SOLUTION Section 4.4 – Equivalence Relations

Exercise

Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations?
Determine the properties of an equivalence relation that the others lack.
What are the equivalence classes of the equivalence relations?

a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$

b) $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$

c) $\{(0, 0), (1, 1), (1, 2), (2, 1), (3, 2), (3, 3)\}$

d) $\{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

e) $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

Solution

a) This is an equivalence relation. It is reflexive, symmetric and transitive.
The equivalence classes all have just one element.
Each element is in an equivalence class by itself.

b) This is not reflexive since pair (1, 1) is missing. It is symmetric and it is not transitive since the pairs (0, 2) and (2, 3) are there, but not (0, 3).
This is not an equivalence relation.

c) This is an equivalence relation. It is reflexive, symmetric and transitive.
The elements 1 and 2 are in one equivalence class, and 0 and 3 are each in their own equivalence class.

d) This is reflexive and symmetric and it is not transitive since the pairs (1, 3) and (3, 2) are there, but not (1, 2).
This is not an equivalence relation.

e) This is reflexive, it is not symmetric since (2, 1) is missing and it is not transitive since the pairs (2, 0) and (0, 1) are there, but not (2, 1).
This is not an equivalence relation.

Exercise

Which of these relations on the set of all people are equivalence relations?
Determine the properties of an equivalence relation that the others lack.
What are the equivalence classes of the equivalence relations?

a) $\{(a, b) \mid a \text{ and } b \text{ are the same age}\}$

b) $\{(a, b) \mid a \text{ and } b \text{ have the same parents}\}$

c) $\{(a, b) \mid a \text{ and } b \text{ share a common parent}\}$

d) $\{(a, b) \mid a \text{ and } b \text{ have met}\}$

e) $\{(a, b) \mid a \text{ and } b \text{ speak a common language}\}$

Solution

a) This relation is reflexive, since a is the same person (same age).
If a is the same age as b, then b has to be the same age as a. this relation is symmetric.
If a is the same age as b and b is the same age as c, then a has to be the same age as c. this relation is transitive.
An equivalence class is the set of all people who are the same age. To really identify the equivalence class and the equivalence relation itself, one would need to specify exactly what ine meant by the "same age". For example, we could define two people to be the same age if their official dates of birth were identical.

b) For each pair (m, w) of a man and a woman, the set of offspring of their union, if nonempty, is an equivalence class. In many cases, then, an equivalence class consists of all the children in a nuclear family with children.

c) Let assume the relation is biological parentage. It is possible that a to be the the child of W and X, b is the child of X and Y, and c is the child of Y and Z. Then a is related to b, and b is related to c, but a is not related to c. This is not an equivalence relation, since it is not transitive. Therefore, this is not an equivalence relation.

d) If a met b and b met c, then it is not neccesary that a met c. This is not an equivalence relation, since it is not transitive. Therefore, this is not an equivalence relation.

e) If a speaks the same language (english) as b and b speaks the same language (spanish) as c, then it is not neccesary that a can speak spanish as c. This is not an equivalence relation, since it is not transitive.

Exercise

Which of these relations on the set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$ are equivalence relations?
Determine the properties of an equivalence relation that the others lack.
What are the equivalence classes of the equivalence relations?

a) $\{(f, g) \mid f(1) = g(1)\}$

b) $\{(f, g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$

c) $\{(f, g) \mid f(x) - g(x) = 1 \text{ for all } x \in \mathbb{Z}\}$

d) $\{(f, g) \mid \text{for some } C \in \mathbb{Z}, \text{for all } x \in \mathbb{Z}, f(x) - g(x) = C\}$

e) $\{(f, g) \mid f(0) = g(1) \text{ or } f(1) = g(0)\}$

Solution

a) This is an equivalence relation, one of the general form that 2 things are considered equivalent if they have the same "something" (is 1).
$\{(f, f) \mid f(1) = f(1)\}$ This relation is reflexive.
If $\{(f, g) \mid f(1) = g(1)\}$ then $\{(g, f) \mid g(1) = f(1)\}$ $\therefore$ this relation is symmetric
If $\{(f, g) \mid f(1) = g(1)\}$ and $\{(g, h) \mid g(1) = h(1)\}$, then $\{(f, h) \mid f(1) = h(1)\}$ $\therefore$ this relation is transitive.

