Solution Section 2.7 – Derivatives of Exponential and Logarithmic Functions

Exercise

Find the derivative of $f(x) = e^{3x}$

Solution

$$f'(x) = 3e^{3x}$$

Exercise

Find the derivative of $f(x) = e^{-2x^3}$

Solution

$$\begin{aligned} f'(x) &= e^{-2x^3} \frac{d}{dx}[-2x^3] \\ &= e^{-2x^3}\left[-6x^2\right] \\ &= -\frac{6x^2}{e^{2x^3}} \end{aligned}$$

Exercise

Find the derivative of $f(x) = 4e^{x^2}$

Solution

$$\begin{aligned} f'(x) &= 4e^{x^2} \frac{d}{dx}[x^2] \\ &= 4e^{x^2}(2x) \\ &= 8xe^{x^2} \end{aligned}$$

Exercise

Find the derivative of $f(x) = e^{-2x}$

Solution

$$f'(x) = -2e^{-2x}$$
$$\boxed{= -\frac{2}{e^{2x}}}$$

Exercise

Find the derivative of $f(x) = x^2 e^x$

Solution

$$f'(x) = e^x \frac{d}{dx}[x^2] + x^2 \frac{d}{dx}[e^x]$$
$$= e^x(2x) + x^2 e^x$$
$$\boxed{= xe^x(2+x)}$$

Exercise

Find the derivative of $f(x) = \frac{e^x + e^{-x}}{2}$

Solution

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$
$$f'(x) = \frac{1}{2}\left(\frac{d}{dx}\left[e^x\right] + \frac{d}{dx}\left[e^{-x}\right]\right)$$
$$\boxed{= \frac{1}{2}(e^x - e^{-x})}$$

Exercise

Find the derivative of $f(x)=\dfrac{e^x}{x^2}$

Solution

$$f'(x)=\frac{x^2e^x-e^x(2x)}{x^4}$$

$$=\frac{x^2e^x-2xe^x}{x^4}$$

$$=\frac{xe^x(x-2)}{x^4}$$

$$=\boxed{\frac{e^x(x-2)}{x^3}}$$

Exercise

Find the derivative of $f(x)=x^2e^x-e^x$

Solution

$$f'(x)=e^x\frac{d}{dx}[x^2]+x^2\frac{d}{dx}[e^x]-\frac{d}{dx}[e^x]$$

$$=e^x(2x)+x^2e^x-e^x$$

$$=\boxed{e^x(x^2+2x-1)}$$

Exercise

Find the derivative of $f(x)=(1+2x)e^{4x}$

Solution

$$f'(x)=(2)e^{4x}+(1+2x)(4e^{4x})$$

$$=2e^{4x}+(1+2x)(4e^{4x})$$

$$=2e^{4x}(1+2(1+2x))$$

$$=2e^{4x}(1+2+4x)$$

$$=\boxed{2e^{4x}(3+4x)}$$

Exercise

Find the derivative of $y = x^2 e^{5x}$

Solution

$$y' = x^2\left(5e^{5x}\right) + 2x\left(e^{5x}\right)$$

$$\boxed{= xe^{5x}(5x+2)}$$

Exercise

Find the derivative of $f(x) = \dfrac{100,000}{1+100e^{-0.3x}}$

Solution

$$f'(x) = \frac{(0)\left(1+100e^{-0.3x}\right) - 100,000\left(0+(-0.3)100e^{-0.3x}\right)}{\left(1+100e^{-0.3x}\right)^2}$$

$$= \frac{-100,000\left(-30e^{-0.3x}\right)}{\left(1+100e^{-0.3x}\right)^2}$$

$$\boxed{= \frac{3,000,000e^{-0.3x}}{\left(1+100e^{-0.3x}\right)^2}}$$

Exercise

Find the derivative of $y = x^2 e^{-2x}$

Solution

$$y' = 2xe^{-2x} - 2x^3 e^{-2x}$$

$$\boxed{= 2xe^{-2x}\left(1-x^2\right)}$$

Exercise

Find the derivative of $y=\dfrac{e^{x}+e^{-x}}{x}$

Solution

$$f=e^{x}+e^{-x} \qquad g=x$$

$$f'=e^{x}-e^{-x} \qquad g'=1$$

$$y=\frac{\left(e^{x}-e^{-x}\right)x-\left(e^{x}+e^{-x}\right)}{x^{2}}$$

$$=\frac{xe^{x}-xe^{-x}-e^{x}-e^{-x}}{x^{2}}$$

$$=\boxed{\frac{(x-1)e^{x}-(x+1)e^{-x}}{x^{2}}}$$

Exercise

Find the derivative of $y=\sqrt{e^{2x^{2}}+e^{-2x^{2}}}$

Solution

$$y=\sqrt{e^{2x^{2}}+e^{-2x^{2}}}=\left(e^{2x^{2}}+e^{-2x^{2}}\right)^{1/2}=U^{1/2}$$

$$U=e^{2x^{2}}+e^{-2x^{2}} \qquad \left(e^{2x^{2}}\right)'=\left(2x^{2}\right)'e^{2x^{2}}=4xe^{2x^{2}}$$

$$U'=4xe^{2x^{2}}-4xe^{-2x^{2}}$$

$$y'=\frac{1}{2}\left(4xe^{2x^{2}}-4xe^{-2x^{2}}\right)\left(e^{2x^{2}}+e^{-2x^{2}}\right)^{-1/2}$$

$$=\frac{1}{2}\frac{4x\left(e^{2x^{2}}-e^{-2x^{2}}\right)}{\left(e^{2x^{2}}+e^{-2x^{2}}\right)^{1/2}}$$

$$=\boxed{\frac{2x\left(e^{2x^{2}}-e^{-2x^{2}}\right)}{\sqrt{e^{2x^{2}}+e^{-2x^{2}}}}}$$

Exercise

Find the derivative of $y = \dfrac{x}{e^{2x}}$

Solution

$$f = x \qquad g = e^{2x}$$

$$f' = 1 \qquad g' = 2e^{2x}$$

$$y' = \frac{1\left(e^{2x}\right) - x\left(2e^{2x}\right)}{\left(e^{2x}\right)^2}$$

$$= \frac{e^{2x}(1-2x)}{\left(e^{2x}\right)^2}$$

$$= \frac{1-2x}{e^{2x}}$$

Exercise

Find the second derivative of $y = 3\, e^{5x^3+1}$

Solution

$$y' = 3(15x^2)e^{5x^3+1}$$

$$y' = 45x^2e^{5x^3+1}$$

$$f = x^2 \qquad g = e^{5x^3+1}$$

$$f' = 2x \qquad g' = 15x^2e^{5x^3+1}$$

$$y'' = 45\left(2xe^{5x^3+1} + \left(x^2\right)15x^2e^{5x^3+1}\right)$$

$$= 45e^{5x^3+1}\left(2x + 15x^4\right)$$

$$= 45xe^{5x^3+1}\left(2 + 15x^3\right)$$

Exercise

Find the derivative of $y = \ln\sqrt{x+5}$

Solution

$$y = \ln(x+5)^{1/2}$$

$$= \frac{1}{2}\ln(x+5)$$

$$y' = \frac{1}{2(x+5)}$$

Exercise

Find the Derivatives of $y = (3x+7)\ln(2x-1)$

Solution

$$f = 3x+7 \quad f' = 3$$

$$g = \ln(2x-1) \quad g' = \frac{2}{2x-1}$$

$$y' = 3x\ln(2x-1) + \frac{2(3x+7)}{2x-1}$$

Exercise

Find the Derivatives of $y = e^{x^2}\ln x$

Solution

$$f = e^{x^2} \quad f' = 2xe^{x^2}$$

$$g = \ln x \quad g' = \frac{1}{x}$$

$$y' = 2xe^{x^2}\ln x + \frac{e^{x^2}}{x}$$

Exercise

Find the Derivatives of $y = \log_7 \sqrt{4x-3}$

Solution

$$y' = \frac{1}{\ln 7}\frac{\left(\sqrt{4x-3}\right)'}{\sqrt{4x-3}} \qquad \frac{d}{dx}\left[\log_a |g(x)|\right] = \frac{1}{(\ln a)}\cdot\frac{g'(x)}{g(x)}$$

$$\left(\sqrt{4x-3}\right)' = \left((4x-3)^{1/2}\right)'$$

$$= \tfrac{1}{2}(4)(4x-3)^{-1/2} \qquad \left(U^n\right)' = nU'U^{n-1}$$

$$= 2(4x-3)^{-1/2}$$

$$y' = \frac{1}{\ln 7}\frac{2(4x-3)^{-1/2}}{\sqrt{4x-3}}$$

$$= \frac{1}{\ln 7}\frac{2}{(4x-3)^{1/2}(4x-3)^{1/2}}$$

$$= \frac{1}{\ln 7}\frac{2}{(4x-3)}$$

Exercise

Find the Derivatives of $f(x) = \ln\sqrt[3]{x+1}$

Solution

$$f(x) = \ln(x+1)^{1/3}$$

$$= \frac{1}{3}\ln(x+1)$$

$$u = x+1 \Rightarrow \frac{du}{dx} = 1$$

$$f'(x) = \frac{1}{3}\frac{1}{x+1}$$

$$= \frac{1}{3(x+1)}$$

Exercise

Find the Derivatives of $f(x)=\ln\left[x^2\sqrt{x^2+1}\right]$

Solution

$f(x)=\ln(x^2)+\ln\sqrt{x^2+1}$ *Product Property*

$f(x)=\ln(x^2)+\ln(x^2+1)^{1/2}$

$f(x)=2\ln x+\frac{1}{2}\ln\left(x^2+1\right)$ *Power Property*

$f'(x)=2\frac{1}{x}+\frac{1}{2}\frac{2x}{x^2+1}$ *Differentiate*

$=\frac{2}{x}+\frac{x}{x^2+1}$

Exercise

Find the Derivatives of $y=\ln\frac{1+e^x}{1-e^x}$

Solution

$y=\ln\left(1+e^x\right)-\ln\left(1-e^x\right)$

$y'=\frac{e^x}{1+e^x}-\frac{-e^x}{1-e^x}$

$=\frac{e^x}{1+e^x}+\frac{e^x}{1-e^x}$

$=\frac{e^x-e^{2x}+e^x+e^{2x}}{\left(1+e^x\right)\left(1-e^x\right)}$

$=\frac{2e^x}{\left(1+e^x\right)\left(1-e^x\right)}$

Exercise

Find the Derivatives of $y = \ln \frac{x^2}{x^2+1}$

Solution

$$y = \ln x^2 - \ln x^2 + 1$$

$$y' = \frac{2x}{x^2} - \frac{2x}{x^2+1}$$

$$= \frac{2}{x} - \frac{2x}{x^2+1}$$

Exercise

Find the Derivatives of $y = \ln\left[\frac{x^2(x+1)^3}{(x+3)^{1/2}}\right]$

Solution

$$y = \ln\left[x^2(x+1)^3\right] - \ln(x+3)^{1/2} \qquad \textit{Quotient Rule}$$

$$= \ln x^2 + \ln(x+1)^3 - \ln(x+3)^{1/2} \qquad \textit{Product Rule}$$

$$= 2\ln x + 3\ln(x+1) - \tfrac{1}{2}\ln(x+3) \qquad \textit{Power Rule}$$

$$y' = \frac{2}{x} + \frac{3}{x+1} - \frac{1}{2(x+3)}$$

Exercise

Find the Derivatives of $y = \ln(x^2+1)$

Solution

$$y' = \frac{2x}{x^2+1} \qquad (\ln U)' = \frac{U'}{U}$$

Exercise

Find the Derivatives of $y = \dfrac{\ln x}{e^{2x}}$

Solution

$$y' = \frac{e^{2x}(1/x) - \ln x(2e^{2x})}{e^{4x}}$$

$$= \frac{e^{2x} - 2x\ln x(e^{2x})}{e^{4x}}$$

$$= \frac{e^{2x}\left(1 - 2x\ln x\right)}{e^{4x}}$$

Exercise

Find the Derivatives of $f(x) = \ln(x^2 - 4)$

Solution

Let $\mathrm{u} = \mathrm{x}^2 - 4 \Rightarrow \frac{du}{dx} = 2x$

$$f'(x) = \frac{1}{u}\frac{du}{dx}$$

$$= \frac{1}{x^2-4}(2x)$$

$$= \frac{2x}{x^2-4}$$

Exercise

Find the Derivatives of $f(x) = x^2 \ln x$

Solution

$$f' = x^2 \frac{d}{dx}[\ln x] + \ln x \frac{d}{dx}\left[x^2\right] \qquad (fg)' = f'g + fg'$$

$$= x^2\left(\frac{1}{x}\right) + 2x\ln x$$

$$= x + 2x\ln x$$

$$= x\left(1 + 2\ln x\right)$$

Exercise

Find the Derivatives of $f(x) = -\frac{\ln x}{x^2}$

Solution

$$f' = -\frac{x^2 \frac{d}{dx}[\ln x] - \ln x \frac{d}{dx}\left[x^2\right]}{\left(x^2\right)^2}$$

$$= -\frac{x^2 \frac{1}{x} - 2x\ln x}{x^4}$$

$$= -\frac{x - 2x\ln x}{x^4}$$

$$= -\frac{x(1 - 2\ln x)}{x^4}$$

$$= -\frac{1 - 2\ln x}{x^3}$$

Exercise

Find the Derivative of $f(x) = \frac{e^{\sqrt{x}}}{\ln\left(\sqrt{x} + 1\right)}$

Solution

$f = e^{\sqrt{x}}$ $\quad U = \sqrt{x} = x^{1/2} \Rightarrow U' = \frac{1}{2}x^{-1/2}$ $\quad f' = \frac{1}{2}x^{-1/2}e^{\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

$g = \ln\left(\sqrt{x} + 1\right)$ $\quad U = x^{1/2} + 1 \Rightarrow U' = \frac{1}{2}x^{-1/2}$ $\quad g' = \frac{\frac{1}{2}x^{-1/2}}{\sqrt{x}+1} = \frac{1}{2x^{1/2}\left(\sqrt{x}+1\right)}$

$$f'(x) = \frac{\frac{e^{\sqrt{x}}}{2\sqrt{x}}\ln\left(\sqrt{x}+1\right) - \frac{1}{2\sqrt{x}\left(\sqrt{x}+1\right)}e^{\sqrt{x}}}{\left(\sqrt{x}+1\right)^2}$$

$$= \frac{\frac{\left(\sqrt{x}+1\right)e^{\sqrt{x}}\ln\left(\sqrt{x}+1\right) - e^{\sqrt{x}}}{2\sqrt{x}\left(\sqrt{x}+1\right)}}{\left(\ln\left(\sqrt{x}+1\right)\right)^2}$$

$$= \frac{e^{\sqrt{x}}\left[\left(\sqrt{x}+1\right)\ln\left(\sqrt{x}+1\right) - 1\right]}{2\sqrt{x}\left(\sqrt{x}+1\right)\left(\ln\left(\sqrt{x}+1\right)\right)^2}$$

