Lecture One – Vectors and the Geometry of Space

Solution *Section* 1.1 – Three-Dimensional Coordinate Systems

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2+z^2=4, \quad y=0$

Solution

The circle $x^2+z^2=4$ in the ***xz***-plane

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2+y^2=4, \quad z=-2$

Solution

The circle $x^2+y^2=4$ in the plane $z=-2$

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2+y^2+z^2=1, \quad x=0$

Solution

The circle $y^2+z^2=1$ in the ***yz***-plane

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2+(y-1)^2+z^2=4, \quad y=0$

Solution

$$x^2+(0-1)^2+z^2=4 \quad \Rightarrow x^2+z^2=3$$

The circle $x^2+z^2=3$ in the ***xz***-plane

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $\quad x^2+y^2+z^2=4, \quad y=x$

Solution

The circle formed by the intersection of the sphere $x^2+y^2+z^2=4$ and the plane $y=x$

Exercise

Find the distance between points $P_1(1,\ 1,\ 1), \quad P_2(3,\ 3,\ 0)$

Solution

$$\begin{aligned}\left|P_1P_2\right| &= \sqrt{(3-1)^2+(3-1)^2+(0-1)^2} \\ &= \sqrt{4+4+1} \\ &= \sqrt{9} \\ &= 3\end{aligned}$$

Exercise

Find the distance between points $P_1(-1,\ 1,\ 5), \quad P_2(2,\ 5,\ 0)$

Solution

$$\begin{aligned}\left|P_1P_2\right| &= \sqrt{(2+1)^2+(5-1)^2+(0-5)^2} \\ &= \sqrt{9+16+25} \\ &= \sqrt{50} \\ &= 5\sqrt{2}\end{aligned}$$

Exercise

Find the distance between points $P_1(1,\ 4,\ 5), \quad P_2(4,\ -2,\ 7)$

Solution

$$\begin{aligned}\left|P_1P_2\right| &= \sqrt{(4-1)^2+(-2-4)^2+(7-5)^2} \\ &= \sqrt{9+36+4} \\ &= 7\end{aligned}$$

Exercise

Find the distance between points $P_1(3,\ 4,\ 5),\quad P_2(2,\ 3,\ 4)$

Solution

$$\begin{aligned}\left|P_1P_2\right| &= \sqrt{(2-3)^2+(3-4)^2+(4-5)^2}\\ &= \sqrt{1+1+1}\\ &= \boxed{\sqrt{3}}\end{aligned}$$

Exercise

Find the center and radii of the spheres $\qquad x^2+y^2+z^2+4x-4z=0$

Solution

$$\left(x^2+4x\right)+y^2+\left(z^2-4z\right)=0$$

$$\left(x^2+4x+4\right)+y^2+\left(z^2-4z+4\right)=4+4$$

$$(x+2)^2+y^2+(z-2)^2=8$$

The center is at $(-2,\ 0,\ 2)$ and the radius is $\sqrt{8}=\boxed{2\sqrt{2}}$

Exercise

Find the center and radii of the spheres $\qquad x^2+y^2+z^2-6y+8z=0$

Solution

$$x^2+\left(y^2-6y\right)+\left(z^2+8z\right)=0$$

$$x^2+\left(y^2-6y+\left(-\frac{6}{2}\right)^2\right)+\left(z^2+8z+\left(\frac{8}{2}\right)^2\right)=9+16$$

$$x^2+(y-3)^2+(z+4)^2=25$$

The center is at $(0,\ 3,\ -4)$ and the radius is $\boxed{5}$

Exercise

Find the center and radii of the spheres $2x^2 + 2y^2 + 2z^2 + x + y + z = 9$

Solution

$$x^2 + y^2 + z^2 + \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z = \frac{9}{2}$$

$$\left(x^2 + \frac{1}{2}x + \left(\frac{1}{2}\frac{1}{2}\right)^2\right) + \left(y^2 + \frac{1}{2}y + \left(\frac{1}{4}\right)^2\right) + \left(z^2 + \frac{1}{2}z + \left(\frac{1}{4}\right)^2\right) = \frac{9}{2} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$$

$$\left(x + \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 + \left(z + \frac{1}{4}\right)^2 = \frac{9}{2} + \frac{3}{16}$$

$$\left(x + \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 + \left(z + \frac{1}{4}\right)^2 = \frac{75}{16}$$

The center is at $\left(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$ and the radius is $\boxed{\frac{5\sqrt{3}}{4}}$

Exercise

Find a formula for the distance from the point $P(x, y, z)$ to x-axis

Solution

The distance between (x, y, z) and $(x, 0, 0)$ is:

$$d = \sqrt{(x-x)^2 + (y-0)^2 + (z-0)^2}$$

$$= \sqrt{y^2 + z^2}$$

Exercise

Find a formula for the distance from the point $P(x, y, z)$ to xy-plane

Solution

The distance between (x, y, z) and $(x, 0, z)$ is:

$$d = \sqrt{(x-x)^2 + (y-0)^2 + (z-z)^2}$$

$$= y$$

Solution   Section 1.2 – Vectors

Exercise

Let $\boldsymbol{u} = \langle 3,\ -2\rangle$ *and* $\boldsymbol{v} = \langle -2,\ 5\rangle$. Find the component form and the magnitude if the vector

a) $3\boldsymbol{u}$

b) $\boldsymbol{u} - \boldsymbol{v}$

c) $2\boldsymbol{u} - 3\boldsymbol{v}$

d) $-2\boldsymbol{u} + 5\boldsymbol{v}$

e) $-\frac{5}{13}\boldsymbol{u} + \frac{12}{13}\boldsymbol{v}$

Solution

a) $3\boldsymbol{u} = 3\langle 3,\ -2\rangle = \boxed{\langle 9,\ -6\rangle}$

b) $\boldsymbol{u} - \boldsymbol{v} = \langle 3,\ -2\rangle - \langle -2,\ 5\rangle = \boxed{\langle 5,\ -7\rangle}$

