

$$1 \quad \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = \underline{5}$$

$$2 \quad \begin{vmatrix} 5 & 2 \\ -6 & 3 \end{vmatrix} = 15 + 12 = \underline{27}$$

$$3 \quad \begin{vmatrix} -7 & 6 \\ \frac{1}{2} & 3 \end{vmatrix} = -21 - 3 = \underline{-24}$$

$$4 \quad \begin{vmatrix} 0 & 8 \\ 0 & 4 \end{vmatrix} = 0$$

$$5 \quad \begin{vmatrix} 2-3 & 2 \\ 4 & 2-1 \end{vmatrix} = (2-3)(2-1) - 8 \\ = 2^2 - 4(2) + 2 - 8 \\ = \underline{2^2 - 4(2) - 5}$$

$$10 \quad \begin{vmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{vmatrix} = 1 \begin{vmatrix} 2 & 0 \\ 4 & 3 \end{vmatrix} - 4 \begin{vmatrix} 3 & 0 \\ -1 & 3 \end{vmatrix} - 2 \begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix} \\ = 6 - 4(9) - 2(12 - 2) \\ = 6 - 36 - 20 \\ = \underline{-50}$$

$$13 \quad \begin{vmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{vmatrix} = 2(3)(-5) \\ = \underline{-30}$$

$$14 \quad \begin{vmatrix} x & y & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} \begin{vmatrix} x & y \\ 3 & 2 \\ 1 & 1 \end{vmatrix} = 2x + 0 - 3 + 2 - 3y \\ = \underline{2x - 3y - 1}$$

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$$\begin{vmatrix} e^{2x} & e^{2x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x} \\ = \underline{e^{5x}}$$

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$$\begin{vmatrix} x & \ln x \\ 1 & \frac{1}{x} \end{vmatrix} = 1 - \ln x$$

15

$$\begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \cos^2 \theta - r \sin^2 \theta \\ = r(\cos^2 \theta - \sin^2 \theta) \\ = \underline{r}$$

$$\begin{array}{c} \# \\ \left| \begin{array}{ccc|c} 1 & 7 & -3 & \\ 1 & 3 & 1 & R_2 - R_1 \\ 4 & 8 & 1 & R_3 - 4R_1 \end{array} \right| \end{array}$$

$$\begin{array}{c} \left| \begin{array}{ccc|c} 1 & 7 & -3 & \\ 0 & -4 & 4 & \\ 0 & -20 & 13 & \end{array} \right| = 1 \left| \begin{array}{cc|c} -4 & 4 & \\ -20 & 13 & \end{array} \right| \quad 4 \left| \begin{array}{cc|c} -1 & 1 & \\ -20 & 13 & \end{array} \right| \\ = 4(-13 + 20) \\ = \underline{28} \end{array}$$

$$\# 8/ \left| \begin{array}{cc|c} \cos \theta & \sin \theta & \\ -\sin \theta & \cos \theta & \end{array} \right| = \cos^2 \theta + \sin^2 \theta = \underline{1}$$

$$9/ \left| \begin{array}{cc|c} \sin \theta & 1 & \\ 1 & \sin \theta & \end{array} \right| = \sin^2 \theta - 1 = -\cos^2 \theta$$

$$\# 11/ \left| \begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & k & 0 & \\ 0 & 0 & 1 & \end{array} \right| = k \quad \text{diagonal}$$

$$\# 12/ \left| \begin{array}{ccc|c} 1 & 0 & 0 & \\ k & 1 & 0 & \\ 0 & 0 & 1 & \end{array} \right| = 1 \quad \text{lower triangular}$$

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$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

(a, b, c ≠ 0)

$$= (1+a)(1+b)(1+c) + 1 + 1 - (1+b) - (1+c) - (1+a)$$

$$= (1+a+b+ab)(1+c) - a - b - c - 1$$

$$= 1+c+a+ac+b+bc+ab+abc - a - b - c - 1$$

$$= ac + ab + bc + abc$$

$$= abc \left(\frac{1}{b} + \frac{1}{c} + \frac{1}{a} + 1 \right) \checkmark$$

$$\begin{aligned} \#17 \quad & \begin{vmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{vmatrix} = 5 \begin{vmatrix} 6 & 4 & 12 \\ 2 & -3 & 4 \\ 1 & -2 & 2 \end{vmatrix} - 4 \begin{vmatrix} 3 & 0 & 6 \\ 2 & -3 & 4 \\ 1 & -2 & 2 \end{vmatrix} \\ &= 5(-36 + 16 - 48 + 36 + 48 - 16) \\ &\quad - 4(-18 - 24 + 18 + 20) \\ &= 0. \end{aligned}$$

