

# Lecture Four - Integration

## Section 4.1 – Antiderivatives

### Antiderivatives

$$f(x) = x^3 \quad \Rightarrow \quad f'(x) = 3x^2$$

### Definition of Antiderivatives

A Function  $F$  is an *antiderivative* of a function  $f$  on an interval  $I$  if

$$F'(x) = f(x) \quad \text{for all } x \text{ in } I.$$

### Theorem

If  $F$  is an antiderivative of  $f$  on an interval  $I$ , then the most general antiderivative of  $f$  on  $I$  is

$$F(x) + C$$

Where  $C$  is an arbitrary constant.

### Notation for Antiderivatives and indefinite integrals

The notation  $\int f(x)dx = F(x) + C$

where  $C$  is an arbitrary constant, means that  $F$  is an Antiderivative of  $f$ .

That is  $F'(x) = f(x)$  for all  $x$  in the domain of  $f$ .

$\int f(x)dx$  Indefinite integral

The diagram shows the equation  $\int f(x)dx = F(x) + C$  with several red arrows and labels pointing to its components:

- An arrow labeled "Integral sign" points to the integral symbol  $\int$ .
- An arrow labeled "Integrand" points to the function  $f(x)$ .
- An arrow labeled "Differential" points to the  $dx$  term.
- An arrow labeled "Antiderivative" points to the expression  $F(x) + C$ , which is also grouped by a red bracket.

## Basic Integration Rules

$$\int k dx = kx + C$$

$$\int kf(x) dx = k \int f(x) dx$$

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

$$\int [f(x) - g(x)] dx = \int f(x) dx - \int g(x) dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1$$

### Example

Find each indefinite integral.

$$\begin{aligned} a) \quad \int \frac{1}{x^2} dx &= \int x^{-2} dx \\ &= \frac{x^{-2+1}}{-2+1} + C \\ &= \frac{x^{-1}}{-1} + C \\ &= -\frac{1}{x} + C \end{aligned}$$

$$\begin{aligned} b) \quad \int \sqrt[3]{x} dx &= \int x^{1/3} dx \\ &= \frac{x^{1/3+1}}{1/3+1} + C \\ &= \frac{x^{4/3}}{4/3} + C \\ &= \frac{3}{4} x^{4/3} + C \quad \text{or} \quad = \frac{3}{4} x \sqrt[3]{x} + C \end{aligned}$$

**Example**

Evaluate  $\int (4x^3 - 5x + 2) dx$

**Solution**

$$\begin{aligned}\int (4x^3 - 5x + 2) dx &= \int 4x^3 dx - \int 5x dx + \int 2 dx \\ &= 4 \frac{x^4}{4} - 5 \frac{x^2}{2} + 2x + C \\ &= \underline{x^4 - \frac{5}{2}x^2 + 2x + C}\end{aligned}$$

**Example**

Evaluate  $\int (x^2 - 2x + 5) dx$

**Solution**

$$\int (x^2 - 2x + 5) dx = \underline{\frac{x^3}{3} - x^2 + 5x + C}$$

**Example**

Evaluate  $\int \sin x \, dx$

**Solution**

$$\int \sin x \, dx = \underline{-\cos x + C}$$

**Example**

Evaluate  $\int \cos 3x \, dx$

**Solution**

$$\int \cos 3x \, dx = \underline{\frac{1}{3} \sin 3x + C}$$

## Exercises      Section 4.1 – Antiderivatives

Find each indefinite integral.

1.  $\int v^2 dv$

2.  $\int x^{1/2} dx$

3.  $\int 4y^{-3} dy$

4.  $\int (x^3 - 4x + 2) dx$

5.  $\int (3z^2 - 4z + 5) dz$

6.  $\int (x^2 - 1)^2 dx$

7.  $\int \frac{x^2 + 1}{\sqrt{x}} dx$

8.  $\int \left(4\sqrt{x^3} + 1\right) dx$

9.  $\int \sqrt{x}(x+1) dx$

10.  $\int (1+3t)t^2 dt$

11.  $\int \frac{x^2 - 5}{x^2} dx$

12.  $\int (-40x + 250) dx$

13.  $\int \frac{x+2}{\sqrt{x}} dx$

14.  $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$

15.  $\int (\sqrt{x} + \sqrt[3]{x}) dx$

16.  $\int 2x(1 - x^{-3}) dx$

17.  $\int \left(\frac{4 + \sqrt{t}}{t^3}\right) dt$

18.  $\int (-2 \cos t) dt$

19.  $\int 7 \sin \frac{\theta}{3} d\theta$

20.  $\int \frac{2}{5} \sec \theta \tan \theta d\theta$

21.  $\int (4 \sec x \tan x - 2 \sec^2 x) dx$

22.  $\int (2 \cos 2x - 3 \sin 3x) dx$

23.  $\int (1 + \tan^2 \theta) d\theta$

24.  $\int \frac{\csc \theta}{\csc \theta - \sin \theta} d\theta$

25. Solve the initial value problem:  $\frac{dy}{dx} = 2x - 7, \quad y(2) = 0$

26. Solve the initial value problem:  $\frac{dy}{dx} = \frac{1}{x^2} + x, \quad y(2) = 1; \quad x > 0$

27. Solve the initial value problem:  $\frac{ds}{dt} = 1 + \cos t, \quad s(0) = 4$

28. Solve the initial value problem:  $\frac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1$

29. Derive the position function if a ball is thrown upward with initial velocity of 32 ft per second from an initial height of 48 ft. When does the ball hit the ground? With what velocity does the ball hit the ground?

30. Find the general solution of  $F'(x) = 4x + 2$ , and find the particular solution that satisfies the initial condition  $F(1) = 8$ .

- 31.** Suppose a publishing company has found that the marginal cost at a level of production of  $x$  thousand books is given by

$$\frac{dC}{dx} = \frac{50}{\sqrt{x}}$$

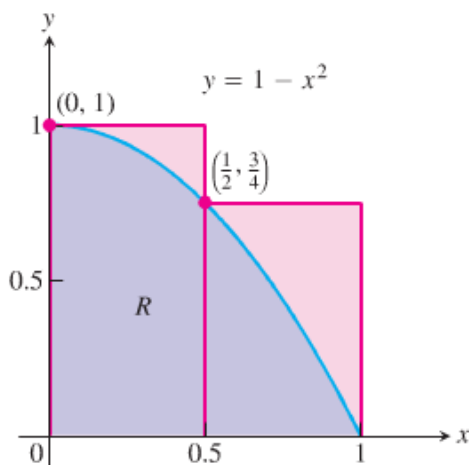
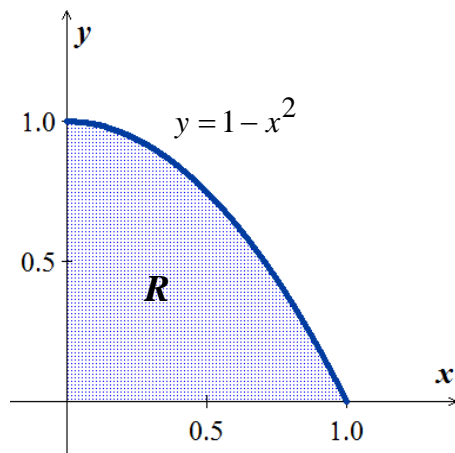
And that the fixed cost (the cost before the first book can be produced) is a \$25,000. Find the cost function  $C(x)$ .

## Section 4.2 – Area and Estimating with Finite Sums

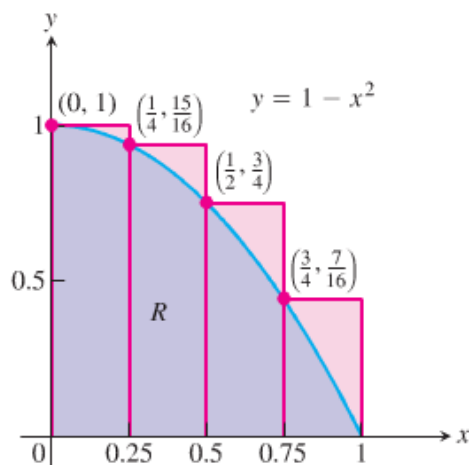
The *definite integral* is the key tool in calculus for defining and calculating quantities important to mathematics and science, such as areas, volumes, lengths, and more...

### Area

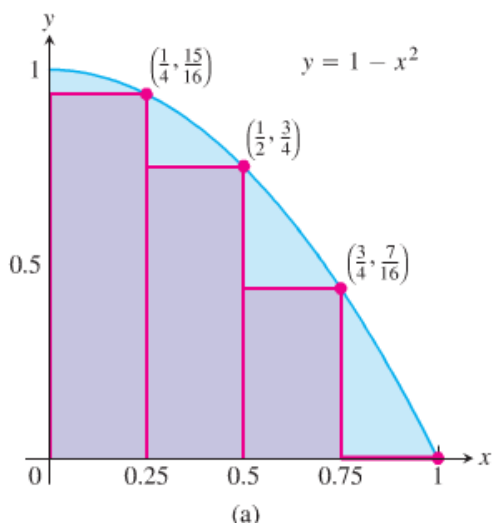
To find the area of the shaded region  $R$  that lies above the  $x$ -axis, below the graph of  $y = 1 - x^2$  and between the vertical lines  $x = 0$  and  $x = 1$ .



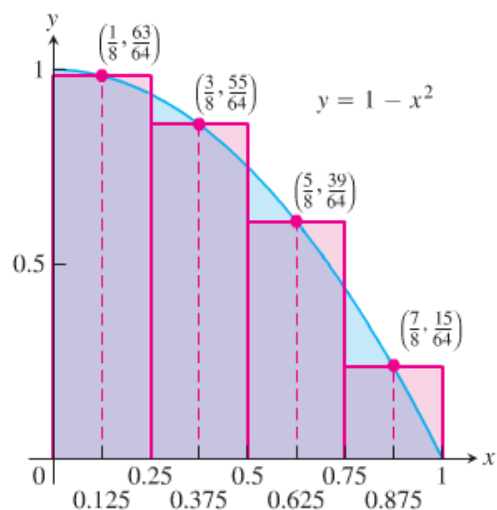
$$\text{Area} \approx 1 \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = 0.875$$



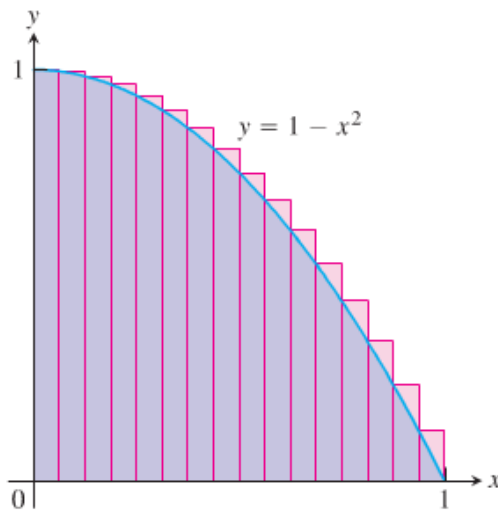
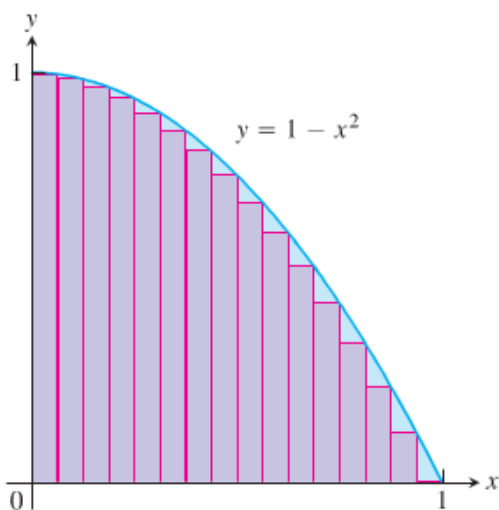
$$\text{Area} \approx 1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} = 0.78125$$



$$\text{Area} \approx \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = 0.53125$$



$$\text{Area} \approx \frac{63}{64} \cdot \frac{1}{4} + \frac{55}{64} \cdot \frac{1}{4} + \frac{39}{64} \cdot \frac{1}{4} + \frac{15}{64} \cdot \frac{1}{4} = 0.671875$$



In each case of the computations, the interval  $[a, b]$  over which the function  $f$  is defined was subdivided into  $n$  equal subintervals (also called **length**)  $\Delta x = \frac{b-a}{n}$ , and  $f$  was evaluated at a point in each subinterval. The finite sums can be given by the form:

$$f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + \cdots + f(c_n)\Delta x$$

## Distance Traveled

The distance formula is given by:  $distance = velocity \times time$

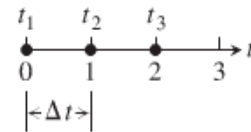
### Example

The velocity function of a projectile fired straight up into the air is  $f(t) = 160 - 9.8t$  m/sec. Use the summation technique to estimate how far the projectile rises during the first 3 sec. How close do the sums come to the exact value of 435.9 m?

