow c = \sin a$; $f'(0) = g'(0) \Rightarrow \cos(a) = b\cos(0) - c\sin(0) \Rightarrow b = \cos a$. Therefore, $g(x) = \sin x\cos a + \cos x\sin a$.

(c) When $f(x) = \cos x$, $f'''(x) = \sin x$ and $f^{(4)}(x) = \cos x$; when $g(x) = 1 - \frac{1}{2}x^2$, $g'''(x) = 0$ and $g^{(4)}(x) = 0$. Thus $f'''(0) = 0 = g'''(0)$ so the third derivatives agree at $x = 0$. However, the fourth derivatives do not agree since $f^{(4)}(0) = 1$ but $g^{(4)}(0) = 0$. In case (b), when $f(x) = \sin(x + a)$ and $g(x) = \sin x\cos a + \cos x\sin a$, notice that $f(x) = g(x)$ for all x, not just $x = 0$. Since this is an identity, we have $f^{(n)}(x) = g^{(n)}(x)$ for any x and any positive integer n.

4. (a) $y = \sin x \Rightarrow y' = \cos x \Rightarrow y'' = -\sin x \Rightarrow y'' + y = -\sin x + \sin x = 0$; $y = \cos x \Rightarrow y' = -\sin x$ $\Rightarrow y'' = -\cos x \Rightarrow y'' + y = -\cos x + \cos x = 0$; $y = a\cos x + b\sin x \Rightarrow y' = -a\sin x + b\cos x$ $\Rightarrow y'' = -a\cos x - b\sin x \Rightarrow y'' + y = (-a\cos x - b\sin x) + (a\cos x + b\sin x) = 0$

(b) $y = \sin(2x) \Rightarrow y' = 2\cos(2x) \Rightarrow y'' = -4\sin(2x) \Rightarrow y'' + 4y = -4\sin(2x) + 4\sin(2x) = 0$. Similarly, $y = \cos(2x)$ and $y = a\cos(2x) + b\sin(2x)$ satisfy the differential equation $y'' + 4y = 0$. In general, $y = \cos(mx)$, $y = \sin(mx)$ and $y = a\cos(mx) + b\sin(mx)$ satisfy the differential equation $y'' + m^2y = 0$.

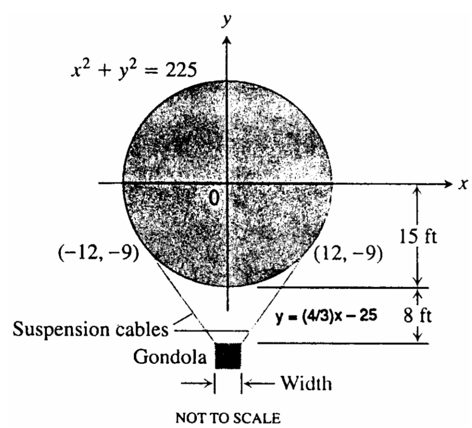
5. If the circle $(x-h)^2 + (y-k)^2 = a^2$ and $y = x^2 + 1$ are tangent at $(1, 2)$, then the slope of this tangent is $m = 2x|_{(1,2)} = 2$ and the tangent line is $y = 2x$. The line containing (h, k) and $(1, 2)$ is perpendicular to $y = 2x \Rightarrow \frac{k-2}{h-1} = -\frac{1}{2} \Rightarrow h = 5 - 2k \Rightarrow$ the location of the center is $(5 - 2k, k)$. Also, $(x-h)^2 + (y-k)^2 = a^2$ $\Rightarrow x - h + (y-k)y' = 0 \Rightarrow 1 + (y')^2 + (y-k)y'' = 0 \Rightarrow y'' = \frac{1+(y')^2}{k-y}$. At the point $(1, 2)$ we know $y' = 2$ from the tangent line and that $y'' = 2$ from the parabola. Since the second derivatives are equal at $(1, 2)$ we obtain $2 = \frac{1+(2)^2}{k-2} \Rightarrow k = \frac{9}{2}$. Then $h = 5 - 2k = -4 \Rightarrow$ the circle is $(x+4)^2 + \left(y - \frac{9}{2}\right)^2 = a^2$. Since $(1, 2)$ lies on the circle we have that $a = \frac{5\sqrt{5}}{2}$.

6. The total revenue is the number of people times the price of the fare: $r(x) = xp = x\left(3 - \frac{x}{40}\right)^2$, where $0 \le x \le 60$. The marginal revenue is $\frac{dr}{dx} = \left(3 - \frac{x}{40}\right)^2 + 2x\left(3 - \frac{x}{40}\right)\left(-\frac{1}{40}\right) \Rightarrow \frac{dr}{dx} = \left(3 - \frac{x}{40}\right)\left[\left(3 - \frac{x}{40}\right) - \frac{2x}{40}\right]$ $= 3\left(3 - \frac{x}{40}\right)\left(1 - \frac{x}{40}\right)$. Then $\frac{dr}{dx} = 0 \Rightarrow x = 40$ (since $x = 120$ does not belong to the domain). When 40 people are on the bus the marginal revenue is zero and the fare is $p(40) = \left(3 - \frac{x}{40}\right)^2\Big|_{x=40} = \$4.00$.

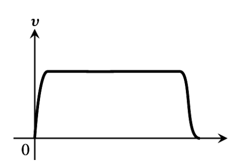
7. (a) $y = uv \Rightarrow \frac{dy}{dt} = \frac{du}{dt}v + u\frac{dv}{dt} = (0.04u)v + u(0.05v) = 0.09uv = 0.09y \Rightarrow$ the rate of growth of the total production is 9% per year.

(b) If $\frac{du}{dt} = -0.02u$ and $\frac{dv}{dt} = 0.03v$, then $\frac{dy}{dt} = (-0.02u)v + (0.03v)u = 0.01uv = 0.01y$, increasing at 1% per year.

8. When $x^2 + y^2 = 225$, then $y' = -\frac{x}{y}$. The tangent line to the balloon at $(12, -9)$ is $y + 9 = \frac{4}{3}(x - 12)$ $\Rightarrow y = \frac{4}{3}x - 25$. The top of the gondola is $15 + 8$ $= 23$ ft below the center of the balloon. The intersection of $y = -23$ and $y = \frac{4}{3}x - 25$ is at the far right edge of the gondola $\Rightarrow -23 = \frac{4}{3}x - 25$ $\Rightarrow x = \frac{3}{2}$. Thus the gondola is $2x = 3$ ft wide.



9. Answers will vary. Here is one possibility.



10. $s(t) = 10\cos\left(t + \frac{\pi}{4}\right) \Rightarrow v(t) = \frac{ds}{dt} = -10\sin\left(t + \frac{\pi}{4}\right) \Rightarrow a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -10\cos\left(t + \frac{\pi}{4}\right)$

(a) $s(0) = 10\cos\left(\frac{\pi}{4}\right) = \frac{10}{\sqrt{2}}$

(b) Left: -10, Right: 10

(c) Solving $10\cos\left(t+\frac{\pi}{4}\right)=-10 \Rightarrow \cos\left(t+\frac{\pi}{4}\right)=-1 \Rightarrow t=\frac{3\pi}{4}$ when the particle is farthest to the left.
Solving $10\cos\left(t+\frac{\pi}{4}\right)=10 \Rightarrow \cos\left(t+\frac{\pi}{4}\right)=1 \Rightarrow t=-\frac{\pi}{4}$, but $t\ge 0 \Rightarrow t=2\pi+\frac{-\pi}{4}=\frac{7\pi}{4}$ when the particle is farthest to the right. Thus, $v\left(\frac{3\pi}{4}\right)=0$, $v\left(\frac{7\pi}{4}\right)=0$, $a\left(\frac{3\pi}{4}\right)=10$, and $a\left(\frac{7\pi}{4}\right)=-10$.

(d) Solving $10\cos\left(t+\frac{\pi}{4}\right)=0 \Rightarrow t=\frac{\pi}{4} \Rightarrow v\left(\frac{\pi}{4}\right)=-10$, $\left|v\left(\frac{\pi}{4}\right)\right|=10$ and $a\left(\frac{\pi}{4}\right)=0$.

11. (a) $s(t)=64t-16t^2 \Rightarrow v(t)=\frac{ds}{dt}=64-32t=32(2-t)$. The maximum height is reached when $v(t)=0$ $\Rightarrow t=2$ sec. The velocity when it leaves the hand is $v(0)=64$ ft/sec.

(b) $s(t)=64t-2.6t^2 \Rightarrow v(t)=\frac{ds}{dt}=64-5.2t$. The maximum height is reached when $v(t)=0 \Rightarrow t\approx 12.31$ sec. The maximum height is about $s(12.31)=393.85$ ft.

