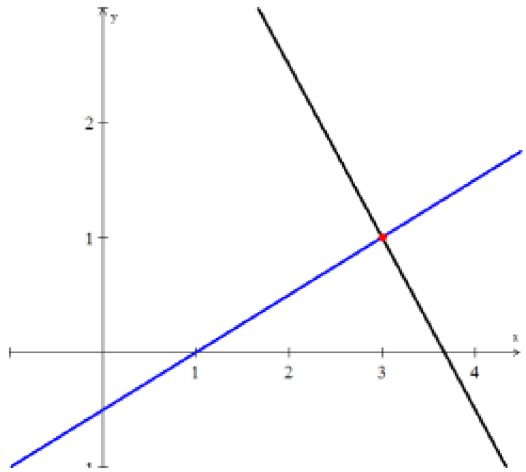
Section 1.2 – Gaussian Elimination

Elimination produces an ***upper triangular system***.

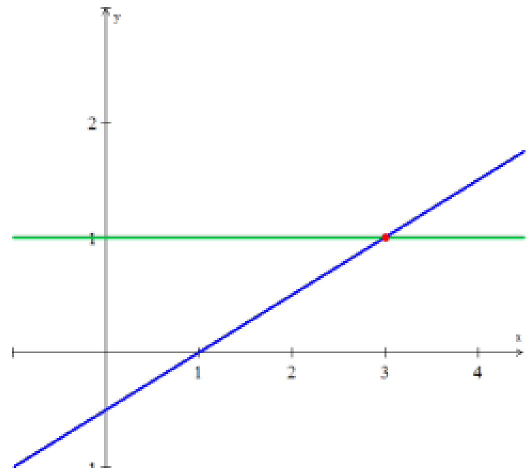
$$\begin{cases} x-2y=1 \\ 3x+2y=11 \end{cases} \Rightarrow \begin{cases} x-2y=1 & \textit{Mutliply by } 3 \\ \quad\;\; 8y=8 & \textit{and subtract} \end{cases}$$

The equation $8y=8$ *reveals* $y=1$

This process is called ***back substitution***.



Before elimination



After elimination

Definitions

Pivot: first nonzero in the row that does the elimination

Multiplier: (entry to eliminate) divide by pivot

$$\begin{array}{lll} 4x-8y=4 & \textit{Multiply equation } 1 \textit{ by } \frac{3}{4} & 4x-8y=4 \\ 3x+2y=11 & \textit{Subtract from equation } 2 & \qquad 8y=8 \end{array}$$

The first pivot is 4 (the coefficient of *x*) and the multiplier is $l=\frac{3}{4}$

The pivots are on the diagonal of the triangle after elimination.

Definition

The operations are the elementary reduction operations, or row operations, or Gaussian operations. They are swapping, multiplying by a scalar or rescaling, and pivoting.

Reduced Row Echelon Form

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

Use the Gaussian elimination method to solve the system

$$\begin{aligned} 3x + \ \ y + 2z &= 31 \\ x + \ \ y + 2z &= 19 \\ x + 3y + 2z &= 25 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 3 & 1 & 2 & 31 \\ 1 & 3 & 2 & 25 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{rrrr} 3 & 1 & 2 & 31 \\ -3 & -3 & -6 & -57 \\ \hline 0 & -2 & -4 & -26 \end{array} \qquad \begin{array}{rrrr} 1 & 2 & 2 & 25 \\ -1 & -1 & -2 & -19 \\ \hline 0 & 2 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -2 & -4 & -26 \\ 0 & 2 & 0 & 6 \end{array}\right] -\tfrac{1}{2}R_2 \qquad 0 \quad 1 \quad 2 \quad 13$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 2 & 0 & 6 \end{array}\right] R_3 - 2R_2 \qquad \begin{array}{rrrr} 0 & 2 & 0 & 6 \\ 0 & -2 & -4 & -26 \\ \hline 0 & 0 & -4 & -20 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -4 & -20 \end{array}\right] -\tfrac{1}{4}R_3 \qquad 0 \quad 0 \quad 1 \quad 5$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array}\right] \Rightarrow \begin{array}{rl} x + y + 2z = 19 & (3) \\ y + 2z = 13 & (2) \\ z = 5 & (1) \end{array}$$

$(2) \Rightarrow y = 13 - 2z = 13 - 2(5) = 3$

$(3) \Rightarrow x = 19 - y - 2z = 19 - 3 - 10 = 6$

$\Rightarrow$ **(6, 3, 5)**

Example

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
5x_3 + 10x_4 \qquad + 15x_6 &= 5 \\
2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}$$

