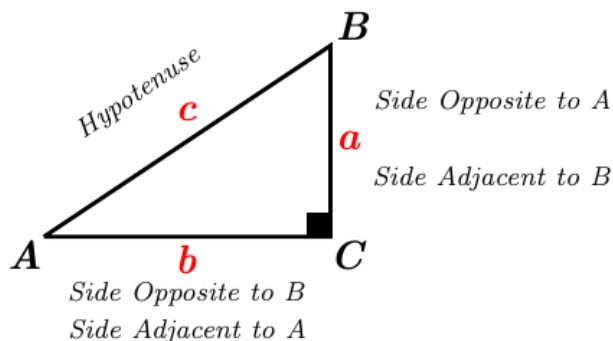
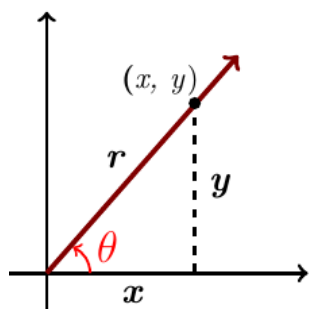


## Section 6.3 – Trigonometric Functions

Let  $(x, y)$  be a point on the terminal side of an angle  $\theta$  in standard position

The distance from the point to the origin is given by:  $r = \sqrt{x^2 + y^2}$

### Six Trigonometry Functions



$$\sin \theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}$$

$$\csc \theta = \frac{\text{hyp}}{\text{opp}} = \frac{1}{\sin \theta} = \frac{r}{y}$$

$$\cos \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$$

$$\sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\cos \theta} = \frac{r}{x}$$

$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{\sin \theta}{\cos \theta} = \frac{y}{x}$$

$$\cot \theta = \frac{\text{adj}}{\text{opp}} = \frac{\cos \theta}{\sin \theta} = \frac{1}{\tan \theta} = \frac{x}{y}$$

$$\sin A = \frac{a}{c} = \cos B$$

$$\csc A = \frac{c}{a} = \sec B$$

$$\cos A = \frac{b}{c} = \sin B$$

$$\sec A = \frac{c}{b} = \csc B$$

$$\tan A = \frac{a}{b} = \cot B$$

$$\cot A = \frac{b}{a} = \tan B$$

### Example

Find the six trigonometry functions of  $\theta$  if  $\theta$  is in the standard position and the point  $(8, 15)$  is on the terminal side of  $\theta$ .

#### Solution

$$r = \sqrt{8^2 + 15^2} \quad r = \sqrt{x^2 + y^2}$$

$$= 17$$

$$\sin \theta = \frac{y}{r} = \frac{15}{17}$$

$$\cos \theta = \frac{x}{r} = \frac{8}{17}$$

$$\tan \theta = \frac{y}{x} = \frac{15}{8}$$

$$\csc \theta = \frac{r}{y} = \frac{17}{15}$$

$$\sec \theta = \frac{r}{x} = \frac{17}{8}$$

$$\cot \theta = \frac{x}{y} = \frac{8}{15}$$

### Example

Which will be greater,  $\tan 30^\circ$  or  $\tan 40^\circ$ ? How large could  $\tan \theta$  be?

### Solution

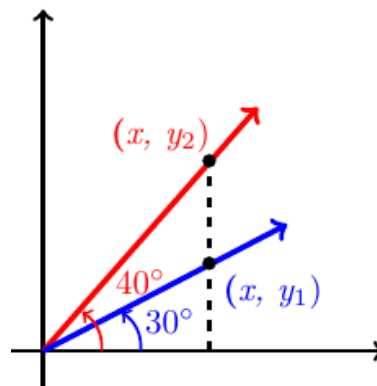
$$\tan 30^\circ = \frac{y_1}{x}$$

$$\tan 40^\circ = \frac{y_2}{x}$$

$$\text{Ratio: } \frac{y_2}{x} > \frac{y_1}{x}$$

$$\rightarrow \tan 40^\circ > \tan 30^\circ$$

**No limit** as to how large  $\tan \theta$  can be.



### Example

If  $\cos \theta = \frac{\sqrt{3}}{2}$ , and  $\theta$  is **QIV**, find  $\sin \theta$  and  $\tan \theta$ .

