Lecture R – Introduction to Differential Equation

Solution *Section* R.1 – Derivative

Exercise

Find the derivative of $f(t) = -3t^2 + 2t - 4$

Solution

$$\boxed{f'(t) = -6t + 2}$$

Exercise

Find the derivative of $g(x) = 4\sqrt[3]{x} + 2$

Solution

$$g(x) = 4x^{1/3} + 2$$

$$g'(x) = \frac{4}{3}x^{-2/3}$$

$$\boxed{= \frac{4}{3\sqrt[3]{x^2}}}$$

Exercise

Find the derivative of $f(x) = x\left(x^2 + 1\right)$

Solution

$$f(x) = x^3 + x$$

$$\boxed{f'(x) = 3x^2 + 1}$$

Exercise

Find the derivative of $f(x) = \frac{2x^2 - 3x + 1}{x}$

Solution

$$f(x) = \frac{2x^2}{x} - \frac{3x}{x} + \frac{1}{x}$$

$$= 2x - 3 + \frac{1}{x}$$

$$\boxed{f'(x) = 2 - \frac{1}{x^2}} \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

Exercise

Find the derivative of $f(x)=\frac{4x^3-3x^2+2x+5}{x^2}$

Solution

$$f(x)=4x-3+\frac{2}{x}+5x^{-2} \qquad \left(\frac{1}{x}\right)'=-\frac{1}{x^2}$$

$$f'(x)=4-\frac{2}{x^2}-10x^{-3}$$

$$=4-\frac{2}{x^2}-\frac{10}{x^3}$$

Exercise

Find the derivative of $f(x)=\frac{-6x^3+3x^2-2x+1}{x}$

Solution

$$f(x)=-6x^2+3x-2+\frac{1}{x} \qquad \left(\frac{1}{x}\right)'=-\frac{1}{x^2}$$

$$f'(x)=-12x+3-\frac{1}{x^2}$$

Exercise

Find the derivative of $f(x)=x\left(1-\frac{2}{x+1}\right)$

Solution

$$f(x)=x-\frac{2x}{x+1} \qquad \left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

$$f'(x)=1-\frac{2}{(x+1)^2}$$

Exercise

Find the derivative of $g(s)=\frac{s^2-2s+5}{\sqrt{s}}$

Solution

$$g(s)=\frac{s^2}{s^{1/2}}-2\frac{s}{s^{1/2}}+\frac{5}{s^{1/2}}$$

$$= s^{3/2} - 2s^{1/2} + 5s^{-1/2}$$

$$g'(s) = \frac{3}{2}s^{1/2} - 2\frac{1}{2}s^{-1/2} + 5\left(-\frac{1}{2}\right)s^{-3/2}$$

$$= \frac{3}{2}s^{1/2} - s^{-1/2} - \frac{5}{2}s^{-3/2}$$

$$= \frac{3}{2}\sqrt{s} - \frac{1}{\sqrt{s}} - \frac{5}{2s^{3/2}}$$

$$\boxed{= \frac{3}{2}\sqrt{s} - \frac{1}{\sqrt{s}} - \frac{5}{2s\sqrt{s}}}$$

Exercise

Find the derivative of $f(x) = \dfrac{x+1}{\sqrt{x}}$

Solution

$$f(x) = \frac{x}{x^{1/2}} + \frac{1}{x^{1/2}}$$

$$= x^{1/2} + x^{-1/2}$$

$$f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$$

$$\boxed{= \frac{1}{2x^{1/2}} - \frac{1}{2x^{3/2}}}$$

Exercise

Find the derivative to the following functions $y = 3x\left(2x^2 + 5x\right)$

Solution

$$y = 6x^3 + 15x^2$$

$$\boxed{y' = 18x^2 + 30x}$$

Exercise

Find the derivative to the following functions $y = 3\left(2x^2 + 5x\right)$

Solution

$$y = 6x^2 + 15x$$

$$\boxed{y' = 12x + 15}$$

Exercise

Find the derivative to the following functions $y = \dfrac{x^2 + 4x}{5}$

Solution

$$\boxed{y' = \frac{1}{5}(2x + 4)}$$

Exercise

Find the derivative to the following functions $y = \dfrac{3x^4}{5}$

Solution

$$\boxed{y' = \frac{12}{5}x^3}$$

Exercise

Find the derivative to the following functions $y = \dfrac{x^2 - 4}{2x + 5}$

Solution

$$\boxed{y' = \frac{2x^2 + 10x + 8}{(2x+5)^2}}$$

$$\frac{d}{dx}\left(\frac{ax^2 + bx + c}{dx^2 + ex + f}\right) = \frac{(ae - bd)x^2 + 2(af - dd)x + (bf - ce)}{\left(dx^2 + ex + f\right)^2}$$

Exercise

Find the derivative to the following functions $y = \dfrac{(1+x)(2x-1)}{x-1}$

Solution

$$\begin{aligned}
y' &= \frac{(x-1)\frac{d}{dx}\left[(1+x)(2x-1)\right] - (1+x)(2x-1)\frac{d}{dx}[x-1]}{(x-1)^2} \\
&= \frac{(x-1)\left[(1)(2x-1) + 2(1+x)\right] - (1+x)(2x-1)(1)}{(x-1)^2} \\
&= \frac{(x-1)(2x-1+2+2x) - (2x-1+2x^2-x)}{(x-1)^2} \\
&= \frac{(x-1)(4x+1) - 2x + 1 - 2x^2 + x}{(x-1)^2}
\end{aligned}$$

$$= \frac{4x^2 + x - 4x - 1 - 2x + 1 - 2x^2 + x}{(x-1)^2}$$

$$\boxed{= \frac{2x^2 - 4x}{(x-1)^2}}$$

Or

$$y = \frac{(1+x)(2x-1)}{x-1}$$

$$= \frac{2x - 1 + 2x^2 - x}{x-1}$$

$$= \frac{2x^2 + x - 1}{x-1}$$

$$\boxed{y' = \frac{2x^2 - 4x}{(x-1)^2}}$$

$$\frac{d}{dx}\left(\frac{ax^2 + bx + c}{dx^2 + ex + f}\right) = \frac{(ae - bd)x^2 + 2(af - dd)x + (bf - ce)}{\left(dx^2 + ex + f\right)^2}$$

