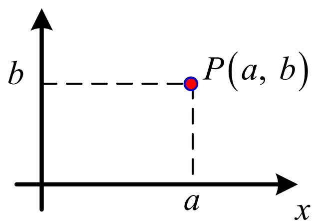
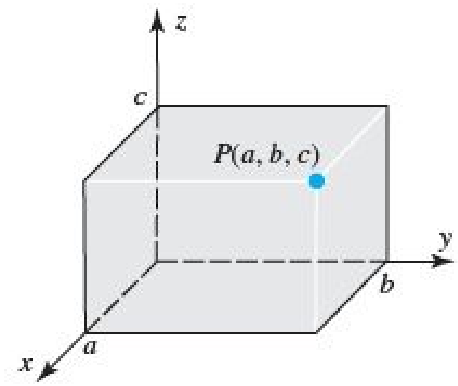
Section 2.7 – Coordinates, Basis and Dimension

Coordinate Systems in Linear Algebra

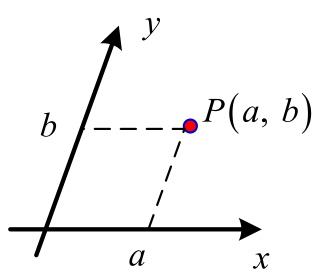
In ***analytic geometry***, we use rectangular coordinate systems to create a point either in 2-space or 3-space



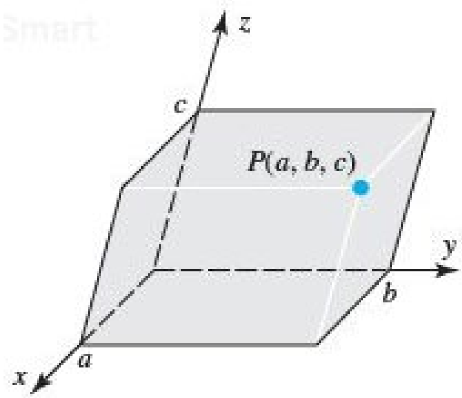
Coordinates of P in a rectangular coordinate system in 2-space



Coordinates of P in a rectangular coordinate system in 3-space

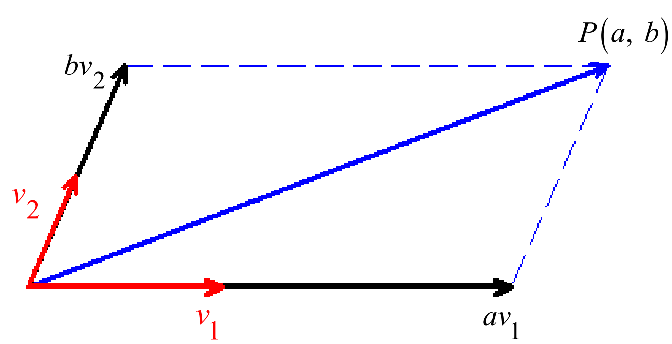


Coordinates of P in a nonrectangular coordinate system in 2-space



Coordinates of P in a nonrectangular coordinate system in 3-space

In ***linear algebra*** coordinate systems are commonly specified using vectors rather than coordinate axes.



Basis

Definition

If V is any vector space and $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a finite set of vectors in V, then S is called a ***basis*** for V if the following two conditions hold:

1. S is linearly independent.
2. S spans V.

Example

The columns of $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ produce the "standard basis" for $\mathbb{R}^2$.

Solution

The basis vectors: $\hat{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $\hat{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are independent. They span $\mathbb{R}^2$.

Example

The columns of any invertible n by n matrix give a basis for $\mathbb{R}^n$.

$$\underbrace{A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}}_{\text{Basis}} \quad \underbrace{A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{\text{Basis}} \quad \underbrace{A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}}_{\text{Not basis}}$$

Example

The standard unit vectors $\hat{e}_1 = (1, 0, 0, \cdots, 0)$, $\hat{e}_2 = (0, 1, 0, \cdots, 0)$, $\ldots$, $\hat{e}_n = (0, 0, \cdots, 0, 1)$ form a basis in $\mathbb{R}^n$.

Solution

1. $k_1\hat{e}_1 + k_2\hat{e}_2 + \ldots + k_n\hat{e}_n = 0 \rightarrow (k_1, k_2, \ldots, k_n) = (0, 0, \ldots, 0)$ it follows that $k_1 = k_2 = \ldots = k_n = 0$. That implies they are linearly independent.

2. Every vector $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ can be expressed as $\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + \ldots + v_n\hat{e}_n$ which is linear combination of $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n$. Thus, the standard vector span $\mathbb{R}^n$

Thus, they form a basis for $\mathbb{R}^n$ that we call the ***standard basis*** for $\mathbb{R}^n$.

Example

Show that the vectors $\vec{v}_1 = (1,\ 2,\ 1),\quad \vec{v}_2 = (2,\ 9,\ 0),\quad and\quad \vec{v}_3 = (3,\ 3,\ 4)$ form a basis in $\mathbb{R}^3$

Solution

1. We need to show that the vectors are linearly independent.

   $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0}$

   $k_1(1,\ 2,\ 1) + k_2(2,\ 9,\ 0) + k_3(3,\ 3,\ 4) = (0,\ 0,\ 0)$

   Which yields the homogeneous linear system

   $$\rightarrow \begin{cases} k_1 + 2k_2 + 3k_3 = 0 \\ 2k_1 + 9k_2 + 3k_3 = 0 \\ k_1 \qquad\ \ + 4k_3 = 0 \end{cases}$$

   $$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} = \underline{-1 \neq 0}$$

   $k_1 = 0,\quad k_2 = 0,\quad k_3 = 0$ has a trivial solution.

   The vectors $\vec{v}_1,\ \vec{v}_2,\ and\ \vec{v}_3$ are linearly independent.

2. Every vector can be expressed as $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{b}$ which is linear combination. Thus, the standard vector span $\mathbb{R}^3$

That proves that the vectors $\vec{v}_1,\ \vec{v}_2,\ and\ \vec{v}_3$ form a basis in $\mathbb{R}^3$

- The vectors $\vec{v}_1,\ \cdots,\ \vec{v}_n$ are a basis for $\mathbb{R}^n$ exactly when they are the ***columns*** of an ***n* by *n* *invertible matrix***. Thus $\mathbb{R}^n$ has infinitely many different bases.

- ***The pivots columns of A are a basis for its column space***. The pivot rows of *A* are a basis for its row space. So are the pivot rows of its echelon form *R*.

Example

Find bases for the column and row spaces of a rank two matrices: $R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution

Columns 1 and 3 are the pivot columns. They are a basis for the column space. It is a subspace of $\mathbb{R}^3$.

Column 2 and 4 are a basis for the same column space.

Coordinates Relative to a Basis

Theorem – Uniqueness of Basis Representation

If $S = \left\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \right\}$ is a basis for a vector space *V*, then every vector $\vec{v}$ in *V* can be expressed in the form $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$ in exactly one way.

Proof

Suppose that some vector $\vec{v}$ can be written as

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$$

Also $\vec{v} = k_1\vec{v}_1 + k_2\vec{v}_2 + \ldots + k_n\vec{v}_n$

Subtracting the second from the first equation

$$0 = \left(c_1 - k_1\right)\vec{v}_1 + \left(c_2 - k_2\right)\vec{v}_2 + \ldots + \left(c_n - k_n\right)\vec{v}_n$$

Since the right side of this equation is a linear combination of vectors in *S*, the linear independence of *S* implies that

$$c_1 - k_1 = 0, \quad c_2 - k_2 = 0, \ \ldots, \quad c_n - k_n = 0$$

That implies $c_1 = k_1, \quad c_2 = k_2, \ \ldots, \quad c_n = k_n$

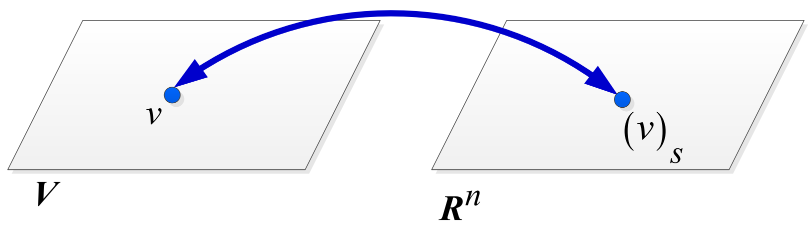
Thus, the two expressions for $\vec{v}$ are the same.

