

## ***Solution***      **Section 1.1 – Introduction to System of Linear Equations**

### ***Exercise***

Find a solution for  $x, y, z$  to the system of equations

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3e + 2\sqrt{2} + \pi \\ 6e + 5\sqrt{2} + 4\pi \\ 9e + 8\sqrt{2} + 7\pi \end{pmatrix}$$

### **Solution**

$$\begin{pmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{pmatrix} = \begin{pmatrix} 3e + 2\sqrt{2} + \pi \\ 6e + 5\sqrt{2} + 4\pi \\ 9e + 8\sqrt{2} + 7\pi \end{pmatrix}$$

$$\Rightarrow \begin{cases} x + 2y + 3z = \pi + 2\sqrt{2} + 3e \\ 4x + 5y + 6z = 4\pi + 5\sqrt{2} + 6e \\ 7x + 8y + 9z = 7\pi + 8\sqrt{2} + 9e \end{cases}$$

$$\text{Solution: } \underline{x = \pi \quad y = \sqrt{2} \quad z = e}$$

### ***Exercise***

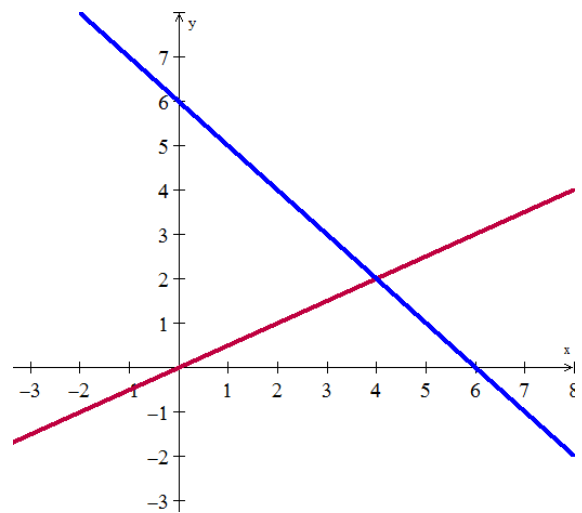
Draw the two pictures in two planes for the equations:  $x - 2y = 0$ ,  $x + y = 6$

### **Solution**

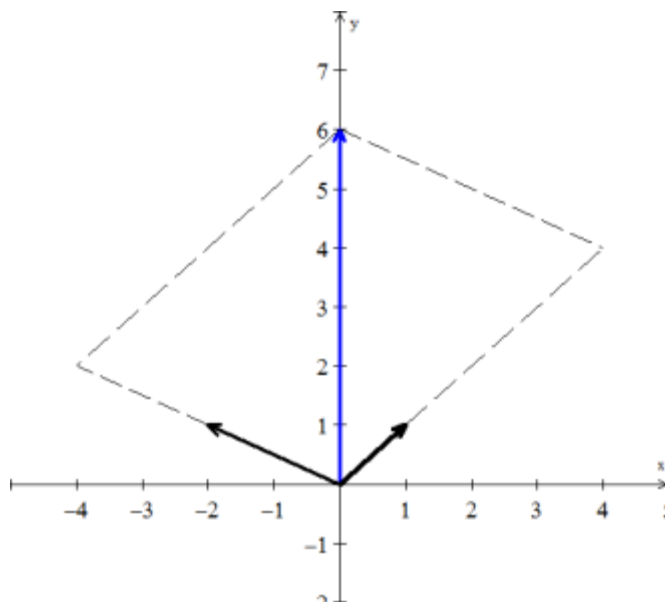
The matrix form of the 2 equations:

$$\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

**Row picture** is the 2 lines from the given equations and their intersection is the point  $(4, 2)$  which is the solution for the system.



**Column Picture** is the column vectors  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$



$$x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

The parallelogram shows how the solution vector  $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$  can be written as the linear combination of the column vectors.

### Exercise

Normally 4 planes in 4-dimensional space meet at a \_\_\_\_\_. Normally 4 column vectors in 4-dimensional space can combine to produce  $b$ . what combinations of

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$  produces  $b = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 2 \end{pmatrix}$ ?

What 4 equations for  $x$ ,  $y$ ,  $z$ ,  $w$  are you solving?

### Solution

Normally 4 planes in 4-dimensional space meet at a **point**.

The combination of the vectors producing  $b$  is:

$$0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 2 \end{pmatrix}$$

$$1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 2 \end{pmatrix}$$

The system of equations that satisfies the given vectors is:

$$\begin{cases} x + y + z + w = 3 \\ y + z + w = 3 \\ z + w = 3 \\ w = 2 \end{cases}$$

### Exercise

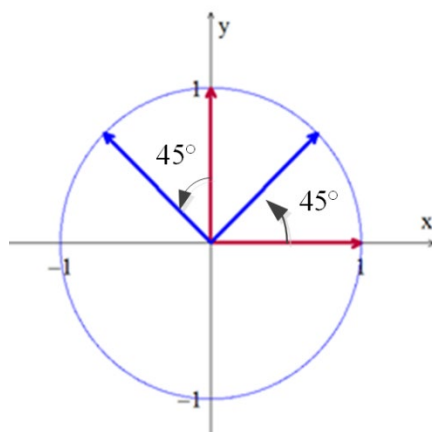
What 2 by 2 matrix  $A$  rotates every vector through  $45^\circ$  ?

The vector  $(1, 0)$  goes to  $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ . The vector  $(0, 1)$  goes to  $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ .

Those determine the matrix. Draw these particular vectors in the  $xy$ -plane and find  $A$ .

### Solution

$$A = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$



### Exercise

What two vectors are obtained by rotating the plane vectors  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  by  $30^\circ$  (cw) ?

Write a matrix  $A$  such that for every vector  $v$  in the plane,  $Av$  is the vector obtained by rotating  $v$  clockwise by  $30^\circ$ .

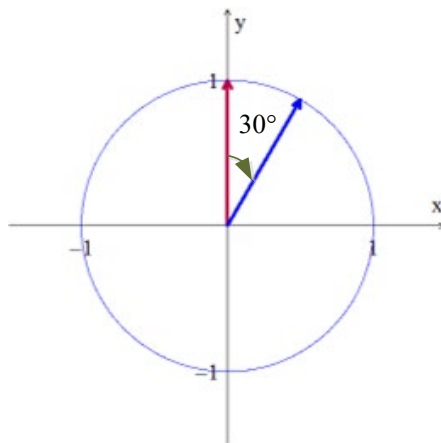
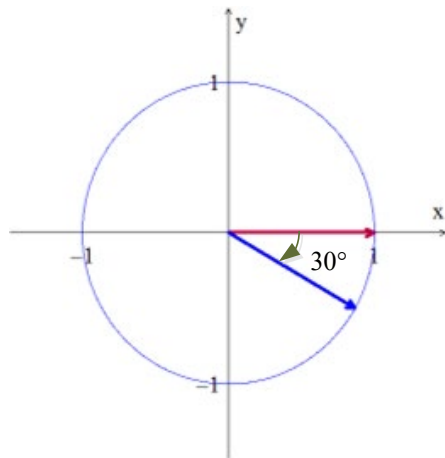
Find a matrix  $B$  such that for every 3-dimensional vector  $v$ , the vector  $Bv$  is the reflection of  $v$  through the plane  $x + y + z = 0$ . *Hint* :  $v = (1, 0, 0)$

### Solution

Rotating the vectors by  $30^\circ$  (cw) yields:

$$\text{For the vector } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ yields to } \begin{pmatrix} \cos(-30^\circ) \\ \sin(-30^\circ) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}$$

And for the vector  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  yields to  $\begin{pmatrix} \sin(30^\circ) \\ \cos(30^\circ) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$



The desired matrix is:  $A = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

To get 1 from  $\frac{\sqrt{3}}{2}$  is to multiply by  $\frac{2}{\sqrt{3}} = 2 \frac{1}{\sqrt{3}}$

The unit vector to the plane  $x + y + z = 0$  is  $\hat{u} = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$

$$\begin{aligned}
 Bv &= B \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{3}} \hat{u} \\
 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{3}} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \\
 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 B \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{\sqrt{3}} \hat{u} \\
 &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 B \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{3}} \hat{u} \\
 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \\
 &= \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix} \\
 &= \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}
 \end{aligned}$$

$$\text{The solution: } \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

**Exercise**

Find a system of linear equation corresponding to the given augmented matrix

$$\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$$

**Solution**

$$\begin{cases} 3x_2 - x_3 - x_4 = -1 \\ 5x_1 + 2x_2 - 3x_4 = -6 \end{cases}$$

**Exercise**

Find a system of linear equation corresponding to the given augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \\ 5 & -6 & 1 & 1 \\ -8 & 0 & 0 & 3 \end{bmatrix}$$

**Solution**

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ -4x_1 - 3x_2 - 2x_3 = -1 \\ 5x_1 - 6x_2 + x_3 = 1 \\ -8x_1 = 3 \end{cases}$$

**Exercise**

Find the augmented matrix for the given system of linear equations.

$$\begin{cases} -2x_1 = 6 \\ 3x_1 = 8 \\ 9x_1 = -3 \end{cases}$$

**Solution**

$$\left[ \begin{array}{c|c} -2 & 6 \\ 3 & 8 \\ 9 & -3 \end{array} \right]$$

**Exercise**

Find the augmented matrix for the given system of linear equations.

$$\begin{cases} 3x_1 - 2x_2 = -1 \\ 4x_1 + 5x_2 = 3 \\ 7x_1 + 3x_2 = 2 \end{cases}$$

**Solution**

$$\left[ \begin{array}{cc|c} 3 & -2 & -1 \\ 4 & 5 & 3 \\ 7 & 3 & 2 \end{array} \right]$$

**Exercise**

Find the augmented matrix for the given system of linear equations.

$$\begin{cases} 2x_1 + 2x_3 = 1 \\ 3x_1 - x_2 + 4x_3 = 7 \\ 6x_1 + x_2 - x_3 = 0 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 2 & 0 & 2 & 1 \\ 3 & -1 & 4 & 7 \\ 6 & 1 & -1 & 0 \end{array} \right]$$

**Exercise**

Determine the size of the matrix

$$\begin{bmatrix} 1 & 2 & -4 \\ 3 & 4 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

**Solution**

3 Rows & 3 columns

Size  $3 \times 3$

***Exercise***

Determine the size of the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

**Solution**

3 Rows &amp; 4 columns

Size  $3 \times 4$ ***Exercise***

Determine the size of the matrix

$$\begin{bmatrix} 2 & 1 & -1 & -1 \\ -6 & 2 & 0 & 1 \end{bmatrix}$$

**Solution**

2 Rows &amp; 4 columns

Size  $2 \times 4$



## ***Solution***      **Section 1.2 – Gaussian Elimination**

### ***Exercise***

When elimination is applied to the matrix  $A = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix}$

- a) What are the first and second pivots?
- b) What is the multiplier  $l_{21}$  in the first step ( $l_{21}$  times row 1 is subtracted from row 2)?
- c) What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3?
- d) What is the multiplier  $l_{31} = 0$ , subtracting 0 times row 1 from row 3?

### **Solution**

- a) The first pivot is 3 and when 2 times row 1 is subtracted from row 2, the second pivot is revealed as 7.

$$\begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix} \begin{array}{l} \text{subtract 2 times row.1} \\ \text{from row.2} \end{array} \begin{bmatrix} 3 & 1 & 0 \\ 0 & \boxed{7} & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

- b) The multiplier  $l_{21}$  in the first step is  $\frac{6}{3} = 2$ .
- c) If we reduce the entry 9 to 2, that drop of 7 in the  $a_{22}$  position would force a row exchange.

$$\begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix} \begin{array}{l} \text{subtract 7 times row.1} \\ \text{from row.2} \end{array} \begin{bmatrix} 3 & 1 & 0 \\ -15 & \boxed{2} & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

- d) The multiplier  $l_{31}$  is already zero because  $a_{31} = 0$  and no needs row elimination.

### ***Exercise***

Use elimination to reach upper triangular matrices  $U$ . Solve by back substitution or explain why this impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the  $-x$  in equation (3).

$$\begin{cases} x + y + z = 7 \\ x + y - z = 5 \\ x - y + z = 3 \end{cases} \qquad \begin{cases} x + y + z = 7 \\ x + y - z = 5 \\ -x - y + z = 3 \end{cases}$$

### **Solution**

For the *first* system:

$$\begin{array}{lll}
x + y + z = 7 & \text{subtract eqn.1} & x + y + z = 7 \\
x + y - z = 5 & \text{from eqn.2} & 0y - 2z = -2 \\
x - y + z = 3 & \text{from eqn.3} & -2y - 0z = -4 \\
\\ 
x + y + z = 7 & & 1x + y + z = 7 \\
x + y - z = 5 & \text{Exchange eqn.2} & -2y - 0z = -4 \\
x - y + z = 3 & \text{and eqn.3} & -2z = -2
\end{array}$$

The solutions are:  $z = 1$   $y = 2$   $x = 4$  and the pivots are 1, -2, -2.

For the *second* system:

$$\begin{array}{lll}
x + y + z = 7 & \text{Subtract eqn.1} & x + y + z = 7 \\
x + y - z = 5 & \text{from eqn.2} & 0y - 2z = -2 \\
-x - y + z = 3 & \text{Add eqn.1} & 0y + 2z = 10 \\
& \text{to eqn.3} & \\
\\ 
x + y + z = 7 & & x + y + z = 7 \\
0y - 2z = -2 & \text{Add eqn.2} & 0y - 2z = -2 \\
0y + 2z = 10 & \text{to eqn.3} & \boxed{0z = 8}
\end{array}$$

The three planes don't meet. But if we change '3' in the last equation to '-5'

$$\begin{array}{lll}
x + y + z = 7 & \text{Subtract eqn.1} & x + y + z = 7 \\
x + y - z = 5 & \text{from eqn.2} & 0y - 2z = -2 \\
-x - y + z = -5 & \text{Add eqn.1} & 0y + 2z = 2 \\
& \text{to eqn.3} & \\
\\ 
x + y + z = 7 & & x + y = 6 \\
0y - 2z = -2 & & \\
0y + 2z = 10 & & z = 1
\end{array}$$

There are unique infinite many solutions!

The three planes now meet along a whole line.

**Exercise**

For which numbers  $a$  does the elimination break down (1) permanently (2) temporarily

$$ax + 3y = -3$$

$$4x + 6y = 6$$

Solve for  $x$  and  $y$  after fixing the second breakdown by a row change.

**Solution**

The matrix form is:  $\begin{pmatrix} a & 3 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$

If  $a = 0$ , the elimination brakes down temporarily.

$$\begin{pmatrix} 4 & 6 \\ 0 & \boxed{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

The system is in upper triangular form and entry row 2 column 2 is not equal to zero, therefore the system has a solution.

If  $a \neq 0$ ,

$$\begin{pmatrix} a & 3 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} \quad R_2 - \frac{4}{a}R_1$$

$$\begin{pmatrix} a & 3 \\ 0 & 6 - \frac{12}{a} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 + \frac{12}{a} \end{pmatrix}$$

$$6 - \frac{12}{a} = 0$$

$$\frac{12}{a} = 6$$

$$a = \frac{12}{6}$$

$$\boxed{= 2}$$

If  $a = 2$ ,

$$\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \end{pmatrix}$$

The system will fail and has no solution.

If  $a \neq 2$ ;

$$\begin{pmatrix} a & 3 \\ 0 & 6 - \frac{12}{a} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 + \frac{12}{a} \end{pmatrix}$$

The system has a unique solution.

**Exercise**

Find the pivots and the solution for these four equations:

$$2x + y = 0$$

$$x + 2y + z = 0$$

$$y + 2z + t = 0$$

$$z + 2t = 5$$

**Solution**

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} \quad R_2 - \frac{1}{2}R_1$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 1.5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} \quad R_3 - \frac{2}{3}R_2$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} \quad R_4 - \frac{3}{4}R_3$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 5 \end{pmatrix} \quad \begin{array}{l} 2x = -y \text{ (1)} \\ \frac{3}{2}y + z = 0 \\ \frac{4}{3}z + t = 0 \\ \frac{5}{4}t = 5 \text{ (4)} \end{array}$$

$$\left\{ \begin{array}{ll} \text{(1)} & \rightarrow x = -2\frac{1}{2} = \underline{-1} \\ y = -z\frac{2}{3} = -(-3)\frac{2}{3} & \rightarrow y = \underline{2} \\ \frac{4}{3}z = -t & \rightarrow z = -4\frac{3}{4} = \underline{-3} \\ \text{(4)} & \rightarrow t = \underline{4} \end{array} \right.$$

The pivots are diagonal entries and the solution is:  $(-1, 2, -3, 4)$

**Exercise**

Look for a matrix that has row sums 4 and 8, and column sums 2 and  $s$ .

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{array}{ll} a + b = 4 & a + c = 2 \\ c + d = 8 & b + d = s \end{array}$$

The four equations are solvable only if  $s = \underline{\hspace{1cm}}$ . Then find two different matrices that have the correct row and column sums.

**Solution**

$$\begin{array}{r} a + b = 4 \\ + \quad c + d = 8 \\ \hline a + c + b + d = 12 \end{array}$$

$$2 + s = 12$$

$$\underline{s = 10}$$

**Exercise**

Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of  $A$  is a \_\_\_\_\_ of the first two rows. Find a third equation that can't be solved together with  $x + y + z = 0$  and  $x - 2y - z = 1$

**Solution**

The system is singular if row 3 of  $A$  is a **linear combination** of the first two rows.

There are many possible of a third equation that can't be solved together with  $x + y + z = 0$  and  $x - 2y - z = 1$ .

$$\begin{array}{rcl} 3 \text{ times } 1^{\text{st}} \text{ equation} & 3x + 3y + 3z & \\ \text{minus } 2^{\text{nd}} & -x + 2y + z & \\ \hline & 2x + 5y + 4z = 1 & \end{array}$$

**Exercise**

Use the Gauss-Jordan method to solve the system

$$x - y + 5z = -6$$

$$3x + 3y - z = 10$$

$$x + 3y + 2z = 5$$

**Solution**

$$\left[ \begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array} \right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{ccc|c} 3 & 3 & -1 & 10 \\ -3 & 3 & -15 & 18 \\ \hline 0 & 6 & -16 & 28 \end{array} \quad \begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ -1 & 1 & -5 & 6 \\ \hline 0 & 4 & -3 & 11 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 4 & -3 & 11 \end{array} \right] \frac{1}{6}R_2$$

$$0 \quad 1 \quad -\frac{8}{3} \quad \frac{14}{3}$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 4 & -3 & 11 \end{array} \right] \begin{array}{l} R_1 + R_2 \\ \\ R_3 - 4R_2 \end{array}$$

$$\begin{array}{ccc|c} 0 & 4 & -3 & 11 \\ 0 & -4 & \frac{32}{3} & -\frac{56}{3} \\ \hline 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array} \quad \begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ \hline 1 & 0 & \frac{7}{3} & -\frac{4}{3} \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array} \right] \frac{3}{23}R_3$$

$$0 \quad 0 \quad 1 \quad -1$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & 1 & -1 \end{array} \right] \begin{array}{l} R_1 - \frac{7}{3}R_3 \\ R_2 + \frac{8}{3}R_3 \\ \end{array}$$

$$\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 0 & -\frac{7}{3} & \frac{7}{3} \\ \hline 1 & 0 & 0 & 1 \end{array} \quad \begin{array}{ccc|c} 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & \frac{8}{3} & -\frac{8}{3} \\ \hline 0 & 1 & 0 & 2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

**Solution:**  $\underline{(1, 2, -1)}$

**Exercise**

Use the Gauss-Jordan method to solve the system

$$\begin{cases} 2x - y + 4z = -3 \\ x - 2y - 10z = -6 \\ 3x \quad \quad + 4z = 7 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 2 & -1 & 4 & -3 \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array} \right] \frac{1}{2}R_1$$

$$1 \quad -\frac{1}{2} \quad 2 \quad -\frac{3}{2}$$

$$\left[ \begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array} \right] \begin{array}{l} R_2 - R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{ccc|c} 1 & -2 & -10 & -6 \\ -1 & \frac{1}{2} & -2 & \frac{3}{2} \\ \hline 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \end{array} \quad \begin{array}{ccc|c} 3 & 0 & 4 & 7 \\ -3 & \frac{3}{2} & -6 & \frac{9}{2} \\ \hline 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array} \right] -\frac{2}{3}R_2$$

$$0 \quad 1 \quad 8 \quad 3$$

$$\left[ \begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & 1 & 8 & 3 \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array} \right] \begin{array}{l} R_1 + \frac{1}{2}R_2 \\ R_3 - \frac{3}{2}R_2 \end{array}$$

$$\begin{array}{ccc|c} 0 & \frac{3}{2} & -2 & \frac{23}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ \hline 0 & 0 & -14 & 7 \end{array} \quad \begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & \frac{1}{2} & 4 & \frac{3}{2} \\ \hline 1 & 0 & 6 & 0 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & -14 & 7 \end{array} \right] -\frac{1}{14}R_3$$

$$0 \quad 0 \quad 1 \quad -\frac{1}{2}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array} \right] \begin{array}{l} R_1 - 6R_3 \\ R_2 - 8R_3 \end{array}$$

$$\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 0 & -6 & 3 \\ \hline 1 & 0 & 0 & 3 \end{array} \quad \begin{array}{ccc|c} 0 & 1 & 8 & 3 \\ 0 & 0 & -8 & 4 \\ \hline 0 & 1 & 0 & 7 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array} \right]$$

**Solution:**  $\left( 3, 7, -\frac{1}{2} \right)$

**Exercise**

Use the Gauss-Jordan method to solve the system 
$$\begin{cases} 4x + 3y - 5z = -29 \\ 3x - 7y - z = -19 \\ 2x + 5y + 2z = -10 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 4 & 3 & -5 & -29 \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array} \right] \frac{1}{4}R_1$$

$$1 \quad \frac{3}{4} \quad -\frac{5}{4} \quad -\frac{29}{4}$$

$$\left[ \begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array} \right] \begin{array}{l} R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & -7 & -1 & -19 \\ -3 & -\frac{9}{4} & \frac{15}{4} & \frac{87}{4} \\ 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \end{array} \quad \begin{array}{cccc} 2 & 5 & 2 & -10 \\ -2 & -\frac{3}{2} & \frac{5}{2} & \frac{29}{2} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array} \right] -\frac{4}{37}R_2$$

$$0 \quad 1 \quad -\frac{11}{37} \quad -\frac{11}{37}$$

$$\left[ \begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array} \right] \begin{array}{l} R_1 - \frac{3}{4}R_2 \\ R_3 - \frac{7}{2}R_2 \end{array}$$

$$\begin{array}{cccc} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{3}{4} & \frac{33}{148} & \frac{33}{148} \\ 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \end{array} \quad \begin{array}{cccc} 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \\ 0 & -\frac{7}{2} & \frac{77}{72} & \frac{77}{72} \\ 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array} \right] \frac{72}{401}R_3$$

$$0 \quad 0 \quad 1 \quad 1$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & 1 & 1 \end{array} \right] \begin{array}{l} R_1 + \frac{38}{37}R_3 \\ R_2 + \frac{11}{37}R_3 \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 0 & \frac{38}{37} & \frac{38}{37} \\ 1 & 0 & 0 & -6 \end{array} \quad \begin{array}{cccc} 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{11}{37} & \frac{11}{37} \\ 0 & 1 & 0 & 0 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

**Solution:**  $\underline{(-6, 0, 1)}$



**Exercise**

Use the Gauss-Jordan method to solve the system 
$$\begin{cases} x + 2y - 3z = -15 \\ 2x - 3y + 4z = 18 \\ -3x + y + z = 1 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 3R_1 \end{array}$$

$$\begin{array}{cccc} -2 & -4 & 6 & 30 \\ 2 & -3 & 4 & 18 \\ 0 & -7 & 10 & 48 \end{array} \quad \begin{array}{cccc} 3 & 6 & -9 & -45 \\ -3 & 1 & 1 & 1 \\ 0 & 7 & -8 & -44 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \end{array} \right] -\frac{1}{7}R_2$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 7 & -8 & -44 \end{array} \right] \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 - 7R_2 \end{array}$$

$$\begin{array}{cccc} 1 & 2 & -3 & -15 \\ 0 & -2 & \frac{20}{7} & \frac{96}{7} \\ 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \end{array} \quad \begin{array}{cccc} 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \\ 0 & 0 & 2 & 4 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 2 & 4 \end{array} \right] \frac{1}{2}R_3$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 1 & 2 \end{array} \right] \begin{array}{l} R_1 + \frac{1}{7}R_3 \\ R_2 + \frac{10}{7}R_3 \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 0 & \frac{1}{7} & \frac{2}{7} \\ 1 & 0 & 0 & -1 \end{array} \quad \begin{array}{cccc} 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & \frac{10}{7} & \frac{20}{7} \\ 0 & 1 & 0 & -4 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

**Solution:**  $\underline{(-1, -4, 2)}$

**Exercise**

Use the Gauss-Jordan method to solve the system 
$$\begin{cases} x + 2y + 3z = 10 \\ 4x + 5y + 6z = 11 \\ 7x + 8y + 9z = 12 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \end{array} \right] \begin{array}{l} \\ R_2 - 4R_1 \\ R_3 - 7R_1 \end{array} \quad \begin{array}{cccc} -4 & -8 & -12 & -40 \\ 4 & 5 & 6 & 11 \\ 0 & -3 & -6 & -29 \end{array} \quad \begin{array}{cccc} -7 & -14 & -21 & -70 \\ 7 & 8 & 9 & 12 \\ 0 & -6 & -12 & -58 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -3 & -6 & -29 \\ 0 & -6 & -12 & -58 \end{array} \right] \frac{1}{3} R_2$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & \frac{29}{3} \\ 0 & -6 & -12 & -58 \end{array} \right] R_3 + 6R_2 \quad \begin{array}{cccc} 0 & -6 & -12 & -58 \\ 0 & 6 & 12 & 58 \\ 0 & 0 & 0 & 0 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & \frac{29}{3} \\ 0 & 0 & 0 & 0 \end{array} \right]$$

let  $z$  be the variable

$$\text{From Row 1} \Rightarrow y + 2z = \frac{29}{3}$$

$$\underline{y = \frac{29}{3} - 2z}$$

$$\text{From Row 1} \Rightarrow x + 2y + 3z = 10$$

$$x = 10 - 2y - 3z$$

$$x = 10 - 2\left(\frac{29}{3} - 2z\right) - 3z$$

$$x = 10 - \frac{58}{3} + 4z - 3z$$

$$\underline{x = z - \frac{28}{3}}$$

$$\text{Solution: } \underline{\left( z - \frac{28}{3}, \frac{29}{3} - 2z, z \right)}$$

**Exercise**

Use the Gauss-Jordan method to solve the system 
$$\begin{cases} 2x + y + 2z = 4 \\ 2x + 2y = 5 \\ 2x - y + 6z = 2 \end{cases}$$

**Solution**

$$\begin{aligned} & \left[ \begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array} \right] \begin{array}{l} \frac{1}{2}R_1 \\ \\ \end{array} \quad \begin{array}{cccc} & & 1 & \frac{1}{2} \\ & & & 1 \\ & & & 2 \end{array} \\ & \left[ \begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array} \quad \begin{array}{cccc} 2 & 2 & 0 & 5 \\ -2 & -1 & -2 & -4 \\ 0 & 1 & -2 & 1 \end{array} \quad \begin{array}{cccc} 2 & -1 & 6 & 2 \\ -2 & -1 & -2 & -4 \\ 0 & -2 & 4 & -2 \end{array} \\ & \left[ \begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & -2 \end{array} \right] \begin{array}{l} R_1 - \frac{1}{2}R_2 \\ \\ R_3 + 2R_2 \end{array} \quad \begin{array}{cccc} 1 & \frac{1}{2} & 1 & 2 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 1 & 0 & 2 & \frac{3}{2} \end{array} \quad \begin{array}{cccc} 0 & -2 & 4 & -2 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array} \\ & \left[ \begin{array}{ccc|c} 1 & 0 & 2 & \frac{3}{2} \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

From Row 3:  $0 = 0$  is a true statement. Let  $z$  be the variable.

From Row 2:  $y - 2z = 1$

$$\underline{y = 1 + 2z}$$

From Row 1:  $x + 2z = \frac{3}{2}$

$$\underline{x = -2z + \frac{3}{2}}$$

***∴ Solution:***  $\underline{\left( -2z + \frac{3}{2}, 2z + 1, z \right)}$

**Exercise**

Use the Gauss-Jordan method to solve the system

$$\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array} \right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array} \quad \begin{array}{cccc} -1 & -2 & 3 & 1 \\ 1 & 1 & 2 & 8 \\ \hline 0 & -1 & 5 & 9 \end{array} \quad \begin{array}{cccc} 3 & -7 & 4 & 10 \\ -3 & -3 & -6 & -24 \\ \hline 0 & -10 & -2 & -14 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array} \right] -R_2$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array} \right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 10R_2 \end{array} \quad \begin{array}{cccc} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ \hline 1 & 0 & 7 & 17 \end{array} \quad \begin{array}{cccc} 0 & -10 & -2 & -14 \\ 0 & 10 & -50 & -90 \\ \hline 0 & 0 & -52 & -104 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array} \right] -\frac{1}{52}R_3$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array} \right] \begin{array}{l} R_1 - 7R_3 \\ R_2 + 5R_3 \\ \end{array} \quad \begin{array}{cccc} 1 & 0 & 7 & 17 \\ 0 & 0 & -7 & -14 \\ \hline 1 & 0 & 0 & 3 \end{array} \quad \begin{array}{cccc} 0 & 1 & -5 & -9 \\ 0 & 0 & 5 & 10 \\ \hline 0 & 1 & 0 & 1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

 **$\therefore$  Solution:** (3, 1, 2)**Exercise**

Use the Gauss-Jordan method to solve the system

$$\begin{cases} x + 2y + z = 8 \\ -x + 3y - 2z = 1 \\ 3x + 4y - 7z = 10 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ -1 & 3 & -2 & 1 \\ 3 & 4 & -7 & 10 \end{array} \right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & -2 & -10 & -14 \end{array} \right] \begin{array}{l} 5R_1 - 2R_2 \\ \\ 5R_3 + 2R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 5 & 0 & 7 & 22 \\ 0 & 5 & -1 & 9 \\ 0 & 0 & -52 & -52 \end{array} \right] \begin{array}{l} (2) \\ (1) \\ \rightarrow \underline{z=1} \end{array}$$

$$(1) \rightarrow 5y = 10$$

$$\underline{y=2}$$

$$(1) \rightarrow 5x = 22 - 7$$

$$\underline{x=3}$$

$$\therefore \text{Solution: } \underline{(3, 2, 1)}$$

### Exercise

Use augmented elimination to solve linear system

$$2x - 5y + 3z = 1$$

$$x - 2y - 2z = 8$$

### Solution

$$\left[ \begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 2 & -5 & 3 & 1 \end{array} \right] R_2 - 2R_1$$

$$\begin{array}{cccc} 2 & -5 & 3 & 1 \\ -2 & 4 & 4 & -16 \\ \hline 0 & -1 & 7 & -15 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & -1 & 7 & -15 \end{array} \right] -R_2$$

$$\left[ \begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & 1 & -7 & 15 \end{array} \right] R_1 + 2R_2$$

$$\begin{array}{cccc} 1 & -2 & -2 & 8 \\ 0 & 2 & -14 & 30 \\ \hline 1 & 0 & -16 & 38 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -16 & 38 \\ 0 & 1 & -7 & 15 \end{array} \right] \begin{array}{l} \rightarrow x - 16z = 38 \\ \rightarrow y - 7z = 15 \end{array}$$

$$\begin{cases} x = 16z + 38 \\ y = 7z + 15 \end{cases}$$

$$\therefore \text{Solution: } \underline{(16z + 38, 7z + 15, z)}$$

**Exercise**

Use augmented elimination to solve linear system 
$$\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 5 \\ 1 & -1 & 1 & -2 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array} \quad \begin{array}{cccc} 2 & 1 & -1 & 5 \\ -2 & -2 & -2 & -4 \\ 0 & -1 & -3 & 1 \end{array} \quad \begin{array}{cccc} 1 & -1 & 1 & -2 \\ -1 & -1 & -1 & -2 \\ 0 & -2 & 0 & -4 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & 1 \\ 0 & -2 & 0 & -4 \end{array} \right] \begin{array}{l} (2) \\ (1) \\ -2y = -4 \end{array}$$

$$\underline{y = 2}$$

$$(1) \rightarrow -y - 3z = 1$$

$$3z = -1 - 2$$

$$\underline{z = -1}$$

$$(2) \rightarrow x + y + z = 2$$

$$x = 2 - 2 + 1$$

$$\underline{x = 1}$$

$$\therefore \text{Solution: } \underline{(1, 2, -1)}$$

**Exercise**

Use augmented elimination to solve linear system 
$$\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ -1 & -1 & 1 & 1 \\ 3 & -1 & 1 & 9 \end{array} \right] \begin{array}{l} \\ 2R_2 + R_1 \\ 2R_3 - 3R_1 \end{array} \quad \begin{array}{cccc} -2 & -2 & 2 & 2 \\ 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \end{array} \quad \begin{array}{cccc} 6 & -2 & 2 & 18 \\ -6 & -3 & -3 & -27 \\ 0 & -5 & -1 & -9 \end{array}$$

$$\left[ \begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \\ 0 & -5 & -1 & -9 \end{array} \right] \begin{array}{l} \\ \\ R_3 - 5R_2 \end{array} \quad \begin{array}{cccc} 0 & -5 & -1 & -9 \\ 0 & 5 & -15 & -55 \\ 0 & 0 & -16 & -64 \end{array}$$

$$\left[ \begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \\ 0 & 0 & -16 & -64 \end{array} \right] \quad \begin{array}{l} (2) \\ (1) \end{array}$$

$$-16z = -64$$

$$\underline{z = 4}$$

$$(1) \rightarrow -y + 3z = 11$$

$$y = 12 - 11$$

$$\underline{y = 1}$$

$$(2) \rightarrow 2x + y + z = 9$$

$$2x = 9 - 1 - 4$$

$$\underline{x = 2}$$

$$\therefore \text{Solution: } \underline{(2, 1, 4)}$$

### Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$$

### Solution

$$\left[ \begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ -3 & 6 & 2 & 11 \end{array} \right] \quad R_3 + 3R_1$$

$$\begin{array}{cccc} -3 & 6 & 2 & 11 \\ 3 & 15 & -3 & -12 \\ \hline 0 & 21 & -1 & -1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ 0 & 21 & -1 & -1 \end{array} \right] \quad R_3 - 7R_2$$

$$\begin{array}{cccc} 0 & 21 & -1 & -1 \\ 0 & -21 & 7 & 7 \\ \hline 0 & 0 & 6 & 6 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ 0 & 0 & 6 & 6 \end{array} \right] \quad \begin{array}{l} \rightarrow x + 5y - z = -4 \quad (2) \\ \rightarrow 3y - z = -1 \quad (1) \\ \rightarrow 6z = 6 \end{array}$$

$$\underline{z = 1}$$

$$(1) \rightarrow 3y = -1 + 1$$

$$\underline{y = 0}$$

$$(2) \rightarrow x = -4 + 1$$

$$\underline{x = -3}$$

$$\therefore \text{Solution: } \underline{(-3, 0, 1)}$$

**Exercise**

Use augmented elimination to solve linear system 
$$\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 2 & -3 & 2 & 10 \\ 3 & -1 & 1 & 9 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \quad \begin{array}{cccc} 2 & -3 & 2 & 10 \\ -2 & -6 & -8 & -28 \\ \hline 0 & -9 & -6 & -18 \end{array} \quad \begin{array}{cccc} 3 & -1 & 1 & 9 \\ -3 & -9 & -12 & -42 \\ \hline 0 & -10 & -11 & -33 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -9 & -6 & -18 \\ 0 & -10 & -11 & -33 \end{array} \right] \begin{array}{l} \\ \\ 9R_3 - 10R_2 \end{array} \quad \begin{array}{cccc} 0 & -90 & -99 & -297 \\ 0 & 90 & 60 & 180 \\ \hline 0 & 0 & -39 & -117 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -9 & -6 & -18 \\ 0 & 0 & -39 & -117 \end{array} \right] \begin{array}{l} x + 3y + 4z = 14 \quad (3) \\ -9y - 6z = -18 \quad (2) \\ -39z = -117 \quad (1) \end{array}$$

$$(1) \rightarrow z = \frac{117}{39} \\ \quad \quad \quad \underline{= 3}$$

$$(2) \rightarrow 9y = 18 - 6(3) \\ \quad \quad \quad 9y = 0 \\ \quad \quad \quad \underline{y = 0}$$

$$(3) \rightarrow x = 14 - 12 \\ \quad \quad \quad \underline{x = 2}$$

$$\therefore \text{Solution: } \underline{(2, 0, 3)}$$

**Exercise**

Use augmented elimination to solve linear system 
$$\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 3 & 2 & 1 & 8 \\ 2 & -3 & 2 & -16 \end{array} \right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \quad \begin{array}{cccc} 3 & 2 & 1 & 8 \\ -3 & -12 & 3 & -60 \\ \hline 0 & -10 & 4 & -52 \end{array} \quad \begin{array}{cccc} 2 & -3 & 2 & -16 \\ -2 & -8 & 2 & -40 \\ \hline 0 & -11 & 4 & -56 \end{array}$$



$$\left[ \begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 0 & -10 & 4 & -52 \\ 0 & -11 & 4 & -56 \end{array} \right] \quad 10R_3 - 11R_2 \quad \begin{array}{cccc} 0 & -110 & 40 & -560 \\ 0 & 110 & -44 & 572 \\ \hline 0 & 0 & -4 & 12 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 0 & -10 & 4 & -52 \\ 0 & 0 & -4 & 12 \end{array} \right] \quad \begin{array}{l} x + 4y - z = 20 \quad (3) \\ -10y + 4z = -52 \quad (2) \\ -4z = 12 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -3}$$

$$(2) \rightarrow -10y = -52 + 12$$

$$-10y = -40$$

$$\underline{y = 4}$$

$$(3) \rightarrow x = 20 - 16 - 3$$

$$\underline{x = 1}$$

$$\therefore \text{Solution: } \underline{(1, 4, -3)}$$

### Exercise

Use augmented elimination to solve linear system 
$$\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$$

### Solution

$$\left[ \begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 2 & -1 & 7 \\ 2 & -3 & 2 & -1 \end{array} \right] \quad R_3 - 2R_1 \quad \begin{array}{cccc} 2 & -3 & 2 & -1 \\ -2 & -4 & -2 & -34 \\ \hline 0 & -7 & 0 & -35 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 2 & -1 & 7 \\ 0 & -7 & 0 & -35 \end{array} \right] \quad \begin{array}{l} x + 2y + z = 17 \quad (3) \\ 2y - z = 7 \quad (2) \\ -7y = -35 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{y = 5}$$

$$(2) \rightarrow z = 10 - 7$$

$$\underline{z = 3}$$

$$(3) \rightarrow x = 17 - 10 - 3$$

$$\underline{x = 4}$$

$$\therefore \text{Solution: } \underline{(4, 5, 3)}$$

**Exercise**

Use augmented elimination to solve linear system 
$$\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ -4 & 5 & 3 & 7 \\ -6 & 3 & 5 & -4 \end{array} \right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \quad \begin{array}{cccc} -4 & 5 & 3 & 7 \\ 4 & -12 & -14 & -6 \\ 0 & -7 & -11 & 1 \end{array} \quad \begin{array}{cccc} -6 & 3 & 5 & -4 \\ 6 & -18 & -21 & -9 \\ 0 & -15 & -16 & -13 \end{array}$$

$$\left[ \begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ 0 & -7 & -11 & 1 \\ 0 & -15 & -16 & -13 \end{array} \right] 7R_3 - 15R_1 \quad \begin{array}{cccc} 0 & -105 & -112 & -91 \\ 0 & 105 & 165 & -15 \\ 0 & 0 & 53 & -106 \end{array}$$

$$\left[ \begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ 0 & -7 & -11 & 1 \\ 0 & 0 & 53 & -106 \end{array} \right] \begin{array}{l} -2x + 6y + 7z = 3 \quad (3) \\ -7y - 11z = 1 \quad (2) \\ 53z = -106 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -2}$$

$$(2) \rightarrow -7y = 1 - 22$$

$$-7y = -21$$

$$\underline{y = 3}$$

$$(3) \rightarrow -2x = 3 - 18 + 14$$

$$-2x = -1$$

$$\underline{x = \frac{1}{2}}$$

$$\therefore \text{Solution: } \underline{\left( \frac{1}{2}, 3, -2 \right)}$$

**Exercise**

Use augmented elimination to solve linear system 
$$\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 3 & -3 & 4 & 5 \\ 4 & -2 & 3 & 4 \end{array} \right] \begin{array}{l} 2R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \quad \begin{array}{cccc} 6 & -6 & 8 & 10 \\ -6 & 3 & -3 & -3 \\ 0 & -3 & 5 & 7 \end{array} \quad \begin{array}{cccc} 4 & -2 & 3 & 4 \\ -4 & 2 & -2 & -2 \\ 0 & 0 & 1 & 2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 0 & -3 & 5 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right] \quad \begin{array}{l} 2x - y + z = 1 \quad (2) \\ -3y + 5z = 7 \quad (1) \\ \underline{z = 2} \end{array}$$

$$(1) \rightarrow -3y = 7 - 10$$

$$-3y = -3$$

$$\underline{y = 1}$$

$$(3) \rightarrow 2x = 1 + 1 - 2$$

$$\underline{x = 0}$$

$$\therefore \text{Solution: } \underline{(0, 1, 2)}$$

### Exercise

Use augmented elimination to solve linear system 
$$\begin{cases} 3x - 4y + 4z = 7 \\ x - y - 2z = 2 \\ 2x - 3y + 6z = 5 \end{cases}$$

### Solution

$$\left[ \begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 3 & -4 & 4 & 7 \\ 2 & -3 & 6 & 5 \end{array} \right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \quad \begin{array}{cccc} 3 & -4 & 4 & 7 \\ -3 & 3 & 6 & -6 \\ \hline 0 & -1 & 10 & 1 \end{array} \quad \begin{array}{cccc} 2 & -3 & 6 & 5 \\ -2 & 2 & 4 & -4 \\ \hline 0 & -1 & 10 & 1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & -1 & 10 & 1 \\ 0 & -1 & 10 & 1 \end{array} \right] \quad \begin{array}{l} \rightarrow x - y - 2z = 2 \quad (2) \\ \rightarrow -y + 10z = 1 \quad (1) \\ R_3 = R_2 \end{array}$$

$$(1) \rightarrow \underline{y = 10z - 1}$$

$$(2) \rightarrow x = 2 + 10z - 1 + 2z \\ \underline{= 12z + 1}$$

$$\therefore \text{Solution: } \underline{(12z + 1, 10z - 1, z)}$$

### Exercise

Use augmented elimination to solve linear system 
$$\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$$

### Solution

$$\left[ \begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 2 & -1 & 1 & 4 \\ -1 & 1 & 1 & 4 \end{array} \right] \begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \end{array} \quad \begin{array}{cccc} 2 & -1 & 1 & 4 \\ -2 & 4 & 2 & -4 \\ \hline 0 & 3 & 3 & 0 \end{array} \quad \begin{array}{cccc} -1 & 1 & 1 & 4 \\ 1 & -2 & -1 & 2 \\ \hline 0 & -1 & 0 & 6 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & 0 & 6 \end{array} \right] \begin{array}{l} x - 2y - z = 2 \quad (3) \\ 3y + 3z = 0 \quad (2) \\ -y = 6 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{y = -6}$$

$$(2) \rightarrow \underline{z = -y}$$

$$\underline{= 6}$$

$$(3) \rightarrow \underline{x = 2 - 12 + 6}$$

$$\underline{= -4}$$

$$\therefore \text{Solution: } \underline{(-4, -6, 6)}$$

### Exercise

Use augmented elimination to solve linear system  $\begin{cases} x + y + z = 3 \\ -y + 2z = 1 \\ -x + z = 0 \end{cases}$

### Solution

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ -1 & 0 & 1 & 0 \end{array} \right] R_3 + R_1 \quad \begin{array}{cccc} -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 3 \\ \hline 0 & 1 & 2 & 3 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{array} \right] R_3 + R_2 \quad \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 0 & -1 & 2 & 1 \\ \hline 0 & 0 & 4 & 4 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 4 & 4 \end{array} \right] \begin{array}{l} x + y + z = 3 \quad (3) \\ -y + 2z = 1 \quad (2) \\ \underline{z = 1} \quad (1) \end{array}$$

$$(2) \rightarrow \underline{-y = 1 - 2}$$

$$\underline{y = 1}$$

$$(3) \rightarrow \underline{x = 3 - 1 - 1}$$

$$\underline{= 1}$$

$$\therefore \text{Solution: } \underline{(1, 1, 1)}$$

**Exercise**

Use augmented elimination to solve linear system 
$$\begin{cases} 3x + y + 3z = 14 \\ 7x + 5y + 8z = 37 \\ x + 3y + 2z = 9 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 3 & 1 & 3 & 14 \\ 7 & 5 & 8 & 37 \end{array} \right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 7R_1 \end{array} \quad \begin{array}{cccc} 3 & 1 & 3 & 14 \\ -3 & -9 & -6 & -27 \\ 0 & -8 & -3 & -13 \end{array} \quad \begin{array}{cccc} 7 & 5 & 8 & 37 \\ -7 & -21 & -14 & -63 \\ 0 & -16 & -6 & -26 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -8 & -3 & -13 \\ 0 & -16 & -6 & -26 \end{array} \right] \begin{array}{l} \\ \\ R_3 - 2R_2 \end{array} \quad \begin{array}{cccc} 0 & -16 & -6 & -26 \\ 0 & 16 & 6 & 26 \\ 0 & 0 & 0 & 0 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -8 & -3 & -13 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} x + 3y + 2z = 9 \quad (2) \\ -8y - 3z = -13 \quad (1) \\ \end{array}$$

$$(1) \rightarrow -8y = 3z - 13$$

$$\underline{y = -\frac{3}{8}z + \frac{13}{8} \quad |}$$

$$(3) \rightarrow x = 9 - 3\left(\frac{13}{8} - \frac{3}{8}z\right) - 2z$$

$$= 9 - \frac{39}{8} + \frac{9}{8}z - 2z$$

$$\underline{= \frac{33}{8} - \frac{7}{8}z \quad |}$$

$$\therefore \text{Solution: } \underline{\left( \frac{33}{8} - \frac{7}{8}z, \frac{13}{8} - \frac{3}{8}z, z \right) \quad |}$$

**Exercise**

Use augmented elimination to solve linear system 
$$\begin{cases} 4x - 2y + z = 7 \\ x + y + z = -2 \\ 4x + 2y + z = 3 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 4 & -2 & 1 & 7 \\ 4 & 2 & 1 & 3 \end{array} \right] \begin{array}{l} \\ R_2 - 4R_1 \\ R_3 - 4R_1 \end{array} \quad \begin{array}{cccc} 4 & -2 & 1 & 7 \\ -4 & -4 & -4 & 8 \\ 0 & -6 & -3 & 15 \end{array} \quad \begin{array}{cccc} 4 & 2 & 1 & 3 \\ -4 & -4 & -4 & 8 \\ 0 & -2 & -3 & 11 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 15 \\ 0 & -2 & -3 & 11 \end{array} \right] \quad -3R_3 + R_2 \quad \begin{array}{ccc|c} 0 & 6 & 9 & -33 \\ 0 & -6 & -3 & 15 \\ \hline 0 & 0 & 6 & -18 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 15 \\ 0 & 0 & 6 & -18 \end{array} \right] \quad \begin{array}{l} x + y + z = -2 \quad (3) \\ -6y - 3z = 15 \quad (2) \\ 6z = -18 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -3}$$

$$(2) \rightarrow -6y = 15 - 9$$

$$\underline{y = -1}$$

$$(3) \rightarrow x = -2 + 1 + 3$$

$$\underline{= 2}$$

$$\therefore \text{Solution: } \underline{(2, -1, -3)}$$

### Exercise

Use augmented elimination to solve linear system  $\begin{cases} 2x - 2y + z = -4 \\ 6x + 4y - 3z = -24 \\ x - 2y + 2z = 1 \end{cases}$

### Solution

$$\left[ \begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 2 & -2 & 1 & -4 \\ 6 & 4 & -3 & -24 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 6R_1 \end{array} \quad \begin{array}{ccc|c} 2 & -2 & 1 & -4 \\ -2 & 4 & -4 & -2 \\ \hline 0 & 2 & -3 & -6 \end{array} \quad \begin{array}{ccc|c} 6 & 4 & -3 & -24 \\ -6 & 12 & -12 & -6 \\ \hline 0 & 16 & -15 & -30 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 16 & -15 & -30 \end{array} \right] \quad R_3 - 8R_2 \quad \begin{array}{ccc|c} 0 & 16 & -15 & -30 \\ 0 & -16 & 24 & 48 \\ \hline 0 & 0 & 9 & 18 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 0 & 9 & 18 \end{array} \right] \quad \begin{array}{l} x - 2y + 2z = 1 \quad (3) \\ 2y - 3z = -6 \quad (2) \\ 9z = 18 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = 2}$$

$$(2) \rightarrow 2y = -6 + 6$$

$$\underline{y = 0}$$

$$(3) \rightarrow x = 1 - 4$$

$$= -3 \mid$$

$$\therefore \text{Solution: } \underline{(-3, 0, 2) \mid}$$

### Exercise

Use augmented elimination to solve linear system 
$$\begin{cases} 9x + 3y + z = 4 \\ 16x + 4y + z = 2 \\ 25x + 5y + z = 2 \end{cases}$$

### Solution

$$\begin{cases} z + 9x + 3y = 4 \\ z + 16x + 4y = 2 \\ z + 25x + 5y = 2 \end{cases}$$

$$\left[ \begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 1 & 16 & 4 & 2 \\ 1 & 25 & 5 & 2 \end{array} \right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \quad \begin{array}{cccc} 1 & 16 & 4 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 7 & 1 & -2 \end{array} \quad \begin{array}{cccc} 1 & 25 & 5 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 16 & 2 & -2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 16 & 2 & -2 \end{array} \right] \begin{array}{l} \\ \\ 7R_3 - 16R_2 \end{array} \quad \begin{array}{cccc} 0 & 112 & 14 & -14 \\ 0 & -112 & -16 & 32 \\ \hline 0 & 0 & -2 & 18 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 0 & -2 & 18 \end{array} \right] \begin{array}{l} z + 9x + 3y = 4 \quad (3) \\ 7x + y = -2 \quad (2) \\ -2y = 18 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{y = -9 \mid}$$

$$(2) \rightarrow 7x = -2 + 9 \\ = 1 \mid$$

$$(3) \rightarrow z = 4 - 9 + 27 \\ = 22 \mid$$

$$\therefore \text{Solution: } \underline{(1, -9, 22) \mid}$$

**Exercise**

Use augmented elimination to solve linear system 
$$\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 2 & -8 \\ 3 & -1 & -4 & 3 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \quad \begin{array}{cccc} 2 & -1 & 2 & -8 \\ -2 & -4 & 6 & -18 \\ 0 & -5 & 8 & -26 \end{array} \quad \begin{array}{cccc} 3 & -1 & -4 & 3 \\ -3 & -6 & 9 & -27 \\ 0 & -7 & 5 & -24 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & -7 & 5 & -24 \end{array} \right] \begin{array}{l} \\ \\ 5R_3 - 7R_2 \end{array} \quad \begin{array}{cccc} 0 & -35 & 25 & -120 \\ 0 & 35 & -56 & 182 \\ 0 & 0 & -31 & 62 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & 0 & -31 & 62 \end{array} \right] \begin{array}{l} x + 2y - 3z = 9 \\ -5y + 8z = -26 \\ -31z = 62 \end{array} \quad \begin{array}{l} (3) \\ (2) \\ (1) \end{array}$$

$$(1) \rightarrow \underline{z = -2}$$

$$\begin{aligned} (2) \rightarrow -5y &= -26 + 16 \\ -5y &= 10 \\ \underline{y} &= 2 \end{aligned}$$

$$\begin{aligned} (3) \rightarrow x &= 9 - 4 - 6 \\ &= -1 \end{aligned}$$

$$\therefore \text{Solution: } \underline{(-1, 2, -2)}$$

**Exercise**

Use augmented elimination to solve linear system 
$$\begin{cases} x - 3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 2 & -1 & 2 & 16 \\ 7 & -3 & -5 & 19 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 7R_1 \end{array} \quad \begin{array}{cccc} 2 & -1 & 2 & 16 \\ -2 & 0 & 6 & 10 \\ 0 & -1 & 8 & 26 \end{array} \quad \begin{array}{cccc} 7 & -3 & -5 & 19 \\ -7 & 0 & 21 & 35 \\ 0 & -3 & 16 & 54 \end{array}$$



$$\left[ \begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & -3 & 16 & 54 \end{array} \right] R_3 - 3R_2 \quad \begin{array}{ccc|c} 0 & -3 & 16 & 54 \\ 0 & 3 & -24 & -78 \\ \hline 0 & 0 & -8 & -24 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & 0 & -8 & -24 \end{array} \right] \quad \begin{array}{l} x - 3z = -5 \quad (3) \\ -y + 8z = 26 \quad (2) \\ -8z = -24 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = 3}$$

$$(2) \rightarrow -y = 26 - 24$$

$$\underline{y = -2}$$

$$(3) \rightarrow x = -5 + 9$$

$$\underline{x = 4}$$

$$\therefore \text{Solution: } \underline{(4, -2, 3)}$$

### Exercise

Use augmented elimination to solve linear system  $\begin{cases} x + 2y - z = 5 \\ 2x - y + 3z = 0 \\ 2y + z = 1 \end{cases}$

### Solution

$$\left[ \begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 2 & -1 & 3 & 0 \\ 0 & 2 & 1 & 1 \end{array} \right] R_2 - 2R_1 \quad \begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ -2 & -4 & 2 & -10 \\ \hline 0 & -5 & 5 & -10 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 2 & 1 & 1 \end{array} \right] 5R_3 + 2R_2 \quad \begin{array}{ccc|c} 0 & 10 & 5 & 5 \\ 0 & -10 & 10 & -20 \\ \hline 0 & 0 & 15 & -15 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 0 & 15 & -15 \end{array} \right] \quad \begin{array}{l} x + 2y - z = 5 \quad (3) \\ -5y + 5z = -10 \quad (2) \\ 15z = -15 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -1}$$

$$(2) \rightarrow -5y = -10 + 5$$

$$\underline{y = 1}$$

$$(3) \rightarrow x = 5 - 2 - 1$$

$$= 2 \mid$$

$$\therefore \text{Solution: } \underline{(2, 1, -1)}$$

### Exercise

Use augmented elimination to solve linear system  $\begin{cases} x + y + z = 6 \\ 3x + 4y - 7z = 1 \\ 2x - y + 3z = 5 \end{cases}$

### Solution

$$\begin{bmatrix} 1 & 1 & 1 & \mid & 6 \\ 3 & 4 & -7 & \mid & 1 \\ 2 & -1 & 3 & \mid & 5 \end{bmatrix} \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \quad \begin{array}{cccc} 3 & 4 & -7 & 1 \\ -3 & -3 & -3 & -18 \\ \hline 0 & 1 & -10 & -17 \end{array} \quad \begin{array}{cccc} 2 & -1 & 3 & 5 \\ -2 & -2 & -2 & -12 \\ \hline 0 & -3 & 1 & -7 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & \mid & 6 \\ 0 & 1 & -10 & \mid & -17 \\ 0 & -3 & 1 & \mid & -7 \end{bmatrix} \begin{array}{l} \\ \\ R_3 + 3R_2 \end{array} \quad \begin{array}{cccc} 0 & -3 & 1 & -7 \\ 0 & 3 & -30 & -51 \\ \hline 0 & 0 & -29 & -58 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & \mid & 6 \\ 0 & 1 & -10 & \mid & -17 \\ 0 & 0 & -29 & \mid & -58 \end{bmatrix} \begin{array}{l} x + y + z = 6 \quad (3) \\ y - 10z = -17 \quad (2) \\ -29z = -58 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = 2}$$

$$(2) \rightarrow y = -17 + 20 \\ = 3 \mid$$

$$(3) \rightarrow x = 6 - 3 - 2 \\ = 1 \mid$$

$$\therefore \text{Solution: } \underline{(1, 3, 2)}$$

### Exercise

Use augmented elimination to solve linear system  $\begin{cases} 3x + 2y + 3z = 3 \\ 4x - 5y + 7z = 1 \\ 2x + 3y - 2z = 6 \end{cases}$

### Solution

$$\begin{bmatrix} 3 & 2 & 3 & \mid & 3 \\ 4 & -5 & 7 & \mid & 1 \\ 2 & 3 & -2 & \mid & 6 \end{bmatrix} \begin{array}{l} \\ 3R_2 - 4R_1 \\ 3R_3 - 2R_1 \end{array} \quad \begin{array}{cccc} 12 & -15 & 21 & 3 \\ -12 & -8 & -12 & -12 \\ \hline 0 & -23 & 9 & -9 \end{array} \quad \begin{array}{cccc} 6 & 9 & -6 & 18 \\ -6 & -4 & -6 & -6 \\ \hline 0 & 5 & -12 & 12 \end{array}$$

$$\left[ \begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 0 & -23 & 9 & -9 \\ 0 & 5 & -12 & 12 \end{array} \right] 23R_3 + 5R_2 \quad \begin{array}{cccc} 0 & 115 & -276 & 276 \\ 0 & -115 & 45 & -45 \\ \hline 0 & 0 & -231 & 231 \end{array}$$

$$\left[ \begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 0 & -23 & 9 & -9 \\ 0 & 0 & -231 & 231 \end{array} \right] \begin{array}{l} 3x + 2y + 3z = 3 \quad (3) \\ -23y + 9z = -9 \quad (2) \\ -231z = 231 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -1}$$

$$(2) \rightarrow -23y = -9 + 9$$

$$\underline{y = 0}$$

$$(3) \rightarrow 3x = 3 + 3$$

$$\underline{x = 2}$$

$$\therefore \text{Solution: } \underline{(2, 0, -1)}$$

### Exercise

Use augmented elimination to solve linear system

$$\begin{cases} x - 3y + z = 2 \\ 4x - 12y + 4z = 8 \\ -2x + 6y - 2z = -4 \end{cases}$$

### Solution

$$\begin{cases} x - 3y + z = 2 \\ \frac{1}{4} \times 4x - 12y + 4z = 8 \\ -\frac{1}{2} \times -2x + 6y - 2z = -4 \end{cases}$$

$$\begin{cases} x - 3y + z = 2 \\ x - 3y + z = 2 \\ x - 3y + z = 2 \end{cases}$$

Since all three equations are the same.

$$\therefore \text{Solution: is the plane } \underline{x - 3y + z = 2}$$

**Exercise**

Use augmented elimination to solve linear system 
$$\begin{cases} 2x - 2y + z = -1 \\ x + 2y - z = 2 \\ 6x + 4y + 3z = 5 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & -2 & 1 & -1 \\ 6 & 4 & 3 & 5 \end{array} \right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 6R_1 \end{array} \quad \begin{array}{cccc} 2 & -2 & 1 & -1 \\ -2 & -4 & 2 & -4 \\ \hline 0 & -6 & 3 & -5 \end{array} \quad \begin{array}{cccc} 6 & 4 & 3 & 5 \\ -6 & -12 & 6 & -12 \\ \hline 0 & -8 & 9 & -7 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -6 & 3 & -5 \\ 0 & -8 & 9 & -7 \end{array} \right] 3R_3 - 4R_2 \quad \begin{array}{cccc} 0 & -24 & 27 & -21 \\ 0 & 24 & -12 & 20 \\ \hline 0 & 0 & 15 & -1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -6 & 3 & -5 \\ 0 & 0 & 15 & -1 \end{array} \right] \begin{array}{l} x + 2y - z = 2 \quad (3) \\ -6y + 3z = -5 \quad (2) \\ 15z = -1 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{z = -\frac{1}{15}}$$

$$(2) \rightarrow -6y = -5 + \frac{1}{5}$$

$$-6y = -\frac{24}{5}$$

$$\underline{y = \frac{4}{5}}$$

$$(3) \rightarrow x = 2 - \frac{8}{5} - \frac{1}{15}$$

$$\underline{= \frac{1}{3}}$$

$$\therefore \text{Solution: } \underline{\left( \frac{1}{3}, \frac{4}{5}, -\frac{1}{15} \right)}$$

**Exercise**

Use augmented elimination to solve linear system 
$$\begin{cases} x_1 - 3x_3 = -2 \\ 3x_1 + x_2 - 2x_3 = 5 \\ 2x_1 + 2x_2 + x_3 = 4 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{array} \right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{array} \right] \quad R_3 - 2R_2$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{array} \right] \quad \begin{array}{l} (2) \\ (1) \\ \underline{z = 2} \end{array}$$

$$(1) \rightarrow y = 11 - 14$$

$$\underline{y = -3}$$

$$(2) \rightarrow x = -2 + 6$$

$$\underline{x = 4}$$

$$\therefore \text{Solution: } \underline{(4, -3, 2)}$$

### Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 2x_1 + 3x_3 = 3 \\ 4x_1 - 3x_2 + 7x_3 = 5 \\ 8x_1 - 9x_2 + 15x_3 = 10 \end{cases}$$

### Solution

$$\left[ \begin{array}{ccc|c} 2 & 0 & 3 & 3 \\ 4 & -3 & 7 & 5 \\ 8 & -9 & 15 & 10 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 4R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 2 & 0 & 3 & 3 \\ 0 & -3 & 1 & -1 \\ 0 & -9 & 3 & -2 \end{array} \right] \quad R_3 - 3R_2$$

$$\left[ \begin{array}{ccc|c} 2 & 0 & 3 & 3 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad 0 \neq 1$$

$$\therefore \text{Solution: } \underline{\text{No Solution}}$$

**Exercise**

Use augmented elimination to solve linear system

$$\begin{cases} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 \\ x_2 - 3x_3 - x_4 = 0 \\ 3x_1 + 2x_3 - x_4 = 6 \\ -4x_1 + x_2 + 4x_3 + 2x_4 = -3 \end{cases}$$

