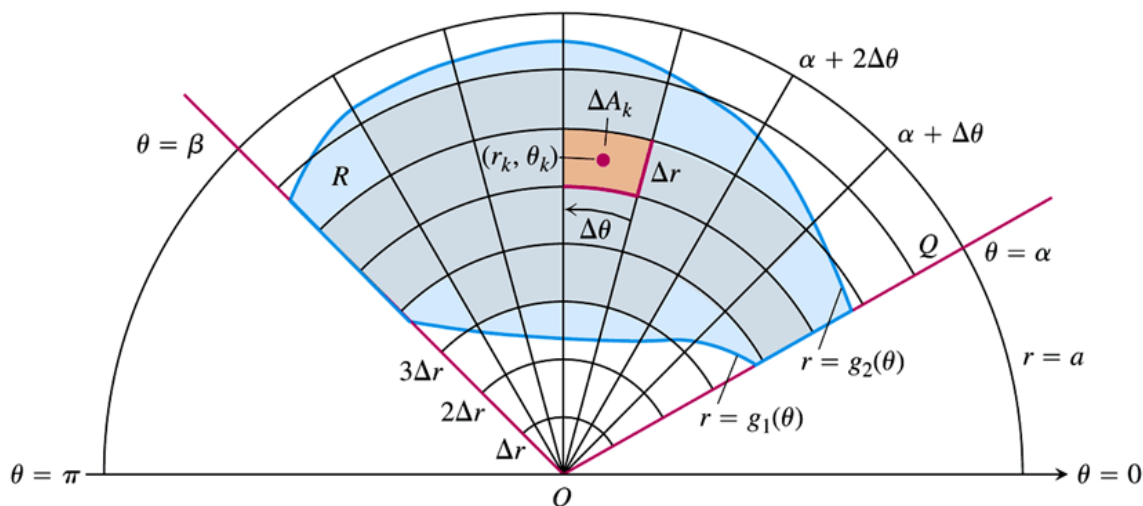


## Section 3.3 – Double Integrals in Polar Coordinates

### Integrals in Polar Coordinates



$$S_n = \sum_{k=1}^n f(r_k, \theta_k) \Delta A_k$$

If  $f$  is continuous throughout  $R$ , this sum will approach a limit as  $\Delta r$  and  $\Delta \theta$  go to zero. The limit is called the double integral of  $f$  over  $R$ .

$$\lim_{n \rightarrow \infty} S_n = \iint_R f(r, \theta) dA$$

However, the area of a wedge-shaped sector of a circle having radius  $r$  and angle  $\theta$  is

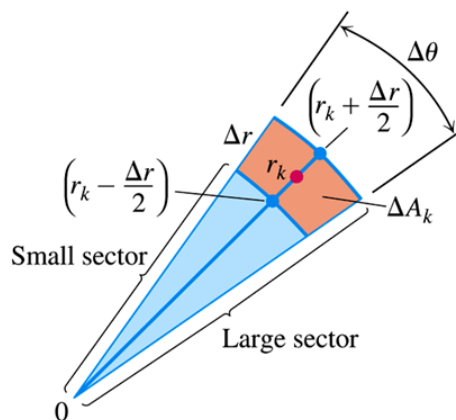
$$A = \frac{1}{2} \theta \cdot r^2$$

Inner radius:  $\frac{1}{2} \left( r_k - \frac{\Delta r}{2} \right)^2 \cdot \Delta \theta$

outer radius:  $\frac{1}{2} \left( r_k + \frac{\Delta r}{2} \right)^2 \cdot \Delta \theta$

$$\Delta A_k = \left( \text{area of large sector} \right) - \left( \text{area of small sector} \right)$$

