

## ***Solution***

## **Section 2.1 – Functions and Graphs**

### ***Exercise***

$$f(x) = \begin{cases} 2+x & \text{if } x < -4 \\ -x & \text{if } -4 \leq x \leq 2 \\ 3x & \text{if } x > 2 \end{cases}$$

**Find:**  $f(-5)$ ,  $f(-1)$ ,  $f(0)$ , and  $f(3)$

### **Solution**

a)  $f(-5) = 2 - 5 = -3$

b)  $f(-1) = -(-1) = 1$

c)  $f(0) = -0 = 0$

d)  $f(3) = 3(3) = 9$

### ***Exercise***

$$f(x) = \begin{cases} -2x & \text{if } x < -3 \\ 3x-1 & \text{if } -3 \leq x \leq 2 \\ -4x & \text{if } x > 2 \end{cases}$$

**Find:**  $f(-5)$ ,  $f(-1)$ ,  $f(0)$ , and  $f(3)$

### **Solution**

a)  $f(-5) = -2(-5) = 10$

b)  $f(-1) = 3(-1) - 1 = -4$

c)  $f(0) = 3(0) - 1 = -1$

d)  $f(3) = -4(3) = -12$

### ***Exercise***

$$f(x) = \begin{cases} x^3 + 3 & \text{if } -2 \leq x \leq 0 \\ x + 3 & \text{if } 0 < x < 1 \\ 4 + x - x^2 & \text{if } 1 \leq x \leq 3 \end{cases}$$

**Find:**  $f(-5)$ ,  $f(-1)$ ,  $f(0)$ , and  $f(3)$

### **Solution**

a)  $f(-5) = \text{doesn't exist}$

b)  $f(-1) = (-1)^3 + 3$   
 $\quad \quad \quad = 2$

c)  $f(0) = (0)^3 + 3$

$$\underline{= 3}$$

$$d) \quad f(3) = 4 + (3) - (3)^2 \\ \underline{= -2}$$

### Exercise

$$h(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases} \quad \text{Find: } h(5), h(0), \text{ and } h(3)$$

### Solution

$$a) \quad h(5) = \frac{5^2 - 9}{5 - 3} \\ \underline{= 8}$$

$$b) \quad h(0) = \frac{0^2 - 9}{0 - 3} \\ \underline{= 3}$$

$$c) \quad \underline{h(3) = 6}$$

### Exercise

$$f(x) = \begin{cases} 3x + 5 & \text{if } x < 0 \\ 4x + 7 & \text{if } x \geq 0 \end{cases} \quad \text{Find}$$

$$a) \quad f(0) \quad b) \quad f(-2) \quad c) \quad f(1) \quad d) \quad f(3) + f(-3) \quad e) \quad \text{Graph } f(x)$$

### Solution

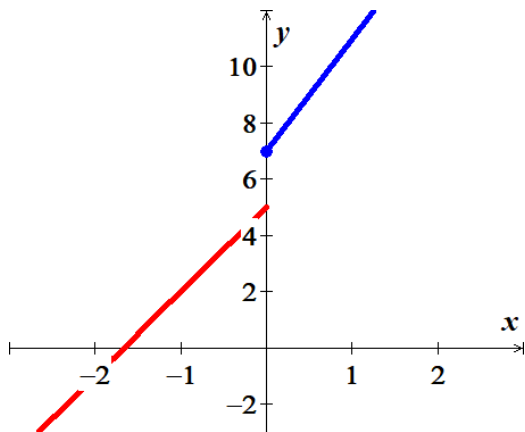
$$a) \quad f(0) = 4(0) + 7 \\ \underline{= 7}$$

$$b) \quad f(-2) = 3(-2) + 5 \\ \underline{= -1}$$

$$c) \quad f(1) = 4(1) + 7 \\ \underline{= 11}$$

$$d) \quad f(3) + f(-3) = 4(3) + 7 + 3(-3) + 5 \\ = 12 + 7 - 9 + 5 \\ \underline{= 15}$$

$$e)$$



### Exercise

$$f(x) = \begin{cases} 6x - 1 & \text{if } x < 0 \\ 7x + 3 & \text{if } x \geq 0 \end{cases} \quad \text{Find}$$

- a)  $f(0)$       b)  $f(-1)$       c)  $f(4)$       d)  $f(2) + f(-2)$       e) Graph  $f(x)$

