Solution   *Section* 3.2 – Double Integrals over General Regions

Exercise

Sketch the region of integration and evaluate the integral $\int_0^{\pi}\int_0^{x} x\sin y\, dydx$

Solution

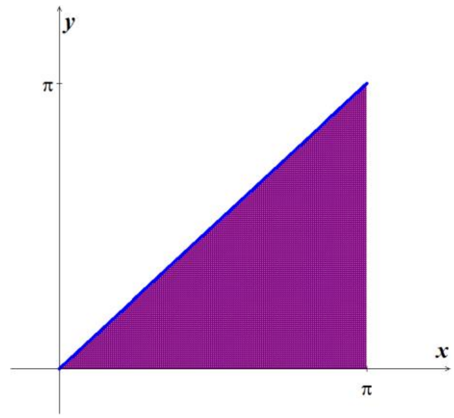
$$\int_0^{\pi}\int_0^{x} x\sin y\, dydx = \int_0^{\pi}\left[-x\cos y\right]_0^{x} dx$$

$$= \int_0^{\pi}\left[-x\cos x + x\right] dx$$

$$= \left[-\left(x\sin x + \cos x\right) + \frac{1}{2}x^2\right]_0^{\pi}$$

$$= -(-1) + \frac{1}{2}\pi^2 - (-1)$$

$$= \frac{\pi^2}{2} + 2$$



		$\int \cos x$
+	x	$\sin x$
–	1	$-\cos x$

Exercise

Sketch the region of integration and evaluate the integral $\int_0^{\pi}\int_0^{\sin x} ydydx$

Solution

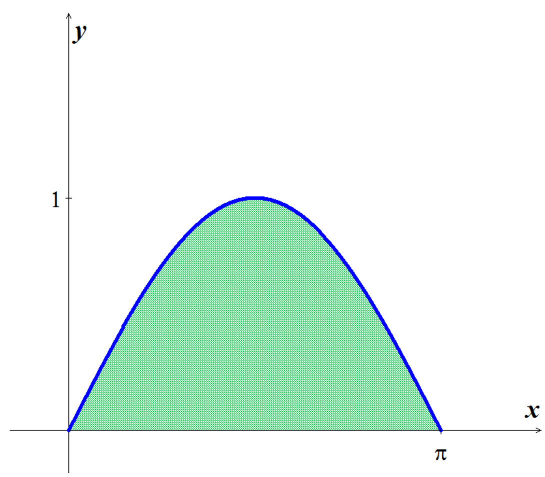
$$\int_0^{\pi}\int_0^{\sin x} ydydx = \int_0^{\pi}\left[\frac{1}{2}y^2\right]_0^{\sin x} dx$$

$$= \int_0^{\pi}\frac{1}{2}\sin^2 x\, dx$$

$$= \frac{1}{4}\int_0^{\pi}\left(1-\cos 2x\right) dx$$

$$= \frac{1}{4}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}$$

$$= \frac{\pi}{4}$$



Exercise

Sketch the region of integration and evaluate the integral $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}dxdy$

Solution

$$\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}dxdy = \int_1^{\ln 8}\left[e^{x+y}\right]_0^{\ln y} dy$$

$$= \int_1^{\ln 8}\left(e^{\ln y+y} - e^y\right)dy$$

$$= \int_1^{\ln 8}\left(e^{\ln y}e^y - e^y\right)dy$$

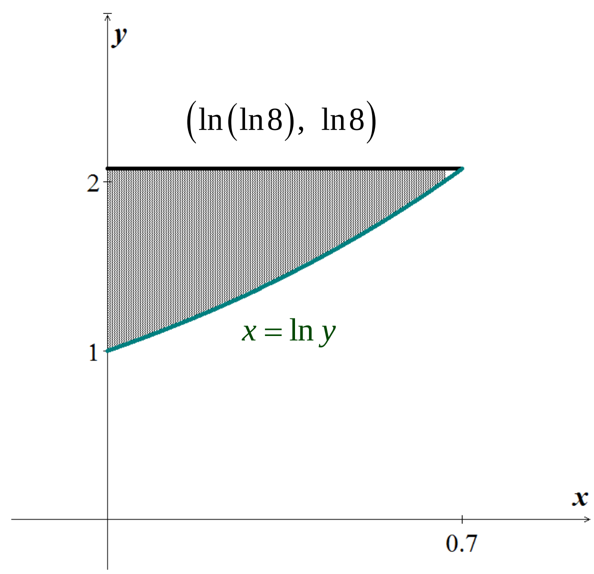
$$= \int_1^{\ln 8}\left(ye^y - e^y\right)dy$$

$$= \left[ye^y - e^y - e^y\right]_1^{\ln 8}$$

$$= (\ln 8)e^{\ln 8} - 2e^{\ln 8} - (e - 2e)$$

$$= 8\ln 8 - 16 - e$$

		$\int e^y$
+	y	e^y
−	1	e^y

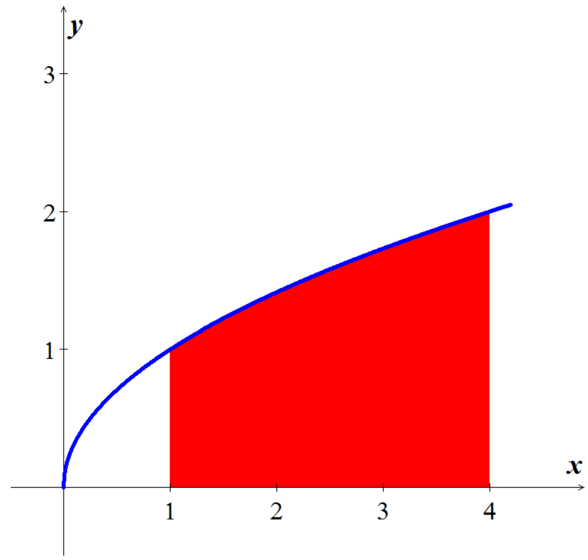


Exercise

Sketch the region of integration and evaluate the integral $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}} dydx$

Solution

$$\begin{aligned}
\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}} dydx &= \frac{3}{2} \int_1^4 \left[\sqrt{x} e^{y/\sqrt{x}} \right]_0^{\sqrt{x}} dx \\
&= \frac{3}{2} \int_1^4 \sqrt{x}(e-1) dx \\
&= \frac{3}{2}(e-1) \int_1^4 x^{1/2} dx \\
&= \frac{3}{2}(e-1) \left[\frac{2}{3} x^{3/2} \right]_1^4 \\
&= (e-1) \left[x^{3/2} \right]_1^4 \\
&= (e-1)[8-1] \\
&= 7(e-1)
\end{aligned}$$



