

## Section 2.7 – Derivatives of Exponential and Logarithmic Functions

### Derivative of the Natural Exponential Function

$$\frac{d}{dx}[e^x] = e^x$$

$$\frac{d}{dx}[e^U] = e^U \frac{dU}{dx}$$

### Derivative of $a^x$

$$\frac{d}{dx}(a^x) = (\ln a)a^x$$

$$\frac{d}{dx}[a^{g(x)}] = \ln(a)a^{g(x)}g'(x)$$

### Example

Find the derivative of each function

a)  $y = e^{5x}$

$$\begin{aligned}y' &= (5x)' e^{5x} \\ &= 5e^{5x}\end{aligned}$$

b)  $s = 3^t$

$$\frac{ds}{dt} = (\ln 3)3^t$$

c)  $y = 10e^{3x^2}$

$$\begin{aligned}y' &= 10e^{3x^2} (3x^2)' \\ &= 10e^{3x^2} (6x) \\ &= 60x e^{3x^2}\end{aligned}$$

d)  $s = 8 \cdot 10^{1/t}$

$$\begin{aligned}\frac{ds}{dt} &= 8(\ln 10)10^{1/t} (t^{-1})' \\ &= -\frac{8(\ln 10)10^{1/t}}{t^2}\end{aligned}$$

**Example**

Find the derivative of  $y = e^{x^2+1}\sqrt{5x+2}$

Solution

$$\begin{aligned}
 f &= e^{x^2+1} & g &= (5x+2)^{1/2} \\
 f' &= 2xe^{x^2+1} & g' &= \frac{1}{2}5(5x+2)^{-1/2} = \frac{5}{2\sqrt{5x+2}} \\
 y' &= (2x)e^{x^2+1}\sqrt{5x+2} + e^{x^2+1}\frac{5}{2\sqrt{5x+2}} \\
 &= 2xe^{x^2+1}\sqrt{5x+2}\frac{2\sqrt{5x+2}}{2\sqrt{5x+2}} + \frac{5e^{x^2+1}}{2\sqrt{5x+2}} \\
 &= \frac{4xe^{x^2+1}(5x+2)}{2\sqrt{5x+2}} + \frac{5e^{x^2+1}}{2\sqrt{5x+2}} \\
 &= \frac{20x^2e^{x^2+1} + 8xe^{x^2+1} + 5e^{x^2+1}}{2\sqrt{5x+2}} \\
 &= \frac{e^{x^2+1}(20x^2 + 8x + 5)}{2\sqrt{5x+2}}
 \end{aligned}$$

**Example**

The demand function for the product is modeled by  $p = 50e^{-0.0000125x}$  where  $p$  is the price per unit in dollars and  $x$  is the number of units. What price will yield maximum revenue?

Solution

$$\begin{aligned}
 R &= xp = 50xe^{-0.0000125x} \\
 R' &= 50e^{-0.0000125x} + (-0.0000125)50xe^{-0.0000125x} \\
 R' &= 50e^{-0.0000125x} - 0.000625xe^{-0.0000125x} \\
 R' &= e^{-0.0000125x}(50 - 0.000625x) = 0 \\
 50 - 0.000625x &= 0 \\
 -0.000625x &= -50 \\
 x &= \frac{-50}{-0.000625} = 80000 \\
 p(x = 80000) &= 50e^{-0.0000125(80000)} \\
 &\approx \$18.39 / unit
 \end{aligned}$$

**Example**

A company sells 990 units of a new product in the first year and 3213 units in the fourth year. They expect that sales can be approximated by a logistic function, leveling off at around 100,000 in the long run given by the formula

