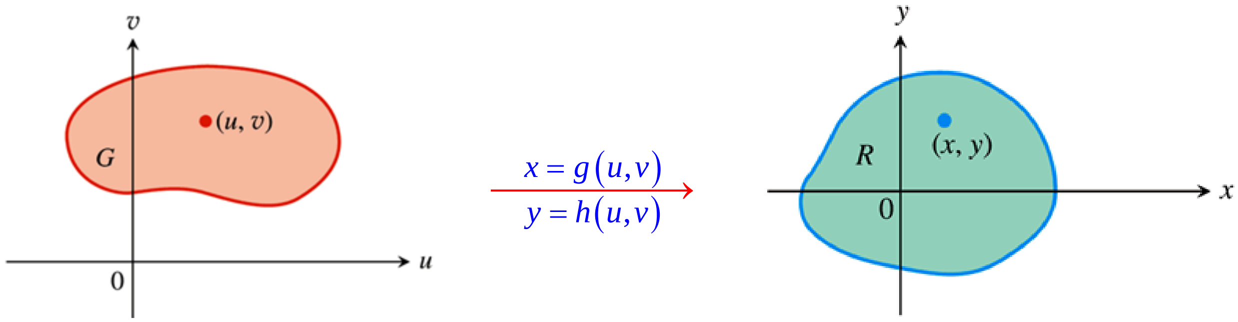
Section 3.7 – Change of variables in Multiple Integrals

Substitution in Double Integrals

Suppose that a region G in the uv-plane is transformed one-to-one into the region R in the xy-plane by equations of the form

$$x = g(u,v), \quad y = h(u,v)$$



Cartesian uv-plane

Cartesian xy - plane

R is the image of G under the transformation, and G the ***preimage*** of R.

$$\iint_R f(x,y)\,dx\,dy = \iint_G f\big(g(u,v),h(u,v)\big)\left|J(u,v)\right|du\,dv$$

Definition

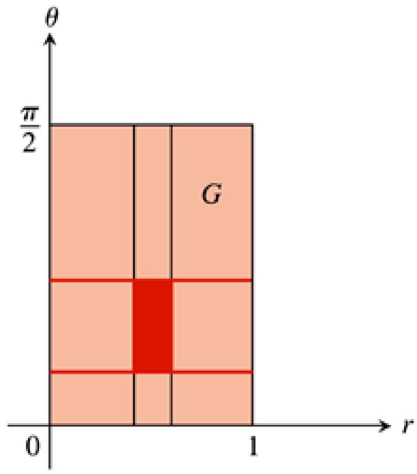
The ***Jacobian determinant*** or ***Jacobian*** of the coordinate transformation $x = g(u,v), \quad y = h(u,v)$ is

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Example

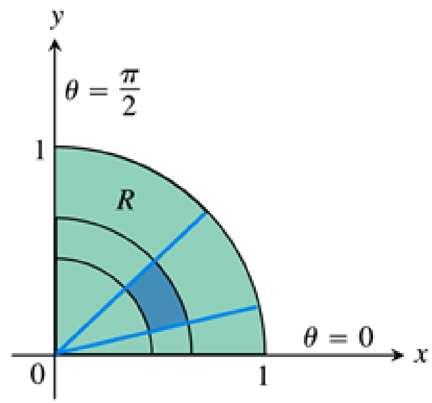
Find the Jacobian for the polar coordinate transformation $x = r\cos\theta, \quad y = r\sin\theta$, write the Cartesian integral $\iint\limits_R f(x,y)\,dxdy$ as a polar integral.

Solution



Cartesian $r\theta$–plane

$$\xrightarrow[y = r\sin\theta]{x = r\cos\theta}$$



Cartesian xy –plane

$x = r\cos\theta, \quad y = r\sin\theta$ transform the rectangle G: $0 \le r \le 1,\ 0 \le \theta \le 2\pi$, into the quarter circle R bounded by $x^2 + y^2 = 1$ in QI.

$$J(r,\ \theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r\left(\cos^2\theta + \sin^2\theta\right)$$

