Solution Section 3.1 – Inverse Functions

Exercise

Find the inverse relation of the given set: $A=\{(-2,\ 2),\ (1,\ -1),\ (0,\ 4),\ (1,\ 3)\}$

Solution

$$\boxed{A^{-1}=\{(2,\ -2),\ (-1,\ 1),\ (4,\ 0),\ (3,\ 1)\}}$$

Exercise

Find the inverse relation of the given set: $B=\{(1,\ -1),\ (2,\ -2),\ (3,\ -3),\ (4,\ -4)\}$

Solution

$$\boxed{B^{-1}=\{(-1,\ 1),\ (-2,\ 2),\ (-3,\ 3),\ (-4,\ 4)\}}$$

Exercise

Find the inverse relation of the given set: $C=\{(a,\ -a),\ (b,\ -b),\ (c,\ -c)\}$

Solution

$$\boxed{C^{-1}=\{(-a,\ a),\ (-b,\ b),\ (-c,\ c)\}}$$

Exercise

Find the inverse relation of the given set: $D=\{(0,\ 0),\ (1,\ 1),\ (2,\ 2),\ (3,\ 3),\ (4,\ 4)\}$

Solution

$$\boxed{D^{-1}=\{(0,\ 0),\ (1,\ 1),\ (2,\ 2),\ (3,\ 3),\ (4,\ 4)\}}$$

Exercise

Find the inverse relation of the given set: $E=\{(-a,\ a),\ (-b,\ b),\ (-c,\ c),\ (-d,\ d)\}$

Solution

$$\boxed{E^{-1}=\{(a,\ -a),\ (b,\ -b),\ (c,\ -c),\ (d,\ -d)\}}$$

Exercise

Determine whether the function is one-to-one: $f(x) = 3x - 7$

Solution

$$f(a) = f(b)$$
$$3a - 7 = 3b - 7$$
$$3a = 3b - 7 + 7$$
$$3a = 3b$$

Divide both sides by 3

$$a = b$$

$\therefore$ The function is one-to-one

Exercise

Determine whether the function is one-to-one: $f(x) = x^2 - 9$

Solution

$1 \neq -1$ $1^2 - 9 \neq (-1)^2 - 9$ $-8 = -8 \rightarrow$ Contradict the definition	$f(a) = f(b)$ $a^2 - 9 = b^2 - 9$ $a^2 = b^2$ $a = \pm b$

$\therefore$ The function is ***not*** one-to-one

Exercise

Determine whether the function is one-to-one: $f(x) = \sqrt{x}$

Solution

$$f(a) = f(b)$$
$$\sqrt{a} = \sqrt{b}$$
$$\left(\sqrt{a}\right)^2 = \left(\sqrt{b}\right)^2$$

Square both sides

$$a = b$$

$\therefore$ The function is one-to-one

Exercise

Determine whether the function is one-to-one: $f(x)=\sqrt[3]{x}$

Solution

$f(a)=f(b)$

$\sqrt[3]{a}=\sqrt[3]{b}$

$\left(\sqrt[3]{a}\right)^3=\left(\sqrt[3]{b}\right)^3$ ***cube both sides***

$a=b$

$\therefore$ The function is one-to-one

Exercise

Determine whether the function is one-to-one: $f(x)=|x|$

Solution

$1\neq -1$

$|1|\neq |-1|$

$1\neq 1\ (\mathit{false})$

$\therefore$ The function is ***not*** one-to-one

Exercise

Determine whether the function is one-to-one $f(x)=\dfrac{2}{x+3}$

Solution

$f(a)=f(b)$

$\dfrac{2}{a+3}=\dfrac{2}{b+3}$

$2(b+3)=2(a+3)$

$b+3=a+3$

$a=b$

$\therefore f$ is one-to-one

Exercise

Determine whether the function is one-to-one $f(x)=(x-2)^3$

Solution

$f(a)=f(b)$

$$(a-2)^3 = (b-2)^3$$

$$\left[(a-2)^3\right]^{1/3} = \left[(b-2)^3\right]^{1/3}$$

$a-2 = b-2$ *Add 2 on both sides*

$a = b$

$\therefore$ Function is one-to-one

Exercise

Determine whether the function is one-to-one $y = x^2 + 2$

Solution

$f(a) = f(b)$

$a^2 + 2 = b^2 + 2$ *Subtract 2*

$a^2 = b^2$

$a = \pm\sqrt{b^2}$

$\therefore$ Function is ***not*** a one-to-one

The inverse function doesn't exist.

Exercise

Determine whether the function is one-to-one $f(x) = \dfrac{x+1}{x-3}$

Solution

$f(a) = f(b)$

$\dfrac{a+1}{a-3} = \dfrac{b+1}{b-3}$ ***Cross multiplication***

$(a+1)(b-3) = (b+1)(a-3)$

$ab - 3a + b - 3 = ab - 3b + a - 3$

$-4a = -4b$ ***Divide by -4***

$a = b$

$\therefore$ Function is one-to-one

Exercise

Given that $f(x)=5x+8$, use composition of functions to show that $f^{-1}(x)=\frac{x-8}{5}$

Solution

$$\left(f^{-1}\circ f\right)(x)=f^{-1}(f(x))$$
$$=f^{-1}(5x+8)$$
$$=\frac{(5x+8)-8}{5}$$
$$=\frac{5x+8-8}{5}$$
$$=\frac{5x}{5}$$
$$=x$$

$$\left(f\circ f^{-1}\right)(x)=f(f^{-1}(x))$$
$$=f^{-1}\left(\frac{x-8}{5}\right)$$
$$=5\left(\frac{x-8}{5}\right)+8$$
$$=x-8+8$$
$$=x$$

