Solution Section 2.4 – Integration of Rational Functions by Partial Fractions

Exercise

Evaluate $\displaystyle\int \frac{dx}{x^2+2x}$

Solution

$$\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} = \frac{Ax+2A+Bx}{x^2+2x}$$

$$1 = (A+B)x + 2A \quad \Rightarrow \begin{cases} 2A = 1 & \to A = \frac{1}{2} \\ A+B=0 & \to B = -\frac{1}{2} \end{cases}$$

$$\int \frac{1}{x^2+2x}\,dx = \frac{1}{2}\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{1}{x+2}\,dx$$

$$= \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + C$$

Exercise

Evaluate $\displaystyle\int \frac{2x+1}{x^2-7x+12}\,dx$

Solution

$$\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3} = \frac{(A+B)x - 3A - 4B}{(x-4)(x-3)}$$

$$\to \begin{cases} A+B=2 \\ -3A-4B=1 \end{cases} \Rightarrow \boxed{A=9} \quad \boxed{B=-7}$$

$$\int \frac{2x+1}{x^2-7x+12}\,dx = 9\int \frac{dx}{x-4} - 7\int \frac{dx}{x-3}$$

$$= 9\ln|x-4| - 7\ln|x-3| + C$$

$$= \ln\left|\frac{(x-4)^9}{(x-3)^7}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{x+3}{2x^3-8x}\,dx$

Solution

$$\frac{x+3}{2x^3-8x}=\frac{1}{2}\frac{x+3}{x\left(x^2-4\right)}=\frac{1}{2}\left(\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-2}\right)$$

$$=\frac{1}{2}\frac{A(x+2)(x-2)+Bx(x-2)+Cx(x+2)}{x(x+2)(x-2)}$$

$$(A+B+C)x^2+(2C-2B)x-4A=x+3$$

$$\begin{cases}A+B+C=0\\ 2C-2B=1\\ -4A=3\end{cases}\rightarrow \boxed{A=-\frac{3}{4}}\quad \boxed{B=\frac{1}{8}}\quad \boxed{C=\frac{5}{8}}$$

$$\int \frac{x+3}{2x^3-8x}\,dx=\frac{1}{2}\int -\frac{3}{4}\frac{dx}{x}+\frac{1}{2}\int \frac{1}{8}\frac{dx}{x+2}+\frac{1}{2}\int \frac{5}{8}\frac{dx}{x-2}$$

$$=-\frac{3}{8}\ln|x|+\frac{1}{16}\ln|x+2|+\frac{5}{16}\ln|x-2|+K$$

$$=\frac{1}{16}\left(\ln|x+2|+5\ln|x-2|-6\ln|x|\right)+K$$

$$=\frac{1}{16}\ln\left|\frac{(x+2)(x-2)^5}{x^6}\right|+K$$

Exercise

Evaluate $\displaystyle\int \frac{x^2}{(x-1)\left(x^2+2x+1\right)}dx$

Solution

$$\frac{x^2}{(x-1)\left(x^2+2x+1\right)}=\frac{x^2}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$x^2=A(x+1)^2+B(x-1)(x+1)+C(x-1)$$

$$x^2=(A+B)x^2+(2A+C)x+A-B-C$$

$$\begin{cases}A+B=1\\ 2A+C=0\\ A-B-C=0\end{cases}\rightarrow \boxed{A=\frac{1}{4}}\quad \boxed{B=\frac{3}{4}}\quad \boxed{C=-\frac{1}{2}}$$

$$\int \frac{x^2}{(x-1)\left(x^2+2x+1\right)}dx=\frac{1}{4}\int \frac{dx}{x-1}+\frac{3}{4}\int \frac{dx}{x+1}-\frac{1}{2}\int \frac{dx}{(x+1)^2}$$

$$= \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2}\frac{1}{(x+1)} + K$$

$$= \frac{1}{4}\left(\ln|x-1| + \ln|x+1|^3\right) + \frac{1}{2(x+1)} + K$$

$$\boxed{= \frac{1}{4}\ln\left|(x-1)(x+1)^3\right| + \frac{1}{2(x+1)} + K}$$

Exercise

Evaluate $\displaystyle\int \frac{8x^2+8x+2}{\left(4x^2+1\right)^2}dx$

Solution

$$\frac{8x^2+8x+2}{\left(4x^2+1\right)^2} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{\left(4x^2+1\right)^2} = \frac{(Ax+B)\left(4x^2+1\right)+Cx+D}{\left(4x^2+1\right)^2}$$

$$8x^2+8x+2 = 4Ax^3+4Bx^2+(A+C)x+B+D$$

$$\begin{cases} A=0 \\ 4B=8 \\ A+C=8 \\ B+D=2 \end{cases} \to \boxed{A=0} \quad \boxed{B=2} \quad \boxed{C=8} \quad \boxed{D=0}$$

$$\int \frac{8x^2+8x+2}{\left(4x^2+1\right)^2}dx = \int \frac{2}{4x^2+1}dx + \int \frac{8x}{\left(4x^2+1\right)^2}dx \qquad d\left(4x^2+1\right) = 8xdx$$

$$= \int \frac{2}{4x^2+1}dx + \int \frac{d\left(4x^2+1\right)}{\left(4x^2+1\right)^2} \qquad \int \frac{du}{u^2} = -\frac{1}{u} \quad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\boxed{= \tan^{-1}2x - \frac{1}{4x^2+1} + K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2+x}{x^4-3x^2-4}dx$

Solution

$$\frac{x^2+x}{x^4-3x^2-4} = \frac{x^2+x}{\left(x^2-4\right)\left(x^2+1\right)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1}$$

$$x^2+x = A(x+2)\left(x^2+1\right) + B(x-2)\left(x^2+1\right) + (Cx+D)\left(x^2-4\right)$$

$$= Ax^3 + Ax + 2Ax^2 + 2A + Bx^3 + Bx - 2Bx^2 - 2B + Cx^3 - 4Cx + Dx^2 - 4D$$

$$= (A+B+C)x^3 + (2A-2B+D)x^2 + (A+B-4C)x + 2A - 2B - 4D$$

$$\begin{cases} A+B+C=0 \\ 2A-2B+D=1 \\ A+B-4C=1 \\ 2A-2B-4D=0 \end{cases} \Rightarrow \boxed{A=\frac{3}{10}} \quad \boxed{B=-\frac{1}{10}} \quad \boxed{C=-\frac{1}{5}} \quad \boxed{D=\frac{1}{5}}$$

$$\int \frac{x^2+x}{x^4-3x^2-4}dx = \frac{3}{10}\int \frac{1}{x-2}dx - \frac{1}{10}\int \frac{1}{x+2}dx + \frac{1}{5}\int \frac{-x+1}{x^2+1}dx$$

$$= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{5}\int \frac{x}{x^2+1}dx + \frac{1}{5}\int \frac{1}{x^2+1}dx \qquad d\left(x^2+1\right) = 2xdx$$

$$= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{10}\int \frac{d\left(x^2+1\right)}{x^2+1} + \frac{1}{5}\int \frac{1}{x^2+1}dx \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{10}\ln\left(x^2+1\right) + \frac{1}{5}\tan^{-1}x + K$$

Exercise

Evaluate $\displaystyle\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{\left(\theta^2+1\right)^3}d\theta$

Solution

$$\frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{\left(\theta^2+1\right)^3} = \frac{A\theta+B}{\theta^2+1} + \frac{C\theta+D}{\left(\theta^2+1\right)^2} + \frac{E\theta+F}{\left(\theta^2+1\right)^3}$$

$$\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1 = (A\theta+B)\left(\theta^2+1\right)^2 + (C\theta+D)\left(\theta^2+1\right) + E\theta + F$$

$$= (A\theta+B)\left(\theta^4+2\theta^2+1\right) + C\theta^3 + C\theta + D\theta^2 + D + E\theta + F$$

$$= A\theta^5 + B\theta^4 + (2A+C)\theta^3 + (2B+D)\theta^2 + (A+C+E)\theta + B + D + F$$

$$\begin{cases} \boxed{A=0} \\ \boxed{B=1} \\ 2A+C=-4 \\ 2B+D=2 \\ A+C+E=-3 \\ B+D+F=1 \end{cases} \rightarrow \boxed{C=-4} \quad \boxed{D=0} \quad \boxed{E=1} \quad \boxed{F=0}$$

$$\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{\left(\theta^2+1\right)^3}d\theta = \int \frac{1}{\theta^2+1}d\theta - 4\int \frac{\theta}{\left(\theta^2+1\right)^2}d\theta + \int \frac{\theta}{\left(\theta^2+1\right)^3}d\theta$$

$$= \int \frac{1}{\theta^2+1}\,d\theta - 2\int \frac{d\left(\theta^2+1\right)}{\left(\theta^2+1\right)^2} + \frac{1}{2}\int \frac{d\left(\theta^2+1\right)}{\left(\theta^2+1\right)^3} \qquad d\left(\theta^2+1\right) = 2\theta d\theta$$

$$\boxed{= \tan^{-1}\theta + 2\frac{1}{\theta^2+1} - \frac{1}{4}\frac{1}{\left(\theta^2+1\right)^2} + K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^4}{x^2-1}\,dx$

Solution

$$\frac{x^4}{x^2-1} = x^2+1+\frac{1}{(x-1)(x+1)}$$

$$\begin{array}{r|l} & x^2+1 \\ \hline x^2-1 & x^4 \\ & \underline{x^4-x^2} \\ & x^2 \\ & \underline{x^2-1} \\ & 1 \end{array}$$

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1}+\frac{B}{x+1} = \frac{(A+B)x+A-B}{(x-1)(x+1)}$$

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \to \boxed{A=\frac{1}{2}} \quad \boxed{B=-\frac{1}{2}}$$

$$\int \frac{x^4}{x^2-1}\,dx = \int \left(x^2+1\right)dx + \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx$$

$$= \frac{1}{3}x^3+x+\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+C$$

$$= \frac{1}{3}x^3+x+\frac{1}{2}\left(\ln|x-1|-\ln|x+1|\right)+C$$

$$\boxed{= \frac{1}{3}x^3+x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{16x^3}{4x^2-4x+1}\,dx$

Solution

$$\frac{16x^3}{4x^2-4x+1} = 4x+4+\frac{12x-4}{(2x-1)^2}$$

$$\begin{array}{r|l} & 4x+4 \\ \hline 4x^2-4x+1 & 16x^3 \\ & \underline{16x^3-16x^2+4x} \\ & 16x^2-4x \\ & \underline{16x^2-16x+4} \\ & 12x-4 \end{array}$$

$$= 4x+4+\frac{A}{2x-1}+\frac{B}{(2x-1)^2}$$

$$12x-4 = 2Ax-A+B$$

$$\begin{cases} 2A = 12 \\ -A + B = -4 \end{cases} \rightarrow \boxed{A = 6} \quad \boxed{B = 2}$$

$$\int \frac{16x^3}{4x^2 - 4x + 1} dx = \int (4x + 4) dx + 6\int \frac{dx}{2x - 1} + 2\int \frac{dx}{(2x - 1)^2}$$

$$= 2x^2 + 4x + 6\left(\frac{1}{2}\right)\ln|2x - 1| + 2\left(-\frac{1}{2}\right)\frac{1}{2x - 1} + C$$

$$= 2x^2 + 4x + 3\ln|2x - 1| - \frac{1}{2x - 1} + C$$

Exercise

Evaluate $\displaystyle\int \frac{e^{4x} + 2e^{2x} - e^x}{e^{2x} + 1} dx$

Solution

$$\int \frac{e^{4x} + 2e^{2x} - e^x}{e^{2x} + 1} dx = \int \frac{e^x\left(e^{3x} + 2e^x - 1\right)}{e^{2x} + 1} dx \qquad y = e^x \Rightarrow dy = e^x dx$$

$$= \int \frac{y^3 + 2y - 1}{y^2 + 1} dy$$

$$= \int \left(y + \frac{y - 1}{y^2 + 1}\right) dy$$

$$= \int y dy + \int \frac{y}{y^2 + 1} dy - \int \frac{1}{y^2 + 1} dy$$

$$= \int y dy + \frac{1}{2}\int \frac{1}{y^2 + 1} d\left(y^2 + 1\right) - \int \frac{1}{y^2 + 1} dy \qquad d\left(y^2 + 1\right) = 2y dy$$

$$= \frac{1}{2}y^2 + \frac{1}{2}\ln\left(y^2 + 1\right) - \tan^{-1} y + C$$

$$= \frac{1}{2}e^{2x} + \frac{1}{2}\ln\left(e^{2x} + 1\right) - \tan^{-1} e^x + C$$

Exercise

Evaluate $\displaystyle\int \frac{\sin\theta d\theta}{\cos^2\theta + \cos\theta - 2}$

Solution

Let $y = \cos\theta \Rightarrow dy = -\sin\theta d\theta$

$$\int \frac{\sin\theta d\theta}{\cos^2\theta + \cos\theta - 2} = -\int \frac{dy}{y^2 + y - 2}$$

$$\frac{1}{y^2+y-2}=\frac{1}{(y+2)(y-1)}=\frac{A}{y+2}+\frac{B}{y-1}$$

$$1=(A+B)y-A+2B$$

$$\begin{cases} A+B=0 \\ -A+2B=1 \end{cases} \rightarrow \boxed{A=-\frac{1}{3}} \quad \boxed{B=\frac{1}{3}}$$

$$\int \frac{\sin\theta d\theta}{\cos^2\theta+\cos\theta-2}=-\left(-\frac{1}{3}\int\frac{dy}{y+2}+\frac{1}{3}\int\frac{dy}{y-1}\right)$$

$$=\frac{1}{3}\ln|y+2|-\frac{1}{3}\ln|y-1|+C$$

$$=\frac{1}{3}\left(\ln|y+2|-\ln|y-1|\right)+C$$

$$=\frac{1}{3}\ln\left|\frac{y+2}{y-1}\right|+C$$

$$=\frac{1}{3}\ln\left|\frac{\cos\theta+2}{\cos\theta-1}\right|+C$$

Exercise

Evaluate $\int \frac{(x-2)^2\tan^{-1}(2x)-12x^3-3x}{\left(4x^2+1\right)(x-2)^2}\,dx$

