Section 4.2 – General Linear Transformations

Definition

A transformation T assigns an output $T(\vec{v})$ to each input vector $\vec{v}$. The transformation is ***linear*** if it meets these requirements for all $\vec{v}$ and $\vec{w}$:

$$\begin{cases} T(c\vec{v}) = cT(\vec{v}) \\ T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w}) \end{cases}$$

We can combine both into one: $\boxed{T(c\vec{v}+d\vec{w}) = cT(\vec{v}) + dT(\vec{w})}$

Theorem

If $T: V \to W$ is a linear transformation, then:

1. $T(0) = 0$
2. $T(\vec{u}-\vec{v}) = T(\vec{u}) - T(\vec{v})$ for all $\vec{u}$ and $\vec{v}$ in *V*.

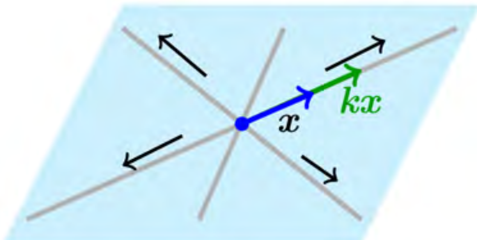
Example

If *V* is a vector space and *k* is any scalar, then the mapping $T: V \to W$ given by $T(\vec{x}) = k\vec{x}$ is a linear operator on *V*, for if *c* is any scalar and if $\vec{u}$ and $\vec{v}$ are any vectors in *V*, then

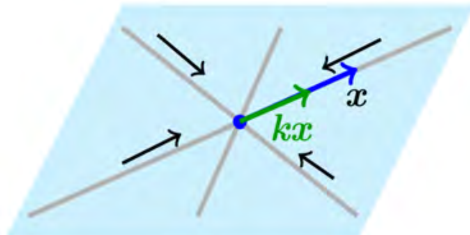
$$\begin{aligned} T(c\vec{u}) &= k(c\vec{u}) \\ &= c(k\vec{u}) \\ &= cT(\vec{u}) \end{aligned}$$

$$\begin{aligned} T(\vec{u}+\vec{v}) &= k(\vec{u}+\vec{v}) \\ &= k\vec{u} + k\vec{v} \\ &= T(\vec{u}) + T(\vec{v}) \end{aligned}$$

If $0 < k < 1$, then *T* is called ***contraction*** of *V* with factor *k*, and if $k > 1$, then *T* is called ***dilation*** of *V* with factor *k*



Dilation of V



Contraction of V

Example

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T = f_A$. $T(x, y, z) = (z - x, z - y)$

Solution

Let $\vec{u} = (x_1, y_1, z_1)$ *and* $\vec{v} = (x_2, y_2, z_2)$

$$\begin{aligned} T(\vec{u}+\vec{v}) &= T(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2) \\ &= \left(z_1 + z_2 - (x_1 + x_2),\ z_1 + z_2 - (y_1 + y_2)\right) \\ &= (z_1 + z_2 - x_1 - x_2,\ z_1 + z_2 - y_1 - y_2) \\ &= (z_1 - x_1,\ z_1 - y_1) + (z_2 - x_2,\ z_2 - y_2) \\ &= T(x_1, y_1, z_1) + T(x_2, y_2, z_2) \\ &= T(\vec{u}) + T(\vec{v}) \end{aligned}$$

$$\begin{aligned} T(r\vec{u}) &= T(rx_1,\ ry_1,\ rz_1) \\ &= (rz_1 - rx_1,\ rz_1 - ry_1) \\ &= r(z_1 - x_1,\ z_1 - y_1) \\ &= rT(\vec{u}) \end{aligned}$$

Since $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{u}) = rT(\vec{v})$

Then function T is a linear transformation.