Solution

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \quad \textit{Adding } (-2) \textit{ times the 1st row to the 2nd} \\ \\ R_4 - 2R_1 \quad \textit{Adding } (-2) \textit{ times the 1st row to the 4th} \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] -R_2$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} R_3 - 5R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right] \tfrac{1}{6}R_4 \quad \textit{then interchanging row3 and row4}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] R_2 - 3R_3$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{cases} x_1 + 3x_2 \quad + 4x_4 + 2x_5 = 0 \\ x_3 + 2x_4 = 0 \\ + x_6 = \frac{1}{3} \end{cases}$$

The general solution of the system: $\underline{x_6 = \frac{1}{3}, \quad x_3 = -2x_4, \quad x_1 = -3x_2 - 4x_4 - 2x_5}$

Example

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$\begin{aligned} 2x+8y-z+w&=0\\ 4x+16y-3z-w&=-10\\ -2x+4y-z+3w&=-6\\ -6x+2y+5z+w&=3 \end{aligned}$$

Solution

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0\\ 4 & 16 & -3 & -1 & -10\\ -2 & 4 & -1 & 3 & -6\\ -6 & 2 & 5 & 1 & 3 \end{array}\right] \begin{array}{l} \\ R_2-2R_1\\ R_3+R_1\\ R_4+3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0\\ 0 & 12 & -2 & 4 & -6\\ 0 & 0 & -1 & -3 & -10\\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4-\frac{13}{6}R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0\\ 0 & 0 & -1 & -3 & -10\\ 0 & 12 & -2 & 4 & -6\\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \qquad \textit{Interchange } R_2 \textit{ and } R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0\\ 0 & 12 & -2 & 4 & -6\\ 0 & 0 & -1 & -3 & -10\\ 0 & 0 & \frac{19}{3} & -\frac{14}{3} & 16 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4+\frac{19}{3}R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0\\ 0 & 12 & -2 & 4 & -6\\ 0 & 0 & -1 & -3 & -10\\ 0 & 0 & 0 & -\frac{71}{3} & -\frac{142}{3} \end{array}\right] \begin{array}{lcll} 2x+8y-z+w=0 & \to & 2x=-8y+z-w=6 & \Rightarrow \boxed{x=3}\\ 12y-2z+4w=-6 & \to & 12y=2z-4w-6=-6 & \Rightarrow \boxed{y=-\frac{1}{2}}\\ -z-3w=-10 & \to & z=10-3w=4 & \\ -\frac{71}{3}w=-\frac{142}{3} & \to & \boxed{w=2} & \end{array}$$

Solution: $\left(3,\ -\frac{1}{2},\ 4,\ 2\right)$

Theorem: Free Variable Theorem for Homogeneous Systems

If a *homogeneous linear* system has ***n*** unknowns, and if the reduced row echelon form of its augmented matrix has ***r*** nonzero rows, then the system has $n - r$ free variables.

Theorem

A *homogeneous linear* system with more unknowns than equations has ***infinitely many*** *unknowns*.

Breakdown Elimination

Permanent failure with no solution

$$\begin{array}{rcr} x-2y=1 & \textit{Subtract } 3 \textit{ times} & x-2y=1 \\ 3x-6y=11 & \textit{eqn. } 1 \textit{ from eqn. } 2 & 0y=8 \end{array}$$

The last equation $0y=8$; therefore, there is *no* solution.
This system has no second pivot, since no zero allowed as a pivot.

Permanent failure with infinitely many solutions

$$\begin{array}{rcr} x-2y=1 & \textit{Subtract } 3 \textit{ times} & x-2y=1 \\ 3x-6y=3 & \textit{eqn. } 1 \textit{ from eqn. } 2 & 0y=0 \end{array}$$

Every *y* satisfies $0y=0$. There is only one equation $x-2y=1$.
There are ***unique infinitely*** many solutions.

Three Equations in Three Unknowns

To understand Gaussian elimination, you have to go beyond 2 by 2 systems.

Consider the system equations: $\begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases}$

$$\begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases} \quad \begin{array}{c} \textit{subtract } 2 \textit{ times eqn.}1 \\ \textit{from eqn.}2 \\ \ \end{array} \quad \begin{array}{r} 2x+4y-2z=2 \\ y+z=4 \\ -2x-3y+7z=10 \end{array}$$

$$\begin{cases} 2x+4y-2z=2 \\ y+z=4 \\ -2x-3y+7z=10 \end{cases} \quad \begin{array}{c} \textit{Add eqn.}1 \\ \ \\ \textit{and eqn.}3 \end{array} \quad \begin{array}{r} 2x+4y-2z=2 \\ y+z=4 \\ y+5z=12 \end{array}$$

$$\begin{cases} 2x + 4y - 2z = 2 & & 2x + 4y - 2z = 2 & \Rightarrow & x = 1 - 2y + z = -1 \\ y + z = 4 & \textit{Subtract eqn.2} & y + z = 4 & \Rightarrow & y = 4 - z = 2 \\ y + 5z = 12 & \textit{from eqn.3} & 4z = 8 & \Rightarrow & \boxed{z = 2} \end{cases}$$

The solution is $(-1,\ 2,\ 2)$

Definition

A square matrix is nonsingular if it is the matrix of coefficient of a homogeneous system, with a unique solution. It is singular otherwise, that is, if it is the matrix of coefficients of a homogeneous system, with infinitely many solutions.

