

Solution **Section 4.1 – Antiderivatives**

Exercise

Find indefinite integral $\int v^2 dv$

Solution

$$\int v^2 dv = \underline{\frac{v^3}{3} + C}$$

Exercise

Find indefinite integral $\int x^{1/2} dx$

Solution

$$\int x^{1/2} dx = \underline{\frac{2}{3} x^{3/2} + C}$$

Exercise

Find indefinite integral $\int 4y^{-3} dy$

Solution

$$\begin{aligned} \int 4y^{-3} dy &= 4 \frac{y^{-2}}{-2} + C \\ &= \underline{-\frac{2}{y^2} + C} \end{aligned}$$

Exercise

Find indefinite integral $\int (x^3 - 4x + 2) dx$

Solution

$$\begin{aligned} \int (x^3 - 4x + 2) dx &= \frac{x^4}{4} - 4 \frac{x^2}{2} + 2x + C \\ &= \underline{\frac{1}{4} x^4 - 2x^2 + 2x + C} \end{aligned}$$

Exercise

Find indefinite integral $\int (3z^2 - 4z + 5) dz$

Solution

$$\begin{aligned}\int (3z^2 - 4z + 5) dz &= 3 \frac{z^3}{3} - 4 \frac{z^2}{2} + 5z + C \\ &= \underline{z^3 - 2z^2 + 5z + C}\end{aligned}$$

Exercise

Find indefinite integral $\int (x^2 - 1)^2 dx$

Solution

$$\begin{aligned}\int (x^2 - 1)^2 dx &= \int (x^4 - 2x^2 + 1) dx \\ &= \underline{\frac{1}{5}x^5 - \frac{2}{3}x^3 + x + C}\end{aligned}$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

Exercise

Find indefinite integral $\int \frac{x^2 + 1}{\sqrt{x}} dx$

Solution

$$\begin{aligned}\int \frac{x^2 + 1}{\sqrt{x}} dx &= \int \frac{x^2}{x^{1/2}} + \frac{1}{x^{1/2}} dx \\ &= \int (x^{3/2} + x^{-1/2}) dx \\ &= \frac{x^{5/2}}{5/2} - \frac{x^{1/2}}{1/2} + C \\ &= \underline{\frac{2}{5}x^{5/2} - 2x^{1/2} + C}\end{aligned}$$

Exercise

Find indefinite integral $\int (\sqrt[4]{x^3} + 1) dx$

Solution

$$\int \left(\sqrt[4]{x^3} + 1 \right) dx = \int \left(x^{3/4} + 1 \right) dx$$

$$= \underline{\underline{\frac{4}{7} x^{7/4} + x + C}}$$

Exercise

Find indefinite integral $\int \sqrt{x}(x+1)dx$

Solution

$$\int \sqrt{x}(x+1)dx = \int x^{1/2}(x+1)dx$$

$$= \int \left(x^{3/2} + x^{1/2} \right) dx$$

$$= \underline{\underline{\frac{2}{5} x^{5/2} + \frac{2}{3} x^{3/2} + C}}$$

Exercise

Find indefinite integral $\int (1+3t)t^2 dt$

Solution

$$\int (1+3t)t^2 dt = \int \left(t^2 + 3t^3 \right) dt$$

$$= \underline{\underline{\frac{1}{3} t^3 + \frac{3}{4} t^4 + C}}$$

Exercise

Find indefinite integral $\int \frac{x^2-5}{x^2} dx$

Solution

$$\int \frac{x^2-5}{x^2} dx = \int \left(1 - \frac{5}{x^2} \right) dx$$

$$= \int \left(1 - 5x^{-2} \right) dx$$

$$= \underline{\underline{x + \frac{5}{x} + C}}$$

Exercise

Find indefinite integral $\int (-40x + 250)dx$

Solution

$$\int (-40x + 250)dx = \underline{-20x^2 + 250x + C}$$

Exercise

Find indefinite integral $\int \frac{x+2}{\sqrt{x}} dx$

Solution

$$\begin{aligned} \int \frac{x+2}{\sqrt{x}} dx &= \int \left[\frac{x}{x^{1/2}} + \frac{2}{x^{1/2}} \right] dx \\ &= \int \frac{x}{x^{1/2}} dx + \int \frac{2}{x^{1/2}} dx \\ &= \int x^{1/2} dx + 2 \int x^{-1/2} dx \\ &= \frac{x^{3/2}}{3/2} + 2 \frac{x^{1/2}}{1/2} + C \\ &= \underline{\frac{2}{3}x^{3/2} + 4x^{1/2} + C} \end{aligned}$$

Exercise

Find indefinite integral $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x \right) dx$

Solution

$$\begin{aligned} \int \left(\frac{1}{5} - \frac{2}{x^3} + 2x \right) dx &= \int \frac{1}{5} dx - \int 2x^{-3} dx + \int 2x dx \\ &= \frac{x}{5} - 2 \frac{x^{-2}}{-2} + x^2 + C \\ &= \underline{\frac{x}{5} + \frac{1}{x^2} + x^2 + C} \end{aligned}$$

Exercise

Find indefinite integral $\int (\sqrt{x} + \sqrt[3]{x}) dx$

Solution

$$\begin{aligned} \int (\sqrt{x} + \sqrt[3]{x}) dx &= \int (x^{1/2} + x^{1/3}) dx \\ &= \frac{x^{3/2}}{3/2} + \frac{x^{4/3}}{4/3} + C \\ &= \frac{2}{3} x^{3/2} + \frac{3}{4} x^{4/3} + C \end{aligned}$$

Exercise

Find indefinite integral $\int 2x(1 - x^{-3}) dx$

Solution

$$\begin{aligned} \int 2x(1 - x^{-3}) dx &= \int (2x - 2x^{-2}) dx \\ &= x^2 - 2 \frac{x^{-1}}{-1} + C \\ &= x^2 + \frac{2}{x} + C \end{aligned}$$

Exercise

Find indefinite integral $\int \left(\frac{4 + \sqrt{t}}{t^3} \right) dt$

Solution

$$\begin{aligned} \int \left(\frac{4 + \sqrt{t}}{t^3} \right) dt &= \int \left(\frac{4}{t^3} + \frac{t^{1/2}}{t^3} \right) dt \\ &= \int (4t^{-3} + t^{-5/2}) dt \\ &= 4 \frac{t^{-2}}{-2} + \frac{t^{-3/2}}{-3/2} + C \\ &= -\frac{2}{t^2} - \frac{2}{3t^{3/2}} + C \end{aligned}$$

Exercise

Find indefinite integral $\int (-2 \cos t) dt$

Solution

$$\int (-2 \cos t) dt = \underline{-2 \sin t + C}$$

Exercise

Find indefinite integral $\int 7 \sin \frac{\theta}{3} d\theta$

Solution

$$\begin{aligned} \int 7 \sin \frac{\theta}{3} d\theta &= 7 \frac{-\cos\left(\frac{\theta}{3}\right)}{\frac{1}{3}} + C \\ &= \underline{-21 \cos\left(\frac{\theta}{3}\right) + C} \end{aligned}$$

Exercise

Find indefinite integral $\int \frac{2}{5} \sec \theta \tan \theta d\theta$

Solution

$$\int \frac{2}{5} \sec \theta \tan \theta d\theta = \underline{\frac{2}{5} \sec \theta + C}$$

Exercise

Find indefinite integral $\int (4 \sec x \tan x - 2 \sec^2 x) dx$

Solution

$$\begin{aligned} \int (4 \sec x \tan x - 2 \sec^2 x) dx &= 4 \int (\sec x \tan x) dx - 2 \int (\sec^2 x) dx \\ &= \underline{4 \sec x - 2 \tan x + C} \end{aligned}$$

Exercise

Find indefinite integral $\int (2 \cos 2x - 3 \sin 3x) dx$

Solution

$$\int (2 \cos 2x - 3 \sin 3x) dx = \underline{\sin 2x + \cos 3x + C}$$

Exercise

Find indefinite integral $\int (1 + \tan^2 \theta) d\theta$

Solution

$$\int (1 + \tan^2 \theta) d\theta = \int (\sec^2 \theta) d\theta$$

$$= \underline{\tan \theta + C}$$

Exercise

Find indefinite integral $\int \frac{\csc \theta}{\csc \theta - \sin \theta} d\theta$

Solution

$$\int \frac{\csc \theta}{\csc \theta - \sin \theta} d\theta = \int \frac{1}{1 - \frac{\sin \theta}{\csc \theta}} d\theta \quad \text{divide by } \csc \theta \text{ \& } \csc \theta = \frac{1}{\sin \theta}$$

$$= \int \frac{1}{1 - \sin^2 \theta} d\theta \quad \sin^2 \theta + \cos^2 \theta = 1 \Rightarrow 1 - \sin^2 \theta = \cos^2 \theta$$

$$= \int \frac{1}{\cos^2 \theta} d\theta$$

$$= \int \sec^2 \theta d\theta$$

$$= \underline{\tan \theta + C}$$

Exercise

Evaluate the integral $\int (2e^x - 3e^{-2x}) dx$

Solution

$$\int (2e^x - 3e^{-2x}) dx = \underline{2e^x + \frac{3}{2}e^{-2x} + C}$$

Exercise

Evaluate $\int \frac{dx}{\sqrt{9-x^2}}$

Solution

$$\int \frac{dx}{\sqrt{9-x^2}} = \underline{\sin^{-1}\left(\frac{x}{3}\right) + C}$$

Exercise

Evaluate $\int \frac{dx}{9+3x^2}$

Solution

$$\begin{aligned} \int \frac{dx}{9+3x^2} &= \frac{1}{3} \int \frac{dx}{3+x^2} & a^2 = 3 \rightarrow a = \sqrt{3} \\ &= \frac{1}{3} \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C \\ &= \underline{\frac{\sqrt{3}}{9} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C} \end{aligned}$$

Exercise

Find the integral $\int \frac{4x^2 - 3x + 2}{x^2} dx$

Solution

$$\begin{aligned} \int \frac{4x^2 - 3x + 2}{x^2} dx &= \int \left(\frac{4x^2}{x^2} - \frac{3x}{x^2} + \frac{2}{x^2} \right) dx \\ &= \int \left(4 - \frac{3}{x} + 2x^{-2} \right) dx \\ &= 4x - 3\ln|x| - 2x^{-1} + C \\ &= \underline{4x - 3\ln|x| - \frac{2}{x} + C} \end{aligned}$$

Exercise

Find the integral $\int (x^8 - 3x^3 + 1) dx$

Solution

$$\int (x^8 - 3x^3 + 1) dx = \underline{\frac{1}{9}x^9 - \frac{3}{4}x^4 + x + C}$$

Exercise

Find the integral $\int (2x + 1)^2 dx$

Solution

$$\begin{aligned} \int (2x + 1)^2 dx &= \int (4x^2 + 4x + 1) dx \\ &= \underline{\frac{4}{3}x^3 + 2x^2 + x + C} \end{aligned}$$

Exercise

Find the integral $\int \frac{x+1}{x} dx$

Solution

$$\begin{aligned} \int \frac{x+1}{x} dx &= \int \left(1 + \frac{1}{x}\right) dx \\ &= \underline{x + \ln|x| + C} \end{aligned}$$

Exercise

Find the integral $\int \left(\frac{1}{x^2} - \frac{2}{x^{5/2}} \right) dx$

Solution

$$\begin{aligned} \int \left(\frac{1}{x^2} - \frac{2}{x^{5/2}} \right) dx &= \int \left(\frac{1}{x^2} - 2x^{-5/2} \right) dx \\ &= -\frac{1}{x} + \frac{4}{3}x^{-3/2} + C \\ &= \underline{-\frac{1}{x} + \frac{4}{3x^{3/2}} + C} \end{aligned}$$

Exercise

Find the integral $\int \frac{x^4 - 2\sqrt{x} + 2}{x^2} dx$

Solution

$$\begin{aligned} \int \frac{x^4 - 2\sqrt{x} + 2}{x^2} dx &= \int (x^2 - 2x^{-3/2} + 2x^{-2}) dx \\ &= \underline{\frac{1}{3}x^3 + 4x^{-1/2} - \frac{2}{x} + C} \end{aligned}$$

Exercise

Find the integral $\int (1 + \cos 3\theta) d\theta$

Solution

$$\int (1 + \cos 3\theta) d\theta = \underline{\theta + \frac{1}{3}\sin 3\theta + C}$$

Exercise

Find the integral $\int 2\sec^2 \theta d\theta$

Solution

$$\int 2\sec^2 \theta d\theta = \underline{2\tan \theta + C}$$

Exercise

Find the integral $\int \sec 2x \tan 2x dx$

Solution

$$\int \sec 2x \tan 2x dx = \underline{\frac{1}{2}\sec 2x + C}$$

Exercise

Find the integral $\int 2e^{2x} dx$

Solution

$$\int 2e^{2x} dx = \underline{e^{2x} + C}$$

Exercise

Find the integral $\int \frac{12}{x} dx$

Solution

$$\int \frac{12}{x} dx = \underline{12 \ln|x| + C}$$

Exercise

Find the integral $\int \frac{dx}{\sqrt{1-x^2}}$

Solution

$$\int \frac{dx}{\sqrt{1-x^2}} = \underline{\sin^{-1} x + C}$$

Exercise

Find the integral $\int \frac{dx}{x^2 + 1}$

Solution

$$\int \frac{dx}{x^2 + 1} = \underline{\tan^{-1} x + C}$$

Exercise

Find the integral $\int \frac{1 + \tan \theta}{\sec \theta} d\theta$

Solution

$$\int \frac{1 + \tan \theta}{\sec \theta} d\theta = \int \left(\frac{1}{\sec \theta} + \frac{\tan \theta}{\sec \theta} \right) d\theta$$

$$= \int (\cos \theta + \sin \theta) d\theta$$

$$= \sin \theta - \cos \theta + C$$

Exercise

Find the integral $\int (\sqrt[4]{x^3} + \sqrt{x^5}) dx$

Solution

$$\int (\sqrt[4]{x^3} + \sqrt{x^5}) dx = \int (x^{3/4} + x^{5/2}) dx$$

$$= \frac{4}{7} x^{7/4} + \frac{2}{7} x^{7/2} + C$$

$$= \frac{4}{7} x \sqrt[4]{x^3} + \frac{2}{7} x^3 \sqrt{x} + C$$

Exercise

Find the integral $\int (x^{-3} + 7e^{5x} + \frac{4}{x}) dx$

Solution

$$\int (x^{-3} + 7e^{5x} + \frac{4}{x}) dx = \frac{x^{-2}}{-2} + \frac{7}{5} e^{5x} + 4 \ln|x| + C$$

$$= -\frac{1}{2x^2} + \frac{7}{5} e^{5x} + 4 \ln|x| + C$$

Exercise

Find the integral $\int (\frac{2}{x} + \frac{x}{2}) dx$

Solution

$$\int (\frac{2}{x} + \frac{x}{2}) dx = 2 \ln|x| + \frac{1}{4} x^2 + C$$

Exercise

Find the integral $\int \frac{1}{ax} dx$

Solution

$$\int \frac{1}{ax} dx = \underline{\frac{1}{a} \ln|x| + C}$$

Exercise

Find the integral $\int x\sqrt{x} dx$

Solution

$$\begin{aligned} \int x\sqrt{x} dx &= \int x^{3/2} dx \\ &= \underline{\frac{2}{5} x^{5/2} + C} \end{aligned}$$

Exercise

Find the integral $\int \left(\frac{2}{\sqrt{x}} + 2\sqrt{x} \right) dx$

Solution

$$\begin{aligned} \int \left(\frac{2}{\sqrt{x}} + 2\sqrt{x} \right) dx &= \int \left(2x^{-1/2} + 2x^{1/2} \right) dx \\ &= 4x^{1/2} + \frac{4}{3} x^{3/2} + C \\ &= \underline{4\sqrt{x} + \frac{4}{3} x\sqrt{x} + C} \end{aligned}$$

Exercise

Find the integral $\int \left(x - 2x^2 + \frac{1}{2x} \right) dx$

Solution

$$\int \left(x - 2x^2 + \frac{1}{2x} \right) dx = \underline{\frac{1}{2} x^2 - \frac{2}{3} x^3 + \frac{1}{2} \ln|x| + C}$$

Exercise

Find the integral $\int \left(\frac{7}{2x^3} - \sqrt[3]{x} \right) dx$

Solution

$$\begin{aligned} \int \left(\frac{7}{2x^3} - \sqrt[3]{x} \right) dx &= \int \left(\frac{7}{2} x^{-3} - x^{1/3} \right) dx \\ &= -\frac{7}{4} x^{-2} - \frac{3}{4} x^{4/3} + C \\ &= \underline{-\frac{7}{4x^2} - \frac{3}{4} x \sqrt[3]{x} + C} \end{aligned}$$

Exercise

Find the integral $\int 3e^{-2x} dx$

Solution

$$\int 3e^{-2x} dx = \underline{-\frac{3}{2}e^{-2x} + C}$$

Exercise

Find the integral $\int e^{-x} dx$

Solution

$$\int e^{-x} dx = \underline{-e^{-x} + C}$$

Exercise

Find the integral $\int e dx$

Solution

$$\int e dx = \underline{ex + C}$$

Exercise

Find the integral $\int \frac{7}{2e^{2x}} dx$

Solution

$$\begin{aligned} \int \frac{7}{2e^{2x}} dx &= \int \frac{7}{2} e^{-2x} dx \\ &= -\frac{7}{4} e^{-2x} + C \\ &= \underline{-\frac{7}{4e^{2x}} + C} \end{aligned}$$

Exercise

Find the integral $\int -3(e^{2x} + 1) dx$

Solution

$$\int -3(e^{2x} + 1) dx = \underline{-3\left(\frac{1}{2}e^{2x} + x\right) + C}$$

Exercise

Find the integral $\int \left(-3e^{-x} + 2x - \frac{1}{2}e^{5x}\right) dx$

Solution

$$\int \left(-3e^{-x} + 2x - \frac{1}{2}e^{5x}\right) dx = \underline{3e^{-x} + x^2 - \frac{1}{10}e^{5x} + C}$$

Exercise

Find the integral $\int \left(\sqrt[4]{x^3} + 1\right) dx$

Solution

$$\begin{aligned} \int \left(\sqrt[4]{x^3} + 1\right) dx &= \int \left(x^{3/4} + 1\right) dx \\ &= \underline{\frac{4}{7}x^{7/4} + x + C} \end{aligned}$$

Exercise

Find the integral $\int (5x^4 + 3x^2 + 2x + 5) dx$

Solution

$$\int (5x^4 + 3x^2 + 2x + 5) dx = \underline{x^5 + x^3 + x^2 + 5x + C}$$

Exercise

Find the integral $\int (5x^{4/3} + 3x^{2/3} + 2x^{1/3}) dx$

Solution

$$\int (5x^{4/3} + 3x^{2/3} + 2x^{1/3}) dx = \underline{\frac{15}{7}x^{7/3} + \frac{9}{5}x^{5/3} + \frac{3}{2}x^{4/3} + C}$$

Exercise

Find the integral $\int (5x^{-4/3} + 3x^{-2/3} + 2x^{-1/3}) dx$

Solution

$$\int (5x^{-4/3} + 3x^{-2/3} + 2x^{-1/3}) dx = \underline{-15x^{-1/3} + 9x^{1/3} + 3x^{2/3} + C}$$

Exercise

Find the integral $\int \frac{x^4 - 3x^2 + 5}{x^4} dx$

Solution

$$\begin{aligned} \int \frac{x^4 - 3x^2 + 5}{x^4} dx &= \int \left(1 - \frac{3}{x^2} + 5x^{-4} \right) dx \\ &= \underline{x + \frac{3}{x} - \frac{5}{3x^3} + C} \end{aligned}$$

Exercise

Find the integral $\int \left(\frac{3}{x^7} - \frac{5}{x^6} \right) dx$

Solution

$$\begin{aligned} \int \left(\frac{3}{x^7} - \frac{5}{x^6} \right) dx &= \int (3x^{-7} - 5x^{-6}) dx \\ &= -\frac{1}{2}x^{-6} + x^{-5} + C \\ &= \underline{-\frac{1}{2x^6} + \frac{1}{x^5} + C} \end{aligned}$$

Exercise

Find the integral $\int \frac{x+8}{\sqrt{x}} dx$

Solution

$$\begin{aligned} \int \frac{x+8}{\sqrt{x}} dx &= \int (x^{1/2} + 8x^{-1/2}) dx \\ &= \underline{\frac{2}{3}x^{3/2} + 16x^{1/2} + C} \end{aligned}$$

Exercise

Find the integral $\int \frac{x^2+8}{\sqrt[3]{x}} dx$

Solution

$$\begin{aligned} \int \frac{x^2+8}{\sqrt[3]{x}} dx &= \int (x^{5/3} + 8x^{-1/3}) dx \\ &= \underline{\frac{3}{8}x^{8/3} + 12x^{2/3} + C} \end{aligned}$$

Exercise

Find the integral $\int \cos\left(\frac{5\pi}{3}x\right) dx$

Solution

$$\int \cos\left(\frac{5\pi}{3}x\right) dx = \underline{\frac{3}{5\pi} \sin \frac{5\pi}{3}x + C}$$

Exercise

Find the integral $\int \sin\left(\frac{2x}{3}\right) dx$

Solution

$$\int \sin\left(\frac{2x}{3}\right) dx = \underline{-\frac{3}{2} \cos \frac{2x}{3} + C}$$

Exercise

Find the integral $\int (5 \cos x + 4 \sin x + 3 \sec^2 x) dx$

Solution

$$\int (5 \cos x + 4 \sin x + 3 \sec^2 x) dx = \underline{5 \sin x - 4 \cos x + 3 \tan x + C}$$

Exercise

Find the integral $\int \sec \theta (\sec \theta + \tan \theta) d\theta$

Solution

$$\int \sec \theta (\sec \theta + \tan \theta) d\theta = \int (\sec^2 \theta + \sec \theta \tan \theta) d\theta$$

$$= \underline{\tan \theta + \sec \theta + C}$$

Exercise

Find the integral $\int (\tan^2 \theta + 1) d\theta$

Solution

$$\int (\tan^2 \theta + 1) d\theta = \int \sec^2 \theta d\theta$$

$$= \underline{\tan \theta + C}$$

Exercise

Find the integral $\int (\cos^4 \theta - \sin^4 \theta) d\theta$

Solution

$$\begin{aligned}
 \int (\cos^4 \theta - \sin^4 \theta) d\theta &= \int (\cos^2 \theta - \sin^2 \theta)(\cos^2 \theta + \sin^2 \theta) d\theta \\
 &= \int \cos 2\theta d\theta && \cos^2 \theta - \sin^2 \theta = \cos 2\theta \quad \cos^2 \theta + \sin^2 \theta = 1 \\
 &= \underline{\frac{1}{2} \sin 2\theta + C}
 \end{aligned}$$

Exercise

Find the integral $\int (\cos^2 \theta - \sin^2 \theta) d\theta$

Solution

$$\begin{aligned}
 \int (\cos^2 \theta - \sin^2 \theta) d\theta &= \int \cos 2\theta d\theta && \cos^2 \theta - \sin^2 \theta = \cos 2\theta \\
 &= \underline{\frac{1}{2} \sin 2\theta + C}
 \end{aligned}$$

Exercise

Find the integral $\int (\cos^2 \theta + \sin^2 \theta) d\theta$

Solution

$$\begin{aligned}
 \int (\cos^2 \theta + \sin^2 \theta) d\theta &= \int (1) d\theta \\
 &= \underline{\theta + C}
 \end{aligned}$$

Exercise

Find the integral $\int (\cos 2x \cos 4x - \sin 2x \sin 4x) dx$

Solution

$$\begin{aligned}
 \int (\cos 2x \cos 4x - \sin 2x \sin 4x) dx &= \int \cos 6x dx \\
 &= \underline{\frac{1}{6} \sin 6x + C}
 \end{aligned}$$

Exercise

Find the integral $\int (\sin 2x \cos 4x - \cos 2x \sin 4x) dx$

Solution

$$\begin{aligned}\int (\sin 2x \cos 4x - \cos 2x \sin 4x) dx &= \int \sin(-2x) dx \\ &= -\int \sin 2x dx \\ &= \underline{\underline{\frac{1}{2} \cos 2x + C}}\end{aligned}$$

Exercise

Find the integral $\int (\sin 3x \cos 2x + \cos 3x \sin 2x) dx$

Solution

$$\begin{aligned}\int (\sin 3x \cos 2x + \cos 3x \sin 2x) dx &= \int \sin 5x dx \\ &= \underline{\underline{-\frac{1}{5} \cos 5x + C}}\end{aligned}$$

Exercise

Find the integral $\int \cos 2x \sin 2x dx$

Solution

$$\begin{aligned}\int \cos 2x \sin 2x dx &= \frac{1}{2} \int \sin 4x dx && \sin \alpha = 2 \sin \alpha \cos \alpha \\ &= \underline{\underline{-\frac{1}{8} \cos 4x + C}}\end{aligned}$$

Exercise

Find the integral $\int (2 \cos^2 x - 1) dx$

Solution

$$\begin{aligned}\int (2 \cos^2 x - 1) dx &= \int \cos 2x dx && \cos 2x = 2 \cos^2 x - 1 \\ &= \underline{\underline{\frac{1}{2} \sin 2x + C}}\end{aligned}$$

Exercise

Find the integral $\int (1 - 2\sin^2 x) dx$

Solution

$$\int (1 - 2\sin^2 x) dx = \int \cos 2x dx \qquad \cos 2x = 1 - 2\sin^2 x$$

$$\underline{= \frac{1}{2} \sin 2x + C}$$

Exercise

Find the integral $\int e^{-5x} dx$

Solution

$$\int e^{-5x} dx = \underline{-\frac{1}{5} e^{-5x} + C}$$

Exercise

Find the integral $\int 4e^{4x} dx$

Solution

$$\int 4e^{4x} dx = \underline{e^{4x} + C}$$

Exercise

Find the integral $\int (2\sin \theta - 5e^\theta) d\theta$

Solution

$$\int (2\sin \theta - 5e^\theta) d\theta = \underline{-2\cos \theta - 5e^\theta + C}$$

Exercise

Find the integral $\int \left(\frac{3}{x} + \sec^2 x \right) dx$

Solution

$$\int \left(\frac{3}{x} + \sec^2 x \right) dx = \underline{3 \ln|x| + \tan x + C}$$

Exercise

Find the integral $\int (\sin x + 2^x) dx$

Solution

$$\int (\sin x +) dx = \underline{-\cos x + \frac{2^x}{\ln 2} + C} \quad \left(a^x \right)' = a^x \ln a$$

Exercise

Find the integral $\int (2x - 3^x) dx$

Solution

$$\int (2x - 3^x) dx = \underline{x^2 - \frac{3^x}{\ln 3} + C} \quad \left(a^x \right)' = a^x \ln a$$

Exercise

Find the integral $\int \left(4x - \frac{3}{x} - \csc^2 x \right) dx$

Solution

$$\int \left(4x - \frac{3}{x} - \csc^2 x \right) dx = \underline{2x^2 - 3 \ln|x| + \cot x + C}$$

Exercise

Find the integral $\int \left(e^{4x} - \frac{3}{x} + 2 \csc x \cot x \right) dx$

Solution

$$\int \left(e^{4x} - \frac{3}{x} + 2 \csc x \cot x \right) dx = \underline{\frac{1}{4} e^{4x} - 3 \ln|x| - 2 \csc x + C}$$

Exercise

Find the integral $\int (a+b)e^{(a+b)x} dx$

Solution

$$\int (a+b)e^{(a+b)x} dx = \underline{e^{(a+b)x} + C}$$

Exercise

Find the integral $\int (a^2 - b^2)e^{(a-b)x} dx$

Solution

$$\begin{aligned} \int (a^2 - b^2)e^{(a-b)x} dx &= \frac{a^2 - b^2}{a-b} e^{(a-b)x} + C \\ &= \underline{(a+b)e^{(a-b)x} + C} \end{aligned}$$

Exercise

Find the function with the following property: $\frac{dy}{dx} = 2x - 7, \quad y(2) = 0$

Solution

$$\frac{dy}{dx} = 2x - 7 \Rightarrow dy = (2x - 7)dx$$

$$\int dy = \int (2x - 7)dx$$

$$y = x^2 - 7x + C$$

At point (2, 0): $0 = 2^2 - 7(2) + C$

$$0 = 4 - 14 + C \rightarrow 0 = -10 + C \quad \boxed{C = 10}$$

$$y(x) = \underline{x^2 - 7x + 10}$$

Exercise

Find the function with the following property: $\frac{dy}{dx} = \frac{1}{x^2} + x, \quad y(2) = 1; \quad x > 0$

Solution

$$\frac{dy}{dx} = \frac{1}{x^2} + x \rightarrow dy = (x^{-2} + x)dx$$

$$\int dy = \int (x^{-2} + x) dx$$

$$y = -x^{-1} + \frac{1}{2}x^2 + C$$

$$1 = -(2)^{-1} + \frac{1}{2}(2)^2 + C$$

$$1 + \frac{1}{2} - 2 = C \Rightarrow \boxed{C = -\frac{1}{2}}$$

$$\underline{y(x) = -\frac{1}{x} + \frac{1}{2}x^2 - \frac{1}{2} \mid}$$

Exercise

Find the function with the following property: $\frac{ds}{dt} = 1 + \cos t, \quad s(0) = 4$

Solution

$$\frac{ds}{dt} = 1 + \cos t \Rightarrow ds = (1 + \cos t) dt$$

$$\int ds = \int (1 + \cos t) dt$$

$$s = t + \sin t + C$$

$$4 = 0 + \sin(0) + C \Rightarrow \boxed{C = 4}$$

$$\underline{s(t) = t + \sin t + 4 \mid}$$

Exercise

Find the function with the following property: $\frac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1$

Solution

$$\frac{ds}{dt} = \cos t + \sin t \rightarrow ds = (\cos t + \sin t) dt$$

$$\int ds = \int (\cos t + \sin t) dt$$

$$s = \sin t - \cos t + C$$

$$1 = \sin \pi - \cos \pi + C$$

$$1 = 0 - (-1) + C$$

$$1 = 1 + C$$

$$\boxed{C = 0}$$

$$\underline{s(t) = \sin t - \cos t \mid}$$

Exercise

Find the function with the following property: $f'(x) = 3x^2 - 1$ & $f(0) = 10$

Solution

$$\begin{aligned} f(x) &= \int (3x^2 - 1) dx \\ &= x^3 - x + C \\ f(0) &= \underline{C=10} \\ f(x) &= \underline{x^3 - x + 10} \end{aligned}$$

Exercise

Find the function with the following property: $f'(t) = \sin t + 2t$ & $f(0) = 5$

Solution

$$\begin{aligned} f(t) &= \int (\sin t + 2t) dt \\ &= -\cos t + t^2 + C \\ f(0) &= -1 + C = 5 \rightarrow \underline{C=6} \\ f(t) &= \underline{-\cos t + t^2 + 6} \end{aligned}$$

Exercise

Find the function with the following property: $f'(x) = x^2 + x^{-2}$ & $f(1) = 1$

Solution

$$\begin{aligned} f(x) &= \int (x^2 + x^{-2}) dx \\ &= \frac{1}{3}x^3 - \frac{1}{x} + C \\ f(1) &= \frac{1}{3} - 1 + C = 1 \rightarrow \underline{C = \frac{5}{3}} \\ f(x) &= \underline{\frac{1}{3}x^3 - \frac{1}{x} + \frac{5}{3}} \end{aligned}$$

Exercise

Find the function with the following property: $f'(x) = \sin^2 x$ & $f(1) = 1$

Solution

$$f(x) = \int \sin^2 x \, dx \qquad \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$= \frac{1}{2} \int (1 - \cos 2x) \, dx$$

$$= \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + C$$

$$f(1) = \frac{1}{2} - \frac{1}{4} \sin 2 + C = 1 \rightarrow \underline{C = \frac{1}{2} + \frac{1}{4} \sin 2}$$

$$\underline{f(x) = \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + \frac{1}{2} + \frac{1}{4} \sin 2}$$

Exercise

Derive the position function if a ball is thrown upward with initial velocity of 32 *feet* per second from an initial height of 48 *feet*. When does the ball hit the ground? With what velocity does the ball hit the ground?

Solution

$$s(t) = -16t^2 + 32t + 48$$

$$s(0) = 48$$

$$s'(0) = 32$$

$$s''(t) = -32$$

$$s'(t) = \int -32 \, dt$$

$$= -32t + C_1$$

$$s'(0) = -32(0) + C_1 = 32$$

$$\Rightarrow C_1 = 32$$

$$s'(t) = -32t + 32$$

$$s(t) = \int (-32t + 32) \, dt$$

$$= -32 \frac{t^2}{2} + 32t + C_2$$

$$s(0) = -32 \frac{0^2}{2} + 32(0) + C_2 = 48 \Rightarrow C_2 = 48$$

$$s(t) = -16t^2 + 32t + 48$$

$$s(t) = -16t^2 + 32t + 48 = 0$$

$$-t^2 + 2t + 3 = 0 \Rightarrow t = -1, t = 3$$

The ball hits the ground in 3 seconds

The velocity: $v(t) = s'(t) = -32t + 32$

$$v(t = 3) = -32(3) + 32 = \underline{-64 \text{ ft / sec}^2}$$

Exercise

Suppose a publishing company has found that the marginal cost at a level of production of x thousand books is given by

$$\frac{dC}{dx} = \frac{50}{\sqrt{x}}$$

And that the fixed cost (the cost before the first book can be produced) is a \$25,000. Find the cost function $C(x)$.

Solution

$$\frac{dC}{dx} = \frac{50}{\sqrt{x}} = 50x^{-1/2}$$

$$dC = 50x^{-1/2} dx$$

$$\int dC = \int 50x^{-1/2} dx$$

$$\begin{aligned} C(x) &= 50 \frac{x^{1/2}}{1/2} + C \\ &= 50(2)x^{1/2} + C \\ &= 100\sqrt{x} + C \end{aligned}$$

$$25000 = 100\sqrt{0} + C$$

Before the first ($x = 0$) costs 25,000

$$\boxed{25000 = C}$$

$$\underline{C(x) = 100\sqrt{x} + 25,000}$$

Exercise

Find the general solution of $F'(x) = 4x + 2$, and find the particular solution that satisfies the initial condition $F(1) = 8$.

Solution

$$\begin{aligned}
 F(x) &= \int (4x + 2)dx \\
 &= 4 \frac{x^2}{2} + 2x + C \\
 &= 2x^2 + 2x + C
 \end{aligned}$$

$$\begin{aligned}
 F(x) &= 2(1)^2 + 2(1) + C = 8 \\
 2 + 2 + C &= 8 \\
 4 + C &= 8
 \end{aligned}$$

$$\boxed{C = 4} \quad \Rightarrow F(x) = 2x^2 + 2x + 4$$

Exercise

The marginal cost function for producing x units of a product is modeled by

$$\frac{dC}{dx} = 28 - 0.02x$$

It costs \$40 to produce one unit. Find the cost of producing 200 units.