Exercise

Integrate $f(x,y) = \frac{x}{y}$ over the region in the first quadrant bounded by the lines

$y = x, \quad y = 2x, \quad x = 1, \quad and \quad x = 2$

Solution

$$\begin{aligned}
\int_1^2 \int_x^{2x} \frac{x}{y} dydx &= \int_1^2 [x \ln y]_x^{2x} dx \\
&= \int_1^2 x(\ln 2x - \ln x) dx \\
&= \int_1^2 x\left(\ln \frac{2x}{x}\right) dx \qquad \text{Quotient Rule: } \ln M - \ln P = \ln \frac{M}{P} \\
&= \ln 2 \int_1^2 x \, dx
\end{aligned}$$

$$= (\ln 2)\left[\frac{1}{2}x^2\right]_1^2$$

$$= (\ln 2)\left[\frac{1}{2}(4-1)\right]$$

$$= \frac{3}{2}\ln 2$$

Exercise

Integrate $f(x,y) = x^2 + y^2$ over the triangular region with vertices $(0,\ 0)$ $(1,\ 0)$ *and* $(0,\ 1)$

Solution

$$\int_0^1 \int_0^{1-x} \left(x^2 + y^2\right) dydx = \int_0^1 \left[x^2 y + \frac{1}{3}y^3\right]_0^{1-x} dx$$

$$= \int_0^1 \left[x^2(1-x) + \frac{1}{3}(1-x)^3\right] dx$$

$$= \int_0^1 \left[x^2 - x^3 + \frac{1}{3}(1-x)^3\right] dx$$

$$= \left[\frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{12}(1-x)^4\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4} - \left(-\frac{1}{12}\right)$$

$$= \frac{1}{6}$$

Exercise

Integrate $f(s,t) = e^s \ln t$ over the region in the first quadrant of the *st*-plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$.

Solution

$$\int_1^2 \int_0^{\ln t} e^s \ln t \ dsdt = \int_1^2 \left[e^s \ln t\right]_0^{\ln t} dt$$

$$= \int_1^2 (t \ln t - \ln t) dt$$

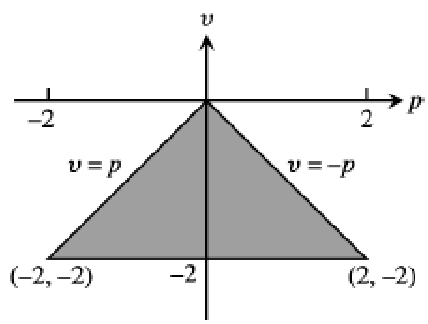
$$u = \ln t \qquad dv = dt$$
$$du = \frac{1}{t}dt \qquad v = t \qquad \rightarrow \quad \int \ln t = t\ln t - \int t\frac{1}{t}dt = t\ln t - t$$
$$\int t\ln t = \frac{1}{2}t^2\ln t - \frac{1}{4}t^2$$

$$= \left[\frac{1}{2}t^2\ln t - \frac{1}{4}t^2 - t\ln t + t\right]_1^2$$
$$= 2\ln 2 - 1 - 2\ln 2 + 2 - \left(0 - \frac{1}{4} - 0 + 1\right)$$
$$= \frac{1}{4}$$

Exercise

Evaluate $\displaystyle\int_{-2}^{0}\int_{v}^{-v} 2dpdv$

Solution

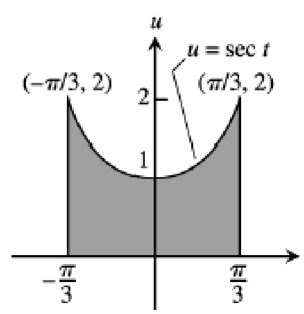


$$\int_{-2}^{0}\int_{v}^{-v} 2dpdv = 2\int_{-2}^{0}\left[p\right]_{v}^{-v} dv$$
$$= -4\int_{-2}^{0} v\ dv$$
$$= -2\left[v^2\right]_{-2}^{0}$$
$$= 8$$

Exercise

Evaluate $\displaystyle\int_{-\pi/3}^{\pi/3}\int_{0}^{\sec t} 3\cos t\ dudt$

Solution



$$\int_{-\pi/3}^{\pi/3}\int_{0}^{\sec t} 3\cos t\ dudt = \int_{-\pi/3}^{\pi/3}(3\cos t)\left[u\right]_{0}^{\sec t} dt$$
$$= \int_{-\pi/3}^{\pi/3}(3\cos t\sec t)dt \qquad \cos t\sec t = \cos t\frac{1}{\cos t} = 1$$
$$= \int_{-\pi/3}^{\pi/3} 3dt$$

$$= 3t\Big|_{-\pi/3}^{\pi/3}$$

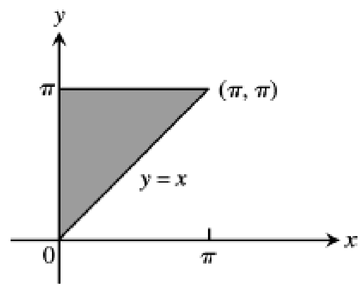
$$= 3\frac{2\pi}{3}$$

$$\boxed{= 2\pi}$$

Exercise

Sketch the region of integration, reverse the order of integration, and evaluate the integral

$$\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dydx$$



Solution

$$\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dydx = \int_0^{\pi}\int_0^{y} \frac{\sin y}{y}\, dxdy$$

$$= \int_0^{\pi} \frac{\sin y}{y}\left[x\right]_0^y\, dy$$

$$= \int_0^{\pi} \frac{\sin y}{y}(y)\, dy$$

$$= \int_0^{\pi} \sin y\ dy$$

$$= -\cos y\Big|_0^{\pi}$$

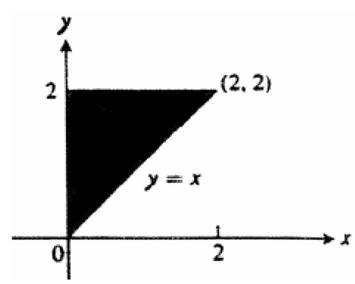
$$= -(-1-1)$$

$$\boxed{= 2}$$

Exercise

Sketch the region of integration, reverse the order of integration, and evaluate the integral

$$\int_0^{2}\int_x^{2} 2y^2 \sin xy\ dydx$$



Solution

$$\int_0^{2}\int_x^{2} 2y^2 \sin xy\ dydx = \int_0^{2}\int_0^{y} 2y^2 \sin xy\ dxdy$$

$$= -2\int_0^2 \left[y\cos xy\right]_0^y \ dy$$

$$= -2\int_0^2 \left(y\cos y^2 - y\right)dy \qquad u = y^2 \ \Rightarrow \ du = 2ydy$$