$$\begin{aligned} \#18 \quad & \begin{vmatrix} x+3 & 2 \\ 1 & x+2 \end{vmatrix} = 0 \\ & (x+3)(x+2) - 2 = 0 \\ & x^2 + 5x + 4 = 0 \\ & \Rightarrow x = -1, -4 \end{aligned}$$

$$\begin{aligned} \#18 \quad & \begin{vmatrix} \lambda+2 & 2 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 2\lambda - 2 = 0 \\ & \lambda_{1,2} = -1 \pm \sqrt{3} \end{aligned}$$

$$\begin{aligned} \#19 \quad & \begin{vmatrix} \lambda & 2 & 0 \\ 0 & \lambda+1 & 2 \\ 0 & 1 & \lambda \end{vmatrix} = \lambda(\lambda(\lambda+1) - 2) \\ &= \lambda(\lambda^2 + \lambda - 2) = 0 \\ & \lambda_{1,2,3} = 0, 1, -2 \end{aligned}$$

$$20/ A = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$a) |A| = \begin{vmatrix} -2 & 1 \\ 4 & -2 \end{vmatrix} = 0$$

$$b) |B| = \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1$$

$$c) AB = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \\ = \begin{pmatrix} -2 & -3 \\ 4 & 6 \end{pmatrix}$$

$$d) |AB| = \begin{vmatrix} -2 & -3 \\ 4 & 6 \end{vmatrix} \quad -12 + 12 \\ = 0$$

$$\Rightarrow |A| \cdot |B| = |AB| = 0 \checkmark$$

$$21/ A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$a) A = \begin{vmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 2$$

$$b) B = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = -6$$

$$c) AB = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \\ = \begin{pmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}$$

$$d) |AB| = \begin{vmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 0 & 2 & 0 \end{vmatrix} = -12$$

$$|A| \cdot |B| = (2)(-6) = -12 = |AB| \checkmark$$

22/ $A = \begin{pmatrix} 5 & 15 \\ 10 & -20 \end{pmatrix}$ $2 \times 2 \Rightarrow n = 2$

$$cA = \begin{pmatrix} 5c & 15c \\ 10c & -20c \end{pmatrix}$$

$$|cA| = \begin{vmatrix} 5c & 15c \\ 10c & -20c \end{vmatrix}$$

$$= -100c^2 - 150c^2$$

$$= -250c^2$$

$$|A| = \begin{vmatrix} 5 & 15 \\ 10 & -20 \end{vmatrix} = -250$$

$$|A| = 5 \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix}$$

$$= 25(-10)$$

$$= -250$$

$$c = 5$$

$$\Rightarrow |cA| = (-250)c^2$$

$$= c^2 |A| \quad \checkmark$$

23/ $A = \begin{pmatrix} -3 & 6 & 9 \\ 6 & 9 & 12 \\ 9 & 12 & 15 \end{pmatrix}$ $3 \times 3 \Rightarrow n = 3$

$$= 3 \begin{pmatrix} -1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

$$|A| = c^3 \begin{vmatrix} -1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= 3^3 (-15 + 24 + 24 - 27 + 16 - 20)$$

$$= 27(2)$$

$$= 54$$

$$24/ \quad A = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$$

$$a) |A| = \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} = -2$$

$$b) |B| = \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -2$$

$$c) A+B = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix} \\ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$d) |A+B| = 0$$

$$\Rightarrow |A| + |B| = |A+B| \\ -2 - 2 \neq 0 \\ -4 \neq 0 \quad \checkmark$$

$$25/ \quad \begin{vmatrix} 5 & 4 \\ 10 & 8 \end{vmatrix} = 40 - 40 = 0 \quad \text{Singular}$$

$$26/ \quad \begin{vmatrix} \frac{1}{2} & \frac{3}{2} & 2 \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ 1 & 1 & 1 \end{vmatrix} = -\frac{1}{6} + \frac{4}{3} + \frac{2}{3} = 1 \\ = \frac{5}{6} \quad \text{Non singular}$$

$$27/ \quad A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \Rightarrow |A| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 5$$

$$A^{-1} = \frac{1}{5} \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix} \Rightarrow \underline{2 \times 2} \subseteq \mathbb{R}^n$$

$$|A^{-1}| = \left(\frac{1}{5}\right)^2 \begin{vmatrix} 4 & -3 \\ -1 & 2 \end{vmatrix} \\ = \frac{5}{25} \\ = \frac{1}{5} \\ = \frac{1}{|A|} \quad \checkmark$$

