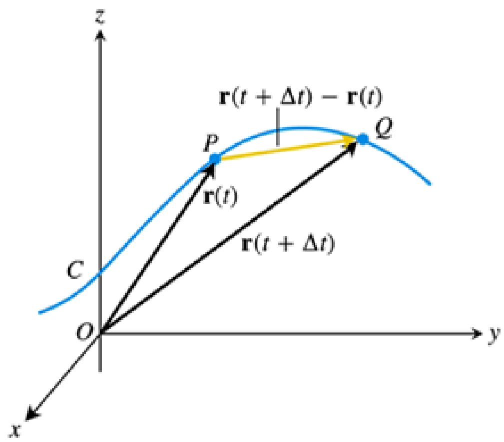
Section 1.5 – Calculus of Vector-Valued Functions

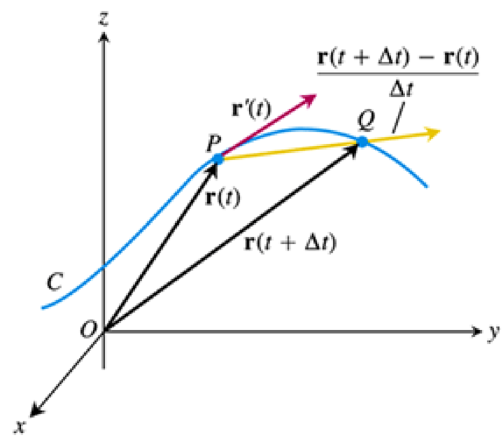
Derivative

Definition

The vector function $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$ has a derivative (is differentiable) at *t* if *f*, *g*, and *h* have derivatives at *t*. The derivative is the vector function

$$\boldsymbol{r}'(t) = \frac{d\boldsymbol{r}}{dt} = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)}{\Delta t} = \frac{df}{dt}\boldsymbol{i} + \frac{dg}{dt}\boldsymbol{j} + \frac{dh}{dt}\boldsymbol{k}$$





Definitions

If $\boldsymbol{r}$ is the position vector of a particle moving along a smooth curve in space, then

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt}$$

is the particle's ***velocity vector***, tangent to the curve. At any time *t*, the direction of $\vec{v}$ is the ***direction of motion***, the magnitude of $\vec{v}$ is the particle's ***speed***, and the derivative $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}$, when it exists, is the particle's ***acceleration vector***. In summary,

1. Velocity is the derivative of position: $\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt}$

2. Speed is the magnitude of velocity: $Speed = |\boldsymbol{v}|$

3. Acceleration is the derivative of velocity: $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt^2}$

4. The unit vector $\frac{\boldsymbol{v}}{|\boldsymbol{v}|}$ is the direction of motion at time *t*.

Example

Find the velocity, speed, and acceleration of a particle whose motion in space is given by the position vector $\boldsymbol{r}(t) = 2\cos t\ \boldsymbol{i} + 2\sin t\ \boldsymbol{j} + 5\cos^2 t\ \boldsymbol{k}$. Sketch the velocity vector $\boldsymbol{v}\left(\frac{7\pi}{4}\right)$

Solution

The velocity vector at time *t* is:

$$\boldsymbol{v}(t) = \boldsymbol{r}'(t) = -2\sin t\ \boldsymbol{i} + 2\cos t\ \boldsymbol{j} - 10\cos t \sin t\ \boldsymbol{k}$$

$$= -2\sin t\ \boldsymbol{i} + 2\cos t\ \boldsymbol{j} - 5\sin 2t\ \boldsymbol{k}$$

The acceleration vector at time *t* is:

$$\boldsymbol{a}(t) = \boldsymbol{r}''(t) = -2\cos t\ \boldsymbol{i} - 2\sin t\ \boldsymbol{j} - 10\cos 2t\ \boldsymbol{k}$$