$$= -\int_0^2 \cos u du + \int_0^2 2ydy$$

$$= \left[-\sin y^2 + y^2\right]_0^2$$

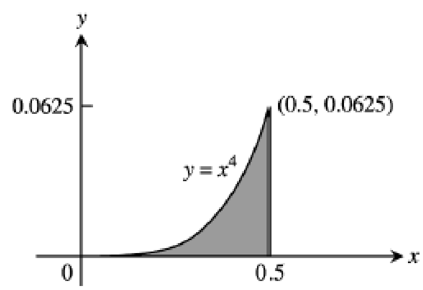
$$= -\sin 4 + 4$$

Exercise

Sketch the region of integration, reverse the order of integration, and evaluate the integral

$$\int_0^{1/16}\int_{y^{1/4}}^{1/2} \cos\left(16\pi x^5\right)dxdy$$

Solution



$$x = y^{1/4} \quad \Rightarrow \quad y = x^4$$

$$\int_0^{1/16}\int_{y^{1/4}}^{1/2} \cos\left(16\pi x^5\right)dxdy = \int_0^{1/2}\int_0^{x^4} \cos\left(16\pi x^5\right)dydx$$

$$= \int_0^{1/2} \cos\left(16\pi x^5\right)\left[y\right]_0^{x^4} \ dx$$

$$= \int_0^{1/2} x^4\cos\left(16\pi x^5\right)dx \qquad u=16\pi x^5 \ \rightarrow \ du=80\pi x^4 dx$$

$$= \frac{1}{80\pi}\int_0^{1/2} \cos u \ du$$

$$= \frac{1}{80\pi}\left[\sin 16\pi x^5\right]_0^{1/2}$$

$$= \frac{1}{80\pi}\left(\sin\frac{16\pi}{32} - 0\right)$$

$$= \frac{1}{80\pi}$$

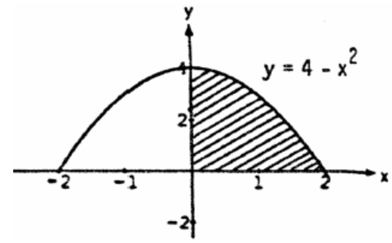
Exercise

Sketch the region of integration, reverse the order of integration, and evaluate the integral

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dydx$$

Solution

$$y = 4 - x^2 \quad \Rightarrow \quad x^2 = 4 - y \to x = \sqrt{4-y}$$



$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dydx = \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\,dxdy$$

$$= \int_0^4 \frac{e^{2y}}{4-y}\left[\frac{1}{2}x^2\right]_0^{\sqrt{4-y}} dy$$

$$= \frac{1}{2}\int_0^4 \frac{e^{2y}}{4-y}(4-y)\,dy$$

$$= \frac{1}{2}\int_0^4 e^{2y}\,dy$$

$$= \frac{1}{4}\left[e^{2y}\right]_0^4$$

$$= \frac{1}{4}\left(e^8 - 1\right)$$

Exercise

Find the volume of the region bounded above the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the *xy*-plane

Solution

$y = x \qquad x + y = 2 \to y = 2 - x$

$x = 0 \qquad y = x \to x + x = 2 \Rightarrow x = 1$

$$V = \int_0^1 \int_x^{2-x} \left(x^2 + y^2\right) dydx$$

$$= \int_0^1 \left[x^2 y + \frac{1}{3}y^3\right]_x^{2-x} dx$$

$$= \int_0^1 \left(x^2(2-x) + \frac{1}{3}(2-x)^3 - x^3 - \frac{1}{3}x^3\right) dx$$

$$= \int_0^1 \left(2x^2 - x^3 + \frac{1}{3}(2-x)^3 - \frac{4}{3}x^3\right) dx$$

$$= \int_0^1 \left(2x^2 - \frac{7}{3}x^3\right)dx + \int_0^1 \frac{1}{3}(2-x)^3\left(-d(2-x)\right)$$

$$= \left[\frac{2}{3}x^3 - \frac{7}{12}x^4 - \frac{1}{12}(2-x)^4\right]_0^1$$

$$= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12}\right) - \left(-\frac{16}{12}\right)$$

$$= \frac{4}{3}$$

Exercise

Find the volume of the solid that is bounded above the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the *xy*-plane

Solution

$$V = \int_{-2}^{1}\int_{x}^{2-x^2} x^2 dydx$$

$$= \int_{-2}^{1} x^2 [y]_x^{2-x^2} dx$$

$$= \int_{-2}^{1} x^2\left(2 - x^2 - x\right)dx$$

$$= \int_{-2}^{1} \left(2x^2 - x^4 - x^3\right)dx$$

$$= \left[\frac{2}{3}x^3 - \frac{1}{5}x^5 - \frac{1}{4}x^4\right]_{-2}^{1}$$

$$= \frac{2}{3} - \frac{1}{5} - \frac{1}{4} - \left(-\frac{15}{3} + \frac{32}{5} - \frac{16}{4}\right)$$

$$= \frac{63}{20}$$

Exercise

Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$ and the plane $z + y = 3$

Solution

$$V = \int_0^2 \int_0^{\sqrt{4-x^2}} (3-y)\,dydx$$

$$= \int_0^2 \left[3y - \tfrac{1}{2}y^2\right]_0^{\sqrt{4-x^2}} dx$$

$$= \int_0^2 \left[3\sqrt{4-x^2} - \tfrac{1}{2}\left(4-x^2\right)\right] dx \qquad \int \sqrt{a^2-x^2}\,dx = \tfrac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$= \left[\frac{3}{2}x\sqrt{4-x^2} + 6\sin^{-1}\left(\frac{x}{2}\right) - 2x + \frac{1}{6}x^3\right]_0^2$$

$$= 0 + 6\frac{\pi}{2} - 4 + \frac{8}{6} - (0)$$

$$= 3\pi - \frac{8}{3}$$

Exercise

Find the volume of the solid that is bounded on the front and back by the planes $x = 2$, and $x = 1$, on the sides by the cylinders $y = \pm\frac{1}{x}$ and above and below the planes $z = x + 1$ and $z = 0$.