Exercise

Given the function $f(x)=(x+8)^3$

a) Find $f^{-1}(x)$

b) Graph f and f^{-1} in the same rectangular coordinate system

c) Find the domain and the range of f and f^{-1}

Solution

a) $y=(x+8)^3$ **Replace f(*x*) with *y***

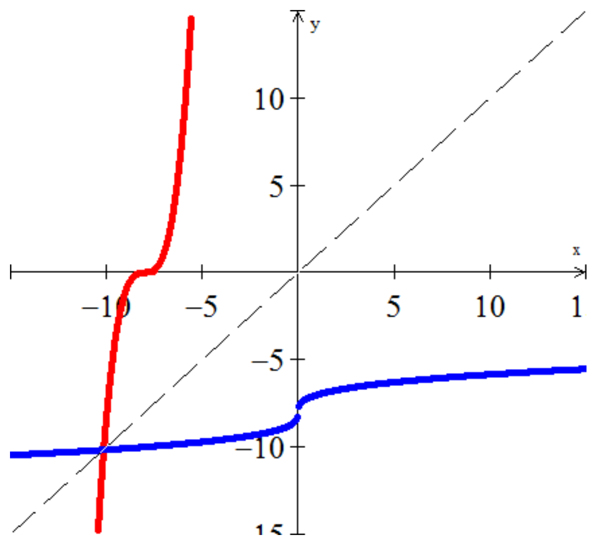
$x=(y+8)^3$ **Interchange *x* and *y***

$(x)^{1/3}=\left((y+8)^3\right)^{1/3}$

$x^{1/3}=y+8$ ***Subtract 8 from both sides***.

$$f^{-1}(x)=x^{1/3}-8$$

b)



c) Domain of f = Range of f^{-1}: $(-\infty,\infty)$

Range of f = Domain of f^{-1}: $(-\infty,\infty)$

Exercise

Prove that $f(x)$ *and* $g(x)$ are inverse functions of each other $\quad f(x)=4x; \quad g(x)=\frac{x}{4}$

Solution

$$f(g(x))=f\left(\frac{x}{4}\right)$$

$$=4\left(\frac{x}{4}\right)$$

$$=x$$

$$g(f(x))=g(4x)$$

$$=\frac{4x}{4}$$

$$=x$$

$\therefore\ f(x)$ *and* $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ *and* $g(x)$ are inverse functions of each other $\quad f(x)=2x; \quad g(x)=\frac{1}{2x}$

Solution

$$f(g(x))=f\left(\frac{1}{2x}\right)$$

$$= 2\left(\frac{1}{2x}\right)$$

$$= \frac{1}{x} \quad \neq x$$

$\therefore\ f(x)\ \ and\ \ g(x)$ are ***not*** inverse functions to each other

Exercise

Prove that $f(x)\ \ and\ \ g(x)$ are inverse functions of each other $\quad f(x) = 4x - 1; \quad g(x) = \frac{x+1}{4}$

Solution

$$f(g(x)) = f\left(\frac{x+1}{4}\right)$$

$$= 4\left(\frac{x+1}{4}\right) - 1$$

$$= x + 1 - 1$$

$$= x$$

$$g(f(x)) = g(4x - 1)$$

$$= \frac{4x - 1 + 1}{4}$$

$$= \frac{4x}{4}$$

$$= x$$

$\therefore\ f(x)\ \ and\ \ g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)\ \ and\ \ g(x)$ are inverse functions of each other $\quad f(x) = \frac{1}{2}x - \frac{3}{2}; \quad g(x) = 2x + 3$

Solution

$$f(g(x)) = f(2x + 3)$$

$$= \frac{1}{2}(2x + 3) - \frac{3}{2}$$

$$= x + \frac{3}{2} - \frac{3}{2}$$

$$= x$$

$$g(f(x)) = g\left(\frac{1}{2}x - \frac{3}{2}\right)$$

$$= 2\left(\frac{1}{2}x - \frac{3}{2}\right) + 3$$

$$= x - 3 + 3$$

$$= x$$

$\therefore\ f(x)\ \ and\ g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)\ \ and\ g(x)$ are inverse functions of each other $\quad f(x) = -\frac{1}{2}x - \frac{1}{2};\quad g(x) = -2x + 1$

Solution

$$f(g(x)) = f(-2x+1)$$
$$= -\frac{1}{2}(-2x+1) - \frac{1}{2}$$
$$= x - \frac{1}{2} - \frac{1}{2}$$
$$= \frac{1}{x} - 1 \quad \neq x$$

$\therefore\ f(x)\ \ and\ g(x)$ are ***not*** inverse functions to each other

Exercise

Prove that $f(x)\ \ and\ g(x)$ are inverse functions of each other $\quad f(x) = 3x + 2;\quad g(x) = \frac{1}{3}(x-2)$

Solution

$$f(g(x)) = f\left(\frac{x-2}{3}\right)$$
$$= 3\left(\frac{x-2}{3}\right) + 2$$
$$= x - 2 + 2$$
$$= x$$

$$g(f(x)) = g(3x+2)$$
$$= \frac{1}{3}(3x + 2 - 2)$$
$$= \frac{1}{3}(3x)$$
$$= x$$

$\therefore\ f(x)\ \ and\ g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ *and* $g(x)$ are inverse functions of each other $\quad f(x)=\frac{5}{x+3}; \quad g(x)=\frac{5}{x}-3$

