

Solution **Section 2.5 – Derivative as Rates of Change**

Exercise

The position $s(t) = t^2 - 3t + 2$, $0 \leq t \leq 2$ of a body moving on a coordinate line, with s in meters and t in seconds.

- a) Find the body's displacement and average velocity for the given time interval.
- b) Find the body's speed and acceleration at the endpoints of the interval.
- c) When, if ever, during the interval does the body change direction?

Solution

$$\begin{aligned} \text{a) Displacement: } \Delta s &= s(2) - s(0) \\ &= 2^2 - 3(2) + 2 - (0^2 - 3(0) + 2) \\ &= -2 \text{ m} \end{aligned}$$

$$\begin{aligned} \text{Average velocity} &= \frac{\Delta s}{\Delta t} \\ &= \frac{-2}{2-0} \\ &= -1 \text{ m/sec} \end{aligned}$$

$$\begin{aligned} \text{b) } v &= \frac{ds}{dt} = 2t - 3 \\ &\Rightarrow \begin{cases} |v(0)| = |-3| = 3 \text{ m/sec} \\ |v(2)| = 1 \text{ m/sec} \end{cases} \end{aligned}$$

$$\begin{aligned} a &= \frac{dv}{dt} = 2 \\ a(0) &= a(2) = 2 \text{ m/sec}^2 \end{aligned}$$

$$\begin{aligned} \text{c) } v &= 0 \\ 2t - 3 &= 0 \\ t &= \frac{3}{2} \end{aligned}$$

v is negative in the interval $0 < t < \frac{3}{2}$

v is positive in the interval $\frac{3}{2} < t < 2$

The body changes direction at $t = \frac{3}{2}$

Exercise

The position $s(t) = \frac{25}{t+5}$, $-4 \leq t \leq 0$ of a body moving on a coordinate line, with s in meters and t in seconds.

- a) Find the body's displacement and average velocity for the given time interval.
- b) Find the body's speed and acceleration at the endpoints of the interval.
- c) When, if ever, during the interval does the body change direction?

Solution

a) Displacement: $\Delta s = s(0) - s(-4)$

$$\begin{aligned} &= \frac{25}{0+5} - \frac{25}{-4+5} \\ &= 5 - 25 \\ &= -20 \text{ m} \end{aligned}$$

$$\text{Average velocity} = \frac{\Delta s}{\Delta t} = \frac{-20}{0 - (-4)} = -5 \text{ m/sec}$$

b) $v = \frac{ds}{dt} = \frac{25(-1)}{(t+5)^2} = -\frac{25}{(t+5)^2}$

$$\Rightarrow \begin{cases} |v(-4)| = \left| -\frac{25}{(-4+5)^2} \right| = 25 \text{ m/sec} \\ |v(0)| = \left| -\frac{25}{(0+5)^2} \right| = 1 \text{ m/sec} \end{cases}$$

$$a = \frac{dv}{dt} = -\frac{-25[2(t+5)(1)]}{(t+5)^4}$$

$$= \frac{50}{(t+5)^3}$$

$$a(-4) = \frac{50}{(-4+5)^3} = 50 \text{ m/sec}^2$$

$$a(0) = \frac{50}{(0+5)^3} = \frac{2}{5} \text{ m/sec}^2$$

c) $v = 0$

$$-\frac{25}{(t+5)^2} = 0 \rightarrow v < 0$$

v is never equal to zero \Rightarrow The body never changes direction.

Exercise

At time t , the position of a body moving along the s -axis is $s = t^3 - 6t^2 + 9t$ m.

- Find the body's acceleration each time the velocity is zero.
- Find the body's speed each time the acceleration is zero.
- Find the total distance traveled by the body from $t = 0$ to $t = 2$.

