Section 4.2 – Representing Relations

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero-one matrix. Suppose that R is a relation from $A = \{a_1, a_2, a_3, \ldots, a_m\}$ to $B = \{b_1, b_2, b_3, \ldots, b_n\}$. The relation R can be represented by the matrix $M_a = \{m_{ij}\}$ where

$$m_{ij} = \begin{cases} 1 & if\left(a_i, b_j\right) \in \boldsymbol{R} \\ 0 & if\left(a_i, b_j\right) \notin \boldsymbol{R} \end{cases}$$

Example

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R the relation from A to B containing (a, b) if $a \in A,\ b \in B$, and $a > b$. What is the matrix representing R is $a_1 = 1,\ a_2 = 2,\ a_3 = 3,$ and $b_1 = 1,\ b_2 = 2$?

Solution

$R = \{(2, 1), (3, 1), (3, 2)\}$

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Example

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} ?$$

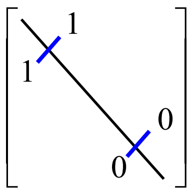
Solution

$$R = \left\{\left(a_1, b_2\right), \left(a_2, b_1\right), \left(a_2, b_3\right), \left(a_2, b_4\right), \left(a_3, b_1\right), \left(a_3, b_3\right), \left(a_3, b_5\right)\right\}$$

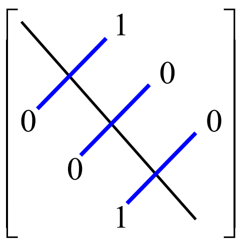
A relation R on A is ***reflexive*** if $(a, a) \in R$ whenever $a \in A$

$$M_R = \left(M_R\right)^t \qquad \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

A relation R on A is ***symmetric***



A relation R on A is ***antisymmetric*** iff $(a,\ b) \in R$ and $(b,\ a) \in R\ \Rightarrow\ a = b$



Example

Suppose that the relation R on the set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, and/or antisymmetric?

<u>Solution</u>

Because the diagonal elements are equal to 1, R is reflexive.

M_R is symmetric and it is not antisymmetric.

Relations Using Diagraphs

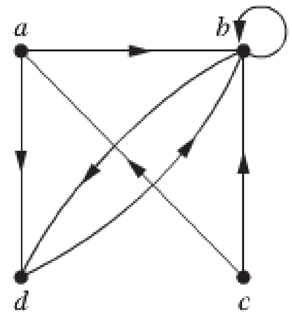
Definition

A directed ***graph***, or ***diagraph***, consists of a set V of ***vertices*** (or *nodes*) together with a set E ordered pairs of elements of V called ***edges*** (or *arcs*). The vertex a is called the ***initial*** vertex of the edge $(a,\ b)$, and the vertex b is called the ***terminal*** vertex of this edge.

Example

Draw the directed graph with vertices *a, b, c*, and *d*, and edges (*a, b*), (*a, d*), (*b, b*), (*b, d*), (*c, a*), (*c, b*), and (*d, b*)

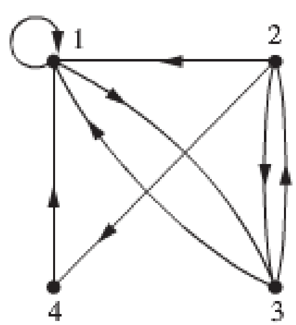
<u>Solution</u>



Example

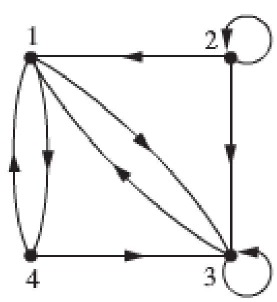
Draw the directed graph of the relation $R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$ on the set $\{1, 2, 3, 4\}$

<u>Solution</u>



Example

What are the ordered pairs in the relation *R* represented by the directed graph shown below



<u>Solution</u>

$R = \{(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}$

Exercises *Section* 4.2 – Representing Relations

1. Represent each of these relations on {1, 2, 3} with a matrix (with the elements of this set listed in increasing order). Then draw the directed graphs representing each relation

a) {(1, 1), (1, 2), (1, 3)}
b) {(1, 2), (2, 1), (2, 2), (3, 3)}
c) {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}
d) {(1, 3), (3, 1)}

2. Represent each of these relations on {1, 2, 3, 4} with a matrix (with the elements of this set listed in increasing order). Then draw the directed graphs representing each relation

a) {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}
b) {(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)}
c) { (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)}
d) {(2, 4), (3, 1), (3, 2), (3, 4)}

3. List the ordered pairs in the relations on {1, 2, 3} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order). Then draw the directed graphs representing each relation

$$a)\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad b)\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad c)\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

4. List the ordered pairs in the relations on {1, 2, 3, 4} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order). Then draw the directed graphs representing each relation

$$a)\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad b)\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad c)\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

5. Let R be the relation represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find: $a)\ R^2 \quad b)\ R^3 \quad c)\ R^4$

6. Draw the directed graph that represents the relation {(*a, a*), (*a, b*), (*b, c*), (*c, b*), (*c, d*), (*d, a*), (*d, b*)}

7. Determine whether the relations represented by the directed graphs are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive