Section 2.4 – Partial Fractions

This section shows how to express a rational; function as a sum of simpler functions, called ***partial fractions***.

Example

Evaluate $\displaystyle\int \frac{5x-3}{x^2-2x-3}\,dx$

Solution

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$5x-3 = (A+B)x - 3A + B$$

$$x \qquad A+B=5$$

$$x^0 \qquad -3A+B=-3$$

$$\begin{array}{rr} & A+B=5 \\ - & 3A-B=3 \\ \hline & 4A=8 \end{array}$$

$$\underline{A=2,\quad B=3}$$

$$\int \frac{5x-3}{x^2-2x-3}\,dx = \int \left(\frac{2}{x+1} + \frac{3}{x-3}\right)dx$$

$$= 2\ln|x+1| + 3\ln|x-3| + C$$

Example

Use partial fractions to evaluate $\displaystyle\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$

Solution

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

$$x^2+4x+1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$$

$$= Ax^2 + 4Ax + 3A + Bx^2 + 2Bx - 3B + Cx^2 - C$$

$$x^2 \qquad A+B+C=1$$

$$x \qquad 4A+2B=4$$

$$x^0 \qquad 3A-3B-C=1$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 3 & -3 & -1 \end{vmatrix} = -16 \qquad \Delta_A = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 1 & -3 & -1 \end{vmatrix} = -12$$

$$\Delta_B = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & 0 \\ 3 & 1 & -1 \end{vmatrix} = -8 \qquad \Delta_C = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 4 \\ 3 & -3 & 1 \end{vmatrix} = 4$$

$$A = \frac{-12}{-16} \underline{= \frac{3}{4}} \quad B = \frac{-8}{-16} \underline{= \frac{1}{2}} \quad C = \frac{4}{-16} \underline{= -\frac{1}{4}}$$

$$\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\, dx = \int \left(\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3} \right) dx$$

$$\underline{= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K}$$

Method of Partial Fractions $\left(f(x)/g(x)\ \textbf{Proper}\right)$

1. Let $(x-r)$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $(x-r)$ that divides $g(x)$. Then,

$$\frac{A_1}{(x-r)}+\frac{A_2}{(x-r)^2}+\cdots+\frac{A_m}{(x-r)^m}$$

2. Let x^2+px+q be an irreducible quadratic function of $g(x)$ has no real roots. Suppose that $\left(x^2+px+q\right)^n$ is the highest power. Then

$$\frac{B_1x+C_1}{\left(x^2+px+q\right)}+\frac{B_2x+C_2}{\left(x^2+px+q\right)^2}+\cdots+\frac{B_nx+C_n}{\left(x^2+px+q\right)^n}$$

3. Set the original fraction $\frac{f(x)}{g(x)}$ equal to the sum of these partial fractions.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Example

Use partial fractions to evaluate $\int \frac{6x+7}{(x+2)^2}dx$

Solution

$$\frac{6x+7}{(x+2)^2}=\frac{A}{x+2}+\frac{B}{(x+2)^2}$$

$$\begin{aligned}6x+7&=A(x+2)+B\\&=Ax+2A+B\end{aligned}$$

$$\to\begin{cases}\boxed{A=6}\\2A+B=7\quad\to\boxed{B=7-12=-5}\end{cases}$$

$$\begin{aligned}\int\frac{6x+7}{(x+2)^2}dx&=\int\left(\frac{6}{x+2}-\frac{5}{(x+2)^2}\right)dx\qquad d(x+2)=dx\\&=\int\frac{6}{x+2}dx-5\int(x+2)^{-2}d(x+2)\\&=6\ln|x+2|+5(x+2)^{-1}+C\\&=\boxed{6\ln|x+2|+\frac{5}{x+2}+C}\end{aligned}$$

Example

Use partial fractions to evaluate $\displaystyle\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}dx$

Solution

$$\begin{array}{r|l} & \quad 2x \\ \hline x^2-2x-3 & 2x^3-4x^2-x-3 \\ & \underline{2x^3-4x^2-6x} \\ & \qquad\qquad 5x-3 \end{array}$$

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3} = 2x + \frac{5x-3}{x^2-2x-3}$$

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$5x-3 = (A+B)x - 3A + B$$

$$\begin{array}{ll} x & A+B=5 \\ x^0 & -3A+B=-3 \end{array}$$

$$\begin{array}{rr} & A+B=5 \\ - & 3A-B=3 \\ \hline & 4A=8 \end{array}$$

$$\boxed{A=2, \quad B=3}$$

$$\begin{aligned} \int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx &= \int 2x\,dx + \int \frac{5x-3}{x^2-2x-3}\,dx \\ &= \int 2x\,dx + \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx \\ &= \boxed{x^2 + 2\ln|x+1| + 3\ln|x-3| + C} \end{aligned}$$