Solution

$$\int \frac{(x-2)^2\tan^{-1}(2x)-12x^3-3x}{\left(4x^2+1\right)(x-2)^2}\,dx=\int\frac{(x-2)^2\tan^{-1}(2x)}{\left(4x^2+1\right)(x-2)^2}\,dx-\int\frac{12x^3+3x}{\left(4x^2+1\right)(x-2)^2}\,dx$$

$$=\int\frac{\tan^{-1}(2x)}{4x^2+1}\,dx-\int\frac{3x\left(4x^2+1\right)}{\left(4x^2+1\right)(x-2)^2}\,dx$$

$$=\int\frac{\tan^{-1}(2x)}{4x^2+1}\,dx-\int\frac{3x}{(x-2)^2}\,dx$$

$$d\left(\tan^{-1}2x\right)=\frac{dx}{(2x)^2+1}=\frac{dx}{4x^2+1}$$

$$\frac{3x}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}=\frac{Ax-2A+B}{(x-2)^2}$$

$$\begin{cases} \boxed{A=3} \\ -2A+B=0 \end{cases} \rightarrow \boxed{B=6}$$

$$\int \frac{(x-2)^2 \tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2}\,dx = \frac{1}{2}\int \tan^{-1}(2x)\,d\left(\tan^{-1}(2x)\right) - 3\int \frac{dx}{x-2} - 6\int \frac{dx}{(x-2)^2}$$

$$= \frac{1}{4}\left(\tan^{-1}(2x)\right)^2 - 3\int \frac{d(x-2)}{x-2} - 6\int \frac{d(x-2)}{(x-2)^2}$$

$$= \frac{1}{4}\left(\tan^{-1}(2x)\right)^2 - 3\ln|x-2| - \frac{6}{x-2} + C$$

Exercise

Evaluate $\displaystyle\int \frac{\sqrt{x+1}}{x}\,dx$

Solution

Let $x+1 = u^2 \quad \Rightarrow \quad dx = 2u\,du$

$$\int \frac{\sqrt{x+1}}{x}\,dx = \int \frac{u}{u^2-1}\,2u\,du$$

$$= 2\int \frac{u^2}{u^2-1}\,du$$

$$\require{enclose}\begin{array}{r} 1 \\ u^2-1 \enclose{longdiv}{u^2} \\ \underline{u^2-1} \\ 1 \end{array}$$

$$= 2\int \left(1 + \frac{1}{u^2-1}\right) du$$

$$= 2\int du + 2\int \frac{1}{u^2-1}\,du$$

$$\frac{1}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1} = \frac{(A+B)u + A - B}{(u-1)(u+1)}$$

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \Rightarrow \boxed{A=\frac{1}{2}} \quad \boxed{B=-\frac{1}{2}}$$

$$= 2\int du + 2\int \left(\frac{1}{2}\frac{1}{u-1} - \frac{1}{2}\frac{1}{u+1}\right) du$$

$$= 2u + \int \frac{1}{u-1}\,du - \int \frac{1}{u+1}\,du$$

$$= 2u + \ln|u-1| - \ln|u+1| + C$$

$$= 2\sqrt{x+1} + \ln\left|\sqrt{x+1}-1\right| - \ln\left|\sqrt{x+1}+1\right| + C$$

$$= 2\sqrt{x+1} + \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{x^3-2x^2+3x-4}{x^2+1}\,dx$

Solution

$$\int \frac{x^3-2x^2+3x-4}{x^2+1}\,dx = \int \left(x-2+\frac{2x-2}{x^2+1}\right)dx$$

$$= \int (x-2)\,dx + \int \frac{2x}{x^2+1}\,dx - 2\int \frac{1}{x^2+1}\,dx$$

$$= \int (x-2)\,dx + \int \frac{d\left(x^2+1\right)}{x^2+1} - 2\int \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{2}x^2 - 2x + \ln\left(x^2+1\right) - 2\tan^{-1}(x) + C$$

$$\begin{array}{r|l} & x-2 \\ \hline x^2+1 & x^3-2x^2+3x-4 \\ & \underline{x^3 \qquad\quad +x} \\ & -2x^2+2x-4 \\ & \underline{-2x^2 \qquad\quad -2} \\ & 2x-2 \end{array}$$

Exercise

Evaluate $\displaystyle\int \frac{4x^2+2x+4}{x+1}\,dx$

Solution

$$\int \frac{4x^2+2x+4}{x+1}\,dx = \int \left(4x+2+\frac{6}{x+1}\right)dx$$

$$= \int (4x-2)dx + \int \frac{6}{x+1}\,dx$$

$$= \int (4x-2)dx + 6\int \frac{d(x+1)}{x+1}$$

$$\int \frac{d(U)}{U} = \ln|U|$$

$$= 2x^2 - 2x + 6\ln|x+1| + C$$

Exercise

Evaluate $\displaystyle\int \frac{3x^2+7x-2}{x^3-x^2-2x}\,dx$

Solution

$$\frac{3x^2+7x-2}{x^3-x^2-2x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

$$3x^2+7x-2 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1)$$

$$= Ax^2 - Ax - 2A$$

$$Bx^2 - 2Bx$$

$$Cx^2 + Cx$$

$$\begin{cases} A+B+C=3 \\ -A-2B+C=7 \\ -2A=-2 \end{cases} \rightarrow \boxed{A=1} \quad \begin{cases} B+C=2 \\ -2B+C=8 \end{cases} \rightarrow \underline{B=-2|} \quad \underline{C=4|}$$

$$\int \frac{3x^2+7x-2}{x^3-x^2-2x}dx = \int \left(\frac{1}{x} - \frac{2}{x+1} + \frac{4}{x-2}\right)dx$$

$$= \ln|x| - 2\ln|x+1| + 4\ln|x-2| + K$$

$$= \ln\frac{|x|(x-2)^4}{(x+1)^2} + K$$

Exercise

Evaluate $\displaystyle\int \frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}dx$

Solution

$$\frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)} = \frac{A}{x-1} + \frac{B}{x-5} + \frac{C}{x+4}$$

$$3x^2+2x+5 = (A+B+C)x^2 + (-A+3B-6C)x - 20A - 4B + 5C$$

$$\begin{cases} A+B+C=3 \\ -A+3B-6C=2 \\ -20A-4B+5C=5 \end{cases} \rightarrow A=\frac{1}{2}, \quad B=\frac{5}{2}, \quad C=1$$

$$\int \frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}dx = \int \left(\frac{1}{2}\frac{1}{x-1} + \frac{5}{2}\frac{1}{x-5} + \frac{1}{x+4}\right)dx$$

$$= \frac{1}{2}\ln|x-1| + \frac{5}{2}\ln|x-5| + \ln|x+4| + K$$

Exercise

Evaluate $\displaystyle\int \frac{5x^2-3x+2}{x^3-2x^2}dx$

Solution

$$\frac{5x^2-3x+2}{x^3-2x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$5x^2-3x+2 = Ax^2 - 2Ax + Bx - 2B + Cx^2$$

$$\begin{cases} A+C=5 \\ -2A+B=-3 \\ -2B=2 \end{cases} \rightarrow \quad \underline{B=-1; \ A=1; \ C=4|}$$

$$\int \frac{5x^2-3x+2}{x^3-2x^2}dx = \int \frac{dx}{x} - \int \frac{dx}{x^2} + 4\int \frac{dx}{x-2}$$

$$= \ln|x| + \frac{1}{x} + 4\ln|x-2| + K$$

Exercise

Evaluate $\displaystyle\int \frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}dx$

Solution

$$\frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)} = \frac{A}{x-2} + \frac{Bx+C}{x^2-2x+3}$$

$$7x^2-13x+13 = Ax^2-2Ax+3A+Bx^2-2Bx+Cx-2C$$

$$\begin{cases} A+B=7 \\ -2A-2B+C=-13 \\ 3A-2C=13 \end{cases} \rightarrow \quad A=5;\ \ B=2;\ C=1$$

$$\int \frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}dx = \int \frac{5dx}{x-2} + \int \frac{2x+1}{x^2-2x+3}dx$$

$$= 5\ln|x-2| + \int \frac{2x-2+3}{x^2-2x+3}dx$$

$$= 5\ln|x-2| + \int \frac{2x-2}{x^2-2x+3}dx + \int \frac{3}{(x-1)^2+3}dx$$

$$= 5\ln|x-2| + \ln\left(x^2-2x+3\right) + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + K$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{1+\sin x}$

Solution

$$\int \frac{dx}{1+\sin x} = \int \frac{1}{1+\frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2}du$$

$$= \int \frac{2}{u^2+2u+1}\, du$$

$$= \int \frac{2}{(u+1)^2}\, d(u+1)$$

$$= -\frac{2}{u+1} + C$$

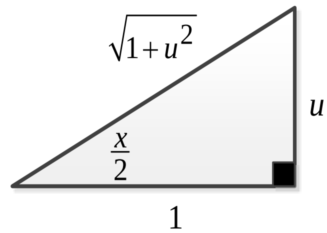
$$= -\frac{2}{\tan\left(\frac{x}{2}\right)+1} + C$$

Let $u = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1}u \quad \to dx = \frac{2du}{1+u^2}$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}}$$

$$= \frac{2u}{1+u^2}$$



Exercise

Evaluate $\displaystyle\int \frac{dx}{2+\cos x}$

Solution

$$\int \frac{dx}{2+\cos x} = \int \frac{1}{2+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}du$$

$$= 2\int \frac{1}{u^2+3}\, du$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C$$

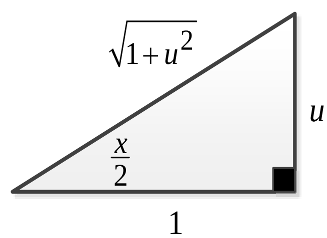
$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) + C$$

Let $u = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1}u \quad \to dx = \frac{2du}{1+u^2}$

$$\cos x = 2\cos^2\frac{x}{2} - 1$$

$$= 2\frac{1}{1+u^2} - 1$$

$$= \frac{1-u^2}{1+u^2}$$



$$\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{1-\cos x}$

Solution

$$\int \frac{dx}{1-\cos x} = \int \frac{1}{1-\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}du$$

$$= \int \frac{1}{u^2}\, du$$

$$= -\frac{1}{u} + C$$

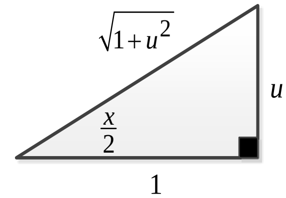
$$= -\frac{1}{\tan\frac{x}{2}} + C \qquad \boxed{= -\cot\frac{x}{2} + C}$$

Let $u = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1} u \quad \rightarrow dx = \frac{2du}{1+u^2}$

$$\cos x = 2\cos^2\frac{x}{2} - 1$$

$$= 2\frac{1}{1+u^2} - 1$$

$$= \frac{1-u^2}{1+u^2}$$



Exercise

Evaluate $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

Solution

$$\int \frac{dx}{1+\sin x+\cos x} = \int \frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} du$$

$$= 2\int \frac{1}{2+2u}\, du$$

$$= \int \frac{1}{1+u}\, d(1+u)$$

$$= \ln|1+u| + C$$

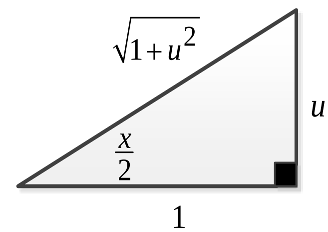
$$\boxed{= \ln\left|1+\tan\frac{x}{2}\right| + C}$$

$u = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1} u \quad \rightarrow dx = \frac{2du}{1+u^2}$

$$\cos x = 2\cos^2\frac{x}{2} - 1$$

$$= 2\frac{1}{1+u^2} - 1$$

$$= \frac{1-u^2}{1+u^2}$$



$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2}$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^2-5x+6}\, dx$

Solution

$$\frac{1}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$Ax - 3A + Bx - 2B = 1 \quad \rightarrow \begin{cases} A+B=0 \\ -3A-2B=1 \end{cases} \quad \rightarrow \quad A=-1 \quad B=1$$

$$\int \frac{1}{x^2-5x+6}\, dx = \int \left(\frac{-1}{x-2} + \frac{1}{x-3}\right) dx$$

$$= \ln|x-3| - \ln|x-2| + C$$

$$\boxed{= \ln\left|\frac{x-3}{x-2}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^2-5x+5}\,dx$

Solution

$$\frac{1}{x^2-5x+5} = \frac{A}{x-\frac{5+\sqrt{5}}{2}} + \frac{B}{x-\frac{5-\sqrt{5}}{2}} \qquad\qquad x=\frac{5\pm\sqrt{5}}{2}$$

$$Ax-\left(\frac{5-\sqrt{5}}{2}\right)A+Bx-\left(\frac{5+\sqrt{5}}{2}\right)B=1$$

$$\begin{cases} A+B=0 \\ -\frac{5-\sqrt{5}}{2}A-\frac{5+\sqrt{5}}{2}B=1 \end{cases} \to \begin{array}{l} \frac{5-\sqrt{5}}{2}A+\frac{5-\sqrt{5}}{2}B=0 \\ -\frac{5-\sqrt{5}}{2}A-\frac{5+\sqrt{5}}{2}B=1 \end{array}$$

$$-\sqrt{5}B=1 \to B=-\frac{1}{\sqrt{5}} \quad\Rightarrow\quad A=\frac{1}{\sqrt{5}}$$

$$\int \frac{1}{x^2-5x+5}\,dx = \int\left(\frac{\sqrt{5}}{5}\frac{2}{2x-5-\sqrt{5}}-\frac{\sqrt{5}}{5}\frac{2}{2x-5+\sqrt{5}}\right)dx$$

$$=\frac{\sqrt{5}}{5}\ln\left|2x-5-\sqrt{5}\right|-\frac{\sqrt{5}}{5}\ln\left|2x-5+\sqrt{5}\right|+C$$

Exercise

Evaluate $\displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$

Solution

$$\frac{5x^2+20x+6}{x^3+2x^2+x} = \frac{5x^2+20x+6}{x(x+1)^2} = \frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$Ax^2+2Ax+A+Bx^2+Bx+Cx=5x^2+20x+6$$

$$\begin{cases} A+B=5 \\ 2A+B+C=20 \quad \to \quad \underline{B=-1} \quad \underline{C=9} \\ \underline{A=6} \end{cases}$$

$$\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx = \int\left(\frac{6}{x}-\frac{1}{x+1}+\frac{9}{(x+1)^2}\right)dx$$

$$=6\ln|x|-\ln|x+1|-\frac{9}{x+1}+C$$

$$=\ln\frac{x^6}{|x+1|}-\frac{9}{x+1}+C$$

Exercise

Evaluate $\displaystyle\int \frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}\,dx$