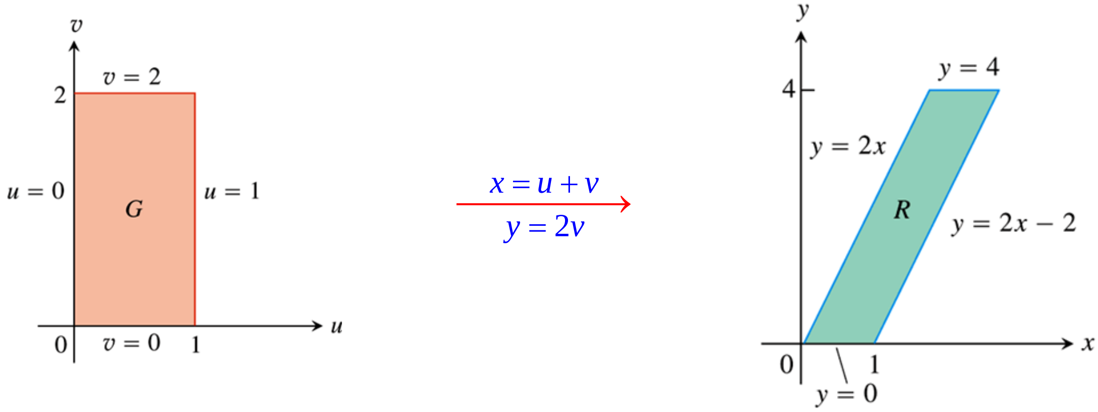
$$\underline{= r|}$$

Example

Evaluate $\displaystyle\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\,dxdy$ by applying the transformation $u=\dfrac{2x-y}{2},\quad v=\dfrac{y}{2}$ and integrating over an appropriate region in the *uv*-plane.

Solution

$$\to \underline{y=2v}\rfloor,\quad 2u=2x-y \Rightarrow \lfloor \underline{x=\frac{2u+y}{2}=\frac{2u+2v}{2}=u+v}\rfloor$$



xy*-eqns for the boundary of *R	**Corresponding *uv*-eqns. for the boundary of *G***	**Simplified *uv*-eqns.**
$x=\dfrac{y}{2}$	$u+v=\dfrac{2v}{2}=v$	$u=0$
$x=\dfrac{y}{2}+1$	$u+v=\dfrac{2v}{2}+1=v+1$	$u=1$
$y=0$	$2v=0$	$v=0$
$y=4$	$2v=4$	$v=2$

$$J(u,v)=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\dfrac{\partial}{\partial u}(u+v) & \dfrac{\partial}{\partial v}(u+v)\\[2mm] \dfrac{\partial}{\partial u}(2v) & \dfrac{\partial}{\partial v}(2v)\end{vmatrix}$$

$$=\begin{vmatrix}1 & 1\\ 0 & 2\end{vmatrix}$$

$$\underline{=2}\rfloor$$

$$\int_0^4 \int_{y/2}^{(y/2)+1}\left(x-\frac{y}{2}\right)dxdy=\int_0^{v=2}\int_{u=0}^{u=1} u\left|J(u,v)\right|dudv$$

