Solution — *Section* 2.2 – Linear, Homogeneous Equations with Constant Coefficients

Exercise

Find the general solution: $y'' + y' = 0$

Solution

The characteristic equation: $\lambda^2 + \lambda = 0 \rightarrow \boxed{\lambda_{1,2} = 0,\ -1}$

$$\boxed{y(x) = C_1 + C_2 e^{-x}}$$

Exercise

Find the general solution: $y'' - 4y = 0$

Solution

The characteristic equation: $\lambda^2 - 4 = 0 \rightarrow \boxed{\lambda_{1,2} = \pm 2}$

$$\boxed{y(x) = C_1 e^{-2x} + C_2 e^{2x}}$$

Exercise

Find the general solution: $y'' + 8y = 0$

Solution

The characteristic equation: $\lambda^2 + 8\lambda = 0 \rightarrow \boxed{\lambda_{1,2} = 0,\ 8}$

$$\boxed{y(x) = C_1 + C_2 e^{8x}}$$

Exercise

Find the general solution: $y'' - 36y = 0$

Solution

The characteristic equation: $\lambda^2 - 36 = 0 \rightarrow \boxed{\lambda_{1,2} = \pm 6}$

$$\boxed{y(x) = C_1 e^{-6x} + C_2 e^{6x}}$$

Exercise

Find the general solution: $y'' + 9y = 0$

Solution

The characteristic equation: $\lambda^2 + 9 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 3i}$

$\boxed{y(x) = C_1 \cos 3x + C_2 \sin 3x}$

Exercise

Find the general solution: $y'' + 16y = 0$

Solution

The characteristic equation: $\lambda^2 + 16 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 4i}$

$\boxed{y(x) = C_1 \cos 4x + C_2 \sin 4x}$

Exercise

Find the general solution: $y'' + 25y = 0$

Solution

The characteristic equation: $\lambda^2 + 25 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 5i}$

$\boxed{y(x) = C_1 \cos 5x + C_2 \sin 5x}$

Exercise

Find the general solution: $y'' - 64y = 0$

Solution

The characteristic equation: $\lambda^2 - 64 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 8}$

$\boxed{y(x) = C_1 e^{-8x} + C_2 e^{8x}}$

Exercise

Find the general solution: $y'' + y' + y = 0$

Solution

The characteristic equation: $\lambda^2 + \lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$\boxed{y(x) = e^{-x/2}\left(C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x\right)}$

Exercise

Find the general solution: $y'' + y' - y = 0$

Solution

The characteristic equation: $\lambda^2 + \lambda - 1 = 0 \quad \to \quad \boxed{\lambda_{1,2} = \frac{-1 \pm \sqrt{5}}{2}}$

$$\boxed{y(x) = C_1 e^{\frac{-1-\sqrt{5}}{2}x} + C_2 e^{\frac{-1+\sqrt{5}}{2}x}}$$

Exercise

Find the general solution: $y'' - y' - 2y = 0$

Solution

The characteristic equation: $\lambda^2 - \lambda - 2 = 0 \quad \to \quad \boxed{\lambda_1 = -1,\ \lambda_2 = 2}$

$$\boxed{y(x) = C_1 e^{-x} + C_2 e^{2x}}$$

Exercise

Find the general solution: $y'' - y' - 6y = 0$

Solution

The characteristic equation: $\lambda^2 - \lambda - 6 = 0 \quad \to \quad \lambda_{1,2} = \frac{1 \pm 5}{2} \ \boxed{= -2,\ 3}$

$$\boxed{y(x) = C_1 e^{-2x} + C_2 e^{3x}}$$

Exercise

Find the general solution: $y'' + y' - 6y = 0$

Solution

The characteristic equation: $\lambda^2 + \lambda - 6 = 0 \quad \to \quad \lambda_{1,2} = \frac{-1 \pm 5}{2} \ \boxed{= -3,\ 2}$

$$\boxed{y(x) = C_1 e^{-3x} + C_2 e^{2x}}$$

Exercise

Find the general solution: $y'' - y' - 11y = 0$

Solution

The characteristic equation: $\lambda^2 - \lambda - 11 = 0 \quad \to \quad \lambda_{1,2} = \frac{1 \pm \sqrt{45}}{2} \ \boxed{= \frac{1 \pm 3\sqrt{5}}{2}}$

$$\boxed{y(x) = C_1 e^{\frac{1-3\sqrt{5}}{2}x} + C_2 e^{\frac{1+3\sqrt{5}}{2}x}}$$

Exercise

Find the general solution: $y'' - y' - 12y = 0$

Solution

The characteristic equation: $\lambda^2 - \lambda - 12 = 0 \quad \Rightarrow \boxed{\lambda_{1,2} = -3,\ 4}$

$\boxed{y(t) = C_1 e^{-3t} + C_2 e^{4t}}$

Exercise

Find the general solution: $y'' + 2y' + y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \boxed{\lambda_{1,2} = -1}$

$\boxed{y(t) = \left(C_1 + C_2 t\right) e^{-t}}$

Exercise

Find the general solution: $y'' + 2y' + 3y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 3 = 0 \quad \Rightarrow \lambda_{1,2} = \frac{-2 \pm 2i\sqrt{2}}{2} \boxed{= -1 \pm i\sqrt{2}}$

$\boxed{y(x) = e^{-x}\left(C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x\right)}$

Exercise

Find the general solution: $y'' + 2y' - 3y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda - 3 = 0 \quad \Rightarrow \boxed{\lambda_{1,2} = 1,\ -3}$

$\boxed{y(x) = C_1 e^{-3x} + C_2 e^{x}}$

Exercise

Find the general solution: $y'' - 2y' - 3y = 0$

Solution

The characteristic equation: $\lambda^2 - 2\lambda - 3 = 0 \quad \Rightarrow \boxed{\lambda_{1,2} = -1,\ 3}$

$\boxed{y(x) = C_1 e^{-x} + C_2 e^{3x}}$

Exercise

Find the general solution: $y'' - 2y' + 3y = 0$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 3 = 0 \quad \Rightarrow \lambda_{1,2} = \frac{2 \pm 2i\sqrt{2}}{2} = 1 \pm i\sqrt{2}$

$$y(x) = e^{x}\left(C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x\right)$$

Exercise

Find the general solution: $y'' + 2y' + 4y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 4 = 0 \quad \Rightarrow \lambda_{1,2} = \frac{-2 \pm 2i\sqrt{3}}{2} = -1 \pm i\sqrt{3}$

$$y(x) = e^{-x}\left(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right)$$

Exercise

Find the general solution: $y'' + 2y' - 15y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda - 15 = 0 \quad \Rightarrow \lambda_{1,2} = \frac{-2 \pm 8}{2} = -5,\ 3$

$$y(x) = C_1 e^{-5x} + C_2 e^{3x}$$

Exercise

Find the general solution: $y'' + 2y' + 17y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 17 = 0 \quad \Rightarrow \lambda_{1,2} = -1 \pm 4i$

$$y(t) = e^{-t}\left(C_1 \cos 4t + C_2 \sin 4t\right)$$

Exercise

Find the general solution: $y'' - 2y' + 5y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 5 = 0 \quad \Rightarrow \lambda_{1,2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

$$y(x) = e^{-x}\left(C_1 \cos 2x + C_2 \sin 2x\right)$$

Exercise

Find the general solution: $y'' - 3y' + 2y = 0$

Solution

The characteristic equation: $\lambda^2 - 3\lambda + 2 = 0 \quad \rightarrow \quad \lambda_{1,2} = 1,\ 2$

$y(x) = C_1 e^{x} + C_2 e^{2x}$

Exercise

Find the general solution: $y'' + 3y' - 4y = 0$

Solution

The characteristic equation: $\lambda^2 + 3\lambda - 4 = 0 \quad \rightarrow \quad \lambda_{1,2} = 1,\ -4$

$y(x) = C_1 e^{x} + C_2 e^{-4x}$

Exercise

Find the general solution: $y'' + 4y' - y = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda - 1 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$

$y(x) = C_1 e^{\left(-2-\sqrt{5}\right)x} + C_2 e^{\left(-2+\sqrt{5}\right)x}$

Exercise

Find the general solution: $y'' - 4y' + 4y = 0$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 4 = 0 \quad \Rightarrow \lambda_{1,2} = 2$

$y(t) = \left(C_1 + C_2 t\right)e^{2t}$

Exercise

Find the general solution: $y'' + 4y' + 4y = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 4 = 0 \quad \Rightarrow \lambda_{1,2} = -2$

$y(t) = \left(C_1 + C_2 t\right)e^{-2t}$

Exercise

Find the general solution: $y'' - 4y' + 5y = 0$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 5 = 0 \quad \to \quad \lambda_{1,2} = \frac{4 \pm 2i}{2} = 2 \pm i$

$y(x) = e^{2x}\left(C_1 \cos x + C_2 \sin x\right)$

Exercise

Find the general solution: $y'' + 4y' + 5y = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 5 = 0 \quad \to \quad \lambda_{1,2} = \frac{-4 \pm 2i}{2} = -2 \pm i$

$y(x) = e^{-2x}\left(C_1 \cos x + C_2 \sin x\right)$

Exercise

Find the general solution: $y'' + 4y' - 5y = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda - 5 = 0 \quad \to \quad \lambda_{1,2} = -5,\ 1$

$y(x) = C_1 e^{-5x} + C_2 e^{x}$

Exercise

Find the general solution: $y'' + 4y' + 7y = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 7 = 0 \quad \to \quad \lambda_{1,2} = \frac{-4 \pm 2i\sqrt{3}}{2} = -2 \pm i\sqrt{3}$

$y(x) = e^{-2x}\left(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right)$

Exercise

Find the general solution: $y'' + 4y' + 9y = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 9 = 0 \quad \to \quad \lambda_{1,2} = \frac{-4 \pm 2i\sqrt{5}}{2} = -2 \pm i\sqrt{5}$

$y(x) = e^{-2x}\left(C_1 \cos\sqrt{5}x + C_2 \sin\sqrt{5}x\right)$

Exercise

Find the general solution: $y'' + 5y' = 0$

Solution

The characteristic equation: $\lambda^2 + 5\lambda = 0 \quad \rightarrow \quad \lambda_1 = -5, \ \lambda_2 = 0$

$$y(x) = C_1 e^{-5x} + C_2$$

Exercise

Find the general solution: $y'' + 5y' + 6y = 0$

Solution

The characteristic equation: $\lambda^2 + 5\lambda + 6 = 0 \quad \rightarrow \quad \lambda_1 = -3, \ \lambda_2 = -2$

$$y(x) = C_1 e^{-3x} + C_2 e^{-2x}$$

Exercise

Find the general solution: $y'' + 6y' + 9y = 0$

Solution

The characteristic equation: $\lambda^2 + 6\lambda + 9 = 0 \quad \rightarrow \quad \lambda_{1,2} = -3$

$$y(x) = \left(C_1 + C_2 x\right) e^{-3x}$$

Exercise

Find the general solution: $y'' - 6y' + 9y = 0$

Solution

The characteristic equation: $\lambda^2 - 6\lambda + 9 = 0 \quad \Rightarrow \lambda_{1,2} = 3$

$$y(t) = \left(C_1 + C_2 t\right) e^{3t}$$

Exercise

Find the general solution: $y'' - 6y' + 25y = 0$

Solution

The characteristic equation: $\lambda^2 - 6\lambda + 25 = 0 \quad \Rightarrow \lambda_{1,2} = \frac{6 \pm 8i}{2} = 3 \pm 4i$

$$y(x) = e^{3x}\left(C_1 \cos 4x + C_2 \sin 4x\right)$$

Exercise

Find the general solution: $y'' + 8y' + 16y = 0$

Solution

The characteristic equation: $\lambda^2 + 8\lambda + 16 = (\lambda + 4)^2 = 0 \rightarrow \lambda_{1,2} = -4$

$y(x) = (C_1 + C_2 x)e^{-4x}$

Exercise

Find the general solution: $y'' + 8y' - 16y = 0$

Solution

The characteristic equation: $\lambda^2 + 8\lambda - 16 = 0 \rightarrow \lambda_{1,2} = \frac{-8 \pm 8\sqrt{2}}{2} = -4 \pm 4\sqrt{2}$

$y(x) = C_1 e^{(-4-4\sqrt{2})x} + C_2 e^{(-4+4\sqrt{2})x}$

Exercise

Find the general solution: $y'' - 9y' + 20y = 0$

Solution

The characteristic equation: $\lambda^2 - 9\lambda + 20 = 0 \rightarrow \lambda_{1,2} = \frac{9 \pm 1}{2} = 4,\ 5$

$y(x) = C_1 e^{4x} + C_2 e^{5x}$

Exercise

Find the general solution: $y'' - 10y' + 25y = 0$

Solution

The characteristic equation: $\lambda^2 - 10\lambda + 25 = (\lambda - 5)^2 = 0 \rightarrow \lambda_{1,2} = 5$

$y(x) = (C_1 + C_2 x)e^{5x}$

Exercise

Find the general solution: $y'' + 14y' + 49y = 0$

Solution

The characteristic equation: $\lambda^2 + 14\lambda + 49 = (\lambda + 7)^2 = 0 \Rightarrow \lambda_{1,2} = -7$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{-7x}}$$

Exercise

Find the general solution: $2y'' - y' - 3y = 0$

Solution

The characteristic equation: $2\lambda^2 - \lambda - 3 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -1,\ \frac{3}{2}}$

$$\boxed{y(x) = C_1 e^{-x} + C_2 e^{3x/2}}$$

Exercise

Find the general solution: $2y'' + y' - y = 0$

Solution

The characteristic equation: $2\lambda^2 + \lambda - 1 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -1,\ \frac{1}{2}}$

$$\boxed{y(x) = C_1 e^{-x} + C_2 e^{x/2}}$$

Exercise

Find the general solution: $2y'' + 2y' + y = 0$

Solution

The characteristic equation: $2\lambda^2 + 2\lambda + 1 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-2 \pm 2i}{4} \boxed{= \frac{1}{2} \pm \frac{1}{2} i}$

$$\boxed{y(x) = e^{x/2}\left(C_1 \cos\frac{1}{2}x + C_2 \sin\frac{1}{2}x\right)}$$

Exercise

Find the general solution: $2y'' + 2y' + 3y = 0$

Solution

The characteristic equation: $2\lambda^2 + 2\lambda + 3 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-2 \pm 2i\sqrt{5}}{4} \boxed{= -\frac{1}{2} \pm \frac{\sqrt{5}}{2} i}$

$$\boxed{y(x) = e^{-x/2}\left(C_1 \cos\frac{\sqrt{5}}{2}x + C_2 \sin\frac{\sqrt{5}}{2}x\right)}$$

Exercise

Find the general solution: $2y'' - 3y' - 2y = 0$

Solution

The characteristic equation: $2\lambda^2 - 3\lambda - 2 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{3 \pm 5}{4} = -\frac{1}{2},\ 2$

$y(x) = C_1 e^{-x/2} + C_2 e^{2x}$

Exercise

Find the general solution: $2y'' - 3y' + 4y = 0$

Solution

The characteristic equation: $2\lambda^2 - 3\lambda + 4 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{3 \pm i\sqrt{23}}{4} = \frac{3}{4} \pm \frac{\sqrt{23}}{4} i$

$y(x) = e^{3x/4}\left(C_1 \cos\frac{\sqrt{23}}{4}x + C_2 \sin\frac{\sqrt{23}}{4}x\right)$

Exercise

Find the general solution: $2y'' - 4y' + 8y = 0$

Solution

The characteristic equation: $2\lambda^2 - 4\lambda + 8 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{4 \pm 4i\sqrt{3}}{4} = 1 \pm i\sqrt{3}$

$y(x) = e^{x}\left(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right)$

Exercise

Find the general solution: $2y'' + 5y' = 0$

Solution

The characteristic equation: $2\lambda^2 + 5\lambda = \lambda(2\lambda + 5) = 0 \quad \Rightarrow \quad \lambda_1 = 0, \quad \lambda_2 = -\frac{5}{2}$

$y(x) = C_1 + C_2 e^{-5x/2}$

Exercise

Find the general solution: $2y'' - 5y' - 3y = 0$

Solution

The characteristic equation: $2\lambda^2 - 5\lambda - 3 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-5 \pm 7}{4} = -3,\ \frac{1}{2}$

