Solution Section 1.9 – Functions

Exercise

Why is f not a function from $\mathbb{R}$ to $\mathbb{R}$ if

a) $f(x) = \frac{1}{x}$?

b) $f(x) = \sqrt{x}$?

c) $f(x) = \pm\sqrt{x^2+1}$?

Solution

a) Because for $x = 0$ the value of $f(x)$ is not defined by the given rule.

b) Because for $x < 0$ the value of $f(x)$ is not defined in $\mathbb{R}$

c) Because for $f(1) = \sqrt{2}$ *or* $f(1) = -\sqrt{2}$

Exercise

Determine whether f is a function from $\mathbb{Z}$ to $\mathbb{R}$ if

a) $f(x) = \pm x$?

b) $f(x) = \sqrt{x^2+1}$?

c) $f(x) = \frac{1}{x^2-4}$?

Solution

a) This is not a function because $f(1) = 1$ *or* $f(1) = -1$,

b) This is a function for all integers x.

c) This is not a function since for $x = \pm 2$ the value of $f(x)$ is not defined by the given rule.

Exercise

Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.

a) The function that assigns to each bit string the number of ones in the string minus the number of zeros in the string.

b) The function that assigns to each bit string twice the number of zeros in that string.

c) The function that assigns the number of bits over when a bit string is split into bytes (which are blocks of 8 bits).

Solution

a) The domain is the set of all bit strings. Depending on how we read the word "difference" the output values can be either integers or natural numbers. For example. If the input is 1010000, then one might read the rule as stating that the function value is 3, since there are three more 0s than 1s; but most people would probably consider the function value to be –3, obtained by subtracting in the order stated: $2-5=-3$. Then the range is $\mathbb{Z}$.

b) The domain is the set of all bit strings. Since there can be any natural number of 0s in a bit string, the value of the function can be 0, 2, 4, …. Therefore the range is the set of even natural numbers.

c) The domain is the set of all bit strings. Since the number of leftover bits can be any whole number from 0 to 7 (if it were more, then we could form another byte), the range is (0, 1, 2, 3, 4, 5, 6, 7).

Exercise

Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one and onto.

a) $f(a)=b,\ f(b)=a,\ f(c)=c,\ f(d)=d$

b) $f(a)=b,\ f(b)=b,\ f(c)=d,\ f(d)=c$

c) $f(a)=d,\ f(b)=b,\ f(c)=c,\ f(d)=d$

Solution

a) This is one–to–one.

b) This is not one–to–one, since *b* is the image of both *a* and *b*.

c) This is not one–to–one, since *d* is the image of both *a* and *d*.

Exercise

Determine whether the function $f:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ is onto if

a) $f(m,\ n)=m+n$

b) $f(m,\ n)=m^2+n^2$

c) $f(m,\ n)=m$

d) $f(m,\ n)=|n|$

e) $f(m,\ n)=m-n$

f) $f(m,\ n)=2m-n$

g) $f(m,\ n)=m^2-n^2$

h) $f(m,\ n)=m+n+1$

i) $f(m,\ n)=|m|-|n|$

j) $f(m,\ n)=m^2-4$

Solution

a) Given any integer *n*, we have $f(0,\ n)=n$, so the function is onto.

b) The range contains no negative integers, so the function is not onto.

c) Given any integer *m*, we have $f(m,\ 4)=m$, so the function is onto.

d) The range contains no negative integers, so the function is not onto.

e) Given any integer *m*, we have $f(m,\ 0)=m$, so the function is onto.

f) For any integer *n*, we have $f(0,\ -n)=n$, so the function is onto.

g) $m^2-n^2=(m-n)(m+n)\neq 2$, since 2 is not in the range, so the function is not onto.

h) For any integer *n*, we have $f(0,\ n-1)=n$, so the function is onto.

i) This is onto. To achieve negative values we set $m=0$, and to achieve nonnegative value we set $n=0$.

j) $m^2-4=(m-2)(m+2)\neq 2$ since 2 is not in the range, so the function is not onto.

Exercise

Determine whether each of these functions is a bijection from $\mathbb{R}\to\mathbb{R}$

a) $f(x)=2x+1$

b) $f(x)=x^2+1$

c) $f(x)=x^3$

d) $f(x)=\dfrac{x^2+1}{x^2+2}$

e) $f(x)=x^5+1$

Solution

a) $f(x)=2x+1 \quad\Rightarrow\quad f^{-1}(x)=\frac{x-1}{2}$. Therefore the function is a bijection.

b) It is not one-to-one since $f(1)=f(-1)=2$ and it is also not onto since the range is the interval $[1,\ \infty)$.

c) $f(x)=x^3 \quad\Rightarrow\quad f^{-1}(x)=\sqrt[3]{x}$. Therefore the function is a bijection.

d) It is not one-to-one since $f(1)=f(-1)=\frac{2}{3}$ and it is also not onto since the range is the interval $\left[\frac{1}{2},\ \infty\right)$.

e) $f(x)=x^5+1 \quad\Rightarrow\quad f^{-1}(x)=\sqrt[5]{x-1}$. Therefore the function is a bijection.

Exercise

Suppose that g is a function from A to B and f is a function from B to C.

a) Show that if both f and g are one–to–one functions, then $f \circ g$ is also one–to–one.

b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

Solution

a) Assume that both f and g are one–to–one. We need to show that $f \circ g$ is also one–to–one. If x and y are two distinct elements of A, then $f\left(g(x)\right) \neq f\left(g(y)\right)$. First, since g is one–to–one, by definition $g(x) \neq g(y)$. Second, since now $g(x)$ and $g(y)$ are distinct elements of B, and since f is one–to–one, we conclude that $f\left(g(x)\right) \neq f\left(g(y)\right)$ as desired.

b) Assume that both f and g are onto. We need to show that $f \circ g$ is onto. If z is any element of C, then there is some element $x \in A$ such that $f\left(g(x)\right) = z$. First, since f is onto, we can conclude that there is an element $y \in B$ such that $f(y) = z$. Second, since g is onto and $y \in B$, we can conclude that there is an element $x \in A$ such that $g(x) = y$. therefore $z = f(y) = f\left(g(x)\right)$ as desired.