Section 2.4 – Integer Representations and Algorithms

Representations of integers

Theorem

Let b be an integer greater that 1. Then if n is a positive integer, it can be expressed uniquely in the form

$$n = a_k b_k + a_{k-1} b_{k-1} + \cdots + a_1 b + a_0$$

Where k is a nonnegative integer $a_0,\ a_1,\ \ldots,\ a_k$ are nonnegative integers less than b, and $a_k \neq 0$

Example

What is the decimal expansion of the integer that has $(1\ 0101\ 1111)_2$ as its binary expansion?

Solution

$$\begin{aligned}(1\ 0101\ 1111)_2 &= 1\cdot 2^8 + 0\cdot 2^7 + 1\cdot 2^6 + 0\cdot 2^5 + 1\cdot 2^4 + 1\cdot 2^3 + 1\cdot 2^2 + 1\cdot 2^1 + 1\cdot 2^0 \\ &= 351\end{aligned}$$

Octal and Hexadecimal Expansions

Base 8 expansions are called ***octal*** expansions.
Base 16 expansions are called ***hexadecimal*** expansions.

Example

What is the decimal expansion of the number with octal expansion $(7016)_8$?

Solution

$$\begin{aligned}(7016)_8 &= 7\cdot 8^3 + 0\cdot 8^2 + 1\cdot 8^1 + 6\cdot 8^0 \\ &= 3598\end{aligned}$$

Example

What is the decimal expansion of the number with hexadecimal expansion $(2AE0B)_{16}$?

Solution

$$\begin{aligned}(2AE0B)_{16} &= 2\cdot 16^4 + 10\cdot 16^3 + 14\cdot 16^2 + 0\cdot 16^1 + 11\cdot 16^0 \\ &= 175627\end{aligned}$$

Base Conversion

The algorithm for constructing the base b expansion of an integer n, divide n by b to obtain a quotient and remainder, that is,

$$n = bq_0 + a_0, \qquad 0 \le a_0 \le b$$

$$q_0 = bq_1 + a_1, \qquad 0 \le a_1 \le b$$

Example

Find the octal expansion of $(12345)_{10}$

Solution

$$12345 = 8 \cdot 1543 + 1$$
$$1543 = 8 \cdot 192 + 7$$
$$192 = 8 \cdot 24 + 0$$
$$24 = 8 \cdot 3 + 0$$
$$3 = 8 \cdot 0 + 3$$
$$(12345)_{10} = (30071)_8$$

Example

Find the hexadecimal expansion of $(177130)_{10}$

Solution

$$177130 = 16 \cdot 11070 + 10 \qquad (10 = A)$$
$$11070 = 16 \cdot 691 + 14 \qquad (14 = E)$$
$$691 = 16 \cdot 43 + 3$$
$$43 = 16 \cdot 2 + 11 \qquad (11 = B)$$
$$2 = 16 \cdot 0 + 2$$
$$(177130)_{10} = (2B3EA)_{16}$$

Example

Find the bianry expansion of $(241)_{10}$

Solution

$$241 = 2 \cdot 120 + 1$$
$$120 = 2 \cdot 60 + 0$$
$$60 = 2 \cdot 30 + 0$$
$$30 = 2 \cdot 15 + 0$$

$15 = 2 \cdot 7 + 1$

$7 = 2 \cdot 3 + 1$

$3 = 2 \cdot 1 + 1$

$1 = 2 \cdot 0 + 1$

$(241)_{10} = (11110001)_2$

Representation of the Integers 0 through 15.																
Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Hexadecimal	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Octal	0	1	2	3	4	5	6	7	10	11	12	13	14	15	16	17
Binary	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111

Example

Find the octal and hexadecimal expansions of $(11\,1110\,1011\,1100)_2$

Solution

Octal: $(11\,1110\,1011\,1100)_2 = (011\,111\,010\,111\,100)_2$

$= (37274)_8$

Hexadecimal: $(11\,1110\,1011\,1100)_2 = (0011\,1110\,1011\,1100)_2$

$= (3EBC)_{16}$

Example

Find the binary expansions of $(765)_8$ and $(A8D)_{16}$

Solution

$(765)_8 = (111\,110\,101)_2$

$(A8D)_{16} = (1010\,1000\,1101)_2$

Algorithms for Integer Operations

Addition Algorithm

To add a and b, first add their rightmost bits. This gives

$$a_0 + b_0 = c_0 \cdot 2 + s_0$$

$$a_1 + b_1 + c_0 = c_1 \cdot 2 + s_1$$

Example

Add $a = (1110)_2$ *and* $b = (1011)_2$

Solution

$$a_0 + b_0 = 0 + 1 = 0 \cdot 2 + 1 \quad \Rightarrow \quad c_0 = 0,\ s_0 = 1$$

$$a_1 + b_1 + c_0 = 1 + 1 + 0 = 1 \cdot 2 + 0 \quad \Rightarrow \quad c_1 = 1,\ s_1 = 0$$

$$a_2 + b_2 + c_1 = 1 + 0 + 1 = 1 \cdot 2 + 0 \quad \Rightarrow \quad c_2 = 1,\ s_2 = 0$$

$$a_3 + b_3 + c_2 = 1 + 1 + 1 = 1 \cdot 2 + 1 \quad \Rightarrow \quad c_3 = 1,\ s_3 = 1$$

Therefore, $s = a + b = \underline{(11001)_2}|$

$$\begin{array}{llccccc}
(carry) & c & 1 & 1 & 1 & & \\
 & & & 1 & 1 & 1 & 0 \\
 & & + & 1 & 0 & 1 & 1 \\
\hline
 & s & 1 & 1 & 0 & 0 & 1
\end{array}$$

Example

How many additions of bits are required to use Algorithm 2 to add two integers with n bits (or less) in their binary representations?

Solution

Two integers are added by successively adding pairs of bits. Adding each pair of bits and the carry requires two additions of bits. Thus, the total number of additions of bits used is less than twice the number of bits in the expansion. Hence, the number of additions of bits used by Algorithm 2 to add two n-bit integers is $O(n)$.

Multiplication Algorithm

$$ab = a\left(b_0 2^0 + b_1 2^1 + \cdots + b_{n-1} 2^{n-1}\right)$$
$$= a\left(b_0 2^0\right) + a\left(b_1 2^1\right) + \cdots + a\left(b_{n-1} 2^{n-1}\right)$$

Algorithm: Multiplication of Integers

for $j := 0$ ***to*** $n - 1$

  if $b_j = 1$ **then** $c_j := a$ shifted j places

  else $c_j := 0$

$p := 0$

for $j := 0$ **to** $n - 1$

  $p := p + c_j$

Return p {p is the value of ab}

Example

Find the product of $a = (110)_2$ and $b = (101)_2$

Solution

$$\begin{array}{r} 110 \\ \times\, 101 \\ \hline 110 \\ 000 \\ 110 \\ \hline 11110 \end{array}$$

Modular Exponential

It is important to find b^n **mod** m efficiently, where *b*, *n* and *m* are large integers.

$$b^n = b^{a_{k-1}\cdot 2^{k-1}+\cdots+a_1\cdot 2+a_0} = b^{a_{k-1}\cdot 2^{k-1}}\cdots b^{a_1\cdot 2}\cdot b^{a_0}$$

Example

Compute 3^{11}

Solution

$$11=(1011)_2 \quad \rightarrow \quad 3^{11}=3^8 3^2 3^1$$

$$3^2=9, \quad 3^4=81, \quad 3^8=(81)^2=6561$$

$$\begin{aligned} 3^{11} &= 3^8 3^2 3^1 \\ &= 6561\cdot 9\cdot 3 \\ &= 177{,}147 \end{aligned}$$

Example

Use Algorithm 5 to find 3^{644} **mod** 645

Solution

$i=0$	$a_0=0$	$x=1$	$Power = 3^2 \textbf{ mod } 645 = 9 \textbf{ mod } 645 = 9$
$i=1$	$a_1=0$	$x=1$	$Power = 9^2 \textbf{ mod } 645 = 81 \textbf{ mod } 645 = 81$
$i=2$	$a_2=1$	$x=1\cdot 81 \textbf{ mod } 645 = 81$	$Power = 81^2 \textbf{ mod } 645 = 6561 \textbf{ mod } 645 = 111$
$i=3$	$a_3=0$	$x=81$	$Power = 111^2 \textbf{ mod } 645 = 12{,}321 \textbf{ mod } 645 = 66$
$i=4$	$a_4=0$	$x=81$	$Power = 66^2 \textbf{ mod } 645 = 4356 \textbf{ mod } 645 = 486$
$i=5$	$a_5=0$	$x=81$	$Power = 486^2 \textbf{ mod } 645 = 236{,}196 \textbf{ mod } 645 = 126$
$i=6$	$a_6=0$	$x=81$	$Power = 126^2 \textbf{ mod } 645 = 15{,}876 \textbf{ mod } 645 = 396$
$i=7$	$a_7=1$	$x=(81\cdot 396) \textbf{ mod } 645 = 471$	$Power = 396^2 \textbf{ mod } 645 = 156{,}816 \textbf{ mod } 645 = 81$
$i=8$	$a_8=0$	$x=471$	$Power = 81^2 \textbf{ mod } 645 = 6561 \textbf{ mod } 645 = 111$
$i=9$	$a_9=1$	$x=(471\cdot 111) \textbf{ mod } 645 = 36$	

This shows that following steps of Algorithm 5 produces the result 3^{644} **mod** $645=36$

Exercises *Section* 2.4 – Integer Representations and Algorithms

1. Convert the decimal expansion of each of these integers to a binary expansion

a) 321 *b)* 1023 *c)* 100632 *d)* 231 *e)* 4532

2. Convert binary the expansion of each of these integers to a decimal expansion

a) $(1\,1011)_2$ *b)* $(10\,1011\,0101)_2$ *c)* $(11\,1011\,1110)_2$ *d)*

e) $(111\,1100\,0001\,1111)_2$ *f)* $(1\,1111)_2$ *g)* $(10\,0000\,0001)_2$ *h)*

i) $(10\,0101\,0101)_2$ *j)* $(110\,1001\,0001\,0000)_2$ *k)*

3. Convert the binary expansion of each of these integers to an octal expansion

a) $(1111\,0111)_2$ *b)* $(1010\,1010\,1010)_2$

c) $(111\,0111\,0111\,0111)_2$ *d)* $(101\,0101\,0101\,0101)_2$

4. Convert the octal expansion of each of these integers to a binary expansion

a) $(572)_8$ *b)* $(1604)_8$ *c)* $(423)_8$ *d)* $(2417)_8$

5. Convert the hexadecimal expansion of each of these integers to a binary expansion

a) $(80E)_{16}$ *b)* $(135AB)_{16}$ *c)* $(ABBA)_{16}$

d) $(DEFACED)_{16}$ *e)* $(BADFACED)_{16}$ *f)* $(ABCDEF)_{16}$

6. Show that the binary expansion of a positive integer can be obtained from its hexadecimal expansion by translating each hexadecimal digit into a block of four binary digits.

7. Show that the binary expansion of a positive integer can be obtained from its octal expansion by translating each octal digit into a block of three binary digits.

8. Explain how to convert from binary to base 64 expansions and from base 64 expansions to binary expansions and from octal to base 64 expansions and from base 64 expansions to octal expansions

9. Find the sum and product of each of these pairs of numbers. Express your answers as a base 3 expansions

a) $(112)_3$, $(210)_3$ *b)* $(2112)_3$, $(12021)_3$

c) $(20001)_3$, $(1111)_3$ *d)* $(120021)_3$, $(2002)_3$

10. Find the sum and product of each of these pairs of numbers. Express your answers as an octal expansion.

a) $(763)_8$, $(147)_8$ *b)* $(6001)_8$, $(272)_8$

c) $(1111)_8$, $(777)_8$ *d)* $(54321)_8$, $(3456)_8$

11. Find the sum and product of each of these pairs of numbers. Express your answers as an hexadecimal expansion.

a) $(1AE)_{16}$, $(BBC)_{16}$ *b)* $(20CBA)_{16}$, $(A01)_{16}$

c) $(ABCDE)_{16}$, $(1111)_{16}$ *d)* $(E0000E)_{16}$, $(BAAA)_{16}$