Solution Section 3.4 - Concavity and the Second Derivative Test

Exercise

Determine the intervals on which the graph of the function is concave upward or concave downward.

$$f(x) = \frac{x^2 - 1}{2x + 1}$$

Solution

$$f'(x) = \frac{(2x+1)(2x) - (x^2 - 1)(2)}{(2x+1)^2}$$

$$= \frac{4x^2 + 2x - 2x^2 + 2}{(2x+1)^2}$$

$$= \frac{2x^2 + 2x + 2}{(2x+1)^2}$$

$$= \frac{2(x^2 + x + 1)}{(2x+1)^2}$$

$$f''(x) = 2\frac{(2x+1)^2(2x+1) - (x^2 + x + 1)(2)(2x+1)(2)}{(2x+1)^4}$$

$$= 2\frac{(2x+1)^3 - 4\left(x^2 + x + 1\right)(2x+1)}{(2x+1)^4}$$

$$= 2\frac{(2x+1)[(2x+1)^2 - 4(x^2 + x + 1)]}{(2x+1)^4}$$

$$= 2\frac{4x^2 + 4x + 1 - 4x^2 - 4x - 4}{(2x+1)^3}$$

$$= 2\frac{-3}{(2x+1)^3}$$

$$= -\frac{6}{(2x+1)^3}$$

$2x = 1 = 0 \ \Rightarrow \ x = -\frac{1}{2}$

$-\infty$ $-\frac{1}{2}$	∞
$f''(-1) > 0$	$f''(0) < 0$
Upward	***Downward***

f is concave upward on $\left(-\infty, -\frac{1}{2}\right)$

f is concave downward on $\left(-\frac{1}{2}, \infty\right)$

Exercise

Find the points of inflection. $f(x) = x^3 - 9x^2 + 24x - 18$

Solution

$$f'(x) = 3x^2 - 18x + 24$$

$$f''(x) = 6x - 18 = 0 \Rightarrow x = 3$$

$$x = 3 \Rightarrow f(3) = 0$$

$\rightarrow$ Point of inflection (3, 0)

Exercise

Find the second derivative of $f(x) = -2\sqrt{x}$ and discuss the concavity of the graph

Solution

$$f'(x) = -x^{-1/2}$$

$$\Rightarrow f''(x) = \frac{1}{2}x^{-3/2}$$

$$= \frac{1}{2x^{3/2}} > 0 \text{ for all } x > 0$$

f is concave upward for all *x* > *0*.

Exercise

Determine the intervals on which the graph of the function $f(x) = -4x^3 - 8x^2 + 32$ is concave upward or concave downward

Solution

$$f'(x) = -12x^2 - 16x$$

$$f''(x) = -24x - 16$$

$$f''(x) = -24x - 16 = 0$$

$$\Rightarrow -24x = 16$$

$$\Rightarrow x = \frac{16}{-24} = -\frac{2}{3}$$

$-\infty$	$-\frac{2}{3}$	∞
$f''(-1) > 0$		$f''(0) < 0$
Upward		***Downward***

Concave upward on $(-\infty, -2/3)$ and concave downward on $(-2/3, \infty)$

Exercise

Determine the intervals on which the graph of the function $f(x)=\frac{12}{x^2+4}$ is concave upward or concave downward.

Solution

$$f(x)=12(x^2+4)^{-1}$$

$$f'(x)=-12(x^2+4)^{-2}(2x)=-\frac{12x}{(x^2+4)^2}$$

$$f''(x)=-\frac{12(x^2+4)^2-12x(2)(x^2+4)(2x)}{(x^2+4)^4}$$

$$=-\frac{12(x^2+4)^2-48x^2(x^2+4)}{(x^2+4)^4}$$

$$=-\frac{12(x^2+4)\left[(x^2+4)-4x^2\right]}{(x^2+4)^4}$$

$$=-\frac{12(x^2+4)\left[x^2+4-4x^2\right]}{(x^2+4)^4}$$

$$=-\frac{12(x^2+4)(-3x^2+4)}{(x^2+4)^4}$$

$$=-\frac{12(-3x^2+4)}{(x^2+4)^3}$$

Solve for x:

$$f''(x)=-\frac{12(-3x^2+4)}{(x^2+4)^3}=0$$

$$\Rightarrow -3x^2+4=0$$

$$\rightarrow -3x^2=-4$$

$$\rightarrow x^2=\frac{4}{3}$$

$$\Rightarrow x=\pm\sqrt{\frac{4}{3}}$$

$$=\pm\frac{\sqrt{4}}{\sqrt{3}}$$

$$=\pm\frac{2}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}}$$

$$=\pm\frac{2\sqrt{3}}{3}$$

$-\infty$	$-\frac{2\sqrt{3}}{3}$	$\frac{2\sqrt{3}}{3}$	∞

$f''(-2) > 0$	$f''(0) < 0$	$f''(2) > 0$
upward	***downward***	***upward***

f is concave upward on $\left(-\infty, -\frac{2\sqrt{3}}{3}\right)$ *and* $\left(\frac{2\sqrt{3}}{3}, \infty\right)$

f is concave downward on $\left(-\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right)$

Exercise

Find the extrema using the second derivative test $f(x) = \frac{4}{x^2+1}$

Solution

$f'(x) = \frac{-8x}{\left(x^2+1\right)^2}$ CN is $x = 0$

$f''(x) = \frac{8(3x^2-1)}{\left(x^2+1\right)^3}$

$f''(0) = -8 < 0 \Rightarrow f(0) = 4$ is a relative maximum *(RMAX)*

Exercise

Find all relative extrema of $f(x) = x^4 - 4x^3 + 1$

Solution

$f'(x) = 4x^3 - 12x^2$

$f'(x) = 4x^2(x-3) = 0 \rightarrow \boxed{x = 0,\ 3}$

$f''(x) = 12x^2 - 12x$

Points: (0, 1) $f''(0) = 0$ Test fails

(3, -26) $f''(3) > 0 \Rightarrow$ relative Minimum *(RMIN)*

Exercise

Discuss the concavity of the graph of f and find its points of inflection. $f(x) = x^4 - 2x^3 + 1$

Solution

$f'(x) = 4x^3 - 6x^2$

$f''(x) = 12x^2 - 12x = 0$

$12x(x-1) = 0 \Rightarrow x = 0,1$

For $x = 0 \Rightarrow f(0) = 0^4 - 2(0)^3 + 1 = 1 \to (0,1)$

For $x = 0 \Rightarrow f(1) = 1^4 - 2(1)^3 + 1 = 0 \to (1,0)$

-∞	0	1	∞
$f''(-1) > 0$ **upward**	$f''(1/2) < 0$ **downward**	$f''(2) > 0$ **upward**	

f is concave upward on $(-\infty, 0)$ *and* $(1, \infty)$

f is concave downward on $(0,1)$

Points of inflection: (0, 1), (1, 0)

Exercise

The revenue R generated from sales of a certain product is related to the amount x spent on advertising by

$$R(x) = \frac{1}{15{,}000}\left(600x^2 - x^3\right), \qquad 0 \le x \le 600$$

Where x and R are in thousands of dollars. Is there a point of diminishing returns for this function?

Solution

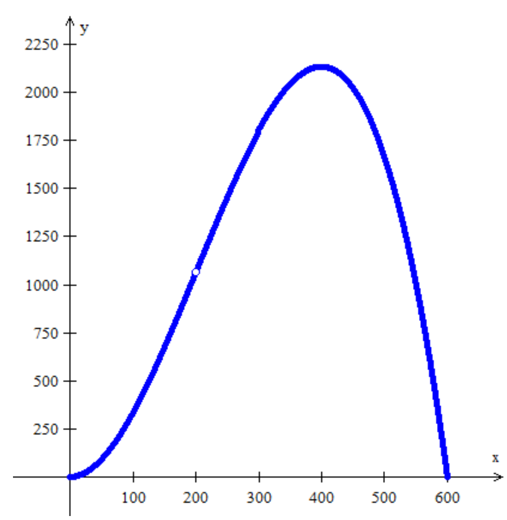
$R' = \frac{1}{15{,}000}\left(1200x - 3x^2\right)$

$R' = \frac{1}{15{,}000}(1200 - 6x) = 0$

$\Rightarrow x = \frac{1200}{6} = 200$

$x = 200$ (or \$200,000) is a ***diminishing point***

An increased investment beyond this point is usually considered a poor use of capital



Exercise

Find the point of diminishing returns (x, y) for the function

$$R(x) = -x^3 + 45x^2 + 400x + 8000, \quad 0 \le x \le 20$$

where $R(x)$ represents revenue in thousands of dollars and x represents the amount spent on advertising in tens of thousands of dollars.

