Lecture Four – *Parametric Equations and Polar Coordinates*

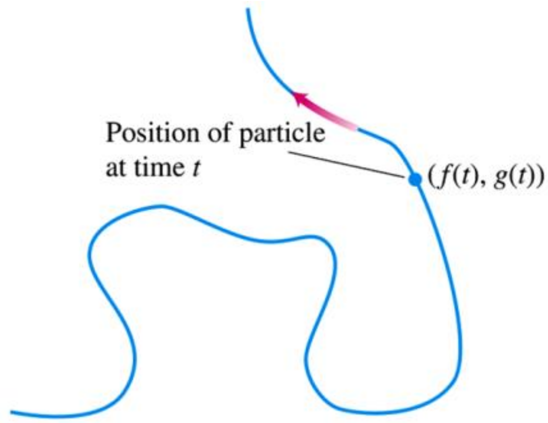
Section 4.1 – Parameterizations of Plane Curves

Definition

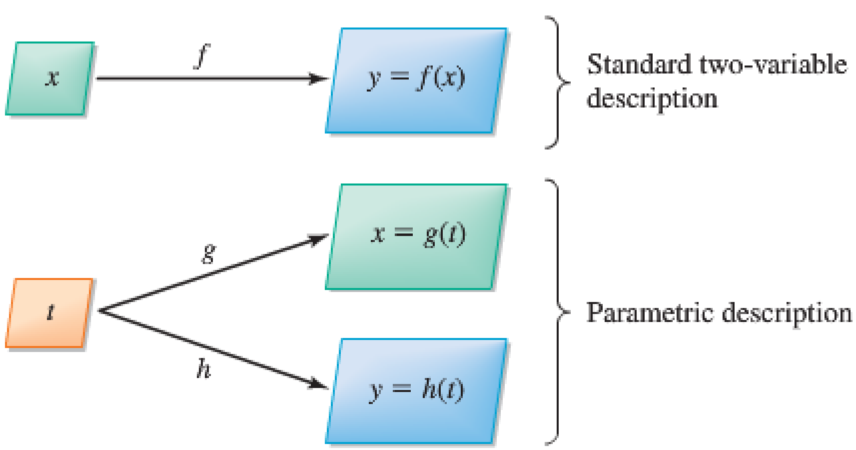
If x and y are given as functions

$$x = f(t), \quad y = g(t)$$

Over an interval I as t–values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a ***parametric curve***. The equations are ***parametric equations*** for the curve



The variable t is a parameter for the curve, and its domain I is the parameter interval.



Definition

The direction in which a parametric curve is generated as the parameter increases is called the forward or positive orientation of the curve.

Example

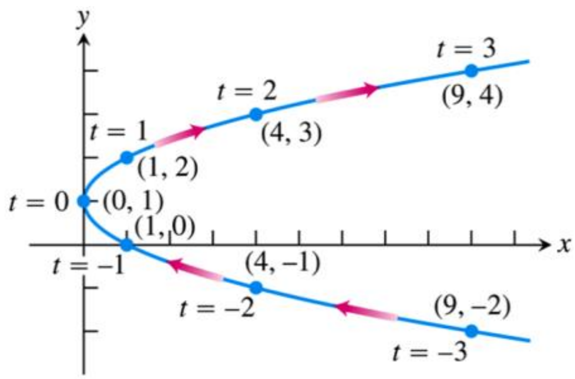
Sketch the curve defined by the parametric equations

$$x = t^2, \quad y = t + 1, \quad -\infty < t < \infty$$

Then obtain an algebraic equation in *x* and *y*.

Solution

t	$x = t^2$	$y = t + 1$
–3	9	–2
–2	4	–1
–1	1	0
0	0	1
1	1	2
2	4	3
3	9	4



$$y = t + 1 \Rightarrow t = y - 1$$

$$x = t^2 = (y-1)^2 = y^2 - 2y + 1$$ ***represents a parabola***

Example

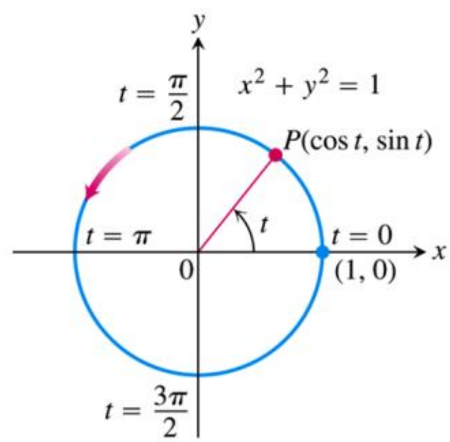
Graph the parametric curve $x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$

Solution

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

This parametric curve lies along the unit circle $x^2 + y^2 = 1$.

