Section 4.4 – The Binomial Theorem

A binomial is a sum $a+b$, where a and b represent numbers. If n is a positive integer, then a general formula for expanding $(a+b)^n$ is given by the ***binomial theorem***.

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

The expansions of $(a+b)^n$ for n = 2, 3, 4, and 5 have the following properties:

- ✓ There are n + 1 terms, the first being a^n and the last b^n
- ✓ The power of a decreases by 1 and the power of b increases by 1. For each term, the sum of the exponents of a and b is n.
- ✓ Each term has the form $(c)a^{n-k}b^k$, where the coefficient c is an integer and $k = 0,\ 1,\ 2,\ \ldots,\ n$.
- ✓ The following formula is true for each of the first n terms of the expansion:

$$\frac{(coefficient\ of\ \ term).(exponent\ of\ \ a)}{number\ of\ \ term} = coefficient\ of\ \ next\ term$$

Coefficient of the $(k+1)st$ **Term in the Expansion of** $(a+b)^n$

$$\frac{n.(n-1).(n-2).(n-3)....(n-k+1)}{k.(k-1).....3.2.1}, \quad k = 1, 2, \ldots, n$$

Factorial Notation

Definition of $n!$ (*n* factorial)

$$\begin{cases} n! = n(n-1)(n-2)\cdots\cdots 1 & if \quad n > 0 \\ 0! = 1 \end{cases}$$

Calculators: ***Math*** $\to$ ***Prob*** $\to 4$

Illustration

1! = 1
2! = 2.1 = 2
3! = 3.2.1 = 6
4! = 4.3.2.1 = 24

Example

Simplify the quotient of factorial: $\frac{7!}{5!}$

Solution

$$\frac{7!}{5!} = \frac{7.6.5.4.3.2.1}{5.4.3.2.1} = 7.6 = 42$$

Example

Simplify the quotient of factorial: $\frac{10!}{6!}$

Solution

$$\frac{10!}{6!} = \frac{10.9.8.7.6!}{6!} = 10.9.8.7 = 5040$$

Coefficient of the $(k+1)st$ Term in the Expansion of $(a+b)^n$ (*Alternative Form*)

$$\boxed{\binom{n}{k} = C(n,\ k) = \frac{n!}{k!(n-k)!}}, \quad k = 0,1,2,\ldots,n$$

Example

Find $\binom{5}{2}, \binom{5}{0}$

Solution

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{1.2.3.4.5}{(1.2)(1.2.3)} = \frac{20}{2} \quad \underline{= 10|}$$

$$\binom{5}{0} = \frac{5!}{0!(5-0)!} = \frac{5!}{1(5!)} \quad \underline{= 1|}$$

Binomial Theorem

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{k}a^{n-k}b^k + \ldots + \binom{n}{n-1}ab^{n-1} + b^n$$

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \ldots + \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}a^{n-k}b^k + \ldots + nab^{n-1} + b^n$$

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

Example

Find the binomial expansion of $\left(2x+3y^2\right)^4$

Solution

$$\begin{aligned}\left(2x+3y^2\right)^4 &= (2x)^4 + \binom{4}{1}(2x)^3\left(3y^2\right)^1 + \binom{4}{2}(2x)^2\left(3y^2\right)^2 + \binom{4}{3}(2x)^1\left(3y^2\right)^3 + \left(3y^2\right)^4 \\ &= 16x^4 + 4\left(8x^3\right)\left(3y^2\right) + 6\left(4x^2\right)\left(9y^4\right) + 4(2x)\left(27y^6\right) + 81y^8 \\ &= 16x^4 + 96x^3y^2 + 216x^2y^4 + 216xy^6 + 81y^8\end{aligned}$$

Example

Find the binomial expansion of $\left(\frac{1}{x} - 2\sqrt{x}\right)^5$

Solution

$$\begin{aligned}\left(\frac{1}{x} - 2\sqrt{x}\right)^5 &= \left(\frac{1}{x}\right)^5 + 5\left(\frac{1}{x}\right)^4\left(-2\sqrt{x}\right)^1 + 10\left(\frac{1}{x}\right)^3\left(-2\sqrt{x}\right)^2 + 10\left(\frac{1}{x}\right)^2\left(-2\sqrt{x}\right)^3 \\ &\quad + 5\left(\frac{1}{x}\right)^1\left(-2\sqrt{x}\right)^4 + \left(-2\sqrt{x}\right)^5 \\ &= \frac{1}{x^5} - 10\frac{1}{x^4}\left(\sqrt{x}\right) + 10\frac{1}{x^3}(4x) - 10\frac{1}{x^2}\left(8x\sqrt{x}\right) + 5\left(\frac{1}{x}\right)\left(16x^2\right) - 32x^{5/2} \\ &= \frac{1}{x^5} - 10\frac{1}{x^{7/2}} + 40\frac{1}{x^2} - 80\frac{1}{x^{1/2}} + 80x - 32x^{5/2}\end{aligned}$$

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 140 210 252 210 140 45 10 1

Example

Find the eighth row of the Pascal's triangle, and use it to expand $(a+b)^7$

Solution

1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

$$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

Exercises *Section* 4.4 – The Binomial Theorem

1. Find the fifth term in the expansion $\left(x^3+\sqrt{y}\right)^{13}$

2. Find the term involving q^{10} in the binomial expansion $\left(\frac{1}{3}p+q^2\right)^{12}$

Expand and simplify:

3. $(4x-y)^3$

4. $(x+y)^6$

5. $(x-y)^7$

6. $(3t-5x)^4$

7. $\left(\frac{1}{3}x+y^2\right)^5$

8. $\left(\frac{1}{x^2}+3x\right)^6$

9. $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^5$

10. $(2y-3)^4$

11. $(x+2)^5$

12. $\left(x^2-y^2\right)^6$

13. $(ax-by)^4$

14. $(ax+by)^5$

15. $\left(\sqrt{x}-\sqrt{3}\right)^4$

16. $\left(\sqrt{x}-\sqrt{2}\right)^6$

17. $(2x-1)^{12}$

18. $\left(x-\frac{1}{x^2}\right)^9$

19. $\left(\frac{2}{x}-3y\right)^5$

20. $\left(3\sqrt{x}+\sqrt[4]{x}\right)^4$

21. $(x+1)^5$

22. $(x-1)^5$

23. $(x-2)^6$

24. $\left(\frac{1}{x^3}-2x\right)^5$