Section 4.8 – Divergence Theorem

Divergence Theorem

The Divergence Theorem is the 3-dimensional version of the flux form of Green's Theorem.
If *R* is a region in the *xy*-plane, *C* is the simple closed piecewise-smooth oriented boundary of *R*, and $\vec{F} = \langle f,\ g \rangle$ is a vector field, Green's Theorem says that

$$\underbrace{\oint_C \vec{F} \bullet \vec{n}\ dS}_{\text{flux across } C} = \iint_R \underbrace{\left(f_x + g_y\right)}_{\text{divergence}} dA$$

Theorem

Let $\vec{F}$ be a vector field whose components have continuous first partial derivatives in a connected and simply connected region *D* enclosed by a smooth oriented surface *S*. Then

$$\iint_S \vec{F} \bullet \vec{n}\ dS = \iiint_D \nabla \bullet \vec{F}\ dV$$

Where $\vec{n}$ is the unit outward normal vector on *S*.

Example

Consider the radial field $\vec{F} = \langle x,\ y,\ z \rangle$ and let *S* be the sphere $x^2 + y^2 + z^2 = a^2$ that encloses the region *D*. Assume $\vec{n}$ is the outward normal vector on the sphere. Evaluate both integrals of the Divergence Theorem.

Solution

The divergence of $\vec{F}$:

$$\nabla \bullet \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 3$$

$$\iiint_D \nabla \bullet \vec{F}\ dV = \iiint_D (3)\ dV$$

$$= 3 \cdot volume(D)$$

$$= 3 \cdot \frac{4\pi}{3} a^3$$

$$= 4\pi a^3$$

$$\vec{r} = \langle x,\ y,\ z\rangle \qquad\qquad R = \{(\phi,\ \theta):\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}$$

$$= \langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi\rangle$$

$$t_\phi = \langle a\cos\phi\cos\theta,\ a\cos\phi\sin\theta,\ -a\sin\phi\rangle$$

$$t_\theta = \langle -a\sin\phi\sin\theta,\ a\sin\phi\cos\theta,\ 0\rangle$$

The required vector normal to the surface is

$$t_\phi \times t_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \left(a^2\sin^2\phi\cos\theta\right)\hat{i} + \left(a^2\sin^2\phi\sin\theta\right)\hat{j} + \left(a^2\sin\phi\cos\phi\cos^2\theta + a^2\sin\phi\cos\phi\sin^2\theta\right)\hat{k}$$

$$= \left\langle a^2\sin^2\phi\cos\theta,\ \ a^2\sin^2\phi\sin\theta,\ \ a^2\sin^2\phi\cos\phi\right\rangle$$

$$\vec{F} \bullet \left(t_\phi \times t_\theta\right) = \langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi\rangle \bullet \left\langle a^2\sin^2\phi\cos\theta,\ a^2\sin^2\phi\sin\theta,\ a^2\sin^2\phi\cos\phi\right\rangle$$

$$= a^3\sin^3\phi\cos^2\theta + a^3\sin^3\phi\sin^2\theta + a^3\sin\phi\cos^2\phi \qquad\qquad \cos^2\theta + \sin^2\theta = 1$$

$$= a^3\sin\phi\left(\sin^2\phi + \cos^2\phi\right)$$

$$= \boxed{a^3\sin\phi}$$

$$\iint\limits_S \vec{F} \bullet \vec{n}\ dS = \iint\limits_R \nabla\vec{F} \bullet \left(t_\phi \times t_\theta\right)\ dA$$

$$= \int_0^{2\pi} d\theta\ \int_0^{\pi} a^3\sin\phi\ d\phi$$

$$= a^3\ \left(\theta\ \Big|_0^{2\pi}\ \ \left(-\cos\theta\ \Big|_0^{\pi}\right.\right.$$

$$\boxed{= 4\pi a^3}$$

∴ The two integral of the Divergence Theorem are equal.

Example

Consider the rotation field:

$$\begin{aligned}\vec{F} &= \vec{a}\times\vec{r}\\ &= \langle 1,\ 0,\ 1\rangle \times \langle x,\ y,\ z\rangle\\ &= \langle -y,\ \ x-z,\ \ y\rangle\end{aligned}$$

Let S be the sphere $x^2+y^2+z^2=a^2$ for $z\ge 0$, together with its base in the *xy*-plane.
Find the net outward flux across S.