Domain: $T:\ \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{aligned} T(x, y, z) &= (z - x, z - y) \\ &= \begin{pmatrix} -x + z \\ -y + z \end{pmatrix} \end{aligned}$$

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ A = & \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$

Example – the Zero Transformations

Let V and W be any vector spaces. The mapping $T:V\to W$ such that $T(v)=0$ for every $\vec{v}$ in V is a linear transformation called the zero transformation. To see that T is linear, observe that:

$$T(\vec{u}+\vec{v})=0, \quad T(\vec{u})=0, \quad T(\vec{v})=0, \quad \text{and} \quad T(k\vec{u})=0$$

Therefore; $\quad T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v}) \quad \text{and} \quad T(k\vec{u})=kT(\vec{u})$

Example

Choose a fixed vector $\vec{a}=(1,\ 3,\ 4)$, and let $T(v)$ be the dot product $\vec{a}\bullet\vec{v}$:

Solution

Let $\vec{v}=(v_1,\ v_2,\ v_3)$

$$\begin{aligned} T(\vec{v}) &= \vec{a}\bullet\vec{v} \\ &= (1,\ 3,\ 4)\bullet(v_1,\ v_2,\ v_3) \\ &= v_1+3v_2+4v_3 \end{aligned}$$

This is linear. The inputs v come from three–dimensional space, so $V=\mathbb{R}^3$. The output just numbers, so the output space is $W=\mathbb{R}^1$. We are multiplying by the row matrix $A = [1, 3, 4]$.
Then $T(\vec{v})=A\vec{v}$

Example

Show that the length $T(\vec{v})=\|\vec{v}\|$ is not linear.

Solution

$$\|\vec{v}+\vec{w}\|\overset{?}{=}\|\vec{v}\|+\|\vec{w}\|$$

There are not equal because the sides of a triangle satisfy an inequality $\|\vec{v}+\vec{w}\|\le\|\vec{v}\|+\|\vec{w}\|$

$$\|c\vec{v}\|\overset{?}{=}c\|\vec{v}\|$$

Not - because the length $\|-\vec{v}\|\ne-\|\vec{v}\|$

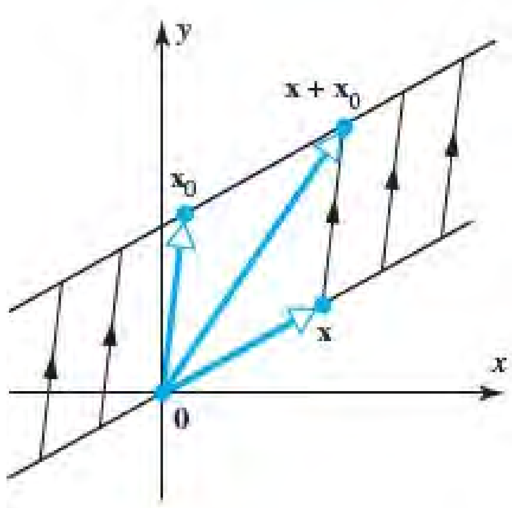
Example

If $\vec{x}_0$ is a fixed nonzero vector in $\mathbb{R}^2$, then the transformation

$$T(\vec{x}) = \vec{x} + \vec{x}_0$$

It has a geometric effect of translating each point $\vec{x}$ in a direction parallel to $\vec{x}_0$ through a distance of $\left\|\vec{x}_0\right\|$.

This cannot be a linear transformation since $T(0) = \vec{x}_0$



Theorem

Let $T: V \to W$ be the linear transformation, where *V* is finite dimensional. If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for *V*, then the image of any vector $\vec{v}$ in *V* can be expressed as

$$T(\vec{v}) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \cdots + c_n T(\vec{v}_n)$$

Where $c_1, c_2, \cdots, c_n$ are the coefficients required to express $\vec{v}$ as a linear combination of the vectors in *S*.