c)
$$\begin{aligned} 2\boldsymbol{u} - 3\boldsymbol{v} &= 2\langle 3,\ -2\rangle - 3\langle -2,\ 5\rangle \\ &= \langle 6,\ -4\rangle - \langle -6,\ 15\rangle \\ &= \boxed{\langle 12,\ -19\rangle} \end{aligned}$$

d)
$$\begin{aligned} -2\boldsymbol{u} + 5\boldsymbol{v} &= -2\langle 3,\ -2\rangle + 5\langle -2,\ 5\rangle \\ &= \langle -6,\ 4\rangle + \langle -10,\ 25\rangle \\ &= \boxed{\langle -14,\ 29\rangle} \end{aligned}$$

e)
$$\begin{aligned} -\frac{5}{13}\boldsymbol{u} + \frac{12}{13}\boldsymbol{v} &= -\frac{5}{13}\langle 3,\ -2\rangle + \frac{12}{13}\langle -2,\ 5\rangle \\ &= \langle -6,\ 4\rangle - \langle -10,\ 25\rangle \\ &= \boxed{\langle 4,\ -21\rangle} \end{aligned}$$

Exercise

Find the component form of the vector: The sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ where

$A = (1,-1),\quad B = (2,0),\quad C = (-1,3),\quad$ *and* $\quad D = (-2,2)$

Solution

$$\overrightarrow{AB} = \langle 2-1,\ 0-(-1)\rangle = \langle 1,\ 1\rangle$$

$$\overrightarrow{CD} = \langle -2-(-1),\ 2-3\rangle = \langle -1,\ -1\rangle$$

$$\overrightarrow{AB} + \overrightarrow{CD} = \langle 1,\ 1\rangle + \langle -1,\ -1\rangle = \boxed{\langle 0,\ 0\rangle}$$

Exercise

Find the component form of the vector: The unit vector that makes an angle $\theta = \frac{2\pi}{3}$ with the positive x-axis

Solution

$$\left\langle \cos\frac{2\pi}{3},\ \sin\frac{2\pi}{3} \right\rangle = \boxed{\left\langle -\frac{1}{2},\ \frac{\sqrt{3}}{2} \right\rangle}$$

Exercise

Find the component form of the vector: The unit vector obtained by rotating the vector $\langle 0,\ 1 \rangle$ $120°$ counterclockwise about the origin

Solution

The angle of unit vector $\langle 0,\ 1 \rangle$ is 90°, this unit vector rotates 120° which makes an angle of $90° + 120° = 210°$ with the positive x-axis

$$\langle \cos 210°,\ \sin 210° \rangle = \boxed{\left\langle -\frac{\sqrt{3}}{2},\ -\frac{1}{2} \right\rangle}$$

Exercise

Find the component form of the vector: The unit vector obtained by rotating the vector $\langle 1,\ 0 \rangle$ $135°$ counterclockwise about the origin

Solution

The angle of unit vector $\langle 1,\ 0 \rangle$ is 0°, this unit vector rotates 135° which makes an angle of $0° + 135° = 135°$ with the positive x-axis

$$\langle \cos 135°,\ \sin 135° \rangle = \boxed{\left\langle -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}} \right\rangle}$$

Exercise

Sketch the indicated vector

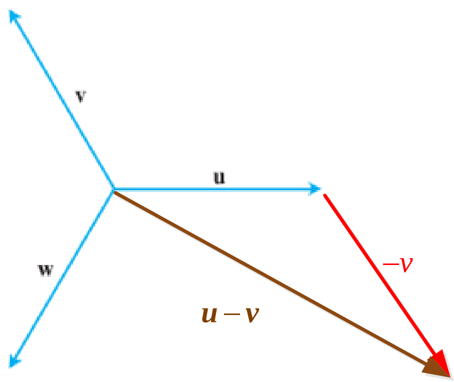
a) $\boldsymbol{u} - \boldsymbol{v}$

b) $2\boldsymbol{u} - \boldsymbol{v}$

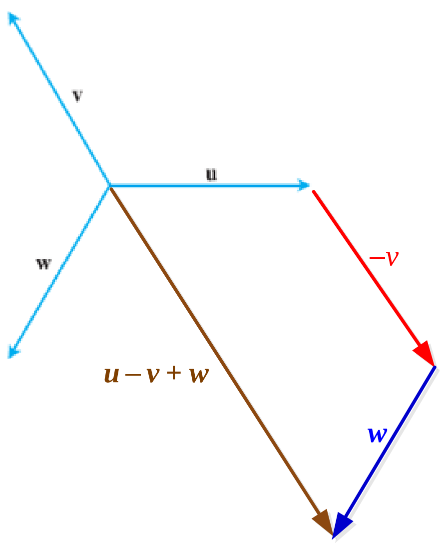
c) $\boldsymbol{u} - \boldsymbol{v} + \boldsymbol{w}$

Solution

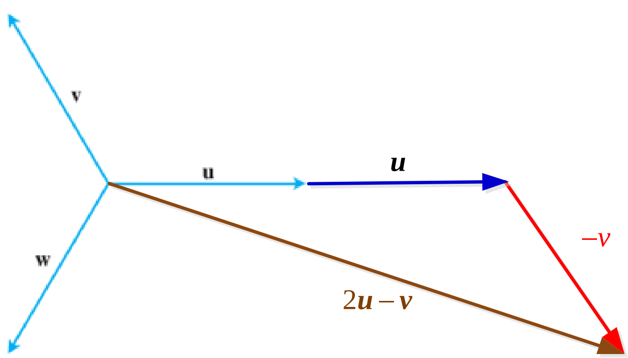
a)



b)



c)



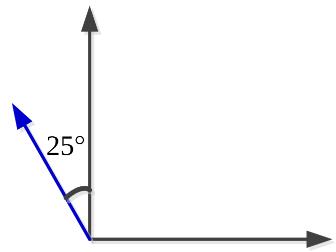
Exercise

An Airplane is flying in the direction 25° west of north at 800 *km/h*. Find the component form of the velocity of the airplane, assuming that the positive *x*-axis represents due east and the positive *y*-axis represents due north.