Solution

$$\begin{aligned}\nabla\cdot\vec{F} &= \frac{\partial}{\partial x}(-y)+\frac{\partial}{\partial y}(x-z)+\frac{\partial}{\partial z}(y)\\ &= 0\end{aligned}$$

$\therefore$ The flux across the hemisphere is zero.

However, with the Divergence Theorem, radial fields are interesting and have many physical applications

Example

Find the net outward flux of the field $\vec{F}=xyz\langle 1,\ 1,\ 1\rangle$ across the boundaries of the cube $D=\{(x,y,z):\ 0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1\}$

Solution

$$\begin{aligned}\nabla\cdot\vec{F} &= \frac{\partial}{\partial x}(xyz)+\frac{\partial}{\partial y}(xyz)+\frac{\partial}{\partial z}(xyz)\\ &= yz+xz+xy\end{aligned}$$

$$\begin{aligned}\iiint\limits_D \nabla\cdot\vec{F}\ dV &= \int_0^1\int_0^1\int_0^1 (yz+xz+xy)\,dxdydz\\ &= \int_0^1\int_0^1 \left(yzx+\frac{1}{2}x^2z+\frac{1}{2}x^2y\ \Big|_0^1\right. dydz\\ &= \int_0^1\int_0^1 \left(yz+\frac{1}{2}z+\frac{1}{2}y\right)dydz\\ &= \int_0^1 \left(\frac{1}{2}y^2z+\frac{1}{2}zy+\frac{1}{4}y^2\ \Big|_0^1\right. dz\end{aligned}$$

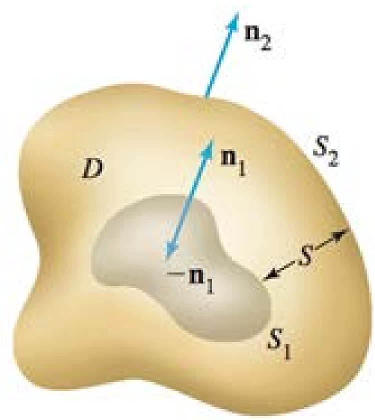
$$= \int_0^1 \left(z + \frac{1}{4} \right) dz$$

$$= \frac{1}{2} z^2 + \frac{1}{4} z \Big|_0^1$$

$$\boxed{= \frac{3}{4}}$$

Divergence *Theorem* for Hollow Regions

Suppose the vector field $\vec{F}$ satisfies the conditions of the Divergence Theorem on a region D bounded by two smooth oriented surfaces S_1 and S_2, where S_1 lies within S_2.



Let S be the entire boundary of $D\left(S = S_1 \cup S_2\right)$ and let $\vec{n}_1$ and $\vec{n}_2$ be the outward unit normal vectors for S_1 and S_2, respectively.

$$\iiint_D \nabla \bullet \vec{F}\, dV = \iint_S \vec{F} \bullet \vec{n}\, dS$$

$$= \iint_{S_2} \vec{F} \bullet \vec{n}_2\, dS - \iint_{S_1} \vec{F} \bullet \vec{n}_1\, dS$$

Interpretation of the Divergence Using Mass Transport

Suppose that $\vec{v}$ is the velocity field of a material, such as water or molasses, and ρ is its constant density. The vector field $\vec{F} = \rho\vec{v} = \langle f,\ g,\ h \rangle$ describes the ***mass transport*** of the material, with units of

$\frac{mass}{vol.} \times \frac{length}{time} = \frac{mass}{area - time}$ typical units of mass transport are $g / m^2 / s$.

This means that $\vec{F}$ gives the mass material flowing past a point (in each of the three coordinates direction) per unit of surface area per unit of time.

When $\vec{F}$ is multiplied by an area, the result in the *flux*, with units of mass/unit time.

Example

Consider the inverse square vector field $\quad \vec{F} = \frac{\vec{r}}{|\vec{r}|^3} = \frac{\langle x,\ y,\ z \rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$

a) Find the net outward flux of $\vec{F}$ across the surface of the region $D = \left\{(x, y, z):\ a^2 \le x^2 + y^2 + z^2 \le b^2\right\}$ that lies between concentric spheres with radii *a* and *b*.

b) Find the outward flux of $\vec{F}$ across any sphere that encloses the origin,

