Section 2.4 – Multiplication Rule and Conditional

Definition

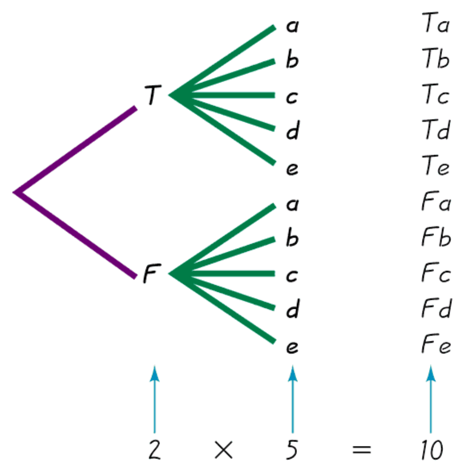
Two events *A* and *B* are ***independent*** if the occurrence of one does not affect the *probability* of the occurrence of the other. (Several events are similarly independent if the occurrence of any does not affect the probabilities of the occurrence of the others.)
If *A* and *B* are not independent, they are said to be ***dependent***.

Two events are ***dependent*** if the occurrence of one of them affects the *probability* of the occurrence of the other, but this does not necessarily mean that one of the events is a ***cause*** of the other.

Tree Diagrams

A tree diagram is a picture of the possible outcomes of a procedure, shown as line segments emanating from one starting point. These diagrams are sometimes helpful in determining the number of possible outcomes in a sample space, if the number of possibilities is not too large.

The figure summarizes the possible outcomes for a true/false question followed by a multiple choice question.
Note *that there are 10 possible combinations.*



Multiplication Rule for Independent Events

If *E* and *F* are independent events, then $P(E \text{ and } F) = P(E) \cdot P(F)$

Example

Assume that we have a batch of 100,000 heart pacemakers, including 99,950 that are good (*G*) and 50 that are defective (*D*).

a) If two of those 100,000 pacemakers are randomly selected without replacement, find the probability that they are both good.

b) If 20 of those 100,000 pacemakers are randomly selected without replacement, find the probability that they are both good.

Solution

a) $P(1st\ good) = \frac{99,950}{100,000}$ $\qquad$ $P(2nd\ good) = \frac{99,949}{100,000}$

$$P(1st\ good\ and\ 2nd\ good) = \frac{99,950}{100,000} \cdot \frac{99,949}{100,000} \quad \underline{= 0.999}$$

b) $P(all\ 20\ pacemakers\ are\ good) = \frac{99,950}{100,000} \cdot \frac{99,949}{100,000} \cdot \frac{99,948}{100,000} \cdot \ldots \cdot \frac{99,931}{100,000}$

Example

Assume that two people are randomly selected and also assume that birthdays occur on the same days of the week with equal frequencies.

a) Find the probability that the two people are born on the same day of the week.

b) Find the probability that the two people are born on Monday.

Solution

a) Because no particular day of the week is specified, the first person can be born on any one of the seven week days.

The probability that the second person is born on the same day as the first person is $\frac{1}{7}$.

Probability that 2 people are born on the same day of the week is $\frac{1}{7}$

b) $P(1st\ born\ on\ Monday) = \frac{1}{7}$

$P(2nd\ born\ on\ Monday) = \frac{1}{7}$

$$P(both\ born\ on\ Monday) = \frac{1}{7} \cdot \frac{1}{7}$$

$$= \frac{1}{49}$$

Example

A geneticist developed a procedure for increasing the likelihood of female babies. In an initial test, 20 couples use the method and the results consist of 20 females among 20 babies. Assuming that the gender-selection procedure has no effect, find the probability of getting 20 females among 20 babies by chance. Does the resulting provide strong evidence to support the geneticist's claim that the procedure is effective in increasing the likelihood that babies will be females?