Solution

25° west of north is 25° + 90° = 115° north of east

$$800\langle \cos 115°,\ \sin 115° \rangle \approx \langle -338.095,\ 725.046 \rangle$$



Exercise

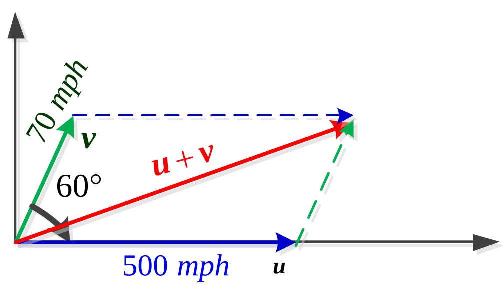
A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What speed and direction should the jetliner have in order for the resultant vector to be 500 *mph* due east?

Solution

$\boldsymbol{u} = \langle x,\ y \rangle$ = the velocity of the airplane

$\boldsymbol{v}$ = the velocity of the tailwind

$$\boldsymbol{v} = \langle 70\cos 60°,\ 70\sin 60° \rangle$$
$$= \langle 35,\ 35\sqrt{3} \rangle$$



$$\boldsymbol{u} + \boldsymbol{v} = \langle 500,\ 0 \rangle$$

$$\langle x,\ y \rangle + \langle 35,\ 35\sqrt{3} \rangle = \langle 500,\ 0 \rangle$$

$$\langle x,\ y \rangle = \langle 500,\ 0 \rangle - \langle 35,\ 35\sqrt{3} \rangle$$
$$= \langle 765,\ -35\sqrt{3} \rangle$$

$$\boxed{\boldsymbol{u} = \langle 765,\ -35\sqrt{3} \rangle}$$

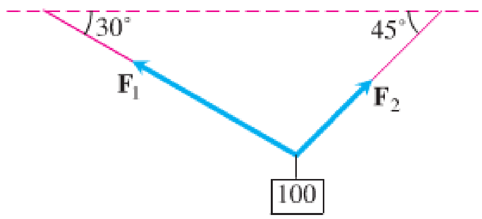
$$|\boldsymbol{u}| = \sqrt{465^2 + \left(-35\sqrt{3}\right)^2}$$
$$\approx 468.9\ mph$$

$$\theta = \tan^{-1}\frac{-35\sqrt{3}}{465} \approx -7.4°$$

The direction is **7.4°** south of east

Exercise

Consider a 100-N weight suspended by two wires. Find the magnitudes and components of the force vectors F_1 *and* F_2



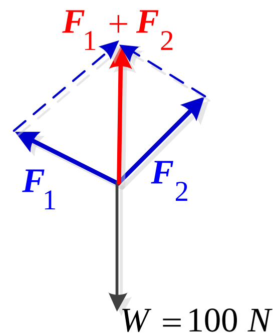
Solution

$$F_1 = \left\langle -\left|F_1\right|\cos 30°,\ \left|F_1\right|\sin 30°\right\rangle$$

$$= \left\langle -\frac{\sqrt{3}}{2}\left|F_1\right|,\ \frac{1}{2}\left|F_1\right|\right\rangle$$

$$F_2 = \left\langle \left|F_2\right|\cos 45°,\ \left|F_2\right|\sin 45°\right\rangle$$

$$= \left\langle \frac{\sqrt{2}}{2}\left|F_2\right|,\ \frac{\sqrt{2}}{2}\left|F_2\right|\right\rangle$$



$$F_1 + F_2 = \langle 0,\ 100\rangle$$

$$\left\langle -\frac{\sqrt{3}}{2}\left|F_1\right|,\ \frac{1}{2}\left|F_1\right|\right\rangle + \left\langle \frac{\sqrt{2}}{2}\left|F_2\right|,\ \frac{\sqrt{2}}{2}\left|F_2\right|\right\rangle = \langle 0,\ 100\rangle$$

$$\left\langle -\frac{\sqrt{3}}{2}\left|F_1\right| + \frac{\sqrt{2}}{2}\left|F_2\right|,\ \frac{1}{2}\left|F_1\right| + \frac{\sqrt{2}}{2}\left|F_2\right|\right\rangle = \langle 0,\ 100\rangle$$

$$\begin{cases} -\frac{\sqrt{3}}{2}\left|F_1\right| + \frac{\sqrt{2}}{2}\left|F_2\right| = 0 \\ \frac{1}{2}\left|F_1\right| + \frac{\sqrt{2}}{2}\left|F_2\right| = 100 \end{cases} \Rightarrow \boxed{\left|F_1\right| \approx 73.205\ N} \quad \boxed{\left|F_2\right| \approx 89.658\ N}$$

$$F_1 = \left\langle -\frac{\sqrt{3}}{2}(73.205),\ \frac{1}{2}(73.205)\right\rangle$$

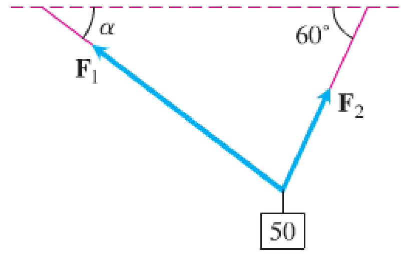
$$\boxed{F_1 \approx \langle -63.397,\ 36.603\rangle}$$

$$F_2 = \left\langle \frac{\sqrt{2}}{2}(89.658),\ \frac{\sqrt{2}}{2}(89.658)\right\rangle$$

$$\boxed{F_2 \approx \langle 63.397,\ 63.397\rangle}$$

Exercise

Consider a 50-N weight suspended by two wires, If the magnitude of vector $F_1 = 35\ N$, find the angle α and the magnitude of vector F_2