Solution

$$\frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}=\frac{2x^3-4x-8}{x(x-1)\left(x^2+4\right)}=\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+4}$$

$$Ax^3-Ax^2+4Ax-4A+Bx^3+4Bx+Cx^3-Cx^2+Dx^2-Dx=2x^3-4x-8$$

$$\begin{cases} x^3 & A+B+C=2 \\ x^2 & -A-C+D=0 \\ x^1 & 4A+4B-D=-4 \\ x^0 & -4A=-8 \end{cases} \to \begin{cases} B+C=0 \\ -C+D=2 \\ 4B-D=-12 \\ \underline{A=2} \end{cases} \Rightarrow \begin{cases} B+D=2 \\ 4B-D=-12 \end{cases} \to \begin{cases} A=2 \\ B=-2 \\ C=2 \\ D=4 \end{cases}$$

$$\int \frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}\,dx=\int\left(\frac{2}{x}-\frac{2}{x-1}+\frac{2x}{x^2+4}+\frac{4}{x^2+4}\right)dx \qquad \int\frac{dx}{x^2+a^2}=\frac{1}{a}\arctan\frac{x}{a}$$

$$\boxed{=2\ln|x|-2\ln|x-1|+\ln\left(x^2+4\right)+2\tan^{-1}\frac{x}{2}+C}$$

Exercise

Evaluate $\displaystyle\int \frac{8x^3+13x}{\left(x^2+2\right)^2}\,dx$

Solution

$$\frac{8x^3+13x}{\left(x^2+2\right)^2}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{\left(x^2+2\right)^2}$$

$$Ax^3+2Ax+Bx^2+2B+Cx+D=8x^3+13x \qquad \begin{cases} x^3 & A=8 \\ x^2 & B=0 \\ x^1 & 2A+C=13 \\ x^0 & D=0 \end{cases} \to C=-3$$

$$\int \frac{8x^3+13x}{\left(x^2+2\right)^2}\,dx=\int\frac{8x}{x^2+2}\,dx-\int\frac{3x}{\left(x^2+2\right)^2}\,dx$$

$$=2\int\frac{1}{x^2+2}\,d\left(x^2+2\right)-\frac{3}{2}\int\frac{1}{\left(x^2+2\right)^2}\,d\left(x^2+2\right)$$

$$\boxed{=2\ln\left(x^2+2\right)+\frac{3}{2}\frac{1}{x^2+2}+C}$$

Exercise

Evaluate $\displaystyle\int \frac{\sin x}{\cos x + \cos^2 x}\, dx$

Solution

$$\frac{\sin x}{\cos x + \cos^2 x} = \frac{A}{\cos x} + \frac{B}{1+\cos x}$$

$$A + A\cos x + B\cos x = \sin x \qquad \begin{cases} \underline{A = \sin x} \\ A + B = 0 \end{cases} \to \underline{B = -\sin x}$$

$$\int \frac{\sin x}{\cos x + \cos^2 x}\, dx = \int \frac{\sin x}{\cos x} dx - \int \frac{\sin x}{1+\cos x} dx$$

$$= -\int \frac{1}{\cos x}\, d(\cos x) + \int \frac{1}{1+\cos x}\, d(1+\cos x)$$

$$= -\ln|\cos x| + \ln|1+\cos x| + C$$

$$\boxed{= \ln\left|\frac{1+\cos x}{\cos x}\right| + C} \qquad \boxed{= \ln|\sec x + 1| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{5\cos x}{\sin^2 x + 3\sin x - 4}\, dx$

Solution

$$\frac{5\cos x}{\sin^2 x + 3\sin x - 4} = \frac{A}{\sin x - 1} + \frac{B}{\sin x + 4}$$

$$A\sin x + 4A + B\sin x - B = 5\cos x \qquad \begin{cases} 4A - B = 5\cos x \\ A + B = 0 \end{cases} \quad \underline{A = \cos x} \quad \underline{B = -\cos x}$$

$$\int \frac{5\cos x}{\sin^2 x + 3\sin x - 4}\, dx = \int \frac{\cos x}{\sin x - 1} dx - \int \frac{\cos x}{\sin x + 4} dx$$

$$= \int \frac{1}{\sin x - 1} d(\sin x - 1) - \int \frac{1}{\sin x + 4} d(\sin x + 4)$$

$$= \ln|\sin x - 1| - \ln|\sin x + 4| + C$$

$$\boxed{= \ln\left|\frac{\sin x - 1}{\sin x + 4}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{e^x}{\left(e^x - 1\right)\left(e^x + 4\right)}\, dx$

Solution

Let $\quad u = e^x \quad \to \quad du = e^x dx$

$$\int \frac{e^x}{\left(e^x-1\right)\left(e^x+4\right)}\, dx = \int \frac{du}{(u-1)(u+4)}$$

$$\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}$$

$$Au + 4A + Bu - B = 1 \quad \Rightarrow \begin{cases} A+B=0 \\ 4A-B=1 \end{cases} \quad \rightarrow \quad \underline{A = \frac{1}{5}, \ B = -\frac{1}{5}}$$

$$\int \frac{du}{(u-1)(u+4)} = \frac{1}{5}\int \frac{1}{u-1}\, du + \frac{4}{5}\int \frac{1}{u+4}\, du$$

$$= \frac{1}{5}\int \frac{1}{u-1}\, d(u-1) + \frac{4}{5}\int \frac{1}{u+4}\, d(u+4)$$

$$= \frac{1}{5}\ln\left|e^x-1\right| - \frac{1}{5}\ln\left(e^x+4\right) + C$$

$$\boxed{= \frac{1}{5}\ln\left|\frac{e^x-1}{e^x+4}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{e^x}{\left(e^{2x}+1\right)\left(e^x-1\right)}\, dx$

Solution

Let $u = e^x \quad \rightarrow \quad du = e^x dx$

$$\int \frac{e^x}{\left(e^{2x}+1\right)\left(e^x-1\right)}\, dx = \int \frac{du}{\left(u^2+1\right)(u-1)}$$

$$\frac{1}{\left(u^2+1\right)(u-1)} = \frac{Au+B}{u^2+1} + \frac{C}{u-1}$$

$$Au^2 - Au + Bu - B + Cu^2 + C = 1$$

$$\begin{cases} u^2 & A+C=0 \\ u^1 & -A+B=0 \rightarrow \begin{cases} B+C=0 \\ -B+C=1 \end{cases} \\ u^0 & -B+C=1 \end{cases} \qquad \underline{C = \frac{1}{2} \quad B = -\frac{1}{2} \quad A = -\frac{1}{2}}$$

$$\int \frac{du}{\left(u^2+1\right)(u-1)} = -\frac{1}{2}\int \frac{u}{u^2+1} du - \frac{1}{2}\int \frac{du}{u^2+1} + \frac{1}{2}\int \frac{du}{u-1}$$

$$= -\frac{1}{4}\int \frac{1}{u^2+1} d\left(u^2+1\right) - \frac{1}{2}\arctan u + \frac{1}{2}\ln|u-1|$$

$$\boxed{= -\frac{1}{4}\ln\left(e^{2x}+1\right) - \frac{1}{2}\arctan e^x + \frac{1}{2}\ln\left|e^x-1\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{\sqrt{x}}{x-4}\,dx$

Solution

Let $u=\sqrt{x} \quad \to \quad du=\frac{1}{2\sqrt{x}}dx \;\Rightarrow\; 2udu=dx$

$$\int \frac{\sqrt{x}}{x-4}\,dx=\int \frac{u}{u^2-4}\,2u\;du$$

$$=\int \frac{2u^2}{u^2-4}\,du$$

$$=\int \left(2+\frac{8}{u^2-4}\right)du$$

$$\frac{8}{u^2-4}=\frac{A}{u-2}+\frac{B}{u+2}$$

$$Au+2A+Bu-2B=8 \qquad \to \begin{cases} A+B=0 \\ 2A-2B=8 \end{cases} \Rightarrow \underline{A=2 \quad B=-2}$$

$$=\int \left(2+\frac{2}{u-2}-\frac{2}{u+2}\right)du$$

$$=2\sqrt{x}+2\ln\left|\sqrt{x}-2\right|-2\ln\left|\sqrt{x}+2\right|+C$$

$$=2\sqrt{x}+2\ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right|+C$$

Exercise

Evaluate $\displaystyle\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$

Solution

Let $u=x^{1/6} \to u^6=x \quad \to \quad 6u^5du=dx$

$u^2=x^{1/3} \quad u^3=x^{1/2}$

$$\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx=\int \frac{6u^5}{u^3-u^2}\,du$$

$$=\int \frac{6u^3}{u-1}\,du$$

$$\begin{array}{r|l} & 6u^2+6u+6 \\ \hline u-1 & 6u^3 \\ & \underline{-6u^3+6u^2} \\ & 6u^2 \\ & \underline{-6u^2+6u} \\ & 6u \\ & \underline{-6u+6} \\ & 6 \end{array}$$

$$=\int \left(6u^2+6u+6+\frac{6}{u-1}\right)du$$

$$=2u^3+3u^2+6u+6\ln|u-1|+C$$

$$=2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+6\ln\left|\sqrt[6]{x}-1\right|+C$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^2-9}\,dx$

Solution

$$\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}$$

$$Ax + 3A + Bx - 3B = 1 \quad \Rightarrow \begin{cases} A+B=0 \\ 3A-3B=1 \end{cases} \to \boxed{A = \tfrac{1}{6} \quad B = -\tfrac{1}{6}}$$

$$\int \frac{1}{x^2-9}\,dx = \frac{1}{6}\int \frac{1}{x-3}\,dx - \frac{1}{6}\int \frac{1}{x+3}\,dx$$

$$= \frac{1}{6}\ln|x-3| - \frac{1}{6}\ln|x+3| + C$$

$$\boxed{= \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{2}{9x^2-1}\,dx$

Solution

$$\frac{2}{9x^2-1} = \frac{A}{3x-1} + \frac{B}{3x+1}$$

$$3Ax + A + 3Bx - B = 2 \quad \Rightarrow \begin{cases} 3A+3B=0 \\ A-B=2 \end{cases} \to \boxed{A = 1 \quad B = -1}$$

$$\int \frac{2}{9x^2-1}\,dx = \int \frac{1}{3x-1}\,dx - \int \frac{1}{3x+1}\,dx$$

$$= \frac{1}{3}\ln|3x-1| - \frac{1}{3}\ln|3x+1| + C$$

$$\boxed{= \frac{1}{3}\ln\left|\frac{3x-1}{3x+1}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{5}{x^2+3x-4}\,dx$

Solution

$$\frac{5}{x^2+3x-4} = \frac{A}{x-1} + \frac{B}{x+4}$$

$$Ax + 4A + Bx - B = 5 \quad \Rightarrow \begin{cases} A+B=0 \\ 4A-B=5 \end{cases} \to \boxed{A = 1 \quad B = -1}$$

$$\int \frac{5}{x^2+3x-4}\,dx = \int \frac{1}{x-1}\,dx - \int \frac{1}{x+4}\,dx$$

$$= \ln|x-1| - \ln|x+4| + C$$

$$\boxed{= \ln\left|\frac{x-1}{x+4}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{3-x}{3x^2-2x-1}\,dx$

Solution

$$\frac{3-x}{3x^2-2x-1} = \frac{A}{x-1} + \frac{B}{3x+1}$$

$$3Ax + A + Bx - B = 3 - x \qquad \Rightarrow \begin{cases} 3A + B = -1 \\ A - B = 3 \end{cases} \rightarrow \boxed{A = \frac{1}{2} \quad B = -\frac{5}{2}}$$

$$\int \frac{3-x}{3x^2-2x-1}\,dx = \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{5}{2}\int \frac{1}{3x+1}\,dx$$

$$\boxed{= \frac{1}{2}\ln|x-1| - \frac{5}{6}\ln|3x+1| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2+12x+12}{x^3-4x}\,dx$

Solution

$$\frac{x^2+12x+12}{x^3-4x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

$$Ax^2 - 4A + Bx^2 + 2Bx + Cx^2 - 2Cx = x^2 + 12x + 12$$

$$\begin{cases} x^2 & A + B + C = 1 \\ x^1 & 2B - 2C = 12 \quad \rightarrow A = -3 \quad B = 5 \quad C = -1 \\ x^0 & -4A = 12 \end{cases}$$

$$\int \frac{x^2+12x+12}{x^3-4x}\,dx = -\frac{3}{x} + \frac{5}{x-2} - \frac{1}{x+2}$$

$$\boxed{= -3\ln|x| + 5\ln|x-2| - \ln|x+2| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x^3-x+3}{x^2+x-2}\,dx$

Solution

$$\frac{x^3-x+3}{x^2+x-2}=x-1+\frac{2x+1}{x^2+x-2}$$

$$\begin{array}{r} x-1 \\ x^2+x-2\overline{)\,x^3-x+3} \\ \underline{-x^3-x^2+2x} \\ -x^2+x+3 \\ \underline{x^2+x-2} \\ 2x-1 \end{array}$$

$$\frac{2x+1}{x^2+x-2}=\frac{A}{x-1}+\frac{B}{x+2}$$

$$Ax+2A+Bx-B=2x+1 \quad \Rightarrow \begin{cases} A+B=2 \\ 2A-B=1 \end{cases} \rightarrow \underline{A=1 \quad B=1|}$$

$$\int \frac{x^3-x+3}{x^2+x-2}\,dx=\int\left(x-1+\frac{1}{x-1}+\frac{1}{x+2}\right)dx$$

$$\underline{=\frac{1}{2}x^2-x+\ln|x-1|+\ln|x+2|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{5x-2}{(x-2)^2}\,dx$