Solution

$$a) \quad v = s' = 3t^2 - 12t + 9 = 0$$

$$\underline{t_1 = 1 \quad \& \quad t_2 = 3}$$

$$a = v' = 6t - 12$$

$$\begin{cases} a(1) = 6 - 12 = -6 \text{ m/sec}^2 \\ a(3) = 6(3) - 12 = 6 \text{ m/sec}^2 \end{cases}$$

The body is motionless but being accelerated left when $t = 1$, and motionless but being accelerated right when $t = 3$.

$$b) \quad a = 0 = 6t - 12$$

$$\Rightarrow \underline{t = 2}$$

$$\begin{aligned} |v(2)| &= |3(2)^2 - 12(2) + 9| \\ &= 3 \text{ m/sec} \end{aligned}$$

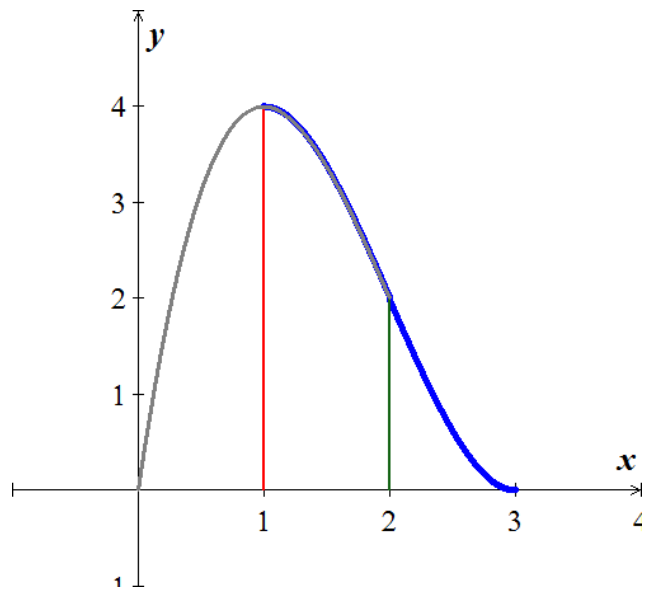
$$c) \quad \text{The body moves forward on } 0 \leq t < 1$$

$$\begin{aligned} d_1 &= s(1) - s(0) \\ &= 1 - 6 + 9 \\ &= 4 \end{aligned}$$

The body moves backward on $1 \leq t < 2$

$$\begin{aligned} d_2 &= |s(2) - s(1)| \\ &= |2 - 4| \\ &= 2 \end{aligned}$$

$$\text{Total distance} = d_1 + d_2 = 4 + 2 = 6 \text{ m}$$



Exercise

A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s(t) = 24t - 0.8t^2 \text{ m}$ in $t \text{ sec}$.

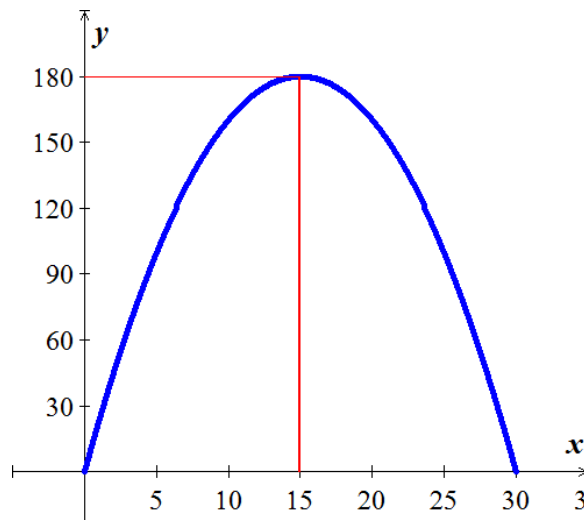
- Find the rock's velocity and acceleration at time t . (The acceleration in this case is the acceleration of gravity on the moon.)
- How long does it take the rock to reach its highest point?
- How high does the rock go?
- How long does it take the rock to reach half its maximum height?
- How long is the rock aloft?

