Section 3.6 – Planar Systems – *Distinct, Complex, and Repeated Eigenvalues – Eigenvectors*

Consider the system equation: $x' = Ax$

Where A is a matrix with constant entries $\left[a_{ij}\right]$

The 1st-order homogeneous equation can be written as $x' = ax$

The solution to this system is given by: $x = Ce^{at}$

We can rewrite the solution in form of vector: $x = ve^{\lambda t}$

The first derivative of the solution: $x' = \lambda ve^{\lambda t}$

$$x' = Ax$$
$$\lambda ve^{\lambda t} = Ave^{\lambda t}$$
$$\lambda v = Av$$

Definition

Suppose A is an nxn matrix and $Av = \lambda v$

The values of λ are called eigenvalues of the matrix $\boldsymbol{A}$ and the nonzero vectors $\boldsymbol{v}$ are called the eigenvectors corresponding to that eigenvalue.

Eigenvalues

Let's change the form of the system to a general matrix form and is defined by the form: $X' = AX(t)$
Where A is a square matrix (n x n)

The behavior of a system can be determined from equilibrium point(s) by finding the eigenvalues and the eigenvectors of the system.

Therefore; the equation can be rewritten into the form: $\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

Let's rewrite the equation $Av = \lambda v$.

$Av - \lambda v = 0$ $\qquad$ λ: are the eigenvalues and not a vector

$$Av - \lambda Iv = 0$$

$$(A - \lambda I)v = 0$$

$$\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} a_{11}-\lambda_1 & a_{12} \\ a_{21} & a_{22}-\lambda_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}=0$$

Since v is a nonzero vector that implies that the matrix $A-\lambda I$ has a nontrivial null space.

This exists if and only if (*iff*):

$$\det(A-\lambda I)=0$$

Therefore, the eigenvalues (λ's) are the roots which can be determined by solving the determinant:

$$\begin{vmatrix} a_{11}-\lambda_1 & a_{12} \\ a_{21} & a_{22}-\lambda_2 \end{vmatrix}=0$$

$$(a_{11}-\lambda_1)(a_{22}-\lambda_2)-a_{12}a_{21}=0$$

Example

Find the eigenvalues of the matrix $A=\begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$

Solution

$$|A-\lambda I|=0$$

$$\begin{aligned} \begin{vmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{vmatrix} &= (-4-\lambda)(5-\lambda)+18 \\ &= -20+4\lambda-5\lambda+\lambda^2+18 \\ &= \lambda^2-\lambda-2 \end{aligned}$$

The characteristic polynomial is: $\lambda^2-\lambda-2=0$

Thus, the eigenvalues of $\boldsymbol{A}$ are 2 and –1.

Eigenvectors

From the eigenvalues, the eigenvectors, in the form $V_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$, of the system can be determined by letting:

$$\left(A - \lambda_1 I\right)V_1 = 0 \qquad \text{and} \left(A - \lambda_2 I\right)V_2 = 0$$

The general solution can be written as:

$$x(t) = V_i e^{\lambda t}$$

Example

Find the eigenvectors of the matrix $\qquad A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$

Solution

The eigenvalues of $\boldsymbol{A}$ are 2 and -1.

For $\lambda = 2$, we have $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -4-2 & 6 \\ -3 & 5-2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -6x + 6y = 0 \\ -3x + 3y = 0 \end{cases} \Rightarrow x = y \qquad \text{If } x = c \quad \Rightarrow y = c$$

$$V_1 = c\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda = -1$, we have $\left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -3x + 6y = 0 \\ -3x + 6y = 0 \end{cases} \Rightarrow x = 2y \qquad \text{If } y = c \quad \Rightarrow x = 2c$$

$$V_2 = c\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \Rightarrow V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x(t) = V_1 e^{\lambda_1 t} + V_2 e^{\lambda_2 t}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}$$

Summary

In general, a polynomial of degree n has n roots. Each root λ is an eigenvalue, and for each we can find an eigenvector v. From these, we can form the solution $y(t) = ve^{\lambda t}$.
However, the numbers of the eigenvalue solutions are as follow:

1. Two Distinct real roots $\left(T^2 - 4D > 0\right)$
2. Two complex conjugate roots $\left(T^2 - 4D < 0\right)$
3. One real Repeated roots $\left(T^2 - 4D = 0\right)$

Planar Systems

2-dimension linear systems are also called planar systems, we will enable to solve the system

$$y' = Ay$$

$$where \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix}$$

$$= \lambda^2 - \left(a_{11} + a_{22}\right)\lambda + a_{11}a_{22} - a_{12}a_{21}$$

$$D = \det(A) = a_{11}a_{22} - a_{12}a_{21}$$

$$T = tr(A) = a_{11} + a_{22} \qquad tr(A) : trace$$

$$\lambda^2 - T\lambda + D = 0$$

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

Proposition

Suppose λ_1 and λ_2 are eigenvalues of an $n \times n$ matrix A. Suppose $V_1 \neq 0$ is an eigenvector for λ_1 and $V_2 \neq 0$ is an eigenvector for λ_2. If $\lambda_1 \neq \lambda_2$ then V_1 and V_2 are linearly independent.

Distinct Real Eigenvalues

If $T^2 - 4D > 0$, then the solutions of the characteristic equation are:

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} \quad \text{and} \quad \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

Then $\lambda_1 < \lambda_2$, and both are real eigenvalues of A.