Example

Consider the basis $S = \left\{\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n\right\}$ for ∞^3, where

$$\vec{v}_1 = (1,\ 1,\ 1) \quad \vec{v}_2 = (1,\ 1,\ 0) \quad \vec{v}_3 = (1,\ 0,\ 0)$$

Let $T:\ \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation for which

$$T\left(\vec{v}_1\right) = (1,\ 0), \quad T\left(\vec{v}_2\right) = (2,\ -1), \quad T\left(\vec{v}_3\right) = (4,\ 3)$$

Find a formula for $T\left(\vec{x}_1,\ \vec{x}_2,\ \vec{x}_3\right)$, and then use that formula to compute $T(2,\ -3,\ 5)$

Solution

$$\left(\vec{x}_1,\ \vec{x}_2,\ \vec{x}_3\right) = c_1 (1,\ 1,\ 1) + c_2 (1,\ 1,\ 0) + c_3 (1,\ 0,\ 0)$$

$$\begin{cases} c_1 + c_2 + c_3 = x_1 \\ c_1 + c_2 \qquad = x_2 \\ c_1 \qquad\qquad = x_3 \end{cases}$$

$$\begin{cases} c_3 = x_1 - x_2 \\ c_2 = x_2 - x_3 \\ \quad c_1 = x_3 \end{cases}$$

$$\begin{aligned} \left(\vec{x}_1,\ \vec{x}_2,\ \vec{x}_3\right) &= x_3 (1,\ 1,\ 1) + \left(x_2 - x_3\right)(1,\ 1,\ 0) + \left(x_1 - x_2\right)(1,\ 0,\ 0) \\ &= x_3 \vec{v}_1 + \left(x_2 - x_3\right)\vec{v}_2 + \left(x_1 - x_2\right)\vec{v}_3 \end{aligned}$$

$$\begin{aligned} T\left(\vec{x}_1,\ \vec{x}_2,\ \vec{x}_3\right) &= x_3 T\left(\vec{v}_1\right) + \left(x_2 - x_3\right) T\left(\vec{v}_2\right) + \left(x_1 - x_2\right) T\left(\vec{v}_3\right) \\ &= x_3 (1,\ 0) + \left(x_2 - x_3\right)(2,\ -1) + \left(x_1 - x_2\right)(4,\ 3) \\ &= \left(4x_1 - 2x_2 - x_3,\ \ 3x_1 - 4x_2 + x_3\right) \end{aligned}$$

$$\begin{aligned} T(2,-3,5) &= \left(4(2) - 2(-3) - 5,\ \ 3(2) - 4(-3) + 5\right) \\ &= (9,\ 23) \end{aligned}$$

Example

T is the transformation that rotates every vector by 30°, the domain is the xy-plane (where the input vector $\vec{v}$ is). The range is also the xy-plane (where the rotated $T(\vec{v})$ is). Is the rotation linear?

Solution

Yes it is. We can rotate two vectors and add the results. The sum of rotation $T(\vec{v})+T(\vec{w})$ is the same as the rotation $T(\vec{v}+\vec{w})$ of the sum.

The whole plane is turning together, in this linear transformation.

Definition

If $T:V \to W$ is a linear transformation, then the set of vectors in V that T maps into $\vec{0}$ is called ***kernel*** of T and is denoted by ***ker***(T). The set of all vectors in W that are images under T of at least one vector in V is called the ***range*** of T and is denoted by ***R***(T).

Note:

Transformations have a language of their own. Where there is no matrix, we can't talk about a column space. But the idea can be rescued and used. The column space consisted of all ouputs $A\vec{v}$.

The nullspace consisted of all inputs for which $A\vec{v}-\vec{0}$. Translate those into "range" and "kernel"

Range of T = set of all outputs $T(\vec{v})$: corresponds to column space

Kernel of T = set of all outputs for which $T(\vec{v})=0$: corresponds to nullspace

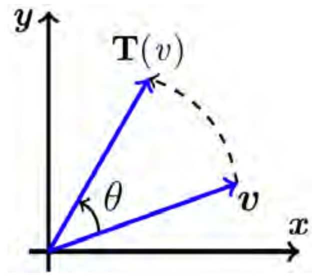
Example

Project every 3-dimensional vector down onto the xy plane.
The range is that plane, which contains every $T(\vec{v})$.

The kernel is the ***z*** axis (which projects down to zero). This projection is linear.