Solution

$$\begin{aligned}
 C &= \int (28 - 0.02x)dx \\
 &= 28x - 0.02 \frac{x^2}{2} + K
 \end{aligned}$$

$$\text{Cost \$40 for one unit} \Rightarrow C(x=1) = 40$$

$$C(x=1) = 28(1) - 0.01(1)^2 + K = 40$$

$$K = 12.01$$

$$C(x) = -0.01x^2 + 28x + 12.01$$

$$\begin{aligned}
 C(200) &= -0.01(200)^2 + 28(200) + 12.01 \\
 &= \underline{\$5212.01}
 \end{aligned}$$

Solution **Section 4.2 – Area under Curves**

Exercise

Use finite approximations to estimate the area under the graph of the function using

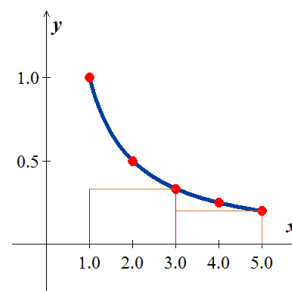
$$f(x) = \frac{1}{x} \quad \text{between } x=1 \quad \text{and } x=5$$

- a) A lower sum with two rectangles of equal width
- b) A lower sum with four rectangles of equal width
- c) An upper sum with two rectangles of equal width
- d) An upper sum with four rectangles of equal width

Solution

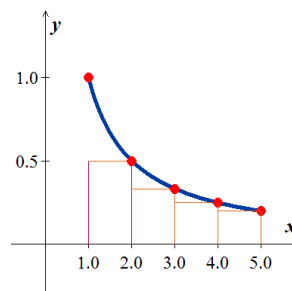
- a) Using 2 lower rectangles: $\Delta x = \frac{5-1}{2} = 2$

$$\begin{aligned} A &\approx \Delta x \left(f(x_1) + f(x_2) \right) \\ &\approx 2 \cdot \left(f(3) + f(5) \right) \\ &\approx 2 \cdot \left(\frac{1}{3} + \frac{1}{5} \right) \\ &\approx \frac{16}{15} \end{aligned}$$



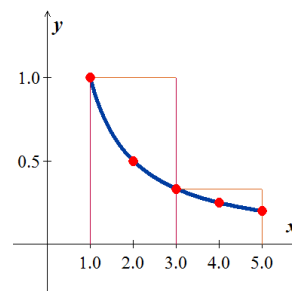
- b) Using 4 lower rectangles: $\Delta x = \frac{5-1}{4} = 1$

$$\begin{aligned} A &\approx 1 \cdot \left(f(2) + f(3) + f(4) + f(5) \right) \\ &\approx 1 \cdot \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \right) \\ &\approx \frac{77}{60} \end{aligned}$$



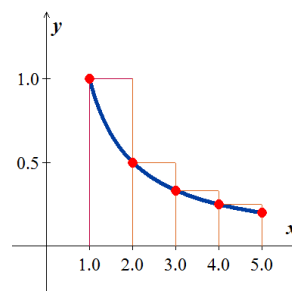
- c) Using 2 upper rectangles: $\Delta x = \frac{5-1}{2} = 2$

$$\begin{aligned} A &\approx 2 \cdot \left(f(1) + f(3) \right) \\ &\approx 2 \cdot \left(1 + \frac{1}{3} \right) \\ &\approx \frac{8}{3} \end{aligned}$$



- d) Using 4 upper rectangles: $\Delta x = \frac{5-1}{4} = 1$

$$\begin{aligned} A &\approx 1 \cdot \left(f(1) + f(2) + f(3) + f(4) \right) \\ &\approx 1 \cdot \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right) \\ &\approx \frac{25}{12} \end{aligned}$$



Exercise

Use finite approximations to estimate the area under the graph of the function using

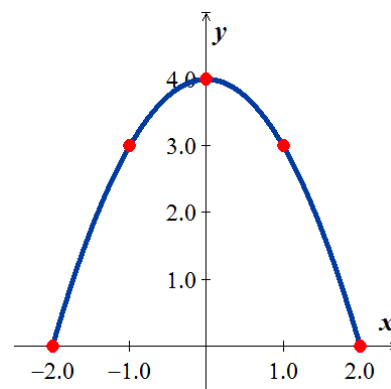
$$f(x) = 4 - x^2 \quad \text{between } x = -2 \quad \text{and} \quad x = 2$$

- a) A lower sum with two rectangles of equal width
- b) A lower sum with four rectangles of equal width
- c) An upper sum with two rectangles of equal width
- d) An upper sum with four rectangles of equal width

Solution

a) Using 2 lower rectangles: $\Delta x = \frac{2 - (-2)}{2} = 2$

$$\begin{aligned} A &\approx \Delta x (f(x_1) + f(x_2)) \\ &\approx 2 \cdot (f(-2) + f(2)) \\ &\approx 2 \cdot \left[(4 - (-2)^2) + (4 - 2^2) \right] \\ &= 0 \end{aligned}$$



b) Using 4 lower rectangles: $\Delta x = \frac{2 - (-2)}{4} = 1$

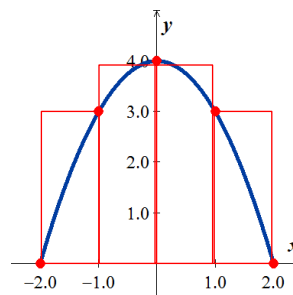
$$\begin{aligned} A &\approx 1 \cdot (f(-2) + f(-1) + f(1) + f(2)) \\ &\approx 1 \cdot (0 + 3 + 3 + 0) \\ &= 6 \end{aligned}$$

c) Using 2 upper rectangles: $\Delta x = \frac{2 - (-2)}{2} = 2$

$$\begin{aligned} A &\approx 2 \cdot (f(0) + f(0)) \\ &\approx 2 \cdot (4 + 4) \\ &= 16 \end{aligned}$$

d) Using 4 lower rectangles: $\Delta x = \frac{2 - (-2)}{4} = 1$

$$\begin{aligned} A &\approx 1 \cdot (f(-1) + f(0) + f(1) + f(2)) \\ &\approx 1 \cdot (3 + 4 + 4 + 3) \\ &= 14 \end{aligned}$$



Exercise

Use finite approximations to estimate the average value of f on the given interval by partitioning the interval into four subintervals of equal length and evaluating f at the subinterval midpoints.

$$f(t) = \frac{1}{2} + \sin^2 \pi t \quad \text{on} \quad [0, 2]$$

Solution

$$\Delta x = \frac{2-0}{4} = 0.5$$

$$f(t = .25) = \frac{1}{2} + \sin^2(.25\pi) = 1$$

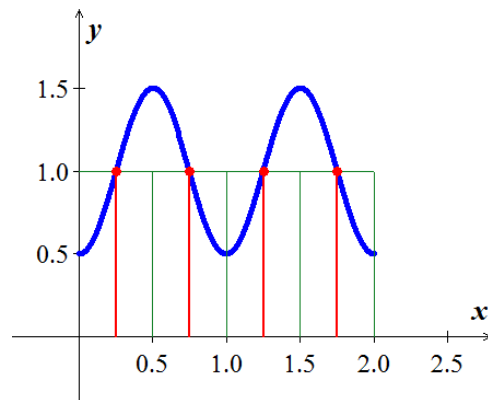
$$f(t = .75) = \frac{1}{2} + \sin^2(.75\pi) = 1$$

$$f(t = 1.25) = \frac{1}{2} + \sin^2(1.25\pi) = 1$$

$$f(t = 1.75) = \frac{1}{2} + \sin^2(1.75\pi) = 1$$

$$\begin{aligned} A &\approx .5 \cdot (f(.25) + f(.75) + f(1.25) + f(1.75)) \\ &= .5(1 + 1 + 1 + 1) \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{Average value} &\approx \frac{\text{Area}}{\text{Length } [0, 2]} \\ &= \frac{2}{2} \\ &= 1 \end{aligned}$$



Exercise

Use finite approximations to estimate the average value of f on the given interval by partitioning the interval into four subintervals of equal length and evaluating f at the subinterval midpoints.

$$f(t) = 1 - \left(\cos \frac{\pi t}{4}\right)^4 \quad \text{on} \quad [0, 4]$$

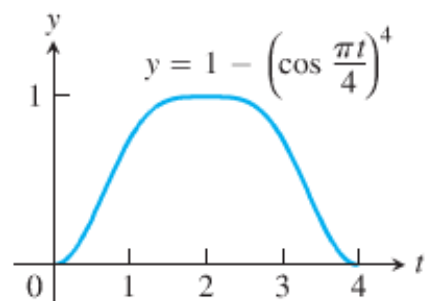
Solution

$$\Delta x = \frac{4-0}{4} = 1$$

$$f(t = 0.5) = 1 - \left(\cos \frac{0.5\pi}{4}\right)^4 = 0.27145$$

$$f(t = 1.5) = 1 - \left(\cos \frac{1.5\pi}{4}\right)^4 = 0.97855$$

$$f(t = 2.5) = 1 - \left(\cos \frac{2.5\pi}{4}\right)^4 = 0.97855$$



$$f(t = 3.5) = 1 - \left(\cos \frac{3.5\pi}{4}\right)^4 = 0.27145$$

$$\begin{aligned} A &\approx 1 \cdot (f(.5) + f(1.5) + f(2.5) + f(3.5)) \\ &= 1(0.27145 + 0.97855 + 0.97855 + 0.27145) \\ &= \underline{2.5} \end{aligned}$$

$$\begin{aligned} \text{Average value} &\approx \frac{\text{Area}}{\text{Length } [0, 2]} \\ &= \frac{2.5}{4} \\ &= \underline{0.625} \end{aligned}$$

Exercise

Write the sums without sigma notation. Then evaluate:

$$\sum_{k=1}^2 \frac{6k}{k+1}$$

Solution

$$\sum_{k=1}^2 \frac{6k}{k+1} = \frac{6}{2} + \frac{12}{3} = \underline{7}$$

Exercise

Write the sums without sigma notation. Then evaluate:

$$\sum_{k=1}^3 \frac{k-1}{k}$$

Solution

$$\sum_{k=1}^3 \frac{k-1}{k} = \frac{0}{1} + \frac{1}{2} + \frac{2}{3} = \underline{\frac{7}{6}}$$

Exercise

Write the sums without sigma notation. Then evaluate:

$$\sum_{k=1}^5 \sin k\pi$$

Solution

$$\sum_{k=1}^5 \sin k\pi = \sin \pi + \sin 2\pi + \sin 3\pi + \sin 4\pi + \sin 5\pi = 0 + 0 + 0 + 0 + 0 = \underline{0}$$

Exercise

Write the sums without sigma notation. Then evaluate: $\sum_{k=1}^4 (-1)^k \cos k\pi$

Solution

$$\sum_{k=1}^4 (-1)^k \cos k\pi = -\cos \pi + \cos 2\pi - \cos 3\pi + \cos 4\pi = -(-1) + 1 - (-1) + 1 = \underline{4}$$

Exercise

Write the following expression $1 + 2 + 4 + 8 + 16 + 32$ in sigma notation

Solution

$$1 + 2 + 4 + 8 + 16 + 32 = \sum_{k=1}^6 2^{k-1}$$

$$1 + 2 + 4 + 8 + 16 + 32 = \sum_{k=0}^5 2^k$$

Exercise

Write the following expression $1 - 2 + 4 - 8 + 16 - 32$ in sigma notation

Solution

$$1 - 2 + 4 - 8 + 16 - 32 = \sum_{k=1}^6 (-2)^{k-1}$$

$$1 - 2 + 4 - 8 + 16 - 32 = \sum_{k=0}^5 (-1)^k 2^k$$

Exercise

Write the following expression $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ in sigma notation

Solution

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = \sum_{k=1}^4 \frac{1}{2^k}$$

Exercise

Write the following expression $-\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5}$ in sigma notation

Solution

$$-\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5} = \sum_{k=1}^5 (-1)^k \frac{k}{5}$$

Exercise

Suppose that $\sum_{k=1}^n a_k = -5$ and $\sum_{k=1}^n b_k = 6$. Find the value of $\sum_{k=1}^n (b_k - 2a_k)$

Solution

$$\begin{aligned} \sum_{k=1}^n (b_k - 2a_k) &= \sum_{k=1}^n b_k - 2 \sum_{k=1}^n a_k \\ &= 6 - 2(-5) \\ &= 16 \end{aligned}$$

Exercise

Evaluate the sums $\sum_{k=1}^{10} k^3$

Solution

$$\begin{aligned} \sum_{k=1}^{10} k^3 &= \left(\frac{10(10+1)}{2} \right)^2 \\ &= 55^2 \\ &= 3025 \end{aligned}$$

Exercise

Evaluate the sums $\sum_{k=1}^7 (-2k)$

Solution

$$\begin{aligned}
 \sum_{k=1}^7 (-2k) &= -2 \sum_{k=1}^7 k \\
 &= -2 \left(\frac{7(7+1)}{2} \right) \\
 &= \underline{-56}
 \end{aligned}$$

Exercise

Evaluate the sums $\sum_{k=1}^5 \frac{\pi k}{15}$

Solution

$$\begin{aligned}
 \sum_{k=1}^5 \frac{\pi k}{15} &= \frac{\pi}{15} \sum_{k=1}^5 k \\
 &= \frac{\pi}{15} \left(\frac{5(5+1)}{2} \right) \\
 &= \underline{\pi}
 \end{aligned}$$

Exercise

Evaluate the sums $\sum_{k=1}^5 k(3k+5)$

Solution

$$\begin{aligned}
 \sum_{k=1}^5 k(3k+5) &= \sum_{k=1}^5 (3k^2 + 5k) \\
 &= 3 \sum_{k=1}^5 k^2 + 5 \sum_{k=1}^5 k \\
 &= 3 \left(\frac{5(5+1)(2(5)+1)}{6} \right) + 5 \frac{5(5+1)}{2} \\
 &= \underline{240}
 \end{aligned}$$

Exercise

Evaluate the sums $\sum_{k=1}^5 \frac{k^3}{225} + \left(\sum_{k=1}^5 k \right)^3$

Solution

$$\begin{aligned} \sum_{k=1}^5 \frac{k^3}{225} + \left(\sum_{k=1}^5 k \right)^3 &= \frac{1}{225} \sum_{k=1}^5 k^3 + \left(\sum_{k=1}^5 k \right)^3 \\ &= \frac{1}{225} \left(\frac{5(5+1)}{2} \right)^2 + \left(\frac{5(5+1)}{2} \right)^3 \\ &= \underline{3376} \end{aligned}$$

Exercise

Evaluate the sums $\sum_{k=1}^{500} 7$

Solution

$$\sum_{k=1}^{500} 7 = 7(500) = \underline{3500}$$

Exercise

Evaluate the sums $\sum_{k=18}^{71} k(k-1)$

Solution

$$\text{Let } n = (k-18) + 1 = k-17 \begin{cases} k=18 & \rightarrow n=1 \\ k=71 & \rightarrow n=54 \end{cases} \Rightarrow k = n+17$$

$$\begin{aligned} \sum_{k=18}^{71} k(k-1) &= \sum_{n=1}^{54} (n+17)(n+17-1) \\ &= \sum_{n=1}^{54} (n+17)(n+16) \\ &= \sum_{n=1}^{54} (n^2 + 33n + 272) \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{54} n^2 + 33 \sum_{n=1}^{54} n + \sum_{n=1}^{54} 272 \\
&= \frac{54(54+1)(54(2)+1)}{6} + 33 \cdot \frac{54(54+1)}{6} + 272(54) \\
&= \underline{117648}
\end{aligned}$$

Exercise

Evaluate the sums $\sum_{k=1}^n \left(\frac{1}{n} + 2n \right)$

Solution

$$\sum_{k=1}^n \left(\frac{1}{n} + 2n \right) = n \cdot \left(\frac{1}{n} + 2n \right) = \underline{1 + 2n^2}$$

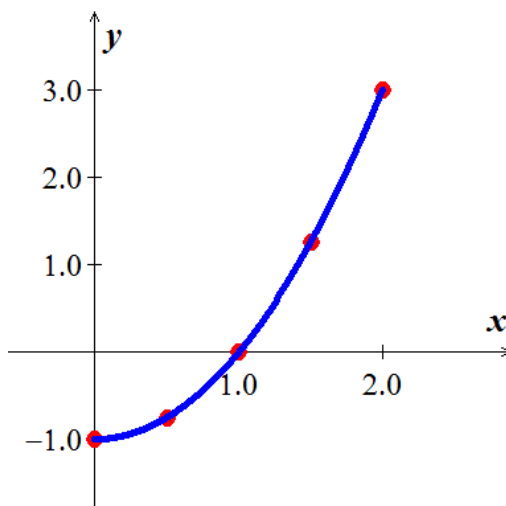
Exercise

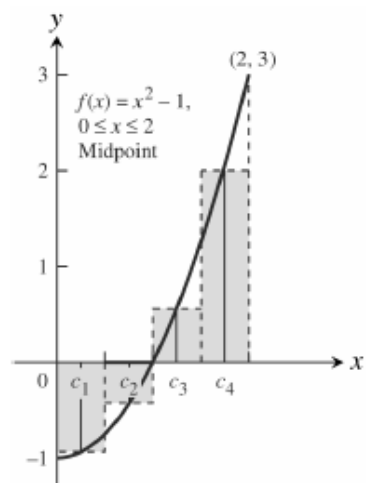
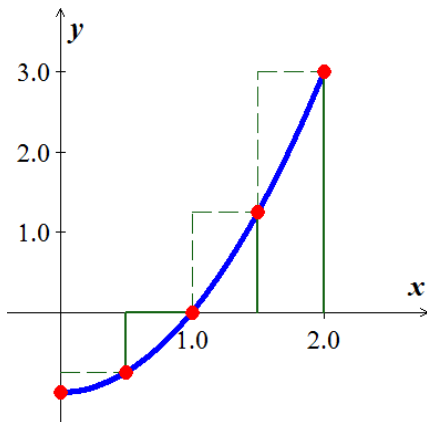
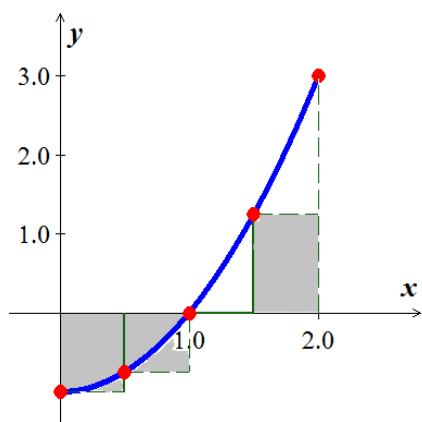
Graph the function $f(x) = x^2 - 1$ over the given interval $[0, 2]$. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum

$$\sum_{k=1}^4 f(c_k) \Delta x_k, \text{ given } c_k \text{ is the}$$

- a) Left-hand endpoint
- b) Right-hand endpoint
- c) Midpoint of k^{th} subinterval.

Solution





Solution

Section 4.4 – Fundamental Theorem of Calculus

Exercise

Evaluate the integral $\int_0^3 (2x+1)dx$

Solution

$$\begin{aligned}\int_0^3 (2x+1)dx &= x^2 + x \Big|_0^3 \\ &= 3^2 + 3 - (0+0) \\ &= \underline{12}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^2 x(x-3)dx$

Solution

$$\begin{aligned}\int_0^2 x(x-3)dx &= \int_0^2 (x^2 - 3x)dx \\ &= \left[\frac{x^3}{3} - \frac{3x^2}{2} \right]_0^2 \\ &= \left(\frac{2^3}{3} - \frac{3(2)^2}{2} \right) - \left(\frac{0^3}{3} - \frac{3(2)^2}{2} \right) \\ &= \underline{-\frac{10}{3}}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^4 \left(3x - \frac{x^3}{4} \right) dx$

Solution

$$\int_0^4 \left(3x - \frac{x^3}{4} \right) dx = \left[3\frac{x^2}{2} - \frac{x^4}{16} \right]_0^4$$

$$= \left(3 \frac{(4)^2}{2} - \frac{(4)^4}{16} \right) - 0$$

$$= 8$$

Exercise

Evaluate the integral $\int_{-2}^2 (x^3 - 2x + 3) dx$

Solution

$$\int_{-2}^2 (x^3 - 2x + 3) dx = \left[\frac{x^4}{4} - x^2 + 3x \right]_{-2}^2$$

$$= \left(\frac{(2)^4}{4} - (2)^2 + 3(2) \right) - \left(\frac{(-2)^4}{4} - (-2)^2 + 3(-2) \right)$$

$$= 12$$

Exercise

Evaluate the integral $\int_0^1 (x^2 + \sqrt{x}) dx$

Solution

$$\int_0^1 (x^2 + \sqrt{x}) dx = \left[\frac{x^3}{3} + \frac{2}{3} x^{3/2} \right]_0^1$$

$$= \left(\frac{(1)^3}{3} + \frac{2}{3} (1)^{3/2} \right) - 0$$

$$= 1$$

Exercise

Evaluate the integral $\int_0^{\pi/3} 4 \sec u \tan u \, du$

Solution

$$\int_0^{\pi/3} 4 \sec u \tan u \, du = 4 \sec u \Big|_0^{\pi/3}$$

$$\begin{aligned}
&= 4\left(\sec \frac{\pi}{3} - \sec 0\right) \\
&= 4(2 - 1) \\
&= 4
\end{aligned}$$

Exercise

Evaluate the integral $\int_{\pi/4}^{3\pi/4} \csc \theta \cot \theta d\theta$

Solution

$$\begin{aligned}
\int_{\pi/4}^{3\pi/4} \csc \theta \cot \theta d\theta &= -\csc \theta \Big|_{\pi/4}^{3\pi/4} \\
&= -\left(\csc \frac{3\pi}{4} - \csc \frac{\pi}{4}\right) \\
&= -(\sqrt{2} - \sqrt{2}) \\
&= 0
\end{aligned}$$

Exercise

Evaluate the integral $\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt$

Solution

$$\begin{aligned}
\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt &= \int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \pi t^{-2}\right) dt \\
&= \left[4 \tan t - \pi t^{-1}\right]_{-\pi/3}^{-\pi/4} \\
&= \left(4 \tan\left(-\frac{\pi}{4}\right) - \pi\left(-\frac{4}{\pi}\right)\right) - \left(4 \tan\left(-\frac{\pi}{3}\right) - \pi\left(-\frac{3}{\pi}\right)\right) \\
&= (4(-1) + 4) - (4(-\sqrt{3}) + 3) \\
&= -(-4\sqrt{3} + 3) \\
&= 4\sqrt{3} - 3
\end{aligned}$$

Exercise

Evaluate the integral $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy$

Solution

$$\begin{aligned}\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy &= \int_{-3}^{-1} \left(\frac{y^5}{y^3} - \frac{2y}{y^3} \right) dy \\&= \int_{-3}^{-1} (y^2 - 2y^{-2}) dy \\&= \left[\frac{1}{3} y^3 + 2y^{-1} \right]_{-3}^{-1} \\&= \left(\frac{1}{3} (-1)^3 + \frac{2}{-1} \right) - \left(\frac{1}{3} (-3)^3 + \frac{2}{-3} \right) \\&= \frac{22}{3}\end{aligned}$$

Exercise

Evaluate the integral $\int_1^8 \frac{(x^{1/3} + 1)(2 - x^{2/3})}{x^{1/3}} dx$

Solution

$$\begin{aligned}\int_1^8 \frac{(x^{1/3} + 1)(2 - x^{2/3})}{x^{1/3}} dx &= \int_1^8 \frac{2x^{1/3} - x + 2 - x^{2/3}}{x^{1/3}} dx \\&= \int_1^8 (2 - x^{2/3} + 2x^{-1/3} - x^{1/3}) dx \\&= \left[2x - \frac{3}{5} x^{5/3} + 3x^{2/3} - \frac{3}{4} x^{4/3} \right]_1^8 \\&= \left(2(8) - \frac{3}{5} (8)^{5/3} + 3(8)^{2/3} - \frac{3}{4} (8)^{4/3} \right) - \left(2(1) - \frac{3}{5} (1)^{5/3} + 3(1)^{2/3} - \frac{3}{4} (1)^{4/3} \right) \\&= \left(-\frac{16}{5} \right) - \left(\frac{73}{20} \right) \\&= -\frac{137}{20}\end{aligned}$$

Exercise

Evaluate the integral $\int_{\pi/2}^{\pi} \frac{\sin 2x}{2 \sin x} dx$

Solution

$$\begin{aligned}\int_{\pi/2}^{\pi} \frac{\sin 2x}{2 \sin x} dx &= \int_{\pi/2}^{\pi} \frac{2 \sin x \cos x}{2 \sin x} dx \\&= \int_{\pi/2}^{\pi} \cos x dx \\&= \sin x \Big|_{\pi/2}^{\pi} \\&= \sin \pi - \sin \frac{\pi}{2} \\&= -1\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/3} (\cos x + \sec x)^2 dx$

Solution

$$\begin{aligned}\int_0^{\pi/3} (\cos x + \sec x)^2 dx &= \int_0^{\pi/3} (\cos^2 x + 2 + \sec^2 x) dx \\&= \int_0^{\pi/3} \left(\frac{1}{2} + \frac{1}{2} \cos 2x + 2 + \sec^2 x \right) dx \\&= \int_0^{\pi/3} \left(\frac{5}{2} + \frac{1}{2} \cos 2x + \sec^2 x \right) dx \\&= \left[\frac{5}{2}x + \frac{1}{4} \sin 2x + \tan x \right]_0^{\pi/3} \\&= \left(\frac{5}{2} \frac{\pi}{3} + \frac{1}{4} \sin \frac{2\pi}{3} + \tan \frac{\pi}{3} \right) - \left(\frac{5}{2}(0) + \frac{1}{4} \sin(2 \cdot 0) + \tan(0) \right) \\&= \frac{5\pi}{6} + \frac{1}{4} \frac{\sqrt{3}}{2} + \sqrt{3} \\&= \frac{5\pi}{6} + \frac{9\sqrt{3}}{8}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi} \frac{1}{2}(\cos x + |\cos x|) dx$

Solution

$$\begin{aligned}\int_0^{\pi} \frac{1}{2}(\cos x + |\cos x|) dx &= \int_0^{\pi/2} \frac{1}{2}(\cos x + \cos x) dx + \int_{\pi/2}^{\pi} \frac{1}{2}(\cos x - \cos x) dx \\ &= \int_0^{\pi/2} \cos x dx \\ &= \sin x \Big|_0^{\pi/2} \\ &= \underline{1}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 2x(4 - x^2) dx$

Solution

$$\begin{aligned}\int_0^1 2x(4 - x^2) dx &= \int_0^1 (8x - 2x^3) dx \\ &= 4x^2 - \frac{1}{2}x^4 \Big|_0^1 \\ &= 4 - \frac{1}{2} \\ &= \underline{\frac{7}{2}}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^4 (8 - 2x) dx$

Solution

$$\begin{aligned}\int_0^4 (8 - 2x) dx &= 8x - x^2 \Big|_0^4 \\ &= 8(4) - (4)^2 - \mathbf{0} \\ &= \underline{16}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^4 \frac{1}{\sqrt{16-x^2}} dx$

Solution

$$\begin{aligned}
 \int_0^4 \frac{1}{\sqrt{16-x^2}} dx &= \sin^{-1} \frac{x}{4} \Big|_0^4 \\
 &= \sin^{-1} \frac{4}{4} - \sin^{-1} 0 \\
 &= \frac{\pi}{2}
 \end{aligned}
 \qquad \sin^{-1} 1 = \frac{\pi}{2}$$

Exercise

Evaluate the integral $\int_{-4}^2 (2x+4) dx$

Solution

$$\begin{aligned}
 \int_{-4}^2 (2x+4) dx &= x^2 + 4x \Big|_{-4}^2 \\
 &= 2^2 + 4(2) - \left((-4)^2 + 4(-4) \right) \\
 &= 4 + 8 - (16 - 16) \\
 &= 12
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^2 (1-x) dx$

Solution

$$\begin{aligned}
 \int_0^2 (1-x) dx &= x - \frac{1}{2}x^2 \Big|_0^2 \\
 &= 2 - \frac{1}{2}(2)^2 - 0 \\
 &= 0
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^2 (x^2 - 2) dx$

Solution

$$\begin{aligned}\int_0^2 (x^2 - 2) dx &= \left. \frac{1}{3}x^3 - 2x \right|_0^2 \\ &= \frac{1}{3}(2)^3 - 2(2) - 0 \\ &= \frac{8}{3} - 4 \\ &= \underline{-\frac{4}{3}}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/2} \cos x \, dx$

Solution

$$\begin{aligned}\int_0^{\pi/2} \cos x \, dx &= \left. \sin x \right|_0^{\pi/2} \\ &= \sin \frac{\pi}{2} - \sin 0 \\ &= \underline{1}\end{aligned}$$

Exercise

Evaluate the integral $\int_1^7 \frac{dx}{x}$

Solution

$$\begin{aligned}\int_1^7 \frac{dx}{x} &= \left. \ln|x| \right|_1^7 \\ &= \ln 7 - \ln 1 \\ &= \underline{\ln 7}\end{aligned}$$

Exercise

Evaluate the integral $\int_4^9 3\sqrt{x} \, dx$

Solution

$$\begin{aligned}
 \int_4^9 3\sqrt{x} \, dx &= 2x^{3/2} \Big|_4^9 \\
 &= 2\left((9)^{3/2} - (4)^{3/2}\right) \\
 &= 2(27 - 8) \\
 &= \underline{38}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_{-2}^3 (x^2 - x - 6) dx$

Solution

$$\begin{aligned}
 \int_{-2}^3 (x^2 - x - 6) dx &= \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x \Big|_{-2}^3 \\
 &= \frac{1}{3}3^3 - \frac{1}{2}3^2 - 6(3) - \left(\frac{1}{3}2^3 - \frac{1}{2}2^2 - 6(2)\right) \\
 &= 9 - \frac{9}{2} - 18 - \left(\frac{8}{3} - 2 - 12\right) \\
 &= -\frac{27}{2} + \frac{46}{3} \\
 &= \underline{\frac{11}{6}}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 (1 - \sqrt{x}) dx$

Solution

$$\begin{aligned}
 \int_0^1 (1 - \sqrt{x}) dx &= \int_0^1 (1 - x^{1/2}) dx \\
 &= x - \frac{2}{3}x^{3/2} \Big|_0^1 \\
 &= 1 - \frac{2}{3} \\
 &= \underline{\frac{1}{3}}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/4} 2 \cos x \, dx$

Solution

$$\begin{aligned}\int_0^{\pi/4} 2 \cos x \, dx &= 2 \sin x \Big|_0^{\pi/4} \\ &= 2 \left(\sin \frac{\pi}{4} - \sin 0 \right) \\ &= \sqrt{2}\end{aligned}$$

Exercise

Evaluate the integral $\int_{-\pi/4}^{7\pi/4} (\sin x + \cos x) \, dx$

Solution

$$\begin{aligned}\int_{-\pi/4}^{7\pi/4} (\sin x + \cos x) \, dx &= -\cos x + \sin x \Big|_{-\pi/4}^{7\pi/4} \\ &= -\cos\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right) - \left(-\cos\left(\frac{7\pi}{4}\right) + \sin\left(\frac{7\pi}{4}\right)\right) \\ &= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) \\ &= -\sqrt{2} + \sqrt{2} \\ &= 0\end{aligned}$$

or since $-\frac{\pi}{4} = \frac{7\pi}{4}$ $\int_a^a f(x) \, dx = 0$

Exercise

Evaluate the integral $\int_0^{\ln 8} e^x \, dx$

Solution

$$\begin{aligned}\int_0^{\ln 8} e^x \, dx &= e^x \Big|_0^{\ln 8} \\ &= e^{\ln 8} - e^0 \\ &= 8 - 1 \\ &= 7\end{aligned}$$

Exercise

Evaluate the integral $\int_1^4 \left(\frac{x-1}{x} \right) dx$

Solution

$$\begin{aligned}
 \int_1^4 \left(\frac{x-1}{x} \right) dx &= \int_1^4 \left(1 - \frac{1}{x} \right) dx \\
 &= x - \ln|x| \Big|_1^4 \\
 &= 4 - \ln 4 - (1 - \ln 1) \\
 &= 4 - \ln 2^2 - 1 \\
 &= \underline{3 - 2\ln 2}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_{-2}^{-1} \left(3e^{3x} + \frac{2}{x} \right) dx$

Solution

$$\begin{aligned}
 \int_{-2}^{-1} \left(3e^{3x} + \frac{2}{x} \right) dx &= e^{3x} + 2\ln|x| \Big|_{-2}^{-1} \\
 &= e^{-3} + 2\ln|-1| - \left(e^{-6} - 2\ln|-2| \right) \\
 &= e^{-3} + 2\ln 1 - e^{-6} + 2\ln 2 \\
 &= \underline{e^{-3} - e^{-6} + 2\ln 2}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^2 \frac{dx}{x^2 + 4}$

Solution

$$\begin{aligned}
 \int_0^2 \frac{dx}{x^2 + 4} &= \frac{1}{2} \tan^{-1} \frac{x}{2} \Big|_0^2 \\
 &= \frac{1}{2} \left(\tan^{-1} 1 - \tan^{-1} 0 \right) \\
 &= \frac{1}{2} \left(\frac{\pi}{4} - 0 \right) \\
 &= \underline{\frac{\pi}{8}}
 \end{aligned}$$

Exercise

Find the total area between the region and the x -axis $y = -x^2 - 2x, \quad -3 \leq x \leq 2$

Solution

$$-x^2 - 2x = 0 \rightarrow -x(x+2) = 0 \Rightarrow \boxed{x = -2, 0}$$

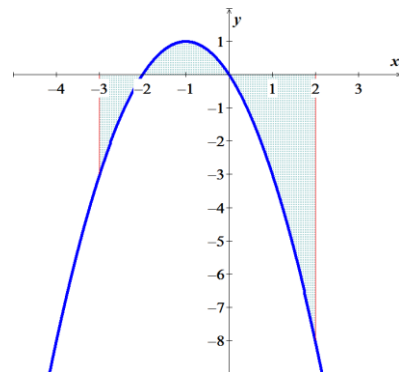
$$A = - \int_{-3}^{-2} (-x^2 - 2x) dx + \int_{-2}^0 (-x^2 - 2x) dx - \int_0^2 (-x^2 - 2x) dx$$

$$= - \left[-\frac{1}{3}x^3 - x^2 \right]_{-3}^{-2} + \left[-\frac{1}{3}x^3 - x^2 \right]_{-2}^0 - \left[-\frac{1}{3}x^3 - x^2 \right]_0^2$$

$$= - \left[\left(-\frac{1}{3}(-2)^3 - (-2)^2 \right) - \left(-\frac{1}{3}(-3)^3 - (-3)^2 \right) \right] + \left[-\left(-\frac{1}{3}(-2)^3 - (-2)^2 \right) \right] - \left[\left(-\frac{1}{3}(2)^3 - (2)^2 \right) \right]$$

$$= \frac{4}{3} + \frac{4}{3} + \frac{20}{3}$$

$$= \frac{28}{3} \text{ unit}^2$$



Exercise

Find the total area between the region and the x -axis $y = x^3 - 3x^2 + 2x, \quad 0 \leq x \leq 2$

Solution

$$x^3 - 3x^2 + 2x = 0 \quad \boxed{x = 0, 1, 2}$$

$$A = \int_0^1 (x^3 - 3x^2 + 2x) dx - \int_1^2 (x^3 - 3x^2 + 2x) dx$$

$$= \left[\frac{1}{4}x^4 - x^3 + x^2 \right]_0^1 - \left[\frac{1}{4}x^4 - x^3 + x^2 \right]_1^2$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2} \text{ unit}^2$$

Exercise

Find the total area between the region and the x -axis $y = x^{1/3} - x, \quad -1 \leq x \leq 8$

Solution

$$x^{1/3} - x = 0 \rightarrow x^{1/3}(1 - x^{2/3}) = 0 \quad \boxed{x = 0, \pm 1}$$

$$\begin{aligned}
A &= -\int_{-1}^0 (x^{1/3} - x) dx + \int_0^1 (x^{1/3} - x) dx - \int_1^8 (x^{1/3} - x) dx \\
&= -\left[\frac{3}{4}x^{4/3} - \frac{1}{2}x^2\right]_{-1}^0 + \left[\frac{3}{4}x^{4/3} - \frac{1}{2}x^2\right]_0^1 - \left[\frac{3}{4}x^{4/3} - \frac{1}{2}x^2\right]_1^8 \\
&= \left(\frac{3}{4} - \frac{1}{2}\right) + \left(\frac{3}{4} - \frac{1}{2}\right) - \left[(12 - 32) - \left(\frac{3}{4} - \frac{1}{2}\right)\right] \\
&= \frac{1}{4} + \frac{1}{4} + \frac{81}{4} \\
&= \frac{83}{4} \text{ unit}^2
\end{aligned}$$

Exercise

Find the total area between the region and the x -axis $f(x) = x^2 + 1$, $2 \leq x \leq 3$

Solution

$$\begin{aligned}
\text{Area} &= \int_2^3 (x^2 + 1) dx \\
&= \left[\frac{1}{3}x^3 + x\right]_2^3 \\
&= \left(\frac{1}{3}3^3 + 3\right) - \left(\frac{1}{3}2^3 + 2\right) \\
&= (9 + 3) - \left(\frac{8}{3} + 2\right) \\
&= 12 - \left(\frac{14}{3}\right) \\
&= \frac{22}{3} \text{ unit}^2
\end{aligned}$$

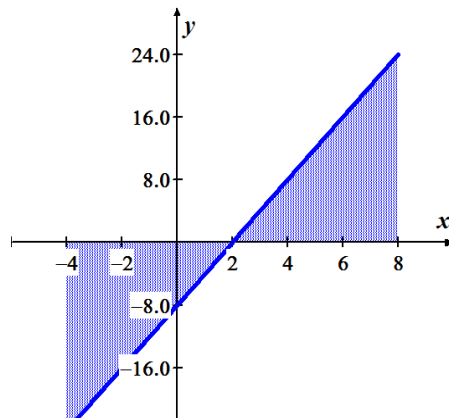
Exercise

Find the area of the region between the graph of $y = 4x - 8$ and the x -axis, for $-4 \leq x \leq 8$

Solution

$$y = 4x - 8 = 0 \Rightarrow x = 2$$

$$\begin{aligned}
\text{Area} &= \int_{-4}^2 (4x - 8) dx + \int_2^8 (4x - 8) dx \\
&= \left|2x^2 - 8x\right|_{-4}^2 + \left|2x^2 - 8x\right|_2^8 \\
&= |8 - 16 - (32 + 32)| + (128 - 64 - (8 - 16))
\end{aligned}$$



$$= |-72| + 72$$

$$= 144 \text{ unit}^2$$

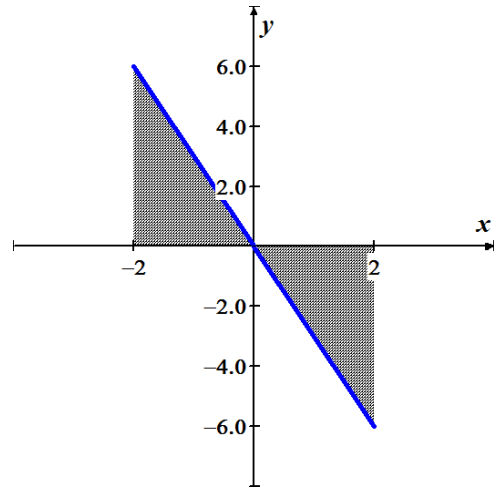
Exercise

Find the area of the region between the graph of $y = -3x$ and the x -axis, for $-2 \leq x \leq 2$

Solution

$$y = -3x = 0 \rightarrow x = 0$$

$$\begin{aligned} \text{Area} &= \int_{-2}^0 (-3x) dx + \int_0^2 (-3x) dx \\ &= 2 \int_{-2}^0 (-3x) dx \\ &= 3 \left(-x^2 \right) \Big|_{-2}^0 \\ &= 3(0 + 4) \\ &= 12 \text{ unit}^2 \end{aligned}$$



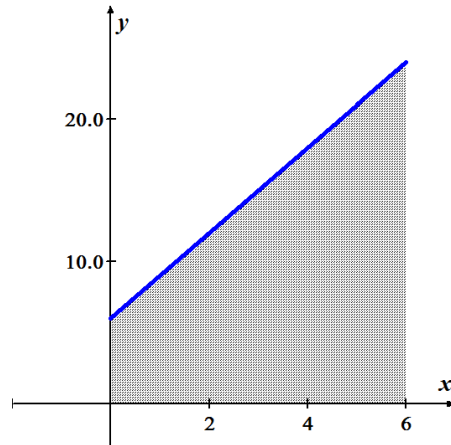
Exercise

Find the area of the region between the graph of $y = 3x + 6$ and the x -axis, for $0 \leq x \leq 6$

Solution

$$y = 3x + 6 = 0 \rightarrow x = -2$$

$$\begin{aligned} \text{Area} &= \int_0^6 (3x + 6) dx \\ &= \frac{3}{2}x^2 + 6x \Big|_0^6 \\ &= \frac{3}{2}(36) + 36 - 0 \\ &= 90 \text{ unit}^2 \end{aligned}$$



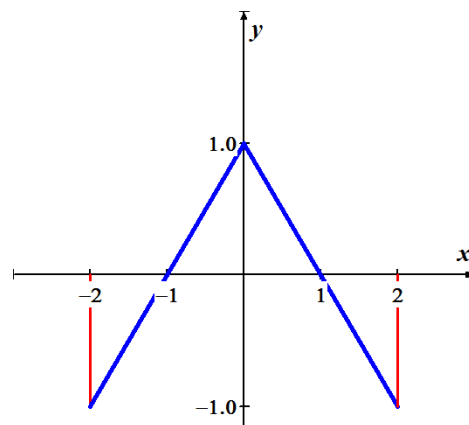
Exercise

Find the area of the region between the graph of $y = 1 - |x|$ and the x -axis, for $-2 \leq x \leq 2$

Solution

$$y = 1 - x = 0 \rightarrow x = 1$$

$$\begin{aligned}
 \text{Area} &= 2 \int_0^1 (1-x) dx + 2 \int_1^2 (1-x) dx \\
 &= 2 \left(x - \frac{1}{2} x^2 \right) \Big|_0^1 + 2 \left(x - \frac{1}{2} x^2 \right) \Big|_1^2 \\
 &= 4 \left(1 - \frac{1}{2} \right) \\
 &= \underline{2 \text{ unit}^2}
 \end{aligned}$$



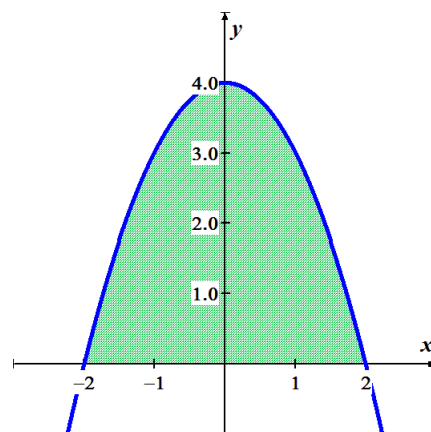
Exercise

Find the area of the region above the x -axis bounded by $y = 4 - x^2$

Solution

$$y = 4 - x^2 = 0 \Rightarrow \underline{x = \pm 2}$$

$$\begin{aligned}
 \text{Area} &= \int_{-2}^2 (4 - x^2) dx \\
 &= 4x - \frac{1}{3} x^3 \Big|_{-2}^2 \\
 &= 8 - \frac{8}{3} - \left(-8 + \frac{8}{3} \right) \\
 &= 2 \left(\frac{16}{3} \right) \\
 &= \underline{\frac{32}{3} \text{ unit}^2}
 \end{aligned}$$



