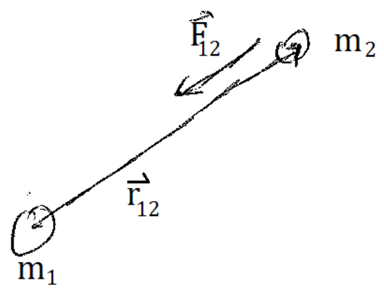
Chapter 14 Section 1 - Gravitation

Newton's Law of gravitation: states that any two objects in the universe attract each other with a gravitational force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers. The direction of the force is along the line joining their centers.

Consider the masses m_1 & m_2 shown. Let $\vec{r}_{12}$ be the position vector of m_2 with respect to m_1. Then the direction of the gravitational force exerted by m_1 on m_2 $(\vec{F}_{12})$ is opposite to $\vec{r}_{12}$. If $\vec{e}_r$ is a unit vector in the direction of $\vec{r}_{12}$, the the force exerted by m_1 on m_2 may be written as



$$\vec{F}_{12} = \frac{-G\, m_1\, m_2}{r_{12}^2}\, \vec{e}_r$$

And since $\vec{e}_r = \frac{\vec{r}_{12}}{r_{12}}$, this can also be written as

$$\vec{F}_{12} = \frac{-G\, m_1\, m_2}{r_{12}^2} \cdot \frac{\vec{r}_{12}}{r_{12}}$$

or

$$\boxed{\vec{F}_{12} = \frac{-G\, m_1\, m_2}{r_{12}^3} \cdot \vec{r}_{12}}$$

Where G is a constant called gravitational constant. Its value is

$$\boxed{G = 6.67 \times 10^{-11}\, \frac{N\, m^2}{kg^2}}$$

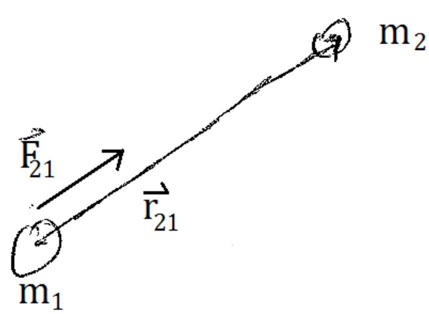
Similarly the gravitational force exerted by m_2 on m_1 $(\vec{F}_{21})$ can be written as

$$\boxed{\vec{F}_{21} = \frac{-G\, m_1\, m_2}{r_{21}^3} \cdot \vec{r}_{21}}$$

Where $\vec{r}_{21}$ is the position vector of m_1 with respect to m_2

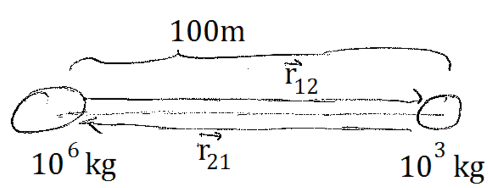
$\vec{F}_{12}$ and $\vec{F}_{21}$ are action reaction forces

$$\vec{F}_{12} = -\vec{F}_{21}$$



Example: Consider the masses shown.

a) Calculate the gravitational force exerted by the $10^6 kg$ object on the $10^3 kg$ object



Solution

Let $m_1 = 10^6 kg$ & $m_2 = 10^3 kg$, then $r_{12} = 100m$, $\vec{r}_{12} = 100m\,\hat{\imath}$

$$\therefore \quad \vec{F}_{12} = \frac{-G\, m_1\, m_2}{r_{12}^3} \cdot \vec{r}_{12}$$

$$= \frac{-(6.67 \times 10^{-11})(10^6)(10^3)}{(100)^3} \cdot [100\hat{\imath}]$$

$$\vec{F}_{12} = -6.67 \times 10^{-6} N\, \hat{\imath}$$

b) Calculate the gravitational force exerted by the $10^3 kg$ object on the $10^6 kg$ object.

