

Lecture Three – Infinite Sequences and Series

Section 3.1 – Sequences

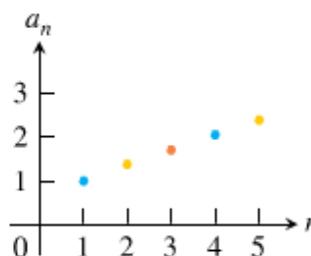
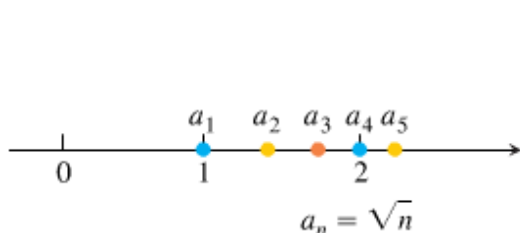
A sequence is a list of numbers

$$a_1, a_2, a_3, \dots, a_n, \dots$$

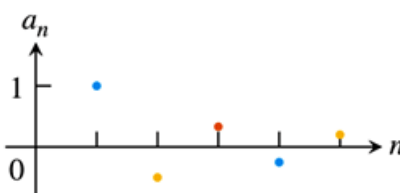
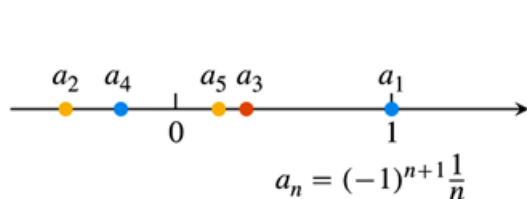
An **infinite sequence** of numbers is a function whose domain is the set of positive integers. These are the **terms** of the sequence. The integer ***n*** is called the **index** of a_n .

Sequences can be described by writing rules that specify their terms such as

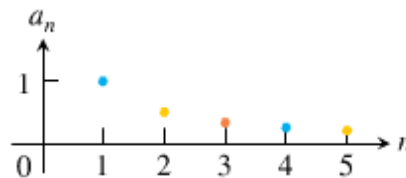
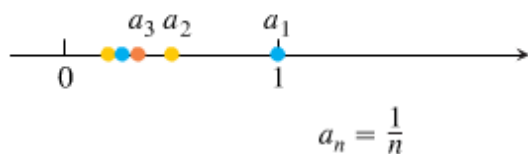
$$a_n = \sqrt{n} \Rightarrow \{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots\}$$



$$a_n = (-1)^{n+1} \frac{1}{n} \Rightarrow \{a_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, (-1)^{n+1} \frac{1}{n}, \dots\right\}$$

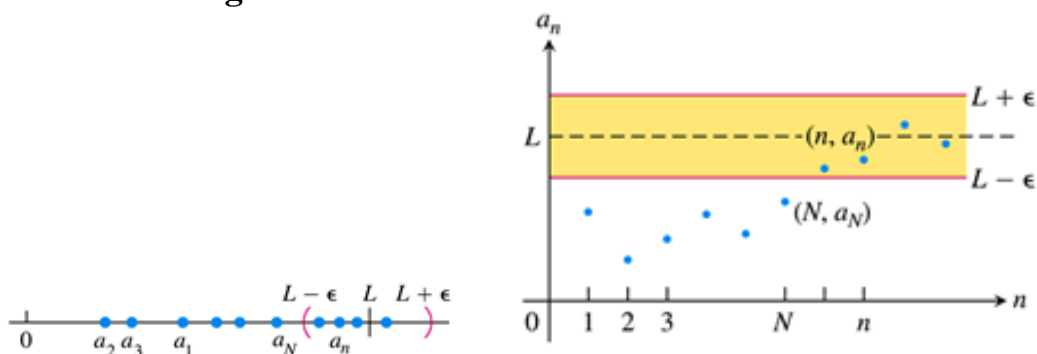


$$a_n = \frac{1}{n} \Rightarrow \{a_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\right\}$$



Also, we can write: $\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}$

Convergence and Divergence



$\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, 1 - \frac{1}{n}, \dots\right\}$ Terms approach 1.