Exercises *Section* 1.2 – Gaussian Elimination

1. When elimination is applied to the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix}$

a) What are the first and second pivots?

b) What is the multiplier l_{21} in the first step (l_{21} times row 1 is subtracted from row 2)?

c) What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3?

d) What is the multiplier $l_{31} = 0$, subtracting 0 times row 1 from row 3?

2. Use elimination to reach upper triangular matrices *U*. Solve by back substitution or explain why this impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the $-x$ in equation (3).

$$\begin{cases} x+y+z=7 \\ x+y-z=5 \\ x-y+z=3 \end{cases} \qquad \begin{cases} x+y+z=7 \\ x+y-z=5 \\ -x-y+z=3 \end{cases}$$

3. For which numbers *a* does the elimination break down (1) permanently (2) temporarily

$$\begin{aligned} ax+3y&=-3 \\ 4x+6y&=6 \end{aligned}$$

Solve for *x* and *y* after fixing the second breakdown by a row change.

4. Find the pivots and the solution for these four equations:

$$\begin{aligned} 2x+y \qquad &= 0 \\ x+2y+z \quad &= 0 \\ y+2z+t &= 0 \\ z+2t &= 5 \end{aligned}$$

5. Look for a matrix that has row sums 4 and 8, and column sums 2 and *s*.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{matrix} a+b=4 & a+c=2 \\ c+d=8 & b+d=s \end{matrix}$$

The four equations are solvable only if *s* = ____. Then find two different matrices that have the correct row and column sums.

6. Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of *A* is a ________ of the first two rows. Find a third equation that can't be solved together with $x+y+z=0$ and $x-2y-z=1$

7. Solve the linear system by Gauss-Jordan elimination.

$$a)\ \begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases} \qquad b)\ \begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \qquad\qquad - 3w = -3 \end{cases}$$

$$c)\ \begin{cases} x + 2y + z = 8 \\ -x + 3y - 2z = 1 \\ 3x + 4y - 7z = 10 \end{cases} \qquad d)\ \begin{cases} 2u - 3v + w - x + y = 0 \\ 4u - 6v + 2w - 3x - y = -5 \\ -2u + 3v - 2w + 2x - y = 3 \end{cases}$$

8. Solve the given linear system by any method

$$a)\ \begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 \qquad = 0 \\ \qquad x_2 + x_3 = 0 \end{cases} \qquad d)\ \begin{cases} 4y + z = 20 \\ 2x - 2y + z = 0 \\ x \qquad + z = 5 \\ x + y - z = 10 \end{cases}$$

$$b)\ \begin{cases} 2x + 2y + 4z \qquad = 0 \\ \quad - y - 3z + w = 0 \\ 3x + y + z + 2w = 0 \\ x + 3y - 2z - 2w = 0 \end{cases} \qquad e)\ \begin{cases} 6x_3 + 2x_4 - 4x_5 - 8x_6 = 8 \\ 3x_3 + x_4 - 2x_5 - 4x_6 = 4 \\ 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 + x_6 = 2 \\ 6x_1 - 9x_2 + 11x_4 - 19x_5 + 3x_6 = 1 \end{cases}$$

$$c)\ \begin{cases} 2x \qquad + z + w = 5 \\ \quad y \qquad - w = -1 \\ 3x \qquad - z - w = 0 \\ 4x + y + 2z + w = 9 \end{cases}$$

9. Add 3 times the second row to the first of $\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$

10. Solve the system using Gaussian elimination $\begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases}$

11. For what value(s) of k, if any, does the system $\begin{cases} x + y - z = 1 \\ 2x + 3y + kz = 3 \\ x + ky + 3z = 2 \end{cases}$ have

a) A unique solution?

b) Infinitely many solutions?
c) No solution?

12. Choose a coefficient b that makes the system singular.

$$\begin{cases} 3x + 4y = 16 \\ 4x + by = g \end{cases}$$

Then choose a right-hand side g that makes it solvable.
Find 2 solutions in that singular case.

13. This system us not linear, in some sense,

$$\begin{cases} 2\sin\alpha - \cos\beta + 3\tan\theta = 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\theta = 10 \\ 6\sin\alpha - 3\cos\beta + \tan\theta = 9 \end{cases}$$

Does the system have a solution?