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$$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & -1 & 2 \\ 3 & 0 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & 2 \\ 3 & 0 & 3 \end{vmatrix} = -3$$

$$A^{-1} = \frac{1}{-3} \begin{pmatrix} -3 & 6 & -1 \\ 3 & -3 & -1 \\ 3 & -6 & 0 \end{pmatrix}$$

$$|A^{-1}| = \left(\frac{1}{-3}\right)^3 \begin{vmatrix} -3 & 6 & -1 \\ 3 & -3 & -1 \\ 3 & -6 & 0 \end{vmatrix}$$

$$= -\frac{1}{27} (-18 + 9 - 9 + 18)$$

$$= -\frac{9}{27}$$

$$= -\frac{1}{3}$$

$$= \frac{1}{|A|}$$

29 A is singular $\Rightarrow |A| = 0$

$$A = \begin{bmatrix} k-1 & 3 \\ 2 & k-2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} k-1 & 3 \\ 2 & k-2 \end{vmatrix}$$

$$= k^2 - 3k - 4 = 0$$

$$\therefore \underline{k = -1, 4}$$

29/ $A, B (n \times n) \ni AB = I$
 Prove $|A| \neq 0$ & $|B| \neq 0$.

$$|AB| = |I|$$

$$|A||B| = 1$$

$$\therefore |A| \neq 0 \text{ \& } |B| \neq 0$$

30/ AB singular

Prove A or B is singular.

$$AB \text{ is singular} \Rightarrow |AB| = 0$$

$$|AB| = |A||B| = 0$$

\Rightarrow Either $|A|$ or $|B|$ is equal to zero

\therefore Either A or B is singular.

31/ $|A| + |B| = |A+B|$ Find $A+B$.

$$\text{Assume: } A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$|B| = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$A+B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$|A+B| = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$\Rightarrow |A| + |B| = |A+B| \quad \checkmark$$

$$0 + 0 = 0$$

32/ $\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = b^2(3a+b)$

$$\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = (a+b)^3 + a^3 + a^3 - 3a^2(a+b) \\ = \underline{a^3} + \underline{3a^2b} + \underline{3ab^2} + \underline{b^3} + \underline{2a^3} \\ \quad - \underline{3a^3} - \underline{3a^2b} \\ = 3ab^2 + b^3 \\ = b^2(3a+b) \checkmark$$

33/ $A(n \times n)$
 sum of the entries of each row = 0
 \Rightarrow One column will result = 0.

$$|A| = 0 \quad A = 0$$

$$A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix} \Rightarrow |A| = -ab + ab = 0.$$

34/

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & -4 & -12 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & -4 & -12 \end{vmatrix} = 0$$

$$\Rightarrow A^{-1} \nexists$$

35/

$$A = \begin{bmatrix} -3 & -5 & -7 \\ 2 & 4 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -3 & -5 & -7 \\ 2 & 4 & 3 \\ 0 & 1 & -1 \end{vmatrix} = -3$$

$$A^{-1} = -\frac{1}{3} \begin{bmatrix} -7 & -12 & 13 \\ 2 & 3 & -5 \\ 2 & 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{3} & 4 & -\frac{13}{3} \\ -\frac{2}{3} & -1 & \frac{5}{3} \\ -\frac{2}{3} & -1 & \frac{2}{3} \end{bmatrix}$$

$$38/ \begin{cases} 20x + 8y = 11 \\ 12x - 24y = 21 \end{cases}$$

$$\Delta = \begin{vmatrix} 20 & 8 \\ 12 & -24 \end{vmatrix} = -573$$

$$\Delta_x = \begin{vmatrix} 11 & 8 \\ 21 & -24 \end{vmatrix} = -432$$

$$\Delta_y = \begin{vmatrix} 20 & 11 \\ 12 & 21 \end{vmatrix} = 288$$

$$x = \frac{-432}{-573} = \frac{3}{4}$$

$$y = \frac{288}{-573} = -\frac{1}{2}$$

$$\therefore \text{soln} : \left(\frac{3}{4}, -\frac{1}{2} \right)$$

$$39/ \begin{cases} 4x - y - z = 1 \\ 2x + 2y + 3z = 10 \\ 5x - 2y - 2z = -1 \end{cases}$$

$$\Delta = \begin{vmatrix} 4 & -1 & -1 \\ 2 & 2 & 3 \\ 5 & -2 & -2 \end{vmatrix} = 3$$