Solution

$$F_1 = \left\langle -\left|F_1\right|\cos\alpha,\ \left|F_1\right|\sin\alpha \right\rangle$$
$$= \langle -35\cos\alpha,\ 35\sin\alpha \rangle$$

$$F_2 = \left\langle \left|F_2\right|\cos 60°,\ \left|F_2\right|\sin 60° \right\rangle$$
$$= \left\langle \frac{1}{2}\left|F_2\right|,\ \frac{\sqrt{3}}{2}\left|F_2\right| \right\rangle$$

$$w = \langle 0,\ -50\rangle \quad\Rightarrow\quad F_1 + F_2 = \langle 0,\ 50\rangle$$

$$\langle -35\cos\alpha,\ 35\sin\alpha\rangle + \left\langle \frac{1}{2}\left|F_2\right|,\ \frac{\sqrt{3}}{2}\left|F_2\right| \right\rangle = \langle 0,\ 50\rangle$$

$$\left\langle -35\cos\alpha + \frac{1}{2}\left|F_2\right|,\ 35\sin\alpha + \frac{\sqrt{3}}{2}\left|F_2\right| \right\rangle = \langle 0,\ 50\rangle$$

$$\to \begin{cases} -35\cos\alpha + \frac{1}{2}\left|F_2\right| = 0 \\ 35\sin\alpha + \frac{\sqrt{3}}{2}\left|F_2\right| = 50 \end{cases} \qquad \to \begin{cases} \left|F_2\right| = 70\cos\alpha \\ \end{cases}$$

$$35\sin\alpha + \frac{\sqrt{3}}{2}(70\cos\alpha) = 50$$
$$35\sqrt{3}\cos\alpha = 50 - 35\sin\alpha$$
$$\sqrt{3}\cos\alpha = \frac{10}{7} - \sin\alpha$$

$$\left(\sqrt{3}\cos\alpha\right)^2 = \left(\frac{10}{7} - \sin\alpha\right)^2$$
$$3\cos^2\alpha = \frac{100}{49} - \frac{20}{7}\sin\alpha + \sin^2\alpha$$
$$3\left(1 - \sin^2\alpha\right) = \frac{100}{49} - \frac{20}{7}\sin\alpha + \sin^2\alpha$$
$$3 - 3\sin^2\alpha - \frac{100}{49} + \frac{20}{7}\sin\alpha - \sin^2\alpha = 0$$
$$-4\sin^2\alpha + \frac{20}{7}\sin\alpha + \frac{47}{49} = 0$$

$$-196\sin^2\alpha + 140\sin\alpha + 47 = 0 \quad\Rightarrow\quad \sin\alpha = \frac{5 \pm 6\sqrt{2}}{14}$$

Since α > 0 $\Rightarrow$ $\sin\alpha > 0$

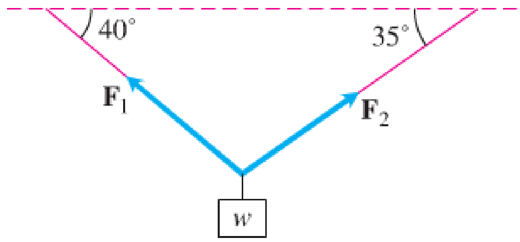
$$\rightarrow \sin\alpha = \frac{5+6\sqrt{2}}{14} \approx 0.963$$

$$\alpha \approx \sin^{-1}(0.963) = 74.42°$$

$$\begin{aligned} |F_2| &= 70\cos\alpha \\ &= 70\cos 74.42° \\ &\approx 18.81\ N \end{aligned}$$

Exercise

Consider a ***w***-N weight suspended by two wires, If the magnitude of vector $F_2 = 100\ N$, find ***w*** and the magnitude of vector F_1



Solution

$$F_1 = \left\langle -|F_1|\cos 40°,\ |F_1|\sin 40° \right\rangle$$

$$\begin{aligned} F_2 &= \left\langle |F_2|\cos 35°,\ |F_2|\sin 35° \right\rangle \\ &= \langle 100(0.819),\ 100(0.5736) \rangle \\ &= \langle 81.915,\ 57.358 \rangle \end{aligned}$$

$$F_1 + F_2 = \langle 0,\ w \rangle$$

$$\left\langle -|F_1|\cos 40°,\ |F_1|\sin 40° \right\rangle + \langle 81.915,\ 57.358 \rangle = \langle 0,\ w \rangle$$

$$\left\langle -|F_1|\cos 40° + 81.915,\ |F_1|\sin 40° + 57.358 \right\rangle = \langle 0,\ w \rangle$$

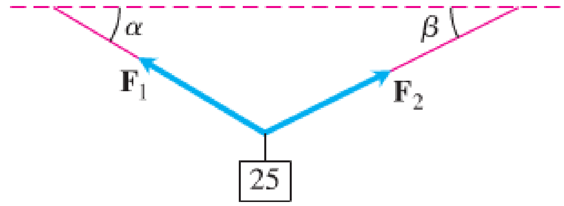
$$-|F_1|\cos 40° + 81.915 = 0 \quad \Rightarrow \quad |F_1|\cos 40° = 81.915$$

$$|F_1| = \frac{81.915}{\cos 40°} \approx 106.933\ N$$

$$\begin{aligned} w &= |F_1|\sin 40° + 57.358 \\ &= 106.933\sin 40° + 57.358 \\ &\approx 126.093\ N \end{aligned}$$

Exercise

Consider a 25-N weight suspended by two wires, If the magnitude of vector F_1 *and* F_2 are both 75 N, then angles α and β are equal. Find α.



Solution

$$F_1 = \left\langle -\left|F_1\right|\cos\alpha,\ \left|F_1\right|\sin\alpha\right\rangle$$
$$= \left\langle -75\cos\alpha,\ 75\sin\alpha\right\rangle$$

$$F_2 = \left\langle \left|F_2\right|\cos\beta,\ \left|F_2\right|\sin\beta\right\rangle$$
$$= \left\langle 75\cos\beta,\ 75\sin\beta\right\rangle$$

$$w = \langle 0,\ -25\rangle \quad \Rightarrow \quad F_1 + F_2 = \langle 0,\ 25\rangle$$

$$\langle -75\cos\alpha,\ 75\sin\alpha\rangle + \langle 75\cos\beta,\ 75\sin\beta\rangle = \langle 0,\ 25\rangle \qquad \textit{since} \quad \alpha = \beta$$
$$\langle -75\cos\alpha + 75\cos\alpha,\ 75\sin\alpha + 75\sin\alpha\rangle = \langle 0,\ 25\rangle$$