As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ starts at (1, 0) and traces the entire circle once counterclockwise.



Example

Graph the parametric curve $x = a\cos t, \quad y = a\sin t, \quad 0 \le t \le 2\pi$

Solution

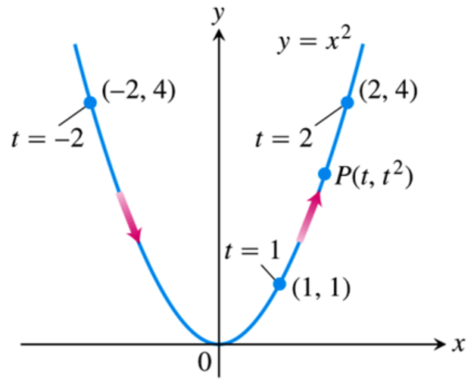
$$x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2\left(\cos^2 t + \sin^2 t\right) = a^2$$

This parametric curve lies along the unit circle $x^2 + y^2 = a^2$.

As *t* increases from 0 to 2π, the point $(x, y) = (a\cos t, a\sin t)$ starts at (*a*, 0) with a radius $r = a$ and traces the entire circle once counterclockwise.

Example

Graph the parametric curve $x = t, \quad y = t^2, \quad -\infty < t < \infty$



Solution

$y = x^2$

Example

Find a parameterization for the line through the point (a, b) having slope m.

Solution

A Cartesian equation of the line is $y - b = m(x - a)$

If $t = x - a \quad \Rightarrow \quad x = t + a$

$y - b = mt \quad \Rightarrow \quad y = mt + b$

That is, $x = t + a \quad y = mt + b, \quad -\infty < t < \infty$

Example

Sketch the curve defined by the parametric equations $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0$

Then obtain an algebraic equation in x and y.

Solution

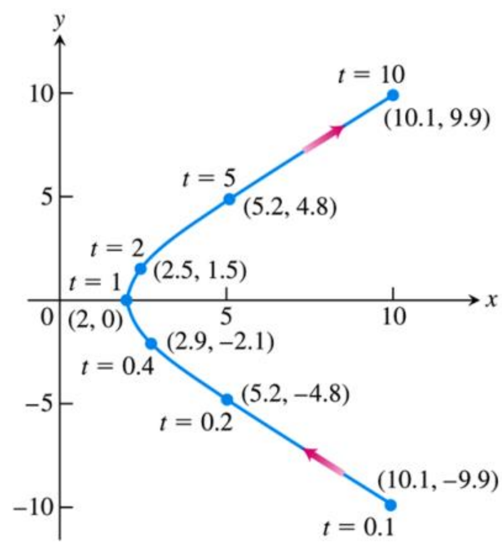
t	$x = t + \frac{1}{t}$	$y = t - \frac{1}{t}$
.2	5.2	–4.8
.5	2.5	–1.5
1	2	0
2	2.5	1.5
5	5.2	4.8

$$x + y = t + \frac{1}{t} + t - \frac{1}{t} = 2t$$

$$\underline{x - y} = t + \frac{1}{t} - t + \frac{1}{t} = \underline{\frac{2}{t}}$$