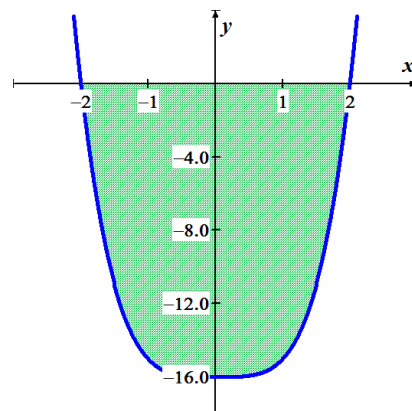
Exercise

Find the area of the region above the x -axis bounded by $y = x^4 - 16$

Solution

$$y = x^4 - 16 = 0 \Rightarrow \underline{x = \pm 2}$$

$$\begin{aligned}
 \text{Area} &= \left| \int_{-2}^2 (x^4 - 16) dx \right| = - \int_{-2}^2 (x^4 - 16) dx \\
 &= - \frac{1}{5} x^5 + 16x \Big|_{-2}^2 \\
 &= - \frac{32}{5} + 32 - \left(\frac{32}{5} - 32 \right) \\
 &= \underline{\frac{256}{5} \text{ unit}^2}
 \end{aligned}$$



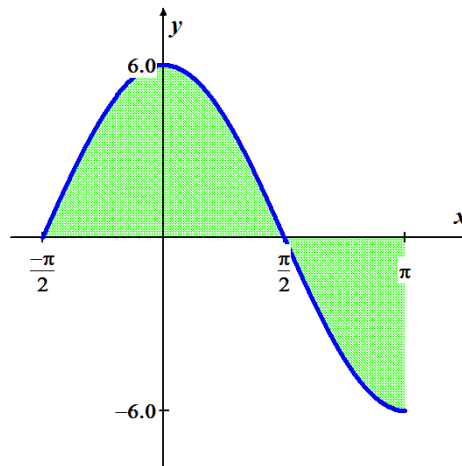
Exercise

Find the area of the region between the graph of $y = 6\cos x$ and the x -axis, for $-\frac{\pi}{2} \leq x \leq \pi$

Solution

$$y = 6\cos x = 0 \Rightarrow x = \pm \frac{\pi}{2}$$

$$\begin{aligned} \text{Area} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (6\cos x) dx + \int_{\frac{\pi}{2}}^{\pi} (-6\cos x) dx \\ &= 6\sin x \Big|_{-\pi/2}^{\pi/2} - 6\sin x \Big|_{\pi/2}^{\pi} \\ &= 6\left(\sin \frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right) - 6\left(\sin \pi - \sin\left(\frac{\pi}{2}\right)\right) \\ &= 6(1+1) - 6(0-1) \\ &= 18 \text{ unit}^2 \end{aligned}$$

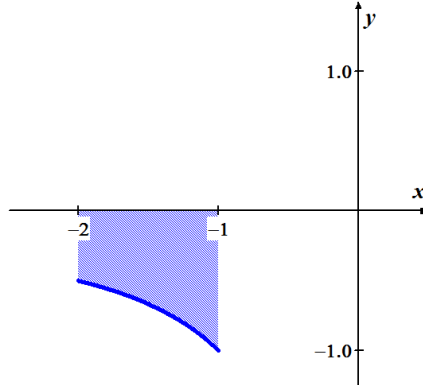


Exercise

Find the area of the region between the graph of $f(x) = \frac{1}{x}$ and the x -axis, for $-2 \leq x \leq -1$

Solution

$$\begin{aligned} \text{Area} &= -\int_{-2}^{-1} \frac{1}{x} dx \\ &= -\ln|x| \Big|_{-2}^{-1} \\ &= -\ln|-1| + \ln|-2| \\ &= -\ln 1 + \ln 2 \\ &= \ln 2 \text{ unit}^2 \end{aligned}$$



Exercise

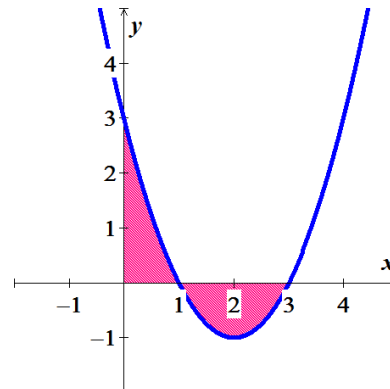
Find the area of the region bounded by the graph of

$$f(x) = x^2 - 4x + 3 \quad x\text{-axis on } 0 \leq x \leq 3$$

Solution

$$f(x) = x^2 - 4x + 3 = 0 \rightarrow x = 1, 3$$

$$A = \int_0^1 (x^2 - 4x + 3) dx - \int_1^3 (x^2 - 4x + 3) dx$$



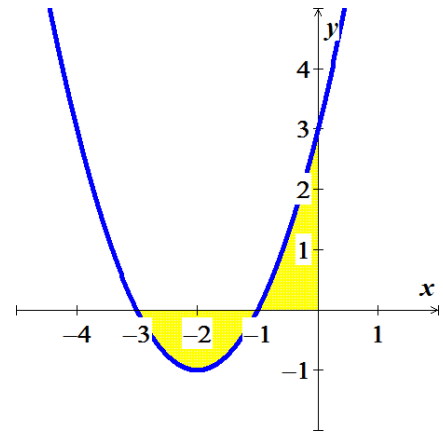
$$\begin{aligned}
&= \frac{1}{3}x^3 - 2x^2 + 3x \Big|_0^1 - \left(\frac{1}{3}x^3 - 2x^2 + 3x \right) \Big|_1^3 \\
&= \frac{1}{3} - 2 + 3 - \left(9 - 18 + 9 - \frac{1}{3} + 2 - 3 \right) \\
&= \underline{\underline{\frac{8}{3} \text{ unit}^2}}
\end{aligned}$$

Exercise

Find the area of the region bounded by the graph of $f(x) = x^2 + 4x + 3$ x -axis on $-3 \leq x \leq 0$

Solution

$$\begin{aligned}
f(x) = x^2 + 4x + 3 = 0 &\rightarrow \underline{x = -1, -3} \\
A &= -\int_{-3}^{-1} (x^2 + 4x + 3) dx + \int_{-1}^0 (x^2 + 4x + 3) dx \\
&= -\left(\frac{1}{3}x^3 + 2x^2 + 3x \right) \Big|_{-3}^{-1} + \left(\frac{1}{3}x^3 + 2x^2 + 3x \right) \Big|_{-1}^0 \\
&= -\left(-\frac{1}{3} + 2 - 3 + 9 - 18 + 9 \right) + \frac{1}{3} - 2 + 3 \\
&= \underline{\underline{\frac{8}{3} \text{ unit}^2}}
\end{aligned}$$

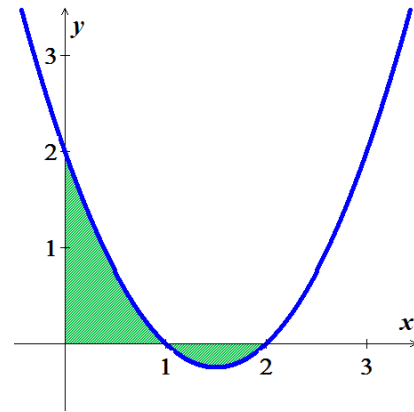


Exercise

Find the area of the region bounded by the graph of $f(x) = x^2 - 3x + 2$ x -axis on $0 \leq x \leq 2$

Solution

$$\begin{aligned}
f(x) = x^2 - 3x + 2 = 0 &\rightarrow \underline{x = 1, 2} \\
A &= \int_0^1 (x^2 - 3x + 2) dx - \int_1^2 (x^2 - 3x + 2) dx \\
&= \left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x \right) \Big|_0^1 - \left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x \right) \Big|_1^2 \\
&= \frac{1}{3} - \frac{3}{2} + 2 - \left(\frac{8}{3} - 6 + 4 - \frac{1}{3} + \frac{3}{2} - 2 \right) \\
&= -\frac{7}{6} + 2 - \left(\frac{8}{3} - 2 + \frac{7}{6} - 2 \right) \\
&= \underline{\underline{1 \text{ unit}^2}}
\end{aligned}$$



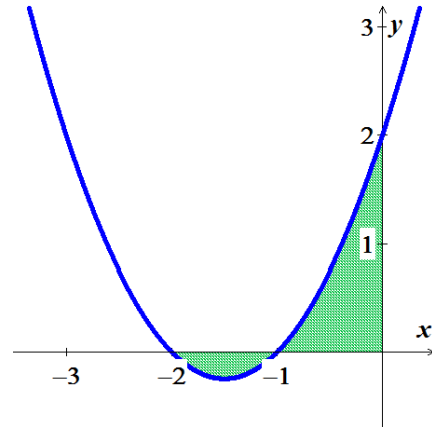
Exercise

Find the area of the region bounded by the graph of $f(x) = x^2 + 3x + 2$ x -axis on $-2 \leq x \leq 0$

Solution

$$f(x) = x^2 + 3x + 2 = 0 \rightarrow \underline{x = -1, -2}$$

$$\begin{aligned} A &= -\int_{-2}^{-1} (x^2 + 3x + 2) dx + \int_{-1}^0 (x^2 + 3x + 2) dx \\ &= -\left(\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x\right)\Big|_{-2}^{-1} + \left(\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x\right)\Big|_{-1}^0 \\ &= -\left(\frac{8}{3} - 6 + 4 - \frac{1}{3} + \frac{3}{2} - 2\right) + \frac{1}{3} - \frac{3}{2} + 2 \\ &= \underline{1 \text{ unit}^2} \end{aligned}$$



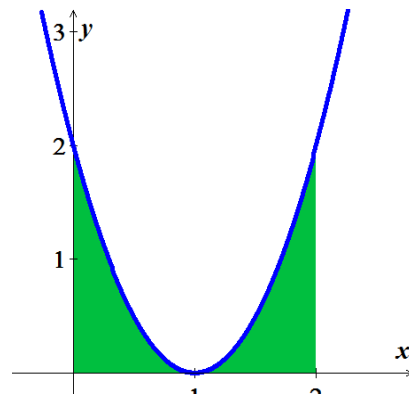
Exercise

Find the area of the region bounded by the graph of $f(x) = 2x^2 - 4x + 2$ x -axis on $0 \leq x \leq 2$

Solution

$$f(x) = 2x^2 - 4x + 2 = 0 \rightarrow \underline{x = 1}$$

$$\begin{aligned} A &= \int_0^1 (2x^2 - 4x + 2) dx + \int_1^2 (2x^2 - 4x + 2) dx \\ &= \left(\frac{2}{3}x^3 - 2x^2 + 2x\right)\Big|_0^1 + \left(\frac{2}{3}x^3 - 2x^2 + 2x\right)\Big|_1^2 \\ &= \frac{2}{3} - 2 + 2 + \frac{16}{3} - 8 + 4 - \frac{2}{3} + 2 - 2 \\ &= \underline{\frac{4}{3} \text{ unit}^2} \end{aligned}$$



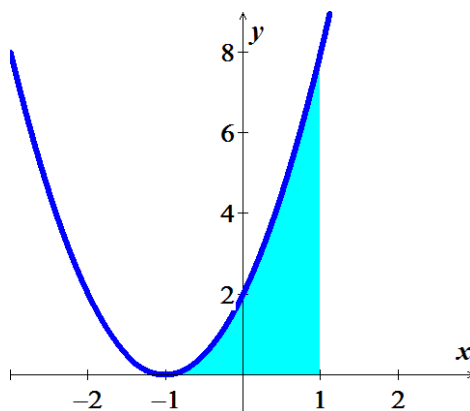
Exercise

Find the area of the region bounded by the graph of $f(x) = 2x^2 + 4x + 2$ x -axis on $-1 \leq x \leq 1$

Solution

$$f(x) = 2x^2 + 4x + 2 = 0 \rightarrow \underline{x = -1}$$

$$\begin{aligned} A &= \int_{-1}^1 (2x^2 + 4x + 2) dx \\ &= \left(\frac{2}{3}x^3 + 2x^2 + 2x \right) \Big|_{-1}^1 \\ &= \frac{2}{3} + 2 + 2 + \frac{2}{3} - 2 + 2 \\ &= \underline{\underline{\frac{16}{3} \text{ unit}^2}} \end{aligned}$$



Exercise

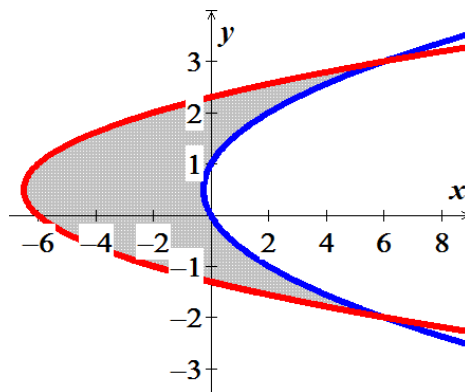
Find the area of the region bounded by the graphs of $x = y^2 - y$ and $x = 2y^2 - 2y - 6$

Solution

$$x = 2y^2 - 2y - 6 = y^2 - y$$

$$y^2 - y - 6 = 0 \rightarrow \underline{y = -2, 3}$$

$$\begin{aligned} A &= \int_{-2}^3 (y^2 - y - 2y^2 + 2y + 6) dy \\ &= \int_{-2}^3 (-y^2 + y + 6) dy \\ &= \left(-\frac{1}{3}y^3 + \frac{1}{2}y^2 + 6y \right) \Big|_{-2}^3 \\ &= -9 + \frac{9}{2} + 18 - \frac{8}{3} - 2 + 12 \\ &= 19 + \frac{11}{6} \\ &= \underline{\underline{\frac{125}{6} \text{ unit}^2}}} \end{aligned}$$



Exercise

Find the area of the region bounded by the graphs of $y = x^2 - 4$ & $y = -x^2 - 2x$ on $-3 \leq x \leq 1$

Solution

$$y = x^2 - 4 = -x^2 - 2x$$

$$2x^2 + 2x - 4 = 0 \rightarrow \underline{x = 1, -2}$$

$$A = \int_{-3}^{-2} (x^2 - 4 + x^2 + 2x) dx + \int_{-2}^1 (-x^2 - 2x - x^2 + 4) dx$$

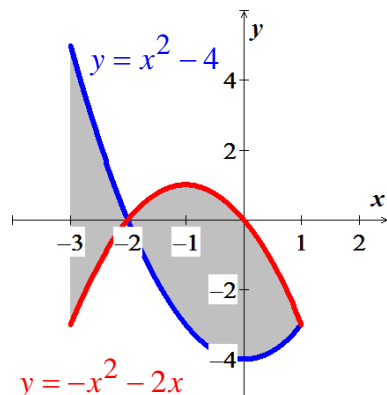
$$= \int_{-3}^{-2} (2x^2 + 2x - 4) dx + \int_{-2}^1 (-2x^2 - 2x + 4) dx$$

$$= \left(\frac{2}{3}x^3 + x^2 - 4x \right) \Big|_{-3}^{-2} + \left(-\frac{2}{3}x^3 - x^2 + 4x \right) \Big|_{-2}^1$$

$$= -\frac{16}{3} + 12 + 18 - 21 + \left(-\frac{2}{3} + 3 \right) - \frac{16}{3} + 12$$

$$= -\frac{34}{3} + 24$$

$$= \underline{\underline{\frac{38}{3} \text{ unit}^2}}$$



Exercise

Compute the area of the region bounded by the graph of f and the x -axis on the given interval.

$$f(x) = \frac{1}{x^2 + 1} \quad \text{on} \quad [-1, \sqrt{3}]$$

Solution

$$A = \int_{-1}^{\sqrt{3}} \frac{1}{x^2 + 1} dx$$

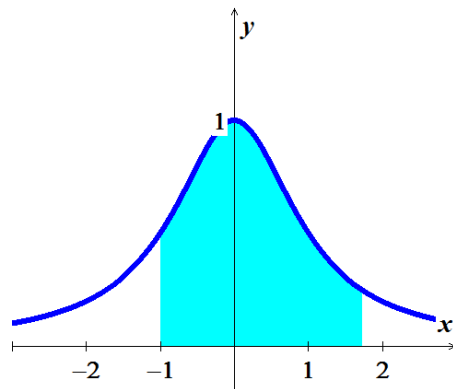
$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$= \tan^{-1} x \Big|_{-1}^{\sqrt{3}}$$

$$= \tan^{-1}(\sqrt{3}) - \tan^{-1}(1)$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \underline{\underline{\frac{\pi}{12} \text{ unit}^2}}$$



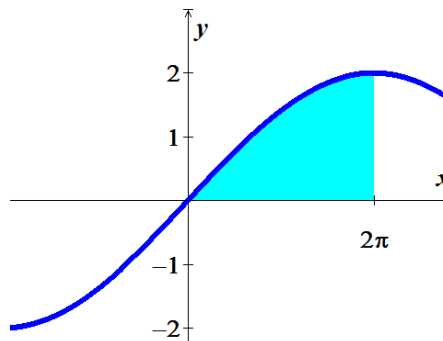
Exercise

Compute the area of the region bounded by the graph of f and the x -axis on the given interval.

$$f(x) = 2 \sin \frac{x}{4} \quad \text{on } [0, 2\pi]$$

Solution

$$\begin{aligned} A &= \int_0^{2\pi} 2 \sin \frac{x}{4} dx \\ &= -8 \cos \frac{x}{4} \Big|_0^{2\pi} \\ &= -8(0 - 1) \\ &= \underline{8 \text{ unit}^2} \end{aligned}$$



Exercise

Archimedes, inventor, military engineer, physicist, and the greatest mathematician of classical times in the Western world, discovered that the area under a parabolic arch is two-thirds the base times the height.

Sketch the parabolic arch $y = h - \left(\frac{4h}{b^2}\right)x^2$ $-\frac{b}{2} \leq x \leq \frac{b}{2}$, assuming that h and b are positive. Then use calculus to find the area of the region enclosed between the arch and the x -axis

Solution

$$\begin{aligned} A &= \int_{-b/2}^{b/2} \left(h - \left(\frac{4h}{b^2} \right) x^2 \right) dx \\ &= \left[hx - \frac{4h}{b^2} \frac{x^3}{3} \right]_{-b/2}^{b/2} \\ &= \left(\frac{hb}{2} - \frac{4h}{3b^2} \frac{b^3}{8} \right) - \left(-\frac{hb}{2} + \frac{4h}{3b^2} \frac{b^3}{8} \right) \\ &= \left(\frac{hb}{2} - \frac{hb}{6} \right) - \left(-\frac{hb}{2} + \frac{hb}{6} \right) \\ &= \frac{hb}{3} + \frac{hb}{3} \\ &= \underline{\frac{2}{3}bh \text{ unit}^2} \end{aligned}$$

Exercise

Suppose that a company's marginal revenue from the manufacture and sale of eggbeaters is

$$\frac{dr}{dx} = 2 - \frac{2}{(x+1)^2}$$

Where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand eggbeaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

Solution

$$\begin{aligned} r &= \int_0^3 \left(2 - 2(x+1)^{-2} \right) dx & d(x+1) &= dx \\ &= \int_0^3 2dx - \int_0^3 2(x+1)^{-2} d(x+1) \\ &= 2x \Big|_0^3 + 2(x+1)^{-1} \Big|_0^3 \\ &= 6 + 2(4)^{-1} - 2 \\ &= 4.5 \quad \rightarrow \quad \$4500.00 \end{aligned}$$

Exercise

The height H (feet) of a palm tree after growing for t years is given by

$$H = \sqrt{t+1} + 5t^{1/3} \quad \text{for } 0 \leq t \leq 8$$

- a) Find the tree's height when $t = 0$, $t = 4$, and $t = 8$.
- b) Find the tree's average height for $0 \leq t \leq 8$

Solution

$$\begin{aligned} \text{a) } t=0 &\Rightarrow H = 1 \text{ ft} \\ t=4 &\Rightarrow H = 10.17 \text{ ft} \\ t=8 &\Rightarrow H = 13 \text{ ft} \\ \text{b) Average height} &= \frac{1}{8-0} \int_0^8 \left(\sqrt{t+1} + 5t^{1/3} \right) dt & d(t+1) &= dt \\ &= \frac{1}{8} \int_0^8 (t+1)^{1/2} d(t+1) + \frac{5}{8} \int_0^8 t^{1/3} dt \\ &= \left[\frac{1}{12} (t+1)^{3/2} + \frac{15}{32} t^{4/3} \right]_0^8 \end{aligned}$$

$$= \frac{1}{12}(9)^{3/2} + \frac{15}{32}(8)^{4/3} - \frac{1}{12}$$

$$\underline{\approx 9.67 \text{ ft}}$$

Exercises Section 4.5 – Working with Integrals

Exercise

If f is an odd function, why is $\int_{-a}^a f(x) dx = 0$?

Solution

If f is an odd function then it is symmetric about the origin, which the region between $-a$ and a , there is as much area above the axis and under f as there is below the axis and above f .
Therefore, the net area must be 0.

Exercise

If f is an even function, why is $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$?

Solution

If f is an even function then it is symmetric about the y -axis, which the region that between $-a$ and 0 has the same net area as the region between 0 and a .

$$\text{So } \int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx = 2 \int_0^a f(x) dx$$

Exercise

Is x^{12} an even or odd function? Is $\sin(x^2)$ an even or odd function?

Solution

$$f(x) = x^{12} \rightarrow f(-x) = (-x)^{12} = x^{12} = f(x); \text{ therefore } f(x) \text{ is an even function.}$$

$$g(x) = \sin(x^2) \rightarrow g(-x) = \sin((-x)^2) = \sin(x^2) = g(x) \text{ is also an even function.}$$

Exercise

Use symmetry to evaluate the following integrals $\int_{-2}^2 x^9 dx$

Solution

$$\text{Because } x^9 \text{ is an odd function, then } \int_{-2}^2 x^9 dx = 0$$

Exercise

Use symmetry to evaluate the following integrals $\int_{-200}^{200} 2x^5 dx$

Solution

Because $2x^5$ is an odd function, then $\int_{-200}^{200} 2x^5 dx = 0$

Exercise

Use symmetry to evaluate the following integrals $\int_{-\pi/4}^{\pi/4} \cos x dx$

Solution

Because $\cos x$ is an even function, then

$$\begin{aligned}\int_{-\pi/4}^{\pi/4} \cos x dx &= 2 \int_0^{\pi/4} \cos x dx \\ &= 2 \sin x \Big|_0^{\pi/4} \\ &= 2 \left(\frac{\sqrt{2}}{2} \right) \\ &= \sqrt{2}\end{aligned}$$

Exercise

Use symmetry to evaluate the following integrals $\int_{-2}^2 (x^9 - 3x^5 + 2x^2 - 10) dx$

Solution

$$\begin{aligned}\int_{-2}^2 (x^9 - 3x^5 + 2x^2 - 10) dx &= \overset{\text{Odd}}{\int_{-2}^2 (x^9 - 3x^5) dx} + \overset{\text{Even}}{\int_{-2}^2 (2x^2 - 10) dx} \\ &= 0 + 2 \int_0^2 (2x^2 - 10) dx \\ &= 2 \left[\frac{2}{3} x^3 - 10x \right]_0^2 \\ &= 2 \left(\frac{16}{3} - 20 \right) \\ &= -\frac{88}{3}\end{aligned}$$

Exercise

Use symmetry to evaluate the following integrals $\int_{-\pi/2}^{\pi/2} (\cos 2x + \cos x \sin x - 3 \sin x^5) dx$

Solution

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} (\cos 2x + \cos x \sin x - 3 \sin x^5) dx &= \overset{\text{Even}}{\int_{-\pi/2}^{\pi/2} \cos 2x dx} + \overset{\text{Odd}}{\int_{-\pi/2}^{\pi/2} (\cos x \sin x - 3 \sin x^5) dx} \\ &= 2 \int_0^{\pi/2} \cos 2x dx \\ &= \sin 2x \Big|_0^{\pi/2} \\ &= \underline{0} \end{aligned}$$

Exercise

Find the average value of the following functions on the given interval. $f(x) = x^3$ on $[-1, 1]$

Solution

$$\text{Average value} = \frac{1}{1 - (-1)} \int_{-1}^1 x^3 dx = \frac{1}{2} \frac{1}{4} x^4 \Big|_{-1}^1 = \underline{0}$$

Exercise

Find the average value of the following functions on the given interval. $f(x) = \frac{1}{x^2 + 1}$ on $[-1, 1]$

Solution

$$\begin{aligned} \text{Average value} &= \frac{1}{1 - (-1)} \int_{-1}^1 \frac{1}{x^2 + 1} dx \\ &= \frac{1}{2} \tan^{-1} x \Big|_{-1}^1 \\ &= \frac{1}{2} \left(\frac{\pi}{4} + \frac{\pi}{4} \right) \\ &= \underline{\frac{\pi}{4}} \end{aligned}$$

Exercise

Find the average value of the following functions on the given interval. $f(x) = \frac{1}{x}$ on $[1, e]$

Solution

$$\begin{aligned}\text{Average value} &= \frac{1}{e-1} \int_1^e \frac{1}{x} dx \\ &= \frac{1}{e-1} [\ln|x|]_1^e \\ &= \frac{1}{e-1} (\ln e - 0) \\ &= \frac{1}{e-1}\end{aligned}$$

Exercise

Find the average value of the following functions on the given interval. $f(x) = e^{2x}$ on $[0, \ln 2]$

Solution

$$\begin{aligned}\text{Average value} &= \frac{1}{\ln 2} \int_0^{\ln 2} e^{2x} dx \\ &= \frac{1}{2 \ln 2} e^{2x} \Big|_0^{\ln 2} \\ &= \frac{1}{2 \ln 2} (e^{2 \ln 2} - 1) \\ &= \frac{1}{2 \ln 2} (4 - 1) \\ &= \frac{3}{2 \ln 2}\end{aligned}$$

Exercise

Suppose that $\int_0^4 f(x) dx = 10$ and $\int_0^4 g(x) dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_{-4}^4 f(x) dx$$

Solution

f is an even function.

$$\begin{aligned}\int_{-4}^4 f(x) dx &= 2 \int_0^4 f(x) dx \\ &= 2(10) \\ &= 20\end{aligned}$$

Exercise

Suppose that $\int_0^4 f(x) dx = 10$ and $\int_0^4 g(x) dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_{-4}^4 3g(x) dx$$

Solution

g is an odd function

$$\begin{aligned}\int_0^4 g(x) dx &= - \int_{-4}^0 g(x) dx \\ \int_{-4}^4 3g(x) dx &= 3 \int_{-4}^0 g(x) dx + 3 \int_0^4 g(x) dx \\ &= 3 \int_{-4}^0 g(x) dx - 3 \int_{-4}^0 g(x) dx \\ &= 0\end{aligned}$$

Exercise

Suppose that $\int_0^4 f(x) dx = 10$ and $\int_0^4 g(x) dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_0^1 8xf(4x^2) dx$$

Solution

$$\int_0^1 8xf(4x^2) dx = \int_0^1 f(4x^2) d(4x^2) \qquad d(4x^2) = 8x dx$$

$$\begin{cases} x=1 & \rightarrow 4x^2=4 \\ x=0 & \rightarrow 4x^2=0 \end{cases}$$

$$\int_0^1 8xf(4x^2)dx = \int_0^4 f(4x^2)d(4x^2)$$

$$\underline{=10}$$

Exercise

Suppose that $\int_0^4 f(x)dx = 10$ and $\int_0^4 g(x)dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_{-2}^2 3xf(x)dx$$

Solution

f is an even function, that implies $xf(x)$ is an odd function.

$$\int_{-2}^2 3xf(x)dx \underline{=0}$$

Exercise

Suppose that $\int_0^4 f(x)dx = 10$ and $\int_0^4 g(x)dx = 20$. Furthermore, suppose that f is an even function and g is an odd function. Evaluate the integral

$$\int_{-4}^4 (4f(x) - 3g(x))dx$$

Solution

f is an even function and g is an odd function.

$$\begin{aligned} \int_{-4}^4 (4f(x) - 3g(x))dx &= 4 \int_{-4}^4 f(x)dx - 3 \int_{-4}^4 g(x)dx \\ &= 8 \int_0^4 f(x)dx - 3(0) \\ &\underline{=80} \end{aligned}$$

Exercise

Suppose that f is an even function with $\int_0^8 f(x) dx = 9$. Evaluate the integral $\int_{-1}^1 x f(x^2) dx$

Solution

f is an even function, that implies $xf(x)$ is an odd function.

$$\int_{-1}^1 x f(x^2) dx = \underline{0}$$

Exercise

Suppose that f is an even function with $\int_0^8 f(x) dx = 9$. Evaluate the integral $\int_{-2}^2 x^2 f(x^3) dx$

Solution

$$d(x^3) = 3x^2 dx$$

$$\begin{cases} x = 2 & \rightarrow x^3 = 8 \\ x = -2 & \rightarrow x^3 = -8 \end{cases}$$

$$\begin{aligned} \int_{-2}^2 x^2 f(x^3) dx &= \frac{1}{3} \int_{-8}^8 f(x^3) d(x^3) \\ &= \frac{1}{3} \int_{-8}^0 f(x^3) d(x^3) + \frac{1}{3} \int_0^8 f(x^3) d(x^3) && f \text{ is an even function} \\ &= \frac{2}{3} \int_0^8 f(x^3) d(x^3) \\ &= \frac{2}{3} \cdot 9 \\ &= \underline{6} \end{aligned}$$

Exercise

Suppose that p is a nonzero real number and f is an odd integrable function with $\int_0^1 f(x) dx = \pi$.

Evaluate the integral $\int_0^{\frac{\pi}{2p}} (\cos px) f(\sin px) dx$

Solution

$$\begin{cases} x = \frac{\pi}{2p} & \rightarrow \sin px = 1 \\ x = 0 & \rightarrow \sin px = 0 \end{cases}$$

$$\int_0^{\frac{\pi}{2p}} (\cos px) f(\sin px) dx = \frac{1}{p} \int_0^1 f(\sin px) d(\sin px)$$

$$= \frac{\pi}{p}$$

$$d(\sin px) = p \cos px \, dx$$

Exercise

Suppose that p is a nonzero real number and f is an odd integrable function with $\int_0^1 f(x) dx = \pi$.

Evaluate the integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x) f(\sin x) dx$

Solution

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos px) f(\sin px) dx = \frac{1}{p} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\sin px) d(\sin px)$$

f is an odd function

$$= 0$$

Exercise

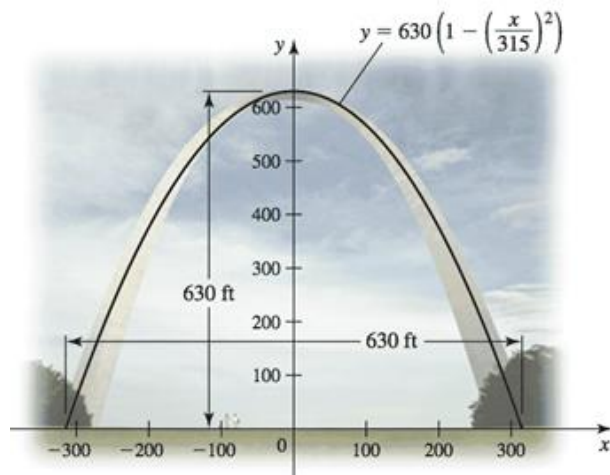
The Gateway Arch in St. Louis is 630 feet high and has a 630-ft base. Its shape can be modeled by the parabola

$$y = 630 \left(1 - \left(\frac{x}{315} \right)^2 \right)$$

Find the average height of the arch above the ground.

Solution

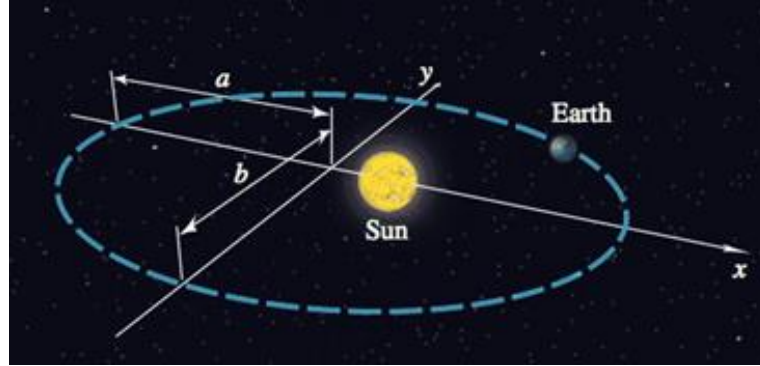
$$\begin{aligned} \text{Average height} &= \frac{1}{630} \int_{-315}^{315} 630 \left(1 - \frac{1}{315^2} x^2 \right) dx \\ &= x - \frac{1}{315^2} \frac{x^3}{3} \Big|_{-315}^{315} \\ &= 315 - 105 + 315 - 105 \\ &= 420 \text{ ft} \end{aligned}$$



Exercise

The planets orbit the Sun in elliptical orbits with the Sun at one focus. The equation of an ellipse whose dimensions are $2a$ in the x -direction and $2b$ in the y -direction is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



- a) Let d^2 denote the square of the distance from a planet to the center of the ellipse at $(0, 0)$. Integrate over the interval $[-a, a]$ to show that the average value of d^2 is $\frac{a^2 + 2b^2}{3}$.
- b) Show that in the case of a circle ($a = b = R$), the average value in part (a) is R^2 .
- c) Assuming $0 < b < a$, the coordinates of the Sun are $(\sqrt{a^2 - b^2}, 0)$. Let D^2 denote the square of the distance from the planet to the Sun. Integrate over the interval $[-a, a]$ to show that the average value of D^2 is $\frac{4a^2 - b^2}{3}$.

Solution

$$a) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \rightarrow y^2 = b^2 - \frac{b^2}{a^2}x^2$$

$$d^2 = x^2 + y^2 = x^2 + b^2 - \frac{b^2}{a^2}x^2$$

$$= b^2 + \left(1 - \frac{b^2}{a^2}\right)x^2$$

$$\text{The average value of } d^2 = \frac{1}{2a} \int_{-a}^a \left(b^2 + \left(1 - \frac{b^2}{a^2}\right)x^2 \right) dx$$

$$= \frac{1}{2a} \left[b^2 x + \frac{1}{3} \left(1 - \frac{b^2}{a^2}\right) x^3 \right]_{-a}^a$$

$$= \frac{1}{2a} \left(ab^2 + \frac{1}{3} \left(1 - \frac{b^2}{a^2}\right) a^3 + ab^2 + \frac{1}{3} \left(1 - \frac{b^2}{a^2}\right) a^3 \right)$$

$$= \frac{1}{2a} \left(2ab^2 + \frac{2}{3}a^3 - \frac{2}{3}b^2a \right)$$

$$= \frac{\frac{2}{3}b^2 + \frac{a^2}{3}}{\quad}$$

b) If $a = b = R$

The average value of $d^2 = \frac{2R^2}{3} + \frac{R^2}{3} = R^2$

c) $D^2 = \left(x - \sqrt{a^2 - b^2} \right)^2 + y^2$

$$= x^2 - 2x\sqrt{a^2 - b^2} + a^2 - b^2 + b^2 - \frac{b^2}{a^2}x^2$$

$$= \left(1 - \frac{b^2}{a^2} \right) x^2 - 2x\sqrt{a^2 - b^2} + a^2$$

The average value of $D^2 = \frac{1}{2a} \int_{-a}^a \left(\left(1 - \frac{b^2}{a^2} \right) x^2 - 2x\sqrt{a^2 - b^2} + a^2 \right) dx$

$$= \frac{1}{2a} \left(\frac{1}{3} \left(1 - \frac{b^2}{a^2} \right) x^3 - x^2 \sqrt{a^2 - b^2} + a^2 x \right)_{-a}^a$$

$$= \frac{1}{2a} \left(\frac{1}{3} \left(1 - \frac{b^2}{a^2} \right) a^3 - a^2 \sqrt{a^2 - b^2} + a^3 + \frac{1}{3} \left(1 - \frac{b^2}{a^2} \right) a^3 + a^2 \sqrt{a^2 - b^2} + a^3 \right)$$

$$= \frac{1}{3} \left(1 - \frac{b^2}{a^2} \right) a^2 + a^2$$

$$= \frac{1}{3} a^2 - \frac{1}{3} b^2 + a^2$$

$$= \frac{\frac{4}{3}a^2 - \frac{1}{3}b^2}{\quad}$$

Exercise

A particle moves along a line with a velocity given by $v(t) = 5 \sin \pi t$ starting with an initial position $s(0) = 0$. Find the displacement of the particle between $t = 0$ and $t = 2$, which is given by

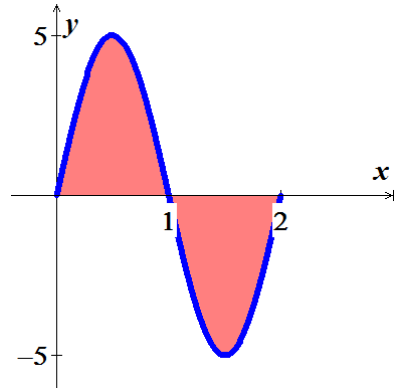
$s(t) = \int_0^2 v(t) dt$. Find the distance traveled by the particle during this interval, which is $\int_0^2 |v(t)| dt$.

Solution

$$s(t) = \int_0^2 5 \sin \pi t \, dt$$

$$\begin{aligned}
 &= -\frac{5}{\pi} \cos \pi t \Big|_0^2 \\
 &= -\frac{5}{\pi} (1-1) \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 s(t) &= \int_0^2 |5 \sin \pi t| \, dt \\
 &= 5 \int_0^1 \sin \pi t \, dt + 5 \int_1^2 (-\sin \pi t) \, dt \\
 &= -\frac{5}{\pi} \cos \pi t \Big|_0^1 + \frac{5}{\pi} \cos \pi t \Big|_1^2 \\
 &= -\frac{5}{\pi} (-1-1) + \frac{5}{\pi} (1+1) \\
 &= \frac{10}{\pi} + \frac{10}{\pi} \\
 &= \frac{20}{\pi}
 \end{aligned}$$



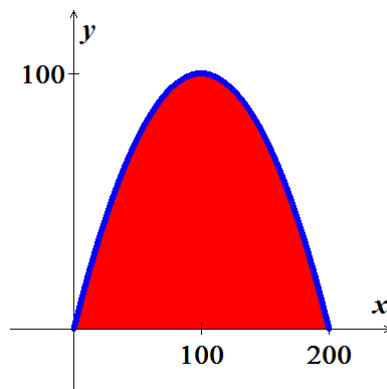
Exercise

A baseball is launched into the outfield on a parabolic trajectory given by $y = 0.01x(200 - x)$. Find the average height of the baseball over the horizontal extent of its flight.