Solution

$$f(g(x))=f\left(\frac{5}{x}-3\right)$$

$$=\frac{5}{\frac{5}{x}-3+3}$$

$$=\frac{5}{\frac{5}{x}}$$

$$=5\frac{x}{5}$$

$$\underline{=x}$$

$$g(f(x))=g\left(\frac{5}{x+3}\right)$$

$$=\frac{5}{\frac{5}{x+3}}-3$$

$$=5\left(\frac{x+3}{5}\right)-3$$

$$=x+3-3$$

$$\underline{=x}$$

$\therefore\ f(x)$ *and* $g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)$ *and* $g(x)$ are inverse functions of each other $\quad f(x)=\frac{2x}{x+1}; \quad g(x)=\frac{-x}{x-2}$

Solution

$$f(g(x))=f\left(\frac{-x}{x-2}\right)$$

$$=2\left(\frac{-x}{x-2}\right)\frac{1}{\frac{-x}{x-2}+1}$$

$$=\left(\frac{-2x}{x-2}\right)\frac{x-2}{-x+x-2}$$

$$=\frac{-2x}{-2}$$

$$\underline{=x}$$

$$g(f(x))=g\left(\frac{2x}{x+1}\right)$$

$$=-\left(\frac{2x}{x+1}\right)\frac{1}{\frac{2x}{x+1}-2}$$

$$=-\left(\frac{2x}{x+1}\right)\frac{x+1}{2x-2x-2}$$

$$=\frac{-2x}{-2}$$

$$\underline{=x}\,|$$

$\therefore\ f(x)\ \ and\ g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)\ \ and\ g(x)$ are inverse functions of each other $\quad f(x)=\frac{3x}{x-1};\quad g(x)=\frac{x}{x-3}$

Solution

$$f(g(x))=f\left(\frac{x}{x-3}\right)$$

$$=3\left(\frac{x}{x-3}\right)\frac{1}{\frac{x}{x-3}-1}$$

$$=\left(\frac{3x}{x-3}\right)\frac{x-3}{x-x+3}$$

$$=\frac{3x}{3}$$

$$\underline{=x}\,|$$

$$g(f(x))=g\left(\frac{3x}{x-1}\right)$$

$$=\left(\frac{3x}{x-1}\right)\frac{1}{\frac{3x}{x-1}-3}$$

$$=\left(\frac{3x}{x-1}\right)\frac{x-1}{3x-3x+3}$$

$$=\frac{3x}{3}$$

$$\underline{=x}\,|$$

$\therefore\ f(x)\ \ and\ g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)\ \ and\ g(x)$ are inverse functions of each other $\quad f(x)=x^3+2;\quad g(x)=\sqrt[3]{x-2}$

Solution

$$f(g(x)) = f\left(\sqrt[3]{x-2}\right)$$
$$= \left(\sqrt[3]{x-2}\right)^3 + 2$$
$$= x - 2 + 2$$
$$\underline{= x}$$

$$g(f(x)) = g\left(x^3 + 2\right)$$
$$= \sqrt[3]{x^3 + 2 - 2}$$
$$= \sqrt[3]{x^3}$$
$$\underline{= x}$$

$\therefore\ f(x)\ \ and\ g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)\ \ and\ g(x)$ are inverse functions of each other $\quad f(x) = (x+4)^3; \quad g(x) = \sqrt[3]{x} - 4$

Solution

$$f(g(x)) = f\left(\sqrt[3]{x} - 4\right)$$
$$= \left(\sqrt[3]{x} - 4 + 4\right)^3$$
$$= \left(\sqrt[3]{x}\right)^3$$
$$\underline{= x}$$

$$g(f(x)) = g\left((x+4)^3\right)$$
$$= \sqrt[3]{(x+4)^3} - 4$$
$$= x + 4 - 4$$
$$\underline{= x}$$

$\therefore\ f(x)\ \ and\ g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)\ \ and\ g(x)$ are inverse functions of each other $\quad f(x) = x^3 - 1; \ g(x) = \sqrt[3]{x+1}$

Solution

$$f(g(x)) = f\left(\sqrt[3]{x+1}\right)$$

$$= \left(\sqrt[3]{x+1}\right)^3 - 1$$
$$= x + 1 - 1$$
$$\underline{= x}$$

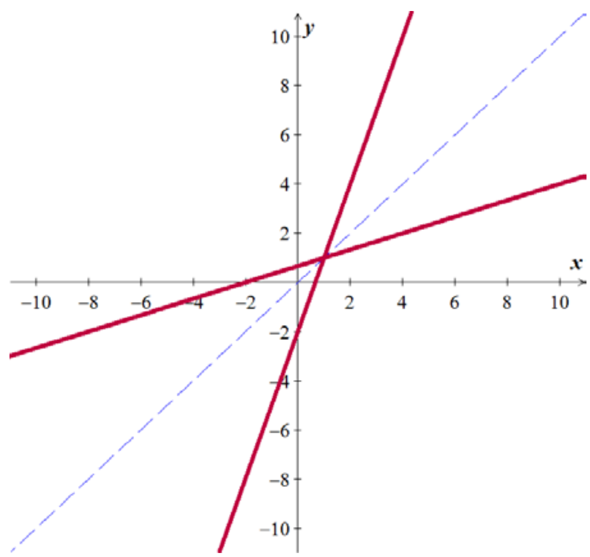
$$g\left(f(x)\right) = g\left(x^3 - 1\right)$$
$$= \sqrt[3]{x^3 - 1 + 1}$$
$$= \sqrt[3]{x^3}$$
$$\underline{= x}$$

$\therefore\ f(x)\ \ and\ \ g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)\ \ and\ \ g(x)$ are inverse functions of each other $\quad f(x) = 3x - 2; \quad g(x) = \dfrac{x+2}{3}$