Definition

If $S=\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for a vector space *V*, and $\vec{v}=c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$ is the expression for a vector $\vec{v}$ in terms of the basis ***S***, then the scalars $c_1, c_2, \ldots, c_n$ are called coordinates of $\vec{v}$ relative to the basis *S*. The vector $(c_1, c_2, \ldots, c_n)$ in $\mathbb{R}^n$ constructed from these coordinates is called ***coordinate vector of v relative to S***; it is denoted by

$$(\vec{v})_S=(c_1, c_2, \ldots, c_n)$$

A *one-to-one correspondence*



Example

a) Given the vectors $\vec{v}_1=(1,\ 2,\ 1)$, $\vec{v}_2=(2,\ 9,\ 0)$, *and* $\vec{v}_3=(3,\ 3,\ 4)$ form a basis for $\mathbb{R}^3$. Find the coordinate vector of $\vec{v}=(5,\ -1,\ 9)$ relative to the basis $S=\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

b) Find the coordinate vector of $\vec{v}$ in $\mathbb{R}^3$ whose coordinate relative to *S* is $(\vec{v})_S=(-1,3,2)$.

<u>Solution</u>

a) To find $(\vec{v})_S$ we must first express $\vec{v}$ as a linear combination of the vectors in *S*;

$$\vec{v}=c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$$

$$(5,\ -1,\ 9)=c_1(1,\ 2,\ 1)+ c_2(2,\ 9,\ 0)+ c_3(3,\ 3,\ 4)$$

Which gives: $\begin{cases} c_1 + 2c_2 + 3c_3 = 5 \\ 2c_1 + 9c_2 + 3c_3 = -1 \\ c_1 \qquad\quad + 4c_3 = 9 \end{cases}$

Solving this system, we obtain $c_1=1,\quad c_2=-1,\quad c_3=2$.

Therefore $(\vec{v})_S=(1,\ -1,\ 2)$

b) $\vec{v}=(-1)\vec{v}_1+3\vec{v}_2+ 2\vec{v}_3$

$=(-1)(1,\ 2,\ 1)+3(2,\ 9,\ 0)+ 2(3,\ 3,\ 4)$

$=(11,\ 31,\ 7)$

Dimension

If $\vec{v}_1, \ldots, \vec{v}_m$ and $\vec{w}_1, \ldots, \vec{w}_n$ are both bases for the same vector space, then $m = n$.

Note

V may have many different bases, but they all must have the same number of elements.

Proof

Let **S** and **W** be bases of **V** can be written as a linear combination of vectors in **S**.

$$\begin{aligned} \vec{w}_1 &= a_{11}\vec{s}_1 + \ldots + a_{1m}s_m \\ \vec{w}_2 &= a_{21}\vec{s}_1 + \ldots + a_{2m}\vec{s}_m \\ &\vdots \\ \vec{w}_n &= a_{n1}\vec{s}_1 + \ldots + a_{nm}\vec{s}_m \end{aligned}$$

But since **V** is a basis, $c_1\vec{w}_1 + \ \ldots + c_n\vec{w}_n = 0 \ \Leftrightarrow \ c_i = 0$ (to be linearly independent, otherwise to be linearly dependent with at least 1 of t $c_i \neq 0$)

$$c_1\left(a_{11}s_1 + \ldots + a_{1m}s_m\right) + c_2\left(a_{21}s_1 + \ldots + a_{2m}s_m\right) + \ldots + c_n\left(a_{n1}s_1 + \ldots + a_{nm}s_m\right) = 0$$

$$\left(c_1a_{11} + c_2a_{21} + \ldots + c_na_{n1}\right)s_1 + \ldots + \left(c_1a_{1m} + c_2a_{2m} + \ldots + c_na_{nm}\right)s_m = 0$$

$$c_1a_{11} + c_2a_{21} + \ldots + c_na_{n1} = 0$$

$\Leftrightarrow$

$$c_1a_{1m} + c_2a_{2m} + \ldots + c_na_{nm} = 0$$

$\therefore \ S_i$'s linear independent.

Now all bases of **V** have some number of elements, we can define the dimension (is # of vectors in a basis)

Definition

The dimension of a finite-dimensional vector space V is denoted by dim(V) and is defined to be the number of vectors in a basis for V. in addition, the zero-vector space is defined to have dimension zero.

1. Dim(**V**) = # elements in basis. If **V** is finite.
2. If $V = \{\vec{0}\}$, then Dim(**V**) = 0, even though there is no basis.
3. Dim(**V**) may be infinite.

- $\dim(\mathbb{R}^n) = n$ **The standard basis has *n* vectors**.
- $\dim(P_n) = n + 1$ **The standard basis has *n* + 1 vectors**.
- $\dim(M_{mn}) = mn$ **The standard basis has *mn* vectors**.

Bases for Matrix Spaces *and Function Spaces*

Independence, basis, and dimension are not all restricted to column vectors.

- The dimension of the whole n by n space is n^2
- The dimension of the subspace of upper triangular matrices is $\frac{1}{2}n^2 + \frac{1}{2}n$
- The dimension of the subspace of diagonal matrices is n
- The dimension of the subspace of symmetric matrices is $\frac{1}{2}n^2 + \frac{1}{2}n$

Function Spaces

The equations:

$y'' = 0$	is solved by any linear function	$y = cx + d$
$y'' = -y$	is solved by any combination	$y = c\sin x + d\cos x$
$y'' = y$	is solved by any combination	$y = ce^x + de^{-x}$

Example

Find a basis for and the dimension of the solution space of the homogeneous system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 - 10x_4 \qquad + 15x_6 &= 0 \\ 2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

Solution

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & -10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{pmatrix} \quad \begin{matrix} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 5 & -10 & 0 & 15 & 0 \\ 0 & 0 & 4 & 8 & 0 & 18 & 0 \end{pmatrix} \quad \begin{matrix} \\ -R_2 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 5 & -10 & 0 & 15 & 0 \\ 0 & 0 & 4 & 8 & 0 & 18 & 0 \end{pmatrix} \quad \begin{matrix} R_1 + 2R_2 \\ \\ R_3 - 5R_2 \\ R_4 - 4R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & -20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \end{pmatrix} \quad \begin{matrix} \\ \\ -\frac{1}{20}R_3 \\ \frac{1}{6}R_4 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{matrix} R_1 - 4R_3 \\ R_2 - 2R_3 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{matrix} R_1 - 6R_4 \\ R_2 - 3R_4 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad x_1 = -3x_2 - 2x_5$$

The solution $\left(x_1, x_2, x_3, x_4, x_5, x_6\right) = \left(-3x_2 - 2x_5, x_2, 0, 0, x_5, 0\right)$

$$= x_2(-3, 1, 0, 0, 0, 0) + x_5(-2, 0, 0, 0, 1, 0)$$

The solution space has dimension 2.

Plus/Minus Theorem

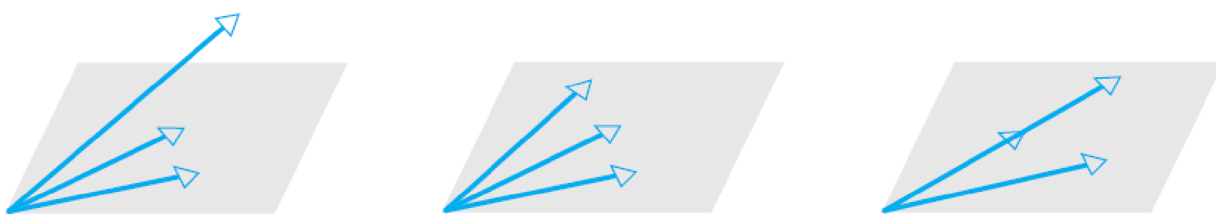
Theorem

Let S be a nonempty set of vector space V.

a) If S is a linearly independent set, and if $\vec{v}$ is a vector in V that is outside of span(S), the set $S \cup \{\vec{v}\}$ that results by inserting $\vec{v}$ into S is still linearly independent.