Solution

a) $\nabla \bullet \vec{F} = \nabla \bullet \left(\frac{\vec{r}}{|\vec{r}|^3}\right)$

$$= -\nabla \bullet \vec{r}\,\frac{1}{|\vec{r}|^3} - \vec{r} \bullet \nabla \frac{1}{|\vec{r}|^3}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$= -3x\left(x^2 + y^2 + z^2\right)^{-5/2}$$

$$= -\frac{3x}{|\vec{r}|^5}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$= -3y\left(x^2 + y^2 + z^2\right)^{-5/2}$$

$$= -\frac{3y}{|\vec{r}|^5}$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial}{\partial z}\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$= -3z\left(x^2 + y^2 + z^2\right)^{-5/2}$$

$$= -\frac{3z}{|\vec{r}|^5}$$

$$\nabla\frac{1}{|\vec{r}|^3} = -3\frac{x\hat{i} + y\hat{j} + z\hat{k}}{|\vec{r}|^5}$$

$$= -3\frac{\vec{r}}{|\vec{r}|^5}$$

$$\nabla \bullet \vec{F} = \nabla \bullet \left(-\frac{\vec{r}}{|\vec{r}|^3}\right)$$

$$= -\frac{1}{|\vec{r}|^3}\nabla \bullet \vec{r} + 3\vec{r} \bullet \frac{\vec{r}}{|\vec{r}|^5}$$

$$= -\frac{3}{|\vec{r}|^3} + \frac{3}{|\vec{r}|^3}$$

$$\underline{= 0}$$

Let $S = S_2\left(\text{with radius } b \text{ larger}\right) \cup S_1\left(\text{with radius } a\right)$

Because $\iiint\limits_D \nabla \bullet \vec{F}\ dV = 0$, the divergence Theorem implies that

$$\iint\limits_S \vec{F} \bullet \vec{n}\ dS = \iint\limits_{S_2} \vec{F} \bullet \vec{n}_2\ dS - \iint\limits_{S_1} \vec{F} \bullet \vec{n}_1\ dS = 0$$

Therefore, the net flux across S is zero.

b) $$\iint\limits_{S_2} \vec{F} \bullet \vec{n}_2\ dS = \iint\limits_{S_1} \vec{F} \bullet \vec{n}_1\ dS$$

$$\iint\limits_{S_1} \vec{F} \bullet \vec{n}_1\ dS = \iint\limits_{S_1} \frac{\vec{r}}{|\vec{r}|^3} \bullet \frac{\vec{r}}{|\vec{r}|}\ dS$$

$$= \iint\limits_{S_1} \frac{\vec{r}^2}{|\vec{r}|^4}\ dS \qquad \left(|\vec{r}| = a\right)$$

$$= \iint_{S_1} \frac{1}{a^2}\, dS \qquad \qquad Surface\ Area = 4\pi a^2$$

$$= \frac{4\pi a^2}{a^2}$$

$$\underline{= 4\pi}$$

$$\iint_{S_2} \vec{F} \bullet \vec{n}_2\, dS = \iint_{S_2} \frac{\vec{r}}{|\vec{r}|^3} \bullet \frac{\vec{r}}{|\vec{r}|}\, dS$$

$$= \frac{4\pi b^2}{b^2}$$

$$\underline{= 4\pi}$$

The flux of the inverse square field across any surface enclosing the origin is 4π .

***Gauss'* Law**

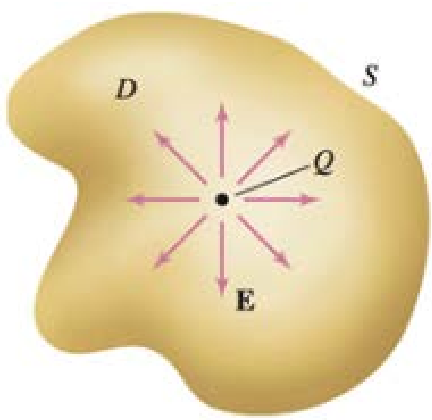
Applying the Divergence Theorem to electric fields leads to one of the fundamental laws of physics. The electric field due to a point charge Q located at the origin is given by the inverse square law.

$$\vec{E}(x,\ y,\ z) = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}}{|\vec{r}|^3}$$

Where $\vec{r}(x,\ y,\ z)$ and ε_0 is a physical constant called the ***permittivity of free square***.

