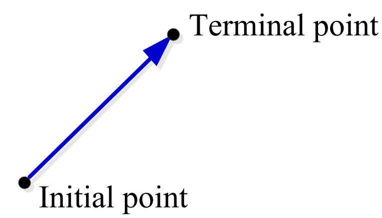
Lecture Two

Section 2.1 – Vectors in 2-Space, 3-Space, and *n*-Space

Vectors in two dimensions are also called **2–*space***

Vectors in three dimensions are also called **3–*space*** by arrow

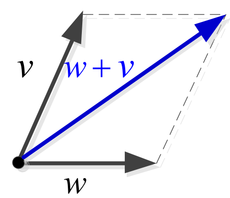
The direction of the arrowhead specifies the ***direction*** of the vector and the ***length*** of the arrow specifies the *magnitude*.



The tail of the arrow is called the ***initial point*** of the vector and the tip the ***terminal point***.

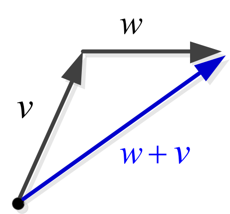
Parallelogram Rule for Vector Addition

If $\vec{v}$ and $\vec{w}$ are vectors in 2-space or 3-space that are positioned so their initial points coincides, then the vectors form adjacent sides of a parallelogram, and then the sum $\vec{v}+\vec{w}$ is the vector represented by the arrow from the common initial point of $\vec{v}$ and $\vec{w}$ to the opposite vertex of the parallelogram.

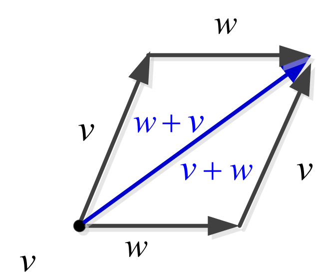


Triangle Rule for Vector Addition

If $\vec{v}$ and $\vec{w}$ are vectors in 2-space or 3-space that are positioned so the initial point of w is at the terminal point of $\vec{v}$, then the sum $\vec{v}+\vec{w}$ is represented by the arrow from the initial point of $\vec{v}$ to the terminal point of $\vec{w}$.

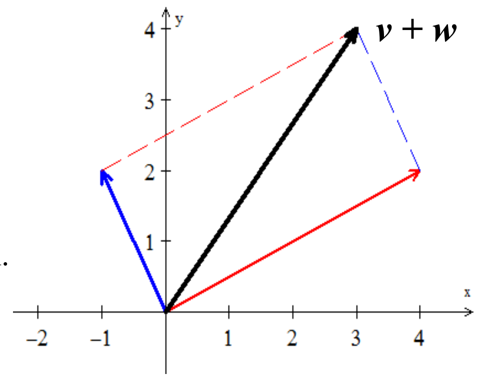


$$\boxed{\vec{v}+\vec{w}=\vec{w}+\vec{v}}$$



Example of Sum and Difference of vectors

Consider the vector $\vec{v}$ is given by the component $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and represented by an arrow. The arrow goes from 4 units to the right and 2 units up.

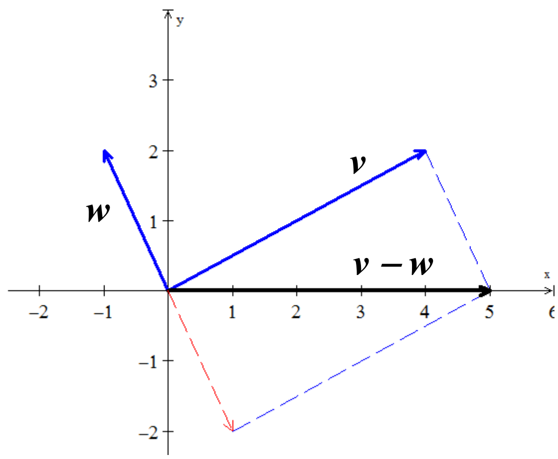


Consider anther vector $\vec{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Vector addition (head to tail) at the end of $\vec{v}$, place the start of $\vec{w}$.

The vector addition and w produces the diagonal of a parallelogram.

$$\vec{v} + \vec{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



$$\vec{v} - \vec{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

In 3-dimensional space, the arrow starts at the origin $(0,\ 0,\ 0)$, where the xyz axis meet.

$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is also written as $(1,\ 2,\ 2)$

Notes:

1. The picture of the combinations $c\vec{u}$ fills a line
2. The picture of the combinations $c\vec{u} + d\vec{v}$ fills a plane
3. The picture of the combinations $c\vec{u} + d\vec{v} + e\vec{w}$ fills a 3-dimensional space.

Linear Combination

Definition

The sum of $c\vec{v}$ and $d\vec{w}$ is a linear combination of vectors $\vec{v}$ and $\vec{w}$; c, d are constants.

4-Special Linear Combinations:

$$\begin{aligned} 1\vec{v} + 1\vec{w} &= \textit{sum of vectors} \\ 1\vec{v} - 1\vec{w} &= \textit{difference of vectors} \\ 0\vec{v} + 0\vec{w} &= \textit{zero vectors} \\ c\vec{v} + 0\vec{w} &= \textit{vector } c\vec{v} \textit{ in the direction of } \vec{v} \end{aligned}$$

Vectors in Coordinate Systems

It is sometimes necessary to consider vectors whose initial are not at the origin. If $\overrightarrow{P_1P_2}$ denotes the vector with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$, then the components of this vector are given by the formula

$$\boxed{\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)}$$

If $P_1(x_1, y_1, z_1)$ *and* $P_2(x_2, y_2, z_2)$

$$\boxed{\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)}$$

Example

The components of the vector $\vec{v} = \overrightarrow{P_1P_2}$ with initial point $P_1(2, -1, 4)$ and terminal point $P_2(7, 5, -8)$, find $\vec{v}$?

Solution

$$\vec{v} = (7-2,\ 5-(-1),\ -8-4)$$

$$= (5,\ 6,\ -12)$$

$n-$Space

The vector spaces are denoted by $\boldsymbol{R}^1, \boldsymbol{R}^2, \boldsymbol{R}^3, \boldsymbol{R}^4 \cdots$. Each space $\boldsymbol{R}^n$ consists of a whole collection of vectors.

Definition

The space $\boldsymbol{R}^n$ consists of all column vectors v with n components.

Example

$$\underset{\boldsymbol{R}^3}{\begin{bmatrix}1\\2\\3\end{bmatrix}} \qquad \underset{\boldsymbol{R}^5}{(1,\ 2,\ 3,\ 0,\ 1)} \qquad \underset{\boldsymbol{C}^2}{\begin{bmatrix}1+i\\1-i\end{bmatrix}}$$

The one-dimensional space $\boldsymbol{R}^1$ is a line (like the x-axis)

The two essential vector operations go on inside the vector space that we can add any vectors in $\boldsymbol{R}^n$, and we can multiply any vector by any scalar. The ***result*** stays in the space.

A real vector space is a set of "***vectors***" together with rules for vector addition and for multiplication by real numbers. The addition and the multiplication must produce vectors that are in the space.

Here are three other spaces other than $\boldsymbol{R}^n$:

M The vector space of ***all real 2 by 2 matrices***.

F The vector space of ***all real functions*** $f(x)$.

Z The vector space that consists only of a ***zero vector***.

The zero vector in $\boldsymbol{R}^3$ is the vector (0, 0, 0).

Operation on Vectors in $\mathbb{R}^n$

Definition

If n is a positive integer, then an ordered ***n-tuple*** is a sequence of real numbers $\left(v_1,\ v_2,\ \ldots,\ v_n\right)$. The set of all ordered n-tuples is called ***n-space*** and is denoted by $\mathbb{R}^n$

Definition

Vectors $\vec{v}=\left(v_1,\ v_2,\ \ldots,\ v_n\right)$ and $\vec{w}=\left(w_1,\ w_2,\ \ldots,\ w_n\right)$ in $\mathbb{R}^n$ are said to be ***equivalent*** (also called ***equal***) if

$$v_1=w_1,\quad v_2=w_2,\quad \ldots\quad v_n=w_n$$

We indicate this by $\vec{v}=\vec{w}$

Example

$$(a,b,c,d)=(1,-4,2,7)$$

Solution

$$Iff\ a=1,\quad b=-4,\quad c=2,\quad d=7$$

Vector Space of Infinite Sequences of Real Numbers

If $\vec{v}=\left(v_1,\ v_2,\ \ldots,\ v_n\right)$ and $\vec{w}=\left(w_1,\ w_2,\ \ldots,\ w_n\right)$ are vectors in $\mathbb{R}^n$, and if k is any scalar, then we defined

$$\begin{aligned}\vec{v}+\vec{w}&=\left(u_1,\ u_2,\ \ldots,\ u_n\right)+\left(w_1,\ w_2,\ \ldots,\ w_n\right)\\&=\left(v_1+w_1,\ v_2+w_2,\ \ldots,\ v_n+w_n\right)\end{aligned}$$

$$k\vec{v}=\left(kv_1,\ kv_2,\ \ldots,\ kv_n\right)$$

$$-\vec{v}=\left(-v_1,\ -v_2,\ \ldots,\ -v_n\right)$$

$$\vec{w}-\vec{v}=\vec{w}+(-\vec{v})=\left(w_1-v_1,\ w_2-v_2,\ \ldots,\ w_n-v_n\right)$$

The *Zero* Vector Space

Let V consist of a single object, which we denote by $\vec{0}$, and define

$$\vec{0}+\vec{0}=\vec{0}\quad and\quad k\vec{0}=\vec{0}$$

Theorem

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$, and if *k* and *m* are scalars, then

a) $\vec{u}+\vec{v}=\vec{v}+\vec{u}$

b) $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$

c) $\vec{u}+\vec{0}=\vec{0}+\vec{u}=\vec{u}$

d) $\vec{u}+(-\vec{u})=\vec{0}$

e) $k(\vec{u}+\vec{v})=k\vec{u}+k\vec{v}$

f) $(k+m)\vec{u}=k\vec{u}+m\vec{u}$

g) $k(m\vec{u})=(km)\vec{u}$

h) $1\vec{u}=\vec{u}$

i) $0\vec{v}=0$

j) $k\vec{0}=\vec{0}$

k) $(-1)\vec{v}=-\vec{v}$

Proof: $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$

Let $\vec{u}=(u_1,\ u_2,\ \ldots,\ u_n)$

$\vec{v}=(v_1,\ v_2,\ \ldots,\ v_n)$

$\vec{w}=(w_1,\ w_2,\ \ldots,\ w_n)$

$$\begin{aligned}(\vec{u}+\vec{v})+\vec{w}&=\left((u_1,\ u_2,\ \ldots,\ u_n)+(v_1,\ v_2,\ \ldots,\ v_n)\right)+(w_1,\ w_2,\ \ldots,\ w_n)\\
&=(u_1+v_1,\ u_2+v_2,\ \ldots,\ u_n+v_n)+(w_1,\ w_2,\ \ldots,\ w_n)\\
&=\left((u_1+v_1)+w_1,\ (u_2+v_2)+w_2,\ \ldots,\ (u_n+v_n)+w_n\right)\\
&=\left(u_1+(v_1+w_1),\ u_2+(v_2+w_2),\ \ldots,\ u_n+(v_n+w_n)\right)\\
&=(u_1,\ u_2,\ \ldots,\ u_n)+\left((v_1,\ v_2,\ \ldots,\ v_n)+(w_1,\ w_2,\ \ldots,\ w_n)\right)\\
&=\vec{u}+(\vec{v}+\vec{w})\end{aligned}$$

Exercises *Section* 2.1 – Vectors in 2-Space, 3-Space, and *n*-Space

1. Sketch the following vectors with initial points located at the origin

a) $P_1(4,8)\quad P_2(3,7)$ *b)* $P_1(-1,0,2)\quad P_2(0,-1,0)$ *c)* $P_1(3,-7,2)\quad P_2(-2,5,-4)$

2. Find the components of the vector $\overrightarrow{P_1P_2}$

a) $P_1(3,5)\quad P_2(2,8)$ *b)* $P_1(5,-2,1)\quad P_2(2,4,2)$ *c)* $P_1(0,\ 0,\ 0)\quad P_2(-1,\ 6,\ 1)$

3. Find the terminal point of the vector that is equivalent to $\vec{u} = (1, 2)$ and whose initial point is $A(1,1)$

4. Find the initial point of the vector that is equivalent to $\vec{u} = (1, 1, 3)$ and whose terminal point is $B(-1,-1,2)$

5. Find a nonzero vector $\vec{u}$ with initial point $P(-1, 3, -5)$ such that

a) $\vec{u}$ has the same direction as $\vec{v} = (6, 7, -3)$

b) $\vec{u}$ is oppositely directed as $\vec{v} = (6, 7, -3)$

6. Let $\vec{u} = (-3, 1, 2)$, $\vec{v} = (4, 0, -8)$, and $\vec{w} = (6, -1, -4)$. Find the components

a) $\vec{v}-\vec{w}$

b) $6\vec{u}+2\vec{v}$

c) $5(\vec{v}-4\vec{u})$

d) $-3(\vec{v}-8\vec{w})$

e) $(2\vec{u}-7\vec{w})-(8\vec{v}+\vec{u})$

f) $-\vec{u}+(\vec{v}-4\vec{w})$

7. Let $\vec{u} = (2, 1, 0, 1, -1)$ and $\vec{v} = (-2, 3, 1, 0, 2)$. Find scalars *a* and *b* so that $a\vec{u}+b\vec{v}=(-8,\ 8,\ 3,\ -1,\ 7)$

8. Find all scalars $c_1,\ c_2,\ and\ c_3$ such that $c_1(1, 2, 0)+c_2(2, 1, 1)+c_3(0, 3, 1)=(0, 0, 0)$

9. Find the distance between the given points $\begin{bmatrix}5 & 1 & 8 & -1 & 2 & 9\end{bmatrix},\ \begin{bmatrix}4 & 1 & 4 & 3 & 2 & 8\end{bmatrix}$

10. Let *V* be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operation on $\vec{u}=(u_1,\ u_2)\quad \vec{v}=(v_1,\ v_2)$

$$\vec{u}+\vec{v}=(u_1+v_1+1,\ u_2+v_2+1)\qquad k\vec{u}=(ku_1,\ ku_2)$$

a) Compute $\vec{u}+\vec{v}$ and $k\vec{u}$ for $\vec{u} = (0, 4)$, $\vec{v} = (1, -3)$, and $k = 2$.

b) Show that $(0, 0) \neq \vec{0}$.

c) Show that $(-1, -1) = 0$.

d) Show that $\vec{u}+(-\vec{u})=0$ for $\vec{u}=(u_1,\ u_2)$

e) Find two vector space axioms that fail to hold.

11. Find $\vec{w}$ given that $10\vec{u}+3\vec{w}=4\vec{v}-2\vec{w}$, $\vec{u}=\begin{pmatrix}1\\-6\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-20\\5\end{pmatrix}$

12. Find $\vec{w}$ given that $\vec{u}+3\vec{v}-2\vec{w}=5\vec{u}+\vec{v}-4\vec{w}$, $\vec{u}=\begin{pmatrix}1\\-1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-2\\3\end{pmatrix}$

13. Find $\vec{w}$ given that $2\vec{u}+\vec{v}-3\vec{w}=5\vec{u}+7\vec{v}+3\vec{w}$, $\vec{u}=\begin{pmatrix}1\\-1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-2\\3\end{pmatrix}$

(14 – 17) Draw $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, and $\vec{u}+2\vec{v}$

14. $\vec{u}=\begin{pmatrix}1\\-1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-2\\3\end{pmatrix}$

15. $\vec{u}=\begin{pmatrix}-1\\1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}2\\1\end{pmatrix}$

16. $\vec{u}=\begin{pmatrix}1\\0\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}1\\2\end{pmatrix}$

17. $\vec{u}=\begin{pmatrix}-1\\0\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}1\\-1\end{pmatrix}$

Section 2.2 – Norm, Dot product, and distance in R^n

Norm of a Vector

The ***length*** (or ***norm***) of a vector $\vec{v}$ is the square root of $\vec{v}\cdot\vec{v}$

$$Length = \|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$$

$$= \sqrt{x^2 + y^2} \qquad \textbf{2-dimension}$$

$$= \sqrt{x^2 + y^2 + z^2} \qquad \textbf{3-dimension}$$

Definition

If $\vec{v} = \left(v_1,\ v_2,\ \dots,\ v_n\right)$ is a vector in $\boldsymbol{R}^n$, then the norm of $\vec{v}$ (also called the length of $\vec{v}$ or the magnitude of $\vec{v}$) is denoted by $\|\vec{v}\|$, and is defined by the formula

$$\boxed{\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \dots + v_n^2}}$$

Example

Find the length of the vector $\vec{v} = (1,\ 2,\ 3)$

Solution

$$\|\vec{v}\| = \sqrt{1^2 + 2^2 + 3^2}$$

$$= \sqrt{14}$$

Theorem

If $\vec{v}$ is a vector in $\boldsymbol{R}^n$, and if $\boldsymbol{k}$ is any scalar, then:

a) $\|\vec{v}\| \ge 0$

b) $\|\vec{v}\| = 0 \quad iff \quad \vec{v} = \vec{0}$

c) $\|k\vec{v}\| = |k|\cdot\|\vec{v}\|$

Unit Vectors

Definition

A ***unit vector*** $\vec{u}$ is a vector whose length equals to one. Then $\vec{u}\cdot\vec{u}=1$

Divide any nonzero vector $\vec{v}$ by its length. Then $\vec{u}=\frac{\vec{v}}{\|\vec{v}\|}$ is a unit vector in the same direction as v.

Example

Find the unit vector $\vec{u}$ that has the same direction as $\vec{v}=(2,2,-1)$

Solution

$$\|\vec{v}\|=\sqrt{2^2+2^2+(-1)^2}$$

$$=3$$

$$\vec{u}=\frac{\vec{v}}{\|\vec{v}\|}$$

$$=\frac{1}{3}(2,\ 2,\ -1)$$

$$=\left(\frac{2}{3},\ \frac{2}{3},\ -\frac{1}{3}\right)$$

$$\|\vec{u}\|=\sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^2+\left(\frac{-1}{3}\right)^2}$$

$$=\sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}}$$

$$=\sqrt{\frac{9}{9}}$$

$$=1 \quad \checkmark$$

Example of unit vectors

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \hat{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

In $\boldsymbol{R}^3$

$$\hat{i} = (1,\ 0,\ 0) \quad \hat{j} = (0,\ 1,\ 0) \quad \textit{and} \quad \hat{k} = (0,\ 0,\ 1)$$

In general, these formulas can be defined as ***standard unit vector*** in $\boldsymbol{R}^n$

$$\hat{e}_1 = (1,\ 0,\ \cdots,\ 0), \quad \hat{e}_2 = (0,\ 1,\ \cdots,\ 0), \quad \ldots, \quad \hat{e}_n = (0,\ 0,\ \cdots,\ 1)$$

$$\begin{aligned} \vec{v} &= \left(v_1,\ v_2,\ \ldots,\ v_n\right) \\ &= v_1\hat{e}_1 + v_2\hat{e}_2 + \ldots + v_n\hat{e}_n \end{aligned}$$

Example

$$(7,\ 3,\ -4,\ 5) = 7\hat{e}_1 + 3\hat{e}_2 - 4\hat{e}_3 + 5\hat{e}_4$$

Distance in $\boldsymbol{R}^n$

In $\boldsymbol{R}^2$ $\quad d = \left\| \overrightarrow{P_1 P_2} \right\| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$

In $\boldsymbol{R}^3$ $\quad d\left(\vec{u},\ \vec{v}\right) = \left\| \overrightarrow{P_1 P_2} \right\| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$

Definition

If $\vec{u} = \left(u_1,\ u_2,\ \ldots,\ u_n\right)$ and $\vec{v} = \left(v_1,\ v_2,\ \ldots,\ v_n\right)$ are points in $\boldsymbol{R}^n$, then we denote the distance between u and v by $d\left(\vec{u},\ \vec{v}\right)$ and define it to be

$$d\left(\vec{u},\ \vec{v}\right) = \left\|\vec{u} - \vec{v}\right\|$$

$$= \sqrt{\left(u_1 - v_1\right)^2 + \left(u_2 - v_2\right)^2 + \ldots + \left(u_n - v_n\right)^2}$$

Dot Product

If $\vec{u}$ and $\vec{v}$ are nonzero vectors in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$, and if θ is he angle between $\vec{u}$ and $\vec{v}$, then the ***dot product*** (also called the ***Euclidean inner product***) of $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u} \bullet \vec{v}$ and is defined as

$$\vec{u} \bullet \vec{v} = \left\|\vec{u}\right\| \ \left\|\vec{v}\right\| \ \cos\theta$$

Cosine Formula

If $\vec{u}$ and $\vec{v}$ are nonzero vectors that implies

$$\Rightarrow \ \cos\theta = \frac{\vec{u} \bullet \vec{v}}{\left\|\vec{u}\right\| . \left\|\vec{v}\right\|}$$

Example

Find the dot product of the vectors $\vec{u} = \left(0,\ 0,\ 1\right)$ and $\vec{v} = \left(0,\ 2,\ 2\right)$ and have an angle of 45°.

Solution

$$\left\|\vec{u}\right\| = 1$$

$$\left\|\vec{v}\right\| = \sqrt{0 + 2^2 + 2^2}$$

$$= \sqrt{8}$$

$$= 2\sqrt{2}$$

$$\begin{aligned}\vec{u} \bullet \vec{v} &= \|\vec{u}\|\ \|\vec{v}\|\cos\theta \\ &= (1)\left(2\sqrt{2}\right)\cos 45° \\ &= \left(2\sqrt{2}\right)\frac{1}{\sqrt{2}} \\ &= 2\end{aligned}$$

Component Form of the Dot Product

The ***dot product*** or ***inner product*** of $\vec{v} = \left(v_1,\ v_2\right)$ and $\vec{w} = \left(w_1,\ w_2\right)$ is the number

$$\vec{v} \bullet \vec{w} = v_1 w_1 + v_2 w_2$$

Example

Find the dot product of $\vec{v} = (4,\ 2)$ and $\vec{w} = (-1,\ 2)$

Solution

$$\begin{aligned}\vec{v} \bullet \vec{w} &= 4.(-1) + 2(2) \\ &= 0\end{aligned}$$

- ***For dot products, zero means that the 2 vectors are perpendicular*** (= 90°).

Example

Put a weight of 4 at the point $x = -1$ and weight of 2 at the point $x = 2$. The *x*-axis will balance on the center point $x = 0$.

Solution

The weight balance is $4(-1) + 2(2) = 0$ (*dot product*).

In 3-dimensionals the dot product:

$$\left(v_1,\ v_2,\ v_3\right) \bullet \left(w_1,\ w_2,\ w_3\right) = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Theorem

a) $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$

b) $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$

c) $\vec{u} \bullet (\vec{v} - \vec{w}) = \vec{u} \bullet \vec{v} - \vec{u} \bullet \vec{w}$

d) $(\vec{u} + \vec{v}) \bullet \vec{w} = \vec{u} \bullet \vec{w} + \vec{v} \bullet \vec{w}$

e) $(\vec{u} - \vec{v}) \bullet \vec{w} = \vec{u} \bullet \vec{w} - \vec{v} \bullet \vec{w}$

f) $k(\vec{u} \bullet \vec{v}) = (k\vec{u}) \bullet \vec{v}$

g) $k(\vec{u} \bullet \vec{v}) = \vec{u} \bullet (k\vec{v})$

h) $\vec{v} \bullet \vec{v} \geq 0 \quad \textit{and} \quad \vec{v} \bullet \vec{v} = 0 \quad \textit{iff} \quad \vec{v} = 0$

i) $\vec{0} \bullet \vec{v} = \vec{v} \bullet \vec{0} = 0$

Right Angles

The dot product is $\vec{v} \bullet \vec{w} = 0$ when $\vec{v}$ is perpendicular to $\vec{w}$

Proof

Perpendicular vectors: $\|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{v} - \vec{w}\|^2$

Let $\vec{v} = (v_1, v_2)$ & $\vec{w} = (w_1, w_2)$

$$\begin{aligned}
\|\vec{v} - \vec{w}\|^2 &= (v_1 - w_1)^2 + (v_2 - w_2)^2 \\
&= v_1^2 - 2v_1 w_1 + w_1^2 + v_2^2 - 2v_2 w_2 + w_2^2 \\
&= v_1^2 + w_1^2 + v_2^2 + w_2^2 - 2(v_1 w_1 + v_2 w_2) \qquad v_1 w_1 + v_2 w_2 = 0 \quad \textit{dot product} \\
&= v_1^2 + w_1^2 + v_2^2 + w_2^2 \\
&= v_1^2 + v_2^2 + w_1^2 + w_2^2 \\
&= \|\vec{v}\|^2 + \|\vec{w}\|^2
\end{aligned}$$

If $\vec{u}$ and U are unit vectors, then $\vec{u} \bullet \vec{U} = \cos\theta$

Certainly,

$$\left|\vec{u} \bullet \vec{U}\right| \le 1$$

$$-1 \le \cos\theta \le 1$$

$$-1 \le dot\ product \le 1$$

Schwarz Inequality

If $\vec{v}$ and $\vec{w}$ are any vectors $\Rightarrow$ $\|\vec{v} \bullet \vec{w}\| \le \|\vec{v}\|.\|\vec{w}\|$

Proof

The dot product of $\vec{v} = (a,\ b)$ and $\vec{w} = (b,\ a)$ is $2ab$ and both lengths are $\sqrt{a^2 + b^2}$.

