Solution *Section* 2.2 – Differentiation Rules

Exercise

Find the derivative of $y = \dfrac{1}{x^3}$

Solution

$$y = x^{-3}$$

$$y' = -3x^{-3-1}$$

$$\underline{= -3x^{-4}|} \qquad \text{or} \quad -\frac{3}{x^4}$$

Exercise

Find the derivative of $D_x\left(x^{4/3}\right)$

Solution

$$D_x\left(x^{4/3}\right) = \frac{4}{3}x^{1/3}$$

Exercise

Find the derivative of $y = \sqrt{z}$

Solution

$$\frac{dy}{dz} = \frac{d}{dz}\left[z^{1/2}\right]$$

$$= \frac{1}{2}z^{1/2-1}$$

$$= \frac{1}{2}z^{-1/2} \qquad \frac{1}{2z^{1/2}} \qquad \frac{1}{2\sqrt{z}}$$

Exercise

Find the derivative of $D_t(-8t)$

Solution

$$D_t(-8t) \ \underline{= -8|}$$

Exercise

Find the derivative of $y = \frac{9}{4x^2}$

Solution

$$y = \frac{9}{4}x^{-2}$$

$$y' = \frac{9}{4}(-2)x^{-3}$$

$$\boxed{= -\frac{9}{2x^3}}$$

Exercise

Find the derivative of $y = 6x^3 + 15x^2$

Solution

$$\boxed{y' = 18x^2 + 30x}$$

Exercise

Find the first derivative of $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$

Solution

$$y' = 3(4)x^3 - 6(3)x^2 + \frac{2}{8}x + 0$$

$$\boxed{= 12x^3 - 18x^2 + \frac{1}{4}x}$$

Exercise

Find the derivative of $p(t) = 12t^4 - 6\sqrt{t} + \frac{5}{t}$

Solution

$$p(t) = 12t^4 - 6t^{1/2} + 5t^{-1}$$

$$p' = 48t^3 - 3t^{-1/2} - 5t^{-2}$$

$$\boxed{= 48t^3 - \frac{3}{t^{1/2}} - \frac{5}{t^2}}$$

Exercise

Find the derivative of $f(x)=\dfrac{x^3+3\sqrt{x}}{x}$

Solution

$$f(x)=\frac{x^3}{x}+3\frac{x^{1/2}}{x}=x^2+3x^{-1/2}$$

$$f'(x)=2x-\tfrac{3}{2}x^{-3/2}$$

$$=2x-\frac{3}{2x^{3/2}}$$

$$\boxed{=2x-\frac{3}{2\sqrt{x^3}}}$$

Exercise

Find the derivative of $y=\dfrac{x^3-4x}{\sqrt{x}}$

Solution

$$y=\frac{x^3}{x^{1/2}}-4\frac{x}{x^{1/2}}=x^{5/2}-4x^{1/2}$$

$$y'=\tfrac{5}{2}x^{3/2}-4\tfrac{1}{2}x^{-1/2}$$

$$\boxed{=\tfrac{5}{2}x\sqrt{x}-2\frac{2}{\sqrt{x}}}$$

Exercise

Find the derivative of $f(x)=\left(4x^2-3x\right)^2$

Solution

$$f(x)=\left(4x^2-3x\right)^2$$

$$(a+b)^2=a^2+2ab+b^2$$

$$=16x^4-24x^3+9x^2$$

$$\boxed{f'(x)=64x^3-72x^2+18x}$$

Exercise

Find the derivative of $y=3x\left(2x^2+5x\right)$

Solution

$$y=6x^3+15x^2 \quad \Rightarrow \boxed{y'=18x^2+30x}$$

Exercise

Find the derivative of $y = 3\left(2x^2 + 5x\right)$

Solution

$$y = 6x^2 + 15x$$

$$\boxed{y' = 12x + 15}$$

Exercise

Find the derivative of $y = \dfrac{x^2 + 4x}{5}$

Solution

$$\boxed{y' = \tfrac{1}{5}(2x + 4)}$$

Exercise

Find the derivative of $y = \dfrac{3x^4}{5}$

Solution

$$\boxed{y' = \frac{12}{5}x^3}$$

Exercise

Find the derivative of $g(s) = \dfrac{s^2 - 2s + 5}{\sqrt{s}}$

Solution

$$\begin{aligned} g(s) &= \frac{s^2}{s^{1/2}} - 2\frac{s}{s^{1/2}} + \frac{5}{s^{1/2}} \\ &= s^{3/2} - 2s^{1/2} + 5s^{-1/2} \end{aligned}$$