$y(x) = C_1 e^{-3x} + C_2 e^{x/2}$

Exercise

Find the general solution: $2y'' + 7y' - 4y = 0$

Solution

The characteristic equation: $2\lambda^2 + 7\lambda - 4 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-7 \pm 9}{4} \boxed{= -4,\ \frac{1}{2}}$

$$\boxed{y(x) = C_1 e^{-4x} + C_2 e^{x/2}}$$

Exercise

Find the general solution: $3y'' + y = 0$

Solution

The characteristic equation: $3\lambda^2 + 1 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm \frac{1}{\sqrt{3}} i}$

$$\boxed{y(x) = C_1 \cos \frac{\sqrt{3}}{3} x + C_2 \sin \frac{\sqrt{3}}{3} x}$$

Exercise

Find the general solution: $3y'' - y' = 0$

Solution

The characteristic equation: $\lambda^2 + \lambda = \lambda(\lambda + 1) = 0 \quad \Rightarrow \quad \lambda_1 = 0, \quad \lambda_2 = -1$

$$\boxed{y(x) = C_1 + C_2 e^{-x}}$$

Exercise

Find the general solution: $3y'' + 2y' + y = 0$

Solution

The characteristic equation: $3\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 12}}{6} \boxed{= -\frac{1}{3} \pm i \frac{\sqrt{2}}{3}}$

$$\boxed{y(x) = e^{-x/3} \left(C_1 \cos \frac{\sqrt{2}}{3} x + C_2 \sin \frac{\sqrt{2}}{3} x \right)}$$

Exercise

Find the general solution: $3y'' + 11y' - 7y = 0$

Solution

The characteristic equation: $3\lambda^2 + 11\lambda - 7 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \frac{-11 \pm \sqrt{205}}{6}}$

$$\boxed{y(x) = C_1 e^{\frac{-11-\sqrt{205}}{6}x} + C_2 e^{\frac{-11+\sqrt{205}}{6}x}}$$

Exercise

Find the general solution: $3y'' - 20y' + 12y = 0$

Solution

The characteristic equation: $3\lambda^2 - 20\lambda + 13 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{20 \pm \sqrt{244}}{6} = \boxed{\frac{10 \pm \sqrt{61}}{3}}$

$$\boxed{y(x) = C_1 e^{\frac{10-\sqrt{61}}{3}x} + C_2 e^{\frac{10+\sqrt{61}}{3}x}}$$

Exercise

Find the general solution: $4y'' + y' = 0$

Solution

The characteristic equation: $4\lambda^2 + \lambda = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 0, \ -\frac{1}{4}}$

$$\boxed{y(x) = C_1 + C_2 e^{-x/4}}$$

Exercise

Find the general solution: $4y'' + 4y' + y = 0$

Solution

The characteristic equation: $4\lambda^2 + 4\lambda + 1 = (2\lambda + 1)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -\frac{1}{2}}$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{-x/2}}$$

Exercise

Find the general solution: $4y'' - 4y' + y = 0$

Solution

The characteristic equation: $4\lambda^2 - 4\lambda + 1 = (2\lambda - 1)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \frac{1}{2}}$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{x/2}}$$

Exercise

Find the general solution: $4y'' + 4y' + 2y = 0$

Solution

The characteristic equation: $4\lambda^2 + 4\lambda + 2 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-4 \pm 4i}{8} \underline{= -\frac{1}{2} \pm \frac{1}{2} i}$

$$\underline{y(x) = e^{-x/2}\left(C_1 \cos\frac{x}{2} + C_2 \sin\frac{x}{2}\right)}$$

Exercise

Find the general solution: $4y'' - 4y' + y = 0$

Solution

The characteristic equation: $4\lambda^2 - 4\lambda + 1 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = \frac{1}{2}}$

$$\underline{y(x) = \left(C_1 + C_2 x\right)e^{x/2}}$$

Exercise

Find the general solution: $4y'' - 4y' + 13y = 0$

Solution

The characteristic equation: $4\lambda^2 - 4\lambda + 13 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{4 \pm 8i\sqrt{3}}{8} \underline{= \frac{1}{2} \pm i\sqrt{3}}$

$$\underline{y(x) = \left(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right)e^{x/2}}$$

Exercise

Find the general solution: $4y'' - 8y' + 7y = 0$

Solution

The characteristic equation: $4\lambda^2 - 8\lambda + 7 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{8 \pm 4i\sqrt{3}}{8} \underline{= 1 \pm \frac{1}{2} i\sqrt{3}}$

$$\underline{y(x) = e^{x}\left(C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right)}$$

Exercise

Find the general solution: $4y'' - 12y' + 9y = 0$

Solution

The characteristic equation: $4\lambda^2 - 12\lambda + 9 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{12 \pm 0}{8} \ \boxed{= \frac{3}{2}}$

$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{3x/2}}$

Exercise

Find the general solution: $4y'' + 20y' + 25y = 0$

Solution

The characteristic equation: $4\lambda^2 + 20\lambda + 25 = (2\lambda + 5)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -\frac{5}{2}}$

$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{-5x/2}}$

Exercise

Find the general solution: $6y'' + y' - 2y = 0$

Solution

The characteristic equation: $6\lambda^2 + \lambda - 2 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-1 \pm 7}{2} \ \boxed{= -4,\ 3}$

$\boxed{y(x) = C_1 e^{-4x} + C_2 e^{3x}}$

Exercise

Find the general solution: $6y'' + 5y' - 6y = 0$

Solution

The characteristic equation: $6\lambda^2 + 5\lambda - 6 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = -3,\ \frac{4}{3}}$

$\boxed{y(t) = C_1 e^{-3t} + C_2 e^{4t/3}}$

Exercise

Find the general solution: $6y'' - 7y' - 20y = 0$

Solution

The characteristic equation: $6\lambda^2 - 7\lambda - 20 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{7 \pm \sqrt{529}}{12} = \frac{7 \pm 23}{12}$

$\boxed{\lambda_1 = -\frac{4}{3},\ \lambda_2 = \frac{5}{2}}$

$\boxed{y(t) = C_1 e^{-4t/3} + C_2 e^{5t/2}}$

Exercise

Find the general solution: $6y'' + 13y' - 5y = 0$

Solution

The characteristic equation: $6\lambda^2 + 13\lambda - 5 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-13 \pm 17}{12} = \boxed{-\frac{5}{2},\ \frac{1}{3}}$

$$\boxed{y(x) = C_1 e^{-5x/2} + C_2 e^{x/3}}$$

Exercise

Find the general solution: $6y'' + 13y' + 7y = 0$

Solution

The characteristic equation: $6\lambda^2 + 13\lambda + 7 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -1,\ -\frac{7}{6}}$

$$\boxed{y(x) = C_1 e^{-x} + C_2 e^{-7x/6}}$$

Exercise

Find the general solution: $6y'' - 13y' + 7y = 0$

Solution

The characteristic equation: $6\lambda^2 - 13\lambda + 7 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 1,\ \frac{7}{6}}$

$$\boxed{y(x) = C_1 e^{x} + C_2 e^{7x/6}}$$

Exercise

Find the general solution: $8y'' - 10y' - 3y = 0$

Solution

The characteristic equation: $8\lambda^2 - 10\lambda - 3 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{10 \pm 14}{16} = \boxed{-\frac{1}{4},\ \frac{3}{2}}$

$$\boxed{y(x) = C_1 e^{-x/4} + C_2 e^{3x/2}}$$

Exercise

Find the general solution: $9y'' - y = 0$

Solution

The characteristic equation: $9\lambda^2 - 1 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm\frac{1}{3}}$

$$\boxed{y(x) = C_1 e^{-x/3} + C_2 e^{x/3}}$$

Exercise

Find the general solution: $9y''+6y'+y=0$

Solution

The characteristic equation: $9\lambda^2+6\lambda+1=(3\lambda+1)^2=0 \rightarrow \lambda_{1,2}=-\frac{1}{3}$

$$y(x)=\left(C_1+C_2x\right)e^{-x/3}$$

Exercise

Find the general solution: $9y''-12y'+4y=0$

Solution

The characteristic equation: $9\lambda^2-12\lambda+4=(3\lambda-2)^2=0 \rightarrow \lambda_{1,2}=\frac{2}{3}$

$$y(x)=\left(C_1+C_2x\right)e^{2x/3}$$

Exercise

Find the general solution: $9y''+24y'+16y=0$

Solution

The characteristic equation: $9\lambda^2+24\lambda+16=(3\lambda+4)^2=0 \rightarrow \lambda_{1,2}=-\frac{4}{3}$

$$y(x)=\left(C_1+C_2x\right)e^{-4x/3}$$

Exercise

Find the general solution: $12y''-5y'-2y=0$

Solution

The characteristic equation: $12\lambda^2-5\lambda-2=0 \rightarrow \lambda_{1,2}=\frac{5\pm 11}{24}=-\frac{1}{4},\ \frac{2}{3}$

$$y(x)=C_1e^{-x/4}+C_2e^{2x/3}$$

Exercise

Find the general solution: $16y''-8y'+7y=0$

Solution

The characteristic equation: $16\lambda^2-8\lambda+7=0 \rightarrow \lambda_{1,2}=\frac{8\pm 8i\sqrt{6}}{32}=\frac{1}{4}\pm i\frac{\sqrt{6}}{4}$

$$y(x)=e^{x/4}\left(C_1\cos\frac{\sqrt{6}}{4}x+C_2\sin\frac{\sqrt{6}}{4}x\right)$$

Exercise

Find the general solution: $16y'' - 12y' - 4y = 0$

Solution

The characteristic equation: $16\lambda^2 - 12\lambda - 4 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{12 \pm 20}{32} = \boxed{-\frac{1}{4},\ 1}$

$$\boxed{y(x) = C_1 e^{-x/4} + C_2 e^{x}}$$

Exercise

Find the general solution: $16y'' - 24y' + 9y = 0$

Solution

The characteristic equation: $16\lambda^2 - 24\lambda + 9 = (4\lambda - 3)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \frac{3}{4}}$

$$\boxed{y(x) = \left(C_1 + C_2 x\right) e^{3x/4}}$$

Exercise

Find the general solution: $25y'' + 10y' + y = 0$

Solution

The characteristic equation: $25\lambda^2 + 10\lambda + 1 = (5\lambda + 1)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -\frac{1}{5}}$

$$\boxed{y(x) = \left(C_1 + C_2 x\right) e^{-x/5}}$$

Exercise

Find the general solution: $25y'' - 10y' + y = 0$

Solution

The characteristic equation: $25\lambda^2 - 10\lambda + 1 = (5\lambda - 1)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \frac{1}{5}}$

$$\boxed{y(x) = \left(C_1 + C_2 x\right) e^{x/5}}$$

Exercise

Find the general solution: $35y'' - y' - 12y = 0$

Solution

The characteristic equation: $35\lambda^2 - \lambda - 12 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{1 \pm \sqrt{1681}}{70} = \frac{1 \pm 41}{70}$

$$\boxed{\lambda_1 = -\frac{4}{7},\ \lambda_2 = \frac{3}{5}}$$

$$\boxed{y(x) = C_1 e^{-4x/5} + C_2 e^{3x/5}}$$

Exercise

Find the general solution of the given higher-order differential equation: $y''' + 3y'' + 3y' + y = 0$

Solution

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 3)^3 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2,3} = -3}$$

$$\boxed{y(x) = \left(C_1 + C_2 x + C_3 x^2\right)e^{-3x}}$$

Exercise

Find the general solution of the given higher-order differential equation: $y''' + 3y'' - y' - 3y = 0$

Solution

$$\lambda^3 + 3\lambda^2 - \lambda - 3 = 0$$

$$\lambda^2(\lambda + 3) - (\lambda + 3) = 0$$

$$(\lambda + 3)\left(\lambda^2 - 1\right) = 0$$

$$\boxed{\lambda_{1,2,3} = -3, \ \pm 1}$$

$$\boxed{y(x) = C_1 e^{-3x} + C_2 e^{-x} + C_3 e^{x}}$$

Exercise

Find the general solution of the given higher-order differential equation: $y^{(3)} + 3y'' - 4y = 0$

Solution

$$\lambda^3 + 3\lambda^2 - 4 = 0 \quad \to \quad \lambda_1 = 1$$

$$\begin{array}{c|cccc} 1 & 1 & 3 & 0 & -4 \\ & & 1 & 4 & 4 \\ \hline & 1 & 4 & 4 & 0 \end{array} \qquad \lambda^2 + 4\lambda + 4 = 0 = (\lambda + 2)^2$$

$$\boxed{\lambda_1 = 1, \ \lambda_{2,3} = -2}$$

$$\boxed{y(x) = C_1 e^{x} + \left(C_2 + C_3 x\right)e^{-2x}}$$

Exercise

Find the general solution of the given higher-order differential equation: $3y''' - 19y'' + 36y' - 10y = 0$

Solution

$$3\lambda^3 - 19\lambda^2 + 36\lambda - 10 = 0 \qquad \boxed{\lambda_1 = \frac{1}{3}}, \ \boxed{\lambda_{2,3} = 3 \pm i}$$

$$\boxed{y(x) = C_1 e^{x/3} + e^{3x}\left(C_2 \cos x + C_3 \sin x\right)}$$

Exercise

Find the general solution of the given higher-order differential equation: $y''' - 6y'' + 12y' - 8y = 0$

Solution

The characteristic equation: $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = (\lambda - 2)^3 = 0 \quad \Rightarrow \quad \boxed{\lambda_{1,2,3} = 2}$

$$\boxed{y(x) = \left(C_1 + C_2 x + C_3 x^2\right)e^{2x}}$$

Exercise

Find the general solution of the given higher–ODE: $y''' + 5y'' + 7y' + 3y = 0$

Solution

The characteristic equation: $\lambda^3 + 5\lambda^2 + 7\lambda + 3 = 0 \quad \Rightarrow \boxed{\lambda_1 = -3}$

$$\begin{array}{r|rrrr} -3 & 1 & 5 & 7 & 3 \\ & & -3 & -6 & -3 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array} \quad \lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \boxed{\lambda_{2,3} = -1}$$

The general solution is: $\boxed{y(x) = C_1 e^{-3x} + C_2 e^{-x} + C_3 x e^{-x}}$

Exercise

Find the general solution of the given higher–ODE: $y^{(3)} + y' - 10y = 0$

Solution

The characteristic equation:

$\lambda^3 + \lambda - 10 = 0 \quad \Rightarrow \boxed{\lambda_1 = 2}$ $\left(\textit{Rational Zero Theorem}\right)$

$$\begin{array}{r|rrrr} 2 & 1 & 0 & 1 & -10 \\ & & 2 & 4 & 10 \\ \hline & 1 & 2 & 5 & \boxed{0} \end{array} \quad \lambda^2 + 2\lambda + 5 = 0 \quad \Rightarrow \boxed{\lambda_{2,3}} = \frac{-2 \pm \sqrt{-16}}{2} = \boxed{-1 \pm 2i}$$

The general solution is: $\boxed{y(x) = C_1 e^{2x} + e^{-x}\left(C_2 \cos 2x + C_3 \sin 2x\right)}$

Exercise

Find the general solution of the given higher ODE: $y''' + y'' - 6y' + 4y = 0$

Solution

The characteristic equation: $\lambda^3 + \lambda^2 - 6\lambda + 4 = 0$

$$\lambda_1 = 1 \qquad \begin{array}{r|rrrr} 1 & 1 & 1 & -6 & 4 \\ & & 1 & 2 & -4 \\ \hline & 1 & 2 & -4 & \boxed{0} \end{array}$$

$$\lambda^2 + 2\lambda - 4 = 0 \quad \rightarrow \lambda_{2,3} = \frac{-2 \pm 2\sqrt{5}}{2} = -1 \pm \sqrt{5}$$

$$y(x) = C_1 e^x + C_2 e^{\left(-1-\sqrt{5}\right)x} + C_3 e^{\left(-1+\sqrt{5}\right)x}$$

Exercise

Find the general solution of the given higher ODE: $y''' - 6y'' - y' + 6y = 0$

Solution

The characteristic equation: $\lambda^3 - 6\lambda^2 - \lambda + 6 = 0$

$$\lambda_1 = 1 \qquad \begin{array}{r|rrrr} 1 & 1 & -6 & -1 & 6 \\ & & 1 & -5 & -6 \\ \hline & 1 & -5 & -6 & \boxed{0} \end{array}$$

$$\lambda^2 - 5\lambda - 6 = 0 \quad \rightarrow \lambda_{2,3} = -1,\ 6$$

$$y(x) = C_1 e^{-x} + C_2 e^{x} + C_3 e^{6x}$$

Exercise

Find the general solution of the given higher ODE: $y''' + 2y'' - 4y' - 8y = 0$

Solution

The characteristic equation: $\lambda^3 - 2\lambda^2 - 4\lambda - 8 = 0$

$$\lambda_1 = -2 \qquad \begin{array}{r|rrrr} -2 & 1 & 2 & -4 & -8 \\ & & -2 & 0 & 8 \\ \hline & 1 & 0 & -4 & \boxed{0} \end{array}$$

$$\lambda^2 - 4 = 0 \quad \rightarrow \lambda_{2,3} = \pm 2$$

$$y(x) = \left(C_1 + C_2 x\right)e^{-2x} + C_3 e^{2x}$$

Exercise

Find the general solution of the given higher ODE: $y''' - 7y'' + 7y' + 15y = 0$

Solution

The characteristic equation: $\lambda^3 - 7\lambda^2 + 7\lambda + 15 = 0$

$\lambda_1 = -1$

$$\begin{array}{r|rrrr} -1 & 1 & -7 & 7 & 15 \\ & & -1 & 8 & -15 \\ \hline & 1 & -8 & 15 & \boxed{0} \end{array}$$

$\lambda^2 - 8\lambda + 15 = 0 \quad \rightarrow \quad \underline{\lambda_{2,3} = 3,\ 5}$

$\underline{y(x) = C_1 e^{-x} + C_2 e^{3x} + C_3 e^{5x}}$

Exercise

Find the general solution of the given higher ODE: $y''' + 3y'' - 4y' - 12y = 0$

Solution

The characteristic equation: $\lambda^3 + 3\lambda^2 - 4\lambda - 12 = 0$

$\lambda^2(\lambda + 3) - 4(\lambda + 3) = 0$

$(\lambda + 3)\left(\lambda^2 - 4\right) = 0 \quad \rightarrow \quad \underline{\lambda_1 = -3,\ \lambda_2 = -2,\ \lambda_3 = 2}$

$\underline{y(x) = C_1 e^{-3x} + C_2 e^{-2x} + C_3 e^{2x}}$

Exercise

Find the general solution of the given higher ODE: $y''' - 4y'' - 5y' = 0$

Solution

The characteristic equation: $\lambda^3 - 4\lambda^2 - 5\lambda = \lambda\left(\lambda^2 - 4\lambda - 5\right) = 0$

$\underline{\lambda_1 = 0,\ \lambda_2 = -1,\ \lambda_3 = 5}$

$\underline{y(x) = C_1 + C_2 e^{-x} + C_3 e^{5x}}$

Exercise

Find the general solution of the given higher ODE: $y''' - y = 0$

Solution

The characteristic equation: $\lambda^3 - 1 = (\lambda - 1)\left(\lambda^2 + \lambda + 1\right) = 0$

$$\boxed{\lambda_1 = 2,\ \ \lambda_{2,3} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}}$$

$$\boxed{y(x) = C_1 e^{2x} + e^{-x/2}\left(C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x\right)}$$

Exercise

Find the general solution of the given higher ODE: $\quad y''' - 5y'' + 3y' + 9y = 0$

Solution

The characteristic equation: $\quad \lambda^3 - 5\lambda^2 + 3\lambda + 9 = 0$

$$\lambda_1 = -1 \qquad \begin{array}{r|rrrr} -1 & 1 & -5 & 3 & 9 \\ & & -1 & 6 & -9 \\ \hline & 1 & -6 & 9 & \boxed{0} \end{array}$$

$$\lambda^2 - 6\lambda + 9 = 0 \quad \to \boxed{\lambda_{2,3} = 3,\ 3}$$

$$\boxed{y(x) = C_1 e^{-x} + \left(C_2 + C_3 x\right)e^{3x}}$$

Exercise

Find the general solution of the given higher ODE: $\quad y''' + 3y'' - 4y' - 12y = 0$

Solution

The characteristic equation: $\quad \lambda^3 + 3\lambda^2 - 4\lambda - 12 = 0$

$$\lambda^2(\lambda + 3) - 4(\lambda + 3) = 0$$

$$(\lambda + 3)\left(\lambda^2 - 4\right) = 0 \quad \to \boxed{\lambda_{2,3} = -3,\ \pm 2}$$

$$\boxed{y(x) = C_1 e^{-3x} + C_2 e^{-2x} + C_3 e^{2x}}$$

Exercise

Find the general solution of the given higher ODE: $\quad y''' + y'' - 2y = 0$

Solution

The characteristic equation: $\quad \lambda^3 + \lambda^2 - 2 = 0$

$$\lambda_1 = 1 \qquad \begin{array}{r|rrrr} 1 & 1 & 1 & 0 & -2 \\ & & 1 & 2 & 2 \\ \hline & 1 & 2 & 2 & \boxed{0} \end{array} \qquad \lambda^2 + 2\lambda + 2 = 0 \quad \to \lambda_{2,3} = \frac{-2 \pm 2i}{2} \boxed{= -1 \pm i}$$

$$\boxed{y(x) = C_1 e^{x} + e^{-x}\left(C_2 \cos x + C_3 \sin x\right)}$$

Exercise

Find the general solution of the given higher ODE: $y''' - y'' - 4y = 0$

Solution

The characteristic equation: $\lambda^3 - \lambda^2 - 4 = 0$

$\lambda_1 = 2$

$$\begin{array}{r|rrrr} 2 & 1 & -1 & 0 & -4 \\ & & 2 & 2 & 4 \\ \hline & 1 & 1 & 2 & \lfloor 0 \end{array}$$

$\lambda^2 + \lambda + 2 = 0 \rightarrow \underline{\lambda_{2,3} = \frac{-1 \pm i\sqrt{7}}{2}}$

$$\underline{y(x) = C_1 e^{2x} + e^{-x/2}\left(C_2 \cos\frac{\sqrt{7}}{2}x + C_3 \sin\frac{\sqrt{7}}{2}x\right)}$$

Exercise

Find the general solution of the given higher ODE: $y''' + 3y'' + 3y' + y = 0$

Solution

The characteristic equation: $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

$\lambda_1 = -1$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 3 & 1 \\ & & -1 & -2 & -1 \\ \hline & 1 & 2 & 1 & \lfloor 0 \end{array}$$

$\lambda^2 + 2\lambda + 1 = 0 \rightarrow \underline{\lambda_{2,3} = -1}$

$$\underline{y(x) = \left(C_1 + C_2 x + C_3 x^2\right)e^{-x}}$$

Exercise

Find the general solution of the given higher ODE: $y''' - 6y'' + 12y' - 8y = 0$

Solution

The characteristic equation: $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$

$\lambda_1 = 2$

$$\begin{array}{r|rrrr} 2 & 1 & -6 & 12 & -8 \\ & & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & \lfloor 0 \end{array}$$

$\lambda^2 - 4\lambda + 4 = 0 \rightarrow \underline{\lambda_{2,3} = 2}$

$$\underline{y(x) = \left(C_1 + C_2 x + C_3 x^2\right)e^{2x}}$$

Exercise

Find the general solution of the given higher ODE: $y^{(4)} + y''' + y'' = 0$

Solution

The characteristic equation: $\lambda^4+\lambda^3+\lambda^2=\lambda^2\left(\lambda^2+\lambda+1\right)=0$

$\lambda_{1,2}=0 \qquad \lambda^2+\lambda+1=0 \quad \to \boxed{\lambda_{3,4}=\dfrac{-1\pm i\sqrt{3}}{2}}$

$$\boxed{y(x)=C_1+C_2x+e^{-x/2}\left(C_3\cos\frac{\sqrt{3}}{2}x+C_4\sin\frac{\sqrt{3}}{2}x\right)}$$

Exercise

Find the general solution of the given higher ODE: $y^{(4)}-2y''+y=0$

Solution

The characteristic equation: $\lambda^4-2\lambda^2+1=\left(\lambda^2-1\right)^2=0$

$\lambda^2=1 \qquad \boxed{\lambda_{1,2}=-1 \quad \lambda_{3,4}=1}$

$$\boxed{y(x)=\left(C_1+C_2x\right)e^{-x}+\left(C_3+C_4x\right)e^{x}}$$

Exercise

Find the general solution of the given higher ODE: $16y^{(4)}+24y''+9y=0$

Solution

The characteristic equation: $16\lambda^4+24\lambda^2+9=\left(4\lambda^2+3\right)^2=0$

$\lambda^2=-\frac{3}{4} \qquad \boxed{\lambda_{1,2}=\pm\frac{\sqrt{3}}{2}i \quad \lambda_{3,4}=\pm\frac{\sqrt{3}}{2}i}$

$$\boxed{y(x)=C_1\cos\frac{\sqrt{3}}{2}x+C_2\sin\frac{\sqrt{3}}{2}x+x\left(C_3\cos\frac{\sqrt{3}}{2}x+C_4\sin\frac{\sqrt{3}}{2}x\right)}$$

Exercise

Find the general solution of the given higher ODE: $y^{(4)}-7y''-18y=0$

Solution

The characteristic equation: $\lambda^4-7\lambda^2-18=0$

$\lambda^2=\frac{7\pm11}{2} \qquad \boxed{\lambda_{1,2}=\pm2i \quad \lambda_{3,4}=\pm3}$

$$\boxed{y(x)=C_1\cos2x+C_2\sin2x+C_3e^{-3x}+C_4e^{3x}}$$

Exercise

Find the general solution of the given higher-order differential equation: $y^{(4)}+2y''+y=0$

Solution

$$\lambda^4+2\lambda^2+1=\left(\lambda^2+1\right)^2=0$$

$$\lambda^2=-1 \quad\Rightarrow\quad \lambda=\pm i \quad\Rightarrow \lambda_{1,2}=-i,\ \lambda_{3,4}=i$$

$$\boxed{y(x)=\left(C_1+C_2x\right)e^{-ix}+\left(C_3+C_4x\right)e^{ix}}$$

OR $\quad \boxed{y(x)=C_1\cos x+C_2\sin x+C_3x\cos x+C_4x\sin x}$

Exercise

Find the general solution of the given higher-order differential equation: $y^{(4)}+y'''+y''=0$

Solution

$$\lambda^4+\lambda^3+\lambda^2=\lambda^2\left(\lambda^2+\lambda+1\right)=0$$

$$\lambda^2=0\to\lambda_{1,2}=0 \quad\Rightarrow\quad \lambda=\frac{-1\pm\sqrt{1-4}}{2} \quad\Rightarrow\ \lambda_{3,4}=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$$

$$\boxed{y(x)=C_1+C_2x+e^{-x/2}\left(C_3\cos\left(\frac{\sqrt{3}}{2}x\right)+C_4\sin\left(\frac{\sqrt{3}}{2}x\right)\right)}$$

Exercise

Find the general solution of the given higher–ODE: $y^{(4)}+4y=0$

Solution

The characteristic equation: $\lambda^4+4=0 \quad\Rightarrow\lambda^2=\pm 2i \quad\to\lambda_{1,2,3,4}=\pm\sqrt{\pm 2i}$

Since $i=e^{\frac{\pi}{2}i} \quad -i=e^{\frac{3\pi}{2}i}$

$$\sqrt{2i}=\left(2e^{i\pi/2}\right)^{1/2}=\sqrt{2}e^{i\pi/4}=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=1+i$$

$$\sqrt{-2i}=\left(2e^{i3\pi/2}\right)^{1/2}=\sqrt{2}e^{i3\pi/4}=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=-1+i$$

$$\lambda=\pm(\pm 1+i)=\begin{cases}1\pm i\\ -1\pm i\end{cases}$$

The general solution is: $\boxed{y(x)=e^{x}\left(C_1\cos x+C_2\sin x\right)+e^{-x}\left(C_3\cos x+C_4\sin x\right)}$

Exercise

Find the general solution of the given higher–ODE: $y^{(4)}+2y'''+9y''-2y'-10y=0$

Solution

The characteristic equation:

$\lambda^4+2\lambda^3+9\lambda^2-2\lambda-10=0 \quad \Rightarrow \boxed{\lambda_1=1}$ (*Rational Zero Theorem*)

$$\begin{array}{r|rrrrr} 1 & 1 & 2 & 9 & -2 & -10 \\ & & 1 & 3 & 12 & 10 \\ \hline & 1 & 3 & 12 & 10 & \boxed{0} \end{array} \qquad \lambda^3+3\lambda^2+12\lambda+10=0 \quad \Rightarrow \boxed{\lambda_2=-1}$$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 12 & 10 \\ & & -1 & -2 & -10 \\ \hline & 1 & 2 & 10 & \boxed{0} \end{array} \qquad \lambda^2+2\lambda+10=0 \quad \Rightarrow \boxed{\lambda_{3,4}=-1\pm 3i}$$

The general solution is: $\boxed{y(x)=C_1e^{x}+C_2e^{-x}+e^{-x}\left(C_3\cos 3x+C_4\sin 3x\right)}$

Exercise

Find the solution of the given initial value problem $x^{(4)}-4x^{(3)}+7x''-4x'+6x=0$

Solution

The characteristic equation: $\lambda^4-4\lambda^3+7\lambda^2-4\lambda+6=0 \quad \rightarrow \boxed{\lambda_{1,2,3,4}=\pm i,\ 2\pm i\sqrt{2}}$

$$\boxed{x(t)=C_1e^{\left(2+i\sqrt{2}\right)t}+C_2e^{\left(2-i\sqrt{2}\right)t}+C_3\cos t+C_4\sin t}$$

Exercise

Find the solution of the given initial value problem $x^{(4)}+8x^{(3)}+24x''+32x'+16x=0$

Solution

The characteristic equation: $\lambda^4+8\lambda^3+24\lambda^2+32\lambda+16=0 \quad \rightarrow \boxed{\lambda_1=-2}$

$$\begin{array}{r|rrrrr} -2 & 1 & 8 & 24 & 32 & 16 \\ & & -2 & -12 & -24 & -16 \\ \hline & 1 & 6 & 12 & 8 & \boxed{0} \end{array} \qquad \lambda^3+6\lambda^2+12\lambda+8=0 \quad \Rightarrow \boxed{\lambda_2=-2}$$

$$\begin{array}{r|rrrr} -2 & 1 & 6 & 12 & 8 \\ & & -2 & -8 & -8 \\ \hline & 1 & 4 & 4 & \boxed{0} \end{array} \qquad \lambda^2+4\lambda+4=0 \quad \Rightarrow \boxed{\lambda_{3,4}=-2}$$

The eigenvalues: $\boxed{\lambda_{1,2,3,4}=-2}$

$$\boxed{x(t)=\left(C_1+C_2t+C_3t^2+C_4t^3\right)e^{-2t}}$$

Exercise

Find the solution of the given initial value problem $x^{(4)} - 4x'' + 16x' + 32x = 0$

Solution

The characteristic equation: $\lambda^4 - 4\lambda^2 + 16\lambda + 32 = 0 \quad \to \quad \underline{\lambda_1 = -2}$

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & -4 & 16 & 32 \\ & & -2 & 4 & 0 & -32 \\ \hline & 1 & -2 & 0 & 16 & \boxed{0} \end{array} \qquad \lambda^3 - 2\lambda^2 + 16 = 0 \quad \Rightarrow \underline{\lambda_2 = -2}$$

$$\begin{array}{r|rrrr} -2 & 1 & -2 & 0 & 16 \\ & & -2 & 8 & -16 \\ \hline & 1 & -4 & 8 & \boxed{0} \end{array} \qquad \lambda^2 - 4\lambda + 8 = 0 \quad \Rightarrow \underline{\lambda_{3,4} = 2 \pm 2i}$$

The eigenvalues: $\underline{\lambda = -2,\ -2,\ 2 \pm 2i}$

$$\underline{x(t) = \left(C_1 + C_2 t\right)e^{-2t} + e^{2t}\left(C_3 \cos 2t + C_4 \sin 2t\right)}$$

Exercise

Find the solution of the given initial value problem $x^{(4)} + 4x^{(3)} + 6x'' + 4x' + x = 0$

Solution

The characteristic equation: $\lambda^4 + 4\lambda^3 + 6\lambda^2 + 4\lambda + 1 = (\lambda + 1)^4 = 0$

$\to \quad \underline{\lambda_{1,2,3,4} = -1}$

$$\underline{x(t) = \left(C_1 + C_2 t + C_3 t^2 + C_4 t^3\right)e^{-t}}$$

Exercise

Find the solution of the given initial value problem $y^{(4)} - y^{(3)} + y'' - 3y' - 6y = 0$

Solution

The characteristic equation: $\lambda^4 - \lambda^3 + \lambda^2 - 3\lambda - 6 = 0 \quad \to \quad \lambda_1 = -1$

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 1 & -3 & -6 \\ & & -1 & 2 & -3 & 6 \\ \hline & 1 & -2 & 3 & -6 & \boxed{0} \end{array} \qquad \lambda^3 - 2\lambda^2 + 3\lambda - 6 = 0 \quad \Rightarrow \underline{\lambda_2 = 2}$$

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 3 & -6 \\ & & 2 & 0 & 6 \\ \hline & 1 & 0 & 3 & \boxed{0} \end{array} \qquad \lambda^2 + 3 = 0 \quad \Rightarrow \underline{\lambda_{3,4} = \pm i\sqrt{3}}$$

The eigenvalues: $\boxed{\lambda = -1,\ 2,\ \pm i\sqrt{3}}$

$$\boxed{y(t) = C_1 e^{-t} + C_2 e^{2t} + C_3 \cos\sqrt{3}\,t + C_4 \sin\sqrt{3}\,t}$$

Exercise

Find the solution of the given initial value problem $\quad y^{(4)} + y^{(3)} - 3y'' - 5y' - 2y = 0$

Solution

The characteristic equation: $\lambda^4 + \lambda^3 - 3\lambda^2 - 5\lambda - 2 = 0 \quad \rightarrow \quad \lambda_1 = -1$

$$\begin{array}{r|rrrrr} -1 & 1 & 1 & -3 & -5 & -2 \\ & & -1 & 0 & 3 & 2 \\ \hline & 1 & 0 & -3 & -2 & \boxed{0} \end{array} \qquad \lambda^3 - 3\lambda - 2 = 0 \quad \Rightarrow \boxed{\lambda_2 = 2}$$

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -3 & -2 \\ & & 2 & 4 & 2 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array} \qquad \lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \boxed{\lambda_{3,4} = -1}$$

The eigenvalues: $\boxed{\lambda_{1,2,3} = -1,\ \lambda_4 = 2}$

$$\boxed{y(t) = \left(C_1 + C_2 t + C_3 t^2\right)e^{-t} + C_4 e^{2t}}$$

Exercise

Find the solution of the given initial value problem $\quad x^{(5)} - x^{(4)} - 2x^{(3)} + 2x'' + x' - x = 0$

Solution

The characteristic equation: $\lambda^5 - \lambda^4 - 2\lambda^3 + 2\lambda^2 + \lambda - 1 = 0 \quad \rightarrow \quad \boxed{\lambda_1 = 1}$

$$\begin{array}{r|rrrrrr} 1 & 1 & -1 & -2 & 2 & 1 & -1 \\ & & 1 & 0 & -2 & 0 & 1 \\ \hline & 1 & 0 & -2 & 0 & 1 & \boxed{0} \end{array}$$

$$\lambda^4 - 2\lambda^2 + 1 = \left(\lambda^2 - 1\right)^2 = 0 \quad \rightarrow \quad \lambda^2 = 1,\ 1$$

The eigenvalues: $\boxed{\lambda = 1,\ 1,\ 1,\ -1,\ -1}$

$$\boxed{x(t) = \left(C_1 + C_2 t + C_3 t^2\right)e^{t} + \left(C_4 + C_5 t\right)e^{-t}}$$

Exercise

Find the solution of the given initial value problem $\quad x^{(5)}+5x^{(4)}+10x^{(3)}+10x''+5x'+x=0$

Solution

The characteristic equation: $\lambda^5+5\lambda^4+10\lambda^3+10\lambda^2+5\lambda+1=(\lambda+1)^5=0$

The eigenvalues: $\lambda=1,\ 1,\ 1,\ -1,\ -1$

$$x(t)=\left(C_1+C_2t+C_3t^2\right)e^t+\left(C_4+C_5t\right)e^{-t}$$

Exercise

Find the general solution of the given higher ODE: $\quad y^{(5)}+5y^{(4)}-2y'''-10y''+y'+5y=0$

Solution

The characteristic equation: $\quad \lambda^5+5\lambda^4-2\lambda^3-10\lambda^2+\lambda+5=0$

$\lambda_1=1$

$$\begin{array}{r|rrrrrr} 1 & 1 & 5 & -2 & -10 & 1 & 5 \\ & & 1 & 6 & 4 & -6 & -5 \\ \hline & 1 & 6 & 4 & -6 & -5 & \boxed{0} \end{array} \quad \to \lambda^4+6\lambda^3+4\lambda^2-6\lambda-5=0 \quad \Rightarrow \lambda_2=1$$

$$\begin{array}{r|rrrrr} 1 & 1 & 6 & 4 & -6 & -5 \\ & & 1 & 7 & 11 & 5 \\ \hline & 1 & 7 & 11 & 5 & \boxed{0} \end{array} \quad \to \lambda^3+7\lambda^2+11\lambda+5=0 \quad \Rightarrow \lambda_3=-1$$

$$\begin{array}{r|rrrr} -1 & 1 & 7 & 11 & 5 \\ & & -1 & -6 & -5 \\ \hline & 1 & 6 & 5 & \boxed{0} \end{array} \quad \to \lambda^2+6\lambda+5=0 \quad \Rightarrow \lambda_{4,5}=-1,\ -5$$

$\lambda=-5,\ -1,\ -1,\ 1,\ 1$

$$y(x)=C_1e^{-x}+\left(C_2+C_3x\right)e^{-x}+\left(C_4+C_5x\right)e^{x}$$

Exercise

Find the general solution of the given higher ODE: $\quad 2y^{(5)}-7y^{(4)}+12y'''+8y''=0$

Solution

The characteristic equation: $\quad 2\lambda^5-7\lambda^4+12\lambda^3+8\lambda^2=\lambda^2\left(2\lambda^3-7\lambda^2+12\lambda+8\right)=0$

$\lambda_{1,2}=0,\ \lambda_3=-\frac{1}{2}$

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -7 & 12 & 8 \\ & & -1 & 4 & -8 \\ \hline & 2 & -8 & 16 & \boxed{0} \end{array} \quad \rightarrow 2\lambda^2 - 8\lambda + 16 = 0 \quad \Rightarrow \lambda_{4,5} = 2 \pm \frac{8i}{4} = \boxed{2 \pm 2i}$$

$$\boxed{\lambda = 0,\ 0,\ -\frac{1}{2},\ 2 \pm 2i}$$

$$\boxed{y(x) = C_1 + C_2 x + C_3 e^{-x/2} + e^{2x}\left(C_4 \cos 2x + C_5 \sin 2x\right)}$$

Exercise

Find the general solution of the given higher-order differential equation: $y^{(5)} - 2y^{(4)} + 17y''' = 0$

Solution

$$\lambda^5 - 2\lambda^4 + 17\lambda^3 = \lambda^3\left(\lambda^2 - 2\lambda + 17\right) = 0$$

$$\lambda^3 = 0 \rightarrow \lambda_{1,2,3} = 0 \quad \Rightarrow \quad \lambda = \frac{2 \pm \sqrt{4-68}}{2} \quad \Rightarrow \lambda_{4,5} = 1 \pm 4i$$

$$\boxed{y(x) = C_1 + C_2 x + C_3 x^2 + e^{x}\left(C_4 \cos 4x + C_5 \sin 4x\right)}$$

Exercise

Find the solution of the given initial value problem $\quad x^{(6)} - 5x^{(4)} + 16x^{(3)} + 36x'' - 16x' - 32x = 0$

Solution

The characteristic equation: $\lambda^6 - 5\lambda^4 + 16\lambda^3 + 36\lambda^2 - 16\lambda - 32 = 0 \quad \rightarrow \quad \boxed{\lambda_1 = 1}$

$$\begin{array}{r|rrrrrrr} 1 & 1 & 0 & -5 & 16 & 36 & -16 & -32 \\ & & 1 & 1 & -4 & 12 & 48 & 32 \\ \hline & 1 & 1 & -4 & 12 & 48 & 32 & \boxed{0} \end{array} \quad \lambda^5 + \lambda^4 - 4\lambda^3 + 12\lambda^2 + 48\lambda + 32 = 0 \quad \rightarrow \quad \boxed{\lambda_2 = -1}$$

$$\begin{array}{r|rrrrrr} -1 & 1 & 1 & -4 & 12 & 48 & 32 \\ & & -1 & 0 & 4 & -16 & -32 \\ \hline & 1 & 0 & -4 & 16 & 32 & \boxed{0} \end{array} \quad \lambda^4 - 4\lambda^2 + 16\lambda + 32 = 0 \quad \rightarrow \quad \lambda_3 = -2$$

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & -4 & 16 & 32 \\ & & -2 & 4 & 0 & -32 \\ \hline & 1 & -2 & 0 & 16 & \boxed{0} \end{array} \quad \lambda^3 - 2\lambda^2 + 16 = 0 \quad \Rightarrow \boxed{\lambda_4 = -2}$$

$$\begin{array}{r|rrrr} -2 & 1 & -2 & 0 & 16 \\ & & -2 & 8 & -16 \\ \hline & 1 & -4 & 8 & \boxed{0} \end{array} \quad \lambda^2 - 4\lambda + 8 = 0 \quad \Rightarrow \boxed{\lambda_{3,4} = 2 \pm 2i}$$

The eigenvalues: $\underline{\lambda = 1,\ -1,\ -2,\ -2,\ 2 \pm 2i}$

$$\underline{x(t) = C_1 e^t + C_2 e^{-t} + \left(C_3 + C_4 t\right)e^{-2t} + e^{2t}\left(C_5 \cos 2t + C_6 \sin 2t\right)}$$

Exercise

Find the general solution of the given higher-order differential equation: $\left(D^2 + 6D + 13\right)^2 y = 0$

Solution

The characteristic equation: $\left(\lambda^2 + 6\lambda + 13\right) = 0$

$$\Rightarrow \lambda = \frac{-6 \pm \sqrt{-16}}{2} = -3 \pm 2i \quad \textit{multiplicity } k = 2$$

$$\underline{y(x) = e^{-3x}\left(C_1 \cos 2x + C_2 \sin 2x\right) + xe^{-3x}\left(C_3 \cos 2x + C_4 \sin 2x\right)}$$

Exercise

Find the general solution of the given higher-order differential equation $\lambda^3(\lambda - 1)(\lambda - 2)^3\left(\lambda^2 + 9\right) = 0$

Solution

$$\lambda^2 + 9 = 0 \quad \Rightarrow \quad \lambda^2 = -9 \ \rightarrow \ \lambda = \pm 3i$$

The solution: $\lambda = 0,\ 0,\ 0,\ 1,\ 2,\ 2,\ 2,\ \pm 3i$

$$\underline{y(x) = C_1 + C_2 x + C_3 x^2 + C_4 e^x + \left(C_5 + C_6 x + C_7 x^2\right)e^{2x} + C_8 \cos 3x + C_9 \sin 3x}$$

Exercise

Find the solution of the given initial value problem $\quad y'' + y = 0, \quad y\left(\frac{\pi}{3}\right) = 0, \quad y'\left(\frac{\pi}{3}\right) = 2$

Solution

The characteristic equation: $\lambda^2 + 1 = 0 \ \rightarrow \ \underline{\lambda_{1,2} = \pm i}$

$$y(x) = C_1 \cos x + C_2 \sin x$$

$$y\left(\frac{\pi}{3}\right) = 0 \ \rightarrow \ \frac{1}{2}C_1 + \frac{\sqrt{3}}{2}C_2 = 0 \ \Rightarrow \underline{C_1 + \sqrt{3}C_2 = 0}$$

$$y'(x) = -C_1 \sin x + C_2 \cos x$$

$$y'\left(\frac{\pi}{3}\right) = 2 \ \rightarrow \ -\frac{\sqrt{3}}{2}C_1 + \frac{1}{2}C_2 = 2 \ \Rightarrow \ \underline{-\sqrt{3}C_1 + C_2 = 4}$$

$$\begin{cases} C_1 + \sqrt{3}C_2 = 0 \\ -\sqrt{3}C_1 + C_2 = 4 \end{cases} \rightarrow \begin{cases} C_1 = -\sqrt{3}C_2 \\ 3C_2 + C_2 = 4 \end{cases} \Rightarrow \boxed{C_2 = 1,\ C_1 = -\sqrt{3}}$$

$$\boxed{y(x) = -\sqrt{3}\cos x + \sin x}$$

Exercise

Find the solution of the given initial value problem $\quad y'' + y = 0; \quad y(0) = 0, \quad y'\left(\frac{\pi}{2}\right) = 0$

Solution

The characteristic equation: $\lambda^2 + 1 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm i}$

$$y(x) = C_1 \cos x + C_2 \sin x$$

$$y(0) = 0 \quad \rightarrow \quad \boxed{C_1 = 0}$$

$$y'(x) = -C_1 \sin x + C_2 \cos x$$

$$y'\left(\frac{\pi}{2}\right) = 0 \quad \rightarrow \quad \boxed{C_1 = 0}$$

$$\boxed{y(x) = C_2 \sin x}$$

Exercise

Find the solution of the given initial value problem $\quad y'' + y' = 0; \quad y(0) = 2, \quad y'(0) = 1$

Solution

The characteristic equation: $\lambda^2 + \lambda = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 0,\ -1}$

$$\boxed{y(x) = C_1 + C_2 e^{-x}}$$

$$y(0) = 2 \quad \rightarrow \quad C_1 + C_2 = 2$$

$$y'(x) = -C_2 e^{-x}$$

$$y'(0) = 1 \quad \rightarrow \quad \boxed{C_2 = -1}$$

$$C_1 + C_2 = 2 \quad \rightarrow \quad \boxed{C_1 = 3}$$

$$\boxed{y(x) = 3e^{-4x}}$$

Exercise

Find the general solution: $\quad y'' - y' - 2y = 0; \quad y(0) = -1, \quad y'(0) = 2$

Solution

The characteristic equation: $\lambda^2 - \lambda - 2 = 0 \qquad \Rightarrow \boxed{\lambda_1 = 2;\ \lambda_2 = -1}$

$y(t) = C_1 e^{2t} + C_2 e^{-t} \qquad\qquad y(0) = C_1 + C_2 = -1$

$y'(t) = 2C_1 e^{2t} - C_2 e^{-t} \qquad\qquad y'(0) = 2C_1 - C_2 = 2 \qquad \boxed{C_1 = \frac{1}{3} \quad C_2 = -\frac{4}{3}}$

$\boxed{y(t) = \frac{1}{3} e^{2t} - \frac{4}{3} e^{-t}}$

Exercise

Find the solution of the given initial value problem $\quad y'' + y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 0$

Solution

The characteristic equation: $\lambda^2 + \lambda + 2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \frac{-1 \pm i\sqrt{7}}{2}}$

$$y(x) = e^{-x/2}\left(C_1 \cos\frac{\sqrt{7}}{2}x + C_2 \sin\frac{\sqrt{7}}{2}x \right)$$

$y(0) = 0 \quad \rightarrow \quad \boxed{C_1 = 0}$

$$y'(x) = e^{-x/2}\left(-\frac{1}{2}C_1 \cos\frac{\sqrt{7}}{2}x - \frac{1}{2}C_2 \sin\frac{\sqrt{7}}{2}x - \frac{\sqrt{7}}{2}C_1 \sin\frac{\sqrt{7}}{2}x + \frac{\sqrt{7}}{2}C_2 \cos\frac{\sqrt{7}}{2}x \right)$$

$y'(0) = 0 \quad \rightarrow \quad -\frac{1}{2}C_1 + \frac{\sqrt{7}}{2}C_2 = 0 \Rightarrow \boxed{C_2 = 0}$

$\boxed{y(x) = 0}$

Exercise

Find the solution of the given initial value problem $\quad y'' + 2y' + y = 0; \quad y(0) = 1, \quad y'(0) = -3$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -1}$

$\boxed{y(x) = \left(C_1 + C_2 x \right) e^{-x}}$

$y(0) = 1 \quad \rightarrow \quad \boxed{C_1 = 1}$

$y'(x) = \left(C_2 - C_1 - C_2 x \right) e^{-x}$

$y'(0) = -3 \quad \rightarrow C_2 - C_1 = -3 \quad \Rightarrow \boxed{C_2 = -2}$

$\boxed{y(x) = (1 - 2x) e^{-x}}$

Exercise

Find the solution of the given initial value problem $y'' - 2y' + y = 0$; $y(0) = 5$, $y'(0) = 10$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0 \rightarrow \boxed{\lambda_{1,2} = 1}$

$$y(x) = (C_1 + C_2 x)e^x$$

$$y(0) = 5 \rightarrow \boxed{C_1 = 5}$$

$$y'(x) = (C_2 + C_1 + C_2 x)e^x$$

$$y'(0) = 10 \rightarrow C_2 + C_1 = 10 \Rightarrow \boxed{C_2 = 5}$$

$$\boxed{y(x) = 5(1 + x)e^x}$$

Exercise

Find the solution of the given initial value problem $y'' - 2y' - 2y = 0$; $y(0) = 0$, $y'(0) = 3$

Solution

The characteristic equation: $\lambda^2 - 2\lambda - 2 = 0 \rightarrow \lambda_{1,2} = \frac{2 \pm 2\sqrt{3}}{2} \boxed{= 1 \pm \sqrt{3}}$

$$\boxed{y(x) = C_1 e^{(1-\sqrt{3})x} + C_2 e^{(1+\sqrt{3})x}}$$

$$y(0) = 0 \rightarrow C_1 + C_2 = 0$$

$$y'(x) = (1 - \sqrt{3})C_1 e^{(1-\sqrt{3})x} + (1 + \sqrt{3})C_2 e^{(1+\sqrt{3})x}$$

$$y'(0) = 3 \rightarrow (1 - \sqrt{3})C_1 + (1 + \sqrt{3})C_2 = 3$$

$$\begin{cases} C_1 + C_2 = 0 \rightarrow C_1 = -C_2 \\ (1 - \sqrt{3})C_1 + (1 + \sqrt{3})C_2 = 3 \end{cases} \rightarrow C_1(1 - \sqrt{3} - 1 - \sqrt{3}) = 3 \quad C_1 = -\frac{\sqrt{3}}{2},\ C_2 = \frac{\sqrt{3}}{2}$$

$$\boxed{y(x) = -\frac{\sqrt{3}}{2} e^{(1-\sqrt{3})x} + \frac{\sqrt{3}}{2} e^{(1+\sqrt{3})x}}$$

Exercise

Find the solution of the given initial value problem $y'' - 2y' + 2y = 0$; $y(0) = 1$, $y(\pi) = 1$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 2 = 0 \rightarrow \lambda_{1,2} = \frac{2 \pm 2i}{2} \boxed{= 1 \pm i}$

$$y(x) = e^x (C_1 \cos x + C_2 \sin x)$$

$y(0)=1 \quad \rightarrow \quad \boxed{C_1 = 1}$

$y(\pi)=1 \quad \rightarrow \quad -e^{\pi}C_1 = 1 \; \boxed{C_1 = -e^{-\pi}}$

$C_1 = -e^{-\pi} \neq 1$

There is No solution the *ODE* under the given conditions.

Exercise

Find the solution of the given initial value problem. $y'' - 2y' - 3y = 0; \quad y(0) = 2, \quad y'(0) = -3$

Solution

The characteristic equation: $\lambda^2 - 2\lambda - 3 = 0 \Rightarrow \lambda_1 = -1;\ \lambda_2 = 3$

$y(t) = C_1 e^{-t} + C_2 e^{3t}$

$y(0) = C_1 + C_2 = 2$

$y'(t) = -C_1 e^{-t} + 3C_2 e^{3t}$

$y'(0) = -C_1 + 3C_2 = -3$

$\begin{cases} C_1 + C_2 = 2 \\ -C_1 + 3C_2 = -3 \end{cases} \rightarrow \quad C_1 = \frac{9}{4} \quad C_2 = -\frac{1}{4}$

$\boxed{y(t) = \frac{9}{4}e^{-t} - \frac{1}{4}e^{3t}}$

Exercise

Find the solution of the given initial value problem $y'' + 2y' - 8y = 0\ ; \quad y(0) = 3, \quad y'(0) = -12$

Solution

The characteristic equation: $\lambda^2 + 2\lambda - 8 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -4,\ 2}$

$\boxed{y(x) = C_1 e^{-4x} + C_2 e^{2x}}$

$y(0) = 3 \quad \rightarrow \quad C_1 + C_2 = 3$

$y'(x) = -4C_1 e^{-4x} + 2C_2 e^{2x}$

$y'(0) = -12 \quad \rightarrow \quad -4C_1 + 2C_2 = -12$

$\begin{cases} C_1 + C_2 = 3 \\ -4C_1 + 2C_2 = -12 \end{cases} \rightarrow \quad \boxed{C_1 = 3,\ C_2 = 0}$

$\boxed{y(x) = 3e^{-4x}}$

Exercise

Find the general solution: $y'' - 2y' + 17y = 0;$ $y(0) = -2,$ $y'(0) = 3$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 17 = 0$

$\Rightarrow \lambda_{1,2} = 1 \pm 4i$

$y(t) = e^t \left(C_1 \cos 4t + C_2 \sin 4t\right)$

$y(t) = e^t \left(C_1 \cos 4t + C_2 \sin 4t\right) \quad \to y(0) = \boxed{C_1 = -2}$

$y'(t) = e^t \left(C_1 \cos 4t + C_2 \sin 4t\right) + e^t \left(-4C_1 \sin 4t + 4C_2 \cos 4t\right)$

$\Rightarrow y'(0) = C_1 + 4C_2 = 3$

$\Rightarrow \boxed{C_2 = \frac{5}{4}}$

$y(t) = e^t \left(-2\cos 4t + \frac{5}{4}\sin 4t\right)$

Exercise

Find the general solution: $y'' + 2\sqrt{2}y' + 2y = 0$; $y(0) = 1,$ $y'(0) = 0$

Solution

The characteristic equation: $\lambda^2 + 2\sqrt{2}\,\lambda + 2 = \left(\lambda + \sqrt{2}\right)^2 = 0 \quad \Rightarrow \lambda_{1,2} = \pm\sqrt{2}\,i$

$y(t) = C_1 \cos\sqrt{2}\,t + C_2 \sin\sqrt{2}\,t$

$y(0) = 1 \quad \to \quad C_1 = 1$

$y'(t) = -\sqrt{2}C_1 \sin\sqrt{2}\,t + \sqrt{2}C_2 \cos\sqrt{2}\,t$

$y'(0) = 0 \quad \to \sqrt{2}C_2 = 0 \quad \Rightarrow \quad C_2 = 0$

$y(t) = \cos\sqrt{2}\,t$

Exercise

Find the general solution: $y'' + 3y' - 10y = 0;$ $y(0) = 4,$ $y'(0) = -2$

Solution

The characteristic equation: $\lambda^2 + 3\lambda + 10 = 0 \quad \to \quad \lambda_{1,2} = -5,\ 2$

$$\boxed{y(x)=C_1e^{-5x}+C_2e^{2x}}$$

$$y(0)=4 \quad \rightarrow C_1+C_2=4$$

$$y'=-5C_1e^{-5x}+2C_2e^{2x}$$

$$y'(0)=-2 \quad \rightarrow \ -5C_1+2C_2=-2$$

$$\begin{cases} C_1+C_2=4 \\ -5C_1+2C_2=-2 \end{cases} \quad \rightarrow \quad \boxed{C_1=\frac{10}{7},\ C_2=\frac{18}{7}}$$

$$\boxed{y(x)=\frac{10}{7}e^{-5x}+\frac{18}{7}e^{2x}}$$

Exercise

Find the solution of the given initial value problem $y''+4y=0$; $y(0)=0,\quad y(\pi)=0$

Solution

The characteristic equation: $\lambda^2+4=0 \quad \rightarrow \quad \boxed{\lambda_{1,2}=\pm 2i}$

$$y(x)=C_1\cos 2x+C_2\sin 2x$$

$$y(0)=0 \quad \rightarrow \boxed{C_1=0}$$

$$y(\pi)=0 \quad \rightarrow \boxed{C_1=0}$$

$$\boxed{y(x)=C_2\sin 2x}$$

Exercise

Find the solution of the given initial value problem $y''+4y=0$; $y\left(\frac{\pi}{4}\right)=-2,\quad y'\left(\frac{\pi}{4}\right)=1$

Solution

The characteristic equation: $\lambda^2+4=0 \quad \rightarrow \quad \boxed{\lambda_{1,2}=\pm 2i}$

$$\boxed{y(x)=C_1\cos 2x+C_2\sin 2x}$$

$$y\left(\frac{\pi}{4}\right)=-2 \quad \rightarrow \boxed{C_2=-2}$$

$$y'=-2C_1\sin 2x+2C_2\cos 2x$$

$$y'\left(\frac{\pi}{4}\right)=1 \quad \rightarrow \ -2C_1=1 \quad \Rightarrow \boxed{C_1=-\frac{1}{2}}$$

$$\boxed{y(x)=-2\cos 2x-\frac{1}{2}\sin 2x}$$

Exercise

Find the solution of the given initial value problem $y'' + 4y' + 2y = 0; \quad y(0) = -1, \quad y'(0) = 2$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 2 = 0 \rightarrow \lambda_{1,2} = \frac{-4 \pm 2\sqrt{2}}{2} = \underline{-2 \pm \sqrt{2}}$

$$y(x) = C_1 e^{-(2+\sqrt{2})x} + C_2 e^{(-2+\sqrt{2})x}$$

$$y(0) = -1 \rightarrow \underline{C_1 + C_2 = -1}$$

$$y'(x) = -(2+\sqrt{2})C_1 e^{-(2+\sqrt{2})x} + (-2+\sqrt{2})C_2 e^{(-2+\sqrt{2})x}$$

$$y'(0) = 2 \rightarrow \underline{(-2-\sqrt{2})C_1 + (-2+\sqrt{2})C_2 = 2}$$

$$\begin{cases} C_1 + C_2 = -1 \\ (-2-\sqrt{2})C_1 + (-2+\sqrt{2})C_2 = 2 \end{cases}$$

$$\rightarrow \Delta = \begin{vmatrix} 1 & 1 \\ -2-\sqrt{2} & -2+\sqrt{2} \end{vmatrix} = 2\sqrt{2} \qquad \Delta_{C_c} = \begin{vmatrix} -1 & 1 \\ 2 & -2+\sqrt{2} \end{vmatrix} = -\sqrt{2}$$

$$\underline{C_1 = -\tfrac{1}{2}, \quad C_2 = -\tfrac{1}{2}}$$

$$\underline{y(x) = -\tfrac{1}{2} e^{-(2+\sqrt{2})x} - \tfrac{1}{2} e^{(-2+\sqrt{2})x}}$$

Exercise

Find the solution of the given initial value problem $y'' - 4y' + 3y = 0, \quad y(0) = 1, \quad y'(0) = \frac{1}{3}$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 3 = 0 \rightarrow \underline{\lambda_{1,2} = 1,\ 3}$

$$\underline{y(x) = C_1 e^{x} + C_2 e^{3x}}$$

$$y(0) = 1 \rightarrow C_1 + C_2 = 1$$

$$y'(x) = C_1 e^{x} + 3C_2 e^{3x}$$

$$y'(0) = \tfrac{1}{3} \rightarrow C_1 + 3C_2 = \tfrac{1}{3}$$

$$\begin{cases} C_1 + C_2 = 1 \\ C_1 + 3C_2 = \tfrac{1}{3} \end{cases} \rightarrow C_2 = -\tfrac{1}{3} \quad C_1 = \tfrac{4}{3}$$

$$\boxed{y(x)=\frac{4}{3}e^{x}-\frac{1}{3}e^{3x}}$$

Exercise

Find the solution of the given initial value problem $y''-4y'+4y=0;\quad y(1)=1,\quad y'(1)=1$

Solution

The characteristic equation: $\lambda^2-4\lambda+4=(\lambda-2)^2=0 \rightarrow \boxed{\lambda_{1,2}=2}$

$$\boxed{y(x)=\left(C_1+C_2x\right)e^{2x}}$$

$$y(1)=1 \rightarrow \left(C_1+C_2\right)e^{2}=1 \Rightarrow C_1+C_2=e^{-2}$$

$$y'(x)=\left(C_2+2C_1+2C_2x\right)e^{2x}$$

$$y'(1)=1 \rightarrow \left(2C_1+3C_2\right)e^{2}=1 \Rightarrow 2C_1+3C_2=e^{-2}$$

$$\begin{cases} C_1+C_2=e^{-2} \\ 2C_1+3C_2=e^{-2} \end{cases} \rightarrow \Delta=\begin{vmatrix}1 & 1\\ 2 & 3\end{vmatrix}=1 \quad \Delta_{C_1}=\begin{vmatrix}e^{-2} & 1\\ e^{-2} & 3\end{vmatrix}=2e^{-2}$$

$$\Rightarrow \boxed{C_1=2e^{-2}} \quad \boxed{C_2=-e^{-2}}$$

$$y(x)=\left(2e^{-2}-e^{-2}x\right)e^{-x}$$

$$\boxed{=2e^{-x-2}-xe^{-x-2}}$$

Exercise

Find the solution of the given initial value problem $y''+4y'+4y=0,\quad y(0)=1,\quad y'(0)=3$

Solution

The characteristic equation: $\lambda^2+4\lambda+4=(\lambda+2)^2=0 \rightarrow \boxed{\lambda_{1,2}=-2}$

$$\boxed{y(x)=\left(C_1+C_2x\right)e^{-2x}}$$

$$y(0)=1 \rightarrow \boxed{C_1=1}$$

$$y'(x)=\left(C_2-2C_1-2C_2x\right)e^{-2x}$$

$$y'(0)=3 \rightarrow C_2-2=3 \Rightarrow \boxed{C_2=5}$$

$$\boxed{y(x)=(1+5x)e^{-2x}}$$

Exercise

Find the solution of the given initial value problem: $y'' - 4y' + 5y = 0; \quad y(0) = 1, \quad y'(0) = 5$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 5 = 0 \quad \Rightarrow \quad \lambda_{1,2} = 2 \pm i$

$y(x) = e^{2x}\left(C_1 \cos x + C_2 \sin x\right)$

$$y(x) = e^{2x}\left(C_1 \cos x + C_2 \sin x\right) \quad \Rightarrow y(0) = \boxed{C_1 = 1}$$

$$y'(x) = 2e^{2x}\left(C_1 \cos x + C_2 \sin x\right) + e^{2x}\left(-C_1 \sin x + C_2 \cos x\right)$$

$$\Rightarrow y'(0) = 2C_1 + C_2 = 5 \qquad \Rightarrow \boxed{C_2 = 3}$$

$$y(x) = e^{2x}\left(\cos x + 3\sin x\right)$$

Exercise

Find the solution of the given initial value problem $y'' + 4y' + 5y = 0$; $y(0) = 1, \quad y'(0) = 0$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 5 = 0 \quad \to \lambda_{1,2} = \frac{-4 \pm 2i}{2} = -2 \pm i$

$$y(x) = e^{-2x}\left(C_1 \cos x + C_2 \sin x\right)$$

$$y(0) = 1 \quad \to \quad C_1 = 1$$

$$y'(x) = e^{-2x}\left(-2C_1 \cos x - 2C_2 \sin x - C_1 \sin x + C_2 \cos x\right)$$

$$y'(0) = 0 \quad \to \quad -2C_1 + C_2 = 0 \quad \Rightarrow \quad C_2 = 2$$

$$y(x) = e^{-2x}\left(\cos x + 2\sin x\right)$$

Exercise

Find the solution of the given initial value problem $y'' + 4y' + 5y = 0; \quad y\left(\frac{\pi}{2}\right) = \frac{1}{2}, \quad y'\left(\frac{\pi}{2}\right) = -2$

Solution

The characteristic equation: $\lambda^2 + 4\lambda + 5 = 0 \quad \to \lambda_{1,2} = \frac{-4 \pm 2i}{2} = -2 \pm i$

$y(x) = e^{-2x}\left(C_1 \cos x + C_2 \sin x\right)$

$$y\left(\frac{\pi}{2}\right) = \frac{1}{2} \quad \to e^{-\pi} C_2 = \frac{1}{2} \quad \Rightarrow \quad C_2 = \frac{e^{\pi}}{2}$$

$$y'(x)=e^{-2x}\left(-2C_1\cos x-2C_2\sin x-C_1\sin x+C_2\cos x\right)$$

$$y'\left(\frac{\pi}{2}\right)=-2 \quad\rightarrow\; e^{-\pi}\left(-2\frac{e^{\pi}}{2}-C_1\right)=-2$$

$$\Rightarrow 1+e^{-\pi}C_1=2 \;\rightarrow\; \underline{C_1=e^{\pi}|}$$

$$y(x)=e^{-2x}\left(e^{\pi}\cos x+\tfrac{1}{2}e^{\pi}\sin x\right)$$

$$\underline{=e^{\pi-2x}\left(\cos x+\tfrac{1}{2}\sin x\right)\Big|}$$

Exercise

Find the solution of the given initial value problem $\quad y''-4y'-5y=0\,;\quad y(1)=0,\quad y'(1)=2$

Solution

The characteristic equation: $\lambda^2-4\lambda-5=0 \;\rightarrow\; \underline{\lambda_{1,2}=-1,\;5|}$

$$y(x)=C_1e^{-x}+C_2e^{5x}$$

$$y(1)=0 \;\rightarrow\; C_1e^{-1}+C_2e^{5}=0$$

$$y'(x)=-C_1e^{-x}+5C_2e^{5x}$$

$$y'(1)=2 \;\rightarrow\; e^{-1}C_1+5e^{5}C_2=2$$

$$\begin{cases} e^{-1}C_1+e^{5}C_2=0\\ -e^{-1}C_1+5e^{5}C_2=2\end{cases} \rightarrow\; \Delta=\begin{vmatrix} e^{-1} & e^{5}\\ -e^{-1} & 5e^{5}\end{vmatrix}=6e^{4} \quad \Delta_{C_1}=\begin{vmatrix} 0 & e^{5}\\ 2 & 5e^{5}\end{vmatrix}=-2e^{5}$$

$$\Rightarrow C_1=-\tfrac{1}{3}e,\;\; C_2=\tfrac{1}{3}e^{-5},$$

$$\underline{y(x)=-\tfrac{1}{3}e^{1-x}+\tfrac{1}{3}e^{-5x-5}\Big|}$$

Exercise

Find the solution of the given initial value problem $\quad y''-4y'-5y=0,\quad y(-1)=3,\quad y'(-1)=9$

Solution

The characteristic equation: $\lambda^2-4\lambda-5=0 \;\rightarrow\; \underline{\lambda_{1,2}=-1,\;5|}$

$$\underline{y(x)=C_1e^{-x}+C_2e^{5x}\Big|}$$

$$y(-1)=3 \;\rightarrow\; C_1e+C_2e^{-5}=3$$

$$y'(x) = -C_1 e^{-x} + 5C_2 e^{5x}$$

$$y'(-1) = 9 \quad \rightarrow \quad -C_1 e + 5C_2 e^{-5} = 9$$

$$\begin{cases} eC_1 + e^{-5}C_2 = 3 \\ -eC_1 + 5e^{-5}C_2 = 9 \end{cases} \rightarrow \Delta = \begin{vmatrix} e & e^{-5} \\ -e & 5e^{-5} \end{vmatrix} = 6e^{-4} \quad \Delta_{C_1} = \begin{vmatrix} 3 & e^{-5} \\ 9 & 5e^{-5} \end{vmatrix} = 6e^{-5}$$

$$\Rightarrow \boxed{C_1 = e^{-1}, \ C_2 = 2e^5}$$

$$\boxed{y(x) = e^{-1-x} + 2e^{5x+5}}$$

Exercise

Find the solution of the given initial value problem $\quad y'' - 4y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = -8$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 9 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{4 \pm 2i\sqrt{5}}{2} = \boxed{2 \pm i\sqrt{5}}$

$$y(t) = e^{2t}\left(C_1 \cos\sqrt{5}t + C_2 \sin\sqrt{5}t\right)$$

$$y(0) = 0 \quad \rightarrow \quad \boxed{C_1 = 0}$$

$$y' = e^{2t}\left(2C_1 \cos\sqrt{5}t + 2C_2 \sin\sqrt{5}t - \sqrt{5}C_1 \sin\sqrt{5}t + \sqrt{5}C_2 \cos\sqrt{5}t\right)$$

$$y'(0) = 8 \quad \rightarrow \quad -\sqrt{5}C_2 = 8 \quad \Rightarrow \quad \boxed{C_2 = -\frac{8}{\sqrt{5}}}$$

$$\boxed{y(t) = -\frac{8}{\sqrt{5}} e^{2t} \sin\sqrt{5}t}$$

Exercise

Find the solution of the given initial value problem. $\quad y'' - 4y' + 13y = 0; \quad y(0) = -1, \quad y'(0) = 2$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 13 = 0$

$$\Rightarrow \lambda_{1,\,2} = \frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i$$

$$\boxed{y(x) = e^{2x}\left(C_1 \cos 3x + C_2 \sin 3x\right)}$$

$$y(0) = e^0\left(C_1 \cos(0) + C_2 \sin(0)\right) \quad \rightarrow \quad \boxed{C_1 = -1}$$

$$y'(x) = 2e^{2x}\left(C_1 \cos 3x + C_2 \sin 3x\right) + e^{2x}\left(-3C_1 \sin 3x + 3C_2 \cos 3x\right)$$

$$y'(0) = 2C_1 + 3C_2 = 2 \Rightarrow \boxed{C_2 = \frac{2-2(-1)}{3} = \frac{4}{3}}$$

$$\boxed{y(x) = e^{2x}\left(-\cos 3x + \tfrac{4}{3}\sin 3x\right)}$$

Exercise

Find the solution of the given initial value problem $y'' - 5y' + 6y = 0;\quad y(1) = e^2,\quad y'(1) = 3e^2$

Solution

The characteristic equation: $\lambda^2 - 5\lambda + 6 = 0 \rightarrow \boxed{\lambda_{1,2} = 2,\ 3}$

$$y(x) = C_1 e^{2x} + C_2 e^{3x}$$

$$y(1) = e^2 \rightarrow C_1 e^2 + C_2 e^3 = e^2 \Rightarrow \boxed{C_1 + eC_2 = 1}$$

$$y'(x) = 2C_1 e^{2x} + 3C_2 e^{3x}$$

$$y'(1) = 3e^2 \rightarrow 2C_1 e^2 + 3C_2 e^3 = 3e^2 \Rightarrow \boxed{2C_1 + 3eC_2 = 3}$$

$$-2\times\begin{cases} C_1 + eC_2 = 1 \\ 2C_1 + 3eC_2 = 3 \end{cases} \rightarrow eC_2 = 1 \quad \boxed{C_2 = e^{-1};\ C_1 = 0}$$

$$\boxed{y(x) = e^{3x-1}}$$

Exercise

Find the solution of the given initial value problem $y'' + 6y' + 5y = 0,\quad y(1) = 0,\quad y'(0) = 3$

Solution

The characteristic equation: $\lambda^2 + 6\lambda + 5 = 0 \rightarrow \boxed{\lambda_{1,2} = -1,\ -5}$

$$y(x) = C_1 e^{-x} + C_2 e^{-5x}$$

$$y(1) = 0 \rightarrow C_1 e^{-1} + C_2 e^{-5} = 0 \Rightarrow C_1 e^4 + C_2 = e^5$$

$$y'(x) = -C_1 e^{-x} - 5C_2 e^{-5x}$$

$$y'(0) = 3 \rightarrow -C_1 - 5C_2 = 3$$

$$\begin{cases} e^4 C_1 + C_2 = e^5 \\ C_1 + 5C_2 = -3 \end{cases}$$

$$\rightarrow \Delta = \begin{vmatrix} e^4 & 1 \\ 1 & 5 \end{vmatrix} = 5e^4 - 1 \quad \Delta_{C_1} = \begin{vmatrix} e^5 & 1 \\ -3 & 5 \end{vmatrix} = 5e^5 + 3 \quad \Delta_{C_2} = \begin{vmatrix} e^4 & e^5 \\ 1 & -3 \end{vmatrix} = -3e^4 - e^5$$

$$C_1 = \frac{5e^5+3}{5e^4-1}; \quad C_2 = -\frac{3e^4+e^5}{5e^4-1}$$

$$y(x) = \frac{5e^5+3}{5e^4-1}e^{-x} - \frac{3e^4+e^5}{5e^4-1}e^{-5x}$$

Exercise

Find the solution of the given initial value problem $y'' - 6y' + 5y = 0; \quad y(0) = 3, \quad y'(0) = 11$

Solution

The characteristic equation: $\lambda^2 - 6\lambda + 5 = 0 \quad \rightarrow \quad \lambda_{1,2} = 1, 5$

$$y(x) = C_1 e^x + C_2 e^{5x}$$

$$y(0) = 3 \quad \rightarrow \quad C_1 + C_2 = 3$$

$$y'(x) = C_1 e^x + 5C_2 e^{5x}$$

$$y'(0) = 11 \quad \rightarrow \quad C_1 + 5C_2 = 11$$

$$\begin{cases} C_1 + C_2 = 3 \\ C_1 + 5C_2 = 11 \end{cases} \quad \rightarrow \quad C_2 = 2; \; C_1 = 1$$

$$y(x) = e^x + 2e^{5x}$$

Exercise

Find the solution of the given initial value problem $y'' - 6y' + 9y = 0, \quad y(0) = 2, \quad y'(0) = \frac{25}{3}$

Solution

The characteristic equation: $\lambda^2 - 6\lambda + 9 = 0 \quad \rightarrow \quad \lambda_{1,2} = 3$

$$y(x) = \left(C_1 + C_2 x\right)e^{3x}$$

$$y(0) = 2 \quad \rightarrow \quad C_1 = 2$$

$$y'(x) = \left(C_2 + 3C_1 + 3C_2 x\right)e^{3x}$$

$$y'(0) = \frac{25}{3} \quad \rightarrow C_2 + 3C_1 = \frac{25}{3} \quad C_2 = \frac{7}{3}$$

$$y(x) = \left(2 + \frac{7}{3}x\right)e^{3x}$$

Exercise

Find the solution of the given initial value problem $y'' - 6y' + 9y = 0; \quad y(0) = 0, \quad y'(0) = 5$

Solution

The characteristic equation: $\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \rightarrow \boxed{\lambda_{1,2} = 3}$

$y(x) = (C_1 + C_2 x)e^{3x}$

$y(0) = 0 \rightarrow \boxed{C_1 = 0}$

$y'(x) = (C_2 + 3C_1 + 3C_2 x)e^{3x}$

$y'(0) = 5 \rightarrow 3C_1 + C_2 = 5 \Rightarrow \boxed{C_2 = 5}$

$\boxed{y(x) = 5xe^{3x}}$

Exercise

Find the solution of the given initial value problem $y'' + 6y' + 9y = 0; \quad y(0) = 2, \quad y'(0) = -2$

Solution

The characteristic equation: $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0 \rightarrow \boxed{\lambda_{1,2} = -3}$

$y(x) = (C_1 + C_2 x)e^{-3x}$

$y(0) = 2 \rightarrow \boxed{C_1 = 2}$

$y'(x) = (C_2 - 3C_1 - 3C_2 x)e^{-3x}$

$y'(0) = -2 \rightarrow C_2 - 6 = -2 \Rightarrow \boxed{C_2 = 4}$

$\boxed{y(x) = (2 + 4x)e^{-3x}}$

Exercise

Find the solution of the given initial value problem $y'' + 8y' - 9y = 0; \quad y(1) = 2, \quad y'(1) = 0$

Solution

The characteristic equation: $\lambda^2 + 8\lambda - 9 = 0 \rightarrow \boxed{\lambda_{1,2} = 1, \ -9}$

$y(x) = C_1 e^{-9x} + C_2 e^{x}$

$y(1) = 2 \rightarrow C_1 e^{-9} + C_2 e = 2 \Rightarrow \boxed{C_1 + e^{10}C_2 = 2e^9}$

$y'(x) = -9C_1 e^{-9x} + C_2 e^{x}$

$$y'(1)=0 \quad \to \quad -9C_1e^{-9}+C_2e=0 \Rightarrow \boxed{C_2=9e^{-10}C_1}$$

$$C_1+e^{10}\left(9e^{-10}C_1\right)=2e^9 \quad \Rightarrow \quad \boxed{C_1=\frac{e^9}{5}} \quad \boxed{C_2=\frac{9}{5e}}$$

$$\boxed{y(x)=\frac{1}{5}e^{9-9x}+\frac{9}{5}e^{x-1}}$$

Exercise

Find the solution of the given initial value problem. $y''-8y'+17y=0; \quad y(0)=4, \quad y'(0)=-1$

Solution

The characteristic equation: $\lambda^2-8\lambda+17=0$

$$\Rightarrow \lambda_{1,\,2}=\frac{8\pm\sqrt{64-68}}{2}=\frac{8\pm i2}{2}=4\pm i$$

$$\boxed{y(x)=e^{4x}\left(C_1\cos x+C_2\sin x\right)}$$

$$y(0)=e^0\left(C_1\cos(0)+C_2\sin(0)\right) \quad \to \quad \boxed{4=C_1}$$

$$y'(x)=4e^{4x}\left(C_1\cos x+C_2\sin x\right)+e^{4x}\left(-C_1\sin x+C_2\cos x\right)$$

$$y'(0)=4C_1+C_2=-1 \quad \Rightarrow \quad \boxed{C_2=-1-16=-17}$$

$$\boxed{y(x)=e^{4x}\left(4\cos x-17\sin x\right)}$$

Exercise

Find the solution of the given initial value problem $y''-9y=0, \quad y(0)=2, \quad y'(0)=-1$

Solution

The characteristic equation: $\lambda^2-9=0 \quad \to \quad \boxed{\lambda_{1,2}=\pm 3}$

$$y(x)=C_1e^{-3x}+C_2e^{3x}$$

$$y(0)=2 \quad \to \quad \Rightarrow \boxed{C_1+C_2=2}$$

$$y'(x)=-3C_1e^{-3x}+3C_2e^{3x}$$

$$y'(0)=-1 \quad \to \quad \boxed{-3C_1+3C_2=-1}$$

$$\begin{cases} C_1+C_2=2 \\ -3C_1+3C_2=-1 \end{cases} \quad \to \quad \boxed{C_1=\frac{7}{6}, \; C_2=\frac{5}{6}}$$

$$\boxed{y(x)=\frac{7}{6}e^{-3x}+\frac{5}{6}e^{3x}}$$

Exercise

Find the solution of the given initial value problem $y'' - 10y' + 25y = 0, \quad y(0) = 1, \quad y'(1) = 0$

Solution

The characteristic equation: $\lambda^2 - 10\lambda + 25 = (\lambda - 5)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = 5}$

$$y(x) = \left(C_1 + C_2 x\right)e^{5x}$$

$$y(0) = 1 \quad \rightarrow \quad \boxed{C_1 = 1}$$

$$y'(x) = \left(C_2 + 5C_1 + 5C_2 x\right)e^{5x}$$

$$y'(1) = 0 \quad \rightarrow \quad \left(6C_2 + 5C_1\right)e^5 = 0 \Rightarrow 6C_2 + 5C_1 = 0 \quad \Rightarrow \boxed{C_2 = -\frac{5}{6}}$$

$$\boxed{y(x) = \left(1 - \frac{5}{6}x\right)e^{5x}}$$

Exercise

Find the general solution: $y'' + 10y' + 25y = 0; \quad y(0) = 2, \quad y'(0) = -1$

Solution

The characteristic equation: $\lambda^2 + 10\lambda + 25 = 0$

$\Rightarrow \lambda_{1,2} = -5$

$$y(t) = \left(C_1 + C_2 t\right)e^{-5t}$$

$$y(t) = \left(C_1 + C_2 t\right)e^{-5t} \qquad y(0) = \boxed{C_1 = 2}$$

$$y' = C_2 e^{-5t} - 5\left(C_1 + C_2 t\right)e^{-5t} \quad y'(0) = C_2 - 5C_1 = -1 \quad \Rightarrow \boxed{C_2 = 9}$$

$$\boxed{y(t) = (2 + 9t)e^{-5t}}$$

Exercise

Find the general solution: $y'' + 11y' + 24y = 0; \quad y(0) = 0, \quad y'(0) = -7$

Solution

The characteristic equation: $\lambda^2 + 11\lambda + 24 = 0 \quad \rightarrow \lambda_{1,2} = \frac{-11 \pm 5}{2} \quad \boxed{\lambda_{1,2} = -8, \ -3}$

$$y(x) = C_1 e^{-8x} + C_2 e^{-3x}$$

$$y(0) = 0 \quad \rightarrow \quad \Rightarrow \boxed{C_1 + C_2 = 0}$$

$$y' = -8C_1 e^{-8x} - 3C_2 e^{-3x}$$

$y'(0) = -7 \quad \rightarrow \quad \boxed{-8C_1 - 3C_2 = -7}$

$$\begin{cases} C_1 + C_2 = 0 \\ -8C_1 - 3C_2 = -7 \end{cases} \quad \rightarrow \quad \boxed{C_1 = \frac{7}{5}, \ C_2 = -\frac{7}{5}}$$

$$\boxed{y(x) = \frac{7}{5}e^{-8x} - \frac{7}{5}e^{-3x}}$$

Exercise

Find the solution of the given initial value problem $\quad y'' + 12y = 0, \quad y(0) = 0, \quad y'(0) = 1$

Solution

The characteristic equation: $\lambda^2 + 12 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 2i\sqrt{3}}$

$y(x) = C_1 \cos 2\sqrt{3}x + C_2 \sin 2\sqrt{3}x$

$y(0) = 0 \quad \rightarrow \boxed{C_1 = 0}$

$y'(x) = -2\sqrt{3}C_1 \sin 2\sqrt{3}x + 2\sqrt{3}C_2 \cos 2\sqrt{3}x$

$y'(0) = 2 \quad \rightarrow \ 2\sqrt{3}C_2 = 2 \quad \Rightarrow \ \boxed{C_2 = \frac{\sqrt{3}}{3}}$

$$\boxed{y(x) = \frac{\sqrt{3}}{3}\sin 2\sqrt{3}x}$$

Exercise

Find the solution of the given initial value problem $\quad y'' + 16y = 0, \quad y(\pi) = 2, \quad y'(0) = -2$

Solution

The characteristic equation: $\lambda^2 + 16 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 4i}$

$y(x) = C_1 \cos 4x + C_2 \sin 4x$

$y(0) = 2 \quad \rightarrow \boxed{C_1 = 2}$

$y'(x) = -4C_1 \sin 4x + 4C_2 \cos 4x$

$y'(0) = -2 \quad \rightarrow \ 4C_2 = -2 \quad \Rightarrow \ \boxed{C_2 = -\frac{1}{2}}$

$$\boxed{y(x) = 2\cos 4x - \frac{1}{2}\sin 4x}$$

Exercise

Find the solution of the given initial value problem $\quad y''+16y=0, \quad y\left(\frac{\pi}{2}\right)=-10, \quad y'\left(\frac{\pi}{2}\right)=3$

Solution

The characteristic equation: $\lambda^2+16=0 \rightarrow \underline{\lambda_{1,2}=\pm 4i}$

$y(x)=C_1\cos 4x+C_2\sin 4x$

$$y\left(\frac{\pi}{2}\right)=-10 \quad \rightarrow \underline{C_1=-10}$$

$$y'=-4C_1\sin 4x+4C_2\cos 4x$$

$$y'\left(\frac{\pi}{2}\right)=3 \quad \rightarrow 4C_2=3 \quad \Rightarrow \underline{C_2=\frac{3}{4}}$$

$$\underline{y(x)=-10\cos 4x+\frac{3}{4}\sin 4x}$$

Exercise

Find the solution of the given initial value problem $\quad y''+16y=0, \quad y(\pi)=2, \quad y'(0)=-2$

Solution

The characteristic equation: $\lambda^2+16=0 \rightarrow \underline{\lambda_{1,2}=\pm 4i}$

$y(x)=C_1\cos 4x+C_2\sin 4x$

$$y(\pi)=2 \quad \rightarrow \underline{C_1=2}$$

$$y'(x)=-4C_1\sin 4x+4C_2\cos 4x$$

$$y'(0)=-2 \quad \rightarrow 4C_2=-2 \quad \Rightarrow \underline{C_2=-\frac{1}{2}}$$

$$\underline{y(x)=2\cos 4x-\frac{1}{2}\sin 4x}$$

Exercise

Find the general solution: $\quad y''+25y=0; \quad y(0)=1, \quad y'(0)=-1$

Solution

The characteristic equation: $\lambda^2+25=0$

$\Rightarrow \lambda_{1,2}=\pm 5i$

$y(t)=e^{0t}\left(C_1\cos 5t+C_2\sin 5t\right)$

$y(t)=C_1\cos 5t+C_2\sin 5t$

$$y(t)=C_1\cos 5t+C_2\sin 5t \quad \rightarrow y(0)=\boxed{C_1=1}$$

$$y'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t$$

$$\Rightarrow y'(0) = 5C_2 = -1 \rightarrow \boxed{C_2 = -\frac{1}{5}}$$

$$y(t) = 5\cos 5t - \frac{1}{5}\sin 5t$$

Exercise

Find the solution of the given initial value problem $2y'' - 2y' + y = 0$; $y(-\pi) = 1, \quad y'(-\pi) = -1$

Solution

The characteristic equation: $2\lambda^2 - 2\lambda + 1 = 0 \rightarrow \lambda_{1,2} = \frac{2 \pm 2i}{4} = \frac{1}{2} \pm \frac{1}{2}i$

$$y(x) = e^{x/2}\left(C_1 \cos\frac{x}{2} + C_2 \sin\frac{x}{2}\right)$$

$$y(-\pi) = 1 \rightarrow e^{-\pi/2}(-C_2) = 1 \Rightarrow C_2 = -e^{\pi/2}$$

$$y'(x) = e^{x/2}\left(\frac{1}{2}C_1 \cos\frac{x}{2} + \frac{1}{2}C_2 \sin\frac{x}{2} - \frac{1}{2}C_1 \sin\frac{x}{2} + \frac{1}{2}C_2 \cos\frac{x}{2}\right)$$

$$y'(-\pi) = -1 \rightarrow e^{-\pi/2}\left(\frac{1}{2}e^{\pi/2} + \frac{1}{2}C_1\right) = -1$$

$$\frac{1}{2} + \frac{1}{2}e^{-\pi/2}C_1 = -1 \rightarrow C_1 = -3e^{\pi/2}$$

$$y(x) = e^{x/2}\left(-3e^{\pi/2}\cos\frac{x}{2} - e^{\pi/2}\sin\frac{x}{2}\right)$$

$$= -e^{(x+\pi)/2}\left(3\cos\frac{x}{2} + \sin\frac{x}{2}\right)$$

Exercise

Find the solution of the given initial value problem $3y'' + y' - 14y = 0, \quad y(0) = 2, \quad y'(0) = -1$

Solution

The characteristic equation: $3\lambda^2 + \lambda - 14 = 0 \rightarrow \lambda_{1,2} = \frac{-1 \pm 13}{6} = -\frac{7}{3}, 2$

$$y(x) = C_1 e^{-7x/3} + C_2 e^{2x}$$

$$y(0) = 2 \rightarrow C_1 + C_2 = 2$$

$$y'(x) = -\frac{7}{3}C_1 e^{-7x/3} + 2C_2 e^{2x}$$

$$y'(0) = -1 \rightarrow -\frac{7}{3}C_1 + 2C_2 = -1$$

$$\begin{cases} C_1 + C_2 = 2 \\ -7C_1 + 6C_2 = -3 \end{cases} \to \underline{C_2 = \frac{11}{13}, \ C_1 = \frac{15}{13}}$$

$$\underline{y(x) = \frac{11}{13}e^{-7x/3} + \frac{15}{13}e^{2x}}$$

Exercise

Find the solution of the given initial value problem $3y'' + 2y' - 8y = 0, \quad y(0) = -6, \quad y'(0) = -18$

Solution

The characteristic equation: $3\lambda^2 + 2\lambda - 8 = 0 \quad \to \quad \lambda_{1,2} = \frac{-2 \pm 10}{6} \underline{= -2, \ \frac{4}{3}}$

$$y(x) = C_1 e^{-2x} + C_2 e^{4x/3}$$

$$y(0) = -6 \quad \to C_1 + C_2 = -6$$

$$y' = -2C_1 e^{-2x} + \frac{4}{3}C_2 e^{4x/3}$$

$$y'(0) = -18 \quad \to \ -2C_1 + \frac{4}{3}C_2 = -18$$

$$\begin{cases} C_1 + C_2 = -6 \\ -3C_1 + 2C_2 = -27 \end{cases} \to \underline{C_1 = \frac{15}{5} = 3, \ C_2 = -9}$$

$$\underline{y(x) = 3e^{-2x} - 9e^{4x/3}}$$

Exercise

Find the solution of the given initial value problem $4y'' - 4y' + y = 0, \quad y(0) = 4, \quad y'(0) = 4$

Solution

The characteristic equation: $4\lambda^2 - 4\lambda + 1 = (2\lambda - 1)^2 = 0 \quad \to \quad \underline{\lambda_{1,2} = \frac{1}{2}}$

$$y(x) = (C_1 + C_2 x)e^{x/2}$$

$$y(0) = 4 \quad \to \underline{C_1 = 4}$$

$$y'(x) = \left(C_2 + \frac{1}{2}C_1 + \frac{1}{2}C_2 x\right)e^{x/2}$$

$$y'(0) = 4 \quad \to C_2 + \frac{1}{2}C_1 = 4 \quad \Rightarrow \underline{C_2 = 2}$$

$$\underline{y(x) = 2(1 + x)e^{x/2}}$$

Exercise

Find the solution of the given initial value problem $\quad 4y'' - 4y' + y = 0; \quad y(1) = -4, \quad y'(1) = 0$

Solution

The characteristic equation: $\quad 4\lambda^2 - 4\lambda + 1 = (2\lambda - 1)^2 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = \frac{1}{2}}$

$$\underline{y(x) = (C_1 + C_2 x)e^{x/2}}$$

$$y(1) = -4 \quad \rightarrow \quad (C_1 + C_2)e^{1/2} = -4$$

$$y'(x) = \left(C_2 + \frac{1}{2}C_1 + \frac{1}{2}C_2 x\right)e^{x/2}$$

$$y'(1) = 0 \quad \rightarrow \quad \left(C_2 + \frac{1}{2}C_1 + \frac{1}{2}C_2\right)e^{1/2} = 0 \quad \Rightarrow \quad C_1 + 3C_2 = 0$$

$$\begin{cases} C_1 + C_2 = -4e^{-1/2} \\ C_1 + 3C_2 = 0 \end{cases} \quad \underline{\rightarrow \quad C_2 = 2e^{-1/2} \quad C_1 = -6e^{-1/2}}$$

$$y(x) = \left(-6e^{-1/2} + 2e^{-1/2}x\right)e^{x/2}$$

$$\underline{= 2(x-3)e^{(x-1)/2}}$$

Exercise

Find the solution of the given initial value problem $\quad 4y'' - 4y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 5$

Solution

The characteristic equation: $\quad 4\lambda^2 - 4\lambda - 3 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{4 \pm 8}{8} \underline{= -\frac{1}{2}, \ \frac{3}{2}}$

$$y(x) = C_1 e^{-x/2} + C_2 e^{3x/2}$$

$$y(0) = 1 \quad \rightarrow \quad C_1 + C_2 = 1$$

$$y'(x) = -\frac{1}{2}C_1 e^{-x/2} + \frac{3}{2}C_2 e^{3x/2}$$

$$y'(0) = 5 \quad \rightarrow \quad -\frac{1}{2}C_1 + \frac{3}{2}C_2 = 5$$

$$\begin{cases} C_1 + C_2 = 1 \\ -C_1 + 3C_2 = 10 \end{cases} \quad \rightarrow \quad C_2 = \frac{11}{4} \quad C_1 = -\frac{7}{4}$$

$$\underline{y(x) = -\frac{7}{4}e^{-x/2} + \frac{11}{4}e^{3x/2}}$$

Exercise

Find the solution of the given initial value problem $4y''+4y'+5y=0, \quad y(\pi)=1, \quad y'(\pi)=0$

Solution

The characteristic equation: $4\lambda^2+4\lambda+5=0 \rightarrow \lambda_{1,2}=\frac{-4\pm 8i}{8}=-\frac{1}{2}\pm i$

$$y(x)=e^{-x/2}\left(C_1\cos x+C_2\sin x\right)$$

$$y(\pi)=1 \rightarrow -C_1e^{-\pi/2}=1 \quad \boxed{C_1=-e^{\pi/2}}$$

$$y'(x)=e^{-x/2}\left(-C_1\sin x+C_2\cos x-\tfrac{1}{2}C_1\cos x-\tfrac{1}{2}C_2\sin x\right)$$

$$y'(\pi)=0 \rightarrow \left(-C_2+\tfrac{1}{2}C_1\right)e^{-\pi/2}=0$$

$$C_2=\tfrac{1}{2}C_1=-\tfrac{1}{2}e^{\pi/2}$$

$$y(x)=e^{-x/2}\left(-e^{\pi/2}\cos x-\tfrac{1}{2}e^{\pi/2}\sin x\right)$$

$$=-\tfrac{1}{2}e^{(\pi-x)/2}\left(2\cos x+\sin x\right)$$

Exercise

Find the solution of the given initial value problem $4y''+4y'+17y=0, \quad y(0)=-1, \quad y'(0)=2$

Solution

The characteristic equation: $4\lambda^2+4\lambda+17=0 \rightarrow \lambda_{1,2}=\frac{-4\pm 16i}{8}=-\frac{1}{2}\pm 2i$

$$y(x)=e^{-x/2}\left(C_1\cos 2x+C_2\sin 2x\right)$$

$$y(0)=-1 \rightarrow \boxed{-1=C_1}$$

$$y'(x)=e^{-x/2}\left(-\tfrac{1}{2}C_1\cos 2x-\tfrac{1}{2}C_2\sin 2x-2C_1\sin 2x+2C_2\cos 2x\right)$$

$$y'(0)=2 \rightarrow 2=\frac{1}{2}+2C_2 \quad \Rightarrow \boxed{C_2=\frac{3}{4}}$$

$$y(x)=e^{-x/2}\left(-\cos 2x+\tfrac{3}{4}\sin 2x\right)$$

Exercise

Find the solution of the given initial value problem $4y''-5y'=0, \quad y(-2)=0, \quad y'(-2)=7$

Solution

The characteristic equation: $4\lambda^2-5\lambda=0 \rightarrow \lambda_{1,2}=0, \frac{5}{4}$

$y(t) = C_1 + C_2 e^{5t/4}$

$y(-2) = 0 \quad \to \quad \boxed{C_1 + C_1 e^{-5/2} = 0}$

$y' = \frac{5}{4} C_2 e^{5t/4}$

$y'(-2) = 7 \quad \to \frac{5}{4} C_2 e^{-5/2} = 7 \quad \Rightarrow \quad \boxed{C_2 = \frac{28}{5} e^{5/2}}$

$\boxed{C_1 = -\frac{28}{5}}$

$\boxed{y(t) = -\frac{28}{5} + \frac{28}{5} e^{5t/4}}$

Exercise

Find the solution of the given initial value problem $\quad 4y'' + 12y' + 9y = 0, \quad y(0) = 2, \quad y'(0) = 1$

Solution

The characteristic equation: $\quad 4\lambda^2 + 12\lambda + 9 = (2\lambda + 3)^2 = 0 \quad \to \quad \boxed{\lambda_{1,2} = -\frac{3}{2}}$

$y(x) = \left(C_1 + C_2 x\right) e^{-3x/2}$

$y(0) = 2 \quad \to \quad \boxed{C_1 = 2}$

$y'(x) = \left(C_2 - \frac{3}{2} C_1 - \frac{3}{2} C_2 x\right) e^{-3x/2}$

$y'(0) = 1 \quad \to \quad C_2 - \frac{3}{2} C_1 = 1 \quad \Rightarrow \boxed{C_2 = 4}$

$\boxed{y(x) = (2 + 4x) e^{-3x/2}}$

Exercise

Find the solution of the given initial value problem $\quad 4y'' + 24y' + 37y = 0, \quad y(\pi) = 1, \quad y'(\pi) = 0$

Solution

The characteristic equation: $\quad 4\lambda^2 + 24\lambda + 37 = 0 \quad \to \quad \lambda_{1,2} = \frac{-24 \pm 4i}{8} \ \boxed{= -3 \pm \frac{1}{2} i}$

$y(t) = e^{-3t}\left(C_1 \cos\frac{1}{2}t + C_2 \sin\frac{1}{2}t\right)$

$y(\pi) = 1 \quad \to e^{-3\pi} C_2 = 1 \quad \Rightarrow \quad \boxed{C_2 = e^{3\pi}}$

$y' = e^{-3t}\left(-3C_1 \cos\frac{1}{2}t - 3C_2 \sin\frac{1}{2}t - \frac{1}{2} C_1 \sin\frac{1}{2}t + \frac{1}{2} C_2 \cos\frac{1}{2}t\right)$

$y'(\pi) = 0 \quad \to e^{-3\pi}\left(-3e^{3\pi} - \frac{1}{2} C_1\right) = 0 \quad \Rightarrow \quad \boxed{C_1 = -6e^{3\pi}}$

$$y(t)=e^{-3t}\left(-6e^{3\pi}\cos\frac{1}{2}t+e^{3\pi}\sin\frac{1}{2}t\right)$$

$$\boxed{=-6e^{3(\pi-t)}\cos\frac{t}{2}+e^{3(\pi-t)}\sin\frac{t}{2}}$$

Exercise

Find the solution of the given initial value problem $\quad 9y''+y=0;\quad y\left(\frac{\pi}{2}\right)=4,\quad y'\left(\frac{\pi}{2}\right)=0$

Solution

The characteristic equation: $\quad 9\lambda^2+1=0 \quad\rightarrow\quad \boxed{\lambda_{1,2}=\pm\frac{1}{3}i}$

$$\boxed{y(x)=C_1\cos\frac{x}{3}+C_2\sin\frac{x}{3}}$$

$$y\left(\frac{\pi}{2}\right)=4 \quad\rightarrow\quad \frac{\sqrt{3}}{2}C_1+\frac{1}{2}C_2=4$$

$$y'(x)=-\frac{1}{3}C_1\sin\frac{x}{3}+\frac{1}{3}C_2\cos\frac{x}{3}$$

$$y'\left(\frac{\pi}{2}\right)=0 \quad\rightarrow\quad -\frac{1}{2}C_1+\frac{\sqrt{3}}{2}C_2=0$$

$$\begin{cases}\sqrt{3}C_1+C_2=8\\ -C_1+\sqrt{3}C_2=0\end{cases}\qquad \Delta=\begin{vmatrix}\sqrt{3}&1\\-1&\sqrt{3}\end{vmatrix}=4\quad \Delta_1=\begin{vmatrix}8&1\\0&\sqrt{3}\end{vmatrix}=8\sqrt{3}\quad \Delta_2=\begin{vmatrix}\sqrt{3}&8\\-1&0\end{vmatrix}=8$$

$$\boxed{C_1=2\sqrt{3}\quad C_2=2}$$

$$\boxed{y(x)=2\sqrt{3}\cos\frac{x}{3}+2\sin\frac{x}{3}}$$

Exercise

Find the solution of the given initial value problem $\quad 9y''+\pi^2y=0;\quad y(3)=2,\quad y'(3)=-\pi$

Solution

The characteristic equation: $\quad 9\lambda^2+\pi^2=0 \quad\rightarrow\quad \boxed{\lambda_{1,2}=\pm\frac{\pi}{3}i}$

$$\boxed{y(x)=C_1\cos\frac{\pi}{3}x+C_2\sin\frac{\pi}{3}x}$$

$$y(3)=2 \quad\rightarrow\quad \boxed{C_1=-2}$$

$$y'(x)=-\frac{\pi}{3}C_1\sin\frac{\pi}{3}x+\frac{\pi}{3}C_2\cos\frac{\pi}{3}x$$

$$y'(3)=-\pi \quad\rightarrow\quad -\frac{\pi}{3}C_2=-\pi \quad\Rightarrow\quad \boxed{C_2=3}$$

$$\boxed{y(x)=-2\cos\frac{\pi}{3}x+3\sin\frac{\pi}{3}x}$$

Exercise

Find the solution of the given initial value problem $9y'' - 6y' + y = 0;\quad y(3) = -2,\quad y'(3) = -\frac{5}{3}$

Solution

The characteristic equation: $9\lambda^2 - 6\lambda + 1 = 0 = (3\lambda - 1)^2 \quad \to \quad \boxed{\lambda_{1,2} = \frac{1}{3}}$

$$\boxed{y(x) = (C_1 + C_2 x)e^{x/3}}$$

$$y(3) = -2 \quad \to \quad (C_1 + 3C_2)e = -2$$

$$y'(x) = \left(C_2 + \tfrac{1}{3}C_1 + \tfrac{1}{3}C_2 x\right)e^{x/3}$$

$$y'(3) = -\tfrac{5}{3} \quad \to \quad \left(2C_2 + \tfrac{1}{3}C_1\right)e = -\tfrac{5}{3}$$

$$\begin{cases} C_1 + 3C_2 = -\frac{2}{e} \\ C_1 + 6C_2 = -\frac{5}{e} \end{cases} \qquad \Delta = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \quad \Delta_1 = \begin{vmatrix} -\frac{2}{e} & 3 \\ -\frac{5}{e} & 6 \end{vmatrix} = \frac{3}{e} \quad \Delta_2 = \begin{vmatrix} 1 & -\frac{2}{e} \\ 1 & -\frac{5}{e} \end{vmatrix} = -\frac{3}{e}$$

$$\boxed{C_1 = \frac{1}{e} \quad C_2 = -\frac{1}{e}}$$

$$\boxed{y(x) = \frac{1}{e}(1 - x)e^{x/3}}$$

Exercise

Find the solution of the given initial value problem $9y'' + 6y' + 2y = 0\ ;\quad y(3\pi) = 0,\quad y'(3\pi) = \frac{1}{3}$

Solution

The characteristic equation: $9\lambda^2 + 6\lambda + 2 = 0 \quad \to \quad \lambda_{1,2} = \frac{-6 \pm 6i}{18} = -\frac{1}{3} \pm \frac{1}{3}i$

$$\boxed{y(x) = \left(C_1 \cos\frac{x}{3} + C_2 \sin\frac{x}{3}\right)e^{-x/3}}$$

$$y(3\pi) = 0 \quad \to \quad -C_1 e^{-\pi} = 0 \quad \Rightarrow \quad \boxed{C_1 = 0}$$

$$y'(x) = \left(-\tfrac{1}{3}C_1 \sin\tfrac{x}{3} + \tfrac{1}{3}C_2 \cos\tfrac{x}{3} - \tfrac{1}{3}C_1 \cos\tfrac{x}{3} - \tfrac{1}{3}C_2 \sin\tfrac{x}{3}\right)e^{-x/3}$$

$$y'(3\pi) = -\tfrac{5}{3} \quad \to \quad \left(-\tfrac{1}{3}C_2\right)e^{-\pi} = \frac{1}{3} \quad \Rightarrow \quad \boxed{C_2 = -e^{\pi}}$$

$$\boxed{y(x) = -e^{\pi} \sin\tfrac{x}{3}\, e^{-x/3}}$$

Exercise

Find the solution of the given initial value problem $9y'' - 12y' + 4y = 0, \quad y(0) = -1, \quad y'(0) = 1$

Solution

The characteristic equation: $9\lambda^2 - 12\lambda + 4 = (3\lambda - 2)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \frac{2}{3}}$

$$y(x) = \left(C_1 + C_2 x\right)e^{2x/3}$$

$$y(0) = -1 \quad \rightarrow \quad \boxed{C_1 = -1}$$

$$y'(x) = \left(C_2 + \frac{2}{3}C_1 + \frac{2}{3}C_2 x\right)e^{2x/3}$$

$$y'(0) = 1 \quad \rightarrow \quad C_2 + \frac{2}{3}C_1 = 1 \quad \Rightarrow \boxed{C_2 = \frac{5}{3}}$$

$$\boxed{y(x) = \left(-1 + \frac{5}{3}x\right)e^{2x/3}}$$

Exercise

Find the solution of the given initial value problem $12y'' + 5y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = -1$

Solution

The characteristic equation: $12\lambda^2 + 5\lambda - 2 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-5 \pm 11}{24} \boxed{= -\frac{2}{3}, \ \frac{1}{4}}$

$$y(x) = C_1 e^{2x/3} + C_2 e^{x/4}$$

$$y(0) = 1 \quad \rightarrow \quad C_1 + C_2 = 1$$

$$y'(x) = \frac{2}{3}C_1 e^{2x/3} + \frac{1}{4}C_2 e^{x/4}$$

$$y'(0) = -1 \quad \rightarrow \quad \frac{2}{3}C_1 + \frac{1}{4}C_2 = -1$$

$$\begin{cases} C_1 + C_2 = 1 \\ 8C_1 + 3C_2 = -12 \end{cases} \quad \rightarrow \quad \boxed{C_1 = -3, \ \ C_2 = 4}$$

$$\boxed{y(x) = -3e^{2x/3} + 4e^{x/4}}$$

Exercise

Find the solution of the given initial value problem $16y'' - 8y' + y = 0 \ ; \quad y(0) = -4, \quad y'(0) = 3$

Solution

The characteristic equation: $16\lambda^2 - 8\lambda + 1 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{8 \pm 0}{32} \boxed{= \frac{1}{4}}$

$$y(x) = \left(C_1 + C_2 x\right)e^{x/4}$$

$y(0) = -4 \quad \rightarrow \quad \boxed{C_1 = -4}$

$y'(x) = \left(C_2 + \frac{1}{4}C_1 + \frac{1}{4}C_2 x\right)e^{x/4}$

$y'(0) = 3 \quad \rightarrow C_2 - 1 = 3 \quad \Rightarrow \boxed{C_2 = 4}$

$\boxed{y(x) = (-4 + 4x)e^{x/4}}$

Exercise

Find the solution of the given initial value problem

$$25y'' + 20y' + 4y = 0\ ; \quad y(5) = 4e^{-2}, \quad y'(5) = -\frac{3}{5}e^{-2}$$

Solution

The characteristic equation: $\quad 25\lambda^2 + 20\lambda + 4 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-20 \pm 0}{50} = \boxed{-\frac{2}{5}}$

$y(x) = \left(C_1 + C_2 x\right)e^{-2x/5}$

$y(5) = 4e^{-2} \quad \rightarrow \quad \left(C_1 + 5C_2\right)e^{-2} = 4e^{-2} \quad \Rightarrow \boxed{C_1 + 5C_2 = 4}$

$y'(x) = \left(C_2 - \frac{2}{5}C_1 - \frac{2}{5}C_2 x\right)e^{-2x/5}$

$y'(5) = -\frac{3}{5}e^{-2} \quad \rightarrow \quad \left(C_2 - \frac{2}{5}C_1 - 2C_2\right)e^{-2} = -\frac{3}{5}e^{-2} \quad \Rightarrow \boxed{2C_1 + 5C_2 = 3}$

$\begin{cases} C_1 + 5C_2 = 4 \\ 2C_1 + 5C_2 = 3 \end{cases} \quad \rightarrow \quad \Delta = \begin{vmatrix} 1 & 5 \\ 2 & 5 \end{vmatrix} = -5 \quad \Delta_1 = \begin{vmatrix} 4 & 5 \\ 3 & 5 \end{vmatrix} = 5$

$\boxed{C_1 = -1, \quad C_2 = 1}$

$\boxed{y(x) = (-1 + x)e^{-2x/5}}$

Exercise

Find the solution of the given initial value problem

$$y''' + 12y'' + 36y' = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = -7$$

Solution

$\lambda^3 + 12\lambda^2 + 36\lambda = 0$

$\lambda(\lambda + 6)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_1 = 0, -6, -6}$

$y(x) = C_1 + \left(C_2 + C_3 x\right)e^{-6x}$

$y(0) = 0 \quad \rightarrow \quad C_1 + C_2 = 0$

$$y'(x) = \left(C_3 - 6C_2 - 6C_3 x\right)e^{-6x}$$

$$y'(0) = 1 \quad \rightarrow \ C_3 - 6C_2 = 1$$

$$y'' = \left(-12C_3 + 36C_2 + 36C_3 x\right)e^{-6x}$$

$$y''(0) = -7 \quad \rightarrow \ -12C_3 + 36C_2 = -7$$

$$\begin{cases} C_3 - 6C_2 = 1 \\ -12C_3 + 36C_2 = -7 \end{cases} \quad \rightarrow \quad C_3 = \frac{1}{6}, \ C_2 = -\frac{5}{36}, \ C_1 = \frac{5}{36}$$

$$\boxed{y(x) = \frac{5}{36} - \frac{5}{36}e^{-6x} + \frac{1}{6}xe^{-6x}}$$

Exercise

Find the solution of the given initial value problem

$$y''' + 2y'' - 5y' - 6y = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$$

Solution

The characteristic equation: $\lambda^3 + 2\lambda^2 - 5\lambda - 6 = 0 \quad \rightarrow \quad \boxed{\lambda_1 = -1}$

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & & -1 & -1 & 6 \\ \hline & 1 & 1 & -6 & \boxed{0} \end{array} \qquad \lambda^2 + \lambda - 6 = 0 \quad \Rightarrow \boxed{\lambda_3 = -3, \ \lambda_4 = 2}$$

$$y(x) = C_1 e^{-3x} + C_2 e^{-x} + C_3 e^{2x}$$

$$y(0) = 0 \quad \rightarrow \ \boxed{C_1 + C_2 + C_3 = 0}$$

$$y'(x) = -3C_1 e^{-3x} - C_2 e^{-x} + 2C_3 e^{2x}$$

$$y'(0) = 0 \quad \rightarrow \ \boxed{-3C_1 - C_2 + 2C_3 = 0}$$

$$y''(x) = 9C_1 e^{-3x} + C_2 e^{-x} + 4C_3 e^{2x}$$

$$y''(0) = 1 \quad \rightarrow \ \boxed{9C_1 + C_2 + 4C_3 = 1}$$

$$\begin{cases} C_1 + C_2 + C_3 = 0 \\ -3C_1 - C_2 + 2C_3 = 0 \\ 9C_1 + C_2 + 4C_3 = 1 \end{cases} \ \rightarrow \ \Delta = \begin{vmatrix} 1 & 1 & 1 \\ -3 & -1 & 2 \\ 9 & 1 & 4 \end{vmatrix} = 30 \quad \Delta_{C_1} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = 3 \quad \Delta_{C_2} = \begin{vmatrix} 1 & 0 & 1 \\ -3 & 0 & 2 \\ 9 & 1 & 4 \end{vmatrix} = 5$$

$$\boxed{C_1 = \frac{1}{10}, \ C_2 = \frac{1}{6}, \ C_3 = -\frac{4}{15}}$$

$$\boxed{y(x) = \frac{1}{10}e^{-3x} + \frac{1}{6}e^{-x} - \frac{4}{15}e^{2x}}$$

Exercise

The roots of the characteristic equation of a certain differential equation are:

$3, -5, 0, 0, 0, 0, -5, \ 2 \pm 3i$ and $2 \pm 3i$

Write a general solution of this homogeneous differential equation.

Solution

For $\lambda = 0, \ 0, \ 0, \ 0 \quad \Rightarrow \quad y_1 = C_1 + C_2 x + C_3 x^2 + C_4 x^3$

For $\lambda = 3 \quad \Rightarrow \quad y_2 = C_5 e^{3x}$

For $\lambda = -5, \ -5 \quad \Rightarrow \quad y_3 = C_6 e^{-5x} + C_7 x e^{-5x}$

For $\lambda = 2 \pm 3i, \ 2 \pm 3i$

$$\Rightarrow \quad y_4 = e^{2x}\left(C_8 \cos 3x + C_9 \sin 3x\right) + xe^{-5x}\left(C_{10} \cos 3x + C_{11} \sin 3x\right)$$

$$\begin{aligned} y(x) = &\ C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 e^{3x} + C_6 e^{-5x} + C_7 x e^{-5x} \\ &+ e^{2x}\left(C_8 \cos 3x + C_9 \sin 3x\right) + xe^{-5x}\left(C_{10} \cos 3x + C_{11} \sin 3x\right) \end{aligned}$$

Exercise

$y(x) = C_1 e^{2x} + C_2 e^{-2x} + C_3 \cos 2x + C_4 \sin 2x$ is the general solution of a homogeneous equation. What is the equation?

Solution

$\lambda_1 = 2, \quad \lambda_2 = -2, \quad \lambda_{3,4} = \pm 2i$

$(\lambda - 2)(\lambda + 2)(\lambda - 2i)(\lambda + 2i) = 0$

$\left(\lambda^2 - 4\right)\left(\lambda^2 + 4\right) = 0$

$\lambda^4 - 16 = 0 \quad \Rightarrow \quad \boxed{y^{(4)} - 16y = 0}$

Exercise

Show that the second differential equation $y'' + 4y = 0$

a) Has no solution to the boundary value $y(0) = 0, \quad y(\pi) = 1$

b) There are infinitely many solutions to the boundary value $y(0) = 0, \quad y(\pi) = 0$

Solution

The characteristic equation: $\lambda^2 + 4 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 2i}$

$\boxed{y(x) = C_1 \cos 2x + C_2 \sin 2x}$

a) $y(0) = 0 \quad \rightarrow \boxed{C_1 = 0}$

$y(\pi)=1 \quad \rightarrow \boxed{C_1=1}$

Therefore, there is no solution since $\cancel{C_1=0=1}$

b) $y(0)=0 \quad \rightarrow \boxed{C_1=0}$

$y(\pi)=0 \quad \rightarrow \boxed{C_1=0}$

$\boxed{y(x)=C_2\sin 2x}$

∴ There are infinite many solutions.

Exercise

Show that the general solution of the equation/ $\quad y''+Py'+Qy=0$

(where *P* and *Q* are constant) approaches 0 as $x\to\infty$ if and only if *P* and *Q* are both positive.

Solution

$$\lambda^2+P\lambda+Q=0$$

The solutions: $\lambda_{1,2}=\dfrac{-P\pm\sqrt{P^2-4Q}}{2}$

If $P^2-4Q<0 \quad \rightarrow \quad P<2\sqrt{Q}$ (P & Q are positives)

$$\lambda_{1,2}=-\frac{P}{2}\pm i\frac{\sqrt{4Q-P^2}}{2}$$

$$y(x)=e^{-Px/2}\left(C_1\cos\frac{1}{2}\sqrt{4Q-P^2}\,x+C_2\sin\frac{1}{2}\sqrt{4Q-P^2}\,x\right)$$

$$\lim_{x\to\infty}y(x)=\lim_{x\to\infty}\left[e^{-Px/2}\left(C_1\cos\frac{1}{2}\sqrt{4Q-P^2}\,x+C_2\sin\frac{1}{2}\sqrt{4Q-P^2}\,x\right)\right]$$

$$\boxed{=0} \qquad\qquad \lim_{x\to\infty}\left(e^{-Px/2}\right)=\lim_{x\to\infty}\left(\frac{1}{e^{Px/2}}\right)=\frac{1}{\infty}=0 \quad (P>0)$$

If $P^2-4Q=0 \quad \rightarrow \quad \lambda_{1,2}=-\frac{1}{2}P$

$$y(x)=\left(C_1+C_2x\right)e^{-Px/2}$$

$$\lim_{x\to\infty}y(x)=\lim_{x\to\infty}\left(C_1+C_2x\right)e^{-Px/2}\ \boxed{=0}$$

If $P^2-4Q>0 \quad \rightarrow \quad \lambda_{1,2}=\dfrac{-P\pm\sqrt{P^2-4Q}}{2}$

$$y(x)=C_1e^{\frac{-P-\sqrt{P^2-4Q}}{2}x}+C_2e^{\frac{-P+\sqrt{P^2-4Q}}{2}x}$$

$$\sqrt{P^2-4Q}<\sqrt{P^2}=P \quad \rightarrow \quad \frac{-P+\sqrt{P^2-4Q}}{2}<0$$

$$\lim_{x\to\infty} e^{\frac{-P+\sqrt{P^2-4Q}}{2}x}=0$$

$$\lim_{x\to\infty} y(x)=\lim_{x\to\infty}\left(C_1e^{\frac{-P-\sqrt{P^2-4Q}}{2}x}+C_2e^{\frac{-P+\sqrt{P^2-4Q}}{2}x}\right)$$

$$=0$$