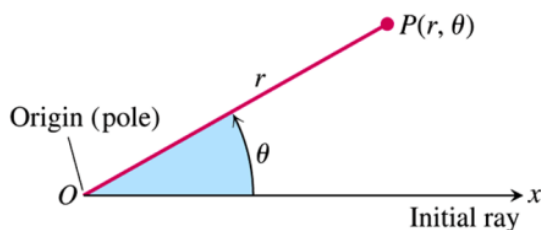


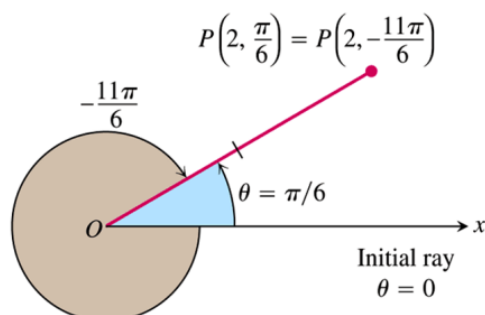
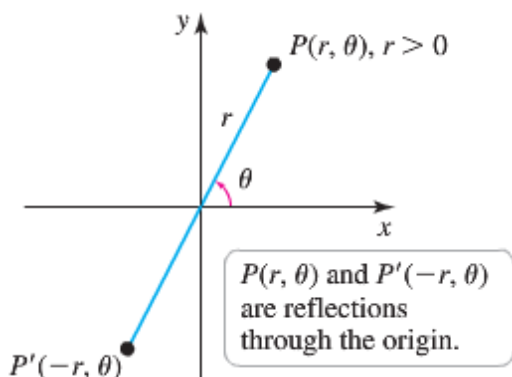
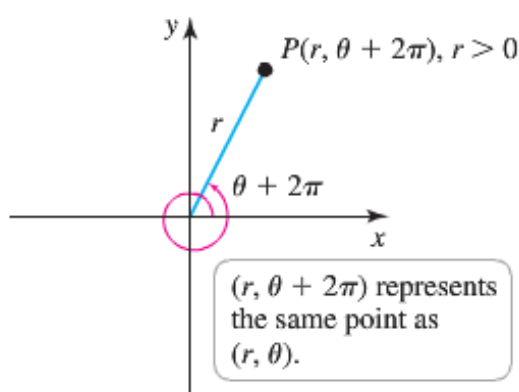
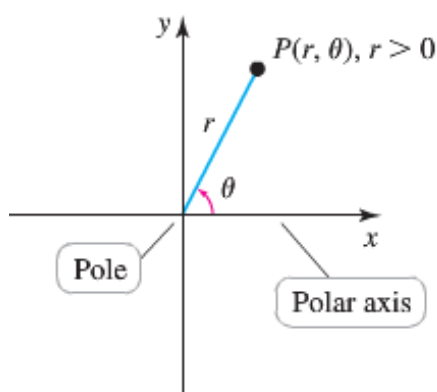
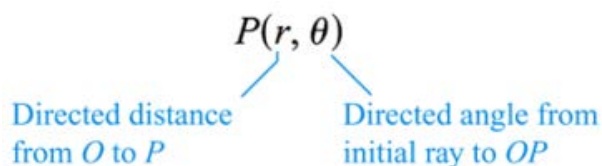
Section 4.3 – Polar Coordinates and Graphs

Definition of Polar Coordinates

To define polar coordinates, let an **origin** O (called the **pole**) and an **initial ray** from O . Then each point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP .



Polar Coordinates



Example

Find all the polar coordinates of the point $P\left(2, \frac{\pi}{6}\right)$

Solution

For $r = 2 \Rightarrow \theta = \frac{\pi}{6}, \frac{\pi}{6} \pm 2\pi, \frac{\pi}{6} \pm 4\pi, \dots$

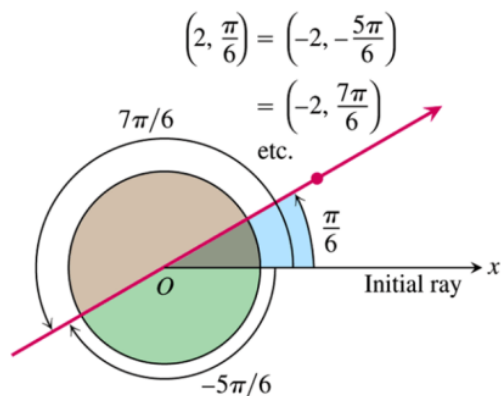
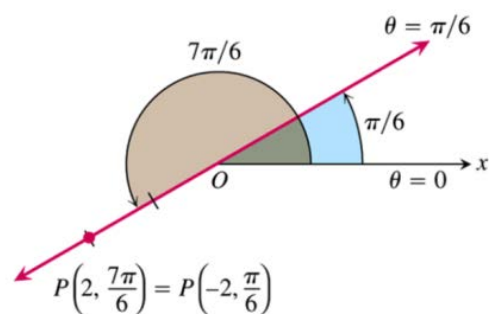
For $r = -2 \Rightarrow \theta = -\frac{5\pi}{6}, -\frac{5\pi}{6} \pm 2\pi, -\frac{5\pi}{6} \pm 4\pi, \dots$

The corresponding coordinate pairs of P are

$$\left(2, \frac{\pi}{6} + 2n\pi\right), \quad n = 0, \pm 1, \pm 2, \dots$$

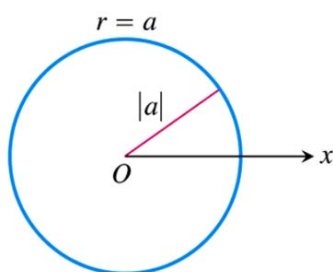
And

$$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \quad n = 0, \pm 1, \pm 2, \dots$$



Polar Equations and Graphs

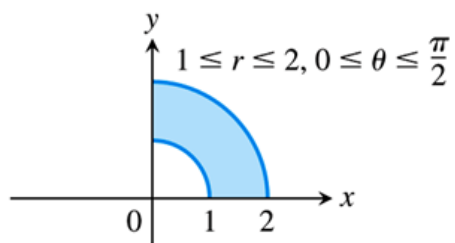
Equation	Graph
$r = a$	Circle of radius $ a $ centered at O
$\theta = \theta_0$	Line through O making an angle θ_0 with the initial ray



Example

Graph the polar coordinate $1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{2}$

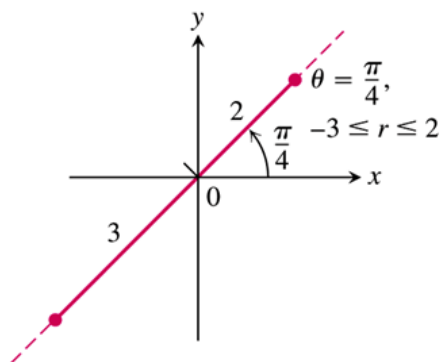
Solution



Example

Graph the polar coordinate $-3 \leq r \leq 2$ and $\theta = \frac{\pi}{4}$

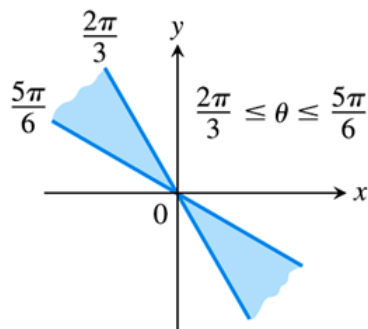
Solution



Example

Graph the polar coordinate $\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$ (no restriction on r)

