Section 4.3 – LU-Decompositions

The goal is to describe Gaussian elimination in the most useful way by looking at them closely, which are factorizations of a matrix.

> ***The factors are triangular matrices.***
>
> ***The factorization that comes from elimination is*** $A = LU$ **.**

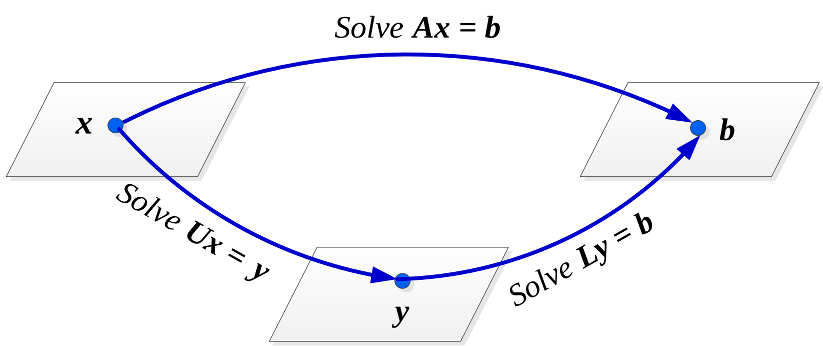
The Method of *LU*–Decomposition

Step 1: Rewrite the system $A\mathbf{x} = b$ as $LU\mathbf{x} = b$

Step 2: Define a new $n \times 1$ matrix $\boldsymbol{y}$ by $U\mathbf{x} = \boldsymbol{y}$

Step 3: Use $U\mathbf{x} = \boldsymbol{y}$ to rewrite $LU\mathbf{x} = b$ as $L\boldsymbol{y} = b$ and solve this system for $\boldsymbol{y}$.

Step 4: Substitute $\boldsymbol{y}$ in $U\mathbf{x} = \boldsymbol{y}$ and solve for $\boldsymbol{x}$.



Example

Given 2 by 2 matrix $A = \begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix}$

To make ***row*** 2 ***column*** 1 is ***zero*** then we need to subtract 3 times *row* 2 from *row* 2

$$\begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix} R_2 - 3R_1$$

That step is $E_{21} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$ in the forward direction such that:

$$E_{21}A = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix} = U$$

The return step from U to A is $L = E_{21}^{-1}$

Back from U to A: $\boxed{E_{21}^{-1}U = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix} = A}$

Therefore; $A = LU$

Example

What matrix L and U puts A into triangular form $A = LU$ where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} R_2 - \tfrac{1}{2}R_1 : \ell_{21} \quad \rightarrow \quad \begin{pmatrix} 2 & 1 & 0 \\ 0 & \tfrac{3}{2} & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & \tfrac{3}{2} & 1 \\ 0 & 1 & 2 \end{pmatrix} R_3 - \tfrac{2}{3}R_2 : \ell_{32} \quad \rightarrow \quad \begin{pmatrix} 2 & 1 & 0 \\ 0 & \tfrac{3}{2} & 1 \\ 0 & 0 & \tfrac{4}{3} \end{pmatrix} = U$$

The lower triangular L has all 1's on its diagonal. The multipliers ℓ_{ij} are below the diagonal of L

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \tfrac{1}{2} & 1 & 0 \\ 0 & \tfrac{2}{3} & 1 \end{pmatrix}$$

$$\overbrace{\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}}^{A} = \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ \tfrac{1}{2} & 1 & 0 \\ 0 & \tfrac{2}{3} & 1 \end{pmatrix}}^{L}\overbrace{\begin{pmatrix} 2 & 1 & 0 \\ 0 & \tfrac{3}{2} & 1 \\ 0 & 0 & \tfrac{4}{3} \end{pmatrix}}^{U}$$

❖ $\left(E_{32}E_{31}E_{21}\right)A=U$ *becomes* $A=\left(E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}\right)U$ *which is* $A=LU$

The inverses go in opposite order.

❖ $(A=LU)$ This is ***elimination without row exchanges***. The *upper triangular* ***U*** has the pivots on its diagonal. The *lower triangular* **L** has all 1's on its diagonal. ***The multipliers*** ℓ_{ij} ***are below the diagonal of L***.

One Square System = *Two* Triangular Systems

Factor: into L and U, by forward elimination on A.

Solve: forward on ***b*** using L, then back substitution using U.

Solve $Lc=b$ and then solve $Ux=c$

Example

Forward elimination on $Ax=b$ ends at $Ux=c$

$$\begin{aligned} x+2y&=5 \\ 4x+9y&=21 \end{aligned} \quad \textit{becomes} \quad \begin{aligned} x+2y&=5 \\ y&=1 \end{aligned}$$

Solution

The multiplier was 4. $\left(R_2-4R_1\right)$

The lower triangular system: $Lc=b$

$$\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}[c]=\begin{bmatrix} 5 \\ 21 \end{bmatrix} \quad \Rightarrow \quad c=\begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

The upper triangular system: $Ux=c$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}[x]=\begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad \Rightarrow \quad x=\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

To solve 1000 equations on a PC

❖ Elimination on A requires about $\frac{1}{3}n^3$ multiplications and $\frac{1}{3}n^3$ subtractions.

❖ Each right side needs n^2 multiplications and n^2 subtractions.

Exercises Section 4.3 – LU-Decompositions

1. What matrix E puts A into triangular form $EA=U$? Multiply by $E^{-1}=L$ to factor A into LU:

$$A=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$$

2. Solve $Lc=b$ to find $\boldsymbol{c}$. Then solve $Ux=c$ to find $\boldsymbol{x}$. What was A?

$$L=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad U=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad b=\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

3. Find L and U for the symmetric matrix

$$A=\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get $A=LU$ with four pivots

4. For which c is $A=LU$ impossible – with three pivots?

$$A=\begin{pmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

5. Find an LU-decomposition of the coefficient matrix, and then use to solve the system

a) $\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} -2 \\ -2 \end{bmatrix}$

b) $\begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} -10 \\ 19 \end{bmatrix}$

c) $\begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$

d) $\begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$

e) $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 5 \\ -1 \\ 3 \\ 7 \end{bmatrix}$