Exercise

Find the Derivative of $f(x)=e^{2x}\ln\left(xe^{x}+1\right)$

Solution

$$f=e^{2x} \qquad U=2x \to U'=2 \qquad f'=2e^{2x}$$

$$g=\ln\left(xe^{x}+1\right) \qquad U=xe^{x}+1 \to U'=e^{x}+xe^{x} \qquad g'=\frac{e^{x}+xe^{x}}{xe^{x}+1}$$

$$f'(x)=2e^{2x}\ln\left(xe^{x}+1\right)+e^{2x}\frac{e^{x}+xe^{x}}{xe^{x}+1}$$

$$=e^{2x}\left[2\ln\left(xe^{x}+1\right)+\frac{e^{x}(1+x)}{xe^{x}+1}\right]$$

Exercise

Find the Derivative of $f(x)=\dfrac{xe^{x}}{\ln\left(x^{2}+1\right)}$

Solution

$$f=xe^{x} \qquad f'=e^{x}+xe^{x}$$

$$g=\ln\left(x^{2}+1\right) \qquad g'=\frac{2x}{x^{2}+1}$$

$$f'(x)=\frac{e^{x}(1+x)\ln\left(x^{2}+1\right)-\frac{2x}{x^{2}+1}xe^{x}}{\left[\ln\left(x^{2}+1\right)\right]^{2}}$$

$$=\frac{e^{x}\left[(1+x)\ln\left(x^{2}+1\right)-\frac{2x^{2}}{x^{2}+1}\right]}{\left[\ln\left(x^{2}+1\right)\right]^{2}}$$

$$=\frac{e^{x}\left[\frac{\left(x^{2}+1\right)(1+x)\ln\left(x^{2}+1\right)-2x^{2}}{x^{2}+1}\right]}{\left[\ln\left(x^{2}+1\right)\right]^{2}}$$

$$=\frac{e^{x}\left[\left(x^{2}+1\right)(1+x)\ln\left(x^{2}+1\right)-2x^{2}\right]}{\left(x^{2}+1\right)\left[\ln\left(x^{2}+1\right)\right]^{2}}$$

Exercise

Find the derivative $f(x)=2\ln\left(x^2-3x+4\right)$

Solution

$$f'(x)=2\frac{2x-3}{x^2-3x+4}$$
$$=\boxed{\frac{4x-6}{x^2-3x+4}}$$

Exercise

Find the derivative $f(x)=e^{x^2+3x+1}$

Solution

$$f'(x)=\boxed{(2x+3)e^{x^2+3x+1}}$$

Exercise

Find the derivative $f(x)=3\ln\left(1+x^2\right)$

Solution

$$f'(x)=3\frac{2x}{1+x^2}$$
$$=\boxed{\frac{6x}{1+x^2}}$$

Exercise

Find the derivative $f(x)=(1+\ln x)^3$

Solution

$$f'(x)=3(1+\ln x)^2(1+\ln x)'$$
$$=3(1+\ln x)^2\left(\frac{1}{x}\right)$$
$$=\frac{3}{x}(1+\ln x)^2$$

Exercise

Find the derivative $f(x) = (x - 2\ln x)^4$

Solution

$$f'(x) = 4(x - 2\ln x)^3 (x - 2\ln x)'$$

$$= 4(x - 2\ln x)^3 \left(1 - \frac{2}{x}\right)$$

$$= 4(x - 2\ln x)^3 \left(\frac{x-2}{x}\right)$$

$$\boxed{= \frac{4x-8}{x}(x - 2\ln x)^3}$$

Exercise

Find the derivative $f(x) = \dfrac{e^x}{x^2+1}$

Solution

$u = e^x \qquad v = x^2 + 1$

$u' = e^x \qquad v' = 2x$

$$f'(x) = \frac{e^x\left(x^2+1\right) - 2xe^x}{\left(x^2+1\right)^2}$$

$$\boxed{= \frac{\left(x^2+1-2x\right)e^x}{\left(x^2+1\right)^2}}$$

Exercise

Find the derivative $f(x) = \dfrac{1-e^x}{1+e^x}$

Solution

$u = 1 - e^x \qquad v = 1 + e^x$

$u' = -e^x \qquad v' = e^x$

$$f'(x) = \frac{-e^x\left(1+e^x\right) - e^x\left(1-e^x\right)}{\left(1+e^x\right)^2}$$

