Solution Section 3.1 – Proving Identities

Exercise

Prove the identity $\cos\theta\cot\theta+\sin\theta=\csc\theta$

Solution

$$\begin{aligned}\cos\theta\cot\theta+\sin\theta &= \cos\theta\frac{\cos\theta}{\sin\theta}+\sin\theta\\ &= \frac{\cos^2\theta}{\sin\theta}+\sin\theta\\ &= \frac{\cos^2\theta}{\sin\theta}+\sin\theta\frac{\sin\theta}{\sin\theta}\\ &= \frac{\cos^2\theta}{\sin\theta}+\frac{\sin^2\theta}{\sin\theta}\\ &= \frac{\cos^2\theta+\sin^2\theta}{\sin\theta}\\ &= \frac{1}{\sin\theta}\\ &= \csc\theta\end{aligned}$$

Exercise

Prove the identity $\sec\theta\cot\theta-\sin\theta=\dfrac{\cos^2\theta}{\sin\theta}$

Solution

$$\begin{aligned}\sec\theta\cot\theta-\sin\theta &= \frac{1}{\cos\theta}\frac{\cos\theta}{\sin\theta}-\sin\theta\\ &= \frac{1}{\sin\theta}-\sin\theta\\ &= \frac{1-\sin^2\theta}{\sin\theta}\\ &= \frac{\cos^2\theta}{\sin\theta}\end{aligned}$$

Exercise

Prove the identity $\dfrac{\csc\theta\tan\theta}{\sec\theta}=1$

Solution

$$\begin{aligned}\frac{\csc\theta\tan\theta}{\sec\theta} &= \csc\theta\tan\theta\frac{1}{\sec\theta}\\ &= \frac{1}{\sin\theta}\frac{\sin\theta}{\cos\theta}\cos\theta\\ &= 1\end{aligned}$$

Exercise

Prove the identity $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

Solution

$$\begin{aligned}(\sin\theta + \cos\theta)^2 &= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta \\ &= 1 + 2\sin\theta\cos\theta\end{aligned}$$

Exercise

Prove the identity $\sin\theta(\sec\theta + \cot\theta) = \tan\theta + \cos\theta$

Solution

$$\begin{aligned}\sin\theta(\sec\theta + \cot\theta) &= \sin\theta\left(\frac{1}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right) \\ &= \sin\theta\frac{1}{\cos\theta} + \sin\theta\frac{\cos\theta}{\sin\theta} \\ &= \frac{\sin\theta}{\cos\theta} + \cos\theta \\ &= \tan\theta + \cos\theta \quad \checkmark\end{aligned}$$

Exercise

Prove the identity $\cos\theta(\csc\theta + \tan\theta) = \cot\theta + \sin\theta$

Solution

$$\begin{aligned}\cos\theta(\csc\theta + \tan\theta) &= \cos\theta\frac{1}{\sin\theta} + \cos\theta\frac{\sin\theta}{\cos\theta} \\ &= \cot\theta + \sin\theta \quad \checkmark\end{aligned}$$

Exercise

Prove the identity $\cos\theta(\sec\theta - \cos\theta) = \sin^2\theta$

Solution

$$\begin{aligned}\cos\theta(\sec\theta - \cos\theta) &= \cos\theta\frac{1}{\cos\theta} - \cos^2\theta \\ &= 1 - \cos^2\theta \\ &= \sin^2\theta \quad \checkmark\end{aligned}$$

Exercise

Prove the identity $\cot\theta + \tan\theta = \csc\theta\sec\theta$

Solution

$$\cot\theta + \tan\theta = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta}\frac{1}{\cos\theta}$$

$$\underline{= \csc\theta\sec\theta} \quad \checkmark$$

Exercise

Prove $\tan x(\cos x + \cot x) = \sin x + 1$

Solution

$$\tan x(\cos x + \cot x) = \frac{\sin x}{\cos x}\left(\cos x + \frac{\cos x}{\sin x}\right)$$

$$= \cos x\frac{\sin x}{\cos x} + \frac{\sin x}{\cos x}\frac{\cos x}{\sin x}$$

$$\underline{= \sin x + 1} \quad \checkmark$$

Exercise

Prove $\dfrac{1-\cos^4\theta}{1+\cos^2\theta} = \sin^2\theta$

Solution

$$\frac{1-\cos^4\theta}{1+\cos^2\theta} = \frac{(1+\cos^2\theta)(1-\cos^2\theta)}{1+\cos^2\theta}$$

$$= 1-\cos^2\theta$$

$$\underline{= \sin^2\theta} \quad \checkmark$$

Exercise

Prove $\dfrac{1-\sec x}{1+\sec x} = \dfrac{\cos x - 1}{\cos x + 1}$

Solution

$$\frac{1-\sec x}{1+\sec x} = \frac{1-\frac{1}{\cos x}}{1+\frac{1}{\cos x}}$$

$$= \frac{\frac{\cos x - 1}{\cos x}}{\frac{\cos x + 1}{\cos x}}$$

$$= \boxed{\frac{\cos x - 1}{\cos x + 1}} \quad \checkmark$$

Exercise

Prove $\quad \dfrac{\cos x}{1+\sin x} - \dfrac{1-\sin x}{\cos x} = 0$

Solution

$$\frac{\cos x}{1+\sin x} - \frac{1-\sin x}{\cos x} = \frac{\cos x}{\cos x}\frac{\cos x}{1-\sin x} - \frac{1+\sin x}{1+\sin x}\frac{1-\sin x}{\cos x}$$

