SOLUTION *Section* **4.3 – Eigenvalue Method for Linear System**

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = x_1 + 2x_2, \quad x_2' = 2x_1 + x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4$$

$$= \lambda^2 - 2\lambda - 3 = 0$$

The distinct real eigenvalues: $\underline{\lambda_1 = -1, \quad \lambda_2 = 3}$

For $\lambda_1 = -1 \Rightarrow (A+I)V_1 = 0$

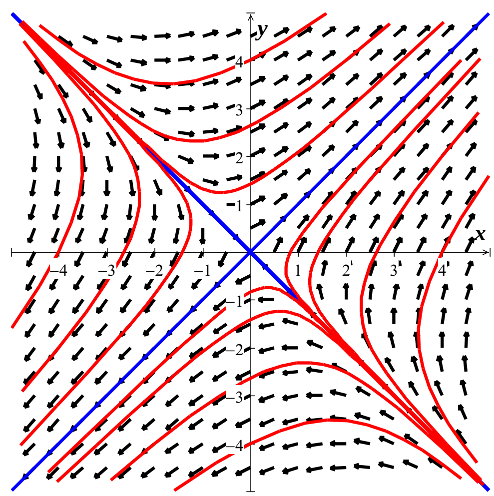
$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2x_1 + 2y_1 = 0 \to y_1 = -x_1$$

$$\to V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow x_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

For $\lambda_2 = 3 \Rightarrow (A-3I)V_2 = 0$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -2x_2 + 2y_2 = 0 \to x_2 = y_2$$

$$\to V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow x_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$



$$x_1(t) = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} \quad x_2(t) = \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$$

Using Wronskian: $\begin{vmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{vmatrix} = 2e^{2t} \neq 0$

The general solution: $\boxed{x(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}}$ ***OR*** $\begin{cases} x_1(t) = C_1 e^{-t} + C_2 e^{3t} \\ x_2(t) = -C_1 e^{-t} + C_2 e^{3t} \end{cases}$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = 2x_1 + 3x_2, \quad x_2' = 2x_1 + x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 6$$

$$= \lambda^2 - 3\lambda - 4 = 0$$

The distinct real eigenvalues: $\lambda_1 = -1, \quad \lambda_2 = 4$

For $\lambda_1 = -1 \;\Rightarrow\; (A+I)V_1 = 0$

$$\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\Rightarrow\; 3x_1 + 3y_1 = 0 \to y_1 = -x_1$$

$$\to V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \;\Rightarrow\; x_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

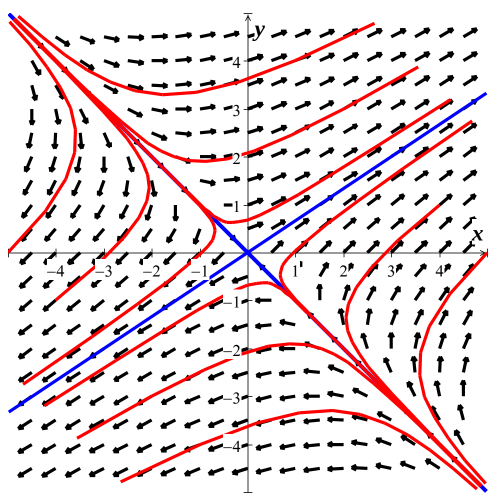
For $\lambda_2 = 4 \;\Rightarrow\; (A-4I)V_2 = 0$

$$\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\Rightarrow\; 2x_2 = 3y_2$$

$$\to V_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \;\Rightarrow\; x_2(t) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$$

$$x_1(t) = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} \quad x_2(t) = \begin{pmatrix} 3e^{4t} \\ 2e^{4t} \end{pmatrix}$$

The general solution: $x(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$



OR $\begin{cases} x_1(t) = C_1 e^{-t} + 3C_2 e^{4t} \\ x_2(t) = -C_1 e^{-t} + 2C_2 e^{4t} \end{cases}$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = 6x_1 - 7x_2, \quad x_2' = x_1 - 2x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 6-\lambda & -7 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 5 = 0$$

The distinct real eigenvalues: $\underline{\lambda_1 = -1, \quad \lambda_2 = 5}$

For $\lambda_1 = -1 \quad \Rightarrow \quad (A+I)V_1 = 0$

$$\begin{pmatrix} 7 & -7 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad x_1 = y_1$$

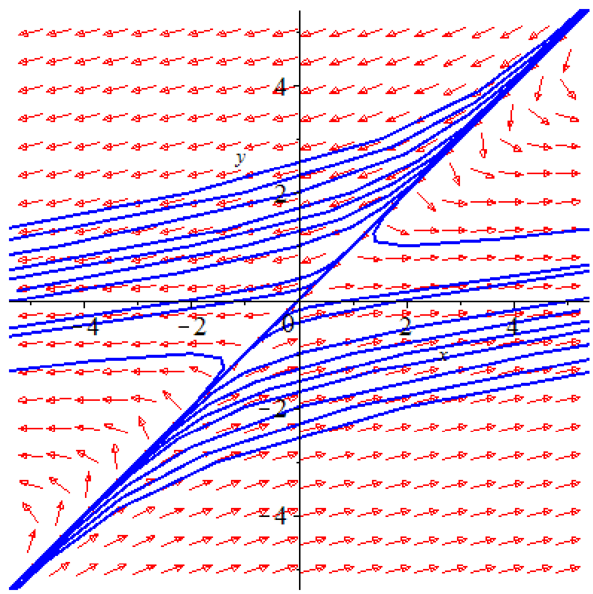
$$\to V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

For $\lambda_2 = 5 \quad \Rightarrow \quad (A-5I)V_2 = 0$

$$\begin{pmatrix} 1 & -7 \\ 1 & -7 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad x_2 = 7y_2$$

$$\to V_2 = \begin{pmatrix} 7 \\ 1 \end{pmatrix} \quad \Rightarrow \quad x_2(t) = \begin{pmatrix} 7 \\ 1 \end{pmatrix} e^{5t}$$

The general solution: $\underline{x(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 7 \\ 1 \end{pmatrix} e^{5t}}$



OR $\begin{cases} x_1(t) = C_1 e^{-t} + 7C_2 e^{5t} \\ x_2(t) = C_1 e^{-t} + C_2 e^{5t} \end{cases}$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = -3x_1 + 4x_2, \quad x_2' = 6x_1 - 5x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -3 & 4 \\ 6 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} -3-\lambda & 4 \\ 6 & -5-\lambda \end{vmatrix} = \lambda^2 + 8\lambda - 9 = 0$$

The distinct real eigenvalues: $\underline{\lambda_1 = -9, \quad \lambda_2 = 1}$

For $\lambda_1 = -9 \implies (A + 9I)V_1 = 0$

$$\begin{pmatrix} 6 & 4 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies 6x_1 = -4y_1$$

$$\rightarrow V_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \implies x_1(t) = \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{-9t}$$

For $\lambda_2 = 1 \implies (A - I)V_2 = 0$

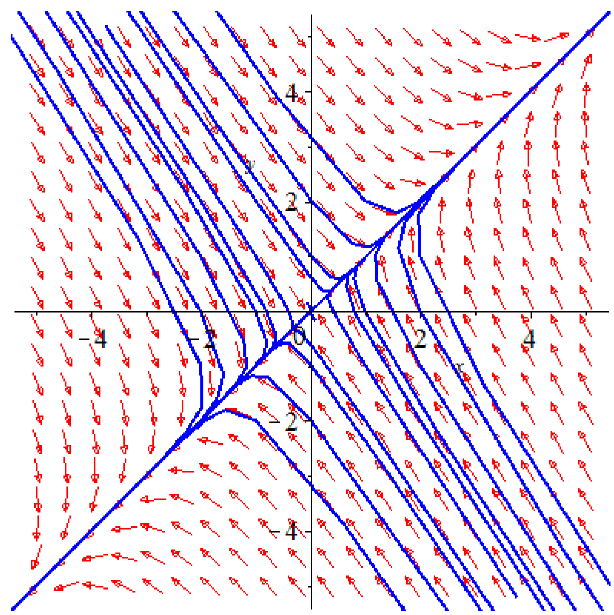
$$\begin{pmatrix} -4 & 4 \\ 6 & -6 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x_2 = y_2$$

$$\rightarrow V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies x_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$$

The general solution:

$$x(t) = C_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{-9t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$$

$$\begin{cases} x_1(t) = 2C_1 e^{-9t} + C_2 e^{t} \\ x_2(t) = -3C_1 e^{-9t} + C_2 e^{t} \end{cases}$$



Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = x_1 - 5x_2, \quad x_2' = x_1 - x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + 4 = 0$$

The distinct real eigenvalues: $\lambda_{1,2} = \pm 2i$

For $\lambda = 2i \quad \Rightarrow \quad (A - \lambda I)V = 0$

$$\begin{pmatrix} 1-2i & -5 \\ 1 & -1-2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad (1-2i)x - 5y = 0 \ \rightarrow \ (1-2i)x = 5y$$

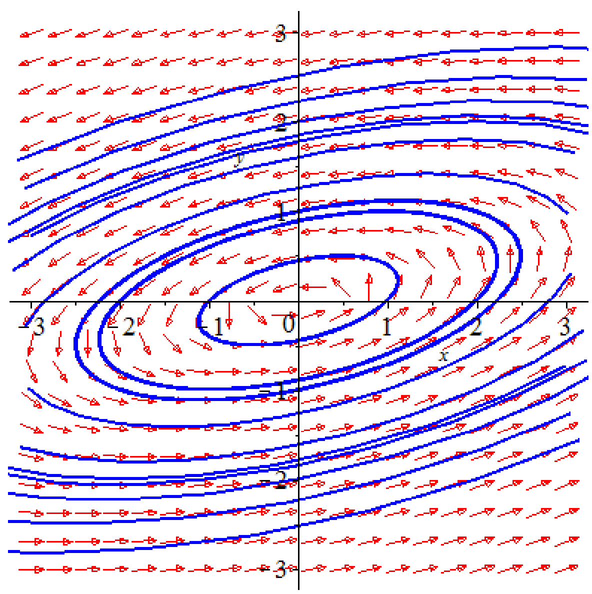
$$\rightarrow V = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix}$$

$$x(t) = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix} e^{2it} \qquad\qquad e^{ait} = \cos at + i \sin at$$

$$= \begin{pmatrix} 5 \\ 1-2i \end{pmatrix}(\cos 2t + i \sin 2t)$$

$$= \begin{pmatrix} 5\cos 2t + 5i \sin 2t \\ \cos 2t + 2\sin 2t + i(\sin 2t - 2\cos 2t) \end{pmatrix}$$

$$\begin{cases} x_1(t) = 5C_1 \cos 2t + 5C_2 \sin 2t \\ x_2(t) = C_1(\cos 2t + 2\sin 2t) + C_2(\sin 2t - 2\cos 2t) \\ \qquad\quad = (C_1 - 2C_2)\cos 2t + (2C_1 + C_2)\sin 2t \end{cases}$$



Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = -3x_1 - 2x_2, \quad x_2' = 9x_1 + 3x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -3 & -2 \\ 9 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} -3-\lambda & -2 \\ 9 & 3-\lambda \end{vmatrix} = \lambda^2 + 9 = 0$$

