Section 1.3 – Infinite Limits

Definitions

We say that $f(x)$ has the **limit *L* as *x* approaches infinity** and write $\lim_{x\to\infty} f(x) = L$

If, $\forall \varepsilon > 0 \; \exists \, N \ni \forall x, \quad x > M \quad \Rightarrow \quad |f(x) - L| < \varepsilon$

We say that $f(x)$ has the **limit *L* as *x* approaches *minus* infinity** and write $\lim_{x\to-\infty} f(x) = L$

If, $\forall \varepsilon > 0 \; \exists \, N \ni \forall x, \quad x < M \quad \Rightarrow \quad |f(x) - L| < \varepsilon$

***Basic Facts*:** $\boxed{\lim_{x\to\pm\infty} k = k \quad \textit{and} \quad \lim_{x\to\pm\infty} \frac{1}{x} = 0}$

Example

Find $\lim_{x\to\infty} \dfrac{5x^2 + 8x - 3}{3x^2 + 2}$

Solution

$$\lim_{x\to\infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim_{x\to\infty} \frac{5 + \frac{8}{x} - \frac{3}{x^2}}{3 + \frac{2}{x^2}}$$

Divide by x^2

$$= \frac{5 + 0 - 0}{3 + 0}$$

$\lim_{x\to\pm\infty} \frac{1}{x} = 0$

$$= \boxed{\frac{5}{3}}$$

Example

Find $\lim_{x\to\infty} \dfrac{11x + 2}{2x^3 - 1}$

Solution

$$\lim_{x\to\infty} \frac{11x + 2}{2x^3 - 1} = \lim_{x\to\infty} \frac{\frac{11}{x^2} + \frac{2}{x^3}}{2 - \frac{1}{x^3}}$$

$\lim_{x\to\pm\infty} \frac{1}{x} = 0$

$$= \frac{0 + 0}{2 - 0}$$

$$= \boxed{0}$$

Vertical Asymptote (*VA*) - ***Think Domain***

The line $x = a$ is a ***vertical asymptote*** for the graph of a function f if

$$\lim_{x\to a^+} f(x) \to \pm\infty \quad or \quad \lim_{x\to a^-} f(x) \to \pm\infty$$

As ***x*** approaches ***a*** from either the left or the right

$$\lim_{x\to 0^+} \frac{1}{x} \to \infty \quad or \quad \lim_{x\to 0^-} \frac{1}{x} \to -\infty$$

Example

Find $\lim_{x\to 3^+} \frac{2-5x}{x-3}$ *and* $\lim_{x\to 3^-} \frac{2-5x}{x-3}$

Solution

$$\lim_{x\to 3^+} \frac{2-5x}{x-3} = \frac{2-5(3)}{3^+-3} \to -13 \to \textit{positive and approaches } 0$$

$$= -\infty$$

$$\lim_{x\to 3^-} \frac{2-5x}{x-3} = \frac{2-5(3)}{3^--3} \to -13 \to \textit{negative and approaches } 0$$

$$= \infty$$

Example

Find $\lim_{x\to -4^+} \frac{-x^3+5x^2-6x}{-x^3-4x^2}$

Solution

$$\lim_{x\to -4^+} \frac{-x^3+5x^2-6x}{-x^3-4x^2} = \frac{168}{0} \qquad \frac{-x^3+5x^2-6x}{-x^3-4x^2} = \frac{(x-2)(x-3)}{x(x+4)} \to \textit{positive}$$

$$= -\infty \to \textit{negative and approaches } 0$$

Example

Let $f(x) = \frac{x^2-4x+3}{x^2-1}$, determine the following limits and find the vertical asymptotes of *f*.

a) $\lim_{x\to 1} f(x)$ *b)* $\lim_{x\to -1^-} f(x)$ *c)* $\lim_{x\to -1^+} f(x)$