Solution

$$\frac{5x-2}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}$$

$$Ax-2A+B=5x-2 \quad \Rightarrow \begin{cases} \underline{A=5|} \\ -2A+B=-2 \end{cases} \rightarrow \underline{B=8|}$$

$$\int \frac{5x-2}{(x-2)^2}\,dx=\frac{5}{x-2}+\frac{8}{(x-2)^2}$$

$$\underline{=5\ln|x-2|-\frac{8}{x-2}+C}$$

Exercise

Evaluate $\displaystyle\int \frac{2x^3-4x^2-15x+4}{x^2-2x-8}\,dx$

$$\begin{array}{r} 2x \\ x^2-2x-8\overline{)\,2x^3-4x^2-15x+4} \\ \underline{2x^3-4x^2-16x} \\ x+4 \end{array}$$

Solution

$$\int \frac{2x^3-4x^2-15x+4}{x^2-2x-8}\,dx=\int 2x\,dx+\int\frac{x+4}{x^2-2x-8}\,dx$$

$$\frac{x+4}{x^2-2x-8}=\frac{A}{x-4}+\frac{B}{x+2}$$

$$Ax+2A+Bx-4B=x+4 \quad \Rightarrow \begin{cases} A+B=1 \\ 2A-4B=4 \end{cases} \rightarrow \underline{A=\tfrac{4}{3} \quad B=-\tfrac{1}{3}|}$$

$$\int \frac{2x^3-4x^2-15x+4}{x^2-2x-8}\,dx=x^2+\frac{4}{3}\int\frac{1}{x-4}\,dx-\frac{1}{3}\int\frac{1}{x+2}\,dx$$

$$\underline{=x^2+\frac{4}{3}\ln|x-4|-\frac{1}{3}\ln|x+2|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{x+2}{x^2+5x}\,dx$

Solution

$$\frac{x+2}{x^2+5x} = \frac{A}{x} + \frac{B}{x+5}$$

$$Ax + 5A + Bx = x + 2 \qquad \Rightarrow \begin{cases} A+B=1 \\ 5A=2 \end{cases} \rightarrow \underline{A=\frac{2}{5} \quad B=\frac{3}{5}}$$

$$\int \frac{x+2}{x^2+5x}\,dx = \frac{2}{5}\int \frac{1}{x}\,dx + \frac{3}{5}\int \frac{1}{x+5}\,dx$$

$$= \frac{2}{5}\ln|x| + \frac{3}{5}\ln|x+5| + C$$

Exercise

Evaluate $\displaystyle\int_0^2 \frac{3}{4x^2+5x+1}\,dx$

Solution

$$\frac{3}{4x^2+5x+1} = \frac{A}{x+1} + \frac{B}{4x+1}$$

$$4Ax + A + Bx + B = 3 \qquad \Rightarrow \begin{cases} 4A+B=0 \\ A+B=3 \end{cases} \rightarrow \underline{A=-1 \quad B=4}$$

$$\int_0^2 \frac{3}{4x^2+5x+1}\,dx = -\int_0^2 \frac{1}{x+1}\,dx + \int_0^2 \frac{4}{4x+1}\,dx$$

$$= -\ln(x+1) + \ln(4x+1)\Big|_0^2$$

$$= \ln\frac{4x+1}{x+1}\Big|_0^2$$

$$= \ln 3$$

Exercise

Evaluate $\displaystyle\int_1^5 \frac{x-1}{x^2(x+1)}\,dx$

Solution

$$\frac{x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$Ax^2 + Ax + Bx + B + Cx^2 = x - 1$$

$$\begin{cases} x^2 & A+C=0 \\ x^1 & A+B=1 \quad \rightarrow A=2 \quad C=-2 \\ x^0 & \underline{B=-1} \end{cases}$$

$$\int_1^5 \frac{x-1}{x^2(x+1)}\, dx = \int_1^5 \left(\frac{2}{x} - \frac{1}{x^2} - \frac{2}{x+1} \right) dx$$

$$= 2\ln x + \frac{1}{x} - 2\ln(x+1) \Big|_1^5$$

$$= 2\ln 5 + \frac{1}{5} - 2\ln 6 - 1 + 2\ln 2$$

$$\boxed{= 2\ln\frac{5}{3} - \frac{4}{5}}$$

Exercise

Evaluate $\displaystyle\int_1^2 \frac{x+1}{x(x^2+1)}\, dx$

Solution

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$Ax^2 + A + Bx^2 + Cx = x+1$$

$$\begin{cases} x^2 & A+B=0 \\ x^1 & \underline{C=1} \quad \rightarrow \underline{B=-1} \\ x^0 & \underline{A=1} \end{cases}$$

$$\int_1^2 \frac{x+1}{x(x^2+1)}\, dx = \int_1^2 \frac{1}{x}\, dx - \int_1^2 \frac{x}{x^2+1}\, dx + \int_1^2 \frac{1}{x^2+1}\, dx$$

$$= \int_1^2 \frac{1}{x}\, dx - \frac{1}{2}\int_1^2 \frac{1}{x^2+1}\, d(x^2+1) + \int_1^2 \frac{1}{x^2+1}\, dx$$

$$= \ln x - \frac{1}{2}\ln(x^2+1) + \arctan x \Big|_1^2$$

$$= \ln 2 - \frac{1}{2}\ln 5 + \arctan 2 + \frac{1}{2}\ln 2 - \frac{\pi}{4}$$

$$= \frac{1}{2}(3\ln 2 - \ln 5) - \frac{\pi}{4} + \arctan 2$$

$$\boxed{= \frac{1}{2}\ln\frac{8}{5} - \frac{\pi}{4} + \arctan 2}$$

Exercise

Evaluate $\displaystyle\int_0^1 \frac{x^2-x}{x^2+x+1}\,dx$

Solution

$$\int_0^1 \frac{x^2-x}{x^2+x+1}\,dx = \int_0^1 \left(1-\frac{2x+1}{x^2+x+1}\right)dx$$

$$= \int_0^1 dx - \int_0^1 \frac{1}{x^2+x+1}\,d\left(x^2+x+1\right)$$

$$= x - \ln\left(x^2+x+1\right)\Big|_0^1$$

$$\underline{= 1-\ln 3}$$

Exercise

Evaluate $\displaystyle\int_4^8 \frac{ydy}{y^2-2y-3}$

Solution

$$\frac{y}{y^2-2y-3} = \frac{A}{y-3}+\frac{B}{y+1} = \frac{(A+B)y+A-3B}{(y-3)(y+1)} \qquad \to \begin{cases} A+B=1 \\ A-3B=0 \end{cases} \Rightarrow \boxed{A=\frac{3}{4}} \quad \boxed{B=\frac{1}{4}}$$

$$\int_4^8 \frac{ydy}{y^2-2y-3} = \frac{3}{4}\int_4^8 \frac{dy}{y-3}+\frac{1}{4}\int_4^8 \frac{dy}{y+1}$$

$$= \left[\tfrac{3}{4}\ln|y-3|+\tfrac{1}{4}\ln|y+1|\right]_4^8$$

$$= \tfrac{3}{4}\ln|5|+\tfrac{1}{4}\ln|9|-\left(\tfrac{3}{4}\ln|1|+\tfrac{1}{4}\ln|5|\right)$$

$$= \frac{3}{4}\ln 5+\frac{1}{4}\ln 9-\frac{1}{4}\ln 5$$

$$= \frac{1}{2}\ln 5+\frac{1}{4}\ln 3^2 \qquad \textit{Power Rule}$$

$$= \frac{1}{2}\ln 5+\frac{1}{2}\ln 3$$

$$= \frac{1}{2}(\ln 5+\ln 3) \qquad \textit{Product Rule}$$

$$\underline{= \frac{1}{2}\ln 15}$$

Exercise

Evaluate $\displaystyle\int_1^{\sqrt{3}} \frac{3x^2+x+4}{x^3+x}\,dx$

Solution

$$\frac{3x^2+x+4}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{(A+B)x^2+Cx+A}{x\left(x^2+1\right)}$$

$$\begin{cases} A+B=3 \\ C=1 \\ A=4 \end{cases} \to \boxed{A=4} \quad \boxed{B=-1} \quad \boxed{C=1}$$

$$\int_1^{\sqrt{3}} \frac{3x^2+x+4}{x^3+x}\,dx = \int_1^{\sqrt{3}} \frac{4}{x}\,dx + \int_1^{\sqrt{3}} \frac{-x+1}{x^2+1}\,dx$$

$$= 4\int_1^{\sqrt{3}} \frac{1}{x}\,dx - \int_1^{\sqrt{3}} \frac{x}{x^2+1}\,dx + \int_1^{\sqrt{3}} \frac{1}{x^2+1}\,dx \qquad d\left(x^2+1\right) = 2xdx$$

$$= 4\int_1^{\sqrt{3}} \frac{1}{x}\,dx - \frac{1}{2}\int_1^{\sqrt{3}} \frac{d\left(x^2+1\right)}{x^2+1} + \int_1^{\sqrt{3}} \frac{1}{x^2+1}\,dx \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \left[4\ln|x| - \tfrac{1}{2}\ln\left(x^2+1\right) + \tan^{-1}x\right]_1^{\sqrt{3}}$$

$$= 4\ln\sqrt{3} - \tfrac{1}{2}\ln 4 + \tan^{-1}\sqrt{3} - \left(4\ln 1 - \tfrac{1}{2}\ln 2 + \tan^{-1}1\right)$$

$$= 4\ln 3^{1/2} - \tfrac{1}{2}\ln 2^2 + \tfrac{\pi}{3} + \tfrac{1}{2}\ln 2 - \tfrac{\pi}{4}$$

$$= 2\ln 3 - \ln 2 + \tfrac{\pi}{12} + \tfrac{1}{2}\ln 2$$

$$= 2\ln 3 - \tfrac{1}{2}\ln 2 + \tfrac{\pi}{12}$$

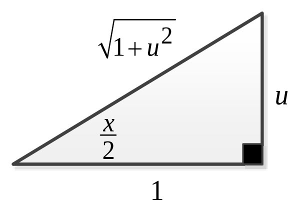
$$= \ln\left(\frac{9}{\sqrt{2}}\right) + \frac{\pi}{12}$$

Exercise

Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$

Solution

$$\int_0^{\pi/2} \frac{dx}{\sin x + \cos x} = \int_0^{\pi/2} \frac{1}{\frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}\,du$$



$$= 2\int_0^{\pi/2} \frac{du}{2u+1-u^2}$$

$$u = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1}u \quad \to dx = \frac{2du}{1+u^2}$$

$$= -2\int_0^{\pi/2} \frac{du}{u^2 - 2u - 1}$$

$\cos x = 2\cos^2 \frac{x}{2} - 1 = 2\frac{1}{1+u^2} - 1 = \frac{1-u^2}{1+u^2}$

$\sin x = 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2}$

$$= -\frac{1}{\sqrt{2}}\int_0^{\pi/2}\left(\frac{1}{u-1-\sqrt{2}} - \frac{1}{u-1+\sqrt{2}}\right)du$$

$\frac{2}{u^2-2u-1} = \frac{A}{u-1-\sqrt{2}} + \frac{B}{u-1+\sqrt{2}}$

$2 = Au + \left(-1+\sqrt{2}\right)A + Bu + \left(-1-\sqrt{2}\right)B$

$\begin{cases} A + B = 0 \\ \left(-1+\sqrt{2}\right)A - \left(1+\sqrt{2}\right)B = 2 \end{cases} \rightarrow \begin{cases} B = -A = -\frac{1}{\sqrt{2}} \\ 2\sqrt{2}A = 2 \end{cases}$

$$= -\frac{1}{\sqrt{2}}\left(\ln\left|\frac{1}{u-1-\sqrt{2}}\right| - \ln\left|\frac{1}{u-1+\sqrt{2}}\right|\right)\Bigg|_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}}\left(\ln\left|\frac{u-1+\sqrt{2}}{u-1-\sqrt{2}}\right|\right)\Bigg|_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}}\left(\ln\left|\frac{\tan\frac{x}{2}-1+\sqrt{2}}{\tan\frac{x}{2}-1-\sqrt{2}}\right|\right)\Bigg|_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}}\left(\ln|-1| - \ln\left|\frac{-1+\sqrt{2}}{-1-\sqrt{2}}\right|\right)$$

$$= \frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

Exercise

Evaluate $\displaystyle\int_0^{\pi/3}\frac{\sin\theta}{1-\sin\theta}d\theta$

Solution

$$\int_0^{\pi/3}\frac{\sin\theta}{1-\sin\theta}d\theta = \int_0^{\pi/3}\frac{1}{\csc\theta - 1}d\theta$$

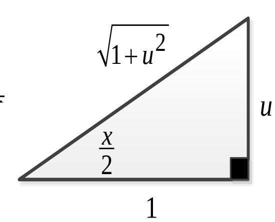
$$= \int_0^{\pi/3}\frac{1}{\frac{1+u^2}{2u}-1}\cdot\frac{2}{1+u^2}du$$

$u = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1}u \quad \rightarrow dx = \frac{2du}{1+u^2}$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$= 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}}$

$= \frac{2u}{1+u^2}$



$$= \int_0^{\pi/3} \frac{4u}{\left(1+u^2-2u\right)\left(1+u^2\right)}\,du$$

$$= \int_0^{\pi/3} \frac{4u}{(u-1)^2\left(1+u^2\right)}\,du$$

$$\frac{4u}{(u-1)^2\left(1+u^2\right)} = \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{Cu+D}{1+u^2}$$

$$4u = Au + Au^3 - A - Au^2 + B + Bu^2 + Cu^3 - 2Cu^2 + Cu + Du^2 - 2Du + D$$

$$\begin{cases} A + C = 0 \\ -A + B - 2C + D = 0 \\ C - 2D = 4 \\ -A + B + D = 0 \end{cases} \rightarrow \begin{cases} A = 0; \quad B = 2 \\ C = 0; \quad D = -2 \end{cases}$$

$$= \int_0^{\pi/3} \left(\frac{2}{(u-1)^2} - \frac{2}{1+u^2} \right) du$$

$$= \frac{-2}{u-1} - 2\tan^{-1} u \Big|_0^{\pi/3}$$

$$= \frac{-2}{\tan\frac{x}{2} - 1} - 2\tan^{-1}\left(\tan\frac{x}{2}\right) \Big|_0^{\pi/3}$$

$$= \frac{-2}{\tan\frac{x}{2} - 1} - x \Big|_0^{\pi/3}$$

$$= \frac{-2}{\frac{1}{\sqrt{3}} - 1} - \frac{\pi}{3} - 2$$

$$= \frac{-2\sqrt{3}}{1-\sqrt{3}} - \frac{\pi}{3} - 2$$

$$= \frac{-2}{1-\sqrt{3}} - \frac{\pi}{3} \qquad\qquad = \frac{-2}{1-\sqrt{3}}\,\frac{1+\sqrt{3}}{1+\sqrt{3}} - \frac{\pi}{3}$$

$$\boxed{= 1 + \sqrt{3} - \frac{\pi}{3}}$$

Exercise

Find the volume of the solid generated by the revolving the shaded region about x-axis

Solution

$$V = \pi \int_{0.5}^{2.5} y^2 dx$$

$$= \pi \int_{0.5}^{2.5} \frac{9}{3x - x^2} dx$$

$$= 9\pi \int_{0.5}^{2.5} \frac{1}{3x - x^2} dx$$

$$\frac{1}{3x - x^2} = \frac{1}{x(3-x)} = \frac{A}{x} + \frac{B}{3-x} = \frac{(B-A)x + 3A}{x(3-x)}$$

$$\begin{cases} B - A = 0 \\ 3A = 1 \end{cases} \Rightarrow \boxed{A = \frac{1}{3}} \quad \boxed{B = \frac{1}{3}}$$

$$= 9\pi \int_{0.5}^{2.5} \frac{1}{3}\left(\frac{1}{x} + \frac{1}{3-x}\right) dx$$

$$= 3\pi \int_{0.5}^{2.5} \left(\frac{1}{x} - \frac{1}{x-3}\right) dx$$

$$= 3\pi \left[\int_{0.5}^{2.5} \frac{1}{x} dx - \int_{0.5}^{2.5} \frac{1}{x-3} dx\right]$$

$$= 3\pi \left[\ln|x| - \ln|x-3|\right]_{0.5}^{2.5}$$

$$= 3\pi \left[\ln\left|\frac{x}{x-3}\right|\right]_{0.5}^{2.5}$$

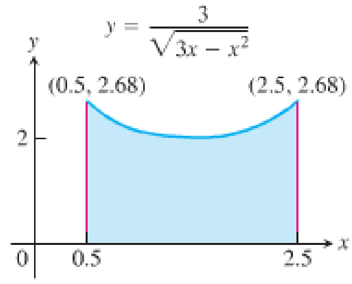
$$= 3\pi \left[\ln\left|\frac{2.5}{-.5}\right| - \ln\left|\frac{0.5}{-2.5}\right|\right]$$

$$= 3\pi \left[\ln 5 - \ln \frac{1}{5}\right]$$

$$= 3\pi \left[\ln 5 + \ln 5\right]$$

$$= 3\pi \left[2 \ln 5\right]$$

$$= 3\pi \ln 25$$



Solution Section 2.5 – Numerical Integration

Exercise

Find the Midpoint Rule approximations to: $\int_0^1 \sin \pi x \, dx$ *using* $n = 6$ *subintervals*

Solution

$$\Delta x = \frac{1-0}{6} = \frac{1}{6}$$

$$x_0 = 0,\ x_1 = 0 + \frac{1}{6} = \frac{1}{6},\ x_2 = \frac{1}{3},\ x_3 = \frac{1}{2},\ x_4 = \frac{2}{3},\ x_5 = \frac{5}{6},\ x_6 = 1$$

$$m_1 = \frac{1}{2}\left(0 + \frac{1}{6}\right) = \frac{1}{12},\ m_2 = \frac{1}{4},\ m_3 = \frac{5}{12},\ m_4 = \frac{7}{12},\ m_5 = \frac{9}{12},\ m_6 = \frac{11}{12}$$

$$M(6) = \left(\sin\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{5\pi}{12}\right) + \sin\left(\frac{7\pi}{12}\right) + \sin\left(\frac{9\pi}{12}\right) + \sin\left(\frac{11\pi}{12}\right)\right)\left(\frac{1}{6}\right)$$

$$\approx 0.6439505509$$

Exercise

Find the Midpoint Rule approximations to: $\int_0^1 e^{-x} dx$ *using* $n = 8$ *subintervals*

Solution

$$\Delta x = \frac{1-0}{8} = \frac{1}{8}$$

$$x_0 = 0,\ x_1 = \frac{1}{8},\ x_2 = \frac{1}{4},\ x_3 = \frac{3}{8},\ x_4 = \frac{1}{2},\ x_5 = \frac{5}{8},\ x_6 = \frac{3}{4},\ x_7 = \frac{7}{8},\ x_8 = 1$$

$$m_1 = \frac{1}{2}\left(0 + \frac{1}{8}\right) = \frac{1}{16},\ m_2 = \frac{3}{16},\ m_3 = \frac{5}{16},\ m_4 = \frac{7}{16},\ m_5 = \frac{9}{16},\ m_6 = \frac{11}{16},\ m_7 = \frac{13}{16},\ m_8 = \frac{15}{16}$$

$$M(8) = \frac{1}{8}\left(e^{-1/16} + e^{-3/16} + e^{-5/16} + e^{-7/16} + e^{-9/16} + e^{-11/16} + e^{-13/16} + e^{-15/16}\right)$$

$$\approx 0.6317092095$$

Exercise

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of 10^{-4} by (***a***) the Trapezoid Rule and (***b***) Simpson's Rule. $\int_1^3 (2x-1)\, dx$

Solution

a) ***i***) $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$

$$T = \frac{1}{2}\Delta x\left(m\, f\left(x_i\right)\right)$$

$$= \frac{1}{2}\frac{1}{2}(24) \;\; = \underline{6|}$$

$$f(x) = 2x - 1 \quad \Rightarrow \quad f'(x) = 2$$

$$\Rightarrow \quad f''(x) = 0 = M$$

$$\Rightarrow Error = 0$$

	x_i	$f\left(x_i\right) = 2x_i - 1$	m	$mf\left(x_i\right)$
x_0	1	1	1	1
x_1	$1 + \frac{1}{2} = \frac{3}{2}$	2	2	4
x_2	2	3	2	6
x_3	$\frac{5}{2}$	4	2	8
x_4	3	5	1	5
				24

ii) $$\int_1^3 (2x-1)\,dx = \left[x^2 - x\right]_1^3$$

$$= \left(3^2 - 3\right) - \left(1^2 - 1\right)$$

$$\underline{= 6|}$$

iii) $$Error = \frac{|E_T|}{True\ \ Value} \times 100 = 0\%$$

b) ***i***) $$\underline{|\Delta x} = \frac{b-a}{n} = \frac{3-1}{4} = \underline{\frac{1}{2}|}$$

$$S = \frac{1}{3}\Delta x\left(\sum m\, f\left(x_i\right)\right)$$

$$= \frac{1}{3}\frac{1}{2}(36) \;\; = \underline{6|}$$

$$f(x) = 2x - 1 \quad \Rightarrow \quad f^{(4)}(x) = 0 = M$$

$$\Rightarrow \left|E_s\right| = 0$$

	x_i	$f\left(x_i\right) = 2x_i - 1$	m	$mf\left(x_i\right)$
x_0	1	1	1	1
x_1	$\frac{3}{2}$	2	4	8
x_2	2	3	2	6
x_3	$\frac{5}{2}$	4	4	16
x_4	3	5	1	5
				36

ii) $$\int_1^3 (2x-1)\,dx = 6$$

$$\left|E_s\right| = \int_1^3 (2x-1)\,dx - S = 6 - 6 = 0$$

iii) $$Error = \frac{|E_T|}{True\ \ Value} \times 100 = 0\%$$

Exercise

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of 10^{-4} by (***a***) the Trapezoid Rule and (***b***) Simpson's Rule. $\int_{-1}^{1}\left(x^2+1\right)dx$

Solution

a) ***i***) $\Delta x=\frac{b-a}{n}=\frac{1+1}{4}=\frac{1}{2}$

	x_i	$f(x_i)$	m	$m\,f(x_i)$
x_0	-1	2	1	2
x_1	$-\frac{1}{2}$	$\frac{5}{4}$	2	$\frac{5}{2}$
x_2	0	1	2	2
x_3	$\frac{1}{2}$	$\frac{5}{4}$	2	$\frac{5}{2}$
x_4	1	2	1	2
				11

$$T=\frac{1}{2}\Delta x\left(m\,f\left(x_i\right)\right)=\frac{1}{2}\frac{1}{2}(11)=\frac{11}{4}$$

$$f(x)=x^2+1 \quad\Rightarrow\quad f'(x)=2x$$

$$\Rightarrow\quad f''(x)=2=M$$

$$\left|E_T\right|=\frac{1-(-1)}{12}\left(\frac{1}{2}\right)^2(2)=0.0833\ldots$$

ii) $$\int_{-1}^{1}\left(x^2+1\right)dx=\left[\frac{1}{3}x^3+x\right]_{-1}^{1}=\left(\frac{1}{3}+1\right)-\left(-\frac{1}{3}-1\right)=\frac{8}{3}$$

$$E_T=\int_{-1}^{1}\left(x^2+1\right)dx-T=\frac{8}{3}-\frac{11}{4}=-\frac{1}{12}$$

iii) $$Error=\frac{\left|E_T\right|}{True\ \ Value}\times 100=\frac{\frac{1}{12}}{\frac{8}{3}}\approx 3\%$$

b) ***i***) $\Delta x=\frac{b-a}{n}=\frac{-1-(-1)}{4}=\frac{1}{2}$

	x_i	$f(x_i)$	m	$m\,f(x_i)$
x_0	-1	2	1	2
x_1	$-\frac{1}{2}$	$\frac{5}{4}$	4	5
x_2	0	1	2	2
x_3	$\frac{1}{2}$	$\frac{5}{4}$	4	5
x_4	1	2	1	2
				16

$$S=\frac{1}{3}\Delta x\left(\sum m\,f\left(x_i\right)\right)=\frac{1}{3}\frac{1}{2}(16)=\frac{8}{3}$$

$$f(x)=x^2+1 \quad\Rightarrow\quad f^{(4)}(x)=0=M$$

$$\Rightarrow\left|E_s\right|=0$$

ii) $$\int_{-1}^{1}\left(x^2+1\right)dx=\frac{8}{3}$$

$$E_S=\int_{-1}^{1}\left(x^2+1\right)dx-S=\frac{8}{3}-\frac{8}{3}=0$$

iii) $$Error=\frac{\left|E_T\right|}{True\ \ Value}\times 100=0\%$$

Exercise

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of 10^{-4} by (***a***) the Trapezoid Rule and (***b***) Simpson's Rule. $\displaystyle\int_2^4 \frac{1}{(s-1)^2}\,ds$

Solution

a) $\boxed{\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2}}$

$x_0 = 2 \qquad x_1 = 2 + \frac{1}{2} = \frac{5}{2} \qquad x_2 = 2 + 2\left(\frac{1}{2}\right) = 3 \qquad x_3 = 2 + 3\left(\frac{1}{2}\right) = \frac{7}{2} \qquad x_4 = 4$

$$T = \frac{1}{2}\Delta x\left(m\, f\left(x_i\right)\right)$$

$$= \frac{1}{2}\frac{1}{2}\left(\frac{1}{(2-1)^2} + 2\frac{1}{\left(\frac{5}{2}-1\right)^2} + 2\frac{1}{(3-1)^2} + 2\frac{1}{\left(\frac{7}{2}-1\right)^2} + \frac{1}{(4-1)^2}\right)$$

$$= \frac{1}{4}\left(\frac{1}{1} + \frac{8}{9} + \frac{1}{2} + \frac{8}{25} + \frac{1}{9}\right)$$

$$= \frac{1269}{1800}$$

$$\approx 0.705$$

$f(s) = (s-1)^{-2} \quad \Rightarrow f'(s) = -2(s-1)^{-3}$

$\qquad \Rightarrow f''(s) = 6(s-1)^{-4} = \dfrac{6}{(s-1)^4} \Rightarrow M = 6$

$$\int_2^4 \frac{1}{(s-1)^2}\,ds = \int_2^4 (s-1)^{-2}\,d(s-1)$$

$$= -\left[(s-1)^{-1}\right]_2^4$$

$$= -\left(3^{-1} - 1^{-1}\right)$$

$$= \frac{2}{3}$$

The percentage error: $\dfrac{|0.705 - .6667|}{.6667} \approx 0.0575 \quad 5.75\%$

b) $\boxed{\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2}}$

$x_0 = 2 \qquad x_1 = 2 + \frac{1}{2} = \frac{5}{2} \qquad x_2 = 2 + 2\left(\frac{1}{2}\right) = 3 \qquad x_3 = 2 + 3\left(\frac{1}{2}\right) = \frac{7}{2} \qquad x_4 = 4$

$$S = \frac{1}{3}\Delta x\left(m\, f\left(x_i\right)\right)$$

$$= \frac{1}{3}\frac{1}{2}\left(\frac{1}{(2-1)^2} + 4\frac{1}{\left(\frac{5}{2}-1\right)^2} + 2\frac{1}{(3-1)^2} + 4\frac{1}{\left(\frac{7}{2}-1\right)^2} + \frac{1}{(4-1)^2} \right)$$

$$= \frac{1}{6}\left(\frac{1}{1} + \frac{16}{9} + \frac{1}{2} + \frac{16}{25} + \frac{1}{9}\right)$$

$$= \frac{1813}{450}$$

$$\approx 0.67148$$

$$\int_2^4 \frac{1}{(s-1)^2}\, ds = \frac{2}{3}$$

The percentage error: $\frac{|0.67148 - .6667|}{.6667} \approx 0.0072$ $\quad 0.72\%$

Exercise

Find the Trapezoid & Simpson's Rule approximations and error: $\int_0^1 \sin \pi x \, dx \quad n = 6 \;\; subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1666666667	0.5000000000	1.0000000000
2	0.3333333333	0.8660254000	1.7320508000
3	0.5000000000	1.0000000000	2.0000000000
4	0.6666666667	0.8660254000	1.7320508000
5	0.8333333333	0.5000000000	1.0000000000
6	1.0000000000	0.0000000000	0.0000000000

Trapezoid Rule approximation ≈ 0.62200847

Simpson's Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.1666666667	0.5000000000	2.0000000000
2	0.3333333333	0.8660254000	1.7320508000
3	0.5000000000	1.0000000000	2.0000000000
4	0.6666666667	0.8660254000	1.7320508000
5	0.8333333333	0.5000000000	1.0000000000
6	1.0000000000	0.0000000000	0.0000000000