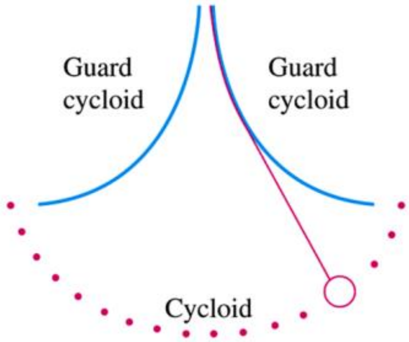
$$(x + y)(x - y) = 2t\left(\frac{2}{t}\right) = 4$$

$$\boxed{x^2 - y^2 = 4}$$



Cycloids

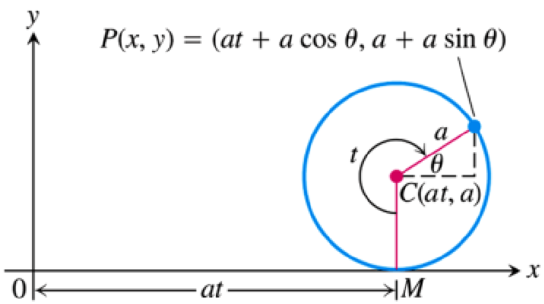
The problem with a pendulum clock whose bob swings in a circular arc is that the frequency of the swing depends on the amplitude of the swing.

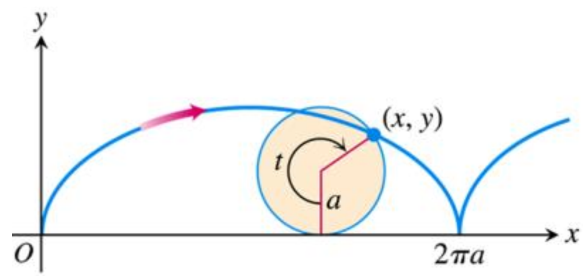


Example

A wheel of a radius *a* rolls along a horizontal straight line. Find parametric equations for the path traced by a point P on the wheel's circumference. The path is called a cycloid.

Solution





We take the line to be on the *x*-axis, mark a point *P* on the wheel, start the wheel with *P* at the origin, and roll the wheel. As parameter, we use the angle *t* through which the wheel turns, measured in radians. The wheel's center C lies at $(at,\ a)$ and the coordinates of *P* are

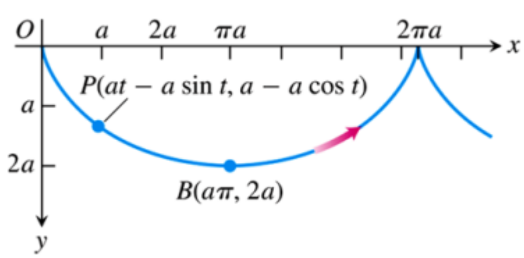
$$x = at + a\cos\theta, \quad y = a + a\sin\theta$$

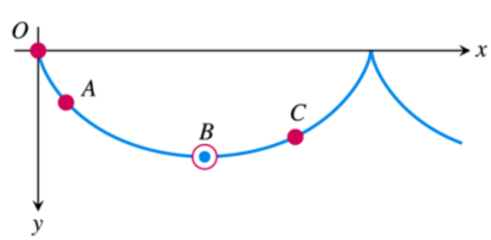
To express θ in terms of *t*, we observe that $t + \theta = \frac{3\pi}{2}$, so that $\theta = \frac{3\pi}{2} - t$

That makes $\cos\theta = \cos\left(\frac{3\pi}{2} - t\right) = -\sin t, \quad \sin\theta = \sin\left(\frac{3\pi}{2} - t\right) = -\cos t$

$x = at - a\sin t, \quad y = a - a\cos t$

That implies to: $\boxed{x = a(t - \sin t), \quad y = a(1 - \cos t)}$





Exercises Section 4.1 – Parameterizations of Plane Curves

Give parametric equations and parameter intervals for the motion of a particle in the *xy*-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