Leads to the formula:  $\Delta A_k = r_k \Delta r \Delta \theta$

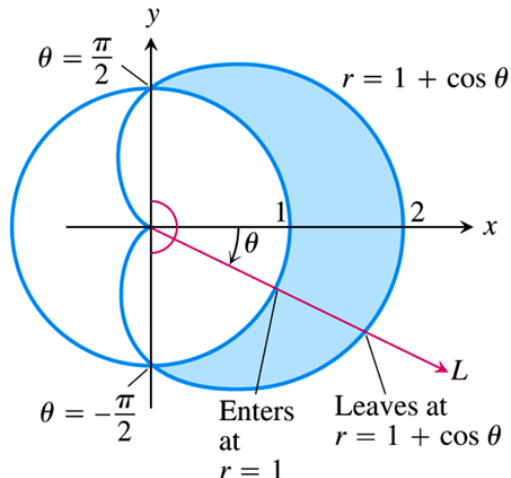


### Example

Find the limits of integration for integrating  $f(r, \theta)$  over the region  $R$  that lies inside the cardioid  $r = 1 + \cos \theta$  and outside the circle  $r = 1$ .

### Solution

The sketch of the region:



From the graph, we can find the  $r$  - limits of integration. A typical ray from the origin enters  $R$  where  $r = 1$  and leaves where  $r = 1 + \cos \theta$

$\theta$  - limits of integration: The rays from the origin that intersect  $R$  run from  $\theta = -\frac{\pi}{2}$  to  $\theta = \frac{\pi}{2}$ . The integral is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^{1+\cos \theta} f(r, \theta) r \, dr \, d\theta$$

### Area in Polar Coordinates

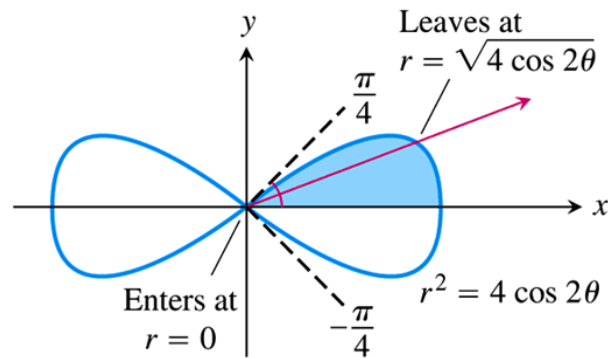
The area of a closed and bounded region  $R$  in the polar coordinate plane is

$$A = \iint_R r \, dr \, d\theta$$

### Example

Find the area enclosed by the lemniscate  $r^2 = 4\cos 2\theta$

### Solution



From the graph, we can determine the lemniscate limits of integration, and the total area is 4 times the first-quadrant portion, since it has a form of symmetry.

$$\begin{aligned} A &= 4 \int_0^{\pi/4} \int_0^{\sqrt{4\cos 2\theta}} r \, dr \, d\theta \\ &= 4 \int_0^{\pi/4} \left. \frac{1}{2} r^2 \right|_0^{\sqrt{4\cos 2\theta}} d\theta \\ &= 4 \int_0^{\pi/4} (2\cos 2\theta) d\theta \\ &= 4 \int_0^{\pi/4} \cos 2\theta \, d(2\theta) \\ &= 4 \sin 2\theta \Big|_0^{\pi/4} \\ &= 4 \sin \frac{\pi}{2} \\ &= \underline{4 \text{ unit}^2} \end{aligned}$$

## Changing Cartesian Integrals into Polar Integrals

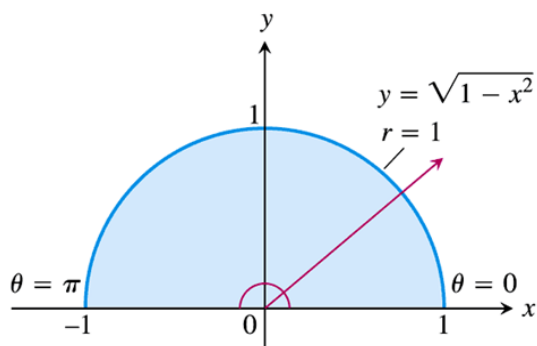
$$\iint_R f(x, y) dx dy = \iint_G f(r \cos \theta, r \sin \theta) \color{red}{r} dr d\theta$$

### Example

Evaluate  $\iint_R e^{x^2+y^2} dy dx$

Where  $R$  is the semicircular region bounded by the  $x$ -axis and the curve  $y = \sqrt{1-x^2}$

