Solution *Section* 4.4 – The Binomial Theorem

Exercise

Find the fifth term in the expansion $\left(x^3+\sqrt{y}\right)^{13}$

Solution

$$\binom{13}{4}\left(x^3\right)^9\left(\sqrt{y}\right)^4=\frac{13!}{4!(13-4)!}x^{27}y^2$$

$$=715x^{27}y^2$$

Exercise

Find the term involving q^{10} in the binomial expansion $\left(\frac{1}{3}p+q^2\right)^{12}$

Solution

Given: $a=\frac{1}{3}p,\quad b=q^2,\quad n=12$

$$q^{10}=\left(q^2\right)^5=b^5$$

$$\binom{n}{k}a^{n-k}b^k=\binom{12}{5}\left(\frac{1}{3}p\right)^{12-5}\left(q^2\right)^5$$

$$=\frac{12!}{5!(12-5)!}\left(\frac{1}{3}p\right)^7q^{10}$$

$$=\frac{88}{243}p^7q^{10}$$

Exercise

Use the binomial theorem to expand and simplify: $(4x-y)^3$

Solution

$$(4x-y)^3=\binom{3}{0}(4x)^3(-y)^0+\binom{3}{1}(4x)^2(-y)^1+\binom{3}{2}(4x)^1(-y)^2+\binom{3}{3}(4x)^0(-y)^3$$

$$=64x^3+3\left(16x^2\right)(-y)+3(4x)y^2-y^3$$

$$=64x^3-48x^2y+12xy^2-y^3$$

Exercise

Use the binomial theorem to expand and simplify: $(x+y)^6$

Solution

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

Exercise

Use the binomial theorem to expand and simplify: $(a-b)^6$

Solution

$$(a-b)^6 = a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$$

Exercise

Use the binomial theorem to expand and simplify: $(x-y)^7$

Solution

$$(x-y)^7 = x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$$

Exercise

Use the binomial theorem to expand and simplify: $(a+b)^8$

Solution

$$(a+b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$$

Exercise

Use the binomial theorem to expand and simplify: $(3t-5x)^4$

Solution

$$(3t-5x)^4 = (3t)^4 + 4(3t)^3(-5x)^1 + 6(3t)^2(-5x)^2 + 4(3t)^1(-5x)^3 + (-5x)^4$$

$$= 81t^4 - 540t^3x + 1350t^2x^2 - 1500tx^3 + 625x^4$$

Exercise

Use the binomial theorem to expand and simplify: $\left(\frac{1}{3}x+y^2\right)^5$

Solution

$$\left(\frac{1}{3}x+y^2\right)^5=\left(\frac{1}{3}x\right)^5+5\left(\frac{1}{3}x\right)^4y^2+10\left(\frac{1}{3}x\right)^3\left(y^2\right)^2+10\left(\frac{1}{3}x\right)^2\left(y^2\right)^3+5\frac{1}{3}x\left(y^2\right)^4+\left(y^2\right)^5$$

$$=\frac{1}{243}x^5+\frac{5}{81}x^4y^2+\frac{10}{27}x^3y^4+\frac{10}{9}x^2y^6+\frac{5}{3}xy^8+y^{10}$$

Exercise

Use the binomial theorem to expand and simplify: $\left(\frac{1}{x^2}+3x\right)^6$

Solution

$$\left(\frac{1}{x^2}+3x\right)^6=\left(x^{-2}+3x\right)^6$$

$$=\left(x^{-2}\right)^6+6\left(x^{-2}\right)^5 3x+15\left(x^{-2}\right)^4(3x)^2+20\left(x^{-2}\right)^3(3x)^3$$

$$+15\left(x^{-2}\right)^2(3x)^4+15x^{-2}(3x)^5+(3x)^6$$

$$=x^{-12}+18x^{-9}+135x^{-6}+540x^{-3}+1215+1458x^3+729x^6$$

Exercise

Use the binomial theorem to expand and simplify: $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^5$

Solution

$$\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^5=\left(x^{1/2}+x^{-1/2}\right)^5$$

$$=\left(x^{1/2}\right)^5+5\left(x^{1/2}\right)^4x^{-1/2}+10\left(x^{1/2}\right)^3\left(x^{-1/2}\right)^2+10\left(x^{1/2}\right)^2\left(x^{-1/2}\right)^3$$

$$+5x^{1/2}\left(x^{-1/2}\right)^4+\left(x^{-1/2}\right)^5$$

$$=x^{5/2}+5x^2x^{-1/2}+10x^{3/2}x^{-1}+10xx^{-3/2}+5x^{1/2}x^{-2}+x^{-5/2}$$

$$=x^{5/2}+5x^{3/2}+10x^{1/2}+10x^{-1/2}+5x^{-3/2}+x^{-5/2}$$

Exercise

Expand and simplify: $(2y-3)^4$

Solution

$$(2y-3)^4 = (2y)^4 + 4(2y)^3(-3) + 6(2y)^2(-3)^2 + 4(2y)(-3)^3 + (-3)^4$$
$$= 16y^4 - 96y^3 + 216y^2 - 216y + 81$$

