Solution — Section 1.7 – Properties of Determinants: Cramer's Rule

Exercise

Use Cramer's Rule with ratios $\dfrac{\det B_j}{\det A}$ to solve $A\boldsymbol{x} = b$. Also find the inverse matrix $A^{-1} = \dfrac{C^T}{\det A}$. Why is the solution $\boldsymbol{x}$ is the first part the same as column 3 of A^{-1}? Which cofactors are involved in computing that column $\boldsymbol{x}$?

$$Ax = b \quad is \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Find the volumes of the boxes whose edges are columns of $\boldsymbol{A}$ and then rows of A^{-1}.

Solution

$$|A| = \begin{vmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{vmatrix} = 2$$

$$\left|B_1\right| = \begin{vmatrix} 0 & 6 & 2 \\ 0 & 4 & 2 \\ 1 & 9 & 0 \end{vmatrix} = 4 \qquad \left|B_2\right| = \begin{vmatrix} 2 & 0 & 2 \\ 1 & 0 & 2 \\ 5 & 1 & 0 \end{vmatrix} = -2 \qquad \left|B_1\right| = \begin{vmatrix} 2 & 6 & 0 \\ 1 & 4 & 0 \\ 5 & 9 & 1 \end{vmatrix} = 2$$

$$x = \frac{4}{2} = 2; \quad y = \frac{-2}{2} = -1; \quad z = \frac{2}{2} = 1$$

The solution is: $\boxed{(2, \ -1, \ 1)}$

$$C_{11} = \begin{vmatrix} 4 & 2 \\ 9 & 0 \end{vmatrix} = -18 \qquad C_{12} = -\begin{vmatrix} 1 & 2 \\ 5 & 0 \end{vmatrix} = 10 \qquad C_{13} = \begin{vmatrix} 1 & 4 \\ 5 & 9 \end{vmatrix} = -11$$

$$C_{21} = -\begin{vmatrix} 6 & 2 \\ 9 & 0 \end{vmatrix} = 18 \qquad C_{22} = \begin{vmatrix} 2 & 2 \\ 5 & 0 \end{vmatrix} = -10 \qquad C_{23} = -\begin{vmatrix} 2 & 6 \\ 5 & 9 \end{vmatrix} = 12$$

$$C_{31} = \begin{vmatrix} 6 & 2 \\ 4 & 2 \end{vmatrix} = 4 \qquad C_{32} = -\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = -2 \qquad C_{33} = \begin{vmatrix} 2 & 6 \\ 1 & 4 \end{vmatrix} = 2$$

$$C = \begin{pmatrix} -18 & 10 & -11 \\ 18 & -10 & 12 \\ 4 & -2 & 2 \end{pmatrix}$$

$$C^T = \begin{pmatrix} -18 & 18 & 4 \\ 10 & -10 & -2 \\ -11 & 12 & 2 \end{pmatrix}$$

$$A^{-1}=\frac{1}{2}\begin{pmatrix}-18 & 18 & 4\\ 10 & -10 & -2\\ -11 & 12 & 2\end{pmatrix} \qquad A^{-1}=\frac{C^T}{\det A}$$

$$=\begin{pmatrix}-9 & 9 & 2\\ 5 & -5 & -1\\ -\frac{11}{2} & 6 & 1\end{pmatrix}$$

The solution $\boldsymbol{x}$ is the third column of A^{-1} because $\boldsymbol{b}=(0, 0, 1)$ is the third column of I.

The volume of the boxes whose edges are columns of $\boldsymbol{A}=\det(\boldsymbol{A})=2$.

Since $\left|A^T\right|=|A|$. The box from rows of A^{-1} has volume $\left|A^{-1}\right|=\frac{1}{|A|}\ \underline{=\frac{1}{2}}$

Exercise

Verify that $\det(AB)=\det(BA)$ and determine whether the equality $\det(A+B)=\det(A)+\det(B)$ holds

$$A=\begin{bmatrix}2 & 1 & 0\\ 3 & 4 & 0\\ 0 & 0 & 2\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix}1 & -1 & 3\\ 7 & 1 & 2\\ 5 & 0 & 1\end{bmatrix}$$

Solution

$$AB=\begin{bmatrix}2 & 1 & 0\\ 3 & 4 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & -1 & 3\\ 7 & 1 & 2\\ 5 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}9 & -1 & 8\\ 31 & 1 & 17\\ 10 & 0 & 2\end{bmatrix}$$

$$\det(AB)=\begin{vmatrix}9 & -1 & 8\\ 31 & 1 & 17\\ 10 & 0 & 2\end{vmatrix}$$

$$\underline{=-170}$$

$$BA=\begin{bmatrix}1 & -1 & 3\\ 7 & 1 & 2\\ 5 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\ 3 & 4 & 0\\ 0 & 0 & 2\end{bmatrix}$$

$$=\begin{bmatrix}-1 & -3 & 6\\ 17 & 11 & 4\\ 10 & 5 & 2\end{bmatrix}$$

$$\det(BA) = \begin{vmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{vmatrix}$$

$$\underline{= -170}$$

Thus, $\underline{\det(AB) = \det(BA)}$

$$\det(A) = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$$\underline{= 10}$$

$$\det(B) = \begin{vmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{vmatrix}$$

$$\underline{= -17}$$

$$A + B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 3 \\ 10 & 5 & 2 \\ 5 & 0 & 3 \end{bmatrix}$$

$$\det(A+B) = \begin{vmatrix} 3 & 0 & 3 \\ 10 & 5 & 2 \\ 5 & 0 & 3 \end{vmatrix}$$

$$\underline{= -30}$$

$$\det(A) + \det(B) = 10 - 17$$

$$= -7 \quad \neq -30$$

$$\underline{\neq \det(A+B)}$$

Exercise

Verify that $\det(kA) = k^n \det(A)$ $\quad A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}, \quad k = 2$

Solution

$$\det(A) = \begin{vmatrix} -1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$= -10$$

$$\det(2A) = \begin{vmatrix} -2 & 4 \\ 6 & 8 \end{vmatrix}$$

$$= -40$$

$$= 4(-10)$$

$$= 2^2(-10)$$

$$= k^2 \det(A)$$

Exercise

Verify that $\det(kA) = k^n \det(A)$ $\quad A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}, \quad k = -2$

Solution

$$\det(A) = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{vmatrix}$$

$$= 56$$

$$\det(-2A) = \begin{vmatrix} -4 & 2 & -6 \\ -6 & -4 & -2 \\ -2 & -8 & -0 \end{vmatrix}$$

$$= -448$$

$$= (-2)^3(56)$$

$$= k^3 \det(A)$$

Exercise

Verify that $\det(kA) = k^n \det(A)$ $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}, \quad k = 3$

Solution

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= -7$$

$$\det(3A) = \begin{vmatrix} 3 & 3 & 3 \\ 0 & 6 & 9 \\ 0 & 3 & -6 \end{vmatrix}$$

$$= -189$$

$$= 3^3(-7)$$

$$= k^3 \det(A)$$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$

Solution

$$D = \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -3 - 4 = -7$$

$$D_x = \begin{vmatrix} -4 & 2 \\ -5 & -1 \end{vmatrix} = 4 - (-10) = 14$$

$$D_y = \begin{vmatrix} 3 & -4 \\ 2 & -5 \end{vmatrix} = -15 - (-8) = -7$$

$$x = \frac{D_x}{D} = \frac{14}{-7} = -2$$

$$y = \frac{D_y}{D} = \frac{-7}{-7} = 1$$

$\therefore$ ***Solution***: $(-2,\ 1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$

Solution

$D = \begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix} = -29 \qquad D_x = \begin{vmatrix} 7 & 5 \\ -3 & -2 \end{vmatrix} = 1 \qquad D_y = \begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix} = -41$

$x = \frac{1}{-29} = -\frac{1}{29} \qquad x = \frac{D_x}{D}$

$y = \frac{41}{29} \qquad y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $\left(-\frac{1}{29},\ \frac{41}{29}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$

Solution

$D = \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -7 \qquad D_x = \begin{vmatrix} -4 & -5 \\ 2 & -1 \end{vmatrix} = 14 \qquad D_y = \begin{vmatrix} 3 & -4 \\ 2 & -5 \end{vmatrix} = -7$

$x = -\frac{14}{7} = -2 \qquad x = \frac{D_x}{D}$

$y = \frac{7}{7} = 1 \qquad y = \frac{D_y}{D}$

Solution: $(-2,\ 1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$

Solution

$D = \begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix} = -29 \qquad D_x = \begin{vmatrix} 7 & 5 \\ -3 & -2 \end{vmatrix} = 1 \qquad D_y = \begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix} = -41$

$x = -\frac{1}{29} \qquad x = \frac{D_x}{D}$

$$y = \frac{41}{29}$$ $$y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\left(-\frac{1}{29},\ \frac{41}{29}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$

Solution

$$D = \begin{vmatrix} 4 & -7 \\ 2 & 5 \end{vmatrix} = 34 \qquad D_x = \begin{vmatrix} -16 & -7 \\ 9 & 5 \end{vmatrix} = -17 \qquad D_y = \begin{vmatrix} 4 & -16 \\ 2 & 9 \end{vmatrix} = 68$$

$$x = -\frac{17}{34} = -\frac{1}{2} \qquad x = \frac{D_x}{D}$$

$$y = \frac{68}{34} = 2 \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\left(-\frac{1}{2},\ 2\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$

Solution

$$D = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = -1 \qquad D_x = \begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} = 2 \qquad D_y = \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -5$$

$$x = -2 \qquad x = \frac{D_x}{D}$$

$$y = 5 \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $(-2,\ 5)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$

Solution

$$D=\begin{vmatrix}3 & 4\\ 2 & 5\end{vmatrix}=7 \qquad D_x=\begin{vmatrix}2 & 4\\ -1 & 5\end{vmatrix}=14 \qquad D_y=\begin{vmatrix}3 & 2\\ 2 & -1\end{vmatrix}=-7$$

$$x=\frac{14}{7}=2 \qquad x=\frac{D_x}{D}$$

$$y=-\frac{7}{7}=-1 \qquad y=\frac{D_y}{D}$$

$\therefore$ ***Solution***: $\quad (2,\ -1)$

Exercise

Use Cramer's rule to solve the system $\quad \begin{cases} 5x-2y=4 \\ -10x+4y=7 \end{cases}$

Solution

$$D=\begin{vmatrix}5 & -2\\ -10 & 4\end{vmatrix}=0 \qquad D_y=\begin{vmatrix}5 & 4\\ -10 & 7\end{vmatrix}=75\neq 0$$

$\therefore$ ***No Solution***

Exercise

Use Cramer's rule to solve the system $\quad \begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$

Solution

$$D=\begin{vmatrix}1 & -4\\ 5 & -20\end{vmatrix}=0 \qquad D_y=\begin{vmatrix}1 & -8\\ 5 & -40\end{vmatrix}=0$$

$$\begin{array}{l} \\ \frac{1}{5}\times \end{array}\begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$$

$$\begin{cases} x-4y=-8 \\ x-4y=-8 \end{cases}$$

$\therefore$ ***Solution***: $\quad (4y-8,\ y)$

Exercise

Use Cramer's rule to solve the system $\quad \begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3 \qquad D_x = \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} = -6 \qquad D_y = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 3$$

$$x = \frac{6}{3} = 2 \qquad x = \frac{D_x}{D}$$

$$y = -\frac{3}{3} = -1 \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $(2,\ -1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & 10 \\ 7 & -2 \end{vmatrix} = -74 \qquad D_x = \begin{vmatrix} -14 & 10 \\ -16 & -2 \end{vmatrix} = 188 \qquad D_y = \begin{vmatrix} 2 & -14 \\ 7 & -16 \end{vmatrix} = 66$$

$$x = -\frac{188}{74} = -\frac{94}{37} \qquad x = \frac{D_x}{D}$$

$$y = -\frac{66}{74} = -\frac{33}{37} \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\left(-\frac{94}{37},\ -\frac{33}{37}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$

Solution

$$D = \begin{vmatrix} 4 & -3 \\ -3 & 9 \end{vmatrix} = 27 \qquad D_x = \begin{vmatrix} 24 & -3 \\ -1 & 9 \end{vmatrix} = 213 \qquad D_y = \begin{vmatrix} 4 & 24 \\ -3 & -1 \end{vmatrix} = 68$$

$$x = \frac{213}{27} = \frac{71}{9} \qquad x = \frac{D_x}{D}$$

$$y = \frac{68}{27} \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\left(\frac{71}{9},\ \frac{68}{27}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$

Solution

$D = \begin{vmatrix} 4 & 2 \\ 3 & -2 \end{vmatrix} = -14 \qquad D_x = \begin{vmatrix} 12 & 2 \\ 16 & -2 \end{vmatrix} = -56 \qquad D_y = \begin{vmatrix} 4 & 12 \\ 3 & 16 \end{vmatrix} = 28$

$x = \frac{56}{14} = 4 \qquad x = \frac{D_x}{D}$

$y = -\frac{28}{14} = -2 \qquad y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $(4,\ -2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$

Solution

$D = \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} = -10 \qquad D_x = \begin{vmatrix} -1 & 2 \\ 6 & -2 \end{vmatrix} = -10 \qquad D_y = \begin{vmatrix} 1 & -1 \\ 4 & 6 \end{vmatrix} = 10$

$x = 1 \qquad x = \frac{D_x}{D}$

$y = -1 \qquad y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $(1,\ -1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$

Solution

$D = \begin{vmatrix} 1 & -2 \\ -10 & 2 \end{vmatrix} = -18 \qquad D_x = \begin{vmatrix} 5 & -2 \\ 4 & 2 \end{vmatrix} = 18 \qquad D_y = \begin{vmatrix} 1 & 5 \\ -10 & 4 \end{vmatrix} = 54$

$x = -1 \qquad x = \frac{D_x}{D}$

$y = -\frac{54}{18} = -3 \qquad y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $(-1,\ -3)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$

Solution

$$\begin{array}{l} \frac{1}{3}\times \\ \frac{1}{15}\times \end{array} \begin{cases} 12x+15y=-27 \\ 30x-15y=-15 \end{cases}$$

$$\begin{cases} 4x+5y=-9 \\ 2x-y=-1 \end{cases}$$

$$D=\begin{vmatrix} 4 & 5 \\ 2 & -1 \end{vmatrix}=-14 \qquad D_x=\begin{vmatrix} -9 & 5 \\ -1 & -1 \end{vmatrix}=14 \qquad D_y=\begin{vmatrix} 4 & -9 \\ 2 & -1 \end{vmatrix}=14$$

$\underline{x=-1}$ $\quad x=\dfrac{D_x}{D}$

$\underline{y=-1}$ $\quad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(-1,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x-4y=-12 \\ 4x+4y=-20 \end{cases}$

