3.5 Calculus Topics

3.5.1 Limits

The study of limits of functions is a fundamental activity in calculus. Finding the limit of a function f, as x approaches c, means that we analyze the values of $f(x)$ for values of x near c. If the values of $f(x)$ appear to be close to a value L, for values of x near c, then it is possible that the limit of the function is L. This analysis of a function can be carried out by examining the graph and values of a function as illustrated in the examples below. It is important to realize that the exact computation of a limit requires knowledge and techniques of calculus.

For example, enter the function $f(x) = \frac{\sin(5x)}{3x}$ into the `Y= Editor`. We know that when $x = 0$ both numerator and denominator are zero. Does f have a limit as x goes to zero? How do the values of this function behave when x is near zero? We can begn to analyze these and other questions using a table and by graphing. Remember, though, that we can only be sure of our answers by using the analytical techniques of calculus.

Using a Table. Construct a table of values for the function: press $\boxed{\Diamond}$ $\boxed{\text{F4}}$ to reach the `Tblset` menu and set `tblStart=-0.03,` `Δtbl=0.01`, and `Independent` to `AUTO`. Next create the table of values by pressing $\boxed{\Diamond}$ $\boxed{\text{F5}}$. The calculator produces the table given in Figure 101.

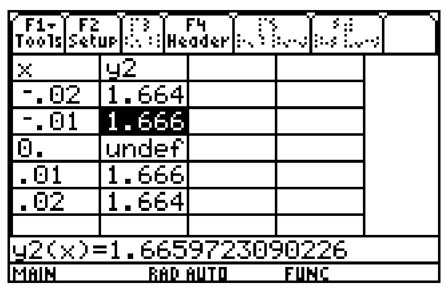


Figure 101: Values of the function near zero

As suspected, the calculator cannot compute the value of the function at $x = 0$, but it can compute the values of the function for x near zero. As the values of x approach zero, either from the negative numbers or the positive numbers, the values of the function are getting closer to $1.66666 \approx \frac{5}{3}$.

Using a Graph. Now let's look at the graph of the function in Figures 102. Note that the viewing window is set at `xmin=-2.5, xmax=2.5, xscl=1, ymin=-2.5, ymax=2.5, yscl=1`.

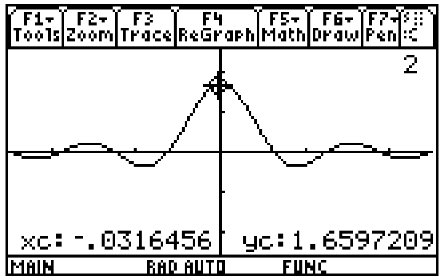
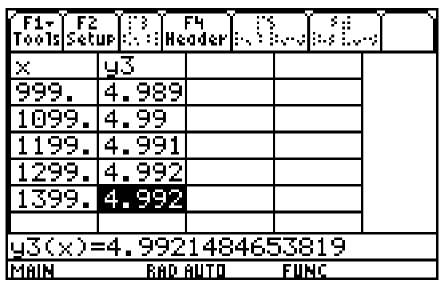


Figure 102: Graph of the function near zero

By using `Trace`, as in Figure 102, we see that the graph also indicates that the limit of $f(x)$ as x approaches zero is $\frac{5}{3}$. This is definitely the correct answer and can be verified analytically. (Don't skip this step!) The table of values and the graph of a function help you to understand limits. As this example shows, the calculator may only compute an approximate value of a limit (1.66666).

You can also study the limit of a function, f, as x goes to infinity, that is, x is moving away from the origin in the positive direction, and its values become very large. Use the same techniques described above, but this time to examine a function for large values of x. Using the `Y= Editor`, enter the function $f(x) = \frac{5x-1}{x+2}$. Construct a table for large values of x by setting `tblStart=999,` Δ`tbl=100`, and `Independent` to `AUTO`. The calculator should produce the table given in Figure 103. As you continue to scroll down, observe that the values of the function approach 5.



x	y3
999.	4.989
1099.	4.99
1199.	4.991
1299.	4.992
1399.	4.992

Figure 103: Values of the function for large x

To look at the graph for large values of x, use the viewing window `xmin=999, xmax=1999, xscl=100, ymin=0, ymax=10, yscl=2`. Graph the function and use the `Trace` feature to verify that the values of the function gets closer to 5 as the cursor moves away from the origin in the positive direction (Figure 104). The limit of this function as x goes to infinity is indeed 5 and the value can be obtained using analytical methods.

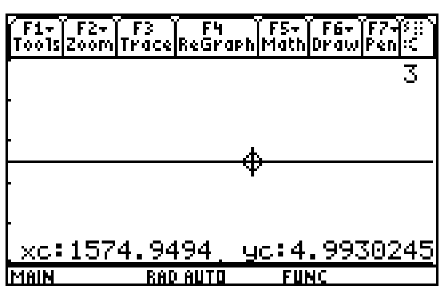


Figure 104: Graph of the function for large x

We can use similar techniques to examine functions for extreme negative values of x, but these will not be discussed here.

