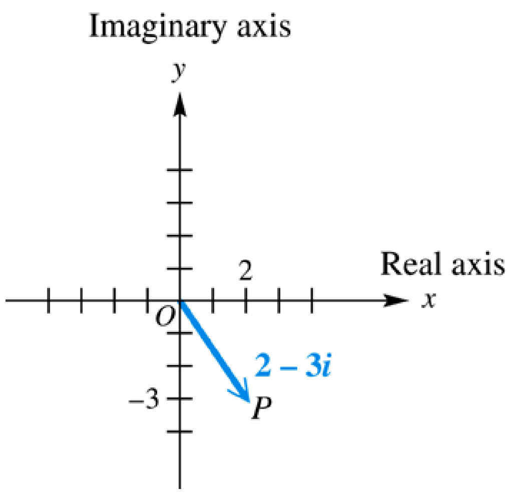
Section 3.7 – Trigonometric Form

$\sqrt{-1} = i$

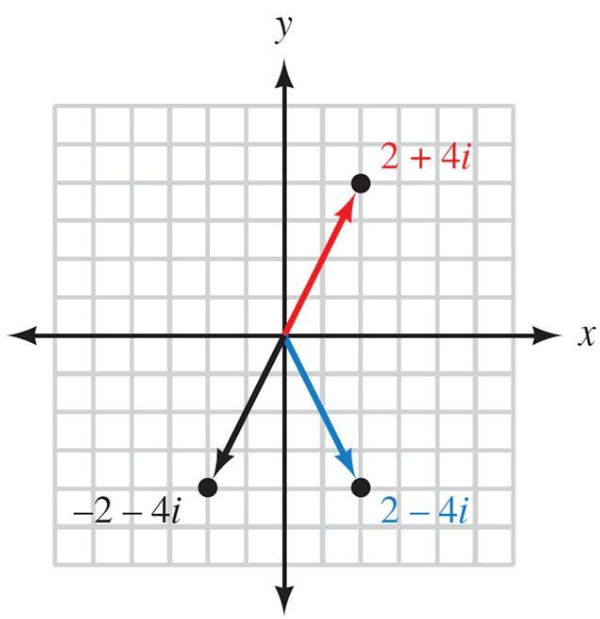
The graph of the complex number $x = yi$ is a vector (arrow) that extends from the origin out to the point (x, y)

- Horizontal axis: ***real axis***
- Vertical axis: ***imaginary axis***



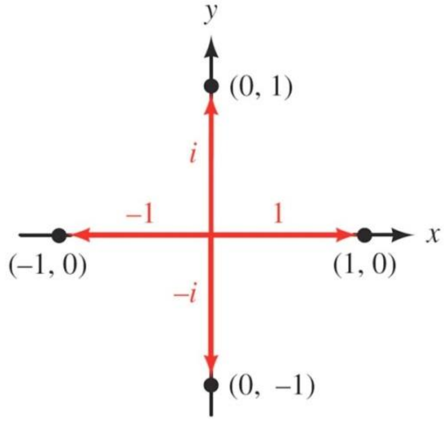
Example

Graph each complex number: $2+4i$, $-2-4i$, and $2-4i$



Example

Graph each complex number: $1,\ i,\ -1,\ and\ \ -i$

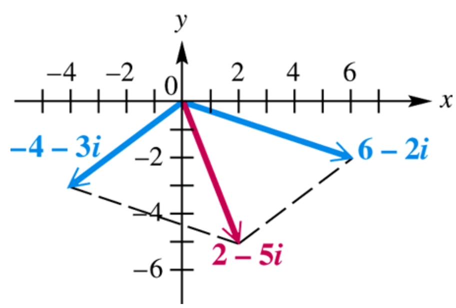


Example

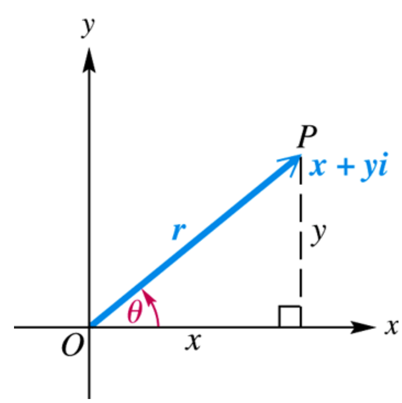
Find the sum of 6 – 2*i* and –4 – 3*i*. Graph both complex numbers and their resultant.

