Exercise 1

Ox and Oy are bisector to 2 adjacent acute angles, $\widehat{AOB}$ and $\widehat{BOC}$ where the difference is $36°$ and $\widehat{AOC} = 90°$. Oz is the bisector of the angle $\widehat{xOy}$. Determine the angle $\widehat{BOz}$

Solution

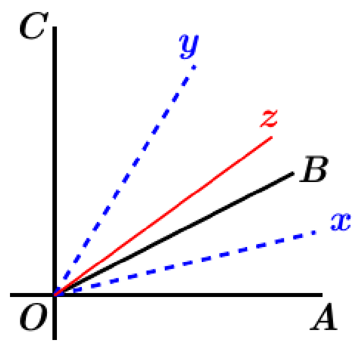
$$\widehat{BOC} - \widehat{AOB} = 36°$$

$$\underline{\widehat{BOC} + \widehat{AOB} = 90°}$$

$$2\ \widehat{BOC} = 126°$$

$$\widehat{BOC} = 63°$$

$$\widehat{AOB} = 27°$$



$$\widehat{xOB} = \frac{1}{2}\widehat{AOB}$$

$$= \frac{27}{2}^{\circ}$$

$$\widehat{BOy} = \frac{63}{2}^{\circ}$$

$$\widehat{xOy} = \widehat{xOB} + \widehat{BOy}$$

$$= \frac{1}{2}(63° + 27°)$$

$$= 45°$$

$$\widehat{xOz} = \frac{45}{2}^{\circ}$$

$$\widehat{BOZ} = \widehat{xOz} - \widehat{xOB}$$

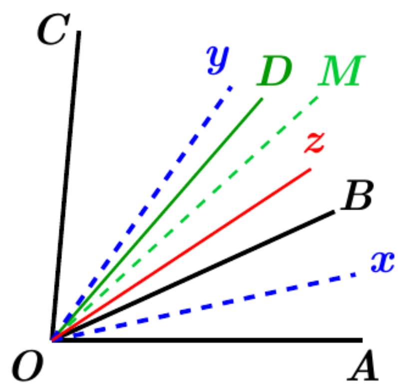
$$= \frac{1}{2}(45° - 27°)$$

$$= 9°$$

Exercise 2

Ox and Oy are bisector to 2 adjacent acute angles, $\widehat{AOB}$ and $\widehat{BOC}$ where the difference is $36°$. Oz is the bisector of the angle $\widehat{xOy}$. Determine the angle $\widehat{BOz}$

Solution



Ox is the bisector $\widehat{AOB}$ $\quad(1)$

OB is the bisector $\widehat{AOD}$ $\quad(2)$

OM is the bisector $\widehat{AOC}$ $\quad(3)$

Oz is the bisector $\widehat{xOy}$ $\quad(4)$

Oy is the bisector $\widehat{BOC}$ $\quad(5)$

$$\widehat{BOC}-\widehat{AOB}=36°$$
$$\widehat{BOC}-\widehat{BOD}=36°$$
$$\boxed{\widehat{DOC}=36°}$$

$$(3)\rightarrow \quad \widehat{AOM}=\frac{1}{2}\widehat{AOC}$$
$$=\frac{1}{2}\left(2\widehat{AOB}+\widehat{DOC}\right)$$
$$=\frac{1}{2}\left(2\widehat{AOB}+36°\right)$$
$$=\widehat{AOB}+18°$$

$$\widehat{BOM}=\widehat{AOM}-\widehat{AOB}$$
$$=\widehat{AOB}+18°-\widehat{AOB}$$
$$\boxed{=18°}$$

$$(1)\rightarrow \quad \widehat{BOx}=\frac{1}{2}\widehat{AOB}$$
$$(4)\rightarrow \quad \widehat{BOy}=\frac{1}{2}\widehat{BOC}$$
$$(1)+(4)\rightarrow \quad \widehat{xOy}=\frac{1}{2}\widehat{AOC}$$
$$(3)\rightarrow \quad \widehat{AOM}=\frac{1}{2}\widehat{AOC}=\widehat{xOy}$$

$$\widehat{BOz}=\widehat{xOz}-\widehat{xOB}$$
$$=\frac{1}{2}\left(\widehat{xOy}-\widehat{AOB}\right)$$
$$=\frac{1}{2}\left(\widehat{AOM}-\widehat{AOB}\right)$$
$$=\frac{1}{2}\widehat{BOM}$$
$$\boxed{=9°}$$

Exercise 3

Four consecutive half-lines (segments): *OA*, *OB*, *OC*, and *OD* formed angles such as

$$\widehat{DOA} = \widehat{COB} = 2\widehat{AOB} \qquad \text{and} \qquad \widehat{COD} = 3\widehat{AOB}$$

Calculate the angles to demonstrate that the bisectors of $\widehat{AOB}$ and $\widehat{COD}$ are in a straight line.