Then, the Schwarz inequality says that: $2ab \le a^2 + b^2$

$$a^2 + b^2 - 2ab = (a - b)^2 \ge 0$$

$$a^2 + b^2 - 2ab \ge 0$$

$$a^2 + b^2 \ge 2ab$$

This proves the Schwarz inequality:

$$2ab \le a^2 + b^2$$

$$\Rightarrow \|\vec{v} \bullet \vec{w}\| \le \|\vec{v}\|.\|\vec{w}\|$$

Theorem – Parallelogram Equation for Vectors

If $\vec{u}$ and $\vec{v}$ are vectors in $\boldsymbol{R}^n$, then

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right)$$

Proof

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) + (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) \\
&= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} + \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} \\
&= 2(\vec{u}\cdot\vec{u}) + 2(\vec{v}\cdot\vec{v}) \\
&= 2\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right)
\end{aligned}$$

Theorem

If $\vec{u}$ and $\vec{v}$ are vectors in $\boldsymbol{R}^n$ with the Euclidean Inner product, then

$$\vec{u} \bullet \vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2$$

Exercises *Section* 2.2 – Norm, Dot product, and distance in R^n

1. If $\|\vec{v}\| = 5$ and $\|\vec{w}\| = 3$, what are the smallest and largest possible values of $\|\vec{v} - \vec{w}\|$ and $\vec{v}.\vec{w}$?

2. If $\|\vec{v}\| = 7$ and $\|\vec{w}\| = 3$, what are the smallest and largest possible values of $\|\vec{v} + \vec{w}\|$ and $\vec{v} \bullet \vec{w}$?

3. Given that $\cos(\alpha) = \frac{\vec{v}_1}{\|\vec{v}\|}$ and $\sin(\alpha) = \frac{\vec{v}_2}{\|\vec{v}\|}$. Similarly, $\cos(\beta) =$ ______ and $\sin(\beta) =$ ______ . The angle θ is $\beta - \alpha$. Substitute into the trigonometry formula $\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ for $\cos(\beta - \alpha)$ to find $\cos\theta = \frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\|.\|\vec{w}\|}$

4. Can three vectors in the xy plane have $\vec{u} \bullet \vec{v} < 0$, $\vec{v} \bullet \vec{w} < 0$ and $\vec{u} \bullet \vec{w} < 0$?

5. Find the norm of $\vec{v}$, a unit vector that has the same direction as $\vec{v}$, and a unit vector that is oppositely directed.

a) $\vec{v} = (4, -3)$

b) $\vec{v} = (1, -1, 2)$

c) $\vec{v} = (-2, 3, 3, -1)$

6. Evaluate the given expression with $\vec{u}$ = (2, –2, 3), $\vec{v}$ = (1, –3, 4), and $\vec{w}$ = (3, 6, –4)

a) $\|\vec{u} + \vec{v}\|$

b) $\|-2\vec{u} + 2\vec{v}\|$

c) $\|3\vec{u} - 5\vec{v} + \vec{w}\|$

d) $\|3\vec{v}\| - 3\|\vec{v}\|$

e) $\|\vec{u}\| + \|-2\vec{v}\| + \|-3\vec{w}\|$

7. Let $\boldsymbol{v}$ = (1, 1, 2, –3, 1). Find all scalars k such that $\|k\vec{v}\| = 5$

8. Find $\vec{u} \bullet \vec{v}$, $\vec{u} \bullet \vec{u}$, *and* $\vec{v} \bullet \vec{v}$

a) $\vec{u} = (3, 1, 4)$, $\vec{v} = (2, 2, -4)$

b) $\vec{u} = (1, 1, 4, 6)$, $\vec{v} = (2, -2, 3, -2)$

c) $\vec{u} = (2, -1, 1, 0, -2)$, $\vec{v} = (1, 2, 2, 2, 1)$

9. Find the Euclidean distance between $\vec{u}$ and $\vec{v}$, then find the angle between them

a) $\vec{u} = (3, 3, 3)$, $\vec{v} = (1, 0, 4)$

b) $\vec{u} = (1, 2, -3, 0)$, $\vec{v} = (5, 1, 2, -2)$

c) $\vec{u} = (0, 1, 1, 1, 2)$, $\vec{v} = (2, 1, 0, -1, 3)$

10. Find a unit vector that has the same direction as the given vector

a) $(-4, -3)$

b) $\left(-3,\ 2,\ \sqrt{3}\right)$

c) $(1, 2, 3, 4, 5)$

11. Find a unit vector that is oppositely to the given vector

a) $(-12, -5)$ *b)* $(3, -3, 3)$ *c)* $\left(-3,\ 1,\ \sqrt{6},\ 3\right)$

12. Verify that the Cauchy-Schwarz inequality holds

a) $\vec{u} = (-3, 1, 0),\ \vec{v} = (2, -1, 3)$

b) $\vec{u} = (0, 2, 2, 1),\ \vec{v} = (1, 1, 1, 1)$

c) $\vec{u} = (1, 3, 5, 2, 0, 1),\ \vec{v} = (0, 2, 4, 1, 3, 5)$

13. Find $\vec{u} \bullet \vec{v}$ and then the angle θ between $\vec{u}$ and $\vec{v}$ $\vec{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$

14. Find the norm: $\|\vec{u}\| + \|\vec{v}\|,\ \|\vec{u} + \vec{v}\|$ for $\vec{u} = (3,\ -1,\ -2,\ 1,\ 4)$ $\vec{v} = (1,\ 1,\ 1,\ 1,\ 1)$

15. Find all numbers *r* such that: $\left\|r(1,\ 0,\ -3,\ -1,\ 4,\ 1)\right\| = 1$

16. Find the distance between $P_1(7, -5, 1)$ and $P_2(-7, -2, -1)$

17. Given $\vec{u} = (1, -5, 4),\ \vec{v} = (3, 3, 3)$

a) Find $\vec{u} \bullet \vec{v}$

b) Find the cosine of the angle θ between $\vec{u}$ and $\vec{v}$.

18. Let $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Find $\left\| \frac{1}{\|2\vec{u} + \vec{v}\|}(2\vec{u} + \vec{v}) \right\|$

19. Let $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$. Find $\left\| \frac{1}{\|\vec{u} - \vec{v}\|}(\vec{u} - \vec{v}) \right\|$

20. Let $\vec{u} = \begin{pmatrix} 18 \\ 6 \end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix} -11 \\ 12 \end{pmatrix}$. Find $\left\| \frac{1}{\|5\vec{u} + 3\vec{v}\|}(5\vec{u} + 3\vec{v}) \right\|$

21. Let $\vec{u} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$. Calculate the following:

a) $\vec{u} + \vec{v}$ *b)* $2\vec{u} + 3\vec{v}$ *c)* $\vec{v} + (2\vec{u} - 3\vec{v})$ *d)* $\|\vec{u}\|$ *e)* $\|\vec{v}\|$ *f)* *unit vector of* $\vec{v}$

22. Let $\vec{u}=\begin{pmatrix}2\\-1\\0\\1\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}1\\4\\3\\1\end{pmatrix}$. Calculate the following:

a) $\vec{u}-\vec{v}$ *b)* $3\vec{u}-2\vec{v}$ *c)* $2(\vec{u}-\vec{v})+3\vec{u}$ *d)* $\|\vec{u}\|$ *e) unit vector of* $\vec{v}$

23. Let $\vec{u}=\begin{pmatrix}2\\1\\3\\0\\-1\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}1\\0\\2\\1\\-1\end{pmatrix}$. Calculate the following:

a) $\vec{v}-\vec{u}$ *b)* $\vec{u}+3\vec{v}$ *c)* $3(\vec{u}+\vec{v})-3\vec{u}$ *d)* $\|\vec{v}\|$ *e) unit vector of* $\vec{v}$

24. Let $\vec{u}=\begin{pmatrix}2\\-1\\1\end{pmatrix}$, $\vec{v}=\begin{pmatrix}1\\2\\-2\end{pmatrix}$ and $\vec{w}=\begin{pmatrix}3\\2\\1\end{pmatrix}$. Calculate the following:

a) $\vec{u}\bullet\vec{v}$ *b)* $\vec{u}\bullet(\vec{v}+\vec{w})$ *c)* $(\vec{u}+2\vec{v})\bullet\vec{w}$ *d)* $\|(\vec{w}\bullet\vec{v})\vec{u}\|$

25. Let $\vec{u}=\begin{pmatrix}1\\3\\2\\1\end{pmatrix}$, $\vec{v}=\begin{pmatrix}-2\\5\\2\\-6\end{pmatrix}$ and $\vec{w}=\begin{pmatrix}4\\-1\\0\\-2\end{pmatrix}$. Calculate the following:

a) $\vec{u}\bullet\vec{v}$ *b)* $\vec{u}\bullet(\vec{v}+\vec{w})$ *c)* $(\vec{u}+\vec{v})\bullet(\vec{u}-\vec{v})$ *d)* $\|(\vec{w}\bullet\vec{v})\vec{u}\|$

26. Let $\vec{u}=\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}$, $\vec{v}=\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$ and $\vec{w}=\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$. Calculate the following:

a) $\vec{u}\bullet(\vec{v}+\vec{w})$ *b)* $\|(\vec{w}\bullet\vec{v})\vec{u}\|$ *c)* $(\vec{u}\bullet\vec{w})\vec{v}+(\vec{v}\bullet\vec{w})\vec{u}$ *d)* $(\vec{u}+2\vec{v})\bullet(\vec{u}-\vec{v})$

27. Suppose $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^n$ such that $\vec{u}\bullet\vec{v}=2$, $\vec{u}\bullet\vec{w}=-3$, and $\vec{v}\bullet\vec{w}=5$. If possible, calculate the following values:

a) $\vec{u}\bullet(\vec{v}+\vec{w})$ *d)* $\vec{w}\bullet(2\vec{v}-4\vec{u})$ *g)* $\vec{w}\bullet((\vec{u}\bullet\vec{w})\vec{u})$

b) $(\vec{u}+\vec{v})\bullet\vec{w}$ *e)* $(\vec{u}+\vec{v})\bullet(\vec{v}+\vec{w})$ *h)* $\vec{u}\bullet((\vec{u}\bullet\vec{v})\vec{v}+(\vec{u}\bullet\vec{w})\vec{w})$

c) $\vec{u}\bullet(2\vec{v}-\vec{w})$ *f)* $\vec{w}\bullet(5\vec{v}+\pi\vec{u})$

28. You are in an airplane flying from Chicago to Boston for a job interview. The compass in the cockpit of the plane shows that your plane is pointed due East, and the airspeed indicator on the plane shows that the plane is traveling through the air at 400 *mph*. there is a crosswind that affects your plane however, and the crosswind is blowing due South at 40 *mph*.
Given the crosswind you wonder; relative to the ground, in what direction are you really flying and how fast are you really traveling?

29. A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What speed and direction should the jetliner have in order for the resultant vector to be 500 *mph* due east?

30. A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?

31. A bird flies from its nest 5 *km* in the direction 60° north east, where it stops to rest on a tree. It then flies 10 *km* in the direction due southeast and lands atop a telephone pole. Place an *xy*-coordinate system so that the origin is the bird's nest, the *x*-axis points east, and the *y*-axis points north.

a) At what point is the tree located?
b) At what point is the telephone pole?

32. Prove $\vec{u} \bullet \vec{u} = \|\vec{u}\|^2 \ \geq 0$

33. Prove, for any vectors and $\vec{v}$ in $\mathbb{R}^2$ and any scalars c and d,
$(c\vec{u} + d\vec{v}) \bullet (c\vec{u} + d\vec{v}) = c^2 \|\vec{u}\|^2 + 2cd\vec{u} \bullet \vec{v} + d^2 \|\vec{v}\|^2$

34. Prove $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$

35. Prove Minkowski theorem: $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

Section 2.3 – Orthogonality

Definition

Two nonzero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are said to be ***orthogonal*** (or ***perpendicular***) if their dot product is zero $\vec{u} \bullet \vec{v} = 0$.

We will also agree that he zero vector in $\mathbb{R}^n$ is orthogonal to every vector in $\mathbb{R}^n$. A nonempty set of vectors $\mathbb{R}^n$ is called an ***orthogonal set*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set of unit vectors is called an ***orthonormal set***.

Example

The floor of your room (extended to infinity) is a subspace ***V***. The line where two walls meet is a subspace ***W*** (one-dimensional). Those subspaces are orthogonal. Every vector up the meeting line is perpendicular to every vector on the floor. The origin (0, 0, 0) is in the corner.

Example

Show that $\vec{u} = (-2, 3, 1, 4)$ and $\vec{v} = (1, 2, 0, -1)$ are orthogonal in $\mathbb{R}^4$

Solution

$$\begin{aligned}\vec{u} \bullet \vec{v} &= (-2)(1) + (3)(2) + (1)(0) + (4)(-1) \\ &= -2 + 6 + 0 - 4 \\ &= 0\end{aligned}$$

These vectors are orthogonal in $\mathbb{R}^4$

Standard Unit Vectors

$$\hat{i} \bullet \hat{j} = \hat{i} \bullet \hat{k} = \hat{j} \bullet \hat{k} = \mathbf{0}$$

Proof

$$\begin{aligned}\hat{i} \bullet \hat{j} &= (1,\ 0,\ 0) \bullet (0,\ 1,\ 0) \\ &= 0\end{aligned}$$

Normal

To specify slope and inclination is to use a nonzero vector $\vec{n}$, called a ***normal***, that is orthogonal to the line or plane.

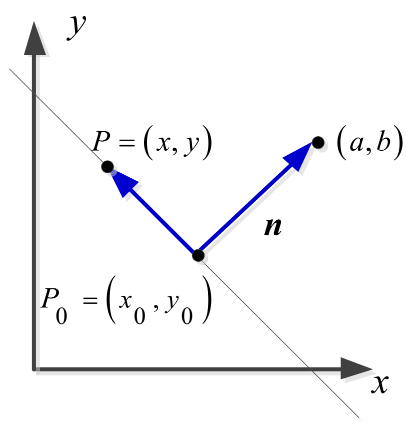
The line passes through a point $P_0\left(x_0, y_0\right)$ that has a normal $\vec{n} = (a,\ b)$

The plane through $P_0\left(x_0, y_0, z_0\right)$ that has a normal $\vec{n} = (a,\ b,\ c)$.

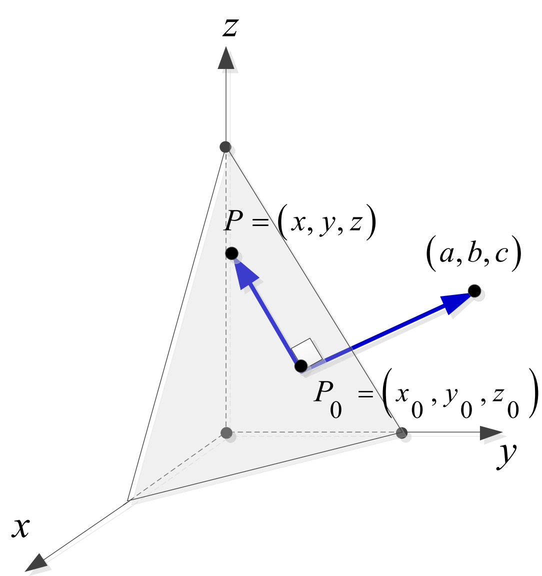
Both the line and the plane are represented by the vector equation

$$\vec{n} \bullet \overrightarrow{P_0 P} = 0$$

The line equation: $\quad a\left(x - x_0\right) + b\left(y - y_0\right) = 0$



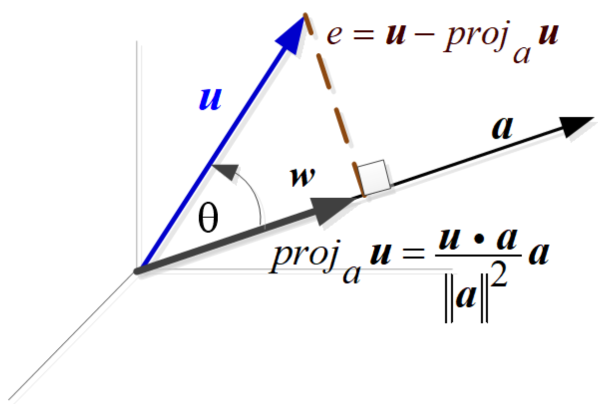
The plane equation: $\quad a\left(x - x_0\right) + b\left(y - y_0\right) + c\left(z - z_0\right) = 0$



Projections

Theorem **Projection onto a line**

If $\vec{u}$ and $\vec{a}$ are vectors in $\boldsymbol{R}^n$, and if $\vec{a} \neq 0$, then $\vec{u}$ can be expressed in exactly one way in the form $\vec{u} = \vec{w} + \vec{e}$, where $\vec{w}$ is a scalar multiple of $\vec{a}$ and $\vec{e}$ is orthogonal to $\vec{a}$.



The vector $\vec{w}$ is called the ***orthogonal projection*** of $\vec{u}$ on $\vec{a}$ or sometimes ***component*** of $\vec{u}$ along $\vec{a}$. The vector $\vec{e}$ is called the vector ***component*** of $\vec{u}$ ***orthogonal*** to $\vec{a}$ (error vector and should be perpendicular to $\vec{a}$)

$$proj_{\vec{a}}\vec{u} = \frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a} = \vec{p} \qquad (\textit{vector component of } \vec{u} \textit{ along } \vec{a})$$

$$\vec{u} - proj_{\vec{a}}\vec{u} = \vec{u} - \frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a} \qquad (\textit{vector component of } \vec{u} \textit{ orthogonal to } \vec{a})$$

The length is $\left\|proj_{\vec{a}}\vec{u}\right\| = \|\vec{u}\| |\cos\theta|$

$$\left\|proj_{\vec{a}}\vec{u}\right\| = \frac{|\vec{u} \bullet \vec{a}|}{\|\vec{a}\|}$$

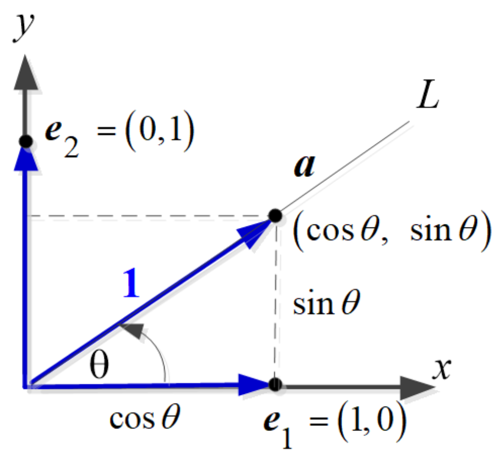
Special case: If $\vec{u} = \vec{a}$ then $\frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2} = 1$. The projection of $\vec{a}$ onto $\vec{a}$ is itself.

Special case: If $\vec{u}$ is perpendicular to $\vec{a}$ then $\vec{u} \bullet \vec{a} = 0$. The projection is $\vec{p} = \vec{0}$.

Example

Find the orthogonal projections of the vectors $\hat{e}_1 = (1,\ 0)$ and $\hat{e}_2 = (0,\ 1)$ on the line L that makes an angle θ with the positive x-axis in $\mathbb{R}^2$

Solution



Let $\vec{a} = (\cos\theta,\ \sin\theta)$ be the unit vector along the line L.

$$\|\vec{a}\| = \sqrt{\cos^2\theta + \sin^2\theta}$$

$$= 1$$

$$\hat{e}_1 \bullet \vec{a} = (1,0)(\cos\theta, \sin\theta)$$

$$= (1)\cos\theta + (0)\sin\theta$$

$$= \cos\theta$$

$$proj_{\vec{a}}\hat{e}_1 = \frac{\hat{e}_1 \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a}$$

$$= \frac{\cos\theta}{1}(\cos\theta,\ \sin\theta)$$

$$= \left(\cos^2\theta,\ \cos\theta\sin\theta\right)$$

$$proj_{\vec{a}}\hat{e}_2 = \frac{\hat{e}_2 \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a}$$

$$= \frac{(0,\ 1)(\cos\theta,\ \sin\theta)}{1}(\cos\theta,\ \sin\theta)$$

$$= \sin\theta(\cos\theta,\ \sin\theta)$$

$$= \left(\sin\theta\cos\theta,\ \sin^2\theta\right)$$

Example

Let $\vec{u} = (2, -1, 3)$ and $\vec{a} = (4, -1, 2)$. Find the vector component of $\vec{u}$ along $\vec{a}$ and the vector component of $\vec{u}$ orthogonal to $\vec{a}$.

Solution

$$\begin{aligned}
proj_{\vec{a}}\vec{u} &= \frac{\vec{u} \bullet \vec{a}}{\|\vec{a}\|^2}\vec{a} \\
&= \frac{(2, \ -1, \ 3)\bullet(4, \ -1, \ 2)}{\left(\sqrt{4^2+(-1)^2+2^2}\right)^2}(4, \ -1, \ 2) \\
&= \frac{8+1+6}{21}(4, \ -1, \ 2) \\
&= \frac{15}{21}(4, \ -1, \ 2) \\
&= \frac{5}{7}(4, \ -1, \ 2) \\
&= \left(\frac{20}{7}, \ -\frac{5}{7}, \ \frac{10}{7}\right)
\end{aligned}$$

The vector component of $\vec{u}$ orthogonal to $\vec{a}$ is

$$\begin{aligned}
\vec{u} - proj_{\vec{a}}\vec{u} &= (2, \ -1, \ 3) - \left(\frac{20}{7}, \ -\frac{5}{7}, \ \frac{10}{7}\right) \\
&= \left(-\frac{6}{7}, \ -\frac{2}{7}, \ \frac{11}{7}\right)
\end{aligned}$$

Theorem of *Pythagoras* in $\mathbb{R}^n$

If $\vec{u}$ and $\vec{v}$ are orthogonal vectors in $\mathbb{R}^n$ with the Euclidean inner product, then

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Proof

Since $\vec{u}$ and $\vec{v}$ are orthogonal, then $\vec{u} \bullet \vec{v} = 0$

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v}) \bullet (\vec{u}+\vec{v}) \\
&= \|\vec{u}\|^2 + 2(\vec{u} \bullet \vec{v}) + \|\vec{v}\|^2 \\
&= \|\vec{u}\|^2 + \|\vec{v}\|^2
\end{aligned}$$

Distance

Theorem

In $\mathbb{R}^2$ the distance D between the point $P_0 = (x_0,\ y_0)$ and the line $ax + by + c = 0$ is

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

In $\mathbb{R}^3$ the distance D between the point $P_0 = (x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Exercises Section 2.3 – Orthogonality

1. Determine whether $\vec{u}$ and $\vec{v}$ are orthogonal

a) $\vec{u}=(-6,\ -2),\quad \vec{v}=(5,\ -7)$

b) $\vec{u}=(6,\ 1,\ 4),\quad \vec{v}=(2,\ 0,\ -3)$

c) $\vec{u}=(1,\ -5,\ 4),\quad \vec{v}=(3,\ 3,\ 3)$

d) $\vec{u}=(-2,\ 2,\ 3),\quad \vec{v}=(1,\ 7,\ -4)$

2. Determine whether the vectors form an orthogonal set

a) $\vec{v}_1=(2,\ 3),\quad \vec{v}_2=(3,\ 2)$

b) $\vec{v}_1=(1,\ -2),\quad \vec{v}_2=(-2,\ 1)$

c) $\vec{u}=(-4,\ 6,\ -10,\ 1)\quad \vec{v}=(2,\ 1,\ -2,\ 9)$

d) $\vec{u}=(a,\ b)\quad \vec{v}=(-b,\ a)$

e) $\vec{v}_1=(-2,\ 1,\ 1),\quad \vec{v}_2=(1,\ 0,\ 2),\quad \vec{v}_3=(-2,\ -5,\ 1)$

f) $\vec{v}_1=(1,\ 0,\ 1),\quad \vec{v}_2=(1,\ 1,\ 1),\quad \vec{v}_3=(-1,\ 0,\ 1)$

g) $\vec{v}_1=(2,\ -2,\ 1),\quad \vec{v}_2=(2,\ 1,\ -2),\quad \vec{v}_3=(1,\ 2,\ 2)$

3. Find a unit vector that is orthogonal to both $\vec{u}=(1, 0, 1)$ and $\vec{v}=(0, 1, 1)$

4. *a*) Show that $\vec{v}=(a, b)$ and $\vec{w}=(-b, a)$ are orthogonal vectors.

b) Use the result to find two vectors that are orthogonal to $\vec{v}=(2, -3)$.

c) Find two unit vectors that are orthogonal to $(-3, 4)$

5. Find the vector component of $\vec{u}$ along $\vec{a}$ and the vector component of $\vec{u}$ orthogonal to $\vec{a}$.

a) $\vec{u}=(6,\ 2),\quad \vec{a}=(3,\ -9)$

b) $\vec{u}=(3,\ 1,\ -7),\quad \vec{a}=(1,\ 0,\ 5)$

c) $\vec{u}=(1,\ 0,\ 0),\quad \vec{a}=(4,\ 3,\ 8)$

d) $\vec{u}=(1,\ 1,\ 1),\quad \vec{a}=(0,\ 2,\ -1)$

e) $\vec{u}=(2,\ 1,\ 1,\ 2),\quad \vec{a}=(4,\ -4,\ 2,\ -2)$

f) $\vec{u}=(5,\ 0,\ -3,\ 7),\quad \vec{a}=(2,\ 1,\ -1,-1)$

6. Project the vector $\vec{v}$ onto the line through $\vec{a}$, check that $\vec{e}=\vec{u}-proj_{\vec{a}}\vec{u}$ is perpendicular to $\vec{a}$:

a) $\vec{v}=\begin{pmatrix}1\\2\\2\end{pmatrix}$ *and* $\vec{a}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ *b)* $\vec{v}=\begin{pmatrix}1\\3\\1\end{pmatrix}$ *and* $\vec{a}=\begin{pmatrix}-1\\-3\\-1\end{pmatrix}$ *c)* $\vec{v}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ *and* $\vec{a}=\begin{pmatrix}1\\2\\2\end{pmatrix}$

7. Find the projection matrix $proj_{\vec{a}}\vec{u}=\dfrac{\vec{u}\bullet\vec{a}}{\|\vec{a}\|^2}\vec{a}$ onto the line through $\vec{a}=\begin{bmatrix}1\\2\\2\end{bmatrix}$

(8 – 9) Draw the projection of $\vec{b}$ onto $\vec{a}$ and also compute it from $proj_{\vec{a}}\vec{u} = \dfrac{\vec{u}\bullet\vec{a}}{\|\vec{a}\|^2}\vec{a}$

8. $\vec{b} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ *and* $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

9. $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and* $\vec{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

10. Show that if $\vec{v}$ is orthogonal to both $\vec{w}_1$ and $\vec{w}_2$, then $\boldsymbol{v}$ is orthogonal to $k_1\vec{w}_1 + k_2\vec{w}_2$ for all scalars k_1 and k_2.

11. *a*) Project the vector $\vec{v}$ = (3, 4, 4) onto the line through $\vec{a}$ = (2, 2, 1) and then onto the plane that also contains $\vec{a}^* = (1,\ 0,\ 0)$.

b) Check that the first error vector $\vec{v} - \vec{p}$ is perpendicular to $\vec{a}$, and the second error vector $\vec{v} - \vec{p}^*$ is also perpendicular to $\vec{a}^*$.

12. Compute the projection matrices $\vec{a}\vec{a}^T / \vec{a}^T\vec{a}$ onto the lines through $\vec{a}_1 = (-1,\ 2,\ 2)$ and $\vec{a}_2 = (2,\ 2,\ -1)$. Multiply those projection matrices and explain why their product P_1P_2 is what it is. Project $\vec{v} = (1,\ 0,\ 0)$ onto the lines $\vec{a}_1$, $\vec{a}_2$, and also onto $\vec{a}_3 = (2,\ -1,\ 2)$.Add up the three projections $p_1 + p_2 + p_3$.

13. If $P^2 = P$ show that $(I-P)^2 = I - P$. When P projects onto the column space of A, $I - P$ projects onto the ____.

14. What linear combination of $(1,\ 2,\ -1)$ and $(1,\ 0,\ 1)$ is closest to $\vec{v} = (2,\ 1,\ 1)$?

15. Show that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$ if and only if $\|\vec{u}\| = \|\vec{v}\|$

16. Given $\vec{u} = (3,\ -1,\ 2)$ $\vec{v} = (4,\ -1,\ 5)$ *and* $\vec{w} = (8,\ -7,\ -6)$

a) Find $3\vec{v} - 4(5\vec{u} - 6\vec{w})$

b) Find $\vec{u}\bullet\vec{v}$ and then the angle θ between $\vec{u}$ and $\vec{v}$.

17. Given: $\vec{u} = (3,\ 1,\ 3)$ $\vec{v} = (4,\ 1,\ -2)$

a) Compute the projection $\vec{w}$ of $\vec{u}$ on $\vec{v}$

b) Find $\vec{p} = \vec{u} - \vec{v}$ and show that $\vec{p}$ is perpendicular to $\vec{v}$.

18. *a*) Show that $\vec{v}$ = (*a, b*) and $\vec{w}$ = (–*b, a*) are orthogonal vectors

b) Use the result in part (*a*) to find two vectors that are orthogonal to $\vec{v}$ = (2, –3)

c) Find two unit vectors that are orthogonal to (–3, 4)

19. Show that *A* (3, 0, 2), *B* (4, 3, 0), and *C* (8, 1, −1) are vertices of a right triangle. At which vertex is the right angle?

20. Establish the identity: $\vec{u} \bullet \vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2$

21. Find the Euclidean inner product $\vec{u} \bullet \vec{v}$: $\vec{u} = (-1,\ 1,\ 0,\ 4,\ -3)$ $\vec{v} = (-2,\ -2,\ 0,\ 2,\ -1)$

22. Find the Euclidean distance between $\vec{u}$ and $\vec{v}$: $\vec{u} = (3,\ -3,\ -2,\ 0,\ -3)$ $\vec{v} = (-4,\ 1,\ -1,\ 5,\ 0)$

(*Exercises* **22 – 26**) Find

a) $\vec{v} \bullet \vec{u},\ |\vec{v}|,\ |\vec{u}|$
b) The cosine of the angle between $\vec{v}$ and $\vec{u}$
c) The scalar component of $\vec{u}$ in the direction of $\vec{v}$
d) The vector $proj_{\vec{v}}\vec{u}$

23. $\vec{v} = 2\hat{i} - 4\hat{j} + \sqrt{5}\,\hat{k}, \quad \vec{u} = -2\hat{i} + 4\hat{j} - \sqrt{5}\,\hat{k}$

24. $\vec{v} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{k}, \quad \vec{u} = 5\hat{i} + 12\hat{j}$

25. $\vec{v} = 2\hat{i} + 10\hat{j} - 11\hat{k}, \quad \vec{u} = 2\hat{i} + 2\hat{j} + \hat{k}$

26. $\vec{v} = 5\hat{i} + \hat{j}, \quad \vec{u} = 2\hat{i} + \sqrt{17}\,\hat{j}$

27. $\vec{v} = \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}}\right), \quad \vec{u} = \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{3}}\right)$

28. Suppose Ted weighs 180 *lb*. and he is sitting on an inclined plane that drops 3 *units* for every 4 horizontal units. The gravitational force vector is $\vec{F}_g = \begin{pmatrix} 0 \\ -180 \end{pmatrix}$.

a) Find the force pushing Ted down the slope.
b) Find the force acting to hold Ted against the slope

29. Prove that is two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ are orthogonal to nonzero vector $\vec{w}$ in $\mathbb{R}^2$, then $\vec{u}$ and $\vec{v}$ are scalar multiples of each other.

Section 2.4 – Cross Product

The *Cross* Product

To find a vector in 3-space that is perpendicular to two vectors; the type of vector multiplication that facilities this construction is the cross product.

Definition

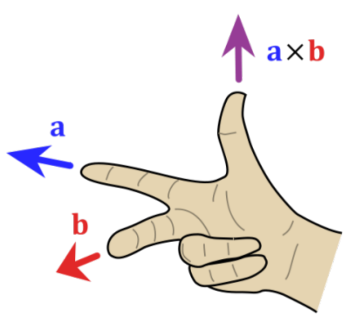
The cross product of $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ is the vector

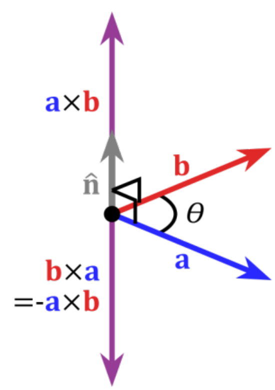
$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

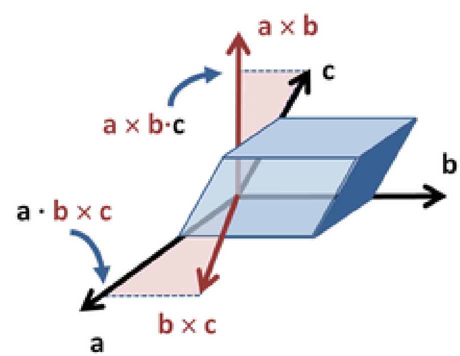
$$= \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \hat{i} - \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} \hat{j} + \begin{vmatrix} u_1 & v_2 \\ u_2 & v_1 \end{vmatrix} \hat{k}$$

$$= (u_2 v_3 - u_3 v_2)\hat{i} - (u_1 v_3 - u_3 v_1)\hat{j} + (u_1 v_2 - u_2 v_1)\hat{k}$$

$$= \left(u_2 v_3 - u_3 v_2, \; u_3 v_1 - u_1 v_3, \; u_1 v_2 - u_2 v_1\right)$$







- In 1773, ***Joseph Louis Lagrange*** introduced the component form of both the dot and cross products in order to study the tetrahedron in three dimensions. In 1843 the Irish mathematical physicist Sir ***William Rowan Hamilton*** introduced the quaternion product, and with it the terms "*vector*" and "*scalar*". Given two quaternions $[0, \vec{u}\,]$ and $[0, \vec{v}\,]$, where $\vec{u}$ and $\vec{v}$ are vectors in $\mathbf{R}^3$, their quaternion product can be summarized as $\left[-\vec{u} \bullet \vec{v}, \; \vec{u} \times \vec{v}\right]$. ***James Clerk Maxwell*** used Hamilton's quaternion tools to develop his famous ***electromagnetism*** equations, and for this and other reasons quaternions for a time were an essential part of physics education.

Example

Find $\vec{u}\times\vec{v}$, where $\vec{u}=(1,\ 2,\ -2)$ and $\vec{v}=(3,\ 0,\ 1)$

Solution

$$\begin{bmatrix} 1 & 2 & -2 \\ 3 & 0 & 1 \end{bmatrix}$$

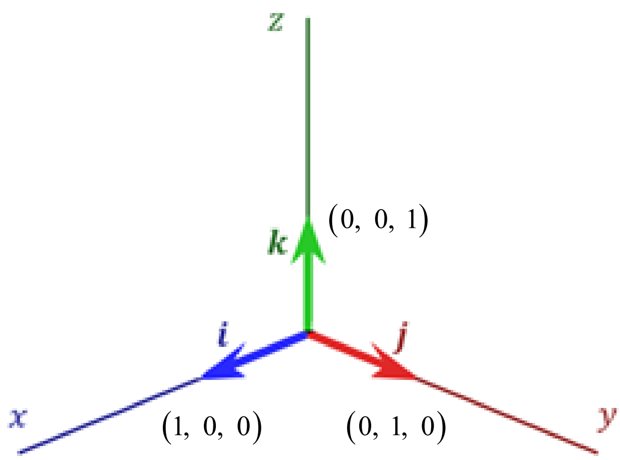
$$\vec{u}\times\vec{v}=\left(\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix},\ -\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix},\ \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}\right)$$

$$=(2,\ -7,\ -6)$$

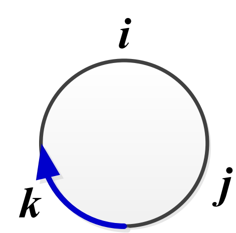
Example

Consider the vectors $\hat{\boldsymbol{i}}=(1,\ 0,\ 0)$ $\hat{\boldsymbol{j}}=(0,\ 1,\ 0)$ $\hat{\boldsymbol{k}}=(0,\ 0,\ 1)$

These vectors each have length of 1 and lie along the coordinate axes. They are called the ***standard unit vectors*** in 3-space.



For example: $(2,\ 3,\ -4)=2\hat{i}+3\hat{j}-4\hat{k}$



Note:

- ✓ $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{\mathbf{0}}$
- ✓ $\hat{j}\times\hat{i}=-\hat{k},\quad \hat{k}\times\hat{j}=-\hat{i},\quad \hat{i}\times\hat{k}=-\hat{j}$
- ✓ $\hat{i}\times\hat{j}=\hat{k},\quad \hat{j}\times\hat{k}=\hat{i},\quad \hat{k}\times\hat{i}=\hat{j}$

Properties

1. $\vec{u}\times\vec{v}$ reverses rows 2 and 3 in the determinant so it is equals $-(\vec{u}\times\vec{v})$

2. The cross product $\vec{u}\times\vec{v}$ is perpendicular to $\vec{u}$, then $\vec{u}\bullet(\vec{u}\times\vec{v})=0$

3. The cross product $\vec{u}\times\vec{v}$ is perpendicular to $\vec{v}$, then $(\vec{u}\times\vec{v})\bullet\vec{v}=0$

4. The cross product of any vector with itself (two equal rows) is $\vec{u}\times\vec{u}=0$.

5. Lagrange's identity: $\|\vec{u}\times\vec{v}\|^2=\|\vec{u}\|^2\ \|\vec{v}\|^2-(\vec{u}\bullet\vec{v})^2$
$$=\|\vec{u}\|\ \|\vec{v}\|\ |\sin\theta|$$

$$|\vec{u}\bullet\vec{v}|=\|\vec{u}\|\ \|\vec{v}\|\ |\cos\theta|$$

Theorem

a) $\vec{u}\times\vec{v}=-(\vec{v}\times\vec{u})$

b) $\vec{u}\times(\vec{v}+\vec{w})=(\vec{u}\times\vec{v})+(\vec{u}\times\vec{w})$

c) $(\vec{u}+\vec{v})\times\vec{w}=(\vec{u}\times\vec{w})+(\vec{v}\times\vec{w})$

d) $k(\vec{u}\times\vec{v})=(k\vec{u})\times\vec{v}=\vec{u}\times(k\vec{v})$

e) $\vec{u}\times\vec{0}=\vec{0}\times\vec{u}=\vec{0}$

f) $\vec{u}\times\vec{u}=0$

Definition

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in 3-space, then $\boxed{\vec{u}\bullet(\vec{v}\times\vec{w})}$ is called the ***scalar triple product*** of $\vec{u}$, $\vec{v}$, and $\vec{w}$.

Example

Calculate the scalar triple product $\vec{u}\bullet(\vec{u}\times\vec{v})$ of the vectors:

$$\vec{u}=-2\hat{i}+6\hat{k}\quad \vec{v}=\hat{i}-3\hat{j}+\hat{k}\quad \vec{w}=-5\hat{i}-\hat{j}+\hat{k}$$

Solution

$$\vec{u}\bullet(\vec{u}\times\vec{v})=\begin{vmatrix}-2 & 0 & 6\\ 1 & -3 & 1\\ -5 & -1 & 1\end{vmatrix}$$

$$\underline{=-92}$$

Area of a Parallelogram

Theorem

If $\vec{u}$ and $\vec{v}$ are vectors in 3-space, then $\|\vec{u}\times\vec{v}\|$ is equal to the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.

Example

Find the area of the triangle determined by the points $P_1(2,\ 2,\ 0),\ P_2(-1,\ 0,\ 2),\ \textit{and}\ \ P_3(0,\ 4,\ 3)$.

Solution

The area of the triangle is $\frac{1}{2}$ the area of the parallelogram determined by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$

$$\begin{aligned}\overrightarrow{P_1P_2} &= (-1,\ 0,\ 2)-(2,\ 2,\ 0)\\ &= (-3,\ -2,\ 2)\end{aligned}$$

$$\begin{aligned}\overrightarrow{P_1P_3} &= (0,\ 4,\ 3)-(2,\ 2,\ 0)\\ &= (-2,\ 2,\ 3)\end{aligned}$$

$$\begin{aligned}\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3} &= \left(\begin{vmatrix}-2 & 2\\ 2 & 3\end{vmatrix},\ -\begin{vmatrix}-3 & 2\\ -2 & 3\end{vmatrix},\ \begin{vmatrix}-3 & -2\\ -2 & 2\end{vmatrix}\right)\\ &= (-10,\ 5,\ -10)\end{aligned}$$

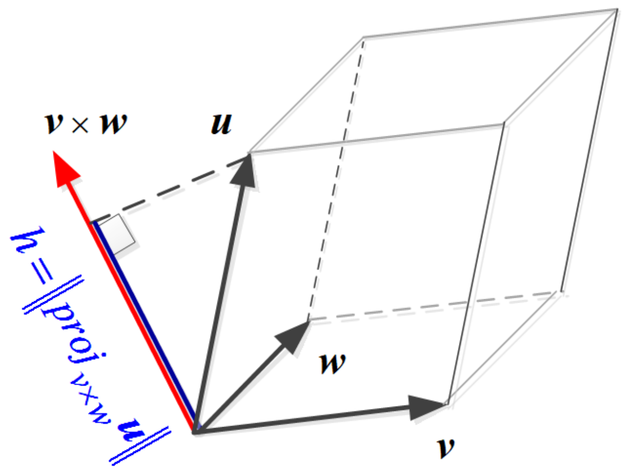
$$\begin{aligned}Area &= \frac{1}{2}\left\|\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}\right\|\\ &= \frac{1}{2}\sqrt{(-10)^2+5^2+(-10)^2}\\ &= \frac{15}{2}\end{aligned}$$

Volume

The Volume of the Parallelepiped is

$$V = (area\ of\ base).(height) = \|\vec{v}\times\vec{w}\| \frac{|\vec{u} \bullet (\vec{v}\times\vec{w})|}{\|\vec{v}\times\vec{w}\|} = |\vec{u} \bullet (\vec{u}\times\vec{v})|$$

$$V = \left| \det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \right|$$



Theorem

If the vectors $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, and $\vec{w} = (w_1, w_2, w_3)$ have the initial point, then they lie in the same plane if and only if

$$\vec{u} \bullet (\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$$

Example

Find the volume of the parallelepiped with sides $\vec{u} = (2, -6, 2)$, $\vec{v} = (0, 4, -2)$, and $\vec{w} = (2, 2, -4)$

Solution

$$V = \left| \det\begin{bmatrix} 2 & -6 & 2 \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{bmatrix} \right|$$

$$= 16$$

Exercises Section 2.4 – Cross Product

1. Prove when the cross product $\vec{u}\times\vec{v}$ is perpendicular to $\vec{u}$, then $\vec{u}\bullet(\vec{u}\times\vec{v})=0$

2. Find $\vec{u}\times\vec{v}$, where $\vec{u}=(1,\ 2,\ -2)$ *and* $\vec{v}=(3,\ 0,\ 1)$ and show that $\vec{u}\times\vec{v}$ is perpendicular to $\vec{u}$ and to $\vec{v}$.

3. Given $\vec{u}=(3,\ 2,\ -1)$, $\vec{v}=(0,\ 2,\ -3)$, *and* $\vec{w}=(2,\ 6,\ 7)$ Compute the vectors

a) $\vec{u}\times\vec{v}$

b) $\vec{v}\times\vec{w}$

c) $\vec{u}\times(\vec{v}\times\vec{w})$

d) $(\vec{u}\times\vec{v})\times\vec{w}$

e) $\vec{u}\times(\vec{v}-2\vec{w})$

4. Use the cross product to find a vector that is orthogonal to both

a) $\vec{u}=(-6,\ 4,\ 2)$, $\vec{v}=(3,\ 1,\ 5)$

b) $\vec{u}=(1,\ 1,\ -2)$, $\vec{v}=(2,\ -1,\ 2)$

c) $\vec{u}=(-2,\ 1,\ 5)$, $\vec{v}=(3,\ 0,\ -3)$

5. Find the area of the parallelogram determined by the given vectors

a) $\vec{u}=(1,\ -1,\ 2)$ *and* $\vec{v}=(0,\ 3,\ 1)$

b) $\vec{u}=(3,\ -1,\ 4)$ *and* $\vec{v}=(6,\ -2,\ 8)$

c) $\vec{u}=(2,\ 3,\ 0)$ *and* $\vec{v}=(-1,\ 2,\ -2)$

6. Find the area of the parallelogram with the given vertices

$P_1(3,\ 2)$, $P_2(5,\ 4)$, $P_3(9,\ 4)$, $P_4(7,\ 2)$

7. Find the area of the triangle with the given vertices:

a) $A(2,\ 0)$ $B(3,\ 4)$ $C(-1,\ 2)$

b) $A(1,\ 1)$ $B(2,\ 2)$ $C(3,\ -3)$

c) $P(2,\ 6,\ -1)$ $Q(1,\ 1,\ 1)$ $R=(4,\ 6,\ 2)$

8. *a*) Find the area of the parallelogram with edges $\vec{v}=(3,\ 2)$ and $\vec{w}=(1,\ 4)$

b) Find the area of the triangle with sides $\vec{v}$, $\vec{w}$, and $\vec{v}+\vec{w}$. Draw it.

c) Find the area of the triangle with sides $\vec{v}$, $\vec{w}$, and $\vec{v}-\vec{w}$. Draw it.

9. Find the volume of the parallelepiped with sides $\vec{u}$, $\vec{v}$, and $\vec{w}$.

a) $\vec{u}=(2,\ -6,\ 2)$, $\vec{v}=(0,\ 4,\ -2)$, $\vec{w}=(2,\ 2,\ -4)$

b) $\vec{u}=(3,\ 1,\ 2)$, $\vec{v}=(4,\ 5,\ 1)$, $\vec{w}=(1,\ 2,\ 4)$

10. Compute the scalar triple product $\vec{u} \bullet (\vec{v} \times \vec{w})$

a) $\vec{u} = (-2,\ 0,\ 6),\quad \vec{v} = (1,\ -3,\ 1),\quad \vec{w} = (-5,\ -1,\ 1)$

b) $\vec{u} = (-1,\ 2,\ 4),\quad \vec{v} = (3,\ 4,\ -2),\quad \vec{w} = (-1,\ 2,\ 5)$

c) $\vec{u} = (a,\ 0,\ 0),\quad \vec{v} = (0,\ b,\ 0),\quad \vec{w} = (0,\ 0,\ c)$

d) $\vec{u} = 3\hat{i} - 2\hat{j} - 5\hat{k},\quad \vec{v} = \hat{i} + 4\hat{j} - 4\hat{k},\quad \vec{w} = 3\hat{j} + 2\hat{k}$

e) $\vec{u} = (3,\ -1,\ 6)\quad \vec{v} = (2,\ 4,\ 3)\quad \vec{w} = (5,\ -1,\ 2)$

11. Use the cross product to find the sine of the angle between the vectors
$\vec{u} = (2,\ 3,\ -6),\quad \vec{v} = (2,\ 3,\ 6)$

12. Simplify $(\vec{u} + \vec{v}) \times (\vec{u} - \vec{v})$

13. Prove Lagrange's identity: $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2\ \|\vec{v}\|^2 - (\vec{u} \bullet \vec{v})^2$

14. Polar coordinates satisfy $x = r\cos\theta$ *and* $y = \sin\theta$. Polar area $J\ dr\ d\theta$ includes J:

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

The two columns are orthogonal. Their lengths are ______. Thus $J =$ ______.

15. Prove that $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$ if and only if $\vec{u}$ and $\vec{v}$ are parallel vectors.

16. State the following statements as True or False

a) The cross product of two nonzero vectors $\vec{u}$ and $\vec{v}$ is a nonzero vector if and only if $\vec{u}$ and $\vec{v}$ are not parallel.

b) A normal vector to a plane can be obtained by taking the cross product of two nonzero and noncollinear vectors lying in the plane.

c) The scalar triple product of $\vec{u}$, $\vec{v}$, and $\vec{w}$ determines a vector whose length is equal to the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$.

d) If $\vec{u}$ and $\vec{v}$ are vectors in 3-space, then $\|\vec{u} \times \vec{v}\|$ is equal to the area of the parallelogram determine by $\vec{u}$ and $\vec{v}$.

e) For all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in R^3, the vectors $(\vec{u} \times \vec{v}) \times \vec{w}$ and $\vec{u} \times (\vec{v} \times \vec{w})$ are the same.

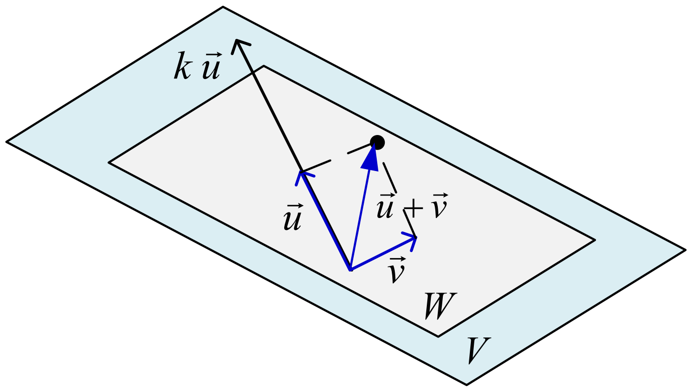
f) If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in R^3, where $\vec{u}$ is nonzero and $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, then $\vec{v} = \vec{w}$

Section 2.5 – Subspaces, Span and Null Spaces

Subspaces

Definition

A subset W of a vector space V is called a ***subspace*** of V if W itself a vector space under the addition and scalar multiplication defined in V.