### Solution

$$\cos \theta = \frac{\sqrt{3}}{2} = \frac{x}{r} \rightarrow x = \sqrt{3}, r = 2$$

$$\begin{aligned} y &= \sqrt{r^2 - x^2} \\ &= \sqrt{2^2 - (\sqrt{3})^2} \\ &= \sqrt{4 - 3} \\ &= 1 \end{aligned}$$

Since  $\theta$  is **Q IV**  $\Rightarrow y = -1$

$$\sin \theta = -\frac{1}{2} \quad \sin \theta = \frac{y}{r}$$

$$\tan \theta = \frac{-1}{\sqrt{3}} \quad \tan \theta = \frac{y}{x}$$

$$= -\frac{1}{\sqrt{3}} \frac{\sqrt{3}}{\sqrt{3}}$$

$$= -\frac{\sqrt{3}}{3}$$

## ***Reciprocal Identities***

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\sin \theta = \frac{1}{\csc \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cos \theta = \frac{1}{\sec \theta}$$

$$\tan \theta = \frac{1}{\cot \theta}$$

## ***Ratio Identities***

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

## ***Pythagorean Identities***

$$x^2 + y^2 = r^2$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$(\cos \theta)^2 + (\sin \theta)^2 = 1 \Rightarrow \boxed{\cos^2 \theta + \sin^2 \theta = 1}$$

Solving for  $\cos \theta$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

$$\cos \theta = \pm \sqrt{1 - \sin^2 \theta}$$

Solving for  $\sin \theta$

$$\sin^2 \theta = 1 - \cos^2 \theta$$

$$\boxed{\sin \theta = \pm \sqrt{1 - \cos^2 \theta}}$$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

$$\frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

$$\left(\frac{\cos \theta}{\cos \theta}\right)^2 + \left(\frac{\sin \theta}{\cos \theta}\right)^2 = \left(\frac{1}{\cos \theta}\right)^2$$

$$\boxed{1 + \tan^2 \theta = \sec^2 \theta}$$

$\cos^2 \theta + \sin^2 \theta = 1$
$\cos \theta = \pm \sqrt{1 - \sin^2 \theta}$
$\sin \theta = \pm \sqrt{1 - \cos^2 \theta}$
$1 + \tan^2 \theta = \sec^2 \theta$
$1 + \cot^2 \theta = \csc^2 \theta$

### Example

Write  $\tan \theta$  in terms of  $\sin \theta$ .

#### Solution

$$\begin{aligned} \tan \theta &= \frac{\sin \theta}{\cos \theta} \\ &= \frac{\sin \theta}{\pm \sqrt{1 - \sin^2 \theta}} \\ &= \pm \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} \end{aligned}$$

### Example

Simplify the expression  $\sqrt{x^2 + 9}$  as much as possible after substituting  $3 \tan \theta$  for  $x$

#### Solution

$$\begin{aligned} x &= 3 \tan \theta \\ \sqrt{x^2 + 9} &= \sqrt{(3 \tan \theta)^2 + 9} \\ &= \sqrt{9 \tan^2 \theta + 9} \\ &= \sqrt{9(\tan^2 \theta + 1)} \\ &= 3\sqrt{\sec^2 \theta} \\ &= \boxed{3 \sec \theta} \end{aligned}$$

### Example

Triangle  $ABC$  is a right triangle with  $C = 90^\circ$ . If  $a = 6$  and  $c = 10$ , find the six trigonometric functions of  $A$ .