$$-75\cos\alpha + 75\cos\beta = 0 \quad \Rightarrow \quad \cos\alpha = \cos\beta$$

$$150\sin\alpha = 25$$

$$\sin\alpha = \frac{25}{150}$$

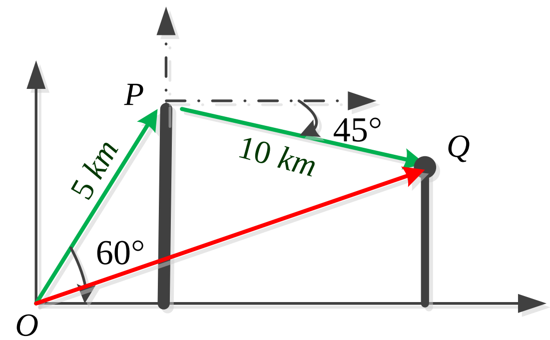
$$\underline{\left|\alpha = \sin^{-1}\frac{25}{150} \approx 9.59^\circ\right|}$$

Exercise

A bird flies from its nest 5 km in the direction 60° north east, where it stops to rest on a tree. It then flies 10 km in the direction due southeast and lands atop a telephone pole. Place an *xy*-coordinate system so that the origin is the bird's nest, the *x*-axis points east, and the *y*-axis points north.

a) At what point is the tree located?

b) At what point is the telephone pole?



Solution

a) $\overrightarrow{OP} = (5\cos 60°)\boldsymbol{i} + (5\sin 60°)\boldsymbol{j}$

$$= \frac{5}{2}\boldsymbol{i} + \frac{5\sqrt{3}}{2}\boldsymbol{j}$$

The tree is located at the point

$$\boxed{P = \left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)}$$

b) $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$

$$= \frac{5}{2}\boldsymbol{i} + \frac{5\sqrt{3}}{2}\boldsymbol{j} + (10\cos 315°)\boldsymbol{i} + (10\sin 315°)\boldsymbol{j}$$

$$= \frac{5}{2}\boldsymbol{i} + \frac{5\sqrt{3}}{2}\boldsymbol{j} + \left(10\frac{\sqrt{2}}{2}\right)\boldsymbol{i} + \left(10\left(-\frac{\sqrt{2}}{2}\right)\right)\boldsymbol{j}$$

$$= \left(\frac{5}{2} + 5\sqrt{2}\right)\boldsymbol{i} + \left(\frac{5\sqrt{3}}{2} - \frac{10\sqrt{2}}{2}\right)\boldsymbol{j}$$

$$= \left(\frac{5+10\sqrt{2}}{2}\right)\boldsymbol{i} + \left(\frac{5\sqrt{3}-10\sqrt{2}}{2}\right)\boldsymbol{j}$$

The pole is located at the point $\boxed{Q = \left(\frac{5+10\sqrt{2}}{2}, \frac{5\sqrt{3}-10\sqrt{2}}{2}\right)}$

Exercise

Suppose that *A*, *B*, and *C* are the corner points of the thin triangular plate of constant density.

a) Find the vector from *C* to the midpoint *M* of side *AB*.

b) Find the vector from *C* to the point that lies two-thirds of the way from *C* to *M* on the median *CM*.

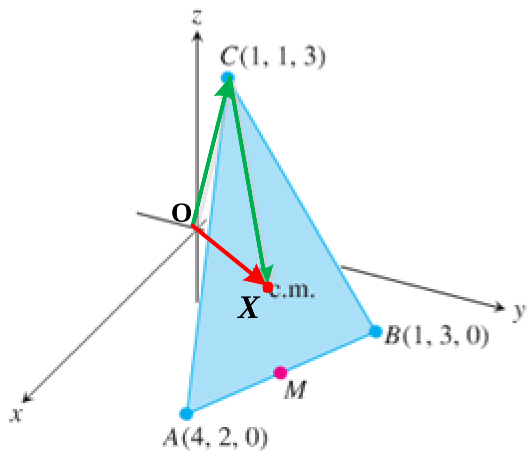
c) Find the coordinates of the point in which the medians of $\triangle ABC$ intersect (this point is the plate's center of mass).

Solution

a) The midpoint of AB is: $M=\left(\frac{4+1}{2},\frac{2+3}{2},0\right)=\left(\frac{5}{2},\ \frac{5}{2},\ 0\right)$

$$\overrightarrow{CM}=\left(\frac{5}{2}-1\right)\boldsymbol{i}+\left(\frac{5}{2}-1\right)\boldsymbol{j}+(0-3)\boldsymbol{k}$$

$$=\frac{3}{2}\boldsymbol{i}+\frac{3}{2}\boldsymbol{j}-3\boldsymbol{k}$$



b) The desired vector is $\overrightarrow{CX}=\frac{2}{3}\overrightarrow{CM}$

$$=\frac{2}{3}\left(\frac{3}{2}\boldsymbol{i}+\frac{3}{2}\boldsymbol{j}-3\boldsymbol{k}\right)$$

$$=\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}$$

c) The vector whose sum is the vector from the origin to C and the result of part (b) will terminate at the center of mass.

$$\overrightarrow{OX}=\overrightarrow{OC}+\overrightarrow{CX}$$

$$=\boldsymbol{i}+\boldsymbol{j}+3\boldsymbol{k}+\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}$$

$$=2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$$

Therefore; the center of mass point is $(2,\ 2,\ 1)$

Exercise

Show that a unit vector in the plane can be expressed as $\boldsymbol{u} = (\cos\theta)\boldsymbol{i} + (\sin\theta)\boldsymbol{j}$, obtained by rotating $\boldsymbol{i}$ through an angle θ in the counterclockwise direction. Explain why this form gives ***every*** *unit vector* in the plane.

Solution

Let $\boldsymbol{u}$ be any unit vector in the plane.
If $\boldsymbol{u}$ is positioned so that its initial point and terminal point is at (x, y), then $\boldsymbol{u}$ makes an angle θ with $\boldsymbol{i}$, measured in the *ccw* direction.