Solution

$$\begin{array}{l} \frac{1}{4}\times \\ \frac{1}{4}\times \end{array} \begin{cases} 4x-4y=-12 \\ 4x+4y=-20 \end{cases}$$

$$\begin{cases} x-y=-3 \\ x+y=-5 \end{cases}$$

$$D=\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}=2 \qquad D_x=\begin{vmatrix} -3 & -1 \\ -5 & 1 \end{vmatrix}=-8 \qquad D_y=\begin{vmatrix} 1 & -3 \\ 1 & -5 \end{vmatrix}=-2$$

$\underline{x=-4}$ $\quad x=\dfrac{D_x}{D}$

$\underline{y=-1}$ $\quad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(-4,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+y=7 \\ x-y=3 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}=-2$$

$$D_x=\begin{vmatrix} 7 & 1 \\ 3 & -1 \end{vmatrix}=-10 \qquad D_y=\begin{vmatrix} 1 & 7 \\ 1 & 3 \end{vmatrix}=-4$$

$\underline{x=5}$ $\qquad x=\frac{D_x}{D}$

$\underline{y=2}$ $\qquad y=\frac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(5,\ 2)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+y=3 \\ x-y=3 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}=-3$$

$$D_x=\begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix}=-6 \qquad D_y=\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix}=3$$

$\underline{x=2}$ $\qquad x=\frac{D_x}{D}$

$\underline{y=-1}$ $\qquad y=\frac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(2,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 12x+3y=15 \\ 2x-3y=13 \end{cases}$

Solution

$$D=\begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix}=-42$$

$$D_x = \begin{vmatrix} 15 & 3 \\ 13 & -3 \end{vmatrix} = -84 \qquad D_y = \begin{vmatrix} 12 & 15 \\ 2 & 13 \end{vmatrix} = 126$$

$\underline{x = 2}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = -3}$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\quad \underline{(2,\ -3)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x - 2y = 5 \\ 5x - y = -2 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & -2 \\ 5 & -1 \end{vmatrix} = 9$$

$$D_x = \begin{vmatrix} 5 & -2 \\ -2 & -1 \end{vmatrix} = -9 \qquad D_y = \begin{vmatrix} 1 & 5 \\ 5 & -2 \end{vmatrix} = -27$$

$\underline{x = -1}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = -3}$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\quad \underline{(-1,\ -3)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x - 5y = 17 \\ 2x + 3y = 3 \end{cases}$

Solution

$$D = \begin{vmatrix} 4 & -5 \\ 2 & 3 \end{vmatrix} = 22$$

$$D_x = \begin{vmatrix} 17 & -5 \\ 3 & 3 \end{vmatrix} = 66 \qquad D_y = \begin{vmatrix} 4 & 17 \\ 2 & 3 \end{vmatrix} = -22$$

$\underline{x = 3}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = -1}$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\quad \underline{(3,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y=2 \\ 2x+2y=3 \end{cases}$

Solution

$$D = \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 2$$

$$D_x = \begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = -2 \qquad D_y = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5$$

$$\underline{x=-1} \qquad x = \frac{D_x}{D}$$

$$\underline{y=\frac{5}{2}} \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\underline{\left(-1,\ \frac{5}{2}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-3y=4 \\ 3x-4y=12 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & -3 \\ 3 & -4 \end{vmatrix} = 5 \qquad D_x = \begin{vmatrix} 4 & -3 \\ 12 & -4 \end{vmatrix} = 20 \qquad D_y = \begin{vmatrix} 1 & 4 \\ 3 & 12 \end{vmatrix} = 0$$

$$\underline{x=4} \qquad x = \frac{D_x}{D}$$

$$\underline{y=0} \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\underline{(4,\ 0)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x-9y=5 \\ 3x-3y=11 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & -9 \\ 3 & -3 \end{vmatrix} = 21 \qquad D_x = \begin{vmatrix} 5 & -9 \\ 11 & -3 \end{vmatrix} = 84 \qquad D_y = \begin{vmatrix} 2 & 5 \\ 3 & 11 \end{vmatrix} = 7$$

$$\underline{x=4} \qquad x = \frac{D_x}{D}$$

$y = \frac{1}{3}$ $\qquad y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $\left(4,\ \frac{1}{3}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x - 4y = 4 \\ x + y = 6 \end{cases}$

Solution

$$D = \begin{vmatrix} 3 & -4 \\ 1 & 1 \end{vmatrix} = 7$$

$$D_x = \begin{vmatrix} 4 & -4 \\ 6 & 1 \end{vmatrix} = 28 \qquad D_y = \begin{vmatrix} 3 & 4 \\ 1 & 6 \end{vmatrix} = 14$$

$x = 4$ $\qquad x = \frac{D_x}{D}$

$y = 2$ $\qquad y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $(4,\ 2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x = 7y + 1 \\ 2x = 3y - 1 \end{cases}$

Solution

$$\begin{cases} 3x - 7y = 1 \\ 2x - 3y = -1 \end{cases}$$

$$D = \begin{vmatrix} 3 & -7 \\ 2 & -3 \end{vmatrix} = 5$$

$$D_x = \begin{vmatrix} 1 & -7 \\ -1 & -3 \end{vmatrix} = -10 \qquad D_y = \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} = -5$$

$x = -2$ $\qquad x = \frac{D_x}{D}$

$y = -1$ $\qquad y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $(-2,\ -1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x = 3y + 2 \\ 5x = 51 - 4y \end{cases}$

Solution

$$\begin{cases} 2x - 3y = 2 \\ 5x + 4y = 51 \end{cases}$$

$$D = \begin{vmatrix} 2 & -3 \\ 5 & 4 \end{vmatrix} = 23$$

$$D_x = \begin{vmatrix} 2 & -3 \\ 51 & 4 \end{vmatrix} = 161 \qquad D_y = \begin{vmatrix} 2 & 2 \\ 5 & 51 \end{vmatrix} = 92$$

$$\underline{x = 7} \qquad x = \frac{D_x}{D}$$

$$\underline{y = 4} \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\underline{(7,\ 4)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} y = -4x + 2 \\ 2x = 3y - 1 \end{cases}$

Solution

$$\begin{cases} 4x + y = 2 \\ 2x - 3y = -1 \end{cases}$$

$$D = \begin{vmatrix} 4 & 1 \\ 2 & -3 \end{vmatrix} = -14$$

$$D_x = \begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix} = -5 \qquad D_y = \begin{vmatrix} 4 & 2 \\ 2 & -1 \end{vmatrix} = -8$$

$$\underline{x = \frac{5}{14}} \qquad x = \frac{D_x}{D}$$

$$\underline{y = \frac{4}{7}} \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\underline{\left(\frac{15}{4},\ \frac{4}{7}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x = 2 - 3y \\ 2y = 3 - 2x \end{cases}$

Solution

$$\begin{cases} 3x + 3y = 2 \\ 2x + 2y = 3 \end{cases}$$

$$D = \begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix} = 0 \qquad D_y = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5 \neq 0$$

$\therefore$ ***No Solution***

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + 2y - 3 = 0 \\ 12 = 8y + 4x \end{cases}$

Solution

$$\begin{cases} x + 2y = 3 \\ 4x + 8y = 12 \end{cases}$$

$$\begin{cases} x + 2y = 3 \\ x + 2y = 3 \end{cases}$$

$\therefore$ ***Solution:*** $\quad (3 - 2y,\ y)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 7x - 2y = 3 \\ 3x + y = 5 \end{cases}$