Solution



$$f\left(g(x)\right) = f\left(\frac{x+2}{3}\right)$$
$$= 3\left(\frac{x+2}{3}\right) - 2$$
$$= x + 2 - 2$$
$$\underline{= x}$$

$$g\left(f(x)\right) = g(3x - 2)$$
$$= \frac{3x - 2 + 2}{3}$$
$$= \frac{3x}{x}$$
$$\underline{= x}$$

$\therefore\ f(x)\ \ and\ \ g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)\ \ and\ \ g(x)$ are inverse functions of each other

$f(x) = x^2 + 5,\ x \le 0 \quad g(x) = -\sqrt{x-5}, x \ge 5$

Solution

$$f\left(g(x)\right) = f\left(-\sqrt{x-5}\right)$$
$$= \left(-\sqrt{x-5}\right)^2 + 5$$

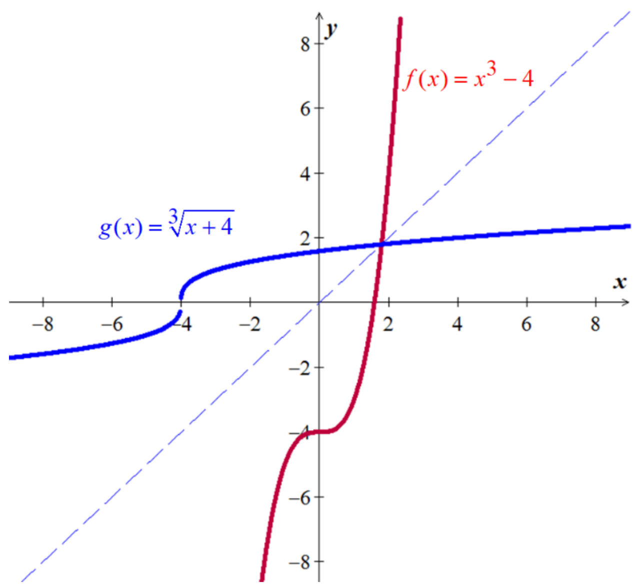
$$= x - 5 + 5$$
$$\underline{= x}$$
$$g\left(f(x)\right) = g\left(x^2 + 5\right)$$
$$= -\sqrt{x^2 + 5 - 5}$$
$$= -\sqrt{x^2}$$
$$= -|x| \quad x \le 0$$
$$= -(-x)$$
$$\underline{= x}$$

$\therefore\ f(x)\ \ and\ g(x)$ are inverse functions to each other

Exercise

Prove that $f(x)\ \ and\ g(x)$ are inverse functions of each other $\quad f(x) = x^3 - 4;\quad g(x) = \sqrt[3]{x+4}$

Solution



$$f\left(g(x)\right) = f\left(\sqrt[3]{x+4}\right)$$
$$= \left(\sqrt[3]{x+4}\right)^3 - 4$$
$$= x + 4 - 4$$
$$\underline{= x}$$
$$g\left(f(x)\right) = g\left(x^3 - 4\right)$$
$$= \sqrt[3]{x^3 - 4 + 4}$$
$$= \sqrt[3]{x^3}$$
$$\underline{= x}$$

$\therefore\ f(x)\ \ and\ g(x)$ are inverse functions to each other

Exercise

Find the inverse of $f(x) = (x-2)^3$

Solution

$$y = (x-2)^3$$
$$x = (y-2)^3$$

$$x^{1/3} = \left[(y-2)^3 \right]^{1/3}$$

$$x^{1/3} = y - 2$$

$$\sqrt[3]{x} + 2 = y$$

$$\boxed{f^{-1}(x) = \sqrt[3]{x} + 2}$$

Exercise

Find the inverse of $f(x) = \dfrac{x+1}{x-3}$

Solution

$$y = \frac{x+1}{x-3}$$

$$x = \frac{y+1}{y-3}$$

$$x(y-3) = y+1$$

$$xy - 3x = y + 1$$

$$xy - y = 3x + 1$$

$$y(x-1) = 3x + 1$$

$$\boxed{f^{-1}(x) = \frac{3x+1}{x-1}}$$

Exercise

Find the inverse of $f(x) = \dfrac{2x+1}{x-3}$

Solution

$$y = \frac{2x+1}{x-3}$$

$$x = \frac{2y+1}{y-3}$$

$$xy - 3x = 2y + 1$$

$$y(x-2) = 3x + 1$$

$$\boxed{f^{-1}(x) = \frac{3x+1}{x-2}}$$

Exercise

Determine the domain and range of f^{-1}: $f(x) = -\dfrac{2}{x-1}$ (*Hint*: first find the domain and range of f)

Solution

$x - 1 \neq 0 \Rightarrow x \neq 1$

Range of f^{-1} = Domain of f: $\mathbb{R} - \{1\}$ $\qquad (-\infty,\ 1) \cup (1,\ \infty)$

Domain of f^{-1} = Range of f: $\mathbb{R} - \{0\}$ $\qquad (-\infty,\ 0) \cup (0,\ \infty)$

Exercise

Determine the domain and range of f^{-1}: $f(x) = \dfrac{5}{x+3}$ (*Hint*: first find the domain and range of *f*)

Solution

Domain of f^{-1} = Range of f: $\mathbb{R} - \{0\}$ $\qquad (-\infty,\ 0) \cup (0,\ \infty)$

Range of f^{-1} = Domain of f: $\mathbb{R} - \{-3\}$ $\qquad (-\infty,\ -3) \cup (-3,\ \infty)$