The distinct real eigenvalues: $\lambda_{1,2} = \pm 3i$

For $\lambda = 3i \Rightarrow (A - \lambda I)V = 0$

$$\begin{pmatrix} -3-3i & -2 \\ 9 & 3-3i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow (-3-3i)x - 2y = 0 \rightarrow (3+3i)x = -2y$$

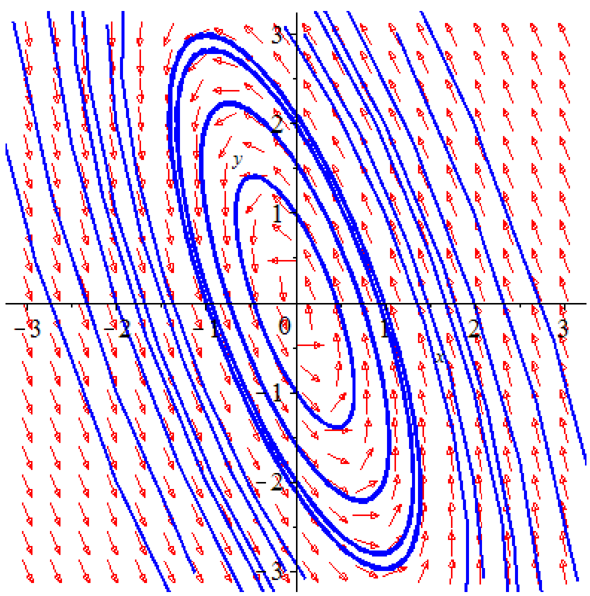
$$\rightarrow V = \begin{pmatrix} -2 \\ 3+3i \end{pmatrix}$$

$$x(t) = \begin{pmatrix} -2 \\ 3+3i \end{pmatrix} e^{3it} \qquad\qquad e^{ait} = \cos at + i\sin at$$

$$= \begin{pmatrix} -2 \\ 3+3i \end{pmatrix}(\cos 3t + i\sin 3t)$$

$$= \begin{pmatrix} -2\cos 3t - 2i\sin 3t \\ 3\cos 3t - 3\sin 3t + i(3\sin 3t + 3\cos 3t) \end{pmatrix}$$

$$\begin{cases} x_1(t) = -2C_1\cos 3t - 2C_2\sin 3t \\ x_2(t) = 3C_1(\cos 3t - \sin 3t) + 3C_2(\sin 3t + \cos 3t) \\ \qquad = 3(C_1 + C_2)\cos 3t + 3(C_2 - C_1)\sin 3t \end{cases}$$



Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = x_1 - 5x_2, \quad x_2' = x_1 + 3x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & -5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

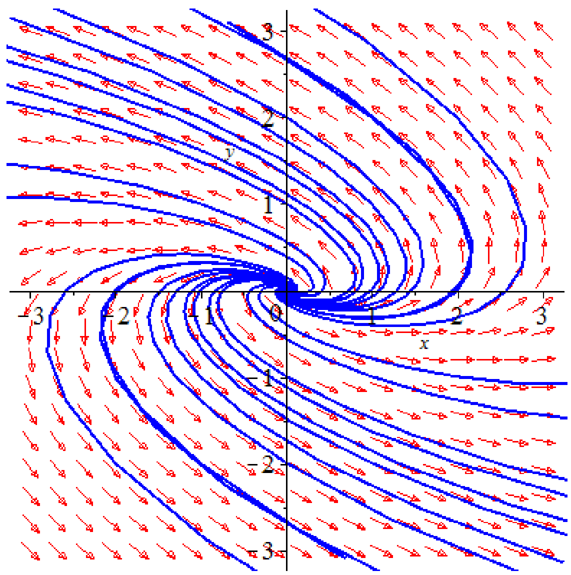
$$\begin{vmatrix} 1-\lambda & -5 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 8 = 0$$

The distinct real eigenvalues: $\underline{\lambda_{1,2} = 2 \pm 2i|}$

For $\lambda = 2+2i \quad \Rightarrow \quad (A - \lambda I)V = 0$

$$\begin{pmatrix} -1-2i & -5 \\ 1 & 1-2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad (1+2i)x = -5y$$

$$\rightarrow V = \begin{pmatrix} -5 \\ 1+2i \end{pmatrix}$$



$$\begin{aligned} x(t) &= \begin{pmatrix} -5 \\ 1+2i \end{pmatrix} e^{(2+2i)t} \\ &= \begin{pmatrix} -5 \\ 1+2i \end{pmatrix} e^{2t} e^{2it} \\ &= \begin{pmatrix} -5 \\ 1+2i \end{pmatrix} e^{2t}(\cos 2t + i\sin 2t) \\ &= \underline{\begin{pmatrix} -5\cos 2t - 5i\sin 2t \\ \cos 2t - 2\sin 2t + i(2\cos 2t + \sin 2t) \end{pmatrix} e^{2t}|} \end{aligned}$$

$$\begin{cases} x_1(t) = (-5C_1\cos 2t - 2C_2\sin 2t)\ e^{2t} \\ x_2(t) = \left[C_1(\cos 2t - 2\sin 2t) + C_2(2\cos 2t + \sin 2t)\right] e^{2t} \\ \qquad\quad = \left[(C_1 + 2C_2)\cos 2t + (C_2 - 2C_1)\sin 2t\right] e^{2t} \end{cases}$$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = 5x_1 - 9x_2, \quad x_2' = 2x_1 - x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 5 & -9 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

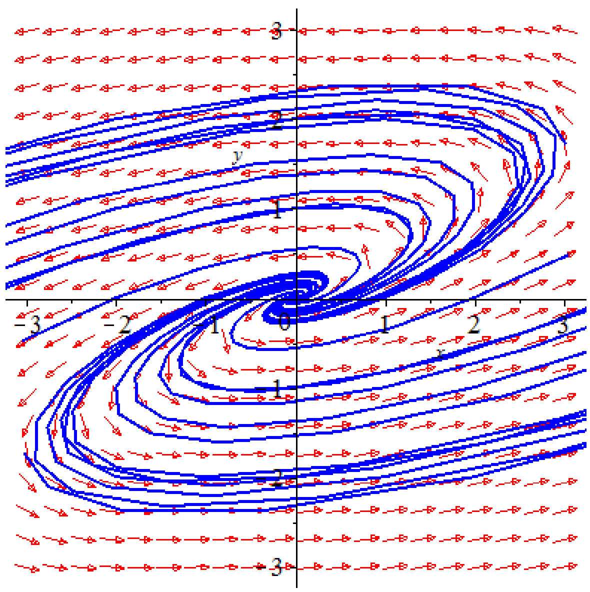
$$\begin{vmatrix} 5-\lambda & -9 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 13 = 0$$

The distinct real eigenvalues: $\underline{\lambda_{1,2} = 2 \pm 3i}$

For $\lambda = 2+3i \quad \Rightarrow \quad (A - \lambda I)V = 0$

$$\begin{pmatrix} 3-3i & -9 \\ 2 & -3-3i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 3(1-i)x = 9y$$

$$\rightarrow V = \begin{pmatrix} 3 \\ 1-i \end{pmatrix}$$



$$\begin{aligned} x(t) &= \begin{pmatrix} 3 \\ 1-i \end{pmatrix} e^{(2+3i)t} \\ &= \begin{pmatrix} 3 \\ 1-i \end{pmatrix} e^{2t} e^{3it} \\ &= \begin{pmatrix} 3 \\ 1-i \end{pmatrix} e^{2t}(\cos 3t + i\sin 3t) \\ &= \underline{\begin{pmatrix} 3\cos 3t + 3i\sin 3t \\ \cos 3t + \sin 3t + i(\sin 3\text{t} - \cos 3t) \end{pmatrix} e^{2t}} \end{aligned}$$

$$\begin{cases} x_1(t) = \left(3C_1 \cos 3t + 3C_2 \sin 3t\right) e^{2t} \\ x_2(t) = \left[C_1(\cos 3\text{t} + \sin 3t) + C_2(\sin 3t - \cos 3t)\right] e^{2t} \\ \qquad = \left[\left(C_1 - C_2\right)\cos 3t + \left(C_1 + C_2\right)\sin 3t\right] e^{2t} \end{cases}$$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = 3x_1 + 4x_2, \quad x_2' = 3x_1 + 2x_2; \quad x_1(0) = x_2(0) = 1$

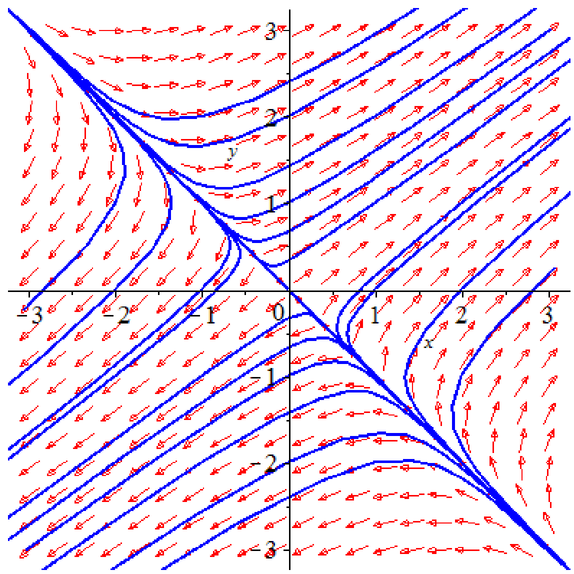
Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 5\lambda - 6 = 0$$

The distinct real eigenvalues: $\underline{\lambda_1 = -1, \quad \lambda_2 = 6}|$



For $\lambda_1 = -1 \Rightarrow (A+I)V_1 = 0$

$$\begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = -y_1$$

$$\rightarrow V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow x_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

For $\lambda_2 = 6 \Rightarrow (A-6I)V_2 = 0$

$$\begin{pmatrix} -3 & 4 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 3x_2 = 4y_2$$

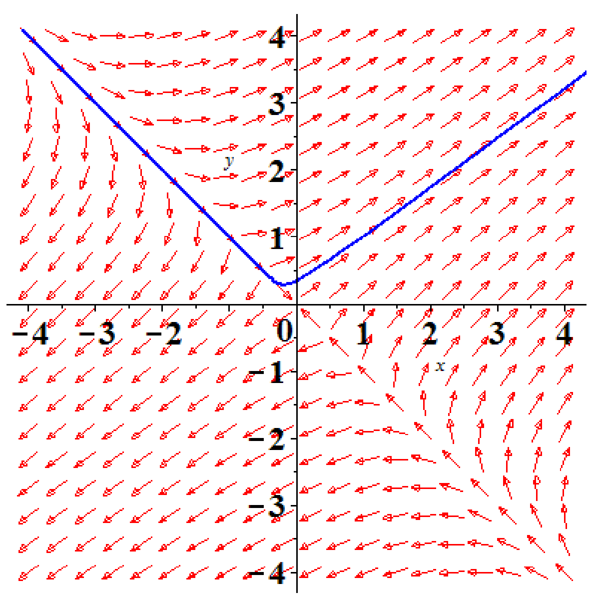
$$\rightarrow V_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \Rightarrow x_2(t) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{6t}$$

The general solution: $x(t) = C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2\begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{6t}$

$$\begin{cases} x_1(t) = C_1 e^{-t} + 4C_2 e^{6t} \\ x_2(t) = -C_1 e^{-t} + 3C_2 e^{6t} \end{cases}$$

Given: $\begin{cases} x_1(0) = C_1 + 4C_2 = 1 \\ x_2(0) = -C_1 + 3C_2 = 1 \end{cases}$

$$\rightarrow \underline{C_2 = \tfrac{2}{7}, \ C_1 = -\tfrac{1}{7}}|$$



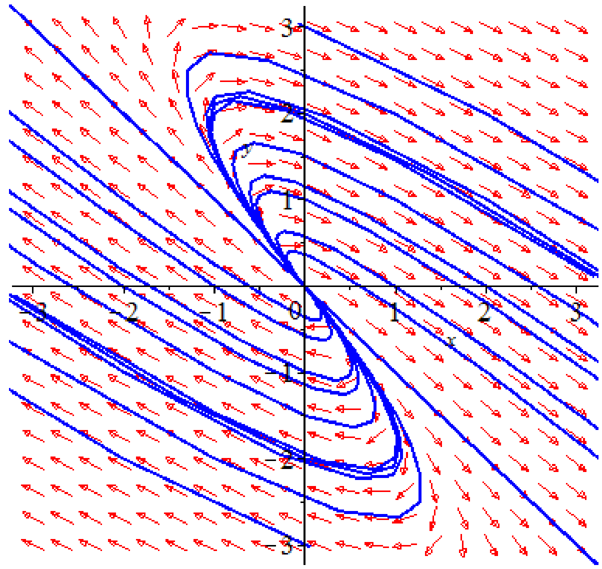
$$\begin{cases} x_1(t) = -\frac{1}{7}e^{-t} + \frac{8}{7}e^{6t} \\ x_2(t) = \frac{1}{7}e^{-t} + \frac{6}{7}e^{6t} \end{cases}$$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = 9x_1 + 5x_2, \quad x_2' = -6x_1 - 2x_2; \quad x_1(0) = 1,\ x_2(0) = 0$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 9 & 5 \\ -6 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



The characteristic equation:

$$\begin{vmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 12 = 0$$

The distinct real eigenvalues: $\underline{\lambda_1 = 3, \quad \lambda_2 = 4|}$

For $\lambda_1 = 3 \ \Rightarrow\ (A-3I)V_1 = 0$

$$\begin{pmatrix} 6 & 5 \\ -6 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \Rightarrow\ 6x_1 = -5y_1$$

$$\rightarrow V_1 = \begin{pmatrix} 5 \\ -6 \end{pmatrix} \ \Rightarrow\ x_1(t) = \begin{pmatrix} 5 \\ -6 \end{pmatrix}e^{3t}$$

For $\lambda_2 = 4 \ \Rightarrow\ (A-4I)V_2 = 0$

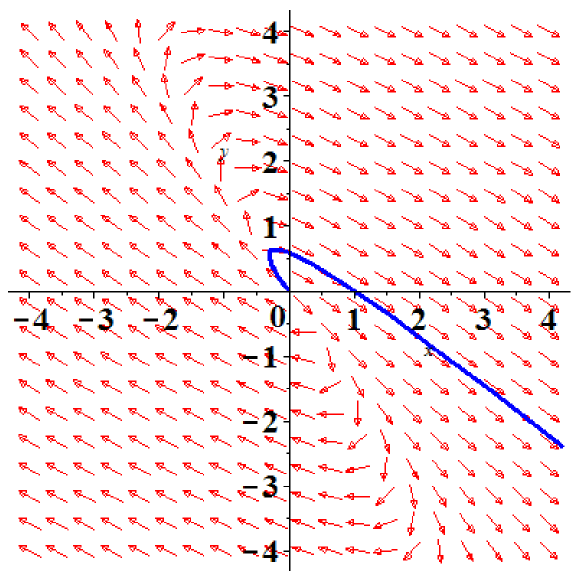
$$\begin{pmatrix} 5 & 5 \\ -6 & -6 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \Rightarrow\ x_2 = -y_2$$

$$\rightarrow V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \ \Rightarrow\ x_2(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{4t}$$

The general solution:

$$x(t) = C_1\begin{pmatrix} 5 \\ -6 \end{pmatrix}e^{3t} + C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{4t}$$

$$\begin{cases} x_1(t) = 5C_1e^{3t} + C_2e^{4t} \\ x_2(t) = -6C_1e^{3t} - C_2e^{4t} \end{cases}$$



Given: $\begin{cases} x_1(0) = 5C_1 + C_2 = 1 \\ x_2(0) = -6C_1 - C_2 = 0 \end{cases}$

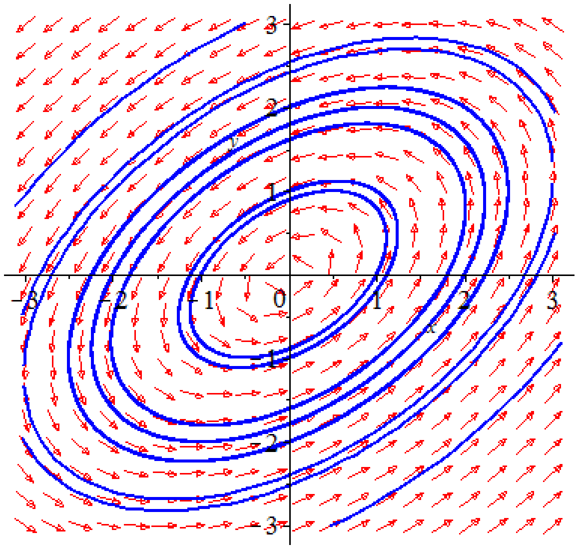
$$\rightarrow \underline{C_1 = -1, \ C_2 = 6|}$$

$$\begin{cases} x_1(t) = -5e^{3t} + 6e^{4t} \\ x_2(t) = 6e^{3t} - 6e^{4t} \end{cases}$$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = 2x_1 - 5x_2, \quad x_2' = 4x_1 - 2x_2; \quad x_1(0) = 2,\ x_2(0) = 3$

Solution



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 2-\lambda & -5 \\ 4 & -2-\lambda \end{vmatrix} = \lambda^2 + 16 = 0$$

The distinct real eigenvalues: $\underline{\lambda = \pm 4i}|$

For $\lambda = 4i \quad \Rightarrow \quad (A - \lambda I)V = 0$

$$\begin{pmatrix} 2-4i & -5 \\ 4 & -2-4i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad (2-4i)x = 5y$$

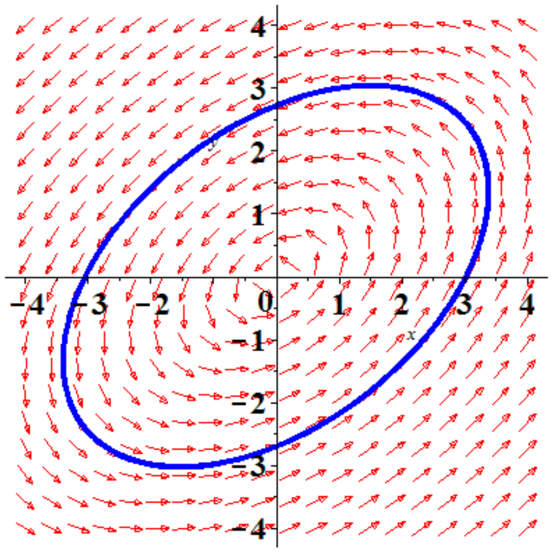
$$\to V = \begin{pmatrix} 5 \\ 2-4i \end{pmatrix}$$

$$x(t) = \begin{pmatrix} 5 \\ 2-4i \end{pmatrix} e^{4it} \qquad\qquad e^{ait} = \cos at + i \sin at$$

$$= \begin{pmatrix} 5 \\ 2-4i \end{pmatrix}(\cos 4t + i \sin 4t)$$

$$= \begin{pmatrix} 5\cos 4t + 5i \sin 4t \\ 2\cos 4t + 4\sin 4t + i(2\sin 4t - 4\cos 4t) \end{pmatrix}$$

$$\begin{cases} x_1(t) = 5C_1 \cos 4t + 5C_2 \sin 4t \\ x_2(t) = C_1(2\cos 4t + 4\sin 4t) + C_2(2\sin 4t - 4\cos 4t) \end{cases}$$



Given: $x_1(0) = 2, \quad x_2(0) = 3$

$$\begin{cases} x_1(0) = 5C_1 = 2 \\ x_2(0) = 2C_1 - 4C_2 = 3 \end{cases} \to \underline{C_1 = \tfrac{2}{5}, \quad C_2 = -\tfrac{11}{20}}|$$

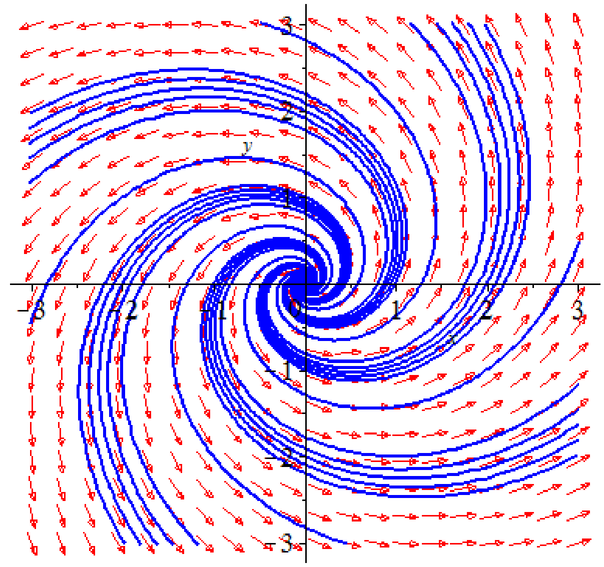
$$\begin{cases} x_1(t) = 2\cos 4t - \frac{11}{4}\sin 4t \\ x_2(t) = \frac{2}{5}(2\cos 4t + 4\sin 4t) - \frac{11}{20}(2\sin 4t - 4\cos 4t) \end{cases}$$

$$\begin{cases} x_1(t) = 2\cos 4t - \frac{11}{4}\sin 4t \\ x_2(t) = 3\cos 4t + \frac{1}{2}\sin 4t \end{cases}$$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = x_1 - 2x_2, \quad x_2' = 2x_1 + x_2; \quad x_1(0) = 0,\ x_2(0) = 4$

Solution



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5 = 0$$

The distinct real eigenvalues: $\underline{\lambda = 1 \pm 2i}$

For $\lambda = 1 - 2i \Rightarrow (A - \lambda I)V = 0$

$$\begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow (2i)x = 2y$$

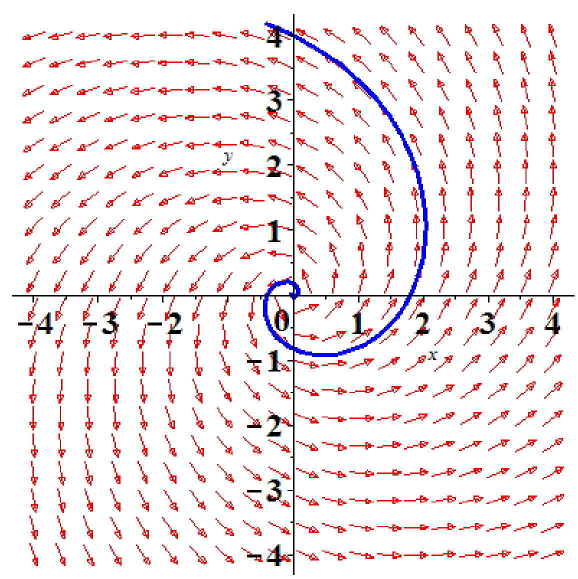
$$\to V = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$x(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(1-2i)t} \qquad\qquad e^{ait} = \cos at + i\sin at$$

$$= \begin{pmatrix} 1 \\ i \end{pmatrix}(\cos 2t - i\sin 2t)\, e^t$$

$$= \underline{\begin{pmatrix} \cos 2t - i\sin 2t \\ \sin 2t + i\cos 2t \end{pmatrix} e^t}$$

$$\begin{cases} x_1(t) = \left(C_1\cos 2t - C_2\sin 2t\right) e^t \\ x_2(t) = \left(C_1\sin 2t + C_2\cos 2t\right) e^t \end{cases}$$



Given: $x_1(0) = 0, \quad x_2(0) = 4$

$$\begin{cases} x_1(0) = C_1 = 0 \\ x_2(0) = C_2 = 4 \end{cases}$$

$$\begin{cases} x_1(t) = -4e^t \sin 2t \\ x_2(t) = 4e^t \cos 2t \end{cases}$$

Exercise

Find the general solution of the given system.

$$x_1' = 4x_1 + x_2 + 4x_3, \quad x_2' = x_1 + 7x_2 + x_3, \quad x_3' = 4x_1 + x_2 + 4x_3$$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 4-\lambda & 1 & 4 \\ 1 & 7-\lambda & 1 \\ 4 & 1 & 4-\lambda \end{vmatrix} = (4-\lambda)^2(7-\lambda) + 8 - 112 + 16\lambda - 8 + 2\lambda$$

$$= \left(16 - 8\lambda + \lambda^2\right)(7-\lambda) + 18\lambda - 112$$

$$= -\lambda^3 + 15\lambda^2 - 54\lambda = 0$$

The distinct real eigenvalues: $\lambda_1 = 0; \quad \lambda_2 = 6; \quad \lambda_3 = 9$

For $\lambda_1 = 0 \quad \Rightarrow \quad (A - 0I)V_1 = 0$

$$\begin{pmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} 4a_1 + b_1 + 4c_1 = 0 \\ a_1 + 7b_1 + c_1 = 0 \end{cases}$$

$$\text{Let } b_1 = 0 \quad \Rightarrow \quad a_1 = -c_1 = 1 \quad \rightarrow V_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $\lambda_2 = 6 \quad \Rightarrow \quad (A - 6I)V_2 = 0$

$$\begin{pmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathit{rref} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} a_2 = c_2 \\ b_2 = -2c_2 \end{cases} \rightarrow V_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 9 \quad \Rightarrow \quad (A - 9I)V_3 = 0$

$$\begin{pmatrix} -5 & 1 & 4 \\ 1 & -2 & 1 \\ 4 & 1 & -5 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathit{rref} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} a_3 = c_3 \\ b_3 = c_3 \end{cases} \rightarrow V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x_1(t) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad x_2(t) = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} e^{6t} \quad x_3(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{9t}$$

$$\begin{cases} x_1(t) = C_1 + C_2 e^{6t} + C_3 e^{9t} \\ x_2(t) = -2C_2 e^{6t} + C_3 e^{9t} \\ x_3(t) = -C_1 - C_2 e^{6t} + C_3 e^{9t} \end{cases}$$

Exercise

Find the general solution of the given system.

$$x_1' = x_1 + 2x_2 + 2x_3, \quad x_2' = 2x_1 + 7x_2 + x_3, \quad x_3' = 2x_1 + x_2 + 7x_3$$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 7-\lambda & 1 \\ 2 & 1 & 7-\lambda \end{vmatrix} = (1-\lambda)(7-\lambda)^2 + 8 - 28 + 4\lambda - 1 + \lambda - 28 + 4\lambda$$

$$= (1-\lambda)\left(49 - 14\lambda + \lambda^2\right) + 9\lambda - 49$$

$$= -\lambda^3 + 15\lambda^2 - 54\lambda = 0$$

The distinct real eigenvalues: $\underline{\lambda_1 = 0; \quad \lambda_2 = 6; \quad \lambda_3 = 9}$

For $\lambda_1 = 0 \quad \Rightarrow \quad (A - 0I)V_1 = 0$

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \Rightarrow \quad \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_1 = -4c_1 \\ b_1 = c_1 \end{cases} \quad \to V_1 = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 6 \quad \Rightarrow \quad (A - 6I)V_2 = 0$

$$\begin{pmatrix} -5 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_2 = 0 \\ b_2 = -c_2 \end{cases} \quad \to V_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

For $\lambda_3 = 9 \quad \Rightarrow \quad (A - 9I)V_3 = 0$

$$\begin{pmatrix} -8 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_3 = \frac{1}{2}c_3 \\ b_3 = c_3 \end{cases} \to V_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$x_1(t)=\begin{pmatrix}-4\\1\\1\end{pmatrix}\quad x_2(t)=\begin{pmatrix}0\\1\\-1\end{pmatrix}e^{6t}\quad x_3(t)=\begin{pmatrix}1\\2\\2\end{pmatrix}e^{9t}$$

$$\begin{cases}x_1(t)=-4C_1+C_3e^{9t}\\ x_2(t)=C_1+C_2e^{6t}+2C_3e^{9t}\\ x_3(t)=C_1-C_2e^{6t}+2C_3e^{9t}\end{cases}$$

Exercise

Find the general solution of the given system.

$$x_1'=4x_1+x_2+x_3,\quad x_2'=x_1+4x_2+x_3,\quad x_3'=x_1+x_2+4x_3$$

Solution

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}'=\begin{pmatrix}4&1&1\\1&4&1\\1&1&4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix}4-\lambda&1&1\\1&4-\lambda&1\\1&1&4-\lambda\end{vmatrix}=(4-\lambda)^3+1+1-3(4-\lambda)$$

$$=64-48\lambda+12\lambda^2-\lambda^3-10+3\lambda$$

$$=-\lambda^3+12\lambda^2-45\lambda+54=0$$

The distinct real eigenvalues: $\lambda_{1,2}=3;\quad \lambda_3=6$

For $\lambda_1=3 \quad\Rightarrow\quad (A-3I)V_1=0$

$$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad\Rightarrow\quad a_1+b_1+c_1=0\qquad\to V_1=\begin{pmatrix}1\\-1\\0\end{pmatrix}\ \to V_2=\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

For $\lambda_3=6 \quad\Rightarrow\quad (A-6I)V_3=0$

$$\begin{pmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{pmatrix}\begin{pmatrix}a_3\\b_3\\c_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad rref\quad\Rightarrow\quad\begin{pmatrix}1&0&-1\\0&1&-1\\0&0&0\end{pmatrix}\to\begin{cases}a_3=c_3\\b_3=c_3\end{cases}\to V_3=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$x_1(t)=\begin{pmatrix}1\\-1\\0\end{pmatrix}e^{3t}\quad x_2(t)=\begin{pmatrix}1\\0\\-1\end{pmatrix}e^{3t}\quad x_3(t)=\begin{pmatrix}1\\1\\1\end{pmatrix}e^{6t}$$

$$\begin{cases}x_1(t)=C_1e^{3t}+C_2e^{3t}+C_3e^{6t}\\ x_2(t)=-C_1e^{3t}\qquad\quad +C_3e^{6t}\\ x_3(t)=\qquad\quad -C_2e^{3t}+C_3e^{6t}\end{cases}$$

Exercise

Find the general solution of the given system.

$$x_1'=5x_1+x_2+3x_3,\quad x_2'=x_1+7x_2+x_3,\quad x_3'=3x_1+x_2+5x_3$$

Solution

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}'=\begin{pmatrix}5&1&3\\1&7&1\\3&1&5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix}5-\lambda&1&3\\1&7-\lambda&1\\3&1&5-\lambda\end{vmatrix}=(7-\lambda)(5-\lambda)^2+6-9(7-\lambda)-5+\lambda-5+\lambda$$

$$=(7-\lambda)\left(25-10\lambda+\lambda^2\right)-67+11\lambda$$

$$=-\lambda^3+17\lambda^2-84\lambda+108=0$$

The distinct real eigenvalues: $\lambda_1=2;\quad \lambda_2=6;\quad \lambda_3=9$

For $\lambda_1=2 \quad\Rightarrow\quad (A-2I)V_1=0$

$$\begin{pmatrix}3&1&3\\1&5&1\\3&1&3\end{pmatrix}\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad rref\Rightarrow\quad\begin{pmatrix}1&0&1\\0&1&0\\0&0&0\end{pmatrix}\to\begin{cases}a_1=-c_1\\b_1=0\end{cases}\quad\to V_1=\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

For $\lambda_2=6 \quad\Rightarrow\quad (A-6I)V_2=0$

$$\begin{pmatrix}-1&1&3\\1&1&1\\3&1&-1\end{pmatrix}\begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad rref\quad\Rightarrow\quad\begin{pmatrix}1&0&-1\\0&1&2\\0&0&0\end{pmatrix}\to\begin{cases}a_2=c_2\\b_2=-2c_2\end{cases}\quad\to V_2=\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

For $\lambda_3=9 \quad\Rightarrow\quad (A-9I)V_3=0$

$$\begin{pmatrix} -4 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} a_3 = c_3 \\ b_3 = c_3 \end{cases} \rightarrow V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x_1(t) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{2t} \quad x_2(t) = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{6t} \quad x_3(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{9t}$$

$$\begin{cases} x_1(t) = -C_1 e^{2t} + C_2 e^{6t} + C_3 e^{9t} \\ x_2(t) = \qquad\quad -2C_2 e^{6t} + C_3 e^{9t} \\ x_3(t) = C_1 e^{2t} + C_2 e^{6t} + C_3 e^{9t} \end{cases}$$

Exercise

Find the general solution of the given system.

$$x_1' = 5x_1 - 6x_3, \quad x_2' = 2x_1 - x_2 - 2x_3, \quad x_3' = 4x_1 - 2x_2 - 4x_3$$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 5-\lambda & 0 & -6 \\ 2 & -1-\lambda & -2 \\ 4 & -2 & -4-\lambda \end{vmatrix} = (-1-\lambda)(5-\lambda)(-4-\lambda) + 24 + 24(-1-\lambda) - 4(5-\lambda)$$

$$= (-1-\lambda)\left(-20 - \lambda + \lambda^2\right) - 24\lambda - 20 + 4\lambda$$

$$= -\lambda^3 + \lambda = 0$$

The distinct real eigenvalues: $\lambda_1 = -1; \quad \lambda_2 = 0; \quad \lambda_3 = 1$

