Lecture Three – *Infinite Sequences and Series*

Section 3.1 – Sequences

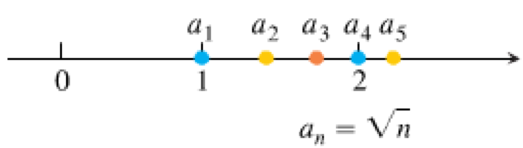
A sequence is a list of numbers

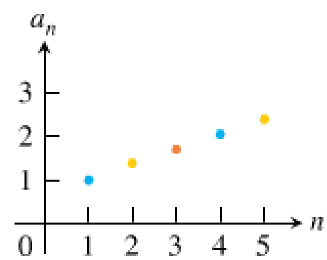
$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$$

An ***infinite sequence*** of numbers is a function whose domain is the set of positive integers. These are the ***terms*** of the sequence. The integer ***n*** is called the ***index*** of a_n.

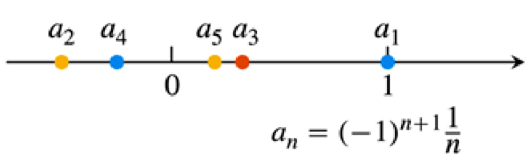
Sequences can be described by writing rules that specify their terms such as

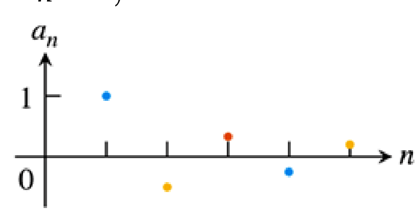
$$a_n = \sqrt{n} \quad \Rightarrow \quad \{a_n\} = \left\{\sqrt{1},\ \sqrt{2},\ \sqrt{3},\ \ldots,\ \sqrt{n},\ \ldots\right\}$$



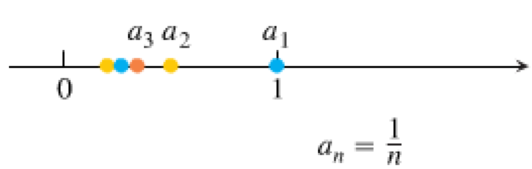


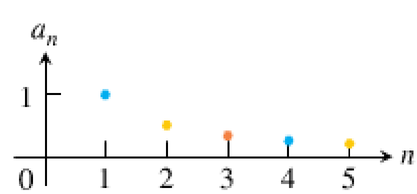
$$a_n = (-1)^{n+1}\frac{1}{n} \quad \Rightarrow \quad \{a_n\} = \left\{1,\ -\frac{1}{2},\ \frac{1}{3},\ -\frac{1}{4},\ \ldots,\ (-1)^{n+1}\frac{1}{n},\ \ldots\right\}$$





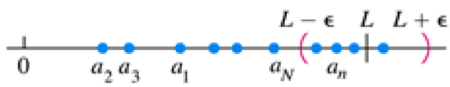
$$a_n = \frac{1}{n} \quad \Rightarrow \quad \{a_n\} = \left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots,\ \frac{1}{n},\ \ldots\right\}$$

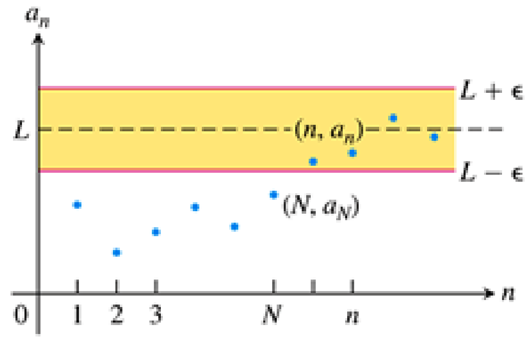




Also, we can write: $\{a_n\} = \left\{\sqrt{n}\right\}_{n=1}^{\infty}$

Convergence and Divergence





$\left\{0,\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \ldots,\ 1-\frac{1}{n},\ \ldots\right\}$ Terms approach 1.

$\left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots,\ \frac{1}{n},\ \ldots\right\}$ Terms approach 0.

