1-a. $\lim\limits_{(x,y)\to(1,2)} \dfrac{2x^2-xy}{4x^2-y^2} = \dfrac{2-2}{4-4} = \dfrac{0}{0}$

$= \lim\limits_{(x,y)\to(1,2)} \dfrac{x(2x-y)}{(2x-y)(2x+y)}$

$= \lim\limits_{(x,y)\to(1,2)} \dfrac{x}{2x+y}$

$= \boxed{\dfrac{1}{4}}$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin x+\sin y}{x+y} = \dfrac{0}{0}$

$= \lim\limits_{(x,y)\to(0,0)} \dfrac{2\sin\left(\frac{x+y}{2}\right)\overbrace{\cos\left(\frac{x-y}{2}\right)}^{=1}}{x+y}$

$= \lim\limits_{(x,y)\to(0,0)} \dfrac{\sin\left(\frac{x+y}{2}\right)}{\frac{x+y}{2}}$ $\qquad \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta}$ $\quad \theta = \dfrac{x+y}{2}$

$= \boxed{1}$

or let $z = x+y \to y = z-x$ $\qquad (x,y)\to(0,0) \Rightarrow z\to 0$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin x+\sin y}{x+y} = \lim\limits_{z\to 0}\dfrac{\sin x}{z} + \lim\limits_{z\to 0}\dfrac{\sin(z-x)}{z}$

$= \lim\limits_{z\to 0} \dfrac{1}{z}\left(\sin x + \sin z\cos x - \cos z\sin x\right)$

$= \lim\limits_{\substack{z\to 0\\ x\to 0}} \left(\sin x\dfrac{(1-\cos z)}{z} + \underbrace{\dfrac{\sin z}{z}}_{=1}\underbrace{\cos x}_{=1}\right)$

$= \lim\limits_{x\to 0}\sin x \cdot \lim\limits_{z\to 0}\dfrac{1-\cos z}{z} + 1$

$= 0 \cdot \lim\limits_{z\to 0}\dfrac{\sin z}{1} + 1$

$= \boxed{1}$

c) $\lim\limits_{(x,y)\to(0,\frac{\pi}{2})} \dfrac{1-\cos xy}{4x^2y^3} = \dfrac{0}{0}$ Let $u = xy \to 0$

$y = \frac{\pi}{2}$

$= \dfrac{1}{4} \lim\limits_{u\to 0} \dfrac{1-\cos u}{u^2 \cdot y}$

$= \dfrac{1}{2\pi} \lim\limits_{u\to 0} \dfrac{1-\cos u}{u^2} = \dfrac{0}{0}$

$= \dfrac{1}{2\pi} \lim\limits_{u\to 0} \dfrac{\sin u}{2u}$ $\dfrac{\sin u}{u} \xrightarrow[u\to 0]{} 1$

$= \dfrac{1}{4\pi}$

d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-2xy+y^2}{x-y} = \dfrac{0}{0}$

$= \lim\limits_{(x,y)\to(0,0)} \dfrac{(x-y)(x-y)}{x-y}$

$= \lim\limits_{(x,y)\to(0,0)} (x-y)$

$= 0$

2- $f(x,y,z) = x^2y + 2xz^2 - 3y^2z$ f_x f_y f_z

a) f_{xx} f_{xy} f_{xz}

f_{yx} f_{yy} f_{yz}

f_{zx} f_{zy} f_{zz}

b) $f_x = 2xy + 2z^2 \qquad f_y = x^2 - 6yz \qquad f_z = 4xz - 3y^2$

$f_{xx} = 2y \qquad f_{yx} = 2x \qquad f_{zx} = 4z$

$f_{xy} = 2x \qquad f_{yy} = -6z \qquad f_{zy} = -6y$

$f_{xz} = 4z \qquad f_{yz} = -6y \qquad f_{zz} = 4x$

c) $f_{xy} = f_{yx} = 2x$

$f_{xz} = f_{zx} = 4z$

$f_{yz} = f_{zy} = -6y$

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3) $f(x,y) = \frac{1}{2}\ln(x^2+y^2) + \tan^{-1}\frac{y}{x}$

$$f_x = \frac{1}{2}\frac{2x}{x^2+y^2} + \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}$$

$$= \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}$$

$$= \frac{x-y}{x^2+y^2}$$

$$f_y = \frac{1}{2}\frac{2y}{x^2+y^2} + \frac{\frac{1}{x}}{1+\frac{y^2}{x^2}}$$

$$= \frac{y}{x^2+y^2} + \frac{x}{x^2+y^2}$$

$$= \frac{x+y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} + f_{yy} + f_{zz} = 0.$$