b) If $\vec{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\vec{v}\}$ denotes the set obtained by removing $\vec{v}$ from S, then S and $S - \{\vec{v}\}$ span the same space; that is,

$$span(S) = span(S - \{\mathbf{v}\})$$



The vector outside the plane can be adjoined to the other two without affecting their linear independence	Any of the vectors can be removed, and the remaining two still span the plane	Either of the collinear vectors can be removed, and the remaining two will still span the plane

Theorem

If W is a subspace of a finite-dimensional vector space V, then

a) W is finite-dimensional

b) $\dim(W) \le \dim(V)$

c) $W = V$ if and only if $\dim(W) = \dim(V)$

Exercises Section 2.7 – Coordinates, Basis and Dimension

1. Suppose $\vec{v}_1, \ldots, \vec{v}_n$ is a basis for $\mathbb{R}^n$ and the n by n matrix A is invertible. Show that $A\vec{v}_1, \ldots, A\vec{v}_n$ is also a basis for $\mathbb{R}^n$.

2. Consider the matrix $A = \begin{pmatrix} 1 & 0 & a \\ 2 & -1 & b \\ 1 & 1 & c \\ -2 & 1 & d \end{pmatrix}$

a) Which vectors $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ will make the columns of ***A*** linearly dependent?

b) Which vectors $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ will make the columns of ***A*** a basis for $\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : y + w = 0 \right\}$?

c) For $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$, compute a basis for the four subspaces.

3. Find a basis for $x - 2y + 3z = 0$ in $\mathbb{R}^3$.

Find a basis for the intersection of that plane with *xy* plane. Then find a basis for all vectors perpendicular to the plane.

4. **U** comes from ***A*** by subtracting row 1 from row 3:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad \textit{and} \quad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Find the bases for the two column spaces. Find the bases for the two row spaces. Find bases for the two nullspaces.

5. Write a 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives $c_1P_1 + \ldots + c_5P_5 = 0$, and check entries to prove c_i is zero.) The five permutation matrices are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

6. Choose three independent columns of $A = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix}$. Then choose a different three independent columns. Explain whether either of these choices forms a basis for $C(A)$.

7. Which of the following sets of vectors are bases for $\mathbb{R}^2$?

a) $\{(2,\ 1),\ (3,\ 0)\}$ *b)* $\{(0,\ 0),\ (1,\ 3)\}$

8. Which of the following sets of vectors are bases for $\mathbb{R}^3$?

a) $\{(1,\ 0,\ 0),\ (2,\ 2,\ 0),\ (3,\ 3,\ 3)\}$ *c)* $\{(2,\ -3,\ 1),\ (4,\ 1,\ 1),\ (0,\ -7,\ 1)\}$

b) $\{(3,\ 1,\ -4),\ (2,\ 5,\ 6),\ (1,\ 4,\ 8)\}$ *d)* $\{(1,\ 6,\ 4),\ (2,\ 4, -1),\ (-1,\ 2,\ 5)\}$

9. Let V be the space spanned by $\vec{v}_1 = \cos^2 x,\quad \vec{v}_2 = \sin^2 x,\quad \vec{v}_3 = \cos 2x$

a) Show that $S = \{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\}$ is not a basis for V.

b) Find a basis for V.

10. Find the coordinate vector of $\vec{w}$ relative to the basis $S = \{\vec{u}_1,\ \vec{u}_2\}$ for $\mathbb{R}^2$

a) $\vec{u}_1 = (1,\ 0),\quad \vec{u}_2 = (0,\ 1),\quad \vec{w} = (3,\ -7)$ *d)* $\vec{u}_1 = (1,\ -1),\quad \vec{u}_2 = (1,\ 1),\quad \vec{w} = (0,\ 1)$

b) $\vec{u}_1 = (2,\ -4),\quad \vec{u}_2 = (3,\ 8),\quad \vec{w} = (1,\ 1)$ *e)* $\vec{u}_1 = (1,\ -1),\quad \vec{u}_2 = (1,\ 1),\quad \vec{w} = (1,\ 1)$

c) $\vec{u}_1 = (1,\ 1),\quad \vec{u}_2 = (0,\ 2),\quad \vec{w} = (a,\ b)$

11. Find the coordinate vector of $\vec{v}$ relative to the basis $S = \{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\}$

a) $\vec{v} = (2,\ -1,\ 3),\quad \vec{v}_1 = (1,\ 0,\ 0),\quad \vec{v}_2 = (2,\ 2,\ 0),\quad \vec{v}_3 = (3,\ 3,\ 3)$

b) $\vec{v} = (5,\ -12,\ 3),\quad \vec{v}_1 = (1,\ 2,\ 3),\quad \vec{v}_2 = (-4,\ 5,\ 6),\quad \vec{v}_3 = (7,\ -8,\ 9)$

