

2.4

$$y'' + p y' + q y = f(x) \quad (IV) \text{ non-homogeneous}$$

in homogeneous

Homogeneous:  $y'' + p y' + q y = 0$

$$\lambda^2 + p\lambda + q = 0$$

$$\left\{ \begin{array}{l} 2 \mathbb{R} \Rightarrow y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \\ 2 \mathbb{C} \Rightarrow y(x) = e^{ax} (C_1 \cos bt + C_2 \sin bt) \\ 2 \text{ repeated} \Rightarrow y(x) = (C_1 + C_2 x) e^{ax} \end{array} \right.$$

$$y(x) = \underbrace{y_p(x)}_{\text{Particular}} + \underbrace{C_1 y_1(x) + C_2 y_2(x)}_{\text{homogeneous}}$$

Ex  $y'' - y' - 2y = \underbrace{2e^{-2t}}_{\text{Forcing Term}} \quad y_p(t)$

Let  $y = a e^{-2t}$

$$y' = -2a e^{-2t}$$

$$y'' = 4a e^{-2t}$$

$$4a e^{-2t} + 2a e^{-2t} - 2a e^{-2t} = 2e^{-2t}$$

$$4a = 2$$

$$a = \frac{1}{2}$$

$$y_p(t) = \frac{1}{2} e^{-2t}$$

$$f(t) = A \cos \omega t + B \sin \omega t \quad (\text{either/both})$$

$$y_p(t) = a \cos \omega t + b \sin \omega t.$$

Ex.  $y'' + 2y' - 3y = 5 \sin 3t$   $y_p(t) = ?$

$$\text{let } y_p = a \sin 3t + b \cos 3t$$

$$\therefore y' = 3a \cos 3t - 3b \sin 3t$$

$$y'' = -9a \sin 3t - 9b \cos 3t$$

$$\begin{array}{rcl} & \cos 3t & \sin 3t \\ \hline & -3b + 6a - 9b = 0 & -3a - 6b - 9a = 5 \end{array}$$

$$\left\{ \begin{array}{l} 6a - 12b = 0 \\ -12a - 6b = 5 \end{array} \right. \rightarrow \left\{ \begin{array}{l} a - 2b = 0 \\ 12a + 6b = -5 \end{array} \right.$$