Solution

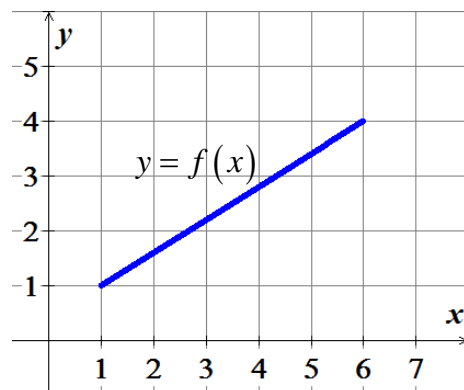
$$y = 0.01x(200 - x) = 0 \rightarrow x = 0, 200$$

$$\begin{aligned}
 Avg &= \frac{1}{200} \int_0^{200} (2x - 0.01x^2) \, dx \\
 &= \frac{1}{200} \left(x^2 - \frac{0.01}{3} x^3 \right) \Big|_0^{200} \\
 &= \frac{1}{200} \left(4 \times 10^4 - \frac{1}{300} 8 \times 10^6 \right) \\
 &= \frac{4 \times 10^4}{200} \left(1 - \frac{2}{3} \right) \\
 &= \frac{200}{3}
 \end{aligned}$$



Exercise

Find the average value of f shown in the figure on the interval $[1, 6]$ and then find the point(s) c in $(1, 6)$ guaranteed to exist by the Mean Value Theorem for Integrals



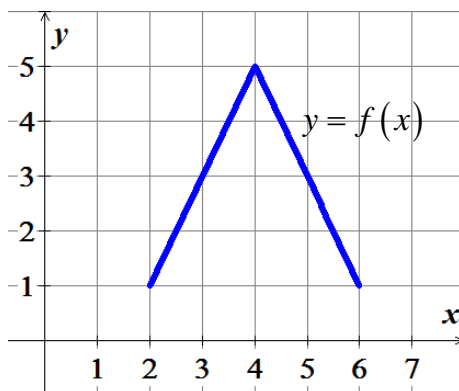
Solution

Since it is a straight line, then the average value is 2.5

The average value occurs at the midpoint of the interval which is $(3.5, 2.5)$

Exercise

Find the average value of f shown in the figure on the interval $[2, 6]$ and then find the point(s) c in $(2, 6)$ guaranteed to exist by the Mean Value Theorem for Integrals



Solution

Over interval $[2, 4]$; it is a straight line, then the average value is 3

Over interval $[4, 6]$; it is a straight line, then the average value is 3 .

Therefore; the overage is 3 over $[2, 6]$

Solution **Section 4.6 – Substitution Rule**

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int 2(2x+4)^5 dx, \quad u = 2x+4$$

Solution

$$\text{Let } u = 2x+4 \Rightarrow du = 2x dx$$

$$\begin{aligned} \int 2(2x+4)^5 dx &= \int u^5 du \\ &= \frac{1}{6}u^6 + C \\ &= \frac{1}{6}(2x+4)^6 + C \end{aligned}$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int \frac{4x^3}{(x^4+1)^2} dx, \quad u = x^4+1$$

Solution

$$\text{Let } u = x^4+1 \Rightarrow du = 4x^3 dx$$

$$\begin{aligned} \int \frac{4x^3}{(x^4+1)^2} dx &= \int \frac{du}{u^2} \\ &= \int u^{-2} du \\ &= \frac{u^{-1}}{-1} + C \\ &= \frac{-1}{x^4+1} + C \end{aligned}$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int x \sin(2x^2) dx, \quad u = 2x^2$$

Solution

$$\text{Let } u = 2x^2 \Rightarrow du = 4x dx \rightarrow \frac{1}{4} du = x dx$$

$$\begin{aligned} \int x \sin(2x^2) dx &= \int \frac{1}{4} \sin u du \\ &= -\frac{1}{4} \cos u + C \\ &= \underline{-\frac{1}{4} \cos(2x^2) + C} \end{aligned}$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int 12(y^4 + 4y^2 + 1)^2 (y^3 + 2y) dx, \quad u = y^4 + 4y^2 + 1$$

Solution

$$\text{Let } u = y^4 + 4y^2 + 1 \Rightarrow du = (4y^3 + 8y) dx \rightarrow du = 4(y^3 + 2y) dx$$

$$\begin{aligned} \int 12(y^4 + 4y^2 + 1)^2 (y^3 + 2y) dx &= \int 12u^2 \left(\frac{1}{4} du\right) \\ &= 3 \int u^2 du \\ &= 3 \frac{u^3}{3} + C \\ &= \underline{(y^4 + 4y^2 + 1)^3 + C} \end{aligned}$$

Exercise

Evaluate the indefinite integrals by using the given substitutions to reduce the integrals to standard form

$$\int \csc^2 2\theta \cot 2\theta d\theta \rightarrow \begin{cases} a) \text{ Using } u = \cot 2\theta \\ b) \text{ Using } u = \csc 2\theta \end{cases}$$

Solution

$$\text{Let } u = \cot 2\theta \Rightarrow du = -2 \csc^2 2\theta d\theta \rightarrow -\frac{1}{2} du = \csc^2 2\theta dx$$

$$\begin{aligned} \int \csc^2 2\theta \cot 2\theta d\theta &= -\int \frac{1}{2} u du \\ &= -\frac{1}{2} \frac{u^2}{2} + C \\ &= \underline{-\frac{1}{4} \cot^2 2\theta + C} \end{aligned}$$

$$\text{Let } u = \csc 2\theta \Rightarrow du = -2 \csc 2\theta \cot 2\theta d\theta \rightarrow -\frac{1}{2} du = \csc 2\theta \cot 2\theta dx$$

$$\begin{aligned} \int \csc^2 2\theta \cot 2\theta d\theta &= \int \csc 2\theta (\csc 2\theta \cot 2\theta d\theta) \\ &= -\int \frac{1}{2} u du \\ &= -\frac{1}{2} \frac{u^2}{2} + C \\ &= \underline{-\frac{1}{4} \csc^2 2\theta + C} \end{aligned}$$

Exercise

Evaluate the integrals $\int \frac{1}{\sqrt{5s+4}} ds$

Solution

$$\text{Let } u = 5s + 4 \Rightarrow du = 5ds \rightarrow \frac{1}{5} du = ds$$

$$\begin{aligned} \int \frac{1}{\sqrt{5s+4}} ds &= \frac{1}{5} \int u^{-1/2} du \\ &= \frac{1}{5} \frac{u^{1/2}}{1/2} + C \\ &= \underline{\frac{2}{5} \sqrt{5s+4} + C} \end{aligned}$$

Exercise

Evaluate the integrals $\int \theta \sqrt[4]{1-\theta^2} d\theta$

Solution

$$\text{Let } u = 1 - \theta^2 \Rightarrow du = -2\theta d\theta \rightarrow -\frac{1}{2} du = \theta d\theta$$

$$\begin{aligned}
 \int \theta \sqrt[4]{1-\theta^2} \, d\theta &= -\frac{1}{2} \int u^{1/4} \, du \\
 &= -\frac{1}{2} \frac{u^{5/4}}{5/4} + C \\
 &= \underline{-\frac{2}{5} (1-\theta^2)^{5/4} + C}
 \end{aligned}$$

Exercise

Evaluate the integrals $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2} \, dx$

Solution

$$\text{Let } u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} \, dx \rightarrow 2du = \frac{1}{\sqrt{x}} \, dx$$

$$\begin{aligned}
 \int \frac{1}{\sqrt{x}(1+\sqrt{x})^2} \, dx &= \int \frac{2}{u^2} \, du \\
 &= 2 \int u^{-2} \, du \\
 &= 2 \frac{u^{-1}}{-1} + C \\
 &= \underline{-\frac{2}{1+\sqrt{x}} + C}
 \end{aligned}$$

Exercise

Evaluate the integrals $\int \tan^2 x \sec^2 x \, dx$

Solution

$$\text{Let } u = \tan x \Rightarrow du = \sec^2 x \, dx$$

$$\begin{aligned}
 \int \tan^2 x \sec^2 x \, dx &= \int u^2 \, du \\
 &= \frac{1}{3} u^3 + C \\
 &= \underline{\frac{1}{3} \tan^3 x + C}
 \end{aligned}$$

Exercise

Evaluate the integrals $\int \sin^5 \frac{x}{3} \cos \frac{x}{3} dx$

Solution

$$\text{Let } u = \sin\left(\frac{x}{3}\right) \Rightarrow du = \frac{1}{3} \cos\left(\frac{x}{3}\right) dx \rightarrow 3du = \cos\left(\frac{x}{3}\right) dx$$

$$\begin{aligned} \int \sin^5 \frac{x}{3} \cos \frac{x}{3} dx &= \int u^5 (3du) \\ &= 3 \frac{u^6}{6} + C \\ &= \frac{1}{2} \sin^6\left(\frac{x}{3}\right) + C \end{aligned}$$

Exercise

Evaluate the integrals $\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2} dx$

Solution

$$\text{Let } u = \tan\left(\frac{x}{2}\right)$$

$$du = \frac{1}{2} \sec^2\left(\frac{x}{2}\right) dx \rightarrow 2du = \sec^2\left(\frac{x}{2}\right) dx$$

$$\begin{aligned} \int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2} dx &= 2 \int u^7 du \\ &= 2 \frac{1}{8} u^8 + C \\ &= \frac{1}{4} \tan^8 \frac{x}{2} + C \end{aligned}$$

$$\begin{aligned} \int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2} dx &= 2 \int \tan^7 \frac{x}{2} d\left(\tan \frac{x}{2}\right) \\ &= \frac{1}{4} \tan^8 \frac{x}{2} + C \end{aligned}$$

Exercise

Evaluate the integrals $\int r^4 \left(7 - \frac{r^5}{10}\right)^3 dr$

Solution

$$\text{Let } u = 7 - \frac{r^5}{10}$$

$$du = -\frac{1}{10} 5r^4 dr \rightarrow -2du = r^4 dr$$

$$\int r^4 \left(7 - \frac{r^5}{10}\right)^3 dr = \int u^3 (-2du)$$

$$\begin{aligned}
&= -2 \int u^3 du \\
&= -2 \frac{u^4}{4} + C \\
&= -\frac{1}{2} \left(7 - \frac{r^5}{10} \right)^4 + C
\end{aligned}$$

Exercise

Evaluate the integrals $\int x^{1/2} \sin(x^{3/2} + 1) dx$

Solution

$$\text{Let } u = x^{3/2} + 1 \Rightarrow du = \frac{3}{2} x^{1/2} dx \rightarrow \frac{2}{3} du = x^{1/2} dx$$

$$\begin{aligned}
\int x^{1/2} \sin(x^{3/2} + 1) dx &= \int \sin u \left(\frac{2}{3} du \right) \\
&= \frac{2}{3} \int \sin u \, du \\
&= \frac{2}{3} (-\cos u) + C \\
&= -\frac{2}{3} \cos(x^{3/2} + 1) + C
\end{aligned}$$

Exercise

Evaluate the integrals $\int \csc\left(\frac{v-\pi}{2}\right) \cot\left(\frac{v-\pi}{2}\right) dv$

Solution

$$\text{Let } u = \csc\left(\frac{v-\pi}{2}\right) \Rightarrow du = -\frac{1}{2} \csc\left(\frac{v-\pi}{2}\right) \cot\left(\frac{v-\pi}{2}\right) dv$$

$$-2du = \csc\left(\frac{v-\pi}{2}\right) \cot\left(\frac{v-\pi}{2}\right) dv$$

$$\begin{aligned}
\int \csc\left(\frac{v-\pi}{2}\right) \cot\left(\frac{v-\pi}{2}\right) dv &= \int -2du \\
&= -2u + C \\
&= -2 \csc\left(\frac{v-\pi}{2}\right) + C
\end{aligned}$$

Exercise

Evaluate the integrals $\int \frac{\sin(2t+1)}{\cos^2(2t+1)} dt$

Solution

$$\begin{aligned}\int \frac{\sin(2t+1)}{\cos^2(2t+1)} dt &= -\frac{1}{2} \int \frac{d(\cos(2t+1))}{\cos^2(2t+1)} \\ &= \underline{\underline{\frac{1}{2\cos(2t+1)} + C}}\end{aligned}$$

Exercise

Evaluate the integrals $\int \frac{\sec z \tan z}{\sqrt{\sec z}} dz$

Solution

$$\text{Let } u = \sec z \Rightarrow du = \sec z \tan z dz$$

$$\begin{aligned}\int \frac{\sec z \tan z}{\sqrt{\sec z}} dz &= \int \frac{du}{u^{1/2}} \\ &= \int u^{-1/2} du \\ &= 2u^{1/2} + C \\ &= \underline{\underline{2\sqrt{\sec z} + C}}\end{aligned}$$

$$\begin{aligned}\int \frac{\sec z \tan z}{\sqrt{\sec z}} dz &= \int (\sec z)^{-1/2} d(\sec z) \\ &= \underline{\underline{2\sqrt{\sec z} + C}}\end{aligned}$$

Exercise

Evaluate the integrals $\int \frac{1}{\sqrt{t}} \cos(\sqrt{t}+3) dt$

Solution

$$u = \sqrt{t} + 3 \Rightarrow du = \frac{1}{2\sqrt{t}} dt \rightarrow 2du = \frac{1}{\sqrt{t}} dt$$

$$\begin{aligned}\int \frac{1}{\sqrt{t}} \cos(\sqrt{t}+3) dt &= \int (\cos u)(2du) \\ &= 2 \int \cos u du \\ &= 2 \sin u + C \\ &= \underline{\underline{2 \sin(\sqrt{t}+3) + C}}\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\sqrt{t}} \cos(\sqrt{t}+3) dt &= 2 \int \cos(\sqrt{t}+3) d(\sqrt{t}+3) \\ &= \underline{\underline{2 \sin(\sqrt{t}+3) + C}}\end{aligned}$$

Exercise

Evaluate the integrals $\int \frac{1}{\theta^2} \sin \frac{1}{\theta} \cos \frac{1}{\theta} d\theta$

Solution

$$\text{Let } u = \sin \frac{1}{\theta} \Rightarrow du = \left(\cos \frac{1}{\theta} \right) \left(\frac{1}{\theta} \right)' = \left(\cos \frac{1}{\theta} \right) \left(-\frac{1}{\theta^2} \right) d\theta$$

$$-du = \frac{1}{\theta^2} \cos \frac{1}{\theta} d\theta$$

$$\begin{aligned} \int \frac{1}{\theta^2} \sin \frac{1}{\theta} \cos \frac{1}{\theta} d\theta &= -\int u du \\ &= -\frac{1}{2} u^2 + C \\ &= \underline{-\frac{1}{2} \sin^2 \frac{1}{\theta} + C} \end{aligned}$$

Exercise

Evaluate the integrals $\int t^3 (1+t^4)^3 dt$

Solution

$$u = 1+t^4 \Rightarrow du = 4t^3 dt \rightarrow \frac{1}{4} du = t^3 dt$$

$$\begin{aligned} \int t^3 (1+t^4)^3 dt &= \frac{1}{4} \int u^3 du \\ &= \frac{1}{4} \left(\frac{u^4}{4} \right) + C \\ &= \underline{\frac{1}{16} (1+t^4)^4 + C} \end{aligned}$$

$$d(1+t^4) = 4t^3 dt$$

$$\begin{aligned} \int t^3 (1+t^4)^3 dt &= \frac{1}{4} \int (1+t^4)^3 d(1+t^4) \\ &= \underline{\frac{1}{16} (1+t^4)^4 + C} \end{aligned}$$

Exercise

Evaluate the integrals $\int \frac{1}{x^3} \sqrt{\frac{x^2-1}{x^2}} dx$

Solution

$$\text{Let } u = \frac{x^2-1}{x^2} = 1 - \frac{1}{x^2} = 1 - x^{-2}$$

$$du = 2x^{-3} dx \rightarrow \frac{1}{2} du = \frac{1}{x^3} dx$$

$$\begin{aligned}
 \int \frac{1}{x^3} \sqrt{\frac{x^2-1}{x^2}} dx &= \int u^{1/2} \left(\frac{1}{2} du \right) \\
 &= \frac{1}{2} \left(\frac{2}{3} u^{3/2} \right) + C \\
 &= \frac{1}{3} \left(1 - \frac{1}{x^2} \right)^{3/2} + C
 \end{aligned}$$

Exercise

Evaluate the integrals $\int x^3 \sqrt{x^2 + 1} dx$

Solution

$$\text{Let } u = x^2 + 1 \Rightarrow du = 2x dx \rightarrow \frac{1}{2} du = x dx$$

$$x^2 = u - 1$$

$$\begin{aligned}
 \int x^3 \sqrt{x^2 + 1} dx &= \int x^2 \sqrt{x^2 + 1} x dx \\
 &= \int (u - 1) u^{1/2} \left(\frac{1}{2} du \right) \\
 &= \frac{1}{2} \int \left(u^{3/2} - u^{1/2} \right) du \\
 &= \frac{1}{2} \left(\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) + C \\
 &= \frac{1}{5} (x^2 + 1)^{5/2} - \frac{1}{3} (x^2 + 1)^{3/2} + C
 \end{aligned}$$

Exercise

Evaluate the integrals $\int \frac{x}{(x^2 - 4)^3} dx$

Solution

$$u = x^2 - 4 \Rightarrow du = 2x dx \rightarrow \frac{1}{2} du = x dx$$

$$\begin{aligned}
 \int \frac{x}{(x^2 - 4)^3} dx &= \frac{1}{2} \int u^{-3} du \\
 &= -\frac{1}{4} u^{-2} + C
 \end{aligned}$$

$$d(x^2 - 4) = 2x dx$$

$$\int \frac{x}{(x^2 - 4)^3} dx = \frac{1}{2} \int (x^2 - 4)^{-3} d(x^2 - 4)$$

$$\left| = -\frac{1}{4(x^2-4)^2} + C \right| \quad \left| = -\frac{1}{4(x^2-4)^2} + C \right|$$

Exercise

Evaluate the integrals $\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}} dr$

Solution

$$\text{Let } u = \sqrt{3(2r-1)^2+6} \Rightarrow du = \frac{1}{2} \left(3(2r-1)^2+6 \right)^{-1/2} (6(2r-1)(2)) dr$$

$$= \frac{6(2r-1)}{\left(3(2r-1)^2+6 \right)^{1/2}} dr$$

$$\rightarrow \frac{1}{6} du = \frac{2r-1}{\sqrt{3(2r-1)^2+6}} dr$$

$$\begin{aligned} \int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}} dr &= \int \cos u \left(\frac{1}{6} du \right) \\ &= \frac{1}{6} \sin u + C \\ &= \frac{1}{6} \sin \sqrt{3(2r-1)^2+6} + C \end{aligned}$$

Exercise

Evaluate the integrals $\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\cos^3\sqrt{\theta}} d\theta$

Solution

$$\text{Let } u = \cos\sqrt{\theta} \Rightarrow du = (-\sin\sqrt{\theta}) \left(\frac{1}{2\sqrt{\theta}} \right) d\theta$$

$$-2du = \frac{1}{\sqrt{\theta}} \sin\sqrt{\theta} d\theta$$

$$\begin{aligned} \int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\cos^3\sqrt{\theta}} d\theta &= \int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\sqrt{\cos^3\sqrt{\theta}}} d\theta \\ &= \int \frac{1}{u^{3/2}} (-2du) \end{aligned}$$

$$\begin{aligned}
 &= -2 \int u^{-3/2} du \\
 &= -2 \frac{u^{-1/2}}{-1/2} + C \\
 &= \frac{4}{\sqrt{\cos \sqrt{\theta}}} + C
 \end{aligned}$$

Exercise

Evaluate the integrals. $\int 2x\sqrt{x^2 - 2} \, dx$

Solution

$$\begin{aligned}
 \int 2x\sqrt{x^2 - 2} \, dx &= \int (x^2 - 2)^{1/2} d(x^2 - 2) \\
 &= \frac{2}{3} (x^2 - 2)^{3/2} + C
 \end{aligned}$$

Exercise

Evaluate the integrals $\int x^3(3x^4 + 1)^2 \, dx$

Solution

$$\begin{aligned}
 \int x^3(3x^4 + 1)^2 \, dx &= \int (3x^4 + 1)^2 d(3x^4 + 1) \\
 &= \frac{1}{36} (3x^4 + 1)^3 + C
 \end{aligned}$$

Exercise

Evaluate the integrals $\int 2(3x^4 + 1)^2 \, dx$

Solution

$$\begin{aligned}
 \int 2(3x^4 + 1)^2 \, dx &= \int 2(9x^8 + 6x^4 + 1) \, dx \\
 &= \int (18x^8 + 12x^4 + 2) \, dx \\
 &= 2x^9 + \frac{12}{5}x^5 + 2x + C
 \end{aligned}$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

Exercise

Evaluate the integrals $\int 5x\sqrt{x^2-1} \, dx$

Solution

$$u = x^2 - 1 \Rightarrow du = 2x dx \Rightarrow \frac{1}{2} du = x dx$$

$$\begin{aligned} \int 5x(x^2-1)^{1/2} \, dx &= 5 \int u^{1/2} \frac{1}{2} du && \text{Substitute for } x \text{ and } dx \\ &= 5 \int u^{1/2} \frac{1}{2} du \\ &= \frac{5}{2} \int u^{1/2} du \\ &= \frac{5}{2} \frac{u^{3/2}}{3/2} + C \\ &= \frac{5}{3} u^{3/2} + C \\ &= \frac{5}{3} (x^2-1)^{3/2} + C \end{aligned}$$

Exercise

Find the integral $\int (x^2-1)^3 (2x) dx$

Solution

$$\begin{aligned} \int (x^2-1)^3 (2x) dx &= \int (x^2-1)^3 d(x^2-1) \\ &= \frac{1}{4} (x^2-1)^4 + C \end{aligned}$$

Exercise

Find the integral $\int \frac{6x}{(1+x^2)^3} dx$

Solution

$$\int \frac{6x}{(1+x^2)^3} dx = 3 \int (1+x^2)^{-3} d(1+x^2)$$

$$\begin{aligned}
 &= -\frac{3}{2} \left(1+x^2\right)^{-2} + C \\
 &= -\frac{3}{2} \frac{1}{\left(1+x^2\right)^2} + C
 \end{aligned}$$

Exercise

Find the integral $\int u^3 \sqrt{u^4 + 2} \, du$

Solution

$$\begin{aligned}
 \int u^3 \sqrt{u^4 + 2} \, du &= \frac{1}{4} \int \left(u^4 + 2\right)^{1/2} d\left(u^4 + 2\right) \\
 &= \frac{1}{6} \left(u^4 + 2\right)^{3/2} + C
 \end{aligned}$$

Exercise

Find the integral $\int \frac{t+2t^2}{\sqrt{t}} \, dt$

Solution

$$\begin{aligned}
 \int \frac{t+2t^2}{\sqrt{t}} \, dt &= \int \left(\frac{t}{t^{1/2}} + 2 \frac{t^2}{t^{1/2}} \right) dt \\
 &= \int \left(t^{1/2} + 2t^{3/2} \right) dt \\
 &= \frac{2}{3} t^{3/2} + 2 \frac{2}{5} t^{5/2} + C \\
 &= \frac{2}{3} t^{3/2} + \frac{4}{5} t^{5/2} + C
 \end{aligned}$$

Exercise

Find the integral $\int \left(1 + \frac{1}{t}\right)^3 \frac{1}{t^2} \, dt$

Solution

$$\begin{aligned}
 \int \left(1 + \frac{1}{t}\right)^3 \frac{1}{t^2} \, dt &= - \int \left(1 + \frac{1}{t}\right)^3 d\left(1 + \frac{1}{t}\right) \\
 &= -\frac{1}{4} \left(1 + \frac{1}{t}\right)^4 + C
 \end{aligned}$$

Exercise

Find the integral $\int (7 - 3x - 3x^2)(2x + 1) dx$

Solution

$$d(7 - 3x - 3x^2) = (-3 - 6x)dx = -3(1 + 2x)dx$$

$$\begin{aligned} \int (7 - 3x - 3x^2)(2x + 1) dx &= -\frac{1}{3} \int (7 - 3x - 3x^2) d(7 - 3x - 3x^2) \\ &= -\frac{1}{6} (7 - 3x - 3x^2)^2 + C \end{aligned}$$

Exercise

Find the integral $\int \sqrt{x} (4 - x^{3/2})^2 dx$

Solution

$$u = 4 - x^{3/2} \Rightarrow du = -\frac{3}{2} x^{1/2} dx$$

$$\rightarrow -\frac{2}{3} du = \sqrt{x} dx$$

$$\begin{aligned} \int \sqrt{x} (4 - x^{3/2})^2 dx &= \int u^2 \left(-\frac{2}{3}\right) du \\ &= -\frac{2}{3} \int u^2 du \\ &= -\frac{2}{9} u^3 + C \\ &= -\frac{2}{9} (4 - x^{3/2})^3 + C \end{aligned}$$

$$d(4 - x^{3/2}) = -\frac{3}{2} x^{1/2} dx$$

$$\begin{aligned} \int \sqrt{x} (4 - x^{3/2})^2 dx &= -\frac{2}{3} \int (4 - x^{3/2})^2 d(4 - x^{3/2}) \\ &= -\frac{2}{9} (4 - x^{3/2})^3 + C \end{aligned}$$

Exercise

Find the integral $\int \frac{1}{\sqrt{x} + \sqrt{x+1}} dx$

Solution

$$\begin{aligned} \int \frac{1}{\sqrt{x} + \sqrt{x+1}} dx &= \int \frac{1}{\sqrt{x} + \sqrt{x+1}} \frac{\sqrt{x} - \sqrt{x+1}}{\sqrt{x} - \sqrt{x+1}} dx \\ &= \int \frac{\sqrt{x} - \sqrt{x+1}}{x - (x+1)} dx \end{aligned}$$

$$\begin{aligned}
&= \int \frac{\sqrt{x} - \sqrt{x+1}}{-1} dx \\
&= - \int \left(x^{1/2} - (x+1)^{1/2} \right) dx \\
&= - \left(\frac{2}{3} x^{3/2} - \frac{2}{3} (x+1)^{3/2} \right) + C \\
&= \underline{\frac{2}{3} (x+1)^{3/2} - \frac{2}{3} x^{3/2} + C}
\end{aligned}$$

Exercise

Find the integral $\int \sqrt{1-x} \, dx$

Solution

$$\begin{aligned}
\int \sqrt{1-x} \, dx &= - \int (1-x)^{1/2} \, d(1-x) & d(1-x) &= -dx \\
&= \underline{-\frac{2}{3} (1-x)^{3/2} + C}
\end{aligned}$$

Exercise

Find the integral $\int x\sqrt{x^2+4} \, dx$

Solution

$$\begin{aligned}
\int \sqrt{x^2+4} \, x dx &= \frac{1}{2} \int (x^2+4)^{1/2} \, d(x^2+4) & d(x^2+4) &= 2x dx \\
&= \underline{\frac{1}{3} (x^2+4)^{3/2} + C}
\end{aligned}$$

Exercise

Find the integral $\int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta$

Solution

$$\begin{aligned}
\int \sin^2\left(\theta + \frac{\pi}{6}\right) d\theta &= \frac{1}{2} \int \left(1 - \cos\left(2\theta + \frac{\pi}{3}\right)\right) d\theta \\
&= \frac{1}{2} \left(\theta - \frac{1}{2} \sin\left(2\theta + \frac{\pi}{3}\right) \right) + C \\
&= \underline{\frac{\theta}{2} - \frac{1}{4} \sin\left(2\theta + \frac{\pi}{3}\right) + C}
\end{aligned}$$

Exercise

Find the integral $\int \cos^2(8\theta) d\theta$

Solution

$$\begin{aligned}\int \cos^2(8\theta) d\theta &= \frac{1}{2} \int (1 + \cos(16\theta)) d\theta \\ &= \frac{1}{2} \left(1 + \frac{1}{16} \sin(16\theta) \right) + C \\ &= \underline{\frac{1}{2} + \frac{1}{32} \sin(16\theta) + C}\end{aligned}$$

Exercise

Find the integral $\int \sin^2(2\theta) d\theta$

Solution

$$\begin{aligned}\int \sin^2(2\theta) d\theta &= \frac{1}{2} \int (1 - \cos(4\theta)) d\theta \\ &= \frac{1}{2} \left(1 - \frac{1}{4} \sin(4\theta) \right) + C \\ &= \underline{\frac{1}{2} - \frac{1}{8} \sin(4\theta) + C}\end{aligned}$$

Exercise

Evaluate the integral $\int 8\cos^4 2\pi x dx$

Solution

$$\begin{aligned}\int 8\cos^4 2\pi x dx &= 8 \int (\cos 2\pi x)^4 dx \\ &= 8 \int \left(\frac{1 + \cos 4\pi x}{2} \right)^2 dx \\ &= 2 \int (1 + \cos 4\pi x)^2 dx \\ &= 2 \int (1 + 2\cos 4\pi x + \cos^2 4\pi x) dx \\ &= 2 \int dx + 4 \int \cos 4\pi x dx + 2 \int \cos^2 4\pi x dx\end{aligned}$$

$$\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}$$

$$\begin{aligned}
&= 2x + 4 \frac{1}{4\pi} \cos 4\pi x + 2 \int \frac{1 + \cos 8\pi x}{2} dx \\
&= 2x + \frac{1}{\pi} \cos 4\pi x + \int (1 + \cos 8\pi x) dx \\
&= 2x + \frac{1}{\pi} \sin 4\pi x + x + \frac{1}{8\pi} \sin 8\pi x + C \\
&= \underline{3x + \frac{1}{\pi} \sin 4\pi x + \frac{1}{8\pi} \sin 8\pi x + C}
\end{aligned}$$

Exercise

Evaluate the integral $\int \sec x dx$

Solution

$$\begin{aligned}
\int \sec x dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx & d(\sec x + \tan x) &= (\sec x \tan x + \sec^2 x) dx \\
&= \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} \\
&= \underline{\ln|\sec x + \tan x| + C}
\end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{\sqrt{1-4x^2}}$

Solution

$$\text{Let } u = 2x \Rightarrow du = 2dx \rightarrow \underline{\frac{1}{2} du = dx}$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{1-4x^2}} &= \int \frac{dx}{\sqrt{1-(2x)^2}} \\
&= \frac{1}{2} \int \frac{du}{\sqrt{1-u^2}} \\
&= \frac{1}{2} \sin^{-1} u + C \\
&= \underline{\frac{1}{2} \sin^{-1}(2x) + C}
\end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{\sqrt{3-4x^2}}$

Solution

$$a^2 = 3 \rightarrow a = \sqrt{3} \quad u^2 = 4x^2 = (2x)^2 \rightarrow u = 2x \quad du = 2dx$$

$$\begin{aligned} \int \frac{dx}{\sqrt{3-4x^2}} &= \frac{1}{2} \int \frac{dx}{\sqrt{a^2 - u^2}} \\ &= \frac{1}{2} \sin^{-1}\left(\frac{u}{a}\right) + C \\ &= \frac{1}{2} \sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C \end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{\sqrt{e^{2x} - 6}}$

Solution

$$a^2 = 6 \rightarrow a = \sqrt{6} \quad u^2 = e^{2x} \rightarrow u = e^x \quad du = e^x dx \rightarrow dx = \frac{du}{e^x} = \frac{du}{u}$$

$$\begin{aligned} \int \frac{dx}{\sqrt{e^{2x} - 6}} &= \int \frac{du}{u\sqrt{u^2 - a^2}} \\ &= \frac{1}{a} \sec^{-1}\left|\frac{u}{a}\right| + C \\ &= \frac{1}{\sqrt{6}} \sec^{-1}\left|\frac{e^x}{\sqrt{6}}\right| + C \end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{\sqrt{4x - x^2}}$

Solution

$$\begin{aligned} 4x - x^2 &= -(x^2 - 4x) - 4 + 4 \\ &= -(x^2 - 4x + 4) + 4 \\ &= 4 - (x - 2)^2 \end{aligned}$$

$$a = 2 \quad u = x - 2 \rightarrow du = dx$$

Using Completing the Square

$$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}}$$

$$= \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

Exercise

Evaluate $\int \frac{dx}{4x^2 + 4x + 2}$

Solution

$$4x^2 + 4x + 2 = 4\left(x^2 + x\right) + 2$$

$$= 4\left(x^2 + x + \frac{1}{4}\right) + 2 - 4\left(\frac{1}{4}\right)$$

$$= 4\left(x + \frac{1}{2}\right)^2 + 1$$

$$= (2x+1)^2 + 1$$

$$a=1 \quad u=2x+1 \quad \rightarrow du=2dx$$

$$\int \frac{dx}{4x^2 + 4x + 2} = \int \frac{dx}{(2x+1)^2 + 1}$$

$$= \frac{1}{2} \int \frac{du}{u^2 + 1}$$

$$= \frac{1}{2} \cdot \frac{1}{1} \tan^{-1}\left(\frac{2x+1}{1}\right) + C$$

$$= \frac{1}{2} \tan^{-1}(2x+1) + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Exercise

Find the integral $\int \frac{1}{6x-5} dx$

Solution

$$\int \frac{1}{6x-5} dx = \frac{1}{6} \int \frac{d(6x-5)}{6x-5}$$

$$= \frac{1}{6} \ln|6x-5| + C$$

Exercise

Find the integral $\int \frac{x^2 + 2x + 3}{x^3 + 3x^2 + 9x + 1} dx$

Solution

$$d(x^3 + 3x^2 + 9x + 1) = (3x^2 + 6x + 9)dx$$

$$\begin{aligned} \int \frac{x^2 + 2x + 3}{x^3 + 3x^2 + 9x + 1} dx &= \frac{1}{3} \int \frac{d(x^3 + 3x^2 + 9x + 1)}{x^3 + 3x^2 + 9x + 1} \\ &= \frac{1}{3} \ln|x^3 + 3x^2 + 9x + 1| + C \end{aligned}$$

Exercise

Find the integral $\int \frac{1}{x(\ln x)^2} dx$

Solution

$$\begin{aligned} \int \frac{1}{x(\ln x)^2} dx &= \int \frac{1}{(\ln x)^2} d(\ln x) \\ &= -\frac{1}{\ln x} + C \end{aligned}$$

Exercise

Find the integral $\int \frac{x-3}{x+3} dx$

Solution

$$\begin{aligned} \int \frac{x-3}{x+3} dx &= \int \left(1 - \frac{6}{x+3}\right) dx \\ &= x - 6\ln|x+3| + C \end{aligned}$$

Exercise

Find the indefinite integral. $\int \frac{3x}{x^2 + 4} dx$

Solution

$$u = x^2 + 4 \rightarrow du = 2x dx \rightarrow \frac{1}{2} du = x dx$$

$$\begin{aligned}
 \int \frac{3x}{x^2+4} dx &= \int \frac{3}{u} \frac{1}{2} du \\
 &= \frac{3}{2} \int \frac{1}{u} du \\
 &= \frac{3}{2} \ln|u| + C \\
 &= \frac{3}{2} \ln(x^2+4) + C
 \end{aligned}
 \quad \left| \quad
 \begin{aligned}
 \int \frac{3x}{x^2+4} dx &= \frac{3}{2} \int \frac{d(x^2+4)}{x^2+4} \\
 &= \frac{3}{2} \ln(x^2+4) + C
 \end{aligned}
 \right|$$

Exercise

Evaluate the integral $\int \frac{dx}{2\sqrt{x}+2x}$

Solution

$$\begin{aligned}
 \int \frac{dx}{2\sqrt{x}+2x} &= \int \frac{dx}{2\sqrt{x}(1+\sqrt{x})} \\
 &= \int \frac{du}{u} \\
 &= \ln u + C \\
 &= \ln(1+\sqrt{x}) + C
 \end{aligned}
 \quad u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} dx$$

Exercise

Evaluate the integral $\int \frac{\sec x dx}{\sqrt{\ln(\sec x + \tan x)}}$

Solution

$$\begin{aligned}
 \text{Let } u = \sec x + \tan x &\Rightarrow du = (\sec x \tan x + \sec^2 x) dx = \sec x (\tan x + \sec x) dx \\
 \sec x dx &= \frac{du}{\tan x + \sec x} = \frac{du}{u}
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{\sec x dx}{\sqrt{\ln(\sec x + \tan x)}} &= \int \frac{du}{u \sqrt{\ln u}} \\
 &= \int (\ln u)^{-1/2} d(\ln u) \\
 &= 2(\ln u)^{1/2} + C \\
 &= 2\sqrt{\ln(\sec x + \tan x)} + C
 \end{aligned}
 \quad d(\ln u) = \frac{1}{u} du$$

Exercise

Evaluate the integral $\int 8e^{(x+1)} dx$

Solution

$$\int 8e^{(x+1)} dx = \underline{8e^{(x+1)} + C}$$

Exercise

Find the indefinite integral. $\int 4xe^{x^2} dx$

Solution

$$\begin{aligned} \int 4xe^{x^2} dx &= 2 \int e^{x^2} d(x^2) \\ &= \underline{2e^{x^2} + C} \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{e^{-\sqrt{r}}}{\sqrt{r}} dr$

Solution

$$\begin{aligned} u &= -r^{1/2} \rightarrow du = -\frac{1}{2}r^{-1/2} dr \\ \Rightarrow -2du &= \frac{1}{r^{1/2}} dr \end{aligned}$$

$$\begin{aligned} \int \frac{e^{-\sqrt{r}}}{\sqrt{r}} dr &= \int e^u (-2du) \\ &= -2e^u + C \\ &= \underline{-2e^{-\sqrt{r}} + C} \end{aligned}$$

Exercise

Evaluate the integral $\int t^3 e^{t^4} dt$

Solution

$$\int t^3 e^{t^4} dt = \frac{1}{4} \int e^{t^4} d(t^4)$$

$$= \frac{1}{4} e^{t^4} + C$$

Exercise

Evaluate the integral $\int e^{\sec \pi t} \sec \pi \tan \pi t dt$

Solution

$$u = \sec \pi t \rightarrow du = \pi \sec \pi t \tan \pi t dt$$

$$\frac{1}{\pi} du = \sec \pi t \tan \pi t dt$$

$$\int e^{\sec \pi t} \sec \pi \tan \pi t dt = \frac{1}{\pi} \int e^u du$$

$$= \frac{1}{\pi} e^u + C$$

$$= \frac{1}{\pi} e^{\sec \pi t} + C$$

$$d(\sec \pi t) = \pi \sec \pi t \tan \pi t dt$$

$$\int e^{\sec \pi t} \sec \pi \tan \pi t dt = \frac{1}{\pi} \int e^{\sec \pi t} d(\sec \pi t)$$

$$= \frac{1}{\pi} e^{\sec \pi t} + C$$

Exercise

Find the integral $\int (2x+1)e^{x^2+x} dx$

Solution

$$\int (2x+1)e^{x^2+x} dx = \int e^{x^2+x} d(x^2+x)$$

$$= e^{x^2+x} + C$$

Exercise

Evaluate the integral $\int \frac{dx}{1+e^x}$

Solution

$$\int \frac{dx}{1+e^x} = \int \frac{e^{-x}}{e^{-x} + 1} dx$$

$$\begin{aligned}
&= \int \frac{e^{-x} dx}{e^{-x} + 1} \\
&= - \int \frac{1}{e^{-x} + 1} d(e^{-x} + 1) \\
&= \underline{-\ln(e^{-x} + 1) + C}
\end{aligned}$$

Exercise

Find the integral $\int \frac{e^x}{1 + e^x} dx$

Solution

$$u = 1 + e^x \Rightarrow du = e^x dx$$

$$\begin{aligned}
\int \frac{1}{u} du &= \ln|u| + C \\
&= \underline{\ln(1 + e^x) + C}
\end{aligned}$$

$$\left| \int \frac{e^x}{1 + e^x} dx = \int \frac{1}{1 + e^x} d(1 + e^x) \right. \\
\left. \qquad \qquad \qquad = \underline{\ln(1 + e^x) + C} \right|$$

Exercise

Find the integral $\int \frac{2}{e^{-x} + 1} dx$

Solution

$$\begin{aligned}
\int \frac{2}{e^{-x} + 1} dx &= \int \frac{2}{e^{-x} + 1} \frac{e^x}{e^x} dx \\
&= 2 \int \frac{e^x}{1 + e^x} dx \\
&= 2 \int \frac{d(e^x + 1)}{1 + e^x} \\
&= \underline{2 \ln(e^x + 1) + C}
\end{aligned}$$

Exercise

Find the integral $\int \frac{1}{x^3} e^{1/4x^2} dx$

Solution

$$\begin{aligned} u = \frac{1}{4x^2} = \frac{1}{4} x^{-2} &\Rightarrow du = -\frac{1}{2} x^{-3} dx \\ &\Rightarrow -2du = \frac{1}{x^3} dx \\ \int e^u (-2) du &= -2 \int e^u du \\ &= -2e^u + C \\ &= \underline{-2e^{1/4x^2} + C} \end{aligned} \quad \left| \begin{aligned} d\left(\frac{1}{4} x^{-2}\right) &= -\frac{1}{2} x^{-3} dx \\ \int \frac{1}{x^3} e^{1/4x^2} dx &= -2 \int e^{1/4x^2} d\left(\frac{1}{4x^2}\right) \\ &= \underline{-2e^{1/4x^2} + C} \end{aligned} \right|$$

Exercise

Find the integral $\int \frac{e^{1/\sqrt{x}}}{x^{3/2}} dx$

Solution

$$\begin{aligned} u = \frac{1}{\sqrt{x}} = x^{-1/2} &\Rightarrow du = -\frac{1}{2} x^{-3/2} dx \Rightarrow -2du = \frac{1}{x^{3/2}} dx \\ \int \frac{e^{1/\sqrt{x}}}{x^{3/2}} dx &= \int e^u (-2du) \\ &= -2 \int e^u du \\ &= -2e^u + C \\ &= \underline{-2e^{1/\sqrt{x}} + C} \end{aligned}$$

Exercise

Find the integral $\int \frac{-e^{3x}}{2 - e^{3x}} dx$

Solution

$$\begin{aligned} \int \frac{-e^{3x}}{2 - e^{3x}} dx &= \frac{1}{3} \int \frac{1}{2 - e^{3x}} d(2 - e^{3x}) \\ &= \underline{\frac{1}{3} \ln|2 - e^{3x}| + C} \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{7e^{7x}}{3+e^{7x}} dx$

Solution

$$u = 3 + e^{7x} \rightarrow du = 7e^{7x} dx$$

$$\begin{aligned} \int \frac{7e^{7x}}{3+e^{7x}} dx &= \int \frac{du}{u} \\ &= \ln|u| \\ &= \ln(3 + e^{7x}) + C \end{aligned}$$

$$\begin{aligned} \int \frac{7e^{7x}}{3+e^{7x}} dx &= \int \frac{1}{3+e^{7x}} d(3+e^{7x}) \\ &= \ln(3 + e^{7x}) + C \end{aligned}$$

Exercise

Find the integral $\int \frac{2(e^x - e^{-x})}{(e^x + e^{-x})^2} dx$

Solution

$$u = e^x + e^{-x} \Rightarrow du = (e^x - e^{-x}) dx$$

$$\begin{aligned} \int \frac{2(e^x - e^{-x})}{(e^x + e^{-x})^2} dx &= 2 \int \frac{1}{u^2} du \\ &= 2 \int u^{-2} du \\ &= 2 \frac{u^{-1}}{-1} + C \\ &= -2 \frac{1}{u} + C \\ &= -\frac{2}{e^x + e^{-x}} + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{3^x}{3 - 3^x} dx$

Solution

$$\text{Let } u = 3 - 3^x \rightarrow du = -3^x \ln 3 dx$$

$$\Rightarrow -\frac{1}{\ln 3} du = 3^x dx$$

$$\begin{aligned}
 \int \frac{3^x}{3-3^x} dx &= -\frac{1}{\ln 3} \int \frac{du}{u} \\
 &= -\frac{1}{\ln 3} \ln|u| + C \\
 &= \underline{-\frac{1}{\ln 3} \ln|3-3^x| + C}
 \end{aligned}$$

Exercise

Find the integral $\int (6x + e^x) \sqrt{3x^2 + e^x} dx$

Solution

$$\begin{aligned}
 u = 3x^2 + e^x &\Rightarrow du = (6x + e^x) dx \\
 \frac{du}{6x + e^x} &= dx
 \end{aligned}$$

$$\begin{aligned}
 \int (6x + e^x) \sqrt{u} \frac{du}{6x + e^x} &= \int u^{1/2} du \\
 &= \frac{u^{3/2}}{3/2} + C \\
 &= \frac{2}{3} u^{3/2} + C \\
 &= \underline{\frac{2}{3} (3x^2 + e^x)^{3/2} + C}
 \end{aligned}$$

$$\begin{aligned}
 \int (6x + e^x) \sqrt{3x^2 + e^x} dx &= \int (3x^2 + e^x)^{1/2} d(3x^2 + e^x) \\
 &= \underline{\frac{2}{3} (3x^2 + e^x)^{3/2} + C}
 \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x 2^{x^2}}{1 + 2^{x^2}} dx$

Solution

$$\begin{aligned}
 \text{Let } u = 1 + 2^{x^2} &\Rightarrow du = 2x 2^{x^2} \ln(2) dx \\
 \Rightarrow \frac{du}{2 \ln 2} &= x 2^{x^2} dx
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{x 2^{x^2}}{1 + 2^{x^2}} dx &= \frac{1}{2 \ln 2} \int \frac{du}{u} \\
 &= \frac{1}{2 \ln 2} \ln u + C \\
 &= \underline{\frac{1}{2 \ln 2} \ln(1 + 2^{x^2}) + C}
 \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{x(\log_8 x)^2}$

Solution

$$\begin{aligned}
 \int \frac{dx}{x(\log_8 x)^2} &= \int \frac{dx}{x\left(\frac{\ln x}{\ln 8}\right)^2} \\
 &= (\ln 8)^2 \int \frac{dx}{x(\ln x)^2} \\
 &= (\ln 8)^2 \int \frac{d(\ln x)}{(\ln x)^2} \\
 &= -(\ln 8)^2 \frac{1}{\ln x} + C
 \end{aligned}$$

$$d(\ln x) = \frac{1}{x} dx$$

Exercise

Evaluate $\int \frac{dx}{x\sqrt{25x^2 - 2}}$

Solution

$$\text{Let } u = 5x \Rightarrow du = 5dx \rightarrow \frac{1}{5} du = dx$$

$$\begin{aligned}
 \int \frac{dx}{x\sqrt{25x^2 - 2}} &= \int \frac{du/5}{\frac{u}{5}\sqrt{u^2 - 2}} \\
 &= \int \frac{du}{u\sqrt{u^2 - (\sqrt{2})^2}} \\
 &= \frac{1}{\sqrt{2}} \sec^{-1} \left| \frac{u}{\sqrt{2}} \right| + C \\
 &= \frac{1}{\sqrt{2}} \sec^{-1} \left| \frac{5x}{\sqrt{2}} \right| + C
 \end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

Exercise

Evaluate $\int \frac{6dr}{\sqrt{4 - (r+1)^2}}$

Solution

$$u = r + 1 \Rightarrow du = dr$$

$$a^2 = 4 \rightarrow a = 2$$

$$\begin{aligned} \int \frac{6dr}{\sqrt{4-(r+1)^2}} &= 6 \int \frac{du}{\sqrt{4-u^2}} \\ &= 6 \sin^{-1} \frac{u}{2} + C \\ &= \underline{6 \sin^{-1} \left(\frac{r+1}{2} \right) + C} \end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

Exercise

Evaluate $\int \frac{dx}{2+(x-1)^2}$

Solution

$$u = x - 1 \Rightarrow du = dx$$

$$a^2 = 2 \rightarrow a = \sqrt{2}$$

$$\begin{aligned} \int \frac{dx}{2+(x-1)^2} &= \int \frac{du}{2+u^2} \\ &= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) + C \\ &= \underline{\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x-1}{\sqrt{2}} \right) + C} \end{aligned}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Exercise

Evaluate $\int \frac{\sec^2 y \, dy}{\sqrt{1 - \tan^2 y}}$

Solution

$$\begin{aligned} \int \frac{\sec^2 y \, dy}{\sqrt{1 - \tan^2 y}} &= \int \frac{du}{\sqrt{1 - u^2}} \\ &= \sin^{-1}(u) + C \\ &= \underline{\sin^{-1}(\tan y) + C} \end{aligned}$$

$$\begin{aligned} u = \tan y &\Rightarrow du = \sec^2 y \, dy \\ a^2 = 1 &\rightarrow a = 1 \end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

Exercise

Evaluate $\int \frac{dx}{\sqrt{-x^2 + 4x - 3}}$

Solution

$$\begin{aligned}
 \int \frac{dx}{\sqrt{-x^2 + 4x - 3}} &= \int \frac{dx}{\sqrt{1 - x^2 + 4x - 3 - 1}} \\
 &= \int \frac{dx}{\sqrt{1 - (x^2 - 4x + 4)}} \\
 &= \int \frac{dx}{\sqrt{1 - (x + 2)^2}} \\
 &= \int \frac{du}{\sqrt{1 - u^2}} \int \frac{dx}{\sqrt{2x - x^2}} \\
 &= \sin^{-1} u + C \\
 &= \sin^{-1}(x - 2) + C
 \end{aligned}$$

$$u = x + 2 \Rightarrow du = dx$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

Exercise

Evaluate $\int \frac{dx}{\sqrt{2x - x^2}}$

Solution

$$\begin{aligned}
 \int \frac{dx}{\sqrt{2x - x^2}} &= \int \frac{dx}{\sqrt{1 + 2x - x^2 - 1}} \\
 &= \int \frac{dx}{\sqrt{1 - (x^2 - 2x + 1)}} \\
 &= \int \frac{dx}{\sqrt{1 - (x - 1)^2}} \\
 &= \int \frac{du}{\sqrt{1 - u^2}} \\
 &= \sin^{-1}(x - 1) + C
 \end{aligned}$$

$$u = x - 1 \Rightarrow du = dx$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

Exercise

Evaluate $\int \frac{x-2}{x^2-6x+10} dx$

Solution

$$\begin{aligned}\int \frac{x-2}{x^2-6x+10} dx &= \int \frac{x-2}{x^2-6x+9+1} dx \\&= \int \frac{x-2-\mathbf{1+1}}{(x-3)^2+1} dx \\&= \int \frac{x-3+1}{(x-3)^2+1} dx & u = x-3 \Rightarrow du = dx \\&= \int \frac{u+1}{u^2+1} du \\&= \int \frac{u}{u^2+1} du + \int \frac{1}{u^2+1} du & w = u^2+1 \Rightarrow dw = 2udu \rightarrow \frac{1}{2}dw = udu \\&= \frac{1}{2} \int \frac{dw}{w} + \int \frac{1}{u^2+1} du \\&= \frac{1}{2} \ln w + \tan^{-1} u + C \\&= \frac{1}{2} \ln((x-3)^2+1) + \tan^{-1}(x-3) + C \\&= \frac{1}{2} \ln(x^2-6x+10) + \tan^{-1}(x-3) + C\end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

Solution

$$\begin{aligned}\int \frac{dx}{(x+1)\sqrt{x^2+2x}} &= \int \frac{dx}{(x+1)\sqrt{x^2+2x+\mathbf{1-1}}} \\&= \int \frac{dx}{(x+1)\sqrt{(x+1)^2-1}} & \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| \\&= \sec^{-1}|x+1| + C\end{aligned}$$

Exercise

Evaluate $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

Solution

$$\begin{aligned}
 \int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} &= \int \frac{dx}{(x-2)\sqrt{x^2-4x+4-1}} \\
 &= \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}} \\
 &= \int \frac{du}{u\sqrt{u^2-1}} \\
 &= \sec^{-1} u + C \\
 &= \sec^{-1} |x-2| + C
 \end{aligned}$$

$$\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} |u|$$

Exercise

Evaluate $\int \frac{e^{\cos^{-1} x} dx}{\sqrt{1-x^2}}$

Solution

$$\begin{aligned}
 \int \frac{e^{\cos^{-1} x} dx}{\sqrt{1-x^2}} &= - \int e^{\cos^{-1} x} d(\cos^{-1} x) \\
 &= -e^{\cos^{-1} x} + C
 \end{aligned}$$

$$d(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} dx$$

Exercise

Evaluate $\int \frac{(\sin^{-1} x)^2 dx}{\sqrt{1-x^2}}$

Solution

$$\begin{aligned}
 \int \frac{(\sin^{-1} x)^2 dx}{\sqrt{1-x^2}} &= \int (\sin^{-1} x)^2 d(\sin^{-1} x) \\
 &= \frac{1}{3} (\sin^{-1} x)^3 + C
 \end{aligned}$$

$$d(\sin^{-1} x) = \frac{dx}{\sqrt{1-x^2}}$$

Exercise

Evaluate $\int \frac{dy}{(\sin^{-1} y)\sqrt{1+y^2}}$

Solution

$$\begin{aligned} \int \frac{dy}{(\sin^{-1} y)\sqrt{1+y^2}} &= \int \frac{1}{\sin^{-1} y} d(\sin^{-1} y) & d(\sin^{-1} y) &= \frac{dy}{\sqrt{1-y^2}} \\ &= \ln|\sin^{-1} y| + C \end{aligned}$$

Exercise

Evaluate $\int \frac{1}{\sqrt{x}(x+1)\left((\tan^{-1} \sqrt{x})^2 + 9\right)} dx$

Solution

$$\begin{aligned} d(\tan^{-1} \sqrt{x}) &= \frac{1}{2\sqrt{x}} \frac{1}{1+(\sqrt{x})^2} dx \\ &= \frac{1}{2\sqrt{x}(1+x)} dx \\ \int \frac{1}{\sqrt{x}(x+1)\left((\tan^{-1} \sqrt{x})^2 + 9\right)} dx &= 2 \int \frac{1}{\left((\tan^{-1} \sqrt{x})^2 + 9\right)} d(\tan^{-1} \sqrt{x}) \\ &= \frac{2}{3} \tan^{-1} \left(\frac{\tan^{-1} \sqrt{x}}{3} \right) + C \end{aligned}$$

Exercise

Evaluate the integral $\int 2x(x^2+1)^4 dx$

Solution

$$\begin{aligned} \int 2x(x^2+1)^4 dx &= \int (x^2+1)^4 d(x^2+1) & d(x^2+1) &= 2x dx \\ &= \frac{1}{5} (x^2+1)^5 + C \end{aligned}$$

Exercise

Evaluate the integral $\int 8x \cos(4x^2 + 3) dx$

Solution

$$\int 8x \cos(4x^2 + 3) dx = \int \cos(4x^2 + 3) d(4x^2 + 3) \qquad d(4x^2 + 3) = 8x dx$$

$$\underline{= \sin(4x^2 + 3) + C}$$

Exercise

Evaluate the integral $\int \sin^3 x \cos x dx$

Solution

$$\int \sin^3 x \cos x dx = \int \sin^3 x d(\sin x) \qquad d(\sin x) = \cos x dx$$

$$\underline{= \frac{1}{4} \sin^4 x + C}$$

Exercise

Evaluate the integral $\int (6x + 1) \sqrt{3x^2 + x} dx$

Solution

$$\int (6x + 1) \sqrt{3x^2 + x} dx = \int (3x^2 + x)^{1/2} d(3x^2 + x) \qquad d(3x^2 + x) = (6x + 1) dx$$

$$\underline{= \frac{2}{3} (3x^2 + x)^{3/2} + C}$$

Exercise

Evaluate the integral $\int 2x(x^2 - 1)^{99} dx$

Solution

$$\int 2x(x^2 - 1)^{99} dx = \int (x^2 - 1)^{99} d(x^2 - 1) \qquad d(x^2 - 1) = 2x dx$$

$$\underline{= \frac{1}{100} (x^2 - 1)^{100} + C}$$

Exercise

Evaluate the integral $\int x e^{x^2} dx$

Solution

$$\begin{aligned} \int x e^{x^2} dx &= \frac{1}{2} \int e^{x^2} d(x^2) & d(x^2) &= 2x dx \\ &= \frac{1}{2} e^{x^2} + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{2x^2}{\sqrt{1-4x^3}} dx$

Solution

$$\begin{aligned} \int \frac{2x^2}{\sqrt{1-4x^3}} dx &= -\frac{1}{6} \int (1-4x^3)^{-1/2} d(1-4x^3) & d(1-4x^3) &= -12x^2 dx \\ &= -\frac{1}{3} \sqrt{1-4x^3} + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{(\sqrt{x}+1)^4}{2\sqrt{x}} dx$

Solution

$$\begin{aligned} \int \frac{(\sqrt{x}+1)^4}{2\sqrt{x}} dx &= \int (\sqrt{x}+1)^4 d(\sqrt{x}+1) & d(\sqrt{x}+1) &= \frac{1}{2\sqrt{x}} dx \\ &= \frac{1}{5} (\sqrt{x}+1)^5 + C \end{aligned}$$

Exercise

Evaluate the integral $\int (x^2+x)^{10} (2x+1) dx$

Solution

$$\begin{aligned} \int (x^2+x)^{10} (2x+1) dx &= \int (x^2+x)^{10} d(x^2+x) & d(x^2+x) &= (2x+1) dx \\ &= \frac{1}{11} (x^2+x)^{11} + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{10x-3}$

Solution

$$\int \frac{dx}{10x-3} = \frac{1}{10} \int \frac{d(10x-3)}{10x-3} \quad d(10x-3) = 10dx$$

$$= \ln|10x-3| + C$$

Exercise

Evaluate the integral $\int x^3 (x^4 + 16)^6 dx$

Solution

$$\int x^3 (x^4 + 16)^6 dx = \frac{1}{4} \int (x^4 + 16)^6 d(x^4 + 16) \quad d(x^4 + 16) = 4x^3 dx$$

$$= \frac{1}{28} (x^4 + 16)^7 + C$$

Exercise

Evaluate the integral $\int \sin^{10} \theta \cos \theta d\theta$

Solution

$$\int \sin^{10} \theta \cos \theta d\theta = \int \sin^{10} \theta d(\sin \theta) \quad d(\sin \theta) = \cos \theta d\theta$$

$$= \frac{1}{11} \sin^{11} \theta + C$$

Exercise

Evaluate the integral $\int \frac{dx}{\sqrt{1-9x^2}}$

Solution

$$\int \frac{dx}{\sqrt{1-9x^2}} = \frac{1}{3} \int \frac{d(3x)}{\sqrt{1-(3x)^2}} \quad d(3x) = 3dx \quad \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

$$= \frac{1}{3} \arcsin 3x + C$$

Exercise

Evaluate the integral $\int x^9 \sin x^{10} dx$

Solution

$$\begin{aligned}\int x^9 \sin x^{10} dx &= \frac{1}{10} \int \sin x^{10} d(x^{10}) & d(x^{10}) &= 10x^9 dx \\ &= \underline{-\frac{1}{10} \cos x^{10} + C}\end{aligned}$$

Exercise

Evaluate the integral $\int (x^6 - 3x^2)^4 (x^5 - x) dx$

Solution

$$\begin{aligned}\int (x^6 - 3x^2)^4 (x^5 - x) dx &= \frac{1}{6} \int (x^6 - 3x^2)^4 d(x^6 - 3x^2) & d(x^6 - 3x^2) &= 6(x^5 - x) dx \\ &= \underline{\frac{1}{30} (x^6 - 3x^2)^5 + C}\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x}{x-2} dx$

Solution

$$\begin{aligned}\int \frac{x}{x-2} dx &= \int \left(1 + \frac{2}{x-2}\right) dx & x-2 &\overline{)x} \\ & & &\underline{-x+2} \\ & & &2 \\ &= \underline{x + 2 \ln|x-2| + C}\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{1+4x^2}$

Solution

$$\begin{aligned}\int \frac{dx}{1+4x^2} &= \frac{1}{2} \int \frac{d(2x)}{1+(2x)^2} & d(2x) &= 2dx & \int \frac{dx}{a^2+x^2} &= \frac{1}{a} \tan^{-1} \frac{x}{a} \\ &= \underline{\frac{1}{2} \arctan 2x + C}\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{3}{1+25y^2} dy$

Solution

$$\int \frac{3}{1+25y^2} dy = \frac{3}{5} \int \frac{d(5y)}{1+(5y)^2} \quad d(5y) = 5dy \quad \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$= \underline{\underline{\frac{3}{5} \arctan 5y + C}}$$

Exercise

Evaluate the integral $\int \frac{2}{x \sqrt{4x^2 - 1}} dx \quad \left(x > \frac{1}{2}\right)$

Solution

$$\int \frac{2}{x \sqrt{4x^2 - 1}} dx = \int \frac{d(2x)}{x \sqrt{(2x)^2 - 1}} \quad \int \frac{dx}{x \sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{x}{a} \right|$$

$$= \underline{\underline{\operatorname{arcsec}(2x) + C}}$$

Exercise

Evaluate the integral $\int \frac{8x+6}{2x^2+3x} dx$

Solution

$$\int \frac{8x+6}{2x^2+3x} dx = 2 \int \frac{1}{2x^2+3x} d(2x^2+3x) \quad 2d(2x^2+3x) = 2(4x+3)dx$$

$$= \underline{\underline{2 \ln |2x^2+3x| + C}}$$

Exercise

Evaluate the integral $\int \frac{x}{\sqrt{x-4}} dx$

Solution

$$u = x - 4 \rightarrow x = u + 4$$

$$dx = du$$

$$\int \frac{x}{\sqrt{x-4}} dx = \int \frac{u+4}{u^{1/2}} du$$

$$\begin{aligned}
&= \int \left(u^{1/2} + 4u^{-1/2} \right) du \\
&= \frac{2}{3} u^{3/2} + 8u^{1/2} + C \\
&= \frac{2}{3} (x-4)^{3/2} + 8(x-4)^{1/2} + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x^2}{(x+1)^4} dx$

Solution

$$u = x + 1 \rightarrow x = u - 1$$

$$dx = du$$

$$\begin{aligned}
\int \frac{x^2}{(x+1)^4} dx &= \int \frac{(u-1)^2}{u^4} du \\
&= \int \frac{u^2 - 2u + 1}{u^4} du \\
&= \int \left(\frac{1}{u^2} - 2u^{-3} + u^{-4} \right) du \\
&= -\frac{1}{u} + u^{-2} - \frac{1}{3} u^{-3} + C \\
&= -\frac{1}{x+1} + \frac{1}{(x+1)^2} - \frac{1}{3(x+1)^3} + C
\end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x}{\sqrt[3]{x+4}} dx$

Solution

$$u = x + 4 \rightarrow x = u - 4$$

$$dx = du$$

$$\begin{aligned}
\int \frac{x}{\sqrt[3]{x+4}} dx &= \int \frac{u-4}{u^{1/3}} du \\
&= \int \left(u^{2/3} - 4u^{-1/3} \right) du \\
&= \frac{3}{5} u^{5/3} - 6u^{2/3} + C
\end{aligned}$$

$$= \frac{3}{5}(x+4)^{5/3} - 6(x+4)^{2/3} + C \quad |$$

Exercise

Evaluate the integral $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

Solution

$$\begin{aligned} \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx &= \int \frac{1}{e^x + e^{-x}} d(e^x + e^{-x}) & d(e^x + e^{-x}) &= (e^x - e^{-x}) dx \\ &= \ln(e^x + e^{-x}) + C & & \end{aligned}$$

Exercise

Evaluate the integral $\int x \sqrt[3]{2x+1} dx$

Solution

$$\begin{aligned} u = 2x+1 &\rightarrow x = \frac{1}{2}(u-1) \\ dx &= \frac{1}{2} du \\ \int x \sqrt[3]{2x+1} dx &= \int \frac{1}{2}(u-1)u^{1/3} \left(\frac{1}{2} du\right) \\ &= \frac{1}{4} \int (u^{4/3} - u^{1/3}) du \\ &= \frac{1}{4} \left(\frac{3}{7} u^{7/3} - \frac{3}{4} u^{4/3} \right) + C \\ &= \frac{3}{28} (2x+1)^{7/3} - \frac{3}{16} (2x+1)^{4/3} + C \quad | \end{aligned}$$

Exercise

Evaluate the integral $\int (x+1)\sqrt{3x+2} dx$

Solution

$$\begin{aligned} u = 3x+2 &\rightarrow x = \frac{1}{3}(u-2) \\ dx &= \frac{1}{3} du \end{aligned}$$

$$\begin{aligned}
\int (x+1)\sqrt{3x+2} \, dx &= \int \left(\frac{1}{3}u - 2 + 1\right)u^{1/2} \frac{1}{3}du \\
&= \frac{1}{3} \int \left(\frac{1}{3}u - 1\right)u^{1/2} \, du \\
&= \frac{1}{3} \int \left(\frac{1}{3}u^{3/2} - u^{1/2}\right) \, du \\
&= \frac{1}{3} \left(\frac{2}{15}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C \\
&= \underline{\underline{\frac{2}{45}(3x+2)^{5/2} - \frac{2}{9}(3x+2)^{3/2} + C}}
\end{aligned}$$

Exercise

Evaluate the integral $\int \sin^2 x \, dx$

Solution

$$\begin{aligned}
\int \sin^2 x \, dx &= \frac{1}{2} \int (1 - \cos 2x) \, dx \\
&= \underline{\underline{\frac{1}{2} \left(x - \frac{1}{2} \sin 2x\right) + C}}
\end{aligned}
\qquad \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

Exercise

Evaluate the integral $\int \sin^2 \left(\theta + \frac{\pi}{6}\right) d\theta$

Solution

$$\begin{aligned}
\int \sin^2 \left(\theta + \frac{\pi}{6}\right) d\theta &= \int \frac{1}{2} \left(1 - \cos 2\left(\theta + \frac{\pi}{6}\right)\right) d\theta \\
&= \frac{1}{2} \int d\theta - \frac{1}{4} \int \cos \left(2\theta + \frac{\pi}{3}\right) d\left(2\theta + \frac{\pi}{3}\right) \\
&= \underline{\underline{\frac{1}{2}\theta - \frac{1}{4} \sin \left(2\theta + \frac{\pi}{3}\right) + C}}
\end{aligned}
\qquad \begin{aligned} \sin^2 \theta &= \frac{1}{2}(1 - \cos 2\theta) \\ d\left(2\theta + \frac{\pi}{3}\right) &= 2d\theta \end{aligned}$$

Exercise

Evaluate the integral $\int x \cos^2(x^2) \, dx$

Solution

$$\begin{aligned}
 \int x \cos^2(x^2) dx &= \frac{1}{2} \int \cos^2(x^2) d(x^2) & d(x^2) &= 2x dx \\
 &= \frac{1}{4} \int (1 + \cos(2x^2)) d(x^2) \\
 &= \frac{1}{4} \int d(x^2) + \frac{1}{8} \int \cos(2x^2) d(2x^2) & d(x^2) &= 2d(x^2) \\
 &= \frac{1}{4} x^2 + \frac{1}{8} \sin(2x^2) + C
 \end{aligned}$$

Or

$$\begin{aligned}
 \int x \cos^2(x^2) dx &= \frac{1}{2} \int x (1 + \cos(2x^2)) dx & \cos^2 \theta &= \frac{1}{2}(1 + \cos 2\theta) \\
 &= \frac{1}{2} \int x dx + \frac{1}{2} \int x \cos(2x^2) dx \\
 &= \frac{1}{4} x^2 + \frac{1}{8} \int \cos(2x^2) d(2x^2) \\
 &= \frac{1}{4} x^2 + \frac{1}{8} \sin(2x^2) + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \sec 4x \tan 4x dx$

Solution

$$\begin{aligned}
 \int \sec 4x \tan 4x dx &= \frac{1}{4} \int d(\sec 4x) & d(\sec 4x) &= 4 \sec 4x \tan 4x \\
 &= \frac{1}{4} \sec 4x + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int \sec^2 10x dx$

Solution

$$\begin{aligned}
 \int \sec^2 10x dx &= \frac{1}{10} \int \sec^2 10x d(10x) \\
 &= \frac{1}{10} \tan 10x + C
 \end{aligned}$$

Exercise

Evaluate the integral $\int (\sin^5 x + 3\sin^3 x - \sin x) \cos x \, dx$

Solution

$$\begin{aligned} \int (\sin^5 x + 3\sin^3 x - \sin x) \cos x \, dx &= \int (\sin^5 x + 3\sin^3 x - \sin x) d(\sin x) & d(\sin x) &= \cos x \, dx \\ &= \frac{1}{6} \sin^6 x + \frac{3}{4} \sin^4 x - \frac{1}{2} \sin^2 x + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\csc^2 x}{\cot^3 x} \, dx$

Solution

$$\begin{aligned} \int \frac{\csc^2 x}{\cot^3 x} \, dx &= - \int \cot^{-3} x \, d(\cot x) & d(\cot x) &= -\csc^2 x \, dx \\ &= \frac{1}{2} \cot^{-2} x + C \\ &= \frac{1}{2 \cot^2 x} + C \\ &= \frac{1}{2} \tan^2 x + C \end{aligned}$$

Exercise

Evaluate the integral $\int (x^{3/2} + 8)^5 \sqrt{x} \, dx$

Solution

$$\begin{aligned} \int (x^{3/2} + 8)^5 \sqrt{x} \, dx &= \frac{2}{3} \int (x^{3/2} + 8)^5 d(x^{3/2} + 8) & d(x^{3/2} + 8) &= \frac{3}{2} x^{1/2} \, dx \\ &= \frac{1}{9} (x^{3/2} + 8)^6 + C \end{aligned}$$

Exercise

Evaluate the integral $\int \sin x \sec^8 x \, dx$

Solution

$$\begin{aligned} \int \sin x \sec^8 x \, dx &= - \int \cos^{-8} x \, d(\cos x) & d(\cos x) &= -\sin x \, dx; \quad \sec x = \frac{1}{\cos x} \end{aligned}$$

$$= \frac{1}{7} \cos^{-7} x + C$$

$$= \frac{1}{7} \sec^7 x + C$$

Exercise

Evaluate the integral $\int \frac{e^{2x}}{e^{2x} + 1} dx$

Solution

$$\int \frac{e^{2x}}{e^{2x} + 1} dx = \frac{1}{2} \int \frac{1}{e^{2x} + 1} d(e^{2x} + 1)$$

$$= \frac{1}{2} \ln(e^{2x} + 1) + C$$

$$d(e^{2x} + 1) = 2e^{2x} dx$$

Exercise

Evaluate the integral $\int \sec^3 \theta \tan \theta d\theta$

Solution

$$\int \sec^3 \theta \tan \theta d\theta = \int \sec^2 \theta \sec \theta \tan \theta d\theta$$

$$= \int \sec^2 \theta d(\sec \theta)$$

$$= \frac{1}{3} \sec^3 \theta + C$$

$$d(\sec \theta) = \tan \theta \sec \theta d\theta$$

Exercise

Evaluate the integral $\int x \sin^4 x^2 \cos x^2 dx$

Solution

$$\int x \sin^4 x^2 \cos x^2 dx = \frac{1}{2} \int \sin^4 x^2 d(\sin x^2)$$

$$= \frac{1}{10} \sin^5(x^2) + C$$

$$d(\sin x^2) = 2x \cos x^2 dx$$

Exercise

Evaluate the integral $\int \frac{dx}{\sqrt{1+\sqrt{1+x}}}$

Solution

$$u = 1 + \sqrt{1+x} \rightarrow \sqrt{1+x} = u - 1$$

$$du = \frac{1}{2\sqrt{1+x}} dx \rightarrow dx = 2(u-1)du$$

$$\begin{aligned} \int \frac{dx}{\sqrt{1+\sqrt{1+x}}} &= \int \frac{2(u-1)}{u^{1/2}} du \\ &= 2 \int (u^{1/2} - u^{-1/2}) du \\ &= 2 \left(\frac{2}{3} u^{3/2} - 2u^{1/2} \right) + C \\ &= \frac{4}{3} (1+\sqrt{1+x})^{3/2} - 4(1+\sqrt{1+x})^{1/2} + C \end{aligned}$$

Exercise

Evaluate the integral $\int \tan^{10} 4x \sec^2 4x \, dx$

Solution

$$\begin{aligned} \int \tan^{10} 4x \sec^2 4x \, dx &= \frac{1}{4} \int \tan^{10} 4x \, d(\tan 4x) & d(\tan 4x) &= 4 \sec^2(4x) dx \\ &= \frac{1}{44} \tan^{11} 4x + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{x^2}{x^3+27} dx$

Solution

$$\begin{aligned} \int \frac{x^2}{x^3+27} dx &= \frac{1}{3} \int \frac{1}{x^3+27} d(x^3+27) & d(x^3+27) &= 3x^2 dx \\ &= \frac{1}{3} \ln |x^3+27| + C \end{aligned}$$

Exercise

Evaluate the integral $\int y^2(3y^3+1)^4 dy$

Solution

$$\begin{aligned} \int y^2(3y^3+1)^4 dy &= \frac{1}{9} \int (3y^3+1)^4 d(3y^3+1) & d(3y^3+1) &= 9y^2 dy \\ &= \frac{1}{45} (3y^3+1)^5 + C \end{aligned}$$

Exercise

Evaluate the integral $\int x \sin x^2 \cos^8 x^2 dx$

Solution

$$\begin{aligned} \int x \sin x^2 \cos^8 x^2 dx &= -\frac{1}{2} \int \cos^8 x^2 d(\cos x^2) & d(\cos x^2) &= -2x \sin x^2 dx \\ &= -\frac{1}{18} \cos^9(x^2) + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\sin 2x}{1+\cos^2 x} dx$

Solution

$$\begin{aligned} \int \frac{\sin 2x}{1+\cos^2 x} dx &= -\int \frac{1}{1+\cos^2 x} d(1+\cos^2 x) & d(1+\cos^2 x) &= -2\cos x \sin x dx = -\sin 2x dx \\ &= -\ln|1+\cos^2 x| + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

Solution

$$\begin{aligned} \int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx &= \int \sin^{-1} x d(\sin^{-1} x) & d(\sin^{-1} x) &= \frac{dx}{\sqrt{1-x^2}} \\ &= \frac{1}{2} (\sin^{-1} x)^2 + C \end{aligned}$$

Exercise

Evaluate the integral $\int \frac{dx}{(\tan^{-1} x)(1+x^2)}$

Solution

$$\int \frac{dx}{(\tan^{-1} x)(1+x^2)} = \int \frac{1}{\tan^{-1} x} d(\tan^{-1} x) \qquad d(\tan^{-1} x) = \frac{dx}{1+x^2}$$
$$\underline{= \ln |\tan^{-1} x| + C}$$

Exercise

Evaluate the integral $\int \frac{(\tan^{-1} x)^5}{1+x^2} dx$

Solution

$$\int \frac{(\tan^{-1} x)^5}{1+x^2} dx = \int (\tan^{-1} x)^5 d(\tan^{-1} x) \qquad d(\tan^{-1} x) = \frac{dx}{1+x^2}$$
$$\underline{= \frac{1}{6} (\tan^{-1} x)^6 + C}$$

Exercise

Evaluate the integral $\int \frac{1}{x^2} \sin \frac{1}{x} dx$

Solution

$$\int \frac{1}{x^2} \sin \frac{1}{x} dx = - \int \sin \frac{1}{x} d\left(\frac{1}{x}\right) \qquad d\left(\frac{1}{x}\right) = -\frac{1}{x^2} dx$$
$$\underline{= \cos \frac{1}{x} + C}$$

Exercise

Evaluate the integral $\int_{-1}^2 x^2 e^{x^3+1} dx$

Solution

$$\int_{-1}^2 x^2 e^{x^3+1} dx = \frac{1}{3} \int_{-1}^2 e^{x^3+1} d(x^3+1) \qquad d(x^3+1) = 3x^2 dx$$

$$= \frac{1}{3} e^{x^3+1} \Big|_{-1}^2$$

$$= \frac{1}{3} (e^9 - 1) \Big|$$

Exercise

Evaluate the integral $\int_0^2 x^2 e^{x^3} dx$

Solution

$$\int_0^2 x^2 e^{x^3} dx = \frac{1}{3} \int_0^2 e^{x^3} d(x^3)$$

$$= \frac{1}{3} e^{x^3} \Big|_0^2$$

$$= \frac{1}{3} (e^8 - e) \Big|$$

$$d(x^3) = 3x^2 dx$$

Exercise

Evaluate the integral $\int_0^4 \frac{x}{x^2+1} dx$

Solution

$$\int_0^4 \frac{x}{x^2+1} dx = \frac{1}{2} \int_0^4 \frac{1}{x^2+1} d(x^2+1)$$

$$= \frac{1}{2} \ln(x^2+1) \Big|_0^4$$

$$= \frac{1}{2} (\ln 17 - \ln 1)$$

$$= \frac{1}{2} \ln 17 \Big|$$

Exercise

Evaluate the integrals $\int \frac{18 \tan^2 x \sec^2 x}{(2 + \tan^3 x)^2} dx$

a) $u = \tan x$, followed by $v = u^3$ then by $w = 2 + v$

b) $u = \tan^3 x$, followed by $v = 2 + u$

c) $u = 2 + \tan^3 x$

Solution

a) Let $u = \tan x \Rightarrow du = \sec^2 x dx$

$$v = u^3 \Rightarrow dv = 3u^2 du$$

$$w = 2 + v \Rightarrow dw = dv$$

$$\begin{aligned} \int \frac{18 \tan^2 x \sec^2 x}{(2 + \tan^3 x)^2} dx &= \int \frac{18u^2 du}{(2 + u^3)^2} \\ &= \int \frac{6dv}{(2 + v)^2} \\ &= \int \frac{6dw}{w^2} \\ &= 6 \int w^{-2} dw \\ &= 6 \frac{w^{-1}}{-1} + C \\ &= -\frac{6}{w} + C \\ &= -\frac{6}{2 + v} + C \\ &= -\frac{6}{2 + u^3} + C \\ &= \underline{-\frac{6}{2 + \tan^3 x} + C} \end{aligned}$$

b) Let $u = \tan^3 x \Rightarrow du = 3 \tan^2 x \sec^2 x dx$

$$v = 2 + u \Rightarrow dv = du$$

$$\int \frac{18 \tan^2 x \sec^2 x}{(2 + \tan^3 x)^2} dx = \int \frac{6du}{(2 + u)^2}$$

$$\begin{aligned}
&= \int \frac{6dv}{v^2} \\
&= \int 6v^{-2} dv \\
&= -6v^{-1} + C \\
&= -\frac{6}{v} + C \\
&= -\frac{6}{2+u} + C \\
&= -\frac{6}{2+\tan^3 x} + C
\end{aligned}$$

c) Let $u = 2 + \tan^3 x \Rightarrow du = 3 \tan^2 x \sec^2 x dx \rightarrow \frac{1}{3} du = \tan^2 x \sec^2 x dx$

$$du = 3 \tan^2 x \sec^2 x dx \rightarrow \frac{1}{3} du = \tan^2 x \sec^2 x dx$$

$$\begin{aligned}
\int \frac{18 \tan^2 x \sec^2 x}{(2 + \tan^3 x)^2} dx &= \int \frac{18}{u^2} \left(\frac{1}{3} du \right) \\
&= 6 \int u^{-2} du \\
&= -6u^{-1} + C \\
&= -\frac{6}{u} + C \\
&= -\frac{6}{2 + \tan^3 x} + C
\end{aligned}$$

