

Section 3.4 – Orthogonal Matrices

Definition

A square matrix A is said to be orthogonal if its transpose is the same as its inverse, that is, if

$$A^{-1} = A^T$$

or, equivalently, if

$$AA^T = A^T A = I$$

Example

The matrix $A = \begin{pmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{pmatrix}$

Solution

$$\begin{aligned} A^T A &= \begin{pmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{pmatrix} \begin{pmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Example

The matrix $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

Solution

$$\begin{aligned} A^T A &= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Theorem

The following are equivalent for $n \times n$ matrix A .

- a) A is orthogonal.
- b) The row vectors of A form an orthonormal set in R^n with the Euclidean inner product.
- c) The column vectors of A form an orthonormal set in R^n with the Euclidean inner product.

Theorem

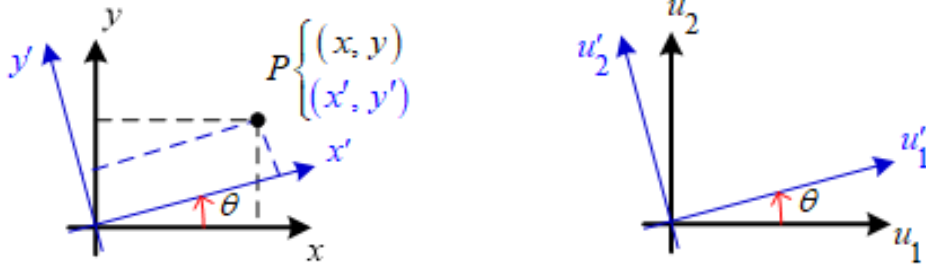
- a) The inverse of an orthogonal matrix is orthogonal
- b) A product of orthogonal matrices is orthogonal
- c) If A is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$

Theorem

If A is an $n \times n$ matrix, then the following are equivalent

- a) A is orthogonal.
- b) $\|A\vec{x}\| = \|\vec{x}\|$ for all \vec{x} in R^n .
- c) $A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$ for all \vec{x} and \vec{y} in R^n .

Let \vec{u}_1 and \vec{u}_2 be the unit vectors along the x - and y -axes and unit vectors \vec{u}'_1 and \vec{u}'_2 along the x' - and y' -axes.

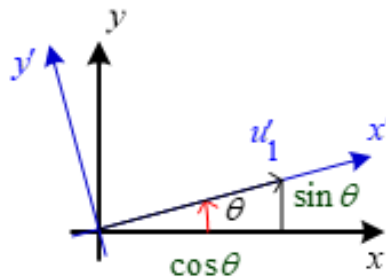


The new coordinates (x', y') and the old coordinates (x, y) of a point P will be related by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

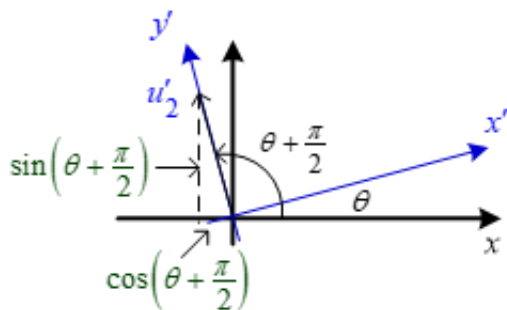
$$P = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$P^{-1} = P^T = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$



$$\begin{bmatrix} \vec{x}' \\ \vec{y}' \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix}$$

$$\rightarrow \begin{cases} x' = x \cos \theta + y \sin \theta \\ y' = -x \sin \theta + y \cos \theta \end{cases}$$



These are sometimes called the *rotation equations*.

Example

Use the form $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ to find the new coordinates of the point $Q(2, 1)$ if the coordinate axes of a rectangular coordinate system are rotated through an angle of $\theta = \frac{\pi}{4}$.

Solution

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} \cos \frac{\pi}{4} & \sin \frac{\pi}{4} \\ -\sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

The new coordinates of Q are $(x', y') = \left(\frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$

Exercises Section 3.4 – Orthogonal Matrices

(1 – 2) Show that the matrix is orthogonal

$$1. \quad A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$$

$$2. \quad A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

(3 – 12) Determine if the matrix is orthogonal. For those that is orthogonal find the inverse.

$$3. \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$8. \quad \begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$11. \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$$

$$4. \quad \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$9. \quad \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$12. \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$$

$$5. \quad \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

$$6. \quad \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$

$$7. \quad \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$$

$$10. \quad \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

13. Find a last column so that the resulting matrix is orthogonal

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \dots \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \dots \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & \dots \end{bmatrix}$$

14. Determine if the given matrix is orthogonal. If it is, find its inverse

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{5} & \frac{3}{7} \\ \frac{4}{9} & \frac{3}{5} & -\frac{2}{7} \\ \frac{8}{9} & -\frac{2}{5} & \frac{3}{7} \end{bmatrix}$$

15. Prove that if A is orthogonal, then A^T is orthogonal.
16. Prove that if A is orthogonal, then A^{-1} is orthogonal.
17. Prove that if A and B are orthogonal, then AB is orthogonal.
18. Let Q be an $n \times n$ orthogonal matrix, and let A be an $n \times n$ matrix. Show that $\det(QAQ^T) = \det(A)$

19. Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$

- a) Is matrix A an orthogonal matrix?
- b) Let B be the matrix obtained by normalizing each row of A , find B .
- c) Is B an orthogonal matrix?
- d) Are the columns of B orthogonal?