Simpson's Rule approximation ≈ 0.63689453

	Exact	Trapezoid	Simpson
Value:	0.63661977	0.62200847	0.63689453
Error:		2.2951 %	0.0432 %

Exercise

Find the Trapezoid & Simpson's Rule approximations to and error to $\int_0^1 e^{-x}dx \quad n=8 \ \ subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.1250000000	0.8824969000	1.7649938000
2	0.2500000000	0.7788007800	1.5576015600
3	0.3750000000	0.6872892800	1.3745785600
4	0.5000000000	0.6065306600	1.2130613200
5	0.6250000000	0.5352614300	1.0705228600
6	0.7500000000	0.4723665500	0.9447331000
7	0.8750000000	0.4168620200	0.8337240400
8	1.0000000000	0.3678794400	0.3678794400

Trapezoid Rule approximation ≈ 0.63294342

Simpson's Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	1.0000000000	1.0000000000
1	0.1250000000	0.8824969000	3.5299876000
2	0.2500000000	0.7788007800	1.5576015600
3	0.3750000000	0.6872892800	2.7491571200
4	0.5000000000	0.6065306600	1.2130613200
5	0.6250000000	0.5352614300	2.1410457200
6	0.7500000000	0.4723665500	0.9447331000
7	0.8750000000	0.4168620200	1.6674480800
8	1.0000000000	0.3678794400	0.3678794400

Simpson's Rule approximation ≈ 0.63212141

	Exact	Trapezoid	Simpson
Value:	0.63212056	0.63294342	0.63212141
Error:		0.1302 %	0.0001 %

Exercise

Find the Trapezoid & Simpson's Rule approximations and error to:

$$\int_1^5 \left(3x^2 - 2x\right) dx \quad n = 8 \;\; subintervals$$

Solution

Trapezoid Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.5000000000	3.7500000000	7.5000000000
2	2.0000000000	8.0000000000	16.0000000000
3	2.5000000000	13.7500000000	27.5000000000
4	3.0000000000	21.0000000000	42.0000000000
5	3.5000000000	29.7500000000	59.5000000000
6	4.0000000000	40.0000000000	80.0000000000
7	4.5000000000	51.7500000000	103.500000000
8	5.0000000000	65.0000000000	65.0000000000

Trapezoid Rule approximation ≈ 100.50000000

Simpson's Rule Method

n	x_n	$f\left(x_n\right)$	$m \cdot f\left(x_n\right)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.5000000000	3.7500000000	15.0000000000
2	2.0000000000	8.0000000000	16.0000000000
3	2.5000000000	13.7500000000	55.0000000000
4	3.0000000000	21.0000000000	42.0000000000
5	3.5000000000	29.7500000000	119.000000000
6	4.0000000000	40.0000000000	80.0000000000
7	4.5000000000	51.7500000000	207.000000000
8	5.0000000000	65.0000000000	65.0000000000

Simpson's Rule approximation ≈ 100.00000000

	Exact	***Trapezoid***	***Simpson***
Value:	100.000000	100.500000	100.00000000
Error:		0.5000%	0.0000 %

Exercise

Find the Trapezoid & Simpson's Rule approximations and error: $\int_0^{\pi/4} 3\sin 2x\, dx \quad n=8 \ \ subintervals$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m\cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.0981747704	0.5852709700	1.1705419400
2	0.1963495408	1.1480503000	2.2961006000
3	0.2945243113	1.6667107000	3.3334214000
4	0.3926990817	2.1213203400	4.2426406800
5	0.4908738521	2.4944088400	4.9888176800
6	0.5890486225	2.7716386000	5.5432772000
7	0.6872233930	2.9423558400	5.8847116800
8	0.7853981634	3.0000000000	3.0000000000

Trapezoid Rule approximation ≈ 1.49517776

Simpson's Rule Method

n	x_n	$f(x_n)$	$m\cdot f(x_n)$
0	0.0000000000	0.0000000000	0.0000000000
1	0.0981747704	0.5852709700	2.3410838800
2	0.1963495408	1.1480503000	2.2961006000
3	0.2945243113	1.6667107000	6.6668428000
4	0.3926990817	2.1213203400	4.2426406800
5	0.4908738521	2.4944088400	9.9776353600
6	0.5890486225	2.7716386000	5.5432772000
7	0.6872233930	2.9423558400	11.7694233600
8	0.7853981634	3.0000000000	3.0000000000

Simpson's Rule approximation ≈ 1.50001244

	Exact	***Trapezoid***	***Simpson***
Value:	1.500000	1.49517776	1.50001244
Error:		0.3215 %	0.0008 %

Exercise

Find the Trapezoid & Simpson's Rule approximations and error: $\int_0^8 e^{-2x}\,dx \quad n=8 \ \textit{subintervals}$

Solution

Trapezoid Rule Method

n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.0000000000	0.1353352800	0.2706705600
2	2.0000000000	0.0183156400	0.0366312800
3	3.0000000000	0.0024787500	0.0049575000
4	4.0000000000	0.0003354600	0.0006709200
5	5.0000000000	0.0000454000	0.0000908000
6	6.0000000000	0.0000061400	0.0000122800
7	7.0000000000	0.0000008300	0.0000016600
8	8.0000000000	0.0000001100	0.0000001100

Trapezoid Rule approximation ≈ 0.65651755

Simpson's Rule Method

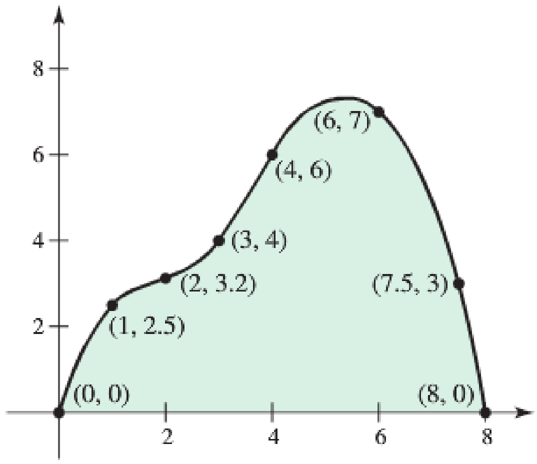
n	x_n	$f(x_n)$	$m \cdot f(x_n)$
0	0.0000000000	1.0000000000	1.0000000000
1	1.0000000000	0.1353352800	0.5413411200
2	2.0000000000	0.0183156400	0.0366312800
3	3.0000000000	0.0024787500	0.0099150000
4	4.0000000000	0.0003354600	0.0006709200
5	5.0000000000	0.0000454000	0.0001816000
6	6.0000000000	0.0000061400	0.0000122800
7	7.0000000000	0.0000008300	0.0000033200
8	8.0000000000	0.0000001100	0.0000001100

Simpson's Rule approximation ≈ 0.52958521

	Exact	***Trapezoid***	***Simpson***
Value:	0.49999994	0.65651755	0.52958521
Error:		31.3035 %	5.9171 %

Exercise

A piece of wood paneling must be cut in the shape shown below. The coordinates of several point on its curved surface are also shown (with units of inches).



a) Estimate the surface area of the paneling using the Trapezoid Rule

b) Estimate the surface area of the paneling using a left Riemann sum.

c) Could two identical pieces be cut from a 9-in by 9-in piece of wood?

Solution

a) The *trapezoid* Rule gives

$$\frac{(0+.25)\cdot 1}{2}+\frac{(2.5+3.2)\cdot 1}{2}+\frac{(3.2+4)\cdot 1}{2}+\frac{(4+6)\cdot 1}{2}+\frac{(6+7)\cdot 2}{2}+\frac{(7+5.3)\cdot 1.5}{2}+\frac{(3+0)\cdot 0.5}{2}\underline{=35.675|}$$

b) The left *Riemann* sum gives

$$0\cdot 1+2.5\cdot 1+3.2\cdot 1+4\cdot 1+6\cdot 2+7\cdot 1.5+5.3\cdot 0.5\underline{=34.85|}$$

c) Although the surface area of the piece appears to be less than half of $81=9^2$ (area of 9×9 piece of wood), the shape prohibits the creation of two identical pieces.

Solution   *Section* 2.6 – Improper Integrals

Exercise

Evaluate the integral $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$

Solution

$$\int_0^\infty \frac{dx}{x^2+1} = \lim_{b\to\infty}\int_0^b \frac{dx}{x^2+1}$$

$$= \lim_{b\to\infty}\left[\tan^{-1}x\right]_0^b$$

$$= \lim_{b\to\infty}\left(\tan^{-1}b - \tan^{-1}0\right)$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

Exercise

Evaluate the integral $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$

Solution

$$\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{b\to4^-}\int_0^b (4-x)^{-1/2}\,dx$$

$$= \lim_{b\to4^-}\int_0^b -(4-x)^{-1/2}\,d(4-x)$$

$$= -2\lim_{b\to4^-}\left[(4-x)^{1/2}\right]_0^b$$

$$= -2\lim_{b\to4^-}\left[(4-b)^{1/2} - (4)^{1/2}\right]$$

$$= -2(0-2)$$

$$= 4$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{2} \frac{2dx}{x^2+4}$

Solution

$$\int_{-\infty}^{2} \frac{2dx}{x^2+4} = 2\lim_{b\to-\infty}\int_{b}^{2} \frac{dx}{x^2+2^2}$$

$$= 2\lim_{b\to-\infty}\frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_b^2$$

$$= \lim_{b\to-\infty}\left[\tan^{-1}1-\tan^{-1}\frac{b}{2}\right]$$

$$= \frac{\pi}{4}-\left(-\frac{\pi}{2}\right)$$

$$= \frac{3\pi}{4}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{xdx}{\left(x^2+4\right)^{3/2}}$

Solution

$$\int_{-\infty}^{\infty} \frac{xdx}{\left(x^2+4\right)^{3/2}} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{d\left(x^2+4\right)}{\left(x^2+4\right)^{3/2}}$$

$u = x^2+4 \quad \to du = 2xdx$

$$= \frac{1}{2}\left[-2\left(x^2+4\right)^{-1/2}\right]_{-\infty}^{\infty}$$

$$= -\left[\frac{1}{\sqrt{x^2+4}}\right]_{-\infty}^{\infty}$$

$$= -(0-0)$$

$$= 0$$

Exercise

Evaluate the integral $\displaystyle\int_{1}^{\infty} \frac{dx}{x\sqrt{x^2-1}}$

Solution

$$\int_{1}^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \int_{1}^{2} \frac{dx}{x\sqrt{x^2-1}} + \int_{2}^{\infty} \frac{dx}{x\sqrt{x^2-1}}$$

$$= \lim_{b\to 1^+} \int_b^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{c\to\infty} \int_2^c \frac{dx}{x\sqrt{x^2-1}}$$

$$= \lim_{b\to 1^+} \left[\sec^{-1}|x|\right]_b^2 + \lim_{c\to\infty} \left[\sec^{-1}|x|\right]_2^c$$

$$= \lim_{b\to 1^+} \left(\sec^{-1}2 - \sec^{-1}b\right) + \lim_{c\to\infty} \left(\sec^{-1}c - \sec^{-1}2\right)$$

$$= \left(\frac{\pi}{3} - 0\right) + \left(\frac{\pi}{2} - \frac{\pi}{3}\right)$$

$$\underline{= \frac{\pi}{2}}$$

Exercise

Evaluate the integral $\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx$

Solution

$$\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx = \int_{-\infty}^{0} 2xe^{-x^2}\,dx + \int_{0}^{\infty} 2xe^{-x^2}\,dx \qquad d\left(-x^2\right) = -2xdx$$

$$= -\lim_{b\to-\infty} \int_b^0 e^{-x^2}\,d\left(-x^2\right) - \lim_{c\to\infty} \int_0^c e^{-x^2}\,d\left(-x^2\right)$$

$$= -\lim_{b\to-\infty} \left[e^{-x^2}\right]_b^0 - \lim_{c\to\infty} \left[e^{-x^2}\right]_0^c$$

$$= -\lim_{b\to-\infty} \left(1 - e^{-b^2}\right) - \lim_{c\to\infty} \left(e^{-c^2} - 1\right) \qquad = -(1-0) - (0-1)$$

$$\underline{= 0}$$

Exercise

Evaluate the integral $\int_0^1 (-\ln x)\,dx$

Solution

$$\int_0^1 (-\ln x)\,dx = -\lim_{b\to 0^+} \int_b^1 (\ln x)\,dx$$

$$= -\lim_{b\to 0^+} \left[x\ln x - x\right]_b^1$$

$$= -\lim_{b\to 0^+} \left(\ln 1 - 1 - (b\ln b - b)\right)$$

$$= -(0 - 1 - 0 + 0)$$

$$\underline{= 1}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{4} \frac{dx}{\sqrt{|x|}}$

Solution

$$\int_{-1}^{4} \frac{dx}{\sqrt{|x|}} = \lim_{b \to 0^-} \int_{-1}^{b} \frac{dx}{\sqrt{-x}} + \lim_{c \to 0^+} \int_{c}^{4} \frac{dx}{\sqrt{x}}$$

$$= \lim_{b \to 0^-} \left[-2\sqrt{-x}\right]_{-1}^{b} + \lim_{c \to 0^+} \left[2\sqrt{x}\right]_{c}^{4}$$

$$= \lim_{b \to 0^-} \left(-2\sqrt{-b} + 2\right) + \lim_{c \to 0^+} \left(2\sqrt{4} - 2\sqrt{c}\right)$$

$$= 2 + 4$$

$$\underline{= 6}$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{\infty} e^{-3x}\, dx$

Solution

$$\int_{0}^{\infty} e^{-3x}\, dx = -\frac{1}{3} e^{-3x} \Big|_{0}^{\infty}$$

$$= -\frac{1}{3}\left(e^{-\infty} - 1\right)$$

$$\underline{= \frac{1}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

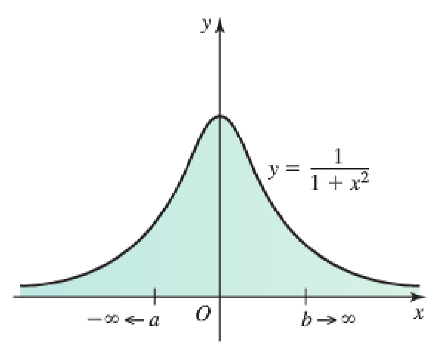
Solution

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \tan^{-1} x \Big|_{-\infty}^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1}(-\infty)$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$\underline{= \pi}$$



Area of region under the curve $y = \frac{1}{1+x^2}$ on $(-\infty, \infty)$ has finite value π.

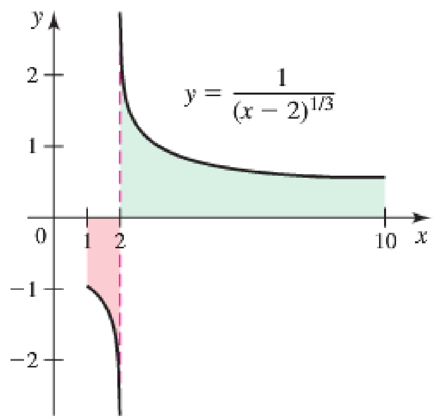
Exercise

Evaluate the integral $\displaystyle\int_1^{10} \frac{dx}{(x-2)^{1/3}}$

Solution

$$\begin{aligned}\int_1^{10} (x-2)^{-1/3}\, dx &= \frac{3}{2}(x-2)^{2/3}\Big|_1^{10} \\ &= \frac{3}{2}\left(8^{2/3} - (-1)^{2/3}\right) \\ &= \frac{3}{2}(4-1) \\ &= \frac{9}{2}\end{aligned}$$

$$\begin{aligned}\int_1^{10} (x-2)^{-1/3}\, dx &= \int_1^{2} (x-2)^{-1/3}\, dx + \int_2^{10} (x-2)^{-1/3}\, dx \\ &= \frac{3}{2}(x-2)^{2/3}\Big|_1^{2} + (x-2)^{2/3}\Big|_2^{10} \\ &= \frac{3}{2}\left(0 - (-1)^{2/3}\right) + \frac{3}{2}\left(8^{2/3} - 0\right) \\ &= \frac{3}{2}(-1+4) \\ &= \frac{9}{2}\end{aligned}$$



Exercise

Evaluate the integral $\displaystyle\int_1^{\infty} \frac{dx}{x^2}$

Solution

$$\begin{aligned}\int_1^{\infty} \frac{dx}{x^2} &= -\frac{1}{x}\Big|_1^{\infty} \\ &= -\left(\frac{1}{\infty} - 1\right) \\ &= -(0-1) \\ &= 1\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\infty} \frac{dx}{(x+1)^3}$

Solution

$$\int_0^{\infty} (x+1)^{-3} dx = -\frac{2}{(x+1)^2}\Big|_0^{\infty}$$

$$= -2\left(\frac{1}{\infty} - 1\right)$$

$$= -2(0-1)$$

$$\boxed{= 2}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{0} e^x \, dx$

Solution

$$\int_{-\infty}^{0} e^x \, dx = e^x \Big|_{-\infty}^{0}$$

$$= \left(1 - e^{-\infty}\right)$$

$$\boxed{= 1}$$

Exercise

Evaluate the integral $\displaystyle\int_1^{\infty} 2^{-x} \, dx$

Solution

$$\int_1^{\infty} 2^{-x} \, dx = -\int_1^{\infty} 2^{-x} \, d(-x)$$

$$\int a^x dx = \frac{a^x}{\ln a}$$

$$= -\frac{2^{-x}}{\ln 2}\Big|_1^{\infty}$$

$$= -\frac{1}{\ln 2}\left(0 - \frac{1}{2}\right)$$

$$\boxed{= \frac{1}{2\ln 2}}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\infty}^{0} \frac{dx}{\sqrt[3]{2-x}}$

Solution

$$\int_{-\infty}^{0} \frac{dx}{\sqrt[3]{2-x}} = -\int_{-\infty}^{0} (2-x)^{-1/3}\, d(2-x)$$

$$= -\frac{3}{2}(2-x)^{2/3}\Big|_{-\infty}^{0}$$

$$= -\frac{3}{2}\left(2^{2/3} - \infty\right)$$

$$\underline{=\infty|} \quad \textit{diverges}$$

Exercise

Evaluate the integral $\displaystyle\int_{4/\pi}^{\infty} \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)dx$

Solution

$$\int_{4/\pi}^{\infty} \frac{1}{x^2}\sec^2\left(\frac{1}{x}\right)dx = -\int_{4/\pi}^{\infty} \sec^2\left(\frac{1}{x}\right)d\left(\frac{1}{x}\right) \qquad d\left(\frac{1}{x}\right) = -\frac{1}{x^2}dx$$

$$= -\tan\left(\frac{1}{x}\right)\Big|_{4/\pi}^{\infty}$$

$$= -\left(\tan 0 - \tan\frac{\pi}{4}\right)$$

$$\underline{=1|}$$

Exercise

Evaluate the integral $\displaystyle\int_{e^2}^{\infty} \frac{dx}{x\ln^p x} \quad p>1$

Solution

$$\int_{e^2}^{\infty} \frac{dx}{x\ln^p x} = \int_{e^2}^{\infty} (\ln x)^{-p}\, d(\ln x)$$

$$= \frac{1}{1-p}(\ln x)^{1-p}\Big|_{e^2}^{\infty}$$

$$= \frac{1}{1-p}\left((\ln x)^{-\infty} - \left(\ln e^2\right)^{1-p}\right)$$

$$= \frac{-1}{1-p} 2^{1-p}$$

$$\boxed{= \frac{1}{(p-1)2^{p-1}}}$$

Exercise

Evaluate the integral $\int_0^\infty \frac{p}{\sqrt[5]{p^2+1}}\, dp$

Solution

$$\int_0^\infty \frac{p}{\sqrt[5]{p^2+1}}\, dp = \frac{1}{2}\int_0^\infty \left(p^2+1\right)^{-1/5}\, d\left(p^2+1\right) \qquad d\left(p^2+1\right) = 2p\,dp$$

$$= \frac{5}{8}\left(p^2+1\right)^{4/5}\Big|_0^\infty$$

$$\boxed{= \infty} \quad \textit{diverges}$$

Exercise

Evaluate the integral $\int_{-1}^{1} \ln y^2\, dy$

Solution

$$\int_{-1}^{1} \ln y^2\, dy = 2\int_0^1 \ln y^2\, dy$$

$$= 4\left(y\ln y - y\right)\Big|_0^1$$

$$= 4\left[-1-0\right]$$

$$\boxed{= -4}$$

$$\int \ln x^2 dx = 2\int \ln x dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx \qquad v = \int dx = x$$

$$= 2\left[x\ln x - \int dx\right]$$

$$\boxed{= 2(x\ln x - x) + C}$$

Exercise

Evaluate the integral $\int_{-2}^{6} \frac{dx}{\sqrt{|x-2|}}$

Solution

$$\int_{-2}^{6} \frac{dx}{\sqrt{|x-2|}} = \int_{-2}^{2} \frac{dx}{\sqrt{2-x}} + \int_{2}^{6} \frac{dx}{\sqrt{x-2}}$$

$$= -\int_{-2}^{2} (2-x)^{-1/2}\, d(2-x) + \int_2^6 (x-2)^{-1/2}\, d(x-2)$$

$$= -2\sqrt{2-x}\,\Big|_{-2}^{2} + 2\sqrt{x-2}\,\Big|_{2}^{6}$$

$$= -2(0-2) + 2(2-0)$$

$$\underline{= 8}$$

Exercise

Evaluate $\displaystyle\int_0^{\infty} xe^{-x}dx$

Solution

$$\int_0^{\infty} xe^{-x}dx = -xe^{-x} - e^{-x}\Big|_0^{\infty}$$

$$= 0 - (-1)$$

$$\underline{= 1}$$

		$\int e^{-x}$
+	x	$-e^{-x}$
−	1	e^{-x}

Exercise

Evaluate $\displaystyle\int_0^{1} x\ln x\, dx$

Solution

$u = \ln x \qquad dv = x\, dx$

$du = \dfrac{dx}{x} \qquad v = \dfrac{1}{2}x^2$

$$\int x\ln x\, dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int xdx \;\; \underline{= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2}$$

$$\int_0^1 x\ln x\, dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2\Big|_0^1$$

$$\underline{= -\frac{1}{4}}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}dx$

Solution

$u = \ln x \qquad dv = \dfrac{1}{x^2}\, dx$

$du = \dfrac{dx}{x} \qquad v = -\dfrac{1}{x}$

$$\int \frac{\ln x}{x^2}\, dx = -\frac{1}{x}\ln x + \int \frac{1}{x^2}dx$$

$$= -\frac{1}{x}\ln x - \frac{1}{x}$$

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = -\frac{1}{x}(\ln x + 1)\Big|_1^{\infty}$$

$$\boxed{= 1}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} (1-x)e^{-x}\,dx$

Solution

$$\int_1^{\infty} (1-x)e^{-x}\,dx = \left[-e^{-x} - (-x-1)e^{-x}\right]_1^{\infty}$$

$$= \left[xe^{-x}\right]_1^{\infty}$$

$$= 0 - e^{1}$$

$$\boxed{= \frac{1}{e}}$$

Exercise

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$

Solution

$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \int_{-\infty}^{\infty} \frac{du}{1+u^2} \qquad u = e^x \rightarrow du = e^x dx$$

$$= \arctan e^x \Big|_{-\infty}^{\infty} \qquad = \arctan\infty - \arctan 0$$

$$\boxed{= \frac{\pi}{2}}$$

Exercise

Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$

Solution

$$\int_0^1 x^{-1/3}dx = \frac{3}{2}x^{2/3}\Big|_0^1 \quad \boxed{= \frac{3}{2}}$$

Exercise

Evaluate $\int_1^\infty \frac{4}{\sqrt[4]{x}}\, dx$

Solution

$$\int_1^\infty 4x^{-1/4} dx = \frac{16}{3}x^{3/4}\Big|_1^\infty$$

$$= \infty \qquad \textbf{\textit{Diverges}}$$

Exercise

Evaluate $\int_0^2 \frac{dx}{x^3}$

Solution

$$\int_0^2 \frac{dx}{x^3} = -\frac{1}{2x^2}\Big|_0^2$$

$$= -\frac{1}{8} + \infty$$

$$= \infty \qquad \textbf{\textit{Diverges}}$$

Exercise

Evaluate $\int_1^\infty \frac{dx}{x^3}$

Solution

$$\int_1^\infty \frac{dx}{x^3} = -\frac{1}{2x^2}\Big|_1^\infty \quad = \frac{1}{2}$$

Exercise

Evaluate $\int_1^\infty \frac{6}{x^4}\, dx$

Solution

$$\int_1^\infty 6x^{-4}\, dx = -2\frac{1}{x^3}\Big|_1^\infty \quad = 2$$

Exercise

Evaluate $\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$

Solution

$$u = \sqrt{x} \ \to \ u^2 = x \quad \Rightarrow dx = 2udu$$

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = \int_0^\infty \frac{2u}{u\left(u^2+1\right)}du$$

$$= 2\int_0^\infty \frac{1}{u^2+1}du$$

$$= 2\arctan\sqrt{x}\Big|_0^\infty$$

$$= 2\left(\frac{\pi}{2} - 0\right)$$

$$= \pi$$

Exercise

Evaluate $\int_{-\infty}^0 xe^{-4x}dx$

Solution

$$\int_{-\infty}^0 xe^{-4x}dx = \left(-\frac{x}{4} - \frac{1}{16}\right)e^{-4x}\Big|_{-\infty}^0$$

$$= -\frac{1}{16} - \infty$$

$$= -\infty$$ ***Diverges***

Exercise

Evaluate $\int_0^\infty xe^{-x/3}dx$

Solution

$$\int_0^\infty xe^{-x/3}dx = (-3x-9)e^{-x/3}\Big|_0^\infty$$

$$= 9$$

Exercise

Evaluate $\int_0^{\infty} x^2 e^{-x} dx$

Solution

$$\int_0^{\infty} x^2 e^{-x} dx = \left(-x^2 - 2x - 2\right)e^{-x}\Big|_0^{\infty} \underline{= 2}$$

Exercise

Evaluate $\int_0^{\infty} e^{-x}\cos x\, dx$

Solution

$$\int e^{-x}\cos x\, dx = e^{-x}(\sin x - \cos x) - \int e^{-x}\cos x\, dx$$

$$2\int e^{-x}\cos x\, dx = e^{-x}(\sin x - \cos x)$$

$$\int_0^{\infty} e^{-x}\cos x\, dx = \frac{1}{2}e^{-x}(\sin x - \cos x)\Big|_0^{\infty}$$

$$= \frac{1}{2}(0 - (-1))$$

$$\underline{= \frac{1}{2}}$$

		$\int \cos x$
+	e^{-x}	$\sin x$
−	$-e^{-x}$	$-\cos x$
+	e^{-x}	$-\int \cos x$

Exercise

Evaluate $\int_4^{\infty} \frac{1}{x(\ln x)^3} dx$

Solution

$$\int_4^{\infty} \frac{1}{x(\ln x)^3} dx = \int_4^{\infty} (\ln x)^{-3}\, d(\ln x)$$

$$= -\frac{1}{2}\frac{1}{(\ln x)^2}\Big|_4^{\infty}$$

$$= \frac{1}{2}\left(0 - \frac{1}{(\ln 4)^2}\right)$$

$$\underline{= \frac{1}{2(\ln 4)^2}}$$

Exercise

Evaluate $\displaystyle\int_1^{\infty} \frac{\ln x}{x}\,dx$

Solution

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \int_1^{\infty} \ln x \; d(\ln x)$$

$$= \frac{1}{2}(\ln x)^2 \Big|_1^{\infty}$$

$$= \infty \qquad \textit{Diverges}$$

Exercise

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{4}{16+x^2}\,dx$

Solution

$$\int_{-\infty}^{\infty} \frac{4}{16+x^2}\,dx = \arctan\left(\frac{x}{4}\right)\Big|_{-\infty}^{\infty}$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$

$$= \pi$$

Exercise

Find the area of the region R between the graph of $f(x) = \dfrac{1}{\sqrt{9-x^2}}$ and the x-*axis* on the interval $(-3,\ 3)$ (if it exists)

Solution

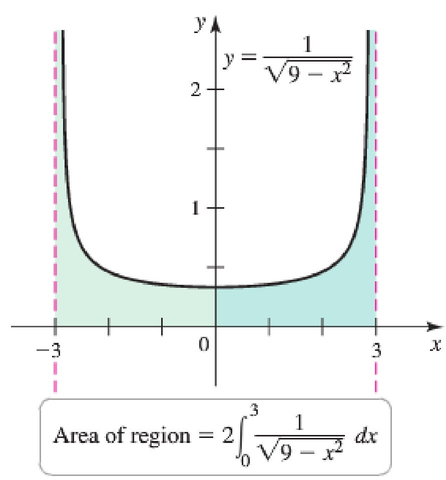
$$A = \int_{-3}^{3} \frac{dx}{\sqrt{9-x^2}}$$

$$= 2\int_{0}^{3} \frac{dx}{\sqrt{9-x^2}}$$

$$= 2\sin^{-1}\frac{x}{3}\Big|_0^3$$

$$= 2\left(\sin^{-1}1 - \sin^{-1}0\right)$$

$$= \pi \ \ unit^2$$



Exercise

Find the volume of the region bounded by $f(x) = \left(x^2+1\right)^{-1/2}$ and the *x-axis* on the interval $[2, \infty)$ is revolved about the *x-axis*.