Let V_1 and V_2 be the associated eigenvectors. Then we have two exponential solutions:

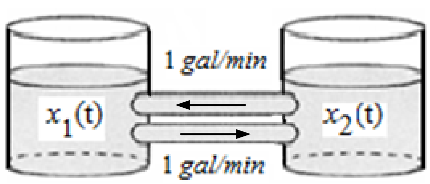
$$y_1(t) = V_1 e^{\lambda_1 t} \quad \text{and} \quad y_2(t) = V_2 e^{\lambda_2 t}$$

The general solution is:

$$\begin{aligned} y(t) &= C_1 y_1(t) + C_2 y_2(t) \\ &= C_1 V_1 e^{\lambda_1 t} + C_2 V_2 e^{\lambda_2 t} \end{aligned}$$

Example

Two tanks are connected by two pipes. Each tank contains 500 *gallons* of a salt solution. Through on pipe solution is pumped from the first tank to the second at 1 *gal/min*. Through the other pipe, solution is pumped at the same rate from the second to the first tank. Suppose that at time $t = 0$ there is no salt in the tank on the right and 100 *lb* in the tank on the left. Show the salt content in each tank varies with time.



Solution

$$\text{Rate in } = 1 gal/min \frac{x_2}{500} lb/gal = \frac{x_2}{500} lb/min$$

$$\text{Rate out } = 1 gal/min \frac{x_1}{500} lb/gal = \frac{x_1}{500} lb/min$$

$$\begin{aligned} \frac{dx_1}{dt} &= \text{Rate in} - \text{Rate out} = \frac{x_2}{500} - \frac{x_1}{500} \\ &= -\frac{x_1}{500} + \frac{x_2}{500} \end{aligned}$$

$$\frac{dx_2}{dt} = \frac{x_1}{500} - \frac{x_2}{500}$$

$$\begin{cases} x_1' = -\dfrac{x_1}{500} + \dfrac{x_2}{500} \\ x_2' = \dfrac{x_1}{500} - \dfrac{x_2}{500} \end{cases}$$

The system is: $x' = Ax(t)$

Where $A = \begin{pmatrix} -\frac{1}{500} & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500} \end{pmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} -\frac{1}{500} - \lambda & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500} - \lambda \end{vmatrix}$$

$$= \left(-\frac{1}{500} - \lambda\right)\left(-\frac{1}{500} - \lambda\right) - \frac{1}{500}\frac{1}{500}$$

$$= \frac{1}{500^2} + \frac{2}{500}\lambda + \lambda^2 - \frac{1}{500^2}$$

$$= \lambda^2 + \frac{1}{250}\lambda$$

$$= \lambda\left(\lambda + \frac{1}{250}\right)$$

The eigenvalues are: $\lambda_1 = -\frac{1}{250} \qquad \lambda_2 = 0$

For $\lambda_1 = -\frac{1}{250}$, we have

$$(A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} -\frac{1}{500} + \frac{1}{250} & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500} + \frac{1}{250} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{500} & \frac{1}{500} \\ \frac{1}{500} & \frac{1}{500} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \frac{1}{500}x + \frac{1}{500}y = 0 \\ \frac{1}{500}x + \frac{1}{500}y = 0 \end{cases} \Rightarrow \begin{matrix} x + y = 0 \\ x + y = 0 \end{matrix} \Rightarrow x = -y$$

$$V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t/250}$$

For $\lambda_2 = 0$, we have

$$(A - \lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} -\frac{1}{500} & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -\frac{1}{500}x + \frac{1}{500}y = 0 \\ \frac{1}{500}x - \frac{1}{500}y = 0 \end{cases} \Rightarrow \begin{matrix} -x + y = 0 \\ x - y = 0 \end{matrix} \Rightarrow x = y$$

$$V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad x_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{0t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The general solution:

$$x(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t/250} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} C_1 e^{-t/250} + C_2 \\ -C_1 e^{-t/250} + C_2 \end{pmatrix}$$

$$x(0) = \begin{pmatrix} C_1 e^{-0/250} + C_2 \\ -C_1 e^{-0/250} + C_2 \end{pmatrix}$$

$$\begin{pmatrix} 100 \\ 0 \end{pmatrix} = \begin{pmatrix} C_1 + C_2 \\ -C_1 + C_2 \end{pmatrix}$$

$$\to \begin{cases} C_1 + C_2 = 100 \\ -C_1 + C_2 = 0 \end{cases} \Rightarrow \quad C_1 = 50 \quad C_2 = 50$$

$$x(t) = \begin{pmatrix} 50 + 50e^{-t/250} \\ 50 - 50e^{-t/250} \end{pmatrix}$$

Complex Eigenvalues

If $T^2 - 4D < 0$, then the solutions of the characteristic equation are the complex conjugate:

$$\lambda_1 = \frac{T + i\sqrt{4D - T^2}}{2} \qquad \textit{and} \qquad \lambda_2 = \frac{T - i\sqrt{4D - T^2}}{2}$$

Example

Find the eigenvalues and eigenvectors for the matrix $\quad A = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$

Solution

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -\lambda & 1 \\ -2 & 2-\lambda \end{vmatrix} \\ &= -\lambda(2-\lambda) + 2 \\ &= \lambda^2 - 2\lambda + 2 \\ &= 0 \end{aligned}$$

$$\lambda^2 - 2\lambda + 2 = 0 \quad \Rightarrow \lambda = 1 + i \quad \bar{\lambda} = 1 - i$$

