

Solution **Section 2.4 –Derivatives of Trigonometric Functions**

Exercise

Find the derivative of $y = -10x + 3\cos x$

Solution

$$\underline{y' = -10 - 3\sin x}$$

Exercise

Find the derivative of $y = \csc x - 4\sqrt{x} + 7$

Solution

$$\begin{aligned} y' &= -\csc x \cot x - 4\left(\frac{1}{2}x^{-1/2}\right) \\ &= -\csc x \cot x - \frac{2}{\sqrt{x}} \end{aligned}$$

Exercise

Find the derivative of $y = x^2 \cos x$

Solution

$$\begin{aligned} y &= 2x \cos x + x^2(-\sin x) \\ &= 2x \cos x - x^2 \sin x \end{aligned}$$

$$(uv)' = u'v + v'u$$

Exercise

Find the derivative of $y = \csc x \cot x$

Solution

$$\begin{aligned} y' &= (-\csc x \cot x) \cot x + \csc x(-\csc^2 x) \\ &= -\csc x \cot^2 x - \csc^3 x \\ &= -\csc x(\cot^2 x + \csc^2 x) \end{aligned}$$

$$(uv)' = u'v + uv'$$

Exercise

Find the derivative of $y = (\sin x + \cos x)\sec x$

Solution

$$u = \sin x + \cos x \quad v = \sec x$$

$$u' = \cos x - \sin x \quad v' = \sec x \tan x$$

$$y' = (\cos x - \sin x)\sec x + (\sin x + \cos x)(\sec x \tan x)$$

$$= \sec x \left[\cos x - \sin x + (\sin x + \cos x) \frac{\sin x}{\cos x} \right]$$

$$= \sec x \left(\cos x - \sin x + \frac{\sin^2 x}{\cos x} + \sin x \right)$$

$$= \sec x \left(\cos x + \frac{\sin^2 x}{\cos x} \right)$$

$$= \sec x \left(\frac{\cos^2 x + \sin^2 x}{\cos x} \right)$$

$$= \sec x \left(\frac{1}{\cos x} \right)$$

$$= \sec x \sec x$$

$$= \sec^2 x$$

$$y = (\sin x + \cos x) \frac{1}{\cos x}$$

$$= \tan x + 1$$

$$y' = \sec^2 x$$

Exercise

Find the derivative of $y = (\sec x + \tan x)(\sec x - \tan x)$

Solution

$$y = (\sec x + \tan x)(\sec x - \tan x)$$

$$= \sec^2 x - \tan^2 x$$

$$= 1 + \tan^2 x - \tan^2 x$$

$$= 1$$

$$y' = 0$$

$$y' = (\sec x + \tan x)'(\sec x - \tan x) + (\sec x + \tan x)(\sec x - \tan x)'$$

$$= (\sec x \tan x + \sec^2 x)(\sec x - \tan x)$$

$$+ (\sec x + \tan x)(\sec x \tan x - \sec^2 x)$$

$$= \sec^2 x \tan x - \sec x \tan^2 x + \sec^3 x - \sec^2 x \tan x$$

$$+ \sec^2 x \tan x - \sec^3 x + \sec x \tan^2 x - \sec^2 x \tan x$$

$$= 0$$

Exercise

Find the derivative of $y = \frac{\cos x}{x} + \frac{x}{\cos x}$

Solution

$$y = \frac{\cos^2 x + x^2}{x \cos x}$$

$$u = \cos^2 x + x^2 \quad v = x \cos x$$

$$u' = 2 \cos x (-\sin x) + 2x \quad v' = \cos x - x \sin x$$

$$\begin{aligned} y' &= \frac{(-2 \cos x \sin x + 2x)x \cos x - (\cos x - x \sin x)(\cos^2 x + x^2)}{(x \cos x)^2} & \left(\frac{u}{v}\right)' &= \frac{u'v - v'u}{v^2} \\ &= \frac{-2x \sin x \cos^2 x + 2x^2 \cos x - \cos^3 x - x^2 \cos x + x \sin x \cos^2 x + x^3 \sin x}{(x \cos x)^2} \\ &= \frac{-x \sin x \cos^2 x - x^2 \cos x - \cos^3 x + x^3 \sin x}{(x \cos x)^2} \end{aligned}$$