Solution

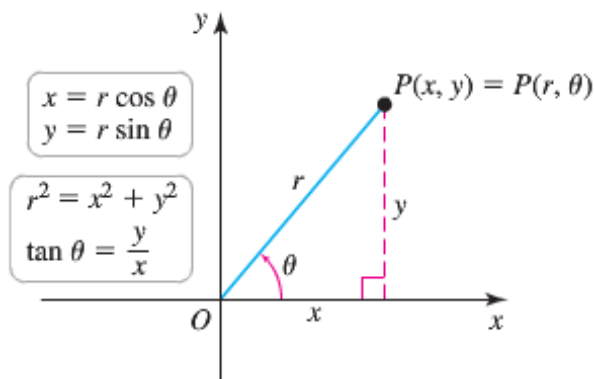


Relating Polar and Cartesian Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial polar ray as the positive x -axis. The ray $\theta = \frac{\pi}{2}$, $r > 0$ becomes the positive y -axis. The two coordinate systems are then related by the following equations

Equations Relating Polar and Cartesian Coordinates

$$\begin{cases} x = r \cos \theta, & y = r \sin \theta \\ r^2 = x^2 + y^2, & \tan \theta = \frac{y}{x} \end{cases}$$



Polar equation	Cartesian equation
$r \cos \theta = 2$	$x = 2$
$r^2 \cos \theta \sin \theta = 4$	$xy = 4$
$r^2 \cos^2 \theta - r^2 \sin^2 \theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r \cos \theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos \theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

Example

Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$

Solution

$$x^2 + (y - 3)^2 = 9$$

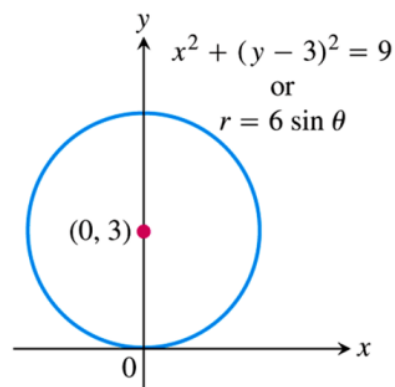
$$x^2 + y^2 - 6y + 9 = 9$$

$$x^2 + y^2 - 6y = 0$$

$$x^2 + y^2 = r^2$$

$$r^2 - 6r \sin \theta = 0$$

$$r(r - 6 \sin \theta) = 0 \Rightarrow \boxed{r = 0} \quad \boxed{r = 6 \sin \theta}$$



Example

Replace the polar equation by equivalent Cartesian equation and identify the graph: $r \cos \theta = -4$

Solution

$$r \cos \theta = -4 \Rightarrow x = -4$$

The graph: Vertical line through $x = -4$

Example

Replace the polar equation by equivalent Cartesian equation and identify the graph: $r^2 = 4r \cos \theta$

Solution

$$r^2 = 4r \cos \theta$$

$$x^2 + y^2 = 4x$$

$$x^2 - 4x + y^2 = 0$$

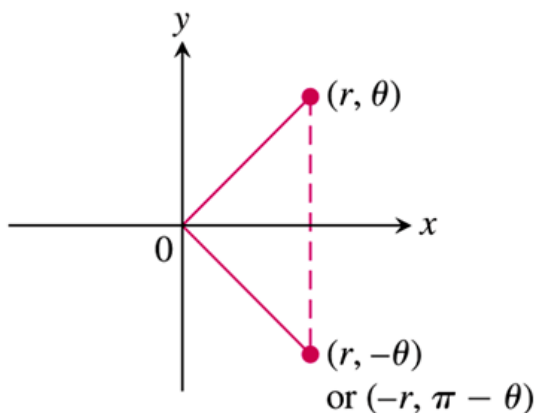
$$x^2 - 4x + 4 + y^2 = 4$$

$$(x - 2)^2 + y^2 = 4$$

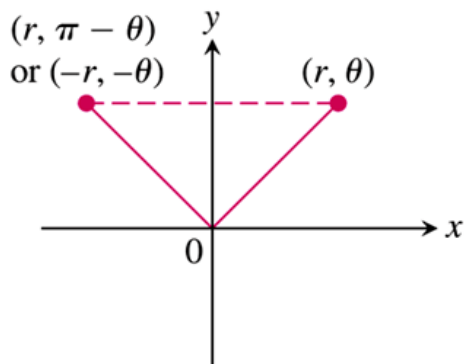
The **graph**: Circle with center (2, 0) and radius 2.

