Solutions   Section 3.6 – Planar Systems – *Distinct, Complex, and Repeated Eigenvalues – Eigenvectors*

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices. $A = \begin{pmatrix} 12 & 14 \\ -7 & -9 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 12-\lambda & 14 \\ -7 & -9-\lambda \end{vmatrix}$$

$$= (12-\lambda)(-9-\lambda) - (14)(-7)$$

$$= -108 - 12\lambda + 9\lambda + \lambda^2 + 98$$

$$= \lambda^2 - 3\lambda - 10 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -2$ *and* $\lambda_2 = 5$

For $\lambda_1 = -2$, we have: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 12+2 & 14 \\ -7 & -9+2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 14 \\ -7 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \rightarrow \begin{cases} 14x + 14y = 0 \\ -7x - 7y = 0 \end{cases} \Rightarrow x = -y$$

$$\Rightarrow V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 5$, we have $(A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 7 & 14 \\ -7 & -14 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \rightarrow \begin{cases} 7x + 14y = 0 \\ -7x - 14y = 0 \end{cases} \Rightarrow x = -2y$$

$$\Rightarrow V_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices. $A = \begin{pmatrix} -4 & 1 \\ -2 & 1 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix}$$

$$= (-4-\lambda)(1-\lambda) + 2$$

$$= \lambda^2 + 3\lambda - 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = \frac{-3-\sqrt{17}}{2}$ *and* $\lambda_2 = \frac{-3+\sqrt{17}}{2}$

For $\lambda_1 = \frac{-3-\sqrt{17}}{2}$, we have: $(A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -4-\frac{-3-\sqrt{17}}{2} & 1 \\ -2 & 1-\frac{-3-\sqrt{17}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-5+\sqrt{17}}{2} & 1 \\ -2 & \frac{5+\sqrt{17}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} \frac{-5+\sqrt{17}}{2}x + y = 0 \\ -2x + \frac{5+\sqrt{17}}{2}y = 0 \end{cases} \Rightarrow x = \left(\frac{5+\sqrt{17}}{4}\right)y$$

$$\Rightarrow V_1 = \boxed{\begin{pmatrix} \frac{5+\sqrt{17}}{4} \\ 1 \end{pmatrix}}$$

For $\lambda_2 = \frac{-3+\sqrt{17}}{2}$, we have: $(A-\lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -4-\frac{-3+\sqrt{17}}{2} & 1 \\ -2 & 1-\frac{-3+\sqrt{17}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-5-\sqrt{17}}{2} & 1 \\ -2 & \frac{5-\sqrt{17}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} \frac{-5-\sqrt{17}}{2}x + y = 0 \\ -2x + \frac{5-\sqrt{17}}{2}y = 0 \end{cases} \Rightarrow x = \left(\frac{5-\sqrt{17}}{4}\right)y$$

$$\Rightarrow V_2 = \boxed{\begin{pmatrix} \frac{5-\sqrt{17}}{4} \\ 1 \end{pmatrix}}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices. $A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$

Solution

$$\det(A-\lambda I) = \begin{vmatrix} 5-\lambda & 3 \\ -6 & -4-\lambda \end{vmatrix}$$
$$= (-4-\lambda)(5-\lambda) + 18$$
$$= \lambda^2 - \lambda - 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1$ *and* $\lambda_2 = 2$

For $\lambda_1 = -1$, we have: $(A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 6 & 3 \\ -6 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} 6x + 3y = 0 \\ -6x - 3y = 0 \end{cases} \Rightarrow y = -2x$$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 3 & 3 \\ -6 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 3x + 3y = 0 \\ -6x - 6y = 0 \end{cases} \Rightarrow y = -x$$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices. $A = \begin{pmatrix} -2 & 3 \\ 0 & -5 \end{pmatrix}$

Solution

$$\det\left(A - \lambda I\right) = \begin{vmatrix} -2-\lambda & 3 \\ 0 & -5-\lambda \end{vmatrix}$$

$$= \left(-2-\lambda\right)\left(-5-\lambda\right) - 0$$

$$= \left(2+\lambda\right)\left(5+\lambda\right) = 0$$

Thus, the eigenvalues are: $\lambda_1 = -5$ *and* $\lambda_2 = -2$

For $\lambda_1 = -5$, we have: $\left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow 3x + 3y = 0 \Rightarrow y = -x$$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

For $\lambda_2 = -2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 0 & 3 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 3y = 0 \\ -3y = 0 \end{cases} \Rightarrow y = 0$$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices. $A = \begin{pmatrix} 6 & 10 \\ -5 & -9 \end{pmatrix}$

Solution

$$\det\left(A - \lambda I\right) = \begin{vmatrix} 6-\lambda & 10 \\ -5 & -9-\lambda \end{vmatrix}$$

$$= (6-\lambda)(-9-\lambda)+50$$

$$= -54+3\lambda+\lambda^2+50$$

$$= \lambda^2+3\lambda-4 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -4$ *and* $\lambda_2 = 1$

For $\lambda_1 = -4$, we have: $(A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 10 & 10 \\ -5 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 10x+10y=0 \\ -5x-5y=0 \end{cases} \Rightarrow y=-x$$

$$\Rightarrow \underline{V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

For $\lambda_2 = 1 \ \Rightarrow (A-\lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 5 & 10 \\ -5 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 5x+10y=0 \\ -5x-10y=0 \end{cases} \Rightarrow x=-2y$$

$$\Rightarrow \underline{V_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices. $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

Solution

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(-1-\lambda)-0$$

$$= \lambda^2-2\lambda-3$$

The characteristic equation: $\lambda^2-2\lambda-3$

$\lambda^2-2\lambda-3=0$ The eigenvalues are $\lambda_1 = -1$ *and* $\lambda_2 = 3$

$$\lambda_1 = -1 \quad \rightarrow \quad (A-\lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 4 & 0 \\ 8 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 4x=0 \\ 8x=0 \end{cases} \Rightarrow x=0$$

Therefore, the eigenvector $\underline{V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

$$\lambda_2 = 3 \quad \rightarrow \quad (A-\lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} 0 & 0 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} 0 = 0 \\ 8x - 4y = 0 \end{cases} \Rightarrow 8x = 4y \to \boxed{2x = y}$$

Therefore, the eigenvector $V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices. $A = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 10-\lambda & -9 \\ 4 & -2-\lambda \end{vmatrix}$$

$$= (10-\lambda)(-2-\lambda) + 36$$

$$= \lambda^2 - 8\lambda + 16$$

$\Rightarrow$ The characteristic equation: $\lambda^2 - 8\lambda + 16$

$\lambda^2 - 8\lambda + 16 = 0 \quad \Rightarrow$ The eigenvalues are $\lambda_{1,2} = 4$

$\lambda_1 = 4 \quad \to \quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} 6x - 9y = 0 \\ 4x - 6y = 0 \end{cases} \Rightarrow \boxed{2x = 3y}$$

Therefore the eigenvector $V_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices. $A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix}$$

$$= (4-\lambda)(1-\lambda)(1-\lambda) + 2(1-\lambda)$$

$$= (1-\lambda)\left[(4-\lambda)(1-\lambda) + 2\right]$$

$$= (1-\lambda)\left(\lambda^2 - 5\lambda + 6\right)$$

$\Rightarrow$ The characteristic equation: $-\lambda^3 + 6\lambda^2 - 11\lambda + 6$

$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$ The eigenvalues are $\boxed{\lambda = 1,\ 2,\ 3}$

$$\lambda_1 = 1 \quad \rightarrow \quad \begin{pmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 3x_1 + x_3 = 0 \\ x_1 = 0 \end{cases} \Rightarrow x_1 = x_3 = 0$$

Therefore; the eigenvector $V_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\lambda_2 = 2 \quad \rightarrow \quad \begin{pmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 2x_1 + x_3 = 0 \\ -2x_1 - x_2 = 0 \\ -2x_1 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = -2x_1 \\ x_2 = -2x_1 \end{cases}$$

Therefore; the eigenvector $V_2 = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$

$$\lambda_3 = 3 \quad \rightarrow \quad \begin{pmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} x_1 + x_3 = 0 \\ -2x_1 - 2x_2 = 0 \\ -2x_1 - 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = -x_1 \\ x_2 = -x_1 \end{cases}$$

Therefore; the eigenvector $V_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices. $A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 4 & 3-\lambda & 2 \\ -8 & -4 & -3-\lambda \end{vmatrix}$$

$$= (1-\lambda)(3-\lambda)(-3-\lambda) + 8(1-\lambda)$$

$$= -9 + 9\lambda + \lambda^2 - \lambda^3 + 8 - 8\lambda$$

$$= -\lambda^3 + \lambda^2 + \lambda - 1 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1 \quad and \quad \lambda_{2,3} = 1$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 4 & 2 & 2 \\ -8 & -4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 0 = 0 \\ 4x + 2y + 2z = 0 \\ -8x - 4y - 4z = 0 \end{cases} \Rightarrow \begin{cases} \\ 2x = -y - z \\ 2x = -y - z \end{cases} \rightarrow \begin{matrix} If \ \boxed{x = 0} \\ \boxed{y = -z} \end{matrix}$$

$$\Rightarrow V_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda_{2,3} = -1 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 2 & 0 & 0 \\ 4 & 4 & 2 \\ -8 & -4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 2x = 0 \\ 4x + 4y + 2z = 0 \\ -8x - 4y - 2z = 0 \end{cases} \Rightarrow \begin{cases} \boxed{x = 0} \\ 4y = -2z \rightarrow \boxed{z = -2y} \\ 4y = -2z \end{cases}$$

$$\Rightarrow V_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices $A = \begin{pmatrix} -1 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & -12 & 2 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & -4 & -2 \\ 0 & 1-\lambda & 1 \\ -6 & -12 & 2-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(1-\lambda)(2-\lambda) + 24 - 12(1-\lambda) + 12(-1-\lambda)$$

$$= -\lambda^3 + 2\lambda^2 + \lambda - 2 + 24 - 12 + 12\lambda - 12 - 12\lambda$$

$$= -\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1 \quad \lambda_2 = 1 \quad and \quad \lambda_3 = 2$

For $\lambda_1 = -1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & -4 & -2 \\ 0 & 2 & 1 \\ -6 & -12 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -4y-2z=0 \\ 2y+z=0 \\ -6x-12y+3z=0 \end{cases} \Rightarrow \begin{cases} -4y=2z \\ 2y=-z \\ -6x=12y-3z \end{cases} \rightarrow \begin{array}{l} \boxed{y=-\frac{1}{2}z} \\ x=\frac{-9z}{-6}=\frac{3}{2}z \end{array}$$

$$\Rightarrow V_1 = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1 \quad \Rightarrow (A-\lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -2 & -4 & -2 \\ 0 & 0 & 1 \\ -6 & -12 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -2x-4y-2z=0 \\ \boxed{z=0} \\ -6x-12y+2z=0 \end{cases} \Rightarrow \begin{cases} -2x-4y=0 \\ \\ -6x-12y=0 \end{cases} \rightarrow \boxed{x=-2y}$$

$$\Rightarrow V_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_3 = 2 \quad \Rightarrow (A-\lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -3 & -4 & -2 \\ 0 & -1 & 1 \\ -6 & -12 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -3x-4y-2z=0 \\ -y+z=0 \\ -6x-12y=0 \end{cases} \Rightarrow \boxed{y=z} \rightarrow \begin{cases} -3x=6z \\ \\ -6x=12z \end{cases} \rightarrow \begin{cases} \\ \boxed{x=-2z} \end{cases}$$

$$\Rightarrow V_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices $A = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

Solution

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(4-\lambda)(-1-\lambda) - 4 - 8 + 4(4-\lambda) + 4(3-\lambda) + 2\lambda + 2$$

$$= -\lambda^3 + 6\lambda^2 - 5\lambda - 12 - 12 + 16 - 4\lambda + 12 - 4\lambda + 2\lambda + 2$$
$$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6 \underline{= 0}$$

Thus, the eigenvalues are: $\lambda_1 = 1 \quad \lambda_2 = 2 \quad and \quad \lambda_3 = 3$

For $\lambda_1 = 1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x + 2y + 2z = 0 \\ x + 3y + z = 0 \\ -2x - 4y - 2z = 0 \end{cases} \Rightarrow (1)\ \&\ (3) \rightarrow \boxed{y = 0} \rightarrow \begin{cases} 2x + 2z = 0 \\ x + z = 0 \end{cases} \Rightarrow \boxed{x = -z}$$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x + 2y + 2z = 0 \\ x + 2y + z = 0 \\ -2x - 4y - 3z = 0 \end{cases} \Rightarrow \begin{cases} 2x + 4y + 2z = 0 \\ -2x - 4y - 3z = 0 \end{cases} \rightarrow \boxed{z = 0} \rightarrow \boxed{x = -2y}$$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}$$

For $\lambda_3 = 3 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ -2 & -4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2y + 2z = 0 \\ x + y + z = 0 \\ -2x - 4y - 4z = 0 \end{cases} \Rightarrow \boxed{y = -z} \rightarrow \boxed{x = 0}$$

$$\Rightarrow \boxed{V_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices $A = \begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}$

Solution

$$\left|A - \lambda I\right| = \begin{vmatrix} -6-\lambda & 4 & 4 \\ -4 & 2-\lambda & 4 \\ -10 & 8 & 4-\lambda \end{vmatrix}$$

$$= (-6-\lambda)(2-\lambda)(4-\lambda) - 160 - 128 + 40(2-\lambda) + 32(6+\lambda) + 16(4-\lambda)$$

$$= -\lambda^3 + 4\lambda \; = 0$$

Thus, the eigenvalues are: $\lambda_1 = 0 \quad \lambda_2 = -2 \quad and \quad \lambda_3 = 2$

For $\lambda_1 = 0 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -6x + 4y + 4z = 0 \\ -4x + 2y + 4z = 0 \\ -10x + 8y + 4z = 0 \end{cases} \Rightarrow \; 2x - 2y = 0 \to \boxed{x = y} \to \begin{matrix} -2x + 4z = 0 \\ -2x + 4z = 0 \end{matrix} \Rightarrow \boxed{x = 2z = y}$$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = -2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -4 & 4 & 4 \\ -4 & 4 & 4 \\ -10 & 8 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -4x + 4y + 4z = 0 \\ -4x + 4y + 4z = 0 \\ -10x + 8y + 6z = 0 \end{cases} \Rightarrow \; \begin{matrix} -x + y + z = 0 \\ -5x + 4y + 3z = 0 \end{matrix} \; \Rightarrow \; \begin{matrix} 4x - 4y = 4z \\ -5x + 4y = -3z \end{matrix} \; \to \; \begin{matrix} \boxed{x = -z} \\ \boxed{y = -2z} \end{matrix}$$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}}$$

For $\lambda_3 = 2 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -8 & 4 & 4 \\ -4 & 0 & 4 \\ -10 & 8 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} -8x + 4y + 4z = 0 \\ -4x + 4z = 0 \\ -10x + 8y + 2z = 0 \end{cases} \Rightarrow \boxed{x = z} \Rightarrow \begin{matrix} 4y - 4z = 0 \\ \\ 8y - 8z = 0 \end{matrix} \Rightarrow \boxed{y = z}$$

$$\Rightarrow V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices. $A = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{pmatrix} -\lambda & 0 & 2 & 0 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -2-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & 0 & 2 \\ 1 & -\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left(\lambda^2(-2-\lambda) + 2 + \lambda\right)$$

$$= (1-\lambda)\left(-\lambda^3 - 2\lambda^2 + \lambda + 2\right)$$

$$= \lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2$$

$\Rightarrow$ The characteristic equation: $\lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2$

$\lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2 = 0 \quad \Rightarrow \quad$ The eigenvalues are $\boxed{\lambda = -2, -1, 1, 1}$

$$\lambda_1 = -2 \quad \rightarrow \quad \begin{pmatrix} 2 & 0 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} 2x_1 + 2x_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \\ x_2 = 0 \\ x_4 = 0 \end{cases} \rightarrow \begin{cases} x_1 = -x_3 \\ x_1 = -x_3 \\ x_2 = 0 \\ x_4 = 0 \end{cases}$$

Therefore; the eigenvector $V_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$\lambda_2 = -1 \quad \rightarrow \quad \begin{pmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x_1 + 2x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \\ x_2 - x_3 = 0 \\ x_4 = 0 \end{cases} \rightarrow \begin{cases} x_1 = -2x_3 \\ x_1 = -x_2 - x_3 \\ x_2 = x_3 \end{cases}$$

Therefore; the eigenvector $V_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

$$\lambda_3 = 1 \quad \rightarrow \quad \begin{pmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -x_1 + 2x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_2 - 3x_3 = 0 \end{cases} \rightarrow \begin{cases} x_1 = 2x_3 \\ x_1 = x_2 - x_3 \\ x_2 = 3x_3 \end{cases}$$

Therefore; the eigenvector $V_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_4 = 1 \quad \rightarrow$ Therefore; the eigenvector $V_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Exercise

Find a fundamental set of solutions for the system $x' = Ax$, where A is the given matrices.

$$A = \begin{pmatrix} 2 & 0 \\ -4 & -2 \end{pmatrix}$$

Solution

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 \\ -4 & -2-\lambda \end{vmatrix}$$

$$= (2-\lambda)(-2-\lambda) - 0$$

$$= \lambda^2 - 4 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -2$ *and* $\lambda_2 = 2$

For $\lambda_1 = -2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 0 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \rightarrow \begin{cases} 4x = 0 \\ -4x = 0 \end{cases} \Rightarrow \boxed{x = 0} \quad \boxed{y = 1}$$

The eigenvector is: $V_1 = (0,\ 1)^T$

The solution is: $x_1(t) = e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 \\ -4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \to \begin{cases} 0 = 0 \\ -4x - 4y = 0 \end{cases} \Rightarrow \boxed{x = -y}$$

The eigenvector is: $V_2 = (-1,\ 1)^T$

The solution is: $\boxed{x_2(t) = e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

Since the vectors V_1 and V_2 are independent, the solutions $x_1(t)$ and $x_2(t)$ are independent for all t and for a fundamental set of solutions.

Exercise

Find a fundamental set of solutions for the system $x' = Ax$, where A is the given matrices.

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -6 \\ -2 & 3 & 4 \end{pmatrix}$$

Solution

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 0 & 0 \\ 2 & -5-\lambda & -6 \\ -2 & 3 & 4-\lambda \end{vmatrix}$$

$$= \lambda^3 + 2\lambda^2 - \lambda - 2 \underline{= 0}$$

Thus, the eigenvalues are: $\lambda_1 = -2 \quad \lambda_2 = -1 \quad \textit{and} \quad \lambda_3 = 1$

For $\lambda_1 = -2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & -6 \\ -2 & 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} \boxed{x = 0} \\ 2x - 3y - 6z = 0 \Rightarrow 3y = -6z \to \boxed{y = -2z} \\ -2x + 3y + 6z = 0 \end{cases}$$

$$\Rightarrow V_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -1 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & -4 & -6 \\ -2 & 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 0=0 \\ 2x-4y-6z=0 \Rightarrow -4y=6z \to \boxed{y=-z} \to 2x=4y+6z=2z \Rightarrow \boxed{x=z} \\ -2x+3y+5z=0 \quad 3y=-5z \end{cases}$$

$$\Rightarrow V_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 1 \quad \Rightarrow (A-\lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -2 & 0 & 0 \\ 2 & -6 & -6 \\ -2 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} -2x=0 & \boxed{x=0} \\ 2x-6y-6z=0 \Rightarrow -6y=6z \to \boxed{y=-z} \\ -2x+3y+3z=0 \quad 3y=-3z \end{cases}$$

$$\Rightarrow V_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

The vectors are given by: $V = \begin{pmatrix} 0 & 1 & 0 \\ -2 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

$$\det(V) = \begin{vmatrix} 0 & 1 & 0 \\ -2 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 1 \underline{\neq 0}$$

The solutions are independent for all t and form a fundamental set of solutions.

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = x_1 + 2x_2 \\ x_2'(t) = 4x_1 + 3x_2 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix}$$

$$= \lambda^2 - 4\lambda - 5 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1$ *and* $\lambda_2 = 5$

For $\lambda_1 = -1 \quad \Rightarrow (A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \underline{x=-y} \Rightarrow V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$

For $\lambda_2 = 5 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \underline{2x = y|} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The solution is: $x_2(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t}$

$$\therefore \; \boxed{x(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t}}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 2x_1 + 2x_2 \\ x_2'(t) = x_1 + 3x_2 \end{cases}$

Solution

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= \lambda^2 - 5\lambda + 4 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 1 \;\; and \;\; \lambda_2 = 4$

For $\lambda_1 = 1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \underline{x = -2y|} \quad \Rightarrow \quad V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{t}$

For $\lambda_2 = 4 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \underline{x = y|} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The solution is: $x_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

$$\therefore \; \boxed{x(t) = C_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = -4x_1 + 2x_2 \\ x_2'(t) = -\frac{5}{2}x_1 + 2x_2 \end{cases}$

Solution

$$A = \begin{pmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -4-\lambda & 2 \\ -\frac{5}{2} & 2-\lambda \end{vmatrix}$$

$$= \lambda^2 + 2\lambda - 3 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 1$ *and* $\lambda_2 = -3$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -5 & 2 \\ -\frac{5}{2} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{5x = 2y|} \quad \Rightarrow V_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^t$

For $\lambda_2 = -3 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -1 & 2 \\ -\frac{5}{2} & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = 2y|} \quad \Rightarrow V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The solution is: $x_2(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-3t}$

$$\therefore \ \underline{x(t) = C_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^t + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-3t}|}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = -\frac{5}{2}x_1 + 2x_2 \\ x_2'(t) = \frac{3}{4}x_1 - 2x_2 \end{cases}$

Solution

$$A = \begin{pmatrix} -\frac{5}{2} & 2 \\ \frac{3}{4} & -2 \end{pmatrix}$$

$$\left|A-\lambda I\right| = \begin{vmatrix} -\frac{5}{2}-\lambda & 2 \\ \frac{3}{4} & -2-\lambda \end{vmatrix}$$

$$= \lambda^2 + \frac{9}{2}\lambda + \frac{7}{2} = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1$ *and* $\lambda_2 = -\frac{7}{2}$

For $\lambda_1 = -1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -\frac{3}{2} & 2 \\ \frac{3}{4} & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{\frac{3}{2}x = 2y} \quad \Rightarrow V_1 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{-t}$

For $\lambda_2 = -\frac{7}{2} \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 1 & 2 \\ \frac{3}{4} & \frac{3}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = -2y} \quad \Rightarrow V_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

The solution is: $x_2(t) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-7t/2}$

$$\therefore \underline{x(t) = C_1 \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-7t/2}}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 3x_1 - x_2 \\ x_2'(t) = 9x_1 - 3x_2 \end{cases}$

Solution

$$A = \begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}$$

$$\left|A-\lambda I\right| = \begin{vmatrix} 3-\lambda & -1 \\ 9 & -3-\lambda \end{vmatrix}$$

$$= \lambda^2 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = 0$

For $\lambda_1 = 0 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{3x = y} \quad \Rightarrow V_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

For the second eigenvector $V_2 \quad \Rightarrow \quad AV_2 = V_1$

$$\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \rightarrow \quad 3x - y = 1$$

$\rightarrow if \ \ x = 1 \quad \Rightarrow \quad y = 2 \qquad V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

The solution is: $x_2(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} t$ $\qquad x_2(t) = e^{\lambda t}\left(V_2 + tV_1\right)$

$$\therefore \ x(t) = C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} t \right)$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = -6x_1 + 5x_2 \\ x_2'(t) = -5x_1 + 4x_2 \end{cases}$

Solution

$$A = \begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -6-\lambda & 5 \\ -5 & 4-\lambda \end{vmatrix}$$

$$= \lambda^2 + 2\lambda + 1 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = -1$

For $\lambda_1 = -1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x = y \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For the second eigenvector $V_2 \quad \Rightarrow \quad AV_2 = V_1$

$$\begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \rightarrow \quad -6x + 5y = 1$$

$\rightarrow if \ \ x = 0 \quad \Rightarrow \quad y = \frac{1}{5} \qquad V_2 = \begin{pmatrix} 0 \\ \frac{1}{5} \end{pmatrix}$

The solution is: $x_2(t) = \begin{pmatrix} 0 \\ \frac{1}{5} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t$ $\qquad x_2(t) = e^{\lambda t}\left(V_2 + tV_1\right)$

$$\therefore\ \underline{x(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \left(\begin{pmatrix} 0 \\ \frac{1}{5} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t \right)}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 6x_1 - x_2 \\ x_2'(t) = 5x_1 + 2x_2 \end{cases}$

Solution

$$A = \begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix}$$

$$\left|A - \lambda I\right| = \begin{vmatrix} 6-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix}$$

$$= \lambda^2 - 8\lambda + 17 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = 4 \pm i$

For $\lambda_1 = 4 + i \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{(2-i)x = y} \quad \Rightarrow V_1 = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$

$$z(t) = \begin{pmatrix} 1 \\ 2-i \end{pmatrix} e^{(4+i)t}$$

$$= \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right) (\cos t + i \sin t) e^{4t}$$

$$= \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin t + i \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos t \right) \right) e^{4t}$$

$$= \left(\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix} \right) e^{4t}$$

$$\therefore\ \underline{x(t) = C_1 \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix} e^{4t}}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = x_1 + x_2 \\ x_2'(t) = -2x_1 - x_2 \end{cases}$

Solution

$$A = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$$

$$\left|A - \lambda I\right| = \begin{vmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix}$$

$$= \lambda^2 + 1 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = \pm i$

For $\lambda_1 = i \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{(-1+i)x = y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}$

$$z(t) = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{it}$$

$$= \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)(\cos t + i\sin t)$$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t + i\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t\right)$$

$$= \begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + i\begin{pmatrix} \sin t \\ -\sin t + \cos t \end{pmatrix}$$

$$\therefore \ \underline{x(t) = C_1\begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + C_2\begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix}}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 5x_1 + x_2 \\ x_2'(t) = -2x_1 + 3x_2 \end{cases}$

Solution

$$A = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}$$

$$\left|A - \lambda I\right| = \begin{vmatrix} 5-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix}$$

$$= \lambda^2 - 8\lambda + 17 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = 4 \pm i$

For $\lambda_1 = 4 + i \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{(-1+i)x = y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}$

$$\begin{aligned} z(t) &= \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{(4+i)t} \\ &= \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) (\cos t + i \sin t) e^{4t} \\ &= \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t + i \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t \right) \right) e^{4t} \\ &= \left(\begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ 2\sin t + \cos t \end{pmatrix} \right) e^{4t} \end{aligned}$$

$$\therefore \ \underline{x(t) = C_1 \begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} \sin t \\ 2\sin t + \cos t \end{pmatrix} e^{4t}}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 4x_1 + 5x_2 \\ x_2'(t) = -2x_1 + 6x_2 \end{cases}$

Solution

$$A = \begin{pmatrix} 4 & 5 \\ -2 & 6 \end{pmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 4-\lambda & 5 \\ -2 & 6-\lambda \end{vmatrix} \\ &= \lambda^2 - 10\lambda + 34 = 0 \end{aligned}$$

