Solution Section 4.3 – Vectors and Dot Product

Exercise

An arrow is shot into the air so that its horizontal velocity is 25 feet per second and its vertical is 15 feet per second. Find the velocity of the arrow.

Solution

The magnitude of the velocity is:

$$|V| = \sqrt{25^2 + 15^2}$$
$$= 29\ ft/\sec$$

$$\tan\theta = \frac{|V_y|}{|V_x|} = \frac{15}{25} = 0.6$$

$$\underline{|\theta} = \tan^{-1} 0.6 \approx 31^\circ|$$

Exercise

A boat travels 72 miles on a course of bearing N 27° E and then changes its course to travel 37 miles at N 55° E. How far north and how far east has the boat traveled on this 109-mile trip?

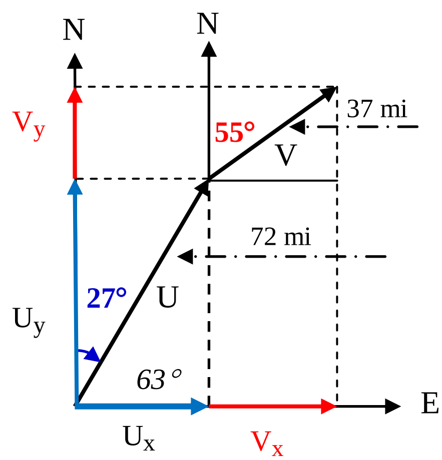
Solution

Total distance traveled east $= |U_x| + |V_x|$

$$= 72\cos 63^\circ + 37\cos 35^\circ$$
$$= 63\ mi$$

Total distance traveled North $= |U_y| + |V_y|$

$$= 72\sin 63^\circ + 37\sin 35^\circ$$
$$= 85\ mi$$



Exercise

Let **u** = $\langle -2, 1\rangle$ and **v** = $\langle 4, 3\rangle$. Find the following.

a) **u** + **v**
b) –2**u**
c) 4**u** – 3**v**

Solution

a) $\mathbf{u} + \mathbf{v} = \langle -2, 1\rangle + \langle 4, 3\rangle$

$= \langle 2, 4\rangle$

b) $-2\mathbf{u} = -2\langle -2, 1\rangle$

$= \langle 4, -2\rangle$

c) $4\mathbf{u} - 3\mathbf{v} = 4\langle -2, 1\rangle - 3\langle 4, 3\rangle$

$= \langle -8, 4\rangle - \langle 12, 9\rangle$

$= \langle -20, -5\rangle$

Exercise

Given: $|V| = 13.8,\ \theta = 24.2°$, find the magnitudes of the horizontal and vertical vector components of V, V_x *and* V_y, respectively

Solution

$$\left|V_x\right| = |V|\cos\theta = 13.8\cos 24.2° \approx 12.6$$

$$\left|V_y\right| = |V|\sin\theta = 13.8\sin 24.2° \approx 5.7$$

Exercise

Find the angle θ between the two vectors **u** = $\langle 3, 4\rangle$ and **v** = $\langle 2, 1\rangle$.

Solution

$$\theta = \cos^{-1}\frac{u.v}{|u|.|v|}$$

$$= \cos^{-1}\left(\frac{(3)(2)+(4)(1)}{\sqrt{3^2+4^2}\sqrt{2^2+1^2}}\right)$$

$$\approx 26.57°$$

Exercise

A bullet is fired into the air with an initial velocity of 1,800 feet per second at an angle of 60° from the horizontal. Find the magnitude of the horizontal and vertical vector component as of the velocity vector.

Solution

$$\boxed{\left|V_x\right| = 1800\cos 60^\circ = 900\ ft/\sec}$$

$$\boxed{\left|V_y\right| = 1800\sin 60^\circ = 1600\ ft/\sec}$$

Exercise

A bullet is fired into the air with an initial velocity of 1,200 feet per second at an angle of 45° from the horizontal.

a) Find the magnitude of the horizontal and vertical vector component as of the velocity vector.

b) Find the horizontal distance traveled by the bullet in 3 seconds. (Neglect the resistance of air on the bullet).

