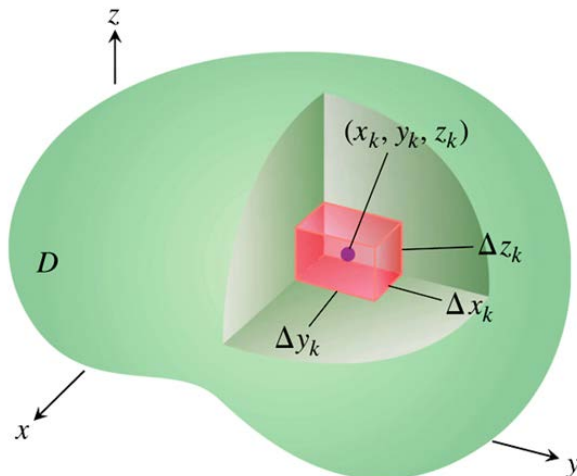


Section 3.4 – Triple Integrals

Triple Integrals

If $F(x, y, z)$ is a function defined on a closed, bounded region D in space, such a solid ball or a lump of clay, then the integral of F over D may be defined in the following way.



$$\Delta V_k = \Delta x_k \Delta y_k \Delta z_k \quad \rightarrow \quad S_n = \sum_{k=1}^n F(x_k, y_k, z_k) \Delta V_k$$

The limit of this summation is the triple integral of F over D

$$\lim_{n \rightarrow \infty} S_n = \iiint_D F(x, y, z) \, dV \quad \text{or} \quad \lim_{\|P\| \rightarrow 0} S_n = \iiint_D F(x, y, z) \, dx dy dz$$

Volume of a region in Space

Definition

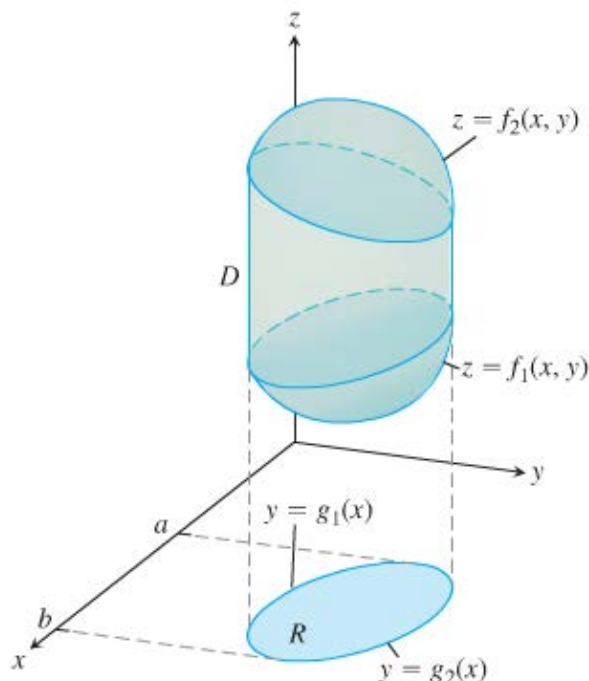
The volume of a closed, bounded region D in space is

$$V = \iiint_D dV$$

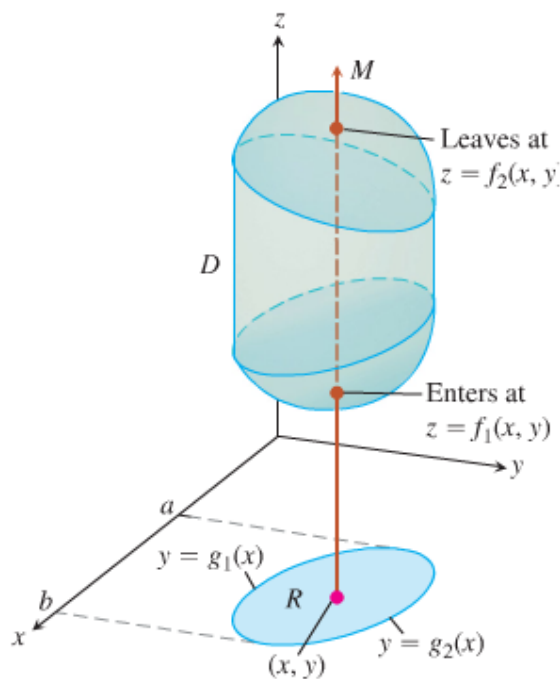
Find Limits of Integration in the Order $dz\,dy\,dx$

To evaluate $\iiint_D F(x, y, z) \, dV$

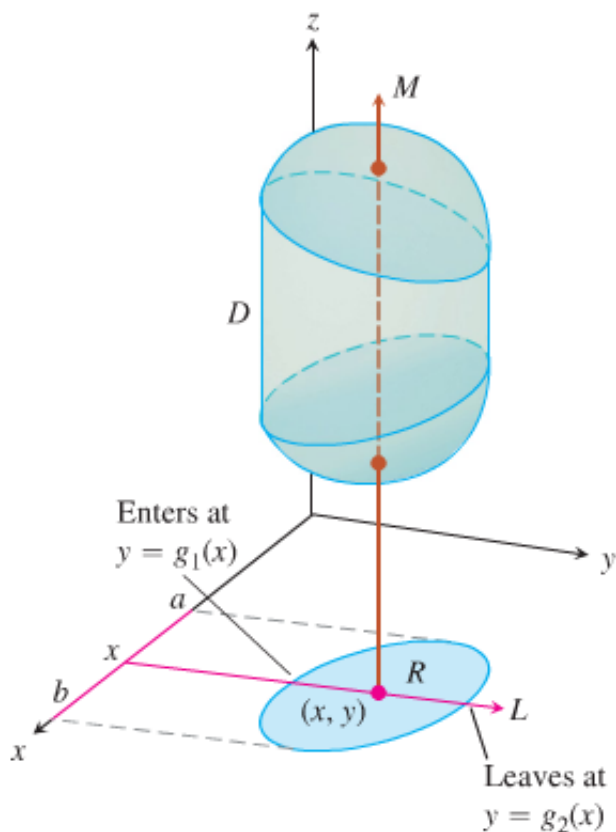
1. **Sketch:** Sketch the region D along with its “shadow” R (vertical projection) in the xy -plane. Label the upper and lower bounding surfaces of D and R .



2. **Find the z -limits of integration:** Draw a line M passing through (x, y) in R parallel to the z -axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$.



3. **Find the y-limits of integration:** Draw a line L passing through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$.