Example – *Kernel and Range of a Rotation*

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator that rotates each vector in the *xy*–plane through the angle θ. Since every vector in the *xy*–plane can be obtained by rotating some vector through the angle θ, it follows that $R(T) = \mathbb{R}^2$.



Moreover, the only vector that rotates into $\vec{0}$ is $\vec{0}$, so $\ker(T) = \{0\}$

Theorem

If $T: V \to W$ is a linear transformation, then:

1. The *kernel* of *T* is a subspace of *V*
2. The *range* of *T* is a subspace of *W*

Theorem

If $T: V \to W$ is a linear transformation from an *n*-dimensional vector space *V* to a vector space *W*, then

$$rank(T) + nullity(T) = n$$

Example

Project every 3-dimensional vector down onto horizontal plane ***z*** = 1.
The vector $\vec{v} = (x,\ y,\ z)$ is transformed to $T(\vec{v}) = (x,\ y,\ 1)$. This transformation is not linear, it doesn't even transform $\vec{v} = \vec{0}$ into $T(\vec{v}) = \vec{0}$.

Multiply every 3-dimensional vector by a 3 by 3 matrix *A*. This is definitely a linear transformation $T(\vec{v} + \vec{w}) = A(\vec{v} + \vec{w})$ which does equal $A\vec{v} + A\vec{w} = T(\vec{v}) + T(\vec{w})$

Example

Suppose A is an invertible matrix. The kernel of T is the zero vector; the range $\boldsymbol{W}$ equals the domain $\boldsymbol{V}$. Another linear transformation is multiplication by A^{-1}.
This is the inverse transformation T^{-1}, which brings every vector $T(\vec{v})$ back to $\vec{v}$:

$$T^{-1}(T(\vec{v})) = \vec{v} \quad \text{matches the matrix multiplication } A^{-1}(A\vec{v}) = \vec{v}$$

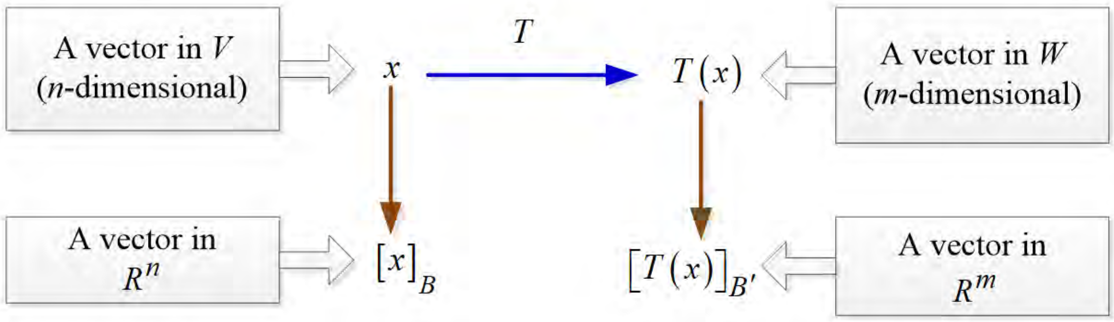
Are all linear transformation produced by matrices?

Each m by n matrix does produce a linear transformation from $V = \mathbb{R}^n$ to $W = \mathbb{R}^m$. When a linear T is described as a "rotation" or "projection" or "..." is there always a matrix hiding behind T?

The answer is yes. This is an approach to linear algebra that doesn't start with matrices. The next section shows that we still end up with matrices.

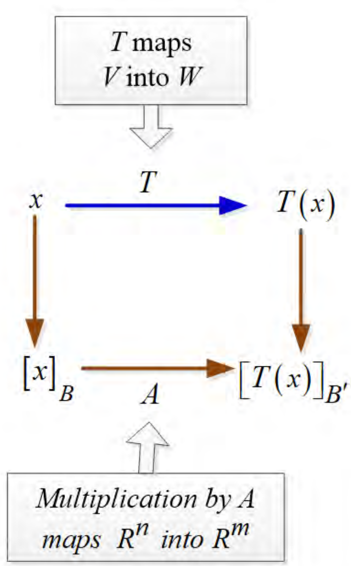
Matrices for General Linear Transformations

