Lecture Four – Series

Section 4.1 – Introduction and Review of Power Series

Definition

A ***power series*** *about the point* x_0 is a series of the form

$$\sum_{n=0}^{\infty} a_n\left(x-x_0\right)^n = a_0 + a_1\left(x-x_0\right) + a_2\left(x-x_0\right)^2 + a_3\left(x-x_0\right)^3 + \cdots$$

The series is said to converge at x if the sequence of partial sums

$$S_N(x) = \sum_{n=0}^{N} a_n\left(x-x_0\right)^n$$

$$= a_0 + a_1\left(x-x_0\right) + a_2\left(x-x_0\right)^2 + \cdots + a_N\left(x-x_0\right)^N$$

Converges as $N \to \infty$. The sum of the series at the point x is defined to be the limit at the partial sums,

$$\sum_{n=0}^{N} a_n\left(x-x_0\right)^n = \lim_{N\to\infty} S_N(x)$$

Example

Show that the geometric series $\sum_{n=0}^{\infty} x^n$ converges for $|x|<1$ and that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ *for* $|x|<1$

Show that the series diverges for $|x| \ge 1$.

Solution

The partial sums $S_N(x) = \sum_{n=0}^{N} x^n$ can be evaluates as follows.

$$(1-x)S_N(x) = (1-x)\left(1+x+x^2+\cdots+x^N\right)$$

$$= \left(1+x+x^2+\cdots+x^N\right) - \left(x+x^2+\cdots+x^N+x^{N+1}\right)$$

$$= 1-x^{N+1}$$

$$S_N(x) = \sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x} \qquad x \ne 1$$

If $|x|<1$, then $x^{N+1} \to 0 \quad as\ N \to \infty \Rightarrow S_N(x) \to \frac{1}{1-x}$

If $|x|>1$, then x^{N+1} diverges and therefore the $S_N(x)$ diverges

If $|x|=1$, then $S_N(1) = N+1$

Radius of Convergence of a Power Series

Corollary to Theorem

The convergence of the series $\sum c_n (x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x $(R = \infty)$.
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$)

R is called the ***radius of convergence*** of the power series, and the interval of radius R centered at $x = a$ is called the ***interval of convergence***.

Interval of convergence

Theorem

For any power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ there is an ***R***, either a nonnegative number or ∞, such that the series converges if $|x-x_0| < R$ and diverges if $|x-x_0| > R$

The ratio Test

Theorem

Suppose the terms of the series $\sum_{n=0}^{\infty} A_n$ have the property that

$$\lim_{n\to\infty} \frac{|A_{n+1}|}{|A_n|} = L$$

exists. If $L<1$ the series converges, while if $L>1$ the series diverges

Definition

Suppose that $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists or is ∞. Then the power series $\sum c_n (x-a)^n$ has radius of convergence $R = \frac{1}{L}$. (If $L = 0$, then $R = \infty$; if $L = \infty$, then $R = 0$) and $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$

How to Test a Power Series for Convergence

1. Use the Ratio Test (or Root Test) to find the interval where the series converges. Ordinarily, this is an open interval

$$|x-a| < R \quad \text{or} \quad a-R < x < a+R$$

2. If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint. Use the Comparison Test, the Integral Test, or the Alternating Series Test.

3. If the interval of absolute convergence is $a-R < x < a+R$, the series diverges for $|x-a| > R$ (it does not even converge conditionally) because the nth term does not approach zero for those values of x.

Example

Find the radius of convergence for the series. $\sum_{n=0}^{\infty}\frac{2^n x^{2n}}{2n(n+1)}$

Solution

$$\frac{|A_{n+1}|}{|A_n|} == \frac{2^{n+1}x^{2(n+1)}}{2(n+1)(n+2)}\cdot\frac{2n(n+1)}{2^n x^{2n}}$$

$$= \frac{2n}{(n+2)}x^2$$

$$\lim_{n\to\infty}\frac{|A_{n+1}|}{|A_n|} = \lim_{n\to\infty}\frac{2n}{n+2}x^2$$

$$\to 2x^2$$

By the ratio test, the series converges if $2x^2 < 1$, so the radius of convergence is $R = \frac{1}{\sqrt{2}}$

$x^2 < \frac{1}{2} \qquad -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

Example

Determine the centre, radius, and interval of convergence of $\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n}$

Solution

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \frac{1}{n^2+1}\left(x+\frac{5}{2}\right)^n$$

The centre of convergence is $x+\frac{5}{2}=0 \quad \Rightarrow \quad \underline{x=-\frac{5}{2}}$

$$L = \lim_{n\to\infty} \left| \frac{\left(\frac{2}{3}\right)^{n+1} \frac{1}{(n+1)^2+1}}{\left(\frac{2}{3}\right)^n \frac{1}{n^2+1}} \right|$$

$$= \lim_{n\to\infty} \frac{2}{3} \frac{n^2+1}{(n+1)^2+1}$$

$$\underline{= \frac{2}{3}}$$

$$R = \frac{1}{L} \ \underline{= \frac{3}{2}}$$

The series converges absolutely on ***interval*** $\left(-\frac{5}{2}-\frac{3}{2},\ -\frac{5}{2}+\frac{3}{2}\right) = \underline{(-4,\ -1)}$ $\quad a-R<x<a+R$

It diverges on $(-\infty,\ -4)\cup(-1,\ \infty)$

At $x=-4 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$

At $x=-1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{3^n}{(n^2+1)3^n} = \sum_{n=0}^{\infty} \frac{1}{n^2+1}$

Both series converge (absolutely).