4. **Find the x-limits of integration:** Choose x -limits that include all lines through R parallel to the y-axis ($x = a$ and $x = b$).

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z) \, dz \, dy \, dx$$

Example

Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

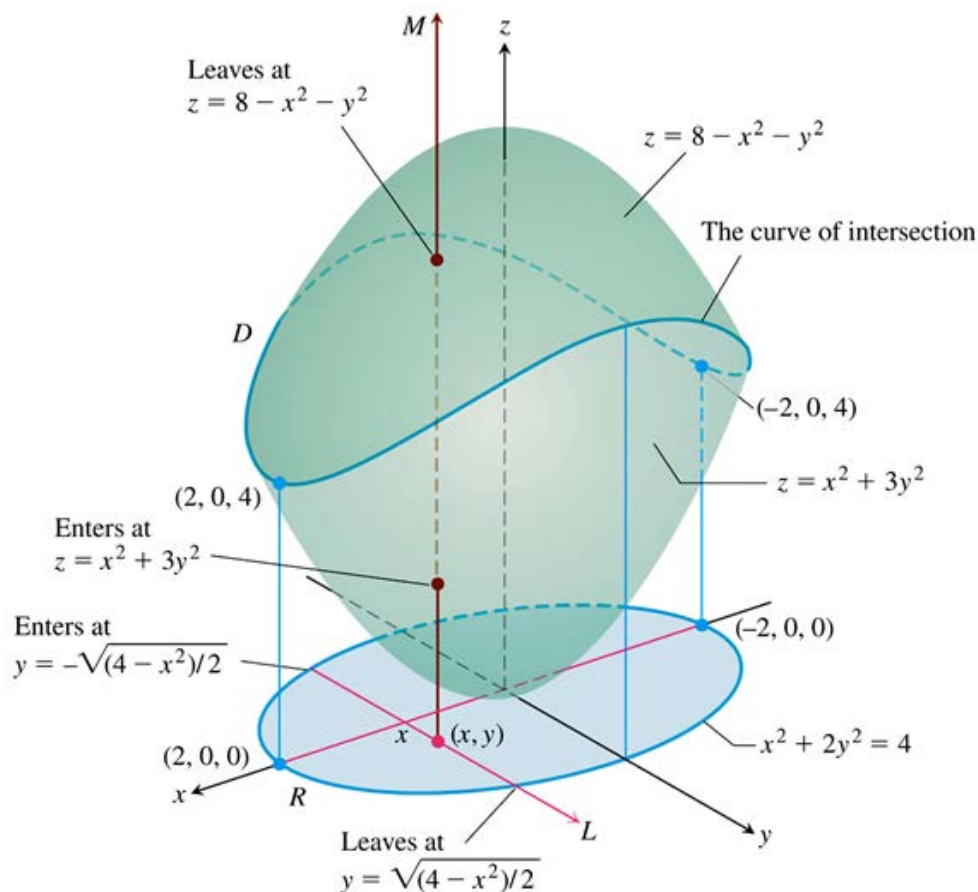
Solution

z -limits: $x^2 + 3y^2 \leq z \leq 8 - x^2 - y^2$

y -limits: $z = x^2 + 3y^2 = 8 - x^2 - y^2 \rightarrow 2x^2 + 4y^2 = 8 \Rightarrow x^2 + 2y^2 = 4$

$$y^2 = \frac{4-x^2}{2} \Rightarrow y = \pm \sqrt{\frac{4-x^2}{2}} \rightarrow -\sqrt{\frac{4-x^2}{2}} \leq y \leq \sqrt{\frac{4-x^2}{2}}$$

x -limits: $x^2 + 2y^2 = 4$ ($y = 0$) $\rightarrow x = \pm 2$



$$V = \int_{-2}^2 \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz dy dx$$

$$= \int_{-2}^2 \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} z \Big|_{x^2+3y^2}^{8-x^2-y^2} dy dx$$

$$\begin{aligned}
&= \int_{-2}^2 \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8-x^2-y^2-x^2-3y^2) dy dx \\
&= \int_{-2}^2 \left((8-2x^2)y - \frac{4}{3}y^3 \right) \bigg|_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} dx \\
&= \int_{-2}^2 \left[(8-2x^2)\sqrt{\frac{4-x^2}{2}} - \frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2} + (8-2x^2)\sqrt{\frac{4-x^2}{2}} - \frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right] dx \\
&= \int_{-2}^2 \left[2(8-2x^2)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right] dx \\
&= \int_{-2}^2 \left[2\left(\frac{2}{2}\right)(2)(4-x^2)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right] dx \\
&= \int_{-2}^2 \left[8\left(\frac{4-x^2}{2}\right)\left(\frac{4-x^2}{2}\right)^{1/2} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right] dx \\
&= \int_{-2}^2 \left[8\left(\frac{4-x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right] dx \\
&= \frac{16}{3} \int_{-2}^2 \left(\frac{4-x^2}{2}\right)^{3/2} dx \\
&= \frac{16}{3(2)^{3/2}} \int_{-2}^2 (4-x^2)^{3/2} dx \qquad \frac{16}{3(2)^{3/2}} \frac{2^{1/2}}{2^{1/2}} = \frac{16\sqrt{2}}{3 \cdot 4} = \frac{4\sqrt{2}}{3}
\end{aligned}$$

$$x = 2 \sin u \quad dx = 2 \cos u du \quad (4-x^2 = 4-4\sin^2 u = 4\cos^2 u)$$

$$\begin{cases} x = 2 & \rightarrow u = \sin^{-1} \frac{x}{2} = \sin^{-1} 1 = \frac{\pi}{2} \\ x = -2 & \rightarrow u = \sin^{-1}(-1) = -\frac{\pi}{2} \end{cases}$$

$$= \frac{4\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} (4\cos^2 u)^{3/2} (2\cos u du)$$

$$\begin{aligned}
&= \frac{4\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} 16(\cos u)^3 (\cos u) du \\
&= \frac{64\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \cos^4 u du \\
&= \frac{64\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos 2u}{2} \right)^2 du \\
&= \frac{16\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(1 + 2\cos 2u + \cos^2 2u \right) du \\
&= \frac{16\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(1 + 2\cos 2u + \frac{1}{2} + \frac{1}{2}\cos 4u \right) du \\
&= \frac{16\sqrt{2}}{3} \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} + 2\cos 2u + \frac{1}{2}\cos 4u \right) du \\
&= \frac{16\sqrt{2}}{3} \left(\frac{3}{2}u + \sin 2u + \frac{1}{8}\sin 4u \right) \Big|_{-\pi/2}^{\pi/2} \\
&= \frac{16\sqrt{2}}{3} \left[\frac{3\pi}{4} + \sin \pi + \frac{1}{8}\sin 2\pi - \left(-\frac{3\pi}{4} - \sin \pi - \frac{1}{8}\sin 2\pi \right) \right] \\
&= \frac{16\sqrt{2}}{3} \left(\frac{3\pi}{2} \right) \\
&= \underline{8\pi\sqrt{2} \text{ unit}^3}
\end{aligned}$$

Example

Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D with vertices $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 1, 1)$. Use the order of integration $dydzdx$.

