Section 1.8 – Applications

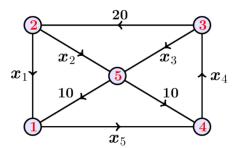
Network Analysis

Networks composed of branches and junctions are used as models in such fields as economics, traffic analysis, and electrical engineering.

In a network model, you assume that the total flow into a junction is equal to the total flow out of the junction

Example

Set up a system of linear equations to represent the network shown below. Then solve the system for x_i , i = 1, 2, 3, 4, 5.



Solution

$$1 \rightarrow x_1 + 10 = x_5 \implies x_1 - x_5 = -10$$

$$2 \rightarrow x_1 + x_2 = 20$$

$$3 \rightarrow x_4 = x_3 + 20 \implies -x_3 + x_4 = 20$$

$$4 \rightarrow x_4 = x_5 + 10 \implies x_4 - x_5 = 10$$

$$5 \rightarrow x_2 + x_3 = 10 + 10 = 20$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & | & -10 \\ 1 & 1 & 0 & 0 & 0 & | & 20 \\ 0 & 0 & -1 & 1 & 0 & | & 20 \\ 0 & 0 & 0 & 1 & -1 & | & 10 \\ 0 & 1 & 1 & 0 & 0 & | & 20 \\ \end{pmatrix} \quad R_2 - R_1$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & | & -10 \\ 0 & 1 & 0 & 0 & 1 & | & 30 \\ 0 & 0 & -1 & 1 & 0 & | & 20 \\ 0 & 0 & 0 & 1 & -1 & | & 10 \\ 0 & 1 & 1 & 0 & 0 & | & 20 \end{pmatrix} \quad R_5 - R_2$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & | & -10 \\ 0 & 1 & 0 & 0 & 1 & | & 30 \\ 0 & 0 & -1 & 1 & 0 & | & 20 \\ 0 & 0 & 0 & 1 & -1 & | & 10 \\ 0 & 0 & 1 & 0 & -1 & | & -10 \end{pmatrix} \quad R_5 + R_3$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & | & -10 \\ 0 & 1 & 0 & 0 & 1 & | & 30 \\ 0 & 0 & -1 & 1 & 0 & | & 20 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & | & -10 \\ 0 & 1 & 0 & 0 & 1 & | & 30 \\ 0 & 0 & -1 & 1 & 0 & | & 20 \\ 0 & 0 & 0 & 1 & -1 & | & 10 \\ 0 & 0 & 0 & 1 & -1 & | & 10 \end{pmatrix} \quad R_5 - R_4$$

Solution:
$$(x_5 - 10, 30 - x_5, x_5 - 10, 10 + x_5, x_5)$$

2nd Method

$$\begin{vmatrix} 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{vmatrix} - 1 \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \end{vmatrix}$$
$$= 1 \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix}$$
$$= -1 + 1$$
$$= 0$$

Infinite solution:

$$\begin{array}{cccc}
1 & \xrightarrow{x_1 = x_5 - 10} \\
2 & \xrightarrow{x_2 = 20 - x_1} & = 30 - x_5 \\
4 & \xrightarrow{x_4 = x_5 + 10} \\
3 & \xrightarrow{x_3 = x_4 - 20} & = x_5 - 10
\end{array}$$

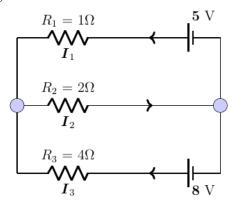
Electrical network

An electrical network is another type of network where analysis is commonly applied. An analysis of such a system uses two properties of electrical networks known as Kirchhoff's Laws.

- All the current flowing into a junction must flow out of it.
- The sum of the products IR (I is current and R is resistance) around a closed path is equal to the total voltage in the path.

Example

Determine the currents I_1 , I_2 , and I_3 for the electrical network



Solution

$$\begin{split} I_2 &= I_1 + I_3 \\ I_1 + 2I_2 &= 5 \\ 2I_2 + 4I_3 &= 8 \\ \begin{cases} I_1 - I_2 + I_3 &= 0 \\ I_1 + 2I_2 &= 5 \\ I_2 + 2I_3 &= 4 \\ \end{cases} \end{split}$$

$$\begin{vmatrix} 1 \\ 0 \end{vmatrix} = 7$$

$$R_{1} = 1\Omega$$

$$I_{1}$$

$$R_{2} = 2\Omega$$

$$I_{2}$$

$$R_{3} = 4\Omega$$

$$I_{3}$$

$$8 \text{ V}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 7 \qquad D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 5 & 2 & 0 \\ 4 & 1 & 2 \end{vmatrix} = 7 \qquad D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 5 & 0 \\ 0 & 4 & 2 \end{vmatrix} = 14 \qquad D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 5 \\ 0 & 1 & 4 \end{vmatrix} = 7$$

$$\underline{I_1 = 1 \ A} \quad \underline{I_2 = 2 \ A} \quad \underline{I_3 = 1 \ A}$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 5 & 0 \\ 0 & 4 & 2 \end{vmatrix} = 14$$

$$D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 5 \\ 0 & 1 & 4 \end{vmatrix} = 7$$

Cryptography

A *cryptogram* is a message written according to a secret code (the Greek word *kryptos* means "hidden"). One method of using matrix multiplication to *encode* and *decode* messages.