Solution

$$V = \int_1^2 \int_{-1/x}^{1/x} (x+1)\,dydx$$

$$= \int_1^2 (x+1)\left[y\right]_{-1/x}^{1/x} dx$$

$$= \int_1^2 (x+1)\left(\frac{2}{x}\right) dx$$

$$= 2\int_1^2 \left(1 + \frac{1}{x}\right) dx$$

$$= 2\left[x + \ln x\right]_1^2$$

$$= 2\left[2 + \ln 2 - 1\right]$$

$$= 2(1 + \ln 2)$$

Exercise

Find the volume under the parabolic cylinder $z = x^2$ above the region enclosed by the parabola $y = 6 - x^2$ and the line $y = x$ in the *xy*-plane

Solution

$$y = 6 - x^2 = x$$

$$x^2 - x - 6 = 0 \quad \rightarrow \quad \underline{x = -3,\ 2}$$

$$V = \int_{-3}^{2} \int_{x}^{6-x^2} z\ dydx$$

$$= \int_{-3}^{2} \int_{x}^{6-x^2} x^2\ dydx$$

$$= \int_{-3}^{2} x^2 y \Big|_{x}^{6-x^2} dx$$

$$= \int_{-3}^{2} x^2 \left(6 - x^2 - x\right) dx$$

$$= \int_{-3}^{2} \left(6x^2 - x^4 - x^3\right) dx$$

$$= 2x^3 - \frac{1}{5}x^5 - \frac{1}{4}x^4 \Big|_{-3}^{2}$$

$$= 16 - \frac{32}{5} - 4 - \left(-54 + \frac{3^5}{5} - \frac{81}{4}\right)$$

$$= \frac{125}{4}\ unit^3$$

Exercise

Find the area of the region enclosed by the line $y = 2x + 4$ and the parabola $y = 4 - x^2$ in the *xy*-plane.

Solution

$$y = 2x + 4 = 4 - x^2$$

$$x^2 + 2x = 0 \quad \rightarrow \quad \underline{x = 0,\ -2}$$

$$A = \int_{-2}^{0} \int_{2x+4}^{4-x^2} dydx$$

$$= \int_{-2}^{0} y \Big|_{2x+4}^{4-x^2} dx$$

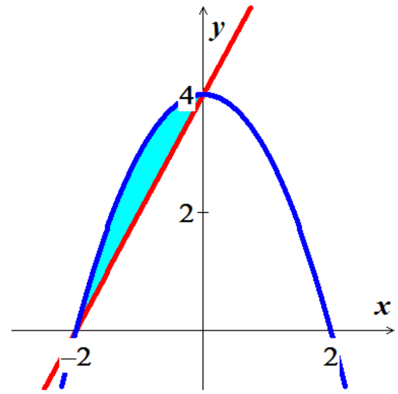
$$= \int_{-2}^{0} \left(4 - x^2 - 2x - 4\right) dx$$

$$= \int_{-2}^{0} \left(-x^2 - 2x\right) dx$$

$$= -\frac{1}{3}x^3 - x^2 \Big|_{-2}^{0}$$

$$= -\frac{8}{3} + 4$$

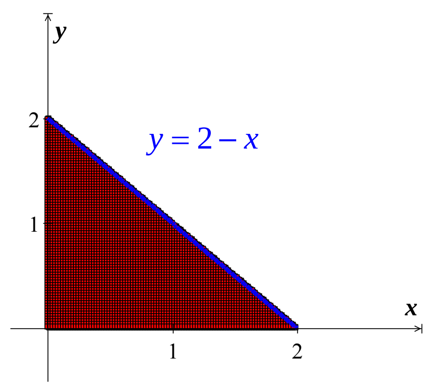
$$\boxed{= \frac{4}{3} \ \ unit^2}$$



Exercise

Find the area of the region enclosed by the coordinate axes and the line $x + y = 2$.

Solution



$$\int_0^2 \int_0^{2-x} dydx = \int_0^2 [y]_0^{2-x} dx$$

$$= \int_0^2 (2 - x) dx$$

$$= \left[2x - \frac{1}{2}x^2\right]_0^2$$

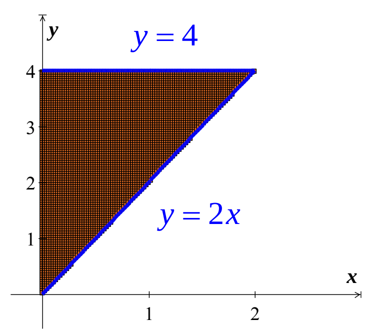
$$= 4 - \frac{1}{2}(4)$$

$$\underline{= 2}$$

Exercise

Find the area of the region enclosed by the lines $x = 0$, $y = 2x$, and $y = 4$

Solution



$$\int_0^2 \int_{2x}^{4} dydx = \int_0^2 [y]_{2x}^{4} dx$$

$$= \int_0^2 (4-2x)\,dx$$

$$= \left[4x - x^2 \right]_0^2$$

$$= 4$$

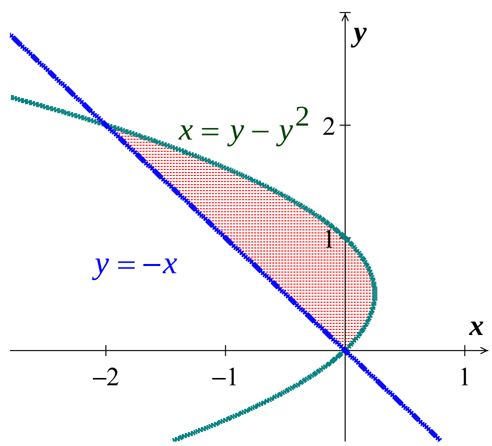
Exercise

Find the area of the region enclosed by the parabola $x = y - y^2$ and the line $y = -x$.

Solution

$$x = y - y^2 = -y \quad \rightarrow \quad 2y - y^2 = 0 \Rightarrow \boxed{y = 0,\ 2}$$

$$\int_0^2 \int_{-y}^{y-y^2} dxdy = \int_0^2 [x]_{-y}^{y-y^2}\, dy$$



$$= \int_0^2 \left(y - y^2 + y \right) dy$$

$$= \int_0^2 \left(2y - y^2 \right) dy$$

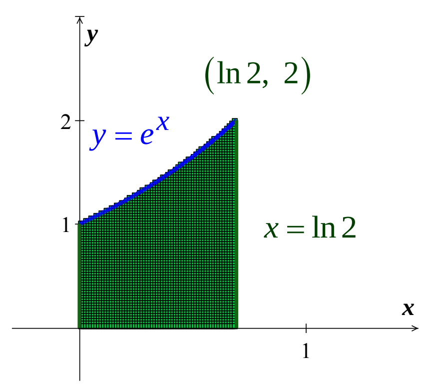
$$= \left[y^2 - \frac{1}{3} y^3 \right]_0^2$$

$$= 4 - \frac{8}{3}$$

$$= \frac{4}{3}$$

Exercise

Find the area of the region enclosed by the curve $y = e^x$ and the lines $y = 0$, $x = 0$ and $x = \ln 2$.



