Solution *Section* 3.1 – Mathematical Induction

Exercise

Prove that $1^2+3^2+5^2+\cdots+(2n+1)^2=\frac{1}{3}(n+1)(2n+1)(2n+3)$ whenever n is a nonnegative integer.

Solution

Since n is a nonnegative integer that implies to $n\geq 0$

(1) For $\boldsymbol{n}$ **= 0** $\Rightarrow 1^2=\frac{1}{3}(0+1)(0+1)(0+3)$

$$1=\frac{1}{3}(1)(2)(3)=1 \quad \checkmark$$

Hence P_1 is true.

(1) Assume that $1^2+3^2+\cdots+(2k+1)^2=\frac{1}{3}(k+1)(2k+1)(2k+3)$ is true

$$1^2+3^2+\cdots+(2k+1)^2+(2(k+1)+1)^2=\frac{1}{3}((k+1)+1)(2(k+1)+1)(2(k+1)+3)$$

$$1^2+3^2+\cdots+(2k+1)^2+(2k+3)^2=\frac{1}{3}(k+2)(2k+3)(2k+5)$$

$$\begin{aligned}1^2+3^2+\cdots+(2k+1)^2+(2k+3)^2&=\frac{1}{3}(k+1)(2k+1)(2k+3)+(2k+3)^2\\&=\frac{1}{3}(2k+3)\left[(k+1)(2k+1)+3(2k+3)\right]\\&=\frac{1}{3}(2k+3)\left(2k^2+k+2k+1+6k+9\right)\\&=\frac{1}{3}(2k+3)\left(2k^2+9k+10\right)\\&=\frac{1}{3}(2k+3)(k+2)(2k+5) \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that $1\cdot 1!+2\cdot 2!+\cdots+n\cdot n!=(n+1)!-1$ whenever n is a positive integer.

Solution

Since n is a positive integer that implies to $n\geq 1$

(2) For $\boldsymbol{n}$ **= 1**

$$1\cdot 1!=(1+1)!-1$$

$$1=1 \quad \checkmark$$

Hence P_1 is true.

(3) Assume that $1\cdot 1!+2\cdot 2!+\cdots+k\cdot k!=(k+1)!-1$ is true

$$1\cdot 1!+2\cdot 2!+\cdots+k\cdot k!+(k+1)\cdot(k+1)!=((k+1)+1)!-1=(k+2)!-1$$

$$\begin{aligned}1\cdot 1!+2\cdot 2!+\cdots+k\cdot k!+(k+1)\cdot(k+1)! &= (k+1)!-1+(k+1)\cdot(k+1)! \\ &= (k+1)\cdot(k+1)!+(k+1)!-1 \\ &= (k+1)!(k+1+1)-1 \\ &= (k+1)!(k+2)-1 \\ &= (k+2)!-1 \ \surd\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that $3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^n=\frac{3}{4}\left(5^{n+1}-1\right)$ whenever n is a nonnegative integer.

Solution

(1) For ***n* = 0** $\Rightarrow$ $3=\frac{3}{4}(5-1)$

$$3=3 \ \surd$$

Hence P_1 is true.

(4) Assume that $3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^k=\frac{3}{4}\left(5^{k+1}-1\right)$ is true

$$3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^k+3\cdot 5^{k+1}=\frac{3}{4}\left(5^{k+2}-1\right)$$

$$\begin{aligned}3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^k+3\cdot 5^{k+1} &= \frac{3}{4}\left(5^{k+1}-1\right)+3\cdot 5^{k+1} \\ &= \frac{3}{4}\left[5^{k+1}-1+4\cdot 5^{k+1}\right] \\ &= \frac{3}{4}\left(5\cdot 5^{k+1}-1\right) \\ &= \frac{3}{4}\left(5^{k+2}-1\right) \ \surd\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that $2-2\cdot 7+2\cdot 7^2-\cdots+2\cdot(-7)^n=\dfrac{1-(-7)^{n+1}}{4}$ whenever n is a nonnegative integer.

Solution

(1) For $\boldsymbol{n = 0} \Rightarrow\ 2=\dfrac{1-(-7)^1}{4}$

$$2=\frac{8}{4}=2 \quad \checkmark$$

Hence P_1 is true.

(2) Assume that $2-2\cdot 7+2\cdot 7^2-\cdots+2\cdot(-7)^k=\dfrac{1-(-7)^{k+1}}{4}$ is true

We need to prove that P_{k+1} is also true

$$2-2\cdot 7+2\cdot 7^2-\cdots+2\cdot(-7)^k+2\cdot(-7)^{k+1}=\frac{1-(-7)^{(k+1)+1}}{4}$$

$$2-2\cdot 7+2\cdot 7^2-\cdots+2\cdot(-7)^k+2\cdot(-7)^{k+1}=\frac{1-(-7)^{k+2}}{4}$$

$$\begin{aligned}2-2\cdot 7+2\cdot 7^2-\cdots+2\cdot(-7)^k+2\cdot(-7)^{k+1}&=\frac{1-(-7)^{k+1}}{4}+2\cdot(-7)^{k+1}\\&=\frac{1-(-7)^{k+1}+8\cdot(-7)^{k+1}}{4}\\&=\frac{1-(-7)^{k+1}(1-8)}{4}\\&=\frac{1-(-7)^{k+1}(-7)}{4}\\&=\frac{1-(-7)^{k+2}}{4} \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Find a formula for the sum of the first *n* even positive integers. Prove the formula.

Solution

$$\begin{array}{c} 1+2+\cdots+(n-1)+n \\ n+(n-1)+\cdots+2+1 \\ \hline (n+1)+(n+1)+\cdots+(n+1) \end{array}$$

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

(1) For ***n* = 1**

$$1=\frac{1(2)}{2}$$

$$1=1 \quad \checkmark$$

Hence P_1 is true.

(2) Assume that $1+2+\cdots+k=\frac{k(k+1)}{2}$ is true

We need to prove that P_{k+1} is also true $1+2+\cdots+k+(k+1)=\frac{(k+1)((k+1)+1)}{2}=\frac{(k+1)(k+2)}{2}$

$$\begin{aligned} 1+2+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

a) Find a formula for $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}$ by examining the values of this expression for values of this expression for small values of *n*.

b) Prove the formula.

Solution

a) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

b) For $n=1 \quad \Rightarrow \quad \frac{1}{1\cdot 2}=\frac{1}{1+1}$

$$\frac{1}{2} = \frac{1}{2} \quad \surd$$

Hence P_1 is true.

Assume that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}$ is true

We need to prove that P_{k+1} is also true

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{(k+1)+1} = \frac{k+1}{k+2}$$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2} \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that $1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2}$ whenever n is a positive integer.

Solution

(1) For ***n* = 1**

$$1^2 = (-1)^0 \frac{1(2)}{2}$$

$$1 = 1 \quad \surd$$

Hence P_1 is true.

(2) Assume that $1^2 - 2^2 + 3^2 - \cdots + (-1)^{k-1} k^2 = (-1)^{k-1} \frac{k(k+1)}{2}$ is true

We need to prove that $1^2 - 2^2 + 3^2 - \cdots + (-1)^{k-1} k^2 + (-1)^k (k+1)^2 = (-1)^k \frac{(k+1)(k+2)}{2}$ is also true

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{k-1} k^2 + (-1)^k (k+1)^2 = (-1)^{k-1} \frac{k(k+1)}{2} + (-1)^k (k+1)^2$$

$$=(-1)^k(k+1)\left[(-1)^{-1}\frac{1}{2}k+(k+1)\right]$$
$$=(-1)^k(k+1)\left(-\frac{k}{2}+k+1\right)$$
$$=(-1)^k(k+1)\left(\frac{k}{2}+1\right)$$
$$=(-1)^k(k+1)\left(\frac{k+2}{2}\right) \quad \surd$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that for very positive integer n $\qquad \sum_{k=1}^{n} k2^k=(n-1)2^{n+1}+2$

Solution

For $\boldsymbol{n=1} \Rightarrow 1\cdot 2^1=(1-1)^0\, 2^2+2$

$$2=2 \quad \surd$$

Hence P_1 is true

Assume that $\sum_{k=1}^{n} k\cdot 2^k=(n-1)2^{n+1}+2$ is true

We need to prove that $\sum_{k=1}^{n+1} k\cdot 2^k=n\cdot 2^{n+2}+2$ is also true

$$\sum_{k=1}^{n+1} k\cdot 2^k=\sum_{k=1}^{n} k\cdot 2^k+(n+1)\cdot 2^{n+1}$$
$$=(n-1)\cdot 2^{n+1}+2+(n+1)\cdot 2^{n+1}$$
$$=(n-1+n+1)\cdot 2^{n+1}+2$$
$$=2n\cdot 2^{n+1}+2$$
$$=n\cdot 2^{n+2}+2 \quad \surd$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that for very positive integer n $\quad 1\cdot 2+2\cdot 3+\cdots+n(n+1)=\frac{1}{3}n(n+1)(n+2)$.

Solution

For $\boldsymbol{n=1}$

$$1\cdot 2=\frac{1}{3}1(1+1)(1+2)$$

$$2=\frac{1}{3}(2)(3)=2 \quad \checkmark$$

Hence P_1 is true

Assume that $1\cdot 2+2\cdot 3+\cdots+k(k+1)=\frac{1}{3}k(k+1)(k+2)$ is true

We need to prove that $1\cdot 2+2\cdot 3+\cdots+k(k+1)+(k+1)(k+2)=\frac{1}{3}(k+1)(k+2)(k+3)$ is also true

$$\begin{aligned}1\cdot 2+2\cdot 3+\cdots+k(k+1)+(k+1)(k+2)&=\frac{1}{3}k(k+1)(k+2)+(k+1)(k+2)\\&=(k+1)(k+2)\left(\frac{1}{3}k+1\right)\\&=(k+1)(k+2)\left(\frac{k+3}{3}\right)\\&=\frac{1}{3}(k+1)(k+2)(k+3) \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that for very positive integer n $\quad 1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+n(n+1)(n+2)=\frac{1}{4}n(n+1)(n+2)(n+3)$

Solution

For $\boldsymbol{n=1}$

$$1\cdot 2\cdot 3=\frac{1}{4}1(1+1)(1+2)(1+3)$$

$$2=\frac{1}{3}(2)(3)=2 \quad \checkmark$$

Hence P_1 is true

Assume that $1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+k(k+1)(k+2)=\frac{1}{4}k(k+1)(k+2)(k+3)$ is true

$$1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+k(k+1)(k+2)+(k+1)(k+2)(k+3)=\frac{1}{4}(k+1)(k+2)(k+3)(k+4)$$

$$\begin{aligned}1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+k(k+1)(k+2)+(k+1)(k+2)(k+3)&\\=\frac{1}{4}k(k+1)(k+2)(k+3)+(k+1)(k+2)(k+3)&\\=\frac{1}{4}(k+1)(k+2)(k+3)[k+4] \quad \checkmark&\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Let $P(n)$ be the statement that $1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}<2-\frac{1}{n}$ where *n* is an integer greater than 1.

a) Show is the statement $P(2)$?

b) Show that $P(2)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step.

f) Explain why these steps show that this inequality is true whenever *n* is an integer greater than 1.

Solution

a) $P(2)$: $1+\frac{1}{4}<2-\frac{1}{2}$

b) $1+\frac{1}{4}<2-\frac{1}{2}$

$\frac{5}{4}<\frac{3}{2}$

$10<12$ √

Exercise

Prove that $3^n<n!$ if *n* is an integer greater than 6.

Solution

For $\boldsymbol{n=7} \Rightarrow 3^7<7! \quad \Rightarrow \quad 2187<5040$; Hence P_7 is true

Assume that $3^k<k!$ is true, we need to prove that $3^{k+1}<(k+1)!$

$$\begin{aligned} 3^{k+1} &= 3^k 3 \\ &< k!\cdot 3 \qquad \text{Since } k>6 \Rightarrow 6<k \rightarrow 3<k+1 \\ &< k!\cdot(k+1) \\ &= (k+1)! \ \surd \end{aligned}$$

$\therefore$ The statement $3^n<n!$ is true

Exercise

Prove that for every positive integer n: $\quad 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}>2\left(\sqrt{n+1}-1\right)$

Solution

For $\boldsymbol{n}$ **= 1**

$$1>2\left(\sqrt{1+1}-1\right)$$

$$1>2\left(\sqrt{2}-1\right)\approx 0.828$$

Hence P_1 is true

Assume that $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}>2\left(\sqrt{k+1}-1\right)$ is true.

We need to prove that $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}>2\left(\sqrt{(k+1)+1}-1\right)=2\left(\sqrt{k+2}-1\right)$

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}>2\left(\sqrt{k+1}-1\right)+\frac{1}{\sqrt{k+1}}$$

$$2\left(\sqrt{k+1}-1\right)+\frac{1}{\sqrt{k+1}}>2\left(\sqrt{k+2}-1\right)$$

$$2\sqrt{k+1}-2+\frac{1}{\sqrt{k+1}}>2\sqrt{k+2}-2$$

$$2\sqrt{k+1}+\frac{1}{\sqrt{k+1}}>2\sqrt{k+2}$$

$$\frac{1}{\sqrt{k+1}}>2\sqrt{k+2}-2\sqrt{k+1}$$

$$\frac{1}{\sqrt{k+1}}>2\left(\sqrt{k+2}-\sqrt{k+1}\right)$$

$$\left(\sqrt{k+2}+\sqrt{k+1}\right)\frac{1}{\sqrt{k+1}}>2\left(\sqrt{k+2}-\sqrt{k+1}\right)\left(\sqrt{k+2}+\sqrt{k+1}\right)$$

$$\frac{\sqrt{k+2}}{\sqrt{k+1}}+1>2\left(k+2-k-1\right)$$

$$\frac{\sqrt{k+2}}{\sqrt{k+1}}+1>2$$

Which is clearly true since $\frac{\sqrt{k+2}}{\sqrt{k+1}}>1$

Exercise

Use mathematical induction to prove that 2 divides n^2+n whenever n is a positive integer.

Solution

For $\boldsymbol{n}$ **= 1**

$$1^2+1=2$$

since 2 divides 2;

Hence P_1 is true

Assume that 2 divides k^2+k is true, we need to prove that 2 divides $(k+1)^2+(k+1)$ is true

$$(k+1)^2+(k+1)=k^2+2k+1+k+1$$
$$=k^2+k+2k+2$$
$$=k^2+k+2(k+1)\quad \checkmark$$

2 divides k^2+k and certainly 2 divides $2(k+1)$, so 2 divides their sum.

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Use mathematical induction to prove that 3 divides n^3+2n whenever n is a positive integer.

Solution

For $\boldsymbol{n}$ **= 1**

$1^3+2(1)=3$

since 3 divides 3 $\checkmark$

Hence P_1 is true

Assume that 3 divides k^3+2k is true.

We need to prove that 3 divides $(k+1)^3+2(k+1)$ is also true

$$(k+1)^3+2(k+1)=k^3+3k^2+3k+1+2k+2$$
$$=k^3+2k+3k^2+3k+3$$
$$=k^3+2k+3\left(k^2+k+1\right)\quad \checkmark$$

By the inductive hypothesis, 3 divides k^3+2k and certainly 3 divides $3\left(k^2+k+1\right)$, so 3 divides their sum.

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Use mathematical induction to prove that 5 divides $n^5 - n$ whenever n is a positive integer.

Solution

For $\boldsymbol{n = 1}$

$1^5 - 1 = 0$, which is divisible by 5

Hence P_1 is true

Assume that 5 divides $k^5 - k$ is true.

We need to prove that 5 divides $(k+1)^5 - (k+1)$ is also true

$$\begin{aligned}(k+1)^5 - (k+1) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 \\ &= k^5 - k + 5k^4 + 10k^3 + 10k^2 + 5k \\ &= k^5 - k + 5\left(k^4 + 2k^3 + 2k^2 + k\right) \qquad \surd\end{aligned}$$

By the inductive hypothesis, 5 divides $k^5 - k$ and certainly 5 divides $5\left(k^4 + 2k^3 + 2k^2 + k\right)$, so 5 divides their sum.

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Use mathematical induction to prove that $n^2 - 1$ is divisible by 8 whenever n is an odd positive integer.

Solution

For $\boldsymbol{n = 1}$

$1^2 - 1 = 0$, which is divisible by 8

Hence P_1 is true

Assume that 8 divides $k^2 - 1$ is true, than implies to $k^2 - 1 = 8p$.

We need to prove that 8 divides $(k+1)^2 - 1$ is also true

$$\begin{aligned}(k+1)^2 - 1 &= k^2 + 2k + 1 - 1 \\ &= \left(k^2 - 1\right) + 2k + 1\end{aligned}$$

By the inductive hypothesis, 8 divides $k^2 - 1$ and certainly 8 divides $2k + 1$, so 8 divides their sum.