$$\begin{aligned} g'(s) &= \tfrac{3}{2}s^{1/2} - 2\tfrac{1}{2}s^{-1/2} + 5\left(-\tfrac{1}{2}\right)s^{-3/2} \\ &= \tfrac{3}{2}s^{1/2} - s^{-1/2} - \tfrac{5}{2}s^{-3/2} \\ &= \frac{3}{2}\sqrt{s} - \frac{1}{\sqrt{s}} - \frac{5}{2s^{3/2}} \\ &= \boxed{\frac{3}{2}\sqrt{s} - \frac{1}{\sqrt{s}} - \frac{5}{2s\sqrt{s}}} \end{aligned}$$

Exercise

Find the derivative of $f(x) = \dfrac{x+1}{\sqrt{x}}$

Solution

$$f(x) = \frac{x}{x^{1/2}} + \frac{1}{x^{1/2}}$$

$$= x^{1/2} + x^{-1/2}$$

$$f'(x) = \tfrac{1}{2}x^{-1/2} - \tfrac{1}{2}x^{-3/2}$$

$$= \frac{1}{2x^{1/2}} - \frac{1}{2x^{3/2}}$$

Exercise

Find the derivative of $f(x) = 4x^{5/3} + 6x^{-3/2} - 11x$

Solution

$$f'(x) = \frac{20}{3}x^{2/3} - 9x^{-5/2} - 11$$

Exercise

Find the derivative of $f(x) = \frac{2}{3}x^3 + \pi x^2 + 7x + 1$

Solution

$$f'(x) = 2x^2 + 2\pi x + 7$$

Exercise

Find the derivative of $f(x) = \dfrac{x^5 - x^3}{15}$

Solution

$$f(x) = \tfrac{1}{15}x^5 - \tfrac{1}{15}x^3$$

$$f'(x) = \tfrac{1}{3}x^4 - \tfrac{1}{5}x^2$$

Exercise

Find the derivative of $f(x) = x^{1/3} + 2x^{1/4} - 3x^{1/5}$

Solution

$$f'(x) = \tfrac{1}{3}x^{-2/3} + \tfrac{1}{2}x^{-3/4} - \tfrac{3}{5}x^{-4/5}$$

Exercise

Find the derivative of $f(t) = 3\sqrt[3]{t^2} - \dfrac{2}{\sqrt{t^3}}$

Solution

$$f(t) = 3t^{2/3} - 2t^{-1/3}$$

$$\boxed{f'(t) = 2t^{-1/3} + \frac{2}{3}2t^{-4/3}}$$

Exercise

Find the derivative of $f(t) = \sqrt{t}\left(5 - t - \frac{1}{3}t^2\right)$

Solution

$$f(t) = 5t^{1/2} - t^{3/2} - \frac{1}{3}t$$

$$\boxed{f(t) = \frac{5}{2}t^{-1/2} - \frac{3}{2}t^{1/2} - \frac{1}{3}}$$

Exercise

Find the derivative of $f(x) = \frac{3}{5}x^{5/3} + \frac{5}{3}x^{-3/5}$

Solution

$$\boxed{f(x) = x^{2/3} - x^{-8/5}}$$

Exercise

Find the derivative of $f(x) = x^{23} - x^{-23}$

Solution

$$\boxed{f'(x) = 23x^{22} + 23x^{-24}}$$

Exercise

Find the ***first*** and ***second*** derivatives $y = -x^3 + 3$

Solution

$$\boxed{y' = -3x^2}$$

$$\boxed{y'' = -6x}$$

Exercise

Find the ***first*** and ***second*** derivatives $y = 3x^7 - 7x^3 + 21x^2$

Solution

$$\boxed{y' = 21x^6 - 21x^2 + 42x}$$

$$\boxed{y'' = 126x^5 - 42x + 42}$$