### Solution

$$b = \sqrt{c^2 - a^2} = \sqrt{10^2 - 6^2} = 8$$

$$6, b \rightarrow 10 \rightarrow 2(3, b \rightarrow 5)$$

$$b = 4 \mid$$

$\sin A = \frac{a}{c} = \frac{3}{5}$	$\cos A = \frac{b}{c} = \frac{8}{10} = \frac{4}{5}$	$\tan A = \frac{a}{b} = \frac{6}{8} = \frac{3}{4}$
$\csc A = \frac{5}{3}$	$\sec A = \frac{5}{4}$	$\cot A = \frac{4}{3}$

$$\text{if } A + B = 90^\circ \Rightarrow \begin{cases} \sin A = \cos B \\ \sec A = \csc B \\ \tan A = \cot B \end{cases}$$

### Cofunction Theorem

A trigonometric function of an angle is always equal to the cofunction of the complement of the angle.

### Example

Write each function in terms of its cofunction

a)  $\cos 52^\circ$

### Solution

$$\cos 52^\circ = \sin(90^\circ - 52^\circ) = \sin 38^\circ$$

b)  $\tan 71^\circ$

### Solution

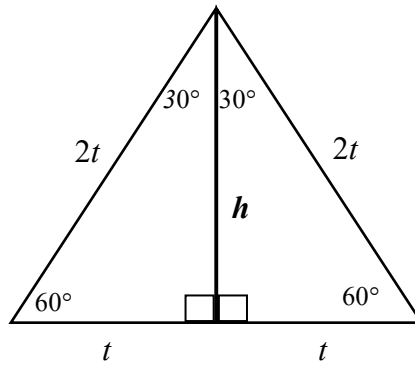
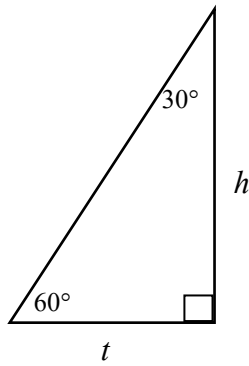
$$\tan 71^\circ = \cot(90^\circ - 71^\circ) = \cot 19^\circ$$

c)  $\sec 24^\circ$

### Solution

$$\sec 24^\circ = \csc(90^\circ - 24^\circ) = \csc 66^\circ$$

### 30° - 60° - 90° Triangle



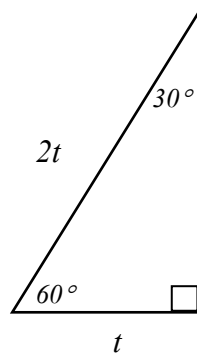
$$t^2 + h^2 = (2t)^2$$

$$t^2 + h^2 = 4t^2$$

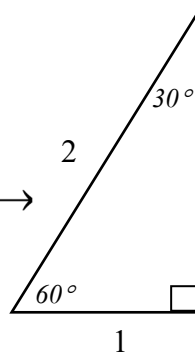
$$h^2 = 4t^2 - t^2$$

$$h^2 = 3t^2$$

$$h = t\sqrt{3}$$



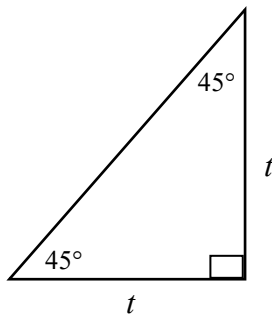
$$t\sqrt{3} \rightarrow$$



$$\sqrt{3} \Rightarrow$$

$$\Rightarrow \sin 60^\circ = \frac{\sqrt{3}}{2}$$

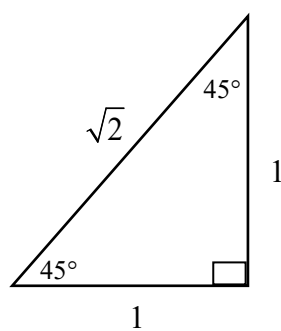
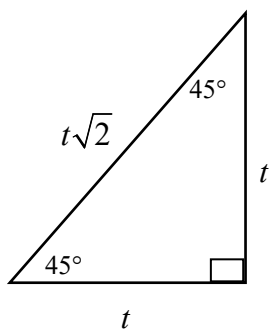
### 45° - 45° - 90° Triangle