The speed is:

$$|\boldsymbol{v}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-5\sin 2t)^2}$$

$$= \sqrt{4\sin^2 t + 4\cos^2 t + 25\sin^2 2t}$$

$$= \sqrt{4\left(\sin^2 t + \cos^2 t\right) + 25\sin^2 2t} \qquad \sin^2 t + \cos^2 t = 1$$

$$= \sqrt{4 + 25\sin^2 2t}$$

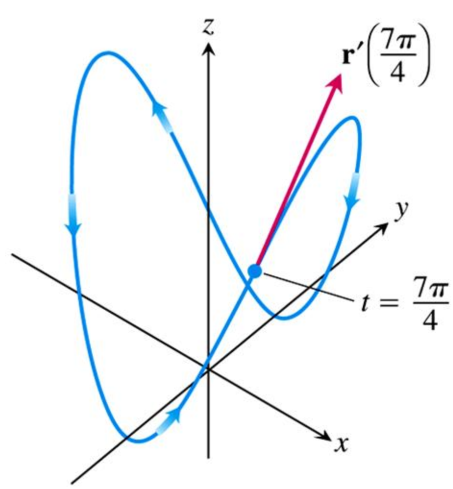
$$\boldsymbol{v}\left(\frac{7\pi}{4}\right) = -2\sin\left(\frac{7\pi}{4}\right)\boldsymbol{i} + 2\cos\left(\frac{7\pi}{4}\right)\boldsymbol{j} - 5\sin\left(\frac{7\pi}{2}\right)\boldsymbol{k}$$

$$\underline{= \sqrt{2}\boldsymbol{i} + \sqrt{2}\boldsymbol{j} + 5\boldsymbol{k}}$$

$$\boldsymbol{a}\left(\frac{7\pi}{4}\right) = -2\cos\left(\frac{7\pi}{4}\right)\boldsymbol{i} - 2\sin\left(\frac{7\pi}{4}\right)\boldsymbol{j} - 10\cos\left(\frac{7\pi}{2}\right)\boldsymbol{k}$$

$$\underline{= -\sqrt{2}\boldsymbol{i} + \sqrt{2}\boldsymbol{j}}$$

$$\left|\boldsymbol{v}\left(\frac{7\pi}{4}\right)\right| = \sqrt{4 + 25\sin^2\left(\frac{7\pi}{2}\right)} \ \underline{= \sqrt{29}}$$



Differentiation Rules for vector Functions

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be differentiable vector functions of t, C a constant vector, c any scalar and f any differentiable scalar function.

1. *Constant Function Rule*: $\frac{d}{dt}\boldsymbol{C}=\mathbf{0}$
2. *Scalar Multiple Rules*: $\frac{d}{dt}\left[c\boldsymbol{u}(t)\right]=c\boldsymbol{u}'(t)$

   $\frac{d}{dt}\left[f(t)\boldsymbol{u}(t)\right]=f'(t)\boldsymbol{u}(t)+f(t)\boldsymbol{u}'(t)$
3. *Sum Rule*: $\frac{d}{dt}\left[\boldsymbol{u}(t)+\boldsymbol{v}(t)\right]=\boldsymbol{u}'(t)+\boldsymbol{v}'(t)$
4. *Difference Rule*: $\frac{d}{dt}\left[\boldsymbol{u}(t)-\boldsymbol{v}(t)\right]=\boldsymbol{u}'(t)-\boldsymbol{v}'(t)$
5. *Dot Product Rule*: $\frac{d}{dt}\left[\boldsymbol{u}(t)\bullet\boldsymbol{v}(t)\right]=\boldsymbol{u}'(t)\bullet\boldsymbol{v}(t)+\boldsymbol{u}(t)\bullet\boldsymbol{v}'(t)$
6. *Cross Product Rule*: $\frac{d}{dt}\left[\boldsymbol{u}(t)\times\boldsymbol{v}(t)\right]=\boldsymbol{u}'(t)\times\boldsymbol{v}(t)+\boldsymbol{u}(t)\times\boldsymbol{v}'(t)$
7. *Chain Rule*: $\frac{d}{dt}\left[\boldsymbol{u}(f(t))\right]=f'(t)\boldsymbol{u}'(f(t))$