Since $|\boldsymbol{u}| = 1 \quad \Rightarrow \quad x = \cos\theta \quad \textit{and} \quad y = \sin\theta$

That implies to: $\boldsymbol{u} = (\cos\theta)\boldsymbol{i} + (\sin\theta)\boldsymbol{j}$

Since $\boldsymbol{u}$ is any unit vector in the plane; this holds for every unit vector in the plane.

Solution *Section* **1.3 – The Dot Product**

Exercise

Find for $\boldsymbol{v} = 2\boldsymbol{i} - 4\boldsymbol{j} + \sqrt{5}\boldsymbol{k}, \quad \boldsymbol{u} = -2\boldsymbol{i} + 4\boldsymbol{j} - \sqrt{5}\boldsymbol{k}$

a) $\boldsymbol{v} \bullet \boldsymbol{u},\ |\boldsymbol{v}|,\ |\boldsymbol{u}|$

b) The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

c) The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

d) The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

Solution

a)

$$\begin{aligned} \boldsymbol{v} \bullet \boldsymbol{u} &= \left(2\boldsymbol{i} - 4\boldsymbol{j} + \sqrt{5}\boldsymbol{k}\right) \cdot \left(-2\boldsymbol{i} + 4\boldsymbol{j} - \sqrt{5}\boldsymbol{k}\right) \\ &= -4 - 16 - 5 \\ &= -25 \end{aligned}$$

$$\begin{aligned} |\boldsymbol{v}| &= \sqrt{2^2 + (-4)^2 + \left(\sqrt{5}\right)^2} \\ &= \sqrt{4 + 16 + 5} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

$$\begin{aligned} |\boldsymbol{u}| &= \sqrt{(-2)^2 + 4^2 + \left(-\sqrt{5}\right)^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

b) $$\cos\theta = \frac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} = \frac{-25}{(5)(5)} = -1$$

c) $$|\boldsymbol{u}|\cos\theta = (5)(-1) = -5$$

d)

$$\begin{aligned} proj_{\boldsymbol{v}}\boldsymbol{u} &= \left(\frac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v} \\ &= \left(\frac{-25}{5^2}\right)\left(2\boldsymbol{i} - 4\boldsymbol{j} + \sqrt{5}\boldsymbol{k}\right) \\ &= -\left(2\boldsymbol{i} - 4\boldsymbol{j} + \sqrt{5}\boldsymbol{k}\right) \\ &= -2\boldsymbol{i} + 4\boldsymbol{j} - \sqrt{5}\boldsymbol{k} \end{aligned}$$

Exercise

Find for $\boldsymbol{v}=\frac{3}{5}\boldsymbol{i}+\frac{4}{5}\boldsymbol{k}, \quad \boldsymbol{u}=5\boldsymbol{i}+12\boldsymbol{j}$

a) $\boldsymbol{v}\cdot\boldsymbol{u},\ |\boldsymbol{v}|,\ |\boldsymbol{u}|$

b) The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

c) The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

d) The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

Solution

a) $$\boldsymbol{v}\cdot\boldsymbol{u}=\left(\frac{3}{5}\boldsymbol{i}+\frac{4}{5}\boldsymbol{k}\right)\cdot(5\boldsymbol{i}+12\boldsymbol{j})$$
$$=3$$

$$|\boldsymbol{v}|=\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}$$
$$=\sqrt{\frac{9}{25}+\frac{16}{25}}$$
$$=\sqrt{\frac{25}{25}}$$
$$=1$$

$$|\boldsymbol{u}|=\sqrt{5^2+12^2}$$
$$=13$$

b) $$\cos\theta=\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$$
$$=\frac{3}{(1)(13)}$$
$$=\frac{3}{13}$$

c) $$|\boldsymbol{u}|\cos\theta=(13)\left(\frac{3}{13}\right)=3$$

d) $$proj_{\boldsymbol{v}}\boldsymbol{u}=\left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$$
$$=\left(\frac{3}{1^2}\right)\left(\frac{3}{5}\boldsymbol{i}+\frac{4}{5}\boldsymbol{k}\right)$$
$$=\frac{9}{5}\boldsymbol{i}+\frac{12}{5}\boldsymbol{k}$$

Exercise

Find for $\boldsymbol{v} = 2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k}, \quad \boldsymbol{u} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

a) $\boldsymbol{v} \bullet \boldsymbol{u}, \ |\boldsymbol{v}|, \ |\boldsymbol{u}|$

b) The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

c) The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

d) The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

Solution

a)
$$\begin{aligned}\boldsymbol{v} \bullet \boldsymbol{u} &= (2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k}) \cdot (2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) \\ &= 4 + 20 - 11 \\ &= 13\end{aligned}$$

$$\begin{aligned}|\boldsymbol{v}| &= \sqrt{2^2 + 10^2 + (-11)^2} \\ &= \sqrt{4 + 100 + 121} \\ &= \sqrt{225} \\ &= 15\end{aligned}$$

$$\begin{aligned}|\boldsymbol{u}| &= \sqrt{2^2 + 2^2 + 1^2} \\ &= 3\end{aligned}$$

b)
$$\begin{aligned}\cos\theta &= \frac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} \\ &= \frac{13}{(3)(15)} \\ &= \frac{13}{45}\end{aligned}$$

c)
$$|\boldsymbol{u}|\cos\theta = (3)\left(\frac{13}{45}\right) = \frac{13}{15}$$

d)
$$\begin{aligned}proj_{\boldsymbol{v}}\boldsymbol{u} &= \left(\frac{\boldsymbol{u} \bullet \boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v} \\ &= \left(\frac{13}{15^2}\right)(2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k}) \\ &= \frac{13}{225}(2\boldsymbol{i} + 10\boldsymbol{j} - 11\boldsymbol{k})\end{aligned}$$