1. $x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty$

2. $x = -\sqrt{t}, \quad y = t, \quad t \ge 0$

3. $x = 3 - 3t, \quad y = 2t, \quad 0 \le t \le 1$

4. $x = \cos 2t, \quad y = \sin 2t, \quad 0 \le t \le \pi$

5. $x = \cos(\pi - t), \quad y = \sin(\pi - t), \quad 0 \le t \le \pi$

6. $x = 4\sin t, \quad y = 5\cos t, \quad 0 \le t \le 2\pi$

7. $x = 1 + \sin t, \quad y = \cos t - 2, \quad 0 \le t \le 2\pi$

8. $x = t^2, \quad y = t^6 - 2t^4, \quad -\infty < t < \infty$

9. $x = \frac{t}{t-1}, \quad y = \frac{t-2}{t+1}, \quad -1 < t < 1$

10. $x = \sqrt{t+1}, \quad y = \sqrt{t}, \quad t \ge 0$

11. $x = 2\sinh t, \quad y = 2\cosh t, \quad -\infty < t < \infty$

12. $x = 4\cos 2\pi t, \quad y = 4\sin 2\pi t, \quad 0 \le t \le 1$

13. $x = \sqrt{t} + 4, \quad y = 3\sqrt{t}; \quad 0 \le t \le 16$

14. $x = (t+1)^2, \quad y = t + 2; \quad -10 \le t \le 10$

15. $x = t - 1, \quad y = t^3; \quad -4 \le t \le 4$

16. $x = e^{2t}, \quad y = e^t + 1; \quad 0 \le t \le 25$

17. $x = 3\cos t, \quad y = 3\sin t; \quad \pi \le t \le 2\pi$

18. $x = -7\cos 2t, \quad y = -7\sin 2t; \quad 0 \le t \le \pi$

19. $x = 1 - 3\sin 4\pi t, \quad y = 2 + 3\cos 4\pi t; \quad 0 \le t \le \frac{1}{2}$

20. Find parametric equation for the left half of the parabola $y = x^2 + 1$, originating at $(0,\ 1)$

21. Find parametric equation for the line that passes through the points $(1,\ 1)$ and $(3,\ 5)$, oriented in the direction of increasing *x*.

22. Find parametric equation for the lower half of the circle centered at $(-2,\ 2)$ with radius 6, oriented in the counterclockwise direction.

23. Find parametric equation for the upper half of the parabola $x = y^2$, originating at $(0,\ 0)$

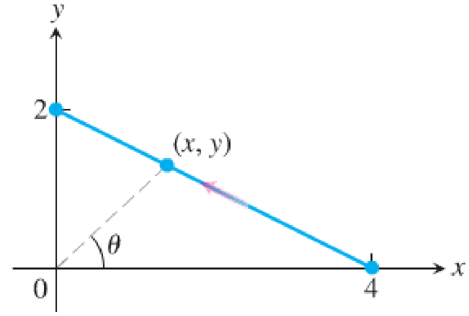
24. Find parametric equations (not unique) of an ellipse centered at the origin with major axis of length 6 on the *x-axis* and minor axis of length 3 on the *y-axis*, generated counterclockwise. Graph the ellipse and find a description in terms of *x* and *y*.

25. Find parametric equations (not unique) of an ellipse centered at the origin with major axis of length 12 on the *x-axis* and minor axis of length 2 on the *y-axis*, generated clockwise. Graph the ellipse and find a description in terms of *x* and *y*.

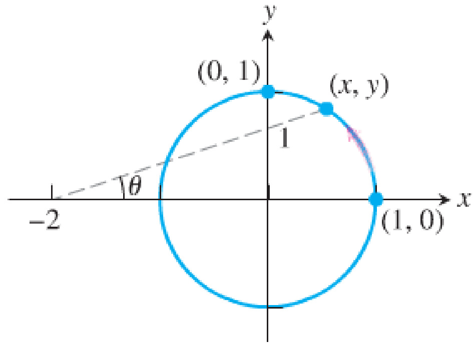
26. Find parametric equations (not unique) of an ellipse centered at $(-2,\ -3)$ with major and minor axes of lengths 30 and 20, parallel to the *x-axis* and *y-axis*, respectively. Graph the ellipse and find a description in terms of *x* and *y*.

27. Find a parametric equations and a parameter interval for the motion of a particle starting at the point (2, 0) and tracing the top half of the circle $x^2 + y^2 = 4$ four times.

28. Find a parametrization for the line segment joining points (0, 2) and (4, 0) using the angle θ in the accompanying figure as the parameter.



29. Find a parametrization for the circle $x^2 + y^2 = 1$ starting at (1, 0) and moving counterclockwise to the terminal point (0, 1), using the angle θ in the accompanying figure as the parameter.