Solution

$\widehat{AOB} + \widehat{BOC} + \widehat{COD} + \widehat{DOA} = 360°$

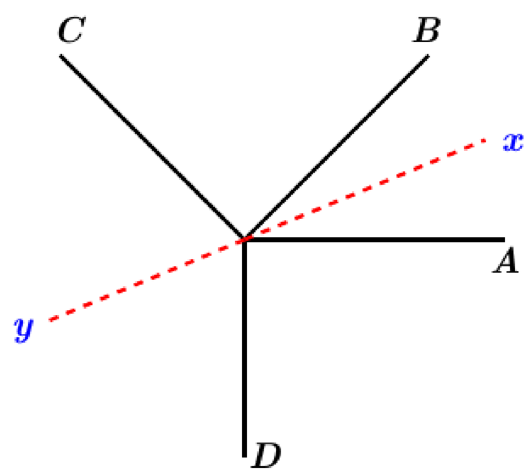
$\widehat{AOB} + 2\widehat{AOB} + 3\widehat{AOB} + 2\widehat{AOB} = 360°$

$8\widehat{AOB} = 360°$

$\boxed{\widehat{AOB} = 45°}$

$\boxed{\widehat{DOA} = \widehat{COB} = 90°}$

$\boxed{\widehat{COD} = 135°}$



Let:

Ox is the bisector $\widehat{AOB}$

Oy is the bisector $\widehat{COD}$

$$\begin{aligned}\widehat{xOy} &= \widehat{xOB} + \widehat{BOC} + \widehat{COy} \\ &= \frac{1}{2}\widehat{AOB} + 90° + \frac{1}{2}\widehat{COD} \\ &= \frac{1}{2}(45° + 135°) + 90° \\ &\boxed{= 180°}\end{aligned}$$

Therefore; the bisectors of $\widehat{AOB}$ and $\widehat{COD}$ are in a straight line

Exercise 4

The segments OA and OB formed with OX the angles α and β.

a) Demonstrate that the bisector OC of the angle $\widehat{AOB}$ made with OX an angle $\frac{\alpha+\beta}{2}$.

b) Examine the cases where

   i. $\alpha+\beta=90°$

   ii. $\alpha+\beta=180°$

Solution

Given:

$$\widehat{XOA}=\alpha \quad \& \quad \widehat{XOB}=\beta$$

$$\widehat{AOC}=\frac{1}{2}\widehat{AOB}$$

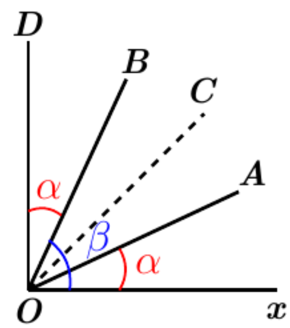
$$=\frac{\beta-\alpha}{2}$$

a) $\widehat{XOC}=\widehat{XOA}+\widehat{AOC}$

$$=\alpha+\frac{\beta-\alpha}{2}$$

$$\boxed{=\frac{\alpha+\beta}{2}}$$

b) ***i.*** If $\alpha+\beta=90°$, then



$\widehat{XOC}=45°$

Let: $\widehat{XOD}=90°$ that implies OC is the bisector of $\widehat{XOD}$

Since OC is the bisector of $\widehat{AOB}$, then

$\widehat{BOD}=90°-\beta$ $\quad\quad\quad\quad$ $\beta=90°-\alpha$

$=90°-90°+\alpha$

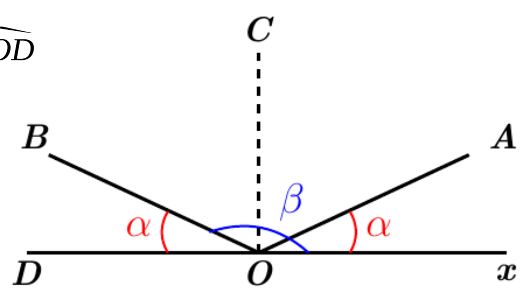
$\boxed{=\alpha}$

ii. If $\alpha+\beta=180°$, then

$\widehat{XOC}=90°$

Let: $\widehat{XOD}=180°$ that implies OC is the bisector of $\widehat{XOD}$

Since OC is the bisector of $\widehat{AOB}$, then



$\widehat{BOD}=180°-\beta$ $\quad\quad\quad\quad$ $\beta=180°-\alpha$

$=180°-180°+\alpha$

$\boxed{=\alpha}$

Exercise 5

A point O takes on an infinite right $x'Ox$ be conducted the same side half-lines OA and OB, as well as the bisectors of angles $\widehat{xOA}$, $\widehat{AOB}$, and $\widehat{BOx'}$.

Calculate the angles of the figure such that the bisector of the angle $\widehat{AOB}$ is perpendicular to $x'Ox$ and the bisectors of the extreme angles formed an angle of $100°$.

Solution

Given: $\widehat{zOz'} = 100°$

$\widehat{xOC} = 90°$

OC is the bisector $\widehat{AOB}$

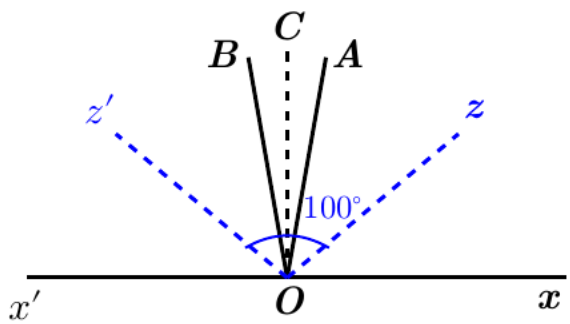
$\widehat{AOC} = \widehat{COB}$

Oz is the bisector $\widehat{xOA}$

$\widehat{xOz} = \widehat{zOA}$

Oz' is the bisector $\widehat{x'OB}$

$\widehat{x'Oz'} = \widehat{z'OB}$



$$\widehat{xOz} = \frac{180° - 100°}{2}$$

$$= 40°$$

$$\widehat{AOB} = 2\widehat{AOC}$$

$$= 2\left(90° - 2\widehat{xOz}\right)$$

$$= 2\left(90° - 80°\right)$$

$$= 20°$$

Exercise 6

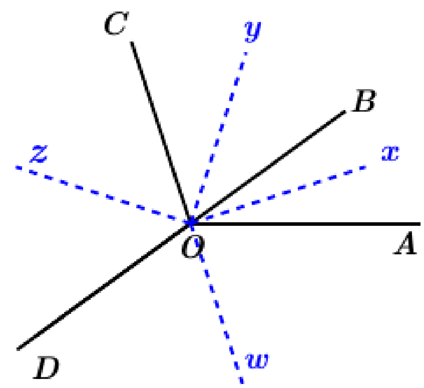
Four consecutive half-lines *OA*, *OB*, *OC*, and *OD* formed four adjacent consecutive angles which are between them like 1, 2, 3, 4.
Calculate the angles and the adjacent consecutive angles formed by their bisectors.