### Solution

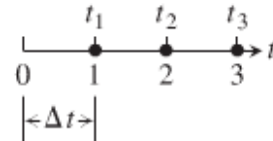
i.  $\Delta t = 1$  sec  $\rightarrow t = 0, 1, 2$

$$\begin{aligned} D &\approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t \\ &= f(0)\Delta t + f(1)\Delta t + f(2)\Delta t \\ &= (160 - 9.8(0))(1) + (160 - 9.8(1))(1) + (160 - 9.8(2))(1) \\ &= \underline{450.6} \end{aligned}$$



ii.  $\Delta t = 1$  sec  $\rightarrow t = 1, 2, 3$

$$\begin{aligned} D &\approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t \\ &= (160 - 9.8(1))(1) + (160 - 9.8(2))(1) + (160 - 9.8(3))(1) \\ &= \underline{421.2} \end{aligned}$$



iii.  $\Delta t = 0.5$  sec  $\rightarrow t = 0, 0.5, 1, 1.5, 2, 2.5$

$$\begin{aligned} D &\approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t + f(t_4)\Delta t + f(t_5)\Delta t + f(t_6)\Delta t \\ &= (160 - 9.8(0))(1) + (160 - 9.8(0.5))(1) + (160 - 9.8(1))(1) + (160 - 9.8(1.5))(1) \\ &\quad + (160 - 9.8(2))(1) + (160 - 9.8(2.5))(1) \\ &= \underline{\approx 443.25} \end{aligned}$$

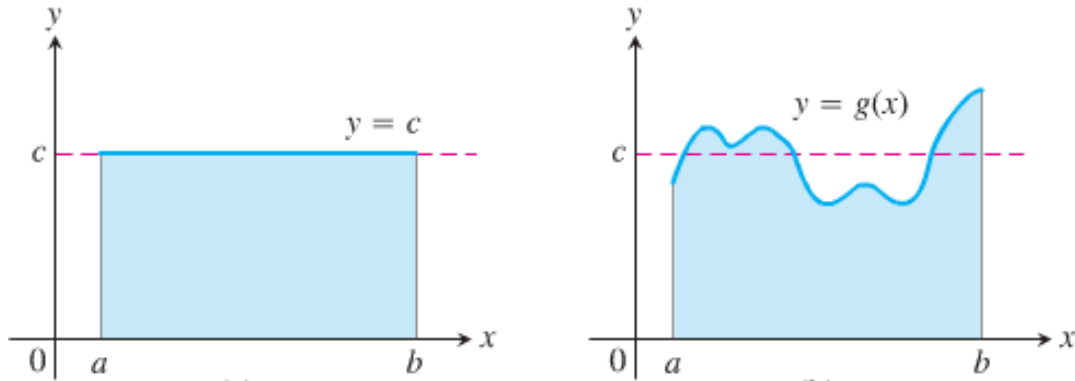
iv.  $\Delta t = 0.5$  sec  $\rightarrow t = 0.5, 1, 1.5, 2, 2.5, 3$

$$\begin{aligned} D &\approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t + f(t_4)\Delta t + f(t_5)\Delta t + f(t_6)\Delta t \\ &= (160 - 9.8(0.5))(1) + (160 - 9.8(1))(1) + (160 - 9.8(1.5))(1) + (160 - 9.8(2))(1) \\ &\quad + (160 - 9.8(2.5))(1) + (160 - 9.8(3))(1) \\ &= \underline{\approx 428.55} \end{aligned}$$

The true value is 435.9 if you use more subintervals  $\Delta t = 0.25$  sec, the interval 436.13 & 435.67  
The projectile rose about 436 m during the first 3 sec of flight.



## Average Value of a Nonnegative Continuous Function

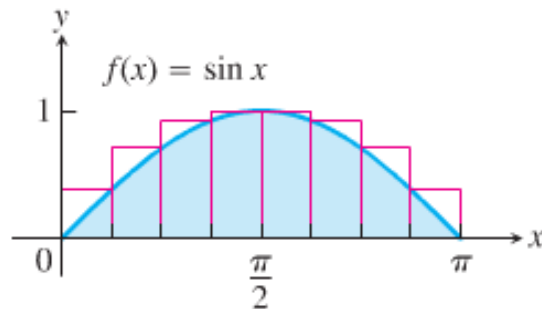


The average value of a collection of  $n$  numbers  $x_1, x_2, \dots, x_n$  is obtained by adding them together and dividing by  $n$ .

### Example

Estimate the average value of the function  $f(x) = \sin x$  on the interval  $[0, \pi]$ .

### Solution



To get the upper sum approximation with 8 rectangles of equal width  $\Delta x = \frac{\pi}{8}$ .

$$A \approx \left( \sin \frac{\pi}{8} + \sin \frac{\pi}{4} + \sin \frac{3\pi}{8} + \sin \frac{\pi}{2} + \sin \frac{\pi}{2} + \sin \frac{5\pi}{8} + \sin \frac{3\pi}{4} + \sin \frac{7\pi}{8} \right) \cdot \frac{\pi}{8}$$

$\approx 2.365$

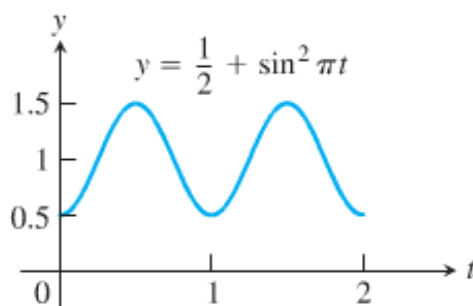
## Exercises      Section 4.2 – Area and Estimating with Finite Sums

Use finite approximations to estimate the area under the graph of the function using

- a) A lower sum with two rectangles of equal width
- b) A lower sum with four rectangles of equal width
- c) A upper sum with two rectangles of equal width
- d) A upper sum with four rectangles of equal width

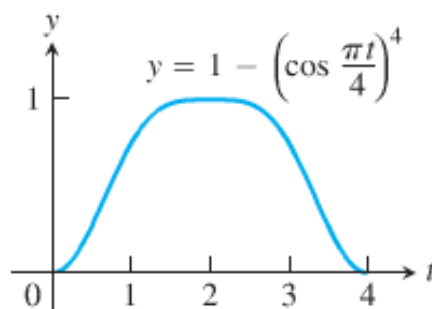
1.  $f(x) = \frac{1}{x}$  between  $x=1$  and  $x=5$
2.  $f(x) = 4 - x^2$  between  $x=-2$  and  $x=2$
3. Use finite approximations to estimate the average value of  $f$  on the given interval by partitioning the interval into four subintervals of equal length and evaluating  $f$  at the subinterval midpoints.

$$f(t) = \frac{1}{2} + \sin^2 \pi t \quad \text{on } [0, 2]$$



4. Use finite approximations to estimate the average value of  $f$  on the given interval by partitioning the interval into four subintervals of equal length and evaluating  $f$  at the subinterval midpoints.

$$f(t) = 1 - \left(\cos \frac{\pi t}{4}\right)^4 \quad \text{on } [0, 4]$$



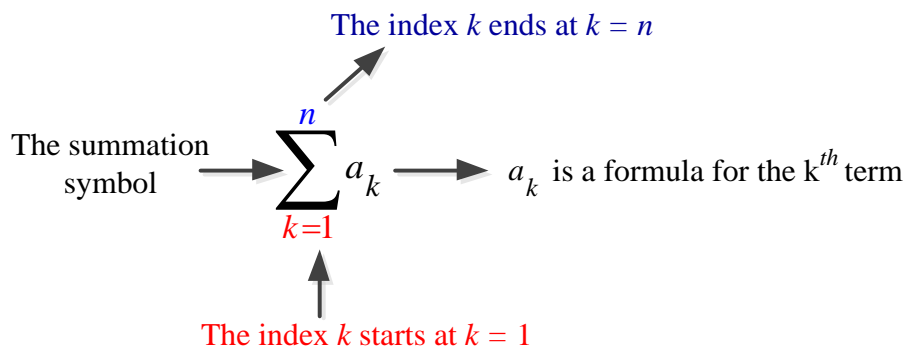
## Section 4.3 – Sigma Notation and Limits of Finite Sums

### Finite Sums and Sigma Notation

*Sigma notation* enables us to write a sum with many terms in the compact form

$$\sum_{k=1}^n a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

The Greek letter  $\Sigma$  (capital *sigma*, corresponding to our letter S)



### Example

<i>Sigma Notation</i>	<i>Written</i>	<i>Value of the Sum</i>
$\sum_{k=1}^5 k$	$1 + 2 + 3 + 4 + 5$	15
$\sum_{k=1}^4 (-1)^k \cdot k$	$(-1)^1 \cdot 1 + (-1)^2 \cdot 2 + (-1)^3 \cdot 3 + (-1)^4 \cdot 4$	$-1 + 2 - 3 + 4 = \underline{2}$
$\sum_{k=1}^3 \frac{k}{k+1}$	$\frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1}$	$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \underline{\frac{23}{12}}$
$\sum_{k=4}^5 \frac{k^2}{k-1}$	$\frac{4^2}{4-1} + \frac{5^2}{5-1}$	$\frac{16}{3} + \frac{25}{4} = \underline{\frac{139}{12}}$

### ***Example***

We can write:

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 = \sum_{k=1}^{10} k^2$$

### ***Example***

Express the sum  $1 + 3 + 5 + 7 + 9$  in sigma notation.

#### **Solution**

$$\text{Starting with } k = 0: 1 + 3 + 5 + 7 + 9 = \sum_{k=0}^4 (2k + 1)$$

$$\text{Starting with } k = 1: 1 + 3 + 5 + 7 + 9 = \sum_{k=1}^5 (2k - 1)$$

### ***Theorem on Sums***

If  $a_1, a_2, a_3, \dots, a_n, \dots$  and  $b_1, b_2, b_3, \dots, b_n, \dots$  are infinite sequences, then for every positive integer  $n$ ,

$$(1) \quad \sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$$

$$(2) \quad \sum_{k=1}^n (a_k - b_k) = \sum_{k=1}^n a_k - \sum_{k=1}^n b_k$$

$$(3) \quad \sum_{k=1}^n c a_k = c \left( \sum_{k=1}^n a_k \right)$$

$$(4) \quad \sum_{k=1}^n c = n \cdot c$$

**Proof**

$$\begin{aligned}\sum_{k=1}^n (a_k + b_k) &= (a_1 + b_1) + (a_2 + b_2) + \dots + (a_n + b_n) \\ &= (a_1 + a_2 + \dots + a_n) + (b_1 + b_2 + \dots + b_n) \\ &= \sum_{k=1}^n a_k + \sum_{k=1}^n b_k\end{aligned}$$

***Example***

$$\begin{aligned}(1) \quad \sum_{k=1}^n (k + 4) &= \sum_{k=1}^n k + \sum_{k=1}^n 4 = \sum_{k=1}^n k + 4 \cdot n \\ (2) \quad \sum_{k=1}^n (3k - k^2) &= 3 \sum_{k=1}^n k - \sum_{k=1}^n k^2 \\ (3) \quad \sum_{k=1}^n (-a_k) &= \sum_{k=1}^n (-1) \cdot (a_k) = (-1) \cdot \sum_{k=1}^n (a_k) = - \sum_{k=1}^n a_k \\ (4) \quad \sum_{k=1}^n \frac{1}{n} &= n \cdot \frac{1}{n} = 1\end{aligned}$$

***Example***

Show that the sum of the first  $n$  integers is  $\sum_{k=1}^n k = \frac{n(n+1)}{2}$

**Solution**

The sum of the first 4 integers is:  $\sum_{k=1}^4 k = \frac{4(5)}{2} = 10$

To prove the formula in general:

$$\begin{array}{ccccccc} 1 & + & 2 & + & 3 & + & \dots & + & n \\ n & + & (n-1) & + & (n-2) & + & \dots & + & 1 \\ \hline n+1 & + & n+1 & + & n+1 & + & \dots & + & n+1 & \rightarrow n(n+1) \end{array}$$

Since it is twice the desired quantity, the sum of the first  $n$  integers is  $\frac{n(n+1)}{2}$

$$\sum_{k=1}^n k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^n k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} = (1+2+3+\cdots+n)^2$$

## Limits of Finite Sums

### Example

Find the limiting value of lower sum approximations to the area of the region  $R$  below the graph of  $y = 1 - x^2$  and above the interval  $[0, 1]$  on the  $x$ -axis using equal-width rectangles whose width approach zero and whose number approaches infinity.

