Section 4.2 - Exponential Functions

Definition

The exponential function f with base $\boldsymbol{b}$ is defined by

$$f(x) = b^x \qquad \text{or} \qquad y = b^x$$

Base

where $b > 0$, $b \neq 1$ and $\boldsymbol{x}$ is any real number.

$$f(x) = 2^x \qquad f(x) = \left(\tfrac{1}{2}\right)^{2x+1} \qquad f(x) = 3^{-x} \qquad f(x) = (-2)^x$$

Example

Given: $f(x) = 13.49\ (0.967)^x - 1$, find $f(60)$

Solution

$$f(\mathbf{60}) = 13.49\ (0.967)^{\mathbf{60}} - 1$$
$$= 0.8014$$

Example

If $f(x) = 2^x$, find each of the following. $f(-1)$, $f(3)$, $f\left(\tfrac{5}{2}\right)$

Solution

a) $f(-1) = 2^{-1} = 0.5$

b) $f(3) = 2^3 = 8$

c) $f\left(\tfrac{5}{2}\right) = 2^{\frac{5}{2}} = 5.6569$

Graphing Exponential

1. Define the Horizontal Asymptote $f(x) = b^x \pm d$

$y = 0 \pm d$

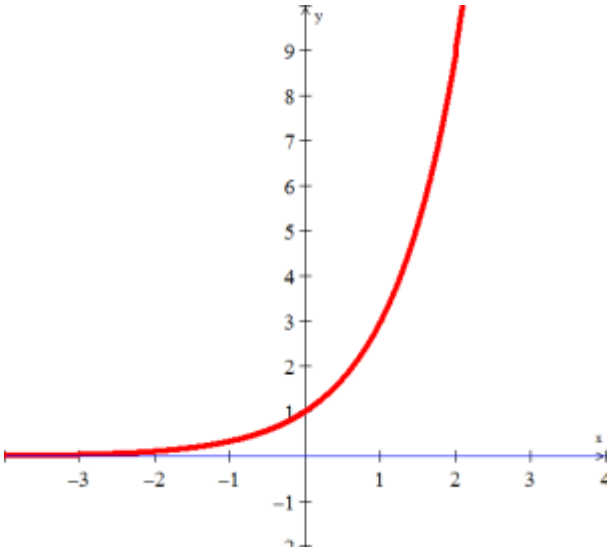
Example

$f(x) = 3^x$

Asymptote: $y = 0$

The exponential function always equals to 0

$x \to \infty$ *or* $x \to -\infty \Rightarrow f(x) \to 0$



2. Define/Make a table

(Force your exponential to = 0, then solve for x)

x	*f(x)*
$x - 2$	
$x - 1$	
x	
$x + 1$	
$x + 2$	

x	*f(x)*
−2	1/9
−1	1/3
0	1
1	3
2	9

Domain: $(-\infty, \infty)$

Range: (d, ∞)

Example

$f(x) = \left(\frac{1}{3}\right)^x$

$= \left(3^{-1}\right)^x$

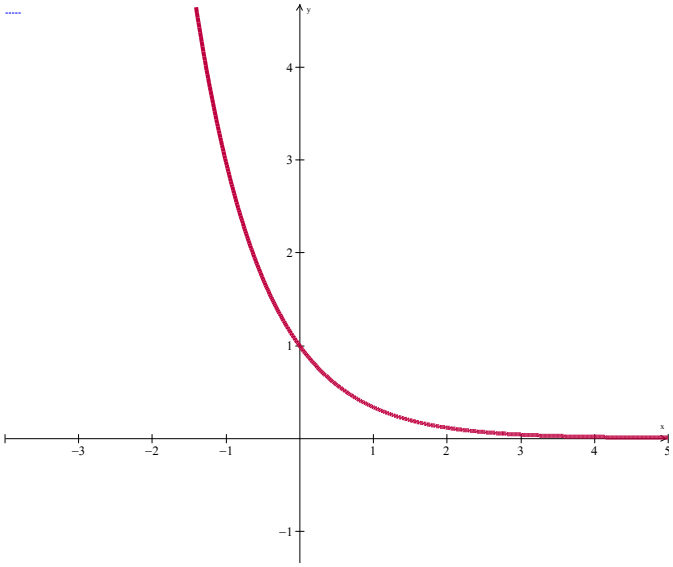
$= 3^{-x}$

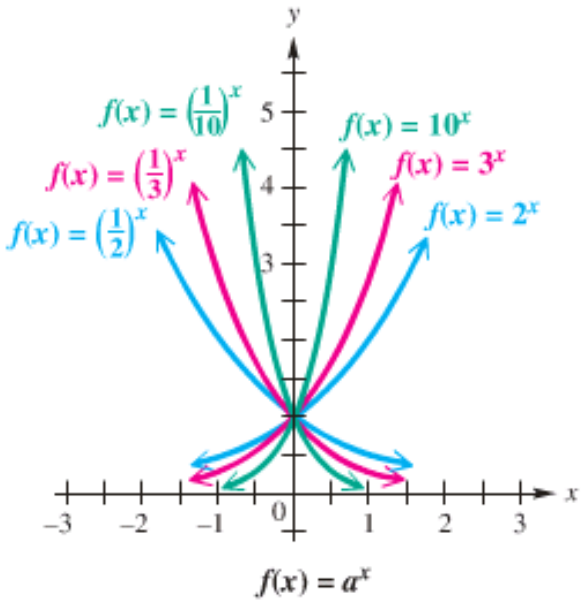
Reflected across y-axis

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$





Example

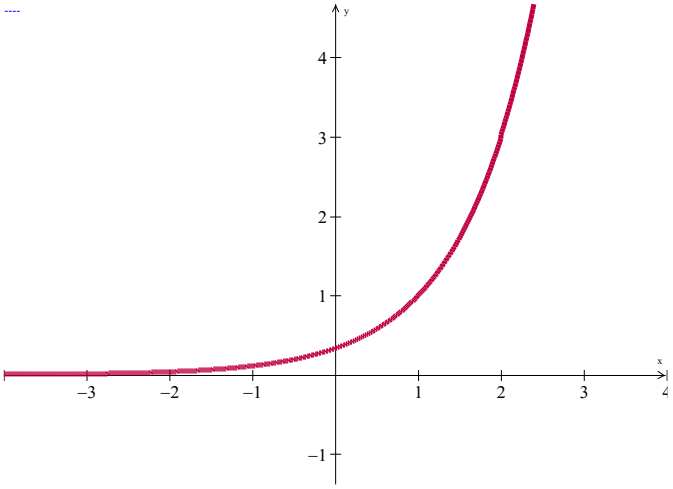
$f(x) = 3^{x-1}$

Shift right 1 unit

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$



Example

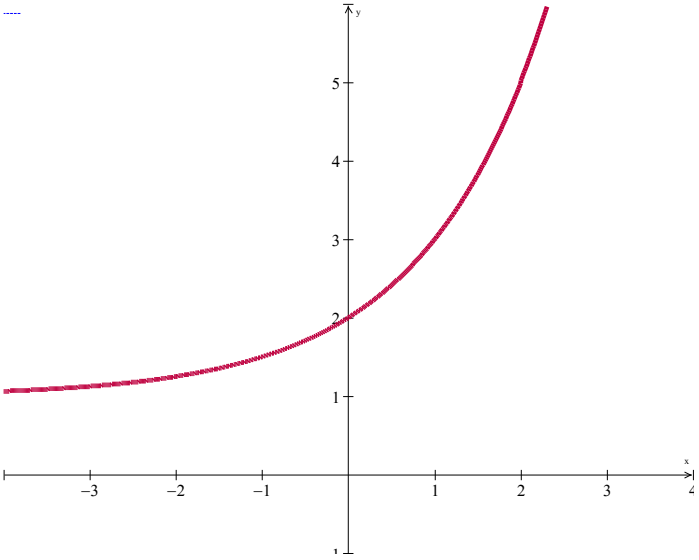
$f(x) = 2^x + 1$

Shift up 1 unit

Asymptote: $y = 1$

Domain: $(-\infty, \infty)$

Range: $(1, \infty)$

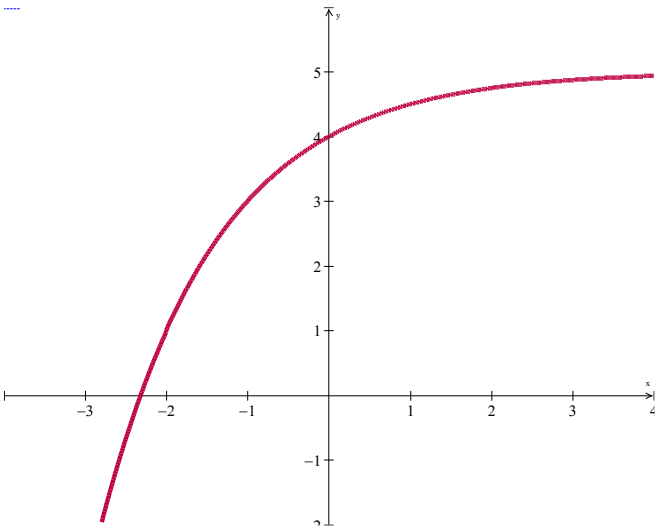


$f(x) = 5 - 2^{-x}$

Shifted up 5 units

Reflected across x-axis and y-axis

Asymptote: $y = 5$



Example

Give the domain and range.

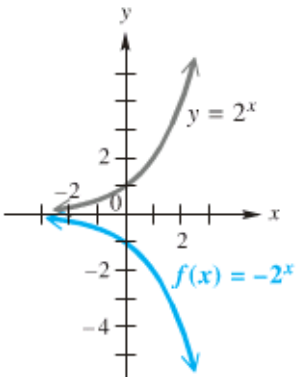
a) $f(x) = -2^x$

Reflected across x-axis

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(-\infty, 0)$