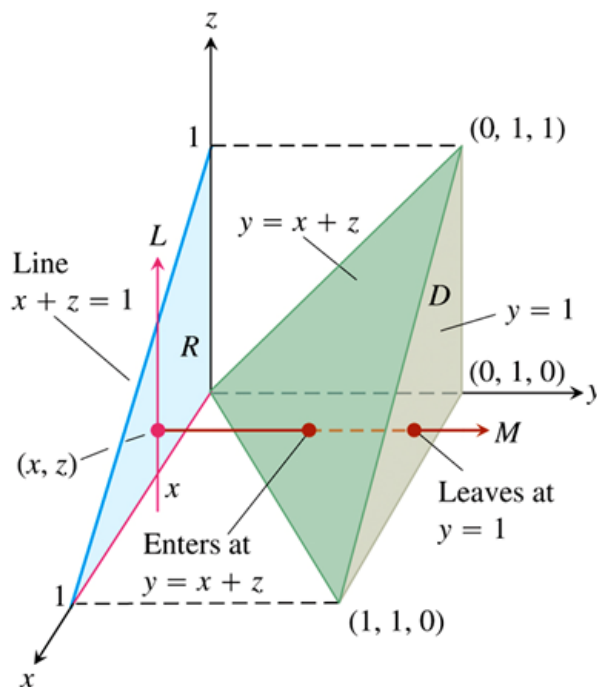
Solution

From the sketch, the upper (right-hand) bounding surface of D lies in the plane $y = 1$.

The lower (left-hand) bounding surface lies in the plane $y = x + z$.

The upper boundary of R is the line $z = 1 - x$.

The lower boundary is the line $z = 0$.



y-limits: The line through (x, z) in R parallel to the y -axis enters D at $y = x + z$ and leaves at $y = 1$.

z-limits: The line through (x, z) in R parallel to the z -axis enters R at $z = 0$ and leaves at $z = 1 - x$.

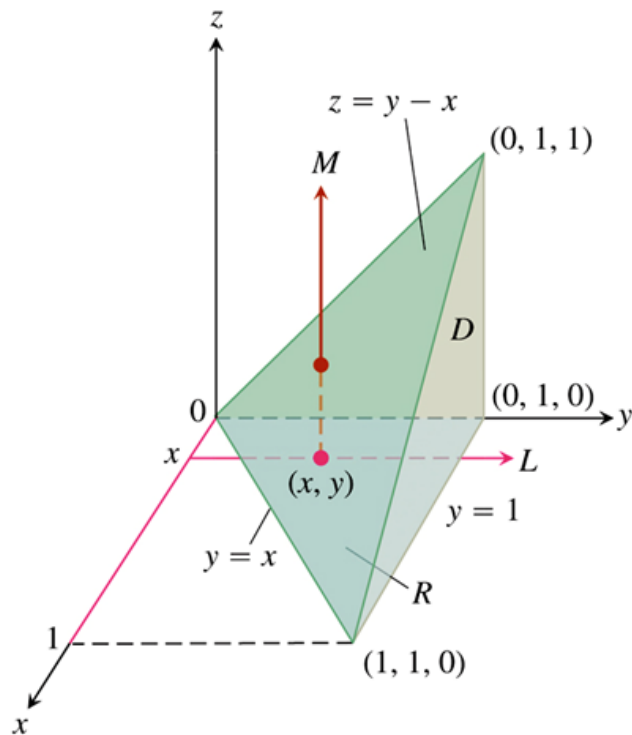
x-limits: $0 \leq x \leq 1$

$$\int_0^1 \int_0^{1-x} \int_{x+z}^1 F(x, y, z) \, dydzdx$$

Example

Integrate $F(x, y, z) = 1$ over the tetrahedron D in the previous example in the order $dz \, dy \, dx$, and then integrate in the order $dy \, dz \, dx$.

Solution



z-limits of integration: A line M parallel to the z -axis through a typical point (x, y) in the xy -plane “shadow” enters the tetrahedron at $z = 0$ and exists through the upper plane where $z = y - x$. $0 \leq z \leq y - x$

Line is given by: $ax + by + cz = 0$ passes through the 2 points:

$$(1, 1, 0) \rightarrow a + b = 0 \Rightarrow a = -b$$

$$\text{and } (0, 1, 1) \rightarrow b + c = 0 \Rightarrow c = -b$$

$$\rightarrow -bx + by - bz = 0$$

$$-x + y - z = 0 \Rightarrow z = y - x$$

y-limits of integration: On the xy -plane, where $z = 0$, the sloped side of the tetrahedron crosses the plane along the line $y = x$. A line L through (x, y) parallel to the y -axis enters the shadow in the xy -plane at $y = x$ and exists at $y = 1$. $x \leq y \leq 1$

x-limits of integration: A line L parallel to the y -axis through a typical point (x, y) in the xy -plane sweeps out the shadow, where $0 \leq x \leq 1$ at the point $(1, 1, 0)$