Vector Functions of Constant Length

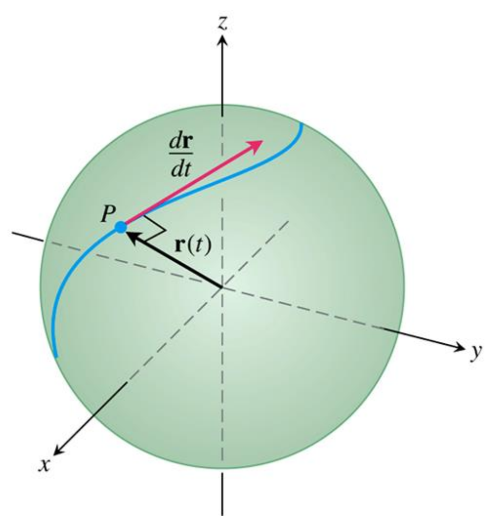
The position vector, of a particle that is moving on a sphere, has a constant length equal to the radius of the sphere. The velocity vector $\frac{d\boldsymbol{r}}{dt}$, tangent to the path of motion, is tangent to the sphere and hence perpendicular to $\boldsymbol{r}$. the vector and its first derivative are orthogonal.

$\boldsymbol{r}(t)\bullet\boldsymbol{r}(t)=c^2$ $\qquad$ $|\boldsymbol{r}(t)|=c$ *is constant*

$\frac{d}{dt}\left[\boldsymbol{r}(t)\bullet\boldsymbol{r}(t)\right]=0$ $\qquad$ ***Differentiate both sides***

$\boldsymbol{r}'(t)\bullet\boldsymbol{r}(t)+\boldsymbol{r}(t)\bullet\boldsymbol{r}'(t)=0$

$2\boldsymbol{r}'(t)\bullet\boldsymbol{r}(t)=0$



If $\boldsymbol{r}$ is a differentiable vector function of t of constant length, then

$$\boldsymbol{r}\bullet\frac{d\boldsymbol{r}}{dt}=0$$

Exercises *Section* 1.5 – Calculus of Vector-Valued Functions

(*Exercises* 1 - 4) $\boldsymbol{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find an equation in *x* and *y* whose is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of *t*.

1. $\boldsymbol{r}(t)=(t+1)\boldsymbol{i}+\left(t^2-1\right)\boldsymbol{j}, \quad t=1$

2. $\boldsymbol{r}(t)=\frac{t}{t+1}\boldsymbol{i}+\frac{1}{t}\boldsymbol{j}, \quad t=-\frac{1}{2}$

3. $\boldsymbol{r}(t)=e^t\boldsymbol{i}+\frac{2}{9}e^{2t}\boldsymbol{j}, \quad t=\ln 3$

4. $\boldsymbol{r}(t)=(\cos 2t)\boldsymbol{i}+(3\sin 2t)\boldsymbol{j}, \quad t=0$

Give the position vectors of particles moving along various curves in the *xy*-plane. Find the particle's velocity and acceleration vectors at the stated times and sketch them as vectors on the curve

5. Motion on the circle $x^2+y^2=1$ $\quad \boldsymbol{r}(t)=(\sin t)\boldsymbol{i}+(\cos t)\boldsymbol{j}, \quad t=\frac{\pi}{4}$ *and* $\frac{\pi}{2}$

6. Motion on the cycloid $x=t-\sin t, \ \ y=1-\cos t;$ $\quad \boldsymbol{r}(t)=(1-\sin t)\boldsymbol{i}+(1-\cos t)\boldsymbol{j}; \quad t=\pi$ & $\frac{3\pi}{2}$

$\boldsymbol{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of *t*. Write the particle's velocity at that time as the product of its speed and direction.

7. $\vec{r}(t)=(t+1)\hat{i}+\left(t^2-1\right)\hat{j}+2t\hat{k}, \quad t=1$

8. $\vec{r}(t)=(t+1)\hat{i}+\frac{t^2}{\sqrt{2}}\hat{j}+\frac{t^3}{3}\hat{k}, \quad t=1$

9. $\vec{r}(t)=(2\cos t)\hat{i}+(3\sin t)\hat{j}+4t\hat{k}, \quad t=\frac{\pi}{2}$

10. $\vec{r}(t)=(2\ln(t+1))\hat{i}+t^2\hat{j}+\frac{t^2}{2}\hat{k}, \quad t=1$

11. $\vec{r}(t)=\left(e^{-t}\right)\hat{i}+(2\cos 3t)\hat{j}+(2\sin 3t)\hat{k}, \quad t=0$

12. Find all points on the ellipse $\vec{r}(t)=\langle 1,\ 8\sin t,\ \cos t\rangle$, for $0\le t\le 2\pi$, at which $\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal.