Reduced Row Echelon Form

main diagonal ↔

/ diagonal

$$\begin{cases} 3x + y + 2z = 31 \\ x + y + 2z = 19 \\ x + 3y + 2z = 25 \end{cases}$$

matrix [ ] ( )

| | | determinant

$$\begin{array}{c} x \quad y \quad z = \\ \left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 3 & 1 & 2 & 31 \\ 1 & 3 & 2 & 25 \end{array}\right] \end{array} \rightarrow \begin{array}{l} R_2 - 3R_1 \leftarrow \\ R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 19 \\ 0 & -2 & -4 & -26 \\ 0 & 2 & 0 & 6 \end{array}\right]$$

$x + y + 2z = 19$ (2)

$-2y - 4z = -26$ (1)

$2y = 6 \rightarrow y = 3$

(1) $\rightarrow 4z = -2(3) + 26 \Rightarrow z = 5$

(2) $x = 19 - 3 - 10$

$= 6$

$\therefore (6, 3, 5)$

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$

$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1$$

$$5x_3 + 10x_4 + 15x_6 = 5$$

$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & +1 & +2 & 0 & +3 & +1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 5R_2 \\ R_4 - 4R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix} \begin{matrix} \\ \rightarrow (2) \\ \rightarrow (1) \\ \\ 6x_6 = 2 \end{matrix}$$

$$x_6 = \frac{1}{3}$$

(1) $x_3 + 2x_4 + 3x_6 = 1$

$x_3 = 1 - 2x_4 - 1$

$= -2x_4$

$x_3 + 3x_2 - 2x_3 + 2x_4 = 0$

(2) $x_1 = -3x_2 + 2x_3 - 2x_5$

$= -3x_2 - 4x_4 - 2x_5$

$$\therefore \left(-3x_2 - 4x_1 - 2x_5,\ x_2,\ -2x_4,\ x_4,\ x_5,\ \frac{1}{3}\right)$$

$$\begin{cases} 2x + 8y - z + w = 0 \\ 4x + 16y - 3z - w = -10 \\ -2x + 4y - z + 3w = -6 \\ -6x + 2y + 5z + w = 3 \end{cases}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 4 & 16 & -3 & -1 & -10 \\ -2 & 4 & -1 & 3 & -6 \\ -6 & 2 & 5 & 1 & 3 \end{array}\right] \quad \begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{matrix}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \quad \begin{matrix} 6 \\ 12 \end{matrix} R_4 - \frac{13}{6} R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \quad \begin{matrix} R_4 - \frac{13}{6} R_2 \\ 6R_4 - 13R_2 \end{matrix}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \end{array}\right.$$

$$\left[\begin{array}{cccc|c} 0 & 0 & +1 & +3 & +10 \\ 0 & 0 & 38 & -28 & 96 \end{array}\right] \quad R_4 - 38R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & 1 & 3 & 10 \\ 0 & 0 & 0 & -142 & -284 \\ x & y & z & w & \end{array}\right] \begin{array}{l} (4) \\ (3) \\ (2) \\ (1) \end{array}$$

(1) $w = \frac{-284}{-142} = 2$

(2) $z = 10 - 6 = 4$

(3) $\rightarrow y = \frac{1}{12}(8 - 8 - 6) = -\frac{1}{2}$

(4) $\rightarrow x = \frac{1}{2}(4 + 4 - 2) = 3$

Solution: $(3, -\frac{1}{2}, 4, 2)$

$AX = B \Rightarrow AX = 0$

1.3 Algebra of Matrices

$+$, $-$, $*$ (product)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{array}{l} \leftarrow \text{row 1 } R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array}$$

Columns ↑ ↑ ↑
C_1 C_2 C_3

$A = [a_{ij}]$ $\quad i = j = 3$

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- # rows = # columns
  matrix is square matrix

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad$ #R #C → size / order
2×3

Zero ~~#~~

Zero matrix: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$ Augmented matrix

$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ $\quad B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$AX = B$ matrix form

Equality of Matrices

$a_{ij} = b_{ij}$ "Same size" or order

Ex $\begin{pmatrix} 2 & 1 \\ p & q \end{pmatrix} = \begin{pmatrix} x & y \\ -1 & 0 \end{pmatrix}$

$2\times 2 \qquad 2\times 2$

$x=2,\ y=1,\ p=-1,\ q=0$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$ can't be true

$2\times 1 \neq 3\times 1$

Addition & Subtraction

Same (size) order $m \times n$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 4 & 4 \\ 9 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 7 & 8 \\ 9 & 9 \end{bmatrix}$$

scalar multiplication

$$k\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{pmatrix}$$

Multiplication (product)