Ex

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \quad\quad + 2x_5 \quad\quad &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\
5x_3 - 10x_4 \quad\quad + 15x_6 &= 0 \\
2x_1 + 6x_2 \quad\quad + 8x_4 + 4x_5 + 18x_6 &= 0
\end{aligned}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & -10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix} \longrightarrow$$

$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6
\end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & 3 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$x_1 = -3x_2 - 2x_5$

$\rightarrow x_3 = 0$

$\rightarrow x_4 = 0$

$\rightarrow x_6 = 0$

$$\begin{aligned}
&(x_1, x_2, x_3, x_4, x_5, x_6) \\
&= (-3x_2 - 2x_5, x_2, 0, 0, x_5, 0) \\
&= x_2(-3, 1, 0, 0, 0, 0) + x_5(-2, 0, 0, 0, 1, 0)
\end{aligned}$$

dim = 2

2.8 Row & Column Spaces

$$
\begin{array}{cc}
 & \begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & & \end{array} \\
A = & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{array}{l} \leftarrow \vec{r}_1 \\ \leftarrow \vec{r}_2 \\ \\ \\ \end{array}
\end{array}
$$

$$\vec{r}_1 = [a_{11} \quad a_{12} \quad \cdots \quad a_{1n}]$$

row vector

$$\vdots$$

$$\vec{r}_m = [a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn}]$$

$$\vec{v}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \qquad \vec{v}_2 \; - - - \; \vec{v}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Column vector

Defn $A(m \times n)$ subspace of $\mathbb{R}^n$ spanned the rows vectors of A is called row space, $R(A)$ or $\underline{RS(A)}$

column space $\quad CS(A) \quad C(A)$

The soln of the homogeneous system $Ax = 0$ is called null space of A $(NS(A)$

$N(A)$

Column space of A

of $Ax = b$

$$A = \begin{pmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 2 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$Ax = b$$

$$\begin{pmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = b$$

$$b = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} x_2$$

$$\boxed{= (1, 4, 2)\, x_1 + (0, 3, 2)\, x_2}$$

Ex

$$\begin{pmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -9 \\ -3 \end{pmatrix}$$

$$x_1 = 2, \quad x_2 = -1, \quad x_3 = 3$$

$$2 \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -9 \\ -3 \end{pmatrix}$$

Ex

$$I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Column space of I in R^2

CS(I) Space language

Ex

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$\begin{matrix} 1 & 2 \\ 0 & 0 \end{matrix}$

$x_1 + 2x_2 =$

$\downarrow$

$2 * C_1 = C_2$

$CS(A)$ is a line

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \end{pmatrix}$$

$x_1 + 2x_2 = + x_3$

$1 + 2 = 3$

$0 + 4 = 4$

$CS(B)$ is all $\mathbb{R}^2$

3 vector = sum 1 & 2

Pivot Columns & Free

$$\begin{array}{c} R = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \qquad\quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \end{array}$$

pivot columns: 1 & 3

Pivot variables: x_1 & x_3

Free variables: x_2, x_4, x_5

Free columns: 2, 4, 5 $\quad C_2, C_4, C_5$

Complete Soln $\quad Ax = B$

$$A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 6 \\ 7 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 7 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 = 1 - 3x_2 - 2x_4 \\ x_3 = 6 - 4x_4 \end{cases} \Bigg/ \quad \underbrace{\begin{matrix} x_1 = -3x_2 - 2x_4 \\ x_3 = -4x_4 \end{matrix}}_{\text{homogeneous}}$$

$$\begin{array}{cccccc} \downarrow & x_2 & \downarrow & x_4 & x_5 & \\ & & & & & \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \to (1) \\ \to (2) \\ \end{matrix}$$

pivot variables: x_1, x_3

free var: $x_2 \; x_4, x_5 \Leftarrow$

$$(1) \to \begin{cases} x_1 = -3x_2 - 2x_4 + x_5 \\ x_3 = -4x_4 + 3x_5 \end{cases}$$

Special solution

$x_2 = 1 \quad x_4 = 0 \quad x_5 = 0 \Rightarrow S_1 = (-3, 1, 0, 0, 0)$

$x_2 = 0 \quad x_4 = 1 \quad x_5 = 0 \Rightarrow S_2 = (-2, 0, -4, 1, 0)$

$x_2 = 0 \quad x_4 = 0 \quad x_5 = 1 \qquad S_3 = (1, 0, 3, 0, 1)$

$\dim(A) = 3$

Ex

$$A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 1 \\ 6 \\ 7 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad x_p = \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \end{pmatrix}$$

particular solution

- Pivot var: x_1, x_3
- free var.: x_2, x_4
- dim $(A) = 2$ + $r(A) = 2$

e) $\rightarrow$ $x_1 = -3x_2 - 2x_4$

$x_3 = -4x_4$

$x_2 = 1 \quad x_4 = 0 \rightarrow s_2 = (-3, 1, 0, 0)$

$x_2 = 0 \quad x_4 = 1 \rightarrow s_4 = (-2, 0, -4, 1)$

f) $NS(A) = \begin{pmatrix} -3 & -2 \\ 1 & 0 \\ 0 & -4 \\ 0 & 1 \end{pmatrix}$

g) $x_p = \begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix}$ $\qquad x_p = (1, 0, 6, 0)$

h) complete soln: $X = x_p + x_n$

$$= \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix}$$

$$b = A \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \end{pmatrix}$$

$b = AX \rightarrow$ particular is given

2.9 Rank (A) = r

Defn rank of A (m x n) is the # of non zero rows (RREF)

$$R = \begin{pmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$R(R) = 2$

$\dim(R) = 3$

rank + dim = # columns

rank of a matrix is # pivots

dim " " is # of vectors in a basis

+ A full row rank : $r = m$ (no zero in R)
- " " col " $r = n$ (no free var.)

$r(A) + N(A) = n$

Fundamental Theorem of L.A.