Solution

(6 – 2*i*) + (–4 – 3*i*) = 6 – 4 – 2i – 3i = 2 – 5*i*



Definition



The *absolute value* or ***modulus*** of the complex number $z = x + yi$ is the distance from the origin to the point (*x*, *y*). If this distance is denoted by *r*, then

$$r = |z| = |x + yi| = \sqrt{x^2 + y^2}$$

The ***argument*** of the complex number $z = x + yi$ denoted $\arg(z)$ is the smallest possible angle θ from the positive real axis to the graph of *z*.

$$\cos\theta = \frac{x}{r} \qquad \Rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \qquad \Rightarrow y = r\sin\theta$$

$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + (r\sin\theta)\ i \\ &= r(\cos\theta + i\ \sin\theta) \end{aligned}$$

$\rightarrow$ is called the *trigonometric* from of *z*.

Definition

If $z = x + y\ i$ is a complex number in standard form then the ***trigonometric form*** for z is given by

$$z = r(\cos\theta + i\ \sin\theta) = r\ cis\theta$$

Where $\boldsymbol{r}$ is the modulus or absolute value of z and

$\boldsymbol{\theta}$ is the argument of z.

We can convert back and forth between standard form and trigonometric form by using the relationships that follow

For $z = x + y\ i = r(\cos\theta + i\ \sin\theta) = r\ cis\theta$

$$r = \sqrt{x^2 + y^2}$$

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}, \quad and \quad \tan\theta = \frac{y}{x}$$

Example

Write $z = -1 + i$ in trigonometric form

Solution

The modulus r:

$$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$\cos\theta = \frac{x}{r} = \frac{-1}{\sqrt{2}}$$

$$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}}$$

$$\to \theta = 135°$$

$$z = x + y\ i$$

$$= \sqrt{2}(\cos 135° + i\ \sin 135°)$$

$$= \sqrt{2}\ cis 135°$$

In radians: $z = \sqrt{2}\ cis\left(\frac{3\pi}{4}\right)$

Example

Write $z = 2\ cis60°$ in rectangular form.

Solution

$$
\begin{aligned}
z &= 2\ cis60° \\
&= 2(\cos 60° + i\ \sin 60°) \\
&= 2\left(\frac{1}{2} + i\ \frac{\sqrt{3}}{2}\right) \\
&= 1 + i\ \sqrt{3}
\end{aligned}
$$

Example

Express $2(\cos 300° + i \sin 300°)$ in rectangular form.

Solution

$$
\begin{aligned}
2(\cos 300° + i \sin 300°) &= 2\left(\tfrac{1}{2} - i\frac{\sqrt{3}}{2}\right) \\
&= 1 - i\sqrt{3}
\end{aligned}
$$

Example

Find the modulus of each of the complex numbers $5i$, 7, and $3 + 4i$

Solution

For $z = 5i = 0 + 5i \Rightarrow r = |z| = \sqrt{0^2 + 5^2} = 5$

For $z = 7 = 7 + 0i \Rightarrow r = |z| = \sqrt{7^2 + 0^2} = 7$

For $3 + 4i \Rightarrow r = \sqrt{3^2 + 4^2} = 5$

Product Theorem

If $r_1\left(\cos\theta_1 + i\sin\theta_1\right)$ and $r_2\left(\cos\theta_2 + i\sin\theta_2\right)$ are any two complex numbers, then

$$\left[r_1\left(\cos\theta_1 + i\sin\theta_1\right)\right]\left[r_2\left(\cos\theta_2 + i\sin\theta_2\right)\right] = r_1 r_2\left[\cos\left(\theta_1+\theta_2\right) + i\sin\left(\theta_1+\theta_2\right)\right]$$

$$\left(r_1 cis\theta_1\right)\left(r_2 cis\theta_2\right) = r_1 r_2 cis\left(\theta_1+\theta_2\right)$$

$$\boxed{(a+bi)(a-bi) = a^2+b^2}$$

$$\boxed{\left(\sqrt{a}+\sqrt{b}i\right)\left(\sqrt{a}-\sqrt{b}i\right) = a+b}$$

Example

Find the product of $3(\cos 45° + i\sin 45°)$ and $2(\cos 135° + i\sin 135°)$. Write the result in rectangular form.