Exercise

Determine the domain and range of f^{-1}: $f(x) = \dfrac{4x+5}{3x-8}$ (*Hint*: first find the domain and range of *f*)

Solution

Domain of f^{-1} = Range of f: $\mathbb{R} - \left\{\frac{8}{3}\right\}$ $\qquad \left(-\infty,\ \frac{8}{3}\right) \cup \left(\frac{8}{3},\ \infty\right)$

Range of f^{-1} = Domain of f: $\mathbb{R} - \left\{\frac{4}{3}\right\}$ $\qquad \left(-\infty,\ \frac{4}{3}\right) \cup \left(\frac{4}{3},\ \infty\right)$

Exercise

For the given function $\qquad f(x) = \dfrac{2x}{x-1}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{2a}{a-1} = \frac{2b}{b-1}$$

$$2ab - 2a = 2ab - 2b$$
$$-2a = -2b$$
$$\underline{a = b} \quad \checkmark$$

$\therefore\ f(x)$ is one-to-one function.

b) $y = \dfrac{2x}{x-1}$

$x = \dfrac{2y}{y-1}$

$xy - x = 2y$

$(x-2)y = x$

$y = \dfrac{x}{x-2} = f^{-1}(x)$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\mathbb{R} - \{1\}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\mathbb{R} - \{2\}$

Exercise

For the given function $f(x) = \dfrac{x}{x-2}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{a}{a-2} = \frac{b}{b-2}$$
$$ab - 2a = ab - 2b$$
$$-2a = -2b$$
$$\underline{a = b} \quad \checkmark$$

$\therefore\ f(x)$ is one-to-one function.

b) $y = \dfrac{x}{x-2}$

$x = \dfrac{y}{y-2}$

$xy - 2x = y$

$(x-1)y = 2x$

$$\boxed{f^{-1}(x)=\frac{2x}{x-1}}$$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\boxed{\mathbb{R}-\{2\}}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\boxed{\mathbb{R}-\{1\}}$

Exercise

For the given function $f(x)=\dfrac{x+1}{x-1}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$$\frac{a+1}{a-1}=\frac{b+1}{b-1}$$

$$ab-a+b-1=ab-b+a-1$$

$$-2a=-2b$$

$$\boxed{a=b} \quad \checkmark$$

$\therefore$ $f(x)$ is one-to-one function.

b) $y=\dfrac{x+1}{x-1}$

$$x=\frac{y+1}{y-1}$$

$$xy-x=y+1$$

$$(x-1)y=x+1$$

$$\boxed{f^{-1}(x)=\frac{x+1}{x-1}}$$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\boxed{\mathbb{R}-\{1\}}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\boxed{\mathbb{R}-\{1\}}$

Exercise $f(x)=\dfrac{2x+1}{x+3}$

For the given function

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$\frac{2a+1}{a+3} = \frac{2b+1}{b+3}$

$2ab + 6a + b + 3 = 2ab + 6b + a + 3$

$5a = 5b$

$\underline{a = b}|$ ✓

$\therefore$ $f(x)$ is one-to-one function.

b) $y = \frac{2x+1}{x+3}$

$x = \frac{2y+1}{y+3}$

$xy + 3x = 2y + 1$

$(x-2)y = -3x + 1$

$\underline{f^{-1}(x) = \frac{-3x+1}{x-2}}|$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\underline{\mathbb{R} - \{-3\}}|$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\underline{\mathbb{R} - \{2\}}|$

Exercise

For the given function $f(x) = \frac{3x-1}{x-2}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$\frac{3a-1}{a-2} = \frac{3b-1}{b-2}$

$3ab - 6a - b + 2 = 3ab - 6b - a + 2$

$-5a = -5b$

$\underline{a = b}|$ ✓

$\therefore$ $f(x)$ is one-to-one function.

b) $y = \dfrac{3x-1}{x-2}$

$x = \dfrac{3y-1}{y-2}$

$xy - 2x = 3y - 1$

$(x-3)y = 2x - 1$

$\boxed{f^{-1}(x) = \dfrac{2x-1}{x-3}}$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\boxed{\mathbb{R} - \{2\}}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\boxed{\mathbb{R} - \{3\}}$

Exercise

For the given function $f(x) = \dfrac{3x-2}{x+4}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$\dfrac{3a-2}{a+4} = \dfrac{3b-2}{b+4}$

$3ab + 12a - 2b - 8 = 3ab + 12b - 2a - 8$

$14a = 14b$

$\boxed{a = b}$ ✓

$\therefore$ $f(x)$ is one-to-one function.

b) $y = \dfrac{3x-2}{x+4}$

$x = \dfrac{3y-2}{y+4}$

$xy + 4x = 3y - 2$

$(x-3)y = -4x - 2$

$\boxed{f^{-1}(x) = \dfrac{-4x-2}{x-3}}$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\boxed{\mathbb{R} - \{-4\}}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\boxed{\mathbb{R} - \{3\}}$

Exercise

For the given function $f(x)=\dfrac{-3x-2}{x+4}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$$\frac{-3a-2}{a+4}=\frac{-3b-2}{b+4}$$

$$-3ab-12a-2b-8=-3ab-12b-2a-8$$

$$-10a=-10b$$

$$\underline{a=b} \quad \checkmark$$

$\therefore f(x)$ is one-to-one function.

b) $y=\dfrac{-3x-2}{x+4}$

$$x=\frac{-3y-2}{y+4}$$

$$xy+4x=-3y-2$$

$$(x+3)y=-4x-2$$

$$f^{-1}(x)=\frac{-4x-2}{x+3}$$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\mathbb{R}-\{-4\}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\mathbb{R}-\{-3\}$

