Solution Section 1.2 – Propositional Equivalences

Exercise

Use the truth table to verify these equivalences

a) $p \wedge \boldsymbol{T} \equiv p$ *b)* $p \vee \boldsymbol{F} \equiv p$ *c)* $p \wedge \boldsymbol{F} \equiv \boldsymbol{F}$

d) $p \vee \boldsymbol{T} \equiv \boldsymbol{T}$ *e)* $p \vee p \equiv p$ *f)* $p \wedge p \equiv p$

Solution

p	$p \wedge \boldsymbol{T}$	$p \vee \boldsymbol{F}$	$p \wedge \boldsymbol{F}$	$p \vee \boldsymbol{T}$	$p \vee p$	$p \wedge p$
T	*T*	*T*	*F*	*T*	*T*	*T*
F	*F*	*F*	*F*	*T*	*F*	*F*
	hold	***hold***	***hold***	***hold***	***hold***	***hold***

Exercise

Show that $\neg(\neg p)$ and p are logically equivalent

Solution

p	$\neg p$	$\neg(\neg p)$
T	*F*	*T*
F	*T*	*F*

Therefore, $\neg(\neg p)$ and p are logically equivalent

Exercise

Use the truth table to verify the commutative laws

a) $p \vee q \equiv q \vee p$

b) $p \wedge q \equiv q \wedge p$

Solution

p	q	$p \vee q$	$q \vee p$
T	T	***T***	***T***
T	F	***T***	***T***
F	T	***T***	***T***
F	F	***F***	***F***

Identical

p	q	$p \wedge q$	$q \wedge p$
T	T	***T***	***T***
T	F	***F***	***F***
F	T	***F***	***F***
F	F	***F***	***F***

Identical

Exercise

Use the truth table to verify the associative laws

a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Solution

a)

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$
T	T	T	**T**	***T***	**T**	***T***
T	T	F	**T**	***T***	**T**	***T***
T	F	T	**T**	***T***	**T**	***T***
T	F	F	**T**	***T***	**F**	***T***
F	T	T	**T**	***T***	**T**	***T***
F	T	F	**T**	***T***	**T**	***T***
F	F	T	**F**	***T***	**T**	***T***
F	F	F	**F**	***F***	**F**	***F***

$(p \vee q) \vee r \equiv p \vee (q \vee r)$ is true

b)

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	T	**T**	***T***	**T**	***T***
T	T	F	**T**	***F***	**F**	***F***
T	F	T	**F**	***F***	**F**	***F***
T	F	F	**F**	***F***	**F**	***F***
F	T	T	**F**	***F***	**T**	***F***
F	T	F	**F**	***F***	**F**	***F***
F	F	T	**F**	***F***	**F**	***F***
F	F	F	**F**	***F***	**F**	***F***

$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ is true

Exercise

Show that each of these conditional statements is a tautology by using truth result tables.

a) $(p \wedge q) \to p$

b) $p \to (p \vee q)$

c) $\neg p \to (p \to q)$

d) $(p \wedge q) \to (p \to q)$

e) $\neg(p \to q) \to p$

f) $[\neg p \wedge (p \vee q)] \to q$

g) $[(p \to q) \wedge (q \to r)] \to (p \to r)$

h) $[p \wedge (p \to q)] \to q$

Solution

a)

p	q	$p \vee q$	$p \to (p \vee q)$
T	T	***T***	***T***
T	F	***T***	***T***
F	T	***T***	***T***
F	F	***F***	***T***

b)

p	q	$p \wedge q$	$(p \wedge q) \to p$
T	T	***T***	***T***
T	F	***F***	***T***
F	T	***F***	***T***
F	F	***F***	***T***

c)

p	q	$\neg p$	$p \to q$	$\neg p \to (p \to q)$
T	T	***F***	***T***	***T***
T	F	***F***	***F***	***T***
F	T	***T***	***T***	***T***
F	F	***T***	***T***	***T***

d)

p	q	$p \wedge q$	$p \to q$	$(p \wedge q) \to (p \to q)$
T	T	***T***	***T***	***T***
T	F	***F***	***F***	***T***
F	T	***F***	***T***	***T***
F	F	***F***	***T***	***T***

e)

p	q	$p \to q$	$\neg(p \to q)$	$\neg(p \to q) \to p$
T	T	***T***	***F***	***T***
T	F	***F***	***T***	***T***
F	T	***T***	***F***	***T***
F	F	***T***	***F***	***T***

f)

p	q	$p \to q$	$\neg(p \to q)$	$[\neg p \wedge (p \vee q)] \to q$
T	T	***T***	***F***	***T***
T	F	***F***	***T***	***T***
F	T	***T***	***F***	***T***
F	F	***T***	***F***	***T***

g)