Solution

a) $\boxed{\left|V_x\right| = 1200\cos 45^\circ = 848.53 \approx 850\ ft/\sec}$

$$\boxed{\left|V_y\right| = 1200\sin 45^\circ \approx 850\ ft/\sec}$$

b) $x = 850 * 3 = 2{,}550\ ft$ $\boxed{x = \left|V_x\right|.t = 850 * 3 = 2{,}550\ ft}$

Exercise

A ship travels 130 km on a bearing of S 42° E. How far east and how far south has it traveled?

Solution

$$\boxed{V_{east} = 130\cos 42^\circ = 96.6\ km}$$

$$\boxed{V_{south} = 130\sin 42^\circ \approx 87\ km}$$

Exercise

An arrow is shot into the air with so that its horizontal velocity is 15.0 ft./sec and its vertical velocity is 25.0 ft./sec. Find the velocity of the arrow?

Solution

$$|V| = \sqrt{V_x^2 + V_y^2}$$

$$V = \sqrt{15^2 + 25^2}$$

$$V \approx 29.2\ ft/\sec$$

$$\tan\theta = \frac{25}{15}$$

$$\theta = \tan^{-1}\left(\frac{25}{15}\right)$$

$$= 60°$$

Exercise

A plane travels 170 miles on a bearing of N 18 E and then changes its course to N 49 E and travels another 120 miles. Find the total distance traveled north and the total distance traveled east.

Solution

Total distance traveled east $= |U_x| + |V_x|$

$$= 170\cos 72° + 120\cos 41°$$

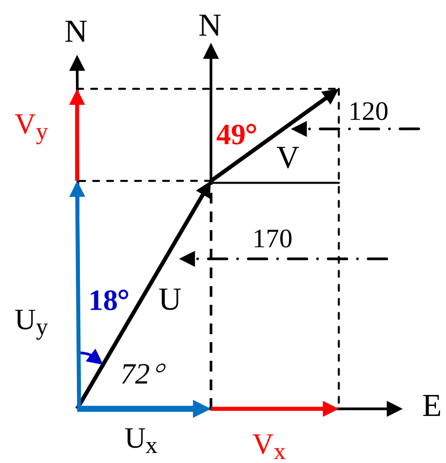
$$= 143.1\ mi$$

$$= 140\ mi\text{ - }East$$

Total distance traveled North $= |U_y| + |V_y|$

$$= 170\sin 72° + 120\sin 41°$$

$$= 240\ mi\text{ - }South$$

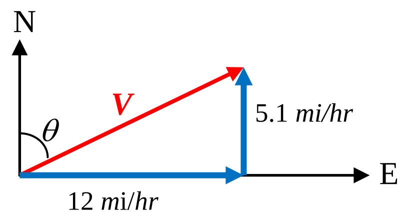


Exercise

A boat is crossing a river that run due north. The boat is pointed due east and is moving through the water at 12 miles per hour. If the current of the river is a constant 5.1 miles per hour, find the actual course of the boat through the water to two significant digits.

Solution

$$\tan\theta = \frac{12}{5.1}$$
$$= 2.3529$$



$$\theta = \tan^{-1} 2.3529$$
$$= 67°$$

$$|V| = \sqrt{12^2 + 5.1^2}$$ ***(using Pythagorean Theorem)***

$$= 13$$

Exercise

Two forces of 15 and 22 Newtons act on a point in the plane. (A **Newton** is a unit of force that equals .225 lb.) If the angle between the forces is 100°, find the magnitude of the resultant vector.