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Use mathematical induction to prove that 21 divides $4^{n+1}+5^{2n-1}$ whenever *n* is a positive integer.

Solution

For $\boldsymbol{n = 1}$

$4^2+5^1=21$, which is divisible by 21

Hence P_1 is true.

Assume that 21 divides $4^{k+1}+5^{2k-1}$ is true.

We need to prove that 21 divides $4^{(k+1)+1}+5^{2(k+1)-1}$ is also true

$$\begin{aligned}
4^{(k+1)+1}+5^{2(k+1)-1} &= 4\cdot 4^{(k+1)}+5^{2k+2-1} \\
&= 4\cdot 4^{(k+1)}+5^2\cdot 5^{2k-1} \\
&= 4\cdot 4^{(k+1)}+25\cdot 5^{2k-1} \\
&= 4\cdot 4^{(k+1)}+(4+21)\cdot 5^{2k-1} \\
&= 4\cdot 4^{(k+1)}+4\cdot 5^{2k-1}+21\cdot 5^{2k-1} \\
&= 4\cdot\left(4^{(k+1)}+5^{2k-1}\right)+21\cdot 5^{2k-1}
\end{aligned}$$

By the inductive hypothesis, 21 divides $4^{k+1}+5^{2k-1}$ and certainly 21 divides 5^{2k-1}, so 21 divides their sum.

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer *n*. $1+2.2+3.2^2+\ldots+n.2^{n-1}=1+(n-1).2^n$

Solution

For $\boldsymbol{n = 1}$

$1\overset{?}{=}1+(1-1)2^1=1-0=1$

Hence P_1 is true.

$1+2.2+3.2^2+\ldots+k.2^{k-1}=1+(k-1).2^k$ is true

$1+2.2+3.2^2+\ldots+k.2^{k-1}+(k+1).2^{(k+1)-1}=1+((k+1)-1).2^{k+1}$?

$$\begin{aligned}
1+2.2+3.2^2+\ldots+k.2^{k-1}+(k+1).2^{(k+1)-1} &= 1+(k-1).2^k+(k+1).2^{k+1-1} \\
&= 1+k.2^k-1.2^k+(k+1).2^k
\end{aligned}$$

$$=1+k.2^k-1.2^k+k.2^k+1.2^k$$
$$=1+2^1 k.2^k$$
$$=1+(k+0).2^{k+1}$$
$$=1+((k+1)-1).2^{k+1} \quad \surd$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. $1^2+2^2+3^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$

Solution

For $\boldsymbol{n}$ **= 1**

$$1^2 \stackrel{?}{=} \frac{1(1+1)(2(1)+1)}{6}=\frac{1(2)(3)}{6}=\frac{6}{6}=1 \ \surd$$

Hence P_1 is true.

$1^2+2^2+3^2+\ldots+k^2=\dfrac{k(k+1)(2k+1)}{6}$ is true

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} ?$$

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$
$$=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$
$$=\frac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6}$$
$$=\frac{(k+1)\left[2k^2+k+6k+6\right]}{6}$$
$$=\frac{(k+1)\left[2k^2+7k+6\right]}{6}$$
$$=\frac{(k+1)((k+2)(2k+3))}{6}$$
$$=\frac{(k+1)((k+1+1)(2k+2+1))}{6}$$
$$=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \quad \surd$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. $\frac{1}{1\bullet 2}+\frac{1}{2\bullet 3}+\frac{1}{3\bullet 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

Solution

For $\boldsymbol{n}$ **= 1**

$$\frac{1}{1.2}\overset{?}{=}\frac{1}{1+1}=\frac{1}{2}=\frac{1}{1.2}\ \checkmark$$

Hence P_1 is true.

$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$ is true

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{(k+1)+1}?$$

$$\begin{aligned}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}&=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}\\&=\frac{k(k+2)+1}{(k+1)(k+2)}\\&=\frac{k^2+2k+1}{(k+1)(k+2)}\\&=\frac{(k+1)(k+1)}{(k+1)(k+2)}\\&=\frac{k+1}{(k+1+1)}\\&=\frac{k+1}{(k+1)+1}\ \checkmark\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$

Solution

For $\boldsymbol{n}$ **= 1**

$$\frac{1}{2}\overset{?}{=}1-\frac{1}{2}=\frac{1}{2} \quad \checkmark$$

Hence, P_1 is true.

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}=1-\frac{1}{2^k} \text{ is true}$$

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^{k+1}}?$$

$$\begin{aligned}
\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}} &= 1-\frac{1}{2^k}+\frac{1}{2^{k+1}}\\
&= 1-\frac{1}{2^k}+\frac{1}{2^k\cdot 2}\\
&= \frac{2^{k+1}-2+1}{2^{k+1}}\\
&= \frac{2^{k+1}-1}{2^{k+1}}\\
&= \frac{2^{k+1}}{2^{k+1}}-\frac{1}{2^{k+1}}\\
&= 1-\frac{1}{2^{k+1}} \quad \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n-2)\cdot(3n+1)}=\frac{n}{3n+1}$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$$\frac{1}{1\cdot 4}\overset{?}{=}\frac{1}{3(1)+1}=\frac{1}{4} \quad \checkmark$$

Hence, P_1 is true.

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1} \text{ is true}$$

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)}\overset{?}{=}\frac{k+1}{3(k+1)+1}$$

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3(k+1)-2)(3(k+1)+1)}=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$=\frac{(3k+1)(k+1)}{(3k+1)(3k+3+1)}$$

$$=\frac{k+1}{3(k+1)+1} \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $\frac{4}{5}+\frac{4}{5^2}+\frac{4}{5^3}+\cdots+\frac{4}{5^n}=1-\frac{1}{5^n}$

Solution

For $\boldsymbol{n}$ **= 1**

$$\frac{4}{5}\overset{?}{=}1-\frac{1}{5}=\frac{4}{5} \quad \surd$$

Hence, P_1 is true.

$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}=1-\frac{1}{5^k}$ is true

$$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}+\frac{4}{5^{k+1}}\overset{?}{=}1-\frac{1}{5^{k+1}}$$

$$\frac{4}{5}+\frac{4}{5^2}+\cdots+\frac{4}{5^k}+\frac{4}{5^{k+1}}=1-\frac{1}{5^k}+\frac{4}{5^{k+1}}$$

$$=1-\left(\frac{1}{5^k}-\frac{4}{5^{k+1}}\right)$$

$$=1-\frac{5-4}{5^{k+1}}$$

$$=1-\frac{1}{5^{k+1}} \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$

Solution

For $\boldsymbol{n}$ **= 1**

$$1^3 \overset{?}{=} \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1 \quad \checkmark$$

Hence, P_1 is true.

$$\frac{4}{5} + \frac{4}{5^2} + \cdots + \frac{4}{5^k} = 1 - \frac{1}{5^k} \text{ is true}$$

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 \overset{?}{=} \frac{(k+1)^2((k+1)+1)^2}{4}$$

$$\begin{aligned}
1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2\left[k^2 + 4(k+1)\right]}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2((k+1)+1)^2}{4} \quad \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. $3 + 3^2 + 3^3 + \ldots + 3^n = \frac{3}{2}(3^n - 1)$

Solution

For ***n* = 1**

$$3 \overset{?}{=} \frac{3}{2}(3^1 - 1) = \frac{3}{2}2 = 3 \quad \checkmark$$

Hence, P_1 is true.

$$3 + 3^2 + \cdots + 3^k = \frac{3}{2}(3^k - 1) \text{ is true}$$

$$3 + 3^2 + \cdots + 3^k + 3^{k+1} \overset{?}{=} \frac{3}{2}(3^{k+1} - 1)$$

$$3 + 3^2 + \cdots + 3^k + 3^{k+1} = \frac{3}{2}(3^k - 1) + 3^{k+1}$$

$$=\frac{1}{2}3^{k+1}-\frac{3}{2}+3^{k+1}$$
$$=\frac{3}{2}3^{k+1}-\frac{3}{2}$$
$$=\frac{3}{2}\left(3^{k+1}-1\right) \quad \surd$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $x^{2n}+x^{2n-1}y+\cdots+xy^{2n-1}+y^{2n}=\frac{x^{2n+1}-y^{2n+1}}{x-y}$

Solution

For ***n* = 1**

$$x^2+xy+y^2\overset{?}{=}\frac{x^3-y^3}{x-y}$$
$$=\frac{(x-y)\left(x^2+xy+y^2\right)}{x-y}$$
$$=x^2+xy+y^2 \quad \surd$$

Hence, P_1 is true.

$x^{2k}+x^{2k-1}y+\cdots+xy^{2k-1}+y^{2k}=\frac{x^{2k+1}-y^{2k+1}}{x-y}$ is true

$$x^{2(k+1)}+x^{2(k+1)-1}y+\cdots+xy^{2(k+1)-1}+y^{2(k+1)}\overset{?}{=}\frac{x^{2(k+1)+1}-y^{2(k+1)+1}}{x-y}$$

$$x^{2k+2}+x^{2k+1}y+\cdots+xy^{2k+1}+y^{2k+2}=x^2\left(x^{2k}+x^{2k-1}y+\cdots+y^{2k}\right)+xy^{2k+1}+y^{2k+2}$$
$$=x^2\left(\frac{x^{2k+1}-y^{2k+1}}{x-y}\right)+xy^{2k+1}+y^{2k+2}$$
$$=\frac{x^{2k+3}-x^2y^{2k+1}+x^2y^{2k+1}+xy^{2k+2}-xy^{2k+2}-y^{2(k+1)+1}}{x-y}$$
$$=\frac{x^{2(k+1)+1}-y^{2(k+1)+1}}{x-y} \quad \surd$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $5 \cdot 6 + 5 \cdot 6^2 + 5 \cdot 6^3 + \cdots + 5 \cdot 6^n = 6\left(6^n - 1\right)$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$$5 \cdot 6 \overset{?}{=} 6\left(6^1 - 1\right) = 6(5) \ \checkmark$$

Hence, P_1 is true.

$5 \cdot 6 + 5 \cdot 6^2 + \cdots + 5 \cdot 6^k = 6\left(6^k - 1\right)$ is true

$$5 \cdot 6 + 5 \cdot 6^2 + \cdots + 5 \cdot 6^k + 5 \cdot 6^{k+1} \overset{?}{=} 6\left(6^{k+1} - 1\right)$$

$$\begin{aligned}
5 \cdot 6 + 5 \cdot 6^2 + \cdots + 5 \cdot 6^k + 5 \cdot 6^{k+1} &= 6\left(6^k - 1\right) + 5 \cdot 6^{k+1} \\
&= 6^{k+1} - 6 + 5 \cdot 6^{k+1} \\
&= 6^{k+1}(1+5) - 6 \\
&= 6 \cdot 6^{k+1} - 6 \\
&= 6\left(6^{k+1} - 1\right) \ \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $7 \cdot 8 + 7 \cdot 8^2 + 7 \cdot 8^3 + \cdots + 7 \cdot 8^n = 8\left(8^n - 1\right)$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$$7 \cdot 8 \overset{?}{=} 8\left(8^1 - 1\right) = 8(7) \ \checkmark$$

Hence, P_1 is true.

$7 \cdot 8 + 7 \cdot 8^2 + \cdots + 7 \cdot 8^k = 8\left(8^k - 1\right)$ is true

$$7 \cdot 8 + 7 \cdot 8^2 + \cdots + 7 \cdot 8^k + 7 \cdot 8^{k+1} \overset{?}{=} 8\left(8^{k+1} - 1\right)$$

$$\begin{aligned}
7 \cdot 8 + 7 \cdot 8^2 + \cdots + 7 \cdot 8^k + 7 \cdot 8^{k+1} &= 8\left(8^k - 1\right) + 7 \cdot 8^{k+1} \\
&= 8^{k+1} - 8 + 7 \cdot 8^{k+1}
\end{aligned}$$

$$=8^{k+1}(1+7)-8$$
$$=8\left(8^{k+1}-1\right) \quad \surd$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $3+6+9+\cdots+3n=\frac{3n(n+1)}{2}$

Solution

For ***n* = 1**

$$3\overset{?}{=}\frac{3(1)(1+1)}{2}=3 \quad \surd$$

Hence, P_1 is true.

$3+6+9+\cdots+3k=\frac{3k(k+1)}{2}$ is true

$$3+6+9+\cdots+3k+3(k+1)\overset{?}{=}\frac{3(k+1)(k+2)}{2}$$

$$\begin{aligned}3+6+9+\cdots+3k+3(k+1)&=\frac{3k(k+1)}{2}+3(k+1)\\&=\frac{3k(k+1)+6(k+1)}{2}\\&=\frac{(k+1)(3k+6)}{2}\\&=\frac{3(k+1)(k+2)}{2} \quad \surd\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $5+10+15+\cdots+5n=\frac{5n(n+1)}{2}$

Solution

For ***n* = 1**

$$5\overset{?}{=}\frac{5(1)(1+1)}{2}=5 \quad \surd$$

Hence, P_1 is true.

$$5+10+15+\cdots+5k=\frac{5k(k+1)}{2} \text{ is true}$$

$$5+10+15+\cdots+5k+5(k+1)\overset{?}{=}\frac{5(k+1)(k+2)}{2}$$

$$\begin{aligned}5+10+15+\cdots+5k+5(k+1)&=\frac{5k(k+1)}{2}+5(k+1)\\&=\frac{5k(k+1)+10(k+1)}{2}\\&=\frac{(k+1)(5k+10)}{2}\\&=\frac{5(k+1)(k+2)}{2} \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $1+3+5+\cdots+(2n-1)=n^2$

Solution

For ***n* = 1**

$$1\overset{?}{=}1^2=1 \quad \checkmark$$

Hence, P_1 is true.

$$1+3+5+\cdots+(2k-1)=k^2 \text{ is true}$$

$$1+3+5+\cdots+(2k-1)+(2(k+1)-1)\overset{?}{=}(k+1)^2$$

$$\begin{aligned}1+3+5+\cdots+(2k-1)+(2(k+1)-1)&=k^2+2k+2-1\\&=k^2+2k+1\\&=(k+1)^2 \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true: $4+7+10+\cdots+(3n+1)=\dfrac{n(3n+5)}{2}$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$$4\overset{?}{=}\frac{1(3+5)}{2}=4 \quad \surd$$

Hence, P_1 is true.

$$4+7+10+\cdots+(3k+1)=\frac{k(3k+5)}{2} \text{ is true}$$

$$4+7+10+\cdots+(3k+1)+(3(k+1)+1)\overset{?}{=}\frac{(k+1)(3(k+1)+5)}{2}=\frac{(k+1)(3k+8)}{2}$$

$$\begin{aligned}4+7+10+\cdots+(3k+1)+(3k+4)&=\frac{k(3k+5)}{2}+3k+4\\&=\frac{3k^2+5k+6k+8}{2}\\&=\frac{3k^2+5k+3k+3k+8}{2}\\&=\frac{k(3k+8)+(3k+8)}{2}\\&=\frac{(3k+8)(k+1)}{2} \quad \surd\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction:

$$2+4+6+\cdots+2(n-1)+2n=n(n+1)$$

Solution

For $\boldsymbol{n} = \mathbf{1}$

$$2\overset{?}{=}1(1+1)$$

$$2=2 \quad \surd$$

Hence, P_1 is true.

For $\boldsymbol{k}$: $2+4+6+\cdots+2(k-1)+2k=k(k+1)$

$$2+4+\cdots+2k+2(k+1)=(k+1)(k+2)$$

$$2+4+\cdots+2k+2(k+1)=k(k+1)+2(k+1)$$
$$=(k+1)(k+2) \quad \checkmark$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction:

$$1+(1+2)+(1+2+3)+\cdots+(1+2+\cdots+n)=\frac{n(n+1)(n+2)}{6}$$

Solution

For $\boldsymbol{n}$ **= 1**

$$1\overset{?}{=}\frac{1(1+1)(1+2)}{6}$$

$$1\overset{?}{=}\frac{(2)(3)}{6}$$

$$1=1 \quad \checkmark$$

Hence, P_1 is true.