**Solution**

$$\left[ \begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 3 & 0 & 2 & -1 & 6 \\ -4 & 1 & 4 & 2 & -3 \end{array} \right] \begin{array}{l} R_3 - 3R_1 \\ R_4 + 4R_1 \end{array} \quad \begin{array}{ccccc} 3 & 0 & 2 & -1 & 6 \\ -3 & 15 & -6 & 6 & -12 \\ \hline 0 & 15 & -4 & 5 & -6 \end{array} \quad \begin{array}{ccccc} -4 & 1 & 4 & 2 & -3 \\ 4 & -20 & 8 & -8 & 16 \\ \hline 0 & -19 & 12 & -6 & 13 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 15 & -4 & 5 & -6 \\ 0 & -19 & 12 & -6 & 13 \end{array} \right] \begin{array}{l} R_3 - 15R_2 \\ R_4 + 19R_2 \end{array} \quad \begin{array}{ccccc} 0 & 15 & -4 & 5 & -6 \\ 0 & -15 & 45 & 15 & 0 \\ \hline 0 & 0 & 41 & 20 & -6 \end{array} \quad \begin{array}{ccccc} 0 & -19 & 12 & -6 & 13 \\ 0 & 19 & -57 & -19 & 0 \\ \hline 0 & 0 & -45 & -25 & 13 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & -45 & -25 & 13 \end{array} \right] 41R_4 + 45R_2 \quad \begin{array}{ccccc} 0 & 0 & -1845 & -1025 & 533 \\ 0 & 0 & 1845 & 900 & -270 \\ \hline 0 & 0 & 0 & -125 & 263 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & 0 & -125 & 263 \end{array} \right] \begin{array}{l} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 \quad (4) \\ x_2 - 3x_3 - x_4 = 0 \quad (3) \\ 41x_3 + 20x_4 = -6 \quad (2) \\ -125x_4 = 263 \quad (1) \end{array}$$

$$(1) \rightarrow x_4 = -\frac{263}{125}$$

$$(2) \rightarrow 41x_3 = -6 + \frac{1,052}{25} \\ = \frac{902}{25}$$

$$x_3 = \frac{22}{25}$$

$$(3) \rightarrow x_2 = \frac{66}{25} - \frac{263}{125} \\ = \frac{67}{125}$$

$$(4) \rightarrow x_1 = 4 + \frac{67}{25} - \frac{44}{25} - \frac{526}{125}$$

$$\begin{aligned}
 &= 4 + \frac{23}{25} - \frac{526}{125} \\
 &= \frac{500 + 115 - 526}{125} \\
 &= \frac{89}{125}
 \end{aligned}$$

$$\therefore \text{Solution: } \left( \frac{89}{125}, \frac{67}{125}, \frac{22}{25}, -\frac{263}{125} \right)$$

### Exercise

Use augmented elimination to solve linear system

$$\begin{cases}
 x_1 + x_2 + x_3 + x_4 = 5 \\
 x_1 + 2x_2 - x_3 - 2x_4 = -1 \\
 x_1 - 3x_2 - 3x_3 - x_4 = -1 \\
 2x_1 - x_2 + 2x_3 - x_4 = -2
 \end{cases}$$

### Solution

$$\left[ \begin{array}{cccc|c}
 1 & 1 & 1 & 1 & 5 \\
 1 & 2 & -1 & -2 & -1 \\
 1 & -3 & -3 & -1 & -1 \\
 2 & -1 & 2 & -1 & -2
 \end{array} \right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - 2R_1 \end{array}$$

$$\begin{array}{ccccc}
 1 & 2 & -1 & -2 & -1 \\
 -1 & -1 & -1 & -1 & -5 \\
 \hline
 0 & 1 & -2 & -3 & -6
 \end{array}$$

$$\begin{array}{ccccc}
 1 & -3 & -3 & -1 & -1 \\
 -1 & -1 & -1 & -1 & -5 \\
 \hline
 0 & -4 & -4 & -2 & -6
 \end{array}$$

$$\begin{array}{ccccc}
 2 & -1 & 2 & -1 & -2 \\
 -2 & -2 & -2 & -2 & -10 \\
 \hline
 0 & -3 & 0 & -3 & -12
 \end{array}$$

$$\left[ \begin{array}{cccc|c}
 1 & 1 & 1 & 1 & 5 \\
 0 & 1 & -2 & -3 & -6 \\
 0 & -4 & -4 & -2 & -6 \\
 0 & -3 & 0 & -3 & -12
 \end{array} \right] \begin{array}{l} \\ \\ R_3 + 4R_2 \\ R_4 + 3R_2 \end{array}$$

$$\begin{array}{ccccc}
 0 & -4 & -4 & -2 & -6 \\
 0 & 4 & -8 & -12 & -24 \\
 \hline
 0 & 0 & -12 & -14 & -30
 \end{array}$$

$$\begin{array}{ccccc}
 0 & -3 & 0 & -3 & -12 \\
 0 & 3 & -6 & -9 & -18 \\
 \hline
 0 & 0 & -6 & -12 & -30
 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & -12 & -14 & -30 \\ 0 & 0 & -6 & -12 & -30 \end{array} \right] \quad -2R_4 + R_3 \quad \begin{array}{ccccc} 0 & 0 & 12 & 24 & 60 \\ 0 & 0 & -12 & -14 & -30 \\ \hline 0 & 0 & 0 & 10 & 30 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & -12 & -14 & -30 \\ 0 & 0 & 0 & 10 & 30 \end{array} \right] \quad \begin{array}{l} x_1 + x_2 + x_3 + x_4 = 5 \quad (4) \\ x_2 - 2x_3 - 3x_4 = -6 \quad (3) \\ -12x_3 - 14x_4 = -30 \quad (2) \\ 10x_4 = 30 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{x_4 = 3}$$

$$(2) \rightarrow -12x_3 = -30 + 42 \\ = 12$$

$$\underline{x_3 = -1}$$

$$(3) \rightarrow x_2 = -6 - 2 + 9 \\ = 1$$

$$(4) \rightarrow x_1 = 5 - 1 + 1 - 3 \\ = 2$$

$$\therefore \text{Solution: } \underline{(2, 1, -1, 3)}$$

### Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 2x + 8y - z + w = 0 \\ 4x + 16y - 3z - w = -10 \\ -2x + 4y - z + 3w = -6 \\ -6x + 2y + 5z + w = 3 \end{cases}$$

### Solution

$$\left[ \begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 4 & 16 & -3 & -1 & -10 \\ -2 & 4 & -1 & 3 & -6 \\ -6 & 2 & 5 & 1 & 3 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{array}$$



$$\left[ \begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 26 & 2 & 4 & 3 \end{array} \right] R_4 - \frac{13}{6}R_2$$

$$\left[ \begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 26 & 2 & 4 & 3 \end{array} \right] \text{Interchange } R_2 \text{ and } R_3$$

$$\left[ \begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & \frac{19}{3} & -\frac{14}{3} & 16 \end{array} \right] R_4 + \frac{19}{3}R_3$$

$$\left[ \begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & 0 & -\frac{71}{3} & -\frac{142}{3} \end{array} \right] \begin{array}{l} 2x + 8y - z + w = 0 \quad (3) \\ 12y - 2z + 4w = -6 \quad (2) \\ -z - 3w = -10 \quad (1) \\ -\frac{71}{3}w = -\frac{142}{3} \rightarrow \underline{w = 2} \end{array}$$

$$(1) \rightarrow z = 10 - 3w = \underline{4}$$

$$(2) \rightarrow 12y = 2z - 4w - 6$$

$$\underline{y = -\frac{1}{2}}$$

$$(3) \rightarrow 2x = -8y + z - w$$

$$2x = 4 + 4 - 2$$

$$\underline{x = 3}$$

$$\therefore \text{Solution: } \underline{\left( 3, -\frac{1}{2}, 4, 2 \right)}$$

### Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

### Solution

$$\begin{cases} x_1 = -2x_2 \\ x_3 = -x_2 \end{cases}$$

$$2x_1 + x_2 + 3x_3 = 0$$

$$-4x_2 + x_2 - 3x_2 = 0 \rightarrow \underline{x_2 = 0}$$

$$\therefore \text{Solution: } \underline{(0, 0, 0)}$$

### Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 2x + 2y + 4z = 0 \\ -y - 3z + w = 0 \\ 3x + y + z + 2w = 0 \\ x + 3y - 2z - 2w = 0 \end{cases}$$

### Solution

$$\left[ \begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & -1 & -3 & 1 & 0 \\ 3 & 1 & 1 & 2 & 0 \\ 1 & 3 & -2 & -2 & 0 \end{array} \right] \begin{array}{l} \\ -R_2 \\ 2R_3 - 3R_1 \\ 2R_4 - R_1 \end{array}$$

$$\left[ \begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & -4 & -10 & 4 & 0 \\ 0 & 4 & -8 & -4 & 0 \end{array} \right] \begin{array}{l} \\ R_3 + 4R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[ \begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -20 & 0 & 0 \end{array} \right] \begin{array}{l} 2x + 2y - 4z = 0 \quad (1) \\ y + 3z - w = 0 \quad (2) \\ \rightarrow \underline{z = 0} \\ \end{array}$$

$$(2) \rightarrow \underline{y = w}$$

$$(1) \rightarrow 2x = -2y \quad \underline{x = -w}$$

$$\therefore \text{Solution: } \underline{(-w, w, 0, w)}$$

**Exercise**

Use augmented elimination to solve linear system

$$\begin{cases} 2x & + z + w = 5 \\ & y & - w = -1 \\ 3x & & - z - w = 0 \\ 4x + y + 2z + w = 9 \end{cases}$$

**Solution**

$$\left[ \begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 3 & 0 & -1 & -1 & 0 \\ 4 & 1 & 2 & 1 & 9 \end{array} \right] \quad \begin{array}{l} \\ 2R_3 - 3R_1 \\ 2R_4 - 4R_1 \end{array}$$

$$\left[ \begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 & -15 \\ 0 & 2 & 0 & -2 & -2 \end{array} \right] \quad R_4 - 2R_2$$

$$\left[ \begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 & -15 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} 2x + z + w = 5 \quad (1) \\ y - w = -1 \quad (2) \\ -5z - 5w = -15 \quad (3) \end{array}$$

$$(2) \rightarrow \underline{y = 1 + w}$$

$$(3) \rightarrow \underline{z = 3 - w}$$

$$(1) \rightarrow 2x = 5 - (3 - w) - w$$

$$\underline{x = 1}$$

$$\therefore \text{Solution: } \underline{(1, 1 + w, 3 - w, w)}$$

**Exercise**

Use augmented elimination to solve linear system

$$\begin{cases} & 4y + z = 20 \\ 2x - 2y + z = 0 \\ x & + z = 5 \\ x + y - z = 10 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 1 & 5 \\ 0 & 4 & 1 & 20 \end{array} \right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{l}
 \left[ \begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 0 & -4 & 3 & -20 \\ 0 & -1 & 2 & -5 \\ 0 & 4 & 1 & 20 \end{array} \right] \begin{array}{l} \\ 4R_3 - R_2 \\ R_4 + R_2 \end{array} \\
 \\
 \left[ \begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 0 & -4 & 3 & -20 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 4 & 0 \end{array} \right] \begin{array}{l} x + y = 10 \\ \rightarrow -4y = -20 \\ \rightarrow z = 0 \end{array}
 \end{array}$$

**$\therefore$  Solution:**  $(5, 5, 0)$

### Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \qquad \qquad -3w = -3 \end{cases}$$

### Solution

$$\left[ \begin{array}{cccc|c} 1 & 2 & 1 & 0 & 8 \\ -1 & 3 & -2 & 0 & 1 \\ 3 & 4 & -7 & 0 & 10 \end{array} \right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 2 & 1 & 0 & 8 \\ 0 & 5 & -1 & 0 & 9 \\ 0 & -2 & -10 & 0 & -14 \end{array} \right] 5R_3 + 2R_2$$

$$\left[ \begin{array}{cccc|c} 1 & 2 & 1 & 0 & 8 \\ 0 & 5 & -1 & 0 & 9 \\ 0 & 0 & -52 & 0 & -52 \end{array} \right] \begin{array}{l} x + 2y + z = 8 \quad (3) \\ 5y - z = 9 \quad (2) \\ -52z = -52 \quad (1) \end{array}$$

$$(1) \Rightarrow z = 1$$

$$(2) \Rightarrow 5y = 9 + 1 = 10 \rightarrow y = 2$$

$$(3) \Rightarrow x = 8 - 4 - 1 = 3$$

**$\therefore$  Solution:**  $(3, 2, 1)$

**Exercise**

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} 2u - 3v + w - x + y = 0 \\ 4u - 6v + 2w - 3x - y = -5 \\ -2u + 3v - 2w + 2x - y = 3 \end{cases}$$

**Solution**

$$\left[ \begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 4 & -6 & 2 & -3 & -1 & -5 \\ -2 & 3 & -2 & 2 & -1 & 3 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left[ \begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -3 & -5 \\ 0 & 0 & -1 & 1 & 0 & 3 \end{array} \right] \begin{array}{l} 2u - 3v + w - x + y = 0 \quad (3) \\ -x - 3y = -5 \quad (2) \\ -w + x = 3 \quad (1) \end{array}$$

$$(2) \Rightarrow \underline{x = 5 - 3y}$$

$$(1) \Rightarrow \underline{w = x - 3} \\ \underline{= 2 - 3y}$$

$$(3) \Rightarrow 2u = 3v - 2 + 3y + 5 - 3y - y \\ = 3v - y + 3$$

$$u = \frac{3}{2}v - \frac{1}{2}y + \frac{3}{2}$$

$$\therefore \textbf{Solution: } \underline{\left( \frac{3}{2}v - \frac{1}{2}y + \frac{3}{2}, v, 2 - 3y, 5 - 3y, y \right)}$$

**Exercise**

Use augmented elimination to solve linear system

$$\begin{cases} 6x_3 + 2x_4 - 4x_5 - 8x_6 = 8 \\ 3x_3 + x_4 - 2x_5 - 4x_6 = 4 \\ 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 + x_6 = 2 \\ 6x_1 - 9x_2 + 11x_4 - 19x_5 + 3x_6 = 1 \end{cases}$$

**Solution**

$$\left[ \begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 6 & -9 & 0 & 11 & -19 & 3 & 1 \end{array} \right] R_4 - 3R_1$$

$$\left[ \begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & -3 & -1 & 2 & 0 & -5 \end{array} \right] \quad \begin{array}{l} R_3 - 2R_2 \\ R_4 + R_2 \end{array}$$

$$\left[ \begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & -1 \end{array} \right] \quad \begin{array}{l} 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 = 2 - x_6 \\ 3x_3 + x_4 - 2x_5 = 4 + 4x_6 \\ \rightarrow x_6 = \frac{1}{4} \end{array}$$

$$\begin{cases} 3x_3 = 5 - x_4 + 2x_5 \\ 2x_1 = \frac{7}{4} + 3x_2 - \frac{1}{3}(5 - x_4 + 2x_5) - 4x_4 + 7x_5 \end{cases}$$

$$\begin{cases} x_3 = \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5 \\ 2x_1 = \frac{1}{12} + 3x_2 - \frac{11}{3}x_4 + \frac{19}{3}x_5 \end{cases}$$

$$x_1 = \frac{1}{24} + \frac{3}{2}x_2 - \frac{11}{6}x_4 + \frac{19}{6}x_5$$

$$\therefore \text{Solution: } \left( \frac{1}{24} + \frac{3}{2}x_2 - \frac{11}{6}x_4 + \frac{19}{6}x_5, \quad x_2, \quad \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5, \quad x_4, \quad x_5, \quad \frac{1}{4} \right)$$

### Exercise

Use augmented elimination to solve linear system

$$\begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases}$$

### Solution

$$\left[ \begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array} \right] \quad \begin{array}{l} 3R_2 - 5R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & -1 & 4 & 26 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad R_3 - R_2$$

$$\left[ \begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} 3x_1 + 2x_2 - x_3 = -15 \quad (3) \\ -x_2 + 11x_3 = 75 \quad (2) \\ -7x_3 = -49 \quad (1) \end{array}$$

$$(1) \rightarrow \underline{x_3 = 7}$$

$$(2) \rightarrow x_2 = 77 - 75 \\ = 2$$

$$(1) \rightarrow 3x_1 = -15 - 4 + 7 = 12 \\ \underline{x_1 = -4}$$

$$\therefore \text{Solution: } \underline{(-4, 2, 7)}$$

### Exercise

Use augmented elimination to solve linear system

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ 5x_3 + 10x_4 + 15x_6 = 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$$

### Solution

$$\left[ \begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_4 - 2R_1 \end{array}$$

$$\left[ \begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array} \right] \quad -R_2$$

$$\left[ \begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array} \right] \quad \begin{array}{l} R_3 - 5R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[ \begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array} \right] \frac{1}{6}R_4 \text{ then interchanging row3 and row4}$$

$$\left[ \begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] R_2 - 3R_3$$

$$\left[ \begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \begin{cases} x_1 + 3x_2 + 4x_4 + 2x_5 = 0 \\ x_3 + 2x_4 = 0 \\ + x_6 = \frac{1}{3} \end{cases}$$

The general solution of the system:  $x_6 = \frac{1}{3}, \quad x_3 = -2x_4, \quad x_1 = -3x_2 - 4x_4 - 2x_5$

**∴ Solution:**  $\left( -3x_2 - 4x_4 - 2x_5, x_2, -2x_4, x_4, x_5, \frac{1}{3} \right)$

### Exercise

Add 3 times the second row to the first of

$$\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

### Solution

$$E = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$EA = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$



$$= \begin{bmatrix} 27 & 8 & -1 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix} \xrightarrow{R_1 + 3R_2} \begin{bmatrix} 27 & 8 & -1 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

### Exercise

For what value(s) of  $k$ , if any, does the system  $\begin{cases} x + y - z = 1 \\ 2x + 3y + kz = 3 \\ x + ky + 3z = 2 \end{cases}$  have

- A unique solution?
- Infinitely many solutions?
- No solution?

### Solution

$$\left[ \begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & k & 3 \\ 1 & k & 3 & 2 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & k-1 & 4 & 1 \end{array} \right] \begin{array}{l} \\ \\ R_3 - (k-1)R_2 \end{array}$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & 0 & 4-(k-1)(k+2) & 2-k \end{array} \right] \begin{array}{l} x = 1 - y + z \\ y = 1 - (k+2)z \\ \rightarrow (6 - k^2 - k)z = -(k-2) \end{array}$$

$$\begin{cases} z = -\frac{k-2}{-(k-2)(k+3)} = \frac{1}{k+3} \quad (k \neq 2, -3) \\ y = 1 - \frac{k+2}{k+3} = \frac{1}{k+3} \\ x = \frac{k+2}{k+3} + \frac{1}{k+3} = 1 \end{cases}$$

- Unique solution if  $k \neq 2, -3$
- Infinitely solution if  $k = 2$
- No solution if  $k = -3$

**Exercise**

Assume that the matrix is the augmented matrix of a system of linear equations, and

$$\left[ \begin{array}{ccc} 1 & k & 2 \\ -3 & 4 & 1 \end{array} \right]$$

- Determine the number of equations and the number of variables.
- Find the value(s) of  $k$  such that the system is consistent.
- Determine the number of equations and the number of variables but if the matrix is the coefficient matrix of a *homogeneous* system of linear equations.
- Find the value(s) of  $k$  from part (c).

**Solution**

a) 2 equations & 2 variables

$$b) \left[ \begin{array}{cc|c} 1 & k & 2 \\ -3 & 4 & 1 \end{array} \right] \quad R_2 + 3R_1$$

$$\left[ \begin{array}{cc|c} 1 & k & 2 \\ 0 & 4+3k & 7 \end{array} \right]$$

For the system to be consistent, then  $4+3k \neq 0$

$$k \in \mathbb{R} - \left\{ -\frac{4}{3} \right\}$$

$$c) \left[ \begin{array}{ccc|c} 1 & k & 2 & 0 \\ -3 & 4 & 1 & 0 \end{array} \right]$$

2 equations & 3 variables

$$d) \left[ \begin{array}{ccc|c} 1 & k & 2 & 0 \\ -3 & 4 & 1 & 0 \end{array} \right] \quad R_2 + 3R_1$$

$$\left[ \begin{array}{ccc|c} 1 & k & 2 & 0 \\ 0 & 4+k & 7 & 0 \end{array} \right] \quad \begin{array}{l} x_1 = -2x_3 - kx_2 \\ (4+k)x_2 = -7x_3 \end{array}$$

$$\underline{x_2 = -\frac{7}{4+k}x_3}$$

$$x_1 = -2x_3 + \frac{7k}{4+k}x_3$$

$$\underline{= \frac{5k-8}{4+k}x_3}$$

$$k \in \mathbb{R}$$

If  $k = -4$ , then no solution.

**Exercise**

Choose a coefficient  $b$  that makes the system singular.

$$\begin{cases} 3x + 4y = 16 \\ 4x + by = g \end{cases}$$

Then choose a right-hand side  $g$  that makes it solvable.

Find 2 solutions in that singular case.

**Solution**

$$\left( \begin{array}{cc|c} 3 & 4 & 16 \\ 4 & b & g \end{array} \right) \quad 3R_2 - 4R_1$$

$$\left( \begin{array}{cc|c} 3 & 4 & 16 \\ 0 & 3b-16 & 3g-64 \end{array} \right)$$

So, the system is singular if

$$3b - 16 = 0 \Rightarrow b = \frac{16}{3}$$

$$\& \ 3g - 64 = 0 \Rightarrow g = \frac{64}{3}$$

$$\begin{cases} 3x + 4y = 16 \\ 4x + \frac{16}{3}y = \frac{64}{3} \end{cases} \rightarrow \underline{3x + 4y = 16}$$

$$\text{If } \begin{cases} x = 0 \rightarrow y = 4 \\ x = 4 \rightarrow y = 1 \end{cases}$$

**Exercise**

This system is not linear, in some sense,

$$\begin{cases} 2\sin\alpha - \cos\beta + 3\tan\theta = 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\theta = 10 \\ 6\sin\alpha - 3\cos\beta + \tan\theta = 9 \end{cases}$$

Does the system have a solution?

**Solution**

$$\left[ \begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 10 \\ 6 & -3 & 1 & 9 \end{array} \right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 0 & 4 & -8 & 4 \\ 0 & 0 & -8 & 0 \end{array} \right] \quad \begin{array}{l} 2 \sin \alpha = 3 + \cos \beta \\ 4 \cos \beta = 4 \\ \tan \theta = 0 \end{array}$$

$$\left\{ \begin{array}{l} \sin \alpha = \frac{3}{2} + \frac{1}{2} = 2 \\ \cos \beta = 1 \\ \tan \theta = 0 \end{array} \right.$$

The system has *no solution* since  $\sin \alpha$  cannot be equal 2. ( $-1 \leq \sin \alpha \leq 1$ )

### ***Exercise***

Determine whether the matrix is in row-echelon form. If it is, determine whether it is also in reduced row-echelon form.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

### **Solution**

The given matrix is in reduced row-echelon form.

## ***Solution***      **Section 1.3 – Matrices and Matrix operations**

### ***Exercise***

For the matrices:  $A = \begin{bmatrix} p & 0 \\ q & r \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ , when does  $AB = BA$

### **Solution**

$$\begin{aligned} AB &= \begin{pmatrix} p & 0 \\ q & r \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} p & p \\ q & q+r \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & 0 \\ q & r \end{pmatrix} \\ &= \begin{pmatrix} p+q & r \\ q & r \end{pmatrix} \end{aligned}$$

$$AB = BA$$

$$\begin{pmatrix} p & p \\ q & q+r \end{pmatrix} = \begin{pmatrix} p+q & r \\ q & r \end{pmatrix}$$

$$\begin{cases} p = p+q \\ p = r \\ q+r = r \end{cases} \Rightarrow \begin{cases} q = 0 \\ p = r \\ q = 0 \end{cases}$$

### ***Exercise***

Find values for the variables so that the matrices are equal.  $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

### **Solution**

$$\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$$

$$\Rightarrow \begin{cases} w = 9 & x = 17 \\ y = 8 & z = -12 \end{cases}$$

**Exercise**

Find values for the variables so that the matrices are equal.  $\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$

**Solution**

$$\begin{cases} \underline{x=12} \\ y+3=5 \rightarrow \underline{y=2} \\ 2z=6 \rightarrow \underline{z=3} \end{cases}$$

**Exercise**

Find values for the variables so that the matrices are equal.  $\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$

**Solution**

$$\begin{bmatrix} 5 = y+3 & x-4 = 2 & 9 = 9 \\ 2 = z+4 & -3 = -3 & 8 = 8 \\ 6 = 6 & 0 = 0 & 5 = w \end{bmatrix}$$

$$\rightarrow \begin{cases} y=2 & z=-2 \\ x=6 & w=5 \end{cases}$$

**Exercise**

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

**Solution**

$$\begin{bmatrix} a-5 & 5b & 4c+6 \\ -2d & 7f-6 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

$$\begin{cases} a-5=15 \rightarrow a=20 \\ 5b=25 \rightarrow b=5 \\ 4c+6=6 \rightarrow 4c=0 \rightarrow c=0 \\ -2d=-8 \rightarrow d=4 \\ 7f-6=1 \rightarrow 7f=7 \rightarrow f=1 \end{cases}$$

**Exercise**

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+11 & 12z+1 & 5m \\ 11k & 3 & 1 \end{bmatrix} + \begin{bmatrix} 9a & 9z & 4m \\ 12k & 5 & 3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

**Solution**

$$\begin{bmatrix} a+11+9a & 12z+1+9z & 5m+4m \\ 11k+12k & 3+5 & 1+3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 10a+11 & 21z+1 & 9m \\ 23k & 8 & 4 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

$$10a+11=41 \rightarrow 10a=30$$

$$\underline{a=3}$$

$$21z+1=-62 \rightarrow 21z=-63$$

$$\underline{z=-3}$$

$$9m=72 \rightarrow \underline{m=8}$$

$$23k=92$$

$$k=\frac{92}{23}$$

$$\underline{=4}$$

**Exercise**

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} x+2 & 3y+1 & 5z \\ 8w & 2 & 3 \end{bmatrix} + \begin{bmatrix} 3x & 2y & 5z \\ 2w & 5 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$$

**Solution**

$$\begin{bmatrix} 4x+2 & 5y+1 & 10z \\ 10w & 7 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$$

$$\begin{cases} 4x+2=10 & \rightarrow \underline{x=2} \\ 5y+1=-14 & \rightarrow \underline{y=-3} \\ 10z=80 & \rightarrow \underline{z=8} \\ 10w=10 & \rightarrow \underline{w=1} \end{cases}$$

**Exercise**

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} 2x-3 & y-2 & 2z+1 \\ 5 & 2w & 7 \end{bmatrix} + \begin{bmatrix} 3x-3 & y+2 & z-1 \\ -5 & 5w+1 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$$

**Solution**

$$\begin{bmatrix} 5x-6 & 2y & 3z \\ 0 & 7w+1 & 10 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$$

$$\begin{cases} 5x-6=20 & \rightarrow \underline{x=\frac{26}{5}} \\ 2y=8 & \rightarrow \underline{y=4} \\ 3z=9 & \rightarrow \underline{z=3} \\ 7w+1=8 & \rightarrow \underline{w=1} \end{cases}$$

**Exercise**

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} 16 & 4 & 5 & 4 \\ -3 & 13 & 15 & 6 \\ 0 & 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 4 & 2x+1 & 4 \\ -3 & 13 & 16 & 3x \\ 0 & 2 & 3y-5 & 0 \end{bmatrix}$$

**Solution**

$$5 = 2x + 1 \rightarrow \underline{x=2}$$

$$6 = 3x \rightarrow x=2 \quad \checkmark$$

$$4 = 3y - 5 \rightarrow \underline{y=3}$$

**Exercise**

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 19 & 2 \end{bmatrix}$$

**Solution**

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 19 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 19 & 2 \end{bmatrix}$$



$$\begin{cases} a + 2c = 6 \\ 3a + 4c = 19 \end{cases}$$

$$-3a - 6c = -18$$

$$\begin{array}{r} 3a + 4c = 19 \\ -3a - 6c = -18 \\ \hline -2c = 1 \end{array}$$

$$c = -\frac{1}{2}$$

$$a = 7$$

$$\begin{cases} b + 2d = 3 \\ 3b + 4d = 2 \end{cases}$$

$$-3b - 6d = -9$$

$$\begin{array}{r} 3b + 4d = 2 \\ -3b - 6d = -9 \\ \hline -2d = -7 \end{array}$$

$$d = \frac{7}{2}$$

$$b = -4$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 7 & -4 \\ -\frac{1}{2} & \frac{7}{2} \end{pmatrix}$$

### Exercise

Find a combination  $x_1 w_1 + x_2 w_2 + x_3 w_3$  that gives the zero vector:

$$w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \quad w_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}; \quad w_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are independent or dependent?

The vectors lie in a \_\_\_\_\_.

The matrix  $W$  with those columns is not invertible.

### Solution

$$w_1 - 2w_2 + w_3 = 0$$

Therefore, those vectors are dependent

The vectors lie in a plane.

**Exercise**

The very last words say that the 5 by 5 centered difference matrix is not invertible, Write down the 5 equations  $Cx = b$ . Find a combination of left sides that gives zero. What combination of  $b_1, b_2, b_3, b_4, b_5$  must be zero?

**Solution**

The 5 by 5 centered difference matrix is

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

The five equations  $Cx = b$  are:

$$x_2 = b_1, \quad -x_1 + x_3 = b_2, \quad -x_2 + x_4 = b_3, \quad -x_3 + x_5 = b_4, \quad -x_4 = b_5.$$

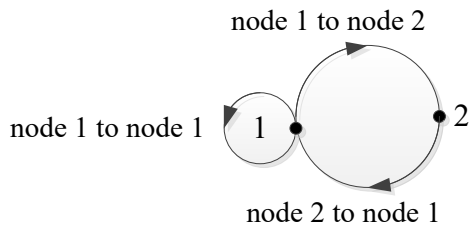
Observe that the sum of the first

$$x_2 - x_2 + x_4 - x_4 = b_1 + b_2 + b_5$$

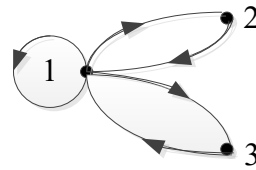
$$0 = b_1 + b_2 + b_5$$

**Exercise**

A direct graph starts with  $n$  nodes. There are  $n^2$  possible edges, each edge leaves one of the  $n$  nodes and enters one of the  $n$  nodes (possibly itself). The  $n$  by  $n$  adjacency matrix has  $a_{ij} = 1$  when edge leaves node  $i$  and enter node  $j$ ; if no edge then  $a_{ij} = 0$ . Here are directed graphs and their adjacency matrices:



$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The  $i, j$  entry of  $A^2$  is  $a_{i1}a_{1j} + \dots + a_{in}a_{nj}$ .

Why does that sum count the two-step paths from  $i$  to any node to  $j$ ?

The  $i, j$  entry of  $A^k$  counts  $k$ -steps paths:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{array}{l} \text{counts the paths} \\ \text{with two edges} \end{array} \quad \begin{bmatrix} 1 \text{ to } 2 \text{ to } 1, 1 \text{ to } 1 \text{ to } 1 & 1 \text{ to } 1 \text{ to } 2 \\ 2 \text{ to } 1 \text{ to } 1 & 2 \text{ to } 1 \text{ to } 2 \end{bmatrix}$$

List all 3-step paths between each pair of nodes and compare with  $A^3$ . When  $A^k$  has **no zeros**, that number  $k$  is the diameter of the graph – the number of edges needed to connect the most pair of nodes. What is the diameter of the second graph?

### Solution

The number  $a_{ik}a_{kj}$  will be “1” if there is an edge from node  $i$  to  $k$  and an edge from  $k$  to  $j$ .

This is a 2-step path. The number  $a_{ik}a_{kj}$  will be “0” if either of those edge (from node  $i$  to  $k$  and from  $k$  to  $j$ ) is missing.

The sum of  $a_{ik}a_{kj}$  is the number of 2-step paths leaving  $i$  and entering  $j$ .

Matrix multiplication is right for this count.

The 3-step paths are counted by  $A^3$ ; we look at paths to node 2:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{array}{l} \text{counts the paths} \\ \text{with three steps} \end{array} \quad \begin{bmatrix} \dots & 1 \text{ to } 1 \text{ to } 1 \text{ to } 2, 1 \text{ to } 2 \text{ to } 1 \text{ to } 2 \\ \dots & 2 \text{ to } 1 \text{ to } 1 \text{ to } 2 \end{bmatrix}$$

The  $A^k$  contain Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, 13, ....

Fibonacci’s rule  $F_{k+2} = F_{k+1} + F_k$  show up in  $(A)(A^k) = A^{k+1}$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} = \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix} = A^{k+1}$$

There are **13 six-step** paths from node one to node 1.

### **Exercise**

$A$  is 3 by 5,  $B$  is 5 by 3,  $C$  is 5 by 1, and  $D$  is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

- $AB$
- $BA$
- $ABD$
- $DBA$
- $ABC$
- $ABCD$
- $A(B+C)$

### Solution

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

**a)**  $AB : (3 \times 5)(5 \times 3) = (3 \times 3)$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{pmatrix}$$

**b)**  $BA : (5 \times 3)(3 \times 5) = (5 \times 5)$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}$$

**c)**  $ABD : (3 \times 5)(5 \times 3)(3 \times 1) = (3 \times 1)$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ 15 \\ 15 \end{pmatrix}$$

**d)**  $DBA : (3 \times 1)(5 \times 3)(3 \times 5) = NA$

**e)**  $ABC : (3 \times 5)(5 \times 3)(5 \times 1) = NA$

**f)**  $ABCD : (3 \times 5)(5 \times 3)(5 \times 1)(3 \times 1) = NA$

**g)**  $A(B + C) : (3 \times 5)((5 \times 3) + (5 \times 1)) = NA$

Matrices  $B$  and  $C$  are not the same size.

**Exercise**

What rows or columns or matrices do you multiply to find.

- a) The third column of  $AB$ ?
- b) The second column of  $AB$ ?
- c) The first row of  $AB$ ?
- d) The second row of  $AB$ ?
- e) The entry in row 3, column 4 of  $AB$ ?
- f) The entry in row 2, column 3 of  $AB$ ?

**Solution**

- a)  $A$  (column 3 of  $B$ )
- b)  $A$  (column 2 of  $B$ )
- c) (Row 1 of  $A$ )  $B$
- d) (Row 2 of  $A$ )  $B$
- e) (Row 3 of  $A$ ) (Column 4 of  $B$ )
- f) (Row 2 of  $A$ ) (Column 3 of  $B$ )

**Exercise**

Add  $AB$  to  $AC$  and compare with  $A(B + C)$ :

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

**Solution**

$$AB = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 0 & 7 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$$

$$\begin{aligned} A(B + C) &= \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \left( \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix} \end{aligned}$$

$$AB + AC = \begin{bmatrix} 0 & 7 \\ 0 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix}$$

$$\underline{A(B+C) = AB + AC}$$

### Exercise

True or False

- a) If  $A^2$  is defined then  $A$  is necessarily square.
- b) If  $AB$  and  $BA$  are defined then  $A$  and  $B$  are square.
- c) If  $AB$  and  $BA$  are defined then  $AB$  and  $BA$  are square.
- d) If  $AB = B$ , then  $A = I$

### Solution

- a) True
- b) False, if  $A$  has an order  $m$  by  $n$  and  $B$   $n$  by  $m$ :  $AB : m \times m$      $BA : n \times n$
- c) True;  $AB : m \times m$      $BA : n \times n$
- d) False, if  $B$  is the matrix of all zeros.

### Exercise

- a) Find a nonzero matrix  $A$  such that  $A^2 = 0$
- b) Find a matrix that has  $A^2 \neq 0$  but  $A^3 = 0$

### Solution

- a) A nonzero matrix  $A$  such that  $A^2 = 0$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{aligned} A^2 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

- b) A matrix that has  $A^2 \neq 0$  but  $A^3 = 0$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = A^2 A$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

### Exercise

Suppose you solve  $Ax = b$  for three special right sides  $b$ :

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If the three solutions  $x_1, x_2, x_3$  are the columns of a matrix  $X$ , what is  $A$  times  $X$ ?

### Solution

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Therefore,  $Ax = I$

### Exercise

Show that  $(A + B)^2$  is different from  $A^2 + 2AB + B^2$ , when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

Write down the correct rule for  $(A + B)(A + B) = A^2 + \underline{\hspace{2cm}} + B^2$

### Solution

$$\begin{aligned}A+B &= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}(A+B)^2 &= \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix}\end{aligned}$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}$$

$$2AB = 2 \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned}A^2 + 2AB + B^2 &= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 14 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}\end{aligned}$$

$$\begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix} \neq \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$$

$$\boxed{(A+B)^2 \neq A^2 + 2AB + B^2}$$

$$\begin{aligned}BA &= \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}A^2 + AB + BA + B^2 &= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 4 \\ 5 & 6 \end{bmatrix}\end{aligned}$$

$$\boxed{(A+B)(A+B) = A^2 + \underline{AB} + \underline{BA} + B^2}$$



**Exercise**

Let  $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$        $B = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$

Show the computations that

a)  $(A + B)(A + B) \neq A^2 + 2AB + B^2$

b)  $(A + B)(A - B) \neq A^2 - B^2$

c)  $(A + B)(A + B) = A^2 + AB + BA + B^2$

**Solution**

$$\begin{aligned} \text{a) } A + B &= \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} (A + B)(A + B) &= \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -6 \\ 6 & 15 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A^2 &= \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} 2AB &= 2 \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \\ &= 2 \begin{pmatrix} 0 & -2 \\ 2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -4 \\ 4 & 8 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} B^2 &= \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A^2 + 2AB + B^2 &= \begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 0 & -4 \\ 4 & 8 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -7 \\ 7 & 16 \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} 3 & -6 \\ 6 & 15 \end{pmatrix} \neq \begin{pmatrix} 2 & -7 \\ 7 & 16 \end{pmatrix}$$

Therefore,  $(A+B)(A+B) \neq A^2 + 2AB + B^2$

**b)**  $(A+B)(A-B) \neq A^2 - B^2$

From part (a)

$$A+B = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$$

$$\begin{aligned} A-B &= \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} (A+B)(A-B) &= \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 \\ -4 & -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A^2 - B^2 &= \begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} 1 & -2 \\ -4 & -1 \end{pmatrix} \neq \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$$

Therefore,  $(A+B)(A-B) \neq A^2 - B^2$

**c)**  $(A+B)(A+B) = A^2 + AB + BA + B^2$

From part (a)

$$(A+B)(A+B) = \begin{pmatrix} 3 & -6 \\ 6 & 15 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$$

$$\begin{aligned} AB &= \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2 \\ 2 & 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A^2 + AB + BA + B^2 &= \begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -6 \\ 6 & 15 \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} 3 & -6 \\ 6 & 15 \end{pmatrix} = \begin{pmatrix} 3 & -6 \\ 6 & 15 \end{pmatrix}$$

$$\text{Therefore, } (A+B)(A+B) = A^2 + AB + BA + B^2$$

### Exercise

Find the product of the 2 matrices by rows or by columns:  $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

### Solution

$$\begin{aligned} \text{By rows: } \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} &= \begin{pmatrix} (2 \quad 3) \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ (5 \quad 1) \begin{pmatrix} 4 \\ 2 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 14 \\ 22 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{By columns: } \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} &= 4 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 14 \\ 22 \end{pmatrix} \end{aligned}$$

**Exercise**

Find the product of the 2 matrices by rows or by columns:  $\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

**Solution**

$$\begin{aligned} \text{By rows: } \begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} &= \begin{pmatrix} (3 \ 6)(2 \ -1) \\ (6 \ 12)(2 \ -1) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{By columns: } \begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} &= 2 \begin{pmatrix} 3 \\ 6 \end{pmatrix} - 1 \begin{pmatrix} 6 \\ 12 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

**Exercise**

Find the product of the 2 matrices by rows or by columns:  $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

**Solution**

By rows:

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} (1 \ 2 \ 4)(3 \ 1 \ 1) \\ (2 \ 0 \ 1)(3 \ 1 \ 1) \end{pmatrix} \\ &= \begin{pmatrix} 1(3) + 2(1) + 4(1) \\ 2(3) + 0(1) + 1(1) \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ 7 \end{pmatrix} \end{aligned}$$

By columns:

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} &= 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ 7 \end{pmatrix} \end{aligned}$$

**Exercise**

Find the product of the 2 matrices by rows or by columns:  $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

**Solution**

By rows:

$$\begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} (1 \ 2 \ 4)(2 \ 2 \ 3) \\ (-2 \ 3 \ 1)(2 \ 2 \ 3) \\ (-4 \ 1 \ 2)(2 \ 2 \ 3) \end{pmatrix} \\ = \begin{pmatrix} 18 \\ 5 \\ 0 \end{pmatrix}$$

By columns:

$$\begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} \\ = \begin{pmatrix} 18 \\ 5 \\ 0 \end{pmatrix}$$

**Exercise**

Given  $A = \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$   $B = \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix}$  Find  $A + B$ ,  $2A$ , and  $-B$

**Solution**

$$A + B = \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix} + \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix} \\ = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -2 \\ 8 & -2 & 0 \end{bmatrix}$$

$$2A = 2 \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 2 & 6 \\ 6 & -2 & -4 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\begin{aligned} -B &= - \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 2 & 3 \\ 1 & 0 & 0 \\ -8 & 2 & 4 \end{bmatrix} \end{aligned}$$

### ***Exercise***

Given  $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$  and  $B = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}$ . Find  $AB$  and  $BA$ .

### **Solution**

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 13 \\ 4 & -4 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -16 & 29 \\ -4 & 10 \end{bmatrix} \end{aligned}$$

**Note:**  $AB \neq BA$

### ***Exercise***

Given  $A = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$   $B = \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

### **Solution**

$$\begin{aligned} AB &= \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix} \\ &= \begin{pmatrix} -6 & 17 \\ 6 & -8 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
 BA &= \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & 14 \\ 1 & -20 \end{pmatrix}
 \end{aligned}$$

*Note:*  $AB \neq BA$

### Exercise

Given  $A = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$   $B = \begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

#### Solution

$$\begin{aligned}
 AB &= \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix} \\
 &= \begin{pmatrix} -3 & -11 \\ 4 & 0 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 BA &= \begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} -7 & 1 \\ 16 & 4 \end{pmatrix}
 \end{aligned}$$

*Note:*  $AB \neq BA$

### Exercise

Given  $A = \begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}$   $B = \begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

#### Solution

$$\begin{aligned}
 AB &= \begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix} \\
 &= \begin{pmatrix} 10 & 6 \\ 14 & -7 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 BA &= \begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix} \\
 &= \begin{pmatrix} 14 & -1 \\ 0 & -11 \end{pmatrix}
 \end{aligned}$$

*Note:*  $AB \neq BA$

**Exercise**

Given  $A = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}$   $B = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

**Solution**

$$\begin{aligned} AB &= \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 2 \\ 4 & -4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -4 \\ 14 & -12 \end{pmatrix} \end{aligned}$$

**Note:**  $AB \neq BA$

**Exercise**

Given  $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}$   $B = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

**Solution**

$$\begin{aligned} AB &= \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -4 & 5 \\ 6 & 0 & 3 \\ -3 & -2 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 5 & -13 \\ 3 & -4 \end{pmatrix} \end{aligned}$$

**Note:**  $AB \neq BA$



**Exercise**

Given  $A = \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}$   $B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

**Solution**

$$AB = \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 5 & 4 \\ 2 & -3 & 8 \\ -3 & 8 & -5 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -14 & 7 \\ -4 & 5 \end{pmatrix}$$

**Note:**  $AB \neq BA$

**Exercise**

Given  $A = \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix}$   $B = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

**Solution**

$$AB = \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 24 & 4 \\ 2 & -6 & -2 \\ -13 & 12 & 10 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 8 \\ -10 & 10 \end{pmatrix}$$

**Note:**  $AB \neq BA$

**Exercise**

Given  $A = \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix}$   $B = \begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$  Find  $AB$  and  $BA$  if possible

**Solution**

$$\begin{aligned} AB &= \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3(3) + 2(0) - 3(1) & 3(-4) + 2(1) - 3(0) \\ 0(3) + 1(0) + 0(1) & 0(-4) + 1(1) + 0(0) \end{bmatrix} \\ &= \begin{bmatrix} 6 & -10 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3(3) - 4(0) & 3(2) - 4(1) & 3(-3) - 4(0) \\ 0(3) + 1(0) & 0(2) + 1(1) & 0(-3) + 1(0) \\ 1(3) + 0(0) & 1(2) + 0(1) & 1(-3) + 0(0) \end{bmatrix} \\ &= \begin{bmatrix} 9 & 2 & -9 \\ 0 & 1 & 0 \\ 3 & 2 & -3 \end{bmatrix} \end{aligned}$$

**Note:**  $AB \neq BA$

**Exercise**

Given  $A = \begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix}$   $B = \begin{bmatrix} 4 & -2 \\ -2 & 0 \\ 9 & 1 \end{bmatrix}$  Find  $AB$  and  $BA$  if possible

**Solution**

$AB = \text{Undefined}$

$$\begin{aligned} BA &= \begin{bmatrix} 4 & -2 \\ -2 & 0 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 22 & 12 \\ -10 & -6 \\ 44 & 27 \end{bmatrix} \end{aligned}$$

**Note:**  $AB \neq BA$

**Exercise**

Given  $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$   $B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$  Find  $AB$  and  $BA$  if possible

**Solution**

$$AB = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}$$

$$BA = \text{Undefined}$$

**Exercise**

Given  $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$   $B = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

**Solution**

$$AB = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -2 & -6 \\ 0 & -1 & 2 \\ 5 & -3 & -4 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -3 & 9 \\ 2 & -3 & 4 \\ 4 & -6 & 3 \end{pmatrix}$$

**Note:**  $AB \neq BA$

**Exercise**

Given  $A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix}$   $B = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

**Solution**

$$\begin{aligned} AB &= \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 8 & -2 \\ 3 & -8 & 4 \\ -2 & 13 & -5 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 5 & -1 \\ 11 & -5 & 6 \\ -8 & 7 & -4 \end{pmatrix} \end{aligned}$$

**Note:**  $AB \neq BA$

**Exercise**

Given  $A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}$   $B = \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

**Solution**

$$\begin{aligned} AB &= \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -5 & -7 & 2 \\ -6 & 2 & 2 \\ -8 & -6 & 0 \end{pmatrix} \end{aligned}$$

$$BA = \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 6 & 1 \\ 7 & 0 & 5 \\ 4 & -4 & -2 \end{pmatrix}$$

*Note:*  $AB \neq BA$

### Exercise

Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Compute the following (where possible):

$$a) D + E \quad b) D - E \quad c) 5A \quad d) -7C \quad e) 2B - C \quad f) -3(D + 2E)$$

### Solution

$$\begin{aligned} a) \quad D + E &= \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 6 & 5 \\ -2 & 1 & 3 \\ 7 & 3 & 7 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} b) \quad D - E &= \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} c) \quad 5A &= 5 \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 15 & 0 \\ -5 & 10 \\ 5 & 5 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}d) \quad -7C &= -7 \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -7 & -28 & -14 \\ -21 & -7 & -35 \end{bmatrix}\end{aligned}$$

e) Since  $B$  and  $C$  are not the same size

$2B - C$ : *can't be calculated*

$$\begin{aligned}f) \quad -3(D + 2E) &= -3 \left( \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 12 & 2 & 6 \\ -2 & 2 & 4 \\ 8 & 2 & 6 \end{bmatrix} \right) \\ &= -3 \begin{bmatrix} 13 & 7 & 8 \\ -3 & 2 & 5 \\ 11 & 4 & 10 \end{bmatrix} \\ &= \begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix}\end{aligned}$$

### Exercise

Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} -3 & -1 \\ 2 & 1 \\ 4 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$$

Compute the following (where possible):

- a)  $A + B$       b)  $A + C$       c)  $AB$       d)  $BA$       e)  $CD$       f)  $DC$   
g)  $BD$       h)  $DB$       i)  $A^2$       j)  $B^2$       k)  $D^2$

### Solution

a) Since  $A$  and  $B$  are not the same size, then

$A + B =$  *can't be calculated*

$$b) \quad A + C = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -3 & -1 \\ 2 & 1 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 3 \\ 5 & 4 \end{bmatrix}$$

c)  $A: 3 \times 2$   $B: 3 \times 3$

$AB$  *can't be calculated*, since the inner are not equal.

d)  $B: 3 \times 3$   $A: 3 \times 2$

$$BA = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 12 \\ -1 & 2 \\ -10 & 5 \end{bmatrix}$$

e)  $C: 3 \times 2$   $D: 2 \times 2$

$$CD = \begin{bmatrix} -3 & -1 \\ 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 3 \\ 10 & -2 \\ 22 & -4 \end{bmatrix}$$

f)  $D: 2 \times 2$   $C: 3 \times 2$

$DC$  *can't be calculated*, since the inner are not equal.

g)  $B: 3 \times 3$   $D: 2 \times 2$

$BD$  *can't be calculated*, since the inner are not equal.

h)  $D: 2 \times 2$   $B: 3 \times 3$

$DB$  *can't be calculated*, since the inner are not equal.

i)  $A^2$  *can't be calculated*, since  $A$  is not square matrix.

j)  $B^2 = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -12 & 12 & 8 \\ -2 & -4 & -2 \\ -17 & -16 & 1 \end{bmatrix}$$

$$\begin{aligned}k) \quad D^2 &= \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 14 & -4 \\ 8 & -2 \end{bmatrix}\end{aligned}$$

### Exercise

Find if possible, a)  $A + B$  b)  $A - B$  c)  $2A$  d)  $2A - B$  e)  $B + \frac{1}{2}A$  f)  $AB$  g)  $BA$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix}$$

### Solution

$$\begin{aligned}a) \quad A + B &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 0 \\ 6 & 3 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}b) \quad A - B &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 4 \\ -2 & -1 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}c) \quad 2A &= 2 \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}d) \quad 2A - B &= \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2 \\ 8 & 4 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}e) \quad B + \frac{1}{2}A &= \begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{5}{2} & -1 \\ 6 & \frac{33}{2} \end{pmatrix}\end{aligned}$$

$$f) \quad AB = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix}$$



$$= \begin{pmatrix} 5 & 2 \\ 2 & 6 \end{pmatrix}$$

$$\begin{aligned} \text{g) } BA &= \begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -7 & -8 \\ 8 & 10 \end{pmatrix} \end{aligned}$$

### Exercise

Find if possible, a)  $A + B$  b)  $A - B$  c)  $2A$  d)  $2A - B$  e)  $B + \frac{1}{2}A$  f)  $AB$  g)  $BA$

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -1 \\ -1 & 8 \end{pmatrix}$$

### Solution

$$\begin{aligned} \text{a) } A + B &= \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -1 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 1 \\ 3 & 10 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{b) } A - B &= \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ -1 & 8 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 3 \\ 5 & -6 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{c) } 2A &= 2 \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 4 \\ 8 & 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{d) } 2A - B &= \begin{pmatrix} 2 & 4 \\ 8 & 4 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -1 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 3 \\ 7 & 12 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{e) } B + \frac{1}{2}A &= \begin{pmatrix} 2 & -1 \\ -1 & 8 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 1 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{5}{2} & 0 \\ 1 & 9 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} f) \quad AB &= \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 15 \\ 6 & 12 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} g) \quad BA &= \begin{pmatrix} 2 & -1 \\ -1 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 2 \\ 31 & 14 \end{pmatrix} \end{aligned}$$

### Exercise

Find if possible, a)  $A + B$  b)  $A - B$  c)  $2A$  d)  $2A - B$  e)  $B + \frac{1}{2}A$  f)  $AB$  g)  $BA$

$$A = \begin{pmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 8 & -1 \\ 4 & -3 \end{pmatrix}$$

### Solution

$$\begin{aligned} a) \quad A + B &= \begin{pmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{pmatrix} + \begin{pmatrix} 8 & -1 \\ 4 & -3 \end{pmatrix} \\ &= \text{can't be calculated} \end{aligned}$$

$$\begin{aligned} b) \quad A - B &= \begin{pmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{pmatrix} - \begin{pmatrix} 8 & -1 \\ 4 & -3 \end{pmatrix} \\ &= \text{can't be calculated} \end{aligned}$$

$$\begin{aligned} c) \quad 2A &= 2 \begin{pmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 12 & 0 & 6 \\ -2 & -8 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} d) \quad 2A - B &= \begin{pmatrix} 12 & 0 & 6 \\ -2 & -8 & 0 \end{pmatrix} - \begin{pmatrix} 8 & -1 \\ 4 & -3 \end{pmatrix} \\ &= \text{can't be calculated} \end{aligned}$$

$$\begin{aligned} e) \quad B + \frac{1}{2}A &= \begin{pmatrix} 8 & -1 \\ 4 & -3 \end{pmatrix} + \begin{pmatrix} 3 & 0 & \frac{3}{2} \\ -\frac{1}{2} & -2 & 0 \end{pmatrix} \\ &= \text{can't be calculated} \end{aligned}$$

$$\begin{aligned} f) \quad AB &= \begin{pmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{pmatrix} \begin{pmatrix} 8 & -1 \\ 4 & -3 \end{pmatrix} \\ &= \text{can't be calculated} \end{aligned}$$

$$\begin{aligned} \text{g) } BA &= \begin{pmatrix} 8 & -1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 19 & 4 & 24 \\ 27 & 12 & 12 \end{pmatrix} \end{aligned}$$

### Exercise

Find if possible, a)  $A + B$  b)  $A - B$  c)  $2A$  d)  $2A - B$  e)  $B + \frac{1}{2}A$  f)  $AB$  g)  $BA$

$$A = \begin{pmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{pmatrix}$$

### Solution

$$\begin{aligned} \text{a) } A + B &= \begin{pmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{pmatrix} \\ &= \text{can't be calculated} \end{aligned}$$

$$\begin{aligned} \text{b) } A - B &= \begin{pmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{pmatrix} \\ &= \text{can't be calculated} \end{aligned}$$

$$\begin{aligned} \text{c) } 2A &= 2 \begin{pmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 2 \\ -2 & 8 \\ 2 & 12 \end{pmatrix} \end{aligned}$$

$$\text{d) } 2A - B = \text{can't be calculated}$$

$$\text{e) } B + \frac{1}{2}A = \text{can't be calculated}$$

$$\begin{aligned} \text{f) } AB &= \begin{pmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{pmatrix} \\ &= \text{can't be calculated} \end{aligned}$$

$$\text{g) } BA = \begin{pmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -4 \\ 10 & 16 \\ 26 & 38 \end{pmatrix}$$

**Exercise**

Consider the matrices  $A = \begin{pmatrix} x^2 & x \\ -3 & 2 \end{pmatrix}$        $B = \begin{pmatrix} x & x^2 \\ 4 & x \end{pmatrix}$

Compute the following (where possible):

a)  $A + B$       b)  $A - B$       c)  $AB$       d)  $BA$       e)  $2A - 3B$

**Solution**

$$\begin{aligned} \text{a) } A + B &= \begin{pmatrix} x^2 & x \\ -3 & 2 \end{pmatrix} + \begin{pmatrix} x & x^2 \\ 4 & x \end{pmatrix} \\ &= \begin{pmatrix} x^2 + x & x + x^2 \\ 1 & 2 + x \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{b) } A - B &= \begin{pmatrix} x^2 & x \\ -3 & 2 \end{pmatrix} - \begin{pmatrix} x & x^2 \\ 4 & x \end{pmatrix} \\ &= \begin{pmatrix} x^2 - x & x - x^2 \\ -7 & 2 - x \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{c) } AB &= \begin{pmatrix} x^2 & x \\ -3 & 2 \end{pmatrix} \begin{pmatrix} x & x^2 \\ 4 & x \end{pmatrix} \\ &= \begin{pmatrix} x^3 + 4x & x^4 + x^2 \\ -3x + 8 & -3x^2 + 2x \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{d) } BA &= \begin{pmatrix} x & x^2 \\ 4 & x \end{pmatrix} \begin{pmatrix} x^2 & x \\ -3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} x^3 - 3x^2 & 3x^2 \\ 4x^2 - 3x & 6x \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{e) } 2A - 3B &= 2 \begin{pmatrix} x^2 & x \\ -3 & 2 \end{pmatrix} - 3 \begin{pmatrix} x & x^2 \\ 4 & x \end{pmatrix} \\ &= \begin{pmatrix} 2x^2 - 3x & 2x - 3x^2 \\ -18 & 4 - 3x \end{pmatrix} \end{aligned}$$

**Exercise**

Consider the matrices

$$A = \begin{pmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{pmatrix}$$

Compute the following (where possible):

$$a) A + B \quad b) A - B \quad c) AB \quad d) BA \quad e) 2A - 3B$$

**Solution**

$$\begin{aligned} a) \quad A + B &= \begin{pmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} + \begin{pmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 2x & -1 \\ 0 & 2 & 1 \\ 2 & -1 & 6 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} b) \quad A - B &= \begin{pmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} - \begin{pmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & -3 \\ 6 & 0 & 1 \\ -2 & -3 & -2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} c) \quad AB &= \begin{pmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -3x-2 & 2x-2 & -7 \\ 5 & 3x+2 & 7 \\ 10 & 0 & 4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} d) \quad BA &= \begin{pmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2+3x & 3x-2 & x-2 \\ 0 & -3x+1 & 7 \\ 5 & 2x-7 & 5 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} e) \quad 2A - 3B &= 2 \begin{pmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} - 3 \begin{pmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -4 & -x & -7 \\ 15 & -1 & 2 \\ -6 & -9 & -8 \end{pmatrix} \end{aligned}$$

### Exercise

Let  $B = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$ , show that  $B^4 = \begin{pmatrix} a^4 & 0 \\ a^3 + a^2b + ab^2 + b^3 & b^4 \end{pmatrix}$

### Solution

$$\begin{aligned} B^2 &= \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} \\ &= \begin{pmatrix} a^2 & 0 \\ a+b & b^2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} B^4 &= B^2 B^2 \\ &= \begin{pmatrix} a^2 & 0 \\ a+b & b^2 \end{pmatrix} \begin{pmatrix} a^2 & 0 \\ a+b & b^2 \end{pmatrix} \\ &= \begin{pmatrix} a^4 & 0 \\ a^3 + a^2b + ab^2 + b^3 & b^4 \end{pmatrix} \quad \checkmark \end{aligned}$$

### Exercise

Let  $B = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$ , show that  $B^n = \begin{pmatrix} a^n & 0 \\ \sum_{k=0}^{n-1} a^{n-1-k} b^k & b^n \end{pmatrix}$

### Solution

$$\begin{aligned} n=2 \quad \rightarrow \quad B^2 &= \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} \\ &= \begin{pmatrix} a^2 & 0 \\ a+b & b^2 \end{pmatrix} \quad \checkmark \end{aligned}$$

Let assume  $B^n = \begin{pmatrix} a^n & 0 \\ \sum_{k=0}^{n-1} a^{n-1-k} b^k & b^n \end{pmatrix}$  is true

We need to also prove that it is true for  $B^{n+1} = \begin{pmatrix} a^{n+1} & 0 \\ \sum_{k=0}^n a^{n-k} b^k & b^{n+1} \end{pmatrix}$

$$B^{n+1} = B^n B = \begin{pmatrix} a^n & 0 \\ \sum_{k=0}^{n-1} a^{n-1-k} b^k & b^n \end{pmatrix} \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$

$$= \begin{pmatrix} a^{n+1} & 0 \\ b^n + a \sum_{k=0}^{n-1} a^{n-1-k} b^k & b^{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} a^{n+1} & 0 \\ b^n + \sum_{k=0}^n a^{n-k} b^k & b^{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} a^{n+1} & 0 \\ \sum_{k=0}^n a^{n-k} b^k & b^{n+1} \end{pmatrix} \quad \checkmark$$

### Exercise

Let  $A = \begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix}$ . Prove that  $A^n = \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$  if  $n \geq 1$

### Solution

Using the principle of mathematical induction.

For  $n = 1$

$$\begin{aligned} A &= \begin{bmatrix} 1+6 & 4 \\ -9 & 1-6 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix} \quad \checkmark \quad P_1 \text{ is true} \end{aligned}$$

Assume that  $P_n$  is true,  $A^n = \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$

We need to prove that  $P_{n+1}$ :

$$\begin{aligned} A^{n+1} &= \begin{bmatrix} 1+6(n+1) & 4(n+1) \\ -9(n+1) & 1-6(n+1) \end{bmatrix} \\ &= \begin{bmatrix} 7+6n & 4n+4 \\ -9n-9 & -6n-5 \end{bmatrix} \quad \text{is also true.} \end{aligned}$$

$$\begin{aligned} A^{n+1} &= AA^n \\ &= \begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix} \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix} \\ &= \begin{bmatrix} 7+42n-36n & 28n+4-24n \\ -9-54n+45n & -36n-5+30n \end{bmatrix} \\ &= \begin{bmatrix} 7+6n & 4n+4 \\ -9n-9 & -6n-5 \end{bmatrix} \quad \checkmark \quad P_{n+1} \text{ is also true} \end{aligned}$$

$\therefore$  By mathematical induction, the proof of  $A^n = \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$  is completed.

### Exercise

Let  $A = \begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix}$ . Prove that  $A^n = \begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix}$  if  $n \geq 1$

### Solution

Using the principle of mathematical induction.

For  $n = 1$

$$\begin{aligned} A^1 &= \begin{bmatrix} (1+1)a & -a^2 \\ 1a^0 & (1-1)a \end{bmatrix} \\ &= \begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix} \quad \checkmark \quad P_1 \text{ is true} \end{aligned}$$

Assume that  $P_n$  is true,  $A^n = \begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix}$



We need to prove that  $P_{n+1}$  :

$$A^{n+1} = \begin{bmatrix} (n+2)a^{n+1} & -(n+1)a^{n+2} \\ (n+1)a^n & -na^{n+1} \end{bmatrix} \text{ is also } \textcolor{red}{true}.$$

$$\begin{aligned} A^{n+1} &= AA^n \\ &= \begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix} \\ &= \begin{bmatrix} 2(n+1)a^{n+1} - na^{n+1} & -2na^{n+2} - (1-n)a^{n+2} \\ (n+1)a^n & -na^{n+1} \end{bmatrix} \\ &= \begin{bmatrix} (2n+2-n)a^{n+1} & -(2n+1-n)a^{n+2} \\ (n+1)a^n & -na^{n+1} \end{bmatrix} \\ &= \begin{bmatrix} (n+2)a^{n+1} & -(n+1)a^{n+2} \\ (n+1)a^n & -na^{n+1} \end{bmatrix} \checkmark P_{n+1} \text{ is also true} \end{aligned}$$

$\therefore$  By mathematical induction, the proof of  $A^n = \begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix}$  is completed.

### **Exercise**

The following system of recurrence relations holds for all  $n \geq 0$

$$\begin{cases} x_{n+1} = 7x_n + 4y_n \\ y_{n+1} = -9x_n - 5y_n \end{cases}$$

Solve the system for  $x_n$  and  $y_n$  in terms of  $x_0$  and  $y_0$

### **Solution**

$$\begin{cases} x_{n+1} = 7x_n + 4y_n \\ y_{n+1} = -9x_n - 5y_n \end{cases}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$X_{n+1} = AX_n$$

$$A = \begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix} \quad X_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$X_1 = AX_0$$

$$X_2 = AX_1 = A(AX_0) = A^2X_0$$

$$X_3 = AX_2 = A(A^2X_0) = A^3X_0$$

$$\vdots \quad \vdots$$

$$X_n = A^nX_0$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix}^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Since, when  $\begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix}$  that implies  $A^n = \begin{pmatrix} 1+6n & 4n \\ -9n & 1-6n \end{pmatrix}$  (from previous prove).

$$\begin{aligned} \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= \begin{pmatrix} 1+6n & 4n \\ -9n & 1-6n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \begin{pmatrix} (1+6n)x_0 + 4ny_0 \\ -9nx_0 + (1-6n)y_0 \end{pmatrix} \end{aligned}$$

$$\therefore \begin{cases} x_n = (1+6n)x_0 + 4ny_0 \\ y_n = -9nx_0 + (1-6n)y_0 \end{cases}$$

### Exercise

If  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , prove that  $A^2 - (a+d)A + (ad-bc)I_{2 \times 2} = 0$

### Solution

$$\begin{aligned} A^2 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} \end{aligned}$$

$$A^2 - (a+d)A + (ad-bc)I = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} - (a+d) \begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} - \begin{pmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{pmatrix} + \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\
&= \begin{pmatrix} a^2 + bc - a^2 - ad + ad - bc & ab + bd - ab - bd \\ ac + cd - ac - cd & bc + d^2 - ad - d^2 + ad - bc \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \underline{0} \quad \checkmark
\end{aligned}$$

### Exercise

If  $A = \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}$ , use the fact  $A^2 = 4A - 3I$  and mathematical induction, to prove that

$$A^n = \frac{3^n - 1}{2} A + \frac{3 - 3^n}{2} I \quad \text{if } n \geq 1$$

### Solution

$$\begin{aligned}
A^2 &= \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 13 & -12 \\ 4 & -3 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
4A - 3I &= 4 \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 16 & -12 \\ 4 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 13 & -12 \\ 4 & -3 \end{pmatrix} \\
&= \underline{A^2}
\end{aligned}$$

Using mathematical induction model

$$\text{For } n=1 \rightarrow A^1 = \frac{3^1 - 1}{2} A + \frac{3 - 3^1}{2} I$$

$$A = A + 0 = A \quad \checkmark \quad \text{is true for } P_1$$

$$\text{Assume is true for } P_k \rightarrow A^k = \frac{3^k - 1}{2} A + \frac{3 - 3^k}{2} I$$

$$\text{We need to prove that is also true for } P_{k+1} \rightarrow A^{k+1} = \frac{3^{k+1} - 1}{2} A + \frac{3 - 3^{k+1}}{2} I$$

$$A^{k+1} = AA^k$$

$$\begin{aligned}
&= A \left( \frac{3^k - 1}{2} A + \frac{3 - 3^k}{2} I \right) \\
&= \frac{3^k - 1}{2} A^2 + \frac{3 - 3^k}{2} (AI) \quad A^2 = 4A - 3I \\
&= \frac{3^k - 1}{2} (4A - 3I) + \frac{3 - 3^k}{2} A \\
&= 2(3^k - 1)A - \frac{3(3^k - 1)}{2} I + \frac{3 - 3^k}{2} A \\
&= \left( 2 \cdot 3^k - 2 + \frac{3 - 3^k}{2} \right) A - \frac{3^{k+1} - 3}{2} I \\
&= \left( \frac{4 \cdot 3^k - 4 + 3 - 3^k}{2} \right) A - \frac{3^{k+1} - 3}{2} I \\
&= \left( \frac{3 \cdot 3^k - 1}{2} \right) A - \frac{3^{k+1} - 3}{2} I \\
&= \frac{3^{k+1} - 1}{2} A + \frac{3 - 3^{k+1}}{2} I \quad \checkmark \text{ is also true for } P_{k+1}
\end{aligned}$$

By mathematical induction, the proof that  $A^n = \frac{3^n - 1}{2} A + \frac{3 - 3^n}{2} I$  if  $n \geq 1$  is completed.

### Exercise

A sequence of numbers  $x_1, x_2, \dots, x_n, \dots$  satisfies the recurrence relation  $x_{n+1} = ax_n + bx_{n-1}$  for  $n \geq 1$ , where  $a$  and  $b$  are constants. Prove that

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = A \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

Where  $A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$  and hence express  $\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$  in terms of  $\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$ .

If  $a = 4$  and  $b = -3$ , use the previous question to find a formula for  $x_n$  in terms  $x_1$  and  $x_0$

### Solution

$$\begin{aligned}
x_{n+1} &= ax_n + bx_{n-1} \\
&= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}
\end{aligned}$$

$$x_n = x_n$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

$$= A \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

$$x_{n+1} = ax_n + bx_{n-1}$$

$$\underline{= 4x_n - 3x_{n-1} \quad |}$$

$$n=1 \rightarrow x_2 = 4x_1 - 3x_0$$

$$n=2 \rightarrow x_3 = 4x_2 - 3x_1$$

$$= 4(4x_1 - 3x_0) - 3x_1 = (4^2 - 3)x_1 - 3x_0$$

$$= 13x_1 - 12x_0$$

$$n=3 \rightarrow x_4 = 4x_3 - 3x_2$$

$$= 4(13x_1 - 12x_0) - 3(4x_1 - 3x_0)$$

$$= 40x_1 - 39x_0$$

$$n=4 \rightarrow x_5 = 4x_4 - 3x_3$$

$$= 4(40x_1 - 39x_0) - 3(13x_1 - 12x_0)$$

$$= 121x_1 - 120x_0$$

	$x_1$	$x_0$
$n=2 \rightarrow$	4	-3
$n=3 \rightarrow$	13	-12
$n=4 \rightarrow$	40	-39
$n=5 \rightarrow$	121	-120

$$\underline{x_n = \frac{3^n - 1}{2}x_1 + \frac{3 - 3^n}{2}x_0 \quad |}$$

**Exercise**

Prove  $(A + B)C = AC + BC$

**Solution**

$$\text{Let } A = \begin{bmatrix} a_{ij} \end{bmatrix}, \quad B = \begin{bmatrix} b_{ij} \end{bmatrix} \quad \& \quad C = \begin{bmatrix} c_{jk} \end{bmatrix}$$

$$\begin{aligned} \text{Let } D &= A + B \\ &= \begin{bmatrix} a_{ij} + b_{ij} \end{bmatrix} \\ &= \begin{bmatrix} d_{ij} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{Let } E &= AC \\ &= \sum a_{ij} c_{jk} \end{aligned}$$

$$\begin{aligned} \text{Let } F &= BC \\ &= \sum b_{ij} c_{jk} \end{aligned}$$

$$\begin{aligned} AC + BC &= E + F \\ &= \sum a_{ij} c_{jk} + \sum b_{ij} c_{jk} \\ &= \sum (a_{ij} c_{jk} + b_{ij} c_{jk}) \\ &= \sum (a_{ij} + b_{ij}) c_{jk} \\ &= \sum d_{ij} c_{jk} \\ &= DC \\ &= (A + B) C \quad \checkmark \end{aligned}$$

Thus,  $(A + B)C = AC + BC$

**Exercise**

Prove  $(A + B) + C = A + (B + C)$

**Solution**

$$\text{Let } A = \begin{bmatrix} a_{ij} \end{bmatrix}, \quad B = \begin{bmatrix} b_{ij} \end{bmatrix} \quad \& \quad C = \begin{bmatrix} c_{ij} \end{bmatrix}$$

$$\begin{aligned}
(A+B)+C &= \begin{bmatrix} a_{ij} \end{bmatrix} + \left( \begin{bmatrix} b_{ij} \end{bmatrix} + \begin{bmatrix} c_{ij} \end{bmatrix} \right) \\
&= \begin{bmatrix} a_{ij} \end{bmatrix} + \begin{bmatrix} b_{ij} + c_{ij} \end{bmatrix} \\
&= \begin{bmatrix} a_{ij} + (b_{ij} + c_{ij}) \end{bmatrix} \\
&= \begin{bmatrix} (a_{ij} + b_{ij}) + c_{ij} \end{bmatrix} \\
&= (A+B)+C
\end{aligned}$$

Thus,  $(A+B)+C = A+(B+C)$

### Exercise

Prove  $A(BC) = (AB)C$

### Solution

$$\text{Let } A = \begin{bmatrix} a_{ij} \end{bmatrix}, \quad B = \begin{bmatrix} b_{jk} \end{bmatrix} \quad \& \quad C = \begin{bmatrix} c_{kl} \end{bmatrix}$$

$$\begin{aligned}
\text{Let } E &= BC \\
&= \sum b_{jk} c_{kl} \\
&= \begin{bmatrix} e_{jl} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{Let } F &= AB \\
&= \sum a_{ij} b_{jk} \\
&= \begin{bmatrix} f_{ik} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
A(BC) &= AE \\
&= \begin{bmatrix} a_{ij} \end{bmatrix} \begin{bmatrix} e_{jl} \end{bmatrix} \\
&= \sum a_{ij} e_{jl} \\
&= \sum \left( a_{ij} \left( \sum b_{jk} c_{kl} \right) \right) \\
&= \sum \sum a_{ij} (b_{jk} c_{kl})
\end{aligned}$$

$$\begin{aligned}(AB)C &= FC \\&= \begin{bmatrix} f_{ik} \end{bmatrix} \begin{bmatrix} c_{kl} \end{bmatrix} \\&= \sum f_{ik} c_{kl} \\&= \sum \left( \sum (a_{ij} b_{jk}) \right) c_{kl} \\&= \sum \sum (a_{ij} b_{jk}) c_{kl} \\&= \sum \sum a_{ij} (b_{jk} c_{kl}) = \sum \sum (a_{ij} b_{jk}) c_{kl}\end{aligned}$$

Thus,  $A(BC) = (AB)C$

### ***Exercise***

Prove  $A(B - C) = AB - AC$

### **Solution**

$$\text{Let } A = \begin{bmatrix} a_{ij} \end{bmatrix}, \quad B = \begin{bmatrix} b_{jk} \end{bmatrix} \quad \& \quad C = \begin{bmatrix} c_{jk} \end{bmatrix}$$

$$\begin{aligned}\text{Let } D &= B - C \\&= \begin{bmatrix} b_{jk} - c_{jk} \end{bmatrix} \\&= \begin{bmatrix} d_{ij} \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\text{Let } E &= AB \\&= \sum a_{ij} b_{jk}\end{aligned}$$

$$\begin{aligned}\text{Let } F &= AC \\&= \sum a_{ij} c_{jk}\end{aligned}$$

$$\begin{aligned}AB - AC &= E - F \\&= \sum a_{ij} b_{jk} - \sum a_{ij} c_{jk} \\&= \sum (a_{ij} b_{jk} - a_{ij} c_{jk})\end{aligned}$$



$$\begin{aligned}
&= \sum a_{ij} (b_{jk} - c_{jk}) \\
&= \sum a_{ij} d_{jk} \\
&= AD \\
&= A(B - C) \quad \checkmark
\end{aligned}$$

Thus,  $A(B - C) = AB - AC$

### Exercise

Prove  $(A + B)(A - B) \neq A^2 - B^2$

#### Solution

$$\begin{aligned}
(A + B)(A - B) &= AA - AB + BA - BB \\
&= A^2 - AB + BA - B^2
\end{aligned}$$

Since  $AB \neq BA$

Therefore,  $(A + B)(A - B) \neq A^2 - B^2$

### Exercise

Show that  $AC = BC$ , even though  $A \neq B$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}; \quad C = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$$

#### Solution

$$\begin{aligned}
AC &= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
BC &= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}
\end{aligned}$$

Therefore,  $AC = BC$

**Exercise**

Verify  $AB = BA$  for the matrices below

$$A = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \quad B = \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix}$$

**Solution**

$$\begin{aligned} AB &= \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix} \\ &= \begin{pmatrix} \cos \alpha \cos \beta - \sin \alpha \sin \beta & -\cos \alpha \sin \beta - \sin \alpha \cos \beta \\ \sin \alpha \cos \beta + \cos \alpha \sin \beta & -\sin \alpha \sin \beta + \cos \alpha \cos \beta \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \\ &= \begin{pmatrix} \cos \beta \cos \alpha - \sin \beta \sin \alpha & -\sin \alpha \cos \beta - \cos \alpha \sin \beta \\ \cos \alpha \sin \beta + \sin \alpha \cos \beta & -\sin \alpha \sin \beta + \cos \alpha \cos \beta \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{pmatrix} \end{aligned}$$

Therefore,  $AB = BA$

**Exercise**

Solve for  $A$ :  $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

**Solution**

$$\begin{aligned} \text{Let } A &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} a_{11} + 2a_{21} & a_{12} + 2a_{22} \\ 3a_{11} + 5a_{21} & 3a_{12} + 5a_{22} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{cases} a_{11} + 2a_{21} = 1 \\ 3a_{11} + 5a_{21} = 0 \end{cases}$$

$$\begin{cases} -3a_{11} - 6a_{21} = -3 \\ 3a_{11} + 5a_{21} = 0 \end{cases}$$

$$\begin{cases} a_{11} = -5 \\ a_{21} = 3 \end{cases}$$

$$\begin{cases} a_{12} + 2a_{22} = 0 \\ 3a_{12} + 5a_{22} = 1 \end{cases}$$

$$\begin{cases} a_{12} = 2 \\ a_{22} = -1 \end{cases}$$

$$A = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

### ***Exercise***

Solve for  $X$  in the equation  $3X + 2A = B$ , given

$$A = \begin{pmatrix} -4 & 0 \\ 1 & -5 \\ -3 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{pmatrix}$$

### **Solution**

$$3X + 2A = B$$

$$X = -\frac{2}{3}A + \frac{1}{3}B$$

$$= -\frac{2}{3} \begin{pmatrix} -4 & 0 \\ 1 & -5 \\ -3 & 2 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & \frac{2}{3} \\ -\frac{4}{3} & \frac{11}{3} \\ \frac{10}{3} & 0 \end{pmatrix}$$

**Exercise**

For  $A = \begin{pmatrix} 4 & 1 & 3 \\ 0 & 5 & -2 \end{pmatrix}$ ,  $B = \begin{pmatrix} 0 & -2 & -4 \\ 3 & 0 & 1 \end{pmatrix}$

Solve the equations for  $X$ .

a)  $2X = 3A - B$

c)  $-2X = -5A + 3B$

b)  $3X - 2A + B = 0$

d)  $4X + 5A - 3B = 0$

**Solution**

a)  $2X = 3A - B$

$$\begin{aligned} X &= \frac{1}{2} \left( 3 \begin{pmatrix} 4 & 1 & 3 \\ 0 & 5 & -2 \end{pmatrix} - \begin{pmatrix} 0 & -2 & -4 \\ 3 & 0 & 1 \end{pmatrix} \right) \\ &= \frac{1}{2} \begin{pmatrix} 12 & 5 & 13 \\ -3 & 15 & -7 \end{pmatrix} \\ &= \begin{pmatrix} 6 & \frac{5}{2} & \frac{13}{2} \\ -\frac{3}{2} & \frac{15}{2} & -\frac{7}{2} \end{pmatrix} \end{aligned}$$

b)  $3X - 2A + B = 0$

$$\begin{aligned} X &= \frac{1}{3} \left( 2 \begin{pmatrix} 4 & 1 & 3 \\ 0 & 5 & -2 \end{pmatrix} + \begin{pmatrix} 0 & -2 & -4 \\ 3 & 0 & 1 \end{pmatrix} \right) \\ &= \frac{1}{3} \begin{pmatrix} 8 & 0 & 2 \\ 3 & 5 & -3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{8}{3} & 0 & \frac{2}{3} \\ 1 & \frac{5}{3} & -1 \end{pmatrix} \end{aligned}$$

c)  $-2X = -5A + 3B$

$$\begin{aligned} X &= \frac{5}{2} \begin{pmatrix} 4 & 1 & 3 \\ 0 & 5 & -2 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} 0 & -2 & -4 \\ 3 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 10 & \frac{11}{2} & \frac{27}{2} \\ -\frac{3}{2} & \frac{25}{2} & -\frac{13}{2} \end{pmatrix} \end{aligned}$$

d)  $4X + 5A - 3B = 0$

$$X = -\frac{5}{4} \begin{pmatrix} 4 & 1 & 3 \\ 0 & 5 & -2 \end{pmatrix} + \frac{3}{4} \begin{pmatrix} 0 & -2 & -4 \\ 3 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -\frac{11}{4} & -\frac{27}{4} \\ \frac{9}{4} & -\frac{25}{4} & \frac{13}{4} \end{pmatrix}$$

### Exercise

For  $A = \begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{pmatrix}$ ,  $B = \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 3 \end{pmatrix}$

Solve the equations for  $X$ .

a)  $X = 3A - 2B$

c)  $-2X = -4A + 3B$

b)  $3X + 2A = B$

d)  $2X + 3A - 4B = 0$

### Solution

a)  $X = 3A - 2B$

$$= 3 \begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{pmatrix} - 2 \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -8 & -7 \\ 7 & -2 \\ 1 & -18 \end{pmatrix}$$

b)  $3X + 2A = B$

$$X = -\frac{2}{3}A + \frac{1}{3}B$$

$$= -\frac{2}{3} \begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{3} & \frac{4}{3} \\ -\frac{4}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{11}{3} \end{pmatrix}$$

c)  $-2X = -4A + 3B$

$$X = 2A - \frac{3}{2}B$$

$$= 2 \begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{11}{2} & -5 \\ 5 & -\frac{3}{2} \\ 0 & -\frac{25}{2} \end{pmatrix}$$

$$d) \quad 2X + 3A - 4B = 0$$

$$X = -\frac{3}{2}A + 2B$$

$$= -\frac{3}{2} \begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{pmatrix} + 2 \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & \frac{11}{2} \\ -\frac{11}{2} & 2 \\ \frac{7}{2} & 12 \end{pmatrix}$$

### Exercise

$$\text{For } A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & -3 \\ 1 & 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 0 & 5 \\ 6 & 9 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 4 & 4 \\ 5 & -1 & 0 \\ 7 & 8 & -1 \end{pmatrix}$$

Solve the equations in few steps as you dare

$$e) \quad \frac{1}{2}(X + A) = 3[X + (2X + B)] + C$$

$$f) \quad 2(X + B) = 3\left(X + \left(\frac{1}{2}X + A\right)\right) + C$$

$$g) \quad 40(X + A) = 47(X + B) + 48(X + C)$$

$$h) \quad \sqrt{2}(X + C) = 31(X + \sqrt{2}(X + A - B))$$

### Solution

$$a) \quad \frac{1}{2}(X + A) = 3[X + (2X + B)] + C$$

$$\frac{1}{2}X + \frac{1}{2}A = 3X + 6X + 3B + C$$

$$\frac{1}{2}X - 9X = 3B + C - \frac{1}{2}A$$

$$\left(\frac{1}{2} - 9\right)X = \frac{1}{2}(6B + 2C - A)$$

$$-\frac{17}{2}X = \frac{1}{2}(6B + 2C - A)$$

$$X = \frac{1}{17}(A - 6B - 2C)$$

**OR**

$$A - 6B - 2C = (6 + 12 - 1)X$$

$$\begin{aligned} X &= \frac{1}{17} \left( \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & -3 \\ 1 & 0 & 4 \end{pmatrix} - 6 \begin{pmatrix} 3 & 3 & 3 \\ 3 & 0 & 5 \\ 6 & 9 & -1 \end{pmatrix} - 2 \begin{pmatrix} 4 & 4 & 4 \\ 5 & -1 & 0 \\ 7 & 8 & -1 \end{pmatrix} \right) \\ &= \begin{pmatrix} -\frac{24}{17} & -\frac{24}{17} & -\frac{24}{17} \\ -\frac{26}{17} & \frac{3}{17} & -\frac{33}{17} \\ -\frac{48}{17} & \frac{70}{17} & \frac{12}{17} \end{pmatrix} \end{aligned}$$

$$b) \quad 2(X + B) = 3\left(X + \left(\frac{1}{2}X + A\right)\right) + C$$

$$\left(2 - 3 - \frac{3}{2}\right)X = 3A + C - 2B$$

$$\begin{aligned} X &= -\frac{2}{5} \left( 3 \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & -3 \\ 1 & 0 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 4 & 4 \\ 5 & -1 & 0 \\ 7 & 8 & -1 \end{pmatrix} - 2 \begin{pmatrix} 3 & 3 & 3 \\ 3 & 0 & 5 \\ 6 & 9 & -1 \end{pmatrix} \right) \\ &= \begin{pmatrix} -\frac{8}{5} & -\frac{8}{5} & -\frac{8}{5} \\ -2 & -\frac{4}{5} & \frac{28}{5} \\ \frac{4}{5} & 4 & -\frac{26}{5} \end{pmatrix} \end{aligned}$$

$$c) \quad 40(X + A) = 47(X + B) + 48(X + C)$$

$$-55X = 47B + 48C - 40A$$

$$\begin{aligned} X &= -\frac{1}{55} \left( 47 \begin{pmatrix} 3 & 3 & 3 \\ 3 & 0 & 5 \\ 6 & 9 & -1 \end{pmatrix} + 48 \begin{pmatrix} 4 & 4 & 4 \\ 5 & -1 & 0 \\ 7 & 8 & -1 \end{pmatrix} - 40 \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & -3 \\ 1 & 0 & 4 \end{pmatrix} \right) \\ &= \begin{pmatrix} -\frac{23}{5} & -\frac{23}{5} & -\frac{23}{5} \\ -\frac{301}{55} & \frac{11}{5} & -\frac{71}{11} \\ -\frac{578}{55} & -\frac{807}{55} & \frac{51}{11} \end{pmatrix} \end{aligned}$$

$$d) \quad \sqrt{2}(X + C) = 31\left(X + \sqrt{2}(X + A - B)\right)$$

$$-(31+30\sqrt{2})X = \sqrt{2}(31A - 31B - C)$$

$$X = -\frac{\sqrt{2}}{31+30\sqrt{2}} \left( 31 \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & -3 \\ 1 & 0 & 4 \end{pmatrix} - 31 \begin{pmatrix} 3 & 3 & 3 \\ 3 & 0 & 5 \\ 6 & 9 & -1 \end{pmatrix} - \begin{pmatrix} 4 & 4 & 4 \\ 5 & -1 & 0 \\ 7 & 8 & -1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} \frac{35\sqrt{2}}{31+30\sqrt{2}} & \frac{35\sqrt{2}}{31+30\sqrt{2}} & \frac{35\sqrt{2}}{31+30\sqrt{2}} \\ \frac{36\sqrt{2}}{31+30\sqrt{2}} & -\frac{36\sqrt{2}}{31+30\sqrt{2}} & \frac{248\sqrt{2}}{31+30\sqrt{2}} \\ \frac{162\sqrt{2}}{31+30\sqrt{2}} & \frac{287\sqrt{2}}{31+30\sqrt{2}} & -\frac{156\sqrt{2}}{31+30\sqrt{2}} \end{pmatrix}$$

### Exercise

If  $f$  is the polynomial function is given by  $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$

Then  $f(A)$  is defined by  $f(A) = a_0I + a_1A + a_2A^2 + \dots + a_nA^n$

Find  $f(A)$

$$f(x) = x^2 + 3x - 2, \quad A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

### Solution

$$A^2 = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 9 \\ 0 & 1 \end{pmatrix}$$

$$f(A) = a_0I + a_1A + a_2A^2$$

$$= -2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 3 \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} + 1 \begin{pmatrix} 4 & 9 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 18 \\ 0 & 2 \end{pmatrix}$$



**Exercise**

If  $f$  is the polynomial function is given by  $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

Then  $f(A)$  is defined by  $f(A) = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n$

Find  $f(A)$

$$f(x) = x^2 - 2x + 3, \quad A = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$$

**Solution**

$$\begin{aligned} A^2 &= \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 10 & 9 \\ 9 & 13 \end{pmatrix} \end{aligned}$$

$$f(x) = 3 - 2x + x^2$$

$$\begin{aligned} f(A) &= a_0I + a_1A + a_2A^2 \\ &= 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} + 1 \begin{pmatrix} 10 & 9 \\ 9 & 13 \end{pmatrix} \\ &= \begin{pmatrix} 11 & 3 \\ 3 & 12 \end{pmatrix} \end{aligned}$$

**Exercise**

If  $f$  is the polynomial function is given by  $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

Then  $f(A)$  is defined by  $f(A) = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n$

Find  $f(A)$

$$f(x) = 2x^2 - 3x + 5, \quad A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$$

**Solution**

$$\begin{aligned} A^2 &= \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \\ &= \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix} \end{aligned}$$

$$f(x) = 5 - 3x + 2x^2$$

$$f(A) = a_0I + a_1A + a_2A^2$$

$$\begin{aligned} &= 5 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 3 \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} + 2 \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix} \\ &= \begin{pmatrix} 16 & -18 \\ -27 & 61 \end{pmatrix} \end{aligned}$$

**Exercise**

If  $f$  is the polynomial function is given by  $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

Then  $f(A)$  is defined by  $f(A) = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n$

Find  $f(A)$

$$g(x) = x^2 + 3x - 10, \quad A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$$

**Solution**

$$\begin{aligned} A^2 &= \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \\ &= \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix} \end{aligned}$$

$$g(x) = -10 + 3x + x^2$$

$$\begin{aligned} g(A) &= a_0I + a_1A + a_2A^2 \\ &= -10 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 3 \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} + \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

**Exercise**

If  $f$  is the polynomial function is given by  $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

Then  $f(A)$  is defined by  $f(A) = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n$

Find  $f(A)$

$$f(x) = x^2 - 2x - 5, \quad A = \begin{pmatrix} 2 & -5 & 8 \\ 3 & -6 & -7 \\ 4 & 0 & -1 \end{pmatrix}$$

**Solution**

$$\begin{aligned}
 A^2 &= \begin{pmatrix} 2 & -5 & 8 \\ 3 & -6 & -7 \\ 4 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & -5 & 8 \\ 3 & -6 & -7 \\ 4 & 0 & -1 \end{pmatrix} \\
 &= \begin{pmatrix} 21 & 20 & 43 \\ -40 & 21 & 73 \\ 4 & -20 & 33 \end{pmatrix}
 \end{aligned}$$

$$f(x) = -5 - 2x + x^2$$

$$\begin{aligned}
 f(A) &= a_0 I + a_1 A + a_2 A^2 \\
 &= -5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2 \begin{pmatrix} 2 & -5 & 8 \\ 3 & -6 & -7 \\ 4 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 21 & 20 & 43 \\ -40 & 21 & 73 \\ 4 & -20 & 33 \end{pmatrix} \\
 &= \begin{pmatrix} 7 & 30 & 27 \\ -46 & 28 & 87 \\ -4 & -20 & 30 \end{pmatrix}
 \end{aligned}$$

### Exercise

If  $f$  is the polynomial function is given by  $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

Then  $f(A)$  is defined by  $f(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$

Find  $f(A)$

$$g(x) = x^2 - 3x + 15, \quad A = \begin{pmatrix} 2 & -5 & 8 \\ 3 & -6 & -7 \\ 4 & 0 & -1 \end{pmatrix}$$

### Solution

$$\begin{aligned}
 A^2 &= \begin{pmatrix} 2 & -5 & 8 \\ 3 & -6 & -7 \\ 4 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & -5 & 8 \\ 3 & -6 & -7 \\ 4 & 0 & -1 \end{pmatrix} \\
 &= \begin{pmatrix} 21 & 20 & 43 \\ -40 & 21 & 73 \\ 4 & -20 & 33 \end{pmatrix}
 \end{aligned}$$

$$g(x) = 15 - 3x + x^2$$

$$g(A) = a_0 I + a_1 A + a_2 A^2$$

$$\begin{aligned}
&= 15 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 3 \begin{pmatrix} 2 & -5 & 8 \\ 3 & -6 & -7 \\ 4 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 21 & 20 & 43 \\ -40 & 21 & 73 \\ 4 & -20 & 33 \end{pmatrix} \\
&= \begin{pmatrix} 30 & 35 & 19 \\ -49 & 54 & 94 \\ -8 & -20 & 51 \end{pmatrix}
\end{aligned}$$

### Exercise

If  $f$  is the polynomial function is given by  $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

Then  $f(A)$  is defined by  $f(A) = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n$

Find  $f(A)$

$$f(x) = x^3 - 2x^2 - 3x + 5, \quad A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

### Solution

$$\begin{aligned}
A^2 &= \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 6 & 1 & -3 \\ 0 & 3 & 5 \\ -4 & 2 & 12 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
A^3 &= AA^2 \\
&= \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 6 & 1 & -3 \\ 0 & 3 & 5 \\ -4 & 2 & 12 \end{pmatrix} \\
&= \begin{pmatrix} 16 & 3 & -13 \\ -2 & 5 & 21 \\ -18 & 8 & 44 \end{pmatrix}
\end{aligned}$$

$$f(x) = 5 - 3x - 2x^2 + x^3$$

$$f(A) = a_0I + a_1A + a_2A^2 + a_3A^3$$

$$= 5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 3 \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{pmatrix} - 2 \begin{pmatrix} 6 & 1 & -3 \\ 0 & 3 & 5 \\ -4 & 2 & 12 \end{pmatrix} + \begin{pmatrix} 16 & 3 & -13 \\ -2 & 5 & 21 \\ -18 & 8 & 44 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 & -4 \\ -5 & 4 & 5 \\ 29 & 1 & 16 \end{pmatrix}$$



## ***Solution***      ***Section 1.4 – Inverse Matrices – Finding $A^{-1}$***

### ***Exercise***

Apply Gauss-Jordan method to find the inverse of this triangular “Pascal matrix”

$$\text{Triangular Pascal matrix} \quad A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

### **Solution**


$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 3 & 1 & -1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ \\ R_3 - 2R_2 \\ R_4 - 3R_2 \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & 1 & 2 & -3 & 0 & 1 \end{array} \right] \begin{array}{l} \\ \\ \\ R_4 - 3R_3 \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{array} \right]$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$

 The inverse matrix  $A^{-1}$  looks like  $A$ , except odd-numbered diagonals are multiplied by -1.

**Exercise**

If  $A$  is invertible and  $AB = AC$ , prove that  $B = C$

**Solution**

$$AB = AC$$

*Multiply by  $A^{-1}$  both sides.*

$$A^{-1}(AB) = A^{-1}(AC)$$

*Multiplication is associative*

$$(A^{-1}A)B = (A^{-1}A)C$$

$$A^{-1}A = I$$

$$IB = IC$$

$$B = C \quad \checkmark$$

**Exercise**

If  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ , find two matrices  $B \neq C$  such that  $AB = AC$

**Solution**

$$\text{Let } B = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\underline{B \neq C \Rightarrow AB = AC}$$



**Exercise**

If  $A$  has **row** 1 + **row** 2 = **row** 3, show that  $A$  is not invertible

- Explain why  $Ax = (1, 0, 0)$  can't have a solution.
- Which right sides  $(b_1, b_2, b_3)$  might allow a solution to  $Ax = b$
- What happens to **row** 3 in elimination?

**Solution**

- Let  $A_1, A_2, A_3$  be the row vectors of  $A$  and  $x$  is a solution to  $Ax = (1, 0, 0)$ .

Then  $A_1 \cdot x = 1, A_2 \cdot x = 0, A_3 \cdot x = 0$ .

Since  $A_1 + A_2 = A_3$

Means  $A_1 \cdot x + A_2 \cdot x = A_3 \cdot x$

Implies  $1 + 0 = 0$  a contradiction

- If  $Ax = (b_1, b_2, b_3)$

$A_1 \cdot x = b_1, A_2 \cdot x = b_2, A_3 \cdot x = b_3$

Since  $A_1 + A_2 = A_3$

$A_1 \cdot x + A_2 \cdot x = A_3 \cdot x$

$\Rightarrow b_1 + b_2 = b_3$

- In the elimination matrix, the third row will be zero.

**Exercise**

True or false (with a counterexample if false and a reason if true):

- A 4 by 4 matrix with a row of zeros is not invertible.
- A matrix with 1's down the main diagonal is invertible.
- If  $A$  is invertible then  $A^{-1}$  is invertible.
- If  $A$  is invertible then  $A^2$  is invertible.

**Solution**

- True, because it can have at most 3 pivots.

- False, if the matrix:  $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$  and only has 2 pivots, thus is not invertible.

c) True, If  $A$  is invertible then necessarily  $A^{-1}$  is invertible.

d) True,  $A^2x = 0$  where  $x$  is nonzero matrix.

$$A^{-1}A^2x = (A^{-1}A)Ax = IAx = Ax = 0$$

Since  $A$  is invertible, this can only be true if  $x$  was zero to begin with. Thus  $A^2$  must also be invertible.

### Exercise

Do there exist 2 by 2 matrices  $A$  and  $B$  with real entries such that  $AB - BA = I$ , where  $I$  is the identity matrix?

### Solution

$$\text{Let } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$\begin{aligned} AB &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ &= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{pmatrix} \end{aligned}$$

$$\begin{aligned} AB - BA &= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} - \begin{pmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{pmatrix} \\ &= \begin{pmatrix} bg - cf & af + bh - be - df \\ ce + dg - ag - ch & cf - bg \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{cases} bg - cf = 1 \\ af + bh - be - df = 0 \\ ce + dg - ag - ch = 0 \\ cf - bg = 1 \end{cases}$$

$$\rightarrow \begin{cases} bg - cf = 1 \\ cf - bg = 1 \\ \hline 0 = 2 \end{cases}$$

Therefore,  $AB - BA \neq I$  for any 2 by 2 matrices.

### Exercise

If  $B$  is the inverse of  $A^2$ , show that  $AB$  is the inverse of  $A$ .

### Solution

Since  $B$  is the inverse of  $A^2$  that implies:

$$\begin{aligned} B &= (A^2)^{-1} \\ &= (AA)^{-1} \\ &= \underline{A^{-1}A^{-1}} \end{aligned}$$

Show that  $AB$  is the inverse of  $A$

$$\begin{aligned} (AB)A &= \left( A(A^{-1}A^{-1}) \right) A \\ &= \left( (AA^{-1})A^{-1} \right) A \\ &= (IA^{-1})A \\ &= A^{-1}A \\ &= \underline{I} \end{aligned}$$

Therefore,  $AB$  is the inverse of  $A$ .

### Exercise

Find and check the inverses (assuming they exist) of these block matrices.

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}$$

### Solution

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ C+A & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \Rightarrow C+A=0 \rightarrow \textcolor{red}{A} = -\textcolor{red}{C}$$

$$\begin{pmatrix} \textcolor{blue}{I} & \textcolor{blue}{0} \\ \textcolor{blue}{C} & \textcolor{blue}{I} \end{pmatrix}^{-1} = \begin{pmatrix} \textcolor{blue}{I} & \textcolor{blue}{0} \\ -\textcolor{blue}{C} & \textcolor{blue}{I} \end{pmatrix}$$

$$\begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \begin{bmatrix} E & 0 \\ F & G \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} AE & 0 \\ CE+DF & DG \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\Rightarrow \begin{cases} AE = I \\ CE+DF = 0 \\ DG = I \end{cases} \rightarrow \begin{cases} \textcolor{blue}{E} = \textcolor{blue}{A}^{-1} \\ \textcolor{blue}{G} = \textcolor{blue}{D}^{-1} \end{cases}$$

$$CE+DF=0 \rightarrow CA^{-1}+DF=0$$

$$DF = -CA^{-1}$$

$$D^{-1}DF = -D^{-1}CA^{-1}$$

$$IF = -D^{-1}CA^{-1}$$

$$F = -D^{-1}CA^{-1}$$

$$\begin{pmatrix} \textcolor{blue}{A} & \textcolor{blue}{0} \\ \textcolor{blue}{C} & \textcolor{blue}{D} \end{pmatrix}^{-1} = \begin{pmatrix} \textcolor{blue}{A}^{-1} & \textcolor{blue}{0} \\ -\textcolor{blue}{D}^{-1}\textcolor{blue}{C}\textcolor{blue}{A}^{-1} & \textcolor{blue}{D}^{-1} \end{pmatrix}$$

$$\begin{bmatrix} 0 & I \\ I & D \end{bmatrix} \begin{bmatrix} A & I \\ I & B \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & B \\ A+D & I+DB \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\rightarrow \begin{cases} \textcolor{blue}{B} = \textcolor{blue}{0} \\ A+D=0 \\ I+DB=I \end{cases} \Rightarrow \begin{cases} \textcolor{blue}{A} = -\textcolor{blue}{D} \\ \textcolor{blue}{D}\textcolor{blue}{B} = \textcolor{blue}{0} \end{cases}$$

$$\begin{pmatrix} 0 & I \\ I & D \end{pmatrix}^{-1} = \begin{pmatrix} -D & I \\ I & 0 \end{pmatrix}$$

**Exercise**

For which three numbers  $c$  is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

**Solution**

$$c = 0, A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 7 & 0 \end{bmatrix} \text{ (zero column 2 / row 2)}$$

$$c = 2, A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 8 & 7 & 2 \end{bmatrix} \text{ (equal rows)}$$

$$c = 7, A = \begin{bmatrix} 2 & 7 & 7 \\ 7 & 7 & 7 \\ 8 & 7 & 7 \end{bmatrix} \text{ (equal columns)}$$

**Exercise**

Find  $A^{-1}$  and  $B^{-1}$  (if they exist) by elimination.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

**Solution**

$$\left( \begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array} \right) \frac{1}{2}R_1$$

$$\left( \begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array} \right) \frac{2}{3}R_2$$

$$\left( \begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - \frac{1}{2}R_2 \\ R_3 - \frac{1}{2}R_2 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array} \right) \frac{3}{4}R_3$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array} \right) \begin{array}{l} R_1 - \frac{1}{3}R_3 \\ R_2 - \frac{1}{3}R_3 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$\left( \begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array} \right) \frac{1}{2}R_1$$

$$\left( \begin{array}{ccc|ccc} 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} R_2 + R_1 \\ R_3 + R_1 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array} \right) R_3 + R_2$$

$$\left( \begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 1 & 1 \end{array} \right)$$

$B^{-1}$  *doesn't* exist, and if we add the columns in  $B$ , the result is zero.

### Exercise

Find  $A^{-1}$  using the Gauss-Jordan method, which has a remarkable inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

### Solution

$$\left( \begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) R_1 + R_2$$

$$\left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) R_2 + R_3$$

$$\left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) R_3 + R_4$$

$$\left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

**Exercise**

Find the inverse, if exists of  $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$

**Solution**

$$\begin{aligned} \begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}^{-1} &= \frac{1}{12+12} \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{12} & \frac{1}{6} \\ \frac{1}{8} & \frac{1}{4} \end{bmatrix} \end{aligned}$$

**Exercise**

Find the inverse, if exists of  $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$

**Solution**

$$\begin{aligned} \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}^{-1} &= \frac{1}{7-8} \begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix} \\ &= - \begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix} \end{aligned}$$

**Exercise**

Find the inverse, if exists of  $\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$

**Solution**

$$\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}^{-1} = \frac{1}{-15-24} \begin{bmatrix} 5 & -6 \\ -4 & -3 \end{bmatrix}$$



$$= -\frac{1}{39} \begin{bmatrix} 5 & -6 \\ -4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{39} & \frac{2}{13} \\ \frac{4}{39} & \frac{1}{13} \end{bmatrix}$$

**Exercise**

Find the inverse, if exists, of  $A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

**Solution**

$$A^{-1} = \frac{1}{-4+6} \begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

**Exercise**

Find the inverse, if exists, of  $A = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$

**Solution**

$$A^{-1} = \frac{1}{10-10} \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

$$= \frac{1}{\textcolor{red}{0}} \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

$\therefore$  Inverse *doesn't exist*

**Exercise**

Find the inverse of  $A = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$

**Solution**

$$\left[ \begin{array}{cc|cc} -2 & 3 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{array} \right] \quad -\frac{1}{2}R_1 \quad \quad 1 \quad -\frac{3}{2} \quad -\frac{1}{2} \quad 0$$

$$\begin{array}{l}
 \left[ \begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ -3 & 4 & 0 & 1 \end{array} \right] \quad R_2 + 3R_1 \quad \begin{array}{cccc} -3 & 4 & 0 & 1 \\ 3 & -\frac{9}{2} & -\frac{3}{2} & 0 \\ \hline 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array} \\
 \\
 \left[ \begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array} \right] \quad -2R_2 \quad \begin{array}{cccc} & & & \\ 0 & 1 & 3 & -2 \end{array} \\
 \\
 \left[ \begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 3 & -2 \end{array} \right] \quad R_1 + \frac{3}{2}R_2 \quad \begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & \frac{9}{2} & -3 \\ \hline 1 & 0 & 4 & -3 \end{array} \\
 \\
 \left[ \begin{array}{cc|cc} 1 & 0 & 4 & -3 \\ 0 & 1 & 3 & -2 \end{array} \right] \\
 \\
 A^{-1} = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}
 \end{array}$$

**Exercise**

Find the inverse of  $A = \begin{bmatrix} a & b \\ 3 & 3 \end{bmatrix}$

**Solution**

$$\begin{aligned}
 A^{-1} &= \frac{1}{3a-3b} \begin{bmatrix} 3 & -b \\ -3 & a \end{bmatrix} \\
 &= \begin{bmatrix} \frac{3}{3(a-b)} & \frac{-b}{3(a-b)} \\ \frac{-3}{3(a-b)} & \frac{a}{3(a-b)} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{a-b} & \frac{-b}{3(a-b)} \\ \frac{-1}{a-b} & \frac{a}{3(a-b)} \end{bmatrix}
 \end{aligned}$$

**Exercise**

Find the inverse of  $A = \begin{bmatrix} -2 & a \\ 4 & a \end{bmatrix}$

**Solution**

$$A^{-1} = \frac{1}{-2a-4a} \begin{bmatrix} a & -a \\ -4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{-6a} & \frac{-a}{-6a} \\ \frac{-4}{-6a} & \frac{-2}{-6a} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{6} & \frac{1}{6} \\ \frac{2}{3a} & \frac{1}{3a} \end{bmatrix}$$

**Exercise**

Find the inverse of  $A = \begin{bmatrix} 4 & 4 \\ b & a \end{bmatrix}$

**Solution**

$$A^{-1} = \frac{1}{4a - 4b} \begin{bmatrix} a & -4 \\ -b & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{4(a-b)} & \frac{-4}{4(a-b)} \\ \frac{-b}{4(a-b)} & \frac{4}{4(a-b)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{4(a-b)} & \frac{-1}{a-b} \\ \frac{-b}{4(a-b)} & \frac{1}{a-b} \end{bmatrix}$$

**Exercise**

Find the inverse of  $A = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$

**Solution**

$$A^{-1} = \frac{1}{-1+4} \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

**Exercise**

Find the inverse of  $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

**Solution**

$$\begin{aligned} A^{-1} &= \frac{1}{3} \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \end{aligned}$$

**Exercise**

Find the inverse of  $A = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

**Solution**

$$\begin{aligned} A^{-1} &= -\frac{1}{14} \begin{pmatrix} -1 & -4 \\ -3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{14} & \frac{2}{7} \\ \frac{3}{14} & -\frac{1}{7} \end{pmatrix} \end{aligned}$$

**Exercise**

Find the inverse of  $A = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

**Solution**

$$\begin{aligned} A^{-1} &= -\frac{1}{5} \begin{pmatrix} -1 & -3 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} \end{aligned}$$

**Exercise**

Find the inverse of  $A = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$

**Solution**

$$\begin{aligned}
 A^{-1} &= \frac{1}{11} \begin{pmatrix} 5 & -3 \\ 2 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{5}{11} & -\frac{3}{11} \\ \frac{2}{11} & \frac{1}{11} \end{pmatrix}
 \end{aligned}$$

**Exercise**

Find the inverse of  $A = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$

**Solution**

$$A^{-1} = \frac{1}{\textcolor{red}{0}} \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$$

$\therefore$  Inverse *doesn't exist*

**Exercise**

Find the inverse of  $A = \begin{pmatrix} -6 & 9 \\ 2 & -3 \end{pmatrix}$

**Solution**

$$A^{-1} = \frac{1}{\textcolor{red}{18-18}} \begin{pmatrix} & \\ & \end{pmatrix}$$

$\therefore$  Inverse *doesn't exist*

**Exercise**

Find the inverse of  $A = \begin{pmatrix} -2 & 7 \\ 0 & 2 \end{pmatrix}$

**Solution**

$$\begin{aligned}
 A^{-1} &= \frac{1}{4} \begin{pmatrix} 2 & -7 \\ 0 & -2 \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{2} & -\frac{7}{4} \\ 0 & -\frac{1}{2} \end{pmatrix}
 \end{aligned}$$

**Exercise**

Find the inverse of  $A = \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$

**Solution**

$$A = \frac{1}{-16 + 16} \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$$

$\therefore$  Inverse *doesn't exist*

**Exercise**

Find the inverse of  $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

**Solution**

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

**Exercise**

Find the inverse of  $A = \begin{pmatrix} 2 & 1 \\ a & a \end{pmatrix}$

**Solution**

$$\begin{aligned} A^{-1} &= \frac{1}{a} \begin{pmatrix} a & -1 \\ -a & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\frac{1}{a} \\ -1 & \frac{2}{a} \end{pmatrix} \end{aligned}$$

**Exercise**

Find the inverse of  $A = \begin{pmatrix} b & 3 \\ b & 2 \end{pmatrix}$

**Solution**

$$\begin{aligned} A^{-1} &= -\frac{1}{b} \begin{pmatrix} 2 & -3 \\ -b & b \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2}{b} & \frac{3}{b} \\ 1 & -1 \end{pmatrix} \end{aligned}$$

**Exercise**

Find the inverse of  $A = \begin{pmatrix} 1 & a \\ 3 & a \end{pmatrix}$

**Solution**

$$\begin{aligned} A^{-1} &= -\frac{1}{2a} \begin{pmatrix} a & -a \\ -3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2a} & -\frac{1}{2a} \end{pmatrix} \end{aligned}$$

**Exercise**

Find the inverse of  $A = \begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$

**Solution**

$$\begin{aligned} A^{-1} &= \frac{1}{a^2 - 4} \begin{pmatrix} a & -2 \\ -2 & a \end{pmatrix} \\ &= \begin{pmatrix} \frac{a}{a^2 - 4} & \frac{-2}{a^2 - 4} \\ \frac{-2}{a^2 - 4} & \frac{a}{a^2 - 4} \end{pmatrix} \end{aligned}$$

**Exercise**

Find the inverse of  $A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$

**Solution**

$$A^{-1} = \frac{1}{\textcolor{red}{0}} \begin{pmatrix} & \\ & \end{pmatrix}$$

$\therefore$  Inverse *doesn't exist*

**Exercise**

Find the inverse of  $A = \begin{pmatrix} -3 & \frac{1}{2} \\ 6 & -1 \end{pmatrix}$

**Solution**

$$A^{-1} = \frac{1}{\textcolor{red}{0}} \begin{pmatrix} & \\ & \end{pmatrix}$$

$\therefore$  Inverse *doesn't exist*

**Exercise**

Find the inverse of  $A = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$

**Solution**

$$A^{-1} = \frac{1}{\textcolor{red}{0}} \begin{pmatrix} & \\ & \end{pmatrix}$$

$\therefore$  Inverse *doesn't exist*

**Exercise**

Find the inverse of  $A = \begin{pmatrix} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{pmatrix}$

**Solution**

$$\begin{aligned} A^{-1} &= \frac{1}{\sin^2 \theta + \cos^2 \theta} \begin{pmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{pmatrix} \\ &= \begin{pmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{pmatrix} \end{aligned}$$

**Exercise**

Find  $A^{-1}$  if  $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

**Solution**

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \quad \begin{array}{cccccc} 2 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -2 & 0 & 0 \\ \hline 0 & -2 & -3 & -2 & 1 & 0 \end{array} \quad \begin{array}{cccccc} 3 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & -3 & 0 & 0 \\ \hline 0 & 0 & -3 & -3 & 0 & 1 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array} \right] -\frac{1}{2}R_2$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array} \right] -\frac{1}{3}R_3$$



$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array} \right] \begin{array}{l} R_1 - R_3 \\ R_2 - \frac{3}{2}R_3 \end{array} \quad \begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ \hline 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array} \quad \begin{array}{cccccc} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ \hline 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

### Exercise

Find  $A^{-1}$ , where  $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

### Solution

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 + 2R_1 \end{array} \quad \begin{array}{cccccc} 3 & 5 & 3 & 0 & 1 & 0 \\ -3 & -6 & 3 & -3 & 0 & 0 \\ \hline 0 & -1 & 6 & -3 & 1 & 0 \end{array} \quad \begin{array}{cccccc} 2 & 4 & 3 & 0 & 0 & 1 \\ -2 & -4 & 2 & -2 & 0 & 0 \\ \hline 0 & 0 & 5 & -2 & 0 & 1 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right] -R_2$$

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right] R_1 - 2R_2 \quad \begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 12 & -6 & 2 & 0 \\ \hline 1 & 0 & 11 & -5 & 2 & 0 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right] \frac{1}{5}R_3 \quad \begin{array}{cccccc} 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array} \right] \begin{array}{l} R_1 - 11R_3 \\ R_2 + 6R_3 \end{array}$$

$$\begin{array}{cccccc} 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 6 & -\frac{12}{5} & 0 & \frac{6}{5} \\ \hline 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 0 & -11 & \frac{22}{5} & 0 & -\frac{11}{5} \\ \hline 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{bmatrix}$$

**Exercise**

Find  $A^{-1}$ , where  $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

**Solution**

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccccc} -2 & 0 & 1 & 0 & 1 & 0 \\ 2 & 4 & -2 & 2 & 0 & 0 \\ \hline 0 & 4 & -1 & 2 & 1 & 0 \end{array}$$

$$\begin{array}{cccccc} 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & -2 & 1 & -1 & 0 & 0 \\ \hline 0 & -3 & 1 & -1 & 0 & 1 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 4 & -1 & 2 & 1 & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array} \right] \frac{1}{4}R_2$$

$$0 \quad 1 \quad -\frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{4} \quad 0$$

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\begin{array}{cccccc} 0 & -3 & 1 & -1 & 0 & 1 \\ 0 & 3 & -\frac{3}{4} & \frac{3}{2} & \frac{3}{4} & 0 \\ \hline 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{array}$$

$$\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & \frac{1}{2} & -1 & -\frac{1}{2} & 0 \\ \hline 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{array} \right] 4R_3$$

$$0 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array} \right] \begin{array}{l} R_1 + \frac{1}{2}R_3 \\ R_2 + \frac{1}{4}R_3 \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 \\ \hline 1 & 0 & 0 & 1 & 1 & 2 \end{array}$$

$$\begin{array}{cccccc} 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ \hline 0 & 1 & 0 & 1 & 1 & 1 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

**Exercise**

Find  $A^{-1}$ , where  $A = \begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$

**Solution**

$$\left[ \begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array} \right] \quad \frac{1}{-2}R_1 \quad \begin{array}{cccccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array} \right] \quad R_2 - 4R_1 \quad \begin{array}{cccccc} 4 & -1 & 3 & 0 & 1 & 0 \\ -4 & 10 & 6 & 2 & 0 & 0 \\ \hline 0 & 9 & 9 & 2 & 1 & 0 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array} \right] \quad R_3 - 7R_1 \quad \begin{array}{cccccc} 7 & -2 & 5 & 0 & 0 & 1 \\ -7 & \frac{35}{2} & \frac{21}{2} & \frac{7}{2} & 0 & 0 \\ \hline 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array} \right] \quad \frac{1}{9}R_2 \quad \begin{array}{cccccc} 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array} \right] \quad R_3 - \frac{31}{2}R_2 \quad \begin{array}{cccccc} 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \\ 0 & -\frac{31}{2} & -\frac{31}{2} & -\frac{31}{9} & -\frac{31}{18} & 0 \\ \hline 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array} \right]$$

$\therefore$  The inverse matrix *doesn't exist*

### Exercise

Find the inverse, if exists, of  $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}$

### Solution

$$\left( \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array} \right) \quad R_2 + R_1 \quad \begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ \hline 0 & 4 & 4 & 1 & 1 & 0 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 4 & 1 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array} \right) \quad \frac{1}{4}R_2$$

$$\left( \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array} \right) \quad \begin{array}{l} R_1 - R_2 \\ R_3 - 4R_2 \end{array} \quad \begin{array}{cccccc} 0 & 4 & 3 & 0 & 0 & 1 \\ 0 & -4 & -4 & -1 & -1 & 0 \\ \hline 0 & 0 & -1 & -1 & -1 & 1 \end{array} \quad \begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ \hline 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array} \right) \quad -R_3$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array} \right) \quad \begin{array}{l} R_1 + R_3 \\ R_2 - R_3 \end{array} \quad \begin{array}{cccccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ \hline 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1 \end{array} \quad \begin{array}{cccccc} 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \\ \hline 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1 \\ 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{7}{4} & \frac{3}{4} & -1 \\ -\frac{3}{4} & -\frac{3}{4} & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

### Exercise

Find the inverse, if exists, of  $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$

### Solution

$$\left( \begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ -2 & -3 & 0 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ \\ R_3 + 2R_1 \end{array} \quad \begin{array}{cccccc} -2 & -3 & 0 & 0 & 0 & 1 \\ 2 & -2 & 2 & 2 & 0 & 0 \\ \hline 0 & -5 & 2 & 2 & 0 & 1 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array} \right) \begin{array}{l} \\ \\ -\frac{1}{2}R_2 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 + R_2 \\ \\ R_3 + 5R_2 \end{array} \quad \begin{array}{cccccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ \hline 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \end{array} \quad \begin{array}{cccccc} 0 & -5 & 2 & 2 & 0 & 1 \\ 0 & 5 & -\frac{5}{2} & 0 & -\frac{5}{2} & 0 \\ \hline 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \end{array} \right) \begin{array}{l} \\ \\ -2R_3 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -4 & 5 & -2 \end{array} \right) \begin{array}{l} R_1 - \frac{1}{2}R_3 \\ R_2 + \frac{1}{2}R_3 \\ \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & -4 & 5 & -2 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{pmatrix}$$

**Exercise**

Find the inverse, if exists, of  $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

**Solution**

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array} \right) \frac{1}{2}R_2$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array} \right) R_3 + R_2$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array} \right) 2R_3$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right) \begin{array}{l} R_1 - 2R_3 \\ R_2 - \frac{5}{2}R_3 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix}$$

### Exercise

Find the inverse, if exists, of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 1 & 2 \end{pmatrix}$$

### Solution

$$\left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & -1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -4 & -3 & 1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array} \right) -R_2$$

$$\left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 2R_2 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & 0 & 7 & 3 & -2 & 1 \end{array} \right) \frac{1}{7}R_3$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{array} \right) \begin{array}{l} R_1 + 3R_3 \\ R_2 - 4R_3 \\ \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ 0 & 1 & 0 & \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{pmatrix}$$

**Exercise**

Find the inverse, if exists, of  $A = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$

**Solution**

$$\left( \begin{array}{ccc|ccc} 3 & 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array} \right) \frac{1}{3}R_1$$

$$\left( \begin{array}{ccc|ccc} 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & -3 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - R_2 \\ R_3 + 3R_2 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{7}{3} & -\frac{5}{3} & 3 & 1 \end{array} \right) \frac{3}{7}R_3$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array} \right) \begin{array}{l} R_1 + \frac{1}{3}R_3 \\ R_2 - \frac{2}{3}R_3 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ 0 & 1 & 0 & \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{pmatrix}$$



**Exercise**

Find the inverse, if exists, of  $A = \begin{pmatrix} -3 & 1 & -1 \\ 1 & -4 & -7 \\ 1 & 2 & 5 \end{pmatrix}$

**Solution**

$$\left( \begin{array}{ccc|ccc} -3 & 1 & -1 & 1 & 0 & 0 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array} \right) \quad -\frac{1}{3}R_1$$

$$\left( \begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array} \right) \quad \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{11}{3} & -\frac{22}{3} & \frac{1}{3} & 1 & 0 \\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1 \end{array} \right) \quad -\frac{3}{11}R_2$$

$$\left( \begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0 \\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1 \end{array} \right) \quad R_3 - \frac{7}{3}R_2$$

$$\left( \begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0 \\ 0 & 0 & 0 & -\frac{1}{11} & -\frac{3}{11} & 0 \end{array} \right)$$

$\therefore$  Inverse **does not exist**

**Exercise**

Find the inverse, if exists, of  $A = \begin{pmatrix} 1 & 1 & -3 \\ 2 & -4 & 1 \\ -5 & 7 & 1 \end{pmatrix}$

**Solution**

$$\left( \begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 2 & -4 & 1 & 0 & 1 & 0 \\ -5 & 7 & 1 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 5R_1 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & -6 & 7 & -2 & 1 & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array} \right) -\frac{1}{6}R_2$$

$$\left( \begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 12R_2 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & 0 \end{array} \right)$$

$\therefore$  Inverse *does not exist*

### Exercise

Find the inverse, if exists, of  $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{pmatrix}$

### Solution

$$\left( \begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_3 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right) -R_2$$

$$\left( \begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right) R_1 - 2R_2$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right) \frac{1}{5}R_3$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array} \right) \begin{array}{l} R_1 - 11R_3 \\ R_2 + 6R_3 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{pmatrix}$$

### Exercise

Find the inverse, if exists of  $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

### Solution

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad R_3 - R_1$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array} \right] \quad R_3 - R_2$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array} \right] \quad -\frac{1}{2}R_3$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array} \right] \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

**Exercise**

Find the inverse, if exists of  $A = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{pmatrix}$

**Solution**

$$\left[ \begin{array}{ccc|ccc} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & 1 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} & 0 & 1 & 0 \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} & 0 & 0 & 1 \end{array} \right] \begin{array}{l} 5R_1 \\ 10R_2 \\ 10R_3 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 2 & 2 & 1 & 0 & 10 & 0 \\ 2 & -8 & 1 & 0 & 0 & 10 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 0 & 5 & -10 & 10 & 0 \\ 0 & -10 & 5 & -10 & 0 & 10 \end{array} \right] \begin{array}{l} \\ R_2 - \frac{1}{10}R_3 \\ \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & \frac{9}{2} & -9 & 10 & -1 \\ 0 & -10 & 5 & -10 & 0 & 10 \end{array} \right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 10R_2 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & -\frac{13}{2} & 14 & -10 & 1 \\ 0 & 1 & \frac{9}{2} & -9 & 10 & -1 \\ 0 & 0 & 50 & -100 & 100 & 0 \end{array} \right] \begin{array}{l} \\ \\ \frac{1}{50}R_3 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & -\frac{13}{2} & 14 & -10 & 1 \\ 0 & 1 & \frac{9}{2} & -9 & 10 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array} \right] \quad \begin{array}{l} R_1 + \frac{13}{2}R_3 \\ R_2 - \frac{9}{2}R_3 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{bmatrix}$$

### Exercise

Find the inverse, if exists of  $A = \begin{pmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

### Solution

$$\left[ \begin{array}{ccc|ccc} \sqrt{2} & 3\sqrt{2} & 0 & 1 & 0 & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \quad R_2 + 4R_1$$

$$\left[ \begin{array}{ccc|ccc} \sqrt{2} & 3\sqrt{2} & 0 & 1 & 0 & 0 \\ 0 & 13\sqrt{2} & 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \quad 13R_1 - 3R_2$$

$$\left[ \begin{array}{ccc|ccc} 13\sqrt{2} & 0 & 0 & 1 & -3 & 0 \\ 0 & 13\sqrt{2} & 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \quad \begin{array}{l} \frac{1}{13\sqrt{2}}R_1 \\ \frac{1}{13\sqrt{2}}R_2 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{13\sqrt{2}} & -\frac{3}{13\sqrt{2}} & 0 \\ 0 & 1 & 0 & \frac{4}{13\sqrt{2}} & \frac{1}{13\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$A^{-1} = \begin{pmatrix} \frac{\sqrt{2}}{26} & -\frac{3\sqrt{2}}{26} & 0 \\ \frac{2\sqrt{2}}{13} & \frac{\sqrt{2}}{26} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**Exercise**

Find the inverse, if exists of  $A = \begin{pmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{pmatrix}$

**Solution**

$$\begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}^{-1} = \text{doesn't exist}$$

Since this matrix is *singular*, row 3 all zeros.