### Solution

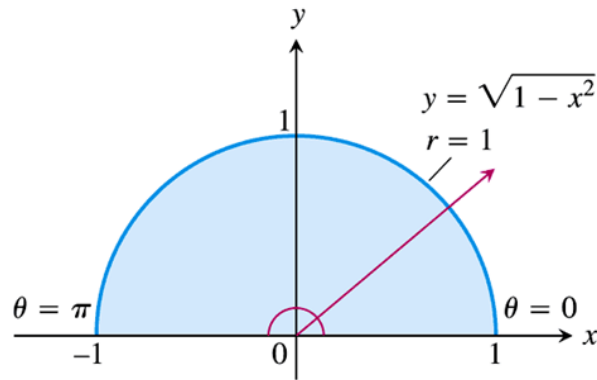


$$\begin{aligned} \iint_R e^{x^2+y^2} dy dx &= \int_0^\pi \int_0^1 e^{r^2} r dr d\theta & d(r^2) &= 2r dr \\ &= \frac{1}{2} \int_0^\pi d\theta \int_0^1 e^{r^2} d(r^2) \\ &= \frac{1}{2} \theta \bigg|_0^\pi e^{r^2} \bigg|_0^1 \\ &= \frac{1}{2} \int_0^\pi (e-1) d\theta \\ &= \underline{\underline{\frac{\pi}{2}(e-1)}} \end{aligned}$$

### Example

Evaluate the integral  $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx$

### Solution



Since:  $0 \leq x \leq 1 \rightarrow$  interior of  $x^2 + y^2 = 1$  and in  $QI$

Let:  $r^2 = x^2 + y^2$  with  $0 \leq r \leq 1$

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx &= \int_0^{\pi/2} \int_0^1 (r^2) r dr d\theta \\ &= \int_0^{\pi/2} d\theta \int_0^1 r^3 dr \\ &= \theta \left|_0^{\pi/2} \frac{1}{4} r^4 \right|_0^1 \\ &= \frac{\pi}{8} \end{aligned}$$

○ Or we can use the integral table to solve it

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx = \int_0^1 \left[ x^2 \sqrt{1-x^2} + \frac{1}{3} (1-x^2)^{3/2} \right] dx$$

### Example

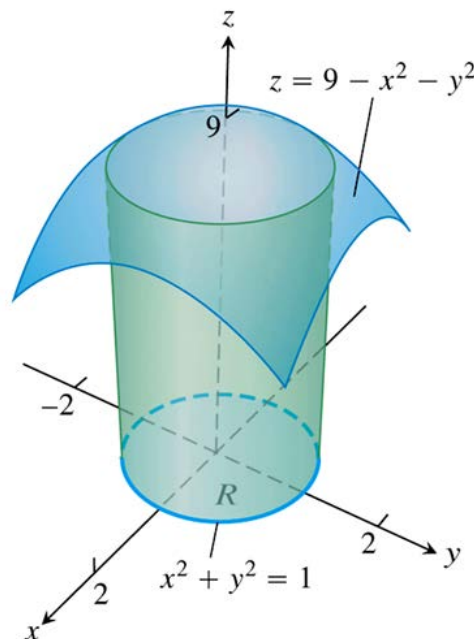
Find the volume of the solid region bounded above by the paraboloid  $z = 9 - x^2 - y^2$  and below by the unit circle in the  $xy$ -plane.

### Solution

The region of integration  $R$  is the unit circle:

$$x^2 + y^2 = 1 \rightarrow r = 1, \quad 0 \leq \theta \leq 2\pi$$

$$\begin{aligned} \text{Volume} &= \int_0^{2\pi} \int_0^1 (9 - r^2) r \, dr \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^1 (9r - r^3) \, dr \\ &= 2\pi \left( \frac{9}{2} r^2 - \frac{1}{4} r^4 \right) \bigg|_0^1 \\ &= 2\pi \left( \frac{9}{2} - \frac{1}{4} \right) \\ &= \frac{17\pi}{2} \text{ unit}^3 \end{aligned}$$



### Example

Using the polar integration, find the area of the region  $R$  in the  $xy$ -plane enclosed by the circle  $x^2 + y^2 = 4$ , above the line  $y = 1$ , and below the line  $y = \sqrt{3}x$ .