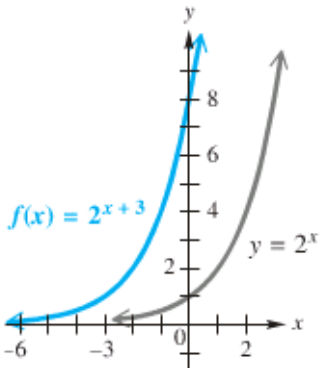
b) $f(x) = 2^{x+3}$

Shifted left 3 units

Asymptote: $y = 0$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$



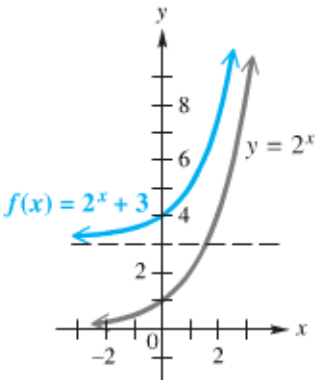
c) $f(x) = 2^x + 3$

Shifted up 3 units

Asymptote: $y = 3$

Domain: $(-\infty, \infty)$

Range: $(3, \infty)$



Natural Base e

The irrational number e is called natural base

$f(x) = e^x$ is called natural exponential function

$e^0 = 1$ $\quad e \approx 2.7183$ $\quad e^2 \approx 7.3891$ $\quad e^{-1} \approx 0.3679$

Example

The exponential function $f(x) = 1066e^{0.042x}$ models the gray wolf population of the Western Great Lakes, $f(x)$, in billions, x years after 1978. Project the gray population in the recovery area in 2012.

Solution

x = 2012 - 1978 = 34

$f(x = 34) = 1066e^{0.042(34)}$ $\quad$ 1066 e ^ (.042 * 34)