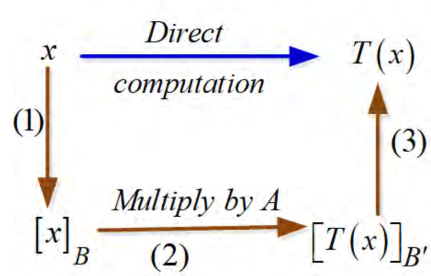
Suppose that V is an n-dimensional vector space, W is an m-dimensional vector space, and that $T: V \to W$ is a linear transformation. Suppose further that B is a basis for V, that B' is a basis for W, and that for each **x** in V, the coordinate matrices for **x** and $T(\vec{x})$ are $[\vec{x}]_B$ and $[T(\vec{x})]_{B'}$, respectively



By using matrix multiplication, we can execute the linear transformation and the following indirect procedure:

1. Compute the coordinate vector $[\vec{x}]_B$
2. Multiply $[\vec{x}]_B$ on the left by A to produce $[T(\vec{x})]_{B'}$
3. Reconstruct $T(\vec{x})$ from its coordinate vector $[T(\vec{x})]_{B'}$





$$A[\vec{x}]_B = [T(\vec{x})]_{B'}$$

Example

Let $T : P_1 \to P_2$ be the linear transformation defined by $T\left(p(x)\right) = xp(x)$

Find the matrix for T with respect to the standard bases

$$B = \left\{\vec{u}_1,\ \vec{u}_2\right\} \quad \textit{and} \quad B' = \left\{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\right\}$$

Where $\vec{u}_1 = 1, \quad \vec{u}_2 = x; \quad \vec{v}_1 = 1, \quad \vec{v}_2 = x, \quad \vec{v}_3 = x^2$

Solution

$$T\left(\vec{u}_1\right) = T(1)$$

$$\boxed{= x}$$

$$T\left(\vec{u}_2\right) = T(x)$$

$$= x(x)$$

$$\boxed{= x^2}$$

$$\left[T\left(\vec{u}_1\right)\right]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\left[T\left(\vec{u}_2\right)\right]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The matrix for T with respect to B and B' is

$$[T]_{B,B'} = \left[\left[T\left(\vec{u}_1\right)\right]_{B'} \middle| \left[T\left(\vec{u}_2\right)\right]_{B'}\right]$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example

Let $T: P_1 \to P_2$ be the linear transformation defined by $T\left(p(x)\right) = xp(x)$ describe in the following figure to perform the computation

$$T(a+bx) = x(a+bx) = ax+bx^2$$

$$\begin{tikzcd}
x \arrow[r, "Direct"', "computation"] \arrow[d, "(1)"'] & T(x) \\
{[x]_B} \arrow[r, "\text{Multiply by } [T]_{B,B'}", "(2)"'] & {[T(x)]_{B'}} \arrow[u, "(3)"']
\end{tikzcd}$$

$B = \{1,\ x\}$ *and* $B' = \left\{1,\ x,\ x^2\right\}$

Solution

Step 1: The coordinates matrix for $\vec{x} = ax+b$ relative to the basis $B = \{1,\ x\}$ is

$$[\boldsymbol{x}]_B = \begin{bmatrix} a \\ b \end{bmatrix}$$

Step 2: Multiply $[\vec{x}]_B$ by the matrix $[T]_{B,B'}$ found in previous example, we obtain

$$\begin{aligned}
[T]_{B,B'}[\vec{x}]_B &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ a \\ b \end{bmatrix} \\
&= \left[T(\vec{x})\right]_{B'}
\end{aligned}$$

Step 3: Reconstructing $T(\vec{x}) = T(ax+b)$ from $\left[T(\vec{x})\right]_{B'}$ we obtain

$$\begin{aligned}
T(ax+b) &= 0+ax+bx^2 \\
&= ax+bx^2
\end{aligned}$$

Exercises *Section* 4.2 – General Linear Transformations

1. The matrix $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ gives a shearing transformation $T(x, y) = (x,\ 3x + y)$.