Solution

$\vec{F}_{21} = ??$

$\vec{r}_{21} = -100m\, \hat{\imath}$

$$\vec{F}_{21} = \frac{-G\, m_1\, m_2}{r_{21}^3} \cdot \vec{r}_{21}$$

$$\vec{F}_{21} = \frac{-(6.67 \times 10^{-11})(10^6)(10^3)}{(100)^3} \cdot [-100\hat{\imath}]$$

$$= 6.67 \times 10^{-6} N\, \hat{\imath}$$

Of course this can also be calculate from

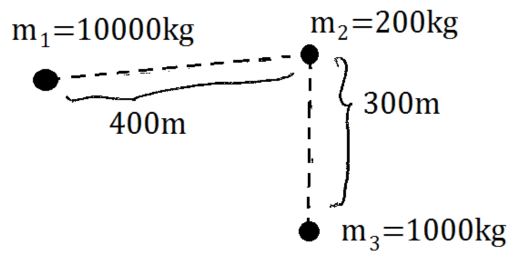
The fact that $\vec{F}_{21} = -\vec{F}_{12}$

Superposition Principle

If an object is in the vicinity of more than one object, then the net gravitational force acting on the object is the vector sum of all the forces due to the objects in its vicinity.

$$\boxed{\vec{F}_{net} = \sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots}$$

Example: Consider the objects shown. Calculate the net gravitational force exerted on m_3 by m_1 & m_2.



$$\vec{F}_3 = \vec{F}_{13} + \vec{F}_{23}$$

$$= \frac{-G\, m_1\, m_3}{r_{13}^3} \cdot \vec{r}_{13} + \frac{-G\, m_2\, m_3}{r_{23}^3} \cdot \vec{r}_{23}$$

Where $m_1 = 10{,}000kg,\ m_2 = 2000kg\ \&\ m_3 = 1000kg$

$r_{13} = \sqrt{300^2 + 400^2} = 500m$

$\vec{r}_{13} = (400\hat{\imath} - 300\hat{\jmath})m$

$r_{23} = 300\, m$

$\vec{r}_{23} = (-300\hat{\jmath})m$

$$\vec{F}_3 = -Gm_3 \left[\frac{m_1}{r_{13}^3} \cdot \vec{r}_{13} + \frac{m_2}{r_{23}^3} \cdot \vec{r}_{23}\right]$$

$$= -(6.67 \times 10^{-11})(1000) \left[\frac{10000}{(500)^3} \cdot [400\hat{\imath} - 300\hat{\jmath}] + \frac{2000}{(300)^3} [-300\hat{\jmath}]\right]$$

$$= -6.67 \times^{-8} [(.032\hat{\imath} - .024\hat{\jmath}) + (-.0222\hat{\jmath})]$$

$$= -6.67 \times^{-8} [.032\hat{\imath} - (.024 + .0222)\hat{\jmath}]$$

$$= -6.67 \times^{-8} [.032\hat{\imath} - .046\hat{\jmath}]$$

$$\vec{F}_3 = [-2.13 \times 10^{-9}\hat{\imath} + 3.07 \times 10^{-9}\hat{\jmath}]N$$

Gravitational Acceleration due to earth

Consider an object of mass *m* located at an altitude *h* above the surface of earth.

$$F_g = \frac{G\, m\, m_E}{(R_E + h)^2}$$

Where m_E is mass of earth and R_E is radius of earth. $R_E + h$ is the distance b/n the object and the center of earth.

According to Newton's 2nd law, this force must be equal to mass times acceleration.

$$F_g = ma_g = \frac{G\,m\,m_E}{(R_E+h)^2}$$ Where a_g is gravitational acceleration

$$\boxed{a_g = \frac{Gm_E}{(R_E+h)^2}}$$

Gravitational acceleration varies inversely with the square of the distance from the center of earth. Gravitational acceleration is taken to be constant at the surface of earth (9.8 m/s^2) because $R_E + h = R_E$ on the surface & radius of earth can be taken to be constant.

Example: Using the fact that gravitational acceleration at the surface of earth is 9.8 m/s^2, calculate the mass of earth. (Radius of earth = 6.37×10^6 meters)

Solution

At the surface of earth $a_g = 9.8$ m/s^2 & $h = 0$

$$\therefore \quad a_g = \frac{Gm_E}{(R_E+h)^2} = \frac{Gm_E}{R_E^2}$$

$$m_E = \frac{a_g R_E^2}{G} = \frac{(9.8)(6.37\times 10^6)^2}{6.67\times 10^{-11}}$$

$$= 5.96\times 10^{24} kg$$

Which is very close to the accepted value $5.98\times 10^{24} kg$

Example: Calculate gravitational acceleration at an altitude of 1000 km.