Definition

The sequence $\{a_n\}$ ***converges*** to the number L if for every positive number ε there corresponds an integer N such that for all n,

$$n > N \quad \Rightarrow \quad \left|a_n - L\right| < \varepsilon$$

If no such number L exists, we say $\{a_n\}$ ***diverges***.

The $\{a_n\}$ ***converges*** to L, we write $\lim\limits_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the ***limit*** of the sequence.

Example

Show that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

Solution

Let ε > 0 be given. We must show that there exists an integer N such that for all n,

$$n > N \quad \Rightarrow \quad \left|\frac{1}{n} - 0\right| < \varepsilon$$

This implication will hold if $\frac{1}{n} < \varepsilon$ or $n > \frac{1}{\varepsilon}$. If N is any integer greater than $\frac{1}{\varepsilon}$, the implication will hold for all $n > N$. This proves that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

Example

Show that $\lim_{n\to\infty} k = k \quad (\textit{any constant } k)$

Solution

Let ε > 0 be given. We must show that there exists an integer *N* such that for all *n*,

$$n > N \quad \Rightarrow \quad |k - k| < \varepsilon$$

Since $k - k = 0$, we can use any positive integer for *N* and the implication will hold for all *n* > *N*. This proves that $\lim_{n\to\infty} k = k$

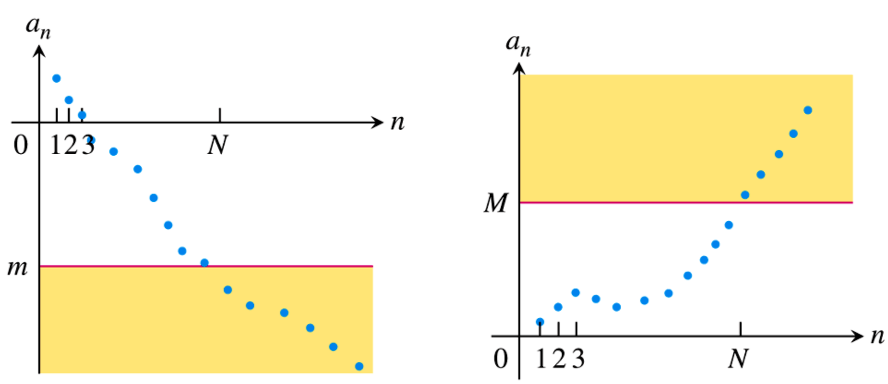
Definition

The sequence $\{a_n\}$ ***diverges*** to infinity if for every number *M* there is an integer *N* such that for all *n* larger than *N*, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \quad or \quad a_n \to \infty$$

Similarly, if for every number *m* there is an integer *N* such that for all *n* > *N* we have $a_n < m$, then we say $\{a_n\}$ ***diverges to negative infinity*** and write

$$\lim_{n\to\infty} a_n = -\infty \quad or \quad a_n \to -\infty$$



Theorem

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let *A* and *B* real numbers. The following rules hold if $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} b_n = B$

Sum Rule: $\lim\limits_{n\to\infty}\left(a_n + b_n\right) = A + B$

Difference Rule: $\lim\limits_{n\to\infty}\left(a_n - b_n\right) = A - B$

Constant Multiple Rule: $\lim\limits_{n\to\infty}\left(ka_n\right) = kA$

Product Rule: $\lim\limits_{n\to\infty}\left(a_n \bullet b_n\right) = A \bullet B$

Quotient Rule: $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{A}{B}$ *if* $B \neq 0$

Example

a) $\lim\limits_{n\to\infty}\left(-\frac{1}{n}\right) = -1\bullet \lim\limits_{n\to\infty}\left(\frac{1}{n}\right) = -1(0) = 0$