### Solution

The lower sum approximation using  $n$  rectangles of equal width:  $\Delta x = \frac{1-0}{n} = \frac{1}{n}$

By subdividing the interval  $[0, 1]$  into  $n$  equal width subintervals:

$$[0, 1] = \left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \left[\frac{2}{n}, \frac{3}{n}\right], \dots, \left[\frac{n-1}{n}, \frac{n}{n}\right] = \left[\frac{k-1}{n}, \frac{k}{n}\right]$$

$$f\left(\frac{k}{n}\right) = 1 - \left(\frac{k}{n}\right)^2$$

$$\left[f\left(\frac{1}{n}\right)\right] \cdot \left(\frac{1}{n}\right) + \left[f\left(\frac{2}{n}\right)\right] \cdot \left(\frac{1}{n}\right) + \cdots + \left[f\left(\frac{n}{n}\right)\right] \cdot \left(\frac{1}{n}\right)$$

We can write this in sigma notation:

$$\begin{aligned} \sum_{k=1}^n f\left(\frac{k}{n}\right) \cdot \left(\frac{1}{n}\right) &= \sum_{k=1}^n \left[1 - \left(\frac{k}{n}\right)^2\right] \cdot \left(\frac{1}{n}\right) \\ &= \sum_{k=1}^n \left(\frac{1}{n} - \frac{k^2}{n^3}\right) \\ &= \sum_{k=1}^n \frac{1}{n} - \sum_{k=1}^n \frac{k^2}{n^3} \\ &= n \cdot \frac{1}{n} - \frac{1}{n^3} \sum_{k=1}^n k^2 \end{aligned}$$

$$\begin{aligned}
&= 1 - \frac{1}{n^3} \cdot \frac{n \cdot (n+1) \cdot (2n+1)}{6} \\
&= 1 - \frac{2n^3 + 3n^2 + n}{6n^3} \\
&= \frac{6n^3 - 2n^3 - 3n^2 - n}{6n^3} \\
&= \frac{4n^3 - 3n^2 - n}{6n^3}
\end{aligned}$$

$$\lim_{x \rightarrow \infty} \left( \frac{4n^3 - 3n^2 - n}{6n^3} \right) = \frac{4}{6} = \frac{2}{3}$$

The lower sum approximation converge to  $\frac{2}{3}$

The upper sum approximation also converge to  $\frac{2}{3}$

## Review

### Definition of Arithmetic Sequence

A sequence  $a_1, a_2, a_3, \dots, a_n, \dots$  is an arithmetic sequence if there is a real number  $d$  such that for every positive integer  $k$ ,

$$a_{k+1} = a_k + d$$

The number  $d = a_{k+1} - a_k$  is called the *common difference* of the sequence.

*The  $n$ th Term of an Arithmetic Sequence:*  $a_n = a_1 + (n-1)d$

### Example

Express the sum in terms of summation notation:  $4 + 11 + 18 + 25 + 32$ . (Answers are not unique)

### Solution

Number of terms:  $n = 5$

Difference in terms:  $d = 11 - 4 = 7$

$$a_n = a_1 + (n-1)d$$

$$a_n = 4 + (n-1)7 = 4 + 7n - 7 = \underline{7n - 3}$$

$$\sum_{n=1}^5 (7n - 3)$$

## Theorem

### Formulas for $S_n$

If  $a_1, a_2, a_3, \dots, a_n, \dots$  is an arithmetic sequence with common difference  $d$ , then the  $n$ th partial sum  $S_n$  (that is, the sum of the first  $n$  terms) is given by either

$$S_n = \frac{n}{2} [2a_1 + (n-1)d] \quad \text{or} \quad S_n = \frac{n}{2} (a_1 + a_n)$$



### **Definition of Geometric Sequence**

A sequence  $a_1, a_2, a_3, \dots, a_n, \dots$  is a geometric sequence if  $a_1 \neq 0$  and if there is a real number  $r \neq 0$  such that for every positive integer  $k$ .

$$a_{k+1} = a_k r$$

The number  $r = \frac{a_{k+1}}{a_k}$  is called the **common ratio** of the sequence.

**The formula for the  $n^{\text{th}}$  Term of a Geometric Sequence:**  $a_n = a_1 r^{n-1}$

The common ratio for: 6, -12, 24, -48, ...,  $(-2)^{n-1}(6)$ , ... is  $= \frac{-12}{6} = -2$

### **Example**

Express the sum in terms of summation notation (Answers are not unique.)

$$\frac{1}{4} - \frac{1}{12} + \frac{1}{36} - \frac{1}{108}$$

### **Solution**

$$\begin{aligned} \frac{1}{4} - \frac{1}{12} + \frac{1}{36} - \frac{1}{108} &= \frac{1}{4} - \frac{1}{4} \frac{1}{3^1} + \frac{1}{4} \frac{1}{3^2} - \frac{1}{4} \frac{1}{3^3} \\ &= \sum_{n=1}^4 (-1)^{n+1} \frac{1}{4} \left(\frac{1}{3}\right)^{n-1} \end{aligned}$$

### **Theorem: Formula for $S_n$**

The  $n$ th partial sum  $S_n$  of a geometric sequence with first term  $a_1$  and common ratio  $r \neq 1$  is

$$S_n = a_1 \frac{1-r^n}{1-r}$$

## Riemann Sums

The theory of limits of finite approximations was made precise by the German mathematician **Bernhard Riemann**.

We introduce the notion of a *Riemann sum*, which underlies the theory of the definite integral.

Let a closed interval  $[a, b]$  be partitioned by points  $a < x_1 < x_2 < \cdots < x_{n-1} < b$

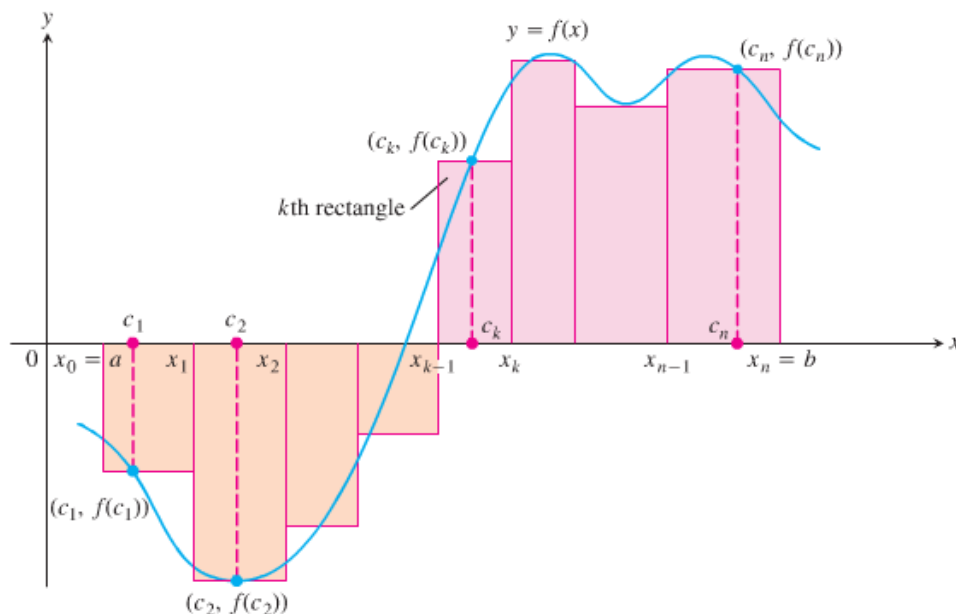
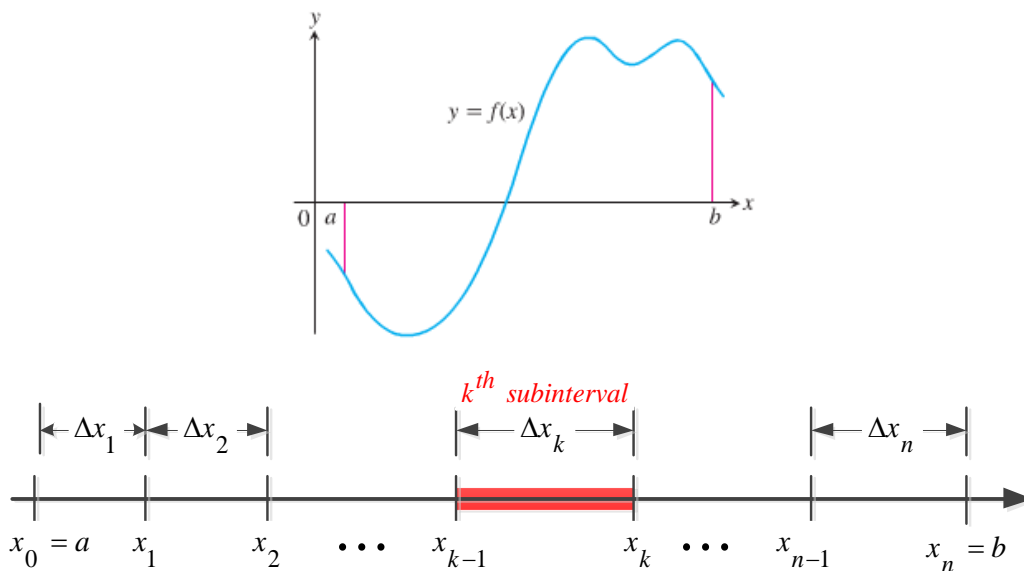
To make the notation consistent, so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

The set:  $P = \{x_0, x_1, x_2, \cdots, x_{n-1}, x_n\}$  is called a partition of  $[a, b]$ .

The partition  $P$  divides  $[a, b]$  into  $n$  closed subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$$



These products are:

$$S_P = \sum_{k=1}^n f(c_k) \Delta x_k$$

The sum  $S_P$  is called a **Riemann sum** for  $f$  on the interval  $[a, b]$ , and  $c_k$  in the subintervals.

If we choose  $n$  subintervals all having equal width  $\Delta x = \frac{b-a}{n}$  to partition  $[a, b]$ , then choose the point  $c_k$  to be the right-hand endpoints of each subintervals when forming the Riemann sum. This choice leads to the Riemann sum formula

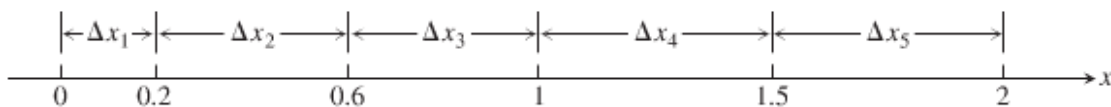
$$S_n = \sum_{k=1}^n f\left(a + k \frac{b-a}{n}\right) \cdot \left(\frac{b-a}{n}\right)$$

### Example

The set  $P = \{0, 0.2, 0.6, 1, 1.5, 2\}$  is a partition of  $[0, 2]$

### Solution

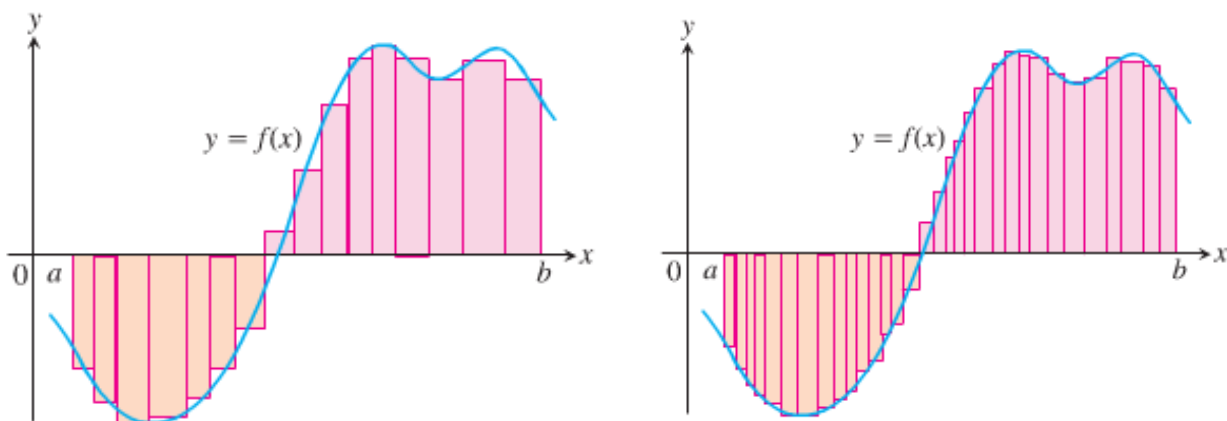
There are five subintervals of  $P$ :  $[0, 0.2]$ ,  $[0.2, 0.6]$ ,  $[0.6, 1]$ ,  $[1, 1.5]$ , and  $[1.5, 2]$



The lengths of the subintervals are:

$$\Delta x_1 = 0.2 \quad \Delta x_2 = 0.4 \quad \Delta x_3 = 0.4 \quad \Delta x_4 = 0.5 \quad \Delta x_5 = 0.5$$

The longest subinterval length is 0.5, so the norm of the partition is  $\|P\| = 0.5$



## Exercises Section 4.3 – Sigma Notation and Limits of Finite Sums

Write the sums without sigma notation. Then evaluate them:

1.  $\sum_{k=1}^2 \frac{6k}{k+1}$
2.  $\sum_{k=1}^3 \frac{k-1}{k}$
3.  $\sum_{k=1}^5 \sin k\pi$
4.  $\sum_{k=1}^4 (-1)^k \cos k\pi$
5. Write the following expression  $1 + 2 + 4 + 8 + 16 + 32$  in sigma notation
6. Write the following expression  $1 - 2 + 4 - 8 + 16 - 32$  in sigma notation
7. Write the following expression  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$  in sigma notation
8. Write the following expression  $-\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5}$  in sigma notation
9. Suppose that  $\sum_{k=1}^n a_k = -5$  and  $\sum_{k=1}^n b_k = 6$ . Find the value of  $\sum_{k=1}^n (b_k - 2a_k)$

Evaluate the sums

10.  $\sum_{k=1}^{10} k^3$
11.  $\sum_{k=1}^7 (-2k)$
12.  $\sum_{k=1}^5 \frac{\pi k}{15}$
13.  $\sum_{k=1}^5 k(3k+5)$
14.  $\sum_{k=1}^5 \frac{k^3}{225} + \left( \sum_{k=1}^5 k \right)^3$
15.  $\sum_{k=1}^{500} 7$
16.  $\sum_{k=18}^{71} k(k-1)$
17.  $\sum_{k=1}^n \left( \frac{1}{n} + 2n \right)$
18. Graph the function  $f(x) = x^2 - 1$  over the given interval  $[0, 2]$ . Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum  $\sum_{k=1}^4 f(c_k) \Delta x_k$ , given  $c_k$  is the
  - a) Left-hand endpoint
  - b) Right-hand endpoint
  - c) Midpoint of  $k^{th}$  subinterval.