Solution

$h = 10^6 m$
$R_E = 6.37\times 10^6$
$m_E = 5.98\times 10^{24}$
$a_g = ??$

$$a_g = \frac{Gm_E}{(h+R_E)^2} = \frac{(6.67\times 10^{-11})(5.98\times 10^{24})}{(10^6 + 6.37\times 10^6)^2}$$

$$a_g = 7.34 \text{ m/s}^2$$

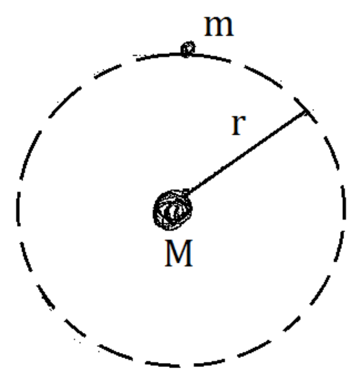
Orbits due to gravitational force

If an object of mass *m* is revolving in a circular orbit around an object of mass *M* due to the gravitational force exerted on *m* by *M*, then the centripetal force responsible for the motion may be written as

$$F_c = \frac{mv^2}{r} = \frac{G\,m\,M}{r^2}$$ Where *v* is its speed and *r* is the distance between *m* & *M*

$$\boxed{v = \sqrt{\frac{GM}{r}}}$$

The speed of an object in orbit due to gravitational force depends only on the mass of the object exerting the gravitational force and the radius of the orbit only.



Example: Consider the motion of earth around the sun. (mass of sun $= 1.991 \times 10^{30}$kg, distance b/n earth & sun $= 1.496 \times 10^{11}$ meters). Assume a circular orbit.

a) Calculate the speed with which earth is revolving around the sun

$M_s = 1.991 \times 10$^30 kg

$r_{Es} = 1.496 \times 10^{11}$ m

$v = ??$

$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{(6.67 \times 10^{-11})(1.991 \times 10^{30})}{1.496 \times 10^{11}}}$$

$$v = 2.97943 \times 10^4 \text{ m/s}$$

b) Calculate the time taken for earth to make one complete revolution around earth in days

$T = ??$ (period)

$$T = \frac{2\pi r_{Es}}{v} = \frac{2\pi(1.496 \times 10^{11})}{2.97943 \times 10^4}$$

$$T = 31548468\ sec.$$

$$T = 31548468 \left(\frac{min}{60s} \cdot \frac{hr}{60min} \cdot \frac{day}{24hr}\right)$$

≈ 365 days as expected

Kepler's laws of planetary motion

Kepler's laws are laws that govern motion of planets around the sun; there are 3 of them.

1. The planets revolve around the sun in elliptical orbits with the sun on one of the foci of the ellipse.

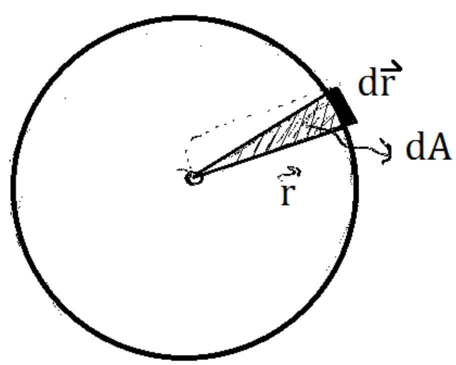
   For a number of planets, the elliptical path can be approximated by a circular orbit without the loss of a lot of accuracy.
2. The radius from the sun to the planet sweeps equal areas in equal intervals of time.

   This law is a direct consequence of the fact that the angular momentum of the planets is conserved. The angular momentum is conserver for any central force because torque of a central force is zero.

$$\text{If } \vec{F} = f(r) \cdot \frac{\vec{r}}{r}$$

$$\text{then } \tau = \vec{r} \times F = \vec{r} \times \left(f(r)\frac{\vec{r}}{r}\right) = \frac{f(r)}{r}\,\vec{r} \times \vec{r} = 0$$

$$\vec{\tau} = 0 \implies \vec{L} = \text{constant} \quad (\text{because } \vec{r} \times \vec{r} = 0)$$



$$\text{But} \quad L = \vec{r} \times m\vec{v}$$

$$= m(\vec{r} \times \vec{v}) \;\&\; \vec{v} = \frac{d\vec{r}}{dt}$$

$$\therefore \quad L = m\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \frac{m}{dt}(\vec{r} \times d\vec{r})$$

