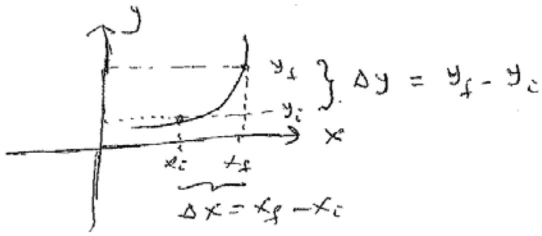
Chapter 2 – Motion in Dimension

Motion in one dimension is motion in a straight line.

Calculus is a branch of mathematics that studies relationships between changes systematically. To be exact, it deals with rate of change of one variable with respect to another variable. If 'y' is a function of 'x' then the average rate of change if 'y' with respect to (wrt) 'x' is equal to the ratio between the change in 'y' and the change in 'x' changes in 'y' & 'x' are represented as Δy & Δx respectively. That is, $\Delta y = y_f - y_i$ and $\Delta x = x_f - x_i$ where (x_i, y_i) is the initial point and (x_f, y_f) is the final point. Graphically the average rate of 'y' change wrt 'x' is equal to the slope of the line joining the initial point $(x_i, y_i)$and the final point $\left(x_f, y_f\right)$.



Average rate of change 'y' wrt 'x'= $\frac{\Delta y}{\Delta x} = \frac{y_f - y_i}{x_f - x_i}$

If we are interested in the instantaneous rate of change of 'y' wrt 'x' (or rate of changes of 'y' wrt 'x' at a given point), then the initial point (x_i, y_i), and the final point $\left(x_f, y_f\right)$ should be infinite small close to each other. That is the values of Δy & Δx should approach zero. Changes whose values approach zero are represented by the operator 'd'.

That is:

$$dy = \lim_{\Delta y \to 0} \Delta y \quad \text{(limiting value of Δy as Δy approaches zero)}$$

$$dx = \lim_{\Delta x \to 0} \Delta x \quad \text{(limiting value of Δx as Δx approaches zero)}$$

Therefore the instantaneous rate of change of 'y' wrt 'x' may be represented as the ratio between dy and dx.

Instantaneous rate of change of 'y' wrt 'x' = $\frac{dy}{dx}$

In calculus this instantaneous rate of change is called a derivative. $\frac{dy}{dx}$ is read as the derivative of 'y' wrt 'x'.

$\frac{dy}{dx}$ =derivative of y wrt x

Mathematically,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$\frac{dy}{dx}$ is equal to the ratio $\frac{dy}{dx}$ as Δx approaches zero

The purpose of the first part of calculus is basically to express the derivative of $\left(\frac{dy}{dx}\right)$ as a function of 'x' provided 'y' is known as a function of 'x'. For example if $y = x^n$ (n is a constant, it can be shown that

$$\boxed{\frac{dy}{dx} = \frac{d}{dx}[x^n] = n * x^{n-1}}$$

Example

Find the derivative of $y = x^5$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}\ [x^5] = 5x^{5-1} = 5x^4$$

The following rules are useful when dealing with combinations of functions.

If 'f' and 'g' are arbitrary functions of 'x', and 'C' is a constant then,

1. $\frac{d}{dx}[\,cf] = C\frac{df}{dx}$
2. $\frac{d}{dx}[f \pm g] = \frac{df}{dx} \pm \frac{dg}{dx}$
3. $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$
4. $\frac{d}{dx}f(g) = \frac{df}{dg} * dg/dx$

In the early parts of the course we will limit ourselves to polynomial functions. That is functions of the form $y = a_n * x^n + a_{n-1} * x^{n-1} + a_{n-1} * x^{n-2} + \cdots$

Also for the early parts we will be using on the first two rules of combinations.

Example

Find the derivative of $y = 4x^3$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}[4x^3] = 4\frac{dx^3}{dx} = 4(3x^{3-1}) = 12x^2$$

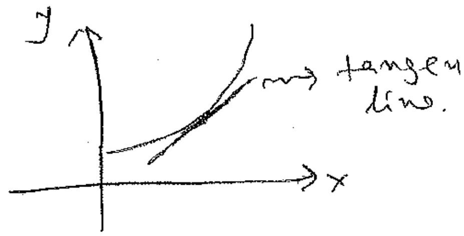
Example

Find the derivative of $y = 2x^4 - 3x^2 + 5$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}[2x^4 - 3x^2 + 5] = \frac{d}{dx}[2x^4] + \frac{d}{dx}[3x^2] + \frac{d}{dx}[5]$$
$$= 2\frac{d}{dx}(x^4) - 3\frac{d}{dx}\ (x^2) + \frac{d[5]}{dx}$$
$$=\ 2 * 4x^3 - 3 * 2x^1 + 0 \quad (\mathit{derivative\ of\ a\ constant\ is\ } 0 \mathit{\ since\ it\ doesn't\ change})$$
$$= 8x^3 - 6x$$

Graphically, the derivative of 'y' <u>wrt</u> 'x' can be obtained from a graph of 'y' versus 'x' as the slope of the line tangent to the curve at the given point.