Exercise

Evaluate: $\int_0^1 (2t+3)^3 dt$

Solution

$$\begin{aligned}
\int_0^1 (2t+3)^3 dt &= \frac{1}{2} \int_0^1 (2t+3)^3 d(2t+3) \\
&= \frac{1}{8} (2t+3)^4 \Big|_0^1 \\
&= \frac{1}{8} \left[(2(1)+3)^4 - (2(0)+3)^4 \right] \\
&= \frac{1}{8} \left[5^4 - 3^4 \right] \\
&= 68
\end{aligned}$$

$$d(2t+3) = 2dt \rightarrow \frac{1}{2} d(2t+3) = dt$$

Exercise

Evaluate the integral $\int_0^2 \sqrt{4-x^2} dx$

Solution

$$\int_0^2 \sqrt{4-x^2} dx = \left[\frac{1}{2} x \sqrt{4-x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} \right]_0^2$$

$\sqrt{4-x^2}$ is a semi-circle with center (0, 0) and radius = 2. Since x from 0 to 2

$$\Rightarrow \text{Area} = \frac{1}{4} (\text{Area of this circle})$$

$$= \frac{1}{4} 2\pi 2^2$$

$$= \underline{2\pi}$$

Exercise

Evaluate the integral $\int_0^3 \sqrt{y+1} dy$

Solution

$$\int_0^3 \sqrt{y+1} dy = \int_0^3 (y+1)^{1/2} d(y+1)$$

$$d(y+1) = dy$$

$$= \frac{2}{3} (y+1)^{3/2} \Big|_0^3$$

$$= \frac{2}{3} \left[(\textcolor{red}{3}+1)^{3/2} - (\textcolor{blue}{0}+1)^{3/2} \right]$$

$$= \frac{2}{3} [8-1]$$

$$= \underline{\frac{14}{3}}$$

Exercise

Evaluate the integral $\int_{-1}^1 r\sqrt{1-r^2} dr$

Solution

$$\int_{-1}^1 r\sqrt{1-r^2} dr = -\frac{1}{2} \int_{-1}^1 (1-r^2)^{1/2} d(1-r^2)$$

$$d(1-r^2) = -2rdr$$

$$\begin{aligned}
&= -\frac{1}{3} \left[(1-r^2)^{3/2} \right]_{-1}^1 \\
&= -\frac{1}{3} \left[(1-(1)^2)^{3/2} - (1-(-1)^2)^{3/2} \right] \\
&= -\frac{1}{3} [0-0] \\
&= \underline{0}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/4} \tan x \sec^2 x \, dx$

Solution

$$\begin{aligned}
\int_0^{\pi/4} \tan x \sec^2 x \, dx &= \int_0^{\pi/4} \tan x \, d(\tan x) \\
&= \frac{1}{2} \tan^2 x \Big|_0^{\pi/4} \\
&= \frac{1}{2} [1^2 - 0^2] \\
&= \underline{\frac{1}{2}}
\end{aligned}$$

$$d(\tan x) = \sec^2 x \, dx$$

Exercise

Evaluate the integral $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x \, dx$

Solution

$$\begin{aligned}
\int_{2\pi}^{3\pi} 3\cos^2 x \sin x \, dx &= - \int_{2\pi}^{3\pi} 3\cos^2 x \, d(\cos x) \\
&= -\cos^3 x \Big|_{2\pi}^{3\pi} \\
&= -[(-1)^3 - 1^3] \\
&= \underline{2}
\end{aligned}$$

$$d(\cos x) = -\sin x \, dx$$

Exercise

Evaluate the integral $\int_0^1 t^3 (1+t^4)^3 dt$

Solution

$$\begin{aligned}\int_0^1 t^3 (1+t^4)^3 dt &= \frac{1}{4} \int_0^1 (1+t^4)^3 d(1+t^4) \\ &= \frac{1}{16} (1+t^4)^4 \Big|_0^1 \\ &= \frac{1}{16} (2^4 - 1^4) \\ &= \frac{15}{16}\end{aligned}$$

$$d(1+t^4) = 4t^3 dt$$

Exercise

Evaluate the integral $\int_0^1 \frac{r}{(4+r^2)^2} dr$

Solution

$$\begin{aligned}\int_0^1 \frac{r}{(4+r^2)^2} dr &= \frac{1}{2} \int_0^1 \frac{d(4+r^2)}{(4+r^2)^2} \\ &= -\frac{1}{2} \left[\frac{1}{4+r^2} \right]_0^1 \\ &= -\frac{1}{2} \left(\frac{1}{5} - \frac{1}{4} \right) \\ &= -\frac{1}{40}\end{aligned}$$

$$d(4+r^2) = 2r dr$$

Exercise

Evaluate the integral $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2} dv$

Solution

$$\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2} dv = \frac{20}{3} \int_0^1 \frac{d(1+v^{3/2})}{(1+v^{3/2})^2}$$

$$d(1+v^{3/2}) = \frac{3}{2} \sqrt{v} dv$$

$$\begin{aligned}
 &= -\frac{20}{3} \left[\frac{1}{1+v^{3/2}} \right]_0^1 \\
 &= -\frac{20}{3} \left(\frac{1}{2} - 1 \right) \\
 &= \underline{\underline{\frac{10}{3}}}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}} dx$

Solution

Let $u = x^2 + 1$

$$du = 2x dx \rightarrow \frac{1}{2} du = x dx$$

$$\rightarrow \begin{cases} x = \sqrt{3} & \rightarrow u = 4 \\ x = -\sqrt{3} & \rightarrow u = 4 \end{cases}$$

$$\begin{aligned}
 \int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}} dx &= \int_4^4 4u^{-1/2} \left(\frac{1}{2} du \right) \\
 &= \underline{\underline{0}}
 \end{aligned}$$

OR

$$\begin{aligned}
 \int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}} dx &= 2 \int_{-\sqrt{3}}^{\sqrt{3}} (x^2+1)^{-1/2} d(x^2+1) \\
 &= 4 \sqrt{x^2+1} \Big|_{-\sqrt{3}}^{\sqrt{3}} \\
 &= 4(2-2) \\
 &= \underline{\underline{0}}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 \frac{x^3}{\sqrt{x^4+9}} dx$

Solution

$$u = x^4 + 9$$

$$du = 4x^3 dx \rightarrow \frac{1}{4} du = x^3 dx \rightarrow \begin{cases} x=1 \rightarrow u=10 \\ x=0 \rightarrow u=9 \end{cases}$$

$$\begin{aligned}
 \int_0^1 \frac{x^3}{\sqrt{x^4+9}} dx &= \frac{1}{4} \int_9^{10} u^{-1/2} du \\
 &= \frac{1}{4} \left[2u^{1/2} \right]_9^{10} \\
 &= \frac{1}{2} \left[\textcolor{red}{10}^{1/2} - \textcolor{blue}{9}^{1/2} \right]
 \end{aligned}$$

$$= \frac{\sqrt{10}-3}{2}$$

Or

$$\begin{aligned} \int_0^1 \frac{x^3}{\sqrt{x^4+9}} dx &= \frac{1}{4} \int_0^1 (x^4+9)^{-1/2} d(x^4+9) \\ &= \frac{1}{2} (x^4+9)^{1/2} \Big|_0^1 \\ &= \frac{1}{2} [\textcolor{red}{10}^{1/2} - \textcolor{blue}{9}^{1/2}] \\ &= \frac{\sqrt{10}-3}{2} \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/6} (1-\cos 3t) \sin 3t \, dt$

Solution

$$\begin{aligned} \int_0^{\pi/6} (1-\cos 3t) \sin 3t \, dt &= \frac{1}{3} \int_0^{\pi/6} (1-\cos 3t) \, d(1-\cos 3t) \\ &= \frac{1}{6} \left[(1-\cos 3t)^2 \right]_0^{\pi/6} \\ &= \frac{1}{6} (1^2 - 0^2) \\ &= \frac{1}{6} \end{aligned}$$

Exercise

Evaluate the integral $\int_{-\pi/2}^{\pi/2} \left(2 + \tan \frac{t}{2}\right) \sec^2 \frac{t}{2} \, dt$

Solution

$$\begin{aligned} d\left(2 + \tan \frac{t}{2}\right) &= \frac{1}{2} \sec^2 \frac{t}{2} \, dt \\ \int_{-\pi/2}^{\pi/2} \left(2 + \tan \frac{t}{2}\right) \sec^2 \frac{t}{2} \, dt &= 2 \int_{-\pi/2}^{\pi/2} \left(2 + \tan \frac{t}{2}\right) d\left(2 + \tan \frac{t}{2}\right) \end{aligned}$$

$$\begin{aligned}
 &= \left(2 + \tan \frac{t}{2} \right)^2 \bigg|_{-\pi/2}^{\pi/2} \\
 &= 3^2 - 1 \\
 &= 8
 \end{aligned}$$

Or

$$u = 2 + \tan \frac{t}{2}$$

$$du = \frac{1}{2} \sec^2 \frac{t}{2} dt \rightarrow 2du = \sec^2 \frac{t}{2} dt$$

$$\begin{cases} t = \frac{\pi}{2} & \rightarrow u = 3 \\ t = -\frac{\pi}{2} & \rightarrow u = 1 \end{cases}$$

$$\begin{aligned}
 \int_{-\pi/2}^{\pi/2} \left(2 + \tan \frac{t}{2} \right) \sec^2 \frac{t}{2} dt &= \int_1^3 u (2du) \\
 &= 2 \left[\frac{u^2}{2} \right]_1^3 \\
 &= [3^2 - 1^2] \\
 &= 8
 \end{aligned}$$

Exercise

Evaluate the integral $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4 + 3 \sin z}} dz$

Solution

Let $u = 4 + 3 \sin z$

$$du = 3 \cos z dz \rightarrow \frac{1}{3} du = \cos z dz \quad \begin{cases} z = \pi & \rightarrow u = 4 \\ z = -\pi & \rightarrow u = 4 \end{cases}$$

$$\begin{aligned}
 \int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4 + 3 \sin z}} dz &= \int_4^4 \frac{1}{\sqrt{u}} \frac{1}{3} du \\
 &= 0
 \end{aligned}$$

Exercise

Evaluate the integral $\int_{-\pi/2}^0 \frac{\sin w}{(3 + 2 \cos w)^2} dw$

Solution

$$\begin{aligned}
\int_{-\pi/2}^0 \frac{\sin w}{(3+2\cos w)^2} dw &= -\frac{1}{2} \int_{-\pi/2}^0 \frac{d(3+2\cos w)}{(3+2\cos w)^2} \\
&= \frac{1}{2} \left[\frac{1}{3+2\cos w} \right]_{-\pi/2}^0 \\
&= \frac{1}{2} \left(\frac{1}{5} - \frac{1}{3} \right) \\
&= -\frac{1}{15}
\end{aligned}$$

$$d\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

Exercise

Evaluate the integral $\int_0^1 \sqrt{t^5 + 2t} (5t^4 + 2) dt$

Solution

$$\begin{aligned}
\int_0^1 \sqrt{t^5 + 2t} (5t^4 + 2) dt &= \int_0^1 (t^5 + 2t)^{1/2} d(t^5 + 2t) \\
&= \frac{2}{3} (t^5 + 2t)^{3/2} \Big|_0^1 \\
&= \frac{2}{3} (3^{3/2}) \\
&= 2\sqrt{3}
\end{aligned}$$

Exercise

Evaluate the integral $\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$

Solution

$$\begin{aligned}
\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2} &= \int_1^4 \frac{1}{(1+\sqrt{y})^2} d(1+\sqrt{y}) \quad d(1+\sqrt{y}) = \frac{1}{2\sqrt{y}} dy \\
&= -\frac{1}{1+\sqrt{y}} \Big|_1^4 \\
&= -\left(\frac{1}{3} - \frac{1}{2}\right) \\
&= \frac{1}{6}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 (4y - y^2 + 4y^3 + 1)^{-2/3} (12y^2 - 2y + 4) dy$

Solution

$$\text{Let } u = 4y - y^2 + 4y^3 + 1 \Rightarrow du = (4 - 2y + 12y^2) dy$$

$$\rightarrow \begin{cases} y = 1 & \rightarrow u = 8 \\ y = 0 & \rightarrow u = 1 \end{cases}$$

$$\begin{aligned} \int_0^1 (4y - y^2 + 4y^3 + 1)^{-2/3} (12y^2 - 2y + 4) dy &= \int_1^8 u^{-2/3} du \\ &= 3u^{1/3} \Big|_1^8 \\ &= 3(8^{1/3} - 1^{1/3}) \\ &= \underline{3} \end{aligned}$$

Exercise

Evaluate the integral $\int_0^5 |x - 2| dx$

Solution

$$|x - 2| = \begin{cases} x - 2 & x > 2 \\ -(x - 2) & x < 2 \end{cases}$$

$$\begin{aligned} \int_0^5 |x - 2| dx &= \int_0^2 -(x - 2) dx + \int_2^5 (x - 2) dx \\ &= -\frac{x^2}{2} + 2x \Big|_0^2 + \left(\frac{x^2}{2} - 2x \right) \Big|_2^5 \\ &= -\frac{4}{2} + 4 - 0 + \left(\frac{25}{2} - 10 - \left(\frac{4}{2} - 4 \right) \right) \\ &= -2 + 4 + \frac{25}{2} - 10 - 2 + 4 \\ &= \frac{25}{2} - 6 \\ &= \underline{\frac{13}{2}} \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/2} e^{\sin x} \cos x \, dx$

Solution

$$\begin{aligned}\int_0^{\pi/2} e^{\sin x} \cos x \, dx &= \int_0^{\pi/2} e^{\sin x} d(\sin x) \\ &= e^{\sin x} \Big|_0^{\pi/2} \\ &= e^{\sin \frac{\pi}{2}} - e^{\sin 0} \\ &= e^1 - e^0 \\ &= e - 1\end{aligned}$$

$$d(\sin x) = \cos x \, dx$$

Exercise

Evaluate $\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

Solution

$$\begin{aligned}\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1} x \Big|_{\sqrt{2}/2}^{\sqrt{3}/2} \\ &= \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) - \sin^{-1} \left(\frac{\sqrt{2}}{2} \right) \\ &= \frac{\pi}{3} - \frac{\pi}{4} \\ &= \frac{\pi}{12}\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

Exercise

Evaluate the integral $\int_0^{\pi/3} \frac{4 \sin \theta}{1 - 4 \cos \theta} d\theta$

Solution

$$\int_0^{\pi/3} \frac{4 \sin \theta}{1 - 4 \cos \theta} d\theta = \int_0^{\pi/3} \frac{d(1 - 4 \cos \theta)}{1 - 4 \cos \theta}$$

$$\begin{aligned}
&= \ln|1 - 4\cos\theta| \Big|_0^{\pi/3} \\
&= \left(\ln\left|1 - 4\cos\frac{\pi}{3}\right| - \ln|1 - 4\cos 0| \right) \\
&= \ln|-1| - \ln|-3| \\
&= \ln 1 - \ln 3 \\
&= -\ln 3 \\
&= \underline{\underline{\frac{1}{\ln 3}}}
\end{aligned}$$

Exercise

Evaluate the integral $\int_1^2 \frac{2\ln x}{x} dx$

Solution

$$u = \ln x \rightarrow du = \frac{1}{x} dx \rightarrow \begin{cases} x=1 & u = \ln 1 = 0 \\ x=2 & u = \ln 2 \end{cases}$$

$$\begin{aligned}
\int_1^2 \frac{2\ln x}{x} dx &= \int_0^{\ln 2} 2u du \\
&= u^2 \Big|_0^{\ln 2} \\
&= \underline{\underline{(\ln 2)^2}}
\end{aligned}$$

Exercise

Evaluate the integral $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$

Solution

$$u = \ln x \rightarrow du = \frac{1}{x} dx \rightarrow \begin{cases} x=2 & u = \ln 2 \\ x=16 & u = \ln 16 = \ln 2^4 \end{cases}$$

$$\begin{aligned}
\int_2^{16} \frac{dx}{2x\sqrt{\ln x}} &= \int_{\ln 2}^{4\ln 2} \frac{1}{2} u^{-1/2} du \\
&= u^{1/2} \Big|_{\ln 2}^{4\ln 2}
\end{aligned}$$

$$\begin{aligned}
 &= (4 \ln 2)^{1/2} - (\ln 2)^{1/2} \\
 &= 2\sqrt{\ln 2} - \sqrt{\ln 2} \\
 &= \sqrt{\ln 2}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/2} \tan \frac{x}{2} dx$

Solution

$$d \cos \frac{x}{2} = -\frac{1}{2} \sin \frac{x}{2} dx$$

$$\begin{aligned}
 \int_0^{\pi/2} \tan \frac{x}{2} dx &= \int_0^{\pi/2} \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} dx \\
 &= \int_0^{\pi/2} \frac{-2}{\cos \frac{x}{2}} d \cos \frac{x}{2} \\
 &= -2 \ln \left| \cos \frac{x}{2} \right| \Big|_0^{\pi/2} \\
 &= -2 \left[\ln \left| \cos \frac{\pi}{4} \right| - \ln |\cos 0| \right] \\
 &= -2 \left[\ln \left| \frac{1}{\sqrt{2}} \right| - \ln |1| \right] \\
 &= -2 \ln 2^{-1/2} \\
 &= \ln 2
 \end{aligned}$$

Exercise

Evaluate the integral $\int_{\pi/4}^{\pi/2} \cot x dx$

Solution

$$\begin{aligned}
 \int_{\pi/4}^{\pi/2} \cot x dx &= \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x} dx \\
 &= \int_{\pi/4}^{\pi/2} \frac{d(\sin x)}{\sin x}
 \end{aligned}$$

$$\begin{aligned}
&= \ln(\sin x) \Big|_{\pi/4}^{\pi/2} \\
&= \ln 1 - \ln \frac{1}{\sqrt{2}} \\
&= -\ln \frac{1}{\sqrt{2}} \\
&= \ln \sqrt{2}
\end{aligned}$$

Exercise

Evaluate the integral $\int_{-\ln 2}^0 e^{-x} dx$

Solution

$$\begin{aligned}
\int_{-\ln 2}^0 e^{-x} dx &= -e^{-x} \Big|_{-\ln 2}^0 \\
&= -(e^0 - e^{\ln 2}) \\
&= -(1 - 2) \\
&= 1
\end{aligned}$$

Exercise

Evaluate the integral $\int_{\pi/4}^{\pi/2} (1 + e^{\cot \theta}) \csc^2 \theta d\theta$

Solution

$$\text{Let } u = \cot \theta \Rightarrow du = -\csc^2 \theta d\theta \rightarrow \begin{cases} \theta = \frac{\pi}{2} \Rightarrow u = 0 \\ \theta = \frac{\pi}{4} \Rightarrow u = 1 \end{cases}$$

$$\begin{aligned}
\int_{\pi/4}^{\pi/2} (1 + e^{\cot \theta}) \csc^2 \theta d\theta &= \int_{\pi/4}^{\pi/2} \csc^2 \theta d\theta + \int_{\pi/4}^{\pi/2} e^{\cot \theta} \csc^2 \theta d\theta \\
&= -\cot \theta \Big|_{\pi/4}^{\pi/2} + \int_0^1 e^u du \\
&= -\left(\cot \frac{\pi}{2} - \cot \frac{\pi}{4}\right) + e^u \Big|_0^1 \\
&= -(0 - 1) + e^1 - 1 \\
&= e
\end{aligned}$$

Exercise

Evaluate the integral $\int_1^4 \frac{2\sqrt{x}}{\sqrt{x}} dx$

Solution

$$\text{Let } u = \sqrt{x} \rightarrow du = \frac{1}{2\sqrt{x}} dx \Rightarrow 2du = \frac{1}{\sqrt{x}} dx \rightarrow \begin{cases} x=1 & u=1 \\ x=4 & u=\sqrt{4}=2 \end{cases}$$

$$\begin{aligned} \int_1^4 \frac{2\sqrt{x}}{\sqrt{x}} dx &= \int_1^2 2^u (2du) \\ &= 2 \int_1^2 2^u du \\ &= 2 \left[\frac{2^u}{\ln 2} \right]_1^2 \\ &= \frac{2}{\ln 2} [2^2 - 2^1] \\ &= \frac{2}{\ln 2} (2) \\ &= \frac{4}{\ln 2} \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} \sec^2 t dt$

Solution

$$u = \tan t \Rightarrow du = \sec^2 t dt \rightarrow \begin{cases} t = \frac{\pi}{4} & \rightarrow u = 1 \\ t = 0 & \rightarrow u = 0 \end{cases}$$

$$\begin{aligned} \int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} \sec^2 t dt &= \int_0^1 \left(\frac{1}{3}\right)^u du \\ &= \left[\frac{1}{\ln \frac{1}{3}} \left(\frac{1}{3}\right)^u \right]_0^1 \\ &= \frac{1}{-\ln 3} \left[\frac{1}{3} - 1 \right] \end{aligned}$$

$$= \frac{1}{-\ln 3} \left(\frac{-2}{3} \right)$$

$$= \frac{2}{3 \ln 3}$$

Exercise

Evaluate the integral $\int_1^e x^{(\ln 2)^{-1}} dx$

Solution

$$\int_1^e x^{(\ln 2)^{-1}} dx = \frac{x^{\ln 2}}{\ln 2} \Big|_1^e$$

$$= \frac{1}{\ln 2} (e^{\ln 2} - 1)$$

$$= \frac{1}{\ln 2} (2 - 1)$$

$$= \frac{1}{\ln 2}$$

Exercise

Evaluate the integral $\int_1^e \frac{2 \ln 10 \log_{10} x}{x} dx$

Solution

$$\int_1^e \frac{2 \ln 10 \log_{10} x}{x} dx = 2 \ln 10 \int_1^e \frac{1}{x} \frac{\ln x}{\ln 10} dx$$

$$= 2 \int_1^e \frac{\ln x}{x} dx$$

$$= 2 \int_1^e \ln x \, d(\ln x)$$

$$= 2 \left[\frac{1}{2} (\ln x)^2 \right]_1^e$$

$$= (\ln e)^2 - (\ln 1)^2$$

$$= 1$$

$$d(\ln x) = \frac{1}{x} dx$$

Exercise

Evaluate the integral $\int_0^9 \frac{2 \log_{10}(x+1)}{x+1} dx$

Solution

$$\begin{aligned} \int_0^9 \frac{2 \log_{10}(x+1)}{x+1} dx &= 2 \int_0^9 \frac{1}{x+1} \frac{\ln(x+1)}{\ln 10} dx \\ &= \frac{2}{\ln 10} \int_0^9 \frac{\ln(x+1)}{x+1} dx & d(\ln(x+1)) &= \frac{1}{x+1} dx \\ &= \frac{2}{\ln 10} \int_0^9 \ln(x+1) d(x+1) \\ &= \frac{2}{\ln 10} \left[\frac{1}{2} (\ln(x+1))^2 \right]_0^9 \\ &= \frac{1}{\ln 10} \left[(\ln 10)^2 - (\ln 1)^2 \right] \\ &= \frac{1}{\ln 10} \left[(\ln 10)^2 \right] \\ &= \ln 10 \end{aligned}$$

Exercise

Evaluate the integral $\int_1^{e^x} \frac{1}{t} dt$

Solution

$$\begin{aligned} \int_1^{e^x} \frac{1}{t} dt &= \ln|t| \Big|_1^{e^x} \\ &= \ln|e^x| - \ln 1 \\ &= x \end{aligned}$$

Exercise

Evaluate the integral $\frac{1}{\ln a} \int_1^x \frac{1}{t} dt \quad x > 0$

Solution

$$\begin{aligned}
 \frac{1}{\ln a} \int_1^x \frac{1}{t} dt &= \frac{1}{\ln a} \left[\ln |t| \right]_1^x \\
 &= \frac{1}{\ln a} (\ln x - \ln 1) \\
 &= \frac{\ln x}{\ln a} \\
 &= \log_a x
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\sqrt{\ln \pi}} 2x e^{x^2} \cos(e^{x^2}) dx$

Solution

$$\begin{aligned}
 \int_0^{\sqrt{\ln \pi}} 2x e^{x^2} \cos(e^{x^2}) dx &= \int_0^{\sqrt{\ln \pi}} \cos(e^{x^2}) d(e^{x^2}) \\
 &= \sin(e^{x^2}) \Big|_0^{\sqrt{\ln \pi}} \\
 &= \sin \pi - \sin 1 \\
 &= -\sin 1 \approx -0.84147
 \end{aligned}$$

Exercise

Evaluate $\int_0^{3\sqrt{2}/4} \frac{dx}{\sqrt{9-4x^2}}$

Solution

Let: $u = 2x \Rightarrow du = 2dx \rightarrow \frac{du}{2} = dx$

$$\begin{cases} x = \frac{3\sqrt{2}}{4} & \rightarrow u = \frac{3\sqrt{2}}{2} \\ x = 0 & \rightarrow u = 0 \end{cases}$$

$$\begin{aligned}
 \int_0^{3\sqrt{2}/4} \frac{dx}{\sqrt{9-4x^2}} &= \frac{1}{2} \int_0^{3\sqrt{2}/2} \frac{du}{\sqrt{9-u^2}} \\
 &= \frac{1}{2} \sin^{-1} \frac{u}{3} \Big|_0^{3\sqrt{2}/2} \\
 &= \frac{1}{2} \left(\sin^{-1} \frac{\sqrt{2}}{2} - \sin^{-1} 0 \right)
 \end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

$$= \frac{1}{2} \left(\frac{\pi}{4} - 0 \right)$$

$$= \frac{\pi}{8}$$

Exercise

Evaluate $\int_{\pi/6}^{\pi/4} \frac{\csc^2 x dx}{1 + (\cot x)^2}$

Solution

$$u = \cot x \Rightarrow du = -\csc^2 x dx$$

$$a^2 = 1 \rightarrow a = 1$$

$$\begin{cases} x = \frac{\pi}{4} & \rightarrow u = \cot \frac{\pi}{4} = 1 \\ x = \frac{\pi}{6} & \rightarrow u = \cot \frac{\pi}{6} = \sqrt{3} \end{cases}$$

$$\int_{\pi/6}^{\pi/4} \frac{\csc^2 x dx}{1 + (\cot x)^2} = - \int_{\sqrt{3}}^1 \frac{du}{1 + u^2}$$

$$= -\tan^{-1} u \Big|_{\sqrt{3}}^1$$

$$= -\left(\tan^{-1} 1 - \tan^{-1} \sqrt{3} \right)$$

$$= -\left(\frac{\pi}{4} - \frac{\pi}{3} \right)$$

$$= \frac{\pi}{12}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Exercise

Evaluate $\int_1^{e^{\pi/4}} \frac{4dt}{t(1 + \ln^2 t)}$

Solution

$$u = \ln t \Rightarrow du = \frac{dt}{t}$$

$$a^2 = 1 \rightarrow a = 1$$

$$\begin{cases} u = e^{\pi/4} & \rightarrow u = \ln e^{\pi/4} = \frac{\pi}{4} \\ u = 1 & \rightarrow u = \ln 1 = 0 \end{cases}$$

$$\begin{aligned}
 \int_1^{e^{\pi/4}} \frac{4dt}{t(1+\ln^2 t)} &= 4 \int_0^{\pi/4} \frac{du}{1+u^2} \\
 &= \tan^{-1} u \Big|_0^{\pi/4} \\
 &= 4 \left(\tan^{-1} \frac{\pi}{4} - \tan^{-1} 0 \right) \\
 &= \underline{4 \tan^{-1} \frac{\pi}{4}}
 \end{aligned}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Exercise

Evaluate $\int_{1/2}^1 \frac{6dx}{\sqrt{-4x^2 + 4x + 3}}$

Solution

$$\begin{aligned}
 -4x^2 + 4x + 3 &= -4x^2 + 4x + 3 + 1 - 1 \\
 &= 4 - 4x^2 + 4x - 1 \\
 &= 4 - (4x^2 - 4x + 1) \\
 &= 2^2 - (2x - 1)^2
 \end{aligned}$$

$$\begin{aligned}
 \int_{1/2}^1 \frac{6dx}{\sqrt{-4x^2 + 4x + 3}} &= \int_{1/2}^1 \frac{6dx}{\sqrt{2^2 - (2x - 1)^2}} \\
 &= \int_{1/2}^1 \frac{3du}{\sqrt{2^2 - u^2}} \\
 &= 3 \sin^{-1} \left(\frac{2x - 1}{2} \right) \Big|_{1/2}^1 \\
 &= 3 \left[\sin^{-1} \left(\frac{1}{2} \right) - \sin^{-1} (0) \right] \\
 &= 3 \left(\frac{\pi}{6} - 0 \right) \\
 &= \underline{\frac{\pi}{2}}
 \end{aligned}$$

$$u = 2x - 1 \Rightarrow du = 2dx \rightarrow \frac{du}{2} = dx$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$$

Exercise

Evaluate
$$\int_{2/\sqrt{3}}^2 \frac{\cos(\sec^{-1} x) dx}{x\sqrt{x^2-1}}$$

Solution

$$u = \sec^{-1} x \Rightarrow du = \frac{dx}{x\sqrt{x^2-1}}$$

$$\begin{cases} x = 2 & \rightarrow u = \sec^{-1} 2 = \frac{\pi}{3} \\ x = \frac{2}{\sqrt{3}} & \rightarrow u = \sec^{-1} \frac{2}{\sqrt{3}} = \frac{\pi}{6} \end{cases}$$

$$\begin{aligned} \int_{2/\sqrt{3}}^2 \frac{\cos(\sec^{-1} x) dx}{x\sqrt{x^2-1}} &= \int_{\pi/6}^{\pi/3} \cos u \, du \\ &= \sin u \Big|_{\pi/6}^{\pi/3} \\ &= \sin \frac{\pi}{3} - \sin \frac{\pi}{6} \\ &= \frac{\sqrt{3}}{2} - \frac{1}{2} \\ &= \frac{\sqrt{3}-1}{2} \end{aligned}$$

Exercise

Evaluate the definite integral
$$\int_0^3 \frac{x}{\sqrt{25-x^2}} dx$$

Solution

$$\begin{aligned} \int_0^3 \frac{x}{\sqrt{25-x^2}} dx &= -\frac{1}{2} \int_0^3 (25-x^2)^{-1/2} d(25-x^2) \\ &= -\left(25-x^2\right)^{1/2} \Big|_0^3 \\ &= -(4-5) \\ &= 1 \end{aligned}$$

$$d(25-x^2) = -2x dx$$

Exercise

Evaluate the definite integral $\int_0^{\pi} \sin^2 5\theta \, d\theta$

Solution

$$\begin{aligned} \int_0^{\pi} \sin^2 5\theta \, d\theta &= \frac{1}{2} \int_0^{\pi} (1 - \cos 10\theta) \, d\theta \\ &= \frac{1}{2} \left(\theta - \frac{1}{10} \sin 10\theta \right) \Big|_0^{\pi} \\ &= \frac{\pi}{2} \end{aligned}$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

Exercise

Evaluate the definite integral $\int_0^{\pi} (1 - \cos^2 3\theta) \, d\theta$

Solution

$$\begin{aligned} \int_0^{\pi} (1 - \cos^2 3\theta) \, d\theta &= \int_0^{\pi} \left(1 - \frac{1}{2} - \cos 6\theta \right) \, d\theta \\ &= \int_0^{\pi} \left(\frac{1}{2} - \cos 6\theta \right) \, d\theta \\ &= \frac{1}{2} \theta - \frac{1}{6} \sin 6\theta \Big|_0^{\pi} \\ &= \frac{\pi}{2} \end{aligned}$$

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

Exercise

Evaluate the definite integral $\int_2^3 \frac{x^2 + 2x - 2}{x^3 + 3x^2 - 6x} \, dx$

Solution

$$\begin{aligned} d(x^3 + 3x^2 - 6x) &= (3x^2 + 6x - 6) \, dx \\ &= 3(x^2 + 2x - 2) \, dx \end{aligned}$$

$$\begin{aligned}
\int_2^3 \frac{x^2 + 2x - 2}{x^3 + 3x^2 - 6x} dx &= \frac{1}{3} \int_2^3 \frac{1}{x^3 + 3x^2 - 6x} d(x^3 + 3x^2 - 6x) \\
&= \frac{1}{3} \ln \left| x^3 + 3x^2 - 6x \right| \Big|_2^3 \\
&= \frac{1}{3} (\ln 36 - \ln 8) \\
&= \frac{1}{3} (\ln 6^2 - \ln 2^3) \\
&= \frac{1}{3} (2 \ln 6 - 3 \ln 2) \\
&= \underline{\underline{\frac{2}{3} \ln 6 - \ln 2}}
\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^{\ln 2} \frac{e^x}{1 + e^{2x}} dx$

Solution

$$\begin{aligned}
\int_0^{\ln 2} \frac{e^x}{1 + e^{2x}} dx &= \int_0^{\ln 2} \frac{1}{1 + (e^x)^2} d(e^x) \\
&= \arctan e^x \Big|_0^{\ln 2} \\
&= \arctan e^{\ln 2} - \arctan 1 \\
&= \underline{\underline{\arctan 2 - \frac{\pi}{4}}}
\end{aligned}$$

$$d(e^x) = e^x dx$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Exercise

Evaluate the definite integral $\int_1^3 x \sqrt[3]{x^2 - 1} dx$

Solution

$$\begin{aligned}
\int_1^3 x \sqrt[3]{x^2 - 1} dx &= \frac{1}{2} \int_1^3 (x^2 - 1)^{1/3} d(x^2 - 1) \\
&= \frac{3}{8} (x^2 - 1)^{4/3} \Big|_1^3
\end{aligned}$$

$$d(x^2 - 1) = 2x dx$$

$$\begin{aligned}
 &= \frac{3}{8} \left(8^{4/3} - 0 \right) \\
 &= \frac{3}{8} 2^4 \\
 &= \underline{6}
 \end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^2 (x+3)^3 dx$

Solution

$$\begin{aligned}
 \int_0^2 (x+3)^3 dx &= \int_0^2 (x+3)^3 d(x+3) & d(x+3) &= dx \\
 &= \frac{1}{4} (x+3)^4 \Big|_0^2 \\
 &= \frac{1}{4} (5^4 - 3^4) \\
 &= \frac{1}{4} (625 - 81) \\
 &= \frac{544}{4} \\
 &= \underline{136}
 \end{aligned}$$

Exercise

Evaluate the definite integral $\int_{-2}^2 e^{4x+8} dx$

Solution

$$\begin{aligned}
 \int_{-2}^2 e^{4x+8} dx &= \frac{1}{4} \int_{-2}^2 e^{4x+8} d(4x+8) & d(4x+8) &= 4dx \\
 &= \frac{1}{4} e^{4x+8} \Big|_{-2}^2 \\
 &= \underline{\frac{1}{4} (e^{16} - 1)}
 \end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^1 \sqrt{x}(\sqrt{x}+1)dx$

Solution

$$\begin{aligned}\int_0^1 \sqrt{x}(\sqrt{x}+1)dx &= \int_0^1 (x+x^{1/2}) dx \\ &= \frac{1}{2}x + \frac{2}{3}x^{3/2} \Big|_0^1 \\ &= \frac{1}{2} + \frac{2}{3} \\ &= \frac{7}{6}\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

Solution

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{4-x^2}} &= \sin^{-1} \frac{x}{2} \Big|_0^1 \\ &= \sin^{-1} \frac{1}{2} - \sin^{-1} 0 \\ &= \frac{\pi}{6}\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a}$$

Exercise

Evaluate the definite integral $\int_0^2 \frac{2x}{(x^2+1)^2} dx$

Solution

$$\begin{aligned}\int_0^2 \frac{2x}{(x^2+1)^2} dx &= \int_0^2 \frac{1}{(x^2+1)^2} d(x^2+1) \\ &= -\frac{1}{x^2+1} \Big|_0^2\end{aligned}$$

$$= -\left(\frac{1}{5} - 1\right)$$

$$\underline{= \frac{4}{5}}$$

Exercise

Evaluate the definite integral $\int_0^{\pi/2} \sin^2 \theta \cos \theta \, d\theta$

Solution

$$\int_0^{\pi/2} \sin^2 \theta \cos \theta \, d\theta = \int_0^{\pi/2} \sin^2 \theta \, d(\sin \theta) \quad d(\sin \theta) = \cos \theta \, d\theta$$

$$= \frac{1}{3} \sin^3 \theta \bigg|_0^{\pi/2}$$

$$\underline{= \frac{1}{3}}$$

Exercise

Evaluate the definite integral $\int_0^{\pi/4} \frac{\sin x}{\cos^2 x} \, dx$

Solution

$$\int_0^{\pi/4} \frac{\sin x}{\cos^2 x} \, dx = - \int_0^{\pi/4} \frac{1}{\cos^2 x} \, d(\cos x)$$

$$= \frac{1}{\cos x} \bigg|_0^{\pi/4}$$

$$\underline{= \sqrt{2} - 1}$$

Exercise

Evaluate the definite integral $\int_{1/3}^{1/\sqrt{3}} \frac{4}{9x^2 + 1} \, dx$

Solution

$$\int_{1/3}^{1/\sqrt{3}} \frac{4}{9x^2 + 1} \, dx = \frac{4}{3} \int_{1/3}^{1/\sqrt{3}} \frac{1}{(3x)^2 + 1} \, d(3x) \quad d(3x) = 3 \, dx$$

$$\begin{aligned}
&= \frac{4}{3} \arctan(3x) \Big|_{1/3}^{1/\sqrt{3}} \\
&= \frac{4}{3} \left(\arctan(\sqrt{3}) - \arctan 1 \right) \\
&= \frac{4}{3} \left(\frac{\pi}{3} - \frac{\pi}{4} \right) \\
&= \frac{\pi}{9}
\end{aligned}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Exercise

Evaluate the definite integral $\int_0^{\ln 4} \frac{e^x}{3 + 2e^x} dx$

Solution

$$\begin{aligned}
\int_0^{\ln 4} \frac{e^x}{3 + 2e^x} dx &= \frac{1}{2} \int_0^{\ln 4} \frac{1}{3 + 2e^x} d(3 + 2e^x) \\
&= \frac{1}{2} \ln(3 + 2e^x) \Big|_0^{\ln 4} \\
&= \frac{1}{2} \left(\ln(3 + 2e^{\ln 4}) - \ln 5 \right) \\
&= \frac{1}{2} (\ln 11 - \ln 5) \\
&= \frac{1}{2} \ln \frac{11}{5}
\end{aligned}$$