12. Show that $\{A_1, A_2, A_3, A_4\}$ is a basis for M_{22}, and express A as a linear combination of the basis vectors

a) $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $A = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$

b) $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

c) $A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$, $A_1 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

13. Consider the eight vectors

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

a) List all of the one-element. Linearly dependent sets formed from these.

b) What are the two-element, linearly dependent sets?

c) Find a three-element set spanning a subspace of dimension three, and dimension of two? One? Zero?

d) Which four-element sets are linearly dependent? Explain why.

(**14 – 18**) Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space

14. $\begin{cases} x_1 + x_2 - x_3 = 0 \\ -2x_1 - x_2 + 2x_3 = 0 \\ -x_1 \qquad + x_3 = 0 \end{cases}$

15. $\begin{cases} 3x_1 + x_2 + x_3 + x_4 = 0 \\ 5x_1 - x_2 + x_3 - x_4 = 0 \end{cases}$

16. $\begin{cases} x_1 - 3x_2 + x_3 = 0 \\ 2x_1 - 6x_2 + 2x_3 = 0 \\ 3x_1 - 9x_2 + 3x_3 = 0 \end{cases}$

17. $\begin{cases} x + y + z = 0 \\ 3x + 2y - 2z = 0 \\ 4x + 3y - z = 0 \\ 6x + 5y + z = 0 \end{cases}$

18. $\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 \qquad + 5x_3 = 0 \\ \qquad x_2 + x_3 = 0 \end{cases}$

19. If $AS = SA$ for the shift matrix S. Show that $\boldsymbol{A}$ must have this special form:

$$If \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$then\ A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

"The subspace of matrices that commute with the shift S has dimension ______."

20. Find bases for the following subspaces of $\mathbb{R}^3$

a) All vectors of the form $(a, b, c, 0)$

b) All vectors of the form (a, b, c, d), where $d = a + b$ and $c = a - b$.

c) All vectors of the form (a, b, c, d), where $a = b = c = d$.

21. Find a basis for the null space of A.

a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

22. Find a basis for the subspace of $\mathbb{R}^4$ spanned by the given vectors

a) $(1, 1, -4, -3),\ (2, 0, 2, -2),\ (2, -1, 3, 2)$

b) $(-1, 1, -2, 0),\ (3, 3, 6, 0),\ (9, 0, 0, 3)$

23. Determine whether the given vectors form a basis for the given vector space

a) $\vec{v}_1(3, -2, 1),\ \vec{v}_2(2, 3, 1),\ \vec{v}_3(2, 1, -3),\ in\ \mathbb{R}^3$

b) $\vec{v}_1 = (1, 1, 0, 0),\ \vec{v}_2 = (0, 1, 1, 0),\ \vec{v}_3 = (0, 0, 1, 1),\ \vec{v}_4 = (1, 0, 0, 1),\ for\ \mathbb{R}^4$

c) $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ M_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ M_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},\ M_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\ M_{22}$

24. Find a basis for, and the dimension of, the null space of the given matrix $\begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & -2 & -2 & 1 \end{bmatrix}$

25. Let $\mathbb{R}$ be the set of all real numbers and let $\mathbb{R}^+$ be the set of all positive real numbers. Show that $\mathbb{R}^+$ is a vector space over $\mathbb{R}$ under the addition

$$\alpha \oplus \beta = \alpha\beta \quad \alpha,\ \beta \in \mathbb{R}^+$$

And the scalar multiplication

$$a \odot \alpha = \alpha^a \quad \alpha \in \mathbb{R}^+,\ a \in \mathbb{R}$$

Find the dimension of the vector space. Is $\mathbb{R}^+$ also a vector space if the scalar multiplication is instead defined as

$$a \otimes \alpha = a^\alpha \quad \alpha \in \mathbb{R}^+,\ a \in \mathbb{R}?$$