Exercise

Find for $\boldsymbol{v} = 5\boldsymbol{i} + \boldsymbol{j}, \quad \boldsymbol{u} = 2\boldsymbol{i} + \sqrt{17}\,\boldsymbol{j}$

a) $\boldsymbol{v}\bullet\boldsymbol{u},\ |\boldsymbol{v}|,\ |\boldsymbol{u}|$

b) The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

c) The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

d) The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

Solution

a) $\boldsymbol{v}\bullet\boldsymbol{u} = (5\boldsymbol{i} + \boldsymbol{j})\cdot\left(2\boldsymbol{i} + \sqrt{17}\boldsymbol{j}\right) \underline{= 10 + \sqrt{17}|}$

$$|\boldsymbol{v}| = \sqrt{25+1} \underline{= \sqrt{26}|}$$

$$|\boldsymbol{u}| = \sqrt{4+17} \underline{= \sqrt{21}|}$$

b) $\cos\theta = \dfrac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$

$$= \frac{10+\sqrt{17}}{\sqrt{21}\sqrt{26}}$$

$$\underline{= \frac{10+\sqrt{17}}{\sqrt{546}}|}$$

c) $|\boldsymbol{u}|\cos\theta = \left(\sqrt{21}\right)\left(\dfrac{10+\sqrt{17}}{\sqrt{546}}\right)$

$$\underline{= \frac{10+\sqrt{17}}{\sqrt{26}}|}$$

d) $proj_{\boldsymbol{v}}\boldsymbol{u} = \left(\dfrac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$

$$\underline{= \left(\frac{10+\sqrt{17}}{26}\right)(5\boldsymbol{i} + \boldsymbol{j})|}$$

Exercise

Find for $\boldsymbol{v} = \left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}} \right\rangle$, $\boldsymbol{u} = \left\langle \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{3}} \right\rangle$

a) $\boldsymbol{v}\cdot\boldsymbol{u}$, $|\boldsymbol{v}|$, $|\boldsymbol{u}|$

b) The cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$

c) The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$

d) The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

Solution

a) $\boldsymbol{v}\cdot\boldsymbol{u} = \left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}} \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{3}} \right\rangle = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$$|\boldsymbol{v}| = \sqrt{\frac{1}{2}+\frac{1}{3}} = \frac{\sqrt{5}}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{30}}{6}$$

$$|\boldsymbol{u}| = \sqrt{\frac{1}{2}+\frac{1}{3}} = \frac{\sqrt{5}}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{30}}{6}$$

b) $\cos\theta = \dfrac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$

$$= \frac{\frac{1}{6}}{\frac{\sqrt{30}}{6}\frac{\sqrt{30}}{6}}$$

$$= \frac{1}{6}\left(\frac{36}{30}\right)$$

$$= \frac{1}{5}$$

c) $|\boldsymbol{u}|\cos\theta = \left(\frac{\sqrt{30}}{6}\right)\left(\frac{1}{5}\right) = \frac{\sqrt{30}}{30} = \frac{1}{\sqrt{30}}$

d) $proj_{\boldsymbol{v}}\boldsymbol{u} = \left(\dfrac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$

$$= \frac{1}{6}\left(\frac{36}{30}\right)\left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}} \right\rangle$$

$$= \frac{1}{5}\left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}} \right\rangle$$

Exercise

Find the angles between the vectors $\boldsymbol{u} = 2\boldsymbol{i} + \boldsymbol{j}, \quad \boldsymbol{v} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

Solution

$$\theta = \cos^{-1}\frac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$$

$$= \cos^{-1}\left(\frac{2+2+0}{\sqrt{4+1}\sqrt{1+4+1}}\right)$$

$$= \cos^{-1}\left(\frac{4}{\sqrt{5}\sqrt{6}}\right)$$

$$= \cos^{-1}\left(\frac{4}{\sqrt{30}}\right)$$

$$\approx 0.84 \ \ rad$$

Exercise

Find the angles between the vectors $\boldsymbol{u} = \sqrt{3}\boldsymbol{i} - 7\boldsymbol{j}, \quad \boldsymbol{v} = \sqrt{3}\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

Solution

$$\theta = \cos^{-1}\frac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$$

$$= \cos^{-1}\left(\frac{3-7+0}{\sqrt{3+49}\sqrt{3+1+1}}\right)$$

$$= \cos^{-1}\left(\frac{-4}{\sqrt{52}\sqrt{5}}\right)$$

$$= \cos^{-1}\left(-\frac{4}{\sqrt{260}}\right)$$

$$\approx 1.82 \ \ rad$$

Exercise

Find the angles between the vectors $\boldsymbol{u} = \boldsymbol{i} + \sqrt{2}\boldsymbol{j} - \sqrt{2}\boldsymbol{k}, \quad \boldsymbol{v} = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

Solution

$$\theta = \cos^{-1}\frac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$$

$$= \cos^{-1}\left(\frac{-1+\sqrt{2}-\sqrt{2}}{\sqrt{1+2+2}\sqrt{1+1+1}}\right)$$

$$= \cos^{-1}\left(\frac{-1}{\sqrt{5}\sqrt{3}}\right)$$

$$= \cos^{-1}\left(-\frac{1}{\sqrt{15}}\right)$$

$$\approx 1.83 \ rad$$

Exercise

The direction angles α, β, and γ of a vector $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ are defined as follows:

is the angle between $\boldsymbol{v}$ and the positive x-axis $(0 \le \alpha \le \pi)$

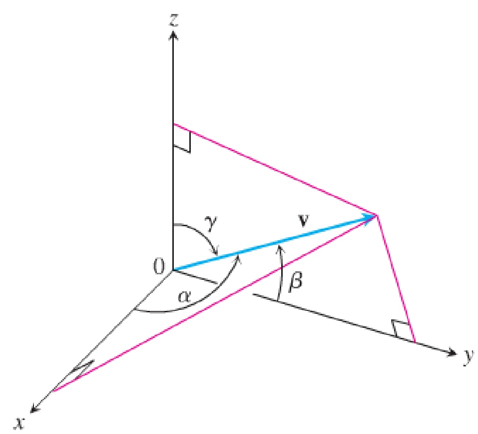
is the angle between $\boldsymbol{v}$ and the positive y-axis $(0 \le \beta \le \pi)$

is the angle between $\boldsymbol{v}$ and the positive z-axis $(0 \le \gamma \le \pi)$

a) Show that $\cos\alpha = \frac{a}{|\mathbf{v}|}$, $\cos\beta = \frac{b}{|\mathbf{v}|}$, $\cos\gamma = \frac{c}{|\mathbf{v}|}$, and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. These cosines are called the direction cosines of $\boldsymbol{v}$.

b) Show that if $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ is a unit vector, then a, b, and c are the direction cosines of $\boldsymbol{v}$.