(Make a separate sketch for each set of rectangles.)

## Section 4.4 – The Definite Integral

### Definition

Let  $f(x)$  be a function defined on a closed interval  $[a, b]$ . We say that a number  $J$  is the **definite integral** of  $f$  over  $[a, b]$  and that  $J$  is the limit of the Riemann sums  $\sum_{k=1}^n f(c_k) \Delta x_k$  if the following condition is satisfied:

Given any number  $\varepsilon > 0$  there is a corresponding number  $\delta > 0$  such that for every partition  $P = \{x_0, x_1, \dots, x_n\}$  of  $[a, b]$  with  $\|P\| < \delta$  and any choice of  $c_k$  in  $[x_{k-1}, x_k]$ , we have

$$\left| \sum_{k=1}^n f(c_k) \Delta x_k - J \right| < \varepsilon$$

**Leibniz** introduced a notation for the definite integral that captures its construction as a limit of Riemann sums.

The diagram shows the notation  $\int_a^b f(x) dx$  with several labels and arrows pointing to its components:

- Upper limit of integration**: A blue arrow points to the  $b$  above the integral sign.
- Function is integrand**: A grey arrow points to the  $f(x)$  part of the expression.
- Integral sign**: A grey arrow points to the large integral symbol  $\int$ .
- Lower limit of integration**: A red arrow points to the  $a$  below the integral sign.
- $x$  is the variable of integration**: A grey arrow points to the  $dx$  part of the expression.

**Integral of  $f$  from  $a$  to  $b$ .**

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k = J = \int_a^b f(x) dx$$

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x_k = J = \int_a^b f(x) dx$$

## **Theorem – Integrability of Continuous Functions**

If a function  $f$  is continuous over the interval  $[a, b]$ , or if  $f$  has at most finitely many jump discontinuities there, then the definite integral  $\int_a^b f(x)dx$  exists and  $f$  is integrable over  $[a, b]$

## **Properties of Definite Integrals**

$$\int_b^a f(x)dx = -\int_a^b f(x)dx \qquad \int_a^a f(x)dx = 0$$

## **Theorem**

When  $f$  and  $g$  are integrable over the interval  $[a, b]$ , the definite integral satisfies the rules:

*Order of Integration:*  $\int_b^a f(x)dx = -\int_a^b f(x)dx$

*Zero Width Interval:*  $\int_a^a f(x)dx = 0$

*Constant Multiple:*  $\int_a^b kf(x)dx = k \int_a^b f(x)dx$

*Sum and Difference:*  $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

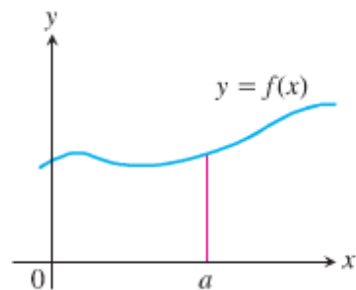
*Additivity:*  $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

*Max-Min Inequality:* If  $f$  has **maximum** value  $\max f$  and **minimum** value  $\min f$  on  $[a, b]$ , then

$$(\min f) \cdot (b - a) \leq \int_a^b f(x)dx \leq (\max f) \cdot (b - a)$$

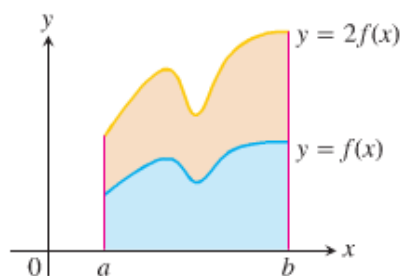
*Domination:*  $f(x) \geq g(x) \text{ on } [a, b] \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx$

$$f(x) \geq 0 \text{ on } [a, b] \Rightarrow \int_a^b f(x)dx \geq 0$$



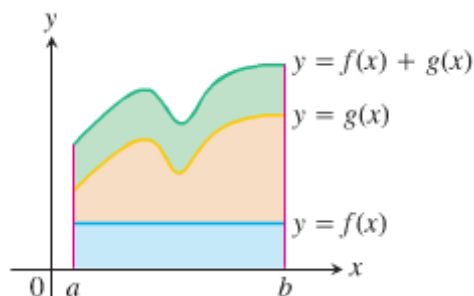
(a) Zero Width Interval:

$$\int_a^a f(x) dx = 0$$



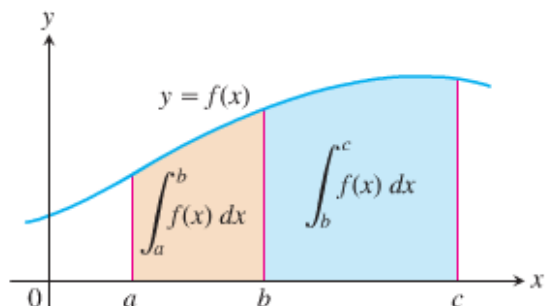
(b) Constant Multiple: ( $k = 2$ )

$$\int_a^b kf(x) dx = k \int_a^b f(x) dx$$



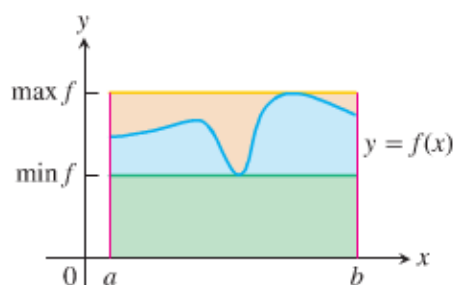
(c) Sum: (areas add)

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$



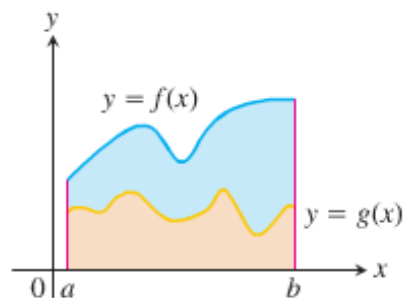
(d) Additivity for definite integrals:

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$



(e) Max-Min Inequality:

$$\begin{aligned} \min f \cdot (b - a) &\leq \int_a^b f(x) dx \\ &\leq \max f \cdot (b - a) \end{aligned}$$



(f) Domination:

$$\begin{aligned} f(x) &\geq g(x) \text{ on } [a, b] \\ \Rightarrow \int_a^b f(x) dx &\geq \int_a^b g(x) dx \end{aligned}$$

**Example**

Suppose that  $\int_{-1}^1 f(x)dx = 5$ ,  $\int_1^4 f(x)dx = -2$ ,  $\int_{-1}^1 h(x)dx = 7$ . Find:

a)  $\int_4^1 f(x)dx$

b)  $\int_{-1}^1 [2f(x) + 3h(x)]dx$

**Solution**

a)  $\int_4^1 f(x)dx = -\int_1^4 f(x)dx = -(-2) = \underline{2}$

b)  $\int_{-1}^1 [2f(x) + 3h(x)]dx = 2\int_{-1}^1 f(x)dx + 3\int_{-1}^1 h(x)dx$   
 $= 2(5) + 3(7)$   
 $= \underline{31}$

**Example**

Show that the value of  $\int_0^1 \sqrt{1 + \cos x} dx$  is less than or equal to  $\sqrt{2}$

**Solution**

$\min f \cdot (b - a)$ : is the lower bound

$\max f \cdot (b - a)$ : is the upper bound

The maximum value of  $\sqrt{1 + \cos x}$  on  $[0, 1]$  is  $\sqrt{1 + 1} = \sqrt{2}$

So,  $\int_0^1 \sqrt{1 + \cos x} dx \leq \sqrt{2} \cdot (1 - 0) = \sqrt{2}$



## Area Under the Graph of a Nonnegative Function

### Definition

If  $y = f(x)$  is nonnegative and integrable over a closed interval  $[a, b]$ , then the area under the curve  $y = f(x)$  over  $[a, b]$  is the integral of  $f$  from  $a$  to  $b$ ,

$$A = \int_a^b f(x) dx$$

### Example

Compute  $\int_0^b x dx$  and find the area  $A$  under  $y = x$  over the interval  $[0, b]$ ,  $b > 0$ .

### Solution

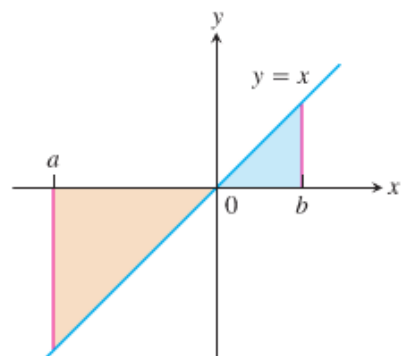
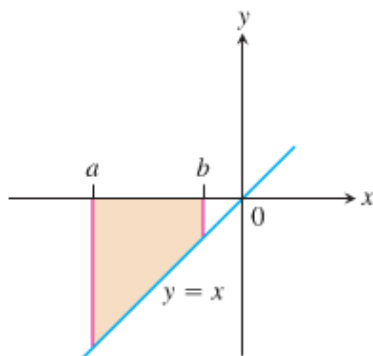
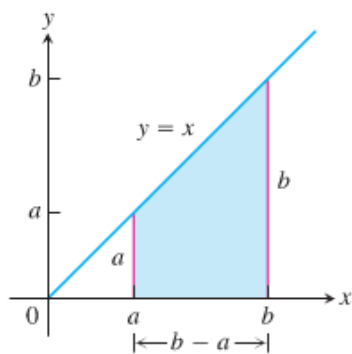
To Compute the definite integral, we consider the partition  $P$  subdivides the interval  $[0, b]$  into  $n$  subintervals of equal width  $\Delta x = \frac{b-0}{n} = \frac{b}{n}$ .

$$P = \left\{ 0, \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}, \dots, \frac{nb}{n} \right\} \quad \text{and} \quad c_k = \frac{kb}{n}$$

$$\begin{aligned} \sum_{k=1}^n f(c_k) \Delta x &= \sum_{k=1}^n \frac{kb}{n} \cdot \frac{b}{n} \\ &= \sum_{k=1}^n \frac{kb^2}{n^2} \\ &= \frac{b^2}{n^2} \sum_{k=1}^n k \\ &= \frac{b^2}{n^2} \cdot \frac{n(n+1)}{2} \\ &= \frac{b^2}{2} \left( 1 + \frac{1}{n} \right) \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left[ \frac{b^2}{2} \left( 1 + \frac{1}{n} \right) \right] = \frac{b^2}{2}$$

$$\int_0^b x dx = \frac{b^2}{2}$$



$$A = \int_0^b x dx = \frac{b^2}{2}$$

$$\begin{aligned} \int_a^b x dx &= \int_a^0 x dx + \int_0^b x dx \\ &= -\int_0^a x dx + \int_0^b x dx \\ &= -\frac{a^2}{2} + \frac{b^2}{2} \end{aligned}$$

$$\boxed{\int_a^b x dx = \frac{b^2}{2} - \frac{a^2}{2} \quad a < b}$$

$$\boxed{\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3} \quad a < b}$$

## Average Value of a Continuous Function Revisited

The average value of a nonnegative continuous function  $f$  over an interval  $[a, b]$ , leading us to define this average as the area under the graph of  $y = f(x)$  divided by  $b - a$ .

$$\text{Average} = \frac{1}{b-a} \int_a^b f(x) dx$$

### Definition

If  $f$  is integrable on  $[a, b]$ , then its *average value* on  $[a, b]$ , also called its **mean**, is

$$av(f) = \frac{1}{b-a} \int_a^b f(x) dx$$

### Example

Find the average value of  $f(x) = \sqrt{4-x^2}$  on  $[-2, 2]$

#### Solution

$f(x) = \sqrt{4-x^2}$  is a function of an upper semicircle with a radius 2 and centered at the origin.