Solution

$$\int_0^{\ln 2} \int_0^{e^x} dydx = \int_0^{\ln 2} [y]_0^{e^x}\, dx$$

$$= \int_0^{\ln 2} e^x dx$$

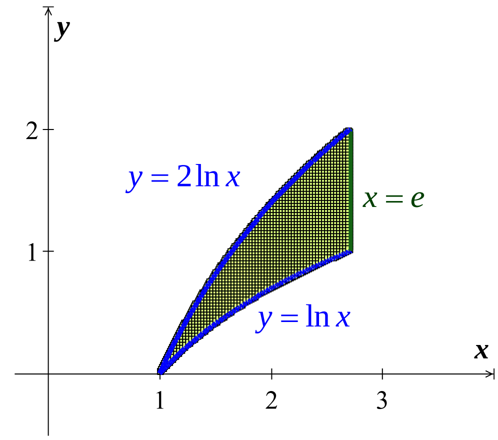
$$= \left[e^x \right]_0^{\ln 2}$$

$$= 2 - 1$$

$$\boxed{= 1}$$

Exercise

Find the area of the region enclosed by the curve $y = \ln x$ and $y = 2\ln x$ and the lines $x = e$ in the first quadrant.



Solution

$$\int_1^e \int_{\ln x}^{2\ln x} dydx = \int_1^e \left[y \right]_{\ln x}^{2\ln x} dx$$

$$= \int_0^{\ln 2} \ln x \, dx$$

$$= \left[x\ln x - x \right]_1^e$$

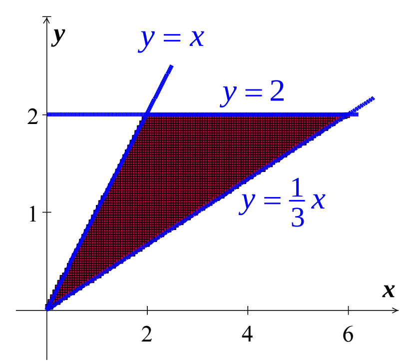
$$= e - e - (0 - 1)$$

$$\boxed{= 1}$$

Exercise

Find the area of the region enclosed by the lines $y = x$, $y = \frac{x}{3}$, and $y = 2$

Solution



$$\int_0^2 \int_y^{3y} dxdy = \int_0^2 x \Big|_y^{3y} dy$$

$$= \int_0^2 (2y) dy$$

$$= y^2 \Big|_0^2$$

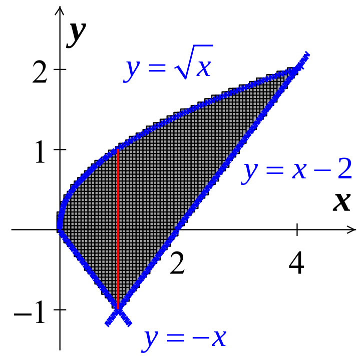
$$\boxed{= 4}$$

Exercise

Find the area of the region enclosed by the lines $y = x - 2$ and $y = -x$ and the curve $y = \sqrt{x}$

Solution

$$\int_0^1 \int_{-x}^{\sqrt{x}} dydx + \int_1^4 \int_{x-2}^{\sqrt{x}} dydx = \int_0^1 y \Big|_{-x}^{\sqrt{x}} dx + \int_1^4 y \Big|_{x-2}^{\sqrt{x}} dx$$



$$= \int_0^1 \left(\sqrt{x} - x\right) dx + \int_1^4 \left(\sqrt{x} - x + 2\right) dx$$

$$= \left[\frac{2}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^1 + \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right]_1^4$$

$$= \frac{2}{3} + \frac{1}{2} + \frac{2}{3}4^{3/2} - 2 + 8 - \frac{2}{3} - \frac{1}{2} + 2$$

$$= \frac{13}{3}$$

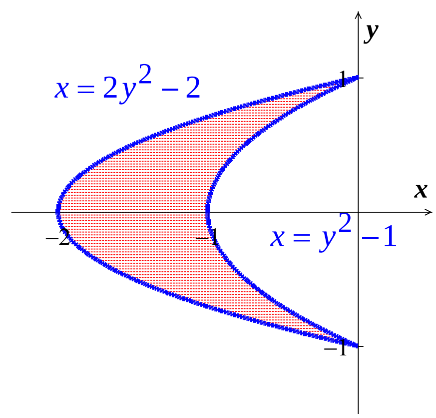
Exercise

Find the area of the region enclosed by the parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$

Solution

$$\int_{-1}^1 \int_{2y^2-2}^{y^2-1} dxdy = \int_{-1}^1 \left[y\right]_{2y^2-2}^{y^2-1} dy$$

$$= \int_{-1}^1 \left(y^2 - 1 - 2y^2 + 2\right) dy$$



$$= \int_{-1}^1 \left(1 - y^2\right) dy$$

$$= \left[y - \frac{1}{3}y^3\right]_{-1}^1$$

$$= 1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)$$

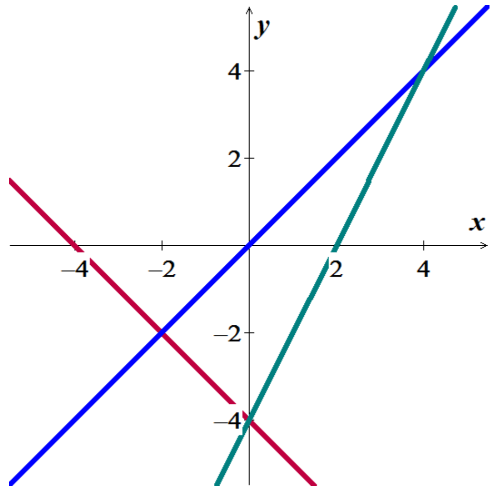
$$= 2 - \frac{2}{3}$$

$$= \frac{4}{3}$$

Exercise

Find the area of the region bounded by the lines $y=-x-4$, $y=x$, and $y=2x-4$. Make a sketch of the region.