$$=\int_0^{v=2}\int_{u=0}^{u=1}(u)(2)\,dudv$$

$$= \int_0^{v=2} u^2 \Big|_0^1 \, dv$$

$$= \int_0^{v=2} dv$$

$$= v \Big|_0^2$$

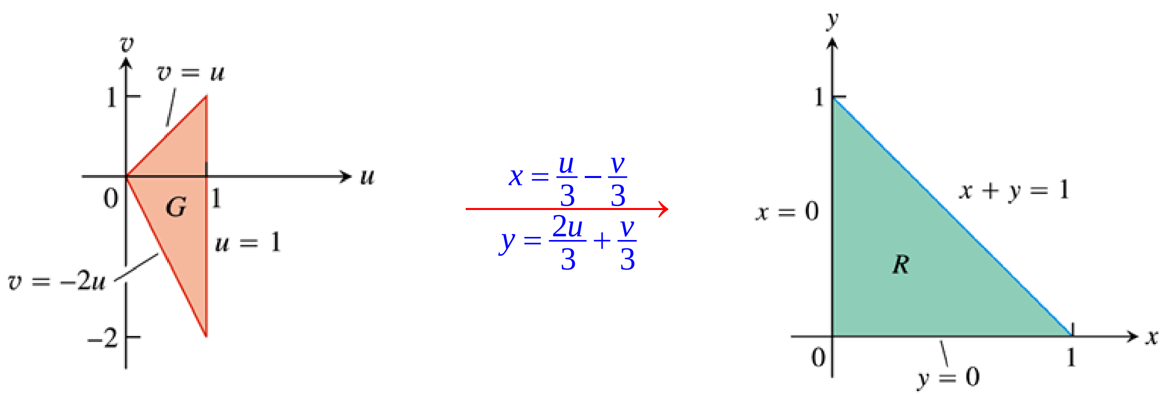
$$\underline{= 2}$$

Example

Evaluate $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2 \, dydx$

Solution

$$\begin{cases} u = x + y \\ v = y - 2x \end{cases} \rightarrow \begin{cases} x = \dfrac{u}{3} - \dfrac{v}{3} \\ y = \dfrac{2u}{3} + \dfrac{v}{3} \end{cases}$$



xy*-eqns for the boundary of *R	**Corresponding *uv*-eqns. for the boundary of *G***	**Simplified *uv*-eqns.**
$x + y = 1$	$\left(\frac{u}{3} - \frac{v}{3}\right) + \left(\frac{2u}{3} + \frac{v}{3}\right) = 1$	$u = 1$
$x = 0$	$\frac{u}{3} - \frac{v}{3} = 0$	$v = u$
$y = 0$	$\frac{2u}{3} + \frac{v}{3} = 0$	$v = -2u$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \underline{\frac{1}{3}}$$

$$\int_0^1\int_0^{1-x}\sqrt{x+y}\,(y-2x)^2\,dydx = \int_{u=0}^{u=1}\int_{v=-2u}^{v=u} u^{1/2}v^2\left|J(u,v)\right|dvdu$$

$$= \int_0^1\int_{-2u}^{u} u^{1/2}v^2\left(\frac{1}{3}\right)dvdu$$

$$= \int_0^1 u^{1/2}\left[\frac{1}{9}v^3\right]_{-2u}^{u}du$$

$$= \frac{1}{9}\int_0^1 u^{1/2}\left(u^3+8u^3\right)du$$

$$= \int_0^1 u^{7/2}du$$

$$= \frac{2}{9}u^{9/2}\Big|_0^1$$

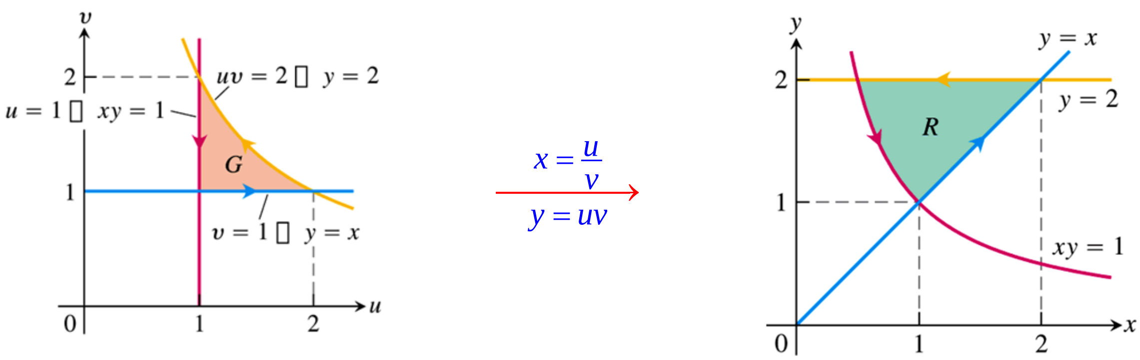
$$= \frac{2}{9}$$

Example

Evaluate $\displaystyle\int_1^2\int_{1/y}^{y}\sqrt{\frac{y}{x}}\,e^{\sqrt{xy}}\,dxdy$