Thus, the eigenvalues are: $\lambda_{1,2} = 5 \pm 3i$

For $\lambda_1 = 5 + 3i \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -1-3i & 5 \\ -2 & 1-3i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{(1+3i)x = 5y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 5 \\ 1+3i \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} 5 \\ 1+3i \end{pmatrix}$

$$z(t) = \begin{pmatrix} 5 \\ 1+3i \end{pmatrix} e^{(5+3i)t}$$

$$= \left(\begin{pmatrix} 5 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right) (\cos 3t + i \sin 3t) e^{5t}$$

$$= \left(\begin{pmatrix} 5 \\ 1 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \sin 3t + i \left(\begin{pmatrix} 5 \\ 1 \end{pmatrix} \sin 3t + \begin{pmatrix} 0 \\ 3 \end{pmatrix} \cos 3t \right) \right) e^{5t}$$

$$= \left(\begin{pmatrix} 5\cos 3t \\ \cos 3t - 3\sin 3t \end{pmatrix} + i \begin{pmatrix} 5\sin 3t \\ \sin 3t + 3\cos 3t \end{pmatrix} \right) e^{5t}$$

$$\therefore\ \boxed{x(t) = C_1 \begin{pmatrix} 5\cos 3t \\ \cos 3t - 3\sin 3t \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} 5\sin 3t \\ \sin 3t + 3\cos 3t \end{pmatrix} e^{5t}}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 5x_1 - 4x_2 \\ x_2'(t) = 2x_1 - x_2 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -4 \\ 2 & -1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$$

$$= \lambda^2 - 4\lambda + 3 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 1$ *and* $\lambda_2 = 3$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 3 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = 2y} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\therefore\ \boxed{x(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t}}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 6x_1 - 6x_2 \\ x_2'(t) = 4x_1 - 4x_2 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 6-\lambda & -6 \\ 4 & -4-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 6 & -6 \\ 4 & -4 \end{pmatrix}$$

$$= \lambda^2 - 2\lambda = 0$$

Thus, the eigenvalues are: $\lambda_1 = 0$ & $\lambda_2 = 2$

For $\lambda_1 = 0 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 6 & -6 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = y} \quad \Rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 4 & -6 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{2x = 3y} \quad \Rightarrow V_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\therefore \ \underline{x(t) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{3t}}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 5x_1 - 3x_2 \\ x_2'(t) = 2x_1 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -3 \\ 2 & -\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix}$$

$$= \lambda^2 - 5\lambda + 6 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 2$ & $\lambda_2 = 3$

For $\lambda_1 = 2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = y} \quad \Rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 3 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{2x = 3y} \quad \Rightarrow V_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\therefore \ \underline{x(t) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t} + C_2\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{3t}}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 5x_1 - 4x_2 \\ x_2'(t) = 3x_1 - 2x_2 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -4 \\ 3 & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix}$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 1$ & $\lambda_2 = 2$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & -4 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = y|} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 3 & -4 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{3x = 4y|} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\therefore \ x(t) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + C_2\begin{pmatrix} 4 \\ 3 \end{pmatrix}e^{2t}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 9x_1 - 8x_2 \\ x_2'(t) = 6x_1 - 5x_2 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 9-\lambda & -8 \\ 6 & -5-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 9 & -8 \\ 6 & -5 \end{pmatrix}$$

$$= \lambda^2 - 4\lambda + 3 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 1$ & $\lambda_2 = 3$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 8 & -8 \\ 6 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = y|} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 3 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 6 & -8 \\ 6 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{3x = 4y|} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\therefore \ x(t) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + C_2\begin{pmatrix} 4 \\ 3 \end{pmatrix}e^{3t}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 10x_1 - 6x_2 \\ x_2'(t) = 12x_1 - 7x_2 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 10-\lambda & -6 \\ 12 & -7-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 10 & -6 \\ 12 & -7 \end{pmatrix}$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 1$ & $\lambda_2 = 2$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 9 & -6 \\ 12 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to 3x = 2y \quad \Rightarrow \quad V_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 8 & -6 \\ 12 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to 4x = 3y \quad \Rightarrow \quad V_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\therefore \ x(t) = C_1\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^t + C_2\begin{pmatrix} 3 \\ 4 \end{pmatrix}e^{2t}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 6x_1 - 10x_2 \\ x_2'(t) = 2x_1 - 3x_2 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 6-\lambda & -10 \\ 2 & -3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 6 & -10 \\ 2 & -3 \end{pmatrix}$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 1$ & $\lambda_2 = 2$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 5 & -10 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = 2y \quad \Rightarrow \quad V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 4 & -10 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to 2x = 5y \quad \Rightarrow \quad V_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\therefore \ x(t) = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^t + C_2\begin{pmatrix} 5 \\ 2 \end{pmatrix}e^{2t}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 11x_1 - 15x_2 \\ x_2'(t) = 6x_1 - 8x_2 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 11-\lambda & -15 \\ 6 & -8-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 11 & -15 \\ 6 & -8 \end{pmatrix}$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 1$ & $\lambda_2 = 2$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 10 & -15 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{2x = 3y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 9 & -15 \\ 6 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{3x = 5y} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$\therefore \; \underline{x(t) = C_1\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{t} + C_2\begin{pmatrix} 5 \\ 3 \end{pmatrix}e^{2t}}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 3x_1 + x_2 \\ x_2'(t) = x_1 + 3x_2 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$= \lambda^2 - 6\lambda + 8 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 2$ & $\lambda_2 = 4$

For $\lambda_1 = 2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = -y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 4 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = y} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \; \underline{x(t) = C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{2t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t}}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 4x_1 + 2x_2 \\ x_2'(t) = 2x_1 + 4x_2 \end{cases}$

Solution

$$|A-\lambda I| = \begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

$$= \lambda^2 - 8\lambda + 12 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 2$ & $\lambda_2 = 6$

For $\lambda_1 = 2 \quad \Rightarrow (A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = -y} \quad \Rightarrow V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 6 \quad \Rightarrow (A-\lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = y} \quad \Rightarrow V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \ \underline{x(t) = C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{2t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{6t}}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 9x_1 + 2x_2 \\ x_2'(t) = 2x_1 + 6x_2 \end{cases}$

Solution

$$|A-\lambda I| = \begin{vmatrix} 9-\lambda & 2 \\ 2 & 6-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 9 & 2 \\ 2 & 6 \end{pmatrix}$$

$$= \lambda^2 - 15\lambda + 50 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 5$ & $\lambda_2 = 10$

For $\lambda_1 = 5 \quad \Rightarrow (A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{2x = y} \quad \Rightarrow V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 10 \quad \Rightarrow (A-\lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = 2y} \quad \Rightarrow V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\therefore \ \underline{x(t) = C_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{5t} + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{10t}}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 13x_1 + 4x_2 \\ x_2'(t) = 4x_1 + 7x_2 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 13-\lambda & 4 \\ 4 & 7-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 13 & 4 \\ 4 & 7 \end{pmatrix}$$

$$= \lambda^2 - 20\lambda + 75 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 5$ & $\lambda_2 = 15$

For $\lambda_1 = 5 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{2x = y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 15 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = 2y} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\therefore \ x(t) = C_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{5t} + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{15t}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 3x_1 - 2x_2 \\ x_2'(t) = 2x_1 - 2x_2 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$$

$$= \lambda^2 - \lambda - 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1$ & $\lambda_2 = 2$

For $\lambda_1 = -1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{2x = y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = 2y} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\therefore \ x(t) = C_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-t} + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{2t}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 2x_1 - x_2 \\ x_2'(t) = 3x_1 - 2x_2 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$$

$$= \lambda^2 - 1 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1$ & $\lambda_2 = 1$

For $\lambda_1 = -1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{3x = y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

For $\lambda_2 = 1 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = y} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \ x(t) = C_1\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 5x_1 - x_2 \\ x_2'(t) = 3x_1 - x_2 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -1 \\ 3 & -1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 5 & -1 \\ 3 & -1 \end{pmatrix}$$

$$= \lambda^2 - 4\lambda - 2 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = 2 \pm \sqrt{6}$

For $\lambda_1 = 2 - \sqrt{6} \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3+\sqrt{6} & -1 \\ 3 & -3+\sqrt{6} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{(3+\sqrt{6})x = y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 3+\sqrt{6} \end{pmatrix}$$

For $\lambda_2 = 2 + \sqrt{6} \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 3-\sqrt{6} & -1 \\ 3 & -3-\sqrt{6} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{(3-\sqrt{6})x = y} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 3-\sqrt{6} \end{pmatrix}$$

$$\therefore \ x(t) = C_1\begin{pmatrix} 1 \\ 3+\sqrt{6} \end{pmatrix}e^{(2-\sqrt{6})t} + C_2\begin{pmatrix} 1 \\ 3-\sqrt{6} \end{pmatrix}e^{(2+\sqrt{6})t}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = x_1 + x_2 \\ x_2'(t) = 4x_1 - 2x_2 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$$

$$= \lambda^2 + \lambda - 6 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -3$ & $\lambda_2 = 2$

For $\lambda_1 = -3 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{4x = y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = y} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \ \underline{x(t) = C_1\begin{pmatrix} 1 \\ 4 \end{pmatrix}e^{-3t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t}}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = -x_1 - 4x_2 \\ x_2'(t) = x_1 - x_2 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$$

$$= \lambda^2 + 2\lambda + 5 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = -1 \pm 2i$

For $\lambda_1 = -1 - 2i \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2i & -4 \\ 1 & 2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = -2iy} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 2i \\ -1 \end{pmatrix} \qquad x_1(t) = \begin{pmatrix} 2i \\ -1 \end{pmatrix}$$

$$z(t) = \begin{pmatrix} 2i \\ -1 \end{pmatrix}e^{(-1-2i)t}$$

$$= \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix} + i\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right)(\cos 2t + i\sin 2t)e^{-t}$$

$$= \left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\sin 2t + i\left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin 2t + \begin{pmatrix} 2 \\ 0 \end{pmatrix}\cos 2t\right)\right)e^{-t}$$

$$= \left(\begin{pmatrix} -2\sin 2t \\ -\cos 2t \end{pmatrix} + i \begin{pmatrix} 2\cos 2t \\ -\sin 2t \end{pmatrix} \right) e^{-t}$$

$$\therefore\ x(t) = C_1 \begin{pmatrix} -2\sin 2t \\ -\cos 2t \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 2\cos 2t \\ -\sin 2t \end{pmatrix} e^{-t}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 2x_1 + 3x_2 - 7 \\ x_2'(t) = -x_1 - 2x_2 + 5 \end{cases}$

Solution

$$A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{vmatrix}$$

$$= \lambda^2 - 1 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1$ *and* $\lambda_2 = 1$

For $\lambda_1 = -1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = -y} \quad \Rightarrow V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$

For $\lambda_2 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x = -3y} \quad \Rightarrow V_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

The solution is: $x_2(t) = \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t}$

$$x_h(t) = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t}$$

$$\begin{cases} 2a_1 + 3a_2 = 7 \\ -a_1 - 2a_2 = -5 \end{cases} \qquad \Delta = \begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} = -1 \quad \Delta_1 = \begin{vmatrix} 7 & 3 \\ -5 & -2 \end{vmatrix} = 1 \quad \Delta_2 = \begin{vmatrix} 2 & 7 \\ -1 & -5 \end{vmatrix} = -3$$

$$a_1 = -1 \quad a_2 = 3 \quad \to \quad x_p = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\therefore\ x(t) = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 5x_1 + 9x_2 + 2 \\ x_2'(t) = -x_1 + 11x_2 + 6 \end{cases}$

Solution

$$A = \begin{pmatrix} 5 & 9 \\ -1 & 11 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 9 \\ -1 & 11-\lambda \end{vmatrix}$$

$$= \lambda^2 - 16\lambda - 64 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = 8$

For $\lambda_1 = 8 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -3 & 9 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \underline{x = 3y} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

The solution is: $x_1(t) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{8t}$

For the second eigenvector $V_2 \quad \Rightarrow \quad AV_2 = V_1$

$$\begin{pmatrix} -3 & 9 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \to \quad -x + 3y = 1$$

$\to if \ y = 1 \quad \Rightarrow \quad x = 2 \qquad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The solution is: $x_2(t) = \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} t\right) e^{8t}$ $\qquad x_2(t) = e^{\lambda t}\left(V_2 + tV_1\right)$

$$x_h(t) = C_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{8t} + C_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} t\right] e^{8t}$$

$$\begin{cases} 5a_1 + 9a_2 = -2 \\ -a_1 11a_2 = -6 \end{cases} \qquad \Delta = \begin{vmatrix} 5 & 9 \\ -1 & 11 \end{vmatrix} = 64 \quad \Delta_1 = \begin{vmatrix} -2 & 9 \\ -6 & 11 \end{vmatrix} = 32 \quad \Delta_2 = \begin{vmatrix} 5 & -2 \\ -1 & -6 \end{vmatrix} = -32$$

$$a_1 = \frac{1}{2} \quad a_2 = -\frac{1}{2} \quad \to \quad x_p = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$\therefore \ x(t) = C_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{8t} + C_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} t\right] e^{8t} + \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

Exercise

Find the general solution of the system $\begin{cases} y_1'(t) = 6y_1 + y_2 + 6t \\ y_2'(t) = 4y_1 + 3y_2 - 10t + 4 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 6-\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}$$

$$= \lambda^2 - 9\lambda + 14 = 0$$

The eigenvalues: $\lambda_{1,2} = 2,\ 7$

For $\lambda_1 = 2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow 4x = -y \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

For $\lambda_2 = 7 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x = y \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$Y_h = C_1\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{2t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{7t}$$

$$\varphi(t) = \begin{pmatrix} e^{2t} & e^{7t} \\ -4e^{2t} & e^{7t} \end{pmatrix}$$

$$\varphi^{-1} = \frac{1}{5e^{9t}}\begin{pmatrix} e^{7t} & -e^{7t} \\ 4e^{2t} & e^{2t} \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} e^{-2t} & -e^{-2t} \\ 4e^{-7t} & e^{-7t} \end{pmatrix}$$

		$\int e^{-7t}$
+	$14t+4$	$-\frac{1}{7}e^{-7t}$
−	14	$\frac{1}{49}e^{-7t}$

		$\int e^{-2t}$
+	$16t-4$	$-\frac{1}{2}e^{-2t}$
−	16	$\frac{1}{4}e^{-2t}$

$$\begin{cases} 6y_1 + y_2 + 6t \\ 4y_1 + 3y_2 - 10t + 4 \end{cases}$$

$$F(t) = \begin{pmatrix} 6t \\ -10t + 4 \end{pmatrix}$$

$$Y_p = \frac{1}{5}\begin{pmatrix} e^{2t} & e^{7t} \\ -4e^{2t} & e^{7t} \end{pmatrix}\int\begin{pmatrix} e^{-2t} & -e^{-2t} \\ 4e^{-7t} & e^{-7t} \end{pmatrix}\begin{pmatrix} 6t \\ -10t+4 \end{pmatrix}dt \qquad X_p = \varphi(t)\int \varphi^{-1}(t)F(t)\,dt$$

$$= \frac{1}{5}\begin{pmatrix} e^{2t} & e^{7t} \\ -4e^{2t} & e^{7t} \end{pmatrix}\int\begin{pmatrix} (16t-4)e^{-2t} \\ (14t+4)e^{-7t} \end{pmatrix}dt$$

$$= \frac{1}{5}\begin{pmatrix} e^{2t} & e^{7t} \\ -4e^{2t} & e^{7t} \end{pmatrix}\begin{pmatrix} (-8t+2-4)e^{-2t} \\ \left(-2t-\frac{4}{7}-\frac{14}{49}\right)e^{-7t} \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} e^{2t} & e^{7t} \\ -4e^{2t} & e^{7t} \end{pmatrix}\begin{pmatrix} (-8t-2)e^{-2t} \\ \left(-2t-\frac{6}{7}\right)e^{-7t} \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} -8t-2-2t-\frac{6}{7} \\ 32t+8-2t-\frac{6}{7} \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} -10t-\frac{20}{7} \\ 30t+\frac{50}{7} \end{pmatrix}$$

$$= \begin{pmatrix} -2t-\frac{4}{7} \\ 6t+\frac{10}{7} \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 6 \end{pmatrix}t + \begin{pmatrix} -\frac{4}{7} \\ \frac{10}{7} \end{pmatrix}$$

$$Y(t) = C_1\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{2t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{7t} + \begin{pmatrix} -2 \\ 6 \end{pmatrix}t + \begin{pmatrix} -\frac{4}{7} \\ \frac{10}{7} \end{pmatrix}$$

$$\begin{cases} y_1(t) = C_1e^{2t} + C_2e^{7t} - 2t - \frac{4}{7} \\ y_2(t) = -4C_1e^{2t} + C_2e^{7t} + 6t + \frac{10}{7} \end{cases}$$

Exercise

Find the general solution of the system $\begin{cases} x'(t) = 5x + 3y - 2e^{-t} + 1 \\ y'(t) = -x + y + e^{-t} - 5t + 7 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 3 \\ -1 & 1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix}$$

$$= \lambda^2 - 6\lambda + 8 = 0$$

The eigenvalues: $\lambda_{1,2} = 2,\ 4$

For $\lambda_1 = 2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x=-y \quad \Rightarrow \quad V_1=\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 4 \quad \Rightarrow (A-\lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x=-3y \quad \Rightarrow \quad V_2=\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$Y_h = C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{2t} + C_2\begin{pmatrix} -3 \\ 1 \end{pmatrix}e^{4t}$$

$$\varphi(t)=\begin{pmatrix} -e^{2t} & -3e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix}$$

$$\varphi^{-1}=\frac{1}{2e^{6t}}\begin{pmatrix} e^{4t} & 3e^{4t} \\ -e^{2t} & -e^{2t} \end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix} e^{-2t} & 3e^{-2t} \\ -e^{-4t} & -e^{-4t} \end{pmatrix}$$

$$F(t)=\begin{pmatrix} 1-2e^{-t} \\ e^{-t}-5t+7 \end{pmatrix}$$

$$\varphi^{-1}(t)F(t)=\frac{1}{2}\begin{pmatrix} e^{-2t} & 3e^{-2t} \\ -e^{-4t} & -e^{-4t} \end{pmatrix}\begin{pmatrix} 1-2e^{-t} \\ e^{-t}-5t+7 \end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix} e^{-2t}-2e^{-3t}+3e^{-3t}-15te^{-2t}+21e^{-2t} \\ -e^{-4t}+2e^{-5t}-e^{-5t}+5te^{-4t}-7e^{-4t} \end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix} e^{-3t}+(-15t+22)e^{-2t} \\ e^{-5t}+(5t-8)e^{-4t} \end{pmatrix}$$

		$\int e^{-4t}$
+	$5t-8$	$-\frac{1}{4}e^{-4t}$
−	5	$\frac{1}{16}e^{-4t}$

$$Y_p = \frac{1}{2}\begin{pmatrix} -e^{2t} & -3e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix}\int\begin{pmatrix} e^{-3t}+(-15t+22)e^{-2t} \\ e^{-5t}+(5t-8)e^{-4t} \end{pmatrix}dt$$

$$X_p = \varphi(t)\int \varphi^{-1}(t)F(t)\,dt$$

$$=\frac{1}{2}\begin{pmatrix} -e^{2t} & -3e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix}\begin{pmatrix} -\frac{1}{3}e^{-3t}+\left(\frac{15}{2}t-11+\frac{15}{4}\right)e^{-2t} \\ -\frac{1}{5}e^{-5t}+\left(-\frac{5}{4}t+2-\frac{5}{16}\right)e^{-4t} \end{pmatrix}$$

		$\int e^{-2t}$
+	$-15t+22$	$-\frac{1}{2}e^{-2t}$
−	-15	$\frac{1}{4}e^{-2t}$

$$=\frac{1}{2}\begin{pmatrix} -e^{2t} & -3e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix}\begin{pmatrix} -\frac{1}{3}e^{-3t}+\left(\frac{15}{2}t-\frac{29}{4}\right)e^{-2t} \\ -\frac{1}{5}e^{-5t}+\left(-\frac{5}{4}t+\frac{27}{16}\right)e^{-4t} \end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix} \frac{1}{3}e^{-t}-\frac{15}{2}t+\frac{29}{4}+\frac{3}{5}e^{-t}+\frac{15}{4}t-\frac{81}{16} \\ -\frac{1}{3}e^{-t}+\frac{15}{2}t-\frac{29}{4}-\frac{1}{5}e^{-t}-\frac{5}{4}t+\frac{27}{16} \end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix} \frac{14}{15}e^{-t}-\frac{15}{4}t+\frac{35}{16} \\ -\frac{8}{15}e^{-t}+\frac{25}{4}t-\frac{89}{16} \end{pmatrix}$$

$$\boxed{=\begin{pmatrix} \frac{14}{30}e^{-t}-\frac{15}{8}t+\frac{35}{32} \\ -\frac{8}{30}e^{-t}+\frac{25}{8}t-\frac{89}{32} \end{pmatrix}}$$

$$Y(t)=C_1\begin{pmatrix}-1\\1\end{pmatrix}e^{2t}+C_2\begin{pmatrix}-3\\1\end{pmatrix}e^{4t}+\begin{pmatrix}\frac{14}{30}\\-\frac{8}{30}\end{pmatrix}e^{-t}+\begin{pmatrix}-\frac{15}{4}\\\frac{25}{8}\end{pmatrix}t+\begin{pmatrix}\frac{35}{32}\\-\frac{89}{32}\end{pmatrix}$$

$$\boxed{\begin{cases} y_1(t)=-C_1e^{2t}-3C_2e^{4t}+\frac{14}{30}e^{-t}-\frac{15}{4}t+\frac{35}{32} \\ y_2(t)=C_1e^{2t}+C_2e^{4t}-\frac{8}{30}e^{-t}+\frac{25}{8}t-\frac{89}{32} \end{cases}}$$

Exercise

Find the general solution of the system $\begin{cases} x'(t)=-3x+y+3t \\ y'(t)=2x-4y+e^{-t} \end{cases}$

Solution

$$|A-\lambda I|=\begin{vmatrix}-3-\lambda & 1\\ 2 & -4-\lambda\end{vmatrix} \qquad A=\begin{pmatrix}-3 & 1\\ 2 & -4\end{pmatrix}$$

$$=\lambda^2+7\lambda+10=0$$

The eigenvalues: $\boxed{\lambda_{1,2}=-2,\ -5}$

For $\lambda_1=-2 \quad \Rightarrow (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}-1 & 1\\ 2 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \quad \to x=y \quad \Rightarrow \quad V_1=\begin{pmatrix}1\\1\end{pmatrix}$$

For $\lambda_2=-5 \quad \Rightarrow (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix}2 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \quad \to 2x=-y \quad \Rightarrow \quad V_2=\begin{pmatrix}1\\-2\end{pmatrix}$$

$$\boxed{Y_h=C_1\begin{pmatrix}1\\1\end{pmatrix}e^{-2t}+C_2\begin{pmatrix}1\\-2\end{pmatrix}e^{-5t}}$$

$$\varphi(t)=\begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}$$

$$\varphi^{-1}=-\frac{1}{3e^{-7t}}\begin{pmatrix} -2e^{-5t} & -e^{-5t} \\ -e^{-2t} & e^{-2t} \end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix} 2e^{2t} & e^{2t} \\ e^{5t} & -e^{5t} \end{pmatrix}$$

$$F(t)=\begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}$$

$$\varphi^{-1}(t)F(t)=\frac{1}{3}\begin{pmatrix} 2e^{2t} & e^{2t} \\ e^{5t} & -e^{5t} \end{pmatrix}\begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix} 6te^{2t}+e^{t} \\ 3te^{5t}-e^{4t} \end{pmatrix}$$

		$\int e^{2t}$
+	$6t$	$\frac{1}{2}e^{2t}$
–	6	$\frac{1}{4}e^{2t}$

$$X_p=\varphi(t)\int \varphi^{-1}(t)F(t)\,dt$$

$$Y_p=\frac{1}{3}\begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\int\begin{pmatrix} 6te^{2t}+e^{t} \\ 3te^{5t}-e^{4t} \end{pmatrix}dt$$

$$=\frac{1}{3}\begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\begin{pmatrix} \left(3t-\frac{3}{2}\right)e^{2t}+e^{t} \\ \left(\frac{3}{5}t-\frac{3}{25}\right)e^{5t}-\frac{1}{4}e^{4t} \end{pmatrix}$$

		$\int e^{5t}$
+	$3t$	$\frac{1}{5}e^{5t}$
–	3	$\frac{1}{25}e^{5t}$

$$=\frac{1}{3}\begin{pmatrix} 3t-\frac{3}{2}+e^{-t}+\frac{3}{5}t-\frac{3}{25}-\frac{1}{4}e^{-t} \\ 3t-\frac{3}{2}+e^{-t}-\frac{6}{5}t+\frac{6}{25}+\frac{1}{2}e^{-t} \end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix} \frac{3}{4}e^{-t}+\frac{18}{5}t-\frac{81}{50} \\ \frac{3}{2}e^{-t}+\frac{9}{5}t-\frac{63}{50} \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{4}e^{-t}+\frac{6}{5}t-\frac{27}{50} \\ \frac{1}{2}e^{-t}+\frac{3}{5}t-\frac{21}{50} \end{pmatrix}$$

$$Y(t)=C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-2t}+C_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-5t}+\begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix}e^{-t}+\begin{pmatrix} \frac{6}{5} \\ \frac{3}{5} \end{pmatrix}t-\begin{pmatrix} \frac{27}{50} \\ \frac{21}{50} \end{pmatrix}$$

$$\boxed{\begin{cases} y_1(t)=C_1e^{-2t}+C_2e^{-5t}+\frac{1}{4}e^{-t}+\frac{6}{5}t-\frac{27}{50} \\ y_2(t)=C_1e^{-2t}-2C_2e^{-5t}+\frac{1}{2}e^{-t}+\frac{3}{5}t-\frac{21}{50} \end{cases}}$$

Exercise

Find the general solution of the system
$$\begin{cases} x'(t) = 2x - y + (\sin 2t)e^{2t} \\ y'(t) = 4x + 2y + (2\cos 2t)e^{2t} \end{cases}$$