12. $s_1=3t^3-12t^2+18t+5$ and $s_2=-t^3+9t^2-12t \Rightarrow v_1=9t^2-24t+18$ and $v_2=-3t^2+18t-12$; $v_1=v_2$ $\Rightarrow 9t^2-24t+18=-3t^2+18t-12 \Rightarrow 2t^2-7t+5=0 \Rightarrow (t-1)(2t-5)=0 \Rightarrow t=1$ sec and $t=2.5$ sec.

13. $m\left(v^2-v_0^2\right)=k\left(x_0^2-x^2\right) \Rightarrow m\left(2v\frac{dv}{dt}\right)=k\left(-2x\frac{dx}{dt}\right) \Rightarrow m\frac{dv}{dt}=k\left(-\frac{2x}{2v}\right)\frac{dx}{dt} \Rightarrow m\frac{dv}{dt}=-kx\left(\frac{1}{v}\right)\frac{dx}{dt}$. Then substituting $\frac{dx}{dt}=v \Rightarrow m\frac{dv}{dt}=-kx$, as claimed.

14. (a) $x=At^2+Bt+C$ on $[t_1,t_2] \Rightarrow v=\frac{dx}{dt}=2At+B \Rightarrow v\left(\frac{t_1+t_2}{2}\right)=2A\left(\frac{t_1+t_2}{2}\right)+B=A\left(t_1+t_2\right)+B$ is the instantaneous velocity at the midpoint. The average velocity over the time interval is $v_{av}=\frac{\Delta x}{\Delta t}$
$=\frac{\left(At_2^2+Bt_2+C\right)-\left(At_1^2+Bt_1+C\right)}{t_2-t_1}=\frac{\left(t_2-t_1\right)\left[A\left(t_2+t_1\right)+B\right]}{t_2-t_1}=A\left(t_2+t_1\right)+B.$

(b) On the graph of the parabola $x=At^2+Bt+C$, the slope of the curve at the midpoint of the interval $[t_1,t_2]$ is the same as the average slope of the curve over the interval.

15. (a) To be continuous at $x=\pi$ requires that $\lim_{x\to\pi^-}\sin x=\lim_{x\to\pi^+}(mx+b) \Rightarrow 0=m\pi+b \Rightarrow m=-\frac{b}{\pi}$;

(b) If $y'=\begin{cases}\cos x, & x<\pi\\ m, & x\ge\pi\end{cases}$ is differentiable at $x=\pi$, then $\lim_{x\to\pi^-}\cos x=m \Rightarrow m=-1$ and $b=\pi$.

16. $f(x)$ is continuous at 0 because $\lim_{x\to 0}\frac{1-\cos x}{x}=0=f(0)$. $f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{\frac{1-\cos x}{x}-0}{x}$
$=\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)\left(\frac{1+\cos x}{1+\cos x}\right)=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\left(\frac{1}{1+\cos x}\right)=\frac{1}{2}$. Therefore $f'(0)$ exists with value $\frac{1}{2}$.

17. (a) For all a, b and for all $x\ne 2$, f is differentiable at x. Next, f differentiable at $x=2 \Rightarrow$ f continuous at $x=2$ $\Rightarrow \lim_{x\to 2^-}f(x)=f(2) \Rightarrow 2a=4a-2b+3 \Rightarrow 2a-2b+3=0$. Also, f differentiable at $x\ne 2$
$\Rightarrow f'(x)=\begin{cases}a, & x<2\\ 2ax-b, & x>2\end{cases}$. In order that $f'(2)$ exist we must have $a=2a(2)-b \Rightarrow a=4a-b \Rightarrow 3a=b$.
Then $2a-2b+3=0$ and $3a=b \Rightarrow a=\frac{3}{4}$ and $b=\frac{9}{4}$.

(b) For $x<2$, the graph of f is a straight line having a slope of $\frac{3}{4}$ and passing through the origin; for $x\ge 2$, the graph of f is a parabola. At $x=2$, the value of the y-coordinate on the parabola is $\frac{3}{2}$ which matches the y-coordinate of the point on the straight line at $x=2$. In addition, the slope of the parabola at the match up point is $\frac{3}{4}$ which is equal to the slope of the straight line. Therefore, since the graph is differentiable at the match up point, the graph is smooth there.