Solution

$$\begin{cases} u=\sqrt{xy} \\ v=\sqrt{\dfrac{y}{x}} \end{cases} \to \begin{cases} u^2 = xy \\ v^2=\dfrac{y}{x} \end{cases} \quad\Rightarrow\quad x=\frac{u}{v},\quad y=uv$$



$$J(u,v)=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix} = \begin{vmatrix}\dfrac{1}{v} & -\dfrac{u}{v^2}\\ v & u\end{vmatrix} = \frac{2u}{v}$$

xy-eqns for the boundary of ***R***	Corresponding *uv*-eqns. for the boundary of ***G***	Simplified *uv*-eqns.
$x = y$	$\frac{u}{v} = uv$	$v = 1$
$x = \frac{1}{y}$	$\frac{u}{v} = \frac{1}{uv}$	$u = 1$
$y = 1$	$uv = 1$	
$y = 2$	$uv = 2$	$u = 2 \quad v = \frac{2}{u}$

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx dy = \int_1^2 \int_1^{2/u} 2u\ e^{u} dv du$$

$$= 2\int_1^2 u\ e^{u} \left[v\right]_1^{2/u} du$$

$$= 2\int_1^2 u\ e^{u} \left(\frac{2}{u} - 1\right) du$$

$$= 2\int_1^2 (2-u) e^{u} du$$

$$= 2\left[(2-u+1)e^{u}\right]_1^2$$

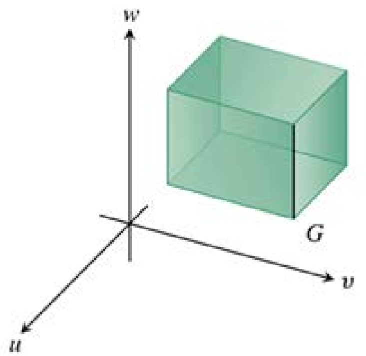
$$= 2\left[(1)e^2 - 2e\right]$$

$$\underline{= 2e(e-2)}$$

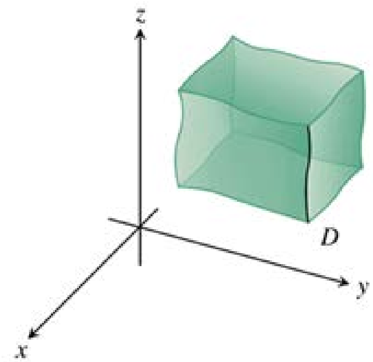
		e^u
(+)	$2-u$	e^u
(−)	-1	e^u
	0	

Substitutions in Triple Integrals

$$x = g(u,v,w), \quad y = h(u,v,w), \quad z = k(u,v,w)$$



$$\xrightarrow[\;]{\substack{x=g(u,v,w)\\ y=h(u,v,w)\\ z=k(u,v,w)}}$$



Cartesian *uvw* - plane

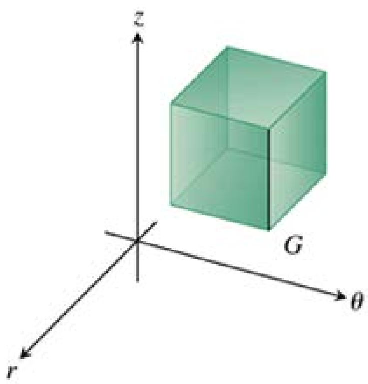
Cartesian *xyz* - plane

$$\iiint\limits_R f(x,y)\,dxdy = \iiint\limits_R H(u,v,w)\left|J(u,v,w)\right| dudvdw$$

The ***Jacobian determinant*** is

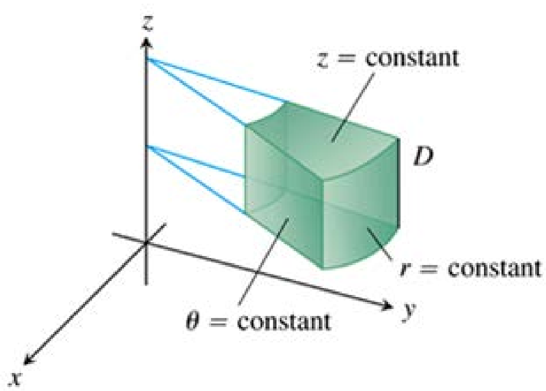
$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

Cube with sides parallel to the axes



$$\xrightarrow[\;]{\substack{x=r\cos\theta\\ y=r\sin\theta\\ z=z}}$$

Cube with sides parallel to the axes



Cartesian $r\theta z$ - plane

Cartesian *xyz* - plane

$$J(r,\theta,z) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta \underline{= r}$$