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	**T**	**T**	**T**	**T**	***T***
T	T	F	**T**	**F**	**F**	**F**	***T***
T	F	T	**F**	**T**	**F**	**T**	***T***
T	F	F	**F**	**T**	**F**	**F**	***T***
F	T	T	**T**	**T**	**T**	**T**	***T***
F	T	F	**T**	**F**	**F**	**T**	***T***
F	F	T	**T**	**T**	**T**	**T**	***T***
F	F	F	**T**	**T**	**T**	**T**	***T***

h)

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
T	T	***T***	***T***	***T***
T	F	***F***	***F***	***T***
F	T	***T***	***F***	***T***
F	F	***T***	***F***	***T***

Exercise

Show that $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent

Solution

The proposition $p \leftrightarrow q$ is true when p and q have the same true or false value. Since p and q are truth, then $p \wedge q$ only true. When p and q are false, then the negation $\neg p$ and $\neg q$ are true, then $\neg p \wedge \neg q$ is true. Therefore $(p \wedge q) \vee (\neg p \wedge \neg q)$ is true only when both are true. Therefore these two expressions are logically equivalent.

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$	$p \leftrightarrow q$
T	T	***T***	F	F	***F***	***T***	***T***
T	F	***F***	F	T	***F***	***F***	***F***
F	T	***F***	T	F	***F***	***F***	***F***
F	F	***F***	T	T	***T***	***T***	***T***

Exercise

Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent

Solution

The proposition $\neg(p \leftrightarrow q)$ is true when $p \leftrightarrow q$ is false. Since $p \leftrightarrow q$ is true when p and q have the same truth value, it is false when p and q have different truth values (either p is true and q is false, or vice versa). These are precisely the cases in which $p \leftrightarrow \neg q$ is true. Therefore these two expressions are logically equivalent.

p	q	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$\neg q$	$p \leftrightarrow \neg q$
T	T	***T***	***F***	F	***F***
T	F	***F***	***T***	T	***T***
F	T	***F***	***T***	F	***T***
F	F	***T***	***F***	T	***F***

Exercise

Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent

Solution

It is easy to see from the definitions of conditional statement and negation of these propositions is false in the case which p is true and q is false the proposition is false, and true in the other three cases. Therefore these two expressions are logically equivalent.

p	q	$p \rightarrow q$	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
T	T	***T***	F	F	***T***
T	F	***F***	T	F	***F***
F	T	***T***	F	T	***T***
F	F	***T***	T	T	***T***

Exercise

Show that $\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$ are logically equivalent

Solution

The proposition $\neg p \leftrightarrow q$ is true when $\neg p$ and q have the same truth values, which means that p and q have different truth values (either p is true and q is false, or vice versa). By the same reasoning, these are exactly the cases in which $p \leftrightarrow \neg q$ is true. Therefore these two expressions are logically equivalent.

Exercise

Show that $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$ are logically equivalent

Solution

$(p \to q) \vee (p \to r)$ will be true when either of the conditional statements is true. The conditional statement will be true if p is false, or if q in one case or r in the other case is true, when $q \vee r$ is true, which is precisely $p \to (q \vee r)$ is true. Since the two propositions are true in exactly the same situation, they are logically equivalent.

Exercise

Show that $(p \to r) \vee (q \to r)$ and $(p \wedge q) \to r$ are logically equivalent

Solution

In order for $(p \to r) \vee (q \to r)$ to be false, we must have both of the two implications false, which happens exactly when r is false and both p and q are true. But this precisely the case in which $p \wedge q$ is true and r is false, which is $(p \wedge q) \to r$ is false. Therefore these two expressions are logically equivalent.

Exercise

Show that $(p \to q) \wedge (q \to r) \to (p \to r)$ is a tautology

Solution

Given that p and $p \to q$ are both true, we conclude that q is true; from that and $q \to r$ we conclude that r is true.

Exercise

Show that $(p \vee q) \vee (\neg p \vee r) \to (q \vee r)$ is a tautology

Solution

The conclusion $q \vee r$ will be true in every case except when q and r are both false. But if q and r are both false, then one of $p \vee q$ or $\neg p \vee r$ is false, because one of p or $\neg p$ is false. Thus in this case $(p \vee q) \wedge (\neg p \vee r)$ is false. An conditional statement in which the conclusion is true or the hypothesis is false.

Exercise

Show that | (NAND) is functionally complete

Solution

Equivalence of NOT:

$p|p \equiv \neg p$

$\neg(p \wedge p) \equiv \neg p$ ***Equivalence of*** NAND

$\neg(p) \equiv \neg p$ ***Idempotent law***

Equivalence of AND:

$p \wedge q \equiv \neg(p|q)$ ***Definition of*** NAND

$p|p$

$(p|q)|(p|p)q$ ***Negation of*** $(p|q)$

Equivalence of OR:

$p \vee q \equiv \neg(\neg p \wedge \neg q)$ **DeMorgan's *equivalence of*** OR

We can do AND and OR with NANDs, also do ORs with NANDs

Thus, NAND is functionally complete.