Solution

$$\begin{aligned} a) \quad v(t) &= s' = 24 - 1.6t \text{ m/sec} \\ a(t) &= v' = s'' = -1.6 \text{ m/sec}^2 \end{aligned}$$

$$\begin{aligned} b) \quad v(t) &= 0 = 24 - 1.6t \\ t &= \frac{24}{1.6} \\ &= 15 \text{ sec} \end{aligned}$$

$$\begin{aligned} c) \quad s(15) &= 24(15) - 0.8(15)^2 \\ &= 180 \text{ m} \end{aligned}$$



- Since the maximum high is 180 m , then half is 90 m :

$$s(t) = 24t - 0.8t^2 = 90$$

$$-0.8t^2 + 24t - 90 = 0 \Rightarrow t = 4.39 \quad t = 25.61$$

$$2t^2 - 60t + 225 = 0$$

$$\begin{aligned} t_{1,2} &= \frac{60 \pm \sqrt{3,600 - 1,800}}{4} \\ &= 15 \pm \frac{15\sqrt{2}}{2} \end{aligned}$$

$$t_1 = 4.39 \quad t_2 = 25.61$$

It took 4.39 sec going up and 25.6 sec going down.

- The rock took 30 sec to reach its highest point.

Exercise

Had Galileo dropped a cannonball from the Tower of Pisa, 179 *ft* above the ground, the ball's height above the ground t *sec* into the fall would have been $s = 179 - 16t^2$.

- a) What would have been the ball's velocity, speed, and acceleration at time t ?
- b) About how long would it have taken the ball to hit the ground?
- c) What would have been the ball's velocity at the moment of impact?

Solution

a) $v = s' = -32t$ |

$$\text{speed} = |v| = 32t \text{ ft/sec}$$

$$a = -32 \text{ ft/sec}^2$$

b) $s = 0 = 179 - 16t^2$

$$16t^2 = 179$$

$$t = \sqrt{\frac{179}{16}}$$

$$= \frac{\sqrt{179}}{4} \text{ sec}$$

$$\approx 3.3 \text{ sec}$$

c) When $t = 3.3 \text{ sec}$

$$\Rightarrow v = -32t$$

$$= -32(3.3)$$

$$= -107 \text{ ft/sec}$$

Exercise

A toy rocket fired straight up into the air has height $s(t) = 160t - 16t^2$ *feet* after t seconds.

- a) What is the rocket's initial velocity (when $t = 0$)?
- b) What is the acceleration when $t = 3$?
- c) At what time will the rocket hit the ground?
- d) At what velocity will the rocket be traveling just as it smashes into the ground?

Solution

a) $v(t) = s'(t) = 160 - 32t$

$$v(0) = 160$$

b) $a(t) = v'(t) = -32$

$$a(t=3) = -32 \text{ ft/sec}^2$$

$$c) \quad s(t) = 160t - 16t^2 = 0$$

The rocket hit the ground at $t = 0$

$$t = \frac{160}{16}$$

$$= 10 \text{ sec}$$

Exercise

A helicopter is rising straight up in the air. Its distance from the ground t seconds after takeoff is

$$s(t) = t^2 + t \text{ feet}$$

- How long will it take for the helicopter to rise 20 feet ?
- Find the velocity and the acceleration of the helicopter when it is 20 feet above the ground.

Solution

$$a) \quad s(t) = t^2 + t = 20$$

$$t^2 + t - 20 = 0$$

$$t = -5, \quad t = 4$$

It will take 10 sec. for the helicopter to rise 20 feet.

$$b) \quad v(t) = s'(t) = 2t + 1$$

$$v(t = 10) = 21 \text{ ft/sec}$$

$$a(t) = v'(t) = 2$$

$$a(t = 10) = 2 \text{ ft}^2/\text{sec}$$

Exercise

The position of a particle moving on a line is given by $s(t) = 2t^3 - 21t^2 + 60t$, $t \geq 0$, where t is measured in seconds and s in feet.

- What is the velocity after 3 seconds and after 6 seconds?
- When the particle moving in the positive direction?
- Find the total distance traveled by the particle during the first 7 seconds.

