Solution Section 2.8 – Row and Column Spaces

Exercise

List the row vectors and column vectors of the matrix

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ 3 & 5 & 7 & -1 \\ 1 & 4 & 2 & 7 \end{bmatrix}$$

Solution

Row vectors: $r_1 = \begin{bmatrix} 2 & -1 & 0 & 1 \end{bmatrix}$, $r_2 = \begin{bmatrix} 3 & 5 & 7 & -1 \end{bmatrix}$, $r_3 = \begin{bmatrix} 1 & 4 & 2 & 7 \end{bmatrix}$

Column vectors: $c_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $c_2 = \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}$, $c_3 = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$, $c_4 = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$

Exercise

Express the product A$\boldsymbol{x}$ as a linear combination of the column vectors of *A*.

a) $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

b) $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$

c) $\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$

Solution

a) $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

b) $\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = -1\begin{bmatrix} -3 \\ 5 \\ 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 6 \\ -4 \\ 3 \\ 8 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}$

c) $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = -2\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 6 \\ -1 \end{bmatrix} + 5\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$

Exercise

Determine whether $\boldsymbol{b}$ is in the column space of A, and if so, express $\boldsymbol{b}$ as a linear combination of the column vectors of A.

a) $A=\begin{bmatrix}1 & 3\\ 4 & -6\end{bmatrix},\quad b=\begin{bmatrix}-2\\ 10\end{bmatrix}$

b) $A=\begin{bmatrix}1 & -1 & 1\\ 9 & 3 & 1\\ 1 & 1 & 1\end{bmatrix},\quad b=\begin{bmatrix}5\\ 1\\ -1\end{bmatrix}$

c) $A=\begin{bmatrix}1 & 1 & 2\\ 1 & 0 & 1\\ 2 & 1 & 3\end{bmatrix}\quad \boldsymbol{b}=\begin{bmatrix}-1\\ 0\\ 2\end{bmatrix}$

d) $A=\begin{bmatrix}1 & 2 & 0 & 1\\ 0 & 1 & 2 & 1\\ 1 & 2 & 1 & 3\\ 0 & 1 & 2 & 2\end{bmatrix},\quad b=\begin{bmatrix}4\\ 3\\ 5\\ 7\end{bmatrix}$

Solution

a) $\left[\begin{array}{cc|c}1 & 3 & -2\\ 4 & -6 & 10\end{array}\right]\xrightarrow{rref}\left[\begin{array}{cc|c}1 & 0 & 1\\ 0 & 1 & -1\end{array}\right]$

$$\begin{bmatrix}-2\\ 10\end{bmatrix}=\begin{bmatrix}1\\ 4\end{bmatrix}-\begin{bmatrix}3\\ -6\end{bmatrix}$$

b) $\left[\begin{array}{ccc|c}1 & -1 & 1 & 5\\ 9 & 3 & 1 & 1\\ 1 & 1 & 1 & -1\end{array}\right]\xrightarrow{rref}\left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right]$

The system $A\boldsymbol{x}=\boldsymbol{b}$ is inconsistent and $\boldsymbol{b}$ is not in the column space of A.

c) $\left[\begin{array}{ccc|c}1 & 1 & 2 & -1\\ 1 & 0 & 1 & 0\\ 2 & 1 & 3 & 2\end{array}\right]\xrightarrow{rref}\left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right]$

The system $A\boldsymbol{x}=\boldsymbol{b}$ is inconsistent and $\boldsymbol{b}$ is not in the column space of A.

d) $\left[\begin{array}{cccc|c}1 & 2 & 0 & 1 & 4\\ 0 & 1 & 2 & 1 & 3\\ 1 & 2 & 1 & 3 & 5\\ 0 & 1 & 2 & 2 & 7\end{array}\right]\xrightarrow{rref}\left[\begin{array}{cccc|c}1 & 0 & 0 & 0 & -26\\ 0 & 1 & 0 & 0 & 13\\ 0 & 0 & 1 & 0 & -7\\ 0 & 0 & 0 & 1 & 4\end{array}\right]$

$$\begin{bmatrix}4\\ 3\\ 5\\ 7\end{bmatrix}=-26\begin{bmatrix}1\\ 0\\ 1\\ 0\end{bmatrix}+13\begin{bmatrix}2\\ 1\\ 2\\ 1\end{bmatrix}-7\begin{bmatrix}0\\ 2\\ 1\\ 2\end{bmatrix}+4\begin{bmatrix}1\\ 1\\ 3\\ 2\end{bmatrix}$$