Solution

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ 4 & 2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}$$

$$= \lambda^2 - 4\lambda + 8 = 0$$

The eigenvalues: $\lambda_{1,2} = 2 \pm 2i$

For $\lambda_1 = 2 - 2i \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2i & -1 \\ 4 & 2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow 2ix = y \quad \Rightarrow V_1 = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

$$z(t) = \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{(-2-2i)t}$$

$$= \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 2 \end{pmatrix}\right)(\cos 2t - i\sin 2t)e^{-2t}$$

$$= \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos 2t + \begin{pmatrix} 0 \\ 2 \end{pmatrix}\sin 2t + i\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix}\cos 2t - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin 2t\right)\right)e^{-2t}$$

$$= \left(\begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix} + i\begin{pmatrix} -\sin 2t \\ 2\cos 2t \end{pmatrix}\right)e^{2t}$$

$$Y_h = C_1\begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix}e^{2t} + C_2\begin{pmatrix} -\sin 2t \\ 2\cos 2t \end{pmatrix}e^{2t}$$

$$\varphi(t) = \begin{pmatrix} e^{2t}\cos 2t & -e^{2t}\sin 2t \\ 2e^{2t}\sin 2t & 2e^{2t}\cos 2t \end{pmatrix}$$

$$\varphi^{-1} = \frac{1}{2e^{4t}}\begin{pmatrix} 2e^{2t}\cos 2t & e^{2t}\sin 2t \\ -2e^{2t}\sin 2t & e^{2t}\cos 2t \end{pmatrix}$$

$$= \begin{pmatrix} e^{-2t}\cos 2t & \frac{1}{2}e^{-2t}\sin 2t \\ -e^{-2t}\sin 2t & \frac{1}{2}e^{-2t}\cos 2t \end{pmatrix}$$

$$F(t) = \begin{pmatrix} (\sin 2t)e^{2t} \\ (2\cos 2t)e^{2t} \end{pmatrix}$$

$$\varphi^{-1}(t)F(t)=\begin{pmatrix} e^{-2t}\cos 2t & \frac{1}{2}e^{-2t}\sin 2t \\ -e^{-2t}\sin 2t & \frac{1}{2}e^{-2t}\cos 2t \end{pmatrix}\begin{pmatrix} (\sin 2t)e^{2t} \\ (2\cos 2t)e^{2t} \end{pmatrix}$$

$$=\begin{pmatrix} 2\cos 2t\sin 2t \\ \cos^2 2t-\sin^2 2t \end{pmatrix}$$

$$=\begin{pmatrix} \sin 4t \\ \cos 4t \end{pmatrix}$$

$$Y_p=\begin{pmatrix} e^{2t}\cos 2t & -e^{2t}\sin 2t \\ 2e^{2t}\sin 2t & 2e^{2t}\cos 2t \end{pmatrix}\int\begin{pmatrix} \sin 4t \\ \cos 4t \end{pmatrix}dt \qquad X_p=\varphi(t)\int\varphi^{-1}(t)F(t)\,dt$$

$$=\begin{pmatrix} e^{2t}\cos 2t & -e^{2t}\sin 2t \\ 2e^{2t}\sin 2t & 2e^{2t}\cos 2t \end{pmatrix}\begin{pmatrix} -\frac{1}{4}\cos 4t \\ \frac{1}{4}\sin 4t \end{pmatrix}$$

$$=\begin{pmatrix} -\frac{1}{4}\cos 2t\cos 4t-\frac{1}{4}\sin 2t\sin 4t \\ -\frac{1}{2}\sin 2t\cos 4t+\frac{1}{2}\cos 2t\sin 4t \end{pmatrix}e^{2t}$$

$$Y(t)=C_1\begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix}e^{2t}+C_2\begin{pmatrix} -\sin 2t \\ 2\cos 2t \end{pmatrix}e^{2t}+\begin{pmatrix} -\frac{1}{4}\cos 2t\cos 4t-\frac{1}{4}\sin 2t\sin 4t \\ -\frac{1}{2}\sin 2t\cos 4t+\frac{1}{2}\cos 2t\sin 4t \end{pmatrix}e^{2t}$$

$$\begin{cases} x(t)=\left(C_1\cos 2t-C_2\sin 2t-\frac{1}{4}\cos 2t\cos 4t-\frac{1}{4}\sin 2t\sin 4t\right)e^{2t} \\ y(t)=\left(2C_1\sin 2t+2C_2\cos 2t-\frac{1}{2}\sin 2t\cos 4t+\frac{1}{2}\cos 2t\sin 4t\right)e^{2t} \end{cases}$$

Exercise

Find the general solution of the system $\begin{cases} x'(t)= \quad 2y+e^t \\ y'(t)=-x+3y-e^t \end{cases}$

Solution

$$|A-\lambda I|=\begin{vmatrix} -\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} \qquad A=\begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$$

$$=\lambda^2-3\lambda+2=0$$

The eigenvalues: $\lambda_{1,2}=1,\ 2$

For $\lambda_1=1 \quad \Rightarrow (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x=2y \qquad \Rightarrow \ V_1=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = y \qquad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{Y_h = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^t + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t}}$$

$$\varphi(t) = \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}$$

$$\varphi^{-1} = \frac{1}{e^{3t}}\begin{pmatrix} e^{2t} & -e^{2t} \\ -e^t & 2e^t \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}$$

$$F(t) = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

$$\varphi^{-1}(t)F(t) = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}\begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -3e^{-t} \end{pmatrix}$$

$$Y_p = \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}\int\begin{pmatrix} 2 \\ -3e^{-t} \end{pmatrix}dt \qquad\qquad X_p = \varphi(t)\int \varphi^{-1}(t)F(t)\,dt$$

$$= \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix}\begin{pmatrix} 2t \\ 3e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} 4te^t + 3 \\ 2te^t + 3 \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} 4 \\ 2 \end{pmatrix}te^t + \begin{pmatrix} 3 \\ 3 \end{pmatrix}}$$

$$Y(t) = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^t + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t} + \begin{pmatrix} 4 \\ 2 \end{pmatrix}te^t + \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\boxed{\begin{cases} x(t) = 2C_1e^t + C_2e^{2t} + 4te^t + 3 \\ y(t) = C_1e^t + C_2e^{2t} + 2te^t + 3 \end{cases}}$$

Exercise

Find the general solution of the system $\begin{cases} x'(t) = \qquad 2y + 2 \\ y'(t) = -x + 3y + e^{-3t} \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

The eigenvalues: $\lambda_{1,2} = 1,\ 2$

For $\lambda_1 = 1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = 2y \qquad \Rightarrow \ V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

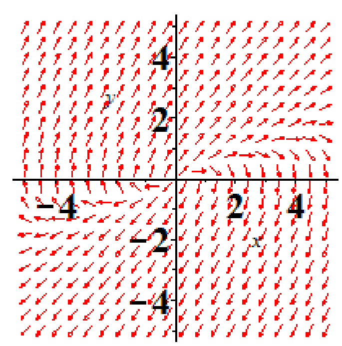
$$\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = y \qquad \Rightarrow \ V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$Y_h = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t}$$

$$\varphi(t) = \begin{pmatrix} 2e^{t} & e^{2t} \\ e^{t} & e^{2t} \end{pmatrix}$$

$$\varphi^{-1} = \frac{1}{e^{3t}}\begin{pmatrix} e^{2t} & -e^{2t} \\ -e^{t} & 2e^{t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}$$



$$F(t) = \begin{pmatrix} 2 \\ e^{-3t} \end{pmatrix}$$

$$\varphi^{-1}(t)F(t) = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}\begin{pmatrix} 2 \\ e^{-3t} \end{pmatrix}$$

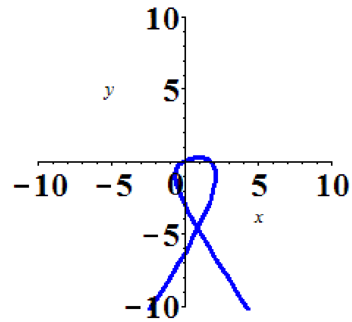
$$= \begin{pmatrix} 2e^{-t} - e^{-4t} \\ -2e^{-2t} + 2e^{-5t} \end{pmatrix}$$

$$Y_p = \begin{pmatrix} 2e^{t} & e^{2t} \\ e^{t} & e^{2t} \end{pmatrix}\int \begin{pmatrix} 2e^{-t} - e^{-4t} \\ -2e^{-2t} + 2e^{-5t} \end{pmatrix}dt \qquad X_p = \varphi(t)\int \varphi^{-1}(t)F(t)\,dt$$

$$= \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix} \begin{pmatrix} -2e^{-t} + \frac{1}{4}e^{-4t} \\ e^{-2t} - \frac{2}{5}e^{-5t} \end{pmatrix}$$

$$= \begin{pmatrix} -4 + \frac{1}{2}e^{-3t} + 1 - \frac{2}{5}e^{-3t} \\ -2 + \frac{1}{4}e^{-3t} + 1 - \frac{2}{5}e^{-3t} \end{pmatrix}$$

$$= \begin{pmatrix} -3 + \frac{1}{10}e^{-3t} \\ -1 - \frac{3}{20}e^{-3t} \end{pmatrix}$$



$$Y(t) = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} \frac{1}{10} \\ -\frac{3}{20} \end{pmatrix} e^{-3t} - \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\boxed{\begin{cases} x(t) = 2C_1 e^t + C_2 e^{2t} + \frac{1}{10}e^{-3t} - 3 \\ y(t) = C_1 e^t + C_2 e^{2t} - \frac{3}{20}e^{-3t} - 1 \end{cases}}$$

Exercise

Find the general solution of the system $\begin{cases} x'(t) = x + 8y + 12t \\ y'(t) = x - y + 12t \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 8 \\ 1 & -1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix}$$

$$= \lambda^2 - 9 = 0$$

The eigenvalues: $\boxed{\lambda_{1,2} = \pm 3}$

For $\lambda_1 = -3 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & 8 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = -2y \quad \Rightarrow \quad V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 3 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

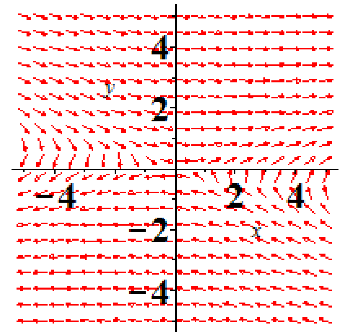
$$\begin{pmatrix} -2 & 8 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = 4y \quad \Rightarrow \quad V_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\boxed{Y_h = C_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t} + C_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{3t}}$$

$$\varphi(t)=\begin{pmatrix}-2e^{-3t} & 4e^{3t}\\ e^{-3t} & e^{3t}\end{pmatrix}$$

$$\varphi^{-1}=-\frac{1}{6}\begin{pmatrix}e^{3t} & -4e^{3t}\\ -e^{-3t} & -2e^{-3t}\end{pmatrix}$$

$$F(t)=\begin{pmatrix}12t\\ 12t\end{pmatrix}$$



$$\varphi^{-1}(t)F(t)=-\frac{1}{6}\begin{pmatrix}e^{3t} & -4e^{3t}\\ -e^{-3t} & -2e^{-3t}\end{pmatrix}\begin{pmatrix}12t\\ 12t\end{pmatrix}$$

$$=\begin{pmatrix}-e^{3t} & 4e^{3t}\\ e^{-3t} & 2e^{-3t}\end{pmatrix}\begin{pmatrix}2t\\ 2t\end{pmatrix}$$

$$=\begin{pmatrix}6te^{3t}\\ 6te^{-3t}\end{pmatrix}$$

$$Y_p=\begin{pmatrix}-2e^{-3t} & 4e^{3t}\\ e^{-3t} & e^{3t}\end{pmatrix}\int\begin{pmatrix}6te^{3t}\\ 6te^{-3t}\end{pmatrix}dt$$

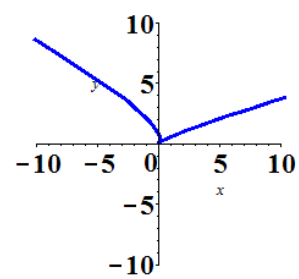
$$X_p=\varphi(t)\int\varphi^{-1}(t)F(t)\,dt$$

$$=\begin{pmatrix}-2e^{-3t} & 4e^{3t}\\ e^{-3t} & e^{3t}\end{pmatrix}\begin{pmatrix}\left(2t-\frac{2}{3}\right)e^{3t}\\ \left(-2t-\frac{2}{3}\right)e^{-3t}\end{pmatrix}$$

		$\int e^{3t}$
+	$6t$	$\frac{1}{3}e^{3t}$
−	6	$\frac{1}{9}e^{3t}$

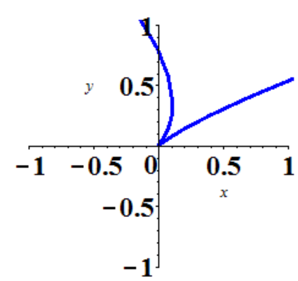
$$=\begin{pmatrix}-4t+\frac{4}{3}-8t-\frac{8}{3}\\ 2t-\frac{2}{3}-2t-\frac{2}{3}\end{pmatrix}$$

$$=\begin{pmatrix}-12t-\frac{4}{3}\\ -\frac{4}{3}\end{pmatrix}$$



$$Y(t)=C_1\begin{pmatrix}-2\\ 1\end{pmatrix}e^{-3t}+C_2\begin{pmatrix}4\\ 1\end{pmatrix}e^{3t}+\begin{pmatrix}-12\\ 0\end{pmatrix}t-\begin{pmatrix}\frac{4}{3}\\ \frac{4}{3}\end{pmatrix}$$

$$\begin{cases}x(t)=-2C_1e^{-3t}+4C_2e^{3t}-12t-\frac{4}{3}\\ y(t)=C_1e^{-3t}+C_2e^{3t}-\frac{4}{3}\end{cases}$$



Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = x_1 + x_2 - x_3 \\ x_2'(t) = 2x_2 \\ x_3'(t) = x_2 - x_3 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & -1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= -\left(1-\lambda^2\right)(2-\lambda) = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1, \ \ \lambda_2 = 1 \ \ and \ \ \lambda_2 = 2$

For $\lambda_1 = -1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} 2x + y - z = 0 \\ \quad y = 0 \end{cases} \qquad 2x - z = 0 \quad \Rightarrow \ V_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 1 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} y - z = 0 \qquad z = 0 \\ \quad y = 0 \end{cases} \qquad \Rightarrow \ V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda_3 = 2 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} -x + y - z = 0 \\ \\ \quad y = 3z \end{cases} \qquad \Rightarrow \ V_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$x(t) = C_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + C_3 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} e^{2t}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 2x_1 - 7x_2 \\ x_2'(t) = 5x_1 + 10x_2 + 4x_3 \\ x_3'(t) = 5x_2 + 2x_3 \end{cases}$

Solution

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & -7 & 0 \\ 5 & 10-\lambda & 4 \\ 0 & 5 & 2-\lambda \end{vmatrix} \qquad\qquad A = \begin{pmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{pmatrix}$$

$$= (2-\lambda)^2 (10-\lambda) - 20(2-\lambda) + 35(2-\lambda)$$

$$= (2-\lambda)\left((10-\lambda)(2-\lambda)+15\right)$$

$$= (2-\lambda)\left(35-12\lambda+\lambda^2\right) = 0$$

Thus, the eigenvalues are: $\lambda_1 = 2,\ \ \lambda_2 = 5\ \ and\ \ \lambda_2 = 7$

For $\lambda_1 = 2 \quad \Rightarrow \left(A-\lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 0 & -7 & 0 \\ 5 & 8 & 4 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} y=0 \\ 5x=-4z \end{cases} \quad \Rightarrow \ V_1 = \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix}$$

For $\lambda_2 = 5 \quad \Rightarrow \left(A-\lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -5 & -7 & 0 \\ 5 & 3 & 4 \\ 0 & 5 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} 5x=-7y \\ \\ y=z \end{cases} \quad \Rightarrow \ V_2 = \begin{pmatrix} -7 \\ 5 \\ 5 \end{pmatrix}$$

For $\lambda_3 = 7 \quad \Rightarrow \left(A-\lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} 0 & -7 & 0 \\ 5 & 8 & 4 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} y=0 \\ 5x=-4z \end{cases} \quad \Rightarrow \ V_1 = \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix}$$

$$x(t) = C_1\begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}e^{2t} + C_2\begin{pmatrix} -7 \\ 3 \\ 5 \end{pmatrix}e^{5t} + C_3\begin{pmatrix} -7 \\ 5 \\ 5 \end{pmatrix}e^{7t}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 3x_1 - x_2 - x_3 \\ x_2'(t) = x_1 + x_2 - x_3 \\ x_3'(t) = x_1 - x_2 + x_3 \end{cases}$

Solution

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & -1 & -1 \\ 1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} \qquad\qquad A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \left(1-2\lambda+\lambda^2\right)(3-\lambda) + 2 + 2 - 2\lambda - 3 + \lambda$$

$$= 3 - 7\lambda + 5\lambda^2 - \lambda^3 + 1 - \lambda$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

$$\begin{array}{r|rrrr} 1 & -1 & 5 & -8 & 4 \\ & & -1 & 4 & -4 \\ \hline & -1 & 4 & -4 & 0 \end{array} \rightarrow \boxed{-\lambda^2 + 4\lambda - 4 = 0}$$

Thus, the eigenvalues are: $\lambda_1 = 1$ *and* $\lambda_{2,3} = 2$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} y = 0 \\ x = y \\ x = z \end{cases} \Rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x = y + z \\ \\ \end{cases} \Rightarrow V_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x = y + z \\ \\ \end{cases} \Rightarrow V_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\boxed{x(t) = C_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^t + C_2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}e^{2t} + C_3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{2t}}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 3x_1 + 2x_2 + 4x_3 \\ x_2'(t) = 2x_1 + 2x_3 \\ x_3'(t) = 4x_1 + 2x_2 + 3x_3 \end{cases}$

Solution

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$$

$$= -9\lambda + 6\lambda^2 - \lambda^3 + 32 + 16\lambda - 12 + 4\lambda - 12 + 4\lambda$$

$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0$$

$$\begin{array}{r|rrrr} -1 & -1 & 6 & 15 & 8 \\ & & 1 & -7 & -8 \\ \hline & -1 & 7 & 8 & 0 \end{array} \rightarrow \boxed{-\lambda^2 + 7\lambda + 8 = 0}$$

Thus, the eigenvalues are: $\lambda_1 = 8$ *and* $\lambda_{2,3} = -1$

For $\lambda_1 = 8 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 5x - 2y - 4z = 0 \\ x - 4y + z = 0 \\ 4x + 2y - 5z = 0 \end{cases} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = -1 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 2x + y + 2z = 0 \\ 2x + y + 2z = 0 \\ 4x + 2y + 4z = 0 \end{cases} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \qquad \Rightarrow \quad V_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$x(t) = C_1\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}e^{8t} + C_2\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}e^{-t} + C_3\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}e^{-t}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = x_1 + x_2 + x_3 \\ x_2'(t) = 2x_1 + x_2 - x_3 \\ x_3'(t) = -8x_1 - 5x_2 - 3x_3 \end{cases}$

Solution

$$\left|A - \lambda I\right| = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -8 & -5 & -3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}$$

$$= \left(1 - 2\lambda + \lambda^2\right)\left(-3-\lambda\right) - 2 + 3 - 3\lambda + 6 + 2\lambda$$

$$= -\lambda^3 - \lambda^2 + 4\lambda + 4 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1, \;\; \lambda_2 = -2, \;\; and \;\; \lambda_3 = 2$

For $\lambda_1 = -1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ -8 & -5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 2x + y + z = 0 \\ 2x + 2y - z = 0 \\ -8x - 5y - 2z = 0 \end{cases}$$

$$z = 1 \quad \rightarrow \begin{cases} 2x + y = -1 \\ 2x + 2y = 1 \end{cases} \qquad \Delta = \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 2 \quad \Delta_x = \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} = -3 \quad \Delta_y = \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = 4$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & -1 \\ -8 & -5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} 3x + y + z = 0 \\ 2x + 3y - z = 0 \\ -8x - 5y - z = 0 \end{cases}$$

$$z = 1 \quad \to \begin{cases} 3x + y = -1 \\ 2x + 3y = 1 \end{cases} \qquad \Delta = \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} = 7 \quad \Delta_x = \begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix} = -4 \quad \Delta_y = \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = 5$$

$$\Rightarrow \quad V_2 = \begin{pmatrix} -\frac{4}{7} \\ \frac{5}{7} \\ 1 \end{pmatrix}$$

For $\lambda_3 = 2 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -8 & -5 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} -x + y + z = 0 \\ 2x - y - z = 0 \\ -8x - 5y - 5z = 0 \end{cases}$$

$$x = 0 \quad \to \begin{cases} y + z = 0 \\ -y - z = 0 \end{cases} \qquad \Rightarrow \quad V_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$x(t) = C_1\begin{pmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{pmatrix}e^{-t} + C_2\begin{pmatrix} -\frac{4}{7} \\ \frac{5}{7} \\ 1 \end{pmatrix}e^{-2t} + C_3\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}e^{2t}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = x_1 - x_2 + 4x_3 \\ x_2'(t) = 3x_1 + 2x_2 - x_3 \\ x_3'(t) = 2x_1 + x_2 - x_3 \end{cases}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

$$= -\left(1 - \lambda^2\right)(2 - \lambda) + 14 - 16 + 8\lambda + 1 - \lambda - 3 - 3\lambda$$

$$= -\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$$

$$\begin{array}{c|cccc} 1 & -1 & 2 & 5 & -6 \\ & & -1 & 1 & 6 \\ \hline & -1 & 1 & 6 & 0 \end{array} \quad \to \quad -\lambda^2 + \lambda + 6 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -2, \quad \lambda_2 = 1, \quad and \quad \lambda_3 = 3$

For $\lambda_1 = -2 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} 3x - y + 4z = 0 \\ 3x + 4y - z = 0 \\ 2x + y + z = 0 \end{cases}$$

$$z = 1 \quad \rightarrow \begin{cases} 3x - y = -4 \\ 3x + 4y = 1 \end{cases} \qquad \Delta = \begin{vmatrix} 3 & -1 \\ 3 & 4 \end{vmatrix} = 15 \quad \Delta_x = \begin{vmatrix} -4 & -1 \\ 1 & 4 \end{vmatrix} = -15 \quad \Delta_y = \begin{vmatrix} 3 & -4 \\ 3 & 1 \end{vmatrix} = 15$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} -y + 4z = 0 \\ 3x + y - z = 0 \\ 2x + y - 2z = 0 \end{cases}$$

$$z = 1 \quad \rightarrow \begin{cases} y = 4 \\ 3x + y = 1 \end{cases} \qquad \Rightarrow \quad V_2 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 3 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} -2x - y + 4z = 0 \\ 3x - y - z = 0 \\ 2x + y - 4z = 0 \end{cases}$$

$$z = 1 \quad \rightarrow \begin{cases} -2x - y = -4 \\ 3x - y = 1 \end{cases} \qquad \Delta = \begin{vmatrix} -2 & -1 \\ 3 & -1 \end{vmatrix} = 5 \quad \Delta_x = \begin{vmatrix} -4 & -1 \\ 1 & -1 \end{vmatrix} = 5 \quad \Delta_y = \begin{vmatrix} -2 & -4 \\ 3 & 1 \end{vmatrix} = 10$$

$$\Rightarrow \quad V_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$x(t) = C_1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} e^{t} + C_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{3t}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = x_1 + x_2 + e^{t} \\ x_2'(t) = x_1 + x_2 + e^{2t} \\ x_3'(t) = 3x_3 + te^{3t} \end{cases}$

Solution

$$|A-\lambda I|=\begin{vmatrix}1-\lambda & 1 & 0\\ 1 & 1-\lambda & 0\\ 0 & 0 & 3-\lambda\end{vmatrix} \qquad A=\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 3\end{pmatrix}$$

$$=\left(1-2\lambda+\lambda^2\right)(3-\lambda)-(3-\lambda)$$

$$=(3-\lambda)\left(\lambda^2-2\lambda\right)=0$$

The eigenvalues: $\underline{\lambda_{1,2,3}=0,\ 2,\ 3|}$

For $\lambda_1=0 \quad \Rightarrow \left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \quad \to \begin{cases}x=-y\\ z=0\end{cases} \quad \Rightarrow \quad V_1=\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}$$

For $\lambda_2=2 \quad \Rightarrow \left(A-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix}-1 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \quad \to \begin{cases}x=y\\ z=0\end{cases} \quad \Rightarrow \quad V_2=\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}$$

For $\lambda_3=3 \quad \Rightarrow \left(A-\lambda_3 I\right)V_3=0$

$$\begin{pmatrix}-2 & 1 & 0\\ 1 & -2 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \quad \to \begin{cases}2x=y\\ x=2y\end{cases} \quad \Rightarrow \quad V_3=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

$$\underline{X_h=C_1\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}+C_2\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}e^{2t}+C_3\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}e^{3t}|}$$

$$\varphi(t)=\begin{pmatrix}-1 & e^{2t} & 0\\ 1 & e^{2t} & 0\\ 0 & 0 & e^{3t}\end{pmatrix}$$

$$\varphi^{-1}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2} & 0\\ \frac{1}{2}e^{-2t} & \frac{1}{2}e^{-2t} & 0\\ 0 & 0 & e^{-3t}\end{pmatrix}$$

$$F(t)=\begin{pmatrix} e^t \\ e^{2t} \\ te^{3t} \end{pmatrix}$$

$$\varphi^{-1}(t)F(t)=\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2}e^{-2t} & \frac{1}{2}e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{pmatrix}\begin{pmatrix} e^t \\ e^{2t} \\ te^{3t} \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{2}e^t-\frac{1}{2}e^{2t} \\ \frac{1}{2}e^{-t}+\frac{1}{2} \\ t \end{pmatrix}$$

$$\int\left(\frac{1}{2}e^t-\frac{1}{2}e^{2t}\right)dt=\frac{1}{2}e^t-\frac{1}{4}e^{2t}$$

$$\int\left(\frac{1}{2}e^{-t}+\frac{1}{2}\right)dt=-\frac{1}{2}e^{-t}+\frac{1}{2}t$$

$$\int t\,dt=\frac{1}{2}t^2$$

$$X_p=\begin{pmatrix} -1 & e^{2t} & 0 \\ 1 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}\begin{pmatrix} \frac{1}{2}e^t-\frac{1}{4}e^{2t} \\ -\frac{1}{2}e^{-t}+\frac{1}{2}t \\ \frac{1}{2}t^2 \end{pmatrix} \qquad X_p=\varphi(t)\int\varphi^{-1}(t)F(t)\,dt$$

$$=\begin{pmatrix} -\frac{1}{2}e^t+\frac{1}{4}e^{2t}-\frac{1}{2}e^t+\frac{1}{2}te^{2t} \\ \frac{1}{2}e^t-\frac{1}{4}e^{2t}-\frac{1}{2}e^t+\frac{1}{2}te^{2t} \\ \frac{1}{2}t^2e^{3t} \end{pmatrix}$$

$$=\begin{pmatrix} -e^t+\frac{1}{4}e^{2t}+\frac{1}{2}te^{2t} \\ \frac{1}{2}te^{2t}-\frac{1}{4}e^{2t} \\ \frac{1}{2}t^2e^{3t} \end{pmatrix}$$

$$X(t)=C_1\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}+C_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{2t}+C_3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}e^{3t}+\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}e^t+\begin{pmatrix} \frac{1}{2}t+\frac{1}{4} \\ \frac{1}{2}t-\frac{1}{4} \\ 0 \end{pmatrix}e^{2t}+\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}t^2 \end{pmatrix}e^{3t}$$

$$\boxed{\begin{cases} x_1(t) = -C_1 - e^t + \left(C_2 + \frac{1}{2}t + \frac{1}{4}\right)e^{2t} + C_3e^{3t} \\ x_2(t) = C_1 + \left(C_2 + \frac{1}{2}t - \frac{1}{4}\right)e^{2t} + 2C_3e^{3t} \\ x_3(t) = \left(C_3 + \frac{1}{2}t^2\right)e^{3t} \end{cases}}$$

Exercise

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = -y_1 + 6y_2 \\ y_2'(t) = -3y_1 + 8y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 6 \\ -3 & 8-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} -1 & 6 \\ -3 & 8 \end{pmatrix}$$

$$= (-1-\lambda)(8-\lambda) + 18$$

$$= -8 - 7\lambda + \lambda^2 + 18$$

$$= \lambda^2 - 7\lambda + 10$$

Thus, the eigenvalues are: $\lambda_1 = 2$ *and* $\lambda_2 = 5$

For $\lambda_1 = 2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -3x + 6y = 0 \\ -3x + 6y = 0 \end{cases} \Rightarrow \boxed{x = 2y} \qquad V_1 = (2,\ 1)^T$$

The solution is: $y_1(t) = e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

For $\lambda_2 = 5 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \rightarrow \begin{cases} -6x + 6y = 0 \\ -3x + 3y = 0 \end{cases} \Rightarrow \boxed{x = y} \qquad V_2 = (1,\ 1)^T$$

The solution is: $y_2(t) = e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The general solution is given by: $y(t) = C_1y_1(t) + C_2y_2(t)$

$$y(t) = C_1e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$y(0) = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2C_1 + C_2 \\ C_1 + C_2 \end{pmatrix} \rightarrow \begin{cases} 2C_1 + C_2 = 1 \\ C_1 + C_2 = -2 \end{cases} \rightarrow \boxed{C_1 = 3} \qquad \boxed{C_2 = -5}$$

The particular solution is:

$$\boxed{y(t) = 3e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 5e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \quad \to \quad \begin{cases} y_1(t) = 6e^{2t} - 5e^{5t} \\ y_2(t) = 3e^{2t} - 5e^{5t} \end{cases}$$

Exercise

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = y_1 + 2y_2 \\ y_2'(t) = -y_1 + 4y_2 \end{cases} \quad y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Solution