$$a = \frac{-10}{30} = -\frac{1}{3} \quad b = -\frac{1}{6}$$

$$y_p(t) = -\frac{1}{3} \sin 3t - \frac{1}{6} \cos 3t$$

$$f(t) = a_0 t^n + a_1 t^{n-1} + \dots + a_n$$

Ex

$$y'' + 2y' - 3y = 3t + 4$$

$$y_p = at + b$$

$$y' = a$$

$$y'' = 0$$

$$2a - \underline{3at} - 3b = \underline{3t + 4}$$

$$-3a = 3 \Rightarrow \underline{a = -1}$$

$$2(-1) - 3b = 4 \Rightarrow b = -2$$

$$\underline{y_p(t) = -t - 2}$$

Ex

$$y'' - y' - 2y = 3e^{-t}$$

$$y = ae^{-t}$$

$$y' = -ae^{-t}$$

$$y'' = ae^{-t}$$

$$ae^{-t} + ae^{-t} - 2ae^{-t} = 3e^{-t}$$

$$0 = 3e^{-t} \neq$$

$$y = ate^{-t} \rightarrow y = at^2e^{-t}$$

$$(H): \lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda = -1, 2$$

$$y = ate^{-t}$$

$$y' = (a - at)e^{-t} = ae^{-t} - ate^{-t}$$

$$y'' = (-a - a + at)e^{-t} \\ = (at - 2a)e^{-t}$$

$$ate^{-t} - 2ae^{-t} - ae^{-t} + ate^{-t} - 2ae^{-t} = 3e^{-t}$$

$$at - 2a - a + at - 2at = 3 \\ -3a = 3 \Rightarrow \underline{a = -1}$$

$$e^{-t} \neq 0$$

$$\underline{y(H) = -te^{-t}}$$

Ex  $2y'' - 5y' + 3y = 4e^{3t}$

$y(0) = 1, y'(0) = 0$

soln  $2\lambda^2 - 5\lambda + 3 = 0 \Rightarrow \lambda_{1,2} = 1, \frac{3}{2}$

$y_h = C_1 e^t + C_2 e^{3t/2}$

$y_p = a e^{3t}$

$y' = 3a e^{3t}$

$y'' = 9a e^{3t}$

$(18a - 15a + 3a)e^{3t} = 4e^{3t}$   
 $6a = 4 \Rightarrow a = \frac{2}{3}$

$y(t) = C_1 e^t + C_2 e^{3t/2} + \frac{2}{3} e^{3t}$

$y(0) = C_1 + C_2 + \frac{2}{3} = 1$

$C_1 + C_2 = \frac{1}{3} \quad (1)$

$y'(t) = C_1 e^t + \frac{3}{2} C_2 e^{3t/2} + 2e^{3t}$

$y'(0) = C_1 + \frac{3}{2} C_2 + 2 = 0$

$2C_1 + 3C_2 = -4 \quad (2)$

$-2 \times (1) \Rightarrow -2C_1 - 2C_2 = -\frac{2}{3}$

$(2) \Rightarrow 2C_1 + 3C_2 = -4$

$C_2 = -\frac{14}{3}$

$(1) \Rightarrow C_1 = \frac{1}{3} + \frac{14}{3} = 5$

$y(t) = 5e^t - \frac{14}{3} e^{3t/2} + \frac{2}{3} e^{3t}$

# 73  $y'' - 3y' - 10y = 2x - 3$

$$\lambda^2 - 3\lambda - 10 = 0$$

$$\lambda_{1,2} = -2, 5$$

$$y_h = C_1 e^{-2t} + C_2 e^{5t}$$

$$y_p = ax + b$$

$$y' = a$$

$$y'' = 0$$

$$-3a - 10ax - 10b = 2x - 3$$

$$-10a = 2$$

$$a = -\frac{1}{5}$$

$$+3a + 10b = -3$$

$$10b = 3 - \frac{3}{5} = \frac{12}{5}$$

$$b = \frac{6}{25}$$

$$y(t) = C_1 e^{-2t} + C_2 e^{5t} + \frac{1}{5}x + \frac{6}{25}$$



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$$y'' + 4y' + 8y = \sin t$$

$$y(0) = 1$$

$$y'(0) = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda^2 + 4\lambda + 8 = 0 \Rightarrow \lambda = -2 \pm 2i$$

$$y_h(t) = e^{-2t} (C_1 \cos 2t + C_2 \sin 2t)$$

$$y_p = a \cos t + b \sin t$$

$$y' = -a \sin t + b \cos t$$

$$y'' = -a \cos t - b \sin t$$

$$\cos t \quad \sin t$$

$$8a + 4b - a = 0$$

$$8b - 4a - b = 1$$

$$\begin{cases} 7a + 4b = 0 \\ -4a + 7b = 1 \end{cases}$$

$$a = \frac{-4}{65}$$

$$b = \frac{7}{4} \left( \frac{-4}{65} \right) = \frac{7}{65}$$

$$y(t) = (C_1 \cos 2t + C_2 \sin 2t) e^{-2t} - \frac{4}{65} \cos t + \frac{7}{65} \sin t$$

$$y(0) = C_1 - \frac{4}{65} = 1 \Rightarrow C_1 = \frac{69}{65}$$

$$y' = (-2C_1 \sin 2t + 2C_2 \cos 2t - 2C_1 \cos 2t - 2C_2 \sin 2t) e^{-2t} + \frac{4}{65} \sin t + \frac{7}{65} \cos t$$

$$y'(0) = 2C_2 - 2 \left( \frac{69}{65} \right) + \frac{7}{65} = 0$$

$$2C_2 = \frac{-7 + 138}{65} \Rightarrow C_2 = \frac{131}{130}$$

$$y(t) = \left( \frac{69}{65} \cos 2t + \frac{131}{130} \sin 2t \right) e^{-2t} - \frac{4}{65} \cos t + \frac{7}{65} \sin t$$