Let assign a number to each letter in the alphabet (with 0 assigned to a blank space), as shown

77

Example

Consider the invertible matrix:
$$A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix}$$

The message: **MEET ME MONDAY**

- a) Write the uncoded row matrices 1×3 for the message.
- b) Use the matrix A to encode the message.
- c) Decode a message from part b) given the matrix A.

Solution

b) Let encode the message **MEET ME MONDAY**

$$\begin{bmatrix} 13 & 5 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -26 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 33 & -53 & -12 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 18 & -23 & -42 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 14 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 5 & -20 & 56 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 25 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} -24 & 23 & 77 \end{bmatrix}$$

The sequence of coded row matrices is

$$\begin{bmatrix} 13 & -26 & -21 \end{bmatrix}$$
 $\begin{bmatrix} 33 & -53 & -12 \end{bmatrix}$ $\begin{bmatrix} 18 & -23 & -42 \end{bmatrix}$ $\begin{bmatrix} 5 & -20 & 56 \end{bmatrix}$ $\begin{bmatrix} -24 & 23 & 77 \end{bmatrix}$

The cryptogram:

$$13 - 26 - 21 \ 33 - 53 - 12 \ 18 - 23 - 42 \ 5 - 20 \ 56 - 24 \ 23 \ 77$$

c) To decode a message given the matrix A.

$$|A| = \begin{vmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{vmatrix} = 1$$

$$A^{-1} = \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$$

With the cryptogram:

$$\begin{bmatrix} 13 & -26 & -21 \end{bmatrix} \ \begin{bmatrix} 33 & -53 & -12 \end{bmatrix} \ \begin{bmatrix} 18 & -23 & -42 \end{bmatrix} \ \begin{bmatrix} 5 & -20 & 56 \end{bmatrix} \ \begin{bmatrix} -24 & 23 & 77 \end{bmatrix}$$

$$\begin{bmatrix} 13 & -26 & 21 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 13 & 5 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 33 & -53 & -1 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 18 & -23 & -42 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 13 \end{bmatrix}$$

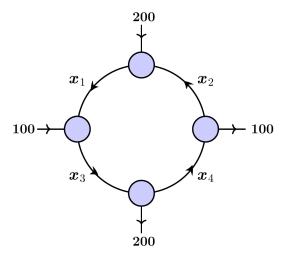
$$\begin{bmatrix} 5 & -20 & 56 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 15 & 14 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -24 & 23 & 77 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 25 & 0 \end{bmatrix}$$

The message is:

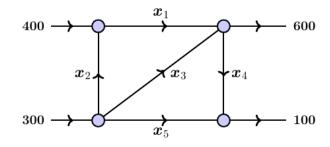
Exercises Section 1.8 – Applications

1. The flow of traffic, in vehicles per hour, through a network of streets as is shown below

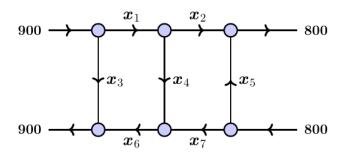


- a) Solve this system for x_i , i = 1, 2, 3, 4.
- b) Find the traffic flow when $x_4 = 0$.
- c) Find the traffic flow when $x_4 = 100$.
- d) Find the traffic flow when $x_1 = 2x_2$.

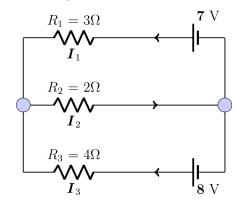
2. Through a network, Express x_n 's in terms of the parameters s and t.



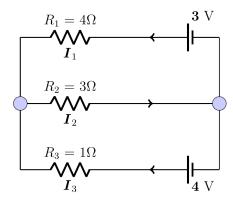
3. Water is flowing through a network of pipes. Express x_n 's in terms of the parameters s and t.



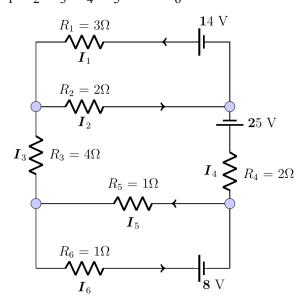
4. Determine the currents I_1 , I_2 , and I_3 for the electrical network shown below



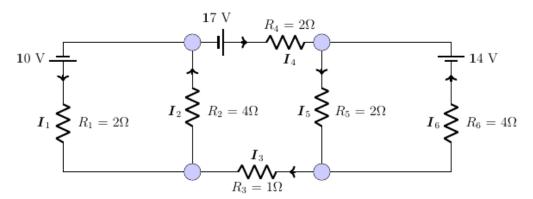
5. Determine the currents I_1 , I_2 , and I_3 for the electrical network shown below



6. Determine the currents I_1 , I_2 , I_3 , I_4 , I_5 , and I_6 for the electrical network shown below



7. Determine the currents I_1 , I_2 , I_3 , I_4 , I_5 , and I_6 for the electrical network shown below



8. Consider the invertible matrix: $A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -2 & -2 & 7 \end{pmatrix}$

The message: ICEBERG DEAD AHEAD

- a) Write the uncoded row matrices 1×3 for the message.
- b) Use the matrix A to encode the message.
- c) Decode a message from part b) given the matrix A.