### Solution

The  $y = \sqrt{3}x$  has a slope of  $\sqrt{3} = \tan \theta \Rightarrow \theta = \frac{\pi}{3}$

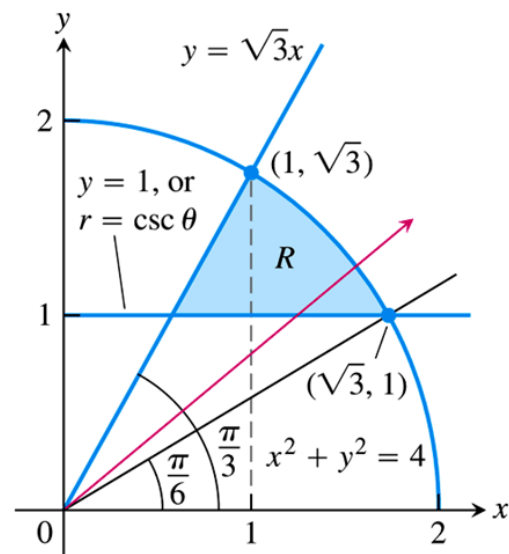
Line  $y = 1$  intersects  $x^2 + y^2 = 4$

when  $x^2 + 1 = 4 \rightarrow x = \sqrt{3}$ .

A line from origin to  $(\sqrt{3}, 1)$  has a slope of

$$\frac{1}{\sqrt{3}} = \tan \theta \rightarrow \theta = \frac{\pi}{6}$$

$$\therefore \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$$



The polar coordinate  $r$  varies from the horizontal line  $y = 1$  to the circle  $x^2 + y^2 = 4$ .

Substituting  $r \sin \theta$  for  $y$ :

$$y = 1 \rightarrow r \sin \theta = 1$$

$$r = \frac{1}{\sin \theta} = \csc \theta$$

The radius of the circle is 2.

$$\therefore \csc \theta \leq r \leq 2$$

$$\begin{aligned} \text{Area} &= \int_{\pi/6}^{\pi/3} \int_{\csc \theta}^2 r \, dr \, d\theta \\ &= \int_{\pi/6}^{\pi/3} \left. \frac{1}{2} r^2 \right|_{\csc \theta}^2 d\theta \\ &= \frac{1}{2} \int_{\pi/6}^{\pi/3} (4 - \csc^2 \theta) d\theta \\ &= \frac{1}{2} (4\theta + \cot \theta) \Big|_{\pi/6}^{\pi/3} \\ &= \frac{1}{2} \left[ \frac{4\pi}{3} + \frac{1}{\sqrt{3}} - \left( \frac{4\pi}{6} + \sqrt{3} \right) \right] \\ &= \frac{1}{2} \left( \frac{2\pi}{3} + \frac{\sqrt{3}}{3} - \sqrt{3} \right) \\ &= \frac{1}{2} \left( \frac{2\pi - 2\sqrt{3}}{3} \right) \\ &= \frac{\pi - \sqrt{3}}{3} \text{ unit}^2 \end{aligned}$$

### Example

Evaluate the double integral:  $\iint e^{-x^2-y^2} dA$

In the first quadrant and bounded by the circle  $x^2 + y^2 = a^2$  and the coordinate axes.

### Solution

$$x^2 + y^2 = r^2$$

$$0 \leq r \leq a$$

$$\text{In QI: } 0 \leq \theta \leq \frac{\pi}{2}$$

$$\begin{aligned}
\iint e^{-x^2-y^2} dA &= \int_0^{\frac{\pi}{2}} \int_0^a e^{-r^2} r \, dr d\theta \\
&= -\frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta \int_0^a e^{-r^2} d(-r^2) \\
&= -\frac{1}{2} \theta \left|_0^{\frac{\pi}{2}} e^{-r^2} \right|_0^a \\
&= -\frac{1}{2} \left( \frac{\pi}{2} \right) \left( e^{-a^2} - 1 \right) \\
&= \frac{\pi}{4} \left( 1 - e^{-a^2} \right)
\end{aligned}$$