There is one equivalence class for each $n \in \mathbb{Z}$ and it contains all those functions whose value at 1 is n.

b) Let $f(x)=0, \quad g(x)=x, \quad and \quad h(x)=1$ for all $x \in \mathbb{Z}$. Then *f* is related to *g* since $f(0)=g(0)$ and *g* is related to *h* since $g(1)=h(1)$, but $f(0)\neq h(1)$, therefore *f* is related to *h* since that have no values in common.
Hence, this is not an equivalence relation because it is not transitive.

c) It is not reflexive relation since $f(x)-f(x)=0\neq 1$.
It is not symmetric since if $f(x)-g(x)=1$, then $g(x)-f(x)=-1\neq 1$
It is not transitive since $f(x)-g(x)=1$ and $g(x)-h(x)=1 \quad \Rightarrow \quad f(x)-h(x)=2\neq 1$,
This is not an equivalence relation.

d) This relation is reflexive, $f(x)-f(x)=0\in \mathbb{Z}$
$f(x)-g(x)=C \quad \Rightarrow \quad g(x)-f(x)=-C\in \mathbb{Z}$, this relation is symmetric
$f(x)-g(x)=C_1 \quad g(x)-h(x)=C_2 \quad f(x)-h(x)=C_1+C_2 \in \mathbb{Z}$, this relation is transitive.
This is an equivalence relation.
The set of equivalence classes is uncountable. For each function $f:\mathbb{Z}\to\mathbb{Z}$, there is the equivalence class consisting of all functions *g* for which there is a constant *C* such that $g(n)=f(n)+C$ for all $n\in\mathbb{Z}$.

e) It is not reflexive relation since $f(0)\neq f(1)$ since it not given.
This is not an equivalence relation.

Exercise

Define three equivalence relations on the set of students in your discrete mathematics class different from the relations discussed in the text. Determine the equivalence classes for each of these equivalence relations.

Solution

One relation is that *a* and *b* are related if they were born in the same state. Here the equivalence classes are the nonempty sets of students from each state.
Another example is for *a* to be related to *b* if *a* and *b* have lived the same number of complete decades. The equivalence classes are the set of all 10 to 19 years olds. The set of all 20 to 29 year olds, and so on.
A Third example is *a* to be related to *b* if 10 is a divisor of the difference between *a*'s age and *b*'s age, where "age" means the whole number of years since birth, as of the first day of class.
For each $i = 0, 1, 2, \ldots, 9$, there is the equivalence class (if it is nonempty) of those students whose age ends with the digit i.

Exercise

Define three equivalence relations on the set of buildings on a college campus. Determine the equivalence classes for each of these equivalence relations.

Solution

Two buildings are equivalent, if they were opened during the same year; an equivalence class consists of the set of buildings opened in a given year.
For another example, we can define 2 buildings to be equivalent if they have the same number of stories; the equivalence classes are the set of 1-story buildings, the set of 2-story buildings, and so on.
The third example, partition the set of all buildings into 2 classes – those in which you do have a class this semester and those in which you don't. Every building in which you have a class is equivalent to every building in which you have a class (including itself), and every building in which you don't have a class is equivalent to every building in which you don't have a class.

Exercise

Let R be the relation on the set of all sets of real numbers such that $S\ R\ T$ if and only if S and T have the same cardinality. Show that R is an equivalence relation. What are the equivalence classes of the sets $\{0, 1, 2\}$ and $\boldsymbol{Z}$?

Solution

Two sets have the same cardinality if there is a bijection (1−1 and onto function) from one set to the other.
We need to prove that R is reflexive, symmetric, and transitive.
Every set has the same cardinality as itself because of the identity function.
If f is a bijection from S to T, then f^{-1} is a bijection from T tot S, so R is symmetric.
If f is a bijection from S to T, and g is a bijection from T to U, then $g \circ f$ is a bijection from S to U, so R is transitive.
The equivalence class $\{1, 2, 3\}$ is the set of all 3-element sets of real numbers, including such sets as $\{4, 25, 1948\}$ and $\left(e,\ \pi,\ \sqrt{2}\right)$.
Similarly, $[\boldsymbol{Z}]$ is the set of all infinite countable sets of real numbers, such as the set of natural numbers, the set of rational numbers, and the set of the prime numbers, but not including the set $\{1, 2, 3\}$ (it's too small) or the set of all real numbers (too big).