$$= \frac{-e^x - e^{2x} - e^x + e^{2x}}{\left(1+e^x\right)^2}$$

$$\boxed{= -\frac{2e^x}{\left(1+e^x\right)^2}}$$

Exercise

Find the derivative $\quad f(x) = \frac{\ln x}{1+x}$

Solution

$$u = \ln x \qquad v = 1 + x$$

$$u' = \frac{1}{x} \qquad v' = 1$$

$$f'(x) = \frac{\left(\frac{1}{x}\right)(1+x) - \ln x}{(1+x)^2}$$

$$= \frac{1}{x}\frac{1+x-x\ln x}{(1+x)^2}$$

$$\boxed{= \frac{1+x-x\ln x}{x(1+x)^2}}$$

Exercise

Find the derivative $\quad f(x) = \frac{2x}{1+\ln x}$

Solution

$$u = 2x \qquad v = 1 + \ln x$$

$$u' = 2 \qquad v' = \frac{1}{x}$$

$$f'(x) = \frac{2(1+\ln x) - (2x)\frac{1}{x}}{(1+\ln x)^2}$$

$$= \frac{2+2\ln x - 2}{(1+\ln x)^2}$$

$$\boxed{= \frac{2\ln x}{(1+\ln x)^2}}$$

Exercise

Find the derivative $f(x) = x^2 e^x$

Solution

$u = x^2 \qquad v = e^x$

$u' = 2x \qquad v' = e^x$

$$f'(x) = 2xe^x + x^2 e^x$$

$$= \boxed{\left(2x + x^2\right)e^x}$$

Exercise

Find the derivative $f(x) = x^3 \ln x$

Solution

$u = x^3 \qquad v = \ln x$

$u' = 3x^2 \qquad v' = \frac{1}{x}$

$$f'(x) = 3x^2 \ln x + x^3 \frac{1}{x}$$

$$= 3x^2 \ln x + x^2$$

$$= \boxed{(3\ln x + 1)x^2}$$

Exercise

Find the derivative $f(x) = 6x^4 \ln x$

Solution

$u = 6x^4 \qquad v = \ln x$

$u' = 24x^3 \qquad v' = \frac{1}{x}$

$$f'(x) = 24x^3 \ln x + 6x^4 \frac{1}{x}$$

$$= 24x^3 \ln x + 6x^3$$

$$= \boxed{6x^3(4\ln x + 1)}$$

Exercise

Find the derivative $f(x)=2x^3e^x$

Solution

$u=2x^3 \quad v=e^x$

$u'=6x^2 \quad v'=e^x$

$$f'(x)=6x^2e^x+2x^3e^x$$

$$\boxed{=2x^2e^x(3+x)}$$

Exercise

Find the derivative $f(x)=\dfrac{3e^x}{1+e^x}$

Solution

$u=3e^x \quad v=1+e^x$

$u'=3e^x \quad v'=e^x$

$$f'(x)=\frac{3e^x\left(1+e^x\right)-3e^xe^x}{\left(1+e^x\right)^2}$$

$$=\frac{3e^x+3e^{2x}-3e^{2x}}{\left(1+e^x\right)^2}$$

$$\boxed{=\frac{3e^x}{\left(1+e^x\right)^2}}$$

Exercise

Find the derivative $f(x)=5e^x+3x+1$

Solution

$$f'(x)\boxed{=5e^x+3}$$

Exercise

Find the derivative $f(x)=\dfrac{\ln x}{2x+5}$

Solution

$u=\ln x \quad v=2x+5$

$u'=\dfrac{1}{x} \qquad v'=2$

$$f'(x)=\dfrac{\frac{1}{x}(2x+5)-(2)\ln x}{(2x+5)^2}\cdot\dfrac{x}{x}$$

$$=\boxed{\dfrac{2x+5-2x\ln x}{x(2x+5)^2}}$$

Exercise

Find the derivative $f(x)=-2\ln x+x^2-4$

Solution

$$f'(x)=\boxed{-\dfrac{2}{x}+2x}$$

Exercise

Find the derivative $f(x)=e^x+x-\ln x$

Solution

$$f'(x)=\boxed{e^x+1-\dfrac{1}{x}}$$

Exercise

Find the derivative $f(x)=\ln x+2e^x-3x^2$

Solution

$$f'(x)=\boxed{\dfrac{1}{x}+2e^x-6x}$$

Exercise

Find the derivative $f(x) = \ln x^8$

Solution

$$f(x) = \ln x^8 = 8\ln x$$

$$f'(x) = \boxed{\frac{8}{x}}$$

Power Rule

$$(\ln x)' = \frac{1}{x}$$

Exercise

Find the derivative $f(x) = 5x - \ln x^5$

Solution

$$f(x) = 5x - \ln x^5$$

$$= 5x - 5\ln x$$

$$f'(x) = \boxed{5 - \frac{5}{x}}$$

Power Rule

$$(\ln x)' = \frac{1}{x}$$

Exercise

Find the derivative $f(x) = \ln x^2 + 4e^x$

Solution

$$f(x) = 2\ln x + 4e^x$$

$$f'(x) = \boxed{\frac{2}{x} + 4e^x}$$

Power Rule

$$(\ln x)' = \frac{1}{x}$$

Exercise

Find the derivative $f(x) = \ln x^{10} + 2\ln x$

Solution

$$f(x) = 10\ln x + 2\ln x$$

$$= 12\ln x$$

$$f'(x) = \boxed{\frac{12}{x}}$$

Power Rule

$$(\ln x)' = \frac{1}{x}$$

Exercise

The percentage of people of any particular age group that will die in a given year may be approximated by the formula