Theorem

If W is a set of one or more vectors in a vector space V, then W is a subspace of V iff the following conditions holds

1. If $\vec{u}$ and $\vec{v}$ are vectors in W, then $\vec{u}+\vec{v}$ is in W.
2. If k is any scalar and $\vec{v}$ is any vector in W, the $k\vec{v}$ is in the subspace in W.

- The most fundamental ideas in linear algebra are that the plane is a subspace of the full vector space $\mathbb{R}^n$.

- Every subspace contains the zero vector. The plane vector in $\mathbb{R}^3$ has to go through (0, 0, 0). From rule (**2**), if we choose $k=0$ and the rule requires $0v$ to be in the subspace.

The ***axioms*** that are ***not*** inherited by W are

Axiom 1 – Closure of W under addition

Axiom 4 – Existence of a zero vector in W

Axiom 5 – Existence of a negative in W for every vector in W

Axiom 6 – Closure of W under scalar multiplication

Example

Keep only the vectors (x, y) whose components are positive or zero (first quadrant ***"quarter-plane"***).
The vector $(2, 3)$ is included but $(-2, -3)$ is not. So, rule (**2**) is violated when we try $k = -1$. ***The quarter-plane is not a subspace***.

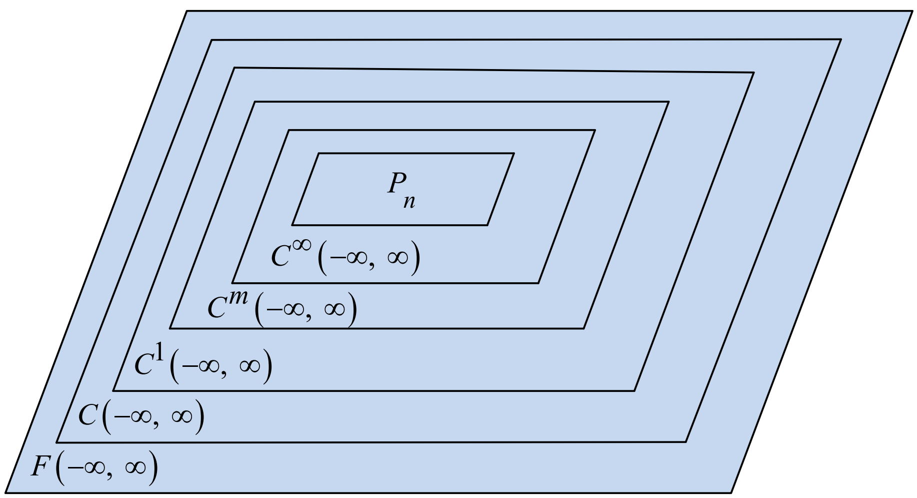
Example

Include also the vectors whose components are both negative. Now we have two quarter-planes. Rule (***ii***) satisfies when we multiply by any c. But rule (***i***) fails. The sum of $v = (2, 3)$ and $w = (-3, -2)$ is $(-1, 1)$ which is outside the quarter-plane. ***Two quarter-planes don't make a subspace***.

Example

The **Subspace** $C(-\infty, \infty)$

There is a theorem in calculus which states that a sum of continuous functions is continuous and than a constant times a continuous fruncition is continuous. In vector word, the set of continuous functions on $(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$. We dente this subspace by $C(-\infty, \infty)$



Theorem

If $W_1, W_2, \ldots, W_n$ are subspaces of a vector space V, then intersection of these subspaces is also a subspace of V.

- ***A subspace containing $\vec{v}$ and $\vec{w}$ must contain all linear combination $c\vec{v} + d\vec{w}$.***

Example

Inside the vector space M of all 2 by 2 matrices, given two subspaces:

U all upper triangular matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

D all diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$

Solution

If we add 2 matrices in **U**: $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 0 & 2d \end{bmatrix}$ is in **U**.

If we add 2 matrices in **D**: $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 0 & 2d \end{bmatrix}$ is in **D**.

In this case **D** is also a subspace of **U**!. The zero matrix is in these subspaces, when a, b, and d all equal zero.

Span

Definition

The subspace of a vector space V that is formed from all possible linear combinations of the vectors in a nonempty set S is called the ***span of S***, and we say that the vectors in S *span* that subspace. If $S = \left\{w_1, w_2, \ldots, w_r\right\}$, then we denoted the span of S by

$$span\left\{w_1, w_2, \ldots, w_r\right\} \quad or \quad span(S)$$

Theorem

Let $\vec{v}_1, \ldots, \vec{v}_n$ be vectors in vector space V and S be their span. Then,

a) S is a subspace of V.

Proof: $\forall\ \vec{u},\ \vec{v} \in S$, $\vec{u} = a_1\vec{v}_1 + \ldots + a_n\vec{v}_n$ and $\vec{v} = b_1\vec{v}_1 + \ldots + b_n\vec{v}_n$

$$\vec{u} + \vec{v} = \left(a_1 + b_1\right)\vec{v}_1 + \ldots + \left(a_n + b_n\right)\vec{v}_n \quad \in \boldsymbol{S}$$

$$k\vec{u} = ka_1\vec{v}_1 + \ldots + ka_n\vec{v}_n \quad \in \boldsymbol{S}$$

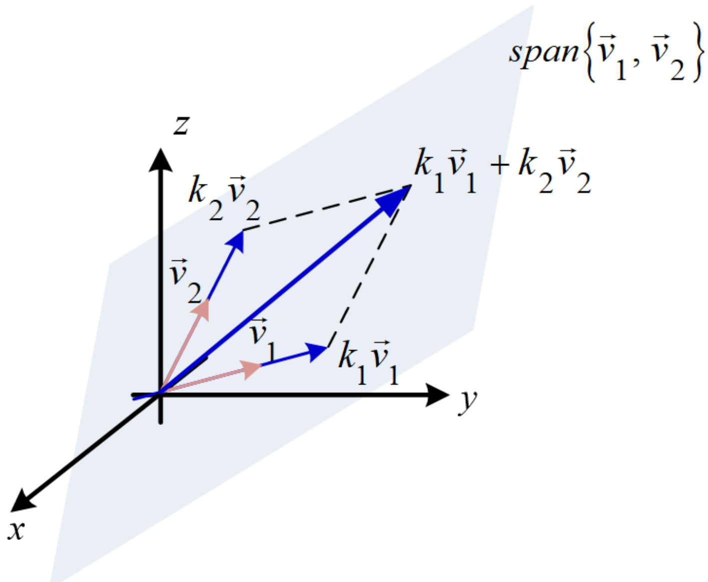
b) S is the smallest subspace of V that contains $\vec{v}_1, \ldots, \vec{v}_k$. i.e. any other subspace $\vec{w}$ containing $\vec{v}_1, \ldots, \vec{v}_n$ also contains S.

Proof: *let* $\vec{u} \in S$, $\vec{u} = a_1\vec{v}_1 + \ldots + a_n\vec{v}_n$

But $a_1\vec{v}_1, \ldots, a_n\vec{v}_n \in \vec{w}$ $\therefore$ $\vec{w}$ *closed under scalar multiplication.*

$a_1\vec{v}_1, \ldots, a_n\vec{v}_n \in \vec{w}$ $\therefore$ $\vec{w}$ *closed under addition.*

$\therefore$ $\underline{\vec{u} \in \vec{w}}$ |



Example

a) $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ span the full two-dimensional space $\mathbb{R}^2$.

b) $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, *and* $\vec{v}_3 = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$ span the full space $\mathbb{R}^2$.

c) $\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *and* $\vec{w}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ only span a line in $\mathbb{R}^2$.

Definition

The ***row space*** of a matrix is the subspace of $\mathbb{R}^n$ spanned by the rows.

Example

Determine whether $\vec{v}_1=(1,\ 1,\ 2),\quad \vec{v}_2=(1,\ 0,\ 1),\quad and\quad \vec{v}_3=(2,\ 1,\ 3)$ span the vector space $\mathbb{R}^3$

Solution

Let $b=\left(b_1,\ b_2,\ b_3\right)$ be the arbitrary vector in $\boldsymbol{R}^3$ can be expressed as a linear combination

$$\vec{b}=k_1\vec{v}_1+k_2\vec{v}_2+k_3\vec{v}_3$$

$$\left(b_1,\ b_2,\ b_3\right)=k_1(1,\ 1,\ 2)+k_2(1,\ 0,\ 1)+k_3(2,\ 1,\ 3)$$

$$\left(b_1,\ b_2,\ b_3\right)=\left(k_1+k_2+2k_3,\ \ k_1+k_3,\ \ 2k_1+k_2+3k_3\right)$$

$$\rightarrow\begin{cases}k_1+k_2+2k_3=b_1\\ k_1\qquad\ +k_3=b_2\\ 2k_1+k_2+3k_3=b_3\end{cases}$$

$$|A|=\begin{vmatrix}1&1&2\\1&0&1\\2&1&3\end{vmatrix}$$

$$\underline{=0\ |}$$

Since the determinant is zero, the $\vec{v}_1,\ \vec{v}_2,\ and\ \vec{v}_3$ ***do not*** span space $\mathbb{R}^3$

Solution Spaces of *Homogeneous* (*Null Space*) Systems

Theorem

The solution set of a homogeneous linear system $A\vec{x}=\vec{0}$ in *n* unknowns is a subspace of $\mathbb{R}^n$

Proof

Let *W* be the solution set for the system. The set *W* is not empty because it contains at least the trivial solution $\vec{x}=\vec{0}$.

To show that *W* is a subspace of $\mathbb{R}^n$, we must show that it is closed under addition and scalar multiplication.

Let $\vec{x}_1$ and $\vec{x}_2$ be vectors in *W* and these vectors are solution of $A\vec{x}=\vec{0}$.

$$A\vec{x}_1=\vec{0}\quad and\quad A\vec{x}_2=\vec{0}$$

Therefore,

$$\begin{aligned} A\left(\vec{x}_1 + \vec{x}_2\right) &= A\vec{x}_1 + A\vec{x}_2 \\ &= \vec{0} + \vec{0} \\ &= \vec{0} \end{aligned}$$

So, W is closed under addition.

$$A\left(k\vec{x}_1\right) = kA\vec{x}_1 = k0 = 0$$

So, W is closed under scalar multiplication.

Null Spaces

Definition

The nullspace of A consists of all solutins to $A\vec{x}=\vec{0}$. These solution vectors $\vec{x}$ are in $\mathbb{R}^n$. The Nullspace containing all solutions is denoted by $N(A)$ or $NS(A)$.

$\left\{\vec{x}\in\mathbb{R}^n \mid A\vec{x}=\vec{0}\right\}$ is the nullspace of A, $NS(A)$

(Can also be called ***Kernel*** of A: $Ker(A)$)

Theorem

Suppose $NS(A)$ is a subspace of $\boldsymbol{R}^n$ for $A_{m\times n}$

- ✓ Let $\vec{x}$ and $\vec{y}$ are in the nullspace $(\vec{x}, \vec{y}\in NS(A))$ then

$$\begin{aligned} A(\vec{x}+\vec{y}) &= A\vec{x}+A\vec{y} \\ &= \vec{0}+\vec{0} \\ &= \vec{0} \end{aligned}$$

- ✓ Let $\vec{x}\in NS(A)$ then $c\vec{x}\in NS(A)$

$$\begin{aligned} \therefore A(c\vec{x}) &= cA\vec{x} \\ &= c\vec{0} \\ &= \vec{0} \end{aligned}$$

Since we can add and multiply without leaving the Nullspace, it is a subspace.

Example

The equation $x+2y+3z=0$ comes from the 1 by 3 matrix $A=\begin{bmatrix}1 & 2 & 3\end{bmatrix}$. This equation produces a plane through the origin. The plane is a subspace of $\mathbb{R}^3$. *It is the Nullspace* of A.

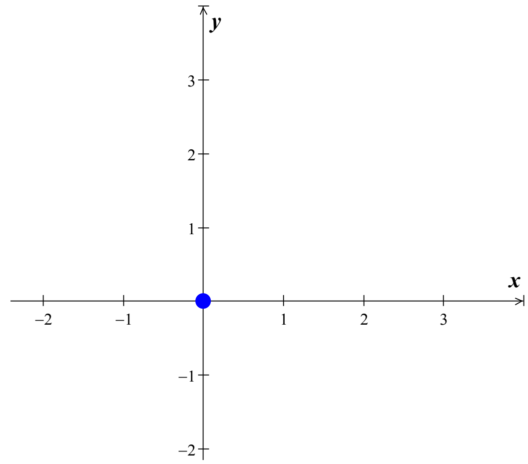
Solution

The solution to $x+2y+3z=6$ also form a plane, but not a subspace.

Example

Find the null space of

$a)\quad A=\begin{bmatrix}1&2\\0&3\end{bmatrix} \qquad b)\quad B=\begin{bmatrix}1&2\\3&6\end{bmatrix}$



Solution

a) $\begin{bmatrix}1&2\\0&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\to \begin{matrix}x_1+x_2=0\\3x_2=0\end{matrix}$

$\Rightarrow x_1=x_2=0$

So $NS(A)=\{\vec{0}\}$

b) $\begin{bmatrix}1&2&0\\3&6&0\end{bmatrix} \quad R_2-3R_1$

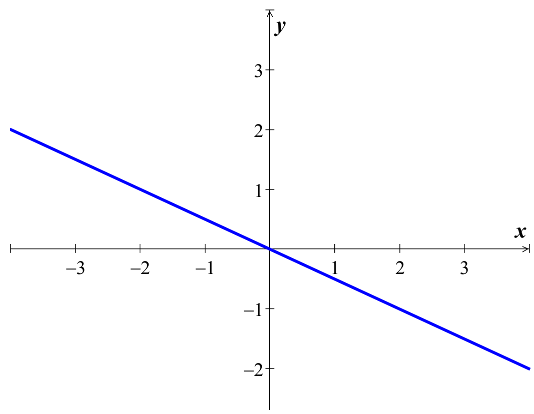
$\begin{bmatrix}1&2&0\\0&0&0\end{bmatrix}$

$\Rightarrow x_1=-2x_2$

If we let $x_2=s$, then

$\begin{bmatrix}x_1\\x_2\end{bmatrix}$ is in $NS(B)$ if and only if

$\begin{bmatrix}x_1\\x_2\end{bmatrix}=s\begin{bmatrix}-2\\1\end{bmatrix}$



Example

Describe the nullspace of $A=\begin{bmatrix}1&2\\3&6\end{bmatrix}$

Solution

Apply the elimination to the linear equations $Ax=0$:

$$\begin{bmatrix}x_1+2x_2=0\\3x_1+6x_2=0\end{bmatrix}\xrightarrow[R_2-3R_1]{}\begin{bmatrix}x_1+2x_2=0\\0=0\end{bmatrix}$$

There is only one equation $(x_1+2x_2=0)$, this line is the Nullspace $N(A)$.

Example

Consider the linear system $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution

$z = t, \quad y = s, \quad x = 2s - 3t$

$\Rightarrow x - 2y + 3z = 0$

This is the equation of a plane through the origin that has $\vec{n}$ = (1, −2, 3) as a normal.

Example

Consider the linear system $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution

$x = 0, \quad y = 0, \quad z = 0$

The solution space is $\{\vec{0}\}$

Exercises Section 2.5 – Subspaces, Span and Null Spaces

1. Suppose S and T are two subspaces of a vector space **V**.

a) The sum $S+T$ contains all sums $\vec{s}+\vec{t}$ of a vector $\vec{s}$ in S and a vector $\vec{t}$ in T. Show that $S+T$ satisfies the requirements (addition and scalar multiplication) for a vector space.

b) If S and T are lines in $\mathbb{R}^m$, what is the difference between $S+T$ and $S\cup T$? That union contains all vectors from S and T or both. Explain this statement: The span of $S\cup T$ is $S+T$.

2. Determine which of the following are subspaces of $\mathbb{R}^3$?

a) All vectors of the form $(a, 0, 0)$

b) All vectors of the form $(a, 1, 1)$

c) All vectors of the form (a, b, c), where $b = a + c$

d) All vectors of the form (a, b, c), where $b = a + c + 1$

e) All vectors of the form $(a, b, 0)$

3. Determine which of the following are subspaces of $\boldsymbol{R}^\infty$?

a) All sequences $\vec{v}$ in $\mathbb{R}^\infty$ of the form $\vec{v} = (v, 0, v, 0, \dots)$

b) All sequences $\vec{v}$ in $\mathbb{R}^\infty$ of the form $\vec{v} = (v, 1, v, 1, \dots)$

c) All sequences $\vec{v}$ in $\mathbb{R}^\infty$ of the form $\vec{v} = (v,\ 2v,\ 4v,\ 8v,\ 16v,\ \dots)$

4. Which of the following are linear combinations of $\vec{u} = (0, -2, 2)$ and $\vec{v} = (1, 3, -1)$?

a) $(2, 2, 2)$ *b)* $(3, 1, 5)$ *c)* $(0, 4, 5)$ *d)* $(0, 0, 0)$

5. Which of the following are linear combinations of $\vec{u} = (2, 1, 4)$, $\vec{v} = (1, -1, 3)$ and $\vec{w} = (3, 2, 5)$?

a) $(-9, -7, -15)$ *b)* $(6, 11, 6)$ *c)* $(0, 0, 0)$

6. Determine whether the given vectors span $\mathbb{R}^3$

a) $\vec{v}_1 = (2,\ 2,\ 2),\quad \vec{v}_2 = (0,\ 0,\ 3),\quad \vec{v}_3 = (0,\ 1,\ 1)$

b) $\vec{v}_1 = (2,\ -1,\ 3),\quad \vec{v}_2 = (4,\ 1,\ 2),\quad \vec{v}_3 = (8,\ -1,\ 8)$

c) $\vec{v}_1 = (3,\ 1,\ 4),\quad \vec{v}_2 = (2,\ -3,\ 5),\quad \vec{v}_3 = (5,\ -2,\ 9),\quad \vec{v}_4 = (1,\ 4,\ -1)$

7. Which of the following are linear combinations of $A = \begin{pmatrix} 4 & 0 \\ -2 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$

a) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ *b)* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *c)* $\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$

8. Suppose that $\vec{v}_1 = (2,\ 1,\ 0,\ 3)$, $\vec{v}_2 = (3,\ -1,\ 5,\ 2)$, $\vec{v}_3 = (-1,\ 0,\ 2,\ 1)$. Which of the following vectors are in span $\{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\}$

a) (2, 3, −7, 3) *b)* (0, 0, 0, 0) *c)* (1, 1, 1, 1) *d)* (−4, 6, −13, 4)

9. Let $\boldsymbol{f} = \cos^2 x$ and $\boldsymbol{g} = \sin^2 x$. Which of the following lie in the space spanned by $\boldsymbol{f}$ and $\boldsymbol{g}$

a) $\cos 2x$ *b)* $3 + x^2$ *c)* $\sin x$ *d)* 0

10. Let $S = \left\{ (x,\ y) \middle| \ x^2 + y^2 = 0;\ x,\ y \in \mathbb{R} \right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^2$?

11. Let $S = \left\{ (x,\ y) \middle| \ x^2 + y^2 = 0;\ x,\ y \in \mathbb{C} \right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{C}^2$?

12. Let $S = \left\{ (x,\ y) \middle| \ x^2 - y^2 = 0;\ x,\ y \in \mathbb{R} \right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^2$?

13. Let $S = \{ (x,\ y) |\ x - y = 0;\ x,\ y \in \mathbb{R} \}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^2$?

14. Let $S = \{ (x,\ y) |\ x - y = 1;\ x,\ y \in \mathbb{R} \}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^2$?

15. $V = \mathbb{R}^3, \quad S = \{(0, s, t) | \; s, t \; are \; real \; numbers\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V?

16. $V = \mathbb{R}^3, \quad S = \{(x, y, z) | \; x, y, z \geq 0\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V?

17. $V = \mathbb{R}^3, \quad S = \{(x, y, z) | \; z = x + y + 1\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V?

18. Let $S = \left\{ \left(a_1, a_2, a_3\right) \in \mathbb{R}^3 : \; a_1 = 3a_2 \; and \; a_3 = -a_2 \right\}$, Determine:

d) Is S closed under addition?
e) Is S closed under scalar multiplication?
f) Is S a subspace of $\mathbb{R}^3$?

19. Let $S = \left\{ \left(a_1, a_2, a_3\right) \in \mathbb{R}^3 : \; a_1 = a_3 + 2 \right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^3$?

20. Let $S = \left\{ \left(a_1, a_2, a_3\right) \in \mathbb{R}^3 : \; 2a_1 - 7a_2 + a_3 = 0 \right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^3$?

21. Let $S = \left\{ \left(a_1, a_2, a_3\right) \in \mathbb{R}^3 : \; a_1 - 4a_2 - a_3 = 0 \right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^3$?

22. Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ a_1+2a_2-3a_3=0\right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

23. Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ a_1+2a_2-3a_3=1\right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

24. Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ 5a_1^2-3a_2^2+6a_3^2=0\right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

25. Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ a_3=a_1+a_2\right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

26. Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ a_1+a_2+a_3=0\right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

27. $S=\left\{\left(x_1,\ x_2,\ 1\right):\ x_1\ \textit{and}\ x_2\ \textit{are real numbers}\right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

28. $S=\left\{\left(x_1,\ x_2,\ x_3\right)\in\mathbb{R}^3:\ x_2=x_1+2x_3\right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

29. $S=\left\{\begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \in M_{2\times 2} \mid a,\ b,\ c \in \mathbb{R}\right\}$ and $V = M_{2,2}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V ?

30. $S=\left\{\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in M_{2\times 2} \mid a,\ b,\ c \in \mathbb{R}\right\}$ and $V = M_{2,2}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V ?

31. Let $S=\left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in M_{2\times 2} \mid a,\ d \in \mathbb{R}\ \ \&\ \ ad \geq 0\right\}$ and $V = M_{2,2}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V ?

32. $V = M_{33}$, $S=\left\{A \middle| A \text{ is invertible}\right\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V?

33. Let $S=\left\{p(t)=a+2at+3at^3 \mid a \in \mathbb{R}\ \ \&\ \ p(t) \in P_2\right\}$ and $V = P_2$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V ?

34. Let $S=\left\{p(t) \mid p(t) \in \boldsymbol{P}[t] \text{ has degree } 3\right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\boldsymbol{P}[t]$?

35. Let $S=\left\{p(t) \mid p(0)=0,\ p(t) \in \boldsymbol{P}[t]\right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\boldsymbol{P}[t]$?

36. Given: $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix}$

a) Find $NS(A)$

b) For which *n* is $NS(A)$ a subspace of $\mathbb{R}^n$

c) Sketch $NS(A)$ in $\mathbb{R}^2$ or $\mathbb{R}^3$

37. Determine which of the following are subspaces of M_{22}

a) All 2×2 matrices with integer entries

b) All matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a+b+c+d=0$

38. Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \ ad - bc = 1 \right\}$. Is *V* a vector space?

39. Let $V = \{(x, 0, y): \ x \,\&\, y \text{ are arbitrary } \mathbb{R}\}$. Define addition and scalar multiplication as follows:

$$\begin{cases} (x_1,\ 0,\ y_1) + (x_2,\ 0,\ y_2) = (x_1 + x_2,\ y_1 + y_2) \\ \quad c(x,\ 0,\ y) = (cx,\ cy) \end{cases}$$

Is *V* a vector space?

40. Construct a matrix whose column space contains $(1,\ 1,\ 0)$ and $(0,\ 1,\ 1)$ and whose nullspace contains $(1,\ 0,\ 1)$ and $(0,\ 0,\ 1)$

41. How is the nullspace $N(C)$ related to the spaces $N(A)$ and $N(B)$, is $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

42. True or False (check addition or give a counterexample)

a) If *V* is a vector space and *W* is a subset of *V* that is a vector space, then *W* is s subspace of *V*.
b) The empty set is a subspace of every vector space.
c) If *V* is a vector space other than the zero vector space, then *V* contains a subspace *W* such that $W \neq V$.
d) The intersection of any two subsets of *V* is a subspace of *V*.
e) Let *W* be the *xy*-plane in $\mathbb{R}^3$; that is, $W = \{(a_1,\ a_2,\ 0): \ a_1,\ a_2 \in \mathbb{R}\}$. Then $W = \mathbb{R}^2$

43. Let $A\vec{x} = \vec{0}$ be a homogeneous system of *n* linear equations in *n* unknowns that has only the trivial solution. Show that of *k* is any positive integer, then the system $A^k \vec{x} = \vec{0}$ also has only trivial solution.

44. Let $A\vec{x} = \vec{0}$ be a homogeneous system of n linear equations in n unknowns and let Q be an invertible $n \times n$ matrix. Show that of $A\vec{x} = \vec{0}$ has just trivial solution if and only if $(QA)\vec{x} = \vec{0}$ has just trivial solution.

