Section 2.3 – Orthogonality

Definition

Two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\boldsymbol{R}^n$ are said to be ***orthogonal*** (or ***perpendicular***) if their dot product is zero $\boldsymbol{u}.\boldsymbol{v} = 0$.

We will also agree that he zero vector in $\boldsymbol{R}^n$ is orthogonal to every vector in $\boldsymbol{R}^n$. A nonempty set of vectors $\boldsymbol{R}^n$ is called an ***orthogonal set*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set of unit vectors is called an ***orthonormal set***.

Example

The floor of your room (extended to infinity) is a subspace $\boldsymbol{V}$. The line where two walls meet is a subspace $\boldsymbol{W}$ (one-dimensional). Those subspaces are orthogonal. Every vector up the meeting line is perpendicular to every vector on the floor. The origin (0, 0, 0) is in the corner.

Example

Show that $\boldsymbol{u} = (-2, 3, 1, 4)$ and $\boldsymbol{v} = (1, 2, 0, -1)$ are orthogonal in $\boldsymbol{R}^4$

Solution

$$\begin{aligned}\boldsymbol{u}.\boldsymbol{v} &= (-2)(1) + (3)(2) + (1)(0) + (4)(-1)\\ &= -2 + 6 + 0 - 4\\ &= 0\end{aligned}$$

These vectors are orthogonal in $\boldsymbol{R}^4$

Standard Unit Vectors

$$\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{j}} = \hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{k}} = \hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{k}} = \mathbf{0}$$

Proof

$$\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{j}} = (1,\ 0,\ 0) \cdot (0,\ 1,\ 0) = \mathbf{0}$$

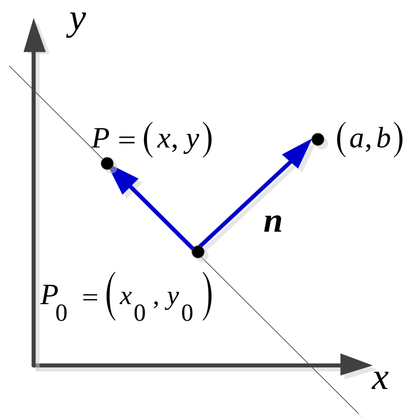
Normal

To specify slope and inclination is to use a nonzero vector $\boldsymbol{n}$, called a ***normal***, that is orthogonal to the line or plane.

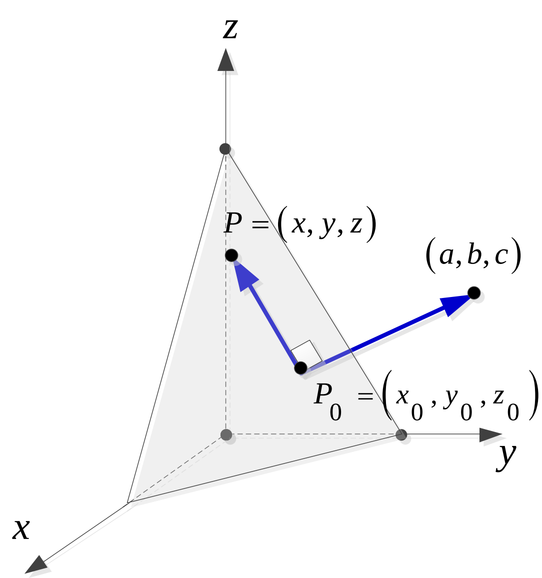
The line passes through a point $P_0\left(x_0, y_0\right)$ that has a normal $\boldsymbol{n} = (a, b)$ and the plane through $P_0\left(x_0, y_0, z_0\right)$ that has a normal $\boldsymbol{n} = (a, b, c)$. Both the line and the plane are represented by the vector equation

$$\boldsymbol{n} \bullet \overrightarrow{P_0 P} = 0$$

The line equation: $a\left(x - x_0\right) + b\left(y - y_0\right) = 0$



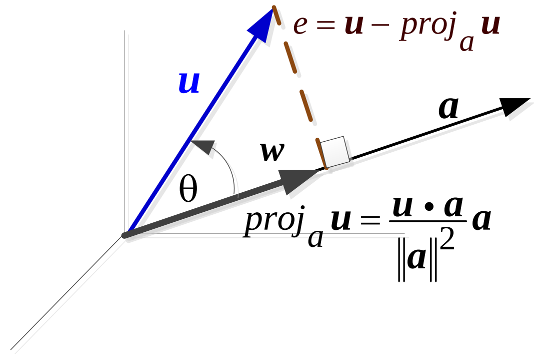
The plane equation: $a\left(x - x_0\right) + b\left(y - y_0\right) + c\left(z - z_0\right) = 0$