For $\boldsymbol{k}$: $\quad 1+(1+2)+\cdots+(1+2+\cdots+k)=\dfrac{k(k+1)(k+2)}{6}$

Is P_{k+1}: $\quad 1+(1+2)+\cdots+(1+2+\cdots+k)+\big(1+2+\cdots+k+(k+1)\big)\overset{?}{=}\dfrac{(k+1)(k+2)(k+3)}{6}$

$$1+(1+2)+\cdots+(1+2+\cdots+k)+\big(1+2+\cdots+k+(k+1)\big)=\frac{k(k+1)(k+2)}{6}+\big(1+2+\cdots+k+(k+1)\big)$$

$$1+2+\cdots+n=\frac{1}{2}n(n+1)$$

$$1+2+\cdots+k+(k+1)=\frac{1}{2}k(k+1)+(k+1)$$
$$=(k+1)\left(\frac{1}{2}k+1\right)$$
$$=\frac{1}{2}(k+1)(k+2)$$

$$1+(1+2)+\cdots+(1+2+\cdots+k)+\big(1+2+\cdots+k+(k+1)\big)=\frac{k(k+1)(k+2)}{6}+\frac{1}{2}(k+1)(k+2)$$
$$=(k+1)(k+2)\left(\frac{k}{6}+\frac{1}{2}\right)$$
$$=\frac{(k+1)(k+2)(k+3)}{6} \quad \checkmark$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction:

$$1+2+3+\cdots+n < \frac{(2n+3)^2}{7}$$

Solution

For $\boldsymbol{n}$ **= 1**

$$1 \overset{?}{<} \frac{(2+3)^2}{7}$$

$$1 < \frac{25}{7} \quad >1 \quad \checkmark$$

Hence, P_1 is true.

For $\boldsymbol{k}$: $\quad 1+2+\cdots+k < \frac{(2k+3)^2}{7}$

Is P_{k+1}: $\quad 1+2+\cdots+k+(k+1) < \frac{(2(k+1)+3)^2}{7}$

$$< \frac{(2k+5)^2}{7} \ ? \qquad \frac{4k^2+20k+25}{7}$$

$$\begin{aligned}
1+2+\cdots+k+(k+1) &< \frac{(2k+3)^2}{7}+(k+1) \\
&= \frac{4k^2+12k+9+7k+7}{7} \\
&= \frac{1}{7}\left(4k^2+19k+16+k+9-k-9\right) \\
&= \frac{1}{7}\left(4k^2+20k+25-(k+9)\right) \\
&= \frac{(2k+5)^2}{7}-\frac{k+9}{7} \\
&< \frac{(2k+5)^2}{7} \quad \checkmark
\end{aligned}$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction:

$$\frac{1}{2n} \le \frac{1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot(2n)}$$

Solution

For $\boldsymbol{n}$ **= 1**

$$\frac{1}{2} \le \frac{1}{2} \checkmark$$

Hence, P_1 is true.

For $\boldsymbol{k}$: $\quad \dfrac{1}{2k} \le \dfrac{1\cdots(2k-3)\cdot(2k-1)}{2\cdot\ \cdots\ (2k-2)\cdot(2k)}$

Is P_{k+1}: $\quad \dfrac{1}{2(k+1)} \le \dfrac{1\cdots(2k-1)\cdot(2k+1)}{2\cdot\ \cdots\ (2k)\cdot(2k+2)}$

$$\frac{1\cdots(2k-1)\cdot(2k+1)}{2\cdot\ \cdots\ (2k)\cdot(2k+2)} \ge \frac{1}{2(k+1)} \quad ?$$

$$\begin{aligned}\frac{1\cdots(2k-1)\cdot(2k+1)}{2\cdot\ \cdots\ (2k)\cdot(2k+2)} &= \frac{1\cdots(2k-1)}{2\cdot\ \cdots\ (2k)}\frac{2k+1}{2k+2} \\ &\ge \frac{1}{2k}\cdot\frac{2k+1}{2k+2} \\ &= \frac{2k+1}{2k}\cdot\frac{1}{2(k+1)} \\ &= \left(1+\frac{1}{2k}\right)\cdot\frac{1}{2(k+1)} \\ &\ge \frac{1}{2(k+1)} \quad \checkmark\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction: $\quad \dfrac{2n+1}{2n+2} \le \dfrac{\sqrt{n+1}}{\sqrt{n+2}}$

Solution

For $\boldsymbol{n}$ **= 1**

$$\frac{2+1}{2+2} \overset{?}{\le} \frac{\sqrt{1+1}}{\sqrt{1+2}}$$

$$\frac{3}{4} \overset{?}{\le} \frac{\sqrt{2}}{\sqrt{3}}$$

$$3\sqrt{3} \quad ? \quad 4\sqrt{2}$$ *Square both sides*

$$27 \le 32 \quad \checkmark$$

Hence, P_1 is true.

For ***k***: $\frac{2k+1}{2k+2} \le \frac{\sqrt{k+1}}{\sqrt{k+2}}$

$$(2k+1)\sqrt{k+2} \le (2k+2)\sqrt{k+1}$$

Is P_{k+1}: $\frac{2k+3}{2k+4} \overset{?}{\le} \frac{\sqrt{k+2}}{\sqrt{k+3}}$

$$(2k+3)\sqrt{k+3} \overset{?}{\le} (2k+4)\sqrt{k+2}$$

$$(2k+3) \le (2k+4)$$

$$\underline{\sqrt{k+3} \le \sqrt{k+2}}$$

$$(2k+3)\sqrt{k+3} \le (2k+4)\sqrt{k+2} \quad \checkmark$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction: $n! < n^n$ *for* $n > 1$

Solution

For ***n* = 2**

$$2! \overset{?}{<} 2^2$$

$$2 < 4 \quad \checkmark$$

Hence, P_1 is true.

For ***k***: $k! < k^k$

Is P_{k+1}: $(k+1)! < (k+1)^{k+1}$

$$(k+1)! = k!\,(k+1)$$

$$< k^k\,(k+1)$$

$$k < k+1$$

$$k^k < (k+1)^k$$

$$< (k+1)^k (k+1)$$
$$= (k+1)^{k+1} \quad \checkmark$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true by mathematical induction: $\left(a^m\right)^n = a^{mn}$ (a and m are constant)

Solution

For ***n* = 1**

$$\left(a^m\right)^1 \stackrel{?}{=} a^{m(1)}$$
$$a^m = a^m \quad \checkmark$$

Hence, P_1 is true.

$\left(a^m\right)^k = a^{mk}$ is true

$$\left(a^m\right)^{(k+1)} \stackrel{?}{=} a^{m(k+1)}$$

$$\left(a^m\right)^{(k+1)} = \left(a^m\right)^k a^m$$
$$= a^{km}a^m$$
$$= a^{km+m}$$
$$= a^{m(k+1)} \quad \checkmark$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. $n < 2^n$

Solution

For ***n* = 1**

$$1 \stackrel{?}{<} 2^1 \quad \checkmark$$

Hence, P_1 is true.

Assume that P_k is true $k < 2^k$

We need to prove that P_{k+1} is true, that is $k+1 < 2^{k+1}$

$$\begin{aligned} k+1 < k+k &= 2k \\ &< 2 \cdot 2^k \\ &= 2^{k+1} \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. 3 is a factor of $n^3 - n + 3$

Solution

For ***n* = 1**

$1^3 - 1 + 3 = 3 = 3(1)$ ✓

Hence, P_1 is true.

Assume that P_k is true 3 is a factor of $k^3 - k + 3$

We need to prove that P_{k+1} is true, that is $(k+1)^3 - (k+1) + 3$

$$\begin{aligned} (k+1)^3 - (k+1) + 3 &= k^3 + 3k^2 + 3k + 1 - k - 1 + 3 \\ &= \left(k^3 - k + 3\right) + 3k^2 + 3k \qquad\qquad k^3 - k + 3 = 3K \\ &= 3K + 3k^2 + 3k \\ &= 3\left(K + k^2 + k\right) \quad \checkmark \end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement is true for every positive integer n. 4 is a factor of $5^n - 1$

Solution

For ***n* = 1**

$5^1 - 1 = 4 = 4(1)$ ✓

Hence, P_1 is true.

Assume that P_k is true 4 is a factor of $5^k - 1$

We need to prove that P_{k+1} is true, that is $5^{k+1} - 1$

$$5^{k+1} - 1 = 5^k 5^1 - 5 + 4$$
$$= 5\left(5^k - 1\right) + 4$$
$$= 5\left(5^k - 1\right) + 4$$

By the induction hypothesis, 4 is a factor of $5^k - 1$ and 4 is a factor of 4, so 4 is a factor of the $(k+1)$ term. √

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $2^n > 2n \quad if \quad n \geq 3$

Solution

For ***n* = 3**

$$2^3 \geq 2(3)$$
$$8 \geq 6 \quad √$$

Hence, P_3 is true.

Assume that P_k is true: $2^k > 2k$

We need to prove that P_{k+1}: $2^{k+1} > 2(k+1)$ is true

$$2^k > 2k$$
$$2^k \cdot 2 > 2k \cdot 2$$
$$2^{k+1} > 4k = 2k + 2k \qquad k \geq 3$$
$$> 2k + 2$$
$$= 2(k+1) \quad √$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: If $0 < a < 1$, then $a^n < a^{n-1}$

Solution

For $\boldsymbol{n = 1}$

$a^1 < a^{1-1}$

$a < 1$ √

since $0 < a < 1 \Rightarrow P_1$ is true.

Assume that P_k is true: $a^k < a^{k-1}$

We need to prove that $P_{k+1}: a^{k+1} < a^k$ is true

$$a^k < a^{k-1} \quad \to \quad a^k \cdot a < a^{k-1} \cdot a$$

$$a^{k+1} < a^k \quad √$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: If $n \geq 4$, then $n! > 2^n$

Solution

For $\boldsymbol{n = 4}$

$4! > 2^4$

$24 > 16$ √

Hence, P_4 is true.

Assume that P_k is true: $k! > 2^k$

We need to prove that $P_{k+1}: (k+1)! > 2^{k+1}$ is true

$$\begin{aligned}(k+1)! &= k! \cdot (k+1) \\ &> 2^k (k+1) \qquad \textit{Since } \ k \geq 4 \Rightarrow k+1 > 2 \\ &> 2^k \cdot 2 \\ &= 2^{k+1} \quad √\end{aligned}$$

Hence P_{k+1} is true.

∴ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $3^n > 2n+1 \quad if \quad n \ge 2$

Solution

For $\boldsymbol{n} = \mathbf{2}$

$3^2 > 2(2)+1$

$9 > 5 \ \checkmark$

Hence, P_2 is true.

Assume that P_k is true: $3^k > 2k+1;$

We need to prove that $P_{k+1}: \ 3^{k+1} > 2(k+1)+1$ is true

$$3^k > 2k+1 \quad \Rightarrow \quad 3^k \cdot 3 > (2k+1)\cdot 3$$

$$3^{k+1} > 6k+3$$

$$> 2k+2+1$$

$$= 2(k+1)+1 \quad \checkmark$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $2^n > n^2 \quad for \quad n > 4$

Solution

For $\boldsymbol{n} = \mathbf{5}$

$2^5 > 5^2$

$32 > 25 \ \checkmark$

Hence, P_5 is true.

Assume that P_k is true: $2^k > k^2$

We need to prove that $P_{k+1}: \ 2^{k+1} > (k+1)^2$ is true

$$2^k > k^2$$

$$2^k \cdot 2 > k^2 \cdot 2$$

$$2^{k+1} > 2k^2$$

$$= k^2 + k^2 \qquad\qquad k < k+1 \ \Rightarrow \ k\cdot k > k+k+1 \Rightarrow \ k^2 > 2k+1$$

$$> k^2 + 2k+1$$

$$=(k+1)^2 \quad \checkmark$$

Hence P_{k+1} is true.

$\therefore$ By the mathematical induction, the given statement is true.

Exercise

Prove that the statement by mathematical induction: $4^n > n^4 \quad for \quad n \ge 5$

Solution

For ***n*** **= 5**

$$4^5 > 5^4$$

$$1024 > 625 \quad \checkmark$$

Hence, P_5 is true.

Assume that P_k is true: $4^k > k^4$

We need to prove that P_{k+1}: $4^{k+1} > (k+1)^4$ is true

$$4^k > k^4$$

$$4^k \cdot 4 > k^4 \cdot 4$$

$$4^{k+1} > 4k^4$$

$$k < k+1$$

$$4k > k+1$$

$$4k^4 > (k+1)^4$$

$$> (k+1)^4 \quad \checkmark$$

Hence P_{k+1} is true.

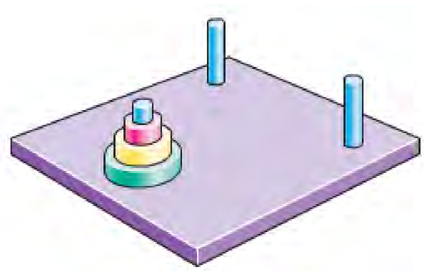
$\therefore$ By the mathematical induction, the given statement is true.

Exercise

A pile of *n* rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must be moved, one at a time, to a different peg with no ring ever placed on top of a smaller ring.

Find the least number of moves that would be required.

Prove your result by mathematical induction.



Solution

With 1 ring, 1 move is required.

With 2 rings, 3 moves are required $\Rightarrow \quad 3 = 2 + 1$

With 3 rings, 7 moves are required $\Rightarrow \quad 7 = 2^2 + 2 + 1$

With n rings, $2^{n-1} + \cdots + 2^2 + 2^1 + 2^0 = 2^n - 1$ moves are required

For ***n* = 1**

$$2^0 = 2^1 - 1 = 1 \; \surd$$

Hence, P_1 is true.

Assume that P_k is true: $2^{k-1} + \cdots + 2^2 + 2^1 + 2^0 = 2^k - 1$

$$2^k + 2^{k-1} + \cdots + 2^2 + 2^1 + 1 \stackrel{?}{=} 2^{k+1} - 1$$

$$\begin{aligned} 2^k + 2^{k-1} + \cdots + 2^2 + 2^1 + 1 &= 2^k + 2^k - 1 \\ &= 2 \cdot 2^k - 1 \\ &= 2^{k+1} - 1 \quad \surd \end{aligned}$$

$\therefore$ By the mathematical induction, the given statement is true.

Solution *Section* 3.2 – Recursive Definitions and Structural Induction

Exercise

Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$ if $f(n)$ is defined recursively by $f(0)=1$ and for $n=0,\ 1,\ 2,\ldots$

a) $f(n+1)=f(n)+2$ *b)* $f(n+1)=3f(n)$

c) $f(n+1)=2^{f(n)}$ *d)* $f(n+1)=f(n)^2+f(n)+1$

Solution

a) $f(1)=f(0)+2$

$=1+2$

$\underline{=3}$

$f(2)=f(1)+2$

$=3+2$

$\underline{=5}$

$f(3)=f(2)+2$

$=5+2$

$\underline{=7}$

$f(4)=f(3)+2$

$=7+2$

$\underline{=9}$

b) $f(n+1)=3f(n)$

$f(1)=3\cdot f(0)$

$=3(1)$

$\underline{=3}$

$f(2)=3\cdot f(1)$

$=3(3)$

$\underline{=9}$

$f(3)=3\cdot f(2)$

$=3(9)$

$\underline{=27}$

$f(4)=3\cdot f(3)$

$=3(27)$

$\underline{=81}$

c) $f\left(n+1\right)=2^{f\left(n\right)}$

$f\left(1\right)=2^{f\left(0\right)}$

$=2^1$

$\underline{=2}$

$f\left(2\right)=2^{f\left(1\right)}$

$=2^2$

$\underline{=4}$

$f\left(3\right)=2^{f\left(2\right)}$

$=2^4$

$\underline{=16}$

$f\left(4\right)=2^{f\left(3\right)}$

$=2^{16}$

$\underline{=65536}$

d) $f\left(n+1\right)=f\left(n\right)^2+f\left(n\right)+1$

$f\left(1\right)=f\left(0\right)^2+f\left(0\right)+1$

$=1^2+1+1$

$\underline{=3}$

$f\left(2\right)=f\left(1\right)^2+f\left(1\right)+1$

$=3^2+3+1$

$\underline{=13}$

$f\left(3\right)=f\left(2\right)^2+f\left(2\right)+1$

$=13^2+13+1$

$\underline{=183}$

$f\left(4\right)=f\left(3\right)^2+f\left(3\right)+1$

$=183^2+183+1$

$\underline{=33673}$

Exercise

Find $f(1)$, $f(2)$, $f(3)$, $f(4)$ and $f(5)$ if $f(n)$ is defined recursively by $f(0)=3$ and for $n=0,\ 1,\ 2,\dots$

a) $f(n+1)=-2f(n)$

b) $f(n+1)=3f(n)+7$

c) $f(n+1)=3^{f(n)/3}$

d) $f(n+1)=f(n)^2-2f(n)-2$

Solution

a) $f(n+1)=-2f(n)$

$$f(1)=-2f(0)$$
$$=-2(3)$$
$$\underline{=-6}$$

$$f(2)=-2f(1)$$
$$=-2(-6)$$
$$\underline{=12}$$

$$f(3)=-2f(2)$$
$$=-2(12)$$
$$\underline{=-24}$$

$$f(4)=-2f(3)$$
$$=-2(-24)$$
$$\underline{=48}$$

$$f(5)=-2f(4)$$
$$=-2(48)$$
$$\underline{=-96}$$

b) $f(1)=3\cdot f(0)+7$

$$=3(3)+7$$
$$\underline{=16}$$

$$f(2)=3\cdot f(1)+7$$
$$=3(16)+7$$
$$\underline{=55}$$

$$f(3)=3\cdot f(2)+7$$
$$=3(55)+7$$
$$\underline{=172}$$

$$f(4) = 3 \cdot f(3) + 7$$
$$= 3(172) + 7$$
$$\underline{= 523}$$
$$f(5) = 3 \cdot f(4) + 7$$
$$= 3(523) + 7$$
$$\underline{= 1576}$$

c) $f(1) = 3^{f(0)/3}$
$$= 3^{3/3}$$
$$= 3^1$$
$$\underline{= 3}$$
$$f(2) = 3^{f(1)/3}$$
$$= 3^{3/3}$$
$$= 3^1$$
$$\underline{= 3}$$
$$f(3) = 3^{f(2)/3}$$
$$= 3^{3/3}$$
$$= 3^1$$
$$\underline{= 3}$$
$$f(4) = 3^{f(3)/3}$$
$$= 3^{3/3}$$
$$= 3^1$$
$$\underline{= 3}$$
$$f(5) = 3^{f(3)/3}$$
$$= 3^{3/3}$$
$$= 3^1$$
$$\underline{= 3}$$

d) $f(1) = f(0)^2 - 2f(0) - 2$
$$= 3^2 - 2(3) - 2$$
$$\underline{= 1}$$
$$f(2) = f(1)^2 - 2f(1) - 2$$

$$=1^2-2(1)-2$$
$$\underline{=-3}$$
$$f(3)=f(2)^2-2f(2)-2$$
$$=(-3)^2-2(-3)-2$$
$$\underline{=13}$$
$$f(4)=f(3)^2-2f(3)-2$$
$$=(13)^2-2(13)-2$$
$$\underline{=141}$$
$$f(5)=f(4)^2-2f(4)-2$$
$$=(141)^2-2(141)-2$$
$$\underline{=19,597}$$