For $\lambda = 1 + i$; $\qquad (A - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = 0$

$$\begin{pmatrix} -1-i & 1 \\ -2 & 2-i-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} -1-i & 1 \\ -2 & 1-i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\rightarrow \begin{cases} -(1+i)x + y = 0 \\ -2x + (1-i)y = 0 \end{cases}$$

$$\rightarrow \begin{cases} -(1+i)(1-i)x + (1-i)y = 0 \\ \qquad -2x + (1-i)y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -2x + (1-i)y = 0 \\ -2x + (1-i)y = 0 \end{cases} \rightarrow x = 1 \Rightarrow y = \frac{2}{1-i}\frac{1+i}{1+i} = 1 + i$$

$$V_1 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{(1+i)t}$$

For $\bar{\lambda} = 1 - i$; $\quad V_2 = \bar{V}_1 \quad \Rightarrow \quad x_2(t) = V_2\, e^{\bar{\lambda} t}$

$$V_2 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} \quad \Rightarrow \quad x_2(t) = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{(1-i)t}$$

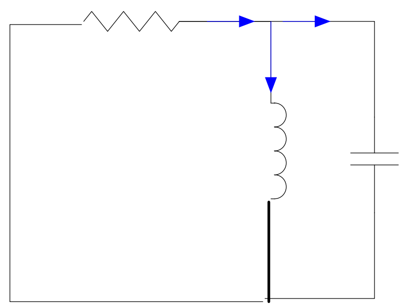
Theorem

Suppose that A is a 2×2 matrix with complex conjugate eigenvalues λ and $\bar{\lambda}$. Suppose that V is an eigenvector associated with λ. Then the general solution to the system $x' = Ax$ is

$$x(t) = C_1 V_1 e^{\lambda t} + C_2 V_2 e^{\bar{\lambda} t}$$

Example

Find the current I_1 and I_2 for the circuit below, where $R = 1\,\Omega$, $L = 1\,henry$, and $C = \frac{5}{4}\,farad$. Assume that $I_1(0) = 5A$ and $I_2(0) = 1A$.



Solution

Kirchoff's current law: $I = I_1 + I_2$

Since there is no voltage: $0 = RI + LI_1' \quad \Rightarrow LI_1' = -RI = -R\left(I_1 + I_2\right)$

$$I_1' = -\frac{R}{L}I_1 - \frac{R}{L}I_2 \quad (1)$$

Kirchoff's law applied to the large loop: $0 = RI + \frac{1}{C}Q \quad \Rightarrow \quad \frac{1}{C}Q + R\left(I_1 + I_2\right) = 0$

$$\frac{1}{C}Q' + R\left(I_1' + I_2'\right) = 0$$

$$\frac{1}{C}I_2 + R\left(I_1' + I_2'\right) = 0$$

$$RI_2' = -\frac{1}{C}I_2 - RI_1'$$

$$I_2' = -\frac{1}{RC}I_2 - \left(-\frac{R}{L}I_1 - \frac{R}{L}I_2\right)$$

$$= \frac{R}{L}I_1 + \left(\frac{R}{L} - \frac{1}{RC}\right)I_2 \quad (2)$$

$$\begin{cases} I_1' = -\frac{R}{L}I_1 - \frac{R}{L}I_2 \\ I_2' = \frac{R}{L}I_1 + \left(\frac{R}{L} - \frac{1}{RC}\right)I_2 \end{cases}$$

$$\Rightarrow \begin{cases} I_1' = -I_1 - I_2 \\ I_2' = I_1 + \left(1 - \frac{1}{5/4}\right) I_2 \end{cases}$$

$$\begin{cases} I_1' = -I_1 - I_2 \\ I_2' = I_1 + \frac{1}{5} I_2 \end{cases}$$

The system can be written as: $I' = AI$, where $A = \begin{pmatrix} -1 & -1 \\ 1 & \frac{1}{5} \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & -1 \\ 1 & \frac{1}{5} - \lambda \end{vmatrix} = (-1-\lambda)\left(\tfrac{1}{5} - \lambda\right) + 1 = 0$$

$$\lambda^2 + \tfrac{4}{5}\lambda + \tfrac{4}{5} = 0$$

$$5\lambda^2 + 4\lambda + 4 = 0 \qquad \lambda = \frac{-2 \pm 4i}{5}$$

For $\lambda = \dfrac{-2+4i}{5}$;

$$\begin{pmatrix} -1 + \frac{2}{5} - \frac{4i}{5} & -1 \\ 1 & \frac{1}{5} + \frac{2}{5} - \frac{4i}{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} -\frac{3}{5} - \frac{4i}{5} & -1 \\ 1 & \frac{3}{5} - \frac{4i}{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} -\frac{1}{5}(3+4i)x - y = 0 \\ x + \frac{1}{5}(3-4i)y = 0 \end{cases} \rightarrow x = -5 \qquad y = 3 + 4i$$

$$V_1 = \begin{pmatrix} -5 \\ 3+4i \end{pmatrix}$$

$$x_1(t) = V_1 e^{\lambda t}$$

$$= \begin{pmatrix} -5 \\ 3+4i \end{pmatrix} e^{\left(\frac{-2+4i}{5}\right)t}$$

$$= \begin{pmatrix} -5 \\ 3+4i \end{pmatrix} e^{-\frac{2}{5}t} e^{\frac{4}{5}it}$$