Solution

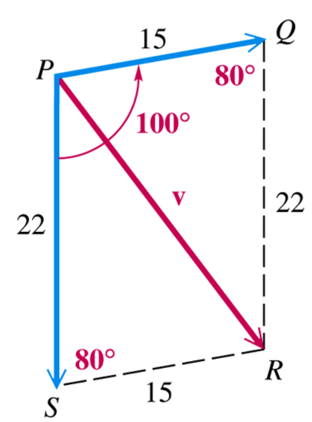
$$P + Q + R + S = 360°$$

$$2P + 2Q = 360°$$

$$2Q = 360° - 2(100°)$$

$$2Q = 160°$$

$$\boxed{Q = 80°}$$



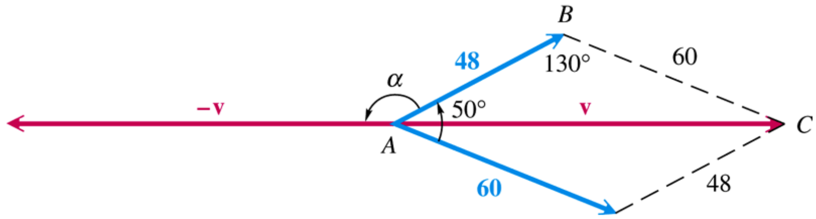
$$|V|^2 = 15^2 + 22^2 - 2(15)(22)\cos 80°$$
$$\approx 594.39$$

$$\boxed{|V| \approx 24.4\ N}$$

Exercise

Find the magnitude of the equilibrant of forces of 48 N and 60 N acting on a point *A*, if the angle between the forces is 50°. Then find the angle between the equilibrant and the 48-newton force.

Solution



The equilibrant is $-\mathbf{v}$.

$2A + 2B = 360°$

$2B = 360° - 2(50°)$

$2B = 260°$

$B = 130°$

$$|V|^2 = 48^2 + 60^2 - 2(48)(60)\cos 130°$$

$$\boxed{|V| \approx 98\ N}$$

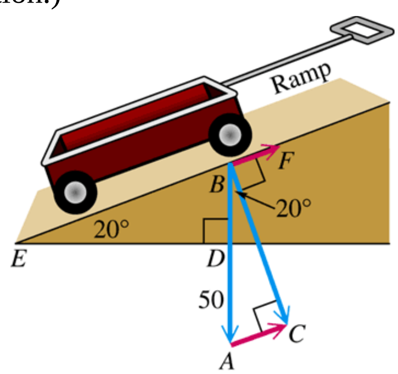
$$\frac{\sin BAC}{60} = \frac{\sin 130°}{98} \quad \Rightarrow \sin BAC = \frac{60\sin 130°}{98}$$

$$\angle BAC = \sin^{-1}\left(\frac{60\sin 130°}{98}\right) \approx 28°$$

$$\alpha = 180° - 28° = 152°$$

Exercise

Find the force required to keep a 50-lb wagon from sliding down a ramp inclined at 20° to the horizontal. (Assume there is no friction.)



Solution

Vector **BC** represents the force with which the weight pushes against the ramp.
Vector **BF** represents the force that would pull the weight up the ramp.
The vertical force **BA** represents the force of gravity.

$$\sin 20° = \frac{|AC|}{50} \quad \Rightarrow \quad |AC| = 50\sin 20° \approx 17$$

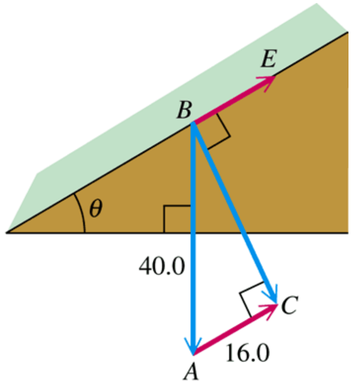
A force of approximately 17 lbs. will keep the wagon from sliding down the ramp.

Exercise

A force of 16.0 lbs. is required to hold a 40.0 lbs. lawn mower on an incline. What angle does the incline make with the horizontal?