The second part of calculus is the inverse of a derivative. Its purpose is to express 'y' as a function of 'x', provided dy/dx as known as a function of x. This inverse process is called integration. The symbol of integration is $'\int'$.

$$\boxed{\begin{array}{c}\text{If } \dfrac{dy}{dx} = f(x) \text{ then} \\ y(x) = \displaystyle\int f(x)\,dx + C\end{array}}$$

y(x) is equal to the integral of f(x) <u>wrt</u> to x

The arbitrary constant C is called constant of integration. It is needed because all functions of the form y(x) + C have the same derivative) because the derivative of C is zero). Because of this, this kind of integral is called indefinite integral. An additional condition is needed to determine C.

Integral of $x^n = \int x^n * dx$

Since $x^n = \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{\cancel{n+1}}{\cancel{n+1}}\, x^n$

It follows that $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

The following rules of combination are useful:

If f and g arbitrary functions of 'x', and 'C' is a constant

1. $\int C\, f(x) dx = C \int f(c)\, dx$
2. $\int [f(x) \pm g(x)] dx = \int f(x)\, dx \pm \int g(x) dx$

<u>Example:</u>

Find the integral of $y = x^2$

<u>Solution</u>

$$\int x^2 dx = \frac{x^{2+1}}{2+1} + C = 5\left(\frac{x^5}{5} + C\right)$$
$$= x^5 + C^1$$

<u>Example:</u>

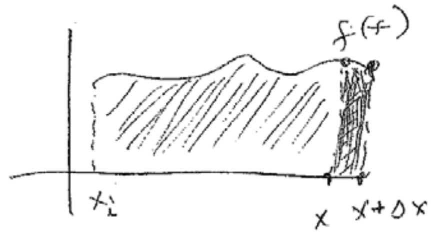
Find the integral of $y = 3x^3 - 4x + 2$

<u>Solution</u>

$$\int (3x^3 - 4x + 2) dx = \int 3x\text{^}3\, dx - \int 4x\, dx + \int 2\, dx + C$$
$$= 3\int x^3\, dx + 4\int x\, dx + 2\int dx + C$$
$$= \frac{3 * x^4}{4} - \frac{4x^2}{2} + 2x + C \ (note\ \int x^0 dx = \frac{x^1}{1})$$

For the early part of the course we will limit ourselves to polynomial functions (Functions of the form $a_n * x^n + a_n * x^{n-1} + \cdots + a_0$)

To understand the graphical meaning of the integral, consider the area enclosed between the f(x) curve and the x-axis.



Let the area between x_1 & x be A(x) and the area between x_I& $x+\Delta x$ be $A(x+\Delta x)$. The difference between $A(x+\Delta x)$ and $A(x)$is equal to the area of the shaded rectangle. The shaded rectangle has base Δx and height f(x). Therefore the area of the shaded rectangle is also equal to f(x)*Δx

Therefore $A(x+\Delta x)-A(x)=f(x)*\Delta x$

$$f(x)=\frac{A(x+\Delta x)-A(x)}{\Delta x}=\frac{\Delta A}{\Delta x}$$

If Δx is taken to be infinitely small:

$$f(x)=\lim_{\Delta x\to 0}\frac{\Delta A}{\Delta x}=\frac{dA}{dx}$$

And $f(x)=\frac{dA}{dx}\ \rightarrow A(x)=\int f(x)dx+C$

Therefore it follows that the graphical meaning of the integral of f(x) is the area enclosed between the f(x) versus x curve and the x-axis.

<u>Definite Integral</u>

The area enclosed between two known values of x, say $x_i and\, x_f$ is unique (without and arbitrary constant) and is called definitely integral. The definite integral between $x=x_i$& $x=x_f$ is written as$\int_{x=x_i}^{x=x_f} f(x)\,dx$.