## Exercises Section 3.3 – Double Integrals in Polar Coordinates

(1 – 16) Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

1.  $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} dy dx$

2.  $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) dx dy$

3.  $\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy dx$

4.  $\int_0^6 \int_0^y x dx dy$

5.  $\int_{-1}^0 \int_{-\sqrt{1-x^2}}^0 \frac{2}{1+\sqrt{x^2+y^2}} dy dx$

6.  $\int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2+y^2}} dx dy$

7.  $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1) dx dy$

8.  $\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2 + y^2)^2} dy dx$

9.  $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2 dy dx}{(1+x^2+y^2)^2}$

10.  $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1) dx dy$

11.  $\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2} dx dy$

12.  $\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2} dy dx$

13.  $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2 + y^2)^{3/2} dy dx$

14.  $\int_{-4}^4 \int_0^{\sqrt{16-y^2}} (16-x^2-y^2) dx dy$

15.  $\int_0^{\frac{\pi}{4}} \int_0^{\sec \theta} r^3 dr d\theta$

16.  $\int_0^{\frac{\pi}{2}} \int_1^\infty \frac{\cos \theta}{r^3} r dr d\theta$

17. Find the area of the region cut from the first quadrant by the curve  $r = 2(2 - \sin 2\theta)^{1/2}$

18. Find the area of the region lies inside the cardioid  $r = 1 + \cos \theta$  and outside the circle  $r = 1$

19. Find the area enclosed by one leaf of the rose  $r = 12 \cos 3\theta$

20. Find the area of the region common to the interiors of the cardioids  $r = 1 + \cos \theta$  and  $r = 1 - \cos \theta$

21. Find the area of the region bounded by all leaves of the rose  $r = 3 \cos 2\theta$

22. Find the area of the region inside both the circles  $r = 2$  and  $r = 4 \cos \theta$

23. Find the area of the region that lies inside both the cardioids  $r = 2 - 2 \cos \theta$  and  $r = 2 + 2 \cos \theta$
24. Find the area of the annular region  $\{(r, \theta): 1 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$
25. Find the area of the region bounded by the cardioid  $r = 2(1 - \sin \theta)$
26. Find the area of the region bounded by all leaves of the rose  $r = 2 \cos 3\theta$
27. Find the area of the region inside both the cardioid  $r = 1 - \cos \theta$  and the circle  $r = 1$
28. Find the area of the region inside both the cardioid  $r = 1 + \sin \theta$  and the cardioid  $r = 1 + \cos \theta$
29. Find the area of the region bounded by the spiral  $r = 2\theta$ , for  $0 \leq \theta \leq \pi$ , and the  $x$ -axis.
30. Find the area of the region inside the limaçon  $r = 1 + \frac{1}{2} \cos \theta$
31. Find the area of the region bounded by  $r = 2 \sin 2\theta$
32. Find the area of the region bounded by  $r^2 = 2 \sin 2\theta$
33. Find the area of the region outside the circle  $r = 1$  and inside the rose  $r = 2 \sin 3\theta$  in  $QI$
34. Find the area of the region outside the circle  $r = \frac{1}{2}$  and inside the circle  $r = 1 + \cos \theta$
35. Integrate  $f(x, y) = \frac{\ln(x^2 + y^2)}{\sqrt{x^2 + y^2}}$  over the region  $1 \leq x^2 + y^2 \leq e$
36. The region enclosed by the lemniscates  $r^2 = 2 \cos 2\theta$  is the base of a solid right cylinder whose top is bounded by the sphere  $z = \sqrt{2 - r^2}$ . Find the cylinder's volume.
37. Evaluate  $\iint_R (x + y) dA$ ;  $R$  is the disk bounded by circle  $r = 4 \sin \theta$
38. Find the volume of the solid bounded above by the paraboloid  $z = 2 - x^2 - y^2$  and below by the plane  $z = 1$
39. Find the volume of the solid bounded above by the paraboloid  $z = 8 - x^2 - 3y^2$  and below by the hyperbolic paraboloid  $z = x^2 - y^2$
- (40 – 51) Evaluate the integral over  $R$  using polar coordinates
40.  $\iint_R (x^2 + y^2) dA$ ;  $R = \{(r, \theta): 0 \leq r \leq 4, 0 \leq \theta \leq 2\pi\}$