The area between the semicircle and the x-axis from  $-2$  to  $2$  can be computed using the geometry formula:

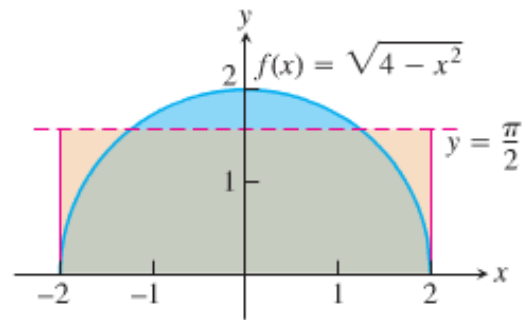
$$\text{Area} = \frac{1}{2} \pi r^2 = \frac{1}{2} \pi (2)^2 = 2\pi$$

$$\text{Area} = \int_{-2}^2 \sqrt{4-x^2} dx = 2\pi$$

$$av(f) = \frac{1}{2-(-2)} \int_{-2}^2 \sqrt{4-x^2} dx$$

$$= \frac{1}{4} (2\pi)$$

$$= \frac{\pi}{2}$$

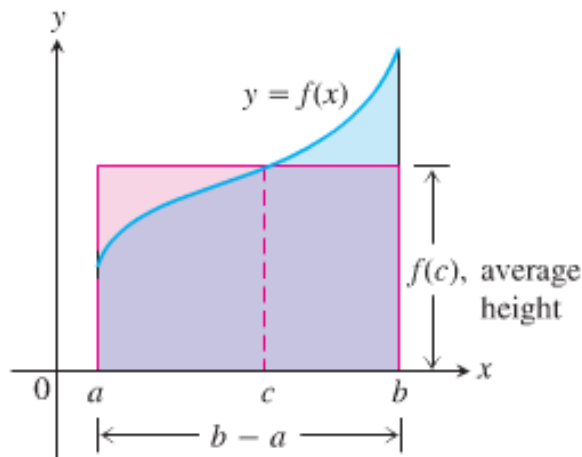


## Section 4.5 – The Fundamental Theorem of Calculus

### Mean Value Theorem for Definite Integrals

If  $f$  is continuous on  $[a, b]$ , then some point  $c$  in  $[a, b]$ ,

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx$$



### **Theorem** – The Fundamental Theorem of Calculus, P-1

If  $f$  is continuous on  $[a, b]$ , then  $F(x) = \int_a^x f(t) dt$  is continuous on  $[a, b]$ , and differentiable on  $(a, b)$  and its derivative is  $f(x)$ :

$$F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x)$$

### **Theorem** – The Fundamental Theorem of Calculus, P-2

If  $f$  is continuous at every point in  $[a, b]$ , then  $F$  is any antiderivative of  $f$  on  $[a, b]$ , then

$$\int_a^b f(x) dx = F(b) - F(a)$$

**Example**

$$\begin{aligned} a) \int_0^{\pi} \cos x \, dx &= \sin x \Big|_{\substack{\pi \\ 0}} \\ &= \sin \pi - \sin 0 \\ &= \underline{0} \end{aligned}$$

$$\begin{aligned} b) \int_{-\frac{\pi}{4}}^0 \sec x \tan x \, dx &= \sec x \Big|_{\substack{0 \\ -\frac{\pi}{4}}} \\ &= \sec 0 - \sec \left( -\frac{\pi}{4} \right) \\ &= \underline{1 - \sqrt{2}} \end{aligned}$$

$$\begin{aligned} c) \int_1^4 \left( \frac{3}{2} \sqrt{x} - \frac{4}{x^2} \right) dx &= \left[ x^{3/2} + \frac{4}{x} \right]_{\substack{4 \\ 1}} \\ &= \left( (4)^{3/2} + \frac{4}{4} \right) - \left( (1)^{3/2} + \frac{4}{1} \right) \\ &= (9) - (5) \\ &= \underline{4} \end{aligned}$$

## ***Theorem – The Net Change Theorem***

The net change in a function  $F(x)$  over an interval  $a \leq x \leq b$  is the integral of its rate of change:

$$F(\textcolor{red}{b}) - F(\textcolor{blue}{a}) = \int_{\textcolor{blue}{a}}^{\textcolor{red}{b}} F'(x) dx$$

### ***Example***

Consider the analysis of a heavy rock blown straight up from the ground by a dynamite blast. The velocity of the rock at any time  $t$  during its motion was given as  $v(t) = 160 - 32t$  *ft / sec*

- a) Find the displacement of the rock during the time period  $0 \leq t \leq 8$
- b) Find the total distance traveled during this time period.

### **Solution**

$$\begin{aligned} \text{a) displacement: } s(t) &= \int_0^8 v(t) dt \\ &= \int_0^8 (160 - 32t) dt \\ &= \left[ 160t - 16t^2 \right]_{\textcolor{blue}{0}}^{\textcolor{red}{8}} \\ &= \left( 160(\textcolor{red}{8}) - 16(\textcolor{red}{8})^2 \right) - \left( 160(\textcolor{blue}{0}) - 16(\textcolor{blue}{0})^2 \right) \\ &= \underline{\underline{256}} \end{aligned}$$

The height of the rock is 256 *ft* above the ground 8 *sec* after the explosion.

$$\text{b) } v(t) = 160 - 32t = 0 \rightarrow \boxed{t = 5 \text{ sec}}$$

The velocity is positive over the time  $[0, 5]$  and negative over  $[5, 8]$

$$\begin{aligned} \int_0^8 |v(t)| dt &= \int_0^5 |v(t)| dt + \int_5^8 |v(t)| dt \\ &= \int_0^5 (160 - 32t) dt - \int_5^8 (160 - 32t) dt \\ &= \left[ 160t - 16t^2 \right]_0^5 - \left[ 160t - 16t^2 \right]_5^8 \\ &= \left[ \left( 160(\textcolor{red}{5}) - 16(\textcolor{red}{5})^2 \right) - \left( 160(\textcolor{blue}{0}) - 16(\textcolor{blue}{0})^2 \right) \right] \\ &\quad - \left[ \left( 160(\textcolor{red}{8}) - 16(\textcolor{red}{8})^2 \right) - \left( 160(\textcolor{blue}{5}) - 16(\textcolor{blue}{5})^2 \right) \right] \\ &= 400 - (-144) \\ &= \underline{\underline{544}} \end{aligned}$$

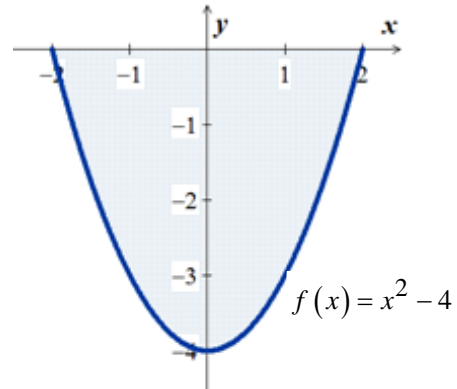
### Example

Shows the graph of  $f(x) = x^2 - 4$  and its mirror image  $g(x) = 4 - x^2$  are reflected across the  $x$ -axis. For each function, compute

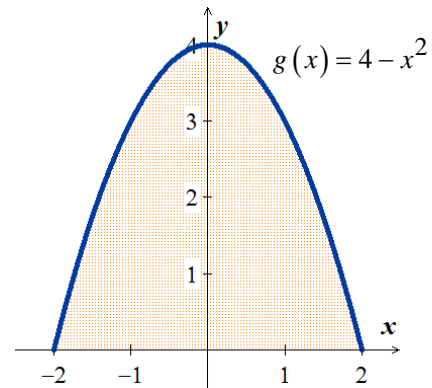
- a) The definite integral over the interval  $[-2, 2]$
- b) The area between the graph and the  $x$ -axis over  $[-2, 2]$

### Solution

$$\begin{aligned} \text{a) } \int_{-2}^2 f(x) dx &= \int_{-2}^2 (x^2 - 4) dx \\ &= \left[ \frac{x^3}{3} - 4x \right]_{-2}^2 \\ &= \left[ \frac{(2)^3}{3} - 4(2) \right] - \left[ \frac{(-2)^3}{3} - 4(-2) \right] \\ &= \left( \frac{8}{3} - 8 \right) - \left( -\frac{8}{3} + 8 \right) \\ &= -\frac{32}{3} \end{aligned}$$



$$\begin{aligned} \int_{-2}^2 g(x) dx &= \int_{-2}^2 (4 - x^2) dx \\ &= \left[ 4x - \frac{x^3}{3} \right]_{-2}^2 \\ &= \left[ 4(2) - \frac{(2)^3}{3} \right] - \left[ 4(-2) - \frac{(-2)^3}{3} \right] \\ &= \frac{32}{3} \end{aligned}$$



- b) In both cases, the area between the curve and the  $x$ -axis over  $[-2, 2]$  is  $\frac{32}{3}$  units.

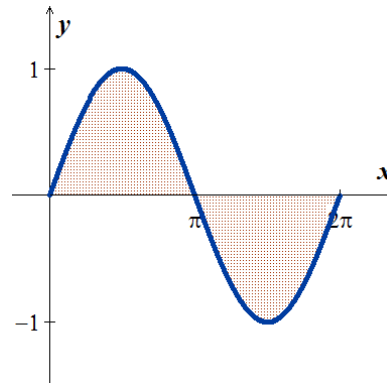
### Example

Shows the graph of  $f(x) = \sin x$  between  $x = 0$  and  $x = 2\pi$ . Compute

- a) The definite integral of  $f(x)$  over  $[0, 2\pi]$
- b) The area between the graph and the  $x$ -axis over  $[0, 2\pi]$

### Solution

$$\begin{aligned} \text{a) } \int_0^{2\pi} \sin x dx &= -\cos x \Big|_0^{2\pi} \\ &= -(\cos 2\pi - \cos 0) \\ &= -(1 - 1) \\ &= \underline{0} \end{aligned}$$



- b) The area between the graph and the axis is obtained by adding the absolute values

$$\begin{aligned} \text{Area} &= \left| \int_0^{\pi} \sin x dx \right| + \left| \int_{\pi}^{2\pi} \sin x dx \right| \\ &= \left| -\cos x \right|_0^{\pi} + \left| -\cos x \right|_{\pi}^{2\pi} \\ &= |-(\cos \pi - \cos 0)| + |-(\cos 2\pi - \cos \pi)| \\ &= | -(-1 - 1) | + | -(1 - (-1)) | \\ &= |2| + |-2| \\ &= \underline{4} \end{aligned}$$

### Summary

To find the area between the graph of  $y = f(x)$  and the  $x$ -axis over the interval  $[a, b]$ :

1. Subdivide  $[a, b]$  at the zeros of  $f$ .
2. Integrate  $f$  over each subinterval.
3. Add the absolute values of the integrals.