$$= \frac{\cos^2 x - (1-\sin^2 x)}{\cos x(1+\sin x)}$$

$$= \frac{\cos^2 x - 1 + \sin^2 x)}{\cos x(1+\sin x)}$$

$$= \frac{1-1}{\cos x(1+\sin x)}$$

$$= \frac{0}{\cos x(1+\sin x)}$$

$$= \boxed{0} \quad \checkmark$$

Exercise

Prove $\quad \dfrac{1+\cot^3 t}{1+\cot t} = \csc^2 t - \cot t$

Solution

$$\frac{1+\cot^3 t}{1+\cot t} = \frac{1+\frac{\cos^3 t}{\sin^3 t}}{1+\frac{\cos t}{\sin t}}$$

$$= \frac{\frac{\sin^3 t+\cos^3 t}{\sin^3 t}}{\frac{\sin t+\cos t}{\sin t}}$$

$$= \frac{\sin^3 t+\cos^3 t}{\sin^3 t} \cdot \frac{\sin t}{\sin t+\cos t}$$

$$= \frac{(\sin t+\cos t)(\sin^2 t - \sin t\cos t+\cos^2 t)}{\sin^2 t} \cdot \frac{1}{\sin t+\cos t}$$

$$= \frac{1-\sin t\cos t}{\sin^2 t}$$

$$= \frac{1}{\sin^2 t} - \frac{\sin t \cos t}{\sin^2 t}$$

$$= \csc^2 t - \frac{\cos t}{\sin t}$$

$$\underline{= \csc^2 t - \cot t}\,\Big| \quad \checkmark$$

Exercise

Prove: $\tan x + \cot x = \sec x \csc x$

Solution

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x}$$

$$= \frac{1}{\cos x \sin x}$$

$$= \frac{1}{\cos x} \frac{1}{\sin x}$$

$$\underline{= \sec x \ \csc x}\,\Big| \quad \checkmark$$

Exercise

Prove: $\dfrac{\tan x - \cot x}{\sin x \cos x} = \sec^2 x - \csc^2 x$

Solution

$$\frac{\tan x - \cot x}{\sin x \cos x} = \frac{\tan x}{\sin x \cos x} - \frac{\cot x}{\sin x \cos x}$$

$$= \tan x \frac{1}{\sin x \cos x} - \cot x \frac{1}{\sin x \cos x}$$

$$= \frac{\sin x}{\cos x} \frac{1}{\sin x \cos x} - \frac{\cos x}{\sin x} \frac{1}{\sin x \cos x}$$

$$= \frac{1}{\cos^2 x} - \frac{1}{\sin^2 x}$$

$$\underline{= \sec^2 x - \csc^2 x}\,\Big| \quad \checkmark$$

Exercise

Prove: $\dfrac{\sec x + \tan x}{\sec x - \tan x} = \dfrac{1 + 2\sin x + \sin^2 x}{\cos^2 x}$

Solution

$$\frac{\sec x + \tan x}{\sec x - \tan x} = \frac{\frac{1}{\cos x} + \frac{\sin x}{\cos x}}{\frac{1}{\cos x} - \frac{\sin x}{\cos x}} \frac{\cos x}{\cos x}$$

$$= \frac{\frac{1}{\cos x}\cos x + \frac{\sin x}{\cos x}\cos x}{\frac{1}{\cos x}\cos x - \frac{\sin x}{\cos x}\cos x}$$

$$= \frac{1+\sin x}{1-\sin x}$$

$$= \frac{1+\sin x}{1-\sin x}\frac{1+\sin x}{1+\sin x}$$

$$= \frac{(1+\sin x)^2}{1-\sin^2 x}$$

$$\boxed{= \frac{1+2\sin x+\sin^2 x}{\cos^2 x}} \quad \checkmark$$

Exercise

Prove the identity: $\sin^2 x - \cos^2 x = 2\sin^2 x - 1$

Solution

$$\begin{aligned} \sin^2 x - \cos^2 x &= \sin^2 x - \left(1-\sin^2 x\right) \\ &= \sin^2 x - 1 + \sin^2 x \\ &\boxed{= 2\sin^2 x - 1} \quad \checkmark \end{aligned}$$

Exercise

Prove the identity: $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$

Solution

$$\begin{aligned} \sin^4 x - \cos^4 x &= \left(\sin^2 x + \cos^2 x\right)\left(\sin^2 x - \cos^2 x\right) \\ &= (1)\left(\sin^2 x - \cos^2 x\right) \\ &\boxed{= \sin^2 x - \cos^2 x} \quad \checkmark \end{aligned}$$

Exercise

Prove the identity: $\dfrac{\cos\alpha}{1+\sin\alpha} = \sec\alpha - \tan\alpha$

Solution

$$\begin{aligned} \frac{\cos\alpha}{1+\sin\alpha} &= \frac{\cos\alpha}{1+\sin\alpha}\frac{1-\sin\alpha}{1-\sin\alpha} \\ &= \frac{\cos\alpha - \cos\alpha\sin\alpha}{1-\sin^2\alpha} \end{aligned}$$

$$= \frac{\cos\alpha - \cos\alpha\sin\alpha}{\cos^2\alpha}$$

$$= \frac{\cos\alpha}{\cos^2\alpha} - \frac{\cos\alpha\sin\alpha}{\cos^2\alpha}$$

$$= \frac{1}{\cos\alpha} - \frac{\sin\alpha}{\cos\alpha}$$

$$= \sec\alpha - \tan\alpha \quad \checkmark$$

Exercise

Prove the identity: $\dfrac{\sin\alpha}{1-\sin\alpha} - \dfrac{\cos\alpha}{1-\sin\alpha} = \dfrac{1-\cot\alpha}{\csc\alpha - 1}$

Solution

$$\frac{\sin\alpha}{1-\sin\alpha} - \frac{\cos\alpha}{1-\sin\alpha} = \frac{\sin\alpha - \cos\alpha}{1-\sin\alpha}$$

$$= \frac{\dfrac{\sin\alpha - \cos\alpha}{\sin\alpha}}{\dfrac{1-\sin\alpha}{\sin\alpha}}$$

$$= \frac{\dfrac{\sin\alpha}{\sin\alpha} - \dfrac{\cos\alpha}{\sin\alpha}}{\dfrac{1}{\sin\alpha} - \dfrac{\sin\alpha}{\sin\alpha}}$$

$$= \frac{1-\cot\alpha}{\csc\alpha - 1} \quad \checkmark$$

Exercise

Prove the identity: $\dfrac{\dfrac{1}{\tan x} + \cot x}{\dfrac{1}{\tan x} + \tan x} = \dfrac{2}{\sec^2 x}$

Solution

$$\frac{\dfrac{1}{\tan x} + \cot x}{\dfrac{1}{\tan x} + \tan x} = \frac{\dfrac{1}{\tan x} + \dfrac{1}{\tan x}}{\dfrac{1+\tan^2 x}{\tan x}}$$

$$= \frac{\dfrac{2}{\tan x}}{\dfrac{\sec^2 x}{\tan x}}$$

$$= \frac{2}{\sec^2 x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cot^2\theta+3\cot\theta-4}{\cot\theta+4}=\cot\theta-1$

Solution

$$\frac{\cot^2\theta+3\cot\theta-4}{\cot\theta+4}=\frac{(\cot\theta+4)(\cot\theta-1)}{\cot\theta+4}$$

$$=\cot\theta-1 \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin\theta}{1+\cos\theta}=\dfrac{1-\cos\theta}{\sin\theta}$

Solution

$$\frac{\sin\theta}{1+\cos\theta}=\frac{\sin\theta}{1+\cos\theta}\frac{1-\cos\theta}{1-\cos\theta}$$

$$=\frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta}$$

$$=\frac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$$

$$=\frac{1-\cos\theta}{\sin\theta} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\tan x(\csc x-\sin x)=\cos x$

Solution

$$\tan x(\csc x-\sin x)=\frac{\sin x}{\cos x}\left(\frac{1}{\sin x}-\sin x\right)$$

$$=\frac{\sin x}{\cos x}\left(\frac{1-\sin^2 x}{\sin x}\right)$$

$$=\frac{1}{\cos x}\left(\frac{\cos^2 x}{1}\right)$$

$$=\cos x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\sin x(\tan x\cos x-\cot x\cos x)=1-2\cos^2 x$

Solution

$$\sin x(\tan x\cos x-\cot x\cos x)=\sin x\cos x\left(\frac{\sin x}{\cos x}-\frac{\cos x}{\sin x}\right)$$

$$= \sin x \cos x \left(\frac{\sin^2 x - \cos^2 x}{\cos x \sin x} \right)$$

$$= 1 - \cos^2 x - \cos^2 x$$

$$\underline{= 1 - 2\cos^2 x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $(1 + \tan x)^2 + (\tan x - 1)^2 = 2\sec^2 x$

Solution

$$(1 + \tan x)^2 + (\tan x - 1)^2 = 1 + 2\tan x + \tan^2 x + 1 - 2\tan x + \tan^2 x$$

$$= 2 + 2\tan^2 x$$

$$= 2\left(1 + \tan^2 x\right)$$

$$\underline{= 2\sec^2 x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\sec x + \tan x = \dfrac{\cos x}{1 - \sin x}$

Solution

$$\sec x + \tan x = \frac{1}{\cos x} + \frac{\sin x}{\cos x}$$

$$= \frac{1 + \sin x}{\cos x} \frac{1 - \sin x}{1 - \sin x}$$

$$= \frac{1 - \sin^2 x}{\cos x (1 - \sin x)}$$

$$= \frac{\cos^2 x}{\cos x (1 - \sin x)}$$

$$\underline{= \frac{\cos x}{1 - \sin x}} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\tan x - 1}{\tan x + 1} = \dfrac{1 - \cot x}{1 + \cot x}$

Solution

$$\frac{\tan x - 1}{\tan x + 1} = \frac{\frac{1}{\cot x} - 1}{\frac{1}{\cot x} + 1}$$

$$=\frac{\frac{1-\cot x}{\cot x}}{\frac{1+\cot x}{\cot x}}$$

$$=\frac{1-\cot x}{1+\cot x}\quad \checkmark$$

Exercise

Prove the following equation is an identity: $7\csc^2 x-5\cot^2 x=2\csc^2 x+5$

Solution

$$\begin{aligned}7\csc^2 x-5\cot^2 x&=7\csc^2 x-5\left(\csc^2 x-1\right)\\&=7\csc^2 x-5\csc^2 x+5\\&=2\csc^2 x+5\quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $1-\dfrac{\cos^2 x}{1-\sin x}=-\sin x$

Solution

$$\begin{aligned}1-\frac{\cos^2 x}{1-\sin x}&=1-\frac{1-\sin^2 x}{1-\sin x}\\&=1-\frac{(1-\sin x)(1+\sin x)}{1-\sin x}\\&=1-(1+\sin x)\\&=1-1-\sin x\\&=-\sin x\quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{1-\cos x}{1+\cos x}=\dfrac{\sec x-1}{\sec x+1}$

Solution

$$\begin{aligned}\frac{1-\cos x}{1+\cos x}&=\frac{\frac{1}{\cos x}-\frac{\cos x}{\cos x}}{\frac{1}{\cos x}+\frac{\cos x}{\cos x}}\\&=\frac{\sec x-1}{\sec x+1}\quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\sec x-1}{\tan x}=\frac{\tan x}{\sec x+1}$

Solution

$$\frac{\sec x-1}{\tan x}=\frac{\sec x-1}{\tan x}\frac{\sec x+1}{\sec x+1}$$

$$=\frac{\sec^2 x-1}{\tan x(\sec x+1)}$$

$$=\frac{\tan^2 x}{\tan x(\sec x+1)}$$

$$=\frac{\tan x}{\sec x+1} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\cos x}{\cos x-\sin x}=\frac{1}{1-\tan x}$

Solution

$$\frac{\cos x}{\cos x-\sin x}=\frac{\frac{\cos x}{\cos x}}{\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}}$$

$$=\frac{1}{1-\tan x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $(\sec x+\tan x)^2=\frac{1+\sin x}{1-\sin x}$

Solution

$$(\sec x+\tan x)^2=\left(\frac{1}{\cos x}+\frac{\sin x}{\cos x}\right)^2$$

$$=\left(\frac{1+\sin x}{\cos x}\right)^2$$

$$=\frac{(1+\sin x)^2}{\cos^2 x}$$

$$=\frac{(1+\sin x)^2}{1-\sin^2 x}$$

$$=\frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)}$$

$$=\frac{1+\sin x}{1-\sin x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\cos x}{1+\tan x}-\frac{\sin x}{1+\cot x}=\cos x-\sin x$

Solution

$$\begin{aligned}
\frac{\cos x}{1+\tan x}-\frac{\sin x}{1+\cot x} &= \frac{\cos x}{1+\frac{\sin x}{\cos x}}-\frac{\sin x}{1+\frac{\cos x}{\sin x}} \\
&= \frac{\cos x}{\frac{\cos x+\sin x}{\cos x}}-\frac{\sin x}{\frac{\sin x+\cos x}{\sin x}} \\
&= \frac{\cos^2 x}{\cos x+\sin x}-\frac{\sin^2 x}{\sin x+\cos x} \\
&= \frac{\cos^2 x-\sin^2 x}{\cos x+\sin x} \\
&= \frac{(\cos x+\sin x)(\cos x-\sin x)}{\cos x+\sin x} \\
&= \cos x-\sin x \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\cot x+\csc x-1}{\cot x-\csc x+1}=\csc x+\cot x$

Solution

$$\begin{aligned}
\frac{\cot x+\csc x-1}{\cot x-\csc x+1} &= \frac{\cot x+\csc x-\left(\csc^2 x-\cot^2 x\right)}{\cot x-\csc x+1} \\
&= \frac{\cot x+\csc x-(\csc x-\cot x)(\csc x+\cot x)}{\cot x-\csc x+1} \\
&= \frac{(\csc x+\cot x)(1-(\csc x-\cot x))}{\cot x-\csc x+1} \\
&= \frac{(\csc x+\cot x)(1-\csc x+\cot x)}{\cot x-\csc x+1} \\
&= \csc x+\cot x \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\frac{\tan x+\cot x}{\tan x-\cot x}=\frac{1}{\sin^2 x-\cos^2 x}$

Solution

$$\frac{\tan x+\cot x}{\tan x-\cot x}=\frac{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}}{\frac{\sin x}{\cos x}-\frac{\cos x}{\sin x}}$$

$$= \frac{\frac{\sin^2 x+\cos^2 x}{\cos x \sin x}}{\frac{\sin^2 x-\cos^2 x}{\cos x \sin x}}$$

$$\underline{= \frac{1}{\sin^2 x-\cos^2 x}} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{1-\cot^2 x}{1+\cot^2 x}+1=2\sin^2 x$

Solution

$$\frac{1-\cot^2 x}{1+\cot^2 x}+1=\frac{1-\cot^2 x+1+\cot^2 x}{1+\cot^2 x}$$

$$=\frac{2}{\csc^2 x}$$

$$\underline{=2\sin^2 x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{1+\cos x}{1-\cos x}-\dfrac{1-\cos x}{1+\cos x}=4\cot x\csc x$