### Solution

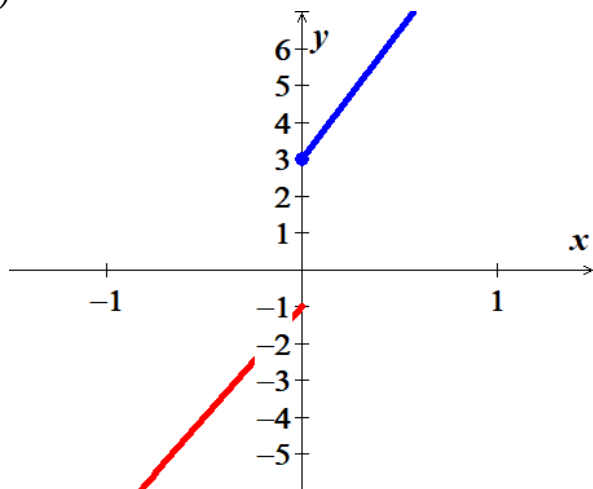
$$\begin{aligned} \text{a) } f(0) &= 7(0) + 3 \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{b) } f(-2) &= 6(-1) - 1 \\ &= -7 \end{aligned}$$

$$\begin{aligned} \text{c) } f(4) &= 7(4) + 3 \\ &= 31 \end{aligned}$$

$$\begin{aligned} \text{d) } f(2) + f(-2) &= 7(2) + 3 + 6(-2) - 1 \\ &= 14 + 3 - 12 - 1 \\ &= 4 \end{aligned}$$

e)



### Exercise

$$f(x) = \begin{cases} 2x+1 & \text{if } x \leq 1 \\ 3x-2 & \text{if } x > 1 \end{cases} \quad \text{Find}$$

- a)  $f(0)$       b)  $f(2)$       c)  $f(-2)$       d)  $f(1) + f(-1)$       e) Graph  $f(x)$

### Solution

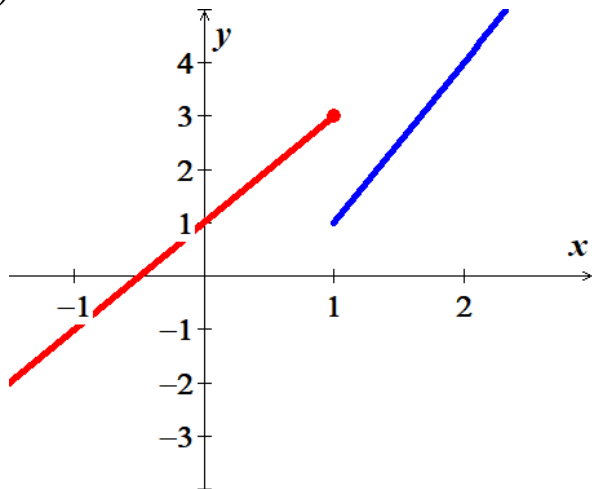
$$\begin{aligned} \text{a) } f(0) &= 2(0) + 1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{b) } f(2) &= 3(2) - 2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{c) } f(-2) &= 2(-2) + 1 \\ &= -3 \end{aligned}$$

$$\begin{aligned} \text{d) } f(1) + f(-1) &= (2(1) + 1) + (2(-1) + 1) \\ &= 2 + 1 - 2 + 1 \\ &= 2 \end{aligned}$$

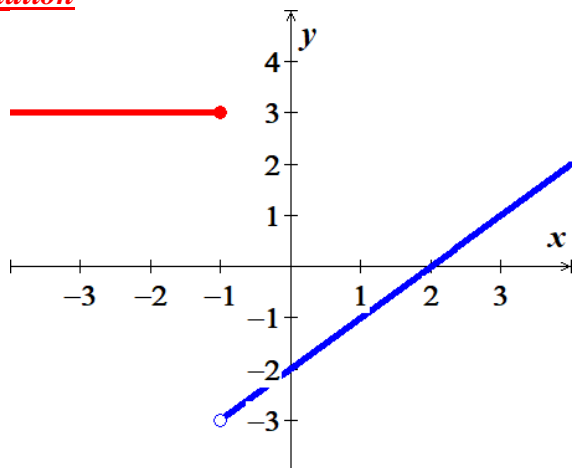
e)



### Exercise

Graph the piecewise function defined by  $f(x) = \begin{cases} 3 & \text{if } x \leq -1 \\ x-2 & \text{if } x > -1 \end{cases}$

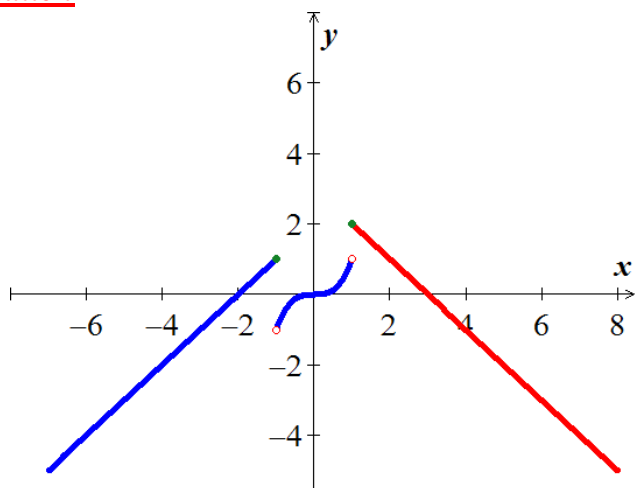
### Solution



### Exercise

Sketch the graph  $f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ x^3 & \text{if } -1 < x < 1 \\ -x+3 & \text{if } x \geq 1 \end{cases}$

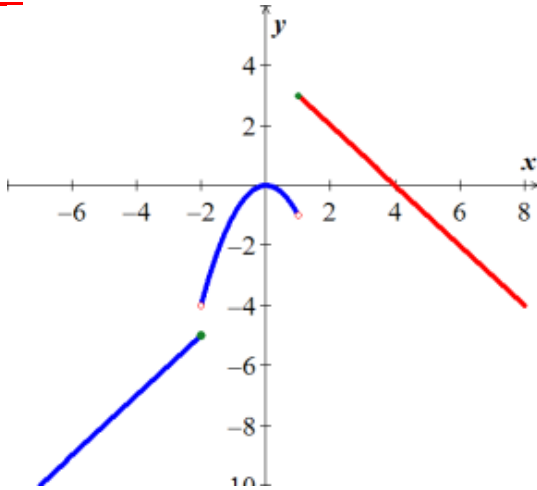
### Solution



### Exercise

Sketch the graph  $f(x) = \begin{cases} x-3 & \text{if } x \leq -2 \\ -x^2 & \text{if } -2 < x < 1 \\ -x+4 & \text{if } x \geq 1 \end{cases}$

### Solution



### Exercise

Determine any **relative maximum** or **minimum** of the function, determine the intervals on which the function **increasing** or **decreasing**, and then find the **domain** and the **range**.

$$f(x) = x^2 - 2x + 3$$

### Solution

**Relative Maximum:** None

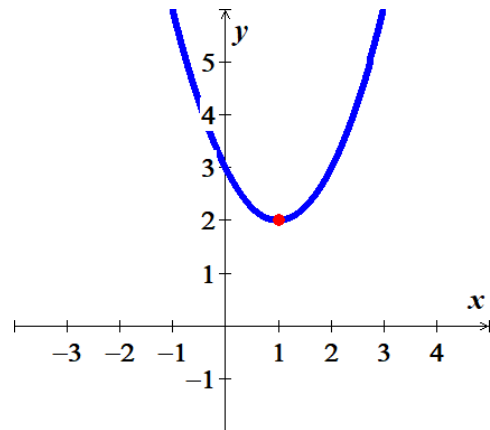
**Minimum Point:** (1, 2)

**Increasing:** (1, ∞)

**Decreasing:** (-∞, 1)

**Domain:** ℝ

**Range:** [2, ∞)



### Exercise

Determine any **relative maximum** or **minimum** of the function, determine the intervals on which the function **increasing** or **decreasing**, and then find the **domain** and the **range**.

$$f(x) = -x^2 - 2x + 3$$

### Solution

**Maximum Point:**  $(-1, 4)$

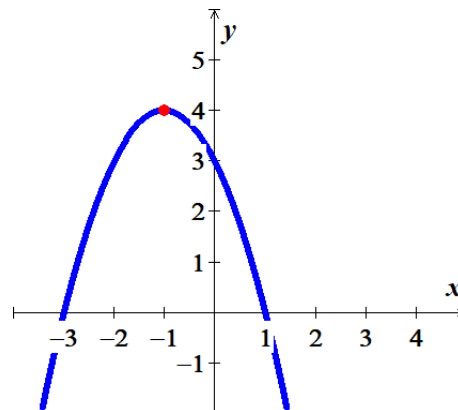
**Relative Minimum:** *None*

**Increasing:**  $(-\infty, -1)$

**Decreasing:**  $(-1, \infty)$

**Domain:**  $\mathbb{R}$

**Range:**  $(-\infty, 4]$



### Exercise

Determine any **relative maximum** or **minimum** of the function, determine the intervals on which the function **increasing** or **decreasing**, and then find the **domain** and the **range**.

$$f(x) = -x^3 + 3x^2$$

### Solution

**Relative Maximum:**  $(2, 4)$

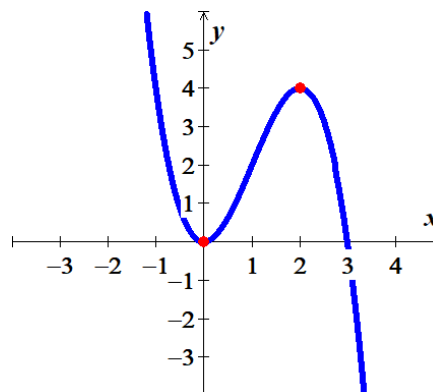
**Relative Minimum:**  $(0, 0)$

**Increasing:**  $(0, 2)$

**Decreasing:**  $(-\infty, 0) \cup (2, \infty)$

**Domain:**  $\mathbb{R}$

**Range:**  $\mathbb{R}$



### Exercise

Determine any **relative maximum** or **minimum** of the function, determine the intervals on which the function **increasing** or **decreasing**, and then find the **domain** and the **range**.

$$f(x) = x^3 - 3x^2$$

#### Solution

**Relative Maximum:**  $(0, 0)$

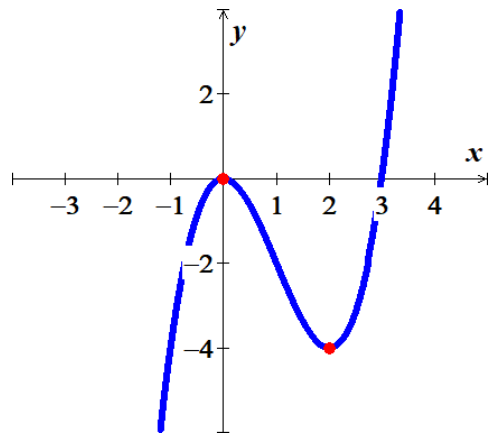
**Relative Minimum:**  $(2, -4)$

**Increasing:**  $(-\infty, 0) \cup (2, \infty)$

**Decreasing:**  $(0, 2)$

**Domain:**  $\mathbb{R}$

**Range:**  $\mathbb{R}$



### Exercise

Determine any **relative maximum** or **minimum** of the function, determine the intervals on which the function **increasing** or **decreasing**, and then find the **domain** and the **range**.

$$f(x) = \frac{1}{4}x^4 - 2x^2$$

#### Solution

**Relative Maximum:**  $(0, 0)$

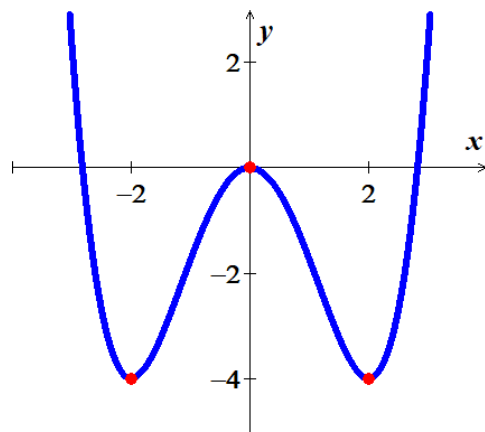
**Minimum Points:**  $(-2, -4)$  &  $(2, -4)$

**Increasing:**  $(-2, 0) \cup (2, \infty)$

**Decreasing:**  $(-\infty, -2) \cup (0, 2)$

**Domain:**  $\mathbb{R}$

**Range:**  $[-4, \infty)$





## Exercise

Determine any **relative maximum** or **minimum** of the function, determine the intervals on which the function **increasing** or **decreasing**, and then find the **domain** and the **range**.

$$f(x) = \frac{4}{81}x^4 - \frac{8}{9}x^2 + 4$$

### Solution

**Relative Maximum:** (0, 4)

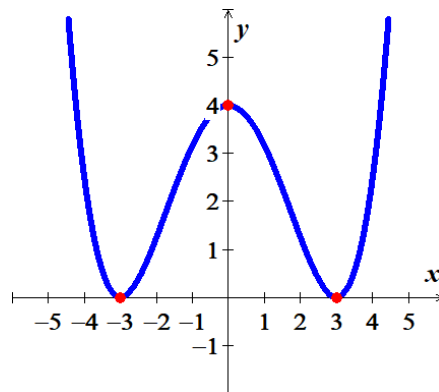
**Minimum Points:** (-3, 0) & (3, 0)

**Increasing:** (-3, 0)  $\cup$  (3,  $\infty$ )

**Decreasing:** ( $-\infty$ , -3)  $\cup$  (0, 3)

**Domain:**  $\mathbb{R}$

**Range:**  $[0, \infty)$



## Exercise

The elevation  $H$ , in *meters*, above sea level at which the boiling point of water is in  $t$  degrees Celsius is given by the function

$$H(t) = 1000(100 - t) + 580(100 - t)^2$$

At what elevation is the boiling point  $99.5^\circ$ .

### Solution

$$\begin{aligned} H(99.5) &= 1000(100 - 99.5) + 580(100 - 99.5)^2 \\ &= \underline{645 \text{ m}} \end{aligned}$$

## Exercise

A hot-air balloon rises straight up from the ground at a rate of  $120 \text{ ft./min}$ . The balloon is tracked from a rangefinder on the ground at point P, which is  $400 \text{ ft.}$  from the release point Q of the balloon. Let  $d$  = the distance from the balloon to the rangefinder and  $t$  – the time, in minutes, since the balloon was released. Express  $d$  as a function of  $t$ .

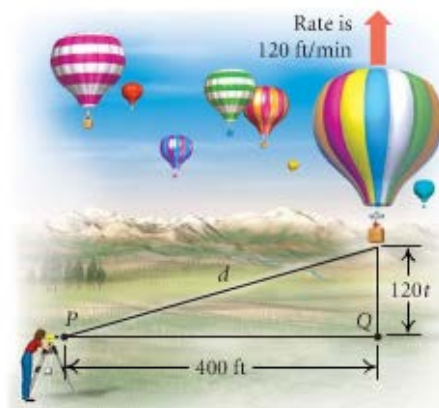
### Solution

$$d^2 = (120t)^2 + 400^2$$

$$d = \sqrt{14400t^2 + 160000}$$

$$d = \sqrt{1600(9t^2 + 100)}$$

$$d(t) = \underline{40\sqrt{9t^2 + 100}}$$



### Exercise

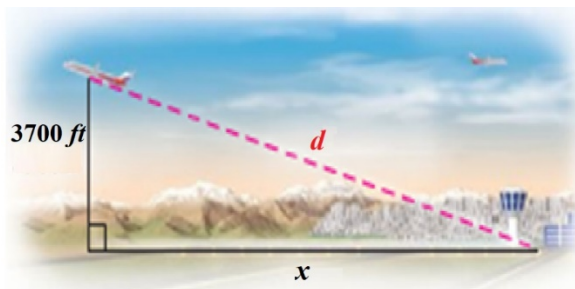
An airplane is flying at an altitude of 3700 *feet*. The slanted distance directly to the airport is  $d$  *feet*. Express the horizontal distance  $x$  as a function of  $d$ .

#### Solution

$$d^2 = (3,700)^2 + x^2$$

$$h^2 = d^2 - (3700)^2$$

$$h(t) = \sqrt{d^2 - (3,700)^2}$$



### Exercise

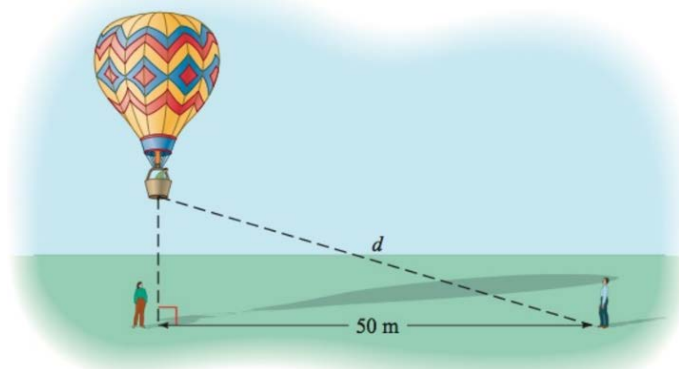
For the first minute of flight, a hot air balloon rises vertically at a rate of 3 *m/sec*. If  $t$  is the time in *seconds* that the balloon has been airborne, write the distance  $d$  between the balloon and a point on the ground 50 *meters* from the point to lift off as a function of  $t$ .

#### Solution

$$h = 3t \quad v = \frac{h}{t}$$

$$d^2 = h^2 + 50^2$$

$$d(t) = \sqrt{9t^2 + 2,500}$$



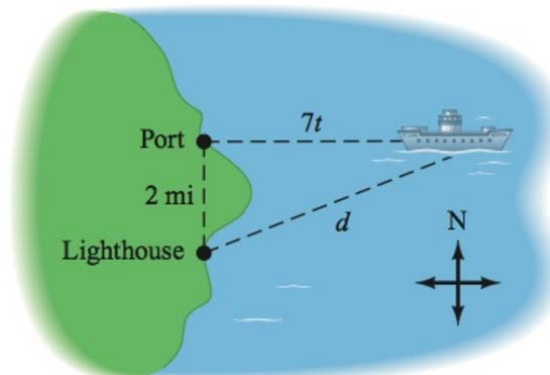
### Exercise

A light house is 2 *miles* south of a port. A ship leaves port and sails east at a rate of 7 *miles per hour*. Express the distance  $d$  between the ship and the lighthouse as a function of time, given that the ship has been sailing for  $t$  hours.