a) $f(x,y,z) = x^2 + y^2 + z^2$

$f_x = 2x \qquad f_y = 2y \qquad f_z = 2z$

$f_{xx} = 2 \qquad f_{yy} = 2 \qquad f_{zz} = 2$

$$f_{xx} + f_{yy} + f_{zz} = 2 + 2 + 2$$
$$= 6 \neq 0.$$

it's not satisfied Laplace eqn

b) $f(x,y,z) = 2z^3 - 3(x^2+y^2)z$

$f_x = -6xz \qquad f_y = -6yz \qquad f_z = 6z^2 - 3x^2 - 3y^2$

$f_{xx} = -6z \qquad f_{yy} = -6z \qquad f_{zz} = 12z$

$$f_{xx} + f_{yy} + f_{zz} = -6z - 6z + 12z$$
$$= 0 \checkmark.$$

$\therefore$ The fctn satisfies Laplace eqn.

c) $f(x,y,z) = e^{-2y}\cos 2x$

$f_x = -2e^{-2y}\sin 2x \qquad f_y = -2e^{-2y}\cos 2x \qquad f_z = 0$

$f_{xx} = -4e^{-2y}\cos 2x \qquad f_{yy} = 4e^{-2y}\cos 2x \qquad f_{zz} = 0$

$$f_{xx} + f_{yy} + f_{zz} = -4e^{-2y}\cos 2x + 4e^{-2y}\cos 2x + 0$$
$$= 0 \checkmark$$

$\therefore$ The fctn satisfies Laplace eqn

d) $f(x,y,z) = \ln\sqrt{x^2+y^2}$

$= \frac{1}{2}\ln(x^2+y^2)$

$f_x = \frac{x}{x^2+y^2}$ $\quad f_y = \frac{y}{x^2+y^2}$ $\quad f_z = 0.$

$f_{xx} = \frac{-x^2+y^2}{(x^2+y^2)^2}$ $\quad f_{yy} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{-y^2+x^2}{(x^2+y^2)^2}$

$f_{zz} = 0$

$$f_{xx} + f_{yy} + f_{zz} = \frac{-x^2+y^2}{(x^2+y^2)^2} + \frac{-y^2+x^2}{(x^2+y^2)^2} + 0$$

$$= \frac{-x^2+y^2-y^2+x^2}{(x^2+y^2)^2}$$

$= 0$

∴ The given fctn satisfies Laplace eqn.

e) $f(x,y,z) = \tan^{-1}\frac{x}{y}$

$f_x = \frac{\frac{1}{y}}{1+\frac{x^2}{y^2}} = \frac{y}{x^2+y^2}$ $\quad f_y = \frac{-\frac{x}{y^2}}{1+\frac{x^2}{y^2}} = -\frac{x}{x^2+y^2}$

$f_{xx} = -\frac{2xy}{(x^2+y^2)^2}$ $\quad f_{yy} = \frac{2xy}{(x^2+y^2)^2}$ $\quad f_{zz} = 0$

$$f_{xx} + f_{yy} + f_{zz} = -\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} + 0$$

$= 0$

∴ The given fctn satisfies Laplace eqn.

Defn Gradient

$$\nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$

directional Derivative

$$\left(\frac{df}{ds}\right)_{\vec{u}, P_0} = (\nabla f)_{P_0} \cdot \vec{u}$$

Ex $f(x,y) = xe^y + \cos(xy)$
$@ (2,0)$ $\vec{v} = 3\hat{i} - 4\hat{j}$

$$\vec{u} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

$$\nabla f = f_x \hat{i} + f_y \hat{j}$$

$$= (e^y - y \sin xy)\hat{i} + (xe^y - x \sin xy)\hat{j} \Big|_{(2,0)}$$

$$= \hat{i} + 2\hat{j}$$

$$(D_{\vec{u}} f)_{(2,0)} = (\hat{i} + 2\hat{j}) \cdot \left(\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}\right)$$

$$= \frac{3}{5} - \frac{8}{5}$$

$$\underline{= -1}$$

Ex $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$

$$\nabla f = x\hat{i} + y\hat{j} \Big|_{(1,1)}$$

$$= \hat{i} + \hat{j}$$

$$\vec{u} = \frac{\hat{i}+\hat{j}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

$$\nabla f = -\hat{i} - \hat{j} \qquad \vec{u} = -\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$$

zero change

$$\vec{n} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} \quad \text{or } \vec{n} = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