**Exercise**

Find the inverse, if exists, of  $A = \begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -6 & 1 \\ -2 & -2 & 5 & 1 \end{bmatrix}$

**Solution**

$$\left[ \begin{array}{cccc|cccc} -2 & -3 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad -\frac{1}{2}R_1$$

$$\left[ \begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad R_4 + 2R_1$$

$$\left[ \begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array} \right] R_4 - R_2$$

$$\left[ \begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

$\therefore$  Inverse *does not exist*

### Exercise

Find the inverse, if exists, of  $A = \begin{bmatrix} 1 & -14 & 7 & 38 \\ -1 & 2 & 1 & -2 \\ 1 & 2 & -1 & -6 \\ 1 & -2 & 3 & 6 \end{bmatrix}$

### Solution

$$\left[ \begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & -6 & 0 & 0 & 1 & 0 \\ 1 & -2 & 3 & 6 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_2 + R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & -12 & 8 & 36 & 1 & 1 & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array} \right] -\frac{1}{12}R_2$$

$$\left[ \begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 + 14R_2 \\ R_3 - 16R_2 \\ R_4 - 12R_2 \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & \frac{8}{3} & 4 & \frac{1}{3} & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array} \right] \frac{3}{8}R_3$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 + \frac{7}{3}R_3 \\ R_2 + \frac{2}{3}R_3 \\ \\ R_4 - 4R_3 \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & -2 & -\frac{1}{2} & -1 & -\frac{3}{2} & 1 \end{array} \right] -\frac{1}{2}R_4$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array} \right] \begin{array}{l} R_1 + \frac{1}{2}R_4 \\ R_2 + 2R_4 \\ R_3 - \frac{3}{2}R_4 \\ \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ 0 & 0 & 1 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{bmatrix}$$



**Exercise**

Find the inverse, if exists, of  $A = \begin{bmatrix} 10 & 20 & -30 & 15 \\ 3 & -7 & 14 & -8 \\ -7 & -2 & -1 & 2 \\ 4 & 4 & -3 & 1 \end{bmatrix}$

**Solution**

$$\left[ \begin{array}{cccc|cccc} 10 & 20 & -30 & 15 & 1 & 0 & 0 & 0 \\ 3 & -7 & 14 & -8 & 0 & 1 & 0 & 0 \\ -7 & -2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & -3 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad \frac{1}{10}R_1$$

$$\left[ \begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 3 & -7 & 14 & -8 & 0 & 1 & 0 & 0 \\ -7 & -2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & -3 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 + 7R_1 \\ R_4 - 4R_1 \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 0 & -13 & 23 & -\frac{25}{2} & -\frac{3}{10} & 1 & 0 & 0 \\ 0 & 12 & -22 & \frac{25}{2} & \frac{7}{10} & 0 & 1 & 0 \\ 0 & -4 & 9 & -5 & -\frac{2}{5} & 0 & 0 & 1 \end{array} \right] \quad -\frac{1}{13}R_2$$

$$\left[ \begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 12 & -22 & \frac{25}{2} & \frac{7}{10} & 0 & 1 & 0 \\ 0 & -4 & 9 & -5 & -\frac{2}{5} & 0 & 0 & 1 \end{array} \right] \quad \begin{array}{l} R_1 - 2R_2 \\ R_3 - 12R_2 \\ R_4 + 4R_2 \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 0 & -\frac{10}{13} & \frac{25}{26} & \frac{11}{26} & \frac{12}{13} & 1 & 0 \\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1 \end{array} \right] \quad -\frac{13}{10}R_3$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1 \end{array} \right] \begin{array}{l} R_1 - \frac{7}{13}R_3 \\ R_2 + \frac{23}{13}R_3 \\ R_4 - \frac{25}{13}R_3 \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{19}{20} & -\frac{11}{5} & -\frac{23}{10} & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array} \right] \begin{array}{l} R_2 + R_4 \\ R_3 + R_4 \end{array}.$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array} \right] \frac{4}{5}R_4$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array} \right] R_1 - \frac{1}{4}R_4$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{bmatrix}$$



$$\begin{aligned}
 b) \quad & \left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array} \right] \quad R_3 - 2R_1 \\
 & \left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 3 & 3 & -2 & 0 & 1 \end{array} \right] \quad R_3 - 3R_2 \\
 & \left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & * & * & * \end{array} \right]
 \end{aligned}$$

The inverse matrix *doesn't exist*

### Exercise

Show that the inverse of  $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$  is  $\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$

### Solution

$$\begin{aligned}
 & \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} \\
 &= \begin{bmatrix} (\cos \theta)\cos(-\theta) - (\sin \theta)\sin(-\theta) & (\cos \theta)\sin(-\theta) - (\sin \theta)\cos(-\theta) \\ (-\sin \theta)\cos(-\theta) - (\cos \theta)\sin(-\theta) & (-\sin \theta)\sin(-\theta) + (\cos \theta)\cos(-\theta) \end{bmatrix} \\
 &= \begin{bmatrix} \cos \theta \cos \theta + \sin \theta \sin \theta & -\cos \theta \sin \theta - \sin \theta \cos \theta \\ -\sin \theta \cos \theta + \cos \theta \sin \theta & \sin \theta \sin \theta + \cos \theta \cos \theta \end{bmatrix} \quad \begin{cases} \cos(-\theta) = \cos \theta & (\text{even}) \\ \sin(-\theta) = -\sin \theta & (\text{odd}) \end{cases} \\
 &= \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & 0 \\ 0 & \sin^2 \theta + \cos^2 \theta \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
 &= I
 \end{aligned}$$

### Exercise

If the product  $C = AB$  is invertible (and  $A$  &  $B$  are square matrices), find a formula for  $A^{-1}$  that involves  $C^{-1}$  and  $B$ .

Hence, it is not possible to multiply a non-invertible matrix by another matrix and obtain an invertible matrix as a result.

### Solution

Since  $C = AB$  is invertible, the  $CC^{-1} = C^{-1}C = I$

$$CC^{-1} = I$$

$$(AB)C^{-1} = I$$

$$A(BC^{-1}) = I$$

$$A^{-1}A(BC^{-1}) = A^{-1}I$$

$$I(BC^{-1}) = A^{-1}$$

$$\underline{BC^{-1} = A^{-1}} \quad \checkmark$$

### Exercise

Prove that if  $A$  is an  $m \times n$  matrix, there is an invertible matrix  $C$  such that  $CA$  is in reduced row-echelon form.

#### Solution

The reduced row-echelon form of  $A$  can be written in the form  $E_n \dots E_2 E_1 A$  . where  $E_1, E_2, \dots, E_n$  are elementary matrices.

Let  $C = E_n \dots E_2 E_1$ , then  $C$  is invertible since  $E_1, E_2, \dots, E_n$  are invertible.

Hence, there exists such a matrix  $C$ .

### Exercise

Prove that 2  $m \times n$  matrices  $A$  and  $B$  are row equivalent if and only if there exists a nonsingular matrix  $P$  such that  $B = PA$

#### Solution

Suppose that  $A \sim B$ , then there exist elementary matrices  $E_1, E_2, \dots, E_n$  such that  $B = E_n \dots E_2 E_1 A$ .

Let  $P = E_n \dots E_2 E_1 \Rightarrow$  by the theorem,  $P$  is nonsingular.

Suppose that  $B = PA$ , for some nonsingular matrix  $P$ . By the theorem,  $P$  is row equivalent to  $I_k$ . That is,  $I_k = E_n \dots E_2 E_1 P$ .

Thus,  $B = E_n^{-1} E_{n-1}^{-1} \dots E_1^{-1} I_k A$  and this implies that  $A$  is row equivalent to  $B$ .

**Exercise**

Let  $A$  and  $B$  be  $m \times n$  matrices. Suppose  $A$  is row equivalent to  $B$ . Prove that  $A$  is nonsingular if and only if  $B$  is nonsingular.

**Solution**

Suppose that  $A$  is row equivalent to  $B$ . Then, there exists a nonsingular matrix  $P$  such that  $B = PA$ .

If  $A$  is nonsingular then  $B$  is nonsingular.

Conversely, if  $B$  is nonsingular then  $A = P^{-1}B$  is nonsingular.

**Exercise**

Show that if  $A$  and  $B$  are two  $n \times n$  invertible matrices then  $A$  is row equivalent to  $B$ .

**Solution**

Since  $A$  is invertible, then  $A$  is a row equivalent to  $I_n$ . That is, there exist elementary matrices

$E_1, E_2, \dots, E_k$  such that  $I_n = E_k E_{k-1} \cdots E_1 A$ .

Similarly, there exist elementary matrices  $F_1, F_2, \dots, F_k$  such that  $I_n = F_i F_{i-1} \cdots F_1 B$ .

$$\begin{aligned} \text{Hence, } A &= E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n \\ &= E_1^{-1} E_2^{-1} \cdots E_k^{-1} (F_i F_{i-1} \cdots F_1 B) \\ &= (E_1^{-1} E_2^{-1} \cdots E_k^{-1} F_i F_{i-1} \cdots F_1 B) \end{aligned}$$

That is,  $A$  row equivalent to  $B$ .

**Exercise**

Prove that a square matrix  $A$  is nonsingular if and only if  $A$  is a product of elementary matrices.

**Solution**

Suppose that  $A$  is nonsingular. Then  $A$  is row equivalent to  $I_n$ . That is, there exist elementary matrices

$E_1, E_2, \dots, E_k$  such that  $I_n = E_k E_{k-1} \cdots E_1 A \rightarrow A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n$ .

But each  $E_i^{-1}$  is an elementary matrix.

Conversely, suppose that  $A = E_1 E_2 \cdots E_k$ , then  $(E_1 E_2 \cdots E_k)^{-1} A = I_n$

That is,  $A$  is nonsingular.

**Exercise**

Show that if  $A \sim B$  (that is, if they are row equivalent), then  $EA = B$  for some matrix  $E$  which is a product of elementary matrices.

**Solution**

If  $A \sim B$ , there is some sequence of elementary row operations which, when performed on  $A$ , produce  $B$ . Further, multiplying on the left by the corresponding elementary matrix is the same as performing that row operation. So we have

$$\begin{aligned} A &\sim E_1 A \\ &\sim E_2 E_1 A \\ &\sim E_k E_{k-1} \dots E_2 E_1 A \\ &= B \end{aligned}$$

Thus, if  $E = E_k \dots E_1$ , then we have  $EA = B$

**Exercise**

Show that if  $EA = B$  for some matrix  $E$  which is a product of elementary matrices, then  $AC \sim BC$  for every  $n \times n$  matrix  $C$ .

**Solution**

Let  $E = E_k E_{k-1} \dots E_1$ , where each  $E_i$  is an elementary matrix.

$$\begin{aligned} AC &\sim E_1 AC \\ &\sim E_2 E_1 AC \\ &\sim E_k E_{k-1} \dots E_2 E_1 AC \\ &= EAC && \text{since } EA = B \\ &= BC \end{aligned}$$

Therefore;  $AC \sim BC$

**Exercise**

Let  $A\vec{x} = 0$  be a homogeneous system of  $n$  linear equations in  $n$  unknowns that has only the trivial solution. Show that if  $k$  is any positive integer, then the system  $A^k \vec{x} = 0$  also has only trivial solution.

**Solution**

Since  $A$  is a square matrix, thus  $A$  has only the trivial solution. That implies that  $A$  is invertible.

But  $A^k$  is also invertible so  $A^k \vec{x} = 0$  has only trivial solution.

**Exercise**

Let  $A\vec{x} = 0$  be a homogeneous system of  $n$  linear equations in  $n$  unknowns, and let  $Q$  be an invertible  $n \times n$  matrix. Show that  $A\vec{x} = 0$  has just trivial solution if and only if  $(QA)\vec{x} = 0$  has just trivial solution.

### **Solution**

$A$  is a square  $(n \times n)$  matrix. If  $A\vec{x} = 0$  has just a trivial solution, then  $A$  is invertible. Since  $Q$  is an invertible  $n \times n$  matrix, then  $QA$  is also invertible.

Thus,  $(QA)\vec{x} = 0$  has trivial solution.

On the other hand, if  $(QA)\vec{x} = 0$  has trivial solution, then  $QA$  is also invertible.

Since  $Q$  is invertible, then  $Q^{-1}$  is also invertible.

Thus,  $A = Q^{-1}QA$  is invertible, i.e.  $A\vec{x} = 0$  has just trivial solution, equivalent  $A\vec{x} = 0$  has just trivial solution if and only if  $(QA)\vec{x} = 0$  has just trivial solution.

### ***Exercise***

Let  $A\vec{x} = b$  be any consistent system of linear equations, and let  $\vec{x}_1$  be a fixed solution. Show that every solution to the system can be written in the form  $\vec{x} = \vec{x}_1 + \vec{x}_0$  where  $\vec{x}_0$  is a solution to  $A\vec{x} = 0$ . Show also that every matrix of this form is a solution.

### **Solution**

Since  $\vec{x}_0$  is a solution to  $A\vec{x} = 0$ , we have  $A\vec{x}_0 = 0$ .

Adding  $A\vec{x}_0 = 0$  to  $A\vec{x} = b$ , then

$$A\vec{x} + A\vec{x}_0 = b + 0$$

$$A(\vec{x} + \vec{x}_0) = b$$

As adding an equation to the original equation does not affect the solution.

If we let  $\vec{x}_1$  be a fixed solution, then every solution to  $A\vec{x} = b$  is  $\vec{x} = \vec{x}_1 + \vec{x}_0$ .

Besides,

$$\begin{aligned} A(\vec{x} + \vec{x}_0) &= A\vec{x} + A\vec{x}_0 \\ &= b + 0 \\ &= b \end{aligned}$$

So, every matrix (vector) in the form  $\vec{x}_1 + \vec{x}_0$  is a solution to  $A\vec{x} = b$



**Exercise**

If  $A$  and  $B$  are  $n \times n$  matrices satisfying  $A^2 = B^2 = (AB)^2 = I_n$ . Prove that  $AB = BA$ .

**Solution**

Since  $A^2 = B^2 = (AB)^2 = I_n$ , then  $A, B, AB$  are nonsingular.

$$A^2 = I \rightarrow A = A^{-1}$$

$$B^2 = I \rightarrow B = B^{-1}$$

$$(AB)^2 = I \rightarrow AB = (AB)^{-1}$$

$$AB = (AB)^{-1}$$

$$= B^{-1}A^{-1}$$

$$= BA \quad \checkmark$$

**Exercise**

Let  $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix}$ . Verify that  $A^3 = 5I$ , then find  $A^{-1}$  in term of  $A$ .

**Solution**

$$A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 5 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}$$

$$A^3 = AA^2$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 5 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= 5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 5I \quad |$$

$$\text{Since } A^3 = AA^2 = 5I$$

$$\frac{1}{5}(AA^2) = I$$

$$A\left(\frac{1}{5}A^2\right) = I$$

$$\underline{A^{-1} = \frac{1}{5}A^2} \quad |$$

### Exercise

Consider  $B(A, I) = (BA, B)$ , thus if  $B$  is the inverse of  $A$ , then  $(BA, B)$  becomes  $(I, A^{-1})$ . On the other hand  $B$  is a product of elementary matrices since it is invertible. This indicates that the inverse of  $A$  can be obtained by applying elementary row operations to  $(A, I)$  to get  $(I, A^{-1})$ .

Find the inverses of

$$a) \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix}$$

$$b) \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{pmatrix}$$

### Solution

$$a) \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix} \quad R_3 - aR_1 \quad E_{31} = -a$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{pmatrix} \quad R_3 - bR_2 \quad E_{32} = -b$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & -b & 1 \end{pmatrix}$$

b) First, we have move row 4 to row 1, for the calculation

$$\begin{pmatrix} a & b & c & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\frac{1}{a}R_1} \quad E_{11} = \frac{1}{a}$$

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{c}{a} & \frac{d}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - \frac{b}{a}R_2} \quad E_{12} = -\frac{b}{a}$$

$$\begin{pmatrix} 1 & 0 & \frac{c}{a} & \frac{d}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - \frac{c}{a}R_3} \quad E_{13} = -\frac{c}{a}$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{d}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - \frac{d}{a}R_4} \quad E_{14} = -\frac{d}{a}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$E = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} & -\frac{c}{a} & -\frac{d}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since we move Row 4 to Row 1, we must move Column 1 to Column 4 to get the inverse matrix.

$$B^{-1} = \begin{pmatrix} -\frac{b}{a} & -\frac{c}{a} & -\frac{d}{a} & \frac{1}{a} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

**Exercise**

Let  $A, B, C, X, Y, Z \in M_n(\mathbb{C})$ ,  $A$  and  $C$  are invertible. Find

$$a) \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}^{-1} \qquad b) \begin{pmatrix} I & X & Y \\ 0 & I & Z \\ 0 & 0 & I \end{pmatrix}^{-1}$$

**Solution**

$$a) \left( \begin{array}{cc|cc} A & B & I & 0 \\ 0 & C & 0 & I \end{array} \right) \quad \textcolor{red}{A}^{-1}R_1$$

$$\left( \begin{array}{cc|cc} \textcolor{red}{A}^{-1}A & \textcolor{red}{A}^{-1}B & \textcolor{red}{A}^{-1}I & 0 \\ 0 & C & 0 & I \end{array} \right)$$

$$\left( \begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ 0 & C & 0 & I \end{array} \right) \quad \textcolor{red}{C}^{-1}R_2$$

$$\left( \begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ 0 & C^{-1}C & 0 & C^{-1}I \end{array} \right)$$

$$\left( \begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ 0 & I & 0 & C^{-1} \end{array} \right) \quad R_1 - \textcolor{red}{A}^{-1}BR_2$$

$$\left( \begin{array}{cc|cc} I & A^{-1}B - A^{-1}BI & A^{-1} & -A^{-1}BC^{-1} \\ 0 & I & 0 & C^{-1} \end{array} \right)$$

$$\left( \begin{array}{cc|cc} I & A^{-1}B - A^{-1}B & A^{-1} & -A^{-1}BC^{-1} \\ 0 & I & 0 & C^{-1} \end{array} \right)$$

$$\left( \begin{array}{cc|cc} I & 0 & A^{-1} & -A^{-1}BC^{-1} \\ 0 & I & 0 & C^{-1} \end{array} \right)$$

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}^{-1} = \begin{pmatrix} \textcolor{blue}{A}^{-1} & -\textcolor{blue}{A}^{-1}BC^{-1} \\ \textcolor{blue}{0} & \textcolor{blue}{C}^{-1} \end{pmatrix}$$

$$b) \left( \begin{array}{ccc|ccc} I & X & Y & I & 0 & 0 \\ 0 & I & Z & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 & I \end{array} \right) \quad R_1 - \textcolor{red}{X}R_2$$

$$\left( \begin{array}{ccc|ccc} I & 0 & Y-XZ & I & -X & 0 \\ 0 & I & Z & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 & I \end{array} \right) \begin{array}{l} R_1 - (Y-XZ)R_3 \\ R_2 - ZR_3 \end{array}$$

$$\left( \begin{array}{ccc|ccc} I & 0 & 0 & I & -X & XZ-Y \\ 0 & I & 0 & 0 & I & -Z \\ 0 & 0 & I & 0 & 0 & I \end{array} \right)$$

$$\left( \begin{array}{ccc} I & X & Y \\ 0 & I & Z \\ 0 & 0 & I \end{array} \right)^{-1} = \left( \begin{array}{ccc} I & -X & XZ-Y \\ 0 & I & -Z \\ 0 & 0 & I \end{array} \right)$$

### Exercise

Suppose that  $A$ ,  $B$ , and  $A - B$  are invertible  $n \times n$  matrices. Show that

$$(A - B)^{-1} = A^{-1} + A^{-1}(B^{-1} - A^{-1})^{-1}A^{-1}$$

### Solution

$A$ ,  $B$ , and  $A - B$  are invertible Then

$$AA^{-1} = A^{-1}A = I \quad BB^{-1} = B^{-1}B = I$$

$$(A - B)(A - B)^{-1} = (A - B)^{-1}(A - B) = I$$

Let:

$$(A - B)^{-1}(A - B) = I$$

Then, we need to prove that

$$\left( A^{-1} + A^{-1}(B^{-1} - A^{-1})^{-1}A^{-1} \right)(A - B) \stackrel{?}{=} I$$

$$\left( A^{-1} + A^{-1}(B^{-1} - A^{-1})^{-1}A^{-1} \right)(A - B) = \left( A^{-1} + A^{-1} \left( A(B^{-1} - A^{-1}) \right)^{-1} \right)(A - B)$$

$$\left( A(B^{-1} - A^{-1}) \right)^{-1} = (B^{-1} - A^{-1})^{-1}A^{-1}$$

$$= \left( A^{-1} + A^{-1} \left( AB^{-1} - AA^{-1} \right)^{-1} \right)(A - B)$$

$$= \left( A^{-1} + A^{-1} \left( AB^{-1} - I \right)^{-1} \right)(A - B)$$

$$= \left( A^{-1} + A^{-1} \left( AB^{-1} - BB^{-1} \right)^{-1} \right)(A - B)$$

$$\begin{aligned}
&= \left( A^{-1} + A^{-1} \left( (A-B)B^{-1} \right)^{-1} \right) (A-B) \\
&\quad \left( (A-B)B^{-1} \right)^{-1} = B(A-B)^{-1} \\
&= \left( A^{-1} + A^{-1} \left( B(A-B)^{-1} \right) \right) (A-B) \\
&= \left( A^{-1} + A^{-1}B(A-B)^{-1} \right) (A-B) \\
&= A^{-1}(A-B) + A^{-1}B(A-B)^{-1}(A-B) \\
&= A^{-1}A - A^{-1}B + A^{-1}BI \\
&= I - A^{-1}B + A^{-1}B \\
&= I \quad \checkmark
\end{aligned}$$

Therefore;  $(A-B)^{-1} = A^{-1} + A^{-1} \left( B^{-1} - A^{-1} \right)^{-1} A^{-1}$

### Exercise

Suppose  $P$  is invertible and  $A = PBP^{-1}$ . Solve for  $B$  in terms of  $A$ .

### Solution

Since  $P$  is invertible, then  $PP^{-1} = P^{-1}P = I$

$$A = PBP^{-1}$$

$$P^{-1}AP = P^{-1}PBP^{-1}P \quad PP^{-1} = P^{-1}P = I$$

$$P^{-1}AP = IBI \quad BI = B$$

$$\underline{P^{-1}AP = B}$$

### Exercise

Suppose  $(A-B)C = 0$ , where  $A$  and  $B$  are  $m \times n$  matrices and  $C$  is invertible. Show that  $A = B$ .

### Solution

Since  $C$  is invertible, then  $CC^{-1} = C^{-1}C = I$

$$(A-B)C = 0$$

$$(A-B)CC^{-1} = 0C^{-1}$$

$$(A-B)I = 0$$

$$A - B = 0$$

$$A - B + B = 0 + B$$

$$\underline{A = B} \quad \checkmark$$

### Exercise

Prove  $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$  where  $A$ ,  $B$ , and  $C$  are invertible

### Solution

Since  $A$  is invertible, then  $AA^{-1} = A^{-1}A = I$

$B$  is invertible, then  $BB^{-1} = B^{-1}B = I$

$C$  is invertible, then  $CC^{-1} = C^{-1}C = I$

$$(ABC)(ABC)^{-1} = I$$

$$\begin{aligned} (ABC)(ABC)^{-1} &= (ABC)C^{-1}B^{-1}A^{-1} \\ &= AB(CC^{-1})B^{-1}A^{-1} \\ &= AB(I)B^{-1}A^{-1} \\ &= A(BIB^{-1})A^{-1} \\ &= A(BB^{-1})A^{-1} \\ &= A(I)A^{-1} \\ &= (AI)A^{-1} \\ &= AA^{-1} \\ &= I \end{aligned}$$

$$\begin{aligned} (ABC)^{-1}(ABC) &= (C^{-1}B^{-1}A^{-1})(ABC) \\ &= C^{-1}B^{-1}(A^{-1}A)BC \\ &= C^{-1}B^{-1}(I)BC \\ &= C^{-1}(B^{-1}B)C \\ &= C^{-1}(I)C \\ &= C^{-1}C \\ &= I \end{aligned}$$

Thus,  $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

**Exercise**

Prove  $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$  where  $A$ ,  $B$ ,  $C$  and  $D$  are invertible

**Solution**

Since  $A$  is invertible, then  $AA^{-1} = A^{-1}A = I$

$B$  is invertible, then  $BB^{-1} = B^{-1}B = I$

$C$  is invertible, then  $CC^{-1} = C^{-1}C = I$

$D$  is invertible, then  $DD^{-1} = D^{-1}D = I$

$$(ABCD)(ABCD)^{-1} = I$$

$$\begin{aligned} (ABCD)(ABCD)^{-1} &= (ABCD)D^{-1}C^{-1}B^{-1}A^{-1} \\ &= ABC(DD^{-1})C^{-1}B^{-1}A^{-1} \\ &= ABC(I)C^{-1}B^{-1}A^{-1} \\ &= AB(CC^{-1})B^{-1}A^{-1} \\ &= AB(I)B^{-1}A^{-1} \\ &= A(BIB^{-1})A^{-1} \\ &= A(BB^{-1})A^{-1} \\ &= A(I)A^{-1} \\ &= (AI)A^{-1} \\ &= AA^{-1} \\ &= I \end{aligned}$$

$$\begin{aligned} (ABCD)^{-1}(ABCD) &= (D^{-1}C^{-1}B^{-1}A^{-1})(ABCD) \\ &= D^{-1}C^{-1}B^{-1}(A^{-1}A)BCD \\ &= D^{-1}C^{-1}B^{-1}(I)BCD \\ &= D^{-1}C^{-1}(B^{-1}B)CD \\ &= D^{-1}C^{-1}(I)CD \\ &= D^{-1}(C^{-1}C)D \\ &= D^{-1}(I)D \\ &= D^{-1}D \end{aligned}$$



$$= I$$

$$\text{Thus, } (ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$$

### Exercise

Prove that if  $A^2 = A$ , then  $I - 2A = (I - 2A)^{-1}$

### Solution

$$\begin{aligned} (I - 2A)(I - 2A) &= I^2 - 2IA - 2AI + 4A^2 \\ &= I - 2A - 2A + 4A^2 & A^2 = A \\ &= I - 4A + 4A \\ &= I \end{aligned}$$

$$\text{Therefore, } I - 2A = (I - 2A)^{-1}$$

### Exercise

$$\text{Let } A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$a) \text{ Show that } A^2 - 2A + 5I = 0$$

$$b) \text{ Show that } A^{-1} = \frac{1}{5}(2I - A)$$

$$c) \text{ Show that for any square matrix satisfying } A^2 - 2A + 5I = 0, \text{ the inverse of } A \text{ is } A^{-1} = \frac{1}{5}(2I - A)$$

### Solution

$$\begin{aligned} a) \quad A^2 &= \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -3 & 4 \\ -4 & -3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A^2 - 2A + 5I &= \begin{pmatrix} -3 & 4 \\ -4 & -3 \end{pmatrix} - 2 \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} + 5 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &= 0 \end{aligned}$$

$$b) \quad A^{-1} = \frac{1}{5}(2I - A)$$

$$A^{-1} = \frac{1}{5} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$\begin{aligned} \frac{1}{5}(2I - A) &= \frac{1}{5} \left( \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \right) \\ &= \frac{1}{5} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \end{aligned}$$

$$A^{-1} = \frac{1}{5}(2I - A) = \frac{1}{5} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \quad \checkmark$$

$$c) \quad A^2 - 2A + 5I = 0 \quad \& \quad A^{-1} = \frac{1}{5}(2I - A)$$

$$\begin{aligned} AA^{-1} &= A \left( \frac{1}{5}(2I - A) \right) \\ &= \frac{1}{5}(2IA - A^2) \\ &= \frac{1}{5}(2A - A^2) \end{aligned}$$

$$A^2 - 2A + 5I = 0$$

$$5I = 2A - A^2$$

$$I = \frac{1}{5}(2A - A^2)$$

$$\frac{1}{5}(2A - A^2) = AA^{-1} = I \quad \checkmark$$

### Exercise

find the inverse of the  $3 \times 3$  Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{pmatrix}$$

When  $a_1$ ,  $a_2$ , and  $a_3$  are distinct from each other

### Solution

$$\left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ a_1 & a_2 & a_3 & 0 & 1 & 0 \\ a_1^2 & a_2^2 & a_3^2 & 0 & 0 & 1 \end{array} \right) \quad \begin{array}{l} \\ R_2 - a_1 R_1 \\ R_3 - a_1^2 R_1 \end{array}$$

$$\left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & a_2 - a_1 & a_3 - a_1 & -a_1 & 1 & 0 \\ 0 & a_2^2 - a_1^2 & a_3^2 - a_1^2 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} (a_2 - a_1)R_1 - R_2 \\ R_3 - (a_2 + a_1)R_2 \end{array}$$

$$\left( \begin{array}{ccc|ccc} a_2 - a_1 & 0 & a_2 - a_3 & a_2 & -1 & 0 \\ 0 & a_2 - a_1 & a_3 - a_1 & -a_1 & 1 & 0 \\ 0 & 0 & (a_2^2 - a_1^2) - (a_2 + a_1)(a_3 - a_1) & a_1(a_2 + a_1) & -(a_2 + a_1) & 1 \end{array} \right)$$

$$\left( \begin{array}{ccc|ccc} a_2 - a_1 & 0 & a_2 - a_3 & a_2 & -1 & 0 \\ 0 & a_2 - a_1 & a_3 - a_1 & -a_1 & 1 & 0 \\ 0 & 0 & (a_3 - a_1)(a_3 - a_2) & a_1(a_2 + a_1) & -(a_2 + a_1) & 1 \end{array} \right) \begin{array}{l} (a_3 - a_1)R_1 + R_3 \\ (a_3 - a_2)R_2 - R_3 \end{array}$$

$$\left( \begin{array}{ccc|ccc} (a_2 - a_1)(a_3 - a_1) & 0 & 0 & a_2(a_3 + a_2) & -(a_3 + a_2) & 1 \\ 0 & (a_2 - a_1)(a_3 - a_2) & 0 & -a_1(a_3 + a_1) & a_3 - a_1 & -1 \\ 0 & 0 & (a_3 - a_1)(a_3 - a_2) & a_1(a_2 + a_1) & -(a_2 + a_1) & 1 \end{array} \right)$$

$$\frac{1}{(a_2 - a_1)(a_3 - a_1)} R_1$$

$$\frac{1}{(a_2 - a_1)(a_3 - a_2)} R_2$$

$$\frac{1}{(a_3 - a_2)(a_3 - a_1)} R_3$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{a_2(a_3 + a_2)}{(a_2 - a_1)(a_3 - a_1)} & -\frac{a_3 + a_2}{(a_2 - a_1)(a_3 - a_1)} & \frac{1}{(a_2 - a_1)(a_3 - a_1)} \\ 0 & 1 & 0 & \frac{-a_1(a_3 + a_1)}{(a_2 - a_1)(a_3 - a_2)} & \frac{a_3 - a_1}{(a_2 - a_1)(a_3 - a_2)} & -\frac{1}{(a_2 - a_1)(a_3 - a_2)} \\ 0 & 0 & 1 & \frac{a_1(a_2 + a_1)}{(a_3 - a_1)(a_3 - a_2)} & -\frac{a_2 + a_1}{(a_3 - a_1)(a_3 - a_2)} & \frac{1}{(a_3 - a_1)(a_3 - a_2)} \end{array} \right)$$

$$V^{-1} = \begin{pmatrix} \frac{a_2(a_3 + a_2)}{(a_2 - a_1)(a_3 - a_1)} & -\frac{a_3 + a_2}{(a_2 - a_1)(a_3 - a_1)} & \frac{1}{(a_2 - a_1)(a_3 - a_1)} \\ \frac{-a_1(a_3 + a_1)}{(a_2 - a_1)(a_3 - a_2)} & \frac{a_3 - a_1}{(a_2 - a_1)(a_3 - a_2)} & -\frac{1}{(a_2 - a_1)(a_3 - a_2)} \\ \frac{a_1(a_2 + a_1)}{(a_3 - a_1)(a_3 - a_2)} & -\frac{a_2 + a_1}{(a_3 - a_1)(a_3 - a_2)} & \frac{1}{(a_3 - a_1)(a_3 - a_2)} \end{pmatrix}$$

## ***Solution***      **Section 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices**

### ***Exercise***

Consider the matrices

$$A = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix} \quad B = \begin{pmatrix} -7 & -3 \\ 6 & 4 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

Find:

- |                |                    |                   |                    |
|----------------|--------------------|-------------------|--------------------|
| a) $A^T$       | e) $(A\vec{v})^T$  | h) $A^T B^T$      | k) $B^T \vec{v}^T$ |
| b) $B^T$       | f) $A^T \vec{v}^T$ | i) $B^T A^T$      | l) $(2A)^T$        |
| c) $A^T + B^T$ | g) $(AB)^T$        | j) $(B\vec{v})^T$ | m) $2A^T$          |
| d) $(A+B)^T$   |                    |                   |                    |

### **Solution**

$$a) \quad A^T = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$$

$$b) \quad B^T = \begin{pmatrix} -7 & 6 \\ -3 & 4 \end{pmatrix}$$

$$c) \quad A^T + B^T = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix} + \begin{pmatrix} -7 & 6 \\ -3 & 4 \end{pmatrix} \\ = \begin{pmatrix} -6 & 9 \\ -5 & 9 \end{pmatrix}$$

$$d) \quad A + B = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix} + \begin{pmatrix} -7 & -3 \\ 6 & 4 \end{pmatrix} \\ = \begin{pmatrix} -6 & -5 \\ 9 & 9 \end{pmatrix}$$

$$(A+B)^T = \begin{pmatrix} -6 & 9 \\ -5 & 9 \end{pmatrix}$$

*Note*  $(A+B)^T = A^T + B^T$

$$e) \quad A\vec{v} = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \end{pmatrix} \\ = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$$

$$(A\vec{v})^T = [6 \quad 7]$$

$$f) \quad \vec{v}^T = [4 \quad -1]$$

$$A^T \vec{v}^T = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix} [6 \quad 7]$$

Can't be *found*, inner not equal  $2 \neq 1$

$$g) \quad AB = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} -7 & -3 \\ 6 & 4 \end{pmatrix} \\ = \begin{pmatrix} -19 & -11 \\ 9 & 11 \end{pmatrix}$$

$$(AB)^T = \begin{pmatrix} -19 & 9 \\ -11 & 11 \end{pmatrix}$$

$$h) \quad A^T B^T = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} -7 & 6 \\ -3 & 4 \end{pmatrix} \\ = \begin{pmatrix} -13 & 18 \\ -1 & 8 \end{pmatrix}$$

$$\text{Note } (AB)^T \neq A^T B^T$$

$$i) \quad B^T A^T = \begin{pmatrix} -7 & 6 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix} \\ = \begin{pmatrix} -19 & 9 \\ -11 & 11 \end{pmatrix}$$

$$\text{Note } (AB)^T = B^T A^T$$

$$j) \quad B\vec{v} = \begin{pmatrix} -7 & -3 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \end{pmatrix} \\ = \begin{pmatrix} -25 \\ 20 \end{pmatrix} \\ (B\vec{v})^T = [-25 \quad 20]$$

$$k) \quad B^T = \begin{pmatrix} -7 & 6 \\ -3 & 4 \end{pmatrix} \quad \vec{v}^T = [4 \quad -1]$$

$$B^T \vec{v}^T = \begin{pmatrix} -7 & 6 \\ -3 & 4 \end{pmatrix} [4 \quad -1]$$

Can't be *found*, inner not equal  $2 \neq 1$

$$l) \quad 2A = \begin{pmatrix} 2 & -4 \\ 6 & 10 \end{pmatrix}$$

$$(2A)^T = \begin{pmatrix} 2 & 6 \\ -4 & 10 \end{pmatrix}$$

$$m) \quad 2A^T = 2 \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 6 \\ -4 & 10 \end{pmatrix}$$

$$\text{Note } (2A)^T = 2A^T$$

### Exercise

Solve  $Lc = b$  to find  $c$ . Then solve  $Ux = c$  to find  $x$ . What was  $A$ ?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

### Solution

$$Lc = b$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

$$\rightarrow \begin{cases} \underline{c_1 = 4} \\ c_1 + c_2 = 5 \\ c_1 + c_2 + c_3 = 6 \end{cases}$$

$$\begin{cases} \underline{c_2 = 5 - 4 = 1} \\ \underline{c_3 = 6 - 4 - 1 = 1} \end{cases}$$

$$c = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

$$Ux = c$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x + y + z = 4 \\ y + z = 1 \\ z = 1 \end{cases} \rightarrow \begin{cases} x = 3 \\ y = 0 \\ z = 1 \end{cases}$$

$$x = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

$$L\vec{c} = b$$

$$LU\vec{x} = \vec{b}$$

$$A\vec{x} = \vec{b}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_A \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_x \underbrace{\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}}_b = \underbrace{\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}}_b$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

### Exercise

Find  $L$  and  $U$  for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on  $a, b, c, d$  to get  $A = LU$  with four pivots

### Solution

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$



**Exercise**

Determine whether the given matrix is invertible

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

**Solution**

The matrix is a diagonal matrix with nonzero entries on the diagonal, so it is invertible.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

**Exercise**

Find  $A^2$ ,  $A^{-2}$ , and  $A^{-k}$  by inspection  $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

**Solution**

$$\begin{aligned} A^2 &= \begin{bmatrix} 1^2 & 0 \\ 0 & (-2)^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A^{-2} &= \begin{bmatrix} 1^{-2} & 0 \\ 0 & (-2)^{-2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A^{-k} &= \begin{bmatrix} 1^{-k} & 0 \\ 0 & (-2)^{-k} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{(-2)^k} \end{bmatrix} \end{aligned}$$

**Exercise**

Find  $A^2$ ,  $A^{-2}$ , and  $A^{-k}$  by inspection  $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$

**Solution**

$$A^2 = \begin{bmatrix} \left(\frac{1}{2}\right)^2 & 0 & 0 \\ 0 & \left(\frac{1}{3}\right)^2 & 0 \\ 0 & 0 & \left(\frac{1}{4}\right)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{16} \end{bmatrix}$$

$$A^{-2} = \begin{bmatrix} \left(\frac{1}{2}\right)^{-2} & 0 & 0 \\ 0 & \left(\frac{1}{3}\right)^{-2} & 0 \\ 0 & 0 & \left(\frac{1}{4}\right)^{-2} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

$$A^{-k} = \begin{bmatrix} \left(\frac{1}{2}\right)^{-k} & 0 & 0 \\ 0 & \left(\frac{1}{3}\right)^{-k} & 0 \\ 0 & 0 & \left(\frac{1}{4}\right)^{-k} \end{bmatrix}$$

$$= \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix}$$

**Exercise**

Find  $A^2$ ,  $A^{-2}$ , and  $A^{-k}$  by inspection  $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

**Solution**

$$A^2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$A^{-2} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{16} & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$A^{-k} = \begin{bmatrix} (-2)^{-k} & 0 & 0 & 0 \\ 0 & (-4)^{-k} & 0 & 0 \\ 0 & 0 & (-3)^{-k} & 0 \\ 0 & 0 & 0 & (2)^{-k} \end{bmatrix}$$

**Exercise**

Find  $A^3$ , and  $A^{-1}$   $A = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$

**Solution**

$$A^3 = \begin{pmatrix} -8 & 0 & 0 & 0 & 0 \\ 0 & 64 & 0 & 0 & 0 \\ 0 & 0 & 125 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

***Exercise***

Find  $A^3$ , and  $A^{-1}$        $B = \begin{pmatrix} 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{5} \end{pmatrix}$

**Solution**

$$B^3 = \begin{pmatrix} 512 & 0 & 0 & 0 & 0 & 0 \\ 0 & 64 & 0 & 0 & 0 & 0 \\ 0 & 0 & -125 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{64} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{125} \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{pmatrix}$$

**Exercise**

Find the power of  $A^{16}$  for the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

**Solution**

$$\begin{aligned} A^{16} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ &= I_{5 \times 5} \end{aligned}$$

**Exercise**

Decide whether the given matrix is symmetric  $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

**Solution**

Not symmetric, since  $a_{12} \neq a_{21}$  ( $1 \neq -1$ )

**Exercise**

Decide whether the given matrix is symmetric  $\begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$

**Solution**

Symmetric

**Exercise**

Decide whether the given matrix is symmetric  $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

**Solution**

Not symmetric, since  $a_{13} = 1 \neq 3 = a_{31}$

**Exercise**

Find all values of the unknown constant(s) in order for  $A$  to be symmetric

$$A = \begin{bmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 7 \end{bmatrix}$$

**Solution**

$$\begin{cases} a-2b+2c=3 \\ 2a+b+c=0 \\ a+c=-2 \end{cases}$$

$$\underline{a=11, \quad b=9, \quad c=-13}$$

**Exercise**

Find  $x$  and  $B$ , if the given matrix  $B$  is symmetric

$$B = \begin{pmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{pmatrix}$$

**Solution**

For matrix  $B$  is symmetric, then  $B = B^T$

$$\begin{pmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{pmatrix} = \begin{pmatrix} 4 & 2x-3 \\ x+2 & x+1 \end{pmatrix}$$

$$\rightarrow x+2 = 2x-3$$

$$\underline{x=5}$$

$$B = \begin{pmatrix} 4 & 7 \\ 7 & 6 \end{pmatrix}$$

**Exercise**

Find a diagonal matrix  $A$  that satisfies the given condition  $A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**Solution**

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^{-2} = \begin{pmatrix} a^{-2} & 0 & 0 \\ 0 & b^{-2} & 0 \\ 0 & 0 & c^{-2} \end{pmatrix}$$

$$= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{cases} a^{-2} = 9 \Rightarrow a = \pm 9^{-1/2} = \pm \frac{1}{3} \\ b^{-2} = 4 \Rightarrow b = \pm 2^{-1/2} = \pm \frac{1}{2} \\ c^{-2} = 1 \Rightarrow c = \pm 1^{-1/2} = \pm 1 \end{cases}$$

$$A = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \dots$$

$$A = \begin{pmatrix} \pm \frac{1}{3} & 0 & 0 \\ 0 & \pm \frac{1}{2} & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$$

### Exercise

Let  $A$  be an  $n \times n$  symmetric matrix

- Show that  $A^2$  is symmetric
- Show that  $2A^2 - 3A + I$  is symmetric
- Show that  $\frac{1}{2}(A + A^T)$  is symmetric

### Solution

- The property of the transpose states that  $(AB)^T = B^T A^T$

Since  $A$  is symmetric then  $A^T = A$

$$\begin{aligned} (A^2)^T &= (AA)^T \\ &= A^T A^T \\ &= (A^T)^2 \quad \text{\textit{A is symmetric}} \\ &= A^2 \end{aligned}$$

$\therefore A^2$  is symmetric

$$\begin{aligned}
 b) \quad (2A^2 - 3A + I)^T &= 2(A^2)^T - 3(A)^T + (I)^T \\
 &= 2(A^T)^2 - 3A^T + (I)^T && \text{A and I are symmetric} \\
 &= 2A^2 - 3A + I \\
 \therefore 2A^2 - 3A + I &\text{ is } \textbf{Symmetric}
 \end{aligned}$$

$$c) \quad \text{Then we need to prove that } \frac{1}{2}(A + A^T) = \left(\frac{1}{2}(A + A^T)\right)^T$$

$$\begin{aligned}
 \left(\frac{1}{2}(A + A^T)\right)^T &= \frac{1}{2}(A + A^T)^T \\
 &= \frac{1}{2}\left(A^T + (A^T)^T\right) \\
 &= \frac{1}{2}(A^T + A) \\
 &= \frac{1}{2}(A + A^T)
 \end{aligned}$$

$$\therefore \frac{1}{2}(A + A^T) \text{ is symmetric}$$

### Exercise

Prove if  $A^T A = A$ , then  $A$  is symmetric and  $A = A^2$

#### Solution

If  $A^T A = A$ , then

$$\begin{aligned}
 A^T &= (A^T A)^T \\
 &= A^T (A^T)^T \\
 &= A^T A \\
 &= A
 \end{aligned}$$

So,  $A$  is symmetric.

Since  $A = A^T$

$$\begin{aligned}
 AA &= A^T A && A^T A = A \\
 A^2 &= A
 \end{aligned}$$



**Exercise**

A square matrix  $A$  is called **skew-symmetric** if  $A^T = -A$ . Prove

- If  $A$  is an invertible skew-symmetric matrix, then  $A^{-1}$  is skew-symmetric.
- If  $A$  and  $B$  are skew-symmetric matrices, then so are  $A^T$ ,  $A + B$ ,  $A - B$ , and  $kA$  for any scalar  $k$ .
- Every square matrix  $A$  can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

$$\left[ \text{Hint: Note the identity } A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) \right]$$

**Solution**

$$\begin{aligned} a) \quad (A^{-1})^T &= (A^T)^{-1} \\ &= (-A)^{-1} \quad \text{skew-symmetric} \\ &= -A^{-1} \end{aligned}$$

$\therefore A^{-1}$  is also skew-symmetric

- b) Let  $A$  and  $B$  are skew-symmetric matrices

$$\begin{aligned} (A^T)^T &= (-A)^T \\ &= -A^T \end{aligned}$$

$$\begin{aligned} (A + B)^T &= A^T + B^T \\ &= -A - B \\ &= -(A + B) \end{aligned}$$

$$\begin{aligned} (A - B)^T &= A^T - B^T \\ &= -A + B \\ &= -(A - B) \end{aligned}$$

$$\begin{aligned} (kA)^T &= k(A)^T \\ &= k(-A) \\ &= -kA \end{aligned}$$

- c) We need to prove from the hint that  $\frac{1}{2}(A + A^T)$  is symmetric and  $\frac{1}{2}(A - A^T)$  is skew-symmetric

$$\begin{aligned} \frac{1}{2}(A + A^T)^T &= \frac{1}{2}\left(A^T + (A^T)^T\right) \\ &= \frac{1}{2}(A + A^T) \end{aligned}$$

Thus  $\frac{1}{2}(A + A^T)$  is symmetric

$$\begin{aligned}\frac{1}{2}(A - A^T)^T &= \frac{1}{2}\left(A^T - (A^T)^T\right) \\ &= \frac{1}{2}(A^T - A) \\ &= -\frac{1}{2}(A - A^T)\end{aligned}$$

Thus  $\frac{1}{2}(A - A^T)$  is skew-symmetric

### Exercise

Suppose  $R$  is rectangular ( $m$  by  $n$ ) and  $A$  is symmetric ( $m$  by  $m$ )

- Transpose  $R^T AR$  to show its symmetric
- Show why  $R^T R$  has no negative numbers on its diagonal.

### Solution

$$\begin{aligned}a) \quad (R^T AR)^T &= \left((R^T A)R\right)^T \\ &= R^T (R^T A)^T \\ &= R^T A^T (R^T)^T \\ &= R^T AR\end{aligned}$$

$$\begin{aligned}b) \quad (R^T R)_{jj} &= (\text{column } j \text{ of } R) \cdot (\text{column } j \text{ of } R) \\ &= \text{Product of the diagonal entry by itself.} \\ &= \text{length squared of column } j.\end{aligned}$$

### Exercise

If  $L$  is a lower-triangular matrix, then  $(L^{-1})^T$  is \_\_\_\_\_ Triangular

### Solution

$(L^{-1})^T$  is **upper** triangular.

$L^{-1}$  is a lower-triangular because  $L$  is.

The transpose carries the lower-triangular matrices to the upper-triangular (and vice versa).

**Exercise**

True or False

- a) The block matrix  $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$  is automatically symmetric
- b) If  $A$  and  $B$  are symmetric then their product is symmetric
- c) If  $A$  is not symmetric then  $A^{-1}$  is not symmetric
- d) When  $A, B, C$  are symmetric, the transpose of  $ABC$  is  $CBA$ .
- e) The transpose of a diagonal matrix is a diagonal.
- f) The transpose of an upper triangular matrix is an upper triangular matrix.
- g) The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.
- h) All entries of a symmetric matrix are determined by the entries occurring on and above the main diagonal.
- i) All entries of an upper triangular matrix are determined by the entries occurring on and above the main diagonal.
- j) The inverse of an invertible lower triangular matrix is an upper triangular matrix.
- k) A diagonal matrix is invertible if and only if all of its diagonal entries are positive.
- l) The sum of a diagonal matrix and a lower triangular matrix is a lower triangular matrix.
- m) A matrix that is both symmetric and upper triangular must be a diagonal matrix.
- n) If  $A$  and  $B$  are  $n \times n$  matrices such that  $A + B$  is symmetric, then  $A$  and  $B$  are symmetric.
- o) If  $A$  and  $B$  are  $n \times n$  matrices such that  $A + B$  is upper triangular, then  $A$  and  $B$  are upper triangular.
- p) If  $A^2$  is a symmetric matrix, then  $A$  is a symmetric matrix.
- q) If  $kA$  is a symmetric matrix for some  $k \neq 0$ , then  $A$  is a symmetric matrix.

**Solution**

a) **False:**  $\left( \begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array} \right)$

b) **False**  $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$   
 $A$   $B$

c) **True** by definition.

d) **True**  $(ABC)^T = C^T (AB)^T = C^T B^T A^T = CBA$  Since  $A^T = A, B^T = B, C^T = C$

e) **True** Since a diagonal matrix must be square and have zeros off the main diagonal, its transpose is also diagonal.

f) **False** The transpose of an upper triangular matrix is lower triangular.

g) **False**  $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix}$

h) **True** The entries above the main diagonal determine the entries below the main diagonal in a symmetric matrix.

i) **True** in an upper triangular matrix, the series below the main diagonal are all zeros.

j) **False** The inverse of an invertible lower triangular matrix is lower triangular.

k) **False** The diagonal entries may be negative, as long as they are nonzero.

l) **True** Adding a diagonal matrix to a lower triangular matrix will not create nonzero entries above the main diagonal.

m) **True** Since the entries below the main diagonal must be zero, so also must be the entries above the main diagonal.

n) **False**  $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$  which is symmetric

o) **False**  $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}$  which is upper triangular.

p) **False**  $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

q) **True**  $(kA)^T = kA$  then

$$(kA)^T - kA = 0$$

$$kA^T - kA = 0$$

$$k(A^T - A) = 0 \text{ since } k \neq 0 \text{ then } A^T = A$$

Therefore,  $A$  is a symmetric matrix.

## Exercise

Find 2 by 2 symmetric matrices  $A = A^T$  with these properties

- a)  $A$  is not invertible
- b)  $A$  is invertible but cannot be factored into  $LU$  (row exchanges needed)
- c)  $A$  can be factored into  $LDL^T$  but not into  $LL^T$  (because of negative  $D$ )

## Solution

a)  $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

b)  $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$  only need a *zero* in the diagonal.

c)  $A = LDL^T$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} a & 0 \\ a & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & a \\ a & a+d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \rightarrow \begin{cases} a=1 \\ d=1 \end{cases}$$

$LL^T$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\rightarrow D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

### Exercise

A group of matrices includes  $AB$  and  $A^{-1}$  if it includes  $A$  and  $B$  “Products and inverses stay in the group.” Which of these sets are groups?

Lower triangular matrices  $L$  with 1’s on the diagonal, symmetric matrices  $S$ , positive matrices  $M$ , diagonal invertible matrices  $D$ , permutation matrices  $P$ , matrices with  $Q^T = Q^{-1}$ . **Invent two more matrix groups.**

### Solution

The lower triangular matrices  $L$  with 1’s on the diagonal form a group.

Clearly the product of two is a third. The Gauss-Jordan method shows that the inverse of one is another.

The symmetric matrices don’t form a group. An example of the 2 symmetric matrices  $A$  and  $B$  whose product is not symmetric

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \quad AB = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix}$$

The positive matrices do not form a group.

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad M^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \text{ the inverse is not symmetric.}$$

The diagonal invertible matrices form a group.

The permutation matrices form a group.

The matrices with  $Q^T = Q^{-1}$  form a group. If  $A$  and  $B$  are two matrices, then so are  $AB$  and  $A^{-1}$ , as

$$\begin{aligned}(AB)^T &= B^T A^T \\ &= B^{-1} A^{-1} \\ &= (AB)^{-1}\end{aligned}$$

$$\begin{aligned}(A^{-1})^T &= (A^T)^{-1} \\ &= A^{-1}\end{aligned}$$

There are many more matrix groups. For example, given two, the block matrices  $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$  form a third as  $A$  ranges over the first group and  $B$  ranges over the second.

Another example is the set of all products  $cP$  where  $c$  is a nonzero scalar and  $P$  is a permutation matrix of given size.

### Exercise

Write  $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$  as the product  $EH$  of an elementary row operation matrix  $E$  and a symmetric matrix  $H$ .

### Solution

$$A = EH$$

$$E^{-1}A = E^{-1}EH$$

$$E^{-1}A = H$$

An elementary row operation matrix has the form  $E = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$

The inverse is:  $E^{-1} = \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}$

$$\begin{aligned}H &= \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ -x+4 & -2x+9 \end{pmatrix}\end{aligned}$$

Since matrix  $H$  is symmetric, therefore:

$$-x+4=2$$

$$\underline{x=2}$$

$$\begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}}_{\text{Elementary}} \underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}}_{\text{Symmetric}}$$

**Exercise**

When is the product of two symmetric matrices symmetric? Explain your answer.

**Solution**

$$AB \text{ is symmetric iff } AB = (AB)^T$$

$$\begin{aligned} AB &= (AB)^T \\ &= B^T A^T && \text{A and B are symmetric} \\ &= BA \end{aligned}$$

$AB$  is symmetric iff  $A$  and  $B$  commute

**Exercise**

Express  $\left((AB)^{-1}\right)^T$  in terms of  $\left(A^{-1}\right)^T$  and  $\left(B^{-1}\right)^T$

**Solution**

$$\begin{aligned} \left((AB)^{-1}\right)^T &= \left(B^{-1}A^{-1}\right)^T \\ &= \left(A^{-1}\right)^T \left(B^{-1}\right)^T \end{aligned}$$

**Exercise**

Find the transpose of the given matrix:

$$\begin{bmatrix} 8 & -1 \\ 3 & 5 \\ -2 & 5 \\ 1 & 2 \\ -3 & -5 \end{bmatrix}$$
**Solution**

$$A^T = \begin{bmatrix} 8 & 3 & -2 & 1 & -3 \\ -1 & 5 & 5 & 2 & -5 \end{bmatrix}$$

**Exercise**

Find the transpose of the given matrix:  $D = \begin{pmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -1 \end{pmatrix}$

**Solution**

$$D^T = \begin{pmatrix} 1 & -3 & 5 \\ -2 & 4 & -1 \end{pmatrix}$$

**Exercise**

Show that if  $A$  is symmetric and invertible, then  $A^{-1}$  is also symmetric.

**Solution**

$A$  is symmetric and invertible, then  $A = A^T$   $AA^{-1} = I$

$$\begin{aligned} (A^{-1})^T &= (A^T)^{-1} \\ &= A^{-1} \end{aligned}$$

$\Rightarrow A^{-1}$  is symmetric.

**Exercise**

Prove that  $(AB)^T = B^T A^T$

**Solution**

Let  $A = [a_{ik}]$  and  $B = [b_{kj}]$

Then the  $ij$ -entry of  $AB$  is:

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{im}b_{mj}$$

The reverse order,  $ji$ -entry of  $(AB)^T$

Column  $j$  of  $B$  becomes row  $j$  of  $B^T$ , and row  $i$  of  $A$  becomes column  $i$  of  $A^T$ .

Thus, the  $ij$ -entry of  $B^T A^T$  is:

$$\left( (b_{1j}, b_{2j}, \dots, b_{mj}) (a_{i1}, a_{i2}, \dots, a_{im}) \right)^T = b_{1j}a_{i1} + b_{2j}a_{i2} + \dots + b_{mj}a_{im}$$

Thus  $(AB)^T = B^T A^T$



**Exercise**

Prove that  $(A + B)^T = A^T + B^T$

**Solution**

For adding matrices, the matrices must have the size  $(m \times n)$

$$\text{Let } A = \begin{bmatrix} a_{ij} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{ij} \end{bmatrix}$$

$$\text{Let } C = \begin{bmatrix} c_{ij} \end{bmatrix} \text{ such that } C = A + B$$

Then the  $ij$ -entry of  $(A + B)$  is:

$$C = A + B$$

$$= \begin{bmatrix} a_{ij} \end{bmatrix} + \begin{bmatrix} b_{ij} \end{bmatrix}$$

$$= \begin{bmatrix} a_{ij} + b_{ij} \end{bmatrix}$$

$$\begin{bmatrix} c_{ij} \end{bmatrix} = \begin{bmatrix} a_{ij} + b_{ij} \end{bmatrix}$$

$$(A + B)^T = \begin{bmatrix} c_{ij} \end{bmatrix}^T$$

$$= \begin{bmatrix} c_{ji} \end{bmatrix}$$

$$A^T + B^T = \begin{bmatrix} a_{ij} \end{bmatrix}^T + \begin{bmatrix} b_{ij} \end{bmatrix}^T$$

$$= \begin{bmatrix} a_{ji} \end{bmatrix} + \begin{bmatrix} b_{ji} \end{bmatrix}$$

$$= \begin{bmatrix} a_{ji} + b_{ji} \end{bmatrix}$$

$$= \begin{bmatrix} c_{ji} \end{bmatrix}$$

$$= (A + B)^T$$

$$(A + B)^T = A^T + B^T \quad \checkmark$$

**Exercise**

Prove that if  $A$ ,  $B$ , and  $C$  are square symmetric matrices and  $ABC = I$ , then  $B$  is an invertible and  $B^{-1} = CA$ .

**Solution**

**Given:**  $A$ ,  $B$ , and  $C$  are square symmetric matrices, then

$$A = A^T, \quad B = B^T, \quad C = C^T \quad \& \quad ABC = I$$

$$CABC = CI$$

$$CABCA = CA$$

$$(CAB)CA = CA$$

$$CA(BCA) = CA$$

$$B(CA) = I$$

That implies to  $B$  in the invertible of  $CA$ ,  $B^{-1} = CA$

$$BB^{-1} = I$$

Therefore,  $B$  is invertible.

**Exercise**

Prove that each statement is true when  $A$  and  $B$  are square matrices of order  $n$  and  $c$  is a scalar.

$$a) \quad Tr(A + B) = Tr(A) + Tr(B)$$

$$b) \quad Tr(cA) = c Tr(A)$$

**Solution**

$$\text{Let } A = \begin{bmatrix} a_{ij} \end{bmatrix} \quad \& \quad B = \begin{bmatrix} b_{ij} \end{bmatrix}$$

$$\begin{aligned} a) \quad A + B &= \begin{bmatrix} a_{ij} \end{bmatrix} + \begin{bmatrix} b_{ij} \end{bmatrix} \\ &= \begin{bmatrix} a_{ij} + b_{ij} \end{bmatrix} \end{aligned}$$

$$Tr(A + B) = Tr\left(\begin{bmatrix} a_{ij} + b_{ij} \end{bmatrix}\right)$$

$$= \sum_{k=1}^n \begin{bmatrix} a_{kk} + b_{kk} \end{bmatrix}$$

$$\begin{aligned}
&= \sum_{k=1}^n [a_{kk}] + \sum_{k=1}^n [b_{kk}] \\
&= \text{Tr}(A) + \text{Tr}(B)
\end{aligned}$$

Therefore,  $\text{Tr}(A+B) = \text{Tr}(A) + \text{Tr}(B)$

**b)** Let  $c$  be any scalar

$$\begin{aligned}
\text{Tr}(cA) &= \sum_{k=1}^n [ca_{kk}] \\
&= c \sum_{k=1}^n [a_{kk}] \\
&= c \text{Tr}(A)
\end{aligned}$$

Therefore,  $\text{Tr}(cA) = c \text{Tr}(A)$

### Exercise

Verify that  $(AB)^T = B^T A^T$  given  $A = \begin{bmatrix} -1 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix}$ ;  $B = \begin{bmatrix} -3 & 0 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$

### Solution

$$\begin{aligned}
AB &= \begin{bmatrix} -1 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 1 & 2 \\ 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 4 \\ -5 & -1 \end{bmatrix}
\end{aligned}$$

$$(AB)^T = \begin{bmatrix} 2 & -5 \\ 4 & -1 \end{bmatrix}$$

$$\begin{aligned}
B^T A^T &= \begin{bmatrix} -3 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 0 \\ -2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -5 \\ 4 & -1 \end{bmatrix}
\end{aligned}$$

Therefore,  $(AB)^T = B^T A^T$

**Exercise**

For the given matrix, compute  $A^T$ ,  $(A^T)^{-1}$ ,  $A^{-1}$ , and  $(A^{-1})^T$ , then compare  $(A^T)^{-1}$  and  $(A^{-1})^T$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

**Solution**

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 - 2R_2 \\ \\ \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 + R_3 \\ R_1 - 2R_3 \\ \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$(A^T)^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 & 1 \end{array} \right] \begin{array}{l} \\ \\ R_3 - 2R_2 \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$(A^{-1})^T = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(A^T)^{-1} = (A^{-1})^T$$

**Exercise**

Show that a  $2 \times 2$  lower triangular matrix is invertible if and only if  $a_{11}a_{22} \neq 0$  and in this case the inverse is also lower triangular.

**Solution**

Let  $A$  be the lower triangular matrix

$$A = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}$$

$\det(A) = a_{11}a_{22} \neq 0$  is invertible iff  $a_{11}a_{22} \neq 0$  and then

$$\begin{aligned} A^{-1} &= \frac{1}{a_{11}a_{22}} \begin{pmatrix} a_{22} & 0 \\ -a_{21} & a_{11} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{a_{11}} & 0 \\ -\frac{a_{21}}{a_{11}a_{22}} & \frac{1}{a_{22}} \end{pmatrix} \end{aligned}$$

**Exercise**

Let  $A$  be any  $(2 \times 2)$  diagonal matrix. Give a necessary and sufficient condition on the diagonal entries so that  $A$  has an inverse. Compute the inverse of any such matrix.

**Solution**

$$\text{Let } A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{a_{11}} & 0 \\ 0 & \frac{1}{a_{22}} \end{pmatrix}$$

So,  $A^{-1}$  exists when both entries on the main diagonal are nonzero.

**Exercise**

Find  $(2 \times 2)$  matrices  $A$  and  $B$  such that  $A$  and  $B$  are symmetric, but  $AB$  is not symmetric.

**Solution**

Since  $A$  and  $B$  are symmetric, then

$$A = A^T \quad \& \quad B = B^T$$

$$\begin{aligned}
 (AB)^T &= B^T A^T \\
 &= BA \\
 &\neq AB
 \end{aligned}$$

Since the product is not commutative.



Since we going to prove  $AB$  is **not** symmetric, then we can use an example.

$$\text{Let } A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix}$$

$$\begin{aligned}
 AB &= \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & 9 \\ 1 & 13 \end{pmatrix}
 \end{aligned}$$

$AB$  is not symmetric even though  $A$  and  $B$  are symmetric

### Exercise

Let  $A$  and  $B$  be  $(n \times n)$  symmetric matrices. Give a necessary and sufficient condition for  $AB$  to be symmetric.