Solution

$$\left[3(\cos 45° + i\sin 45°)\right]\left[2(\cos 135° + i\sin 135°)\right]$$

$$= (3)(2)\left[\cos(45°+135°) + i\sin(45°+135°)\right]$$

$$= 6(\cos 180° + i\sin 180°)$$

$$= 6(-1 + i.0)$$

$$= -6$$

Quotient Theorem

If $r_1\left(\cos\theta_1 + i\sin\theta_1\right)$ and $r_2\left(\cos\theta_2 + i\sin\theta_2\right)$ are any two complex numbers, then

$$\frac{r_1\left(\cos\theta_1 + i\sin\theta_1\right)}{r_2\left(\cos\theta_2 + i\sin\theta_2\right)} = \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right]$$

$$\frac{r_1 cis\theta_1}{r_2 cis\theta_2} = \frac{r_1}{r_2} cis\left(\theta_1 - \theta_2\right)$$

Example

Find the quotient $\frac{10cis\left(-60°\right)}{5cis\left(150°\right)}$. Write the result in rectangular form.

Solution

$$\begin{aligned}
\frac{10cis\left(-60°\right)}{5cis\left(150°\right)} &= \frac{10}{5} cis\left(-60° - 150°\right) \\
&= 2cis\left(-210°\right) \\
&= 2\left[\cos\left(-210°\right) + i\sin\left(-210°\right)\right] \\
&= 2\left[-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right] \\
&= -\sqrt{3} + i
\end{aligned}$$

De Moivre's *Theorem*

If $r(\cos\theta + i\sin\theta)$ is a complex number, then

$$\left[r(\cos\theta + i\sin\theta)\right]^{n} = r^{n}(\cos n\theta + i\sin n\theta)$$

$$\boxed{\left[r cis\theta\right]^{n} = r^{n}(cis n\theta)}$$

Example

Find $\left(1+i\sqrt{3}\right)^{8}$ and express the result in rectangular form.

Solution

$$1+i\sqrt{3} \Rightarrow \begin{cases} x=1 \\ y=\sqrt{3} \end{cases}$$

$$r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$$

$$\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$$

θ is in QI, that implies: $\theta = 60°$

$$1+i\sqrt{3} = 2cis60°$$

Apply De Moivre's theorem:

$$\begin{aligned}
\left(1+i\sqrt{3}\right)^{8} &= \left(2cis60°\right)^{8} \\
&= 2^{8}\left[cis(8.60°)\right] \\
&= 256\left[cis(480°)\right] \qquad 480° - 360° = 120° \\
&= 256\left[cis(120°)\right] \\
&= 256\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\
&= -128 + 128i\sqrt{3}
\end{aligned}$$

n^{th} Root Theorem

For a positive integer n, the complex number $a + bi$ is an $\boldsymbol{n^{th}}$ **root** of the complex number $x + iy$ if

$$(a+bi)^n = x + yi$$

If n is any positive integer, r is a positive real number, and θ is in degrees, then the nonzero complex number $r(\cos\theta + i\sin\theta)$ has exactly n distinct nth roots, given by

$$\sqrt[n]{r}\left(\cos\alpha + i\sin\alpha\right) \text{ or } \sqrt[n]{r}\ cis\alpha$$

Where $\alpha = \dfrac{\theta + 360°k}{n},\quad k = 0,1,2,\cdots,n-1 \qquad \boxed{\alpha = \dfrac{\theta}{n} + \dfrac{360°k}{n}}$

$$\alpha = \frac{\theta + 2\pi k}{n},\quad k = 0,1,2,\cdots,n-1 \qquad \boxed{\alpha = \frac{\theta}{n} + \frac{2\pi k}{n}}$$

Example

Find the two square root of $\boldsymbol{4i}$. Write the roots in rectangular form.

