Exercise

Prove the identity $\cos(A+B)+\cos(A-B)=2\cos A\cos B$

Solution

$$\cos(A+B)+\cos(A-B)=\cos A\cos B-\sin A\sin B+\cos A\cos B+\sin A\sin B$$

$$=\cos A\cos B+\cos A\cos B$$

$$=2\cos A\cos B$$

Exercise

Prove the identity $\sec(A+B)=\dfrac{\cos(A-B)}{\cos^2 A-\sin^2 B}$

Solution

$$\sec(A+B)=\frac{1}{\cos(A+B)}$$

$$=\frac{1}{\cos A\cos B-\sin A\sin B}$$

$$=\frac{1}{\cos A\cos B-\sin A\sin B}\frac{\cos(A-B)}{\cos(A-B)}$$

$$=\frac{1}{\cos A\cos B-\sin A\sin B}\frac{\cos(A-B)}{\cos A\cos B+\sin A\sin B}$$

$$=\frac{\cos(A-B)}{\cos^2 A\cos^2 B-\sin^2 A\sin^2 B}$$

$$=\frac{\cos(A-B)}{\cos^2 A(1-\sin^2 B)-(1-\cos^2 A)\sin^2 B}$$

$$=\frac{\cos(A-B)}{\cos^2 A-\cos^2 A\sin^2 B-\sin^2 B+\cos^2 A\sin^2 B}$$

$$=\frac{\cos(A-B)}{\cos^2 A-\sin^2 B}$$

Exercise

Prove the identity $\quad \dfrac{\cos 4\alpha}{\sin\alpha}-\dfrac{\sin 4\alpha}{\cos\alpha}=\dfrac{\cos 5\alpha}{\sin\alpha\cos\alpha}$

Solution

$$\begin{aligned}\frac{\cos 4\alpha}{\sin\alpha}-\frac{\sin 4\alpha}{\cos\alpha}&=\frac{\cos 4\alpha\cos\alpha-\sin 4\alpha\sin\alpha}{\sin\alpha\cos\alpha}\\&=\frac{\cos(4\alpha+\alpha)}{\sin\alpha\cos\alpha}\\&=\frac{\cos 5\alpha}{\sin\alpha\cos\alpha}\end{aligned}$$

Exercise

Show that $\sin\left(x-\frac{\pi}{2}\right)=-\cos x$

Solution

$$\begin{aligned}\sin\left(x-\frac{\pi}{2}\right)&=\sin x\cos\frac{\pi}{2}-\cos x\sin\frac{\pi}{2}\\&=\sin x\cdot(0)-\cos x\cdot(1)\\&=-\cos x\end{aligned}$$

Exercise

Prove the identity $\quad \sin\left(\frac{\pi}{4}+x\right)+\sin\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$

Solution

$$\begin{aligned}\sin\left(\frac{\pi}{4}+x\right)+\sin\left(\frac{\pi}{4}-x\right)&=\sin\frac{\pi}{4}\cos x+\sin x\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\cos x-\sin x\cos\frac{\pi}{4}\\&=\sin\frac{\pi}{4}\cos x+\sin\frac{\pi}{4}\cos x\\&=2\sin\frac{\pi}{4}\cos x\\&=2\frac{\sqrt{2}}{2}\cos x\\&=\sqrt{2}\cos x\end{aligned}$$

Exercise

Prove the identity $\dfrac{\sin(A-B)}{\cos A\cos B}=\tan A-\tan B$

Solution

$$\begin{aligned}\frac{\sin(A-B)}{\cos A\cos B}&=\frac{\sin A\cos B-\sin B\cos A}{\cos A\cos B}\\&=\frac{\sin A\cos B}{\cos A\cos B}-\frac{\sin B\cos A}{\cos A\cos B}\\&=\frac{\sin A}{\cos A}-\frac{\sin B}{\cos B}\\&=\tan A-\tan B\end{aligned}$$

Exercise

Write the expression as a single trigonometric function $\sin 8x\cos x-\cos 8x\sin x$

Solution

$$\begin{aligned}\sin 8x\cos x-\cos 8x\sin x&=\sin(8x-x)\\&=\sin 7x\end{aligned}$$

Exercise

If $\sin A = \frac{4}{5}$ with A in QII, and $\cos B = -\frac{5}{13}$ with B in QIII, find $\sin(A+B)$, $\cos(A+B)$, and $\tan(A+B)$