The integral is:
$$\int_0^1 \int_x^1 \int_0^{y-x} F(x, y, z) \, dz \, dy \, dx$$

$$\begin{aligned}
V &= \int_0^1 \int_x^1 \int_0^{y-x} dz dy dx \\
&= \int_0^1 \int_x^1 z \Big|_0^{y-x} dy dx \\
&= \int_0^1 \int_x^1 (y-x) dy dx \\
&= \int_0^1 \left(\frac{1}{2} y^2 - xy \Big|_x^1 \right) dx \\
&= \int_0^1 \left[\frac{1}{2} - x - \left(\frac{1}{2} x^2 - x^2 \right) \right] dx \\
&= \int_0^1 \left(\frac{1}{2} - x + \frac{1}{2} x^2 \right) dx \\
&= \left(\frac{1}{2} x - \frac{1}{2} x^2 + \frac{1}{6} x^3 \Big|_0^1 \right) \\
&= \frac{1}{2} - \frac{1}{2} + \frac{1}{6} \\
&= \frac{1}{6} \text{ unit}^3
\end{aligned}$$

$$\begin{aligned}
V &= \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy dz dx \\
&= \int_0^1 \int_0^{1-x} y \Big|_{x+z}^1 dz dx \\
&= \int_0^1 \int_0^{1-x} (1-x-z) dz dx \\
&= \int_0^1 \left(z - xz - \frac{1}{2} z^2 \Big|_0^{1-x} \right) dx \\
&= \int_0^1 \left(1-x - x(1-x) - \frac{1}{2} (1-x)^2 \right) dx \\
&= \int_0^1 \left((1-x)(1-x) - \frac{1}{2} (1-x)^2 \right) dx
\end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \left((1-x)^2 - \frac{1}{2}(1-x)^2 \right) dx \\
&= \frac{1}{2} \int_0^1 (1-x)^2 dx \\
&= -\frac{1}{6}(1-x)^3 \Big|_0^1 \\
&= \frac{1}{6} \text{ unit}^3 \Big|
\end{aligned}$$

Example

Evaluate the integral $\int_0^1 \int_x^{x^2} \int_{xy}^{x^2 y^3} xy \, dz dy dx$

Solution

$$\begin{aligned}
\int_0^1 \int_x^{x^2} \int_{xy}^{x^2 y^3} xy \, dz dy dx &= \int_0^1 \int_x^{x^2} xy \Big|_{xy}^{x^2 y^3} dy dx \\
&= \int_0^1 \int_x^{x^2} xy (x^2 y^3 - xy) dy dx \\
&= \int_0^1 \int_x^{x^2} (x^3 y^4 - x^2 y^2) dy dx \\
&= \int_0^1 \left(\frac{1}{5} x^3 y^5 - \frac{1}{3} x^2 y^3 \right) \Big|_x^{x^2} dx \\
&= \int_0^1 \left(\frac{1}{5} x^{13} - \frac{1}{3} x^8 - \frac{1}{5} x^8 + \frac{1}{3} x^5 \right) dx \\
&= \int_0^1 \left(\frac{1}{5} x^{13} - \frac{8}{15} x^8 + \frac{1}{3} x^5 \right) dx \\
&= \left(\frac{1}{70} x^{14} - \frac{8}{135} x^9 + \frac{1}{18} x^6 \right) \Big|_0^1 \\
&= \frac{1}{70} - \frac{8}{135} + \frac{1}{18} \\
&= \frac{2}{189} \Big|
\end{aligned}$$

Example

Evaluate the integral $\int_0^a \int_0^{a-z} \int_0^{a-y-z} yz \, dx dy dz$

Solution

$$\begin{aligned} \int_0^a \int_0^{a-z} \int_0^{a-y-z} yz \, dx dy dz &= \int_0^a \int_0^{a-z} yzx \Big|_0^{a-y-z} dy dz \\ &= \int_0^a \int_0^{a-z} (ayz - y^2z - yz^2) dy dz \\ &= \int_0^a \left(\frac{1}{2}azy^2 - \frac{1}{3}zy^3 - \frac{1}{2}z^2y^2 \right) \Big|_0^{a-z} dz \\ &= \int_0^a \left(\frac{1}{2}(az - z^2)(a-z)^2 - \frac{1}{3}z(a-z)^3 \right) dz \\ &= \frac{1}{6} \int_0^a (a-z)^2 \left(3(az - z^2) - 2z(a-z) \right) dz \\ &= \frac{1}{6} \int_0^a (a-z)^2 (3az - 3z^2 - 2az + 2z^2) dz \\ &= \frac{1}{6} \int_0^a (a-z)^2 (az - z^2) dz \\ &= \frac{1}{6} \int_0^a z(a-z)^3 dz \\ &= \frac{1}{6} \int_0^a (a^3z - 3a^2z^2 + 3az^3 - z^4) dz \\ &= \frac{1}{6} \left(\frac{1}{2}a^3z^2 - a^2z^3 + \frac{3}{4}az^4 - \frac{1}{5}z^5 \right) \Big|_0^a \\ &= \frac{1}{6} \left(\frac{1}{2}a^5 - a^5 + \frac{3}{4}a^5 - \frac{1}{5}a^5 \right) \\ &= \frac{a^5}{6} \left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5} \right) \end{aligned}$$

$$= \frac{a^5}{6} \left(\frac{1}{20} \right)$$

$$\underline{= \frac{a^5}{120}}$$

Example

Find the volume bounded by the cylinder $z = \frac{4}{y^2 + 1}$, bounded by the planes

$$y = x, \quad y = 3, \quad x = 0, \quad z = 0$$

Solution

$$0 \leq z \leq \frac{4}{y^2 + 1}$$

$$0 \leq x \leq y$$

$$0 \leq y \leq 3$$

$$V = \int_0^3 \int_0^y \int_0^{\frac{4}{y^2 + 1}} dz dx dy$$

$$= \int_0^3 \int_0^y z \bigg|_0^{\frac{4}{y^2 + 1}} dx dy$$

$$= \int_0^3 \int_0^y \frac{4}{y^2 + 1} dx dy$$

$$= \int_0^3 \frac{4}{y^2 + 1} x \bigg|_0^y dy$$

$$= \int_0^3 \frac{4y}{y^2 + 1} dy$$

$$= 4 \int_0^3 \frac{1}{y^2 + 1} d(y^2 + 1)$$

$$= 4 \ln(y^2 + 1) \bigg|_0^3$$

$$\underline{= 2 \ln(10) \text{ unit}^3}$$

Average Value of a Function in Space

The average value of a function F over a region D in space is defined by the formula

$$\text{Average value of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint_D F dV$$

Example

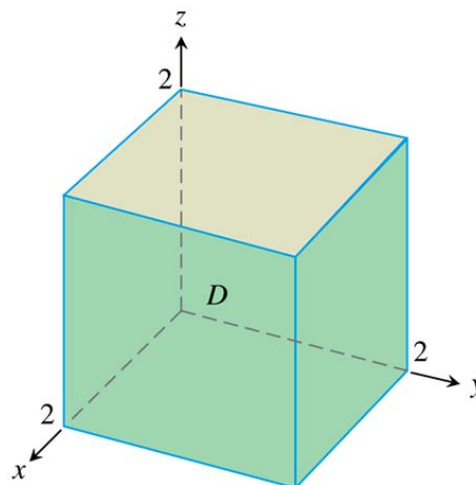
Find the average of $F(x, y, z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.