$= 4445.6$

≈ 4446

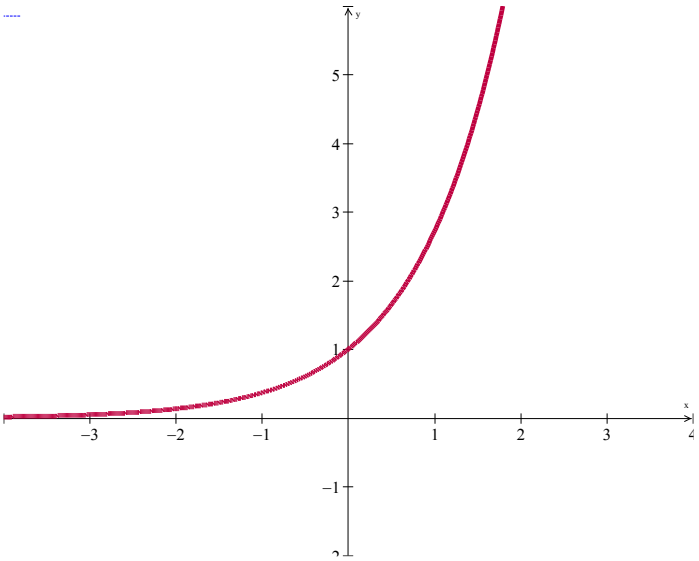
Example

Graph $f(x) = e^x$

Solution

Asymptote: $y = 0$

x	*f(x)*
-2	.14
-1	.4
0	1
1	2.7
2	7.4



Example

$f(x) = e^{x+3}$

Solution

Shifted left 3 units

Asymptote: $y = 0$

Formulas for Compound Interest

1. For n compounding per year: $A = P\left(1+\frac{r}{n}\right)^{nt}$
2. For Continuous compounding: $A = Pe^{rt}$

P: Principal, initial value
n: number of period per year
t: number of years
r: interest rate

A is also called ***Future value***

P is also called ***Present value***

Example

A sum of $10,000 is invested at an annual rate of 8%. Find the balance in the account after 5 years subject to quarterly compounding

Solution

Given:
P = $10,000
r = 8% = 0.08
t = 5

Quarterly $n = 4 \Rightarrow A = P\left(1+\frac{r}{n}\right)^{nt} = 10000\left(1+\frac{0.08}{4}\right)^{4(5)}$ 10000(1 + .08 / 4) ^ (4 * 5)

$= \$14,859.47$

Example

Suppose $5000 is deposited in an account paying 3% interest compounded continuously for 5 *yrs*. Find the total amount on deposit at the end of 5 *yrs*.

Solution

$A = Pe^{rt} = 5000e^{.03(5)} \approx \$5,809.17$ 5000 *e*(.03 * 5)

Exercises *Section* 4.2 - Exponential Functions

Find

1. $2^{3.4}$ **2.** $5^{\sqrt{3}}$ **3.** $6^{-1.2}$

Evaluate to four decimal places using a calculator

4. $e^{-0.75}$ **5.** $e^{2.3}$ **6.** $e^{-0.95}$

Sketch the graph

7. $f(x) = 2^x + 3$

8. $f(x) = 2^{3-x}$

9. $f(x) = \left(\frac{2}{5}\right)^{-x}$

10. $f(x) = -\left(\frac{1}{2}\right)^x + 4$

11. $f(x) = e^{x+4}$

12. The exponential function $f(x) = 1066e^{0.042x}$ models the gray wolf population of the Western Great Lakes, $f(x)$, in billions, x years after 1978. Project the gray population in the recovery area in 2012.

13. The function $f(x) = 6.4e^{0.0123x}$ describes world population, $f(x)$, in billions, x years after 2004 subject to a growth rate of 1.23% annually. Use the function to predict world population in 2050.

14. Find the accumulated value of an investment of $10,000 for 5 years at an interest rate of 5.5% if the money is

a) Compounded semiannually

b) Compounded quarterly

c) Compounded monthly

15. Suppose $1000 is deposited in an account paying 4% interest per year compounded quarterly.

a) Find the amount in the account after 10 years with no withdraws.

b) How much interest is earned over the 10 years period?

16. An investment of 1,000 increased to $13,464 in 20 years. If the interest was compounded continuously, find the interest rate.

17. Becky must pay a lump sum of $6000 in 5 yrs.

a) What amount deposited today at 3.1% compounded annually will grow to $6000 in 5 yrs.?

b) If only $5000 is available to deposit now, what annual interest rate is necessary for the money to increase to $6000 in 5 yrs.?

18. Find the present value of $4,000 if the annual interest rate is 3.5% compounded quarterly for 6 years.

19. How much money will there be in an account at the end of 8 years if $18,000 is deposited at 3% interest compounded semi-annually?

20. The function defined by $P(x) = 908e^{-0.0001348x}$ approximates the atmospheric pressure (in millibars) at an altitude of x meters. Use P to predict the pressure:

a) At 0 meters

b) At 12,000 meters