$$41. \iint_R 2xy dA; \quad R = \left\{ (r, \theta) : 1 \leq r \leq 3, \quad 0 \leq \theta \leq \frac{\pi}{2} \right\}$$

$$42. \iint_R 2xy \, dA; \quad R = \left\{ (x, y) : x^2 + y^2 \leq 9, \quad y \geq 0 \right\}$$

$$43. \iint_R \frac{dA}{1 + x^2 + y^2}; \quad R = \left\{ (r, \theta) : 1 \leq r \leq 2, \quad 0 \leq \theta \leq \pi \right\}$$

$$44. \iint_R \frac{dA}{\sqrt{16 - x^2 - y^2}}; \quad R = \left\{ (x, y) : x^2 + y^2 \leq 4, \quad y \geq 0 \right\}$$

$$45. \iint_R \frac{dA}{\sqrt{16 - x^2 - y^2}}; \quad R = \left\{ (x, y) : x^2 + y^2 \leq 4, \quad x, y \geq 0 \right\}$$

$$46. \iint_R e^{-x^2 - y^2} dA; \quad R = \left\{ (x, y) : x^2 + y^2 \leq 9 \right\}$$

$$47. \iint_R \sqrt{x^2 + y^2} \, dA; \quad R = \left\{ (x, y) : y \leq x \leq 1, \quad 0 \leq y \leq 1 \right\}$$

$$48. \iint_R \sqrt{x^2 + y^2} \, dA; \quad R = \left\{ (x, y) : 1 \leq x^2 + y^2 \leq 2 \right\}$$

$$49. \iint_R \frac{dA}{(x^2 + y^2)^{5/2}}; \quad R = \left\{ (r, \theta) : 1 \leq r \leq \infty, \quad 0 \leq \theta \leq 2\pi \right\}$$

$$50. \iint_R e^{-x^2 - y^2} dA; \quad R = \left\{ (r, \theta) : 0 \leq r \leq \infty, \quad 0 \leq \theta \leq \frac{\pi}{2} \right\}$$

$$51. \iint_R \frac{dA}{(1 + x^2 + y^2)^2}; \quad R \in QI$$

52. Which bowl holds more water if it is filled to a depth of four units?

a) The paraboloid  $z = x^2 + y^2$ , for  $0 \leq z \leq 4$

b) The cone  $z = \sqrt{x^2 + y^2}$ , for  $0 \leq z \leq 4$

c) The hyperboloid  $z = \sqrt{1 + x^2 + y^2}$ , for  $1 \leq z \leq 5$

d) To what weight (above the bottom of the bowl) must the cone and paraboloid bowls be filled to hold the same volume of water as the hyperboloid bowl filled to a depth of 4 units ( $1 \leq z \leq 5$ )

53. Consider the surface  $z = x^2 - y^2$

a) Find the region in the  $xy$ -plane in polar coordinates for which  $z \geq 0$ .

b) Let  $R = \left\{ (r, \theta) : 0 \leq r \leq a, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \right\}$ , which is a sector of a circle of radius  $a$ . Find the volume of the region below the hyperbolic paraboloid and above the region  $R$ .

54. A cake is shaped like a hemisphere of radius 4 with its base on the  $xy$ -plane. A wedge of the cake is removed by making two slices from the center of the cake outward, perpendicular to the  $xy$ -plane and separated by an angle of  $\varphi$ .

a) Use a double integral to find the volume of the slice for  $\varphi = \frac{\pi}{4}$ .

b) Suppose the cake is sliced by a plane perpendicular to the  $xy$ -plane at  $x = a > 0$ . Let  $D$  be the smaller of the two pieces produced. For what value of  $a$  is the volume of  $D$  equal to the volume in part (a)?