Example

Use partial fractions to evaluate $\displaystyle\int \frac{-2x+4}{\left(x^2+1\right)(x-1)^2}\,dx$

Solution

$$\frac{-2x+4}{\left(x^2+1\right)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$-2x+4 = (Ax+B)(x-1)^2 + C(x-1)\left(x^2+1\right) + D\left(x^2+1\right)$$

$$= (Ax+B)\left(x^2-2x+1\right)+C\left(x^3-x^2+x-1\right)+Dx^2+D$$

$$= (A+C)x^3+(-2A+B-C+D)x^2+(A-2B+C)x+B-C+D$$

$$\begin{array}{lll} x^3 & A+C=0 & \to C=-A \\ x^2 & -2A+B-C+D=0 & \\ x & A-2B+C=-2 & \to -2B=-2 \quad \underline{B=1} \\ x^0 & B-C+D=4 & \end{array}$$

$$\begin{array}{l} -A+D=-1 \\ \underline{\ \ A+D=3\ \ } \\ \quad 2D=2 \end{array} \to \underline{D=1 \quad A=2}$$

$$\to \underline{C=-2}$$

$$\int \frac{-2x+4}{\left(x^2+1\right)(x-1)^2}dx = \int\left(\frac{2x+1}{x^2+1}-\frac{2}{x-1}+\frac{1}{(x-1)^2}\right)dx$$

$$= \int\left(\frac{2x}{x^2+1}+\frac{1}{x^2+1}-\frac{2}{x-1}+\frac{1}{(x-1)^2}\right)dx$$

$$\underline{= \ln\left(x^2+1\right)+\tan^{-1}x-2\ln|x-1|-\frac{1}{x-1}+K}$$

Example

Use partial fractions to evaluate $\displaystyle\int \frac{dx}{x\left(x^2+1\right)^2}$

Solution

$$\frac{1}{x\left(x^2+1\right)^2} = \frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{\left(x^2+1\right)^2}$$

$$1 = A\left(x^2+1\right)^2+(Bx+C)x\left(x^2+1\right)+x(Dx+E)$$

$$1 = (A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A$$

$$\begin{cases} x^4 & A+B=0 & \to \underline{B=-1} \\ x^3 & \underline{C=0} & \\ x^2 & 2A+B+D=0 & \to \underline{D=-1} \\ x & C+E=0 & \to \underline{E=0} \\ x^0 & \underline{A=1} & \end{cases}$$

$$\int \frac{dx}{x\left(x^2+1\right)^2} = \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{\left(x^2+1\right)^2}$$

$$= \int \frac{dx}{x} - \frac{1}{2}\int \frac{1}{x^2+1}\,d\left(x^2+1\right) - \frac{1}{2}\int \frac{1}{\left(x^2+1\right)^2}\,d\left(x^2+1\right)$$

$$= \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{2}\frac{1}{x^2+1} + K$$

$$= \ln|x| - \ln\sqrt{x^2+1} + \frac{1}{2}\frac{1}{x^2+1} + K$$

$$\boxed{= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2\left(x^2+1\right)} + K}$$

Exercises Section 2.4 – Partial Fractions

Express the integrand as a sum of partial fractions and evaluate the integrals

1. $\displaystyle\int \frac{dx}{x^2+2x}$

2. $\displaystyle\int \frac{2x+1}{x^2-7x+12}dx$

3. $\displaystyle\int \frac{x+3}{2x^3-8x}dx$

4. $\displaystyle\int \frac{x^2}{(x-1)\left(x^2+2x+1\right)}dx$

5. $\displaystyle\int \frac{8x^2+8x+2}{\left(4x^2+1\right)^2}dx$

6. $\displaystyle\int \frac{x^2+x}{x^4-3x^2-4}dx$

7. $\displaystyle\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{\left(\theta^2+1\right)^3}d\theta$

8. $\displaystyle\int \frac{x^4}{x^2-1}dx$

9. $\displaystyle\int \frac{16x^3}{4x^2-4x+1}dx$

10. $\displaystyle\int \frac{e^{4x}+2e^{2x}-e^x}{e^{2x}+1}dx$

11. $\displaystyle\int \frac{\sin\theta\, d\theta}{\cos^2\theta+\cos\theta-2}$

12. $\displaystyle\int \frac{(x-2)^2\tan^{-1}(2x)-12x^3-3x}{\left(4x^2+1\right)(x-2)^2}dx$

13. $\displaystyle\int \frac{\sqrt{x+1}}{x}dx$

14. $\displaystyle\int \frac{x^3-2x^2+3x-4}{x^2+1}dx$

15. $\displaystyle\int \frac{4x^2+2x+4}{x+1}dx$

16. $\displaystyle\int \frac{3x^2+7x-2}{x^3-x^2-2x}dx$

17. $\displaystyle\int \frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}dx$