Solution

$\cos A = -\frac{3}{5}$ $\quad$ $\tan A = \dfrac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$	$\sin B = -\frac{12}{13}$ $\quad$ $\tan B = \dfrac{-\frac{12}{13}}{-\frac{5}{13}} = \frac{12}{5}$
$\sin(A+B) = \sin A\cos B + \sin B\cos A$ $= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) + \left(-\frac{12}{13}\right)\left(-\frac{3}{5}\right)$ $= -\frac{20}{65} + \frac{36}{65}$ $= \frac{16}{65}$	$\cos(A+B) = \cos A\cos B - \sin A\sin B$ $= \left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)$ $= \frac{15}{65} + \frac{48}{65}$ $= \frac{63}{65}$
$\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$ $= \dfrac{\frac{16}{65}}{\frac{63}{65}}$ $= \frac{16}{63}$	$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$ $= \dfrac{-\frac{4}{3} + \frac{12}{5}}{1 - \left(-\frac{4}{3}\right)\left(\frac{12}{5}\right)}$ $= \dfrac{\frac{-20+36}{15}}{1 + \frac{48}{15}}$ $= \dfrac{\frac{16}{15}}{\frac{63}{15}}$ $= \frac{16}{63}$

Exercise

If $\sin A = \frac{1}{\sqrt{5}}$ with A in QI, and $\tan B = \frac{3}{4}$ with B in QI, find $\sin(A+B)$, $\cos(A+B)$, and $\tan(A+B)$

Solution

$\cos A = \sqrt{1-\sin^2 A} \quad A \in \text{ QI}$ $\cos A = \sqrt{1-\frac{1}{5}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$	$\sin B = \frac{3}{5}$ $\cos B = \frac{4}{5}$
$\sin(A+B) = \sin A\cos B + \sin B\cos A$ $= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{4}{5}\right) + \left(\frac{3}{5}\right)\left(\frac{2}{\sqrt{5}}\right)$ $= \frac{4}{5\sqrt{5}} + \frac{6}{5\sqrt{5}}$ $= \frac{10}{5\sqrt{5}}$ $= \frac{2}{\sqrt{5}}$	$\cos(A+B) = \cos A\cos B - \sin A\sin B$ $= \left(\frac{2}{\sqrt{5}}\right)\left(\frac{4}{5}\right) - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{3}{5}\right)$ $= \frac{8}{5\sqrt{5}} - \frac{3}{5\sqrt{5}}$ $= \frac{5}{5\sqrt{5}}$ $= \frac{1}{\sqrt{5}}$
$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$ $= \frac{\frac{2}{\sqrt{5}}}{\frac{1}{\sqrt{5}}}$ $= 2$	

Exercise

If $\sec A = \sqrt{5}$ with A in QI, and $\sec B = \sqrt{10}$ with B in QI, find $\sec(A+B)$

Solution

$$\sec(A+B) = \frac{1}{\cos(A+B)}$$

$$\sec A = \sqrt{5} \Rightarrow \cos A = \frac{1}{\sqrt{5}} \quad \sin A = \frac{2}{\sqrt{5}}$$

$$\sec B = \sqrt{10} \Rightarrow \cos B = \frac{1}{\sqrt{10}} \quad \sin B = \sqrt{1-\frac{1}{10}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{5}}$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$= \frac{1}{\sqrt{5}}\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{5}}\frac{3}{\sqrt{10}}$$

$$= \frac{1}{\sqrt{50}} - \frac{6}{\sqrt{50}}$$

$$=\frac{5}{\sqrt{50}}$$

$$=\frac{5}{5\sqrt{2}}$$

$$=\frac{1}{\sqrt{2}}$$

$$\boxed{\sec(A+B)=\frac{1}{\frac{1}{\sqrt{2}}}=\sqrt{2}}$$

Exercise

Prove the following equation is an identity: $\sin(x-y)-\sin(y-x)=2\sin x\cos y-2\cos x\sin y$

Solution

$$\begin{aligned}\sin(x-y)-\sin(y-x)&=\sin x\cos y-\sin y\cos x-(\sin y\cos x-\sin x\cos y)\\&=\sin x\cos y-\sin y\cos x-\sin y\cos x+\sin x\cos y\\&=2\sin x\cos y-2\sin y\cos x\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cos(x-y)+\cos(y-x)=2\cos x\cos y+2\sin x\sin y$

Solution

$$\begin{aligned}\cos(x-y)+\cos(y-x)&=\cos x\cos y+\sin x\sin y+\cos y\cos x+\sin y\sin x\\&=2\cos x\cos y+2\sin x\sin y\end{aligned}$$