#### Solution

$$d^2 = 4^2 + (7t)^2$$

$$d(t) = \sqrt{16 + 49t^2}$$



### Exercise

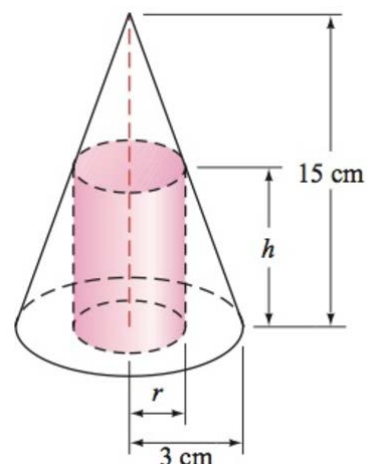
A cone has an altitude of 15 cm and a radius of 3 cm. A right circular cylinder of radius  $r$  and height  $h$  is inscribed in the cone. Use similar triangles to write  $h$  as a function of  $r$ .

### Solution

$$\frac{15-h}{15} = \frac{r}{3}$$

$$15 - h = 5r$$

$$\underline{h(r) = 15 - 5r}$$



### Exercise

Water is flowing into a conical drinking cup with an altitude of 4 inches and a radius of 2 inches.

- Write the radius  $r$  of the surface of the water as a function of its depth  $h$ .
- Write the volume  $V$  of the water as a function of its depth  $h$ .

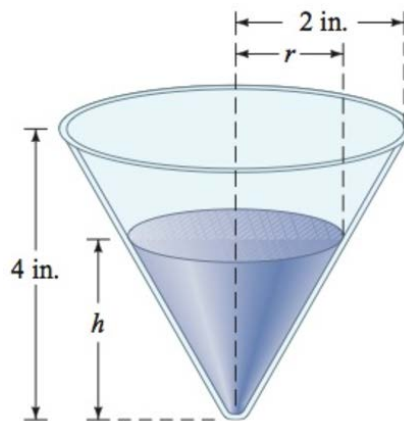
### Solution

$$a) \quad \frac{h}{4} = \frac{r}{2}$$

$$\underline{r(h) = \frac{1}{2}h}$$

$$b) \quad \text{Area} = \pi r^2$$

$$\begin{aligned} V &= \frac{1}{3} \pi r^2 h \\ &= \frac{1}{3} \pi \left( \frac{h^2}{4} \right) h \\ &= \frac{1}{12} \pi h^3 \end{aligned}$$



### Exercise

A water tank has the shape of a right circular cone with height 16 feet and radius 8 feet. Water is running into the tank so that the radius  $r$  (in feet) of the surface of the water is given by  $r = 1.5t$ , where  $t$  is the time (in minutes) that the water has been running.

- The area  $A$  of the surface of the water is  $A = \pi r^2$ . Find  $A(t)$  and use it to determine the area of the surface of the water when  $t = 2$  minutes.
- The volume  $V$  of the water is given by  $V = \frac{1}{3} \pi r^2 h$ . Find  $V(t)$  and use it to determine the volume of the water when  $t = 3$  minutes.

### Solution

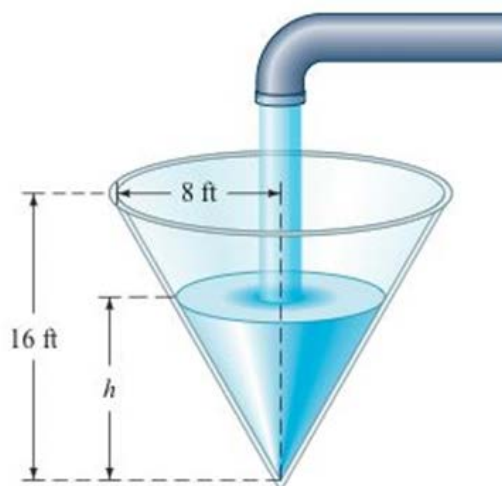
c)  $Area = \pi r^2$

$$A(t) = \pi \left( \frac{3}{2}t \right)^2$$
$$= \frac{9\pi}{4}t^2$$

d)  $\frac{h}{16} = \frac{r}{8}$

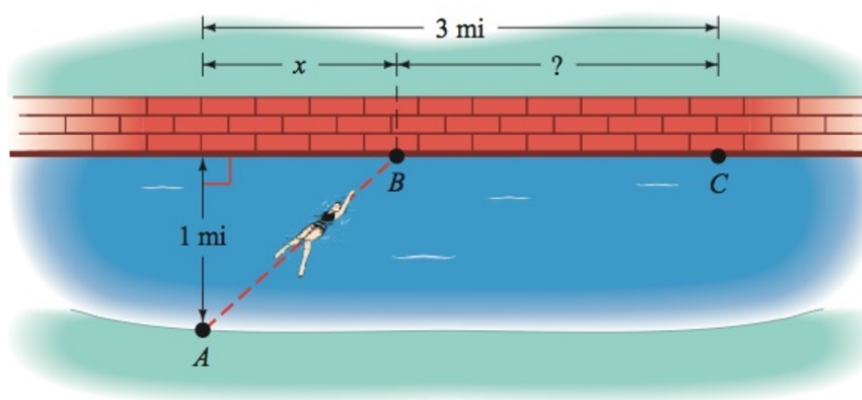
$$h = 2r$$

$$V(t) = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi r^2 (2r)$$
$$= \frac{2}{3}\pi r^3$$
$$= \frac{2}{3}\pi \left( \frac{3}{2}t \right)^3$$
$$= \frac{9}{4}\pi t^3$$



### **Exercise**

An athlete swims from point **A** to point **B** at a rate of 2 miles per hour and runs from point **B** to point **C** at a rate of 8 miles per hour. Use the dimensions in the figure to write the time  $t$  required to reach point **C** as a function of  $x$ .