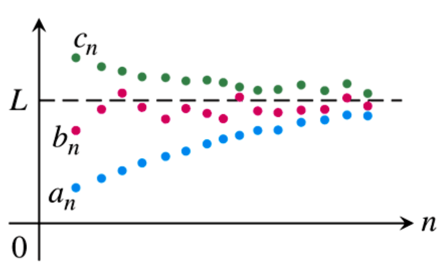
b) $\lim\limits_{n\to\infty}\left(\frac{n-1}{n}\right) = \lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right) = \lim\limits_{n\to\infty} 1 - \lim\limits_{n\to\infty}\frac{1}{n} = 1 - 0 = 1$

c) $\lim\limits_{n\to\infty}\left(\frac{5}{n^2}\right) = 5\bullet \lim\limits_{n\to\infty}\left(\frac{1}{n^2}\right) = 5\bullet \lim\limits_{n\to\infty}\left(\frac{1}{n}\right)\bullet \lim\limits_{n\to\infty}\left(\frac{1}{n}\right) = -1\bullet 0\bullet 0 = 0$

d) $\lim\limits_{n\to\infty}\left(\frac{4-7n^6}{n^6+3}\right) = \lim\limits_{n\to\infty}\left(\frac{\frac{4}{n^6}-7}{1+\frac{3}{n^6}}\right) = \frac{0-7}{1+0} = -7$

Theorem – *The Sandwich Theorem for Sequences*

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all *n* beyond some index *N*, and if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$ also.



Example

Since $\frac{1}{n} \to 0$, we know that

a) $\frac{\cos n}{n} \to 0$ *because* $-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$

b) $\frac{1}{2^n} \to 0$ *because* $0 \le \frac{1}{2^n} \le \frac{1}{n}$

c) $(-1)^n \frac{1}{n} \to 0$ *because* $-\frac{1}{n} \le (-1)^n \le \frac{1}{n}$

Theorem – The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Example

Show that $\sqrt{\frac{n+1}{n}} \to 1$

Solution

We know that $\frac{n+1}{n} \to 1$. Taking $f(x) = \sqrt{x}$ and $L = 1$ that gives $\sqrt{\frac{n+1}{n}} \to \sqrt{1} = 1$

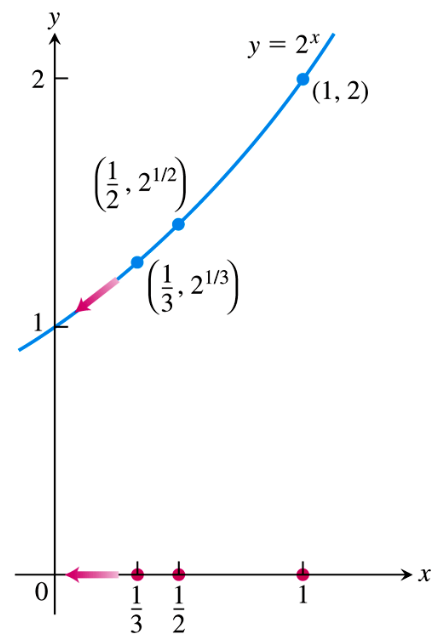
Example

The sequence $\left\{\frac{1}{n}\right\}$ converges to 0.

By taking $a_n = \frac{1}{n}$, $f(x) = 2^x$, and $L = 0$.

We see that $2^{1/n} = f\left(\frac{1}{n}\right) \to f(L) = 2^0 = 1$.

The sequence $\left\{2^{1/n}\right\}$ converges to 1.



Using L'Hôpital's Rule

Theorem

Suppose that $f(x)$ is a function for all $x \ge n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \ge n_0$. Then

$$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L$$

Example

Show that $\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$

Solution

The function $\frac{\ln x}{x}$ is defined for all $x \ge 1$ and agrees with the given sequence at positive integers.

Therefore;

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$\underline{= 0}$$

Example

Does the sequence whose nth term is $a_n = \left(\frac{n+1}{n-1}\right)^n$ converge? If so, find $\lim\limits_{n\to\infty} a_n$

Solution

The limit leads to the indeterminate form 1^∞.

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n$$

$$= n\ln\left(\frac{n+1}{n-1}\right) \qquad \textbf{∞.0 } \textit{form}$$

$$= \frac{\ln\left(\frac{n+1}{n-1}\right)}{\frac{1}{n}} \qquad \textbf{0.0 } \textit{form}$$

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{\frac{1}{n}} \qquad \left(\ln\frac{n+1}{n-1}\right)' = \frac{\frac{n-1-(n+1)}{(n-1)^2}}{\frac{n+1}{n-1}} = \frac{-2}{(n+1)(n-1)}$$

$$= \lim_{n\to\infty} \frac{\frac{-2}{n^2-1}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2-1}$$

$$= 2$$

$$\lim_{n\to\infty} a_n = e^2$$

Theorem

The following six sequences converge to the limits listed below:

1. $\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim\limits_{n\to\infty} x^{1/n} = 1 \quad x > 0$

4. $\lim\limits_{n\to\infty} x^n = 1 \quad |x| < 1$

5. $\lim\limits_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (any\ x)$

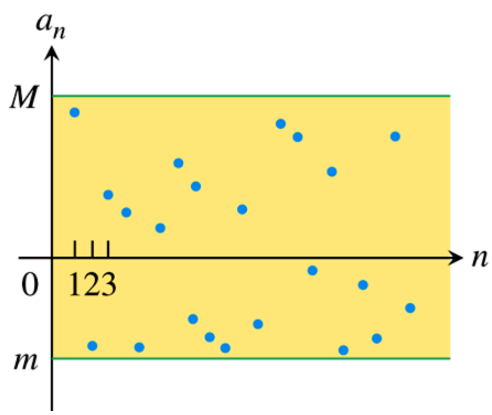
6. $\lim\limits_{n\to\infty} \frac{x^n}{n!} = 0 \quad (any\ x)$

Bounded *Monotonic* Sequences

Definitions

A sequence $\{a_n\}$ is ***bounded from above*** if there exists a number M such that $a_n \leq M$ for all n. The number M is an ***upper bound*** for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the ***least upper bound*** for $\{a_n\}$.

A sequence $\{a_n\}$ is ***bounded from below*** if there exists a number m such that $a_n \geq m$ for all n. The number m is an ***lower bound*** for $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the ***greatest lower bound*** for $\{a_n\}$.



If $\{a_n\}$ is bounded from above and below, the $\{a_n\}$ is ***bounded***.

If $\{a_n\}$ is not bounded, then $\{a_n\}$ is an ***unbounded*** sequence.

Definition

A sequence $\{a_n\}$ is ***nondecreasing*** if $a_n \le a_{n+1}$ for all n. That is $a_1 \le a_2 \le a_3 \le \ldots$

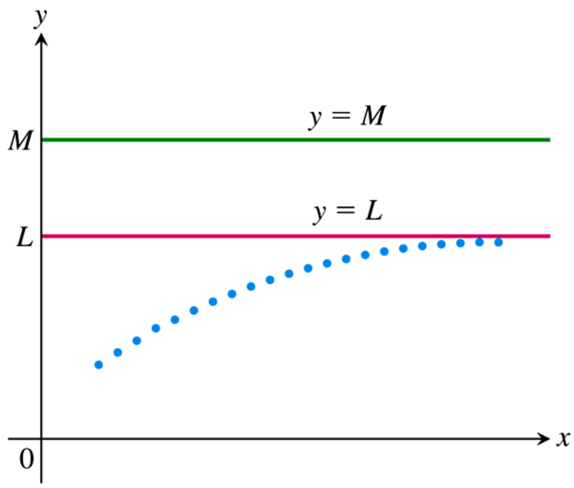
Which each term is greater than or equal to its predecessor $\left(a_{n+1} \ge a_n\right)$

Example: $\left\{1-\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{0,\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \ldots\right\}$

A sequence $\{a_n\}$ is ***nonincreasing*** if $a_n \ge a_{n+1}$ for all n, which each term is less than or equal to its predecessor $\left(a_{n+1} \le a_n\right)$

Example: $\left\{1+\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{2,\ \frac{3}{2},\ \frac{4}{3},\ \frac{5}{4},\ \ldots\right\}$

The sequence $\{a_n\}$ is ***monotonic*** if it is either nondecreasing or nonincreasing.



Theorem

If a sequence $\{a_n\}$ is both *bounded* and *monotonic*, then the sequence converges.