Exercise

Find the derivative to the following functions $y = \dfrac{4}{2x+1}$

Solution

$$\boxed{y' = -\frac{8}{(2x+1)^2}}$$

$$\left(\frac{ax+b}{cx+d}\right)' = \frac{ad - bc}{(cx+d)^2}$$

Exercise

Find the derivative to the following functions $y = \dfrac{2}{(x-1)^3}$

Solution

$$y = 2(x-1)^{-3}$$

$$y' = 2(-3)(x-1)^{-4}(1)$$

$$\boxed{= -\frac{6}{(x-1)^4}}$$

Exercise

Find the derivative to the following functions $y = \sqrt[3]{(x+4)^2}$

Solution

$$y = (x+4)^{2/3}$$

$$y' = \tfrac{2}{3}(x+4)^{-1/3}$$

$$= \frac{2}{3}\frac{1}{(x+4)^{1/3}}$$

$$= \frac{2}{3\sqrt[3]{x+4}}$$

Exercise

Find the derivative of $f(x) = \sqrt{2t^2 + 5t + 2}$

Solution

$$f(t) = \left(2t^2 + 5t + 2\right)^{1/2} \qquad U = 2t^2 + 5t + 2 \;\rightarrow\; U' = 4t + 5$$

$$f'(t) = \tfrac{1}{2}(4t+5)\left(2t^2 + 5t + 2\right)^{-1/2} \qquad \left(U^n\right)' = nU'U^{n-1}$$

$$= \frac{1}{2}\frac{4t+5}{\sqrt{2t^2 + 5t + 2}}$$

Exercise

Find the derivative of $f(x) = \dfrac{1}{\left(x^2 - 3x\right)^2}$

Solution

$$f'(x) = -\frac{2(2x-3)}{\left(x^2 - 3x\right)^3} \qquad \left(\frac{1}{U^n}\right)' = -\frac{nU'}{U^{n+1}}$$

Exercise

Find the derivative of $y = t^2\sqrt{t-2}$

Solution

$$f = t^2 \qquad f' = 2t$$

$$g = (t-2)^{1/2} \qquad g' = \tfrac{1}{2}(t-2)^{-1/2}$$

$$y' = 2t\sqrt{t-2} + t^2\,\tfrac{1}{2}(t-2)^{-1/2}$$

$$= \left[2t(t-2)^{1/2} + t^2\,\tfrac{1}{2}(t-2)^{-1/2}\right]\frac{2(t-2)^{1/2}}{2(t-2)^{1/2}}$$

$$= \frac{4t(t-2)+t^2}{2(t-2)^{1/2}}$$

$$= \frac{4t^2-8t+t^2}{2\sqrt{t-2}}$$

$$= \frac{5t^2-4t}{2\sqrt{t-2}}$$

Exercise

Find the derivative of $y = \left(\frac{-5x+6}{x^2-1}\right)^2$

Solution

$$\left(U^n\right)' = nU'\,U^{n-1} \qquad \frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right) = \frac{(ae-bd)x^2+2(af-dd)x+(bf-ce)}{\left(dx^2+ex+f\right)^2}$$

$$y' = 2\frac{5x^2-12x+5}{\left(x^2-1\right)^2}\left(\frac{6-5x}{x^2-1}\right)$$

$$= \frac{2\left(5x^2-12x+5\right)(6-5x)}{\left(x^2-1\right)^3}$$

Exercise

Find the derivative to the following functions $y = x^2\sqrt{x^2+1}$

Solution

$$y = x^2\left(x^2+1\right)^{1/2}$$

$$y' = x^2\frac{d}{dx}\left[(x^2+1)^{1/2}\right] + (x^2+1)^{1/2}\frac{d}{dx}\left[x^2\right]$$

$$= x^2\left[\frac{1}{2}(x^2+1)^{-1/2}(2x)\right] + (x^2+1)^{1/2}[2x]$$

$$= x^3(x^2+1)^{-1/2} + 2x(x^2+1)^{1/2}$$

$$= \frac{(x^2+1)^{1/2}}{(x^2+1)^{1/2}}\left[x^3(x^2+1)^{-1/2} + 2x(x^2+1)^{1/2}\right]$$

$$= \frac{x^3(x^2+1)^{-1/2}(x^2+1)^{1/2}+2x(x^2+1)^{1/2}(x^2+1)^{1/2}}{(x^2+1)^{1/2}}$$

$$= \frac{x^3+2x(x^2+1)}{(x^2+1)^{1/2}}$$

$$= \frac{x^3+2x^3+2x}{\sqrt{x^2+1}}$$

$$= \frac{3x^3+2x}{\sqrt{x^2+1}}$$

$$= \frac{x(3x^2+2)}{\sqrt{x^2+1}}$$

Exercise

Find the derivative to the following functions $y = \left(\frac{x+1}{x-5}\right)^2$