3.5.2 Maximum and Minimum

To find the maximum of a function on your calculator, you can approach the subject from a graphical or numeric point of view. In the examples below use the function $f(x) = 2x^3 - 5x^2 + x - 3$ and the viewing window `xmin=-4, xmax=4, xscl=1, ymin=-10, ymax=10, yscl=2`. Geometrically, you can graph the function, then use the `Trace` feature to move the cursor to the peaks and valleys of the graph and determine the x- and y-values at those points (Figures 105 and 106).

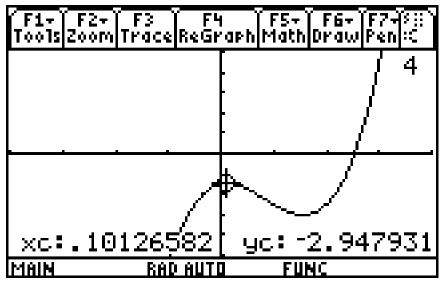


Figure 105: The maximum

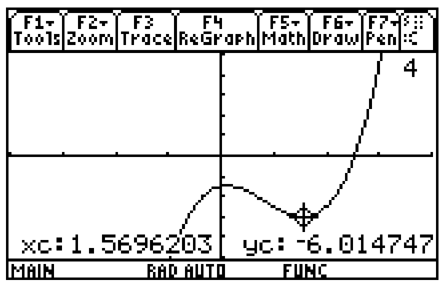


Figure 106: The minimum

Numerically, you can compute a table of values for the function and analyze the outputs of the function (Figures 107 and 108).

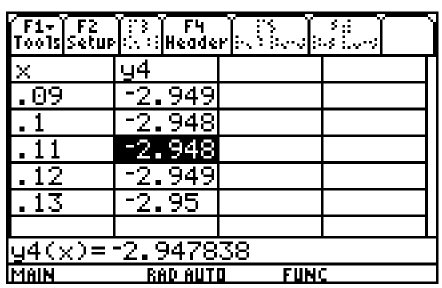


Figure 107: The maximum

Notice that answers can be different. Graphically, we find the maximum of -2.947931 to occur at $x = .10126582$, and numerically, we find that the maximum of -2.947838 occurs at $x = 0.11$. Although both values are good approximations, the exact value is only obtained analytically. (The exact value of x is $\frac{5-\sqrt{19}}{6}$, and the exact value of y is $\frac{19\sqrt{19}}{54} - \frac{121}{27}$.)

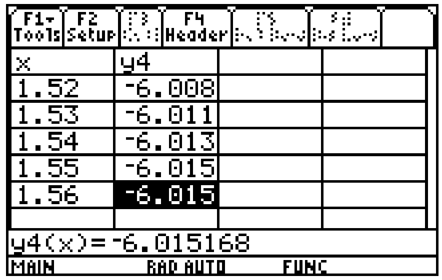


Figure 108: The minimum

Using Built-in Functions. The TI-89 also has built-in functions that allow you to find the maximum and minimum of a function: the `Minimum` and `Maximum` and the `fMin(` and `fMax(` features. We will describe these below with the function $f(x) = 2x^3 - 5x^2 + x - 3$ using the viewing window `xmin=-4, xmax=4, xscl=1, ymin=-10, ymax=10, yscl=2`.

To use the `Minimum` and `Maximum` features in the `GRAPH Math` menu, you must first view the graph of the function. Select `Minimum` from the `Math` menu (press $\boxed{\texttt{F5}}$). Use the arrows to move the cursor to select the lower bound and the upper bound, as prompted by the calculator. Press $\boxed{\texttt{ENTER}}$ to save each of your selections. The cursor will move to the lowest point of the function within the bounds selected and the calculator will display the values of x and y at that point (Figures 109–111). The commands for finding the maximum of the function are similar and will not be discussed here.

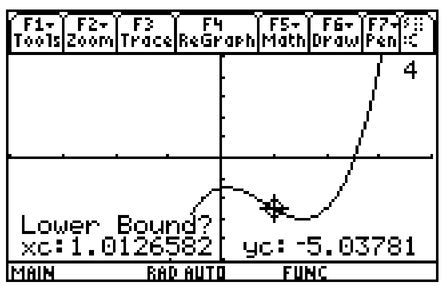


Figure 109: Lower bound

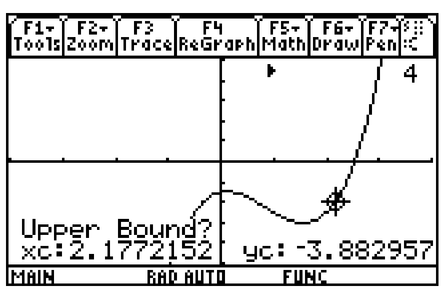


Figure 110: Upper bound

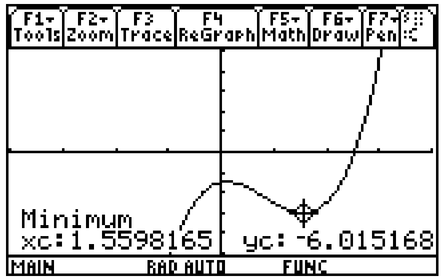


Figure 111: The minimum

For `fMin(` and `fMax(`, press $\boxed{\texttt{HOME}}$ $\boxed{\texttt{F3}}$ to select the `Calc` menu and select `fMax(`. The command `fMax(` will be copied into the entry line. Finish the command as shown below and in Figure 112. You can either write out the equation, or, if you have the function entered in `y1`, press $\boxed{\texttt{Y}}$ $\boxed{\texttt{1}}$ $\boxed{\texttt{(}}$ $\boxed{\texttt{X}}$ $\boxed{\texttt{)}}$. Your completed command should contain a search area, and will look like:

$$\texttt{fMax(y1(x), x)} \,|\, \texttt{x} > -1\ \texttt{and}\ \texttt{x} < 1.$$

The symbols '>' and '<' are in the bottom row of the calculator and require the $\boxed{\texttt{2nd}}$ key. '`and`' can either be typed in directly, or can be found in the `MATH Test` menu. See Figure 112.

In this example, if you do not explicitly give a lower and upper bound for the x search area, the calculator will return $x = \infty$. In general, the command `fMax(` needs a lower bound and an upper bound for the search area; you must enter these using the '`with`' key, $\boxed{|}$, and a logical statement as in this example.

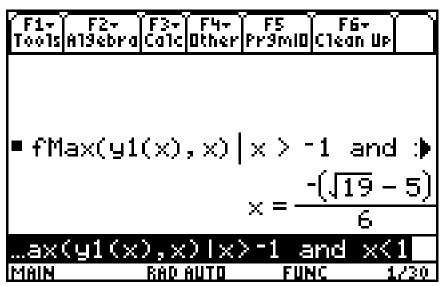


Figure 112: The x-value where the maximum of the function occurs

The TI-89 returns the value of x at which the maximum value of the function occurs. In this case, it is the exact value. Once the calculator gives you the x-value you can compute the y-value by having the calculator compute the value of the function at the given x-value (Figures 112 and 113). The computations for the minimum are similar and will not be discussed here.

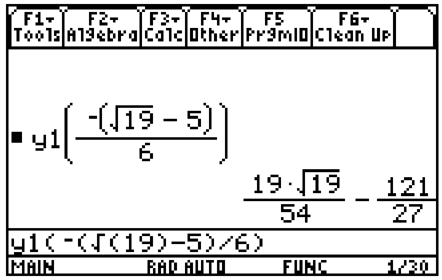


Figure 113: The maximum value of the function

3.5.3 Derivative

The TI-89 has built-in functions that allow you to find either the numerical derivative or the exact derivative of a function. The `d( differentiate` feature gives the exact value (even symbolically) and is found in the `Calc` (press HOME F3) and `MATH Calculus` (press 2nd 5 and scroll to `B`) menus. The `dy/dx` feature is in the `GRAPH Math Derivatives` menu and will return an approximation for the derivative (the 2nd [d] sequence from the keypad can also be used to compute derivatives). We illustrate these features below with the function $f(x) = x^4 + 2x^3 - x^2 + 1$ with the viewing window `xmin=-4, xmax=4, xscl=1, ymin=-6, ymax=6, yscl=1`.

To use the `dy/dx` feature, you must enter and graph the function. Press `Math Derivatives dy/dx` (press F5 6 1). Use the arrows to move the cursor to select the point at which the derivative will be computed, or enter a value for x. Press ENTER. The cursor will move to the point and the calculator will display the value of `dy/dx` at the selected point (Figure 114).

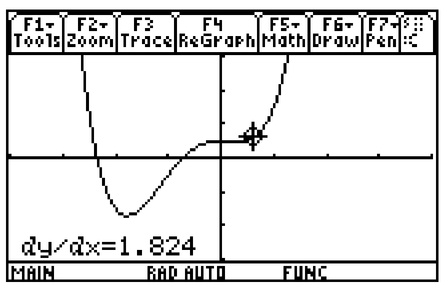


Figure 114: `dy/dx` computed at $x = 0.6$

To use the `d( differentiate` command, go to the Home Screen, and select F3 to get to `Calc d( differentiate`. The command `d(` will be copied onto the screen. Complete the command by entering the equation and the variable `x` (if you have the function entered press Y 1 (X) , X) ENTER). The calculator will return the symbolic derivative of the function. You can use this feature to obtain a numerical value for the derivative at a given value of x. To achieve this attach `|x=0.6` at the end of the derivative command, press ENTER and the derivative of the function will be calculated at $x = 0.6$ (Figure 115).