18. (a) For any a, b and for any $x\ne -1$, g is differentiable at x. Next, g differentiable at $x=-1 \Rightarrow$ g continuous at $x=-1 \Rightarrow \lim_{x\to -1^+}g(x)=g(-1) \Rightarrow -a-1+2b=-a+b \Rightarrow b=1$. Also, g differentiable at $x\ne -1$
$\Rightarrow g'(x)=\begin{cases}a, & x<-1\\ 3ax^2+1, & x>-1\end{cases}$. In order that $g'(-1)$ exist we must have $a=3a(-1)^2+1 \Rightarrow a=3a+1$
$\Rightarrow a=-\frac{1}{2}$.

(b) For $x \le -1$, the graph of g is a straight line having a slope of $-\frac{1}{2}$ and a y-intercept of 1. For $x > -1$, the graph of g is a cubic. At $x = -1$, the value of the y-coordinate on the cubic is $\frac{3}{2}$ which matches the y-coordinate of the point on the straight line at $x = -1$. In addition, the slope of the cubic at the match up point is $-\frac{1}{2}$ which is equal to the slope of the straight line. Therefore, since the graph is differentiable at the match up point, the graph is smooth there.

19. f odd $\Rightarrow f(-x) = -f(x) \Rightarrow \frac{d}{dx}(f(-x)) = \frac{d}{dx}(-f(x)) \Rightarrow f'(-x)(-1) = -f'(x) \Rightarrow f'(-x) = f'(x) \Rightarrow f'$ is even.

20. f even $\Rightarrow f(-x) = f(x) \Rightarrow \frac{d}{dx}(f(-x)) = \frac{d}{dx}(f(x)) \Rightarrow f'(-x)(-1) = f'(x) \Rightarrow f'(-x) = -f'(x) \Rightarrow f'$ is odd.

21. Let $h(x) = (fg)(x) = f(x)\,g(x) \Rightarrow h'(x) = \lim\limits_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} = \lim\limits_{x \to x_0} \frac{f(x)\,g(x) - f(x_0)\,g(x_0)}{x - x_0}$
$= \lim\limits_{x \to x_0} \frac{f(x)\,g(x) - f(x)\,g(x_0) + f(x)\,g(x_0) - f(x_0)\,g(x_0)}{x - x_0} = \lim\limits_{x \to x_0} \left[f(x)\left[\frac{g(x) - g(x_0)}{x - x_0}\right]\right] + \lim\limits_{x \to x_0} \left[g(x_0)\left[\frac{f(x) - f(x_0)}{x - x_0}\right]\right]$
$= f(x_0) \lim\limits_{x \to x_0} \left[\frac{g(x) - g(x_0)}{x - x_0}\right] + g(x_0)\,f'(x_0) = 0 \cdot \lim\limits_{x \to x_0} \left[\frac{g(x) - g(x_0)}{x - x_0}\right] + g(x_0)\,f'(x_0) = g(x_0)\,f'(x_0)$, if g is continuous at x_0. Therefore $(fg)(x)$ is differentiable at x_0 if $f(x_0) = 0$, and $(fg)'(x_0) = g(x_0)\,f'(x_0)$.

22. From Exercise 21 we have that fg is differentiable at 0 if f is differentiable at 0, $f(0) = 0$ and g is continuous at 0.
(a) If $f(x) = \sin x$ and $g(x) = |x|$, then $|x| \sin x$ is differentiable because $f'(0) = \cos(0) = 1$, $f(0) = \sin(0) = 0$ and $g(x) = |x|$ is continuous at $x = 0$.
(b) If $f(x) = \sin x$ and $g(x) = x^{2/3}$, then $x^{2/3} \sin x$ is differentiable because $f'(0) = \cos(0) = 1$, $f(0) = \sin(0) = 0$ and $g(x) = x^{2/3}$ is continuous at $x = 0$.
(c) If $f(x) = 1 - \cos x$ and $g(x) = \sqrt[3]{x}$, then $\sqrt[3]{x}\,(1 - \cos x)$ is differentiable because $f'(0) = \sin(0) = 0$, $f(0) = 1 - \cos(0) = 0$ and $g(x) = x^{1/3}$ is continuous at $x = 0$.
(d) If $f(x) = x$ and $g(x) = x \sin\left(\frac{1}{x}\right)$, then $x^2 \sin\left(\frac{1}{x}\right)$ is differentiable because $f'(0) = 1$, $f(0) = 0$ and $\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right) = \lim\limits_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim\limits_{t \to \infty} \frac{\sin t}{t} = 0$ (so g is continuous at $x = 0$).