Solution

Vector **BE** represents the force required to hold the mower on the incline.



$$\sin B = \frac{16}{40}$$

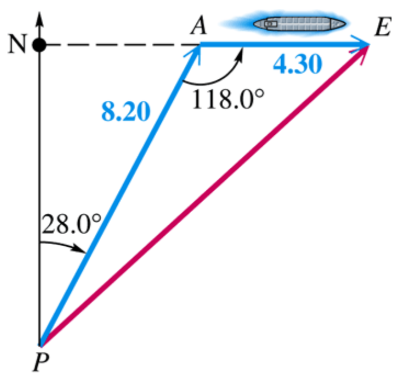
$$\underline{|B} = \sin^{-1}\left(\frac{16}{40}\right) \underline{\approx 23.6^\circ|}$$

The hill makes an angle of about 23.6° with the horizontal.

Exercise

A ship leaves port on a bearing of 28.0° and travels 8.20 mi. The ship then turns due east and travels 4.30 mi. How far is the ship from port? What is its bearing from port?

Solution



$$\angle NAP = 90^\circ - 28^\circ = 62^\circ$$

$$\angle PAE = 180^\circ - 62^\circ = 118^\circ$$

$$|PE|^2 = 8.20^2 + 4.30^2 - 2(8.2)(4.3)\cos 118^\circ$$

$$\approx 120.6$$

$$|PE| = 10.9$$

The ship is about 10.9 miles from port.

The bearing is given by:

$$\frac{\sin APE}{4.3} = \frac{\sin 118}{10.9}$$

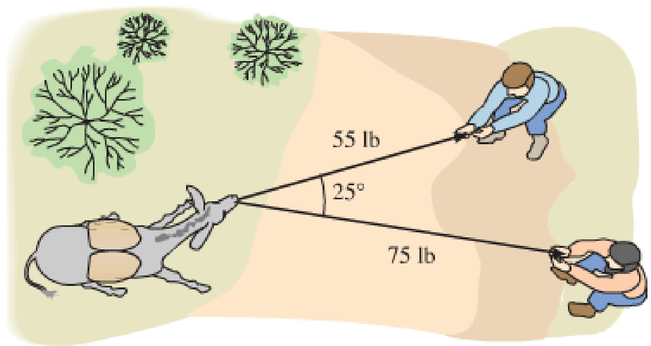
$$\sin APE = \frac{4.3\sin 118}{10.9}$$

$$\underline{|\angle APE} = \sin^{-1}\left(\frac{4.3\sin 118}{10.9}\right) \underline{\approx 20.4^\circ|}$$

The bearing: $28^\circ + 20.4^\circ = 48.4^\circ$

Exercise

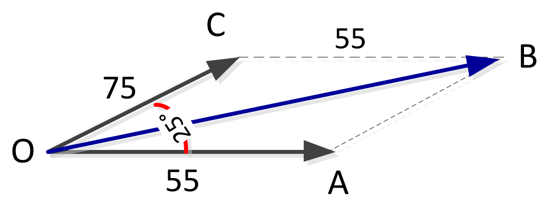
Two prospectors are pulling on ropes attached around the neck of a donkey that does not want to move. One prospector pulls with a force of 55 lb, and the other pulls with a force of 75 lb. If the angle between the ropes is 25°, then how much force must the donkey use in order to stay put? (The donkey knows the proper direction in which to apply his force.)



Solution

$$\angle OCB = \angle OAB = \frac{1}{2}\left(360° - 2(25°)\right) = 155°$$

By the cosine law, the magnitude of the resultant force is:



$$OB = \sqrt{OC^2 + CB^2 - 2(OC)(OB)\cos\left(OCB\right)}$$

$$= \sqrt{75^2 + 55^2 - 2(75)(55)\cos\left(155°\right)}$$

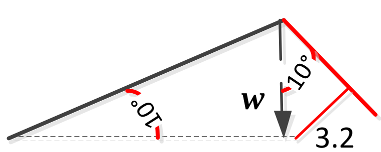
$$\approx 127\ lb$$

So the donkey must pull a force of 127 pounds in the direction opposite that of the resultant's.