45. Let $A\vec{x} = \vec{b}$ be a consistent system of linear equations and let $\vec{x}_1$ be a fixed solution. Show that every solution to the system can be written in the form $\vec{x} = \vec{x}_1 + \vec{x}_0$ where $\vec{x}_0$ is a solution to $A\vec{x} = \vec{0}$. Show also that every matrix of this form is a solution.

Section 2.6 – Linear Independence

There are n columns in an m by n matrix, and each column has m components. But the true ***dimension*** of the column space is not necessarily m or n. The dimension is measured by counting ***independent columns***.

- **Independent vectors** (*not too many*)
- **Spanning a space** (*not too few*)

Linear Independence (LI)

The columns of A are ***linearly independent*** when the only solution to $A\vec{x}=0$ is $\vec{x}=\vec{0}$. ***No other combination*** Ax ***of the columns gives the zero vector***.

Definitions

- A set of two or more vectors is ***linearly dependent*** if one vector in the set is a linear combination of the others. A set of one vector is ***linearly dependent*** if that one vector is the zero vector.

$$\vec{0}=0\vec{v}_1+0\vec{v}_2+\cdots+0\vec{v}_k$$

- The sequence of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is ***linearly independent*** if the only combination that gives the zero vector is $0\vec{v}_1+0\vec{v}_2+\cdots+0\vec{v}_k$. Thus, linear independence means that:

$x_1\vec{v}_1+x_2\vec{v}_2+\cdots+x_k\vec{v}_k=0$ only happens when all x's are zero.

- A (nonempty) set of vectors is ***linearly independent*** if it is not linearly dependent.
- If three vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3$ are in the same plane, they are dependent.
- The empty set is linearly independent, for linearly dependent sets must be nonempty.
- A set consisting of a single nonzero vector is linearly independent. For if $\{\vec{v}\}$ is linearly dependent, then $a\vec{v}=\vec{0}$ for some nonzero scalar a. Thus,

$$\vec{v}=a^{-1}(a\vec{v})=a^{-1}\vec{0}=\vec{0}$$

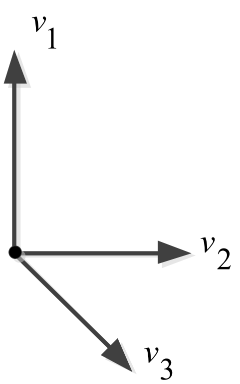
Theorem

A set S with two or more vectors $S = \{\vec{v}_1, \cdots, \vec{v}_k\}$ is

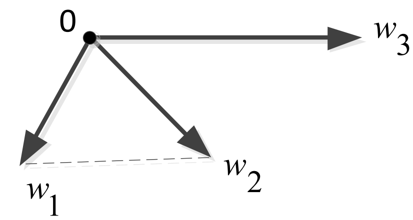
a) Linearly dependent *iff* at least one of the vectors in S is expressible as a linear combination of the other vectors in S. There are numbers $c_1, \ldots, c_k$ at least one of which is nonzero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = 0$$

b) Linearly independent *iff* no vector in S is expressible as a linear combination of the other vectors in S.



Independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$



Dependent vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3$

The combination $\vec{w}_1 - \vec{w}_2 + \vec{w}_3$ is (0, 0, 0)

Example

a) The vectors (1, 0) and (0, 1) are ***independent***.

b) The vectors (1, 1) and (1, 0.0001) are ***independent***.

c) The vectors (1, 1) and (2, 2) are ***dependent***.

d) The vectors (1, 1) and (0, 0) are ***dependent***.

Theorem

a) A finite set that contains $\vec{0}$ is linearly dependent.

b) A set with exactly one vector is linearly independent if and only if that vector is not $\vec{0}$.

c) A set with exactly two vectors is linearly independent *iff* neither vector is a scalar multiple of the other.

Theorem

Let S be a set $\boldsymbol{k}$ vectors in $\mathbb{R}^n$, then if $k > n$, S is ***linearly dependent***.

Example

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are 3 vectors in $\mathbb{R}^2 \Rightarrow$ *Linearly dependent*.

Example

Determine whether the vectors $\vec{v}_1 = (1,\ -2,\ 3)\quad \vec{v}_2 = (5,\ 6,\ -1)\quad \vec{v}_3 = (3,\ 2,\ 1)$ are linearly dependent or linearly independent in $\mathbb{R}^3$

Solution

$$k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \mathbf{0}$$

$$k_1(1,\ -2,\ 3) + k_2(5,\ 6,\ -1) + k_3(3,\ 2,\ 1) = (0,\ 0,\ 0)$$

$$\rightarrow \begin{cases} k_1 + 5k_2 + 3k_3 = 0 \\ -2k_1 + 6k_2 + 2k_3 = 0 \\ 3k_1 - k_2 + k_3 = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{bmatrix} \quad \begin{matrix} R_2 + 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & -16 & -8 & 0 \end{bmatrix} \quad R_3 + R_2$$

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \tfrac{1}{16}R_2$$

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 - 5R_2$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} k_1 + \frac{1}{2}k_3 = 0 \\ k_2 + \frac{1}{2}k_3 = 0 \end{matrix}$$

Solve the system equations: $k_1 = -\frac{1}{2}t, \quad k_2 = -\frac{1}{2}t, \quad k_3 = t$

This shows that the system has nontrivial solutions and hence that the vectors are linearly dependent.

2nd method to determine the linearly is to compute the determinant of the coefficient matrix

$$A = \begin{pmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{pmatrix}$$

$|A| = 0$ Which has nontrivial solutions and the vectors are *linearly dependent*.

Example

Determine whether the vectors are linearly dependent or linearly independent in $\mathbb{R}^4$

$$\vec{v}_1 = (1,\ 2,\ 2,\ -1) \quad \vec{v}_2 = (4,\ 9,\ 9,\ -4) \quad \vec{v}_3 = (5,\ 8,\ 9,\ -5)$$

Solution

$k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0}$

$k_1(1,\ 2,\ 2,\ -1) + k_2(4,\ 9,\ 9,\ -4) + k_3(5,\ 8,\ 9,\ -5) = (0,\ 0,\ 0,\ 0)$

Which yields the homogeneous linear system

$$\rightarrow \begin{cases} k_1 + 4k_2 + 5k_3 = 0 \\ 2k_1 + 9k_2 + 8k_3 = 0 \\ 2k_1 + 9k_2 + 9k_3 = 0 \\ -k_1 - 4k_2 - 5k_3 = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 4 & 5 & 0 \\ 2 & 9 & 8 & 0 \\ 2 & 9 & 9 & 0 \\ -1 & -4 & -5 & 0 \end{pmatrix} \quad \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \\ R_4 + R_1 \end{matrix}$$

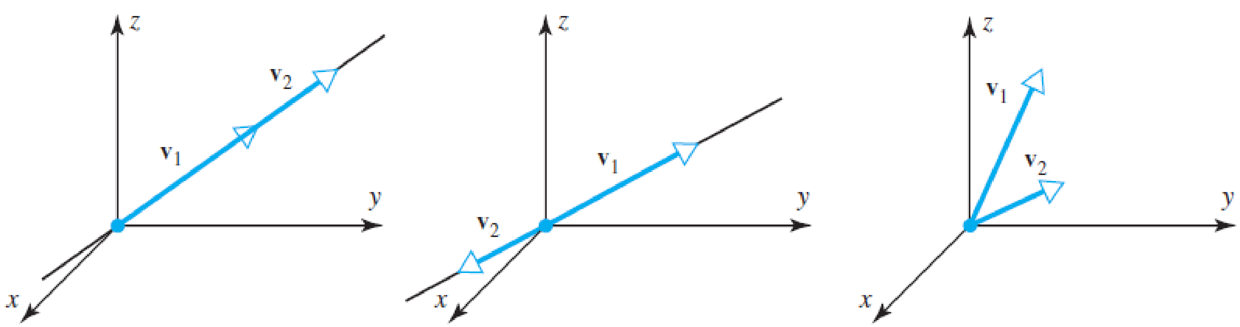
$$\begin{pmatrix} 1 & 4 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} R_1 - 4R_2 \\ \\ R_3 - R_2 \\ \\ \end{matrix}$$

$$\begin{array}{c} \begin{pmatrix} 1 & 0 & 13 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} R_1 - 13R_3 \\ R_2 + 2R_3 \\ \\ \\ \end{array}$$

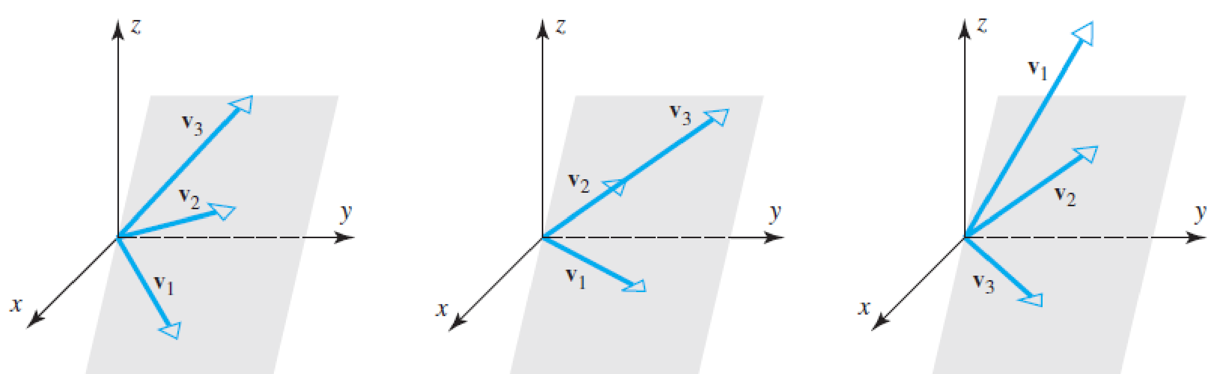
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \rightarrow\ k_1 = 0 \\ \rightarrow\ k_2 = 0 \\ \rightarrow\ k_3 = 0 \\ \\ \end{array}$$

Solve the system equations: $k_1 = 0, \quad k_2 = 0, \quad k_3 = 0$ has a trivial solution.

The vectors $\vec{v}_1, \vec{v}_2,$ *and* $\vec{v}_3$ are *linearly independent*.



(*a*) Linearly dependent (*b*) Linearly dependent (*c*) Linearly independent



(*a*) Linearly dependent (*b*) Linearly dependent (*c*) Linearly independent

Linear independence of Functions

Definition

If $\mathbf{f}_1 = f_1(x),\ \mathbf{f}_2 = f_2(x),\ \ldots,\ \mathbf{f}_n = f_n(x)$ are functions that are $n-1$ times differentiable on the interval $(-\infty, \infty)$, the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the ***Wronskian*** of $f_1, f_2, \ldots, f_n$

$$\begin{cases} if\ \ W = 0 & \Rightarrow \quad Linearly\ Dependent \\ if\ \ W \neq 0 & \Rightarrow \quad Linearly\ Independent \end{cases}$$

Example

Use the Wronskian to show that $\mathbf{f}_1 = x,\quad \mathbf{f}_2 = \sin x$ are linearly independence

Solution

The Wronskian is

$$W(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix}$$

$$= x\cos x - \sin x \neq 0$$

This function is not identically zero. Thus, the functions are linearly independent.

Example

Use the Wronskian to show that $\mathbf{f}_1 = 1$, $\mathbf{f}_2 = e^x$, $\mathbf{f}_3 = e^{2x}$ are linearly independence

Solution

The Wronskian is

$$W(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix}$$

$$= e^x 4e^{2x} - 2e^{2x}e^x$$

$$= 2e^{3x} \neq 0$$

Thus, the functions are linearly independent.

Theorem

Let S be a linearly independent subset of a vector space V, and let $\vec{v}$ be a vector in V that is not in S. Then $S\cup\{\vec{v}\}$ is linearly dependent if and only if $\vec{v}\in span(S)$

Proof

If $S\cup\{\vec{v}\}$ is linearly dependent, then there are vectors $\vec{u}_1,\ \vec{u}_2,\ \ldots,\ \vec{u}_n$ in $S\cup\{\vec{v}\}$ such that $a_1\vec{u}_1+a_2\vec{u}_2+\ \ldots+a_n\vec{u}_n=\vec{0}$ for some nonzero scalars $a_1,\ a_2,\ \ldots,\ a_n$.

Because S is linearly independent, one of the $\vec{u}_i$'s say $\vec{u}_1$, equal $\vec{v}$. Thus $a_1\vec{v}+a_2\vec{u}_2+\ \ldots+a_n\vec{u}_n=\vec{0}$, and so

$$\begin{aligned}
a_1\vec{v}&=-\left(a_2\vec{u}_2+\ \ldots+a_n\vec{u}_n\right)\\
\vec{v}&=-a_1^{-1}\left(a_2\vec{u}_2+\ \ldots+a_n\vec{u}_n\right)\\
&=-\left(a_1^{-1}a_2\right)\vec{u}_2-\ \ldots-\left(a_1^{-1}a_n\right)\vec{u}_n
\end{aligned}$$

Since $\vec{v}$ is linear combination of $\vec{u}_2,\ \ldots,\ \vec{u}_n$, which are in S, we have $\vec{v}\in span(S)$.

Conversely, let $\vec{v}\in span(S)$.
Then there exist vectors $\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_m$ in S and scalars $b_1,\ b_2,\ \ldots,\ b_m$ such that $\vec{v}=b_1\vec{v}_1+b_2\vec{v}_2+\ \ldots+b_m\vec{v}_m$. Hence,

$$b_1\vec{v}_1+b_2\vec{v}_2+\ \ldots+b_m\vec{v}_m+(-1)\vec{v}=\vec{0}$$

Since $\vec{v}\neq\vec{v}_i$ for $i=1,\ 2,\ \ldots,\ m$, the coefficient of $\vec{v}$ in this linear combination is nonzero, and so the set $\left\{\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_m,\ \vec{v}\right\}$ is linearly dependent.

Therefore $S\cup\{\vec{v}\}$ is *linearly dependent.*

Exercises *Section* 2.6 – Linear Independence

1. State the following statements as ***true*** or ***false***

a) If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.

b) Any set containing the zero vector is linearly dependent.

c) The empty set is linearly dependent.

d) Subsets of linearly dependent sets are linearly dependent.

e) Subsets of linearly independent sets are linearly independent.

f) If $a_1x_1 + a_2x_2 + \ \ldots + a_nx_n = \vec{0}$ and $x_1, x_2, \ldots, x_n$ are linearly independent, the null the scalars a_i are zero

2. Given three independent vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3$. Take combinations of those vectors to produce $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Write the combinations in a matrix form as $V = WM$.

$$\begin{aligned}
\vec{v}_1 &= \vec{w}_1 + \ \vec{w}_2 \\
\vec{v}_2 &= \vec{w}_1 + 2\vec{w}_2 + \vec{w}_3 \\
\vec{v}_1 &= \qquad \vec{w}_2 + c\vec{w}_3
\end{aligned}$$

which is $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{bmatrix}$

What is the test on a matrix **V** to see if its columns are linearly independent?

If $c \ne 1$ show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

If $c = 1$ show that $\vec{v}\,'s$ are linearly *dependent*.

3. Find the largest possible number of independent vectors among

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \vec{v}_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \vec{v}_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad \vec{v}_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

4. Show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent but $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are dependent:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{v}_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

Solve either $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = 0$ *or* $A\vec{x} = 0$. The v's go in the columns of $\boldsymbol{A}$.

5. Decide the dependence or independence of

a) The vectors (1, 3, 2), (2, 1, 3), and (3, 2, 1).

b) The vectors $(1,\ -3,\ 2)$, $(2,\ 1,\ -3)$, and $(-3,\ 2,\ 1)$.

6. Find two independent vectors on the plane $x+2y-3z-t=0$ in $\mathbb{R}^4$. Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

7. Determine whether the vectors are linearly dependent or linearly independent in $\mathbb{R}^3$

a) (4, –1, 2), (–4, 10, 2)

b) (8, –1, 3), (4, 0, 1)

c) (–3, 0, 4), (5, –1, 2), (1, 1, 3)

d) (–2, 0, 1), (3, 2, 5), (6, –1, 1), (7, 0, –2)

8. Determine whether the vectors are linearly dependent or linearly independent in $\mathbb{R}^4$

a) $\{(3,\ 8,\ 7,\ -3),\ (1,\ 5,\ 3,\ -1),\ (2,\ -1,\ 2,\ 6),\ (1,\ 4,\ 0,\ 3)\}$

b) $\{(0,\ 0,\ 2,\ 2),\ (3,\ 3,\ 0,\ 0),\ (1,\ 1,\ 0,-1)\}$

c) $\{(0,\ 3,\ -3,\ -6),\ (-2,\ 0,\ 0,\ -6),\ (0,\ -4,\ -2,\ -2),\ (0,\ -8,\ 4,\ -4)\}$

d) $\{(3,\ 0,\ -3,\ 6),\ (0,\ 2,\ 3,\ 1),\ (0,\ -2,\ -2,\ 0),\ (-2,\ 1,\ 2,\ 1)\}$

e) $\{(1,\ 3,\ -4,\ 2),\ (2,\ 2,\ -4,\ 0),\ (2,\ 3,\ 2,\ -4),\ (-1,\ 0,\ 1,\ 0)\}$

f) $\{(1,\ 3,\ -4,\ 2),\ (2,\ 2,\ -4,\ 0),\ (1,\ -3,\ 2,\ -4),\ (-1,\ 0,\ 1,\ 0)\}$

g) $\{(1,\ 0,\ 0,\ -1),\ (0,\ 1,\ 0,\ -1),\ (0,\ 1,\ 0,\ -1),\ (0,\ 0,\ 0,\ 1)\}$

9. *a*) Show that the three vectors $\vec{v}_1=(1,\ 2,\ 3,\ 4)$ $\vec{v}_2=(0,\ 1,\ 0,\ -1)$ $\vec{v}_3=(1,\ 3,\ 3,\ 3)$ form a linearly dependent set in $\mathbb{R}^4$.

b) Express each vector in part (*a*) as a linear combination of the other two.

10. For which real values of λ do the following vectors form a linearly dependent set in $\mathbb{R}^3$

$$\vec{v}_1=\left(\lambda,\ -\tfrac{1}{2},\ -\tfrac{1}{2}\right)\quad \vec{v}_2=\left(-\tfrac{1}{2},\ \lambda,\ -\tfrac{1}{2}\right)\quad \vec{v}_3=\left(-\tfrac{1}{2},\ -\tfrac{1}{2},\ \lambda\right)$$

11. Show that if $S=\{\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_n\}$ is a linearly independent set of vectors, then so is every nonempty subset of S.

12. Show that if $S=\{\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_r\}$ is a linearly dependent set of vectors in a vector space *V*, and if $\vec{v}_{r+1},\ \ldots,\ \vec{v}_n$ are vectors in *V* that are not in *S*, then $\{\vec{v}_1,\ \vec{v}_2,\ \ldots,\ \vec{v}_r,\ \vec{v}_{r+1},\ \ldots,\ \vec{v}_n\}$ is also linearly dependent.

13. Show that $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent and $\vec{v}_3$ does not lie in span $\{\vec{v}_1, \vec{v}_2\}$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent.

14. By using the appropriate identities, where required, determine $F(-\infty, \infty)$ are linearly dependent.

a) $6,\ 3\sin^2 x,\ 2\cos^2 x$

b) $x,\ \cos x$

c) $1,\ \sin x,\ \sin 2x$

d) $(3-x)^2,\ x^2-6x,\ 5$

e) $\cos 2x,\ \sin^2 x,\ \cos^2 x$

15. $f_1(x)=\sin x,\quad f_2(x)=\cos x$ are linearly independent in $F(-\infty, \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using Wrońskian's test.

16. Show $f_1(x)=e^x,\quad f_2(x)=xe^x \quad f_3(x)=x^2e^x$ are linearly independent in $F(-\infty, \infty)$.

17. Use the Wronskian to show that $f_1(x)=\sin x,\quad f_2(x)=\cos x,\quad f_3(x)=x\cos x$ span a three-dimensional subspace of $F(-\infty, \infty)$

18. Show by inspection that the vectors are linearly dependent.

$\vec{v}_1(4,\ -1,\ 3),\ \ \vec{v}_2(2,\ 3,\ -1),\ \ \vec{v}_3(-1,\ 2,\ -1),\ \ \vec{v}_4(5,\ 2,\ 3),\ \ in\ \mathbb{R}^3$

(**19 – 37**) Determine if the given vectors are linearly dependent or independent, (any method)

19. $(2,\ -1,\ 3),\ \ (3,\ 4,\ 1),\ \ (2,\ -3,\ 4),\ \ in\ \mathbb{R}^3$

20. $(1,\ 0,\ 0,\ 0),\ \ (1,\ 1,\ 0,\ 0),\ \ (1,\ 1,\ 1,\ 0),\ \ (1,\ 1,\ 1,\ 1),\ \ in\ \mathbb{R}^4$

21. $A_1\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix},\ \ A_2\begin{bmatrix}-2 & 4\\ 1 & 0\end{bmatrix},\ \ A_3\begin{bmatrix}3 & -1\\ 2 & 0\end{bmatrix},\ \ in\ M_{22}$

22. $\left\{\begin{pmatrix}1 & -3 & 2\\ -4 & 0 & 5\end{pmatrix}, \begin{pmatrix}-3 & 7 & 4\\ 6 & -2 & -7\end{pmatrix}, \begin{pmatrix}-2 & 3 & 11\\ -1 & -3 & 2\end{pmatrix}\right\}$ in $M_{2\times 3}(\mathbb{R})$

23. $\left\{\begin{pmatrix}1 & -3\\ -2 & 4\end{pmatrix}, \begin{pmatrix}-2 & 6\\ 4 & -8\end{pmatrix}\right\}$ in $M_{2\times 2}(\mathbb{R})$

24. $\left\{\begin{pmatrix}1 & -2\\ -1 & 4\end{pmatrix}, \begin{pmatrix}-1 & 1\\ 2 & -4\end{pmatrix}\right\}$ in $M_{2\times 2}(\mathbb{R})$

25. $\left\{\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}-1 & 2\\ 1 & 0\end{pmatrix}, \begin{pmatrix}2 & 1\\ -4 & 4\end{pmatrix}\right\}$ *in* $M_{2\times 2}(\mathbb{R})$

26. $\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & -2 \end{pmatrix} \right\}$ *in* $M_{2\times 2}(\mathbb{R})$

27. $\left\{ e^{x},\ \ln x \right\}$ *in* $\mathbb{R}$

28. $\left\{ x,\ \frac{1}{x} \right\}$ *in* $\mathbb{R}$

29. $\{1+x,\ 1-x\}$ *in* $P_2(\mathbb{R})$

30. $\left\{ 9x^2-x+3,\ 3x^2-6x+5,\ -5x^2+x+1 \right\}$ *in* $P_3(\mathbb{R})$

31. $\left\{ -x^2,\ 1+4x^2 \right\}$ *in* $P_3(\mathbb{R})$

32. $\left\{ 7x^2+x+2,\ 2x^2-x+3,\ -3x^2 4 \right\}$ *in* $P_3(\mathbb{R})$

33. $\left\{ 3x^2+3x+8,\ 2x^2+x,\ 2x^2+2x+2,\ 5x^2-2x+8 \right\}$ *in* $P_3(\mathbb{R})$

34. $\left\{ x^3+2x^2,\ -x^2+3x+1,\ x^3-x^2+2x-1 \right\}$ *in* $P_3(\mathbb{R})$

35. $\left\{ x^3-x,\ 2x^2+4,\ -2x^3+3x^2+2x+6 \right\}$ *in* $P_3(\mathbb{R})$

36. $\left\{ \begin{matrix} x^4-x^3+5x^2-8x+6, \quad -x^4+x^3-5x^2+5x-3, \quad x^4+3x^2-3x+5, \\ 2x^4+3x^3+4x^2-x+1,\ x^3-x+2 \end{matrix} \right\}$ *in* $P_4(\mathbb{R})$

37. $\left\{ \begin{matrix} x^4-x^3+5x^2-8x+6, \quad -x^4+x^3-5x^2+5x-3, \\ x^4+3x^2-3x+5, \quad 2x^4+x^3+4x^2+8x \end{matrix} \right\}$ *in* $P_4(\mathbb{R})$

38. Suppose that the vectors $\vec{u}_1,\ \vec{u}_2,\ and\ \vec{u}_3$ are linearly dependent. Are the vectors $\vec{v}_1 = \vec{u}_1 + \vec{u}_2$, $\vec{v}_2 = \vec{u}_1 + \vec{u}_3$, and $\vec{v}_3 = \vec{u}_2 + \vec{u}_3$ also linearly dependent?

(***Hint***: Assume that $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = 0$, and see what the $a_i's$ can be.)

39. Show that the set $F = \left\{ 1+t,\ t^2,\ t-2 \right\}$ is a linearly independent subset of P_2

40. Suppose that *A* is linearly dependent set of vectors and *B* is any set containing *A*. Show that *B* must be linearly dependent.