18. $\displaystyle\int \frac{5x^2-3x+2}{x^3-2x^2}dx$

19. $\displaystyle\int \frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}dx$

20. $\displaystyle\int \frac{1}{x^2-5x+6}dx$

21. $\displaystyle\int \frac{1}{x^2-5x+5}dx$

22. $\displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx$

23. $\displaystyle\int \frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}dx$

24. $\displaystyle\int \frac{8x^3+13x}{\left(x^2+2\right)^2}dx$

25. $\displaystyle\int \frac{\sin x}{\cos x+\cos^2 x}dx$

26. $\displaystyle\int \frac{5\cos x}{\sin^2 x+3\sin x-4}dx$

27. $\displaystyle\int \frac{e^x}{\left(e^x-1\right)\left(e^x+4\right)}dx$

28. $\displaystyle\int \frac{e^x}{\left(e^{2x}+1\right)\left(e^x-1\right)}dx$

29. $\int \frac{\sqrt{x}}{x-4}\, dx$

30. $\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\, dx$

31. $\int \frac{dx}{1+\sin x}$

32. $\int \frac{dx}{2+\cos x}$

33. $\int \frac{dx}{1-\cos x}$

34. $\int \frac{dx}{1+\sin x+\cos x}$

35. $\int \frac{1}{x^2-9}\, dx$

36. $\int \frac{5}{x^2+3x-4}\, dx$

37. $\int \frac{2}{9x^2-1}\, dx$

38. $\int \frac{3-x}{3x^2-2x-1}\, dx$

39. $\int \frac{x^2+12x+12}{x^3-4x}\, dx$

40. $\int \frac{x^3-x+3}{x^2+x-2}\, dx$

41. $\int \frac{5x-2}{(x-2)^2}\, dx$

42. $\int \frac{2x^3-4x^2-15x+4}{x^2-2x-8}\, dx$

43. $\int \frac{x+2}{x^2+5x}\, dx$

44. $\int \frac{\sec^2 x}{\tan^2 x+5\tan x+6}\, dx$

45. $\int \frac{\sec^2 x}{\tan x(\tan x+1)}\, dx$

46. $\int \frac{x\, dx}{x^2+4x+3}$

47. $\int \frac{x+1}{x^2(x-1)}\, dx$

48. $\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\, dx$

49. $\int \frac{8x+5}{2x^2+3x+1}\, dx$

50. $\int \frac{2x^2+7x+4}{x^3+2x^2+2x}\, dx$

51. $\int \frac{3x^3+4x^2+6x}{(x+1)^2(x^2+4)}\, dx$

52. $\int \frac{x^2-4}{x^2+4}\, dx$

53. $\int \frac{dx}{x^2-2x-15}$

54. $\int \frac{3x^2+x-3}{x^2-1}\, dx$

55. $\int \frac{2x^2-4x}{x^2-4}\, dx$

56. $\int \frac{dx}{x^3-2x^2}$

57. $\int \frac{dx}{x^2-x-2}$

58. $\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\, dx$

59. $\displaystyle\int \frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3}\,dx$

60. $\displaystyle\int \frac{x^2-x-21}{2x^3-x^2+8x-4}\,dx$

61. $\displaystyle\int \frac{5x^3-3x^2+7x-3}{\left(x^2+1\right)^2}\,dx$

62. $\displaystyle\int \frac{2x^4-2x^3+6x^2-5x+1}{x^3-x^2+x-1}\,dx$

63. $\displaystyle\int \frac{81}{x^3-9x^2}\,dx$

64. $\displaystyle\int \frac{10x}{x^2-2x-24}\,dx$

65. $\displaystyle\int \frac{x+1}{x^2\left(x^2+4\right)}\,dx$

66. $\displaystyle\int \frac{1+x^2}{(x+1)^3}\,dx$

67. $\displaystyle\int \frac{6}{x^2-1}\,dx$

68. $\displaystyle\int \frac{21x^2}{x^3-x^2-12x}\,dx$

69. $\displaystyle\int \frac{x+1}{x^3+3x^2-18x}\,dx$

70. $\displaystyle\int \frac{x^2+12x-4}{x^3-4x}\,dx$

71. $\displaystyle\int \frac{6x^2}{x^4-5x^2+4}\,dx$

72. $\displaystyle\int \frac{4x-2}{x^3-x}\,dx$

73. $\displaystyle\int \frac{16x^2}{(x-6)(x+2)^2}\,dx$

74. $\displaystyle\int \frac{8\left(x^2+4\right)}{x\left(x^2+8\right)}\,dx$

75. $\displaystyle\int \frac{x^2+x+2}{(x+1)\left(x^2+1\right)}\,dx$

76. $\displaystyle\int \frac{2}{x\left(x^2+1\right)^2}\,dx$

77. $\displaystyle\int \frac{1}{(x+1)\left(x^2+2x+2\right)^2}\,dx$

78. $\displaystyle\int \frac{2-x}{x^2+x}\,dx$

79. $\displaystyle\int \frac{3x+11}{(x+2)(x+3)}\,dx$

80. $\displaystyle\int \frac{1}{x^2-a^2}\,dx$

81. $\displaystyle\int \frac{1}{x^2+5x+6}\,dx$

82. $\displaystyle\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}\,dx$

83. $\displaystyle\int \frac{2x^2+5x-1}{x^3+x^2-2x}\,dx$

84. $\displaystyle\int \frac{3x+6}{x^3+2x^2-3x}\,dx$

85. $\displaystyle\int \frac{3x^2+2x-2}{x^3-1}\,dx$

86. $\displaystyle\int \frac{x^3+5x^2+2x-4}{x^4-1}\,dx$

87. $\displaystyle\int \frac{x^2+4x}{\left(x^2+4\right)(x-2)^2}\,dx$

88. $\displaystyle\int \frac{x^2+2x+3}{(x-1)(x+1)^2}\,dx$

89. $\displaystyle\int \frac{x^4-x^3+3x^2-x+2}{(x-1)\left(x^2+2\right)^2}\,dx$

90. $\displaystyle\int \frac{-x^2+11x+18}{(x-1)(x+1)\left(x^2+3x+3\right)}\,dx$

91. $\displaystyle\int \frac{x^3+5x^2+2x-4}{x\left(x^2+4\right)^2}\,dx$

92. $\displaystyle\int_{-1}^{2} \frac{5x}{x^2-x-6}\,dx$

93. $\displaystyle\int_{0}^{5} \frac{2}{x^2-4x-32}\,dx$

94. $\displaystyle\int_{0}^{1} \frac{dx}{(x+1)\left(x^2+1\right)}$

95. $\displaystyle\int_{-1/2}^{1/2} \frac{x^2+1}{x^2-1}\,dx$

96. $\displaystyle\int_{0}^{2} \frac{3}{4x^2+5x+1}\,dx$

97. $\displaystyle\int_{1}^{5} \frac{x-1}{x^2(x+1)}\,dx$

98. $\displaystyle\int_{0}^{1} \frac{x^2-x}{x^2+x+1}\,dx$

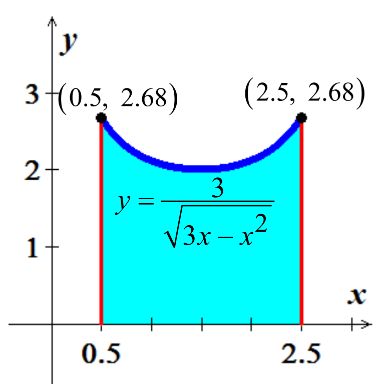
99. $\displaystyle\int_{4}^{8} \frac{y\,dy}{y^2-2y-3}$

100. $\displaystyle\int_{1}^{\sqrt{3}} \frac{3x^2+x+4}{x^3+x}\,dx$

101. $\displaystyle\int_{0}^{\pi/2} \frac{dx}{\sin x+\cos x}$

102. $\displaystyle\int_{0}^{\pi/3} \frac{\sin\theta}{1-\sin\theta}\,d\theta$

103. Find the volume of the solid generated by the revolving the shaded region about *x*-axis



Find the area of the region bounded by the graphs of

104. $y=\dfrac{12}{x^2+5x+6},\quad y=0,\quad x=0,\quad and\quad x=1$

105. $y=\dfrac{7}{16-x^2}\quad and\quad y=1$

106. Find the length of the graph of the function $y = \ln\left(1-x^2\right)$, $0 \le x \le \frac{1}{2}$

107. The region in the first quadrant that is enclosed by the x-axis, the curve $y = \dfrac{5}{x\sqrt{5-x}}$, and the lines $x = 1$ and $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

108. Consider the region bounded by the graphs $y = \dfrac{2x}{x^2+1}$, $y = 0$, $x = 0$, *and* $x = 3$.

a) Find the volume of the solid generated by revolving the region about the x-*axis*
b) Find the centroid of the region.

109. Consider the region bounded by the graph $y^2 = \dfrac{(2-x)^2}{(1+x)^2}$ $0 \le x \le 1$.

Find the volume of the solid generated by revolving this region about the x-*axis*.

110. A single infected individual enters a community of *n* susceptible individuals. Let *x* be the number of newly infected individuals at time *t*. The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. So,

$\dfrac{dx}{dt} = k(x+1)(n-x)$ and you obtain

$$\int \frac{1}{(x+1)(n-x)}\,dx = \int k\,dt$$

Solve for *x* as a function of *t*.

111. Evaluate $\displaystyle\int_0^1 \frac{x}{1+x^4}\,dx$ in ***two*** different ways.