Solution

$$\frac{1+\cos x}{1-\cos x}-\frac{1-\cos x}{1+\cos x}=\frac{(1+\cos x)^2-(1-\cos x)^2}{1-\cos^2 x}$$

$$=\frac{(1+\cos x+1-\cos x)(1+\cos x-1+\cos x)}{\sin^2 x} \qquad a^2-b^2=(a-b)(a+b)$$

$$=\frac{(2)(2\cos x)}{\sin^2 x}$$

$$=4\frac{\cos x}{\sin x}\frac{1}{\sin x}$$

$$\underline{=4\cot x\csc x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin^3 x-\cos^3 x}{\sin x-\cos x}=1+\sin x\cos x$

Solution

$$\frac{\sin^3 x-\cos^3 x}{\sin x-\cos x}=\frac{(\sin x-\cos x)\left(\sin^2 x+\sin x\cos x+\cos^2 x\right)}{\sin x-\cos x} \qquad a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$$

$$\underline{=1+\sin x\cos x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $1+\sec^2 x\sin^2 x=\sec^2 x$

Solution

$$1+\sec^2 x\sin^2 x=1+\frac{1}{\cos^2 x}\sin^2 x$$

$$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}$$

$$=\frac{1}{\cos^2 x}$$

$$=\sec^2 x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{1+\csc x}{\sec x}=\cos x+\cot x$

Solution

$$\frac{1+\csc x}{\sec x}=\frac{1}{\sec x}+\frac{\csc x}{\sec x}$$

$$=\cos x+\frac{\frac{1}{\sin x}}{\frac{1}{\cos x}}$$

$$=\cos x+\frac{\cos x}{\sin x}$$

$$=\cos x+\cot x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\tan^2 x=\sec^2 x-\sin^2 x-\cos^2 x$

Solution

$$\sec^2 x-\sin^2 x-\cos^2 x=\frac{1}{\cos^2 x}-\left(\sin^2 x+\cos^2 x\right)$$

$$=\frac{1}{\cos^2 x}-1$$

$$=\frac{1-\cos^2 x}{\cos^2 x}$$

$$=\frac{\sin^2 x}{\cos^2 x}$$

$$=\tan^2 x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\sin x}{1-\cos x}+\frac{\sin x}{1+\cos x}=2\csc x$

Solution

$$\begin{aligned}\frac{\sin x}{1-\cos x}+\frac{\sin x}{1+\cos x} &= \sin x\left(\frac{1}{1-\cos x}+\frac{1}{1+\cos x}\right)\\ &= \sin x\left(\frac{1+\cos x+1-\cos x}{1-\cos^2 x}\right)\\ &= \sin x\left(\frac{2}{\sin^2 x}\right)\\ &= \frac{2}{\sin x}\\ &= 2\csc x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cos^2(\alpha-\beta)-\cos^2(\alpha+\beta)=\sin^2(\alpha+\beta)-\sin^2(\alpha-\beta)$

Solution

$$\begin{aligned}\cos^2(\alpha-\beta)-\cos^2(\alpha+\beta) &= 1-\sin^2(\alpha-\beta)-\left[1-\sin^2(\alpha+\beta)\right]\\ &= 1-\sin^2(\alpha-\beta)-1+\sin^2(\alpha+\beta)\\ &= \sin^2(\alpha+\beta)-\sin^2(\alpha-\beta) \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\tan x\csc x-\sec^2 x\cos x=0$

Solution

$$\begin{aligned}\tan x\csc x-\sec^2 x\cos x &= \frac{\sin x}{\cos x}\frac{1}{\sin x}-\frac{1}{\cos^2 x}\cos x\\ &= \frac{1}{\cos x}-\frac{1}{\cos x}\\ &= 0 \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $(1+\tan x)^2-2\tan x=\frac{1}{(1-\sin x)(1+\sin x)}$

Solution

$$(1+\tan x)^2-2\tan x=1+2\tan x+\tan^2 x-2\tan x$$

$$= 1 + \tan^2 x$$

$$= \sec^2 x$$

$$= \frac{1}{\cos^2 x}$$

$$= \frac{1}{1 - \sin^2 x}$$

$$= \boxed{\frac{1}{(1 - \sin x)(1 + \sin x)}} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{3\csc^2 x - 5\csc x - 28}{\csc x - 4} = \dfrac{3}{\sin x} + 7$

Solution

$$\frac{3\csc^2 x - 5\csc x - 28}{\csc x - 4} = \frac{(3\csc x + 7)(\csc x - 4)}{\csc x - 4}$$

$$= 3\csc x + 7$$

$$= \boxed{\frac{3}{\sin x} + 7} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\left(\sec^2 x - 1\right)\left(\sec^2 x + 1\right) = \tan^4 x + 2\tan^2 x$

Solution

$$\left(\sec^2 x - 1\right)\left(\sec^2 x + 1\right) = \sec^4 x - 1 \qquad (a-b)(a+b) = a^2 - b^2 \quad a = \sec^2 x$$

$$= \left(\sec^2 x\right)^2 - 1$$

$$= \left(1 + \tan^2 x\right)^2 - 1$$

$$= 1 + 2\tan^2 x + \tan^4 x - 1$$

$$= \boxed{\tan^4 x + 2\tan^2 x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\csc x}{\cot x} - \dfrac{\cot x}{\csc x} = \dfrac{\sin x}{\cot x}$