Solution

$$y' = 2\frac{-6}{(x-5)^2}\left(\frac{x+1}{x-5}\right)$$

$$\left(U^n\right)' = nU'\,U^{n-1} \qquad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

$$= -\frac{12(x+1)}{(x-5)^3}$$

Exercise

Find the derivative to the following functions $y = x^2 \sin x$

Solution

$$y' = 2x\sin x + x^2\cos x$$

$$u = x^2 \qquad v = \sin x$$
$$u' = 2x \qquad v' = \cos x$$

Exercise

Find the derivative to the following functions $y = \frac{\sin x}{x}$

Solution

$$y' = \frac{x\cos x - \sin x}{x^2}$$

$$u = \sin x \qquad v = x$$
$$u' = \cos x \qquad v' = 1$$

Exercise

Find the derivative to the following functions $y = \frac{\cot x}{1+\cot x}$

Solution

$$y' = \frac{-\csc^2 x(1+\cot x)+\csc^2 x\cot x}{(1+\cot x)^2}$$

$$u = \cot x \qquad v = 1+\cot x$$

$$u' = -\csc^2 x \quad v' = -\csc^2 x$$

$$= \frac{-\csc^2 x - \csc^2 x\cot x + \csc^2 x\cot x}{(1+\cot x)^2}$$

$$= \frac{-\csc^2 x}{(1+\cot x)^2}$$

Exercise

Find the derivative to the following functions $y = x^2\sin x + 2x\cos x - 2\sin x$

Solution

$$y' = 2x\sin x + x^2\cos x + 2\cos x - 2x\sin x - 2\cos x$$

$$= x^2\cos x$$

Exercise

Find the derivative to the following functions $y = x^3\sin x\cos x$

Solution

$$y' = \left(x^3\right)'\sin x\cos x + x^3(\sin x)'\cos x + x^3\sin x(\cos x)'$$

$$= 3x^2\sin x\cos x + x^3\cos^2 x - x^3\sin^2 x$$

Exercise

Find the derivative to the following functions $y = \frac{4}{\cos x} + \frac{1}{\tan x}$

Solution

$$y' = \frac{-4\sin x}{\cos^2 x} - \frac{\sec^2 x}{\tan^2 x}$$

$$\left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$$

$$= -4\frac{\sin x}{\cos x}\frac{1}{\cos x} - \frac{1}{\cos^2 x}\frac{\cos^2 x}{\sin^2 x}$$

$$= -4\tan x\sec x - \csc^2 x$$

Exercise

Find the derivative to the following functions $f(x) = x^2 e^x$

Solution

$$f'(x) = e^x \frac{d}{dx}\left(x^2\right) + x^2 \frac{d}{dx}\left(e^x\right)$$

$$= e^x (2x) + x^2 e^x$$

$$\boxed{= xe^x (2 + x)}$$

Exercise

Find the derivative to the following functions $f(x) = \frac{e^x + e^{-x}}{2}$

Solution

$$f(x) = \frac{e^x + e^{-x}}{2}$$

$$= \frac{1}{2}(e^x + e^{-x})$$

$$f'(x) = \frac{1}{2}\left(\frac{d}{dx}\left[e^x\right] + \frac{d}{dx}\left[e^{-x}\right]\right)$$

$$\boxed{= \frac{1}{2}\left(e^x - e^{-x}\right)}$$

Exercise

Find the derivative to the following functions $f(x) = \frac{e^x}{x^2}$

Solution

$$f'(x) = \frac{x^2 e^x - e^x (2x)}{x^4}$$

$$= \frac{x^2 e^x - 2xe^x}{x^4}$$

$$= \frac{xe^x (x-2)}{x^4}$$

$$\boxed{= \frac{e^x (x-2)}{x^3}}$$

Exercise

Find the derivative to the following functions $f(x) = x^2 e^x - e^x$

Solution

$$f'(x) = e^x \frac{d}{dx}[x^2] + x^2 \frac{d}{dx}[e^x] - \frac{d}{dx}[e^x]$$

$$= e^x(2x) + x^2 e^x - e^x$$

$$\boxed{= e^x\left(x^2 + 2x - 1\right)}$$

Exercise

Find the derivative to the following functions $f(x) = (1+2x)e^{4x}$

Solution

$$f'(x) = (2)e^{4x} + (1+2x)(4e^{4x})$$

$$= 2e^{4x} + (1+2x)(4e^{4x})$$

$$= 2e^{4x}(1+2(1+2x))$$

$$= 2e^{4x}(1+2+4x)$$

$$\boxed{= 2e^{4x}(3+4x)}$$

Exercise

Find the derivative to the following functions $y = x^2 e^{5x}$

Solution

$$y' = x^2\left(5e^{5x}\right) + 2x\left(e^{5x}\right)$$

$$\boxed{= xe^{5x}(5x+2)}$$

Exercise

Find the derivative to the following functions $y = e^{x^2+1}\sqrt{5x+2}$

Solution

$$y = (2x)e^{x^2+1}\sqrt{5x+2} + e^{x^2+1}\frac{5}{2\sqrt{5x+2}}$$

$$= 2xe^{x^2+1}\sqrt{5x+2}\,\frac{2\sqrt{5x+2}}{2\sqrt{5x+2}} + \frac{5e^{x^2+1}}{2\sqrt{5x+2}}$$

$$= \frac{4xe^{x^2+1}(5x+2)}{2\sqrt{5x+2}} + \frac{5e^{x^2+1}}{2\sqrt{5x+2}}$$

$$= \frac{20x^2e^{x^2+1}+8xe^{x^2+1}+5e^{x^2+1}}{2\sqrt{5x+2}}$$

$$= \boxed{\frac{e^{x^2+1}\left(20x^2+8x+5\right)}{2\sqrt{5x+2}}}$$

Exercise

Find the derivative to the following functions $f(x) = \ln\sqrt[3]{x+1}$

Solution

$$f(x) = \ln(x+1)^{1/3}$$

$$= \frac{1}{3}\ln(x+1)$$

$$f'(x) = \frac{1}{3}\frac{1}{x+1}$$

$$= \boxed{\frac{1}{3(x+1)}}$$

Exercise

Find the derivative to the following functions $f(x) = \ln\left\lfloor x^2\sqrt{x^2+1}\right\rfloor$