Solution

$$\begin{aligned} \text{Volume} &= 2 \cdot 2 \cdot 2 \\ &= 8 \text{ unit}^3 \end{aligned}$$

The value of the integral of F over the cube is

$$\begin{aligned} V &= \int_0^2 \int_0^2 \int_0^2 xyz \, dx \, dy \, dz \\ &= \int_0^2 z dz \int_0^2 y dy \int_0^2 x dx \\ &= \left(\frac{1}{2} z^2 \right) \Big|_0^2 \left(\frac{1}{2} y^2 \right) \Big|_0^2 \left(\frac{1}{2} x^2 \right) \Big|_0^2 \\ &= \frac{1}{8} (4)(4)(4) \\ &= 8 \text{ unit}^3 \end{aligned}$$



$$\begin{aligned} \text{Average value of } xyz \text{ over cube} &= \frac{1}{\text{volume of } D} \iiint_{\text{cube}} xyz \, dV \\ &= \left(\frac{1}{8} \right) (8) \\ &= 1 \end{aligned}$$

Exercises Section 3.4 – Triple Integrals

(1 – 31) Evaluate the integral

$$1. \int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2) dz dy dx$$

$$2. \int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz dx dy$$

$$3. \int_0^{\pi/6} \int_0^1 \int_{-2}^3 y \sin z \, dx dy dz$$

$$4. \int_{-1}^1 \int_0^1 \int_0^2 (x + y + z) dy dx dz$$

$$5. \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz dy dx$$

$$6. \int_0^1 \int_0^{1-x^2} \int_0^{4-x^2-y} x dz dy dx$$

$$7. \int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \cos(u + v + w) du dv dw$$

$$8. \int_0^{\pi/4} \int_0^{\ln \sec v} \int_{-\infty}^{2t} e^x dx dt dv$$

$$9. \int_0^1 \int_{-z}^z \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy dx dz$$

$$10. \int_0^{\pi} \int_0^y \int_0^{\sin x} dz dx dy$$

$$11. \int_0^9 \int_0^1 \int_{2y}^2 \frac{4 \sin x^2}{\sqrt{z}} dx dy dz$$

$$12. \int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \cos(x + y + z) dx dy dz$$

$$13. \int_1^e \int_1^x \int_0^z \frac{2y}{z^3} dy dz dx$$

$$14. \int_{\ln 6}^{\ln 7} \int_0^{\ln 2} \int_{\ln 4}^{\ln 5} e^{(x+y+z)} dz dy dx$$

$$15. \int_0^1 \int_0^{x^2} \int_0^{x+y} (2x - y - z) dz dy dx$$

$$16. \int_{-2}^2 \int_3^6 \int_0^2 dx dy dz$$

$$17. \int_{-1}^1 \int_{-1}^2 \int_0^1 6xyz \, dy dx dz$$

$$18. \int_{-2}^2 \int_1^2 \int_1^e \frac{xy^2}{z} dz dx dy$$

$$19. \int_0^{\ln 4} \int_0^{\ln 3} \int_0^{\ln 2} e^{-x+y+z} dx dy dz$$

$$20. \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\frac{\pi}{2}} \sin \pi x \cos y \sin 2z \, dy dx dz$$

$$21. \int_0^2 \int_1^2 \int_0^1 yze^x dx dz dy$$

$$22. \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz dy dx$$

$$23. \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 2xz \, dz dy dx$$

$$24. \int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \int_0^{16-\frac{1}{4}x^2-y^2} dz dx dy$$

$$25. \int_1^6 \int_0^{4-\frac{2}{3}y} \int_0^{12-2y-3z} \frac{1}{y} dx dz dy$$

$$26. \int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{1+x^2+z^2}} dy dx dz$$

$$27. \int_0^\pi \int_0^\pi \int_0^{\sin x} \sin y dz dx dy$$

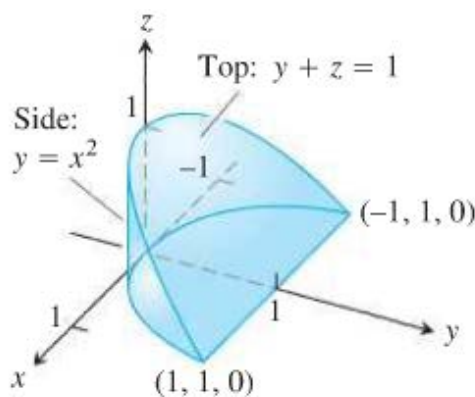
$$28. \int_0^{\ln 8} \int_1^{\sqrt{z}} \int_{\ln y}^{\ln 2y} e^{x+y^2-z} dx dy dz$$

$$29. \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{2-x} 4yz dz dy dx$$

$$30. \int_0^2 \int_0^4 \int_{y^2}^4 \sqrt{x} dz dx dy$$

$$31. \int_0^1 \int_y^{2-y} \int_0^{2-x-y} xy dz dx dy$$

32. Here is the region of integration of the integral $\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz dy dx$



a) $dydzdx$ b) $dydx dz$ c) $dx dy dz$ d) $dx dz dy$ e) $dz dx dy$

(33 – 37) Use another order to evaluate

$$33. \int_0^5 \int_{-1}^0 \int_0^{4x+4} dy dx dz$$

$$36. \int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-z^2}} dy dz dx$$

$$34. \int_0^1 \int_{-2}^2 \int_0^{\sqrt{4-y^2}} dz dy dx$$

$$37. \int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin \sqrt{yz}}{x^{3/2}} dy dx dz$$

$$35. \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dy dz dx$$

(38 – 39) Evaluate

38. $\iiint_D (xy + xz + yz) dV$; $D = \{(x, y, z): -1 \leq x \leq 1, -2 \leq y \leq 2, -3 \leq z \leq 3\}$

39. $\iiint_D xyz e^{-x^2 - y^2} dV$; $D = \{(x, y, z): 0 \leq x \leq \sqrt{\ln 2}, 0 \leq y \leq \sqrt{\ln 4}, 0 \leq z \leq 1\}$

40. Let $D = \{(x, y, z): 0 \leq x \leq y^2, 0 \leq y \leq z^3, 0 \leq z \leq 2\}$

a) Use a triple integral to find the volume of D .

b) In theory, how many other possible orderings of the variables (besides the one used in part (a)) can be used to find the volume of D ? Verify the result of part (a) using one of these other ordering.

c) What is the volume of the region $D = \{(x, y, z): 0 \leq x \leq y^p, 0 \leq y \leq z^q, 0 \leq z \leq 2\}$, where p and q are positive real numbers?