$$P(t) = 0.00239e^{0.0957t}$$

Where t is the age of the person in years

Solution

a) Find $P(25)$

$$P(25) = 0.00239e^{0.0957(25)} = \underline{0.02615}$$

b) Find $P'(25)$

$$P'(t) = 0.000228723e^{0.0957t}$$

$$P'(25) = 0.000228723e^{0.0957(25)} = \underline{0.0025}$$

Exercise

Find the equations of the tangent lines to $f(x) = e^x$ at the points (0, 1)

Solution

$$f'(x) = e^x$$

$$(0,\ 1) \Rightarrow m = f'(x=0)$$

$$= e^0$$

$$= 1$$

$$y - y_1 = m\left(x - x_1\right)$$

$$y - 1 = 1(x - 0)$$

$$y - 1 = x$$

$$\underline{y = x + 1}$$

Exercise

Find the equations of the tangent lines to $f(x) = e^x$ at the points (1, e)

Solution

$$f'(x) = e^x$$

$(1, e) \Rightarrow m = f'(x=1) = e^1 = e$

$$y - e = e(x-1)$$

$$y - e = ex - e$$

$$\underline{y = ex|}$$

Exercise

Find the equations of the tangent lines to $y = 4xe^{-x} + 5$ at $x = 1$

Solution

$$y' = 4e^{-x} - 4xe^{-x} = 4e^{-x}(1-x)$$

$$= 4e^{-x}(1-x)$$

$$m = y'(x=1)$$

$$= 4e^{-1}(1-1) = 0$$

$$\Rightarrow x = 1 \rightarrow y = 4e^{-1} + 5$$

$$\left(1, 4e^{-1} + 5\right)$$

$$y - (4e^{-1} + 5) = 0(x-1)$$

$$\underline{y = 4e^{-1} + 5|}$$

Exercise

Find the equation of the tangent lines to $f(x) = 4e^{-8x}$ at the points (0, 4)

Solution

$$f'(x) = -32e^{-8x}$$

$$m = f'(0) = -32e^{-8(0)} = -32$$

$$y - 4 = -32(x-0) \qquad\qquad y - y_1 = m\left(x - x_1\right)$$

$$y - 4 = -32x$$

$$\boxed{y = -32x + 4}$$

Exercise

Assume the cost of a gallon of milk is \$2.90. With continuous compounding, find the time it would take the cost to be 5 times as much (to the nearest tenth of a year), at an annual inflation rate of 6 %.

Solution

$A = Pe^{rt}$

$5(2.90) = 2.9e^{.06t}$ ***Divide both sides by 2.9***

$5 = e^{.06t}$ ***"ln" both sides***

$\ln 5 = \ln e^{.06t}$ ***use inverse property***

$.06t = \ln 5$

$\boxed{t = \frac{\ln 5}{.06} = 26.8 \text{ years}}$

Exercise

The sales in thousands of a new type of product are given by $S(t) = 30 - 90e^{-0.5t}$, where t represents time in years. Find the rate of change of sales at the time when $t = 3$

Solution

$S' = -90(-.5)e^{-0.5t}$

$= 45e^{-0.5t}$

$S'(t = 3) = 45e^{-0.5(3)}$

$\boxed{= 10.04}$

The rate of change of sales at the time when $t = 3$ is 10,040.

Exercise

A company's total cost, in millions of dollars, is given by $C(t) = 300 - 70e^{-t}$ where t = time in years. Find the marginal cost when $t = 3$.

Solution

$C'(t) = -70(-1)e^{-t}$

$= 70e^{-t}$

$C'(t = 3) = 70e^{-3}$ $\boxed{= 3.485}$

The marginal cost is \$3,485,000.

Exercise

A company's total cost, in millions of dollars, is given by $C(t) = 280 - 30e^{-t}$ where *t* = time in years. Find the marginal cost when $t = 2$.

Solution

$$C'(t) = 30e^{-t}$$

$$C'(t = 2) = 30e^{-2} \underline{\left. = 4.06 \right|}$$

The marginal cost is $4,060,000.