For spherical coordinates, ρ, ϕ, and θ take the place of u, v, and w. The transformation from Cartesian $\rho\phi\theta$-space to Cartesian xyz –space is given by

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

The Jacobian of the transformation

$$J(\rho,\phi,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \theta} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \rho^2\cos^2\phi\sin\phi\cos^2\theta + \rho^2\sin^3\phi\sin^2\theta + \rho^2\sin\phi\cos^2\phi\sin^2\theta + \rho^2\sin^3\phi\cos^2\theta$$

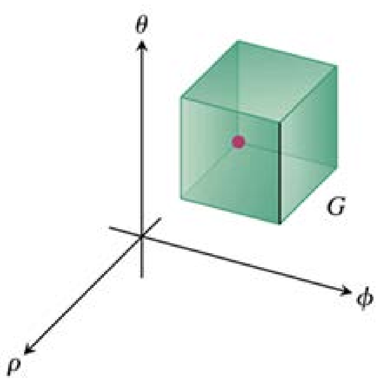
$$= \rho^2\cos^2\phi\sin\phi\left(\cos^2\theta + \sin^2\theta\right) + \rho^2\sin^3\phi\left(\sin^2\theta + \cos^2\theta\right)$$

$$= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi$$

$$= \rho^2\sin\phi\left(\cos^2\phi + \sin^2\phi\right)$$

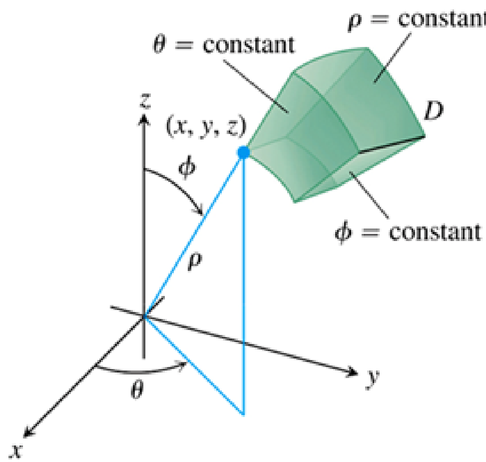
$$\boxed{= \rho^2\sin\phi}$$

$$\iiint\limits_D F(x,y,z)\,dxdydz = \iiint\limits_G H(\rho,\phi,\theta)\left|\rho^2\sin\phi\right| d\rho d\phi d\theta$$



Cartesian $r\theta z$ - plane

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$



Cartesian xyz - plane

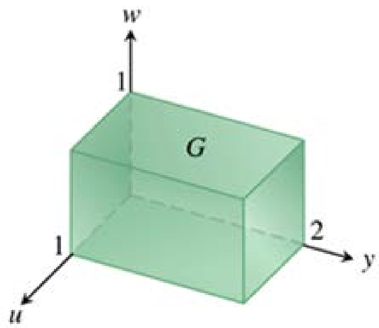
Example

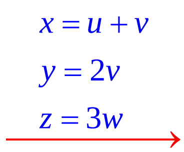
Evaluate $\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dxdydz$ by applying the transformation

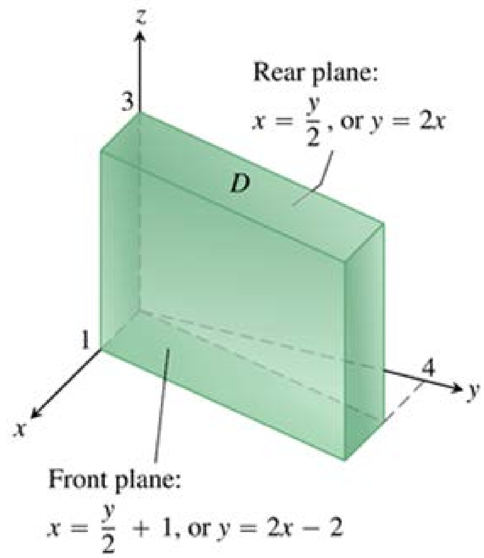
$u = \frac{2x-y}{2}, \quad v = \frac{y}{2}, \quad w = \frac{z}{3}$ and integrating over an appropriate region in the *uvw*-plane.