$$= e^{-\frac{2}{5}t} \begin{pmatrix} -5 \\ 3+4i \end{pmatrix} \left(\cos\left(\tfrac{4}{5}t\right) + \boldsymbol{i} \sin\left(\tfrac{4}{5}t\right)\right)$$

$$= e^{-\frac{2}{5}t}\begin{pmatrix} -5\cos\left(\frac{4}{5}t\right) - i5\sin\left(\frac{4}{5}t\right) \\ 3\cos\left(\frac{4}{5}t\right) - 4\sin\left(\frac{4}{5}t\right) + i\left(4\cos\left(\frac{4}{5}t\right) + 3\sin\left(\frac{4}{5}t\right)\right) \end{pmatrix}$$

$$x_1(t) = e^{-\frac{2}{5}t}\begin{pmatrix} -5\cos\left(\frac{4}{5}t\right) \\ 3\cos\left(\frac{4}{5}t\right) - 4\sin\left(\frac{4}{5}t\right) \end{pmatrix}$$

$$x_2(t) = e^{-\frac{2}{5}t}\begin{pmatrix} -5\sin\left(\frac{4}{5}t\right) \\ 4\cos\left(\frac{4}{5}t\right) + 3\sin\left(\frac{4}{5}t\right) \end{pmatrix}$$

$$I(t) = C_1 x_1 + C_2 x_2$$

$$I(0) = C_1 e^{-\frac{2}{5}(0)}\begin{pmatrix} -5\cos\left(\frac{4}{5}(0)\right) \\ 3\cos\left(\frac{4}{5}(0)\right) - 4\sin\left(\frac{4}{5}(0)\right) \end{pmatrix} + C_2 e^{-\frac{2}{5}(0)}\begin{pmatrix} -5\sin\left(\frac{4}{5}(0)\right) \\ 4\cos\left(\frac{4}{5}(0)\right) + 3\sin\left(\frac{4}{5}(0)\right) \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ 1 \end{pmatrix} = C_1\begin{pmatrix} -5 \\ 3 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -5C_1 \\ 3C_1 + 4C_2 \end{pmatrix}$$

$$\begin{cases} -5C_1 = 5 \\ 3C_1 + 4C_2 = 1 \end{cases} \Rightarrow C_1 = -1 \quad C_2 = 1$$

The general solution is:

$$I(t) = -e^{-\frac{2}{5}t}\begin{pmatrix} -5\cos\left(\frac{4}{5}t\right) \\ 3\cos\left(\frac{4}{5}t\right) - 4\sin\left(\frac{4}{5}t\right) \end{pmatrix} + e^{-\frac{2}{5}t}\begin{pmatrix} -5\sin\left(\frac{4}{5}t\right) \\ 4\cos\left(\frac{4}{5}t\right) + 3\sin\left(\frac{4}{5}t\right) \end{pmatrix}$$

$$I(t) = e^{-\frac{2}{5}t}\begin{pmatrix} 5\cos\left(\frac{4}{5}t\right) - 5\sin\left(\frac{4}{5}t\right) \\ \cos\left(\frac{4}{5}t\right) + 7\sin\left(\frac{4}{5}t\right) \end{pmatrix}$$

One Real Eigenvalue of Multiplicity 2

If $T^2 - 4D = 0 \Rightarrow T^2 = 4D$, then the solutions of the characteristic equation: $\lambda_1 = \lambda_2 = \frac{T}{2}$

$$\left.\begin{aligned} x_1(t) &= V_1 e^{\lambda t} \\ x_2(t) &= \left(V_1 t + V_2\right) e^{\lambda t} \end{aligned}\right\} \Rightarrow x(t) = C_1 x_1(t) + C_2 x_2(t)$$

Example

Find all exponential solutions for $A = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$

Solution

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix} \\ &= (-1-\lambda)(-3-\lambda) + 1 \\ &= \lambda^2 + 4\lambda + 4 \\ &= (\lambda + 2)^2 = 0 \end{aligned}$$

$$\lambda_{1,2} = -2 \ \rightarrow \ (A - \lambda I)^2 V_2 = 0$$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} V_2 = 0$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} V_2 = 0 \qquad \Rightarrow \ V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(A + 2I) V_2 = V_1$$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = V_1 \qquad \Rightarrow \ \begin{pmatrix} 1 \\ 1 \end{pmatrix} = V_1$$

$$x_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}$$

$$\begin{aligned} x_2(t) &= \left(V_1 t + V_2\right) e^{-2t} \\ &= \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) e^{-2t} \\ &= \begin{pmatrix} t+1 \\ t \end{pmatrix} e^{-2t} \end{aligned}$$

$$\begin{aligned} x(t) &= C_1 x_1(t) + C_2 x_2(t) \\ &= C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} t+1 \\ t \end{pmatrix} e^{-2t} \end{aligned}$$

Exercises Section 3.6 – Planar Systems – *Distinct, Complex, and Repeated Eigenvalues – Eigenvectors*

Find the eigenvalues and the eigenvectors for each of the matrices.