Exercise

Prove the following equation is an identity: $\tan(x+y)\tan(x-y)=\dfrac{\tan^2 x-\tan^2 y}{1-\tan^2 x\tan^2 y}$

Solution

$$\tan(x+y)\tan(x-y)=\frac{\tan x+\tan y}{1-\tan x\tan y}\frac{\tan x-\tan y}{1+\tan x\tan y}\qquad (a+b)(a-b)=a^2-b^2$$

$$=\frac{\tan^2 x+\tan^2 y}{1-\tan x^2\tan^2 y}$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)} = \dfrac{1-\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$

Solution

$$\frac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)} = \frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\cos\beta+\sin\beta\cos\alpha}$$

$$= \frac{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\beta\cos\alpha}{\cos\alpha\cos\beta}}$$

$$= \frac{1+\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$$

Exercise

Prove the following equation is an identity: $\sec(x+y) = \dfrac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\sin^2 y}$

Solution

$$\sec(x+y) = \frac{1}{\cos(x+y)}\frac{\cos(x-y)}{\cos(x-y)}$$

$$= \frac{\cos x\cos y+\sin x\sin y}{(\cos x\cos y-\sin x\sin y)(\cos x\cos y+\sin x\sin y)}$$

$$= \frac{\cos x\cos y+\sin x\sin y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$$

$$= \frac{\cos x\cos y+\sin x\sin y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$$

$$= \frac{\cos x\cos y+\sin x\sin y}{\cos^2 x\left(1-\sin^2 y\right)-\left(1-\cos^2 x\right)\sin^2 y}$$

$$= \frac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\cos^2 x\sin^2 y-\sin^2 y+\cos^2 x\sin^2 y}$$

$$= \frac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\sin^2 y}$$

Exercise

Prove the following equation is an identity: $\csc(x-y)=\dfrac{\sin x\cos y+\cos x\sin y}{\sin^2 x-\sin^2 y}$

Solution

$$\csc(x-y)=\frac{1}{\sin(x-y)}\frac{\sin x\cos y+\cos x\sin y}{\sin x\cos y+\cos x\sin y}$$

$$=\frac{\sin x\cos y+\cos x\sin y}{(\sin x\cos y-\cos x\sin y)(\sin x\cos y+\cos x\sin y)}$$

$$=\frac{\sin x\cos y+\cos x\sin y}{\sin^2 x\cos^2 y-\cos^2 x\sin^2 y}$$

$$=\frac{\sin x\cos y+\cos x\sin y}{\sin^2 x\left(1-\sin^2 y\right)-\left(1-\sin^2 x\right)\sin^2 y}$$

$$=\frac{\sin x\cos y+\cos x\sin y}{\sin^2 x-\sin^2 x\sin^2 y-\sin^2 y+\sin^2 x\sin^2 y}$$

$$=\frac{\sin x\cos y+\cos x\sin y}{\sin^2 x-\sin^2 y}$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos(x+y)}{\cos(x-y)}=\dfrac{\cot y-\tan x}{\cot y+\tan x}$

Solution

$$\frac{\cos(x+y)}{\cos(x-y)}=\frac{\cos x\cos y-\sin x\sin y}{\cos x\cos y+\sin x\sin y}$$

$$=\frac{\frac{\cos x\cos y}{\cos x\sin y}-\frac{\sin x\sin y}{\cos x\sin y}}{\frac{\cos x\cos y}{\cos x\sin y}+\frac{\sin x\sin y}{\cos x\sin y}}$$

$$=\frac{\cot y-\tan x}{\cot y+\tan x}$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin(x+y)}{\sin(x-y)}=\dfrac{\cot y+\cot x}{\cot y-\cot x}$

Solution

$$\frac{\sin(x+y)}{\sin(x-y)}=\frac{\sin x\cos y+\sin y\cos x}{\sin x\cos y-\sin y\cos x}$$

$$=\frac{\frac{\sin x\cos y}{\sin x\sin y}+\frac{\sin y\cos x}{\sin x\sin y}}{\frac{\sin x\cos y}{\sin x\sin y}-\frac{\sin y\cos x}{\sin x\sin y}}$$

$$=\frac{\cot y+\cot x}{\cot y-\cot x}$$

Exercise

Prove the following equation is an identity: $\frac{\cos(x+y)}{\cos(x-y)}=\frac{\cot y-\tan x}{\cot y+\tan x}$