The derivative of a function is a function itself and therefore can be entered into the calculator where it can be graphed and analyzed. However, you do not need to actually compute the derivative. Enter the function $y1 = x^3 - 5x^2 + 6x - 4$ into the calculator, then enter the derivative in y_2 as follows. Pressing 2nd `[MATH]`, scrolling to `B`, and choosing `d( differentiate`, will cause the entry line to read $y_2(x) = d($. Finish the command by typing Y 1 (X) and pressing ENTER. See Figure 116. The graphs of the original function and its derivative are shown in Figure 117 using the viewing

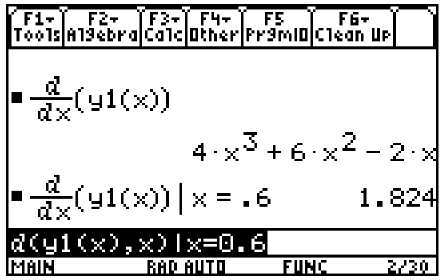


Figure 115: `dy/dx` computed at $x = 0.6$

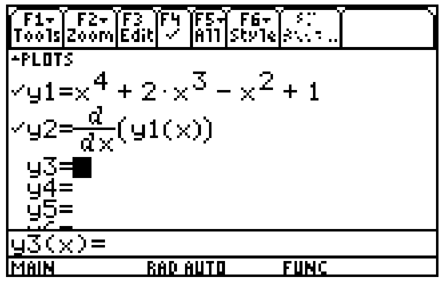


Figure 116: The derivative entered

window `xmin=-5, xmax=5, xscl=1, ymin=-6, ymax=5, yscl=1`. This graph may take some time to generate.

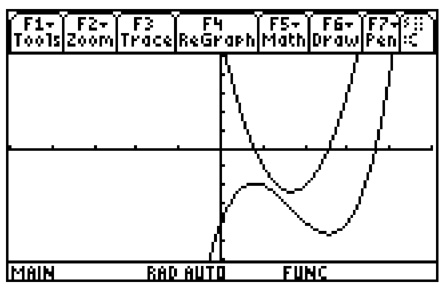


Figure 117: The graphs of the function and its derivative

3.5.4 Integrals

The TI-89 has built-in functions that allow you to find definite and indefinite integrals of a function, both numerically and exactly. The `∫f(x)dx` feature is in the `GRAPH Math` menu (press F5 from the graph screen), and both the `∫( integrate` and `nInt(` features are in the `Calc` (HOME F3) and in the `MATH Calculus` menu (HOME 2nd 5 and scroll to `B`). (The 2nd `[∫]` sequence from the keypad can also be used to compute integrals). We illustrate these features below with the function $f(x) = 5e^{-(x-3)^2}$, and the viewing window `xmin=-2, xmax=6, xscl=1, ymin=-2, ymax=6, yscl=1`.

To use the `∫f(x)dx` feature, enter and graph the function. Press `Math ∫f(x)dx` (F5 7). Use the arrows to move the cursor to select the lower and upper limits of integration as prompted by the calculator or enter values for these limits, and press ENTER. The calculator will shade the area represented by the definite integral and display the value of `∫f(x)dx` (Figures 118–120).
To use the `∫f( integrate` or the `nInt(` features, go to the Home Screen and select `Calc ∫f( integrate`. (Press F3 2.) The command `∫(` will be copied onto the screen. Complete it by entering

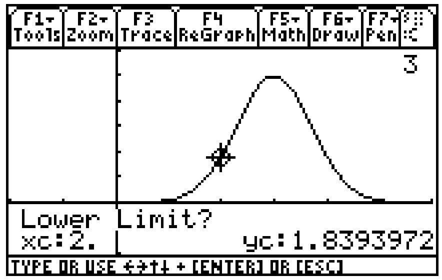


Figure 118: Lower limit

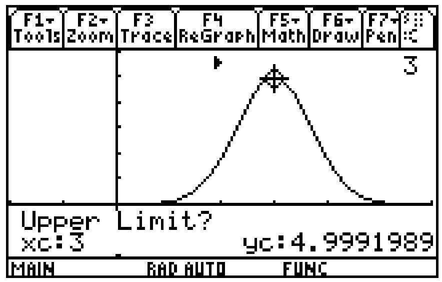


Figure 119: Upper limit

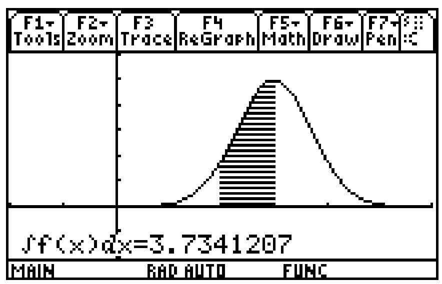


Figure 120: The integral

the equation, the variable (`x`), and limits of integration for a definite integral, separated by commas, close the parenthesis, and press $\boxed{\texttt{ENTER}}$. The calculator will return a value for the integral of the function. If you do not supply limits of integration, the calculator will attempt to find an antiderivative for the function. The steps are similar when applying `nInt(`, only this feature always will always return a numerical value (Figure 121).