Solution

$$\frac{\csc x}{\cot x} - \frac{\cot x}{\csc x} = \frac{\csc^2 x - \cot^2 x}{\cot x \csc x}$$

$$=\frac{\csc^2 x-\left(\csc^2 x-1\right)}{\cot x\csc x}$$

$$=\frac{\csc^2 x-\csc^2 x+1}{\cot x\csc x}$$

$$=\frac{1}{\cot x\frac{1}{\sin x}}$$

$$=\boxed{\frac{\sin x}{\cot x}}\quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{1-\cos^2 x}{1+\cos x}=\frac{\sec x-1}{\sec x}$

Solution

$$\frac{1-\cos^2 x}{1+\cos x}=\frac{(1-\cos x)(1+\cos x)}{1+\cos x}$$

$$=1-\cos x$$

$$=1-\frac{1}{\sec x}$$

$$=\boxed{\frac{\sec x-1}{\sec x}}\quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\cos x}{1+\cos x}=\frac{\sec x-1}{\tan^2 x}$

Solution

$$\frac{\cos x}{1+\cos x}=\frac{\cos x}{1+\cos x}\frac{1-\cos x}{1-\cos x}$$

$$=\frac{\cos x-\cos^2 x}{\cos^2 x-1}$$

$$=\frac{\cos x-\cos^2 x}{\sin^2 x}\frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}}$$

$$=\frac{\frac{1}{\cos x}-1}{\frac{\sin^2 x}{\cos^2 x}}$$

$$=\boxed{\frac{\sec x-1}{\tan^2 x}}\quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{1-2\sin^2 x}{1+2\sin x\cos x}=\dfrac{\cos x-\sin x}{\cos x+\sin x}$

Solution

$$\begin{aligned}
\frac{1-2\sin^2 x}{1+2\sin x\cos x} &= \frac{\cos^2 x+\sin^2 x-2\sin^2 x}{\cos^2 x+\sin^2 x+2\sin x\cos x}\\
&= \frac{\cos^2 x-\sin^2 x}{(\cos x+\sin x)^2}\\
&= \frac{(\cos x-\sin x)(\cos x+\sin x)}{(\cos x+\sin x)^2}\\
&= \frac{\cos x-\sin x}{\cos x+\sin x} \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $(\cos x-\sin x)^2+(\cos x+\sin x)^2=2$

Solution

$$\begin{aligned}
(\cos x-\sin x)^2+(\cos x+\sin x)^2 &= \cos^2 x-2\sin x\cos x+\sin^2 x+\cos^2 x+2\sin x\cos x+\sin^2 x\\
&= \cos^2 x+\sin^2 x+\cos^2 x+\sin^2 x\\
&= 1+1\\
&= 2 \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin x}{1+\cos x}+\dfrac{1+\cos x}{\sin x}=2\csc x$

Solution

$$\begin{aligned}
\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x} &= \frac{\sin x\sin x+(1+\cos x)(1+\cos x)}{(1+\cos x)\sin x}\\
&= \frac{\sin^2 x+1+2\cos x+\cos^2 x}{(1+\cos x)\sin x}\\
&= \frac{1+1+2\cos x}{(1+\cos x)\sin x}\\
&= \frac{2+2\cos x}{(1+\cos x)\sin x}\\
&= \frac{2(1+\cos x)}{(1+\cos x)\sin x}\\
&= \frac{2}{\sin x}\\
&= 2\csc x \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\sin x+\tan x}{\cot x+\csc x}=\sin x\tan x$

Solution

$$\begin{aligned}
\frac{\sin x+\tan x}{\cot x+\csc x} &= \frac{\sin x+\tan x}{\frac{1}{\tan x}+\frac{1}{\sin x}} \\
&= \frac{\sin x+\tan x}{\frac{\sin x+\tan x}{\tan x\sin x}} \\
&= (\sin x+\tan x)\frac{\tan x\sin x}{\sin x+\tan x} \\
&= \underline{\tan x\sin x} \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\csc^2 x\sec^2 x=\sec^2 x+\csc^2 x$

Solution

$$\begin{aligned}
\csc^2 x\sec^2 x &= \frac{1}{\sin^2 x}\frac{1}{\cos^2 x} \\
&= \frac{1}{\sin^2 x\cos^2 x} \\
&= \frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x} \\
&= \frac{\sin^2 x}{\sin^2 x\cos^2 x}+\frac{\cos^2 x}{\sin^2 x\cos^2 x} \\
&= \frac{1}{\cos^2 x}+\frac{1}{\sin^2 x} \\
&= \underline{\sec^2 x+\csc^2 x} \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cos^2 x+1=2\cos^2 x+\sin^2 x$

Solution

$$\begin{aligned}
\cos^2 x+1 &= \cos^2 x+\cos^2 x+\sin^2 x \\
&= \underline{2\cos^2 x+\sin^2 x} \quad \checkmark
\end{aligned}$$

Exercise

Prove the following equation is an identity: $1-\dfrac{\cos^2 x}{1+\sin x}=\sin x$

Solution

$$\begin{aligned}1-\frac{\cos^2 x}{1+\sin x}&=1-\frac{1-\sin^2 x}{1+\sin x}\\&=1-\frac{(1-\sin x)(1+\sin x)}{1+\sin x}\\&=1-(1-\sin x)\\&=1-1+\sin x\\&=\sin x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\cot^2 x=(\csc x-1)(\csc x+1)$