Solution

$$R'(x) = -3x^2 + 90x + 400$$

$$R''(x) = -6x + 90 = 0$$

$$-6x = -90$$

$$\boxed{x = \frac{-90}{-6} = 15}$$

$$R(x = 15) = -(15)^3 + 45(15)^2 + 400(15) + 8000$$

$$\boxed{= 20,750}$$

The point of diminishing returns is $\boxed{(15, \;\; 20,750)}$

Exercise

The population of a certain species of fish introduced into a lake is described by the logistic equation

$$G(t) = \frac{12,000}{1 + 19e^{-1.2t}}$$

where $G(t)$ is the population after t years. Find the point at which the growth rate of this population begins to decline.

Solution

$$G'(t) = -\frac{12,000\left((-1.2)19e^{-1.2t}\right)}{\left(1 + 19e^{-1.2t}\right)^2} \qquad G = \frac{1}{U} \quad G' = -\frac{U'}{U^2}$$

$$= 273,600 \frac{e^{-1.2t}}{\left(1 + 19e^{-1.2t}\right)^2}$$

$$f = e^{-1.2t} \qquad f' = -1.2e^{-1.2t}$$

$$g = \left(1 + 19e^{-1.2t}\right)^2 \qquad g' = 2(-1.2)\left(19e^{-1.2t}\right)\left(1 + 19e^{-1.2t}\right)$$

$$= -45.6e^{-1.2t}\left(1 + 19e^{-1.2t}\right)$$

$$G''(t) = 273,600\frac{-1.2e^{-1.2t}\left(1+19e^{-1.2t}\right)^2 - e^{-1.2t}\left(-45.6e^{-1.2t}\left(1+19e^{-1.2t}\right)\right)}{\left(1+19e^{-1.2t}\right)^4}$$

$$= 273,600\frac{-1.2e^{-1.2t}\left(1+19e^{-1.2t}\right)^2 + 45.6e^{-2.4t}\left(1+19e^{-1.2t}\right)}{\left(1+19e^{-1.2t}\right)^4}$$

$$= 273,600\frac{\left(1+19e^{-1.2t}\right)\left[-1.2e^{-1.2t}\left(1+19e^{-1.2t}\right)+45.6e^{-2.4t}\right]}{\left(1+19e^{-1.2t}\right)^4}$$

$$= 273,600\frac{\left[-1.2e^{-1.2t} - 22.8e^{-2.4t} + 45.6e^{-2.4t}\right]}{\left(1+19e^{-1.2t}\right)^3}$$

$$= 273,600\frac{\left(-1.2e^{-1.2t} + 22.8e^{-2.4t}\right)}{\left(1+19e^{-1.2t}\right)^3}$$

$$= 273,600\frac{1.2e^{-1.2t}\left(-1+19e^{-1.2t}\right)}{\left(1+19e^{-1.2t}\right)^3} = 0$$

$$\to \begin{cases} e^{-1.2t} = 0 & \textit{never equals to zero} \\ -1+19e^{-1.2t} = 0 & e^{-1.2t} = \frac{1}{19} \end{cases}$$

$$\ln e^{-1.2t} = \ln\frac{1}{19}$$

$$-1.2t = \ln\frac{1}{19}$$

$$\underline{\left| t = \frac{\ln\left(\frac{1}{19}\right)}{-1.2} = 2.45 \right|}$$

$$G(t = 2.45) = \frac{12,000}{1+19e^{-1.2(2.45)}} = 5,986.68$$

The point at which the growth rate of this population begins to decline: $\boxed{(2.45,\ \ 5,987)}$