30. A common task is to parameterize curves given either by either Cartesian equations or by graphs. Find a parametric representation of the following curves.

a) The segment of the parabola $y = 9 - x^2$, *for* $-1 \le x \le 3$

b) The complete curve $x = (y-5)^2 + \sqrt{y}$

c) The piecewise linear path that connects $P(-2,\ 0)$ to $Q(0,\ 3)$ to $R(4,\ 0)$ (in that order), where the parameter varies over the interval $0 \le t \le 2$

31. A projectile launched from the ground with an initial speed of 20 *m/s* and a launch angle θ follows a trajectory approximated by

$$x = (20\cos\theta)t, \quad y = -4.9t^2 + (20\sin\theta)t$$

Where x and y are the horizontal and vertical positions of the projectile relative to the launch point $(0,\ 0)$.

a) Graph the trajectory for various of θ in the range $0 < \theta < \frac{\pi}{2}$.

b) Based on your observations, what value of θ gives the greatest range (the horizontal distance between the launch and landing points)?

32. Many fascinating curves are generated by points on rolling wheels. The path of a light on the rim of a rolling when is a cycloid, which has the parametric equations

$$x = a(t - \sin t), \quad y = a(1 - \cos t), \quad \text{for } \ t \geq 0$$

Where $a > 0$. Graph the cycloid with $a = 1$. On what interval does the parameter generate one arch of the cycloid?

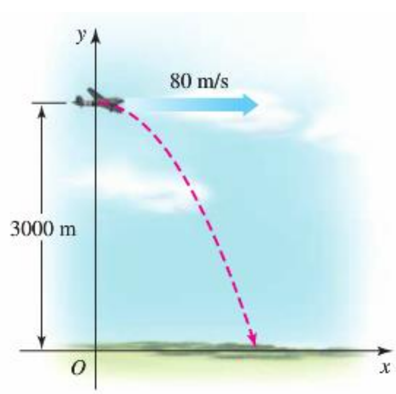
(18−20) Find parametric equations that describe the circular path of the objects. Assume (x, y) denotes the position of the object relative to the origin at the center of the circle.

33. A go-cart moves counterclockwise with constant speed around a circular track of radius 400 *m*, completing in 1.5 *min*.

34. The tip of the 15-*in* second hand of a clock completes one revolution in 60 *sec*.

35. A Ferris wheel has a radius of 20 *m* and completes a revolution in the clockwise direction at constant speed in 3 *min*. Assume that *x* and *y* measure the horizontal and vertical positions of a seat on the Ferris wheel relative to a coordinate system whose origin is at the low point of the wheel. Assume the seat begins moving at the origin.

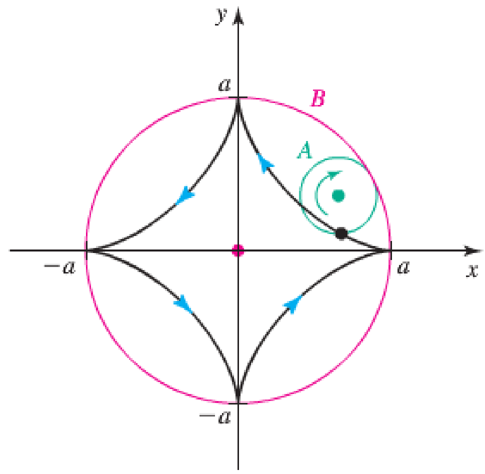
36. A plane traveling horizontally at 80 *m/s* over flat ground at an elevation of 3000 *m* releases an emergency packet. The trajectory of the packet is given by



$$x = 80t, \quad y = -4.9t^2 + 3000, \quad \text{for } \ t \geq 0$$

Where the origin is the point on the ground directly beneath the plane at the moment of the release. Graph the trajectory of the packet and find the coordinates of the point where the packet lands.

37. The path of a point on circle A with radius $\frac{a}{4}$ that rolls on the inside of circle B with a radius a is an asteroid or hypocycloid. Its parametric equations are



$$x = a\cos^3 t, \quad y = a\sin^3 t, \quad 0 \le t \le 2\pi$$

Graph the asteroid with $a = 1$ and find its equation in terms of x and y.