Solution

$$D = \begin{vmatrix} 7 & -2 \\ 3 & 1 \end{vmatrix} = 13$$

$$D_x = \begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 13 \qquad D_y = \begin{vmatrix} 7 & 3 \\ 3 & 5 \end{vmatrix} = 26$$

$x = 1 \qquad x = \dfrac{D_x}{D}$

$y = 2 \qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\quad (1,\ 2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y-z=4 \\ 3x-2y+z=5 \\ 4x-5y-z=-1 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 2 & -1 \\ 3 & -2 & 1 \\ 4 & -5 & -1 \end{vmatrix}=42 \qquad D_x=\begin{vmatrix} 4 & 2 & -1 \\ 5 & -2 & 1 \\ -1 & -5 & -1 \end{vmatrix}=63$$

$$D_y=\begin{vmatrix} 3 & 4 & -1 \\ 3 & 5 & 1 \\ 4 & -1 & -1 \end{vmatrix}=39 \qquad D_z=\begin{vmatrix} 3 & 2 & 4 \\ 3 & -2 & 5 \\ 4 & -5 & -1 \end{vmatrix}=99$$

$x=\frac{63}{42}=\frac{3}{2}$ $\qquad x=\frac{D_x}{D}$

$y=\frac{39}{42}=\frac{13}{14}$ $\qquad y=\frac{D_y}{D}$

$z=\frac{99}{42}=\frac{33}{14}$ $\qquad z=\frac{D_z}{D}$

Solution: $\left(\frac{3}{2},\ \frac{13}{14},\ \frac{33}{14}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}=-6 \qquad D_x=\begin{vmatrix} 2 & 1 & 1 \\ 5 & 1 & -1 \\ -2 & -1 & 1 \end{vmatrix}=-6$$

$$D_y=\begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & -1 \\ 1 & -2 & 1 \end{vmatrix}=-12 \qquad D_z=\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{vmatrix}=6$$

$x=1$ $\qquad x=\frac{D_x}{D}$

$y=2$ $\qquad y=\frac{D_y}{D}$

$z = -1$ $\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $(1,\ 2,\ -1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & 1 & 1 \\ -1 & -1 & 1 \\ 3 & -1 & 1 \end{vmatrix} \begin{matrix} 2 & 1 \\ -1 & -1 \\ 3 & -1 \end{matrix} = -2 + 3 + 1 + 3 + 2 + 1$$

$= 8$

$$D_x = \begin{vmatrix} 9 & 1 & 1 \\ 1 & -1 & 1 \\ 9 & -1 & 1 \end{vmatrix} \begin{matrix} 9 & 1 \\ 1 & -1 \\ 9 & -1 \end{matrix}$$

$= -9 + 9 - 1 + 9 + 9 - 1$

$= 16$

$$D_y = \begin{vmatrix} 2 & 9 & 1 \\ -1 & 1 & 1 \\ 3 & 9 & 1 \end{vmatrix} \begin{matrix} 2 & 9 \\ -1 & 1 \\ 3 & 9 \end{matrix}$$

$= 2 + 27 - 9 - 3 - 18 + 9$

$= 8$

$$D_z = \begin{vmatrix} 2 & 1 & 9 \\ -1 & -1 & 1 \\ 3 & -1 & 9 \end{vmatrix} \begin{matrix} 2 & 1 \\ -1 & -1 \\ 3 & -1 \end{matrix}$$

$= -18 + 3 + 9 + 27 + 2 + 9$

$= 32$

$x = 2$ $\qquad x = \dfrac{D_x}{D}$

$y = 1$ $\qquad y = \dfrac{D_y}{D}$

$z = \dfrac{32}{8} = 4$ $\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $(2,\ 1,\ 4)$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$$

Solution

$$D = \begin{vmatrix} 0 & 3 & -1 \\ 1 & 5 & -1 \\ -3 & 6 & 2 \end{vmatrix} \begin{matrix} 0 & 3 \\ 1 & 5 \\ -3 & 6 \end{matrix}$$

$$= 9 - 6 - 15 - 6$$

$$= -18$$

$$D_x = \begin{vmatrix} -1 & 3 & -1 \\ -4 & 5 & -1 \\ 11 & 6 & 2 \end{vmatrix} \begin{matrix} -1 & 3 \\ -4 & 5 \\ 11 & 6 \end{matrix}$$

$$= -10 - 33 + 24 + 55 - 6 + 24$$

$$= 54$$

$$D_y = \begin{vmatrix} 0 & -1 & -1 \\ 1 & -4 & -1 \\ -3 & 11 & 2 \end{vmatrix} \begin{matrix} 0 & -1 \\ 1 & -4 \\ -3 & 11 \end{matrix}$$

$$= -3 - 11 + 12 + 2$$

$$= 0$$

$$D_z = \begin{vmatrix} 0 & 3 & -1 \\ 1 & 5 & -4 \\ -3 & 6 & 11 \end{vmatrix} \begin{matrix} 0 & 3 \\ 1 & 5 \\ -3 & 6 \end{matrix}$$

$$= 36 - 6 - 15 - 33$$

$$= -18$$

$x = -3$ $\qquad x = \dfrac{D_x}{D}$

$y = 0$ $\qquad y = \dfrac{D_y}{D}$

$z = 1$ $\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $(-3,\ 0,\ 1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+3y+4z=14 \\ 2x-3y+2z=10 \\ 3x-y+z=9 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & 3 & 4 \\ 2 & -3 & 2 \\ 3 & -1 & 1 \end{vmatrix}\begin{matrix} 1 & 3 \\ 2 & -3 \\ 3 & -1 \end{matrix}$$

$$=-3+18-8+36+2-6$$

$$=39$$

$$D_x=\begin{vmatrix} 14 & 3 & 4 \\ 10 & -3 & 2 \\ 9 & -1 & 1 \end{vmatrix}\begin{matrix} 14 & 3 \\ 10 & -3 \\ 9 & -1 \end{matrix}$$

$$=-42+54-40+108+28-30$$

$$=78$$

$$D_y=\begin{vmatrix} 1 & 14 & 4 \\ 2 & 10 & 2 \\ 3 & 9 & 1 \end{vmatrix}\begin{matrix} 1 & 14 \\ 2 & 10 \\ 3 & 9 \end{matrix}$$

$$=10+84+72-120-18-28$$

$$=0$$

$$D_z=\begin{vmatrix} 1 & 3 & 14 \\ 2 & -3 & 10 \\ 3 & -1 & 9 \end{vmatrix}\begin{matrix} 1 & 3 \\ 2 & -3 \\ 3 & -1 \end{matrix}$$

$$=-27+90-28+126+10-54$$

$$=117$$

$x=\frac{78}{39}=2$ $\qquad$ $x=\frac{D_x}{D}$

$y=0$ $\qquad$ $y=\frac{D_y}{D}$

$z=\frac{117}{39}=3$ $\qquad$ $z=\frac{D_z}{D}$

$\therefore$ ***Solution***: $(2,\ 0,\ 3)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+4y-z=20 \\ 3x+2y+z=8 \\ 2x-3y+2z=-16 \end{cases}$

Solution

$$D=\begin{vmatrix}1 & 4 & -1\\ 3 & 2 & 1\\ 2 & -3 & 2\end{vmatrix}\begin{matrix}1 & 4\\ 3 & 2\\ 2 & -3\end{matrix}$$

$$=4+8+9+4+3-24$$

$$=4$$

$$D_x=\begin{vmatrix}20 & 4 & -1\\ 8 & 2 & 1\\ -16 & -3 & 2\end{vmatrix}\begin{matrix}20 & 4\\ 8 & 2\\ -16 & -3\end{matrix}$$

$$=80-64+24-32+60-64$$

$$=4$$

$$D_y=\begin{vmatrix}1 & 20 & -1\\ 3 & 8 & 1\\ 2 & -16 & 2\end{vmatrix}\begin{matrix}1 & 20\\ 3 & 8\\ 2 & -16\end{matrix}$$

$$=16+40+48+16+16-120$$

$$=16$$

$$D_z=\begin{vmatrix}1 & 4 & 20\\ 3 & 2 & 8\\ 2 & -3 & -16\end{vmatrix}\begin{matrix}1 & 4\\ 3 & 2\\ 2 & -3\end{matrix}$$

$$=-32+64-180-80+24+192$$

$$=-12$$

$x=\frac{4}{4}=1$ $\qquad x=\frac{D_x}{D}$

$y=\frac{16}{4}=4$ $\qquad y=\frac{D_y}{D}$

$z=-\frac{12}{4}=-3$ $\qquad z=\frac{D_z}{D}$

$\therefore$ ***Solution***: $(1,\ 4,\ -3)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} -2x+6y+7z=3 \\ -4x+5y+3z=7 \\ -6x+3y+5z=-4 \end{cases}$

Solution

$$D=\begin{vmatrix} -2 & 6 & 7 \\ -4 & 5 & 3 \\ -6 & 3 & 5 \end{vmatrix}\begin{matrix} -2 & 6 \\ -4 & 5 \\ -6 & 3 \end{matrix}$$

$$=-50-108-84+210+18+120$$

$$=106$$

$$D_x=\begin{vmatrix} 3 & 6 & 7 \\ 7 & 5 & 3 \\ -4 & 3 & 5 \end{vmatrix}\begin{matrix} 3 & 6 \\ 7 & 5 \\ -4 & 3 \end{matrix}$$

$$=75-72+147+140-27-210$$

$$=53$$

$$D_y=\begin{vmatrix} -2 & 3 & 7 \\ -4 & 7 & 3 \\ -6 & -4 & 5 \end{vmatrix}\begin{matrix} -2 & 3 \\ -4 & 7 \\ -6 & -4 \end{matrix}$$

$$=-70-54+112+294-24+60$$

$$=318$$

$$D_z=\begin{vmatrix} -2 & 6 & 3 \\ -4 & 5 & 7 \\ -6 & 3 & -4 \end{vmatrix}\begin{matrix} -2 & 6 \\ -4 & 5 \\ -6 & 3 \end{matrix}$$

$$=40-252-36+90+42-96$$

$$=-212$$

$$x=\frac{53}{106}=\frac{1}{2} \qquad x=\frac{D_x}{D}$$

$$y=\frac{318}{106}=3 \qquad y=\frac{D_y}{D}$$

$$z=-\frac{212}{106}=-2 \qquad z=\frac{D_z}{D}$$

$\therefore$ ***Solution***: $\left(\frac{1}{2},\ 3,\ -2\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x-y+z=1 \\ 3x-3y+4z=5 \\ 4x-2y+3z=4 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & -1 & 1 \\ 3 & -3 & 4 \\ 4 & -2 & 3 \end{vmatrix}\begin{matrix} 2 & -1 \\ 3 & -3 \\ 4 & -2 \end{matrix}$$

$$=-18-16-6+12+16+9$$

$$=-3$$

$$D_x=\begin{vmatrix} 1 & -1 & 1 \\ 5 & -3 & 4 \\ 4 & -2 & 3 \end{vmatrix}\begin{matrix} 1 & -1 \\ 5 & -3 \\ 4 & -2 \end{matrix}$$

$$=-9-16-10+12+8+15$$

$$=0$$

$$D_y=\begin{vmatrix} 2 & 1 & 1 \\ 3 & 5 & 4 \\ 4 & 4 & 3 \end{vmatrix}\begin{matrix} 2 & 1 \\ 3 & 5 \\ 4 & 4 \end{matrix}$$

$$=30+16+12-20-32-9$$

$$=-3$$

$$D_z=\begin{vmatrix} 2 & -1 & 1 \\ 3 & -3 & 5 \\ 4 & -2 & 4 \end{vmatrix}\begin{matrix} 2 & -1 \\ 3 & -3 \\ 4 & -2 \end{matrix}$$

$$=-24-20-6+12+20+12$$

$$=-6$$

$$x=-\frac{0}{3}=0 \qquad x=\frac{D_x}{D}$$

$$y=\frac{-3}{-3}=1 \qquad y=\frac{D_y}{D}$$

$$z=\frac{-6}{-3}=2 \qquad z=\frac{D_z}{D}$$

$\therefore$ ***Solution***: $(0,\ 1,\ 2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x-4y+4z=7 \\ x-y-2z=2 \\ 2x-3y+6z=5 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & -4 & 4 \\ 1 & -1 & -2 \\ 2 & -3 & 6 \end{vmatrix}\begin{matrix} 3 & -4 \\ 1 & -1 \\ 2 & -3 \end{matrix}$$

$$=-18+16-12+8-18+24$$

$$\underline{=0}$$

$$D_z=\begin{vmatrix} 3 & -4 & 7 \\ 1 & -1 & 2 \\ 2 & -3 & 5 \end{vmatrix}\begin{matrix} 3 & -4 \\ 1 & -1 \\ 2 & -3 \end{matrix}$$

$$=-15-16-21+14+18+20$$

$$\underline{=0}$$

$$-3\times(2)\quad \begin{cases} -3x+3y+6z=-6 \\ \quad 2x-3y+6z=5 \end{cases}$$

$$-x+12z=-1$$

$$\underline{x=12z+1}$$

$$(2)\rightarrow \quad y=12z+1-2z-2$$

$$\underline{=10z-1}$$

$\therefore$ ***Solution***: $\underline{(12z+1,\ 10z-1,\ z)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix}\begin{matrix} 1 & -2 \\ 2 & -1 \\ -1 & 1 \end{matrix}$$

$$=-1+2-2+1-1+4$$

$$\underline{=3}$$

$$D_x = \begin{vmatrix} 2 & -2 & -1 \\ 4 & -1 & 1 \\ 4 & 1 & 1 \end{vmatrix} \begin{matrix} 2 & -2 \\ 4 & -1 \\ 4 & 1 \end{matrix}$$

$$= -2 - 8 - 4 - 4 - 2 + 8$$

$$\underline{= -12}$$

$$D_y = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 4 & 1 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & 4 \\ -1 & 4 \end{matrix}$$

$$= 4 - 2 - 8 - 4 - 4 - 4$$

$$\underline{= -18}$$

$$D_z = \begin{vmatrix} 1 & -2 & 2 \\ 2 & -1 & 4 \\ -1 & 1 & 4 \end{vmatrix} \begin{matrix} 1 & -2 \\ 2 & -1 \\ -1 & 1 \end{matrix}$$

$$= -4 + 8 + 4 - 2 - 4 + 16$$

$$\underline{= 18}$$

$x = -\frac{12}{3} \ \underline{= -4}$ $\qquad x = \frac{D_x}{D}$

$y = -\frac{18}{3} \ \underline{= -6}$ $\qquad y = \frac{D_y}{D}$

$z = \frac{18}{3} \ \underline{= 6}$ $\qquad z = \frac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(-4,\ -6,\ 6)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + y + z = 3 \\ -y + 2z = 1 \\ -x + z = 0 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & 1 \end{vmatrix} \begin{matrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{matrix}$$

$$\underline{= -4}$$

$$D_x = \begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} 3 & 1 \\ 1 & -1 \\ 0 & 0 \end{matrix}$$