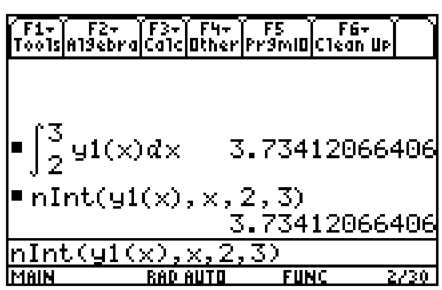


Figure 121: `∫f(` and `nInt(` computed in [2, 3]

The integral $\int_0^x f(t)dt$ of a function f is a function itself and therefore can be entered into the calculator, where it can be graphed and analyzed, but you do not need to actually compute the integral. Enter the function $y_1 = \frac{1}{1+x^2}$ into the calculator, then enter the integral as follows. Press $\boxed{\texttt{2nd}}$ [MATH], scroll to B, and choose ∫(integrate. Finish the command with the key sequence $\boxed{\texttt{Y}}$ $\boxed{\texttt{1}}$ $\boxed{\texttt{(}}$ $\boxed{\texttt{X}}$ $\boxed{\texttt{)}}$ $\boxed{\texttt{,}}$ $\boxed{\texttt{X}}$ $\boxed{\texttt{)}}$. Press $\boxed{\texttt{ENTER}}$. See Figure 122.

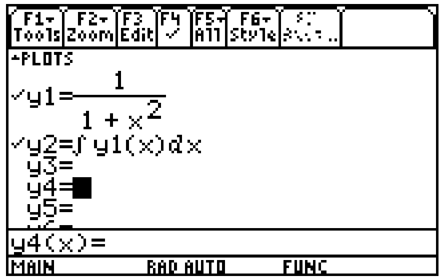


Figure 122: The integral entered

The graphs of the original function and its integral are shown in Figure 123 in the viewing window `xmin=-3, xmax=3, xscl=1, ymin=-3, ymax=3, yscl=1`.

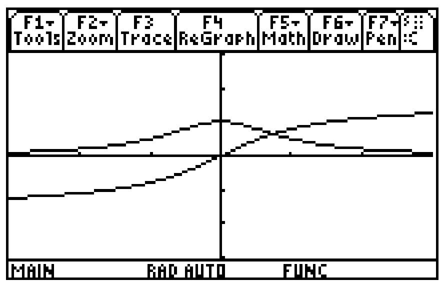


Figure 123: The graphs of the function and its integral

3.5.5 `GRAPH Draw` menu

The TI-89 has the `Inverse` feature in the `GRAPH Draw` menu that is useful in calculus applications. For additional topics in the `GRAPH Draw` menu, consult the guidebook that came with your calculator. First, press $\boxed{\diamond}$ `[GRAPH] Draw ClrDraw`. This will erase any drawings in the window and graph only selected functions and plots.

Inverse. To draw the graph of the inverse of $f(x) = \ln(2x - 1)$, enter the function (make sure this function is the only one selected) and set the viewing window to `xmin=-10.6, xmax=10.6, xscl=1, ymin=-8, ymax=8, yscl=1`. Graph the function. While you are viewing the graph select `DrawInv` from the `Draw` menu (press $\boxed{\texttt{F6}}$ $\boxed{\texttt{3}}$). The command `DrawInv` will appear in the entry line. Complete the command with the name of the function, as in Figure 124, then press $\boxed{\texttt{ENTER}}$. The graph will reappear and the inverse function will be drawn on the screen (Figure 125).

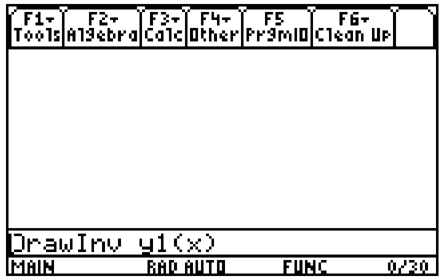


Figure 124: The `DrawInv` command

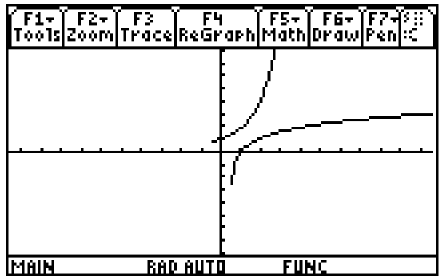


Figure 125: A function and its inverse

3.5.6 `GRAPH Math` menu

The TI-89 has four features in the `GRAPH Math` menu that are useful in calculus applications; we describe these here. For additional topics in the `GRAPH Math` menu, consult the guidebook that came with your calculator. First, press $\boxed{\Diamond}$ `[GRAPH] Draw ClrDraw`. This will erase any drawings in the window and the calculator will graph only selected functions and plots.

Inflection Point. To find an inflection point for the function $f(x) = x^4 - 3x^3 + 2x^2 - x - 3$, enter the function with the `Y= Editor`, set the viewing window to `xmin=-4, xmax=4, xscl=1, ymin=-8, ymax=6, yscl=1`, and graph the function. While you are viewing the graph select `Inflection` from the `Math` menu (press $\boxed{\text{F5}}$ $\boxed{8}$). Enter a lower bound and an upper bound as prompted by the calculator, by using the arrows to move the cursor, and pressing $\boxed{\text{ENTER}}$ each time to save your choice. The cursor will position itself at a possible point of inflection of the graph within the search area, and the coordinates of the point will be displayed on the screen (Figure 126).