41. Find the volume the parallelepiped (slanted box) with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, $(0, 1, 1)$, $(1, 1, 1)$, $(0, 2, 1)$, $(1, 2, 1)$

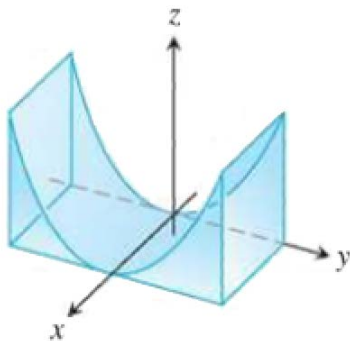
42. Find the volume the larger of two solids formed when the parallelepiped with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, $(2, 2, 0)$, $(0, 1, 1)$, $(2, 1, 1)$, $(0, 3, 1)$, $(2, 3, 1)$ is sliced by the plane $y = 2$.

43. Find the volume of the pyramid with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(2, 2, 0)$, $(0, 2, 0)$, $(0, 0, 4)$

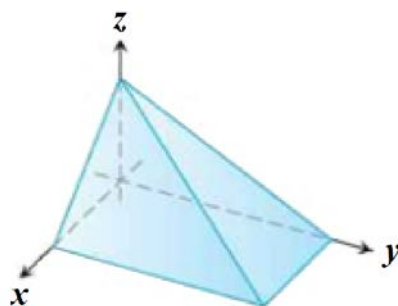
44. Two different tetrahedrons fill the region in the first octant bounded by the coordinate planes and the plane $x + y + z = 4$. Both solids have densities that vary in the z -direction between $\rho = 4$ and $\rho = 8$, according to the functions $\rho_1 = 8 - z$ and $\rho_2 = 4 + z$. Find the mass of each solid

45. Suppose a wedge of cheese fills the region in the first octant bounded by the planes $y = z$, $y = 4$ and $x = 4$. You could divide the wedge into two equal pieces (by volume) if you sliced the wedge with the plane $x = 2$. Instead find a with $0 < a < 1$ such that slicing the wedge with the plane $y = a$ divides the wedge into two pieces of equal volume

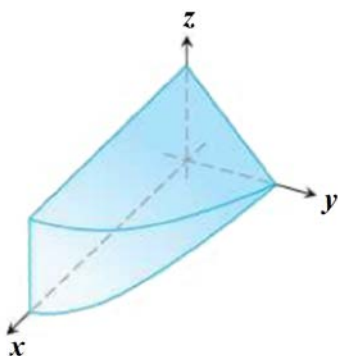
46. Find the volumes of the region between the cylinder $z = y^2$ and the xy -plane that is bounded by the planes $x = 0$, $x = 1$, $y = -1$, $y = 1$



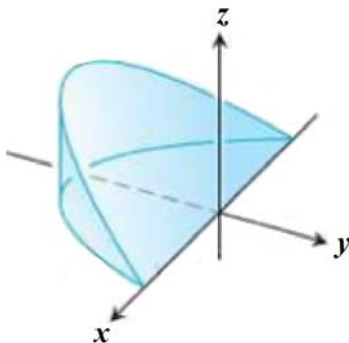
47. Find the volumes of the region in the first octant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$



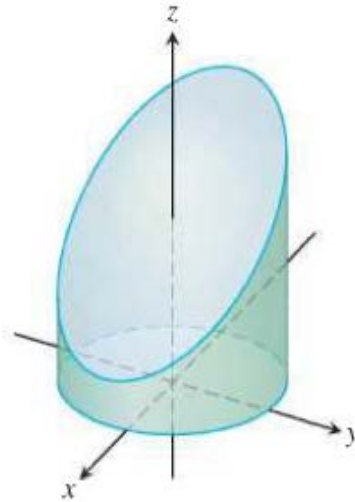
48. Find the volumes of the region in the first octant bounded by the coordinate planes and the plane $y + z = 2$, and the cylinder $x = 4 - y^2$



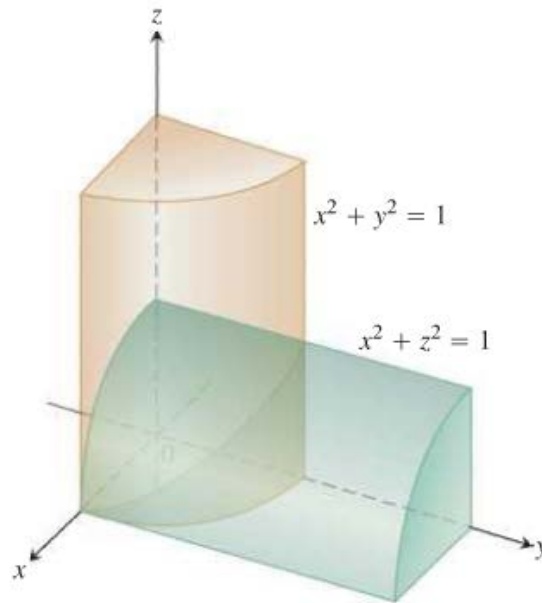
49. Find the volumes of the wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y$, $z = 0$



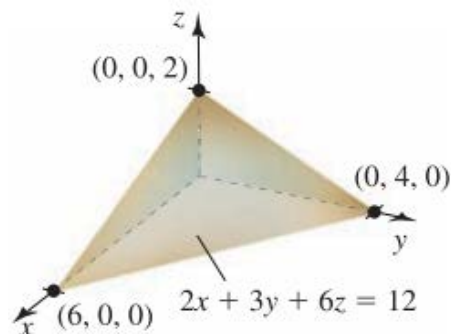
50. Find the volumes of the region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$



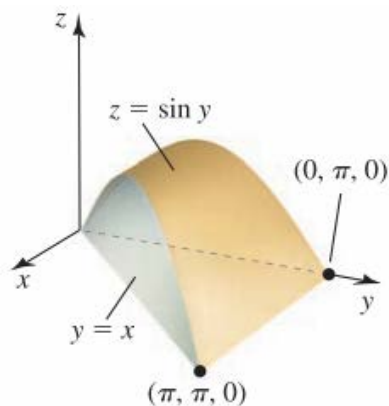
51. Find the volumes of the region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown below



