

Linear Algebra Glossary

Adjacency matrix of a graph: Square matrix with $a_{ij} = 1$ when there is an edge from node i to node j ; otherwise $a_{ij} = 0$, $A = A^T$ for an undirected graph.

Affine Transformation: $T(v) = Av + v_0$ = linear transformation plus shift.

Back substitution: Upper triangular systems are solved in reverse order x_n to x_1 .

Basis for V : Independent vectors v_1, \dots, v_n whose linear combinations give every v in V . A vector space has many bases.

Block matrix: A matrix can be partitioned into matrix blocks, by cuts between rows and/or between columns, **Block multiplication** of AB is allowed if the block shapes permit (the columns of A and rows of B must be matching blocks).

Cayley-Hamilton Theorem: $p(\lambda) = \det(A - \lambda I) = \text{zero matrix}$.

Change of basis matrix M : The old basis vectors v_i are combinations $\sum m_{ij} w_i$ of the new basis vectors. The coordinates of $c_1 v_1 + \dots + c_n v_n = d_1 w_1 + \dots + d_n w_n$ are related by $d = Mc$. (For $n = 2$ set $v_1 = m_{11} w_1 + m_{21} w_2$, $v_2 = m_{12} w_1 + m_{22} w_2$.)

Characteristic equation: $\det(A - \lambda I) = 0$. The n roots are the eigenvalues of A .

Cholesky factorization: $A = CC^T = (L\sqrt{D})(L\sqrt{D})^T$ for positive eigenvalues of A .

Circulant matrix C : Constant diagonals wrap around as in cyclic shift S . Every Circulant is $c_0 I + c_1 S + \dots + c_{n-1} S_{n-1}$. $Cx = \text{convolution } c * x$. Eigenvectors in F .

Cofactor C_{ij} : Remove row i and column j ; multiply the determinant by $(-1)^{i+j}$

Column picture of $Ax = b$: The vector b becomes a combination of the columns of A . The system is solvable only when b is in the column space $C(A)$.

Column space $C(A)$: consists of all linear combinations of the columns. The combinations are all possible vectors Ax .

Commuting matrices $AB = BA$: If diagonalizable, they share n eigenvectors.

Companion matrix: Put c_1, \dots, c_n in row n and put $n - 1$ 1's along diagonal 1. Then

$$\det(A - \lambda I) = \pm (c_1 + c_2 \lambda + c_3 \lambda^2 + \dots)$$

Complete solution: $x = x_p + x_n$ to $Ax = b$. $\left(\text{Particular } x_p \right) + \left(x_n \text{ in nullspace} \right)$

Complex conjugate: $\bar{z} = a - ib$ for any complex number $z = a + ib$. Then

$$z\bar{z} = |z|^2 \Rightarrow (a - ib)(a + ib) = a^2 + b^2$$

Covariance matrix Σ : When random variables x_i have mean = average value = 0, their covariances

\sum_{ij} are the averages of $x_i x_j$. With means \bar{x}_i , the matrix $\Sigma = \text{mean of } (x - \bar{x})(x - \bar{x})^T$ is positive (semi) definite; it is diagonal if the x_i are independent.

Cramer's Rule for $Ax = b$: B_j has b replacing column j of A , and $x_j = \frac{|B_j|}{|A|}$.

Cross product $u \times v$ in \mathbf{R}^3 : Vector perpendicular to u and v , length $\|u\|\|v\|\sin\theta$ = parallelogram area, computed as the “determinant” of $\begin{bmatrix} i & j & k; & u_1 & u_2 & u_3; & v_1 & v_2 & v_3 \end{bmatrix}$

Diagonal matrix D : $d_{ij} = 0$ if $i \neq j$. **Block diagonal:** zero outside square blocks D_{ij} .

Dimension of a vector space: $\dim(V)$ = number of vectors in any basis for V .

Dot Product: $x^T y = x_1 y_1 + \dots + x_n y_n$. Complex dot product is $\bar{x}^T y$. Perpendicular vectors have zero dot product. $(AB)_{ij} = (\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B)$

Echelon matrix U : The first nonzero entry (the pivot) in each row comes after the pivot in the previous row. All zero rows come last.

Eigenvalue λ and eigenvector x : $Ax = \lambda x$ with $x \neq 0$ so $\det(A - \lambda I) = 0$

Elimination: A sequence of row operations that reduces A to an upper triangular U or to the reduced form $R = \text{rref}(A)$. Then $A = LU$ with multipliers ℓ_{ij} in L , or $PA = LU$ with row exchanges in P , or $EA = R$ with an invertible E .

Factorization: $A = LU$. If elimination takes A to U without row exchanges, then the lower triangular L with multipliers ℓ_{ij} (and $\ell_{ii} = 1$) brings U back to A .