1. $A=\begin{pmatrix}12 & 14\\ -7 & -9\end{pmatrix}$

2. $A=\begin{pmatrix}-4 & 1\\ -2 & 1\end{pmatrix}$

3. $A=\begin{pmatrix}5 & 3\\ -6 & -4\end{pmatrix}$

4. $A=\begin{pmatrix}-2 & 3\\ 0 & -5\end{pmatrix}$

5. $A=\begin{pmatrix}6 & 10\\ -5 & -9\end{pmatrix}$

6. $A=\begin{pmatrix}3 & 0\\ 8 & -1\end{pmatrix}$

7. $A=\begin{pmatrix}10 & -9\\ 4 & -2\end{pmatrix}$

8. $A=\begin{pmatrix}4 & 0 & 1\\ -2 & 1 & 0\\ -2 & 0 & 1\end{pmatrix}$

9. $A=\begin{pmatrix}1 & 0 & 0\\ 4 & 3 & 2\\ -8 & -4 & -3\end{pmatrix}$

10. $A=\begin{pmatrix}-1 & -4 & -2\\ 0 & 1 & 1\\ -6 & -12 & 2\end{pmatrix}$

11. $A=\begin{pmatrix}3 & 2 & 2\\ 1 & 4 & 1\\ -2 & -4 & -1\end{pmatrix}$

12. $A=\begin{pmatrix}-6 & 4 & 4\\ -4 & 2 & 4\\ -10 & 8 & 4\end{pmatrix}$

13. $A=\begin{pmatrix}0 & 0 & 2 & 0\\ 1 & 0 & 1 & 0\\ 0 & 1 & -2 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$

Find a fundamental set of solutions for the system $x' = Ax$, where A is the given matrices.

14. $A=\begin{pmatrix}2 & 0\\ -4 & -1\end{pmatrix}$

15. $A=\begin{pmatrix}-1 & 0 & 0\\ 2 & -5 & -6\\ -2 & 3 & 4\end{pmatrix}$

Find the general solution of the system

16. $\begin{cases} x_1'(t) = x_1 + 2x_2 \\ x_2'(t) = 4x_1 + 3x_2 \end{cases}$

17. $\begin{cases} x_1'(t) = 2x_1 + 2x_2 \\ x_2'(t) = x_1 + 3x_2 \end{cases}$

18. $\begin{cases} x_1'(t) = -4x_1 + 2x_2 \\ x_2'(t) = -\frac{5}{2}x_1 + 2x_2 \end{cases}$

19. $\begin{cases} x_1'(t) = -\frac{5}{2}x_1 + 2x_2 \\ x_2'(t) = \frac{3}{4}x_1 - 2x_2 \end{cases}$

20. $\begin{cases} x_1'(t) = 3x_1 - x_2 \\ x_2'(t) = 9x_1 - 3x_2 \end{cases}$

21. $\begin{cases} x_1'(t) = -6x_1 + 5x_2 \\ x_2'(t) = -5x_1 + 4x_2 \end{cases}$

22. $\begin{cases} x_1'(t) = 6x_1 - x_2 \\ x_2'(t) = 5x_1 + 2x_2 \end{cases}$

23. $\begin{cases} x_1'(t) = x_1 + x_2 \\ x_2'(t) = -2x_1 - x_2 \end{cases}$

24. $\begin{cases} x_1'(t) = 5x_1 + x_2 \\ x_2'(t) = -2x_1 + 3x_2 \end{cases}$

25. $\begin{cases} x_1'(t) = 4x_1 + 5x_2 \\ x_2'(t) = -2x_1 + 6x_2 \end{cases}$

26. $\begin{cases} x_1'(t) = 5x_1 - 4x_2 \\ x_2'(t) = 2x_1 - x_2 \end{cases}$

27. $\begin{cases} x_1'(t) = 6x_1 - 6x_2 \\ x_2'(t) = 4x_1 - 4x_2 \end{cases}$

28. $\begin{cases} x_1'(t) = 5x_1 - 3x_2 \\ x_2'(t) = 2x_1 \end{cases}$

29. $\begin{cases} x_1'(t) = 5x_1 - 4x_2 \\ x_2'(t) = 3x_1 - 2x_2 \end{cases}$

30. $\begin{cases} x_1'(t) = 9x_1 - 8x_2 \\ x_2'(t) = 6x_1 - 5x_2 \end{cases}$

31. $\begin{cases} x_1'(t) = 10x_1 - 6x_2 \\ x_2'(t) = 12x_1 - 7x_2 \end{cases}$

32. $\begin{cases} x_1'(t) = 6x_1 - 10x_2 \\ x_2'(t) = 2x_1 - 3x_2 \end{cases}$

33. $\begin{cases} x_1'(t) = 11x_1 - 15x_2 \\ x_2'(t) = 6x_1 - 8x_2 \end{cases}$

34. $\begin{cases} x_1'(t) = 3x_1 + x_2 \\ x_2'(t) = x_1 + 3x_2 \end{cases}$

35. $\begin{cases} x_1'(t) = 4x_1 + 2x_2 \\ x_2'(t) = 2x_1 + 4x_2 \end{cases}$

36. $\begin{cases} x_1'(t) = 9x_1 + 2x_2 \\ x_2'(t) = 2x_1 + 6x_2 \end{cases}$

37. $\begin{cases} x_1'(t) = 13x_1 + 4x_2 \\ x_2'(t) = 4x_1 + 7x_2 \end{cases}$

38. $\begin{cases} x_1'(t) = 3x_1 - 2x_2 \\ x_2'(t) = 2x_1 - 2x_2 \end{cases}$

39. $\begin{cases} x_1'(t) = 2x_1 - x_2 \\ x_2'(t) = 3x_1 - 2x_2 \end{cases}$

40. $\begin{cases} x_1'(t) = 5x_1 - x_2 \\ x_2'(t) = 3x_1 - x_2 \end{cases}$

41. $\begin{cases} x_1'(t) = x_1 + x_2 \\ x_2'(t) = 4x_1 - 2x_2 \end{cases}$

42. $\begin{cases} x_1'(t) = -x_1 - 4x_2 \\ x_2'(t) = x_1 - x_2 \end{cases}$

43. $\begin{cases} x_1'(t) = 2x_1 + 3x_2 - 7 \\ x_2'(t) = -x_1 - 2x_2 + 5 \end{cases}$

44. $\begin{cases} x_1'(t) = 5x_1 + 9x_2 + 2 \\ x_2'(t) = -x_1 + 11x_2 + 6 \end{cases}$

45. $\begin{cases} y_1'(t) = 6y_1 + y_2 + 6t \\ y_2'(t) = 4y_1 + 3y_2 - 10t + 4 \end{cases}$

46. $\begin{cases} x'(t) = 5x + 3y - 2e^{-t} + 1 \\ y'(t) = -x + y + e^{-t} - 5t + 7 \end{cases}$

47. $\begin{cases} x'(t) = -3x + y + 3t \\ y'(t) = 2x - 4y + e^{-t} \end{cases}$

48. $\begin{cases} x'(t) = 2x - y + (\sin 2t)e^{2t} \\ y'(t) = 4x + 2y + (2\cos 2t)e^{2t} \end{cases}$

49. $\begin{cases} x'(t) = 2y + e^{t} \\ y'(t) = -x + 3y - e^{t} \end{cases}$

50. $\begin{cases} x'(t) = 2y + 2 \\ y'(t) = -x + 3y + e^{-3t} \end{cases}$

51. $\begin{cases} x'(t) = x + 8y + 12t \\ y'(t) = x - y + 12t \end{cases}$

52. $\begin{cases} x_1'(t) = x_1 + x_2 - x_3 \\ x_2'(t) = 2x_2 \\ x_3'(t) = x_2 - x_3 \end{cases}$

53. $\begin{cases} x_1'(t) = 2x_1 - 7x_2 \\ x_2'(t) = 5x_1 + 10x_2 + 4x_3 \\ x_3'(t) = 5x_2 + 2x_3 \end{cases}$

54. $\begin{cases} x_1'(t) = 3x_1 - x_2 - x_3 \\ x_2'(t) = x_1 + x_2 - x_3 \\ x_3'(t) = x_1 - x_2 + x_3 \end{cases}$

55. $\begin{cases} x_1'(t) = 3x_1 + 2x_2 + 4x_3 \\ x_2'(t) = 2x_1 + 2x_3 \\ x_3'(t) = 4x_1 + 2x_2 + 3x_3 \end{cases}$

56. $\begin{cases} x_1'(t) = x_1 + x_2 + x_3 \\ x_2'(t) = 2x_1 + x_2 - x_3 \\ x_3'(t) = -8x_1 - 5x_2 - 3x_3 \end{cases}$

57. $\begin{cases} x_1'(t) = x_1 - x_2 + 4x_3 \\ x_2'(t) = 3x_1 + 2x_2 - x_3 \\ x_3'(t) = 2x_1 + x_2 - x_3 \end{cases}$

58. $\begin{cases} x_1'(t) = x_1 + x_2 + e^t \\ x_2'(t) = x_1 + x_2 + e^{2t} \\ x_3'(t) = 3x_3 + te^{3t} \end{cases}$

59. $\begin{cases} x_1'(t) = 3x_1 - x_2 - x_3 \\ x_2'(t) = x_1 + x_2 - x_3 + t \\ x_3'(t) = x_1 - x_2 + x_3 + 2e^t \end{cases}$

Find the general solution of the system $y' = Ay$

60. $\begin{cases} y_1'(t) = -y_1 + 6y_2 \\ y_2'(t) = -3y_1 + 8y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

61. $\begin{cases} y_1'(t) = y_1 + 2y_2 \\ y_2'(t) = -y_1 + 4y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

62. $\begin{cases} y_1'(t) = -4y_1 - 8y_2 \\ y_2'(t) = 4y_1 + 4y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

63. $\begin{cases} y_1'(t) = -y_1 - 2y_2 \\ y_2'(t) = 4y_1 + 3y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

64. $\begin{cases} y_1'(t) = 3y_1 - y_2 \\ y_2'(t) = y_1 + y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

65. $\begin{cases} y_1'(t) = -3y_1 + y_2 \\ y_2'(t) = -y_1 - y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

66. $\begin{cases} y_1'(t) = 2y_1 + 4y_2 \\ y_2'(t) = -y_1 + 6y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

67. $\begin{cases} y_1'(t) = -8y_1 - 10y_2 \\ y_2'(t) = 5y_1 + 7y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

68. $\begin{cases} y_1'(t) = -3y_1 + 2y_2 \\ y_2'(t) = -3y_1 + 4y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

69. $\begin{cases} y_1'(t) = 3y_1 - y_2 \\ y_2'(t) = 5y_1 - 3y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

70. $\begin{cases} y_1'(t) = y_1 + 9y_2 \\ y_2'(t) = -2y_1 - 5y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

71. $\begin{cases} y_1'(t) = 4y_1 + y_2 \\ y_2'(t) = -2y_1 + y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