$\underline{=-4}$

$$D_y = \begin{vmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{vmatrix} \begin{matrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{matrix}$$

$\underline{=-4}$

$$D_z = \begin{vmatrix} 1 & 1 & 3 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \end{vmatrix} \begin{matrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{matrix}$$

$\underline{=-4}$

$x = \frac{4}{4} \underline{=1}$ $\qquad x = \frac{D_x}{D}$

$y = \frac{4}{4} \underline{=1}$ $\qquad y = \frac{D_y}{D}$

$z = \frac{4}{4} \underline{=1}$ $\qquad z = \frac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(1,\ 1,\ 1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x + y + 3z = 14 \\ 7x + 5y + 8z = 37 \\ x + 3y + 2z = 9 \end{cases}$

Solution

$$D = \begin{vmatrix} 3 & 1 & 3 \\ 7 & 5 & 8 \\ 1 & 3 & 2 \end{vmatrix} \begin{matrix} 3 & 1 \\ 7 & 5 \\ 1 & 3 \end{matrix}$$

$= 30 + 8 + 62 - 15 - 72 - 14$

$\underline{=0}$

$$D_z = \begin{vmatrix} 3 & 1 & 14 \\ 7 & 5 & 37 \\ 1 & 3 & 9 \end{vmatrix} \begin{matrix} 3 & 1 \\ 7 & 5 \\ 1 & 3 \end{matrix}$$

$= 135 + 37 + 294 - 70 - 333 - 63$

$\underline{=0}$

$$\begin{array}{ll} -3\times(1) & \begin{cases} -9x - 3y - 9z = -42 \\ \quad x + 3y + 2z = 9 \end{cases} \\ (3) & \\ \hline & \quad -8x - 7z = -33 \end{array}$$

$$\underline{x = -\frac{7}{8}z + \frac{33}{8}}$$

$$(1) \rightarrow \quad y = 14 - 3z - 3\left(-\frac{7}{8}z + \frac{33}{8}\right)$$

$$= \frac{13}{8} - \frac{3}{8}z$$

$\therefore$ ***Solution***: $\left(\frac{33}{8} - \frac{7}{8}z, \ \ \frac{13}{8} - \frac{3}{8}z, \ \ z\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x - 2y + z = 7 \\ x + y + z = -2 \\ 4x + 2y + z = 3 \end{cases}$

Solution

$$D = \begin{vmatrix} 4 & -2 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix} \begin{matrix} 4 & -2 \\ 1 & 1 \\ 4 & 2 \end{matrix}$$

$$= -12$$

$$D_x = \begin{vmatrix} 7 & -2 & 1 \\ -2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} \begin{matrix} 7 & -2 \\ -2 & 1 \\ 3 & 2 \end{matrix}$$

$$= -24$$

$$D_y = \begin{vmatrix} 4 & 7 & 1 \\ 1 & -2 & 1 \\ 4 & 3 & 1 \end{vmatrix} \begin{matrix} 4 & 7 \\ 1 & -2 \\ 4 & 3 \end{matrix}$$

$$= 12$$

$$D_z = \begin{vmatrix} 4 & -2 & 7 \\ 1 & 1 & -2 \\ 4 & 2 & 3 \end{vmatrix} \begin{matrix} 4 & -2 \\ 1 & 1 \\ 4 & 2 \end{matrix}$$

$$= 36$$

$$x = \frac{24}{12} = 2 \qquad\qquad x = \frac{D_x}{D}$$

$$y = -\frac{12}{12} = -1 \qquad\qquad y = \frac{D_y}{D}$$

$$z = -\frac{36}{12} = -3 \qquad\qquad z = \frac{D_z}{D}$$

$\therefore$ ***Solution***: $(2, \ -1, \ -3)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

Solution

$$D = \begin{vmatrix} 0 & 2 & -1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix} \begin{matrix} 0 & 2 \\ 1 & 2 \\ 2 & 3 \end{matrix}$$

$$\underline{= 1|}$$

$$D_x = \begin{vmatrix} 7 & 2 & -1 \\ 17 & 2 & 1 \\ -1 & 3 & 2 \end{vmatrix} \begin{matrix} 7 & 2 \\ 17 & 2 \\ -1 & 3 \end{matrix}$$

$$\underline{= -116|}$$

$$D_y = \begin{vmatrix} 0 & 7 & -1 \\ 1 & 17 & 1 \\ 2 & -1 & 2 \end{vmatrix} \begin{matrix} 0 & 7 \\ 1 & 17 \\ 2 & -1 \end{matrix}$$

$$\underline{= 35|}$$

$$D_z = \begin{vmatrix} 0 & 2 & 7 \\ 1 & 2 & 17 \\ 2 & 3 & -1 \end{vmatrix} \begin{matrix} 0 & 2 \\ 1 & 2 \\ 2 & 3 \end{matrix}$$

$$\underline{= 63|}$$

$x \underline{= -116|}$ $\qquad x = \dfrac{D_x}{D}$

$y \underline{= 35|}$ $\qquad y = \dfrac{D_y}{D}$

$z \underline{= 63|}$ $\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(-116,\ 35,\ 63)\ |}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & -2 & 1 \\ 6 & 4 & -3 \\ 1 & -2 & 2 \end{vmatrix}\begin{matrix} 2 & -2 \\ 6 & 4 \\ 1 & -2 \end{matrix}$$

$$\underline{=18}$$

$$D_x=\begin{vmatrix} -4 & -2 & 1 \\ -24 & 4 & -3 \\ 1 & -2 & 2 \end{vmatrix}\begin{matrix} -4 & -2 \\ -24 & 4 \\ 1 & -2 \end{matrix}$$

$$\underline{=-54}$$

$$D_y=\begin{vmatrix} 2 & -4 & 1 \\ 6 & -24 & -3 \\ 1 & 1 & 2 \end{vmatrix}\begin{matrix} 2 & -4 \\ 6 & -24 \\ 1 & 1 \end{matrix}$$

$$\underline{=0}$$

$$D_z=\begin{vmatrix} 2 & -2 & -4 \\ 6 & 4 & -24 \\ 1 & -2 & 1 \end{vmatrix}\begin{matrix} 2 & -2 \\ 6 & 4 \\ 1 & -2 \end{matrix}$$

$$\underline{=36}$$

$$x=-\frac{54}{18} \qquad\qquad x=\frac{D_x}{D}$$

$$\underline{=-3}$$

$$y\ \underline{=0} \qquad\qquad y=\frac{D_y}{D}$$

$$z\ \underline{=2} \qquad\qquad z=\frac{D_z}{D}$$

$\therefore$ ***Solution***: $\underline{(-3,\ 0,\ 2)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