55. Suppose the density of a thin plate represented by the region  $R$  is  $\rho(r, \theta)$  (in units of mass per

area). The mass of the plate is  $\iint_R \rho(r, \theta) dA$ . Find the mass of the thin half annulus

$R = \left\{ (r, \theta) : 1 \leq r \leq 4, 0 \leq \theta \leq \pi \right\}$  with a density  $\rho(r, \theta) = 4 + r \sin \theta$

56. An important integral in statistics associated with the normal distribution is  $I = \int_{-\infty}^{\infty} e^{-x^2} dx$ . It is evaluated in the following steps.

a) Assume that 
$$I = \left( \int_{-\infty}^{\infty} e^{-x^2} dx \right) \left( \int_{-\infty}^{\infty} e^{-y^2} dy \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2 - y^2} dx dy$$

Where we have chosen the variables of integration to be  $x$  and  $y$  and then written the product as an iterated integral. Evaluate this integral in polar coordinates and show that  $I = \sqrt{\pi}$ . Why is the solution  $I = -\sqrt{\pi}$  rejected?

b) Evaluate  $\int_0^{\infty} e^{-x^2} dx$ ,  $\int_0^{\infty} x e^{-x^2} dx$ , and  $\int_0^{\infty} x^2 e^{-x^2} dx$ .

57. For what values of  $p$  does the integral  $\iint_R \frac{k}{(x^2 + y^2)^p} dA$  exist in the following cases?

a)  $R = \{(r, \theta): 1 \leq r \leq \infty, 0 \leq \theta \leq 2\pi\}$

b)  $R = \{(r, \theta): 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$

58. Consider the integral  $\iint_R \frac{k}{(1 + x^2 + y^2)^2} dA$  where  $R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq a\}$

a) Evaluate  $I$  for  $a = 1$ .

b) Evaluate  $I$  for arbitrary  $a > 0$ .

c) Let  $a \rightarrow \infty$  in part (b) to find  $I$  over the infinite strip  $R = \{(x, y): 0 \leq x \leq 1, 0 \leq y \leq \infty\}$

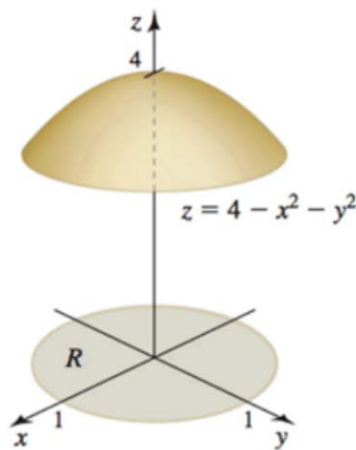
59. In polar coordinates an equation of an ellipse with eccentricity  $0 < e < 1$  and semimajor axis  $a$  is

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta}$$

a) Write the integral that gives the area of the ellipse.

b) Show that the area of an ellipse is  $\pi ab$ , where  $b^2 = a^2(1 - e^2)$

(60 – 63) Find the volume of the solid below the paraboloid  $z = 4 - x^2 - y^2$  and above



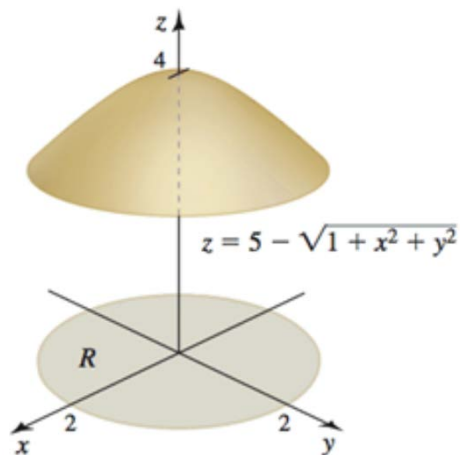
60.  $R = \{(r, \theta): 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$

61.  $R = \{(r, \theta): 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$

62.  $R = \{(r, \theta): 1 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$

63.  $R = \{(r, \theta): 1 \leq r \leq 2, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$

(64 – 67) Find the volume of the solid below the hyperboloid  $z = 5 - \sqrt{1 + x^2 + y^2}$  and above