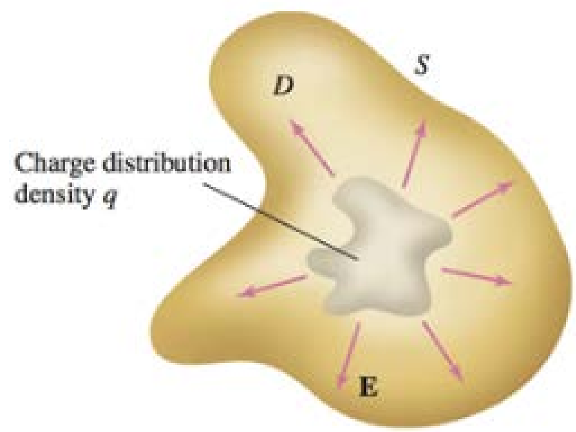
This is one statement of Gauss' Law: If S is a surface that encloses a point charge Q, then the flux of the electric field across S is

$$\iint_S \boldsymbol{E} \bullet \vec{n}\ dS = \frac{Q}{\varepsilon_0}$$



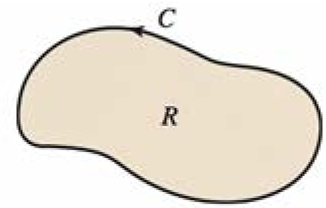
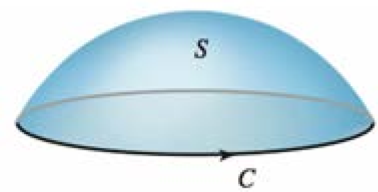
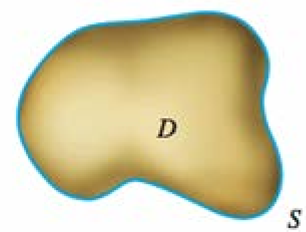
Gauss' Law: Flux of electric field across S due to point charge Q =

$$\iint_S \boldsymbol{E} \bullet \vec{n}\ dS = \frac{Q}{\varepsilon_0}$$



Gauss' Law: Flux of electric field across S due to charge distribution q =

$$\iint_S \boldsymbol{E} \bullet \vec{n}\ dS = \frac{1}{\varepsilon_0}\iiint_D q\ dV$$

Fundamental Theorem of Calculus	$\int_a^b f'(x)\,dx = f(b) - f(a)$	
Fundamental Theorem of Line Integrals	$\int_C \nabla f \bullet d\mathbf{x} = f(B) - f(A)$	
Green's Theorem (Circulation Form)	$\iint\limits_R \left(g_x - f_y\right) dA = \oint\limits_C fdx + gdy$	
Stokes' Theorem	$\iint\limits_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS = \oint_C \vec{F} \bullet d\vec{r}$	
Divergence Theorem	$\iint\limits_S \vec{F} \bullet \vec{n}\ dS = \iiint\limits_D \nabla \bullet \vec{F}\ dV$	

Exercises *Section* 4.8 – Divergence Theorem

(1–4) Evaluate both integrals of the Divergence Theorem for the following vector fields and region. Check for agreement.

1. $\vec{F}=\langle 2x,\ 3y,\ 4z\rangle \quad D=\left\{(x,y,z):\ x^2+y^2+z^2\leq 4\right\}$

2. $\vec{F}=\langle -x,\ -y,\ -z\rangle \quad D=\{(x,y,z):\ |x|\leq 1,\ |y|\leq 1,\ |z|\leq 1\}$

3. $\vec{F}=\langle z-y,\ x,\ -x\rangle \quad D=\left\{(x,y,z):\ \dfrac{x^2}{4}+\dfrac{y^2}{8}+\dfrac{z^2}{12}\leq 1\right\}$

4. $\vec{F}=\left\langle x^2,\ y^2,\ z^2\right\rangle \quad D=\{(x,y,z):\ |x|\leq 1,\ |y|\leq 2,\ |z|\leq 3\}$

5. Find the net outward flux of the field $\vec{F}=\langle 2z-y,\ x,\ -2x\rangle$ across the sphere of radius 1 centered at the origin.

6. Find the net outward flux of the field $\vec{F}=\langle bz-cy,\ cx-az,\ ay-bx\rangle$ across any smooth closed surface $\mathbb{R}^3$, where *a*, *b*, and *c* are constants.

7. Find the net outward flux of the field $\vec{F}=\langle z-y,\ x-z,\ y-x\rangle$ across the boundary of the cube $\{(x,\ y,\ z):\ |x|\leq 1,\ \ |y|\leq 1\ \ |z|\leq 1\}$

(8–47) Use the Divergence Theorem to compute the net outward flux of the following fields across the given surface *S or D*.