Solution

$\widehat{AOB}+\widehat{BOC}+\widehat{COD}+\widehat{DOA}=360°$

$\widehat{AOB}+2\widehat{AOB}+3\widehat{AOB}+4\widehat{AOB}=360°$

$10\widehat{AOB}=360°$



$\boxed{\widehat{AOB}=36°}$

$\boxed{\widehat{BOC}=72°}$

$\boxed{\widehat{COD}=108°}$

$\boxed{\widehat{DOA}=144°}$

$$\begin{aligned}\widehat{xOy}&=\frac{1}{2}\widehat{AOB}+\frac{1}{2}\widehat{BOC}\\&=\frac{1}{2}36°+\frac{1}{2}72°\\&=18°+36°\\&\boxed{=54°}\end{aligned}$$

$$\begin{aligned}\widehat{yOz}&=\frac{1}{2}\widehat{BOC}+\frac{1}{2}\widehat{COD}\\&=\frac{1}{2}72°+\frac{1}{2}108°\\&=36°+54°\\&\boxed{=90°}\end{aligned}$$

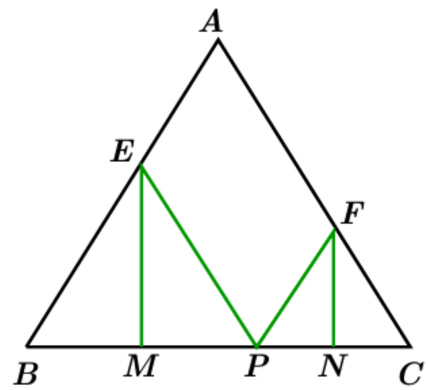
$$\begin{aligned}\widehat{zOw}&=\frac{1}{2}\widehat{COD}+\frac{1}{2}\widehat{DOA}\\&=\frac{1}{2}108°+\frac{1}{2}144°\\&=54°+72°\\&\boxed{=126°}\end{aligned}$$

$$\begin{aligned}\widehat{wOx}&=\frac{1}{2}\widehat{DOA}+\frac{1}{2}\widehat{AOB}\\&=\frac{1}{2}144°+\frac{1}{2}36°\\&=72°+18°\\&\boxed{=90°}\end{aligned}$$

Exercise 7

A point P is on the base BC of an isosceles triangle ABC. The two points M and N are the middle points of the segments PB and PC, respectively, which lead the perpendicular to the base BC; these perpendiculars meet AB in E, AC in F.
Demonstrate that the angle EPF is equal to A.

Solution



$$\widehat{BAC} = 180° - \widehat{ABC} - \widehat{ACB}$$

M is the middle of the segment BP and $EM \perp$ to BP, therefore

$$EB = EP \quad \& \quad \widehat{EBP} = \widehat{EPB}$$

N is the middle of the segment CP and $FN \perp$ to CP, therefore

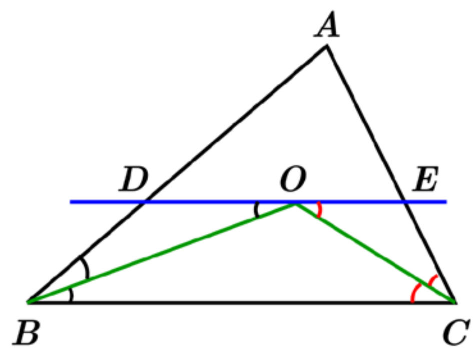
$$FP = FP \quad \& \quad \widehat{FPC} = \widehat{FCP}$$

$$\begin{aligned}\widehat{EPF} &= 180° - \widehat{CPF} - \widehat{BPE} \\ &= 180° - \widehat{PFC} - \widehat{PBE} \\ &= 180° - \widehat{ABC} - \widehat{ACB} \\ &= \widehat{A} \quad \checkmark\end{aligned}$$

Exercise 8

Given the triangle ABC and the bisectors BO and CO of the angles of the base, where the point O is the intersection of the 2 bisectors. A line DOE passes through the point O parallel to base BC.
Prove that $DE = DB + CE$

Solution



CO is the bisector of $\widehat{BCE} \Rightarrow \widehat{BCO} = \widehat{OCE}$

$OE \parallel BC \Rightarrow \widehat{COE} = \widehat{BOC}$

$\therefore \widehat{EOC} = \widehat{OCE} \rightarrow \underline{OE = EC}$

Similar; BO is the bisector of $\widehat{DBC} \Rightarrow \widehat{DBO} = \widehat{OBC}$

$DO \parallel BC \Rightarrow \widehat{DOB} = \widehat{OBC}$

$\therefore \widehat{DOB} = \widehat{OBC} \rightarrow \underline{DO = DB}$

$DE = DO + OE$

$\underline{= DB + CE}$

Exercise 9

A right triangle *ABC* at *A* with a height *AH*. We drop perpendiculars *HE* and *HD* from *H* to sides *AB* and *AC* respectively.

a) Prove that $DE = AH$

b) Prove that *AM* is perpendicular to *DE*, where *M* is the middle point of *BC*.

c) Prove that *MN* (*N* is the middle point of *AB*) and the segment *Bx* (parallel to *DE*) are intersect on *AH*.

d) Prove that *AM* and *HD* are intersect on *Bx*.