If $\int f(x)dx=f(x)+C, then$

$\int_{x=x_i}^{x=x_f} f(x)\,dx=[f(x)+C]|_{x=x_f}-[f(x)+C)|_{x=x_i}$

The constant cancels out and

$$\boxed{\begin{gathered}\int_{x=x_i}^{x=x_f} f(x)=F(x_f)-F(x_i)\\ \text{where }\int f(x)dx=F(x)+C\end{gathered}}$$

<u>Example:</u>

Calculate the area enclosed between x=2 , x=4 , and $y=x^2$

<u>Solution</u>

$$\int_{x=2}^{x=4} x^2=\frac{x^3}{3}|_{x=4}-\frac{x^3}{3}|_{x=2}=\frac{64}{8}-\frac{8}{3}=\frac{56}{3}$$

<u>Motion Variables</u>

<u>Position(x)</u>: is a physical quantity used to represent the location of a particle. Specification of position requires a reference point or origin. Positions to the right of the origin are taken to be positive while those left are taken to be negative. (For vertical motion up is positive & down is negative) SI unit of position is meter (m).

Displacement **(Δx)**: is defined to be change in position.

$$\boxed{\Delta x = x_f - x_i}$$

Where x_i & x_f are initial and final positions, respectively.

Displacement does not depend on the choice of a reference point. Displacement to the right (left) is positive (negative). Infinitely small displacement is represented by dx.

Average Velocity ($\bar{v}$) defined to be displacement per unit time.

$$\boxed{\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}}$$

Instantaneous Velocity (v): is velocity at a given instant of time or it is average velocity evaluated at an infinitely small interval of time.

$$\boxed{v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}}$$

Instantaneous velocity is equal to the derivative of position with respect to time.

Average acceleration ($\bar{a}$): is defined to be change in velocity per a unit time.

$$\boxed{\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}}$$

Unit of measurement for acceleration is $meter/second^2$ ($\frac{m}{s^2}$)

Instantaneous acceleration (a) is acceleration at a given instant of time; or it is average acceleration evaluated at infinitely small interval of time.

$$\boxed{a = \lim_{\Delta a \to 0} \frac{\Delta a}{\Delta t} = \frac{da}{dt}}$$

Instantaneous acceleration is equal to the derivative of velocity with respect to time.

Since $= \frac{dx}{dt}$, we may also write $a = \frac{d}{dt}(\frac{dx}{dt})$

In calculus this is written as $a = \frac{d^2x}{dt^2}\ \frac{d}{dt}\left(\frac{dx}{dt}\right)$

Example

The position of a certain particle varies with time according to the equation

$$x = 4t^2 - 2t^2 + t + 1$$

a)Where is the particle after 2 seconds?

$$\begin{aligned} x|_{t=2} &= 4(2)^3 - 2(2)^2 + 2 + 1 \\ &= 32 - 8 + 2 + 1 \\ &= 27 \end{aligned}$$

b)Calculate its displacement between t=1 and t=2

$$\Delta x = x_f - x_i = x|_{t=2} - x|_{t=1}$$
$$= [4t^3 - 2t^2 + t + 1]_{t=2} - [4t^3 - 2t^2 + t + 1]_{t=1}$$
$$= [4(2)^3 - 2(2)^2 + 2 + 1] - [4(1)^3 - 2(1)^2 + 1 + 1$$
$$= 27 - 4$$
$$= 23 \text{ meters}$$

c) Calculate its average velocity between t=1 and t=2

$t_1 = 1; t_2 = 2; x_f = x_2 = 27; x_i = x_1 = 4$

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{27 - 4}{2 - 1} = 23\, m/s$$

d) Calculate its instantaneous velocity after 4 seconds.

$$= 4\frac{d}{dt}(t^3) - 2\frac{d}{dt}(t^2) + \frac{d}{dt}(t) + \cancel{\frac{d}{d}(1)^0}$$
$$= 4 * 3 * t^2 - 2(2 * t) + 1$$
$$v|_t = 12t^2 - 4t + 1$$
$$v|_{t=4} = 12(4)^2 - 4(4) + 1$$
$$= 96 - 16 + 1 = 81\, m/s$$

e) Calculate its average acceleration between t=1 & t=2

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i} \qquad t_i = 1; \ t_f = 3; \ v_f = v|_{t=3}; v_i = v|_{t=1}$$
$$a = \frac{v|_{t=3} - v|_{t=1}}{3 - 1} = \frac{[12t^2 - 4t + 1]|_{t=3} - [12t^2 - 4t + 1]|_{t=1}}{2}$$
$$= \frac{[12(3)^2 - 4(2) + 1]| - [12(1)^2 - 4(1) + 1]|}{2} = \boxed{44\, m/s^2}$$

f) Calculate its instantaneous acceleration after 10 seconds

$$a = \frac{d}{dt}(12t^2 - 4t + 1)$$
$$= 24t - 4$$
$$a|_{t=10} = 24(10) - 4 = \boxed{236\, m/s^2}$$

g) At what time is its acceleration zero?

$$a(t) = 0 \qquad 24t - 4 = 0$$
$$24t = 4$$
$$t = \frac{4}{24} = \frac{1}{6} s$$

Example:

The velocity of a particle varies with time according to the equation v= 2t-4. Assuming at t=0 the particle was located at the origin (x=0), find its location after 5 seconds.