Exercise

For the given function $f(x)=\sqrt{x-1} \quad x\ge 1$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$$\sqrt{a-1}=\sqrt{b-1}$$

$$\left(\sqrt{a-1}\right)^2 = \left(\sqrt{b-1}\right)^2$$
$a-1=b-1$

$\boxed{a=b}$ ✓

$\therefore\ f(x)$ is one-to-one function.

b) $y=\sqrt{x-1}$

$x=\sqrt{y-1}$

$x^2=y-1$

$y=x^2+1$

$\boxed{f^{-1}(x)=x^2+1 \quad x\geq 0}$

c) Domain of $f(x)$ = Range of $f^{-1}(x)$: $\boxed{[1,\ \infty)}$

Range of $f(x)$ = Domain of $f^{-1}(x)$: $\boxed{[0,\ \infty)}$

Exercise

For the given function $f(x)=\sqrt{2-x} \quad x\leq 2$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$\sqrt{2-a}=\sqrt{2-b}$

$\left(\sqrt{2-a}\right)^2=\left(\sqrt{2-b}\right)^2$

$2-a=2-b$

$\boxed{a=b}$ ✓

$\therefore\ f(x)$ is one-to-one function.

b) $y=\sqrt{2-x}$

$x=\sqrt{2-y}$

$x^2=2-y$

$y=2-x^2$

$\boxed{f^{-1}(x) = 2 - x^2 \quad x \ge 0}$

c) Domain of $f(x)$ = Range of $f^{-1}(x)$: $\boxed{(-\infty,\ 2]}$

Range of $f(x)$ = Domain of $f^{-1}(x)$: $\boxed{[0,\ \infty)}$

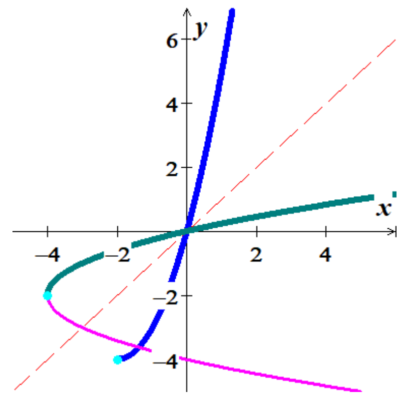
Exercise

For the given function $\quad f(x) = x^2 + 4x \quad x \ge -2$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$



Solution

$x_{vertex} = -\frac{4}{2}$

$\boxed{= -2}$

$f(-2) = 4 - 8$

$\boxed{= -4}$

$Vertex = (-2,\ -4)$

a) Since, $f(x)$ is a restricted function with $x \ge -2$.

$x = -2$ is the line symmetry, therefore; $f(x)$ is one-to-one function.

b) $y = x^2 + 4x$

$x = y^2 + 4y$

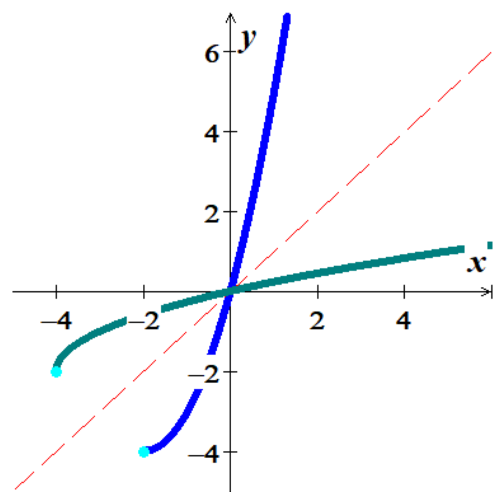
$y^2 + 4y - x = 0$

$y = \dfrac{-4 \pm \sqrt{16 + 4x}}{2}$

$= \dfrac{-4 \pm 2\sqrt{4 + x}}{2}$

$= -2 + \sqrt{x + 4}$

$\boxed{f^{-1}(x) = -2 + \sqrt{x + 4} \quad x \ge 0}$



c) Domain of $f(x)$ = Range of $f^{-1}(x)$: $\boxed{[-2,\ \infty)}$

Range of $f(x)$ = Domain of $f^{-1}(x)$: $\boxed{[-4,\ \infty)}$

Exercise

For the given function $f(x) = 3x + 5$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$3a + 5 = 3b + 5$

$3a = 3b$

$a = b$

$\therefore f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = 3x + 5$

$x = 3y + 5$ *Interchange x and y*

$x - 5 = 3y$ *Solve for y*

$\frac{x-5}{3} = y$

$f^{-1}(x) = \frac{x-5}{3}$

c) Domain of f^{-1} = Range of f: $\mathbb{R}$

Range of f^{-1} = Domain of f: $\mathbb{R}$

Exercise

For the given function $f(x) = \frac{1}{3x-2}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$\frac{1}{3a-2} = \frac{1}{3b-2}$

$3b - 2 = 3a - 2$

$3b = 3a$

$a = b$

$\therefore f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = \dfrac{1}{3x-2}$

$x = \dfrac{1}{3y-2}$ ***Interchange x and y***

$x(3y-2) = 1$ ***Solve for y***

$3xy - 2x = 1$

$3xy = 1 + 2x$

$\boxed{f^{-1}(x) = \dfrac{1+2x}{3x}}$

c) Domain of f^{-1} = Range of f: $\boxed{\mathbb{R} - \left\{\dfrac{2}{3}\right\}}$

