SOLUTION *Section* 4.2 – Representing Relations

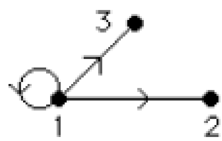
Exercise

Represent each of these relations on {1, 2, 3} with a matrix (with the elements of this set listed in increasing order). Then draw the directed graphs representing each relation

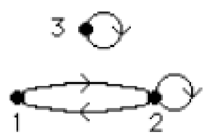
a) {(1, 1), (1, 2), (1, 3)}
b) {(1, 2), (2, 1), (2, 2), (3, 3)}
c) {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}
d) {(1, 3), (3, 1)}

Solution

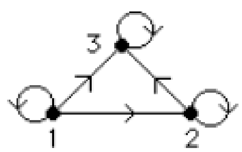
a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$



b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



c) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$



d) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

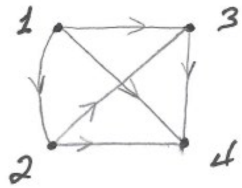
Exercise

Represent each of these relations on {1, 2, 3, 4} with a matrix (with the elements of this set listed in increasing order). Then draw the directed graphs representing each relation

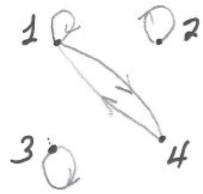
a) {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}
b) {(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)}
c) { (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)}
d) {(2, 4), (3, 1), (3, 2), (3, 4)}

Solution

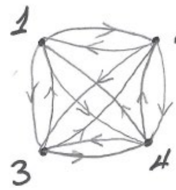
a) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$



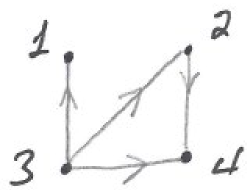
b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$



c) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$



d) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$



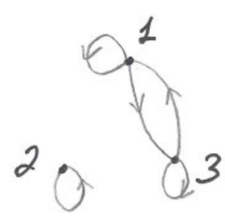
Exercise

List the ordered pairs in the relations on {1, 2, 3} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order). Then draw the directed graphs representing each relation

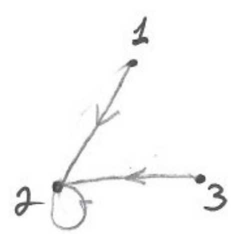
$a)\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad b)\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad c)\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Solution

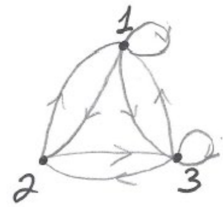
a) $\{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$



b) $\{(1, 2), (2, 2), (3, 2)\}$



c) $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (3, 3)\}$



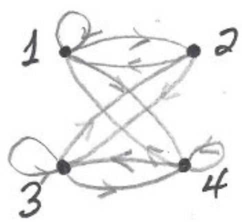
Exercise

List the ordered pairs in the relations on $\{1, 2, 3, 4\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order). Then draw the directed graphs representing each relation

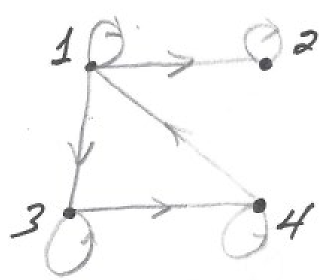
$a) \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad b) \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad c) \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

Solution

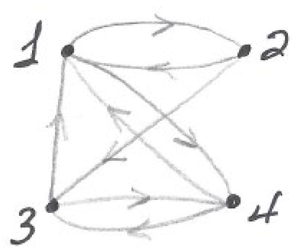
a) $\{(1, 1), (1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4)\}$



b) $\{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3), (3, 4), (4, 1), (4, 4)\}$



c) $\{(1, 2), (1, 4), (2, 1), (2, 3), (3, 1), (3, 4), (4, 1), (4, 3)\}$



Exercise

Let R be the relation represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find: $a)\ R^2$ $\quad b)\ R^3$ $\quad c)\ R^4$