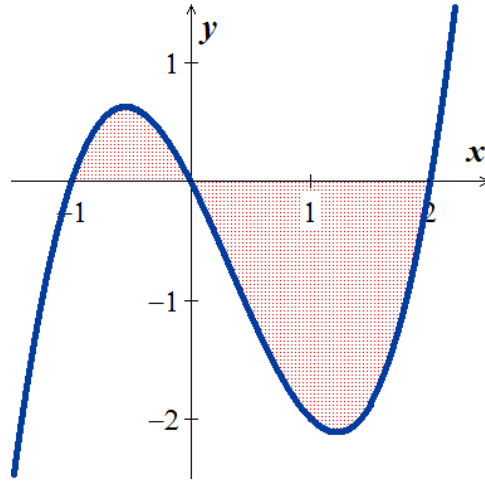
### Example

Find the area of the region between the  $x$ -axis and the graph of  $f(x) = x^3 - x^2 - 2x$ ,  $-1 \leq x \leq 2$

### Solution

The zeros of:  $f(x) = x^3 - x^2 - 2x = 0$

$$x(x^2 - x - 2) = 0 \Rightarrow x = 0, -1, 2$$



$$\begin{aligned}\int_{-1}^0 (x^3 - x^2 - 2x) dx &= \left[ \frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_{-1}^0 \\ &= \left[ 0 - \left( \frac{(-1)^4}{4} - \frac{(-1)^3}{3} - (-1)^2 \right) \right] \\ &= -\left( \frac{1}{4} + \frac{1}{3} - 1 \right) \\ &= \frac{5}{12}\end{aligned}$$

$$\begin{aligned}\int_0^2 (x^3 - x^2 - 2x) dx &= \left[ \frac{x^4}{4} - \frac{x^3}{3} - x^2 \right]_0^2 \\ &= \left[ \left( \frac{(2)^4}{4} - \frac{(2)^3}{3} - (2)^2 \right) - 0 \right] \\ &= \left( 4 - \frac{8}{3} - 4 \right) \\ &= -\frac{8}{3}\end{aligned}$$

$$\begin{aligned}\text{Area} &= \left| \int_{-1}^0 (x^3 - x^2 - 2x) dx \right| + \left| \int_0^2 (x^3 - x^2 - 2x) dx \right| \\ &= \frac{5}{12} + \left| -\frac{8}{3} \right| \\ &= \frac{5}{12} + \frac{8}{3} \\ &= \frac{37}{12}\end{aligned}$$

## Exercises      Section 4.5 – The Fundamental Theorem of Calculus

Evaluate the integrals

- |  |   |   |
|--|---|---|
| 1. $\int_0^2 x(x-3)dx$                           | 5. $\int_0^{\pi/3} 4\sec u \tan u \, du$                                | 9. $\int_1^8 \frac{(x^{1/3}+1)(2-x^{2/3})}{x^{1/3}} dx$ |
| 2. $\int_0^4 \left(3x - \frac{x^3}{4}\right) dx$ | 6. $\int_{\pi/4}^{3\pi/4} \csc \theta \cot \theta d\theta$              | 10. $\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x} dx$     |
| 3. $\int_{-2}^2 (x^3 - 2x + 3) dx$               | 7. $\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt$ | 11. $\int_0^{\pi/3} (\cos x + \sec x)^2 dx$             |
| 4. $\int_0^1 (x^2 + \sqrt{x}) dx$                | 8. $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy$                             | 12. $\int_0^{\pi} \frac{1}{2} (\cos x +  \cos x ) dx$   |

Find the total area between the region and the  $x$ -axis

- |  |   |
|--|---|
| 13. $y = -x^2 - 2x, \quad -3 \leq x \leq 2$      | 15. $y = x^{1/3} - x, \quad -1 \leq x \leq 8$ |
| 14. $y = x^3 - 3x^2 + 2x, \quad 0 \leq x \leq 2$ | 16. $f(x) = x^2 + 1, \quad 2 \leq x \leq 3$   |
17. Archimedes, inventor, military engineer, physicist, and the greatest mathematician of classical times in the Western world, discovered that the area under a parabolic arch is two-thirds the base times the height. Sketch the parabolic arch  $y = h - \left(\frac{4h}{b^2}\right)x^2 \quad -\frac{b}{2} \leq x \leq \frac{b}{2}$ , assuming that  $h$  and  $b$  are positive. Then use calculus to find the area of the region enclosed between the arch and the  $x$ -axis
18. Suppose that a company's marginal revenue from the manufacture and sale of eggbeaters is

$$\frac{dr}{dx} = 2 - \frac{2}{(x+1)^2}$$

Where  $r$  is measured in thousands of dollars and  $x$  in thousands of units. How much money should the company expect from a production run of  $x = 3$  thousand eggbeaters? To find out, integrate the marginal revenue from  $x = 0$  to  $x = 3$ .

19. The height  $H$  (ft) of a palm tree after growing for  $t$  years is given by

$$H = \sqrt{t+1} + 5t^{1/3} \quad \text{for } 0 \leq t \leq 8$$

- a) Find the tree's height when  $t = 0$ ,  $t = 4$ , and  $t = 8$ .
- b) Find the tree's average height for  $0 \leq t \leq 8$

## Section 4.6 – Indefinite Integrals and the Substitution Method

The indefinite integral of the function  $f$  with respect to  $x$  as the set of all antiderivatives of  $f$ :

$$\int f(x) dx = F(x) + C$$

Where  $C$  is any arbitrary constant.

### ***Substitution:*** Running the Chain Rule Backwards

The Chain rule formula is:  $\frac{d}{dx} \left( \frac{u^{n+1}}{n+1} \right) = u^n \frac{du}{dx}$

We can see that  $\frac{u^{n+1}}{n+1}$  is an antiderivative of the function  $u^n \frac{du}{dx}$ . Therefore, if we integrate both sides

$$\int u^n \frac{du}{dx} dx = \int \frac{d}{dx} \left( \frac{u^{n+1}}{n+1} \right) dx$$

$$\boxed{\int u^n du = \frac{u^{n+1}}{n+1} + C}$$

### ***Example***

Find the integral  $\int (x^3 + x)^5 (3x^2 + 1) dx$

### **Solution**

$$\text{Let: } u = x^3 + x \Rightarrow du = \frac{du}{dx} dx = (3x^2 + 1) dx$$

$$\begin{aligned} \int (x^3 + x)^5 (3x^2 + 1) dx &= \int u^5 du \\ &= \frac{u^6}{6} + C \\ &= \frac{(x^3 + x)^6}{6} + C \end{aligned}$$

### Example

Find the integral  $\int \sqrt{2x+1} dx$

### Solution

$$\text{Let: } u = 2x+1 \Rightarrow du = \frac{du}{dx} dx = 2dx$$

$$dx = \frac{1}{2} du$$

$$\begin{aligned} \int \sqrt{2x+1} dx &= \frac{1}{2} \int u^{1/2} du && \text{or } \frac{1}{2} \int (2x+1)^{1/2} 2dx \\ &= \frac{1}{2} \frac{u^{3/2}}{3/2} + C \\ &= \frac{1}{3} (2x+1)^{3/2} + C \end{aligned}$$

### Theorem – The Substitution Rule

If  $u = g(x)$  is differentiable function whose range is an interval  $I$ , and  $f$  is continuous on  $I$ , then

$$\int f(g(x)) g'(x) dx = \int f(u) du$$

### Proof

By the Chain Rule,  $F(g(x))$  is an antiderivative of  $f(g(x)) \cdot g'(x)$  whenever  $F$  is an antiderivative of  $f$ :

$$\begin{aligned} \frac{d}{dx} F(g(x)) &= F'(g(x)) \cdot g'(x) \\ &= f(g(x)) \cdot g'(x) \end{aligned}$$

If we make the substitution  $u = g(x)$ , then

$$\begin{aligned} \int f(g(x)) g'(x) dx &= \int \frac{d}{dx} F(g(x)) dx \\ &= F(g(x)) + C \\ &= F(u) + C \\ &= \int F'(u) du \\ &= \int f(u) du \end{aligned}$$

### Example

Find the integral  $\int \sec^2(5t+1) \cdot 5dt$

#### Solution

$$\text{Let: } u = 5t + 1 \Rightarrow du = \frac{du}{dt} dt = 5dt$$

$$\int \sec^2(5t+1) \cdot 5dt = \int \sec^2 u \, du$$

$$= \tan u + C$$

$$= \tan(5t+1) + C$$

$$\frac{d}{du} \tan u = \sec^2 u$$

### Example

Find the integral  $\int \cos(7\theta+3) d\theta$

#### Solution

$$\text{Let: } u = 7\theta + 3 \Rightarrow du = \frac{du}{d\theta} d\theta = 7d\theta$$

$$d\theta = \frac{1}{7} du$$

$$\int \cos(7\theta+3) d\theta = \frac{1}{7} \int \cos u \, du$$

$$= \frac{1}{7} \sin u + C$$

$$= \frac{1}{7} \sin(7\theta+3) + C$$

### Example

Find the integral  $\int x^2 \sin(x^3) dx$

#### Solution

$$\text{Let: } u = x^3 \Rightarrow du = \frac{du}{dx} dx = 3x^2 dx$$

$$x^2 dx = \frac{1}{3} du$$

$$\int x^2 \sin(x^3) dx = \int \sin u \cdot \frac{1}{3} du$$

$$\begin{aligned}
&= \frac{1}{3} \int \sin u \, du \\
&= \frac{1}{3} (-\cos u) + C \\
&= \underline{-\frac{1}{3} \cos(x^3) + C}
\end{aligned}$$

### ***Example***

Find the integral  $\int x\sqrt{2x+1} \, dx$

### **Solution**

$$\text{Let: } u = 2x + 1 \Rightarrow du = 2dx$$

$$dx = \frac{1}{2} du$$

$$u = 2x + 1 \rightarrow 2x = u - 1 \Rightarrow x = \frac{u-1}{2}$$

$$\begin{aligned}
\int x\sqrt{2x+1} \, dx &= \int \frac{1}{2}(u-1)\sqrt{u} \, \frac{1}{2} du \\
&= \frac{1}{4} \int (u-1)u^{1/2} \, du \\
&= \frac{1}{4} \int \left(u^{3/2} - u^{1/2}\right) \, du \\
&= \frac{1}{4} \left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C \\
&= \underline{\frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C}
\end{aligned}$$

### Example

Find the integral  $\int \frac{2z \, dz}{\sqrt[3]{z^2 + 1}}$

### Solution

$$\text{Let: } u = z^2 + 1 \Rightarrow du = 2z \, dz$$

$$\begin{aligned} \int \frac{2z \, dz}{\sqrt[3]{z^2 + 1}} &= \int \frac{du}{u^{1/3}} \\ &= \int u^{-1/3} \, du \\ &= \frac{3}{2} u^{2/3} + C \\ &= \frac{3}{2} (z^2 + 1)^{2/3} + C \end{aligned}$$

$$\text{Or let: } u = \sqrt[3]{z^2 + 1} \rightarrow u^3 = z^2 + 1$$
$$3u^2 \, du = 2z \, dz$$

$$\begin{aligned} \int \frac{2z \, dz}{\sqrt[3]{z^2 + 1}} &= \int \frac{3u^2 \, du}{u} \\ &= 3 \int u \, du \\ &= 3 \cdot \frac{u^2}{2} + C \\ &= \frac{3}{2} (z^2 + 1)^{2/3} + C \end{aligned}$$

***Integrals*** of  $\sin^2 x$  and  $\cos^2 x$

***Example***

Find the integral  $\int \sin^2 x \, dx$

**Solution**

$$\begin{aligned}\int \sin^2 x \, dx &= \int \frac{1 - \cos 2x}{2} \, dx \\ &= \frac{1}{2} \int (1 - \cos 2x) \, dx \\ &= \frac{1}{2} \left[ x - \frac{1}{2} \sin 2x \right] + C \\ &= \frac{1}{2} x - \frac{1}{4} \sin 2x + C\end{aligned}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

***Example***

Find the integral  $\int \cos^2 x \, dx$

**Solution**

$$\begin{aligned}\int \cos^2 x \, dx &= \int \frac{1 + \cos 2x}{2} \, dx \\ &= \frac{1}{2} \left[ x + \frac{1}{2} \sin 2x \right] + C \\ &= \frac{x}{2} + \frac{\sin 2x}{4} + C\end{aligned}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$



### Example

We can model the voltage in the electrical wiring of a typical home with the sine function

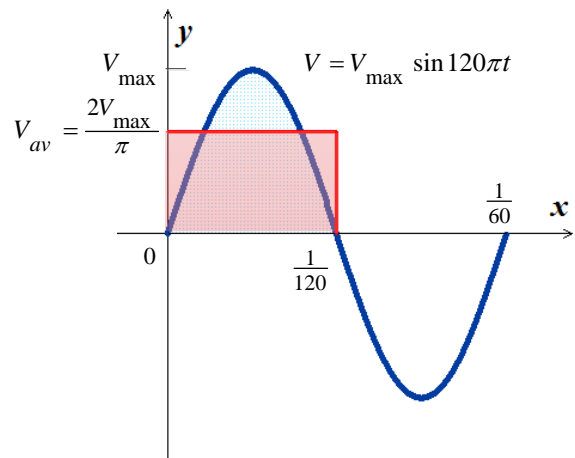
$$V = V_{\max} \sin 120\pi t$$

Which express the voltage  $V$  in volts as a function of time  $t$  in seconds. The function runs through 60 cycles each second (its frequency is 60 Hz (*hertz*)). The positive constant  $V_{\max}$  is the **peak voltage**.

### Solution

The average value of  $V$  over the half-cycle from 0 to  $\frac{1}{120}$  sec is

$$\begin{aligned} V_{av} &= \frac{1}{\frac{1}{120} - 0} \int_0^{1/120} V_{\max} \sin 120\pi t \, dt \\ &= 120V_{\max} \int_0^{1/120} \sin 120\pi t \, dt \\ &= 120V_{\max} \left[ -\frac{1}{120\pi} \cos 120\pi t \right]_0^{1/120} \\ &= -\frac{1}{\pi} V_{\max} \left( \cos \left( 120\pi \frac{1}{120} \right) - \cos(120\pi \cdot 0) \right) \\ &= -\frac{1}{\pi} V_{\max} (\cos \pi - \cos 0) \\ &= -\frac{1}{\pi} V_{\max} (-1 - 1) \\ &= \frac{2}{\pi} V_{\max} \end{aligned}$$



The average value of the voltage over a full cycle is zero.