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \qquad \left|A - \lambda I\right| = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda)(4-\lambda) + 2$$

$$= 4 - 5\lambda + \lambda^2 + 2$$

$$= \lambda^2 - 5\lambda + 6$$

Thus, the eigenvalues are: $\lambda_1 = 2$ *and* $\lambda_2 = 3$

For $\lambda_1 = 2 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} -x + 2y = 0 \\ -x + 2y = 0 \end{cases} \Rightarrow \boxed{x = 2y}$$

The eigenvector is: $V_1 = (2,\ 1)^T$

The solution is: $y_1(t) = e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

For $\lambda_2 = 3 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} -2x + 2y = 0 \\ -x + y = 0 \end{cases} \quad \Rightarrow \quad \boxed{x = y}$$

The eigenvector is: $V_2 = (1,\ 1)^T$

The solution is: $y_2(t) = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The general solution is given by: $y(t) = C_1 y_1(t) + C_2 y_2(t)$

$$y(t) = C_1 e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$y(0) = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2C_1 + C_2 \\ C_1 + C_2 \end{pmatrix} \quad \to \begin{cases} 2C_1 + C_2 = 3 \\ C_1 + C_2 = 2 \end{cases} \to \boxed{C_1 = 1} \quad \boxed{C_2 = 1}$$

The particular solution is: $y(t) = e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Exercise

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = -4y_1 - 8y_2 \\ y_2'(t) = 4y_1 + 4y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Solution

$$A = \begin{pmatrix} -4 & -8 \\ 4 & 4 \end{pmatrix} \qquad |A - \lambda I| = \begin{vmatrix} -4-\lambda & -8 \\ 4 & 4-\lambda \end{vmatrix}$$

$$= (-4-\lambda)(4-\lambda) + 32$$

$$= -16 + \lambda^2 + 32$$

$$= \lambda^2 + 16 = 0$$

$$\lambda^2 = -16 \Rightarrow \lambda = \pm 4i$$

Thus, the eigenvalues are: $\lambda_1 = -4i$ *and* $\lambda_2 = 4i$

For $\lambda = 4i \quad \Rightarrow (A - \lambda I)V = 0$

$$\begin{pmatrix} -4-4i & -8 \\ 4 & 4-4i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} (-4-4i)x - 8y = 0 \\ 4x + (4-4i)y = 0 \end{cases} \Rightarrow \text{divide by } 4 \begin{cases} -x - ix - 2y = 0 \\ x + y - iy = 0 \end{cases}$$

$$\overline{\quad -ix - y - iy = 0 \quad}$$

$$ix = (-1-i)y \Rightarrow x = \frac{-1-i}{i} y \frac{i}{i} = \frac{-i+1}{-1} y = (-1+i)y$$

The eigenvector is: $V = (-1+i,\ 1)^T$

$$z(t) = e^{4it}\begin{pmatrix} -1+i \\ 1 \end{pmatrix}$$

$$= (\cos 4t + i\sin 4t)\left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$$

$$= \cos 4t\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \sin 4t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\left(\sin 4t\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \cos 4t\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$= \begin{pmatrix} -\cos 4t - \sin 4t \\ \cos 4t \end{pmatrix} + i\begin{pmatrix} -\sin 4t + \cos 4t \\ \sin 4t \end{pmatrix}$$

$$y_1(t) = \begin{pmatrix} -\cos 4t - \sin 4t \\ \cos 4t \end{pmatrix} \ \& \ y_2(t) = \begin{pmatrix} -\sin 4t + \cos 4t \\ \sin 4t \end{pmatrix}$$

The general solution is given by: $y(t)=C_1y_1(t)+C_2y_2(t)$

$$y(t)=C_1\begin{pmatrix}-\cos 4t-\sin 4t\\ \cos 4t\end{pmatrix}+C_2\begin{pmatrix}\cos 4t-\sin 4t\\ \sin 4t\end{pmatrix}$$

$$y(0)=C_1\begin{pmatrix}-1\\ 1\end{pmatrix}+C_2\begin{pmatrix}1\\ 0\end{pmatrix}$$

$$\begin{pmatrix}0\\ 2\end{pmatrix}=\begin{pmatrix}-C_1+C_2\\ C_1\end{pmatrix}$$

$$\Rightarrow \boxed{C_1=2} \quad \boxed{C_2=2}$$

$$y(t)=2\begin{pmatrix}-\cos 4t-\sin 4t\\ \cos 4t\end{pmatrix}+2\begin{pmatrix}\cos 4t-\sin 4t\\ \sin 4t\end{pmatrix}$$

$$=\begin{pmatrix}-4\sin 4t\\ 2\cos 4t+2\sin 4t\end{pmatrix}$$

Exercise

Find the general solution of the system $y'=Ay$ $\begin{cases}y_1'(t)=-y_1-2y_2\\ y_2'(t)=4y_1+3y_2\end{cases}$ $y(0)=\begin{pmatrix}0 & 1\end{pmatrix}^T$

Solution

$$A=\begin{pmatrix}-1 & -2\\ 4 & 3\end{pmatrix} \qquad |A-\lambda I|=\begin{vmatrix}-1-\lambda & -2\\ 4 & 3-\lambda\end{vmatrix}$$

$$=(-1-\lambda)(3-\lambda)+8$$

$$=-3-2\lambda+\lambda^2+8$$

$$=\lambda^2-2\lambda+5=0$$

$$\Rightarrow \lambda=1\pm 2i$$

For $\lambda=1+2i \quad \Rightarrow (A-\lambda I)V=0$

$$\begin{pmatrix}-2-2i & -2\\ 4 & 2-2i\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{cases}(-2-2i)x-2y=0\\ 4x+(2-2i)y=0\end{cases} \Rightarrow \textit{divide by } 2\begin{cases}-x-ix-y=0\\ 2x+y-iy=0\end{cases}$$

$$x-ix-iy=0$$

$$(1-i)x=iy \Rightarrow \frac{i}{i}\frac{1-i}{i}x=y$$

$$\Rightarrow y=-(i+1)x$$

The eigenvector is: $V=(1,\ -1-i)^T$

$$z(t) = e^{(1+2i)t}\begin{pmatrix} 1 \\ -1-i \end{pmatrix}$$

$$= e^t e^{2it}\begin{pmatrix} 1 \\ -1-i \end{pmatrix}$$

$$= e^t\left(\cos 2t + i\sin 2t\right)\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right)$$

$$= e^t\left[\cos 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\cos 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix} + i\sin 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]$$

$$= e^t\left[\cos 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] + ie^t\left[\left(\cos 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)\right]$$

$$= e^t\begin{pmatrix} \cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix} + ie^t\begin{pmatrix} \sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix}$$

$$y_1(t) = e^t\begin{pmatrix} \cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix} \qquad \& \qquad y_2(t) = e^t\begin{pmatrix} \sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix}$$

Form a fundamental equation:

$$y(t) = C_1 e^t\begin{pmatrix} \cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix} + C_2 e^t\begin{pmatrix} \sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix}$$

$$y(0) = C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 \\ -C_1 - C_2 \end{pmatrix} \quad \Rightarrow \quad \boxed{C_1 = 0} \quad \boxed{C_2 = -1}$$

$$y(t) = -e^t\begin{pmatrix} \sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix}$$

Exercise

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = 3y_1 - y_2 \\ y_2'(t) = y_1 + y_2 \end{cases} \quad y(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= (3-\lambda)(1-\lambda) + 1$$

$$= 3 - 4\lambda + \lambda^2 + 1$$

$$= \lambda^2 - 4\lambda + 4 = 0$$

Thus, the eigenvalues are: $\lambda_1 = \lambda_2 = 2$

For $\lambda = 2 \quad \Rightarrow (A - 2I)V_1 = 0$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} x - y = 0 \\ x - y = 0 \end{cases} \Rightarrow \boxed{x = y}$$

The eigenvector is: $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The solution is: $y_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$

For the second eigenvector $V_2 \quad \Rightarrow (A - 2I)V_2 = V_1$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \rightarrow \begin{cases} x - y = 1 \\ x - y = 1 \end{cases} \Rightarrow if \ \ y = 0 \Rightarrow \boxed{x = 1}$$

The eigenvector is: $V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The solution is: $y_2(t) = e^{2t}\left(V_2 + tV_1\right)$

$$= e^{2t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

Therefore, the final solution can be written as: $y(t) = C_1 y_1(t) + C_2 y_2(t)$

$$y(t) = C_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$$= e^{2t}\left(C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$$y(0) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} C_1 + C_2 \\ C_1 \end{pmatrix} \quad \rightarrow \begin{cases} C_1 + C_2 = 2 \\ \boxed{C_1 = -1} \end{cases} \Rightarrow \boxed{C_2 = 2 - C_1 = 3}$$

$$y(t) = e^{2t}\left(-\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$$\boxed{= e^{2t}\begin{pmatrix} 2 + 3t \\ -1 + 3t \end{pmatrix}}$$

Exercise

Find the general solution of the system $y' = Ay \quad \begin{cases} y_1'(t) = -3y_1 + y_2 \\ y_2'(t) = -y_1 - y_2 \end{cases} \quad y(0) = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

Solution

$$A=\begin{pmatrix}-3 & 1\\ -1 & -1\end{pmatrix} \qquad |A-\lambda I|=\begin{vmatrix}-3-\lambda & 1\\ -1 & -1-\lambda\end{vmatrix}$$

$$=(-3-\lambda)(-1-\lambda)+1$$

$$=3+4\lambda+\lambda^2+1$$

$$=\lambda^2+4\lambda+4 \underline{=0}$$

Thus, the eigenvalues are: $\lambda_1=\lambda_2=-2$

For $\lambda=-2 \quad \Rightarrow (A+2I)V_1=0$

$$\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \quad \rightarrow \begin{cases}-x+y=0\\ -x+y=0\end{cases} \Rightarrow \boxed{x=y}$$

The eigenvector is: $V_1=\begin{pmatrix}1\\ 1\end{pmatrix} \quad \Rightarrow \quad y_1(t)=e^{-2t}\begin{pmatrix}1\\ 1\end{pmatrix}$

For the second eigenvector $V_2 \quad \Rightarrow (A+2I)V_2=V_1$

$$\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix} \quad \rightarrow \begin{cases}-x+y=1\\ -x+y=1\end{cases} \Rightarrow if \ y=0 \Rightarrow \boxed{x=-1}$$

The eigenvector is: $V_2=\begin{pmatrix}-1\\ 0\end{pmatrix}$

The solution is: $y_2(t)=e^{-2t}\left(V_2+tV_1\right)$

$$=e^{-2t}\left(\begin{pmatrix}-1\\ 0\end{pmatrix}+t\begin{pmatrix}1\\ 1\end{pmatrix}\right)$$

Therefore, the final solution can be written as: $y(t)=C_1y_1(t)+C_2y_2(t)$

$$y(t)=C_1e^{-2t}\begin{pmatrix}1\\ 1\end{pmatrix}+C_2e^{-2t}\left(\begin{pmatrix}-1\\ 0\end{pmatrix}+t\begin{pmatrix}1\\ 1\end{pmatrix}\right)$$

$$=e^{-2t}\left(C_1\begin{pmatrix}1\\ 1\end{pmatrix}+C_2\begin{pmatrix}-1\\ 0\end{pmatrix}+C_2t\begin{pmatrix}1\\ 1\end{pmatrix}\right)$$

$$y(0)=C_1\begin{pmatrix}1\\ 1\end{pmatrix}+C_2\begin{pmatrix}-1\\ 0\end{pmatrix}$$

$$\begin{pmatrix}0\\ -3\end{pmatrix}=\begin{pmatrix}C_1-C_2\\ C_1\end{pmatrix} \qquad \rightarrow \begin{cases}C_1-C_2=0\\ \boxed{C_1=-3}\end{cases} \Rightarrow C_2=C_1=-3$$

$$y(t)=e^{-2t}\left(-3\begin{pmatrix}1\\ 1\end{pmatrix}-3\begin{pmatrix}-1\\ 0\end{pmatrix}-3t\begin{pmatrix}1\\ 1\end{pmatrix}\right)$$

$$=e^{-2t}\begin{pmatrix}-3t\\ -3-3t\end{pmatrix}$$

Exercise

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = 2y_1 + 4y_2 \\ y_2'(t) = -y_1 + 6y_2 \end{cases} \quad y(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Solution

$$A = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} \qquad |A - \lambda I| = \begin{vmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{vmatrix}$$

$$= (2-\lambda)(6-\lambda) + 4$$

$$= 12 - 2\lambda - 6\lambda + \lambda^2 + 4$$

$$= \lambda^2 - 8\lambda + 16 = 0$$

Thus, the eigenvalues are: $\lambda_1 = \lambda_2 = 4$

For $\lambda = 4 \quad \Rightarrow (A - 4I)V_1 = 0$

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} -2x + 4y = 0 \\ -x + 2y = 0 \end{cases} \Rightarrow \boxed{x = 2y}$$

The eigenvector is: $V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \ y_1(t) = e^{4t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

For the second eigenvector $V_2 \quad \Rightarrow (A - 4I)V_2 = V_1$

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \rightarrow \begin{cases} -2x + 4y = 2 \\ -x + 2y = 1 \end{cases} \Rightarrow \text{if } \ y = 0 \Rightarrow \boxed{x = -1}$$

The eigenvector is: $V_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

The solution is: $y_2(t) = e^{4t}\left(V_2 + tV_1\right)$

$$= e^{4t}\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$$

Therefore, the final solution can be written as: $y(t) = C_1 y_1(t) + C_2 y_2(t)$

$$y(t) = C_1 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t}\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$$= e^{4t}\left(C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ 0 \end{pmatrix} + C_2 t\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$$

$$y(0) = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2C_1 - C_2 \\ C_1 \end{pmatrix} \quad \rightarrow \begin{cases} 2C_1 - C_2 = 3 \\ \boxed{C_1 = 1} \end{cases} \Rightarrow C_2 = 2C_1 - 3 = -1$$

$$y(t) = e^{4t}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} - t\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$$

$$\underline{= e^{4t}\begin{pmatrix} 3-2t \\ 1-t \end{pmatrix}}$$

Exercise

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = -8y_1 - 10y_2 \\ y_2'(t) = 5y_1 + 7y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Solution

$$A = \begin{pmatrix} -8 & -10 \\ 5 & 7 \end{pmatrix} \qquad |A - \lambda I| = \begin{vmatrix} -8-\lambda & -10 \\ 5 & 7-\lambda \end{vmatrix}$$

$$= (-8-\lambda)(7-\lambda) + 50$$

$$= -56 + 8\lambda - 7\lambda + \lambda^2 + 50$$

$$= \lambda^2 + \lambda - 6 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -3$ *and* $\lambda_2 = 2$

For $\lambda_1 = -3 \quad \Rightarrow (A + 3I)V_1 = 0$

$$\begin{pmatrix} -5 & -10 \\ 5 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} -5x - 10y = 0 \\ 5x + 10y = 0 \end{cases} \Rightarrow 5x = -10y \rightarrow \boxed{x = -2y}$$

The eigenvector is: $V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \qquad \Rightarrow \; y_1(t) = e^{-3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

For $\lambda_2 = 2 \quad \Rightarrow (A - 2I)V_2 = 0$

$$\begin{pmatrix} -10 & -10 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} -10x - 10y = 0 \\ 5x + 5y = 0 \end{cases} \Rightarrow 5x = -5y \rightarrow \boxed{x = -y}$$

The eigenvector is: $V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \Rightarrow \; y_2(t) = e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Therefore, the final solution can be written as: $y(t) = C_1 y_1(t) + C_2 y_2(t)$

$$y(t) = C_1 e^{-3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$y(0) = C_1\begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2C_1 - C_2 \\ C_1 + C_2 \end{pmatrix}$$

$$\begin{cases} -2C_1 - C_2 = 3 \\ \;\;C_1 + C_2 = 1 \end{cases} \quad \underline{-C_1 = 4} \rightarrow \boxed{C_1 = -4} \Rightarrow C_2 = 1 - C_1 = 5$$

$$y(t) = -4e^{-3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + 5e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8e^{-3t} - 5e^{2t} \\ -4e^{-3t} + 5e^{2t} \end{pmatrix}$$

Exercise

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = -3y_1 + 2y_2 \\ y_2'(t) = -3y_1 + 4y_2 \end{cases} \quad y(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} -3-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} -3 & 2 \\ -3 & 4 \end{pmatrix}$$

$$= \lambda^2 - \lambda - 6 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -2 \quad and \quad \lambda_2 = 3$

For $\lambda_1 = -2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -1 & 2 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x = 2y \quad V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 3 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -6 & 2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow 3x = y \quad V_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$y(t) = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-2t} + C_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{3t}$$

$$y(0) = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\begin{cases} 2C_1 + C_2 = 0 \\ C_1 + 3C_2 = 2 \end{cases} \qquad \Delta = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5 \quad \Delta_1 = \begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} = -2 \quad \Delta_2 = \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = 4$$

$$C_1 = -\tfrac{2}{5}, \quad C_2 = \tfrac{4}{5}$$

$$y(t) = -\tfrac{2}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-2t} + \tfrac{4}{5}\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{3t}$$

$$= \begin{pmatrix} -\frac{4}{5} \\ -\frac{2}{5} \end{pmatrix} e^{-2t} + \begin{pmatrix} \frac{4}{5} \\ \frac{12}{5} \end{pmatrix} e^{3t}$$

Exercise

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) = 3y_1 - y_2 \\ y_2'(t) = 5y_1 - 3y_2 \end{cases}$ $y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix}$$

$$= \lambda^2 - 4 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = \pm 2$

For $\lambda_1 = -2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 5 & -1 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to 5x = y \qquad V_1 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & -1 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = y \qquad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$y(t) = C_1 \begin{pmatrix} 5 \\ 1 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

$$y(0) = C_1 \begin{pmatrix} 5 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} 5C_1 + C_2 = 1 \\ C_1 + C_2 = -1 \end{cases} \qquad \Delta = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} = 4 \quad \Delta_1 = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 \quad \Delta_2 = \begin{vmatrix} 5 & 1 \\ 1 & -1 \end{vmatrix} = -6$$

$$C_1 = \frac{1}{2}, \quad C_2 = -\frac{3}{2}$$

$$y(t) = \frac{1}{2}\begin{pmatrix} 5 \\ 1 \end{pmatrix} e^{-2t} - \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

$$= \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \end{pmatrix} e^{-2t} - \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} e^{2t}$$

Exercise

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = y_1 + 9y_2 \\ y_2'(t) = -2y_1 - 5y_2 \end{cases} \qquad y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 9 \\ -2 & -5-\lambda \end{vmatrix} \qquad\qquad A = \begin{pmatrix} 1 & 9 \\ -2 & -5 \end{pmatrix}$$

$$= \lambda^2 + 4\lambda + 13 = 0$$

Thus, the eigenvalues are: $\boxed{\lambda_{1,2} = -2 \pm 3i}$

For $\lambda_1 = -2 - 3i \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3+3i & 9 \\ -2 & -3+3i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad (1+i)x = -3y \qquad V_1 = \begin{pmatrix} -3 \\ 1+i \end{pmatrix}$$

$$z(t) = \begin{pmatrix} -3 \\ 1+i \end{pmatrix} e^{(-2-3i)t}$$

$$= \left(\begin{pmatrix} -3 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)(\cos 3t - i\sin 3t)e^{-2t}$$

$$= \left[\begin{pmatrix} -3 \\ 1 \end{pmatrix}\cos 3t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin 3t + i\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos 3t - \begin{pmatrix} -3 \\ 1 \end{pmatrix}\sin 3t\right)\right]e^{-2t}$$

$$= \left[\begin{pmatrix} -3\cos 3t \\ \cos 3t + \sin 3t \end{pmatrix} + i\begin{pmatrix} 3\sin 3t \\ \cos 3t - \sin 3t \end{pmatrix}\right]e^{-2t}$$

$$\boxed{y(t) = C_1\begin{pmatrix} -3\cos 3t \\ \cos 3t + \sin 3t \end{pmatrix}e^{-2t} + C_2\begin{pmatrix} 3\sin 3t \\ \cos 3t - \sin 3t \end{pmatrix}e^{-2t}}$$

$$y(0) = C_1\begin{pmatrix} -3 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{cases} -3C_1 = 3 \\ C_1 + C_2 = 2 \end{cases} \qquad \boxed{C_1 = -1, \quad C_2 = 3}$$

$$y(t) = \begin{pmatrix} 3\cos 3t \\ -\cos 3t - \sin 3t \end{pmatrix}e^{-2t} + \begin{pmatrix} 9\sin 3t \\ 3\cos 3t - 3\sin 3t \end{pmatrix}e^{-2t}$$

$$\boxed{= \begin{pmatrix} 3\cos 3t + 9\sin 3t \\ 2\cos 3t - 4\text{in}\,3t \end{pmatrix}e^{-2t}}$$

Exercise

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = 4y_1 + y_2 \\ y_2'(t) = -2y_1 + y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}$$

$$= \lambda^2 - 5\lambda + 6 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 2$ & $\lambda_2 = 3$

For $\lambda_1 = 2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to 2x = -y \quad V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

For $\lambda_2 = 3 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = -y \quad V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\boxed{y(t) = C_1\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{2t} + C_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{3t}}$$

$$y(0) = C_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} C_1 - C_2 = 1 \\ -2C_1 + C_2 = 0 \end{cases} \qquad \boxed{C_1 = -1, \quad C_2 = -2}$$

$$\boxed{\begin{cases} y_1(t) = -e^{2t} + 2e^{3t} \\ y_2(t) = 2e^{2t} - 2e^{3t} \end{cases}}$$

Exercise

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = 2y_1 + y_2 - e^{2t} \\ y_2'(t) = y_1 + 2y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \lambda^2 - 4\lambda + 3 = 0$$

Thus, the eigenvalues are: $\boxed{\lambda_1 = 1 \quad \& \quad \lambda_2 = 3}$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x=-y \quad V_1=\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 3 \quad \Rightarrow (A-\lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x=y \quad V_2=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{y_h = C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^t + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t}}$$

$$\varphi(t)=\begin{pmatrix} -e^t & e^{3t} \\ e^t & e^{3t} \end{pmatrix}$$

$$\varphi^{-1}(t)=\frac{1}{-2e^{4t}}\begin{pmatrix} e^{3t} & -e^{3t} \\ -e^t & -e^t \end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix} -e^{-t} & e^{-t} \\ e^{-3t} & e^{-3t} \end{pmatrix}$$

$$\varphi^{-1}\cdot\begin{pmatrix} -e^{2t} \\ 0 \end{pmatrix}=\frac{1}{2}\begin{pmatrix} -e^{-t} & e^{-t} \\ e^{-3t} & e^{-3t} \end{pmatrix}\begin{pmatrix} -e^{2t} \\ 0 \end{pmatrix}=\frac{1}{2}\begin{pmatrix} e^t \\ -e^{-t} \end{pmatrix}$$

$$X=\frac{1}{2}\int\begin{pmatrix} e^t \\ -e^{-t} \end{pmatrix}dt$$

$$=\frac{1}{2}\begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}$$

$$X_p(t)=\varphi X=\frac{1}{2}\begin{pmatrix} -e^t & e^{3t} \\ e^t & e^{3t} \end{pmatrix}\begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}$$

$$=\begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$$

$$y(t)=\begin{pmatrix} -C_1e^t + C_2e^{3t} \\ C_1e^t + C_2e^{3t} \end{pmatrix}+\begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$$

$$y(0)=\begin{pmatrix} -C_1 + C_2 \\ C_1 + C_2 + 1 \end{pmatrix}=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} -C_1 + C_2 = 1 \\ C_1 + C_2 = -2 \end{cases} \quad \rightarrow \boxed{C_1 = -\frac{3}{2}, \quad C_2 = -\frac{1}{2}}$$

$$\boxed{\begin{cases} y_1(t) = \frac{3}{2}e^t - \frac{1}{2}e^{3t} \\ y_1(t) = -\frac{3}{2}e^t - \frac{1}{2}e^{3t} + e^{2t} \end{cases}}$$

Exercise

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) + 2y_2'(t) = 4y_1 + 5y_2 \\ 2y_1'(t) - y_2'(t) = 3y_1 \end{cases}$ $y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Solution

$$\begin{aligned} &y_1'(t) + 2y_2'(t) = 4y_1 + 5y_2 \\ &4y_1'(t) - 2y_2'(t) = 6y_1 \end{aligned} \quad \rightarrow \quad \begin{cases} y_1'(t) = 2y_1 + y_2 \\ y_2'(t) = y_1 + 2y_2 \end{cases}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \lambda^2 - 4\lambda + 3 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 1$ *and* $\lambda_2 = 3$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x = -y \qquad V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x = y \qquad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{y(t) = C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^t + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t}}$$

$$y(0) = C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} -C_1 + C_2 = 1 \\ C_1 + C_2 = -1 \end{cases} \qquad \boxed{C_1 = -1, \quad C_2 = 0}$$

$$\boxed{y(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^t}$$

Exercise

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) = 3y_1 - 2y_2 \\ y_2'(t) = 2y_1 - 2y_2 \end{cases}$ $y(0) = \begin{pmatrix} 3 \\ \frac{1}{2} \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$$

$$= \lambda^2 - \lambda - 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1$ *and* $\lambda_2 = 2$

For $\lambda_1 = -1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to 2x = y \quad V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = 2y \quad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\boxed{y(t) = C_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-t} + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{2t}}$$

$$y(0) = C_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{1}{2} \end{pmatrix}$$

$$\begin{cases} C_1 + 2C_2 = 3 \\ 2C_1 + C_2 = \frac{1}{2} \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3 \quad \Delta_1 = \begin{vmatrix} 3 & 2 \\ \frac{1}{2} & 1 \end{vmatrix} = 2 \quad \Delta_2 = \begin{vmatrix} 1 & 3 \\ 2 & \frac{1}{2} \end{vmatrix} = -\frac{11}{2}$$

$$\boxed{C_1 = -\frac{2}{3}, \quad C_2 = \frac{11}{6}}$$

$$\boxed{\begin{cases} y_1(t) = -\frac{2}{3}e^{-t} + \frac{11}{3}e^{2t} \\ y_2(t) = -\frac{4}{3}e^{-t} + \frac{11}{6}e^{2t} \end{cases}}$$

Exercise

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) = y_1 - 2y_2 \\ y_2'(t) = 3y_1 - 4y_2 \end{cases}$ $y(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$$

$$= \lambda^2 + 3\lambda + 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1 \quad and \quad \lambda_2 = -2$

For $\lambda_1 = -1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x = y \qquad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow 3x = 2y \qquad V_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\boxed{y(t) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + C_2\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{-2t}}$$

$$y(0) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\begin{cases} C_1 + 2C_2 = -1 \\ C_1 + 3C_2 = 2 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1 \quad \Delta_1 = \begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix} = -7 \quad \Delta_2 = \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 3$$

$$\boxed{C_1 = -7, \quad C_2 = 3}$$

$$y(t) = -7\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + 3\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{-2t}$$

$$\boxed{\begin{cases} y_1(t) = -7e^{-t} + 6e^{-2t} \\ y_2(t) = -7e^{-t} + 9e^{-2t} \end{cases}}$$

Exercise

Find the general solution of the system $y' = Ay \quad \begin{cases} y_1'(t) = y_1 - 4y_2 \\ y_2'(t) = 4y_1 - 7y_2 \end{cases} \qquad y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{vmatrix} \qquad\qquad A = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}$$

$$= \lambda^2 + 6\lambda + 9 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = -3$

For $\lambda_1 = -3 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow x = y \qquad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \Rightarrow y_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-3t}$$