Solution



$(y=-x-4)\cap(y=x)$

$\rightarrow y=x=-x-4$

$2x=-4 \quad \rightarrow x=-2 \quad \underline{(-2,\ -2)|}$

$(y=-x-4)\cap(y=2x-4)$

$\rightarrow y=2x-4=-x-4$

$x=0 \quad \rightarrow \quad \underline{(0,\ -4)|}$

$(y=x)\cap(y=2x-4)$

$\rightarrow y=2x-4=x$

$x=4 \quad \rightarrow \quad \underline{(4,\ 4)|}$

$$A=\int_{-2}^{0}\int_{-x-4}^{x} dydx+\int_{0}^{4}\int_{2x-4}^{x} dydx$$

$$=\int_{-2}^{0} y\Big|_{-x-4}^{x} dx+\int_{0}^{4} y\Big|_{2x-4}^{x} dx$$

$$=\int_{-2}^{0}(2x+4)\,dx+\int_{0}^{4}(-x+4)\,dx$$

$$=\left(x^2+4x\right)\Big|_{-2}^{0}+\left(-\frac{1}{2}x^2+4x\right)\Big|_{0}^{4}$$

$$=(-4+8)+(-8+16)$$

$$\underline{=12\ \ unit^2\ |}$$

Exercise

Find the area of the region bounded by the lines $y=|x|$ and $y=20-x^2$. Make a sketch of the region.

Solution

$y=20-x^2=x$

$x^2+x-20=0 \quad \rightarrow \quad x=\cancel{-5},\ 4$

$$A = 2\int_0^4 \int_x^{20-x^2} dydx$$

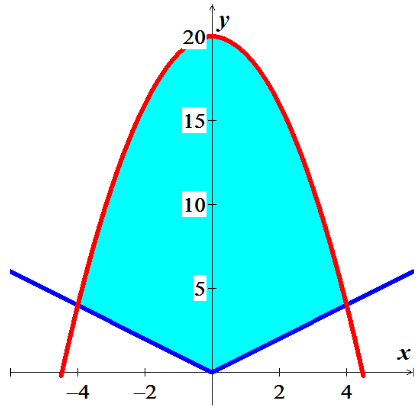
$$= 2\int_0^4 y \Big|_x^{20-x^2} dx$$

$$= 2\int_0^4 \left(20 - x^2 - x\right) dx$$

$$= 2\left(20x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right)\Big|_0^4$$

$$= 2\left(80 - \frac{64}{3} - 8\right)$$

$$= \boxed{\frac{304}{3} \ \ unit^2}$$



Exercise

Find the area of the region bounded by the lines $y = x^2$ and $y = 1 + x - x^2$. Make a sketch of the region.

Solution

$$y = 1 + x - x^2 = x^2$$

$$2x^2 - x - 1 = 0 \quad \rightarrow \quad \boxed{x = 1, \ -\frac{1}{2}}$$

$$A = \int_{-\frac{1}{2}}^1 \int_{x^2}^{1+x-x^2} dydx$$

$$= \int_{-\frac{1}{2}}^1 \left(1 + x - 2x^2\right) dx$$

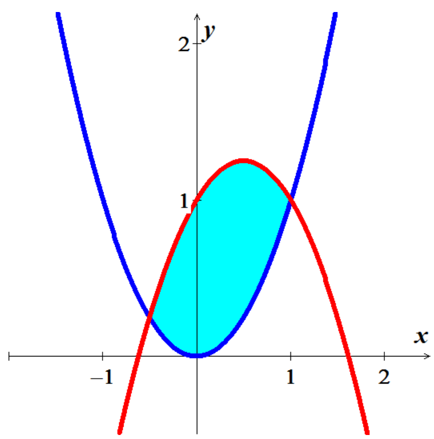
$$= x + \frac{1}{2}x^2 - \frac{2}{3}x^3 \Big|_{-\frac{1}{2}}^1$$

$$= 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{2} - \frac{1}{8} - \frac{1}{12}$$

$$= 2 - \frac{21}{24}$$

$$= \frac{27}{24}$$

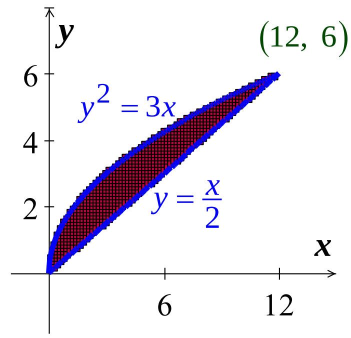
$$= \boxed{\frac{9}{8} \ \ unit^2}$$



Exercise

Find the area of the region $\displaystyle\int_0^6 \int_{y^2/3}^{2y} dxdy$

Solution



$$\int_0^6 \int_{y^2/3}^{2y} dxdy = \int_0^6 \left[x\right]_{y^2/3}^{2y} dy$$

$$= \int_0^6 \left(2y - \frac{1}{3}y^2\right) dy$$

$$= \left[y^2 - \frac{1}{9}y^3\right]_0^6$$

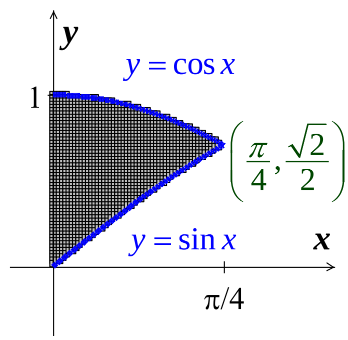
$$= 36 - \frac{1}{9}(216)$$

$$\underline{= 12}$$

Exercise

Find the area of the region $\displaystyle\int_0^{\pi/4} \int_{\sin x}^{\cos x} dydx$

Solution



$$\int_0^{\pi/4} \int_{\sin x}^{\cos x} dydx = \int_0^{\pi/4} \left[y\right]_{\sin x}^{\cos x} dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x) dx$$

$$= \left[\sin x + \cos x\right]_0^{\pi/4}$$

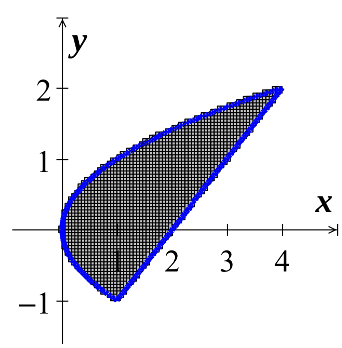
$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1)$$

$$\underline{= \sqrt{2} - 1}$$

Exercise

Find the area of the region $\displaystyle\int_{-1}^2 \int_{y^2}^{y+2} dxdy$

Solution



$$\int_{-1}^{2}\int_{y^2}^{y+2} dxdy = \int_{-1}^{2}\left(y+2-y^2\right)dy$$

$$= \left[\frac{1}{2}y^2+2y-\frac{1}{3}y^3\right]_{-1}^{2}$$

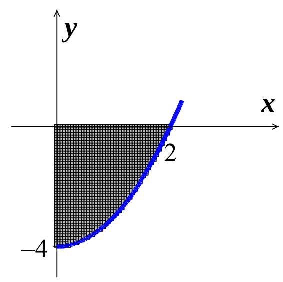
$$= \left(2+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)$$