Solution

$f(x) = \ln(x^2) + \ln\sqrt{x^2+1}$ ***Product Property***

$f(x) = \ln(x^2) + \ln(x^2+1)^{1/2}$

$f(x) = 2\ln x + \frac{1}{2}\ln(x^2+1)$ ***Power Property***

$f'(x) = 2\frac{1}{x} + \frac{1}{2}\frac{2x}{x^2+1}$ ***Differentiate***

$$= \boxed{\frac{2}{x} + \frac{x}{x^2+1}}$$

Exercise

Find the derivative to the following functions $y = \ln\frac{x^2}{x^2+1}$

Solution

$$y = \ln x^2 - \ln x^2 + 1$$

$$y' = \frac{2x}{x^2} - \frac{2x}{x^2+1}$$

$$= \boxed{\frac{2}{x} - \frac{2x}{x^2+1}}$$

Exercise

Find the derivative to the following functions $y = \ln\frac{1+e^x}{1-e^x}$

Solution

$$y = \ln\left(1+e^x\right) - \ln\left(1-e^x\right)$$

$$y' = \frac{e^x}{1+e^x} - \frac{-e^x}{1-e^x}$$

$$= \frac{e^x}{1+e^x} + \frac{e^x}{1-e^x}$$

$$= \frac{e^x - e^{2x} + e^x + e^{2x}}{\left(1+e^x\right)\left(1-e^x\right)}$$

$$= \boxed{\frac{2e^x}{\left(1+e^x\right)\left(1-e^x\right)}}$$

Exercise

Find the derivative to the following functions $y = x\ 3^{x+1}$

Solution

$$y' = 3^{x+1} + x\ 3^{x+1}\ln 3$$

$$= \boxed{3^{x+1}\left(1 + x\ \ln 3\right)}$$

Exercise

Find the derivative to the following functions $f(t) = \frac{\log_8\left(t^{3/2}+1\right)}{t}$

Solution

$$f' = \frac{\frac{1}{\ln 8}\frac{\frac{3}{2}t^{1/2}}{t^{3/2}+1} \cdot t - \log_8\left(t^{3/2}+1\right)}{t}$$

$$= \frac{\frac{1}{\ln 8}\frac{\frac{3}{2}t^{3/2}}{t^{3/2}+1} - \log_8\left(t^{3/2}+1\right)}{t^2} \cdot \frac{2\ln 8\left(t^{3/2}+1\right)}{2\ln 8\left(t^{3/2}+1\right)}$$

$$= \boxed{\frac{3t^{3/2} - 2\left(t^{3/2}+1\right)(\ln 8)\log_8\left(t^{3/2}+1\right)}{t^2\left(t^{3/2}+1\right)\ln 8}}$$

Exercise

Find the derivative of $\quad y = \dfrac{2x-3}{x+1}$

Solution

$$y' = \frac{2+3}{(x+1)^2} \qquad\qquad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

$$= \boxed{\frac{5}{(x+1)^2}}$$

Exercise

Find the derivative of $\quad y = \dfrac{3x}{3x-2}$

Solution

$$y' = \boxed{\frac{-6}{(3x-2)^2}} \qquad\qquad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $\quad y = \dfrac{x-3}{2x+5}$

Solution

$$y' = \boxed{\frac{11}{(2x+5)^2}} \qquad\qquad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

Exercise

Find the derivative of $\quad y = \dfrac{x^2-4}{5x^2-2}$

Solution

$$y' = \frac{2(-2+20)x}{\left(5x^2-2\right)^2} \qquad\qquad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$= \boxed{\frac{36x}{\left(5x^2-2\right)^2}}$$

Exercise

Find the derivative of $y = \dfrac{3x^2 - 4}{2x^2 - 1}$

Solution

$$y' = \frac{2(-3+8)x}{\left(2x^2-1\right)^2} \qquad \left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}$$

$$= \boxed{\frac{10x}{\left(2x^2-1\right)^2}}$$

Exercise

Find the derivative of $y = \dfrac{x^2 - 4x + 1}{5x^2 - 2x - 1}$

Solution

$$y' = \frac{\begin{vmatrix} 1 & -4 \\ 5 & -2 \end{vmatrix} x^2 + 2\begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} x + \begin{vmatrix} -4 & 1 \\ -2 & -1 \end{vmatrix}}{\left(5x^2-2x-1\right)^2} \qquad \frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2\begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

$$= \boxed{\frac{18x^2-12x+6}{\left(5x^2-2x-1\right)^2}}$$

Exercise

Find the derivative of $y = \dfrac{3x^2 - 4x + 2}{2x^2 + x - 1}$

Solution

$$y' = \frac{\begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} x^2 + 2\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} x + \begin{vmatrix} -4 & 2 \\ 1 & -1 \end{vmatrix}}{\left(2x^2+x-1\right)^2} \qquad \frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2\begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

$$= \boxed{\frac{11x^2-14x+6}{\left(2x^2+x-1\right)^2}}$$

Exercise

Find the derivative of $f(x)=\dfrac{x^2+3}{(2x-1)^3(3x+1)^4}$

Solution

$$f(x)=\left(x^2+3\right)(2x-1)^{-3}(3x+1)^{-4} \qquad \left(U^mV^nW^p\right)'=U^{m-1}V^{n-1}W^{p-1}\left(mU'VW+nUV'W+pUVW'\right)$$

$$f'(x)=(2x-1)^{-4}(3x+1)^{-5}$$

$$\left[2x(2x-1)(3x+1)-6\left(x^2+3\right)(3x+1)-12\left(x^2+3\right)(2x-1)\right]$$

$$=\frac{1}{(2x-1)^4(3x+1)^5}\left(\left(4x^2-2x\right)(3x+1)-6\left(3x^3+x^2+9x+3\right)-12\left(2x^3-x^2+6x-3\right)\right)$$

