

Section 3.4 – Comparison Tests

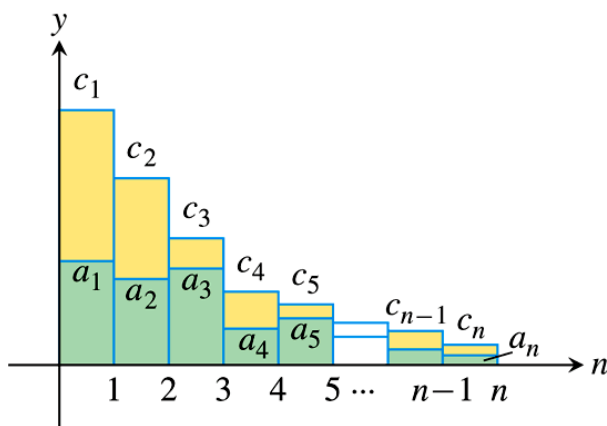
Theorem

Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with nonnegative terms. Suppose that for some integer N .

$$d_n \leq a_n \leq c_n \quad \text{for all } n > N$$

a) If $\sum c_n$ converges, then $\sum a_n$ also converges.

b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges.



Example

Use the comparison Test to determine if $\sum_{n=1}^{\infty} \frac{5}{5n-1}$ converges or diverges.

Solution

$$\begin{aligned} \frac{5}{5n-1} &= \frac{1}{n-\frac{1}{5}} \\ &> \frac{1}{n} \end{aligned}$$

The series **diverges** because its n th term is greater than the n th term of the divergent harmonic series.

Example

Use the comparison Test to determine if $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges or diverges.

Solution

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{n!} &< 1 + \sum_{n=0}^{\infty} \frac{1}{2^n} \\ &= 1 + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n\end{aligned}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \text{ is a geometric series } |r| = \frac{1}{2} < 1$$

$$\begin{aligned}1 + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n &= 1 + \frac{1}{1 - \frac{1}{2}} \\ &= 3\end{aligned}$$

The series ***converges***.

Limit Comparison Test

Theorem

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer)

1. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge
2. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges
3. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges

Example

Does the series $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \dots$ converge or diverge?

Solution

$$\begin{aligned} \frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \dots &= \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} \\ &= \sum_{n=1}^{\infty} \frac{2n+1}{n^2 + 2n + 1} \end{aligned}$$

$$\text{Let } a_n = \frac{2n+1}{n^2 + 2n + 1} \rightarrow \frac{2n}{n^2} = \frac{2}{n}$$

$$\frac{2}{n} > b_n = \frac{1}{n}$$

$$\text{Since } \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{2n+1}{n^2 + 2n + 1} \cdot \frac{n}{1} \\ &= 2 \end{aligned}$$

By the limit Comparison test $\sum a_n$ diverges

Example

Does the series $\frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \dots = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converge or diverge?

Solution

$$\text{Let } a_n = \frac{1}{2^n - 1} \rightarrow b_n = \frac{1}{2^n}$$

$$\text{Since } \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{2^n}{2^n - 1} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1 - \frac{1}{2^n}} \end{aligned}$$

$$= 1$$

$\Rightarrow \sum a_n$ converges by the Limit Comparison Test.

Example

Does the series $\frac{1+2\ln 2}{9} + \frac{1+3\ln 3}{14} + \frac{1+4\ln 4}{21} + \dots = \sum_{n=2}^{\infty} \frac{1+n\ln n}{n^2+5}$ converge or diverge?

Solution

$$\text{Let } a_n = \frac{1+n\ln n}{n^2+5} \rightarrow b_n = \frac{n\ln n}{n^2} = \frac{\ln n}{n} > \frac{1}{n}$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{1+n\ln n}{n^2+5} \cdot \frac{n}{1} \\ &= \lim_{n \rightarrow \infty} \frac{n+n^2\ln n}{n^2+5} \end{aligned}$$

$$= \infty$$

$\Rightarrow \sum a_n$ diverges by the Limit Comparison Test.

Example

Does the series $\sum_{n=2}^{\infty} \frac{\ln n}{n^{3/2}}$ converge?

Solution

$$\begin{aligned}\text{Let } a_n &= \frac{\ln n}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}} \\ &= \frac{1}{n^{5/4}} = b_n\end{aligned}$$

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{\ln n}{n^{3/2}} \cdot \frac{n^{5/4}}{1} \\ &= \lim_{n \rightarrow \infty} \frac{\ln n}{n^{1/4}} \\ &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{1}{4}n^{-3/4}} \\ &= \lim_{n \rightarrow \infty} \frac{4}{n^{1/4}} \\ &= 0\end{aligned}$$

L'hôpital Rule

$\Rightarrow \sum a_n$ converges by the Limit Comparison Test.