To measure the voltage effectively, we can use an instrument the square root of the average value of the square of the voltage, namely:

$$V_{rms} = \sqrt{(V^2)_{av}}$$

“rms” : root mean square.

$$V_{av}^2 = \frac{1}{\frac{1}{60} - 0} \int_0^{1/60} (V_{\max})^2 \sin^2 120\pi t \, dt = \frac{(V_{\max})^2}{2}$$

$$\text{The rms voltage is: } V_{rms} = \sqrt{(V^2)_{av}} = \frac{V_{\max}}{\sqrt{2}}$$

The “115 volts ac” means that the rms voltage is 115. The peak voltage is:

$$V_{\max} = \sqrt{2} V_{rms} = \sqrt{2} (115) \approx \underline{163 \text{ volts}}$$

## Exercises      Section 4.6 – Indefinite Integrals and the Substitution Method

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

1.  $\int 2(2x+4)^5 dx, \quad u = 2x+4$
2.  $\int \frac{4x^3}{(x^4+1)^2} dx, \quad u = x^4+1$
3.  $\int x \sin(2x^2) dx, \quad u = 2x^2$
4.  $\int 12(y^4+4y^2+1)^2 (y^3+2y) dy, \quad u = y^4+4y^2+1$
5.  $\int \csc^2 2\theta \cot 2\theta d\theta \rightarrow \begin{cases} a) U \sin g & u = \cot 2\theta \\ b) U \sin g & u = \csc 2\theta \end{cases}$

Evaluate the integrals

- |   |   |
|---|---|
| 6. $\int \frac{1}{\sqrt{5s+4}} ds$                  | 14. $\int \csc\left(\frac{v-\pi}{2}\right) \cot\left(\frac{v-\pi}{2}\right) dv$   |
| 7. $\int \theta \sqrt[4]{1-\theta^2} d\theta$       | 15. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)} dt$                                     |
| 8. $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx$       | 16. $\int \frac{\sec z \tan z}{\sqrt{\sec z}} dz$                                 |
| 9. $\int \tan^2 x \sec^2 x dx$                      | 17. $\int \frac{1}{\sqrt{t}} \cos(\sqrt{t}+3) dt$                                 |
| 10. $\int \sin^5 \frac{x}{3} \cos \frac{x}{3} dx$   | 18. $\int \frac{1}{\theta^2} \sin \frac{1}{\theta} \cos \frac{1}{\theta} d\theta$ |
| 11. $\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2} dx$ | 19. $\int t^3 (1+t^4)^3 dt$   |
| 12. $\int r^4 \left(7 - \frac{r^5}{10}\right)^3 dr$ | 20. $\int \frac{1}{x^3} \sqrt{\frac{x^2-1}{x^2}} dx$                              |
| 13. $\int x^{1/2} \sin(x^{3/2}+1) dx$               | 21. $\int x^3 \sqrt{x^2+1} dx$  |

$$22. \int \frac{x}{(x^2 - 4)^3} dx$$

$$23. \int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}} dr$$

$$24. \int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\cos^3\sqrt{\theta}} d\theta$$

$$25. \int 2x\sqrt{x^2-2} dx$$

$$26. \int x^3(3x^4+1)^2 dx$$

$$27. \int 2(3x^4+1)^2 dx$$

$$28. \int 5x\sqrt{x^2-1} dx$$

$$29. \int (x^2-1)^3 (2x) dx$$

$$30. \int \frac{6x}{(1+x^2)^3} dx$$

$$31. \int u^3\sqrt{u^4+2} du$$

$$32. \int \frac{t+2t^2}{\sqrt{t}} dt$$

$$33. \int \left(1+\frac{1}{t}\right)^3 \frac{1}{t^2} dt$$

$$34. \int (7-3x-3x^2)(2x+1) dx$$

$$35. \int \sqrt{x}(4-x^{3/2})^2 dx$$

$$36. \int \frac{1}{\sqrt{x}+\sqrt{x+1}} dx$$

$$37. \int \sqrt{1-x} dx$$

$$38. \int x\sqrt{x^2+4} dx$$

$$39. \int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta$$

$$40. \int \cos^2(8\theta) d\theta$$

$$41. \int \sin^2(2\theta) d\theta$$

$$42. \int 8\cos^4 2\pi x dx$$

43. Evaluate the integral  $\int \frac{18\tan^2 x \sec^2 x}{(2+\tan^3 x)^2} dx$

a)  $u = \tan x$ , followed by  $v = u^3$  then by  $w = 2 + v$

b)  $u = \tan^3 x$ , followed by  $v = 2 + u$

c)  $u = 2 + \tan^3 x$

## Section 4.7 – Substitution and Area between Curves

### Substitution Formula

#### *Theorem*

If  $g'$  is continuous on the interval  $[a, b]$  and  $f$  is continuous on the range of  $g(x) = u$ , then

$$\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

#### *Proof*

Let  $F$  denote any antiderivative of  $f$ . Then

$$\begin{aligned} \int_a^b f(g(x)) \cdot g'(x) dx &= F(g(x)) \Big|_{x=a}^{x=b} \\ &= F(g(b)) - F(g(a)) \\ &= F(u) \Big|_{u=g(a)}^{u=g(b)} \\ &= \int_{g(a)}^{g(b)} f(u) du \end{aligned}$$

#### *Example*

Evaluate  $\int_{-1}^1 3x^2 \sqrt{x^3 + 1} dx$

#### *Solution*

Let  $u = x^3 + 1 \Rightarrow du = 3x^2 dx$

$$\rightarrow \begin{cases} x = -1 & u = (-1)^3 + 1 = 0 \\ x = 1 & u = (1)^3 + 1 = 2 \end{cases}$$

$$\begin{aligned} \int_{-1}^1 3x^2 \sqrt{x^3 + 1} dx &= \int_0^2 \sqrt{u} du \\ &= \int_0^2 u^{1/2} du \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{3} u^{3/2} \Big|_0^2 \\
&= \frac{2}{3} \left[ 2^{3/2} - 0^{3/2} \right] \\
&= \frac{2}{3} 2^{3/2} \\
&= \frac{2^{5/2}}{3} \\
&= \frac{4\sqrt{2}}{3}
\end{aligned}$$

$$2^{5/2} = \sqrt{2^5} = 2^2 \sqrt{2}$$

### Example

Evaluate  $\int_{\pi/4}^{\pi/2} \cot \theta \csc^2 \theta \, d\theta$

### Solution

Let  $u = \cot \theta \Rightarrow du = -\csc^2 \theta d\theta \rightarrow -du = \csc^2 \theta d\theta$

$$\rightarrow \begin{cases} \theta = \frac{\pi}{4} & u = \cot \frac{\pi}{4} = 1 \\ \theta = \frac{\pi}{2} & u = \cot \frac{\pi}{2} = 0 \end{cases}$$

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} \cot \theta \csc^2 \theta \, d\theta &= \int_1^0 u \cdot (-du) \\
&= - \int_1^0 u \, du \\
&= - \left[ \frac{u^2}{2} \right]_1^0 \\
&= - \left[ \frac{0^2}{2} - \frac{1^2}{2} \right] \\
&= \frac{1}{2}
\end{aligned}$$

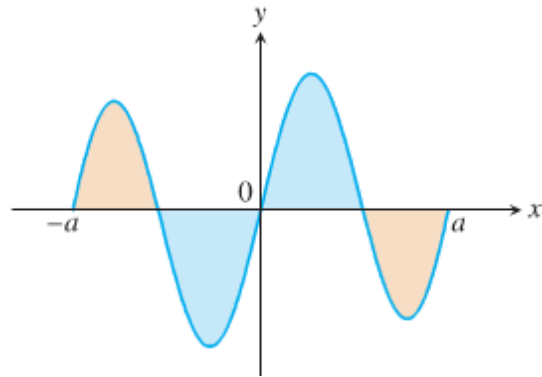
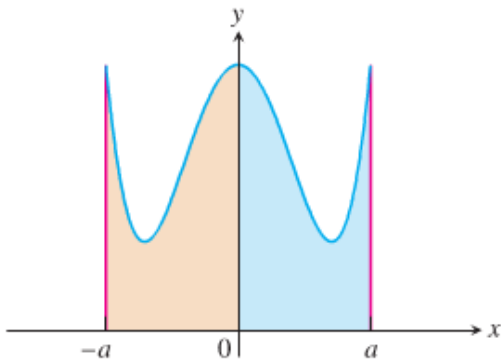
## Definite Integrals of Symmetric Functions

### Theorem

Let  $f$  be continuous on the symmetric interval  $[-a, a]$

✓ If  $f$  is even, then  $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

✓ If  $f$  is odd, then  $\int_{-a}^a f(x) dx = 0$



### Example

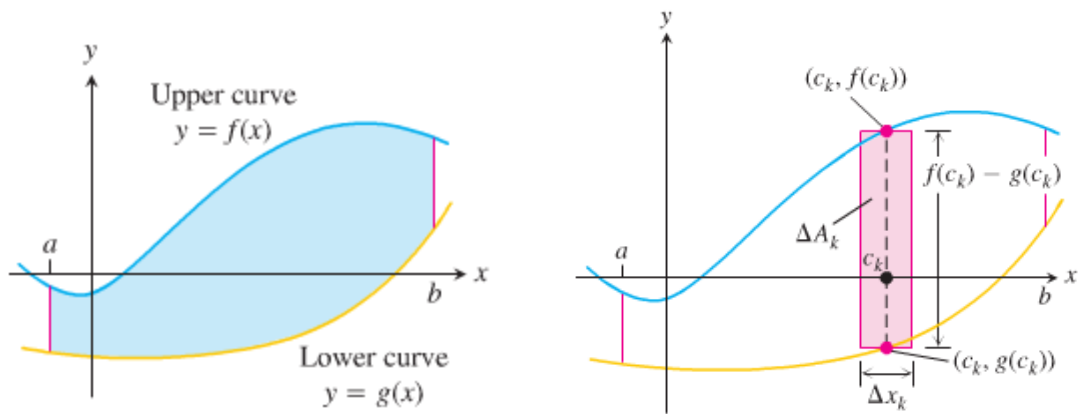
Evaluate  $\int_{-2}^2 (x^4 - 4x^2 + 6) dx$

### Solution

Since  $f(-x) = f(x) \Rightarrow f(x)$  is even

$$\begin{aligned} \int_{-2}^2 (x^4 - 4x^2 + 6) dx &= 2 \int_0^2 (x^4 - 4x^2 + 6) dx \\ &= 2 \left[ \frac{x^5}{5} - \frac{4}{3}x^3 + 6x \right]_0^2 \\ &= 2 \left[ \left( \frac{2^5}{5} - \frac{4}{3}2^3 + 6(2) \right) - 0 \right] \\ &= \underline{\underline{\frac{232}{15}}} \end{aligned}$$

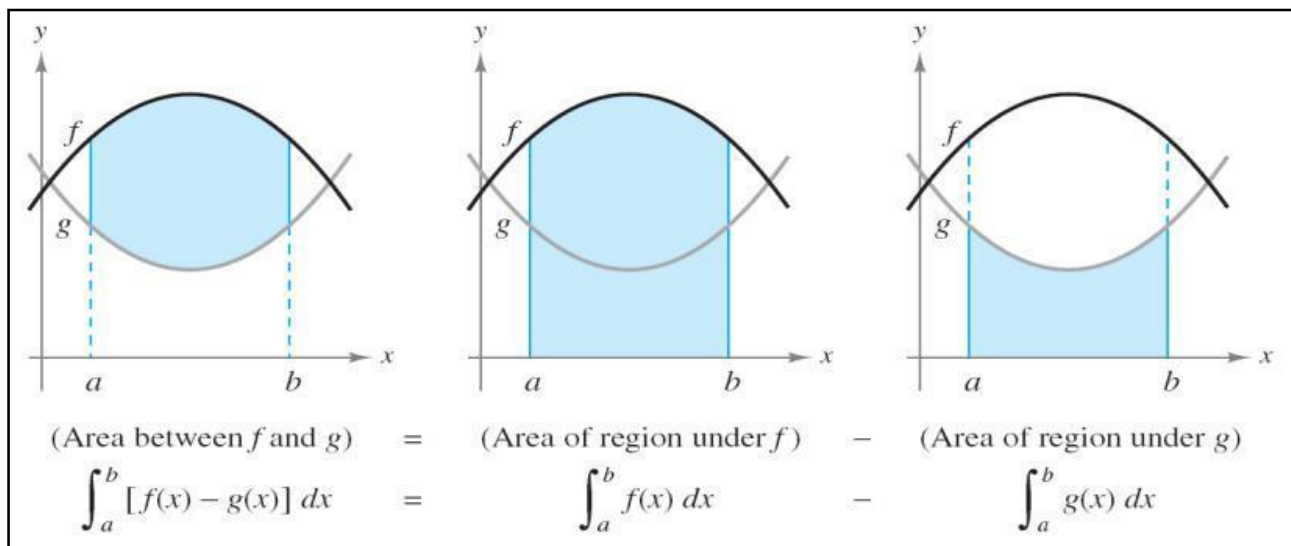
## Areas between Curves



### Definition

If  $f$  and  $g$  are continuous with  $f(x) \geq g(x)$  throughout  $[a, b]$ , then the **area of the region between the curves**  $y = f(x)$  and  $y = g(x)$  **from  $a$  to  $b$**  is:

$$A = \int_a^b [f(x) - g(x)] dx$$



### Example

Find the area of the region enclosed by the parabola  $y = 2 - x^2$  and the line  $y = -x$ .