For $\lambda_1 = -1 \quad \Rightarrow \quad (A+I)V_1 = 0$

$$\begin{pmatrix} 6 & 0 & -6 \\ 2 & 0 & -2 \\ 4 & -2 & -3 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \Rightarrow \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} a_1 = c_1 \\ b_1 = \frac{1}{2}c_1 \end{cases} \rightarrow V_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 0 \quad \Rightarrow \quad (A - 0I)V_2 = 0$

$$\begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \begin{pmatrix} 1 & 0 & -\frac{6}{5} \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_2 = \frac{6}{5}c_2 \\ b_2 = \frac{2}{5}c_2 \end{cases} \quad \to V_2 = \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix}$$

For $\lambda_3 = 1 \quad \Rightarrow \quad (A - I)V_3 = 0$

$$\begin{pmatrix} 4 & 0 & -6 \\ 2 & -2 & -2 \\ 4 & -2 & -5 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_3 = \frac{3}{2}c_3 \\ b_3 = \frac{1}{2}c_3 \end{cases} \to V_3 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$x_1(t) = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{-t} \quad x_2(t) = \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} \quad x_3(t) = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} e^{t}$$

$$\begin{cases} x_1(t) = 2C_1e^{-t} + 6C_2 + 3C_3e^{t} \\ x_2(t) = C_1e^{-t} + 2C_2 + C_3e^{t} \\ x_3(t) = 2C_1e^{-t} + 5C_2 + 2C_3e^{t} \end{cases}$$

Exercise

Find the general solution of the given system.

$$x_1' = 3x_1 + 2x_2 + 2x_3, \quad x_2' = -5x_1 - 4x_2 - 2x_3, \quad x_3' = 5x_1 + 5x_2 + 3x_3$$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 3 & 2 & 2 \\ -5 & -4 & -2 \\ 5 & 5 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 3-\lambda & 2 & 2 \\ -5 & -4-\lambda & -2 \\ 5 & 5 & 3-\lambda \end{vmatrix} = (3-\lambda)^2(-4-\lambda) - 20 - 50 - 10(-4-\lambda) + 20(3-\lambda)$$

$$= \left(9 - 6\lambda + \lambda^2\right)(-4-\lambda) + 30 - 10\lambda$$

$$= -\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$$

The distinct real eigenvalues: $\lambda_1 = -2; \quad \lambda_2 = 1; \quad \lambda_3 = 3$

For $\lambda_1 = -2 \quad \Rightarrow \quad (A + 2I)V_1 = 0$

$$\begin{pmatrix}5&2&2\\-5&-2&-2\\5&5&5\end{pmatrix}\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad rref\Rightarrow\quad\begin{pmatrix}1&0&0\\0&1&1\\0&0&0\end{pmatrix}\to\begin{cases}a_1=0\\b_1=-c_1\end{cases}\quad\to V_1=\begin{pmatrix}0\\-1\\1\end{pmatrix}$$

For $\lambda_2=1\ \Rightarrow\ (A-I)V_2=0$

$$\begin{pmatrix}2&2&2\\-5&-5&-2\\5&5&2\end{pmatrix}\begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad rref\quad\Rightarrow\quad\begin{pmatrix}1&1&0\\0&0&1\\0&0&0\end{pmatrix}\to\begin{cases}a_2=-b_2\\c_2=0\end{cases}\quad\to V_2=\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

For $\lambda_3=3\ \Rightarrow\ (A-3I)V_3=0$

$$\begin{pmatrix}0&2&2\\-5&-7&-2\\5&5&0\end{pmatrix}\begin{pmatrix}a_3\\b_3\\c_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad rref\quad\Rightarrow\quad\begin{pmatrix}1&0&-1\\0&1&1\\0&0&0\end{pmatrix}\to\begin{cases}a_3=c_3\\b_3=-c_3\end{cases}\to V_3=\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

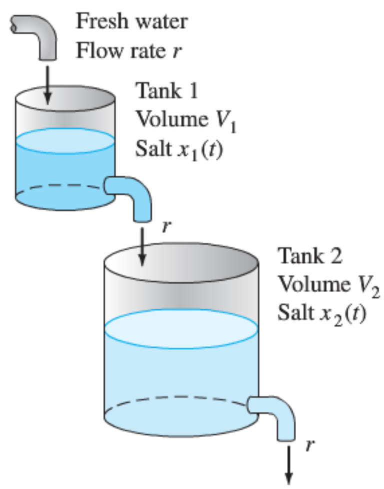
$$x_1(t)=\begin{pmatrix}0\\-1\\1\end{pmatrix}e^{-2t}\quad x_2(t)=\begin{pmatrix}1\\-1\\0\end{pmatrix}e^{t}\quad x_3(t)=\begin{pmatrix}1\\-1\\1\end{pmatrix}e^{3t}$$

$$\begin{cases}x_1(t)= & C_2e^{t}+C_3e^{3t}\\x_2(t)=-C_1e^{-2t}-C_2e^{t}-C_3e^{3t}\\x_3(t)=C_1e^{-2t} & +C_3e^{3t}\end{cases}$$

Exercise

Find the amount $x_1(t),\ x_2(t)$ of salt in each tank at time $t\ge 0$, with $x_1(0)=15\ lb\quad x_2(0)=0$. If $V_1=50\ gal,\quad V_2=25\ gal,\quad r=10\ gal/\min$

Solution



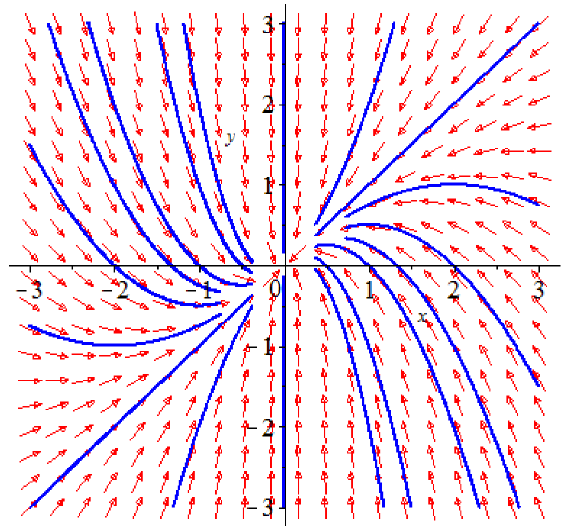
$$\begin{cases}x_1'=-k_1x_1\\x_2'=k_1x_1-k_2x_2\end{cases}\qquad \text{where } k_i=\frac{r}{v_i}\quad i=1,2$$

$$k_1=\frac{10}{50}=.2\quad k_2=\frac{10}{25}=.4\ \to\begin{cases}x_1'=-.2x_1\\x_2'=.2x_1-.4x_2\end{cases}$$

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}'=\begin{pmatrix}-.2&0\\.2&-.4\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}\quad with\quad x(0)=\begin{pmatrix}15\\0\end{pmatrix}$$

$$|A-\lambda I|=\begin{vmatrix}-.2-\lambda&0\\.2&-.4-\lambda\end{vmatrix}=(-.2-\lambda)(-.4-\lambda)=0$$

The eigenvalues are: $\boxed{\lambda_1 = -.4 \quad \lambda_2 = -.2}$



For $\lambda_1 = -.4 \quad \Rightarrow \quad (A + .4I)V_1 = 0$

$$\begin{pmatrix} .2 & 0 \\ .2 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a_1 = 0 \quad \rightarrow \quad V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -.2 \quad \Rightarrow \quad (A + .2I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 \\ .2 & -.2 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a_2 = b_2 \quad \rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow x(t) = C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-.4t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-.2t}$$

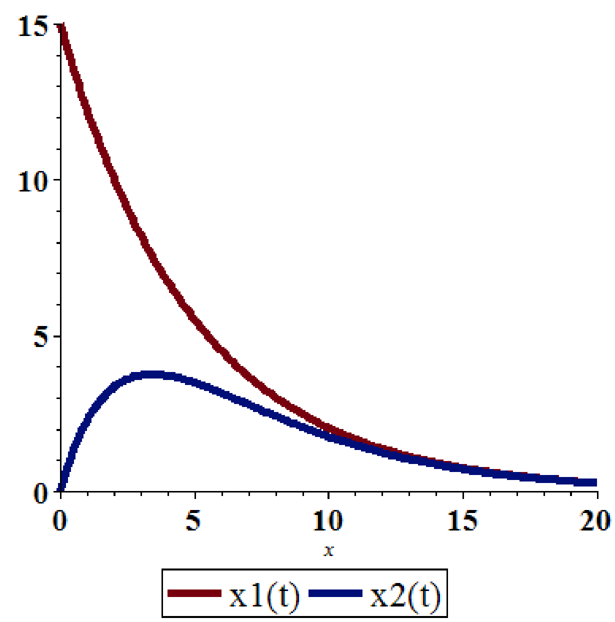
The general solution:

$$\begin{cases} x_1(t) = C_2 e^{-.2t} \\ x_2(t) = C_1 e^{-.4t} + C_2 e^{-.2t} \end{cases}$$

$$\begin{cases} x_1(0) = C_2 = 15 \\ x_2(0) = C_1 + C_2 = 0 \end{cases} \quad \Rightarrow \quad \boxed{C_2 = 15, \ C_1 = -15}$$

$$\begin{cases} x_1(t) = 15e^{-.2t} \\ x_2(t) = 15e^{-.2t} - 15e^{-.4t} \end{cases}$$

***Tank* 2**: $x_2'(t) = -3e^{-.2t} + 6e^{-.4t} = 0$



$$e^{-.2t} = 2e^{-.4t}$$

$$\ln e^{-.2t} = \ln\left(2e^{-.4t}\right)$$

$$-.2t = \ln(2) - .4t$$

$$\boxed{t = \frac{1}{.2}\ln 2 = 5\ln 2}$$

The maximum values of salt in tank 2 is:

$$x_2(t = 5\ln 2) = 15e^{-.2(5\ln 2)} - 15e^{-.4(5\ln 2)}$$

$$= 15\left(2^{-1} - 2^{-2}\right)$$

$$\boxed{= 3.75 \ \ lb.}$$

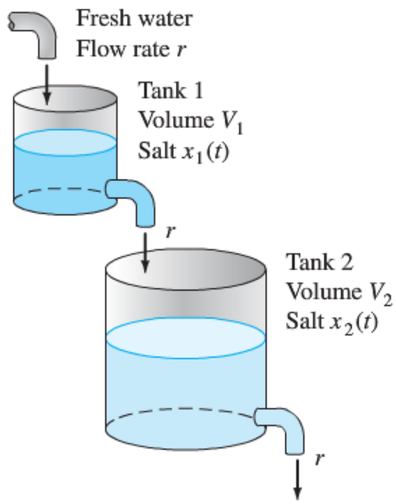
There is no maximum values of salt in tank 1.

$$x_1'(t) = -3e^{-.2t} \neq 0$$

Exercise

Find the amount $x_1(t),\ x_2(t)$ of salt in each tank at time $t \ge 0$, with $x_1(0)=15\ lb \quad x_2(0)=0$. If $V_1=25\ gal, \quad V_2=40\ gal, \quad r=10\ gal/\min$