Exercise

Evaluate the definite integral $\int_{-\pi}^{\pi} \cos^2 x \, dx$

Solution

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos^2 x \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} (1 + \cos 2x) \, dx \\
&= \frac{1}{2} \left(x + \frac{1}{2} \sin 2x \right) \Big|_{-\pi}^{\pi} \\
&= \frac{1}{2} (\pi + \pi) \\
&= \pi
\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^{\pi/4} \cos^2 8\theta \, d\theta$

Solution

$$\begin{aligned} \int_0^{\pi/4} \cos^2 8\theta \, d\theta &= \frac{1}{2} \int_0^{\pi/4} (1 + \cos 16\theta) \, d\theta \\ &= \frac{1}{2} \left(\theta + \frac{1}{16} \sin 16\theta \right) \bigg|_0^{\pi/4} \\ &= \frac{1}{2} \left(\frac{\pi}{4} \right) \\ &= \frac{\pi}{8} \end{aligned}$$

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

Exercise

Evaluate the definite integral $\int_{-\pi/4}^{\pi/4} \sin^2 2\theta \, d\theta$

Solution

$$\begin{aligned} \int_{-\pi/4}^{\pi/4} \sin^2 2\theta \, d\theta &= \frac{1}{2} \int_{-\pi/4}^{\pi/4} (1 - \cos 4\theta) \, d\theta \\ &= \frac{1}{2} \left(\theta - \frac{1}{4\theta} \sin 4\theta \right) \bigg|_{-\pi/4}^{\pi/4} \\ &= \frac{1}{2} \left(\frac{\pi}{4} + \frac{\pi}{4} \right) \\ &= \frac{\pi}{4} \end{aligned}$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

Exercise

Evaluate the definite integral $\int_0^{\pi/6} \frac{\sin 2x}{\sin^2 x + 2} \, dx$

Solution

$$\begin{aligned} d(\sin^2 x + 2) &= 2 \sin x \cos x \, dx \\ &= \sin 2x \, dx \\ \int_0^{\pi/6} \frac{\sin 2x}{\sin^2 x + 2} \, dx &= \int_0^{\pi/6} \frac{1}{\sin^2 x + 2} d(\sin^2 x + 2) \end{aligned}$$

$$\begin{aligned}
&= \ln \left| \sin^2 x + 2 \right| \Bigg|_0^{\pi/6} \\
&= \ln \frac{9}{4} - \ln 2 \\
&= \ln \frac{9}{8}
\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^{\pi/2} \sin^4 \theta \, d\theta$

Solution

$$\begin{aligned}
\int_0^{\pi/2} \sin^4 \theta \, d\theta &= \int_0^{\pi/2} \left(\frac{1 - \cos 2\theta}{2} \right)^2 d\theta & \sin^2 \theta &= \frac{1}{2}(1 - \cos 2\theta) \\
&= \frac{1}{4} \int_0^{\pi/2} (1 - 2\cos 2\theta + \cos^2 2\theta) \, d\theta \\
&= \frac{1}{4} \int_0^{\pi/2} \left(1 - 2\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta \right) d\theta \\
&= \frac{1}{4} \int_0^{\pi/2} \left(\frac{3}{2} - 2\cos 2\theta + \frac{1}{2}\cos 4\theta \right) d\theta \\
&= \frac{1}{4} \left(\frac{3}{2}\theta - \sin 2\theta + \frac{1}{8}\sin 4\theta \right) \Bigg|_0^{\pi/2} \\
&= \frac{1}{4} \left(\frac{3}{2} \frac{\pi}{2} \right) \\
&= \frac{3\pi}{16}
\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^1 x\sqrt{1-x^2} \, dx$

Solution

$$\int_0^1 x\sqrt{1-x^2} \, dx = -\frac{1}{2} \int_0^1 (1-x^2)^{1/2} d(1-x^2) \qquad d(1-x^2) = -2x dx$$

$$\begin{aligned}
&= -\frac{1}{3} \left(1 - x^2\right)^{3/2} \Big|_0^1 \\
&= -\frac{1}{3} (0 - 1) \\
&= \frac{1}{3}
\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^{1/4} \frac{x}{\sqrt{1-16x^2}} dx$

Solution

$$\begin{aligned}
\int_0^{1/4} \frac{x}{\sqrt{1-16x^2}} dx &= -\frac{1}{32} \int_0^{1/4} \left(1-16x^2\right)^{-1/2} d\left(1-16x^2\right) \\
&= -\frac{1}{16} \left(1-16x^2\right)^{1/2} \Big|_0^{1/4} \\
&= -\frac{1}{16} (0 - 1) \\
&= \frac{1}{16}
\end{aligned}$$

Exercise

Evaluate the definite integral $\int_2^3 \frac{x}{\sqrt[3]{x^2-1}} dx$

Solution

$$\begin{aligned}
\int_2^3 \frac{x}{\sqrt[3]{x^2-1}} dx &= \frac{1}{2} \int_2^3 \left(x^2-1\right)^{-1/3} d\left(x^2-1\right) & d\left(x^2-1\right) &= 2x dx \\
&= \frac{3}{4} \left(x^2-1\right)^{2/3} \Big|_2^3 \\
&= \frac{3}{4} \left(8^{2/3} - 1\right) \\
&= \frac{3}{4} (4 - 1) \\
&= \frac{9}{4}
\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^{6/5} \frac{dx}{25x^2 + 36}$

Solution

$$\begin{aligned}
 \int_0^{6/5} \frac{dx}{25x^2 + 36} &= \int_0^{6/5} \frac{dx}{25\left(x^2 + \frac{36}{25}\right)} \\
 &= \int_0^{6/5} \frac{dx}{25\left(x^2 + \left(\frac{6}{5}\right)^2\right)} \\
 &= \frac{1}{25} \left(\frac{5}{6}\right) \tan^{-1} \frac{5x}{6} \Bigg|_0^{6/5} \\
 &= \frac{1}{30} \left(\tan^{-1} 1 - \tan^{-1} 0\right) \\
 &= \frac{1}{30} \left(\frac{\pi}{4}\right) \\
 &= \frac{\pi}{120}
 \end{aligned}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Exercise

Evaluate the definite integral $\int_0^2 x^3 \sqrt{16 - x^4} \, dx$

Solution

$$\begin{aligned}
 \int_0^2 x^3 \sqrt{16 - x^4} \, dx &= -\frac{1}{4} \int_0^2 \left(16 - x^4\right)^{1/2} d\left(16 - x^4\right) \\
 &= -\frac{1}{6} \left(16 - x^4\right)^{3/2} \Bigg|_0^2 \\
 &= -\frac{1}{6} \left(0 - 4^3\right) \\
 &= \frac{32}{3}
 \end{aligned}$$

$$d(16 - x^4) = -4x^3 dx$$

Exercise

Evaluate the definite integral $\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^2 x} dx$

Solution

$$\begin{aligned}
 \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^2 x} dx &= \int_{\pi/4}^{\pi/2} \frac{1}{\sin^2 x} d(\sin x) & d(\sin x) &= \cos x \, dx \\
 &= -\frac{1}{\sin x} \Big|_{\pi/4}^{\pi/2} \\
 &= -(1 - \sqrt{2}) \\
 &= \underline{\sqrt{2} - 1}
 \end{aligned}$$

Exercise

Evaluate the definite integral $\int_{-1}^1 (x-1)(x^2-2x)^7 dx$

Solution

$$\begin{aligned}
 d(x^2 - 2x) &= (2x - 2) dx \\
 &= 2(x - 1) dx \\
 \int_{-1}^1 (x-1)(x^2-2x)^7 dx &= \frac{1}{2} \int_{-1}^1 (x^2-2x)^7 d(x^2-2x) \\
 &= \frac{1}{16} (x^2-2x)^8 \Big|_{-1}^1 \\
 &= \frac{1}{16} (1 - 3^8) \\
 &= \frac{6560}{16} \\
 &= \underline{410}
 \end{aligned}$$

Exercise

Evaluate the definite integral $\int_{-\pi}^0 \frac{\sin x}{2 + \cos x} dx$

Solution

$$\begin{aligned}
 \int_{-\pi}^0 \frac{\sin x}{2 + \cos x} dx &= - \int_{-\pi}^0 \frac{1}{2 + \cos x} d(2 + \cos x) & d(2 + \cos x) &= -\sin x \, dx \\
 &= -\ln|2 + \cos x| \Big|_{-\pi}^0 \\
 &= -(\ln 3 - \ln 1) \\
 &= \underline{-\ln 3}
 \end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^1 \frac{(x+1)(x+2)}{2x^3 + 9x^2 + 12x + 36} dx$

Solution

$$\begin{aligned}
 d(2x^3 + 9x^2 + 12x + 36) &= (6x^2 + 18x + 12) dx \\
 &= 6(x^2 + 3x + 2) dx \\
 \int_0^1 \frac{(x+1)(x+2)}{2x^3 + 9x^2 + 12x + 36} dx &= \int_0^1 \frac{x^2 + 3x + 2}{2x^3 + 9x^2 + 12x + 36} dx \\
 &= \frac{1}{6} \int_0^1 \frac{1}{2x^3 + 9x^2 + 12x + 36} d(2x^3 + 9x^2 + 12x + 36) \\
 &= \frac{1}{6} \ln|2x^3 + 9x^2 + 12x + 36| \Big|_0^1 \\
 &= \frac{1}{6} (\ln 59 - \ln 36) \\
 &= \underline{\frac{1}{6} \ln \frac{59}{36}}
 \end{aligned}$$

Exercise

Evaluate the definite integral $\int_1^2 \frac{4}{9x^2 + 6x + 1} dx$

Solution

$$\int_1^2 \frac{4}{9x^2 + 6x + 1} dx = \int_1^2 \frac{4}{(3x+1)^2} dx$$

$$\begin{aligned}
&= \frac{4}{3} \int_1^2 \frac{1}{(3x+1)^2} d(3x+1) & d(3x+1) = 3x dx \\
&= -\frac{4}{3} \frac{1}{3x+1} \Big|_1^2 \\
&= -\frac{4}{3} \left(\frac{1}{7} - \frac{1}{4} \right) \\
&= -\frac{4}{3} \left(-\frac{3}{28} \right) \\
&= \frac{1}{7}
\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^{\pi/4} e^{\sin^2 x} \sin 2x \, dx$

Solution

$$\begin{aligned}
d(\sin^2 x) &= 2 \sin x \cos x \, dx \\
&= \sin 2x \, dx
\end{aligned}$$

$$\begin{aligned}
\int_0^{\pi/4} e^{\sin^2 x} \sin 2x \, dx &= \int_0^{\pi/4} e^{\sin^2 x} d(\sin^2 x) \\
&= e^{\sin^2 x} \Big|_0^{\pi/4} \\
&= e^{\sin^2 \frac{\pi}{4}} - e^{\sin^2 0} \\
&= e^{\frac{1}{2}} - 1 \\
&= \sqrt{e} - 1
\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^1 x \sqrt{x+a} \, dx \quad (a > 0)$

Solution

$$\text{Let } u = x + a \rightarrow x = u - a \Rightarrow du = dx$$

$$\int_0^1 x \sqrt{x+a} \, dx = \int_0^1 (u-a) u^{1/2} \, du$$

$$\begin{aligned}
&= \int_0^1 \left(u^{3/2} - au^{1/2} \right) du \\
&= \left. \frac{2}{5} u^{5/2} - \frac{2}{3} au^{3/2} \right|_0^1 \\
&= \left. \frac{2}{5} (x+a)^{5/2} - \frac{2}{3} a(x+a)^{3/2} \right|_0^1 \\
&= \frac{2}{5} (1+a)^{5/2} - \frac{2}{3} a(1+a)^{3/2} - \frac{2}{5} a^{5/2} + \frac{2}{3} a(a^{3/2}) \\
&= \frac{2}{5} (1+a)^{5/2} - \frac{2}{3} a(1+a)^{3/2} - \frac{2}{5} a^{5/2} + \frac{2}{3} a^{5/2} \\
&= \frac{2}{5} (1+a)^{5/2} - \frac{2}{3} a(1+a)^{3/2} + \frac{4}{15} a^{5/2} \\
&= \frac{2}{5} (1+a)^2 \sqrt{1+a} - \frac{2}{3} a(1+a) \sqrt{1+a} + \frac{4}{15} a^2 \sqrt{a} \\
&= \left(\frac{2}{5} (1+a)^2 - \frac{2}{3} (a+a^2) \right) \sqrt{1+a} + \frac{4}{15} a^2 \sqrt{a}
\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^1 x \sqrt[p]{x+a} \, dx \quad (a > 0)$

Solution

Let $u = x + a \rightarrow x = u - a$

$$du = dx$$

$$\begin{aligned}
\int_0^1 x \sqrt[p]{x+a} \, dx &= \int_0^1 (u-a) u^{1/p} \, du \\
&= \int_0^1 \left(u^{1+1/p} - au^{1/p} \right) du \\
&= \left. \frac{p}{2p+1} u^{2+1/p} - \frac{p}{p+1} au^{1+1/p} \right|_0^1 \\
&= \left. \frac{p}{2p+1} (x+a)^{2+1/p} - \frac{p}{p+1} a(x+a)^{1+1/p} \right|_0^1 \\
&= \frac{p}{2p+1} (1+a)^{2+1/p} - \frac{p}{p+1} a(1+a)^{1+1/p} - \frac{p}{2p+1} a^{2+1/p} + \frac{p}{p+1} a(a)^{1+1/p}
\end{aligned}$$

$$\begin{aligned}
&= \frac{p}{2p+1}(1+a)^2 \sqrt[p]{1+a} - \frac{p}{p+1}a(1+a)\sqrt[p]{1+a} - \frac{p}{2p+1}a^{2+1/p} + \frac{p}{p+1}a^{2+1/p} \\
&= \left(\frac{p}{2p+1}(1+a)^2 - \frac{p}{p+1}a(1+a) \right) \sqrt[p]{1+a} + \left(\frac{p}{p+1} - \frac{p}{2p+1} \right) a^{2+1/p} \\
&= \left(\frac{p}{2p+1}(1+a)^2 - \frac{p}{p+1}(a+a^2) \right) \sqrt[p]{1+a} + \left(\frac{2p^2+p-p^2-p}{(p+1)(2p+1)} \right) a^{2+1/p} \\
&= \left(\frac{p}{2p+1}(1+a)^2 - \frac{p}{p+1}(a+a^2) \right) \sqrt[p]{1+a} + \frac{p^2}{(p+1)(2p+1)} a^{2+1/p} \Bigg|
\end{aligned}$$

Or

$$\begin{aligned}
\text{Let } u &= \sqrt[p]{x+a} \rightarrow u^p = x+a \\
x &= u^p - a \rightarrow dx = pu^{p-1} du
\end{aligned}$$

$$\begin{aligned}
\int_0^1 x \sqrt[p]{x+a} dx &= \int_0^1 (u^p - a) \cdot u \cdot (pu^{p-1}) du \\
&= p \int_0^1 (u^p - a) \cdot u^p du \\
&= p \int_0^1 (u^{2p} - au^p) du \\
&= p \left(\frac{1}{2p+1} \left(\sqrt[p]{x+a} \right)^{2p+1} - \frac{1}{p+1} a \left(\sqrt[p]{x+a} \right)^{p+1} \right) \Bigg|_0^1 \\
&= p \left(\frac{1}{2p+1} \left(\sqrt[p]{1+a} \right)^{2p+1} - \frac{1}{p+1} a \left(\sqrt[p]{1+a} \right)^{\frac{p+1}{p}} - \frac{1}{2p+1} \left(\sqrt[p]{a} \right)^{2p+1} + \frac{1}{p+1} a \left(\sqrt[p]{a} \right)^{p+1} \right) \\
&= p \left(\frac{1}{2p+1} (1+a)^{(2p+1)/p} - \frac{1}{p+1} a (1+a)^{(p+1)/p} \right. \\
&\quad \left. - \frac{1}{2p+1} (a)^{(2p+1)/p} + \frac{1}{p+1} a (a)^{(p+1)/p} \right) \\
&= p \left(\frac{1}{2p+1} (1+a)^{(2p+1)/p} - \frac{1}{p+1} a (1+a)^{(p+1)/p} \right. \\
&\quad \left. - \frac{1}{2p+1} (a)^{(2p+1)/p} + \frac{1}{p+1} (a)^{(2p+1)/p} \right)
\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^1 x \sqrt{1-\sqrt{x}} \, dx$

Solution

$$u = 1 - \sqrt{x} \rightarrow x = (1-u)^2$$

$$dx = -2(1-u) du$$

$$\begin{aligned} \int_0^1 x \sqrt{1-\sqrt{x}} \, dx &= -2 \int_0^1 (1-u)^2 u^{1/2} (1-u) du \\ &= -2 \int_0^1 (1-u)^3 u^{1/2} du \\ &= -2 \int_0^1 (1-3u+3u^2-u^3) u^{1/2} du \\ &= -2 \int_0^1 (u^{1/2} - 3u^{3/2} + 3u^{5/2} - u^{7/2}) du \\ &= -2 \left(\frac{2}{3}(1-\sqrt{x})^{3/2} - \frac{6}{5}(1-\sqrt{x})^{5/2} + \frac{6}{7}(1-\sqrt{x})^{7/2} - \frac{2}{9}(1-\sqrt{x})^{9/2} \right) \Big|_0^1 \\ &= -2 \left(0 - \frac{2}{3} + \frac{6}{5} - \frac{6}{7} + \frac{2}{9} \right) \\ &= -2 \left(-\frac{32}{315} \right) \\ &= \frac{34}{315} \end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^1 \sqrt{x-x\sqrt{x}} \, dx$

Solution

$$u = 1 - \sqrt{x} \rightarrow x = (1-u)^2 \Rightarrow dx = -2(1-u) du$$

$$\begin{aligned} \int_0^1 \sqrt{x-x\sqrt{x}} \, dx &= \int_0^1 \sqrt{x(1-\sqrt{x})} \, dx \\ &= -2 \int_0^1 \sqrt{(1-u)^2 u} (1-u) du \end{aligned}$$

$$\begin{aligned}
&= -2 \int_0^1 \sqrt{(1-u)^2 u} (1-u) du \\
&= -2 \int_0^1 (1-u)^2 \sqrt{u} du \\
&= -2 \int_0^1 (1-2u+u^2) u^{1/2} du \\
&= -2 \int_0^1 \left(u^{1/2} - 2u^{3/2} + u^{5/2} \right) du \\
&= -2 \left(\frac{2}{3} (1-\sqrt{x})^{3/2} - \frac{4}{5} (1-\sqrt{x})^{5/2} + \frac{2}{7} (1-\sqrt{x})^{7/2} \right) \Big|_0^1 \\
&= -2 \left(0 - \frac{2}{3} + \frac{4}{5} - \frac{2}{7} \right) \\
&= -2 \left(\frac{-16}{105} \right) \\
&= \underline{\underline{\frac{32}{105}}}
\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^{\pi/2} \frac{\cos \theta \sin \theta}{\sqrt{\cos^2 \theta + 16}} d\theta$

Solution

$$d(\cos^2 \theta + 16) = -2 \cos \theta \sin \theta d\theta$$

$$\begin{aligned}
\int_0^{\pi/2} \frac{\cos \theta \sin \theta}{\sqrt{\cos^2 \theta + 16}} d\theta &= -\frac{1}{2} \int_0^{\pi/2} (\cos^2 \theta + 16)^{-1/2} d(\cos^2 \theta + 16) \\
&= -\sqrt{\cos^2 \theta + 16} \Big|_0^{\pi/2} \\
&= -(4 - \sqrt{17}) \\
&= \underline{\underline{\sqrt{17} - 4}}
\end{aligned}$$

Exercise

Evaluate the definite integral $\int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{dx}{x\sqrt{25x^2-1}}$

Solution

$$\begin{aligned}\int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{dx}{x\sqrt{25x^2-1}} &= \int_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \frac{d(5x)}{(5x)\sqrt{(5x)^2-1}} \\ &= \sec^{-1}(5x) \Big|_{\frac{2}{5\sqrt{3}}}^{\frac{2}{5}} \\ &= \sec^{-1}(2) - \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) \\ &= \frac{\pi}{3} - \frac{\pi}{6} \\ &= \frac{\pi}{6}\end{aligned}$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{x}{a} \right|$$

Exercise

Evaluate the definite integral $\int_0^4 \frac{x}{\sqrt{9+x^2}} dx$

Solution

$$\begin{aligned}\int_0^4 \frac{x}{\sqrt{9+x^2}} dx &= \frac{1}{2} \int_0^4 (9+x^2)^{-1/2} d(9+x^2) \\ &= \sqrt{9+x^2} \Big|_0^4 \\ &= 5 - 3 \\ &= 2\end{aligned}$$

$$d(9+x^2) = 2x dx$$

Exercise

Evaluate the definite integral $\int_0^{\pi/4} \frac{\sin \theta}{\cos^3 \theta} d\theta$

Solution

$$\begin{aligned}
\int_0^{\pi/4} \frac{\sin \theta}{\cos^3 \theta} d\theta &= -\int_0^{\pi/4} \cos^{-3} \theta d(\cos \theta) \\
&= \frac{1}{2} \frac{1}{\cos^2 \theta} \Big|_0^{\pi/4} \\
&= \frac{1}{2} (2 - 1) \\
&= \frac{1}{2}
\end{aligned}$$

$$d(\cos \theta) = -\sin \theta$$

Exercise

Evaluate the definite integral $\int_0^1 2x(4-x^2) dx$

Solution

$$\begin{aligned}
\int_0^1 2x(4-x^2) dx &= -\int_0^1 (4-x^2) d(4-x^2) \\
&= -\frac{1}{2} (4-x^2)^2 \Big|_0^1 \\
&= -\frac{1}{2} (9-16) \\
&= \frac{7}{2}
\end{aligned}$$

$$d(4-x^2) = -2x dx$$

Exercise

Evaluate the definite integral $\int_0^3 \frac{x^2+1}{\sqrt{x^3+3x+4}} dx$

Solution

$$\begin{aligned}
\int_0^3 \frac{x^2+1}{\sqrt{x^3+3x+4}} dx &= \frac{1}{3} \int_0^3 (x^3+3x+4)^{-1/2} d(x^3+3x+4) \\
&= \frac{2}{3} \sqrt{x^3+3x+4} \Big|_0^3 \\
&= \frac{2}{3} (\sqrt{40} - 2) \\
&= \frac{4}{3} (\sqrt{10} - 1)
\end{aligned}$$

$$d(x^3+3x+4) = (3x^2+3) dx$$

Exercise

Evaluate the definite integral $\int_0^4 \frac{x}{x^2+1} dx$

Solution

$$\begin{aligned}\int_0^4 \frac{x}{x^2+1} dx &= \frac{1}{2} \int_0^4 \frac{1}{x^2+1} d(x^2+1) & d(x^2+1) &= 2x dx \\ &= \frac{1}{2} \ln(x^2+1) \Big|_0^4 \\ &= \frac{1}{2} (\ln 17 - \ln 1) \\ &= \frac{1}{2} \ln 17\end{aligned}$$

Exercise

Evaluate the definite integral $\int_1^{e^2} \frac{\ln x}{x} dx$

Solution

$$\begin{aligned}\int_1^{e^2} \frac{\ln x}{x} dx &= \int_1^{e^2} \ln x \, d(\ln x) & d(\ln x) &= \frac{1}{x} dx \\ &= \frac{1}{2} (\ln x)^2 \Big|_1^{e^2} \\ &= \frac{1}{2} \left((\ln e^2)^2 - (\ln 1)^2 \right) \\ &= \frac{1}{2} (2)^2 \\ &= 2\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^3 \frac{x^2+1}{\sqrt{x^3+3x+4}} dx$

Solution

$$d(x^3+3x+4) = (3x^2+3) dx$$

$$= 3(x^2 + 1)dx$$

$$\begin{aligned}\int_0^3 \frac{x^2 + 1}{\sqrt{x^3 + 3x + 4}} dx &= \frac{1}{3} \int_0^3 (x^3 + 3x + 4)^{-1/2} d(x^3 + 3x + 4) \\ &= \frac{2}{3} \sqrt{x^3 + 3x + 4} \Big|_0^3 \\ &= \frac{2}{3} (\sqrt{40} - 2) \\ &= \frac{2}{3} (\sqrt{10} - 1)\end{aligned}$$

Exercise

Evaluate the definite integral $\int_{-\pi/4}^{\pi/4} \sin^2 2\theta \, d\theta$

Solution

$$\begin{aligned}\int_{-\pi/4}^{\pi/4} \sin^2 2\theta \, d\theta &= \frac{1}{2} \int_{-\pi/4}^{\pi/4} (1 - \cos 4\theta) \, d\theta & \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta) \\ &= \frac{1}{2} \left(\theta - \frac{1}{4} \sin 4\theta \right) \Big|_{-\pi/4}^{\pi/4} \\ &= \frac{1}{2} \left(\frac{\pi}{4} + \frac{\pi}{4} \right) \\ &= \frac{\pi}{4}\end{aligned}$$

Exercise

Evaluate the definite integral $\int_0^1 (y^3 + 6y^2 - 12y + 9)^{-1/2} (y^2 + 4y - 4) dy$

Solution

$$\begin{aligned}d(y^3 + 6y^2 - 12y + 9) &= (3y^2 + 12y - 12) dy \\ &= 3(y^2 + 4y - 4) dy \\ \int_0^1 (y^3 + 6y^2 - 12y + 9)^{-1/2} (y^2 + 4y - 4) dy\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3} \int_0^1 (y^3 + 6y^2 - 12y + 9)^{-1/2} (y^3 + 6y^2 - 12y + 9) dy \\
&= \frac{2}{3} \sqrt{y^3 + 6y^2 - 12y + 9} \Big|_0^1 \\
&= \frac{2}{3} (2 - 3) \\
&= \underline{-\frac{2}{3}}
\end{aligned}$$

Exercise

Solve the initial value problem $\frac{dy}{dt} = e^t \sin(e^t - 2)$, $y(\ln 2) = 0$

Solution

$$\frac{dy}{dt} = e^t \sin(e^t - 2) \Rightarrow y = \int e^t \sin(e^t - 2) dt$$

$$\text{Let } u = e^t - 2 \rightarrow du = e^t dt$$

$$y = \int \sin u \, du$$

$$= -\cos u + C$$

$$= -\cos(e^t - 2) + C$$

$$y(\ln 2) = -\cos(e^{\ln 2} - 2) + C = 0$$

$$C = \cos(2 - 2)$$

$$[C = \cos(0) = 1]$$

$$\underline{y(t) = -\cos(e^t - 2) + 1}$$

Exercise

Solve the initial value problem $\frac{dy}{dt} = e^{-t} \sec^2(\pi e^{-t})$, $y(\ln 4) = \frac{2}{\pi}$

Solution

$$\frac{dy}{dt} = e^{-t} \sec^2(\pi e^{-t}) \Rightarrow y = \int e^{-t} \sec^2(\pi e^{-t}) dt$$

$$\text{Let } u = \pi e^{-t} \rightarrow du = -\pi e^{-t} dt \Rightarrow -\frac{1}{\pi} du = e^{-t} dt$$

$$y = \int e^{-t} \sec^2(\pi e^{-t}) dt = -\frac{1}{\pi} \int \sec^2 u \, du$$

$$= -\frac{1}{\pi} \tan u + C$$

$$= -\frac{1}{\pi} \tan(\pi e^{-t}) + C$$

$$y(\ln 4) = -\frac{1}{\pi} \tan(\pi e^{-\ln 4}) + C = \frac{2}{\pi}$$

$$C = \frac{2}{\pi} + \frac{1}{\pi} \tan\left(\pi \cdot \frac{1}{4}\right)$$

$$= \frac{2}{\pi} + \frac{1}{\pi}$$

$$= \frac{3}{\pi}$$

$$\underline{y(t) = -\frac{1}{\pi} \tan(\pi e^{-t}) + \frac{3}{\pi}}$$

Exercise

Verify the integration formula: $\int \frac{\tan^{-1} x}{x^2} dx = \ln x - \frac{1}{2} \ln(1+x^2) - \frac{\tan^{-1} x}{x} + C$

Solution

$$\text{If } y = \ln x - \frac{1}{2} \ln(1+x^2) - \frac{\tan^{-1} x}{x} + C$$

$$dy = \left[\frac{1}{x} - \frac{1}{2} \frac{2x}{1+x^2} - \frac{\frac{x}{1+x^2} - \tan^{-1} x}{x^2} \right] dx$$

$$dy = \left[\frac{1}{x} - \frac{x}{1+x^2} - \frac{x - (1+x^2)\tan^{-1} x}{x^2(1+x^2)} \right] dx$$

$$dy = \left(\frac{x(1+x^2) - x^3 - x + (1+x^2)\tan^{-1} x}{x^2(1+x^2)} \right) dx$$

$$dy = \left(\frac{x + x^3 - x^3 - x + (1+x^2)\tan^{-1} x}{x^2(1+x^2)} \right) dx$$

$$dy = \frac{(1+x^2)\tan^{-1}x}{x^2(1+x^2)}dx$$

$$dy = \frac{\tan^{-1}x}{x^2}dx \quad \checkmark \quad \text{Which verifies the formula}$$

Exercise

Verify the integration formula: $\int \ln(a^2 + x^2)dx = x\ln(a^2 + x^2) - 2x + 2a \tan^{-1} \frac{x}{a} + C$

Solution

$$\text{If } y = x\ln(a^2 + x^2) - 2x + 2a \tan^{-1} \frac{x}{a} + C$$

$$dy = \left[\ln(a^2 + x^2) + x \frac{2x}{a^2 + x^2} - 2 + 2a \frac{\frac{1}{a}}{1 + \frac{x^2}{a^2}} \right] dx$$

$$dy = \left[\ln(a^2 + x^2) + \frac{2x^2}{a^2 + x^2} - 2 + \frac{2}{\frac{a^2 + x^2}{a^2}} \right] dx$$

$$dy = \left[\ln(a^2 + x^2) + \frac{2x^2}{a^2 + x^2} - 2 + \frac{2a^2}{a^2 + x^2} \right] dx$$

$$dy = \left[\ln(a^2 + x^2) + \frac{2x^2 + 2a^2}{a^2 + x^2} - 2 \right] dx$$

$$dy = \left[\ln(a^2 + x^2) + \frac{2(x^2 + a^2)}{a^2 + x^2} - 2 \right] dx$$

$$dy = \left[\ln(a^2 + x^2) + 2 - 2 \right] dx$$

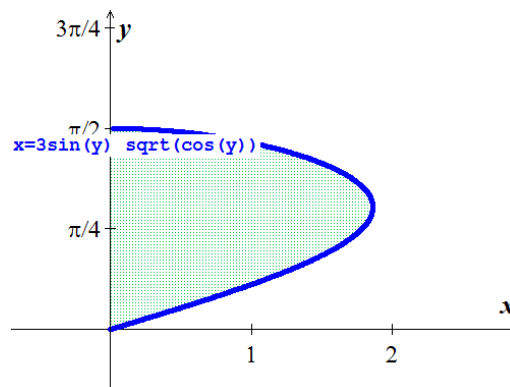
$$dy = \ln(a^2 + x^2)dx \quad \checkmark \quad \text{Which verifies the formula}$$

Exercise

Find the area of the region bounded by the graphs of $x = 3\sin y\sqrt{\cos y}$, and $x = 0$, $0 \leq y \leq \frac{\pi}{2}$

Solution

$$\begin{aligned} A &= \int_0^{\pi/2} (3\sin y\sqrt{\cos y} - 0) dy & d(\cos y) &= -\sin y dy \\ &= -3 \int_0^{\pi/2} \cos^{1/2} y d(\cos y) \\ &= -3 \left[\frac{2}{3} \cos^{3/2} y \right]_0^{\pi/2} \\ &= -2(0 - 1) \\ &= 2 \text{ unit}^2 \end{aligned}$$



Exercise

Find the area of the region bounded by the graph of $f(x) = \frac{x}{\sqrt{x^2 - 9}}$ on $3 \leq x \leq 4$

Solution

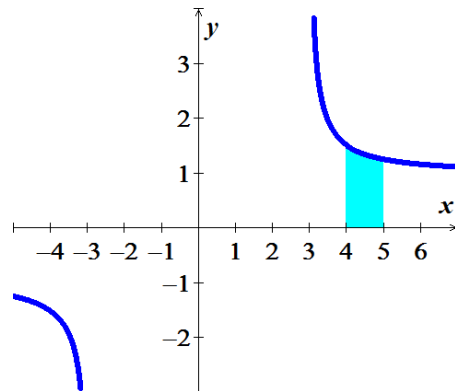
$$\begin{aligned} A &= \int_3^4 \frac{x}{\sqrt{x^2 - 9}} dx \\ &= \frac{1}{2} \int_3^4 (x^2 - 9)^{-1/2} d(x^2 - 9) & d(x^2 - 9) &= 2x dx \\ &= \sqrt{x^2 - 9} \Big|_3^4 \\ &= \sqrt{7} - 0 \\ &= \sqrt{7} \text{ unit}^2 \end{aligned}$$

Exercise

Find the area of the region bounded by the graph of $f(x) = \frac{x}{\sqrt{x^2 - 9}}$ and the x -axis between $x = 4$ and $x = 5$.

Solution

$$\begin{aligned}
 A &= \int_4^5 \frac{x}{\sqrt{x^2-9}} dx \\
 &= \frac{1}{2} \int_4^5 (x^2-9)^{-1/2} d(x^2-9) \quad d(x^2-9) = 2x dx \\
 &= \sqrt{x^2-9} \Big|_4^5 \\
 &= 4 - \sqrt{7} \text{ unit}^2
 \end{aligned}$$

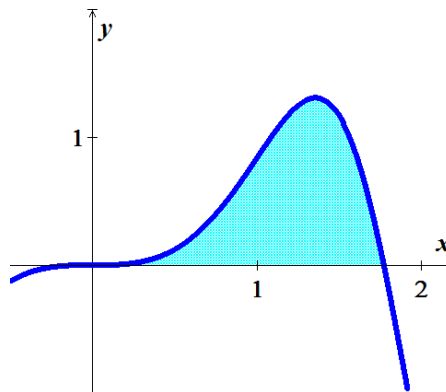
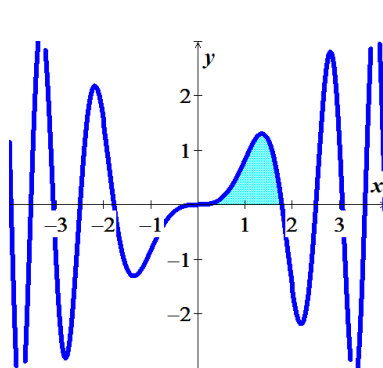


Exercise

Find the area of the region bounded by the graph of $f(x) = x \sin x^2$ and the x -axis between $x=0$ and $x=\sqrt{\pi}$.

Solution

$$\begin{aligned}
 A &= \int_0^{\sqrt{\pi}} x \sin x^2 dx \\
 &= \frac{1}{2} \int_0^{\sqrt{\pi}} \sin x^2 d(x^2) \\
 &= -\frac{1}{2} \cos(x^2) \Big|_0^{\sqrt{\pi}} \\
 &= -\frac{1}{2}(-1-1) \\
 &= 1 \text{ unit}^2
 \end{aligned}$$



Exercise

Find the area of the region bounded by the graph of $f(\theta) = \cos \theta \sin \theta$ and the θ -axis between $\theta=0$ and $\theta=\frac{\pi}{2}$.

Solution

$$\begin{aligned}
 A &= \int_0^{\frac{\pi}{2}} \cos \theta \sin \theta d\theta \\
 &= \int_0^{\frac{\pi}{2}} \sin \theta d(\sin \theta)
 \end{aligned}$$

$$= \frac{1}{2} \sin^2 \theta \Big|_0^{\pi/2}$$

$$= \underline{1 \text{ unit}^2}$$

Exercise

Find the area of the region bounded by the graph of $f(x) = (x-4)^4$ and the x -axis between $x=2$ and $x=6$.

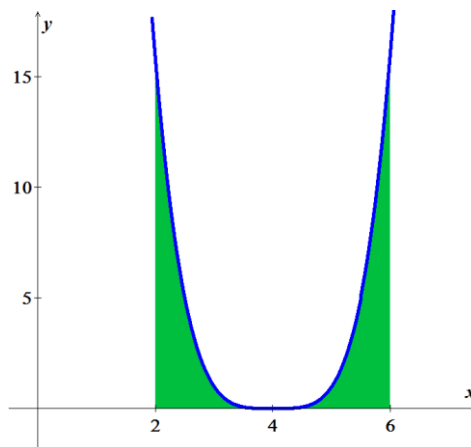
Solution

$$A = \int_2^6 (x-4)^4 dx$$

$$= \int_2^6 (x-4)^4 d(x-4)$$

$$= 2 \times \frac{1}{5} (x-4)^5 \Big|_2^6$$

$$= \underline{\frac{64}{5} \text{ unit}^2}$$



Exercise

Perhaps the simplest change of variables is the shift or translation given by $u = x + c$, where c is a real number.

a) Prove that shifting a function does not change the net area under the curve, in the sense that

$$\int_a^b f(x+c) dx = \int_{a+c}^{b+c} f(u) du$$

b) Draw a picture to illustrate this change of variables in the case that $f(x) = \sin x$, $a=0$, $b=\pi$, and

$$c = \frac{\pi}{2}$$

Solution

a) Let $u = x + c \rightarrow du = dx$

$$\begin{cases} x=b & \rightarrow u=b+c \\ x=a & \rightarrow u=a+c \end{cases}$$

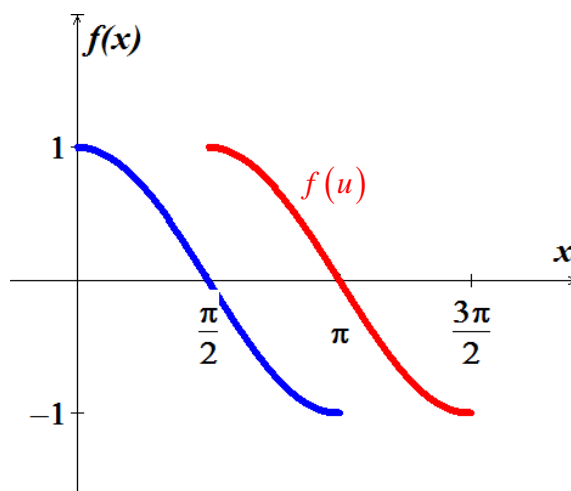
$$\int_a^b f(x+c) dx = \int_{a+c}^{b+c} f(u) du$$

b) Given: $f(x) = \sin x$, $a=0$, $b=\pi$, & $c = \frac{\pi}{2}$

$$f(x+c) = \sin\left(x + \frac{\pi}{2}\right)$$

$$\begin{cases} b = \pi & \rightarrow f\left(\pi + \frac{\pi}{2}\right) = \sin \frac{3\pi}{2} = -1 \\ a = 0 & \rightarrow f\left(0 + \frac{\pi}{2}\right) = \sin \frac{\pi}{2} = 1 \end{cases}$$

$$f(u) \rightarrow \begin{cases} b+c = \frac{3\pi}{2} \\ a+c = \frac{\pi}{2} \end{cases}$$



Exercise

Another change of variables that can be interpreted geometrically is the scaling $u = cx$, where c is a real number. Prove and interpret the fact that

$$\int_a^b f(cx) dx = \frac{1}{c} \int_{ac}^{bc} f(u) du$$

Draw a picture to illustrate this change of variables in the case that $f(x) = \sin x$, $a = 0$, $b = \pi$, and

$$c = \frac{1}{2}$$

Solution

$$\text{Let } u = cx \rightarrow du = c dx$$

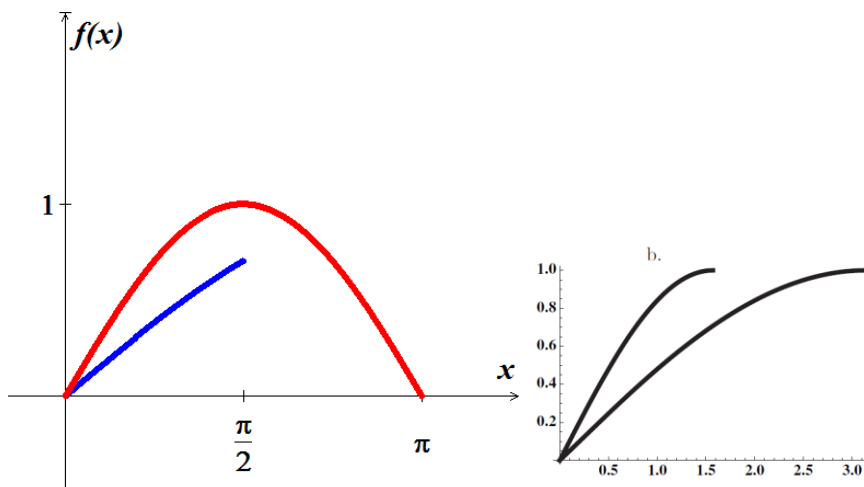
$$\begin{cases} x = b & \rightarrow u = bc \\ x = a & \rightarrow u = ac \end{cases}$$

$$\int_a^b f(cx) dx = \frac{1}{c} \int_{ac}^{bc} f(u) du$$

$$\text{Given: } f(x) = \sin x, \quad a = 0, \quad b = \pi, \quad \& \quad c = \frac{1}{2}$$

$$f(cx) = f\left(\frac{x}{2}\right) = \sin \frac{x}{2}$$

$$\begin{cases} a = 0 & \rightarrow ac = 0 \\ b = \pi & \rightarrow bc = \frac{\pi}{2} \end{cases}$$



Exercise

The function f satisfies the equation $3x^4 - 48 = \int_2^x f(t) dt$. Find f and check your answer by substitution.