Solution

$$a) \quad v(t) = s'(t) = 6t^2 - 42t + 60$$

$$v(t = 3) = 6(9) - 42(3) + 60$$

$$= -12 \text{ ft/sec}$$

$$v(t=6) = 6(36) - 42(6) + 60$$

$$= 24 \text{ ft/sec}$$

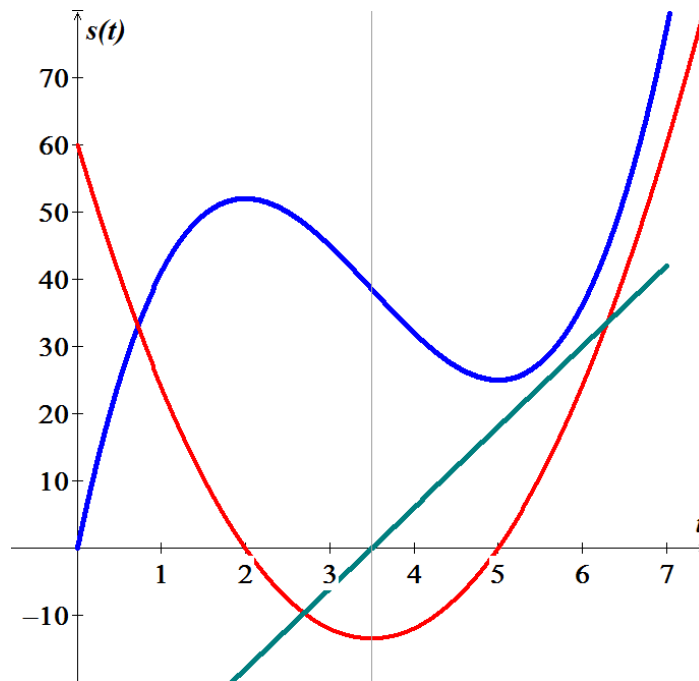
$$b) \quad a(t) = v'(t) = 12t - 42 = 0$$

$$t = \frac{7}{2} \text{ sec}$$

The particle is moving in the positive direction at 3.5 sec

$$c) \quad s(t=7) = 2(7)^3 - 21(7)^2 + 60(7)$$

$$= 77 \text{ ft}$$



Exercise

A small probe is launched vertically from the ground. After it reaches its high point, a parachute deploys, and the probe descends to Earth. The height of the probe the ground is

$$s(t) = \frac{300t - 50t^2}{t^3 + 2} \quad \text{for } 0 \leq t \leq 6$$

- Graph the height function and describe the motion of the probe.
- Find the velocity of the probe.
- Graph the velocity function and determine the approximate time at which the velocity is a maximum.

Solution

$$a) \quad s'(t) = \frac{(300 - 100t)(t^3 + 2) - 3t^2(300t - 50t^2)}{(t^3 + 2)^2}$$

$$= \frac{300t^3 - 100t^4 + 600 - 200t - 900t^3 + 150t^4}{(t^3 + 2)^2}$$

$$= \frac{50t^4 - 600t^3 - 200t + 600}{(t^3 + 2)^2}$$

$$50t^4 - 600t^3 - 200t + 600 = 0$$

$$t^4 - 12t^3 - 4t + 12 = 0$$

$$t = 0.91, \quad \cancel{12.02} > 6 \quad |$$

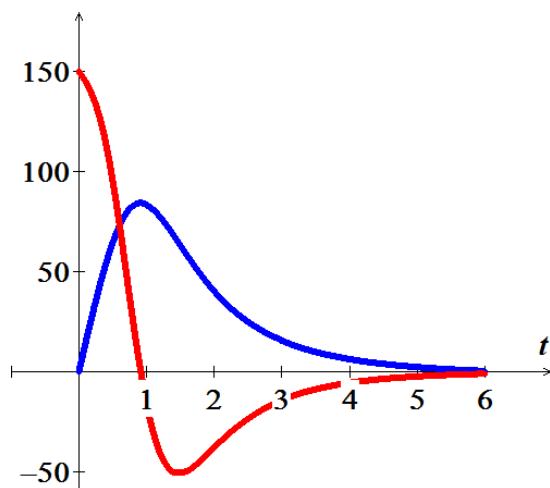
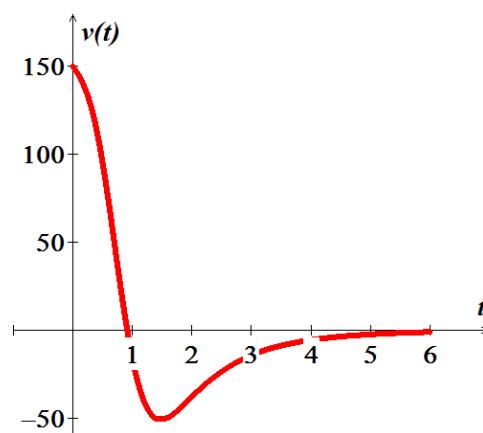
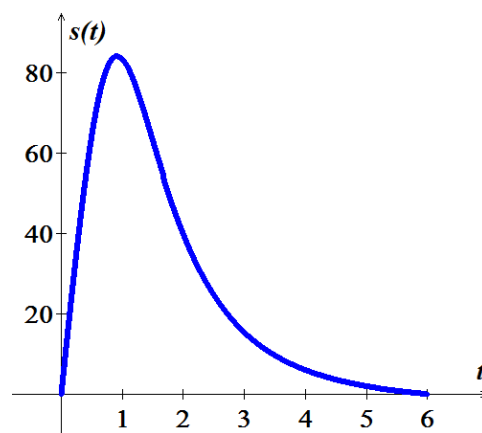
$$s(t = .91) = 84.107 \quad |$$

The maximum height is 84.107 at $t = 0.91$

$$b) \quad v(t) = s'(t) = \frac{50t^4 - 600t^3 - 200t + 600}{(t^3 + 2)^2}$$

$$c) \quad v(t = 0) = \frac{600}{4} \\ = 150 \quad |$$

The maximum velocity is 150



Exercise

Suppose the cost of producing x lawn mowers is $C(x) = -0.02x^2 + 400x + 5000$

- a) Determine the average and marginal costs for $x = 3000$ lawn mowers.
- b) Interpret the meaning of your results in part (a)

Solution

$$\begin{aligned} \text{a) Average Cost} &= \frac{C(3,000)}{3,000} \\ &= \frac{-0.02(9 \times 10^6) + 1,200,000 + 5,000}{3,000} \\ &= \frac{1,025,000}{3,000} \\ &= \$341.67 \end{aligned}$$

$$\text{Marginal Cost} = C'(x) = -0.04x + 400$$

$$\begin{aligned} C'(3,000) &= -0.04(3,000) + 400 \\ &= \$280.00 \end{aligned}$$

- b) The average cost of producing 3,000 lawnmowers is \$341.67 per mower.

The cost of producing the 3,001st lawnmower is about \$280.00

Exercise

Suppose a company produces fly rods. Assume $C(x) = -0.0001x^3 + 0.05x^2 + 60x + 800$ represents the cost of making x fly rods.

- a) Determine the average and marginal costs for $x = 400$ fly rods.
- b) Interpret the meaning of your results in part (a)

Solution

$$\begin{aligned} \text{a) Average Cost} &= \frac{C(400)}{400} \\ &= \frac{-0.0001(400)^3 + 0.05(400)^2 + 24,000 + 800}{400} \\ &= \frac{26,400}{400} \\ &= \$66.00 \end{aligned}$$

$$\text{Marginal Cost} = C'(x) = -0.0003x^2 + 0.1x + 60$$

$$\begin{aligned} C'(400) &= -0.0003(160,000) + 40 + 60 \\ &= \$52.00 \end{aligned}$$

- c) The average cost of producing 400 fly rods is \$66.00 per fly rod.

The cost of producing the 401st flying rod is about \$52.00

Exercise

Suppose $p(t) = -1.7t^3 + 72t^2 + 7200t + 80,000$ is the population of a city t years after 1950.

- a) Determine the average rate of growth of the city from 1950 to 2000.
- b) What was the rate of growth of the city in 1990?

Solution

From 1950 to 2000 $\rightarrow 0 \leq t \leq 50$

$$\begin{aligned} \text{a) Average growth rate} &= \frac{P(50) - P(0)}{50 - 0} \\ &= \frac{407,500 - 80,000}{50} \\ &= \underline{6,550 \text{ ppl/yr}} \end{aligned}$$

$$\text{b) } p'(t) = -5.1t^2 + 144t + 7200$$

$$\begin{aligned} p'(40) &= -5.1(1,600) + 144(40) + 7200 \\ &= \underline{4,800 \text{ ppl/yr}} \end{aligned}$$