23. If $f(x) = x$ and $g(x) = x \sin\left(\frac{1}{x}\right)$, then $x^2 \sin\left(\frac{1}{x}\right)$ is differentiable at $x = 0$ because $f'(0) = 1$, $f(0) = 0$ and $\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right) = \lim\limits_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim\limits_{t \to \infty} \frac{\sin t}{t} = 0$ (so g is continuous at $x = 0$). In fact, from Exercise 21, $h'(0) = g(0)\,f'(0) = 0$. However, for $x \ne 0$, $h'(x) = \left[x^2 \cos\left(\frac{1}{x}\right)\right]\left(-\frac{1}{x^2}\right) + 2x \sin\left(\frac{1}{x}\right)$. But $\lim\limits_{x \to 0} h'(x) = \lim\limits_{x \to 0} \left[-\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right)\right]$ does not exist because $\cos\left(\frac{1}{x}\right)$ has no limit as $x \to 0$. Therefore, the derivative is not continuous at $x = 0$ because it has no limit there.

24. From the given conditions we have $f(x + h) = f(x)\,f(h)$, $f(h) - 1 = hg(h)$ and $\lim\limits_{h \to 0} g(h) = 1$. Therefore,
$f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \frac{f(x)\,f(h) - f(x)}{h} = \lim\limits_{h \to 0} f(x)\left[\frac{f(h) - 1}{h}\right] = f(x)\left[\lim\limits_{h \to 0} g(h)\right] = f(x) \cdot 1 = f(x)$
$\Rightarrow f'(x) = f(x)$ and $f'(x)$exists at every value of x.

25. Step 1: The formula holds for $n = 2$ (a single product) since $y = u_1 u_2 \Rightarrow \frac{dy}{dx} = \frac{du_1}{dx} u_2 + u_1 \frac{du_2}{dx}$.
Step 2: Assume the formula holds for $n = k$:
$$y = u_1 u_2 \cdots u_k \Rightarrow \frac{dy}{dx} = \frac{du_1}{dx} u_2 u_3 \cdots u_k + u_1 \frac{du_2}{dx} u_3 \cdots u_k + \ldots + u_1 u_2 \cdots u_{k-1} \frac{du_k}{dx}.$$
If $y = u_1 u_2 \cdots u_k u_{k+1} = (u_1 u_2 \cdots u_k)\,u_{k+1}$, then $\frac{dy}{dx} = \frac{d(u_1 u_2 \cdots u_k)}{dx} u_{k+1} + u_1 u_2 \cdots u_k \frac{du_{k+1}}{dx}$
$= \left(\frac{du_1}{dx} u_2 u_3 \cdots u_k + u_1 \frac{du_2}{dx} u_3 \cdots u_k + \cdots + u_1 u_2 \cdots u_{k-1} \frac{du_k}{dx}\right) u_{k+1} + u_1 u_2 \cdots u_k \frac{du_{k+1}}{dx}$
$= \frac{du_1}{dx} u_2 u_3 \cdots u_{k+1} + u_1 \frac{du_2}{dx} u_3 \cdots u_{k+1} + \cdots + u_1 u_2 \cdots u_{k-1} \frac{du_k}{dx} u_{k+1} + u_1 u_2 \cdots u_k \frac{du_{k+1}}{dx}$.
Thus the original formula holds for $n = (k+1)$ whenever it holds for $n = k$.

26. Recall $\binom{m}{k} = \frac{m!}{k!\,(m-k)!}$. Then $\binom{m}{1} = \frac{m!}{1!\,(m-1)!} = m$ and $\binom{m}{k} + \binom{m}{k+1} = \frac{m!}{k!\,(m-k)!} + \frac{m!}{(k+1)!\,(m-k-1)!}$ $= \frac{m!\,(k+1) + m!\,(m-k)}{(k+1)!\,(m-k)!} = \frac{m!\,(m+1)}{(k+1)!\,(m-k)!} = \frac{(m+1)!}{(k+1)!\,((m+1)-(k+1))!} = \binom{m+1}{k+1}$. Now, we prove Leibniz's rule by mathematical induction.