$x^3 \quad 12-18-24$

$x^2 \quad 4-6-6+12$

$x \quad -2-54-72$

$x^0 \quad -18+36$

$$\boxed{f'(x)=\frac{-30x^3+4x^2-128x+18}{(2x-1)^4(3x+1)^5}}$$

Exercise

Find the derivative of $f(x)=\dfrac{\left(x^3-3x\right)^3\left(x^2+4x\right)^4}{\left(x^2+4x+1\right)^2}$

Solution

$$f(x)=\left(x^3-3x\right)^3\left(x^2+4x\right)^4\left(x^2+4x+1\right)^{-2}$$

$$\left(U^mV^nW^p\right)'=U^{m-1}V^{n-1}W^{p-1}\left(mU'VW+nUV'W+pUVW'\right)$$

$$f'(x)=\left(x^3-3x\right)^2\left(x^2+4x\right)^3\left(x^2+4x+1\right)^{-3}\left[3\left(3x^2-3\right)\left(x^2+4x\right)\left(x^2+4x+1\right)\right.$$

$$\left.+4\left(x^3-3x\right)(2x+4)\left(x^2+4x+1\right)-2\left(x^3-3x\right)\left(x^2+4x\right)(2x+4)\right]$$

$$f'(x)=\left(x^3-3x\right)^2\left(x^2+4x\right)^3\left(x^2+4x+1\right)^{-3}\left[\left(9x^2-9\right)\left(x^4+8x^3+9x^2+4x\right)\right.$$

$$\left.+\left(4x^3-12x\right)\left(2x^3+12x^2+18x+4\right)+\left(-2x^3+6x\right)\left(2x^3+12x^2+16x\right)\right]$$

x^6	9+8−4
x^5	72+48−24
x^4	81−9+72−24−32+12
x^3	36−72+16−144+72
x^2	−81−216+96
x^1	−36−48

$$f'(x) = \frac{\left(13x^6 + 96x^5 + 100x^4 - 92x^3 - 201x^2 - 84x\right)\left(x^3 - 3x\right)^2 \left(x^2 + 4x\right)^3}{\left(x^2 + 4x + 1\right)^3}$$

Solution Section R.2 – Integration

Exercise

Find each indefinite integral. $\int \frac{x+2}{\sqrt{x}}dx$

Solution

$$\int \frac{x+2}{\sqrt{x}}dx = \int \left[\frac{x}{x^{1/2}} + \frac{2}{x^{1/2}}\right]dx$$

$$= \int \frac{x}{x^{1/2}}dx + \int \frac{2}{x^{1/2}}dx$$

$$= \int x^{1/2}dx + 2\int x^{-1/2}dx$$

$$= \frac{x^{3/2}}{3/2} + 2\frac{x^{1/2}}{1/2} + C$$

$$\boxed{= \frac{2}{3}x^{3/2} + 4x^{1/2} + C}$$

Exercise

Find each indefinite integral $\int 4y^{-3}dy$

Solution

$$\int 4y^{-3}dy = 4\frac{y^{-2}}{-2} + C$$

$$\boxed{= -\frac{2}{y^2} + C}$$

Exercise

Find each indefinite integral $\int (x^3 - 4x + 2)dx$

Solution

$$\int \left(x^3 - 4x + 2\right)dx = \boxed{\frac{1}{4}x^4 - 2x^2 + 2x + C}$$

Exercise

Find each indefinite integral $\int\left(\sqrt[4]{x^3}+1\right)dx$

Solution

$$\int\left(x^{3/4}+1\right)dx=\boxed{\frac{4}{7}x^{7/4}+x+C}$$

Exercise

Find each indefinite integral $\int\sqrt{x}(x+1)dx$

Solution

$$\begin{aligned}\int x^{1/2}(x+1)dx&=\int\left(x^{3/2}+x^{1/2}\right)dx\\&=\boxed{\frac{2}{5}x^{5/2}+\frac{2}{3}x^{3/2}+C}\end{aligned}$$

Exercise

Find each indefinite integral $\int(1+3t)t^2dt$

Solution

$$\int\left(t^2+3t^3\right)dt=\boxed{\frac{1}{3}t^3+\frac{3}{4}t^4+C}$$

Exercise

Find each indefinite integral $\int\frac{x^2-5}{x^2}dx$

Solution

$$\begin{aligned}\int\frac{x^2-5}{x^2}dx&=\int\left(1-\frac{5}{x^2}\right)dx\\&=\int\left(1-5x^{-2}\right)dx\\&=\boxed{x+\frac{5}{x}+C}\end{aligned}$$

Exercise

Find each indefinite integral $\int(-40x+250)\,dx$

Solution

$$\int(-40x+250)\,dx = \boxed{-20x^2+250x+C}$$

Exercise

Find each indefinite integral $\int\left(7-3x-3x^2\right)(2x+1)\,dx$

Solution

$$\begin{aligned}\int\left(7-3x-3x^2\right)(2x+1)\,dx &= \int\left(14x+7-6x^2-3x-6x^3-3x^2\right)dx\\ &= \int\left(-6x^3-9x^2+11x+7\right)dx\\ &= \boxed{-\tfrac{3}{2}x^4-3x^3+\tfrac{11}{2}x^2+7x+C}\end{aligned}$$

