Section 4.6 – Exponential Growth and Decay

Exponential Growth and Decay

The mathematical model for exponential growth or decay is given by

$$A(t) = A_0 e^{kt}$$

$A(t)$: Exponential Function (After time t)

A_0 : At time zero (initial value).

t: Time

k: Exponential rate.

- If $k > 0$, the function models of a growing entity
- If $k < 0$, the function models of a decay entity.

Example

In 1990, the population of Africa was 643 million and by 2000 it had grown to 813 million

a. Use the exponential growth function $A(t) = A_0 e^{kt}$, in which t is the number of years after 1990, to find the exponential growth function that models data

b. By which year will Africa's population reach 2000 million, or two billion?

Solution

a. $A(t) = A_0 e^{kt}$

From 1990 to 2000, is 10 years, that implies in 10 years the population grows from 643 to 813

$$813 = 643e^{k(10)}$$

$$\frac{813}{643} = e^{10k}$$

$$\ln\frac{813}{643} = \ln e^{10k}$$

$$\ln\frac{813}{643} = 10k$$

$$\frac{1}{10}\ln\frac{813}{643} = k$$

$$k \approx 0.023$$

$$\Rightarrow \boxed{A(t) = 643e^{0.023t}}$$

b. $2000 = 643e^{0.023t}$

$$\frac{2000}{643} = e^{0.023t}$$

$$\ln\frac{2000}{643} = \ln e^{0.023t}$$

$$\ln\frac{2000}{643} = 0.023t$$

$$\frac{\ln\frac{2000}{643}}{0.023} = t$$

$$\Rightarrow t \approx 49 \quad \rightarrow \boxed{Year:\ 2039}$$

Doubling Time

$$P(t) = P_0 e^{kt}$$

$$2P_0 = P_0 e^{kt} \qquad P(t) = 2P_0$$

$$2 = e^{kt}$$

$$\ln 2 = \ln e^{kt}$$

$$\ln 2 = kt \ln e \qquad \ln e = 1$$

$$\ln 2 = kt$$

$$t = \frac{\ln 2}{k}$$

Growth Rate and Doubling Time

$$\boxed{Tk = \ln 2} \qquad or \ \ T = \frac{\ln 2}{k} \qquad or \ \ k = \frac{\ln 2}{T}$$

Example

A country's population doubled in 45 years. What was the exponential growth rate?

Solution

$$k = \frac{\ln 2}{t}$$

$$= \frac{\ln 2}{45}$$

$$\approx 0.0154$$

Example

According to the U.S. Census Bureau, the world population reached 6 billion people on July 18, 1999, and was growing exponentially. By the end of 2000, the population had grown to 6.079 billion. The projected world population (in billion of people) t years after 2000, is given by the function defined by

$$f(t) = 6.079e^{.0126t}$$

a) Based on this model, what will the world population be in 2010?

b) In what year will the world population reach 7 billion?

Solution

a) In 2010 $\rightarrow t = 10$

$$f(t=10) = 6.079e^{.0126(10)}$$

$$\approx 6.895$$

b) $7 = 6.079e^{.0126t}$

$$\frac{7}{6.079} = e^{.0126t}$$

$$\ln\frac{7}{6.079} = \ln e^{.0126t}$$

$$\ln\left(\frac{7}{6.079}\right) = .0126t \qquad \ln e = 1$$

$$t = \frac{1}{.0126}\ln\left(\frac{7}{6.079}\right) \qquad \ln(7/6.079)/.0126$$

$$\approx 11.2$$

$$\boxed{Year:\ 2011}$$

Example

Strontium-90 is a waste product from nuclear reactors. As a consequence of fallout from atmosphere nuclear tests, we all have a measurable amount of strontium-90 in our bones.

a. The half-life of Strontium-90 is 28 years, meaning that all after 28 years a given amount of the substance will have decayed to half the original amount. Find the exponential decay model for Strontium-90.

b. Suppose the nuclear accident occurs and releases 60 grams of Strontium-90 into the atmosphere. How long will it take for Strontium-90 to decay to a level of 10 grams?

Solution

a. $A = A_0 e^{kt}$

$$\frac{1}{2}A_0 = A_0 e^{k(28)}$$

$$\frac{1}{2} = e^{28k}$$

$$\ln\frac{1}{2} = \ln e^{28k}$$

$$\ln\frac{1}{2} = 28k$$

$$k = \frac{1}{28}\ln\frac{1}{2} \approx -0.0248$$

$$\boxed{A = A_0 e^{-0.0248t}}$$

b. $A = A_0 e^{-0.0248t}$

$$10 = 60e^{-0.0248t}$$

$$\frac{1}{6} = e^{-0.0248t}$$

$$\ln\frac{1}{6} = \ln e^{-0.0248t}$$

$$\ln\frac{1}{6} = -0.0248t$$

$$\boxed{t = \frac{\ln\frac{1}{6}}{-0.0248} \approx 72.25\ \ yrs}$$

Exercises *Section* 4.6 – Exponential Growth and Decay

1. Suppose that \$10,000 is invested at interest rate of 5.4% per year, compounded continuously.

a) Find the exponential growth function
b) What will the balance be after, 1 *yr*. 10 *yrs*.?
c) What will the balance be after, 1 *yr*. 10 *yrs*.?

2. In 1990, the population of Africa was 643 million and by 2000 it had grown to 813 million

a) Use the exponential growth function $A(t) = A_0 e^{kt}$, in which t is the number of years after 1990, to find the exponential growth function that models data
b) By which year will Africa's population reach 2000 million, or two billion?

3. The radioactive element carbon-14 has a half-life of 5750 yr. The percentage of carbon-14 present in the remains of organic matter can be used to determine the age of that organic matter. Archaeologists discovered that the linen wrapping from one of the Dead Sea Scrolls had lost 22.3% of its carbon-14 at the time it was found. How old was the linen wrapping?

4. Suppose that \$2000 is invested at interest rate ***k***, compounded continuously, and grows to \$2983.65 in 5 yrs.

a) What is the interest rate?
b) Find the exponential growth function
c) What will the balance be after 10 yrs?
d) After how long will the \$2000 have doubled?

5. In 2005, the population of China was about 1.306 billion, and the exponential growth rate was 0.6% per year.

a) Find the exponential growth function
b) Estimate the population in 2008
c) After how long will the population be double what it was in 2005?

6. How long will it take for the money in an account that is compounded continuously at 3% interest to double?

7. If 600 *g* of radioactive substance are present initially and 3 *yrs*. later only 300 *g* remain, how much of the substance will be present after 6 *yrs*.?

8. The population of an endangered species of bird was 4200 in 1990. Thirteen years later, in 2003, the bird population declined to 3000. The population of the birds is decreasing exponentially according to the function $A(t) = 4200e^{kt}$ where *A* is the bird population *t* years after 1990. Use this information to find the value of ***k***.