$$S(t) = \frac{100,000}{1 + 100e^{-100,000kt}}$$

a) Find  $k$  and rewrite the function

Solution

$$S(4) = 3213$$

$$3213 = \frac{100,000}{1 + 100e^{-100,000k \cdot 4}}$$

$$3213 = \frac{100,000}{1 + 100e^{-400,000k}}$$

$$3213 + 321300e^{-400,000k} = 100,000$$

*Subtract 3213 from both sides*

$$321300e^{-400,000k} = 96787$$

*Divide both sides by 321300*

$$e^{-400,000k} = 0.3012$$

*ln both sides*

$$-400,000k = \ln 0.3012$$

$$k = \frac{\ln 0.3012}{-400,000}$$

$$\approx 3 \times 10^{-6}$$

$$S(t) = \frac{100,000}{1 + 100e^{-0.3t}}$$

b) Find the rate of change of sales after 4 years

Solution

$$S' = \frac{3,000,000e^{-0.3t}}{(1 + 100e^{-0.3t})^2}$$

$$S'(4) = \frac{3,000,000e^{-0.3(4)}}{(1 + 100e^{-0.3(4)})^2}$$

$$3000000e((-)0.3 * 4) / (1 + 100e((-)0.3 * 4))^2$$

$$\approx 933$$

## Derivatives of Logarithmic

### Derivative of $\log_a x$

$$\frac{d}{dx} \left[ \log_a |x| \right] = \frac{1}{(\ln a)x}$$

$$\frac{d}{dx} \left[ \log_a |g(x)| \right] = \frac{1}{(\ln a)} \cdot \frac{g'(x)}{g(x)}$$

### Derivative of $\ln$

$$\frac{d}{dx} (\ln x) = \frac{1}{x}$$

$$\frac{d}{dx} (\ln u) = \frac{1}{u} \cdot \frac{du}{dx} = \frac{u'}{u}$$

### Other Bases

$$\frac{d}{dx} \left[ a^x \right] = a^x \ln a \quad \frac{d}{dx} \left[ a^u \right] = a^u (\ln a) \frac{du}{dx}$$

### Example

Find the derivative of each function

a)  $f(x) = \ln 6x$

$$\begin{aligned} f'(x) &= \frac{(6x)'}{6x} \\ &= \frac{6}{6x} \\ &= \frac{1}{x} \end{aligned}$$

b)  $f(x) = \log x$

$$f' = \frac{1}{(\ln 10)x}$$

c)  $f(x) = \ln(x^2 + 1)$

$$f' = \frac{2x}{x^2 + 1}$$

$$\frac{d}{dx} (\ln u) = \frac{u'}{u}$$

**Example**

Find the derivative of function  $f(x) = \log_2(3x^2 - 4x)$

Solution

$$f' = \frac{1}{\ln 2} \frac{6x - 4}{3x^2 - 4x}$$

$$= \frac{6x - 4}{\ln 2(3x^2 - 4x)}$$

$$\frac{d}{dx} \left[ \log_a |g(x)| \right] = \frac{1}{(\ln a)} \cdot \frac{g'(x)}{g(x)}$$

**Example**

Find the derivative of function  $y = \ln|5x|$

Solution

$$y' = \frac{5}{5x}$$

$$= \frac{1}{x}$$

**Example**

Find the derivative of function  $y = 3x \ln x^2$

Solution

$$y' = 3 \ln x^2 + 3x \frac{2x}{x^2}$$

$$= 3 \ln x^2 + 6$$

**Example**

Find the derivative of function  $y = x 3^{x+1}$

Solution

$$f = x \quad g = 3^{x+1}$$

$$f' = 1 \quad g' = 3^{x+1} \ln(3)$$

$$y' = 3^{x+1} + x 3^{x+1} \ln 3$$

$$= 3^{x+1} [1 + x \ln 3]$$

**Example**

Find the derivative of function  $s(t) = \frac{\log_8(t^{3/2} + 1)}{t}$

Solution

$$f = \log_8(t^{3/2} + 1) \quad g = t$$

$$f' = \frac{1}{\ln 8} \frac{\frac{3}{2} t^{1/2}}{t^{3/2} + 1} \quad g' = 1$$

$$s' = \frac{\frac{1}{\ln 8} \frac{\frac{3}{2} t^{1/2}}{t^{3/2} + 1} \cdot t - \log_8(t^{3/2} + 1)}{t^2}$$

$$= \frac{\frac{1}{\ln 8} \frac{\frac{3}{2} t^{3/2}}{t^{3/2} + 1} - \log_8(t^{3/2} + 1)}{t^2} \cdot \frac{2 \ln 8 (t^{3/2} + 1)}{2 \ln 8 (t^{3/2} + 1)}$$

$$= \frac{3t^{3/2} - 2(t^{3/2} + 1)(\ln 8) \log_8(t^{3/2} + 1)}{t^2 (t^{3/2} + 1) \ln 8}$$