For $V_2 \quad \Rightarrow (A-\lambda I)V_2 = V_1$

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \to 4x-4y=1 \qquad V_2 = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$$

$$y_2(t) = \left(V_2 + tV_1\right)e^{-3t}$$

$$= \begin{pmatrix} \frac{1}{4}+t \\ t \end{pmatrix}e^{-3t}$$

$$y(t) = \left(C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} \frac{1}{4}+t \\ t \end{pmatrix}\right)e^{-3t}$$

$$y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \to \quad C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{cases} C_1 + \frac{1}{4}C_2 = 3 & \underline{C_2 = 4|} \\ \quad \underline{C_1 = 2|} \end{cases}$$

$$y(t) = \left(\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 1+4t \\ 4t \end{pmatrix}\right)e^{-3t}$$

$$\boxed{\begin{cases} y_1(t) = (3+4t)e^{-3t} \\ y_2(t) = (2+4t)e^{-3t} \end{cases}}$$

Exercise

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) = 3y_1 + 9y_2 \\ y_2'(t) = -y_1 - 3y_2 \end{cases}$ $\quad y(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Solution

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & 9 \\ -1 & -3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}$$

$$= \lambda^2 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = 0$

For $\lambda_1 = 0 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = -3y \qquad V_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \Rightarrow y_1(t) = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

For $V_2 \quad \Rightarrow (A-\lambda I)V_2 = V_1$

$$\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \to x + 3y = -1 \qquad V_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{aligned} y_2(t) &= V_2 + tV_1 \\ &= \begin{pmatrix} -1-3t \\ t \end{pmatrix} \end{aligned}$$

$$y(t) = C_1\begin{pmatrix} -3 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} -1-3t \\ t \end{pmatrix}$$

$$y(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \quad \to \quad C_1\begin{pmatrix} -3 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\begin{cases} -3C_1 - C_2 = 2 & \underline{C_2 = -14|} \\ \underline{C_1 = 4|} \end{cases}$$

$$y(t) = \begin{pmatrix} -12 \\ 4 \end{pmatrix} + \begin{pmatrix} 14+42t \\ -14t \end{pmatrix}$$

$$\boxed{\begin{cases} y_1(t) = 2 + 42t \\ y_2(t) = 4 - 14t \end{cases}}$$

Exercise

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) = 2y_1 + \frac{3}{2}y_2 \\ y_2'(t) = -\frac{3}{2}y_1 - y_2 \end{cases}$ $y(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & \frac{3}{2} \\ -\frac{3}{2} & -1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 2 & \frac{3}{2} \\ -\frac{3}{2} & -1 \end{pmatrix}$$

$$= \lambda^2 - \lambda + \frac{1}{4} = 0 \quad \to \quad \left(\lambda - \frac{1}{2}\right)^2 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = \frac{1}{2}$

For $\lambda_1 = \frac{1}{2} \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} \frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = -y \qquad V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \; y_1(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{t/2}$$

For $V_2 \quad \Rightarrow (A - \lambda I)V_2 = V_1$

$$\begin{pmatrix} \frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \rightarrow \ \frac{3}{2}x+\frac{3}{2}y=-1 \qquad V_2=\begin{pmatrix} -\frac{2}{3} \\ 0 \end{pmatrix}$$

$$y_2(t)=\left(V_2+tV_1\right)e^{t/2}$$

$$=\begin{pmatrix} -\frac{2}{3}-t \\ t \end{pmatrix}e^{t/2}$$

$$y(t)=\left(C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}+C_2\begin{pmatrix} -\frac{2}{3}-t \\ t \end{pmatrix}\right)e^{t/2}$$

$$y(0)=\begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad \rightarrow \quad C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}+C_2\begin{pmatrix} -\frac{2}{3} \\ 0 \end{pmatrix}=\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\begin{cases} -C_1-\frac{2}{3}C_2=3 & \underline{C_2=-\frac{3}{2}} \\ \quad \underline{C_1=-2} \end{cases}$$

$$y(t)=\left(\begin{pmatrix} 2 \\ -2 \end{pmatrix}+\begin{pmatrix} 1+\frac{3}{2}t \\ -\frac{3}{2}t \end{pmatrix}\right)e^{t/2}$$

$$\boxed{\begin{cases} y_1(t)=\left(3+\frac{3}{2}t\right)e^{t/2} \\ y_2(t)=-\left(2+\frac{3}{2}t\right)e^{t/2} \end{cases}}$$

Exercise

Find the general solution of the system $y'=Ay$ $\begin{cases} y_1'(t)=-5y_1+12y_2 \\ y_2'(t)=-2y_1+5y_2 \end{cases}$ $\quad y(0)=\begin{pmatrix} 8 \\ 3 \end{pmatrix}$

Solution

$$|A-\lambda I|=\begin{vmatrix} -5-\lambda & 12 \\ -2 & 5-\lambda \end{vmatrix} \qquad A=\begin{pmatrix} -5 & 12 \\ -2 & 5 \end{pmatrix}$$

$$=\lambda^2-1=0$$

Thus, the eigenvalues are: $\lambda_{1,2}=\pm 1$

For $\lambda_1=-1 \quad \Rightarrow \left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix} -4 & 12 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \ x=3y \qquad V_1=\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

For $\lambda_2=1 \quad \Rightarrow \left(A-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix} -6 & 12 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = 2y \qquad V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$y(t) = C_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t}$$

$$y(0) = \begin{pmatrix} 8 \\ 3 \end{pmatrix} \quad \to \quad C_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$$

$$\begin{cases} 3C_1 + 2C_2 = 8 \\ C_1 + C_2 = 3 \end{cases} \quad \to \quad \underline{C_1 = 2, \ C_2 = 1|}$$

$$y(t) = \begin{pmatrix} 6 \\ 2 \end{pmatrix} e^{-t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t}$$

$$\boxed{\begin{cases} y_1(t) = 6e^{-t} + 2e^{t} \\ y_2(t) = 2e^{-t} + e^{t} \end{cases}}$$

Exercise

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) = -4y_1 + 6y_2 \\ y_2'(t) = -3y_1 + 5y_2 \end{cases}$ $y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$$

$$= \lambda^2 - \lambda - 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1$ & $\lambda_2 = 2$

For $\lambda_1 = -1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = 2y \qquad V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = y \qquad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$y(t) = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

$$y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \to \quad C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{cases} 2C_1 + C_2 = 3 \\ C_1 + C_2 = 2 \end{cases} \quad \to \quad \underline{C_1 = 1, \ C_2 = 1|}$$

$$y(t)=\begin{pmatrix}2\\1\end{pmatrix}e^{-t}+\begin{pmatrix}1\\1\end{pmatrix}e^{2t}$$

$$\boxed{\begin{cases}y_1(t)=2e^{-t}+e^{2t}\\ y_2(t)=e^{-t}+e^{2t}\end{cases}}$$

Exercise

Find the general solution of the system $y'=Ay$ $\begin{cases}y_1'(t)=y_1+2y_2\\ y_2'(t)=3y_1+2y_2\end{cases}$ $y(0)=\begin{pmatrix}0\\-4\end{pmatrix}$

Solution

$$|A-\lambda I|=\begin{vmatrix}1-\lambda & 2\\ 3 & 2-\lambda\end{vmatrix} \qquad A=\begin{pmatrix}1&2\\3&2\end{pmatrix}$$

$$=\lambda^2-3\lambda-4=0$$

Thus, the eigenvalues are: $\boxed{\lambda_{1,2}=-1,\ 4}$

For $\lambda_1=-1 \quad \Rightarrow (A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}2&2\\3&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \quad \to x=-y \qquad V_1=\begin{pmatrix}-1\\1\end{pmatrix}$$

For $\lambda_2=4 \quad \Rightarrow (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix}-3&2\\3&-2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \quad \to 3x=2y \qquad V_2=\begin{pmatrix}2\\3\end{pmatrix}$$

$$\boxed{y(t)=C_1\begin{pmatrix}-1\\1\end{pmatrix}e^{-t}+C_2\begin{pmatrix}2\\3\end{pmatrix}e^{4t}}$$

$$y(0)=C_1\begin{pmatrix}-1\\1\end{pmatrix}+C_2\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}0\\-4\end{pmatrix}$$

$$\begin{cases}-C_1+2C_2=0\\ C_1+3C_2=-4\end{cases} \qquad \Delta=\begin{vmatrix}-1&2\\1&3\end{vmatrix}=-5 \quad \Delta_1=\begin{vmatrix}0&2\\-4&3\end{vmatrix}=8 \quad \Delta_2=\begin{vmatrix}-1&0\\1&-4\end{vmatrix}=4$$

$$\boxed{C_1=-\tfrac{8}{5},\quad C_2=-\tfrac{4}{5}}$$

$$y(t)=-\frac{8}{5}\begin{pmatrix}-1\\1\end{pmatrix}e^{-t}-\frac{4}{5}\begin{pmatrix}2\\3\end{pmatrix}e^{4t}$$

$$\boxed{\begin{cases}y_1(t)=\frac{8}{5}e^{-t}-\frac{8}{5}e^{4t}\\ y_2(t)=-\frac{8}{5}e^{-t}-\frac{12}{5}e^{4t}\end{cases}}$$

Exercise

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) = -5y_1 + y_2 \\ y_2'(t) = 4y_1 - 2y_2 \end{cases}$ $y(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} -5-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix}$$

$$= \lambda^2 + 7\lambda + 6 = 0$$

Thus, the eigenvalues are: $\boxed{\lambda_{1,2} = -1, \ -6}$

For $\lambda_1 = -1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -4 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to 4x = y \quad V_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

For $\lambda_2 = -6 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = -y \quad V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\boxed{y(t) = C_1\begin{pmatrix} 1 \\ 4 \end{pmatrix}e^{-t} + C_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{-6t}}$$

$$y(0) = C_1\begin{pmatrix} 1 \\ 4 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{cases} C_1 - C_2 = 1 \\ 4C_1 + C_2 = 2 \end{cases} \quad \to \boxed{C_1 = \frac{3}{5}, \quad C_2 = -\frac{2}{5}}$$

$$y(t) = \frac{3}{5}\begin{pmatrix} 1 \\ 4 \end{pmatrix}e^{-t} - \frac{2}{5}\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{-6t}$$

$$\boxed{\begin{cases} y_1(t) = \frac{3}{5}e^{-t} + \frac{2}{5}e^{-6t} \\ y_2(t) = \frac{12}{5}e^{-t} - \frac{2}{5}e^{-6t} \end{cases}}$$

Exercise

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) = 3y_1 - 9y_2 \\ y_2'(t) = 4y_1 - 3y_2 \end{cases}$ $y(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -9 \\ 4 & -3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & -9 \\ 4 & -3 \end{pmatrix}$$

$$=\lambda^2+27=0$$

Thus, the eigenvalues are: $\boxed{\lambda_{1,2}=\pm 3\sqrt{3}i}$

For $\lambda_1=3\sqrt{3}\,i \quad \Rightarrow \left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix}3-3\sqrt{3}\,i & -9\\ 4 & -3-3\sqrt{3}\,i\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \quad \to \boxed{\left(1-\sqrt{3}\,i\right)x=3y}$$

$$\Rightarrow\ V_1=\begin{pmatrix}3\\ 1-\sqrt{3}\,i\end{pmatrix}$$

$$y(t)=\begin{pmatrix}3\\ 1-\sqrt{3}\,i\end{pmatrix}e^{3\sqrt{3}\,it}$$

$$=\begin{pmatrix}3\\ 1-\sqrt{3}\,i\end{pmatrix}\left(\cos\left(3\sqrt{3}\,t\right)+i\sin\left(3\sqrt{3}\,t\right)\right)$$

$$=\begin{pmatrix}3\cos 3\sqrt{3}t+3i\sin 3\sqrt{3}t\\ \cos 3\sqrt{3}t+\sqrt{3}\sin 3\sqrt{3}t+i\left(\sin 3\sqrt{3}t-\sqrt{3}\cos 3\sqrt{3}t\right)\end{pmatrix}$$

$$\begin{cases}y_1(t)=3C_1\cos 3\sqrt{3}t+3C_2\sin 3\sqrt{3}t\\ y_2(t)=C_1\left(\cos 3\sqrt{3}t+\sqrt{3}\sin 3\sqrt{3}t\right)+C_2\left(\sin 3\sqrt{3}t-\sqrt{3}\cos 3\sqrt{3}t\right)\end{cases}$$

Given: $y_1(0)=2,\ \ y_2(0)=-4$

$$\begin{cases}y_1(0)=3C_1=2 & \to \boxed{C_1=\frac{2}{3}}\\ y_2(0)=C_1-\sqrt{3}C_2=-4 & \to \boxed{C_2=\frac{14}{3\sqrt{3}}}\end{cases}$$

$$\begin{cases}y_1(t)=2\cos 3\sqrt{3}t+\frac{14}{\sqrt{3}}\sin 3\sqrt{3}t\\ y_2(t)=\frac{2}{3}\cos 3\sqrt{3}t+\frac{2\sqrt{3}}{3}\sin 3\sqrt{3}t+\frac{14}{\sqrt{3}}\sin 3\sqrt{3}t-14\cos 3\sqrt{3}t\end{cases}$$

$$\boxed{\begin{cases}y_1(t)=2\cos 3\sqrt{3}t+\frac{14}{\sqrt{3}}\sin 3\sqrt{3}t\\ y_2(t)=\frac{16\sqrt{3}}{3}\sin 3\sqrt{3}t-40\cos 3\sqrt{3}t\end{cases}}$$

Exercise

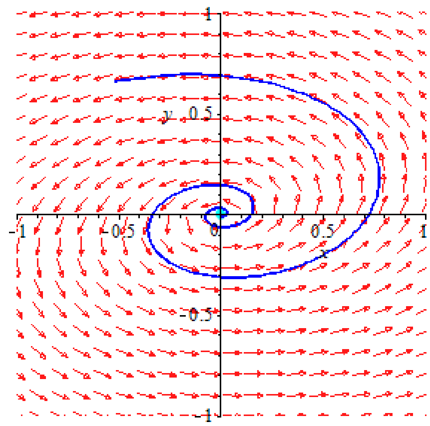
Find the general solution of the system $y'=Ay$ $\quad \begin{cases}y_1'(t)=3y_1-13y_2\\ y_2'(t)=5y_1+y_2\end{cases} \quad y(0)=\begin{pmatrix}3\\ -10\end{pmatrix}$

Solution

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & -13 \\ 5 & 1-\lambda \end{vmatrix} \qquad\qquad A = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix}$$

$$= \lambda^2 - 4\lambda + 68 = 0$$

Thus, the eigenvalues are: $\boxed{\lambda_{1,2} = 2 \pm 8i}$



For $\lambda_1 = 2+8i \quad \Rightarrow (A-\lambda_1 I)V_1 = 0$

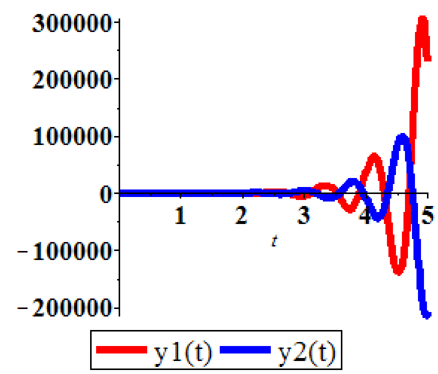
$$\begin{pmatrix} 1-8i & -13 \\ 5 & -1-8i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{(1-8i)x = 13y}$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 13 \\ 1-8i \end{pmatrix}$$

$$y(t) = \begin{pmatrix} 13 \\ 1-8i \end{pmatrix} e^{(2+8i)t}$$

$$= \begin{pmatrix} 13 \\ 1-8i \end{pmatrix} (\cos 8t + i\sin 8t) e^{2t}$$

$$= \begin{pmatrix} 13\cos 8t + 13i\sin 8t \\ \cos 8t + 8\sin 8t + i(\sin 8t - 8\cos 8t) \end{pmatrix} e^{2t}$$

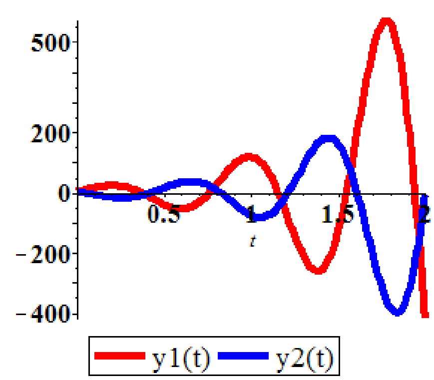


$$\begin{cases} y_1(t) = (13C_1\cos 8t + 13C_2\sin 8t)e^{2t} \\ y_2(t) = \big(C_1(\cos 8t + 8\sin 8t) + C_2(\sin 8t - 8\cos 8t)\big)e^{2t} \end{cases}$$

Given: $y_1(0) = 3, \quad y_2(0) = -10$

$$\begin{cases} y_1(0) = 13C_1 = 3 & \rightarrow \boxed{C_1 = \frac{3}{13}} \\ y_2(0) = C_1 - 8C_2 = -10 & \rightarrow \boxed{C_2 = \frac{133}{104}} \end{cases}$$

$$\begin{cases} y_1(t) = \left(3\cos 8t + \frac{133}{8}\sin 8t\right)e^{2t} \\ y_2(t) = \left(\frac{3}{13}\cos 8t + \frac{24}{13}\sin 8t + \frac{133}{104}\sin 8t - \frac{133}{13}\cos 8t\right)e^{2t} \end{cases}$$



$$\boxed{\begin{cases} y_1(t) = \left(3\cos 8t + \frac{133}{8}\sin 8t\right)e^{2t} \\ y_2(t) = \left(\frac{325}{104}\sin 8t - 10\cos 8t\right)e^{2t} \end{cases}}$$

Exercise

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = 7y_1 + y_2 \\ y_2'(t) = -4y_1 + 3y_2 \end{cases} \quad y(0) = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$

Solution

$$|A-\lambda I| = \begin{vmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} \qquad\qquad A = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}$$

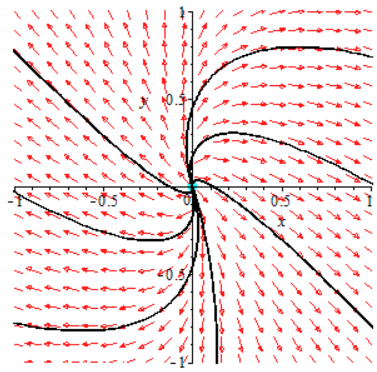
$$= \lambda^2 - 10\lambda + 25 = 0$$

Thus, the eigenvalues are: $\boxed{\lambda_{1,2} = 5}$

For $\lambda_1 = 5 \quad \Rightarrow (A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \boxed{2x = -y}$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \rightarrow \quad y_1(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{5t}$$



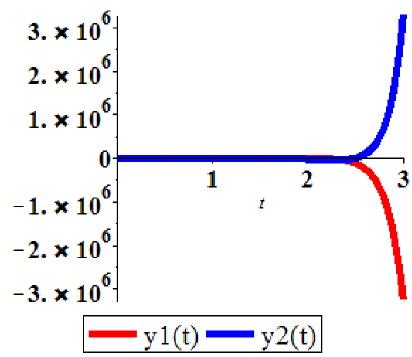
For $V_2 \quad \Rightarrow (A-\lambda I)V_2 = V_1$

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \rightarrow \quad 2x + y = 1 \qquad V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$y_2(t) = V_2 + tV_1$$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

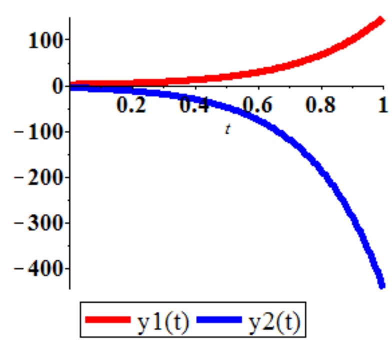
$$= \begin{pmatrix} 1+t \\ -1-2t \end{pmatrix} e^{5t}$$



$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

$$= C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} 1+t \\ -1-2t \end{pmatrix} e^{5t}$$

$$y(0) = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$



$$\begin{cases} C_1 + C_2 = 2 \\ -2C_1 - C_2 = -5 \end{cases} \quad \rightarrow \quad \boxed{C_1 = 3, \quad C_2 = -1}$$

$$y(t) = \left(\begin{pmatrix} 3 \\ -6 \end{pmatrix} + \begin{pmatrix} -1-t \\ 1+2t \end{pmatrix} \right) e^{5t}$$

$$\boxed{= \begin{pmatrix} 2-t \\ -5+2t \end{pmatrix} e^{5t}}$$

$$\boxed{\begin{cases} y_1(t) = (2-t)e^{5t} \\ y_2(t) = (-5+2t)e^{5t} \end{cases}}$$

Exercise

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) = -y_1 + \frac{3}{2}y_2 \\ y_2'(t) = -\frac{1}{6}y_1 - 2y_2 \end{cases}$ $y(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & \frac{3}{2} \\ -\frac{1}{6} & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} -1 & \frac{3}{2} \\ -\frac{1}{6} & -2 \end{pmatrix}$$

$$= \lambda^2 + 3\lambda + \frac{9}{4} = \left(\lambda + \frac{3}{2}\right)^2 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = -\frac{3}{2}$

For $\lambda_1 = -\frac{3}{2} \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad x = -3y$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \to \quad y_1(t) = \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-3t/2}$$

For $V_2 \quad \Rightarrow (A - \lambda I)V_2 = V_1$

$$\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \to x + 3y = -6 \qquad V_2 = \begin{pmatrix} -9 \\ 1 \end{pmatrix}$$

$$\begin{aligned} y_2(t) &= V_2 + tV_1 \\ &= \begin{pmatrix} -9 \\ 1 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -9-3t \\ 1+t \end{pmatrix} e^{-3t/2} \end{aligned}$$

$$y(t) = C_1\begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-3t/2} + C_2\begin{pmatrix} -9-3t \\ 1+t \end{pmatrix} e^{-3t/2} \qquad y(t) = C_1 y_1(t) + C_2 y_2(t)$$

Given: $y(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$y(2) = C_1\begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-3} + C_2\begin{pmatrix} -15 \\ 3 \end{pmatrix} e^{-3} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{cases}\left(-3C_1-15C_2\right)e^{-3}=1\\ \left(C_1+3C_2\right)e^{-3}=0\end{cases} \to \begin{cases}3C_1+15C_2=-e^3\\ C_1+3C_2=0\end{cases}$$

$$\Delta=\begin{vmatrix}3 & 15\\ 1 & 3\end{vmatrix}=-6 \quad \Delta_1=\begin{vmatrix}-e^3 & 15\\ 0 & 3\end{vmatrix}=-3e^3 \quad \Delta_2=\begin{vmatrix}3 & -e^3\\ 1 & 0\end{vmatrix}=e^3$$

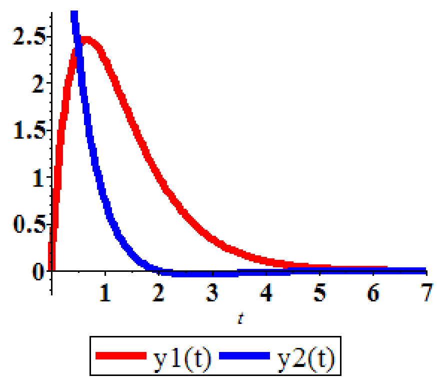
$$\to \boxed{C_1=\frac{-3e^3}{-6}=\frac{1}{2}e^3,\ C_2=-\frac{e^3}{6}}$$

$$y(t)=\frac{1}{2}e^3\begin{pmatrix}-3\\ 1\end{pmatrix}e^{-3t/2}-\frac{1}{6}e^3\begin{pmatrix}-9-3t\\ 1+t\end{pmatrix}e^{-3t/2}$$

$$=\begin{pmatrix}-\frac{3}{2}+\frac{3}{2}+\frac{1}{2}t\\ \frac{1}{2}-\frac{1}{6}-\frac{1}{6}t\end{pmatrix}e^{-\frac{3t}{2}+3}$$

$$=\begin{pmatrix}\frac{1}{2}t\\ \frac{1}{3}-\frac{1}{6}t\end{pmatrix}e^{-\frac{3t}{2}+3}$$

$$=\boxed{\begin{pmatrix}0\\ \frac{1}{3}\end{pmatrix}e^{-\frac{3t}{2}+3}+t\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{6}\end{pmatrix}e^{-\frac{3t}{2}+3}}$$



$$\boxed{\begin{cases}y_1(t)=\frac{1}{2}te^{-\frac{3t}{2}+3}\\ y_2(t)=\left(\frac{1}{3}-\frac{1}{6}t\right)e^{-\frac{3t}{2}+3}\end{cases}}$$

Exercise

Find the general solution of the system $y'=Ay$ $\begin{cases}y_1'(t)=3y_1-3y_2+2\\ y_2'(t)=-6y_1-t\end{cases}$ $y(0)=\begin{pmatrix}1\\ -1\end{pmatrix}$

Solution

$$\left|A-\lambda I\right|=\begin{vmatrix}3-\lambda & -3\\ -6 & -\lambda\end{vmatrix} \qquad A=\begin{pmatrix}3 & -3\\ -6 & 0\end{pmatrix}$$

$$=\lambda^2-3\lambda-18=0$$

Thus, the eigenvalues are: $\boxed{\lambda_{1,2}=-3,\ 6}$

For $\lambda_1=-3 \quad \Rightarrow \left(A-\lambda_1 I\right)V_1=0$

$$\begin{pmatrix}6 & -3\\ -6 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \to \boxed{2x=y} \quad \Rightarrow V_1=\begin{pmatrix}1\\ 2\end{pmatrix}$$

For $\lambda_2=6 \quad \Rightarrow \left(A-\lambda_2 I\right)V_2=0$

$$\begin{pmatrix} -3 & -3 \\ -6 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{x=-y} \quad \Rightarrow V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\underline{y_h = C_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-3t} + C_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{6t}}$$

$$\varphi(t) = \begin{pmatrix} e^{-3t} & -e^{6t} \\ 2e^{-3t} & e^{6t} \end{pmatrix}$$

$$\varphi^{-1}(t) = \frac{1}{3e^{3t}}\begin{pmatrix} e^{6t} & e^{6t} \\ -2e^{-3t} & e^{-3t} \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} e^{3t} & e^{3t} \\ -2e^{-6t} & e^{-6t} \end{pmatrix}$$

$$\varphi^{-1}\cdot\begin{pmatrix} 2 \\ -t \end{pmatrix} = \frac{1}{3}\begin{pmatrix} e^{3t} & e^{3t} \\ -2e^{-6t} & e^{-6t} \end{pmatrix}\begin{pmatrix} 2 \\ -t \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2e^{3t} - te^{3t} \\ -4e^{-6t} - te^{-6t} \end{pmatrix}$$