Range of f^{-1} = Domain of f: $\boxed{\mathbb{R} - \{0\}}$

Exercise

For the given function $f(x) = \dfrac{3x+2}{2x-5}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$\dfrac{3a+2}{2a-5} = \dfrac{3b+2}{2b-5}$

$6ab - 15a + 4b - 10 = 6ab - 15b + 4a - 10$

$19a = 19b$

$a = b$

$\therefore$ $f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = \dfrac{3x+2}{2x-5}$

$x = \dfrac{3y+2}{2y-5}$ ***Interchange x and y***

$2xy - 5x = 2y + 2$ ***Solve for y***

$(2x-3)y = 5x + 2$

$\boxed{f^{-1}(x) = \dfrac{5x+2}{2x-3}}$

c) Domain of f^{-1} = Range of f: $\boxed{\mathbb{R} - \left\{\dfrac{5}{2}\right\}}$

Range of f^{-1} = Domain of f: $\boxed{\mathbb{R} - \left\{\dfrac{3}{2}\right\}}$

Exercise

For the given function $f(x)=\dfrac{4x}{x-2}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$\dfrac{4a}{a-2}=\dfrac{4b}{b-2}$

$4ab-8a=4ab-8b$

$-8a=-8b$

$a=b$

$\therefore\ f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y=\dfrac{4x}{x-2}$

$x=\dfrac{4y}{y-2}$

$xy-2x=4y$

$(x-4)y=2x$

$f^{-1}(x)=\dfrac{2x}{x-4}$

c) Domain of f^{-1} = Range of f: $\mathbb{R}-\{2\}$

Range of f^{-1} = Domain of f: $\mathbb{R}-\{4\}$

Exercise

For the given function $f(x)=2-3x^2;\quad x\le 0$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$2-3a^2=2-3b^2$

$-3a^2=-3b^2$

$a^2=b^2$

$a=b$ since $x\le 0$

$\therefore\ f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = 2 - 3x^2$

$x = 2 - 3y^2$

$3y^2 = 2 - x$

$y^2 = \dfrac{2-x}{3}$

$\boxed{f^{-1}(x) = -\sqrt{\dfrac{2-x}{3}}}$ ***Since x < 0***

c) Domain of f^{-1} = Range of f: $\mathbb{R}$

Range of f^{-1} = Domain of f: $\mathbb{R}$

Exercise

For the given function $f(x) = 2x^3 - 5$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$2a^3 - 5 = 2b^3 - 5$

$a^3 = b^3$

$a = b$

$\therefore\ f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = 2x^3 - 5$

$y + 5 = 2x^3$

$\dfrac{y+5}{2} = x^3$

$x = \sqrt[3]{\dfrac{y+5}{2}}$

$\boxed{f^{-1}(x) = \sqrt[3]{\dfrac{x+5}{2}}}$

c) Domain of f^{-1} = Range of f: $\mathbb{R}$

Range of f^{-1} = Domain of f: $\mathbb{R}$

Exercise

For the given function $f(x)=\sqrt{3-x}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$$\left(\sqrt{3-a}\right)^2=\left(\sqrt{3-b}\right)^2$$

$$3-a=3-b$$

$$a=b$$

$\therefore\ f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y=\sqrt{3-x}$ $\qquad y\geq 0$

$$y=\sqrt{3-x}$$

$$y^2=3-x$$

$$x=3-y^2 \qquad x\geq 0$$

$$\boxed{f^{-1}(x)=3-x^2}$$

c) Domain of f^{-1} = Range of f: $\boxed{(-\infty,\ 3]}$

Range of f^{-1} = Domain of f: $\boxed{[0,\ \infty)}$

Exercise

For the given function $f(x)=\sqrt[3]{x}+1$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$$\sqrt[3]{a}+1=\sqrt[3]{b}+1$$

$$\left(\sqrt[3]{a}\right)^3=\left(\sqrt[3]{b}\right)^3$$

$$a=b$$

$\therefore\ f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y=\sqrt[3]{x}+1$

$$y = \sqrt[3]{x} + 1$$

$$y - 1 = \sqrt[3]{x}$$

$$(y-1)^3 = x$$

$$f^{-1}(x) = (x-1)^3$$

c) Domain of f^{-1} = Range of f: $\mathbb{R}$

Range of f^{-1} = Domain of f: $\mathbb{R}$

Exercise

For the given function $f(x) = \left(x^3 + 1\right)^5$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\left(a^3 + 1\right)^5 = \left(b^3 + 1\right)^5$$

$$a^3 + 1 = b^3 + 1$$

$$a^3 = b^3$$

$$a = b$$

$\therefore$ $f(x)$ is **1–1** **&** $f^{-1}(x)$ exists

b) $y = \left(x^3 + 1\right)^5$

$$y = \left(x^3 + 1\right)^5$$

$$\sqrt[5]{y} = x^3 + 1$$

$$\sqrt[5]{y} - 1 = x^3$$

$$x = \sqrt[3]{\sqrt[5]{y} - 1}$$

$$f^{-1}(x) = \sqrt[3]{\sqrt[5]{x} - 1}$$

c) Domain of f^{-1} = Range of f: $\mathbb{R}$

Range of f^{-1} = Domain of f: $\mathbb{R}$

Exercise

For the given function $f(x) = x^2 - 6x; \quad x \ge 3$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$a^2 - 6a = b^2 - 6b$