72. $\begin{cases} y_1'(t) = 2y_1 + y_2 - e^{2t} \\ y_2'(t) = y_1 + 2y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

73. $\begin{cases} y_1'(t) + 2y_2'(t) = 4y_1 + 5y_2 \\ 2y_1'(t) - y_2'(t) = 3y_1 \end{cases}$ $\quad y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

74. $\begin{cases} y_1'(t) = 3y_1 - 2y_2 \\ y_2'(t) = 2y_1 - 2y_2 \end{cases}$ $y(0) = \begin{pmatrix} 3 \\ \frac{1}{2} \end{pmatrix}$

75. $\begin{cases} y_1'(t) = y_1 - 2y_2 \\ y_2'(t) = 3y_1 - 4y_2 \end{cases}$ $y(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

76. $\begin{cases} y_1'(t) = y_1 - 4y_2 \\ y_2'(t) = 4y_1 - 7y_2 \end{cases}$ $y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

77. $\begin{cases} y_1'(t) = 3y_1 + 9y_2 \\ y_2'(t) = -y_1 - 3y_2 \end{cases}$ $y(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

78. $\begin{cases} y_1'(t) = 2y_1 + \frac{3}{2}y_2 \\ y_2'(t) = -\frac{3}{2}y_1 - y_2 \end{cases}$ $y(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

79. $\begin{cases} y_1'(t) = -5y_1 + 12y_2 \\ y_2'(t) = -2y_1 + 5y_2 \end{cases}$ $y(0) = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$

80. $\begin{cases} y_1'(t) = -4y_1 + 6y_2 \\ y_2'(t) = -3y_1 + 5y_2 \end{cases}$ $y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

81. $\begin{cases} y_1'(t) = y_1 + 2y_2 \\ y_2'(t) = 3y_1 + 2y_2 \end{cases}$ $y(0) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$

82. $\begin{cases} y_1'(t) = -5y_1 + y_2 \\ y_2'(t) = 4y_1 - 2y_2 \end{cases}$ $y(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

83. $\begin{cases} y_1'(t) = 3y_1 - 9y_2 \\ y_2'(t) = 4y_1 - 3y_2 \end{cases}$ $y(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

84. $\begin{cases} y_1'(t) = 3y_1 - 13y_2 \\ y_2'(t) = 5y_1 + y_2 \end{cases}$ $y(0) = \begin{pmatrix} 3 \\ -10 \end{pmatrix}$

85. $\begin{cases} y_1'(t) = 7y_1 + y_2 \\ y_2'(t) = -4y_1 + 3y_2 \end{cases}$ $y(0) = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$

86. $\begin{cases} y_1'(t) = -y_1 + \frac{3}{2}y_2 \\ y_2'(t) = -\frac{1}{6}y_1 - 2y_2 \end{cases}$ $y(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

87. $\begin{cases} y_1'(t) = 3y_1 - 3y_2 + 2 \\ y_2'(t) = -6y_1 - t \end{cases}$ $y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

88. $\begin{cases} y_1'(t) = -5y_1 + y_2 + 6e^{2t} \\ y_2'(t) = 4y_1 - 2y_2 - e^{2t} \end{cases}$ $y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

89. $\begin{cases} y_1'(t) = y_1 + 2y_2 + 2t \\ y_2'(t) = 3y_1 + 2y_2 - 4t \end{cases}$ $y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

90. $\begin{cases} x_1'(t) = 3x_1 - x_2 + 4e^{2t} \\ x_2'(t) = -x_1 + 3x_2 + 4e^{4t} \end{cases}$ $X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

91. $\begin{cases} x_1'(t) = x_1 - x_2 + \frac{1}{t} \\ x_2'(t) = x_1 - x_2 + \frac{1}{t} \end{cases}$ $X(1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

92. $\begin{cases} x_1'(t) = 3x_1 - 2x_2 - 2e^{-t} \\ x_2'(t) = x_1 \quad - 2e^{-t} \end{cases}$ $X(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

93. $\begin{cases} y_1'(t) = y_1 \\ y_2'(t) = -4y_1 + y_2 \\ y_3'(t) = 3y_1 + 6y_2 + 2y_3 \end{cases}$ $y(0) = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}$

94. $\begin{cases} y_1'(t) = -\frac{5}{2}y_1 + y_2 + y_3 \\ y_2'(t) = y_1 - \frac{5}{2}y_2 + y_3 \\ y_3'(t) = y_1 + y_2 - \frac{5}{2}y_3 \end{cases}$ $y(0) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

95. $\begin{cases} x_1'(t) = 3x_1 - x_2 - x_3 \\ x_2'(t) = x_1 + x_2 - x_3 + t \\ x_3'(t) = x_1 - x_2 + x_3 + 2e^{t} \end{cases}$ $X(0) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

96. $\begin{cases} x_1'(t) = x_1 + x_2 \quad + e^t \\ x_2'(t) = x_1 + x_2 \quad + e^{2t} \\ x_3'(t) = \qquad\qquad 3x_3 + te^{3t} \end{cases}$ $\quad X(0) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

Find the general solution of the system

97. $x'' + x = 3$; $\quad x(\pi) = 1, \quad x'(\pi) = 2$

98. $\begin{cases} x'' = x - y \\ y'' = x - y \end{cases}$ $\quad \begin{cases} x(3) = 5, & x'(3) = 2 \\ y(3) = 1, & y'(3) = -1 \end{cases}$