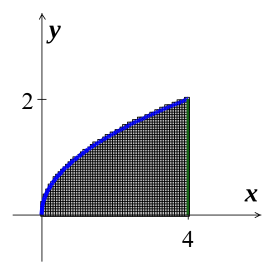
$$= \frac{9}{2}$$

Exercise

Find the area of the region $\int_0^2\int_{x^2-4}^{0} dydx + \int_0^4\int_0^{\sqrt{x}} dydx$

Solution





$$\int_0^2\int_{x^2-4}^{0} dydx + \int_0^4\int_0^{\sqrt{x}} dydx = \int_0^2\left(4-x^2\right)dx + \int_0^4\sqrt{x}\ dx$$

$$= \left[4x-\frac{1}{3}x^3\right]_0^2 + \frac{2}{3}\left[x^{3/2}\right]_0^4$$

$$= \left(8-\frac{8}{3}\right)+\frac{2}{3}\left(4^{3/2}\right)$$

$$= \frac{16}{3}+\frac{16}{3}$$

$$= \frac{32}{3}$$

Exercise

Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2, \quad 0 \le y \le 2$

Solution

$$\text{Average height } = \frac{1}{4}\int_0^2\int_0^2\left(x^2+y^2\right)dydx$$

$$= \frac{1}{4}\int_0^2\left[x^2y+\frac{1}{3}y^3\right]_0^2 dx$$

$$= \frac{1}{4}\int_0^2\left(2x^2+\frac{8}{3}\right)dx$$

$$=\frac{1}{4}\left[\frac{2}{3}x^3+\frac{8}{3}x\right]_0^2$$

$$=\frac{1}{4}\left[\frac{2}{3}(8)+\frac{8}{3}(2)\right]$$

$$=\frac{1}{4}\left[\frac{16}{3}+\frac{16}{3}\right]$$

$$=\frac{8}{3}$$

Exercise

Find the average height of $f(x,y)=\frac{1}{xy}$ over the square $\ln 2\le x\le 2\ln 2,\quad \ln 2\le y\le 2\ln 2$

Solution

$$\text{Average height } =\frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2}\int_{\ln 2}^{2\ln 2}\frac{1}{xy}dydx$$

$$=\frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2}\frac{1}{x}\left[\ln y\right]_{\ln 2}^{2\ln 2}dx$$

$$=\frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2}\frac{1}{x}(2\ln 2-\ln 2)dx$$

$$=\frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2}\frac{1}{x}(\ln 2)dx$$

$$=\frac{1}{\ln 2}\left[\ln x\right]_{\ln 2}^{2\ln 2}$$

$$=\frac{1}{\ln 2}(2\ln 2-\ln 2)$$

$$=1$$

Exercise

Evaluate the integral over the given region

$$\iint_R y\,dA \quad R=\left\{(x,\ y):\ \ 0\le x\le\frac{\pi}{3},\quad 0\le y\le \sec x\right\}$$

Solution

$$\iint\limits_R y\,dA = \int_0^{\frac{\pi}{3}}\int_0^{\sec x} y\ dydx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} y^2 \Big|_0^{\sec x}\ dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} \sec^2 x dx$$

$$= \frac{1}{2}\tan x \Big|_0^{\frac{\pi}{3}}$$

$$\underline{= \frac{\sqrt{3}}{2}}$$

Exercise

Evaluate the integral over the given region

$$\iint\limits_R (x+y)\,dA \quad R \textit{ is the region bounded by } y = \frac{1}{x} \textit{ and } \; y = \frac{5}{2} - x$$

Solution

$$y = \frac{1}{x} = \frac{5}{2} - x$$

$$2x^2 - 5x + 2 = 0 \quad \rightarrow \quad \underline{x = \frac{1}{2},\ 2}$$

$$\iint\limits_R (x+y)\,dA = \int_{\frac{1}{2}}^{2}\int_{1/x}^{\frac{5}{2}-x} (x+y)\,dydx$$

$$= \int_{\frac{1}{2}}^{2}\left(xy + \frac{1}{2}y^2\right)\Big|_{1/x}^{\frac{5}{2}-x}\ dx$$

$$= \int_{\frac{1}{2}}^{2}\left(x\left(\frac{5}{2}-x\right) + \frac{1}{2}\left(\frac{5}{2}-x\right)^2 - 1 - \frac{1}{2x^2}\right)dx$$

$$= \int_{\frac{1}{2}}^{2}\left(\frac{5}{2}x - x^2 + \frac{25}{8} - \frac{5}{2}x + \frac{1}{2}x^2 - 1 - \frac{1}{2x^2}\right)dx$$

$$= \int_{\frac{1}{2}}^{2} \left(-\frac{1}{2}x^2 + \frac{17}{8} - \frac{1}{2x^2} \right) dx$$

$$= -\frac{1}{6}x^3 + \frac{17}{8}x + \frac{1}{2x} \Big|_{\frac{1}{2}}^{2}$$

$$= -\frac{4}{3} + \frac{17}{4} + \frac{1}{4} + \frac{1}{48} - \frac{17}{16} - 1$$

$$= \frac{27}{24}$$

$$= \frac{9}{8}$$

Exercise

Evaluate the integral over the given region

$$\iint_R \frac{xy}{1 + x^2 + y^2} dA \quad R = \{(x, y): \ 0 \le y \le x, \quad 0 \le x \le 2\}$$

Solution

$$\iint_R \frac{xy}{1 + x^2 + y^2} dA = \int_0^2 \int_0^x \frac{xy}{1 + x^2 + y^2} dydx$$

$$= \frac{1}{2} \int_0^2 \int_0^x \frac{x}{1 + x^2 + y^2} d\left(1 + x^2 + y^2\right) dx \qquad d\left(1 + x^2 + y^2\right) = 2ydy$$

$$= \frac{1}{2} \int_0^2 x \ln\left(1 + x^2 + y^2\right) \Big|_0^x dx$$

$$= \frac{1}{2} \int_0^2 \left(x \ln\left(1 + 2x^2\right) - x \ln\left(1 + x^2\right) \right) dx$$

$$= \frac{1}{2} \int_0^2 x \ln\left(1 + 2x^2\right) dx - \frac{1}{2} \int_0^2 x \ln\left(1 + x^2\right) dx$$

$$= \frac{1}{8} \int_0^2 \ln\left(1 + 2x^2\right) d\left(1 + 2x^2\right) - \frac{1}{4} \int_0^2 \ln\left(1 + x^2\right) d\left(1 + x^2\right)$$

$$\int \ln y \ dy = y(\ln y - 1)$$

$$= \frac{1}{8}\left(1 + 2x^2\right)\left(\ln\left(1 + 2x^2\right) - 1\right) \Big|_0^2 - \frac{1}{4}\left(1 + x^2\right)\left(\ln\left(1 + x^2\right) - 1\right) \Big|_0^2$$

$$= \frac{1}{8}(9\ln 9 - 9 + 1) - \frac{1}{4}(5\ln 5 - 5 + 1)$$

$$= \frac{9}{8}\ln 9 + 1 - \frac{5}{4}\ln 5 - 1$$

$$= \frac{9}{8}\ln 9 - \frac{5}{4}\ln 5$$

Exercise

Evaluate the integral over the given region

$$\iint_R x\sec^2 y\, dA \quad R = \left\{(x,\ y):\ \ 0 \le y \le x^2,\ \ \ 0 \le x \le \frac{\sqrt{\pi}}{2}\right\}$$