### Solution

The limits of integrations are found by letting:

$$2 - x^2 = -x$$

$$x^2 - x - 2 = 0 \rightarrow x = -1, 2$$

$$A = \int_{-1}^2 [f(x) - g(x)] dx$$

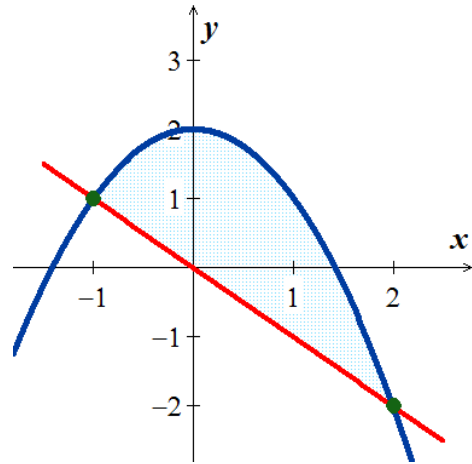
$$= \int_{-1}^2 [2 - x^2 - (-x)] dx$$

$$= \int_{-1}^2 (2 - x^2 + x) dx$$

$$= \left[ 2x - \frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^2$$

$$= \left( 4 - \frac{8}{3} + \frac{4}{2} \right) - \left( -2 + \frac{1}{3} + \frac{1}{2} \right)$$

$$= \frac{9}{2}$$





### Example

Find the area of the region in the first quadrant that is bounded above by  $y = \sqrt{x}$  and below the  $x$ -axis and the line  $y = x - 2$ .

### Solution

$$(y = \sqrt{x}) \cap (y = 0) \rightarrow (0, 0)$$

$$(y = \sqrt{x}) \cap (y = x - 2) \rightarrow \sqrt{x} = x - 2$$

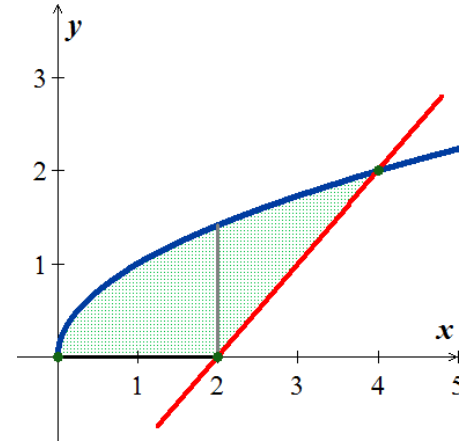
$$(\sqrt{x})^2 = (x - 2)^2$$

$$x = x^2 - 4x + 4$$

$$x^2 - 5x + 4 = 0$$

$$\rightarrow x = \cancel{X}, 4$$

$$(y = 0) \cap (y = x - 2) \rightarrow x = 2$$



$$\text{Total Area} = \int_0^2 [\sqrt{x} - 0] dx + \int_2^4 [\sqrt{x} - (-x + 2)] dx$$

$$= \left[ \frac{2}{3} x^{3/2} \right]_0^2 + \left[ \frac{2}{3} x^{3/2} - \frac{x^2}{2} + 2x \right]_2^4$$

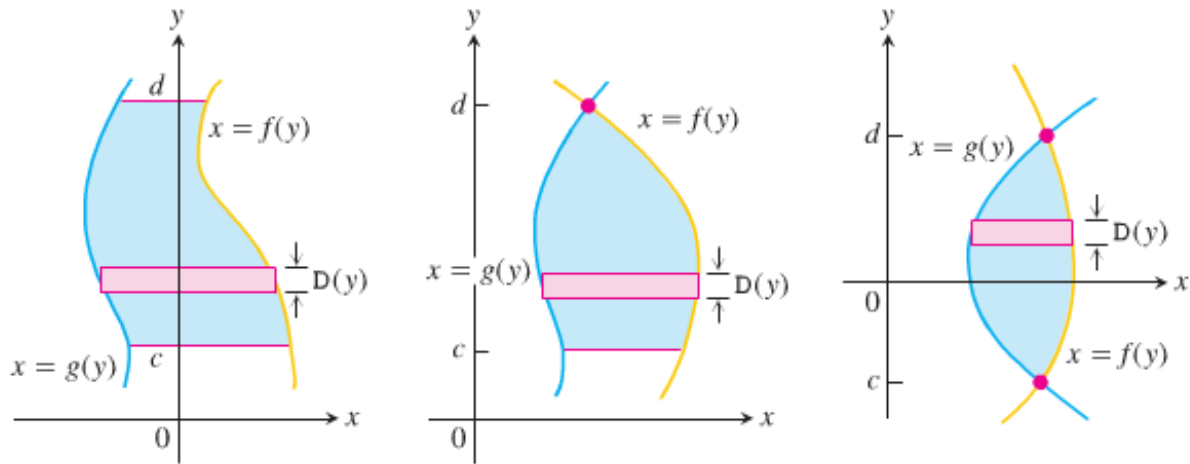
$$= \left[ \frac{2}{3} (2^{3/2}) - 0 \right] + \left( \frac{2}{3} 4^{3/2} - \frac{4^2}{2} + 2(4) \right) - \left( \frac{2}{3} 2^{3/2} - \frac{2^2}{2} + 2(2) \right)$$

$$= \frac{2}{3} (2^{3/2}) + \frac{2}{3} 4^{3/2} - \frac{16}{2} + 8 - \frac{2}{3} 2^{3/2} + \frac{4}{2} - 4$$

$$= \frac{2}{3} (8) - 2$$

$$= \frac{10}{3}$$

## Integration with Respect to y



$$A = \int_c^d [f(y) - g(y)] dy \quad (\text{From right hand to left hand})$$

### Example

Find the area of the region by integrating with respect to y, in the first quadrant that is bounded above by  $y = \sqrt{x}$  and below the x-axis and the line  $y = x - 2$ .

#### Solution

$$y = \sqrt{x} \rightarrow x = y^2$$

$$y = x - 2 \rightarrow x = y + 2$$

$$(x = y^2) \cap (y = 0) \rightarrow (0, 0)$$

$$(x = y^2) \cap (x = y + 2) \rightarrow y^2 = y + 2$$

$$y^2 - y - 2 = 0 \rightarrow y = -1, 2$$

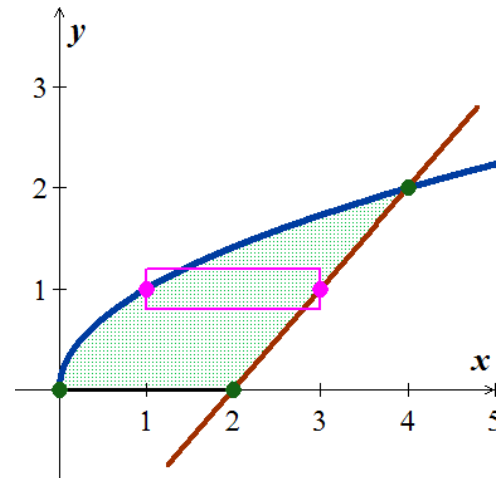
$$(y = 0) \cap (x = y + 2) \rightarrow y = 0$$

$$A = \int_0^2 [y + 2 - y^2] dy$$

$$= \left[ \frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_0^2$$

$$= \frac{2^2}{2} + 2(2) - \frac{2^3}{3} - 0$$

$$= \frac{10}{3}$$



## Exercises      Section 4.7 – Substitution and Area between Curves

Evaluate the integrals

1.  $\int_0^1 (2t+3)^3 dt$

2.  $\int_0^3 (2x+1)dx$

3.  $\int_0^2 \sqrt{4-x^2} dx$

4.  $\int_0^3 \sqrt{y+1} dy$

5.  $\int_{-1}^1 r\sqrt{1-r^2} dr$

6.  $\int_0^{\pi/4} \tan x \sec^2 x dx$

7.  $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x dx$

8.  $\int_0^1 t^3(1+t^4)^3 dt$

9.  $\int_0^1 \frac{r}{(4+r^2)^2} dr$

10.  $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2} dv$

11.  $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}} dx$

12.  $\int_0^1 \frac{x^3}{\sqrt{x^4+9}} dx$

13.  $\int_0^{\pi/6} (1-\cos 3t)\sin 3t dt$

14.  $\int_{-\pi/2}^{\pi/2} \left(2 + \tan \frac{t}{2}\right) \sec^2 \frac{t}{2} dt$

15.  $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}} dz$

16.  $\int_{-\pi/2}^0 \frac{\sin w}{(3+2\cos w)^2} dw$

17.  $\int_0^1 \sqrt{t^5+2t} (5t^4+2) dt$

18.  $\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$

19.  $\int_0^1 (4y-y^2+4y^3+1)^{-2/3} (12y^2-2y+4) dy$

20.  $\int_0^5 |x-2| dx$

21. Find the area of the region bounded by the graphs of  $y = 2x - x^2$  and  $y = -3$

22. Find the area of the region bounded by the graphs of  $y = 7 - 2x^2$  and  $y = x^2 + 4$

23. Find the area of the region bounded by the graphs of  $y = x^4 - 4x^2 + 4$  and  $y = x^2$

24. Find the area of the region bounded by the graphs of  $x = 2y^2$ ,  $x = 0$ , and  $y = 3$
25. Find the area of the region bounded by the graphs of  $x = y^3 - y^2$  and  $x = 2y$
26. Find the area of the region bounded by the graphs of  $4x^2 + y = 4$  and  $x^4 - y = 1$
27. Find the area of the region bounded by the graphs of  $x + 4y^2 = 4$  and  $x + y^4 = 1$ , for  $x \geq 0$
28. Find the area of the region bounded by the graphs of  $y = 2\sin x$ , and  $y = \sin 2x$ ,  $0 \leq x \leq \pi$
29. Find the area of the region bounded by the graphs of  $y = \sin \frac{\pi x}{2}$  and  $y = x$
30. Find the area of the region bounded by the graphs of  

$$y = \sec^2 x, \quad y = \tan^2 x, \quad x = -\frac{\pi}{4}, \quad \text{and} \quad x = \frac{\pi}{4}$$
31. Find the area of the region bounded by the graphs of  $x = 3\sin y\sqrt{\cos y}$ , and  $x = 0$ ,  $0 \leq y \leq \frac{\pi}{2}$
32. Find the area of the region bounded by the graphs of  $y = x^2 + 1$  and  $y = x$  for  $0 \leq x \leq 2$
33. Find the area of the region bounded by the graphs of  $y = 3 - x^2$  and  $y = 2x$
34. Find the area of the region bounded by the graphs of  $y = x^2 - x - 2$  and  $x$ -axis
35. Find the area of the region bounded by the graphs of  $f(x) = x^3 + 2x^2 - 3x$  and  $g(x) = x^2 + 3x$
36. Find the area bounded by  $f(x) = -x^2 + 1$ ,  $g(x) = 2x + 4$ ,  $x = -1$ , and  $x = 2$
37. Find the area between the curves  $y = x^{1/2}$  and  $y = x^3$
38. Find the area of the region bounded by the graphs of  $y = x^2 - 2x$  and  $y = x$  on  $[0, 4]$ .
39. Find the area between the curves  $x = 1$ ,  $x = 2$ ,  $y = x^3 + 2$ ,  $y = 0$
40. Find the area between the curves  $y = x^2 - 18$ ,  $y = x - 6$
41. Find the area between the curves  $y = \sqrt{x}$ ,  $y = x\sqrt{x}$
42. A company is considering a new manufacturing process in one of its plants. The new process provides substantial initial savings, with the savings declining with time  $t$  (in years) according to the rate-of-savings function

$$S'(t) = 100 - t^2$$

where  $S'(t)$  is in thousands of dollars per year. At the same time, the cost of operating the new process increases with time  $t$  (in years), according to the rate-of-cost function (in thousands of dollars per year)

$$C'(t) = t^2 + \frac{14}{3}t$$

- a) For how many years will the company realize savings?
- b) What will be the net total savings during this period?

- 43.** Find the producers' surplus if the supply function for pork bellies is given by

$$S(x) = x^{5/2} + 2x^{3/2} + 50$$

Assume supply and demand are in equilibrium at  $x = 16$ .