$$\text{hypotenuse}^2 = t^2 + t^2$$

$$\text{hypotenuse} = \sqrt{2t^2}$$

$$\text{hypotenuse} = t\sqrt{2}$$

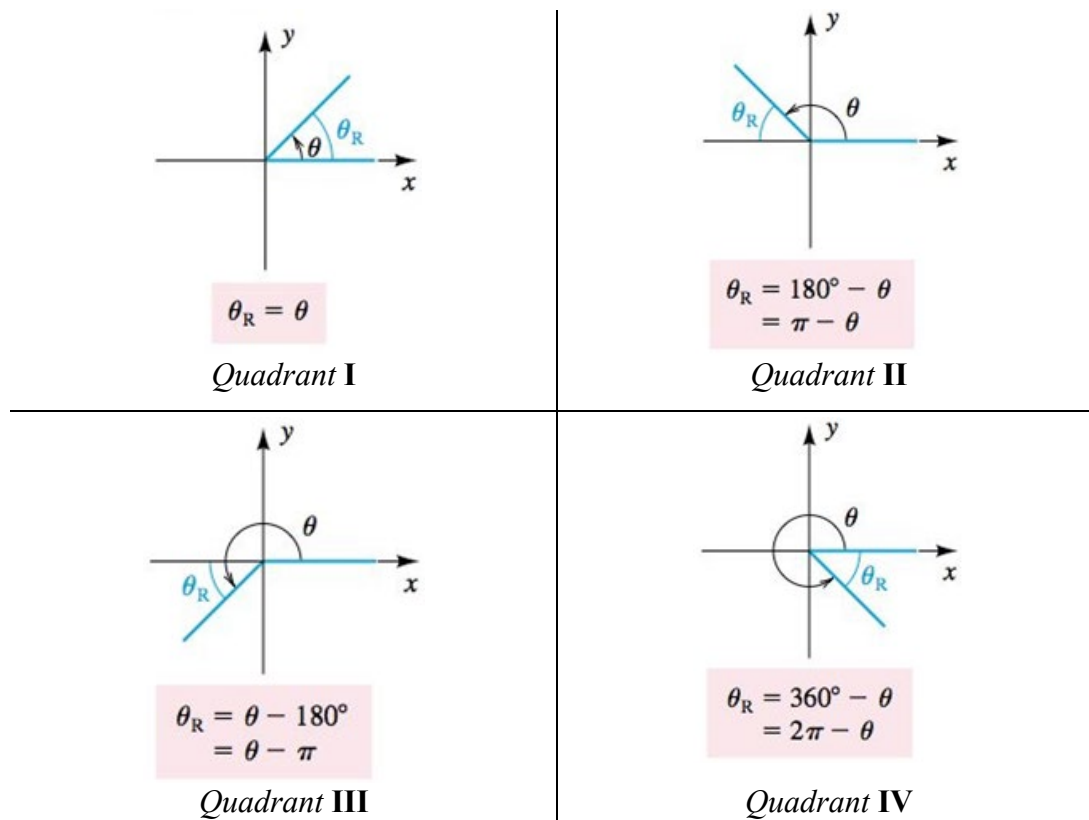


$$\Rightarrow \cos 45^\circ = \frac{1}{\sqrt{2}}$$

## Reference Angle

### Definition

The reference angle or related angle for any angle  $\theta$  in standard position is the positive acute angle between the terminal side of  $\theta$  and the  $x$ -axis, and it is denoted  $\hat{\theta}$