Solution

$$D = \begin{vmatrix} 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{vmatrix} \begin{matrix} 9 & 3 \\ 16 & 4 \\ 25 & 5 \end{matrix}$$

$$= -2$$

$$D_x = \begin{vmatrix} 4 & 3 & 1 \\ 2 & 4 & 1 \\ 2 & 5 & 1 \end{vmatrix} \begin{matrix} 4 & 3 \\ 2 & 4 \\ 2 & 5 \end{matrix}$$

$$= -2$$

$$D_y = \begin{vmatrix} 9 & 4 & 1 \\ 16 & 2 & 1 \\ 25 & 2 & 1 \end{vmatrix} \begin{matrix} 9 & 4 \\ 16 & 2 \\ 25 & 2 \end{matrix}$$

$$= 18$$

$$D_z = \begin{vmatrix} 9 & 3 & 4 \\ 16 & 4 & 2 \\ 25 & 5 & 2 \end{vmatrix} \begin{matrix} 9 & 3 \\ 16 & 4 \\ 25 & 5 \end{matrix}$$

$$= -44$$

$x = \frac{-2}{-2}$ $\qquad$ $x = \frac{D_x}{D}$

$= 1$

$y = \frac{18}{-2}$ $\qquad$ $y = \frac{D_y}{D}$

$= -9$

$z = \frac{-44}{-2}$ $\qquad$ $z = \frac{D_z}{D}$

$= 22$

$\therefore$ ***Solution***: $(1,\ -9,\ 22)$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$$

Solution

$$D = \begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & -3 \\ 3 & -1 & -4 \end{vmatrix} \begin{matrix} 2 & -1 \\ 1 & 2 \\ 3 & -1 \end{matrix}$$

$$= -31$$

$$D_x = \begin{vmatrix} -8 & -1 & 2 \\ 9 & 2 & -3 \\ 3 & -1 & -4 \end{vmatrix} \begin{matrix} -8 & -1 \\ 9 & 2 \\ 3 & -1 \end{matrix}$$

$$= 31$$

$$D_y = \begin{vmatrix} 2 & -8 & 2 \\ 1 & 9 & -3 \\ 3 & 3 & -4 \end{vmatrix} \begin{matrix} 2 & -8 \\ 1 & 9 \\ 3 & 3 \end{matrix}$$

$$= -62$$

$$D_z = \begin{vmatrix} 2 & -1 & -8 \\ 1 & 2 & 9 \\ 3 & -1 & 3 \end{vmatrix} \begin{matrix} 2 & -1 \\ 1 & 2 \\ 3 & -1 \end{matrix}$$

$$= 62$$

$$x = -\frac{31}{31} \qquad x = \frac{D_x}{D}$$

$$= -1$$

$$y = \frac{62}{31} \qquad y = \frac{D_y}{D}$$

$$= 2$$

$$z = -\frac{62}{31} \qquad z = \frac{D_z}{D}$$

$$= -2$$

$\therefore$ ***Solution***: $(-1,\ 2,\ -2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x \qquad -3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 0 & -3 \\ 2 & -1 & 2 \\ 7 & -3 & -5 \end{vmatrix} \begin{matrix} 1 & 0 \\ 2 & -1 \\ 7 & -3 \end{matrix}$$

$$= 8$$

$$D_x = \begin{vmatrix} -5 & 0 & -3 \\ 16 & -1 & 2 \\ 19 & -3 & -5 \end{vmatrix} \begin{matrix} -5 & 0 \\ 16 & -1 \\ 19 & -3 \end{matrix}$$

$$= 32$$

$$D_y = \begin{vmatrix} 1 & -5 & -3 \\ 2 & 16 & 2 \\ 7 & 19 & -5 \end{vmatrix} \begin{matrix} 1 & -5 \\ 2 & 16 \\ 7 & 19 \end{matrix}$$

$$= -16$$

$$D_z = \begin{vmatrix} 1 & 0 & -5 \\ 2 & -1 & 16 \\ 7 & -3 & 19 \end{vmatrix} \begin{matrix} 1 & 0 \\ 2 & -1 \\ 7 & -3 \end{matrix}$$

$$= 24$$

$$x = \frac{32}{8} \qquad\qquad x = \frac{D_x}{D}$$

$$= 4$$

$$y = -\frac{16}{8} \qquad\qquad y = \frac{D_y}{D}$$

$$= -2$$

$$z = \frac{24}{8} \qquad\qquad z = \frac{D_z}{D}$$

$$= 3$$

$\therefore$ ***Solution***: $(4,\ -2,\ 3)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+2y-z=5 \\ 2x-y+3z=0 \\ 2y+z=1 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 0 & 2 & 1 \end{vmatrix}\begin{matrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{matrix}$$

$$=-15$$

$$D_x=\begin{vmatrix} 5 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 2 & 1 \end{vmatrix}\begin{matrix} 5 & 2 \\ 0 & -1 \\ 1 & 2 \end{matrix}$$

$$=-30$$

$$D_y=\begin{vmatrix} 1 & 5 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix}\begin{matrix} 1 & 5 \\ 2 & 0 \\ 0 & 1 \end{matrix}$$

$$=-15$$

$$D_z=\begin{vmatrix} 1 & 2 & 5 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{vmatrix}\begin{matrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{matrix}$$

$$=15$$

$$x=\frac{30}{15} \qquad x=\frac{D_x}{D}$$

$$=2$$

$$y=\frac{15}{15} \qquad y=\frac{D_y}{D}$$

$$=1$$

$$z=-\frac{15}{15} \qquad z=\frac{D_z}{D}$$

$$=-1$$

$\therefore$ ***Solution***: $(2,\ 1,\ -1)$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} x+y+z=6 \\ 3x+4y-7z=1 \\ 2x-y+3z=5 \end{cases}$$

Solution

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & -7 \\ 2 & -1 & 3 \end{vmatrix} \begin{matrix} 1 & 1 \\ 3 & 4 \\ 2 & -1 \end{matrix}$$

$$= -29$$

$$D_x = \begin{vmatrix} 6 & 1 & 1 \\ 1 & 4 & -7 \\ 5 & -1 & 3 \end{vmatrix} \begin{matrix} 6 & 1 \\ 1 & 4 \\ 5 & -1 \end{matrix}$$

$$= -29$$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 3 & 1 & -7 \\ 2 & 5 & 3 \end{vmatrix} \begin{matrix} 1 & 6 \\ 3 & 1 \\ 2 & 5 \end{matrix}$$

$$= -87$$

$$D_z = \begin{vmatrix} 1 & 2 & 6 \\ 2 & -1 & 1 \\ 0 & 2 & 5 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{matrix}$$

$$= -58$$

$$x = \frac{29}{29} \qquad x = \frac{D_x}{D}$$

$$= 1$$

$$y = \frac{87}{29} \qquad y = \frac{D_y}{D}$$

$$= 3$$

$$z = \frac{58}{29} \qquad z = \frac{D_z}{D}$$

$$= 2$$

∴ ***Solution***: $(1,\ 3,\ 2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y+3z=3 \\ 4x-5y+7z=1 \\ 2x+3y-2z=6 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 2 & 3 \\ 4 & -5 & 7 \\ 2 & 3 & -2 \end{vmatrix}\begin{matrix} 3 & 2 \\ 4 & -5 \\ 2 & 3 \end{matrix}$$

$$=77$$

$$D_x=\begin{vmatrix} 3 & 2 & 3 \\ 1 & -5 & 7 \\ 6 & 3 & -2 \end{vmatrix}\begin{matrix} 3 & 2 \\ 1 & -5 \\ 6 & 3 \end{matrix}$$

$$=154$$

$$D_y=\begin{vmatrix} 3 & 3 & 3 \\ 4 & 1 & 7 \\ 2 & 6 & -2 \end{vmatrix}\begin{matrix} 3 & 3 \\ 4 & 1 \\ 2 & 6 \end{matrix}$$

$$=0$$

$$D_z=\begin{vmatrix} 3 & 2 & 3 \\ 4 & -5 & 1 \\ 2 & 3 & 6 \end{vmatrix}\begin{matrix} 3 & 2 \\ 4 & -5 \\ 2 & 3 \end{matrix}$$

$$=-77$$

$x=\frac{154}{77}=2$ $\qquad\qquad x=\frac{D_x}{D}$

$=2$

$y=0$ $\qquad\qquad y=\frac{D_y}{D}$

$z=-\frac{77}{77}$ $\qquad\qquad z=\frac{D_z}{D}$

$=-1$

$\therefore$ ***Solution***: $(2,\ 0,\ -1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x+5y \qquad = 2 \\ 11x+y+2z=3 \\ x+5y+2z=1 \end{cases}$