Exercise

Find $f(2)$, $f(3)$, $f(4)$ and $f(5)$ if $f(n)$ is defined recursively by $f(0)=f(1)=1$ and for $n=1,\ 2,\ldots$

a) $f(n+1)=f(n)-f(n-1)$ *b)* $f(n+1)=f(n)\,f(n-1)$

c) $f(n+1)=f(n)^2+f(n-1)^3$ *d)* $f(n+1)=f(n)/f(n-1)$

Solution

a) $f(2)=f(1)-f(0)$
$$=1-1$$
$$\underline{=0}$$
$$f(3)=f(2)-f(1)$$
$$=0-1$$
$$\underline{=-1}$$
$$f(4)=f(3)-f(2)$$
$$=-1-0$$
$$\underline{=-1}$$
$$f(5)=f(4)-f(3)$$
$$=-1-(-1)$$
$$\underline{=0}$$

b) $f(2) = f(1)\,f(0)$

$= (1)(1)$

$\underline{= 1}$

$f(3) = f(2)\,f(1)$

$= (1)(1)$

$\underline{= 1}$

$f(4) = f(3)\,f(2)$

$= (1)(1)$

$\underline{= 1}$

$f(5) = f(4)\,f(3)$

$= (1)(1)$

$\underline{= 1}$

c) $f(2) = f(1)^2 + f(0)^3$

$= 1^2 + 1^3$

$\underline{= 2}$

$f(3) = f(2)^2 + f(1)^3$

$= 2^2 + 1^3$

$\underline{= 5}$

$f(4) = f(3)^2 + f(2)^3$

$= 5^2 + 2^3$

$\underline{= 33}$

$f(5) = f(4)^2 + f(3)^3$

$= 33^2 + 5^3$

$\underline{= 1,214}$

d) $f(2) = \dfrac{f(1)}{f(0)}$

$= \dfrac{1}{1}$

$\underline{= 1}$

$f(3) = \dfrac{f(2)}{f(1)}$

$= \dfrac{1}{1}$

$\underline{=1}$

$$f(4) = \frac{f(3)}{f(2)}$$

$$= \frac{1}{1}$$

$\underline{=1}$

$$f(5) = \frac{f(4)}{f(3)}$$

$$= \frac{1}{1}$$

$\underline{=1}$

Exercise

Determine whether each of these proposed definitions is a valid recursive definition of a function *f* from the set of nonnegative integers to the set of integers. If f is well defined, fund a formula for $f(n)$ when *n* is nonnegative integer and prove that your formula is valid.

a) $f(0)=0, \quad f(n)=2f(n-2) \quad for \ \ n \ge 1$

b) $f(0)=1, \quad f(n)=-f(n-1) \quad for \ \ n \ge 1$

c) $f(0)=1, \quad f(n)=f(n-1)-1 \quad for \ \ n \ge 1$

d) $f(0)=2, \quad f(1)=3, \quad f(n)=f(n-1)-1 \quad for \ \ n \ge 2$

e) $f(0)=1, \quad f(1)=2, \quad f(n)=2f(n-2) \quad for \ \ n \ge 2$

f) $f(0)=1, \quad f(1)=0, \quad f(2)=2, \quad f(n)=2f(n-3) \quad for \ \ n \ge 3$

g) $f(0)=0, \quad f(1)=1, \quad f(n)=2f(n+1) \quad for \ \ n \ge 2$

h) $f(0)=0, \quad f(1)=1, \quad f(n)=2f(n-1) \quad for \ \ n \ge 2$

i) $f(0)=2, \quad f(n)=f(n-1)$ if *n* is odd and $n \ge 1$ and $f(n)=2f(n-2)$ if *n* is even and $n \ge 2$

j) $f(0)=1, \quad f(n)=3f(n-1)$ if *n* is odd and $n \ge 1$ and $f(n)=9f(n-2)$ if *n* is even and $n \ge 2$

Solution

a) This is invalid, since $f(1)=2f(1-2)=\cancel{2f(-1)} \quad for \ \ n \ge 1, \ f(-1)$ is not defined.

b) $f(1)=-f(0)=-1$, this is a valid, since $n=0$ is provided and each subsequent value is determined by the previous one. $f(n)=(-1)^n$, this is true for $n=0 \quad since \quad (-1)^0=1$.

Assume it is true for $n=k$, then

$$f(k+1)=-f(k+1-1)=-f(k)=-(-1)^k=(-1)^{k+1}$$

c) $f(1)=f(0)-1=1-1=0$, this is a valid.

$f(2) = f(1) - 1 = 0 - 1 = -1$

The sequence: $1,\ 0,\ -1,\ -2,\ -3,\ \ldots \Rightarrow f(n) = 1 - n$

By induction:

The basis step: $f(0) = 1 - 0 = 1$

If $\quad f(k) = 1 - k$

Then $\quad f(k+1) = f(k) - 1 = 1 - k - 1 = 1 - (k+1)$

d) $f(2) = f(1) - 1 = 3 - 1 = 2$

$f(3) = f(2) - 1 = 2 - 1 = 1$

Given: $f(0) = 2, \quad f(1) = 3$

Then the sequence: 2, 3, 2, 1, 0, … $\Rightarrow f(n) = 4 - n$

By induction: Basis step: $f(0) = 2$ and $f(1) = 4 - 1 = 3$

If $\quad f(k) = 4 - k$

Then $\quad f(k+1) = f(k) - 1 = 4 - k - 1 = 4 - (k+1)$

e) $f(2) = 2f(0) = 2 \qquad f(1) = 2$

$f(3) = 2f(1) = 2(2) = 4 \qquad f(4) = 2f(2) = 2(2) = 4$

$f(5) = 2f(3) = 2(4) = 8 \qquad f(6) = 2f(4) = 2(4) = 8$

Then the sequence: 1, 2, 2, 4, 4, 8, 8, … $\Rightarrow f(n) = 2^{(n+1)/2}$

By induction: Basis step: $f(0) = 2^{(0+1)/2} = 1 \quad \textit{and} \quad f(1) = 2^{(1+1)/2} = 2$ and

If $\quad f(k) = 2^{(k+1)/2}$

Then

$f(k+1) = 2f(k-1) = 2 \cdot 2^{(k-1+1)/2} = 2 \cdot 2^{k/2} = 2^{(k/2)+1} = 2^{(k+2)/2} \underline{= 2^{((k+1)+1)/2}}$

f) $f(3) = 2f(0) = 2(1) = 2 \qquad f(4) = 2f(1) = 2(0) = 0 \qquad f(5) = 2f(2) = 2(2) = 4$

$f(6) = 2f(3) = 2(2) = 4 \qquad f(7) = 2f(4) = 2(0) = 0 \qquad f(8) = 2f(5) = 2(4) = 8$

This is valid, since the values n = 0, 1, 2 are given. The sequence: 1, 0, 2, 2, 0, 4, 4, 0, 8, …

We conjecture the formula	***Prove***
$f(n) = 2^{n/3}$ when $n \equiv 0(\boldsymbol{mod}\,3)$	$f(0) = 2^{0/3} = 1$
$f(n) = 0$ when $n \equiv 1(\boldsymbol{mod}\,3)$	$f(1) = 0$
$f(n) = 2^{(n+1)/3}$ when $n \equiv 2(\boldsymbol{mod}\,3)$	$f(2) = 2^{(2+1)/3} = 2^1 = 2$

Assume the inductive hypothesis that the formula is valid for smaller inputs. Then

For $n \equiv 0(\boldsymbol{mod}\,3)$ we have $f(n) = 2f(n) = 2 \cdot 2^{(n-3)/3} = 2 \cdot 2^{n/3} \cdot 2^{-1} = 2^{n/3}$ as desired

For $n \equiv 1(\boldsymbol{mod}\,3)$ we have $f(n) = 2f(n-3) = 2 \cdot 0 = 0$ as desired

For $n \equiv 2(\boldsymbol{mod}\,3)$ we have $f(n) = 2f(n-3) = 2 \cdot 2^{(n-3+1)/3} = 2 \cdot 2^{(n+1)/3} \cdot 2^{-1} = 2^{(n+1)/3}$ as desired

g) $f(2) = 2f(3)$ This is not valid, since $f(3)$ has not been defined

h) $f(2) = 2 \cdot f(1) = 2(1) = 2$ $\qquad f(3) = 2f(2) = 2(2) = 4$

This is ***invalid***, because the value at $n = 1$ is defined in 2 conflicting ways, first as $f(1) = 1$ and then as $f(1) = 2f(1-1) = 2f(0) = 2(0) = 0$

i) $f(1) = f(0) = 2$ $\qquad f(2) = 2f(0) = 2(2) = 4$

$f(3) = f(2) = 4$ $\qquad f(4) = 2f(2) = 2(4) = 8$

This is ***invalid***, since we have a conflict for odd $n \ge 3$.

On one hand $f(3) = f(2)$, but the other hand $f(3) = 2f(1)$.

However, $f(1) = f(0) = 2$ and $f(2) = 2f(0) = 4$, so these apparently conflicting rules tell us that $f(3) = 2 \cdot 2 = 4$ on the other hand. We got the same answer either way.

The sequence: 2, 2, 4, 4, 8, 8, …

j) $f(1) = 3f(0) = 3(1) = 3$ $\qquad f(2) = 9f(0) = 9(1) = 9$

$f(3) = 3f(2) = 3(9) = 27$ $\qquad f(4) = 9f(2) = 9(9) = 81$

The sequence: 1, 3, 9, 27, 81, …

This is a valid, since we conjecture the formula $f(n) = 3^n$

By induction: Basis step: $f(0) = 3^0 = 1$

If $\quad f(k) = 3^k$

Then $f(k+1) = 3f(k) = 3 \cdot 3^k = 3^{k+1}$

Exercise

Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$ if

a) $a_n = 6n$ $\qquad$ *b)* $a_n = 2n+1$ $\qquad$ *c)* $a_n = 10^n$ $\qquad$ *d)* $a_n = 5$

e) $a_n = 4n-2$ $\qquad$ *f)* $a_n = 1+(-1)^n$ $\qquad$ *g)* $a_n = n(n+1)$ $\qquad$ *h)* $a_n = n^2$

Solution

a) $a_1 = 6$

$a_2 = 12 = 6 + 6$

$a_3 = 18 = 12 + 6$

$\vdots \qquad \vdots$

$\rightarrow \boxed{a_{n+1} = a_n + 6}$ *with* $a_1 = 6$ *for all* $n \geq 1$

b) $a_1 = 3$

$a_2 = 5 = 3 + 2$

$a_3 = 7 = 5 + 2$

$\vdots \quad \vdots$

$\rightarrow \boxed{a_{n+1} = a_n + 2}$ *with* $a_1 = 3$ *for all* $n \geq 1$

c) $a_1 = 10$

$a_2 = 10^2 = 10 \cdot 10$

$a_3 = 10^3 = 10 \cdot 10^2$

$\vdots \quad \vdots$

$\boxed{a_{n+1} = 10a_n}$ *with* $a_1 = 10$ *for all* $n \geq 1$

d) $a_1 = 5$

$a_2 = 5$

$a_3 = 5$

$\vdots \quad \vdots$

$\boxed{a_{n+1} = a_1}$ *with* $a_1 = 5,$ *for all* $n \geq 1$

e) $a_1 = 2$

$a_2 = 6 = 2 + 4$

$a_3 = 10 = 6 + 4$

$\vdots \quad \vdots$

$\boxed{a_{n+1} = a_n + 4}$ *with* $a_1 = 2,$ *for all* $n \geq 1$

f) $a_1 = 1 - 1 = 0,$

$a_2 = 1 + 1 = 2$

$a_3 = 1 - 1 = 0$

$\vdots \quad \vdots$

The sequence alternate: 0,2, 0, 2, …

$\boxed{a_n = a_{n-2}}$ *with* $a_1 = 0,\ a_2 = 2,$ *for all* $n \geq 3$

g) $a_1 = 1(2) = 2$

$a_2 = 2(3) = 6$

$a_3 = 12$

$\vdots \quad \vdots$

The sequence alternate: 2,6, 12, 20, 30, …

The difference between successive terms are 4, 6, 8, 10, ….

$\underline{a_n = a_{n-1} + 2n}|$ *with* $a_1 = 2,$ *for all* $n \ge 2$

h) $a_1 = 1^2 = 1$

$a_2 = 2^2 = 4$

$a_3 = 3^2 = 9$

$\vdots \quad \vdots$

The sequence alternate: 1, 4, 9, 16, 25, …

The difference between successive terms are 3, 5, 7, 9, ….

$\underline{a_n = a_{n-1} + 2n - 1}|$ *with* $a_1 = 1,$ *for all* $n \ge 2$

Exercise

Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ when n is a positive integer and f_n is the nth Fibonacci number.

Solution

For $n = 1$: $f_1^2 = f_1 f_2 = 1 \cdot 1 = 1$ is true since both values are 1

Assume the inductive hypothesis. Then

$$\begin{aligned} f_1^2 + f_2^2 + \cdots + f_n^2 + f_{n+1}^2 &= f_n f_{n+1} + f_{n+1}^2 \\ &= f_{n+1}\left(f_n + f_{n+1}\right) \\ &= f_{n+1} f_{n+2} \end{aligned}$$

Exercise

Prove that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ when n is a positive integer and f_n is the nth Fibonacci number.

Solution

Using the principle of mathematical induction

For $n=1$: $f_1 = f_2$ is true since both values are 1

Let assume that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$

We need to prove that $f_1 + f_3 + \cdots + f_{2n-1} + f_{2n+1} = f_{2(n+1)}$

$$f_1 + f_3 + \cdots + f_{2n-1} + f_{2n+1} = f_{2n} + f_{2n+1}$$

$$= f_{2n+2}$$ **(by the *definition* of the Fibonacci numbers)**

Exercise

Give a recursive definition of

a) The set of odd positive integers
b) The set of positive integers powers of 3
c) The set of polynomial with integer coefficients
d) The set of even integers
e) The set of positive integers congruent to 2 modulo 3.
f) The set of positive integers not divisible by 5

Solution

a) Off integers are obtained from other odd integers by adding 2.
Thus, we can define this set S as follows $1 \in S$; and if $n \in S,\ then\ \ n+2 \in S$.

b) Powers of 3 are obtained from other powers of 3 by multiplying by 3.
Thus, we can define this set S as follows $3 \in S$; and if $n \in S,\ then\ \ 3n \in S$.

c) There are several ways to do this. One that is suggested by Horner's method is as follows. We assume that the variable for these polynomials is the letter x. All integers are in S; if $p(x) \in S$ and n is any integer, then $xp(x)+n$ is in S.
Another method constructs the polynomials term by term. Its base case is to let 0 be in S; and its inductive step is to say that if $p(x) \in S$, c is an integer, and n is a nonnegative integer, then $p(x)+cx^n$ is in S.

d) Off integers are obtained from other even integers by adding 2.
Thus, we can define this set S as follows $2 \in S$; and if $n \in S,\ then\ \ n-2 \in S\ \ and\ \ n+2 \in S$.

e) The smallest positive integer congruent to 2 modulo 3 is 2, so $2 \in S$. All the others can be obtained by adding multiples of 3, so the inductive step is that $n \in S,\ then\ \ n+3 \in S$

f) The positive integers no divisible by 5 are the ones congruent to 1, 2, 3, or 4 *modulo* 5.
Thus, we can define this set S as follows $1 \in S,\ 2 \in S,\ 3 \in S,\ and\ 4 \in S$; and if $n \in S,\ then\ \ n+5 \in S$

Exercise

Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0, 0) \in S$.