Solution

$$V = \pi \int_2^{\infty} \frac{1}{x^2+1}\, dx \qquad\qquad V = \pi \int_a^b \left(f(x)\right)^2 dx$$

$$= \pi \tan^{-1} x \Big|_2^{\infty}$$

$$= \pi \left(\tan^{-1}\infty - \tan^{-1} 2\right)$$

$$= \pi \left(\frac{\pi}{2} - \tan^{-1} 2\right) \quad unit^3$$

Exercise

Find the volume of the region bounded by $f(x) = \sqrt{\frac{x+1}{x^3}}$ and the *x-axis* on the interval $[1, \infty)$ is revolved about the *x-axis*.

Solution

$$V = \pi \int_1^{\infty} \frac{x+1}{x^3}\, dx \qquad\qquad V = \pi \int_a^b \left(f(x)\right)^2 dx$$

$$= \pi \int_1^{\infty} \left(\frac{1}{x^2} + x^{-3}\right) dx$$

$$= \pi \left(-\frac{1}{x} - \frac{1}{2}\frac{1}{x^2}\right)\Big|_1^{\infty}$$

$$= \pi \left(1 + \frac{1}{2}\right)$$

$$= \frac{3\pi}{2} \quad unit^3$$

Exercise

Find the volume of the region bounded by $f(x) = (x+1)^{-3}$ and the *x-axis* on the interval $[0, \infty)$ is revolved about the *y-axis*.

Solution

$$V = 2\pi \int_0^{\infty} x \frac{1}{(x+1)^3}\, dx \qquad\qquad V = 2\pi \int_a^b x \cdot f(x)\, dx \quad (\text{Shell method})$$

$$\frac{x}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$x = Ax^2 + 2Ax + A + Bx + B + C$$

$$= 2\pi \int_0^{\infty} \left(\frac{1}{(x+1)^2} - \frac{1}{(x+1)^3}\right) d(x+1)$$

$$= 2\pi\left(\frac{-1}{x+1} + \frac{1}{2}\frac{1}{(x+1)^2}\right)\Bigg|_0^\infty$$

$$= 2\pi\left(1 - \frac{1}{2}\right)$$

$$\boxed{= \pi \ \ unit^3}$$

$$\begin{cases} \boxed{A = 0} \\ 2A + B = 1 \ \to \boxed{B = 1} \quad \boxed{C = -1} \\ B + C = 0 \end{cases}$$

Exercise

Find the volume of the region bounded by $f(x) = \dfrac{1}{\sqrt{x}\ln x}$ and the *x-axis* on the interval $[2, \infty)$ is revolved about the *x-axis*.

Solution

$$V = \pi\int_2^\infty \frac{1}{x\ln^2 x}\,dx \qquad\qquad V = \pi\int_a^b (f(x))^2\,dx$$

$$= \pi\int_2^\infty \frac{1}{\ln^2 x}\,d(\ln x)$$

$$= \pi\left(-\frac{1}{\ln x}\right)\Bigg|_2^\infty$$

$$= \pi\left(-0 + \frac{1}{\ln 2}\right)$$

$$\boxed{= \frac{\pi}{\ln 2} \ \ unit^3}$$

Exercise

Find the volume of the region bounded by $f(x) = \dfrac{\sqrt{x}}{\sqrt[3]{x^2+1}}$ and the *x-axis* on the interval $[0, \infty)$ is revolved about the *x-axis*.

Solution

$$V = \pi\int_0^\infty \frac{x}{\left(x^2+1\right)^{2/3}}\,dx \qquad\qquad V = \pi\int_a^b (f(x))^2\,dx$$

$$= \frac{\pi}{2}\int_0^\infty \left(x^2+1\right)^{-2/3}\,d\left(x^2+1\right)$$

$$= \frac{3\pi}{2}\left(x^2+1\right)^{1/3}\Bigg|_0^\infty$$

$$= \frac{3\pi}{2}(\infty - 1)$$

$\boxed{= \infty \quad diverges}$ So the volume doesn't exist

Exercise

Find the volume of the region bounded by $f(x) = \left(x^2 - 1\right)^{-1/4}$ and the *x-axis* on the interval $(1,\ 2]$ is revolved about the *y-axis*.

Solution

$$V = 2\pi \int_1^2 x\left(x^2 - 1\right)^{-1/4} dx \qquad V = 2\pi \int_a^b x \cdot f(x)\, dx \quad (\textit{Shell method})$$

$$= \pi \int_1^2 \left(x^2 - 1\right)^{-1/4} d\left(x^2 - 1\right)$$

$$= \frac{4\pi}{3}\left(x^2 - 1\right)^{3/4} \Big|_1^2$$

$$= \frac{4\pi}{3}(3)^{3/4}$$

$$\boxed{= \frac{4\pi}{3^{1/4}}\ \ unit^3}$$

Exercise

Find the volume of the region bounded by $f(x) = \tan x$ and the *x-axis* on the interval $\left[0,\ \frac{\pi}{2}\right)$ is revolved about the *x-axis*.

Solution

$$V = \pi \int_0^{\pi/2} \tan^2 x\, dx \qquad V = \pi \int_a^b (f(x))^2\, dx$$

$$= \pi \int_0^{\pi/2} \left(\sec^2 x - 1\right) dx$$

$$= \pi(\tan x - x) \Big|_0^{\pi/2} \qquad \left(\tan \frac{\pi}{2} = \infty\right)$$

$\boxed{= \infty \quad diverges}$ So the volume doesn't exist

Exercise

Find the volume of the region bounded by $f(x) = -\ln x$ and the *x-axis* on the interval $(0,\ 1]$ is revolved about the *x-axis*.

Solution

$$V = \pi \int_0^1 \ln^2 x\, dx \qquad V = \pi \int_a^b (f(x))^2\, dx$$

$$u = \ln x \quad dv = \ln x\, dx \qquad u = \ln x \quad dv = dx$$

$$du = \frac{dx}{x} \quad v = x\ln x - x \qquad du = \frac{dx}{x} \quad v = x \quad \to \int \ln x dx = x\ln x - \int dx = x\ln x - x$$

$$\int \ln^2 x\, dx = \ln x\left(x\ln x - x\right) - \int (\ln x - 1)\, dx$$

$$= x\ln^2 x - x\ln x - \left(x\ln x - x - x\right)$$

$$\underline{= x\ln^2 x - 2x\ln x + 2x}$$

$$V = \pi\left(x\ln^2 x - 2x\ln x + 2x\right)\Big|_0^1$$

$$\underline{= 2\pi \ \ unit^3}$$

Exercise

Let R be the region bounded by the graph of $f(x) = x^{-p}$ and the x-*axis*

a) Let S be the solid generated when R is revolved about the x-*axis*. For what values of p is the volume of S finite for $0 < x \le 1$?

b) Let S be the solid generated when R is revolved about the y-*axis*. For what values of p is the volume of S finite for $0 < x \le 1$?

c) Let S be the solid generated when R is revolved about the x-*axis*. For what values of p is the volume of S finite for $x \ge 1$?

d) Let S be the solid generated when R is revolved about the y-*axis*. For what values of p is the volume of S finite for $x \ge 1$?

Solution

a) $V = \pi\int_0^1 \left(x^{-p}\right)^2 dx \qquad V = \pi\int_a^b f(x)^2 dx$

$$= \pi\int_0^1 x^{-2p} dx$$

$$= \pi \frac{x^{-2p+1}}{1-2p}\Big|_0^1$$

$$= \frac{\pi}{1-2p}\left(1 - 0^{-2p+1}\right)$$

The volume of S finite when $1 - 2p > 0 \ \Rightarrow \ \underline{p < \frac{1}{2}}$

b) $V = 2\pi\int_0^1 x\cdot x^{-p} dx \qquad V = 2\pi\int_a^b xf(x)\, dx$

$$= 2\pi\int_0^1 x^{1-p} dx$$

$$= \frac{2\pi}{2-p} x^{2-p} \Big|_0^1$$

$$= \frac{2\pi}{2-p}\left(1 - 0^{2-p}\right)$$

The volume of S finite when $2 - p > 0 \Rightarrow \boxed{p < 2}$

c) $V = \pi \int_1^\infty \left(x^{-p}\right)^2 dx$ $\qquad V = \pi \int_a^b f(x)^2\, dx$

$$= \pi \int_1^\infty x^{-2p} dx$$

$$= \pi \frac{x^{-2p+1}}{1-2p} \Big|_1^\infty$$

$$= \frac{\pi}{1-2p}\left(\infty^{1-2p} - 1\right)$$

The volume of S finite when $1 - 2p < 0 \Rightarrow \boxed{p > \frac{1}{2}}$ $\left(\frac{1}{\infty} = 0\right)$

d) $V = 2\pi \int_0^1 x \cdot x^{-p} dx$ $\qquad V = 2\pi \int_a^b x f(x)\, dx$

$$= 2\pi \int_0^1 x^{1-p} dx$$

$$= \frac{2\pi}{2-p} x^{2-p} \Big|_0^1$$

$$= \frac{2\pi}{2-p}\left(1 - 0^{2-p}\right)$$

The volume of S finite when $2 - p > 0 \Rightarrow \boxed{p < 2}$

Exercise

The magnetic potential P at a point on the axis of a circular coil is given by

$$P = \frac{2\pi NIr}{k} \int_c^\infty \frac{1}{\left(r^2 + x^2\right)^{3/2}} dx$$

Where N, I, r, k, and c are constants. Find P.

Solution

$$P = \frac{2\pi NIr}{k} \int_c^\infty \frac{1}{\left(r^2 + x^2\right)^{3/2}} dx$$

$x = r \tan\theta \qquad x^2 + r^2 = \left(r \sec\theta\right)^2$

$dx = r \sec^2\theta\, d\theta$

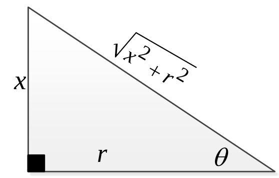
$$= \frac{2\pi NIr}{k}\int_c^{\infty} \frac{1}{r^3 \sec^3\theta} r\sec^2\theta \, d\theta$$

$$= \frac{2\pi NI}{kr}\int_c^{\infty} \cos\theta \, d\theta$$

$$= \frac{2\pi NI}{kr} \sin\theta \Big|_c^{\infty}$$

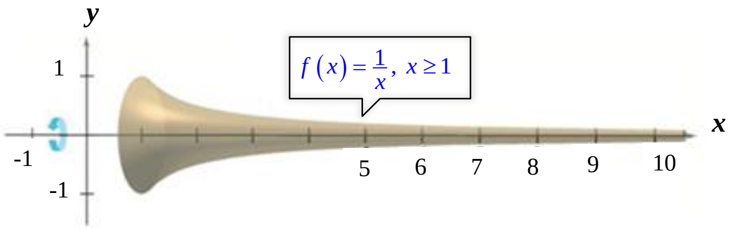
$$= \frac{2\pi NI}{kr} \frac{x}{\sqrt{x^2+r^2}} \Big|_c^{\infty}$$

$$\boxed{= \frac{2\pi NI}{kr}\left(1 - \frac{c}{\sqrt{c^2+r^2}}\right)}$$



Exercise

The solid formed by revolving (about the *x-axis*) the unbounded region lying between the graph of $f(x) = \frac{1}{x}$ and the *x-axis* $(x \ge 1)$ is called ***Gabriel's Horn***.



Show that this solid has a finite volume and an infinite surface area

Solution

$$V = \pi\int_1^{\infty} \frac{1}{x^2} dx \qquad\qquad V = \pi\int_x^b (f(x))^2 \, dx \ \ (\textit{disk method})$$

$$= -\pi \frac{1}{x}\Big|_1^{\infty}$$

$$= -\pi(0-1)$$

$$\boxed{= \pi \ \ unit^3}$$

$$f'(x) = -\frac{1}{x^2}$$

$$S = 2\pi\int_1^{\infty} \frac{1}{x}\sqrt{1+\frac{1}{x^4}} \, dx \qquad\qquad S = 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2} \, dx$$

Since $\quad 1+\frac{1}{x^4} > 1$ and $\int_1^{\infty} \frac{1}{x} dx$ diverges

Therefore the surface area in infinite.

Exercise

Water is drained from a 3000-*gal* tank at a rate that starts at 100 *gal/hr*. and decreases continuously by 5% /*hr*. If the drain left open indefinitely, how much water drains from the tank? Can a full tank be emptied at this rate?

Solution

Rate of the drain water: $r(t) = 100(1-.05)^t$

$$= 100(0.95)^t$$

$$\boxed{= 100e^{(\ln 0.95)t}}$$

Total water amount drained:

$$D = \int_0^\infty 100e^{(\ln 0.95)t}dt$$

$$= \frac{100}{\ln 0.95}e^{(\ln 0.95)t}\Big|_0^\infty$$

$$= \frac{100}{\ln 0.95}(0-1) \qquad \ln 0.95 < 0 \xrightarrow[t\to\infty]{} e^{(\ln 0.95)t} = e^{-\infty} = 0$$

$$\boxed{= -\frac{100}{\ln 0.95} \approx 1950 \ \ gal}$$

Since 1950 *gal* $<$ 3000 *gal* which it takes infinite time.

Therefore, the full 3,000–gallon tank cannot be emptied at this rate.