### Solution

Since  $A$  and  $B$  are symmetric, then

$$A = A^T \quad \& \quad B = B^T$$

$$\begin{aligned}
 (AB)^T &= B^T A^T \\
 &= BA
 \end{aligned}$$

For  $AB$  to be symmetric, then  $AB = BA$

For the product to be commutative, then  $A$  is inverse of matrix  $B$ .



$$\text{Let } A = \begin{pmatrix} a_{11} & a \\ a & a_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b \\ b & b_{22} \end{pmatrix}$$

$$\begin{aligned}
 AB &= \begin{pmatrix} a_{11} & a \\ a & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b \\ b & b_{22} \end{pmatrix} \\
 &= \begin{pmatrix} a_{11}b_{11} + ab & b a_{11} + a b_{22} \\ a b_{11} + b a_{22} & ab + a_{22}b_{22} \end{pmatrix}
 \end{aligned}$$

$$= \begin{pmatrix} * & b a_{11} + a b_{22} \\ a b_{11} + b a_{22} & * \end{pmatrix}$$

For  $AB$  to be symmetric, then

$$b a_{11} + a b_{22} = a b_{11} + b a_{22}$$

$$a (b_{22} - b_{11}) = b (a_{22} - a_{11})$$

*If*  $b_{22} - b_{11} = 0$  &  $a_{22} - a_{11} = 0$

$$b_{22} = b_{11} \quad \& \quad a_{22} = a_{11}$$

$$\begin{aligned} AB &= \begin{pmatrix} * & b a_{11} + a b_{11} \\ a b_{11} + b a_{11} & * \end{pmatrix} \\ &= \begin{pmatrix} * & b a_{11} + a b_{11} \\ a b_{11} + b a_{11} & * \end{pmatrix} \end{aligned}$$

*If*  $b_{22} - b_{11} \neq 0$  &  $a_{22} - a_{11} \neq 0$

$$b_{22} \neq b_{11} \quad \& \quad a_{22} \neq a_{11}$$

$$a = b \frac{a_{22} - a_{11}}{b_{22} - b_{11}}$$

$$\begin{aligned} AB &= \begin{pmatrix} * & b a_{11} + a b_{22} \\ a b_{11} + b a_{22} & * \end{pmatrix} \\ &= \begin{pmatrix} * & b a_{11} + b \frac{a_{22} - a_{11}}{b_{22} - b_{11}} b_{22} \\ b \frac{a_{22} - a_{11}}{b_{22} - b_{11}} b_{11} + b a_{22} & * \end{pmatrix} \\ &= \begin{pmatrix} * & b \frac{a_{11} b_{22} - a_{11} b_{11} + a_{22} b_{22} - a_{11} b_{22}}{b_{22} - b_{11}} \\ b \frac{a_{22} b_{11} - a_{11} b_{11} + a_{22} b_{22} - a_{22} b_{11}}{b_{22} - b_{11}} & * \end{pmatrix} \\ &= \begin{pmatrix} * & b \frac{-a_{11} b_{11} + a_{22} b_{22}}{b_{22} - b_{11}} \\ b \frac{-a_{11} b_{11} + a_{22} b_{22}}{b_{22} - b_{11}} & * \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} * & \frac{b(a_{22}b_{22} - a_{11}b_{11})}{b_{22} - b_{11}} \\ \frac{b(a_{22}b_{22} - a_{11}b_{11})}{b_{22} - b_{11}} & * \end{pmatrix}$$

If  $b_{22} - b_{11} = 0$  &  $a_{22} - a_{11} \neq 0$

$$b_{22} = b_{11} \quad \& \quad a_{22} \neq a_{11}$$

$$AB = \begin{pmatrix} * & ba_{11} + ab_{22} \\ ab_{11} + ba_{22} & * \end{pmatrix}$$

$$= \begin{pmatrix} * & ba_{11} + ab_{11} \\ ab_{11} + ba_{22} & * \end{pmatrix}$$

$$ba_{11} = ba_{22}$$

$$a_{11} = a_{22} \text{ Contradict the assumption.}$$

Therefore, for  $AB$  to be symmetric only and only if when

$$b_{22} = b_{11} \quad \& \quad a_{22} = a_{11}$$

Or

$$b_{22} \neq b_{11} \quad \& \quad a_{22} \neq a_{11}$$

$$\rightarrow a = b \frac{a_{22} - a_{11}}{b_{22} - b_{11}}$$

### Exercise

Let  $A$  be any  $(2 \times 2)$  matrix, and let  $B$  and  $C$  be given by

$$B = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

Find the matrix  $A$  if:

$$a) \quad A^T + B = C$$

$$b) \quad A^T B = C$$

### Solution

$$a) \quad A^T + B = C$$

$$A^T = C - B$$



$$= \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

**b)**  $A^T B = C$

$$A^T B B^{-1} = C B^{-1}$$

$$A^T = C B^{-1}$$

$$B^{-1} = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -3 \\ 11 & -7 \end{pmatrix}$$

$$A = \begin{pmatrix} 5 & 11 \\ -3 & -7 \end{pmatrix}$$

### Exercise

Given the matrix  $A = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

a) Find  $A^T A$ , show it is symmetric

b) Find  $AA^T$ , show it is symmetric

### Solution

**a)**  $A^T A = \begin{bmatrix} 4 & 0 \\ 2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 16 & 8 & 4 \\ 8 & 8 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

Therefore, from the solution the  $A^T A$  is symmetric

**b)**  $AA^T = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 2 & 2 \\ 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 21 & 3 \\ 3 & 5 \end{bmatrix}$$

Therefore, from the solution the  $AA^T$  is symmetric

### Exercise

Prove each of the following:

- a) If  $A$  is any  $(2 \times 2)$  matrix, then  $A^T A$  and  $AA^T$  are symmetric
- b) If  $A$ ,  $B$ , and  $C$  are matrices such that the product  $ABC$  is defined then  $(ABC)^T = C^T B^T A^T$

### Solution

- a) For  $A^T A$  to be symmetric

$$\begin{aligned} A^T A &= (A^T A)^T \\ (A^T A)^T &= A^T (A^T)^T \\ &= A^T A \quad \checkmark \end{aligned}$$

For  $AA^T$  to be symmetric, then  $AA^T = (AA^T)^T$

$$\begin{aligned} (AA^T)^T &= (A^T)^T A^T \\ &= AA^T \quad \checkmark \end{aligned}$$

Therefore,  $A^T A$  and  $AA^T$  are symmetric

-----

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$A^T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\begin{aligned} A^T A &= \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= \begin{pmatrix} * & a_{11}a_{12} + a_{21}a_{22} \\ a_{12}a_{11} + a_{22}a_{21} & * \end{pmatrix} \end{aligned}$$

Hence,  $A^T A$  is symmetric.

$$\begin{aligned}
 AA^T &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \\
 &= \begin{pmatrix} * & a_{11}a_{21} + a_{12}a_{22} \\ a_{21}a_{11} + a_{22}a_{12} & * \end{pmatrix}
 \end{aligned}$$

Hence,  $AA^T$  is symmetric.

**b)** For  $(ABC)^T = C^T B^T A^T$

Given that the product  $ABC$  is defined

Let  $BC = D$

$$\begin{aligned}
 (ABC)^T &= (AD)^T \\
 &= D^T A^T && \text{From part (a) prove} \\
 &= (BC)^T A^T \\
 &= C^T B^T A^T \quad \checkmark
 \end{aligned}$$

Therefore,  $(ABC)^T = C^T B^T A^T$

### Exercise

Let  $A$  be any  $(n \times n)$  real matrix

- a) Show that  $A + A^T$  is symmetric
- b) Show that  $A - A^T$  is skew symmetric

### Solution

$$\begin{aligned}
 \text{a)} \quad (A + A^T)^T &= A^T + (A^T)^T \\
 &= A^T + A \\
 &= A + A^T
 \end{aligned}$$

Therefore,  $A + A^T$  is symmetric

$$\begin{aligned}
 \text{b)} \quad (A - A^T)^T &= A^T - (A^T)^T \\
 &= A^T - A \\
 &= -A + A^T \\
 &= -(A - A^T)
 \end{aligned}$$

Therefore,  $A - A^T$  is skew symmetric

**Exercise**

Suppose that  $L$  is unit lower triangular and  $U$  is the upper triangular. Show that if  $L = U$ , then  $L = U = I$

**Solution**

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \quad \& \quad U = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

If  $L = U$ , then the only common between the lower and upper triangular is the main diagonal with a unit entries.

Therefore, for  $L = U = I$

## ***Solution***      **Section 1.6 – The Properties of Determinants**

### ***Exercise***

Verify that  $\det(AB) = \det(A)\det(B)$  when:  $A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{pmatrix}$

### ***Solution***

$$AB = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{pmatrix}$$

$$\det(AB) = \begin{vmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{vmatrix} \begin{matrix} \textcolor{red}{9} & \textcolor{red}{-1} \\ \textcolor{red}{31} & \textcolor{red}{1} \\ \textcolor{red}{10} & \textcolor{red}{0} \end{matrix}$$

$$\underline{\underline{= -170}}$$

$$\det(A) = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$$\underline{\underline{= 10}}$$

$$\det(B) = \begin{vmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{vmatrix}$$

$$\underline{\underline{= -17}}$$

$$\det(AB) = \det(A)\det(B) = -170 \quad \checkmark$$

### ***Exercise***

For which value(s) of  $k$  does  $A$  fail to be invertible?  $A = \begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix}$

### ***Solution***

For  $A$  to have an invertible the determinant cannot be equal to zero. To **fail**  $\det(A) = 0$ .

$$|A| = \begin{vmatrix} k-3 & -2 \\ -2 & k-2 \end{vmatrix} = 0$$

$$(k-3)(k-2)-4=0$$

$$k^2-5k+6-4=0$$

$$k^2-5k+2=0$$

$$k = \frac{5 \pm \sqrt{17}}{2}$$

### Exercise

Without directly evaluating, show that  $\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$

### Solution

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \xrightarrow{R_2 + R_1} \begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$

It is equal to zero, since first row and third row are proportional.

$$\begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \xrightarrow{R_3 - \frac{1}{a+b+c}R_1} \begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 0 & 0 & 0 \end{vmatrix} = 0$$

### Exercise

If the entries in every row of  $A$  add to zero, solve  $A\vec{x} = 0$  to prove  $\det A = 0$ . If those entries add to one, show that  $\det(A - I) = 0$ . Does this mean  $\det A = I$ ?

### Solution

If  $\mathbf{x} = (1, 1, \dots, 1)$ , then  $A\vec{x}$  = the sums of the rows of  $A$ . Since every row of  $A$  add to zero, that implies  $A\vec{x} = 0$ . Since  $A$  has non-zero nullspace, it is not invertible and  $\det A = 0$ . If the entries in every row of  $A$  sum to one, then the entries in every row of  $A - I$  sum to zero.  $A - I$  has a non-zero nullspace and  $\det(A - I) = 0$ . This does not mean that  $\det A = I$ .

*Example:*

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ every row of } A \text{ adds up to zero}$$

$$\Rightarrow \det A = -1 \neq 1 = \det I$$

**Exercise**

Does  $\det(AB) = \det(BA)$  in general?

- a) True or false if  $A$  and  $B$  are square  $n \times n$  matrices?
- b) True or false if  $A$  is  $m \times n$  and  $B$  is  $n \times m$  with  $m \neq n$ ?

**Solution**

- a) Matrices  $A$  and  $B$  are square matrices, then by the property:

$$\begin{aligned}\det(AB) &= \det(A)\det(B) \\ &= \det(B)\det(A) \\ &= \det(BA)\end{aligned}$$

Therefore; it is true for any  $A$  and  $B$  square matrices.

- b) False, example if  $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$   $B = \begin{pmatrix} 1 & 1 \end{pmatrix}$

$$\begin{aligned}AB &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\det(AB) &= \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \\ &= 0\end{aligned}$$

$$BA = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \end{pmatrix}$$

$$\det(BA) = 2$$

$$\det(AB) \neq \det(BA)$$

**Exercise**

True or false, with a reason if true or a counterexample if false:

- a) The determinant of  $I + A$  is  $1 + \det A$ .
- b) The determinant of  $ABC$  is  $|A||B||C|$ .
- c) The determinant of  $4A$  is  $4|A|$
- d) The determinant of  $AB - BA$  is zero. (try an example)
- e) If  $A$  is not invertible then  $AB$  is not invertible.
- f) The determinant of  $A - B$  equals to  $\det A - \det B$ .

**Solution**

a) **False**, if  $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\det(I + A) = \det \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ = 0$$

$$\det A = 1$$

$$1 + \det A = 1 + 1 \\ = 2$$

$$\neq \det(I + A)$$

b) **True**,  $\det(ABC) = \det(A)\det(BC) = \det(A)\det(B)\det(C)$ .

c) **False**, in general  $\det(4A) = 4^n \det(A)$  if  $A$  is  $n \times n$ .

d) **False**,  $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$AB - BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\det(AB - BA) = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \\ = 1 \neq 0$$

e) **False**, any matrix is invertible, iff its determinant is nonzero. So  $\det A = 0$  which  
 $\det(AB) = \det(A)\det(B) = 0$ . Therefore,  $AB$  can't be invertible.

f)  $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow |A| = 0$

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow |B| = -1$$

$$\det(A) - \det(B) = 0 - (-1) = 1$$

$$\det(A - B) = \det \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} = -1$$

$$\Rightarrow \det(A - B) \neq \det(A) - \det(B)$$



**Exercise**

Use row operations to show the 3 by 3 “Vandermonde determinant” is

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$$

**Solution**

$$\begin{aligned} \det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} &= \det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array} \\ &= \det \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix} \text{factor } (b-a) \\ &= (b-a) \det \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & c-a & c^2-a^2 \end{bmatrix} \begin{array}{l} R_2 - (c-a)R_2 \\ (c-a)(c+a) - (b+a)(c-a) = (c-a)(c+a-b-a) \end{array} \\ &= (b-a) \det \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & (c-a)(c-b) \end{bmatrix} \text{Multiply the main diagonal by } (b-a) \\ &= \underline{(b-a)(c-a)(c-b)} \end{aligned}$$

**Exercise**

The inverse of a 2 by 2 matrix seems to have determinant = 1:

$$\begin{aligned} \det A^{-1} &= \det \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{ad-bc}{ad-bc} \\ &= 1 \end{aligned}$$

What is wrong with this calculation? What is the correct  $\det A^{-1}$

**Solution**

The  $\det \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} (ad-bc)$  it is part of the determinant and it is not the solution.

$$\begin{aligned} \det \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} &= \frac{1}{ad-bc} \det \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{ad-bc} \frac{1}{ad-bc} (ad-bc) \\ &= \frac{1}{ad-bc} \end{aligned}$$

### Exercise

A *Hessenberg* matrix is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci's rule  $|H_4| = |H_3| + |H_2|$ . The same rule will continue for all sizes  $|H_n| = |H_{n-1}| + |H_{n-2}|$ . Which Fibonacci number is  $|H_n|$ ?

$$H_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad H_3 = \begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad H_4 = \begin{bmatrix} 2 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

### Solution

$$|H_4| = 2C_{11} + 1C_{12}$$

The cofactor  $C_{11}$  for  $H_4$  is the determinant  $|H_3|$ .

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$\text{The cofactor } C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= - \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$|H_2| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= -|H_3| + |H_2|$$

$$\begin{aligned} |H_4| &= 2C_{11} + 1C_{12} \\ &= 2|H_3| - |H_3| + |H_2| \\ &= |H_3| + |H_2| \end{aligned}$$

The actual number:  $|H_2| = 3$ ,  $|H_3| = 5$ ,  $H_4 = 8$ .

Since  $|H_n|$  follows Fibonacci's rule  $|H_{n-1}| + |H_{n-2}|$ , it must be  $|H_n| = F_{n+2}$ .

### Exercise

Evaluate  $\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix}$

### Solution

$$\begin{aligned} \begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix} &= -9 - (-6) \\ &= -3 \end{aligned}$$

### Exercise

Evaluate  $\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix}$

### Solution

$$\begin{aligned} \begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix} &= -6 - (0) \\ &= -6 \end{aligned}$$

### Exercise

Evaluate  $\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix}$

### Solution

$$\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix} = x(8x) - 4x(2x)$$

$$\begin{aligned} &= 8x^2 - 8x^2 \\ &= 0 \end{aligned}$$

***Exercise***

Evaluate  $\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix}$

**Solution**

$$\begin{aligned} \begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix} &= 3x - 2x(4) \\ &= 3x - 8x \\ &= -5x \end{aligned}$$

***Exercise***

Evaluate  $\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix} = \underline{-3x^4 - 2x}$$

***Exercise***

Evaluate  $\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix}$

**Solution**

$$\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix} = \underline{-8a + 5b}$$

***Exercise***

Evaluate  $\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}$

**Solution**

$$\begin{aligned} \begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix} &= 15 - 14 \\ &= 1 \end{aligned}$$

**Exercise**

Evaluate  $\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix} = 5 - 20 \\ = -16$$

**Exercise**

Evaluate  $\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix} = 15 + 6 \\ = 21$$

**Exercise**

Evaluate  $\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix} = -24 + 5 \\ = -19$$

**Exercise**

Evaluate  $\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix}$

**Solution**

$$\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix} = 3 - 6 \\ = -3$$

**Exercise**

Evaluate  $\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix}$

**Solution**

$$\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix} = 7 + 18$$
$$\underline{= 25}$$

**Exercise**

Evaluate  $\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix} = \underline{2\sqrt{5} + 6}$$

**Exercise**

Evaluate  $\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix}$

**Solution**

$$\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix} = -\frac{3}{8} - \frac{1}{16}$$
$$\underline{= -\frac{7}{16}}$$

**Exercise**

Evaluate  $\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix} = -1 + 1$$
$$\underline{= 0}$$

**Exercise**

Evaluate  $\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix}$

**Solution**

$$\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix} = \frac{1}{2} + \frac{1}{6}$$

$$= \frac{2}{3}$$

**Exercise**

Evaluate  $\begin{vmatrix} \lambda - 3 & 2 \\ 4 & \lambda - 1 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} \lambda - 3 & 2 \\ 4 & \lambda - 1 \end{vmatrix} = \lambda^2 - 4\lambda - 5$$

**Exercise**

Evaluate  $\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix}$

**Solution**

$$\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix} = x^2 - 4x^2$$

$$= -3x^2$$

**Exercise**

Evaluate  $\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix} = 9x - x^3$$

**Exercise**

Evaluate  $\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix} = \underline{2x^2 + 3x}$$

**Exercise**

Evaluate  $\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix} = 4(x+2) - 6(x-2) \\ = \underline{-2x + 20}$$

**Exercise**

Evaluate  $\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix} = -3x - 3 + 6x + 18 \\ = \underline{-2x + 20}$$

**Exercise**

Evaluate  $\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = \underline{5}$$

**Exercise**

Evaluate  $\begin{vmatrix} 5 & 3 \\ -6 & 3 \end{vmatrix}$

**Solution**



$$\begin{vmatrix} 5 & 3 \\ -6 & 3 \end{vmatrix} = \underline{33}$$

**Exercise**

Evaluate  $\begin{vmatrix} \sin \theta & 1 \\ 1 & \sin \theta \end{vmatrix}$

**Solution**

$$\begin{vmatrix} \sin \theta & 1 \\ 1 & \sin \theta \end{vmatrix} = \underline{\sin^2 \theta - 1}$$

**Exercise**

Evaluate  $\begin{vmatrix} 0 & 8 \\ 0 & 4 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} 0 & 8 \\ 0 & 4 \end{vmatrix} = \underline{0}$$

**Exercise**

Evaluate  $\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$

**Solution**

$$\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x} \\ = \underline{e^{5x}}$$

**Exercise**

Evaluate  $\begin{vmatrix} x & \ln x \\ 1 & \frac{1}{x} \end{vmatrix}$

**Solution**

$$\begin{vmatrix} x & \ln x \\ 1 & \frac{1}{x} \end{vmatrix} = \underline{1 - \ln x}$$

**Exercise**

Evaluate  $\begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix}$

**Solution**

$$\begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix} = \cos^2 \theta + \sin^2 \theta \\ \underline{= 1}$$

**Exercise**

Evaluate  $\begin{vmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{vmatrix} \underline{= -58}$$

**Exercise**

Evaluate  $\begin{vmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{vmatrix} \underline{= k}$$

**Exercise**

Evaluate  $\begin{vmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \underline{= 1}$$

**Exercise**

Evaluate  $\begin{vmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{vmatrix} = \underline{-30}$$

**Exercise**

Evaluate  $\begin{vmatrix} x & y & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} x & y & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \underline{2x - 1 - 3y}$$

**Exercise**

Evaluate  $\begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta$$

$$= \underline{r}$$

**Exercise**

Evaluate  $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

**Solution**

$$\begin{aligned}
 \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} &= (a+1)(b+1)(c+1) + 2 - (b+1) - (a+1) - (c+1) \\
 &= abc + ab + ac + a + bc + b + c + 1 - b - a - c - 1 \\
 &= \underline{abc + ab + ac + bc} \\
 &= \underline{abc \left( 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)}
 \end{aligned}$$

**Exercise**

Evaluate  $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix} \begin{matrix} 3 & 0 \\ 2 & 1 \\ 2 & 5 \end{matrix} = \underline{72}$$

**Exercise**

Evaluate  $\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix} \begin{matrix} 4 & 0 \\ 3 & -1 \\ 2 & -3 \end{matrix} = \underline{24}$$

$$\text{or} = 4 \begin{vmatrix} -1 & 4 \\ -3 & 6 \end{vmatrix}$$

**Exercise**

Evaluate  $\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix} \begin{matrix} 3 & 1 \\ -3 & -4 \\ -1 & 3 \end{matrix} = \underline{-45}$$

**Exercise**

Evaluate  $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix} \begin{matrix} 1 & 1 \\ 2 & 2 \\ 3 & -4 \end{matrix}$$

$$= 10 + 6 - 8 - 6 + 8 - 10$$

$$= 0$$

**Exercise**

Evaluate  $\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix} \begin{matrix} x & 0 \\ 2 & 1 \\ -3 & x \end{matrix}$$

$$= x - 2x - 3 - x^4$$

$$= -x^4 - x - 3$$

**Exercise**

Evaluate  $\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix} \begin{matrix} x & 1 \\ x^2 & x \\ 0 & x \end{matrix}$$

$$= x^2 - x^3 - x^3 - x^2$$

$$= -2x^3$$

***Exercise***

$$\text{Evaluate } \begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$$

**Solution**

$$\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix} = 0 + 126 + 48 - (-48 + 36 + 0) \\ = \underline{90}$$

***Exercise***

$$\text{Evaluate } \begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix}$$

**Solution**

$$\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix} = \underline{10}$$

***Exercise***

$$\text{Evaluate } \begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix}$$

**Solution**

$$\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix} = \begin{matrix} 3 & 1 \\ -2 & 3 \\ 3 & 4 \end{matrix} \\ = -54 + 3 - 16 - 18 - 12 - 12 \\ = \underline{-109}$$

**Exercise**

Evaluate  $\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix} \begin{array}{l} 2x \quad 1 \\ 0 \quad 4 \\ 3 \quad 0 \end{array}$$

$$= 16x + 3x + 12$$

$$= \underline{19x + 12}$$

**Exercise**

Evaluate  $\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix} \begin{array}{l} 0 \quad x \\ x \quad x^2 \\ x \quad 7 \end{array}$$

$$= 5x^2 + 7x^2 - x^4 + 5x^2$$

$$= \underline{17x^2 - x^4}$$

**Exercise**

Evaluate  $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix} \begin{array}{l} 2 \quad x \\ -3 \quad 1 \\ 2 \quad 1 \end{array}$$

$$= 8 - 3 - 2 + 12x$$

$$= \underline{12x + 3}$$

**Exercise**

Evaluate  $\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} \begin{matrix} 1 & x \\ 3 & 1 \\ 0 & -2 \end{matrix}$$
$$= 2 + 12 + 2 - 6x$$
$$= \underline{-6x + 16}$$

**Exercise**

Evaluate  $\begin{vmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{vmatrix}$

**Solution**

$$\begin{matrix} + \\ - \end{matrix} \begin{vmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{vmatrix} = 5 \begin{vmatrix} 6 & 4 & 12 \\ 2 & -3 & 4 \\ 1 & -2 & 2 \end{vmatrix} - 4 \begin{vmatrix} 3 & 0 & 6 \\ 2 & -3 & 4 \\ 1 & -2 & 2 \end{vmatrix}$$
$$= 5(0) - 4(0)$$
$$= \underline{0}$$

**Exercise**

Evaluate  $\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix}$

**Solution**

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{vmatrix}$$



$$\begin{aligned}
&= \begin{vmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} - (-) \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} \\
&= 3 + 2 + 3 + 0 \\
&= \underline{0}
\end{aligned}$$

### Exercise

Find a)  $|A|$  b)  $|B|$  c)  $AB$  d)  $|AB|$  e)  $|A+B|$ .

Then verify that  $|A| |B| = |AB|$  &  $|A| + |B| \neq |A+B|$

$$A = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

### Solution

$$\begin{aligned}
a) \quad |A| &= \begin{vmatrix} -2 & 1 \\ 4 & -2 \end{vmatrix} \\
&= \underline{0}
\end{aligned}$$

$$\begin{aligned}
b) \quad |B| &= \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} \\
&= \underline{-1}
\end{aligned}$$

$$\begin{aligned}
c) \quad AB &= \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \\
&= \underline{\begin{pmatrix} -2 & -3 \\ 4 & 6 \end{pmatrix}}
\end{aligned}$$

$$\begin{aligned}
d) \quad |AB| &= \begin{vmatrix} -2 & -3 \\ 4 & 6 \end{vmatrix} \\
&= \underline{0}
\end{aligned}$$

$$\begin{aligned}
e) \quad A+B &= \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 2 \\ 4 & -3 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
|A+B| &= \begin{vmatrix} -1 & 2 \\ 4 & -3 \end{vmatrix} \\
&= \underline{-4}
\end{aligned}$$

$$|A| |B| = |AB| = 0$$

$$|A| + |B| = -1$$

$$|A + B| = -4$$

$$|A| + |B| \neq |A + B|$$

### ***Exercise***

Find    a)  $|A|$    b)  $|B|$    c)  $AB$    d)  $|AB|$    e)  $|A + B|$

Then verify that  $|A| |B| = |AB|$  &  $|A| + |B| \neq |A + B|$

$$A = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$$

### **Solution**

$$\begin{aligned} \text{a)} \quad |A| &= \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} \\ &= -2 \end{aligned}$$

$$\begin{aligned} \text{b)} \quad |B| &= \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} \\ &= -2 \end{aligned}$$

$$\begin{aligned} \text{c)} \quad AB &= \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{d)} \quad |AB| &= \begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{e)} \quad A + B &= \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} |A + B| &= \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \\ &= 0 \end{aligned}$$

$$|A| |B| = |AB| = 4$$

$$|A| + |B| = -4$$

$$|A + B| = 0$$

$$|A| + |B| \neq |A + B|$$

### Exercise

Find a)  $|A|$  b)  $|B|$  c)  $AB$  d)  $|AB|$  e)  $|A + B|$

Then verify that  $|A| |B| = |AB|$  &  $|A| + |B| \neq |A + B|$

$$A = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

### Solution

$$\begin{aligned} \text{a) } |A| &= \begin{vmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \\ &= \underline{2} \end{aligned}$$

$$\begin{aligned} \text{b) } |B| &= \begin{vmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} \\ &= \underline{-6} \end{aligned}$$

$$\begin{aligned} \text{c) } AB &= \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \\ &= \underline{\begin{pmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}} \end{aligned}$$

$$\begin{aligned} \text{d) } |AB| &= \begin{vmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 0 & 2 & 0 \end{vmatrix} \\ &= \underline{-12} \end{aligned}$$

$$\text{e) } A + B = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 2 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$|A+B| = \begin{vmatrix} -2 & 2 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{vmatrix}$$
$$\underline{= -15}$$

$$|A| |B| = |AB| = -12$$

$$|A| + |B| = -4$$

$$|A+B| = -15$$

$$|A| + |B| \neq |A+B|$$

### Exercise

Find all the values of  $\lambda$  for which  $\det(A) = 0$ :  $A = \begin{bmatrix} \lambda-1 & -2 \\ 1 & \lambda-4 \end{bmatrix}$

#### Solution

$$\begin{vmatrix} \lambda-1 & -2 \\ 1 & \lambda-4 \end{vmatrix} = (\lambda-1)(\lambda-4) + 2$$
$$= \lambda^2 - 5\lambda + 4 + 2$$
$$= \lambda^2 - 5\lambda + 6 = 0$$

*Solve for  $\lambda$ .*

$$\underline{\lambda_{1,2} = -1, 6}$$

### Exercise

Find all the values of  $\lambda$  for which  $\det(A) = 0$ :  $A = \begin{pmatrix} \lambda+2 & 2 \\ 1 & \lambda \end{pmatrix}$

#### Solution

$$|A| = \begin{vmatrix} \lambda+2 & 2 \\ 1 & \lambda \end{vmatrix}$$
$$= \lambda^2 + 2\lambda - 2 = 0 \quad \text{Solve for } \lambda$$

$$\lambda = \frac{-2 \pm \sqrt{12}}{2}$$

$$\underline{\lambda_{1,2} = -1 \pm \sqrt{3}}$$

**Exercise**

Find all the values of  $\lambda$  for which  $\det(A) = 0$ :  $A = \begin{pmatrix} \lambda & 2 & 0 \\ 0 & \lambda + 1 & 2 \\ 0 & 1 & \lambda \end{pmatrix}$

**Solution**

$$\begin{aligned}
 |A| &= \begin{vmatrix} \lambda & 2 & 0 \\ 0 & \lambda + 1 & 2 \\ 0 & 1 & \lambda \end{vmatrix} \\
 &= \lambda^2(\lambda + 1) - 2\lambda \\
 &= \lambda(\lambda^2 + \lambda - 2) = 0 \\
 \lambda_{1, 2} &= 0, 1, 6 - 2
 \end{aligned}$$

**Exercise**

Find all the values of  $\lambda$  for which  $\det(A) = 0$ :  $A = \begin{bmatrix} \lambda - 6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda - 4 \end{bmatrix}$

**Solution**

$$\begin{aligned}
 \begin{vmatrix} \lambda - 6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda - 4 \end{vmatrix} &= \lambda(\lambda - 6)(\lambda - 4) + 4(\lambda - 6) \\
 &= \lambda(\lambda^2 - 10\lambda + 24) + 4\lambda - 24 \\
 &= \lambda^3 - 10\lambda^2 + 24\lambda + 4\lambda - 24 \\
 &= \lambda^3 - 10\lambda^2 + 28\lambda - 24 \\
 \lambda^3 - 10\lambda^2 + 28\lambda - 24 &= 0 \\
 \lambda_{1,2,3} &= 2, 2, 6
 \end{aligned}$$

**Exercise**

Use the fact that  $|cA| = c^n |A|$  to evaluate the determinant of the  $n \times n$  matrix

$$A = \begin{pmatrix} 5 & 15 \\ 10 & -20 \end{pmatrix}$$

**Solution**

$$A: 2 \times 2 \rightarrow n = 2$$

$$\begin{aligned}|A| &= \begin{vmatrix} 5 & 15 \\ 10 & -20 \end{vmatrix} \\ &= -250\end{aligned}$$

$$\begin{aligned}|cA| &= \begin{vmatrix} 5c & 15c \\ 10c & -20c \end{vmatrix} \\ &= c \begin{vmatrix} 5 & 15 \\ 10 & -20 \end{vmatrix} \\ &= -250c^2 \\ &= c^2|A|\end{aligned}$$

$$|cA| = c^n |A| \quad \checkmark$$

### ***Exercise***

Use the fact that  $|cA| = c^n |A|$  to evaluate the determinant of the  $n \times n$  matrix

$$A = \begin{pmatrix} -3 & 6 & 9 \\ 6 & 9 & 12 \\ 9 & 12 & 15 \end{pmatrix}$$

### ***Solution***

$$A: 3 \times 3 \rightarrow n = 3$$

$$\begin{aligned}|A| &= \begin{vmatrix} -3 & 6 & 9 \\ 6 & 9 & 12 \\ 9 & 12 & 15 \end{vmatrix} \\ &= 2\end{aligned}$$

$$\begin{aligned}|cA| &= \begin{vmatrix} -3c & 6c & 9c \\ 6c & 9c & 12c \\ 9c & 12c & 15c \end{vmatrix} \\ &= c^3 \begin{vmatrix} -3 & 6 & 9 \\ 6 & 9 & 12 \\ 9 & 12 & 15 \end{vmatrix} \\ &= (2) c^3 \\ &= c^3 |A|\end{aligned}$$

$$|cA| = c^n |A| \quad \checkmark$$

**Exercise**

Verify that  $|A^{-1}| = \frac{1}{|A|}$        $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

**Solution**

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} \\ = 5$$

$$A^{-1} = \frac{1}{5} \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$$

$$|A^{-1}| = \left(\frac{1}{5}\right)^2 \begin{vmatrix} 4 & -3 \\ -1 & 2 \end{vmatrix} \\ = \frac{1}{25}(5) \\ = \frac{1}{5}$$

Therefore,  $|A^{-1}| = \frac{1}{|A|}$

**Exercise**

Verify that  $|A^{-1}| = \frac{1}{|A|}$        $A = \begin{pmatrix} 2 & -2 & 3 \\ 1 & -1 & 2 \\ 3 & 0 & 3 \end{pmatrix}$

**Solution**

$$|A| = \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & 2 \\ 3 & 0 & 3 \end{vmatrix} \\ = -3$$

$$a_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} = -3$$

$$a_{12} = - \begin{vmatrix} -2 & 3 \\ 0 & 3 \end{vmatrix} = 6$$

$$a_{13} = \begin{vmatrix} -2 & 3 \\ -1 & 2 \end{vmatrix} = -1$$

$$a_{21} = - \begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} = 3$$

$$a_{22} = \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} = -3$$

$$a_{23} = - \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -1$$

$$a_{31} = \begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} = 3$$

$$a_{32} = - \begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} = -6$$

$$a_{33} = \begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} = 0$$

$$\begin{aligned}
 A^{-1} &= -\frac{1}{3} \begin{pmatrix} -3 & 6 & -1 \\ 3 & -3 & -1 \\ 3 & -6 & 0 \end{pmatrix} \\
 |A^{-1}| &= \left(-\frac{1}{3}\right)^3 \begin{vmatrix} -3 & 6 & -1 \\ 3 & -3 & -1 \\ 3 & -6 & 0 \end{vmatrix} \\
 &= -\frac{1}{27}(9) \\
 &= \underline{-\frac{1}{3}}
 \end{aligned}$$

### Exercise

Prove that if a square matrix  $A$  has a column of zeros, then  $\det(A) = 0$

### Solution

Consider a 3 by 3 matrix with a zero column, however to find the determinant we can interchange any column of that matrix; therefore:

$$A = [a_{ij}] = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$$

By definition, the determinant of  $A$  using the cofactor:

$$\begin{aligned}
 |A| &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
 &= 0 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} 0 & a_{23} \\ 0 & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} 0 & a_{22} \\ 0 & a_{32} \end{vmatrix} \\
 &= \underline{0}
 \end{aligned}$$

### Exercise

With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D| \quad \text{but} \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|$$

- Why is the first statement true? Somehow  $B$  doesn't enter.
- Show by example that equality fails (as shown) when  $C$  enters.



c) Show by example that the answer  $\det(AD - CB)$  is also wrong.

### Solution

a) If we don't pick any 0 entries, then the first two columns are picked from  $A$  and the last two rows are from  $D$ . We can't pick any columns or rows from  $B$ , because there isn't any left.

$$b) \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \\ = -1$$

$$\text{and } A = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0, \quad B = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0, \quad C = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0, \quad D = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

c) Use the example from part (b):  $1 \neq 0$

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|$$

### **Exercise**

Show that the value of the following determinant is independent of  $\theta$ .

$$\begin{vmatrix} \sin \theta & \cos \theta & 0 \\ -\cos \theta & \sin \theta & 0 \\ \sin \theta - \cos \theta & \sin \theta + \cos \theta & 1 \end{vmatrix}$$

### Solution

$$\begin{vmatrix} \sin \theta & \cos \theta & 0 \\ -\cos \theta & \sin \theta & 0 \\ \sin \theta - \cos \theta & \sin \theta + \cos \theta & 1 \end{vmatrix} = \begin{vmatrix} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{vmatrix} \\ = \sin^2 \theta + \cos^2 \theta \\ = \sin^2 \theta - (-\cos^2 \theta) \\ = 1$$

Therefore, the determinant is independent of  $\theta$ .

**Exercise**

Show that the matrices  $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$  and  $B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$  commute if and only if  $\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$

**Solution**

$$\begin{aligned} AB &= \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} \\ &= \begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \\ &= \begin{pmatrix} da & db+ec \\ 0 & fc \end{pmatrix} \end{aligned}$$

$$AB = BA \Rightarrow \begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix} = \begin{pmatrix} da & db+ec \\ 0 & fc \end{pmatrix}$$

Iff  $ae+bf = db+ec$

$$\begin{aligned} \begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} &= b(d-f) - e(a-c) \\ &= bd - bf - ea + ec = 0 \end{aligned}$$

$$\underline{bd + ec = bf + ae} \quad \checkmark$$

**Exercise**

Show that  $\det(A) = \frac{1}{2} \begin{vmatrix} \text{tr}(A) & 1 \\ \text{tr}(A^2) & \text{tr}(A) \end{vmatrix}$  for every  $2 \times 2$  matrix A.

**Solution**

$$\text{Let } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \text{tr}(A) = a + d$$

$$\begin{aligned} A^2 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} \end{aligned}$$

$$\text{tr}(A^2) = a^2 + bc + bc + d^2.$$

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \\ = \underline{ad - bc}$$

$$\begin{aligned} \frac{1}{2} \begin{vmatrix} \text{tr}(A) & 1 \\ \text{tr}(A^2) & \text{tr}(A) \end{vmatrix} &= \frac{1}{2} \begin{vmatrix} a+d & 1 \\ a^2+bc+bc+d^2 & a+d \end{vmatrix} \\ &= \frac{1}{2} \left[ (a+d)^2 - (a^2+bc+bc+d^2) \right] \\ &= \frac{1}{2} (a^2 + 2ad + d^2 - a^2 - bc - bc - d^2) \\ &= \frac{1}{2} (2ad - 2bc) \\ &= ad - bc \\ &= \underline{\det(A)} \end{aligned}$$

### Exercise

What is the maximum number of zeros that a  $4 \times 4$  matrix can have without a zero determinant? Explain your reasoning.

### Solution

The maximum number of zeros that a  $4 \times 4$  matrix can have without a zero determinant is 12 zeros. If the main diagonal has nonzero entries and the rest are zero, then the determinant of the matrix is equal to the product of the main diagonal entries.

### Exercise

Evaluate  $\det(A)$ ,  $\det(E)$ , and  $\det(AE)$ . Then verify that  $\det(A) \cdot \det(E) = \det(AE)$

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & & \\ & 3 & \\ & & 1 \end{bmatrix}$$

### Solution

$$\det(A) = \begin{vmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{vmatrix} \\ = \underline{-22}$$

$$\det(E) = \begin{vmatrix} 1 & & \\ & 3 & \\ & & 1 \end{vmatrix} = \underline{3}$$

$$\begin{aligned}
 AE &= \begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 4 & 3 & 3 \\ 0 & -6 & 0 \\ 3 & 3 & 5 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \det(AE) &= \begin{vmatrix} 4 & 3 & 3 \\ 0 & -6 & 0 \\ 3 & 3 & 5 \end{vmatrix} \\
 &= -120 + 54 \\
 &= \underline{-66}
 \end{aligned}$$

$$\begin{aligned}
 \det(A)\det(E) &= (-22)(3) \\
 &= \underline{-66}
 \end{aligned}$$

$$\det(A)\det(E) = \det(AE) \quad \checkmark$$

### Exercise

Show that  $\begin{bmatrix} \sin^2 \alpha & \sin^2 \beta & \sin^2 \gamma \\ \cos^2 \alpha & \cos^2 \beta & \cos^2 \gamma \\ 1 & 1 & 1 \end{bmatrix}$  is not invertible for any values of  $\alpha, \beta, \gamma$

### Solution

$$\begin{aligned}
 &\begin{vmatrix} \sin^2 \alpha & \sin^2 \beta & \sin^2 \gamma \\ \cos^2 \alpha & \cos^2 \beta & \cos^2 \gamma \\ 1 & 1 & 1 \end{vmatrix} = \sin^2 \alpha \cos^2 \beta + \sin^2 \beta \cos^2 \gamma + \sin^2 \gamma \cos^2 \alpha \\
 &\quad - \sin^2 \gamma \cos^2 \beta - \sin^2 \alpha \cos^2 \gamma - \cos^2 \alpha \sin^2 \beta \\
 &= \sin^2 \alpha (\cos^2 \beta - \cos^2 \gamma) + \cos^2 \alpha (\sin^2 \gamma - \sin^2 \beta) + \sin^2 \beta \cos^2 \gamma - \sin^2 \gamma \cos^2 \beta \\
 &= \sin^2 \alpha (\cos^2 \beta - \cos^2 \gamma) + \cos^2 \alpha (1 - \cos^2 \gamma - 1 + \cos^2 \beta) + (1 - \cos^2 \beta) \cos^2 \gamma - (1 - \cos^2 \gamma) \cos^2 \beta \\
 &= \sin^2 \alpha (\cos^2 \beta - \cos^2 \gamma) + \cos^2 \alpha (\cos^2 \beta - \cos^2 \gamma) + \cos^2 \gamma - \cos^2 \gamma \cos^2 \beta - \cos^2 \beta + \cos^2 \gamma \cos^2 \beta \\
 &= (\sin^2 \alpha + \cos^2 \alpha) (\cos^2 \beta - \cos^2 \gamma) + \cos^2 \gamma - \cos^2 \beta \\
 &= \cos^2 \beta - \cos^2 \gamma + \cos^2 \gamma - \cos^2 \beta \\
 &= \underline{0}
 \end{aligned}$$

Therefore, this matrix is not invertible.

**Exercise**

The determinant of a  $2 \times 2$  matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is  $\det(A) = ad - bc$ .

Assuming no rows swaps are required, perform elimination on  $A$  and show explicitly that  $ad - bc$  is the product of the pivots.

**Solution**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{aR_2 - cR_1} \begin{pmatrix} a & b \\ 0 & ad - bc \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{R_2 - \frac{c}{a}R_1} \begin{pmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{pmatrix}$$

$$\begin{vmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{vmatrix} = a \left( d - \frac{bc}{a} \right) \\ = ad - bc \\ = \det(A)$$

**Exercise**

If  $A$  is a  $7 \times 7$  matrix and let  $\det(A) = 17$ . What is  $\det(3A^2)$ ?

**Solution**

$$\det(A^2) = \det(A)\det(A) \\ = 17^2$$

Multiplying a single row by 3 multiplies the determinant by 3.

Multiplying the whole  $7 \times 7$  matrix by 3 multiplies all 7 rows by 3  $\Rightarrow 3^7$ .

$$\therefore \det(3A^2) = 3^7 \cdot 17^2 \\ = 632043$$

**Exercise**

Let  $A$  be an  $n \times n$  real matrix.

- Show that if  $A^t = -A$  and  $n$  is odd, then  $|A| = 0$ .
- Show that if  $A^2 + I = 0$ , then  $n$  must be even.
- Does part (b) remain true for complex matrices?

**Solution**

a) Given:  $A^t = -A$  and  $n$  is odd

$$\begin{aligned} |A| &= |A^t| \\ &= |-A| \\ &= (-1)^n |A| \quad \text{Since } n \text{ is odd} \\ &= -|A| \end{aligned}$$

$$|A| = -|A| \text{ only when } |A| = 0$$

b)  $A^2 + I = 0$

$$\begin{aligned} A^2 &= -I \\ |A|^2 &= |-I| \\ &= |-I| \\ &= (-1)^n \end{aligned}$$

If  $n$  is odd, then  ~~$|A|^2 = -1$~~  *impossible*

If  $n$  is even, then  $|A|^2 = 1$

c) It can't be true because  $|I| = -1 \in \mathbb{R}$

And  $A$  is real matrix, the determinant has to be a real number.

## Exercise

Explain without computations why the following determinant is equal to zero

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & 0 & 0 & 0 \\ d_1 & d_2 & 0 & 0 & 0 \\ e_1 & e_2 & 0 & 0 & 0 \end{vmatrix}$$

## Solution

The determinant is equal to zero because there are too many zeros (as block  $3 \times 3$ ).

**Or**

$$d_1 R_5 - e_1 R_4 \rightarrow R_5 \Rightarrow \begin{matrix} 0 & d_1 e_2 - e_1 d_2 & 0 & 0 & 0 \end{matrix}$$

Since row 5 is has zero entries, therefore the determinant is zero.

**Exercise**

Let  $A$  and  $C$  be  $m \times m$  and  $n \times n$  matrices, respectively.

a) Show that  $\begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = \begin{vmatrix} A & 0 \\ B & C \end{vmatrix} = |A||C|$

b) Evaluate

i.  $\begin{vmatrix} I_m & 0 \\ 0 & I_n \end{vmatrix}$

ii.  $\begin{vmatrix} 0 & I_m \\ I_n & 0 \end{vmatrix}$

iii.  $\begin{vmatrix} I_m & B \\ 0 & I_n \end{vmatrix}$

iv.  $\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix}$

c) Find a formula for  $\begin{vmatrix} 0 & A \\ C & B \end{vmatrix}_{n \times n}$

**Solution**

a) If we let matrices  $B$  be  $m \times n$  and  $0$  be  $n \times m$ , so the determinant of the matrix size will be  $(m+n) \times (m+n)$ , then

$$\begin{aligned} \begin{vmatrix} A & B \\ 0 & C \end{vmatrix} &= \begin{vmatrix} A_{m \times m} & B_{m \times n} \\ 0_{n \times m} & C_{n \times n} \end{vmatrix} \\ &= |A||C| - |B||0| \\ &= |A||C| - 0 \\ &= |A||C| \end{aligned}$$

If we let matrices  $B$  be  $n \times m$  and  $0$  be  $m \times n$ , so the determinant of the matrix size will be  $(m+n) \times (m+n)$ , then

$$\begin{aligned} \begin{vmatrix} A & 0 \\ B & C \end{vmatrix} &= \begin{vmatrix} A & 0 \\ B & C \end{vmatrix} \\ &= |A||C| - |B||0| \\ &= |A||C| - 0 \\ &= |A||C| \end{aligned}$$

$$\begin{aligned}
 b) \quad i - \quad & \left| \begin{array}{cc} I_m & 0 \\ 0 & I_n \end{array} \right| = \left| \begin{array}{c|c} I_m & 0_{m \times n} \\ \hline 0_{n \times m} & I_n \end{array} \right| \\
 & = |I_m| |I_n| - 0 \\
 & = 1 \cdot 1 \\
 & = 1
 \end{aligned}$$

$$\begin{aligned}
 ii - \quad & \left| \begin{array}{cc} 0 & I_m \\ I_n & 0 \end{array} \right| = \left| \begin{array}{c|c} 0_{m \times n} & I_m \\ \hline I_n & 0_{n \times m} \end{array} \right| \\
 & = -|I_m| \cdot (-1) |I_n| \\
 & = (-1)^{mn}
 \end{aligned}$$

$$\begin{aligned}
 iii - \quad & \left| \begin{array}{cc} I_m & B \\ 0 & I_n \end{array} \right| = \left| \begin{array}{c|c} I_m & B_{m \times n} \\ \hline 0_{n \times m} & I_n \end{array} \right| \\
 & = |I_m| |I_n| - 0 \\
 & = 1 \cdot 1 \\
 & = 1
 \end{aligned}$$

$$iv - \quad \left| \begin{array}{ccccc} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{array} \right|_{n \times n}$$

$$\left| \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right| = -1 \qquad \left| \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right| = -1$$

$$\begin{aligned}
 & \begin{array}{cccc} + & - & + & - \end{array} \\
 & \left| \begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right|_{4 \times 4} = - \left| \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right| = -(-1) = 1
 \end{aligned}$$



$$\begin{array}{ccccc}
 + & - & + & - & + \\
 \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix}_{5 \times 5} & = & + & \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}_{4 \times 4} & = & 1
 \end{array}$$

From that we can see that the signs are:  $- \quad - \quad + \quad + \quad - \quad - \quad + \quad +$

$$\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix}_{n \times n} = (-1)^{\frac{n^2+3n}{2}}$$

$$\begin{aligned}
 c) \quad \begin{vmatrix} 0 & A \\ C & B \end{vmatrix}_{n \times n} &= \begin{vmatrix} 0_{m \times n} & A_{m \times m} \\ C_{n \times n} & B_{n \times m} \end{vmatrix} \\
 &= -|A_{m \times m}| \cdot (-1) |C_{n \times n}| \\
 &= (-1)^{mn} |A| |C|
 \end{aligned}$$

### Exercise

Let  $f(x) = (p_1 - x)(p_2 - x) \cdots (p_n - x)$  and let

$$\Delta_n = \begin{vmatrix} p_1 & a & a & \cdots & a & a \\ b & p_2 & a & \cdots & a & a \\ b & b & p_3 & \cdots & a & a \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b & b & b & \cdots & p_{n-1} & a \\ b & b & b & \cdots & b & p_n \end{vmatrix}$$

a) Show that, if  $a \neq b$ ,

$$\Delta_n = \frac{bf(a) - af(b)}{b - a}$$

b) Show that, if  $a = b$ ,

$$\Delta_n = a \sum_{i=1}^n f_i(a) + p_n f_n(a)$$

Where  $f_i(a)$  means  $f(a)$  with factor  $(p_i - a)$  missing.

c) Use part (b) to evaluate

$$\begin{vmatrix} a & b & b & \dots & b & b \\ b & a & b & \dots & b & b \\ b & b & a & \dots & b & b \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & b & \dots & a & b \\ b & b & b & \dots & b & a \end{vmatrix}_{n \times n}$$

### Solution

$$a) \Delta_n = \frac{bf(a) - af(b)}{b-a}; \text{ with } a \neq b$$

Using the mathematical Induction to prove the equality.

For  $n = 2$ :

$$\begin{aligned} \Delta_2 &= \begin{vmatrix} p_1 & a \\ b & p_2 \end{vmatrix} \\ &= p_1 p_2 - ab \\ \Delta_2 &= \frac{bf(a) - af(b)}{b-a} \\ &= \frac{b(p_1 - a)(p_2 - a) - a(p_1 - b)(p_2 - b)}{b-a} \\ &= \frac{bp_1 p_2 - abp_1 - abp_2 + a^2 b - ap_1 p_2 + abp_1 + abp_2 - ab^2}{b-a} \\ &= \frac{bp_1 p_2 + a^2 b - ap_1 p_2 - ab^2}{b-a} \\ &= \frac{(b-a)p_1 p_2 + ab(a-b)}{b-a} \\ &= \frac{(b-a)(p_1 p_2 - ab)}{b-a} \\ &= p_1 p_2 - ab \end{aligned}$$

For  $n = 2$ , the proof is true.

Assume that is true for  $\Delta_k$

$$\Delta_k = \frac{bf(a) - af(b)}{b-a}$$

$$= \frac{b((p_1 - a) \dots (p_{k-1} - a)) - a((p_1 - b) \dots (p_{k-1} - b))}{b - a}$$

$$f(x) = (p_1 - x)(p_2 - x) \dots (p_k - x)$$

We need to prove it is also true for  $\Delta_{k+1} \Rightarrow \Delta_{k+1} = \frac{bF(a) - aF(b)}{b - a}$

$$F(x) = (p_1 - x)(p_2 - x) \dots (p_k - x)(p_{k+1} - x)$$

$$= f(x)(p_{k+1} - x)$$

$$\begin{aligned} \Delta_{k+1} &= \frac{b((p_1 - a) \dots (p_k - a)(p_{k+1} - a)) - a((p_1 - b) \dots (p_k - b)(p_{k+1} - b))}{b - a} \\ &= \frac{bf(a)(p_{k+1} - a) - af(b)(p_{k+1} - b)}{b - a} \\ &= \frac{bF(a) - aF(b)}{b - a} \quad \checkmark \end{aligned}$$

$\Delta_{k+1}$  is also true.

$\therefore$  by the mathematical induction, the proof is completed.

$$b) \text{ If } a = b \rightarrow \Delta_n = a \sum_{i=1}^n f_i(a) + p_n f_n(a)$$

Where  $f_i(a)$  means  $f(a)$  with factor  $(p_i - a)$  missing.

$$\begin{aligned} \Delta_n &= (p_1 - a)\Delta_{n-1} + a(p_2 - b) \dots (p_n - b) && a = b \\ &= (p_1 - a)\Delta_{n-1} + a(p_2 - a) \dots (p_n - a) && (p_1 - a) \text{ missing} \\ &= (p_1 - a)[(p_2 - a)\Delta_{n-2} + aF_2(a)] + af_1(a) \\ &= (p_1 - a)(p_2 - a)\Delta_{n-2} + a(p_1 - a)F_2(a) + af_1(a) \\ &= (p_1 - a)(p_2 - a)\Delta_{n-2} + af_2(a) + af_1(a) \\ &= (p_1 - a)(p_2 - a)(p_3 - a)\Delta_{n-3} + af_3(a) + af_2(a) + af_1(a) \\ &= (p_1 - a)(p_2 - a)(p_3 - a)\Delta_{n-3} + a(f_3(a) + f_2(a) + f_1(a)) \\ &\quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ &= (p_1 - a) \dots (p_{n-2} - a)\Delta_2 + a(f_{n-2}(a) + \dots + f_1(a)) \end{aligned}$$

$$\begin{aligned}
\Delta_2 &= \begin{vmatrix} p_{n-1} & a \\ a & p_n \end{vmatrix} \\
&= p_{n-1}p_n - a^2 \\
&= p_{n-1}p_n - \textcolor{red}{a}p_n + \textcolor{red}{a}p_n - a^2 \\
&= p_n(p_{n-1} - a) + a(p_n - a) \\
\Delta_n &= \left[ (p_1 - a) \cdots (p_{n-2} - a) \right] \left( p_n(p_{n-1} - a) + a(p_n - a) \right) + a(f_{n-2}(a) + \cdots + f_1(a)) \\
&= p_n(p_1 - a) \cdots (p_{n-2} - a) (p_{n-1} - a) + a(p_1 - a) \cdots (p_{n-2} - a) (p_n - a) + a \sum_{i=1}^{n-1} f_i(a) \\
&= p_n f_n(a) + a f_{n-1}(a) + a \sum_{i=1}^{n-1} f_i(a) \\
&= p_n f_n(a) + a \sum_{i=1}^n f_i(a)
\end{aligned}$$

$$c) \quad f_n(x) = (p_1 - x)(p_2 - x) \cdots (p_{n-1} - x)$$

$$f_n(b) = (a - b)(a - b) \cdots (a - b)$$

$$p_n = a$$

$$\begin{vmatrix} a & b & b & \cdots & b & b \\ b & a & b & \cdots & b & b \\ b & b & a & \cdots & b & b \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b & b & b & \cdots & a & b \\ b & b & b & \cdots & b & a \end{vmatrix}_{n \times n} = p_n f_n(b) + b \sum_{i=1}^n f_i(b)$$

$$= a(a-b)^{n-1} + b \left( \underbrace{(a-b)^{n-1} + \cdots + (a-b)^{n-1}}_{\textcolor{red}{n-1}} \right)$$

$$= a(a-b)^{n-1} + b(n-1)(a-b)^{n-1}$$

$$= \underline{\underline{[a + (n-1)b](a-b)^{n-1}}}$$

**Exercise**

Let  $A, B, C, D \in M_n(\mathbb{C})$

- a) Show that when  $A$  is invertible:  $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A| |D - CA^{-1}B|$
- b) Show that when  $AC = CA$ :  $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|$
- c) Can  $B$  and  $C$  on the right-hand side of the identity be switched?
- d) Does part (b) remain true if the condition  $AC = CA$  is dropped?

**Solution**

- a) Since  $A$  is invertible, then  $A^{-1}$  exists and  $AA^{-1} = A^{-1}A = I$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \xrightarrow{A^{-1}R_1} \begin{pmatrix} I & A^{-1}B \\ C & D \end{pmatrix} \xrightarrow{R_2 - CR_1} \begin{pmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix} \xrightarrow{AR_1} \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} A & B \\ 0 & D - CA^{-1}B \end{vmatrix} = |A| |D - CA^{-1}B|$$

- b) When  $AC = CA$

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A| |D - CA^{-1}B|$$

$$= |AD - ACA^{-1}B| \quad AC = CA$$

$$= |AD - C(AA^{-1})B|$$

$$= |AD - CIB|$$

$$= |AD - CB|$$

- c) To switch  $B$  and  $C$  it is not necessary that  $BC = CB$

$$\text{Let } A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{aligned} AD &= \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} CB &= \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BC &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} |AD - CB| &= \left| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right| \\ &= \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \\ &= \underline{0} \end{aligned}$$

$$\begin{aligned} |AD - BC| &= \left| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \right| \\ &= \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} \\ &= \underline{1} \end{aligned}$$

$$|AD - CB| \neq |AD - BC|$$

No,  $B$  and  $C$  on the right-hand side of the identity cannot be switched since

$$|AD - CB| \neq |AD - BC|$$

**d)** No, since from previous part (c)  $D$  doesn't commute necessarily.

**Exercise**

Show that the matrix  $A$  is invertible for all values of  $\theta$ , then find  $A^{-1}$  using  $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$

$$A = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution**

$$\begin{aligned} |A| &= \begin{vmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \cos^2 \theta + \sin^2 \theta \\ &= 1 \end{aligned}$$

$$a_{11} = \begin{vmatrix} \cos \theta & 0 \\ 0 & 1 \end{vmatrix} = \cos \theta$$

$$a_{12} = - \begin{vmatrix} \sin \theta & 0 \\ 0 & 1 \end{vmatrix} = -\sin \theta$$

$$a_{13} = \begin{vmatrix} \sin \theta & 0 \\ \cos \theta & 0 \end{vmatrix} = 0$$

$$a_{21} = - \begin{vmatrix} -\sin \theta & 0 \\ 0 & 1 \end{vmatrix} = \sin \theta$$

$$a_{22} = \begin{vmatrix} \cos \theta & 0 \\ 0 & 1 \end{vmatrix} = \cos \theta$$

$$a_{23} = - \begin{vmatrix} \cos \theta & 0 \\ -\sin \theta & 0 \end{vmatrix} = 0$$

$$a_{31} = \begin{vmatrix} -\sin \theta & \cos \theta \\ 0 & 0 \end{vmatrix} = 0$$

$$a_{32} = - \begin{vmatrix} \cos \theta & \sin \theta \\ 0 & 0 \end{vmatrix} = 0$$

$$a_{33} = \begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix} = 1$$

$$A^{-1} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**Exercise**

Find the inverse matrix of using its adjoint.  $A = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$

**Solution**

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{vmatrix} \\ &= 7 \end{aligned}$$

$$a_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$a_{12} = - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1$$

$$a_{13} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$a_{21} = - \begin{vmatrix} -2 & 0 \\ -4 & 1 \end{vmatrix} = 2$$

$$a_{22} = \begin{vmatrix} 1 & 1 \\ -4 & 1 \end{vmatrix} = 5$$

$$a_{23} = - \begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = -2$$

$$a_{31} = \begin{vmatrix} -2 & 1 \\ -4 & 0 \end{vmatrix} = 4$$

$$a_{32} = - \begin{vmatrix} 1 & 1 \\ -4 & 0 \end{vmatrix} = -4$$

$$a_{33} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3$$

$$A^{-1} = \frac{1}{7} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 5 & -2 \\ 4 & -4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{7} & -\frac{1}{7} & \frac{1}{7} \\ \frac{2}{7} & \frac{5}{7} & -\frac{2}{7} \\ \frac{4}{7} & -\frac{4}{7} & \frac{3}{7} \end{pmatrix}$$

### Exercise

Find the inverse matrix of using its adjoint.  $A = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$

### Solution

$$|A| = \begin{vmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{vmatrix} \\ = 7$$

$$a_{11} = \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = 3$$

$$a_{12} = - \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} = -4$$

$$a_{13} = \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 1$$

$$a_{21} = - \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1$$

$$a_{22} = \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 1$$

$$a_{23} = - \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

$$a_{31} = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5$$

$$a_{32} = - \begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix} = 9$$

$$a_{33} = \begin{vmatrix} 3 & 3 \\ 1 & 2 \end{vmatrix} = 3$$

$$A^{-1} = \frac{1}{7} \begin{pmatrix} 3 & -4 & 1 \\ 1 & 1 & -2 \\ -5 & 9 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{pmatrix}$$



**Exercise**

Find the inverse matrix of using its adjoint.  $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

**Solution**

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{vmatrix} \\ = 1$$

$$a_{11} = \begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} = 3$$

$$a_{12} = - \begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix} = -2$$

$$a_{13} = \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} = -4$$

$$a_{21} = - \begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix} = 3$$

$$a_{22} = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = -2$$

$$a_{23} = - \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = -5$$

$$a_{31} = \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = -1$$

$$a_{32} = - \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 1$$

$$a_{33} = \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 2$$

$$A^{-1} = \begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix}$$

**Exercise**

Find the inverse matrix of using its adjoint.  $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$

**Solution**

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{vmatrix} \\ = 1$$

$$a_{11} = \begin{vmatrix} -2 & 1 \\ -3 & 0 \end{vmatrix} = 3$$

$$a_{12} = - \begin{vmatrix} -1 & 1 \\ -3 & 0 \end{vmatrix} = -3$$

$$a_{13} = \begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} = 1$$

$$a_{21} = - \begin{vmatrix} 0 & 1 \\ -2 & 0 \end{vmatrix} = -2$$

$$a_{22} = \begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = 2$$

$$a_{23} = - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1$$

$$a_{31} = \begin{vmatrix} 0 & -2 \\ -2 & -3 \end{vmatrix} = -4$$

$$a_{32} = - \begin{vmatrix} 1 & -1 \\ -2 & -3 \end{vmatrix} = 5$$

$$a_{33} = \begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix} = -2$$

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{pmatrix}$$

**Exercise**

Find the inverse matrix of using its adjoint.  $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{pmatrix}$

**Solution**

$$|A| = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{vmatrix} \\ = -5$$

$$a_{11} = \begin{vmatrix} 5 & 3 \\ 4 & 3 \end{vmatrix} = 3$$

$$a_{12} = - \begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} = -10$$

$$a_{13} = \begin{vmatrix} 2 & -1 \\ 5 & 3 \end{vmatrix} = 11$$

$$a_{21} = - \begin{vmatrix} 3 & 3 \\ 2 & 3 \end{vmatrix} = -3$$

$$a_{22} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5$$

$$a_{23} = - \begin{vmatrix} 1 & -1 \\ 3 & 3 \end{vmatrix} = -6$$

$$a_{31} = \begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} = 2$$

$$a_{32} = - \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$$

$$a_{33} = \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = -1$$

$$A^{-1} = -\frac{1}{5} \begin{pmatrix} 3 & -10 & 11 \\ -3 & 5 & -6 \\ 2 & 0 & -1 \end{pmatrix} \\ = \begin{pmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{pmatrix}$$

**Exercise**

Find the inverse matrix of using its adjoint.  $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

**Solution**

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 4$$

$$a_{11} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

$$a_{12} = - \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1$$

$$a_{13} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1$$

$$a_{21} = - \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1$$

$$a_{22} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

$$a_{23} = - \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$$a_{31} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$$

$$a_{32} = - \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$$a_{33} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

$$A^{-1} = \frac{1}{4} \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

### Exercise

Find the inverse matrix of using its adjoint.  $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & -4 & -12 \end{pmatrix}$

### Solution

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & -4 & -12 \end{vmatrix}$$

$$= 0$$

$$A^{-1} \text{ } \nexists$$

**Exercise**

Find the inverse matrix of using its adjoint.  $A = \begin{pmatrix} -3 & -5 & -7 \\ 2 & 4 & 3 \\ 0 & 1 & -1 \end{pmatrix}$

**Solution**

$$|A| = \begin{vmatrix} -3 & -5 & -7 \\ 2 & 4 & 3 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= -3$$

$$a_{11} = \begin{vmatrix} 4 & 3 \\ 1 & -1 \end{vmatrix} = -7$$

$$a_{12} = - \begin{vmatrix} -5 & -7 \\ 1 & -1 \end{vmatrix} = -12$$

$$a_{13} = \begin{vmatrix} -5 & -7 \\ 4 & 3 \end{vmatrix} = -13$$

$$a_{21} = - \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} = 2$$

$$a_{22} = \begin{vmatrix} -3 & -7 \\ 0 & -1 \end{vmatrix} = 3$$

$$a_{23} = - \begin{vmatrix} -3 & -7 \\ 2 & 3 \end{vmatrix} = -5$$

$$a_{31} = \begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} = 2$$

$$a_{32} = - \begin{vmatrix} -3 & -5 \\ 0 & 1 \end{vmatrix} = 3$$

$$a_{33} = \begin{vmatrix} -3 & -5 \\ 2 & 4 \end{vmatrix} = -2$$

$$A^{-1} = -\frac{1}{3} \begin{pmatrix} -7 & -12 & 13 \\ 2 & 3 & -5 \\ 2 & 3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{7}{3} & 4 & -\frac{13}{3} \\ -\frac{2}{3} & -1 & \frac{5}{3} \\ -\frac{2}{3} & -1 & \frac{2}{3} \end{pmatrix}$$

**Exercise**

Let  $A$  and  $B$  be  $n \times n$  matrices such that  $AB = I$ . Prove that  $|A| \neq 0$  and  $|B| \neq 0$

**Solution**

$$AB = I$$

$$|AB| = |I|$$

$$|A| |B| = 1$$

Therefore,  $|A| \neq 0$  and  $|B| \neq 0$

**Exercise**

Let  $A$  and  $B$  be  $n \times n$  matrices such that  $AB$  is singular. Prove that  $A$  or  $B$  is singular.