8. $\vec{F}=\langle x,\ -2y,\ 3z\rangle$; *S* is the sphere $\left\{(x,y,z):\ x^2+y^2+z^2=6\right\}$

9. $\vec{F}=\left\langle x^2,\ 2xz,\ y^2\right\rangle$; *S* is surface of the cube cut from the first octant by the planes *x* = 1, *y* = 1, and $z=1$

10. $\vec{F}=\langle x,\ 2y,\ z\rangle$; *S* is boundary of the tetrahedron in the first octant formed by the plane $x+y+z=1$

11. $\vec{F}=\left\langle x^2,\ y^2,\ z^2\right\rangle$; *S* is the sphere $\left\{(x,y,z):\ x^2+y^2+z^2=25\right\}$

12. $\vec{F}=\left\langle y-2x,\ x^3-\text{y},\ y^2-z\right\rangle$; *S* is the sphere $\left\{(x,y,z):\ x^2+y^2+z^2=4\right\}$

13. $\vec{F}=\langle x,\ y,\ z\rangle$; *S* is the surface of the paraboloid $z=4-x^2-y^2$, for $z\geq 0$, plus its base in the *xy*-plane.

14. $\vec{F}=\langle x,\ y,\ z\rangle$; *S* is the surface of the cone $z^2=x^2+y^2$, for $0\leq z\leq 4$, plus its top surface in the plane $z=4$

15. $\vec{F} = \langle z - x,\ x - y,\ 2y - z\rangle$; D is the region between the spheres of radius 2 and 4 centered at origin.

16. $\vec{F} = \vec{r}\,|\vec{r}| = \langle x,\ y,\ z\rangle\sqrt{x^2 + y^2 + z^2}$; D is the region between the spheres of radius 1 and 2 centered at origin.

17. $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|} = \dfrac{\langle x,\ y,\ z\rangle}{\sqrt{x^2 + y^2 + z^2}}$; D is the region between the spheres of radius 1 and 2 centered at origin.

18. $\vec{F} = \langle z - y,\ x - z,\ 2y - x\rangle$; $D = \{(x, y, z):\ 1 \le |x| \le 3,\ 1 \le |y| \le 3,\ 1 \le |z| \le 3\}$ is the region between two cubes

19. $\vec{F} = \langle y + z,\ x + z,\ x + y\rangle$; S consists of the faces of the cube $\{(x,\ y,\ z):\ \ |x| \le 1,\ \ |y| \le 1\ \ |z| \le 1\}$

20. $\vec{F} = \left\langle x^2,\ -y^2,\ z^2\right\rangle$; D is the region in the first octant between the planes $z = 4 - x - y$ and $z = 2 - x - y$

21. $\vec{F} = \langle x,\ 2y,\ 3z\rangle$; D is the region between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ for $0 \le z \le 8$

22. $\vec{F} = \langle -x,\ x - y,\ x - z\rangle$ across S is the surface of the cube cut from the first octant by the planes $x = 1,\ y = 1,\ \textit{and}\ z = 1$

23. $\vec{F} = \frac{1}{3}\left\langle x^3,\ y^3,\ z^3\right\rangle$ across S is the sphere $\left\{(x,\ y,\ z):\ x^2 + y^2 + z^2 = 9\right\}$

24. $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^3} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ across D is the region between two spheres of radius 1 and 2 centered at $(5,\ 5,\ 5)$

25. $\vec{F} = \left\langle x^2,\ y^2,\ z^2\right\rangle$; S is the cylinder $\left\{(x, y, z):\ x^2 + y^2 = 4,\ 0 \le z \le 8\right\}$

26. $\vec{F} = \left\langle x^2 e^y \cos z,\ -4xe^y \cos z,\ 2xe^y \sin z\right\rangle$; S is the boundary of the ellipsoid $\dfrac{x^2}{4} + y^2 + z^2 = 1$

27. $\vec{F} = \langle -yz,\ xz,\ 1\rangle$; S is the boundary of the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{4} + z^2 = 1$

28. $\vec{F} = \langle x \sin y,\ -\cos y,\ z \sin y\rangle$; S is the boundary of the region bounded by the planes $x = 1$, $y = 0$, $y = \frac{\pi}{2}$, $z = 0$, and $z = x$

29. $\vec{F} = \left\langle x^2 + x \sin y,\ y^2 + 2\cos y,\ z^2 + z \sin y\right\rangle$ across the surface S that is the boundary of the prism bounded by the planes $y = 1 - x,\ x = 0,\ y = 0,\ z = 0,\ z = 4$