Exercise

Evaluate the integral $\int xe^{2x}dx$

Solution

Let: $u=x \Rightarrow du=dx$

$$dv=e^{2x}dx \Rightarrow v=\int dv=\int e^{2x}dx=\tfrac{1}{2}e^{2x}$$

$$\int udv \;=\; uv-\int vdu$$

$$\begin{aligned}\int xe^{2x}dx &= \tfrac{1}{2}xe^{2x}-\int\tfrac{1}{2}e^{2x}dx\\ &= \tfrac{1}{2}xe^{2x}-\tfrac{1}{2}\tfrac{1}{2}e^{2x}+C\\ &= \boxed{\tfrac{1}{2}xe^{2x}-\tfrac{1}{4}e^{2x}+C}\end{aligned}$$

Exercise

Evaluate the integral $\int x\ln x\,dx$

Solution

Let: $u=\ln x \Rightarrow du=\frac{1}{x}dx$

$dv=xdx \Rightarrow v=\int dv=\int xdx=\frac{1}{2}x^2$

$$\int x\ln x\,dx=\frac{1}{2}x^2\ln x-\frac{1}{2}\int x^2\frac{dx}{x}$$

$$=\frac{1}{2}x^2\ln x-\frac{1}{2}\int xdx$$

$$=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C$$

Exercise

Evaluate the integral $\int x^2\sin x\,dx$

Solution

		$\int \sin x$
x^2	$(+)$	$-\cos x$
$2x$	$(-)$	$-\sin x$
2	$(+)$	$\cos x$

$$\int x^2\sin x\,dx=-x^2\cos x-2x\sin x+2\cos x+C$$

Exercise

Evaluate the integral $\int \left(x^2-2x+1\right)e^{2x}dx$

Solution

		$\int e^{2x}$
$+$	x^2-2x+1	$\frac{1}{2}e^{2x}$
$-$	$2x-2$	$\frac{1}{4}e^{2x}$
$+$	2	$\frac{1}{8}e^{2x}$

$$\int \left(x^2-2x+1\right)e^{2x}dx=\frac{1}{2}\left(x^2-2x+1\right)e^{2x}-\frac{1}{4}(2x-2)e^{2x}+\frac{1}{8}(2)e^{2x}+C$$

$$=\left(\frac{1}{2}x^2-x+\frac{1}{2}-\frac{1}{2}x+\frac{1}{2}+\frac{1}{4}\right)e^{2x}+C$$

$$=\left(\frac{1}{2}x^2-\frac{3}{2}x+\frac{5}{5}\right)e^{2x}+C$$

Exercise

Evaluate the integral $\int e^{2x}\cos 3x dx$

Solution

		$\int \cos 3x$
+	e^{2x}	$\frac{1}{3}\sin 3x$
-	$\frac{1}{2}e^{2x}$	$-\frac{1}{9}\cos 3x$
+	$\frac{1}{4}e^{2x}$	

$$\int e^{2x}\cos 3x dx = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}\int e^{2x}\cos 3x dx$$

$$\int e^{2x}\cos 3x dx + \frac{9}{4}\int e^{2x}\cos 3x dx = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x + C_1$$

$$\frac{13}{4}\int e^{2x}\cos 3x dx = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x + C_1$$

$$\frac{4}{13}\frac{13}{4}\int e^{2x}\cos 3x dx = \frac{4}{13}\frac{1}{2}e^{2x}\cos 3x + \frac{4}{13}\frac{3}{4}e^{2x}\sin 3x + \frac{4}{13}C_1$$

$$\int e^{2x}\cos 3x dx = \frac{e^{2x}}{13}\left(2\cos 3x + 3\sin 3x\right) + C$$

Exercise

Find the general solution of the differential equation $y' = 2t + 3$

Solution

$$\int dy = \int (2t+3)dt$$

$$y(t) = t^2 + 3t + C$$

Exercise

Find the general solution of the differential equation $y' = 3t^2 + 2t + 3$

Solution

$$\int dy = \int \left(3t^2 + 2t + 3\right)dt$$

$$y(t) = t^3 + t^2 + 3t + C$$

Exercise

Find the general solution of the differential equation $y' = \sin 2t + 2\cos 3t$

Solution

$$\int dy = \int (\sin 2t + 2\cos 3t)\,dt$$

$$\boxed{y(t) = -\frac{1}{2}\cos 2t + \frac{2}{3}\sin 3t + C}$$

Exercise

Find the general solution of the differential equation: $y' = x^3(3x^4+1)^2$

Solution

$$\int x^3\left(3x^4+1\right)^2 dx$$

$$u = 3x^4 + 1 \Rightarrow du = 12x^3 dx$$

$$\int x^3(3x^4+1)^2 dx = \int \frac{1}{12}u^2 du$$

$$= \frac{1}{12}\frac{(3x^4+1)^3}{3} + C$$

$$= \frac{1}{36}(3x^4+1)^3 + C$$

$$\boxed{y(x) = \frac{1}{36}\left(3x^4+1\right)^3 + C}$$

Exercise

Find the general solution of the differential equation: $y' = 5x\sqrt{x^2-1}$

Solution

$$\int 5x\sqrt{x^2-1}\,dx$$

$$u = x^2 - 1 \quad \Rightarrow du = 2xdx$$

$$\int 5x\left(x^2-1\right)^{1/2}\,dx$$

$$= 5\int u^{1/2}\,\frac{1}{2}du$$

Substitute for x and dx

$$= \frac{5}{2}\int u^{1/2}du$$

$$= \frac{5}{2}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{5}{3}u^{3/2} + C$$

$$\boxed{= \frac{5}{3}(x^2 - 1)^{3/2} + C}$$

Exercise

Find the general solution of the differential equation: $y' = x\sqrt{x^2 + 4}$

Solution

$$u = x^2 + 4 \quad \Rightarrow du = 2xdx$$

$$xdx = \frac{1}{2}du$$

$$\int \sqrt{x^2 + 4}\, xdx = \int u^{1/2}\ \frac{1}{2}du$$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{3}(x^2 + 4)^{3/2} + C$$