## **Exercises**     **Section 2.7 – Derivatives of Exponential and Logarithmic Functions**

Find the derivative:

1.  $f(x) = e^{3x}$

2.  $f(x) = e^{-2x^3}$

3.  $f(x) = 4e^{x^2}$

4.  $f(x) = e^{-2x}$

5.  $f(x) = x^2 e^x$

6.  $f(x) = \frac{e^x + e^{-x}}{2}$

7.  $f(x) = \frac{e^x}{x^2}$

8.  $f(x) = x^2 e^x - e^x$

9.  $f(x) = (1 + 2x)e^{4x}$

10.  $y = x^2 e^{5x}$

11.  $f(x) = \frac{100,000}{1 + 100e^{-0.3x}}$

12.  $y = x^2 e^{-2x}$

13.  $y = \frac{e^x + e^{-x}}{x}$

14.  $y = \sqrt{e^{2x^2} + e^{-2x^2}}$

15.  $y = \frac{x}{e^{2x}}$

16.  $y = \ln \sqrt{x+5}$

17.  $y = (3x+7)\ln(2x-1)$

18.  $y = e^{x^2} \ln x$

19.  $y = \log_7 \sqrt{4x-3}$

20.  $f(x) = \ln \sqrt[3]{x+1}$

21.  $f(x) = \ln \left[ x^2 \sqrt{x^2 + 1} \right]$

22.  $y = \ln \frac{1+e^x}{1-e^x}$
23.  $y = \ln \frac{x^2}{x^2+1}$
24.  $y = \ln \left[ \frac{x^2(x+1)^3}{(x+3)^{1/2}} \right]$
25.  $y = \ln(x^2+1)$
26.  $y = \frac{\ln x}{e^{2x}}$
27.  $f(x) = \ln(x^2-4)$
28.  $f(x) = x^2 \ln x$
29.  $f(x) = -\frac{\ln x}{x^2}$
30.  $f(x) = \frac{e^{\sqrt{x}}}{\ln(\sqrt{x}+1)}$
31.  $f(x) = e^{2x} \ln(xe^x+1)$
32.  $f(x) = \frac{xe^x}{\ln(x^2+1)}$
33.  $f(x) = 2 \ln(x^2-3x+4)$
34.  $f(x) = e^{x^2+3x+1}$
35.  $f(x) = 3 \ln(1+x^2)$
36.  $f(x) = (1+\ln x)^3$
37.  $f(x) = (x-2 \ln x)^4$
38.  $f(x) = \frac{e^x}{x^2+1}$
39.  $f(x) = \frac{1-e^x}{1+e^x}$
40.  $f(x) = \frac{\ln x}{1+x}$
41.  $f(x) = \frac{2x}{1+\ln x}$



42.  $f(x) = x^2 e^x$

43.  $f(x) = x^3 \ln x$

44.  $f(x) = 6x^4 \ln x$

45.  $f(x) = 2x^3 e^x$

46.  $f(x) = \frac{3e^x}{1+e^x}$

47.  $f(x) = 5e^x + 3x + 1$

48.  $f(x) = \frac{\ln x}{2x+5}$

49.  $f(x) = -2\ln x + x^2 - 4$

50.  $f(x) = e^x + x - \ln x$

51.  $f(x) = \ln x + 2e^x - 3x^2$

52.  $f(x) = \ln x^8$

53.  $f(x) = 5x - \ln x^5$

54.  $f(x) = \ln x^2 + 4e^x$

55.  $f(x) = \ln x^{10} + 2\ln x$

56. Find the second derivative of  $y = 3e^{5x^3+1}$

57. Find the equation of the tangent line to  $f(x) = e^x$  at the point  $(0, 1)$

58. Find the equation of the tangent line to  $f(x) = e^x$  at the point  $(1, e)$

59. Find the equation of the tangent lines to  $f(x) = 4e^{-8x}$  at the points  $(0, 4)$

60. Find the equation of the tangent line to  $y = 4xe^{-x} + 5$  at  $x = 1$

61. The percentage of people of any particular age group that will die in a given year may be approximated by the formula

$$P(t) = 0.00239e^{0.0957t}$$

where  $t$  is the age of the person in years

a) Find  $P(25)$

b) Find  $P'(25)$

62. Assume the cost of a gallon of milk is \$2.90. With continuous compounding, find the time it would take the cost to be 5 times as much (to the nearest tenth of a year), at an annual inflation rate of 6 %.