### Solution

$$\text{Swimming distance} = \sqrt{x^2 + 1}$$

$$t_{\text{swim}} = \frac{\sqrt{x^2 + 1}}{2} \quad t = \frac{d}{v}$$

$$\text{Running distance} = 3 - x$$

$$t_{run} = \frac{3-x}{8} \qquad t = \frac{d}{v}$$

$$t_{total} = \frac{\sqrt{x^2+1}}{2} + \frac{3-x}{8}$$

### Exercise

A device used in golf to estimate the distance  $d$ , in yards, to a hole measures the size  $s$ , in inches, that the 7-foot pin appears to be in a viewfinder. Express the distance  $d$  as a function of  $s$ .

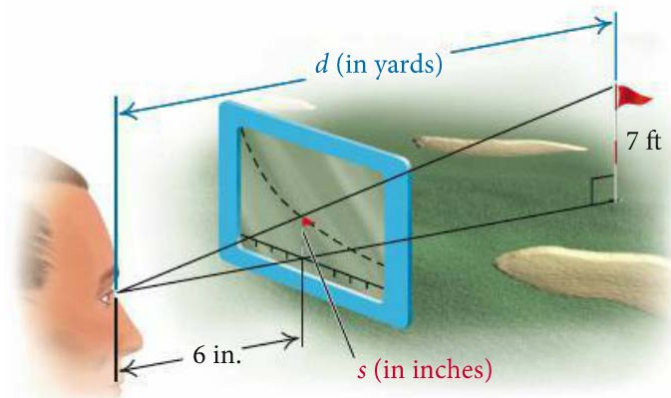
### Solution

$$\frac{d}{6} = \frac{7 \text{ ft}}{s \text{ in}}$$

$$d = \frac{7 \text{ ft}}{s \text{ in}} 6 \text{ in}$$

$$d = \frac{42 \text{ ft}}{s} \frac{1 \text{ yd}}{3 \text{ ft}}$$

$$d(s) = \frac{14}{s}$$



### Exercise

A rhombus is inscribed in a rectangle that is  $w$  meters wide with a perimeter of 40 m. Each vertex of the rhombus is a midpoint of a side of the rectangle. Express the area of the rhombus as a function of the rectangle's width.

### Solution

The area of the rhombus =  $\frac{1}{2}$  area of the rectangle, since each vertex of the rhombus is a midpoint of a side of the rectangle.

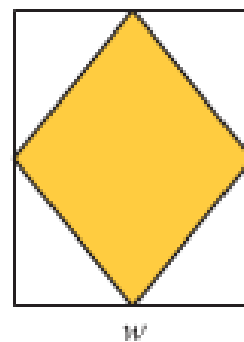
$$\text{Perimeter: } 2l + 2w = 40 \qquad \text{Divide both sides by 2}$$

$$l + w = 20$$

$$l = 20 - w$$

$$\text{Area of the rectangle} = lw = (20 - w)w$$

$$\begin{aligned} \text{Area of the rhombus} &= \frac{1}{2}(20w - w^2) \\ &= -\frac{1}{2}w^2 + 10w \end{aligned}$$



### Exercise

The surface area  $S$  of a right circular cylinder is given by the formula  $S = 2\pi rh + 2\pi r^2$ . If the height is twice the radius, find each of the following.

- a) A function  $S(r)$  for the surface area as a function of  $r$ .
- b) A function  $S(h)$  for the surface area as a function of  $h$ .