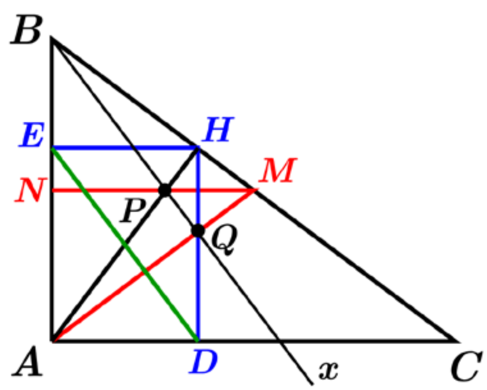
Solution

a) The triangles *AEH* and *ADH* are right triangles and angle *A* is right angle.
Then *AEHD* is a rectangle.
Therefore, $DE = AH$

b) A middle point of a hypotenuse of a right triangle is the center of the circle of that triangle.
Therefore, $MC = MA = MB$

That implies to: $\widehat{MAC} = \widehat{MCA}$

From the rectangle *ADHE*: $\widehat{EAH} = \widehat{EDH}$



$\widehat{EAH} + \widehat{HAM} + \widehat{MAC} = 90°$ $\qquad \widehat{HAM} + \widehat{MAC} = \widehat{HAC}$

$\widehat{EAH} + \widehat{HAC} = 90°$

$\widehat{EAH} + 90° - \widehat{MCA} = 90°$

$\widehat{EAH} = \widehat{MCA} = \widehat{EDH} = \widehat{MAC}$

$\widehat{ADE} + \widehat{EDH} = 90°$

$\widehat{ADE} + \widehat{MAD} = 90°$

Therefore, *AM* is perpendicular to *DE*.

c) *N* is the middle point of *AB* $\Rightarrow$ $NA = NB$

Bx parallel to *DE* $\Rightarrow$ $\widehat{ABx} = \widehat{AED} = \widehat{EDH} = \widehat{EAH}$

Let point *P* the intersection of B*x* and *AH*. Since $\widehat{ABP} = \widehat{BAP}$, then the triangle *BPA* is an isosceles. *PN* is the perpendicular to *AB* as well *MN*. Which gives us that points *M*, *P*, *N* are on the same line.
Therefore, segment *MN* and *AH* intersect at point *P*.

d) Let Point *Q* be the intersection of *AM* and *Bx*.

$\widehat{ABQ} = \widehat{BAH}$ $\quad$ & $\quad$ $\widehat{BAQ} = \widehat{ABH}$

Then, the triangles *BHA* and *BQA* are equivalent, therefore $AQ \perp BQ$ with hypotenuse *AB*.

$HQ \parallel AB$, line *HQ* has to be perpendicular to *AC*.

AM and *HD* are intersect on *Bx* at *Q*.

Exercise 10

Given an isosceles triangle ABC with a peak at A. Extend base BC the length $CD = AB$, then extend AB of a length $BE = \frac{1}{2}BC$, at the end draw a line EHF, H is the middle point of BC and F is located on AD.

a) Prove that $\widehat{ADB} = \frac{1}{2}\widehat{ABC}$

b) Prove that $EA = HD$

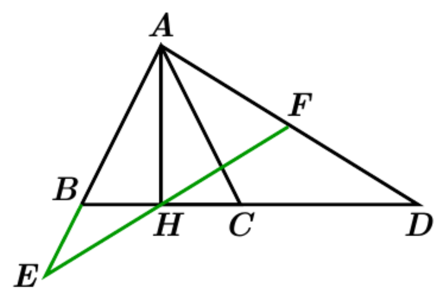
c) Prove that $FA = FD = FH$

d) Calculate the value of the angles $\widehat{AFH}$ and $\widehat{ADB}$ where $\widehat{BAC} = 58°$.

Solution

a) Triangle ABC is isosceles, then $\widehat{ABC} = \widehat{ACB}$

Since, $CD = AB = AC$, then $\widehat{CAD} = \widehat{ADC}$



$$2\widehat{ADC} = 180° - \widehat{ACD}$$
$$2\widehat{ADC} = 180° - \left(180° - \widehat{ACB}\right)$$
$$2\widehat{ADC} = \widehat{ACB}$$
$$\widehat{ADB} = \frac{1}{2}\widehat{ACB}$$
$$= \frac{1}{2}\widehat{ABC}$$

b) $BE = \frac{1}{2}BC$ $\quad$ *H* the middle point of *BC*

$= HC$

$CD = AB$

$HC + CD = BE + AB$

$EA = HD$ ✓

c) $\widehat{ADH} = \frac{1}{2}\widehat{ABD}$

$$= \frac{1}{2}\left(180° - \widehat{HBE}\right)$$
$$= \frac{1}{2}\left(180° - 180° + 2\widehat{BHE}\right)$$
$$= \widehat{BHE}$$

$\Rightarrow$ $FD = FH$

$\widehat{AHF} = 90° - \widehat{FHD}$

$= 90° - \widehat{ADH}$ $\quad$ $(\Delta\ HDA)$

$= 90° - \left(90° - \widehat{HAF}\right)$

$= \widehat{HAF}$

$\Rightarrow \quad \underline{FA = FH}$

$FA = FD = FH$ ✓

d) $\widehat{BAC} = 58°$

$$
\begin{aligned}
\widehat{ADB} &= \frac{1}{2}\widehat{ACB} \\
&= \frac{1}{2}\left(\frac{1}{2}\left(180° - \widehat{BAC}\right)\right) \\
&= \frac{1}{4}(180° - 58°) \\
&= \frac{122}{4}^{\circ} \\
&= \frac{61}{2}^{\circ} \qquad = 30.5°
\end{aligned}
$$