Symmetry Test for Polar Graphs

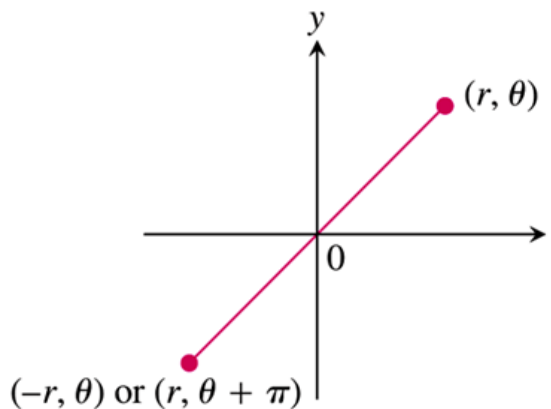
1. **Symmetry about the x-axis:** If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph.



2. **Symmetry about the y-axis:** If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph.



3. **Symmetry about the origin:** If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph.



Example

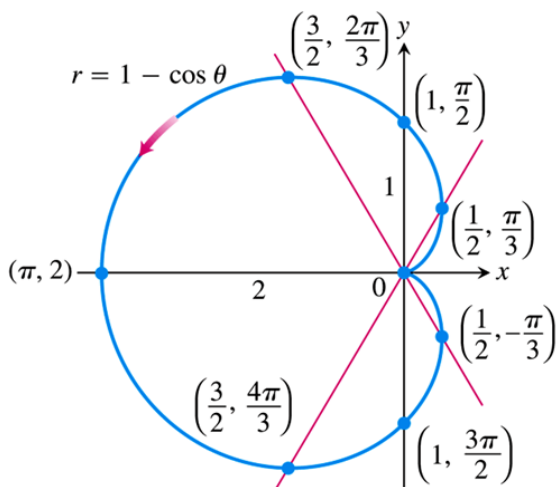
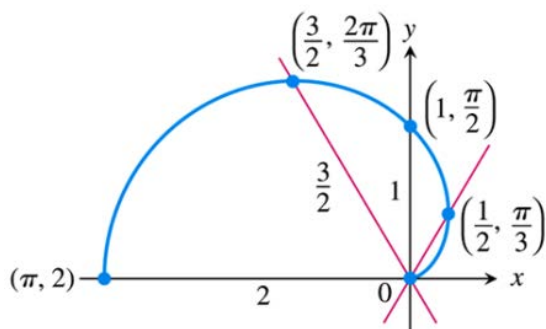
Graph the curve $r = 1 - \cos \theta$

Solution

The curve is symmetric about the x -axis:

$$1 - \cos(-\theta) = 1 - \cos \theta = r$$

θ	$r = 1 - \cos \theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2



Example

Graph the curve $r^2 = 4\cos\theta$

Solution

The curve is symmetric about the x -axis:

$$r^2 = 4\cos\theta$$

$$r^2 = 4\cos(-\theta)$$

$$(r, -\theta)$$

The curve is symmetric about the *origin*:

$$r^2 = 4\cos\theta$$

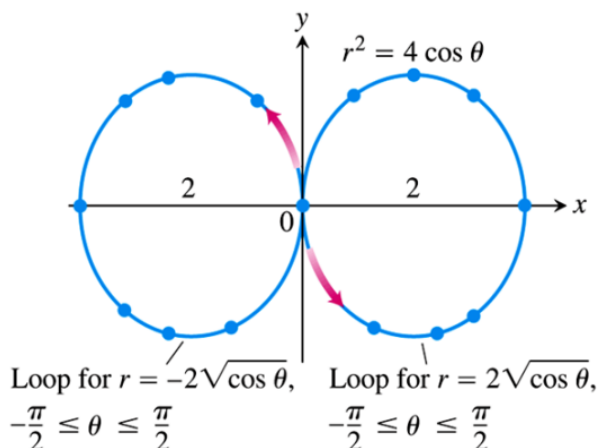
$$(-r)^2 = 4\cos\theta$$

$$(-r, \theta)$$

Therefore, the curve is also symmetric about the y -axis.