Solutions

v= 2t-4 $\quad x|_{t=0} = 0 \quad x|_{t=5} = ?$

$$v = 2t - 4 = \frac{dx}{dt}$$

$$dx = (2t - 4)$$

$$\int dx = \int (2t - 4)\, dt + C$$

$$x = \int 2t\ \ dt - \int 4dt + C$$

$$2 * \frac{t^2}{2} - 4t + C$$

$$x(t) = t^2 + 4t + C$$

The value of C can be determined from the condition $x|_{t=0} = 0$

$$x|_{t=0} = \ 0^2 - 4(0) + C = \ 0$$

$$C = 0$$

Therefore $x(t) = t^2 - 4t$

$$x(t = 5) = 5^2 - 4(5) = 25 - 20 = 5m$$

Example:

A particle is moving with a constant acceleration of $2\, m/s^2$. If its initial speed is 5 m/s and it is initially located at x= 2m

a)Find its velocity after 8 seconds

$a = 2\, m/s^2 \qquad v|_{t=0} = 5\, m/s \qquad x|_{t=0} = 2\, m \qquad v|_{t=8} = ?$

$$a = \frac{dv}{dt} = 2$$

$$dv = 2dt$$

$$\int dv = \int 2\ dt + C$$

$v = 2t + C$ $v\vert_{t=0} = 5\, m/s$	$5 = 2(0) + C$ $C_1 = 5\, m/s$

$$v = 2t + 5$$

$$v|_{t=8} = 2(8) + 5 = 21\, m/s$$

b) Where is the particle located after 10 seconds? $\quad x|_{t=10}$ =?

$$v = \frac{dx}{dt} = 2t + 5$$

$$dx = (2t + 5)dt$$

$$\int dx = \int (2t + 5)\, dt + C_2$$

$$x = 2 * \frac{t^2}{2} + 5t + C_2$$

$$x|_{t=0} = 2$$

$$2 = (0)^2 + 5(0) + C$$

$$C_2 = 2\, m$$

Therefore $x = t^2 + 5t + 2$

$x|_{t=8} = 8^2 + 5(2) + 2 = 106\ m$

Uniformly accelerated motion is motion with constant acceleration.

$$\boxed{a = constant}$$

$$a = \frac{dv}{dt}$$

$$dv = a\ dt$$

$$\int dv = \int a\ dt + C$$

$$v = at + C$$

If the initial velocity is v_i $(i.e\ v|_{t=0} = v_i)$

$$v_i = a(0) + c$$

$$C_1 = v_i$$

$$v = at + v_i$$

This is customarily written as:

$$v_f = v_i + at \ \ldots\ldots.(1)$$

Where $v_i(v_f)$ is the initial (final) speed

Again $v = \frac{dx}{dt}$

Therefore $\frac{dx}{dt} = v_i + at$

$$dx = (v_i + at)dt$$

$$\int dx = \int (v_i + at)\ dt + C_2$$

$$\boxed{x = v_i t + \tfrac{1}{2}at^2 + C_2} \Longrightarrow$$

If the particle is initially located at $x_i (ie\ \ x|_{t=0} = x_i)$

$$x_i = v_i(o) + \frac{1}{2}a(0)^2 + C_2$$

$$C_2 = x_i$$

$$\Rightarrow\ \ v_i t + \tfrac{1}{2}at^2 + x_i$$

$$\Rightarrow\ \ x - x_i = v_i t + \tfrac{1}{2}at^2$$

but $x - x_i = \Delta x$ (displacement)

$$\Delta x = v_i t + \tfrac{1}{2}at^2 \ldots..(2)$$

Equations (1) and (2) are the only independent equations of a uniformly accelerated motion, which means we can solve only for two unknowns. But other dependent equations can be obtained by manipulating there equations.