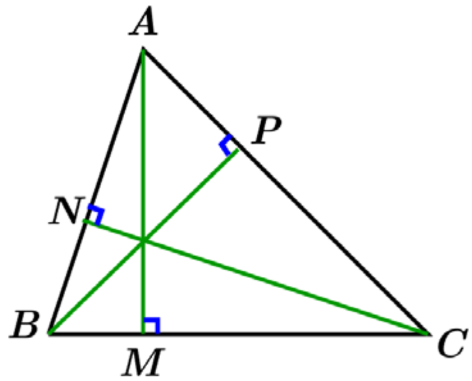
Triangle *AFH* is isosceles then,

$$
\begin{aligned}
\widehat{AFH} &= 180° - \widehat{HFD} \\
&= 180° - \left(180° - 2\widehat{FDH}\right) \\
&= 2\widehat{FDH} \\
&= 2\frac{61°}{2} \\
&= 61°
\end{aligned}
$$

Exercise 11

Demonstrate that the heights of a triangle share the angles of triangle that equal to each other.

Solution



Consider the 2 right triangles *APB* and *ANC*, which they have the same angle *A*.

Therefore, $\widehat{ABP} = \widehat{ACN}$.

Similar, consider the 2 right triangles *BPC* and *AMC*, which they have the same angle *C*.

Therefore, $\widehat{MAC} = \widehat{CBP}$.

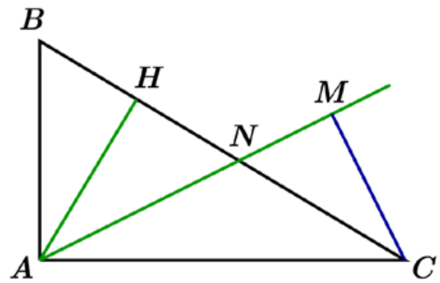
Similar, consider the 2 right triangles *BNC* and *AMB*, which they have the same angle *B*.

Therefore, $\widehat{BCN} = \widehat{BAM}$.

Exercise 12

A right triangle *ABC* at *A* where $AB < AC$, drop a perpendicular *AH* from *A* to the hypotenuse *BC* where $HN = HB$. From *C* drops a perpendicular *CM* at *AN*. Demonstrate that *BC* is the bisector of the angle $\widehat{ACM}$.

Solution



Consider the 2 right triangles *ABC* and *ABH* with a common angle *B*, then

$\widehat{BAH} = \widehat{ACB}$

Given: $HN = HB$, then $\widehat{HAN} = \widehat{BAH} = \widehat{ACB}$

$$\widehat{NAC} = 90° - \widehat{HAB} - \widehat{HAN}$$
$$= 90° - 2\widehat{HCA}$$

Consider the 2 right triangles *AHN* and *CMC*, where $\widehat{HNA} = \widehat{MNC}$

Therefore, $\widehat{HAN} = \widehat{NCM}$

Since $\widehat{HAN} = \widehat{ACB}$

Then $\widehat{ACB} = \widehat{MCB}$

Therefore, *BC* is the bisector of the angle $\widehat{ACM}$

Exercise 13

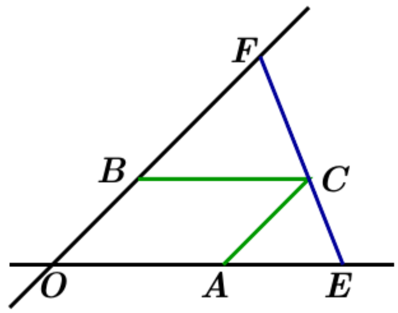
On the sides of an angle that it takes the length *OA* and *OB*, so that $OA + OB = \ell$ (is given) and construct a parallelogram *OABC*. What is the place of the summit *C* of parallelogram?

Solution

Let segment *OE* extension of segment *OA* such that $OE = \ell$
Let segment *OF* extension of segment *OB* such that $OF = \ell$

Then, the triangle *OEF* is an isosceles.

$$\widehat{OEF} = \widehat{OFE} = 90^\circ - \frac{1}{2}\widehat{EOF}$$



$$OA + OB = \ell$$

$$\begin{cases} OA + AC = \ell \\ OA + AE = \ell \end{cases} \Rightarrow \quad AC = AE$$

$$\begin{cases} OB + BC = \ell \\ OB + BF = \ell \end{cases} \Rightarrow \quad BC = BF$$

$$\widehat{OEF} = \widehat{OFE} = \widehat{FCB} = \widehat{ACE}$$

Therefore, the point *C*, *E*, and *F* are aligned.

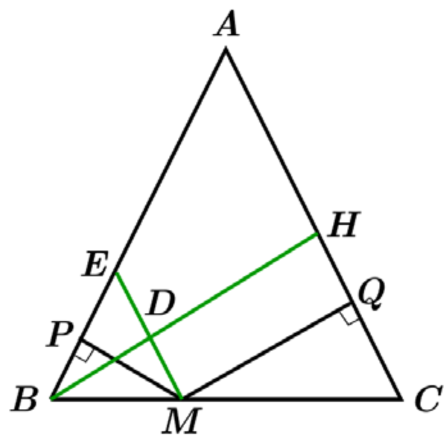
Exercise 14

Demonstrate that the sum of distances from a point M on the base BC of an isosceles triangle ABC to the sides equal a constant.