30. $\vec{F} = \langle x,\ -2y,\ 4z \rangle$ out of the sphere S with $x^2 + y^2 + z^2 = a^2, \quad a > 0$

31. $\vec{F} = \left\langle ye^z,\ x^2e^z,\ xy \right\rangle$ out of the sphere S with $x^2 + y^2 + z^2 = a^2, \quad a > 0$

32. $\vec{F} = \left\langle x^2 + y^2,\ y^2 - z^2,\ z \right\rangle$ of the sphere S with $x^2 + y^2 + z^2 = a^2, \quad a > 0$

33. $\vec{F} = \left\langle x^3,\ \ 3yz^2,\ \ 3y^2z + x^2 \right\rangle$ out of the sphere S with $x^2 + y^2 + z^2 = a^2, \quad a > 0$

34. $\vec{F} = \left\langle 2z,\ x,\ y^2 \right\rangle$; S is the surface of the paraboloid $z = 4 - x^2 - y^2$, for $z \ge 0$, and the *xy*-plane.

35. $\vec{F} = \left\langle x,\ y^2,\ z \right\rangle$; S is the solid region bounded by the coordinate planes and the plane $2x + 2y + z = 6$.

36. $\vec{F} = \left\langle x^2 + \sin z,\ xy + \cos z,\ e^y \right\rangle$; S is the solid region bounded by the cylinder $x^2 + y^2 = 4$, the plane $x + z = 6$, and the *xy*-plane.

37. $\vec{F} = \left\langle 2x^3,\ \ 2y^3,\ \ 2z^3 \right\rangle$ out of the sphere S with $x^2 + y^2 + z^2 = 4$

38. $\vec{F} = \langle x,\ \ y,\ \ z \rangle$ out of the sphere S with $x^2 + y^2 + z^2 = 1$

39. $\vec{F} = \langle z,\ \ y,\ \ x \rangle$ out of the sphere S with $x^2 + y^2 + z^2 = 1$

40. $\vec{F} = \left\langle xy,\ \ y^2 + e^{xz^2},\ \ \sin(xy) \right\rangle$; S is the solid region bounded by the cylinder $z = 1 - x^2$, the planes $y + z = 2$, $z = 0$, and $y = 0$.

41. $\vec{F} = \left\langle x^2,\ y^2,\ z^2 \right\rangle$; across the boundary of an ellipsoid $x^2 + y^2 + 4(z-1)^2 \le 4$

42. $\vec{F} = \left\langle x^2,\ y^2,\ z^2 \right\rangle$; across the boundary of the tetrahedron $x + y + z \le 3 \quad \& \quad x,\ y,\ z \ge 0$

43. $\vec{F} = \left\langle x^2,\ y^2,\ z^2 \right\rangle$; across the boundary of the cylinder $x^2 + y^2 \le 2y \quad \& \quad 0 \le z \le 4$

44. $\vec{F} = \left\langle x^2,\ y^2,\ z^2 \right\rangle$; across the boundary of a ball $(x-2)^2 + y^2 + (z-3)^2 \le 9$

45. $\vec{F} = \left\langle x^4,\ \ -x^3z^2,\ \ 4xy^2z \right\rangle$; across the boundary of the cylinder $x^2 + y^2 = 1$ and the planes $z = x + 2 \quad \& \quad z = 0$

46. $\vec{F} = \left\langle x^2z^3,\ 2xyz^3,\ xz^4 \right\rangle$; S is the surface of the box with vertices $(\pm 1,\ \ \pm 2,\ \ \pm 3)$.

47. $\vec{F} = \left\langle z\tan^{-1}\left(y^2\right),\ z^3\ln\left(x^2+1\right),\ z\right\rangle$; across the part of the paraboloid $x^2+y^2+z=2$ that lies above the plane $z=1$ and is oriented upward.

48. Prove that $\nabla\left(\dfrac{1}{|\vec{r}|^4}\right) = -\dfrac{4\vec{r}}{|\vec{r}|^6}$ and use the result to prove that $\nabla\bullet\nabla\left(\dfrac{1}{|\vec{r}|^4}\right) = \dfrac{12}{|\vec{r}|^6}$

49. Consider the radial vector field $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2+y^2+z^2\right)^{p/2}}$. Let S be the sphere of radius a at the origin.

a) Use the surface integral to show that the outward flux of $\vec{F}$ across S is $4\pi a^{3-p}$. Recall that the unit normal to sphere is $\dfrac{\vec{r}}{|\vec{r}|}$.

b) For what values of p does $\vec{F}$ satisfy the conditions of the Divergence Theorem? For these values of p, use the fact the $\nabla\bullet\vec{F} = \dfrac{3-p}{|\vec{r}|^p}$ to compute the flux around S using the Divergence Theorem.