Solution

$$\frac{d}{dx}(3x^4 - 48) = \frac{d}{dx} \int_2^x f(t) dt$$

$$12x^3 = f(x)$$

$$\begin{aligned} \int_2^x 12t^3 dt &= 3t^4 \Big|_2^x \\ &= 3x^4 - 3(2)^4 \\ &= \underline{3x^4 - 48} \quad \checkmark \end{aligned}$$

Exercise

Assume f' is continuous on $[2, 4]$, $\int_1^2 f'(2x) dx = 10$, and $f(2) = 4$. Evaluate $f(4)$.

Solution

$$\int_1^2 f'(2x) dx = f(2x) \Big|_1^2$$

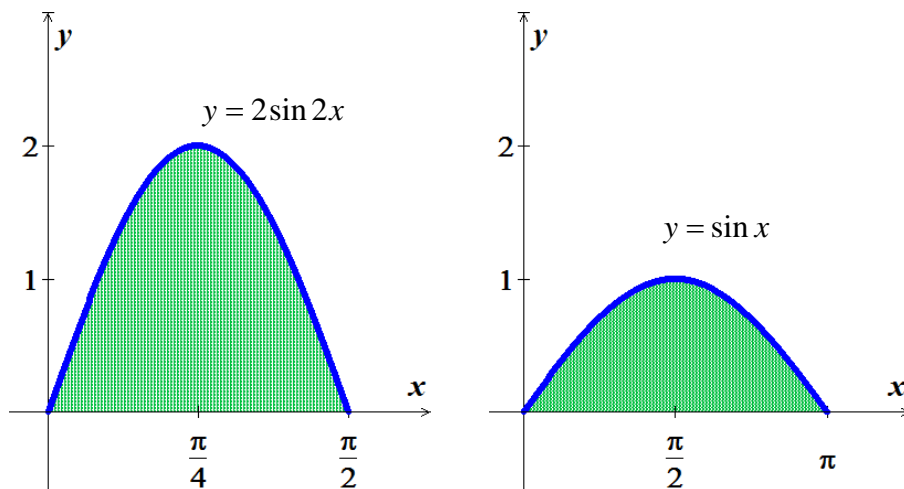
$$= f(4) - f(2) = 10$$

$$f(4) - 4 = 10$$

$$\underline{f(4) = 14}$$

Exercise

The area of the shaded region under the curve $y = 2 \sin 2x$ in



- Equals the area on the shaded region under the curve $y = \sin x$
- Explain why this is true without computation areas.

Solution

$$a) \quad A = \int_0^{\pi/2} 2 \sin 2x \, dx \quad u = 2x \rightarrow du = 2dx$$

$$\begin{cases} x = \frac{\pi}{2} \rightarrow u = \pi \\ x = 0 \rightarrow u = 0 \end{cases}$$

$$= \int_0^{\pi} \sin u \, du$$

$$= \int_0^{\pi} \sin x \, dx$$

$$b) \quad \text{Let } A_1 = \text{area of } \sin x \quad 0 \leq x \leq \pi$$

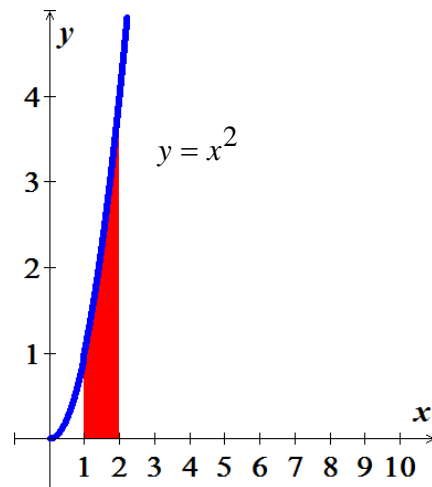
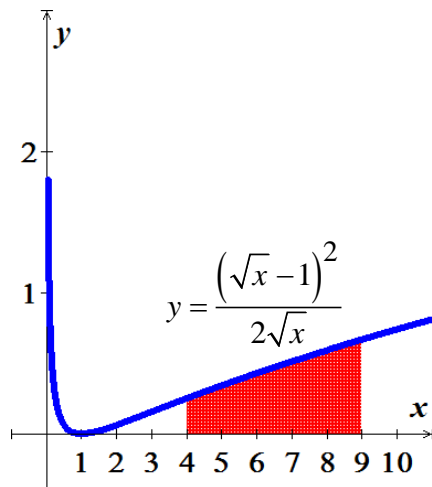
$$A_2 = \text{area of } \sin 2x \quad 0 \leq 2x \leq \pi \rightarrow 0 \leq x \leq \frac{\pi}{2}$$

$$\text{Area of } 0 \leq x \leq \frac{\pi}{2} \text{ is } \frac{1}{2} A_1$$

$$A_2 = 2 \frac{1}{2} A_1 = A_1 \quad \checkmark$$

Exercise

The area of the shaded region under the curve $y = \frac{(\sqrt{x}-1)^2}{2\sqrt{x}}$ on the interval $[4, 9]$



- Equals the area on the shaded region under the curve $y = x^2$ on the interval $[1, 2]$
- Explain why this is true without computation areas.

Solution

a) Let $u = \sqrt{x} - 1 \rightarrow x = (u+1)^2$

$$dx = 2(u+1)du$$

$$\begin{cases} x=9 & \rightarrow u=2 \\ x=4 & \rightarrow u=1 \end{cases}$$

$$\begin{aligned} A_1 &= \int_4^9 \frac{(\sqrt{x}-1)^2}{2\sqrt{x}} dx \\ &= \int_1^2 \frac{u^2}{2(u+1)} 2(u+1) du \\ &= \int_1^2 u^2 du \quad \checkmark \\ &= \frac{1}{3}(\sqrt{x}-1)^3 \Big|_4^9 \\ &= \frac{1}{3}(2^3 - 1) \\ &= \frac{7}{3} \end{aligned}$$

$$\begin{aligned}
 A_2 &= \int_1^2 x^2 \, dx \\
 &= \frac{1}{3}x^3 \Big|_1^2 \\
 &= \frac{1}{3}(2^3 - 1) \\
 &= \frac{7}{3}
 \end{aligned}$$

$$b) \int_4^9 \frac{(\sqrt{x}-1)^2}{2\sqrt{x}} \, dx = \int_1^2 u^2 \, du = \int_1^2 x^2 \, dx \quad \checkmark$$

Exercise

The family of parabolas $y = \frac{1}{a} - \frac{x^2}{a^3}$, where $a > 0$, has the property that for $x \geq 0$, the x -intercept is $(a, 0)$ and the y -intercept is $(0, \frac{1}{a})$. Let $A(a)$ be the area of the region in the first quadrant bounded by the parabola and the x -axis. Find $A(a)$ and determine whether it is increasing, decreasing, or constant function of a .

Solution

$$\text{Given: } y = \frac{1}{a} - \frac{x^2}{a^3} \quad (a, 0) \text{ \& } (0, \frac{1}{a})$$

$$\begin{aligned}
 A &= \int_0^a \left(\frac{1}{a} - \frac{x^2}{a^3} \right) dx \\
 &= \left(\frac{1}{a}x - \frac{1}{3} \frac{x^3}{a^3} \right) \Big|_0^a \\
 &= 1 - \frac{1}{3} \\
 &= \frac{2}{3}
 \end{aligned}$$

$A(a) = \frac{2}{3}$ is a constant function.

Exercise

Consider the right triangle with vertices $(0, 0)$, $(0, b)$, and $(a, 0)$, where $a > 0$ and $b > 0$. Show that the average vertical distance from points on the x -axis to the hypotenuse is $\frac{b}{2}$, for all $a > 0$.

Solution

$$y = \frac{b-0}{0-a}(x-0) + b \qquad y = m(x-x_0) + y_0$$
$$= -\frac{b}{a}x + b \Big|$$

Average vertical distance is:

$$\frac{1}{a-0} \int_0^a \left(-\frac{b}{a}x + b\right) dx = \frac{1}{a} \int_0^a \left(b - \frac{b}{a}x\right) dx$$
$$= \frac{1}{a} \left(bx - \frac{b}{2a}x^2\right) \Big|_0^a$$
$$= \frac{1}{a} \left(ba - \frac{b}{2a}a^2\right)$$
$$= b - \frac{b}{2}$$
$$= \frac{b}{2} \Big|$$

Exercise

Consider the integral $I = \int \sin^2 x \cos^2 x \, dx$

- Find I using the identity $\sin 2x = 2 \sin x \cos x$
- Find I using the identity $\cos^2 x = 1 - \sin^2 x$
- Confirm that the results in part (a) and (b) are consistent and compare the work involved in each method.

Solution

$$a) \quad \sin 2x = 2 \sin x \cos x$$
$$\sin^2 2x = 4 \sin^2 x \cos^2 x$$
$$\sin^2 x \cos^2 x = \frac{1}{4} \sin^2 2x$$

$$I = \int \sin^2 x \cos^2 x \, dx$$

$$= \frac{1}{4} \int \sin^2 2x \, dx$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\begin{aligned}
&= \frac{1}{8} \int (1 - \cos 4x) \, dx \\
&= \frac{1}{8} \left(x - \frac{1}{4} \sin 4x \right) + C \\
&= \frac{1}{8} x - \frac{1}{32} \sin 4x + C \quad |
\end{aligned}$$

$$\begin{aligned}
b) \quad \cos^2 x &= 1 - \sin^2 x & \sin^2 \theta &= \frac{1}{2}(1 - \cos 2\theta) \\
&= 1 - \frac{1}{2} + \frac{1}{2} \cos 2x \\
&= \frac{1}{2} + \frac{1}{2} \cos 2x
\end{aligned}$$

$$\begin{aligned}
I &= \int \frac{1}{2}(1 - \cos 2x) \cdot \frac{1}{2}(1 + \cos 2x) \, dx & \sin^2 \theta &= \frac{1}{2}(1 - \cos 2\theta) \\
&= \frac{1}{4} \int (1 - \cos^2 2x) \, dx & \cos^2 x &= 1 - \sin^2 x \\
&= \frac{1}{4} \int \sin^2 2x \, dx & \text{From part (a)} \\
&= \frac{1}{8} x - \frac{1}{32} \sin 4x + C \quad |
\end{aligned}$$

c) The results from part a & b are consistent.

Exercise

Let $H(x) = \int_0^x \sqrt{4-t^2} \, dt$, for $-2 \leq x \leq 2$.

- Evaluate $H(0)$
- Evaluate $H'(1)$
- Evaluate $H'(2)$
- Use geometry to evaluate $H(2)$
- Find the value of s such that $H(x) = sH(-x)$

Solution

$$a) \quad H(0) = \int_0^0 \sqrt{4-t^2} \, dt = 0 \quad |$$

$$\begin{aligned}
b) \quad H'(x) &= \sqrt{4-x^2} \cdot \frac{d}{dx}(x) \\
&= \sqrt{4-x^2}
\end{aligned}$$

$$H'(1) = \underline{\underline{\sqrt{3}}}$$

$$c) \quad H'(2) = \sqrt{4-4} = \underline{\underline{0}}$$

$$d) \quad H(2) = \int_0^2 \sqrt{4-t^2} \, dt \quad \text{is the area inside a circle in the first quadrant of radius 2}$$

$$= \frac{1}{4} \pi (2)^2$$

$$= \underline{\underline{\pi}}$$

$$e) \quad H(x) = \int_0^{-x} \sqrt{4-t^2} \, dt \quad \sqrt{4-t^2} \text{ is an even function}$$

$$= - \int_{-x}^0 \sqrt{4-t^2} \, dt$$

$$= -H(x)$$

$$\therefore \underline{\underline{s = -1}}$$

$$t = 2 \sin u \rightarrow dt = 2 \cos u \, du$$

$$\sqrt{4-t^2} = 2 \cos u$$

$$H(x) = \int_0^x \sqrt{4-t^2} \, dt$$

$$= \int_0^x 2 \cos u \, 2 \cos u \, du$$

$$= \int_0^x 4 \cos^2 u \, du$$

$$= \int_0^x 2(1 + \cos 2u) \, du$$

$$= 2 \left(u + \frac{1}{2} \sin 2u \right) \Big|_0^x$$

$$t = 2 \sin u \rightarrow u = \sin^{-1} \frac{t}{2}$$

$$= 2 \left(\sin^{-1} \frac{t}{2} + \sin u \cos u \right) \Big|_0^x$$

$$\sqrt{4-t^2} = 2 \cos u \rightarrow \cos u = \frac{1}{2} \sqrt{4-t^2}$$

$$\begin{aligned}
&= 2 \left(\sin^{-1} \frac{t}{2} + \frac{t}{4} \sqrt{4-t^2} \right) \Bigg|_0^x \\
&= 2 \left(\sin^{-1} \frac{x}{2} + \frac{x}{4} \sqrt{4-x^2} \right) \\
&= \underline{2 \sin^{-1} \frac{x}{2} + \frac{x}{2} \sqrt{4-x^2}}
\end{aligned}$$

Exercise

Evaluate the limits $\lim_{x \rightarrow 2} \frac{\int_2^x e^{t^2} dt}{x-2}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 2} \frac{\int_2^x e^{t^2} dt}{x-2} &= \frac{\int_2^2 e^{t^2} dt}{2-2} = \frac{0}{0} \\
&= \lim_{x \rightarrow 2} \frac{e^{x^2} \frac{d}{dx}(x)}{1} \\
&= \lim_{x \rightarrow 2} e^{x^2} \\
&= \underline{e^4}
\end{aligned}$$

Exercise

Evaluate the limits $\lim_{x \rightarrow 1} \frac{\int_1^{x^2} e^{t^3} dt}{x-1}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 1} \frac{\int_1^{x^2} e^{t^3} dt}{x-1} &= \lim_{x \rightarrow 1} \frac{\int_1^1 e^{t^3} dt}{1-1} = \frac{0}{0} \\
&= \lim_{x \rightarrow 1} \frac{2xe^{x^3}}{1} \\
&= \underline{2e}
\end{aligned}$$

Exercise

Prove that for nonzero constants a and b , $\int \frac{dx}{a^2x^2 + b^2} = \frac{1}{ab} \tan^{-1}\left(\frac{ax}{b}\right) + C$

Solution

$$\begin{aligned} \int \frac{dx}{a^2x^2 + b^2} &= \int \frac{dx}{a^2 \left(x^2 + \left(\frac{b}{a}\right)^2 \right)} & \int \frac{dx}{a^2 + x^2} &= \frac{1}{a} \tan^{-1} \frac{x}{a} \\ &= \frac{1}{a^2} \frac{a}{b} \tan^{-1} \frac{x}{\frac{b}{a}} + C \\ &= \frac{1}{ab} \tan^{-1}\left(\frac{ax}{b}\right) + C \quad \checkmark \end{aligned}$$

Exercise

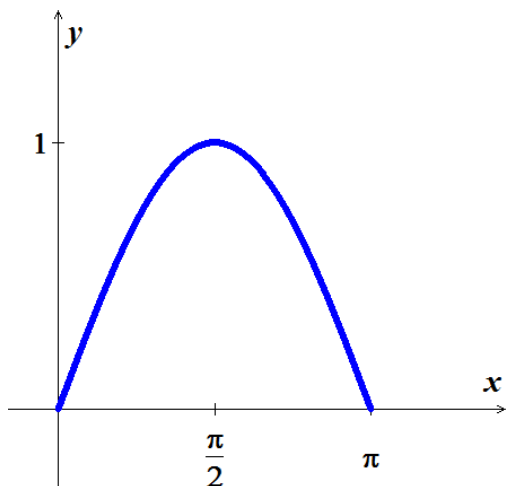
Let $a > 0$ be a real number and consider the family of functions $f(x) = \sin ax$ on the interval $\left[0, \frac{\pi}{a}\right]$.

a) Graph f , for $a = 1, 2, 3$.

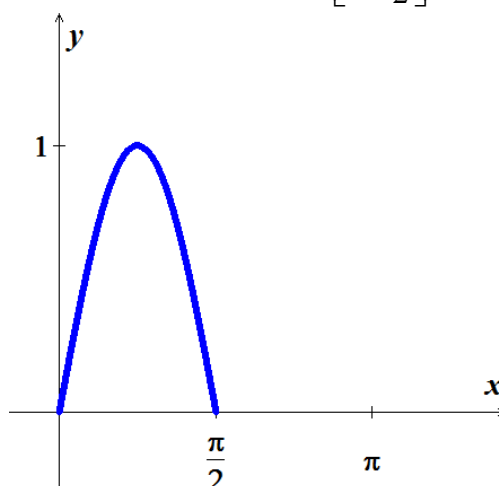
b) Let $g(a)$ be the area of the region bounded by the graph of f and the x -axis on the interval $\left[0, \frac{\pi}{a}\right]$. Graph g for $0 < a < \infty$. Is g an increasing function, a decreasing function, or neither?

Solution

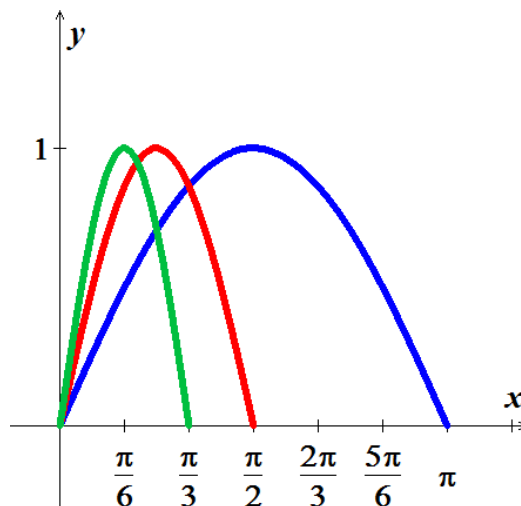
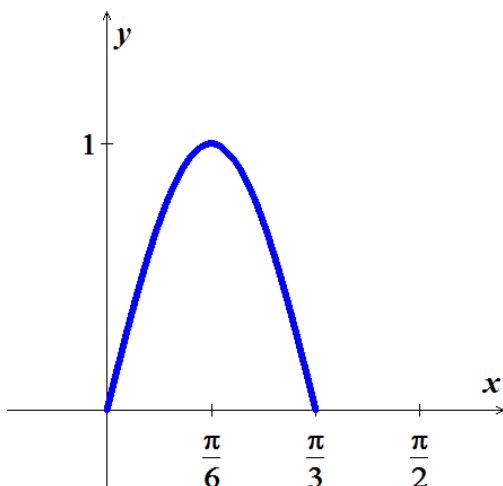
a) $a = 1 \rightarrow f(x) = \sin x \quad x \in [0, \pi]$



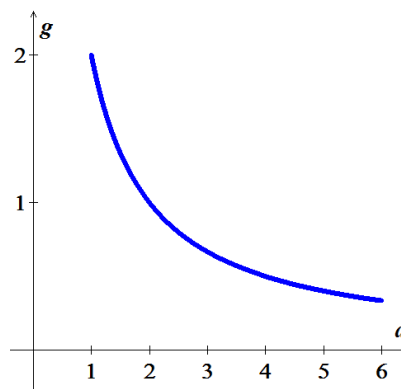
$a = 2 \rightarrow f(x) = \sin 2x \quad x \in \left[0, \frac{\pi}{2}\right]$



$$a = 3 \rightarrow f(x) = \sin 3x \quad x \in \left[0, \frac{\pi}{3}\right]$$



$$\begin{aligned} b) \quad g(x) &= \int_0^{\pi/a} \sin ax \, dx \\ &= -\frac{1}{a} \cos ax \Big|_0^{\pi/a} \\ &= -\frac{1}{a} (\cos \pi - \cos 0) \\ &= -\frac{1}{a} (-1 - 1) \\ &= \frac{2}{a} \end{aligned}$$



The function is decreasing as $a \geq 1$ is increasing.

Exercise

Explain why if a function u satisfies the equation $u(x) + 2 \int_0^x u(t) dt = 10$, then it also satisfies the equation $u'(x) + 2u(x) = 0$. Is it true that if u satisfies the second equation, then it satisfies the first equation?

Solution

$$\frac{d}{dx} u(x) + 2 \frac{d}{dx} \int_0^x u(t) dt = \frac{d}{dx} (10)$$

$$u'(x) + 2 \frac{d}{dx} u(x) \frac{d}{dx} x = 0$$

$$u'(x) + 2u(x) = 0 \quad \checkmark$$

Exercise

Let $f(x) = \int_0^x (t-1)^{15} (t-2)^9 dt$

- Find the interval on which f is increasing and the intervals on which f is decreasing.
- Find the intervals on which f is concave up and the intervals on which f is concave down.
- For what values of x does f have local minima? Local maxima?
- Where are the inflection points of f ?

Solution

a) $f'(x) = (x-1)^{15} (x-2)^9 = 0$

CN: $x=1, 2$

Where $x=1$ is multiplicity of 15
 $x=2$ is multiplicity of 9

0	1	2
+	-	+

Therefore, the sign will change.

f is increasing on $(-\infty, 1) \cup (2, \infty)$

f is decreasing on $(1, 2)$

b) $f''(x) = (x-1)^{14} (x-2)^8 (15(x-2) + 9(x-1))$
 $= (x-1)^{14} (x-2)^8 (24x - 39) = 0$

$x=1, 2, \frac{13}{8}$

0	1	$\frac{13}{8}$	2
-	-	+	+

$(x-1)^{14} (x-2)^8 \geq 0$ (always)

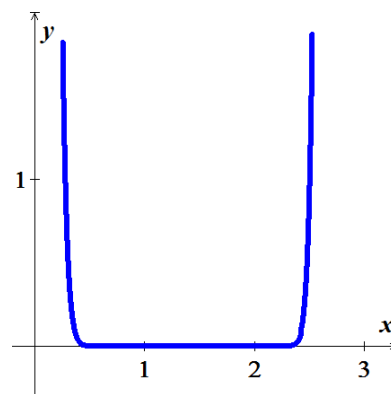
Concave up: $(\frac{13}{8}, \infty)$

Concave down: $(-\infty, \frac{13}{8})$

c) LMIN: $(1, 0)$

LMAX: $(2, 0)$

d) point of inflection: $x = \frac{13}{8}$



Exercise

A company is considering a new manufacturing process in one of its plants. The new process provides substantial initial savings, with the savings declining with time t (in years) according to the rate-of-savings function

$$S'(t) = 100 - t^2$$

where $S'(t)$ is in thousands of dollars per year. At the same time, the cost of operating the new process increases with time t (in years), according to the rate-of-cost function (in thousands of dollars per year)

$$C'(t) = t^2 + \frac{14}{3}t$$

- a) For how many years will the company realize savings?
- b) What will be the net total savings during this period?

Solution

- a) For how many years will the company realize savings?

$$C'(t) = S'(t)$$

$$t^2 + \frac{14}{3}t = 100 - t^2$$

$$2t^2 + \frac{14}{3}t - 100 = 0$$

$$\rightarrow t = -\frac{25}{3} \text{ or } 6$$

$$\boxed{t = 6}$$

The company should use this type for 6 years.

- b) What will be the net total savings during this period?

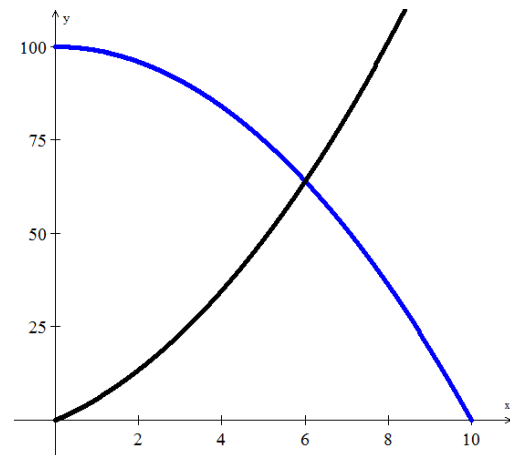
$$\text{Total savings} = \int_0^6 \left[(100 - t^2) - \left(t^2 + \frac{14}{3}t \right) \right] dt$$

$$= \int_0^6 \left[100 - 2t^2 - \frac{14}{3}t \right] dt$$

$$= 100t - \frac{2}{3}t^3 - \frac{7}{3}t^2 \Big|_0^6$$

$$= 100(6) - \frac{2}{3}(6)^3 - \frac{7}{3}(6)^2 - \left(100(0) - \frac{2}{3}(0)^3 - \frac{7}{3}(0)^2 \right)$$

$$= 372 \mid$$



The company will save a total of \$372,000. Over the 6-year period

Exercise

Find the producers' surplus if the supply function for pork bellies is given by

$$S(x) = x^{5/2} + 2x^{3/2} + 50$$

Assume supply and demand are in equilibrium at $x = 16$.

Solution

The equilibrium price:

$$\begin{aligned} p_0 &= S(x = 16) = 16^{5/2} + 2(16)^{3/2} + 50 \\ &= 1202 \end{aligned}$$

$$\begin{aligned} \text{Producer's surplus} &= \int_0^{x_0} [p_0 - S(x)] dx \\ &= \int_0^{16} [1202 - (x^{5/2} + 2x^{3/2} + 50)] dx \\ &= \int_0^{16} [1152 - x^{5/2} - 2x^{3/2}] dx \\ &= 1152x - \frac{2}{7}x^{7/2} - \frac{4}{5}x^{5/2} \Big|_0^{16} \\ &= \left(1152(16) - \frac{2}{7}(16)^{7/2} - \frac{4}{5}(16)^{5/2}\right) - \left(1152(0) - \frac{2}{7}(0)^{7/2} - \frac{4}{5}(0)^{5/2}\right) \\ &= \$12,931.66 \end{aligned}$$

The producers' surplus is \$12,931.66

Exercise

An object moves along a line with a velocity in m/s given by $v(t) = 8\cos\left(\frac{\pi t}{6}\right)$. Its initial position is $s(0) = 0$.

a) Graph the velocity function.

b) The position of the object is given by $s(t) = \int_0^t v(y) dy$, for $t \geq 0$. Find the position function, for $t \geq 0$.

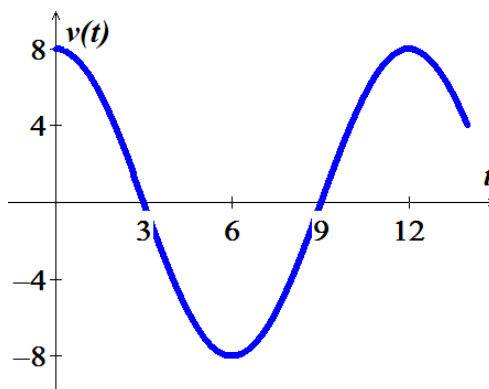
c) What is the period of the motion – that is, starting at any point, how long does it take the object to return to that position?

Solution

$$a) \quad v(t) = 8\cos\left(\frac{\pi t}{6}\right)$$

$$|A| = 8 \quad P = 12$$

t	$v(t)$
0	8
3	0
6	-8
9	0
12	8



$$\begin{aligned}
 b) \quad s(t) &= \int_0^t v(y) dy \\
 &= \int_0^t 8 \cos\left(\frac{\pi}{6} y\right) dy \\
 &= \frac{48}{\pi} \sin\left(\frac{\pi}{6} y\right) \Big|_0^t \\
 &= \frac{48}{\pi} \sin \frac{\pi}{6} t
 \end{aligned}$$

$$c) \text{ Period: } P = \frac{2\pi}{\frac{\pi}{6}} = 12$$

Exercise

The population of a culture of bacteria has a growth rate given by $p'(t) = \frac{200}{(t+1)^r}$ bacteria per hour, for

$t \geq 0$, where $r > 1$ is a real number. It is shown that the increase in the population over time interval

$[0, t]$ is given by $\int_0^t p'(s) ds$. (note that the growth rate decreases in time, reflecting competition for

space and food.)

- Using the population model with $r = 2$, what is the increase in the population over the time interval $0 \leq t \leq 4$?
- Using the population model with $r = 3$, what is the increase in the population over the time interval $0 \leq t \leq 6$?
- Let ΔP be the increase in the population over a fixed time interval $[0, T]$. For fixed T , does ΔP increase or decrease with the parameter r ? Explain.
- A lab technician measures an increase in the population of 350 bacteria over the 10-hr period $[0, 10]$. Estimate the value of r that best fits this data point.

- e) Use the population model in part (b) to find the increase in population over time interval $[0, T]$, for any $T > 0$. If the culture is allowed to grow indefinitely ($T \rightarrow \infty$), does the bacteria population increase without bound? Or does it approach a finite limit?

Solution

a) $r = 2$ & $0 \leq t \leq 4$

$$\begin{aligned}\int_0^4 \frac{200}{(t+1)^2} dt &= \int_0^4 \frac{200}{(t+1)^2} d(t+1) \\ &= -\frac{200}{t+1} \Big|_0^4 \\ &= -(40 - 200) \\ &= \underline{160}\end{aligned}$$

b) $r = 3$ & $0 \leq t \leq 6$

$$\begin{aligned}\int_0^6 \frac{200}{(t+1)^3} dt &= 200 \int_0^6 (t+1)^{-3} d(t+1) \\ &= -100 \frac{1}{(t+1)^2} \Big|_0^6 \\ &= -100 \left(\frac{1}{49} - 1 \right) \\ &= \underline{\frac{4800}{49}}\end{aligned}$$

c) $\Delta P = \int_0^T \frac{200}{(t+1)^r} dt$ decreases as r increases.

Because $\frac{200}{(t+1)^r} > \frac{200}{(t+1)^{r+1}}$

d) $\int_0^{10} \frac{200}{(t+1)^r} dt = 350$

$$200 \int_0^{10} (t+1)^{-r} d(t+1) = 350$$

$$\frac{1}{1-r} (t+1)^{1-r} \Big|_0^{10} = \frac{7}{4}$$

$$\frac{1}{1-r} (11^{1-r} - 1) = \frac{7}{4}$$

$$4(11)^{1-r} - 4 = 7 - r$$

$$4(11)^{1-r} + r - 44 = 0 \xrightarrow{\text{using software}} \boxed{r \approx 1.278}$$

$$\begin{aligned} e) \quad \int_0^T \frac{200}{(t+1)^3} dt &= -100 \frac{1}{(t+1)^2} \Big|_0^T \\ &= -100 \left(\frac{1}{(T+1)^2} - 1 \right) \\ &= 100 - \frac{100}{(T+1)^2} \end{aligned}$$

$$\lim_{T \rightarrow \infty} \left(100 - \frac{100}{(T+1)^2} \right) = \boxed{100}$$

\therefore The bacteria approach a finite limit of 100.

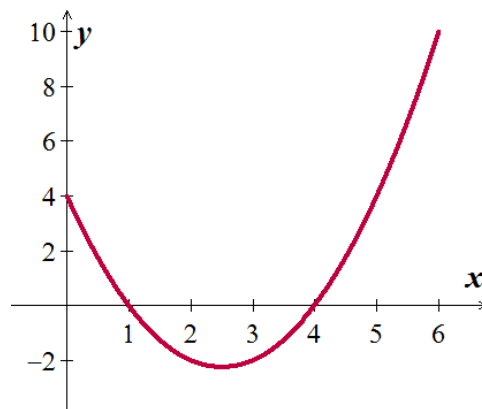
Exercise

Consider the function $f(x) = x^2 - 5x + 4$ and the area function $A(x) = \int_0^x f(t) dt$.

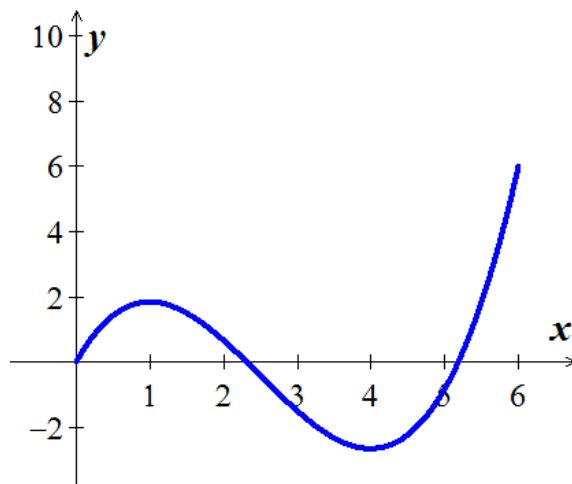
- Graph f on the interval $[0, 6]$.
- Compute and graph A on the interval $[0, 6]$.
- Show that the local extrema of A occur at the zeros of f .
- Give a geometric and analytical explanation for the observation in part (c).
- Find the approximate zeros of A , other than 0, and call them x_1 and x_2 .
- Find b such that the area bounded by the graph of f and the x -axis on the interval $[0, x_1]$ equals the area bounded by the graph of f and the x -axis on the interval $[x_1, b]$.
- If f is an integrable function and $A(x) = \int_0^x f(t) dt$, is it always true that the local extrema of A occur at the zeros of f ? Explain

Solution

a)

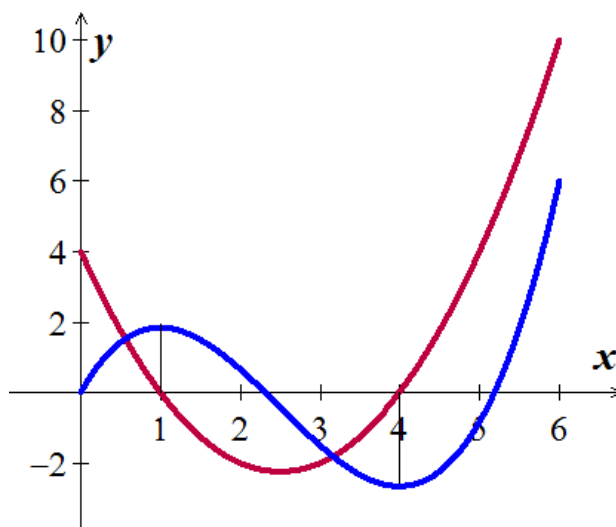


$$\begin{aligned}
 b) \quad A(x) &= \int_0^x f(t) dt = \int_0^x (t^2 - 5t + 4) dt \\
 &= \left(\frac{1}{3}t^3 - \frac{5}{2}t^2 + 4t \right)_0^x \\
 &= \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x
 \end{aligned}$$



$$\begin{aligned}
 c) \quad f(x) &= x^2 - 5x + 4 = 0 \Rightarrow \boxed{x = 0, 4} \\
 A(x) &= \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \rightarrow A'(x) = f(x)
 \end{aligned}$$

The zeros of f are at 1 and 4, and A has a local maximum at $x = 1$ and local minimum at $x = 4$.



d) Since f is above the axis from 0 to 1, the net area A is increasing and switches to decreasing to the right of 1. When x is between 1 and 4, the function f is below x -axis (negative sign), the Area A is decreasing.

By the fundamental Theorem: $A'(x) = f(x)$, the zeros of f are critical points of A .

$$e) \quad A(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x = \frac{1}{6}x(2x^2 - 15x + 24)$$

$$x = \frac{15 \pm \sqrt{33}}{4} \rightarrow \begin{cases} x_1 = \frac{15 - \sqrt{33}}{4} \approx 2.31386 \\ x_2 = \frac{15 + \sqrt{33}}{4} \approx 5.18614 \end{cases}$$

$$\begin{aligned} f) \quad A_1 &= \int_0^{x_1} f(x) dx = \int_0^1 f(x) dx + \left| \int_1^{x_1} f(x) dx \right| \\ &= \left[\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \right]_0^1 + \left| \left[\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \right]_1^{x_1} \right| \\ &= \left[\left(\frac{1}{3} - \frac{5}{2} + 4 \right) - 0 \right] + \left| 0 - \left(\frac{1}{3} - \frac{5}{2} + 4 \right) \right| \\ &= 2 \left(\frac{1}{3} - \frac{5}{2} + 4 \right) \\ &= 2 \left(\frac{11}{6} \right) \\ &= \frac{11}{3} \end{aligned}$$

$$\begin{aligned} A_2 &= \left| \int_{x_1}^{x_2} f(x) dx \right| + \int_{x_2}^b f(x) dx \\ &= \left| \left[\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \right]_{x_1}^{x_2} \right| + \left[\frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x \right]_{x_2}^b \\ &= 0 + \left[\left(\frac{1}{3}b^3 - \frac{5}{2}b^2 + 4b \right) - 0 \right] \\ &= \frac{1}{3}b^3 - \frac{5}{2}b^2 + 4b \end{aligned}$$

Since $A_1 = A_2$

$$\frac{1}{3}b^3 - \frac{5}{2}b^2 + 4b = \frac{11}{3}$$

$$\frac{1}{3}b^3 - \frac{5}{2}b^2 + 4b - \frac{11}{3} = 0 \rightarrow \boxed{b = 5.744348} \text{ (and 2 complex numbers)}$$

g) No, if the function is a piecewise function.

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ -1 & \text{if } 1 \leq x \leq 2 \end{cases}$$

Then $A(x)$ has a maximum at $x = 1$ even though f is never zero.

This is a case where an extreme point occurs at a singular point rather than a stationary point.