### Example

Find the exact value of  $\sin 240^\circ$

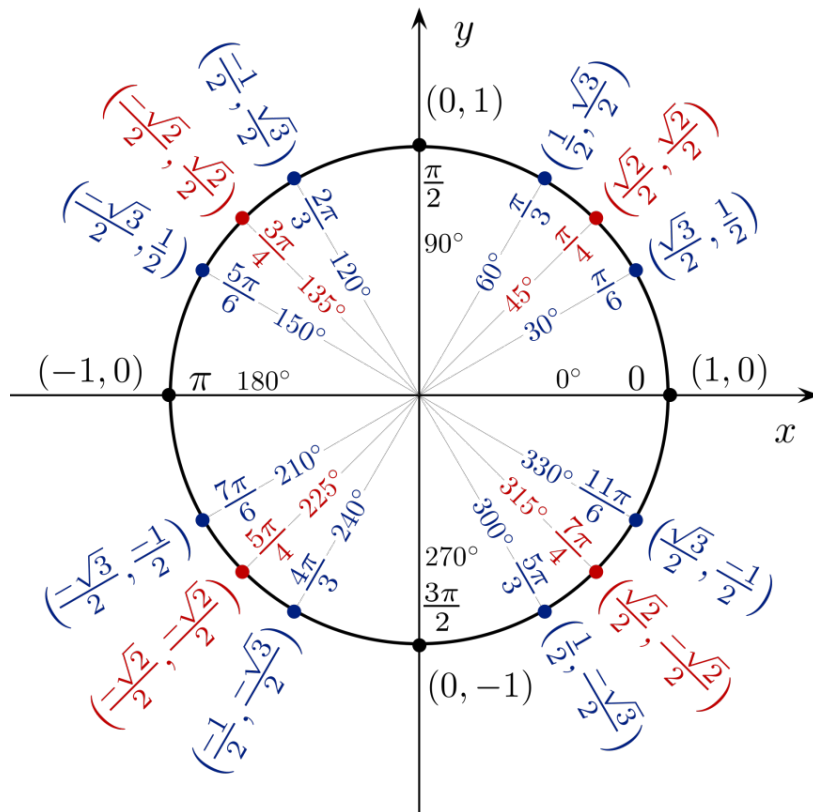
### Solution

$$\hat{\theta} = 240^\circ - 180^\circ = 60^\circ$$

$$\rightarrow 240^\circ \in Q_{III}$$

$$\sin 240^\circ = -\sin 60^\circ$$

$$= -\frac{\sqrt{3}}{2}$$



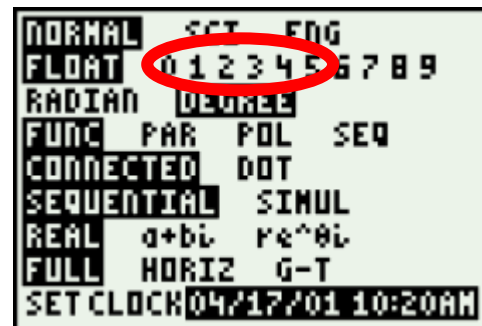
**Approximation**- Simply using calculator

$$\sin 250^\circ \approx -0.9397$$

$$\cos 250^\circ \approx -0.3420$$

$$\tan 250^\circ \approx 2.7475$$

$$\csc 250^\circ = \frac{1}{\sin 250^\circ} \approx -1.0642$$



To find the angle by using the inverse trigonometry functions, always enter a **positive** value.

### Example

Find  $\theta$  if  $\sin \theta = -0.5592$  and  $\theta$  terminates in QIII with  $0^\circ \leq \theta < 360^\circ$ .

### Solution

$$\hat{\theta} = \sin^{-1} 0.5592 \approx 34^\circ$$

$$\theta \in \text{QIII}$$

$$\Rightarrow \boxed{\theta = 180^\circ + 34^\circ = 214^\circ}$$



***Example***

Find  $\theta$  to the nearest degree if  $\cot \theta = -1.6003$  and  $\theta$  terminates in  $QII$  with  $0^\circ \leq \theta < 360^\circ$ .