Exercises Section 3.4 – Comparison Tests

Use the Comparison Test to determine if the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 30}$

7. $\sum_{n=1}^{\infty} \frac{3n+1}{n^3 + 1}$

13. $\sum_{n=2}^{\infty} \frac{\ln n}{n+1}$

2. $\sum_{n=1}^{\infty} \frac{n-1}{n^4 + 2}$

8. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

14. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 1}}$

3. $\sum_{n=2}^{\infty} \frac{n+2}{n^2 - n}$

9. $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

15. $\sum_{n=0}^{\infty} \frac{1}{n!}$

4. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$

10. $\sum_{n=1}^{\infty} \frac{1}{3n^2 + 2}$

16. $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n-1}}$

5. $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$

11. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

17. $\sum_{n=0}^{\infty} e^{-n^2}$

6. $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$

12. $\sum_{n=0}^{\infty} \frac{4^n}{5^n + 3}$

18. $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$

Use the Limit Comparison Test to determine if the series converges or diverges.

19. $\sum_{n=1}^{\infty} \frac{n-2}{n^3 - n^2 + 3}$

24. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

29. $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$

20. $\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2 + 1)(n-1)}$

25. $\sum_{n=1}^{\infty} \frac{1}{1 + \sqrt{n}}$

30. $\sum_{n=1}^{\infty} \frac{2^n + 1}{5^n + 1}$

21. $\sum_{n=1}^{\infty} \frac{2^n}{3 + 4^n}$

26. $\sum_{n=1}^{\infty} \frac{n+5}{n^3 - 2n + 3}$

31. $\sum_{n=1}^{\infty} \frac{2n^2 - 1}{3n^5 + 2n + 1}$

22. $\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}4^n}$

27. $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$

32. $\sum_{n=1}^{\infty} \frac{1}{n^2(n+3)}$

23. $\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4} \right)^n$

28. $\sum_{n=1}^{\infty} \frac{5}{4^n + 1}$

33. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2 + 1}}$

Use any method to determine if the series converges or diverges

$$34. \sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}}$$

$$35. \sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$$

$$36. \sum_{n=1}^{\infty} \frac{n+1}{n^2 \sqrt{n}}$$

$$37. \sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$$

$$38. \sum_{n=1}^{\infty} \left(\frac{n}{3n+1} \right)^n$$

$$39. \sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$$

$$40. \sum_{n=1}^{\infty} \frac{1 + \sin n}{n^2}$$

$$41. \sum_{n=1}^{\infty} \frac{1}{2+3^n}$$

$$42. \sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$$

$$43. \sum_{n=1}^{\infty} \frac{1}{an+b}$$

$$44. \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$$

$$45. \sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$$

$$46. \sum_{n=0}^{\infty} 5\left(-\frac{4}{3}\right)^n$$

$$47. \sum_{n=1}^{\infty} \frac{1}{5^n + 1}$$

$$48. \sum_{n=2}^{\infty} \frac{1}{n^3 - 8}$$

$$49. \sum_{n=1}^{\infty} \frac{2n}{3n-2}$$

$$50. \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$51. \sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$$

$$52. \sum_{n=1}^{\infty} \frac{3}{n(n+3)}$$

$$53. \sum_{n=1}^{\infty} \frac{n 2^n}{4n^3+1}$$

$$54. \sum_{k=1}^{\infty} \frac{|\sin k|}{k^2}$$

$$55. \sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$$

$$56. \sum_{k=1}^{\infty} \sin^2 \frac{1}{k}$$

$$57. \sum_{k=1}^{\infty} \sin \frac{1}{k}$$

$$58. \sum_{k=1}^{\infty} \frac{1}{k} \sin \frac{1}{k}$$

$$59. \sum_{k=1}^{\infty} \frac{1}{k^2} \sin \frac{1}{k}$$

$$60. \sum_{k=1}^{\infty} (-1)^k k \sin \frac{1}{k}$$

$$61. \sum_{k=1}^{\infty} \tan \frac{1}{k}$$

$$62. \sum_{k=1}^{\infty} (-1)^k \tan^{-1} k$$

$$63. \sum_{n=1}^{\infty} \frac{\cos n}{n^3}$$

$$64. \sum_{k=2}^{\infty} \frac{k}{\ln k}$$

$$65. \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{\pi}{2n}$$

$$66. \frac{1}{1+\sqrt{1}} + \frac{1}{1+\sqrt{2}} + \frac{1}{1+\sqrt{3}} + \dots$$