Exercise

Suppose that $x_1 = -1, \quad x_2 = 2, \quad x_3 = 4, \quad x_4 = -3$ is a solution of a nonhomogeneous linear system A$\boldsymbol{x}$ = $\boldsymbol{b}$ and that the solution set of the homogeneous system A$\boldsymbol{x}$ = **0** is given by the formulas

$$x_1 = -3r + 4s, \quad x_2 = r - s, \quad x_3 = r, \quad x_4 = s$$

a) Find a vector form of the general solution of A$\boldsymbol{x}$ = **0**

b) Find a vector form of the general solution of A$\boldsymbol{x}$ = $\boldsymbol{b}$

Solution

a) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

b) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 4 \\ -3 \end{bmatrix} + r\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Exercise

Find the vector form of the general solution of the given linear system A$\boldsymbol{x}$ = $\boldsymbol{b}$; then use that result to find the vector form of the general solution of A$\boldsymbol{x}$ = **0**.

a) $$\begin{cases} x_1 - 3x_2 = 1 \\ 2x_1 - 6x_2 = 2 \end{cases}$$

b) $$\begin{cases} x_1 + x_2 + 2x_3 = 5 \\ x_1 \qquad + x_3 = -2 \\ 2x_1 + x_2 + 3x_3 = 3 \end{cases}$$

c) $$\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 = 4 \\ -2x_1 + x_2 + 2x_3 + x_4 = -1 \\ -x_1 + 3x_2 - x_3 + 2x_4 = 3 \\ 4x_1 - 7x_2 \qquad - 5x_4 = -5 \end{cases}$$

d) $$\begin{cases} x_1 - 2x_2 + x_3 + 2x_4 = -1 \\ 2x_1 - 4x_2 + 2x_3 + 4x_4 = -2 \\ -x_1 + 2x_2 - x_3 - 2x_4 = 1 \\ 3x_1 - 6x_2 + 3x_3 + 6x_4 = -3 \end{cases}$$

Solution

a) $$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & -6 & 2 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = 1 + 3x_2$$

The solution of A$\boldsymbol{x}$ = $\boldsymbol{b}$ is $x_1 = 1 + 3t, \quad x_2 = t$ or $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

The general form of the solution of A$\boldsymbol{x}$ = **0** is $\boldsymbol{x} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

b) $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 1 & 0 & 1 & -2 \\ 2 & 1 & 3 & 3 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow x_1 = -2 - x_3 \\ \Rightarrow x_2 = 7 - x_3 \end{array}$

The solution of A$\boldsymbol{x}$ = $\boldsymbol{b}$ is $x_1 = -2 - t, \quad x_2 = 7 - t, \quad x_3 = t$ or $\boldsymbol{x} = \begin{bmatrix} -2 \\ 7 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

The general form of the solution of A$\boldsymbol{x}$ = **0** is $\boldsymbol{x} = t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

c) $\left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 4 \\ -2 & 1 & 2 & 1 & -1 \\ -1 & 3 & -1 & 2 & 3 \\ 4 & -7 & 0 & -5 & -5 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & -\frac{7}{5} & -\frac{1}{5} & \frac{6}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \Rightarrow x_1 = \frac{6}{5} + \frac{7}{5}x_3 + \frac{1}{5}x_4 \\ \Rightarrow x_2 = \frac{7}{5} + \frac{4}{5}x_3 - \frac{3}{5}x_4 \end{array}$

The solution of A$\boldsymbol{x}$ = $\boldsymbol{b}$ is $\boldsymbol{x} = \begin{bmatrix} \frac{6}{5} \\ \frac{7}{5} \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} \frac{7}{5} \\ \frac{4}{5} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{5} \\ -\frac{3}{5} \\ 0 \\ 1 \end{bmatrix}$

The general form of the solution of A$\boldsymbol{x}$ = **0** is $\boldsymbol{x} = s\begin{bmatrix} \frac{7}{5} \\ \frac{4}{5} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{5} \\ -\frac{3}{5} \\ 0 \\ 1 \end{bmatrix}$

d) $\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & -1 \\ 2 & -4 & 2 & 4 & -2 \\ -1 & 2 & -1 & -2 & 1 \\ 3 & -6 & 3 & 6 & -3 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad x_1 = -1 + 2x_2 - x_3 - 2x_4$

Let $x_2 = s \quad x_3 = t \quad x_4 = r$

The solution of $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

The general form of the solution of $A\boldsymbol{x} = \boldsymbol{0}$ is $\boldsymbol{x} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Exercise

Given the vectors $v_1 = (1,\ 2,\ 0)$ and $v_2 = (2,\ 3,\ 0)$

a) Are they linearly independent?

b) Are they a basis for any space?

c) What space **V** do they span?

d) What is the dimension of that space?

e) What matrices ***A*** have **V** as their column space?

f) Which matrices have **V** as their nullspace?

g) Describe all vectors v_3 that complete a basis v_1, v_2, v_3 for $\boldsymbol{R}^3$.

Solution

a) v_1, v_2 are independent – the only combination to give **0** is $0.v_1 + 0.v_2$.

b) Yes, they are a basis for whatever space **V** they span.

c) That space **V** contains all vectors $(x,\ y,\ 0)$. It is the *xy* plane in $\boldsymbol{R}^3$.

d) The dimension of **V** is 2 since the basis contains 2 vectors.

e) This **V** is the column space of any 3 by *n* matrix ***A*** of rank 2, if every column is a combination of v_1 *and* v_2. In particular ***A*** could just have columns v_1 *and* v_2.

f) This **V** is the nullspace of any *m* by 3 matrix ***B*** of rank 1, if every row is a multiple of $(0,\ 0,\ 1)$. In particular, take $B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. Then $Bv_1 = 0$ *and* $Bv_2 = 0$.

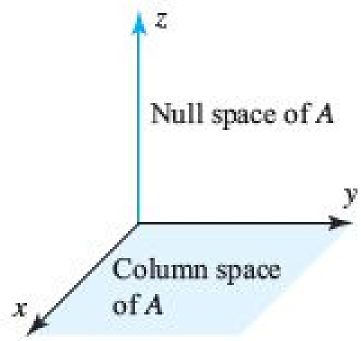
g) Any third vector $v_3 = (a,\ b,\ c)$ will complete a basis for $\boldsymbol{R}^3$ provided $c \neq 0$.

Exercise

a) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Show that relative to an *xyz*-coordinate system in 3-space the null space of *A* consists of all points on the *z*-axis and that the column space consists of all points in the *xy*-plane.

b) Find a 3 x 3 matrix whose null space is the *x*-axis and whose column space is the *yz*-plane.



Solution

a) $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x = 0 \\ y = 0 \\ z = t \end{matrix}$

The general form of the solution of A$\boldsymbol{x}$ = **0** is, $t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ therefore

the null space of *A* is the *z*-axis, and the column space is the span of $c_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

which is all linear combinations of *y* and *x* (*xy*-plane)

b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Exercise

If we add an extra column b to a matrix *A*, then the column space gets larger unless _____. Give an example where the column space gets larger and an example where it doesn't. Why is $Ax = b$ solvable exactly when the column space doesn't get larger – it is the same for *A* and $\begin{bmatrix} A & b \end{bmatrix}$?

Solution

If we add an extra column ***b*** to a matrix ***A***, then the column space gets larger unless ***it contains b*** that is a linear combination of the columns of ***A***.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; then the column space gets larger if $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and it doesn't if $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The equation $Ax = b$ is solvable exactly when ***b*** is a (nontrivial) linear combination of the column of ***A***.

The equation $Ax = b$ is solvable exactly when $\boldsymbol{b}$ lies in the column space, when the column space doesn't get larger.

Exercise

For which right sides (find a condition on b_1, b_2, b_3) are these solvable. (Use the column space $C(A)$ and the equation $Ax = b$)

a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Solution

a) The column space consists of the vectors for $\begin{pmatrix} x_1 + 4x_2 + 2x_3 \\ 2x_1 + 8x_2 + 4x_3 \\ -x_1 - 4x_2 - 2x_3 \end{pmatrix}$ is $\begin{pmatrix} z \\ 2z \\ -z \end{pmatrix} = z\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

They are scalar multiples of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

b) By substituting $x_1 + 4x_2$ with new variable z, then the column space consists of the vectors

$$\begin{pmatrix} x_1 + 4x_2 \\ 2x_1 + 9x_2 \\ -x_1 - 4x_2 \end{pmatrix} = z\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

They are linear combinations of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Exercise

Show that the matrices A and $\left[A \;\; AB\right]$ (with extra columns) have the same column space. But find a square matrix with $C\left(A^2\right)$ smaller than $C(A)$. Important point: An n by n matrix has $C(A) = \boldsymbol{R}^n$ exactly when A is an ______ matrix.

Solution

Each column of $\boldsymbol{AB}$ is a combination of the columns of $\boldsymbol{A}$ (the combining coefficients are the entries in the corresponding column of B). So, any combination of the columns of $\begin{bmatrix} A & AB \end{bmatrix}$ is a combination of the columns of $\boldsymbol{A}$ alone. Thus, $\boldsymbol{A}$ and $\begin{bmatrix} A & AB \end{bmatrix}$ have the same column space.

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; then $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so $C\left(A^2\right) = Z$.

$C(A)$ is the line through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Any n by n matrix has $C(A) = \boldsymbol{R}^n$ exactly when $\boldsymbol{A}$ is an ***invertible*** matrix, because $Ax = b$ is solvable for any given $\boldsymbol{b}$ when $\boldsymbol{A}$ is invertible.

Exercise

The column of AB are combinations of the columns of A. This means: The column space of AB is contained in (possibly equal to) to the column space of A. Give an example where the column spaces A and AB are not equal.

Solution

The column space of $\boldsymbol{AB}$ is contained in (possibly equal to) to the column space of $\boldsymbol{A}$.
$B = 0 \ \textit{and} \ A \neq 0$ is a case when $AB = 0$ has a smaller column space than $\boldsymbol{A}$.

Exercise

Find a square matrix A where $C\left(A^2\right)$ (the column space of A^2 is smaller than $C(A)$.

Solution

For example, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; then $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Thus $C(A)$ is generated by vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is of one dimensional, but $C\left(A^2\right)$ is a zero space.

Hence $C\left(A^2\right)$ is strictly smaller than $C(A)$.

Exercise

Suppose $Ax = b$ and $Cx = b$ have the same (complete) solutions for every $\boldsymbol{b}$. Is true that $A = C$?

Solution

Yes, if $A = C$, let $\boldsymbol{y}$ be any vector of the correct size, and set $b = Ay$. Then $\boldsymbol{y}$ is a solution to $Ax = b$ and it is also a solution to $Cx = b$; $b = Ay = Cy$

Exercise

Apply Gauss-Jordan elimination to $Ux = 0$ and $Ux = c$. Reach $Rx = 0$ and $Rx = d$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \qquad \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

Solve $Rx = 0$ to find x_n (its free variable is $x_2 = 1$).

Solve $Rx = d$ to find x_p (its free variable is $x_2 = 0$).

Solution

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \xrightarrow[\frac{1}{4}R_2]{} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The free variable is x_2, since it is the only one. We have to let $x_2 = 1$

$$\begin{cases} x_1 + 2x_2 = 0 \\ \quad x_3 = 0 \end{cases} \rightarrow x_1 = -2x_2$$

The special solution is $s_1(-2,\ 1,\ 0) \Rightarrow x_n = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix} \xrightarrow[\frac{1}{4}R_2]{} \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The free variable is x_2 that implies to $x_2 = 0$

$$\begin{cases} x_1 + 2x_2 = -1 \\ \quad x_3 = 2 \end{cases} \rightarrow x_1 = -1$$

The particular solution is $x_p = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

Exercise

Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?

a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$

b) The plane of vectors with $b_1 = 1$.

c) The vectors with $b_1 b_2 b_3 = 0$.

d) All linear combinations of $v = (1,\ 4,\ 0)$ and $w = (2,\ 2,\ 2)$.

e) All vectors that satisfies $b_1 + b_2 + b_3 = 0$

f) All vectors with $b_1 \le b_2 \le b_3$.

Solution

a) This is subspace

- For $v=(b_1, b_2, b_3)$ with $b_1=b_2$ and $w=(c_1, c_2, c_3)$ with $c_1=c_2$ the sum $v+w=(b_1+c_1, b_2+c_2, b_3+c_3)$ is in the same set as $b_1+c_1=b_2+c_2$
- For an element $v=(b_1, b_2, b_3)$ with $b_1=b_2$, $cv=(cb_1, cb_2, cb_3)$ and $cb_1=cb_2$, thus it is in the same set.

b) This is not a subspace. For example, for $v=(1,\ 0,\ 0)$ and $cv=-v=(-1,\ 0,\ 0)$ is not in the set.

c) This is not a subspace. For example, for $v=(1,\ 1,\ 0)$ and $w=(1,\ 0,\ 1)$ are in the set, but their sum $v+w=(2,\ 1,\ 1)$ is not in the set.

d) This is subspace, by definition of linear combination.

- For 2 vectors $v_1=\alpha_1 v+\beta_1 w$ and $v_2=\alpha_2 v+\beta_2 w$ the sum

$$\begin{aligned} v_1+v_2 &= \alpha_1 v+\beta_1 w+\alpha_2 v+\beta_2 w \\ &= (\alpha_1+\alpha_2)v+(\beta_1+\beta_2)w \end{aligned}$$

  is still the linear combination of *v* and *w*.
- For an element $v_1=\alpha_1 v+\beta_1 w$, $cv_1=c\alpha_1 v+c\beta_1 w$ is still the linear combination of *v* and *w*, thus it is the same set

e) This is subspace, these are the vectors orthogonal to $(1,\ 1,\ 1)$

- For $v=(b_1, b_2, b_3)$ with $b_1+b_2+b_3=0$ and $w=(c_1, c_2, c_3)$ with $c_1+c_2+c_3=0$ the sum $v+w=(b_1+c_1, b_2+c_2, b_3+c_3)$ is in the same set as $b_1+c_1+b_2+c_2+b_3+c_3=0$
- For an element $v=(b_1, b_2, b_3)$ with $b_1+b_2+b_3=0$, $cv=(cb_1, cb_2, cb_3)$ and $cb_1+cb_2+cb_3=0$, thus it is in the same set.

f) This is not a subspace. For example, for $v=(1,\ 2,\ 3)$ and $-v=(-1,\ -2,\ -3)$ is not in the set.

Exercise

We are given three different vectors b_1, b_2, b_3. Construct a matrix so that the equations $Ax=b_1$ and $Ax=b_2$ are solvable, but $Ax=b_3$ is not solvable.

a) How can you decide if this possible?

b) How could you construct A?

Solution

The equations $Ax = b_1$ and $Ax = b_2$ will be solvable.

$$Ax = \begin{bmatrix} b_1 & b_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_3 \text{ (solvable?)}$$

If $Ax = b_3$ is not solvable, we have the desired matrix A.

If $Ax = b_3$ is solvable, then it is not possible to construct A.

When the column space contains b_1 and b_2, it will have to contain their linear combinations.

So b_3 would necessarily be in that column space and $Ax = b_3$ would necessarily be solvable.

Exercise

For which vectors (b_1, b_2, b_3) do these systems have a solution?