Example

The sequence $\{1, 2, 3, \ldots, n, \ldots\}$ is nondecreasing

The sequence $\left\{\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \ldots,\ \frac{n}{n+1},\ \ldots\right\}$ is nondecreasing

The sequence $\left\{1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \ldots,\ \frac{1}{2^n},\ \ldots\right\}$ is nonincreasing

Exercises **Section 3.1 – Sequences**

1. Find the values of a_1, a_2, a_3, *and* a_4 for $a_n = \dfrac{1-n}{n^2}$

2. Find the values of a_1, a_2, a_3, *and* a_4 for $a_n = \dfrac{1}{n!}$

3. Find the values of a_1, a_2, a_3, *and* a_4 for $a_n = \dfrac{(-1)^{n+1}}{2n-1}$

4. Find the values of a_1, a_2, a_3, *and* a_4 for $a_n = 2+(-1)^n$

5. Find the values of a_1, a_2, a_3, *and* a_4 for $a_n = \dfrac{2^n-1}{2^n}$

6. Write the first ten terms of the sequence $a_1 = 1, \quad a_{n+1} = a_n + \dfrac{1}{2^n}$

7. Write the first ten terms of the sequence $a_1 = 1, \quad a_{n+1} = \dfrac{a_n}{n+1}$

8. Write the first ten terms of the sequence $a_1 = 2, \quad a_2 = -1, \quad a_{n+2} = \dfrac{a_{n+1}}{a_n}$

9. Find a formula for the *n*th term of the sequence $-1,\ 1,\ -1,\ 1,\ -1, \cdots$

10. Find a formula for the *n*th term of the sequence $1,\ -\frac{1}{4},\ \frac{1}{9},\ -\frac{1}{16},\ \frac{1}{25}, \cdots$

11. Find a formula for the *n*th term of the sequence $\frac{1}{9},\ \frac{2}{12},\ \frac{2^2}{15},\ \frac{2^3}{18},\ \frac{2^4}{21}, \cdots$

12. Find a formula for the *n*th term of the sequence $-3,\ -2,\ -1,\ 0,\ 1, \cdots$

13. Find a formula for the *n*th term of the sequence $\frac{1}{25},\ \frac{8}{125},\ \frac{27}{625},\ \frac{64}{3125},\ \frac{125}{15{,}625}, \cdots$

14. Find a formula for the *n*th term of the sequence $0,\ 1,\ 1,\ 2,\ 2,\ 3,\ 3,\ 4,\ \cdots$

(15 – 43) Determine if the sequence converge or diverge? Then find the limit of each convergent sequence.

15. $a_n = \dfrac{n+(-1)^n}{n}$

16. $a_n = \dfrac{1-2n}{1+2n}$

17. $a_n = \dfrac{1-n^3}{70-4n^2}$

18. $a_n = \left(2-\dfrac{1}{2^n}\right)\left(3+\dfrac{1}{2^n}\right)$

19. $a_n = n\pi\cos(n\pi)$

20. $a_n = n-\sqrt{n^2-n}$

21. $a_n = \sqrt{\dfrac{2n}{n+1}}$

22. $a_n = \dfrac{\sin^2 n}{2^n}$

23. $a_n = \dfrac{\ln n}{\ln 2n}$

24. $a_n = \dfrac{3^n \cdot 6^n}{2^{-n} \cdot n!}$

25. $a_n = \sqrt{n}\ \sin\dfrac{1}{\sqrt{n}}$

26. $a_n = \dfrac{n^2}{2^n - 1}$

27. $\{c_n\} = \left\{(-1)^n \dfrac{1}{n!}\right\}$

28. $a_n = \dfrac{5}{n+2}$

29. $a_n = 8 + \dfrac{5}{n}$

30. $a_n = (-1)^n \left(\dfrac{n}{n+1}\right)$

31. $a_n = \dfrac{1 + (-1)^n}{n^2}$

32. $a_n = \dfrac{10n^2 + 3n + 7}{2n^2 - 6}$

33. $a_n = \dfrac{\sqrt[3]{n}}{\sqrt[3]{n} + 1}$

34. $a_n = \dfrac{\ln\left(n^3\right)}{2n}$

35. $a_n = \dfrac{5^n}{3^n}$

36. $a_n = \dfrac{(n+1)!}{n!}$

37. $a_n = \dfrac{(n-2)!}{n!}$

38. $a_n = \dfrac{n^p}{e^n}, \quad p > 0$

39. $a_n = n \sin \dfrac{1}{n}$

40. $a_n = 2^{1/n}$

41. $a_n = -3^{-n}$

42. $a_n = \dfrac{\sin n}{n}$

43. $a_n = \dfrac{\cos \pi n}{n^2}$