What happens to (1, 0) and (2, 0) on the x-axis.
What happens to the points on the vertical lines $x = 0$ and $x = a$?.

2. A nonlinear transformation T is invertible if every $\vec{b}$ in the output space comes from exactly one x in the input space. $T(\vec{x}) = \vec{b}$ always has exactly one solution. Which of these transformation (on real numbers $\vec{x}$ is invertible and what is T^{-1}? None are linear, not even T_3. When you solve $T(\vec{x}) = \vec{b}$, you are inverting T:

$$T_1(\vec{x}) = x^2 \quad T_2(\vec{x}) = x^3 \quad T_3(\vec{x}) = x + 9 \quad T_4(\vec{x}) = e^x \quad T_5(\vec{x}) = \frac{1}{x} \quad \textit{for nonzero } x\text{'s}$$

3. If S and T are linear transformations, is $S(T(\vec{v}))$ linear or quadratic?

a) If $S(\vec{v}) = \vec{v}$ and $T(\vec{v}) = \vec{v}$, then $S(T(\vec{v})) = \vec{v}$ or $\vec{v}^2$?

b) $S(\vec{w}_1 + \vec{w}_2) = S(\vec{w}_1) + S(\vec{w}_2)$ and $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$ combine into

$$S(T(\vec{v}_1 + \vec{v}_2)) = S(\_\_\_\_\_\_) = \_\_\_\_\_ + \_\_\_\_\_\_$$

4. Find the range and kernel (like the column space and nullspace) of T:

a) $T(v_1, v_2) = (v_2, v_1)$

b) $T(v_1, v_2, v_3) = (v_1, v_2)$

c) $T(v_1, v_2) = (0,\ 0)$

d) $T(v_1, v_2) = (v_1, v_1)$

5. M is any 2 by 2 matrix and $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. The transformation T is defined by $T(M) = AM$. What rules of matrix multiplication show that T is linear?

6. Which of these transformations satisfy $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ and which satisfy $T(c\vec{v}) = cT(\vec{v})$?

a) $T(\vec{v}) = \dfrac{\vec{v}}{\|\vec{v}\|}$

b) $T(\vec{v}) = v_1 + v_2 + v_3$

c) $T(\vec{v}) = (v_1,\ 2v_2,\ 3v_3)$

d) $T(\vec{v}) =$ largest component of $\vec{v}$.

7. Consider the basis $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for R^3, where $\vec{v}_1 = (1,\ 1,\ 1)$ $\vec{v}_2 = (1,\ 1,\ 0)$ $\vec{v}_3 = (1,\ 0,\ 0)$ and let $T:\ \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation for which

$$T(\vec{v}_1) = (2,\ -1,\ 4), \quad T(\vec{v}_2) = (3,\ 0,\ 1), \quad T(\vec{v}_3) = (-1,\ 5,\ 1)$$

Find a formula for $T(\vec{x}_1, \vec{x}_2, \vec{x}_3)$, and then use that formula to compute $T(2,\ 4,\ -1)$

8. Consider the basis $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for R^3, where $\vec{v}_1 = (1,\ 2,\ 1)$ $\vec{v}_2 = (2,\ 9,\ 0)$ $\vec{v}_3 = (3,\ 3,\ 4)$ and let $T:\ \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation for which

$$T(\vec{v}_1) = (1,\ 0), \quad T(\vec{v}_2) = (-1,\ 1), \quad T(\vec{v}_3) = (0,\ 1)$$

Find a formula for $T(\vec{x}_1, \vec{x}_2, \vec{x}_3)$, and then use that formula to compute $T(7,\ 13,\ 7)$

9. let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be vectors in a vector space V, and let $T: V \to R^3$ be the linear transformation for which $T(\vec{v}_1) = (1,\ -1,\ 2)$, $T(\vec{v}_2) = (0,\ 3,\ 2)$, $T(\vec{v}_3) = (-3,\ 1,\ 2)$.