$a^2 - b^2 = 6a - 6b$

$(a-b)(a+b) = 6(a-b)$

$a = b$

$\therefore f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = x^2 - 6x$

$x^2 - 6x - y = 0$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-y)}}{2(1)}$$

$$= \frac{6 \pm 4\sqrt{9+y}}{2}$$

$$= 3 \pm \sqrt{9+y}$$

Since $x \ge 3 \Rightarrow$ we can select $x = 3 + \sqrt{y+9}$

$\therefore \ \underline{f^{-1}(x) = 3 + \sqrt{x+9}}$

c) Domain of f^{-1} = Range of f: $\underline{\mathbb{R}: \ \ge 3}$

Range of f^{-1} = Domain of f: $\underline{\ge -9}$

Exercise

For the given function $f(x) = (x-2)^3$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$(a-2)^3 = (b-2)^3$

$a - 2 = b - 2$

$a = b$

$\therefore\ f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = (x-2)^3$

$x = (y-2)^3$

$x^{1/3} = \left[(y-2)^3\right]^{1/3}$

$x^{1/3} = y - 2$

$\sqrt[3]{x} + 2 = y$

$\therefore\ \underline{f^{-1}(x) = \sqrt[3]{x} + 2}$

c) Domain of f^{-1} = Range of f: $\underline{\mathbb{R}}$

Range of f^{-1} = Domain of f: $\underline{\mathbb{R}}$

Exercise

For the given function $f(x) = \dfrac{x+1}{x-3}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

<u>Solution</u>

a) $f(a) = f(b)$

$\dfrac{a+1}{a-3} = \dfrac{b+1}{b-3}$

$ab - 3a + b - 3 = ab - 3b + a - 3$

$-4a = -4b$

$a = b$

$\therefore\ f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = \dfrac{x+1}{x-3}$

$x = \dfrac{y+1}{y-3}$

$x(y-3) = y + 1$

$xy - 3x = y + 1$

$xy - y = 3x + 1$

$y(x-1) = 3x + 1$

$$y = \boxed{\frac{3x+1}{x-1} = f^{-1}(x)}$$

c) Domain of f^{-1} = Range of f : $\boxed{\mathbb{R}-\{3\}}$

Range of f^{-1} = Domain of f : $\boxed{\mathbb{R}-\{1\}}$

Exercise

For the given function $f(x) = \dfrac{2x+1}{x-3}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{2a+1}{a-3} = \frac{2b+1}{b-3}$$

$$2ab-6a+b-3 = 2ab-6b+a-3$$

$$-7a = -7b$$

$$a = b$$

$\therefore$ $f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = \dfrac{2x+1}{x-3}$

$$x = \frac{2y+1}{y-3}$$

$$xy-3x = 2y+1$$

$$y(x-2) = 3x+1$$

$$y = \boxed{\frac{3x+1}{x-2} = f^{-1}(x)}$$

c) Domain of f^{-1} = Range of f : $\boxed{\mathbb{R}-\{3\}}$

Range of f^{-1} = Domain of f : $\boxed{\mathbb{R}-\{2\}}$

Exercise

The function $w(x)=2x+24$ can be used to convert a U.S. women's shoe size into an Italian women's shoe size. Determine the function $w^{-1}(x)$ that can use to convert an Italian women's shoe size to its equivalent U.S. shoe size.

Solution

$x=2w^{-1}(x)+24$

$2w^{-1}(x)=x-24$

$\boxed{w^{-1}(x)=\frac{1}{2}x-12}$

Exercise

The function $m(x)=1.3x-4.7$ can be used to convert a U.S. men's shoe size into an U.K. women's shoe size. Determine the function $m^{-1}(x)$ that can used to convert an U.K. men's shoe size to its equivalent U.S. shoe size.

Solution

$x=1.3m^{-1}(x)-4.7$

$1.3m^{-1}(x)=x+4.7$

$\frac{13}{10}m^{-1}(x)=x+\frac{47}{10}$

$\boxed{m^{-1}(x)=\frac{10}{13}x+\frac{47}{13}}$

$\boxed{w^{-1}(x)=\frac{1}{2}x-12}$

Exercise

A catering service use the function $c(x)=\dfrac{300+12x}{x}$ to determine the amount, in *dollars*, it charges per person for a sit-down dinner, where x is the number of people in attendance.

a) Find $c(30)$ and explain what it represents

b) Find $c^{-1}(x)$

c) Use $c^{-1}(x)$ to determine how many people attended a dinner for which the cost per person was \$15.00

Solution

a) $c(30)=\dfrac{300+12(30)}{30}$

$=\dfrac{30+36}{3}$

$=\dfrac{66}{3}$

$\underline{= \$22}$

Catering service will charge $22 per person to a sit-down dinner.

b) $cx=300+12x$

$(c-12)x=300$

$\underline{c^{-1}(x)=\dfrac{300}{x-12}}$

c) $c^{-1}(15)=\dfrac{300}{15-12}$

$=\dfrac{300}{3}$

$\underline{=100}$

Exercise

A landscaping service use the function $c(x)=\dfrac{600+140x}{x}$ to determine the amount, in *dollars*, it charges per tree to deliver, where x is the number of trees.

a) Find $c(5)$ and explain what it represents

b) Find $c^{-1}(x)$

c) Use $c^{-1}(x)$ to determine how many trees were delivered for which the cost per tree was $160.00

Solution

d) $c(5)=\dfrac{600+140(5)}{5}$

$=\dfrac{1,300}{5}$

$\underline{= \$260}$

Landscaping service will charge $260 per tree to deliver.

e) $y=\dfrac{600+140x}{x}$

$x=\dfrac{600+140y}{y}$

$xy=600+140y$

$(x-140)y=600$

$$\boxed{c^{-1}(x) = \frac{600}{x-140}}$$

f) $c^{-1}(160) = \dfrac{600}{160-140}$

$$= \frac{600}{20}$$

$$\boxed{= 30}$$