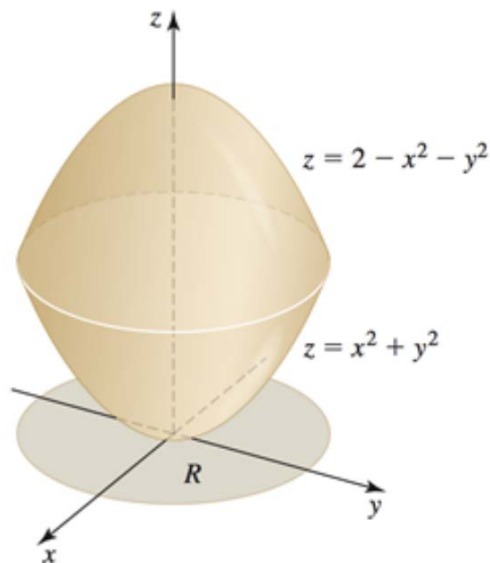
64.  $R = \{(r, \theta) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$

65.  $R = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq \pi\}$

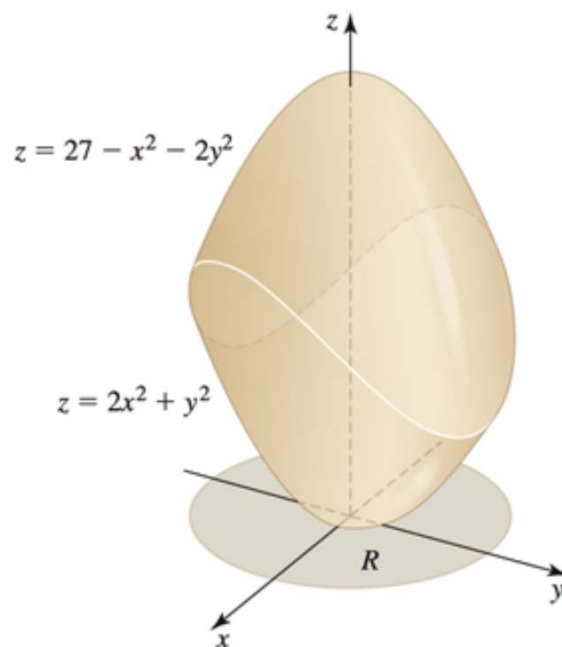
66.  $R = \{(r, \theta) : \sqrt{3} \leq r \leq 2\sqrt{2}, 0 \leq \theta \leq 2\pi\}$

67.  $R = \{(r, \theta) : \sqrt{3} \leq r \leq \sqrt{15}, -\frac{\pi}{2} \leq \theta \leq \pi\}$

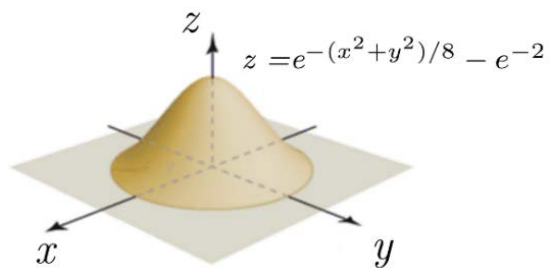
68. Find the volume of the solid between the paraboloids  $z = x^2 + y^2$  and  $z = 2 - x^2 - y^2$



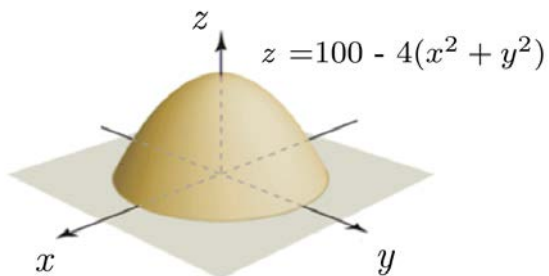
69. Find the volume of the solid between the paraboloids  $z = 2x^2 + y^2$  and  $z = 27 - x^2 - 2y^2$



70. Find the volume of island  $z = e^{-(x^2+y^2)/8} - e^{-2}$



71. Find the volume of island  $z = 100 - 4(x^2 + y^2)$



72. Find the volume of island  $z = 25 - \sqrt{x^2 + y^2}$