Exercise

A solid steel ball is placed on a 10° incline. If a force of 3.2 lb in the direction of the incline is required to keep the ball in place, then what is the weight of the ball?

Solution

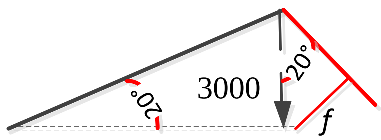


$$\sin 10° = \frac{f}{w} \rightarrow w = \frac{3.2}{\sin 10°} \approx 18.4\ lb$$

Exercise

Find the amount of force required for a winch to pull a 3000-lb car up a ramp that is inclined at 20°.

Solution

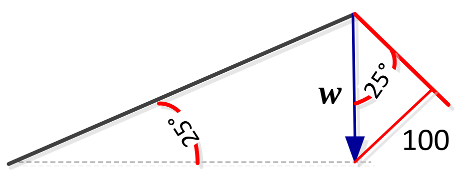


$$\sin 20° = \frac{f}{3000} \rightarrow f = 3000 \sin 20° \underline{\approx 1026.1 \ lb}$$

Exercise

If the amount of force required to push a block of ice up an ice-covered driveway that is inclined at 25° is 100lb, then what is the weight of the block?

Solution

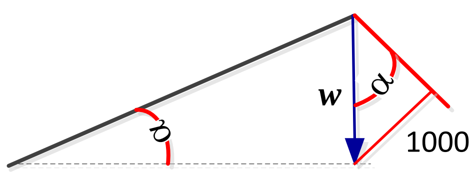


$$\sin 25° = \frac{100}{w} \rightarrow w = \frac{100}{\sin 25°} \underline{\approx 236.6 \ lb}$$

Exercise

If superman exerts 1000 lb of force to prevent a 5000-lb boulder from rolling down a hill and crushing a bus full of children, then what is the angle of inclination of the hill?

Solution



$$\sin \alpha = \frac{1000}{5000} \rightarrow \alpha = \sin^{-1} \frac{1000}{5000} \underline{\approx 11.5°}$$

Exercise

If Sisyphus exerts a 500-lb force in rolling his 4000-lb spherical boulder uphill, then what is the angle of inclination of the hill?

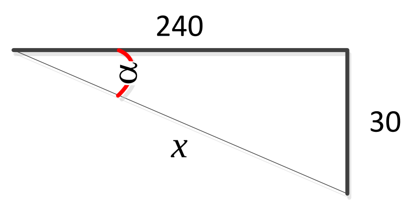
Solution

$$\sin\alpha = \frac{500}{4000} \to \alpha = \sin^{-1}\frac{500}{4000} \approx 7.2^\circ$$

Exercise

A plane is headed due east with an air speed of 240 mph. The wind is from the north at 30 mph. Find the bearing for the course and the ground speed of the plane.

Solution



The bearing is: $\alpha = \tan^{-1}\frac{30}{240} \approx 7.1^\circ$

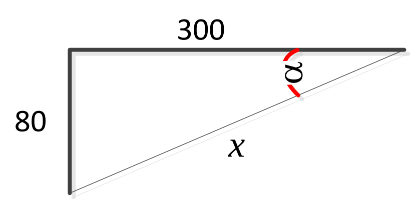
The ground speed can be found by using Pythagorean Theorem:

$$x = \sqrt{240^2 + 30^2} \approx 241.9\ mph$$

Exercise

A plane is headed due west with an air speed of 300 mph. The wind is from the north at 80 mph. Find the bearing for the course and the ground speed of the plane.

Solution



$$\alpha = \tan^{-1}\frac{80}{300} \approx 14.9^\circ$$

The bearing is: $270^\circ - 14.9^\circ = 255.1^\circ$

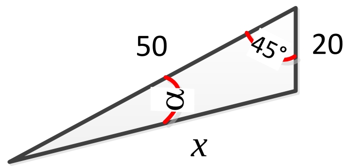
The ground speed can be found by using Pythagorean Theorem:

$$x = \sqrt{300^2 + 80^2} \approx 310.5\ mph$$

Exercise

An ultralight is flying northeast at 50 mph. The wind is from the north at 20 mph. Find the bearing for the course and the ground speed of the ultralight.