Solution

a) $\lim\limits_{x\to 1}\dfrac{x^2-4x+3}{x^2-1}=\dfrac{0}{0}=\lim\limits_{x\to 1}\dfrac{(x-1)(x-3)}{(x-1)(x+1)}$

$$=\lim_{x\to 1}\frac{x-3}{x+1}$$

$$=-1$$

The vertical asymptote: $x=-1$, while the hole is $(1,\ -1)$

b) $\lim\limits_{x\to -1^-} f(x)=\lim\limits_{x\to -1^-}\dfrac{x-3}{x+1}$ $\to$ *negative*

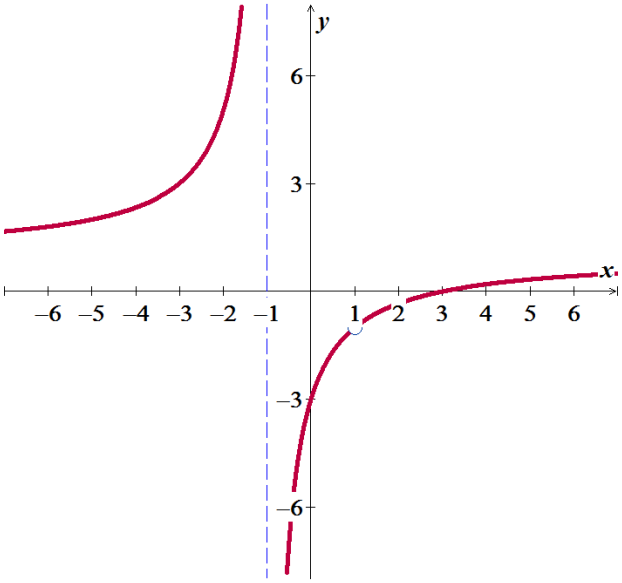
$\to$ *negative and approaches* 0

$$=\infty$$

c) $\lim\limits_{x\to -1^+} f(x)=\lim\limits_{x\to -1^+}\dfrac{x-3}{x+1}$ $\to$ *negative*

$\to$ *positive and approaches* 0

$$=-\infty$$



Example

Find $\lim\limits_{\theta\to 0^+}\cot\theta$ *and* $\lim\limits_{\theta\to 0^-}\cot\theta$

Solution

$$\cot\theta=\frac{\cos\theta}{\sin\theta}\ \Rightarrow\ \cot 0=\frac{1}{0},$$

As $\theta\to 0^+$ $\cos\theta>0;\ \sin\theta>0$ $\lim\limits_{\theta\to 0^+}\cot\theta=\infty$

As $\theta\to 0^-$ $\cos\theta>0;\ \sin\theta<0$ $\lim\limits_{\theta\to 0^+}\cot\theta=-\infty$