But $\vec{r} \times d\vec{r}$ = twice the area of the shaded region-area swept in a time interval dt. (Remember $\vec{r} \times d\vec{r}$ has an area of a parallelogram of sides $\vec{r}$ & $d\vec{r}$)

$$\therefore \quad |\vec{r} \times d\vec{r}| = 2dA$$

$$L = \frac{m}{dt}2dA \implies \frac{dA}{dt} = \frac{L}{2m}$$

Since L is constant, it follows that $\frac{dA}{dt}$ is a constant and hence Kepler's law

3. The square of the period of a planet is proportional to the cube of the distance between the planet and the sun.

This law was deduced by Kepler empirically before the law of gravitation was discovered by Newton. But now it can be shown easily using Newton's law of gravitation.

$$\frac{mv^2}{r} = \frac{G\,m\,m_s}{r^2} \quad \text{but} \quad v = \frac{2\pi r}{T}$$

$$\Rightarrow \left(\frac{2\pi r}{T}\right)^2 \cdot \frac{1}{r} = \frac{GM_s}{r^2} \implies \boxed{T^2 = \left(\frac{4\pi^2}{GM_s}\right) r^3}$$

Since $\frac{4\pi^2}{GM_s}$ is a constant it follows that T^2 is proportional to r^3.

$$\implies \frac{T^2}{r^3} = \text{constant for all planets}$$

$\therefore$ for 2 planets identifies as 1 & 2

$$\boxed{\frac{T_1^2}{r_1^3} = \frac{T_2^2}{r_2^3}}$$

T_1 & r_1 are the period & radius of planet 1
T_2 & r_2 are the period & radius of planet 2

Example: The distance between the Sun and Jupiter is 7.78×10^{11}. How long does Jupiter take to make one complete revolution around the sun in earth days.

Solution

Let's take our 2 planets to be earth and Jupiter. Let *e* represent earth & j represent Jupiter.

$$\frac{T_e^2}{r_e^3} = \frac{T_j^2}{r_j^3}$$

$r_e = 1.496 \times 10^{11}$ m (radius of earth)
$T_e = 365$ days (period of earth)
$r_j = 7.78 \times 10^{11}$ m (radius of Jupiter)
$T_j = ??$ (period of Jupiter)

$$\frac{(365\ days)^2}{(1.496 \times 10^{11})^3} = \frac{T_j^2}{(7.78 \times 10^{11})^3}$$

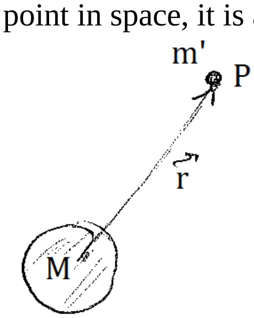
$$T_j^2 = \left(\frac{7.78}{1.496}\right)^3 \cdot 365^2\ day^2$$

$$T_j = \sqrt{\left(\frac{7.78}{1.496}\right)^3} \cdot 365\text{^}2 \approx \underline{\text{4328.8 earth days/11.86 earth years}}$$

Chapter 14 Section 2 - Gravitational Field

Field Theory

Instead of saying an object exerts force on any object placed at any point in space, it is assumed that the object sets up gravitational field throughout space which exerts force on an object placed at ????



A field is a property of space. It depends only on the coordinates of space and possibly spatial derivatives.

The gravitational field due to a certain object at a given point is defined to be the gravitational force per a unit mass exerted on a small mass placed at the given point. Consider the gravitational field due to an object of mass M at point P. Let a small mass object of mass m' be placed at point P, then the gravitational field at point P is defined as

$$\vec{g}_p = \frac{\vec{F}_{Mm'}}{m'}$$

Where $\vec{F}_{Mm'}$ is gravitational force exerted on m' by the object of mass M