Solution

$$4i \Rightarrow \begin{cases} x = 0 \\ y = 4 \end{cases} \qquad \rightarrow r = \sqrt{0^2 + 4^2} = 4$$

$$\tan\theta = \frac{4}{0} = \infty \Rightarrow \boxed{\theta = \frac{\pi}{2}}$$

$$4i = 4cis\frac{\pi}{2}$$

The absolute value: $\sqrt{4} = 2$

Argument: $\alpha = \dfrac{\frac{\pi}{2} + 2\pi k}{2} = \dfrac{\frac{\pi}{2}}{2} + \dfrac{2\pi k}{2} = \dfrac{\pi}{4} + \pi k$

Since there are ***two*** square root, then $k = 0$ and 1.

$$\text{If } k = 0 \Rightarrow \alpha = \frac{\pi}{4} + \pi(0) = \frac{\pi}{4}$$

$$\text{If } k = 1 \Rightarrow \alpha = \frac{\pi}{4} + \pi(1) = \frac{5\pi}{4}$$

The square roots are: $2cis\frac{\pi}{4}$ *and* $2cis\frac{5\pi}{4}$

$$2cis\frac{\pi}{4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2} + i\sqrt{2}$$

$$2cis\frac{5\pi}{4} = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -\sqrt{2} - i\sqrt{2}$$

Example

Find all fourth roots of $-8+8i\sqrt{3}$. Write the roots in rectangular form.

Solution

$$-8+8i\sqrt{3} \Rightarrow \begin{cases} x=-8 \\ y=8\sqrt{3} \end{cases}$$

$$r=\sqrt{(-8)^2+\left(8\sqrt{3}\right)^2}=16$$

$$\tan\theta=\frac{8\sqrt{3}}{-8}=-\sqrt{3} \Rightarrow \boxed{\theta=120^\circ}$$

$$-8+8i\sqrt{3}=16cis120^\circ$$

The fourth roots have absolute value: $\sqrt[4]{16}=2$

$$\alpha=\frac{120^\circ}{4}+\frac{360^\circ k}{4}=30^\circ+90^\circ k$$

Since there are ***four*** roots, then k = 0, 1, 2, and 3.

$If\ \ k=0 \Rightarrow \alpha=30^\circ+90^\circ(0)=30^\circ$

$If\ \ k=1 \Rightarrow \alpha=30^\circ+90^\circ(1)=120^\circ$

$If\ \ k=2 \Rightarrow \alpha=30^\circ+90^\circ(2)=210^\circ$

$If\ \ k=3 \Rightarrow \alpha=30^\circ+90^\circ(3)=300^\circ$

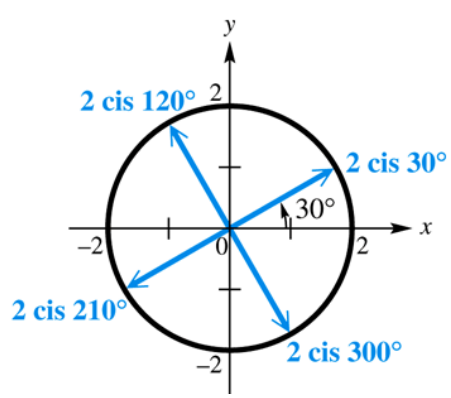
The fourth roots are: $2cis30^\circ,\ \ 2cis120^\circ,\ \ 2cis210^\circ,\ and\ \ 2cis300^\circ$

$$2cis30^\circ=2\left(\cos30^\circ+i\sin30^\circ\right)=2\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)=\sqrt{3}+i$$

$$2cis120^\circ=2\left(\cos120^\circ+i\sin120^\circ\right)=2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=-1+i\sqrt{3}$$

$$2cis210^\circ=2\left(\cos210^\circ+i\sin210^\circ\right)=2\left(-\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)=-\sqrt{3}-i$$

$$2cis300^\circ=2\left(\cos300^\circ+i\sin300^\circ\right)=2\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=1-i\sqrt{3}$$



Example

Find all complex number solutions of $x^5 - 1 = 0$. Graph them as vectors in the complex plane.

Solution

$x^5 - 1 = 0 \Rightarrow x^5 = 1$

There is one real solution, 1, while there are five complex solutions.