Solution

The shortest distance from a point to a line is the perpendicular from that point to the line. Therefore, let:

$MP \perp AB$
$MQ \perp AC$



Let $BH \perp AC$ (Shortest distance from B to side AC.)

Let D be the point of intersection ME and BH.

Let $ME \parallel AC$
Where the point E is the intersection of the lines MD and AB.

Since $MD \parallel AC$ then $\widehat{DMB} = \widehat{ACB}$

Since triangle ABC is an isosceles
$\widehat{DMB} = \widehat{ACB} = \widehat{PBM}$

The right triangles BPM and BDM at P & D and have the same hypotenuse, then
$\Rightarrow \quad |MP| = |BD|$

$MD \parallel HQ$ and DH & $MQ \perp HQ$
$\Rightarrow \quad |MQ| = |DH|$

$$\begin{aligned}|MP| + |MQ| &= |BD| + |DH| \\ &= |BH| \\ &= \textbf{\textit{constant}}\end{aligned}$$

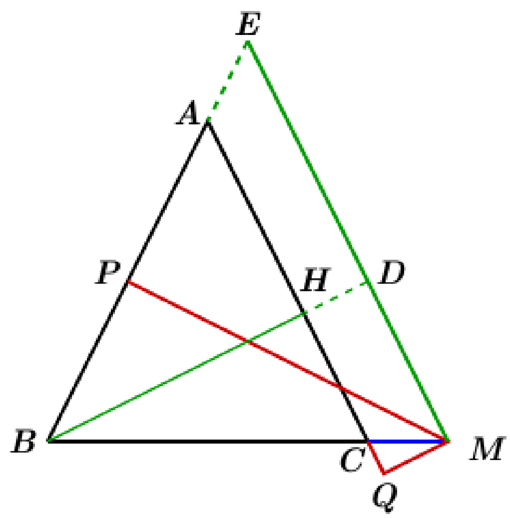
Therefore; the sum of distances from a point M on the base BC of an isosceles triangle ABC to the sides equal a constant.

Exercise 15

Demonstrate that the difference of distances from a point M taken on the extension of the base BC of an isosceles triangle ABC to the sides equal a constant.

Solution

The shortest distance from a point to a line is the perpendicular from that point to the line. Therefore, let:



$MP \perp AB$
$MQ \perp AC$

Let $BH \perp AC$ (Shortest distance from B to side AC.)

Let D be the point of intersection ME and BH.

Let $ME \parallel AC$
Where the point E is the intersection of the extensions of the lines MD and AB.

Since $MD \parallel AC$ then $\widehat{DMB} = \widehat{ACB}$

Since triangle ABC is an isosceles
$\widehat{DMB} = \widehat{ACB} = \widehat{PBM}$

The right triangles BPM and BDM at P & D and have the same hypotenuse, then
$\Rightarrow \quad |MP| = |BD|$

$MD \parallel HQ$ and DH & $MQ \perp HQ$
$\Rightarrow \quad |MQ| = |DH|$

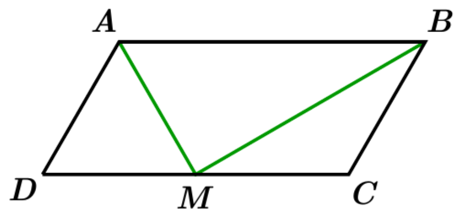
$$|MP| - |MQ| = |BD| - |DH|$$
$$= |BH|$$
$$= \textit{constant}$$

Therefore; the difference of distances from a point M taken on the extension of the base BC of an isosceles triangle ABC to the sides equal a constant.

Exercise 16

Consider a parallelogram $ABCD$ in which $CD = 2AD$. In the joint A and B the middle M of BC. Prove that the angle $\widehat{AMB}$ is a right angle.

Solution



Since the point M is the middle of side BC, then

$$MD = MC = \frac{1}{2}CD$$

$$\Rightarrow \quad MD = AD = BC$$

Therefore; the triangles ADM and BCM are isosceles at D and C respectively.
Which implies that $MA = MB$

Let O be the middle point of the side AB, and $OA = OB = AD$

O and M are middle of the parallelogram $ABCD$, that implies

$$OM = BC = AD$$

$$\Rightarrow \quad OA = OB = OM$$

The triangle MAB inscribed in a circle with center at O and diameter AB, that will imply that is a right triangle at the point M.

Or

$$\widehat{AMD} = \frac{1}{2}\left(180° - \widehat{MDA}\right)$$

$$\widehat{BMC} = \frac{1}{2}\left(180° - \widehat{MCB}\right)$$

$$\widehat{ADM} + \widehat{MCB} = 180°$$

$$\widehat{DMA} + \widehat{AMB} + \widehat{BMC} = 180°$$

$$\widehat{AMB} = 180° - \left(\widehat{BMC} + \widehat{DMA}\right)$$

$$= 180° - \left(90° - \frac{1}{2}\widehat{MDA} + 90° - \frac{1}{2}\widehat{MCB}\right)$$

$$= \frac{1}{2}\left(\widehat{MDA} + \widehat{MCB}\right)$$

$$= \frac{1}{2}(180°)$$

$$\underline{= 90°}$$

Exercise 17

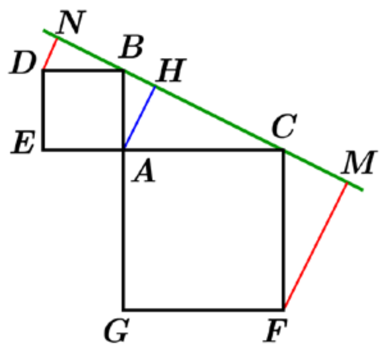
From the sides *AB* and *AC* of a right triangle *ABC* at *A*, draw two squares *ABDE* and *ACFG*. Then lead *DN* and *FM* perpendicular to the line *BC*.

a) Prove that $DN + FM = BC$

b) Prove that the points *D, A, F* on a straight line.

c) Prove that the lines *DE* and *FG* contribute on the extension of the height *AH*.