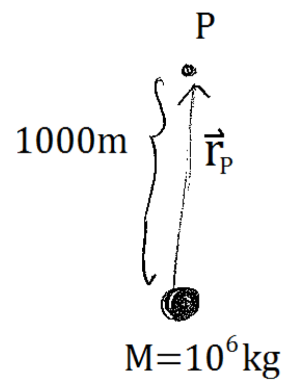
But $\vec{F}_{Mm'} = \frac{-Gm'M}{r_P^3}\vec{r}_P$

$$\vec{g}_p = \frac{\left(\frac{-Gm'M}{r_P^3}\vec{r}_P\right)}{m'}$$

$$\boxed{\vec{g}_p = \frac{-GM}{r_P^3}\vec{r}_P}$$

$\vec{g}_p \rightarrow$ gravitational field at point P
$\vec{r}_P \rightarrow$ position vector of point P wrt the center of M
$r_p \rightarrow$ distance b/n center of M & point P
$M \rightarrow$ mass object setting up gravitational field

Example: Consider the diagram shown. Calculate the field at point P due to the object shown of mass 10^6 kg.



Solution

$M = 10^6 kg$
$\vec{r}_P = 1000\, m\, \hat{j}$
$r_p = 1000\, m$
$\vec{g}_p = ??$

$$\vec{g}_p = \frac{-GM}{r_P^3}\vec{r}_P$$
$$= \frac{-(6.67 \times 10^{-11})(10^6)}{(1000)^3} \cdot 1000\hat{j}$$
$$= -6.67 \times 10^{-11}\, {}^{N}/_{kg}\ \hat{j}$$

If an object of mass m is placed at a point where the gravitational field is $\vec{g}$, then the gravitational force $(\vec{F}_g)$ acting on the object is the product of the mass of the object and the field

$$\boxed{\vec{F}_g = m\vec{g}}$$

In other words, $\vec{g}$ represents acceleration of an object placed at the given point

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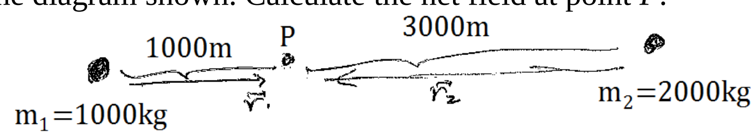
Superposition Principle

If a point is in the vicinity of a number of objects, then the field at the point is the vector sum of all the fields due to the objects in its vicinity.

$$\vec{g}_P = \vec{g}_1 + \vec{g}_2 + \cdots$$

Example: Consider the diagram shown. Calculate the net field at point P.



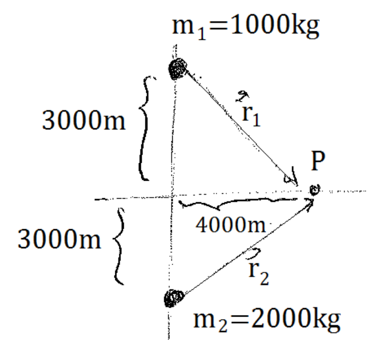
$$\vec{g}_P = \vec{g}_1 + \vec{g}_2$$
$$= \frac{-Gm_1\vec{r}_1}{r_1^3} + \frac{-Gm_2\vec{r}_2}{r_2^3}$$

$$= -G\left[\frac{m_1\vec{r}_1}{r_1^3} + \frac{m_2\vec{r}_2}{r_2^3}\right]$$

$m_1 = 1000kg$	$m_2 = 2000kg$
$\vec{r}_1 = (1000\hat{\imath})m$	$\vec{r}_2 = (-3000\hat{\imath})m$
$r_1 = 1000m$	$r_2 = 3000\ m$

$$\vec{g}_P = -6.67\times10^{-11}\left[\frac{1000}{(1000)^3}(1000\hat{\imath}) + \frac{2000}{(3000)^3}(-3000\hat{\imath})\right]$$
$$= -6.67\times10^{-11}\left[.001\hat{\imath} - \frac{2}{9}(.001)\hat{\imath}\right]$$
$$\vec{g}_P = -5.2\times10^{-14}\ {}^{N}\!/_{kg}$$

Example: Consider the diagram shown. Calculate the net field at point *P*.