$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\right\}$ Terms approach 0.

Definition

The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there corresponds an integer N such that for all n ,

$$n > N \Rightarrow |a_n - L| < \varepsilon$$

If no such number L exists, we say $\{a_n\}$ **diverges**.

The $\{a_n\}$ **converges** to L , we write $\lim_{n \rightarrow \infty} a_n = L$, or simply $a_n \rightarrow L$, and call L the **limit** of the sequence.

Example

Show that $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

Solution

Let $\varepsilon > 0$ be given. We must show that there exists an integer N such that for all n ,

$$n > N \Rightarrow \left| \frac{1}{n} - 0 \right| < \varepsilon$$

This implication will hold if $\frac{1}{n} < \varepsilon$ or $n > \frac{1}{\varepsilon}$. If N is any integer greater than $\frac{1}{\varepsilon}$, the implication will

hold for all $n > N$. This proves that $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

Example

Show that $\lim_{n \rightarrow \infty} k = k$ (any constant k)

Solution

Let $\varepsilon > 0$ be given. We must show that there exists an integer N such that for all n ,

$$n > N \Rightarrow |k - k| < \varepsilon$$

Since $k - k = 0$, we can use any positive integer for N and the implication will hold for all $n > N$. This proves that $\lim_{n \rightarrow \infty} k = k$

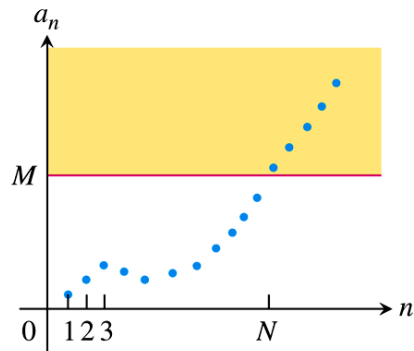
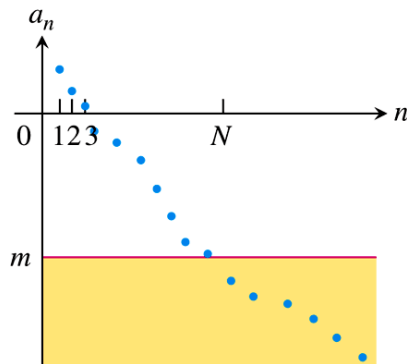
Definition

The sequence $\{a_n\}$ **diverges** to infinity if for every number M there is an integer N such that for all n larger than N , $a_n > M$. If this condition holds we write

$$\lim_{n \rightarrow \infty} a_n = \infty \quad \text{or} \quad a_n \rightarrow \infty$$

Similarly, if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n \rightarrow \infty} a_n = -\infty \quad \text{or} \quad a_n \rightarrow -\infty$$



Theorem

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B real numbers. The following rules hold

if $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$

Sum Rule: $\lim_{n \rightarrow \infty} (a_n + b_n) = A + B$

Difference Rule: $\lim_{n \rightarrow \infty} (a_n - b_n) = A - B$

Constant Multiple Rule: $\lim_{n \rightarrow \infty} (ka_n) = kA$

Product Rule: $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = A \cdot B$

Quotient Rule: $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

Example

a) $\lim_{n \rightarrow \infty} \left(-\frac{1}{n}\right) = -1 \cdot \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) = -1(0) = 0$