Solution



$$\begin{cases} x_1' = -k_1 x_1 \\ x_2' = k_1 x_1 - k_2 x_2 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i=1,2$$

$$k_1 = \frac{10}{25} = .4 \quad k_2 = \frac{10}{40} = .25 \rightarrow \begin{cases} x_1' = -.4x_1 \\ x_2' = .4x_1 - .25x_2 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -.4 & 0 \\ .4 & -.25 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \textit{with} \quad x(0)=\begin{pmatrix} 15 \\ 0 \end{pmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} -.4-\lambda & 0 \\ .4 & -.25-\lambda \end{vmatrix} = (-.25-\lambda)(-.4-\lambda) = 0$$

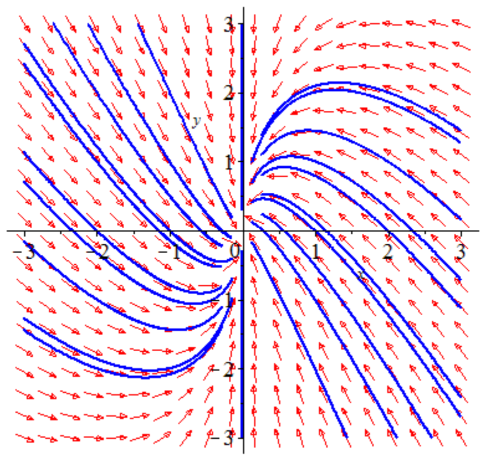
The eigenvalues are: $\lambda_1 = -.4 \quad \lambda_2 = -.25$

For $\lambda_1 = -.4 \quad \Rightarrow \quad (A+.4I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 \\ .4 & .15 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow .4a_1 = -.15b_1 \quad \rightarrow \quad V_1 = \begin{pmatrix} 3 \\ -8 \end{pmatrix}$$

For $\lambda_2 = -.25 \quad \Rightarrow \quad (A+.25I)V_2 = 0$

$$\begin{pmatrix} .15 & 0 \\ .4 & 0 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a_2 = 0 \quad \rightarrow \quad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



$$\Rightarrow x(t) = C_1\begin{pmatrix} 3 \\ -8 \end{pmatrix}e^{-.4t} + C_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-.25t}$$

The general solution:

$$\begin{cases} x_1(t) = 3C_1e^{-.4t} \\ x_2(t) = -8C_1e^{-.4t} + C_2e^{-.25t} \end{cases}$$

$$\begin{cases} x_1(0) = 3C_1 = 15 \\ x_2(0) = -8C_1 + C_2 = 0 \end{cases} \quad \Rightarrow \quad C_1 = 5,\ C_2 = 40$$

$$\begin{cases} x_1(t) = 15e^{-.4t} \\ x_2(t) = -40e^{-.4t} + 40e^{-.25t} \end{cases}$$

There is no maximum values of salt in tank 1.

$$x_1'(t) = -6e^{-.4t} \ne 0$$

***Tank* 2**: $x_2'(t) = 16e^{-.4t} - 10e^{-.25t} = 0$

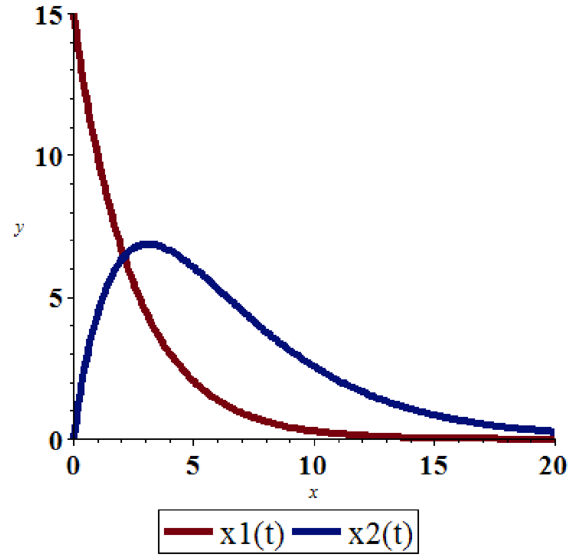
$$8e^{-.4t} = 5e^{-.25t}$$

$$\ln\left(e^{-.4t}\right) = \ln\left(\frac{5}{8}e^{-.25t}\right)$$

$$-.4t = \ln\left(\frac{5}{8}\right) - .25t$$

$$-.15t = \ln\left(\frac{5}{8}\right)$$

$$\underline{t = \frac{1}{.15}\ln\frac{8}{5} = \frac{20}{3}\ln\frac{8}{5}}$$



The maximum values of salt in tank 2 is:

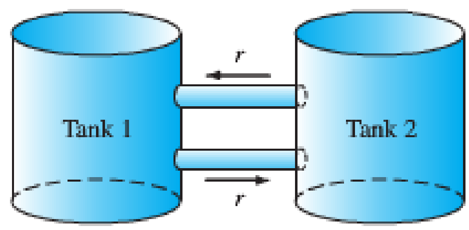
$$x_2\left(t = \frac{20}{3}\ln\frac{8}{5}\right) = -40e^{-.4\left(\frac{20}{3}\ln\frac{8}{5}\right)} + 40e^{-.25\left(\frac{20}{3}\ln\frac{8}{5}\right)}$$

$$\underline{= 6.85 \ \ lb.}$$

Exercise

Find the amount $x_1(t),\ x_2(t)$ of salt in each tank at time $t \ge 0$, with $x_1(0) = 15\ lb \quad x_2(0) = 0$. If $V_1 = 50\ gal, \quad V_2 = 25\ gal, \quad r = 10\ gal/\min$

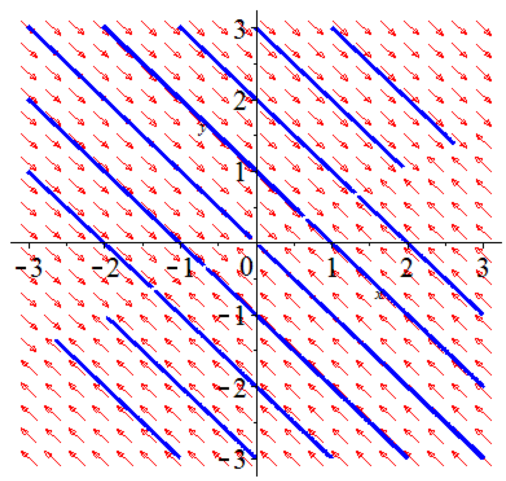
Solution



$$\begin{cases} x_1' = -k_1x_1 + k_2x_2 \\ x_2' = k_1x_1 - k_2x_2 \end{cases} \quad \text{where } k_i = \frac{r}{v_i} \quad i = 1, 2$$

$$k_1 = \frac{10}{50} = .2 \quad k_2 = \frac{10}{25} = .4 \rightarrow \begin{cases} x_1' = -.2x_1 + .4x_2 \\ x_2' = .2x_1 - .4x_2 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -.2 & .4 \\ .2 & -.4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \textit{with} \quad x(0) = \begin{pmatrix} 15 \\ 0 \end{pmatrix}$$



$$|A - \lambda I| = \begin{vmatrix} -.2-\lambda & .4 \\ .2 & -.4-\lambda \end{vmatrix}$$

$$= (-.2-\lambda)(-.4-\lambda) - .08$$

$$= \lambda^2 + .6\lambda = 0$$

The eigenvalues are: $\underline{\lambda_1 = -.6 \quad \lambda_2 = 0}$

For $\lambda_1 = -.6 \quad \Rightarrow \quad (A + .6I)V_1 = 0$

$$\begin{pmatrix} .4 & .4 \\ .2 & .2 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow .4a_1 = -.4b_1 \quad \rightarrow \quad V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = 0 \quad \Rightarrow \quad (A - 0I)V_2 = 0$

$$\begin{pmatrix} -.2 & .4 \\ .2 & -.4 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow .2a_2 = .4b_2 \quad \rightarrow \quad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

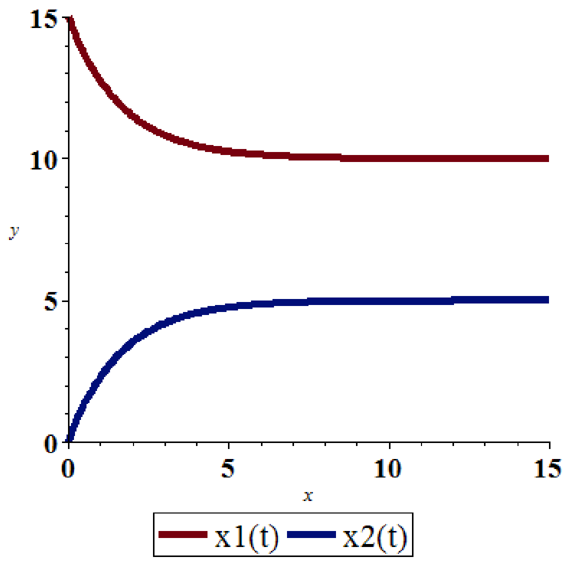
$$\Rightarrow x(t) = C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-.6t} + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The general solution:

$$\begin{cases} x_1(t) = C_1 e^{-0.6t} + 2C_2 \\ x_2(t) = -C_1 e^{-0.6t} + C_2 \end{cases}$$

$$\begin{cases} x_1(0) = C_1 + 2C_2 = 15 \\ x_2(0) = -C_1 + C_2 = 0 \end{cases} \quad \Rightarrow \quad \underline{C_1 = 5,\ C_2 = 5|}$$

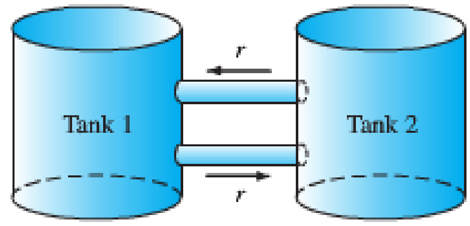
$$\begin{cases} x_1(t) = 10 + 5e^{-0.6t} \\ x_2(t) = 5 - 5e^{-0.6t} \end{cases}$$



Exercise

Find the amount $x_1(t),\ x_2(t)$ of salt in each tank at time $t \ge 0$, with $x_1(0)=15\ lb \quad x_2(0)=0$. If

$V_1=25\ gal,\quad V_2=40\ gal,\quad r=10\ gal\,/\,\text{min}$



Solution

$$\begin{cases} x_1' = -k_1x_1 + k_2x_2 \\ x_2' = k_1x_1 - k_2x_2 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i=1,2$$

$$k_1 = \frac{10}{25} = .4 \quad k_2 = \frac{10}{40} = .25 \rightarrow \begin{cases} x_1' = -.4x_1 + .25x_2 \\ x_2' = .4x_1 - .25x_2 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -.4 & .25 \\ .4 & -.25 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \textit{with} \quad x(0)=\begin{pmatrix} 15 \\ 0 \end{pmatrix}$$

$$\begin{aligned} |A-\lambda I| &= \begin{vmatrix} -.4-\lambda & .25 \\ .4 & -.25-\lambda \end{vmatrix} \\ &= (-.25-\lambda)(-.4-\lambda) - .1 \\ &= \lambda^2 + .65\lambda = 0 \end{aligned}$$

The eigenvalues are: $\underline{\lambda_1 = 0 \quad \lambda_2 = -.65|}$

For $\lambda_1 = 0 \ \Rightarrow \ (A-0I)V_1 = 0$

$$\begin{pmatrix} -.4 & .25 \\ .4 & -.25 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow .4a_1 = .25b_1 \quad \rightarrow \quad V_1 = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