50. Consider the radial vector field $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|} = \dfrac{\langle x,\ y,\ z\rangle}{\sqrt{x^2+y^2+z^2}}$.

a) Evaluate a surface integral to show that $\iint\limits_S \vec{F}\bullet\vec{n}\,dS = 4\pi a^2$, where S is the surface of a sphere of radius a centered at the origin.

b) Note that the first partial derivatives of the components of $\vec{F}$ are undefined at the origin, so the Divergence Theorem does not apply directly. Nevertheless, the flux across the sphere as computed in part (a) is finite. Evaluate the triple integral of the Divergence Theorem as an improper integral as follows. Integrate $div\vec{F}$ over the region between two spheres of radius a and $0<\varepsilon<a$. Then let $\varepsilon\to 0^+$ to obtain the flux computed in part (a).

51. The electric field due to a point charge Q is $\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_0} \bullet \frac{\vec{r}}{|\vec{r}|^3}$, where $\vec{r} = \langle x,\ y,\ z\rangle$ and ε_0 is a constant

a) Show that the flux of the field across a sphere of radius a centered at the origin is $\iint_S \boldsymbol{E} \bullet \vec{n}\ dS = \frac{Q}{\varepsilon_0}$

b) Let S be the boundary of the origin between two spheres centered of radius a and b with $a < b$. Use the Divergence Theorem to show that the net outward flux across S is zero.

c) Suppose there is a distribution of charge within a region D. Let $q(x,\ y,\ z)$ be the charge density (charge per unit volume). Interpret the statement that

$$\iint_S \boldsymbol{E} \bullet \vec{n}\ dS = \frac{1}{\varepsilon_0} \iiint_D q(x,\ y,\ z)\ dV$$

d) Assuming $\boldsymbol{E}$ satisfies the conditions of the Divergence Theorem, conclude from part (*c*) that $\nabla \bullet \boldsymbol{E} = \frac{q}{\varepsilon_0}$

e) Because the electric force is conservative, it has a potential function ϕ. From part (*d*) conclude that $\nabla^2\phi = \nabla \bullet \nabla\phi = \frac{q}{\varepsilon_0}$

(52–55) ***Fourier's Law*** of heat transfer (or heat conduction) states that the heat flow vector $\vec{F}$ at a point is proportional to the negative gradient of the temperature that is, $\vec{F} = -k\nabla T$, which means that heat energy flows from hot regions to cold region. The constant $k > 0$ is called the *conductivity*, which has metric units of $J\ /\ m\text{-}s\text{-}K$. A temperature function for a region D is given. Find the net outward heat flux $\iint\limits_S \vec{F} \bullet \vec{n}\ dS = -k \iint\limits_S \nabla T \bullet \vec{n}\ dS$ across the boundary S of D. In some cases it may be easier to use the Divergence Theorem and evaluate a triple integral. Assume $k = 1$.

52. $T(x,\ y,\ z) = 100 + x + 2y + z;\quad D = \{(x, y, z):\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

53. $T(x,\ y,\ z) = 100 + x^2 + y^2 + z^2;\quad D = \{(x, y, z):\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

54. $T(x,\ y,\ z) = 100 + e^{-z};\quad D = \{(x, y, z):\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

55. $T(x,\ y,\ z) = 100e^{-x^2-y^2-z^2};$ D is the sphere of radius a centered at the origin.

56. Consider the surface S consisting of the quarter-sphere $x^2+y^2+z^2=a^2$, for $z \ge 0$ *and* $x \ge 0$, and the half disk in the yz-plane $y^2+z^2 \le a^2$, for $z \ge 0$. The boundary of S in the xy-plane is C, which consists of the semicircle $x^2+y^2=a^2$, for $x \ge 0$, and the line segment $[-a,\ a]$ on the y-axis, with a counterclockwise orientation. Let $\vec{F}=\langle 2z-y,\ x-z,\ y-2x\rangle$

a) Describe the direction in which the normal vectors point on S.

b) Evaluate $\oint_C \vec{F} \bullet d\vec{r}$

c) Evaluate $\iint_S \left(\nabla \times \vec{F}\right) \bullet \vec{n}\ dS$ and check for segment with part (b).