Solution

$$\iint_R x\sec^2 y\, dA = \int_0^{\frac{\sqrt{\pi}}{2}} \int_0^{x^2} x\sec^2 y\, dydx$$

$$= \int_0^{\frac{\sqrt{\pi}}{2}} x\tan y \Big|_0^{x^2} dx$$

$$= \int_0^{\frac{\sqrt{\pi}}{2}} x\tan x^2\, dx$$

$$= \frac{1}{2}\int_0^{\frac{\sqrt{\pi}}{2}} \tan x^2\, d\left(x^2\right)$$

$$= \frac{1}{2}\ln\left|\sec x^2\right| \Big|_0^{\frac{\sqrt{\pi}}{2}}$$

$$= \frac{1}{2}\ln\left|\sec\frac{\pi}{4}\right|$$

$$= \frac{1}{2}\ln\sqrt{2}$$

$$= \frac{1}{4}\ln 2$$

Exercise

Consider the region $R=\{(x,\ y):\ |x|+|y|\le 1\}$

a) Use a double integral to show that the area of R is 2.

b) Find the volume of the square column whose base is R and whose upper surface is $z=12-3x-4y$.

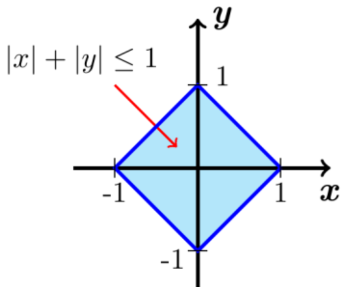
c) Find the volume of the solid above R and beneath the cylinder $x^2+z^2=1$.

d) Find the volume of the pyramid whose base is R and whose vertex is on the z-axis at $(0,\ 0,\ 6)$

Solution

$$-1\le x\le 0 \quad\to\quad \begin{cases} y=x+1 \\ y=-x-1 \end{cases}$$

$$0\le x\le 1 \quad\to\quad \begin{cases} y=x-1 \\ y=-x+1 \end{cases}$$



a)
$$A=\int_{-1}^{0}\int_{-x-1}^{x+1} dydx+\int_{0}^{1}\int_{x-1}^{1-x} dydx$$

$$=\int_{-1}^{0} y\Big|_{-x-1}^{x+1} dx+\int_{0}^{1} y\Big|_{x-1}^{1-x} dx$$

$$=\int_{-1}^{0}(2x+2)\,dx+\int_{0}^{1}(2-2x)\,dx$$

$$=\left(x^2+2x\right)\Big|_{-1}^{0}+\left(2x-x^2\right)\Big|_{0}^{1}$$

$$=-1+2+2-1$$

$$=2$$

b)
$$V=\int_{-1}^{0}\int_{-x-1}^{x+1}(12-3x-4y)\,dydx+\int_{0}^{1}\int_{x-1}^{1-x}(12-3x-4y)\,dydx$$

$$=\int_{-1}^{0}\left(12y-3xy-2y^2\right)\Big|_{-x-1}^{x+1} dx+\int_{0}^{1}\left(12y-3xy-2y^2\right)\Big|_{x-1}^{1-x} dx$$

$$=\int_{-1}^{0}\left(12x+12-3x(x+1)-2(x+1)^2-12(-x-1)+3x(x+1)+2(-x-1)^2\right)dx$$

$$+\int_{0}^{1}\left(12(1-x)-3x(1-x)-2(1-x)^2-12(x-1)+3x(x-1)+2(x-1)^2\right)dx$$

$$=\int_{-1}^{0}\left(18x+24-6x^2\right)dx+\int_{0}^{1}\left(24-30x+6x^2\right)dx$$

$$= \left(9x^2 + 24x - 2x^3\right)\Big|_{-1}^{0} + \left(24x - 15x^2 + 2x^3\right)\Big|_{0}^{1}$$

$$= -9 + 24 - 2 + 24 - 15 + 2$$

$$\underline{= 24}$$

c) $x^2 + z^2 = 1 \quad \rightarrow \quad z = \sqrt{1 - x^2}$

$$V = \int_{-1}^{0}\int_{-x-1}^{x+1} \sqrt{1-x^2}\; dydx + \int_{0}^{1}\int_{x-1}^{1-x} \sqrt{1-x^2}\; dydx \qquad \textit{due to the symmetry}$$

$$= 2\int_{0}^{1}\int_{x-1}^{1-x} \sqrt{1-x^2}\; dydx$$

$$= 2\int_{0}^{1} \sqrt{1-x^2}\; y\Big|_{x-1}^{1-x} dx$$

$$= 2\int_{0}^{1} (2-2x)\sqrt{1-x^2}\; dx$$

$$= 4\int_{0}^{1} \sqrt{1-x^2}\; dx - 4\int_{0}^{1} x\sqrt{1-x^2}\; dx$$

$$= \left(2x\sqrt{1-x^2} + 2\sin^{-1} x\right)\Big|_{0}^{1} + 2\int_{0}^{1}\left(1-x^2\right)^{1/2}\; d\left(1-x^2\right)$$

$$= 2\sin^{-1} 1 + \tfrac{4}{3}\left(1-x^2\right)^{3/2}\Big|_{0}^{1}$$

$$\underline{= \pi - \frac{4}{3}}$$

d) $(1,\ 0,\ 0) \quad (0,\ 1,\ 0) \quad (0,\ 0,\ 6)$

$z = 6(1 - x - y)$

Using symmetry

$$V = 4\int_{0}^{1}\int_{0}^{1-x} 6(1-x-y)\; dydx$$

$$= 24\int_{0}^{1}\left((1-x)\,y - \tfrac{1}{2}y^2\right)\Big|_{0}^{1-x} dx$$

$$= 24\int_{0}^{1}\left((1-x)^2 - \tfrac{1}{2}(1-x)^2\right)dx$$

$$= -12\int_0^1 (1-x)^2 \, d(1-x)$$

$$= -4(1-x)^3 \Big|_0^1$$

$$\underline{= 4}$$