For $\lambda_2 = -.65 \ \Rightarrow \ (A+.65I)V_2 = 0$

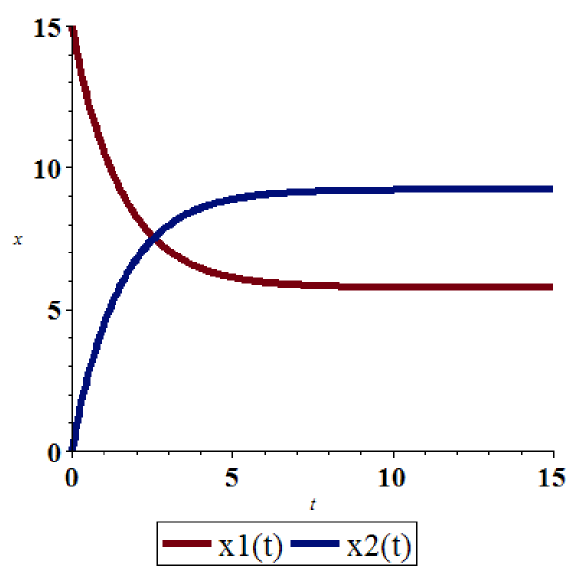
$$\begin{pmatrix} .25 & .25 \\ .4 & .4 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow .25a_2 = -.25b_2 \quad \rightarrow \quad V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow x(t) = C_1\begin{pmatrix} 5 \\ 8 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-.65t}$$

The general solution: $\begin{cases} x_1(t) = 5C_1 + C_2e^{-0.65t} \\ x_2(t) = 8C_1 - C_2e^{-0.65t} \end{cases}$

$$\begin{cases} x_1(0) = 5C_1 + C_2 = 15 \\ x_2(0) = 8C_1 - C_2 = 0 \end{cases} \Rightarrow \underline{C_1 = \frac{15}{13},\ C_2 = \frac{120}{13}|}$$

$$\begin{cases} x_1(t) = \frac{15}{13}\left(5 + 8e^{-0.6t}\right) \\ x_2(t) = \frac{120}{13}\left(1 - e^{-0.6t}\right) \end{cases}$$

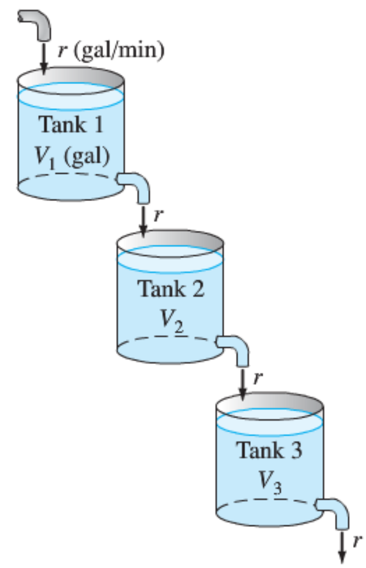


Exercise

Find the amount $x_1(t),\ x_2(t),\ x_3(t)$ of salt in each tank at time $t \ge 0$, if

$V_1 = 30\ gal,\quad V_2 = 15\ gal,\quad V_3 = 10\ gal,\quad r = 30\ gal/\min \quad x_1(0) = 27\ lb \quad x_2(0) = x_3(0) = 0$

Solution



$$\begin{cases} x_1' = -k_1 x_1 \\ x_2' = k_1 x_1 - k_2 x_2 \\ x_3' = \qquad\quad k_2 x_2 - k_3 x_3 \end{cases} \qquad \text{where } k_i = \frac{r}{V_i} \quad i = 1,2,3$$

$$k_1 = \frac{30}{30} = 1 \quad k_2 = \frac{30}{15} = 2 \quad k_3 = \frac{30}{10} = 3$$

$$\to \begin{cases} x_1' = -x_1 \\ x_2' = x_1 - 2x_2 \\ x_3' = \qquad 2x_2 - 3x_3 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \textit{with} \quad x(0) = \begin{pmatrix} 27 \\ 0 \\ 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 0 & 0 \\ 1 & -2-\lambda & 0 \\ 0 & 2 & -3-\lambda \end{vmatrix} = (-1-\lambda)(-2-\lambda)(-3-\lambda) = 0$$

The eigenvalues are: $\underline{\lambda_1 = -3 \quad \lambda_2 = -2 \quad \lambda_3 = -1}$

For $\lambda_1 = -3 \quad \Rightarrow \quad (A + 3I)V_1 = 0$

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} 2a_1 = 0 \ \to\ a_1 = 0 \\ a_1 = -b_1 \ \to\ b_1 = 0 \end{cases} \quad \to V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-3t}$$

For $\lambda_2 = -2 \quad \Rightarrow \quad (A + 2I)V_2 = 0$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} a_2 = 0 \\ 2b_2 = c_2 \end{cases} \quad \to V_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \Rightarrow \quad x_2(t) = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} e^{-2t}$$

For $\lambda_3 = -1 \quad \Rightarrow \quad (A + I)V_3 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} a_3 = b_3 \\ 2b_3 = 2c_3 \end{cases} \quad \to V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad x_3(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-t}$$

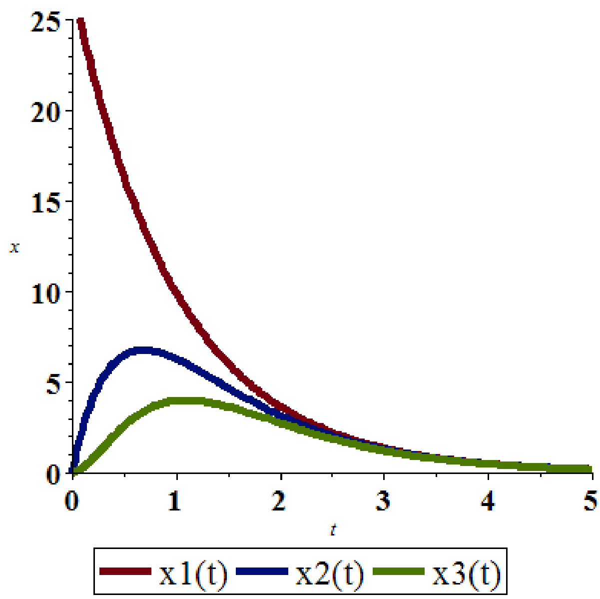
$$\Rightarrow \; x(t) = C_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-3t} + C_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} e^{-2t} + C_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-t}$$

$$\begin{cases} x_1(t) = & & C_3 e^{-t} \\ x_2(t) = & C_2 e^{-2t} + & C_3 e^{-t} \\ x_3(t) = C_1 e^{-3t} + & 2C_2 e^{-2t} + & C_3 e^{-t} \end{cases}$$

With *initial* values

$$\begin{cases} 27 = C_3 \\ 0 = C_2 + C_3 \\ 0 = C_1 + 2C_2 + C_3 \end{cases} \rightarrow \begin{cases} \underline{C_3 = 27} \\ \underline{C_2 = -27} \\ \underline{C_1} = -27 - 2(-27) \underline{= 27} \end{cases}$$

$$\begin{cases} x_1(t) = 27e^{-t} \\ x_2(t) = 27e^{-t} - 27e^{-2t} \\ x_3(t) = 27e^{-t} - 54e^{-2t} + 27e^{-3t} \end{cases}$$



***Tank* 2**: $x_2'(t) = -27e^{-t} + 54e^{-2t} = 0$

$$e^{-t} = 2e^{-2t} \quad \Rightarrow \quad -t = \ln 2 - 2t$$

$$\underline{t = \ln 2}$$

The maximum values of salt in tank 2 is:

$$x_2(\ln 2) = 27\left(e^{-\ln 2} - e^{-2\ln 2}\right) = 27\left(\tfrac{1}{2} - \tfrac{1}{4}\right)$$

$$\underline{= \frac{27}{4} \; lbs}$$

***Tank* 3**: $x_3'(t) = 27\left(-e^{-t} + 4e^{-2t} - 3e^{-3t}\right) = 0$

$$e^{3t}\left(-e^{-t} + 4e^{-2t} - 3e^{-3t}\right) = 0$$

$$e^{2t} - 4e^{t} + 3 = 0 \quad \begin{cases} e^{t} = 1 \rightarrow t = 0 \\ e^{t} = 3 \rightarrow \underline{t = \ln 3} \end{cases}$$

The maximum values of salt in tank 3 is:

$$x_3(\ln 3) = 27\left(e^{-\ln 3} - 2e^{-2\ln 3} + e^{-3\ln 3}\right) = 27\left(\tfrac{1}{3} - \tfrac{2}{9} + \tfrac{1}{27}\right)$$

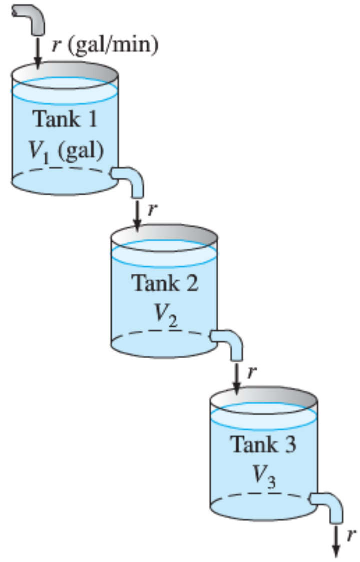
$$\underline{= 4 \; lbs}$$

Exercise

Find the amount $x_1(t),\ x_2(t),\ x_3(t)$ of salt in each tank at time $t \ge 0$, if

$V_1 = 20\ gal,\quad V_2 = 30\ gal,\quad V_3 = 60\ gal,\quad r = 60\ gal/\min \quad x_1(0) = 45\ lb \quad x_2(0) = x_3(0) = 0$

Solution



$$\begin{cases} x_1' = -k_1 x_1 \\ x_2' = k_1 x_1 - k_2 x_2 \\ x_3' = \qquad\quad k_2 x_2 - k_3 x_3 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i = 1,2,3$$

$$k_1 = \frac{60}{20} = 3 \quad k_2 = \frac{60}{30} = 2 \quad k_3 = \frac{60}{60} = 1$$

$$\rightarrow \begin{cases} x_1' = -3x_1 \\ x_2' = 3x_1 - 2x_2 \\ x_3' = \qquad 2x_2 - x_3 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -3 & 0 & 0 \\ 3 & -2 & 0 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad with \quad x(0) = \begin{pmatrix} 45 \\ 0 \\ 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -3-\lambda & 0 & 0 \\ 3 & -2-\lambda & 0 \\ 0 & 2 & -1-\lambda \end{vmatrix} = (-3-\lambda)(-2-\lambda)(-1-\lambda) = 0$$

The eigenvalues are: $\underline{\lambda_1 = -3 \quad \lambda_2 = -2 \quad \lambda_3 = -1}$

For $\lambda_1 = -3 \quad \Rightarrow \quad (A + 3I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 3a_1 = -b_1 & \rightarrow a_1 = 1 \\ 2c_1 = -2b_1 & \rightarrow b_1 = -3 \\ & \quad\ c_1 = 3 \end{cases} \rightarrow V_1 = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} \Rightarrow x_1(t) = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} e^{-3t}$$