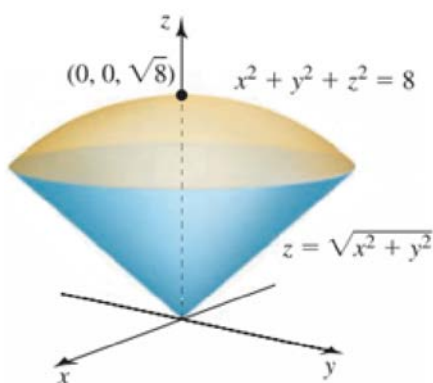
52. Find the volume of the solid in the first octant bounded by the plane $2x + 3y + 6z = 12$ and the coordinate planes



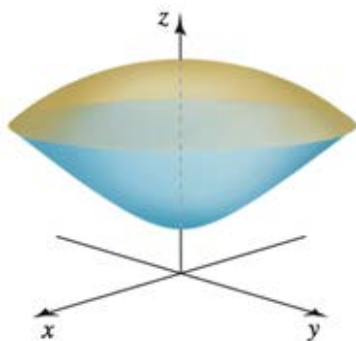
53. Find the volume of the solid in the first octant formed when the cylinder $z = \sin y$, for $0 \leq y \leq \pi$, is sliced by the planes $y = x$ and $x = 0$



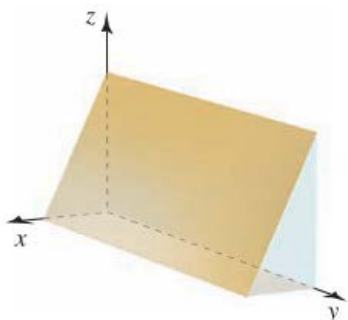
54. Find the volume of the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and bounded above the sphere $x^2 + y^2 + z^2 = 8$



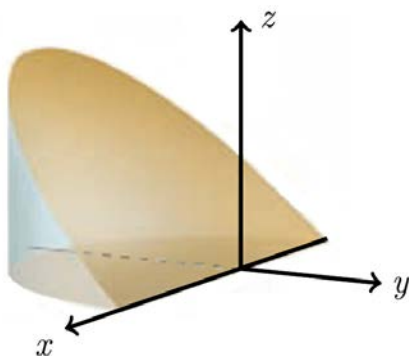
55. The solid between the sphere $x^2 + y^2 + z^2 = 19$ and the hyperboloid $z^2 - x^2 - y^2 = 1$, for $z > 0$



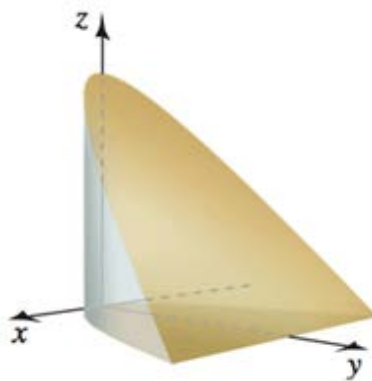
56. Find the volume of the prism in the first octant bounded below by $z = 2 - 4x$ and $y = 8$



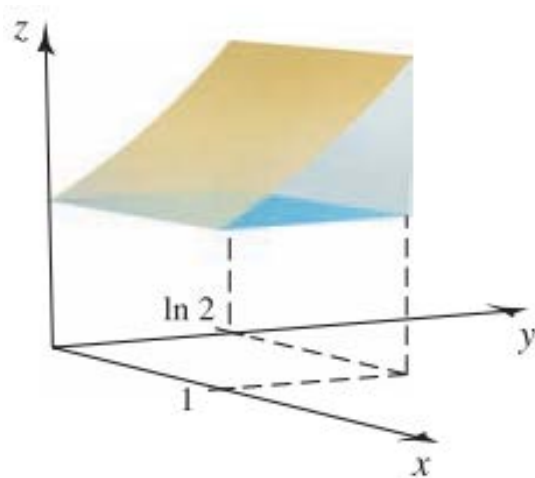
57. Find the volume of the wedge above the xy -plane formed when the cylinder $x^2 + y^2 = 4$ is cut by the planes $z = 0$ and $y = -z$



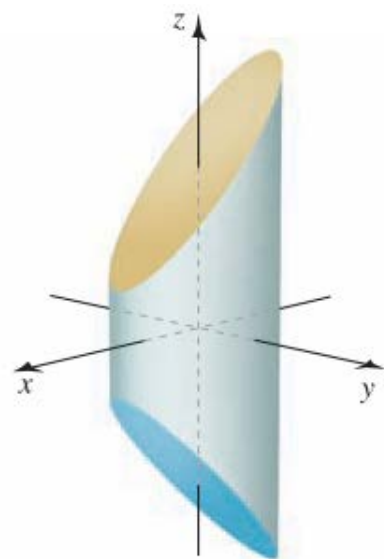
58. The wedge bounded by the parabolic cylinder $y = x^2$ and the planes $z = 3 - y$ and $z = 0$



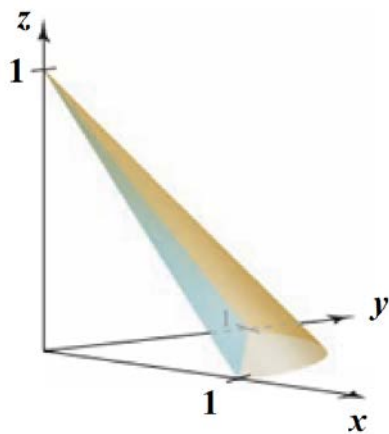
59. Find the volume of the solid bounded by the surfaces $z = e^y$ and $z = 1$ over the rectangle $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq \ln 2\}$



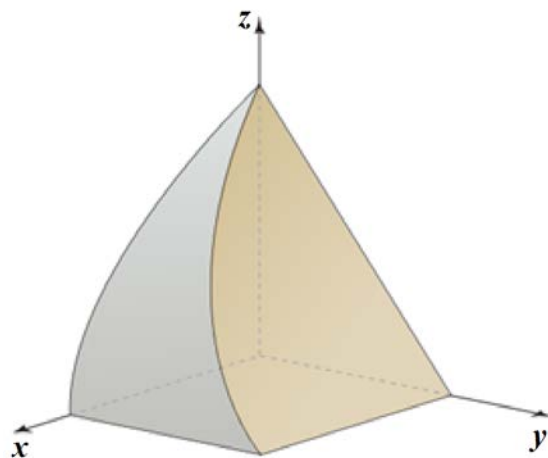
60. Find the volume of the wedge of the cylinder $x^2 + 4y^2 = 4$ created by the planes $z = 3 - x$ and $z = x - 3$