Solution



Using the cosine law, the ground speed is:

$$x = \sqrt{50^2 + 20^2 - 2(50)(20)\cos 45^\circ} \approx 38.5\ mph$$

$$\frac{\sin\alpha}{20} = \frac{\sin 45^\circ}{38.5} \Rightarrow \sin\alpha = \frac{20\sin 45^\circ}{38.5}$$

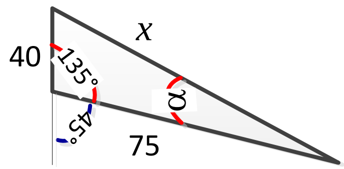
$$\alpha = \sin^{-1}\left(\frac{20\sin 45^\circ}{38.5}\right) \approx 21.5^\circ$$

The bearing is: $45^\circ + 21.5^\circ = 66.5^\circ$

Exercise

A superlight is flying northwest at 75 mph. The wind is from the south at 40 mph. Find the bearing for the course and the ground speed of the superlight.

Solution



Using the cosine law, the ground speed is:

$$x = \sqrt{75^2 + 40^2 - 2(75)(40)\cos 135^\circ} \approx 107.1\ mph$$

$$\frac{\sin\alpha}{40} = \frac{\sin 135^\circ}{107.1} \Rightarrow \sin\alpha = \frac{40\sin 135^\circ}{107.1}$$

$$\alpha = \sin^{-1}\left(\frac{40\sin 135^\circ}{107.1}\right) \approx 15.3^\circ$$

The bearing is: $315^\circ + 15.3^\circ = 330.3^\circ$

Exercise

An airplane is heading on a bearing of 102° with an air speed of 480 mph. If the wind is out of the northeast (bearing 225°) at 58 mph, then what are the bearing of the course and the ground speed of the airplane?

Solution

x: the airplane's vector course and ground speed

$\beta = 225° - 180° = 45°$

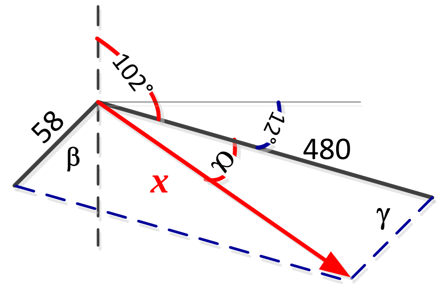
From the parallelogram (from the vectors):

$\gamma = \frac{1}{2}\left(\left(90° - 12°\right) + 45°\right) = 57°$

By the cosine law:

$$x = \sqrt{480^2 + 58^2 - 2(480)(58)\cos 57°}$$

$$= 451\ mph$$



$$\frac{\sin\alpha}{58} = \frac{\sin 57°}{451}$$

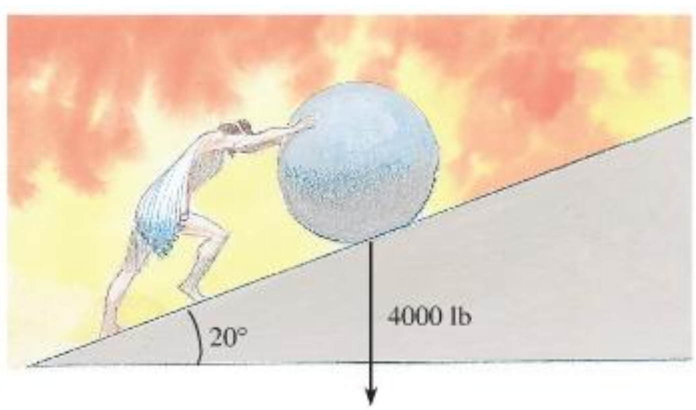
$$\sin\alpha = \frac{58\sin 57°}{451}$$

$$\alpha = \sin^{-1}\left(\frac{58\sin 57°}{451}\right) \approx 6.2°$$

The bearing of the plane: $102° + 6.2° = 108.2°$

Exercise

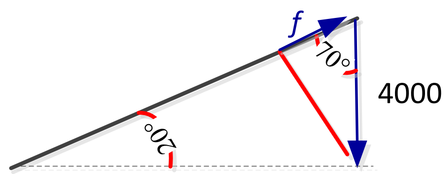
In Roman mythology, Sisyphus revealed a secret of Zeus and thus incurred the god's wrath. As punishment, Zeus banished him to Hades, where he was doomed for eternity to roll a rock uphill, only to have it roll back on him. If Sisyphus stands in front of a 4000-lb spherical rock on a 20° incline, then what force applied in the direction of the incline would keep the rock from rolling down the incline?



Solution

$$\cos 70° = \frac{f}{4000}$$

$$f = 4000\cos 70° \approx 1368\ lb$$



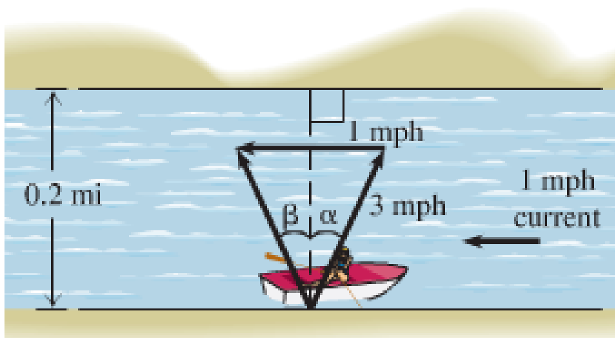
Exercise

A trigonometry student wants to cross a river that is 0.2 mi wide and has a current of 1 mph. The boat goes 3 mph in still water.

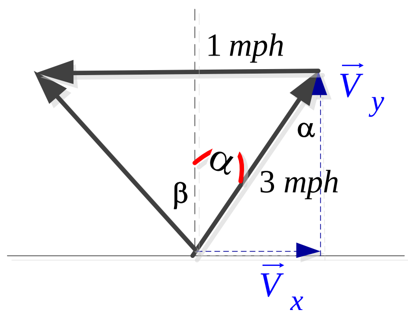
a) Write the distance the boats travels as a function of the angle β.

b) Write the actual speed of the boat as a function of α and β.

c) Write the time for the trip as a function of α. Find the angle α for which the student will cross the river in the shortest amount of time.



Solution



a) The vector in still water is given by:

$$V_x = V_S \cos(90° - \alpha) = 3\sin\alpha$$

$$V_y = 3\cos\alpha$$

$$\vec{V} = 3\sin\alpha\,\boldsymbol{i} + 3\cos\alpha\,\boldsymbol{j}$$

The vector of the current: $\vec{V}_c = -1\boldsymbol{i}$

Therefore, the actual direction and speed is determined by the vector:

$$v = (3\sin\alpha - 1)\,\boldsymbol{i} + 3\cos\alpha\,\boldsymbol{j}$$

The number ***t*** of hours it takes the boat to cross the river:

$$t = \frac{d}{v_y} = \frac{0.2}{3\cos\alpha}$$

The distance of the boat travels:

$$\begin{aligned}
d &= t|v| \\
&= \frac{0.2}{3\cos\alpha}\sqrt{(3\sin\alpha - 1)^2 + (3\cos\alpha)^2} \\
&= 0.2\sqrt{\frac{(3\sin\alpha - 1)^2}{(3\cos\alpha)^2} + \frac{(3\cos\alpha)^2}{(3\cos\alpha)^2}} \\
&= 0.2\sqrt{\left(\frac{3\sin\alpha - 1}{3\cos\alpha}\right)^2 + 1} \qquad \tan\beta = \frac{v_x}{v_y} = \frac{3\sin\alpha - 1}{3\cos\alpha} \\
&= 0.2\sqrt{(\tan\beta)^2 + 1} \\
&= 0.2\sqrt{\tan^2\beta + 1} \qquad \sec^2\beta = \tan^2\beta + 1 \\
&= 0.2\sqrt{\sec^2\beta} \\
&= 0.2|\sec\beta|
\end{aligned}$$