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Solution

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \begin{array}{r} \to x_1 + x_2 + x_3 = b_1 \\ \to x_2 + x_3 = b_2 \\ \to x_3 = b_3 \end{array}$

$$\begin{cases} x_1 = b_1 - b_2 - b_3 \\ x_2 = b_2 - b_3 \\ x_3 = b_3 \end{cases} \Rightarrow \left(b_1 - b_2 - b_3,\ b_2 - b_3,\ b_3\right)$$

Solution for every b.

b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \begin{array}{r} \to x_1 + x_2 + x_3 = b_1 \\ \to x_2 + x_3 = b_2 \\ \to 0x_3 = b_3 \end{array}$

$$\begin{cases} x_1 = b_1 - b_2 \\ x_2 = b_2 \\ 0 = b_3 \end{cases} \Rightarrow \left(b_1 - b_2,\ b_2,\ 0\right)$$

Solvable only if $b_3 = 0$

c) $$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \xrightarrow{R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 0 & b_3 - b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right]$$

$$\Rightarrow b_3 - b_2 = 0 \Rightarrow \boxed{b_3 = b_2}$$

$$\begin{cases} x_1 = b_1 - 2b_3 \\ 0 = 0 \\ x_3 = b_3 \end{cases} \Rightarrow \left(b_1 - 2b_3,\ 0,\ b_3\right)$$

Solvable only if $b_3 = b_2$

Exercise

Find a basis for the null space of A. $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 2 & \frac{4}{3} \\ 0 & 1 & 0 & 0 & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & -\frac{5}{12} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $x_4 = s \quad x_5 = t \rightarrow \begin{cases} x_1 = -2x_4 - \frac{4}{3}x_5 = 2s - \frac{4}{3}t \\ x_2 = \frac{1}{6}x_5 = \frac{1}{6}t \\ x_3 = \frac{5}{12}x_5 = \frac{5}{12}t \end{cases}$

The general form of the solution of A$\boldsymbol{x}$ = $\mathbf{0}$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{4}{3} \\ \frac{1}{6} \\ \frac{5}{12} \\ 0 \\ 1 \end{bmatrix}$

Therefore, the vectors $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -\frac{4}{3} \\ \frac{1}{6} \\ \frac{5}{12} \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

Exercise

Is it true that is $m = n$ then the row space of A equals the column space.

Solution

False

Counterexample, let $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$

We have $m = n = 2$, but the row space of A contains multiple of $(1,\ 2)$ while the column space of A contains multiples of $(1,\ 3)$.

Exercise

If the row space equals the column space the $A^T = A$

Solution

False,

Counterexample, let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Here, the row space and column space are both equal to all of $\boldsymbol{R}^2$ (since A is invertible).

But $A \neq A^T$

Exercise

If $A^T = -A$, then the row space of *A* equals the column space.

Solution

True,

The row space of *A* equals to the column space of A^T, which for this particular *A* equals the column space of $-A$.

Since *A* and $-A$ have the same fundamental subsequences. We conclude that the row space of *A* equals the column space of *A*.

Exercise

Does the matrices *A* and $-A$ share the same 4 subspaces?

Solution

True.

The nullspaces are identical because $A\vec{x} = 0 \quad \Leftrightarrow \quad -A\vec{x} = 0$

The column spaces are identical because any vector $\vec{v}$ that can be expressed as $\vec{v} = A\vec{x}$ for some $\vec{x}$ can also be expressed as $\vec{v} = (-A)(-\vec{x})$

Exercise

Is *A* and *B* share the same 4 subspaces then *A* is multiple of *B*.

Solution

False

Any invertible 2×2 matrix will have R^2 as its column space and row space and zero vector as its (left and right) nullspace.

However, it is easy to produce 2 invertible 2×2 matrices that are not multiples of each other, as example

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Exercise

Suppose $A\vec{x} = b \quad \& \quad C\vec{x} = b$ have the same (complete) solutions for every *b*. Is it true that $A = C$

Solution

If $A\vec{x} = C\vec{x} = b$ for all vectors $\vec{x}$ of the correct size.

Then, it is true that $A = C$

Exercise

A and A^T have the same left nullspace?

Solution

False,

Counterexample, take any a 1×2 matrix, such as $A=\begin{bmatrix}1 & 1\end{bmatrix}$.

The left nullspace of A contains vectors in $\boldsymbol{R}$ while the left nullspace of A^T, which is the right nullspace of A, contains vectors in $\boldsymbol{R}^2$.

So, they can't be the same.