99. $\begin{cases} x'' = x - y \\ y'' = -x + y \end{cases}$ $\quad \begin{cases} x(0) = -1, & x'(0) = 0 \\ y(0) = 1, & y'(0) = 0 \end{cases}$

100. $\begin{cases} \dfrac{d^2x}{dt^2} = y; & x(0) = 3, & x'(0) = 1 \\ \dfrac{d^2y}{dt^2} = x; & y(0) = 1, & y'(0) = -1 \end{cases}$

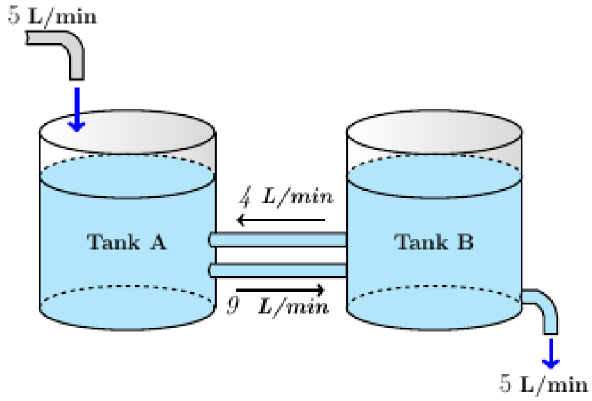
101. $\begin{cases} x'' + 5x - 2y = 0 \\ y'' + 2y - 2x = 3\sin 2t \end{cases}$ $\quad \begin{matrix} x(0) = x'(0) = 0 \\ y(0) = 1, \quad y'(0) = 0 \end{matrix}$

102. $\begin{cases} x'' = -2x' - 5y + 3 \\ y' = x' + 2y \end{cases}$ $\quad x(0) = 0, \ x'(0) = 0, \ y(0) = 1$

103. $\begin{cases} x'' = 2x' + 5y + 3 \\ y' = -x' - 2y \end{cases}$ $\quad x(0) = 0, \ x'(0) = 0, \ y(0) = 1$

104. Find the real and imaginary part of $z(t) = e^{2it}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$

105. Two tanks, each containing 360 *liters* of a salt solution. Pure water pours into tank *A* at a rate of 5 *L/min*. There are two pipes connecting tank *A* to tank *B*. The first pumps salt solution from tank *B* into tank *A* at a rate of 4 *L/min*. The second pumps salt solution from tank *A* into tank *B* at a rate of 9 *L/min*. Finally, there is a drain on tank *B* from which salt solution drains at a rate of 5 *L/min*. Thus, each tank maintains a constant volume of 360 *liters* of salt solution. Initially, there are 60 *kg* of salt present in tank *A*, but tank *B* contains pure water.



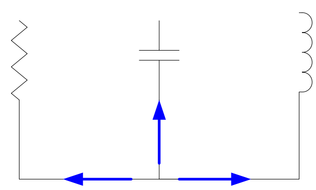
a) Set up, in matrix-vector form, an initial value problem that models the salt content in each tank over time.

b) Find the eigenvalues and eigenvectors of the coefficient matrix in part (*a*), then find the general solution in vector form. Find the solution that satisfies the initial conditions posed in part (*a*).

c) Plot each component of your solution in part (*b*) over a period of four time constants $\left[0,\ 4T_c\right]$. What is the eventual salt content in each tank? Give both a physical and a mathematical reason for your answer.

106. Consider the *RLC* parallel circuit below. Let *V* represent the voltage drop across the capacitor and *I* represent the current across the inductor that satisfied the system.

$$\begin{cases} V' = -\dfrac{V}{RC} - \dfrac{1}{C} \\ I' = \dfrac{V}{L} \end{cases}$$

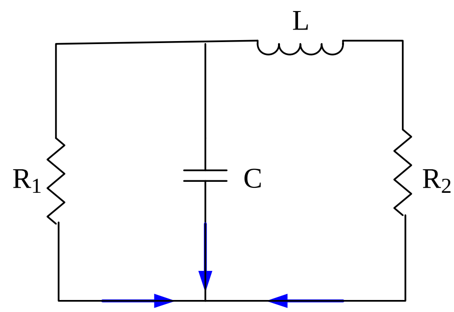


Suppose that the resistance is $R = \frac{1}{2}\Omega$, the capacitor is $C = 1$ *farad*, and the inductance is $L = \frac{1}{2}\ henry$. If the initial voltage across the capacitor is $V(0) = 10\ volts$ and there is no initial current across the inductor, solve the system to determine the voltage and current as a function of time. Plot the voltage and current as a function of time. Assume current flows in the directions indicated.

107. Show that the voltage V across the capacitor and the current I through the inductor satisfy the system

$$\begin{cases} I' = -\dfrac{R_1}{L}I + \dfrac{1}{L}V \\ V' = -\dfrac{1}{C}I - \dfrac{1}{R_2 C}V \end{cases}$$

Suppose that the capacitance is $C = 1$ *farad*, the inductance is $L = 1$ *henry*, the leftmost resistor has resistance $R_2 = 1\,\Omega$, and the rightmost resistor has resistance $R_1 = 5\,\Omega$.



If the initial voltage across the capacitor is 12 *volts* and the initial current through the inductor is zero, determine the voltage V across the capacitor and the current I through the inductor as functions of time. Plot the voltage and current as functions of time. Assume current flows in the directions indicated.