Therefore, the interval of convergence of the given power is $[-4,\ -1]$

Algebraic Operations on Series

The ***sum*** and ***difference*** of two series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots$$

$$\sum_{n=0}^{\infty} a_n x^n \pm \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} \left(a_n \pm b_n\right) x^n$$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{m=0}^{\infty} b_m x^m\right) = \sum_{p=0}^{\infty} c_p x^p \qquad c_p = \sum_{k=0}^{p} a_{p-k} b_k$$

Differentiating Power Series

Theorem

The function

$$f(x) = \sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n = a_0 + a_1\left(x - x_0\right) + a_2\left(x - x_0\right)^2 + a_3\left(x - x_0\right)^3 + \cdots$$

Can be differentiating the series by terms

$$f'(x) = \frac{d}{dx}\left[a_0 + a_1\left(x - x_0\right) + a_2\left(x - x_0\right)^2 + a_3\left(x - x_0\right)^3 + \cdots\right]$$

$$= a_1 + 2a_2\left(x - x_0\right) + 3a_3\left(x - x_0\right)^2 + \cdots$$

$$= \sum_{n=1}^{\infty} n a_n \left(x - x_0\right)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n \left(x - x_0\right)^{n-2}$$

In general: $\boxed{f^{(n)}(x) = n!\,a_n} \quad \Rightarrow \quad a_n = \dfrac{f^{(n)}(a)}{n!}$

Identity Theorem

Suppose that the series $f(x)=\sum_{n=0}^{\infty} a_n\left(x-x_0\right)^n$ has a positive radius of convergence. Then

$$a_n=\frac{f^{(n)}\left(x_0\right)}{n!}$$

The coefficients of a power series are determined by the values of the sum $f(x)$.

If f has a series representation, then the series must be

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

Taylor and Maclaurin Series

Definitions

Let f be a function with derivatives of all orders throughout some interval containing $\boldsymbol{a}$ as an interior point. Then the ***Taylor series generated by*** f at $x=a$ is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

The ***Maclaurin series generated by*** f is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots,$$

The Taylor series generated by f at $x=0$.

Example

Find the Taylor series and the Taylor polynomials generated by $f(x)=\cos x$ at $x=0$

<u>*Solution*</u>

$$\begin{array}{ll} f(x)=\cos x, & f'(x)=-\sin x, \\ f''(x)=-\cos x, & f''(x)=\sin x, \\ \vdots & \vdots \\ f^{(2n)}(x)=(-1)^n\cos x, & f^{(2n+1)}(x)=(-1)^{2n+1}\sin x \end{array}$$

$$f^{(2n)}(0)=(-1)^n, \qquad f^{(2n+1)}(0)=0$$

The Taylor series generated by f at $x=0$ is

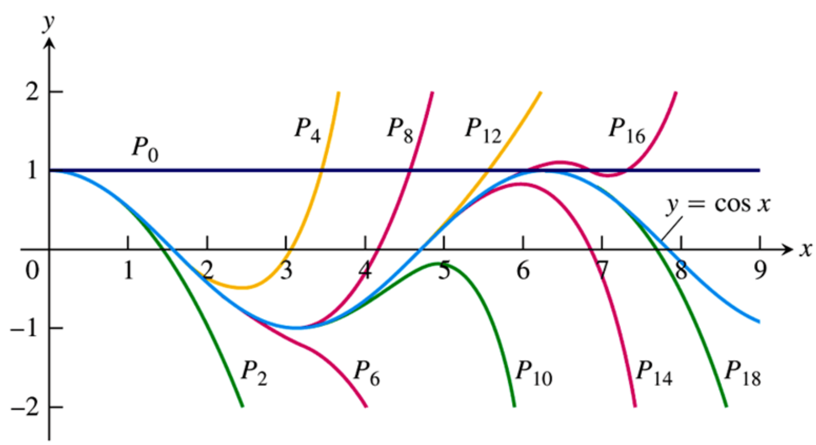
$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots$$

$$=1+0\cdot x-\frac{x^2}{2!}+0\cdot x^3+\frac{x^4}{4!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots$$

$$=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}$$

$$P_{2n}(x)=P_{2n+1}(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}$$



Example

Find the Taylor series for $\ln x$ in powers of $x-2$. Where does the series converge to $\ln x$?