Solution

a) $M_{R^2} = M^2{}_R = M_R \odot M_R$

$$R^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

b) $M_{R^3} = M^3{}_R = M_R \odot M^2{}_R$

$$R^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

c) $M_{R^4} = M^{(4)}{}_R = M_R \odot M^{(3)}{}_R$

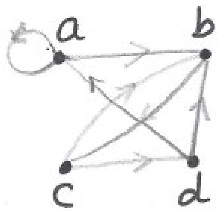
$$R^4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Exercise

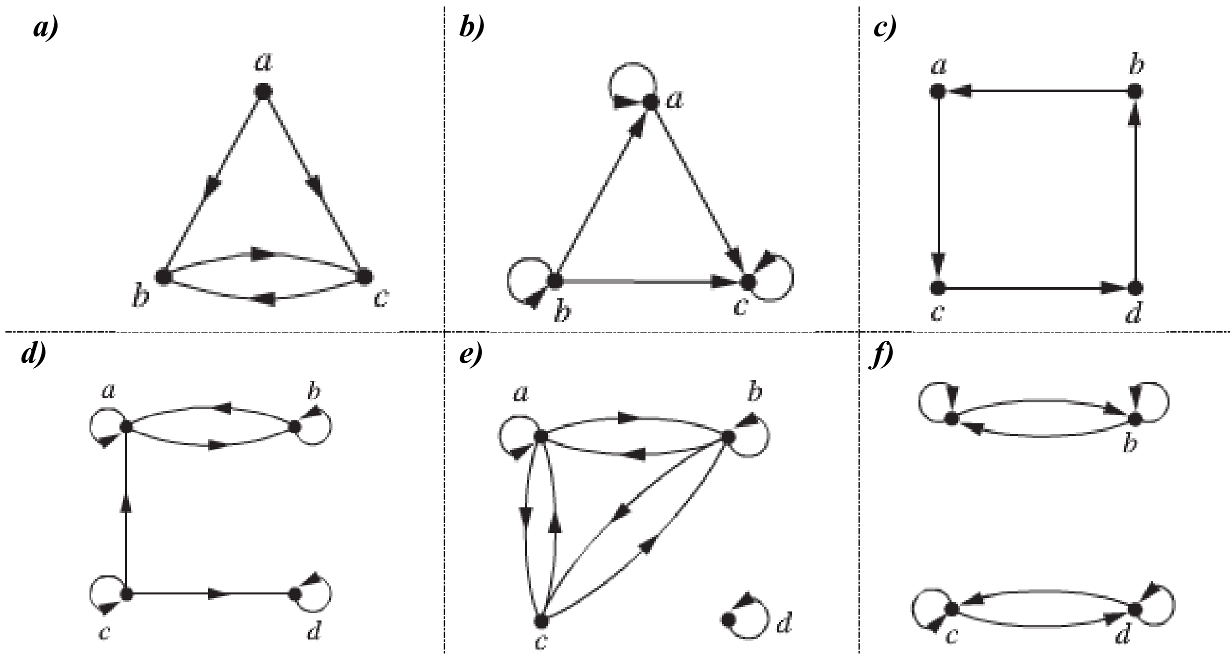
Draw the directed graph that represents the relation $\{(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)\}$

Solution



Exercise

Determine whether the relations represented by the directed graphs are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive



Solution

a) $\{(a, b), (a, c), (b, c), (c, b)\}$

It is not reflexive since (a, a) doesn't exist

It is not symmetric

It is transitive since $(a, b), (b, c) \Rightarrow (a, c)$

b) $\{(a, a), (a, c), (b, a), (b, b), (b, c), (c, c)\}$

It is reflexive

It is not symmetric

It is transitive since $(b, a), (a, c) \Rightarrow (b, c)$

c) $\{(a, c), (b, a), (c, d), (d, b)\}$

It is not reflexive; it is not symmetric, and not transitive since

d) $\{(a, a), (a, b), (b, a), (b, b), (c, a), (c, c), (c, d), (d, d)\}$

It is reflexive, not symmetric (no (a, c)), and not transitive $(c, a), (a, b)$ but no (c, b)

e) $\{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (d, d)\}$

It is not reflexive; it is symmetric and transitive

f) $\{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, d), (d, c)\}$
It is reflexive; it is symmetric and transitive