**Solution**

$$\cot \theta = -1.6003 = \frac{1}{\tan \theta}$$

$$\tan \theta = \frac{1}{-1.6003}$$

$$\hat{\theta} = \tan^{-1} \frac{1}{1.6003} = \underline{32^\circ}$$

$$\theta \in QII \quad \Rightarrow \theta = 180^\circ - 32^\circ = \underline{148^\circ}$$

## Exercise

## Section 6.3 – Trigonometric Functions

(1 – 16) Find the **six** trigonometry functions of  $\theta$  if  $\theta$  is in the standard position and the given point is on the terminal side of  $\theta$ .

- |               |                |                 |                 |
|---------------|----------------|-----------------|-----------------|
| 1. $(-2, 3)$  | 5. $(5, -12)$  | 9. $(-6, 8)$    | 13. $(7, 24)$   |
| 2. $(-3, -4)$ | 6. $(9, -12)$  | 10. $(-15, 8)$  | 14. $(-7, -24)$ |
| 3. $(-3, 0)$  | 7. $(16, -12)$ | 11. $(-7, 24)$  | 15. $(-24, -7)$ |
| 4. $(12, -5)$ | 8. $(15, -8)$  | 12. $(10, -24)$ | 16. $(24, -10)$ |

17. Find the values of the six trigonometric functions for an angle of  $90^\circ$ .

18. Indicate the two quadrants  $\theta$  could terminate in if  $\cos \theta = \frac{1}{2}$

19. Indicate the two quadrants  $\theta$  could terminate in if  $\csc \theta = -2.45$

(20 – 38) Find the remaining trigonometric function of  $\theta$  if

- |  |  |
|--|--|
| 20. $\sin \theta = \frac{12}{13}$ and $\theta$ terminates in $QI$ .          | 29. $\sin \theta = -\frac{3}{5}$ & $\theta \in QIV$    |
| 21. $\cot \theta = -2$ and $\theta$ terminates in $QII$ .                    | 30. $\cos \theta = -\frac{12}{13}$ & $\theta \in QIII$ |
| 22. $\tan \theta = \frac{3}{4}$ and $\theta$ terminates in $QIII$ .          | 31. $\cos \theta = -\frac{5}{13}$ & $\theta \in QII$   |
| 23. $\cos \theta = \frac{24}{25}$ and $\theta$ terminates in $QIV$ .         | 32. $\cos \theta = \frac{12}{13}$ & $\theta \in QIV$   |
| 24. $\cos \theta = \frac{\sqrt{3}}{2}$ and $\theta$ is terminates in $QIV$ . | 33. $\sin \theta = -\frac{8}{17}$ & $\theta \in QIII$  |
| 25. $\tan \theta = -\frac{1}{2}$ and $\cos \theta > 0$                       | 34. $\cos \theta = -\frac{15}{17}$ & $\theta \in QII$  |
| 26. $\cos \theta = \frac{3}{5}$ & $\theta \in QI$                            | 35. $\cos \theta = -\frac{8}{17}$ & $\theta \in QII$   |
| 27. $\cos \theta = -\frac{4}{5}$ & $\theta \in QII$                          | 36. $\cos \theta = -\frac{7}{25}$ & $\theta \in QII$   |
| 28. $\sin \theta = -\frac{3}{5}$ & $\theta \in QIII$                         | 37. $\sin \theta = -\frac{7}{25}$ & $\theta \in QIII$  |
|  | 38. $\sin \theta = -\frac{24}{25}$ & $\theta \in QIV$  |