Exercises *Section* 1.3 – Infinite Limits

Find

1. $\lim_{x\to 5} \dfrac{x-7}{x(x-5)^2}$

2. $\lim_{x\to -5^+} \dfrac{x-5}{x+5}$

3. $\lim_{x\to 3^-} \dfrac{x-4}{x^2-3x}$

4. $\lim_{x\to 0^+} \dfrac{1}{3x}$

5. $\lim_{x\to -5^-} \dfrac{3x}{2x+10}$

6. $\lim_{x\to 0} \dfrac{1}{x^{2/3}}$

7. $\lim_{x\to 0^-} \dfrac{1}{3x^{1/3}}$

8. $\lim_{x\to\left(-\frac{\pi}{2}\right)^+} \sec x$

9. $\lim_{\theta\to 0^-} (1+\csc\theta)$

10. $\lim_{\theta\to 0^+} \csc\theta$

11. $\lim_{x\to 0^+} (-10\cot x)$

12. $\lim_{\theta\to\frac{\pi}{2}^+} \frac{1}{3}\tan\theta$

13. $\lim_{x\to 2^+} \dfrac{1}{x-2}$

14. $\lim_{x\to 2^-} \dfrac{1}{x-2}$

15. $\lim_{x\to 2} \dfrac{1}{x-2}$

16. $\lim_{x\to 3^+} \dfrac{2}{(x-3)^3}$

17. $\lim_{x\to 3^-} \dfrac{2}{(x-3)^3}$

18. $\lim_{x\to 3} \dfrac{2}{(x-3)^3}$

19. $\lim_{x\to 4^+} \dfrac{x-5}{(x-4)^2}$

20. $\lim_{x\to 4^-} \dfrac{x-5}{(x-4)^2}$

21. $\lim_{x\to 4} \dfrac{x-5}{(x-4)^2}$

22. $\lim_{x\to 1^+} \dfrac{x-2}{(x-1)^3}$

23. $\lim_{x\to 1^-} \dfrac{x-2}{(x-1)^3}$

24. $\lim_{x\to 1} \dfrac{x-2}{(x-1)^3}$

25. $\lim_{x\to 3^+} \dfrac{(x-1)(x-2)}{x-3}$

26. $\lim_{x\to 3^-} \dfrac{(x-1)(x-2)}{x-3}$

27. $\lim_{x\to 3} \dfrac{(x-1)(x-2)}{x-3}$

28. $\lim_{x\to 2^+} \dfrac{x-4}{x(x+2)}$

29. $\lim_{x\to 2^-} \dfrac{x-4}{x(x+2)}$

30. $\lim_{x\to 2} \dfrac{x-4}{x(x+2)}$

31. $\lim_{x\to 2^+} \dfrac{x^2-4x+3}{(x-2)^2}$

32. $\lim_{x\to 2^-} \dfrac{x^2-4x+3}{(x-2)^2}$

33. $\lim_{x\to 2} \dfrac{x^2-4x+3}{(x-2)^2}$

34. $\lim_{x\to -2^+} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$

35. $\lim_{x\to -2^-} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$

36. $\lim_{x\to -2} \dfrac{x^3-5x^2+6x}{x^4-4x^2}$

37. $\lim_{u\to 0^+} \dfrac{u-1}{\sin u}$

38. $\lim_{x\to 0^-} \dfrac{2}{\tan x}$

39. $\lim_{x\to 1^+} \dfrac{x^2-5x+6}{x-1}$

40. $\lim_{x\to 4} \dfrac{x-5}{\left(x^2-10x+24\right)^2}$

41. $\lim_{x\to 2\pi^-} \csc x$

42. $\lim_{x\to 0^+} e^{\sqrt{x}}$

43. $\lim_{x\to\frac{\pi}{2}^-} \dfrac{1+\sin x}{\cos x}$

44. $\lim_{x\to\frac{\pi}{2}^+} \dfrac{1+\sin x}{\cos x}$

45. $\lim_{x\to 0^-} \dfrac{e^x}{1-e^x}$

46. $\lim_{x\to 0^+} \dfrac{e^x}{1-e^x}$

47. $\lim_{x\to 1^-} \dfrac{x}{\ln x}$

48. $\lim_{x\to 0^+} \dfrac{x}{\ln x}$

49. $\lim_{x\to 0^-} \dfrac{2e^x+5e^{3x}}{e^{2x}-e^{3x}}$

50. $\lim_{x\to 0^+} \dfrac{2e^x+5e^{3x}}{e^{2x}-e^{3x}}$

51. Let $f(x) = \dfrac{x^2 - 7x + 12}{x - a}$

a) For what values of a, if any, does $\lim\limits_{x \to a^+} f(x)$ equal a finite number?

b) For what values of a, if any, does $\lim\limits_{x \to a^+} f(x) = \infty$?

c) For what values of a, if any, does $\lim\limits_{x \to a^+} f(x) = -\infty$?

52. Analyze $\lim\limits_{x \to 1^+} \sqrt{\dfrac{x-1}{x-3}}$ and $\lim\limits_{x \to 1^-} \sqrt{\dfrac{x-1}{x-3}}$