41. Show that $\{\sin t,\ \sin 2t,\ \cos t\}$ is a linearly independent, subset of $C[0,\ 1]$. Does it span $C[0,\ 1]$

42. Show that the set $\{\sin(t+a),\ \sin(t+b),\ \sin(t+c)\}$ is linearly dependent on $C[0,\ 1]$

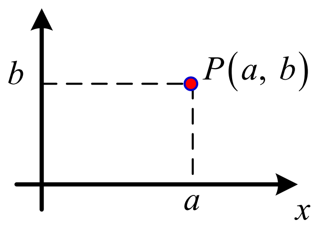
43. Show that if $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n$ are linearly independent and $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n,\ \beta$ are linearly dependent, then β can be uniquely expressed as a linear combination of $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n$.

44. Show that if $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n$ are linearly dependent with $(\alpha_1 \neq 0)$ if and only if there exists an integer k $(1 < k \leq n)$, such that α_k is a linear combination of $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{k-1}$

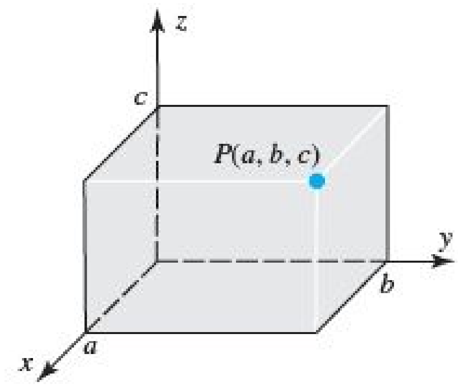
Section 2.7 – Coordinates, Basis and Dimension

Coordinate Systems in Linear Algebra

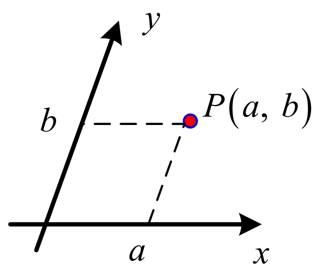
In ***analytic geometry***, we use rectangular coordinate systems to create a point either in 2-space or 3-space



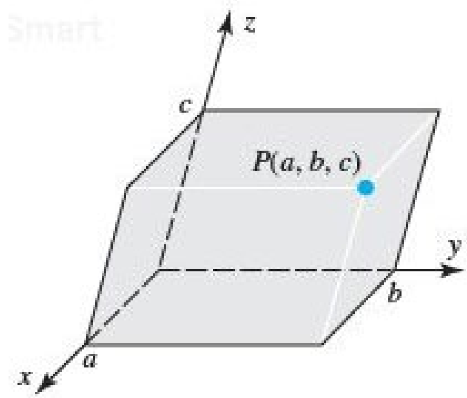
Coordinates of P in a rectangular coordinate system in 2-space



Coordinates of P in a rectangular coordinate system in 3-space

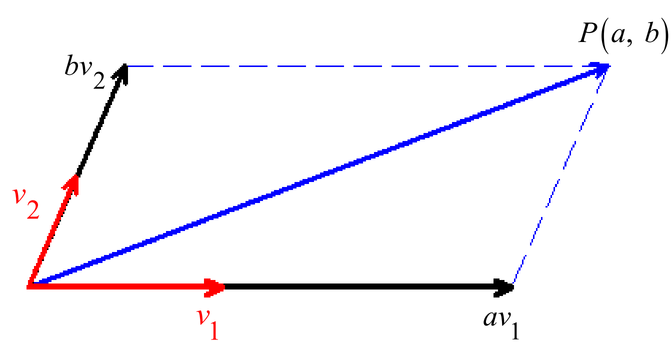


Coordinates of P in a nonrectangular coordinate system in 2-space



Coordinates of P in a nonrectangular coordinate system in 3-space

In ***linear algebra*** coordinate systems are commonly specified using vectors rather than coordinate axes.



Basis

Definition

If V is any vector space and $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a finite set of vectors in V, then S is called a ***basis*** for V if the following two conditions hold:

1. S is linearly independent.
2. S spans V.

Example

The columns of $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ produce the "standard basis" for $\mathbb{R}^2$.

Solution

The basis vectors: $\hat{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $\hat{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are independent. They span $\mathbb{R}^2$.

Example

The columns of any invertible n by n matrix give a basis for $\mathbb{R}^n$.

$$\underbrace{A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}}_{Basis} \quad \underbrace{A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{Basis} \quad \underbrace{A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}}_{Not\ basis}$$

Example

The standard unit vectors $\hat{e}_1 = (1, 0, 0, \cdots, 0)$, $\hat{e}_2 = (0, 1, 0, \cdots, 0)$, $\ldots$, $\hat{e}_n = (0, 0, \cdots, 0, 1)$ form a basis in $\mathbb{R}^n$.

Solution

1. $k_1\hat{e}_1 + k_2\hat{e}_2 + \ldots + k_n\hat{e}_n = 0 \quad \to \quad (k_1, k_2, \ldots, k_n) = (0, 0, \ldots, 0)$ it follows that $k_1 = k_2 = \ldots = k_n = 0$. That implies they are linearly independent.

2. Every vector $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ can be expressed as $\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + \ldots + v_n\hat{e}_n$ which is linear combination of $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n$. Thus, the standard vector span $\mathbb{R}^n$

Thus, they form a basis for $\mathbb{R}^n$ that we call the ***standard basis*** for $\mathbb{R}^n$.

Example

Show that the vectors $\vec{v}_1 = (1,\ 2,\ 1),\quad \vec{v}_2 = (2,\ 9,\ 0),\quad and\quad \vec{v}_3 = (3,\ 3,\ 4)$ form a basis in $\mathbb{R}^3$

Solution

1. We need to show that the vectors are linearly independent.

   $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0}$

   $k_1(1,\ 2,\ 1) + k_2(2,\ 9,\ 0) + k_3(3,\ 3,\ 4) = (0,\ 0,\ 0)$

   Which yields the homogeneous linear system

   $$\rightarrow \begin{cases} k_1 + 2k_2 + 3k_3 = 0 \\ 2k_1 + 9k_2 + 3k_3 = 0 \\ k_1 \qquad + 4k_3 = 0 \end{cases}$$

   $$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} = \underline{-1 \neq 0}$$

   $k_1 = 0,\quad k_2 = 0,\quad k_3 = 0$ has a trivial solution.

   The vectors $\vec{v}_1,\ \vec{v}_2,\ and\ \vec{v}_3$ are linearly independent.

2. Every vector can be expressed as $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{b}$ which is linear combination. Thus, the standard vector span $\mathbb{R}^3$

That proves that the vectors $\vec{v}_1,\ \vec{v}_2,\ and\ \vec{v}_3$ form a basis in $\mathbb{R}^3$

- The vectors $\vec{v}_1,\ \cdots,\ \vec{v}_n$ are a basis for $\mathbb{R}^n$ exactly when they are the ***columns*** of an ***n* by *n* invertible matrix**. Thus $\mathbb{R}^n$ has infinitely many different bases.

- ***The pivots columns of A are a basis for its column space***. The pivot rows of A are a basis for its row space. So are the pivot rows of its echelon form R.

Example

Find bases for the column and row spaces of a rank two matrices: $R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution

Columns 1 and 3 are the pivot columns. They are a basis for the column space. It is a subspace of $\mathbb{R}^3$.

Column 2 and 4 are a basis for the same column space.

Coordinates Relative to a Basis

Theorem – Uniqueness of Basis Representation

If $S = \left\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\right\}$ is a basis for a vector space *V*, then every vector $\vec{v}$ in *V* can be expressed in the form $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$ in exactly one way.

Proof

Suppose that some vector $\vec{v}$ can be written as

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$$

Also $$\vec{v} = k_1\vec{v}_1 + k_2\vec{v}_2 + \ldots + k_n\vec{v}_n$$

Subtracting the second from the first equation

$$0 = \left(c_1 - k_1\right)\vec{v}_1 + \left(c_2 - k_2\right)\vec{v}_2 + \ldots + \left(c_n - k_n\right)\vec{v}_n$$

Since the right side of this equation is a linear combination of vectors in *S*, the linear independence of *S* implies that

$$c_1 - k_1 = 0, \quad c_2 - k_2 = 0, \quad \ldots, \quad c_n - k_n = 0$$

That implies $$c_1 = k_1, \quad c_2 = k_2, \quad \ldots, \quad c_n = k_n$$

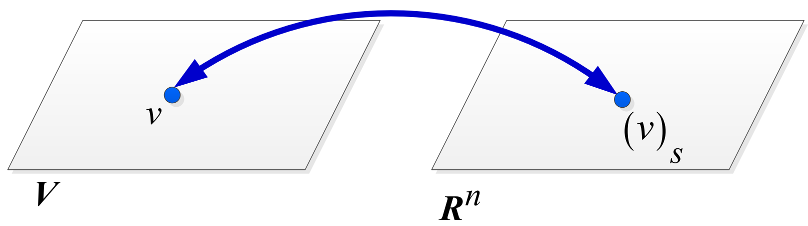
Thus, the two expressions for $\vec{v}$ are the same.

Definition

If $S=\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for a vector space *V*, and $\vec{v}=c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$ is the expression for a vector $\vec{v}$ in terms of the basis ***S***, then the scalars $c_1, c_2, \ldots, c_n$ are called coordinates of $\vec{v}$ relative to the basis *S*. The vector $(c_1, c_2, \ldots, c_n)$ in $\mathbb{R}^n$ constructed from these coordinates is called ***coordinate vector of v relative to S***; it is denoted by

$$(\vec{v})_S=(c_1, c_2, \ldots, c_n)$$

A *one-to-one correspondence*



Example

a) Given the vectors $\vec{v}_1=(1,\ 2,\ 1)$, $\vec{v}_2=(2,\ 9,\ 0)$, *and* $\vec{v}_3=(3,\ 3,\ 4)$ form a basis for $\mathbb{R}^3$. Find the coordinate vector of $\vec{v}=(5,\ -1,\ 9)$ relative to the basis $S=\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

b) Find the coordinate vector of $\vec{v}$ in $\mathbb{R}^3$ whose coordinate relative to *S* is $(\vec{v})_S=(-1,3,2)$.

Solution

a) To find $(\vec{v})_S$ we must first express $\vec{v}$ as a linear combination of the vectors in *S*;

$$\vec{v}=c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$$

$$(5,\ -1,\ 9)=c_1(1,\ 2,\ 1)+ c_2(2,\ 9,\ 0)+ c_3(3,\ 3,\ 4)$$

Which gives: $\begin{cases} c_1+2c_2+3c_3=5 \\ 2c_1+9c_2+3c_3=-1 \\ c_1 \qquad\ \ +4c_3=9 \end{cases}$

Solving this system, we obtain $c_1=1,\quad c_2=-1,\quad c_3=2$.

Therefore $(\vec{v})_S=(1,\ -1,\ 2)$

b) $\vec{v}=(-1)\vec{v}_1+3\vec{v}_2+ 2\vec{v}_3$

$=(-1)(1,\ 2,\ 1)+3(2,\ 9,\ 0)+ 2(3,\ 3,\ 4)$

$=(11,\ 31,\ 7)$

Dimension

If $\vec{v}_1, \ldots, \vec{v}_m$ and $\vec{w}_1, \ldots, \vec{w}_n$ are both bases for the same vector space, then $m = n$.

Note

V may have many different bases, but they all must have the same number of elements.

Proof

Let **S** and **W** be bases of **V** can be written as a linear combination of vectors in **S**.

$$\begin{aligned} \vec{w}_1 &= a_{11}\vec{s}_1 + \ldots + a_{1m}s_m \\ \vec{w}_2 &= a_{21}\vec{s}_1 + \ldots + a_{2m}\vec{s}_m \\ &\vdots \\ \vec{w}_n &= a_{n1}\vec{s}_1 + \ldots + a_{nm}\vec{s}_m \end{aligned}$$

But since **V** is a basis, $c_1\vec{w}_1 + \ \ldots + c_n\vec{w}_n = 0 \iff c_i = 0$ (to be linearly independent, otherwise to be linearly dependent with at least 1 of t $c_i \neq 0$)

$$c_1\left(a_{11}s_1 + \ldots + a_{1m}s_m\right) + c_2\left(a_{21}s_1 + \ldots + a_{2m}s_m\right) + \ldots + c_n\left(a_{n1}s_1 + \ldots + a_{nm}s_m\right) = 0$$

$$\left(c_1a_{11} + c_2a_{21} + \ldots + c_na_{n1}\right)s_1 + \ldots + \left(c_1a_{1m} + c_2a_{2m} + \ldots + c_na_{nm}\right)s_m = 0$$

$$\Leftrightarrow \begin{cases} c_1a_{11} + c_2a_{21} + \ldots + c_na_{n1} = 0 \\ c_1a_{1m} + c_2a_{2m} + \ldots + c_na_{nm} = 0 \end{cases}$$

$\therefore\ S_i$'s linear independent.

Now all bases of **V** have some number of elements, we can define the dimension (is # of vectors in a basis)

Definition

The dimension of a finite-dimensional vector space V is denoted by dim(V) and is defined to be the number of vectors in a basis for V. in addition, the zero-vector space is defined to have dimension zero.

1. Dim(**V**) = # elements in basis. If **V** is finite.
2. If $V = \{\vec{0}\}$, then Dim(**V**) = 0, even though there is no basis.
3. Dim(**V**) may be infinite.

- $\dim(\mathbb{R}^n) = n$ **The standard basis has *n* vectors**.
- $\dim(P_n) = n + 1$ **The standard basis has *n* + 1 vectors**.
- $\dim(M_{mn}) = mn$ **The standard basis has *mn* vectors**.

Bases for Matrix Spaces *and Function Spaces*

Independence, basis, and dimension are not all restricted to column vectors.

- The dimension of the whole n by n space is n^2
- The dimension of the subspace of upper triangular matrices is $\frac{1}{2}n^2 + \frac{1}{2}n$
- The dimension of the subspace of diagonal matrices is n
- The dimension of the subspace of symmetric matrices is $\frac{1}{2}n^2 + \frac{1}{2}n$

Function Spaces

The equations:

$y'' = 0$	is solved by any linear function	$y = cx + d$
$y'' = -y$	is solved by any combination	$y = c\sin x + d\cos x$
$y'' = y$	is solved by any combination	$y = ce^x + de^{-x}$

Example

Find a basis for and the dimension of the solution space of the homogeneous system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \quad\quad\quad + 2x_5 \quad\quad &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 - 10x_4 \quad\quad + 15x_6 &= 0 \\ 2x_1 + 6x_2 \quad\quad + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

Solution

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & -10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{pmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{array}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 5 & -10 & 0 & 15 & 0 \\ 0 & 0 & 4 & 8 & 0 & 18 & 0 \end{pmatrix} \begin{array}{l} \\ -R_2 \\ \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 5 & -10 & 0 & 15 & 0 \\ 0 & 0 & 4 & 8 & 0 & 18 & 0 \end{pmatrix} \begin{array}{l} R_1 + 2R_2 \\ \\ R_3 - 5R_2 \\ R_4 - 4R_2 \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & -20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \end{pmatrix} \begin{array}{l} \\ \\ -\frac{1}{20}R_3 \\ \frac{1}{6}R_4 \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{array}{l} R_1 - 4R_3 \\ R_2 - 2R_3 \\ \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{array}{l} R_1 - 6R_4 \\ R_2 - 3R_4 \\ \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad x_1 = -3x_2 - 2x_5$$

The solution $\left(x_1, x_2, x_3, x_4, x_5, x_6\right) = \left(-3x_2 - 2x_5, x_2, 0, 0, x_5, 0\right)$

$$= x_2\left(-3, 1, 0, 0, 0, 0\right) + x_5\left(-2, 0, 0, 0, 1, 0\right)$$

The solution space has dimension 2.

Plus/Minus Theorem

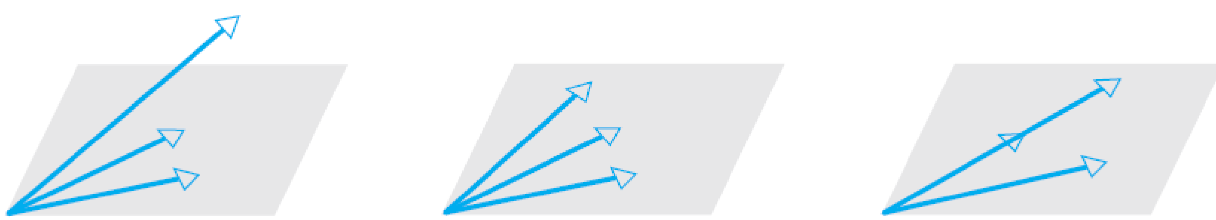
Theorem

Let *S* be a nonempty set of vector space *V*.

a) If *S* is a linearly independent set, and if $\vec{v}$ is a vector in *V* that is outside of span(S), the set $S \cup \{\vec{v}\}$ that results by inserting $\vec{v}$ into *S* is still linearly independent.

b) If $\vec{v}$ is a vector in *S* that is expressible as a linear combination of other vectors in *S*, and if $S - \{\vec{v}\}$ denotes the set obtained by removing $\vec{v}$ from *S*, then *S* and $S - \{\vec{v}\}$ span the same space; that is,

$$span(S) = span(S - \{\mathbf{v}\})$$



The vector outside the plane can be adjoined to the other two without affecting their linear independence | Any of the vectors can be removed, and the remaining two still span the plane | Either of the collinear vectors can be removed, and the remaining two will still span the plane

Theorem

If *W* is a subspace of a finite-dimensional vector space *V*, then

a) *W* is finite-dimensional

b) $\dim(W) \le \dim(V)$

c) $W = V$ if and only if $\dim(W) = \dim(V)$

Exercises Section 2.7 – Coordinates, Basis and Dimension

1. Suppose $\vec{v}_1, \ldots, \vec{v}_n$ is a basis for $\mathbb{R}^n$ and the n by n matrix A is invertible. Show that $A\vec{v}_1, \ldots, A\vec{v}_n$ is also a basis for $\mathbb{R}^n$.

2. Consider the matrix $A = \begin{pmatrix} 1 & 0 & a \\ 2 & -1 & b \\ 1 & 1 & c \\ -2 & 1 & d \end{pmatrix}$

a) Which vectors $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ will make the columns of $\boldsymbol{A}$ linearly dependent?

b) Which vectors $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ will make the columns of $\boldsymbol{A}$ a basis for $\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : y + w = 0 \right\}$?

c) For $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$, compute a basis for the four subspaces.

3. Find a basis for $x - 2y + 3z = 0$ in $\mathbb{R}^3$.

Find a basis for the intersection of that plane with *xy* plane. Then find a basis for all vectors perpendicular to the plane.

4. **U** comes from $\boldsymbol{A}$ by subtracting row 1 from row 3:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad and \quad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Find the bases for the two column spaces. Find the bases for the two row spaces. Find bases for the two nullspaces.

5. Write a 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives $c_1 P_1 + \ldots + c_5 P_5 = 0$, and check entries to prove c_i is zero.) The five permutation matrices are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

6. Choose three independent columns of $A = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix}$. Then choose a different three independent columns. Explain whether either of these choices forms a basis for $C(A)$.

7. Which of the following sets of vectors are bases for $\mathbb{R}^2$?

a) $\{(2, 1), (3, 0)\}$

b) $\{(0, 0), (1, 3)\}$

8. Which of the following sets of vectors are bases for $\mathbb{R}^3$?

a) $\{(1, 0, 0), (2, 2, 0), (3, 3, 3)\}$

b) $\{(3, 1, -4), (2, 5, 6), (1, 4, 8)\}$

c) $\{(2, -3, 1), (4, 1, 1), (0, -7, 1)\}$

d) $\{(1, 6, 4), (2, 4, -1), (-1, 2, 5)\}$

9. Let V be the space spanned by $\vec{v}_1 = \cos^2 x$, $\vec{v}_2 = \sin^2 x$, $\vec{v}_3 = \cos 2x$

a) Show that $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is not a basis for V.

b) Find a basis for V.

10. Find the coordinate vector of $\vec{w}$ relative to the basis $S = \{\vec{u}_1, \vec{u}_2\}$ for $\mathbb{R}^2$

a) $\vec{u}_1 = (1, 0)$, $\vec{u}_2 = (0, 1)$, $\vec{w} = (3, -7)$

b) $\vec{u}_1 = (2, -4)$, $\vec{u}_2 = (3, 8)$, $\vec{w} = (1, 1)$

c) $\vec{u}_1 = (1, 1)$, $\vec{u}_2 = (0, 2)$, $\vec{w} = (a, b)$

d) $\vec{u}_1 = (1, -1)$, $\vec{u}_2 = (1, 1)$, $\vec{w} = (0, 1)$

e) $\vec{u}_1 = (1, -1)$, $\vec{u}_2 = (1, 1)$, $\vec{w} = (1, 1)$

11. Find the coordinate vector of $\vec{v}$ relative to the basis $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

a) $\vec{v} = (2, -1, 3)$, $\vec{v}_1 = (1, 0, 0)$, $\vec{v}_2 = (2, 2, 0)$, $\vec{v}_3 = (3, 3, 3)$

b) $\vec{v} = (5, -12, 3)$, $\vec{v}_1 = (1, 2, 3)$, $\vec{v}_2 = (-4, 5, 6)$, $\vec{v}_3 = (7, -8, 9)$

12. Show that $\{A_1, A_2, A_3, A_4\}$ is a basis for M_{22}, and express A as a linear combination of the basis vectors

a) $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $A = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$

b) $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

c) $A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$, $A_1 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

13. Consider the eight vectors

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

a) List all of the one-element. Linearly dependent sets formed from these.
b) What are the two-element, linearly dependent sets?
c) Find a three-element set spanning a subspace of dimension three, and dimension of two? One? Zero?
d) Which four-element sets are linearly dependent? Explain why.

(**14 – 18**) Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space

14. $\begin{cases} x_1 + x_2 - x_3 = 0 \\ -2x_1 - x_2 + 2x_3 = 0 \\ -x_1 \qquad + x_3 = 0 \end{cases}$

15. $\begin{cases} 3x_1 + x_2 + x_3 + x_4 = 0 \\ 5x_1 - x_2 + x_3 - x_4 = 0 \end{cases}$

16. $\begin{cases} x_1 - 3x_2 + x_3 = 0 \\ 2x_1 - 6x_2 + 2x_3 = 0 \\ 3x_1 - 9x_2 + 3x_3 = 0 \end{cases}$

17. $\begin{cases} x + y + z = 0 \\ 3x + 2y - 2z = 0 \\ 4x + 3y - z = 0 \\ 6x + 5y + z = 0 \end{cases}$

18. $\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 \qquad + 5x_3 = 0 \\ \qquad x_2 + x_3 = 0 \end{cases}$

19. If $AS = SA$ for the shift matrix S. Show that $\boldsymbol{A}$ must have this special form:

$$\textit{If}\ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\textit{then}\ A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

"The subspace of matrices that commute with the shift S has dimension ______."

20. Find bases for the following subspaces of $\mathbb{R}^3$

a) All vectors of the form $(a, b, c, 0)$

b) All vectors of the form (a, b, c, d), where $d = a + b$ and $c = a - b$.

c) All vectors of the form (a, b, c, d), where $a = b = c = d$.

21. Find a basis for the null space of A.

a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

22. Find a basis for the subspace of $\mathbb{R}^4$ spanned by the given vectors

a) $(1, 1, -4, -3),\ (2, 0, 2, -2),\ (2, -1, 3, 2)$

b) $(-1, 1, -2, 0),\ (3, 3, 6, 0),\ (9, 0, 0, 3)$

23. Determine whether the given vectors form a basis for the given vector space

a) $\vec{v}_1(3, -2, 1),\ \vec{v}_2(2, 3, 1),\ \vec{v}_3(2, 1, -3),\ \textit{in}\ \mathbb{R}^3$

b) $\vec{v}_1 = (1, 1, 0, 0),\ \vec{v}_2 = (0, 1, 1, 0),\ \vec{v}_3 = (0, 0, 1, 1),\ \vec{v}_4 = (1, 0, 0, 1),\ \textit{for}\ \mathbb{R}^4$

c) $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ M_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ M_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},\ M_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\ \ M_{22}$

24. Find a basis for, and the dimension of, the null space of the given matrix $\begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & -2 & -2 & 1 \end{bmatrix}$

25. Let $\mathbb{R}$ be the set of all real numbers and let $\mathbb{R}^+$ be the set of all positive real numbers. Show that $\mathbb{R}^+$ is a vector space over $\mathbb{R}$ under the addition

$$\alpha \oplus \beta = \alpha\beta \quad \alpha,\ \beta \in \mathbb{R}^+$$

And the scalar multiplication

$$a \odot \alpha = \alpha^a \quad \alpha \in \mathbb{R}^+,\ a \in \mathbb{R}$$

Find the dimension of the vector space. Is $\mathbb{R}^+$ also a vector space if the scalar multiplication is instead defined as

$$a \otimes \alpha = a^\alpha \quad \alpha \in \mathbb{R}^+,\ a \in \mathbb{R}?$$

Section 2.8 – Row and Column Spaces

Definition

For an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

The vectors

$$\begin{aligned} \vec{v}_1 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ \vec{v}_2 &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \\ \vdots & \qquad\qquad \vdots \\ \vec{v}_{\text{m}} &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{aligned}$$

In $\boldsymbol{R}^n$ that are formed from the rows of *A* are called the ***row vectors*** of *A*, and the vectors

$$\vec{v}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots \quad \vec{v}_3 = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

In $\boldsymbol{R}^m$ that are formed from the rows of *A* are called the ***column vectors*** of *A*.