Exercise

Expand and simplify: $(x+2)^5$

Solution

$$(x+2)^5 = x^5 + 5x^4(2) + 10x^3(2)^2 + 10x^2(2)^3 + 5x(2)^4 + (2)^5$$
$$= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$$

Exercise

Expand and simplify: $\left(x^2 - y^2\right)^6$

Solution

$$\left(x^2 - y^2\right)^6 = \left(x^2\right)^6 + 6\left(x^2\right)^5\left(-y^2\right) + 15\left(x^2\right)^4\left(-y^2\right)^2 + 20\left(x^2\right)^3\left(-y^2\right)^3$$
$$+ 15\left(x^2\right)^2\left(-y^2\right)^4 + 15\left(x^2\right)\left(-y^2\right)^5 + \left(-y^2\right)^6$$
$$= x^{12} - 6x^{10}y^2 + 15x^8y^4 - 20x^6y^6 + 15x^4y^8 - 15x^2y^{10} + y^{12}$$

Exercise

Expand and simplify: $(ax-by)^4$

Solution

$$(ax-by)^4 = (ax)^4 + 4(ax)^3(-by) + 6(ax)^2(-by)^2 + 4(ax)(-by)^3 + (-by)^4$$
$$= a^4x^4 - 4a^3x^3by + 6a^2x^2b^2y^2 - 4axb^3y^3 + b^4y^4$$

Exercise

Expand and simplify: $(ax+by)^5$

Solution

$$(ax+by)^5=(ax)^5+5(ax)^4(by)+10(ax)^3(by)^2+10(ax)^2(by)^3+5(ax)(by)^4+(by)^5$$

$$=a^5x^5+5a^4x^4by+10a^3x^3b^2y^2+10a^2x^2b^3y^3+5axb^4y^4+b^5y^5$$

Exercise

Expand and simplify: $\left(\sqrt{x}-\sqrt{3}\right)^4$

Solution

$$\left(\sqrt{x}-\sqrt{3}\right)^4=\left(\sqrt{x}\right)^4+4\left(\sqrt{x}\right)^3\left(-\sqrt{3}\right)+6\left(\sqrt{x}\right)^2\left(-\sqrt{3}\right)^2+4\left(\sqrt{x}\right)\left(-\sqrt{3}\right)^3+\left(-\sqrt{3}\right)^4$$

$$=x^2-4x\sqrt{3x}+18x^2-13\sqrt{3x}+9$$

Exercise

Expand and simplify: $\left(\sqrt{x}-\sqrt{2}\right)^6$

Solution

$$\left(\sqrt{x}-\sqrt{2}\right)^6=\left(\sqrt{x}\right)^6+6\left(\sqrt{x}\right)^5\left(-\sqrt{2}\right)+15\left(\sqrt{x}\right)^4\left(-\sqrt{2}\right)^2+20\left(\sqrt{x}\right)^3\left(-\sqrt{2}\right)^3$$

$$+15\left(\sqrt{x}\right)^2\left(-\sqrt{2}\right)^4+15\left(\sqrt{x}\right)\left(-\sqrt{2}\right)^5+\left(-\sqrt{2}\right)^6$$

$$=x^3-6x^2\sqrt{2x}+30x^2-40x\sqrt{2x}+60x-60\sqrt{2x}+8$$

Exercise

Expand and simplify: $(2x-1)^{12}$

Solution

$$(2x-1)^{12}=(2x)^{12}+12(2x)^{11}(-1)+66(2x)^{10}(-1)^2+240(2x)^9(-1)^3+535(2x)^8(-1)^4$$