b) The actual speed $= \frac{d}{t}$

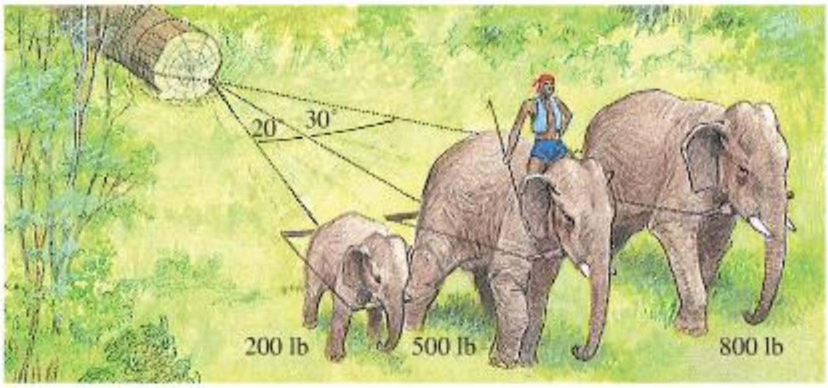
$$\begin{aligned}
&= \frac{0.2|\sec\beta|}{\frac{0.2}{3\cos\alpha}} \\
&= 3\cos\alpha|\sec\beta|
\end{aligned}$$

c) The shortest time for the boat to cross river is straight ahead which $\alpha = 0$

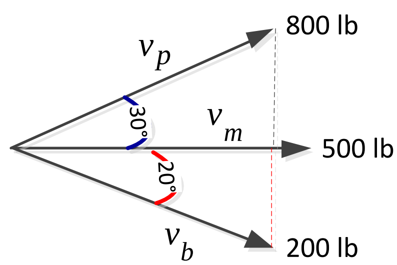
$$t = \frac{0.2}{3\cos 0^\circ} = \frac{0.2}{3} = \frac{2}{30} \; \boxed{= \frac{1}{15}\sec}$$

Exercise

Amal uses three elephants to pull a very large log out of the jungle. The papa elephant pulls with 800 lb of force, the mama elephant pulls with 500 lb of force, and the baby elephant pulls with 200 lb force. The angles between the forces are shown in the figure. What is the magnitude of the resultant of all three forces? If mama is pulling due east, then in what direction will the log move?



Solution



$$v_p = 800\cos 30°\boldsymbol{i} + 800\sin 30°\boldsymbol{j}$$

$$v_m = 500\boldsymbol{i}$$

$$v_b = 200\cos 20°\boldsymbol{i} - 200\sin 20°\boldsymbol{j}$$

The total force is determined by:

$$\begin{aligned} F &= v_p + v_m + v_b \\ &= 800\cos 30°\boldsymbol{i} + 800\sin 30°\boldsymbol{j} + 500\boldsymbol{i} + 200\cos 20°\boldsymbol{i} - 200\sin 20°\boldsymbol{j} \\ &= \left(800\cos 30° + 500 + 200\cos 20°\right)\boldsymbol{i} + \left(800\sin 30° - 200\sin 20°\right)\boldsymbol{j} \\ &= 1380.76\boldsymbol{i} + 331.60\boldsymbol{j} \end{aligned}$$

The magnitude of the resultant:

$$|F| = \sqrt{1380.76^2 + 331.60^2} \underline{\approx 1420\ lbs}$$

The direction: $\tan^{-1}\left(\frac{331.6}{1380.76}\right) \underline{\approx 13.5°}$ $\boxed{E13.5°N}$