4/ $\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$

a) row vectors: $(0\ -2)$ $(1\ -3)$

b) col. vectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} -2 \\ -3 \end{pmatrix}$

5/a) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \frac{1}{2}R_2 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

b) rank = 2

9/ $\begin{bmatrix} 1 & 6 & 18 \\ 7 & 40 & 116 \\ -3 & -12 & -27 \end{bmatrix} \begin{matrix} \\ R_2-7R_1 \\ R_3+3R_1 \end{matrix} \longrightarrow \begin{pmatrix} 1 & 6 & 18 \\ 0 & -2 & -2 \\ 0 & 6 & 25 \end{pmatrix} \begin{matrix} R_1+3R_2 \\ \\ R_3+3R_2 \end{matrix}$

$\begin{pmatrix} 1 & 0 & 12 \\ 0 & -2 & -2 \\ 0 & 0 & 19 \end{pmatrix} \begin{matrix} \\ -\frac{1}{2}R_2 \\ \frac{1}{19}R_3 \end{matrix} \longrightarrow \begin{pmatrix} 1 & 0 & 12 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} R_1-12R_3 \\ R_2-R_3 \\ \\ \end{matrix}$

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow$ basis

rank = 3.

27/ $A = \begin{pmatrix} 2 & -1 \\ -6 & 3 \end{pmatrix}$ R_2+3R_1

$\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \rightarrow$ rank = 1

$2x_1 = x_2 \Rightarrow N(A) = t\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

33/ $A = \begin{pmatrix} 1 & -2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -2 \end{pmatrix}$ $\begin{matrix} R_2 - 2R_1 \\ R_3 - 4R_1 \end{matrix}$

$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & 10 \\ 0 & -5 & 10 \end{pmatrix} \begin{matrix} \\ R_3 - R_1 \end{matrix} \rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & 10 \\ 0 & 0 & 0 \end{pmatrix} -\frac{1}{5}R_2$

$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \rightarrow x_1 = -2x_2 + 3x_3 = -x_3 \\ \rightarrow x_2 = 2x_3 \\ \\ \end{matrix}$

$\left\{ x_3 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$

35/ $A = \begin{pmatrix} 5 & 2 \\ 3 & -1 \\ 2 & 1 \end{pmatrix}$ $\begin{matrix} 5R_2 - 3R_1 \\ 5R_3 - 2R_1 \end{matrix}$

$\begin{pmatrix} 5 & 2 \\ 0 & -11 \\ 0 & 1 \end{pmatrix} \begin{matrix} \rightarrow x_1 = 0 \\ \\ \rightarrow x_2 = 0 \end{matrix}$

$\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

4) $A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 5 & 1 & 1 & 0 \\ 3 & 7 & 2 & 2 & -2 \\ 4 & 9 & 3 & -1 & 4 \end{bmatrix} \longrightarrow B = \begin{bmatrix} 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} ① \\ ② \\ ③ \\ \end{array}$

a) rank (A) = 3
nullity (A) = 2

b) ① $x_1 = -3x_3 + 4x_5$
② $x_2 = x_3 - 2x_5$
③ $x_4 = 2x_5$
let $x_3 = 1, x_5 = 0 \rightarrow (-3, 1, 1, 0, 0)$
$x_3 = 0, x_5 = 1 \rightarrow (4, -2, 0, 2, 1)$

N(A) basis: $\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

c) Basis for row (A)
$\{(1,0,3,0,-4), (0,1,-1,0,2), (0,0,0,1,-2)\}$

d) Basis for CS(A) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix} \right\}$

e) Linearly Dependent

f) i) $\{a_1, a_2, a_4\}$ Yes (LI) (pivot variables)
ii) $\{a_1, a_2, a_3\}$ No $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$ Linearly Depend
iii) $\{a_1, a_3, a_5\}$ $\begin{bmatrix} 1 & 3 & -4 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$
Yes (L.I.)

49/ $x - 4y = 17$
$3x - 12y = 51$
$-2x + 8y = -34$

$$\left(\begin{array}{cc|c} 1 & -4 & 17 \\ 3 & 12 & 51 \\ -2 & 8 & -34 \end{array}\right) \begin{array}{l} R_2 - 3R_1 \\ R_3 + 2R_1 \end{array} \rightarrow \left(\begin{array}{cc|c} 1 & -4 & 17 \\ 0 & 24 & 0 \\ 0 & 0 & 0 \end{array}\right) \tfrac{1}{24} R_2$$

$$\left(\begin{array}{cc|c} 1 & -4 & 17 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightarrow x_1 = 4x_2$$

rank (2) free variable $\cancel{x}_2$
x_1 pivot column.

null space $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$$x_p = \begin{pmatrix} 17 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

57/ $A = \begin{pmatrix} -1 & 2 \\ 4 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$$\left(\begin{array}{cc|c} +1 & -2 & -3 \\ 4 & 0 & 4 \end{array}\right) \rightarrow R_2 - 4R_1 \rightarrow \left(\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 8 & 16 \end{array}\right) \begin{array}{l} \rightarrow 1 \\ \rightarrow x_2 = 2 \end{array}$$