$1 = 1 + 0i$

$r = \sqrt{1^2 + 0^2} = 1$

$\tan\theta = \frac{0}{1} = 0 \Rightarrow \boxed{\theta = 0°}$

$1 = 1 cis 0°$

The fifth roots have absolute value: $\sqrt[5]{1} = 1$

$\underline{|\alpha =} \frac{0°}{5} + \frac{360°k}{5} = 0° + 72°k \underline{= 72°k|}$

Since there are ***fifth*** roots, then k = 0, 1, 2, 3, and 4.

$\quad If\ \ k = 0 \Rightarrow \alpha = 72°(0) = 0°$

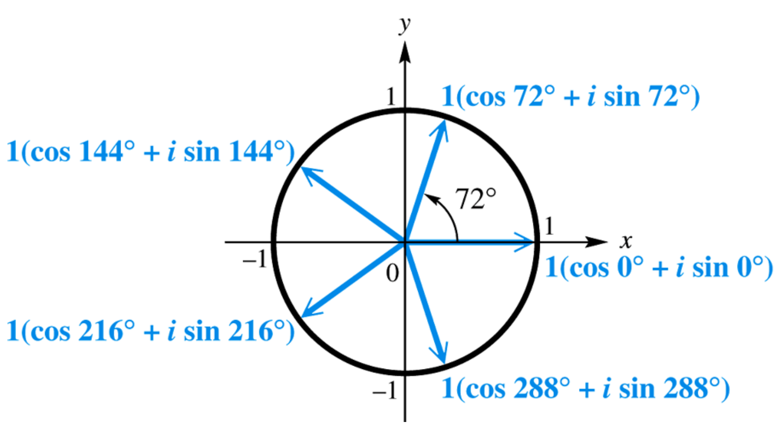
$\quad If\ \ k = 1 \Rightarrow \alpha = 72°(0) = 72°$

$\quad If\ \ k = 2 \Rightarrow \alpha = 72°(2) = 144°$

$\quad If\ \ k = 3 \Rightarrow \alpha = 72°(3) = 216°$

$\quad If\ \ k = 4 \Rightarrow \alpha = 72°(4) = 288°$

Solution: $cis 0°,\ \ cis 72°,\ \ cis 144°,\ cis 216°,\ and\ \ cis 288°$



The graphs of the roots lie on a unit circle. The roots are equally spaced about the circle, 72° apart.

Exercises *Section* 3.7 – Trigonometric Form

Write complex form in trigonometric form

1. $-\sqrt{3}+i$

2. $3-4i$

3. $-21-20i$

4. $11+2i$

5. $\sqrt{3}-i$

6. $1-\sqrt{3}i$

7. $9\sqrt{3}+9i$

8. $-2+3i$

Write in standard form

9. $4\left(\cos 30^\circ + i\sin 30^\circ\right)$

10. $\sqrt{2}\ cis\frac{7\pi}{4}$

11. $3cis210^\circ$

12. $4\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

13. $4cis\frac{\pi}{2}$

14. Find the quotient $\dfrac{20cis\left(75^\circ\right)}{4cis\left(40^\circ\right)}$. Write the result in rectangular form.

15. Divide $z_1 = 1+i\sqrt{3}$ *by* $z_2 = \sqrt{3}+i$. Write the result in rectangular form.

Find and express the result in rectangular form

16. $(1+i)^8$

17. $(1+i)^{10}$

18. $(1-i)^5$

19. $\left(1-\sqrt{5}i\right)^8$

20. $(3cis80^\circ)^3$

21. $\left(\sqrt{3}cis10^\circ\right)^6$

22. $\left(\sqrt{2}-i\right)^6$

23. $(4cis40^\circ)^6$

24. $(2cis30^\circ)^5$

25. $\left(\frac{1}{2}cis72^\circ\right)^5$

26. Find fifth complex roots of $z = 1+i\sqrt{3}$ and express the result in rectangular form.

Find the fourth roots of

27. $z = 16cis60^\circ$

28. $\sqrt{3}-i$

29. $4-4\sqrt{3}i$

30. $-16i$

Find the cube roots of

31. 27

32. $8-8i$

33. -8

34. Find all complex number solutions of $x^3+1=0$.