Exercise

The demand function for a certain book is given by the function $x = D(p) = 70e^{-0.006p}$. Find the marginal demand $D'(p)$

Solution

$$D'(p) = 70(-.006)e^{-0.006p}$$

$$\underline{\left. = -.42e^{-0.006p} \right|}$$

Exercise

Suppose that the amount in grams of a radioactive substance present at time *t* (in years) is given by $A(t) = 840e^{-0.63t}$. Find the rate of change of the quantity present at the time when $t = 2$.

Solution

$$A'(t) = 840(-0.63)e^{-0.63t}$$

$$= -529.2e^{-0.63t}$$

$$A'(t = 2) = -529.2e^{-0.63(2)}$$

$$\underline{\left. = -150.11 \right|}$$

Exercise

Researchers have found that the maximum number of successful trials that a laboratory rat can complete in a week is given by

$$P(t) = 53\left(1 - e^{-0.4t}\right)$$

where t is the number of weeks the rat has been trained. Find the rate of change $P'(t)$.

Solution

$$P'(t) = 53\left(-(-.4)e^{-0.4t}\right)$$
$$\underline{= 21.2e^{-0.4t}}$$

Exercise

When a telephone wire is hung between two poles, the wire forms a U-shape curve called a Catenary. For instance, the function $y = 30\left(e^{x/60} + e^{-x/60}\right) \quad -30 \le x \le 30$ models the shape of the telephone wire strung between two poles that are 60 ft. apart (x & y are measured in ft.). Show that the lowest point on the wire is midway between two poles. How much does the wire sag between the two poles?

Solution

$$y' = 30\left(\frac{1}{60}e^{x/60} - \frac{1}{60}e^{-x/60}\right)$$
$$= \frac{1}{2}\left(e^{x/60} - e^{-x/60}\right)$$

Find the critical number(s)

$$y' = 0$$
$$\frac{1}{2}\left(e^{x/60} - e^{-x/60}\right) = 0$$
$$e^{x/60} - e^{-x/60} = 0$$
$$e^{x/60} = e^{-x/60}$$
$$\frac{x}{60} = -\frac{x}{60}$$
$$\Rightarrow x = 0$$

$$y(x = -30) = 30\left(e^{-30/60} + e^{-(-30)/60}\right) \approx 67.7\ ft$$

$$y(x = 0) = 30\left(e^{0} + e^{0}\right) = 30(2) = 60\ ft$$

$$y(x = 30) = 30\left(e^{30/60} + e^{-(30)/60}\right) \approx 67.7\ ft$$

Sag 7.7 *ft*

Exercise

Find $f''(x)$ for $f(x)=\frac{\ln x}{7x}$, then find $f''(0)$ and $f''(2)$

Solution

$$f(x)=\frac{\ln x}{7x} \qquad \begin{matrix} f=\ln x & f'=\frac{1}{x} \\ g=7x & g'=7 \end{matrix}$$

$$f'(x)=\frac{\frac{1}{x}(7x)-7\ln x}{(7x)^2}$$

$$=\frac{7-7\ln x}{49x^2}$$

$$=\frac{7(1-\ln x)}{49x^2}$$

$$=\frac{1-\ln x}{7x^2}$$

$$\begin{matrix} f=1-\ln x & f'=-\frac{1}{x} \\ g=7x^2 & g'=14x \end{matrix}$$

$$f''(x)=\frac{-\frac{1}{x}\left(7x^2\right)-14x(1-\ln x)}{\left(7x^2\right)^2}$$

$$=\frac{-7x-14x+14x\ln x}{49x^4}$$

$$=\frac{-21x+14x\ln x}{49x^4}$$

$$=\frac{7x(-3+2\ln x)}{49x^4}$$

$$=\frac{-3+2\ln x}{7x^3}$$

$$\boxed{f''(x)=\frac{2\ln x-3}{7x^3}}$$

$$f''(x=0)=\frac{2\boxed{\ln(0)}-3}{7(0)^3}:\textit{undefined}$$ Inside log has to be > 0 Therefore is undefined for ln(0)

$$f''(x=7)=\frac{2\ln(7)-3}{7(7)^3}\approx 0.0004$$

Exercise

Suppose the average test score p and was modeled by $p = 92.3 - 16.9\ln(t+1)$, where t is the time in months. How would the rate at which the average test score changed after 1 year?

Solution

$$\frac{dp}{dt} = -\frac{16.9}{t+1}$$

$t = 1\ yr = 12\ mths$

$$\Rightarrow \frac{dp}{dt} = -\frac{16.9}{12+1}$$

$$= -\frac{16.9}{13}$$

$$\underline{= -1.3|}$$

Exercise

Suppose that the population of a certain collection of rare ants is given by

$$P(t) = (t+100)\ln(t+2)$$

Where t represents the time in days. Find the rates of change of the population on the second day and on the eighth day.

Solution

$$P'(t) = \ln(t+2) + (t+100)\frac{1}{t+2}$$

$$= \ln(t+2) + \frac{t+100}{t+2}$$

$$P'(2) = \ln(2+2) + \frac{2+100}{2+2} = \underline{27.89|}$$

$$P'(8) = \ln(8+2) + \frac{8+100}{8+2} = \underline{13.10|}$$

Exercise

Suppose that the demand function for x units of a certain item is $P(x)=100+\dfrac{180\ln(x+5)}{x}$ where P is the price per unit, in dollars. Find the marginal revenue.