Exercise

Find the ***first*** and ***second*** derivatives $y = 6x^2 - 10x - \frac{1}{x}$

Solution

$$y' = \boxed{12x - 10 + \frac{1}{x^2}}$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$y'' = 12 + \frac{-2x}{x^4}$$

$$\boxed{= 12 - \frac{2}{x^3}}$$

Exercise

Find the ***first*** and ***second*** derivatives $f(x) = \frac{1}{2}x^4 + \pi x^3 - 7x + 1$

Solution

$$\boxed{f'(x) = 2x^3 + 3\pi x^2 - 7}$$

$$\boxed{f''(x) = 6x^2 + 6\pi x}$$

Exercise

Find the ***first*** and ***second*** derivatives $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$

Solution

$$\boxed{y' = 12x^3 - 18x^2 + \frac{x}{4}}$$

$$\boxed{y'' = 36x^2 - 36x + \frac{1}{4}}$$

Exercise

Find the ***first*** and ***second*** derivatives $y = (2x-3)(1-5x)$

Solution

$$y = -10x^2 + 17x - 3$$

$$\boxed{y' = -20x + 17}$$

$$\boxed{y'' = -20}$$

Exercise

Find the derivative $f(x) = 3x^4 - 6x^3 + \frac{x^2}{8} + 5,\quad f^{(4)}(x)$

Solution

$$f^{(4)}(x) = 3(4!)$$

$$\boxed{= 72}$$

Exercise

Find the derivative $f(x) = 3x^4 - 6x^3 + \frac{x^2}{8} + 5,\quad f^{(5)}(x)$

Solution

$$\boxed{f^{(5)}(x) = 0}$$

Exercise

Find the derivative $f(x) = 2x^6 + 4x^4 - x + 2,\quad f^{(6)}(x)$

Solution

$$f^{(6)}(x) = 2(6!)$$

$$\boxed{= 1,440}$$

Exercise

Find the derivative $f(x) = 4x^5 + 4x^4 + x^2 - 2,\quad f^{(5)}(x)$

Solution

$$f^{(5)}(x) = 4(5!)$$

$$\boxed{= 480}$$

Exercise

Find the derivative $f(x) = 4x^5 + 4x^4 + x^2 - 2, \quad f^{(6)}(x)$

Solution

$$f^{(6)}(x) \,\boxed{= 0}$$

Exercise

Find the derivative $f(x) = 4x^4 - 2x^3 + x + 2, \quad f^{(4)}(x)$

Solution

$$f^{(4)}(x) = 4(4!)$$

$$\boxed{= 96}$$

Exercise

Find an equation for the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point (2, 1).

Solution

$$y' = 3x^2 - 4$$

$$m = y'\big|_{x=2} = 3(2)^2 - 4 = 8$$

$$\boxed{m_1 = -\frac{1}{8}}$$

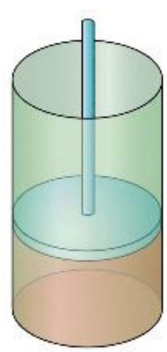
$$y = -\frac{1}{8}(x-2) + 1 \qquad y = m\left(x - x_1\right) + y_1$$

$$\boxed{y = -\frac{1}{8}x - \frac{3}{4}}$$

Exercise

If gas in a cylinder is maintained at a constant temperature T, the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2}$$