Fibonacci numbers: 0, 1, 1, 2, 3, 5, ... satisfy $F_n = F_{n-1} + F_{n-2} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$. Growth rate $\lambda_1 = \frac{1+\sqrt{5}}{2}$ is

the largest eigenvalue of the Fibonacci matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Four Fundamental subspaces of A = $C(A)$, $N(A)$, $C(A^T)$, $N(A^T)$.

Free Columns of A : Columns without pivots; combinations of earlier columns.

Free variables x_i : Column i has no pivot in elimination. We can give the $n - r$ free variables any values, the $Ax = b$ determines the r pivot variables (if solvable!).

Full column rank $r = n$. Independent columns, $N(A) = \{0\}$, no free variables.

Full row rank $r = m$. Independent rows, at least one solution to $Ax = b$, column space is all of \mathbf{R}^m . Full rank means full column rank or full column rank or full row rank.

Fundamental Theorem: the nullspace $N(A)$ and row space $C(A^T)$ are orthogonal complements (perpendicular subspaces of \mathbf{R}^n with dimensions r and $n - r$) from $Ax = 0$. Applied to A^T , the column space $C(A)$ is the orthogonal complement of $N(A^T)$.

Independent vectors: v_1, \dots, v_n . No combination $c_1 v_1 + \dots + c_n v_n = \text{zero vector}$ unless all $c_i = 0$. If the v 's are the columns of A , the only solution to $Ax = 0$ is $x = 0$.

Least squares solution \hat{x} : The vector \hat{x} that minimizes the error $\|e\|^2$ solves $A^T A \hat{x} = A^T b$. Then $e = b - A\hat{x}$ is orthogonal to all columns of A .

Length $\|x\|$: Square root of $x^T x$ (Pythagoras in n dimensions).

Linear combination $cv + dw$ or $\sum c_j v_j$. Vector addition and scalar multiplication.

Linear Transformation T: Each vector v in the input space transforms to $T(v)$ in the output space, and linearity requires $T(cv + dw) = cT(v) + dT(w)$.

Linearly dependent v_1, \dots, v_n . A combination other than all $c_i = 0$ gives $\sum c_i v_i = 0$

Linearly independent when the only solution to $Ax = 0$ is $x = 0$. **No other combination** Ax of the columns gives the zero vector.

Nullspace of A consists of all solutions to $Ax = 0$. These solution vectors x are in \mathbf{R}^n . The Nullspace containing all solutions is denoted by $N(A)$ or $NS(A)$. $\{\vec{x} \in \mathbf{R}^n \mid Ax = 0\}$ is the nullspace of A , $NS(A)$ (Can also be called **Kernel** of A : $Ker(A)$)

Particular solution x_p Any solution to $Ax = b$; often x_p has free variables = 0.

Permutation matrix P. There are $n!$ orders of $1, \dots, n$; the $n!$ P 's have the rows of I in those orders. PA puts the rows of A in the same order. P is a product of row exchanges P_{ij} ; P is *even* or *odd* ($\det P = 1$ or -1) based on the number of exchanges.

Pivot columns of A: Columns that contain pivots after row reduction; not combinations of earlier columns. The pivot columns are a basis for the column space.

Pivot d : The diagonal entry (first nonzero) when a row is used in elimination.

Rank of a matrix A (m by n) is the number of **nonzero rows** in the row-reduced echelon form of A . (it is the number of pivot). $\text{rank}(A) = r$

Reduced Row Echelon Form (rref): is a matrix (R) with each pivot column has only one nonzero entry (the pivots which is always 1).

Row space $C(A^T)$ = all combinations of rows of A . Column vectors by convention.

Schwarz inequality $|v \cdot w| = \|v\| \cdot \|w\|$: Then $\left| v^T A w \right|^2 \leq \left(v^T A v \right) \left(w^T A w \right)$ if $A = C^T C$

Singular matrix A : A square matrix that has no inverse: $\det(A) = 0$.

Spanning set v_1, \dots, v_m for V : Every vector in V is a combination of v_1, \dots, v_m .

Subspace: of a vector space is a set of vectors (including 0) that satisfies two requirements: if v and w are vectors in the subspace and c is any scalar, then $v + w$ is in the subspace and cv is in the subspace

Symmetric matrix A : The transpose is $A^T = A$, and $a_{ij} = a_{ji}$. A^{-1} is also symmetric. All matrices of the form $R^T R$, LDL^T and $Q\Lambda Q^T$ are symmetric. Symmetric matrices have real eigenvalues in Λ and orthonormal eigenvectors in Q .

Trace of A : = sum of diagonal entries = sum of eigenvalues of A . $\text{Tr}(AB) = \text{Tr}(BA)$.

Transpose matrix A^T : Entries $A_{ij}^T = A_{ji}$. A^T is n by m , $A^T A$ is square, symmetric, positive semi-definite. The transposes of AB and A^{-1} are $B^T A^T$ and $(A^T)^{-1}$