Solution

$$\rightarrow \begin{cases} u = \frac{2x-y}{2} \rightarrow x = u + \frac{y}{2} = u + v \\ \qquad v = \frac{y}{2} \rightarrow y = 2v \\ \qquad w = \frac{z}{3} \rightarrow z = 3w \end{cases}$$







xyz-eqns* for the boundary of *D	**Corresponding *uvw- eqns*. for the boundary of *G***	**Simplified *uvw- eqns*.**
$x = \frac{y}{2}$	$u + v = v$	$u = 0$
$x = \frac{y}{2} + 1$	$u + v = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

$$J(u,v,w)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$$\underline{=6}$$

$$\int_0^3\int_0^4\int_{x=y/2}^{x=(y/2)+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dxdydz=\int_0^1\int_0^2\int_0^1(u+w)\left|J(u,v,w)\right|\,dudvdw$$

$$=6\int_0^1\int_0^2\int_0^1(u+w)\,dudvdw$$

$$=6\int_0^1\int_0^2\left[\frac{u^2}{2}+wu\right]_0^1 dvdw$$

$$=6\int_0^1\int_0^2\left(\frac{1}{2}+w\right)dvdw$$

$$=6\int_0^1\left[\frac{1}{2}v+wv\right]_0^2 dw$$

$$=6\int_0^1(1+2w)\,dw$$

$$=6\left[w+w^2\right]_0^1$$

$$=6(1+1)$$

$$\underline{=12}$$

Exercises *Section* 3.7 – Change of Variables in Multiple Integrals

1. *a*) Solve the system $u = x - y, \quad v = 2x + y$ for *x* and *y* in terms of *u* and *v*. Then find the value of the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$

b) Find the image under the transformation $u = x - y, \;\; v = 2x + y$ of the triangular region with vertices (0, 0), (1, 1), and (1, –2) in the *xy*-plane. Sketch the transformed region in the *uv*-plane.

2. Let *R* be the region in the first quadrant of the *xy*-plane bounded by the hyperbolas $xy = 1, \; xy = 9$ and the lines $y = x, \; y = 4x$. Use the transformation $x = \frac{u}{v}, \; y = uv$ with $u > 0, \; and \; v > 0$ to rewrite

$$\iint\limits_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy$$

As an integral over an appropriate region *G* in the uv-plane. Then evaluate the *uv*-integral over *G*.

3. The area πab of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ can be found by integrating the function $f(x,y) = 1$ over the region bounded by the ellipse in the *xy*-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x = au, \; y = bv$ and evaluate the transformed integral over the disk *G*: $u^2 + v^2 \le 1$ in the *uv*-plane. Find the area this way.

4. Use the transformation $x = u + \frac{1}{2}v, \;\; y = v$ to evaluate the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3 (2x - y) e^{(2x-y)^2} dxdy$$

By first writing it as an integral over a region *G* in the *uv*-plane.

5. Use the transformation $x = \frac{u}{v}, \;\; y = uv$ to evaluate the integral

$$\int_1^2 \int_{1/y}^{y} \left(x^2 + y^2\right) dxdy + \int_2^4 \int_{y/4}^{4/y} \left(x^2 + y^2\right) dxdy$$

6. Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ of the transformation

a) $x = u\cos v, \quad y = u\sin v$

b) $x = u\sin v, \quad y = u\cos v$

7. Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ of the transformation

a) $x = u\cos v, \quad y = u\sin v, \quad z = w$

b) $x = 2u-1, \quad y = 3v-4, \quad z = \frac{1}{2}(w-4)$

8. Evaluate the appropriate determinant to show that the Jacobian of the transformation from Cartesian $\rho\phi\theta$–space to Cartesian xyz–space is $\rho^2 \sin\phi$

9. How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.

10. Find the volume of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

(***Hint***: Let $x = au,\ y = bv,\ and\ z = cw$. Then find the volume of an appropriate region in *uvw*-space)

11. Use the transformation $x = u^2 - v^2,\ \ y = 2uv$ to evaluate the integral

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2+y^2}\ dydx$$

(***Hint***: Show that the image of the triangular region ***G*** with vertices (0, 0), (1, 0), (1, 1) in the *uv*-plane is the region of integration ***R*** in the *xy*-plane defined by the limits of integration.)