Recursive step: If $(a, b) \in S$, then $(a+2,\ b+3) \in S$ *and* $(a+3,\ b+2) \in S$

a) List the elements of S produced by the first five applications of the recursive definition.

b) Use strong induction on the number of applications of the recursive step of the definition to show that $5 \mid a+b$ when $(a, b) \in S$.

c) Use structural induction to show that $5 \mid a+b$ when $(a, b) \in S$.

Solution

a) Apply each recursive step rules to the only element given in the basis step, we see that (2, 3) and (3, 2) are in S.

If we apply the recursive step to these we add (4, 6), (5, 5) and (6, 4).

The next round gives us (6, 9), (7, 8), and (9, 6). Add (8, 12), (9, 11), (10, 10), (11, 9), and (12, 8); and a fifth set of applications adds (10, 15), (11, 4), (12, 13), (13, 12), (14, 1), and (15, 10).

b) Let $P(n)$ be the statement that $5 \mid a+b$ when $(a, b) \in S$ is obtained by n applications to the recursive step.

For $n = 0$, $P(0)$ is true, since the only element of S obtained with no applications of the recursive step is (0, 0), and $5 \mid 0+0$ $\checkmark$

Assume the inductive hypothesis that $5 \mid a+b$ whenever $(a, b) \in S$ is obtained by k or fewer applications of the recursive step, and consider an element obtained with $k+1$ applications of the recursive step. Since the final application of the recursive step to an element $(a,\ b)$ must applied to an element, that $5 \mid a+b$.

We need to check that this inequality implies $5 \mid a+2+b+3$ and $5 \mid a+3+b+2$.

This is clear, since each is equivalent to $5 \mid a+b+5$ and 5 divides both $a+b$ and 5.

c) This holds for the basis step, since $5 \mid 0+0$

If this holds for $(a,\ b)$, then it also holds for the elements obtained from $(a,\ b)$ in the recursive step by the same argument as in part (*b*).

Exercise

Let *S* be the subset of the set of ordered pairs of integers defined recursively by

Basis step: (0, 0) ∈ *S*.

Recursive step: If (*a*, *b*) ∈ *S*, then $(a,\ b+1)\in S$, $(a+1,\ b+1)\in S$ *and* $(a+2,\ b+1)\in S$

a) List the elements of *S* produced by the first five applications of the recursive definition.

b) Use strong induction on the number of applications of the recursive step of the definition to show that $a\le 2b$ whenever (*a*, *b*) ∈ *S*.

c) Use structural induction to show that $a\le 2b$ whenever (*a*, *b*) ∈ *S*.

Solution

a) Apply each recursive step rules to the only element given in the basis step, we see that (0, 1), (1, 1) and (2, 1) are in *S*.

2nd step: (0, 2), (1, 2), (2, 2), (3, 2) and (4, 2).

3^{rd} step: (0,3), (1, 3), (2, 3), (3, 3), (4, 3), (5, 3) and (6, 3).

4^{th} step: (0, 4), (1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4), (7, 4) and (8,4)

5^{th} step: (0, 5), (1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (6, 5), (7, 5), (8, 5), (9, 5), and (10, 5)

b) Let $P(n)$ be the statement that $a\le 2b$ whenever (*a*, *b*) ∈ *S* is obtained with no applications of the recursive step.

For the basis step, the only element of *S* obtained with no applications of the recursive step is $(0,\ 0)$, then $0\le 2\cdot 0$ is true. Therefore $P(0)$ is true.

Assume that $a\le 2b$ whenever (*a*, *b*) ∈ *S* is obtained by *k* or fewer applications of the recursive step. Consider an element obtained with *k* + 1 applications of the recursive step.

We know that $a\le 2b$, we need to check this inequality implies $a\le 2(b+1)$, $a+1\le 2(b+1)$, and $a+2\le 2(b+1)$.

Thus is clear that $0\le 2,\ 1\le 2$ *and* $2\le 2$, respectively, to $a\le 2b$ to obtain these inequalities.

c) This holds for the basis step, since $0\le 0$.

If this holds for $(a,\ b)$, then it also holds for the elements obtained from $(a,\ b)$ in the recursive step, since adding $0\le 2,\ 1\le 2$ *and* $2\le 2$, respectively, to $a\le 2b$ yiekds $a\le 2(b+1)$, $a+1\le 2(b+1)$, and $a+2\le 2(b+1)$.

Solution *Section* 3.3 – The Basics of Counting

Exercise

There are 18 mathematics majors and 325 computer science majors at a college

a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?

b) In how many ways can one representative be picked who either a mathematics major or a computer science major?

Solution

a) $18 \cdot 325 \ \underline{= 5850 \ \ ways}$

b) $18 + 325 \ \underline{= 343 \ \ ways}$

Exercise

An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

Solution

Using the product rule: there are $27 \cdot 37 \ \underline{= 999 \ \ offices}$

Exercise

A multiple-choice test contains 10 questions. There are four possible answers for each question

a) In how many ways can a student answer the questions on the test if the student answers every question?

b) In how many ways can a student answer the questions on the test if the student can leave answers blank?

Solution

a) $4 \cdot 4 \cdot 4 \cdots 4 = 4^{10} \ \underline{= 1{,}048{,}576 \ ways}$

b) There are 5 ways to answer each question 0 give any if the 4 answers or give no answer at all

$\underline{5^{10} = 9{,}765{,}625 \ \ ways}$

Exercise

A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of the shirts are made?

Solution

$\boxed{12 \cdot 2 \cdot 3 = 72}$ different types of shirt.

Exercise

How many different three-letter initials can people have?

Solution

$\boxed{26 \cdot 26 \cdot 26 = 17{,}576}$ different initials.

Exercise

How many different three-letter initials with none of the letters repeated can people have?

Solution

$\boxed{26 \cdot 25 \cdot 24 = 15{,}600 \ \ ways}$

Exercise

How many different three-letter initials are there that begin with an A?

Solution

$\boxed{1 \cdot 26 \cdot 26 = 676 \ \ ways}$

Exercise

How many bit strings are there of length eight?

Solution

$\boxed{2^8 = 256 \ \ bit\ strings}$

Exercise

How many bit strings of length ten both begin and end with a 1?

Solution

$\boxed{1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 2^8 = 256 \ \ bit\ strings}$

Exercise

How many bit strings of length n, where n is a positive integer, start and end with 1s?

Solution

$\underline{1 \cdot 2^{n-2} \cdot 1 = 2^{n-2} \quad \text{bit strings}}$ $\qquad$ $1 \cdot 2 \cdot 2 \cdots \cdot 2 \cdot 2 \cdot 1$

Exercise

How many strings are there of lowercase letters of length four or less, not counting the empty string?

Solution

The number of strings of length 4 or less by counting the number of the strings of length $0 \le i \le 4$

There are 26 letters to choose from, and a string of length *i* is specified by choosing its characters, one after another.

The product rules there are 26^i

$$\sum_{i=0}^{4} 26^i = 1 + 26 + 26^2 + 26^3 + 26^4$$

$$= 475,255$$

Exercise

How many strings are there of four lowercase letters that have the letter *x* in them?

Solution

Number of strings of length of 4 lowercase: 26^4

Number of strings of length of 4 lowercase other than ***x***: 25^4

$26^4 - 25^4 = 66,351 \quad \text{strings}$

Exercise

How many positive integers between 50 and 100

a) Are divisible by 7? Which integers are these

b) Are divisible by 11? Which integers are these

c) Are divisible by 7 and 11? Which integers are these

Solution

a) Neither 50 nor 100 is divisible by 7

There are $\frac{50}{7} = 7$ integers less than 50 that are divisible by 7

There are $\frac{100}{7} = 14$ integers less than 100 that are divisible by 7

This leaves $14 - 7 = 7$ numbers between 50 and 100 that are divisible by 7.
They are 56, 63, 70, 77, 84, 91, and 98.

b) Neither 50 nor 100 is divisible by 11

There are $\frac{50}{11}=4$ integers less than 50 that are divisible by 11

There are $\frac{100}{11}=9$ integers less than 100 that are divisible by 1

This leaves $9-4=5$ numbers between 50 and 100 that are divisible by 11

They are 55, 66, 77, 88, and 99.

c) A number is divisible by 7 and 11 which is 77. There is only one such number between 50 and 100, namely 77.

Exercise

How many positive integers less than 100

a) Are divisible by 7?
b) Are divisible by 7 but not by 11?
c) Are divisible by both 7 and 11?
d) Are divisible by either 7 or 11?
e) Are divisible by exactly one of 7 and 11?
f) Are divisible by neither 7 nor 11?

Solution

7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98 *are divisible by* 7
11, 22, 33, 44, 55, 66, 77, 88, 99 *are divisible by* 11

a) Every 7^{th} number is divisible by 7. Therefore, $\frac{99}{7}\approx 14$ such numbers. The k^{th} multiple of 7 does not occur until the number $7k$ has been reached.

b) There are 13 such numbers since 77 is the only one divisible by 11.

c) There is only **1** number (77) divisible by both 7 and 11

d) $\frac{99}{11}\approx 9$ such numbers and 14 such numbers divisible by 7 and only 1 is divisible by 11. Therefore, there are $14+9-1 \underline{=22}$ divisible by either 7 or 11

e) The number of numbers divisible of them: $22-1 \underline{=21}$ (*subtract* (*d*) *from* (*c*))

f) Subtract part (d) from the total number of positive integers less than 100.
$99-22 \underline{=77}$

Exercise

How many positive integers less than 1000

a) Are divisible by 7?
b) Are divisible by 7 but not by 11?
c) Are divisible by both 7 and 11?
d) Are divisible by either 7 or 11?
e) Are divisible by exactly one of 7 and 11?
f) Are divisible by neither 7 nor 11?
g) Have distinct digits?
h) Have distinct digits and are even?

Solution

a) Every 7^{th} number is divisible by 7. Therefore, $\frac{999}{7} \approx 142$ such numbers. The k^{th} multiple of 7 does not occur until the number $7k$ has been reached.

b) Every 11^{th} number is divisible by 11. Therefore, $\frac{999}{11} \approx 90$ numbers.

Since 77 is the first number that is divisible by 7 and 11, and there are $\frac{999}{77} \approx 12$ numbers divisible by 77.

There are $142 - 12 \underline{= 130}$ numbers divisible by 7 but not by 11.

c) There are **12** numbers divisible by both 7 and 11 (from part ***b***)

d) There are $142 + 90 - 12 \underline{= 220}$ divisible by either 7 or 11

e) The number of numbers divisible of them: $220 - 12 \underline{= 208}$ (*subtract* (***d***) *from* (***c***))

f) Subtract part (**d**) from the total number of positive integers less than 1000.
$999 - 220 \underline{= 779}$

g) If we assume that numbers are written without leading 0's, then we can break down this part in three cases: one-digit numbers, two-digit numbers and three-digit numbers.
There are 9 one-digit numbers, and each of them has distinct digits.
There are 90 two-digit numbers (10 – 99), and all but 9 of them have distinct digits, so there are 81 two-digit numbers with distinct digits. Or the first digit 1 through 9 (9 choices), using the product rule: $9 \cdot 9 = 81$ choices in all.
For three-digit numbers there are $9 \cdot 9 \cdot 8 = 648$ distinct digits
Therefore $9 + 81 + 648 = \underline{738}$ total distinct digits.

h) If we use to count the odd numbers with distinct digits and subtract from part (***g***), we can get the numbers distinct digits and are even.
There are 5 odd one-digit numbers.
For two-digit numbers; first the ones digits (5 choices), then the tens digit (8 choices) – neither the ones digit value nor 0 is available, therefore there are 40 such two-digit numbers (half of 81).
For three-digit numbers, first the ones digits (5 choices), the hundreds digit (8 choices), then the tens digit (8 choices). There are $5 \cdot 8 \cdot 8 = 320$ distinct digits

So $5 + 40 + 320 = 365$ total odd numbers with distinct digits.

Therefore $738 - 365 \underline{= 373}$ total distinct digits.

Exercise

A committee is formed consisting of one representative from each of the 50 states in the United States, where the representative from a state is either the governor or one of the two senators from that state. How many ways are there to form this committee?

Solution

There are 50 choices to make each of which can be done in 3 ways, namely by choosing the governor, choosing the senior senator, or choosing the junior senator.

$\underline{3^{50} \approx 7.2 \times 10^{23}}$

Exercise

How many license plates can be made using either three digits followed by three uppercase English letters or three uppercase English letters followed by three digits?

Solution

$10^3 \cdot 26^3 + 26^3 \cdot 10^3 \underline{= 35{,}152{,}000 \quad \textit{license plates}}$

Exercise

How many license plates can be made using either two uppercase English letters followed by four digits or two digits followed by four uppercase English letters?

Solution

Letters		***Digits***			
L	L	D	D	D	D
26	26	10	10	10	10

$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6{,}760{,}000$

Digits		***Letters***			
D	D	L	L	L	L
10	10	26	26	26	26

$10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 45{,}697{,}600$

Therefore: $6{,}760{,}000 + 45{,}697{,}600 \underline{= 52{,}457{,}600 \quad \textit{license plates}}$

Exercise

How many license plates can be made using either three uppercase English letters followed by three digits or four uppercase English letters followed by two digits?

Solution

$$26^3 \cdot 10^3 + 26^4 \cdot 10^2 = 63{,}273{,}600 \quad \textit{license plates}$$

Exercise

How many strings of eight English letter are there.

a) That contains no vowels, if letters can be repeated?
b) That contains no vowels, if letters cannot be repeated?
c) That starts with a vowel, if letters can be repeated?
d) That starts with a vowel, if letters cannot be repeated?
e) That contains at least one vowel, if letters can be repeated?
f) That contains at least one vowel, if letters cannot be repeated?

Solution

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
	NV	**NV**	**NV**	**NV**	**NV**	**NV**	**NV**	**NV**
a	21	21	21	21	21	21	21	21
b	21	20	19	18	17	16	15	14
	V	**L**	**L**	**L**	**L**	**L**	**L**	**L**
c	5	26	26	26	26	26	26	26
d	5	25	24	23	22	21	20	19

a) There are 8 slots which can be filled with $26-5=21$ non-vowels.

By the product rule: $21^8 = 37{,}822{,}859{,}361 \quad \textit{strings}$

b) $21\cdot 20\cdot 19\cdot 18\cdot 17\cdot 16\cdot 15\cdot 14 = 8{,}204{,}716{,}800 \quad \textit{strings}$

c) $5\cdot 26^7 = 40{,}159{,}050{,}880 \quad \textit{strings}$

d) $5\cdot 25\cdot 24\cdot 23\cdot 22\cdot 21\cdot 20\cdot 19 = 12{,}113{,}640{,}000 \quad \textit{strings}$

e) By the product rule: $26^8 - 21^8 = 171{,}004{,}205{,}215 \quad \textit{strings}$

f) $8\cdot 5\cdot 21^7 = 72{,}043{,}541{,}640 \quad \textit{strings}$

Exercise

How many ways are there to seat four of a group of ten people around a circular table where two seatings are considered the same when everyone has the same immediate left and immediate right neighbor?

Solution

The count ordered arrangements of length 4 from the 10 people, then we get $10\cdot 9\cdot 8\cdot 7=5040$ arrangements.

However, we can rotate the people around the table in 4 ways and get the same seating arrangement, so the overcounts by a factor of 4.

Therefore, there are $\frac{5040}{4}=1260\ \ ways$

Exercise

In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and the groom are among these 10 people, if

a) The bride must be in the picture?

b) Both the bride and groom must be in the picture?

c) Exactly one of the bride and the groom is in the picture?

Solution

a) The bride is in any of the 6 positions.

1	**2**	**3**	**4**	**5**	**6**
B	***P***	***P***	***P***	***P***	***P***
1	9	8	7	6	5

Then, it will leave us with 5 remaining positions.