$$\boxed{y(x) = \frac{1}{3}\left(x^2 + 4\right)^{3/2} + C}$$

Exercise

Find the general solution of the differential equation: $y' = (2x+1)e^{x^2+x}$

Solution

$$\int dy = \int (2x+1)e^{x^2+x}\ dx \qquad u = x^2 + x \quad \Rightarrow \quad du = (2x+1)dx$$

$$\int dy = \int e^u\ du$$

$$y = e^u + C$$

$$\boxed{y(x) = e^{x^2+x} + C}$$

Exercise

Find the general solution of the differential equation: $y' = \frac{1}{6x-5}$

Solution

$$\int dy = \int \frac{1}{6x-5}dx$$

$$\int dy = \frac{1}{6}\int \frac{1}{6x-5}d(6x-5)$$

$$\boxed{y(x) = \frac{1}{6}\ln|6x-5| + C}$$

Exercise

Find the general solution of the differential equation: $y' = \frac{x^2+2x+3}{x^3+3x^2+9x+1}$

Solution

$$\int dy = \int \frac{x^2+2x+3}{x^3+3x^2+9x+1}dx \qquad u = x^3+3x^2+9x+1 \quad du = 3\left(x^2+2x+3\right)dx$$

$$\int dy = \frac{1}{3}\int \frac{du}{u}$$

$$y(x) = \frac{1}{3}\ln|u| + C$$

$$\boxed{y(x) = \frac{1}{3}\ln\left|x^3+3x^2+9x+1\right| + C}$$

Exercise

Find the general solution of the differential equation: $y' = \frac{1}{x(\ln x)^2}$

Solution

$$\int dy = \int \frac{1}{x(\ln x)^2}dx \qquad u = \ln x \quad du = \frac{dx}{x}$$

$$\int dy = \int \frac{1}{u^2}du$$

$$y = -\frac{1}{u} + C$$

$$\boxed{y(x) = -\frac{1}{\ln x} + C}$$

Exercise

Evaluate the integrals $\int_{-2}^{2}\left(x^3-2x+3\right)dx$

Solution

$$\int_{-2}^{2}\left(x^3-2x+3\right)dx=\left[\frac{x^4}{4}-x^2+3x\right]_{-2}^{2}$$

$$=\left(\frac{(2)^4}{4}-(2)^2+3(2)\right)-\left(\frac{(-2)^4}{4}-(-2)^2+3(-2)\right)$$

$$\underline{=12}$$

Exercise

Evaluate the integrals $\int_{0}^{1}\left(x^2+\sqrt{x}\right)dx$

Solution

$$\int_{0}^{1}\left(x^2+\sqrt{x}\right)dx=\left[\frac{x^3}{3}+\frac{2}{3}x^{3/2}\right]_{0}^{1}$$

$$=\left(\frac{(1)^3}{3}+\frac{2}{3}(1)^{3/2}\right)-0$$

$$\underline{=1}$$

Exercise

Evaluate the integrals $\int_{0}^{\pi/3}4\sec u\tan u\ du$

Solution

$$\int_{0}^{\pi/3}4\sec u\tan u\ du=4\sec u\Big|_{0}^{\pi/3}$$

$$=4\left(\sec\frac{\pi}{3}-\sec 0\right)$$

$$=4(2-1)$$

$$\underline{=4}$$

Exercise

Evaluate the integrals $\displaystyle\int_{\pi/4}^{3\pi/4} \csc\theta \cot\theta\, d\theta$

Solution

$$\int_{\pi/4}^{3\pi/4} \csc\theta \cot\theta\, d\theta = -\csc\theta \Big|_{\pi/4}^{3\pi/4}$$

$$= -\left(\csc\frac{3\pi}{4} - \csc\frac{\pi}{4}\right)$$

$$= -\left(\sqrt{2} - \sqrt{2}\right)$$

$$\underline{= 0}$$

Exercise

Evaluate the integrals $\displaystyle\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt$

Solution

$$\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt = \int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \pi t^{-2}\right) dt$$

$$= \left[4\tan t - \pi t^{-1}\right]_{-\pi/3}^{-\pi/4}$$

$$= \left(4\tan\left(-\frac{\pi}{4}\right) - \pi\left(-\frac{4}{\pi}\right)\right) - \left(4\tan\left(-\frac{\pi}{3}\right) - \pi\left(-\frac{3}{\pi}\right)\right)$$

$$= \left(4(-1) + 4\right) - \left(4\left(-\sqrt{3}\right) + 3\right)$$

$$= -\left(-4\sqrt{3} + 3\right)$$

$$\underline{= 4\sqrt{3} - 3}$$

Exercise

Evaluate the integrals $\displaystyle\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy$

Solution

$$\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy = \int_{-3}^{-1} \left(\frac{y^5}{y^3} - \frac{2y}{y^3}\right) dy$$

$$= \int_{-3}^{-1} \left(y^2 - 2y^{-2}\right) dy$$

$$= \left[\frac{1}{3}y^3 + 2y^{-1}\right]_{-3}^{-1}$$

$$= \left(\frac{1}{3}(-1)^3 + \frac{2}{-1}\right) - \left(\frac{1}{3}(-3)^3 + \frac{2}{-3}\right)$$

$$\boxed{= \frac{22}{3}}$$

Exercise

Evaluate the integrals $\displaystyle\int_1^8 \frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}}dx$