Solution

$$\begin{aligned} P(all\ 20\ are\ female) &= P(1st\ is\ female\ and\ 2nd\ female\ \cdots\ and\ 20th\ is\ female) \\ &= P(\ female) \bullet P(\ female) \bullet \cdots \bullet P(\ female) \\ &= (0.5) \bullet (0.5) \bullet \cdots \bullet (0.5) \\ &= (0.5)^{20} \\ &= 0.000000954 \end{aligned}$$

The low probability of 0.000000954 indicates that instead of getting 20 females by chance, a more reasonable explanation is that females appear to be more likely with the gender-selection procedure. Because there is such a small probability of getting 20 females in 20 births, we do have to support the geneticist's claim that the gender-selection procedure is effective in increasing the likelihood that babies will be female.

Example

Modern aircraft engines are now highly reliable. One design feature contributing to that reliability is the use of redundancy, whereby critical components are duplicated so that if one fails, the other will work. For example, single-engine aircraft now have two independent electrical systems so that is one electrical system fails, the other can continue to work so that the engine does not fail. For the purposes of this example, we will assume that the probability of an electrical system failure is 0.001.

a) If the engine in an aircraft has one electrical system, what is the probability that it will work?

b) If the engine in an aircraft has 2 independent electrical systems, what is the probability that the engine can function with a working electrical system?

Solution

a) $P(\textit{electrical system failure}) = 0.001 \qquad P(\textit{doe not fail}) = 1 - 0.001 = 0.999$

$$P(\textit{working electrical system}) = P(\textit{electrical system does not fail})$$
$$= 0.999$$

b) $P(\textit{both electrical system fail}) = P(1st\ \textit{electrical system fails and}\ 2nd\ \textit{electrical system fails})$

$$= (.001)(.001)$$
$$= 0.000001$$

There is a 0.000001 probability of both electrical systems failing, so the probability that the engine can function with a working electrical system is 1 – 0.000001 = 0.999999

Complements

Finding the Probability of "*At Least One*"

To find the probability of at least one of something, calculate the probability of none, then subtract that result from 1. That is,

$$P(\textit{at least one}) = 1 - P(\textit{none})$$

- "At least one" is equivalent to "one or more."
- The complement of getting at least one item of a particular type is that you get no items of that type.

Example

Find the probability of a couple having at least 1 girl among 3 children. Assume that boys and girls are equally likely and that the gender of a child is independent of any other child.

Solution

Let A = at least 1 of the 3 children is a girl.

$\overline{A}$ = not getting at least 1 girl among 3 children
= all 3 children are boys
= boy and boy and boy

$$P\left(\bar{A}\right) = P\left(boy\ and\ boy\ and\ boy\right)$$
$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$$
$$= \frac{1}{8}$$

$$P\left(A\right) = 1 - P\left(\bar{A}\right)$$
$$= 1 - \frac{1}{8}$$
$$= \frac{7}{8}$$

There is $\frac{7}{8}$ probability that if a couple has 3 children; at least 1 of them is a girl.

Example

Assume that the probability of a defective Firestone tire is 0.0003 (based on data from Westgard (QC). If the retail outlet Car Stuff buys 100 Firestone tires, find the probability that they get at least 1 that is defective. If that probability is high enough, plans must be made to handle defective tires returned by consumers. Should they make those plans?

Solution

Let A = at least 1 of the 100 tires is defective.

$\bar{A}$ = not getting at least 1 defective among 100 tires

= all 100 tires are good

$$P\left(\bar{A}\right) = \left(1-.0003\right)\left(1-.0003\right)\ldots\left(1-.0003\right)$$
$$= \left(0.9997\right)\left(0.9997\right)\ldots\left(0.9997\right)$$
$$= \left(0.9997\right)^{100}$$
$$= .9704$$

$$P\left(A\right) = 1 - P\left(\bar{A}\right)$$
$$= 1 - .9704$$
$$= .0296$$

There is 0.0296 probability of at least 1 defective tire among the 100 tires. Because the probability is so low, it is not necessary to make plans for dealing with defective tires returned by consumers.

Conditional Probability ***Key Point***

We must adjust the probability of the second event to reflect the outcome of the first event.

The probability for the second event B should take into account the fact that the first event A has already occurred.

Notation for Conditional Probability

$P(B|A)$ represents the probability of event *B* occurring after it is assumed that event *A* has already occurred (read $B|A$ as "*B* given *A*.")