39. If  $\sin \theta = -\frac{5}{13}$ , and  $\theta$  is  $QIII$ , find  $\cos \theta$  and  $\tan \theta$ .

40. If  $\cos \theta = \frac{3}{5}$ , and  $\theta$  is  $QIV$ , find  $\sin \theta$  and  $\tan \theta$ .

41. Use the reciprocal identities if  $\cos \theta = \frac{\sqrt{3}}{2}$  find  $\sec \theta$

42. Find  $\cos \theta$ , given that  $\sec \theta = \frac{5}{3}$

43. Find  $\sin \theta$ , given that  $\csc \theta = -\frac{\sqrt{12}}{2}$
44. Use a ratio identity to find  $\tan \theta$  if  $\sin \theta = \frac{3}{5}$  and  $\cos \theta = -\frac{4}{5}$
45. If  $\cos \theta = -\frac{1}{2}$  and  $\theta$  terminates in QII, find  $\sin \theta$
46. If  $\sin \theta = \frac{3}{5}$  and  $\theta$  terminated in QII, find  $\cos \theta$  and  $\tan \theta$ .
47. Find  $\tan \theta$  if  $\sin \theta = \frac{1}{3}$  and  $\theta$  terminates in QI
48. Find the remaining trigonometric ratios of  $\theta$ , if  $\sec \theta = -3$  and  $\theta \in QIII$
49. Using the calculator and rounding your answer to the nearest hundredth, find the remaining trigonometric ratios of  $\theta$  if  $\csc \theta = -2.45$  and  $\theta \in QIII$ .
50. Write  $\frac{\sec \theta}{\csc \theta}$  in terms of  $\sin \theta$  and  $\cos \theta$ , and then simplify if possible.
51. Write  $\cot \theta - \csc \theta$  in terms of  $\sin \theta$  and  $\cos \theta$ , and then simplify if possible.
52. Write  $\frac{\sin \theta}{\cos \theta} + \frac{1}{\sin \theta}$  in terms of  $\sin \theta$  and/or  $\cos \theta$ , and then simplify if possible.
53. Write  $\sin \theta \cot \theta + \cos \theta$  in terms of  $\sin \theta$  and  $\cos \theta$ , and then simplify if possible.
54. Multiply  $(1 - \cos \theta)(1 + \cos \theta)$
55. Multiply  $(\sin \theta + 2)(\sin \theta - 5)$
56. Simplify the expression  $\sqrt{25 - x^2}$  as much as possible after substituting  $5 \sin \theta$  for  $x$ .
57. Simplify the expression  $\sqrt{4x^2 + 16}$  as much as possible after substituting  $2 \tan \theta$  for  $x$
- (58 – 60) Simplify by using the table
58.  $5 \sin^2 30^\circ$                       59.  $\sin^2 60^\circ + \cos^2 60^\circ$                       60.  $(\tan 45^\circ + \tan 60^\circ)^2$
61. Find  $\theta$  if  $\sin \theta = -\frac{1}{2}$  and  $\theta$  terminates in QIII with  $0^\circ \leq \theta \leq 360^\circ$ .
62. Find  $\theta$  to the nearest degree if  $\sec \theta = 3.8637$  and  $\theta$  terminates in QIV with  $0^\circ \leq \theta < 360^\circ$ .
- (63 – 67) Find the exact value of
63.  $\cos 225^\circ$                       65.  $\tan 315^\circ$                       67.  $\cot 480^\circ$
64.  $\csc 300^\circ$                       66.  $\cos 420^\circ$

(68 – 70) Use the calculator to find the value of

68.  $\csc 166.7^\circ$

69.  $\sec 590.9^\circ$

70.  $\tan 195^\circ 10'$

71. Use the calculator to find  $\theta$  to the nearest degree if  $\sin \theta = -0.3090$  with  $\theta \in \text{QIV}$  with  $0^\circ \leq \theta < 360^\circ$

72. Use the calculator to find  $\theta$  to the nearest degree if  $\cos \theta = -0.7660$  with  $\theta \in \text{QIII}$  with  $0^\circ \leq \theta < 360^\circ$

73. Use the calculator to find  $\theta$  to the nearest degree if  $\sec \theta = -3.4159$  with  $\theta \in \text{QII}$  with  $0^\circ \leq \theta < 360^\circ$

74. Find  $\theta$  to the nearest tenth of a degree if  $\tan \theta = -0.8541$  and  $\theta$  terminates in  $\text{QIV}$  with  $0^\circ \leq \theta < 360^\circ$

75. Use the calculator to find  $\theta$  to the nearest degree if  $\sin \theta = 0.49368329$  with  $\theta \in \text{QII}$  with  $0^\circ \leq \theta < 360^\circ$