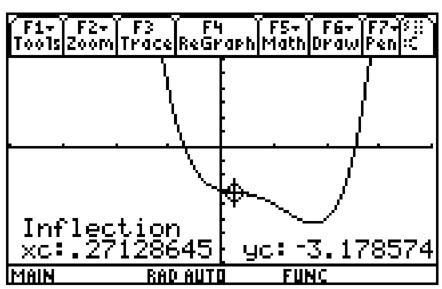


Figure 126: A point of inflection of the graph

Tangent Line. To draw the tangent line to the graph of $f(x) = 2x^3 - 5x^2 + x - 3$ at the point $(2, -5)$, enter the function. Make sure this function is the only one selected, set the viewing window to `xmin=-4, xmax=4, xscl=1, ymin=-10, ymax=6, yscl=1`, and graph. While you are viewing the graph select `Tangent` from the `Math` menu (press $\boxed{\text{F5}}$ and scroll to `A`). Move the cursor to the point

$(2,\ -5)$, or enter 2 for the value of x. Press $\boxed{\texttt{ENTER}}$. The tangent line will appear and the equation of the line will be displayed, as in Figure 127.

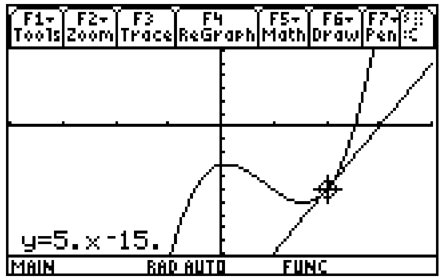


Figure 127: The tangent line

Arc Length. To compute the length of the graph of the function $f(x) = \ln(9 + x^3)$ in the interval $[-1,\ 1]$, enter the function. Make sure it is the only one selected, set the viewing window to `xmin=-3, xmax=5, xscl=1, ymin=-4, ymax=4, yscl=1`, and graph. While you are viewing the graph select `Arc` from the `Math` menu (press $\boxed{\texttt{F5}}$ and scroll to `B`. Input $\boxed{\texttt{-}}$ $\boxed{\texttt{1}}$ for the first point, press $\boxed{\texttt{ENTER}}$, and input $\boxed{\texttt{1}}$ for the second point. The length of the curve in the given interval will be displayed on the screen (Figure 128).

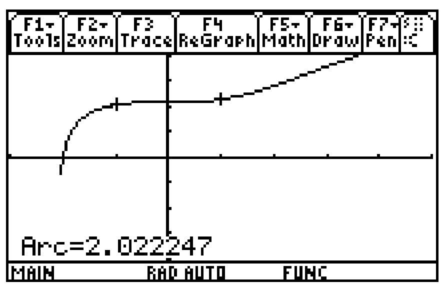


Figure 128: Arc length

Shade. Use the `Shade` command to view the area between the graphs of two functions within a given interval. Let $y_1 = -2\sin(x)$ and $y_2 = -2\cos(x)$. Enter and graph these functions in the viewing window `xmin=-4, xmax=4, xscl=1, ymin=-4, ymax=4, yscl=1`. Note that in the interval $[-\frac{3\pi}{4},\ \frac{\pi}{4}]$, $y_2(x) \leq y_1(x)$. While you are viewing the graphs select `Shade` from the `Math` menu (press $\boxed{\texttt{F5}}$ and scroll to `C`). The calculator will prompt you to select the function to shade above. Use the arrow keys to select `y2`, and press $\boxed{\texttt{ENTER}}$. (This is the lower function as in Figure 129). The calculator will then prompt you to select the function to shade below. Use the arrow keys to select `y1`, and press $\boxed{\texttt{ENTER}}$. (This is the upper function as in Figure 130.)

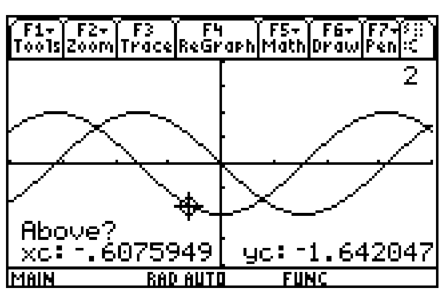


Figure 129: The lower function

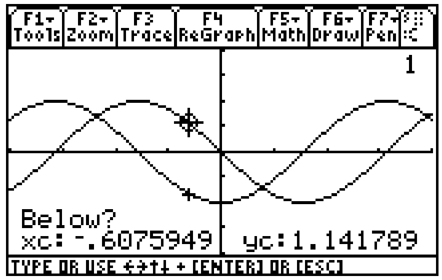


Figure 130: The upper function

Next the calculator will prompt you to select the left endpoint. Input a value (you can literally type in $\boxed{3}$ $\boxed{\pi}$ $\boxed{\div}$ $\boxed{4}$) , and press $\boxed{\texttt{ENTER}}$ (Figure 131). The next prompt will be for the right endpoint. Input the value, and press $\boxed{\texttt{ENTER}}$ (Figure 132). The graphs appear again. The shaded area is above `y2`, below `y1`, and within the interval $[-\frac{3\pi}{4}, \frac{\pi}{4}]$ (Figure 133).