Solution

$m_1 = 1000kg$	$m_2 = 2000kg$
$\vec{r}_1 = (4000\hat{\imath} - 3000\hat{\jmath})m$	$\vec{r}_2 = 4000\hat{\imath} + 3000\hat{\jmath}$
$r_1 = \sqrt{4000^2 + 3000^2}$	$r_2 = \sqrt{4000^2 + 3000^2}$
$= 5000$	$= 5000$

$$\vec{g}_P = \vec{g}_1 + \vec{g}_2$$
$$= \frac{-Gm_1\vec{r}_1}{r_1^3} + \frac{-Gm_2\vec{r}_2}{r_2^3}$$
$$= -G\left[\frac{m_1\vec{r}_1}{r_1^3} + \frac{m_2\vec{r}_2}{r_2^3}\right]$$
$$\vec{g}_P = -6.67\times10^{-11}\left[\frac{1000}{(5000)^3}(4000\hat{\imath} - 3000\hat{\jmath}) + \frac{2000}{(5000)^3}(4000\hat{\imath} + 3000\hat{\jmath})\right]$$
$$= -6.67\times10^{-11}[(3.2\times10^{-5}\hat{\imath} - 2.4\times10^{-5}\hat{\jmath}) + (6.4\times10^{-5}\hat{\imath} + 4.8\times10^{-5}\hat{\jmath})]$$
$$= -6.67\times10^{-11}[9.6\times10^{-5}\hat{\imath} + 2.4\times10^{-5}\hat{\jmath}]$$
$$= [-6.4\times10^{-17} - 1.6\times10^{-17}\hat{\jmath}]\ {}^{N}\!/_{kg}$$

Calculate the gravitational force exerted on an object of mass 100 kg placed at point *P*.

Solution

$F_g = ??$	$\vec{F}_g = m\vec{g}$
$m = 100kg$	$= 100[-6.4\times10^{-17}\hat{\imath} - 1.6\times10^{-17}\hat{\jmath}]$
	$\vec{F}_g = [-6.4\times10^{-15}\hat{\imath} - 1.6\times10^{-15}\hat{\jmath}]N$

Gravitational Potential Energy

Gravitation force is a central force (with $(r) = -\frac{GmM}{r^2}$). As shown in an earlier chapter, for a central force of the form $\vec{F} = f(r)\vec{e}_r$, the change in potential energy between an initial position $\vec{r}_i$ and a final position $\vec{r}_f$ is given by

$$\Delta u = -\int_{r_i}^{r_f} f(r)dr$$

Since $\vec{F}_g = -\frac{GmM}{r^2}\vec{e}_r$, with $f(r) = -\frac{GmM}{r^2}$ the change in gravitational potential energy is given as

$$\Delta u = -\int_{r_i}^{r_f} \left(-\frac{GmM}{r^2}\right) dr$$

$$\int_{r_i}^{r_f} \left(\frac{GmM}{r^2}\right) dr = GmM \int_{r_i}^{r_f} \frac{1}{r^2} dr$$

$$= -GmM \frac{1}{r}\Big|_{r=r_i}^{r=r_f} = -GmM \left[\frac{1}{r_f} - \frac{1}{r_i}\right]$$

$$\therefore \quad \Delta u = u_f - u_i = -GmM \left[\frac{1}{r_f} - \frac{1}{r_i}\right]$$

Let the potential energy at infinity be zero. That is, let $u|_{r_i=\infty} = 0$

$$u_f - u|_{r_i=\infty} = -GmM \left[\frac{1}{r_f} - \underset{\substack{\Downarrow \\ 0}}{\cancel{\frac{1}{\infty}}}\right]$$

$$\Rightarrow u_f = -\frac{GMm}{r_f}$$

Dropping the subscript *f*, the potential energy of the system of the two objects when they are separated by a distance r is given by

$\boxed{u = -\frac{GMm}{r}}$ → provided the potential energy is assumed to be zero when the two objects are separated by an infinite distance

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<u>Example:</u> Calculate the gravitational potential energy stored by earth-satellite system when a satellite of mass 100,000 kg is revolving at an altitude of 1 km. (Mass of earth $= 5.98 \times 10^{24}$ kg; radius of earth $= 6.37 \times 10^6$m)

<u>Solution</u>

$r = h + R_E$ where h is altitude & R_E is the radius of earth

$r = 1000m + 6.37 \times 10^6 m = 7.37 \times 10^6$

$m_s = 100{,}000\ kg$

$m_E = 5.98 \times 10^{24}\ kg$

$u = ??$

$$u = \frac{-Gm_s m_E}{r}$$

$$= \frac{(-6.67 \times 10^{-11})(10^5)(5.98 \times 10^{24})}{7.37 \times 10^6}$$

$$= -5.4 \times 10^{12} J$$

<u>Example:</u> Calculate the work done by gravitational force when an object of mass 1000 kg falls to the surface of earth from an altitude of 1000m.