		$\int e^{3t}$
+	t	$\frac{1}{3}e^{3t}$
–	1	$\frac{1}{9}e^{3t}$

		$\int e^{-6t}$
+	t	$-\frac{1}{6}e^{-6t}$
–	1	$\frac{1}{36}e^{-6t}$

$$X = \frac{1}{3}\int\begin{pmatrix} 2e^{3t} - te^{3t} \\ -4e^{-6t} - te^{-6t} \end{pmatrix}dt$$

$$= \frac{1}{3}\begin{pmatrix} \frac{2}{3}e^{3t} - \frac{1}{3}te^{3t} + \frac{1}{9}e^{3t} \\ \frac{2}{3}e^{-6t} + \frac{1}{6}te^{-6t} + \frac{1}{36}e^{-6t} \end{pmatrix}$$

$$i_p(t) = \varphi X = \frac{1}{3}\begin{pmatrix} e^{-3t} & -e^{6t} \\ 2e^{-3t} & e^{6t} \end{pmatrix}\begin{pmatrix} \frac{2}{3}e^{3t} - \frac{1}{3}te^{3t} + \frac{1}{9}e^{3t} \\ \frac{2}{3}e^{-6t} + \frac{1}{6}te^{-6t} + \frac{1}{36}e^{-6t} \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} \frac{2}{3} - \frac{1}{3}t + \frac{1}{9} - \frac{2}{3} - \frac{1}{6}t - \frac{1}{36} \\ \frac{4}{3} - \frac{2}{3}t + \frac{2}{9} + \frac{2}{3} + \frac{1}{6}t + \frac{1}{36} \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} -\frac{1}{2}t + \frac{1}{12} \\ -\frac{1}{2}t + \frac{81}{36} \end{pmatrix}$$

$$y(t) = \begin{pmatrix} C_1e^{-3t} - C_2e^{6t} \\ 2C_1e^{-3t} + C_2e^{6t} \end{pmatrix} + \begin{pmatrix} -\frac{1}{6}t + \frac{1}{36} \\ -\frac{1}{6}t + \frac{81}{108} \end{pmatrix}$$

$$= \begin{pmatrix} C_1 e^{-3t} - C_2 e^{6t} - \frac{1}{6}t + \frac{1}{36} \\ 2C_1 e^{-3t} + C_2 e^{6t} - \frac{1}{6}t + \frac{81}{108} \end{pmatrix}$$

Given: $y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$y(0) = \begin{pmatrix} C_1 - C_2 + \frac{1}{36} \\ 2C_1 + C_2 + \frac{81}{108} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} C_1 - C_2 + \frac{1}{36} = 1 \\ 2C_1 + C_2 + \frac{81}{108} = -1 \end{cases} \rightarrow \begin{cases} C_1 - C_2 = \frac{35}{36} \\ 2C_1 + C_2 = -\frac{189}{108} \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3 \quad \Delta_1 = \begin{vmatrix} \frac{35}{36} & -1 \\ -\frac{189}{108} & 1 \end{vmatrix} = -\frac{84}{108} \quad \Delta_2 = \begin{vmatrix} 1 & \frac{35}{36} \\ 2 & -\frac{189}{108} \end{vmatrix} = -\frac{133}{36}$$

$$\underline{C_1 = -\frac{7}{27} \quad C_2 = -\frac{133}{108}}$$

$$y(t) = \begin{pmatrix} -\frac{7}{27}e^{-3t} - \frac{133}{108}e^{6t} - \frac{1}{6}t + \frac{1}{36} \\ -\frac{14}{27}e^{-3t} + \frac{133}{108}e^{6t} - \frac{1}{6}t + \frac{81}{108} \end{pmatrix}$$

$$\begin{cases} y_1(t) = -\frac{7}{27}e^{-3t} - \frac{133}{108}e^{6t} - \frac{1}{6}t + \frac{1}{36} \\ y_2(t) = -\frac{14}{27}e^{-3t} + \frac{133}{108}e^{6t} - \frac{1}{6}t + \frac{81}{108} \end{cases}$$

Exercise

Find the general solution of the system $y' = Ay$ $\begin{cases} y_1'(t) = -5y_1 + y_2 + 6e^{2t} \\ y_2'(t) = 4y_1 - 2y_2 - e^{2t} \end{cases}$ $y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} -5-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix}$$

$$= \lambda^2 + 7\lambda + 6 = 0$$

Thus, the eigenvalues are: $\underline{\lambda_{1,2} = -1, \ -6}$

For $\lambda_1 = -1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -4 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \underline{4x = y} \quad \Rightarrow V_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

For $\lambda_2 = -6 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \underline{x = -y|} \quad \Rightarrow \quad V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\boxed{y_h = C_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-6t}}$$

$$\varphi(t) = \begin{pmatrix} e^{-t} & -e^{-6t} \\ 4e^{-t} & e^{-6t} \end{pmatrix}$$

$$\varphi^{-1}(t) = \frac{1}{5e^{-7t}} \begin{pmatrix} e^{-6t} & e^{-6t} \\ -4e^{-t} & e^{-t} \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} e^{t} & e^{t} \\ -4e^{6t} & e^{6t} \end{pmatrix}$$

$$\varphi^{-1} \cdot \begin{pmatrix} 6e^{2t} \\ -e^{2t} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} e^{t} & e^{t} \\ -4e^{6t} & e^{6t} \end{pmatrix} \begin{pmatrix} 6e^{2t} \\ -e^{2t} \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 5e^{3t} \\ -25e^{8t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{3t} \\ -5e^{8t} \end{pmatrix}$$

$$X = \int \begin{pmatrix} e^{3t} \\ -5e^{8t} \end{pmatrix} dt$$

$$= \begin{pmatrix} \frac{1}{3} e^{3t} \\ -\frac{5}{8} e^{8t} \end{pmatrix}$$

$$i_p(t) = \varphi X = \begin{pmatrix} e^{-t} & -e^{-6t} \\ 4e^{-t} & e^{-6t} \end{pmatrix} \begin{pmatrix} \frac{1}{3} e^{3t} \\ -\frac{5}{8} e^{8t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{23}{24} e^{2t} \\ \frac{17}{24} e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{23}{24} \\ \frac{17}{24} \end{pmatrix} e^{2t}$$

$$\boxed{y(t) = C_1\begin{pmatrix}1\\4\end{pmatrix}e^{-t} + C_2\begin{pmatrix}-1\\1\end{pmatrix}e^{-6t} + \begin{pmatrix}\frac{23}{24}\\ \frac{17}{24}\end{pmatrix}e^{2t}}$$

Given: $y(0) = \begin{pmatrix}1\\-1\end{pmatrix}$

$$y(0) = \begin{pmatrix}C_1 - C_2 + \frac{23}{24}\\ 4C_1 + C_2 + \frac{17}{24}\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$$

$$\begin{cases} C_1 - C_2 + \frac{23}{24} = 1\\ 4C_1 + C_2 + \frac{17}{24} = -1\end{cases} \rightarrow \begin{cases} C_1 - C_2 = \frac{1}{24}\\ 4C_1 + C_2 = -\frac{41}{24}\end{cases}$$

$$\Delta = \begin{vmatrix}1 & -1\\ 4 & 1\end{vmatrix} = 5 \quad \Delta_1 = \begin{vmatrix}\frac{1}{24} & -1\\ -\frac{41}{24} & 1\end{vmatrix} = -\frac{5}{3} \quad \Delta_2 = \begin{vmatrix}1 & \frac{1}{24}\\ 4 & -\frac{41}{24}\end{vmatrix} = -\frac{15}{8}$$

$$\boxed{C_1 = -\frac{1}{3} \quad C_2 = -\frac{3}{8}}$$

$$y(t) = -\frac{1}{3}\begin{pmatrix}1\\4\end{pmatrix}e^{-t} - \frac{3}{8}\begin{pmatrix}-1\\1\end{pmatrix}e^{-6t} + \begin{pmatrix}\frac{23}{24}\\ \frac{17}{24}\end{pmatrix}e^{2t}$$

$$\boxed{\begin{cases} y_1(t) = -\frac{1}{3}e^{-t} + \frac{3}{8}e^{-6t} + \frac{23}{24}e^{2t}\\ y_2(t) = -\frac{4}{3}e^{-t} - \frac{3}{8}e^{-6t} + \frac{17}{24}e^{2t}\end{cases}}$$

Exercise

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = y_1 + 2y_2 + 2t\\ y_2'(t) = 3y_1 + 2y_2 - 4t\end{cases} \quad y(0) = \begin{pmatrix}1\\1\end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix}1-\lambda & 2\\ 3 & 2-\lambda\end{vmatrix} \qquad A = \begin{pmatrix}1 & 2\\ 3 & 2\end{pmatrix}$$

$$= \lambda^2 - 3\lambda - 4 = 0$$

Thus, the eigenvalues are: $\boxed{\lambda_{1,2} = -1,\ 4}$

For $\lambda_1 = -1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix}2 & 2\\ 3 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \rightarrow \boxed{x = -y} \Rightarrow V_1 = \begin{pmatrix}-1\\1\end{pmatrix}$$

For $\lambda_2 = 4 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \underline{3x = 2y} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\boxed{y_h = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t}}$$

$$\varphi(t) = \begin{pmatrix} -e^{-t} & 2e^{4t} \\ e^{-t} & 3e^{4t} \end{pmatrix}$$

$$\varphi^{-1}(t) = -\frac{1}{5e^{3t}} \begin{pmatrix} 3e^{4t} & -2e^{4t} \\ -e^{-t} & -e^{-t} \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} -3e^{t} & 2e^{t} \\ e^{-4t} & e^{-4t} \end{pmatrix}$$

		$\int e^t$
+	t	e^t
−	1	e^t

$$\varphi^{-1} \cdot \begin{pmatrix} 2t \\ -4t \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -3e^{t} & 2e^{t} \\ e^{-4t} & e^{-4t} \end{pmatrix} \begin{pmatrix} 2t \\ -4t \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} -14te^{t} \\ -2te^{-4t} \end{pmatrix}$$

		$\int e^{-4t}$
+	t	$-\frac{1}{4}e^{-4t}$
−	1	$\frac{1}{16}e^{-4t}$

$$= -\frac{2}{5} \begin{pmatrix} 7te^{t} \\ te^{-4t} \end{pmatrix}$$

$$X = -\frac{2}{5} \int \begin{pmatrix} 7te^{t} \\ te^{-4t} \end{pmatrix} dt$$

$$= -\frac{2}{5} \begin{pmatrix} 7\left(te^{t} - 1\right)e^{t} \\ -\left(\frac{1}{4}t + \frac{1}{16}\right)e^{-4t} \end{pmatrix}$$

$$i_p(t) = \varphi X = -\frac{2}{5} \begin{pmatrix} -e^{-t} & 2e^{4t} \\ e^{-t} & 3e^{4t} \end{pmatrix} \begin{pmatrix} 7\left(te^{t} - 1\right)e^{t} \\ -\left(\frac{1}{4}t + \frac{1}{16}\right)e^{-4t} \end{pmatrix}$$

$$= -\frac{2}{5} \begin{pmatrix} -7t + 7 - \frac{1}{2}t - \frac{1}{8} \\ 7t - 7 - \frac{3}{4}t - \frac{3}{16} \end{pmatrix}$$

$$= -\frac{2}{5} \begin{pmatrix} -\frac{15}{2}t + \frac{55}{8} \\ \frac{25}{4}t - \frac{115}{16} \end{pmatrix}$$

$$= \begin{pmatrix} 3t - \frac{11}{4} \\ -\frac{5}{2}t + \frac{23}{8} \end{pmatrix}$$

$$\left. y(t) = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t} + \begin{pmatrix} 3t - \frac{11}{4} \\ -\frac{5}{2}t + \frac{23}{8} \end{pmatrix} \right| \qquad y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$y(0) = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -\frac{11}{4} \\ \frac{23}{8} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{cases} -C_1 + 2C_2 - \frac{11}{4} = 1 \\ C_1 + 3C_2 + \frac{23}{8} = 1 \end{cases} \rightarrow \begin{cases} -C_1 + 2C_2 = \frac{15}{4} \\ C_1 + 3C_2 = -\frac{15}{8} \end{cases}$$

$$\Delta = \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} = -5 \quad \Delta_1 = \begin{vmatrix} \frac{15}{4} & 2 \\ -\frac{15}{8} & 3 \end{vmatrix} = 15 \quad \Delta_2 = \begin{vmatrix} -1 & \frac{15}{4} \\ 1 & -\frac{15}{8} \end{vmatrix} = -\frac{15}{8}$$

$$\underline{C_1 = -3 \quad C_2 = \frac{3}{8}}$$

$$y(t) = -3 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + \frac{3}{8} \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t} + \begin{pmatrix} 3t - \frac{11}{4} \\ -\frac{5}{2}t + \frac{23}{8} \end{pmatrix}$$

$$\begin{cases} y_1(t) = -3e^{-t} + \frac{3}{4}e^{4t} + 3t - \frac{11}{4} \\ y_2(t) = 3e^{-t} + \frac{9}{8}e^{4t} - \frac{5}{2}t + \frac{23}{8} \end{cases}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 3x_1 - x_2 + 4e^{2t} \\ x_2'(t) = -x_1 + 3x_2 + 4e^{4t} \end{cases}$ $\quad X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

$$= \lambda^2 - 6\lambda + 8 = 0$$

The eigenvalues: $\lambda_{1,2} = 2,\ 4$

For $\lambda_1 = 2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = y \qquad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 4 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = -y \qquad \Rightarrow \quad V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\boxed{X_h = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t}}$$

$$\varphi(t) = \begin{pmatrix} e^{2t} & -e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix}$$

$$\varphi^{-1} = \frac{1}{2e^{6t}} \begin{pmatrix} e^{4t} & e^{4t} \\ -e^{2t} & e^{2t} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} e^{-2t} & e^{-2t} \\ -e^{-4t} & e^{-4t} \end{pmatrix}$$

$$F(t) = \begin{pmatrix} 4e^{2t} \\ 4e^{4t} \end{pmatrix}$$

$$\varphi^{-1}(t)F(t) = \frac{1}{2} \begin{pmatrix} e^{-2t} & e^{-2t} \\ -e^{-4t} & e^{-4t} \end{pmatrix}\begin{pmatrix} 4e^{2t} \\ 4e^{4t} \end{pmatrix}$$

$$= \begin{pmatrix} 2 + 2e^{2t} \\ 2 - 2e^{-2t} \end{pmatrix}$$

$$X_p = \begin{pmatrix} e^{2t} & -e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix} \int \begin{pmatrix} 2 + 2e^{2t} \\ 2 - 2e^{-2t} \end{pmatrix} dt \qquad\qquad X_p = \varphi(t) \int \varphi^{-1}(t) F(t)\, dt$$

$$= \begin{pmatrix} e^{2t} & -e^{4t} \\ e^{2t} & e^{4t} \end{pmatrix}\begin{pmatrix} 2t + e^{2t} \\ 2t + e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2te^{2t} + e^{4t} - 2te^{4t} - e^{2t} \\ 2te^{2t} + e^{4t} + 2te^{4t} + e^{2t} \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}}$$

$$X(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

$$X(0) = C_1\begin{pmatrix}1\\1\end{pmatrix} + C_2\begin{pmatrix}-1\\1\end{pmatrix} + \begin{pmatrix}-1\\1\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{cases} C_1 - C_2 = 1 \\ C_1 + C_2 = -1 \end{cases} \rightarrow \underline{C_1 = 0,\ C_2 = -1|}$$

$$\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} = -\begin{pmatrix}-1\\1\end{pmatrix}e^{4t} + \begin{pmatrix}2\\2\end{pmatrix}te^{2t} + \begin{pmatrix}-1\\1\end{pmatrix}e^{2t} + \begin{pmatrix}-2\\2\end{pmatrix}te^{4t} + \begin{pmatrix}1\\1\end{pmatrix}e^{4t}$$

$$\begin{cases} x_1(t) = 2e^{4t} - 2te^{4t} + 2te^{2t} - e^{2t} \\ x_2(t) = 2te^{4t} + 2te^{2t} + e^{2t} \end{cases}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = x_1 - x_2 + \frac{1}{t} \\ x_2'(t) = x_1 - x_2 + \frac{1}{t} \end{cases}$ $\quad X(1) = \begin{pmatrix}2\\-1\end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix}1 & -1\\1 & -1\end{pmatrix}$$

$$= \lambda^2 = 0$$

The eigenvalues: $\underline{\lambda_{1,2} = 0,\ 0|}$

For $\lambda_1 = 0 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix}1 & -1\\1 & -1\end{pmatrix}\begin{pmatrix}a_1\\b_1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad \rightarrow a_1 = b_1 \quad \Rightarrow \quad V_1 = \begin{pmatrix}1\\1\end{pmatrix}$$

For $V_2 \quad \Rightarrow (A - \lambda I)V_2 = V_1$

$$\begin{pmatrix}1 & -1\\1 & -1\end{pmatrix}\begin{pmatrix}a_2\\b_2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \quad \rightarrow a_2 - b_2 = 1 \quad \Rightarrow \quad V_2 = \begin{pmatrix}1\\0\end{pmatrix}$$

$$y_2(t) = V_2 + tV_1$$

$$= \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}t$$

$$= \begin{pmatrix}1+t\\t\end{pmatrix}$$

$$\underline{X_h = C_1\begin{pmatrix}1\\1\end{pmatrix} + C_2\begin{pmatrix}1+t\\t\end{pmatrix}|}$$

$$\varphi(t)=\begin{pmatrix}1 & 1+t\\ 1 & t\end{pmatrix}$$

$$\varphi^{-1}=-\begin{pmatrix}t & -1-t\\ -1 & 1\end{pmatrix}$$

$$=\begin{pmatrix}-t & 1+t\\ 1 & -1\end{pmatrix}$$

$$F(t)=\begin{pmatrix}\frac{1}{t}\\ \frac{1}{t}\end{pmatrix}$$

$$\varphi^{-1}(t)F(t)=\begin{pmatrix}-t & 1+t\\ 1 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{t}\\ \frac{1}{t}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{t}\\ 0\end{pmatrix}$$

$$X_p=\begin{pmatrix}1 & 1+t\\ 1 & t\end{pmatrix}\int\begin{pmatrix}\frac{1}{t}\\ 0\end{pmatrix}dt$$

$$X_p=\varphi(t)\int\varphi^{-1}(t)F(t)\,dt$$

$$=\begin{pmatrix}1 & 1+t\\ 1 & t\end{pmatrix}\begin{pmatrix}\ln t\\ 0\end{pmatrix}$$

$$=\boxed{\begin{pmatrix}\ln t\\ \ln t\end{pmatrix}}$$

$$X(t)=C_1\begin{pmatrix}1\\ 1\end{pmatrix}+C_2\begin{pmatrix}1+t\\ t\end{pmatrix}+\begin{pmatrix}\ln t\\ \ln t\end{pmatrix}$$

$$X(1)=C_1\begin{pmatrix}1\\ 1\end{pmatrix}+C_2\begin{pmatrix}2\\ 1\end{pmatrix}+\begin{pmatrix}0\\ 0\end{pmatrix}=\begin{pmatrix}2\\ -1\end{pmatrix}$$

$$\begin{cases}C_1+2C_2=2\\ C_1+C_2=-1\end{cases}\rightarrow\ \boxed{C_1=\frac{4}{-1}=-4,\ \ C_2=3}$$

$$\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}=-4\begin{pmatrix}1\\ 1\end{pmatrix}+3\begin{pmatrix}1+t\\ t\end{pmatrix}+\begin{pmatrix}\ln t\\ \ln t\end{pmatrix}$$

$$\boxed{\begin{cases}x_1(t)=-1+3t+\ln t\\ x_2(t)=-4+3t+\ln t\end{cases}}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 3x_1 - 2x_2 - 2e^{-t} \\ x_2'(t) = x_1 \quad\;\; - 2e^{-t} \end{cases}$ $\quad X(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

The eigenvalues: $\lambda_{1,2} = 1,\ 2$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = y \qquad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to x = 2y \qquad \Rightarrow \quad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$X_h = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{t} + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{2t}$$

$$\varphi(t) = \begin{pmatrix} e^{t} & 2e^{2t} \\ e^{t} & e^{2t} \end{pmatrix}$$

$$\varphi^{-1} = -\frac{1}{e^{3t}}\begin{pmatrix} e^{2t} & -2e^{2t} \\ -e^{t} & e^{t} \end{pmatrix}$$

$$= \begin{pmatrix} -e^{-t} & 2e^{-t} \\ e^{-2t} & -e^{-2t} \end{pmatrix}$$

$$F(t) = \begin{pmatrix} -2e^{-t} \\ -2e^{-t} \end{pmatrix}$$

$$\varphi^{-1}(t)F(t) = \begin{pmatrix} -e^{-t} & 2e^{-t} \\ e^{-2t} & -e^{-2t} \end{pmatrix}\begin{pmatrix} -2e^{-t} \\ -2e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} -2e^{-2t} \\ 0 \end{pmatrix}$$

$$X_p = \begin{pmatrix} e^{t} & 2e^{2t} \\ e^{t} & e^{2t} \end{pmatrix}\int\begin{pmatrix} -2e^{-2t} \\ 0 \end{pmatrix}dt \qquad X_p = \varphi(t)\int \varphi^{-1}(t)F(t)\,dt$$

$$=\begin{pmatrix} e^t & 2e^{2t} \\ e^t & e^{2t} \end{pmatrix}\begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix}$$

$$=\boxed{\begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}}$$

$$X(t) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{2t} + \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$$

$$X(0) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{cases} C_1 + 2C_2 = 1 \\ C_1 + C_2 = -2 \end{cases} \quad \rightarrow \quad \boxed{C_1 = \frac{5}{-1} = -5, \ \ C_2 = 3}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = -5\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + 3\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{2t} + \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$$

$$\boxed{\begin{cases} x_1(t) = -5e^t + 6e^{2t} + e^{-t} \\ x_2(t) = -5e^t + 3e^{2t} + e^{-t} \end{cases}}$$

Exercise

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = y_1 \\ y_2'(t) = -4y_1 + y_2 \\ y_3'(t) = 3y_1 + 6y_2 + 2y_3 \end{cases} \qquad y(0) = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -4 & 1-\lambda & 0 \\ 3 & 6 & 2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix}$$

$$= (1-\lambda)^2(2-\lambda) = 0$$

Thus, the eigenvalues are: $\boxed{\lambda_1 = 2 \ \ \& \ \ \lambda_{2,3} = 1}$

For $\lambda_1 = 2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & 6 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow x = y = 0 \quad V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{matrix} x = 0 \\ 6y = -z \end{matrix} \quad V_2 = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}$$

For $V_3 \quad \Rightarrow (A - \lambda I)V_3 = V_2$

$$\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} \quad \rightarrow \quad \begin{matrix} -4x = 1 \\ 6y + z = -\frac{21}{4} \end{matrix} \quad V_3 = \begin{pmatrix} -\frac{1}{4} \\ -\frac{21}{24} \\ 0 \end{pmatrix}$$

$$y_3(t) = \left(V_3 + tV_2\right)e^t$$

$$= \begin{pmatrix} -\frac{1}{4} \\ -\frac{21}{24} + t \\ -6t \end{pmatrix} e^t$$

$$y(t) = C_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t} + \left(C_2 \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} + C_3 \begin{pmatrix} -\frac{1}{4} \\ -\frac{21}{24} + t \\ -6t \end{pmatrix} \right) e^t$$

$$y(0) = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} -\frac{1}{4}C_3 \\ C_2 - \frac{21}{24}C_3 \\ C_1 - 6C_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix} \qquad C_3 = 4, \quad C_2 = \frac{33}{6}, \quad C_1 = 3$$

$$y(t) = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} e^{2t} + \begin{pmatrix} -1 \\ 2 + 4t \\ -33 - 24t \end{pmatrix} e^t$$

$$\boxed{\begin{cases} y_1(t) = -e^t \\ y_2(t) = (2 + 4t)e^t \\ y_2(t) = 3e^{2t} - (33 + 24t)e^t \end{cases}}$$

Exercise

Find the general solution of the system $y' = Ay$ $\quad \begin{cases} y_1'(t) = -\frac{5}{2}y_1 + y_2 + y_3 \\ y_2'(t) = y_1 - \frac{5}{2}y_2 + y_3 \\ y_3'(t) = y_1 + y_2 - \frac{5}{2}y_3 \end{cases} \quad y(0) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

Solution

$$|A-\lambda I| = \begin{vmatrix} -\frac{5}{2}-\lambda & 1 & 1 \\ 1 & -\frac{5}{2}-\lambda & 1 \\ 1 & 1 & -\frac{5}{2}-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{pmatrix}$$

$$= -\left(\frac{5}{2}+\lambda\right)^3 + 2 + 3\left(\frac{5}{2}+\lambda\right)$$

$$= -\frac{49}{8} - \frac{63}{4}\lambda - \frac{15}{2}\lambda^2 - \lambda^3 = 0$$

$$8\lambda^3 + 60\lambda^2 + 126\lambda + 49 = 0$$

$$\begin{array}{r|rrrr} -\frac{1}{2} & 8 & 60 & 126 & 49 \\ & & -4 & -28 & -49 \\ \hline & 8 & 56 & 98 & 0 \end{array} \rightarrow \underline{8\lambda^2 + 56\lambda + 98 = 0}$$

Thus, the eigenvalues are: $\lambda_1 = -\frac{1}{2}$ & $\lambda_{2,3} = -\frac{7}{2}$

For $\lambda_1 = -\frac{1}{2} \quad \Rightarrow (A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{aligned} -2x+y+z&=0 \\ x-2y+z&=0 \\ x+y-2z&=0 \end{aligned}$$

$$z=1 \quad \rightarrow \begin{cases} -2x+y=-1 \\ x-2y=-1 \end{cases} \quad \Delta = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 3 \quad \Delta_x = \begin{vmatrix} -2 & -1 \\ 1 & -1 \end{vmatrix} = 3$$

$$\rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_1 = -\frac{7}{2} \quad \Rightarrow (A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow x+y+z=1$$

$$z=0 \quad \rightarrow x+y=1 \quad y=1 \Rightarrow x=-1 \qquad \rightarrow \quad V_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$y=0 \quad \rightarrow x+z=1 \quad z=1 \Rightarrow x=-1 \qquad \rightarrow \quad V_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$y(t) = C_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^{-t/2} + \left(C_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + C_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right)e^{-7t/2}$$

$$y(0) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} C_1 - C_2 - C_3 \\ C_1 + C_2 \\ C_1 + C_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$\begin{cases} C_1 - C_2 - C_3 = 2 \\ C_1 + C_2 = 3 \\ C_1 + C_3 = -1 \end{cases} \qquad \Delta = \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 3 \quad \Delta_1 = \begin{vmatrix} 2 & -1 & -1 \\ 3 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 4$$

$$\underline{C_1 = \frac{4}{3}, \quad C_2 = \frac{5}{3}, \quad C_3 = -\frac{7}{3}}$$

$$y(t) = \frac{4}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-t/2} + \left(\frac{5}{3}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \frac{7}{3}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right) e^{-7t/2}$$

$$\begin{cases} y_1(t) = \frac{4}{3}e^{-t/2} + \frac{2}{3}e^{-7t/2} \\ y_2(t) = \frac{4}{3}e^{-t/2} + \frac{5}{3}e^{-7t/2} \\ y_2(t) = \frac{4}{3}e^{-t/2} - \frac{7}{3}e^{-7t/2} \end{cases}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = 3x_1 - x_2 - x_3 \\ x_2'(t) = x_1 + x_2 - x_3 + t \\ x_3'(t) = x_1 - x_2 + x_3 + 2e^t \end{cases}$ $\qquad X(0) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & -1 \\ 1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \left(1 - 2\lambda + \lambda^2\right)(3-\lambda) + 2 + 2(1-\lambda) - (3-\lambda)$$

$$= 3 - 6\lambda + 3\lambda^2 - \lambda + 2\lambda^2 - \lambda^3 + 1 - \lambda$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

$$\begin{array}{c|cccc} 1 & -1 & 5 & -8 & 4 \\ & & -1 & 4 & -4 \\ \hline & -1 & 4 & -4 & 0 \end{array} \quad \to \quad -\lambda^2 + 4\lambda - 4 = 0$$

The eigenvalues: $\underline{\lambda_{1,2,3} = 1, \ 2, \ 2}$

For $\lambda_1 = 1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} 2x = y + z \\ x = z \\ x = y \end{cases} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} x = y + z \\ \end{cases} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

For $V_3 \quad \Rightarrow \left(A - \lambda_2 I\right)V_3 = V_2$

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} V_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad x - y - z = 0 \rightarrow \begin{cases} y = 0 \\ x = z \end{cases} \quad \Rightarrow \quad V_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{X_h = C_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + C_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}}$$

$$\varphi(t) = \begin{pmatrix} e^t & e^{2t} & e^{2t} \\ e^t & e^{2t} & 0 \\ e^t & 0 & e^{2t} \end{pmatrix}$$

$$\varphi^{-1} = \begin{pmatrix} -e^{-t} & e^{-t} & e^{-t} \\ e^{-2t} & 0 & -e^{-2t} \\ e^{-2t} & -e^{-2t} & 0 \end{pmatrix}$$

$$F(t) = \begin{pmatrix} 0 \\ t \\ 2e^t \end{pmatrix}$$

$$\varphi^{-1}(t)F(t) = \begin{pmatrix} -e^{-t} & e^{-t} & e^{-t} \\ e^{-2t} & 0 & -e^{-2t} \\ e^{-2t} & -e^{-2t} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ t \\ 2e^t \end{pmatrix}$$

$$= \begin{pmatrix} te^{-t} + 2 \\ -2e^{-t} \\ -te^{-2t} \end{pmatrix}$$

$$\int \left(te^{-t} + 2\right) dt = \underline{(-t-1)e^{-t} + 2t}$$

$$\int -2e^{-t} dt = \underline{2e^{-t}}$$

$$\int -te^{-2t} dt = \underline{\left(\frac{1}{2}t + \frac{1}{4}\right)e^{-2t}}$$

$$X_p=\begin{pmatrix} e^t & e^{2t} & e^{2t} \\ e^t & e^{2t} & 0 \\ e^t & 0 & e^{2t} \end{pmatrix}\begin{pmatrix} (-t-1)e^{-t}+2t \\ 2e^{-t} \\ \left(\frac{1}{2}t+\frac{1}{4}\right)e^{-2t} \end{pmatrix} \qquad X_p=\varphi(t)\int\varphi^{-1}(t)F(t)\,dt$$

$$=\begin{pmatrix} -t-1+(2t+2)e^t+\frac{1}{2}t+\frac{1}{4} \\ -t-1+2te^t+2e^t \\ -t-1+2te^t+\frac{1}{2}t+\frac{1}{4} \end{pmatrix}$$

$$=\begin{pmatrix} -\frac{1}{2} \\ -1 \\ -\frac{1}{2} \end{pmatrix}t+\begin{pmatrix} -\frac{3}{4} \\ -1 \\ -\frac{3}{4} \end{pmatrix}+\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}e^t+\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}te^t$$

$$X(t)=C_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^t+C_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{2t}+C_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}e^{2t}+\begin{pmatrix} -\frac{1}{2} \\ -1 \\ -\frac{1}{2} \end{pmatrix}t+\begin{pmatrix} -\frac{3}{4} \\ -1 \\ -\frac{3}{4} \end{pmatrix}+\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}e^t+\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}te^t$$

$$X(0)=C_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}+C_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}+C_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}+\begin{pmatrix} -\frac{3}{4} \\ -1 \\ -\frac{3}{4} \end{pmatrix}+\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} C_1+C_2+C_3-\frac{3}{4}+2=1 \\ C_1+C_2-1+2=1 \\ C_1+C_3-\frac{3}{4}=1 \end{cases}$$

$$\begin{cases} C_1+C_2+C_3=-\frac{1}{4} \\ C_1+C_2=0 \\ C_1+C_3=\frac{7}{4} \end{cases} \qquad \Delta=\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}=-1 \quad \Delta_1=\begin{vmatrix} -\frac{1}{4} & 1 & 1 \\ 0 & 1 & 0 \\ \frac{7}{4} & 0 & 1 \end{vmatrix}=-2 \quad \Delta_2=\begin{vmatrix} 1 & -\frac{1}{4} & 1 \\ 1 & 0 & 0 \\ 1 & \frac{7}{4} & 1 \end{vmatrix}=2$$

$$\underline{C_1=2,\quad C_2=-2,\quad C_3=-\frac{1}{4}}$$

$$X(t)=\begin{pmatrix} 2+2 \\ 2+2 \\ 2 \end{pmatrix}e^t+\begin{pmatrix} -2-\frac{1}{4} \\ 2 \\ -\frac{1}{4} \end{pmatrix}e^{2t}+\begin{pmatrix} -\frac{1}{2} \\ -1 \\ -\frac{1}{2} \end{pmatrix}t+\begin{pmatrix} -\frac{3}{4} \\ -1 \\ -\frac{3}{4} \end{pmatrix}+\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}te^t$$

$$\boxed{\begin{cases} x_1(t) = 4e^t - \frac{9}{4}e^{2t} - \frac{3}{4} - \frac{1}{2}t + 2te^t \\ x_2(t) = 4e^t + 2e^{2t} - 1 - t + 2te^t \\ x_3(t) = 2e^t - \frac{1}{4}e^{2t} - \frac{3}{4} - \frac{1}{2}t + 2te^t \end{cases}}$$

Exercise

Find the general solution of the system $\begin{cases} x_1'(t) = x_1 + x_2 \ + e^t \\ x_2'(t) = x_1 + x_2 \ + e^{2t} \\ x_3'(t) = \qquad 3x_3 + te^{3t} \end{cases}$ $\quad X(0) = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \left(1 - 2\lambda + \lambda^2\right)(3-\lambda) - (3-\lambda)$$

$$= (3-\lambda)\left(\lambda^2 - 2\lambda\right) = 0$$

The eigenvalues: $\boxed{\lambda_{1,2,3} = 0,\ 2,\ 3}$

For $\lambda_1 = 0 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} x = -y \\ z = 0 \end{cases} \quad \Rightarrow \quad V_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} x = y \\ z = 0 \end{cases} \quad \Rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_3 = 3 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} 2x = y \\ x = 2y \end{cases} \quad \Rightarrow \quad V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{X_h = C_1\begin{pmatrix}-1\\1\\0\end{pmatrix} + C_2\begin{pmatrix}1\\1\\0\end{pmatrix}e^{2t} + C_3\begin{pmatrix}0\\0\\1\end{pmatrix}e^{3t}}$$

$$\varphi(t) = \begin{pmatrix}-1 & e^{2t} & 0\\ 1 & e^{2t} & 0\\ 0 & 0 & e^{3t}\end{pmatrix}$$

$$\varphi^{-1} = \begin{pmatrix}\frac{1}{2} & -\frac{1}{2} & 0\\ \frac{1}{2}e^{-2t} & \frac{1}{2}e^{-2t} & 0\\ 0 & 0 & e^{-3t}\end{pmatrix}$$

$$F(t) = \begin{pmatrix}e^{t}\\ e^{2t}\\ te^{3t}\end{pmatrix}$$

$$\varphi^{-1}(t)F(t) = \begin{pmatrix}\frac{1}{2} & -\frac{1}{2} & 0\\ \frac{1}{2}e^{-2t} & \frac{1}{2}e^{-2t} & 0\\ 0 & 0 & e^{-3t}\end{pmatrix}\begin{pmatrix}e^{t}\\ e^{2t}\\ te^{3t}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{2}e^{t} - \frac{1}{2}e^{2t}\\ \frac{1}{2}e^{-t} + \frac{1}{2}\\ t\end{pmatrix}$$

$$\int\left(\frac{1}{2}e^{t} - \frac{1}{2}e^{2t}\right)dt = \boxed{\frac{1}{2}e^{t} - \frac{1}{4}e^{2t}}$$

$$\int\left(\frac{1}{2}e^{-t} + \frac{1}{2}\right)dt = \boxed{-\frac{1}{2}e^{-t} + \frac{1}{2}t}$$

$$\int t\,dt = \boxed{\frac{1}{2}t^2}$$

$$X_p = \begin{pmatrix}-1 & e^{2t} & 0\\ 1 & e^{2t} & 0\\ 0 & 0 & e^{3t}\end{pmatrix}\begin{pmatrix}\frac{1}{2}e^{t} - \frac{1}{4}e^{2t}\\ -\frac{1}{2}e^{-t} + \frac{1}{2}t\\ \frac{1}{2}t^2\end{pmatrix} \qquad X_p = \varphi(t)\int \varphi^{-1}(t)F(t)\,dt$$

$$=\begin{pmatrix} -\frac{1}{2}e^{t}+\frac{1}{4}e^{2t}-\frac{1}{2}e^{t}+\frac{1}{2}te^{2t} \\ \frac{1}{2}e^{t}-\frac{1}{4}e^{2t}-\frac{1}{2}e^{t}+\frac{1}{2}te^{2t} \\ \frac{1}{2}t^{2}e^{3t} \end{pmatrix}$$

$$=\begin{pmatrix} -e^{t}+\frac{1}{4}e^{2t}+\frac{1}{2}te^{2t} \\ \frac{1}{2}te^{2t}-\frac{1}{4}e^{2t} \\ \frac{1}{2}t^{2}e^{3t} \end{pmatrix}$$

$$X(t)=C_1\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}+C_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{2t}+C_3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}e^{3t}+\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}e^{t}+\begin{pmatrix} \frac{1}{2}t+\frac{1}{4} \\ \frac{1}{2}t-\frac{1}{4} \\ 0 \end{pmatrix}e^{2t}+\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}t^{2} \end{pmatrix}e^{3t}$$

$$X(0)=C_1\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}+C_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}+C_3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}+\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}+\begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 0 \end{pmatrix}=\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$\begin{cases} -C_1+C_2+C_3=\frac{11}{4} \\ C_1+C_2+2C_3=\frac{13}{4} \\ \boxed{C_3=-1} \end{cases} \rightarrow \begin{cases} -C_1+C_2=\frac{15}{4} \\ C_1+C_2=\frac{21}{4} \end{cases} \quad \boxed{C_1=\frac{3}{4},\ C_2=\frac{9}{2}}$$

$$\begin{cases} x_1(t)=-C_1-e^{t}+\left(C_2+\frac{1}{2}t+\frac{1}{4}\right)e^{2t}+C_3e^{3t} \\ x_2(t)=C_1+\left(C_2+\frac{1}{2}t-\frac{1}{4}\right)e^{2t}+2C_3e^{3t} \\ x_3(t)=\left(C_3+\frac{1}{2}t^{2}\right)e^{3t} \end{cases}$$

$$\boxed{\begin{cases} x_1(t)=-\frac{3}{4}-e^{t}+\left(\frac{1}{2}t+\frac{19}{4}\right)e^{2t}-e^{3t} \\ x_2(t)=\frac{3}{4}+\left(\frac{1}{2}t+\frac{17}{4}\right)e^{2t}-2e^{3t} \\ x_3(t)=\left(\frac{1}{2}t^{2}-1\right)e^{3t} \end{cases}}$$

Exercise

Find the general solution of the system $x'' + x = 3$; $x(\pi) = 1,\quad x'(\pi) = 2$

Solution

Let $x_1 = x \quad x_2 = x' = x_1'$

$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 + 3 \end{cases} \to x(\pi) = x_1(\pi) = 1, \quad x'(\pi) = x_2(\pi) = 2$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \lambda^2 + 1 = 0$$

The eigenvalues: $\boxed{\lambda_{1,2} = \pm i}$

For $\lambda_1 = i \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to ix = y \quad \Rightarrow V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} = \begin{pmatrix} 1 \\ i \end{pmatrix}(\cos t + i \sin t)$$

$$= \begin{pmatrix} \cos t + i \sin t \\ -\sin t + i \cos t \end{pmatrix}$$

$$= \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$$X_h = C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$$\begin{matrix} x_1' \\ x_2' \end{matrix} \begin{cases} a_2 = 0 \\ -a_1 - 3 = 0 \end{cases} \to \begin{cases} a_2 = 0 \\ a_1 = 3 \end{cases} \Rightarrow \boxed{X_p = \begin{pmatrix} 3 \\ 0 \end{pmatrix}}$$

$$X(t) = C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad x_1(\pi) = 1, \quad x_2(\pi) = 2$$

$$X(\pi) = C_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{cases} -C_1 + 3 = 1 & \to \boxed{C_1 = 2} \\ -C_2 = 2 & \to \boxed{C_2 = -2} \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) = 2\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} - 2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1(t) = 2\cos t - 2\sin t + 3 \\ x_2(t) = -2\sin t - 2\cos t \end{cases}$$

$$\underline{x(t) = x_1(t) = 2\cos t - 2\sin t + 3}$$

Exercise

Find the general solution of the system $\begin{cases} x'' = x - y \\ y'' = x - y \end{cases}$ $\begin{cases} x(3) = 5, & x'(3) = 2 \\ y(3) = 1, & y'(3) = -1 \end{cases}$

Solution

$$\left|A - \lambda^2 I\right| = \begin{vmatrix} 1-\lambda^2 & -1 \\ 1 & -1-\lambda^2 \end{vmatrix}$$

$$= \lambda^4 = 0$$

Thus, the eigenvalues are: $\underline{\lambda_{1,2,3,4} = 0}$

$$x(t) = C_1 + C_2 t + C_3 t^2 + C_4 t^3$$

$$x(3) = \underline{C_1 + 3C_2 + 9C_3 + 27C_4 = 5}$$

$$x' = C_2 + 2C_3 t + 3C_4 t^2$$

$$x'(3) = \underline{C_2 + 6C_3 + 27C_4 = 2}$$

$$x'' = x - y \quad \rightarrow \quad y = x - x''$$

$$x'' = 2C_3 + 6C_4 t$$

$$y(t) = C_1 - 2C_3 + \left(C_2 - 6C_4\right)t + C_3 t^2 + C_4 t^3$$

$$y(3) = \underline{C_1 + 3C_2 + 7C_3 + 9C_4 = 1}$$

$$y' = C_2 - 6C_4 + 2C_3 t + 3C_4 t^2$$

$$y'(3) = \underline{C_2 + 6C_3 + 21C_4 = -1}$$

$$\Delta = \begin{vmatrix} 1 & 3 & 9 & 27 \\ 0 & 1 & 6 & 27 \\ 1 & 3 & 7 & 9 \\ 0 & 1 & 6 & 21 \end{vmatrix} = 12 \quad \Delta_1 = \begin{vmatrix} 5 & 3 & 9 & 27 \\ 2 & 1 & 6 & 27 \\ 1 & 3 & 7 & 9 \\ -1 & 1 & 6 & 21 \end{vmatrix} = 42 \quad \Delta_2 = \begin{vmatrix} 1 & 5 & 9 & 27 \\ 0 & 2 & 6 & 27 \\ 1 & 1 & 7 & 9 \\ 0 & -1 & 6 & 21 \end{vmatrix} = 42$$

$$\underline{C_1 = \frac{42}{12} = \frac{7}{2}, \quad C_2 = \frac{7}{2}, \quad C_3 = -\frac{5}{2}, \quad C_4 = \frac{1}{2}}$$

$$\begin{cases} x(t) = \frac{7}{2} + \frac{7}{2}t - \frac{5}{2}t^2 + \frac{1}{2}t^3 \\ y(t) = \frac{17}{2} + \frac{1}{2}t - \frac{5}{2}t^2 + \frac{1}{2}t^3 \end{cases}$$

Exercise

Find the general solution of the system $\begin{cases} x'' = x - y \\ y'' = -x + y \end{cases}$ $\begin{cases} x(0) = -1, & x'(0) = 0 \\ y(0) = 1, & y'(0) = 0 \end{cases}$

Solution

$$\left|A - \lambda^2 I\right| = \begin{vmatrix} 1-\lambda^2 & -1 \\ -1 & 1-\lambda^2 \end{vmatrix}$$

$$= \lambda^4 - 2\lambda^2 = 0$$

Thus, the eigenvalues are: $\lambda_{1,2} = 0$ & $\lambda_{3,4} = \pm\sqrt{2}$

$$x(t) = C_1 + C_2 t + C_3 e^{-\sqrt{2}t} + C_4 e^{\sqrt{2}t}$$

$$x(0) = C_1 + C_3 + C_4 = -1$$

$$x' = C_2 - \sqrt{2}C_3 e^{-\sqrt{2}t} + \sqrt{2}C_4 e^{\sqrt{2}t}$$

$$x'(0) = C_2 - \sqrt{2}C_3 + \sqrt{2}C_4 = 0$$

$$x'' = x - y \quad \to \quad y = x - x''$$

$$x'' = 2C_3 e^{-\sqrt{2}t} + 2C_4 e^{\sqrt{2}t}$$

$$y(t) = C_1 + C_2 t - C_3 e^{-\sqrt{2}t} - C_4 e^{\sqrt{2}t}$$

$$y(0) = C_1 - C_3 - C_4 = 1$$

$$y' = C_2 + \sqrt{2}C_3 e^{-\sqrt{2}t} - \sqrt{2}C_4 e^{\sqrt{2}t}$$

$$y'(0) = C_2 + \sqrt{2}C_3 - \sqrt{2}C_4 = 0$$

$$\Delta = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -\sqrt{2} & \sqrt{2} \\ 1 & 0 & -1 & -1 \\ 0 & 1 & \sqrt{2} & -\sqrt{2} \end{vmatrix} = 8\sqrt{2} \quad \Delta_1 = \begin{vmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & -\sqrt{2} & \sqrt{2} \\ 1 & 0 & -1 & -1 \\ 0 & 1 & \sqrt{2} & -\sqrt{2} \end{vmatrix} = 0 \quad \Delta_2 = \begin{vmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & -\sqrt{2} & \sqrt{2} \\ 1 & 1 & -1 & -1 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{vmatrix} = 0$$

$$\begin{cases} C_3 + C_4 = -1 \\ \sqrt{2}C_3 - \sqrt{2}C_4 = 0 \end{cases}$$

$$C_1 = 0, \;\; C_2 = 0, \;\; C_3 = -\frac{\sqrt{2}}{2\sqrt{2}} = -\frac{1}{2}, \;\; C_4 = -\frac{1}{2}$$

$$\begin{cases} x(t) = -\frac{1}{2}e^{-\sqrt{2}t} - \frac{1}{2}e^{\sqrt{2}t} \\ y(t) = \frac{1}{2}e^{-\sqrt{2}t} + \frac{1}{2}e^{\sqrt{2}t} \end{cases}$$

Exercise

Find the general solution of the system

$$\begin{cases} \dfrac{d^2x}{dt^2} = y; & x(0) = 3, \quad x'(0) = 1 \\ \dfrac{d^2y}{dt^2} = x; & y(0) = 1, \quad y'(0) = -1 \end{cases}$$

Solution

$$\left|A - \lambda^2 I\right| = \begin{vmatrix} -\lambda^2 & 1 \\ 1 & -\lambda^2 \end{vmatrix}$$

$$= \lambda^4 - 1 = 0$$

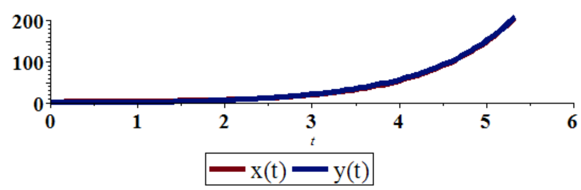
Thus, the eigenvalues are: $\lambda_{1,2} = \pm 1$ & $\lambda_{3,4} = \pm i$

$$x(t) = C_1 e^{-t} + C_2 e^{t} + C_3 \cos t + C_4 \sin t$$

$$x(0) = C_1 + C_2 + C_3 = 3$$

$$x' = -C_1 e^{-t} + C_2 e^{t} - C_3 \sin t + C_4 \cos t$$

$$x'(0) = -C_1 + C_2 + C_4 = 1$$



$$x'' = y$$

$$y(t) = C_1 e^{-t} + C_2 e^{t} - C_3 \cos t - C_4 \sin t$$

$$y(0) = C_1 + C_2 - C_3 = 1$$

$$y' = -C_1 e^{-t} + C_2 e^{t} + C_3 \sin t - C_4 \cos t$$

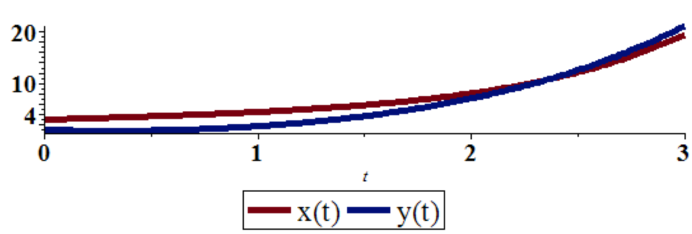
$$y'(0) = -C_1 + C_2 - C_4 = -1$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & -1 \end{vmatrix} = 8 \quad \Delta_1 = \begin{vmatrix} 3 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & -1 \end{vmatrix} = 8 \quad \Delta_2 = \begin{vmatrix} 1 & 3 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & 0 & -1 \end{vmatrix} = 8$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 3 & 0 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ -1 & 1 & -1 & -1 \end{vmatrix} = 8 \quad \Delta_4 = \begin{vmatrix} 1 & 1 & 1 & 3 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 0 & -1 \end{vmatrix} = 8$$

$$C_1 = 1, \;\; C_2 = 1, \;\; C_3 = 1, \;\; C_4 = 1$$

$$\begin{cases} x(t) = e^{-t} + e^{t} + \cos t + \sin t \\ y(t) = e^{-t} + e^{t} - \cos t - \sin t \end{cases}$$



Exercise

Find the general solution of the system $\begin{cases} x'' + 5x - 2y = 0 \\ y'' + 2y - 2x = 3\sin 2t \end{cases}$ $\quad x(0) = x'(0) = 0 \quad y(0) = 1, \quad y'(0) = 0$

Solution

$$\begin{cases} x'' = -5x + 2y \\ y'' = 2x - 2y + 3\sin 2t \end{cases}$$

$$\left|A - \lambda^2 I\right| = \begin{vmatrix} -5-\lambda^2 & 2 \\ 2 & -2-\lambda^2 \end{vmatrix} \qquad\qquad A = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$$

$$= \lambda^4 + 7\lambda^2 + 6 = 0 \qquad \lambda^2 = -1, \ -6$$

The eigenvalues: $\lambda_{1,2} = i \ \ \& \ \ \lambda_{3,4} = \pm i\sqrt{6}$

$$x_h(t) = C_1 \cos t + C_2 \sin t + C_3 \cos\sqrt{6}t + C_4 \sin\sqrt{6}t$$

$$\begin{cases} x_p = A\sin 2t \\ y_p = B\sin 2t \end{cases} \rightarrow \begin{cases} x''_p = -4A\sin 2t \\ y''_p = -4B\sin 2t \end{cases}$$

$$\begin{cases} -4A\sin 2t + 5A\sin 2t - 2B\sin 2t = 0 \\ -4B\sin 2t + 2B\sin t - 2A\sin 2t = 3\sin 2t \end{cases}$$

$$\begin{cases} A - 2B = 0 \\ -2A - 2B = 3 \end{cases} \rightarrow \ \boxed{A = -1, \ \ B = -\frac{1}{2}}$$

$$\begin{cases} x_p = -\sin 2t \\ y_p = -\frac{1}{2}\sin 2t \end{cases}$$

$$x(t) = C_1 \cos t + C_2 \sin t + C_3 \cos\sqrt{6}t + C_4 \sin\sqrt{6}t - \sin 2t$$

$$x(0) = \boxed{C_1 + C_3 = 0} \quad (1)$$

$$x' = -C_1 \sin t + C_2 \cos t - \sqrt{6}C_3 \sin\sqrt{6}t + \sqrt{6}C_4 \cos\sqrt{6}t - 2\cos 2t$$

$$x'(0) = \boxed{C_2 + \sqrt{6}C_4 - 2 = 0} \quad (2)$$

$$x'' + 5x - 2y = 0 \ \rightarrow \ y = \frac{1}{2}\left(x'' + 5x\right)$$

$$x'' = -C_1 \cos t - C_2 \sin t - 6C_3 \cos\sqrt{6}t - 6C_4 \sin\sqrt{6}t + 4\sin 2t$$

$$y(t) = \frac{1}{2}\left(x'' + 5x\right)$$

$$= \frac{1}{2}\left(4C_1 \cos t + 4C_2 \sin t - C_3 \cos\sqrt{6}t - C_4 \sin\sqrt{6}t - \sin 2t\right)$$

$$= 2C_1 \cos t + 2C_2 \sin t - \frac{1}{2}C_3 \cos\sqrt{6}t - \frac{1}{2}C_4 \sin\sqrt{6}t - \frac{1}{2}\sin 2t$$

$$y(0) = \boxed{2C_1 - \frac{1}{2}C_3 = 1} \quad (3)$$

$$y' = -2C_1 \sin t + 2C_2 \cos t + \frac{\sqrt{6}}{2} C_3 \sin \sqrt{6}t - \frac{\sqrt{6}}{2} C_4 \cos \sqrt{6}t - \cos 2t$$

$$y'(0) = \underline{2C_2 - \frac{\sqrt{6}}{2} C_4 - 1 = 0} \quad (4)$$

$$\begin{cases} (1) & C_1 + C_3 = 0 \\ (3) & 4C_1 - C_3 = 2 \end{cases} \qquad \underline{C_1 = \frac{2}{5}, \quad C_3 = -\frac{2}{5}}$$

$$\begin{cases} (2) & C_2 + \sqrt{6}C_4 = 2 \\ (4) & 4C_2 - \sqrt{6}C_4 = 2 \end{cases} \qquad \underline{C_2 = \frac{4\sqrt{6}}{5\sqrt{6}} = \frac{4}{5}, \quad C_4 = \frac{6}{5\sqrt{6}} = \frac{\sqrt{6}}{5}}$$

$$\begin{cases} x(t) = \frac{2}{5}\cos t + \frac{4}{5}\sin t - \frac{2}{5}\cos\sqrt{6}t + \frac{\sqrt{6}}{5}\sin\sqrt{6}t - \sin 2t \\ y(t) = \frac{4}{5}\cos t + \frac{8}{5}\sin t + \frac{1}{5}\cos\sqrt{6}t - \frac{\sqrt{6}}{10}\sin\sqrt{6}t - \frac{1}{2}\sin 2t \end{cases}$$

Exercise

Find the general solution of the system $\begin{cases} x'' = -2x' - 5y + 3 \\ y' = x' + 2y \end{cases}$ $\quad x(0) = 0,\ x'(0) = 0,\ y(0) = 1$

Solution

Let $\begin{matrix} x_1 = x & x_2 = x' = x_1' \\ y_1 = y & y_2 = y' = y_1' \end{matrix}$ $\quad \begin{cases} x(0) = x_1(0) = 0 \\ x'(0) = x_2(0) = 0 \\ y(0) = y_1(0) = 1 \end{cases}$

$$\begin{cases} x_1' = x_2 \\ x_2' = -2x_2 - 5y_1 + 3 \\ y_1' = x_2 + 2y_1 \end{cases}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -2-\lambda & -5 \\ 0 & 1 & 2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & -5 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= \lambda\left(4 - \lambda^2\right) - 5\lambda$$

$$= -\lambda^3 - \lambda = 0$$

The eigenvalues: $\underline{\lambda_1 = 0, \quad \lambda_{2,3} = \pm i}$

For $\lambda_1 = 0 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & -5 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \to \begin{cases} y = 0 \\ z = 0 \end{cases} \quad \Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda_2 = -i \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} i & 1 & 0 \\ 0 & -2+i & -5 \\ 0 & 1 & 2+i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} ix = -y \\ (-2+i)y = 5z \\ y = -(2+i)z \end{cases} \quad \Rightarrow V_2 = \begin{pmatrix} 1 \\ -i \\ \frac{1}{5}(1+2i) \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -i \\ \frac{1}{5}(1+2i) \end{pmatrix} e^{-it} = \begin{pmatrix} 1 \\ -i \\ \frac{1}{5}(1+2i) \end{pmatrix}(\cos t - i\sin t)$$

$$= \begin{pmatrix} \cos t - i\sin t \\ -\sin t - i\cos t \\ \frac{1}{5}\left(\cos t + 2\sin t + i(2\cos t - \sin t)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \cos t \\ -\sin t \\ \frac{1}{5}(\cos t + 2\sin t) \end{pmatrix} + i\begin{pmatrix} -\sin t \\ -\cos t \\ \frac{1}{5}(2\cos t - \sin t) \end{pmatrix}$$

$$X_h = C_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} \cos t \\ -\sin t \\ \frac{1}{5}(\cos t + 2\sin t) \end{pmatrix} + C_3\begin{pmatrix} -\sin t \\ -\cos t \\ \frac{1}{5}(2\cos t - \sin t) \end{pmatrix}$$

$$\begin{cases} -2a_2 - 5a_3 = -3 \\ a_2 + 2a_3 = 0 \end{cases} \qquad \Delta = \begin{vmatrix} -2 & -5 \\ 1 & 2 \end{vmatrix} = 1 \quad \Delta_1 = \begin{vmatrix} -3 & -5 \\ 0 & 2 \end{vmatrix} = -6 \quad \Delta_2 = \begin{vmatrix} -2 & -3 \\ 1 & 0 \end{vmatrix} = 3$$