Solution

$$\begin{aligned}\cot^2 x&=\csc^2 x-1\\&=(\csc x-1)(\csc x+1) \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $10\csc^2 x-6\cot^2 x=4\csc^2 x+6$

Solution

$$\begin{aligned}10\csc^2 x-6\cot^2 x&=10\csc^2 x-6\left(\csc^2 x-1\right)\\&=10\csc^2 x-6\csc^2 x+6\\&=4\csc^2 x+6 \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{\csc x+\cot x}{\tan x+\sin x}=\csc x\cot x$

Solution

$$\begin{aligned}\frac{\csc x+\cot x}{\tan x+\sin x}&=\frac{\csc x+\cot x}{\frac{1}{\cot x}+\frac{1}{\csc x}}\\&=\frac{\csc x+\cot x}{\frac{\csc x+\cot x}{\cot x\csc x}}\\&=\csc x+\cot x\,\frac{\cot x\csc x}{\csc x+\cot x}\\&=\cot x\csc x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $\dfrac{1-\sec x}{\tan x}+\dfrac{\tan x}{1-\sec x}=-2\csc x$

Solution

$$\frac{1-\sec x}{\tan x}+\frac{\tan x}{1-\sec x}=\frac{(1-\sec x)(1-\sec x)+\tan^2 x}{\tan x(1-\sec x)}$$

$$=\frac{(1-\sec x)^2+\sec^2 x-1}{\tan x(1-\sec x)}$$

$$=\frac{(1-\sec x)^2+(\sec x+1)(\sec x-1)}{\tan x(1-\sec x)}$$

$$=\frac{(1-\sec x)^2-(\sec x+1)(1-\sec x)}{\tan x(1-\sec x)}$$

$$=\frac{(1-\sec x)\left[(1-\sec x)-(\sec x+1)\right]}{\tan x(1-\sec x)}$$

$$=\frac{1-\sec x-\sec x-1}{\tan x}$$

$$=\frac{-2\sec x}{\tan x}$$

$$=-2\frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}}$$

$$=-2\frac{1}{\sin x}$$

$$\underline{=-2\csc x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\csc x-\sin x=\cos x\cot x$

Solution

$$\csc x-\sin x=\frac{1}{\sin x}-\sin x$$

$$=\frac{1-\sin^2 x}{\sin x}$$

$$=\frac{\cos^2 x}{\sin x}$$

$$=\cos x\frac{\cos x}{\sin x}$$

$$\underline{=\cos x\cot x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\tan x+\sec x}{\sec x}-\frac{\tan x+\sec x}{\tan x}=-\cos x\cot x$

Solution

$$\frac{\tan x+\sec x}{\sec x}-\frac{\tan x+\sec x}{\tan x}=\frac{(\tan x+\sec x)\tan x-\sec x(\tan x+\sec x)}{\sec x\tan x}$$

$$=\frac{\tan^2 x+\sec x\tan x-\sec x\tan x-\sec^2 x}{\sec x\tan x}$$

$$=\frac{\tan^2 x-\sec^2 x}{\sec x\tan x} \qquad 1+\tan^2\alpha=\sec^2\alpha$$

$$=\frac{-1}{\sec x\tan x}$$

$$=-\frac{1}{\sec x}\frac{1}{\tan x}$$

$$=-\cos x\cot x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\cot^3 x=\cot x\left(\csc^2 x-1\right)$

Solution

$$\cot^3 x=\cot x\cot^2 x$$

$$=\cot x\left(\csc^2 x-1\right) \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\cot^2 x}{\csc x-1}=\frac{1+\sin x}{\sin x}$

Solution

$$\frac{\cot^2 x}{\csc x-1}=\frac{\csc^2 x-1}{\csc x-1}$$

$$=\frac{(\csc x-1)(\csc x+1)}{\csc x-1}$$

$$=\csc x+1$$

$$=\frac{1}{\sin x}+1$$

$$=\frac{1+\sin x}{\sin x} \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\cot^2 x+\csc^2 x=2\csc^2 x-1$

Solution

$$\cot^2 x + \csc^2 x = \csc^2 x - 1 + \csc^2 x$$
$$= 2\csc^2 x - 1 \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cot^2 x}{1+\csc x} = \csc x - 1$

Solution

$$\frac{\cot^2 x}{1+\csc x} = \frac{\csc^2 x - 1}{1+\csc x}$$
$$= \frac{(\csc x - 1)(\csc x + 1)}{1+\csc x}$$
$$= \csc x - 1 \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\sec^4 x - \tan^4 x = \sec^2 x + \tan^2 x$

Solution

$$\sec^4 x - \tan^4 x = \left(\sec^2 x + \tan^2 x\right)\left(\sec^2 x - \tan^2 x\right) \qquad a^2 - b^2 = (a-b)(a+b)$$
$$= \left(\sec^2 x + \tan^2 x\right)(1)$$
$$= \sec^2 x + \tan^2 x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\dfrac{\cos x}{1+\sin x} + \dfrac{1+\sin x}{\cos x} = 2\sec x$