Conditional Probability

A ***conditional probability*** of an event is a probability obtained with the additional information that some other event has already occurred. $P(B|A)$ denotes the conditional probability of event *B* occurring, given that event *A* has already occurred, and it can be found by dividing the probability of events *A* and *B* both occurring by the probability of event *A*:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

The conditional probability of *B* given *A* can be found by assuming that event *A* has occurred, and then calculating the probability that event *B* will occur.

Notation

$$P(A \text{ and } B) = P(\text{event } A \text{ occurs in the first trial and event } B \text{ occurs in a second trial})$$

$$P(A \text{ and } B) = P(A) \bullet P(B)$$

Formal Multiplication Rule

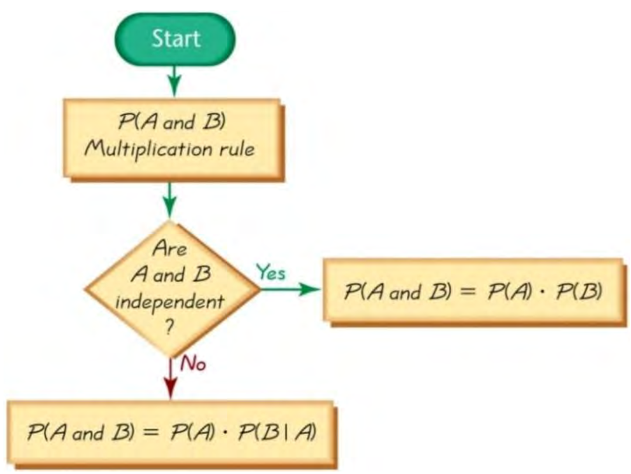
$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

if *A* and *B* are independent events, $P(B|A)$ is really the same as $P(B)$.

Intuitive Multiplication Rule

When finding the probability that event *A* occurs in one trial and event *B* occurs in the next trial, multiply the probability of event *A* by the probability of event *B*, but be sure that the probability of event *B* takes into account the previous occurrence of event *A*.

✓ *When applying the multiplication rule, always consider whether the events are independent or dependent, and adjust the calculations accordingly.*



Example

	No (Did Not Lie)	***Yes*** (Lied)
Positive test result	15 (*false positive*)	42 (*true positive*)
Negative test result	32 (*true negative*)	9 (*false negative*)

a) If 1 of the 98 test subjects is randomly selected, find the probability that the subject had a positive test result, given that the subject actually lied. That is, find *P*(positive test result | subject lied)

b) If 1 of the 98 test subjects is randomly selected, find the probability that the subject actually lied, given that he or she had a positive test result. That is, find *P* (subject lied | positive test result)

c) If two of the subjects are randomly selected without replacement, find the probability that the first selected person had a positive test result and the second selected person had a negative test result.

Solution

a) There are (42 + 9 =) 51 subjects that they lied, 42 has a positive test results.

$$P(\textit{positive test result}\,|\,\textit{lied}) = \frac{42}{51}\ \underline{= 0.824|}$$

Or

$$= \frac{\frac{42}{98}}{\frac{51}{98}}\ \underline{= 0.824|}$$

This indicates that a subject who lies has a 0.824 probability of gettin

$$P(\textit{positive test result}\,|\,\textit{lied}) = \frac{P(\textit{lied and had a positive test result})}{P(\textit{subject lied})}$$ g a positive test result.

b) There are (42 + 15 =) 57 with a positive test results subjects among that 42 lied.

$$P(\textit{subject lied}\,|\,\textit{positive test result}) = \frac{42}{57}\ \underline{= 0.737|}$$

This indicates that for a subject who gets a positive test result, there is a 0.737 probability that this subject actually lied.

c) $P(\textit{positive test result}) = \frac{57}{98}$

After the first selection of a subject, there are 97 subjects remaining

$$P(\textit{negative test result}) = \frac{41}{97}$$

$$P(1st\ \textit{positive and 2nd negative test result}) = \frac{57}{98}\cdot\frac{41}{97}\ \underline{= 0.246|}$$

Confusion of the Inverse

To incorrectly believe that *P*(*A*|*B*) and *P*(*B*|*A*) are the same, or to incorrectly use one value for the other, is often called confusion of the inverse.

Bayes' Theorem

Rare Event Rule for Inferential Statistics

If, under a given assumption, the probability of a particular observed event is extremely small, we conclude that the assumption is probably not correct.

Bayes' Formula

$$P\left(U_1 \mid E\right) = \frac{P\left(U_1 \cap E\right)}{P(E)}$$

$$= \frac{P\left(U_1 \cap E\right)}{P\left(U_1 \cap E\right) + P\left(U_2 \cap E\right) + \cdots}$$

$$= \frac{P\left(E \mid U_1\right) P\left(U_1\right)}{P\left(E \mid U_1\right) P\left(U_1\right) + P\left(E \mid U_2\right) P\left(U_2\right) + \cdots}$$

Exercises Section 2.4 – Multiplication Rule and Conditional

1. Use the data below:

	No (Did Not Lie)	***Yes*** (Lied)
Positive test result	15 (*false positive*)	42 (*true positive*)
Negative test result	32 (*true negative*)	9 (*false negative*)

a) If 2 of the 98 test subjects are randomly selected without replacement find the probability that they both had false positive results. Is it unusual to randomly select 2 subjects without replacement and get 2 results that are both false positive results? Explain.

b) If 3 of the 98 test subjects are randomly selected without replacement, find the probability that all had false positive results. Is it unusual to randomly select 3 subjects without replacement and get 3 results that are all false positive results? Explain.

c) If 4 of the test subjects are randomly selected without replacement find the probability that, in each case, the polygraph indicated that the subject lied. Is such an event unusual?

d) If 4 of the test subjects are randomly selected without replacement find the probability that they all had incorrect test result (either false positive or false negative). Is such an event Likely?

e) Assume that 1 of the 98 test subjects is randomly selected. Find the probability of selecting a subject with a negative test result, given that the subject lied. What does this result suggest about the polygraph test?

f) Find $P(\textit{negative test result} | \textit{subject did not lie})$

g) Find $P(\textit{subject did not lie} | \textit{negative test result})$

2. Use the data in the table below

	Group			
Type	***O***	***A***	***B***	***AB***
Rh^+	39	35	8	4
Rh^-	6	5	2	1

a) If 2 of the 100 subjects are randomly selected, find the probability that they are both group O and type Rh^+

i. Assume that the selections are made with replacement.

ii. Assume that the selections are made without replacement.

b) If 3 of the 100 subjects are randomly selected, find the probability that they are both group B and type Rh^-

i. Assume that the selections are made with replacement.

ii. Assume that the selections are made without replacement.

c) People with blood that is group O and type Rh^- are considered to be universal donors, because they can give blood to anyone. If 4 of the 100 subjects are randomly selected, find the probability that they are all universal donors.
   i. Assume that the selections are made with replacement.
   ii. Assume that the selections are made without replacement.

d) People with blood that is group AB and type Rh^+ are considered to be universal donors, because they can give blood to anyone. If 3 of the 100 subjects are randomly selected, find the probability that they are all universal recipients.
   i. Assume that the selections are made with replacement.
   ii. Assume that the selections are made without replacement.

3. With one method of a procedure called acceptance sampling, a sample of items is randomly selected without replacement and the entire batch is accepted if every item in the sample is okay. The Telektronics Company manufactured a batch of 400 backup power supply units for computers, and 8 of them are defective. If 3 of the units are randomly selected for testing, what is the probability that the entire batch will be accepted?

4. It is common for public opinion polls to have a "confidence level" of 95% meaning that there is a 095 probability that the poll results are accurate within the claimed margins of error. If each of the following organizations conducts an independent poll, find the probability that all of them are accurate within the claim margins of error: Gallup, Roper, Yankelovich, Harris, CNN, ABC, CBS, and NBC, New York Times. Does the result suggest that with a confidence level of 95%, we can expect that almost all polls will be within the claimed margin of error?