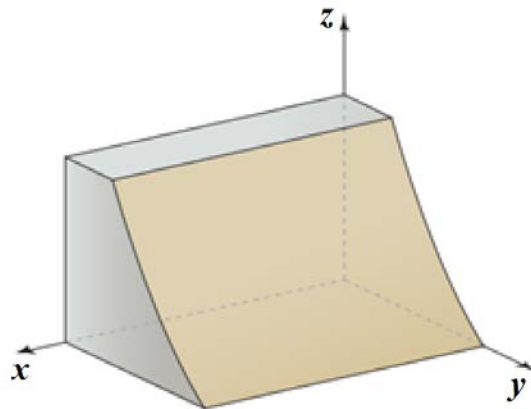
61. Find the volume of the solid in the first octant bounded by the cone $z = 1 - \sqrt{x^2 + y^2}$ and the plane $x + y + z = 1$



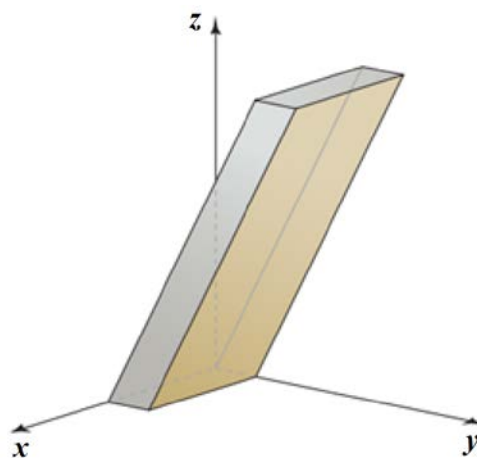
62. Find the volume of the solid bounded by $x = 0$, $x = 1 - z^2$, $y = 0$, $z = 0$, and $z = 1 - y$



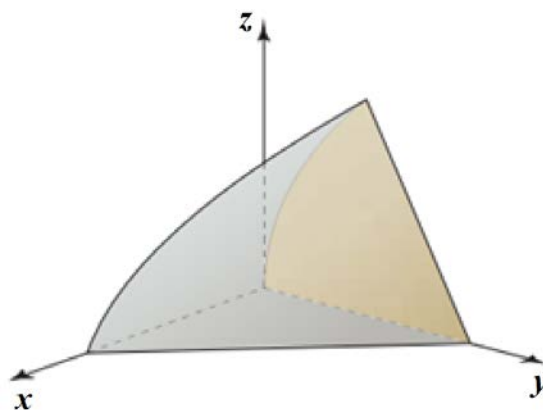
63. Find the volume of the solid bounded by $x = 0$, $x = 2$, $y = 0$, $y = e^{-z}$, $z = 0$, and $z = 1$



64. Find the volume of the solid bounded by $x = 0$, $x = 2$, $y = z$, $y = z + 1$, $z = 0$, and $z = 4$



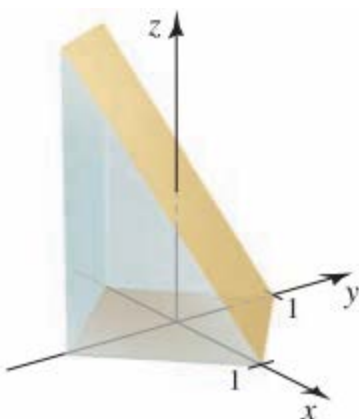
65. Find the volume of the solid bounded by $x = 0$, $y = z^2$, $z = 0$, and $z = 2 - x - y$



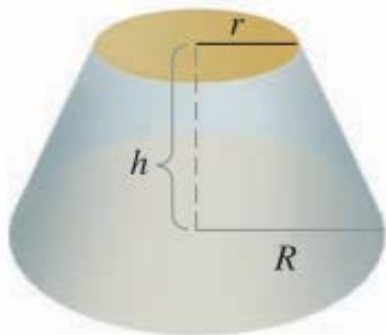
66. Find the volume of the solid common to the cylinders $z = \sin x$ and $z = \sin y$ over the square $R = \{(x, y): 0 \leq x \leq \pi, 0 \leq y \leq \pi\}$



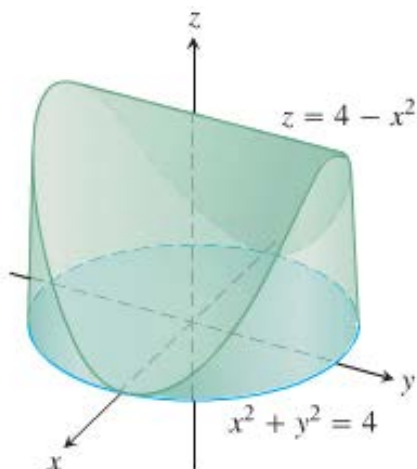
67. Find the volume of the wedge of the square column $|x| + |y| = 1$ created by the planes $z = 0$ and $x + y + z = 1$



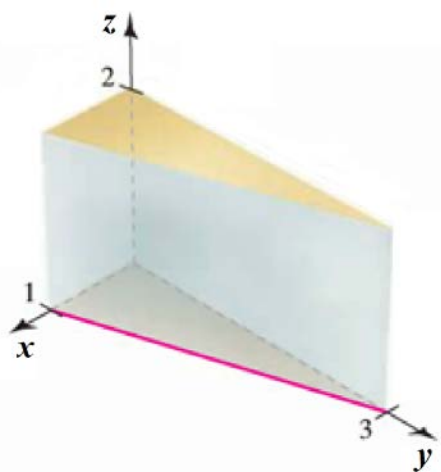
68. Find the volume of a right circular cone with height h and base radius r .
69. Find the volume of a tetrahedron whose vertices are located at $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$
70. Find the volume of a truncated cone of height h whose ends have radii r and R .



71. Find the volume of the solid that is bounded above by the cylinder $z = 4 - x^2$, on the sides by the cylinder $x^2 + y^2 = 4$, and below by the xy -plane.



72. Find the volume of the prism in the first octant bounded by the planes $y = 3 - 3x$ and $z = 2$.



73. Find the volume of the prism in the first octant bounded by the planes $x^2 + y^2 = 4$ and $x^2 + z^2 = 4$.