$$r^2 = 4\cos\theta \Rightarrow r = \pm 2\sqrt{\cos\theta}$$

θ	$r = \pm 2\sqrt{\cos\theta}$
0	± 2
$\pm \frac{\pi}{6}$	$\approx \pm 1.9$
$\pm \frac{\pi}{4}$	$\approx \pm 1.7$
$\pm \frac{\pi}{3}$	$\approx \pm 1.4$
$\pm \frac{\pi}{2}$	0



A Technique for Graphing

One way to graph a polar equation $r = f(\theta)$ is to make a table of (r, θ) values, plot the corresponding points, and connect them in order of increasing.

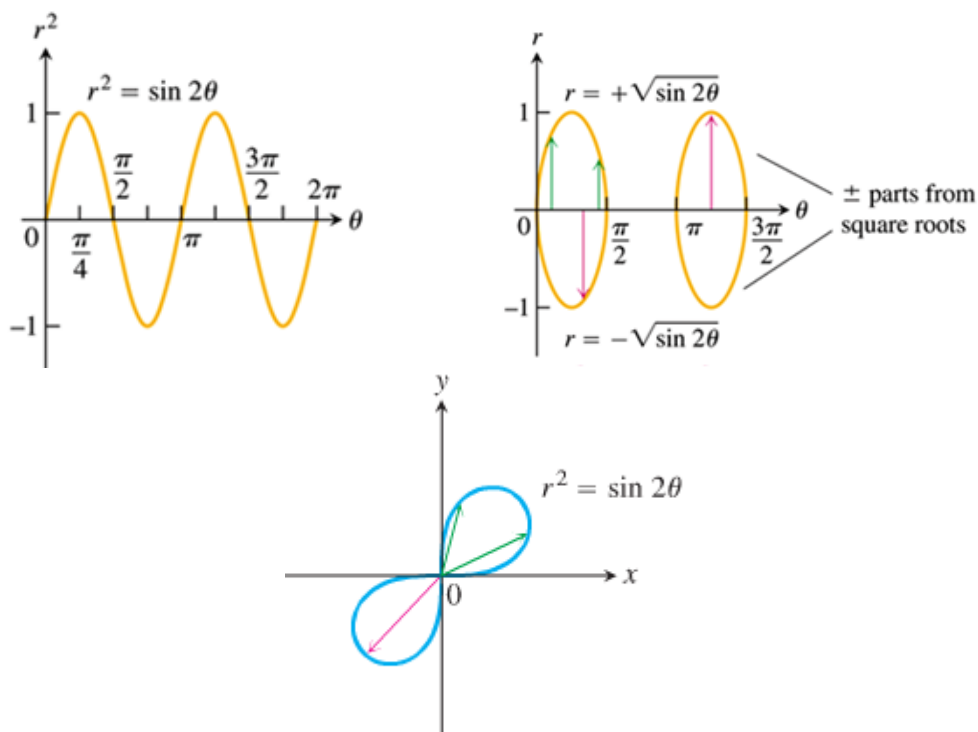
Another method of graphing more reliable is

1. First graph $r = f(\theta)$ in the *Cartesian* $r\theta$ - plane ,
2. Then use the *Cartesian* graph as a table and guide to sketch the *polar coordinate* graph.

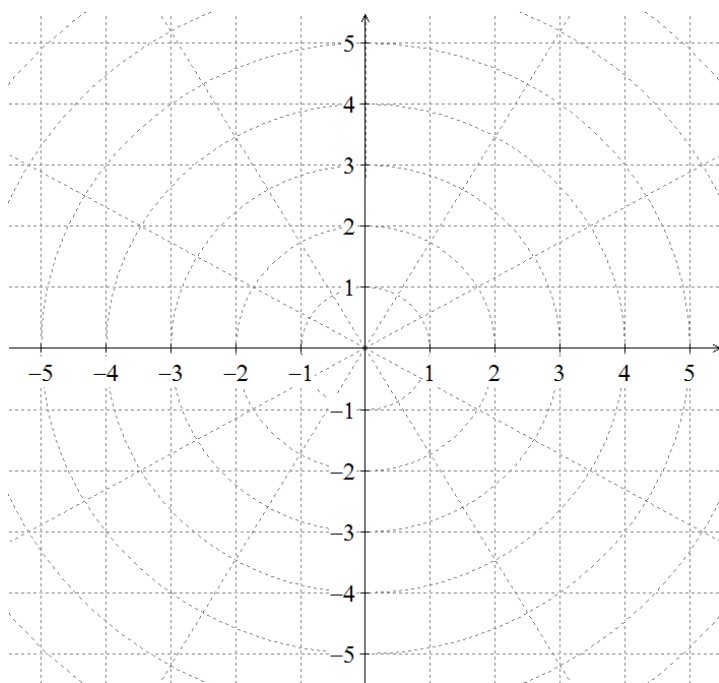
Example

Graph the *lemniscate* curve $r^2 = \sin 2\theta$

Solution



θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{7\pi}{6}$
$r = \pm\sqrt{\sin 2\theta}$	0	$\pm\sqrt{\frac{\sqrt{3}}{2}} \approx \pm.93$	± 1	$\pm\sqrt{\frac{\sqrt{3}}{2}} \approx \pm.93$	0	0	$\pm\sqrt{\frac{\sqrt{3}}{2}} \approx \pm.93$



Exercises Section 4.3 – Polar Coordinates

1. Find the Cartesian coordinates of the following points (given in polar coordinates)

a) $\left(\sqrt{2}, \frac{\pi}{4}\right)$ b) $(1, 0)$ c) $\left(0, \frac{\pi}{2}\right)$ d) $\left(-\sqrt{2}, \frac{\pi}{4}\right)$

2. Find the polar coordinates, $0 \leq \theta < 2\pi$ and $r \geq 0$, of the following points given in Cartesian coordinates

a) $(1, 1)$ b) $(-3, 0)$ c) $(\sqrt{3}, -1)$ d) $(-3, 4)$

3. Find the polar coordinates, $-\pi \leq \theta < \pi$ and $r \geq 0$, of the following points given in Cartesian coordinates

a) $(-2, -2)$ b) $(0, 3)$ c) $(-\sqrt{3}, 1)$ d) $(5, -12)$

(4 – 8) Graph

4. $1 \leq r \leq 2$

7. $-\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}, \quad 0 \leq r \leq 1$

5. $0 \leq \theta \leq \frac{\pi}{6}, \quad r \geq 0$

8. $0 \leq \theta \leq \frac{\pi}{2}, \quad 1 \leq |r| \leq 2$

6. $\theta = \frac{\pi}{2}, \quad r \leq 0$

(9 – 20) Replace the polar equation with equivalent Cartesian equation and identify the graph

9. $r \cos \theta = 2$

14. $r = \frac{5}{\sin \theta - 2 \cos \theta}$

18. $r = 2 \cos \theta + 2 \sin \theta$

10. $r \sin \theta = -1$

15. $r = 4 \tan \theta \sec \theta$

19. $r \sin\left(\frac{2\pi}{3} - \theta\right) = 5$

11. $r = -3 \sec \theta$

16. $r \sin \theta = \ln r + \ln \cos \theta$

20. $r = \frac{4}{2 \cos \theta - \sin \theta}$

12. $r \cos \theta + r \sin \theta = 1$

17. $\cos^2 \theta = \sin^2 \theta$

13. $r^2 = 4r \sin \theta$

(21 – 27) Replace the Cartesian equation with equivalent polar equation

21. $x = y$

24. $xy = 1$

26. $x^2 + (y - 2)^2 = 4$

22. $x^2 - y^2 = 1$

25. $x^2 + xy + y^2 = 1$

27. $(x + 2)^2 + (y - 5)^2 = 16$

23. $\frac{x^2}{9} + \frac{y^2}{4} = 1$

28. a) Show that every vertical line in the xy -plane has a polar equation of the form $r = a \sec \theta$
b) Find the analogous polar equation for horizontal lines in the xy -plane.