Find $T(2\vec{v}_1 - 3\vec{v}_2 + 4\vec{v}_3)$

10. Let $T:\ \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operation given by the formula $T(x,\ y) = (2x - y,\ -8x + 4y)$

Which of the following vectors are in $R(T)$

a) $(1,\ -4)$ ***b***) $(5,\ 0)$ ***c***) $(-3,\ 12)$

11. Let $T:\ \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operation given by the formula $T(x,\ y) = (2x - y,\ -8x + 4y)$

Which of the following vectors are in $\ker(T)$

a) $(5,\ 10)$ ***b***) $(3,\ 2)$ ***c***) $(1,\ 1)$

12. Let $T:\ \mathbb{R}^4 \to \mathbb{R}^3$ be the linear operation given by the formula

$$T(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = (4x_1 + x_2 - 2x_3 - 3x_4,\ \ 2x_1 + x_2 + x_3 - 4x_4,\ \ 6x_1 - 9x_3 + 9x_4)$$

Which of the following vectors are in $R(T)$

a) $(0,\ 0,\ 6)$ ***b***) $(1,\ 3,\ 0)$ ***c***) $(2,\ 4,\ 1)$

13. Let $T:\ \mathbb{R}^4 \to \mathbb{R}^3$ be the linear operation given by the formula

$$T(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = (4x_1 + x_2 - 2x_3 - 3x_4,\ \ 2x_1 + x_2 + x_3 - 4x_4,\ \ 6x_1 - 9x_3 + 9x_4)$$

Which of the following vectors are in $\ker(T)$

a) $(3,\ -8,\ 2,\ 0)$ ***b***) $(0,\ 0,\ 0,\ 1)$ ***c***) $(0,\ -4,\ 1,\ 0)$

14. Determine if the given function T is a linear transformation

$$T : M_{22} \to M_{22} \quad by \quad T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2ab & 3cd \\ 0 & 0 \end{bmatrix}$$

15. Determine if the given function T is a linear transformation

$$T : M_{22} \to M_{22} \quad by \quad T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+d & 0 \\ 0 & b+c \end{bmatrix}$$

16. Determine if the given function T is a linear transformation where A is fixed 2×3 matrix

$$T : M_{22} \to M_{23} \quad by \quad T(B) = BA$$

(17 – 25) Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T = f_A$.

17. $T(x, y) = \left(x^2, y\right)$

18. $T(x, y, z) = (2x + y, x - y + z)$

19. $T(x, y, z) = (z - x, z - y)$

20. $T\left(x_1, x_2, x_3\right) = \left(2x_1 - x_2 + x_3, \; x_2 - 4x_3\right)$

21. $T\left(x_1, x_2\right) = \left(2x_1 - x_2, \; -3x_1 + x_2, \; 2x_1 - 3x_2\right)$

22. $T\left(x_1, x_2\right) = \left(x_1 + 4x_2, \; 0, \; x_1 - 3x_2, \; x_1\right)$

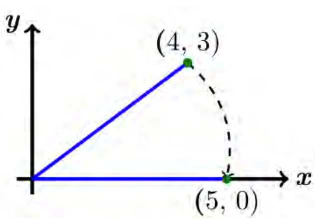
23. $T\left(x_1, x_2, x_3\right) = \left(x_1 - 5x_2 + 4x_3, \; x_2 - 6x_3\right)$

24. $T\left(x_1, x_2, x_3, x_4\right) = \left(x_1 + 2x_2, \; 0, \; 2x_2 + x_4, \; x_2 - x_4\right)$

25. $T\left(x_1, x_2, x_3, x_4\right) = 3x_1 + 4x_3 - 2x_4$

26. A Givens rotation is a linear transformation from $\mathbb{R}^n$ *to* $\mathbb{R}^n$ used in computer to create a zero entry in a vector (usually a column of a matrix). The standard matrix of a Givens rotation in $\mathbb{R}^2$ has the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}; \qquad a^2 + b^2 = 1$$



A Givens rotation in $\mathbb{R}^2$

Find a and b that $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is rotated into $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$.