57. Let S be the hemisphere $x^2+y^2+z^2=a^2$, for $z \ge 0$, and let T be the paraboloid $z=a-\frac{1}{a}\left(x^2+y^2\right)$, for $z \ge 0$, where $a > 0$. Assume the surfaces have outward normal vectors.

a) Verify that S and T have the same base $\left(x^2+y^2 \le a^2\right)$ and the same high point $(0,\ 0,\ a)$.

b) Which surface has the greater area?

c) Show that the flux of the radial field $\vec{F}=\langle x,\ y,\ z\rangle$ across S is $2\pi a^3$.

d) Show that the flux of the radial field $\vec{F}=\langle x,\ y,\ z\rangle$ across T is $\frac{3\pi a^3}{2}$.

58. The gravitational force due to a point mass M is proportional to $\vec{F}=\frac{GM\vec{r}}{|\vec{r}|^3}$, where $\vec{r}=\langle x,\ y,\ z\rangle$ and G is the gravitational constant.

a) Show that the flux force field across a sphere of radius a centered at the origin is

$$\iint_S \vec{F} \bullet \vec{n}\ dS = 4\pi GM$$

b) Let S be the boundary of the region between two spheres centered at the origin of radius a and b with $a < b$. Use the Divergence Theorem to show that the net outward flux across S is zero.

c) Suppose there is a distribution of mass within a region D containing the origin. Let $\rho=(x,\ y,\ z)$ be the mass density (mass per unit volume). Interpret the statement that

$$\iint_S \vec{F} \bullet \vec{n}\ dS = 4\pi G \iiint_D \rho(x,\ y,\ z)\ dV$$

d) Assuming $\vec{F}$ satisfies the conditions of the Divergence Theorem, conclude from part (c) that $\nabla \bullet \vec{F} = 4\pi G\rho$

e) Because the gravitational force is conservative, it has a potential function ϕ. From part (d) conclude that $\nabla^2\phi = 4\pi G\rho$

59. Let $\vec{F}$ be a radial field $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p}$, where p is a real number and $\vec{r} = \langle x,\ y,\ z \rangle$. With $p = 3$, $\vec{F}$ is an inverse square field.

a) Show that the net flux across a sphere centered at the origin is independent of the radius of the sphere only for $p = 3$

b) Explain the observation in part (*a*) by finding the flux of $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p}$ across the boundaries of a spherical box $\left\{(\rho,\ \varphi,\ \theta):\quad a \le \rho \le b,\quad \varphi_1 \le \varphi \le \varphi_2,\quad \theta_1 \le \theta \le \theta_2\right\}$ for various values of *p*.

60. Consider the potential function $\phi(x,\ y,\ z) = G(\rho)$, where *G* is any twice differentiable function and $\rho = \sqrt{x^2 + y^2 + z^2}$; therefore, *G* depends only on the distance from the origin.

a) Show that the gradient vector field associated with ϕ is $\vec{F} = \nabla\phi = G'(\rho)\dfrac{\vec{r}}{\rho}$, where $\vec{r} = \langle x,\ y,\ z \rangle$ and $\rho = |\vec{r}|$.

b) Let *S* be the sphere of radius *a* centered at the origin and let *D* be the region enclosed by *S*. show that the flux of $\vec{F}$ across *S* is $\displaystyle\iint_S \vec{F} \bullet \vec{n}\ dS = 4\pi a^2 G'(a)$.

c) Show that $\nabla \bullet \vec{F} = \nabla \bullet \nabla\phi = \dfrac{2}{\rho} G'(\rho) + G''(\rho)$

d) Use part (*c*) to show that the flux across *S* (as given in part (*b*)) is also obtained by the volume integral $\displaystyle\iiint_D \nabla \bullet \vec{F}\ dV$. (Hint: use spherical coordinates and integrate by parts.)

61. Prove Green's Identity for scalar-valued functions *u* and *v* defined on a region *D*:

$$\iiint_D \left(u\nabla^2 v - v\nabla^2 u\right)\ dV = \iint_S \left(u\nabla v - v\nabla u\right) \bullet \vec{n}\ dS$$

62. Prove the identity: $$\iiint_D \nabla \times \vec{F}\ dV = \iint_S \left(\vec{n} \times \vec{F}\right) dS$$

63. Prove the identity: $$\iint_S \left(\vec{n} \times \nabla\varphi\right) dS = \oint_C \varphi\, d\vec{r}$$