63. The sales in thousands of a new type of product are given by  $S(t) = 30 - 90e^{-0.5t}$ , where  $t$  represents time in years. Find the rate of change of sales at the time when  $t = 3$
64. A company's total cost, in millions of dollars, is given by  $C(t) = 300 - 70e^{-t}$  where  $t =$  time in years. Find the marginal cost when  $t = 3$ .
65. A company's total cost, in millions of dollars, is given by  $C(t) = 280 - 30e^{-t}$  where  $t =$  time in years. Find the marginal cost when  $t = 2$ .
66. The demand function for a certain book is given by the function  $x = D(p) = 70e^{-0.006p}$ . Find the marginal demand  $D'(p)$
67. Suppose that the amount in grams of a radioactive substance present at time  $t$  (in years) is given by  $A(t) = 840e^{-0.63t}$ . Find the rate of change of the quantity present at the time when  $t = 2$ .
68. Researchers have found that the maximum number of successful trials that a laboratory rat can complete in a week is given by

$$P(t) = 53(1 - e^{-0.4t})$$

where  $t$  is the number of weeks the rat has been trained. Find the rate of change  $P'(t)$ .

69. When a telephone wire is hung between two poles, the wire forms a U-shape curve called a Catenary. For instance, the function  $y = 30(e^{x/60} + e^{-x/60}) - 30 \leq x \leq 30$  models the shape of the telephone wire strung between two poles that are 60 ft. apart ( $x$  &  $y$  are measured in ft.). Show that the lowest point on the wire is midway between two poles. How much does the wire sag between the two poles?
70. Find  $f''(x)$  for  $f(x) = \frac{\ln x}{7x}$ , then find  $f''(0)$  and  $f''(2)$
71. Suppose the average test score  $p$  and was modeled by  $p = 92.3 - 16.9 \ln(t + 1)$ , where  $t$  is the time in months. How would the rate at which the average test score changed after 1 year?
72. Suppose that the population of a certain collection of rare ants is given by

$$P(t) = (t + 100) \ln(t + 2)$$

Where  $t$  represents the time in days. Find the rates of change of the population on the second day and on the eighth day.

73. Suppose that the demand function for  $x$  units of a certain item is  $P(x) = 100 + \frac{180 \ln(x + 5)}{x}$  where  $P$  is the price per unit, in dollars. Find the marginal revenue.

74. The population of coyotes in the northwestern portion of Alabama is given by the formula  $P(t) = (t^2 + 100)\ln(t + 2)$ , where  $t$  represents the time in years since 2000 (the year 2000 corresponds to  $t = 0$ ). Find the rate of change of the coyote population in 2013 ( $t = 13$ ).

75. Students in a math class took a final exam. They took equivalent forms of the exam in monthly intervals thereafter. The average score  $S(t)$ , in percent, after  $t$  months was found to be given by

$$S(t) = 73 - 17\ln(t + 1), \quad t \geq 0$$

Find  $S'(t)$ .

76. Suppose that the population of a town is given by  $P(t) = 8\ln\sqrt{8t + 7}$  where  $t$  is the time in years after 1980 and  $P$  is the population of the town in thousands. Find  $P'(t)$ .

77. The following formula accurately models the relationship between the size of a certain type of tumor and the amount of time that it has been growing:

$$V(t) = 450(1 - e - 0.0022t)^3$$

where  $t$  is in months and  $V(t)$  is measured in cubic centimeters. Calculate the rate of change of tumor volume at 80 months.

78. A yeast culture at room temperature ( $68^\circ\text{F}$ ) is placed in a refrigerator set at a constant temperature of  $38^\circ\text{F}$ . After  $t$  hours, the temperature  $T$  of the culture is given approximately by

$$T = 30e^{-0.58t} + 38 \quad t \geq 0$$

What is the rate of change of temperature of the culture at the end of 1 hour? At the end of 4 hours?

79. A mathematical model for the average age of a group of people learning to type is given by

$$N(t) = 10 + 6\ln t \quad t \geq 1$$

Where  $N(t)$  is the number of words per minute typed after  $t$  hours of instruction and practice (2 hours per day, 5 days per week). What is the rate of learning after 10 hours of instruction and practice? After 100 hours?