Ex $\frac{x^2}{4} + y^2 = 2$ @ $(-2,1)$ eqn of tangent

$$f(x,y) = \frac{x^2}{4} + y^2$$

$$\nabla f = \frac{1}{2}x\hat{i} + 2y\hat{j} \Big|_{(-2,1)}$$

$$= \boxed{-\hat{i} + 2\hat{j}}$$

$$f_x(x-x_0) + f_y(y-y_0) = 0$$

$$-(x+2) + 2(y-1) = 0$$

$$\boxed{-x + 2y = 4}$$

Ex $f(x,y,z) = x^3 - xy^2 - z$ $P_0(1,1,0)$

$\vec{v} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

$$\vec{u} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{4+9+36}}$$

$$= \frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

$$\nabla f = (3x^2 - y^2)\hat{i} - 2xy\hat{j} - \hat{k} \Big|_{(1,1,0)}$$

$$= 2\hat{i} - 2\hat{j} - \hat{k}$$

$$(D_{\vec{u}} f)_{P_0} = \nabla f \cdot \vec{u}$$

$$= (2\hat{i} - 2\hat{j} - \hat{k}) \cdot \left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7}$$

$$= \boxed{\frac{4}{7}}$$

2.6

Ex $f(x,y,z) = x^2 + y^2 + z - 9 = 0$

@ $P_0 (1, 2, 4)$

tangent plane? line?

$$\nabla f = 2x\,\hat{i} + 2y\,\hat{j} + \hat{k} \Big|_{(1,2,4)}$$
$$= 2\hat{i} + 4\hat{j} + \hat{k}$$

Tangent plane:

$$2(x-1) + 4(y-2) + z - 4 = 0$$
$$\underline{2x + 4y + z = 14}$$

Tangent line: $\begin{cases} x = 1 + 2t \\ y = 2 + 4t \\ z = 4 + t \end{cases}$

Ex $z = x\cos y - ye^x$ @ $(0,0,0)$

$$f(x,y,z) = x\cos y - ye^x - z$$
$$\nabla f = (\cos y - ye^x)\,\hat{i} + (-x\sin y - e^x)\,\hat{j} - \hat{k} \Big|_{(0,0,0)}$$
$$= \hat{i} - \hat{j} - \hat{k}$$

Tangent plane $x - y - z = 0$

" line: $\begin{cases} x = t \\ y = -t \\ z = -t \end{cases}$

Ex $f(x,y,z) = x^2 + y^2 - 2 = 0$

$g(x,y,z) = x + z - 4 = 0$

$P_0(1,1,3)$

$$\nabla f = 2x\hat{i} + 2y\hat{j} \Big|_{(1,1,3)}$$
$$= 2\hat{i} + 2\hat{j}$$

$$\nabla g = \hat{i} + \hat{k}$$

$$\vec{v} = \nabla f \times \nabla g$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$
$$= 2\hat{i} - 2\hat{j} - 2\hat{k}$$

Tangent line: $\begin{cases} x = 1 + 2t \\ y = 1 - 2t \\ z = 3 - 2t \end{cases}$

Estimating the Change in f in a Direction $\vec{u}$

$$df = \underbrace{\left(\nabla f\Big|_{P_0} \cdot \vec{u}\right)}_{\text{directional derivative}} \cdot \underbrace{ds}_{\text{distance}}$$

<u>Ex</u> $f(x,y,z) = y\sin x + 2yz$

will change @ $P(x,y,z)$ move 0.1 from $P_0(0,1,0)$

$\rightarrow P_1(2,2,-2)$

$$\overrightarrow{P_0P_1} = 2\hat{i} + \hat{j} - 2\hat{k}$$

$$\vec{u} = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{2\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{4+1+4}}$$

$$= \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

$$\nabla f = (y\cos x)\hat{i} + (\sin x + 2z)\hat{j} + 2y\hat{k}\Big|_{(0,1,0)}$$

$$= \hat{i} + 2\hat{k}$$

$$D_{\vec{u}}f = \nabla f \cdot \vec{u}$$

$$= (\hat{i} + 2\hat{k}) \cdot \left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right)$$

$$= \frac{2}{3} - \frac{4}{3}$$

$$= \boxed{-\frac{2}{3}}$$

$$df = \left(-\frac{2}{3}\right)\left(\frac{1}{10}\right) = \boxed{-\frac{1}{15}} \text{ unit}$$