Exercise

Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) such that $f(x) = f(y)$

a) Show that R is an equivalence relation on A.
b) What are the equivalence classes of R?

Solution

a) It is reflexive since $f(x) = f(x)$ for all $x \in A$

It is symmetric since $f(x) = f(y)$, then $f(y) = f(x)$

It is transitive since $f(x) = f(y)$ and $f(y) = f(z)$ then $f(x) = f(z)$

b) The equivalence class of x is the set of all $y \in A$ such that $f(y) = f(x)$ (definition of inverse).

Thus the equivalence classes are precisely the sets $f^{-1}(b)$ for every b in the range of f.

Exercise

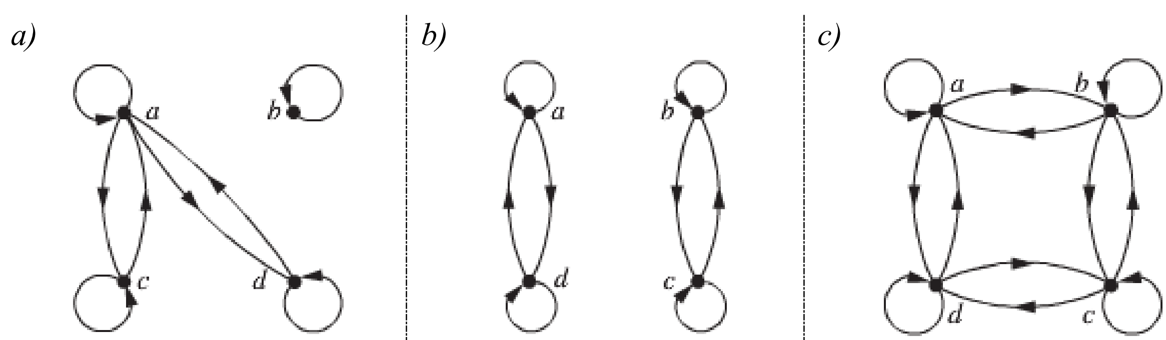
Suppose that A is a nonempty set, and R is an equivalence relation on A. Show that there is a function f with A as its domain such that $(x, y) \in R$ if and only if $f(x) = f(y)$

Solution

The function that sends each $x \in A$ to its equivalence class $[x]$ is obviously such a function.

Exercise

Determine whether the relation with the directed graph shown is an equivalence relation



Solution

a) The relation is reflexive since there is a loop at each vertex.
It is symmetric since every edge has 2 vertices and pointing in the both direction.
It is not transitive since we have {d, a} and {a, c} but not {d, a}

b) The relation is reflexive since there is a loop at each vertex.
It is symmetric since every edge has 2 vertices and pointing in the both direction.
It is transitive since paths of length 2 are accompanied by the path of length 1, edge between the same 2 vertices in the same direction.

This relation is an equivalence relation.
The equivalence classes are $\{a, d\}$ and $\{b, c\}$

c) The relation is reflexive since there is a loop at each vertex.
It is symmetric since every edge has 2 vertices and pointing in the both direction.
It is not transitive (a, b) and (b, c) but not (a, c).

Exercise

Which of these collections of subsets are partitions of $\{1, 2, 3, 4, 5, 6\}$

a) $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}$
b) $\{1\}, \{2, 3, 6\}, \{4\}, \{5\}$
c) $\{2, 4, 6\}, \{1, 3, 5\}$
d) $\{1, 4, 5\}, \{2, 6\}$

Solution

a) This is not a partition, since the sets are not pairwise disjoint. 2 and 4 appear in 2 of the sets.

b) This is a partition

c) This is a partition

d) This is not a partition, since element 3 is missing from the sets

Exercise

Which of these collections of subsets are partitions of $\{-3, -2, -1, 0, 1, 2, 3\}$

a) $\{-3, -1, 1, 3\}, \{-2, 0, 2\}$
b) $\{-3, -2, -1, 0\}, \{0, 1, 2, 3\}$
c) $\{-3, 3\}, \{-2, 2\}, \{-1, 1\}, \{0\}$
d) $\{-3, -2, 2, 3\}, \{-1, 1\}$

Solution

a) This a partition, since it satisfies the definition

b) This is not a partition, since the subsets are not disjoint

c) This a partition, since it satisfies the definition

d) This is not a partition, since the union of the subsets leaves out 0