Solution

$$\frac{\cos x}{1+\sin x} + \frac{1+\sin x}{\cos x} = \frac{\cos^2 x + (1+\sin x)^2}{(1+\sin x)\cos x}$$
$$= \frac{\cos^2 x + 1 + 2\sin x + \sin^2 x}{(1+\sin x)\cos x}$$
$$= \frac{2 + 2\sin x}{(1+\sin x)\cos x}$$
$$= \frac{2(1+\sin x)}{(1+\sin x)\cos x}$$
$$= \frac{2}{\cos x}$$
$$= 2\sec x \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\sin x+\cos x}{\sin x-\cos x}=\frac{1+2\sin x\cos x}{2\sin^2 x-1}$

Solution

$$\frac{\sin x+\cos x}{\sin x-\cos x}=\frac{\sin x+\cos x}{\sin x-\cos x}\frac{\sin x+\cos x}{\sin x+\cos x}$$

$$=\frac{\sin^2 x+2\sin x\cos x+\cos^2 x}{\sin^2 x-\cos^2 x}$$

$$=\frac{1+2\sin x\cos x}{\sin^2 x-\left(1-\sin^2 x\right)}$$

$$=\frac{1+2\sin x\cos x}{\sin^2 x-1+\sin^2 x}$$

$$=\frac{1+2\sin x\cos x}{2\sin^2 x-1}\quad\checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\csc x-1}{\csc x+1}=\frac{\cot^2 x}{\csc^2 x+2\csc x+1}$

Solution

$$\frac{\csc x-1}{\csc x+1}=\frac{\csc x-1}{\csc x+1}\frac{\csc x+1}{\csc x+1}$$

$$=\frac{\csc^2 x-1}{\csc^2 x+2\csc x+1}$$

$$=\frac{\cot^2 x}{\csc^2 x+2\csc x+1}\quad\checkmark$$

Exercise

Prove the following equation is an identity: $\csc^4 x-\cot^4 x=\csc^2 x+\cot^2 x$

Solution

$$\csc^4 x-\cot^4 x=\left(\csc^2 x+\cot^2 x\right)\left(\csc^2 x-\cot^2 x\right)$$

$$=\left(\csc^2 x+\cot^2 x\right)(1)$$

$$=\csc^2 x+\cot^2 x\quad\checkmark$$

Exercise

Prove the following equation is an identity: $\tan\left(\frac{\pi}{4}+x\right)=\cot\left(\frac{\pi}{4}-x\right)$

Solution

$$\tan\left(\frac{\pi}{4}+x\right)=\cot\left[\frac{\pi}{2}-\left(\frac{\pi}{4}+x\right)\right]$$
$$=\cot\left[\frac{\pi}{2}-\frac{\pi}{4}-x\right]$$
$$=\cot\left(\frac{\pi}{4}-x\right) \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\frac{\sin\theta}{1+\sin\theta}-\frac{\sin\theta}{1-\sin\theta}=-2\tan^2\theta$

Solution

$$\frac{\sin\theta}{1+\sin\theta}-\frac{\sin\theta}{1-\sin\theta}=\sin\theta\left[\frac{1-\sin\theta-(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}\right]$$
$$=\sin\theta\left[\frac{1-\sin\theta-1-\sin\theta}{1-\sin^2\theta}\right]$$
$$=\sin\theta\left(\frac{-2\sin\theta}{\cos^2\theta}\right)$$
$$=-2\frac{\sin^2\theta}{\cos^2\theta}$$
$$=-2\tan^2\theta \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\csc^2 x-\cos^2 x\csc^2 x=1$

Solution

$$\csc^2 x-\cos^2 x\csc^2 x=\csc^2 x\left(1-\cos^2 x\right)$$
$$=\frac{1}{\sin^2 x}\left(\sin^2 x\right)$$
$$=1 \quad \checkmark$$

Exercise

Prove the following equation is an identity: $1-2\sin^2 x=2\cos^2 x-1$

Solution

$$1-2\sin^2 x=1-2\left(1-\cos^2 x\right)$$
$$=1-2+2\cos^2 x$$
$$=2\cos^2 x-1 \quad \checkmark$$

Exercise

Prove the following equation is an identity: $\csc^2 x - \cos x \sec x = \cot^2 x$

Solution

$$\begin{aligned}\csc^2 x - \cos x \sec x &= \frac{1}{\sin^2 x} - \cos x \frac{1}{\cos x} \\ &= \frac{1}{\sin^2 x} - 1 \\ &= \frac{1-\sin^2 x}{\sin^2 x} \\ &= \frac{\cos^2 x}{\sin^2 x} \\ &= \cot^2 x \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $(\sec x - \tan x)(\sec x + \tan x) = 1$

Solution

$$\begin{aligned}(\sec x - \tan x)(\sec x + \tan x) &= \sec^2 x - \tan^2 x \\ &= 1 + \tan^2 x - \tan^2 x \\ &= 1 \quad \checkmark\end{aligned}$$

Exercise

Prove the following equation is an identity: $(1+\tan^2 x)(1-\sin^2 x) = 1$

Solution

$$\begin{aligned}(1+\tan^2 x)(1-\sin^2 x) &= \sec^2 x \cos^2 x \\ &= \frac{1}{\cos^2 x}\cos^2 x \\ &= 1 \quad \checkmark\end{aligned}$$