This can be done in $9\cdot 8\cdot 7\cdot 6\cdot 5=15120$ ways.

Therefore $6\cdot 15120=90,720\ \ ways$

b) The bride is in any of the 6 positions.

1	**2**	**3**	**4**	**5**	**6**
B	***G***	***P***	***P***	***P***	***P***
1	1	8	7	6	5

Then place the groom in any of the 5 remaining positions.

Then, it will leave us with 4 remaining positions in the picture.

This can be done in $8\cdot 7\cdot 6\cdot 5=1680$ ways.

Therefore $6\cdot 5\cdot 1680=50,400\ \ ways$

c) For the just the bride to be in the picture: $90720-50400=40,320$ ways.

There are 40,320 ways for just the groom to be in the picture.

Therefore, $40320+40320=80,640\ \ ways$

Exercise

How many different types of homes are available if a builder offers a choice of 6 basic plans, 3 roof styles, and 2 exterior finishes?

Solution

$6.3.2 = \underline{36\,|}$ *different homes types*

Exercise

A menu offers a choice of 3 salads, 8 main dishes, and 7 desserts. How many different meals consisting of one salad, one main dish, and one dessert are possible?

Solution

$3.8.7 = \underline{168\,|}$ ***different meals***

Exercise

A couple has narrowed down the choice of a name for their new baby to 4 first names and 5 middle names. How many different first- and middle-name arrangements are possible?

Solution

$4.5 = \underline{20\,|}$ *possible arrangements*

Exercise

An automobile manufacturer produces 8 models, each available in 7 different exterior colors, with 4 different upholstery fabrics and 5 interior colors. How many varieties of automobile are available?

Solution

$8.7.4.5 = \underline{1120\,|}$

Exercise

A biologist is attempting to classify 52,000 species of insects by assigning 3 initials to each species. Is it possible to classify all the species in this way? If not, how many initials should be used?

Solution

$26^3 = 17,576$ *This would not be enough.*

$26^4 = 456,976$ *Which is more than enough*

Exercise

How many 4-letter code words are possible using the first 10 letters of the alphabet under:

a) No letter can be repeated

b) Letters can be repeated

c) Adjacent can't be alike

Solution

a) 10.9.8.7 = 5040

b) 10.10.10.10 = 10,000

c) 10.9.9.9 = 7290

Exercise

How many 3 letters license plate without repeats

Solution

26.25.24 = 15600 possible

Exercise

How many ways can 2 coins turn up heads, H, or tails, T – if the combined outcome (H, T) is to be distinguished from the outcome (T, H)?

Solution

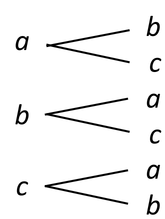
$\underline{2 \times 2 = 4 \quad outcomes}|$

Exercise

How many 2-letter code words can be formed from the first 3 letters of the alphabet if no letter can be used more than once?

Solution

$\underline{3 \times 2 = 6 \quad outcomes}|$



Exercise

A coin is tossed with possible outcomes heads, H, or tails, T. Then a single die is tossed with possible outcomes 1, 2, 3, 4, 5, or 6. How many combined outcomes are there?

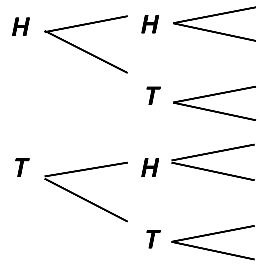
Solution

$\underline{2 \times 6 = 12 \quad outcomes}|$

Exercise

In how many ways can 3 coins turn up heads, H, or tails, T – if combined outcomes such as (H,T,H), (H, H, T), and (T, H, H) are to be considered different?

Solution



$\underline{2\times 2\times 2 = 8 \quad \textit{outcomes}}$

Exercise

An entertainment guide recommends 6 restaurants and 3 plays that appeal to a couple.

a) If the couple goes to dinner or to a play, how many selections are possible?

b) If the couple goes to dinner and then to a play, how many combined selections are possible?

Solution

a) $3 + 6 = 9$

b) $6.3 = 18$

Solution Section 3.4 – Permutations and Combinations

Exercise

Decide whether the situation involves ***permutations*** or ***combinations***

a) A batting order for 9 players for a baseball game
b) An arrangement of 8 people for a picture
c) A committee of 7 delegates chosen from a class of 30 students to bring a petition to the administration
d) A selection of a chairman and a secretary from a committee of 14 people
e) A sample of 5 items taken from 71 items on an assembly line
f) A blend of 3 spices taken from 7 spices on a spice rack
g) From the 7 male and 10 female sales representatives for an insurance company, team of 8 will be selected to attend a national conference on insurance fraud.
h) Marbles are being drawn without replacement from a bag containing 15 marbles.
i) The new university president named 3 new officers a vice-president of finance, a vice-president of academic affairs, and a vice-president of student affairs.
j) A student checked out 4 novels from the library to read over the holiday.
k) A father ordered an ice cream cone (chocolate, vanilla, or strawberry) for each of his 4 children.

Solution

a) Permutation

b) Permutation

c) Combination

d) Permutation

e) Combination

f) Combination

g) Combination

h) Combination

i) Permutation

j) Combination

k) Neither

Exercise

How many different permutations are the of the set $\{a, b, c, d, e, f, g\}$?

Solution

$$P(7,\ 7) = 5040$$

Exercise

How many permutations of $\{a, b, c, d, e, f, g\}$ end with a?

Solution

To find the permutation to with a, then we may forget about the a, and leave us $\{b, c, d, e, f, g\}$

$P(6,\ 6) \ \underline{=720}$

Exercise

Find the number of 5-permutations of a set with nine elements

Solution

$P(9,\ 5) \ \underline{=15{,}120}$ ***by Theorem***

Exercise

In how many different orders can five runners finish a race if no ties are allowed?

Solution

$P(5,\ 5) \ \underline{=120}$

Exercise

A coin flipped eight times where each flip comes up either heads or tails. How many possible outcomes

a) Are there in total?

b) Contain exactly three heads?

c) Contain at least three heads?

d) Contain the same number of heads and tails?

Solution

a) Each flip can be either heads or tails: There are $2^8 \underline{= 256 \ \ possible\ coutcomes}$

b) $C(8,\ 3) \ \underline{=56 \ \ outcomes}$

c) At least three heads means: 3, 4, 5, 6, 7, 8 heads.

$C(8,\ 3)+C(8,\ 4)+C(8,\ 5)+C(8,\ 6)+C(8,\ 7)+C(8,\ 8) \ \underline{=219 \ \ outcomes}$

OR

$256-C(8,\ 0)-C(8,\ 1)-C(8,\ 2)=256-28-8-1 \ \underline{=219 \ \ outcomes}$

d) To have an equal number of heads and tails means 4 heads and 4 tails.

Therefore; $C(8,\ 4) \ \underline{=70 \ \ outcomes}$

Exercise

A coin flipped 10 times where each flip comes up either heads or tails. How many possible outcomes

a) Are there in total?
b) Contain exactly two heads?
c) Contain at most three tails?
d) Contain the same number of heads and tails?

Solution

a) Each flip can be either heads or tails: There are $2^{10} = 1024 \ \ possible\ coutcomes$

b) $C_{10,2} = 45 \ \ outcomes$

c) At most three tails means: 3, 2, 1, 0 tails.

$C_{10,3} + C_{10,2} + C_{10,1} + C_{10,0} = 176 \ \ outcomes$

d) To have an equal number of heads and tails means 5 heads and 5 tails.

Therefore; $C_{10,5} = 252 \ \ outcomes$

Exercise

How many bit strings of length 12 contain?

a) Exactly three 1s?
b) At most three 1s?
c) At least three 1s?
d) An equal number of 0s and 1s?

Solution

a) We need to choose the 3 positions that contains the 1's

$C_{12,\ 3} = 220 \ \ ways$

b) At most three 1's means to contains 3, 2, 1, 0 –1's:

$C_{12,3} + C_{12,2} + C_{12,1} + C_{12,0} = 299 \ \ strings$

c) At least three 1's means to contains 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 –1's:

$C_{12,3} + C_{12,4} + C_{12,5} + C_{12,6} + C_{12,7} + C_{12,8} + C_{12,9} + C_{12,10} + C_{12,11} + C_{12,12} = 4017 \ \ strings$

OR

$2^{12} - C_{12,2} - C_{12,1} - C_{12,0} = 4096 - 66 - 12 - 1 = 4017 \ \ strings$

d) To have an equal number of 0's and 1's means 6 1's.

Therefore; $C(12,\ 6) = 924 \ \ strings$

Exercise

A group contains *n* men and *n* women. How many ways are there to arrange these people in a row if the men and women alternate?

Solution

Consider the order in which the men appear relative to each other. There are *n* men $P(n,n)=n!$ arrangements is allowed.

Consider the order in which the women appear relative to each other. There are *n* women $P(n,n)=n!$ arrangements is allowed.

Men and women must alternate, and there are the same number of men and women; therefore there are exactly 2 possibilities: either the row with a man ends with a woman ***or*** it starts with a woman ends with a man.

By the product rule there are $n!\ n!\ 2 = 2(n!)^2 \ \ ways$

Exercise

In how many ways can a set of two positive integers less than 100 be chosen?

Solution

$C_{99,\ 2} = 4851 \ \ ways$

Exercise

In how many ways can a set of five letters be selected from the English alphabet?

Solution

$C_{26,\ 5} = 65,780 \ \ ways$

Exercise

How many subsets with an odd number of elements does a set with 10 elements have?

Solution

$C_{10,1} + C_{10,3} + C_{10,5} + C_{10,7} + C_{10,9} = 512 \ \ subsets$

Exercise

How many subsets with more than two elements does a set with 100 elements have?

Solution

There are 2^{100} subsets of a set with 100 elements. All of them have more than 2 subsets except the empty set, the 100 subsets consisting of one element each, and $C_{100,\ 2} = 4950$ *subsets* with 2 elements.

Therefore; $2^{100} - 4950 \underline{= 1.26\times 10^{30}}$

Exercise

How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other?

Solution

First position the men relative to each other. Since there are 8 men, there are $P(8,8)$ ways to do this.

This creates 9 slots where a woman may stand: in front of the first man, between the first and second men, …, between the 7th and 8 men, and behind the 8th man.
We need to choose 5 of these positions, in order, for the first through 5th woman to occupy.

Therefore, $P(8,8)\cdot P(9,5) \underline{= 609,638,400\ \ ways}$

Exercise

How many ways are there for six men and 10 women to stand in a line so that no two men stand next to each other?

Solution

16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
W	W	W	W	W	M	W	M	W	M	W	M	W	M	W	M

Since there are 10 women, there are $P(10,10) = 3,628,800$

This creates 11 slots where a man may stand.
This can be done is $P(11,6) = 332,640$

Therefore $P(10,10)\cdot P(11,6) \underline{= 1,207,084,032,000\ \ ways}$

Exercise

A professor writes 40 discrete mathematics true/false questions of the statements in these questions. 17 are true. If the questions can be positioned in any order, how many different answer keys are possible?

Solution

$$C_{40,17} \underline{= 8.9 \times 10^{10} \ \ answers}|$$

Exercise

Thirteen people on a softball team show up for a game.

a) How many ways are there to choose 10 players to take the field?

b) How many ways are there to assign the 10 positions by selecting players from the 13 people who show up?

c) Of the 13 people who show up, there are three women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?

Solution

a) $C_{13,10} \underline{= 286 \ \ ways}|$

b) The order in which the choices are made: $P_{13,10} \underline{= 1,037,836,800 \ \ ways}|$

c) There is only one way to choose the 10 players without choosing a woman, since there are exactly 10 men.

Therefore, there are $286 - 1 \underline{= 285 \ \ ways}|$ to choose the players if at least one of them must be a woman.

Exercise

A club has 25 members

a) How many ways are there to choose four members of the club to serve on an executive committee?

b) How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office?

Solution

a) Since the order of choosing the members is not relevant, we need to use a combination

$C(25,4) \underline{= 12,650 \ \ ways}|$

b) Since the order of choosing the members is matter, we need to use a permutation.

$P(25,4) \underline{= 303,600 \ \ ways}|$

Exercise

How many 4-permutations of the positive integers not exceeding 100 contain three consecutive integers, k, $k + 1$, $k + 2$, in the order

a) Where these consecutive integers can perhaps be separated by other integers in the permutation?
b) Where they are in consecutive positions in the permutation?

Solution

a) The consecutive numbers are 5, 6, 7, since can be separate by other integers the permutation can be written as 5, 6, 32, 7.
In order to specify such 4-permutation, we need to choose 3 consecutive integers. They can be {1, 2, 3} to (98, 99, 100}; thus, there are 98 possibilities. There are 4 possibilities, we need to decide which 97 other positive integers not exceeding 100 is to fill this slot, and there are 97 choices.
In fact, every 4-permutation consisting of 4 consecutive numbers, in order, has been double counted.
Therefore, we need to subtract the number of such 4-permutations. Clearly there are 97 of them.
Further thought shows that every other 4-permutation in our collection arises in a unique way.
Therefore $98 \cdot 4 \cdot 97 - 97 \underline{= 37,927}|$

b) The consecutive numbers be consecutive in the 4-permutation.
There are only 2 places to put the fourth number in slot 1 and slot 4.
Therefore, $98 \cdot 2 \cdot 97 - 97 \underline{= 18,915}|$

Exercise

The English alphabet contains 21 constants and five vowels. How many strings of six lowercase letters of the English alphabet contain?

a) Exactly one vowel?
b) Exactly two vowels?
c) At least one vowel?
d) At least two vowels?

Solution

a) This can be done 6 ways. We need to choose the vowel and this can be done in 5 ways. Each other 5 positions can contain any of the 21 consonants, so there are 21^5 ways to fill the rest of the string.
Therefore, the answer is $6 \cdot 5 \cdot 21^5 \underline{= 122,533,030 \ \ ways}|$

b) The position of the vowels can be done in $C(6,\ 2) = 15\ ways$. We need to choose the 2 vowels in 5^2 ways. Each other 4 positions can contain any of the 21 consonants, so there are 21^4 ways to fill the rest of the string.
Therefore, the answer is $15 \cdot 5^2 \cdot 21^4 \underline{= 72,930,375 \ \ ways}|$

c) Count the number of strings with no vowels and subtract this from the total number of stings.
$26^6 - 21^6 \underline{= 223,149,655 \ \ ways}|$

d) Subtracting the total number of strings from the number of strings with no vowels and the number of strings with one vowel. Answer: $26^6 - 21^6 - 6 \cdot 5 \cdot 21^5 \underline{= 100,626,625 \quad ways}|$

Exercise

Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have

a) The same number of men and women?

b) More women than men?

Solution

a) $C_{10,3} \cdot C_{15,3} \underline{= 54,600 \quad ways}|$

b) There are $C_{15,6}$ *ways* to choose the committee to be composed only of women

$C_{15,5} \cdot C_{10,1}$ *ways* if there are to be five women and one man, and $C_{15,4} \cdot C_{10,2}$ *ways* if there are to be four women and two men.

Therefore, $C_{15,6} + C_{15,5} \cdot C_{10,1} + C_{15,4} \cdot C_{10,2} \underline{= 96,460 \quad ways}|$

Exercise

How many bit strings contain exactly eight 0s and 10 1s if every 0 must be immediately followed by a 1?

Solution

0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	1

There are 8 blocks consisting of the string 01

$C_{10,2} \underline{= 45 \quad ways}|$

Exercise

How many bit strings contain exactly five 0s and 14 1s if every 0 must be immediately followed by two 1s?

Solution

Glue 2 1's to the right of each 0, giving a collection of 9 tokens: five 001's and four 1's.

$C_{9,4} \underline{= 126 \quad ways}|$

Exercise

A concert to raise money for an economics prize is to consist of 5 works; 2 overtures, 2 sonatas, and a piano concerto.

a) In how many ways can the program be arranged?
b) In how many ways can the program be arranged if an overture must come first?

Solution

a) $P(5,5)\ \underline{=120\ \ ways}|$

b) $P(2,1)\cdot P(4,4)\ \underline{=48\ \ ways}|$

Exercise

A zydeco band from Louisiana will play 5 traditional and 3 original Cajun compositions at a concert. In how many ways can they arrange the program if

a) The begin with a traditional piece?
b) An original piece will be played last?

Solution

a) $P(5,1)\cdot P(7,7)\underline{=25,200\ \ ways}|$

b) $P(7,7)\cdot P(3,1)\ \underline{=15,120\ \ ways}|$

Exercise

In an election with 3 candidates for one office and 6 candidates for another office, how many different ballots may be printed?