Solution



a) Let consider the 2 right triangles *DNB* & *BHA* at points *N* & *H* respectively, with $DB = AB$. Then

$$\widehat{HAB} = 90° - \widehat{ABH}$$
$$= 90° - \left(90° - \widehat{NBD}\right)$$
$$= \widehat{NBD}$$
$$\Rightarrow \quad \widehat{BDN} = \widehat{ABH}$$

$\therefore$ The 2 triangles are equals, which implies that $DN = BH$

Similar, for the 2 right triangles *CMF* & *AHC* at points *M* & *H* respectively, with $AC = CF$. Then

$$\widehat{HAC} = 90° - \widehat{ACH}$$
$$= 90° - \left(90° - \widehat{MCF}\right)$$
$$= \widehat{MCF}$$
$$\Rightarrow \quad \widehat{ACH} = \widehat{CFM}$$

$\therefore$ The 2 triangles are equals, which implies that $FM = HC$

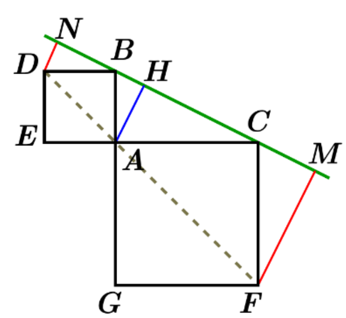
$$DN + FM = BH + HC$$
$$= BC \quad \checkmark$$

b) Since *ABDE* is a square, then $\widehat{BAD} = 45°$

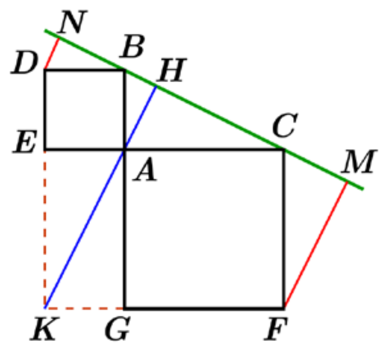
And *ACFG* is a square, then $\widehat{CAF} = 45°$

$$\widehat{DAF} = \widehat{DAB} + \widehat{BAC} + \widehat{CAF}$$
$$= 45° + 90° + 45°$$
$$= 180°$$

$\therefore$ The points *D, A, &* *F* are on a straight line.



c) Let the point K be the intersection of the extension of the sides DE and FG.
Which will result of $GKEA$ is a rectangle with $AE = GK$ & $EK = AG$



Consider the 2 right triangles BAC & KGA at points A & G respectively with $AE = AB = GK$

$\widehat{ACB} = \widehat{GAK} = \widehat{ACH}$

From the right triangle AHC:

$\widehat{HAC} + \widehat{ACH} = 90°$

$\rightarrow \quad \widehat{HAC} + \widehat{KAG} = 90°$

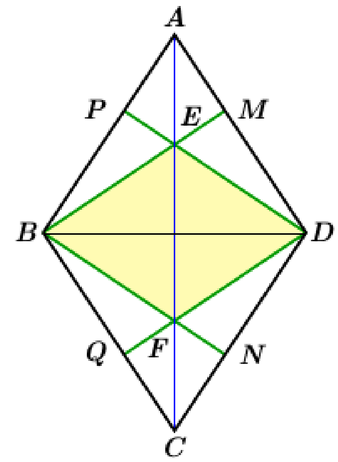
$$\begin{aligned} \widehat{HAC} + \widehat{CAG} + \widehat{KAG} &= \left(\widehat{HAC} + \widehat{KAG}\right) + \widehat{CAG} \\ &= 90° + 90° \\ &= 180° \end{aligned}$$

$\therefore$ The points K, A, & H are on a straight line.

Exercise 18

Given a diamond *ABCD*; the peak *B* and *D*, the same the perpendiculars *BM*, *BN*, *DP*, *DQ* on opposite sides. These perpendiculars are intersected at *E* and *F*.

Demonstrate that the angles of the quadrilateral *BFDE* are equals to the diamond and which is a diamond itself.



Solution

From the right triangles *BPD* & *BMD*, that implies $\widehat{MBD} = \widehat{PDB}$

$\Rightarrow \quad \widehat{EBD} = \widehat{EDB}$

Similar, from the right triangles *BND* & *BQD*, that implies $\widehat{NBD} = \widehat{QDB}$

$\Rightarrow \quad \widehat{FBD} = \widehat{FDB}$

$\widehat{EBD} + \widehat{DBF} = \widehat{EDB} + \widehat{BDF}$

$\widehat{EBF} = \widehat{EDF}$

Since, $AC \perp BD$, then $EF \perp BD$

The 2 triangles *EBF* & *EDF* have *EF* as a common side and $\widehat{EBF} = \widehat{EDF}$, then

$\widehat{BEF} = \widehat{DEF} = \widehat{BFE} = \widehat{DFE}$

$\widehat{BED} = \widehat{BFD}$

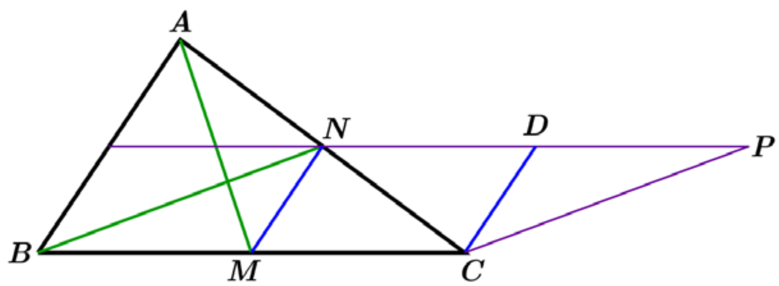
Therefore; the angles of the quadrilateral *BFDE* are equals to the diamond and which is a diamond itself.