For $\lambda_2 = -2 \quad \Rightarrow \quad (A + 2I)V_2 = 0$

$$\begin{pmatrix} -1 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a_2 = 0 \\ 2b_2 = -c_2 \end{cases} \rightarrow V_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \Rightarrow x_2(t) = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} e^{-2t}$$

For $\lambda_3 = -1 \quad \Rightarrow \quad (A + I)V_3 = 0$

$$\begin{pmatrix} -2 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a_3 = b_3 = 0 \rightarrow V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow x_3(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t}$$

$$\Rightarrow\ x(t)=C_1\begin{pmatrix}1\\-3\\3\end{pmatrix}e^{-3t}+C_2\begin{pmatrix}0\\1\\-2\end{pmatrix}e^{-2t}+C_3\begin{pmatrix}0\\0\\1\end{pmatrix}e^{-t}$$

$$\begin{cases}x_1(t)=C_1e^{-3t}\\ x_2(t)=-3C_1e^{-3t}+\ C_2e^{-2t}\\ x_3(t)=3C_1e^{-3t}-2C_2e^{-2t}+C_3e^{-t}\end{cases}$$

With *initial* values

$$\begin{cases}45=C_1\\ 0=-3C_1+C_2\\ 0=3C_1-2C_2+C_3\end{cases}\quad\rightarrow\begin{cases}\underline{C_1=45}\\ \underline{C_2=135}\\ \underline{C_3}=-3(45)+2(-135)\underline{=135}\end{cases}$$

$$\begin{cases}x_1(t)=45e^{-3t}\\ x_2(t)=-135e^{-3t}+135e^{-2t}\\ x_3(t)=135e^{-3t}-270e^{-2t}+135e^{-t}\end{cases}$$

***Tank* 2**: $x_2'(t)=3e^{-3t}-2e^{-2t}=0$

$1.5e^{-3t}=e^{-2t}\quad\Rightarrow\quad \ln1.5-3t=-2t$

$\underline{t=\ln1.5}$

The maximum values of salt in tank 2 is:

$x_2(\ln1.5)=135\left(-e^{-3\ln1.5}+e^{-2\ln1.5}\right)=135\left(-\frac{8}{27}+\frac{4}{9}\right)$

$\underline{=20\ lbs}$

***Tank* 3**: $x_3'(t)=135\left(-3e^{-3t}+4e^{-2t}-e^{-t}\right)=0$

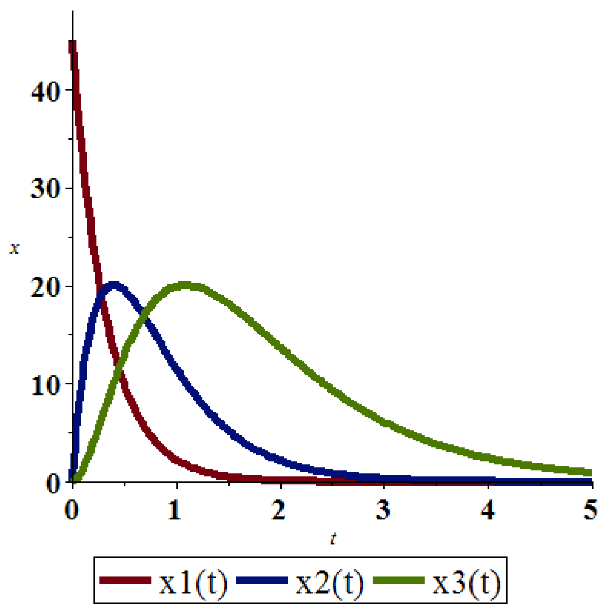
$e^{3t}\left(-3e^{-3t}+4e^{-2t}-e^{-t}\right)=0$

$-3+4e^{t}-e^{2t}=0\quad\begin{cases}e^{t}=1\rightarrow t=0\\ e^{t}=3\rightarrow \underline{t=\ln3}\end{cases}$

The maximum values of salt in tank 3 is:

$x_2(\ln3)=135\left(e^{-3\ln3}-2e^{-2\ln3}+e^{-\ln3}\right)=135\left(\frac{1}{27}-\frac{2}{9}+\frac{1}{3}\right)$

$\underline{=20\ lbs}$



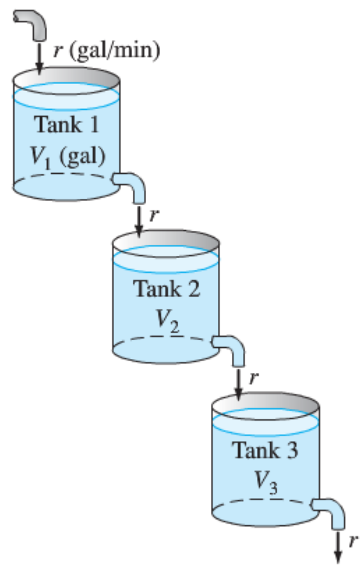
Exercise

Find the amount $x_1(t),\ x_2(t),\ x_3(t)$ of salt in each tank at time $t \ge 0$, if

$V_1 = 15\ gal,\quad V_2 = 10\ gal,\quad V_3 = 30\ gal,\quad r = 60\ gal/\min\quad x_1(0) = 45\ lb\quad x_2(0) = x_3(0) = 0$

Solution

$$\begin{cases} x_1' = -k_1 x_1 \\ x_2' = k_1 x_1 - k_2 x_2 \\ x_3' = \qquad k_2 x_2 - k_3 x_3 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i = 1,2,3$$



$$k_1 = \frac{60}{15} = 4 \quad k_2 = \frac{60}{10} = 6 \quad k_3 = \frac{60}{30} = 2$$

$$\rightarrow \begin{cases} x_1' = -4x_1 \\ x_2' = 4x_1 - 6x_2 \\ x_3' = \qquad 6x_2 - 2x_3 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -4 & 0 & 0 \\ 4 & -6 & 0 \\ 0 & 6 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad with \quad x(0) = \begin{pmatrix} 45 \\ 0 \\ 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -4-\lambda & 0 & 0 \\ 4 & -6-\lambda & 0 \\ 0 & 6 & -2-\lambda \end{vmatrix} = (-4-\lambda)(-6-\lambda)(-2-\lambda) = 0$$

The eigenvalues are: $\underline{\lambda_1 = -4 \quad \lambda_2 = -6 \quad \lambda_3 = -2}$

For $\lambda_1 = -4 \quad \Rightarrow \quad (A + 4I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 4 & -2 & 0 \\ 0 & 6 & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 4a_1 = 2b_1 & \rightarrow a_1 = 1 \\ 2c_1 = -6b_1 & \rightarrow b_1 = 2 \\ & \quad c_1 = -6 \end{cases} \rightarrow V_1 = \begin{pmatrix} 1 \\ 2 \\ -6 \end{pmatrix} \Rightarrow x_1(t) = \begin{pmatrix} 1 \\ 2 \\ -6 \end{pmatrix} e^{-4t}$$

For $\lambda_2 = -6 \quad \Rightarrow \quad (A + 6I)V_2 = 0$

$$\begin{pmatrix} 2 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 6 & 4 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a_2 = 0 \\ 6b_2 = -4c_2 \end{cases} \rightarrow V_2 = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} \Rightarrow x_2(t) = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} e^{-6t}$$

For $\lambda_3 = -2 \quad \Rightarrow \quad (A + 2I)V_3 = 0$

$$\begin{pmatrix} -2 & 0 & 0 \\ 4 & -4 & 0 \\ 0 & 6 & 0 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a_3 = b_3 = 0 \rightarrow V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow x_3(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-2t}$$

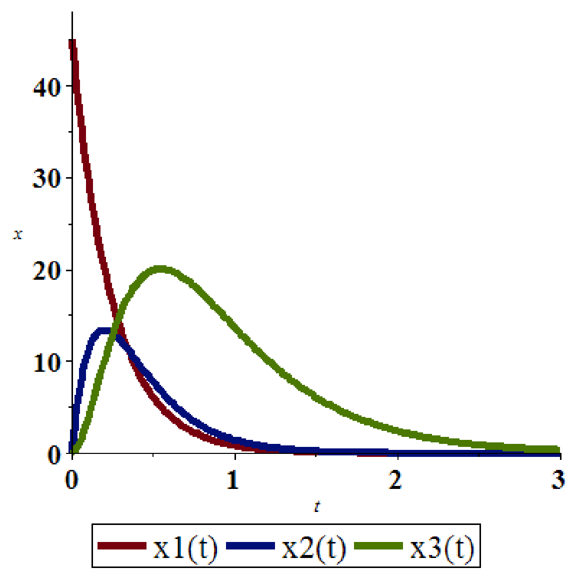
$$\Rightarrow x(t)=C_1\begin{pmatrix}1\\2\\-6\end{pmatrix}e^{-4t}+C_2\begin{pmatrix}0\\2\\-3\end{pmatrix}e^{-6t}+C_3\begin{pmatrix}0\\0\\1\end{pmatrix}e^{-2t}$$

$$\begin{cases}x_1(t)=C_1e^{-4t}\\x_2(t)=2C_1e^{-4t}+2C_2e^{-6t}\\x_3(t)=-6C_1e^{-4t}-3C_2e^{-6t}+C_3e^{-2t}\end{cases}$$

With *initial* values

$$\begin{cases}45=C_1\\0=2C_1+2C_2\\0=-6C_1-3C_2+C_3\end{cases}\to\begin{cases}\boxed{C_1=45}\\\boxed{C_2=-45}\\\boxed{C_3=6(45)+3(-45)=135}\end{cases}$$

$$\begin{cases}x_1(t)=45e^{-4t}\\x_2(t)=90e^{-4t}-90e^{-6t}\\x_3(t)=-270e^{-4t}+135e^{-6t}+135e^{-2t}\end{cases}$$



***Tank* 2**: $x_2'(t)=-360e^{-4t}+540e^{-6t}=0$

$$2e^{-4t}=3e^{-6t}\quad\Rightarrow\quad\ln(2)-4t=\ln(3)-6t$$

$$\boxed{t=\frac{1}{2}\ln 1.5}$$

The maximum values of salt in tank 2 is:

$$x_2\left(\tfrac{1}{2}\ln 1.5\right)=90\left(e^{-2\ln 1.5}-e^{-3\ln 1.5}\right)=90\left(\tfrac{4}{9}-\tfrac{8}{27}\right)$$

$$\underline{=13.3\ lbs}$$

***Tank* 3**: $x_3'(t)=135\left(8e^{-4t}-6e^{-6t}-2e^{-2t}\right)=0$

$$-2e^{-6t}\left(4e^{2t}-3-e^{4t}\right)=0$$

$$e^{4t}-4e^{2t}+3=0\qquad\begin{cases}e^{2t}=1\to t=0\\e^{2t}=3\to \boxed{t=\tfrac{1}{2}\ln 3}\end{cases}$$

The maximum values of salt in tank 3 is:

$$x_2\left(\tfrac{1}{2}\ln 3\right)=135\left(-2e^{-2\ln 3}+e^{-3\ln 3}+e^{-\ln 3}\right)=135\left(-\tfrac{2}{9}+\tfrac{1}{27}+\tfrac{1}{3}\right)$$

$$\underline{=20\ lbs}$$