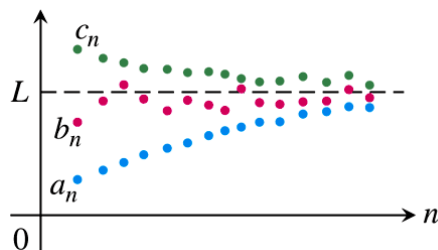
b) $\lim_{n \rightarrow \infty} \left(\frac{n-1}{n}\right) = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = \lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{1}{n} = 1 - 0 = 1$

c) $\lim_{n \rightarrow \infty} \left(\frac{5}{n^2}\right) = 5 \cdot \lim_{n \rightarrow \infty} \left(\frac{1}{n^2}\right) = 5 \cdot \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) \cdot \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) = -1 \cdot 0 \cdot 0 = 0$

d) $\lim_{n \rightarrow \infty} \left(\frac{4-7n^6}{n^6+3}\right) = \lim_{n \rightarrow \infty} \left(\frac{\frac{4}{n^6}-7}{1+\frac{3}{n^6}}\right) = \frac{0-7}{1+0} = -7$

Theorem – The Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N , and if $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$ also.



Example

Since $\frac{1}{n} \rightarrow 0$, we know that

- a) $\frac{\cos n}{n} \rightarrow 0$ because $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$
- b) $\frac{1}{2^n} \rightarrow 0$ because $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$
- c) $(-1)^n \frac{1}{n} \rightarrow 0$ because $-\frac{1}{n} \leq (-1)^n \frac{1}{n} \leq \frac{1}{n}$

Theorem – The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \rightarrow L$ and if f is a function that is continuous at L and defined at all a_n , then $f(a_n) \rightarrow f(L)$.

Example

Show that $\sqrt{\frac{n+1}{n}} \rightarrow 1$

Solution

We know that $\frac{n+1}{n} \rightarrow 1$. Taking $f(x) = \sqrt{x}$ and $L = 1$ that gives $\sqrt{\frac{n+1}{n}} \rightarrow \sqrt{1} = 1$

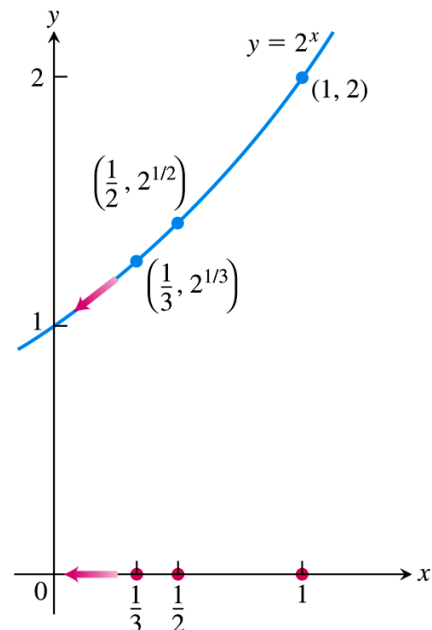
Example

The sequence $\left\{\frac{1}{n}\right\}$ converges to 0.

By taking $a_n = \frac{1}{n}$, $f(x) = 2^x$, and $L = 0$.

We see that $2^{1/n} = f\left(\frac{1}{n}\right) \rightarrow f(L) = 2^0 = 1$.

The sequence $\left\{2^{1/n}\right\}$ converges to 1.



Using L'Hôpital's Rule

Theorem

Suppose that $f(x)$ is a function for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x \rightarrow \infty} f(x) = L \Rightarrow \lim_{n \rightarrow \infty} a_n = L$$

Example

Show that $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$

Solution

The function $\frac{\ln x}{x}$ is defined for all $x \geq 1$ and agrees with the given sequence at positive integers.

$$\text{Therefore, } \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = 0$$

Example

Does the sequence whose n th term is $a_n = \left(\frac{n+1}{n-1}\right)^n$ converge? If so, find $\lim_{n \rightarrow \infty} a_n$

Solution

The limit leads to the indeterminate form 1^∞ .

$$\begin{aligned} \ln a_n &= \ln \left(\frac{n+1}{n-1} \right)^n \\ &= n \ln \left(\frac{n+1}{n-1} \right) && \infty \cdot 0 \text{ form} \\ &= \frac{\ln \left(\frac{n+1}{n-1} \right)}{\frac{1}{n}} && 0/0 \text{ form} \end{aligned}$$

$$\lim_{n \rightarrow \infty} \ln a_n = \lim_{n \rightarrow \infty} \frac{\ln \left(\frac{n+1}{n-1} \right)}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{\frac{-2}{n^2-1}}{-\frac{1}{n^2}}$$

$$= \lim_{n \rightarrow \infty} \frac{2n^2}{n^2-1}$$

$$= 2$$

$$\boxed{\lim_{n \rightarrow \infty} a_n = e^2}$$

$$\left(\ln \frac{n+1}{n-1} \right)' = \frac{\frac{n-1-(n+1)}{(n-1)^2}}{\frac{n+1}{n-1}} = \frac{-2}{(n+1)(n-1)}$$

Theorem

The following six sequences converge to the limits listed below:

1. $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$
2. $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$
3. $\lim_{n \rightarrow \infty} x^{1/n} = 1 \quad x > 0$
4. $\lim_{n \rightarrow \infty} x^n = 1 \quad |x| < 1$
5. $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$
6. $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

Bounded Monotonic Sequences

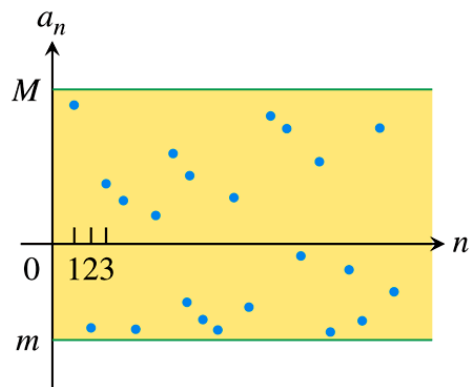
Definitions

A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \leq M$ for all n . The number M is an **upper bound** for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

A sequence $\{a_n\}$ is **bounded from below** if there exists a number m such that $a_n \geq m$ for all n . The number m is a **lower bound** for $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, the $\{a_n\}$ is **bounded**.

If $\{a_n\}$ is not bounded, then $\{a_n\}$ is an **unbounded** sequence.



Definition

A sequence $\{a_n\}$ is **nondecreasing** if $a_n \leq a_{n+1}$ for all n . That is $a_1 \leq a_2 \leq a_3 \leq \dots$

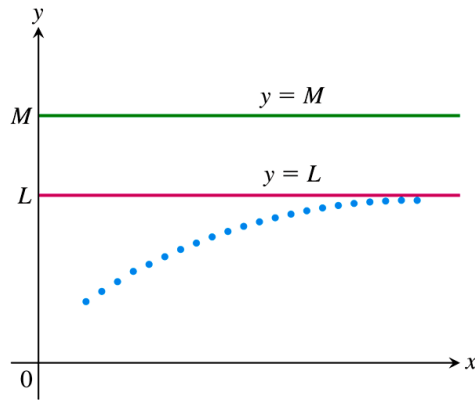
Which each term is greater than or equal to its predecessor $(a_{n+1} \geq a_n)$

Example: $\left\{1 - \frac{1}{n}\right\}_{n=1}^{\infty} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$

A sequence $\{a_n\}$ is **nonincreasing** if $a_n \geq a_{n+1}$ for all n , which each term is less than or equal to its predecessor $(a_{n+1} \leq a_n)$

Example: $\left\{1 + \frac{1}{n}\right\}_{n=1}^{\infty} = \left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots\right\}$

The sequence $\{a_n\}$ is **monotonic** if it is either nondecreasing or nonincreasing.



Theorem

If a sequence $\{a_n\}$ is both *bounded* and *monotonic*, then the sequence converges.

Example

The sequence $\{1, 2, 3, \dots, n, \dots\}$ is nondecreasing

The sequence $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots\right\}$ is nondecreasing

The sequence $\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^n}, \dots\right\}$ is nonincreasing

Exercises Section 3.1 – Sequences

1. Find the values of a_1 , a_2 , a_3 , and a_4 for $a_n = \frac{1-n}{n^2}$
2. Find the values of a_1 , a_2 , a_3 , and a_4 for $a_n = \frac{1}{n!}$
3. Find the values of a_1 , a_2 , a_3 , and a_4 for $a_n = \frac{(-1)^{n+1}}{2n-1}$
4. Find the values of a_1 , a_2 , a_3 , and a_4 for $a_n = 2 + (-1)^n$
5. Find the values of a_1 , a_2 , a_3 , and a_4 for $a_n = \frac{2^n - 1}{2^n}$
6. Write the first ten terms of the sequence $a_1 = 1$, $a_{n+1} = a_n + \frac{1}{2^n}$
7. Write the first ten terms of the sequence $a_1 = 1$, $a_{n+1} = \frac{a_n}{n+1}$
8. Write the first ten terms of the sequence $a_1 = 2$, $a_2 = -1$, $a_{n+2} = \frac{a_{n+1}}{a_n}$
9. Find a formula for the n th term of the sequence $-1, 1, -1, 1, -1, \dots$
10. Find a formula for the n th term of the sequence $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \dots$
11. Find a formula for the n th term of the sequence $\frac{1}{9}, \frac{2}{12}, \frac{2^2}{15}, \frac{2^3}{18}, \frac{2^4}{21}, \dots$
12. Find a formula for the n th term of the sequence $-3, -2, -1, 0, 1, \dots$
13. Find a formula for the n th term of the sequence $\frac{1}{25}, \frac{8}{125}, \frac{27}{625}, \frac{64}{3125}, \frac{125}{15,625}, \dots$
14. Find a formula for the n th term of the sequence $0, 1, 1, 2, 2, 3, 3, 4, \dots$

Determine if the sequence converge or diverge? Then find the limit of each convergent sequence.

- | | | |
|---|---|--|
| 15. $a_n = \frac{n + (-1)^n}{n}$ | 21. $a_n = \sqrt{\frac{2n}{n+1}}$ | 27. $\{c_n\} = \left\{(-1)^n \frac{1}{n!}\right\}$ |
| 16. $a_n = \frac{1-2n}{1+2n}$ | 22. $a_n = \frac{\sin^2 n}{2^n}$ | 28. $a_n = \frac{5}{n+2}$ |
| 17. $a_n = \frac{1-n^3}{70-4n^2}$ | 23. $a_n = \frac{\ln n}{\ln 2n}$ | 29. $a_n = 8 + \frac{5}{n}$ |
| 18. $a_n = \left(2 - \frac{1}{2^n}\right) \left(3 + \frac{1}{2^n}\right)$ | 24. $a_n = \frac{3^n \cdot 6^n}{2^{-n} \cdot n!}$ | 30. $a_n = (-1)^n \left(\frac{n}{n+1}\right)$ |
| 19. $a_n = n\pi \cos(n\pi)$ | 25. $a_n = \sqrt{n} \sin \frac{1}{\sqrt{n}}$ | 31. $a_n = \frac{1 + (-1)^n}{n^2}$ |
| 20. $a_n = n - \sqrt{n^2 - n}$ | 26. $a_n = \frac{n^2}{2^n - 1}$ | 32. $a_n = \frac{10n^2 + 3n + 7}{2n^2 - 6}$ |

$$33. \quad a_n = \frac{\sqrt[3]{n}}{\sqrt[3]{n} + 1}$$

$$34. \quad a_n = \frac{\ln(n^3)}{2n}$$

$$35. \quad a_n = \frac{5^n}{3^n}$$

$$36. \quad a_n = \frac{(n+1)!}{n!}$$

$$37. \quad a_n = \frac{(n-2)!}{n!}$$

$$38. \quad a_n = \frac{n^p}{e^n}, \quad p > 0$$

$$39. \quad a_n = n \sin \frac{1}{n}$$

$$40. \quad a_n = 2^{1/n}$$

$$41. \quad a_n = -3^{-n}$$

$$42. \quad a_n = \frac{\sin n}{n}$$

$$43. \quad a_n = \frac{\cos \pi n}{n^2}$$