Projections

Theorem Projection onto a line

If $\boldsymbol{u}$ and $\boldsymbol{a}$ are vectors in $\boldsymbol{R}^n$, and if $\boldsymbol{a} \neq 0$, then $\boldsymbol{u}$ can be expressed in exactly one way in the form $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{e}$, where $\boldsymbol{w}$ is a scalar multiple of $\boldsymbol{a}$ and $\boldsymbol{e}$ is orthogonal to $\boldsymbol{a}$.



The vector $\boldsymbol{w}$ is called the ***orthogonal projection*** of $\boldsymbol{u}$ on $\boldsymbol{a}$ or sometimes ***component*** of $\boldsymbol{u}$ along $\boldsymbol{a}$. The vector $\boldsymbol{e}$ is called the vector ***component*** of $\boldsymbol{u}$ ***orthogonal*** to $\boldsymbol{a}$ (error vector and should be perpendicular to $\boldsymbol{a}$)

$$proj_{\boldsymbol{a}} \boldsymbol{u} = \frac{\boldsymbol{u} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2} \boldsymbol{a} = p \qquad (\textit{vector component of } \boldsymbol{u} \textit{ along } \boldsymbol{a})$$

$$\boldsymbol{u} - proj_a \boldsymbol{u} = \boldsymbol{u} - \frac{\boldsymbol{u} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2} \boldsymbol{a} \qquad (\textit{vector component of } \boldsymbol{u} \textit{ orthogonal to } \boldsymbol{a})$$

The length is $\left\| proj_{\boldsymbol{a}} \boldsymbol{u} \right\| = \|\boldsymbol{u}\| |\cos\theta|$

$$\left\| proj_a \boldsymbol{u} \right\| = \frac{|\boldsymbol{u} \bullet \boldsymbol{a}|}{\|\boldsymbol{a}\|}$$

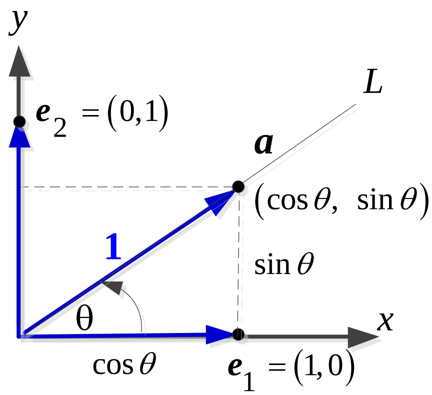
Special case: If $\boldsymbol{u} = \boldsymbol{a}$ then $\frac{\boldsymbol{u} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2} = 1$. The projection of $\boldsymbol{a}$ onto $\boldsymbol{a}$ is itself.

Special case: If $\boldsymbol{u}$ is perpendicular to $\boldsymbol{a}$ then $\boldsymbol{u} \bullet \boldsymbol{a} = 0$. The projection is $\boldsymbol{p} = \boldsymbol{0}$.

Example

Find the orthogonal projections of the vectors $\hat{e}_1 = (1,\ 0)$ and $\hat{e}_2 = (0,\ 1)$ on the line L that makes an angle θ with the positive x-axis in $\boldsymbol{R}^2$

Solution



Let $\boldsymbol{a} = (\cos\theta, \sin\theta)$ be the unit vector along the line L.

$$\|\boldsymbol{a}\| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\boldsymbol{e}_1 \cdot \boldsymbol{a} = (1,0)(\cos\theta, \sin\theta) = (1)\cos\theta + (0)\sin\theta = \underline{\cos\theta}$$

$$proj_{\boldsymbol{a}}\boldsymbol{e_1} = \frac{\boldsymbol{e_1} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

$$= \frac{\cos\theta}{1}(\cos\theta,\ \sin\theta)$$

$$= \left(\cos^2\theta,\ \cos\theta\sin\theta\right)$$

$$proj_{\boldsymbol{a}}\boldsymbol{e_2} = \frac{\boldsymbol{e_2} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} = \frac{(0,1)(\cos\theta,\ \sin\theta)}{1}(\cos\theta,\ \sin\theta)$$

$$= \frac{(0,1)(\cos\theta,\ \sin\theta)}{1}(\cos\theta,\ \sin\theta)$$

$$= \sin\theta(\cos\theta,\ \sin\theta)$$

$$= \left(\sin\theta\cos\theta,\ \sin^2\theta\right)$$

Example

Let $\boldsymbol{u}$ = (2, –1, 3) and $\boldsymbol{a}$ = (4, –1, 2). Find the vector component of $\boldsymbol{u}$ along $\boldsymbol{a}$ and the vector component of $\boldsymbol{u}$ orthogonal to $\boldsymbol{a}$.

Solution

$$\begin{aligned}
proj_{\boldsymbol{a}}\boldsymbol{u} &= \frac{\boldsymbol{u} \bullet \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} \\
&= \frac{(2,\ -1,\ 3)\bullet(4,\ -1,\ 2)}{\left(\sqrt{4^2+(-1)^2+2^2}\right)^2}(4,\ -1,\ 2) \\
&= \frac{8+1+6}{21}(4,\ -1,\ 2) \\
&= \frac{15}{21}(4,\ -1,\ 2) \\
&= \frac{5}{7}(4,\ -1,\ 2) \\
&= \left(\frac{20}{7},\ -\frac{5}{7},\ \frac{10}{7}\right)
\end{aligned}$$

The vector component of $\boldsymbol{u}$ orthogonal to $\boldsymbol{a}$ is

$$\begin{aligned}
\boldsymbol{u} - proj_{\boldsymbol{a}}\boldsymbol{u} &= (2,\ -1,\ 3) - \left(\frac{20}{7},\ -\frac{5}{7},\ \frac{10}{7}\right) \\
&= \left(-\frac{6}{7},\ -\frac{2}{7},\ \frac{11}{7}\right)
\end{aligned}$$

***Theorem* of *Pythagoras* in $\boldsymbol{R}^n$**

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal vectors in $\boldsymbol{R}^n$ with the Euclidean inner product, then

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$$

Proof

Since $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal, then $\boldsymbol{u}\cdot\boldsymbol{v}=0$

$$\begin{aligned}
\|\boldsymbol{u}+\boldsymbol{v}\|^2 &= (\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v}) \\
&= \|\boldsymbol{u}\|^2 + 2(\boldsymbol{u}\bullet\boldsymbol{v}) + \|\boldsymbol{v}\|^2 \\
&= \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2
\end{aligned}$$

Distance

Theorem

In $\boldsymbol{R}^2$ the distance D between the point $P_0 = (x_0,\ y_0)$ and the line $ax + by + c = 0$ is

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

In $\boldsymbol{R}^3$ the distance D between the point $P_0 = (x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Exercises Section 2.3 – Orthogonality

1. Determine whether $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

a) $\boldsymbol{u}=(-6,\ -2),\quad \boldsymbol{v}=(5,\ -7)$

b) $\boldsymbol{u}=(6,\ 1,\ 4),\quad \boldsymbol{v}=(2,\ 0,\ -3)$

c) $\boldsymbol{u}=(1,\ -5,\ 4),\quad \boldsymbol{v}=(3,\ 3,\ 3)$

d) $\boldsymbol{u}=(-2,\ 2,\ 3),\quad \boldsymbol{v}=(1,\ 7,\ -4)$

2. Determine whether the vectors form an orthogonal set

a) $\boldsymbol{v}_1=(2,\ 3),\quad \boldsymbol{v}_2=(3,\ 2)$

b) $\boldsymbol{v}_1=(1,\ -2),\quad \boldsymbol{v}_2=(-2,\ 1)$

c) $\boldsymbol{u}=(-4,\ 6,\ -10,\ 1)\quad \boldsymbol{v}=(2,\ 1,\ -2,\ 9)$

d) $\boldsymbol{u}=(a,\ b)\quad \boldsymbol{v}=(-b,\ a)$

e) $\boldsymbol{v}_1=(-2,\ 1,\ 1),\quad \boldsymbol{v}_2=(1,\ 0,\ 2),\quad \boldsymbol{v}_3=(-2,\ -5,\ 1)$

f) $\boldsymbol{v}_1=(1,\ 0,\ 1),\quad \boldsymbol{v}_2=(1,\ 1,\ 1),\quad \boldsymbol{v}_3=(-1,\ 0,\ 1)$

g) $\boldsymbol{v}_1=(2,\ -2,\ 1),\quad \boldsymbol{v}_2=(2,\ 1,\ -2),\quad \boldsymbol{v}_3=(1,\ 2,\ 2)$

3. Find a unit vector that is orthogonal to both $\boldsymbol{u}$ = (1, 0, 1) and $\boldsymbol{v}$ = (0, 1, 1)

4. *a*) Show that $\boldsymbol{v}$ = (*a*, *b*) and $\boldsymbol{w}$ = (–*b*, *a*) are orthogonal vectors.

b) Use the result to find two vectors that are orthogonal to $\boldsymbol{v}$ = (2, –3).

c) Find two unit vectors that are orthogonal to (–3, 4)

5. Find the vector component of $\boldsymbol{u}$ along $\boldsymbol{a}$ and the vector component of $\boldsymbol{u}$ orthogonal to $\boldsymbol{a}$.

a) $\boldsymbol{u}=(6,\ 2),\quad \boldsymbol{a}=(3,\ -9)$

b) $\boldsymbol{u}=(3,\ 1,\ -7),\quad \boldsymbol{a}=(1,\ 0,\ 5)$

c) $\boldsymbol{u}=(1,\ 0,\ 0),\quad \boldsymbol{a}=(4,\ 3,\ 8)$

d) $\boldsymbol{u}=(1,\ 1,\ 1),\quad \boldsymbol{a}=(0,\ 2,\ -1)$

e) $\boldsymbol{u}=(2,\ 1,\ 1,\ 2),\quad \boldsymbol{a}=(4,\ -4,\ 2,\ -2)$

f) $\boldsymbol{u}=(5,\ 0,\ -3,\ 7),\quad \boldsymbol{a}=(2,\ 1,\ -1,-1)$

6. Project the vector $\boldsymbol{v}$ onto the line through $\boldsymbol{a}$, check that $\boldsymbol{e}=\boldsymbol{u}-proj_a\boldsymbol{u}$ is perpendicular to $\boldsymbol{a}$:

a) $v=\begin{pmatrix}1\\2\\2\end{pmatrix}$ *and* $a=\begin{pmatrix}1\\1\\1\end{pmatrix}$ *b)* $v=\begin{pmatrix}1\\3\\1\end{pmatrix}$ *and* $a=\begin{pmatrix}-1\\-3\\-1\end{pmatrix}$ *c)* $v=\begin{pmatrix}1\\1\\1\end{pmatrix}$ *and* $a=\begin{pmatrix}1\\2\\2\end{pmatrix}$

7. Find the projection matrix $proj_{\boldsymbol{a}}\boldsymbol{u}=\dfrac{\boldsymbol{u}\bullet\boldsymbol{a}}{\|\boldsymbol{a}\|^2}$ onto the line through $a=\begin{bmatrix}1\\2\\2\end{bmatrix}$

8. Draw the projection of $\boldsymbol{v}$ onto $\boldsymbol{a}$ and also compute it from $proj_{\boldsymbol{a}}\boldsymbol{u} = \frac{\boldsymbol{u} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$

a) $v = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ *and* $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *b)* $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $a = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

9. Show that if $\boldsymbol{v}$ is orthogonal to both $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, then $\boldsymbol{v}$ is orthogonal to $k_1\boldsymbol{w}_1 + k_2\boldsymbol{w}_2$ for all scalars k_1 and k_2.

10. *a*) Project the vector $\boldsymbol{v}$ = (3, 4, 4) onto the line through $\boldsymbol{a}$ = (2, 2, 1) and then onto the plane that also contains $\boldsymbol{a}^* = (1,\ 0,\ 0)$.

b) Check that the first error vector $\boldsymbol{v} - \boldsymbol{p}$ is perpendicular to $\boldsymbol{a}$, and the second error vector $\boldsymbol{v} - \boldsymbol{p}^*$ is also perpendicular to $\boldsymbol{a}^*$.

11. Compute the projection matrices $\boldsymbol{a}\boldsymbol{a}^T / \boldsymbol{a}^T\boldsymbol{a}$ onto the lines through $\boldsymbol{a}_1 = (-1,\ 2,\ 2)$ and $\boldsymbol{a}_2 = (2,\ 2,\ -1)$. Multiply those projection matrices and explain why their product P_1P_2 is what it is. Project $\boldsymbol{v} = (1,\ 0,\ 0)$ onto the lines $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and also onto $\boldsymbol{a}_3 = (2,\ -1,\ 2)$.Add up the three projections $p_1 + p_2 + p_3$.

12. If $P^2 = P$ show that $(I - P)^2 = I - P$. When P projects onto the column space of A, $I - P$ projects onto the ____.

13. What linear combination of $(1,\ 2,\ -1)$ and $(1,\ 0,\ 1)$ is closest to $\vec{v} = (2,\ 1,\ 1)$?

14. Show that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$ if and only if $\|\vec{u}\| = \|\vec{v}\|$

15. Given $\boldsymbol{u} = (3,\ -1,\ 2)$ $\boldsymbol{v} = (4,\ -1,\ 5)$ *and* $\boldsymbol{w} = (8,\ -7,\ -6)$

a) Find $3\boldsymbol{v} - 4(5\boldsymbol{u} - 6\boldsymbol{w})$

b) Find $\boldsymbol{u} \bullet \boldsymbol{v}$ and then the angle θ between $\boldsymbol{u}$ and $\boldsymbol{v}$.

16. Given: $\boldsymbol{u} = (3,\ 1,\ 3)$ $\boldsymbol{v} = (4,\ 1,\ -2)$

a) Compute the projection $\boldsymbol{w}$ of $\boldsymbol{u}$ on $\boldsymbol{v}$

b) Find $\boldsymbol{p} = \boldsymbol{u} - \boldsymbol{w}$ and show that $\boldsymbol{p}$ is perpendicular to $\boldsymbol{v}$.

17. *a*) Show that $\boldsymbol{v}$ = (*a*, *b*) and $\boldsymbol{w}$ = (–*b*, *a*) are orthogonal vectors

b) Use the result in part (*a*) to find two vectors that are orthogonal to $\boldsymbol{v}$ = (2, –3)

c) Find two unit vectors that are orthogonal to (–3, 4)

18. Show that A (3, 0, 2), B (4, 3, 0), and C (8, 1, −1) are vertices of a right triangle. At which vertex is the right angle?

19. Establish the identity: $\boldsymbol{u}\bullet\boldsymbol{v} = \frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2$

20. Find the Euclidean inner product $\boldsymbol{u}\bullet\boldsymbol{v}$: $\boldsymbol{u} = (-1,\ 1,\ 0,\ 4,\ -3)$ $\boldsymbol{v} = (-2,\ -2,\ 0,\ 2,\ -1)$

21. Find the Euclidean distance between $\boldsymbol{u}$ and $\boldsymbol{v}$: $\boldsymbol{u} = (3,\ -3,\ -2,\ 0,\ -3)$ $\boldsymbol{v} = (-4,\ 1,\ -1,\ 5,\ 0)$

(*Exercises* 22–26) Find

a) $\vec{v}\bullet\vec{u},\ |\vec{v}|,\ |\vec{u}|$

b) The cosine of the angle between $\vec{v}$ and $\vec{u}$

c) The scalar component of $\vec{u}$ in the direction of $\vec{v}$

d) The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

22. $\vec{v} = 2\hat{i} - 4\hat{j} + \sqrt{5}\hat{k},\quad \vec{u} = -2\hat{i} + 4\hat{j} - \sqrt{5}\hat{k}$

23. $\vec{v} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{k},\quad \vec{u} = 5\hat{i} + 12\hat{j}$

24. $\vec{v} = 2\hat{i} + 10\hat{j} - 11\hat{k},\quad \vec{u} = 2\hat{i} + 2\hat{j} + \hat{k}$

25. $\vec{v} = -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}},\quad \vec{u} = 2\hat{\boldsymbol{i}} + \sqrt{17}\hat{\boldsymbol{j}}$

26. $\vec{v} = \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}}\right),\quad \vec{u} = \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{3}}\right)$

27. Suppose Ted weighs 180 *lb*. and he is sitting on an inclined plane that drops 3 *units* for every 4 horizontal units. The gravitational force vector is $\vec{F}_g = \begin{pmatrix} 0 \\ -180 \end{pmatrix}$.

a) Find the force pushing Ted down the slope.

b) Find the force acting to hold Ted against the slope

28. Prove that is two vectors $\vec{u}$ and $\vec{v}$ in R^2 are orthogonal to nonzero vector $\vec{w}$ in R^2, then $\vec{u}$ and $\vec{v}$ are scalar multiples of each other.