(29 – 34) Identify the symmetries of the curves. Then sketch the curves.

29. $r = 2 - 2\cos\theta$

31. $r = 2 + \sin\theta$

33. $r^2 = -\sin\theta$

30. $r = 1 + \sin\theta$

32. $r^2 = \sin\theta$

34. $r^2 = -\cos\theta$

(35 – 37) Graph the lemniscate. What symmetries do these curves have?

35. $r^2 = 4\cos 2\theta$

36. $r^2 = 4\sin 2\theta$

37. $r^2 = -\cos 2\theta$

(38 – 42) Graph the limaçons is Old French for “snail”. Equations for limaçons have the form

$$r = a \pm b\cos\theta \quad \text{or} \quad r = a \pm b\sin\theta$$

38. $r = \frac{1}{2} + \cos\theta$

40. $r = 1 - \cos\theta$

42. $r = 2 + \cos\theta$

39. $r = \frac{1}{2} + \sin\theta$

41. $r = \frac{3}{2} - \sin\theta$

(43 – 46) Graph the equation

43. $r = 1 - 2\sin 3\theta$

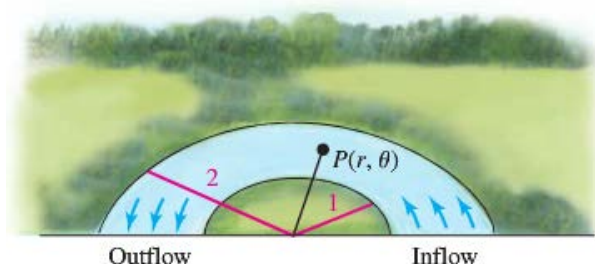
44. $r = \sin^2 \frac{\theta}{2}$

45. $r = 1 - \sin\theta$

46. $r^2 = 4\sin\theta$

47. Graph the *nephroid* of *Freeth* equation $r = 1 + 2\sin\frac{\theta}{2}$

48. Water flows in a shallow semicircular channel with inner and outer radii of 1 m and 2 m. At a point $P(r, \theta)$ in the channel, the flow is in the tangential direction (counterclockwise along circles), and it depends only on r , the distance from the center of the semicircles.



- Express the region formed by the channel as a set in polar coordinates.
- Express the inflow and outflow regions of the channel as sets in polar coordinates.
- Suppose the tangential velocity of the water in m/s is given by $v(r) = 10r$, for $1 \leq r \leq 2$. Is the velocity greater at $(1.5, \frac{\pi}{4})$ or $(1.2, \frac{3\pi}{4})$? Explain.
- Suppose the tangential velocity of the water is given by $v(r) = \frac{20}{r}$, for $1 \leq r \leq 2$. Is the velocity greater at $(1.8, \frac{\pi}{6})$ or $(1.3, \frac{2\pi}{3})$? Explain.

e) The total amount of water that flows through the channel (across a cross section of the channel $\theta = \theta_0$) is proportional to $\int_1^2 v(r) dr$. Is the total flow through the channel greater for the flow in part (c) or (d)?

- 49.** A simplified model assumes that the orbits of Earth and Mars are circular with radii of 2 and 3, respectively, and that Earth completes one orbit in one year while Mars takes two years. When $t = 0$, Earth is at $(2, 0)$ and Mars is at $(3, 0)$; both orbit the Sun (at $(0, 0)$) in the counterclockwise direction.

The position of Mars relative to Earth is given by the parametric equations

$$x = (3 - 4 \cos \pi t) \cos \pi t + 2, \quad y = (3 - 4 \cos \pi t) \sin \pi t$$

- a) Graph the parametric equations, for $0 \leq t \leq 2$
b) Letting $r = 3 - 4 \cos \pi t$, explain why the path of Mars relative to Earth is a limaçon.