Solution   *Section* 4.3 – Integration by Parts

Exercise

Find the integral $\int \ln x^2 dx$

Solution

$$\int \ln x^2 dx = 2\int \ln x dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$

$$dv = dx \Rightarrow v = x$$

$$\int \ln x^2 dx = 2\left[x\ln x - \int x\frac{1}{x}dx\right]$$

$$= 2\left[x\ln x - \int dx\right]$$

$$= 2(x\ln x - x) + C$$

$$= 2x(\ln x - 1) + C$$

Exercise

Find the integral $\int \frac{2x}{e^x}dx$

Solution

$$u = 2x \Rightarrow du = 2dx$$

$$dv = e^{-x}dx \Rightarrow v = -e^{-x}$$

$$\int \frac{2x}{e^x}dx = 2x(-e^{-x}) - \int -e^{-x}2dx$$

$$= -2xe^{-x} + 2\int e^{-x}dx$$

$$= -2xe^{-x} - 2e^{-x} + C$$

$$= -2e^{-x}(x+1) + C$$

$$= -\frac{2(x+1)}{e^x} + C$$

Exercise

Find the integral $\int \ln(3x)dx$

Solution

$$u = \ln 3x \Rightarrow du = \frac{3}{3x}dx = \frac{1}{x}dx$$

$$dv = dx \Rightarrow v = x$$

$$\int \ln(3x)dx = x\ln(3x) - \int x\frac{1}{x}dx$$

$$= x\ln(3x) - \int dx$$

$$= x\ln(3x) - x + C$$

$$= x\left[\ln(3x) - 1\right] + C$$

Exercise

Find the integral $\int \frac{1}{x\ln x}dx$

Solution

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{\ln x}\frac{1}{x}dx$$

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{u}du$$

$$= \ln u + C$$

$$= \ln\left|\ln x\right| + C$$

Exercise

Find the integral $\displaystyle\int \frac{x}{\sqrt{x-1}}dx$

Solution

Let: $u = x \ \Rightarrow du = dx$

$$dv = \frac{dx}{\sqrt{x-1}} \Rightarrow v = \int (x-1)^{-1/2} d(x-1)$$

$$= \frac{(x-1)^{1/2}}{1/2}$$

$$= 2(x-1)^{1/2}$$

$$\int \frac{x}{\sqrt{x-1}}dx = 2x\sqrt{x-1} - 2\int (x-1)^{1/2}dx$$

$$= 2x\sqrt{x-1} - 2\frac{(x-1)^{3/2}}{3/2} + C$$

$$= 2x\sqrt{x-1} - \frac{4}{3}(x-1)\sqrt{x-1} + C$$

$$= \sqrt{x-1}\left[2x - \frac{4}{3}x + \frac{4}{3}\right] + C$$

$$= \sqrt{x-1}\left[\frac{6x-4x+4}{3}\right] + C$$

$$= \sqrt{x-1}\left[\frac{2x+4}{3}\right] + C$$

$$= \frac{2}{3}\sqrt{x-1}(x+2) + C$$

Exercise

Find the integral $\displaystyle\int \frac{x^3 e^{x^2}}{\left(x^2+1\right)^2}dx$

Solution

Let: $u = x^2 e^{x^2} \ \Rightarrow du = \left(2xe^{x^2} + 2xx^2e^{x^2}\right)dx$

$$du = 2xe^{x^2}\left(1+x^2\right)dx$$

$$dv = x\left(x^2+1\right)^{-2}dx \qquad \Rightarrow v = \int x(x^2+1)^{-2}dx$$

$$= \frac{1}{2}\int (x^2+1)^{-2}d(x^2+1)$$

$$= \frac{(x^2+1)^{-1}}{-1}$$

$$= -\frac{1}{2(x^2+1)}$$

$$\int \frac{x^3 e^{x^2}}{(x^2+1)^2}dx = x^2 e^{x^2}\left(-\frac{1}{2(x^2+1)}\right) - \int -\frac{1}{2(x^2+1)}2xe^{x^2}(x^2+1)dx$$

$$= -\frac{x^2 e^{x^2}}{2(x^2+1)} + \int xe^{x^2}dx$$

Let: $u = x^2 \;\Rightarrow du = 2xdx$

$$= -\frac{x^2 e^{x^2}}{2(x^2+1)} + \frac{1}{2}\int e^u du$$

$$= -\frac{x^2 e^{x^2}}{2(x^2+1)} + \frac{1}{2}e^u + C$$

$$= -\frac{x^2 e^{x^2}}{2(x^2+1)} + \frac{1}{2}e^{x^2} + C$$

$$= \frac{1}{2}e^{x^2}\left[-\frac{x^2}{(x^2+1)} + 1\right] + C$$

$$= \frac{1}{2}e^{x^2}\left[\frac{-x^2+x^2+1}{(x^2+1)}\right] + C$$

$$= \frac{e^{x^2}}{2(x^2+1)} + C$$

Exercise

Find the integral $\int x^2 e^{-3x} dx$

Solution

$$u = x^2 \Rightarrow du = 2xdx$$

$$dv = e^{-3x}dx \Rightarrow v = -\frac{1}{3}e^{-3x}$$

$$\int x^2 e^{-3x} dx = -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\int xe^{-3x} dx$$

$$u = x \Rightarrow du = dx$$

$$dv = e^{-3x}dx \Rightarrow v = -\frac{1}{3}e^{-3x}$$

$$\begin{aligned}\int x^2 e^{-3x} dx &= -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\left[-\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x} dx\right] \\ &= -\frac{1}{3}x^2 e^{-3x} + \frac{2}{3}\left[-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right] + C \\ &= -\frac{1}{3}x^2 e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + C \\ &= -\frac{9x^2+6x+2}{27}e^{-3x} + C\end{aligned}$$