Exercise 19

In a triangle *ABC*, we trace the median *AM* and *BN* and from *N* a parallel to *BC*, from *C* a parallel to *BN*; that the two sides intersect at a point *P*. Let *D* be the middle point of the segment *PN*.

Prove that *CD* is parallel to *MN*.

Solution



Since the points *M* & *N* are middle of the sides *BC* & *AC* of the triangle *ABC*, then $MN \parallel AB$

Given: $NP \parallel MC$

$BN \parallel CP$

Since *M* & *D* are the middle points of the segments *BC* and *NP* respectively, then $BN \parallel CP \parallel MD$

Therefore, *BNPC* is a parallelogram, and $MC = ND$.

Since $MC = ND$ & $MN = CD$

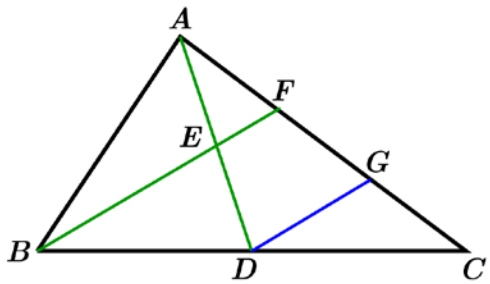
Therefore; *MCDC* is a parallelogram which implies to *CD* parallel to *MN*

Exercise 20

The median *AD* of a given triangle *ABC* to the side *BC*. The same the median *BE* to the side *AD* which intersect *AC* at a point *F*.

Prove that where $AF = \frac{1}{3} AC$

Solution



Let *DG* be parallel to segment *BEF*.

Given: *E* is the middle point of the segment *AD* $\Rightarrow \quad AE = ED$

D is the middle point of the segment *BC* $\Rightarrow \quad BD = DC$

Since $EF \parallel DG$, and $AE = ED$, that implies $AF = FG$

Consider the triangles CDG and CBF:

$EF \parallel DG$, and $CD = DB$, that implies $GC = FG$

That will imply to: $AF = FG = GC$

$$AC = AF + FG + GC$$
$$= 3AF$$

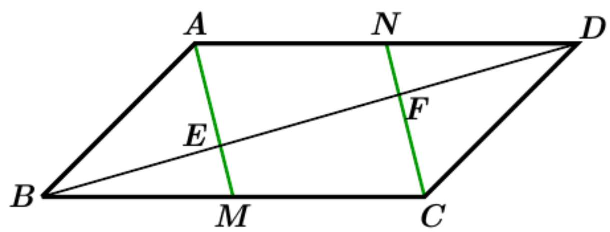
Therefore; $AF = \frac{1}{3} AC$

Exercise 21

In a parallelogram *ABCD*, from the points peak *A* and *C* joint the middle of opposite sides at *M* and *N* respectively.

Prove that the diagonal *BD* is divided in three equal parts.

Solution



M is the middle point of the segment BC, then $BM = CM$
N is the middle point of the segment AD, then $NA = ND$

From these, implies that $AM \,//\, CN$.

From the triangles BEM & BCF, and since $ME \,//\, CF$

It will give us that $BE = EF$

From the triangles DFN & DEA, and since $AE \,//\, FN$

It will give us that $\Rightarrow \quad DF = EF$

Therefore, $BE = EF = DF$

$$BD = BE + EF + FD$$
$$= 3BE$$

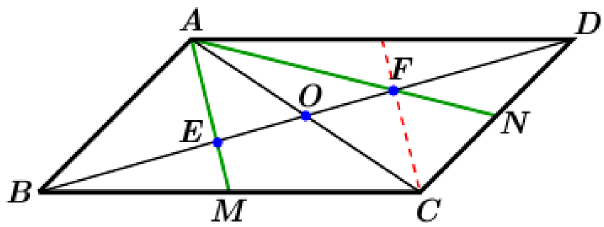
Therefore; the diagonal *BD* is divided in three equal parts

Exercise 22

In a parallelogram *ABCD*, from the point peak *A*, extend to the middle of sides *BC* and *DC* at *M* and *N* respectively.

Prove that the diagonal *BD* is divided in three equal parts.

Solution



Let a point *E* be the intersection of the segments *AM* & *BD*.
Let a point *F* be the intersection of the segments *AN* & *BD*.

Le *O* be the intersection of the both diagonal *AC* & *BD*.
From the triangles *BEM* & *BCF*, and since $ME \parallel CF$

$$\Rightarrow \quad BE = EF$$

Similar, $\Rightarrow \quad DF = EF$

$BO = OF \quad \rightarrow \quad OE = OF$

$$\begin{aligned} BO &= BE + EO \\ &= BE + \frac{1}{2}BE \\ &= \frac{3}{2}BE \end{aligned}$$

$$\begin{aligned} BE &= \frac{2}{3}BO \\ &= \frac{2}{3}\left(\frac{1}{2}BD\right) \\ &= \frac{1}{3}BD \end{aligned}$$

$$\begin{aligned} DF &= \frac{2}{3}DO \\ &= \frac{2}{3}\left(\frac{1}{2}BD\right) \\ &= \frac{1}{3}BD \end{aligned}$$

Therefore; the diagonal *BD* is divided in three equal parts