1 $x_1 = 2x_2 - 3 = 1$

$$\begin{bmatrix} -1 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Sec 4.7

5/ $B = \{(2,-1),(0,1)\}$ $[x]_B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$x = 4(2,-1) + 1(0,1)$
$= (8,-3)$

$[x]_{B'} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$

7/ $B = \{(1,0,1),(1,1,0),(0,1,1)\}$ $[x]_B = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

$x = 2(1,0,1) + 3(1,1,0) + 1(0,1,1)$
$= (5, 4, 3)$

$[x]_{B'} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$

11/ $B' = \{(4,0),(0,3)\}$ $x = (12,6)$

$(12,6) = c_1(4,0) + c_2(0$

$12 = 4c_1 \rightarrow \boxed{c_1 = 3}$
$6 = 3c_2 \rightarrow \boxed{c_2 = 2}$

$[x]_{B'} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

13/ $B' = \{(8,11,0),(7,0,10),(1,4,6)\}$ $x = (3,19,2)$

$(3,9,2) = c_1(8,11,0) + c_2(7,0,10) + c_3(1,4,6)$

$$\begin{cases} 8c_1 + 7c_2 + c_3 = 3 \\ 11c_1 + 4c_3 = 19 \\ 10c_2 + 6c_3 = 2 \end{cases}$$

$\Delta = \begin{vmatrix} 8 & 7 & 1 \\ 11 & 0 & 4 \\ 0 & 10 & 6 \end{vmatrix} = -672$

$\Delta_1 = \begin{vmatrix} 3 & 7 & 1 \\ 19 & 0 & 4 \\ 2 & 10 & 6 \end{vmatrix} = -672$

$4c_3 = 19 - 11 \rightarrow \boxed{c_3 = 2}$

$10c_2 = 2 - 12 \rightarrow \boxed{c_2 = -1}$

$\boxed{c_1 = +\frac{672}{672} = 1}$

$[x]_{B'} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

17/ $B = \{(1,0),(0,1)\}$ $B' = \{(2,4),(1,3)\}$

$$[B' \; B] = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{bmatrix} R_2 - 2R_1$$

$$\sim \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} R_1 - R_2$$

$$\begin{bmatrix} 2 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{bmatrix} \tfrac{1}{2}R_1 \to \begin{pmatrix} 1 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & -2 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{pmatrix}$$

21/ $B = \{(-1,0,0),(0,1,0),(0,0,-1)\}$

$B' = \{(0,0,2),(1,4,0),(5,0,2)\}$

$$[B' \; B] = \left[\begin{array}{ccc|ccc} 0 & 1 & 5 & -1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & -1 \end{array}\right] R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 0 & 0 & -1 \\ 0 & 1 & 5 & -1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \end{array}\right] R_3 - 4R_2$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 0 & 0 & -1 \\ 0 & 1 & 5 & -1 & 0 & 0 \\ 0 & 0 & -20 & 4 & 1 & 0 \end{array}\right] \begin{array}{l} 10R_1 + R_3 \\ 4R_2 + R_3 \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 20 & 0 & 0 & 4 & 1 & -10 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & -20 & 4 & 1 & 0 \end{array}\right] \begin{array}{l} \frac{1}{20} \\ \frac{1}{4} \\ -\frac{1}{20} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & \frac{1}{20} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & -\frac{1}{20} & 0 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{1}{20} & -\frac{1}{2} \\ 0 & \frac{1}{4} & 0 \\ -\frac{1}{5} & -\frac{1}{20} & 0 \end{bmatrix}$$

13/ $\{x, -\sin x\}$

$$W = \begin{vmatrix} x & -\sin x \\ 1 & -\cos x \end{vmatrix} = \underline{-x\cos x + \sin x}$$

15/ $\{e^x, e^{-x}\}$

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1$$
$$= \underline{-2}$$

17/ $\{x, \sin x, \cos x\}$

$$W = \begin{vmatrix} x & \sin x & \cos x \\ 1 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix}$$
$$= -x\cos^2 x - \sin x\cos x - x\sin^2 x + \sin x \cos x$$
$$= -x(\cos^2 x + \sin^2 x)$$
$$= \underline{-x}$$

19/ $\{e^{-x}, xe^{-x}, (x+3)e^{-x}\}$

$$W = \begin{vmatrix} e^{-x} & xe^{-x} & (x+3)e^{-x} \\ -e^{-x} & (1-x)e^{-x} & (-2-x)e^{-x} \\ e^{-x} & (-2+x)e^{-x} & (x+1)e^{-x} \end{vmatrix}$$
$$= (e^{-x})^3 \begin{vmatrix} 1 & x & x+3 \\ -1 & 1-x & -x-2 \\ 1 & x-2 & x+1 \end{vmatrix}$$
$$= (1 - x^2 - x^2 - 2x + x^2 - x + 6 - x^2 + 2x - 3 + x^2 - 4 + x^2 + x)e^{-3x}$$
$$= \underline{0}$$

21) $\{1, e^x, e^{2x}\}$

$$W = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = 4e^{3x} - 2e^{3x}$$

$$= \boxed{2e^{3x}}$$