$v = \frac{\Delta x}{\Delta t} = \frac{\Delta x}{t}$ (assuming the initial time is set to zero)

Therefore $v = \frac{\Delta x}{t} = \frac{v_i t + \frac{1}{2}at^2}{t}$

$$v_i + \tfrac{1}{2}at = \frac{2v_i + at}{2}$$

but since $v_f = v_i + at$ $(eq. 1)$

$at = v_f - v_i$

$$v = \frac{2v_i + (v_f - v_i)}{2} = \frac{v_i + v_f}{2}$$

$$\boxed{v = \frac{v_i + v_f}{2}}$$

Average velocity of a uniformly accelerated motion.

$$\frac{\Delta x}{t} = v \quad ====> \quad \Delta x = vt$$

$$= \left(\frac{v_i + v_f}{2}\right) t$$

$$\boxed{\Delta x = \left(\frac{v_i + v_f}{2}\right) t}$$

Again since $v_f = v_i + at$

$$\boxed{v_f^2 = v_i^2 + 2a\Delta x}$$

$$t = \frac{v_f - v_i}{a}$$

$$\Delta x = \left(\frac{v_i + v_f}{2}\right) t = \left(\frac{v_i + v_f}{2}\right)\left(\frac{v_f - v_i}{a}\right)$$

$$==> \quad \Delta x = \frac{v_f^2 - v_i^2}{2a}$$

This equation is customarily written as

$$v_f^2 = v_i^2 + 2a\Delta x$$

Now let's put the 4 equations together

Equations of a Uniformly Accelerated Motion

$$\boxed{\begin{gathered} v_f = v_i + at \\ \Delta x = v_i t + \frac{1}{2} a t^2 \\ \Delta x = \left(\frac{v_i + v_f}{2}\right) t \\ v_f^2 = v_i^2 + 2a\Delta x \end{gathered}}$$

There are 5 variables in these equations ($v_f, v_i, t, \Delta x, a$). Since only two equations are independent, if we know any 3 variables, we can solve for the other variables. In using these equations you should be careful about signs. Δx is positive(or negative) for the right (or left). $v_i\left(v_f\right)$ are positive if the motion is to the right and negative if the motions is to the left. 'a' is positive if the velocity is increasing and negative if the velocity is decreasing.(remember positive is an increase, negative is a decrease). 't' (time) is ALWAYS positive.

Example:

A car changes its speed from 10 m/s to 30 m/s in 10 sec.

a) Calculate its acceleration

$v_i = 10\, m/s\, ; v_f = 30\, m/s\, ; t = 10\ sec; \quad a = ??$

$v_f = v_i + at$

$30 = 10 + a(10)$

$\boxed{a = 2\, m/s^2}$

b) Calculate the distance traveled

$$\Delta x = \left(\frac{v_i+v_f}{2}\right)t$$

$$= \left(\frac{10+30}{2}\right)(10) = \underline{200\text{ m}}$$

Example:

A car initially moving with a speed of 40 m/s was stopped in a distance of 100 m.

a) Calculate its acceleration

$v_i = 40\, m/s\, ; v_f = 0\ (stopped);\ \Delta x = 100m$

$v_f^2 = v_i^2 + 2a\Delta x$

$0 = 40^2 + 2a\ (100)$

$a = -8\, m/s^2$

Motion Under Gravity

Motion under gravity is a uniformly accelerated motion. Its numerical value is 9.8 m/s^2 and its effect is to decrease speed (it decreases the speeds of objects going up and it increases the negative velocity of objects going down). Therefore its sign must be negative.

Gravitational Acceleration $\boxed{g = -9.8\, m/s^2}$

(We will use $-10\, m/s^2$ for simplicity in examples. $-9.8\, m/s^2$ will need to be used in quizzes and exams)

The equations of motion under gravity can be obtained easily from the equations of a uniformly accelerated motion by replacing $'a'$ by $'g'$ and Δx by Δy (since it is a vertical motion).

<u>Equations of Motion Under Gravity</u>

$$\boxed{\begin{gathered} v_f = v_i + gt \\ \Delta y = v_i t + \frac{1}{2} g t^2 \\ \Delta y = \left(\frac{v_i + v_f}{2}\right) t \\ v_f^2 = v_i^2 + 2g\Delta y \end{gathered}}$$

Δy is positive if the final position is above initial position and negative if final position is below initial position. v_i, v_f are positive for upward motion and negative for downward motion. There are 4 variables $(v_f, v_i, t, \Delta y)$. If we know any 2 of the variables we can solve for the other 2.

Example:

An object is thrown upwards with a speed of 20 m/s.

a) How long will it take to reach its maximum height?

$v_i = 20\, m/s\, ;\quad v_f = 0\ (velocity\ is\ zero\ at\ maximum\ height);\ \ t = ??$

$$v_f = v_i + gt$$

$$0 = 20 + (-10)t$$

$$\underline{t = 2\ sec}$$

b) How high will it rise?

$$\Delta y = v_i t + \frac{1}{2} g t^2$$

$$= (20)(2) + \frac{1}{2}(-10)2^2$$

$$= 40 - 20 = \underline{20m}$$