In which a, b, n, and R are constants. Find $\frac{dP}{dV}$

Solution

$$\frac{dP}{dV} = \frac{d}{dV}\left(\frac{nRT}{V - nb}\right) - \frac{d}{dV}\left(\frac{an^2}{V^2}\right)$$

$$= -nRT\frac{(V-nb)'}{(V-nb)^2} - an^2\left(-\frac{2V}{V^4}\right)$$

$$= -nRT\frac{1}{(V-nb)^2} + an^2\left(\frac{2}{V^3}\right)$$

$$\boxed{= -\frac{nRT}{(V-nb)^2} + \frac{2an^2}{V^3}}$$

Exercise

Show that if $(a, f(a))$ is any point on the graph of $f(x) = x^2$, then the slope of the tangent line at that point is $m = 2a$

Solution

$$m = f'(a) = \lim_{x\to a} \frac{x^2 - a^2}{x - a}$$

$$= \lim_{x\to a} \frac{(x-a)(x+a)}{x-a}$$

$$= \lim_{x\to a} (x+a)$$

$$\boxed{= 2a}$$

Exercise

Show that if $(a, f(a))$ is any point on the graph of $f(x) = bx^2 + cx + d$, then the slope of the tangent line at that point is $m = 2ab + c$

Solution

$$m = f'(a) = \lim_{h\to 0} \frac{b(a+h)^2 + c(a+h) + d - ba^2 - ca - d}{h}$$

$$= \lim_{h\to 0} \frac{ba^2 + 2abh + bh^2 + ch - ba^2}{h}$$

$$= \lim_{h\to 0} \frac{2abh + bh^2 + ch}{h}$$

$$= \lim_{h\to 0} (2ab + bh + c)$$

$$\boxed{= 2ab + c}$$

Exercise

Let $f(x) = x^2$

a) Show that $\dfrac{f(x) - f(y)}{x - y} = f'\left(\dfrac{x+y}{2}\right)$, for all $x \neq y$

b) Is this property true for $f(x) = ax^2$, where a is a nonzero real number?

c) Give a geometrical interpretation of this property.

d) Is this property true for $f(x) = ax^3$?

Solution

a) $f'(x) = 2x$

$$\begin{aligned}\frac{f(x) - f(y)}{x - y} &= \frac{x^2 - y^2}{x - y} \\ &= \frac{(x - y)(x + y)}{x - y} \\ &= x + y\end{aligned}$$

$$\begin{aligned}f'\left(\frac{x+y}{2}\right) &= 2\left(\frac{x+y}{2}\right) \\ &= x + y\end{aligned}$$

$$\frac{f(x) - f(y)}{x - y} = f'\left(\frac{x+y}{2}\right), \text{ for all } x \neq y$$

b) $f(x) = ax^2 \quad \rightarrow \quad f'(x) = 2ax$

$$\begin{aligned}f'\left(\frac{x+y}{2}\right) &= 2a\left(\frac{x+y}{2}\right) \\ &= a(x + y)\end{aligned}$$

$$\begin{aligned}\frac{f(x) - f(y)}{x - y} &= \frac{ax^2 - ay^2}{x - y} \\ &= \frac{a(x - y)(x + y)}{x - y} \\ &= a(x + y)\end{aligned}$$

$$\frac{f(x) - f(y)}{x - y} = f'\left(\frac{x+y}{2}\right), \text{ for all } x \neq y$$

c) Line thru $(x,\ f(x))$ and $(y,\ f(y))$ is parallel to the tangent line and midpoint is between x and y.

d) $f(x) = ax^3 \quad \rightarrow \quad f'(x) = 3ax^2$

$$f'\left(\frac{x+y}{2}\right) = 3a\left(\frac{x+y}{2}\right)^2$$

$$= \frac{3}{4}a(x+y)^2$$

$$\frac{f(x)-f(y)}{x-y} = \frac{ax^3 - ay^3}{x-y}$$

$$= \frac{a(x-y)\left(x^2+xy+y^2\right)}{x-y}$$

$$= a\left(x^2+xy+y^2\right)$$

$$x^2+xy+y^2 \neq (x+y)^2$$

$$\frac{f(x)-f(y)}{x-y} \neq f'\left(\frac{x+y}{2}\right) \qquad (\textbf{No})$$