$$\Delta_x = \begin{vmatrix} 1 & -1 & -1 \\ 10 & 2 & 3 \\ -1 & -2 & -2 \end{vmatrix} = 3$$

$$\Delta_y = \begin{vmatrix} 4 & 1 & -1 \\ 2 & 10 & 3 \\ 5 & -1 & -2 \end{vmatrix} = 3$$

$$\Delta_z = \begin{vmatrix} 4 & -1 & 1 \\ 2 & 2 & 10 \\ 5 & -2 & -1 \end{vmatrix} = 6$$

$$x = \frac{3}{3} = 1 \quad y = \frac{3}{3} = 1 \quad z = \frac{6}{3} = 2$$

$$\therefore \text{soln} (1, 1, 2)$$

$$\begin{cases} x_1 + 2x_2 = 5 \\ -x_1 + x_2 = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = \underline{3}$$

$$\Delta_{x_1} = \begin{vmatrix} 5 & 2 \\ 1 & 1 \end{vmatrix} = \underline{3}$$

$$\Delta_{x_2} = \begin{vmatrix} 1 & 5 \\ -1 & 1 \end{vmatrix} = \underline{6}$$

$$x_1 = \frac{3}{3} = \underline{1} \quad x_2 = \frac{6}{3} = \underline{2}$$

\therefore soln: $(1, 2)$

$$\begin{cases} 3x + 4y = -2 \\ 5x + 3y = 4 \end{cases}$$

$$\Delta = \begin{vmatrix} 3 & 4 \\ 5 & 3 \end{vmatrix} = -11$$

$$\Delta_x = \begin{vmatrix} -2 & 4 \\ 4 & 3 \end{vmatrix} = -22$$

$$\Delta_y = \begin{vmatrix} 3 & -2 \\ 5 & 4 \end{vmatrix} = 22$$

$$x = \frac{-22}{-11} = 2$$

$$y = \frac{22}{-11} = -2$$

\therefore soln: $(2, -2)$

$$\text{u.s.} \begin{cases} 3x + 4y + 4z = 11 \\ 4x - 4y + 6z = 11 \\ 6x - 6y = 3 \end{cases}$$

$$\Delta = \begin{vmatrix} 3 & 4 & 4 \\ 4 & -4 & 6 \\ 6 & -6 & 0 \end{vmatrix} = 252$$

$$\Delta_x = \begin{vmatrix} 11 & 4 & 4 \\ 11 & -4 & 6 \\ 3 & -6 & 0 \end{vmatrix} = 252$$

$$\Delta_y = \begin{vmatrix} 3 & 11 & 4 \\ 4 & 11 & 6 \\ 6 & 3 & 0 \end{vmatrix} = 126$$

$$\Delta_z = \begin{vmatrix} 3 & 4 & 11 \\ 4 & -4 & 11 \\ 6 & -6 & 3 \end{vmatrix} = 378$$

$$x = \frac{252}{252} = 1 \quad y = \frac{126}{252} = \frac{1}{2} \quad z = \frac{378}{252} = \frac{3}{2}$$

$$\text{Soln} : \left(1, \frac{1}{2}, \frac{3}{2} \right),$$

$$\begin{vmatrix} \cos x & 0 & \sin x \\ \sin x & 0 & \cos x \\ \sin x - \cos x & 1 & \sin x + \cos x \end{vmatrix} = 0$$

$$-\cos^2 x + \sin^2 x = 0$$

$$\Rightarrow \cos 2x = 0 \Rightarrow 2x = (2n+1)\frac{\pi}{2}$$

$$x = (2n+1)\frac{\pi}{4}, \quad n \in \mathbb{Z}$$

$$\begin{vmatrix} 1-n & 1 & 1 & \dots & 1 \\ 1 & 1-n & 1 & \dots & 1 \\ 1 & 1 & 1-n & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1-n \end{vmatrix}$$

$$2 \times 2: \begin{vmatrix} 1-n & 1 \\ 1 & 1-n \end{vmatrix} = n^2 - 2n = n(n-2) = 0 \quad \text{since } n=2$$

$$3 \times 3: \begin{vmatrix} 1-n & 1 & 1 \\ 1 & 1-n & 1 \\ 1 & 1 & 1-n \end{vmatrix} = (1-n)^2 + 1 + 1 - 3(1-n) \\ = n^2 - 3n^2 \\ = n^2(n-3) = 0 \quad (n=3).$$

$$n \times n \Rightarrow \begin{vmatrix} \vdots & \vdots \end{vmatrix} = n^{n-1}(n-n) \\ = \underline{0}$$