Exercise

Find the derivative of $y = x^2 \cos x - 2x \sin x - 2 \cos x$

Solution

$$\begin{aligned} y' &= 2x \cos x - x^2 \sin x - 2(\sin x + x \cos x) - 2(-\sin x) \\ &= 2x \cos x - x^2 \sin x - 2 \sin x - 2x \cos x + 2 \sin x \\ &= -x^2 \sin x \end{aligned}$$

Exercise

Find the derivative of $y = (2 - x) \tan^2 x$

Solution

$$\begin{aligned} y' &= -\tan^2 x + (2 - x)(2 \tan x \sec^2 x) \\ &= \tan x(-\tan x + 2(2 - x) \sec^2 x) \\ &= \tan x(2(2 - x) \sec^2 x - \tan x) \end{aligned}$$

Exercise

Find the derivative of $y = t^2 - \sec t + 1$

Solution

$$\underline{y' = 2t - \sec t \tan t}$$

Exercise

Find the derivative of $y = \frac{1 + \csc t}{1 - \csc t}$

Solution

$$\begin{aligned} u &= 1 + \csc t & v &= 1 - \csc t \\ u' &= -\csc x \cot x & v' &= \csc x \cot x \\ y' &= \frac{(-\csc x \cot x)(1 - \csc t) - (1 + \csc t)(\csc x \cot x)}{(1 - \csc t)^2} \\ &= \frac{-\csc x \cot x + \csc^2 x \cot x - \csc x \cot x - \csc^2 x \cot x}{(1 - \csc t)^2} \\ &= \underline{-\frac{2 \csc x \cot x}{(1 - \csc t)^2}} \end{aligned}$$

Exercise

Find the derivative of $r = \theta \sin \theta + \cos \theta$

Solution

$$\begin{aligned} r' &= \sin \theta + \theta \cos \theta - \sin \theta \\ &= \underline{\theta \cos \theta} \end{aligned}$$

Exercise

Find the derivative of $p = \frac{\sin q + \cos q}{\cos q}$

Solution

$$\begin{aligned} u &= \sin q + \cos q & v &= \cos q \\ u' &= \cos q - \sin q & v' &= -\sin q \\ p' &= \frac{(\cos q - \sin q)\cos q - (-\sin q)(\sin q + \cos q)}{\cos^2 q} \\ &= \frac{\cos^2 q - \sin q \cos q + \sin^2 q + \sin q \cos q}{\cos^2 q} \end{aligned}$$

$$\begin{aligned}
&= \frac{\cos^2 q + \sin^2 q}{\cos^2 q} \\
&= \frac{1}{\cos^2 q} \\
&= \sec^2 q \quad |
\end{aligned}$$

Exercise

Find the derivative of $p = \frac{3q + \tan q}{q \sec q}$

Solution

$$u = 3q + \tan q \quad v = q \sec q$$

$$u' = 3 + \sec^2 q \quad v' = \sec q + q \sec q \tan q$$

$$\begin{aligned}
p' &= \frac{(3 + \sec^2 q)(q \sec q) - (3q + \tan q)(\sec q + q \sec q \tan q)}{(q \sec q)^2} & \left(\frac{u}{v}\right)' &= \frac{u'v - v'u}{v^2} \\
&= \frac{3q \sec q + q \sec^3 q - 3q \sec q - 3q^2 \sec q \tan q - \tan q \sec q - q \sec q \tan^2 q}{q^2 \sec^2 q} \\
&= \frac{q \sec^3 q - 3q^2 \sec q \tan q - \tan q \sec q - q \sec q \tan^2 q}{q^2 \sec^2 q} \quad |
\end{aligned}$$

Exercise

Find the derivative of $f(x) = \frac{\sin x + 2x}{x}$

Solution

$$\begin{aligned}
f'(x) &= \frac{x \cos x + 2x - \sin x - 2x}{x^2} \\
&= \frac{x \cos x - \sin x}{x^2} \quad |
\end{aligned}$$