<u>Solution</u>

$h = 1000m$

$R_E = 6.37 \times 10^6$

$r_i = h + R_E$

$\quad = 6.371 \times 10^6$

$r_f = R_E = 6.37 \times 10^6$

$m_E = 5.98 \times 10^{24} kg$

$m_s = 1000kg$

$$w = -\Delta u = -(u_f - u_i)$$

$$= -\left[\left(-\frac{GMm}{r_f}\right) - \left(-\frac{GMm}{r_i}\right)\right]$$

$$= Gm_E m_s \left[\frac{1}{r_f} - \frac{1}{r_i}\right]$$

$$= (6.67 \times 10^{-11})(10^3)(5.98 \times 10^{24}) \left[\frac{1}{6.37 \times 10^6} - \frac{1}{6.37 \times 10^6}\right]$$

$$= 39.89 \times 10^7 \left[\frac{1}{6.37} - \frac{1}{6.371}\right]$$

$$= 9828\, J$$

Conservation of Mechanical Energy

Since gravitational force is conservative, mechanical energy is conserved if only gravitational forces are involved. Consider a system involving two objects of masses *M* and *m*, where *M* is much bigger than *m*. If $M >> m$, then *M* can essentially be treated as a stationary object. The mechanical energy of the system can be approximated as the sum of the potential energy of the system, and the kinetic energy of the smaller object (neglecting the kinetic energy of the bigger object).

$$ME = u + \frac{1}{2}mv^2$$

$$\Longrightarrow \boxed{ME = -\frac{GmM}{r} + \frac{1}{2}mv^2} \text{ provided } M >> m$$

Where *v* is the speed of the smaller object of mass *m*

Therefore, the principle of conservation of mechanical energy for gravitational forces can be written as

$$\boxed{-\frac{GMm}{r_i} + \frac{1}{2}mv_i^2 = -\frac{GMm}{r_f} + \frac{1}{2}mv_f^2}$$

Example: An object of mass 100,000 kg is released from an altitude of 1000 m. Assuming no air resistance, calculate the speed of the object by the time it reaches the surface of earth.

Solution

$r_i = R_E + h$ $= 1000 + 6.37 \times 10^6 m$ $r_i = 6.371 \times 10^6 m$ $r_f = R_E = 6.37 \times 10^6 m$ $v_i = 0$ (released from rest) $m = 100{,}000\ kg$ $M = 5.98 \times 10^{24}\ kg$	$-\frac{GM\cancel{m}}{r_i} + \frac{1}{2}\cancel{m}v_i^2 = -\frac{GM\cancel{m}}{r_f} + \frac{1}{2}\cancel{m}v_f^2$ $v_f^2 = 2\left(\frac{Gm}{r_f} - \frac{GM}{r_i}\right)$ $= 2GM\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$ $v_f = \sqrt{2GM\left(\frac{1}{r_f} - \frac{1}{r_i}\right)}$ $= \sqrt{2(6.67 \times 10^{-11})(5.98 \times 10^{24})\left[\frac{1}{6.37 \times 10^6} - \frac{1}{6.371 \times 10^6}\right]}$ $\boxed{v_f = 140\ {}^{m}/_{s}}$

Kinetic and Mechanical Energy of Objects in Orbit

Consider an object of mass *m* revolving around a massive object of mass *M* ($M >> m$) at a radius of *r* with a speed of *v*. Its kinetic energy is $KE = \frac{1}{2}mv^2$. v^2 can be expressed in terms of *r* by equating the centripetal force of ${}^{mv^2}/_{r}$ to the gravitational force responsible for the centripetal force.

$$\frac{mv^2}{r} = \frac{GmM}{r^2} \quad \Longrightarrow \quad v^2 = \frac{GM}{r}$$