Solution

$$R=x.P(x)$$
$$=x\left(100+\frac{180\ln(x+5)}{x}\right)$$
$$=100x+180\ln(x+5)$$

$$R'(x)=100+180\frac{1}{x+5}$$
$$=\frac{100(x+5)+180}{x+5}$$
$$=\frac{100x+500+180}{x+5}$$
$$=\frac{100x+680}{x+5}$$

Exercise

The population of coyotes in the northwestern portion of Alabama is given by the formula $P(t)=\left(t^2+100\right)\ln(t+2)$, where t represents the time in years since 2000 (the year 2000 corresponds to $(t=0)$ Find the rate of change of the coyote population in $2013\ (t=13)$.

Solution

$$f=t^2+100 \quad \to f'=2t$$
$$g=\ln(t+2) \quad \to g'=\frac{1}{t+2}$$

$$P'=f'g+g'f$$
$$P'(t)=2t\ln(t+2)+\frac{1}{t+2}\left(t^2+100\right)$$
$$=2t\ln(t+2)+\frac{t^2+100}{t+2}$$

$$P'(t=13)=2(13)\ln(13+2)+\frac{13^2+100}{13+2} \qquad 2*13\ln(13+2)+(13\wedge2+100)/(13+2)$$
$$=88.34$$

Exercise

Students in a math class took a final exam. They took equivalent forms of the exam in monthly intervals thereafter. The average score $S(t)$, in percent, after t months was found to be given by

$$S(t) = 73 - 17\ln(t+1), \qquad t \ge 0$$

Find $S'(t)$.

Solution

$$S'(t) = -17\frac{1}{t+1}$$

$$\boxed{= -\frac{17}{t+1}}$$

Exercise

Suppose that the population of a town is given by $P(t) = 8\ln\sqrt{8t+7}$ where t is the time in years after 1980 and P is the population of the town in thousands. Find $P'(t)$.

Solution

$$U = 8t+7 \quad \to U' = 8$$

$$V = \sqrt{8t+7} = (8t+7)^{1/2} = U^{1/2}$$

$$\to V' = nU'U^{n-1} = \tfrac{1}{2}8(8t+7)^{-1/2}$$

$$= \frac{4}{(8t+7)^{1/2}}$$

$$P'(t) = 8\frac{V'}{V}$$

$$= 8\underset{V'}{\frac{4}{(8t+7)^{1/2}}}\ \underset{V}{\frac{1}{(8t+7)^{1/2}}}$$

$$\boxed{= \frac{32}{8t+7}}$$

Exercise

The following formula accurately models the relationship between the size of a certain type of tumor and the amount of time that it has been growing:

$$V(t) = 450(1 - e - 0.0022t)^3$$

where t is in months and $V(t)$ is measured in cubic centimeters. Calculate the rate of change of tumor volume at 80 months.

Solution

$$U = 1 - e - 0.0022t \qquad V = 450U^3$$

$$U' = -.0022 \qquad V' = 450(3)U^2U'$$

$$V'(t) = 450(3)(1 - e - 0.0022t)^2(-.0022) \ \underline{= 2.97(1 - e - 0.0022t)^2}$$

$$V'(t = 80) = 2.97(1 - e - 0.0022(80))^2 \ \underline{= 10.66}$$

Exercise

A yeast culture at room temperature (68° F) is placed in a refrigerator set at a constant temperature of 38° F. After t hours, the temperature T of the culture is given approximately by

$$T = 30e^{-0.58t} + 38 \quad t \ge 0$$

What is the rate of change of temperature of the culture at the end of 1 hour? At the end of 4 hours?

Solution

$$T' = 30(-0.58)e^{-0.58t} \ \underline{= -17.4e^{-0.58t}}$$

$$T'(1) = -17.4e^{-0.58(1)} \ \underline{= -9.74° \ F/hr}$$

$$T'(4) = -17.4e^{-0.58(4)} \ \underline{= -1.71° \ F/hr}$$

Exercise

A mathematical model for the average age of a group of people learning to type is given by

$$N(t) = 10 + 6\ln t \quad t \ge 1$$

Where $N(t)$ is the number of words per minute typed after t hours of instruction and practice (2 hours per day, 5 days per week). What is the rate of learning after 10 hours of instruction and practice? After 100 hours?

Solution

$$N'(t) = \frac{6}{t}$$

$$N'(10) = \frac{6}{10} \ \underline{= 0.6}$$

After 10 hours of instruction and practice, the rate of learning is 0.6 words/minute per hour of instruction and practice.

$$N'(100) = \frac{6}{100} \underline{= 0.06}$$

After 100 hours of instruction and practice, the rate of learning is 0.06 words/minute per hour of instruction and practice.