Solution

$$\int_1^8 \frac{\left(x^{1/3}+1\right)\left(2-x^{2/3}\right)}{x^{1/3}}dx = \int_1^8 \frac{2x^{1/3} - x + 2 - x^{2/3}}{x^{1/3}}dx$$

$$= \int_1^8 \left(2 - x^{2/3} + 2x^{-1/3} - x^{1/3}\right)dx$$

$$= \left[2x - \frac{3}{5}x^{5/3} + 3x^{2/3} - \frac{3}{4}x^{4/3}\right]_1^8$$

$$= \left(2(8) - \frac{3}{5}(8)^{5/3} + 3(8)^{2/3} - \frac{3}{4}(8)^{4/3}\right) - \left(2(1) - \frac{3}{5}(1)^{5/3} + 3(1)^{2/3} - \frac{3}{4}(1)^{4/3}\right)$$

$$= \left(-\frac{16}{5}\right) - \left(\frac{73}{20}\right)$$

$$\boxed{= -\frac{137}{20}}$$

Exercise

Evaluate: $\displaystyle\int_0^1 (2t+3)^3 dt$

Solution

$$\int_0^1 (2t+3)^3 dt = \int_0^1 u^3 \frac{1}{2} du \qquad u = 2t+3 \Rightarrow du = 2dt \to \frac{du}{2} = dt$$

$$= \frac{1}{2}\int_0^1 u^3 du$$

$$= \frac{1}{2}\frac{u^4}{4}\Big|_0^1$$

$$= \frac{1}{8}(2t+3)^4\Big|_0^1$$

$$= \frac{1}{8}\left\lfloor (2(1)+3)^4 - (2(0)+3)^4 \right\rfloor$$

$$= \frac{1}{8}\left(5^4 - 3^4\right)$$

$$\underline{= 68}$$

Exercise

Evaluate the integral $\displaystyle\int_{-1}^{1} r\sqrt{1-r^2}\ dr$

Solution

Let $u = 1 - r^2 \quad \Rightarrow \quad du = -2rdr \quad \to \quad -\frac{1}{2}du = rdr$

$$\int_{-1}^{1} r\sqrt{1-r^2}\ dr = \int_{-1}^{1} u^{1/2}\ \left(-\tfrac{1}{2}du\right)$$

$$= -\frac{1}{2}\frac{2}{3}u^{3/2}\Big|_{-1}^{1}$$

$$= -\frac{1}{3}\left[\left(1-r^2\right)^{3/2}\right]_{-1}^{1}$$

$$= -\frac{1}{3}\left[\left(1-(1)^2\right)^{3/2} - \left(1-(-1)^2\right)^{3/2}\right]$$

$$= -\frac{1}{3}(0-0)$$

$$\underline{= 0}$$

Exercise

Find the general solution of $F'(x) = 4x + 2$, and find the particular solution that satisfies the initial condition $F(1) = 8$.

Solution

$$F(x) = \int (4x+2)dx$$

$$= 2x^2 + 2x + C$$

$$F(x) = 2(1)^2 + 2(1) + C = 8$$

$$2 + 2 + C = 8$$

$$C = 4$$

$$\underline{F(x) = 2x^2 + 2x + 4}$$

Exercise

Find the general solution of the differential equation: $y' = t\cos 3t$

Solution

$$u = t \to du = dt$$

$$dv = \cos 3t \to v = \frac{1}{3}\sin 3t$$

$$y = \frac{1}{3}t\sin 3t - \frac{1}{3}\int \sin 3t\,dt$$

$$= \frac{1}{3}t\sin 3t - \frac{1}{3}\frac{1}{3}\cos 3t + C$$

$$\boxed{y(t) = \frac{1}{3}t\sin 3t - \frac{1}{9}\cos 3t + C}$$

Exercise

A ball is thrown into the air from an initial height of 6 *m* with an initial velocity of 120 *m/s*. What will be the maximum height of the ball and at what time will this event occur?

Solution

$$\frac{dv}{dt} = -g \Rightarrow dv = -g\,dt$$

$$v(t) = -gt + C_1$$

$$v(t=0) = -g(0) + C_1 = 120$$

$$C_1 = 120$$

$$v(t) = -9.8t + 120$$

$$\frac{dx}{dt} = v \Rightarrow dx = v\,dt$$

$$x(t) = \int (-9.8t + 120)\,dt$$

$$= -4.9t^2 + 120t + C_2$$

$$x(0) = -4.9(0)^2 + 120(0) + C_2 = 6$$

$$C_2 = 6$$

$$x(t) = -4.9t^2 + 120t + 6$$

$$v(t) = -9.8t + 120 = 0 \quad \to t = \frac{120}{9.8} = 12.24\ sec$$

$$x(t = 12.24) = -4.9(12.24)^2 + 120(12.24) + 6$$

$$\boxed{= 740.69\ \ m}$$

Exercise

Derive the position function if a ball is thrown upward with initial velocity of 32 *ft* per *second* from an initial height of 48 *ft*. When does the ball hit the ground? With what velocity does the ball hit the ground?

Solution

$s(t) = -16t^2 + 32t + 48$

$s(0) = 48$

$s''(t) = -32$

$s'(t) = \int -32dt \quad s'(0) = 32$

$\quad = -32t + C_1$

$s'(0) = -32(0) + C_1 = 32$

$\quad \Rightarrow C_1 = 32$

$s'(t) = -32t + 32$

$s(t) = \int (-32t + 32)dt$

$\quad = -32\frac{t^2}{2} + 32t + C_2$

$s(0) = -32\frac{0^2}{2} + 32(0) + C_2 = 48 \qquad \Rightarrow C_2 = 48$

$s(t) = -16t^2 + 32t + 48$

$s(t) = -16t^2 + 32t + 48 = 0$

$-t^2 + 2t + 3 = 0 \ \Rightarrow t = -1, \ t = 3$

The ball hits the ground in **3** seconds

The velocity: $v(t) = s'(t) = -32t + 32$

$v(t = 3) = -32(3) + 32 = \boxed{-64 \ \ ft/\sec^2}$