### Solution

**Given:**  $h = 2r$

a)  $S = 2\pi rh + 2\pi r^2$

$$\begin{aligned} S(r) &= 2\pi r(2r) + 2\pi r^2 \\ &= 4\pi r^2 + 2\pi r^2 \\ &= 6\pi r^2 \end{aligned}$$

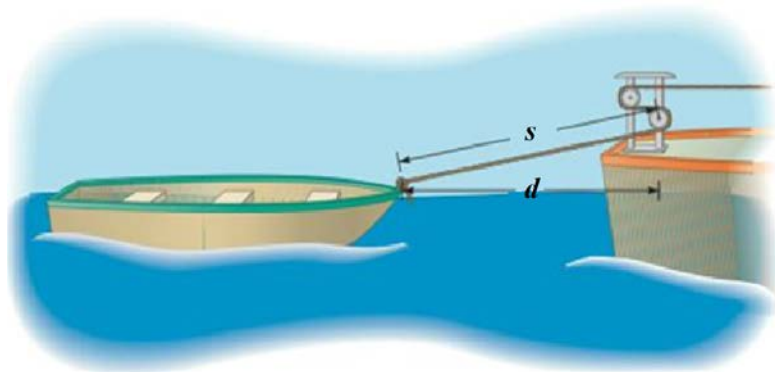
b)  $r = \frac{1}{2}h$

$$\begin{aligned} S(h) &= 2\pi\left(\frac{1}{2}h\right)h + 2\pi\left(\frac{1}{2}h\right)^2 \\ &= \pi h^2 + \frac{1}{2}\pi h^2 \\ &= \frac{3}{2}\pi h^2 \end{aligned}$$



### Exercise

A boat is towed by a rope that runs through a pulley that is 4 feet above the point where the rope is tied to the boat. The length (in feet) of the rope from the boat to the pulley is given by  $s = 48 - t$ , where  $t$  is the time in seconds that the boat has been in tow. The horizontal distance from the pulley to the boat is  $d$ .



- a) Find  $d(t)$
- b) Evaluate  $s(35)$  and  $d(35)$

### Solution

$$\begin{aligned}
 a) \quad s^2 &= d^2 + 4^2 \\
 d^2 &= (48 - t)^2 - 16 \\
 d(t) &= \sqrt{2,304 - 96t + t^2 - 16} \\
 &= \sqrt{t^2 - 96t + 2,288}
 \end{aligned}$$

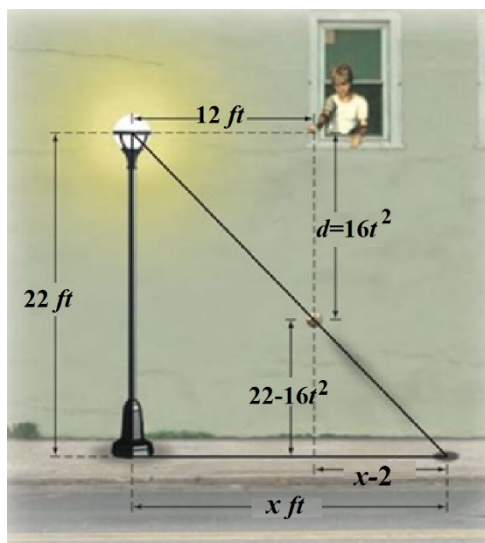
$$\begin{aligned}
 b) \quad s(35) &= 48 - 35 \\
 &= 13 \text{ feet} \\
 d(35) &= \sqrt{(48 - 35)^2 - 16} \\
 &= \sqrt{13^2 - 16} \\
 &= \sqrt{153} \text{ feet}
 \end{aligned}$$

### Exercise

The light from a lamppost casts a shadow from a ball that was dropped from a height of 22 feet above the ground. The distance  $d$ , in feet, the ball has dropped  $t$  seconds after it is released is given by  $d(t) = 16t^2$ . Find the distance  $x$ , in feet, of the shadow from the base of the lamppost as a function of time  $t$ .

### Solution

$$\begin{aligned}
 \frac{22 - 16t^2}{22} &= \frac{x - 12}{x} \\
 (22 - 16t^2)x &= 22(x - 12) \\
 (22 - 16t^2)x &= 22x - 264 \\
 (22 - 16t^2 - 22)x &= -264 \\
 -16t^2x &= -264 \\
 x(t) &= \frac{33}{2t^2}
 \end{aligned}$$



### Exercise

A right circular cylinder of height  $h$  and a radius  $r$  is inscribed in a right circular cone with a height of 10 feet and a base with radius 6 feet.

- Express the height  $h$  of the cylinder as a function of  $r$ .
- Express the volume  $V$  of the cylinder as a function of  $r$ .
- Express the volume  $V$  of the cylinder as a function of  $h$ .

### Solution

$$a) \quad \frac{h}{10} = \frac{6-r}{6}$$

$$\boxed{h(r) = \frac{5}{3}(6-r)}$$

$$b) \quad V = \pi r^2 h$$

$$V(r) = \frac{5}{3} \pi r^2 (6-r)$$

$$\boxed{= \frac{5}{3} \pi (6r^2 - r^3)}$$

$$c) \quad \frac{3}{5}h = 6-r$$

$$r = 6 - \frac{3}{5}h$$

$$V = \pi r^2 h$$

$$V(h) = \pi \left( \frac{30-3h}{5} \right)^2 h$$

$$\boxed{= \frac{1}{25} \pi h (30-3h)^2}$$