## 2.5 Variation of Parameters.

$$y = C_1 y_1 + C_2 y_2 \quad \left\{ \begin{array}{l} y'' + p y' + q y = \underline{g(t)} \end{array} \right.$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

$$\begin{cases} N_1 = - \int \frac{y_2 g(t)}{W} dt \\ N_2 = \int \frac{y_1 g(t)}{W} dt \end{cases}$$

$$y_p = N_1 y_1 + N_2 y_2$$

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Ex 1

$$y_1(x) = x^4$$

$$y_2(x) = x^2$$

$$y'' - \frac{5}{x} y' + \frac{6}{x^2} y = 4x^3$$

$$\begin{aligned} W' &= \begin{vmatrix} x^4 & x^2 \\ 4x^3 & 2x \end{vmatrix} \\ &= 2x^5 - 4x^5 \\ &= -2x^5 \neq 0 \end{aligned}$$

$$\begin{aligned} v_1 &= - \int \frac{x^2(4x^3)}{-2x^5} dx \\ &= \int 2 dx \\ &= \underline{2x} \end{aligned}$$

$$\begin{aligned} v_2 &= \int \frac{x^4(4x^3)}{-2x^5} dx \\ &= -2 \int x^2 dx \\ &= -\frac{2}{3} x^3 \end{aligned}$$

$$\begin{aligned} y_p &= (2x)(x^4) + \left(-\frac{2}{3}x^3\right)(x^2) \\ &= 2x^5 - \frac{2}{3}x^5 \\ &= \frac{4}{3}x^5 \end{aligned}$$

$$y(x) = C_1 x^4 + C_2 x^2 + \frac{4}{3} x^5$$

$$\left\{ \begin{aligned} y_1(x) &= e^{2x} \\ y_2(x) &= xe^{2x} \end{aligned} \right\}$$

$$y'' - 4y' + 4y = \frac{e^{2x}}{x}$$

$$\begin{aligned} W &= \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} \\ &= e^{4x} + 2xe^{4x} - 2xe^{4x} \\ &= e^{4x} \end{aligned}$$

$$\begin{aligned} v_1 &= - \int \frac{xe^{2x}}{e^{4x}} \frac{e^{2x}}{x} dx \\ &= -x \end{aligned}$$

$$\begin{aligned} v_2 &= \int \frac{e^{2x}}{e^{4x}} \frac{e^{2x}}{x} dx \\ &= \int \frac{dx}{x} \\ &= \ln|x| \end{aligned}$$

$$y_{fp} = -xe^{2x} + x(\ln x)e^{2x}$$

$$y(x) = C_1 e^{2x} + \underbrace{C_2 xe^{2x} - xe^{2x}} + x(\ln x)e^{2x}$$

$$C_1 e^{2x} + C_3 xe^{2x} + x(\ln x)e^{2x}$$

$$\boxed{C_3 = C_2 - 1}$$

Ex  $y'' + y = \tan t$

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

$$a=0$$

$$y_h = C_1 \underbrace{\cos t}_{f_1} + C_2 \underbrace{\sin t}_{f_2}$$

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} \\ = \underline{1}$$

$$\begin{aligned} N_1 &= - \int \sin t \tan t \, dt \\ &= - \int \frac{\sin^2 t}{\cos t} \, dt \\ &= - \int \frac{1 - \cos^2 t}{\cos t} \, dt \\ &= - \int (\sec t - \cos t) \, dt \\ &= - (\ln |\sec t + \tan t| - \sin t) \\ &= \underline{\sin t - \ln |\sec t + \tan t|} \end{aligned}$$

$$\begin{aligned} N_2 &= \int \cos t \tan t \, dt \\ &= \int \sin t \, dt \\ &= -\cos t \end{aligned}$$

$$\begin{matrix} C_4 \\ \text{"} \\ (C_2 + 1) \sin t \end{matrix}$$

$$\begin{aligned} y(t) &= C_1 \cos t + C_2 \sin t + \sin t - \ln |\sec t + \tan t| - \cos t \\ &= C_3 \cos t + C_4 \sin t - \ln |\sec t + \tan t| \end{aligned}$$