Solution

Let $t=\frac{x-2}{2}$, then

$$\ln x=\ln\left(2+(x-2)\right)$$

$$=\ln\left[2\left(1+\frac{x-2}{2}\right)\right]$$

$$=\ln 2+\ln(1+t)$$

$f(t)=\ln(1+t)$ $\qquad$ $f(0)=\ln(1)=0$

$f'(t)=\frac{1}{1+t}$ $\qquad$ $f'(0)=1$

$f''(t)=\frac{-1}{(1+t)^2}$ $\qquad$ $f''(0)=-1$

$$f'''(t) = \frac{2}{(1+t)^3} \qquad\qquad f'''(0) = 2$$

$$f^{(4)}(t) = \frac{-6}{(1+t)^4} \qquad\qquad f'''(0) = -6$$

$$f^{(n)}(t) = (-1)^{n+1}\frac{(n-1)!}{(1+t)^n}$$

$$\begin{aligned}
\ln(1+t) &= f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \frac{f'''(0)}{3!}t^3 + \frac{f^{(IV)}(0)}{4!}t^4 + \cdots \\
&= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots
\end{aligned}$$

$$\begin{aligned}
\ln x &= \ln 2 + \ln(1+t) \\
&= \ln 2 + t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \\
&= \ln 2 + \frac{x-2}{2} - \frac{(x-2)^2}{2\times 2^2} + \frac{(x-2)^3}{3\times 2^3} - \frac{(x-2)^4}{4\times 2^4} + \cdots \\
&= \ln 2 + \sum_{n=1}^{\in} \frac{(-1)^{n-1}}{n\times 2^n}(x-2)^n
\end{aligned}$$

Since the series for $\ln(1+t)$ is valid for $-1 < t \le 1$, this series for $\ln x$ is valid for $-1 < \frac{x-2}{2} \le 1$

$-2 < x-2 \le 2 \quad \to \quad 0 < x \le 4$

Integrating Power Series

Theorem

Suppose the power series $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges for $|x-x_0| < R, \quad R > 0$

$$\int f(x)dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-x_0)^{n+1}}{n+1}$$

Exercises *Section* 4.1 – Introduction and Review of Power Series

Determine the *centre*, *radius*, and *interval of convergence* of each of the power series

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{\sqrt{n+1}}$

2. $\displaystyle\sum_{n=0}^{\infty} 3n(x+1)^n$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} x^n$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^3}(4-x)^n$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{1+5^n}{n!} x^n$

Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a

7. $f(x) = e^{2x}, \quad a = 0$

8. $f(x) = \sin x, \quad a = 0$

9. $f(x) = \ln(1+x), \quad a = 0$

10. $f(x) = \frac{1}{x+2}, \quad a = 0$

11. $f(x) = \sqrt{1-x}, \quad a = 0$

12. $f(x) = x^3, \quad a = 1$

13. $f(x) = 8\sqrt{x}, \quad a = 1$

14. $f(x) = \sin x, \quad a = \frac{\pi}{4}$

15. $f(x) = \cos x, \quad a = \frac{\pi}{6}$

16. $f(x) = \sqrt{x}, \quad a = 9$

17. $f(x) = \sqrt[3]{x}, \quad a = 8$

18. $f(x) = \ln x, \quad a = e$

19. $f(x) = \sqrt[4]{x}, \quad a = 8$

20. $f(x) = \tan^{-1} x + x^2 + 1, \quad a = 1$

21. $f(x) = e^x, \quad a = \ln 2$

Find the nth Maclaurin polynomial for the function

22. $f(x) = e^{4x}, \quad n = 4$

23. $f(x) = e^{-x}, \quad n = 5$

24. $f(x) = e^{-x/2}, \quad n = 4$

25. $f(x) = e^{x/3}, \quad n = 4$

26. $f(x) = \sin x, \quad n = 5$

27. $f(x) = \cos \pi x, \quad n = 4$

28. $f(x) = xe^x, \quad n = 4$

29. $f(x) = x^2 e^{-x}, \quad n = 4$

30. $f(x) = \frac{1}{x+1}, \quad n = 5$

31. $f(x) = \frac{x}{x+1}, \quad n = 4$

32. $f(x) = \sec x, \quad n = 2$

33. $f(x) = \tan x, \quad n = 3$

Finding Taylor and Maclaurin Series generated by f *at* $x = a$

34. $f(x) = x^3 - 2x + 4, \quad a = 2$

35. $f(x) = 2x^3 + x^2 + 3x - 8, \quad a = 1$

36. $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, \quad a = -1$

37. $f(x) = \cos\left(2x + \frac{\pi}{2}\right), \quad a = \frac{\pi}{4}$

Find the Maclaurin series for

38. xe^x

39. $5\cos \pi x$

40. $\dfrac{x^2}{x+1}$

41. e^{3x+1}

42. $\cos\left(2x^3\right)$

43. $\cos(2x-\pi)$

44. $x^2 \sin\left(\dfrac{x}{3}\right)$

45. $\cos^2\left(\dfrac{x}{2}\right)$

46. $\sin x \cos x$

47. $\tan^{-1}\left(5x^2\right)$

48. $\ln\left(2+x^2\right)$

49. $\dfrac{1+x^3}{1+x^2}$

50. $\ln\dfrac{1+x}{1-x}$

51. $\dfrac{e^{2x^2}-1}{x^2}$

52. $\cosh x - \cos x$

53. $\sinh x - \sin x$