Exercise

Find the derivative of $f(x) = \frac{\sin x}{x^2}$

Solution

$$f'(x) = \frac{x^2 \cos x - 2x \sin x}{x^4}$$

$$= \frac{x \cos x - 2 \sin x}{x^3}$$

Exercise

Find the derivative of $f(x) = x^3 \cos x$

Solution

$$f'(x) = 3x^2 \cos x - x^3 \sin x$$

Exercise

Find the derivative of $f(x) = \frac{1}{x} - 12 \sec x$

Solution

$$f'(x) = -\frac{1}{x^2} - 12 \sec x \tan x$$

Exercise

Find the derivative of $f(\theta) = 5\theta \sec \theta + \theta \tan \theta$

Solution

$$f'(\theta) = 5 \sec \theta + 5\theta \sec \theta \tan \theta + \tan \theta + \theta \sec^2 \theta$$

Exercise

Find the derivative of $y = \sec \pi x$

Solution

$$y' = \pi \sec \pi x \tan \pi x$$

Exercise

Find the derivative of $y = \cos 5x$

Solution

$$y' = -5 \sin 5x$$

Exercise

Find the derivative of $y = \cos(4 - 3x)$

Solution

$$\underline{y' = 3\sin(4 - 3x) \quad |}$$

Exercise

Find the derivative of $f(x) = \sin(4 - 3x)$

Solution

$$\underline{f'(x) = -3\cos(4 - 3x) \quad |}$$

Exercise

Find the derivative of $f(\theta) = \frac{\sin a\theta}{\cos b\theta}$

Solution

$$u = \sin a\theta \quad v = \cos b\theta$$

$$u' = a \cos a\theta \quad v' = -b \sin b\theta$$

$$\underline{f'(\theta) = \frac{a \cos a\theta \cos b\theta + b \sin a\theta \sin b\theta}{\cos^2 b\theta} \quad |}$$

Exercise

Find the derivative of $f(\theta) = \sin 2\theta - \cos 2\theta$

Solution

$$\underline{f'(\theta) = 2\cos 2\theta + 2\sin 2\theta \quad |}$$

Exercise

Find the derivative of $f(\theta) = \tan \theta - \cot \theta$

Solution

$$\underline{f'(\theta) = \sec^2 \theta + \csc^2 \theta \quad |}$$

Exercise

Find the derivative of $\frac{d}{dx}(5x^2 \sin x)$

Solution

$$\underline{\frac{d}{dx}(5x^2 \sin x) = 10x \sin x + 5x^2 \cos x}$$

Exercise

Find the derivative of $\frac{d}{dx}(2x(\sin x)\sqrt{3x-1})$

Solution

$$\begin{aligned} \frac{d}{dx}(2x(\sin x)\sqrt{3x-1}) &= \underbrace{2}_{u'}(\sin x)\sqrt{3x-1} + 2x(\underbrace{\cos x}_{v'})\sqrt{3x-1} + 2x(\sin x)\underbrace{\frac{1}{2}(3)(3x-1)^{-1/2}}_{w'} \\ &= 2(\sin x)\sqrt{3x-1} + 2x(\cos x)\sqrt{3x-1} + \frac{3x \sin x}{\sqrt{3x-1}} \end{aligned}$$

Exercise

Find $y^{(4)}$ if $y = 9 \cos x$

Solution

$$\underline{y' = -9 \sin x}$$

$$\underline{y'' = -9 \cos x}$$

$$\underline{y''' = 9 \sin x}$$

$$\underline{y^{(4)} = 9 \cos x}$$

Exercise

Find $\frac{d^{999}}{dx^{999}}(\cos x)$

Solution

$$\underline{y' = -\sin x}$$

$$\underline{y'' = -\cos x}$$

$$\underline{y''' = \sin x}$$

$$\underline{y^{(4)} = \cos x}$$

$$999 = 249 \times 4 + 3$$

$$\begin{aligned} \frac{d^{999}}{dx^{999}}(\cos x) &= \frac{d^3}{dx^3}(\cos x) \\ &= \sin x \end{aligned}$$

Exercise

Find y' , y'' , y''' $y = \sin \sqrt{x}$

Solution

$$\underline{y' = \frac{1}{2\sqrt{x}} \cos \sqrt{x}}$$

$$\underline{y'' = -\frac{1}{4x^{3/2}} \cos \sqrt{x} - \frac{1}{4x} \sin \sqrt{x}}$$

$$\begin{aligned} y''' &= \frac{3}{8x^{5/2}} \cos \sqrt{x} + \frac{1}{8x^2} \sin \sqrt{x} + \frac{1}{4x^2} \sin \sqrt{x} - \frac{1}{8x^{3/2}} \cos \sqrt{x} \\ &= \frac{3}{8x^2} \sin \sqrt{x} + \frac{3-x}{8x^{5/2}} \cos \sqrt{x} \end{aligned}$$