Solution

$$D = \begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \begin{matrix} 4 & 5 \\ 11 & 1 \\ 1 & 5 \end{matrix}$$

$$\boxed{=-132}$$

$$D_x = \begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \begin{matrix} 2 & 5 \\ 3 & 1 \\ 1 & 5 \end{matrix}$$

$$\boxed{=-36}$$

$$D_y = \begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} \begin{matrix} 4 & 2 \\ 11 & 3 \\ 1 & 1 \end{matrix}$$

$$\boxed{=-24}$$

$$D_z = \begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} \begin{matrix} 4 & 5 \\ 11 & 1 \\ 1 & 5 \end{matrix}$$

$$\boxed{=12}$$

$$x = \frac{36}{132} \qquad\qquad x = \frac{D_x}{D}$$

$$\boxed{=\frac{3}{11}}$$

$$y = \frac{24}{132} \qquad\qquad y = \frac{D_y}{D}$$

$$\boxed{=\frac{2}{11}}$$

$$z = -\frac{12}{132} \qquad\qquad z = \frac{D_z}{D}$$

$$\boxed{=-\frac{1}{11}}$$

$\therefore$ ***Solution***: $\boxed{\left(\frac{3}{11},\ \frac{2}{11},\ -\frac{1}{11}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-4y+z=6 \\ 4x-y+2z=-1 \\ 2x+2y-3z=-20 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix}\begin{matrix} 1 & -4 \\ 4 & -1 \\ 2 & 2 \end{matrix}$$

$$=-55$$

$$D_x=\begin{vmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix}\begin{matrix} 6 & -4 \\ -1 & -1 \\ -20 & 2 \end{matrix}$$

$$=144$$

$$D_y=\begin{vmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix}\begin{matrix} 1 & 6 \\ 4 & -1 \\ 2 & -20 \end{matrix}$$

$$=61$$

$$D_z=\begin{vmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix}\begin{matrix} 1 & -4 \\ 4 & -1 \\ 2 & 2 \end{matrix}$$

$$=-230$$

$$x=-\frac{144}{55} \qquad x=\frac{D_x}{D}$$

$$y=-\frac{61}{55} \qquad y=\frac{D_y}{D}$$

$$z=\frac{230}{55} \qquad z=\frac{D_z}{D}$$

$$=\frac{46}{11}$$

$\therefore$ ***Solution***: $\left(-\frac{144}{55},\ -\frac{61}{55},\ \frac{46}{11}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x - y + z = -1 \\ 3x + 4y - z = -1 \\ 4x - y + 2z = -1 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 4 & -1 \\ 4 & -1 & 2 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & 4 \\ 4 & -1 \end{matrix}$$

$$= 5$$

$$D_x = \begin{vmatrix} -1 & -1 & 1 \\ -1 & 4 & -1 \\ -1 & -1 & 2 \end{vmatrix} \begin{matrix} -1 & -1 \\ -1 & 4 \\ -1 & -1 \end{matrix}$$

$$= -5$$

$$D_y = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -1 & -1 \\ 4 & -1 & 2 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & -1 \\ 4 & -1 \end{matrix}$$

$$= 5$$

$$D_z = \begin{vmatrix} 2 & -1 & -1 \\ 3 & 4 & -1 \\ 4 & -1 & -1 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & 4 \\ 4 & -1 \end{matrix}$$

$$= 10$$

$$x = \frac{-5}{5} \qquad x = \frac{D_x}{D}$$

$$= -1$$

$$y = \frac{5}{5} \qquad y = \frac{D_y}{D}$$

$$= 1$$

$$z = \frac{10}{5} \qquad z = \frac{D_z}{D}$$

$$= 2$$

$\therefore$ ***Solution***: $(-1,\ 1,\ 2)$

Exercise

Use Cramer's rule to solve the system
$$\begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$$

Solution

$$D = \begin{vmatrix} -1 & -4 & 2 & 1 \\ 2 & -1 & 7 & 9 \\ -1 & 1 & 3 & 1 \\ -1 & -2 & 1 & -4 \end{vmatrix}$$
$$= -243$$

$$D_1 = \begin{vmatrix} -32 & -4 & 2 & 1 \\ 14 & -1 & 7 & 9 \\ 11 & 1 & 3 & 1 \\ -4 & -2 & 1 & -4 \end{vmatrix}$$
$$= -2115$$

$$D_2 = \begin{vmatrix} -1 & -32 & 2 & 1 \\ 2 & 14 & 7 & 9 \\ -1 & 11 & 3 & 1 \\ -1 & -4 & 1 & -4 \end{vmatrix}$$
$$= -1834$$

$$D_3 = \begin{vmatrix} -1 & -4 & -32 & 1 \\ 2 & -1 & 14 & 9 \\ -1 & 1 & 11 & 1 \\ -1 & -2 & -4 & -4 \end{vmatrix}$$
$$= -1279$$

$$D_4 = \begin{vmatrix} -1 & -4 & 2 & -32 \\ 2 & -1 & 7 & 14 \\ -1 & 1 & 3 & 11 \\ -1 & -2 & 1 & -4 \end{vmatrix}$$
$$= 883$$

$$x_1 = \frac{-2115}{-243}$$
$$= \frac{235}{27}$$

$$x_2 = \frac{-1834}{-243}$$

$$= \frac{1834}{243}$$

$$x_3 = \frac{-1279}{-243}$$

$$= \frac{1279}{243}$$

$$x_4 = -\frac{883}{243}$$

∴ ***Solution***: $\left(\frac{235}{27}, \frac{1834}{243}, \frac{1279}{243}, -\frac{883}{243}\right)$

Exercise

Show that the matrix A is invertible for all values of θ, then find A^{-1} using $A^{-1} = \frac{1}{\det(A)} adj(A)$

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

$$\det(A) = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos^2\theta + \sin^2\theta$$

$$= 1$$

$\Rightarrow$ ***A*** is invertible

$$C_{11} = \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta$$

$$C_{12} = -\begin{vmatrix} -\sin\theta & 0 \\ 0 & 1 \end{vmatrix} = \sin\theta$$

$$C_{13} = \begin{vmatrix} -\sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{21} = -\begin{vmatrix} \sin\theta & 0 \\ 0 & 1 \end{vmatrix} = -\sin\theta$$

$$C_{22} = \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta$$

$$C_{23} = -\begin{vmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{31} = \begin{vmatrix} \sin\theta & 0 \\ \cos\theta & 0 \end{vmatrix} = 0$$

$$C_{32} = -\begin{vmatrix} \cos\theta & 0 \\ -\sin\theta & 0 \end{vmatrix} = 0$$

$$C_{33} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1$$

$$adj(A) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} adj(A) \\ &= \frac{1}{1} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$