**Solution**

Since  $AB$  is singular  $\Rightarrow |AB| = 0$

$$|A| |B| = 0$$

That implies that either  $|A| = 0$  or  $|B| = 0$

Therefore, either  $A$  or  $B$  is singular.

**Exercise**

Find two  $2 \times 2$  matrices such that  $|A| + |B| = |A + B|$

**Solution**

Assume  $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$|A| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

$$= 0$$

$$|B| = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}$$

$$= 0$$

$$|A| + |B| = 0$$

$$A + B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$|A + B| = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$$

$$= 0$$

Therefore,  $|A| + |B| = |A + B|$

**Exercise**

Verify the equation  $\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = b^2(3a+b)$

**Solution**

$$\begin{aligned} \begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} &= (a+b)^3 + a^3 + a^3 - 3a^2(a+b) \\ &= a^3 + 3a^2b + 3ab^2 + b^3 + 2a^3 - 3a^3 - 3a^2b \\ &= 3ab^2 + b^3 \\ &= \underline{b^2(3a+b)} \quad \checkmark \end{aligned}$$

**Exercise**

Let  $A$  be an  $n \times n$  matrix in which the entries of each row sum to zero. Find  $|A|$

**Solution**

$$\begin{aligned} \text{If } A &= \begin{pmatrix} a & -a \\ b & -b \end{pmatrix} \\ |A| &= \begin{vmatrix} a & -a \\ b & -b \end{vmatrix} \\ &= \underline{0} \end{aligned}$$

If we added one row and equal to zero, then we will have a zero entry.

If we keep adding each row, then we will have a column with zero entries.

Therefore,  $|A| = 0$

**Exercise**

Show that if  $A$  is an  $n \times n$  matrix with entries 1 and  $-1$ , then  $|A|$  is divisible by  $2^{n-1}$

**Solution**

Let  $A$  is an  $2 \times 2$  matrix with entries 1 and  $-1$ , then

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 \\ -1 & \pm 1 \end{pmatrix} \\ |A| &= \begin{vmatrix} 1 & -1 \\ -1 & \pm 1 \end{vmatrix} \end{aligned}$$

$$\begin{aligned}
 & \begin{cases} = 0 \\ = 2 \end{cases} \\
 & = 2^1 \\
 & \underline{= 2^{2-1}}
 \end{aligned}$$

It is true for  $n = 2$

Let  $B$  is an  $3 \times 3$  matrix with entries 1 and  $-1$ , then

$$\begin{aligned}
 B &= \begin{pmatrix} 1 & 1 & 1 \\ -1 & \pm 1 & \pm 1 \\ -1 & \pm 1 & \pm 1 \end{pmatrix} \\
 |B| &= \begin{vmatrix} 1 & 1 & 1 \\ -1 & \pm 1 & \pm 1 \\ -1 & \pm 1 & \pm 1 \end{vmatrix} \begin{matrix} R_2 + R_1 \\ R_3 + R_1 \end{matrix} \\
 &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2a & 2b \\ 0 & 2c & 2d \end{vmatrix} \\
 &= 1 \begin{vmatrix} 2a & 2b \\ 2c & 2d \end{vmatrix} \\
 &= 2^2 \begin{vmatrix} a & b \\ c & d \end{vmatrix} \\
 & \underline{= 2^{3-1}}
 \end{aligned}$$

It is also true for  $n = 3$

For  $n \times n$ , then  $A$  can be written as:  $A = \begin{pmatrix} 1 & * \\ -1 & B \end{pmatrix}$

Where the matrix  $B$  be with entries  $\pm 1$   $A = \left( \begin{array}{c|c} \pm 1 & * \\ \hline 0 & B' \end{array} \right)$

Where the matrix  $B'$   $(n-1) \times (n-1)$  has with entries  $\pm 2$  or  $0$

Then,  $|A| = \pm 1 |B'|$

If we factor 2 from the matrix  $B'$

$$|A| = 2^{n-1} |B''|$$

Therefore,  $|A|$  is divisible by  $2^{n-1}$

**Exercise**

Show that (the Vandermonde determinant)

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (a_j - a_i)$$

In Particular, If  $V$  is the  $n \times n$  matrix, with  $(i, j)$  – entry  $j^{i-1}$ , then

$$|V| = (n-1)(n-2)^2 \cdots 2^{n-2}$$

**Solution**

Using the Mathematical induction to prove:

For  $n = 2$ :

$$\begin{vmatrix} 1 & 1 \\ a_1 & a_2 \end{vmatrix} = a_2 - a_1 \quad \checkmark$$

It is true for  $V_{2 \times 2}$

Assume it is true for  $V_{k \times k}$

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_k \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_k^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{k-1} & a_2^{k-1} & a_3^{k-1} & \cdots & a_k^{k-1} \end{vmatrix} = \prod_{1 \leq i < j \leq k} (a_j - a_i)$$

$$= (a_2 - a_1)(a_3 - a_1)(a_3 - a_2) \cdots (a_k - a_{k-1})$$

Is  $V_{(k+1) \times (k+1)}$  also true

$$|V|_{(k+1) \times (k+1)} = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & \cdots & a_k & a_{k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{k-1} & a_2^{k-1} & \cdots & a_k^{k-1} & a_{k+1}^{k-1} \\ a_1^k & a_2^k & \cdots & a_k^k & a_{k+1}^k \end{vmatrix}$$



$$\begin{aligned}
&= \prod_{1 \leq i < j \leq k+1} (a_j - a_i) \\
&= (a_2 - a_1)(a_3 - a_1)(a_3 - a_2) \cdots (a_k - a_{k-1})(a_{k+1} - a_1) \cdots (a_{k+1} - a_k) \\
&= \left( \prod_{1 \leq i < j \leq k} (a_j - a_i) \right) (a_{k+1} - a_1) \cdots (a_{k+1} - a_k) \\
&= \left( \prod_{1 \leq i < j \leq k} (a_j - a_i) \right) \prod_{1 < i \leq k} (a_{k+1} - a_i) \\
&= \prod_{1 < i < j \leq k+1} (a_j - a_i) \quad \checkmark
\end{aligned}$$

$V_{(k+1) \times (k+1)}$  is also true

By the mathematical induction, the Vandermonde determinant is completed.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & \cdots & a_k & a_{k+1} \\ a_1^2 & a_2^2 & \cdots & a_k^2 & a_{k+1}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{k-1} & a_2^{k-1} & \cdots & a_k^{k-1} & a_{k+1}^{k-1} \\ a_1^k & a_2^k & \cdots & a_k^k & a_{k+1}^k \end{bmatrix} \quad \begin{array}{l} R_2 - a_1 R_1 \\ R_3 - a_1 R_2 \\ \vdots \\ R_{k-1} - a_1 R_{k-2} \\ R_{k+1} - a_1 R_{k-1} \end{array}$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & a_2 - a_1 & \cdots & a_k - a_1 \\ 0 & a_2^2 - a_1 a_2 & \cdots & a_k^2 - a_1 a_k \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a_2^{k-2} - a_1 a_2^{k-3} & \cdots & a_k^{k-2} - a_1 a_k^{k-3} \\ 0 & a_2^{k-1} - a_1 a_2^{k-2} & \cdots & a_k^{k-1} - a_1 a_k^{k-2} \\ 0 & a_2^k - a_1 a_2^{k-1} & \cdots & a_k^k - a_1 a_k^{k-1} \end{bmatrix}$$

$$\begin{aligned}
& \begin{vmatrix} \textcolor{red}{1} & 1 & \cdots & 1 & 1 \\ 0 & a_2 - a_1 & \cdots & a_k - a_1 & a_{k+1} - a_1 \\ 0 & a_2(a_2 - a_1) & \cdots & a_k(a_k - a_1) & a_{k+1}(a_{k+1} - a_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_2^{k-2}(a_2 - a_1) & \cdots & a_k^{k-2}(a_k - a_1) & a_{k+1}^{k-2}(a_{k+1} - a_1) \\ 0 & a_2^{k-1}(a_2 - a_1) & \cdots & a_k^{k-1}(a_k - a_1) & a_{k+1}^{k-1}(a_{k+1} - a_1) \end{vmatrix} \\
&= \textcolor{red}{1} \begin{vmatrix} a_2 - a_1 & \cdots & a_k - a_1 & a_{k+1} - a_1 \\ (a_2 - a_1)a_2 & \cdots & (a_k - a_1)a_k & (a_{k+1} - a_1)a_{k+1} \\ \vdots & \vdots & \vdots & \vdots \\ (a_2 - a_1)a_2^{k-2} & \cdots & (a_k - a_1)a_k^{k-2} & (a_{k+1} - a_1)a_{k+1}^{k-2} \\ (a_2 - a_1)a_2^{k-1} & \cdots & (a_k - a_1)a_k^{k-1} & (a_{k+1} - a_1)a_{k+1}^{k-1} \end{vmatrix} \\
&= (a_2 - a_1) \cdots (a_{k+1} - a_1) \begin{vmatrix} 1 & \cdots & 1 & 1 \\ a_2 & \cdots & a_k & a_{k+1} \\ \vdots & \vdots & \vdots & \vdots \\ a_2^{k-2} & \cdots & a_k^{k-2} & a_{k+1}^{k-2} \\ a_2^{k-1} & \cdots & a_k^{k-1} & a_{k+1}^{k-1} \end{vmatrix} \\
&= \prod_{1 < j \leq k+1} (a_j - a_1) \left( \prod_{1 \leq i < j \leq k} (a_j - a_i) \right) \\
&= \prod_{1 < i < j \leq k+1} (a_j - a_i) \quad \textcolor{blue}{\checkmark}
\end{aligned}$$

$(i, j)$ -entry  $j^{i-1}$

$$|V|_{n \times n} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & n \\ 1 & 2^2 & 3^2 & \cdots & n^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 2^n & 3^n & \cdots & n^n \end{vmatrix}$$

$$|V| \stackrel{\textcolor{red}{?}}{=} (n-1)(n-2)^2 \cdots (n-(n-2))^{n-2} (n-(n-1))^{n-1}$$

$$\stackrel{?}{=} (n-1) (n-2)^2 \dots (2)^{n-2} (1)^{n-1}$$

$$\stackrel{?}{=} (n-1) (n-2)^2 \dots 2^{n-2}$$

From previous prove of the Vandermonde determinant.

$$\begin{aligned} |V| &= (a_2 - a_1)(a_3 - a_1)(a_3 - a_2) \cdots (a_n - a_1) \cdots (a_n - a_{n-1}) \\ &= (2-1)(3-1)(3-2) \cdots (n-1) \cdots (n-(n-1)) \\ &= (1)(2)(1)(3)(2)(1) \cdots (n-1) \cdots (1) \\ &= (1)(2)^2 (3)^3 \cdots (n-(n-1))^{n-1} \\ &= (n-1) (n-2)^2 \dots (2)^{n-2} (1)^{n-1} \quad \checkmark \end{aligned}$$

-----

$$|V|_{2 \times 2} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$$\stackrel{=}{=} 1$$

$$= 2 - 1$$

$$= (n-1) \quad \checkmark$$

$$|V|_{3 \times 3} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2^2 & 3^2 \end{vmatrix} \quad = (a_2 - a_1)(a_3 - a_1)(a_3 - a_2)$$

$$= (2-1)(3-1)(3-2)$$

$$= (3-1)(2-1)(3-2)$$

$$= (3-1)(3-2)(3-2)$$

$$= (n-1) (n-2)^2 \quad \checkmark$$

$$|V|_{k \times k} = (k-1) (k-2)^2 \dots (k-(k-2))^{k-2} (k-(k-1))^{k-1}$$

$$|V|_{(k+1) \times (k+1)} = (k+1-1) (k+1-2)^2 \dots (k+1-(k+1-2))^{k+1-2} (k+1-(k+1-1))^{k+1-1}$$

$$= (k) (k-1)^2 (k-2)^3 \dots (2)^{k-1} (1)^k$$

$$(k) (k-1)^2 (k-2)^3 \dots (2)^{k-1} (1)^k = \left( (k-1) (k-2)^2 \dots 2^{k-2} \right) (k(k-1) (k-2) \dots 2^1 (1))$$

$$= |V|_{k \times k} \left( ((k+1)-1) ((k+1)-2) ((k+1)-3) \dots ((k+1)-(k-1)) ((k+1)-k) \right)$$

$$= |V|_{k \times k} \left( \prod_{1 < j \leq k} ((k+1) - j) \right)$$

$$|V|_{(k+1) \times (k+1)}$$

### Exercise

Show that if  $a \neq b$ , then the determinant for  $n \times n$  matrix is:

$$\begin{vmatrix} a+b & ab & 0 & \cdots & 0 & 0 \\ 1 & a+b & ab & \cdots & 0 & 0 \\ 0 & 1 & a+b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a+b & ab \\ 0 & 0 & 0 & \cdots & 1 & a+b \end{vmatrix} = \frac{a^{n+1} - b^{n+1}}{a - b}$$

What if  $a = b$  ?

### Solution

Using the Mathematical induction to prove:

For  $n = 2$

$$\begin{vmatrix} a+b & ab \\ 1 & a+b \end{vmatrix} = a^2 + b^2 + ab$$

$$= (a^2 + b^2 + ab) \frac{a-b}{a-b}$$

$$= \frac{a^3 - b^3}{a-b} \quad \checkmark$$

It is true for  $2 \times 2$

Assume that it is true for  $k \times k$

$$\begin{vmatrix} a+b & ab & 0 & \cdots & 0 & 0 \\ 1 & a+b & ab & \cdots & 0 & 0 \\ 0 & 1 & a+b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a+b & ab \\ 0 & 0 & 0 & \cdots & 1 & a+b \end{vmatrix} = \frac{a^{k+1} - b^{k+1}}{a - b}$$

Is it true for  $(k+1) \times (k+1)$

$$\begin{vmatrix}
 a+b & ab & 0 & \cdots & 0 & 0 \\
 1 & a+b & ab & \cdots & 0 & 0 \\
 0 & 1 & a+b & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
 0 & 0 & 0 & \cdots & a+b & ab \\
 0 & 0 & 0 & \cdots & 1 & a+b
 \end{vmatrix}
 \begin{array}{l}
 \\
 \\
 \\
 \\
 = \frac{a^{k+2} - b^{k+2}}{a-b} \\
 \\
 \end{array}
 \begin{array}{l}
 \\
 \\
 \\
 \\
 (k+1) \times (k+1)
 \end{array}$$

$$\begin{array}{cc}
 + & - \\
 \begin{vmatrix}
 a+b & ab & 0 & \cdots & 0 & 0 \\
 1 & a+b & ab & \cdots & 0 & 0 \\
 0 & 1 & a+b & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
 0 & 0 & 0 & \cdots & a+b & ab \\
 0 & 0 & 0 & \cdots & 1 & a+b
 \end{vmatrix}
 \end{array}
 \begin{array}{l}
 \\
 \\
 \\
 \\
 (k+1) \times (k+1)
 \end{array}$$

$$= (a+b) \begin{vmatrix}
 a+b & ab & 0 & \cdots & 0 & 0 \\
 1 & a+b & ab & \cdots & 0 & 0 \\
 0 & 1 & a+b & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\
 0 & 0 & 0 & 1 & a+b & ab \\
 0 & 0 & 0 & \cdots & 1 & a+b
 \end{vmatrix}
 \begin{array}{l}
 \\
 \\
 \\
 \\
 k \times k
 \end{array}$$

$$-ab \begin{vmatrix}
 1 & ab & \cdots & 0 & 0 \\
 0 & a+b & \cdots & 0 & 0 \\
 0 & 1 & \cdots & 0 & 0 \\
 \vdots & \vdots & \cdots & \cdots & \vdots \\
 0 & 0 & 1 & a+b & ab \\
 0 & 0 & \cdots & 1 & a+b
 \end{vmatrix}
 \begin{array}{l}
 \\
 \\
 \\
 \\
 k \times k
 \end{array}$$

$$= (a+b) \frac{a^{k+1} - b^{k+1}}{a-b} - ab(1) \begin{vmatrix}
 a+b & ab & \cdots & 0 \\
 1 & a+b & \cdots & 0 \\
 0 & 1 & \cdots & 0 \\
 \vdots & \vdots & \cdots & \vdots \\
 0 & 0 & \cdots & a+b & ab \\
 0 & 0 & \cdots & 1 & a+b
 \end{vmatrix}
 \begin{array}{l}
 \\
 \\
 \\
 \\
 (k-1) \times (k-1)
 \end{array}$$

$$= (a+b) \frac{a^{k+1} - b^{k+1}}{a-b} - ab \left( \frac{a^k - b^k}{a-b} \right)$$

$$= \frac{1}{a-b} \left( a^{k+2} - ab^{k+1} + ba^{k+1} - b^{k+2} - ba^{k+1} + ab^{k+1} \right)$$

$$= \frac{a^{k+2} - b^{k+2}}{a - b} \quad \checkmark$$

It is also true for  $(k+1) \times (k+1)$

By the mathematical induction, the determinant proof is completed.

if  $a = b$

$$\begin{vmatrix} 2a & a^2 & 0 & \cdots & 0 & 0 \\ 1 & 2a & a^2 & \cdots & 0 & 0 \\ 0 & 1 & 2a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2a & a^2 \\ 0 & 0 & 0 & \cdots & 1 & 2a \end{vmatrix}$$

$$\begin{vmatrix} 2a & a^2 \\ 1 & 2a \end{vmatrix} = 3a^2$$

$$\begin{vmatrix} 2a & a^2 & 0 \\ 1 & 2a & a^2 \\ 0 & 1 & 2a \end{vmatrix} = 4a^3$$

$$\begin{vmatrix} 2a & a^2 & 0 & 0 \\ 1 & 2a & a^2 & 0 \\ 0 & 1 & 2a & a^2 \\ 0 & 0 & 1 & 2a \end{vmatrix} = 5a^4$$

$$\begin{vmatrix} 2a & a^2 & 0 & \cdots & 0 & 0 \\ 1 & 2a & a^2 & \cdots & 0 & 0 \\ 0 & 1 & 2a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2a & a^2 \\ 0 & 0 & 0 & \cdots & 1 & 2a \end{vmatrix} = (n+1)a^n$$

.....

$$\begin{aligned}
& \begin{bmatrix} 2a & a^2 & 0 & \cdots & 0 & 0 \\ 1 & 2a & a^2 & \cdots & 0 & 0 \\ 0 & 1 & 2a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2a & a^2 \\ 0 & 0 & 0 & \cdots & 1 & 2a \end{bmatrix} & 2aR_2 - R_1 \\
\\
& \begin{bmatrix} 2a & a^2 & 0 & \cdots & 0 & 0 \\ 0 & 3a^2 & 2a^3 & \cdots & 0 & 0 \\ 0 & 1 & 2a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2a & a^2 \\ 0 & 0 & 0 & \cdots & 1 & 2a \end{bmatrix} & \begin{aligned} & \frac{1}{2a}R_1 \\ & 3a^2R_3 - R_2 \end{aligned} \\
\\
& \begin{bmatrix} 1 & * & 0 & 0 & \cdots & 0 & 0 \\ 0 & 3a^2 & * & 0 & \cdots & 0 & 0 \\ 0 & 0 & 4a^3 & * & \cdots & 0 & 0 \\ 0 & 0 & 1 & 2a & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 2a & a^2 \\ 0 & 0 & \cdots & \cdots & \cdots & 1 & 2a \end{bmatrix} & \begin{aligned} & \frac{1}{3a^2}R_2 \\ & 4a^3R_4 - R_3 \end{aligned} \\
\\
& \begin{bmatrix} 1 & * & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & * & 0 & \cdots & 0 & 0 \\ 0 & 0 & 4a^3 & * & \cdots & 0 & 0 \\ 0 & 0 & 0 & 5a^4 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 2a & a^2 \\ 0 & 0 & \cdots & \cdots & \cdots & 1 & 2a \end{bmatrix} & \begin{aligned} & \frac{1}{4a^3}R_3 \\ & \frac{1}{5a^4}R_4 \end{aligned} \\
\\
& \begin{bmatrix} 1 & * & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & * & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & * & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & na^{n-1} & (n-1)a^n \\ 0 & 0 & \cdots & \cdots & \cdots & 1 & 2a \end{bmatrix} & (na^{n-1})R_n - R_{n-1}
\end{aligned}$$

$$\begin{bmatrix} 1 & * & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & * & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & * & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & * & 0 \\ 0 & 0 & \cdots & \cdots & 0 & na^{n-1} & * \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & (n+1)a^n \end{bmatrix} \xrightarrow{\frac{1}{na^{n-1}}R_1}$$

$$\begin{bmatrix} 1 & * & 0 & \cdots & 0 & 0 \\ 0 & 1 & * & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & * \\ 0 & 0 & \cdots & \cdots & 0 & (n+1)a^n \end{bmatrix} \xrightarrow{=(n+1)a^n}$$

### Exercise

Let  $a = (-a_0, -a_1, \dots, -a_{n-2})$  and let  $A = \begin{pmatrix} 0 & I_{n-1} \\ a & -a_{n-1} \end{pmatrix}$

Show that  $|\lambda I - A| = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0$

### Solution

$$A = \left( \begin{array}{c|c} 0_{(n-1) \times 1} & I_{(n-1) \times (n-1)} \\ \hline a_{1 \times (n-1)} & -a_{n-1} \end{array} \right)$$

$$A = \left( \begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \vdots & \vdots & 0 & 1 \\ \hline -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{array} \right)_{(n-1) \times (n-1)}$$

For  $A$  to be  $2 \times 2$ , then  $n = 2$ :

$$a = -a_0$$

$$A = \begin{pmatrix} 0 & I_1 \\ a & -a_1 \end{pmatrix}$$



$$= \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ a_0 & \lambda + a_1 \end{vmatrix} \\ = \lambda^2 + a_1\lambda + a_0 \quad \checkmark$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ a_0 & a_1 & \lambda + a_2 \end{vmatrix} \\ = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 \quad \checkmark$$

$$\lambda I - A = \begin{pmatrix} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & \vdots \\ \vdots & 0 & \ddots & \cdots & \vdots \\ 0 & \vdots & \vdots & \lambda & -1 \\ \hline a_0 & a_1 & \cdots & a_{n-2} & \lambda + a_{n-1} \end{pmatrix} \quad \lambda R_{n-1} - a_0 R_1$$

$$\begin{pmatrix} \lambda & * & 0 & 0 & \cdots & 0 \\ 0 & \lambda & * & 0 & \cdots & \vdots \\ \vdots & 0 & \lambda & * & \cdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \lambda & -1 \\ \hline 0 & a_1\lambda + a_0 & a_2\lambda & \cdots & a_{n-2}\lambda & \lambda^2 + a_{n-1}\lambda \end{pmatrix} \quad \frac{1}{\lambda} R_1$$

$$\lambda R_{n-1} - (\lambda a_1 + a_0) R_2$$

$$\begin{pmatrix} 1 & 0 & * & * & \cdots & * \\ 0 & \lambda & * & * & \cdots & \vdots \\ \vdots & 0 & \lambda & * & \cdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \ddots & * \\ 0 & \cdots & \cdots & \cdots & \lambda & -1 \\ \hline 0 & 0 & a_2\lambda^2 + a_1\lambda + a_0 & \cdots & a_{n-2}\lambda^2 & \lambda^3 + a_{n-1}\lambda^2 \end{pmatrix} \quad \frac{1}{\lambda} R_2$$

$$\lambda R_{n-1} - (a_2\lambda^2 + a_1\lambda + a_0) R_3$$

$$\left( \begin{array}{cccccc} 1 & 0 & * & & * & \dots & * \\ 0 & 1 & * & & * & \dots & \vdots \\ \vdots & 0 & 1 & & * & \dots & \vdots \\ \vdots & \vdots & \dots & & \ddots & \ddots & * \\ 0 & \dots & \dots & & \dots & \lambda & -1 \\ \hline 0 & 0 & 0 & a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 & a_{n-2}\lambda^3 & \lambda^4 + a_{n-1}\lambda^3 \end{array} \right)$$

$$\left( \begin{array}{cccccc} 1 & 0 & * & * & \dots & * \\ 0 & 1 & * & * & \dots & \vdots \\ \vdots & 0 & 1 & * & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & \ddots & * \\ 0 & \dots & \dots & \dots & \lambda & -1 \\ \hline 0 & 0 & 0 & \dots & a_{n-2}\lambda^{n-2} + \dots + a_2\lambda^2 + a_1\lambda + a_0 & \lambda^{n-1} + a_{n-1}\lambda^{n-2} \end{array} \right)$$

$$\lambda R_{n-2} - \left( a_{n-2}\lambda^{n-2} + \dots + a_2\lambda^2 + a_1\lambda + a_0 \right) R_{n-1}$$

$$\left( \begin{array}{cccccc} 1 & 0 & * & * & \dots & * \\ 0 & 1 & * & * & \dots & \vdots \\ \vdots & 0 & 1 & * & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & \ddots & * \\ 0 & \dots & \dots & \dots & 1 & * \\ \hline 0 & 0 & 0 & \dots & 0 & \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \dots + a_2\lambda^2 + a_1\lambda + a_0 \end{array} \right)$$

$$\boxed{|\lambda I - A| = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \dots + a_1\lambda + a_0}$$

## Solution

### Section 1.7 – Properties of Determinants: Cramer's Rule

#### Exercise

Use Cramer's Rule with ratios  $\frac{\det B_j}{\det A}$  to solve  $A\mathbf{x} = b$ . Also find the inverse matrix  $A^{-1} = \frac{C^T}{\det A}$ . Why is the solution  $\mathbf{x}$  the first part the same as column 3 of  $A^{-1}$ ? Which cofactors are involved in computing that column  $\mathbf{x}$ ?

$$Ax = b \quad \text{is} \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Find the volumes of the boxes whose edges are columns of  $A$  and then rows of  $A^{-1}$ .

#### Solution

$$|A| = \begin{vmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{vmatrix} = 2$$

$$|B_1| = \begin{vmatrix} 0 & 6 & 2 \\ 0 & 4 & 2 \\ 1 & 9 & 0 \end{vmatrix} = 4$$

$$|B_2| = \begin{vmatrix} 2 & 0 & 2 \\ 1 & 0 & 2 \\ 5 & 1 & 0 \end{vmatrix} = -2$$

$$|B_3| = \begin{vmatrix} 2 & 6 & 0 \\ 1 & 4 & 0 \\ 5 & 9 & 1 \end{vmatrix} = 2$$

$$x = \frac{4}{2} = 2; \quad y = \frac{-2}{2} = -1; \quad z = \frac{2}{2} = 1$$

The solution is: (2, -1, 1)

$$C_{11} = \begin{vmatrix} 4 & 2 \\ 9 & 0 \end{vmatrix} = -18$$

$$C_{12} = -\begin{vmatrix} 1 & 2 \\ 5 & 0 \end{vmatrix} = 10$$

$$C_{13} = \begin{vmatrix} 1 & 4 \\ 5 & 9 \end{vmatrix} = -11$$

$$C_{21} = -\begin{vmatrix} 6 & 2 \\ 9 & 0 \end{vmatrix} = 18$$

$$C_{22} = \begin{vmatrix} 2 & 2 \\ 5 & 0 \end{vmatrix} = -10$$

$$C_{23} = -\begin{vmatrix} 2 & 6 \\ 5 & 9 \end{vmatrix} = 12$$

$$C_{31} = \begin{vmatrix} 6 & 2 \\ 4 & 2 \end{vmatrix} = 4$$

$$C_{32} = -\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = -2$$

$$C_{33} = \begin{vmatrix} 2 & 6 \\ 1 & 4 \end{vmatrix} = 2$$

$$C = \begin{pmatrix} -18 & 10 & -11 \\ 18 & -10 & 12 \\ 4 & -2 & 2 \end{pmatrix}$$

$$C^T = \begin{pmatrix} -18 & 18 & 4 \\ 10 & -10 & -2 \\ -11 & 12 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{pmatrix} -18 & 18 & 4 \\ 10 & -10 & -2 \\ -11 & 12 & 2 \end{pmatrix} \quad A^{-1} = \frac{C^T}{\det A}$$

$$= \begin{pmatrix} -9 & 9 & 2 \\ 5 & -5 & -1 \\ -\frac{11}{2} & 6 & 1 \end{pmatrix}$$

The solution  $\mathbf{x}$  is the third column of  $A^{-1}$  because  $\mathbf{b} = (0, 0, 1)$  is the third column of  $I$ .

The volume of the boxes whose edges are columns of  $A$  is  $\det(A) = 2$ .

Since  $|A^T| = |A|$ .

The box from rows of  $A^{-1}$  has volume  $|A^{-1}| = \frac{1}{|A|} = \frac{1}{2}$

### Exercise

Verify that  $\det(AB) = \det(BA)$  and determine whether the equality  $\det(A+B) = \det(A) + \det(B)$  holds

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

### Solution

$$AB = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{bmatrix}$$

$$\det(AB) = \begin{vmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{vmatrix}$$

$$= -170$$

$$BA = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{bmatrix}$$

$$\det(BA) = \begin{vmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{vmatrix}$$

$$= -170$$

Thus,  $\det(AB) = \det(BA)$

$$\det(A) = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 10$$

$$\det(B) = \begin{vmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{vmatrix}$$

$$= -17$$

$$A + B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 3 \\ 10 & 5 & 2 \\ 5 & 0 & 3 \end{bmatrix}$$

$$\det(A + B) = \begin{vmatrix} 3 & 0 & 3 \\ 10 & 5 & 2 \\ 5 & 0 & 3 \end{vmatrix}$$

$$= -30$$

$$\det(A) + \det(B) = 10 - 17$$

$$= -7 \neq -30$$

$$\neq \det(A + B)$$

**Exercise**

Verify that  $\det(kA) = k^n \det(A)$   $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$ ,  $k = 2$

**Solution**

$$\begin{aligned} \det(A) &= \begin{vmatrix} -1 & 2 \\ 3 & 4 \end{vmatrix} \\ &= -10 \end{aligned}$$

$$\begin{aligned} \det(2A) &= \begin{vmatrix} -2 & 4 \\ 6 & 8 \end{vmatrix} \\ &= -40 \\ &= 4(-10) \\ &= 2^2(-10) \\ &= k^2 \det(A) \end{aligned}$$

**Exercise**

Verify that  $\det(kA) = k^n \det(A)$   $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}$ ,  $k = -2$

**Solution**

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{vmatrix} \\ &= 56 \end{aligned}$$

$$\begin{aligned} \det(-2A) &= \begin{vmatrix} -4 & 2 & -6 \\ -6 & -4 & -2 \\ -2 & -8 & -10 \end{vmatrix} \\ &= -448 \\ &= (-2)^3(56) \\ &= k^3 \det(A) \end{aligned}$$

**Exercise**

Verify that  $\det(kA) = k^n \det(A)$   $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}$ ,  $k = 3$

**Solution**

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{vmatrix} \\ = -7$$

$$\det(3A) = \begin{vmatrix} 3 & 3 & 3 \\ 0 & 6 & 9 \\ 0 & 3 & -6 \end{vmatrix} \\ = -189 \\ = 3^3(-7) \\ = k^3 \det(A)$$

**Exercise**

Use Cramer's rule to solve the system  $\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$

**Solution**

$$D = \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -3 - 4 = -7$$

$$D_x = \begin{vmatrix} -4 & 2 \\ -5 & -1 \end{vmatrix} = 4 - (-10) = 14$$

$$D_y = \begin{vmatrix} 3 & -4 \\ 2 & -5 \end{vmatrix} = -15 - (-8) = -7$$

$$x = \frac{D_x}{D} = \frac{14}{-7} = -2$$

$$y = \frac{D_y}{D} = \frac{-7}{-7} = 1$$

$$\therefore \text{Solution: } \underline{(-2, 1)}$$

**Exercise**

Use Cramer's rule to solve the system 
$$\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix} = -29 \qquad D_x = \begin{vmatrix} 7 & 5 \\ -3 & -2 \end{vmatrix} = 1 \qquad D_y = \begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix} = -41$$

$$x = \frac{1}{-29} = -\frac{1}{29} \qquad x = \frac{D_x}{D}$$

$$y = \frac{41}{29} \qquad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \left( -\frac{1}{29}, \frac{41}{29} \right)$$

**Exercise**

Use Cramer's rule to solve the system 
$$\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -7 \qquad D_x = \begin{vmatrix} -4 & -5 \\ 2 & -1 \end{vmatrix} = 14 \qquad D_y = \begin{vmatrix} 3 & -4 \\ 2 & -5 \end{vmatrix} = -7$$

$$x = -\frac{14}{7} = -2 \qquad x = \frac{D_x}{D}$$

$$y = \frac{7}{7} = 1 \qquad y = \frac{D_y}{D}$$

$$\text{Solution: } (-2, 1)$$

**Exercise**

Use Cramer's rule to solve the system 
$$\begin{cases} 2x + 5y = 7 \\ 5x - 2y = -3 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix} = -29 \qquad D_x = \begin{vmatrix} 7 & 5 \\ -3 & -2 \end{vmatrix} = 1 \qquad D_y = \begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix} = -41$$

$$x = -\frac{1}{29} \qquad x = \frac{D_x}{D}$$



$$y = \frac{41}{29} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \left( -\frac{1}{29}, \frac{41}{29} \right)$$

### Exercise

Use Cramer's rule to solve the system 
$$\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 4 & -7 \\ 2 & 5 \end{vmatrix} = 34 \quad D_x = \begin{vmatrix} -16 & -7 \\ 9 & 5 \end{vmatrix} = -17 \quad D_y = \begin{vmatrix} 4 & -16 \\ 2 & 9 \end{vmatrix} = 68$$

$$x = -\frac{17}{34} = -\frac{1}{2} \quad x = \frac{D_x}{D}$$

$$y = \frac{68}{34} = 2 \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \left( -\frac{1}{2}, 2 \right)$$

### Exercise

Use Cramer's rule to solve the system 
$$\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = -1 \quad D_x = \begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} = 2 \quad D_y = \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -5$$

$$x = -2 \quad x = \frac{D_x}{D}$$

$$y = 5 \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \left( -2, 5 \right)$$

### Exercise

Use Cramer's rule to solve the system 
$$\begin{cases} 3x + 4y = 2 \\ 2x + 5y = -1 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} = 7$$

$$D_x = \begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix} = 14$$

$$D_y = \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -7$$

$$x = \frac{14}{7} = 2$$

$$x = \frac{D_x}{D}$$

$$y = -\frac{7}{7} = -1$$

$$y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(2, -1)}$$

### Exercise

Use Cramer's rule to solve the system 
$$\begin{cases} 5x - 2y = 4 \\ -10x + 4y = 7 \end{cases}$$

#### Solution

$$D = \begin{vmatrix} 5 & -2 \\ -10 & 4 \end{vmatrix} = 0$$

$$D_y = \begin{vmatrix} 5 & 4 \\ -10 & 7 \end{vmatrix} = 75 \neq 0$$

$\therefore$  **No Solution**

### Exercise

Use Cramer's rule to solve the system 
$$\begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$$

#### Solution

$$D = \begin{vmatrix} 1 & -4 \\ 5 & -20 \end{vmatrix} = 0$$

$$D_y = \begin{vmatrix} 1 & -8 \\ 5 & -40 \end{vmatrix} = 0$$

$$\frac{1}{5} \times \begin{cases} x - 4y = -8 \\ 5x - 20y = -40 \end{cases}$$

$$\begin{cases} x - 4y = -8 \\ x - 4y = -8 \end{cases}$$

$$\therefore \text{Solution: } \underline{(4y - 8, y)}$$

### Exercise

Use Cramer's rule to solve the system 
$$\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$$

#### Solution

$$D = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

$$D_x = \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} = -6$$

$$D_y = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 3$$

$$x = \frac{6}{-3} = -2$$

$$x = \frac{D_x}{D}$$

$$y = -\frac{3}{-3} = 1$$

$$y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(2, -1)}$$

### Exercise

Use Cramer's rule to solve the system 
$$\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$$

#### Solution

$$D = \begin{vmatrix} 2 & 10 \\ 7 & -2 \end{vmatrix} = -74$$

$$D_x = \begin{vmatrix} -14 & 10 \\ -16 & -2 \end{vmatrix} = 188$$

$$D_y = \begin{vmatrix} 2 & -14 \\ 7 & -16 \end{vmatrix} = 66$$

$$x = -\frac{188}{-74} = \frac{94}{37}$$

$$x = \frac{D_x}{D}$$

$$y = -\frac{66}{-74} = \frac{33}{37}$$

$$y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{\left(\frac{94}{37}, \frac{33}{37}\right)}$$

### Exercise

Use Cramer's rule to solve the system 
$$\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$$

#### Solution

$$D = \begin{vmatrix} 4 & -3 \\ -3 & 9 \end{vmatrix} = 27$$

$$D_x = \begin{vmatrix} 24 & -3 \\ -1 & 9 \end{vmatrix} = 213$$

$$D_y = \begin{vmatrix} 4 & 24 \\ -3 & -1 \end{vmatrix} = 68$$

$$x = \frac{213}{27} = \frac{71}{9}$$

$$x = \frac{D_x}{D}$$

$$y = \frac{68}{27}$$

$$y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{\left(\frac{71}{9}, \frac{68}{27}\right)}$$

**Exercise**

Use Cramer's rule to solve the system 
$$\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 4 & 2 \\ 3 & -2 \end{vmatrix} = -14 \quad D_x = \begin{vmatrix} 12 & 2 \\ 16 & -2 \end{vmatrix} = -56 \quad D_y = \begin{vmatrix} 4 & 12 \\ 3 & 16 \end{vmatrix} = 28$$

$$x = \frac{56}{14} = 4 \quad x = \frac{D_x}{D}$$

$$y = -\frac{28}{14} = -2 \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(4, -2)}$$

**Exercise**

Use Cramer's rule to solve the system 
$$\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} = -10 \quad D_x = \begin{vmatrix} -1 & 2 \\ 6 & -2 \end{vmatrix} = -10 \quad D_y = \begin{vmatrix} 1 & -1 \\ 4 & 6 \end{vmatrix} = 10$$

$$x = 1 \quad x = \frac{D_x}{D}$$

$$y = -1 \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(1, -1)}$$

**Exercise**

Use Cramer's rule to solve the system 
$$\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 1 & -2 \\ -10 & 2 \end{vmatrix} = -18 \quad D_x = \begin{vmatrix} 5 & -2 \\ 4 & 2 \end{vmatrix} = 18 \quad D_y = \begin{vmatrix} 1 & 5 \\ -10 & 4 \end{vmatrix} = 54$$

$$x = -1 \quad x = \frac{D_x}{D}$$

$$y = -\frac{54}{18} = -3 \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(-1, -3)}$$

**Exercise**

Use Cramer's rule to solve the system  $\begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$

**Solution**

$$\begin{aligned} \frac{1}{3} \times & \begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases} \\ \frac{1}{15} \times & \end{aligned}$$

$$\begin{cases} 4x + 5y = -9 \\ 2x - y = -1 \end{cases}$$

$$D = \begin{vmatrix} 4 & 5 \\ 2 & -1 \end{vmatrix} = -14 \quad D_x = \begin{vmatrix} -9 & 5 \\ -1 & -1 \end{vmatrix} = 14 \quad D_y = \begin{vmatrix} 4 & -9 \\ 2 & -1 \end{vmatrix} = 14$$

$$\underline{x = -1} \quad x = \frac{D_x}{D}$$

$$\underline{y = -1} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(-1, -1)}$$

**Exercise**

Use Cramer's rule to solve the system  $\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$

**Solution**

$$\begin{aligned} \frac{1}{4} \times & \begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases} \\ \frac{1}{4} \times & \end{aligned}$$

$$\begin{cases} x - y = -3 \\ x + y = -5 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \quad D_x = \begin{vmatrix} -3 & -1 \\ -5 & 1 \end{vmatrix} = -8 \quad D_y = \begin{vmatrix} 1 & -3 \\ 1 & -5 \end{vmatrix} = -2$$

$$\underline{x = -4} \quad x = \frac{D_x}{D}$$

$$\underline{y = -1} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(-4, -1)}$$

**Exercise**

Use Cramer's rule to solve the system  $\begin{cases} x + y = 7 \\ x - y = 3 \end{cases}$

**Solution**

$$D = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

$$D_x = \begin{vmatrix} 7 & 1 \\ 3 & -1 \end{vmatrix} = -10 \quad D_y = \begin{vmatrix} 1 & 7 \\ 1 & 3 \end{vmatrix} = -4$$

$$\underline{x = 5} \quad x = \frac{D_x}{D}$$

$$\underline{y = 2} \quad y = \frac{D_y}{D}$$

$$\therefore \textbf{Solution: } \underline{(5, 2)}$$

**Exercise**

Use Cramer's rule to solve the system  $\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$

**Solution**

$$D = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

$$D_x = \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} = -6 \quad D_y = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 3$$

$$\underline{x = 2} \quad x = \frac{D_x}{D}$$

$$\underline{y = -1} \quad y = \frac{D_y}{D}$$

$$\therefore \textbf{Solution: } \underline{(2, -1)}$$

**Exercise**

Use Cramer's rule to solve the system  $\begin{cases} 12x + 3y = 15 \\ 2x - 3y = 13 \end{cases}$

**Solution**

$$D = \begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix} = -42$$

$$D_x = \begin{vmatrix} 15 & 3 \\ 13 & -3 \end{vmatrix} = -84 \quad D_y = \begin{vmatrix} 12 & 15 \\ 2 & 13 \end{vmatrix} = 126$$

$$\underline{x = 2} \quad x = \frac{D_x}{D}$$

$$\underline{y = -3} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(2, -3)}$$

### Exercise

Use Cramer's rule to solve the system  $\begin{cases} x - 2y = 5 \\ 5x - y = -2 \end{cases}$

### Solution

$$D = \begin{vmatrix} 1 & -2 \\ 5 & -1 \end{vmatrix} = 9$$

$$D_x = \begin{vmatrix} 5 & -2 \\ -2 & -1 \end{vmatrix} = -9 \quad D_y = \begin{vmatrix} 1 & 5 \\ 5 & -2 \end{vmatrix} = -27$$

$$\underline{x = -1} \quad x = \frac{D_x}{D}$$

$$\underline{y = -3} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(-1, -3)}$$

### Exercise

Use Cramer's rule to solve the system  $\begin{cases} 4x - 5y = 17 \\ 2x + 3y = 3 \end{cases}$

### Solution

$$D = \begin{vmatrix} 4 & -5 \\ 2 & 3 \end{vmatrix} = 22$$

$$D_x = \begin{vmatrix} 17 & -5 \\ 3 & 3 \end{vmatrix} = 66 \quad D_y = \begin{vmatrix} 4 & 17 \\ 2 & 3 \end{vmatrix} = -22$$

$$\underline{x = 3} \quad x = \frac{D_x}{D}$$

$$\underline{y = -1} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(3, -1)}$$

**Exercise**

Use Cramer's rule to solve the system 
$$\begin{cases} 3x + 2y = 2 \\ 2x + 2y = 3 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 2$$

$$D_x = \begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = -2 \quad D_y = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5$$

$$\underline{x = -1} \quad x = \frac{D_x}{D}$$

$$\underline{y = \frac{5}{2}} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{\left(-1, \frac{5}{2}\right)}$$

**Exercise**

Use Cramer's rule to solve the system 
$$\begin{cases} x - 3y = 4 \\ 3x - 4y = 12 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 1 & -3 \\ 3 & -4 \end{vmatrix} = 5 \quad D_x = \begin{vmatrix} 4 & -3 \\ 12 & -4 \end{vmatrix} = 20 \quad D_y = \begin{vmatrix} 1 & 4 \\ 3 & 12 \end{vmatrix} = 0$$

$$\underline{x = 4} \quad x = \frac{D_x}{D}$$

$$\underline{y = 0} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(4, 0)}$$

**Exercise**

Use Cramer's rule to solve the system 
$$\begin{cases} 2x - 9y = 5 \\ 3x - 3y = 11 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 2 & -9 \\ 3 & -3 \end{vmatrix} = 21 \quad D_x = \begin{vmatrix} 5 & -9 \\ 11 & -3 \end{vmatrix} = 84 \quad D_y = \begin{vmatrix} 2 & 5 \\ 3 & 11 \end{vmatrix} = 7$$

$$\underline{x = 4} \quad x = \frac{D_x}{D}$$



$$\underline{y = \frac{1}{3}} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{\left(4, \frac{1}{3}\right)}$$

### Exercise

Use Cramer's rule to solve the system 
$$\begin{cases} 3x - 4y = 4 \\ x + y = 6 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 3 & -4 \\ 1 & 1 \end{vmatrix} = 7$$

$$D_x = \begin{vmatrix} 4 & -4 \\ 6 & 1 \end{vmatrix} = 28 \quad D_y = \begin{vmatrix} 3 & 4 \\ 1 & 6 \end{vmatrix} = 14$$

$$\underline{x = 4} \quad x = \frac{D_x}{D}$$

$$\underline{y = 2} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(4, 2)}$$

### Exercise

Use Cramer's rule to solve the system 
$$\begin{cases} 3x = 7y + 1 \\ 2x = 3y - 1 \end{cases}$$

### Solution

$$\begin{cases} 3x - 7y = 1 \\ 2x - 3y = -1 \end{cases}$$

$$D = \begin{vmatrix} 3 & -7 \\ 2 & -3 \end{vmatrix} = 5$$

$$D_x = \begin{vmatrix} 1 & -7 \\ -1 & -3 \end{vmatrix} = -10 \quad D_y = \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} = -5$$

$$\underline{x = -2} \quad x = \frac{D_x}{D}$$

$$\underline{y = -1} \quad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } \underline{(-2, -1)}$$

**Exercise**

Use Cramer's rule to solve the system 
$$\begin{cases} 2x = 3y + 2 \\ 5x = 51 - 4y \end{cases}$$

**Solution**

$$\begin{cases} 2x - 3y = 2 \\ 5x + 4y = 51 \end{cases}$$

$$D = \begin{vmatrix} 2 & -3 \\ 5 & 4 \end{vmatrix} = 23$$

$$D_x = \begin{vmatrix} 2 & -3 \\ 51 & 4 \end{vmatrix} = 161 \quad D_y = \begin{vmatrix} 2 & 2 \\ 5 & 51 \end{vmatrix} = 92$$

$$\underline{x = 7} \quad x = \frac{D_x}{D}$$

$$\underline{y = 4} \quad y = \frac{D_y}{D}$$

$$\therefore \textbf{Solution: } \underline{(7, 4)}$$

**Exercise**

Use Cramer's rule to solve the system 
$$\begin{cases} y = -4x + 2 \\ 2x = 3y - 1 \end{cases}$$

**Solution**

$$\begin{cases} 4x + y = 2 \\ 2x - 3y = -1 \end{cases}$$

$$D = \begin{vmatrix} 4 & 1 \\ 2 & -3 \end{vmatrix} = -14$$

$$D_x = \begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix} = -5 \quad D_y = \begin{vmatrix} 4 & 2 \\ 2 & -1 \end{vmatrix} = -8$$

$$\underline{x = \frac{5}{14}} \quad x = \frac{D_x}{D}$$

$$\underline{y = \frac{4}{7}} \quad y = \frac{D_y}{D}$$

$$\therefore \textbf{Solution: } \underline{\left(\frac{15}{4}, \frac{4}{7}\right)}$$

**Exercise**

Use Cramer's rule to solve the system 
$$\begin{cases} 3x = 2 - 3y \\ 2y = 3 - 2x \end{cases}$$

**Solution**

$$\begin{cases} 3x + 3y = 2 \\ 2x + 2y = 3 \end{cases}$$

$$D = \begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix} = 0 \qquad D_y = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5 \neq 0$$

$\therefore$  **No Solution**

**Exercise**

Use Cramer's rule to solve the system 
$$\begin{cases} x + 2y - 3 = 0 \\ 12 = 8y + 4x \end{cases}$$

**Solution**

$$\begin{cases} x + 2y = 3 \\ 4x + 8y = 12 \end{cases}$$

$$\begin{cases} x + 2y = 3 \\ x + 2y = 3 \end{cases}$$

$\therefore$  **Solution:**  $\underline{(3 - 2y, \ y)}$

**Exercise**

Use Cramer's rule to solve the system 
$$\begin{cases} 7x - 2y = 3 \\ 3x + y = 5 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 7 & -2 \\ 3 & 1 \end{vmatrix} = 13$$

$$D_x = \begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 13 \qquad D_y = \begin{vmatrix} 7 & 3 \\ 3 & 5 \end{vmatrix} = 26$$

$$\underline{x = 1} \qquad x = \frac{D_x}{D}$$

$$\underline{y = 2} \qquad y = \frac{D_y}{D}$$

$\therefore$  **Solution:**  $\underline{(1, \ 2)}$

**Exercise**

Use Cramer's rule to solve the system 
$$\begin{cases} x_1 + 2x_2 = 5 \\ -x_1 + x_2 = 1 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3$$

$$D_x = \begin{vmatrix} 5 & 2 \\ 1 & 1 \end{vmatrix} = 3 \qquad D_y = \begin{vmatrix} 1 & 5 \\ -1 & 1 \end{vmatrix} = 6$$

$$x = 1 \qquad x = \frac{D_x}{D}$$

$$y = 2 \qquad y = \frac{D_y}{D}$$

$$\therefore \text{Solution: } (1, 2)$$

**Exercise**

Use Cramer's rule to solve the system 
$$\begin{cases} 3x + 2y - z = 4 \\ 3x - 2y + z = 5 \\ 4x - 5y - z = -1 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 3 & 2 & -1 \\ 3 & -2 & 1 \\ 4 & -5 & -1 \end{vmatrix} = 42 \qquad D_x = \begin{vmatrix} 4 & 2 & -1 \\ 5 & -2 & 1 \\ -1 & -5 & -1 \end{vmatrix} = 63$$

$$D_y = \begin{vmatrix} 3 & 4 & -1 \\ 3 & 5 & 1 \\ 4 & -1 & -1 \end{vmatrix} = 39 \qquad D_z = \begin{vmatrix} 3 & 2 & 4 \\ 3 & -2 & 5 \\ 4 & -5 & -1 \end{vmatrix} = 99$$

$$x = \frac{63}{42} = \frac{3}{2} \qquad x = \frac{D_x}{D}$$

$$y = \frac{39}{42} = \frac{13}{14} \qquad y = \frac{D_y}{D}$$

$$z = \frac{99}{42} = \frac{33}{14} \qquad z = \frac{D_z}{D}$$

$$\text{Solution: } \left( \frac{3}{2}, \frac{13}{14}, \frac{33}{14} \right)$$

**Exercise**

Use Cramer's rule to solve the system 
$$\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = -6 \qquad D_x = \begin{vmatrix} 2 & 1 & 1 \\ 5 & 1 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -6$$

$$D_y = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & -1 \\ 1 & -2 & 1 \end{vmatrix} = -12 \qquad D_z = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{vmatrix} = 6$$

$$\underline{x = 1} \qquad x = \frac{D_x}{D}$$

$$\underline{y = 2} \qquad y = \frac{D_y}{D}$$

$$\underline{z = -1} \qquad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(1, 2, -1)}$$

**Exercise**

Use Cramer's rule to solve the system 
$$\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 2 & 1 & 1 \\ -1 & -1 & 1 \\ 3 & -1 & 1 \end{vmatrix} \begin{matrix} 2 & 1 \\ -1 & -1 \\ 3 & -1 \end{matrix} = -2 + 3 + 1 + 3 + 2 + 1$$

$$\underline{= 8}$$

$$D_x = \begin{vmatrix} 9 & 1 & 1 \\ 1 & -1 & 1 \\ 9 & -1 & 1 \end{vmatrix} \begin{matrix} 9 & 1 \\ 1 & -1 \\ 9 & -1 \end{matrix}$$

$$= -9 + 9 - 1 + 9 + 9 - 1$$

$$\underline{= 16}$$

$$D_y = \begin{vmatrix} 2 & 9 & 1 \\ -1 & 1 & 1 \\ 3 & 9 & 1 \end{vmatrix} \begin{matrix} 2 & 9 \\ -1 & 1 \\ 3 & 9 \end{matrix}$$

$$= 2 + 27 - 9 - 3 - 18 + 9$$

$$= \underline{8}$$

$$D_z = \begin{vmatrix} 2 & 1 & 9 \\ -1 & -1 & 1 \\ 3 & -1 & 9 \end{vmatrix} \begin{matrix} 2 & 1 \\ -1 & -1 \\ 3 & -1 \end{matrix}$$

$$= -18 + 3 + 9 + 27 + 2 + 9$$

$$= \underline{32}$$

$$x = \underline{2}$$

$$x = \frac{D_x}{D}$$

$$y = \underline{1}$$

$$y = \frac{D_y}{D}$$

$$z = \frac{32}{8} = \underline{4}$$

$$z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(2, 1, 4)}$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 0 & 3 & -1 \\ 1 & 5 & -1 \\ -3 & 6 & 2 \end{vmatrix} \begin{matrix} 0 & 3 \\ 1 & 5 \\ -3 & 6 \end{matrix}$$

$$= 9 - 6 - 15 - 6$$

$$= \underline{-18}$$

$$D_x = \begin{vmatrix} -1 & 3 & -1 \\ -4 & 5 & -1 \\ 11 & 6 & 2 \end{vmatrix} \begin{matrix} -1 & 3 \\ -4 & 5 \\ 11 & 6 \end{matrix}$$

$$= -10 - 33 + 24 + 55 - 6 + 24$$

$$= \underline{54}$$

$$D_y = \begin{vmatrix} 0 & -1 & -1 \\ 1 & -4 & -1 \\ -3 & 11 & 2 \end{vmatrix} \begin{matrix} 0 & -1 \\ 1 & -4 \\ -3 & 11 \end{matrix}$$

$$= -3 - 11 + 12 + 2$$

$$= \underline{0}$$

$$D_z = \begin{vmatrix} 0 & 3 & -1 \\ 1 & 5 & -4 \\ -3 & 6 & 11 \end{vmatrix} \begin{matrix} 0 & 3 \\ 1 & 5 \\ -3 & 6 \end{matrix}$$

$$= 36 - 6 - 15 - 33$$

$$= \underline{-18}$$

$$\underline{x = -3} \quad x = \frac{D_x}{D}$$

$$\underline{y = 0} \quad y = \frac{D_y}{D}$$

$$\underline{z = 1} \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(-3, 0, 1)}$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases} x + 3y + 4z = 14 \\ 2x - 3y + 2z = 10 \\ 3x - y + z = 9 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 1 & 3 & 4 \\ 2 & -3 & 2 \\ 3 & -1 & 1 \end{vmatrix} \begin{matrix} 1 & 3 \\ 2 & -3 \\ 3 & -1 \end{matrix}$$

$$= -3 + 18 - 8 + 36 + 2 - 6$$

$$= \underline{39}$$

$$D_x = \begin{vmatrix} 14 & 3 & 4 \\ 10 & -3 & 2 \\ 9 & -1 & 1 \end{vmatrix} \begin{matrix} 14 & 3 \\ 10 & -3 \\ 9 & -1 \end{matrix}$$

$$= -42 + 54 - 40 + 108 + 28 - 30$$

$$= \underline{78}$$

$$D_y = \begin{vmatrix} 1 & 14 & 4 \\ 2 & 10 & 2 \\ 3 & 9 & 1 \end{vmatrix} \begin{matrix} 1 & 14 \\ 2 & 10 \\ 3 & 9 \end{matrix}$$

$$= 10 + 84 + 72 - 120 - 18 - 28$$

$$= \underline{0}$$

$$D_z = \begin{vmatrix} 1 & 3 & 14 \\ 2 & -3 & 10 \\ 3 & -1 & 9 \end{vmatrix} \begin{matrix} 1 & 3 \\ 2 & -3 \\ 3 & -1 \end{matrix}$$

$$= -27 + 90 - 28 + 126 + 10 - 54$$

$$\underline{= 117} \quad |$$

$$x = \frac{78}{39} \underline{= 2} \quad | \quad x = \frac{D_x}{D}$$

$$\underline{y = 0} \quad | \quad y = \frac{D_y}{D}$$

$$z = \frac{117}{39} \underline{= 3} \quad | \quad z = \frac{D_z}{D}$$

$$\therefore \textbf{Solution: } \underline{(2, 0, 3)} \quad |$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases} x + 4y - z = 20 \\ 3x + 2y + z = 8 \\ 2x - 3y + 2z = -16 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 1 & 4 & -1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix} \begin{vmatrix} 1 & 4 \\ 3 & 2 \\ 2 & -3 \end{vmatrix}$$

$$= 4 + 8 + 9 + 4 + 3 - 24$$

$$\underline{= 4} \quad |$$

$$D_x = \begin{vmatrix} 20 & 4 & -1 \\ 8 & 2 & 1 \\ -16 & -3 & 2 \end{vmatrix} \begin{vmatrix} 20 & 4 \\ 8 & 2 \\ -16 & -3 \end{vmatrix}$$

$$= 80 - 64 + 24 - 32 + 60 - 64$$

$$\underline{= 4} \quad |$$

$$D_y = \begin{vmatrix} 1 & 20 & -1 \\ 3 & 8 & 1 \\ 2 & -16 & 2 \end{vmatrix} \begin{vmatrix} 1 & 20 \\ 3 & 8 \\ 2 & -16 \end{vmatrix}$$

$$= 16 + 40 + 48 + 16 + 16 - 120$$

$$\underline{= 16} \quad |$$

$$D_z = \begin{vmatrix} 1 & 4 & 20 \\ 3 & 2 & 8 \\ 2 & -3 & -16 \end{vmatrix} \begin{vmatrix} 1 & 4 \\ 3 & 2 \\ 2 & -3 \end{vmatrix}$$

$$= -32 + 64 - 180 - 80 + 24 + 192$$

$$\underline{= -12} \quad |$$



$$x = \frac{4}{4} = \underline{1} \quad x = \frac{D_x}{D}$$

$$y = \frac{16}{4} = \underline{4} \quad y = \frac{D_y}{D}$$

$$z = -\frac{12}{4} = \underline{-3} \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(1, 4, -3)}$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases} -2x + 6y + 7z = 3 \\ -4x + 5y + 3z = 7 \\ -6x + 3y + 5z = -4 \end{cases}$$