For an object revolving in an orbit $\left(KE = \frac{1}{2}mv^2\right)$

$$\boxed{KE = \frac{1}{2}\frac{GMm}{r}} \rightarrow \quad KE \text{ of an object in orbit}$$

And its mechanical energy is

$$ME = u + \frac{1}{2}mv^2$$
$$= -\frac{GMm}{r} + \frac{1}{2}\frac{GMm}{r}$$

$$\boxed{ME = -\frac{1}{2}\frac{GMm}{r}} \rightarrow \quad ME \text{ of an object in orbit}$$

From these relationships it follows that

$$\boxed{KE = -\frac{1}{2}u \ \& \ ME = \frac{1}{2}u}$$

Example: A satellite of mass 100,000 kg is revolving at an altitude of 1000 m. How much energy is needed to put it in orbit at an altitude of 2000 m.

Solution:

The energy needed is equal to the difference between the mechanical energies of both orbits.

$r_i = R_E + h_i$ $= 6.37 \times 10^6 + 1000$ $= 6.371 \times 10^6\ m$	$r_f = R_E + h_f$ $= 6.37 \times 10^6 + 2000$ $= 6.372 \times 10^6\ m$
$m = 100{,}000\ kg$	$M = 5.98 \times 10^{24}\ kg$

$$\text{Energy needed} = \Delta ME = ME_f - ME_i$$
$$= \left(-\frac{1}{2}\frac{GMm}{r_f}\right) - \left(-\frac{1}{2}\frac{GMm}{r_i}\right)$$
$$= \frac{1}{2}GMm\left(\frac{1}{r_f} - \frac{1}{r_f}\right)$$
$$= \frac{1}{2}(6.67 \times 10^{-11})(10^5)(5.98 \times 10^{24})(10^{-6})\left(\frac{1}{6.37} - \frac{1}{6.372}\right)$$
$$= \underline{4.9 \times 10^8\ J}$$

Escape Velocity

Escape velocity is the minimum velocity for which an object will never return to earth when propelled upwards from the surface of earth.

The minimum velocity is the velocity for which the final speed is zero when the object is free from gravitational field.

When $v_i = v_{min} = v_{esc},\ v_f = 0$

From conservation of mechanical energy

$$-\frac{Gm_Em}{r_i} + \frac{1}{2}mv_{esc}^2 = -\frac{Gm_Em}{r_f} + \cancel{\frac{1}{2}mv_f^2} \Rightarrow 0\ (v_f = 0)$$
$$\frac{1}{2}mv_{esc}^2 = Gm_Em\left[\frac{1}{r_i} - \frac{1}{r_f}\right]$$

$r_i = R_E$

$r_f = R_E + h$

$$\frac{1}{2}\cancel{m}v_{esc}^2 = Gm_E\cancel{m}\left[\frac{1}{R_E} - \frac{1}{R_E + h}\right]$$

The objects will be free from gravitational field when the two objects are separated by an infinite distance ($h \rightarrow \infty$). When $h \rightarrow \infty$ the 2^{nd} term will be zero and we get

$$v_{esc}^2 = \frac{2Gm_E}{R_E}$$

$$\therefore \quad \boxed{v_{esc} = \sqrt{\frac{2Gm_E}{R_E}}}$$

$v_{esc} \rightarrow$ escape velocity (velocity for which the object will never be back)
$m_E \rightarrow$ mass of earth
$R_E \rightarrow$ radius of earth

We see that the escape velocity does not depend on the mass of being propelled. It is a constant for all objects and its value is

$$v_{esc} = \sqrt{\frac{2Gm_E}{R_E}} = \sqrt{\frac{2(6.67 \times 10^{-11})(5.98 \times 10^{24})}{6.37 \times 10^6}}$$

$$\boxed{v_{esc} = 1.12 \times 10^4 \ {}^{m}/_{s}}$$

Example: Calculate the energy needed to propel an object of mass 2×10^{-3} kg for which the object will never return to the surface of earth (assume no air resistance).

Solution

The object has to be propelled with the escape velocity. The kinetic energy need for the escape velocity is .

$m = 2 \times 10^{-3}$

$$KE = \frac{1}{2} m v_{esc}^2 = \frac{1}{2}(2 \times 10^{-3})(1.12 \times 10^4)$$

$$KE = \underline{1.23 \times 10^5 J}$$