$\Rightarrow \quad a_2 = -6 \quad a_3 = 3$

$x_1' = x_2 \quad \rightarrow \quad a_1' = -6 \;\Rightarrow\; \boxed{a_1 = -6t}$

$$X_P = \begin{pmatrix} -6t \\ -6 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \end{pmatrix}(t) = C_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} \cos t \\ -\sin t \\ \frac{1}{5}(\cos t + 2\sin t) \end{pmatrix} + C_3\begin{pmatrix} -\sin t \\ -\cos t \\ \frac{1}{5}(2\cos t - \sin t) \end{pmatrix} + \begin{pmatrix} -6t \\ -6 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \end{pmatrix}(0) = C_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 0 \\ \frac{1}{5} \end{pmatrix} + C_3\begin{pmatrix} 0 \\ -1 \\ \frac{2}{5} \end{pmatrix} + \begin{pmatrix} 0 \\ -6 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} C_1 + C_2 = 0 & \rightarrow \boxed{C_1 = -2} \\ -C_3 - 6 = 0 & \rightarrow \boxed{C_3 = -6} \\ \frac{1}{5}C_2 + \frac{2}{5}C_3 + 3 = 1 & \rightarrow \boxed{C_2 = 2} \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \end{pmatrix}(t) = -2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} \cos t \\ -\sin t \\ \frac{1}{5}(\cos t + 2\sin t) \end{pmatrix} - 6\begin{pmatrix} -\sin t \\ -\cos t \\ \frac{1}{5}(2\cos t - \sin t) \end{pmatrix} + \begin{pmatrix} -6t \\ -6 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 + 2\cos t + 6\sin t - 6t \\ -2\sin t + 6\cos t - 6 \\ \frac{2}{5}\cos t + \frac{4}{5}\sin t - \frac{12}{5}\cos t + \frac{6}{5}\sin t + 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2\cos t + 6\sin t - 6t - 2 \\ -2\sin t + 6\cos t - 6 \\ 2\sin t - 2\cos t + 3 \end{pmatrix}$$

$$\boxed{\begin{cases} x(t) = x_1(t) = 2\cos t + 6\sin t - 6t - 2 \\ y(t) = y_1(t) = 2\sin t - 2\cos t + 3 \end{cases}}$$

Exercise

Find the general solution of the system $\begin{cases} x'' = 2x' + 5y + 3 \\ y' = -x' - 2y \end{cases}$ $\quad x(0) = 0,\ x'(0) = 0,\ y(0) = 1$

Solution

Let $\quad x_1 = x \quad x_2 = x' = x_1'$
$\quad\quad y_1 = y \quad y_2 = y' = y_1'$

$$\begin{cases} x(0) = x_1(0) = 0 \\ x'(0) = x_2(0) = 0 \\ y(0) = y_1(0) = 1 \end{cases}$$

$$\begin{cases} x_1' = x_2 \\ x_2' = 2x_2 5y_1 + 3 \\ y_1' = -x_2 - 2y_1 \end{cases}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & 2-\lambda & 5 \\ 0 & -1 & -2-\lambda \end{vmatrix} \qquad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 5 \\ 0 & -1 & -2 \end{pmatrix}$$

$$= \lambda\left(4 - \lambda^2\right) - 5\lambda$$

$$= -\lambda^3 - \lambda = 0$$

The eigenvalues: $\boxed{\lambda_1 = 0,\ \ \lambda_{2,3} = \pm i}$

For $\lambda_1 = 0 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 5 \\ 0 & -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} y = 0 \\ z = 0 \\ \end{cases} \quad \Rightarrow \ V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda_2 = -i \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} i & 1 & 0 \\ 0 & 2+i & 5 \\ 0 & -1 & -2+i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} ix = -y \\ (2+i)y = -5z \\ y = (-2+i)z \end{cases}$$

$$x = 1 \quad \rightarrow y = -i \quad \Rightarrow -i = (-2+i)z$$

$$z = -\frac{i}{-2+i}\frac{-2-i}{-2-i} = \frac{-1+2i}{5}$$

$$\Rightarrow \ V_2 = \begin{pmatrix} 1 \\ -i \\ \frac{1}{5}(-1+2i) \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -i \\ \frac{1}{5}(-1+2i) \end{pmatrix} e^{-it} = \begin{pmatrix} 1 \\ -i \\ \frac{1}{5}(-1+2i) \end{pmatrix}(\cos t - i\sin t)$$

$$= \begin{pmatrix} \cos t - i\sin t \\ -\sin t - i\cos t \\ \frac{1}{5}\left(-\cos t + 2\sin t + i(2\cos t + \sin t)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \cos t \\ -\sin t \\ \frac{1}{5}(-\cos t + 2\sin t) \end{pmatrix} + i\begin{pmatrix} -\sin t \\ -\cos t \\ \frac{1}{5}(2\cos t + \sin t) \end{pmatrix}$$

$$X_h = C_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} \cos t \\ -\sin t \\ \frac{1}{5}(-\cos t + 2\sin t) \end{pmatrix} + C_3\begin{pmatrix} -\sin t \\ -\cos t \\ \frac{1}{5}(2\cos t + \sin t) \end{pmatrix}$$

$$\begin{cases} 2a_2 + 5a_3 = -3 \\ -a_2 - 2a_3 = 0 \end{cases} \quad \Delta = \begin{vmatrix} 2 & 5 \\ -1 & -2 \end{vmatrix} = 1 \quad \Delta_1 = \begin{vmatrix} -3 & 5 \\ 0 & -2 \end{vmatrix} = 6 \quad \Delta_2 = \begin{vmatrix} 2 & -3 \\ -1 & 0 \end{vmatrix} = -3$$

$\Rightarrow \ \boxed{a_2 = 6 \quad a_3 = -3}$

$$x_1' = x_2 \quad \to \quad a_1' = 6 \;\Rightarrow\; \boxed{a_1 = 6t}$$

$$\boxed{X_P = \begin{pmatrix} 6t \\ 6 \\ -3 \end{pmatrix}}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \end{pmatrix}(t) = C_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} \cos t \\ -\sin t \\ \frac{1}{5}(-\cos t + 2\sin t) \end{pmatrix} + C_3\begin{pmatrix} -\sin t \\ -\cos t \\ \frac{1}{5}(2\cos t + \sin t) \end{pmatrix} + \begin{pmatrix} 6t \\ 6 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \end{pmatrix}(0) = C_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 0 \\ -\frac{1}{5} \end{pmatrix} + C_3\begin{pmatrix} 0 \\ -1 \\ \frac{2}{5} \end{pmatrix} + \begin{pmatrix} 0 \\ 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} C_1 + C_2 = 0 & \to \boxed{C_1 = 8} \\ -C_3 + 6 = 0 & \to \boxed{C_3 = 6} \\ -\frac{1}{5}C_2 + \frac{2}{5}C_3 - 3 = 1 & \to \boxed{C_2 = -8} \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \end{pmatrix}(t) = 8\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 8\begin{pmatrix} \cos t \\ -\sin t \\ \frac{1}{5}(-\cos t + 2\sin t) \end{pmatrix} + 6\begin{pmatrix} -\sin t \\ -\cos t \\ \frac{1}{5}(2\cos t + \sin t) \end{pmatrix} + \begin{pmatrix} 6t \\ 6 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 8 - 8\cos t - 6\sin t + 6t \\ 8\sin t - 6\cos t + 6 \\ \frac{8}{5}\cos t - \frac{16}{5}\sin t + \frac{12}{5}\cos t + \frac{6}{5}\sin t - 3 \end{pmatrix}$$

$$= \begin{pmatrix} -8\cos t - 6\sin t + 6t + 8 \\ 8\sin t - 6\cos t + 6 \\ 4\cos t - 2\sin t - 3 \end{pmatrix}$$

$$\boxed{\begin{cases} x(t) = x_1(t) = -8\cos t - 6\sin t + 6t + 8 \\ y(t) = y_1(t) = 4\cos t - 2\sin t - 3 \end{cases}}$$

Exercise

Find the real and imaginary part of $z(t) = e^{2it}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$

Solution

$$z(t) = (\cos 2t + i\sin 2t)\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$=\begin{pmatrix} \cos 2t + i\sin 2t \\ \cos 2t + i\sin 2t + i\cos 2t - \sin 2t \end{pmatrix}$$

$$=\begin{pmatrix} \cos 2t + i\sin 2t \\ \cos 2t - \sin 2t + i(\sin 2t + \cos 2t) \end{pmatrix}$$

The real part is: $\underline{\left(\cos 2t,\ \cos 2t - \sin 2t\right)^T}$

The imaginary part is: $\underline{\left(\sin 2t,\ \sin 2t + \cos 2t\right)^T}$

Exercise

Two tanks, each containing 360 *liters* of a salt solution. Pure water pours into tank *A* at a rate of 5 *L/min*. There are two pipes connecting tank *A* to tank *B*. The first pumps salt solution from tank *B* into tank *A* at a rate of 4 *L/min*. The second pumps salt solution from tank *A* into tank *B* at a rate of 9 *L/min*. Finally, there is a drain on tank *B* from which salt solution drains at a rate of 5 *L/min*. Thus, each tank maintains a constant volume of 360 *liters* of salt solution. Initially, there are 60 *kg* of salt present in tank *A*, but tank *B* contains pure water.

a) Set up, in matrix-vector form, an initial value problem that models the salt content in each tank over time.

b) Find the eigenvalues and eigenvectors of the coefficient matrix in part (*a*), then find the general solution in vector form. Find the solution that satisfies the initial conditions posed in part (*a*).

c) Plot each component of your solution in part (*b*) over a period of four time constants $\left[0,\ 4T_c\right]$. What is the eventual salt content in each tank? Give both a physical and a mathematical reason for your answer.

Solution

a) Let $x_A(t)$ and $x_A(t)$ represent the number of pounds of salt as a function of time.

*Tank **A***:

$$Rate\ in = (5+4)\ \frac{L}{\min}\frac{x_A}{360}\frac{kg}{L} = \frac{x_A}{40}\,kg / min$$

$$Rate\ out = 4\ \frac{L}{\min}\frac{x_B}{360}\frac{kg}{L} = \frac{x_B}{90}\ kg / min$$

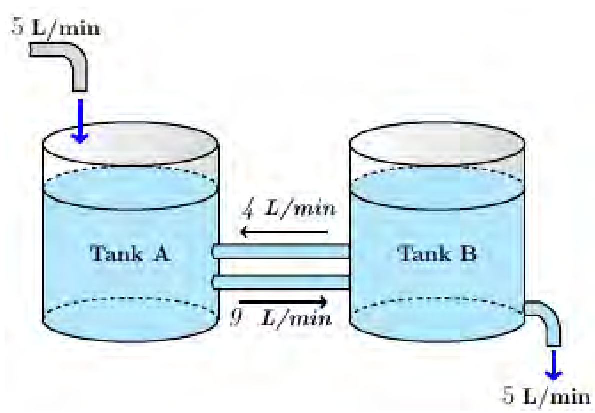
$$\frac{dx_A}{dt} = Rate\ in - Rate\ out = -\frac{x_A}{40} + \frac{x_B}{90}$$

*Tank **B***:

$$Rate\ in = 9\ \frac{L}{\min}\frac{x_A}{360}\frac{kg}{L} = \frac{x_A}{40}\,kg / min$$

$$Rate\ out = (5+4)\ \frac{L}{\min}\frac{x_B}{360}\frac{kg}{L} = \frac{x_B}{40}\ kg / min$$

$$\frac{dx_B}{dt} = Rate\ in - Rate\ out = \frac{x_A}{40} - \frac{x_B}{40}$$



$$\begin{cases} x'_A = -\dfrac{x_A}{40} + \dfrac{x_B}{90} \\ x'_B = \dfrac{x_A}{40} - \dfrac{x_B}{40} \end{cases}$$

The system is: $\begin{pmatrix} x_A \\ x_B \end{pmatrix}' = \begin{pmatrix} -\frac{1}{40} & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{40} \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix}$ $\qquad x' = Ax(t)$

With initial 60 kg of salt in tank A; $\begin{pmatrix} x_A(0) \\ x_B(0) \end{pmatrix} = \begin{pmatrix} 60 \\ 0 \end{pmatrix}$

b) $\det(A - \lambda I) = \begin{vmatrix} -\frac{1}{40} - \lambda & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{40} - \lambda \end{vmatrix}$

$$= \left(-\frac{1}{40} - \lambda\right)\left(-\frac{1}{40} - \lambda\right) - \frac{1}{90}\frac{1}{40}$$

$$= \frac{1}{1600} + \frac{1}{20}\lambda + \lambda^2 - \frac{1}{3600}$$

$$= \lambda^2 + \frac{1}{20}\lambda + \frac{5}{14400}$$

$\therefore$ The eigenvalues are: $\boxed{\lambda_1 = -\frac{1}{120}}$ *and* $\boxed{\lambda_2 = -\frac{1}{24}}$

For $\lambda_1 = -\frac{1}{120} \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$, we have

$$\begin{pmatrix} -\frac{1}{40} + \frac{1}{120} & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{40} + \frac{1}{120} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{60} & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{60} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{60} & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{60} \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad x - \frac{2}{3}y = 0$$

$$V_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \to \quad x_1(t) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-t/120}$$

For $\lambda_2 = -\frac{1}{24} \quad \Rightarrow \quad (A - \lambda_2 I)V_2 = 0$, we have

$$\begin{pmatrix} -\frac{1}{40} + \frac{1}{24} & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{40} + \frac{1}{24} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{60} & \frac{1}{90} \\ \frac{1}{40} & \frac{1}{60} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \frac{1}{60} & \frac{1}{90} \\ \frac{1}{40} & \frac{1}{60} \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & \frac{2}{3} \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad x + \frac{2}{3}y = 0$$

$$V_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad \rightarrow \quad \boxed{x_2(t) = \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{-t/24}}$$

$$x(t) = C_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-t/120} + C_2 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{-t/24}$$

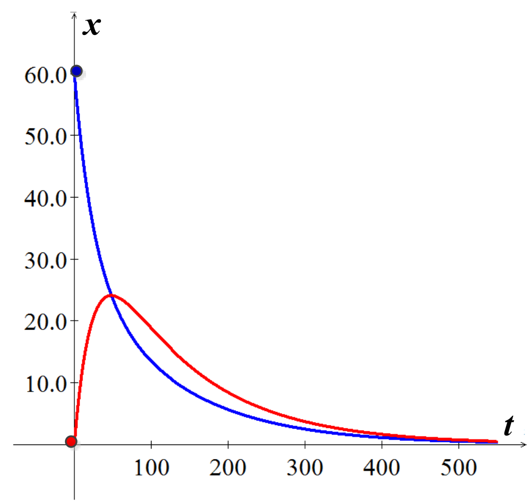
Given $\begin{pmatrix} x_A(0) \\ x_B(0) \end{pmatrix} = \begin{pmatrix} 60 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 60 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 60 \\ 0 \end{pmatrix} = \begin{pmatrix} 2C_1 - 2C_2 \\ 3C_1 + 3C_2 \end{pmatrix}$$

$$\rightarrow \begin{cases} 2C_1 - 2C_2 = 60 \\ 3C_1 + 3C_2 = 0 \end{cases} \quad \rightarrow \quad \boxed{C_1 = 15} \quad \boxed{C_2 = -15}$$

$$\boxed{x(t) = 15 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-t/120} - 15 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{-t/24}}$$



c) $x(t) = \begin{pmatrix} 30 & 30 \\ 45 & -45 \end{pmatrix}\begin{pmatrix} e^{-t/120} \\ e^{-t/24} \end{pmatrix}$

$$\begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} 30e^{-t/120} + 30e^{-t/24} \\ 45e^{-t/120} - 45e^{-t/24} \end{pmatrix}$$

The time constant on $e^{-t/120}$ is $T_c = 120$

The time constant on $e^{-t/24}$ is $T_c = 24$

If we choose the larger of these two time constants over a period of four time constants

$\left[0,\ 4T_c\right] = [0,\ 480]$.

This allows enough time to show both components decaying to zero.

Physically, if we keep pouring pure water into the tank B, eventually the system will purge itself of all salt content.

Mathematically: $\begin{cases} 30e^{-t/120} + 30e^{-t/24} \xrightarrow[t\to\infty]{} 0 \\ 45e^{-t/120} - 45e^{-t/24} \xrightarrow[t\to\infty]{} 0 \end{cases}$

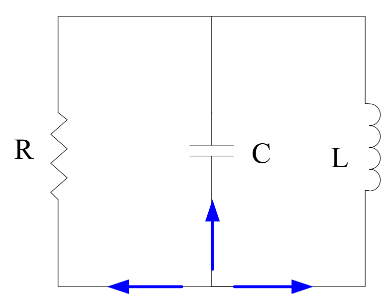
Exercise

Consider the *RLC* parallel circuit below. Let *V* represent the voltage drop across the capacitor and *I* represent the current across the inductor that satisfied the system.

$$\begin{cases} V' = -\dfrac{V}{RC} - \dfrac{1}{C} \\ I' = \dfrac{V}{L} \end{cases}$$

Suppose that the resistance is $R = \frac{1}{2}\Omega$, the capacitor is $C = 1$ *farad*, and the inductance is $L = \frac{1}{2}$ *henry*. If the initial voltage across the capacitor is $V(0) = 10$ *volts* and there is no initial current across the inductor, solve the system to determine the voltage and current as a function of time. Plot the voltage and current as a function of time. Assume current flows in the directions indicated.

Solution



$$\begin{cases} V' = -2V - 1 \\ I' = 2V \end{cases}$$

$$\begin{pmatrix} V \\ I \end{pmatrix}' = \begin{pmatrix} -2 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} V \\ I \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & -1 \\ 2 & -\lambda \end{vmatrix}$$

$$= \lambda^2 + 2\lambda + 2 = 0$$

$\therefore$ The *eigenvalues* are: $\lambda = -1 \pm i$

For $\lambda_1 = -1 + i \quad \Rightarrow (A - \lambda_1 I)V = 0$

$$\begin{pmatrix} -1-i & -1 \\ 2 & 1-i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} -x - y - ix = 0 \\ 2x + y - iy = 0 \end{cases} \to \; 2x = (-1+i)y$$

$$V = \begin{pmatrix} -1+i \\ 2 \end{pmatrix} \quad \to \quad z(t) = \begin{pmatrix} -1+i \\ 2 \end{pmatrix} e^{(-1+i)t}$$

$$z(t) = \begin{pmatrix} -1+i \\ 2 \end{pmatrix} e^{-t} e^{it}$$

$$= e^{-t}(\cos t + i\sin t)\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$= e^{t}\left[\cos t\begin{pmatrix} -1 \\ 2 \end{pmatrix} - \sin t\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] + ie^{t}\left[\left(\sin t\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \cos\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\right]$$

$$= e^{-t}\begin{pmatrix} -\cos t - \sin t \\ 2\cos t \end{pmatrix} + ie^{-t}\begin{pmatrix} \cos t - \sin t \\ 2\sin t \end{pmatrix}$$

$$\boxed{x(t) = C_1 e^{-t}\begin{pmatrix} -\cos t - \sin t \\ 2\cos t \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} \cos t - \sin t \\ 2\sin t \end{pmatrix}}$$

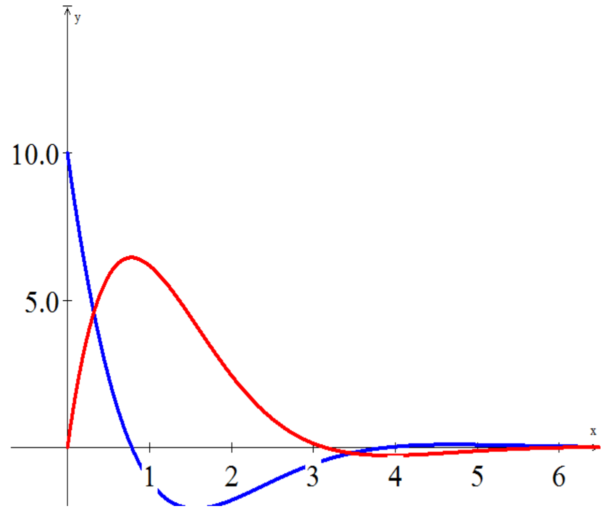
$$x(0) = (1)\begin{pmatrix} -1-0 \\ 2(1) \end{pmatrix} + i(1)\begin{pmatrix} 1-0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 10 \\ 0 \end{pmatrix} = C_1\begin{pmatrix} -1 \\ 2 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 10 \\ 0 \end{pmatrix} = \begin{pmatrix} -1C_1 + C_2 \\ 2C_1 \end{pmatrix} \quad \Rightarrow \quad C_1 = 0 \quad C_2 = 10$$

$$\boxed{x(t) = 10e^{-t}\begin{pmatrix} \cos t - \sin t \\ 2\sin t \end{pmatrix}}$$

$$\begin{pmatrix} V(t) \\ I(t) \end{pmatrix} = x(t) = \boxed{\begin{pmatrix} 10e^{-t}(\cos t - \sin t) \\ 20e^{-t}\sin t \end{pmatrix}}$$



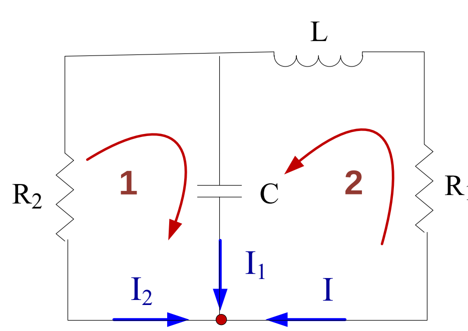
Exercise

Show that the voltage V across the capacitor and the current I through the inductor satisfy the system

$$\begin{cases} I' = -\dfrac{R_1}{L}I + \dfrac{1}{L}V \\ V' = -\dfrac{1}{C}I - \dfrac{1}{R_2 C}V \end{cases}$$

Suppose that the capacitance is $C = 1$ *farad*, the inductance is $L = 1$ *henry*, the leftmost resistor has resistance $R_2 = 1\ \Omega$, and the rightmost resistor has resistance $R_1 = 5\ \Omega$. If the initial voltage across the capacitor is 12 volts and the initial current through the inductor is zero, determine the voltage V across the capacitor and the current I through the inductor as functions of time. Plot the voltage and current as functions of time. Assume current flows in the directions indicated.

Solution



The current coming into the node at a must equal the current coming out,

$$I + I_1 + I_2 = 0$$

$$-R_2 I_2 + V = 0$$

$$-R_2(-I - I_1) + V = 0$$

$$R_2 I + R_2 I_1 = -V$$

The voltage across the capacitor follows the law

$V = \frac{1}{C} q_1$, where q_1 is the charge in the capacitor.

$$CV = q_1$$

$$(CV)' = (q_1)'$$

$$CV' = q_1{}' = I_1$$

$$R_2 I + R_2 I_1 = -V \quad \rightarrow \quad R_2 I + R_2 (CV') = -V$$

$$R_2 CV' = -V - R_2 I$$

$$\boxed{V' = -\frac{1}{R_2 C} V - \frac{1}{C} I}$$

Loop **2**:

$$-V + LI' + R_1 I = 0$$

$$LI' = V - R_1 I$$

$$\boxed{I' = \frac{1}{L} V - \frac{R_1}{L} I}$$

$$\begin{pmatrix} V \\ I \end{pmatrix}' = \begin{pmatrix} -\frac{1}{R_2 C} & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R_1}{L} \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{(1)(1)} & -\frac{1}{1} \\ \frac{1}{1} & -\frac{5}{1} \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -1 \\ 1 & -5-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(-5-\lambda) + 1$$

$$= \lambda^2 + 6\lambda + 6 = 0$$

$\therefore$ The eigenvalues are: $\lambda = -3 \pm \sqrt{3}$

For $\lambda_1 = -3 + \sqrt{3} \quad \Rightarrow (A - \lambda_1 I) V_1 = 0$

$$\begin{pmatrix} 2-\sqrt{3} & -1 \\ 1 & -2-\sqrt{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} (2-\sqrt{3})x - y = 0 \\ x - (2+\sqrt{3})y = 0 \end{cases}$$

$$V_1 = \begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} \quad \rightarrow \quad \boxed{x_1(t) = \begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} e^{\left(-3+\sqrt{3}\right)t}}$$

For $\lambda_2 = -3-\sqrt{3}$

$$V_2 = \begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix} \quad \rightarrow \quad \boxed{x_2(t) = \begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix} e^{\left(-3-\sqrt{3}\right)t}}$$

$$\boldsymbol{x}(t) = C_1 e^{\left(-3+\sqrt{3}\right)t} \begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} + C_2 e^{\left(-3-\sqrt{3}\right)t} \begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix}$$

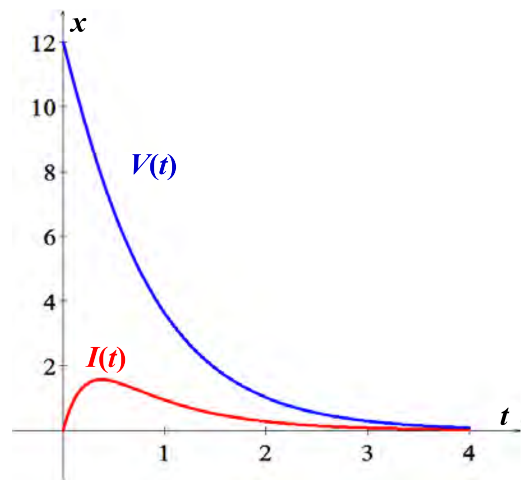
Given: $\quad V_0 = 12\ V \quad I_0 = 0A$

$$\begin{pmatrix} 12 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix}$$

$$\begin{cases} \left(2+\sqrt{3}\right)C_1 + \left(2-\sqrt{3}\right)C_2 = 12 \\ C_1 + C_2 = 0 \end{cases} \quad \Rightarrow \quad C_1 = 2\sqrt{3}, \ \ C_2 = -2\sqrt{3}$$

$$\boldsymbol{x}(t) = 2\sqrt{3}e^{\left(-3+\sqrt{3}\right)t} \begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} - 2\sqrt{3}e^{\left(-3-\sqrt{3}\right)t} \begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} \left(4\sqrt{3}+6\right)e^{\left(-3+\sqrt{3}\right)t} - \left(4\sqrt{3}-6\right)e^{\left(-3-\sqrt{3}\right)t} \\ 2\sqrt{3}e^{\left(-3+\sqrt{3}\right)t} - 2\sqrt{3}e^{\left(-3-\sqrt{3}\right)t} \end{pmatrix}$$



Which leads to the solutions

$$V(t) = \left(4\sqrt{3}+6\right)e^{\left(-3+\sqrt{3}\right)t} - \left(4\sqrt{3}-6\right)e^{\left(-3-\sqrt{3}\right)t}$$

$$I(t) = 2\sqrt{3}e^{\left(-3+\sqrt{3}\right)t} - 2\sqrt{3}e^{\left(-3-\sqrt{3}\right)t}$$