Solution

Office **1**: $P(3,3)$

Office **2**: $P(6,6)$

Multiplication principle: $2\cdot P(3,3)P(6,6)\ \underline{=8640}|$

Exercise

A business school gives courses in typing, shorthand, transcription, business English, technical writing, and accounting. In how many ways can a student arrange a schedule if 3 courses are taken? assume that the order in which courses are schedules matters.

Solution

$P(6,3)\ \underline{=120\ \ ways}|$

Exercise

If your college offers 400 courses, 25 of which are in mathematics, and your counselor arranges your schedule of 4 courses by random selection, how many schedules are possible that do not include a math course? Assume that the order in which courses are scheduled matters.

Solution

$$P(nonmath) = P(375,4) \; \underline{= 1.946 \times 10^{10}}$$

Exercise

A baseball team has 19 players. How many 9-player batting orders are possible?

Solution

$$P(19,9) \; \underline{= 3.352 \times 10^{10}}$$

Exercise

A chapter of union Local 715 has 35 members. In how many different ways can the chapter select a president, a vice-president, a treasurer, and a secretary?

Solution

$$P(35,4) \; \underline{= 1,256,640}$$

Exercise

An economics club has 31 members.

a) If a committee of 4 is to be selected, in how many ways can the selection be made?
b) In how many ways can a committee of at least 1 and at most 3 be selected?

Solution

a) $\underline{C_{31,4} = 31,465}$

b) $$\begin{aligned} P\left(at\ least\ 1\ and\ at\ most\ 3\ be\ selected\right) &= C_{31,1} + C_{31,2} + C_{31,3} \\ &= 31 + 465 + 4495 \\ &\underline{= 4991} \end{aligned}$$

Exercise

In a club with 9 male and 11 female members, how many 5-member committees can be chosen that have

a) All men?
b) All women?
c) 3 men and 2 women?

Solution

a) $C(9,5)\ \boxed{=126}$

b) $C(11,5)\ \boxed{=462}$

c) $C(9,3)\cdot C(11,2)=(84)(55)\ \boxed{=4,620}$

Exercise

In a club with 9 male and 11 female members, how many 5-member committees can be selected that have

a) At least 4 women?
b) No more than 2 men?

Solution

a) $C(11,4)\cdot C(9,1)+C(11,5)\cdot C(9,0)\ \boxed{=3,432}$

b) $C(9,0)\cdot C(11,5)+C(9,1)\cdot C(11,4)+C(9,2)\cdot C(11,3)\ \boxed{=9,372}$

Exercise

In a game of musical chairs, 12 children will sit in 11 chairs arranged in a row (one will be left out). In how many ways can this happen, if we count rearrangements of the children in the chairs as different outcomes?

Solution

$P(12,11)\ \boxed{=479,001,600\quad \textit{different outcomes}}$

Exercise

A group of 3 students is to be selected from a group of 14 students to take part in a class in cell biology.

a) In how many ways can this be done?
b) In how many ways can the group who will not take part be chosen?

Solution

a) $\binom{14}{3}\ \boxed{=364\ \ ways}$

b) $\binom{14}{11}\ \boxed{=364\ \ ways}$

Exercise

Marbles are being drawn without replacement from a bag containing 16 marbles.

a) How many samples of 2 marbles can be drawn?
b) How many samples of 2 marbles can be drawn?
c) If the bag contains 3 yellow, 4 white, and 9 blue marbles, how many samples of 2 marbles can be drawn in which both marbles are blue?

Solution

a) $C(16,2) = 120 \ \ samples$

b) $C(16,4) = 1820 \ \ samples$

c) $C(9,2) = 36 \ \ samples$

Exercise

A bag contains 5 black, 1 red, and 3 yellow jelly beans; you take 3 at random. How many samples are possible in which the jelly beans are

a) All black?
b) All red?
c) All yellow?
d) 2 black and 1 red?
e) 2 black and 1 yellow?
f) 2 yellow and 1 black?
g) 2 red and 1 yellow?

Solution

a) $C_{5,3} = 10$

b) No 3 read. $C_{1,3} = 0$

c) $C_{3,3} = 1$

d) $C_{5,2}C_{1,1} = 10$

e) $C_{5,2}C_{3,1} = 30$

f) $C_{3,2}C_{5,1} = 15$

g) *There is only* **1** ***red***.

Solution Section 3.5 – Applications of Recurrence Relations

Exercise

a) Find a recurrence relation for the number of permutation of a set with *n* elements

b) Use the recurrence relation to find the number of permutations of a set with *n* elements using iteration.

Solution

a) A permutation of a set with *n* elements of a choice of a first element, followed by a permutation of a set of $n-1$ elements. Therefore $P_n = nP_{n-1} \quad with \quad P_0 = 1$

b)
$$\begin{aligned} P_n &= nP_{n-1} \\ &= n(n-1)P_{n-2} \\ &= n(n-1)\cdots 2\cdot 1\cdot P_0 \\ &= n! \end{aligned}$$

Exercise

A vending machine dispensing books of stamps accepts only one-dollar coins, \$1 bills, and \$5 bills.

a) Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the coins and bills are deposited matter.

b) What are the initial conditions?

c) How many ways are there to deposit \$10 for a book of stamps?

Solution

a) Let a_n be the number of ways to deposit *n* dollars in the vending machine. We must express a_n in terms of earlier terms in the sequence. If we want to deposit *n* dollars, we may start with a dollar coin and then deposit *n* – 1 dollars. This gives us a_{n-1} ways to deposit *n* dollars.

We can start with a dollar bill and then deposit *n* – 1 dollars. This gives us a_{n-1} more ways to deposit *n* dollars.

Finally, we can deposit a five-dollar bill and follow that with *n* – 5 dollars; there are a_{n-5} ways to do this, Therefore the recurrence relation is $a_n = 2a_{n-1} + a_{n-5} \quad for \quad n \ge 5$

b) We need initial conditions for all *n* from 0 to 4. Clearly, $a_0 = 1$ (deposit nothing) and $a_1 = 2$ (deposit either the dollar coin or the dollar bill)

$a_2 = 2^2 = 4; \quad a_3 = 2^3 = 8 \quad and \quad a_4 = 2^4 = 16$

c)
$$\begin{aligned} a_5 &= 2a_4 + a_0 = 2(16) + 1 = 33 \\ a_6 &= 2a_5 + a_1 = 2(33) + 2 = 68 \\ a_7 &= 2a_6 + a_2 = 2(68) + 4 = 140 \end{aligned}$$

$a_8 = 2a_7 + a_3 = 2 \cdot 140 + 8 = 288$

$a_9 = 2a_8 + a_4 = 2 \cdot 288 + 16 = 592$

$a_{10} = 2a_8 + a_5 = 2 \cdot 592 + 33 = 1217$

Therefore, there are 1217 ways to deposit \$10.

Exercise

a) Find a recurrence relation for the number of bit strings of length *n* that contain three consecutive 0s.

b) What are the initial conditions?

c) How many bit strings of length seven contain three consecutive 0s?

Solution

a) Let a_n be the number of bit strings of length *n* containing three consecutive 0's. In order to construct a bit string of length *n* containing three consecutive 0's we could start with 1 and follow with a string of length *n* – 1 three consecutive 0's, or we could start with a 01 and follow with a string of length *n* – 2 three consecutive 0's, or we could start with a 001 and follow with a string of length *n* – 3 three consecutive 0's, or we could start with a 000 and follow with a string of length *n* – 3.

These 4 cases are mutually exclusive and exhaust the possibilities for how the string might start. We can write down the recurrence relation, valid for all $n \ge 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$

b) There are no bit strings of length 0, 1, or 2 containing 3 consecutive 0's, so the initial conditions are $a_0 = a_1 = a_2 = 0$

c) $a_3 = a_2 + a_1 + a_0 + 2^0$

$= 0 + 0 + 0 + 1$

$\underline{= 1}$

$a_4 = a_3 + a_2 + a_1 + 2^1$

$= 1 + 0 + 0 + 2$

$\underline{= 3}$

$a_5 = a_4 + a_3 + a_2 + 2^2$

$= 3 + 1 + 0 + 4$

$\underline{= 8}$

$a_6 = a_5 + a_4 + a_3 + 2^3$

$= 8 + 3 + 1 + 8$

$\underline{= 20}$

$a_7 = a_6 + a_5 + a_4 + 2^4$

$= 20 + 8 + 3 + 16$

$\underline{= 47}|$

Therefore, there are 47 bits of length 7 containing three consecutive 0's.

Exercise

a) Find a recurrence relation for the number of bit strings of length *n* that do not contain three consecutive 0s.

b) What are the initial conditions?

c) How many bit strings of length seven do not contain three consecutive 0s?

Solution

a) Let a_n be the number of bit strings of length *n* do not contain three consecutive 0's. In order to construct a bit string of length *n* of this type we could start with 1 and follow with a string of length *n* – 1 not containing three consecutive 0's, or we could start with 01 and follow with a string of length *n* – 2 not containing three consecutive 0's, or we could start with a 001 and follow with a string of length *n* – 3 not containing three consecutive 0's.
These 3 cases are mutually exclusive and exhaust the possibilities for how the string might start since it cannot start 000.
We can write down the recurrence relation, valid for all $n \geq 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3}$

b) There are no bit strings of length 0, 1, or 2 containing 3 consecutive 0's, so the initial conditions are $a_0 = 1$; $a_1 = 2$ *and* $a_2 = 4$

c) $a_3 = a_2 + a_1 + a_0$

$= 4 + 2 + 1$

$\underline{= 7}|$

$a_4 = a_3 + a_2 + a_1$

$= 7 + 4 + 2$

$\underline{= 13}|$

$a_5 = a_4 + a_3 + a_2$

$= 13 + 7 + 4$

$\underline{= 24}|$

$a_6 = a_5 + a_4 + a_3$

$= 24 + 13 + 7$

$\underline{= 44}|$

$a_7 = a_6 + a_5 + a_4$

$= 44 + 24 + 13$

$\underline{= 81}|$

Therefore, there are 81 bits of length 7 that do not contain three consecutive 0's.

Exercise

a) Find a recurrence relation for the number of ways to climb *n* stairs if the person climbing the stairs can take one stair or two stairs at a time.

b) What are the initial conditions?

c) In how many can this person climb a flight of eight stairs

Solution

a) Let a_n be the number of ways to climb *n* stairs. In order to climb *n* stairs, a person must either start with a step of one stair and climb $n-1$ stairs $\left(a_{n-1}\right)$ or else start with a step of two stairs and then climb $n-2$ stairs $\left(a_{n-2}\right)$ or else start with a step of two stairs and then climb $n-3$ stairs $\left(a_{n-3}\right)$.

From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3$: $\quad a_n = a_{n-1} + a_{n-2} + a_{n-3}$

b) The initial conditions are $a_0 = 1, \quad a_1 = 1 \quad and \quad a_2 = 2$, since there is one way to climb no stairs (do nothing), clearly only one way to climb one stair, and two ways to climb stairs (one step twice or two steps at once).

c) Each term in our sequence $\left\{a_n\right\}$ is the sum of the previous three terms, so the sequence begins

$a_0 = 1,\ a_1 = 1,\ a_2 = 2,\ a_3 = 4,\ a_4 = 7,\ a_5 = 13,\ a_6 = 24,\ a_7 = 44,\ a_8 = 81$

Thus, a person can climb a flight of 8 stairs in 81 ways under the restrictions in this problem.

Solution   *Section* 3.6 – Solving Linear Recurrence Relations

Exercise

Determine which of these are linear and homogeneous recurrence relations with constant coefficients. Also find the degree of those that are

a) $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$

b) $a_n = 2na_{n-1} + a_{n-2}$

c) $a_n = a_{n-1} + a_{n-4}$

d) $a_n = a_{n-1} + 2$

e) $a_n = a_{n-1}^2 + a_{n-2}$

f) $a_n = a_{n-2}$

g) $a_n = a_{n-1} + n$

h) $a_n = 3a_{n-2}$

i) $a_n = 3$

j) $a_n = a_{n-1}^2$

k) $a_n = a_{n-1} + 2a_{n-3}$

l) $a_n = \dfrac{a_{n-1}}{n}$

Solution

a) Linear (terms a_i all to the first power), has constant coefficients (3, 4 and 5), and is homogeneous (no terms are functions of just n); has degree 3

b) Linear (terms a_i all to the first power), doesn't have constant coefficients ($2n$), and is homogeneous

c) Linear, homogeneous, with constant coefficients; degree 4

d) Linear with constant coefficients, not homogeneous because of 2

e) Not linear since a_{n-1}^2

f) Linear, homogeneous, with constant coefficients; degree 2

g) Linear but not homogeneous because of the n.

h) Linear, homogeneous, with constant coefficients; degree 2

i) Linear with constant coefficients, not homogeneous because of 3

j) Not linear since a_{n-1}^2

k) Linear, homogeneous, with constant coefficients; degree 3

l) Linear with constant coefficients, not homogeneous

Exercise

Solve these recurrence relations together with the initial conditions given

a) $a_n = 2a_{n-1}$ *for* $n \geq 1,\ a_0 = 3$

b) $a_n = 5a_{n-1} - 6a_{n-2}$ *for* $n \geq 2,\ a_0 = 1,\ a_1 = 0$

c) $a_n = 4a_{n-1} - 4a_{n-2}$ *for* $n \geq 2,\ a_0 = 6,\ a_1 = 8$

d) $a_n = 4a_{n-2}$ *for* $n \geq 2,\ a_0 = 0,\ a_1 = 4$

e) $a_n = \dfrac{a_{n-2}}{4}$ *for* $n \geq 2,\ a_0 = 1,\ a_1 = 0$

f) $a_n = a_{n-1} + 6a_{n-2}$ *for* $n \geq 2,\ a_0 = 3,\ a_1 = 6$

g) $a_n = 7a_{n-1} - 10a_{n-2}$ *for* $n \geq 2,\ a_0 = 2,\ a_1 = 1$

h) $a_n = -6a_{n-1} - 9a_{n-2}$ *for* $n \geq 2,\ a_0 = 3,\ a_1 = -3$

i) $a_{n+2} = -4a_{n-1} + 5a_n$ *for* $n \geq 0,\ a_0 = 2,\ a_1 = 8$

Solution

a) The characteristic polynomial is $r - 2 = 0 \quad \Rightarrow \quad r = 2$

The general solution: $a_n = \alpha_1 2^n$

$3 = \alpha_1 2^0 \quad \rightarrow \quad \alpha_1 = 3$

Therefore, the solution is $\boxed{a_n = 3 \cdot 2^n}$

b) The characteristic polynomial is $r^2 - 5r + 6 = 0 \quad \Rightarrow \quad r = 2,\ 3$

The general solution: $a_n = \alpha_1 2^n + \alpha_2 3^n$

$$\left.\begin{aligned} 1 = \alpha_1 2^0 + \alpha_2 3^0 \quad &\rightarrow \quad 1 = \alpha_1 + \alpha_2 \\ 0 = \alpha_1 2^1 + \alpha_2 3^1 \quad &\rightarrow \quad 0 = 2\alpha_1 + 3\alpha_2 \end{aligned}\right. \quad \Rightarrow \quad \alpha_1 = 3, \quad \alpha_2 = -2$$

Therefore, the solution is $\boxed{a_n = 3 \cdot 2^n - 2 \cdot 3^n}$

c) The characteristic polynomial is $r^2 - 4r + 4 = 0 \quad \Rightarrow \quad r = 2,\ 2$

The general solution: $a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n$

$$6=\alpha_1 2^0+\alpha_2(0)2^0 \quad\to\quad 6=\alpha_1$$
$$8=\alpha_1 2^1+\alpha_2(1)2^1 \quad\to\quad 8=2\alpha_1+2\alpha_2 \qquad \Rightarrow\quad \alpha_1=6,\quad \alpha_2=-2$$

Therefore, the solution is $a_n=6\cdot 2^n-2n\cdot 2^n \ \boxed{=(6-2n)2^n}$

d) The characteristic polynomial is $r^2-4=0 \quad\Rightarrow\quad r=\pm 2$

The general solution: $a_n=\alpha_1(-2)^n+\alpha_2 2^n$

$$0=\alpha_1(-2)^0+\alpha_2 2^0 \quad\to\quad 0=\alpha_1+\alpha_2$$
$$4=\alpha_1(-2)^1+\alpha_2 2^1 \quad\to\quad 4=-2\alpha_1+2\alpha_2 \qquad \Rightarrow\quad \alpha_1=-1,\quad \alpha_2=1$$

Therefore, the solution is $\boxed{a_n=2^n-(-2)^n}$

e) The characteristic polynomial is $r^2-\frac{1}{4}=0 \quad\Rightarrow\quad r=\pm\frac{1}{2}$