Definition

If *A* is $m \times n$ matrix, then the subspace of $\boldsymbol{R}^n$ spanned by the row vectors of *A* is called the ***row space*** of *A* and is denoted by $RS(A)$ *or* $R(A)$, and the subspace $\boldsymbol{R}^m$ spanned by the row vectors of *A* is called the ***column space*** of *A* and is denoted by $CS(A)$ *or* $C(A)$. The solution space of the homogeneous system of equations $Ax = 0$, which is a subspace of $\boldsymbol{R}^n$, is called the null space of *A*.

The *Column Space* of A

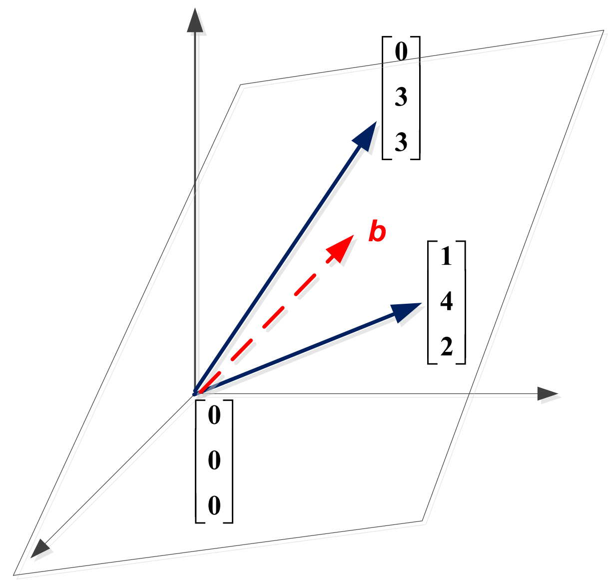
The most important subspaces are tied directly to a matrix A, to solve $A\vec{x} = b$.

Definition

The column space consists of all linear combinations of the columns. The combination are all possible vectors Ax. They fill the column space $C(A)$.

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$$

$$b = x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$



To solve $A\vec{x} = \vec{b}$ is to express b as a combination of the columns.

The column space $CS(A)$ is a plane that containing the two columns. $A\vec{x} = b$ is solvable when b in on that plane.

Theorem

The system $A\vec{x} = b$ is solvable if and only if b is in the column space of A.

Example

Let $A\vec{x} = b$ be the linear system

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Show that ***b*** is in the column space of A by expressing it as a linear combination of the column vectors of A.

Solution

$$\left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 1 & 2 & -3 & -9 \\ 2 & 1 & -2 & -3 \end{array}\right] \quad \begin{array}{l} \\ R_2 + R_1 \\ R_3 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} -1 & 3 & 2 & 1 \\ 0 & 5 & -1 & -8 \\ 0 & 7 & 2 & -1 \end{array}\right] \quad \begin{array}{l} 5R_1 - 3R_2 \\ \\ 5R_3 - 7R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} -5 & 0 & 13 & 29 \\ 0 & 5 & -1 & -8 \\ 0 & 0 & 17 & 51 \end{array}\right] \quad \begin{array}{l} 17R_1 - 13R_3 \\ 17R_2 + R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} -85 & 0 & 0 & -170 \\ 0 & 85 & 0 & -85 \\ 0 & 0 & 17 & 51 \end{array}\right] \quad \begin{array}{l} -\frac{1}{85}R_1 \\ \frac{1}{85}R_2 \\ \frac{1}{17}R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

That implies to $x_1 = 2, \quad x_2 = -1, \quad x_3 = 3$

It follows that

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Example

Describe the column spaces (they are subspaces of $\mathbb{R}^2$) for

$$I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \end{bmatrix}$$

Solution

The column space of I is the whole space $\mathbb{R}^2$. Every vector is a combination of the columns of I. In the space language $CS(I)$ *is* $\mathbb{R}^2$.

The column space of A is only a line, the second column (2, 4) is a multiple of the first column (1, 2) and (2, 4) and all other vectors $(c,\ 2c)$ along that line. The equation $A\vec{x} = \vec{b}$ is only solvable when $\vec{b}$ is on the line.

The column space $C(B)$ is all of $\mathbb{R}^2$. Every b is attainable. The vector $\vec{b} = (3,\ 4)$ is summation of column 1 and 2.

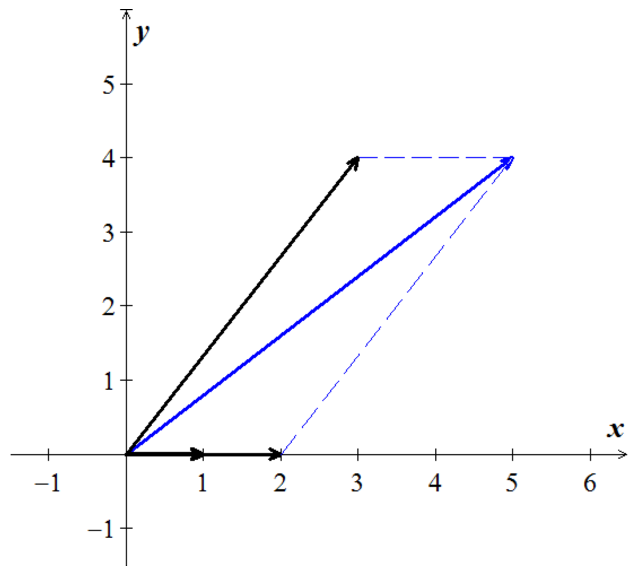
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ \qquad\quad 4x_3 = 4 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 + 2x_2 = 2 \\ \quad x_3 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 1 \end{cases} \quad or \quad \Rightarrow \begin{cases} x_1 = 2 \\ x_2 = 0 \end{cases}$$

$$x = (0,\ 1,\ 1) \quad also \quad x = (2,\ 0,\ 1)$$



This matrix has the same column as I and any $\vec{b}$ is allowed. $\vec{x}$ has an extra component (more solutions).

Pivot Columns

The pivot columns of R have 1's in the pivots and 0's everywhere else.

$$\text{Pivot columns: } A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix}$$

$$\text{Yields to: } R = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- ***The pivot columns are not combinations of earlier columns. The free columns are combinations of columns which are the special solutions!***

Complete **Solution** *to* $AX = B$

To solve $A\vec{x} = \vec{b}$, we need to put into an ***augmented*** form where $\vec{b}$ is not zero.

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \qquad B = \vec{b} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}$$

$$X = \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The augmented matrix is just $\begin{bmatrix} A & B \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} R & \boldsymbol{d} \end{bmatrix}$$

Special Solutions

Each special solution to $A\vec{x}=0$ and $R\vec{x}=0$ has one free variable equal to 1.

$$R\vec{x}=\begin{array}{c}\begin{bmatrix}1 & 3 & 0 & 2 & -1\\ 0 & 0 & 1 & 4 & -3\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}\\ \begin{matrix} & F & & F & F\end{matrix}\end{array}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

The ***free variables*** are $x_2,\ x_4,\ x_5$

$$\to\begin{cases}x_1+3x_2+2x_4-x_5=0\\ \quad x_3+4x_4-3x_5=0\end{cases}$$

1. Set $x_2=1,\ x_4=x_5=0\Rightarrow\begin{cases}x_1=-3\\ x_3=0\end{cases}$ (***Column 2***)

The special solution: $s_1=(-3,\ 1,\ 0,\ 0,\ 0)$

2. Set $x_4=1,\ x_2=x_5=0\Rightarrow\begin{cases}x_1=-2\\ x_3=-4\end{cases}$ (***Column 4***)

The special solution: $s_2=(-2,\ 0,\ -4,\ 1,\ 0)$

3. Set $x_5=1,\ x_2=x_4=0\Rightarrow\begin{cases}x_1=1\\ x_3=3\end{cases}$ (***Column 5***)

The special solution: $s_3=(1,\ 0,\ 3,\ 0,\ 1)$

The nullspace matrix N contains the 3 special solutions in its columns.

$$N=\begin{bmatrix}-3 & -2 & 1\\ 1 & 0 & 0\\ 0 & -4 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{array}{c}\textit{not free}\\ \textit{free}\\ \textit{not free}\\ \textit{free}\\ \textit{free}\end{array}$$

The linear combinations of these three columns give all vectors in the nullspace.

One *Particular* Solution

$$A=\begin{bmatrix}1&3&0&2\\0&0&1&4\\1&3&1&6\end{bmatrix}\qquad \vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\quad \vec{b}=\begin{bmatrix}1\\6\\7\end{bmatrix}$$

$$\left[\begin{array}{cccc|c}1&3&0&2&1\\0&0&1&4&6\\0&0&0&0&0\end{array}\right]=\begin{bmatrix}R & \boldsymbol{d}\end{bmatrix}$$

The ***free variables*** for R to be $x_2=x_4$.

Then the equations give the ***pivot variables*** $x_1=1 \qquad x_3=6$

The ***particular solution*** is: $(1,\ 0,\ 6,\ 0)$

The two special (nullspace) solutions to $Rx=0$:

$$\left[\begin{array}{cccc|c}1&3&0&2&0\\0&0&1&4&0\\0&0&0&0&0\end{array}\right] \Rightarrow \begin{array}{r}x_1+3x_2+x_4=0\\ x_3+4x_4=0\end{array}$$

$$\Rightarrow\begin{cases}x_1=-3x_2-x_4\\ x_3=-4x_4\end{cases}$$

$x_2=1,\ x_4=0$

$$\Rightarrow\ x_1=-3,\ x_3=0\quad\to\ (-3,\ 1,\ 0,\ 0)$$

$x_2=0,\ x_4=1$

$$\Rightarrow\ x_1=-2,\ x_3=-4\ \to\ (-2,\ 0,\ -4,\ 1)$$

The ***complete solution***:

$$x=x_p+x_n$$

$$=\begin{pmatrix}1\\0\\6\\0\end{pmatrix}+x_2\begin{pmatrix}-3\\1\\0\\0\end{pmatrix}+x_4\begin{pmatrix}-2\\0\\-4\\1\end{pmatrix}$$

Example

Find the condition on (b_1, b_2, b_3) for $A\vec{x} = \vec{b}$ to be solvable, if

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Solution

The augmented form:

$$\begin{bmatrix} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{bmatrix} \quad \begin{matrix} \\ R_2 - R_1 \\ R_3 + 2R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & -1 & b_3 + 2b_1 \end{bmatrix} \quad \begin{matrix} R_1 - R_2 \\ \\ R_3 + R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 + b_1 + b_2 \end{bmatrix} \quad \to \quad \boxed{b_1 + b_2 + b_3 = 0}$$

The last equation is 0 = 0 provided $b_1 + b_2 + b_3 = 0$.

There are ***no*** *free variables* and ***no*** *special solutions*.

The nullspace solution: $x_n = 0$

The complete solution:

$$x = x_p + x_n$$

$$= \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $b_1 + b_2 + b_3 \neq 0$, there is no solution to $A\vec{x} = \vec{b}$ and $\vec{x}_p$ doesn't exist.

Example

a) Find a subset of the vectors

$$\vec{v}_1 = (1, -2, 0, 3) \quad \vec{v}_2 = (2, -5, -3, 6), \quad \vec{v}_3 = (0, 1, 3, 0), \quad \vec{v}_4 = (2, -1, 4, -7), \quad \vec{v}_5 = (5, -8, 1, 2)$$

That forms a basis for the space spanned by these vectors

b) Express each vector not in the basis as a linear combination of the basis vectors

Solution

a) Construct the vectors as its column vectors

$$\begin{array}{cc} \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} & \begin{array}{l} \\ R_2 + 2R_1 \\ \\ R_4 - 3R_1 \end{array} \end{array}$$

$$\begin{array}{cc} \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & -3 & 3 & 4 & 1 \\ 0 & 0 & 0 & -13 & -13 \end{bmatrix} & \begin{array}{l} R_1 + 2R_2 \\ \\ R_3 - 3R_2 \\ \\ \end{array} \end{array}$$

$$\begin{array}{cc} \begin{bmatrix} 5 & 0 & 10 & 0 & 5 \\ 0 & -5 & 5 & 0 & -5 \\ 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{l} \frac{1}{5}R_1 \\ -\frac{1}{5}R_2 \\ -\frac{1}{5}R_3 \\ \\ \end{array} \end{array}$$

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 & \vec{w}_4 & \vec{w}_5 \end{array} \end{array}$$

The leading 1's occurs in columns 1, 2, and 4.

$\{\vec{w}_1, \vec{w}_2, \vec{w}_4\}$ is a basis for the column space, and consequently $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$

b) $\vec{w}_1 = (1, 0, 0, 0) \quad \vec{w}_2 = (0, 1, 0, 0), \quad \vec{w}_3 = (2, -1, 0, 0)$

$\vec{w}_4 = (0, 0, 1, 0), \quad \vec{w}_5 = (1, 1, 1, 0)$

$$\vec{w}_3 = 2\vec{w}_1 - \vec{w}_2$$

$$\vec{w}_3 = \vec{w}_1 + \vec{w}_2 + \vec{w}_4$$

We call these ***dependency equations***

The corresponding relationships are:

$$\vec{v}_3 = 2\vec{v}_1 - \vec{v}_2$$
$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 + \vec{v}_4$$

Solving $Ax = 0$ by *elimination*

Matrix A is rectangular and we still use the elimination.

1. Forward elimnation from A to a triangular U.
2. Back substitution in $Ax = 0$ to find x.

Consider the matrix $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix} \quad R_3 - 4R_2$$

Triangular U: $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

P: The ***pivot*** variables are x_1 *and* x_3, since columns 1 and 3 contains pivots.

F: The ***free*** variables are x_2 *and* x_4, since columns 2 and 4 have no pivots.

Special solutions to:

$$\begin{cases} x_1 + x_2 + 2x_3 + 3x_4 = 0 \\ 4x_3 + 4x_4 = 0 \end{cases} \to \begin{cases} x_1 = -x_2 - x_4 \\ x_3 = -x_4 \end{cases}$$

Complete solution: $x = x_2 \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{Special} + x_4 \underbrace{\begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{Special} = \underbrace{\begin{pmatrix} -x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix}}_{Complete}$

The special solution are in the nullspace $NS(A)$, and their combinations fill out the whole Nullspace.

Exercises   *Section* 2.8 – Row and Column Spaces

1. List the row vectors and column vectors of the matrix

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ 3 & 5 & 7 & -1 \\ 1 & 4 & 2 & 7 \end{bmatrix}$$

(2 – 4) Express the product $A\vec{x}$ as a linear combination of the column vectors of A.

2. $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$

4. $\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$

(5 – 8) Determine whether $\vec{b}$ is in the column space of A, and if so, express $\vec{b}$ as a linear combination of the column vectors of A.

5. $A = \begin{bmatrix} 1 & 3 \\ 4 & -6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$

7. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 7 \end{bmatrix}$

9. Suppose that $x_1 = -1, \quad x_2 = 2, \quad x_3 = 4, \quad x_4 = -3$ is a solution of a nonhomogeneous linear system $A\vec{x} = \vec{b}$ and that the solution set of the homogeneous system $A\vec{x} = \vec{0}$ is given by the formulas $\quad x_1 = -3r + 4s, \quad x_2 = r - s, \quad x_3 = r, \quad x_4 = s$

a) Find a vector form of the general solution of $A\vec{x} = \vec{0}$

b) Find a vector form of the general solution of $A\vec{x} = \vec{b}$

(10 – 13) Find the vector form of the general solution of the given linear system $A\vec{x} = \vec{b}$; then use that result to find the vector form of the general solution of $A\vec{x} = \vec{0}$.

10. $\begin{cases} x_1 - 3x_2 = 1 \\ 2x_1 - 6x_2 = 2 \end{cases}$

11. $\begin{cases} x_1 + x_2 + 2x_3 = 5 \\ x_1 \qquad + x_3 = -2 \\ 2x_1 + x_2 + 3x_3 = 3 \end{cases}$

12. $\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 = 4 \\ -2x_1 + x_2 + 2x_3 + x_4 = -1 \\ -x_1 + 3x_2 - x_3 + 2x_4 = 3 \\ 4x_1 - 7x_2 \qquad - 5x_4 = -5 \end{cases}$

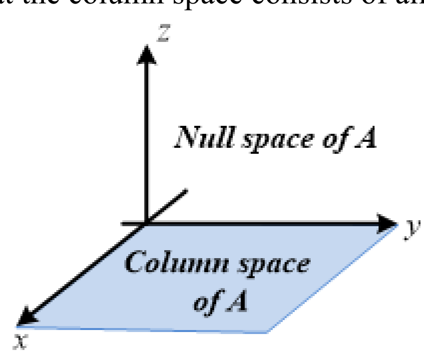
13. $\begin{cases} x_1 - 2x_2 + x_3 + 2x_4 = -1 \\ 2x_1 - 4x_2 + 2x_3 + 4x_4 = -2 \\ -x_1 + 2x_2 - x_3 - 2x_4 = 1 \\ 3x_1 - 6x_2 + 3x_3 + 6x_4 = -3 \end{cases}$

14. Given the vectors $\vec{v}_1 = (1,\ 2,\ 0)$ and $\vec{v}_2 = (2,\ 3,\ 0)$

a) Are they linearly independent?
b) Are they a basis for any space?
c) What space **V** do they span?
d) What is the dimension of that space?
e) What matrices ***A*** have **V** as their column space?
f) Which matrices have **V** as their nullspace?
g) Describe all vectors $\vec{v}_3$ that complete a basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$ for $\mathbb{R}^3$.

15. *a*) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Show that relative to an *xyz*-coordinate system in 3-space the null space of *A* consists of all points on the *z*-axis and that the column space consists of all points in the *xy*-plane.



b) Find a 3 x 3 matrix whose null space is the *x*-axis and whose column space is the *yz*-plane.

16. If we add an extra column $\vec{b}$ to a matrix A, then the column space gets larger unless _____. Give an example where the column space gets larger and an example where it doesn't. Why is $A\vec{x} = \vec{b}$ solvable exactly when the column space doesn't get larger – it is the same for A and $\begin{bmatrix} A & \vec{b} \end{bmatrix}$?

17. For which right sides (find a condition on b_1, b_2, b_3) are these solvable. (Use the column space $C(A)$ and the equation $A\vec{x} = \vec{b}$)

a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

18. Show that the matrices A and $\begin{bmatrix} A & AB \end{bmatrix}$ (with extra columns) have the same column space. But find a square matrix with $C\left(A^2\right)$ smaller than $C\left(A\right)$. Important point: An n by n matrix has $C\left(A\right) = \mathbb{R}^n$ exactly when A is an ______ matrix.

19. The column of AB are combinations of the columns of A. This means: The column space of AB is contained in (possibly equal to) to the column space of A. Give an example where the column spaces A and AB are not equal.

20. Find a square matrix A where $C\left(A^2\right)$ (the column space of A^2 is smaller than $C\left(A\right)$.

21. Suppose $A\vec{x} = \vec{b}$ and $C\vec{x} = \vec{b}$ have the same (complete) solutions for every $\vec{b}$. Is true that $A = C$?

22. Apply Gauss-Jordan elimination to $U\vec{x} = 0$ and $U\vec{x} = c$. Reach $R\vec{x} = 0$ and $R\vec{x} = d$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \qquad \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

Solve $R\vec{x} = 0$ to find x_n (its free variable is $x_2 = 1$).

Solve $R\vec{x} = d$ to find x_p (its free variable is $x_2 = 0$).

The subspace requirements: $x+y$ ***and*** cx ***(and then all linear combinations*** $cx+dy$***) must stay in the subspace.***

23. Which of the following subsets of $\mathbb{R}^3$ are actually subspaces?

a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$

b) The plane of vectors with $b_1 = 1$.

c) The vectors with $b_1 b_2 b_3 = 0$.

d) All linear combinations of $v = (1,\ 4,\ 0)$ and $w = (2,\ 2,\ 2)$.

e) All vectors that satisfies $b_1 + b_2 + b_3 = 0$

f) All vectors with $b_1 \le b_2 \le b_3$.

24. We are given three different vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3$. Construct a matrix so that the equations $A\vec{x} = \vec{b}_1$ and $A\vec{x} = \vec{b}_2$ are solvable, but $A\vec{x} = \vec{b}_3$ is not solvable.

a) How can you decide if this possible?

b) How could you construct A?

25. For which vectors (b_1, b_2, b_3) do these systems have a solution?

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

26. Find a basis for the null space of A.

$$A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$$

27. Is it true that is $m = n$ then the row space of A equals the column space.

28. If the row space equals the column space the $A^T = A$

29. If $A^T = -A$, then the row space of A equals the column space.

30. Does the matrices A and $-A$ share the same 4 subspaces?

31. Is A and B share the same 4 subspaces then A is multiple of B.

32. Suppose $A\vec{x} = \vec{b}$ & $C\vec{x} = \vec{b}$ have the same (complete) solutions for every $\vec{b}$. Is it true that $A = C$

33. A and A^T have the same left nullspace?

Section 2.9 – Rank and the Fundamental Matrix Spaces

The **R**educed **R**ow **E**chelon **F**orm (***rref***) is a matrix (R) with each pivot column has only one nonzero entry (the pivots which is always 1).

$$R = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{R = rref(A)}$$

Rank of a Matrix

The rank of a matrix A (***m*** by ***n***) is the number of ***nonzero rows*** in the row-reduced echelon form of A (it is the number of pivot). The common dimension of the row space and column space of a matrix A is called the ***rank*** of A and is denoted by

$$\boxed{rank(A) = r}$$

Note:

The rank of a matrix is well defined due to the uniqueness of the row-reduced echelon form. No matter what sequence of elementary row operations is performed to put the given matrix in row-reduced echelon form; there will always be the same number of nonzero rows.

Theorem

The row space and column space of a matrix A have the same dimension

The objective is to connect ***rank*** and ***dimension***.

- The ***rank*** of a matrix is the number of pivots.
- The ***dimension*** of a subspace is the number of vectors in a basis.

✓ ***A has full row rank if every row has a pivot:*** $r = m$***. No zero in R.***

✓ ***A has full column rank if every column has a pivot:*** $r = n$***. No free variables.***

Example

Find the rank of $A=\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{bmatrix} \quad R_3 - R_1$$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & -2 & -2 & 4 \end{bmatrix} \quad \begin{matrix} R_1 + R_2 \\ \\ R_3 + 2R_2 \end{matrix}$$

$$R=\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix R has 2 nonzero rows, therefore the $rank(A)=2$

Example

The columns of A are dependent. $A\vec{x}=\vec{0}$ has a nonzero solution.

$$Ax=\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

$$-3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}+1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}+1\begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The rank of A is only $r=2$.
Independent columns would give full column rank $r=n=3$.

- The columns of A are independent exactly when the rank is $r=n$. There are ***n*** pivots and no free variables. Only $\vec{x}=\vec{0}$ is the nullspace.

Example

When all rows are multiplying of one pivot row, the rank is $r = 1$:

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad [6]$$

Solution

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix} \quad R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix} \quad 3R_2 - 5R_1$$

$$\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \quad \tfrac{1}{3}R_2$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad \begin{matrix} \tfrac{1}{5}R_1 \\ 5R_2 - 2R_1 \end{matrix} \quad \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The row-reduced echelon form $R = rref(A)$:

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad [1]$$

These matrices have only one pivot.

Dimension *Theorem* for Matrices

If A is a matrix with n columns, then $\boxed{rank(A) + nullity(A) = n}$

Theorem

If A is an $m \times n$ matrix, then

- $rank(A)$ = the number of leading variables in the general solution of $A\vec{x} = \vec{0}$
- $nullity(A)$ = the number of parameters in the general solution of $A\vec{x} = \vec{0}$

Theorem

If A is any matrix, then $rank(A) = rank\left(A^T\right)$

- ***$Ax = 0$ has $n - r$ free variables and special solutions: n columns minus r pivot columns. The null matrix N has $n - r$ columns (the special solutions).***
- ***The particular solution solves:*** $A\vec{x}_p = \vec{b}$
- ***Full column rank*** $R = \begin{bmatrix} n \text{ by } n \text{ identity matrix} \\ m - n \text{ rows of zeros} \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$

The reduced row echelon form looks like:

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{array}{l} \boldsymbol{r} \textit{ pivot rows} \\ \boldsymbol{m - r} \textit{ zero rows} \end{array}$$

The pivot variables in the $n - r$ special columns come by changing F to $-F$:

Nullspace matrix: $$N = \begin{pmatrix} -F \\ I \end{pmatrix} \begin{array}{l} \boldsymbol{r} \textit{ pivot variables} \\ \boldsymbol{n - r} \textit{ free variables} \end{array}$$

- Every matrix A with ***full column rank*** ($r = n$) has all these properties:
  1. All columns of A are pivot columns
  2. There are no free variables or special solutions.
  3. The nullspace $NS(A)$ contains only the zero vector $\vec{x} = \vec{0}$
  4. If $A\vec{x} = \vec{b}$ has a solution (might not) then it has only one solution.

