SOLUTION

Manual

Lecture 1

Lecture 2

***Lecture* 3**

Lecture 4

Lecture 1

SOLUTION

Solution   *Section* 1.1 – Introduction to System of Linear Equations

Exercise

Find a solution for *x, y, z* to the system of equations:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3e+2\sqrt{2}+\pi \\ 6e+5\sqrt{2}+4\pi \\ 9e+8\sqrt{2}+7\pi \end{pmatrix}$$

Solution

$$\begin{pmatrix} x+2y+3z \\ 4x+5y+6z \\ 7x+8y+9z \end{pmatrix} = \begin{pmatrix} 3e+2\sqrt{2}+\pi \\ 6e+5\sqrt{2}+4\pi \\ 9e+8\sqrt{2}+7\pi \end{pmatrix}$$

$$\Rightarrow \begin{cases} x+2y+3z = \pi+2\sqrt{2}+3e \\ 4x+5y+6z = 4\pi+5\sqrt{2}+6e \\ 7x+8y+9z = 7\pi+8\sqrt{2}+9e \end{cases}$$

Solution: $x=\pi \quad y=\sqrt{2} \quad z=e$

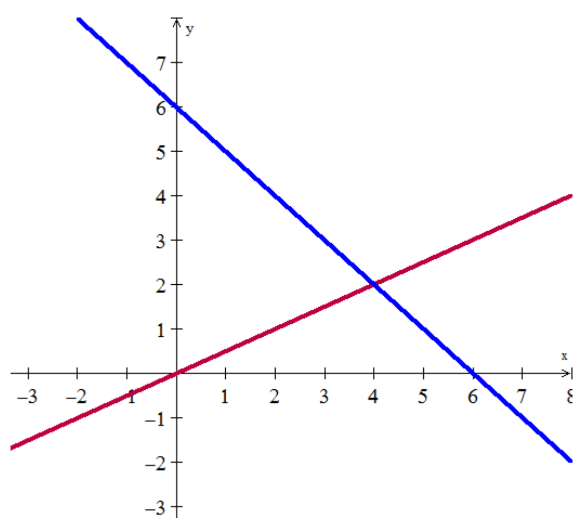
Exercise

Draw the two pictures in two planes for the equations: $x-2y=0, \quad x+y=6$

Solution

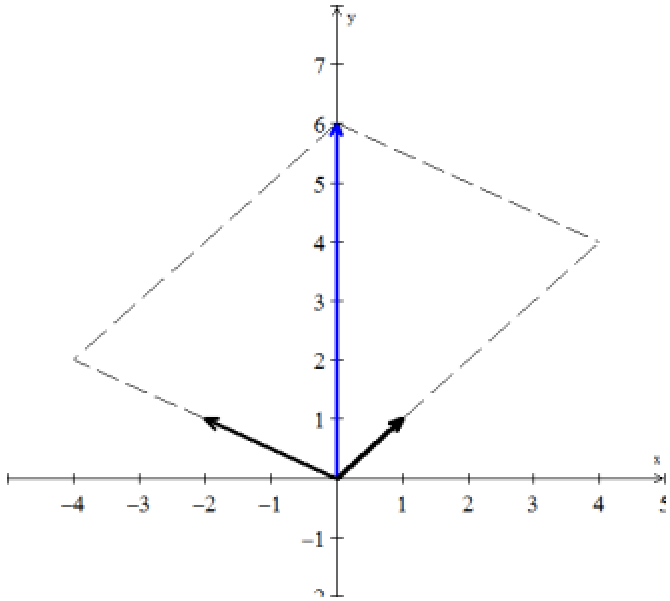
The matrix form of the 2 equations:

$$\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$



Row picture is the 2 lines from the given equations and their intersection is the point (4, 2) which is the solution for the system.

Column Picture is the column vectors $\begin{pmatrix} 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} -2 & 1 \end{pmatrix}$



$$x\begin{pmatrix}1\\1\end{pmatrix}+y\begin{pmatrix}-2\\1\end{pmatrix}=4\begin{pmatrix}1\\1\end{pmatrix}+2\begin{pmatrix}-2\\1\end{pmatrix}=\begin{pmatrix}0\\6\end{pmatrix}$$

The parallelogram shows how the solution vector $\begin{pmatrix}0 & 6\end{pmatrix}$ can be written as the linear combination of the column vectors.

Exercise

Normally 4 planes in 4-dimensional space meet at a ________. Normally 4 column vectors in 4-deimensional space can combine to produce b. what combinations of $(1,\ 0,\ 0,\ 0)$, $(1,\ 1,\ 0,\ 0)$, $(1,\ 1,\ 1,\ 0)$, $(1,\ 1,\ 1,\ 1)$ produces $b=(3,\ 3,\ 3,\ 2)$? What 4 equations for $x,\ y,\ z,\ w$ are you solving?

Solution

Normally 4 planes in 4-dimensional space meet at a ***point***.

The combination of the vectors producing b is:

$$0\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+0\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+1\begin{pmatrix}1\\1\\1\\0\end{pmatrix}+2\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}3\\3\\3\\2\end{pmatrix}$$

$$1\begin{pmatrix}1\\1\\1\\0\end{pmatrix}+2\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}3\\3\\3\\2\end{pmatrix}$$

The system of equations that satisfies the given vectors is:

$$\begin{cases} x+y+z+w=3 \\ \quad y+z+w=3 \\ \qquad z+w=3 \\ \qquad\quad w=2 \end{cases}$$

Exercise

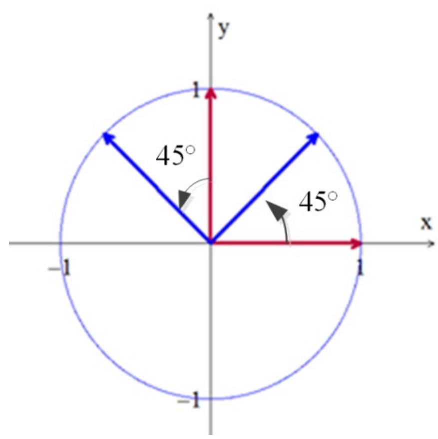
What 2 by 2 matrix A rotates every vector through 45° ?

The vector (1, 0) goes to $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. The vector (0, 1) goes to $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Those determine the matrix. Draw these particular vectors is the xy-plane and find A.

Solution

$$A = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$



Exercise

What two vectors are obtained by rotating the plane vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by 30° (cw) ?

Write a matrix A such that for every vector v in the plane, Av is the vector obtained by rotating v clockwise by 30°.

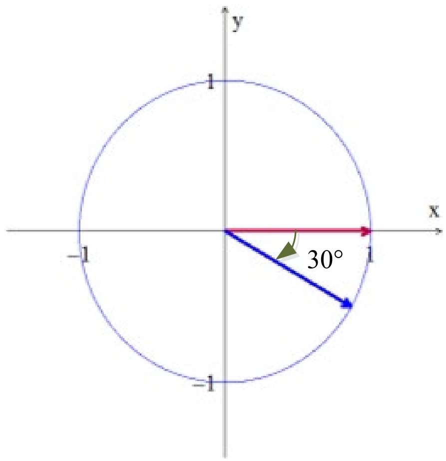
Find a matrix B such that for every 3-dimensional vector v, the vector Bv is the reflection of v through the plane $x+y+z=0$. $Hint: v=(1,\ 0,\ 0)$

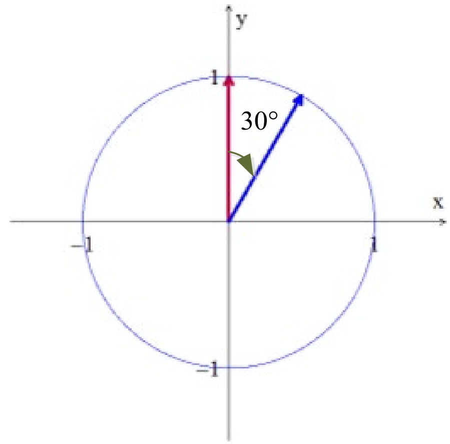
Solution

Rotating the vectors by 30° (cw) yields:

For the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ yields to $\begin{pmatrix} \cos(-30°) \\ \sin(-30°) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}$

And for the vector $\begin{bmatrix}0\\1\end{bmatrix}$ yields to $\begin{pmatrix}\sin(30°)\\ \cos(30°)\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}$





The desired matrix is: $A = \begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$

To get 1 from $\frac{\sqrt{3}}{2}$ is to multiply by $\frac{2}{\sqrt{3}} = 2\frac{1}{\sqrt{3}}$

The unit vector to the plane $x+y+z=0$ is $\hat{u} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$Bv = B\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{2}{\sqrt{3}}\hat{u}$$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{2}{\sqrt{3}}\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}\frac{2}{3}\\ \frac{2}{3}\\ \frac{2}{3}\end{pmatrix}$$

$$B\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}-\frac{2}{\sqrt{3}}\hat{u}$$

$$=\begin{pmatrix}0\\0\\1\end{pmatrix}-\begin{pmatrix}\frac{2}{3}\\\frac{2}{3}\\\frac{2}{3}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{3}\\-\frac{2}{3}\\-\frac{2}{3}\end{pmatrix}$$

$$B\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}-\frac{2}{\sqrt{3}}\hat{u}$$

$$=\begin{pmatrix}0\\1\\0\end{pmatrix}-\begin{pmatrix}\frac{2}{3}\\\frac{2}{3}\\\frac{2}{3}\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{2}{3}\\\frac{1}{3}\\-\frac{2}{3}\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{2}{3}\\-\frac{2}{3}\\\frac{1}{3}\end{pmatrix}$$

The solution: $\begin{pmatrix}\frac{1}{3}&-\frac{2}{3}&-\frac{2}{3}\\-\frac{2}{3}&\frac{1}{3}&-\frac{2}{3}\\-\frac{2}{3}&-\frac{2}{3}&\frac{1}{3}\end{pmatrix}$

Exercise

Find a system of linear equation corresponding to the given augmented matrix

$$\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$$

Solution

$$\begin{cases} \quad\;\; 3x_2 - x_3 - x_4 = -1 \\ 5x_1 + 2x_2 \qquad - 3x_4 = -6 \end{cases}$$

Exercise

Find a system of linear equation corresponding to the given augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \\ 5 & -6 & 1 & 1 \\ -8 & 0 & 0 & 3 \end{bmatrix}$$

Solution

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ -4x_1 - 3x_2 - 2x_3 = -1 \\ 5x_1 - 6x_2 + x_3 = 1 \\ -8x_1 \qquad\qquad = 3 \end{cases}$$

Exercise

Find the augmented matrix for the given system of linear equations.

$$\begin{cases} -2x_1 = 6 \\ 3x_1 = 8 \\ 9x_1 = -3 \end{cases}$$

Solution

$$\left[\begin{array}{c|c} -2 & 6 \\ 3 & 8 \\ 9 & -3 \end{array}\right]$$

Exercise

Find the augmented matrix for the given system of linear equations.

$$\begin{cases} 3x_1 - 2x_2 = -1 \\ 4x_1 + 5x_2 = 3 \\ 7x_1 + 3x_2 = 2 \end{cases}$$

Solution

$$\left[\begin{array}{cc|c} 3 & -2 & -1 \\ 4 & 5 & 3 \\ 7 & 3 & 2 \end{array}\right]$$

Exercise

Find the augmented matrix for the given system of linear equations.

$$\begin{cases} 2x_1 \qquad + 2x_3 = 1 \\ 3x_1 - x_2 + 4x_3 = 7 \\ 6x_1 + x_2 - x_3 = 0 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 1 \\ 3 & -1 & 4 & 7 \\ 6 & 1 & -1 & 0 \end{array}\right]$$

Exercise

Determine the size of the matrix $\begin{bmatrix} 1 & 2 & -4 \\ 3 & 4 & 6 \\ 0 & 1 & 2 \end{bmatrix}$

Solution

3 Rows & 3 columns

Size $\quad 3 \times 3$

Exercise

Determine the size of the matrix $\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

Solution

3 Rows & 4 columns

Size 3×4

Exercise

Determine the size of the matrix $\begin{bmatrix} 2 & 1 & -1 & -1 \\ -6 & 2 & 0 & 1 \end{bmatrix}$

Solution

2 Rows & 4 columns

Size 2×4

Solution *Section* 1.2 – Gaussian Elimination

Exercise

When elimination is applied to the matrix $A=\begin{bmatrix}3&1&0\\6&9&2\\0&1&5\end{bmatrix}$

a) What are the first and second pivots?

b) What is the multiplier l_{21} in the first step (l_{21} times row 1 is subtracted from row 2)?

c) What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3?

d) What is the multiplier $l_{31}=0$, subtracting 0 times row 1 from row 3?

Solution

a) The first pivot is 3 and when 2 times row 1 is subtracted from row 2, the second pivot is revealed as 7.

$$\begin{bmatrix}3&1&0\\6&9&2\\0&1&5\end{bmatrix}\quad \begin{matrix}\textit{subtract 2 times row.1}\\ \textit{from row.2}\end{matrix}\quad \begin{bmatrix}3&1&0\\0&\boxed{7}&2\\0&1&5\end{bmatrix}$$

b) The multiplier l_{21} in the first step is $\frac{6}{3}=2$.

c) If we reduce the entry 9 to 2, that drop of 7 in the a_{22} position would force a row exchange.

$$\begin{bmatrix}3&1&0\\6&9&2\\0&1&5\end{bmatrix}\quad \begin{matrix}\textit{subtract 7 times row.1}\\ \textit{from row.2}\end{matrix}\quad \begin{bmatrix}3&1&0\\-15&\boxed{2}&2\\0&1&5\end{bmatrix}$$

d) The multiplier l_{31} is already zero because $a_{31}=0$ and no needs row elimination.

Exercise

Use elimination to reach upper triangular matrices $\boldsymbol{U}$. Solve by back substitution or explain why this impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the $-x$ in equation (3).

$$\begin{cases}x+y+z=7\\x+y-z=5\\x-y+z=3\end{cases}\qquad\qquad \begin{cases}x+y+z=7\\x+y-z=5\\-x-y+z=3\end{cases}$$

Solution

For the *first* system:

$$\begin{array}{lcl} x+y+z=7 & \textit{subtract eqn.1} & x+y+z=7 \\ x+y-z=5 & \textit{from eqn.2} & 0y-2z=-2 \\ x-y+z=3 & \textit{from eqn.3} & -2y-0z=-4 \end{array}$$

$$\begin{array}{lcl} x+y+z=7 & & 1x+\ y+\ z=7 \\ x+y-z=5 & \textit{Exchange eqn.2} & -2y-0z=-4 \\ x-y+z=3 & \textit{and eqn.3} & -2z=-2 \end{array}$$

The solutions are: $z=1 \quad y=2 \quad x=4$ and the pivots are 1, -2, -2.

For the *second* system:

$$\begin{array}{lcl} x+y+z=7 & \textit{Subtract eqn.1} & x+y+z=7 \\ x+y-z=5 & \textit{from eqn.2} & 0y-2z=-2 \\ -x-y+z=3 & \begin{array}{l}\textit{Add eqn.1} \\ \textit{to eqn.3}\end{array} & 0y+2z=10 \end{array}$$

$$\begin{array}{lcl} x+y+z=7 & & x+y+z=7 \\ 0y-2z=-2 & \textit{Add eqn.2} & 0y-2z=-2 \\ 0y+2z=10 & \textit{to eqn.3} & \boxed{0z=8} \end{array}$$

The three planes don't meet. But if we change '3' in the last equation to '-5'

$$\begin{array}{lcl} x+y+z=7 & \textit{Subtract eqn.1} & x+y+z=7 \\ x+y-z=5 & \textit{from eqn.2} & 0y-2z=-2 \\ -x-y+z=-5 & \begin{array}{l}\textit{Add eqn.1} \\ \textit{to eqn.3}\end{array} & 0y+2z=2 \end{array}$$

$$\begin{array}{ll} x+y+z=7 & x+y=6 \\ 0y-2z=-2 & \\ 0y+2z=10 & z=1 \end{array}$$

There are unique infinite many solutions!

The three planes now meet along a whole line.

Exercise

For which numbers a does the elimination break down (1) permanently (2) temporarily

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= 6 \end{aligned}$$

Solve for x and y after fixing the second breakdown by a row change.

Solution

The matrix form is: $\begin{pmatrix} a & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$

If $a = 0$, the elimination brakes down temporarily.

$$\begin{pmatrix} 4 & 6 \\ 0 & \boxed{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

The system is in upper triangular form and entry row 2 column 2 is not equal to zero, therefore the system has a solution.

If $a \neq 0$,

$$\begin{pmatrix} a & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} \qquad R_2 - \tfrac{4}{a}R_1$$

$$\begin{pmatrix} a & 3 \\ 0 & 6 - \frac{12}{a} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 + \frac{12}{a} \end{pmatrix}$$

$$6 - \frac{12}{a} = 0$$

$$\frac{12}{a} = 6$$

$$a = \frac{12}{6}$$

$$\underline{= 2}\,|$$

If $a = 2$,

$$\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \end{pmatrix}$$

The system will fail and has no solution.

If $a \neq 2$;

$$\begin{pmatrix} a & 3 \\ 0 & 6 - \frac{12}{a} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 6 + \frac{12}{a} \end{pmatrix}$$

The system has a unique solution.

Exercise

Find the pivots and the solution for these four equations:

$$\begin{aligned} 2x + y \qquad\quad &= 0 \\ x + 2y + z \qquad &= 0 \\ y + 2z + t &= 0 \\ z + 2t &= 5 \end{aligned}$$

Solution

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} \quad R_2 - \tfrac{1}{2}R_1$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 1.5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} \quad R_3 - \tfrac{2}{3}R_2$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{pmatrix} \quad R_4 - \tfrac{3}{4}R_3$$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 5 \end{pmatrix} \quad \begin{aligned} & 2x = -y \quad (1) \\ & \tfrac{3}{2}y + z = 0 \\ & \tfrac{4}{3}z + t = 0 \\ & \tfrac{5}{4}t = 5 \quad (4) \end{aligned}$$

$$\begin{cases} (1) & \to & x = -2\frac{1}{2} = -1 \\ y = -z\frac{2}{3} = -(-3)\frac{2}{3} & \to & y = 2 \\ \frac{4}{3}z = -t & \to & z = -4\frac{3}{4} = -3 \\ (4) & \to & t = 4 \end{cases}$$

The pivots are diagonal entries and the solution is: $(-1,\ 2,\ -3,\ 4)$

Exercise

Look for a matrix that has row sums 4 and 8, and column sums 2 and s.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{matrix} a+b=4 & a+c=2 \\ c+d=8 & b+d=s \end{matrix}$$

The four equations are solvable only if $s =$ ____. Then find two different matrices that have the correct row and column sums.

Solution

$$\begin{array}{r} a+b=4 \\ + \quad c+d=8 \\ \hline a+c+b+d=12 \end{array}$$

$2+s=12$

$\underline{s=10}$

Exercise

Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of A is a _______ of the first two rows. Find a third equation that can't be solved together with $x+y+z=0$ and $x-2y-z=1$

Solution

The system is singular if row 3 of A is a ***linear combination*** of the first two rows.

There are many possible of a third equation that can't be solved together with $x+y+z=0$ and $x-2y-z=1$.

$$\begin{array}{rr} 3 \text{ times } 1^{st} \text{ equation} & 3x+3y+3z \\ \text{minus } 2nd & -x+2y+z \\ \hline & 2x+5y+4z=1 \end{array}$$

Exercise

Use the Gauss-Jordan method to solve the system
$$\begin{aligned} x - y + 5z &= -6 \\ 3x + 3y - z &= 10 \\ x + 3y + 2z &= 5 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 3 & -1 & 10 \\ -3 & 3 & -15 & 18 \\ \hline 0 & 6 & -16 & 28 \end{array} \qquad \begin{array}{cccc} 1 & 3 & 2 & 5 \\ -1 & 1 & -5 & 6 \\ \hline 0 & 4 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 4 & -3 & 11 \end{array}\right] \begin{array}{l} \\ \frac{1}{6}R_2 \\ \\ \end{array} \qquad 0 \quad 1 \quad -\frac{8}{3} \quad \frac{14}{3}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 4 & -3 & 11 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ \\ R_3 - 4R_2 \end{array}$$

$$\begin{array}{cccc} 0 & 4 & -3 & 11 \\ 0 & -4 & \frac{32}{3} & -\frac{56}{3} \\ \hline 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array} \qquad \begin{array}{cccc} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ \hline 1 & 0 & \frac{7}{3} & -\frac{4}{3} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array}\right] \begin{array}{l} \\ \\ \frac{3}{23}R_3 \end{array} \qquad 0 \quad 0 \quad 1 \quad -1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 - \frac{7}{3}R_3 \\ R_2 + \frac{8}{3}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & \frac{7}{3} & -\frac{4}{3} \\ 0 & 0 & -\frac{7}{3} & \frac{7}{3} \\ \hline 1 & 0 & 0 & 1 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & \frac{8}{3} & -\frac{8}{3} \\ \hline 0 & 1 & 0 & 2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

S***olution***: $\underline{(1,\ 2,\ -1)}$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 2x-y+4z=-3 \\ x-2y-10z=-6 \\ 3x \qquad +4z=7 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & -3 \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array}\right] \begin{array}{l} \frac{1}{2}R_1 \\ \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 1 & -2 & -10 & -6 \\ 3 & 0 & 4 & 7 \end{array}\right] \begin{array}{l} \\ R_2-R_1 \\ R_3-3R_1 \end{array} \qquad \begin{array}{cccc} 1 & -2 & -10 & -6 \\ -1 & \frac{1}{2} & -2 & \frac{3}{2} \\ \hline 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \end{array} \qquad \begin{array}{cccc} 3 & 0 & 4 & 7 \\ -3 & \frac{3}{2} & -6 & \frac{9}{2} \\ \hline 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}\right] \begin{array}{l} \\ -\frac{2}{3}R_2 \\ \\ \end{array} \qquad \begin{array}{cccc} 0 & 1 & 8 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & 1 & 8 & 3 \\ 0 & \frac{3}{2} & -2 & \frac{23}{2} \end{array}\right] \begin{array}{l} R_1+\frac{1}{2}R_2 \\ \\ R_3-\frac{3}{2}R_2 \end{array} \qquad \begin{array}{cccc} 0 & \frac{3}{2} & -2 & \frac{23}{2} \\ 0 & -\frac{3}{2} & -12 & -\frac{9}{2} \\ \hline 0 & 0 & -14 & 7 \end{array} \qquad \begin{array}{cccc} 1 & -\frac{1}{2} & 2 & -\frac{3}{2} \\ 0 & \frac{1}{2} & 4 & \frac{3}{2} \\ \hline 1 & 0 & 6 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & -14 & 7 \end{array}\right] \begin{array}{l} \\ \\ -\frac{1}{14}R_3 \end{array} \qquad \begin{array}{cccc} 0 & 0 & 1 & -\frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \begin{array}{l} R_1-6R_3 \\ R_2-8R_3 \\ \\ \end{array} \qquad \begin{array}{cccc} 1 & 0 & 6 & 0 \\ 0 & 0 & -6 & 3 \\ \hline 1 & 0 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 1 & 8 & 3 \\ 0 & 0 & -8 & 4 \\ \hline 0 & 1 & 0 & 7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right]$$

Solution: $\left(3,\ 7,\ -\frac{1}{2}\right)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 4x+3y-5z=-29 \\ 3x-7y-z=-19 \\ 2x+5y+2z=-10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 4 & 3 & -5 & -29 \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array}\right] \begin{array}{l} \frac{1}{4}R_1 \\ \\ \\ \end{array} \qquad 1 \quad \frac{3}{4} \quad -\frac{5}{4} \quad -\frac{29}{4}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 3 & -7 & -1 & -19 \\ 2 & 5 & 2 & -10 \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3-2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & -7 & -1 & -19 \\ -3 & -\frac{9}{4} & \frac{15}{4} & \frac{87}{4} \\ \hline 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \end{array} \qquad \begin{array}{cccc} 2 & 5 & 2 & -10 \\ -2 & -\frac{3}{2} & \frac{5}{2} & \frac{29}{2} \\ \hline 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{37}{4} & \frac{11}{4} & \frac{11}{4} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}\right] \begin{array}{l} \\ -\frac{4}{37}R_2 \\ \\ \end{array} \qquad 0 \quad 1 \quad -\frac{11}{37} \quad -\frac{11}{37}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \end{array}\right] \begin{array}{l} R_1-\frac{3}{4}R_2 \\ \\ R_3-\frac{7}{2}R_2 \end{array}$$

$$\begin{array}{cccc} 1 & \frac{3}{4} & -\frac{5}{4} & -\frac{29}{4} \\ 0 & -\frac{3}{4} & \frac{33}{148} & \frac{33}{148} \\ \hline 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \end{array} \qquad \begin{array}{cccc} 0 & \frac{7}{2} & \frac{9}{2} & \frac{9}{2} \\ 0 & -\frac{7}{2} & \frac{77}{72} & \frac{77}{72} \\ \hline 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{401}{72} & \frac{401}{72} \end{array}\right] \begin{array}{l} \\ \\ \frac{72}{401}R_3 \end{array} \qquad 0 \quad 0 \quad 1 \quad 1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1+\frac{38}{37}R_3 \\ R_2+\frac{11}{37}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -\frac{38}{37} & -\frac{260}{37} \\ 0 & 0 & \frac{38}{37} & \frac{38}{37} \\ \hline 1 & 0 & 0 & -6 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{11}{37} & -\frac{11}{37} \\ 0 & 0 & \frac{11}{37} & \frac{11}{37} \\ \hline 0 & 1 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Solution: $(-6,\ 0,\ 1)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x+2y-3z=-15 \\ 2x-3y+4z=18 \\ -3x+y+z=1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 2 & -3 & 4 & 18 \\ -3 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3+3R_1 \end{array}$$

$$\begin{array}{cccc} -2 & -4 & 6 & 30 \\ 2 & -3 & 4 & 18 \\ \hline 0 & -7 & 10 & 48 \end{array} \qquad \begin{array}{cccc} 3 & 6 & -9 & -45 \\ -3 & 1 & 1 & 1 \\ \hline 0 & 7 & -8 & -44 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \end{array}\right] \quad -\frac{1}{7}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -15 \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 7 & -8 & -44 \end{array}\right] \begin{array}{l} R_1-2R_2 \\ \\ R_3-7R_2 \end{array}$$

$$\begin{array}{cccc} 1 & 2 & -3 & -15 \\ 0 & -2 & \frac{20}{7} & \frac{96}{7} \\ \hline 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \end{array} \qquad \begin{array}{cccc} 0 & -7 & 10 & 48 \\ 0 & 7 & -8 & -44 \\ \hline 0 & 0 & 2 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 2 & 4 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{2}R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1+\frac{1}{7}R_3 \\ R_2+\frac{10}{7}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 0 & \frac{1}{7} & \frac{2}{7} \\ \hline 1 & 0 & 0 & -1 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -\frac{10}{7} & -\frac{48}{7} \\ 0 & 0 & \frac{10}{7} & \frac{20}{7} \\ \hline 0 & 1 & 0 & -4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $(-1,\ -4,\ 2)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x+2y+3z=10 \\ 4x+5y+6z=11 \\ 7x+8y+9z=12 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \end{array}\right] \quad \begin{array}{l} R_2 - 4R_1 \\ R_3 - 7R_1 \end{array}$$

$$\begin{array}{cccc} -4 & -8 & -12 & -40 \\ 4 & 5 & 6 & 11 \\ \hline 0 & -3 & -6 & -29 \end{array} \qquad \begin{array}{cccc} -7 & -14 & -21 & -70 \\ 7 & 8 & 9 & 12 \\ \hline 0 & -6 & -12 & -58 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -3 & -6 & -29 \\ 0 & -6 & -12 & -58 \end{array}\right] \quad \tfrac{1}{3}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & \frac{29}{3} \\ 0 & -6 & -12 & -58 \end{array}\right] \quad R_3 + 6R_2$$

$$\begin{array}{cccc} 0 & -6 & -12 & -58 \\ 0 & 6 & 12 & 58 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & \frac{29}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

let z be the variable

From Row 1 $\Rightarrow y + 2z = \frac{29}{3}$

$$\underline{y = \frac{29}{3} - 2z}$$

From Row 1 $\Rightarrow x + 2y + 3z = 10$

$$x = 10 - 2y - 3z$$

$$x = 10 - 2\left(\frac{29}{3} - 2z\right) - 3z$$

$$x = 10 - \frac{58}{3} + 4z - 3z$$

$$\underline{x = z - \frac{28}{3}}$$

Solution: $\underline{\left(z - \frac{28}{3},\ \frac{29}{3} - 2z,\ z\right)}$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} 2x + y + 2z = 4 \\ 2x + 2y \quad\ \ = 5 \\ 2x - y + 6z = 2 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \quad \frac{1}{2}R_1 \qquad 1 \quad \frac{1}{2} \quad 1 \quad 2$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 2 & 2 & 0 & 5 \\ 2 & -1 & 6 & 2 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 2R_1 \end{array} \qquad \begin{array}{cccc} 2 & 2 & 0 & 5 \\ -2 & -1 & -2 & -4 \\ \hline 0 & 1 & -2 & 1 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 6 & 2 \\ -2 & -1 & -2 & -4 \\ \hline 0 & -2 & 4 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & -2 \end{array}\right] \quad \begin{array}{l} R_1 - \frac{1}{2}R_2 \\ \\ R_3 + 2R_2 \end{array} \qquad \begin{array}{cccc} 1 & \frac{1}{2} & 1 & 2 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \hline 1 & 0 & 2 & \frac{3}{2} \end{array} \qquad \begin{array}{cccc} 0 & -2 & 4 & -2 \\ 0 & 2 & -4 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & \frac{3}{2} \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From Row 3: $0 = 0$ is a true statement. Let ***z*** be the variable.

From Row 2: $y - 2z = 1$

$$\boxed{y = 1 + 2z}$$

From Row 1: $x + 2z = \frac{3}{2}$

$$\boxed{x = -2z + \frac{3}{2}}$$

$\therefore$ ***Solution***: $\boxed{\left(-2z + \frac{3}{2},\ 2z + 1,\ z\right)}$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} -1 & -2 & 3 & 1 \\ 1 & 1 & 2 & 8 \\ \hline 0 & -1 & 5 & 9 \end{array} \qquad \begin{array}{cccc} 3 & -7 & 4 & 10 \\ -3 & -3 & -6 & -24 \\ \hline 0 & -10 & -2 & -14 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right] -R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 10R_2 \end{array}$$

$$\begin{array}{cccc} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ \hline 1 & 0 & 7 & 17 \end{array} \qquad \begin{array}{cccc} 0 & -10 & -2 & -14 \\ 0 & 10 & -50 & -90 \\ \hline 0 & 0 & -52 & -104 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right] -\frac{1}{52}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 - 7R_3 \\ R_2 + 5R_3 \\ \\ \end{array}$$

$$\begin{array}{cccc} 1 & 0 & 7 & 17 \\ 0 & 0 & -7 & -14 \\ \hline 1 & 0 & 0 & 3 \end{array} \qquad \begin{array}{cccc} 0 & 1 & -5 & -9 \\ 0 & 0 & 5 & 10 \\ \hline 0 & 1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$\therefore$ ***Solution***: $(3,\ 1,\ 2)$

Exercise

Use the Gauss-Jordan method to solve the system $\begin{cases} x + 2y + z = 8 \\ -x + 3y - 2z = 1 \\ 3x + 4y - 7z = 10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ -1 & 3 & -2 & 1 \\ 3 & 4 & -7 & 10 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & -2 & -10 & -14 \end{array}\right] \begin{array}{l} 5R_1 - 2R_2 \\ \\ 5R_3 + 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 5 & 0 & 7 & 22 \\ 0 & 5 & -1 & 9 \\ 0 & 0 & -52 & -52 \end{array}\right] \begin{array}{l} (2) \\ (1) \\ \rightarrow \ \underline{z=1} \end{array}$$

$(1) \quad \rightarrow \quad 5y = 10$

$\underline{y = 2}$

$(1) \quad \rightarrow \quad 5x = 22 - 7$

$\underline{x = 3}$

$\therefore$ ***Solution***: $\underline{(3,\ 2,\ 1)}$

Exercise

Use augmented elimination to solve linear system
$$\begin{aligned} 2x - 5y + 3z &= 1 \\ x - 2y - 2z &= 8 \end{aligned}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 2 & -5 & 3 & 1 \end{array}\right] \ R_2 - 2R_1$$

2	−5	3	1
−2	4	4	−16
0	−1	7	−15

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & -1 & 7 & -15 \end{array}\right] \ -R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & 1 & -7 & 15 \end{array}\right] \ R_1 + 2R_2$$

1	−2	−2	8
0	2	−14	30
1	0	−16	38

$$\left[\begin{array}{ccc|c} 1 & 0 & -16 & 38 \\ 0 & 1 & -7 & 15 \end{array}\right] \begin{array}{l} \rightarrow \ x - 16z = 38 \\ \rightarrow \ y - 7z = 15 \end{array}$$

$$\begin{cases} x = 16z + 38 \\ y = 7z + 15 \end{cases}$$

$\therefore$ ***Solution***: $\underline{(16z + 38,\ 7z + 15,\ z)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 5 \\ 1 & -1 & 1 & -2 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-R_1 \end{array} \qquad \begin{array}{cccc} 2 & 1 & -1 & 5 \\ -2 & -2 & -2 & -4 \\ \hline 0 & -1 & -3 & 1 \end{array} \qquad \begin{array}{cccc} 1 & -1 & 1 & -2 \\ -1 & -1 & -1 & -2 \\ \hline 0 & -2 & 0 & -4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & 1 \\ 0 & -2 & 0 & -4 \end{array}\right] \begin{array}{c} (2) \\ (1) \\ -2y=-4 \end{array}$$

$\underline{y=2}|$

$(1) \to \quad -y-3z=1$

$\qquad 3z=-1-2$

$\qquad \underline{z=-1}|$

$(2) \to \quad x+y+z=2$

$\qquad x=2-2+1$

$\qquad \underline{=1}|$

$\therefore$ ***Solution***: $\underline{(1,\ 2,\ -1)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x+y+z=9 \\ -x-y+z=1 \\ 3x-y+z=9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ -1 & -1 & 1 & 1 \\ 3 & -1 & 1 & 9 \end{array}\right] \begin{array}{l} \\ 2R_2+R_1 \\ 2R_3-3R_1 \end{array} \qquad \begin{array}{cccc} -2 & -2 & 2 & 2 \\ 2 & 1 & 1 & 9 \\ \hline 0 & -1 & 3 & 11 \end{array} \qquad \begin{array}{cccc} 6 & -2 & 2 & 18 \\ -6 & -3 & -3 & -27 \\ \hline 0 & -5 & -1 & -9 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \\ 0 & -5 & -1 & -9 \end{array}\right] \begin{array}{l} \\ \\ R_3-5R_2 \end{array} \qquad \begin{array}{cccc} 0 & -5 & -1 & -9 \\ 0 & 5 & -15 & -55 \\ \hline 0 & 0 & -16 & -64 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 0 & -1 & 3 & 11 \\ 0 & 0 & -16 & -64 \end{array}\right] \quad \begin{array}{c} (2) \\ (1) \\ -16z = -64 \end{array}$$

$\underline{z = 4}$

$$(1) \rightarrow \quad -y + 3z = 11$$
$$y = 12 - 11$$
$$\underline{= 1}$$

$$(2) \rightarrow \quad 2x + y + z = 9$$
$$2x = 9 - 1 - 4$$
$$\underline{x = 2}$$

$\therefore$ ***Solution***: $\underline{(2,\ 1,\ 4)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ -3 & 6 & 2 & 11 \end{array}\right] \quad R_3 + 3R_1 \qquad \begin{array}{cccc} -3 & 6 & 2 & 11 \\ 3 & 15 & -3 & -12 \\ \hline 0 & 21 & -1 & -1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ 0 & 21 & -1 & -1 \end{array}\right] \quad R_3 - 7R_2 \qquad \begin{array}{cccc} 0 & 21 & -1 & -1 \\ 0 & -21 & 7 & 7 \\ \hline 0 & 0 & 6 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & -4 \\ 0 & 3 & -1 & -1 \\ 0 & 0 & 6 & 6 \end{array}\right] \quad \begin{array}{l} \rightarrow x + 5y - z = -4 \quad (2) \\ \rightarrow 3y - z = -1 \quad (1) \\ \rightarrow 6z = 6 \end{array}$$

$\underline{z = 1}$

$$(1) \rightarrow \quad 3y = -1 + 1$$
$$\underline{y = 0}$$

$$(2) \rightarrow \quad x = -4 + 1$$
$$\underline{x = -3}$$

$\therefore$ ***Solution***: $\underline{(-3,\ 0,\ 1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x+3y+4z=14 \\ 2x-3y+2z=10 \\ 3x-y+z=9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 2 & -3 & 2 & 10 \\ 3 & -1 & 1 & 9 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-3R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -3 & 2 & 10 \\ -2 & -6 & -8 & -28 \\ \hline 0 & -9 & -6 & -18 \end{array} \qquad \begin{array}{cccc} 3 & -1 & 1 & 9 \\ -3 & -9 & -12 & -42 \\ \hline 0 & -10 & -11 & -33 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -9 & -6 & -18 \\ 0 & -10 & -11 & -33 \end{array}\right] \begin{array}{l} \\ \\ 9R_3-10R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -90 & -99 & -297 \\ 0 & 90 & 60 & 180 \\ \hline 0 & 0 & -39 & -117 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -9 & -6 & -18 \\ 0 & 0 & -39 & -117 \end{array}\right] \quad \begin{array}{rl} x+3y+4z=14 & (3) \\ -9y-6z=-18 & (2) \\ -39z=-117 & (1) \end{array}$$

$(1)\rightarrow \quad z=\dfrac{117}{39}$

$\qquad = 3$

$(2)\rightarrow \quad 9y=18-6(3)$

$\qquad 9y=0$

$\qquad y=0$

$(3)\rightarrow \quad x=14-12$

$\qquad x=2$

$\therefore$ ***Solution***: $(2,\ 0,\ 3)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x+4y-z=20 \\ 3x+2y+z=8 \\ 2x-3y+2z=-16 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 3 & 2 & 1 & 8 \\ 2 & -3 & 2 & -16 \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3-2R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 2 & 1 & 8 \\ -3 & -12 & 3 & -60 \\ \hline 0 & -10 & 4 & -52 \end{array} \qquad \begin{array}{cccc} 2 & -3 & 2 & -16 \\ -2 & -8 & 2 & -40 \\ \hline 0 & -11 & 4 & -56 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 0 & -10 & 4 & -52 \\ 0 & -11 & 4 & -56 \end{array}\right] \quad 10R_3 - 11R_2 \qquad \begin{array}{cccc} 0 & -110 & 40 & -560 \\ 0 & 110 & -44 & 572 \\ \hline 0 & 0 & -4 & 12 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 20 \\ 0 & -10 & 4 & -52 \\ 0 & 0 & -4 & 12 \end{array}\right] \quad \begin{aligned} x + 4y - z &= 20 \quad (3) \\ -10y + 4z &= -52 \quad (2) \\ -4z &= 12 \quad (1) \end{aligned}$$

$(1) \rightarrow \quad \underline{z = -3}$

$(2) \rightarrow \quad -10y = -52 + 12$

$-10y = -40$

$\underline{y = 4}$

$(3) \rightarrow \quad x = 20 - 16 - 3$

$\underline{x = 1}$

∴ ***Solution***: $\underline{(1,\ 4,\ -3)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 2 & -1 & 7 \\ 2 & -3 & 2 & -1 \end{array}\right] \quad R_3 - 2R_1 \qquad \begin{array}{cccc} 2 & -3 & 2 & -1 \\ -2 & -4 & -2 & -34 \\ \hline 0 & -7 & 0 & -35 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 17 \\ 0 & 2 & -1 & 7 \\ 0 & -7 & 0 & -35 \end{array}\right] \quad \begin{aligned} x + 2y + z &= 17 \quad (3) \\ 2y - z &= 7 \quad (2) \\ -7y &= -35 \quad (1) \end{aligned}$$

$(1) \rightarrow \quad \underline{y = 5}$

$(2) \rightarrow \quad z = 10 - 7$

$\underline{= 3}$

$(3) \rightarrow \quad x = 17 - 10 - 3$

$\underline{= 4}$

∴ ***Solution***: $\underline{(4,\ 5,\ 3)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} -2x+6y+7z=3 \\ -4x+5y+3z=7 \\ -6x+3y+5z=-4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ -4 & 5 & 3 & 7 \\ -6 & 3 & 5 & -4 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-3R_1 \end{array}$$

$$\begin{array}{cccc} -4 & 5 & 3 & 7 \\ 4 & -12 & -14 & -6 \\ \hline 0 & -7 & -11 & 1 \end{array} \qquad \begin{array}{cccc} -6 & 3 & 5 & -4 \\ 6 & -18 & -21 & -9 \\ \hline 0 & -15 & -16 & -13 \end{array}$$

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ 0 & -7 & -11 & 1 \\ 0 & -15 & -16 & -13 \end{array}\right] \begin{array}{l} \\ \\ 7R_3-15R_1 \end{array}$$

$$\begin{array}{cccc} 0 & -105 & -112 & -91 \\ 0 & 105 & 165 & -15 \\ \hline 0 & 0 & 53 & -106 \end{array}$$

$$\left[\begin{array}{ccc|c} -2 & 6 & 7 & 3 \\ 0 & -7 & -11 & 1 \\ 0 & 0 & 53 & -106 \end{array}\right] \quad \begin{array}{lr} -2x+6y+7z=3 & (3) \\ -7y-11z=1 & (2) \\ 53z=-106 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z=-2}$

$(2) \rightarrow \quad -7y=1-22$

$-7y=-21$

$\underline{y=3}$

$(3) \rightarrow \quad -2x=3-18+14$

$-2x=-1$

$\underline{x=\frac{1}{2}}$

$\therefore$ ***Solution***: $\underline{\left(\frac{1}{2},\ 3,\ -2\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x-y+z=1 \\ 3x-3y+4z=5 \\ 4x-2y+3z=4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 3 & -3 & 4 & 5 \\ 4 & -2 & 3 & 4 \end{array}\right] \begin{array}{l} \\ 2R_2-3R_1 \\ R_3-2R_1 \end{array}$$

$$\begin{array}{cccc} 6 & -6 & 8 & 10 \\ -6 & 3 & -3 & -3 \\ \hline 0 & -3 & 5 & 7 \end{array} \qquad \begin{array}{cccc} 4 & -2 & 3 & 4 \\ -4 & 2 & -2 & -2 \\ \hline 0 & 0 & 1 & 2 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 1 & | & 1 \\ 0 & -3 & 5 & | & 7 \\ 0 & 0 & 1 & | & 2 \end{bmatrix} \quad \begin{aligned} 2x - y + z &= 1 \quad (2) \\ -3y + 5z &= 7 \quad (1) \\ \underline{z = 2} \end{aligned}$$

$(1) \to \quad -3y = 7 - 10$

$-3y = -3$

$\underline{y = 1}$

$(3) \to \quad 2x = 1 + 1 - 2$

$\underline{x = 0}$

$\therefore$ ***Solution***: $\underline{(0,\ 1,\ 2)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x - 4y + 4z = 7 \\ x - y - 2z = 2 \\ 2x - 3y + 6z = 5 \end{cases}$

Solution

$$\begin{bmatrix} 1 & -1 & -2 & | & 2 \\ 3 & -4 & 4 & | & 7 \\ 2 & -3 & 6 & | & 5 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\begin{array}{cccc} 3 & -4 & 4 & 7 \\ -3 & 3 & 6 & -6 \\ \hline 0 & -1 & 10 & 1 \end{array} \qquad \begin{array}{cccc} 2 & -3 & 6 & 5 \\ -2 & 2 & 4 & -4 \\ \hline 0 & -1 & 10 & 1 \end{array}$$

$$\begin{bmatrix} 1 & -1 & -2 & | & 2 \\ 0 & -1 & 10 & | & 1 \\ 0 & -1 & 10 & | & 1 \end{bmatrix} \begin{matrix} \\ \\ R_3 = R_2 \end{matrix} \qquad \begin{aligned} &\to x - y - 2z = 2 \quad (2) \\ &\to -y + 10z = 1 \quad (1) \end{aligned}$$

$(1) \to \quad \underline{y = 10z - 1}$

$(2) \to \quad x = 2 + 10z - 1 + 2z$

$\underline{= 12z + 1}$

$\therefore$ ***Solution***: $\underline{(12z + 1,\ 10z - 1,\ z)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x - 2y - z = 2 \\ 2x - y + z = 4 \\ -x + y + z = 4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 2 & -1 & 1 & 4 \\ -1 & 1 & 1 & 4 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 1 & 4 \\ -2 & 4 & 2 & -4 \\ \hline 0 & 3 & 3 & 0 \end{array} \qquad \begin{array}{cccc} -1 & 1 & 1 & 4 \\ 1 & -2 & -1 & 2 \\ \hline 0 & -1 & 0 & 6 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & 0 & 6 \end{array}\right] \begin{array}{rl} x - 2y - z = 2 & (3) \\ 3y + 3z = 0 & (2) \\ -y = 6 & (1) \end{array}$$

$(1) \rightarrow \ \boxed{y = -6}$

$(2) \rightarrow \ z = -y$

$\qquad = \boxed{6}$

$(3) \rightarrow \ x = 2 - 12 + 6$

$\qquad = \boxed{-4}$

$\therefore$ ***Solution***: $\boxed{(-4,\ -6,\ 6)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + y + z = 3 \\ -y + 2z = 1 \\ -x + z = 0 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ -1 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 + R_1 \end{array} \qquad \begin{array}{cccc} -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 3 \\ \hline 0 & 1 & 2 & 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right] \begin{array}{l} \\ \\ R_3 + R_2 \end{array} \qquad \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 0 & -1 & 2 & 1 \\ \hline 0 & 0 & 4 & 4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 4 & 4 \end{array}\right] \begin{array}{rl} x + y + z = 3 & (3) \\ -y + 2z = 1 & (2) \\ \boxed{z = 1} & (1) \end{array}$$

$(2) \rightarrow \ -y = 1 - 2$

$\qquad \boxed{y = 1}$

$(3) \rightarrow \ x = 3 - 1 - 1$

$\qquad = \boxed{1}$

$\therefore$ ***Solution***: $\boxed{(1,\ 1,\ 1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x+y+3z=14 \\ 7x+5y+8z=37 \\ x+3y+2z=9 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 3 & 1 & 3 & 14 \\ 7 & 5 & 8 & 37 \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3-7R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 1 & 3 & 14 \\ -3 & -9 & -6 & -27 \\ \hline 0 & -8 & -3 & -13 \end{array} \qquad \begin{array}{cccc} 7 & 5 & 8 & 37 \\ -7 & -21 & -14 & -63 \\ \hline 0 & -16 & -6 & -26 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -8 & -3 & -13 \\ 0 & -16 & -6 & -26 \end{array}\right] \begin{array}{l} \\ \\ R_3-2R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -16 & -6 & -26 \\ 0 & 16 & 6 & 26 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 9 \\ 0 & -8 & -3 & -13 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{ll} x+3y+2z=9 & (2) \\ -8y-3z=-13 & (1) \end{array}$$

$(1) \to \quad -8y = 3z - 13$

$$y = -\frac{3}{8}z + \frac{13}{8}$$

$(3) \to \quad x = 9 - 3\left(\frac{13}{8} - \frac{3}{8}z\right) - 2z$

$$= 9 - \frac{39}{8} + \frac{9}{8}z - 2z$$

$$= \frac{33}{8} - \frac{7}{8}z$$

$\therefore$ ***Solution***: $\left(\frac{33}{8} - \frac{7}{8}z,\ \frac{13}{8} - \frac{3}{8}z,\ z\right)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 4x-2y+z=7 \\ x+y+z=-2 \\ 4x+2y+z=3 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 4 & -2 & 1 & 7 \\ 4 & 2 & 1 & 3 \end{array}\right] \begin{array}{l} \\ R_2-4R_1 \\ R_3-4R_1 \end{array}$$

$$\begin{array}{cccc} 4 & -2 & 1 & 7 \\ -4 & -4 & -4 & 8 \\ \hline 0 & -6 & -3 & 15 \end{array} \qquad \begin{array}{cccc} 4 & 2 & 1 & 3 \\ -4 & -4 & -4 & 8 \\ \hline 0 & -2 & -3 & 11 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 15 \\ 0 & -2 & -3 & 11 \end{array}\right] \begin{array}{l} \\ \\ -3R_3+R_2 \end{array} \qquad \begin{array}{cccc} 0 & 6 & 9 & -33 \\ 0 & -6 & -3 & 15 \\ \hline 0 & 0 & 6 & -18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -6 & -3 & 15 \\ 0 & 0 & 6 & -18 \end{array}\right] \quad \begin{array}{rl} x+y+z=-2 & (3) \\ -6y-3z=15 & (2) \\ 6z=-18 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z=-3}|$

$(2) \rightarrow \quad -6y = 15-9$

$\qquad \underline{y=-1}|$

$(3) \rightarrow \quad x = -2+1+3$

$\qquad \underline{=2}|$

∴ ***Solution***: $\underline{(2,\ -1,\ -3)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 2 & -2 & 1 & -4 \\ 6 & 4 & -3 & -24 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-6R_1 \end{array} \qquad \begin{array}{cccc} 2 & -2 & 1 & -4 \\ -2 & 4 & -4 & -2 \\ \hline 0 & 2 & -3 & -6 \end{array} \qquad \begin{array}{cccc} 6 & 4 & -3 & -24 \\ -6 & 12 & -12 & -6 \\ \hline 0 & 16 & -15 & -30 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 16 & -15 & -30 \end{array}\right] \begin{array}{l} \\ \\ R_3-8R_2 \end{array} \qquad \begin{array}{cccc} 0 & 16 & -15 & -30 \\ 0 & -16 & 24 & 48 \\ \hline 0 & 0 & 9 & 18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 2 & -3 & -6 \\ 0 & 0 & 9 & 18 \end{array}\right] \quad \begin{array}{rl} x-2y+2z=1 & (3) \\ 2y-3z=-6 & (2) \\ 9z=18 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z=2}|$

$(2) \rightarrow \quad 2y = -6+6$

$\qquad \underline{y=0}|$

$(3) \rightarrow \quad x = 1-4$

$= -3$

$\therefore$ ***Solution***: $(-3,\ 0,\ 2)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

Solution

$$\begin{cases} z+9x+3y=4 \\ z+16x+4y=2 \\ z+25x+5y=2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 1 & 16 & 4 & 2 \\ 1 & 25 & 5 & 2 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccc} 1 & 16 & 4 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 7 & 1 & -2 \end{array} \qquad \begin{array}{cccc} 1 & 25 & 5 & 2 \\ -1 & -9 & -3 & -4 \\ \hline 0 & 16 & 2 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 16 & 2 & -2 \end{array}\right] \begin{array}{l} \\ \\ 7R_3 - 16R_3 \end{array}$$

$$\begin{array}{cccc} 0 & 112 & 14 & -14 \\ 0 & -112 & -16 & 32 \\ \hline 0 & 0 & -2 & 18 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 9 & 3 & 4 \\ 0 & 7 & 1 & -2 \\ 0 & 0 & -2 & 18 \end{array}\right] \quad \begin{array}{rl} z+9x+3y=4 & (3) \\ 7x+y=-2 & (2) \\ -2y=18 & (1) \end{array}$$

$(1) \to \quad y = -9$

$(2) \to \quad 7x = -2+9$

$= 1$

$(3) \to \quad z = 4-9+27$

$= 22$

$\therefore$ ***Solution***: $(1,\ -9,\ 22)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x - y + 2z = -8 \\ x + 2y - 3z = 9 \\ 3x - y - 4z = 3 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 2 & -8 \\ 3 & -1 & -4 & 3 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 2 & -8 \\ -2 & -4 & 6 & -18 \\ \hline 0 & -5 & 8 & -26 \end{array} \qquad \begin{array}{cccc} 3 & -1 & -4 & 3 \\ -3 & -6 & 9 & -27 \\ \hline 0 & -7 & 5 & -24 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & -7 & 5 & -24 \end{array}\right] \begin{array}{l} \\ \\ 5R_3 - 7R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -35 & 25 & -120 \\ 0 & 35 & -56 & 182 \\ \hline 0 & 0 & -31 & 62 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & 0 & -31 & 62 \end{array}\right] \quad \begin{array}{rl} x + 2y - 3z = 9 & (3) \\ -5y + 8z = -26 & (2) \\ -31z = 62 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{z = -2}$

$(2) \rightarrow \quad -5y = -26 + 16$

$-5y = 10$

$\underline{y = 2}$

$(3) \rightarrow \quad x = 9 - 4 - 6$

$\underline{= -1}$

∴ ***Solution***: $\underline{(-1,\ 2,\ -2)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x \quad\quad - 3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 2 & -1 & 2 & 16 \\ 7 & -3 & -5 & 19 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 7R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -1 & 2 & 16 \\ -2 & 0 & 6 & 10 \\ \hline 0 & -1 & 8 & 26 \end{array} \qquad \begin{array}{cccc} 7 & -3 & -5 & 19 \\ -7 & 0 & 21 & 35 \\ \hline 0 & -3 & 16 & 54 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & -3 & 16 & 54 \end{array}\right] \quad R_3 - 3R_2$$

$$\begin{array}{cccc} 0 & -3 & 16 & 54 \\ 0 & 3 & -24 & -78 \\ \hline 0 & 0 & -8 & -24 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & 0 & -8 & -24 \end{array}\right] \quad \begin{aligned} x - 3z &= -5 & (3) \\ -y + 8z &= 26 & (2) \\ -8z &= -24 & (1) \end{aligned}$$

$(1) \rightarrow \quad \underline{z = 3}$

$(2) \rightarrow \quad -y = 26 - 24$

$\qquad \underline{y = -2}$

$(3) \rightarrow \quad x = -5 + 9$

$\qquad \underline{= 4}$

$\therefore$ ***Solution***: $\underline{(4,\ -2,\ 3)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x + 2y - z = 5 \\ 2x - y + 3z = 0 \\ 2y + z = 1 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 2 & -1 & 3 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right] \quad R_2 - 2R_1$$

$$\begin{array}{cccc} 2 & -1 & 3 & 0 \\ -2 & -4 & 2 & -10 \\ \hline 0 & -5 & 5 & -10 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 2 & 1 & 1 \end{array}\right] \quad 5R_3 + 2R_2$$

$$\begin{array}{cccc} 0 & 10 & 5 & 5 \\ 0 & -10 & 10 & -20 \\ \hline 0 & 0 & 15 & -15 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & -5 & 5 & -10 \\ 0 & 0 & 15 & -15 \end{array}\right] \quad \begin{aligned} x + 2y - z &= 5 & (3) \\ -5y + 5z &= -10 & (2) \\ 15z &= -15 & (1) \end{aligned}$$

$(1) \rightarrow \quad \underline{z = -1}$

$(2) \rightarrow \quad -5y = -10 + 5$

$\qquad \underline{y = 1}$

$(3) \rightarrow \quad x = 5 - 2 - 1$

$=2$

$\therefore$ ***Solution***: $(2,\ 1,\ -1)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x+y+z=6 \\ 3x+4y-7z=1 \\ 2x-y+3z=5 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 4 & -7 & 1 \\ 2 & -1 & 3 & 5 \end{array}\right] \begin{array}{l} \\ R_2-3R_1 \\ R_3-2R_1 \end{array} \qquad \begin{array}{cccc} 3 & 4 & -7 & 1 \\ -3 & -3 & -3 & -18 \\ \hline 0 & 1 & -10 & -17 \end{array} \qquad \begin{array}{cccc} 2 & -1 & 3 & 5 \\ -2 & -2 & -2 & -12 \\ \hline 0 & -3 & 1 & -7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -10 & -17 \\ 0 & -3 & 1 & -7 \end{array}\right] \begin{array}{l} \\ \\ R_3+3R_2 \end{array} \qquad \begin{array}{cccc} 0 & -3 & 1 & -7 \\ 0 & 3 & -30 & -51 \\ \hline 0 & 0 & -29 & -58 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -10 & -17 \\ 0 & 0 & -29 & -58 \end{array}\right] \quad \begin{array}{rl} x+y+z=6 & (3) \\ y-10z=-17 & (2) \\ -29z=-58 & (1) \end{array}$$

$(1) \rightarrow \quad z=2$

$(2) \rightarrow \quad y=-17+20$

$=3$

$(3) \rightarrow \quad x=6-3-2$

$=1$

$\therefore$ ***Solution***: $(1,\ 3,\ 2)$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x+2y+3z=3 \\ 4x-5y+7z=1 \\ 2x+3y-2z=6 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 4 & -5 & 7 & 1 \\ 2 & 3 & -2 & 6 \end{array}\right] \begin{array}{l} \\ 3R_2-4R_1 \\ 3R_3-2R_1 \end{array} \qquad \begin{array}{cccc} 12 & -15 & 21 & 3 \\ -12 & -8 & -12 & -12 \\ \hline 0 & -23 & 9 & -9 \end{array} \qquad \begin{array}{cccc} 6 & 9 & -6 & 18 \\ -6 & -4 & -6 & -6 \\ \hline 0 & 5 & -12 & 12 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 0 & -23 & 9 & -9 \\ 0 & 5 & -12 & 12 \end{array}\right] \; 23R_3 + 5R_2 \qquad \begin{array}{cccc} 0 & 115 & -276 & 276 \\ 0 & -115 & 45 & -45 \\ \hline 0 & 0 & -231 & 231 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 3 & 3 \\ 0 & -23 & 9 & -9 \\ 0 & 0 & -231 & 231 \end{array}\right] \quad \begin{array}{rl} 3x+2y+3z=3 & (3) \\ -23y+9z=-9 & (2) \\ -231z=231 & (1) \end{array}$$

$(1) \to \quad \boxed{z=-1}$

$(2) \to \quad -23y=-9+9$

$\boxed{y=0}$

$(3) \to \quad 3x=3+3$

$\boxed{x=2}$

$\therefore$ ***Solution***: $\boxed{(2,\ 0,\ -1)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x-3y+z=2 \\ 4x-12y+4z=8 \\ -2x+6y-2z=-4 \end{cases}$

Solution

$$\begin{cases} & x-3y+z=2 \\ \frac{1}{4}\times & 4x-12y+4z=8 \\ -\frac{1}{2}\times & -2x+6y-2z=-4 \end{cases}$$

$$\begin{cases} x-3y+z=2 \\ x-3y+z=2 \\ x-3y+z=2 \end{cases}$$

Since all three equations are the same.

$\therefore$ ***Solution***: is the plane $\boxed{x-3y+z=2}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x-2y+z=-1 \\ x+2y-z=2 \\ 6x+4y+3z=5 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & -2 & 1 & -1 \\ 6 & 4 & 3 & 5 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-6R_1 \end{array}$$

$$\begin{array}{cccc} 2 & -2 & 1 & -1 \\ -2 & -4 & 2 & -4 \\ \hline 0 & -6 & 3 & -5 \end{array} \qquad \begin{array}{cccc} 6 & 4 & 3 & 5 \\ -6 & -12 & 6 & -12 \\ \hline 0 & -8 & 9 & -7 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -6 & 3 & -5 \\ 0 & -8 & 9 & -7 \end{array}\right] \begin{array}{l} \\ \\ 3R_3-4R_2 \end{array}$$

$$\begin{array}{cccc} 0 & -24 & 27 & -21 \\ 0 & 24 & -12 & 20 \\ \hline 0 & 0 & 15 & -1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -6 & 3 & -5 \\ 0 & 0 & 15 & -1 \end{array}\right] \begin{array}{ll} x+2y-z=2 & (3) \\ -6y+3z=-5 & (2) \\ 15z=-1 & (1) \end{array}$$

$(1) \to \quad \boxed{z=-\frac{1}{15}}$

$(2) \to \quad -6y=-5+\frac{1}{5}$

$-6y=-\frac{24}{5}$

$\boxed{y=\frac{4}{5}}$

$(3) \to \quad x=2-\frac{8}{5}-\frac{1}{15}$

$\boxed{=\frac{1}{3}}$

$\therefore$ ***Solution***: $\boxed{\left(\frac{1}{3},\ \frac{4}{5},\ -\frac{1}{15}\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x_1 \quad\quad -3x_3 = -2 \\ 3x_1+x_2-2x_3=5 \\ 2x_1+2x_2+x_3=4 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{array}\right] \quad \begin{array}{l} \\ \\ R_3 - 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{array}\right] \quad \begin{array}{l} (2) \\ (1) \\ \underline{z = 2} \end{array}$$

$(1) \rightarrow \quad y = 11 - 14$

$\quad \underline{y = -3}$

$(2) \rightarrow \quad x = -2 + 6$

$\quad \underline{x = 4}$

$\therefore$ ***Solution***: $\underline{(4,\ -3,\ 2)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x_1 \qquad\quad + 3x_3 = 3 \\ 4x_1 - 3x_2 + 7x_3 = 5 \\ 8x_1 - 9x_2 + 15x_3 = 10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 3 \\ 4 & -3 & 7 & 5 \\ 8 & -9 & 15 & 10 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 3 \\ 0 & -3 & 1 & -1 \\ 0 & -9 & 3 & -2 \end{array}\right] \quad \begin{array}{l} \\ \\ R_3 - 3R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 3 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \\ \\ 0 \neq 1 \end{array}$$

$\therefore$ ***Solution***: $\underline{\textit{No Solution}}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 \\ x_2 - 3x_3 - x_4 = 0 \\ 3x_1 + 2x_3 - x_4 = 6 \\ -4x_1 + x_2 + 4x_3 + 2x_4 = -3 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 3 & 0 & 2 & -1 & 6 \\ -4 & 1 & 4 & 2 & -3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_1 \\ R_4 + 4R_1 \end{array}$$

$$\begin{array}{rrrrr} 3 & 0 & 2 & -1 & 6 \\ -3 & 15 & -6 & 6 & -12 \\ \hline 0 & 15 & -4 & 5 & -6 \end{array} \qquad \begin{array}{rrrrr} -4 & 1 & 4 & 2 & -3 \\ 4 & -20 & 8 & -8 & 16 \\ \hline 0 & -19 & 12 & -6 & 13 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 15 & -4 & 5 & -6 \\ 0 & -19 & 12 & -6 & 13 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 15R_2 \\ R_4 + 19R_2 \end{array}$$

$$\begin{array}{rrrrr} 0 & 15 & -4 & 5 & -6 \\ 0 & -15 & 45 & 15 & 0 \\ \hline 0 & 0 & 41 & 20 & -6 \end{array}$$

$$\begin{array}{rrrrr} 0 & -19 & 12 & -6 & 13 \\ 0 & 19 & -57 & -19 & 0 \\ \hline 0 & 0 & -45 & -25 & 13 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & -45 & -25 & 13 \end{array}\right] \begin{array}{l} \\ \\ \\ 41R_4 + 45R_2 \end{array}$$

$$\begin{array}{rrrrr} 0 & 0 & -1845 & -1025 & 533 \\ 0 & 0 & 1845 & 900 & -270 \\ \hline 0 & 0 & 0 & -125 & 263 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -5 & 2 & -2 & 4 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 41 & 20 & -6 \\ 0 & 0 & 0 & -125 & 263 \end{array}\right] \quad \begin{array}{rl} x_1 - 5x_2 + 2x_3 - 2x_4 = 4 & (4) \\ x_2 - 3x_3 - x_4 = 0 & (3) \\ 41x_3 + 20x_4 = -6 & (2) \\ -125x_4 = 263 & (1) \end{array}$$

$(1) \rightarrow \quad \boxed{x_4 = -\frac{263}{125}}$

$(2) \rightarrow \quad 41x_3 = -6 + \frac{1,052}{25}$

$= \frac{902}{25}$

$\boxed{x_3 = \frac{22}{25}}$

$(3) \rightarrow \quad x_2 = \frac{66}{25} - \frac{263}{125}$

$$= \boxed{\frac{67}{125}}$$

$$(4) \to \quad x_1 = 4 + \frac{67}{25} - \frac{44}{25} - \frac{526}{125}$$

$$= 4 + \frac{23}{25} - \frac{526}{125}$$

$$= \frac{500 + 115 - 526}{125}$$

$$= \boxed{\frac{89}{125}}$$

$\therefore$ ***Solution***: $\boxed{\left(\frac{89}{125}, \frac{67}{125}, \frac{22}{25}, -\frac{263}{125}\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x_1 + x_2 + x_3 + x_4 = 5 \\ x_1 + 2x_2 - x_3 - 2x_4 = -1 \\ x_1 - 3x_2 - 3x_3 - x_4 = -1 \\ 2x_1 - x_2 + 2x_3 - x_4 = -2 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 1 & 2 & -1 & -2 & -1 \\ 1 & -3 & -3 & -1 & -1 \\ 2 & -1 & 2 & -1 & -2 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - 2R_1 \end{array}$$

$$\begin{array}{ccccc} 1 & 2 & -1 & -2 & -1 \\ -1 & -1 & -1 & -1 & -5 \\ \hline 0 & 1 & -2 & -3 & -6 \end{array} \qquad \begin{array}{ccccc} 1 & -3 & -3 & -1 & -1 \\ -1 & -1 & -1 & -1 & -5 \\ \hline 0 & -4 & -4 & -2 & -6 \end{array}$$

$$\begin{array}{ccccc} 2 & -1 & 2 & -1 & -2 \\ -2 & -2 & -2 & -2 & -10 \\ \hline 0 & -3 & 0 & -3 & -12 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & -4 & -4 & -2 & -6 \\ 0 & -3 & 0 & -3 & -12 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 4R_2 \\ R_4 + 3R_2 \end{array}$$

$$\begin{array}{ccccc} 0 & -4 & -4 & -2 & -6 \\ 0 & 4 & -8 & -12 & -24 \\ \hline 0 & 0 & -12 & -14 & -30 \end{array}$$

$$\begin{array}{ccccc} 0 & -3 & 0 & -3 & -12 \\ 0 & 3 & -6 & -9 & -18 \\ \hline 0 & 0 & -6 & -12 & -30 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & -12 & -14 & -30 \\ 0 & 0 & -6 & -12 & -30 \end{array}\right] \begin{matrix} \\ \\ \\ -2R_4 + R_3 \end{matrix} \qquad \begin{array}{ccccc} 0 & 0 & 12 & 24 & 60 \\ 0 & 0 & -12 & -14 & -30 \\ \hline 0 & 0 & 0 & 10 & 30 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & -2 & -3 & -6 \\ 0 & 0 & -12 & -14 & -30 \\ 0 & 0 & 0 & 10 & 30 \end{array}\right] \quad \begin{array}{rl} x_1 + x_2 + x_3 + x_4 = 5 & (4) \\ x_2 - 2x_3 - 3x_4 = -6 & (3) \\ -12x_3 - 14x_4 = -30 & (2) \\ 10x_4 = 30 & (1) \end{array}$$

$(1) \rightarrow \quad \underline{x_4 = 3}|$

$(2) \rightarrow \quad -12x_3 = -30 + 42$

$= 12$

$\underline{x_3 = -1}|$

$(3) \rightarrow \quad x_2 = -6 - 2 + 9$

$\underline{= 1}|$

$(4) \rightarrow \quad x_1 = 5 - 1 + 1 - 3$

$\underline{= 2}|$

∴ ***Solution***: $\underline{(2,\ 1,\ -1,\ 3)}|$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x + 8y - z + w = 0 \\ 4x + 16y - 3z - w = -10 \\ -2x + 4y - z + 3w = -6 \\ -6x + 2y + 5z + w = 3 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 4 & 16 & -3 & -1 & -10 \\ -2 & 4 & -1 & 3 & -6 \\ -6 & 2 & 5 & 1 & 3 \end{array}\right] \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{matrix}$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \quad R_4 - \tfrac{13}{6}R_2$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 26 & 2 & 4 & 3 \end{array}\right] \qquad \textit{Interchange } R_2 \textit{ and } R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & \frac{19}{3} & -\frac{14}{3} & 16 \end{array}\right] \quad R_4 + \tfrac{19}{3}R_3$$

$$\left[\begin{array}{cccc|c} 2 & 8 & -1 & 1 & 0 \\ 0 & 12 & -2 & 4 & -6 \\ 0 & 0 & -1 & -3 & -10 \\ 0 & 0 & 0 & -\frac{71}{3} & -\frac{142}{3} \end{array}\right] \quad \begin{array}{ll} 2x+8y-z+w=0 & (3) \\ 12y-2z+4w=-6 & (2) \\ -z-3w=-10 & (1) \\ -\frac{71}{3}w=-\frac{142}{3} & \to \ \underline{w=2} \end{array}$$

$(1) \to \quad z = 10 - 3w \underline{= 4}$

$(2) \to \quad 12y = 2z - 4w - 6$

$\qquad \underline{y = -\frac{1}{2}}$

$(3) \to \quad 2x = -8y + z - w$

$\qquad 2x = 4 + 4 - 2$

$\qquad \underline{x = 3}$

$\therefore$ ***Solution***: $\underline{\left(3,\ -\frac{1}{2},\ 4,\ 2\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 + 2x_2 \qquad = 0 \\ \qquad x_2 + x_3 = 0 \end{cases}$

Solution

$$\begin{cases} x_1 = -2x_2 \\ x_3 = -x_2 \end{cases}$$

$$2x_1 + x_2 + 3x_3 = 0$$

$$-4x_2 + x_2 - 3x_2 = 0 \quad \rightarrow \quad \underline{x_2 = 0}$$

$\therefore$ ***Solution***: $\underline{(0,\ 0,\ 0)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x + 2y + 4z \quad = 0 \\ \quad - y - 3z + w = 0 \\ 3x + y + z + 2w = 0 \\ x + 3y - 2z - 2w = 0 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & -1 & -3 & 1 & 0 \\ 3 & 1 & 1 & 2 & 0 \\ 1 & 3 & -2 & -2 & 0 \end{array}\right] \begin{array}{c} \\ -R_2 \\ 2R_3 - 3R_1 \\ 2R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & -4 & -10 & 4 & 0 \\ 0 & 4 & -8 & -4 & 0 \end{array}\right] \begin{array}{c} \\ \\ R_3 + 4R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 4 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -20 & 0 & 0 \end{array}\right] \begin{array}{ll} 2x + 2y - 4z = 0 & (1) \\ y + 3z - w = 0 & (2) \\ \rightarrow \underline{z = 0} & \\ & \end{array}$$

$$(2) \quad \rightarrow \quad \underline{y = w}$$

$$(1) \quad \rightarrow 2x = -2y \quad \underline{x = -w}$$

$\therefore$ ***Solution***: $\underline{(-w,\ w,\ 0,\ w)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 2x \quad\quad + z + w = 5 \\ \quad\quad y \quad\quad - w = -1 \\ 3x \quad\quad - z - w = 0 \\ 4x + y + 2z + w = 9 \end{cases}$

Solution

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 3 & 0 & -1 & -1 & 0 \\ 4 & 1 & 2 & 1 & 9 \end{array}\right] \begin{array}{l} \\ \\ 2R_3 - 3R_1 \\ 2R_4 - 4R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 & -15 \\ 0 & 2 & 0 & -2 & -2 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - 2R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -5 & -5 & -15 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{rl} 2x + z + w = 5 & (1) \\ y - w = -1 & (2) \\ -5z - 5w = -15 & (3) \\ & \end{array}$$

$(2) \rightarrow \quad \underline{y = 1 + w}$

$(3) \rightarrow \quad \underline{z = 3 - w}$

$(1) \rightarrow \quad 2x = 5 - (3 - w) - w$

$\underline{x = 1}$

$\therefore$ ***Solution***: $\underline{(1,\ 1 + w,\ 3 - w,\ w)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} \quad 4y + z = 20 \\ 2x - 2y + z = 0 \\ x \quad\quad + z = 5 \\ x + y - z = 10 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 1 & 5 \\ 0 & 4 & 1 & 20 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 0 & -4 & 3 & -20 \\ 0 & -1 & 2 & -5 \\ 0 & 4 & 1 & 20 \end{array}\right] \begin{array}{l} \\ \\ 4R_3 - R_2 \\ R_4 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 10 \\ 0 & -4 & 3 & -20 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \begin{array}{r} x + y = 10 \\ \rightarrow -4y = -20 \\ \rightarrow z = 0 \\ \\ \end{array}$$

∴ ***Solution***: $(5,\ 5,\ 0)$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x \qquad\qquad\quad - 3w = -3 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ -1 & 3 & -2 & 1 \\ 3 & 4 & -7 & 10 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & -2 & -10 & -14 \end{array}\right] \begin{array}{l} \\ \\ 5R_3 + 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 5 & -1 & 9 \\ 0 & 0 & -52 & -52 \end{array}\right] \begin{array}{rl} x + 2y + z = 8 & (3) \\ 5y - z = 9 & (2) \\ -52z = -52 & (1) \end{array}$$

$(1) \quad \Rightarrow \quad z = 1$

$(2) \quad \Rightarrow \quad 5y = 9 + 1 = 10 \rightarrow y = 2$

$(3) \quad \Rightarrow \quad x = 8 - 4 - 1 = 3$

∴ ***Solution***: $(3,\ 2,\ 1)$

Exercise

Solve the linear system by Gauss-Jordan elimination.

$$\begin{cases} 2u - 3v + w - x + y = 0 \\ 4u - 6v + 2w - 3x - y = -5 \\ -2u + 3v - 2w + 2x - y = 3 \end{cases}$$

Solution

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 4 & -6 & 2 & -3 & -1 & -5 \\ -2 & 3 & -2 & 2 & -1 & 3 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 2 & -3 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -3 & -5 \\ 0 & 0 & -1 & 1 & 0 & 3 \end{array}\right] \qquad \begin{array}{rl} 2u - 3v + w - x + y = 0 & (3) \\ -x - 3y = -5 & (2) \\ -w + x = 3 & (1) \end{array}$$

$(2) \Rightarrow \underline{x = 5 - 3y}$

$(1) \Rightarrow w = x - 3$

$\qquad = \underline{2 - 3y}$

$(3) \Rightarrow 2u = 3v - 2 + 3y + 5 - 3y - y$

$\qquad = 3v - y + 3$

$u = \frac{3}{2}v - \frac{1}{2}y + \frac{3}{2}$

$\therefore$ ***Solution***: $\underline{\left(\frac{3}{2}v - \frac{1}{2}y + \frac{3}{2}, v, 2 - 3y, 5 - 3y, y\right)}$

Exercise

Use augmented elimination to solve linear system
$$\begin{cases} 6x_3 + 2x_4 - 4x_5 - 8x_6 = 8 \\ 3x_3 + x_4 - 2x_5 - 4x_6 = 4 \\ 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 + x_6 = 2 \\ 6x_1 - 9x_2 + 11x_4 - 19x_5 + 3x_6 = 1 \end{cases}$$

Solution

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 6 & -9 & 0 & 11 & -19 & 3 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - 3R_1 \end{array}$$

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 6 & 2 & -4 & -8 & 8 \\ 0 & 0 & -3 & -1 & 2 & 0 & -5 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_2 \\ R_4 + R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 2 & -3 & 1 & 4 & -7 & 1 & 2 \\ 0 & 0 & 3 & 1 & -2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & -1 \end{array}\right]$$

$$2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 = 2 - x_6$$
$$3x_3 + x_4 - 2x_5 = 4 + 4x_6$$

$$\to \boxed{x_6 = \frac{1}{4}}$$

$$\begin{cases} 3x_3 = 5 - x_4 + 2x_5 \\ 2x_1 = \frac{7}{4} + 3x_2 - \frac{1}{3}\left(5 - x_4 + 2x_5\right) - 4x_4 + 7x_5 \end{cases}$$

$$\begin{cases} \boxed{x_3 = \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5} \\ 2x_1 = \frac{1}{12} + 3x_2 - \frac{11}{3}x_4 + \frac{19}{3}x_5 \end{cases}$$

$$\boxed{x_1 = \frac{1}{24} + 3x_2 - \frac{11}{6}x_4 + \frac{19}{6}x_5}$$

∴ ***Solution***: $\boxed{\left(\frac{1}{24} + \frac{3}{2}x_2 - \frac{11}{6}x_4 + \frac{19}{6}x_5,\quad x_2,\quad \frac{5}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5,\quad x_4,\quad x_5,\quad \frac{1}{4}\right)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} 3x_1 + 2x_2 - x_3 = -15 \\ 5x_1 + 3x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 3x_3 = 11 \\ -6x_1 - 4x_2 + 2x_3 = 30 \end{cases}$

Solution

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{array}\right] \begin{array}{l} \\ 3R_2 - 5R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & -1 & 4 & 26 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -15 \\ 0 & -1 & 11 & 75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} 3x_1 + 2x_2 - x_3 &= -15 & (3) \\ -x_2 + 11x_3 &= 75 & (2) \\ -7x_3 &= -49 & (1) \end{aligned}$$

$(1) \rightarrow \ \underline{x_3 = 7}$

$(2) \rightarrow \ x_2 = 77 - 75$

$\qquad\qquad \underline{= 2}$

$(1) \rightarrow \ 3x_1 = -15 - 4 + 7 = 12$

$\qquad\qquad \underline{x_1 = -4}$

$\therefore$ ***Solution***: $\underline{(-4,\ 2,\ 7)}$

Exercise

Use augmented elimination to solve linear system $\begin{cases} x_1 + 3x_2 - 2x_3 \qquad\qquad + 2x_5 \qquad\quad = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ \qquad\qquad\quad 5x_3 + 10x_4 \qquad\quad + 15x_6 = 5 \\ 2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$

Solution

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 5R_2 \\ R_4 - 4R_2 \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right] \frac{1}{6}R_4 \quad \textit{then interchanging row3 and row4}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] R_2 - 3R_3$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \begin{cases} x_1 + 3x_2 + 4x_4 + 2x_5 = 0 \\ x_3 + 2x_4 = 0 \\ +x_6 = \frac{1}{3} \end{cases}$$

The general solution of the system: $x_6 = \frac{1}{3}, \quad x_3 = -2x_4, \quad x_1 = -3x_2 - 4x_4 - 2x_5$

$\therefore$ ***Solution***: $\left(-3x_2 - 4x_4 - 2x_5,\ x_2,\ -2x_4,\ x_4,\ x_5,\ \frac{1}{3}\right)$

Exercise

Add 3 times the second row to the first of $\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$

Solution

$$E = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$EA = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} 27 & 8 & -1 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 & 5 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix} \quad R_1 + 3R_2 \quad = \begin{bmatrix} 27 & 8 & -1 \\ 7 & 3 & -2 \\ 8 & 1 & 2 \\ 6 & 0 & -1 \end{bmatrix}$$

Exercise

For what value(s) of k, if any, does the system $\begin{cases} x+\ y-\ z=1 \\ 2x+3y+kz=3 \\ x+ky+3z=2 \end{cases}$ have

a) A unique solution?
b) Infinitely many solutions?
c) No solution?

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & k & 3 \\ 1 & k & 3 & 2 \end{array}\right] \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & k-1 & 4 & 1 \end{array}\right] \quad R_3 - (k-1)R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & 0 & 4-(k-1)(k+2) & 2-k \end{array}\right] \qquad \begin{aligned} x &= 1-y+z \\ y &= 1-(k+2)z \\ \to \left(6-k^2-k\right)z &= -(k-2) \end{aligned}$$

$$\begin{cases} z = -\dfrac{k-2}{-(k-2)(k+3)} = \dfrac{1}{k+3} \quad \color{red}{(k \neq 2, -3)} \\ y = 1 - \dfrac{k+2}{k+3} = \dfrac{1}{k+3} \\ x = \dfrac{k+2}{k+3} + \dfrac{1}{k+3} = 1 \end{cases}$$

a) Unique solution if $k \neq 2, -3$

b) Infinitely solution if $k = 2$

c) No solution if $k = -3$

Exercise

Assume that the matrix is the augmented matrix of a system of linear equations, and

$$\begin{bmatrix} 1 & k & 2 \\ -3 & 4 & 1 \end{bmatrix}$$

a) Determine the number of equations and the number of variables.
b) Find the value(s) of ***k*** such that the system is consistent.
c) Determine the number of equations and the number of variables but if the matrix is the coefficient matrix of a *homogeneous* system of linear equations.
d) Find the value(s) of ***k*** from part (*c*).

Solution

a) 2 equations & 2 variables

b) $\left[\begin{array}{cc|c} 1 & k & 2 \\ -3 & 4 & 1 \end{array}\right] \quad R_2 + 3R_1$

$$\left[\begin{array}{cc|c} 1 & k & 2 \\ 0 & 4+3k & 7 \end{array}\right]$$

For the system to be consistent, then $4+3k \neq 0$

$k \in \mathbb{R} - \left\{-\frac{4}{3}\right\}$

c) $\left[\begin{array}{ccc|c} 1 & k & 2 & 0 \\ -3 & 4 & 1 & 0 \end{array}\right]$

2 equations & 3 variables

d) $\left[\begin{array}{ccc|c} 1 & k & 2 & 0 \\ -3 & 4 & 1 & 0 \end{array}\right] \quad R_2 + 3R_1$

$$\left[\begin{array}{ccc|c} 1 & k & 2 & 0 \\ 0 & 4+k & 7 & 0 \end{array}\right] \qquad \begin{aligned} x_1 &= -2x_3 - kx_2 \\ (4+k)x_2 &= -7x_3 \end{aligned}$$

$$x_2 = -\frac{7}{4+k}x_3$$

$$\begin{aligned} x_1 &= -2x_3 + \frac{7k}{4+k}x_3 \\ &= \frac{5k-8}{4+k}x_3 \end{aligned}$$

$k \in \mathbb{R}$

If $k = -4$, then no solution.

Exercise

Choose a coefficient b that makes the system singular.

$$\begin{cases} 3x+4y=16 \\ 4x+by=g \end{cases}$$

Then choose a right-hand side g that makes it solvable.
Find 2 solutions in that singular case.

Solution

$$\left(\begin{array}{cc|c} 3 & 4 & 16 \\ 4 & b & g \end{array}\right) \quad 3R_2-4R_1$$

$$\left(\begin{array}{cc|c} 3 & 4 & 16 \\ 0 & 3b-16 & 3g-64 \end{array}\right)$$

So, the system is singular if

$$3b-16=0 \quad \Rightarrow \quad \boxed{b=\frac{16}{3}}$$

$$\& \ 3g-64=0 \quad \Rightarrow \quad \boxed{g=\frac{64}{3}}$$

$$\begin{cases} 3x+4y=16 \\ 4x+\frac{16}{3}y=\frac{64}{3} \end{cases} \quad \to \quad \boxed{3x+4y=16}$$

$$\text{If } \begin{cases} x=0 & \to \quad \boxed{y=4} \\ x=4 & \to \quad \boxed{y=1} \end{cases}$$

Exercise

This system us not linear, in some sense,

$$\begin{cases} 2\sin\alpha-\cos\beta+3\tan\theta=3 \\ 4\sin\alpha+2\cos\beta-2\tan\theta=10 \\ 6\sin\alpha-3\cos\beta+\tan\theta=9 \end{cases}$$

Does the system have a solution?

Solution

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 10 \\ 6 & -3 & 1 & 9 \end{array}\right] \quad \begin{array}{l} \\ R_2-2R_1 \\ R_3-3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 0 & 4 & -8 & 4 \\ 0 & 0 & -8 & 0 \end{array}\right] \quad \begin{aligned} 2\sin\alpha &= 3+\cos\beta \\ 4\cos\beta &= 4 \\ \tan\theta &= 0 \end{aligned}$$

$$\begin{cases} \sin\alpha = \frac{3}{2} + \frac{1}{2} = 2 \\ \cos\beta = 1 \\ \tan\theta = 0 \end{cases}$$

The system has *no solution* since $\sin\alpha$ cannot be equal 2. $(-1 \le \sin\alpha \le 1)$

Exercise

Determine whether the matrix is in row-echelon form. If it is, determine whether it is also in reduced row-echelon form.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution

The given matrix is in reduced row-echelon form.

Solution Section 1.3 – Matrices and Matrix operations

Exercise

For the matrices: $A = \begin{bmatrix} p & 0 \\ q & r \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, when does $AB = BA$

Solution

$$AB = \begin{pmatrix} p & 0 \\ q & r \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} p & p \\ q & q+r \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & 0 \\ q & r \end{pmatrix}$$

$$= \begin{pmatrix} p+q & r \\ q & r \end{pmatrix}$$

$$AB = BA$$

$$\begin{pmatrix} p & p \\ q & q+r \end{pmatrix} = \begin{pmatrix} p+q & r \\ q & r \end{pmatrix}$$

$$\begin{cases} p = p+q \\ \boxed{p = r} \\ q+r = r \end{cases} \Rightarrow \begin{cases} \boxed{q = 0} \\ \\ q = 0 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal. $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

Solution

$$\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$$

$$\Rightarrow \begin{cases} w = 9 & x = 17 \\ y = 8 & z = -12 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal. $\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$

Solution

$$\begin{cases} \underline{x = 12} \\ y+3=5 \rightarrow \underline{y=2} \\ 2z=6 \rightarrow \underline{z=3} \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal. $\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$

Solution

$$\begin{bmatrix} 5 = y+3 & x-4=2 & 9=9 \\ 2=z+4 & -3=-3 & 8=8 \\ 6=6 & 0=0 & 5=w \end{bmatrix}$$

$$\rightarrow \begin{cases} y=2 & z=-2 \\ x=6 & w=5 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

Solution

$$\begin{bmatrix} a-5 & 5b & 4c+6 \\ -2d & 7f-6 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

$$\begin{cases} a-5=15 & \rightarrow & a=20 \\ 5b=25 & \rightarrow & b=5 \\ 4c+6=6 & \rightarrow & 4c=0 \rightarrow c=0 \\ -2d=-8 & \rightarrow & d=4 \\ 7f-6=1 & \rightarrow & 7f=7 \rightarrow f=1 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+11 & 12z+1 & 5m \\ 11k & 3 & 1 \end{bmatrix} + \begin{bmatrix} 9a & 9z & 4m \\ 12k & 5 & 3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

Solution

$$\begin{bmatrix} a+11+9a & 12z+1+9z & 5m+4m \\ 11k+12k & 3+5 & 1+3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 10a+11 & 21z+1 & 9m \\ 23k & 8 & 4 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

$10a+11=41 \quad \rightarrow \quad 10a=30$

$\qquad a=3$

$21z+1=-62 \quad \rightarrow \quad 21z=-63$

$\qquad z=-3$

$9m=72 \quad \rightarrow \quad m=8$

$23k=92$

$k=\frac{92}{23}$

$\quad =4$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} x+2 & 3y+1 & 5z \\ 8w & 2 & 3 \end{bmatrix} + \begin{bmatrix} 3x & 2y & 5z \\ 2w & 5 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 4x+2 & 5y+1 & 10z \\ 10w & 7 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$$

$$\begin{cases} 4x+2=10 & \rightarrow \quad x=2 \\ 5y+1=-14 & \rightarrow \quad y=-3 \\ 10z=80 & \rightarrow \quad z=8 \\ 10w=10 & \rightarrow \quad w=1 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} 2x-3 & y-2 & 2z+1 \\ 5 & 2w & 7 \end{bmatrix} + \begin{bmatrix} 3x-3 & y+2 & z-1 \\ -5 & 5w+1 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 5x-6 & 2y & 3z \\ 0 & 7w+1 & 10 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$$

$$\begin{cases} 5x-6=20 & \rightarrow \quad \underline{x=\frac{26}{5}} \\ 2y=8 & \rightarrow \quad \underline{y=4} \\ 3z=9 & \rightarrow \quad \underline{z=3} \\ 7w+1=8 & \rightarrow \quad \underline{w=1} \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} 16 & 4 & 5 & 4 \\ -3 & 13 & 15 & 6 \\ 0 & 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 4 & 2x+1 & 4 \\ -3 & 13 & 16 & 3x \\ 0 & 2 & 3y-5 & 0 \end{bmatrix}$$

Solution

$5=2x+1 \quad \rightarrow \quad \underline{x=2}$

$6=3x \quad \rightarrow \quad x=2 \quad \checkmark$

$4=3y-5 \quad \rightarrow \quad \underline{y=3}$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 19 & 2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 19 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 19 & 2 \end{bmatrix}$$

$$\begin{cases} a + 2c = 6 \\ 3a + 4c = 19 \end{cases}$$

$$\begin{array}{r} -3a - 6c = -18 \\ 3a + 4c = 19 \\ \hline -2c = 1 \end{array}$$

$$\underline{c = -\frac{1}{2}}$$

$$\underline{a = 7}$$

$$\begin{cases} b + 2d = 3 \\ 3b + 4d = 2 \end{cases}$$

$$\begin{array}{r} -3b - 6d = -9 \\ 3b + 4d = 2 \\ \hline -2d = -7 \end{array}$$

$$\underline{d = \frac{7}{2}}$$

$$\underline{b = -4}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 7 & -4 \\ -\frac{1}{2} & \frac{7}{2} \end{pmatrix}$$

Exercise

Find a combination $x_1 w_1 + x_2 w_2 + x_3 w_3$ that gives the zero vector:

$$w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \quad w_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}; \quad w_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are independent or dependent?
The vectors lie in a ______.
The matrix W with those columns is not invertible.

Solution

$$w_1 - 2w_2 + w_3 = 0$$

Therefore, those vectors are dependent
The vectors lie in a plane.

Exercise

The very last words say that the 5 by 5 centered difference matrix is not invertible, Write down the 5 equations $Cx = b$. Find a combination of left sides that gives zero. What combination of b_1, b_2, b_3, b_4, b_5 must be zero?

Solution

The 5 by 5 centered difference matrix is

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

The five equations $Cx = b$ are:

$$x_2 = b_1, \quad -x_1 + x_3 = b_2, \quad -x_2 + x_4 = b_3, \quad -x_3 + x_5 = b_4, \quad -x_4 = b_5.$$

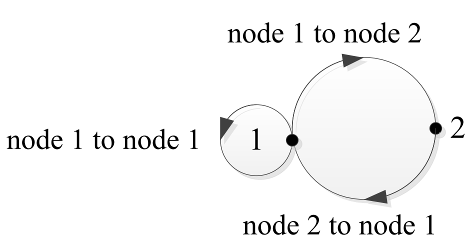
Observe that the sum of the first

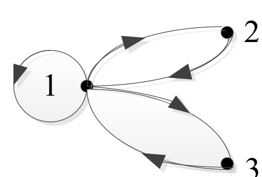
$$x_2 - x_2 + x_4 - x_4 = b_1 + b_2 + b_5$$

$$0 = b_1 + b_2 + b_5$$

Exercise

A direct graph starts with *n* nodes. There are n^2 possible edges, each edge leaves one of the *n* nodes and enters one of the *n* nodes (possibly itself). The *n* by *n* adjacency matrix has $a_{ij} = 1$ when edge leaves node *i* and enter node *j*; if no edge then $a_{ij} = 0$. Here are directed graphs and their adjacency matrices:





$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The *i*, *j* entry of A^2 is $a_{i1}a_{1j} + \ldots + a_{in}a_{nj}$.

Why does that sum count the two-step paths from *i* to any node to *j*?

The i, j entry of A^k counts k-steps paths:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{matrix} \textit{counts the paths} \\ \textit{with two edges} \end{matrix} \quad \begin{bmatrix} 1 \text{ to } 2 \text{ to } 1, 1 \text{ to } 1 \text{ to } 1 & 1 \text{ to } 1 \text{ to } 2 \\ 2 \text{ to } 1 \text{ to } 1 & 2 \text{ to } 1 \text{ to } 2 \end{bmatrix}$$

List all 3-step paths between each pair of nodes and compare with A^3. When A^k has ***no zeros***, that number k is the diameter of the graph – the number of edges needed to connect the most pair of nodes. What is the diameter of the second graph?

Solution

The number $a_{ik}a_{kj}$ will be "**1**" if there is an edge from node i to k and an edge from k to j.

This is a 2-step path. The number $a_{ik}a_{kj}$ will be "**0**" if either of those edge (from node i to k and from k to j) is missing.

The sum of $a_{ik}a_{kj}$ is the number of 2-step paths leaving i and entering j.

Matrix multiplication is right for this count.

The 3-step paths are counted by A^3; we look at paths to node 2:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{matrix} \textit{counts the paths} \\ \textit{with three steps} \end{matrix} \quad \begin{bmatrix} \ldots & 1 \text{ to } 1 \text{ to } 1 \text{ to } 2, \ 1 \text{ to } 2 \text{ to } 1 \text{ to } 2 \\ \ldots & 2 \text{ to } 1 \text{ to } 1 \text{ to } 2 \end{bmatrix}$$

The A^k contain Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, 13, ….

Fibonacci's rule $F_{k+2} = F_{k+1} + F_k$ show up in $(A)\left(A^k\right) = A^{k+1}$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} = \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix} = A^{k+1}$$

There are **13** ***six-step*** paths from node one to node 1.

Exercise

A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

a) AB

b) BA

c) ABD

d) DBA

e) ABC

f) $ABCD$

g) $A(B+C)$

Solution

$$A=\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{pmatrix}\qquad B=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\\1&1&1\\1&1&1\end{pmatrix}\qquad C=\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}\qquad D=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

a) $AB:(3\times 5)(5\times 3)=(3\times 3)$

$$\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\\1&1&1\\1&1&1\end{pmatrix}=\begin{pmatrix}5&5&5\\5&5&5\\5&5&5\end{pmatrix}$$

b) $BA:(5\times 3)(3\times 5)=(5\times 5)$

$$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{pmatrix}=\begin{pmatrix}3&3&3&3&3\\3&3&3&3&3\\3&3&3&3&3\\3&3&3&3&3\\3&3&3&3&3\end{pmatrix}$$

c) $ABD:(3\times 5)(5\times 3)(3\times 1)=(3\times 1)$

$$\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}5&5&5\\5&5&5\\5&5&5\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$=\begin{pmatrix}15\\15\\15\end{pmatrix}$$

d) $DBA:(3\times 1)(5\times 3)(3\times 5)=NA$

e) $ABC:(3\times 5)(5\times 3)(5\times 1)=NA$

f) $ABCD:(3\times 5)(5\times 3)(5\times 1)(3\times 1)=NA$

g) $A(B+C):(3\times 5)\big((5\times 3)+(5\times 1)\big)=NA$

Matrices B and C are not the same size.

Exercise

What rows or columns or matrices do you multiply to find.

a) The third column of AB?

b) The second column of AB?

c) The first row of AB?

d) The second row of AB?

e) The entry in row 3, column 4 of AB?

f) The entry in row 2, column 3 of AB?

Solution

a) A (column 3 of B)

b) A (column 2 of B)

c) (Row 1 of A) B

d) (Row 2 of A) B

e) (Row 3 of A) (Column 4 of B)

f) (Row 2 of A) (Column 3 of B)

Exercise

Add AB to AC and compare with $A(B+C)$:

$$A=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix} \quad and \quad B=\begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix} \quad and \quad C=\begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix}$$

Solution

$$AB=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}\begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 7\\ 0 & 7\end{bmatrix}$$

$$AC=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}\begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix}=\begin{bmatrix}3 & 1\\ 6 & 2\end{bmatrix}$$

$$\begin{aligned}A(B+C)&=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}\left(\begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix}+\begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix}\right)\\ &=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}\begin{bmatrix}3 & 3\\ 0 & 1\end{bmatrix}\\ &=\begin{bmatrix}3 & 8\\ 6 & 9\end{bmatrix}\end{aligned}$$

$$AB+AC=\begin{bmatrix}0 & 7\\ 0 & 7\end{bmatrix}+\begin{bmatrix}3 & 1\\ 6 & 2\end{bmatrix}$$

$$=\begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix}$$

$A(B+C) = AB + AC$

Exercise

True or False

a) If A^2 is defined then A is necessarily square.

b) If AB and BA are defined then A and B are square.

c) If AB and BA are defined then AB and BA are square.

d) If $AB = B$, then $A = I$

Solution

a) True

b) False, if A has an order m by n and B n by m: $AB : m \times m \qquad BA : n \times n$

c) True; $AB : m \times m \qquad BA : n \times n$

d) False, if B is the matrix of all zeros.

Exercise

a) Find a nonzero matrix A such that $A^2 = 0$

b) Find a matrix that has $A^2 \neq 0$ but $A^3 = 0$

Solution

a) A nonzero matrix A such that $A^2 = 0$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

b) A matrix that has $A^2 \neq 0$ but $A^3 = 0$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = A^2 A$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise

Suppose you solve $Ax = b$ for three special right sides b:

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If the three solutions x_1, x_2, x_3 are the columns of a matrix X, what is A times X?

Solution

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Therefore, $Ax = I$

Exercise

Show that $(A+B)^2$ is different from $A^2+2AB+B^2$, when

$$A=\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix} \quad and \quad B=\begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}$$

Write down the correct rule for $(A+B)(A+B)=A^2+$ ________ $+B^2$

Solution

$$A+B=\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}+\begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}$$

$$=\begin{bmatrix}2 & 2\\3 & 0\end{bmatrix}$$

$$(A+B)^2=\begin{bmatrix}2 & 2\\3 & 0\end{bmatrix}\begin{bmatrix}2 & 2\\3 & 0\end{bmatrix}$$

$$=\begin{bmatrix}10 & 4\\6 & 6\end{bmatrix}$$

$$A^2=\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}=\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}$$

$$B^2=\begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}$$

$$AB=\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}=\begin{bmatrix}7 & 0\\0 & 0\end{bmatrix}$$

$$2AB=2\begin{bmatrix}7 & 0\\0 & 0\end{bmatrix}=\begin{bmatrix}14 & 0\\0 & 0\end{bmatrix}$$

$$A^2+2AB+B^2=\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}+\begin{bmatrix}14 & 0\\0 & 0\end{bmatrix}+\begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}$$

$$=\begin{bmatrix}16 & 2\\3 & 0\end{bmatrix}$$

$$\begin{bmatrix}10 & 4\\6 & 6\end{bmatrix}\neq\begin{bmatrix}16 & 2\\3 & 0\end{bmatrix}$$

$$\underline{(A+B)^2\neq A^2+2AB+B^2}$$

$$BA=\begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}$$

$$A^2+AB+BA+B^2=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}+\begin{bmatrix}7 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}+\begin{bmatrix}1 & 0\\ 3 & 0\end{bmatrix}$$

$$=\begin{bmatrix}10 & 4\\ 5 & 6\end{bmatrix}$$

$$(A+B)(A+B)=A^2+\underline{\ AB+BA\ }+B^2$$

Exercise

Let $\quad A=\begin{pmatrix}1 & -1\\ 0 & 2\end{pmatrix} \qquad B=\begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}$

Show the computations that

a) $(A+B)(A+B)\neq A^2+2AB+B^2$

b) $(A+B)(A-B)\neq A^2-B^2$

c) $(A+B)(A+B)=A^2+AB+BA+B^2$

Solution

a) $A+B=\begin{pmatrix}1 & -1\\ 0 & 2\end{pmatrix}+\begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}$

$$=\begin{pmatrix}2 & -1\\ 1 & 4\end{pmatrix}$$

$$(A+B)(A+B)=\begin{pmatrix}2 & -1\\ 1 & 4\end{pmatrix}\begin{pmatrix}2 & -1\\ 1 & 4\end{pmatrix}$$

$$=\begin{pmatrix}3 & -6\\ 6 & 15\end{pmatrix}$$

$$A^2=\begin{pmatrix}1 & -1\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 2\end{pmatrix}$$

$$=\begin{pmatrix}1 & -3\\ 0 & 4\end{pmatrix}$$

$$2AB=2\begin{pmatrix}1 & -1\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 2\end{pmatrix}$$

$$=2\begin{pmatrix}0 & -2\\ 2 & 4\end{pmatrix}$$

$$=\begin{pmatrix} 0 & -4 \\ 4 & 8 \end{pmatrix}$$

$$B^2=\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$$

$$A^2+2AB+B^2=\begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix}+\begin{pmatrix} 0 & -4 \\ 4 & 8 \end{pmatrix}+\begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} 2 & -7 \\ 7 & 16 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -6 \\ 6 & 15 \end{pmatrix} \neq \begin{pmatrix} 2 & -7 \\ 7 & 16 \end{pmatrix}$$

Therefore, $(A+B)(A+B) \neq A^2+2AB+B^2$

b) $(A+B)(A-B) \neq A^2-B^2$

From part (a)

$$A+B=\begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$$

$$A^2=\begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix}$$

$$B^2=\begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$$

$$A-B=\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}-\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$(A+B)(A-B)=\begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & -2 \\ -4 & -1 \end{pmatrix}$$

$$A^2-B^2=\begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix}-\begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ -4 & -1 \end{pmatrix} \neq \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$$

Therefore, $(A+B)(A-B) \neq A^2 - B^2$

c) $(A+B)(A+B) = A^2 + AB + BA + B^2$

From part (a)

$$(A+B)(A+B) = \begin{pmatrix} 3 & -6 \\ 6 & 15 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2 \\ 2 & 4 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

$$A^2 + AB + BA + B^2 = \begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -6 \\ 6 & 15 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -6 \\ 6 & 15 \end{pmatrix} = \begin{pmatrix} 3 & -6 \\ 6 & 15 \end{pmatrix}$$

Therefore, $(A+B)(A+B) = A^2 + AB + BA + B^2$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}$

Solution

By rows: $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 5 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} \end{pmatrix}$

$$= \begin{pmatrix} 14 \\ 22 \end{pmatrix}$$

By columns: $\begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 4\begin{pmatrix} 2 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 14 \\ 22 \end{pmatrix}$$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Solution

By rows: $\begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 3 & 6 \end{pmatrix}\begin{pmatrix} 2 & -1 \end{pmatrix} \\ \begin{pmatrix} 6 & 12 \end{pmatrix}\begin{pmatrix} 2 & -1 \end{pmatrix} \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

By columns: $\begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 6 \end{pmatrix} - 1\begin{pmatrix} 6 \\ 12 \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

Solution

By rows:

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 & 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1(3)+2(1)+4(1) \\ 2(3)+0(1)+1(1) \end{pmatrix}$$

$$= \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

By columns:

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

Solution

By rows:

$$\begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 2 & 2 & 3 \end{pmatrix} \\ \begin{pmatrix} -2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 & 3 \end{pmatrix} \\ \begin{pmatrix} -4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 & 3 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 18 \\ 5 \\ 0 \end{pmatrix}$$

By columns:

$$\begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 18 \\ 5 \\ 0 \end{pmatrix}$$

Exercise

Given $A=\begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$ $\quad B=\begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix}$ $\quad$ Find $A+B,\ \ 2A,\ \ and\ \ -B$

Solution

$$A+B=\begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix}+\begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -2 \\ 8 & -2 & 0 \end{bmatrix}$$

$$2A=2\begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$=\begin{bmatrix} 8 & 2 & 6 \\ 6 & -2 & -4 \\ 0 & 0 & 8 \end{bmatrix}$$

$$-B=-\begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix}$$

$$=\begin{bmatrix} 3 & 2 & 3 \\ 1 & 0 & 0 \\ -8 & 2 & 4 \end{bmatrix}$$

Exercise

Given $A=\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ $\ and\ \ B=\begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}$. Find AB and BA.

Solution

$$AB=\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} -2 & 13 \\ 4 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} -16 & 29 \\ -4 & 10 \end{bmatrix}$$

$Note: \quad AB \neq BA$

Exercise

Given $A = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$ $B = \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}$$
$$= \begin{pmatrix} -6 & 17 \\ 6 & -8 \end{pmatrix}$$

$$BA = \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 14 \\ 1 & -20 \end{pmatrix}$$

$Note: \quad AB \neq BA$

Exercise

Given $A = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$ $B = \begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} -3 & -11 \\ 4 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -7 & 1 \\ 16 & 4 \end{pmatrix}$$

$Note: \quad AB \neq BA$

Exercise

Given $A=\begin{pmatrix}3 & -1\\ 2 & 3\end{pmatrix}$ $B=\begin{pmatrix}4 & 1\\ 2 & -3\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}3 & -1\\ 2 & 3\end{pmatrix}\begin{pmatrix}4 & 1\\ 2 & -3\end{pmatrix}$$
$$=\begin{pmatrix}10 & 6\\ 14 & -7\end{pmatrix}$$
$$BA=\begin{pmatrix}4 & 1\\ 2 & -3\end{pmatrix}\begin{pmatrix}3 & -1\\ 2 & 3\end{pmatrix}$$
$$=\begin{pmatrix}14 & -1\\ 0 & -11\end{pmatrix}$$

Note: $AB \neq BA$

Exercise

Given $A=\begin{pmatrix}-3 & 2\\ 2 & -2\end{pmatrix}$ $B=\begin{pmatrix}0 & 2\\ -2 & 4\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}-3 & 2\\ 2 & -2\end{pmatrix}\begin{pmatrix}0 & 2\\ -2 & 4\end{pmatrix}$$
$$=\begin{pmatrix}-4 & 2\\ 4 & -4\end{pmatrix}$$
$$BA=\begin{pmatrix}0 & 2\\ -2 & 4\end{pmatrix}\begin{pmatrix}-3 & 2\\ 2 & -2\end{pmatrix}$$
$$=\begin{pmatrix}4 & -4\\ 14 & -12\end{pmatrix}$$

Note: $AB \neq BA$

Exercise

Given $A=\begin{pmatrix}2 & -1\\ 0 & 3\\ 1 & -2\end{pmatrix}$ $B=\begin{pmatrix}1 & -2 & 3\\ 2 & 0 & 1\end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -4 & 5 \\ 6 & 0 & 3 \\ -3 & -2 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -13 \\ 3 & -4 \end{pmatrix}$$

$Note: \quad AB \neq BA$

Exercise

Given $A = \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 5 & 4 \\ 2 & -3 & 8 \\ -3 & 8 & -5 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -14 & 7 \\ -4 & 5 \end{pmatrix}$$

$Note: \quad AB \neq BA$

Exercise

Given $A=\begin{pmatrix}2&4\\0&-1\\-3&2\end{pmatrix}$ $B=\begin{pmatrix}3&0&-2\\-2&6&2\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}2&4\\0&-1\\-3&2\end{pmatrix}\begin{pmatrix}3&0&-2\\-2&6&2\end{pmatrix}$$

$$=\begin{pmatrix}-2&24&4\\2&-6&-2\\-13&12&10\end{pmatrix}$$

$$BA=\begin{pmatrix}3&0&-2\\-2&6&2\end{pmatrix}\begin{pmatrix}2&4\\0&-1\\-3&2\end{pmatrix}$$

$$=\begin{pmatrix}12&8\\-10&10\end{pmatrix}$$

Note: $AB\neq BA$

Exercise

Given $A=\begin{bmatrix}3&2&-3\\0&1&0\end{bmatrix}$ $B=\begin{bmatrix}3&-4\\0&1\\1&0\end{bmatrix}$ Find AB *and* BA *if possible*

Solution

$$AB=\begin{bmatrix}3&2&-3\\0&1&0\end{bmatrix}\begin{bmatrix}3&-4\\0&1\\1&0\end{bmatrix}$$

$$=\begin{bmatrix}3(3)+2(0)-3(1)&3(-4)+2(1)-3(0)\\0(3)+1(0)+0(1)&0(-4)+1(1)+0(0)\end{bmatrix}$$

$$=\begin{bmatrix}6&-10\\0&1\end{bmatrix}$$

$$BA=\begin{bmatrix}3&-4\\0&1\\1&0\end{bmatrix}\begin{bmatrix}3&2&-3\\0&1&0\end{bmatrix}$$

$$= \begin{bmatrix} 3(3)-4(0) & 3(2)-4(1) & 3(-3)-4(0) \\ 0(3)+1(0) & 0(2)+1(1) & 0(-3)+1(0) \\ 1(3)+0(0) & 1(2)+0(1) & 1(-3)+0(0) \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 2 & -9 \\ 0 & 1 & 0 \\ 3 & 2 & -3 \end{bmatrix}$$

Note: $AB \neq BA$

Exercise

Given $A = \begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 4 & -2 \\ -2 & 0 \\ 9 & 1 \end{bmatrix}$ Find AB *and* BA *if possible*

Solution

$AB =$ ***Undefined***

$$BA = \begin{bmatrix} 4 & -2 \\ -2 & 0 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 12 \\ -10 & -6 \\ 44 & 27 \end{bmatrix}$$

Note: $AB \neq BA$

Exercise

Given $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$ Find AB *and* BA *if possible*

Solution

$$AB = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}$$

$BA = Undefined$

Exercise

Given $A=\begin{pmatrix}2&-1&3\\0&2&-1\\0&1&2\end{pmatrix}$ $A=\begin{pmatrix}3&0&0\\1&-1&0\\2&-1&-2\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}2&-1&3\\0&2&-1\\0&1&2\end{pmatrix}\begin{pmatrix}3&0&0\\1&-1&0\\2&-1&-2\end{pmatrix}$$

$$=\begin{pmatrix}11&-2&-6\\0&-1&2\\5&-3&-4\end{pmatrix}$$

$$BA=\begin{pmatrix}3&0&0\\1&-1&0\\2&-1&-2\end{pmatrix}\begin{pmatrix}2&-1&3\\0&2&-1\\0&1&2\end{pmatrix}$$

$$=\begin{pmatrix}6&-3&9\\2&-3&4\\4&-6&3\end{pmatrix}$$

Note: $AB \neq BA$

Exercise

Given $A=\begin{pmatrix}-1&2&0\\2&-1&1\\-2&2&-1\end{pmatrix}$ $B=\begin{pmatrix}2&-1&0\\1&5&-1\\0&-1&3\end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix}-1&2&0\\2&-1&1\\-2&2&-1\end{pmatrix}\begin{pmatrix}2&-1&0\\1&5&-1\\0&-1&3\end{pmatrix}$$

$$=\begin{pmatrix}0&8&-2\\3&-8&4\\-2&13&-5\end{pmatrix}$$

$$BA=\begin{pmatrix}2&-1&0\\1&5&-1\\0&-1&3\end{pmatrix}\begin{pmatrix}-1&2&0\\2&-1&1\\-2&2&-1\end{pmatrix}$$

$$= \begin{pmatrix} -4 & 5 & -1 \\ 11 & -5 & 6 \\ -8 & 7 & -4 \end{pmatrix}$$

Note: $AB \neq BA$

Exercise

Given $A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}$ $B = \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}\begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -7 & 2 \\ -6 & 2 & 2 \\ -8 & -6 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 6 & 1 \\ 7 & 0 & 5 \\ 4 & -4 & -2 \end{pmatrix}$$

Note: $AB \neq BA$

Exercise

Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Compute the following (where possible):

a) $D+E$ *b)* $D-E$ *c)* $5A$ *d)* $-7C$ *e)* $2B-C$ *f)* $-3(D+2E)$

Solution

a) $D+E=\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}+\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$

$=\begin{bmatrix} 7 & 6 & 5 \\ -2 & 1 & 3 \\ 7 & 3 & 7 \end{bmatrix}$

b) $D-E=\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}-\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$

$=\begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

c) $5A=5\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$

$=\begin{bmatrix} 15 & 0 \\ -5 & 10 \\ 5 & 5 \end{bmatrix}$

d) $-7C=-7\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$

$=\begin{bmatrix} -7 & -28 & -14 \\ -21 & -7 & -35 \end{bmatrix}$

e) Since B and C are not the same size

$2B-C$: *can't be calculated*

f) $-3(D+2E)=-3\left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}+\begin{bmatrix} 12 & 2 & 6 \\ -2 & 2 & 4 \\ 8 & 2 & 6 \end{bmatrix}\right)$

$=-3\begin{bmatrix} 13 & 7 & 8 \\ -3 & 2 & 5 \\ 11 & 4 & 10 \end{bmatrix}$

$$=\begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix}$$

Exercise

Consider the matrices

$$A=\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \qquad B=\begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix} \qquad C=\begin{bmatrix} -3 & -1 \\ 2 & 1 \\ 4 & 3 \end{bmatrix} \qquad D=\begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$$

Compute the following (where possible):

a) $A+B$ ***b***) $A+C$ ***c***) AB ***d***) BA ***e***) CD ***f***) DC

g) BD ***h***) DB ***i***) A^2 ***j***) B^2 ***k***) D^2

Solution

a) Since A and B are not the same size, then

$A+B=$ *can't be calculated*

b) $A+C=\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}+\begin{bmatrix} -3 & -1 \\ 2 & 1 \\ 4 & 3 \end{bmatrix}$

$$=\begin{bmatrix} 0 & -1 \\ 1 & 3 \\ 5 & 4 \end{bmatrix}$$

c) $A:\ 3\times 2 \quad B:\ 3\times 3$

AB *can't be calculated* , since the inner are not equal.

d) $B:\ 3\times 3 \quad A:\ 3\times 2$

$$BA=\begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 12 \\ -1 & 2 \\ -10 & 5 \end{bmatrix}$$

e) $C:\ 3\times 2 \quad D:\ 2\times 2$

$$CD = \begin{bmatrix} -3 & -1 \\ 2 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -14 & 3 \\ 10 & -2 \\ 22 & -4 \end{bmatrix}$$

f) $D:\ 2\times 2 \quad C:\ 3\times 2$

DC *can't be calculated*, since the inner are not equal.

g) $B:\ 3\times 3 \quad D:\ 2\times 2$

BD *can't be calculated*, since the inner are not equal.

h) $D:\ 2\times 2 \quad B:\ 3\times 3$

DB *can't be calculated*, since the inner are not equal.

i) A^2 *can't be calculated*, since A is not square matrix.

j) $$B^2 = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} -12 & 12 & 8 \\ -2 & -4 & -2 \\ -17 & -16 & 1 \end{bmatrix}$$

k) $$D^2 = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 14 & -4 \\ 8 & -2 \end{bmatrix}$$

Exercise

Find if possible, $a)\ A+B \quad b)\ A-B \quad c)\ 2A \quad d)\ 2A-B \quad e)\ B+\frac{1}{2}A \quad f)\ AB \quad g)\ BA$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix}$$

Solution

a) $$A+B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix}$$

$$=\begin{pmatrix} -2 & 0 \\ 6 & 3 \end{pmatrix}$$

b) $A-B=\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}-\begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix}$

$$=\begin{pmatrix} 4 & 4 \\ -2 & -1 \end{pmatrix}$$

c) $2A=2\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$$=\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$$

d) $2A-B=\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}+\begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix}$

$$=\begin{pmatrix} -1 & 2 \\ 8 & 4 \end{pmatrix}$$

e) $B+\frac{1}{2}A=\begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix}+\begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix}$

$$=\begin{pmatrix} -\frac{5}{2} & -1 \\ 6 & \frac{33}{2} \end{pmatrix}$$

f) $AB=\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix}$

$$=\begin{pmatrix} 5 & 2 \\ 2 & 6 \end{pmatrix}$$

g) $BA=\begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$$=\begin{pmatrix} -7 & -8 \\ 8 & 10 \end{pmatrix}$$

Exercise

Find if possible, $a)\ A+B \quad b)\ A-B \quad c)\ 2A \quad d)\ 2A-B \quad e)\ B+\frac{1}{2}A \quad f)\ AB \quad g)\ BA$

$$A=\begin{pmatrix}1 & 2\\ 4 & 2\end{pmatrix} \qquad B=\begin{pmatrix}2 & -1\\ -1 & 8\end{pmatrix}$$

Solution

a) $A+B=\begin{pmatrix}1 & 2\\ 4 & 2\end{pmatrix}+\begin{pmatrix}2 & -1\\ -1 & 8\end{pmatrix}$

$=\begin{pmatrix}3 & 1\\ 3 & 10\end{pmatrix}$

b) $A-B=\begin{pmatrix}1 & 2\\ 4 & 2\end{pmatrix}-\begin{pmatrix}2 & -1\\ -1 & 8\end{pmatrix}$

$=\begin{pmatrix}-1 & 3\\ 5 & -6\end{pmatrix}$

c) $2A=2\begin{pmatrix}1 & 2\\ 4 & 2\end{pmatrix}$

$=\begin{pmatrix}2 & 4\\ 8 & 4\end{pmatrix}$

d) $2A-B=\begin{pmatrix}2 & 4\\ 8 & 4\end{pmatrix}+\begin{pmatrix}2 & -1\\ -1 & 8\end{pmatrix}$

$=\begin{pmatrix}4 & 3\\ 7 & 12\end{pmatrix}$

e) $B+\frac{1}{2}A=\begin{pmatrix}2 & -1\\ -1 & 8\end{pmatrix}+\begin{pmatrix}\frac{1}{2} & 1\\ 2 & 1\end{pmatrix}$

$=\begin{pmatrix}\frac{5}{2} & 0\\ 1 & 9\end{pmatrix}$

f) $AB=\begin{pmatrix}1 & 2\\ 4 & 2\end{pmatrix}\begin{pmatrix}2 & -1\\ -1 & 8\end{pmatrix}$

$=\begin{pmatrix}0 & 15\\ 6 & 12\end{pmatrix}$

g) $BA=\begin{pmatrix}2 & -1\\ -1 & 8\end{pmatrix}\begin{pmatrix}1 & 2\\ 4 & 2\end{pmatrix}$

$=\begin{pmatrix}-2 & 2\\ 31 & 14\end{pmatrix}$

Exercise

Find if possible, $a)\ A+B \quad b)\ A-B \quad c)\ 2A \quad d)\ 2A-B \quad e)\ B+\frac{1}{2}A \quad f)\ AB \quad g)\ BA$

$$A=\begin{pmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{pmatrix} \qquad B=\begin{pmatrix} 8 & -1 \\ 4 & -3 \end{pmatrix}$$

Solution

a) $A+B=\begin{pmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{pmatrix}+\begin{pmatrix} 8 & -1 \\ 4 & -3 \end{pmatrix}$

$=can't\ \ be\ \ calculated$

b) $A-B=\begin{pmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{pmatrix}-\begin{pmatrix} 8 & -1 \\ 4 & -3 \end{pmatrix}$

$=can't\ \ be\ \ calculated$

c) $2A=2\begin{pmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{pmatrix}$

$=\begin{pmatrix} 12 & 0 & 6 \\ -2 & -8 & 0 \end{pmatrix}$

d) $2A-B=\begin{pmatrix} 12 & 0 & 6 \\ -2 & -8 & 0 \end{pmatrix}-\begin{pmatrix} 8 & -1 \\ 4 & -3 \end{pmatrix}$

$=can't\ \ be\ \ calculated$

e) $B+\frac{1}{2}A=\begin{pmatrix} 8 & -1 \\ 4 & -3 \end{pmatrix}+\begin{pmatrix} 3 & 0 & \frac{3}{2} \\ -\frac{1}{2} & -2 & 0 \end{pmatrix}$

$=can't\ \ be\ \ calculated$

f) $AB=\begin{pmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{pmatrix}\begin{pmatrix} 8 & -1 \\ 4 & -3 \end{pmatrix}$

$=can't\ \ be\ \ calculated$

g) $BA=\begin{pmatrix} 8 & -1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{pmatrix}$

$=\begin{pmatrix} 19 & 4 & 24 \\ 27 & 12 & 12 \end{pmatrix}$

Exercise

Find if possible, $a)\ A+B \quad b)\ A-B \quad c)\ 2A \quad d)\ 2A-B \quad e)\ B+\frac{1}{2}A \quad f)\ AB \quad g)\ BA$

$$A=\begin{pmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{pmatrix} \qquad B=\begin{pmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{pmatrix}$$

Solution

a) $A+B=\begin{pmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{pmatrix}+\begin{pmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{pmatrix}$

$= can't\ \ be\ \ calculated$

b) $A-B=\begin{pmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{pmatrix}-\begin{pmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{pmatrix}$

$= can't\ \ be\ \ calculated$

c) $2A=2\begin{pmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{pmatrix}$

$=\begin{pmatrix} 4 & 2 \\ -2 & 8 \\ 2 & 12 \end{pmatrix}$

d) $2A-B= can't\ \ be\ \ calculated$

e) $B+\frac{1}{2}A= can't\ \ be\ \ calculated$

f) $AB=\begin{pmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{pmatrix}$

$= can't\ \ be\ \ calculated$

g) $BA=\begin{pmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{pmatrix}$

$=\begin{pmatrix} 3 & -4 \\ 10 & 16 \\ 26 & 38 \end{pmatrix}$

Exercise

Consider the matrices $A = \begin{pmatrix} x^2 & x \\ -3 & 2 \end{pmatrix}$ $B = \begin{pmatrix} x & x^2 \\ 4 & x \end{pmatrix}$

Compute the following (where possible):

a) $A + B$ *b*) $A - B$ *c*) AB *d*) BA *e*) $2A - 3B$

Solution

a) $A + B = \begin{pmatrix} x^2 & x \\ -3 & 2 \end{pmatrix} + \begin{pmatrix} x & x^2 \\ 4 & x \end{pmatrix}$

$= \begin{pmatrix} x^2 + x & x + x^2 \\ 1 & 2 + x \end{pmatrix}$

b) $A - B = \begin{pmatrix} x^2 & x \\ -3 & 2 \end{pmatrix} - \begin{pmatrix} x & x^2 \\ 4 & x \end{pmatrix}$

$= \begin{pmatrix} x^2 - x & x - x^2 \\ -7 & 2 - x \end{pmatrix}$

c) $AB = \begin{pmatrix} x^2 & x \\ -3 & 2 \end{pmatrix} \begin{pmatrix} x & x^2 \\ 4 & x \end{pmatrix}$

$= \begin{pmatrix} x^3 + 4x & x^4 + x^2 \\ -3x + 8 & -3x^2 + 2x \end{pmatrix}$

d) $BA = \begin{pmatrix} x & x^2 \\ 4 & x \end{pmatrix} \begin{pmatrix} x^2 & x \\ -3 & 2 \end{pmatrix}$

$= \begin{pmatrix} x^3 - 3x^2 & 3x^2 \\ 4x^2 - 3x & 6x \end{pmatrix}$

e) $2A - 3B = 2\begin{pmatrix} x^2 & x \\ -3 & 2 \end{pmatrix} - 3\begin{pmatrix} x & x^2 \\ 4 & x \end{pmatrix}$

$= \begin{pmatrix} 2x^2 - 3x & 2x - 3x^2 \\ -18 & 4 - 3x \end{pmatrix}$

Exercise

Consider the matrices

$$A = \begin{pmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{pmatrix}$$

Compute the following (where possible):

a) $A+B$ *b)* $A-B$ *c)* AB *d)* BA *e)* $2A-3B$

Solution

a) $$A+B = \begin{pmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} + \begin{pmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2x & -1 \\ 0 & 2 & 1 \\ 2 & -1 & 6 \end{pmatrix}$$

b) $$A-B = \begin{pmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} - \begin{pmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & -3 \\ 6 & 0 & 1 \\ -2 & -3 & -2 \end{pmatrix}$$

c) $$AB = \begin{pmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -3x-2 & 2x-2 & -7 \\ 5 & 3x+2 & 7 \\ 10 & 0 & 4 \end{pmatrix}$$

d) $$BA = \begin{pmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2+3x & 3x-2 & x-2 \\ 0 & -3x+1 & 7 \\ 5 & 2x-7 & 5 \end{pmatrix}$$

e) $$2A-3B=2\begin{pmatrix}1 & x & -2\\ 3 & 1 & 1\\ 0 & -2 & 2\end{pmatrix}-3\begin{pmatrix}2 & x & 1\\ -3 & 1 & 0\\ 2 & 1 & 4\end{pmatrix}$$

$$=\begin{pmatrix}-4 & -x & -7\\ 15 & -1 & 2\\ -6 & -9 & -8\end{pmatrix}$$

Exercise

Let $B=\begin{pmatrix}a & 0\\ 1 & b\end{pmatrix}$, show that $B^4=\begin{pmatrix}a^4 & 0\\ a^3+a^2b+ab^2+b^3 & b^4\end{pmatrix}$

Solution

$$B^2=\begin{pmatrix}a & 0\\ 1 & b\end{pmatrix}\begin{pmatrix}a & 0\\ 1 & b\end{pmatrix}$$

$$=\begin{pmatrix}a^2 & 0\\ a+b & b^2\end{pmatrix}$$

$$B^4=B^2B^2$$

$$=\begin{pmatrix}a^2 & 0\\ a+b & b^2\end{pmatrix}\begin{pmatrix}a^2 & 0\\ a+b & b^2\end{pmatrix}$$

$$=\begin{pmatrix}a^4 & 0\\ a^3+a^2b+ab^2+b^3 & b^4\end{pmatrix} \quad \checkmark$$

Exercise

Let $B=\begin{pmatrix}a & 0\\ 1 & b\end{pmatrix}$, show that $B^n=\begin{pmatrix}a^n & 0\\ \sum_{k=0}^{n-1} a^{n-1-k}b^k & b^n\end{pmatrix}$

Solution

$$n=2 \quad \to \quad B^2=\begin{pmatrix}a & 0\\ 1 & b\end{pmatrix}\begin{pmatrix}a & 0\\ 1 & b\end{pmatrix}$$

$$=\begin{pmatrix}a^2 & 0\\ a+b & b^2\end{pmatrix} \quad \checkmark$$

Let assume $B^n = \begin{pmatrix} a^n & 0 \\ \sum_{k=0}^{n-1} a^{n-1-k} b^k & b^n \end{pmatrix}$ is true

We need to also prove that it is true for $B^{n+1} = \begin{pmatrix} a^{n+1} & 0 \\ \sum_{k=0}^{n} a^{n-k} b^k & b^{n+1} \end{pmatrix}$

$$B^{n+1} = B^n B = \begin{pmatrix} a^n & 0 \\ \sum_{k=0}^{n-1} a^{n-1-k} b^k & b^n \end{pmatrix} \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$

$$= \begin{pmatrix} a^{n+1} & 0 \\ b^n + a\sum_{k=0}^{n-1} a^{n-1-k} b^k & b^{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} a^{n+1} & 0 \\ b^n + \sum_{k=0}^{n} a^{n-k} b^k & b^{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} a^{n+1} & 0 \\ \sum_{k=0}^{n} a^{n-k} b^k & b^{n+1} \end{pmatrix} \quad \checkmark$$

Exercise

Let $A = \begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix}$. Prove that $A^n = \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$ if $n \ge 1$

Solution

Using the principle of mathematical induction.

For $n = 1$

$$A = \begin{bmatrix} 1+6 & 4 \\ -9 & 1-6 \end{bmatrix}$$

$$=\begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix} \qquad \checkmark \quad P_1 \text{ is true}$$

Assume that P_n is true, $A^n = \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$

We need to prove that P_{n+1}:

$$A^{n+1} = \begin{bmatrix} 1+6(n+1) & 4(n+1) \\ -9(n+1) & 1-6(n+1) \end{bmatrix}$$

$$= \begin{bmatrix} 7+6n & 4n+4 \\ -9n-9 & -6n-5 \end{bmatrix} \qquad \text{is also } \textit{true}.$$

$$A^{n+1} = AA^n$$

$$= \begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix}\begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$$

$$= \begin{bmatrix} 7+42n-36n & 28n+4-24n \\ -9-54n+45n & -36n-5+30n \end{bmatrix}$$

$$= \begin{bmatrix} 7+6n & 4n+4 \\ -9n-9 & -6n-5 \end{bmatrix} \checkmark \quad P_{n+1} \text{ is also true}$$

∴ By mathematical induction, the proof of $A^n = \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$ is completed.

Exercise

Let $A = \begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix}$. Prove that $A^n = \begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix}$ if $n \ge 1$

Solution

Using the principle of mathematical induction.

For $n=1$

$$A^1 = \begin{bmatrix} (1+1)a & -a^2 \\ 1a^0 & (1-1)a \end{bmatrix}$$

$$= \begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix} \qquad \checkmark \quad P_1 \text{ is true}$$

Assume that P_n is true, $A^n = \begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix}$

We need to prove that P_{n+1}:

$A^{n+1} = \begin{bmatrix} (n+2)a^{n+1} & -(n+1)a^{n+2} \\ (n+1)a^n & -na^{n+1} \end{bmatrix}$ is also *true*.

$$A^{n+1} = AA^n$$

$$= \begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix}$$

$$= \begin{bmatrix} 2(n+1)a^{n+1} - na^{n+1} & -2na^{n+2} - (1-n)a^{n+2} \\ (n+1)a^n & -na^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} (2n+2-n)a^{n+1} & -(2n+1-n)a^{n+2} \\ (n+1)a^n & -na^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} (n+2)a^{n+1} & -(n+1)a^{n+2} \\ (n+1)a^n & -na^{n+1} \end{bmatrix} \quad \checkmark P_{n+1} \text{ is also true}$$

$\therefore$ By mathematical induction, the proof of $A^n = \begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix}$ is completed.

Exercise

The following system of recurrence relations holds for all $n \ge 0$

$$\begin{cases} x_{n+1} = 7x_n + 4y_n \\ y_{n+1} = -9x_n - 5y_n \end{cases}$$

Solve the system for x_n and y_n in terms of x_0 and y_0

Solution

$$\begin{cases} x_{n+1} = 7x_n + 4y_n \\ y_{n+1} = -9x_n - 5y_n \end{cases}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$X_{n+1} = AX_n$$

$$A = \begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix} \quad X_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$X_1 = AX_0$$

$$X_2 = AX_1 = A\left(AX_0\right) = A^2 X_0$$

$$X_3 = AX_2 = A\left(A^2 X_0\right) = A^3 X_0$$

$$\vdots \qquad \vdots$$

$$X_n = A^n X_0$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix}^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Since, when $\begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix}$ that implies $A^n = \begin{pmatrix} 1+6n & 4n \\ -9n & 1-6n \end{pmatrix}$ (from previous prove).

$$\begin{aligned} \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= \begin{pmatrix} 1+6n & 4n \\ -9n & 1-6n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \begin{pmatrix} (1+6n)x_0 + 4ny_0 \\ -9nx_0 + (1-6n)y_0 \end{pmatrix} \end{aligned}$$

$$\therefore \begin{cases} x_n = (1+6n)x_0 + 4ny_0 \\ y_n = -9nx_0 + (1-6n)y_0 \end{cases}$$

Exercise

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, prove that $A^2 - (a+d)A + (ad-bc)I_{2\times 2} = 0$

Solution

$$\begin{aligned} A^2 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} \end{aligned}$$

$$A^2-(a+d)A+(ad-bc)I=\begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}-(a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix}+(ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}-\begin{pmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{pmatrix}+\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$=\begin{pmatrix} a^2+bc-a^2-ad+ad-bc & ab+bd-ab-bd \\ ac+cd-ac-cd & bc+d^2-ad-d^2+ad-bc \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\underline{=0} \quad \checkmark$$

Exercise

If $A=\begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}$, use the fact $A^2=4A-3I$ and mathematical induction, to prove that

$$A^n=\frac{3^n-1}{2}A+\frac{3-3^n}{2}I \quad if \quad n\ge 1$$

Solution

$$A^2=\begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 13 & -12 \\ 4 & -3 \end{pmatrix}$$

$$4A-3I=4\begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}-3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 16 & -12 \\ 4 & 0 \end{pmatrix}-\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$=\begin{pmatrix} 13 & -12 \\ 4 & -3 \end{pmatrix}$$

$$\underline{=A^2}$$

Using mathematical induction model

For $n=1 \rightarrow A^1=\frac{3^1-1}{2}A+\frac{3-3^1}{2}I$

$A=A+0=A$ ✓ is true for P_1

Assume is true for $P_k \rightarrow A^k=\frac{3^k-1}{2}A+\frac{3-3^k}{2}I$

We need to prove that is also true for $P_{k+1} \rightarrow A^{k+1} = \frac{3^{k+1}-1}{2}A + \frac{3-3^{k+1}}{2}I$

$$\begin{aligned}
A^{k+1} &= AA^k \\
&= A\left(\frac{3^k-1}{2}A + \frac{3-3^k}{2}I\right) \\
&= \frac{3^k-1}{2}A^2 + \frac{3-3^k}{2}(AI) \qquad\qquad A^2 = 4A-3I \\
&= \frac{3^k-1}{2}(4A-3I) + \frac{3-3^k}{2}A \\
&= 2\left(3^k-1\right)A - \frac{3\left(3^k-1\right)}{2}I + \frac{3-3^k}{2}A \\
&= \left(2\cdot 3^k - 2 + \frac{3-3^k}{2}\right)A - \frac{3^{k+1}-3}{2}I \\
&= \left(\frac{4\cdot 3^k - 4 + 3 - 3^k}{2}\right)A - \frac{3^{k+1}-3}{2}I \\
&= \left(\frac{3\cdot 3^k - 1}{2}\right)A - \frac{3^{k+1}-3}{2}I \\
&= \frac{3^{k+1}-1}{2}A + \frac{3-3^{k+1}}{2}I \qquad \checkmark \quad \text{is also true for } P_{k+1}
\end{aligned}$$

By mathematical induction, the proof that $A^n = \frac{3^n-1}{2}A + \frac{3-3^n}{2}I \quad if \quad n \ge 1$ is completed.

Exercise

A sequence of numbers $x_1, x_2, \ldots, x_n, \ldots$ satisfies the recurrence relation $x_{n+1} = ax_n + bx_{n-1}$ for $n \ge 1$, where a and b are constants. Prove that

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = A\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

Where $A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$ and hence express $\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$ in terms of $\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$.

If $a = 4$ and $b = -3$, use the previous question to find a formula for x_n in terms x_1 and x_0

Solution

$$x_{n+1} = ax_n + bx_{n-1}$$

$$= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

$$x_n = x_n$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

$$= A \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

$$x_{n+1} = ax_n + bx_{n-1}$$

$$\underline{= 4x_n - 3x_{n-1}}\Big|$$

$$n = 1 \quad \rightarrow \quad x_2 = 4x_1 - 3x_0$$

$$n = 2 \quad \rightarrow \quad x_3 = 4x_2 - 3x_1$$

$$= 4\left(4x_1 - 3x_0\right) - 3x_1 \qquad\qquad = \left(4^2 - 3\right)x_1 - 3x_0$$

$$= 13x_1 - 12x_0$$

$$n = 3 \quad \rightarrow \quad x_4 = 4x_3 - 3x_2$$

$$= 4\left(13x_1 - 12x_0\right) - 3\left(4x_1 - 3x_0\right)$$

$$= 40x_1 - 39x_0$$

$$n = 4 \quad \rightarrow \quad x_5 = 4x_4 - 3x_3$$

$$= 4\left(40x_1 - 39x_0\right) - 3\left(13x_1 - 12x_0\right)$$

$$= 121x_1 - 120x_0$$

	x_1	x_0
$n = 2 \rightarrow$	4	-3
$n = 3 \rightarrow$	13	-12
$n = 4 \rightarrow$	40	-39
$n = 5 \rightarrow$	121	-120

$$\underline{x_n = \frac{3^n - 1}{2}x_1 + \frac{3 - 3^n}{2}x_0}\Big|$$

Exercise

Prove $\left(A+B\right)C = AC+BC$

Solution

Let $A=\left[a_{ij}\right],\quad B=\left[b_{ij}\right] \quad \& \quad C=\left[c_{jk}\right]$

Let $D = A+B$

$$=\left[a_{ij}+b_{ij}\right]$$

$$=\left[d_{ij}\right]$$

Let $E = AC$

$$=\sum a_{ij}c_{jk}$$

Let $F = BC$

$$=\sum b_{ij}c_{jk}$$

$AC+BC = E+F$

$$=\sum a_{ij}c_{jk}+\sum b_{ij}c_{jk}$$

$$=\sum\left(a_{ij}c_{jk}+b_{ij}c_{jk}\right)$$

$$=\sum\left(a_{ij}+b_{ij}\right)c_{jk}$$

$$=\sum d_{ij}\ c_{jk}$$

$$=DC$$

$$=\left(A+B\right)C \qquad \checkmark$$

Thus, $\left(A+B\right)C = AC+BC$

Exercise

Prove $\left(A+B\right)+C = A+\left(B+C\right)$

Solution

Let $A=\left[a_{ij}\right],\quad B=\left[b_{ij}\right] \quad \& \quad C=\left[c_{ij}\right]$

$$\left(A+B\right)+C=\left[a_{ij}\right]+\left(\left[b_{ij}\right]+\left[c_{ij}\right]\right)$$

$$=\left[a_{ij}\right]+\left[b_{ij}+c_{ij}\right]$$

$$=\left[a_{ij}+\left(b_{ij}+c_{ij}\right)\right]$$

$$=\left[\left(a_{ij}+b_{ij}\right)+c_{ij}\right]$$

$$=\left(A+B\right)+C$$

Thus, $\left(A+B\right)+C=A+\left(B+C\right)$

Exercise

Prove $\quad A\left(BC\right)=\left(AB\right)C$

Solution

Let $A=\left[a_{ij}\right],\quad B=\left[b_{jk}\right] \quad \& \quad C=\left[c_{kl}\right]$

Let $E=BC$

$$=\sum b_{jk}c_{kl}$$

$$=\left[e_{jl}\right]$$

Let $F=AB$

$$=\sum a_{ij}b_{jk}$$

$$=\left[f_{ik}\right]$$

$$A\left(BC\right)=AE$$

$$=\left[a_{ij}\right]\left[e_{jl}\right]$$

$$=\sum a_{ij}e_{jl}$$

$$=\sum \left(a_{ij}\left(\sum b_{jk}c_{kl}\right)\right)$$

$$= \sum \sum a_{ij} \left(b_{jk} c_{kl} \right)$$

$(AB)C = FC$

$$= \left[f_{ik} \right] \left[c_{kl} \right]$$

$$= \sum f_{ik} c_{kl}$$

$$= \sum \left(\sum \left(a_{ij} b_{jk} \right) \right) c_{kl}$$

$$= \sum \sum \left(a_{ij} b_{jk} \right) c_{kl}$$

$$\sum \sum a_{ij} \left(b_{jk} c_{kl} \right) = \sum \sum \left(a_{ij} b_{jk} \right) c_{kl}$$

Thus, $A(BC) = (AB)C$

Exercise

Prove $\quad A(B - C) = AB - AC$

Solution

Let $A = \left[a_{ij} \right]$, $B = \left[b_{jk} \right]$ & $C = \left[c_{jk} \right]$

Let $D = B - C$

$$= \left[b_{jk} - c_{jk} \right]$$

$$= \left[d_{ij} \right]$$

Let $E = AB$

$$= \sum a_{ij} b_{jk}$$

Let $F = AC$

$$= \sum a_{ij} c_{jk}$$

$AB - AC = E - F$

$$= \sum a_{ij} b_{jk} - \sum a_{ij} c_{jk}$$

$$= \sum \left(a_{ij} b_{jk} - a_{ij} c_{jk} \right)$$
$$= \sum a_{ij} \left(b_{jk} - c_{jk} \right)$$
$$= \sum a_{ij}\ d_{jk}$$
$$= AD$$
$$= A(B-C) \quad \checkmark$$

Thus, $A(B-C) = AB - AC$

Exercise

Prove $\quad (A+B)(A-B) \neq A^2 - B^2$

Solution

$$(A+B)(A-B) = AA - AB + BA - BB$$
$$= A^2 - AB + BA - B^2$$

Since $AB \neq BA$

Therefore, $(A+B)(A-B) \neq A^2 - B^2$

Exercise

Show that $AC = BC$, even though $A \neq B$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}; \quad C = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$$

Solution

$$AC = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$$

$$BC = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$$

Therefore, $AC = BC$

Exercise

Verify $AB = BA$ for the matrices below

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad B = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

Solution

$$AB = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

$$BA = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha & -\sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \cos\alpha\sin\beta + \sin\alpha\cos\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

Therefore, $AB = BA$

Exercise

Solve for A: $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} + 2a_{21} & a_{12} + 2a_{22} \\ 3a_{11} + 5a_{21} & 3a_{12} + 5a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} a_{11} + 2a_{21} = 1 \\ 3a_{11} + 5a_{21} = 0 \end{cases}$$

$$\begin{cases} -3a_{11} - 6a_{21} = -3 \\ 3a_{11} + 5a_{21} = 0 \end{cases}$$

$$\begin{cases} a_{11} = -5 \\ a_{21} = 3 \end{cases}$$

$$\begin{cases} a_{12} + 2a_{22} = 0 \\ 3a_{12} + 5a_{22} = 1 \end{cases}$$

$$\begin{cases} a_{12} = 2 \\ a_{22} = -1 \end{cases}$$

$$A = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

Exercise

Solve for X in the equation $3X + 2A = B$, given

$$A = \begin{pmatrix} -4 & 0 \\ 1 & -5 \\ -3 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{pmatrix}$$

Solution

$$3X + 2A = B$$

$$X = -\frac{2}{3}A + \frac{1}{3}B$$

$$= -\frac{2}{3}\begin{pmatrix} -4 & 0 \\ 1 & -5 \\ -3 & 2 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & \frac{2}{3} \\ -\frac{4}{3} & \frac{11}{3} \\ \frac{10}{3} & 0 \end{pmatrix}$$

Exercise

For $A = \begin{pmatrix} 4 & 1 & 3 \\ 0 & 5 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -2 & -4 \\ 3 & 0 & 1 \end{pmatrix}$

Solve the equations for *X*.

a) $2X = 3A - B$

b) $3X - 2A + B = 0$

c) $-2X = -5A + 3B$

d) $4X + 5A - 3B = 0$

Solution

a) $2X = 3A - B$

$$X = \frac{1}{2}\left(3\begin{pmatrix} 4 & 1 & 3 \\ 0 & 5 & -2 \end{pmatrix} - \begin{pmatrix} 0 & -2 & -4 \\ 3 & 0 & 1 \end{pmatrix}\right)$$

$$= \frac{1}{2}\begin{pmatrix} 12 & 5 & 13 \\ -3 & 15 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & \frac{5}{2} & \frac{13}{2} \\ -\frac{3}{2} & \frac{15}{2} & -\frac{7}{2} \end{pmatrix}$$

b) $3X - 2A + B = 0$

$$X = \frac{1}{3}\left(2\begin{pmatrix} 4 & 1 & 3 \\ 0 & 5 & -2 \end{pmatrix} + \begin{pmatrix} 0 & -2 & -4 \\ 3 & 0 & 1 \end{pmatrix}\right)$$

$$= \frac{1}{3}\begin{pmatrix} 8 & 0 & 2 \\ 3 & 5 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{8}{3} & 0 & \frac{2}{3} \\ 1 & \frac{5}{3} & -1 \end{pmatrix}$$

c) $-2X = -5A + 3B$

$$X = \frac{5}{2}\begin{pmatrix} 4 & 1 & 3 \\ 0 & 5 & -2 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 0 & -2 & -4 \\ 3 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & \frac{11}{2} & \frac{27}{2} \\ -\frac{3}{2} & \frac{25}{2} & -\frac{13}{2} \end{pmatrix}$$

d) $4X + 5A - 3B = 0$

$$X = -\frac{5}{4}\begin{pmatrix} 4 & 1 & 3 \\ 0 & 5 & -2 \end{pmatrix} + \frac{3}{4}\begin{pmatrix} 0 & -2 & -4 \\ 3 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -\frac{11}{4} & -\frac{27}{4} \\ \frac{9}{4} & -\frac{25}{4} & \frac{13}{4} \end{pmatrix}$$

Exercise

For $A = \begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 3 \end{pmatrix}$

Solve the equations for X.

a) $X = 3A - 2B$

b) $3X + 2A = B$

c) $-2X = -4A + 3B$

d) $2X + 3A - 4B = 0$

Solution

a) $X = 3A - 2B$

$$= 3\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -8 & -7 \\ 7 & -2 \\ 1 & -18 \end{pmatrix}$$

b) $3X + 2A = B$

$$X = -\frac{2}{3}A + \frac{1}{3}B$$

$$= -\frac{2}{3}\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{3} & \frac{4}{3} \\ -\frac{4}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{11}{3} \end{pmatrix}$$

c) $-2X = -4A + 3B$

$$X = 2A - \frac{3}{2}B$$

$$= 2\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{11}{2} & -5 \\ 5 & -\frac{3}{2} \\ 0 & -\frac{25}{2} \end{pmatrix}$$

d) $2X + 3A - 4B = 0$

$$X = -\frac{3}{2}A + 2B$$

$$= -\frac{3}{2}\begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{pmatrix} + 2\begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & \frac{11}{2} \\ -\frac{11}{2} & 2 \\ \frac{7}{2} & 12 \end{pmatrix}$$

Exercise

For $A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & -3 \\ 1 & 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 0 & 5 \\ 6 & 9 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 4 & 4 \\ 5 & -1 & 0 \\ 7 & 8 & -1 \end{pmatrix}$

Solve the equations in few steps as you dare

e) $\frac{1}{2}(X + A) = 3[X + (2X + B)] + C$

f) $2(X + B) = 3\left(X + \left(\frac{1}{2}X + A\right)\right) + C$

g) $40(X + A) = 47(X + B) + 48(X + C)$

h) $\sqrt{2}(X + C) = 31\left(X + \sqrt{2}(X + A - B)\right)$

Solution

a) $\frac{1}{2}(X + A) = 3[X + (2X + B)] + C$

$$\frac{1}{2}X + \frac{1}{2}A = 3X + 6X + 3B + C$$

$$\frac{1}{2}X - 9X = 3B + C - \frac{1}{2}A$$

$$\left(\frac{1}{2} - 9\right)X = \frac{1}{2}(6B + 2C - A)$$

$$-\frac{17}{2}X = \frac{1}{2}(6B + 2C - A)$$

$$X = \frac{1}{17}(A - 6B - 2C)$$

OR

$A - 6B - 2C = (6 + 12 - 1)X$

$$X = \frac{1}{17}\left(\begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & -3 \\ 1 & 0 & 4 \end{pmatrix} - 6\begin{pmatrix} 3 & 3 & 3 \\ 3 & 0 & 5 \\ 6 & 9 & -1 \end{pmatrix} - 2\begin{pmatrix} 4 & 4 & 4 \\ 5 & -1 & 0 \\ 7 & 8 & -1 \end{pmatrix}\right)$$

$$= \begin{pmatrix} -\frac{24}{17} & -\frac{24}{17} & -\frac{24}{17} \\ -\frac{26}{17} & \frac{3}{17} & -\frac{33}{17} \\ -\frac{48}{17} & \frac{70}{17} & \frac{12}{17} \end{pmatrix}$$

b) $2(X + B) = 3\left(X + \left(\frac{1}{2}X + A\right)\right) + C$

$\left(2 - 3 - \frac{3}{2}\right)X = 3A + C - 2B$

$$X = -\frac{2}{5}\left(3\begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & -3 \\ 1 & 0 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 4 & 4 \\ 5 & -1 & 0 \\ 7 & 8 & -1 \end{pmatrix} - 2\begin{pmatrix} 3 & 3 & 3 \\ 3 & 0 & 5 \\ 6 & 9 & -1 \end{pmatrix}\right)$$

$$= \begin{pmatrix} -\frac{8}{5} & -\frac{8}{5} & -\frac{8}{5} \\ -2 & -\frac{4}{5} & \frac{28}{5} \\ \frac{4}{5} & 4 & -\frac{26}{5} \end{pmatrix}$$

c) $40(X + A) = 47(X + B) + 48(X + C)$

$-55X = 47B + 48C - 40A$

$$X = -\frac{1}{55}\left(47\begin{pmatrix} 3 & 3 & 3 \\ 3 & 0 & 5 \\ 6 & 9 & -1 \end{pmatrix} + 48\begin{pmatrix} 4 & 4 & 4 \\ 5 & -1 & 0 \\ 7 & 8 & -1 \end{pmatrix} - 40\begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & -3 \\ 1 & 0 & 4 \end{pmatrix}\right)$$

$$= \begin{pmatrix} -\frac{23}{5} & -\frac{23}{5} & -\frac{23}{5} \\ -\frac{301}{55} & \frac{11}{5} & -\frac{71}{11} \\ -\frac{578}{55} & -\frac{807}{55} & \frac{51}{11} \end{pmatrix}$$

d) $\sqrt{2}(X + C) = 31\left(X + \sqrt{2}(X + A - B)\right)$

$$-\left(31+30\sqrt{2}\right)X=\sqrt{2}\left(31A-31B-C\right)$$

$$X=-\frac{\sqrt{2}}{31+30\sqrt{2}}\left(31\begin{pmatrix}2&2&2\\2&1&-3\\1&0&4\end{pmatrix}-31\begin{pmatrix}3&3&3\\3&0&5\\6&9&-1\end{pmatrix}-\begin{pmatrix}4&4&4\\5&-1&0\\7&8&-1\end{pmatrix}\right)$$

$$=\begin{pmatrix}\frac{35\sqrt{2}}{31+30\sqrt{2}}&\frac{35\sqrt{2}}{31+30\sqrt{2}}&\frac{35\sqrt{2}}{31+30\sqrt{2}}\\\frac{36\sqrt{2}}{31+30\sqrt{2}}&-\frac{36\sqrt{2}}{31+30\sqrt{2}}&\frac{248\sqrt{2}}{31+30\sqrt{2}}\\\frac{162\sqrt{2}}{31+30\sqrt{2}}&\frac{287\sqrt{2}}{31+30\sqrt{2}}&-\frac{156\sqrt{2}}{31+30\sqrt{2}}\end{pmatrix}$$

Exercise

If *f* is the polynomial function is given by $f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$

Then $f(A)$ is defined by $f(A)=a_0I+a_1A+a_2A^2+\cdots+a_nA^n$

Find $f(A)$

$$f(x)=x^2+3x-2, \qquad A=\begin{pmatrix}2&3\\0&1\end{pmatrix}$$

Solution

$$A^2=\begin{pmatrix}2&3\\0&1\end{pmatrix}\begin{pmatrix}2&3\\0&1\end{pmatrix}$$

$$=\begin{pmatrix}4&9\\0&1\end{pmatrix}$$

$$f(A)=a_0I+a_1A+a_2A^2$$

$$=-2\begin{pmatrix}1&0\\0&1\end{pmatrix}+3\begin{pmatrix}2&3\\0&1\end{pmatrix}+1\begin{pmatrix}4&9\\0&1\end{pmatrix}$$

$$=\begin{pmatrix}8&18\\0&2\end{pmatrix}$$

Exercise

If f is the polynomial function is given by $f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$

Then $f(A)$ is defined by $f(A)=a_0I+a_1A+a_2A^2+\cdots+a_nA^n$

Find $f(A)$

$$f(x)=x^2-2x+3, \qquad A=\begin{pmatrix}1&3\\3&2\end{pmatrix}$$

Solution

$$A^2=\begin{pmatrix}1&3\\3&2\end{pmatrix}\begin{pmatrix}1&3\\3&2\end{pmatrix}$$

$$=\begin{pmatrix}10&9\\9&13\end{pmatrix}$$

$$f(x)=3-2x+x^2$$

$$f(A)=a_0I+a_1A+a_2A^2$$

$$=3\begin{pmatrix}1&0\\0&1\end{pmatrix}-2\begin{pmatrix}1&3\\3&2\end{pmatrix}+1\begin{pmatrix}10&9\\9&13\end{pmatrix}$$

$$=\begin{pmatrix}11&3\\3&12\end{pmatrix}$$

Exercise

If f is the polynomial function is given by $f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$

Then $f(A)$ is defined by $f(A)=a_0I+a_1A+a_2A^2+\cdots+a_nA^n$

Find $f(A)$

$$f(x)=2x^2-3x+5, \qquad A=\begin{pmatrix}1&2\\3&-4\end{pmatrix}$$

Solution

$$A^2=\begin{pmatrix}1&2\\3&-4\end{pmatrix}\begin{pmatrix}1&2\\3&-4\end{pmatrix}$$

$$=\begin{pmatrix}7&-6\\-9&22\end{pmatrix}$$

$$f(x)=5-3x+2x^2$$

$$f(A)=a_0I+a_1A+a_2A^2$$

$$= 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} + 2\begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & -18 \\ -27 & 61 \end{pmatrix}$$

Exercise

If *f* is the polynomial function is given by $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

Then $f(A)$ is defined by $f(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$

Find $f(A)$

$$g(x) = x^2 + 3x - 10, \qquad A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$$

Solution

$$A^2 = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix}$$

$$g(x) = -10 + 3x + x^2$$

$$g(A) = a_0 I + a_1 A + a_2 A^2$$

$$= -10\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 3\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} + \begin{pmatrix} 7 & -6 \\ -9 & 22 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Exercise

If *f* is the polynomial function is given by $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

Then $f(A)$ is defined by $f(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$

Find $f(A)$

$$f(x) = x^2 - 2x - 5, \qquad A = \begin{pmatrix} 2 & -5 & 8 \\ 3 & -6 & -7 \\ 4 & 0 & -1 \end{pmatrix}$$

Solution

$$A^2 = \begin{pmatrix} 2 & -5 & 8 \\ 3 & -6 & -7 \\ 4 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & -5 & 8 \\ 3 & -6 & -7 \\ 4 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 21 & 20 & 43 \\ -40 & 21 & 73 \\ 4 & -20 & 33 \end{pmatrix}$$

$$f(x) = -5 - 2x + x^2$$

$$f(A) = a_0 I + a_1 A + a_2 A^2$$

$$= -5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2\begin{pmatrix} 2 & -5 & 8 \\ 3 & -6 & -7 \\ 4 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 21 & 20 & 43 \\ -40 & 21 & 73 \\ 4 & -20 & 33 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 30 & 27 \\ -46 & 28 & 87 \\ -4 & -20 & 30 \end{pmatrix}$$

Exercise

If *f* is the polynomial function is given by $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

Then $f(A)$ is defined by $f(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$

Find $f(A)$

$$g(x) = x^2 - 3x + 15, \qquad A = \begin{pmatrix} 2 & -5 & 8 \\ 3 & -6 & -7 \\ 4 & 0 & -1 \end{pmatrix}$$

Solution

$$A^2 = \begin{pmatrix} 2 & -5 & 8 \\ 3 & -6 & -7 \\ 4 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & -5 & 8 \\ 3 & -6 & -7 \\ 4 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 21 & 20 & 43 \\ -40 & 21 & 73 \\ 4 & -20 & 33 \end{pmatrix}$$

$$g(x) = 15 - 3x + x^2$$

$$g(A) = a_0 I + a_1 A + a_2 A^2$$

$$= 15\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 3\begin{pmatrix} 2 & -5 & 8 \\ 3 & -6 & -7 \\ 4 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 21 & 20 & 43 \\ -40 & 21 & 73 \\ 4 & -20 & 33 \end{pmatrix}$$

$$= \begin{pmatrix} 30 & 35 & 19 \\ -49 & 54 & 94 \\ -8 & -20 & 51 \end{pmatrix}$$

Exercise

If *f* is the polynomial function is given by $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

Then $f(A)$ is defined by $f(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$

Find $f(A)$

$$f(x) = x^3 - 2x^2 - 3x + 5, \qquad A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

Solution

$$A^2 = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 1 & -3 \\ 0 & 3 & 5 \\ -4 & 2 & 12 \end{pmatrix}$$

$$A^3 = AA^2$$

$$= \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 6 & 1 & -3 \\ 0 & 3 & 5 \\ -4 & 2 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 3 & -13 \\ -2 & 5 & 21 \\ -18 & 8 & 44 \end{pmatrix}$$

$$f(x) = 5 - 3x - 2x^2 + x^3$$

$$f(A) = a_0 I + a_1 A + a_2 A^2 + a_3 A^3$$

$$= 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 3\begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{pmatrix} - 2\begin{pmatrix} 6 & 1 & -3 \\ 0 & 3 & 5 \\ -4 & 2 & 12 \end{pmatrix} + \begin{pmatrix} 16 & 3 & -13 \\ -2 & 5 & 21 \\ -18 & 8 & 44 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 & -4 \\ -5 & 4 & 5 \\ 29 & 1 & 16 \end{pmatrix}$$

Solution *Section* 1.4 – Inverse Matrices – Finding A^{-1}

Exercise

Apply Gauss-Jordan method to find the inverse of this triangular "Pascal matrix"

Triangular Pascal matrix $A=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\1&1&0&0&0&1&0&0\\1&2&1&0&0&0&1&0\\1&3&3&1&0&0&0&1\end{array}\right]\begin{array}{l}\\R_2-R_1\\R_3-R_1\\R_4-R_1\end{array}$$

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&-1&1&0&0\\0&2&1&0&-1&0&1&0\\0&3&3&1&-1&0&0&1\end{array}\right]\begin{array}{l}\\\\R_3-2R_2\\R_4-3R_2\end{array}$$

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&1&-2&1&0\\0&0&3&1&2&-3&0&1\end{array}\right]\begin{array}{l}\\\\\\R_4-3R_3\end{array}$$

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&1&-2&1&0\\0&0&0&1&-1&3&-3&1\end{array}\right]$$

$$A^{-1}=\begin{pmatrix}1&0&0&0\\-1&1&0&0\\1&-2&1&0\\-1&3&-3&1\end{pmatrix}$$

- The inverse matrix A^{-1} looks like A, except odd-numbered diagonals are multiplied by -1.

Exercise

If A is invertible and $AB = AC$, prove that $B = C$

Solution

$AB = AC$ ***Multiply by A^{-1} both sides.***

$A^{-1}(AB) = A^{-1}(AC)$ ***Multiplication is associative***

$(A^{-1}A)B = (A^{-1}A)C$ $A^{-1}A = I$

$IB = IC$

$B = C$ ✓

Exercise

If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find two matrices $B \neq C$ such that $AB = AC$

Solution

Let $B = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ *and* $C = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$\underline{B \neq C \Rightarrow AB = AC}$

Exercise

If A has ***row*** 1 + ***row*** 2 = ***row*** 3, show that A is not invertible

a) Explain why $Ax = (1,\ 0,\ 0)$ can't have a solution.

b) Which right sides $\left(b_1,\ b_2,\ b_3\right)$ might allow a solution to $Ax = b$

c) What happens to ***row*** 3 in elimination?

Solution

a) Let $A_1,\ A_2,\ A_3$ be the row vectors of A and x is a solution to $Ax = (1,\ 0,\ 0)$.

Then $A_1.x = 1,\ A_2.x = 0,\ A_3.x = 0$.

Since $\quad A_1 + A_2 = A_3$

Means $\quad A_1.x + A_2.x = A_3.x$

Implies $\quad 1 + 0 = 0$ a contradiction

b) If $Ax = \left(b_1,\ b_2,\ b_3\right)$

$A_1.x = b_1, \quad A_2.x = b_2, \quad A_3.x = b_3$

Since $\quad A_1 + A_2 = A_3$

$A_1.x + A_2.x = A_3.x$

$\Rightarrow b_1 + b_2 = b_3$

c) In the elimination matrix, the third row will be zero.

Exercise

True or false (with a counterexample if false and a reason if true):

a) A 4 by 4 matrix with a row of zeros is not invertible.

b) A matrix with 1's down the main diagonal is invertible.

c) If A is invertible then A^{-1} is invertible.

d) If A is invertible then A^2 is invertible.

Solution

a) True, because it can have at most 3 pivots.

b) False, if the matrix: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and only has 2 pivots, thus is not invertible.

c) True, If A is invertible then necessarily A^{-1} is invertible.

d) True, $A^2x=0$ where x is nonzero matrix.

$$A^{-1}A^2x=\left(A^{-1}A\right)Ax=IAx=Ax=0$$

Since A is invertible, this can only be true if x was zero to begin with. Thus A^2 must also be invertible.

Exercise

Do there exist 2 by 2 matrices A and B with real entries such that $AB-BA=I$, where I is the identity matrix?

Solution

Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$$\begin{aligned} AB &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ &= \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix} \end{aligned}$$

$$\begin{aligned} AB-BA &= \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}-\begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix} \\ &= \begin{pmatrix} bg-cf & af+bh-be-df \\ ce+dg-ag-ch & cf-bg \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{cases} bg-cf=1 \\ af+bh-be-df=0 \\ ce+dg-ag-ch=0 \\ cf-bg=1 \end{cases}$$

$$\rightarrow \begin{cases} bg - cf = 1 \\ \underline{cf - bg = 1} \\ \quad 0 \;\; = 2 \end{cases}$$

Therefore, $AB - BA \neq I$ for any 2 by 2 matrices.

Exercise

If B is the inverse of A^2, show that AB is the inverse of A.

Solution

Since B is the inverse of A^2 that implies:

$$\begin{aligned} B &= \left(A^2\right)^{-1} \\ &= (AA)^{-1} \\ &= A^{-1}A^{-1} \end{aligned}$$

Show that AB is the inverse of A

$$\begin{aligned} (AB)A &= \left(A\left(A^{-1}A^{-1}\right)\right)A \\ &= \left(\left(AA^{-1}\right)A^{-1}\right)A \\ &= \left(IA^{-1}\right)A \\ &= A^{-1}A \\ &= I \end{aligned}$$

Therefore, AB is the inverse of A.

Exercise

Find and check the inverses (assuming they exist) of these block matrices.

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}$$

Solution

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix}\begin{bmatrix} I & 0 \\ A & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ C+A & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \Rightarrow C+A=0 \to A=-C$$

$$\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -C & I \end{pmatrix}$$

$$\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}\begin{bmatrix} E & 0 \\ F & G \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} AE & 0 \\ CE+DF & DG \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\Rightarrow \begin{cases} AE = I \\ CE+DF=0 \\ DG = I \end{cases} \to \begin{cases} E = A^{-1} \\ \\ G = D^{-1} \end{cases}$$

$$\begin{aligned} CE+DF=0 \to CA^{-1}+DF &= 0 \\ DF &= -CA^{-1} \\ D^{-1}DF &= -D^{-1}CA^{-1} \\ IF &= -D^{-1}CA^{-1} \\ F &= -D^{-1}CA^{-1} \end{aligned}$$

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{pmatrix}$$

$$\begin{bmatrix} 0 & I \\ I & D \end{bmatrix}\begin{bmatrix} A & I \\ I & B \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & B \\ A+D & I+DB \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\to \begin{cases} B=0 \\ A+D=0 \\ I+DB=I \end{cases} \Rightarrow \begin{cases} A=-D \\ DB=0 \end{cases}$$

$$\begin{pmatrix} 0 & I \\ I & D \end{pmatrix}^{-1} = \begin{pmatrix} -D & I \\ I & 0 \end{pmatrix}$$

Exercise

For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

Solution

$$c = 0, \ A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 7 & 0 \end{bmatrix} \text{ (zero column 2 / row 2)}$$

$$c = 2, \ A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 8 & 7 & 2 \end{bmatrix} \text{ (equal rows)}$$

$$c = 7, \ A = \begin{bmatrix} 2 & 7 & 7 \\ 7 & 7 & 7 \\ 8 & 7 & 7 \end{bmatrix} \text{ (equal columns)}$$

Exercise

Find A^{-1} and B^{-1} (if they exist) by elimination.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Solution

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \frac{1}{2}R_1 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right) \begin{array}{l} \\ \frac{2}{3}R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - \frac{1}{2}R_2 \\ \\ R_3 - \frac{1}{2}R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right) \begin{array}{l} \\ \\ \frac{3}{4}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right) \begin{array}{l} R_1 - \frac{1}{3}R_3 \\ R_2 - \frac{1}{3}R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \frac{1}{2}R_1 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 + R_1 \\ R_3 + R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right) R_3 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

B^{-1} ***doesn't*** exist, and if we add the columns in B, the result is zero.

Exercise

Find A^{-1} using the Gauss-Jordan method, which has a remarkable inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) R_1 + R_2$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) R_2 + R_3$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) R_3 + R_4$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise

Find the inverse, if exists of $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$

Solution

$$\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}^{-1} = \frac{1}{12+12}\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{12} & \frac{1}{6} \\ \frac{1}{8} & \frac{1}{4} \end{bmatrix}$$

Exercise

Find the inverse, if exists of $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}^{-1} = \frac{1}{7-8}\begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix}$$

$$= -\begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$$

Exercise

Find the inverse, if exists of $\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$

Solution

$$\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}^{-1} = \frac{1}{-15-24}\begin{bmatrix} 5 & -6 \\ -4 & -3 \end{bmatrix}$$

$$= -\frac{1}{39}\begin{bmatrix} 5 & -6 \\ -4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{39} & \frac{2}{13} \\ \frac{4}{39} & \frac{1}{13} \end{bmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{-4+6}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ -\frac{1}{3} & 1 \end{bmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{10-10}\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

$$= \frac{1}{0}\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$

Solution

$$\left[\begin{array}{cc|cc} -2 & 3 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{array}\right] \quad -\frac{1}{2}R_1$$

$$1 \quad -\frac{3}{2} \quad -\frac{1}{2} \quad 0$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ -3 & 4 & 0 & 1 \end{array}\right] \quad R_2 + 3R_1 \qquad \begin{array}{cccc} -3 & 4 & 0 & 1 \\ 3 & -\frac{9}{2} & -\frac{3}{2} & 0 \\ \hline 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right] \quad -2R_2 \qquad \begin{array}{cccc} 0 & 1 & 3 & -2 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 3 & -2 \end{array}\right] \quad R_1 + \frac{3}{2}R_2 \qquad \begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & \frac{9}{2} & -3 \\ \hline 1 & 0 & 4 & -3 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 4 & -3 \\ 0 & 1 & 3 & -2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} a & b \\ 3 & 3 \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{3a-3b}\begin{bmatrix} 3 & -b \\ -3 & a \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{3(a-b)} & \frac{-b}{3(a-b)} \\ \frac{-3}{3(a-b)} & \frac{a}{3(a-b)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{a-b} & \frac{-b}{3(a-b)} \\ \frac{-1}{a-b} & \frac{a}{3(a-b)} \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} -2 & a \\ 4 & a \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{-2a-4a}\begin{bmatrix} a & -a \\ -4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{-6a} & \frac{-a}{-6a} \\ \frac{-4}{-6a} & \frac{-2}{-6a} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{6} & \frac{1}{6} \\ \frac{2}{3a} & \frac{1}{3a} \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} 4 & 4 \\ b & a \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{4a-4b}\begin{bmatrix} a & -4 \\ -b & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{4(a-b)} & \frac{-4}{4(a-b)} \\ \frac{-b}{4(a-b)} & \frac{4}{4(a-b)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{4(a-b)} & \frac{-1}{a-b} \\ \frac{-b}{4(a-b)} & \frac{1}{a-b} \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{-1+4}\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{3}\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{14}\begin{pmatrix} -1 & -4 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{14} & \frac{2}{7} \\ \frac{3}{14} & -\frac{1}{7} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{5}\begin{pmatrix} -1 & -3 \\ -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{11}\begin{pmatrix} 5 & -3 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{11} & -\frac{3}{11} \\ \frac{2}{11} & \frac{1}{11} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{0}\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} -6 & 9 \\ 2 & -3 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{18-18}\begin{pmatrix} & \\ & \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} -2 & 7 \\ 0 & 2 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -7 \\ 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{7}{4} \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Exercise

Find the inverse of $A=\begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$

Solution

$$A=\frac{1}{-16+16}\begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$$

∴ Inverse ***doesn't exist***

Exercise

Find the inverse of $A=\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Solution

$$A^{-1}=\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Exercise

Find the inverse of $A=\begin{pmatrix} 2 & 1 \\ a & a \end{pmatrix}$

Solution

$$A^{-1}=\frac{1}{a}\begin{pmatrix} a & -1 \\ -a & 2 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & -\frac{1}{a} \\ -1 & \frac{2}{a} \end{pmatrix}$$

Exercise

Find the inverse of $A=\begin{pmatrix} b & 3 \\ b & 2 \end{pmatrix}$

Solution

$$A^{-1}=-\frac{1}{b}\begin{pmatrix} 2 & -3 \\ -b & b \end{pmatrix}$$

$$=\begin{pmatrix} -\frac{2}{b} & \frac{3}{b} \\ 1 & -1 \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & a \\ 3 & a \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{2a}\begin{pmatrix} a & -a \\ -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2a} & -\frac{1}{2a} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{a^2-4}\begin{pmatrix} a & -2 \\ -2 & a \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a}{a^2-4} & \frac{-2}{a^2-4} \\ \frac{-2}{a^2-4} & \frac{a}{a^2-4} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{0}\begin{pmatrix} & \\ & \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} -3 & \frac{1}{2} \\ 6 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{0}\begin{pmatrix} & \\ & \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A=\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$

Solution

$$A^{-1}=\frac{1}{0}\begin{pmatrix} & \\ & \end{pmatrix}$$

∴ Inverse ***doesn't exist***

Exercise

Find the inverse of $A=\begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$

Solution

$$A^{-1}=\frac{1}{\sin^2\theta+\cos^2\theta}\begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$$

$$=\begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$$

Exercise

Find A^{-1} *if* $A=\begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-3R_1 \end{array}$$

$$\begin{array}{cccccc} 2 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -2 & 0 & 0 \\ \hline 0 & -2 & -3 & -2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 3 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & -3 & 0 & 0 \\ \hline 0 & 0 & -3 & -3 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \; -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ -\tfrac{1}{3}R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - \frac{3}{2}R_3 \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ \hline 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array} \qquad \begin{array}{cccccc} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ \hline 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

Exercise

Find A^{-1}, where $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 + 2R_1 \end{array}$$

$$\begin{array}{cccccc} 3 & 5 & 3 & 0 & 1 & 0 \\ -3 & -6 & 3 & -3 & 0 & 0 \\ \hline 0 & -1 & 6 & -3 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 2 & 4 & 3 & 0 & 0 & 1 \\ -2 & -4 & 2 & -2 & 0 & 0 \\ \hline 0 & 0 & 5 & -2 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right] -R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - 2R_2 \\ \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 12 & -6 & 2 & 0 \\ \hline 1 & 0 & 11 & -5 & 2 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{5}R_3 \end{array}$$

$$\begin{array}{cccccc} 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right] \begin{array}{l} R_1 - 11R_3 \\ R_2 + 6R_3 \\ \\ \end{array}$$

$$\begin{array}{cccccc} 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 6 & -\frac{12}{5} & 0 & \frac{6}{5} \\ \hline 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \end{array} \qquad \begin{array}{cccccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 0 & -11 & \frac{22}{5} & 0 & -\frac{11}{5} \\ \hline 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{bmatrix}$$

Exercise

Find A^{-1}, where $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccccc} -2 & 0 & 1 & 0 & 1 & 0 \\ 2 & 4 & -2 & 2 & 0 & 0 \\ \hline 0 & 4 & -1 & 2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & -2 & 1 & -1 & 0 & 0 \\ \hline 0 & -3 & 1 & -1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 4 & -1 & 2 & 1 & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \frac{1}{4}R_2 \\ \\ \end{array}$$

$$\begin{array}{cccccc} 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\begin{array}{cccccc} 0 & -3 & 1 & -1 & 0 & 1 \\ 0 & 3 & -\frac{3}{4} & \frac{3}{2} & \frac{3}{4} & 0 \\ \hline 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{array} \qquad \begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & \frac{1}{2} & -1 & -\frac{1}{2} & 0 \\ \hline 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{array}\right] \begin{array}{l} \\ \\ 4R_3 \end{array}$$

$$\begin{array}{cccccc} 0 & 0 & 1 & 2 & 3 & 4 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}\right] \begin{array}{l} R_1 + \frac{1}{2}R_3 \\ R_2 + \frac{1}{4}R_3 \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 \\ \hline 1 & 0 & 0 & 1 & 1 & 2 \end{array} \qquad \begin{array}{cccccc} 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ \hline 0 & 1 & 0 & 1 & 1 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

Exercise

Find A^{-1}, where $A = \begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \quad \frac{1}{-2}R_1 \qquad 1 \quad -\frac{5}{2} \quad -\frac{3}{2} \quad -\frac{1}{2} \quad 0 \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \quad R_2 - 4R_1 \qquad \begin{array}{cccccc} 4 & -1 & 3 & 0 & 1 & 0 \\ -4 & 10 & 6 & 2 & 0 & 0 \\ \hline 0 & 9 & 9 & 2 & 1 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \quad R_3 - 7R_1 \qquad \begin{array}{cccccc} 7 & -2 & 5 & 0 & 0 & 1 \\ -7 & \frac{35}{2} & \frac{21}{2} & \frac{7}{2} & 0 & 0 \\ \hline 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}\right] \quad \frac{1}{9}R_2 \qquad 0 \quad 1 \quad 1 \quad \frac{2}{9} \quad \frac{1}{9} \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}\right] \quad R_3 - \frac{31}{2}R_2 \quad \begin{array}{cccccc} 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \\ 0 & -\frac{31}{2} & -\frac{31}{2} & -\frac{31}{9} & -\frac{31}{18} & 0 \\ \hline 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array}\right]$$

∴ The inverse matrix ***doesn't exist***

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \quad R_2 + R_1 \quad \begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ \hline 0 & 4 & 4 & 1 & 1 & 0 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 4 & 1 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \quad \frac{1}{4}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{c} R_1 - R_2 \\ \\ R_3 - 4R_2 \end{array} \quad \begin{array}{cccccc} 0 & 4 & 3 & 0 & 0 & 1 \\ 0 & -4 & -4 & -1 & -1 & 0 \\ \hline 0 & 0 & -1 & -1 & -1 & 1 \end{array} \quad \begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ \hline 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right) \quad -R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right) \quad \begin{array}{c} R_1 + R_3 \\ R_2 - R_3 \end{array} \quad \begin{array}{cccccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ \hline 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1 \end{array} \quad \begin{array}{cccccc} 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \\ \hline 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1 \\ 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{7}{4} & \frac{3}{4} & -1 \\ -\frac{3}{4} & -\frac{3}{4} & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $\quad A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ -2 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \quad R_3 + 2R_1$$

$$\begin{array}{cccccc} -2 & -3 & 0 & 0 & 0 & 1 \\ 2 & -2 & 2 & 2 & 0 & 0 \\ \hline 0 & -5 & 2 & 2 & 0 & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array}\right) \quad -\tfrac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 + R_2 \\ \\ R_3 + 5R_2 \end{array}$$

$$\begin{array}{cccccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ \hline 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \end{array} \qquad \begin{array}{cccccc} 0 & -5 & 2 & 2 & 0 & 1 \\ 0 & 5 & -\frac{5}{2} & 0 & -\frac{5}{2} & 0 \\ \hline 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \end{array}\right) \quad -2R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -4 & 5 & -2 \end{array}\right) \quad \begin{array}{l} R_1 - \frac{1}{2}R_3 \\ R_2 + \frac{1}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & -4 & 5 & -2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \frac{1}{2}R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ R_3 + R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right) \begin{array}{l} \\ \\ 2R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right) \begin{array}{l} R_1 - 2R_3 \\ R_2 - \frac{5}{2}R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right)$$

$$A^{-1}=\begin{pmatrix}3 & -2 & -4\\ 3 & -2 & -5\\ -1 & 1 & 2\end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix}1 & 1 & 1\\ 3 & 2 & -1\\ 3 & 1 & 2\end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 3 & 2 & -1 & 0 & 1 & 0\\ 3 & 1 & 2 & 0 & 0 & 1\end{array}\right)\quad \begin{array}{l} \\ R_2-3R_1\\ R_3-3R_1\end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & -1 & -4 & -3 & 1 & 0\\ 0 & -2 & -1 & -3 & 0 & 1\end{array}\right)\quad -R_2$$

$$\left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 1 & 4 & 3 & -1 & 0\\ 0 & -2 & -1 & -3 & 0 & 1\end{array}\right)\quad \begin{array}{l}R_1-R_2\\ \\ R_3+2R_2\end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & -3 & -2 & 1 & 0\\ 0 & 1 & 4 & 3 & -1 & 0\\ 0 & 0 & 7 & 3 & -2 & 1\end{array}\right)\quad \begin{array}{l} \\ \\ \frac{1}{7}R_3\end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & -3 & -2 & 1 & 0\\ 0 & 1 & 4 & 3 & -1 & 0\\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7}\end{array}\right)\quad \begin{array}{l}R_1+3R_3\\ R_2-4R_3\\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & 0 & -\frac{5}{7} & \frac{1}{7} & \frac{3}{7}\\ 0 & 1 & 0 & \frac{9}{7} & \frac{1}{7} & -\frac{4}{7}\\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7}\end{array}\right)$$

$$A^{-1}=\begin{pmatrix}-\frac{5}{7} & \frac{1}{7} & \frac{3}{7}\\ \frac{9}{7} & \frac{1}{7} & -\frac{4}{7}\\ \frac{3}{7} & -\frac{2}{7} & \frac{1}{7}\end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 3 & 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \quad \frac{1}{3}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & -3 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{7}{3} & -\frac{5}{3} & 3 & 1 \end{array}\right) \quad \begin{array}{l} \\ \\ \frac{3}{7}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array}\right) \quad \begin{array}{l} R_1 + \frac{1}{3}R_3 \\ R_2 - \frac{2}{3}R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ 0 & 1 & 0 & \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} -3 & 1 & -1 \\ 1 & -4 & -7 \\ 1 & 2 & 5 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} -3 & 1 & -1 & 1 & 0 & 0 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right) \quad -\frac{1}{3}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{11}{3} & -\frac{22}{3} & \frac{1}{3} & 1 & 0 \\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1 \end{array}\right) \quad -\frac{3}{11}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0 \\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1 \end{array}\right) \quad R_3 - \frac{7}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0 \\ 0 & 0 & 0 & & & \end{array}\right)$$

$\therefore$ Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 1 & -3 \\ 2 & -4 & 1 \\ -5 & 7 & 1 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 2 & -4 & 1 & 0 & 1 & 0 \\ -5 & 7 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 5R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & -6 & 7 & -2 & 1 & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array}\right) \quad -\tfrac{1}{6}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 12R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & & & \end{array}\right)$$

$\therefore$ Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_3 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right) \quad -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 - 2R_2 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ \tfrac{1}{5}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right) \begin{array}{l} R_1 - 11R_3 \\ R_2 + 6R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{pmatrix}$$

Exercise

Find the inverse, if exists of $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \begin{array}{l} \\ \\ -\frac{1}{2}R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Exercise

Find the inverse, if exists of $A = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{pmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & 1 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} & 0 & 1 & 0 \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} & 0 & 0 & 1 \end{array}\right] \begin{array}{l} 5R_1 \\ 10R_2 \\ 10R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 2 & 2 & 1 & 0 & 10 & 0 \\ 2 & -8 & 1 & 0 & 0 & 10 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 0 & 5 & -10 & 10 & 0 \\ 0 & -10 & 5 & -10 & 0 & 10 \end{array}\right] \quad R_2 - \frac{1}{10}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & \frac{9}{2} & -9 & 10 & -1 \\ 0 & -10 & 5 & -10 & 0 & 10 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 10R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{13}{2} & 14 & -10 & 1 \\ 0 & 1 & \frac{9}{2} & -9 & 10 & -1 \\ 0 & 0 & 50 & -100 & 100 & 0 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{50}R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{13}{2} & 14 & -10 & 1 \\ 0 & 1 & \frac{9}{2} & -9 & 10 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right] \begin{array}{l} R_1 + \frac{13}{2}R_3 \\ R_2 - \frac{9}{2}R_3 \\ \ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{bmatrix}$$

Exercise

Find the inverse, if exists of $A = \begin{pmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} \sqrt{2} & 3\sqrt{2} & 0 & 1 & 0 & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_2 + 4R_1$$

$$\left[\begin{array}{ccc|ccc} \sqrt{2} & 3\sqrt{2} & 0 & 1 & 0 & 0 \\ 0 & 13\sqrt{2} & 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad 13R_1 - 3R_2$$

$$\left[\begin{array}{ccc|ccc} 13\sqrt{2} & 0 & 0 & 1 & -3 & 0 \\ 0 & 13\sqrt{2} & 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \frac{1}{13\sqrt{2}}R_1 \\ \frac{1}{13\sqrt{2}}R_2 \\ \ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{13\sqrt{2}} & -\frac{3}{13\sqrt{2}} & 0 \\ 0 & 1 & 0 & \frac{4}{13\sqrt{2}} & \frac{1}{13\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$A^{-1}=\begin{pmatrix} \frac{\sqrt{2}}{26} & -\frac{3\sqrt{2}}{26} & 0 \\ \frac{2\sqrt{2}}{13} & \frac{\sqrt{2}}{26} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise

Find the inverse, if exists of $A=\begin{pmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{pmatrix}$

Solution

$$\begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}^{-1} = \textit{doesn't exist}$$

Since this matrix is ***singular***, row 3 all zeros.

Exercise

Find the inverse, if exists, of $A=\begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -6 & 1 \\ -2 & -2 & 5 & 1 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} -2 & -3 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad -\frac{1}{2}R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_4+2R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

∴ Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 1 & -14 & 7 & 38 \\ -1 & 2 & 1 & -2 \\ 1 & 2 & -1 & -6 \\ 1 & -2 & 3 & 6 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & -6 & 0 & 0 & 1 & 0 \\ 1 & -2 & 3 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & -12 & 8 & 36 & 1 & 1 & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -\frac{1}{12} R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + 14R_2 \\ \\ R_3 - 16R_2 \\ R_4 - 12R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & \frac{8}{3} & 4 & \frac{1}{3} & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \frac{3}{8} R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + \frac{7}{3}R_3 \\ R_2 + \frac{2}{3}R_3 \\ \\ R_4 - 4R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & -2 & -\frac{1}{2} & -1 & -\frac{3}{2} & 1 \end{array}\right] -\frac{1}{2}R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array}\right] \begin{array}{l} R_1 + \frac{1}{2}R_4 \\ R_2 + 2R_4 \\ R_3 - \frac{3}{2}R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ 0 & 0 & 1 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{bmatrix}$$

Exercise

Find the inverse, if exists, of $A=\begin{bmatrix}10&20&-30&15\\3&-7&14&-8\\-7&-2&-1&2\\4&4&-3&1\end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc}10&20&-30&15&1&0&0&0\\3&-7&14&-8&0&1&0&0\\-7&-2&-1&2&0&0&1&0\\4&4&-3&1&0&0&0&1\end{array}\right]\quad \frac{1}{10}R_1$$

$$\left[\begin{array}{cccc|cccc}1&2&-3&\frac{3}{2}&\frac{1}{10}&0&0&0\\3&-7&14&-8&0&1&0&0\\-7&-2&-1&2&0&0&1&0\\4&4&-3&1&0&0&0&1\end{array}\right]\quad \begin{array}{l}R_2-3R_1\\R_3+7R_1\\R_4-4R_1\end{array}$$

$$\left[\begin{array}{cccc|cccc}1&2&-3&\frac{3}{2}&\frac{1}{10}&0&0&0\\0&-13&23&-\frac{25}{2}&-\frac{3}{10}&1&0&0\\0&12&-22&\frac{25}{2}&\frac{7}{10}&0&1&0\\0&-4&9&-5&-\frac{2}{5}&0&0&1\end{array}\right]\quad -\frac{1}{13}R_2$$

$$\left[\begin{array}{cccc|cccc}1&2&-3&\frac{3}{2}&\frac{1}{10}&0&0&0\\0&1&-\frac{23}{13}&\frac{25}{26}&\frac{3}{130}&-\frac{1}{13}&0&0\\0&12&-22&\frac{25}{2}&\frac{7}{10}&0&1&0\\0&-4&9&-5&-\frac{2}{5}&0&0&1\end{array}\right]\quad \begin{array}{l}R_1-2R_2\\ \\R_3-12R_2\\R_4+4R_2\end{array}$$

$$\left[\begin{array}{cccc|cccc}1&0&\frac{7}{13}&-\frac{11}{26}&\frac{7}{130}&\frac{2}{13}&0&0\\0&1&-\frac{23}{13}&\frac{25}{26}&\frac{3}{130}&-\frac{1}{13}&0&0\\0&0&-\frac{10}{13}&\frac{25}{26}&\frac{11}{26}&\frac{12}{13}&1&0\\0&0&\frac{25}{13}&-\frac{15}{13}&-\frac{4}{13}&-\frac{4}{13}&0&1\end{array}\right]\quad -\frac{13}{10}R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 - \frac{7}{13}R_3 \\ R_2 + \frac{23}{13}R_3 \\ \\ R_4 - \frac{25}{13}R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{19}{20} & -\frac{11}{5} & -\frac{23}{10} & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array}\right] \quad \begin{array}{l} R_2 + R_4 \\ R_3 + R_4 \end{array}.$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array}\right] \quad \begin{array}{l} \\ \\ \\ \frac{4}{5}R_4 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array}\right] \quad \begin{array}{l} R_1 - \frac{1}{4}R_4 \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{bmatrix}$$

Exercise

Show that A is not invertible for any values of the entries

$$A=\begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

Solution

Since the matrix A had zero's on its diagonals, therefore A is not invertible.

Exercise

Prove that if A is an invertible matrix and B is row equivalent to A, then B is also invertible.

Solution

Since B is row equivalent to A, there exist some elementary matrices $E_1, E_2, \ldots, E_n$ such that $B = E_n \ \ldots\ E_1 A$. Because $E_1, E_2, \ldots, E_n$ and A are invertible, then B is also invertible.

Exercise

Determine if the given matrix has an inverse, and find the inverse if it exists. Check your answer by multiplying $A \cdot A^{-1} = I$

a) $\begin{bmatrix} 2 & 3 \\ -3 & -5 \end{bmatrix}$ *b)* $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}$

Solution

a) $2(-5)-3(-3) = -10+9$

$$= I$$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} -5 & -3 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 \\ -3 & -2 \end{bmatrix}$$

$$AA^{-1} = \begin{pmatrix} 2 & 3 \\ -3 & -5 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ -3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I$$

b) $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \quad R_3 - 2R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 3 & 3 & -2 & 0 & 1 \end{array}\right] \quad R_3 - 3R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & * & * & * \end{array}\right]$$

The inverse matrix *doesn't exist*

Exercise

Show that the inverse of $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is $\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$

Solution

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\theta)\cos(-\theta) - (\sin\theta)\sin(-\theta) & (\cos\theta)\sin(-\theta) - (\sin\theta)\cos(-\theta) \\ (-\sin\theta)\cos(-\theta) - (\cos\theta)\sin(-\theta) & (-\sin\theta)\sin(-\theta) + (\cos\theta)\cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\theta + \sin\theta\sin\theta & -\cos\theta\sin\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin\theta\sin\theta + \cos\theta\cos\theta \end{bmatrix} \qquad \begin{cases} \cos(-\theta) = \cos\theta & (\mathit{even}) \\ \sin(-\theta) = -\sin\theta & (\mathit{odd}) \end{cases}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

Exercise

If the product $C = AB$ is invertible (and A & B are square matrices), find a formula for A^{-1} that involves C^{-1} and B.

Hence, it is not possible to multiply a non-invertible matrix by another matric and obtain an invertible matrix as a result.

Solution

Since $C = AB$ is invertible, the $CC^{-1} = C^{-1}C = I$

$CC^{-1} = I$

$(AB)C^{-1} = I$

$A\left(BC^{-1}\right) = I$

$A^{-1}A\left(BC^{-1}\right) = A^{-1}I$

$I\left(BC^{-1}\right) = A^{-1}$

$\underline{BC^{-1} = A^{-1}}$ ✓

Exercise

Prove that if A is an $m \times n$ matrix, there is an invertible matrix C such that CA is in reduced row-echelon form.

Solution

The reduced row-echelon form of A can be written in the form $E_n \ldots E_2 E_1 A$. where

$E_1,\ E_2,\ \ldots,\ E_n$ are elementary matrices.

Let $C = E_n \ldots E_2 E_1$, then C is invertible since $E_1,\ E_2,\ \ldots,\ E_n$ are invertible.

Hence, there exists such a matrix C.

Exercise

Prove that 2 $m \times n$ matrices A and B are row equivalent if and only if there exists a nonsingular matrix P such that $B = PA$

Solution

Suppose that $A \sim B$, then there exist elementary matrices $E_1,\ E_2,\ \ldots,\ E_n$ such that

$B = E_n \ldots E_2 E_1 A$.

Let $P = E_n \ldots E_2 E_1$ $\Rightarrow$ by the theorem, P is nonsingular.

Suppose that $B = PA$, for some nonsingular matrix *P*. By the theorem, *P* is row equivalent to I_k.

That is, $I_k = E_n \ldots E_2 E_1 P$.

Thus, $B = E_1^{-1} E_2^{-1} \ldots E_n^{-1} A$ and this implies that *A* is row equivalent to *B*.

Exercise

Let *A* and *B* be 2 $m \times n$ matrices. Suppose *A* is row equivalent to *B*. Prove that *A* is nonsingular if and only if *B* is nonsingular.

Solution

Suppose that *A* is row equivalent to *B*. Then, there exists a nonsingular matrix *P* such that $B = PA$. If *A* is nonsingular then *B* is nonsingular.

Conversely, if *B* is nonsingular then $A = P^{-1}B$ is nonsingular.

Exercise

Show that if *A* and *B* are two $n \times n$ invertible matrices then *A* is row equivalent to *B*.

Solution

Since *A* is invertible, then *A* is a row equivalent to I_n. That is, there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that $I_n = E_k E_{k-1} \cdots E_1 A$.

Similarly, there exist elementary matrices $F_1, F_2, \ldots, F_k$ such that $I_n = F_i F_{i-1} \cdots F_1 B$.

$$\begin{aligned}
\text{Hence, } A &= E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n \\
&= E_1^{-1} E_2^{-1} \cdots E_k^{-1} \left(F_i F_{i-1} \cdots F_1 B \right) \\
&= \left(E_1^{-1} E_2^{-1} \cdots E_k^{-1} F_i F_{i-1} \cdots F_1 B \right)
\end{aligned}$$

That is, *A* row equivalent to *B*.

Exercise

Prove that a square matrix *A* is nonsingular if and only if *A* is a product of elementary matrices.

Solution

Suppose that *A* is nonsingular. Then *A* is row equivalent to I_n. That is, there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that $I_n = E_k E_{k-1} \cdots E_1 A \quad \rightarrow \quad A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n$.

But each E_i^{-1} is an elementary matrix.

Conversely, suppose that $A = E_1 E_2 \ \ldots E_k$, then $\left(E_1 E_2 \ \ldots E_k\right)^{-1} A = I_n$

That is, A is nonsingular.

Exercise

Show that if $A \sim B$ (that is, if they are row equivalent), then $EA = B$ for some matrix E which is a product of elementary matrices.

Solution

If $A \sim B$, there is some sequence of elementary row operations which, when performed on A, produce B.

Further, multiplying on the left by the corresponding elementary matrix is the same as performing that row operation. So we have

$$\begin{aligned} A &\sim E_1 A \\ &\sim E_2 E_1 A \\ &\sim E_k E_{k-1} \ldots E_2 E_1 A \\ &\underline{= B} \end{aligned}$$

Thus, if $E = E_k \ldots E_1$, then we have $EA = B$

Exercise

Show that if $EA = B$ for some matrix E which is a product of elementary matrices, then $AC \sim BC$ for every $n \times n$ matrix C.

Solution

Let $E = E_k E_{k-1} \ldots E_1$, where each E_i is an elementary matrix.

$$\begin{aligned} AC &\sim E_1 AC \\ &\sim E_2 E_1 AC \\ &\sim E_k E_{k-1} \ldots E_2 E_1 AC \\ &= EAC && \text{since } EA = B \\ &= BC \end{aligned}$$

Therefore; $\underline{AC \sim BC}$

Exercise

Let $A\vec{x}=0$ be a homogeneous system of n linear equations in n unknowns that has only the trivial solution. Show that of k is any positive integer, then the system $A^k\vec{x}=0$ also has only trivial solution.

Solution

Since A is a square matrix, thus A has only the trivial solution. That implies that A is invertible.
But A^k is also invertible so $A^k\vec{x}=0$ has only trivial solution.

Exercise

Let $A\vec{x}=0$ be a homogeneous system of n linear equations in n unknowns, and let Q be an invertible $n\times n$ matrix. Show that $A\vec{x}=0$ has just trivial solution if and only if $(QA)\vec{x}=0$ has just trivial solution.

Solution

A is a square $(n\times n)$ matrix. If $A\vec{x}=0$ has just a trivial solution, then A is invertible. Since Q is an invertible $n\times n$ matrix, then QA is also invertible.

Thus, $(QA)\vec{x}=0$ has trivial solution.

On the other hand, if $(QA)\vec{x}=0$ has trivial solution, then QA is also invertible.

Since Q is invertible, then Q^{-1} is also invertible.

Thus, $A=Q^{-1}QA$ is invertible, i.e $A\vec{x}=0$ has just trivial solution, equivalent $A\vec{x}=0$ has just trivial solution if and only if $(QA)\vec{x}=0$ has just trivial solution.

Exercise

Let $A\vec{x}=b$ be any consistent system of linear equations, and let $\vec{x}_1$ be a fixed solution. Show that every solution to the system can be written in the form $\vec{x}=\vec{x}_1+\vec{x}_0$ where $\vec{x}_0$ is a solution to $A\vec{x}=0$. Show also that every matrix of this form is a solution.

Solution

Since $\vec{x}_0$ is a solution to $A\vec{x}=0$, we have $A\vec{x}_0=0$.

Adding $A\vec{x}_0=0$ to $A\vec{x}=b$, then

$$A\vec{x}+Ax_0=b+0$$

$$A\left(\vec{x}+\vec{x}_0\right)=b$$

As adding an equation to the original equation does not affect the solution.
If we let $\vec{x}_1$ be a fixed solution, then every solution to $A\vec{x}=b$ is $\vec{x}=\vec{x}_1+\vec{x}_0$.

Besides,

$$A\left(\vec{x}+\vec{x}_0\right) = A\vec{x} + Ax_0$$
$$= b + 0$$
$$= b$$

So, every matrix (vector) in the form $\vec{x}_1 + \vec{x}_0$ is a solution to $A\vec{x} = b$

Exercise

If A and B are $n \times n$ matrices satisfying $A^2 = B^2 = (AB)^2 = I_n$. Prove that $AB = BA$.

Solution

Since $A^2 = B^2 = (AB)^2 = I_n$, then A, B, AB are nonsingular.

$$A^2 = I \;\rightarrow\; A = A^{-1}$$

$$B^2 = I \;\rightarrow\; B = B^{-1}$$

$$(AB)^2 = I \;\rightarrow\; AB = (AB)^{-1}$$

$$AB = (AB)^{-1}$$
$$= B^{-1}A^{-1}$$
$$= BA \quad \checkmark$$

Exercise

Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix}$. Verify that $A^3 = 5I$, then find A^{-1} in term of A.

Solution

$$A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 1 \\ 5 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}$$

$$A^3 = AA^2$$

$$=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 5 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$=5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 5I$$

Since $A^3 = AA^2 = 5I$

$$\frac{1}{5}\left(AA^2\right) = I$$

$$A\left(\frac{1}{5}A^2\right) = I$$

$$A^{-1} = \frac{1}{5}A^2$$

Exercise

Consider $B(A,\ I) = (BA,\ B)$, thus if B is the inverse of A, then $(BA,\ B)$ becomes $\left(I,\ A^{-1}\right)$. On the other hand B is a product of elementary matrices since it is invertible. This indicates that the inverse of A can be obtained by applying elementary row operations to $(A,\ I)$ to get $\left(I,\ A^{-1}\right)$.

Find the inverses of

a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix}$ *b)* $B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{pmatrix}$

Solution

a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix}$ $R_3 - aR_1$ $E_{31} = -a$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{pmatrix}$ $R_3 - bR_2$ $E_{32} = -b$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & -b & 1 \end{pmatrix}$$

b) First, we have move row 4 to row 1, for the calculation

$$\begin{pmatrix} a & b & c & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \frac{1}{a}R_1 \qquad E_{11} = \frac{1}{a}$$

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{c}{a} & \frac{d}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_1 - \frac{b}{a}R_2 \qquad E_{12} = -\frac{b}{a}$$

$$\begin{pmatrix} 1 & 0 & \frac{c}{a} & \frac{d}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_1 - \frac{c}{a}R_3 \qquad E_{13} = -\frac{c}{a}$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{d}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_1 - \frac{d}{a}R_4 \qquad E_{14} = -\frac{d}{a}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$E = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} & -\frac{c}{a} & -\frac{d}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since we move Row 4 to Row 1, we must move Column 1 to Column 4 to get the inverse matrix.

$$B^{-1}=\begin{pmatrix} -\frac{b}{a} & -\frac{c}{a} & -\frac{d}{a} & \frac{1}{a} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Exercise

Let $A,\ B,\ C,\ X,\ Y,\ Z\ \in M_n(\mathbb{C})$, A and C are invertible. Find

a) $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}^{-1}$ *b)* $\begin{pmatrix} I & X & Y \\ 0 & I & Z \\ 0 & 0 & I \end{pmatrix}^{-1}$

Solution

a) $\left(\begin{array}{cc|cc} A & B & I & 0 \\ 0 & C & 0 & I \end{array}\right)\quad A^{-1}R_1$

$$\left(\begin{array}{cc|cc} A^{-1}A & A^{-1}B & A^{-1}I & 0 \\ 0 & C & 0 & I \end{array}\right)$$

$$\left(\begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ 0 & C & 0 & I \end{array}\right)\quad C^{-1}R_2$$

$$\left(\begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ 0 & C^{-1}C & 0 & C^{-1}I \end{array}\right)$$

$$\left(\begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ 0 & I & 0 & C^{-1} \end{array}\right)\quad R_1-A^{-1}BR_2$$

$$\left(\begin{array}{cc|cc} I & A^{-1}B-A^{-1}BI & A^{-1} & -A^{-1}BC^{-1} \\ 0 & I & 0 & C^{-1} \end{array}\right)$$

$$\left(\begin{array}{cc|cc} I & A^{-1}B-A^{-1}B & A^{-1} & -A^{-1}BC^{-1} \\ 0 & I & 0 & C^{-1} \end{array}\right)$$

$$\left(\begin{array}{cc|cc} I & 0 & A^{-1} & -A^{-1}BC^{-1} \\ 0 & I & 0 & C^{-1} \end{array}\right)$$

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}^{-1}=\begin{pmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{pmatrix}$$

b) $\left(\begin{array}{ccc|ccc} I & X & Y & I & 0 & 0 \\ 0 & I & Z & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 & I \end{array}\right)$ $R_1 - XR_2$

$$\left(\begin{array}{ccc|ccc} I & 0 & Y-XZ & I & -X & 0 \\ 0 & I & Z & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 & I \end{array}\right) \quad \begin{array}{l} R_1 - (Y-XZ)R_3 \\ R_2 - ZR_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} I & 0 & 0 & I & -X & XZ-Y \\ 0 & I & 0 & 0 & I & -Z \\ 0 & 0 & I & 0 & 0 & I \end{array}\right)$$

$$\begin{pmatrix} I & X & Y \\ 0 & I & Z \\ 0 & 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & -X & XZ-Y \\ 0 & I & -Z \\ 0 & 0 & I \end{pmatrix}$$

Exercise

Suppose that *A, B*, and $A - B$ are invertible $n \times n$ matrices. Show that

$$(A-B)^{-1} = A^{-1} + A^{-1}\left(B^{-1} - A^{-1}\right)^{-1} A^{-1}$$

Solution

A, B, and $A - B$ are invertible Then

$AA^{-1} = A^{-1}A = I \quad BB^{-1} = B^{-1}B = I$

$(A-B)(A-B)^{-1} = (A-B)^{-1}(A-B) = I$

Let:

$(A-B)^{-1}(A-B) = I$

Then, we need to prove that

$$\left(A^{-1} + A^{-1}\left(B^{-1} - A^{-1}\right)^{-1} A^{-1}\right)(A-B) \overset{?}{=} I$$

$$\left(A^{-1} + A^{-1}\left(B^{-1} - A^{-1}\right)^{-1} A^{-1}\right)(A-B) = \left(A^{-1} + A^{-1}\left(A\left(B^{-1} - A^{-1}\right)\right)^{-1}\right)(A-B)$$

$$\left(A\left(B^{-1} - A^{-1}\right)\right)^{-1} = \left(B^{-1} - A^{-1}\right)^{-1} A^{-1}$$

$$= \left(A^{-1} + A^{-1}\left(AB^{-1} - AA^{-1}\right)^{-1}\right)(A-B)$$

$$=\left(A^{-1}+A^{-1}\left(AB^{-1}-I\right)^{-1}\right)(A-B)$$

$$=\left(A^{-1}+A^{-1}\left(AB^{-1}-BB^{-1}\right)^{-1}\right)(A-B)$$

$$=\left(A^{-1}+A^{-1}\left((A-B)B^{-1}\right)^{-1}\right)(A-B)$$

$$\left((A-B)B^{-1}\right)^{-1}=B(A-B)^{-1}$$

$$=\left(A^{-1}+A^{-1}\left(B(A-B)^{-1}\right)\right)(A-B)$$

$$=\left(A^{-1}+A^{-1}B(A-B)^{-1}\right)(A-B)$$

$$=A^{-1}(A-B)+A^{-1}B(A-B)^{-1}(A-B)$$

$$=A^{-1}A-A^{-1}B+A^{-1}BI$$

$$=I-A^{-1}B+A^{-1}B$$

$$=I \quad \checkmark$$

Therefore; $(A-B)^{-1}=A^{-1}+A^{-1}\left(B^{-1}-A^{-1}\right)^{-1}A^{-1}$

Exercise

Suppose P is invertible and $A=PBP^{-1}$. Solve for B in terms of A.

Solution

Since P is invertible, then $PP^{-1}=P^{-1}P=I$

$A=PBP^{-1}$

$P^{-1}AP=P^{-1}PBP^{-1}P \qquad PP^{-1}=P^{-1}P=I$

$P^{-1}AP=IBI \qquad BI=B$

$\underline{P^{-1}AP=B}$

Exercise

Suppose $(A-B)C=0$, where A and B are $m\times n$ matrices and C is invertible. Show that $A=B$.

Solution

Since C is invertible, then $CC^{-1} = C^{-1}C = I$

$(A-B)C = 0$

$(A-B)CC^{-1} = 0C^{-1}$

$(A-B)I = 0$

$A-B = 0$

$A-B+B = 0+B$

$\underline{A = B}$ ✓

Exercise

Prove $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ where A, B, and C are invertible

Solution

Since A is invertible, then $AA^{-1} = A^{-1}A = I$

B is invertible, then $BB^{-1} = B^{-1}B = I$

C is invertible, then $CC^{-1} = C^{-1}C = I$

$(ABC)(ABC)^{-1} = I$

$$\begin{aligned}(ABC)(ABC)^{-1} &= (ABC)C^{-1}B^{-1}A^{-1} \\ &= AB\left(CC^{-1}\right)B^{-1}A^{-1} \\ &= AB(I)B^{-1}A^{-1} \\ &= A\left(BIB^{-1}\right)A^{-1} \\ &= A\left(BB^{-1}\right)A^{-1} \\ &= A(I)A^{-1} \\ &= (AI)A^{-1} \\ &= AA^{-1} \\ &= I\end{aligned}$$

$$\begin{aligned}(ABC)^{-1}(ABC) &= \left(C^{-1}B^{-1}A^{-1}\right)(ABC) \\ &= C^{-1}B^{-1}\left(A^{-1}A\right)BC \\ &= C^{-1}B^{-1}(I)BC \\ &= C^{-1}\left(B^{-1}B\right)C\end{aligned}$$

$$= C^{-1}(I)C$$

$$= C^{-1}C$$

$$= I$$

Thus, $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Exercise

Prove $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$ where *A, B*, *C* and *D* are invertible

Solution

Since A is invertible, then $AA^{-1} = A^{-1}A = I$

B is invertible, then $BB^{-1} = B^{-1}B = I$

C is invertible, then $CC^{-1} = C^{-1}C = I$

D is invertible, then $DD^{-1} = D^{-1}D = I$

$$(ABCD)(ABCD)^{-1} = I$$

$$\begin{aligned}(ABCD)(ABCD)^{-1} &= (ABCD)D^{-1}C^{-1}B^{-1}A^{-1}\\ &= ABC\left(DD^{-1}\right)C^{-1}B^{-1}A^{-1}\\ &= ABC(I)C^{-1}B^{-1}A^{-1}\\ &= AB\left(CC^{-1}\right)B^{-1}A^{-1}\\ &= AB(I)B^{-1}A^{-1}\\ &= A\left(BIB^{-1}\right)A^{-1}\\ &= A\left(BB^{-1}\right)A^{-1}\\ &= A(I)A^{-1}\\ &= (AI)A^{-1}\\ &= AA^{-1}\\ &= I\end{aligned}$$

$$\begin{aligned}(ABCD)^{-1}(ABCD) &= \left(D^{-1}C^{-1}B^{-1}A^{-1}\right)(ABCD)\\ &= D^{-1}C^{-1}B^{-1}\left(A^{-1}A\right)BCD\\ &= D^{-1}C^{-1}B^{-1}(I)BCD\end{aligned}$$

$$= D^{-1}C^{-1}\left(B^{-1}B\right)CD$$

$$= D^{-1}C^{-1}(I)CD^{-1}$$

$$= D^{-1}\left(C^{-1}C\right)D$$

$$= D^{-1}(I)D$$

$$= D^{-1}D$$

$$= I$$

Thus, $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$

Exercise

Prove that if $A^2 = A$, then $I - 2A = (I - 2A)^{-1}$

Solution

$$(I - 2A)(I - 2A) = I^2 - 2IA - 2AI + 4A^2$$

$$= I - 2A - 2A + 4A^2 \qquad A^2 = A$$

$$= I - 4A + 4A$$

$$= I$$

Therefore, $I - 2A = (I - 2A)^{-1}$

Exercise

Let $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

a) Show that $A^2 - 2A + 5I = 0$

b) Show that $A^{-1} = \frac{1}{5}(2I - A)$

c) Show that for any square matrix satisfying $A^2 - 2A + 5I = 0$, the inverse of A is $A^{-1} = \frac{1}{5}(2I - A)$

Solution

a) $A^2 = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

$$= \begin{pmatrix} -3 & 4 \\ -4 & -3 \end{pmatrix}$$

$$A^2 - 2A + 5I = \begin{pmatrix} -3 & 4 \\ -4 & -3 \end{pmatrix} - 2\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\underline{= 0} \quad \checkmark$$

b) $A^{-1} = \frac{1}{5}(2I - A)$

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$\frac{1}{5}(2I - A) = \frac{1}{5}\left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\right)$$

$$= \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{5}(2I - A) = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \quad \checkmark$$

c) $A^2 - 2A + 5I = 0$ & $A^{-1} = \frac{1}{5}(2I - A)$

$$AA^{-1} = A\left(\frac{1}{5}(2I - A)\right)$$

$$= \frac{1}{5}\left(2IA - A^2\right)$$

$$= \frac{1}{5}\left(2A - A^2\right)$$

$$A^2 - 2A + 5I = 0$$

$$5I = 2A - A^2$$

$$I = \frac{1}{5}\left(2A - A^2\right)$$

$$\frac{1}{5}\left(2A - A^2\right) = AA^{-1} = I \quad \checkmark$$

Exercise

find the inverse of the 3×3 Vandermonde matrix

$$V=\begin{pmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{pmatrix}$$

When $a_1,\ a_2,\ and\ a_3$ are distinct from each other

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ a_1 & a_2 & a_3 & 0 & 1 & 0 \\ a_1^2 & a_2^2 & a_3^2 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \\ R_2-a_1R_1 \\ R_3-a_1^2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & a_2-a_1 & a_3-a_1 & -a_1 & 1 & 0 \\ 0 & a_2^2-a_1^2 & a_3^2-a_1^2 & -a_1^2 & 0 & 1 \end{array}\right) \quad \begin{array}{l} (a_2-a_1)R_1-R_2 \\ \\ R_3-(a_2+a_1)R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} a_2-a_1 & 0 & a_2-a_3 & a_2 & -1 & 0 \\ 0 & a_2-a_1 & a_3-a_1 & -a_1 & 1 & 0 \\ 0 & 0 & \left(a_3^2-a_1^2\right)-\left(a_2+a_1\right)\left(a_3-a_1\right) & a_1a_2 & -\left(a_2+a_1\right) & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} a_2-a_1 & 0 & a_2-a_3 & a_2 & -1 & 0 \\ 0 & a_2-a_1 & a_3-a_1 & -a_1 & 1 & 0 \\ 0 & 0 & \left(a_3-a_1\right)\left(a_3-a_2\right) & a_1a_2 & -\left(a_2+a_1\right) & 1 \end{array}\right) \quad \begin{array}{l} (a_3-a_1)R_1+R_3 \\ (a_3-a_2)R_2-R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} \left(a_2-a_1\right)\left(a_3-a_1\right) & 0 & 0 & a_2\left(a_3+a_2\right) & -\left(a_3+a_2\right) & 1 \\ 0 & \left(a_2-a_1\right)\left(a_3-a_2\right) & 0 & -a_1\left(a_3+a_1\right) & a_3-a_1 & -1 \\ 0 & 0 & \left(a_3-a_1\right)\left(a_3-a_2\right) & a_1\left(a_2+a_1\right) & -\left(a_2+a_1\right) & 1 \end{array}\right)$$

$$\frac{1}{(a_2 - a_1)(a_3 - a_1)} R_1$$

$$\frac{1}{(a_2 - a_1)(a_3 - a_2)} R_2$$

$$\frac{1}{(a_3 - a_2)(a_3 - a_1)} R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{a_2(a_3 + a_2)}{(a_2 - a_1)(a_3 - a_1)} & -\frac{a_3 + a_2}{(a_2 - a_1)(a_3 - a_1)} & \frac{1}{(a_2 - a_1)(a_3 - a_1)} \\ 0 & 1 & 0 & \frac{-a_1(a_3 + a_1)}{(a_2 - a_1)(a_3 - a_2)} & \frac{a_3 - a_1}{(a_2 - a_1)(a_3 - a_2)} & -\frac{1}{(a_2 - a_1)(a_3 - a_2)} \\ 0 & 0 & 1 & \frac{a_1(a_2 + a_1)}{(a_3 - a_1)(a_3 - a_2)} & -\frac{a_2 + a_1}{(a_3 - a_1)(a_3 - a_2)} & \frac{1}{(a_3 - a_1)(a_3 - a_2)} \end{array}\right)$$

$$V^{-1} = \begin{pmatrix} \frac{a_2(a_3 + a_2)}{(a_2 - a_1)(a_3 - a_1)} & -\frac{a_3 + a_2}{(a_2 - a_1)(a_3 - a_1)} & \frac{1}{(a_2 - a_1)(a_3 - a_1)} \\ \frac{-a_1(a_3 + a_1)}{(a_2 - a_1)(a_3 - a_2)} & \frac{a_3 - a_1}{(a_2 - a_1)(a_3 - a_2)} & -\frac{1}{(a_2 - a_1)(a_3 - a_2)} \\ \frac{a_1(a_2 + a_1)}{(a_3 - a_1)(a_3 - a_2)} & -\frac{a_2 + a_1}{(a_3 - a_1)(a_3 - a_2)} & \frac{1}{(a_3 - a_1)(a_3 - a_2)} \end{pmatrix}$$

Solution Section 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices

Exercise

Consider the matrices

$$A=\begin{pmatrix}1 & -2\\ 3 & 5\end{pmatrix} \qquad B=\begin{pmatrix}-7 & -3\\ 6 & 4\end{pmatrix} \qquad \vec{v}=\begin{pmatrix}4\\ -1\end{pmatrix}$$

Find:

a) A^T
b) B^T
c) A^T+B^T
d) $(A+B)^T$
e) $(A\vec{v})^T$
f) $A^T\vec{v}^T$
g) $(AB)^T$
h) A^TB^T
i) B^TA^T
j) $(B\vec{v})^T$
k) $B^T\vec{v}^T$
l) $(2A)^T$
m) $2A^T$

Solution

a) $A^T=\begin{pmatrix}1 & 3\\ -2 & 5\end{pmatrix}$

b) $B^T=\begin{pmatrix}-7 & 6\\ -3 & 4\end{pmatrix}$

c)
$$\begin{aligned}A^T+B^T&=\begin{pmatrix}1 & 3\\ -2 & 5\end{pmatrix}+\begin{pmatrix}-7 & 6\\ -3 & 4\end{pmatrix}\\ &=\begin{pmatrix}-6 & 9\\ -5 & 9\end{pmatrix}\end{aligned}$$

d)
$$\begin{aligned}A+B&=\begin{pmatrix}1 & -2\\ 3 & 5\end{pmatrix}+\begin{pmatrix}-7 & -3\\ 6 & 4\end{pmatrix}\\ &=\begin{pmatrix}-6 & -5\\ 9 & 9\end{pmatrix}\end{aligned}$$

$$(A+B)^T=\begin{pmatrix}-6 & 9\\ -5 & 9\end{pmatrix}$$

Note $(A+B)^T=A^T+B^T$

e)
$$\begin{aligned}A\vec{v}&=\begin{pmatrix}1 & -2\\ 3 & 5\end{pmatrix}\begin{pmatrix}4\\ -1\end{pmatrix}\\ &=\begin{pmatrix}6\\ 7\end{pmatrix}\end{aligned}$$

$(A\vec{v})^T = [6 \quad 7]$

f) $\vec{v}^T = [4 \quad -1]$

$A^T \vec{v}^T = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix} [6 \quad 7]$

Can't be *found*, inner not equal $2 \neq 1$

g) $AB = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -7 & -3 \\ 6 & 4 \end{pmatrix}$

$= \begin{pmatrix} -19 & -11 \\ 9 & 11 \end{pmatrix}$

$(AB)^T = \begin{pmatrix} -19 & 9 \\ -11 & 11 \end{pmatrix}$

h) $A^T B^T = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} -7 & 6 \\ -3 & 4 \end{pmatrix}$

$= \begin{pmatrix} -13 & 18 \\ -1 & 8 \end{pmatrix}$

Note $(AB)^T \neq A^T B^T$

i) $B^T A^T = \begin{pmatrix} -7 & 6 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$

$= \begin{pmatrix} -19 & 9 \\ -11 & 11 \end{pmatrix}$

Note $(AB)^T = B^T A^T$

j) $B\vec{v} = \begin{pmatrix} -7 & -3 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} -25 \\ 20 \end{pmatrix}$

$(B\vec{v})^T = [-25 \quad 20]$

k) $B^T = \begin{pmatrix} -7 & 6 \\ -3 & 4 \end{pmatrix}$ $\qquad \vec{v}^T = [4 \quad -1]$

$B^T \vec{v}^T = \begin{pmatrix} -7 & 6 \\ -3 & 4 \end{pmatrix} [4 \quad -1]$

Can't be *found*, inner not equal $2 \neq 1$

l) $2A = \begin{pmatrix} 2 & -4 \\ 6 & 10 \end{pmatrix}$

$(2A)^T = \begin{pmatrix} 2 & 6 \\ -4 & 10 \end{pmatrix}$

m) $2A^T = 2\begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$

$= \begin{pmatrix} 2 & 6 \\ -4 & 10 \end{pmatrix}$

Note $(2A)^T = 2A^T$

Exercise

Solve $Lc = b$ to find ***c***. Then solve $Ux = c$ to find ***x***. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Solution

$Lc = b$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

$$\rightarrow \begin{cases} \underline{c_1 = 4} \\ c_1 + c_2 = 5 \\ c_1 + c_2 + c_3 = 6 \end{cases}$$

$$\begin{cases} c_2 = 5 - 4 \underline{= 1} \\ c_3 = 6 - 4 - 1 \underline{= 1} \end{cases}$$

$$c = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

$Ux = c$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x + y + z = 4 \\ \quad y + z = 1 \\ \qquad z = 1 \end{cases} \rightarrow \begin{cases} x = 3 \\ y = 0 \end{cases}$$

$$x = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

$$L\vec{c} = b$$
$$LU\,\vec{x} = \vec{b}$$
$$A\vec{x} = \vec{b}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}}_{b}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise

Find L and U for the symmetric matrix:

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots

Solution

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

Exercise

Determine whether the given matrix is invertible:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Solution

The matrix is a diagonal matrix with nonzero entries on the diagonal, so it is invertible.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Exercise

Find $A^2, A^{-2}, \textit{ and } A^{-k}$ by inspection $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

Solution

$$A^2 = \begin{bmatrix} 1^2 & 0 \\ 0 & (-2)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$A^{-2} = \begin{bmatrix} 1^{-2} & 0 \\ 0 & (-2)^{-2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$A^{-k} = \begin{bmatrix} 1^{-k} & 0 \\ 0 & (-2)^{-k} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{(-2)^k} \end{bmatrix}$$

Exercise

Find $A^2,\ A^{-2},\ and\ A^{-k}$ by inspection $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$

Solution

$$A^2 = \begin{bmatrix} \left(\frac{1}{2}\right)^2 & 0 & 0 \\ 0 & \left(\frac{1}{3}\right)^2 & 0 \\ 0 & 0 & \left(\frac{1}{4}\right)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{16} \end{bmatrix}$$

$$A^{-2} = \begin{bmatrix} \left(\frac{1}{2}\right)^{-2} & 0 & 0 \\ 0 & \left(\frac{1}{3}\right)^{-2} & 0 \\ 0 & 0 & \left(\frac{1}{4}\right)^{-2} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

$$A^{-k} = \begin{bmatrix} \left(\frac{1}{2}\right)^{-k} & 0 & 0 \\ 0 & \left(\frac{1}{3}\right)^{-k} & 0 \\ 0 & 0 & \left(\frac{1}{4}\right)^{-k} \end{bmatrix}$$

$$= \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix}$$

Exercise

Find $A^2,\ A^{-2},\ and\ A^{-k}$ by inspection $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Solution

$$A^2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$A^{-2} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{16} & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$A^{-k} = \begin{bmatrix} (-2)^{-k} & 0 & 0 & 0 \\ 0 & (-4)^{-k} & 0 & 0 \\ 0 & 0 & (-3)^{-k} & 0 \\ 0 & 0 & 0 & (2)^{-k} \end{bmatrix}$$

Exercise

Find $A^3,\ and\ A^{-1}$ $\quad A = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$

Solution

$$A^3 = \begin{pmatrix} -8 & 0 & 0 & 0 & 0 \\ 0 & 64 & 0 & 0 & 0 \\ 0 & 0 & 125 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Exercise

Find A^3, *and* A^{-1} $\quad B = \begin{pmatrix} 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{5} \end{pmatrix}$

Solution

$$B^3 = \begin{pmatrix} 512 & 0 & 0 & 0 & 0 & 0 \\ 0 & 64 & 0 & 0 & 0 & 0 \\ 0 & 0 & -125 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{64} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{125} \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{pmatrix}$$

Exercise

Find the power of A^{16} for the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Solution

$$A^{16} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= I_{5\times 5}$$

Exercise

Decide whether the given matrix is symmetric $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

Solution

Not *symmetric*, since $a_{12} \neq a_{21} \quad (1 \neq -1)$

Exercise

Decide whether the given matrix is symmetric $\begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$

Solution

Symmetric

Exercise

Decide whether the given matrix is symmetric $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

Solution

Not *symmetric*, since $a_{13} = 1 \neq 3 = a_{31}$

Exercise

Find all values of the unknown constant(s) in order for A to be symmetric

$$A = \begin{bmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 7 \end{bmatrix}$$

Solution

$$\begin{cases} a-2b+2c=3 \\ 2a+\ b+c=0 \\ a \qquad\ +c=-2 \end{cases}$$

$$a=11, \quad b=9, \quad c=-13$$

Exercise

Find x and B, if the given matrix B is symmetric

$$B = \begin{pmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{pmatrix}$$

Solution

For matrix B is symmetric, then $B = B^T$

$$\begin{pmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{pmatrix} = \begin{pmatrix} 4 & 2x-3 \\ x+2 & x+1 \end{pmatrix}$$

$\rightarrow \ x+2=2x-3$

$\underline{x=5}$

$$B=\begin{pmatrix} 4 & 7 \\ 7 & 6 \end{pmatrix}$$

Exercise

Find a diagonal matrix A that satisfies the given condition $A^{-2}=\begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^{-2}=\begin{pmatrix} a^{-2} & 0 & 0 \\ 0 & b^{-2} & 0 \\ 0 & 0 & c^{-2} \end{pmatrix}$$

$$=\begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{cases} a^{-2}=9 & \Rightarrow \quad a=\pm 9^{-1/2}=\pm\frac{1}{3} \\ b^{-2}=4 & \Rightarrow \quad b=\pm 2^{-1/2}=\pm\frac{1}{2} \\ c^{-2}=1 & \Rightarrow \quad c=\pm 1^{-1/2}=\pm 1 \end{cases}$$

$$A=\begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A=\begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A=\begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \cdots$$

$$A=\begin{pmatrix} \pm\frac{1}{3} & 0 & 0 \\ 0 & \pm\frac{1}{2} & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$$

Exercise

Let A be an $n \times n$ symmetric matrix

a) Show that A^2 is symmetric

b) Show that $2A^2 - 3A + I$ is symmetric

c) Show that $\frac{1}{2}\left(A + A^T\right)$ is symmetric

Solution

a) The property of the transpose states that $(AB)^T = B^T A^T$

Since A is symmetric then $A^T = A$

$$\begin{aligned}\left(A^2\right)^T &= (AA)^T \\ &= A^T A^T \\ &= \left(A^T\right)^2 \qquad \textit{A is symmetric} \\ &= A^2\end{aligned}$$

$\therefore\ A^2$ is symmetric

b) $$\begin{aligned}\left(2A^2 - 3A + I\right)^T &= 2\left(A^2\right)^T - 3(A)^T + (I)^T \\ &= 2\left(A^T\right)^2 - 3A^T + (I)^T \qquad \textit{A and I are symmetric} \\ &= 2A^2 - 3A + I\end{aligned}$$

$\therefore\ 2A^2 - 3A + I$ is ***Symmetric***

c) Then we need to prove that $\frac{1}{2}\left(A + A^T\right) = \left(\frac{1}{2}\left(A + A^T\right)\right)^T$

$$\begin{aligned}\left(\frac{1}{2}\left(A + A^T\right)\right)^T &= \frac{1}{2}\left(A + A^T\right)^T \\ &= \frac{1}{2}\left(A^T + \left(A^T\right)^T\right) \\ &= \frac{1}{2}\left(A^T + A\right) \\ &= \frac{1}{2}\left(A + A^T\right)\end{aligned}$$

$\therefore\ \frac{1}{2}\left(A + A^T\right)$ is symmetric

Exercise

Prove if $A^T A = A$, then A is symmetric and $A = A^2$

Solution

If $A^T A = A$, then

$$A^T = \left(A^T A\right)^T$$

$$= A^T \left(A^T\right)^T$$

$$= A^T A$$

$$= A$$

So, A is symmetric.

Since $A = A^T$

$$AA = A^T A \qquad\qquad A^T A = A$$

$$A^2 = A$$

Exercise

A square matrix A is called ***skew-symmetric*** if $A^T = -A$. Prove

a) If A is an invertible skew-symmetric matrix, then A^{-1} is skew-symmetric.

b) If A and B are skew-symmetric matrices, then so are A^T, $A+B$, $A-B$, *and* kA for any scalar k.

c) Every square matrix A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

$$\left[\textit{Hint}: \text{ Note the identity } \ A = \tfrac{1}{2}\left(A + A^T\right) + \tfrac{1}{2}\left(A - A^T\right) \right]$$

Solution

a) $$\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$$

$$= (-A)^{-1} \qquad\qquad \textit{skew-symmetric}$$

$$= -A^{-1}$$

$\therefore\ A^{-1}$ is also skew-symmetric

b) Let A and B are skew-symmetric matrices

$$\left(A^T\right)^T = (-A)^T$$

$$= -A^T$$

$$(A+B)^T = A^T + B^T$$

$$= -A - B$$
$$= -(A + B)$$

$$(A - B)^T = A^T - B^T$$
$$= -A + B$$
$$= -(A - B)$$

$$(kA)^T = k(A)^T$$
$$= k(-A)$$
$$= -kA$$

c) We need to prove from the hint that $\frac{1}{2}\left(A + A^T\right)$ is symmetric and $\frac{1}{2}\left(A - A^T\right)$ is skew-symmetric

$$\frac{1}{2}\left(A + A^T\right)^T = \frac{1}{2}\left(A^T + \left(A^T\right)^T\right)$$
$$= \frac{1}{2}\left(A + A^T\right)$$

Thus $\frac{1}{2}\left(A + A^T\right)$ ***is symmetric***

$$\frac{1}{2}\left(A - A^T\right)^T = \frac{1}{2}\left(A^T - \left(A^T\right)^T\right)$$
$$= \frac{1}{2}\left(A^T - A\right)$$
$$= -\frac{1}{2}\left(A - A^T\right)$$

Thus $\frac{1}{2}\left(A - A^T\right)$ ***is skew-symmetric***

Exercise

Suppose R is rectangular (m by n) and A is symmetric (m by m)

a) Transpose $R^T AR$ to show its symmetric

b) Show why $R^T R$ has no negative numbers on its diagonal.

Solution

a) $$\left(R^T AR\right)^T = \left(\left(R^T A\right)R\right)^T$$
$$= R^T\left(R^T A\right)^T$$

$$= R^T A^T \left(R^T\right)^T$$

$$= R^T AR$$

b) $\left(R^T R\right)_{jj} = (column\ j\ of\ R).(column\ j\ of\ R)$

= ***Product of the diagonal entry by itself.***

= length squared of column j.

Exercise

If L is a lower-triangular matrix, then $\left(L^{-1}\right)^T$ is _______Triangular

Solution

$\left(L^{-1}\right)^T$ is ***upper*** triangular.

L^{-1} is a lower-triangular because L is.

The transpose carries the lower-triangular matrices to the upper-triangular (and vice versa).

Exercise

True or False

a) The block matrix $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is automatically symmetric

b) If A and B are symmetric then their product is symmetric

c) If A is not symmetric then A^{-1} is not symmetric

d) When A, B, C are symmetric, the transpose of ABC is CBA.

e) The transpose of a diagonal matrix is a diagonal.

f) The transpose of an upper triangular matrix is an upper triangular matrix.

g) The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.

h) All entries of a symmetric matrix are determined by the entries occurring on and above the main diagonal.

i) All entries of an upper triangular matrix are determined by the entries occurring on and above the main diagonal.

j) The inverse of an invertible lower triangular matrix is an upper triangular matrix.

k) A diagonal matrix is invertible if and only if all of its diagonal entries are positive.

l) The sum of a diagonal matrix and a lower triangular matrix is a lower triangular matrix.

m) A matrix that is both symmetric and upper triangular must be a diagonal matrix.

n) If A and B are $n \times n$ matrices such that $A + B$ is symmetric, then A and B are symmetric.

o) If A and B are $n \times n$ matrices such that $A + B$ is upper triangular, then A and B are upper triangular.

p) If A^2 is a symmetric matrix, then A is a symmetric matrix.

q) If kA is a symmetric matrix for some $k \neq 0$, then A is a symmetric matrix.

Solution

a) False: $\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right)$

b) False $\underset{A}{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}} \underset{B}{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

c) True by definition.

d) True $(ABC)^T = C^T (AB)^T = C^T B^T A^T = CBA$ Since $A^T = A,\ B^T = B,\ C^T = C$

e) True Since a diagonal matrix must be square and have zeros off the main diagonal, its transpose is also diagonal.

f) False The transpose of an upper triangular matrix is lower triangular.

g) False $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 1 & 4 \end{bmatrix}$

h) True The entries above the main diagonal determine the entries below the main diagonal in a symmetric matrix.

i) True in an upper triangular matrix, the series below the main diagonal are all zeros.

j) False The inverse of an invertible lower triangular matrix is lower triangular.

k) False The diagonal entries may be negative, as long as they are nonzero.

l) True Adding a diagonal matrix to a lower triangular matrix will not create nonzero entries above the main diagonal.

m) True Since the entries below the main diagonal must be zero, so also must be the entries above the main diagonal.

n) False $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ which is symmetric

o) False $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}$ which is upper triangular.

p) False $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

q) True $(kA)^T = kA$ then

$$(kA)^T - kA = 0$$

$$kA^T - kA = 0$$

$$k\left(A^T - A\right) = 0 \text{ since } k \neq 0 \text{ then } A^T = A$$

Therefore, A is a symmetric matrix.

Exercise

Find 2 by 2 symmetric matrices $A = A^T$ with these properties

a) A is not invertible

b) A is invertible but cannot be factored into LU (row exchanges needed)

c) A can be factored into LDL^T but not into LL^T (because of negative D)

Solution

a) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

b) $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ only need a *zero* in the diagonal.

c) $A = LDL^T$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} a & 0 \\ a & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & a \\ a & a+d \end{pmatrix} \underset{LL^T}{=} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \rightarrow \begin{cases} a = 1 \\ d = 1 \end{cases}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\rightarrow D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Exercise

A group of matrices includes AB and A^{-1} if it includes A and B "Products and inverses stay in the group." Which of these sets are groups?
Lower triangular matrices L with 1's on the diagonal, symmetric matrices S, positive matrices M, diagonal invertible matrices D, permutation matrices P, matrices with $Q^T = Q^{-1}$. ***Invent two more matrix groups***.

Solution

The lower triangular matrices L with 1's on the diagonal form a group.
Clearly the product of two is a third. The Gauss-Jordan method shows that the inverse of one is another.
The symmetric matrices don't form a group. An example of the 2 symmetric matrices A and B whose product is not symmetric

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \quad AB = \begin{bmatrix} 2 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix}$$

The positive matrices do not form a group.

$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad M^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, the inverse is not symmetric.

The diagonal invertible matrices form a group.
The permutation matrices form a group.
The matrices with $Q^T = Q^{-1}$ form a group. If A and B are two matrices, then so are AB and A^{-1}, as

$$\begin{aligned} (AB)^T &= B^T A^T \\ &= B^{-1} A^{-1} \\ &= (AB)^{-1} \end{aligned}$$

$$\begin{aligned} \left(A^{-1}\right)^T &= \left(A^T\right)^{-1} \\ &= A^{-1} \end{aligned}$$

There are many more matrix groups. For example, given two, the block matrices $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ form a third as A ranges over the first group and B ranges over the second.
Another example is the set of all products cP where c is a nonzero scalar and P is a permutation matrix of given size.

Exercise

Write $A=\begin{bmatrix}1 & 2\\ 4 & 9\end{bmatrix}$ as the product EH of an elementary row operation matrix E and a symmetric matrix H.

Solution

$$A=EH$$
$$E^{-1}A=E^{-1}EH$$
$$E^{-1}A=H$$

An elementary row operation matrix has the form $E=\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}$

The inverse is: $E^{-1}=\begin{pmatrix}1 & 0\\ -x & 1\end{pmatrix}$

$$H=\begin{pmatrix}1 & 0\\ -x & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 4 & 9\end{pmatrix}$$
$$=\begin{pmatrix}1 & 2\\ -x+4 & -2x+9\end{pmatrix}$$

Since matrix H is symmetric, therefore:

$$-x+4=2$$
$$x=2$$

$$\begin{pmatrix}1 & 2\\ 4 & 9\end{pmatrix}=\underbrace{\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}}_{Elementary}\ \underbrace{\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}}_{Symmetric}$$

Exercise

When is the product of two symmetric matrices symmetric? Explain your answer.

Solution

AB is symmetric *iff* $AB=(AB)^T$

$$AB=(AB)^T$$
$$=B^TA^T \qquad \textit{A and B are symmetric}$$
$$=BA$$

AB is symmetric iff A and B commute

Exercise

Express $\left((AB)^{-1}\right)^T$ in terms of $\left(A^{-1}\right)^T$ and $\left(B^{-1}\right)^T$

Solution

$$\begin{aligned}\left((AB)^{-1}\right)^T &= \left(B^{-1}A^{-1}\right)^T \\ &= \left(A^{-1}\right)^T\left(B^{-1}\right)^T\end{aligned}$$

Exercise

Find the transpose of the given matrix: $\begin{bmatrix} 8 & -1 \\ 3 & 5 \\ -2 & 5 \\ 1 & 2 \\ -3 & -5 \end{bmatrix}$

Solution

$$A^T = \begin{bmatrix} 8 & 3 & -2 & 1 & -3 \\ -1 & 5 & 5 & 2 & -5 \end{bmatrix}$$

Exercise

Find the transpose of the given matrix: $D = \begin{pmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -1 \end{pmatrix}$

Solution

$$D^T = \begin{pmatrix} 1 & -3 & 5 \\ -2 & 4 & -1 \end{pmatrix}$$

Exercise

Show that if A is symmetric and invertible, then A^{-1} is also symmetric.

Solution

A is symmetric and invertible, then $A = A^T \quad AA^{-1} = I$

$$\begin{aligned}\left(A^{-1}\right)^T &= \left(A^T\right)^{-1} \\ &= A^{-1}\end{aligned}$$

$\Rightarrow A^{-1}$ is symmetric.

Exercise

Prove that $(AB)^T = B^T A^T$

Solution

Let $A = \left[a_{ik}\right]$ and $B = \left[b_{kj}\right]$

Then the *ij*-entry of *AB* is:

$a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{im}b_{mj}$

The reverse order, *ji*-entry of $(AB)^T$

Column *j* of *B* becomes row *j* of B^T, and row *i* of *A* becomes column *i* of A^T.

Thus, the *ij*-entry of $B^T A^T$ is:

$$\left(\left(b_{1j},\ b_{2j}, \ldots,\ b_{mj}\right)\left(a_{i1},\ a_{i2},\ \ldots,\ a_{im}\right)\right)^T = b_{1j}a_{i1} + b_{2j}a_{i2} + \ldots + b_{mj}a_{im}$$

Thus $(AB)^T = B^T A^T$

Exercise

Prove that $(A+B)^T = A^T + B^T$

Solution

For adding matrices, the matrices must have the size $(m \times n)$

Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$

Let $C = \left[c_{ij}\right]$ such that $C = A + B$

Then the *ij*-entry of $(A+B)$ is:

$$\begin{aligned} C &= A + B \\ &= \left[a_{ij}\right] + \left[b_{ij}\right] \\ &= \left[a_{ij} + b_{ij}\right] \end{aligned}$$

$$\left[c_{ij}\right]=\left[a_{ij}+b_{ij}\right]$$

$$(A+B)^T=\left[c_{ij}\right]^T$$

$$=\left[c_{ji}\right]$$

$$A^T+B^T=\left[a_{ij}\right]^T+\left[b_{ij}\right]^T$$

$$=\left[a_{ji}\right]+\left[b_{ji}\right]$$

$$=\left[a_{ji}+b_{ji}\right]$$

$$=\left[c_{ji}\right]$$

$$=(A+B)^T$$

$$(A+B)^T=A^T+B^T \qquad \checkmark$$

Exercise

Prove that if A, B, and C are square symmetric matrices and $ABC=I$, then B is an invertible and $B^{-1}=CA$.

Solution

Given: A, B, and C are square symmetric matrices, then

$$A=A^T, \quad B=B^T, \quad C=C^T \quad \& \quad ABC=I$$

$$CABC=CI$$
$$CABCA=CA$$
$$(CAB)CA=CA$$
$$CA(BCA)=CA$$
$$B(CA)=I$$

That implies to B in the invertible of CA, $B^{-1}=CA$

$$BB^{-1}=I$$

Therefore, B is invertible.

Exercise

Prove that each statement is true when A and B are square matrices of order n and c is a scalar.

a) $Tr(A+B) = Tr(A) + Tr(B)$

b) $Tr(cA) = c\,Tr(A)$

Solution

Let $A = \left[a_{ij}\right]$ & $B = \left[b_{ij}\right]$

a) $A + B = \left[a_{ij}\right] + \left[b_{ij}\right]$

$$= \left[a_{ij} + b_{ij}\right]$$

$$Tr(A+B) = Tr\left(\left[a_{ij} + b_{ij}\right]\right)$$

$$= \sum_{k=1}^{n} \left[a_{kk} + b_{kk}\right]$$

$$= \sum_{k=1}^{n} \left[a_{kk}\right] + \sum_{k=1}^{n} \left[b_{kk}\right]$$

$$= Tr(A) + Tr(B)$$

Therefore, $Tr(A+B) = Tr(A) + Tr(B)$

b) Let c be any scalar

$$Tr(cA) = \sum_{k=1}^{n} \left[c\,a_{kk}\right]$$

$$= c\sum_{k=1}^{n} \left[a_{kk}\right]$$

$$= c\,Tr(A)$$

Therefore, $Tr(cA) = c\,Tr(A)$

Exercise

Verify that $(AB)^T = B^T A^T$ given $A = \begin{bmatrix} -1 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix}$; $B = \begin{bmatrix} -3 & 0 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$

Solution

$$AB = \begin{bmatrix} -1 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ -5 & -1 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 2 & -5 \\ 4 & -1 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} -3 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -5 \\ 4 & -1 \end{bmatrix}$$

Therefore, $(AB)^T = B^T A^T$

Exercise

For the given matrix, compute A^T, $\left(A^T\right)^{-1}$, A^{-1}, and $\left(A^{-1}\right)^T$, then compare $\left(A^T\right)^{-1}$ and $\left(A^{-1}\right)^T$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

Solution

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_1 - 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 + R_3 \\ R_1 - 2R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \qquad \left(A^T\right)^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \\ \\ R_3 - 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\left(A^{-1}\right)^T = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

Exercise

Show that a 2×2 lower triangular matrix is invertible if and only if $a_{11}a_{22} \neq 0$ and in this case the inverse is also lower triangular.

Solution

Let A to be the lower triangular matrix

$$A = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}$$

$\det(A) = a_{11}a_{22} \neq 0$ is invertible iff $a_{11}a_{22} \neq 0$ and then

$$A^{-1} = \frac{1}{a_{11}a_{22}} \begin{pmatrix} a_{22} & 0 \\ -a_{21} & a_{11} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a_{11}} & 0 \\ -\frac{a_{21}}{a_{11}a_{22}} & \frac{1}{a_{22}} \end{pmatrix}$$

Exercise

Let A be any (2×2) diagonal matrix. Give a necessary and sufficient condition on the diagonal entries so that A has an inverse. Compute the inverse of any such matrix.

Solution

Let $A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$

$$A^{-1} = \begin{pmatrix} \frac{1}{a_{11}} & 0 \\ 0 & \frac{1}{a_{22}} \end{pmatrix}$$

So, A^{-1} exists when both entries on the main diagonal are nonzero.

Exercise

Find (2×2) matrices A and B such that A and B are symmetric, but AB is not symmetric.

Solution

Since A and B are symmetric, then

$$A = A^T \quad \& \quad B = B^T$$

$$\begin{aligned} (AB)^T &= B^T A^T \\ &= BA \\ &\neq AB \end{aligned}$$

Since the product is not commutative.

Since we going to prove AB is ***not*** symmetric, then we can use an example.

Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 9 \\ 1 & 13 \end{pmatrix}$$

AB is not symmetric even though A and B are symmetric

Exercise

Let A and B be $(n \times n)$ symmetric matrices. Give a necessary and sufficient condition for AB to be symmetric.

Solution

Since A and B are symmetric, then

$$A = A^T \quad \& \quad B = B^T$$

$$(AB)^T = B^T A^T$$
$$= BA$$

For AB to be symmetric, then $AB = BA$

For the product to be commutative, then A is inverse of matrix B.

Let $A = \begin{pmatrix} a_{11} & a \\ a & a_{22} \end{pmatrix}$ *and* $B = \begin{pmatrix} b_{11} & b \\ b & b_{22} \end{pmatrix}$

$$AB = \begin{pmatrix} a_{11} & a \\ a & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b \\ b & b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + ab & b\,a_{11} + a\,b_{22} \\ a\,b_{11} + b\,a_{22} & ab + a_{22}b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} * & b\,a_{11} + a\,b_{22} \\ a\,b_{11} + b\,a_{22} & * \end{pmatrix}$$

For AB to be symmetric, then

$$b\,a_{11} + a\,b_{22} = a\,b_{11} + b\,a_{22}$$

$$a\left(b_{22} - b_{11}\right) = b\left(a_{22} - a_{11}\right)$$

If $b_{22} - b_{11} = 0$ & $a_{22} - a_{11} = 0$

$$b_{22} = b_{11} \quad \& \quad a_{22} = a_{11}$$

$$AB = \begin{pmatrix} * & b\,a_{11} + a\,b_{11} \\ a\,b_{11} + b\,a_{11} & * \end{pmatrix}$$

$$= \begin{pmatrix} * & b\,a_{11} + a\,b_{11} \\ a\,b_{11} + b\,a_{11} & * \end{pmatrix}$$

If $b_{22} - b_{11} \neq 0$ & $a_{22} - a_{11} \neq 0$

$b_{22} \neq b_{11}$ & $a_{22} \neq a_{11}$

$$a = b\,\frac{a_{22} - a_{11}}{b_{22} - b_{11}}$$

$$AB = \begin{pmatrix} * & b\,a_{11} + a\,b_{22} \\ a\,b_{11} + b\,a_{22} & * \end{pmatrix}$$

$$= \begin{pmatrix} * & b\,a_{11} + b\,\frac{a_{22} - a_{11}}{b_{22} - b_{11}} b_{22} \\ b\,\frac{a_{22} - a_{11}}{b_{22} - b_{11}} b_{11} + b\,a_{22} & * \end{pmatrix}$$

$$= \begin{pmatrix} * & b\,\frac{a_{11}b_{22} - a_{11}b_{11} + a_{22}\,b_{22} - a_{11}b_{22}}{b_{22} - b_{11}} \\ b\,\frac{a_{22}b_{11} - a_{11}b_{11} + a_{22}b_{22} - a_{22}\,b_{11}}{b_{22} - b_{11}} & * \end{pmatrix}$$

$$= \begin{pmatrix} * & b\,\frac{-\,a_{11}b_{11} + a_{22}\,b_{22}}{b_{22} - b_{11}} \\ b\,\frac{-a_{11}b_{11} + a_{22}b_{22}}{b_{22} - b_{11}} & * \end{pmatrix}$$

$$= \begin{pmatrix} * & \frac{b\left(a_{22}\,b_{22} - a_{11}b_{11}\right)}{b_{22} - b_{11}} \\ \frac{b\left(a_{22}\,b_{22} - a_{11}b_{11}\right)}{b_{22} - b_{11}} & * \end{pmatrix}$$

If $b_{22} - b_{11} = 0$ & $a_{22} - a_{11} \neq 0$

$b_{22} = b_{11}$ & $a_{22} \neq a_{11}$

$$AB = \begin{pmatrix} * & ba_{11} + ab_{22} \\ ab_{11} + ba_{22} & * \end{pmatrix}$$

$$= \begin{pmatrix} * & ba_{11} + ab_{11} \\ ab_{11} + ba_{22} & * \end{pmatrix}$$

$$ba_{11} = ba_{22}$$

$a_{11} = a_{22}$ Contradict the assumption.

Therefore, for AB to be symmetric only and only if when

$$b_{22} = b_{11} \quad \& \quad a_{22} = a_{11}$$

Or

$$b_{22} \neq b_{11} \quad \& \quad a_{22} \neq a_{11}$$

$$\rightarrow \ a = b\frac{a_{22} - a_{11}}{b_{22} - b_{11}}$$

Exercise

Let A be any (2×2) matrix, and let B and C be given by

$$B = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

Find the matrix A if:

a) $A^T + B = C$

b) $A^T B = C$

Solution

a) $A^T + B = C$

$$A^T = C - B$$

$$= \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

b) $A^T B = C$

$$A^T B B^{-1} = C B^{-1}$$

$$A^T = C\,B^{-1}$$

$$B^{-1} = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -3 \\ 11 & -7 \end{pmatrix}$$

$$A = \begin{pmatrix} 5 & 11 \\ -3 & -7 \end{pmatrix}$$

Exercise

Given the matrix $A = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

a) Find $A^T A$, show it is symmetric

b) Find AA^T , show it is symmetric

Solution

a) $A^T A = \begin{bmatrix} 4 & 0 \\ 2 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 16 & 8 & 4 \\ 8 & 8 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

Therefore, from the solution the $A^T A$ is symmetric

b) $AA^T = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 2 & 2 \\ 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 21 & 3 \\ 3 & 5 \end{bmatrix}$$

Therefore, from the solution the AA^T is symmetric

Exercise

Prove each of the following:

a) If A is any (2×2) matrix, then A^TA and AA^T are symmetric

b) If A, B, and C are matrices such that the product ABC is defined then $(ABC)^T = C^TB^TA^T$

Solution

a) For A^TA to be symmetric

$$A^TA = \left(A^TA\right)^T$$

$$\begin{aligned}\left(A^TA\right)^T &= A^T\left(A^T\right)^T \\ &= A^TA \quad \checkmark\end{aligned}$$

For AA^T to be symmetric, then $AA^T = \left(AA^T\right)^T$

$$\begin{aligned}\left(AA^T\right)^T &= \left(A^T\right)^TA^T \\ &= AA^T \quad \checkmark\end{aligned}$$

Therefore, A^TA and AA^T are symmetric

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

$$\begin{aligned}A^TA &= \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= \begin{pmatrix} * & a_{11}a_{12}+a_{21}a_{22} \\ a_{12}a_{11}+a_{22}a_{21} & * \end{pmatrix}\end{aligned}$$

Hence, A^TA is symmetric.

$$\begin{aligned}AA^T &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \\ &= \begin{pmatrix} * & a_{11}a_{21}+a_{12}a_{22} \\ a_{21}a_{11}+a_{22}a_{12} & * \end{pmatrix}\end{aligned}$$

Hence, AA^T is symmetric.

b) For $(ABC)^T = C^T B^T A^T$

Given that the product ABC is defined

Let $BC = D$

$$\begin{aligned}(ABC)^T &= (AD)^T \\ &= D^T A^T \qquad \text{From part } (a) \text{ prove} \\ &= (BC)^T A^T \\ &= C^T B^T A^T \quad \checkmark\end{aligned}$$

Therefore, $(ABC)^T = C^T B^T A^T$

Exercise

Let A be any $(n \times n)$ real matrix

a) Show that $A + A^T$ is symmetric

b) Show that $A - A^T$ is skew symmetric

Solution

a) $$\begin{aligned}\left(A + A^T\right)^T &= A^T + \left(A^T\right)^T \\ &= A^T + A \\ &= A + A^T\end{aligned}$$

Therefore, $A + A^T$ is symmetric

b) $$\begin{aligned}\left(A - A^T\right)^T &= A^T - \left(A^T\right)^T \\ &= A^T - A \\ &= -A + A^T \\ &= -\left(A - A^T\right)\end{aligned}$$

Therefore, $A + A^T$ is skew symmetric

Exercise

Suppose that L is unit lower triangular and U is the upper triangular. Show that if $L = U$, then $L = U = I$

Solution

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \quad \& \quad U = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

If $L = U$, then the only common between the lower and upper triangular is the main diagonal with a unit entries.

Therefore, for $L = U = I$

Solution Section 1.6 – The Properties of Determinants

Exercise

Verify that $\det(AB) = \det(A)\det(B)$ when: $A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{pmatrix}$

Solution

$$AB = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{pmatrix}$$

$$\det(AB) = \begin{vmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{vmatrix}\begin{matrix} 9 & -1 \\ 31 & 1 \\ 10 & 0 \end{matrix}$$

$$= -170$$

$$\det(A) = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 10$$

$$\det(B) = \begin{vmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{vmatrix}$$

$$= -17$$

$$\det(AB) = \det(A)\det(B) \quad = -170 \ \checkmark$$

Exercise

For which value(s) of $\boldsymbol{k}$ does A fail to be invertible? $A = \begin{bmatrix} k-3 & -2 \\ -2 & k-2 \end{bmatrix}$

Solution

For $\boldsymbol{A}$ to have an invertible the determinant cannot be equal to zero. To ***fail*** det(A) = 0.

$$|A| = \begin{vmatrix} k-3 & -2 \\ -2 & k-2 \end{vmatrix} = 0$$

$$(k-3)(k-2) - 4 = 0$$

$$k^2 - 5k + 6 - 4 = 0$$

$$k^2 - 5k + 2 = 0$$

$$k = \frac{5 \pm \sqrt{17}}{2}$$

Exercise

Without directly evaluating, show that $\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$

Solution

$$\begin{vmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \overset{R_2 + R_1}{=} \begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$

It is equal to zero, since first row and third row are proportional.

$$\begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \underset{R_3 - \frac{1}{a+b+c} R_1}{=} \begin{vmatrix} a+b+c & b+c+a & c+b+a \\ a & b & c \\ 0 & 0 & 0 \end{vmatrix}$$

$$= 0$$

Exercise

If the entries in every row of A add to zero, solve $A\vec{x} = 0$ to prove det $A = 0$. If those entries add to one, show that det $(A - I) = 0$. Does this mean det $A = I$?

Solution

If $\boldsymbol{x} = (1, 1, \ldots, 1)$, then $A\vec{x} =$ the sums of the rows of $\boldsymbol{A}$. Since every row of A add to zero, that implies $A\vec{x} = 0$. Since A has non-zero nullspace, it is not invertible and det $A = 0$. If the entries in every row of A sum to one, then the entries in every row of $A - \mathrm{I}$ sum to zero. A – I has a non-zero nullspace and det $(A - \mathrm{I})$ =0. This does not mean that det $A = \mathrm{I}$.

Example:

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ every row of A adds up to zero

$\Rightarrow \det A = -1 \neq 1 = \det I$

Exercise

Does $\det(AB) = \det(BA)$ in general?

a) True or false if A and B are square n x n matrices?

b) True or false if A is m x n and B is n x m with $m \neq n$?

Solution

a) Matrices A and B are square matrices, then by the property:

$$\begin{aligned}\det(AB) &= \det(A)\det(B)\\ &= \det(B)\det(A)\\ &= \det(BA)\end{aligned}$$

Therefore; it is true for any A and B square matrices.

b) False, example if $A = \begin{pmatrix}1\\1\end{pmatrix}$ $\quad B = \begin{pmatrix}1 & 1\end{pmatrix}$

$$\begin{aligned}AB &= \begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix}\\ &= \begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\end{aligned}$$

$$\det(AB) = \begin{vmatrix}1 & 1\\1 & 1\end{vmatrix}$$

$$\underline{= 0}$$

$$BA = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = (2)$$

$$\det(BA) = 2$$

$$\det(AB) \neq \det(BA)$$

Exercise

True or false, with a reason if true or a counterexample if false:

a) The determinant of $I + A$ is 1 + det $\boldsymbol{A}$.

b) The determinant of ABC is $|A||B||C|$.

c) The determinant of 4A is $4|A|$

d) The determinant of $AB - BA$ is zero. (try an example)

e) If A is not invertible then AB is not invertible.

f) The determinant of $A-B$ equals to det A – det B.

Solution

a) ***False***, if $A=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$

$$\det(I+A)=\det\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$
$$=0$$

$$\det A=1$$

$$1+\det A=1+1$$
$$=2$$
$$\neq \det(I+A)$$

b) ***True***, $\det(ABC)=\det(A)\det(BC)=\det(A)\det(B)\det(C)$.

c) ***False***, in general $\det(4A)=4^n\det(A)$ if A is n x n.

d) ***False***, $A=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, $B=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

$$AB-BA=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}-\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$
$$=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}-\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$
$$=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

$$\det(AB-BA)=\begin{vmatrix}0 & 1\\ -1 & 0\end{vmatrix}$$
$$=1\neq 0$$

e) False, any matrix is invertible, iff its determinant is nonzero. So det $A=0$ which $\det(AB)=\det(A)\det(B)=0$. Therefore, AB can't be invertible.

f) $A=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \Rightarrow |A|=0$

$$B=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \Rightarrow |B|=-1$$

$$\det(A)-\det(B)=0-(-1)\quad =1$$

$$\det(A-B)=\det\begin{pmatrix}1 & -1\\ -1 & 0\end{pmatrix}\quad =-1$$

$$\Rightarrow\quad \det(A-B)\neq\det(A)-\det(B)$$

Exercise

Use row operations to show the 3 by 3 "Vandermonde determinant" is

$$\det\begin{bmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{bmatrix}=(b-a)(c-a)(c-b)$$

Solution

$$\det\begin{bmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{bmatrix}=\det\begin{bmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{bmatrix}\begin{matrix}\\ R_2-R_1\\ R_3-R_1\end{matrix}$$

$$=\det\begin{bmatrix}1 & a & a^2\\ 0 & b-a & b^2-a^2\\ 0 & c-a & c^2-a^2\end{bmatrix}\qquad \textit{factor}\ \ (b-a)$$

$$=(b-a)\det\begin{bmatrix}1 & a & a^2\\ 0 & 1 & b+a\\ 0 & c-a & c^2-a^2\end{bmatrix}\ R_2-(c-a)R_2$$

$$(c-a)(c+a)-(b+a)(c-a)=(c-a)(c+a-b-a)$$

$$=(b-a)\det\begin{bmatrix}1 & a & a^2\\ 0 & 1 & b+a\\ 0 & 0 & (c-a)(c-b)\end{bmatrix}\quad \textbf{\textit{Multiply the main diagonal by (b - a)}}$$

$$=(b-a)(c-a)(c-b)$$

Exercise

The inverse of a 2 by 2 matrix seems to have determinant = 1:

$$\det A^{-1} = \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{ad-bc}{ad-bc}$$
$$= 1$$

What is wrong with this calculation? What is the correct $\det A^{-1}$

Solution

The $\det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $(ad - bc)$ it is part of the determinant and it is not the solution.

$$\det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\det\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{ad-bc}\frac{1}{ad-bc}(ad-bc)$$
$$= \frac{1}{ad-bc}$$

Exercise

A *Hessenberg* matrix is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci's rule $|H_4| = |H_3| + |H_2|$. The same rule will continue for all sizes $|H_n| = |H_{n-1}| + |H_{n-2}|$. Which Fibonacci number is $|H_n|$?

$$H_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad H_3 = \begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad H_4 = \begin{bmatrix} 2 & 1 & & \\ \mathbf{1} & 2 & \mathbf{1} & \\ \mathbf{1} & 1 & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

Solution

$$|H_4| = 2C_{11} + 1C_{12}$$

The cofactor C_{11} for H_4 is the determinant $|H_3|$.

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

The cofactor $C_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$

$$= -\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= -\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} \qquad \left|H_2\right| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= -\left|H_3\right| + \left|H_2\right|$$

$$\begin{aligned} \left|H_4\right| &= 2C_{11} + 1C_{12} \\ &= 2\left|H_3\right| - \left|H_3\right| + \left|H_2\right| \\ &= \left|H_3\right| + \left|H_2\right| \end{aligned}$$

The actual number: $\left|H_2\right| = 3, \quad \left|H_3\right| = 5, \quad H_4 = 8.$

Since $\left|H_n\right|$ follows Fibonacci's rule $\left|H_{n-1}\right| + \left|H_{n-2}\right|$, it must be $\left|H_n\right| = F_{n+2}$.

Exercise

Evaluate $\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix}$

Solution

$$\begin{aligned} \begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix} &= -9 - (-6) \\ &= -3 \end{aligned}$$

Exercise

Evaluate $\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix} = -6-(0)$$
$$\boxed{=-6}$$

Exercise

Evaluate $\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix} = x(8x)-4x(2x)$$
$$=8x^2-8x^2$$
$$\boxed{=0}$$

Exercise

Evaluate $\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 2x \\ 4 & 3 \end{vmatrix} = 3x-2x(4)$$
$$=3x-8x$$
$$\boxed{=-5x}$$

Exercise

Evaluate $\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix}$

Solution

$$\begin{vmatrix} x^4 & 2 \\ x & -3 \end{vmatrix} = \boxed{-3x^4-2x}$$

Exercise

Evaluate $\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix}$

Solution

$$\begin{vmatrix} -8 & -5 \\ b & a \end{vmatrix} = \underline{-8a + 5b}$$

Exercise

Evaluate $\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix}$

Solution

$$\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix} = 15 - 14$$

$$\underline{= 1}$$

Exercise

Evaluate $\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix} = 5 - 20$$

$$\underline{= -16}$$

Exercise

Evaluate $\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix}$

Solution

$$\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix} = 15 + 6$$

$$\underline{= 21}$$

Exercise

Evaluate $\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix}$

Solution

$$\begin{vmatrix} -4 & -1 \\ 5 & 6 \end{vmatrix} = -24 + 5$$

$$\underline{= -19}$$

Exercise

Evaluate $\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix}$

Solution

$$\begin{vmatrix} \sqrt{3} & -2 \\ -3 & \sqrt{3} \end{vmatrix} = 3 - 6$$

$$\underline{= -3}$$

Exercise

Evaluate $\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix}$

Solution

$$\begin{vmatrix} \sqrt{7} & 6 \\ -3 & \sqrt{7} \end{vmatrix} = 7 + 18$$

$$\underline{= 25}$$

Exercise

Evaluate $\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} \sqrt{5} & 3 \\ -2 & 2 \end{vmatrix} \underline{= 2\sqrt{5} + 6}$$

Exercise

Evaluate $\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix}$

Solution

$$\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & -\frac{3}{4} \end{vmatrix} = -\frac{3}{8} - \frac{1}{16}$$

$$= -\frac{7}{16}$$

Exercise

Evaluate $\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix}$

Solution

$$\begin{vmatrix} \frac{1}{5} & \frac{1}{6} \\ -6 & -5 \end{vmatrix} = -1 + 1$$

$$= 0$$

Exercise

Evaluate $\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix}$

Solution

$$\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{3}{4} \end{vmatrix} = \frac{1}{2} + \frac{1}{6}$$

$$= \frac{2}{3}$$

Exercise

Evaluate $\begin{vmatrix} \lambda - 3 & 2 \\ 4 & \lambda - 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} \lambda-3 & 2 \\ 4 & \lambda-1 \end{vmatrix} \underline{= \lambda^2 - 4\lambda - 5}$$

Exercise

Evaluate $\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix}$

Solution

$$\begin{vmatrix} x & x^2 \\ 4 & x \end{vmatrix} = x^2 - 4x^2$$

$$\underline{= -3x^2}$$

Exercise

Evaluate $\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & x^2 \\ x & 9 \end{vmatrix} \underline{= 9x - x^3}$$

Exercise

Evaluate $\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} x^2 & x \\ -3 & 2 \end{vmatrix} \underline{= 2x^2 + 3x}$$

Exercise

Evaluate $\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix}$

Solution

$$\begin{vmatrix} x+2 & 6 \\ x-2 & 4 \end{vmatrix} = 4(x+2) - 6(x-2)$$

$$\underline{= -2x + 20}$$

Exercise

Evaluate $\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix}$

Solution

$$\begin{vmatrix} x+1 & -6 \\ x+3 & -3 \end{vmatrix} = -3x-3+6x+18$$

$$\underline{= -2x+20}$$

Exercise

Evaluate $\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} \underline{= 5}$$

Exercise

Evaluate $\begin{vmatrix} 5 & 3 \\ -6 & 3 \end{vmatrix}$

Solution

$$\begin{vmatrix} 5 & 3 \\ -6 & 3 \end{vmatrix} \underline{= 33}$$

Exercise

Evaluate $\begin{vmatrix} \sin\theta & 1 \\ 1 & \sin\theta \end{vmatrix}$

Solution

$$\begin{vmatrix} \sin\theta & 1 \\ 1 & \sin\theta \end{vmatrix} \underline{= \sin^2\theta - 1}$$

Exercise

Evaluate $\begin{vmatrix} 0 & 8 \\ 0 & 4 \end{vmatrix}$

Solution

$$\begin{vmatrix} 0 & 8 \\ 0 & 4 \end{vmatrix} \underline{= 0}$$

Exercise

Evaluate $\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$

Solution

$$\begin{aligned} \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} &= 3e^{5x} - 2e^{5x} \\ &\underline{= e^{5x}} \end{aligned}$$

Exercise

Evaluate $\begin{vmatrix} x & \ln x \\ 1 & \frac{1}{x} \end{vmatrix}$

Solution

$$\begin{vmatrix} x & \ln x \\ 1 & \frac{1}{x} \end{vmatrix} \underline{= 1 - \ln x}$$

Exercise

Evaluate $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$

Solution

$$\begin{aligned} \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} &= \cos^2\theta + \sin^2\theta \\ &\underline{= 1} \end{aligned}$$

Exercise

Evaluate $\begin{vmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{vmatrix} \underline{= -58}$$

Exercise

Evaluate $\begin{vmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{vmatrix} \underline{= k}$$

Exercise

Evaluate $\begin{vmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \underline{= 1}$$

Exercise

Evaluate $\begin{vmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{vmatrix} \underline{= -30}$$

Exercise

Evaluate $\begin{vmatrix} x & y & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & y & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 2x - 1 - 3y$$

Exercise

Evaluate $\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

$$= r$$

Exercise

Evaluate $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

Solution

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = (a+1)(b+1)(c+1) + 2 - (b+1) - (a+1) - (c+1)$$

$$= abc + ab + ac + a + bc + b + c + 1 - b - a - c - 1$$

$$= abc + ab + ac + bc$$

$$= abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

Exercise

Evaluate $\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix} \begin{matrix} 3 & 0 \\ 2 & 1 \\ 2 & 5 \end{matrix} \quad \underline{= 72}$$

Exercise

Evaluate $\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix}$

Solution

$$\begin{vmatrix} 4 & 0 & 0 \\ 3 & -1 & 4 \\ 2 & -3 & 6 \end{vmatrix} \begin{matrix} 4 & 0 \\ 3 & -1 \\ 2 & -3 \end{matrix} \quad \underline{= 24}$$

or $= 4\begin{vmatrix} -1 & 4 \\ -3 & 6 \end{vmatrix}$

Exercise

Evaluate $\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 3 & 1 & 0 \\ -3 & -4 & 0 \\ -1 & 3 & 5 \end{vmatrix} \begin{matrix} 3 & 1 \\ -3 & -4 \\ -1 & 3 \end{matrix}$$

$$\underline{= -45}$$

Exercise

Evaluate $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & -4 & 5 \end{vmatrix} \begin{matrix} 1 & 1 \\ 2 & 2 \\ 3 & -4 \end{matrix}$$

$$= 10 + 6 - 8 - 6 + 8 - 10$$

$$\underline{= 0}$$

Exercise

Evaluate $\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 0 & -1 \\ 2 & 1 & x^2 \\ -3 & x & 1 \end{vmatrix} \begin{matrix} x & 0 \\ 2 & 1 \\ -3 & x \end{matrix}$$

$$= x - 2x - 3 - x^4$$

$$\underline{= -x^4 - x - 3}$$

Exercise

Evaluate $\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix} \begin{matrix} x & 1 \\ x^2 & x \\ 0 & x \end{matrix}$$

$$= x^2 - x^3 - x^3 - x^2$$

$$\underline{= -2x^3}$$

Exercise

Evaluate $\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$

Solution

$$\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix} = 0+126+48-(-48+36+0)$$

$$\underline{=90}$$

Exercise

Evaluate $\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix} \underline{=10}$$

Exercise

Evaluate $\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix}$

Solution

$$\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix} \begin{matrix} 3 & 1 \\ -2 & 3 \\ 3 & 4 \end{matrix}$$

$$=-54+3-16-18-12-12$$

$$\underline{=-109}$$

Exercise

Evaluate $\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix} \begin{matrix} 2x & 1 \\ 0 & 4 \\ 3 & 0 \end{matrix}$$

$$= 16x + 3x + 12$$
$$\boxed{= 19x + 12}$$

Exercise

Evaluate $\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix}$

Solution

$$\begin{vmatrix} 0 & x & x \\ x & x^2 & 5 \\ x & 7 & -5 \end{vmatrix} \begin{matrix} 0 & x \\ x & x^2 \\ x & 7 \end{matrix}$$

$$= 5x^2 + 7x^2 - x^4 + 5x^2$$
$$\boxed{= 17x^2 - x^4}$$

Exercise

Evaluate $\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & x & 1 \\ -3 & 1 & 0 \\ 2 & 1 & 4 \end{vmatrix} \begin{matrix} 2 & x \\ -3 & 1 \\ 2 & 1 \end{matrix}$$

$$= 8 - 3 - 2 + 12x$$
$$\boxed{= 12x + 3}$$

Exercise

Evaluate $\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & x & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} \begin{matrix} 1 & x \\ 3 & 1 \\ 0 & -2 \end{matrix}$$

$$= 2 + 12 + 2 - 6x$$

$$\underline{= -6x + 16}$$

Exercise

Evaluate $\begin{vmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{vmatrix}$

Solution

$$\begin{array}{c} + \\ - \\ \\ \\ \end{array}\begin{vmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{vmatrix} = 5\begin{vmatrix} 6 & 4 & 12 \\ 2 & -3 & 4 \\ 1 & -2 & 2 \end{vmatrix} - 4\begin{vmatrix} 3 & 0 & 6 \\ 2 & -3 & 4 \\ 1 & -2 & 2 \end{vmatrix}$$

$$= 5(0) - 4(0)$$

$$\underline{= 0}$$

Exercise

Evaluate $\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} - (-)\begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= 3 + 2 + 3 + 0$$

$$\underline{= 0}$$

Exercise

Find $a)\ |A|\quad b)\ |B|\quad c)\ AB\quad d)\ |AB|\quad e)\ |A+B|$.

Then verify that $|A|\ |B| = |AB|$ & $|A|+|B| \neq |A+B|$

$$A = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

Solution

a) $|A| = \begin{vmatrix} -2 & 1 \\ 4 & -2 \end{vmatrix}$

$= 0$

b) $|B| = \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix}$

$= -1$

c) $AB = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} -2 & -3 \\ 4 & 6 \end{pmatrix}$

d) $|AB| = \begin{vmatrix} -2 & -3 \\ 4 & 6 \end{vmatrix}$

$= 0$

e) $A+B = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} -1 & 2 \\ 4 & -3 \end{pmatrix}$

$|A+B| = \begin{vmatrix} -1 & 2 \\ 4 & -3 \end{vmatrix}$

$= -4$

$|A|\ |B| = |AB| \quad = 0$

$|A|+|B| \quad = -1$

$|A+B| \quad = -4$

$|A|+|B| \neq |A+B|$

Exercise

Find $\quad a)\ |A| \quad b)\ |B| \quad c)\ AB \quad d)\ |AB| \quad e)\ |A+B|$

Then verify that $|A|\ |B| = |AB|$ & $|A| + |B| \neq |A+B|$

$$A = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$$

Solution

a) $|A| = \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix}$

$\quad = -2$

b) $|B| = \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix}$

$\quad = -2$

c) $AB = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$

$\quad = \begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix}$

d) $|AB| = \begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix}$

$\quad = 4$

e) $A + B = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$

$\quad = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$|A+B| = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$

$\quad = 0$

$|A|\ |B| = |AB| \quad = 4$

$|A| + |B| \quad = -4$

$|A+B| \quad = 0$

$|A| + |B| \neq |A+B|$

Exercise

Find $a)\ |A| \quad b)\ |B| \quad c)\ AB \quad d)\ |AB| \quad e)\ |A+B|$

Then verify that $|A|\ |B| = |AB|$ & $|A|+|B| \neq |A+B|$

$$A = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Solution

a) $|A| = \begin{vmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$

$= 2$

b) $|B| = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$

$= -6$

c) $AB = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}$

d) $|AB| = \begin{vmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 0 & 2 & 0 \end{vmatrix}$

$= -12$

e) $A+B = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} -2 & 2 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

$$|A+B| = \begin{vmatrix} -2 & 2 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= -15$$

$$|A|\,|B| = |AB| \quad = -12$$

$$|A| + |B| \quad = -4$$

$$|A+B| \quad = -15$$

$$|A| + |B| \neq |A+B|$$

Exercise

Find all the values of λ for which det($\boldsymbol{A}$) = 0: $A = \begin{bmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 4 \end{bmatrix}$

Solution

$$\begin{vmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 4 \end{vmatrix} = (\lambda - 1)(\lambda - 4) + 2$$

$$= \lambda^2 - 5\lambda + 4 + 2$$

$$= \lambda^2 - 5\lambda + 6 = 0 \qquad \textit{Solve for } \lambda.$$

$$\lambda_{1,\,2} = -1,\ 6$$

Exercise

Find all the values of λ for which $\det(A) = 0$: $A = \begin{pmatrix} \lambda + 2 & 2 \\ 1 & \lambda \end{pmatrix}$

Solution

$$|A| = \begin{vmatrix} \lambda + 2 & 2 \\ 1 & \lambda \end{vmatrix}$$

$$= \lambda^2 + 2\lambda - 2 = 0 \qquad \textit{Solve for } \lambda.$$

$$\lambda = \frac{-2 \pm \sqrt{12}}{2}$$

$$\lambda_{1,\,2} = -1 \pm \sqrt{3}$$

Exercise

Find all the values of λ for which $\det(A) = 0$: $\quad A = \begin{pmatrix} \lambda & 2 & 0 \\ 0 & \lambda+1 & 2 \\ 0 & 1 & \lambda \end{pmatrix}$

Solution

$$|A| = \begin{vmatrix} \lambda & 2 & 0 \\ 0 & \lambda+1 & 2 \\ 0 & 1 & \lambda \end{vmatrix}$$

$$= \lambda^2(\lambda+1) - 2\lambda$$

$$= \lambda\left(\lambda^2 + \lambda - 2\right) = 0$$

$$\underline{\lambda_{1,\,2} = 0,\ 1,\ 6-2}$$

Exercise

Find all the values of λ for which $\det(A) = 0$: $\quad A = \begin{bmatrix} \lambda-6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda-4 \end{bmatrix}$

Solution

$$\begin{vmatrix} \lambda-6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda-4 \end{vmatrix} = \lambda(\lambda-6)(\lambda-4) + 4(\lambda-6)$$

$$= \lambda\left(\lambda^2 - 10\lambda + 24\right) + 4\lambda - 24$$

$$= \lambda^3 - 10\lambda^2 + 24\lambda + 4\lambda - 24$$

$$= \lambda^3 - 10\lambda^2 + 28\lambda - 24$$

$$\lambda^3 - 10\lambda^2 + 28\lambda - 24 = 0$$

$$\underline{\lambda_{1,2,3} = 2,\ 2,\ 6}$$

Exercise

Use the fact that $|cA| = c^n |A|$ to evaluate the determinant of the $n \times n$ matrix

$$A = \begin{pmatrix} 5 & 15 \\ 10 & -20 \end{pmatrix}$$

Solution

$A: \ 2\times 2 \quad \rightarrow \quad n=2$

$$|A| = \begin{vmatrix} 5 & 15 \\ 10 & -20 \end{vmatrix}$$

$$= -250$$

$$|cA| = \begin{vmatrix} 5c & 15c \\ 10c & -20c \end{vmatrix}$$

$$= c\,c \begin{vmatrix} 5 & 15 \\ 10 & -20 \end{vmatrix}$$

$$= -250\,c^2$$

$$= c^2 |A|$$

$$|cA| = c^n |A| \qquad \checkmark$$

Exercise

Use the fact that $|cA| = c^n |A|$ to evaluate the determinant of the $n\times n$ matrix

$$A = \begin{pmatrix} -3 & 6 & 9 \\ 6 & 9 & 12 \\ 9 & 12 & 15 \end{pmatrix}$$

Solution

$A: \ 3\times 3 \quad \rightarrow \quad n=3$

$$|A| = \begin{vmatrix} -3 & 6 & 9 \\ 6 & 9 & 12 \\ 9 & 12 & 15 \end{vmatrix}$$

$$= 2$$

$$|cA| = \begin{vmatrix} -3c & 6c & 9c \\ 6c & 9c & 12c \\ 9c & 12c & 15c \end{vmatrix}$$

$$= c^3 \begin{vmatrix} -3 & 6 & 9 \\ 6 & 9 & 12 \\ 9 & 12 & 15 \end{vmatrix}$$

$$= (2)\,c^3$$

$$= c^3 |A|$$

$$|cA| = c^n |A| \qquad \checkmark$$

Exercise

Verify that $\left|A^{-1}\right| = \frac{1}{|A|}$ $\qquad A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

Solution

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$$

$$= 5$$

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$$

$$\left|A^{-1}\right| = \left(\frac{1}{5}\right)^2 \begin{vmatrix} 4 & -3 \\ -1 & 2 \end{vmatrix}$$

$$= \frac{1}{25}(5)$$

$$= \frac{1}{5}$$

Therefore, $\left|A^{-1}\right| = \frac{1}{|A|}$

Exercise

Verify that $\left|A^{-1}\right| = \frac{1}{|A|}$ $\qquad A = \begin{pmatrix} 2 & -2 & 3 \\ 1 & -1 & 2 \\ 3 & 0 & 3 \end{pmatrix}$

Solution

$$|A| = \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & 2 \\ 3 & 0 & 3 \end{vmatrix}$$

$$= -3$$

$$a_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} = -3 \qquad a_{12} = -\begin{vmatrix} -2 & 3 \\ 0 & 3 \end{vmatrix} = 6 \qquad a_{13} = \begin{vmatrix} -2 & 3 \\ -1 & 2 \end{vmatrix} = -1$$

$$a_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} = 3 \qquad a_{22} = \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} = -3 \qquad a_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -1$$

$$a_{31} = \begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} = 3 \qquad a_{32} = -\begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} = -6 \qquad a_{33} = \begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} = 0$$

$$A^{-1} = -\frac{1}{3}\begin{pmatrix} -3 & 6 & -1 \\ 3 & -3 & -1 \\ 3 & -6 & 0 \end{pmatrix}$$

$$\left|A^{-1}\right| = \left(-\frac{1}{3}\right)^3 \begin{vmatrix} -3 & 6 & -1 \\ 3 & -3 & -1 \\ 3 & -6 & 0 \end{vmatrix}$$

$$= -\frac{1}{27}(9)$$

$$= -\frac{1}{3}$$

Exercise

Prove that if a square matrix $\boldsymbol{A}$ has a column of zeros, then $\det(A) = 0$

Solution

Consider a 3 by 3 matrix with a zero column, however to find the determinant we can interchange any column of that matrix; therefore:

$$\boldsymbol{A} = [a_{ij}] = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$$

By definition, the determinant of $\boldsymbol{A}$ using the cofactor:

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= 0\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} 0 & a_{23} \\ 0 & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} 0 & a_{22} \\ 0 & a_{32} \end{vmatrix}$$

$$= 0$$

Exercise

With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D| \quad \textit{but} \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|$$

a) Why is the first statement true? Somehow B doesn't enter.

b) Show by example that equality fails (as shown) when C enters.

c) Show by example that the answer $\det(AD - CB)$ is also wrong.

Solution

a) If we don't pick any 0 entries, then the first two columns are picked from A and the last two rows are from D. We can't pick any columns or rows from B, because there isn't any left.

b) $$\left|\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right| = -\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$\underline{= -1}$$

and $A = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0, \quad B = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0, \quad C = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0, \quad D = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$

c) Use the example from part (*b*): $1 \neq 0$

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|$$

Exercise

Show that the value of the following determinant is independent of θ.

$$\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{vmatrix}$$

Solution

$$\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{vmatrix} = \begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix}$$

$$= \sin^2\theta + \cos^2\theta$$

$$= \sin^2\theta - \left(-\cos^2\theta\right)$$

$$\underline{= 1}$$

Therefore, the determinant is independent of θ.

Exercise

Show that the matrices $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ *and* $B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$ commute if and only if $\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$

Solution

$$AB = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$$

$$= \begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix}$$

$$BA = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

$$= \begin{pmatrix} da & db+ec \\ 0 & fc \end{pmatrix}$$

$$AB = BA \quad \Rightarrow \quad \begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix} = \begin{pmatrix} da & db+ec \\ 0 & fc \end{pmatrix}$$

Iff $ae + bf = db + ec$

$$\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = b(d-f) - e(a-c)$$

$$= bd - bf - ea + ec = 0$$

$$\underline{bd + ec = bf + ae} \quad \checkmark$$

Exercise

Show that $\det(A) = \frac{1}{2}\begin{vmatrix} tr(A) & 1 \\ tr(A^2) & tr(A) \end{vmatrix}$ for every 2×2 matrix A.

Solution

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad tr(A) = a + d$

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$$

$tr\left(A^2\right) = a^2 + bc + bc + d^2$.

$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$\quad = ad - bc$

$$\frac{1}{2}\begin{vmatrix} tr(A) & 1 \\ tr\left(A^2\right) & tr(A) \end{vmatrix} = \frac{1}{2}\begin{vmatrix} a+d & 1 \\ a^2+bc+bc+d^2 & a+d \end{vmatrix}$$

$$= \frac{1}{2}\left[(a+d)^2 - \left(a^2+bc+bc+d^2\right)\right]$$

$$= \frac{1}{2}\left(a^2+2ad+d^2-a^2-bc-bc-d^2\right)$$

$$= \frac{1}{2}(2ad-2bc)$$

$$= ad - bc$$

$$= \det(A)$$

Exercise

What is the maximum number of zeros that a 4×4 matrix can have without a zero determinant? Explain your reasoning.

Solution

The maximum number of zeros that a 4×4 matrix can have without a zero determinant is 12 zeros.

If the main diagonal has nonzero entries and the rest are zero, then the determinant of the matrix is equal to the product of the main diagonal entries.

Exercise

Evaluate $\det(A)$, $\det(E)$, and $\det(AE)$. Then verify that $\det(A)\cdot\det(E) = \det(AE)$

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & & \\ & 3 & \\ & & 1 \end{bmatrix}$$

Solution

$\det(A) = \begin{vmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{vmatrix}$

$\quad = -22$

$$\det(E) = \begin{vmatrix} 1 & & \\ & 3 & \\ & & 1 \end{vmatrix} \underline{= 3}$$

$$AE = \begin{bmatrix} 4 & 1 & 3 \\ 0 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 & 3 \\ 0 & -6 & 0 \\ 3 & 3 & 5 \end{bmatrix}$$

$$\det(AE) = \begin{vmatrix} 4 & 3 & 3 \\ 0 & -6 & 0 \\ 3 & 3 & 5 \end{vmatrix}$$

$$= -120 + 54$$

$$\underline{= -66}$$

$$\det(A)\det(E) = (-22)(3)$$

$$\underline{= -66}$$

$$\det(A)\det(E) = \det(AE) \qquad \checkmark$$

Exercise

Show that $\begin{bmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \\ 1 & 1 & 1 \end{bmatrix}$ is not invertible for any values of α, β, γ

Solution

$$\begin{vmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \\ 1 & 1 & 1 \end{vmatrix} = \sin^2\alpha\cos^2\beta + \sin^2\beta\cos^2\gamma + \sin^2\gamma\cos^2\alpha - \sin^2\gamma\cos^2\beta - \sin^2\alpha\cos^2\gamma - \cos^2\alpha\sin^2\beta$$

$$= \sin^2\alpha\left(\cos^2\beta - \cos^2\gamma\right) + \cos^2\alpha\left(\sin^2\gamma - \sin^2\beta\right) + \sin^2\beta\cos^2\gamma - \sin^2\gamma\cos^2\beta$$

$$= \sin^2\alpha\left(\cos^2\beta - \cos^2\gamma\right) + \cos^2\alpha\left(1 - \cos^2\gamma - 1 + \cos^2\beta\right) + \left(1 - \cos^2\beta\right)\cos^2\gamma - \left(1 - \cos^2\gamma\right)\cos^2\beta$$

$$= \sin^2\alpha\left(\cos^2\beta - \cos^2\gamma\right) + \cos^2\alpha\left(\cos^2\beta - \cos^2\gamma\right) + \cos^2\gamma - \cos^2\gamma\cos^2\beta - \cos^2\beta + \cos^2\gamma\cos^2\beta$$

$$=\left(\sin^2\alpha++\cos^2\alpha\right)\left(\cos^2\beta-\cos^2\gamma\right)+\cos^2\gamma-\cos^2\beta$$

$$=\cos^2\beta-\cos^2\gamma+\cos^2\gamma-\cos^2\beta$$

$$\underline{=0\,|}$$

Therefore, this matrix in not invertible.

Exercise

The determinant of a 2×2 matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det(A)=ad-bc$.

Assuming no rows swaps are required, perform elimination on A and show explicitly that $ad-bc$ is the product of the pivots.

Solution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\ aR_2-cR_1 \quad \to \quad \begin{pmatrix} a & b \\ 0 & ad-bc \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\ R_2-\frac{c}{a}R_1 \quad \to \quad \begin{pmatrix} a & b \\ 0 & d-\frac{bc}{a} \end{pmatrix}$$

$$\begin{vmatrix} a & b \\ 0 & d-\frac{bc}{a} \end{vmatrix}=a\left(d-\frac{bc}{a}\right)$$

$$=ad-bc$$

$$\underline{=\det(A)\,|}$$

Exercise

If A is a 7×7 matrix and let $\det(A)=17$. What is $\det\left(3A^2\right)$?

Solution

$$\det\left(A^2\right)=\det(A)\det(A)$$

$$=17^2$$

Multiplying a single row by 3 multiplies the determinant by 3.

Multiplying the whole 7×7 matrix by 3 multiplies all 7 rows by 3 $\Rightarrow$ 3^7.

$$\therefore\ \det\left(3A^2\right)=3^7\cdot 17^2$$

$$\underline{=632043\,|}$$

Exercise

Let A be an $n \times n$ real matrix.

a) Show that if $A^t = -A$ and n is odd, then $|A| = 0$.

b) Show that if $A^2 + I = 0$, then n must be even.

c) Does part (b) remain true for complex matrices?

Solution

a) Given: $A^t = -A$ and n is odd

$$\begin{aligned} |A| &= \left|A^t\right| \\ &= |-A| \\ &= (-1)^n |A| \qquad \text{Since } n \text{ is odd} \\ &= -|A| \end{aligned}$$

$|A| = -|A|$ only when $|A| = 0$

b) $A^2 + I = 0$

$A^2 = -I$

$$\begin{aligned} |A|^2 &= \left|A^2\right| \\ &= |-I| \\ &= (-1)^n \end{aligned}$$

If n is odd, then $\cancel{|A|^2 = -1}$ *impossible*

If n is even, then $|A|^2 = 1$

c) It can't be true because $|I| = -1 \in \mathbb{R}$

And A is real matrix, the determinant has to be a real number.

Exercise

Explain without computations why the following determinant is equal to zero

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & 0 & 0 & 0 \\ d_1 & d_2 & 0 & 0 & 0 \\ e_1 & e_2 & 0 & 0 & 0 \end{vmatrix}$$

Solution

The determinant is equal to zero because there are too many zeros (as block 3×3).

Or

$d_1R_5 - e_1R_4 \quad \to R_5 \Rightarrow \quad 0 \quad d_1e_2 - e_1d_2 \quad 0 \quad 0 \quad 0$

Since row 5 is has zero entries, therefore the determinant is zero.

Exercise

Let A and C be $m\times m$ and $n\times n$ matrices, respectively.

a) Show that $\begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = \begin{vmatrix} A & 0 \\ B & C \end{vmatrix} = |A||C|$

b) Evaluate

i. $\begin{vmatrix} I_m & 0 \\ 0 & I_n \end{vmatrix}$

ii. $\begin{vmatrix} 0 & I_m \\ I_n & 0 \end{vmatrix}$

iii. $\begin{vmatrix} I_m & B \\ 0 & I_n \end{vmatrix}$

iv. $\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix}$

c) Find a formula for $\begin{vmatrix} 0 & A \\ C & B \end{vmatrix}_{n\times n}$

Solution

a) If we let matrices B be $m\times n$ an **0** be $n\times m$, so the determinant of the matrix size will be $(m+n)\times(m+n)$, then

$$\begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = \left|\begin{array}{c|c} A_{m\times m} & B_{m\times n} \\ \hline 0_{n\times m} & C_{n\times n} \end{array}\right|$$

$$= |A||C| - |B||0|$$

$$= |A||C| - 0$$

$$= |A||C|$$

If we let matrices B be $n\times m$ an **0** be $m\times n$, so the determinant of the matrix size will be $(m+n)\times(m+n)$, then

$$\begin{vmatrix} A & 0 \\ B & C \end{vmatrix} = \left|\begin{array}{c|c} A & 0 \\ \hline B & C \end{array}\right|$$

$$= |A||C| - |B||0|$$

$$= |A||C| - 0$$

$$= |A||C|$$

b) i - $\begin{vmatrix} I_m & 0 \\ 0 & I_n \end{vmatrix} = \left|\begin{array}{c|c} I_m & 0_{m\times n} \\ \hline 0_{n\times m} & I_n \end{array}\right|$

$$= |I_m||I_n| - 0$$

$$= 1\bullet 1$$

$$= 1$$

ii - $\begin{vmatrix} 0 & I_m \\ I_n & 0 \end{vmatrix} = \left|\begin{array}{c|c} 0_{m\times n} & I_m \\ \hline I_n & 0_{n\times m} \end{array}\right|$

$$= -|I_m|\bullet(-1)|I_n|$$

$$= (-1)^{mn}$$

iii - $\begin{vmatrix} I_m & B \\ 0 & I_n \end{vmatrix} = \left|\begin{array}{c|c} I_m & B_{m\times n} \\ \hline 0_{n\times m} & I_n \end{array}\right|$

$$= |I_m||I_n| - 0$$

$$= 1\bullet 1$$

$$= 1$$

iv - $\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix}_{n\times n}$

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \qquad \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1$$

$$\begin{array}{cccc} + & - & + & - \end{array}$$

$$\left|\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right|_{4\times 4} = -\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -(-1) = 1$$

$$\begin{array}{ccccc} + & - & + & - & + \end{array}$$

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix}_{5\times 5} = +\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}_{4\times 4} = 1$$

From that we can see that the signs are: $-\quad -\quad +\quad +\quad -\quad -\quad +\quad +$

$$\begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix}_{n\times n} = (-1)^{\frac{n^2+3n}{2}}$$

c) $\begin{vmatrix} 0 & A \\ C & B \end{vmatrix}_{n\times n} = \left|\begin{array}{c|c} 0_{m\times n} & A_{m\times m} \\ \hline C_{n\times n} & B_{n\times m} \end{array}\right|$

$$= -\left|A_{m\times m}\right| \bullet (-1)\left|C_{n\times n}\right|$$

$$\underline{= (-1)^{mn}|A||C|}$$

Exercise

Let $f(x)=(p_1-x)(p_2-x)\ldots(p_n-x)$ and let

$$\Delta_n=\begin{vmatrix} p_1 & a & a & \cdots & a & a \\ b & p_2 & a & \cdots & a & a \\ b & b & p_3 & \cdots & a & a \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b & b & b & \cdots & p_{n-1} & a \\ b & b & b & \cdots & b & p_n \end{vmatrix}$$

a) Show that, if $a \neq b$,

$$\Delta_n=\frac{bf(a)-af(b)}{b-a}$$

b) Show that, if $a=b$,

$$\Delta_n=a\sum_{i=1}^{n} f_i(a)+p_n f_n(a)$$

Where $f_i(a)$ means $f(a)$ with factor (p_i-a) missing.

c) Use part (*b*) to evaluate

$$\begin{vmatrix} a & b & b & \cdots & b & b \\ b & a & b & \cdots & b & b \\ b & b & a & \cdots & b & b \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b & b & b & \cdots & a & b \\ b & b & b & \cdots & b & a \end{vmatrix}_{n\times n}$$

Solution

a) $\Delta_n=\dfrac{bf(a)-af(b)}{b-a}$; with $a \neq b$

Using the mathematical Induction to prove the equality.

For $n=2$:

$$\Delta_2=\begin{vmatrix} p_1 & a \\ b & p_2 \end{vmatrix}$$

$$=p_1p_2-ab$$

$$\Delta_2=\frac{bf(a)-af(b)}{b-a}$$

$$=\frac{b\left(p_1-a\right)\left(p_2-a\right)-a\left(p_1-b\right)\left(p_2-b\right)}{b-a}$$

$$=\frac{bp_1p_2-abp_1-abp_2+a^2b-ap_1p_2+abp_1+abp_2-ab^2}{b-a}$$

$$=\frac{bp_1p_2+a^2b-ap_1p_2-ab^2}{b-a}$$

$$=\frac{(b-a)p_1p_2+ab(a-b)}{b-a}$$

$$=\frac{(b-a)\left(p_1p_2-ab\right)}{b-a}$$

$$=p_1p_2-ab$$

For $n=2$, the proof is true.

Assume that is true for Δ_k

$$\Delta_k=\frac{bf(a)-af(b)}{b-a}$$

$$=\frac{b\left(\left(p_1-a\right)\ldots\left(p_{k-1}-a\right)\right)-a\left(\left(p_1-b\right)\ldots\left(p_{k-1}-b\right)\right)}{b-a}$$

$$f(x)=\left(p_1-x\right)\left(p_2-x\right)\ldots\left(p_k-x\right)$$

We need to prove it is also true for $\Delta_{k+1} \Rightarrow \Delta_{k+1}=\dfrac{bF(a)-aF(b)}{b-a}$

$$F(x)=\left(p_1-x\right)\left(p_2-x\right)\ldots\left(p_k-x\right)\left(p_{k+1}-x\right)$$

$$=f(x)\left(p_{k+1}-x\right)$$

$$\Delta_{k+1}=\frac{b\left(\left(p_1-a\right)\ldots\left(p_k-a\right)\left(p_{k+1}-a\right)\right)-a\left(\left(p_1-b\right)\ldots\left(p_k-b\right)\left(p_{k+1}-b\right)\right)}{b-a}$$

$$=\frac{bf(a)\left(p_{k+1}-a\right)-af(b)\left(p_{k+1}-b\right)}{b-a}$$

$$=\frac{bF(a)-aF(b)}{b-a} \quad \checkmark$$

Δ_{k+1} is also true.

∴ by the mathematical induction, the proof is completed.

b) If $a=b \rightarrow \Delta_n=a\sum_{i=1}^{n} f_i(a)+p_nf_n(a)$

Where $f_i(a)$ means $f(a)$ with factor $(p_i - a)$ missing.

$$\begin{aligned}
\Delta_n &= (p_1 - a)\Delta_{n-1} + a(p_2 - b)\cdots(p_n - b) && a = b \\
&= (p_1 - a)\Delta_{n-1} + a(p_2 - a)\cdots(p_n - a) && (p_1 - a) \text{ missing} \\
&= (p_1 - a)\left[(p_2 - a)\Delta_{n-2} + aF_2(a)\right] + af_1(a) \\
&= (p_1 - a)(p_2 - a)\Delta_{n-2} + a(p_1 - a)F_2(a) + af_1(a) \\
&= (p_1 - a)(p_2 - a)\Delta_{n-2} + af_2(a) + af_1(a) \\
&= (p_1 - a)(p_2 - a)(p_3 - a)\Delta_{n-3} + af_3(a) + af_2(a) + af_1(a) \\
&= (p_1 - a)(p_2 - a)(p_3 - a)\Delta_{n-3} + a\left(f_3(a) + f_2(a) + f_1(a)\right) \\
&\qquad \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \\
&= (p_1 - a)\cdots(p_{n-2} - a)\Delta_2 + a\left(f_{n-2}(a) + \cdots + f_1(a)\right)
\end{aligned}$$

$$\begin{aligned}
\Delta_2 &= \begin{vmatrix} p_{n-1} & a \\ a & p_n \end{vmatrix} \\
&= p_{n-1}p_n - a^2 \\
&= p_{n-1}p_n - ap_n + ap_n - a^2 \\
&= p_n(p_{n-1} - a) + a(p_n - a)
\end{aligned}$$

$$\begin{aligned}
\Delta_n &= \left[(p_1 - a)\cdots(p_{n-2} - a)\right]\left(p_n(p_{n-1} - a) + a(p_n - a)\right) + a\left(f_{n-2}(a) + \cdots + f_1(a)\right) \\
&= p_n(p_1 - a)\cdots(p_{n-2} - a)(p_{n-1} - a) + a(p_1 - a)\cdots(p_{n-2} - a)(p_n - a) + a\sum_{i=1}^{n-1} f_i(a) \\
&= p_n f_n(a) + af_{n-1}(a) + a\sum_{i=1}^{n-1} f_i(a) \\
&= p_n f_n(a) + a\sum_{i=1}^{n} f_i(a)
\end{aligned}$$

c) $f_n(x) = (p_1 - x)(p_2 - x)\ldots(p_{n-1} - x)$

$f_n(b) = (a - b)(a - b)\ldots(a - b)$

$p_n = a$

$$\begin{vmatrix} a & b & b & \dots & b & b \\ b & a & b & \dots & b & b \\ b & b & a & \dots & b & b \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b & b & b & \cdots & a & b \\ b & b & b & \dots & b & a \end{vmatrix}_{n\times n} = p_n f_n(b) + b\sum_{i=1}^{n} f_i(b)$$

$$= a(a-b)^{n-1} + b\left(\underbrace{(a-b)^{n-1} + \cdots + (a-b)^{n-1}}_{n-1}\right)$$

$$= a(a-b)^{n-1} + b(n-1)(a-b)^{n-1}$$

$$\boxed{= [a+(n-1)b](a-b)^{n-1}}$$

Exercise

Let $A, B, C, D \in M_n(\mathbb{C})$

a) Show that when A is invertible: $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\left|D - CA^{-1}B\right|$

b) Show that when $AC = CA$: $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|$

c) Can B and C on the right-hand side of the identity be switched?

d) Does part (b) remain true if the condition $AC = CA$ is dropped?

Solution

a) Since A in invertible, then A^{-1} exists and $AA^{-1} = A^{-1}A = I$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad A^{-1}R_1$$

$$\begin{pmatrix} I & A^{-1}B \\ C & D \end{pmatrix} \quad R_2 - CR_1$$

$$\begin{pmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix} \quad AR_1$$

$$\begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \left|\begin{array}{c|c} A & B \\ \hline 0 & D - CA^{-1}B \end{array}\right|$$

$$= |A| \left| D - CA^{-1}B \right|$$

b) When $AC = CA$

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A| \left| D - CA^{-1}B \right|$$

$$= \left| AD - ACA^{-1}B \right| \qquad AC = CA$$

$$= \left| AD - C\left(AA^{-1}\right)B \right|$$

$$= |AD - CIB|$$

$$= |AD - CB|$$

c) To switch B and C it is not necessary that $BC = CB$

Let $A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$AD = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$CB = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$BC = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$|AD - CB| = \left| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right|$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

$$= 0$$

$$|AD-BC| = \left|\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\right|$$

$$= \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= 1$$

$$|AD-CB| \neq |AD-BC|$$

No, B and C on the right-hand side of the identity cannot be switched since $|AD-CB| \neq |AD-BC|$

d) No, since from previous part (c) D doesn't commute necessarily.

Exercise

Show that the matrix A is invertible for all values of θ, then find A^{-1} using $A^{-1} = \frac{1}{\det(A)} adj(A)$

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

$$|A| = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos^2\theta + \sin^2\theta$$

$$= 1$$

$$a_{11} = \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta \qquad a_{12} = -\begin{vmatrix} \sin\theta & 0 \\ 0 & 1 \end{vmatrix} = -\sin\theta \qquad a_{13} = \begin{vmatrix} \sin\theta & 0 \\ \cos\theta & 0 \end{vmatrix} = 0$$

$$a_{21} = -\begin{vmatrix} -\sin\theta & 0 \\ 0 & 1 \end{vmatrix} = \sin\theta \qquad a_{22} = \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta \qquad a_{23} = -\begin{vmatrix} \cos\theta & 0 \\ -\sin\theta & 0 \end{vmatrix} = 0$$

$$a_{31} = \begin{vmatrix} -\sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix} = 0 \qquad a_{32} = -\begin{vmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{vmatrix} = 0 \qquad a_{33} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1$$

$$A^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise

Find the inverse matrix of using its adjoint. $A=\begin{pmatrix}1&1&1\\-2&1&0\\-4&0&1\end{pmatrix}$

Solution

$$|A|=\begin{vmatrix}1&1&1\\-2&1&0\\-4&0&1\end{vmatrix}$$

$$=7$$

$a_{11}=\begin{vmatrix}1&0\\0&1\end{vmatrix}=1$ $\quad a_{12}=-\begin{vmatrix}1&1\\0&1\end{vmatrix}=-1$ $\quad a_{13}=\begin{vmatrix}1&0\\0&1\end{vmatrix}=1$

$a_{21}=-\begin{vmatrix}-2&0\\-4&1\end{vmatrix}=2$ $\quad a_{22}=\begin{vmatrix}1&1\\-4&1\end{vmatrix}=5$ $\quad a_{23}=-\begin{vmatrix}1&1\\-2&0\end{vmatrix}=-2$

$a_{31}=\begin{vmatrix}-2&1\\-4&0\end{vmatrix}=4$ $\quad a_{32}=-\begin{vmatrix}1&1\\-4&0\end{vmatrix}=-4$ $\quad a_{33}=\begin{vmatrix}1&1\\-2&1\end{vmatrix}=3$

$$A^{-1}=\frac{1}{7}\begin{pmatrix}1&-1&1\\2&5&-2\\4&-4&3\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{7}&-\frac{1}{7}&\frac{1}{7}\\\frac{2}{7}&\frac{5}{7}&-\frac{2}{7}\\\frac{4}{7}&-\frac{4}{7}&\frac{3}{7}\end{pmatrix}$$

Exercise

Find the inverse matrix of using its adjoint. $A=\begin{pmatrix}3&3&1\\1&2&1\\2&-1&1\end{pmatrix}$

Solution

$$|A|=\begin{vmatrix}3&3&1\\1&2&1\\2&-1&1\end{vmatrix}$$

$$=7$$

$a_{11}=\begin{vmatrix}2&1\\-1&1\end{vmatrix}=3$ $\quad a_{12}=-\begin{vmatrix}3&1\\-1&1\end{vmatrix}=-4$ $\quad a_{13}=\begin{vmatrix}3&1\\2&1\end{vmatrix}=1$

$a_{21} = -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1$ $\qquad a_{22} = \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 1$ $\qquad a_{23} = -\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = -2$

$a_{31} = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5$ $\qquad a_{32} = -\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix} = 9$ $\qquad a_{33} = \begin{vmatrix} 3 & 3 \\ 1 & 2 \end{vmatrix} = 3$

$$A^{-1} = \frac{1}{7}\begin{pmatrix} 3 & -4 & 1 \\ 1 & 1 & -2 \\ -5 & 9 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{pmatrix}$$

Exercise

Find the inverse matrix of using its adjoint. $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

Solution

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= 1$$

$a_{11} = \begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} = 3$ $\qquad a_{12} = -\begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix} = -2$ $\qquad a_{13} = \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} = -4$

$a_{21} = -\begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix} = 3$ $\qquad a_{22} = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = -2$ $\qquad a_{23} = -\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = -5$

$a_{31} = \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = -1$ $\qquad a_{32} = -\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 1$ $\qquad a_{33} = \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 2$

$$A^{-1} = \begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix}$$

Exercise

Find the inverse matrix of using its adjoint. $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$

Solution

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{vmatrix}$$

$$= 1$$

$$a_{11} = \begin{vmatrix} -2 & 1 \\ -3 & 0 \end{vmatrix} = 3 \qquad a_{12} = -\begin{vmatrix} -1 & 1 \\ -3 & 0 \end{vmatrix} = -3 \qquad a_{13} = \begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} = 1$$

$$a_{21} = -\begin{vmatrix} 0 & 1 \\ -2 & 0 \end{vmatrix} = -2 \qquad a_{22} = \begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = 2 \qquad a_{23} = -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1$$

$$a_{31} = \begin{vmatrix} 0 & -2 \\ -2 & -3 \end{vmatrix} = -4 \qquad a_{32} = -\begin{vmatrix} 1 & -1 \\ -2 & -3 \end{vmatrix} = 5 \qquad a_{33} = \begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix} = -2$$

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{pmatrix}$$

Exercise

Find the inverse matrix of using its adjoint. $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{pmatrix}$

Solution

$$|A| = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{vmatrix}$$

$$= -5$$

$$a_{11} = \begin{vmatrix} 5 & 3 \\ 4 & 3 \end{vmatrix} = 3 \qquad a_{12} = -\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} = -10 \qquad a_{13} = \begin{vmatrix} 2 & -1 \\ 5 & 3 \end{vmatrix} = 11$$

$$a_{21} = -\begin{vmatrix} 3 & 3 \\ 2 & 3 \end{vmatrix} = -3 \qquad a_{22} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5 \qquad a_{23} = -\begin{vmatrix} 1 & -1 \\ 3 & 3 \end{vmatrix} = -6$$

$a_{31} = \begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} = 2$ $\quad a_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$ $\quad a_{33} = \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = -1$

$$A^{-1} = -\frac{1}{5}\begin{pmatrix} 3 & -10 & 11 \\ -3 & 5 & -6 \\ 2 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{pmatrix}$$

Exercise

Find the inverse matrix of using its adjoint. $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

Solution

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 4$$

$a_{11} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ $\quad a_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1$ $\quad a_{13} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1$

$a_{21} = -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1$ $\quad a_{22} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ $\quad a_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1$

$a_{31} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$ $\quad a_{32} = -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1$ $\quad a_{33} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

Exercise

Find the inverse matrix of using its adjoint. $A=\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 6\\ 0 & -4 & -12\end{pmatrix}$

Solution

$$|A|=\begin{vmatrix}1 & 0 & 0\\ 0 & 2 & 6\\ 0 & -4 & -12\end{vmatrix}$$

$$=0$$

$$A^{-1} \quad \nexists$$

Exercise

Find the inverse matrix of using its adjoint. $A=\begin{pmatrix}-3 & -5 & -7\\ 2 & 4 & 3\\ 0 & 1 & -1\end{pmatrix}$

Solution

$$|A|=\begin{vmatrix}-3 & -5 & -7\\ 2 & 4 & 3\\ 0 & 1 & -1\end{vmatrix}$$

$$=-3$$

$$a_{11}=\begin{vmatrix}4 & 3\\ 1 & -1\end{vmatrix}=-7 \qquad a_{12}=-\begin{vmatrix}-5 & -7\\ 1 & -1\end{vmatrix}=-12 \qquad a_{13}=\begin{vmatrix}-5 & -7\\ 4 & 3\end{vmatrix}=-13$$

$$a_{21}=-\begin{vmatrix}2 & 3\\ 0 & -1\end{vmatrix}=2 \qquad a_{22}=\begin{vmatrix}-3 & -7\\ 0 & -1\end{vmatrix}=3 \qquad a_{23}=-\begin{vmatrix}-3 & -7\\ 2 & 3\end{vmatrix}=-5$$

$$a_{31}=\begin{vmatrix}2 & 4\\ 0 & 1\end{vmatrix}=2 \qquad a_{32}=-\begin{vmatrix}-3 & -5\\ 0 & 1\end{vmatrix}=3 \qquad a_{33}=\begin{vmatrix}-3 & -5\\ 2 & 4\end{vmatrix}=-2$$

$$A^{-1}=-\frac{1}{3}\begin{pmatrix}-7 & -12 & 13\\ 2 & 3 & -5\\ 2 & 3 & -2\end{pmatrix}$$

$$=\begin{pmatrix}\frac{7}{3} & 4 & -\frac{13}{3}\\ -\frac{2}{3} & -1 & \frac{5}{3}\\ -\frac{2}{3} & -1 & \frac{2}{3}\end{pmatrix}$$

Exercise

Let A and B be $n \times n$ matrices such that $AB = I$. Prove that $|A| \neq 0$ and $|B| \neq 0$

Solution

$AB = I$

$|AB| = |I|$

$|A|\ |B| = 1$

Therefore, $|A| \neq 0$ and $|B| \neq 0$

Exercise

Let A and B be $n \times n$ matrices such that AB is singular. Prove that A or B is singular.

Solution

Since AB is singular $\Rightarrow \quad |AB| = 0$

$|A|\ |B| = 0$

That implies that either $|A| = 0$ or $|B| = 0$

Therefore, either A or B is singular.

Exercise

Find two 2×2 matrices such that $|A| + |B| = |A + B|$

Solution

Assume $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$|A| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$

$= 0$

$|B| = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}$

$= 0$

$|A| + |B| = 0$

$A + B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$|A+B| = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$$

$$\underline{= 0}$$

Therefore, $|A| + |B| = |A+B|$

Exercise

Verify the equation $\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = b^2(3a+b)$

Solution

$$\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = (a+b)^3 + a^3 + a^3 - 3a^2(a+b)$$

$$= a^3 + 3a^2b + 3ab^2 + b^3 + 2a^3 - 3a^3 - 3a^2b$$

$$= 3ab^2 + b^3$$

$$\underline{= b^2(3a+b)} \quad \checkmark$$

Exercise

Let A be an $n \times n$ matrix in which the entries of each row sum to zero. Find $|A|$

Solution

If $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$

$$|A| = \begin{vmatrix} a & -a \\ b & -b \end{vmatrix}$$

$$\underline{= 0}$$

If we added one row and equal to zero, then we will have a zero entry.
If we keep adding each row, then we will have a column with zero entries.
Therefore, $|A| = 0$

Exercise

Show that if A is an $n\times n$ matrix with entries 1 and -1, then $|A|$ is divisible by 2^{n-1}

Solution

Let A is an 2×2 matrix with entries 1 and -1, then

$$A=\begin{pmatrix}1 & 1\\ -1 & \pm1\end{pmatrix}$$

$$|A|=\begin{vmatrix}1 & -1\\ -1 & \pm1\end{vmatrix}$$

$$\begin{cases} =0 \\ =2\end{cases}$$

$$=2^1$$

$$\boxed{=2^{2-1}}$$

It is true for $n=2$

Let B is an 3×3 matrix with entries 1 and -1, then

$$B=\begin{pmatrix}1 & 1 & 1\\ -1 & \pm1 & \pm1\\ -1 & \pm1 & \pm1\end{pmatrix}$$

$$|B|=\begin{vmatrix}1 & 1 & 1\\ -1 & \pm1 & \pm1\\ -1 & \pm1 & \pm1\end{vmatrix}\quad \begin{matrix} R_2+R_1\\ R_3+R_1\end{matrix}$$

$$=\begin{vmatrix}1 & 1 & 1\\ 0 & 2a & 2b\\ 0 & 2c & 2d\end{vmatrix}$$

$$=1\begin{vmatrix}2a & 2b\\ 2c & 2d\end{vmatrix}$$

$$=2^2\begin{vmatrix}a & b\\ c & d\end{vmatrix}$$

$$\boxed{=2^{3-1}}$$

It is also true for $n=3$

For $n\times n$, then A can be written as: $\quad A=\begin{pmatrix}1 & *\\ -1 & B\end{pmatrix}$

Where the matrix B be with entries ±1 $\quad A=\left(\begin{array}{c|c}\pm1 & *\\ \hline 0 & B'\end{array}\right)$

Where the matrix B' $(n-1)\times(n-1)$ has with entries ± 2 *or* 0

Then, $|A| = \pm 1\ |B'|$

If we factor 2 from the matrix B'

$|A| = 2^{n-1}\ |B''|$

Therefore, $|A|$ is divisible by 2^{n-1}

Exercise

Show that (the Vandermonde determinant)

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} = \prod_{1\le i<j\le n} \left(a_j - a_i\right)$$

In Particular, If V is the $n\times n$ matrix, with $(i, j)-entry$ j^{i-1}, then

$$|V| = (n-1)\ (n-2)^2\ \cdots\ 2^{n-2}$$

Solution

Using the Mathematical induction to prove:

For $n=2$:

$$\begin{vmatrix} 1 & 1 \\ a_1 & a_2 \end{vmatrix} = a_2 - a_1 \quad \checkmark$$

It is true for $V_{2\times 2}$

Assume it is true for $V_{k\times k}$

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_k \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_k^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{k-1} & a_2^{k-1} & a_3^{k-1} & \cdots & a_k^{k-1} \end{vmatrix} = \prod_{1\le i<j\le k} \left(a_j - a_i\right)$$

$$= \left(a_2 - a_1\right)\left(a_3 - a_1\right)\left(a_3 - a_2\right)\cdots\left(a_k - a_{k-1}\right)$$

Is $V_{(k+1)\times(k+1)}$ also true

$$|V|_{(k+1)\times(k+1)} = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & \cdots & a_k & a_{k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{k-1} & a_2^{k-1} & \cdots & a_k^{k-1} & a_{k+1}^{k-1} \\ a_1^k & a_2^k & \cdots & a_k^k & a_{k+1}^k \end{vmatrix}$$

$$= \prod_{1\le i<j\le k+1} \left(a_j - a_i\right)$$

$$= \left(a_2 - a_1\right)\left(a_3 - a_1\right)\left(a_3 - a_2\right)\cdots\left(a_k - a_{k-1}\right)\left(a_{k+1} - a_1\right)\cdots\left(a_{k+1} - a_k\right)$$

$$= \left(\prod_{1\le i<j\le k} \left(a_j - a_i\right)\right)\left(a_{k+1} - a_1\right)\cdots\left(a_{k+1} - a_k\right)$$

$$= \left(\prod_{1\le i<j\le k} \left(a_j - a_i\right)\right)\prod_{1<i\le k}\left(a_{k+1} - a_i\right)$$

$$= \prod_{1<i<j\le k+1} \left(a_j - a_i\right) \quad \checkmark$$

$V_{(k+1)\times(k+1)}$ is also true

By the mathematical induction, the Vandermonde determinant is completed.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & \cdots & a_k & a_{k+1} \\ a_1^2 & a_2^2 & & a_k^2 & a_{k+1}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{k-1} & a_2^{k-1} & \cdots & a_k^{k-1} & a_{k+1}^{k-1} \\ a_1^k & a_2^k & \cdots & a_k^k & a_{k+1}^k \end{bmatrix} \quad \begin{matrix} \\ R_2 - a_1R_1 \\ R_3 - a_1R_2 \\ \vdots \\ R_{k-1} - a_1R_{k-2} \\ R_{k+1} - a_1R_{k-1} \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & a_2 - a_1 & \cdots & a_k - a_1 \\ 0 & a_2^2 - a_1 a_2 & \cdots & a_k^2 - a_1 a_k \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a_2^{k-2} - a_1 a_2^{k-3} & \cdots & a_k^{k-2} - a_1 a_k^{k-3} \\ 0 & a_2^{k-1} - a_1 a_2^{k-2} & \cdots & a_k^{k-1} - a_1 a_k^{k-2} \\ 0 & a_2^{k} - a_1 a_2^{k-1} & \cdots & a_k^{k} - a_1 a_k^{k-1} \end{bmatrix}$$

$$\begin{vmatrix} {\color{red}1} & 1 & \cdots & 1 & 1 \\ 0 & a_2 - a_1 & \cdots & a_k - a_1 & a_{k+1} - a_1 \\ 0 & a_2\left(a_2 - a_1\right) & \cdots & a_k\left(a_k - a_1\right) & a_{k+1}\left(a_{k+1} - a_1\right) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_2^{k-2}\left(a_2 - a_1\right) & \cdots & a_k^{k-2}\left(a_k - a_1\right) & a_{k+1}^{k-2}\left(a_{k+1} - a_1\right) \\ 0 & a_2^{k-1}\left(a_2 - a_1\right) & \cdots & a_k^{k-1}\left(a_k - a_1\right) & a_{k+1}^{k-1}\left(a_{k+1} - a_1\right) \end{vmatrix}$$

$$= {\color{red}1} \begin{vmatrix} a_2 - a_1 & \cdots & a_k - a_1 & a_{k+1} - a_1 \\ \left(a_2 - a_1\right)a_2 & \cdots & \left(a_k - a_1\right)a_k & \left(a_{k+1} - a_1\right)a_{k+1} \\ \vdots & \vdots & \vdots & \vdots \\ \left(a_2 - a_1\right)a_2^{k-2} & \cdots & \left(a_k - a_1\right)a_k^{k-2} & \left(a_{k+1} - a_1\right)a_{k+1}^{k-2} \\ \left(a_2 - a_1\right)a_2^{k-1} & \cdots & \left(a_k - a_1\right)a_k^{k-1} & \left(a_{k+1} - a_1\right)a_{k+1}^{k-1} \end{vmatrix}$$

$$= \left(a_2 - a_1\right)\cdots\left(a_{k+1} - a_1\right) \begin{vmatrix} 1 & \cdots & 1 & 1 \\ a_2 & \cdots & a_k & a_{k+1} \\ \vdots & \vdots & \vdots & \vdots \\ a_2^{k-2} & \cdots & a_k^{k-2} & a_{k+1}^{k-2} \\ a_2^{k-1} & \cdots & a_k^{k-1} & a_{k+1}^{k-1} \end{vmatrix}$$

$$= \prod_{1 < j \le k+1} \left(a_j - a_1\right) \left(\prod_{1 \le i < j \le k} \left(a_j - a_i\right) \right)$$

$$= \prod_{1 < i < j \le k+1} \left(a_j - a_i\right) \qquad {\color{blue}\checkmark}$$

$(i, j) - entry \quad j^{i-1}$

$$|V|_{n\times n} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & n \\ 1 & 2^2 & 3^2 & \cdots & n^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 2^n & 3^n & \cdots & n^n \end{vmatrix}$$

$$|V| \stackrel{?}{=} (n-1)\,(n-2)^2\ \cdots (n-(n-2))^{n-2}\,(n-(n-1))^{n-1}$$

$$\stackrel{?}{=} (n-1)\,(n-2)^2\ \cdots (2)^{n-2}\,(1)^{n-1}$$

$$\stackrel{?}{=} (n-1)\,(n-2)^2\ \cdots\ 2^{n-2}$$

From previous prove of the Vandermonde determinant.

$$|V| = (a_2 - a_1)(a_3 - a_1)(a_3 - a_2)\cdots(a_n - a_1)\cdots(a_n - a_{n-1})$$

$$= (2-1)(3-1)(3-2)\cdots(n-1)\cdots(n-(n-1))$$

$$= (1)(2)(1)(3)(2)(1)\cdots(n-1)\cdots(1)$$

$$= (1)(2)^2(3)^3\cdots(n-(n-1))^{n-1}$$

$$= (n-1)\,(n-2)^2\ \cdots (2)^{n-2}(1)^{n-1} \quad \checkmark$$

$$|V|_{2\times 2} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 1$$

$$= 2 - 1$$

$$= (n-1) \quad \checkmark$$

$$|V|_{3\times 3} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2^2 & 3^2 \end{vmatrix} \qquad = (a_2 - a_1)(a_3 - a_1)(a_3 - a_2)$$

$$= (2-1)(3-1)(3-2)$$

$$= (3-1)(2-1)(3-2)$$

$$= (3-1)(3-2)(3-2)$$

$$= (n-1)\,(n-2)^2 \quad \checkmark$$

$$|V|_{k\times k}=(k-1)\,(k-2)^2\,\cdots(k-(k-2))^{k-2}(k-(k-1))^{k-1}$$

$$|V|_{(k+1)\times(k+1)}=(k+1-1)\,(k+1-2)^2\,\cdots(k+1-(k+1-2))^{k+1-2}(k+1-(k+1-1))^{k+1-1}$$

$$=(k)\,(k-1)^2\,(k-2)^3\,\cdots(2)^{k-1}(1)^k$$

$$(k)\,(k-1)^2\,(k-2)^3\,\cdots(2)^{k-1}(1)^k=\left((k-1)\,(k-2)^2\,\cdots\,2^{k-2}\right)\left(k(k-1)\,(k-2)\,\cdots\,2^1(1)\right)$$

$$=|V|_{k\times k}\,\left(((k+1)-1)\,((k+1)-2)\,((k+1)-3)\,\cdots\,((k+1)-(k-1))\,((k+1)-k)\right)$$

$$=|V|_{k\times k}\left(\prod_{1<j\le k}((k+1)-j)\right)$$

$$|V|_{(k+1)\times(k+1)}$$

Exercise

Show that if $a\neq b$, then the determinant for $n\times n$ matrix is:

$$\begin{vmatrix} a+b & ab & 0 & \cdots & 0 & 0 \\ 1 & a+b & ab & \cdots & 0 & 0 \\ 0 & 1 & a+b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a+b & ab \\ 0 & 0 & 0 & \cdots & 1 & a+b \end{vmatrix}=\frac{a^{n+1}-b^{n+1}}{a-b}$$

What if $a=b$?

Solution

Using the Mathematical induction to prove:

For $n=2$

$$\begin{vmatrix} a+b & ab \\ 1 & a+b \end{vmatrix}=a^2+b^2+ab$$

$$=\left(a^2+b^2+ab\right)\frac{a-b}{a-b}$$

$$=\frac{a^3-b^3}{a-b}\qquad\checkmark$$

It is true for 2×2

Assume that it is true for $k\times k$

$$\begin{vmatrix} a+b & ab & 0 & \cdots & 0 & 0 \\ 1 & a+b & ab & \cdots & 0 & 0 \\ 0 & 1 & a+b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a+b & ab \\ 0 & 0 & 0 & \cdots & 1 & a+b \end{vmatrix} = \frac{a^{k+1}-b^{k+1}}{a-b}$$

Is it true for $(k+1)\times(k+1)$

$$\begin{vmatrix} a+b & ab & 0 & \cdots & 0 & 0 \\ 1 & a+b & ab & \cdots & 0 & 0 \\ 0 & 1 & a+b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a+b & ab \\ 0 & 0 & 0 & \cdots & 1 & a+b \end{vmatrix}_{(k+1)\times(k+1)} \overset{?}{=} \frac{a^{k+2}-b^{k+2}}{a-b}$$

$$\begin{array}{c} \begin{matrix} + & \quad - & & & & \end{matrix} \\ \begin{vmatrix} a+b & ab & 0 & \cdots & 0 & 0 \\ 1 & a+b & ab & \cdots & 0 & 0 \\ 0 & 1 & a+b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a+b & ab \\ 0 & 0 & 0 & \cdots & 1 & a+b \end{vmatrix}_{(k+1)\times(k+1)} \end{array}$$

$$=(a+b)\begin{vmatrix} a+b & ab & 0 & \cdots & 0 & 0 \\ 1 & a+b & ab & \cdots & 0 & 0 \\ 0 & 1 & a+b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & 1 & a+b & ab \\ 0 & 0 & 0 & \cdots & 1 & a+b \end{vmatrix}_{k\times k}$$

$$-ab\begin{vmatrix} 1 & ab & \cdots & 0 & 0 \\ 0 & a+b & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & 1 & a+b & ab \\ 0 & 0 & \cdots & 1 & a+b \end{vmatrix}_{k\times k}$$

$$= (a+b)\frac{a^{k+1}-b^{k+1}}{a-b} - ab(1)\begin{vmatrix} a+b & ab & \cdots & & 0 \\ 1 & a+b & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ \vdots & \vdots & \cdots & & \vdots \\ 0 & 0 & \cdots & a+b & ab \\ 0 & 0 & \cdots & 1 & a+b \end{vmatrix}_{(k-1)\times(k-1)}$$

$$= (a+b)\frac{a^{k+1}-b^{k+1}}{a-b} - ab\left(\frac{a^k-b^k}{a-b}\right)$$

$$= \frac{1}{a-b}\left(a^{k+2} - ab^{k+1} + ba^{k+1} - b^{k+2} - ba^{k+1} + ab^{k+1}\right)$$

$$= \frac{a^{k+2}-b^{k+2}}{a-b} \qquad \checkmark$$

It is also true for $(k+1)\times(k+1)$

By the mathematical induction, the determinant proof is completed.

if $a=b$

$$\begin{vmatrix} 2a & a^2 & 0 & \cdots & 0 & 0 \\ 1 & 2a & a^2 & \cdots & 0 & 0 \\ 0 & 1 & 2a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2a & a^2 \\ 0 & 0 & 0 & \cdots & 1 & 2a \end{vmatrix}$$

$$\begin{vmatrix} 2a & a^2 \\ 1 & 2a \end{vmatrix} \underline{= 3a^2}$$

$$\begin{vmatrix} 2a & a^2 & 0 \\ 1 & 2a & a^2 \\ 0 & 1 & 2a \end{vmatrix} \underline{= 4a^3}$$

$$\begin{vmatrix} 2a & a^2 & 0 & 0 \\ 1 & 2a & a^2 & 0 \\ 0 & 1 & 2a & a^2 \\ 0 & 0 & 1 & 2a \end{vmatrix} \underline{= 5a^4}$$

$$\begin{vmatrix} 2a & a^2 & 0 & \cdots & 0 & 0 \\ 1 & 2a & a^2 & \cdots & 0 & 0 \\ 0 & 1 & 2a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2a & a^2 \\ 0 & 0 & 0 & \cdots & 1 & 2a \end{vmatrix} \underline{= (n+1)a^n}$$

$$\begin{bmatrix} 2a & a^2 & 0 & \cdots & 0 & 0 \\ 1 & 2a & a^2 & \cdots & 0 & 0 \\ 0 & 1 & 2a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2a & a^2 \\ 0 & 0 & 0 & \cdots & 1 & 2a \end{bmatrix} \quad 2aR_2 - R_1$$

$$\begin{bmatrix} 2a & a^2 & 0 & \cdots & 0 & 0 \\ 0 & 3a^2 & 2a^3 & \cdots & 0 & 0 \\ 0 & 1 & 2a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2a & a^2 \\ 0 & 0 & 0 & \cdots & 1 & 2a \end{bmatrix} \quad \begin{matrix} \frac{1}{2a}R_1 \\ \\ 3a^2R_3 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & * & 0 & 0 & \cdots & 0 & 0 \\ 0 & 3a^2 & * & 0 & \cdots & 0 & 0 \\ 0 & 0 & 4a^3 & * & \cdots & 0 & 0 \\ 0 & 0 & 1 & 2a & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 2a & a^2 \\ 0 & 0 & \cdots & \cdots & \cdots & 1 & 2a \end{bmatrix} \quad \begin{matrix} \frac{1}{3a^2}R_2 \\ \\ 4a^3R_4 - R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & * & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & * & 0 & \cdots & 0 & 0 \\ 0 & 0 & 4a^3 & * & \cdots & 0 & 0 \\ 0 & 0 & 0 & 5a^4 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 2a & a^2 \\ 0 & 0 & \cdots & \cdots & \cdots & 1 & 2a \end{bmatrix} \quad \begin{matrix} \frac{1}{4a^3}R_3 \\ \frac{1}{5a^4}R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & * & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & * & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & * & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & na^{n-1} & (n-1)a^n \\ 0 & 0 & \cdots & \cdots & \cdots & 1 & 2a \end{bmatrix} \quad \left(na^{n-1}\right)R_n - R_{n-1}$$

$$\begin{bmatrix} 1 & * & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & * & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & * & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & * & 0 \\ 0 & 0 & \cdots & \cdots & 0 & na^{n-1} & * \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & (n+1)a^n \end{bmatrix} \quad \frac{1}{na^{n-1}}R_1$$

$$\begin{vmatrix} 1 & * & 0 & \cdots & 0 & 0 \\ 0 & 1 & * & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & * \\ 0 & 0 & \cdots & \cdots & 0 & (n+1)a^n \end{vmatrix} \quad \boxed{= (n+1)a^n}$$

Exercise

Let $a = \left(-a_0,\ -a_1,\ \cdots,\ -a_{n-2}\right)$ and let $A = \begin{pmatrix} 0 & I_{n-1} \\ a & -a_{n-1} \end{pmatrix}$

Show that $\quad |\lambda I - A| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$

Solution

$$A = \left(\begin{array}{c|c} 0_{(n-1)\times 1} & I_{(n-1)\times(n-1)} \\ \hline a_{1\times(n-1)} & -a_{n-1} \end{array}\right)$$

$$A = \left(\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \vdots & \vdots & 0 & 1 \\ \hdashline -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{array}\right)_{(n-1)\times(n-1)}$$

For A to be 2×2, then $n=2$:

$$a = -a_0$$

$$A = \begin{pmatrix} 0 & I_1 \\ a & -a_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ a_0 & \lambda + a_1 \end{vmatrix}$$

$$= \lambda^2 + a_1\lambda + a_0 \quad \checkmark$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ a_0 & a_1 & \lambda + a_2 \end{vmatrix}$$

$$= \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 \quad \checkmark$$

$$\lambda I - A = \left(\begin{array}{ccccc} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & \vdots \\ \vdots & 0 & \ddots & \cdots & \vdots \\ 0 & \vdots & \vdots & \lambda & -1 \\ \hdashline a_0 & a_1 & \cdots & a_{n-2} & \lambda + a_{n-1} \end{array}\right) \qquad \lambda R_{n-1} - a_0 R_1$$

$$\left(\begin{array}{cccccc}
\lambda & * & 0 & 0 & \cdots & 0 \\
0 & \lambda & * & 0 & \cdots & \vdots \\
\vdots & 0 & \lambda & * & \cdots & \vdots \\
\vdots & \vdots & \cdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \lambda & -1 \\
\hdashline
0 & a_1\lambda + a_0 & a_2\lambda & \cdots & a_{n-2}\lambda & \lambda^2 + a_{n-1}\lambda
\end{array}\right)
\qquad
\begin{array}{l}
\frac{1}{\lambda}R_1 \\ \\ \\ \\ \\
\lambda R_{n-1} - \left(\lambda a_1 + a_0\right) R_2
\end{array}$$

$$\left(\begin{array}{cccccc}
1 & 0 & * & * & \cdots & * \\
0 & \lambda & * & * & \cdots & \vdots \\
\vdots & 0 & \lambda & * & \cdots & \vdots \\
\vdots & \vdots & \cdots & \ddots & \ddots & * \\
0 & \cdots & \cdots & \cdots & \lambda & -1 \\
\hdashline
0 & 0 & a_2\lambda^2 + a_1\lambda + a_0 & \cdots & a_{n-2}\lambda^2 & \lambda^3 + a_{n-1}\lambda^2
\end{array}\right)
\qquad
\begin{array}{l}
\frac{1}{\lambda}R_2 \\ \\ \\ \\ \\
\lambda R_{n-1} - \left(a_2\lambda^2 + a_1\lambda + a_0\right) R_3
\end{array}$$

$$\left(\begin{array}{cccccc}
1 & 0 & * & * & \cdots & * \\
0 & 1 & * & * & \cdots & \vdots \\
\vdots & 0 & 1 & * & \cdots & \vdots \\
\vdots & \vdots & \cdots & \ddots & \ddots & * \\
0 & \cdots & \cdots & \cdots & \lambda & -1 \\
\hdashline
0 & 0 & 0 & a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 & a_{n-2}\lambda^3 & \lambda^4 + a_{n-1}\lambda^3
\end{array}\right)$$

$$\left(\begin{array}{cccccc}
1 & 0 & * & * & \cdots & * \\
0 & 1 & * & * & \cdots & \vdots \\
\vdots & 0 & 1 & * & \cdots & \vdots \\
\vdots & \vdots & \cdots & \ddots & \ddots & * \\
0 & \cdots & \cdots & \cdots & \lambda & -1 \\
\hdashline
0 & 0 & 0 & \cdots & a_{n-2}\lambda^{n-2} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 & \lambda^{n-1} + a_{n-1}\lambda^{n-2}
\end{array}\right)$$

$$\lambda R_{n-2} - \left(a_{n-2}\lambda^{n-2} + \cdots + a_2\lambda^2 + a_1\lambda + a_0\right) R_{n-1}$$

$$\left(\begin{array}{cccccc}
1 & 0 & * & * & \cdots & * \\
0 & 1 & * & * & \cdots & \vdots \\
\vdots & 0 & 1 & * & \cdots & \vdots \\
\vdots & \vdots & \cdots & \ddots & \ddots & * \\
0 & \cdots & \cdots & \cdots & 1 & * \\
\hdashline
0 & 0 & 0 & \cdots & 0 & \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_2\lambda^2 + a_1\lambda + a_0
\end{array}\right)$$

$$|\lambda I - A| = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0$$

Solution   *Section* 1.7 – Properties of Determinants: Cramer's Rule

Exercise

Use Cramer's Rule with ratios $\dfrac{\det B_j}{\det A}$ to solve $A\boldsymbol{x} = b$. Also find the inverse matrix $A^{-1} = \dfrac{C^T}{\det A}$.

Why is the solution $\boldsymbol{x}$ is the first part the same as column 3 of A^{-1}? Which cofactors are involved in computing that column $\boldsymbol{x}$?

$$Ax = b \quad is \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Find the volumes of the boxes whose edges are columns of $\boldsymbol{A}$ and then rows of A^{-1}.

Solution

$$|A| = \begin{vmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{vmatrix} = 2$$

$$|B_1| = \begin{vmatrix} 0 & 6 & 2 \\ 0 & 4 & 2 \\ 1 & 9 & 0 \end{vmatrix} = 4 \qquad |B_2| = \begin{vmatrix} 2 & 0 & 2 \\ 1 & 0 & 2 \\ 5 & 1 & 0 \end{vmatrix} = -2 \qquad |B_1| = \begin{vmatrix} 2 & 6 & 0 \\ 1 & 4 & 0 \\ 5 & 9 & 1 \end{vmatrix} = 2$$

$$x = \frac{4}{2} = 2; \quad y = \frac{-2}{2} = -1; \quad z = \frac{2}{2} = 1$$

The solution is: $(2, \ -1, \ 1)$

$$C_{11} = \begin{vmatrix} 4 & 2 \\ 9 & 0 \end{vmatrix} = -18 \qquad C_{12} = -\begin{vmatrix} 1 & 2 \\ 5 & 0 \end{vmatrix} = 10 \qquad C_{13} = \begin{vmatrix} 1 & 4 \\ 5 & 9 \end{vmatrix} = -11$$

$$C_{21} = -\begin{vmatrix} 6 & 2 \\ 9 & 0 \end{vmatrix} = 18 \qquad C_{22} = \begin{vmatrix} 2 & 2 \\ 5 & 0 \end{vmatrix} = -10 \qquad C_{23} = -\begin{vmatrix} 2 & 6 \\ 5 & 9 \end{vmatrix} = 12$$

$$C_{31} = \begin{vmatrix} 6 & 2 \\ 4 & 2 \end{vmatrix} = 4 \qquad C_{32} = -\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = -2 \qquad C_{33} = \begin{vmatrix} 2 & 6 \\ 1 & 4 \end{vmatrix} = 2$$

$$C = \begin{pmatrix} -18 & 10 & -11 \\ 18 & -10 & 12 \\ 4 & -2 & 2 \end{pmatrix}$$

$$C^T = \begin{pmatrix} -18 & 18 & 4 \\ 10 & -10 & -2 \\ -11 & 12 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} -18 & 18 & 4 \\ 10 & -10 & -2 \\ -11 & 12 & 2 \end{pmatrix} \qquad A^{-1} = \frac{C^T}{\det A}$$

$$= \begin{pmatrix} -9 & 9 & 2 \\ 5 & -5 & -1 \\ -\frac{11}{2} & 6 & 1 \end{pmatrix}$$

The solution $\boldsymbol{x}$ is the third column of A^{-1} because $\boldsymbol{b} = (0, 0, 1)$ is the third column of I.

The volume of the boxes whose edges are columns of $\boldsymbol{A} = \det(\boldsymbol{A}) = 2$.

Since $\left|A^T\right| = |A|$.

The box from rows of A^{-1} has volume $\left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{2}$

Exercise

Verify that $\det(AB) = \det(BA)$ and determine whether the equality $\det(A+B) = \det(A) + \det(B)$ holds

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

Solution

$$AB = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{bmatrix}$$

$$\det(AB) = \begin{vmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{vmatrix}$$

$$= -170$$

$$BA = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{bmatrix}$$

$$\det(BA) = \begin{vmatrix} -1 & -3 & 6 \\ 17 & 11 & 4 \\ 10 & 5 & 2 \end{vmatrix}$$

$$\underline{= -170\,|}$$

Thus, $\underline{\det(AB) = \det(BA)\,|}$

$$\det(A) = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$$\underline{= 10\,|}$$

$$\det(B) = \begin{vmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{vmatrix}$$

$$\underline{= -17\,|}$$

$$A + B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 3 \\ 10 & 5 & 2 \\ 5 & 0 & 3 \end{bmatrix}$$

$$\det(A+B) = \begin{vmatrix} 3 & 0 & 3 \\ 10 & 5 & 2 \\ 5 & 0 & 3 \end{vmatrix}$$

$$\underline{= -30\,|}$$

$$\det(A) + \det(B) = 10 - 17$$

$$= -7 \quad \neq -30$$

$$\underline{\neq \det(A+B)\,|}$$

Exercise

Verify that $\det(kA) = k^n \det(A) \quad A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}, \quad k = 2$

Solution

$$\det(A) = \begin{vmatrix} -1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$= -10$$

$$\det(2A) = \begin{vmatrix} -2 & 4 \\ 6 & 8 \end{vmatrix}$$

$$= -40$$

$$= 4(-10)$$

$$= 2^2(-10)$$

$$= k^2 \det(A)$$

Exercise

Verify that $\det(kA) = k^n \det(A) \quad A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}, \quad k = -2$

Solution

$$\det(A) = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{vmatrix}$$

$$= 56$$

$$\det(-2A) = \begin{vmatrix} -4 & 2 & -6 \\ -6 & -4 & -2 \\ -2 & -8 & -0 \end{vmatrix}$$

$$= -448$$

$$= (-2)^3(56)$$

$$= k^3 \det(A)$$

Exercise

Verify that $\det(kA) = k^n \det(A)$ $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}, \quad k = 3$

Solution

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= -7$$

$$\det(3A) = \begin{vmatrix} 3 & 3 & 3 \\ 0 & 6 & 9 \\ 0 & 3 & -6 \end{vmatrix}$$

$$= -189$$

$$= 3^3(-7)$$

$$= k^3 \det(A)$$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x + 2y = -4 \\ 2x - y = -5 \end{cases}$

Solution

$$D = \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -3 - 4 = -7$$

$$D_x = \begin{vmatrix} -4 & 2 \\ -5 & -1 \end{vmatrix} = 4 - (-10) = 14$$

$$D_y = \begin{vmatrix} 3 & -4 \\ 2 & -5 \end{vmatrix} = -15 - (-8) = -7$$

$$x = \frac{D_x}{D} = \frac{14}{-7} = -2$$

$$y = \frac{D_y}{D} = \frac{-7}{-7} = 1$$

$\therefore$ ***Solution***: $(-2,\ 1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix} = -29 \qquad D_x = \begin{vmatrix} 7 & 5 \\ -3 & -2 \end{vmatrix} = 1 \qquad D_y = \begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix} = -41$$

$$x = \frac{1}{-29} = -\frac{1}{29} \qquad x = \frac{D_x}{D}$$

$$y = \frac{41}{29} \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\left(-\frac{1}{29}, \frac{41}{29}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y=-4 \\ 2x-y=-5 \end{cases}$

Solution

$$D = \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -7 \qquad D_x = \begin{vmatrix} -4 & -5 \\ 2 & -1 \end{vmatrix} = 14 \qquad D_y = \begin{vmatrix} 3 & -4 \\ 2 & -5 \end{vmatrix} = -7$$

$$x = -\frac{14}{7} = -2 \qquad x = \frac{D_x}{D}$$

$$y = \frac{7}{7} = 1 \qquad y = \frac{D_y}{D}$$

Solution: $(-2, 1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+5y=7 \\ 5x-2y=-3 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix} = -29 \qquad D_x = \begin{vmatrix} 7 & 5 \\ -3 & -2 \end{vmatrix} = 1 \qquad D_y = \begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix} = -41$$

$$x = -\frac{1}{29} \qquad x = \frac{D_x}{D}$$

$y = \frac{41}{29}$ $\qquad\qquad\qquad$ $y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $\left(-\frac{1}{29}, \frac{41}{29}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x - 7y = -16 \\ 2x + 5y = 9 \end{cases}$

Solution

$D = \begin{vmatrix} 4 & -7 \\ 2 & 5 \end{vmatrix} = 34 \qquad D_x = \begin{vmatrix} -16 & -7 \\ 9 & 5 \end{vmatrix} = -17 \qquad D_y = \begin{vmatrix} 4 & -16 \\ 2 & 9 \end{vmatrix} = 68$

$x = -\frac{17}{34} = -\frac{1}{2}$ $\qquad\qquad\qquad$ $x = \frac{D_x}{D}$

$y = \frac{68}{34} = 2$ $\qquad\qquad\qquad$ $y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $\left(-\frac{1}{2}, 2\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x + 2y = 4 \\ 2x + y = 1 \end{cases}$

Solution

$D = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = -1 \qquad D_x = \begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} = 2 \qquad D_y = \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -5$

$x = -2$ $\qquad\qquad\qquad$ $x = \frac{D_x}{D}$

$y = 5$ $\qquad\qquad\qquad$ $y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $(-2, 5)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+4y=2 \\ 2x+5y=-1 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix}=7 \qquad D_x=\begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix}=14 \qquad D_y=\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}=-7$$

$$x=\frac{14}{7}=2 \qquad x=\frac{D_x}{D}$$

$$y=-\frac{7}{7}=-1 \qquad y=\frac{D_y}{D}$$

∴ ***Solution***: $(2,\ -1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 5x-2y=4 \\ -10x+4y=7 \end{cases}$

Solution

$$D=\begin{vmatrix} 5 & -2 \\ -10 & 4 \end{vmatrix}=0 \qquad D_y=\begin{vmatrix} 5 & 4 \\ -10 & 7 \end{vmatrix}=75\neq 0$$

∴ ***No Solution***

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & -4 \\ 5 & -20 \end{vmatrix}=0 \qquad D_y=\begin{vmatrix} 1 & -8 \\ 5 & -40 \end{vmatrix}=0$$

$$\frac{1}{5}\times \begin{cases} x-4y=-8 \\ 5x-20y=-40 \end{cases}$$

$$\begin{cases} x-4y=-8 \\ x-4y=-8 \end{cases}$$

∴ ***Solution***: $(4y-8,\ y)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$

Solution

$D = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3 \qquad D_x = \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} = -6 \qquad D_y = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 3$

$x = \frac{6}{3} = 2 \qquad x = \frac{D_x}{D}$

$y = -\frac{3}{3} = -1 \qquad y = \frac{D_y}{D}$

∴ ***Solution***: $(2, \ -1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x + 10y = -14 \\ 7x - 2y = -16 \end{cases}$

Solution

$D = \begin{vmatrix} 2 & 10 \\ 7 & -2 \end{vmatrix} = -74 \qquad D_x = \begin{vmatrix} -14 & 10 \\ -16 & -2 \end{vmatrix} = 188 \qquad D_y = \begin{vmatrix} 2 & -14 \\ 7 & -16 \end{vmatrix} = 66$

$x = -\frac{188}{74} = -\frac{94}{37} \qquad x = \frac{D_x}{D}$

$y = -\frac{66}{74} = -\frac{33}{37} \qquad y = \frac{D_y}{D}$

∴ ***Solution***: $\left(-\frac{94}{37}, \ -\frac{33}{37}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x - 3y = 24 \\ -3x + 9y = -1 \end{cases}$

Solution

$D = \begin{vmatrix} 4 & -3 \\ -3 & 9 \end{vmatrix} = 27 \qquad D_x = \begin{vmatrix} 24 & -3 \\ -1 & 9 \end{vmatrix} = 213 \qquad D_y = \begin{vmatrix} 4 & 24 \\ -3 & -1 \end{vmatrix} = 68$

$x = \frac{213}{27} = \frac{71}{9} \qquad x = \frac{D_x}{D}$

$\boxed{y = \frac{68}{27}}$ $\qquad y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $\boxed{\left(\frac{71}{9},\ \frac{68}{27}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x + 2y = 12 \\ 3x - 2y = 16 \end{cases}$

Solution

$D = \begin{vmatrix} 4 & 2 \\ 3 & -2 \end{vmatrix} = -14 \qquad D_x = \begin{vmatrix} 12 & 2 \\ 16 & -2 \end{vmatrix} = -56 \qquad D_y = \begin{vmatrix} 4 & 12 \\ 3 & 16 \end{vmatrix} = 28$

$\boxed{x = \frac{56}{14} = 4} \qquad x = \frac{D_x}{D}$

$\boxed{y = -\frac{28}{14} = -2} \qquad y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $\boxed{(4,\ -2)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + 2y = -1 \\ 4x - 2y = 6 \end{cases}$

Solution

$D = \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} = -10 \qquad D_x = \begin{vmatrix} -1 & 2 \\ 6 & -2 \end{vmatrix} = -10 \qquad D_y = \begin{vmatrix} 1 & -1 \\ 4 & 6 \end{vmatrix} = 10$

$\boxed{x = 1} \qquad x = \frac{D_x}{D}$

$\boxed{y = -1} \qquad y = \frac{D_y}{D}$

$\therefore$ ***Solution***: $\boxed{(1,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x - 2y = 5 \\ -10x + 2y = 4 \end{cases}$

Solution

$D = \begin{vmatrix} 1 & -2 \\ -10 & 2 \end{vmatrix} = -18 \qquad D_x = \begin{vmatrix} 5 & -2 \\ 4 & 2 \end{vmatrix} = 18 \qquad D_y = \begin{vmatrix} 1 & 5 \\ -10 & 4 \end{vmatrix} = 54$

$\underline{x = -1} \qquad x = \dfrac{D_x}{D}$

$\underline{y = -\dfrac{54}{18} = -3} \qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(-1,\ -3)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$

Solution

$\begin{matrix} \frac{1}{3}\times \\ \frac{1}{15}\times \end{matrix} \begin{cases} 12x + 15y = -27 \\ 30x - 15y = -15 \end{cases}$

$\begin{cases} 4x + 5y = -9 \\ 2x - y = -1 \end{cases}$

$D = \begin{vmatrix} 4 & 5 \\ 2 & -1 \end{vmatrix} = -14 \qquad D_x = \begin{vmatrix} -9 & 5 \\ -1 & -1 \end{vmatrix} = 14 \qquad D_y = \begin{vmatrix} 4 & -9 \\ 2 & -1 \end{vmatrix} = 14$

$\underline{x = -1} \qquad x = \dfrac{D_x}{D}$

$\underline{y = -1} \qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(-1,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$

Solution

$\begin{matrix} \frac{1}{4}\times \\ \frac{1}{4}\times \end{matrix} \begin{cases} 4x - 4y = -12 \\ 4x + 4y = -20 \end{cases}$

$\begin{cases} x - y = -3 \\ x + y = -5 \end{cases}$

$D = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \qquad D_x = \begin{vmatrix} -3 & -1 \\ -5 & 1 \end{vmatrix} = -8 \qquad D_y = \begin{vmatrix} 1 & -3 \\ 1 & -5 \end{vmatrix} = -2$

$\underline{x = -4} \qquad x = \dfrac{D_x}{D}$

$\underline{y = -1} \qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(-4,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + y = 7 \\ x - y = 3 \end{cases}$

Solution

$D = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$

$D_x = \begin{vmatrix} 7 & 1 \\ 3 & -1 \end{vmatrix} = -10 \qquad D_y = \begin{vmatrix} 1 & 7 \\ 1 & 3 \end{vmatrix} = -4$

$\underline{x = 5} \qquad x = \dfrac{D_x}{D}$

$\underline{y = 2} \qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(5,\ 2)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x + y = 3 \\ x - y = 3 \end{cases}$

Solution

$D = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$

$D_x = \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} = -6$

$D_y = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 3$

$\underline{x = 2} \qquad x = \dfrac{D_x}{D}$

$\underline{y=-1}\,\big|$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(2,\ -1)}\,\big|$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 12x+3y=15 \\ 2x-3y=13 \end{cases}$

Solution

$$D=\begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix}=-42$$

$$D_x=\begin{vmatrix} 15 & 3 \\ 13 & -3 \end{vmatrix}=-84 \qquad D_y=\begin{vmatrix} 12 & 15 \\ 2 & 13 \end{vmatrix}=126$$

$\underline{x=2}\,\big|$ $\qquad x=\dfrac{D_x}{D}$

$\underline{y=-3}\,\big|$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(2,\ -3)}\,\big|$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-2y=5 \\ 5x-y=-2 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & -2 \\ 5 & -1 \end{vmatrix}=9$$

$$D_x=\begin{vmatrix} 5 & -2 \\ -2 & -1 \end{vmatrix}=-9 \qquad D_y=\begin{vmatrix} 1 & 5 \\ 5 & -2 \end{vmatrix}=-27$$

$\underline{x=-1}\,\big|$ $\qquad x=\dfrac{D_x}{D}$

$\underline{y=-3}\,\big|$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(-1,\ -3)}\,\big|$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x-5y=17 \\ 2x+3y=3 \end{cases}$

Solution

$$D=\begin{vmatrix} 4 & -5 \\ 2 & 3 \end{vmatrix}=22$$

$$D_x=\begin{vmatrix} 17 & -5 \\ 3 & 3 \end{vmatrix}=66 \qquad D_y=\begin{vmatrix} 4 & 17 \\ 2 & 3 \end{vmatrix}=-22$$

$\underline{x=3}$ $\qquad x=\dfrac{D_x}{D}$

$\underline{y=-1}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(3,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y=2 \\ 2x+2y=3 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix}=2$$

$$D_x=\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix}=-2 \qquad D_y=\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}=5$$

$\underline{x=-1}$ $\qquad x=\dfrac{D_x}{D}$

$\underline{y=\frac{5}{2}}$ $\qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{\left(-1,\ \frac{5}{2}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-3y=4 \\ 3x-4y=12 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & -3 \\ 3 & -4 \end{vmatrix} = 5 \qquad D_x = \begin{vmatrix} 4 & -3 \\ 12 & -4 \end{vmatrix} = 20 \qquad D_y = \begin{vmatrix} 1 & 4 \\ 3 & 12 \end{vmatrix} = 0$$

$\underline{x = 4}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = 0}$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(4,\ 0)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x - 9y = 5 \\ 3x - 3y = 11 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & -9 \\ 3 & -3 \end{vmatrix} = 21 \qquad D_x = \begin{vmatrix} 5 & -9 \\ 11 & -3 \end{vmatrix} = 84 \qquad D_y = \begin{vmatrix} 2 & 5 \\ 3 & 11 \end{vmatrix} = 7$$

$\underline{x = 4}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = \frac{1}{3}}$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{\left(4,\ \frac{1}{3}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x - 4y = 4 \\ x + y = 6 \end{cases}$

Solution

$$D = \begin{vmatrix} 3 & -4 \\ 1 & 1 \end{vmatrix} = 7$$

$$D_x = \begin{vmatrix} 4 & -4 \\ 6 & 1 \end{vmatrix} = 28 \qquad D_y = \begin{vmatrix} 3 & 4 \\ 1 & 6 \end{vmatrix} = 14$$

$\underline{x = 4}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = 2}$ $\qquad y = \dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\underline{(4,\ 2)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x = 7y + 1 \\ 2x = 3y - 1 \end{cases}$

Solution

$$\begin{cases} 3x - 7y = 1 \\ 2x - 3y = -1 \end{cases}$$

$$D = \begin{vmatrix} 3 & -7 \\ 2 & -3 \end{vmatrix} = 5$$

$$D_x = \begin{vmatrix} 1 & -7 \\ -1 & -3 \end{vmatrix} = -10 \qquad D_y = \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} = -5$$

$$\underline{x = -2} \qquad x = \frac{D_x}{D}$$

$$\underline{y = -1} \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\underline{(-2,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x = 3y + 2 \\ 5x = 51 - 4y \end{cases}$

Solution

$$\begin{cases} 2x - 3y = 2 \\ 5x + 4y = 51 \end{cases}$$

$$D = \begin{vmatrix} 2 & -3 \\ 5 & 4 \end{vmatrix} = 23$$

$$D_x = \begin{vmatrix} 2 & -3 \\ 51 & 4 \end{vmatrix} = 161 \qquad D_y = \begin{vmatrix} 2 & 2 \\ 5 & 51 \end{vmatrix} = 92$$

$$\underline{x = 7} \qquad x = \frac{D_x}{D}$$

$$\underline{y = 4} \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\underline{(7,\ 4)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} y = -4x + 2 \\ 2x = 3y - 1 \end{cases}$

Solution

$$\begin{cases} 4x + y = 2 \\ 2x - 3y = -1 \end{cases}$$

$$D = \begin{vmatrix} 4 & 1 \\ 2 & -3 \end{vmatrix} = -14$$

$$D_x = \begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix} = -5 \qquad D_y = \begin{vmatrix} 4 & 2 \\ 2 & -1 \end{vmatrix} = -8$$

$$\boxed{x = \frac{5}{14}} \qquad x = \frac{D_x}{D}$$

$$\boxed{y = \frac{4}{7}} \qquad y = \frac{D_y}{D}$$

$\therefore$ ***Solution***: $\boxed{\left(\frac{15}{4},\ \frac{4}{7}\right)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x = 2 - 3y \\ 2y = 3 - 2x \end{cases}$

Solution

$$\begin{cases} 3x + 3y = 2 \\ 2x + 2y = 3 \end{cases}$$

$$D = \begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix} = 0 \qquad D_y = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5 \neq 0$$

$\therefore$ ***No Solution***

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + 2y - 3 = 0 \\ 12 = 8y + 4x \end{cases}$

Solution

$$\begin{cases} x + 2y = 3 \\ 4x + 8y = 12 \end{cases}$$

$$\begin{cases} x+2y=3 \\ x+2y=3 \end{cases}$$

$\therefore$ ***Solution:*** $\quad \underline{(3-2y,\ y)}$

Exercise

Use Cramer's rule to solve the system $\quad \begin{cases} 7x-2y=3 \\ 3x+y=5 \end{cases}$

Solution

$$D=\begin{vmatrix} 7 & -2 \\ 3 & 1 \end{vmatrix}=13$$

$$D_x=\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix}=13 \qquad D_y=\begin{vmatrix} 7 & 3 \\ 3 & 5 \end{vmatrix}=26$$

$\underline{x=1} \qquad x=\dfrac{D_x}{D}$

$\underline{y=2} \qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\quad \underline{(1,\ 2)}$

Exercise

Use Cramer's rule to solve the system $\quad \begin{cases} x_1+2x_2=5 \\ -x_1+x_2=1 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}\underline{=3}$$

$$D_x=\begin{vmatrix} 5 & 2 \\ 1 & 1 \end{vmatrix}\underline{=3} \qquad D_y=\begin{vmatrix} 1 & 5 \\ -1 & 1 \end{vmatrix}\underline{=6}$$

$\underline{x=1} \qquad x=\dfrac{D_x}{D}$

$\underline{y=2} \qquad y=\dfrac{D_y}{D}$

$\therefore$ ***Solution***: $\quad \underline{(1,\ 2)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x+2y-z=4 \\ 3x-2y+z=5 \\ 4x-5y-z=-1 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & 2 & -1 \\ 3 & -2 & 1 \\ 4 & -5 & -1 \end{vmatrix}=42 \qquad D_x=\begin{vmatrix} 4 & 2 & -1 \\ 5 & -2 & 1 \\ -1 & -5 & -1 \end{vmatrix}=63$$

$$D_y=\begin{vmatrix} 3 & 4 & -1 \\ 3 & 5 & 1 \\ 4 & -1 & -1 \end{vmatrix}=39 \qquad D_z=\begin{vmatrix} 3 & 2 & 4 \\ 3 & -2 & 5 \\ 4 & -5 & -1 \end{vmatrix}=99$$

$$x=\frac{63}{42}=\frac{3}{2} \qquad x=\frac{D_x}{D}$$

$$y=\frac{39}{42}=\frac{13}{14} \qquad y=\frac{D_y}{D}$$

$$z=\frac{99}{42}=\frac{33}{14} \qquad z=\frac{D_z}{D}$$

Solution: $\left(\frac{3}{2}, \frac{13}{14}, \frac{33}{14}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+y+z=2 \\ 2x+y-z=5 \\ x-y+z=-2 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}=-6 \qquad D_x=\begin{vmatrix} 2 & 1 & 1 \\ 5 & 1 & -1 \\ -2 & -1 & 1 \end{vmatrix}=-6$$

$$D_y=\begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & -1 \\ 1 & -2 & 1 \end{vmatrix}=-12 \qquad D_z=\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{vmatrix}=6$$

$$x=1 \qquad x=\frac{D_x}{D}$$

$$y=2 \qquad y=\frac{D_y}{D}$$

$\underline{z=-1}$ $\qquad z=\dfrac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(1,\ 2,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x+y+z=9 \\ -x-y+z=1 \\ 3x-y+z=9 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & 1 & 1 \\ -1 & -1 & 1 \\ 3 & -1 & 1 \end{vmatrix}\begin{matrix} 2 & 1 \\ -1 & -1 \\ 3 & -1 \end{matrix}=-2+3+1+3+2+1$$

$\underline{=8}$

$$D_x=\begin{vmatrix} 9 & 1 & 1 \\ 1 & -1 & 1 \\ 9 & -1 & 1 \end{vmatrix}\begin{matrix} 9 & 1 \\ 1 & -1 \\ 9 & -1 \end{matrix}$$

$=-9+9-1+9+9-1$

$\underline{=16}$

$$D_y=\begin{vmatrix} 2 & 9 & 1 \\ -1 & 1 & 1 \\ 3 & 9 & 1 \end{vmatrix}\begin{matrix} 2 & 9 \\ -1 & 1 \\ 3 & 9 \end{matrix}$$

$=2+27-9-3-18+9$

$\underline{=8}$

$$D_z=\begin{vmatrix} 2 & 1 & 9 \\ -1 & -1 & 1 \\ 3 & -1 & 9 \end{vmatrix}\begin{matrix} 2 & 1 \\ -1 & -1 \\ 3 & -1 \end{matrix}$$

$=-18+3+9+27+2+9$

$\underline{=32}$

$\underline{x=2}$ $\qquad x=\dfrac{D_x}{D}$

$\underline{y=1}$ $\qquad y=\dfrac{D_y}{D}$

$z=\dfrac{32}{8}\ \underline{=4}$ $\qquad z=\dfrac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(2,\ 1,\ 4)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3y - z = -1 \\ x + 5y - z = -4 \\ -3x + 6y + 2z = 11 \end{cases}$

Solution

$$D = \begin{vmatrix} 0 & 3 & -1 \\ 1 & 5 & -1 \\ -3 & 6 & 2 \end{vmatrix} \begin{matrix} 0 & 3 \\ 1 & 5 \\ -3 & 6 \end{matrix}$$

$$= 9 - 6 - 15 - 6$$

$$= -18$$

$$D_x = \begin{vmatrix} -1 & 3 & -1 \\ -4 & 5 & -1 \\ 11 & 6 & 2 \end{vmatrix} \begin{matrix} -1 & 3 \\ -4 & 5 \\ 11 & 6 \end{matrix}$$

$$= -10 - 33 + 24 + 55 - 6 + 24$$

$$= 54$$

$$D_y = \begin{vmatrix} 0 & -1 & -1 \\ 1 & -4 & -1 \\ -3 & 11 & 2 \end{vmatrix} \begin{matrix} 0 & -1 \\ 1 & -4 \\ -3 & 11 \end{matrix}$$

$$= -3 - 11 + 12 + 2$$

$$= 0$$

$$D_z = \begin{vmatrix} 0 & 3 & -1 \\ 1 & 5 & -4 \\ -3 & 6 & 11 \end{vmatrix} \begin{matrix} 0 & 3 \\ 1 & 5 \\ -3 & 6 \end{matrix}$$

$$= 36 - 6 - 15 - 33$$

$$= -18$$

$x = -3$ $\qquad$ $x = \dfrac{D_x}{D}$

$y = 0$ $\qquad$ $y = \dfrac{D_y}{D}$

$z = 1$ $\qquad$ $z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $(-3,\ 0,\ 1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+3y+4z=14 \\ 2x-3y+2z=10 \\ 3x-y+z=9 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & 3 & 4 \\ 2 & -3 & 2 \\ 3 & -1 & 1 \end{vmatrix}\begin{matrix} 1 & 3 \\ 2 & -3 \\ 3 & -1 \end{matrix}$$

$=-3+18-8+36+2-6$

$\underline{=39}$

$$D_x=\begin{vmatrix} 14 & 3 & 4 \\ 10 & -3 & 2 \\ 9 & -1 & 1 \end{vmatrix}\begin{matrix} 14 & 3 \\ 10 & -3 \\ 9 & -1 \end{matrix}$$

$=-42+54-40+108+28-30$

$\underline{=78}$

$$D_y=\begin{vmatrix} 1 & 14 & 4 \\ 2 & 10 & 2 \\ 3 & 9 & 1 \end{vmatrix}\begin{matrix} 1 & 14 \\ 2 & 10 \\ 3 & 9 \end{matrix}$$

$=10+84+72-120-18-28$

$\underline{=0}$

$$D_z=\begin{vmatrix} 1 & 3 & 14 \\ 2 & -3 & 10 \\ 3 & -1 & 9 \end{vmatrix}\begin{matrix} 1 & 3 \\ 2 & -3 \\ 3 & -1 \end{matrix}$$

$=-27+90-28+126+10-54$

$\underline{=117}$

$x=\frac{78}{39}\underline{=2}$ $\qquad x=\frac{D_x}{D}$

$\underline{y=0}$ $\qquad y=\frac{D_y}{D}$

$z=\frac{117}{39}\underline{=3}$ $\qquad z=\frac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(2,\ 0,\ 3)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+4y-z=20 \\ 3x+2y+z=8 \\ 2x-3y+2z=-16 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & 4 & -1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix}\begin{matrix} 1 & 4 \\ 3 & 2 \\ 2 & -3 \end{matrix}$$

$$=4+8+9+4+3-24$$

$$\underline{=4}$$

$$D_x=\begin{vmatrix} 20 & 4 & -1 \\ 8 & 2 & 1 \\ -16 & -3 & 2 \end{vmatrix}\begin{matrix} 20 & 4 \\ 8 & 2 \\ -16 & -3 \end{matrix}$$

$$=80-64+24-32+60-64$$

$$\underline{=4}$$

$$D_y=\begin{vmatrix} 1 & 20 & -1 \\ 3 & 8 & 1 \\ 2 & -16 & 2 \end{vmatrix}\begin{matrix} 1 & 20 \\ 3 & 8 \\ 2 & -16 \end{matrix}$$

$$=16+40+48+16+16-120$$

$$\underline{=16}$$

$$D_z=\begin{vmatrix} 1 & 4 & 20 \\ 3 & 2 & 8 \\ 2 & -3 & -16 \end{vmatrix}\begin{matrix} 1 & 4 \\ 3 & 2 \\ 2 & -3 \end{matrix}$$

$$=-32+64-180-80+24+192$$

$$\underline{=-12}$$

$$x=\frac{4}{4}\ \underline{=1} \qquad x=\frac{D_x}{D}$$

$$y=\frac{16}{4}\ \underline{=4} \qquad y=\frac{D_y}{D}$$

$$z=-\frac{12}{4}\ \underline{=-3} \qquad z=\frac{D_z}{D}$$

$\therefore$ ***Solution***: $\underline{(1,\ 4,\ -3)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} -2x+6y+7z=3 \\ -4x+5y+3z=7 \\ -6x+3y+5z=-4 \end{cases}$

Solution

$$D = \begin{vmatrix} -2 & 6 & 7 \\ -4 & 5 & 3 \\ -6 & 3 & 5 \end{vmatrix} \begin{matrix} -2 & 6 \\ -4 & 5 \\ -6 & 3 \end{matrix}$$

$$= -50 - 108 - 84 + 210 + 18 + 120$$

$$= 106$$

$$D_x = \begin{vmatrix} 3 & 6 & 7 \\ 7 & 5 & 3 \\ -4 & 3 & 5 \end{vmatrix} \begin{matrix} 3 & 6 \\ 7 & 5 \\ -4 & 3 \end{matrix}$$

$$= 75 - 72 + 147 + 140 - 27 - 210$$

$$= 53$$

$$D_y = \begin{vmatrix} -2 & 3 & 7 \\ -4 & 7 & 3 \\ -6 & -4 & 5 \end{vmatrix} \begin{matrix} -2 & 3 \\ -4 & 7 \\ -6 & -4 \end{matrix}$$

$$= -70 - 54 + 112 + 294 - 24 + 60$$

$$= 318$$

$$D_z = \begin{vmatrix} -2 & 6 & 3 \\ -4 & 5 & 7 \\ -6 & 3 & -4 \end{vmatrix} \begin{matrix} -2 & 6 \\ -4 & 5 \\ -6 & 3 \end{matrix}$$

$$= 40 - 252 - 36 + 90 + 42 - 96$$

$$= -212$$

$$x = \frac{53}{106} = \frac{1}{2} \qquad x = \frac{D_x}{D}$$

$$y = \frac{318}{106} = 3 \qquad y = \frac{D_y}{D}$$

$$z = -\frac{212}{106} = -2 \qquad z = \frac{D_z}{D}$$

$\therefore$ ***Solution***: $\left(\frac{1}{2},\ 3,\ -2\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x - y + z = 1 \\ 3x - 3y + 4z = 5 \\ 4x - 2y + 3z = 4 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -3 & 4 \\ 4 & -2 & 3 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & -3 \\ 4 & -2 \end{matrix}$$

$$= -18 - 16 - 6 + 12 + 16 + 9$$

$$= -3$$

$$D_x = \begin{vmatrix} 1 & -1 & 1 \\ 5 & -3 & 4 \\ 4 & -2 & 3 \end{vmatrix} \begin{matrix} 1 & -1 \\ 5 & -3 \\ 4 & -2 \end{matrix}$$

$$= -9 - 16 - 10 + 12 + 8 + 15$$

$$= 0$$

$$D_y = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 5 & 4 \\ 4 & 4 & 3 \end{vmatrix} \begin{matrix} 2 & 1 \\ 3 & 5 \\ 4 & 4 \end{matrix}$$

$$= 30 + 16 + 12 - 20 - 32 - 9$$

$$= -3$$

$$D_z = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -3 & 5 \\ 4 & -2 & 4 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & -3 \\ 4 & -2 \end{matrix}$$

$$= -24 - 20 - 6 + 12 + 20 + 12$$

$$= -6$$

$x = 0$ $\qquad$ $x = \dfrac{D_x}{D}$

$y = 1$ $\qquad$ $y = \dfrac{D_y}{D}$

$z = 2$ $\qquad$ $z = \dfrac{D_z}{D}$

∴ ***Solution***: $(0,\ 1,\ 2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x-4y+4z=7 \\ x-y-2z=2 \\ 2x-3y+6z=5 \end{cases}$

Solution

$$D=\begin{vmatrix} 3 & -4 & 4 \\ 1 & -1 & -2 \\ 2 & -3 & 6 \end{vmatrix}\begin{matrix} 3 & -4 \\ 1 & -1 \\ 2 & -3 \end{matrix}$$

$$=-18+16-12+8-18+24$$

$$=0$$

$$D_z=\begin{vmatrix} 3 & -4 & 7 \\ 1 & -1 & 2 \\ 2 & -3 & 5 \end{vmatrix}\begin{matrix} 3 & -4 \\ 1 & -1 \\ 2 & -3 \end{matrix}$$

$$=-15-16-21+14+18+20$$

$$=0$$

$$-3\times(2)\quad \begin{cases} -3x+3y+6z=-6 \\ 2x-3y+6z=5 \end{cases}$$

$$-x+12z=-1$$

$$x=12z+1$$

$$(2)\rightarrow \quad y=12z+1-2z-2$$

$$=10z-1$$

$\therefore$ ***Solution***: $(12z+1,\ 10z-1,\ z)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x-2y-z=2 \\ 2x-y+z=4 \\ -x+y+z=4 \end{cases}$

Solution

$$D=\begin{vmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix}\begin{matrix} 1 & -2 \\ 2 & -1 \\ -1 & 1 \end{matrix}$$

$$=-1+2-2+1-1+4$$

$$=3$$

$$D_x = \begin{vmatrix} 2 & -2 & -1 \\ 4 & -1 & 1 \\ 4 & 1 & 1 \end{vmatrix} \begin{matrix} 2 & -2 \\ 4 & -1 \\ 4 & 1 \end{matrix}$$

$$= -2-8-4-4-2+8$$

$$\underline{= -12}$$

$$D_y = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 4 & 1 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & 4 \\ -1 & 4 \end{matrix}$$

$$= 4-2-8-4-4-4$$

$$\underline{= -18}$$

$$D_z = \begin{vmatrix} 1 & -2 & 2 \\ 2 & -1 & 4 \\ -1 & 1 & 4 \end{vmatrix} \begin{matrix} 1 & -2 \\ 2 & -1 \\ -1 & 1 \end{matrix}$$

$$= -4+8+4-2-4+16$$

$$\underline{= 18}$$

$\underline{x = -4}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = -6}$ $\qquad y = \dfrac{D_y}{D}$

$\underline{z = 6}$ $\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(-4,\ -6,\ 6)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x+y+z=3 \\ -y+2z=1 \\ -x+z=0 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & 1 \end{vmatrix} \begin{matrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{matrix}$$

$$\underline{= -4}$$

$$D_x = \begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} 3 & 1 \\ 1 & -1 \\ 0 & 0 \end{matrix}$$

$$=-4$$

$$D_y=\left|\begin{array}{ccc}1&3&1\\0&1&2\\-1&0&1\end{array}\right|\begin{array}{cc}1&1\\0&-1\\-1&0\end{array}$$

$$=-4$$

$$D_z=\left|\begin{array}{ccc}1&1&3\\0&-1&1\\-1&0&0\end{array}\right|\begin{array}{cc}1&1\\0&-1\\-1&0\end{array}$$

$$=-4$$

$x=1$ $\qquad\qquad x=\dfrac{D_x}{D}$

$y=1$ $\qquad\qquad y=\dfrac{D_y}{D}$

$z=1$ $\qquad\qquad z=\dfrac{D_z}{D}$

∴ ***Solution***: $(1,\ 1,\ 1)$

Exercise

Use Cramer's rule to solve the system $\begin{cases}3x+y+3z=14\\7x+5y+8z=37\\x+3y+2z=9\end{cases}$

Solution

$$D=\left|\begin{array}{ccc}3&1&3\\7&5&8\\1&3&2\end{array}\right|\begin{array}{cc}3&1\\7&5\\1&3\end{array}$$

$$=30+8+62-15-72-14$$

$$=0$$

$$D_z=\left|\begin{array}{ccc}3&1&14\\7&5&37\\1&3&9\end{array}\right|\begin{array}{cc}3&1\\7&5\\1&3\end{array}$$

$$=135+37+294-70-333-63$$

$$=0$$

$$\begin{array}{r}-3\times(1)\\(3)\end{array}\begin{cases}-9x-3y-9z=-42\\\quad x+3y+2z=9\end{cases}$$

$$\overline{\qquad\qquad -8x-7z=-33}$$

$$\underline{x=-\frac{7}{8}z+\frac{33}{8}}$$

$$(1)\rightarrow \quad y=14-3z-3\left(-\frac{7}{8}z+\frac{33}{8}\right)$$

$$=\frac{13}{8}-\frac{3}{8}z$$

$\therefore$ ***Solution***: $\left(\frac{33}{8}-\frac{7}{8}z,\ \frac{13}{8}-\frac{3}{8}z,\ z\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x-2y+z=7 \\ x+y+z=-2 \\ 4x+2y+z=3 \end{cases}$

Solution

$$D=\begin{vmatrix} 4 & -2 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix}\begin{matrix} 4 & -2 \\ 1 & 1 \\ 4 & 2 \end{matrix}$$

$$=-12$$

$$D_x=\begin{vmatrix} 7 & -2 & 1 \\ -2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix}\begin{matrix} 7 & -2 \\ -2 & 1 \\ 3 & 2 \end{matrix}$$

$$=-24$$

$$D_y=\begin{vmatrix} 4 & 7 & 1 \\ 1 & -2 & 1 \\ 4 & 3 & 1 \end{vmatrix}\begin{matrix} 4 & 7 \\ 1 & -2 \\ 4 & 3 \end{matrix}$$

$$=12$$

$$D_z=\begin{vmatrix} 4 & -2 & 7 \\ 1 & 1 & -2 \\ 4 & 2 & 3 \end{vmatrix}\begin{matrix} 4 & -2 \\ 1 & 1 \\ 4 & 2 \end{matrix}$$

$$=36$$

$x=2$ $\qquad x=\dfrac{D_x}{D}$

$y=-1$ $\qquad y=\dfrac{D_y}{D}$

$z=-3$ $\qquad z=\dfrac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(2,\ -1,\ -3)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2y - z = 7 \\ x + 2y + z = 17 \\ 2x - 3y + 2z = -1 \end{cases}$

Solution

$$D = \begin{vmatrix} 0 & 2 & -1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix} \begin{matrix} 0 & 2 \\ 1 & 2 \\ 2 & 3 \end{matrix}$$

$$\underline{=1}$$

$$D_x = \begin{vmatrix} 7 & 2 & -1 \\ 17 & 2 & 1 \\ -1 & 3 & 2 \end{vmatrix} \begin{matrix} 7 & 2 \\ 17 & 2 \\ -1 & 3 \end{matrix}$$

$$\underline{=-116}$$

$$D_y = \begin{vmatrix} 0 & 7 & -1 \\ 1 & 17 & 1 \\ 2 & -1 & 2 \end{vmatrix} \begin{matrix} 0 & 7 \\ 1 & 17 \\ 2 & -1 \end{matrix}$$

$$\underline{=35}$$

$$D_z = \begin{vmatrix} 0 & 2 & 7 \\ 1 & 2 & 17 \\ 2 & 3 & -1 \end{vmatrix} \begin{matrix} 0 & 2 \\ 1 & 2 \\ 2 & 3 \end{matrix}$$

$$\underline{=63}$$

$\underline{x = -116}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = 35}$ $\qquad y = \dfrac{D_y}{D}$

$\underline{z = 63}$ $\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $\underline{(-116,\ 35,\ 63)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x-2y+z=-4 \\ 6x+4y-3z=-24 \\ x-2y+2z=1 \end{cases}$

Solution

$$D=\begin{vmatrix} 2 & -2 & 1 \\ 6 & 4 & -3 \\ 1 & -2 & 2 \end{vmatrix}\begin{matrix} 2 & -2 \\ 6 & 4 \\ 1 & -2 \end{matrix}$$

$$=18$$

$$D_x=\begin{vmatrix} -4 & -2 & 1 \\ -24 & 4 & -3 \\ 1 & -2 & 2 \end{vmatrix}\begin{matrix} -4 & -2 \\ -24 & 4 \\ 1 & -2 \end{matrix}$$

$$=-54$$

$$D_y=\begin{vmatrix} 2 & -4 & 1 \\ 6 & -24 & -3 \\ 1 & 1 & 2 \end{vmatrix}\begin{matrix} 2 & -4 \\ 6 & -24 \\ 1 & 1 \end{matrix}$$

$$=0$$

$$D_z=\begin{vmatrix} 2 & -2 & -4 \\ 6 & 4 & -24 \\ 1 & -2 & 1 \end{vmatrix}\begin{matrix} 2 & -2 \\ 6 & 4 \\ 1 & -2 \end{matrix}$$

$$=36$$

$x=-3$ $\qquad x=\dfrac{D_x}{D}$

$y=0$ $\qquad y=\dfrac{D_y}{D}$

$z=2$ $\qquad z=\dfrac{D_z}{D}$

$\therefore$ ***Solution***: $(-3,\ 0,\ 2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 9x+3y+z=4 \\ 16x+4y+z=2 \\ 25x+5y+z=2 \end{cases}$

Solution

$$D=\begin{vmatrix} 9 & 3 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \end{vmatrix}\begin{matrix} 9 & 3 \\ 16 & 4 \\ 25 & 5 \end{matrix}$$

$$=-2$$

$$D_x=\begin{vmatrix} 4 & 3 & 1 \\ 2 & 4 & 1 \\ 2 & 5 & 1 \end{vmatrix}\begin{matrix} 4 & 3 \\ 2 & 4 \\ 2 & 5 \end{matrix}$$

$$=-2$$

$$D_y=\begin{vmatrix} 9 & 4 & 1 \\ 16 & 2 & 1 \\ 25 & 2 & 1 \end{vmatrix}\begin{matrix} 9 & 4 \\ 16 & 2 \\ 25 & 2 \end{matrix}$$

$$=18$$

$$D_z=\begin{vmatrix} 9 & 3 & 4 \\ 16 & 4 & 2 \\ 25 & 5 & 2 \end{vmatrix}\begin{matrix} 9 & 3 \\ 16 & 4 \\ 25 & 5 \end{matrix}$$

$$=-44$$

$x=1$ $\qquad x=\dfrac{D_x}{D}$

$y=-9$ $\qquad y=\dfrac{D_y}{D}$

$z=22$ $\qquad z=\dfrac{D_z}{D}$

$\therefore$ ***Solution***: $(1,\ -9,\ 22)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x-y+2z=-8 \\ x+2y-3z=9 \\ 3x-y-4z=3 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & -3 \\ 3 & -1 & -4 \end{vmatrix} \begin{matrix} 2 & -1 \\ 1 & 2 \\ 3 & -1 \end{matrix}$$

$$= -31$$

$$D_x = \begin{vmatrix} -8 & -1 & 2 \\ 9 & 2 & -3 \\ 3 & -1 & -4 \end{vmatrix} \begin{matrix} -8 & -1 \\ 9 & 2 \\ 3 & -1 \end{matrix}$$

$$= 31$$

$$D_y = \begin{vmatrix} 2 & -8 & 2 \\ 1 & 9 & -3 \\ 3 & 3 & -4 \end{vmatrix} \begin{matrix} 2 & -8 \\ 1 & 9 \\ 3 & 3 \end{matrix}$$

$$= -62$$

$$D_z = \begin{vmatrix} 2 & -1 & -8 \\ 1 & 2 & 9 \\ 3 & -1 & 3 \end{vmatrix} \begin{matrix} 2 & -1 \\ 1 & 2 \\ 3 & -1 \end{matrix}$$

$$= 62$$

$x = -1$ $\qquad\qquad$ $x = \dfrac{D_x}{D}$

$y = 2$ $\qquad\qquad$ $y = \dfrac{D_y}{D}$

$z = -2$ $\qquad\qquad$ $z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $(-1,\ 2,\ -2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x \qquad - 3z = -5 \\ 2x - y + 2z = 16 \\ 7x - 3y - 5z = 19 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 0 & -3 \\ 2 & -1 & 2 \\ 7 & -3 & -5 \end{vmatrix} \begin{matrix} 1 & 0 \\ 2 & -1 \\ 7 & -3 \end{matrix}$$

$$= 8$$

$$D_x = \begin{vmatrix} -5 & 0 & -3 \\ 16 & -1 & 2 \\ 19 & -3 & -5 \end{vmatrix} \begin{matrix} -5 & 0 \\ 16 & -1 \\ 19 & -3 \end{matrix}$$

$$= 32$$

$$D_y = \begin{vmatrix} 1 & -5 & -3 \\ 2 & 16 & 2 \\ 7 & 19 & -5 \end{vmatrix} \begin{matrix} 1 & -5 \\ 2 & 16 \\ 7 & 19 \end{matrix}$$

$$= -16$$

$$D_z = \begin{vmatrix} 1 & 0 & -5 \\ 2 & -1 & 16 \\ 7 & -3 & 19 \end{vmatrix} \begin{matrix} 1 & 0 \\ 2 & -1 \\ 7 & -3 \end{matrix}$$

$$= 24$$

$x = 4$ $\qquad x = \dfrac{D_x}{D}$

$y = -2$ $\qquad y = \dfrac{D_y}{D}$

$z = 3$ $\qquad z = \dfrac{D_z}{D}$

∴ ***Solution***: $(4,\ -2,\ 3)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + 2y - z = 5 \\ 2x - y + 3z = 0 \\ 2y + z = 1 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 0 & 2 & 1 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{matrix}$$

$$= -15$$

$$D_x = \begin{vmatrix} 5 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 2 & 1 \end{vmatrix} \begin{matrix} 5 & 2 \\ 0 & -1 \\ 1 & 2 \end{matrix}$$

$$= -30$$

$$D_y = \begin{vmatrix} 1 & 5 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix} \begin{matrix} 1 & 5 \\ 2 & 0 \\ 0 & 1 \end{matrix}$$

$$\underline{= -15}$$

$$D_z = \begin{vmatrix} 1 & 2 & 5 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{matrix}$$

$$\underline{= 15}$$

$\underline{x = 2}$ $\qquad x = \dfrac{D_x}{D}$

$\underline{y = 1}$ $\qquad y = \dfrac{D_y}{D}$

$\underline{z = -1}$ $\qquad z = \dfrac{D_z}{D}$

∴ ***Solution***: $\underline{(2,\ 1,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x + y + z = 6 \\ 3x + 4y - 7z = 1 \\ 2x - y + 3z = 5 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & -7 \\ 2 & -1 & 3 \end{vmatrix} \begin{matrix} 1 & 1 \\ 3 & 4 \\ 2 & -1 \end{matrix}$$

$$\underline{= -29}$$

$$D_x = \begin{vmatrix} 6 & 1 & 1 \\ 1 & 4 & -7 \\ 5 & -1 & 3 \end{vmatrix} \begin{matrix} 6 & 1 \\ 1 & 4 \\ 5 & -1 \end{matrix}$$

$$\underline{= -29}$$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 3 & 1 & -7 \\ 2 & 5 & 3 \end{vmatrix} \begin{matrix} 1 & 6 \\ 3 & 1 \\ 2 & 5 \end{matrix}$$

$$\underline{= -87}$$

$$D_z = \begin{vmatrix} 1 & 2 & 6 \\ 2 & -1 & 1 \\ 0 & 2 & 5 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & -1 \\ 0 & 2 \end{matrix}$$

$$= -58$$

$x = 1$ $\qquad x = \dfrac{D_x}{D}$

$y = 3$ $\qquad y = \dfrac{D_y}{D}$

$z = 2$ $\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $(1,\ 3,\ 2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x + 2y + 3z = 3 \\ 4x - 5y + 7z = 1 \\ 2x + 3y - 2z = 6 \end{cases}$

Solution

$$D = \begin{vmatrix} 3 & 2 & 3 \\ 4 & -5 & 7 \\ 2 & 3 & -2 \end{vmatrix} \begin{matrix} 3 & 2 \\ 4 & -5 \\ 2 & 3 \end{matrix}$$

$$= 77$$

$$D_x = \begin{vmatrix} 3 & 2 & 3 \\ 1 & -5 & 7 \\ 6 & 3 & -2 \end{vmatrix} \begin{matrix} 3 & 2 \\ 1 & -5 \\ 6 & 3 \end{matrix}$$

$$= 154$$

$$D_y = \begin{vmatrix} 3 & 3 & 3 \\ 4 & 1 & 7 \\ 2 & 6 & -2 \end{vmatrix} \begin{matrix} 3 & 3 \\ 4 & 1 \\ 2 & 6 \end{matrix}$$

$$= 0$$

$$D_z = \begin{vmatrix} 3 & 2 & 3 \\ 4 & -5 & 1 \\ 2 & 3 & 6 \end{vmatrix} \begin{matrix} 3 & 2 \\ 4 & -5 \\ 2 & 3 \end{matrix}$$

$$= -77$$

$x = 2$ $\qquad x = \dfrac{D_x}{D}$

$\boxed{y=0}$ $\qquad y=\dfrac{D_y}{D}$

$\boxed{z=-1}$ $\qquad z=\dfrac{D_z}{D}$

∴ ***Solution***: $\boxed{(2,\ 0,\ -1)}$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x+5y \qquad = 2 \\ 11x+y+2z=3 \\ x+5y+2z=1 \end{cases}$

Solution

$$D=\begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix}\begin{matrix} 4 & 5 \\ 11 & 1 \\ 1 & 5 \end{matrix}$$

$\boxed{=-132}$

$$D_x=\begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix}\begin{matrix} 2 & 5 \\ 3 & 1 \\ 1 & 5 \end{matrix}$$

$\boxed{=-36}$

$$D_y=\begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix}\begin{matrix} 4 & 2 \\ 11 & 3 \\ 1 & 1 \end{matrix}$$

$\boxed{=-24}$

$$D_z=\begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix}\begin{matrix} 4 & 5 \\ 11 & 1 \\ 1 & 5 \end{matrix}$$

$\boxed{=12}$

$x=\dfrac{36}{132}$ $\qquad x=\dfrac{D_x}{D}$

$\boxed{=\dfrac{3}{11}}$

$y=\dfrac{24}{132}$ $\qquad y=\dfrac{D_y}{D}$

$\boxed{=\dfrac{2}{11}}$

$z = -\dfrac{12}{132}$ $\qquad z = \dfrac{D_z}{D}$

$= -\dfrac{1}{11}$

$\therefore$ ***Solution***: $\left(\dfrac{3}{11},\ \dfrac{2}{11},\ -\dfrac{1}{11}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} x - 4y + z = 6 \\ 4x - y + 2z = -1 \\ 2x + 2y - 3z = -20 \end{cases}$

Solution

$$D = \begin{vmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix} \begin{matrix} 1 & -4 \\ 4 & -1 \\ 2 & 2 \end{matrix}$$

$= -55$

$$D_x = \begin{vmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{vmatrix} \begin{matrix} 6 & -4 \\ -1 & -1 \\ -20 & 2 \end{matrix}$$

$= 144$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{vmatrix} \begin{matrix} 1 & 6 \\ 4 & -1 \\ 2 & -20 \end{matrix}$$

$= 61$

$$D_z = \begin{vmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{vmatrix} \begin{matrix} 1 & -4 \\ 4 & -1 \\ 2 & 2 \end{matrix}$$

$= -230$

$x = -\dfrac{144}{55}$ $\qquad x = \dfrac{D_x}{D}$

$y = -\dfrac{61}{55}$ $\qquad y = \dfrac{D_y}{D}$

$z = \dfrac{46}{11}$ $\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $\left(-\dfrac{144}{55},\ -\dfrac{61}{55},\ \dfrac{46}{11}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 2x - y + z = -1 \\ 3x + 4y - z = -1 \\ 4x - y + 2z = -1 \end{cases}$

Solution

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 4 & -1 \\ 4 & -1 & 2 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & 4 \\ 4 & -1 \end{matrix}$$

$$= 5$$

$$D_x = \begin{vmatrix} -1 & -1 & 1 \\ -1 & 4 & -1 \\ -1 & -1 & 2 \end{vmatrix} \begin{matrix} -1 & -1 \\ -1 & 4 \\ -1 & -1 \end{matrix}$$

$$= -5$$

$$D_y = \begin{vmatrix} 2 & -1 & 1 \\ 3 & -1 & -1 \\ 4 & -1 & 2 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & -1 \\ 4 & -1 \end{matrix}$$

$$= 5$$

$$D_z = \begin{vmatrix} 2 & -1 & -1 \\ 3 & 4 & -1 \\ 4 & -1 & -1 \end{vmatrix} \begin{matrix} 2 & -1 \\ 3 & 4 \\ 4 & -1 \end{matrix}$$

$$= 10$$

$$x = \frac{-5}{5} \qquad x = \frac{D_x}{D}$$

$$= -1$$

$$y = \frac{5}{5} \qquad y = \frac{D_y}{D}$$

$$= 1$$

$$z = \frac{10}{5} \qquad z = \frac{D_z}{D}$$

$$= 2$$

$\therefore$ ***Solution***: $(-1,\ 1,\ 2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 4x - y - z = 1 \\ 2x + 2y + 3z = 10 \\ 5x - 2y - 2z = -1 \end{cases}$

Solution

$$D = \begin{vmatrix} 4 & -1 & -1 \\ 2 & 2 & 3 \\ 5 & -2 & -2 \end{vmatrix}$$

$$= 3$$

$$D_x = \begin{vmatrix} 1 & -1 & -1 \\ 10 & 2 & 3 \\ -1 & -2 & -2 \end{vmatrix}$$

$$= 3$$

$$D_y = \begin{vmatrix} 4 & 1 & -1 \\ 2 & 10 & 3 \\ 5 & -1 & -2 \end{vmatrix}$$

$$= 3$$

$$D_z = \begin{vmatrix} 4 & -1 & -1 \\ 2 & 2 & 10 \\ 5 & -2 & -1 \end{vmatrix}$$

$$= 6$$

$x = 1$ $\qquad x = \dfrac{D_x}{D}$

$y = 1$ $\qquad y = \dfrac{D_y}{D}$

$z = 2$ $\qquad z = \dfrac{D_z}{D}$

$\therefore$ ***Solution***: $(1,\ 1,\ 2)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} 3x + 4y + 4z = 11 \\ 4x - 4y + 6z = 11 \\ 6x - 6y \qquad = 3 \end{cases}$

Solution

$$D = \begin{vmatrix} 3 & 4 & 4 \\ 4 & -4 & 6 \\ 6 & -6 & 0 \end{vmatrix}$$

$$= 252$$

$$D_x = \begin{vmatrix} 11 & 4 & 4 \\ 11 & -4 & 6 \\ 3 & -6 & 0 \end{vmatrix}$$

$$= 252$$

$$D_y = \begin{vmatrix} 3 & 11 & 4 \\ 4 & 11 & 6 \\ 6 & 3 & 0 \end{vmatrix}$$

$$= 126$$

$$D_z = \begin{vmatrix} 3 & 4 & 11 \\ 4 & -4 & 11 \\ 6 & -6 & 3 \end{vmatrix}$$

$$= 378$$

$x = 1$ $\qquad x = \dfrac{D_x}{D}$

$y = \dfrac{1}{2}$ $\qquad y = \dfrac{D_y}{D}$

$z = \dfrac{3}{2}$ $\qquad z = \dfrac{D_z}{D}$

∴ ***Solution***: $\left(1,\ \frac{1}{2},\ \frac{3}{2}\right)$

Exercise

Use Cramer's rule to solve the system $\begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$

Solution

$$D = \begin{vmatrix} -1 & -4 & 2 & 1 \\ 2 & -1 & 7 & 9 \\ -1 & 1 & 3 & 1 \\ -1 & -2 & 1 & -4 \end{vmatrix}$$

$$\underline{= -243 \,|}$$

$$D_1 = \begin{vmatrix} -32 & -4 & 2 & 1 \\ 14 & -1 & 7 & 9 \\ 11 & 1 & 3 & 1 \\ -4 & -2 & 1 & -4 \end{vmatrix}$$

$$\underline{= -2115 \,|}$$

$$D_2 = \begin{vmatrix} -1 & -32 & 2 & 1 \\ 2 & 14 & 7 & 9 \\ -1 & 11 & 3 & 1 \\ -1 & -4 & 1 & -4 \end{vmatrix}$$

$$\underline{= -1834 \,|}$$

$$D_3 = \begin{vmatrix} -1 & -4 & -32 & 1 \\ 2 & -1 & 14 & 9 \\ -1 & 1 & 11 & 1 \\ -1 & -2 & -4 & -4 \end{vmatrix}$$

$$\underline{= -1279 \,|}$$

$$D_4 = \begin{vmatrix} -1 & -4 & 2 & -32 \\ 2 & -1 & 7 & 14 \\ -1 & 1 & 3 & 11 \\ -1 & -2 & 1 & -4 \end{vmatrix}$$

$$\underline{= 883 \,|}$$

$$x_1 = \frac{-2115}{-243}$$

$$\underline{= \frac{235}{27} \,|}$$

$$x_2 = \frac{-1834}{-243}$$

$$\underline{= \frac{1834}{243} \,|}$$

$$x_3 = \frac{-1279}{-243}$$

$$\underline{= \frac{1279}{243} \,|}$$

$$x_4 = -\frac{883}{243}$$

$\therefore$ ***Solution***: $\left(\frac{235}{27},\ \frac{1834}{243},\ \frac{1279}{243},\ -\frac{883}{243} \right)$

Exercise

Determine the polynomial function whose graph through the points and sketch the graph of the polynomial function, showing the points.

$$(2,\ 5),\quad (3,\ 2),\quad (4,\ 5)$$

Solution

Given 3 points, therefore, we must use a quadratic equation:

$$f(x) = ax^2 + bx + c$$

$$(2,\ 5) \rightarrow \quad 4a + 2b + c = 5$$

$$(3,\ 2) \rightarrow \quad 9a + 3b + c = 2$$

$$(4,\ 5) \rightarrow \quad 16a + 4b + c = 5$$

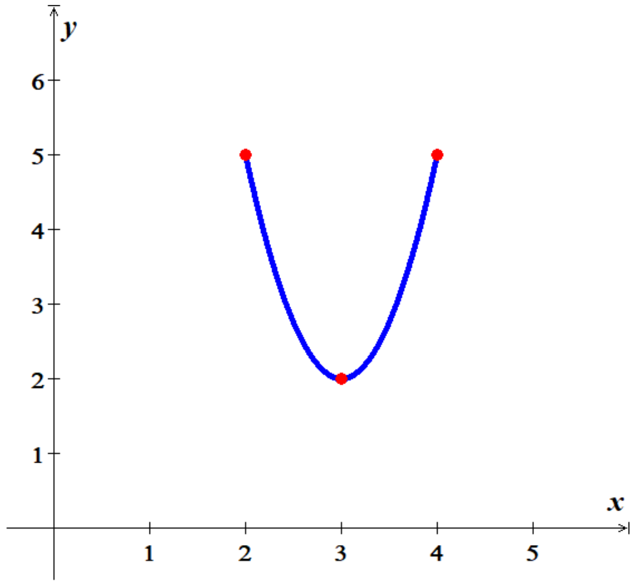
$$\begin{cases} 4a + 2b + c = 5 \\ 9a + 3b + c = 2 \\ 16a + 4b + c = 5 \end{cases}$$

$$D = \begin{vmatrix} 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{vmatrix} = -2 \qquad D_a = \begin{vmatrix} 5 & 2 & 1 \\ 2 & 3 & 1 \\ 5 & 4 & 1 \end{vmatrix} = -6$$

$$D_b = \begin{vmatrix} 4 & 5 & 1 \\ 9 & 2 & 1 \\ 16 & 3 & 1 \end{vmatrix} = 36 \qquad D_c = \begin{vmatrix} 4 & 2 & 5 \\ 9 & 3 & 2 \\ 16 & 4 & 5 \end{vmatrix} = -58$$

$a = 3 \quad b = -18 \quad c = 29$

$$f(x) = 3x^2 - 18x + 29$$



Exercise

Determine the polynomial function whose graph through the points and sketch the graph of the polynomial function, showing the points.

$$(2,\ 4),\quad (3,\ 6),\quad (5,\ 10)$$

Solution

Given 3 points, therefore, we must use a quadratic equation:

$$f(x) = ax^2 + bx + c$$

$$(2,\ 4) \to \quad 4a + 2b + c = 4$$

$$(3,\ 6) \to \quad 9a + 3b + c = 6$$

$$(5,\ 10) \to \quad 25a + 5b + c = 10$$

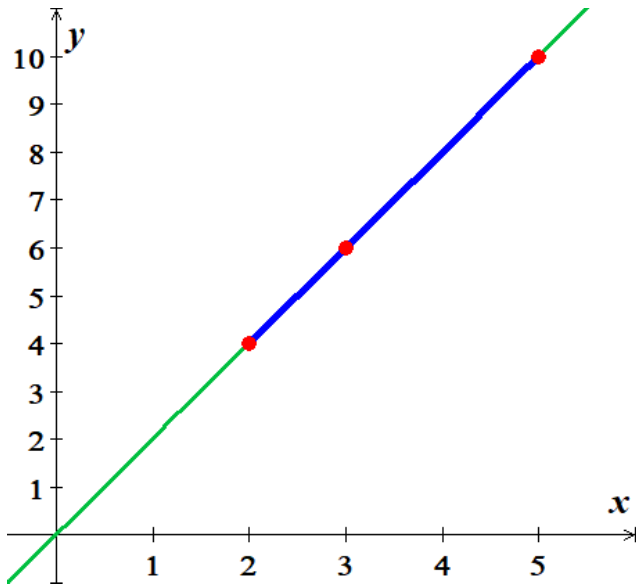
$$\begin{cases} 4a + 2b + c = 4 \\ 9a + 3b + c = 6 \\ 25a + 5b + c = 10 \end{cases}$$

$$D = \begin{vmatrix} 4 & 2 & 1 \\ 9 & 3 & 1 \\ 25 & 5 & 1 \end{vmatrix} = -6 \qquad D_a = \begin{vmatrix} 4 & 2 & 1 \\ 6 & 3 & 1 \\ 10 & 5 & 1 \end{vmatrix} = 0$$

$$D_b = \begin{vmatrix} 4 & 4 & 1 \\ 9 & 6 & 1 \\ 25 & 10 & 1 \end{vmatrix} = -12 \qquad D_c = \begin{vmatrix} 4 & 2 & 4 \\ 9 & 3 & 6 \\ 25 & 5 & 10 \end{vmatrix} = 0$$

$$a = 0 \quad b = 2 \quad c = 0$$

$\boxed{f(x) = 2x}$



Exercise

Determine the polynomial function whose graph through the points and sketch the graph of the polynomial function, showing the points.

$$(-1,\ 3),\quad (0,\ 0),\quad (1,\ 1),\quad (4,\ 58)$$

Solution

Given 3 points, therefore, we must use a third−degree equation:

$$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

$$(-1,\ 3) \rightarrow\ -a_3 + a_2 - a_1 + a_0 = 3$$

$$(0,\ 0) \rightarrow\ \boxed{a_0 = 0}$$

$$(1,\ 1) \rightarrow\ a_3 + a_2 + a_1 + a_0 = 1$$

$$(4,\ 58) \rightarrow\ 64a_3 + 16a_2 + 4a_1 + a_0 = 58$$

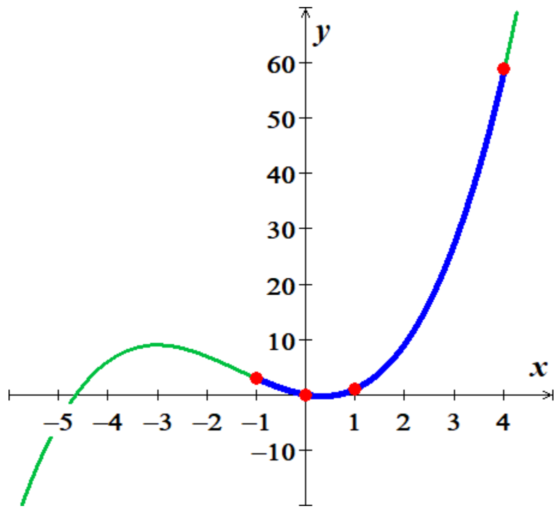
$$\begin{cases} -a_1 + a_2 - a_3 = 3 \\ a_1 + a_2 + a_3 = 1 \\ 4a_1 + 16a_2 + 64a_3 = 58 \end{cases}$$

$$D = \begin{vmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & 16 & 64 \end{vmatrix} \boxed{= -120} \qquad D_{a_1} = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 58 & 16 & 64 \end{vmatrix} \boxed{= 180}$$

$$D_{a_2} = \begin{vmatrix} -1 & 3 & -1 \\ 1 & 1 & 1 \\ 4 & 58 & 64 \end{vmatrix} = -240 \qquad D_{a_3} = \begin{vmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \\ 4 & 16 & 58 \end{vmatrix} = -60$$

$$a_0 = 0 \quad a_1 = -\frac{3}{2} \quad a_2 = 2 \quad a_3 = \frac{1}{2}$$

$$f(x) = \frac{1}{2}x^3 + 2x^2 - \frac{3}{2}x$$



Exercise

The U.S. census lists the population of the United States as 249 million in 1990, 282 million in 2000, and 309 million in 2010. Fit a second-degree polynomial passing through these three points and use it to predict the populations in 2020 and 2030.

Solution

Let assume $x = 0$ at year 1990 $\Rightarrow$ $(0,\ 249)$

$$2000 \rightarrow (10,\ 282)$$
$$2010 \rightarrow (20,\ 309)$$

$$P(x) = a_0 + a_1 x + a_2 x^2$$

$(0,\ 249) \rightarrow$ $a_0 = 249$

$(10,\ 282) \rightarrow$ $a_0 + 10a_1 + 100a_2 = 282$

$(20,\ 309) \rightarrow$ $a_0 + 20a_1 + 400a_2 = 309$

$$\begin{cases} 10a_1 + 100a_2 = 33 \\ 20a_1 + 400a_2 = 60 \end{cases}$$

$$D = \begin{vmatrix} 10 & 100 \\ 20 & 400 \end{vmatrix} \underline{= 2,000}$$

$$D_1 = \begin{vmatrix} 33 & 100 \\ 60 & 400 \end{vmatrix} \underline{= 7,200}$$

$$D_2 = \begin{vmatrix} 10 & 33 \\ 20 & 60 \end{vmatrix} \underline{= -60}$$

$$a_1 = \frac{72}{20} \underline{= \frac{18}{5}}$$

$$a_2 = -\frac{6}{200} \underline{= -\frac{3}{100}}$$

$$\underline{a_0 = 249 \quad a_1 = \frac{18}{5} \quad a_2 = -\frac{3}{100}}$$

$$\underline{P(x) = 249 + \frac{18}{5}x - \frac{3}{100}x^2}$$

Year 2020: $\Rightarrow \quad x = 30$

$$P(30) = 249 + \frac{18}{5}(30) - \frac{3}{100}(900)$$

$$\underline{= 330 \quad \textit{million}}$$

Year 2030: $\Rightarrow \quad x = 40$

$$P(40) = 249 + 18(9) - 3(16)$$

$$\underline{= 345 \quad \textit{million}}$$

Solution Section 1.8 – Applications

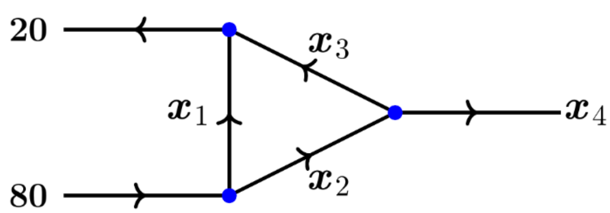
Exercise

Through a network

a) Solve this system for $x_i, \quad i = 1, 2, 3, 4, 5, 6$

b) What is the largest value of x_3 ?

Solution



a) $\begin{cases} x_1 + x_3 = 20 \\ x_1 + x_2 = 80 \\ x_2 = x_3 + x_4 \end{cases}$

$$\begin{cases} x_1 + x_3 = 20 \\ x_1 + x_2 = 80 \\ x_2 - x_3 - x_4 = 0 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 20 \\ 1 & 1 & 0 & 0 & 80 \\ 0 & 1 & -1 & -1 & 0 \end{array}\right) \quad R_2 - R_1$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & 0 & 60 \\ 0 & 1 & -1 & -1 & 0 \end{array}\right) \quad R_3 - R_2$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & 0 & 60 \\ 0 & 0 & 0 & -1 & -60 \end{array}\right) \quad \begin{aligned} x_1 + x_3 &= 20 \\ x_2 + x_3 &= 60 \\ x_4 &= 60 \end{aligned}$$

$$\begin{cases} x_1 = 20 - x_3 \\ x_2 = 60 - x_3 \end{cases}$$

$$\therefore \ \underline{\left(20 - x_3, \ 60 - x_3, \ x_3, \ 60\right)}$$

b) $\begin{cases} 20 - x_3 \ge 0 \\ 60 - x_3 \ge 0 \end{cases}$

$$\begin{cases} x_3 \le 20 \\ x_3 \le 60 \end{cases}$$

The largest value of x_3 is **20**, since 60 will result x_1 with a negative value.

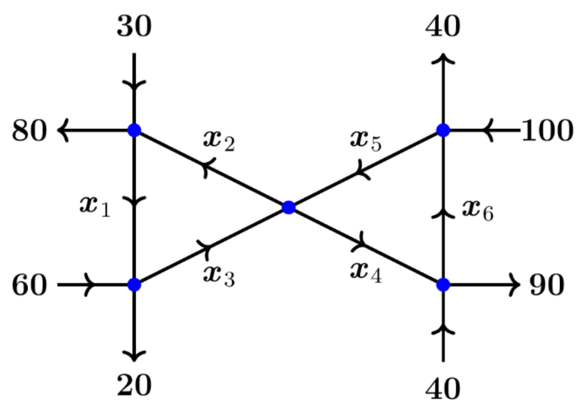
Exercise

The flow of traffic, through a network of streets as is shown below

a) Solve this system for $x_i, \quad i = 1, 2, 3, 4, 5, 6$

b) Find the minimum flows in the branches denoted by $x_2, x_3, x_4,$ *and* x_5

Solution



a) $$\begin{cases} x_1 + 60 = x_3 + 20 \\ x_1 + 80 = x_2 + 30 \\ x_2 + x_4 = x_3 + x_5 \\ x_5 + 40 = x_6 + 100 \\ x_4 + 40 = x_6 + 90 \end{cases}$$

$$\begin{cases} x_1 - x_3 = -40 & (1) \\ x_1 - x_2 = -50 & (2) \\ x_2 - x_3 + x_4 - x_5 = 0 & (3) \\ x_5 - x_6 = 60 & (4) \\ x_4 - x_6 = 50 & (5) \end{cases}$$

$(1) \to \quad x_3 = x_1 + 40$

$(2) \to \quad x_2 = x_1 + 50$

$(4) \to \quad x_5 = 60 + x_6$

$(5) \to \quad x_4 = 50 + x_6$

$\therefore$ ***Solution***: $\underline{\left(x_1, \quad x_1 + 50, \quad x_1 + 40, \quad x_6 + 60, \quad x_6 + 50, \quad x_6\right)}|$

b) The minimum flow for $x_1 = x_6 = 0$

$x_3 = x_1 + 40 \underline{= 40}|$

$x_2 = x_1 + 50 \underline{= 50}|$

$x_4 = x_6 + 50 \underline{= 50}|$

$x_5 = x_6 + 60 \underline{= 60}|$

$\therefore$ ***Solution***: $\underline{(0, \quad 50, \quad 40, \quad 60, \quad 50, \quad 0)}|$

Exercise

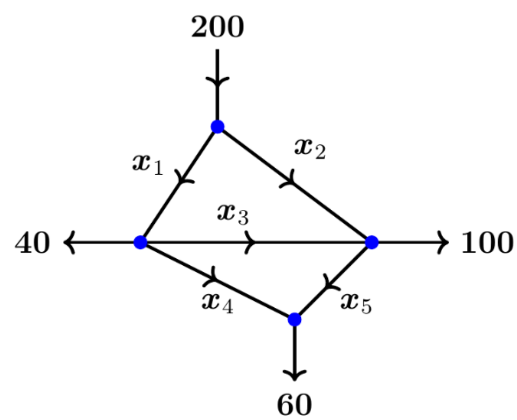
The flow of traffic, through a network of streets as is shown below

a) Solve this system for $x_i, \quad i=1,\ 2,\ 3,\ 4,\ 5$

b) Find the traffic flow when $x_4 = 0$.

c) Find the traffic flow when $x_4 = 100$.

d) Find the traffic flow when $x_1 = 2x_2$.



Solution

a) $$\begin{cases} x_1 = x_3 + x_4 + 40 \\ x_1 + x_2 = 200 \\ x_2 + x_3 = 100 + x_5 \\ x_5 + x_4 = 60 \end{cases}$$

$$\begin{cases} x_1 - x_3 - x_4 = 40 \\ x_1 + x_2 = 200 \\ x_2 + x_3 - x_5 = 100 \\ x_5 + x_4 = 60 \end{cases}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right) \quad \begin{array}{l} \\ R_2 - R_1 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right) \quad \begin{array}{l} \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 0 & 0 & -1 & -1 & -60 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right) \quad \begin{array}{l} R_1 - R_3 \\ R_2 + R_3 \\ \\ R_4 + R_3 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 1 & 100 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & -1 & -1 & -60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} x_1 = 100 + x_3 - x_5 \\ x_2 = 100 - x_3 + x_5 \\ x_4 = 60 - x_5 \\ \end{array}$$

$\therefore$ ***Solution***: $\boxed{\left(100+x_3-x_5,\quad 100-x_3+x_5,\quad x_3,\quad 60-x_5,\quad x_5\right)}$

b) ***Given***: $x_4=0$

$$\begin{cases} x_1-x_3=40 \\ x_1+x_2=200 \\ x_2+x_3-x_5=100 \\ \boxed{x_5=60} \end{cases}$$

$$\begin{cases} x_1-x_3=40 \\ x_1+x_2=200 \\ x_2+x_3=160 \end{cases}$$

$$D=\begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}\boxed{=0} \qquad D_3=\begin{vmatrix} 1 & 0 & 40 \\ 1 & 1 & 200 \\ 0 & 1 & 160 \end{vmatrix}\boxed{=0}$$

$$\begin{cases} \boxed{x_1=40+x_3} \\ \boxed{x_2=160-x_3} \end{cases}$$

$\therefore$ ***Solution***: $\boxed{\left(40+x_3,\quad 160-x_3,\quad x_3,\quad 0,\quad 60\right)}$

c) ***Given***: $x_4=100$

$$\begin{cases} x_1-x_3=140 \\ x_1+x_2=200 \\ x_2+x_3-x_5=100 \\ \boxed{x_5=-40} \end{cases}$$

$$\begin{cases} x_1-x_3=140 \\ x_1+x_2=200 \\ x_2+x_3=60 \end{cases}$$

$$D=\begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}\boxed{=0} \qquad D_3=\begin{vmatrix} 1 & 0 & 140 \\ 1 & 1 & 200 \\ 0 & 1 & 60 \end{vmatrix}\boxed{=0}$$

$$\begin{cases} \boxed{x_1=140+x_3} \\ \boxed{x_2=60-x_3} \end{cases}$$

$\therefore$ ***Solution***: $\boxed{\left(140+x_3,\quad 60-x_3,\quad x_3,\quad 0,\quad -40\right)}$

d) ***Given***: $x_1 = 2x_2$

$$\begin{cases} x_1 - x_3 - x_4 = 40 & (1) \\ x_1 + x_2 = 200 & (2) \\ x_2 + x_3 - x_5 = 100 \\ x_5 + x_4 = 60 \end{cases}$$

$(2)\quad 3x_2 = 200 \quad \rightarrow \quad \boxed{x_2 = \dfrac{200}{3}}$

$\boxed{x_1 = \dfrac{400}{3}}$

$$\begin{cases} -x_3 - x_4 = 40 - \dfrac{400}{3} \\ x_3 - x_5 = 100 - \dfrac{200}{3} \\ x_5 + x_4 = 60 \end{cases}$$

$$\begin{cases} x_3 + x_4 = \dfrac{280}{3} \\ x_3 - x_5 = \dfrac{100}{3} \\ x_4 + x_5 = 60 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 0 \qquad\qquad D_3 = \begin{vmatrix} 1 & 1 & \frac{280}{3} \\ 1 & 0 & \frac{100}{3} \\ 0 & 1 & 60 \end{vmatrix} = 0$$

$$\begin{cases} \boxed{x_3 = \dfrac{100}{3} + x_5} \\ \boxed{x_4 = 60 - x_5} \end{cases}$$

$\therefore$ ***Solution***: $\boxed{\left(\dfrac{400}{3},\quad \dfrac{200}{3},\quad \dfrac{100}{3}+x_5,\quad 60-x_5,\quad x_5\right)}$

Exercise

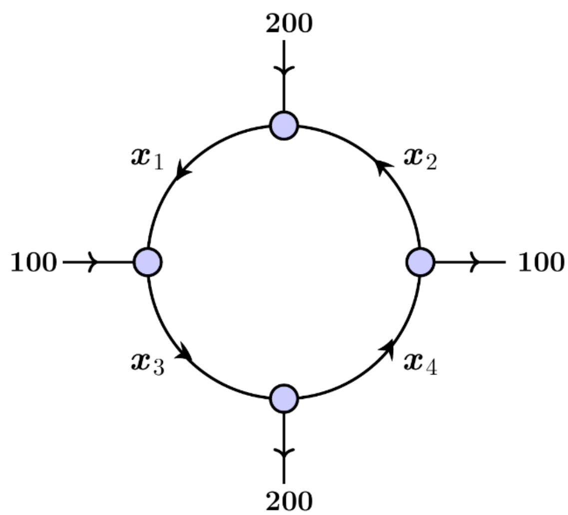
The flow of traffic, in vehicles per hour, through a network of streets as is shown below

a) Solve this system for $x_i, \quad i = 1,\ 2,\ 3,\ 4.$

b) Find the traffic flow when $x_4 = 0$.

c) Find the traffic flow when $x_4 = 100$.

d) Find the traffic flow when $x_1 = 2x_2$.



Solution

a) $\begin{cases} x_1 + 100 = x_3 \\ x_2 + 200 = x_1 \\ x_2 + 100 = x_4 \\ x_4 + 200 = x_3 \end{cases}$

$$\begin{cases} -x_1 + x_3 = 100 \\ x_1 - x_2 = 200 \\ -x_2 + x_4 = 100 \\ x_3 - x_4 = 200 \end{cases}$$

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 1 & -1 & 0 & 0 & 200 \\ 0 & -1 & 0 & 1 & 100 \\ 0 & 0 & 1 & -1 & 200 \end{array}\right) \quad R_2 + R_1$$

$$\begin{vmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1 \begin{vmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 0 & -1 & 1 & 0 & 300 \\ 0 & -1 & 0 & 1 & 100 \\ 0 & 0 & 1 & -1 & 200 \end{array}\right) \quad R_3 - R_2$$

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 0 & -1 & 1 & 0 & 300 \\ 0 & 0 & -1 & 1 & -200 \\ 0 & 0 & 1 & -1 & 200 \end{array}\right) \quad R_4 + R_3$$

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 100 \\ 0 & -1 & 1 & 0 & 300 \\ 0 & 0 & -1 & 1 & -200 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} \rightarrow \ -x_1 + x_3 = 100 \\ \rightarrow \ -x_2 + x_3 = 100 \\ \rightarrow \ -x_3 + x_4 = 100 \end{array}$$

Let x_4 be the free variable

$$\begin{cases} \quad\quad \boxed{x_3 = x_4 + 200} \\ \quad\quad \boxed{x_2 = x_4 - 100} \\ x_1 = 200 + x_2 \ \boxed{= x_4 + 100} \end{cases}$$

Solution: $\left(x_4 + 100, \quad x_4 - 100, \quad x_4 + 200, \quad x_4\right)$

OR

$$\begin{vmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = -1\begin{vmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= -1(1) - 1(-1)$$

$$= -1 + 1$$

$$\boxed{= 0}$$

$$\begin{cases} -x_1 + x_3 = 100 & \to \ x_1 = x_3 - 100 \ \boxed{= x_4 + 100} \\ x_1 - x_2 = 200 & \\ -x_2 + x_4 = 100 & \to \ \boxed{x_2 = x_4 - 100} \\ x_3 - x_4 = 200 & \to \ \boxed{x_3 = x_4 + 200} \end{cases}$$

b) The traffic flow when $x_4 = 0$ is:

$\therefore \ (100, \quad -100, \quad 200, \quad 0)$

c) The traffic flow when $x_4 = 100$ is:

$\therefore \ (200, \quad 0, \quad 300, \quad 100)$

d) The traffic flow when $x_1 = 2x_2$:

$$x_4 + 100 = 2\left(x_4 - 100\right)$$

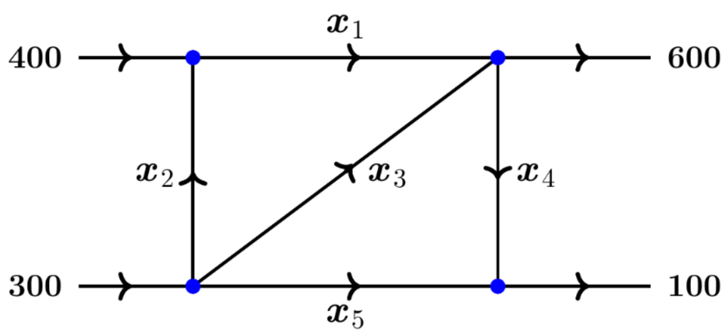
$$x_4 + 100 = 2x_4 - 200$$

$$x_4 = 300$$

$\therefore \ (400, \quad 200, \quad 500, \quad 300)$

Exercise

Through a network, Express x_n's in terms of the parameters ***s*** and ***t***.



Solution

$$\begin{cases} x_1 = x_2 + 400 \\ x_1 + x_3 = x_4 + 600 \\ x_4 + x_5 = 100 \\ x_2 + x_3 + x_5 = 300 \end{cases}$$

$$\begin{cases} x_1 - x_2 = 400 \\ x_2 + x_3 - x_4 = 600 \\ x_4 + x_5 = 100 \\ x_2 + x_3 + x_5 = 300 \end{cases}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 1 & 0 & 1 & -1 & 0 & 600 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 1 & 1 & 0 & 1 & 300 \end{array}\right) \begin{array}{l} \\ R_2 - R_1 \\ \\ R_3 \leftrightarrow R_4 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 1 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & 1 & 100 \end{array}\right) \begin{array}{l} \\ \\ R_3 - R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 100 \end{array}\right) \begin{array}{l} \\ \\ \\ R_4 - R_3 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & 1 & -1 & 0 & 200 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{rl} x_1 - x_2 = 400 & \to x_1 = 400 + x_2 \\ x_2 + x_3 - x_4 = 200 & \to x_2 = 200 - x_3 + x_4 \\ x_4 + x_5 = 100 & \to \underline{x_4 = 100 - t} \end{array}$$

Let $x_5 = t$ & $x_3 = s$

$$x_2 = 200 - s + 100 - t$$

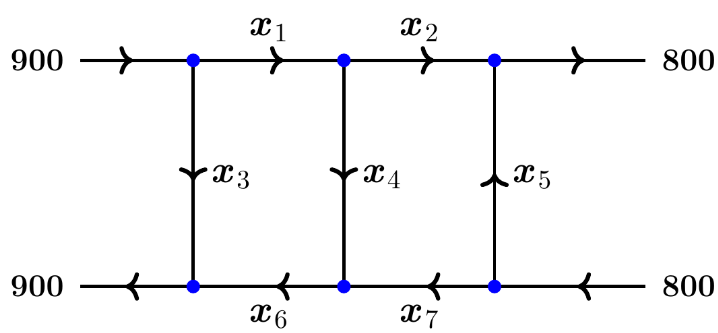
$$\underline{= 300 - s - t}$$

$$x_1 = 400 + 300 - s - t$$

$$\underline{= 700 - s - t}$$

Exercise

Water is flowing through a network of pipes. Express x_n's in terms of the parameters ***s*** and ***t***.



Solution

$$x_1 + x_3 = 900$$

$$x_1 = x_2 + x_4 \quad \to \quad x_1 - x_2 - x_4 = 0$$

$$x_2 + x_5 = 800$$

$$x_5 + x_7 = 800$$

$$x_6 = x_4 + x_7 \quad \to \quad x_4 - x_6 + x_7 = 0$$

$$x_3 + x_6 = 900$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 900 \end{array}\right] \quad \begin{array}{l} \\ R_2 - R_1 \\ \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & -900 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 900 \end{array}\right] \begin{array}{l} \\ \\ R_3 + R_2 \\ R_6 \\ \\ R_4 \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & -900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 900 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ R_4 + R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 800 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ R_5 + R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \\ R_6 - R_5 \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 900 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 900 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 800 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 800 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{rl} x_2 = 900 - x_3 & (5) \\ x_2 = 900 - x_3 - x_4 & (4) \\ x_3 = 100 - x_4 + x_5 & (3) \\ -x_4 = 800 - x_5 - x_6 & (2) \\ x_5 = 800 - x_7 & (1) \end{array}$$

Let $\quad x_6 = s \quad \& \quad x_7 = t$

$(1) \rightarrow \quad x_5 = 800 - t$

$(2) \rightarrow \quad x_4 = s - t$

$(3) \rightarrow \quad x_3 = 900 - s$

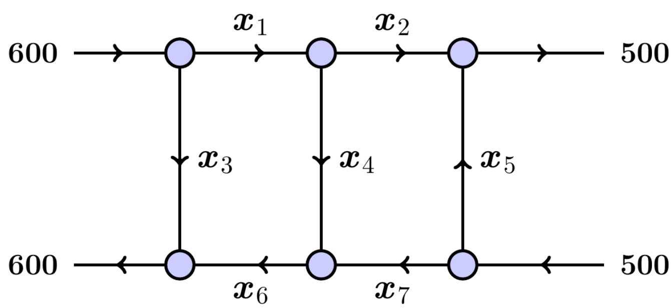
$(2) \to \quad x_2 = t$

$(1) \to \quad x_2 = s$

∴ ***Solution***: $\qquad (s,\ t,\ 900-s,\ s-t,\ 800-t,\ s,\ t)$

Exercise

Water is flowing through a network of pipes (in thousands of cubic meters per hour)



a) Solve this system for the water flow represented by $x_i, \quad i = 1,\ 2,\ \ldots,\ 7.$

b) Find the water flow when $x_1 = x_2 = 100$

c) Find the water flow when $x_6 = x_7 = 0$

d) Find the water flow when $x_5 = 1000 \quad \text{and} \quad x_6 = 0$

Solution

a) $x_1 + x_3 = 600$

$x_1 = x_2 + x_4$

$x_2 + x_5 = 500$

$x_5 + x_7 = 500$

$x_6 = x_4 + x_7$

$x_3 + x_6 = 600$

$$\begin{cases} x_1 + x_3 = 600 \\ x_1 - x_2 - x_4 = 0 \\ x_2 + x_5 = 500 \\ x_5 + x_7 = 500 \\ x_6 - x_4 - x_7 = 0 \\ x_3 + x_6 = 600 \end{cases}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 600 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 500 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 600 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 600 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & -600 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 500 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 600 \end{array}\right] \begin{array}{l} \\ \\ R_3 + R_2 \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 600 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 & -600 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 600 \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \\ \\ \\ \\ R_6 + R_3 \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 700 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & -500 \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & -100 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 500 \end{array}\right] \begin{array}{l} \\ -R_2 \\ -R_3 \\ R_4 \\ R_5 \\ \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 700 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 500 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 100 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 500 \end{array}\right] \begin{array}{l} R_1 - R_4 \\ \\ R_3 - R_4 \\ \\ \\ R_6 + R_4 \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 1 & 1 & -1 & 700 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 500 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 100 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \end{array}\right] \begin{array}{l} R_1 - R_5 \\ R_2 - R_5 \\ R_3 + R_5 \\ \\ \\ R_6 - R_5 \end{array}$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 200 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 600 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 500 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \to x_1 = 200 - x_6 \\ \to x_2 = x_7 \\ \to x_3 = 600 - x_6 \\ \to x_4 = x_6 - x_7 \\ \to x_5 = 500 - x_7 \\ \end{array}$$

∴ ***Solution***: $\left(200 - x_6,\ \ x_7,\ \ 300 - x_6,\ \ x_6 - x_7,\ \ 500 - x_7,\ \ x_6,\ \ x_7\right)$

b) Given: $x_1 = x_2 = 100$

$$\begin{cases} 100 + x_3 = 600 \\ 100 - 100 - x_4 = 0 \\ 100 + x_5 = 500 \end{cases}$$

$$\begin{cases} \underline{x_3 = 500} \\ \underline{x_4 = 0} \\ \underline{x_5 = 400} \end{cases}$$

$$\begin{cases} 400 + x_7 = 500 \\ x_6 - x_7 = 0 \\ 500 + x_6 = 600 \end{cases}$$

$$\begin{cases} \underline{x_7 = 100} \\ \underline{x_6 = 100} \end{cases}$$

∴ ***Solution***: $(100,\ \ 100,\ \ 500,\ \ 0,\ \ 400,\ \ 100,\ \ 100)$

c) Given: $x_6 = x_7 = 0$

$$\begin{cases} \boxed{x_5 = 500} \\ \boxed{x_4 = 0} \\ \boxed{x_3 = 600} \end{cases}$$

$$\begin{cases} x_1 + x_3 = 600 \\ x_1 - x_2 = 0 \\ x_2 + 500 = 500 \end{cases}$$

$$\begin{cases} \boxed{x_2 = 0} \\ \boxed{x_1 = 0} \end{cases}$$

$\therefore$ ***Solution***: $\boxed{(0,\quad 0,\quad 600,\quad 0,\quad 500,\quad 0,\quad 0)}$

d) Given: $x_5 = 1000 \quad and \quad x_6 = 0$

$$\begin{cases} x_1 + x_3 = 600 \\ x_1 - x_2 - x_4 = 0 \\ x_2 + 1000 = 500 \\ 1000 + x_7 = 500 \\ x_4 = -x_7 \\ \boxed{x_3 = 600} \end{cases}$$

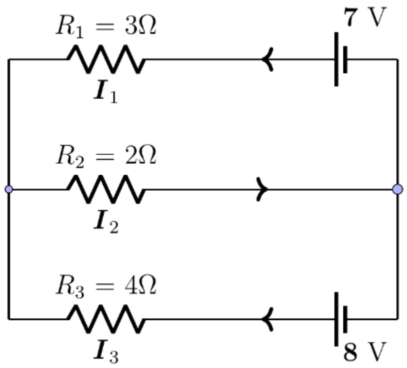
$$\begin{cases} \boxed{x_2 = -500} \\ \boxed{x_7 = -500} \\ \boxed{x_4 = 500} \end{cases}$$

$$\begin{cases} \boxed{x_3 = 600} \\ \boxed{x_1 = 0} \end{cases}$$

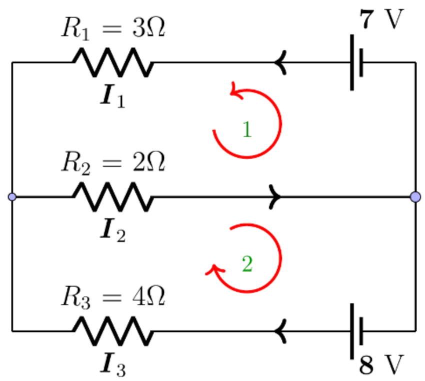
$\therefore$ ***Solution***: $\boxed{(0,\quad -500,\quad 600,\quad 500,\quad 1000,\quad 0,\quad -500)}$

Exercise

Determine the currents I_1, I_2, *and* I_3 for the electrical network shown below



Solution



$$I_2 = I_1 + I_3$$

$$3I_1 + 2I_2 = 7$$

$$2I_2 + 4I_3 = 8$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 3I_1 + 2I_2 \qquad = 7 \\ \qquad I_2 + 2I_3 = 4 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 3 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 13 \qquad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 7 & 2 & 0 \\ 4 & 1 & 2 \end{vmatrix} = 13$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 7 & 0 \\ 0 & 4 & 2 \end{vmatrix} = 26 \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 3 & 2 & 7 \\ 0 & 1 & 4 \end{vmatrix} = 13$$

∴ ***Solution***: $\underline{I_1 = 1\ A}|$ $\underline{I_2 = 2\ A}|$ $\underline{I_3 = 1\ A}|$

OR

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 3 & 2 & 0 & 7 \\ 0 & 1 & 2 & 4 \end{array}\right) \quad R_2 - 3R_1$$

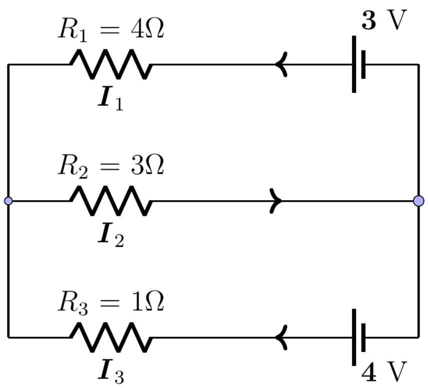
$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -3 & 7 \\ 0 & 1 & 2 & 4 \end{array}\right) \quad -5R_3 + R_2$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 5 & -3 & 7 \\ 0 & 0 & -13 & -13 \end{array}\right) \quad \begin{array}{l} I_1 = I_2 - I_3 \\ 5I_2 = 3I_3 + 7 \\ \boxed{I_3 = 1} \end{array}$$

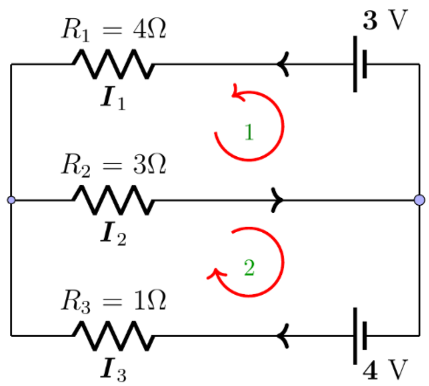
$\boxed{I_2 = 2}$ $\boxed{I_1 = 1}$

Exercise

Determine the currents $I_1, I_2,$ *and* I_3 for the electrical network shown below



Solution



$$I_2 = I_1 + I_3$$

$$4I_1 + 3I_2 = 3$$

$$3I_2 + I_3 = 4$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 4I_1 + 3I_2 \qquad = 3 \\ \qquad 3I_2 + I_3 = 4 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 4 & 3 & 0 \\ 0 & 3 & 1 \end{vmatrix} = 19 \qquad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 3 & 3 & 0 \\ 4 & 3 & 1 \end{vmatrix} = 0$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 4 & 3 & 0 \\ 0 & 4 & 1 \end{vmatrix} = 19 \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 4 & 3 & 3 \\ 0 & 3 & 4 \end{vmatrix} = 19$$

$\therefore$ ***Solution***: $\boxed{I_1 = 0\ A}$ $\boxed{I_2 = 1\ A}$ $\boxed{I_3 = 1\ A}$

OR

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 4 & 3 & 0 & 3 \\ 0 & 3 & 1 & 4 \end{array}\right) \quad R_2 - 4R_1$$

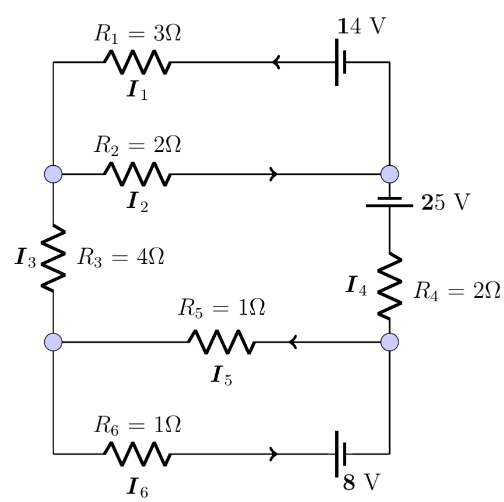
$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 3 \\ 0 & 3 & 1 & 4 \end{array}\right) \quad 7R_3 - 3R_2$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 3 \\ 0 & 0 & 19 & 19 \end{array}\right) \begin{array}{l} \rightarrow I_1 = I_2 - I_3 \quad (2) \\ \rightarrow 7I_2 = 4I_3 + 3 \quad (1) \\ \quad \underline{I_3 = 1|} \end{array}$$

$\underline{I_2 = 1|} \quad \underline{I_1 = 0|}$

Exercise

Determine the currents $I_1, I_2, I_3, I_4, I_5,$ *and* I_6 for the electrical network shown below



Solution

$$I_1 + I_3 = I_2 \quad \rightarrow \quad I_1 - I_2 + I_3 = 0$$

$$I_1 + I_4 = I_2 \quad \rightarrow \quad I_1 - I_2 + I_4 = 0$$

$$I_3 + I_6 = I_5 \quad \rightarrow \quad I_3 - I_5 + I_6 = 0$$

$$\begin{cases} 3I_1 + 2I_2 = 14 \\ 2I_2 + 4I_3 + I_5 + 2I_4 = 25 \\ I_5 + I_6 = 8 \end{cases}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ \\ R_4 - 3R_1 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \begin{array}{l} \\ R_4 \\ \\ R_2 \\ R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \begin{array}{l} \\ \\ 5R_3 - 2R_2 \\ \\ R_5 + R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \begin{array}{l} \\ \\ \\ 26R_4 + R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & 0 & 36 & 5 & 0 & 97 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ 36R_5 - R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & 0 & 36 & 5 & 0 & 97 \\ 0 & 0 & 0 & 0 & -41 & 36 & -97 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{array}\right] \quad 41R_6 + R_5$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 & 0 & 0 & 14 \\ 0 & 0 & 26 & 10 & 5 & 0 & 97 \\ 0 & 0 & 0 & 36 & 5 & 0 & 97 \\ 0 & 0 & 0 & 0 & -41 & 36 & -97 \\ 0 & 0 & 0 & 0 & 0 & 77 & 231 \end{array}\right]$$

$$I_1 = 4 - 2 \quad \rightarrow \boxed{I_1 = 2}$$

$$5I_2 = 14 + 3(2) \quad \rightarrow \boxed{I_2 = 4}$$

$$26I_3 = 97 - 10(2) - 5(5) \quad \rightarrow \boxed{I_3 = 2}$$

$$36I_4 = 97 - 5(5) \quad \rightarrow \boxed{I_4 = 2}$$

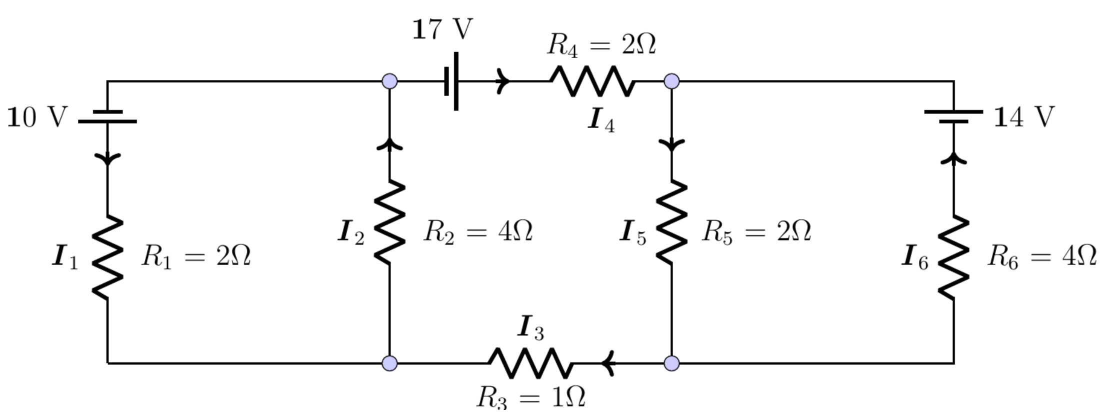
$$-41I_5 = -97 - 36(3) \quad \rightarrow \boxed{I_5 = 5}$$

$$77I_6 = 231 \quad \rightarrow \boxed{I_6 = 3}$$

∴ ***Solution***: $\boxed{(2, \ 4, \ 2, \ 2, \ 5, \ 3)}$

Exercise

Determine the currents $I_1, I_2, I_3, I_4, I_5, \textit{and } I_6$ for the electrical network shown below



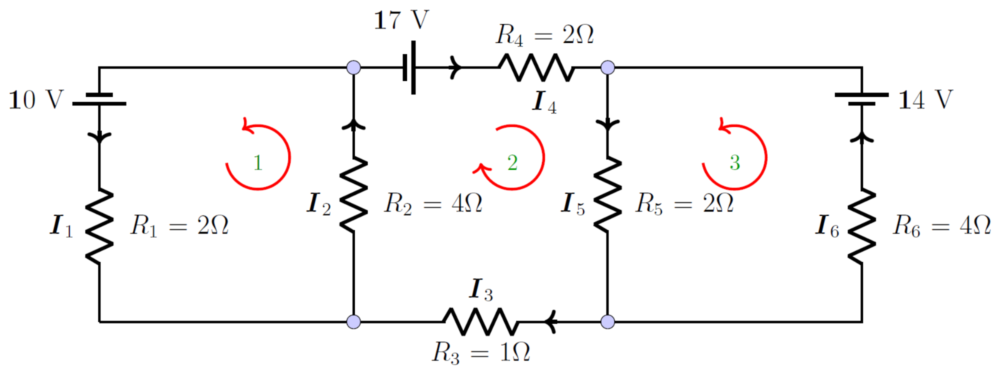
Solution

$1 \rightarrow \quad I_1 + I_3 = I_2$

$2 \rightarrow \quad I_1 + I_4 = I_2$

$3 \rightarrow \quad I_3 + I_6 = I_5$

$4 \rightarrow \quad I_4 + I_6 = I_5$



$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ I_1 - I_2 + I_4 = 0 \\ I_3 - I_5 + I_6 = 0 \\ I_4 - I_5 + I_6 = 0 \\ 2I_1 + 4I_2 = 10 \\ 4I_2 + I_3 + 2I_4 + 2I_5 = 17 \\ 2I_5 + 4I_6 = 14 \end{cases}$$

$$\begin{array}{cc} \left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 2 & 0 & 17 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \end{array}\right) & \begin{array}{l} \\ R_2 - R_1 \\ \\ \\ R_5 - R_1 \\ \\ \\ \end{array} \end{array}$$

$$\begin{array}{cc} \left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 4 & 1 & 2 & 2 & 0 & 17 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \end{array}\right) & \begin{array}{l} \\ \\ R_3 + R_2 \\ \\ \\ 3R_6 - 4R_5 \\ \\ \end{array} \end{array}$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 7 & 6 & 6 & 0 & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \end{array}\right) \quad R_4 - R_3$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 7 & 6 & 6 & 0 & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \end{array}\right) \quad R_6 + 7R_2$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 13 & 6 & 0 & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad R_5 - 13R_4$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 19 & -13 & 31 \\ 0 & 0 & 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad 19R_6 - R_5$$

$$\left(\begin{array}{cccccc|c} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & 0 & 0 & 5 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 19 & -13 & 31 \\ 0 & 0 & 0 & 0 & 0 & 51 & 102 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} (4) \\ (3) \\ (2) \\ (1) \\ I_5 = \frac{1}{19}\left(31 + 13I_6\right) \\ \underline{I_6 = 2} \end{array}$$

$\underline{I_5 = 3}$

$(1) \rightarrow \quad \boxed{I_4 = I_5 - I_6 = 1}$

$(2) \rightarrow \quad \boxed{I_3 = I_4 = 1}$

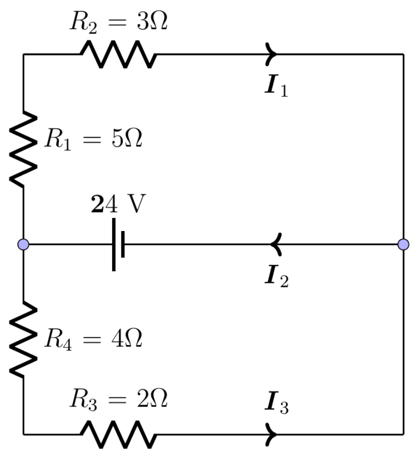
$(3) \rightarrow \quad \boxed{I_2 = \frac{1}{3}\left(I_3 + 5\right) = 2}$

$(4) \rightarrow \quad \boxed{I_1 = I_2 - I_3 = 1}$

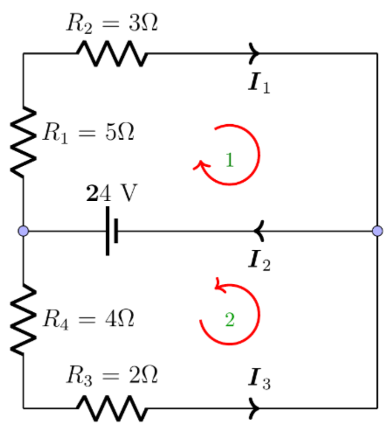
$\therefore$ ***Solution***: $\quad \boxed{(1, \quad 2, \quad 1, \quad 1, \quad 3)}$

Exercise

Determine the currents $I_1, I_2,$ *and* I_3 for the electrical network shown below



Solution



$$\begin{cases} I_1 + I_3 = I_2 \\ 5I_1 + 3I_1 + I_2 = 24 \\ 4I_3 + 2I_3 + I_2 = 24 \end{cases}$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 8I_1 + I_2 = 24 \\ I_2 + 6I_3 = 24 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 8 & 1 & 0 \\ 0 & 1 & 6 \end{vmatrix} = 62 \qquad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 24 & 1 & 0 \\ 24 & 1 & 6 \end{vmatrix} = 144$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 8 & 24 & 0 \\ 0 & 24 & 6 \end{vmatrix} = 336 \qquad D = \begin{vmatrix} 1 & -1 & 0 \\ 8 & 1 & 24 \\ 0 & 1 & 24 \end{vmatrix} = 192$$

$$I_1 = \frac{144}{62} = \frac{72}{31}$$

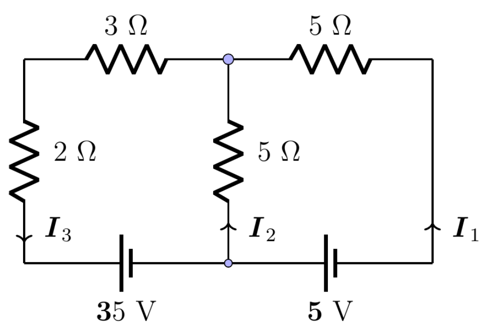
$$I_2 = \frac{336}{62} = \frac{168}{31}$$

$$I_1 = \frac{192}{62} = \frac{96}{31}$$

∴ ***Solution***: $\underline{I_1 = \frac{72}{31}\ A}\,\Big|$ $\quad \underline{I_2 = \frac{168}{31}\ A}\,\Big|$ $\quad \underline{I_3 = \frac{96}{31}\ A}\,\Big|$

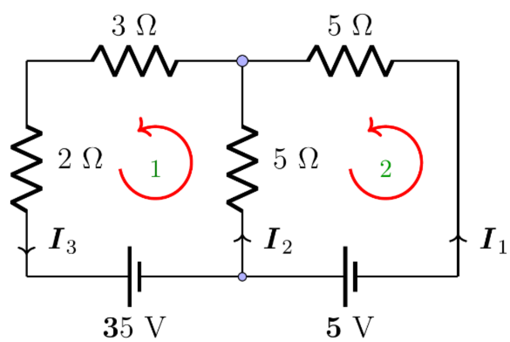
Exercise

Determine the currents I_1, I_2, *and* I_3 for the electrical network shown below



Solution

$$\begin{array}{l} \\ (2) \rightarrow \\ (1) \rightarrow \end{array} \begin{cases} I_1 + I_2 = I_3 \\ 5I_1 - 5I_2 = 5 \\ 5I_2 + 3I_3 + 2I_3 = 35 \end{cases}$$



$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 8I_1 + I_2 = 24 \\ I_2 + 6I_3 = 24 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 8 & 1 & 0 \\ 0 & 1 & 6 \end{vmatrix} \underline{= 62}\,\big|$$

$$D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 24 & 1 & 0 \\ 24 & 1 & 6 \end{vmatrix} \underline{= 144}\,\big|$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 8 & 24 & 0 \\ 0 & 24 & 6 \end{vmatrix} \underline{= 336}\,\big| \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 8 & 1 & 24 \\ 0 & 1 & 24 \end{vmatrix} \underline{= 192}\,\big|$$

$$I_1 = \frac{144}{62} = \frac{72}{31}$$

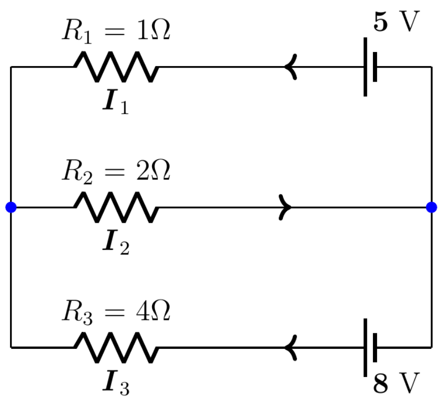
$$I_2 = \frac{336}{62} = \frac{168}{31}$$

$$I_1 = \frac{192}{62} = \frac{96}{31}$$

$\therefore$ ***Solution***: $\boxed{I_1 = \frac{72}{31}\ A}$ $\boxed{I_2 = \frac{168}{31}\ A}$ $\boxed{I_3 = \frac{96}{31}\ A}$

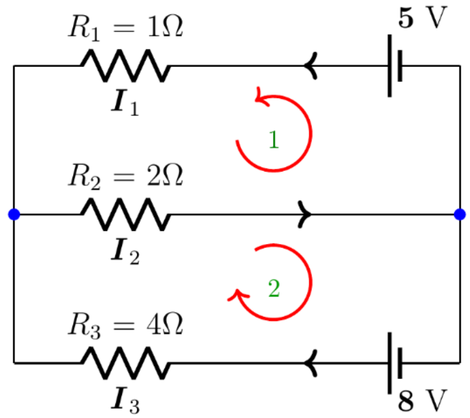
Exercise

Determine the currents $I_1,\ I_2,\ and\ I_3$ for the electrical network shown below



How is the result affected when the 5 V is changed to 2 V and 8V to 6 V?

Solution



$$\begin{cases} I_1 + I_3 = I_2 \\ I_1 + I_2 = 5 & (1) \rightarrow \\ 2I_2 + 4I_3 = 8 & (2) \rightarrow \end{cases}$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ I_1 + I_2 = 5 \\ 2I_2 + 4I_3 = 8 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 4 \end{vmatrix} \boxed{= 10} \qquad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 5 & 1 & 0 \\ 8 & 2 & 4 \end{vmatrix} \boxed{= 22}$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 5 & 0 \\ 0 & 8 & 4 \end{vmatrix} \boxed{= 28} \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 5 \\ 0 & 2 & 8 \end{vmatrix} \boxed{= 6}$$

$\therefore$ ***Solution***: $\boxed{I_1 = \frac{11}{5}\ A}$ $\boxed{I_2 = \frac{19}{5}\ A}$ $\boxed{I_3 = \frac{3}{5}\ A}$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ I_1 + I_2 = 2 \\ 2I_2 + 4I_3 = 6 \end{cases}$$

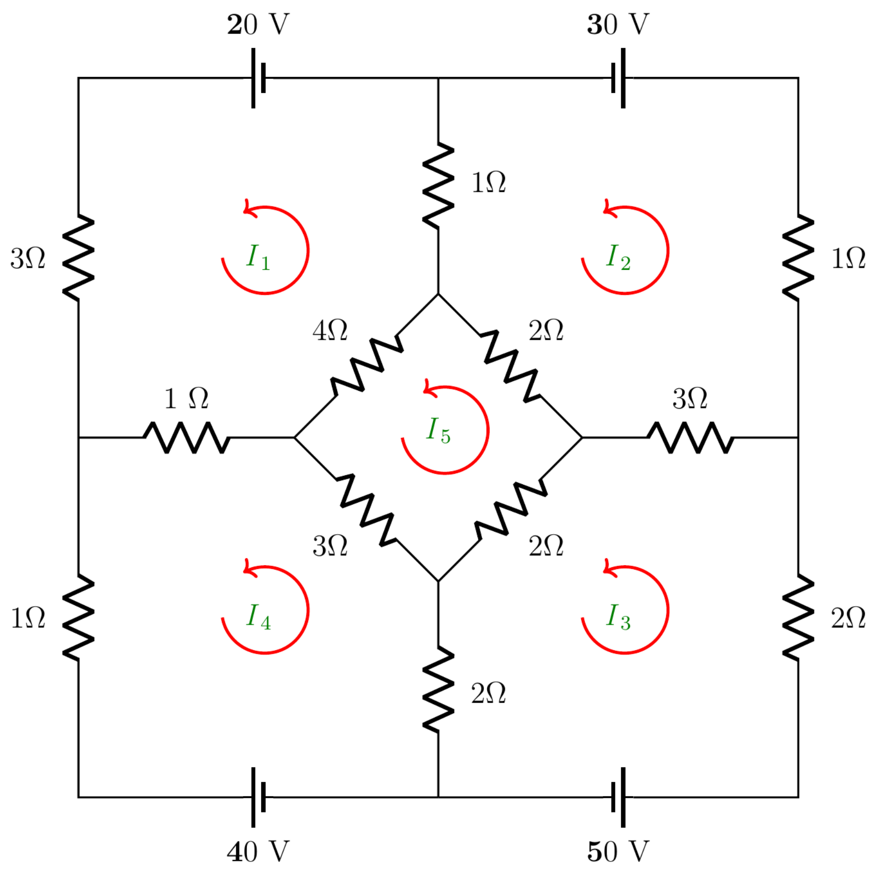
$$D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 2 & 1 & 0 \\ 6 & 2 & 4 \end{vmatrix} = 6 \qquad D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 6 & 4 \end{vmatrix} = 14 \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 6 \end{vmatrix} = 8$$

$\therefore$ ***Solution***: $\boxed{I_1 = \frac{3}{5}\ A}$ $\boxed{I_2 = \frac{7}{5}\ A}$ $\boxed{I_3 = \frac{4}{5}\ A}$

Exercise

Determine the currents for the electrical network:

Solution



$$\begin{cases} 9I_1 - I_2 - I_3 - 4I_5 = 20 \\ 7I_2 - I_1 - 3I_3 - 2I_5 = -30 \\ 9I_3 - 3I_2 - 2I_4 - 2I_5 = 50 \\ 7I_4 - I_1 - 2I_3 - 3I_5 = -40 \\ 11I_5 - 4I_1 - 2I_2 - 2I_3 - 3I_4 = 0 \end{cases}$$

$$\begin{cases} 9I_1 - I_2 - I_3 - 4I_5 = 20 \\ I_1 - 7I_2 + 3I_3 + 2I_5 = 30 \\ -3I_2 + 9I_3 - 2I_4 - 2I_5 = 50 \\ I_1 + 2I_3 - 7I_4 + 3I_5 = 40 \\ 4I_1 + 2I_2 + 2I_3 + 3I_4 - 11I_5 = 0 \end{cases}$$

$$\begin{array}{c} \left[\begin{array}{ccccc|c} 9 & -1 & -1 & 0 & -4 & 20 \\ 1 & -7 & 3 & 0 & 2 & 30 \\ 0 & -3 & 9 & -2 & -2 & 50 \\ 1 & 0 & 2 & -7 & 3 & 40 \\ 4 & 2 & 2 & 3 & -11 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ 9R_2 - R_1 \\ \\ 9R_4 - R_1 \\ 9R_5 - 4R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 9 & -1 & -1 & 0 & -4 & 20 \\ 0 & -62 & 28 & 0 & 22 & 250 \\ 0 & -3 & 9 & -2 & -2 & 50 \\ 0 & 1 & 19 & -63 & 31 & 340 \\ 0 & 22 & 22 & 27 & -83 & -80 \end{array}\right] \begin{array}{l} \\ -\frac{1}{2}R_2 \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} 9 & -1 & -1 & 0 & -4 & 20 \\ 0 & 31 & -14 & 0 & -11 & -125 \\ 0 & -3 & 9 & -2 & -2 & 50 \\ 0 & 1 & 19 & -63 & 31 & 340 \\ 0 & 22 & 22 & 27 & -83 & -80 \end{array}\right] \begin{array}{l} 31R_1 + R_2 \\ \\ 31R_3 + 3R_2 \\ 31R_4 - R_2 \\ 31R_5 - 22R_2 \end{array}$$

$$\left[\begin{array}{ccccc|c} 279 & 0 & -45 & 0 & 135 & 495 \\ 0 & 31 & -14 & 0 & -11 & -125 \\ 0 & 0 & 237 & -62 & -95 & 1{,}175 \\ 0 & 0 & 603 & -1{,}953 & 972 & 10{,}665 \\ 0 & 0 & 990 & 837 & -2{,}331 & 270 \end{array}\right] \begin{array}{l} \frac{1}{9}R_1 \\ \\ \\ \frac{1}{9}R_4 \\ \frac{1}{9}R_5 \end{array}$$

$$\left[\begin{array}{ccccc|c} 31 & 0 & -5 & 0 & 15 & 55 \\ 0 & 31 & -14 & 0 & -11 & -125 \\ 0 & 0 & 237 & -62 & -95 & 1{,}175 \\ 0 & 0 & 67 & -217 & 108 & 1{,}185 \\ 0 & 0 & 110 & 93 & -259 & 30 \end{array}\right] \begin{array}{l} \\ \\ \\ 237R_4 - 67R_3 \\ 237R_5 - 110R_3 \end{array}$$

$$\left[\begin{array}{ccccc|c} 31 & 0 & -5 & 0 & 15 & 55 \\ 0 & 31 & -14 & 0 & -11 & -125 \\ 0 & 0 & 237 & -62 & -95 & 1{,}175 \\ 0 & 0 & 0 & -47{,}275 & 31{,}961 & 202{,}120 \\ 0 & 0 & 0 & 28{,}861 & -50{,}933 & -122{,}140 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ 47275R_5 + 28861R_4 \end{array}$$

$$\left[\begin{array}{ccccc|c} 31 & 0 & -5 & 0 & 15 & 55 \\ 0 & 31 & -14 & 0 & -11 & -125 \\ 0 & 0 & 237 & -62 & -95 & 1{,}175 \\ 0 & 0 & 0 & -47{,}275 & 31{,}961 & 202{,}120 \\ 0 & 0 & 0 & 0 & -1{,}484{,}514{,}360 & 59{,}216{,}820 \end{array}\right]$$

$$\begin{aligned} 31I_1 &= 55 - 15I_5 + 5I_3 \\ 31I_2 &= -125 + 11I_5 + 14I_3 \\ 237I_3 &= 1175 + 95I_5 + 62I_4 \\ -47275I_4 &= 202120 - 31961I_5 \end{aligned}$$

$$\begin{aligned} I_5 &= -\frac{59{,}216{,}820}{1{,}484{,}514{,}360} \\ &= -\frac{986{,}947}{24{,}741{,}906} \\ &\approx -0.04\ A \end{aligned}$$

$$\begin{aligned} I_4 &= \frac{1}{47{,}275}\left(\frac{986{,}947}{24{,}741{,}906} - 202{,}120\right) \\ &= -\frac{1}{47{,}275}\left(\frac{5{,}000{,}833{,}053{,}773}{24{,}741{,}906}\right) \\ &\approx -4.3\ A \end{aligned}$$

$$\begin{aligned} I_3 &= \frac{1}{237}\big(1175 + 95(-.04) + 62(-4.3)\big) \\ &\approx 3.82\ A \end{aligned}$$

$$\begin{aligned} I_2 &= \frac{1}{31}\big(-125 + 11(-.04) + 14(3.82)\big) \\ &\approx -2.323\ A \end{aligned}$$

$$\begin{aligned} I_1 &= \frac{1}{31}\big(55 - 15(-.04) + 5(3.82)\big) \\ &\approx 2.37\ A \end{aligned}$$

Exercise

Use A^{-1} to decode the cryptogram

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$$

11 21 64 112 25 50 29 53 23 46 40 75 55 92

Solution

$$A^{-1} = -\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

$$\begin{bmatrix} 11 & 21 \end{bmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{bmatrix} 8 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 64 & 112 \end{bmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{bmatrix} 16 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 50 \end{bmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{bmatrix} 25 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 29 & 53 \end{bmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{bmatrix} 14 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 23 & 46 \end{bmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{bmatrix} 23 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 40 & 75 \end{bmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{bmatrix} 25 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 55 & 92 \end{bmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{bmatrix} 1 & 18 \end{bmatrix}$$

8	1	16	16	28	00	14	5	23	0	25	5	1	18
H	*A*	*P*	*P*	*Y*	_	*N*	*E*	*W*	_	*Y*	*E*	*A*	*R*

∴ ***Solution***: ***Happy New Year***

Exercise

Use A^{-1} to decode the cryptogram

$$A=\begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{pmatrix}$$

$13 \quad 19 \quad 10 \quad -1 \quad -33 \quad -77 \quad 3 \quad -2 \quad -14 \quad 4 \quad 1 \quad -9 \quad -5 \quad -25 \quad -47 \quad 4 \quad 1 \quad -9$

Solution

$$A^{-1}=\begin{pmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{pmatrix}$$

$$\begin{bmatrix} 13 & 19 & 10 \end{bmatrix}\begin{pmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{pmatrix}=\begin{bmatrix} 9 & 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -33 & -77 \end{bmatrix}\begin{pmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{pmatrix}=\begin{bmatrix} 2 & 5 & 18 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 & -14 \end{bmatrix}\begin{pmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{pmatrix}=\begin{bmatrix} 7 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & -9 \end{bmatrix}\begin{pmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{pmatrix}=\begin{bmatrix} 5 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -5 & -25 & -47 \end{bmatrix}\begin{pmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{pmatrix}=\begin{bmatrix} 0 & 1 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & -9 \end{bmatrix}\begin{pmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{pmatrix}=\begin{bmatrix} 5 & 1 & 4 \end{bmatrix}$$

9	3	5	2	5	18	7	0	4	5	1	4	0	1	8	5	1	4
I	*C*	*E*	*B*	*E*	*R*	*G*	_	*D*	*E*	*A*	*D*	_	*A*	*H*	*E*	*A*	*D*

∴ ***Solution***: ***Iceberg Dead Ahead***

Exercise

Consider the invertible matrix: $A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{pmatrix}$

The message: ***ICEBERG DEAD AHEAD***

a) Write the uncoded row matrices 1×3 for the message.
b) Use the matrix A to encode the message.
c) Decode a message from part *b*) given the matrix A.

Solution

a)

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

$$\begin{matrix} I\ C\ E & B\ E\ R & G\ \_\ D & E\ A\ D & \_\ A\ H & E\ A\ D \\ \begin{bmatrix} 9 & 3 & 5 \end{bmatrix} & \begin{bmatrix} 2 & 5 & 18 \end{bmatrix} & \begin{bmatrix} 7 & 0 & 4 \end{bmatrix} & \begin{bmatrix} 5 & 1 & 4 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 8 \end{bmatrix} & \begin{bmatrix} 5 & 1 & 4 \end{bmatrix} \end{matrix}$$

b) Let encode the message ***ICEBERG DEAD AHEAD***

$$\begin{bmatrix} 9 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 29 & 80 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & 18 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} -37 & 3 & 175 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} -5 & 6 & 42 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} -4 & 9 & 47 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} -21 & -5 & 65 \end{bmatrix}$$

$$\begin{bmatrix}5 & 1 & 4\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 3 & 7 & 9\\ -3 & -2 & 7\end{bmatrix}=\begin{bmatrix}-4 & 9 & 47\end{bmatrix}$$

The sequence of coded row matrices is

$$\begin{bmatrix}3 & 29 & 80\end{bmatrix}\ \begin{bmatrix}-37 & 3 & 175\end{bmatrix}\ \begin{bmatrix}-5 & 6 & 42\end{bmatrix}\ \begin{bmatrix}-4 & 9 & 47\end{bmatrix}\ \begin{bmatrix}-21 & -9 & 65\end{bmatrix}\ \begin{bmatrix}-4 & 9 & 47\end{bmatrix}$$

The cryptogram:

$$3\ \ 29\ \ 80\ \ -37\ \ 3\ \ 175\ \ -5\ \ 6\ \ 42\ \ -4\ \ 9\ \ 47\ \ -21\ \ -9\ \ 65\ \ -4\ \ 9\ \ 47$$

c) To decode a message given the matrix A.

$$|A|=\begin{vmatrix}1 & 2 & 2\\ 3 & 7 & 9\\ -3 & -2 & 7\end{vmatrix}=1$$

$$A^{-1}=\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}$$

With the cryptogram:

$$\begin{bmatrix}3 & 29 & 80\end{bmatrix}\ \begin{bmatrix}-37 & 3 & 175\end{bmatrix}\ \begin{bmatrix}-5 & 6 & 42\end{bmatrix}\ \begin{bmatrix}-4 & 9 & 47\end{bmatrix}\ \begin{bmatrix}-21 & -9 & 65\end{bmatrix}\ \begin{bmatrix}-4 & 9 & 47\end{bmatrix}$$

$$\begin{bmatrix}3 & 29 & 80\end{bmatrix}\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}=\begin{bmatrix}9 & 3 & 5\end{bmatrix}$$

$$\begin{bmatrix}-37 & 3 & 175\end{bmatrix}\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}=\begin{bmatrix}2 & 5 & 18\end{bmatrix}$$

$$\begin{bmatrix}-5 & 6 & 42\end{bmatrix}\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}=\begin{bmatrix}7 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}-4 & 9 & 47\end{bmatrix}\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}=\begin{bmatrix}5 & 1 & 4\end{bmatrix}$$

$$\begin{bmatrix}-21 & -9 & 65\end{bmatrix}\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 8\end{bmatrix}$$

$$\begin{bmatrix}-4 & 9 & 47\end{bmatrix}\begin{bmatrix}67 & -18 & 4\\ -48 & 13 & -3\\ 15 & -4 & 1\end{bmatrix}=\begin{bmatrix}5 & 1 & 4\end{bmatrix}$$

The message is:

9	3	5	2	5	18	7	0	1	5	1	4	0	1	8	5	1	4
I	*C*	*E*	*B*	*E*	*R*	*G*	_	*D*	*E*	*A*	*D*	_	*A*	*H*	*E*	*A*	*D*

Exercise

You want to send the message: ***LINEAR ALGEBRA*** with a key word ***MATH***

a) Write the matrix A.

b) Write the uncoded row matrices 1×2 for the message.

c) Use the matrix A to encode the message.

d) Decode a message from part b) given the matrix A.

Solution

a)

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

M	*A*	*T*	*H*
13	1	20	8

$$A=\begin{pmatrix}13 & 1\\ 20 & 8\end{pmatrix}$$

b)

L	*I*	*N*	*E*	*A*	*R*	_	*A*	*L*	*G*	*E*	*B*	*R*	*A*
12	9	14	5	1	18	0	1	12	7	5	2	18	1

$$\begin{bmatrix}12 & 9\end{bmatrix}\quad\begin{bmatrix}14 & 5\end{bmatrix}\quad\begin{bmatrix}1 & 18\end{bmatrix}\quad\begin{bmatrix}0 & 1\end{bmatrix}\quad\begin{bmatrix}12 & 7\end{bmatrix}\quad\begin{bmatrix}5 & 2\end{bmatrix}\quad\begin{bmatrix}18 & 1\end{bmatrix}$$

c) Encoding the message

$$\begin{bmatrix}12 & 9\end{bmatrix}\begin{bmatrix}13 & 1\\ 20 & 8\end{bmatrix}=\begin{bmatrix}336 & 84\end{bmatrix}$$

$$\begin{bmatrix}14 & 5\end{bmatrix}\begin{bmatrix}13 & 1\\ 20 & 8\end{bmatrix}=\begin{bmatrix}282 & 54\end{bmatrix}$$

$$\begin{bmatrix}1 & 18\end{bmatrix}\begin{bmatrix}13 & 1\\ 20 & 8\end{bmatrix}=\begin{bmatrix}373 & 145\end{bmatrix}$$

$$\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}13 & 1\\ 20 & 8\end{bmatrix}=\begin{bmatrix}20 & 8\end{bmatrix}$$

$$\begin{bmatrix}12 & 7\end{bmatrix}\begin{bmatrix}13 & 1\\ 20 & 8\end{bmatrix}=\begin{bmatrix}296 & 68\end{bmatrix}$$

$$\begin{bmatrix}5 & 2\end{bmatrix}\begin{bmatrix}13 & 1\\ 20 & 8\end{bmatrix}=\begin{bmatrix}105 & 21\end{bmatrix}$$

$$\begin{bmatrix}18 & 1\end{bmatrix}\begin{bmatrix}13 & 1\\ 20 & 8\end{bmatrix}=\begin{bmatrix}254 & 26\end{bmatrix}$$

The cryptogram:

336 84 282 54 373 145 20 8 296 68 105 21 254 26

d) To decode a message given the matrix A.

$$A=\begin{pmatrix}13 & 1\\ 20 & 8\end{pmatrix}$$

$$A^{-1}=\frac{1}{84}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}$$

$$=\begin{pmatrix}\frac{2}{21} & -\frac{1}{84}\\ -\frac{5}{21} & \frac{13}{84}\end{pmatrix}$$

With the cryptogram:

$$\begin{bmatrix}336 & 84\end{bmatrix}\ \begin{bmatrix}282 & 54\end{bmatrix}\ \begin{bmatrix}373 & 145\end{bmatrix}\ \begin{bmatrix}20 & 8\end{bmatrix}\ \begin{bmatrix}296 & 68\end{bmatrix}\ \begin{bmatrix}105 & 21\end{bmatrix}\ \begin{bmatrix}254 & 26\end{bmatrix}$$

$$\begin{bmatrix}336 & 84\end{bmatrix}\begin{bmatrix}\frac{2}{21} & -\frac{1}{84}\\ -\frac{5}{21} & \frac{13}{84}\end{bmatrix}=\begin{bmatrix}12 & 9\end{bmatrix}$$

$$\begin{bmatrix}282 & 54\end{bmatrix}\begin{bmatrix}\frac{2}{21} & -\frac{1}{84}\\ -\frac{5}{21} & \frac{13}{84}\end{bmatrix}=\begin{bmatrix}14 & 5\end{bmatrix}$$

$$\begin{bmatrix}373 & 145\end{bmatrix}\begin{bmatrix}\frac{2}{21} & -\frac{1}{84}\\ -\frac{5}{21} & \frac{13}{84}\end{bmatrix}=\begin{bmatrix}1 & 18\end{bmatrix}$$

$$\begin{bmatrix} 20 & 8 \end{bmatrix} \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 296 & 68 \end{bmatrix} \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = \begin{bmatrix} 12 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 105 & 21 \end{bmatrix} \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = \begin{bmatrix} 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 254 & 26 \end{bmatrix} \begin{bmatrix} \frac{2}{21} & -\frac{1}{84} \\ -\frac{5}{21} & \frac{13}{84} \end{bmatrix} = \begin{bmatrix} 18 & 1 \end{bmatrix}$$

$$\begin{array}{cccccccccccccc} 12 & 9 & 14 & 5 & 1 & 18 & 0 & 1 & 12 & 7 & 5 & 2 & 18 & 1 \\ L & I & N & E & A & R & \_ & A & L & G & E & B & R & A \end{array}$$

The message is: ***Linear Algebra***

Exercise

You want to send the message: ***CRYPTOGRAPHY IS A METHOD OF PROTECTING INFORMATIONS*** with a key word ***CODE***

a) Write the matrix A.

b) Write the uncoded row matrices 1×2 for the message.

c) Use the matrix A to encode the message.

d) Decode a message from part *b*) given the matrix A.

Solution

a)

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

$$\begin{array}{cccc} C & O & D & E \\ 3 & 15 & 4 & 5 \end{array}$$

$$A = \begin{pmatrix} 3 & 15 \\ 4 & 5 \end{pmatrix}$$

b)

C	*R*	*Y*	*P*	*T*	*O*	*G*	*R*	*A*	*P*	*H*	*Y*	_	*I*	*S*	_	*A*	_
3	18	25	16	20	15	7	18	1	16	8	25	0	9	19	0	1	0

M	*E*	*T*	*H*	*O*	*D*	_	*O*	*F*	_	*P*	*R*	*O*	*T*	*E*	*C*	*T*	*I*	*N*	*G*
13	5	20	8	15	4	0	15	6	0	16	18	15	20	5	3	20	9	14	7

_	*I*	*N*	*F*	*O*	*R*	*M*	*A*	*T*	*I*	*O*	*N*	*S*	_
0	9	14	6	15	18	13	1	20	9	15	14	19	0

$$\begin{bmatrix}3 & 18\end{bmatrix} \quad \begin{bmatrix}25 & 16\end{bmatrix} \quad \begin{bmatrix}20 & 15\end{bmatrix} \quad \begin{bmatrix}7 & 18\end{bmatrix} \quad \begin{bmatrix}1 & 16\end{bmatrix} \quad \begin{bmatrix}8 & 25\end{bmatrix} \quad \begin{bmatrix}0 & 9\end{bmatrix} \quad \begin{bmatrix}19 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\end{bmatrix} \quad \begin{bmatrix}13 & 5\end{bmatrix} \quad \begin{bmatrix}20 & 8\end{bmatrix} \quad \begin{bmatrix}15 & 4\end{bmatrix} \quad \begin{bmatrix}0 & 15\end{bmatrix} \quad \begin{bmatrix}6 & 0\end{bmatrix} \quad \begin{bmatrix}16 & 18\end{bmatrix} \quad \begin{bmatrix}15 & 20\end{bmatrix}$$

$$\begin{bmatrix}5 & 3\end{bmatrix} \quad \begin{bmatrix}20 & 9\end{bmatrix} \quad \begin{bmatrix}14 & 7\end{bmatrix} \quad \begin{bmatrix}0 & 9\end{bmatrix} \quad \begin{bmatrix}14 & 6\end{bmatrix} \quad \begin{bmatrix}15 & 18\end{bmatrix} \quad \begin{bmatrix}13 & 1\end{bmatrix} \quad \begin{bmatrix}20 & 9\end{bmatrix}$$

$$\begin{bmatrix}15 & 14\end{bmatrix} \quad \begin{bmatrix}19 & 0\end{bmatrix}$$

c) Encoding the message

$$\begin{bmatrix}3 & 18\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}81 & 135\end{bmatrix}$$

$$\begin{bmatrix}25 & 16\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}139 & 455\end{bmatrix}$$

$$\begin{bmatrix}20 & 15\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}120 & 375\end{bmatrix}$$

$$\begin{bmatrix}7 & 18\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}93 & 195\end{bmatrix}$$

$$\begin{bmatrix}1 & 16\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}67 & 95\end{bmatrix}$$

$$\begin{bmatrix}8 & 25\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}124 & 245\end{bmatrix}$$

$$\begin{bmatrix}0 & 9\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}36 & 45\end{bmatrix}$$

$$\begin{bmatrix}19 & 0\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}57 & 285\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}3 & 15\end{bmatrix}$$

$$\begin{bmatrix} 13 & 5 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 59 & 220 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 8 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 92 & 340 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 4 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 61 & 245 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 15 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 60 & 75 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 18 & 90 \end{bmatrix}$$

$$\begin{bmatrix} 16 & 18 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 120 & 330 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 20 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 125 & 325 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 27 & 90 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 9 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 96 & 345 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 7 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 70 & 245 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 9 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 36 & 45 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 6 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 66 & 240 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 18 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 117 & 315 \end{bmatrix}$$

$$\begin{bmatrix} 13 & 1 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 43 & 200 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 9 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 96 & 345 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 14 \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 101 & 295 \end{bmatrix}$$

$$\begin{bmatrix}19 & 0\end{bmatrix}\begin{bmatrix}3 & 15\\ 4 & 5\end{bmatrix}=\begin{bmatrix}57 & 285\end{bmatrix}$$

The cryptogram:

81 135 139 455 120 375 93 195 67 95 124 245 36 45 57 285
3 15 59 220 92 340 61 245 60 75 18 90 120 330 125 325
27 90 96 345 70 245 36 45 66 240 117 315 43 200 96 345
101 295 57 285

d) To decode a message given the matrix A.

$$A=\begin{pmatrix}3 & 15\\ 4 & 5\end{pmatrix}$$

$$A^{-1}=-\frac{1}{45}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}81 & 135\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-135 & -810\end{bmatrix}$$
$$=\begin{bmatrix}3 & 18\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}139 & 455\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-1{,}125 & -720\end{bmatrix}$$
$$=\begin{bmatrix}25 & 16\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}120 & 375\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-900 & -675\end{bmatrix}$$
$$=\begin{bmatrix}20 & 15\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}93 & 195\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-315 & -810\end{bmatrix}$$
$$=\begin{bmatrix}7 & 18\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}67 & 95\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-45 & -720\end{bmatrix}$$
$$=\begin{bmatrix}1 & 16\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}124 & 245\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-360 & -1{,}125\end{bmatrix}$$
$$=\begin{bmatrix}8 & 25\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}36 & 45\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}0 & -405\end{bmatrix}$$
$$=\begin{bmatrix}0 & 9\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}57 & 285\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-855 & 0\end{bmatrix}$$
$$=\begin{bmatrix}19 & 0\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}3 & 15\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-45 & 0\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}59 & 220\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-585 & -225\end{bmatrix}$$
$$=\begin{bmatrix}13 & 5\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}92 & 340\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-900 & -360\end{bmatrix}$$
$$=\begin{bmatrix}20 & 8\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}61 & 245\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-675 & -180\end{bmatrix}$$
$$=\begin{bmatrix}15 & 4\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}60 & 75\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}0 & -675\end{bmatrix}$$
$$=\begin{bmatrix}0 & 15\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}18 & 90\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-270 & 0\end{bmatrix}$$
$$=\begin{bmatrix}6 & 0\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}120 & 330\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-720 & -810\end{bmatrix}$$
$$=\begin{bmatrix}16 & 18\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}125 & 325\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-675 & -900\end{bmatrix}$$
$$=\begin{bmatrix}15 & 20\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}27 & 90\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-225 & -135\end{bmatrix}$$

$$=\begin{bmatrix}5 & 3\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}96 & 345\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-900 & -405\end{bmatrix}$$
$$=\begin{bmatrix}20 & 9\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}70 & 245\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-630 & -315\end{bmatrix}$$
$$=\begin{bmatrix}14 & 7\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}36 & 45\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}0 & -405\end{bmatrix}$$
$$=\begin{bmatrix}0 & 9\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}66 & 240\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-630 & -270\end{bmatrix}$$
$$=\begin{bmatrix}14 & 6\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}117 & 315\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-675 & -810\end{bmatrix}$$
$$=\begin{bmatrix}15 & 18\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}43 & 200\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-585 & -45\end{bmatrix}$$
$$=\begin{bmatrix}13 & 1\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}96 & 345\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-900 & -405\end{bmatrix}$$
$$=\begin{bmatrix}20 & 9\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}101 & 295\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-675 & -630\end{bmatrix}$$
$$=\begin{bmatrix}15 & 14\end{bmatrix}$$

$$-\frac{1}{45}\begin{bmatrix}57 & 285\end{bmatrix}\begin{pmatrix}5 & -15\\ -4 & 3\end{pmatrix} = -\frac{1}{45}\begin{bmatrix}-855 & 0\end{bmatrix}$$
$$=\begin{bmatrix}19 & 0\end{bmatrix}$$

3	18	25	16	20	15	7	18	1	16	8	25	0	9	19	0	1	0	13	5
C	*R*	*Y*	*P*	*T*	*O*	*G*	*R*	*A*	*P*	*H*	*Y*	_	*I*	*S*	_	*A*	_	*M*	*E*

$$\begin{array}{cccccccccccccccccccc} 20 & 8 & 15 & 4 & 0 & 15 & 6 & 0 & 16 & 18 & 15 & 20 & 5 & 3 & 20 & 9 & 14 & 7 & 0 & 9 \\ T & H & O & D & \_ & O & F & \_ & P & R & O & T & E & C & T & I & N & G & \_ & I \end{array}$$

$$\begin{array}{cccccccccccc} 14 & 6 & 15 & 18 & 16 & 1 & 20 & 9 & 15 & 14 & 19 & 0 \\ N & F & O & R & M & A & T & I & O & N & S & \_ \end{array}$$

The message is: ***Cryptography is a Method of Protecting Informations***

Exercise

Write the matrix A with a key word ***MATH,*** then decode the cryptogram

117 9 456 132 386 62 260 104 413 161 104 8

Solution

$$\begin{array}{cccc} M & A & T & H \\ 13 & 1 & 20 & 8 \end{array}$$

$$A = \begin{pmatrix} 13 & 1 \\ 20 & 8 \end{pmatrix}$$

$$A^{-1} = \frac{1}{84}\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix}$$

With the cryptogram:

$$\begin{bmatrix} 117 & 9 \end{bmatrix} \begin{bmatrix} 456 & 132 \end{bmatrix} \begin{bmatrix} 386 & 62 \end{bmatrix} \begin{bmatrix} 260 & 104 \end{bmatrix} \begin{bmatrix} 413 & 161 \end{bmatrix} \begin{bmatrix} 104 & 8 \end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix} 117 & 9 \end{bmatrix}\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}\begin{bmatrix} 756 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & 0 \end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix} 456 & 132 \end{bmatrix}\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}\begin{bmatrix} 1,008 & 1,260 \end{bmatrix}$$
$$= \begin{bmatrix} 12 & 15 \end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix} 386 & 62 \end{bmatrix}\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}\begin{bmatrix} 1,848 & 420 \end{bmatrix}$$
$$= \begin{bmatrix} 22 & 5 \end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix} 260 & 104 \end{bmatrix}\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}\begin{bmatrix} 0 & 1,092 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 13 \end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix} 413 & 161 \end{bmatrix}\begin{pmatrix} 8 & -1 \\ -20 & 13 \end{pmatrix} = \frac{1}{84}\begin{bmatrix} 84 & 1,680 \end{bmatrix}$$

$$=\begin{bmatrix}1 & 20\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}104 & 8\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}=\frac{1}{84}\begin{bmatrix}672 & 0\end{bmatrix}$$

$$=\begin{bmatrix}8 & 0\end{bmatrix}$$

$$\begin{array}{cccccccccccc} 9 & 0 & 12 & 15 & 22 & 5 & 0 & 13 & 1 & 20 & 8 & 0\\ I & - & L & O & V & E & - & M & A & T & H & - \end{array}$$

The message is: ***I love math***

Exercise

Write the matrix A with a key word ***MATH,*** then decode the cryptogram

438 150 145 37 240 96 635 191 445 157 260 104 413 161 104 8

Solution

$$\begin{array}{cccc} M & A & T & H\\ 13 & 1 & 20 & 8 \end{array}$$

$$A=\begin{pmatrix}13 & 1\\ 20 & 8\end{pmatrix}$$

$$A^{-1}=\frac{1}{84}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}$$

With the cryptogram:

$$\begin{bmatrix}438 & 150\end{bmatrix}\ \begin{bmatrix}145 & 37\end{bmatrix}\ \begin{bmatrix}240 & 96\end{bmatrix}\ \begin{bmatrix}635 & 191\end{bmatrix}\ \begin{bmatrix}445 & 157\end{bmatrix}$$

$$\begin{bmatrix}260 & 104\end{bmatrix}\ \begin{bmatrix}413 & 161\end{bmatrix}\ \begin{bmatrix}104 & 8\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}438 & 150\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}=\frac{1}{84}\begin{bmatrix}504 & 1,512\end{bmatrix}$$

$$=\begin{bmatrix}6 & 18\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}145 & 37\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}=\frac{1}{84}\begin{bmatrix}420 & 336\end{bmatrix}$$

$$=\begin{bmatrix}5 & 4\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}240 & 96\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}=\frac{1}{84}\begin{bmatrix}0 & 1,008\end{bmatrix}$$

$$=\begin{bmatrix}0 & 12\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}635 & 191\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}=\frac{1}{84}\begin{bmatrix}1,260 & 1,848\end{bmatrix}$$
$$=\begin{bmatrix}15 & 22\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}445 & 157\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}=\frac{1}{84}\begin{bmatrix}420 & 1,596\end{bmatrix}$$
$$=\begin{bmatrix}5 & 19\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}260 & 104\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}=\frac{1}{84}\begin{bmatrix}0 & 1,092\end{bmatrix}$$
$$=\begin{bmatrix}0 & 13\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}413 & 161\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}=\frac{1}{84}\begin{bmatrix}84 & 1,680\end{bmatrix}$$
$$=\begin{bmatrix}1 & 20\end{bmatrix}$$

$$\frac{1}{84}\begin{bmatrix}104 & 8\end{bmatrix}\begin{pmatrix}8 & -1\\ -20 & 13\end{pmatrix}=\frac{1}{84}\begin{bmatrix}672 & 0\end{bmatrix}$$
$$=\begin{bmatrix}8 & 0\end{bmatrix}$$

$$\begin{array}{cccccccccccccccc} 6 & 18 & 5 & 4 & 0 & 12 & 15 & 22 & 5 & 19 & 0 & 13 & 1 & 20 & 8 & 0\\ F & R & E & D & - & L & O & V & E & S & - & M & A & T & H & - \end{array}$$

The message is: ***Fred loves math***

Exercise

Consider the invertible matrix: $A=\begin{pmatrix}1 & -2 & 2\\ -1 & 1 & 3\\ 1 & -1 & -4\end{pmatrix}$

Decode the cryptogram:

$$\begin{array}{l} 1\quad -5\quad 11\quad 19\quad -25\quad -45\quad 11\quad -16\quad -28\quad 20\quad -29\quad -27\\ 12\quad -12\quad -53\quad 40\quad -61\quad -35\quad 8\quad -17\quad 7 \end{array}$$

Solution

$$|A|=1$$

$$A^{-1}=\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}$$

With the cryptogram:

$$\begin{bmatrix}1 & -5 & 11\end{bmatrix}\ \begin{bmatrix}19 & -25 & -45\end{bmatrix}\ \begin{bmatrix}11 & -16 & -28\end{bmatrix}\ \begin{bmatrix}20 & -29 & -27\end{bmatrix}$$

$$\begin{bmatrix}12 & -12 & -53\end{bmatrix} \quad \begin{bmatrix}40 & -61 & -35\end{bmatrix} \quad \begin{bmatrix}8 & -17 & 7\end{bmatrix}$$

$$\begin{bmatrix}1 & -5 & 11\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}4 & 9 & 6\end{bmatrix}$$

$$\begin{bmatrix}19 & -25 & -45\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}6 & 5 & 18\end{bmatrix}$$

$$\begin{bmatrix}11 & -16 & -28\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}5 & 14 & 20\end{bmatrix}$$

$$\begin{bmatrix}20 & -29 & -27\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}9 & 1 & 12\end{bmatrix}$$

$$\begin{bmatrix}12 & -12 & -53\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}0 & 5 & 17\end{bmatrix}$$

$$\begin{bmatrix}40 & -61 & -35\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}21 & 1 & 20\end{bmatrix}$$

$$\begin{bmatrix}8 & -17 & 7\end{bmatrix}\begin{pmatrix}-1 & -10 & -8\\ -1 & -6 & -5\\ 0 & -1 & -1\end{pmatrix}=\begin{bmatrix}9 & 15 & 14\end{bmatrix}$$

4	9	6	6	5	18	5	14	20	9	1	12	0	5	17	21	1	20	9	15	14
D	*I*	*F*	*F*	*E*	*R*	*E*	*N*	*T*	*I*	*A*	*L*	_	*E*	*Q*	*U*	*A*	*T*	*I*	*O*	*N*

The message is: ***Differential Equation***.

Exercise

Determine the key word, then decode the given cryptogram

6 18 5 4 15 13 1 20 8

102 649 238 57 324 112 128 622 207

180 613 290 102 360 259 151 580 297

Hint: First row is the key

Solution

The key word from the first row is

6	18	5	4	15	13	1	20	8
f	*r*	*e*	*d*	*o*	*m*	*a*	*t*	*h*

Since it has 9 numbers, then the matrix is $9 = 3^2$ which is 3×3

$$A = \begin{pmatrix} 6 & 18 & 5 \\ 4 & 15 & 13 \\ 1 & 20 & 8 \end{pmatrix}$$

$$|A| = -857$$

$$A^{-1} = -\frac{1}{857}\begin{pmatrix} -140 & -44 & 159 \\ -19 & 43 & -58 \\ 65 & -102 & 18 \end{pmatrix}$$

$$= \frac{1}{857}\begin{pmatrix} 140 & 44 & -159 \\ 19 & -43 & 58 \\ -65 & 102 & -18 \end{pmatrix}$$

With the cryptogram:

$$[102 \quad 649 \quad 238] \quad [57 \quad 324 \quad 112] \quad [128 \quad 622 \quad 207]$$

$$[180 \quad 613 \quad 290] \quad [102 \quad 360 \quad 259] \quad [151 \quad 580 \quad 297]$$

$$\frac{1}{857}[102 \quad 649 \quad 238]\begin{pmatrix} 140 & 44 & -159 \\ 19 & -43 & 58 \\ -65 & 102 & -18 \end{pmatrix} = \frac{1}{857}[11{,}141 \quad 857 \quad 17{,}140]$$

$$= [13 \quad 1 \quad 20]$$

$$\frac{1}{857}[57 \quad 324 \quad 112]\begin{pmatrix} 140 & 44 & -159 \\ 19 & -43 & 58 \\ -65 & 102 & -18 \end{pmatrix} = \frac{1}{857}[6{,}856 \quad 0 \quad 7{,}713]$$

$$= [8 \quad 0 \quad 9]$$

$$\frac{1}{857}\begin{bmatrix}128 & 622 & 207\end{bmatrix}\begin{pmatrix}140 & 44 & -159\\ 19 & -43 & 58\\ -65 & 102 & -18\end{pmatrix}=\frac{1}{857}\begin{bmatrix}16,283 & 0 & 11,998\end{bmatrix}$$

$$=\begin{bmatrix}19 & 0 & 14\end{bmatrix}$$

$$\frac{1}{857}\begin{bmatrix}180 & 613 & 290\end{bmatrix}\begin{pmatrix}140 & 44 & -159\\ 19 & -43 & 58\\ -65 & 102 & -18\end{pmatrix}=\frac{1}{857}\begin{bmatrix}17,997 & 11,141 & 1,714\end{bmatrix}$$

$$=\begin{bmatrix}21 & 13 & 2\end{bmatrix}$$

$$\frac{1}{857}\begin{bmatrix}102 & 360 & 259\end{bmatrix}\begin{pmatrix}140 & 44 & -159\\ 19 & -43 & 58\\ -65 & 102 & -18\end{pmatrix}=\frac{1}{857}\begin{bmatrix}4,285 & 15,426 & 0\end{bmatrix}$$

$$=\begin{bmatrix}5 & 18 & 0\end{bmatrix}$$

$$\frac{1}{857}\begin{bmatrix}151 & 580 & 297\end{bmatrix}\begin{pmatrix}140 & 44 & -159\\ 19 & -43 & 58\\ -65 & 102 & -18\end{pmatrix}=\frac{1}{857}\begin{bmatrix}12,855 & 11,998 & 4,285\end{bmatrix}$$

$$=\begin{bmatrix}15 & 14 & 5\end{bmatrix}$$

13	1	20	8	0	9	19	0	14	21	13	2	5	18	0	15	14	5
M	*A*	*T*	*H*	–	*I*	*S*	–	*N*	*U*	*M*	*B*	*E*	*R*	–	*O*	*N*	*E*

The message is: ***Math is number one***

Exercise

Determine the key word, then decode the given cryptogram

5	17	21	1	20	9	15	14	19
259	863	783	77	378	357	301	448	565
106	266	318	325	365	485	301	522	653
326	653	738	10	566	495	115	640	555
290	791	762	115	474	507	119	332	279
305	454	513	339	645	611	226	341	426
260	338	368	406	657	830	270	649	590
110	337	418	74	318	330	261	561	469
114	426	390	160	543	372	89	535	441
323	842	783	97	344	245	84	601	444
424	851	944	175	262	339	379	698	755
226	341	426	37	454	217	156	694	536

Solution

The key word from the first row, because all the numbers are between 0 and 26, alphabetic letter.

Since it has 9 numbers, then the matrix is $9=3^2$ which is 3×3.

Therefore,

$$A=\begin{pmatrix} 5 & 17 & 21 \\ 1 & 20 & 9 \\ 15 & 14 & 19 \end{pmatrix}$$

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

The key word is:

5	17	21	1	20	9	15	14	19
E	*Q*	*U*	*A*	*T*	*I*	*O*	*N*	*S*

$$A=\begin{pmatrix} 5 & 17 & 21 \\ 1 & 20 & 9 \\ 15 & 14 & 19 \end{pmatrix}$$

$$|A|=-2,764$$

$$A^{-1}=-\frac{1}{2,764}\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix}$$

With the cryptogram:

$$[259\ \ 863\ \ 783]\ [77\ \ 378\ \ 357]\ [301\ \ 448\ \ 565]\ [106\ \ 266\ \ 318]$$

$$[325\ \ 365\ \ 485]\ [301\ \ 522\ \ 653]\ [326\ \ 653\ \ 738]\ [103\ \ 566\ \ 495]$$

$$[115\ \ 640\ \ 555]\ [290\ \ 791\ \ 762]\ [115\ \ 474\ \ 507]\ [119\ \ 332\ \ 279]$$

$$[305\ \ 454\ \ 513]\ [339\ \ 645\ \ 611]\ [226\ \ 341\ \ 426]\ [260\ \ 338\ \ 368]$$

$$[406\ \ 657\ \ 830]\ [270\ \ 649\ \ 590]\ [110\ \ 337\ \ 418]\ [74\ \ 318\ \ 330]$$

$$[261\ \ 561\ \ 469]\ [114\ \ 426\ \ 390]\ [160\ \ 543\ \ 372]\ [89\ \ 535\ \ 441]$$

$$[323\ \ 842\ \ 783]\ [97\ \ 344\ \ 245]\ [84\ \ 601\ \ 444]\ [424\ \ 851\ \ 944]$$

$$[175\ \ 262\ \ 339]\ [379\ \ 698\ \ 755]\ [226\ \ 341\ \ 426]\ [37\ \ 454\ \ 217]$$

$$[156\ \ 694\ \ 536]$$

To decode a message given the matrix A.

$$-\frac{1}{2,764}[259\ \ 863\ \ 783]\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764}[-58,044\ \ -52,516\ \ -24,876]$$

$$= [21\ \ 19\ \ 9]$$

$$-\frac{1}{2,764}[77\ \ 378\ \ 357]\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764}[-38,696\ \ -19,348\ \ 0]$$

$$= [14\ \ 7\ \ 0]$$

$$-\frac{1}{2,764}[301\ \ 448\ \ 565]\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764}[-33,168\ \ -2,764\ \ -44,224]$$

$$= [12\ \ 1\ \ 16]$$

$$-\frac{1}{2,764}[106\ \ 266\ \ 318]\begin{pmatrix} 254 & -29 & -267 \\ 116 & -220 & -24 \\ -286 & 185 & 83 \end{pmatrix} = -\frac{1}{2,764}[-33,168\ \ -2,764\ \ -8,292]$$

$$= [12\ \ 1\ \ 3]$$

$$-\frac{1}{2,764}\begin{bmatrix}325 & 365 & 485\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\116 & -220 & -24\\-286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-13,820 & 0 & -55,280\end{bmatrix}$$
$$=\begin{bmatrix}5 & 0 & 20\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}301 & 522 & 653\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\116 & -220 & -24\\-286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-49,752 & -2,764 & -38,696\end{bmatrix}$$
$$=\begin{bmatrix}18 & 1 & 14\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}326 & 653 & 738\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\116 & -220 & -24\\-286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-52,516 & -16,584 & -41,460\end{bmatrix}$$
$$=\begin{bmatrix}19 & 6 & 15\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}103 & 566 & 495\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\116 & -220 & -24\\-286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-49,752 & -35,932 & 0\end{bmatrix}$$
$$=\begin{bmatrix}18 & 13 & 0\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}115 & 640 & 555\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\116 & -220 & -24\\-286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-55,280 & -41,460 & 0\end{bmatrix}$$
$$=\begin{bmatrix}20 & 15 & 0\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}290 & 791 & 762\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\116 & -220 & -24\\-286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-52,516 & -41,460 & -33,168\end{bmatrix}$$
$$=\begin{bmatrix}19 & 15 & 12\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}115 & 474 & 507\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\116 & -220 & -24\\-286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-60,808 & -13,820 & 0\end{bmatrix}$$
$$=\begin{bmatrix}22 & 5 & 0\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}119 & 332 & 279\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\116 & -220 & -24\\-286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-11,056 & -24,876 & -16,584\end{bmatrix}$$
$$=\begin{bmatrix}4 & 9 & 6\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}305 & 454 & 513\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}-16{,}584 & -13{,}820 & -49{,}752\end{bmatrix}$$
$$= \begin{bmatrix}6 & 5 & 18\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}339 & 645 & 611\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}-13{,}820 & -38{,}696 & -55{,}280\end{bmatrix}$$
$$= \begin{bmatrix}5 & 14 & 20\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}226 & 341 & 426\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}-24{,}876 & -2{,}764 & -33{,}168\end{bmatrix}$$
$$= \begin{bmatrix}9 & 1 & 12\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}260 & 338 & 368\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}0 & 13{,}820 & -46{,}988\end{bmatrix}$$
$$= \begin{bmatrix}0 & 5 & 17\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}406 & 657 & 830\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}-58{,}044 & -2{,}764 & -55{,}280\end{bmatrix}$$
$$= \begin{bmatrix}21 & 1 & 20\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}270 & 649 & 590\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}-24{,}876 & -41{,}460 & -38{,}696\end{bmatrix}$$
$$= \begin{bmatrix}9 & 15 & 14\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}110 & 337 & 418\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}-52{,}516 & 0 & -2{,}764\end{bmatrix}$$
$$= \begin{bmatrix}19 & 0 & 1\end{bmatrix}$$

$$-\frac{1}{2{,}764}\begin{bmatrix}74 & 318 & 330\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2{,}764}\begin{bmatrix}-38{,}696 & -11{,}056 & 0\end{bmatrix}$$
$$= \begin{bmatrix}14 & 4 & 0\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}261 & 561 & 469\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-2,764 & -44,224 & -44,224\end{bmatrix}$$
$$=\begin{bmatrix}1 & 16 & 16\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}114 & 426 & 390\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-33,168 & -24,876 & -8,292\end{bmatrix}$$
$$=\begin{bmatrix}12 & 9 & 3\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}160 & 543 & 372\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-2,764 & -55,280 & -24.876\end{bmatrix}$$
$$=\begin{bmatrix}1 & 20 & 9\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}89 & 535 & 441\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-41,460 & -38,696 & 0\end{bmatrix}$$
$$=\begin{bmatrix}15 & 14 & 0\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}323 & 842 & 783\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-44,224 & -49,752 & -41,460\end{bmatrix}$$
$$=\begin{bmatrix}16 & 18 & 15\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}97 & 344 & 245\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-5,528 & -33,168 & -13,820\end{bmatrix}$$
$$=\begin{bmatrix}2 & 12 & 5\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}84 & 601 & 444\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-35,932 & -52,516 & 0\end{bmatrix}$$
$$=\begin{bmatrix}13 & 19 & 0\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}424 & 851 & 944\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-63,572 & -24,876 & -55,280\end{bmatrix}$$
$$=\begin{bmatrix}23 & 9 & 20\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}175 & 262 & 339\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-22,112 & 0 & -24,876\end{bmatrix}$$
$$=\begin{bmatrix}8 & 0 & 9\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}379 & 698 & 755\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-38,696 & -24,876 & -55,280\end{bmatrix}$$
$$=\begin{bmatrix}14 & 9 & 20\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}226 & 341 & 426\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-24,876 & -2,764 & -33,168\end{bmatrix}$$
$$=\begin{bmatrix}9 & 1 & 12\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}37 & 454 & 217\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}0 & -60,808 & -2,764\end{bmatrix}$$
$$=\begin{bmatrix}0 & 22 & 1\end{bmatrix}$$

$$-\frac{1}{2,764}\begin{bmatrix}156 & 694 & 536\end{bmatrix}\begin{pmatrix}254 & -29 & -267\\ 116 & -220 & -24\\ -286 & 185 & 83\end{pmatrix} = -\frac{1}{2,764}\begin{bmatrix}-33,168 & -58,044 & -13,820\end{bmatrix}$$
$$=\begin{bmatrix}12 & 21 & 5\end{bmatrix}$$

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

21	19	9	14	7	0	12	1	16	12	1	3	5	0	20	18	1	14	19	6	15
U	*S*	*I*	*N*	*G*	–	*L*	*A*	*P*	*L*	*A*	*C*	*E*	–	*T*	*R*	*A*	*N*	*S*	*F*	*O*
18	13	0	20	15	0	19	15	12	22	5	0	4	9	6	6	5	18	5	14	20
R	*M*	–	*T*	*O*	–	*S*	*O*	*L*	*V*	*E*	–	*D*	*I*	*F*	*F*	*E*	*R*	*E*	*N*	*T*
9	1	12	0	5	17	21	1	20	9	15	14	19	0	1	14	4	0	1	16	16
I	*A*	*L*	–	*E*	*Q*	*U*	*A*	*T*	*I*	*O*	*N*	*S*	–	*A*	*N*	*D*	–	*A*	*P*	*P*
12	9	3	1	20	9	15	14	0	16	18	15	2	12	5	13	19	0	23	9	20
L	*I*	*C*	*A*	*T*	*I*	*O*	*N*	–	*P*	*R*	*O*	*B*	*L*	*E*	*M*	*S*	–	*W*	*I*	*T*

8	0	9	14	9	20	9	1	12	0	22	1	12	21	5
H	–	*I*	*N*	*I*	*T*	*I*	*A*	*L*	–	*V*	*A*	*L*	*U*	*E*

The message is:

Using Laplace Transform to Solve Differential Equations and Application Problems with Initial Value.