Solution

$$\frac{\cos(x+y)}{\cos(x-y)}=\frac{\cos x\cos y-\sin x\sin y}{\cos x\cos y+\sin x\sin y}$$

$$=\frac{\frac{\cos x\cos y}{\cos x\sin y}-\frac{\sin x\sin y}{\cos x\sin y}}{\frac{\cos x\cos y}{\cos x\sin y}+\frac{\sin x\sin y}{\cos x\sin y}}$$

$$=\frac{\cot y-\tan x}{\cot y+\tan x}$$

Exercise

Prove the following equation is an identity: $\frac{\sin(x-y)}{\sin x\cos y}=1-\cot x\tan y$

Solution

$$\frac{\sin(x-y)}{\sin x\cos y}=\frac{\sin x\cos y-\cos x\sin y}{\sin x\cos y}$$

$$=\frac{\sin x\cos y}{\sin x\cos y}-\frac{\cos x\sin y}{\sin x\cos y}$$

$$=1-\cot x\tan y$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin(x-y)}{\sin x \sin y} = \cot y - \cot x$

Solution

$$\begin{aligned}
\frac{\sin(x-y)}{\sin x \sin y} &= \frac{\sin x \cos y - \cos x \sin y}{\sin x \sin y} \\
&= \frac{\sin x \cos y}{\sin x \sin y} - \frac{\cos x \sin y}{\sin x \sin y} \\
&= \frac{\cos y}{\sin y} - \frac{\cos x}{\sin x} \\
&= \cot y - \cot x
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos(x+y)}{\cos x \sin y} = \cot y - \tan x$

Solution

$$\begin{aligned}
\frac{\cos(x+y)}{\cos x \sin y} &= \frac{\cos x \cos y - \sin x \sin y}{\cos x \sin y} \\
&= \frac{\cos x \cos y}{\cos x \sin y} - \frac{\sin x \sin y}{\cos x \sin y} \\
&= \frac{\cos y}{\sin y} - \frac{\sin x}{\cos x} \\
&= \cot y - \tan x
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\tan(x+y) + \tan(x-y) = \dfrac{2\tan x}{\cos^2 y\left(1 - \tan^2 x \tan^2 y\right)}$

Solution

$$\begin{aligned}
\tan(x+y) + \tan(x-y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} + \frac{\tan x - \tan y}{1 + \tan x \tan y} \\
&= \frac{(\tan x + \tan y)(1 + \tan x \tan y) + (\tan x - \tan y)(1 - \tan x \tan y)}{(1 - \tan x \tan y)(1 + \tan x \tan y)} \\
&= \frac{\tan x + \tan^2 x \tan y + \tan y + \tan x \tan^2 y + \tan x - \tan^2 x \tan y - \tan y + \tan x \tan^2 y}{\left(1 - \tan^2 x \tan^2 y\right)}
\end{aligned}$$

$$= \frac{2\tan x + 2\tan x \tan^2 y}{\left(1 - \tan^2 x \tan^2 y\right)}$$

$$= \frac{2\tan x\left(1 + \tan^2 y\right)}{\left(1 - \tan^2 x \tan^2 y\right)}$$

$$= \frac{2\tan x \sec^2 y}{\left(1 - \tan^2 x \tan^2 y\right)}$$

$$= \frac{2\tan x}{\cos^2 y\left(1 - \tan^2 x \tan^2 y\right)}$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin(x+y)}{\cos(x-y)} = \dfrac{1+\cot x \tan y}{\cot x + \tan y}$

Solution

$$\frac{\sin(x+y)}{\cos(x-y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y + \sin x \sin y}$$

$$= \frac{\frac{\sin x \cos y}{\sin x \cos y} + \frac{\cos x \sin y}{\sin x \cos y}}{\frac{\cos x \cos y}{\sin x \cos y} + \frac{\sin x \sin y}{\sin x \cos y}}$$

$$= \frac{1+\cot x \tan y}{\cot x + \tan y}$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos(x-y)}{\cos(x+y)} = \dfrac{1+\tan x \tan y}{1-\tan x \tan y}$

Solution

$$\frac{\cos(x-y)}{\cos(x+y)} = \frac{\cos x \cos y + \sin x \sin y}{\cos x \cos y - \sin x \sin y}$$

$$= \frac{\frac{\cos x \cos y}{\cos x \cos y} + \frac{\sin x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} - \frac{\sin x \sin y}{\cos x \cos y}}$$

$$= \frac{1+\tan x \tan y}{1-\tan x \tan y}$$