5. The principle of redundancy is used when system reliability is improved through redundant or back up components. Assume that your alarm clock has a 0.9 probability of working on any given morning.

a) What is the probability that your alarm clock will not work on the morning of an important final exam?
b) If you have 2 such alarm clocks, what is the probability that they both fail on the morning of an important final exam?
c) With one alarm clock, you have a 0.9 probability of being awakened. What is the probability of being awakened if you use two alarm clocks?
d) Does a second alarm clock result in greatly improved reliability?

6. The wheeling Tire Company produced a batch of 5,000 tires that includes exactly 200 that are defective.

a) If 4 tires are randomly selected for installation on a car, what is the probability that they are all good?
b) If 100 tires are randomly selected for shipment to an outlet, what is the probability that they are all good? Should this outlet plan to deal with defective tires returned by consumers?

7. When the 15 players on the LA Lakers basketball team are tested for steroids, at least one of them tests positive. Provide a written description of the complement of this event.

8. If a couple plans to have 6 children, what is the probability that they will have at least one girl? Is that probability high enough for the couple to be very confident that they will get at least one girl in six children?

9. If a couple plans to have 8 children (it could happen), what is the probability that they will have at least one girl? Is the couple eventually has 8 children and they are all boys, what can the couple conclude?

10. If you make guesses for 4 multiple-choice test questions (each with 5 possible answers), what is the probability of getting at least one correct? If a very lenient instructor says that passing test occurs if there is at least one correct answer, can you reasonably expect to pass by guessing?

11. Find the probability of a couple having a baby girl when their fourth child is born, given that the first 3 children were all girls. Is the result the same as the probability of getting 4 girls among 4 children?

12. In China, the probability of a baby being a boy is 0.5845. Couples are allowed to have only one child. If relatives give birth to 5 babies, what is the probability that there is at least one girl? Can that system continue to work indefinitely?

13. An experiment with fruit flies involves one parent with normal wings and one parent with vestigial wings. When these parents have an offspring, there is a $\frac{3}{4}$ probability that the offspring has normal wings and a $\frac{1}{4}$ probability of vestigial wings. If the parents give birth to 10 offspring, what is the probability that at least 1 of the offspring has vestigial wings? If researchers need at least one offspring with vestigial wings, can they be reasonably confident of getting one?

14. According to FBI data, 24.9% of robberies are cleared with arrests. A new detective is assigned to 10 different robberies.
    a) What is the probability that at least 1 of them is cleared with an arrest?
    b) What is the probability that the detective clears all 10 robberies with arrests?
    c) What should we conclude if the detective clears all 10 robberies with arrests?

15. A statistics student wants to ensure that she is not late for an early statistics class because of a malfunctioning alarm clock. Instead of using one alarm clock, she decides to use three. What is the probability that at least one of her alarm clocks works correctly if each individual alarm clock has a 90% chance of working correctly? Does the student really gain much by using three alarm clocks instead on only one? How are the results affected if all of the alarm clocks run on electricity instead of batteries?

16. In a batch of 8,000 clock radios 8% are defective. A sample of 5 clock radios is randomly selected without replacement from the 8,000 and tested. The entire batch will be rejected if at least one of those tested is defective. Find the probability that the entire batch will be rejected.

17. In a blood testing procedure, blood samples from 3 people are combined into one mixture. The mixture will only test negative if all the individual samples are negative. If the probability that an individual sample tests positive is 0.1, find the probability that the mixture will test positive.

18. A sample of 4 different calculators is randomly selected from a group containing 16 that are defective and 36 that have no effects. Find the probability that at least one of the calculator is defective.

19. Among the contestants in a competition are 46 women and 29 men. If 5 winners are randomly selected, find the probability that they are all men?

20. A bin contains 60 lights bubs of which 7 are defective. If 4 light bulbs are randomly selected from the bin with replacement, find the probability that all the bulbs selected are good ones.

21. You are dealt two cards successively (without replacement) from a shuffled deck of 52 playing cards. Find the probability that both cards are black. Express your answer as a simplified fraction.