### Solution

$$\begin{aligned} D &= \begin{vmatrix} -2 & 6 & 7 \\ -4 & 5 & 3 \\ -6 & 3 & 5 \end{vmatrix} \begin{matrix} -2 & 6 \\ -4 & 5 \\ -6 & 3 \end{matrix} \\ &= -50 - 108 - 84 + 210 + 18 + 120 \\ &= \underline{106} \end{aligned}$$

$$\begin{aligned} D_x &= \begin{vmatrix} 3 & 6 & 7 \\ 7 & 5 & 3 \\ -4 & 3 & 5 \end{vmatrix} \begin{matrix} 3 & 6 \\ 7 & 5 \\ -4 & 3 \end{matrix} \\ &= 75 - 72 + 147 + 140 - 27 - 210 \\ &= \underline{53} \end{aligned}$$

$$\begin{aligned} D_y &= \begin{vmatrix} -2 & 3 & 7 \\ -4 & 7 & 3 \\ -6 & -4 & 5 \end{vmatrix} \begin{matrix} -2 & 3 \\ -4 & 7 \\ -6 & -4 \end{matrix} \\ &= -70 - 54 + 112 + 294 - 24 + 60 \\ &= \underline{318} \end{aligned}$$

$$\begin{aligned} D_z &= \begin{vmatrix} -2 & 6 & 3 \\ -4 & 5 & 7 \\ -6 & 3 & -4 \end{vmatrix} \begin{matrix} -2 & 6 \\ -4 & 5 \\ -6 & 3 \end{matrix} \\ &= 40 - 252 - 36 + 90 + 42 - 96 \\ &= \underline{-212} \end{aligned}$$

$$x = \frac{53}{106} = \underline{\frac{1}{2}} \quad x = \frac{D_x}{D}$$

$$y = \frac{318}{106} = \underline{3} \quad y = \frac{D_y}{D}$$

$$z = -\frac{212}{106} = \underline{-2} \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{\left(\frac{1}{2}, 3, -2\right)}$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -3 & 4 \\ 4 & -2 & 3 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & -3 \\ 4 & -2 \end{matrix}$$

$$= -18 - 16 - 6 + 12 + 16 + 9$$

$$= \underline{-3}$$

$$D_x = \begin{vmatrix} 1 & -1 & 1 \\ 5 & -3 & 4 \\ 4 & -2 & 3 \end{vmatrix} \begin{matrix} 1 & -1 \\ 5 & -3 \\ 4 & -2 \end{matrix}$$

$$= -9 - 16 - 10 + 12 + 8 + 15$$

$$= \underline{0}$$

$$D_y = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 5 & 4 \\ 4 & 4 & 3 \end{vmatrix} \begin{matrix} 2 & 1 \\ 3 & 5 \\ 4 & 4 \end{matrix}$$

$$= 30 + 16 + 12 - 20 - 32 - 9$$

$$= \underline{-3}$$

$$D_z = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -3 & 5 \\ 4 & -2 & 4 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & -3 \\ 4 & -2 \end{matrix}$$

$$= -24 - 20 - 6 + 12 + 20 + 12$$

$$= \underline{-6}$$

$$\underline{x = 0} \quad x = \frac{D_x}{D}$$

$$\underline{y = 1} \quad y = \frac{D_y}{D}$$

$$\underline{z = 2} \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(0, 1, 2)}$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 3x - 4y + 4z = 7 \\ x - y - 2z = 2 \\ 2x - 3y + 6z = 5 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 3 & -4 & 4 \\ 1 & -1 & -2 \\ 2 & -3 & 6 \end{vmatrix} \begin{vmatrix} 3 & -4 \\ 1 & -1 \\ 2 & -3 \end{vmatrix}$$

$$= -18 + 16 - 12 + 8 - 18 + 24$$

$$= 0$$

$$D_z = \begin{vmatrix} 3 & -4 & 7 \\ 1 & -1 & 2 \\ 2 & -3 & 5 \end{vmatrix} \begin{vmatrix} 3 & -4 \\ 1 & -1 \\ 2 & -3 \end{vmatrix}$$

$$= -15 - 16 - 21 + 14 + 18 + 20$$

$$= 0$$

$$\begin{array}{l} -3 \times (2) \quad \begin{cases} -3x + 3y + 6z = -6 \\ 2x - 3y + 6z = 5 \end{cases} \\ \hline -x + 12z = -1 \end{array}$$

$$\underline{x = 12z + 1}$$

$$(2) \rightarrow y = 12z + 1 - 2z - 2$$

$$= 10z - 1$$

$$\therefore \text{Solution: } \underline{(12z + 1, 10z - 1, z)}$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} \begin{vmatrix} 1 & -2 \\ 2 & -1 \\ -1 & 1 \end{vmatrix}$$

$$= -1 + 2 - 2 + 1 - 1 + 4$$

$$= 3$$

$$D_x = \begin{vmatrix} 2 & -2 & -1 \\ 4 & -1 & 1 \\ 4 & 1 & 1 \end{vmatrix} \begin{vmatrix} 2 & -2 \\ 4 & -1 \\ 4 & 1 \end{vmatrix}$$

$$= -2 - 8 - 4 - 4 - 2 + 8$$

$$= -12$$

$$D_y = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 4 & 1 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 2 & 4 \\ -1 & 4 \end{vmatrix}$$

$$= 4 - 2 - 8 - 4 - 4 - 4$$

$$= -18$$

$$D_z = \begin{vmatrix} 1 & -2 & 2 \\ 2 & -1 & 4 \\ -1 & 1 & 4 \end{vmatrix} \begin{vmatrix} 1 & -2 \\ 2 & -1 \\ -1 & 1 \end{vmatrix}$$

$$= -4 + 8 + 4 - 2 - 4 + 16$$

$$= 18$$

$$x = -4 \quad x = \frac{D_x}{D}$$

$$y = -6 \quad y = \frac{D_y}{D}$$

$$z = 6 \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } (-4, -6, 6)$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases} x + y + z = 3 \\ -y + 2z = 1 \\ -x + z = 0 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{vmatrix}$$

$$= -4$$

$$D_x = \begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 0 \end{vmatrix}$$

$$= -4$$

$$D_y = \begin{vmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{vmatrix}$$

$$= -4$$

$$D_z = \begin{vmatrix} 1 & 1 & 3 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{vmatrix}$$

$$= -4$$

$$x = 1 \quad x = \frac{D_x}{D}$$

$$y = 1 \quad y = \frac{D_y}{D}$$

$$z = 1 \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } (1, 1, 1)$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 3x + y + 3z = 14 \\ 7x + 5y + 8z = 37 \\ x + 3y + 2z = 9 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 3 & 1 & 3 \\ 7 & 5 & 8 \\ 1 & 3 & 2 \end{vmatrix} \begin{vmatrix} 3 & 1 \\ 7 & 5 \\ 1 & 3 \end{vmatrix}$$

$$= 30 + 8 + 62 - 15 - 72 - 14$$

$$= 0$$

$$\begin{aligned}
 D_z &= \begin{vmatrix} 3 & 1 & 14 \\ 7 & 5 & 37 \\ 1 & 3 & 9 \end{vmatrix} \begin{matrix} 3 & 1 \\ 7 & 5 \\ 1 & 3 \end{matrix} \\
 &= 135 + 37 + 294 - 70 - 333 - 63 \\
 &= 0
 \end{aligned}$$

$$\begin{array}{l}
 -3 \times (1) \\
 (3)
 \end{array}
 \begin{cases}
 -9x - 3y - 9z = -42 \\
 x + 3y + 2z = 9
 \end{cases}$$


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$$-8x - 7z = -33$$

$$x = -\frac{7}{8}z + \frac{33}{8}$$

$$\begin{aligned}
 (1) \rightarrow y &= 14 - 3z - 3\left(-\frac{7}{8}z + \frac{33}{8}\right) \\
 &= \frac{13}{8} - \frac{3}{8}z
 \end{aligned}$$

$$\therefore \text{Solution: } \left( \frac{33}{8} - \frac{7}{8}z, \frac{13}{8} - \frac{3}{8}z, z \right)$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases}
 4x - 2y + z = 7 \\
 x + y + z = -2 \\
 4x + 2y + z = 3
 \end{cases}$$

### Solution

$$\begin{aligned}
 D &= \begin{vmatrix} 4 & -2 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix} \begin{matrix} 4 & -2 \\ 1 & 1 \\ 4 & 2 \end{matrix} \\
 &= -12
 \end{aligned}$$

$$\begin{aligned}
 D_x &= \begin{vmatrix} 7 & -2 & 1 \\ -2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} \begin{matrix} 7 & -2 \\ -2 & 1 \\ 3 & 2 \end{matrix} \\
 &= -24
 \end{aligned}$$

$$\begin{aligned}
 D_y &= \begin{vmatrix} 4 & 7 & 1 \\ 1 & -2 & 1 \\ 4 & 3 & 1 \end{vmatrix} \begin{matrix} 4 & 7 \\ 1 & -2 \\ 4 & 3 \end{matrix} \\
 &= 12
 \end{aligned}$$

$$D_z = \begin{vmatrix} 4 & -2 & 7 \\ 1 & 1 & -2 \\ 4 & 2 & 3 \end{vmatrix} \begin{vmatrix} 4 & -2 \\ 1 & 1 \\ 4 & 2 \end{vmatrix}$$

$$= 36$$

$$x = 2 \quad x = \frac{D_x}{D}$$

$$y = -1 \quad y = \frac{D_y}{D}$$

$$z = -3 \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } (2, -1, -3)$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 0 & 2 & -1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix} \begin{vmatrix} 0 & 2 \\ 1 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= 1$$

$$D_x = \begin{vmatrix} 7 & 2 & -1 \\ 17 & 2 & 1 \\ -1 & 3 & 2 \end{vmatrix} \begin{vmatrix} 7 & 2 \\ 17 & 2 \\ -1 & 3 \end{vmatrix}$$

$$= -116$$

$$D_y = \begin{vmatrix} 0 & 7 & -1 \\ 1 & 17 & 1 \\ 2 & -1 & 2 \end{vmatrix} \begin{vmatrix} 0 & 7 \\ 1 & 17 \\ 2 & -1 \end{vmatrix}$$

$$= 35$$

$$D_z = \begin{vmatrix} 0 & 2 & 7 \\ 1 & 2 & 17 \\ 2 & 3 & -1 \end{vmatrix} \begin{vmatrix} 0 & 2 \\ 1 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= 63$$

$$x = -116 \quad x = \frac{D_x}{D}$$

$$\underline{y = 35} \quad y = \frac{D_y}{D}$$

$$\underline{z = 63} \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(-116, 35, 63)}$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 2x - 2y + z = -4 \\ 6x + 4y - 3z = -24 \\ x - 2y + 2z = 1 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 2 & -2 & 1 \\ 6 & 4 & -3 \\ 1 & -2 & 2 \end{vmatrix} \begin{vmatrix} 2 & -2 \\ 6 & 4 \\ 1 & -2 \end{vmatrix}$$

$$\underline{= 18}$$

$$D_x = \begin{vmatrix} -4 & -2 & 1 \\ -24 & 4 & -3 \\ 1 & -2 & 2 \end{vmatrix} \begin{vmatrix} -4 & -2 \\ -24 & 4 \\ 1 & -2 \end{vmatrix}$$

$$\underline{= -54}$$

$$D_y = \begin{vmatrix} 2 & -4 & 1 \\ 6 & -24 & -3 \\ 1 & 1 & 2 \end{vmatrix} \begin{vmatrix} 2 & -4 \\ 6 & -24 \\ 1 & 1 \end{vmatrix}$$

$$\underline{= 0}$$

$$D_z = \begin{vmatrix} 2 & -2 & -4 \\ 6 & 4 & -24 \\ 1 & -2 & 1 \end{vmatrix} \begin{vmatrix} 2 & -2 \\ 6 & 4 \\ 1 & -2 \end{vmatrix}$$

$$\underline{= 36}$$

$$\underline{x = -3} \quad x = \frac{D_x}{D}$$

$$\underline{y = 0} \quad y = \frac{D_y}{D}$$

$$\underline{z = 2} \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(-3, 0, 2)}$$



**Exercise**

Use Cramer's rule to solve the system

$$\begin{cases} 9x + 3y + z = 4 \\ 16x + 4y + z = 2 \\ 25x + 5y + z = 2 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{vmatrix} \begin{vmatrix} 9 & 3 \\ 16 & 4 \\ 25 & 5 \end{vmatrix}$$

$$= -2$$

$$D_x = \begin{vmatrix} 4 & 3 & 1 \\ 2 & 4 & 1 \\ 2 & 5 & 1 \end{vmatrix} \begin{vmatrix} 4 & 3 \\ 2 & 4 \\ 2 & 5 \end{vmatrix}$$

$$= -2$$

$$D_y = \begin{vmatrix} 9 & 4 & 1 \\ 16 & 2 & 1 \\ 25 & 2 & 1 \end{vmatrix} \begin{vmatrix} 9 & 4 \\ 16 & 2 \\ 25 & 2 \end{vmatrix}$$

$$= 18$$

$$D_z = \begin{vmatrix} 9 & 3 & 4 \\ 16 & 4 & 2 \\ 25 & 5 & 2 \end{vmatrix} \begin{vmatrix} 9 & 3 \\ 16 & 4 \\ 25 & 5 \end{vmatrix}$$

$$= -44$$

$$x = 1 \quad x = \frac{D_x}{D}$$

$$y = -9 \quad y = \frac{D_y}{D}$$

$$z = 22 \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } (1, -9, 22)$$

**Exercise**

Use Cramer's rule to solve the system

$$\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & -3 \\ 3 & -1 & -4 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 3 & -1 \end{vmatrix}$$

$$= -31$$

$$D_x = \begin{vmatrix} -8 & -1 & 2 \\ 9 & 2 & -3 \\ 3 & -1 & -4 \end{vmatrix} \begin{vmatrix} -8 & -1 \\ 9 & 2 \\ 3 & -1 \end{vmatrix}$$

$$= 31$$

$$D_y = \begin{vmatrix} 2 & -8 & 2 \\ 1 & 9 & -3 \\ 3 & 3 & -4 \end{vmatrix} \begin{vmatrix} 2 & -8 \\ 1 & 9 \\ 3 & 3 \end{vmatrix}$$

$$= -62$$

$$D_z = \begin{vmatrix} 2 & -1 & -8 \\ 1 & 2 & 9 \\ 3 & -1 & 3 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 3 & -1 \end{vmatrix}$$

$$= 62$$

$$x = -1 \quad x = \frac{D_x}{D}$$

$$y = 2 \quad y = \frac{D_y}{D}$$

$$z = -2 \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } (-1, 2, -2)$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases} x - 3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 1 & 0 & -3 \\ 2 & -1 & 2 \\ 7 & -3 & -5 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 2 & -1 \\ 7 & -3 \end{vmatrix}$$

$$= 8$$

$$D_x = \begin{vmatrix} -5 & 0 & -3 \\ 16 & -1 & 2 \\ 19 & -3 & -5 \end{vmatrix} \begin{vmatrix} -5 & 0 \\ 16 & -1 \\ 19 & -3 \end{vmatrix}$$

$$= 32$$

$$D_y = \begin{vmatrix} 1 & -5 & -3 \\ 2 & 16 & 2 \\ 7 & 19 & -5 \end{vmatrix} \begin{vmatrix} 1 & -5 \\ 2 & 16 \\ 7 & 19 \end{vmatrix}$$

$$= -16$$

$$D_z = \begin{vmatrix} 1 & 0 & -5 \\ 2 & -1 & 16 \\ 7 & -3 & 19 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 2 & -1 \\ 7 & -3 \end{vmatrix}$$

$$= 24$$

$$x = 4 \quad x = \frac{D_x}{D}$$

$$y = -2 \quad y = \frac{D_y}{D}$$

$$z = 3 \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } (4, -2, 3)$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases} x + 2y - z = 5 \\ 2x - y + 3z = 0 \\ 2y + z = 1 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 0 & 2 & 1 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{vmatrix}$$

$$= -15$$

$$D_x = \begin{vmatrix} 5 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 2 & 1 \end{vmatrix} \begin{vmatrix} 5 & 2 \\ 0 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= -30$$

$$D_y = \begin{vmatrix} 1 & 5 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix} \begin{vmatrix} 1 & 5 \\ 2 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= -15$$

$$D_z = \begin{vmatrix} 1 & 2 & 5 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{vmatrix}$$

$$= 15$$

$$x = 2 \quad x = \frac{D_x}{D}$$

$$y = 1 \quad y = \frac{D_y}{D}$$

$$z = -1 \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } (2, 1, -1)$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases} x + y + z = 6 \\ 3x + 4y - 7z = 1 \\ 2x - y + 3z = 5 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & -7 \\ 2 & -1 & 3 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 3 & 4 \\ 2 & -1 \end{vmatrix}$$

$$= -29$$

$$D_x = \begin{vmatrix} 6 & 1 & 1 \\ 1 & 4 & -7 \\ 5 & -1 & 3 \end{vmatrix} \begin{vmatrix} 6 & 1 \\ 1 & 4 \\ 5 & -1 \end{vmatrix}$$

$$= -29$$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 3 & 1 & -7 \\ 2 & 5 & 3 \end{vmatrix} \begin{vmatrix} 1 & 6 \\ 3 & 1 \\ 2 & 5 \end{vmatrix}$$

$$= -87$$

$$D_z = \begin{vmatrix} 1 & 2 & 6 \\ 2 & -1 & 1 \\ 0 & 2 & 5 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{vmatrix}$$

$$= -58$$

$$x = 1 \quad x = \frac{D_x}{D}$$

$$y = 3 \quad y = \frac{D_y}{D}$$

$$z = 2 \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } (1, 3, 2)$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 3x + 2y + 3z = 3 \\ 4x - 5y + 7z = 1 \\ 2x + 3y - 2z = 6 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 3 & 2 & 3 \\ 4 & -5 & 7 \\ 2 & 3 & -2 \end{vmatrix} \begin{vmatrix} 3 & 2 \\ 4 & -5 \\ 2 & 3 \end{vmatrix}$$

$$= 77$$

$$D_x = \begin{vmatrix} 3 & 2 & 3 \\ 1 & -5 & 7 \\ 6 & 3 & -2 \end{vmatrix} \begin{vmatrix} 3 & 2 \\ 1 & -5 \\ 6 & 3 \end{vmatrix}$$

$$= 154$$

$$D_y = \begin{vmatrix} 3 & 3 & 3 \\ 4 & 1 & 7 \\ 2 & 6 & -2 \end{vmatrix} \begin{vmatrix} 3 & 3 \\ 4 & 1 \\ 2 & 6 \end{vmatrix}$$

$$= 0$$

$$D_z = \begin{vmatrix} 3 & 2 & 3 \\ 4 & -5 & 1 \\ 2 & 3 & 6 \end{vmatrix} \begin{vmatrix} 3 & 2 \\ 4 & -5 \\ 2 & 3 \end{vmatrix}$$

$$= -77$$

$$x = 2 \quad x = \frac{D_x}{D}$$

$$\underline{y = 0} \quad y = \frac{D_y}{D}$$

$$\underline{z = -1} \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(2, 0, -1)}$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases} 4x + 5y = 2 \\ 11x + y + 2z = 3 \\ x + 5y + 2z = 1 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \begin{matrix} 4 & 5 \\ 11 & 1 \\ 1 & 5 \end{matrix}$$

$$\underline{= -132}$$

$$D_x = \begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \begin{matrix} 2 & 5 \\ 3 & 1 \\ 1 & 5 \end{matrix}$$

$$\underline{= -36}$$

$$D_y = \begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} \begin{matrix} 4 & 2 \\ 11 & 3 \\ 1 & 1 \end{matrix}$$

$$\underline{= -24}$$

$$D_z = \begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} \begin{matrix} 4 & 5 \\ 11 & 1 \\ 1 & 5 \end{matrix}$$

$$\underline{= 12}$$

$$x = \frac{36}{132}$$

$$\underline{= \frac{3}{11}}$$

$$x = \frac{D_x}{D}$$

$$y = \frac{24}{132}$$

$$\underline{= \frac{2}{11}}$$

$$y = \frac{D_y}{D}$$

$$z = -\frac{12}{132}$$

$$z = \frac{D_z}{D}$$

$$= -\frac{1}{11}$$

$$\therefore \text{Solution: } \left( \frac{3}{11}, \frac{2}{11}, -\frac{1}{11} \right)$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases} x - 4y + z = 6 \\ 4x - y + 2z = -1 \\ 2x + 2y - 3z = -20 \end{cases}$$

### Solution

$$D = \begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix} \begin{vmatrix} 1 & -4 \\ 4 & -1 \\ 2 & 2 \end{vmatrix}$$

$$= -55$$

$$D_x = \begin{vmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix} \begin{vmatrix} 6 & -4 \\ -1 & -1 \\ -20 & 2 \end{vmatrix}$$

$$= 144$$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix} \begin{vmatrix} 1 & 6 \\ 4 & -1 \\ 2 & -20 \end{vmatrix}$$

$$= 61$$

$$D_z = \begin{vmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix} \begin{vmatrix} 1 & -4 \\ 4 & -1 \\ 2 & 2 \end{vmatrix}$$

$$= -230$$

$$x = -\frac{144}{55}$$

$$x = \frac{D_x}{D}$$

$$y = -\frac{61}{55}$$

$$y = \frac{D_y}{D}$$

$$z = \frac{46}{11}$$

$$z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \left( -\frac{144}{55}, -\frac{61}{55}, \frac{46}{11} \right)$$

**Exercise**

Use Cramer's rule to solve the system

$$\begin{cases} 2x - y + z = -1 \\ 3x + 4y - z = -1 \\ 4x - y + 2z = -1 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 4 & -1 \\ 4 & -1 & 2 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ 3 & 4 \\ 4 & -1 \end{vmatrix} \\ = 5$$

$$D_x = \begin{vmatrix} -1 & -1 & 1 \\ -1 & 4 & -1 \\ -1 & -1 & 2 \end{vmatrix} \begin{vmatrix} -1 & -1 \\ -1 & 4 \\ -1 & -1 \end{vmatrix} \\ = -5$$

$$D_y = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -1 & -1 \\ 4 & -1 & 2 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ 3 & -1 \\ 4 & -1 \end{vmatrix} \\ = 5$$

$$D_z = \begin{vmatrix} 2 & -1 & -1 \\ 3 & 4 & -1 \\ 4 & -1 & -1 \end{vmatrix} \begin{vmatrix} 2 & -1 \\ 3 & 4 \\ 4 & -1 \end{vmatrix} \\ = 10$$

$$x = \frac{-5}{5} \qquad x = \frac{D_x}{D} \\ = -1$$

$$y = \frac{5}{5} \qquad y = \frac{D_y}{D} \\ = 1$$

$$z = \frac{10}{5} \qquad z = \frac{D_z}{D} \\ = 2$$

$$\therefore \text{Solution: } \underline{(-1, 1, 2)}$$



**Exercise**

Use Cramer's rule to solve the system

$$\begin{cases} 4x - y - z = 1 \\ 2x + 2y + 3z = 10 \\ 5x - 2y - 2z = -1 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 4 & -1 & -1 \\ 2 & 2 & 3 \\ 5 & -2 & -2 \end{vmatrix}$$

$$= 3$$

$$D_x = \begin{vmatrix} 1 & -1 & -1 \\ 10 & 2 & 3 \\ -1 & -2 & -2 \end{vmatrix}$$

$$= 3$$

$$D_y = \begin{vmatrix} 4 & 1 & -1 \\ 2 & 10 & 3 \\ 5 & -1 & -2 \end{vmatrix}$$

$$= 3$$

$$D_z = \begin{vmatrix} 4 & -1 & -1 \\ 2 & 2 & 10 \\ 5 & -2 & -1 \end{vmatrix}$$

$$= 6$$

$$\underline{x = 1} \quad x = \frac{D_x}{D}$$

$$\underline{y = 1} \quad y = \frac{D_y}{D}$$

$$\underline{z = 2} \quad z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \underline{(1, 1, 2)}$$

**Exercise**

Use Cramer's rule to solve the system

$$\begin{cases} 3x + 4y + 4z = 11 \\ 4x - 4y + 6z = 11 \\ 6x - 6y = 3 \end{cases}$$

**Solution**

$$D = \begin{vmatrix} 3 & 4 & 4 \\ 4 & -4 & 6 \\ 6 & -6 & 0 \end{vmatrix}$$

$$= 252$$

$$D_x = \begin{vmatrix} 11 & 4 & 4 \\ 11 & -4 & 6 \\ 3 & -6 & 0 \end{vmatrix}$$

$$= 252$$

$$D_y = \begin{vmatrix} 3 & 11 & 4 \\ 4 & 11 & 6 \\ 6 & 3 & 0 \end{vmatrix}$$

$$= 126$$

$$D_z = \begin{vmatrix} 3 & 4 & 11 \\ 4 & -4 & 11 \\ 6 & -6 & 3 \end{vmatrix}$$

$$= 378$$

$$x = 1$$

$$x = \frac{D_x}{D}$$

$$y = \frac{1}{2}$$

$$y = \frac{D_y}{D}$$

$$z = \frac{3}{2}$$

$$z = \frac{D_z}{D}$$

$$\therefore \text{Solution: } \left(1, \frac{1}{2}, \frac{3}{2}\right)$$

### Exercise

Use Cramer's rule to solve the system

$$\begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$$

### Solution

$$D = \begin{vmatrix} -1 & -4 & 2 & 1 \\ 2 & -1 & 7 & 9 \\ -1 & 1 & 3 & 1 \\ -1 & -2 & 1 & -4 \end{vmatrix}$$

$$= -243 \mid$$

$$D_1 = \begin{vmatrix} -32 & -4 & 2 & 1 \\ 14 & -1 & 7 & 9 \\ 11 & 1 & 3 & 1 \\ -4 & -2 & 1 & -4 \end{vmatrix}$$

$$= -2115 \mid$$

$$D_2 = \begin{vmatrix} -1 & -32 & 2 & 1 \\ 2 & 14 & 7 & 9 \\ -1 & 11 & 3 & 1 \\ -1 & -4 & 1 & -4 \end{vmatrix}$$

$$= -1834 \mid$$

$$D_3 = \begin{vmatrix} -1 & -4 & -32 & 1 \\ 2 & -1 & 14 & 9 \\ -1 & 1 & 11 & 1 \\ -1 & -2 & -4 & -4 \end{vmatrix}$$

$$= -1279 \mid$$

$$D_4 = \begin{vmatrix} -1 & -4 & 2 & -32 \\ 2 & -1 & 7 & 14 \\ -1 & 1 & 3 & 11 \\ -1 & -2 & 1 & -4 \end{vmatrix}$$

$$= 883 \mid$$

$$x_1 = \frac{-2115}{-243}$$

$$= \frac{235}{27} \mid$$

$$x_2 = \frac{-1834}{-243}$$

$$= \frac{1834}{243} \mid$$

$$x_3 = \frac{-1279}{-243}$$

$$= \frac{1279}{243} \mid$$

$$x_4 = -\frac{883}{243} \mid$$

$$\therefore \text{Solution: } \left( \frac{235}{27}, \frac{1834}{243}, \frac{1279}{243}, -\frac{883}{243} \right) \mid$$

**Exercise**

Determine the polynomial function whose graph through the points and sketch the graph of the polynomial function, showing the points.

$$(2, 5), (3, 2), (4, 5)$$

**Solution**

**Given** 3 points, therefore, we must use a quadratic equation:

$$f(x) = ax^2 + bx + c$$

$$(2, 5) \rightarrow 4a + 2b + c = 5$$

$$(3, 2) \rightarrow 9a + 3b + c = 2$$

$$(4, 5) \rightarrow 16a + 4b + c = 5$$

$$\begin{cases} 4a + 2b + c = 5 \\ 9a + 3b + c = 2 \\ 16a + 4b + c = 5 \end{cases}$$

$$D = \begin{vmatrix} 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{vmatrix} = -2$$

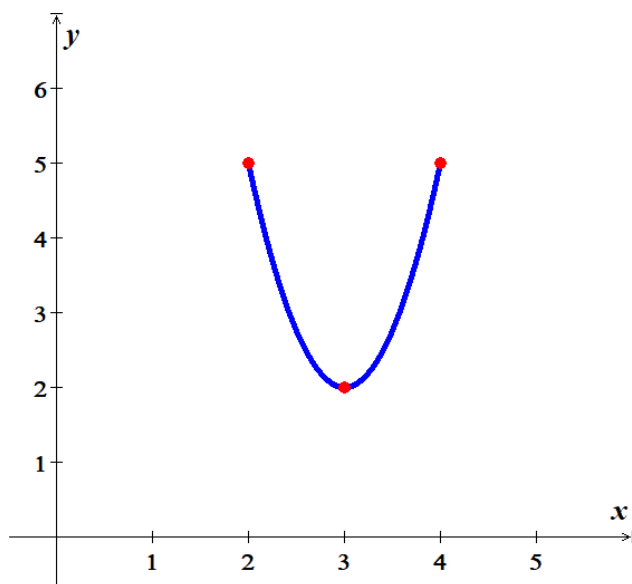
$$D_a = \begin{vmatrix} 5 & 2 & 1 \\ 2 & 3 & 1 \\ 5 & 4 & 1 \end{vmatrix} = -6$$

$$D_b = \begin{vmatrix} 4 & 5 & 1 \\ 9 & 2 & 1 \\ 16 & 3 & 1 \end{vmatrix} = 36$$

$$D_c = \begin{vmatrix} 4 & 2 & 5 \\ 9 & 3 & 2 \\ 16 & 4 & 5 \end{vmatrix} = -58$$

$$a = 3 \quad b = -18 \quad c = 29$$

$$f(x) = 3x^2 - 18x + 29$$



**Exercise**

Determine the polynomial function whose graph through the points and sketch the graph of the polynomial function, showing the points.

$$(2, 4), (3, 6), (5, 10)$$

**Solution**

**Given** 3 points, therefore, we must use a quadratic equation:

$$f(x) = ax^2 + bx + c$$

$$(2, 4) \rightarrow 4a + 2b + c = 4$$

$$(3, 6) \rightarrow 9a + 3b + c = 6$$

$$(5, 10) \rightarrow 25a + 5b + c = 10$$

$$\begin{cases} 4a + 2b + c = 4 \\ 9a + 3b + c = 6 \\ 25a + 5b + c = 10 \end{cases}$$

$$D = \begin{vmatrix} 4 & 2 & 1 \\ 9 & 3 & 1 \\ 25 & 5 & 1 \end{vmatrix} = -6$$

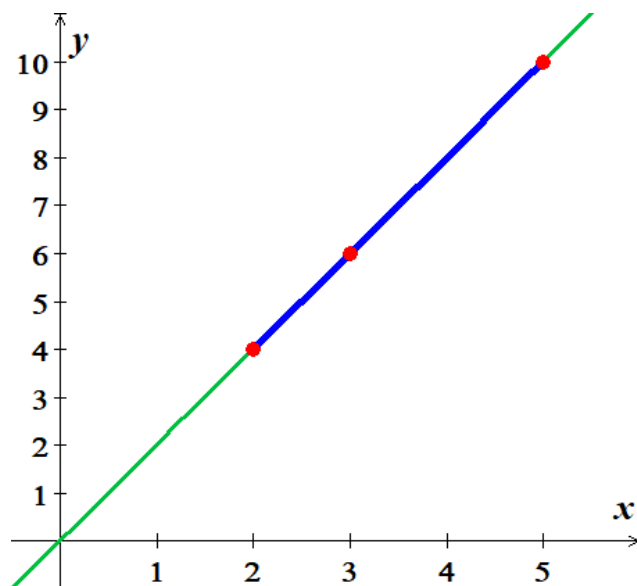
$$D_a = \begin{vmatrix} 4 & 2 & 1 \\ 6 & 3 & 1 \\ 10 & 5 & 1 \end{vmatrix} = 0$$

$$D_b = \begin{vmatrix} 4 & 4 & 1 \\ 9 & 6 & 1 \\ 25 & 10 & 1 \end{vmatrix} = -12$$

$$D_c = \begin{vmatrix} 4 & 2 & 4 \\ 9 & 3 & 6 \\ 25 & 5 & 10 \end{vmatrix} = 0$$

$$a = 0 \quad b = 2 \quad c = 0$$

$$f(x) = 2x$$



**Exercise**

Determine the polynomial function whose graph through the points and sketch the graph of the polynomial function, showing the points.

$$(-1, 3), (0, 0), (1, 1), (4, 58)$$

**Solution**

**Given** 3 points, therefore, we must use a third-degree equation:

$$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

$$(-1, 3) \rightarrow -a_3 + a_2 - a_1 + a_0 = 3$$

$$(0, 0) \rightarrow \underline{a_0 = 0}$$

$$(1, 1) \rightarrow a_3 + a_2 + a_1 + a_0 = 1$$

$$(4, 58) \rightarrow 64a_3 + 16a_2 + 4a_1 + a_0 = 58$$

$$\begin{cases} -a_1 + a_2 - a_3 = 3 \\ a_1 + a_2 + a_3 = 1 \\ 4a_1 + 16a_2 + 64a_3 = 58 \end{cases}$$

$$D = \begin{vmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & 16 & 64 \end{vmatrix} = \underline{-120}$$

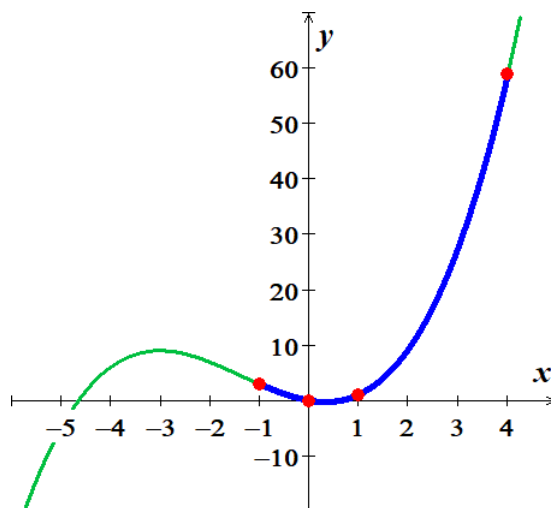
$$D_{a_1} = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 58 & 16 & 64 \end{vmatrix} = \underline{180}$$

$$D_{a_2} = \begin{vmatrix} -1 & 3 & -1 \\ 1 & 1 & 1 \\ 4 & 58 & 64 \end{vmatrix} = \underline{-240}$$

$$D_{a_3} = \begin{vmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \\ 4 & 16 & 58 \end{vmatrix} = \underline{-60}$$

$$\underline{a_0 = 0 \quad a_1 = -\frac{3}{2} \quad a_2 = 2 \quad a_3 = \frac{1}{2}}$$

$$\underline{f(x) = \frac{1}{2}x^3 + 2x^2 - \frac{3}{2}x}$$



**Exercise**

The U.S. census lists the population of the United States as 249 million in 1990, 282 million in 2000, and 309 million in 2010. Fit a second-degree polynomial passing through these three points and use it to predict the populations in 2020 and 2030.

**Solution**

Let assume  $x = 0$  at year 1990  $\Rightarrow (0, 249)$

2000  $\rightarrow (10, 282)$

2010  $\rightarrow (20, 309)$

$$P(x) = a_0 + a_1x + a_2x^2$$

$$(0, 249) \rightarrow \underline{a_0 = 249}$$

$$(10, 282) \rightarrow a_0 + 10a_1 + 100a_2 = 282$$

$$(20, 309) \rightarrow a_0 + 20a_1 + 400a_2 = 309$$

$$\begin{cases} 10a_1 + 100a_2 = 33 \\ 20a_1 + 400a_2 = 60 \end{cases}$$

$$D = \begin{vmatrix} 10 & 100 \\ 20 & 400 \end{vmatrix} = 2,000$$

$$D_1 = \begin{vmatrix} 33 & 100 \\ 60 & 400 \end{vmatrix} = 7,200$$

$$D_2 = \begin{vmatrix} 10 & 33 \\ 20 & 60 \end{vmatrix} = -60$$

$$a_1 = \frac{72}{20} = \frac{18}{5}$$

$$a_2 = -\frac{6}{200} = -\frac{3}{100}$$

$$\underline{a_0 = 249 \quad a_1 = \frac{18}{5} \quad a_2 = -\frac{3}{100}}$$

$$\underline{P(x) = 249 + \frac{18}{5}x - \frac{3}{100}x^2}$$

Year 2020:  $\Rightarrow x = 30$

$$\begin{aligned} P(30) &= 249 + \frac{18}{5}(30) - \frac{3}{100}(900) \\ &= \underline{330 \text{ million}} \end{aligned}$$

Year 2030:  $\Rightarrow x = 40$

$$\begin{aligned} P(40) &= 249 + 18(9) - 3(16) \\ &= \underline{345 \text{ million}} \end{aligned}$$



## Solution

### Section 1.8 – Applications

#### Exercise

Through a network

- a) Solve this system for  $x_i$ ,  $i = 1, 2, 3, 4, 5, 6$
- b) What is the largest value of  $x_3$ ?

#### Solution

$$a) \begin{cases} x_1 + x_3 = 20 \\ x_1 + x_2 = 80 \\ x_2 = x_3 + x_4 \end{cases}$$

$$\begin{cases} x_1 + x_3 = 20 \\ x_1 + x_2 = 80 \\ x_2 - x_3 - x_4 = 0 \end{cases}$$

$$\left( \begin{array}{cccc|c} 1 & 0 & 1 & 0 & 20 \\ 1 & 1 & 0 & 0 & 80 \\ 0 & 1 & -1 & -1 & 0 \end{array} \right) \quad R_2 - R_1$$

$$\left( \begin{array}{cccc|c} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & 0 & 60 \\ 0 & 1 & -1 & -1 & 0 \end{array} \right) \quad R_3 - R_2$$

$$\left( \begin{array}{cccc|c} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & 0 & 60 \\ 0 & 0 & 0 & -1 & -60 \end{array} \right) \quad \begin{array}{l} x_1 + x_3 = 20 \\ x_2 + x_3 = 60 \\ \underline{x_4 = 60} \end{array}$$

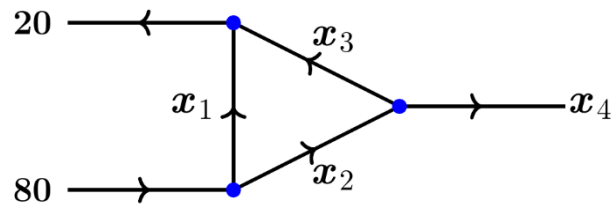
$$\begin{cases} x_1 = 20 - x_3 \\ x_2 = 60 - x_3 \end{cases}$$

$$\therefore \underline{(20 - x_3, 60 - x_3, x_3, 60)}$$

$$b) \begin{cases} 20 - x_3 \geq 0 \\ 60 - x_3 \geq 0 \end{cases}$$

$$\begin{cases} x_3 \leq 20 \\ x_3 \leq 60 \end{cases}$$

The largest value of  $x_3$  is 20, since 60 will result  $x_1$  with a negative value.



**Exercise**

The flow of traffic, through a network of streets as is shown below

- a) Solve this system for  $x_i$ ,  $i = 1, 2, 3, 4, 5, 6$
- b) Find the minimum flows in the branches denoted by  $x_2, x_3, x_4$ , and  $x_5$

**Solution**

$$a) \begin{cases} x_1 + 60 = x_3 + 20 \\ x_1 + 80 = x_2 + 30 \\ x_2 + x_4 = x_3 + x_5 \\ x_5 + 40 = x_6 + 100 \\ x_4 + 40 = x_6 + 90 \end{cases}$$

$$\begin{cases} x_1 - x_3 = -40 & (1) \\ x_1 - x_2 = -50 & (2) \\ x_2 - x_3 + x_4 - x_5 = 0 & (3) \\ x_5 - x_6 = 60 & (4) \\ x_4 - x_6 = 50 & (5) \end{cases}$$

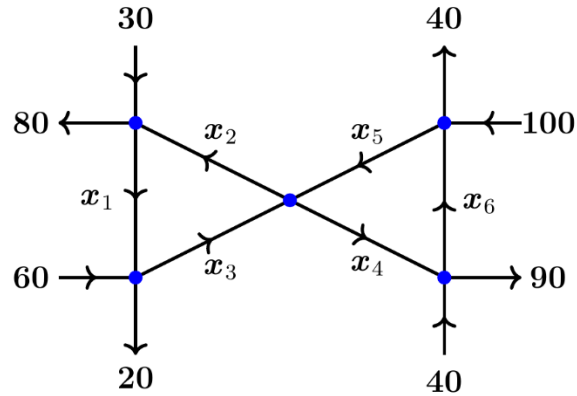
$$(1) \rightarrow x_3 = x_1 + 40$$

$$(2) \rightarrow x_2 = x_1 + 50$$

$$(4) \rightarrow x_5 = 60 + x_6$$

$$(5) \rightarrow x_4 = 50 + x_6$$

$$\therefore \text{Solution: } \underline{(x_1, x_1 + 50, x_1 + 40, x_6 + 60, x_6 + 50, x_6)}$$



- b) The minimum flow for  $x_1 = x_6 = 0$

$$x_3 = x_1 + 40 = \underline{40}$$

$$x_2 = x_1 + 50 = \underline{50}$$

$$x_4 = x_6 + 50 = \underline{50}$$

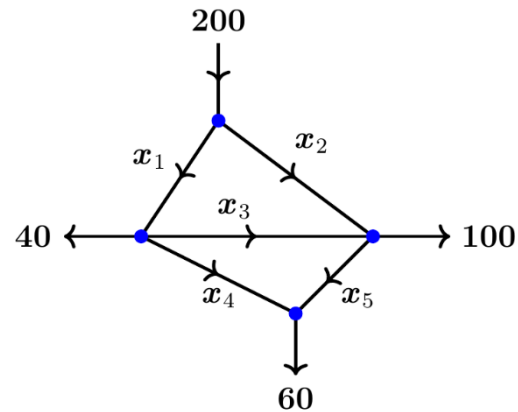
$$x_5 = x_6 + 60 = \underline{60}$$

$$\therefore \text{Solution: } \underline{(0, 50, 40, 60, 50, 0)}$$

**Exercise**

The flow of traffic, through a network of streets as is shown below

- Solve this system for  $x_i$ ,  $i = 1, 2, 3, 4, 5$
- Find the traffic flow when  $x_4 = 0$ .
- Find the traffic flow when  $x_4 = 100$ .
- Find the traffic flow when  $x_1 = 2x_2$ .

**Solution**

$$a) \begin{cases} x_1 = x_3 + x_4 + 40 \\ x_1 + x_2 = 200 \\ x_2 + x_3 = 100 + x_5 \\ x_5 + x_4 = 60 \end{cases}$$

$$\begin{cases} x_1 - x_3 - x_4 = 40 \\ x_1 + x_2 = 200 \\ x_2 + x_3 - x_5 = 100 \\ x_5 + x_4 = 60 \end{cases}$$

$$\left( \begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array} \right) \quad R_2 - R_1$$

$$\left( \begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array} \right) \quad R_3 - R_2$$

$$\left( \begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 0 & 0 & -1 & -1 & -60 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array} \right) \quad \begin{array}{l} R_1 - R_3 \\ R_2 + R_3 \\ R_4 + R_3 \end{array}$$

$$\left( \begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 1 & 100 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & -1 & -1 & -60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 = 100 + x_3 - x_5 \\ x_2 = 100 - x_3 + x_5 \\ x_4 = 60 - x_5 \end{array}$$

$$\therefore \text{Solution: } \underline{(100 + x_3 - x_5, \ 100 - x_3 + x_5, \ x_3, \ 60 - x_5, \ x_5)}$$

b) **Given:**  $x_4 = 0$

$$\begin{cases} x_1 - x_3 = 40 \\ x_1 + x_2 = 200 \\ x_2 + x_3 - x_5 = 100 \\ \underline{x_5 = 60} \end{cases}$$

$$\begin{cases} x_1 - x_3 = 40 \\ x_1 + x_2 = 200 \\ x_2 + x_3 = 160 \end{cases}$$

$$D = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \equiv 0$$

$$D_3 = \begin{vmatrix} 1 & 0 & 40 \\ 1 & 1 & 200 \\ 0 & 1 & 160 \end{vmatrix} \equiv 0$$

$$\begin{cases} \underline{x_1 = 40 + x_3} \\ \underline{x_2 = 160 - x_3} \end{cases}$$

$$\therefore \textbf{Solution: } \underline{(40 + x_3, \ 160 - x_3, \ x_3, \ 0, \ 60)}$$

c) **Given:**  $x_4 = 100$

$$\begin{cases} x_1 - x_3 = 140 \\ x_1 + x_2 = 200 \\ x_2 + x_3 - x_5 = 100 \\ \underline{x_5 = -40} \end{cases}$$

$$\begin{cases} x_1 - x_3 = 140 \\ x_1 + x_2 = 200 \\ x_2 + x_3 = 60 \end{cases}$$

$$D = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \equiv 0$$

$$D_3 = \begin{vmatrix} 1 & 0 & 140 \\ 1 & 1 & 200 \\ 0 & 1 & 60 \end{vmatrix} \equiv 0$$

$$\begin{cases} \underline{x_1 = 140 + x_3} \\ \underline{x_2 = 60 - x_3} \end{cases}$$

$$\therefore \textbf{Solution: } \underline{(140 + x_3, \ 60 - x_3, \ x_3, \ 0, \ -40)}$$

d) **Given:**  $x_1 = 2x_2$

$$\begin{cases} x_1 - x_3 - x_4 = 40 & (1) \\ x_1 + x_2 = 200 & (2) \\ x_2 + x_3 - x_5 = 100 \\ x_5 + x_4 = 60 \end{cases}$$

$$(2) \quad 3x_2 = 200 \rightarrow \underline{x_2 = \frac{200}{3}}$$

$$\underline{x_1 = \frac{400}{3}}$$

$$\begin{cases} -x_3 - x_4 = 40 - \frac{400}{3} \\ x_3 - x_5 = 100 - \frac{200}{3} \\ x_5 + x_4 = 60 \end{cases}$$

$$\begin{cases} x_3 + x_4 = \frac{280}{3} \\ x_3 - x_5 = \frac{100}{3} \\ x_4 + x_5 = 60 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

$$D_3 = \begin{vmatrix} 1 & 1 & \frac{280}{3} \\ 1 & 0 & \frac{100}{3} \\ 0 & 1 & 60 \end{vmatrix} = 0$$

$$\begin{cases} \underline{x_3 = \frac{100}{3} + x_5} \\ \underline{x_4 = 60 - x_5} \end{cases}$$

$$\therefore \text{Solution: } \underline{\left( \frac{400}{3}, \frac{200}{3}, \frac{100}{3} + x_5, 60 - x_5, x_5 \right)}$$

**Exercise**

The flow of traffic, in vehicles per hour, through a network of streets as is shown below

- Solve this system for  $x_i$ ,  $i = 1, 2, 3, 4$ .
- Find the traffic flow when  $x_4 = 0$ .
- Find the traffic flow when  $x_4 = 100$ .
- Find the traffic flow when  $x_1 = 2x_2$ .

**Solution**

$$a) \begin{cases} x_1 + 100 = x_3 \\ x_2 + 200 = x_1 \\ x_2 + 100 = x_4 \\ x_4 + 200 = x_3 \end{cases}$$

$$\begin{cases} -x_1 + x_3 = 100 \\ x_1 - x_2 = 200 \\ -x_2 + x_4 = 100 \\ x_3 - x_4 = 200 \end{cases}$$

$$\left( \begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 1 & -1 & 0 & 0 & 200 \\ 0 & -1 & 0 & 1 & 100 \\ 0 & 0 & 1 & -1 & 200 \end{array} \right)$$

 $R_2 + R_1$ 

$$\left( \begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 1 & -1 & 0 & 0 & 200 \\ 0 & -1 & 0 & 1 & 100 \\ 0 & 0 & 1 & -1 & 200 \end{array} \right) = -1 \left( \begin{array}{ccc|c} -1 & 0 & 0 & 100 \\ -1 & 0 & 1 & 100 \\ 0 & 1 & -1 & 200 \end{array} \right) = -1 \left( \begin{array}{ccc|c} 0 & 1 & 0 & 100 \\ -1 & 0 & 1 & 100 \\ 0 & 1 & -1 & 200 \end{array} \right)$$

$$\left( \begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 0 & -1 & 1 & 0 & 300 \\ 0 & -1 & 0 & 1 & 100 \\ 0 & 0 & 1 & -1 & 200 \end{array} \right)$$

 $R_3 - R_2$ 

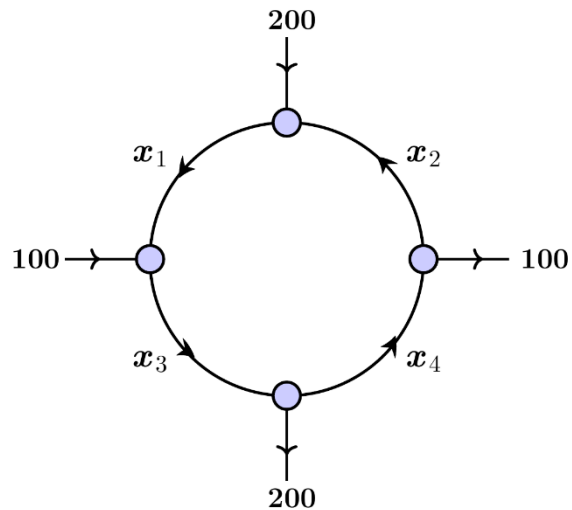
$$\left( \begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 0 & -1 & 1 & 0 & 300 \\ 0 & 0 & -1 & 1 & -200 \\ 0 & 0 & 1 & -1 & 200 \end{array} \right)$$

 $R_4 + R_3$ 

$$\left( \begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 0 & -1 & 1 & 0 & 300 \\ 0 & 0 & -1 & 1 & -200 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{aligned} \rightarrow -x_1 + x_3 &= 100 \\ \rightarrow -x_2 + x_3 &= 100 \\ \rightarrow -x_3 + x_4 &= 100 \end{aligned}$$

Let  $x_4$  be the free variable



$$\begin{cases} \underline{x_3 = x_4 + 200} \\ \underline{x_2 = x_4 - 100} \\ x_1 = 200 + x_2 = \underline{x_4 + 100} \end{cases}$$

**Solution:**  $(x_4 + 100, x_4 - 100, x_4 + 200, x_4)$

**OR**

$$\begin{aligned} \begin{vmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} &= -1 \begin{vmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} \\ &= -1(1) - 1(-1) \\ &= -1 + 1 \\ &= \underline{0} \end{aligned}$$

$$\begin{cases} -x_1 + x_3 = 100 & \rightarrow x_1 = x_3 - 100 = \underline{x_4 + 100} \\ x_1 - x_2 = 200 \\ -x_2 + x_4 = 100 & \rightarrow \underline{x_2 = x_4 - 100} \\ x_3 - x_4 = 200 & \rightarrow \underline{x_3 = x_4 + 200} \end{cases}$$

**b)** The traffic flow when  $x_4 = 0$  is:

$$\therefore (100, -100, 200, 0)$$

**c)** The traffic flow when  $x_4 = 100$  is:

$$\therefore (200, 0, 300, 100)$$

**d)** The traffic flow when  $x_1 = 2x_2$  :

$$x_4 + 100 = 2(x_4 - 100)$$

$$x_4 + 100 = 2x_4 - 200$$

$$x_4 = 300$$

$$\therefore (400, 200, 500, 300)$$

**Exercise**

Through a network, Express  $x_n$ 's in terms of the parameters  $s$  and  $t$ .

**Solution**

$$\begin{cases} x_1 = x_2 + 400 \\ x_1 + x_3 = x_4 + 600 \\ x_4 + x_5 = 100 \\ x_2 + x_3 + x_5 = 300 \end{cases}$$

$$\begin{cases} x_1 - x_2 = 400 \\ x_2 + x_3 - x_4 = 600 \\ x_4 + x_5 = 100 \\ x_2 + x_3 + x_5 = 300 \end{cases}$$

$$\left( \begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 1 & 0 & 1 & -1 & 0 & 600 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 1 & 1 & 0 & 1 & 300 \end{array} \right) \quad \begin{array}{l} R_2 - R_1 \\ R_3 \leftrightarrow R_4 \end{array}$$

$$\left( \begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 1 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & 1 & 100 \end{array} \right) \quad R_3 - R_2$$

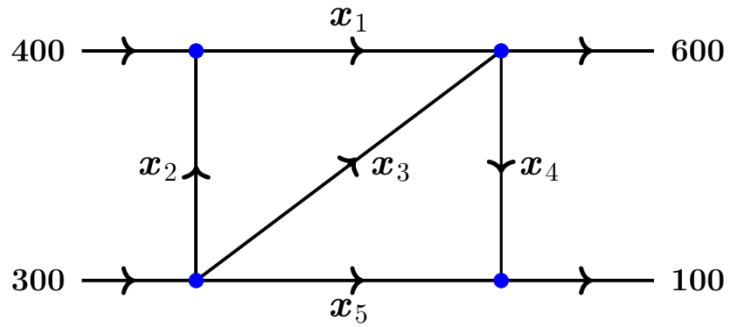
$$\left( \begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 100 \end{array} \right) \quad R_4 - R_3$$

$$\left( \begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} x_1 - x_2 = 400 \quad \rightarrow x_1 = 400 + x_2 \\ x_2 + x_3 - x_4 = 200 \quad \rightarrow x_2 = 200 - x_3 + x_4 \\ x_4 + x_5 = 100 \quad \rightarrow \underline{x_4 = 100 - t} \end{array}$$

Let  $x_5 = t$  &  $x_3 = s$

$$\begin{aligned} x_2 &= 200 - s + 100 - t \\ &= \underline{300 - s - t} \end{aligned}$$

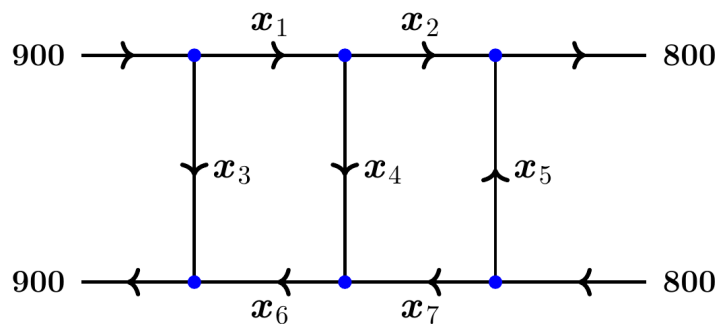
$$\begin{aligned} x_1 &= 400 + 300 - s - t \\ &= \underline{700 - s - t} \end{aligned}$$





**Exercise**

Water is flowing through a network of pipes. Express  $x_n$ 's in terms of the parameters  $s$  and  $t$ .