Exercise

Find $\lim_{x \rightarrow -\frac{\pi}{6}} \sqrt{1 + \cos(\pi \csc x)}$

Solution

$$\begin{aligned} \lim_{x \rightarrow -\frac{\pi}{6}} \sqrt{1 + \cos(\pi \csc x)} &= \sqrt{1 + \cos\left(\pi \csc\left(-\frac{\pi}{6}\right)\right)} \\ &= \sqrt{1 + \cos(\pi(-2))} \\ &= \sqrt{1 + \cos(-2\pi)} \\ &= \sqrt{1+1} \\ &= \underline{\sqrt{2}} \end{aligned}$$

Exercise

Assume that a particle's position on the x -axis is given by $x = 3\cos t + 4\sin t$; ft

a) Find the particle's position when $t = 0$, $t = \frac{\pi}{2}$, and $t = \pi$

b) Find the particle's velocity when $t = 0$, $t = \frac{\pi}{2}$, and $t = \pi$

Solution

a) $t = 0$

$$\begin{aligned}x &= 3\cos 0 + 4\sin 0 \\&= 3 \text{ ft} \end{aligned}$$

$$t = \frac{\pi}{2}$$

$$\begin{aligned}x &= 3\cos \frac{\pi}{2} + 4\sin \frac{\pi}{2} \\&= 0 + 4 \\&= 4 \text{ ft} \end{aligned}$$

$$t = \pi$$

$$\begin{aligned}x &= 3\cos \pi + 4\sin \pi \\&= 3(-1) + 0 \\&= -3 \text{ ft} \end{aligned}$$

b) $v = x' = -3\sin t + 4\cos t$

$$t = 0$$

$$\begin{aligned}x &= -3\sin 0 + 4\cos 0 \\&= 4 \text{ ft/sec} \end{aligned}$$

$$t = \frac{\pi}{2}$$

$$\begin{aligned}x &= -3\sin \frac{\pi}{2} + 4\cos \frac{\pi}{2} \\&= -3 + 0 \\&= -3 \text{ ft/sec} \end{aligned}$$

$$t = \pi$$

$$\begin{aligned}x &= -3\sin \pi + 4\cos \pi \\&= 0 - 4 \\&= -4 \text{ ft/sec} \end{aligned}$$

Exercise

A weight is attached to a spring and reaches its equilibrium position ($x = 0$). It is then set in motion resulting in a displacement of $x = 10 \cos t$

Where x is measured in centimeters and t is measured in seconds.

a) Find the spring's displacement when $t = 0$, $t = \frac{\pi}{3}$, and $t = \frac{3\pi}{4}$

b) Find the spring's velocity when $t = 0$, $t = \frac{\pi}{3}$, and $t = \frac{3\pi}{4}$

Solution

a) $t = 0$

$$x = 10 \cos 0$$

$$= 10 \text{ cm}$$

$$t = \frac{\pi}{3}$$

$$x = 10 \cos \frac{\pi}{3}$$

$$= 10 \left(\frac{1}{2} \right)$$

$$= 5 \text{ cm}$$

$$t = \frac{3\pi}{4}$$

$$x = 10 \cos \frac{3\pi}{4}$$

$$= 10 \frac{\sqrt{2}}{2}$$

$$= 5\sqrt{2} \text{ cm}$$

b) $v = x' = -10 \sin t$

$$t = 0$$

$$x = -10 \sin 0$$

$$= 0 \text{ cm/sec}$$

$$t = \frac{\pi}{3}$$

$$x = -10 \sin \frac{\pi}{3}$$

$$= 10 \left(\frac{\sqrt{3}}{2} \right)$$

$$= 5\sqrt{3} \text{ cm/sec}$$

$$t = \frac{3\pi}{4}$$

$$x = -10 \sin \frac{3\pi}{4}$$

$$= -5\sqrt{2} \text{ cm/sec}$$