The general solution: $a_n=\alpha_1\left(-\frac{1}{2}\right)^n+\alpha_2\left(\frac{1}{2}\right)^n=\alpha_1(-2)^{-n}+\alpha_2(2)^{-n}$

$$1=\alpha_1\left(-\frac{1}{2}\right)^0+\alpha_2\left(\frac{1}{2}\right)^0 \quad\to\quad 1=\alpha_1+\alpha_2$$
$$0=\alpha_1\left(-\frac{1}{2}\right)^1+\alpha_2\left(\frac{1}{2}\right)^1 \quad\to\quad 0=-\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2 \qquad \Rightarrow\quad \alpha_1=\frac{1}{2},\quad \alpha_2=\frac{1}{2}$$

Therefore, the solution is $a_n=\frac{1}{2}\left(-\frac{1}{2}\right)^n+\frac{1}{2}\left(\frac{1}{2}\right)^n$

$$\boxed{=\left(\frac{1}{2}\right)^{n+1}-\left(-\frac{1}{2}\right)^{n+1}}$$

f) The characteristic polynomial is $r^2-r-6=0 \quad\Rightarrow\quad r=-2,\ 3$

The general solution: $a_n=\alpha_1(-2)^n+\alpha_2 3^n$

$$3=\alpha_1(-2)^0+\alpha_2 3^0 \quad\to\quad 3=\alpha_1+\alpha_2$$
$$6=\alpha_1(-2)^1+\alpha_2 3^1 \quad\to\quad 6=-2\alpha_1+3\alpha_2 \qquad \Rightarrow\quad \alpha_1=\frac{3}{5},\quad \alpha_2=\frac{12}{5}$$

Therefore, the solution is $\boxed{a_n=\frac{3}{5}(-2)^n+\frac{12}{5}3^n}$

g) The characteristic polynomial is $r^2-7r+10=0 \quad\Rightarrow\quad r=2,\ 5$

The general solution: $a_n=\alpha_1 2^n+\alpha_2 5^n$

$$2=\alpha_1 2^0+\alpha_2 5^0 \quad\to\quad 2=\alpha_1+\alpha_2$$
$$1=\alpha_1 2^1+\alpha_2 5^1 \quad\to\quad 1=2\alpha_1+5\alpha_2 \qquad \Rightarrow\quad \alpha_1=3,\quad \alpha_2=-1$$

Therefore, the solution is $\boxed{a_n=3\cdot 2^n-5^n}$

h) The characteristic polynomial is $r^2+6r+9=0 \quad \Rightarrow \quad r=-3,\ -3$

The general solution: $a_n=\alpha_1(-3)^n+\alpha_2 n(-3)^n$

$$\left.\begin{aligned} 3=\alpha_1(-3)^0+\alpha_2(0)(-3)^0 \quad &\rightarrow \quad 3=\alpha_1 \\ -3=\alpha_1(-3)^1+\alpha_2(1)(-3)^1 \quad &\rightarrow \quad -3=-3\alpha_1+-3\alpha_2 \end{aligned}\right\} \Rightarrow \quad \alpha_1=3, \quad \alpha_2=-2$$

Therefore, the solution is $a_n=3\cdot(-3)^n-2n(-3)^n=(3-2n)(-3)^n$

i) The characteristic polynomial is $r^2+4r-5=0 \quad \Rightarrow \quad r=-5,\ 1$

The general solution: $a_n=\alpha_1(-5)^n+\alpha_2 1^n=\alpha_1(-5)^n+\alpha_2$

$$\left.\begin{aligned} 2=\alpha_1(-5)^0+\alpha_2 \quad &\rightarrow \quad 2=\alpha_1+\alpha_2 \\ 8=\alpha_1(-5)^1+\alpha_2 \quad &\rightarrow \quad 8=-5\alpha_1+\alpha_2 \end{aligned}\right\} \Rightarrow \quad \alpha_1=-1, \quad \alpha_2=3$$

Therefore, the solution is $a_n=-(-5)^n+3$

Exercise

How many different messages can be transmitted in n microseconds using three different signals if one signal requires 1 microsecond for transmittal, the other two signals require 2 microseconds each for transmittal, and a signal in a message is followed immediately by the next signal?

Solution

The model is the recurrence relation $a_n=a_{n-1}+a_{n-2}+a_{n-2}=a_{n-1}+2a_{n-2}$ with $a_0=a_1=1$

The characteristic polynomial is $r^2-r-2=0$

So, the roots are -1, and 2

The general solution: $a_n=\alpha_1(-1)^n+\alpha_2 2^n$

Plugging in initial conditions gives

$$\left.\begin{aligned} 1=\alpha_1(-1)^0+\alpha_2 2^0 \quad &\rightarrow \quad 1=\alpha_1+\alpha_2 \\ 1=\alpha_1(-1)^1+\alpha_2 2^1 \quad &\rightarrow \quad 1=-\alpha_1+2\alpha_2 \end{aligned}\right. \quad \Rightarrow \quad \alpha_1=\tfrac{1}{3}, \quad \alpha_2=\tfrac{2}{3}$$

Therefore, the solution is in n microseconds $a_n=\frac{1}{3}(-1)^n+\frac{2}{3}2^n$ messages can be transmitted.

Exercise

In how many ways can a $2\times n$ rectangular checkerboard be tiled using 1×2 and 2×2 pieces?

Solution

Let t_n be the number of ways like to tile a $2\times n$ board with 1×2 and 2×2 pieces. To obtain the recurrence relation, imagine what tiles are placed at the left-hand end of the board. We can place a 2×2 tile there, leaving a $2\times(n-2)$ board to be tiled, which of course can be done in t_{n-2} ways.

We can place a 1×2 tile at the edge, oriented vertically, leaving $2\times(n-1)$ board to be tiled, which of course can be done in t_{n-1} ways.

Finally, we can place two 1×2 tiles horizontally, one above the other, leaving a $2\times(n-2)$ board to be tiled, which of course can be done in t_{n-2} ways. These 3 possibilities are disjoint.

Therefore, our recurrence relation is $t_n = t_{n-1} + 2t_{n-2}$

The initial conditions are $t_0 = t_1 = 1$, since there is only one way to tile as 2×0 board and 2×1 board.

This recurrence relation has characteristic roots -1 and 2.

So, the general solution is $t_n = \alpha_1 (-1)^n + \alpha_2 2^n$

Plugging in initial conditions gives

$$\begin{aligned} 1 &= \alpha_1 (-1)^0 + \alpha_2 2^0 \ \rightarrow\ 1 = \alpha_1 + \alpha_2 \\ 1 &= \alpha_1 (-1)^1 + \alpha_2 2^1 \ \rightarrow\ 1 = -\alpha_1 + 2\alpha_2 \end{aligned} \quad \Rightarrow \quad \alpha_1 = \frac{1}{3}, \quad \alpha_2 = \frac{2}{3}$$

Therefore, the solution is $a_n = \frac{1}{3}(-1)^n + \frac{2}{3}\cdot 2^n$

$$= \frac{(-1)^n}{3} + \frac{2^{n+1}}{3}$$

Exercise

Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3} \quad for \quad n \ge 3,\ a_0 = 3,\ a_1 = 6 \ and\ a_2 = 0$

Solution

$$a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$$

The characteristic polynomial is $r^3 - 2r^2 - r + 2 = 0$

That implies to: $r^2(r-2) - (r-2) = (r-2)\left(r^2 - 1\right) = 0$

So, the roots are 1, -1, and 2

The general solution:

$$\begin{aligned} a_n &= \alpha_1 1^n + \alpha_2 (-1)^n + \alpha_3 2^n \\ &= \alpha_1 + \alpha_2 (-1)^n + \alpha_3 2^n \end{aligned}$$

Plugging in initial conditions gives

$$3 = \alpha_1 + \alpha_2 (-1)^0 + \alpha_3 2^0 \ \rightarrow\ 3 = \alpha_1 + \alpha_2 + \alpha_3$$

$$6 = \alpha_1 + \alpha_2 (-1)^1 + \alpha_3 2^1 \quad \to \quad 6 = \alpha_1 - \alpha_2 + 2\alpha_3$$

$$\Rightarrow \ \alpha_1 = 6, \quad \alpha_2 = -2, \quad \alpha_3 = -1$$

$$0 = \alpha_1 + \alpha_2 (-1)^2 + \alpha_3 2^2 \quad \to \quad 0 = \alpha_1 + \alpha_2 + 4\alpha_3$$

Therefore, the solution is $\boxed{a_n = 6 - 2(-1)^n - 2^n}$

Exercise

Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ *with* $a_0 = 9,\ a_1 = 10$ *and* $a_2 = 32$

Solution

This is a third-degree recurrence relation.

The characteristic polynomial is $r^3 - 7r - 6 = 0$

By the rational root test, the possible rational roots are $\pm\left\{\frac{6}{1}\right\} = \pm\{1,\ 2,\ 3,\ 6\}$

We find that $r = -1$ (using calculator).

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array} \qquad Q(x) = r^2 - r - 6 = (r+2)(r-3)$$

$$r^3 - 6r^2 + 12r - 8 = (r+1)(r+2)(r-3) = 0$$

So, the roots are $-2, -1$, and 3.
The general solution:

$$a_n = \alpha_1 (-2)^n + \alpha_2 (-1)^n + \alpha_3 3^n$$

Plugging in initial conditions gives

$$a_0 = 9 = \alpha_1 (-2)^0 + \alpha_2 (-1)^0 + \alpha_3 3^0 \quad \to \quad 9 = \alpha_1 + \alpha_2 + \alpha_3$$

$$a_1 = 10 = \alpha_1 (-2)^1 + \alpha_2 (-1)^1 + \alpha_3 3^1 \quad \to \quad 10 = -2\alpha_1 - \alpha_2 + 3\alpha_3$$

$$a_2 = 32 = \alpha_1 (-2)^2 + \alpha_2 (-1)^2 + \alpha_3 3^2 \quad \to \quad 32 = 4\alpha_1 + \alpha_2 + 9\alpha_3$$

The solution to the system of equations is $\alpha_1 = -3,\ \alpha_2 = 8$ *and* $\alpha_3 = 4$

Therefore, the specific solution is $\boxed{a_n = -3(-2)^n + 8(-1)^n + 4 \cdot 3^n}$

Exercise

Find the solution to $a_n = 5a_{n-2} - 4a_{n-4}$ *with* $a_0 = 3,\ a_1 = 2,\ a_2 = 6$ *and* $a_3 = 8$

Solution

This is a fourth-degree recurrence relation.

The characteristic polynomial is $r^4 - 5r^2 - 4 = 0$

That implies to: $\left(r^2 - 1\right)\left(r^2 - 4\right) = (r-1)(r+1)(r-2)(r+2) = 0$

So, the roots are 1, –1, 2, –2

The general solution: $a_n = \alpha_1 + \alpha_2(-1)^n + \alpha_3 2^n + \alpha_4(-2)^n$

Plugging in initial conditions gives

$$3 = \alpha_1 + \alpha_2(-1)^0 + \alpha_3 2^0 + \alpha_4(-2)^0 \quad \rightarrow \quad 3 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

$$10 = \alpha_1 + \alpha_2(-1)^1 + \alpha_3 2^1 + \alpha_4(-2)^1 \quad \rightarrow \quad 10 = \alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4$$

$$6 = \alpha_1 + \alpha_2(-1)^2 + \alpha_3 2^2 + \alpha_4(-2)^2 \quad \rightarrow \quad 6 = \alpha_1 + \alpha_2 + 4\alpha_3 + 4\alpha_4$$

$$8 = \alpha_1 + \alpha_2(-1)^3 + \alpha_3 2^3 + \alpha_4(-2)^3 \quad \rightarrow \quad 8 = \alpha_1 - \alpha_2 + 8\alpha_3 - 8\alpha_4$$

The solution to the system of equations is $\alpha_1 = \alpha_2 = \alpha_3 = 1$ *and* $\alpha_4 = 0$

Therefore, the solution is $\boxed{a_n = 1 + (-1)^n + 2^n}$

Exercise

Find the recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ *with* $a_0 = -5,\ a_1 = 4$ *and* $a_2 = 88$

Solution

This is a third-degree recurrence relation.

The characteristic polynomial is $r^3 - 6r^2 + 12r - 8 = 0$

By the rational root test, the possible rational roots are $\pm 1,\ \pm 2,\ \pm 4,\ \pm 8$

We find that $r = 2$ (using calculator).

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 12 & -8 \\ & & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array} \qquad Q(x) = r^2 - 4r + 4 = (r-2)^2$$

$$r^3 - 6r^2 + 12r - 8 = (r-2)^3 = 0$$

Hence the only root is 2, with multiplicity 3.

The general solution: $\boxed{a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n + \alpha_3 n^2 \cdot (-2)^n}$

Plugging in initial conditions gives

$$\boxed{-5 = a_0 = \alpha_1}$$

$$4 = a_1 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 \qquad \alpha_1 + \alpha_2 + \alpha_3 = 2 \rightarrow \alpha_2 + \alpha_3 = 7$$

$$88 = a_2 = 4\alpha_1 + 8\alpha_2 + 16\alpha_3 \qquad \alpha_1 + 2\alpha_2 + 4\alpha_3 = 22 \rightarrow 2\alpha_2 + 4\alpha_3 = 27$$

$$\rightarrow \begin{cases} \alpha_2 + \alpha_3 = 7 \\ 2\alpha_2 + 4\alpha_3 = 27 \end{cases} \Rightarrow \boxed{\alpha_2 = \frac{1}{2}} \quad \boxed{\alpha_3 = \frac{13}{2}}$$

Therefore, the solution: $a_n = -5 \cdot 2^n + \frac{1}{2} n \cdot 2^n + \frac{13}{2} n^2 \cdot (-2)^n$

$$\boxed{= -5 \cdot 2^n + n \cdot 2^{n-1} + 13n^2 \cdot (-2)^{n-1}}$$

Exercise

Find the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ *with* $a_0 = 5,\ a_1 = -9$ *and* $a_2 = 15$

Solution

This is a third-degree recurrence relation.

The characteristic polynomial is $r^3 + 3r^2 + 3r + 1 = 0$

$r^3 + 3r^2 + 3r + 1 = 0 = (r+1)^3 = 0$

Hence the only root is -1, with multiplicity 3.

The general solution: $\boxed{a_n = \alpha_1 (-1)^n + \alpha_2 n \cdot (-1)^n + \alpha_3 n^2 \cdot (-1)^n}$

Plugging in initial conditions gives

$$\boxed{5 = a_0 = \alpha_1}$$

$$a_1 = -9 = -\alpha_1 - \alpha_2 - \alpha_3 \rightarrow \alpha_2 + \alpha_3 = 9 - \alpha_1 = 4$$

$$a_2 = 15 = \alpha_1 + 2\alpha_2 + 4\alpha_3 \rightarrow 2\alpha_2 + 4\alpha_3 = 15 - \alpha_1 = 10$$

$$\rightarrow \begin{cases} \alpha_2 + \alpha_3 = 4 \\ 2\alpha_2 + 4\alpha_3 = 10 \end{cases} \Rightarrow \boxed{\alpha_2 = 3} \quad \boxed{\alpha_3 = 1}$$

Therefore, the specific solution is $a_n = 5(-1)^n + 3n \cdot (-1)^n + n^2 \cdot (-1)^n$

$$\boxed{= \left(n^2 + 3n + 5\right)(-1)^n}$$

Exercise

Find the general form of the solutions of the recurrence relation $a_n = 8a_{n-2} - 16a_{n-4}$

Solution

This is a fourth-degree recurrence relation.

The characteristic polynomial is $r^4 - 8r^2 + 16 = \left(r^2 - 4\right)^2$

$\left(r^2 - 4\right)^2 = (r-2)^2(r+2)^2 = 0$

The roots are -2 and 2, each with multiplicity 2.

The general solution: $\quad a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n + \alpha_3 (-2)^n + \alpha_4 n \cdot (-2)^n$

Exercise

What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots 1, 1, 1, 1, –2, –2, –2, 3, 3, –4?

Solution

There are 4 distinct roots, so $t = 4$. The multiplicities are 4, 3, 2, and 1.

The general solution:

$$a_n = \left(\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2 + \alpha_{1,3} n^3\right) + \left(\alpha_{2,0} + \alpha_{2,1} n + \alpha_{2,2} n^2\right)(-2)^n + \left(\alpha_{3,0} + \alpha_{3,1} n\right)3^n + \alpha_{4,0}(-4)^n$$

Exercise

What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots $-1, -1, -1, 2, 2, 5, 5, 7$?

Solution

There are 4 distinct roots, so $t = 4$. The multiplicities are 3, 2, 2, and 1.

The general solution:

$$a_n = \left(\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2\right)(-1)^n + \left(\alpha_{2,0} + \alpha_{2,1} n\right)2^n + \left(\alpha_{3,0} + \alpha_{3,1} n\right)5^n + \alpha_{4,0} 7^n$$