**Solution**

$$x_1 + x_3 = 900$$

$$x_1 = x_2 + x_4 \rightarrow x_1 - x_2 - x_4 = 0$$

$$x_2 + x_5 = 800$$

$$x_5 + x_7 = 800$$

$$x_6 = x_4 + x_7 \rightarrow x_4 - x_6 + x_7 = 0$$

$$x_3 + x_6 = 900$$

$$\left[ \begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 900 \end{array} \right] \quad \begin{array}{l} \\ R_2 - R_1 \\ \\ \\ \\ \end{array}$$

$$\left[ \begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & -900 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 900 \end{array} \right] \quad \begin{array}{l} \\ R_3 + R_2 \\ R_6 \\ \\ R_4 \end{array}$$

$$\left[ \begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & -900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 900 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \end{array} \right] \quad \begin{array}{l} \\ -R_2 \\ \\ R_4 + R_3 \\ \end{array}$$

$$\left[ \begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 800 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \end{array} \right] \quad R_5 + R_4$$

$$\left[ \begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \end{array} \right] \quad R_6 - R_5$$

$$\left[ \begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} x_2 = 900 - x_3 \quad (5) \\ x_2 = 900 - x_3 - x_4 \quad (4) \\ x_3 = 100 - x_4 + x_5 \quad (3) \\ -x_4 = 800 - x_5 - x_6 \quad (2) \\ x_5 = 800 - x_7 \quad (1) \end{array}$$

Let  $x_6 = s$  &  $x_7 = t$

$$(1) \rightarrow x_5 = 800 - t$$

$$(2) \rightarrow x_4 = s - t$$

$$(3) \rightarrow x_3 = 900 - s$$

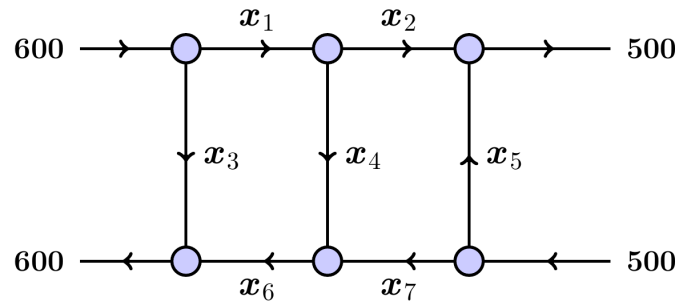
$$(2) \rightarrow x_2 = t$$

$$(1) \rightarrow x_2 = s$$

**∴ Solution:**  $(s, t, 900 - s, s - t, 800 - t, s, t)$

**Exercise**

Water is flowing through a network of pipes (in thousands of cubic meters per hour)



- Solve this system for the water flow represented by  $x_i$ ,  $i = 1, 2, \dots, 7$ .
- Find the water flow when  $x_1 = x_2 = 100$
- Find the water flow when  $x_6 = x_7 = 0$
- Find the water flow when  $x_5 = 1000$  and  $x_6 = 0$

**Solution**

$$a) \quad x_1 + x_3 = 600$$

$$x_1 = x_2 + x_4$$

$$x_2 + x_5 = 500$$

$$x_5 + x_7 = 500$$

$$x_6 = x_4 + x_7$$

$$x_3 + x_6 = 600$$

$$\begin{cases} x_1 + x_3 = 600 \\ x_1 - x_2 - x_4 = 0 \\ x_2 + x_5 = 500 \\ x_5 + x_7 = 500 \\ x_6 - x_4 - x_7 = 0 \\ x_3 + x_6 = 600 \end{cases}$$

$$\left[ \begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 600 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 500 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 600 \end{array} \right] \quad R_2 - R_1$$

$$\left[ \begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 600 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & -600 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 500 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 600 \end{array} \right] \quad \begin{array}{l} \\ R_3 + R_2 \\ \\ \\ \end{array}$$

$$\left[ \begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 600 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & -600 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 600 \end{array} \right] \quad \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \\ \\ \\ R_6 + R_3 \end{array}$$

$$\left[ \begin{array}{ccccccc|c} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 700 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & -500 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 500 \end{array} \right] \quad \begin{array}{l} -R_2 \\ -R_3 \\ R_4 \\ R_5 \end{array}$$

$$\left[ \begin{array}{ccccccc|c} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 700 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 500 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 100 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 500 \end{array} \right] \quad \begin{array}{l} R_1 - R_4 \\ \\ R_3 - R_4 \\ \\ R_6 + R_4 \end{array}$$

$$\left[ \begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 1 & 1 & -1 & 700 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 500 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 100 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \end{array} \right] \quad \begin{array}{l} R_1 - R_5 \\ R_2 - R_5 \\ R_3 + R_5 \\ \\ R_6 - R_5 \end{array}$$

$$\left[ \begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 200 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 600 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} \rightarrow x_1 = 200 - x_6 \\ \rightarrow x_2 = x_7 \\ \rightarrow x_3 = 600 - x_6 \\ \rightarrow x_4 = x_6 - x_7 \\ \rightarrow x_5 = 500 - x_7 \end{array}$$

$$\therefore \text{Solution: } \underline{(200 - x_6, \ x_7, \ 300 - x_6, \ x_6 - x_7, \ 500 - x_7, \ x_6, \ x_7)} \mid$$

b) **Given:**  $x_1 = x_2 = 100$

$$\begin{cases} 100 + x_3 = 600 \\ 100 - 100 - x_4 = 0 \\ 100 + x_5 = 500 \end{cases}$$

$$\begin{cases} \underline{x_3 = 500} \\ \underline{x_4 = 0} \\ \underline{x_5 = 400} \end{cases}$$

$$\begin{cases} 400 + x_7 = 500 \\ x_6 - x_7 = 0 \\ 500 + x_6 = 600 \end{cases}$$

$$\begin{cases} \underline{x_7 = 100} \\ \underline{x_6 = 100} \end{cases}$$

$$\therefore \text{Solution: } \underline{(100, \ 100, \ 500, \ 0, \ 400, \ 100, \ 100)} \mid$$

c) **Given:**  $x_6 = x_7 = 0$

$$\begin{cases} \underline{x_5 = 500} \\ \underline{x_4 = 0} \\ \underline{x_3 = 600} \end{cases}$$

$$\begin{cases} x_1 + x_3 = 600 \\ x_1 - x_2 = 0 \\ x_2 + 500 = 500 \end{cases}$$

$$\begin{cases} \underline{x_2 = 0} \\ \underline{x_1 = 0} \end{cases}$$

$$\therefore \text{Solution: } \underline{(0, \ 0, \ 600, \ 0, \ 500, \ 0, \ 0)} \mid$$

d) **Given:**  $x_5 = 1000$  and  $x_6 = 0$

$$\begin{cases} x_1 + x_3 = 600 \\ x_1 - x_2 - x_4 = 0 \\ x_2 + 1000 = 500 \\ 1000 + x_7 = 500 \\ x_4 = -x_7 \\ x_3 = 600 \end{cases}$$

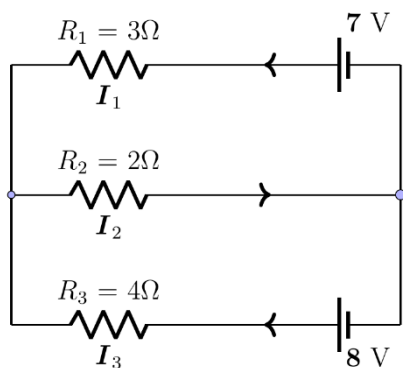
$$\begin{cases} x_2 = -500 \\ x_7 = -500 \\ x_4 = 500 \end{cases}$$

$$\begin{cases} x_3 = 600 \\ x_1 = 0 \end{cases}$$

**∴ Solution:**  $(0, -500, 600, 500, 1000, 0, -500)$

### Exercise

Determine the currents  $I_1$ ,  $I_2$ , and  $I_3$  for the electrical network shown below



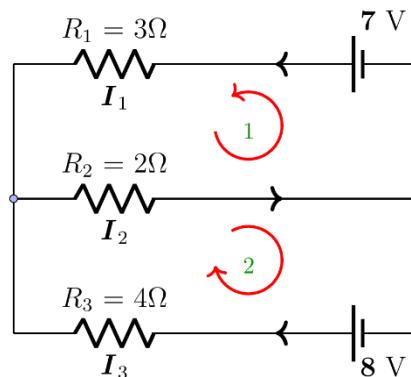
### Solution

$$I_2 = I_1 + I_3$$

$$3I_1 + 2I_2 = 7$$

$$2I_2 + 4I_3 = 8$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 3I_1 + 2I_2 = 7 \\ I_2 + 2I_3 = 4 \end{cases}$$



$$D = \begin{vmatrix} 1 & -1 & 1 \\ 3 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 13 \qquad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 7 & 2 & 0 \\ 4 & 1 & 2 \end{vmatrix} = 13$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 7 & 0 \\ 0 & 4 & 2 \end{vmatrix} = 26 \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 3 & 2 & 7 \\ 0 & 1 & 4 \end{vmatrix} = 13$$

**∴ Solution:**  $\underline{I_1 = 1 \text{ A}} \quad \underline{I_2 = 2 \text{ A}} \quad \underline{I_3 = 1 \text{ A}}$

**OR**

$$\left( \begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 3 & 2 & 0 & 7 \\ 0 & 1 & 2 & 4 \end{array} \right) \quad R_2 - 3R_1$$

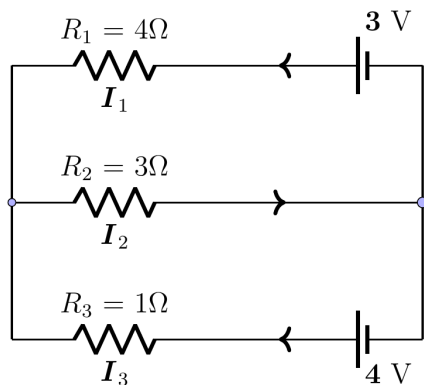
$$\left( \begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -3 & 7 \\ 0 & 1 & 2 & 4 \end{array} \right) \quad -5R_3 + R_2$$

$$\left( \begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -3 & 7 \\ 0 & 0 & -13 & -13 \end{array} \right) \quad \begin{array}{l} I_1 = I_2 - I_3 \\ 5I_2 = 3I_3 + 7 \\ \underline{I_3 = 1} \end{array}$$

$$\underline{I_2 = 2} \quad \underline{I_1 = 1}$$

### Exercise

Determine the currents  $I_1$ ,  $I_2$ , and  $I_3$  for the electrical network shown below

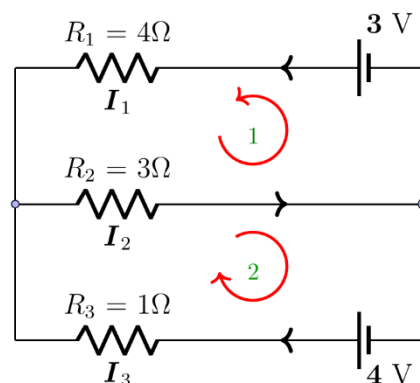


### Solution

$$I_2 = I_1 + I_3$$

$$4I_1 + 3I_2 = 3$$

$$3I_2 + I_3 = 4$$



$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 4I_1 + 3I_2 = 3 \\ 3I_2 + I_3 = 4 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 4 & 3 & 0 \\ 0 & 3 & 1 \end{vmatrix} = 19$$

$$D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 3 & 3 & 0 \\ 4 & 3 & 1 \end{vmatrix} = 0$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 4 & 3 & 0 \\ 0 & 4 & 1 \end{vmatrix} = 19$$

$$D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 4 & 3 & 3 \\ 0 & 3 & 4 \end{vmatrix} = 19$$

$$\therefore \text{Solution: } \quad \underline{I_1 = 0 \quad A} \quad \underline{I_2 = 1 \quad A} \quad \underline{I_3 = 1 \quad A}$$

**OR**

$$\left( \begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 4 & 3 & 0 & 3 \\ 0 & 3 & 1 & 4 \end{array} \right) \quad R_2 - 4R_1$$

$$\left( \begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 3 \\ 0 & 3 & 1 & 4 \end{array} \right) \quad 7R_3 - 3R_2$$

$$\left( \begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 3 \\ 0 & 0 & 19 & 19 \end{array} \right) \quad \begin{array}{l} \rightarrow I_1 = I_2 - I_3 \quad (2) \\ \rightarrow 7I_2 = 4I_3 + 3 \quad (1) \end{array}$$

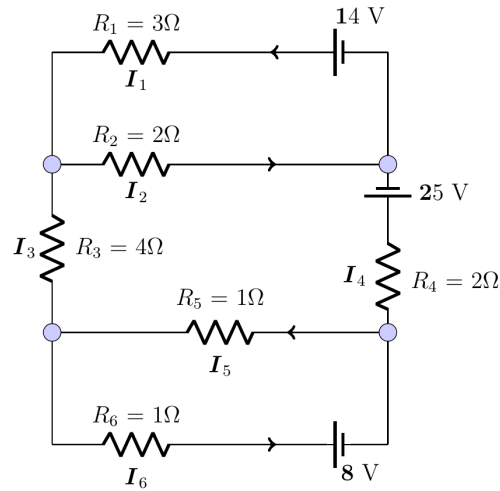
$$\underline{I_3 = 1}$$

$$\underline{I_2 = 1} \quad \underline{I_1 = 0}$$



**Exercise**

Determine the currents  $I_1$ ,  $I_2$ ,  $I_3$ ,  $I_4$ ,  $I_5$ , and  $I_6$  for the electrical network shown below

**Solution**

$$I_1 + I_3 = I_2 \rightarrow I_1 - I_2 + I_3 = 0$$

$$I_1 + I_4 = I_2 \rightarrow I_1 - I_2 + I_4 = 0$$

$$I_3 + I_6 = I_5 \rightarrow I_3 - I_5 + I_6 = 0$$

$$\begin{cases} 3I_1 + 2I_2 = 14 \\ 2I_2 + 4I_3 + I_5 + 2I_4 = 25 \\ I_5 + I_6 = 8 \end{cases}$$

$$\left[ \begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array} \right] \quad \begin{array}{l} R_2 - R_1 \\ R_4 - 3R_1 \end{array}$$

$$\left[ \begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array} \right] \quad \begin{array}{l} R_4 \\ R_2 \\ R_3 \end{array}$$

$$\left[ \begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array} \right] \quad \begin{array}{l} 5R_3 - 2R_2 \\ \\ R_5 + R_4 \end{array}$$

$$\left[ \begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array} \right] \quad 26R_4 + R_3$$

$$\left[ \begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & 0 & 36 & 5 & 0 & 97 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array} \right] \quad 36R_5 - R_4$$

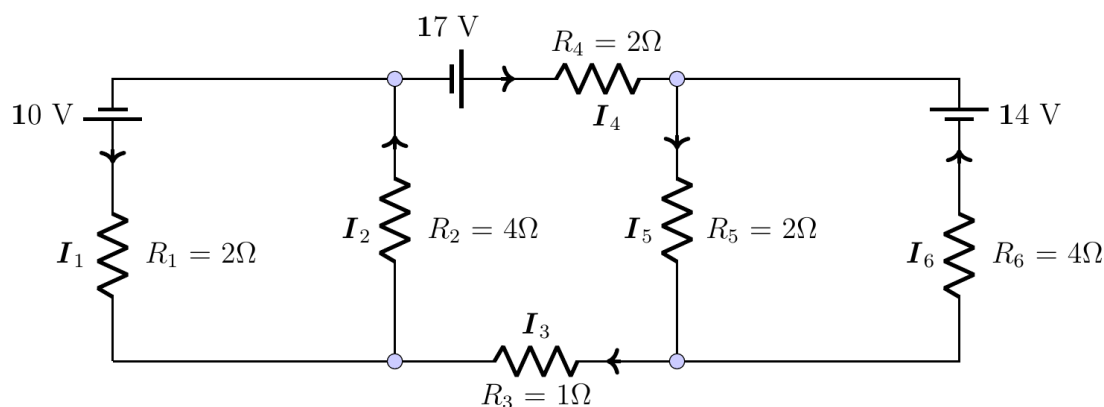
$$\left[ \begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & 0 & 36 & 5 & 0 & 97 \\ 0 & 0 & 0 & 0 & -41 & 36 & -97 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array} \right] \quad 41R_6 + R_5$$

$$\left[ \begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & 0 & 36 & 5 & 0 & 97 \\ 0 & 0 & 0 & 0 & -41 & 36 & -97 \\ 0 & 0 & 0 & 0 & 0 & 77 & 231 \end{array} \right] \quad \begin{array}{ll} I_1 = 4 - 2 & \rightarrow \underline{I_1 = 2} \\ 5I_2 = 14 + 3(2) & \rightarrow \underline{I_2 = 4} \\ 26I_3 = 97 - 10(2) - 5(5) & \rightarrow \underline{I_3 = 2} \\ 36I_4 = 97 - 5(5) & \rightarrow \underline{I_4 = 2} \\ -41I_5 = -97 - 36(3) & \rightarrow \underline{I_5 = 5} \\ 77I_6 = 231 & \rightarrow \underline{I_6 = 3} \end{array}$$

**∴ Solution:**  $\underline{(2, 4, 2, 2, 5, 3)}$

**Exercise**

Determine the currents  $I_1$ ,  $I_2$ ,  $I_3$ ,  $I_4$ ,  $I_5$ , and  $I_6$  for the electrical network shown below

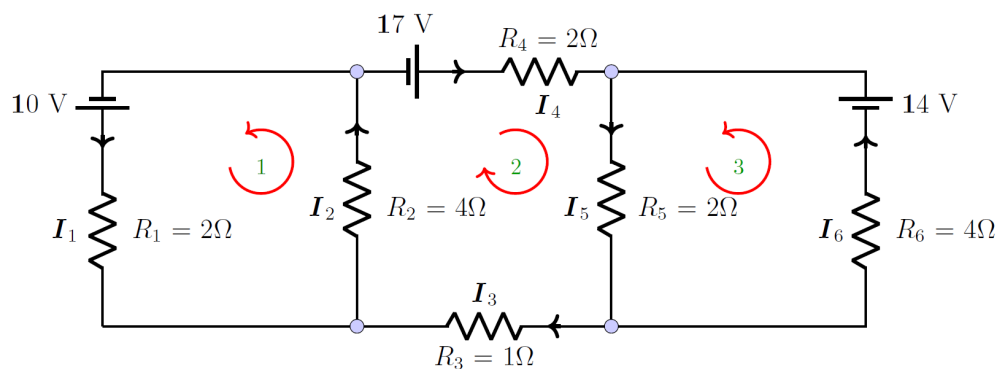
**Solution**

$$1 \rightarrow I_1 + I_3 = I_2$$

$$2 \rightarrow I_1 + I_4 = I_2$$

$$3 \rightarrow I_3 + I_6 = I_5$$

$$4 \rightarrow I_4 + I_6 = I_5$$



$$\left\{ \begin{array}{l} I_1 - I_2 + I_3 = 0 \\ I_1 - I_2 + I_4 = 0 \\ I_3 - I_5 + I_6 = 0 \\ I_4 - I_5 + I_6 = 0 \\ 2I_1 + 4I_2 = 10 \\ 4I_2 + I_3 + 2I_4 + 2I_5 = 17 \\ 2I_5 + 4I_6 = 14 \end{array} \right.$$



$$\left( \begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 19 & -13 & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad 19R_6 - R_5$$

$$\left( \begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 19 & -13 & 31 \\ 0 & 0 & 0 & 0 & 0 & 51 & 102 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} (4) \\ (3) \\ (2) \\ (1) \\ I_5 = \frac{1}{19}(31 + 13I_6) \\ \underline{I_6 = 2} \end{array}$$

$$\underline{I_5 = 3}$$

$$(1) \rightarrow \underline{I_4 = I_5 - I_6 = 1}$$

$$(2) \rightarrow \underline{I_3 = I_4 = 1}$$

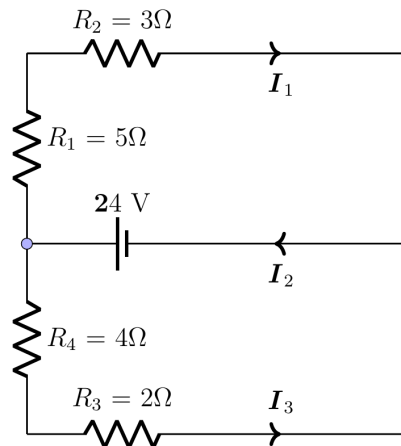
$$(3) \rightarrow \underline{I_2 = \frac{1}{3}(I_3 + 5) = 2}$$

$$(4) \rightarrow \underline{I_1 = I_2 - I_3 = 1}$$

$$\therefore \textbf{Solution:} \quad \underline{(1, \ 2, \ 1, \ 1, \ 3)}$$

**Exercise**

Determine the currents  $I_1$ ,  $I_2$ , and  $I_3$  for the electrical network shown below

**Solution**

$$\begin{cases} I_1 + I_3 = I_2 \\ 5I_1 + 3I_1 + I_2 = 24 \\ 4I_3 + 2I_3 + I_2 = 24 \end{cases}$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 8I_1 + I_2 = 24 \\ I_2 + 6I_3 = 24 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 8 & 1 & 0 \\ 0 & 1 & 6 \end{vmatrix} = 62$$

$$D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 24 & 1 & 0 \\ 24 & 1 & 6 \end{vmatrix} = 144$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 8 & 24 & 0 \\ 0 & 24 & 6 \end{vmatrix} = 336$$

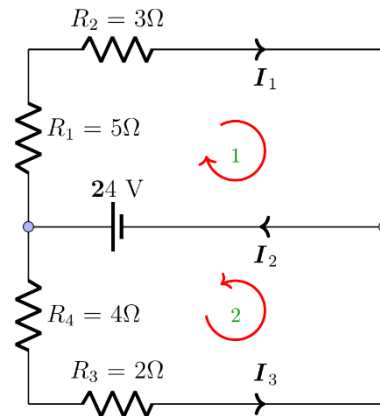
$$D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 8 & 1 & 24 \\ 0 & 1 & 24 \end{vmatrix} = 192$$

$$I_1 = \frac{144}{62} = \frac{72}{31}$$

$$I_2 = \frac{336}{62} = \frac{168}{31}$$

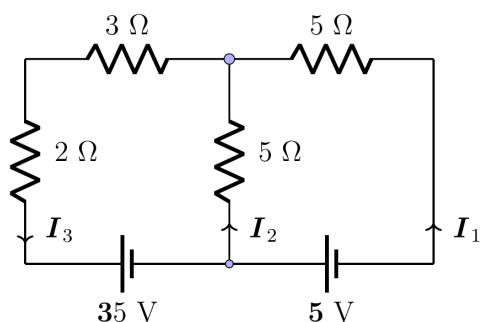
$$I_3 = \frac{192}{62} = \frac{96}{31}$$

$$\therefore \text{Solution: } \underline{I_1 = \frac{72}{31} \text{ A}} \quad \underline{I_2 = \frac{168}{31} \text{ A}} \quad \underline{I_3 = \frac{96}{31} \text{ A}}$$



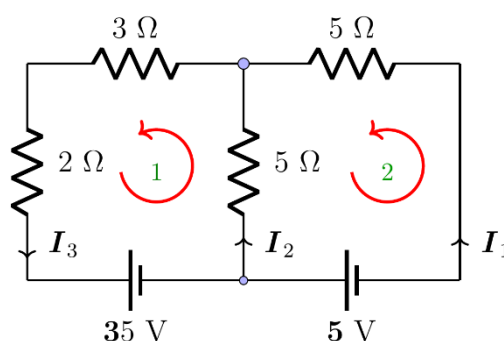
**Exercise**

Determine the currents  $I_1$ ,  $I_2$ , and  $I_3$  for the electrical network shown below

**Solution**

$$\begin{aligned} (2) & \rightarrow \begin{cases} I_1 + I_2 = I_3 \\ 5I_1 - 5I_2 = 5 \end{cases} \\ (1) & \rightarrow 5I_2 + 3I_3 + 2I_3 = 35 \end{aligned}$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 8I_1 + I_2 = 24 \\ I_2 + 6I_3 = 24 \end{cases}$$



$$D = \begin{vmatrix} 1 & -1 & 1 \\ 8 & 1 & 0 \\ 0 & 1 & 6 \end{vmatrix} = 62$$

$$D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 24 & 1 & 0 \\ 24 & 1 & 6 \end{vmatrix} = 144$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 8 & 24 & 0 \\ 0 & 24 & 6 \end{vmatrix} = 336$$

$$D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 8 & 1 & 24 \\ 0 & 1 & 24 \end{vmatrix} = 192$$

$$I_1 = \frac{144}{62} = \frac{72}{31}$$

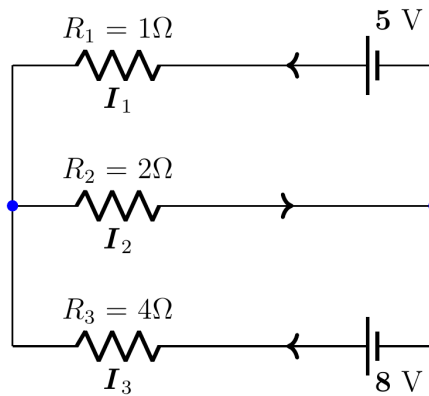
$$I_2 = \frac{336}{62} = \frac{168}{31}$$

$$I_3 = \frac{192}{62} = \frac{96}{31}$$

$$\therefore \text{Solution: } \boxed{I_1 = \frac{72}{31} \text{ A}} \quad \boxed{I_2 = \frac{168}{31} \text{ A}} \quad \boxed{I_3 = \frac{96}{31} \text{ A}}$$

**Exercise**

Determine the currents  $I_1$ ,  $I_2$ , and  $I_3$  for the electrical network shown below



How is the result affected when the 5 V is changed to 2 V and 8 V to 6 V?

**Solution**

$$\begin{aligned} (1) &\rightarrow \begin{cases} I_1 + I_3 = I_2 \\ I_1 + I_2 = 5 \end{cases} \\ (2) &\rightarrow \begin{cases} 2I_2 + 4I_3 = 8 \end{cases} \end{aligned}$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ I_1 + I_2 = 5 \\ 2I_2 + 4I_3 = 8 \end{cases}$$

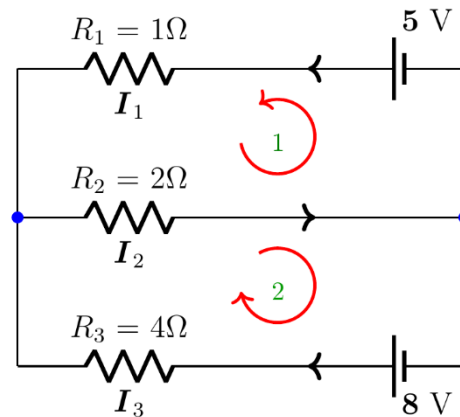
$$D = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 4 \end{vmatrix} = 10$$

$$D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 5 & 1 & 0 \\ 8 & 2 & 4 \end{vmatrix} = 22$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 5 & 0 \\ 0 & 8 & 4 \end{vmatrix} = 28$$

$$D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 5 \\ 0 & 2 & 8 \end{vmatrix} = 6$$

$$\therefore \text{Solution: } \underline{I_1 = \frac{11}{5} \text{ A}} \quad \underline{I_2 = \frac{19}{5} \text{ A}} \quad \underline{I_3 = \frac{3}{5} \text{ A}}$$



$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ I_1 + I_2 = 2 \\ 2I_2 + 4I_3 = 6 \end{cases}$$

$$D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 2 & 1 & 0 \\ 6 & 2 & 4 \end{vmatrix} = 6$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 6 & 4 \end{vmatrix} = 14$$

$$D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 6 \end{vmatrix} = 8$$

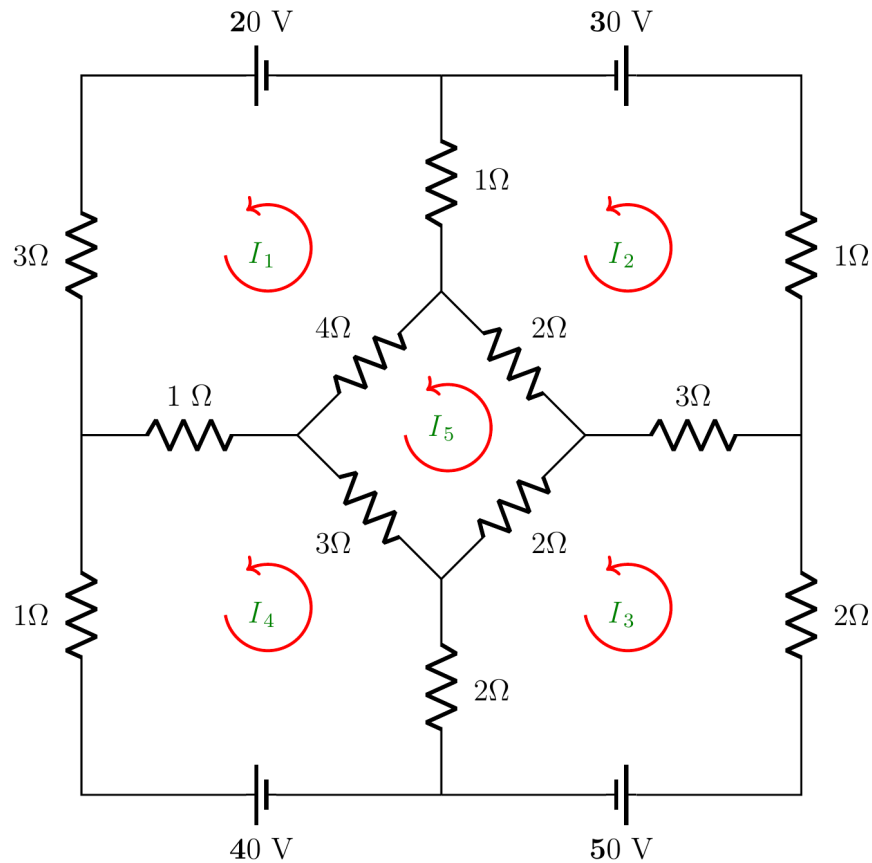


$$\therefore \text{Solution: } \boxed{I_1 = \frac{3}{5} \text{ A}} \quad \boxed{I_2 = \frac{7}{5} \text{ A}} \quad \boxed{I_3 = \frac{4}{5} \text{ A}}$$

### Exercise

Determine the currents for the electrical network

### Solution



$$\begin{cases} 9I_1 - I_2 - I_3 - 4I_5 = 20 \\ 7I_2 - I_1 - 3I_3 - 2I_5 = -30 \\ 9I_3 - 3I_2 - 2I_4 - 2I_5 = 50 \\ 7I_4 - I_1 - 2I_3 - 3I_5 = -40 \\ 11I_5 - 4I_1 - 2I_2 - 2I_3 - 3I_4 = 0 \end{cases}$$

$$\begin{cases} 9I_1 - I_2 - I_3 - 4I_5 = 20 \\ I_1 - 7I_2 + 3I_3 + 2I_5 = 30 \\ -3I_2 + 9I_3 - 2I_4 - 2I_5 = 50 \\ I_1 + 2I_3 - 7I_4 + 3I_5 = 40 \\ 4I_1 + 2I_2 + 2I_3 + 3I_4 - 11I_5 = 0 \end{cases}$$

$$\left[ \begin{array}{ccccc|c} 9 & -1 & -1 & 0 & -4 & 20 \\ 1 & -7 & 3 & 0 & 2 & 30 \\ 0 & -3 & 9 & -2 & -2 & 50 \\ 1 & 0 & 2 & -7 & 3 & 40 \\ 4 & 2 & 2 & 3 & -11 & 0 \end{array} \right] \quad \begin{array}{l} 9R_2 - R_1 \\ 9R_4 - R_1 \\ 9R_5 - 4R_1 \end{array}$$

$$\left[ \begin{array}{ccccc|c} 9 & -1 & -1 & 0 & -4 & 20 \\ 0 & -62 & 28 & 0 & 22 & 250 \\ 0 & -3 & 9 & -2 & -2 & 50 \\ 0 & 1 & 19 & -63 & 31 & 340 \\ 0 & 22 & 22 & 27 & -83 & -80 \end{array} \right] \quad -\frac{1}{2}R_2$$

$$\left[ \begin{array}{ccccc|c} 9 & -1 & -1 & 0 & -4 & 20 \\ 0 & 31 & -14 & 0 & -11 & -125 \\ 0 & -3 & 9 & -2 & -2 & 50 \\ 0 & 1 & 19 & -63 & 31 & 340 \\ 0 & 22 & 22 & 27 & -83 & -80 \end{array} \right] \quad \begin{array}{l} 31R_1 + R_2 \\ 31R_3 + 3R_2 \\ 31R_4 - R_2 \\ 31R_5 - 22R_2 \end{array}$$

$$\left[ \begin{array}{ccccc|c} 279 & 0 & -45 & 0 & 135 & 495 \\ 0 & 31 & -14 & 0 & -11 & -125 \\ 0 & 0 & 237 & -62 & -95 & 1,175 \\ 0 & 0 & 603 & -1,953 & 972 & 10,665 \\ 0 & 0 & 990 & 837 & -2,331 & 270 \end{array} \right] \quad \begin{array}{l} \frac{1}{9}R_1 \\ \frac{1}{9}R_4 \\ \frac{1}{9}R_5 \end{array}$$

$$\left[ \begin{array}{ccccc|c} 31 & 0 & -5 & 0 & 15 & 55 \\ 0 & 31 & -14 & 0 & -11 & -125 \\ 0 & 0 & 237 & -62 & -95 & 1,175 \\ 0 & 0 & 67 & -217 & 108 & 1,185 \\ 0 & 0 & 110 & 93 & -259 & 30 \end{array} \right] \quad \begin{array}{l} 237R_4 - 67R_3 \\ 237R_5 - 110R_3 \end{array}$$

$$\left[ \begin{array}{ccccc|c} 31 & 0 & -5 & 0 & 15 & 55 \\ 0 & 31 & -14 & 0 & -11 & -125 \\ 0 & 0 & 237 & -62 & -95 & 1,175 \\ 0 & 0 & 0 & -47,275 & 31,961 & 202,120 \\ 0 & 0 & 0 & 28,861 & -50,933 & -122,140 \end{array} \right] \quad 47275R_5 + 28861R_4$$

$$\left[ \begin{array}{ccccc|c} 31 & 0 & -5 & 0 & 15 & 55 \\ 0 & 31 & -14 & 0 & -11 & -125 \\ 0 & 0 & 237 & -62 & -95 & 1,175 \\ 0 & 0 & 0 & -47,275 & 31,961 & 202,120 \\ 0 & 0 & 0 & 0 & -1,484,514,360 & 59,216,820 \end{array} \right] \quad \begin{array}{l} 31I_1 = 55 - 15I_5 + 5I_3 \\ 31I_2 = -125 + 11I_5 + 14I_3 \\ 237I_3 = 1175 + 95I_5 + 62I_4 \\ -47275I_4 = 202120 - 31961I_5 \end{array}$$

$$\begin{aligned} I_5 &= -\frac{59,216,820}{1,484,514,360} \\ &= -\frac{986,947}{24,741,906} \\ &\approx -0.04 \text{ A} \end{aligned}$$

$$\begin{aligned} I_4 &= \frac{1}{47,275} \left( \frac{986,947}{24,741,906} - 202,120 \right) \\ &= -\frac{1}{47,275} \left( \frac{5,000,833,053,773}{24,741,906} \right) \\ &\approx -4.3 \text{ A} \end{aligned}$$

$$\begin{aligned} I_3 &= \frac{1}{237} (1175 + 95(-.04) + 62(-4.3)) \\ &\approx 3.82 \text{ A} \end{aligned}$$

$$\begin{aligned} I_2 &= \frac{1}{31} (-125 + 11(-.04) + 14(3.82)) \\ &\approx -2.323 \text{ A} \end{aligned}$$

$$\begin{aligned} I_1 &= \frac{1}{31} (55 - 15(-.04) + 5(3.82)) \\ &\approx 2.37 \text{ A} \end{aligned}$$

### Exercise

Use  $A^{-1}$  to decode the cryptogram

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$$

11 21 64 112 25 50 29 53 23 46 40 75 55 92

### Solution

$$\begin{aligned} A^{-1} &= -\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} \end{aligned}$$

$$\begin{bmatrix} 11 & 21 \end{bmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{bmatrix} 8 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 64 & 112 \end{bmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{bmatrix} 16 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 50 \end{bmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{bmatrix} 25 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 29 & 53 \end{bmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{bmatrix} 14 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 23 & 46 \end{bmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{bmatrix} 23 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 40 & 75 \end{bmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{bmatrix} 25 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 55 & 92 \end{bmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{bmatrix} 1 & 18 \end{bmatrix}$$

8   1   16   16   28   00   14   5   23   0   25   5   1   18  
*H A P P Y \_ N E W \_ Y E A R*

∴ **Solution:**     *Happy New Year*

### Exercise

Use  $A^{-1}$  to decode the cryptogram

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{pmatrix}$$

13   19   10   -1   -33   -77   3   -2   -14   4   1   -9   -5   -25   -47   4   1   -9

### Solution

$$A^{-1} = \begin{pmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{pmatrix}$$

$$\begin{bmatrix} 13 & 19 & 10 \end{bmatrix} \begin{pmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{pmatrix} = \begin{bmatrix} 9 & 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -33 & -77 \end{bmatrix} \begin{pmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{pmatrix} = \begin{bmatrix} 2 & 5 & 18 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 & -14 \end{bmatrix} \begin{pmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{pmatrix} = \begin{bmatrix} 7 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & -9 \end{bmatrix} \begin{pmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{pmatrix} = \begin{bmatrix} 5 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -5 & -25 & -47 \end{bmatrix} \begin{pmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & -9 \end{bmatrix} \begin{pmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{pmatrix} = \begin{bmatrix} 5 & 1 & 4 \end{bmatrix}$$

9 3 5 2 5 18 7 0 4 5 1 4 0 1 8 5 1 4  
 I C E B E R G \_ D E A D \_ A H E A D

**∴ Solution:**     *Iceberg Dead Ahead*

### Exercise

Consider the invertible matrix:  $A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{pmatrix}$

The message: **ICEBERG DEAD AHEAD**

- Write the uncoded row matrices  $1 \times 3$  for the message.
- Use the matrix  $A$  to encode the message.
- Decode a message from part b) given the matrix  $A$ .

### Solution

a)

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

$$\begin{array}{cccccc}
 I & C & E & & B & E & R & & G & & D & & E & A & D & & A & H & & E & A & D \\
 [9 & 3 & 5] & [2 & 5 & 18] & [7 & 0 & 4] & [5 & 1 & 4] & [0 & 1 & 8] & [5 & 1 & 4]
 \end{array}$$

b) Let encode the message **ICEBERG DEAD AHEAD**

$$[9 \ 3 \ 5] \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = [3 \ 29 \ 80]$$

$$[2 \ 5 \ 18] \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = [-37 \ 3 \ 175]$$

$$[7 \ 0 \ 4] \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = [-5 \ 6 \ 42]$$

$$[5 \ 1 \ 4] \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = [-4 \ 9 \ 47]$$

$$[0 \ 1 \ 8] \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = [-21 \ -5 \ 65]$$

$$[5 \ 1 \ 4] \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = [-4 \ 9 \ 47]$$

The sequence of coded row matrices is

$$[3 \ 29 \ 80] \quad [-37 \ 3 \ 175] \quad [-5 \ 6 \ 42] \quad [-4 \ 9 \ 47] \quad [-21 \ -9 \ 65] \quad [-4 \ 9 \ 47]$$

The cryptogram:

$$3 \ 29 \ 80 \ -37 \ 3 \ 175 \ -5 \ 6 \ 42 \ -4 \ 9 \ 47 \ -21 \ -9 \ 65 \ -4 \ 9 \ 47$$

c) To decode a message given the matrix  $A$ .

$$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{vmatrix} = 1$$

$$A^{-1} = \begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix}$$

With the cryptogram:

$$\begin{bmatrix} 3 & 29 & 80 \end{bmatrix} \begin{bmatrix} -37 & 3 & 175 \end{bmatrix} \begin{bmatrix} -5 & 6 & 42 \end{bmatrix} \begin{bmatrix} -4 & 9 & 47 \end{bmatrix} \begin{bmatrix} -21 & -9 & 65 \end{bmatrix} \begin{bmatrix} -4 & 9 & 47 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 29 & 80 \end{bmatrix} \begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} -37 & 3 & 175 \end{bmatrix} \begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 18 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 6 & 42 \end{bmatrix} \begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 9 & 47 \end{bmatrix} \begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -21 & -9 & 65 \end{bmatrix} \begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 8 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 9 & 47 \end{bmatrix} \begin{bmatrix} 67 & -18 & 4 \\ -48 & 13 & -3 \\ 15 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 4 \end{bmatrix}$$

The message is:

9 3 5 2 5 18 7 0 1 5 1 4 0 1 8 5 1 4  
*I C E B E R G \_ D E A D \_ A H E A D*

### Exercise

You want to send the message: **LINEAR ALGEBRA** with a key word **MATH**

- Write the matrix  $A$ .
- Write the uncoded row matrices  $1 \times 2$  for the message.
- Use the matrix  $A$  to encode the message.
- Decode a message from part  $b$ ) given the matrix  $A$ .

### Solution

a)

$$0 = \_ \quad | \quad 4 = D \quad | \quad 8 = H \quad | \quad 12 = L \quad | \quad 16 = P \quad | \quad 20 = T \quad | \quad 24 = X$$

1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	
<i>M</i>	<i>A</i>	<i>T</i>	<i>H</i>			
13	1	20	8			

$$A = \begin{pmatrix} 13 & 1 \\ 20 & 8 \end{pmatrix}$$

b)

<i>L</i>	<i>I</i>	<i>N</i>	<i>E</i>	<i>A</i>	<i>R</i>	<i>_</i>	<i>A</i>	<i>L</i>	<i>G</i>	<i>E</i>	<i>B</i>	<i>R</i>	<i>A</i>
12	9	14	5	1	18	0	1	12	7	5	2	18	1

$$\begin{bmatrix} 12 & 9 \end{bmatrix} \quad \begin{bmatrix} 14 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 18 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 12 & 7 \end{bmatrix} \quad \begin{bmatrix} 5 & 2 \end{bmatrix} \quad \begin{bmatrix} 18 & 1 \end{bmatrix}$$

c) Encoding the message

$$\begin{bmatrix} 12 & 9 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 336 & 84 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 5 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 282 & 54 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 18 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 373 & 145 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 20 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 7 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 296 & 68 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 2 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 105 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 18 & 1 \end{bmatrix} \begin{bmatrix} 13 & 1 \\ 20 & 8 \end{bmatrix} = \begin{bmatrix} 254 & 26 \end{bmatrix}$$

The cryptogram:

$$336 \ 84 \ 282 \ 54 \ 373 \ 145 \ 20 \ 8 \ 296 \ 68 \ 105 \ 21 \ 254 \ 26$$

d) To decode a message given the matrix  $A$ .

$$A = \begin{pmatrix} 13 & 1 \\ 20 & 8 \end{pmatrix}$$



$$A^{-1} = \frac{1}{84} \begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{pmatrix}$$

With the cryptogram:

$$\begin{bmatrix} 336 & 84 \end{bmatrix} \begin{bmatrix} 282 & 54 \end{bmatrix} \begin{bmatrix} 373 & 145 \end{bmatrix} \begin{bmatrix} 20 & 8 \end{bmatrix} \begin{bmatrix} 296 & 68 \end{bmatrix} \begin{bmatrix} 105 & 21 \end{bmatrix} \begin{bmatrix} 254 & 26 \end{bmatrix}$$

$$\begin{bmatrix} 336 & 84 \end{bmatrix} \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = \begin{bmatrix} 12 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 282 & 54 \end{bmatrix} \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = \begin{bmatrix} 14 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 373 & 145 \end{bmatrix} \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = \begin{bmatrix} 1 & 18 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 8 \end{bmatrix} \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 296 & 68 \end{bmatrix} \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = \begin{bmatrix} 12 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 105 & 21 \end{bmatrix} \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = \begin{bmatrix} 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 254 & 26 \end{bmatrix} \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = \begin{bmatrix} 18 & 1 \end{bmatrix}$$

12 9 14 5 1 18 0 1 12 7 5 2 18 1  
*L I N E A R \_ A L G E B R A*

The message is: *Linear Algebra*

**Exercise**

You want to send the message: **CRYPTOGRAPHY IS A METHOD OF PROTECTING INFORMATION** with a key word **CODE**

- Write the matrix  $A$ .
- Write the uncoded row matrices  $1 \times 2$  for the message.
- Use the matrix  $A$  to encode the message.
- Decode a message from part b) given the matrix  $A$ .

**Solution**

a)

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

C O D E  
3 15 4 5

$$A = \begin{pmatrix} 3 & 15 \\ 4 & 5 \end{pmatrix}$$

b)

C R Y P T O G R A P H Y \_ I S \_ A \_  
3 18 25 16 20 15 7 18 1 16 8 25 0 9 19 0 1 0

M E T H O D \_ O F \_ P R O T E C T I N G  
13 5 20 8 15 4 0 15 6 0 16 18 15 20 5 3 20 9 14 7

\_ I N F O R M A T I O N S \_  
0 9 14 6 15 18 13 1 20 9 15 14 19 0

[3 18] [25 16] [20 15] [7 18] [1 16] [8 25] [0 9] [19 0]

[1 0] [13 5] [20 8] [15 4] [0 15] [6 0] [16 18] [15 20]

[5 3] [20 9] [14 7] [0 9] [14 6] [15 18] [13 1] [20 9]

[15 14] [19 0]

c) Encoding the message

$$\begin{bmatrix} 3 & 18 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 81 & 135 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 16 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 139 & 455 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 15 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 120 & 375 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 18 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 93 & 195 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 16 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 67 & 95 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 25 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 124 & 245 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 9 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 36 & 45 \end{bmatrix}$$

$$\begin{bmatrix} 19 & 0 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 57 & 285 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 13 & 5 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 59 & 220 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 8 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 92 & 340 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 4 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 61 & 245 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 15 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 60 & 75 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 18 & 90 \end{bmatrix}$$

$$\begin{bmatrix} 16 & 18 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 120 & 330 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 20 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 125 & 325 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 27 & 90 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 9 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 96 & 345 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 7 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 70 & 245 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 9 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 36 & 45 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 6 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 66 & 240 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 18 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 117 & 315 \end{bmatrix}$$

$$\begin{bmatrix} 13 & 1 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 43 & 200 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 9 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 96 & 345 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 14 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 101 & 295 \end{bmatrix}$$

$$\begin{bmatrix} 19 & 0 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 57 & 285 \end{bmatrix}$$

The cryptogram:

81 135 139 455 120 375 93 195 67 95 124 245 36 45 57 285  
3 15 59 220 92 340 61 245 60 75 18 90 120 330 125 325  
27 90 96 345 70 245 36 45 66 240 117 315 43 200 96 345  
101 295 57 285

**d)** To decode a message given the matrix  $A$ .

$$A = \begin{pmatrix} 3 & 15 \\ 4 & 5 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{45} \begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix}$$

$$-\frac{1}{45} \begin{bmatrix} 81 & 135 \end{bmatrix} \begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45} \begin{bmatrix} -135 & -810 \end{bmatrix} \\ = \begin{bmatrix} 3 & 18 \end{bmatrix}$$

$$-\frac{1}{45}[139 \ 455]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-1,125 \ -720] \\ = [25 \ 16]$$

$$-\frac{1}{45}[120 \ 375]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-900 \ -675] \\ = [20 \ 15]$$

$$-\frac{1}{45}[93 \ 195]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-315 \ -810] \\ = [7 \ 18]$$

$$-\frac{1}{45}[67 \ 95]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-45 \ -720] \\ = [1 \ 16]$$

$$-\frac{1}{45}[124 \ 245]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-360 \ -1,125] \\ = [8 \ 25]$$

$$-\frac{1}{45}[36 \ 45]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[0 \ -405] \\ = [0 \ 9]$$

$$-\frac{1}{45}[57 \ 285]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-855 \ 0] \\ = [19 \ 0]$$

$$-\frac{1}{45}[3 \ 15]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-45 \ 0] \\ = [1 \ 0]$$

$$-\frac{1}{45}[59 \ 220]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-585 \ -225] \\ = [13 \ 5]$$

$$-\frac{1}{45}[92 \ 340]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-900 \ -360] \\ = [20 \ 8]$$

$$-\frac{1}{45}[61 \ 245]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-675 \ -180]$$

$$=[15 \ 4]$$

$$-\frac{1}{45}[60 \ 75]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[0 \ -675] \\ = [0 \ 15]$$

$$-\frac{1}{45}[18 \ 90]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-270 \ 0] \\ = [6 \ 0]$$

$$-\frac{1}{45}[120 \ 330]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-720 \ -810] \\ = [16 \ 18]$$

$$-\frac{1}{45}[125 \ 325]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-675 \ -900] \\ = [15 \ 20]$$

$$-\frac{1}{45}[27 \ 90]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-225 \ -135] \\ = [5 \ 3]$$

$$-\frac{1}{45}[96 \ 345]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-900 \ -405] \\ = [20 \ 9]$$

$$-\frac{1}{45}[70 \ 245]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-630 \ -315] \\ = [14 \ 7]$$

$$-\frac{1}{45}[36 \ 45]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[0 \ -405] \\ = [0 \ 9]$$

$$-\frac{1}{45}[66 \ 240]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-630 \ -270] \\ = [14 \ 6]$$

$$-\frac{1}{45}[117 \ 315]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-675 \ -810] \\ = [15 \ 18]$$

$$-\frac{1}{45}[43 \ 200]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-585 \ -45] \\ = [13 \ 1]$$

$$-\frac{1}{45}[96 \ 345]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-900 \ -405] \\ = [20 \ 9]$$

$$-\frac{1}{45}[101 \ 295]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-675 \ -630] \\ = [15 \ 14]$$

$$-\frac{1}{45}[57 \ 285]\begin{pmatrix} 5 & -15 \\ -4 & 3 \end{pmatrix} = -\frac{1}{45}[-855 \ 0] \\ = [19 \ 0]$$

3 18 25 16 20 15 7 18 1 16 8 25 0 9 19 0 1 0 13 5  
*C R Y P T O G R A P H Y \_ I S \_ A \_ M E*  
 20 8 15 4 0 15 6 0 16 18 15 20 5 3 20 9 14 7 0 9  
*T H O D \_ O F \_ P R O T E C T I N G \_ I*  
 14 6 15 18 16 1 20 9 15 14 19 0  
*N F O R M A T I O N S \_*

The message is: *Cryptography is a Method of Protecting Informations*

### Exercise

Write the matrix  $A$  with a key word *MATH*, then decode the cryptogram

117 9 456 132 386 62 260 104 413 161 104 8

### Solution

$M \ A \ T \ H$   
 13 1 20 8

$$A = \begin{pmatrix} 13 & 1 \\ 20 & 8 \end{pmatrix}$$

$$A^{-1} = \frac{1}{84} \begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix}$$

With the cryptogram:

$$[117 \ 9] \ [456 \ 132] \ [386 \ 62] \ [260 \ 104] \ [413 \ 161] \ [104 \ 8]$$

$$\begin{aligned} \frac{1}{84}[117 \ 9] \begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} &= \frac{1}{84}[756 \ 0] \\ &= [9 \ 0] \end{aligned}$$

$$\begin{aligned} \frac{1}{84}[456 \ 132] \begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} &= \frac{1}{84}[1,008 \ 1,260] \\ &= [12 \ 15] \end{aligned}$$

$$\begin{aligned} \frac{1}{84}[386 \ 62] \begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} &= \frac{1}{84}[1,848 \ 420] \\ &= [22 \ 5] \end{aligned}$$

$$\begin{aligned} \frac{1}{84}[260 \ 104] \begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} &= \frac{1}{84}[0 \ 1,092] \\ &= [0 \ 13] \end{aligned}$$

$$\begin{aligned} \frac{1}{84}[413 \ 161] \begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} &= \frac{1}{84}[84 \ 1,680] \\ &= [1 \ 20] \end{aligned}$$

$$\begin{aligned} \frac{1}{84}[104 \ 8] \begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} &= \frac{1}{84}[672 \ 0] \\ &= [8 \ 0] \end{aligned}$$

$$\begin{array}{cccccccccccc} 9 & 0 & 12 & 15 & 22 & 5 & 0 & 13 & 1 & 20 & 8 & 0 \\ I & - & L & O & V & E & - & M & A & T & H & - \end{array}$$

The message is: *I love math*

### Exercise

Write the matrix  $A$  with a key word **MATH**, then decode the cryptogram

$$438 \ 150 \ 145 \ 37 \ 240 \ 96 \ 635 \ 191 \ 445 \ 157 \ 260 \ 104 \ 413 \ 161 \ 104 \ 8$$

### Solution

$$\begin{array}{cccc} M & A & T & H \\ 13 & 1 & 20 & 8 \end{array}$$

$$A = \begin{pmatrix} 13 & 1 \\ 20 & 8 \end{pmatrix}$$



$$A^{-1} = \frac{1}{84} \begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix}$$

With the cryptogram:

$$\begin{array}{cccccc} [438 & 150] & [145 & 37] & [240 & 96] & [635 & 191] & [445 & 157] \\ [260 & 104] & [413 & 161] & [104 & 8] \end{array}$$

$$\begin{aligned} \frac{1}{84} [438 \quad 150] \begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} &= \frac{1}{84} [504 \quad 1,512] \\ &= [6 \quad 18] \end{aligned}$$

$$\begin{aligned} \frac{1}{84} [145 \quad 37] \begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} &= \frac{1}{84} [420 \quad 336] \\ &= [5 \quad 4] \end{aligned}$$

$$\begin{aligned} \frac{1}{84} [240 \quad 96] \begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} &= \frac{1}{84} [0 \quad 1,008] \\ &= [0 \quad 12] \end{aligned}$$

$$\begin{aligned} \frac{1}{84} [635 \quad 191] \begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} &= \frac{1}{84} [1,260 \quad 1,848] \\ &= [15 \quad 22] \end{aligned}$$

$$\begin{aligned} \frac{1}{84} [445 \quad 157] \begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} &= \frac{1}{84} [420 \quad 1,596] \\ &= [5 \quad 19] \end{aligned}$$

$$\begin{aligned} \frac{1}{84} [260 \quad 104] \begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} &= \frac{1}{84} [0 \quad 1,092] \\ &= [0 \quad 13] \end{aligned}$$

$$\begin{aligned} \frac{1}{84} [413 \quad 161] \begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} &= \frac{1}{84} [84 \quad 1,680] \\ &= [1 \quad 20] \end{aligned}$$

$$\begin{aligned} \frac{1}{84} [104 \quad 8] \begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} &= \frac{1}{84} [672 \quad 0] \\ &= [8 \quad 0] \end{aligned}$$

6 18 5 4 0 12 15 22 5 19 0 13 1 20 8 0  
F R E D - L O V E S - M A T H -

The message is: *Fred loves math*

**Exercise**

Consider the invertible matrix:  $A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix}$

Decode the cryptogram

$$\begin{array}{cccccccccc} 1 & -5 & 11 & 19 & -25 & -45 & 11 & -16 & -28 & 20 & -29 & -27 \\ 12 & -12 & -53 & 40 & -61 & -35 & 8 & -17 & 7 & & & \end{array}$$

**Solution**

$$|A| = 1$$

$$A^{-1} = \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix}$$

With the cryptogram:

$$\begin{array}{cccc} [1 & -5 & 11] & [19 & -25 & -45] & [11 & -16 & -28] & [20 & -29 & -27] \\ [12 & -12 & -53] & [40 & -61 & -35] & [8 & -17 & 7] & & & \end{array}$$

$$[1 \quad -5 \quad 11] \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = [4 \quad 9 \quad 6]$$

$$[19 \quad -25 \quad -45] \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = [6 \quad 5 \quad 18]$$

$$[11 \quad -16 \quad -28] \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = [5 \quad 14 \quad 20]$$

$$[20 \quad -29 \quad -27] \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = [9 \quad 1 \quad 12]$$

$$[12 \quad -12 \quad -53] \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = [0 \quad 5 \quad 17]$$

$$[40 \quad -61 \quad -35] \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = [21 \quad 1 \quad 20]$$

$$\begin{bmatrix} 8 & -17 & 7 \end{bmatrix} \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} = \begin{bmatrix} 9 & 15 & 14 \end{bmatrix}$$

4 9 6 6 5 18 5 14 20 9 1 12 0 5 17 21 1 20 9 15 14  
*D I F F E R E N T I A L \_ E Q U A T I O N*

The message is: *Differential Equation.*

### Exercise

Determine the key word, then decode the given cryptogram

6 18 5 4 15 13 1 20 8  
 102 649 238 57 324 112 128 622 207  
 180 613 290 102 360 259 151 580 297

*Hint:* First row is the key

### Solution

The key word from the first row is

6 18 5 4 15 13 1 20 8  
*f r e d o m a t h*

Since it has 9 numbers, then the matrix is  $9 = 3^2$  which is  $3 \times 3$

$$A = \begin{pmatrix} 6 & 18 & 5 \\ 4 & 15 & 13 \\ 1 & 20 & 8 \end{pmatrix}$$

$$|A| = -857$$

$$\begin{aligned} A^{-1} &= -\frac{1}{857} \begin{pmatrix} -140 & -44 & 159 \\ -19 & 43 & -58 \\ 65 & -102 & 18 \end{pmatrix} \\ &= \frac{1}{857} \begin{pmatrix} 140 & 44 & -159 \\ 19 & -43 & 58 \\ -65 & 102 & -18 \end{pmatrix} \end{aligned}$$

With the cryptogram:

$\begin{bmatrix} 102 & 649 & 238 \end{bmatrix}$   $\begin{bmatrix} 57 & 324 & 112 \end{bmatrix}$   $\begin{bmatrix} 128 & 622 & 207 \end{bmatrix}$

$\begin{bmatrix} 180 & 613 & 290 \end{bmatrix}$   $\begin{bmatrix} 102 & 360 & 259 \end{bmatrix}$   $\begin{bmatrix} 151 & 580 & 297 \end{bmatrix}$

$$\frac{1}{857}[102 \ 649 \ 238] \begin{pmatrix} 140 & 44 & -159 \\ 19 & -43 & 58 \\ -65 & 102 & -18 \end{pmatrix} = \frac{1}{857}[11,141 \ 857 \ 17,140]$$

$$=[13 \ 1 \ 20]$$

$$\frac{1}{857}[57 \ 324 \ 112] \begin{pmatrix} 140 & 44 & -159 \\ 19 & -43 & 58 \\ -65 & 102 & -18 \end{pmatrix} = \frac{1}{857}[6,856 \ 0 \ 7,713]$$

$$=[8 \ 0 \ 9]$$

$$\frac{1}{857}[128 \ 622 \ 207] \begin{pmatrix} 140 & 44 & -159 \\ 19 & -43 & 58 \\ -65 & 102 & -18 \end{pmatrix} = \frac{1}{857}[16,283 \ 0 \ 11,998]$$

$$=[19 \ 0 \ 14]$$

$$\frac{1}{857}[180 \ 613 \ 290] \begin{pmatrix} 140 & 44 & -159 \\ 19 & -43 & 58 \\ -65 & 102 & -18 \end{pmatrix} = \frac{1}{857}[17,997 \ 11,141 \ 1,714]$$

$$=[21 \ 13 \ 2]$$

$$\frac{1}{857}[102 \ 360 \ 259] \begin{pmatrix} 140 & 44 & -159 \\ 19 & -43 & 58 \\ -65 & 102 & -18 \end{pmatrix} = \frac{1}{857}[4,285 \ 15,426 \ 0]$$

$$=[5 \ 18 \ 0]$$

$$\frac{1}{857}[151 \ 580 \ 297] \begin{pmatrix} 140 & 44 & -159 \\ 19 & -43 & 58 \\ -65 & 102 & -18 \end{pmatrix} = \frac{1}{857}[12,855 \ 11,998 \ 4,285]$$

$$=[15 \ 14 \ 5]$$

13   1   20   8   0   9   19   0   14   21   13   2   5   18   0   15   14   5  
M   A   T   H   –   I   S   –   N   U   M   B   E   R   –   O   N   E

The message is: *Math is number one*

**Exercise**

Determine the key word, then decode the given cryptogram

5	17	21	1	20	9	15	14	19
259	863	783	77	378	357	301	448	565
106	266	318	325	365	485	301	522	653
326	653	738	10	566	495	115	640	555
290	791	762	115	474	507	119	332	279
305	454	513	339	645	611	226	341	426
260	338	368	406	657	830	270	649	590
110	337	418	74	318	330	261	561	469
114	426	390	160	543	372	89	535	441
323	842	783	97	344	245	84	601	444
424	851	944	175	262	339	379	698	755
226	341	426	37	454	217	156	694	536

**Solution**

The key word from the first row, because all the numbers are between 0 and 26, alphabetic letter.

Since it has 9 numbers, then the matrix is  $9 = 3^2$  which is  $3 \times 3$ .

Therefore,

$$A = \begin{pmatrix} 5 & 17 & 21 \\ 1 & 20 & 9 \\ 15 & 14 & 19 \end{pmatrix}$$

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

The key word is:

5 17 21 1 20 9 15 14 19  
 E Q U A T I O N S

$$A = \begin{pmatrix} 5 & 17 & 21 \\ 1 & 20 & 9 \\ 15 & 14 & 19 \end{pmatrix}$$

$$|A| = -2,764$$

$$A^{-1} = -\frac{1}{2,764} \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix}$$

With the cryptogram:

$$\begin{aligned}
 & [259 \ 863 \ 783] \ [77 \ 378 \ 357] \ [301 \ 448 \ 565] \ [106 \ 266 \ 318] \\
 & [325 \ 365 \ 485] \ [301 \ 522 \ 653] \ [326 \ 653 \ 738] \ [103 \ 566 \ 495] \\
 & [115 \ 640 \ 555] \ [290 \ 791 \ 762] \ [115 \ 474 \ 507] \ [119 \ 332 \ 279] \\
 & [305 \ 454 \ 513] \ [339 \ 645 \ 611] \ [226 \ 341 \ 426] \ [260 \ 338 \ 368] \\
 & [406 \ 657 \ 830] \ [270 \ 649 \ 590] \ [110 \ 337 \ 418] \ [74 \ 318 \ 330] \\
 & [261 \ 561 \ 469] \ [114 \ 426 \ 390] \ [160 \ 543 \ 372] \ [89 \ 535 \ 441] \\
 & [323 \ 842 \ 783] \ [97 \ 344 \ 245] \ [84 \ 601 \ 444] \ [424 \ 851 \ 944] \\
 & [175 \ 262 \ 339] \ [379 \ 698 \ 755] \ [226 \ 341 \ 426] \ [37 \ 454 \ 217] \\
 & [156 \ 694 \ 536]
 \end{aligned}$$

To decode a message given the matrix  $A$ .

$$\begin{aligned}
 -\frac{1}{2,764} [259 \ 863 \ 783] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} &= -\frac{1}{2,764} [-58,044 \ -52,516 \ -24,876] \\
 &= [21 \ 19 \ 9]
 \end{aligned}$$

$$\begin{aligned}
 -\frac{1}{2,764} [77 \ 378 \ 357] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} &= -\frac{1}{2,764} [-38,696 \ -19,348 \ 0] \\
 &= [14 \ 7 \ 0]
 \end{aligned}$$

$$\begin{aligned}
 -\frac{1}{2,764} [301 \ 448 \ 565] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} &= -\frac{1}{2,764} [-33,168 \ -2,764 \ -44,224] \\
 &= [12 \ 1 \ 16]
 \end{aligned}$$

$$\begin{aligned}
 -\frac{1}{2,764} [106 \ 266 \ 318] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} &= -\frac{1}{2,764} [-33,168 \ -2,764 \ -8,292] \\
 &= [12 \ 1 \ 3]
 \end{aligned}$$

$$\begin{aligned}
 -\frac{1}{2,764} [325 \ 365 \ 485] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} &= -\frac{1}{2,764} [-13,820 \ 0 \ -55,280]
 \end{aligned}$$

$$= [5 \ 0 \ 20]$$

$$-\frac{1}{2,764} [301 \ 522 \ 653] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [-49,752 \ -2,764 \ -38,696]$$

$$= [18 \ 1 \ 14]$$

$$-\frac{1}{2,764} [326 \ 653 \ 738] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [-52,516 \ -16,584 \ -41,460]$$

$$= [19 \ 6 \ 15]$$

$$-\frac{1}{2,764} [103 \ 566 \ 495] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [-49,752 \ -35,932 \ 0]$$

$$= [18 \ 13 \ 0]$$

$$-\frac{1}{2,764} [115 \ 640 \ 555] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [-55,280 \ -41,460 \ 0]$$

$$= [20 \ 15 \ 0]$$

$$-\frac{1}{2,764} [290 \ 791 \ 762] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [-52,516 \ -41,460 \ -33,168]$$

$$= [19 \ 15 \ 12]$$

$$-\frac{1}{2,764} [115 \ 474 \ 507] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [-60,808 \ -13,820 \ 0]$$

$$= [22 \ 5 \ 0]$$

$$-\frac{1}{2,764} [119 \ 332 \ 279] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [-11,056 \ -24,876 \ -16,584]$$

$$= [4 \ 9 \ 6]$$

$$-\frac{1}{2,764} [305 \ 454 \ 513] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [-16,584 \ -13,820 \ -49,752]$$

$$=[6 \ 5 \ 18]$$

$$-\frac{1}{2,764} [339 \ 645 \ 611] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [-13,820 \ -38,696 \ -55,280]$$

$$=[5 \ 14 \ 20]$$

$$-\frac{1}{2,764} [226 \ 341 \ 426] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [-24,876 \ -2,764 \ -33,168]$$

$$=[9 \ 1 \ 12]$$

$$-\frac{1}{2,764} [260 \ 338 \ 368] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [0 \ 13,820 \ -46,988]$$

$$=[0 \ 5 \ 17]$$

$$-\frac{1}{2,764} [406 \ 657 \ 830] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [-58,044 \ -2,764 \ -55,280]$$

$$=[21 \ 1 \ 20]$$

$$-\frac{1}{2,764} [270 \ 649 \ 590] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [-24,876 \ -41,460 \ -38,696]$$

$$=[9 \ 15 \ 14]$$

$$-\frac{1}{2,764} [110 \ 337 \ 418] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [-52,516 \ 0 \ -2,764]$$

$$=[19 \ 0 \ 1]$$

$$-\frac{1}{2,764} [74 \ 318 \ 330] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [-38,696 \ -11,056 \ 0]$$

$$=[14 \ 4 \ 0]$$

$$-\frac{1}{2,764} [261 \ 561 \ 469] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [-2,764 \ -44,224 \ -44,224]$$

$$=[1 \ 16 \ 16]$$



$$-\frac{1}{2,764} \begin{bmatrix} 114 & 426 & 390 \end{bmatrix} \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} \begin{bmatrix} -33,168 & -24,876 & -8,292 \end{bmatrix}$$

$$= [12 \quad 9 \quad 3]$$

$$-\frac{1}{2,764} \begin{bmatrix} 160 & 543 & 372 \end{bmatrix} \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} \begin{bmatrix} -2,764 & -55,280 & -24,876 \end{bmatrix}$$

$$= [1 \quad 20 \quad 9]$$

$$-\frac{1}{2,764} \begin{bmatrix} 89 & 535 & 441 \end{bmatrix} \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} \begin{bmatrix} -41,460 & -38,696 & 0 \end{bmatrix}$$

$$= [15 \quad 14 \quad 0]$$

$$-\frac{1}{2,764} \begin{bmatrix} 323 & 842 & 783 \end{bmatrix} \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} \begin{bmatrix} -44,224 & -49,752 & -41,460 \end{bmatrix}$$

$$= [16 \quad 18 \quad 15]$$

$$-\frac{1}{2,764} \begin{bmatrix} 97 & 344 & 245 \end{bmatrix} \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} \begin{bmatrix} -5,528 & -33,168 & -13,820 \end{bmatrix}$$

$$= [2 \quad 12 \quad 5]$$

$$-\frac{1}{2,764} \begin{bmatrix} 84 & 601 & 444 \end{bmatrix} \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} \begin{bmatrix} -35,932 & -52,516 & 0 \end{bmatrix}$$

$$= [13 \quad 19 \quad 0]$$

$$-\frac{1}{2,764} \begin{bmatrix} 424 & 851 & 944 \end{bmatrix} \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} \begin{bmatrix} -63,572 & -24,876 & -55,280 \end{bmatrix}$$

$$= [23 \quad 9 \quad 20]$$

$$-\frac{1}{2,764} \begin{bmatrix} 175 & 262 & 339 \end{bmatrix} \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} \begin{bmatrix} -22,112 & 0 & -24,876 \end{bmatrix}$$

$$= [8 \quad 0 \quad 9]$$

$$-\frac{1}{2,764} [379 \ 698 \ 755] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [-38,696 \ -24,876 \ -55,280]$$

$$= [14 \ 9 \ 20]$$

$$-\frac{1}{2,764} [226 \ 341 \ 426] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [-24,876 \ -2,764 \ -33,168]$$

$$= [9 \ 1 \ 12]$$

$$-\frac{1}{2,764} [37 \ 454 \ 217] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [0 \ -60,808 \ -2,764]$$

$$= [0 \ 22 \ 1]$$

$$-\frac{1}{2,764} [156 \ 694 \ 536] \begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764} [-33,168 \ -58,044 \ -13,820]$$

$$= [12 \ 21 \ 5]$$

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

21 19 9 14 7 0 12 1 16 12 1 3 5 0 20 18 1 14 19 6 15  
 U S I N G - L A P L A C E - T R A N S F O  
 18 13 0 20 15 0 19 15 12 22 5 0 4 9 6 6 5 18 5 14 20  
 R M - T O - S O L V E - D I F F E R E N T  
 9 1 12 0 5 17 21 1 20 9 15 14 19 0 1 14 4 0 1 16 16  
 I A L - E Q U A T I O N S - A N D - A P P  
 12 9 3 1 20 9 15 14 0 16 18 15 2 12 5 13 19 0 23 9 20  
 L I C A T I O N - P R O B L E M S - W I T  
 8 0 9 14 9 20 9 1 12 0 22 1 12 21 5  
 H - I N I T I A L - V A L U E

The message is:

*Using Laplace Transform to Solve Differential Equations and Application  
Problems with Initial Value*