$$+812(2x)^7(-1)^5+924(2x)^6(-1)^6+812(2x)^5(-1)^7+535(2x)^4(-1)^8$$

$$+240(2x)^3(-1)^9+66(2x)^2(-1)^{10}+12(2x)(-1)^{11}+(-1)^{12}$$

$$=4096x^{12}-24576x^{11}+67584x^{10}-122880x^9+136960x^8-103936x^7$$

$$+59136x^6-25984x^5+8560x^4-1920x^3+264x^2-24x+1$$

Exercise

Expand and simplify: $\left(x-\frac{1}{x^2}\right)^9$

Solution

$$\left(x-\frac{1}{x^2}\right)^9 = x^9+9x^8\left(-\frac{1}{x^2}\right)+36x^7\left(-\frac{1}{x^2}\right)^2+84x^6\left(-\frac{1}{x^2}\right)^3+126x^5\left(-\frac{1}{x^2}\right)^4+126x^4\left(-\frac{1}{x^2}\right)^5$$
$$+84x^3\left(-\frac{1}{x^2}\right)^6+36x^2\left(-\frac{1}{x^2}\right)^7+9x\left(-\frac{1}{x^2}\right)^8+\left(-\frac{1}{x^2}\right)^9$$
$$= x^9-9x^6+36x^3-84+126x^{-3}-126x^{-6}+84x^{-9}-36x^{-12}+9x^{-15}-x^{-18}$$

Exercise

Expand and simplify: $\left(\frac{2}{x}-3y\right)^5$

Solution

$$\left(\frac{2}{x}-3y\right)^5 = \left(\frac{2}{x}\right)^5+5\left(\frac{2}{x}\right)^4(-3y)+10\left(\frac{2}{x}\right)^3(-3y)^2+10\left(\frac{2}{x}\right)^2(-3y)^3+5\left(\frac{2}{x}\right)(-3y)^4+(-3y)^5$$
$$= \frac{32}{x^5}-240\frac{y}{x^4}+720\frac{y^2}{x^3}-1{,}080\frac{y^3}{x^2}+810\frac{y^4}{x}-243y^5$$

Exercise

Expand and simplify: $\left(3\sqrt{x}+\sqrt[4]{x}\right)^4$

Solution

$$\left(3\sqrt{x}+\sqrt[4]{x}\right)^4 = \left(3\sqrt{x}\right)^4+4\left(3\sqrt{x}\right)^3\left(\sqrt[4]{x}\right)+6\left(3\sqrt{x}\right)^2\left(\sqrt[4]{x}\right)^2+4\left(3\sqrt{x}\right)\left(\sqrt[4]{x}\right)^3+\left(\sqrt[4]{x}\right)^4$$
$$= 81x^2+108x^{3/2}x^{1/4}+54x\sqrt{x}+12x^{1/2}x^{3/4}+x$$
$$= 81x^2+108x^{7/4}+54x\sqrt{x}+12x^{5/4}+x$$
$$= 81x^2+108x\sqrt[4]{x^3}+54x\sqrt{x}+12x\sqrt[4]{x}+x$$

Exercise

Expand and simplify: $(x+1)^5$

Solution

$$(x+1)^5 = x^5+5x^4+10x^3+10x^2+5x+1$$

Exercise

Expand and simplify: $(x-1)^5$

Solution

$$(x-1)^5 = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$$

Exercise

Expand and simplify: $(x-2)^6$

Solution

$$(x-2)^6 = x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64$$

Exercise

Expand and simplify: $\left(\frac{1}{x^3} - 2x\right)^5$

Solution

$$\left(\frac{1}{x^3} - 2x\right)^5 = \frac{1}{x^{15}} - 10\frac{x}{x^{12}} + 10\frac{4x^2}{x^9} - 10\frac{8x^3}{x^6} + 5\frac{16x^4}{x^3} - 32x^5$$

$$= \frac{1}{x^{15}} - \frac{10}{x^{11}} + \frac{40}{x^7} - \frac{80}{x^3} + 80x - 32x^5$$

Exercise

Expand and simplify: $\left(\frac{1}{x} - 2x\right)^6$

Solution

$$\left(\frac{1}{x} - 2x\right)^6 = \frac{1}{x^6} - 6\frac{1}{x^5}(2x) + 15\frac{1}{x^4}(2x)^2 - 20\frac{1}{x^3}(2x)^3 + 15\frac{1}{x^2}(2x)^4 - 6\frac{1}{x}(2x)^5 + (2x)^6$$

$$= \frac{1}{x^6} - \frac{12}{x^4} + \frac{60}{x^2} - 160 + 240x^2 - 192x^4 + 64x^6$$