Step 1: If $n = 1$, then $\frac{d(uv)}{dx} = u\,\frac{dv}{dx} + v\,\frac{du}{dx}$. Assume that the statement is true for $n = k$, that is:

$$\frac{d^k(uv)}{dx^k} = \frac{d^k u}{dx^k}\,v + k\,\frac{d^{k-1}u}{dx^{k-1}}\,\frac{dv}{dx} + \binom{k}{2}\frac{d^{k-2}u}{dx^{k-2}}\,\frac{d^2v}{dx^2} + \ldots + \binom{k}{k-1}\frac{du}{dv}\,\frac{d^{k-1}v}{dx^{k-1}} + u\,\frac{d^k v}{dx^k}.$$

Step 2: If $n = k+1$, then $\frac{d^{k+1}(uv)}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k(uv)}{dx^k}\right) = \left[\frac{d^{k+1}u}{dx^{k+1}}\,v + \frac{d^k u}{dx^k}\,\frac{dv}{dx}\right] + \left[k\,\frac{d^k u}{dx^k}\,\frac{dv}{dx} + k\,\frac{d^{k-1}u}{dx^{k-1}}\,\frac{d^2v}{dx^2}\right]$

$+ \left[\binom{k}{2}\frac{d^{k-1}u}{dx^{k-1}}\,\frac{d^2v}{dx^2} + \binom{k}{2}\frac{d^{k-2}u}{dx^{k-2}}\,\frac{d^3v}{dx^3}\right] + \ldots + \left[\binom{k}{k-1}\frac{d^2u}{dx^2}\,\frac{d^{k-1}v}{dx^{k-1}} + \binom{k}{k-1}\frac{du}{dx}\,\frac{d^k u}{dx^k}\,v\right]$

$+ \left[\frac{du}{dx}\,\frac{d^k v}{dx^k} + u\,\frac{d^{k+1}u}{dx^{k+1}}\right] = \frac{d^{k+1}u}{dx^{k+1}}\,v + (k+1)\,\frac{d^k u}{dx^k}\,\frac{dv}{dx} + \left[\binom{k}{1} + \binom{k}{2}\right]\frac{d^{k-1}u}{dx^{k-1}}\,\frac{d^2v}{dx^2} + \ldots$

$+ \left[\binom{k}{k-1} + \binom{k}{k}\right]\frac{du}{dx}\,\frac{d^k v}{dx^k} + u\,\frac{d^{k+1}v}{dx^{k+1}} = \frac{d^{k+1}u}{dx^{k+1}}\,v + (k+1)\,\frac{d^k u}{dx^k}\,\frac{dv}{dx} + \binom{k+1}{2}\frac{d^{k-1}u}{dx^{k-1}}\,\frac{d^2v}{dx^2} + \ldots$

$+ \binom{k+1}{k}\frac{du}{dx}\,\frac{d^k v}{dx^k} + u\,\frac{d^{k+1}v}{dx^{k+1}}$.

Therefore the formula (c) holds for $n = (k+1)$ whenever it holds for $n = k$.

27. (a) $T^2 = \frac{4\pi^2 L}{g} \Rightarrow L = \frac{T^2 g}{4\pi^2} \Rightarrow L = \frac{(1\text{ sec}^2)(32.2\text{ ft/sec}^2)}{4\pi^2} \Rightarrow L \approx 0.8156\text{ ft}$

(b) $T^2 = \frac{4\pi^2 L}{g} \Rightarrow T = \frac{2\pi}{\sqrt{g}}\sqrt{L}$; $dT = \frac{2\pi}{\sqrt{g}} \cdot \frac{1}{2\sqrt{L}}dL = \frac{\pi}{\sqrt{Lg}}dL$; $dT = \frac{\pi}{\sqrt{(0.8156\text{ ft})(32.2\text{ ft/sec}^2)}}(0.01\text{ ft}) \approx 0.00613$ sec.

(c) Since there are 86,400 sec in a day, we have $(0.00613\text{ sec})(86{,}400\text{ sec/day}) \approx 529.6$ sec/day, or 8.83 min/day; the clock will lose about 8.83 min/day.

28. $v = s^3 \Rightarrow \frac{dv}{dt} = 3s^2\frac{ds}{dt} = -k(6s^2) \Rightarrow \frac{ds}{dt} = -2k$. If $s_0 =$ the initial length of the cube's side, then $s_1 = s_0 - 2k$ $\Rightarrow 2k = s_0 - s_1$. Let $t =$ the time it will take the ice cube to melt. Now, $t = \frac{s_0}{2k} = \frac{s_0}{s_0 - s_1} = \frac{(v_0)^{1/3}}{(v_0)^{1/3} - \left(\frac{3}{4}v_0\right)^{1/3}}$ $= \frac{1}{1 - \left(\frac{3}{4}\right)^{1/3}} \approx 11$ hr.