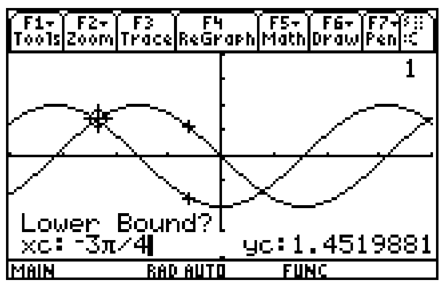


Figure 131: The lower bound

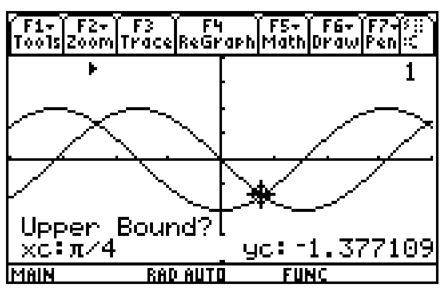


Figure 132: The upper bound

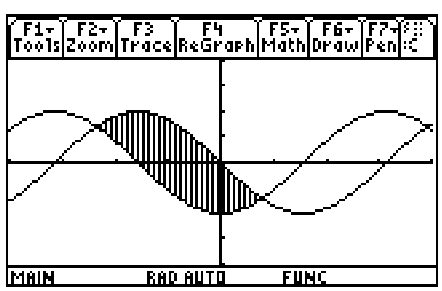


Figure 133: Area between two graphs

3.5.7 Graphing Slope Fields and Solutions to Differential Equations

We can learn a lot about a differential equation by looking at its slope field, and by graphing particular solutions corresponding to certain initial conditions. We first look at the example $y_1' = y_1 * (1-t) + t/2$. Before we begin, we have to change the mode of the `Y= Editor` both to enter the equation and later to graph.

Press the $\boxed{\texttt{MODE}}$ key, and change the `Graph` setting to option `6:DIFF EQUATIONS` as shown in Figure 134. Now when you enter the `Y= Editor`, the new format allows you to enter the differential equation in the line `y1'` and an optional initial condition. Enter the differential equation as shown in Figure 135, leaving the initial condition `yi1` blank for now. Note two important possible problems. First, you must use the independent variable `T` in the differential equation mode. Second, you have to explicitly use the $\boxed{\times}$ key instead of implicit multiplication. If you don't, your parentheses will be interpreted as function calls.

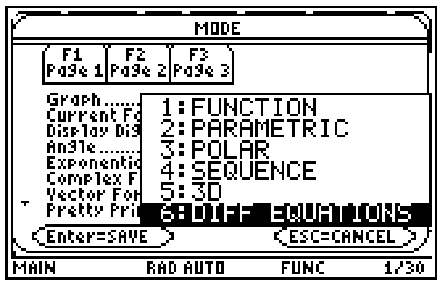


Figure 134: Mode for differential equations

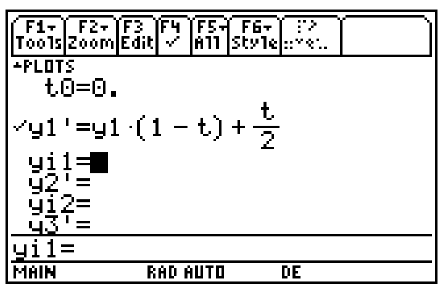


Figure 135: Entering a differential equation

Double check that the `GRAPH FORMATS` dialog box matches the options in Figure 136. To enter the format dialog, press $\boxed{\Diamond}$ $\boxed{|}$ (the `F` key). The important line is at the bottom. `Fields` should be set to `SLPFLD`.

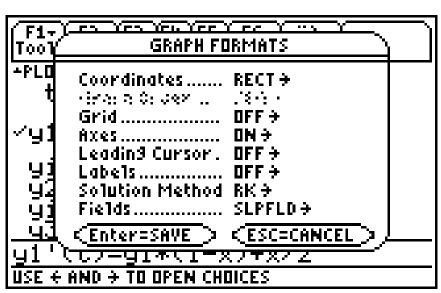


Figure 136: `GRAPH FORMATS` dialog for slope fields

Set the viewing window as follows: t0=0, tmax=10, tsteps=.1, tplot=0, xmin=-3, xmax=3, xscl=1, ymin=-8, ymax=8, yscl=1, ncurves=0, diftol=.001, fldres=20. Then press ◊ F3 to graph the slope field as in Figure 137.

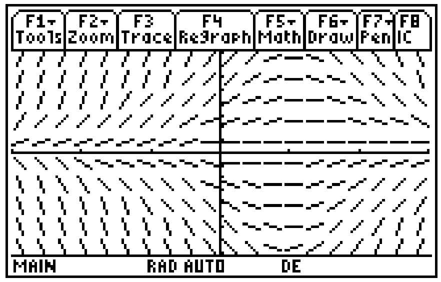


Figure 137: The slope field

To graph a particular solution for a given initial condition, return to the Y= Editor. To graph one solution, enter a number, such as 2, into the line yi1= (See Figure 138). This will graph the solution corresponding to the initial condition $y1(0) = 2$. To graph and compare multiple solutions at once, enter two or more initial conditions in a list as in Figure 139, and graph. The result is shown in Figure 140.