Exercise

Expand and simplify: $\left(x^2-2y\right)^5$

Solution

$$\left(x^2-2y\right)^5 = x^{10}-10x^8y+40x^6y^2-80x^4y^3+80x^2y^4-32y^5$$

Exercise

Expand and simplify: $\left(\frac{2}{x}+3\sqrt{x}\right)^4$

Solution

$$\left(\frac{2}{x}+3\sqrt{x}\right)^4 = \frac{16}{x^4}+\frac{32}{x^3}\left(3\sqrt{x}\right)+\frac{24}{x^2}(9x)+\frac{8}{x}\left(27x\sqrt{x}\right)+81x^2$$
$$= \frac{16}{x^4}+\frac{96\sqrt{x}}{x^3}+\frac{216}{x}+216\sqrt{x}+81x^2$$

Exercise

Expand and simplify: $(2x+5y)^7$

Solution

$$(2x+5y)^7 = 128x^7+7\left(64x^6\right)(5y)+21\left(32x^5\right)\left(25y^2\right)+35\left(16x^4\right)\left(125y^3\right)$$
$$+35\left(8x^3\right)\left(625y^4\right)+21\left(4x^2\right)\left(3{,}125y^5\right)+7(2x)\left(5^6y^6\right)+(5y)^7$$
$$= 128x^7+320x^6y+16{,}800x^5y^2+70{,}000x^4y^3+175{,}000x^3y^4+262{,}500x^2y^5$$
$$+218{,}750xy^6+78{,}125y^7$$

Exercise

Expand and simplify: $(2x-3)^{11}$

Solution

$$(2x-3)^{11} = (2x)^{11}-33(2x)^{10}+495(2x)^9-4{,}995(2x)^8+350(3)^4(2x)^7-462(3)^5(2x)^6$$
$$+462(3)^6(2x)^5-350(3)^7(2x)^4+185(3)^8(2x)^3-55(3)^9(2x)^2+22(3)^{10}x-3^{11}$$
$$= 2{,}048x^{11}-33{,}792x^{10}+253{,}440x^9-1{,}278{,}720x^8+3{,}628{,}800x^7-7{,}185{,}024x^6$$
$$+462(3)^6\,2^5x^5-5{,}600(3)^7x^4+1{,}480(3)^8x^3-220(3)^9x^2+22(3)^{10}x-3^{11}$$

Exercise

Expand and simplify: $(2x-3y)^6$

Solution

$$(2x-3y)^6 = 64x^6 - 576x^5y + 2{,}160x^4y^2 - 4{,}320x^3y^3 + 4{,}860x^2y^4 - 2{,}196xy^5 + 729y^6$$

Exercise

Expand and simplify: $(2x+3y)^5$

Solution

$$(2x+3y)^5 = 32x^5 + 240x^4y + 720x^3y^2 + 1{,}080x^2y^3 + 810xy^4 + 243y^5$$

Exercise

Expand and simplify: $(3x-2y)^4$

Solution

$$(3x-2y)^4 = 81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4$$

Exercise

Expand and simplify: $\left(x^2+y^3\right)^3$

Solution

$$\left(x^2+y^3\right)^3 = x^6 + 3x^4y^3 + 3x^2y^6 + y^9$$

Exercise

Expand and simplify: $\left(x^2-y^2\right)^3$

Solution

$$\left(x^2-y^2\right)^3 = x^6 - 3x^4y^2 + 3x^2y^4 - y^6$$

Exercise

Expand and simplify: $(2+i)^6$

Solution

$$\begin{aligned}(2+i)^6 &= 64+6(32)i+15(16)i^2+20(8)i^3+15(4)i^4+12i^5+i^6\\ &= 64+192i-240-160i+60+12i-1\\ &= -117+44i\end{aligned}$$

Exercise

Expand and simplify: $(2-i)^6$

Solution

$$\begin{aligned}(2-i)^6 &= 64-6(32)i+15(16)i^2-20(8)i^3+15(4)i^4-12i^5+i^6\\ &= 64-192i-240+160i+60-12i-1\\ &= -117-44i\end{aligned}$$

Exercise

Expand and simplify: $\left(\sqrt{2}+i\right)^5$

Solution

$$\begin{aligned}\left(\sqrt{2}+i\right)^5 &= 2\sqrt{2}+20i+20\sqrt{2}\,i^2+20i^3+\sqrt{2}\,i^4+i^5\\ &= 2\sqrt{2}+20i-20\sqrt{2}-20i+\sqrt{2}+i\\ &= -17\sqrt{2}+i\end{aligned}$$

Exercise

Expand and simplify: $(3-i)^4$

Solution

$$\begin{aligned}(3-i)^4 &= 84-108i+54i^2-12i^3+i^4\\ &= 84-108i-54+12i+1\\ &= 31-96i\end{aligned}$$