Example

Suppose A is a square invertible matrix, $m = n = r$. What are $\vec{x}_p$ and $\vec{x}_n$ ***?***

Solution

The particular solution is the one and only solution $A^{-1}\vec{b}$.

There are no special solutions or free variables. $R = I$ has no zero rows.

The only vector in the null space is $\vec{x}_n = \vec{0}$.

The complete solution is

$$\begin{aligned} \vec{x} &= \vec{x}_p + \vec{x}_n \\ &= A^{-1}\vec{b} + \vec{0} \\ &= A^{-1}\vec{b} \end{aligned}$$

Example

Compute $N(A)$ for $A:\ \mathbb{R}^2 \to \mathbb{R}^3$ given by $A = (x + y,\ x,\ 2x - y)$

Solution

To find $N(A)$, we must solve the equation $A(x,\ y) = (0,\ 0,\ 0)$

$$\begin{pmatrix} x+y \\ x \\ 2x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} x + y = 0 \Rightarrow \boxed{y = 0} \\ \boxed{x = 0} \end{array}$$

Thus $NS(A) = \{0\}$, the set that consists solely of the zero vector.

- If $A\vec{x} = \vec{0}$ has more unknowns than equations (more columns than rows) then it has nonzero solutions. There must be free columns, without pivots.

Definition

If W is a subspace of $\mathbb{R}^n$ that are orthogonal to every vector in W is called orthogonal complement of W and is denoted nu the symbol $W^{\perp}$. $N(A)^{\perp}$ is exactly the row space $C\left(A^T\right)$

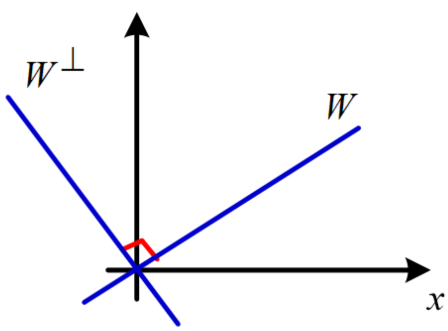
Fundamental Theorem of Linear Algebra

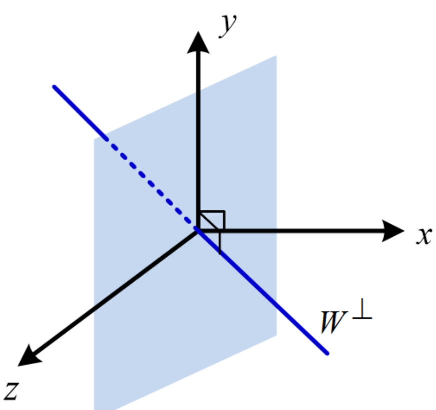
The nullspace is the orthogonal complement of the row space $\left(in\ \mathbb{R}^n\right)$.

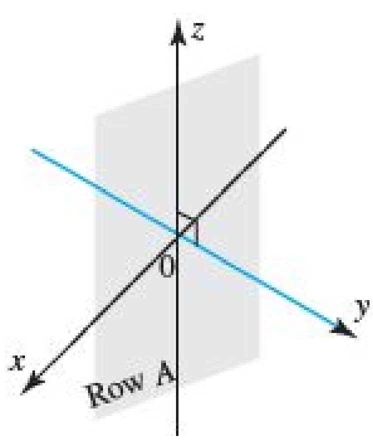
The left nullspace is the orthogonal complement of the column space $\left(in\ \mathbb{R}^m\right)$.

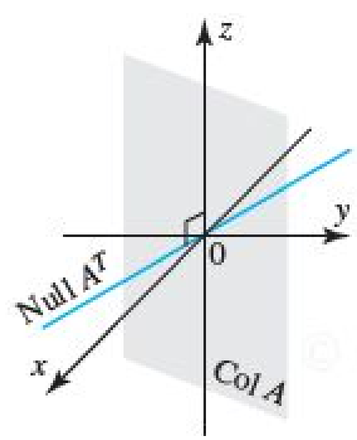
If W is a subspace of $\mathbb{R}^n$

- $W^{\perp}$ is a subspace of $\mathbb{R}^n$`
- The only vector common to W and $W^{\perp}$ is 0.
- The orthogonal complement of $W^{\perp}$ is W.









Left Nullspace

A matrix A^T has m columns and has r ranks, so the number of free columns of A^T must be $m-r$.

$$\dim N\left(A^T\right) = m-r$$

The left nullspace is the collection of vectors $\vec{y}$ for which $A^T\vec{y} = \vec{0}$. Equivalently, $\vec{y}^T A = \vec{0}$, where $\vec{y}$ and $\vec{0}$ are row vectors. We can call "***left nullspace***" because $\vec{y}^T$ is on the left of matrix A in that equation.

To find a basis for the left nullspace we reduce an augmented type of A.

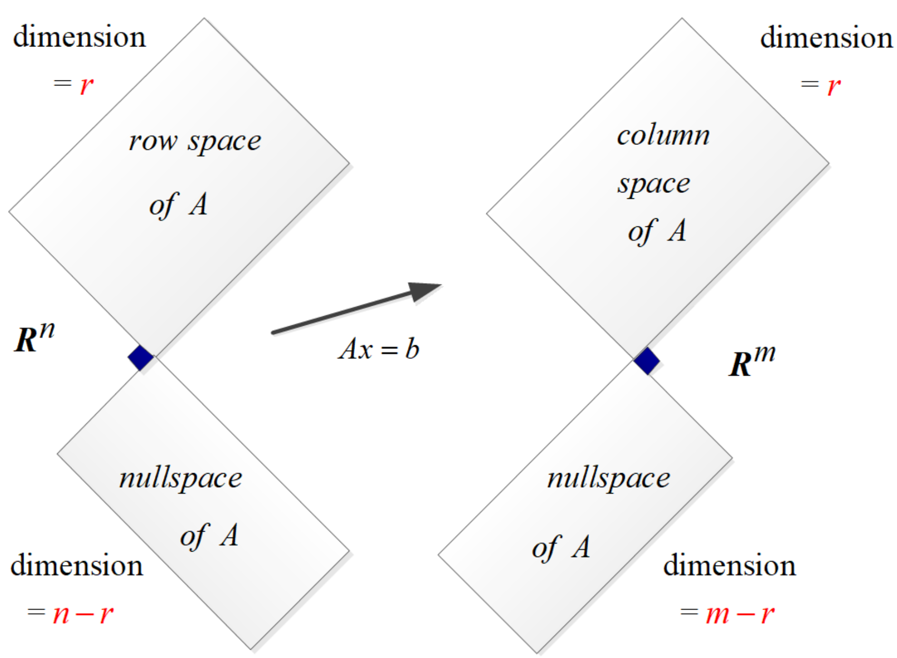
$$\left[A_{m\times n} \mid I_{m\times n}\right] \;\to\; \left[R_{m\times n} \mid E_{m\times n}\right]$$

Where matrix E can be found from $EA = R$.

If matrix A is a square matrix, then $E = A^{-1}$.

The Four Fundamental Subspaces

1. The ***row space*** is $C\left(A^T\right)$, a subspace of $\mathbb{R}^n$.
2. The ***column space*** is $C(A)$, a subspace of $\mathbb{R}^m$.
3. The ***null space*** is $N(A)$, a subspace of $\mathbb{R}^n$.
4. The ***left null space*** is $N\left(A^T\right)$, a subspace of $\mathbb{R}^m$.



Two pairs of orthogonal subspaces.

***For an m x n matrix of rank r*:**

Fundamental Space	***Subspace of***	***Dimension***
Nullspace	$\mathbb{R}^n$	$n-r$
Column Space	$\mathbb{R}^m$	r
Row space	$\mathbb{R}^n$	r
Left nullspace	$\mathbb{R}^m$	$m-r$

Example

Find a basis for each of the four subspaces associated with matrix A:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix} \quad R_2 - 2R_1$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \quad x_1 = -2x_2 - 4x_3 \leftarrow \textit{Row space}$$

1. Basis for ***row space***: $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

2. Basis of the **column spaces**: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Rank($\boldsymbol{A}$) = 1

Dimension of A = 1

The pivots variables are: x_1

The free variables are: x_2, x_3

Set $x_2 = 1 \quad x_3 = 0$

The special solution: $s_1 = (-2,\ 1,\ 0)$

Set $x_2 = 0 \quad x_3 = 1$

The special solution: $s_2 = (-4,\ 0,\ 1)$

3. Basis of the **Null space**: $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$

$$A^T = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{matrix}$$

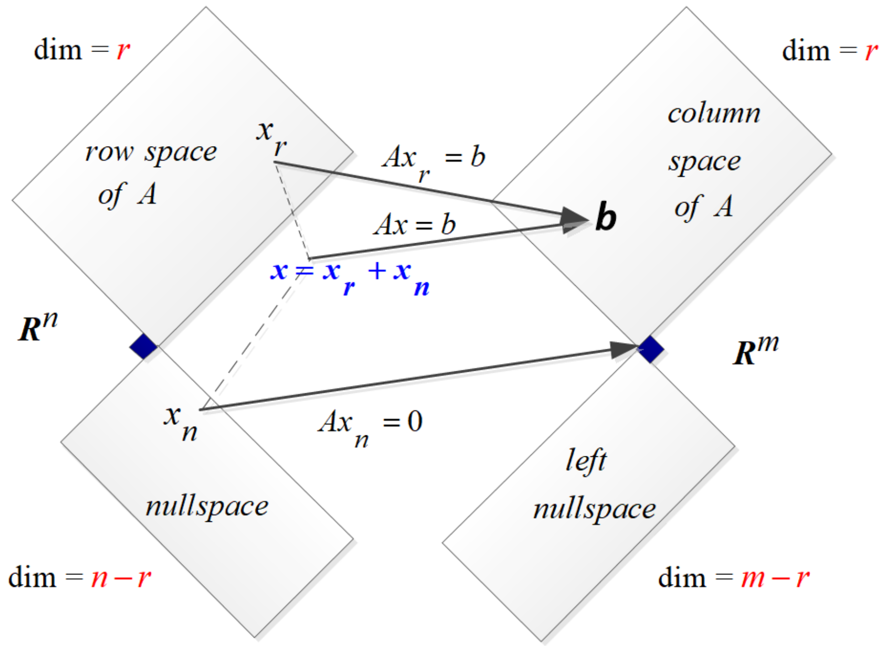
$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad y_1 = -2y_2$$

Set $y_2 = 1 \quad \Rightarrow \quad s^*_1 = (-2,\ 1)$

4. Basis of the **Left Nullspace**: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Combining Bases from Subspaces

- Any n linearly independent vectors in $\mathbb{R}^n$ must span $\mathbb{R}^n$. They are basis. Any n vectors that span $\mathbb{R}^n$ must be independent. They are a basis.
- If the n columns of $\boldsymbol{A}$ are independent, they span $\mathbb{R}^n$, So $A\vec{x} = \vec{b}$ is solvable,
- If the n columns span $\mathbb{R}^n$, they are independent. So $A\vec{x} = \vec{b}$ has only one solution.



- When the orthogonal complement of a subspace S is defined to be the subspace whose vectors pairs to zero with the vectors in S. The larger the S is, the more restriction $S^{\perp}$ has, and hence the smaller $S^{\perp}$ is.

Theorem – *Equivalent Statements*

If A is an $n \times n$ matrix, then the following statements are equivalent.

a) A is invertible

b) $A\vec{x} = \vec{0}$ has only the trivial solution

c) The reduced row echelon form of A is I_n

d) A is expressible as a product of elementary matrices

e) $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ matrix $\vec{b}$

f) $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ matrix $\vec{b}$

g) $\det(A) \neq 0$

h) The column vectors of A are linearly independent

i) The row vectors of A are linearly independent

j) The column vectors of A span $\mathbb{R}^n$

k) The row vectors of A span $\mathbb{R}^n$

l) The column vectors of A form a basis for $\mathbb{R}^n$

m) The row vectors of A form a basis for $\mathbb{R}^n$

n) A has a rank n.

o) A has nullity 0.

p) The orthogonal complement of the null space of A is $\mathbb{R}^n$

q) The orthogonal complement of the row space of A is $\{0\}$

Exercises Section 2.9 – Rank and the Fundamental Matrix Spaces

1. Verify that $rank(A) = rank\left(A^T\right)$

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

2. Find the rank and nullity of the matrix; then verify that the values obtained satisfy $rank(A) + N(A) = n$

a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

3. If A is an $m \times n$ matrix, what is the largest possible value for its rank and the smallest possible value of the nullity of A.

4. Discuss how the rank of A varies with t.

a) $A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{bmatrix}$

5. Are there values of r and s for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

Has rank 1? Has rank 2? If so, find those values.

6. Find the row reduced form R and the rank r of A (those depend on c).
Which are the pivot columns of A? Which variables are free? What are the special solutions and the nullspace matrix N (always depending on c)?

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & c \end{bmatrix} \quad and \quad A = \begin{bmatrix} c & c \\ c & c \end{bmatrix}$$

7. Find the row reduced form R and the rank r of A (those depend on c).
Which are the pivot columns of A? Which variables are free? What are the special solutions and the nullspace matrix N (always depending on c)?

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad and \quad A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$$

8. If A has a rank r, then it has an r by r sub-matrix S that is invertible. Remove $m-r$ rows and $n-r$ columns to find an invertible sub-matrix S inside each A (you could keep the pivot rows and pivot columns of A).

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

9. Suppose that column 3 of 4 x 6 matrix is all zero. Then x_3 must be a ______ variable. Give one special solution for this matrix.

10. Fill in the missing numbers to make A rank 1, rank 2, rank 3. (your solution should be 3 matrices)

$$A = \begin{pmatrix} & -3 & \\ 1 & 3 & -1 \\ & 9 & -3 \end{pmatrix}$$

11. Fill out these matrices so that they have rank 1:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{pmatrix} \quad B = \begin{pmatrix} 2 & & \\ 1 & & \\ 2 & 6 & -3 \end{pmatrix} \quad M = \begin{pmatrix} a & b \\ c & \end{pmatrix}$$

12. Suppose A and B are n by n matrices, and $AB = I$. Prove from $rank(AB) \le rank(A)$ that the $rank(A) = n$. So A is invertible and B must be its two-sided inverse. Therefore $BA = I$ (which is not so obvious!).

13. Every m by n matrix of rank r reduces to (m by r) times (r by n):

$$A = \text{(pivot columns of } A\text{) (first } r \text{ rows of } R\text{)} = (COL)(ROW)^T$$

Write the 3 by 4 matrix $A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix}$ as the product of the 3 by 2 from the pivot columns and the 2 by 4 matrix from R.

14. Suppose R is m by n matrix of rank r, with pivot columns first: $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

a) What are the shapes of those 4 blocks?
b) Find the right-inverse B with $RB = I$ if $r = m$.
c) Find the right-inverse C with $CR = I$ if $r = n$.
d) What is the reduced row echelon form of R^T (with shapes)?
e) What is the reduced row echelon form of $R^T R$ (with shapes)?
Prove that $R^T R$ has the same nullspace as R. Then show that $A^T A$ always has the same nullspace as A (a value fact).
f) Suppose you allow elementary column operations on $\boldsymbol{A}$ as well as elementary row operations (which get to R). What is the "row-and-column reduced form" for an m by n matrix of rank $\boldsymbol{r}$?

15. True or False (check addition or give a counterexample)

a) The symmetric matrices in $M\left(with\ A^T = A\right)$ from a subspace.
b) The skew-symmetric matrices in $M\left(with\ A^T = -A\right)$ from a subspace.
c) The un-symmetric matrices in $M\left(with\ A^T \neq A\right)$ from a subspace.
d) Invertible matrices
e) Singular matrices

16. Let $A = \begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 2 & 4 & -3 & 7 & 0 \\ 3 & 6 & -5 & 10 & -2 \\ 5 & 10 & -9 & 16 & 0 \end{pmatrix}$

a) Reduce A to row-reduced echelon form.
b) What is the rank of A?
c) What are the pivots?
d) What are the free variables?
e) Find the special solutions. What is the nullspace $N(A)$?
f) Exhibit an r x r submatrix of A which is invertible, where $r = rank(A)$. (An r x r submatrix of A is obtained by keeping r rows and r columns of A)

17. Let $A = \begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 2 & 1 & 0 & 0 & -15 \\ 6 & -1 & -8 & -1 & -47 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix}$

a) Reduce A to row-reduced echelon form.
b) What is the rank of A?
c) What the pivots?
d) What are the free variables?
e) Find the special solutions. What is the nullspace $N(A)$?
f) Give the complete solution to $Ax = b,\ where\ \ b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

18. Let $A = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

a) Reduce A to row-reduced echelon form.
b) What is the rank of A?
c) What the pivots?
d) What are the free variables?
e) Find the special solutions.
f) What is the nullspace $N(A)$?

19. Let $A = \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}$

a) Reduce A to row-reduced echelon form.
b) What is the rank of A?
c) What the pivots?
d) What are the free variables?
e) Find the special solutions.
f) What is the nullspace $N(A)$?

20. The 3 by 3 matrix A has rank 2.

$$Ax = b \quad is \quad \begin{matrix} x_1 + 2x_2 + 3x_3 + 5x_4 = b_1 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 = b_2 \\ 3x_1 + 6x_2 + 7x_3 + 13x_4 = b_3 \end{matrix}$$

a) Reduce $\begin{bmatrix} A & b \end{bmatrix}$ to $\begin{bmatrix} U & c \end{bmatrix}$, so that $A\vec{x} = b$ becomes triangular system $U\vec{x} = c$.

b) Find the condition on (b_1, b_2, b_3) for $A\vec{x} = b$ to have a solution

c) Describe the column space of A. Which plane in $\mathbb{R}^3$?

d) Describe the nullspace of A. Which special solutions in $\mathbb{R}^4$?

e) Find a particular solution to $Ax = (0,\ 6,\ -6)$ and then complete solution.

21. Find the special solutions and describe the complete solution to $Ax = 0$ for

$A_1 =$ 3 by 4 zero matrix $\qquad A_2 = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \qquad A_3 = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$

Which are the pivot columns? Which are the free variables? What is the R (Reduced Row Echelon matrix) in each case?

22. Create a 3 by 4 matrix whose special solutions to $A\vec{x} = 0$ are s_1 *and* s_2:

$$\vec{s}_1 = \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad and \quad \vec{s}_2 = \begin{pmatrix} -2 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

You could create the matrix A in row reduced form R. Then describe all possible matrices A with the required Nullspace $N(A) =$ all combinations of $\vec{s}_1$ *and* $\vec{s}_2$.

23. The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - z = 0$. One particular point on this plane is $(12,\ 0,\ 0)$. All points on the plane have the form (fill the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \ \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} \ \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} \ \\ 0 \\ 1 \end{bmatrix}$$

24. Construct a matrix whose column space contains (1, 1, 5) and (0, 3, 1) and whose Nullspace contains $(1,\ 1,\ 2)$.

25. Construct a matrix whose column space contains (1, 1, 0) and (0, 1, 1) and whose Nullspace contains (1, 0, 1) and (0, 0, 1).

26. Construct a matrix whose column space contains (1, 1, 1) and whose Nullspace contains $(1,\ 1,\ 1,\ 1)$.

27. How is the Nullspace N(*C*) related to the spaces N(*A*) and N(*B*), if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

28. Why does no 3 by 3 matrix have a nullspace that equals its column space?

29. If $AB = 0$ then the column space B is contained in the _______ of A. Give an example of A and B.

30. True or false (with reason if true or example to show it is false)
  a) A square matrix has no free variables.
  b) An invertible matrix has no free variables.
  c) An m by n matrix has no more than $\boldsymbol{n}$ pivot variables.
  d) An m by n matrix has no more than $\boldsymbol{m}$ pivot variables.

31. Suppose an m by n matrix has r pivots. The number of special solutions is ______.

The Nullspace contains only $x = 0$ when $r =$ _______.

The column space is all of $\mathbb{R}^m$ when $r =$ ______.

32. Find the complete solution in the form $\vec{x}_p + \vec{x}_n$ to these full rank system:

$$a)\ \ x + y + z = 4 \qquad\qquad b)\ \begin{cases} x + y + z = 4 \\ x - y + z = 4 \end{cases}$$

33. Find the complete solution in the form $\vec{x}_p + \vec{x}_n$ to the system:

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

34. If $\boldsymbol{A}$ is 3 x 7 matrix, its largest possible rank is ________. In this case, there is a pivot in every ________ of $\boldsymbol{U}$ and $\boldsymbol{R}$, the solution to $A\vec{x} = \vec{b}$ _________ (always exists or is unique), and the column space of $\boldsymbol{A}$ is _________. Construct an example of such a matrix $\boldsymbol{A}$.

35. If $\boldsymbol{A}$ is 6 x 3 matrix, its largest possible rank is ________. In this case, there is a pivot in every ________ of $\boldsymbol{U}$ and R, the solution to $A\vec{x} = \vec{b}$ _________ (always exists or is unique), and the nullspace of $\boldsymbol{A}$ is _________. Construct an example of such a matrix $\boldsymbol{A}$.

36. Find the rank of $A, A^T A$ and AA^T for $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix}$

37. Explain why these are all false:

a) The complete solution is any linear combination of $\vec{x}_p$ *and* $\vec{x}_n$.

b) A system $A\vec{x} = \vec{b}$ has at most one particular solution.

c) The solution $\vec{x}_p$ with all free variables zero is the shortest solution (minimum length $\|\vec{x}\|$). Find a 2 by 2 counterexample.

d) If A is invertible there is no solution $\vec{x}_n$ in the null space.

38. Write down all known relation between r and m and n if $A\vec{x} = \vec{b}$ has

a) No solution for some $\vec{b}$.

b) Infinitely many solutions for every $\vec{b}$.

c) Exactly one solution for some $\vec{b}$, no solution for other $\vec{b}$.

d) Exactly one solution for every $\vec{b}$.

39. Find a basis for its row space, find a basis for its column space, and determine its rank

$$a)\ \begin{bmatrix} 0 & 2 & -3 & 4 & 1 & 2 & 1 & 7 \\ 0 & 0 & 3 & -2 & 0 & 4 & -5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 6 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad b)\ \begin{bmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{bmatrix}$$

40. Find a basis for the row space, find a basis for the null space, find $\dim RS$, find $\dim NS$, and verify $\dim RS + \dim NS = n$

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{bmatrix}$$

41. Determine if $\vec{b}$ lies in the column space of the given matrix. If it does, express $\vec{b}$ as linear combination of the column.

$$\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

42. Find the transition matrix from B to C and find $[\vec{x}]_c$

a) $B = \{(3,\ 1), (-1,\ -2)\}, \quad C = \{(1,\ -3), (5,\ 0)\}, \quad [x]_B = [-1 \quad -2]^T$

b) $B = \{(1,\ 1,\ 1),\ (-2,\ -1,\ 0),\ (2,\ 1,\ 2)\}, \quad C = \{(-6,\ -2,\ 1),\ (-1,\ 1,\ 5),\ (-1,\ -1,\ 1)\},$

$[\vec{x}]_B = [-3 \quad 2 \quad 4]^T$

43. Does A and A^T have the same number of pivots.

(44 – 49) For the given matrix A, which is given in row reduction echelon form

a) What is the rank of A?
b) What is the dimension of A?
c) What are the pivots?
d) What are the free variables?
e) Find the special (homogeneous) solutions.
f) What is the nullspace $N(A)$?
g) Find the particular solution $Ax=b$
h) Give the complete solution.

44. $A=\begin{pmatrix}1&3&0&2\\0&0&1&4\\0&0&0&0\end{pmatrix}\qquad where\ \ b=A\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

45. $A=\begin{pmatrix}1&0&2&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\qquad where\ \ b=A\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

46. $A=\begin{pmatrix}1&2&0&4\\0&-3&1&-12\\0&0&0&0\end{pmatrix}\qquad where\ \ b=A\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

47. $A=\begin{pmatrix}1&0&0&\frac{13}{11}\\0&1&0&-\frac{17}{11}\\0&0&1&\frac{6}{11}\\0&0&0&0\end{pmatrix}\qquad where\ \ b=A\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

48. $A=\begin{pmatrix}1&0&0&-\frac{1}{3}\\0&1&0&-1\\0&0&1&-\frac{1}{3}\\0&0&0&0\end{pmatrix}\qquad where\ \ b=A\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$

49. $A = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad$ *where* $b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

(50 – 55) Find a basis for each of the four subspaces associated with each given matrix

50. $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{pmatrix}$

51. $B = \begin{pmatrix} 1 & 3 & 0 & 5 \\ 2 & 6 & 1 & 16 \\ 5 & 15 & 0 & 25 \end{pmatrix}$

52. $C = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$

53. $D = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix}$

54. $M = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{pmatrix}$

55. $N = \begin{pmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{pmatrix}$