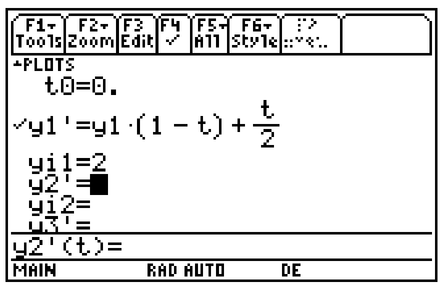


Figure 138: Setting one initial condition

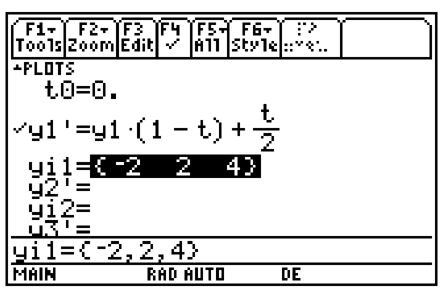


Figure 139: Setting multiple initial conditions for multiple solutions

You can also set an initial condition interactively, while viewing the slope field. To do so press 2nd [F8]. As prompted, type in any set of initial values for t and y1, such as -2 and 4 as shown in Figure 141. Press ENTER each time to save your choices. The graph of the new solution will be drawn. See Figure 142.

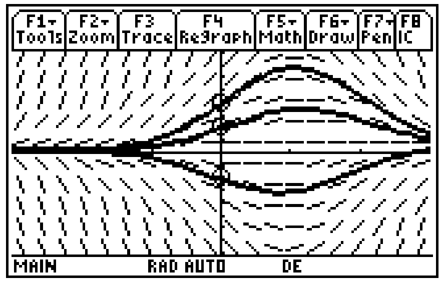


Figure 140: Slope field with graph of three solutions

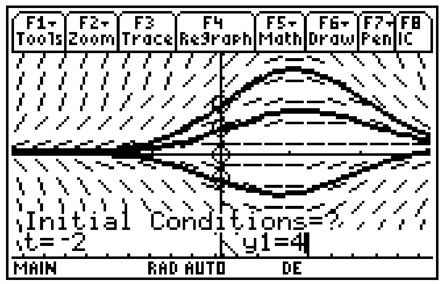


Figure 141: Interactively setting initial conditions

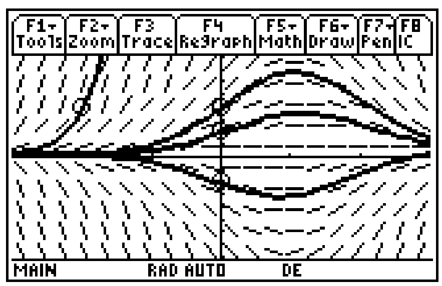


Figure 142: New solution

3.5.8 Solving Differential Equations

To solve a differential equation such as $3xy' - y = \ln x + 1$, we use the command `deSolve` found from the `HOME` screen in the $\boxed{\texttt{F3}}$ menu. Scroll to option `C: deSolve(` and select it. The command requires you to input the whole equation, the independent variable and the dependent variable. For example, finish the `deSolve` command so that your entry lines reads: `deSolve(3x * y' - y = ln(x) + 1, x, y)`. The prime symbol, $'$, is found by pressing $\boxed{\texttt{2nd}}$ $\boxed{=}$. This command finds the general solution to the differential equation given. Press $\boxed{\texttt{ENTER}}$. Notice in Figure 143 the notation `C1` which denotes the constant of integration. Also remember that the form of your answer may not match an answer generated by hand or by other technology. In this example, we could apply a logarithm simplification rule to make our answer look different. The calculator may also give solutions implicitly in terms of y, or include an integral if it is unable to find an explicit solution.

To solve a differential equation with an initial condition, include the condition in the first argument by using the logical command `and` (found in the `MATH Test` menu, or typed out directly). To complete the example using the initial condition $y(1) = -2$, the full entry line should read: `deSolve(3x * y' - y = ln(x) + 1 and y(1) = -2, x, y)`. Press $\boxed{\texttt{ENTER}}$. The results are shown in Figure 144.

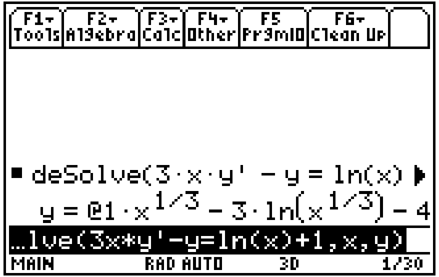


Figure 143: General solution to a differential equation

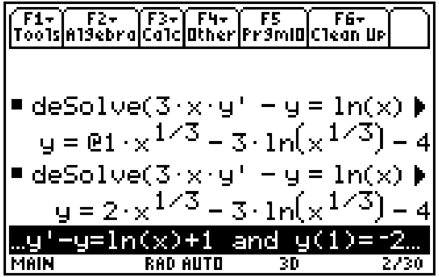


Figure 144: Finding a particular solution