Solution



a)

$$\cos\alpha = \frac{\boldsymbol{i}\bullet\boldsymbol{v}}{|\boldsymbol{i}||\mathbf{v}|} = \frac{\boldsymbol{i}\bullet(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k})}{|\mathbf{v}|} = \frac{a}{|\mathbf{v}|}$$

$$\cos\beta = \frac{\boldsymbol{j}\bullet\boldsymbol{v}}{|\boldsymbol{j}||\mathbf{v}|} = \frac{\boldsymbol{j}\bullet(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k})}{|\mathbf{v}|} = \frac{b}{|\mathbf{v}|}$$

$$\cos\gamma = \frac{\boldsymbol{k}\bullet\boldsymbol{v}}{|\boldsymbol{k}||\mathbf{v}|} = \frac{\boldsymbol{k}\bullet(a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k})}{|\mathbf{v}|} = \frac{c}{|\mathbf{v}|}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{a}{|\mathbf{v}|}\right)^2 + \left(\frac{b}{|\mathbf{v}|}\right)^2 + \left(\frac{c}{|\mathbf{v}|}\right)^2$$

$$= \frac{a^2}{|\mathbf{v}|^2} + \frac{b^2}{|\mathbf{v}|^2} + \frac{c^2}{|\mathbf{v}|^2}$$

$$= \frac{a^2 + b^2 + c^2}{|\mathbf{v}|^2}$$

$$= \frac{a^2 + b^2 + c^2}{a^2 + b^2 + c^2}$$

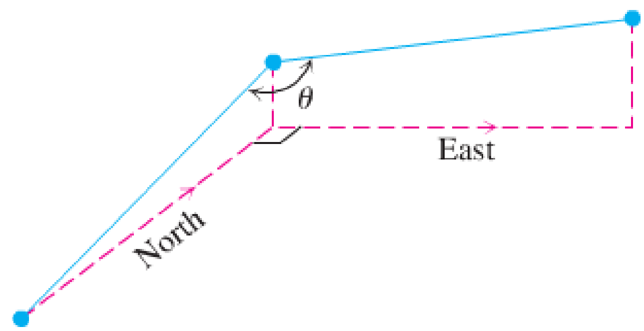
$$= 1$$

b) If $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ is a unit vector $\Rightarrow |\mathbf{v}| = 1$

$\cos\alpha = \frac{a}{|\mathbf{v}|} = a$, $\cos\beta = \frac{b}{|\mathbf{v}|} = b$, $\cos\gamma = \frac{c}{|\mathbf{v}|} = c$ are the direction cosines of $\boldsymbol{v}$.

Exercise

A water main is to be constructed with 20% grade in the north direction and a 10% grade in the east direction. Determine the angle θ required in the water main for the turn from north to east.



Solution

20% grade in the north direction $\Rightarrow zk = 20\%xi = .2xi \quad \rightarrow If\ \ x = 10 \quad z = 2$

Let $\boldsymbol{u} = 10\boldsymbol{i} + 2\boldsymbol{k}$ be parallel to the pipe in the north direction.

$\boldsymbol{v} = 10\boldsymbol{j} + \boldsymbol{k}$ be parallel to the pipe in the east direction.

$$\theta = \cos^{-1}\frac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$$

$$= \cos^{-1}\frac{0+0+2}{\sqrt{100+4}\sqrt{100+1}}$$

$$= \cos^{-1}\frac{2}{\sqrt{104}\sqrt{101}}$$

$$\approx 88.88°$$

Exercise

A gun with muzzle velocity of 1200 ft/sec is fired at an angle of 8° above the horizontal. Find the horizontal and vertical components of the velocity.

Solution

Horizontal component: $1200\cos 8° \approx 1188\ ft/s$

Vertical component: $1200\sin 8° \approx 167\ ft/s$

Exercise

Suppose that a box is being towed up an inclined plane. Find the force ***w*** needed to make the component of the force parallel to the indicated plane equal to 2.5 lb.

Solution

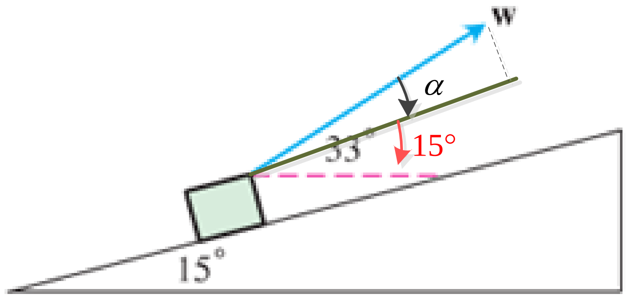
$$2.5 = |w|\cos\alpha$$

$$|\boldsymbol{w}| = \frac{2.5}{\cos(33° - 15°)}$$

$$= \frac{2.5}{\cos 18°}$$

$$\boldsymbol{w} = \frac{2.5}{\cos 18°}\langle \cos 33°,\ \sin 33° \rangle$$

$$\underline{= \langle 2.205,\ 1.432 \rangle}$$



Exercise

Find the work done by a force ***F*** = 5***i*** (magnitude 5 N) in moving an object along the line from the origin to the point (1, 1) (distance in meters)

Solution

$$P(1,\ 1) \quad \Rightarrow \quad \overrightarrow{OP} = \boldsymbol{i} + \boldsymbol{j}$$

$$W = F \cdot \overrightarrow{OP}$$

$$= 5\boldsymbol{i} \cdot (\boldsymbol{i} + \boldsymbol{j})$$

$$\underline{= 5\ J}$$

Exercise

How much work does it take to slide a crate 20 m along a loading dock by pulling on it with a 200 N force at an angle of 30° from the horizontal?

Solution

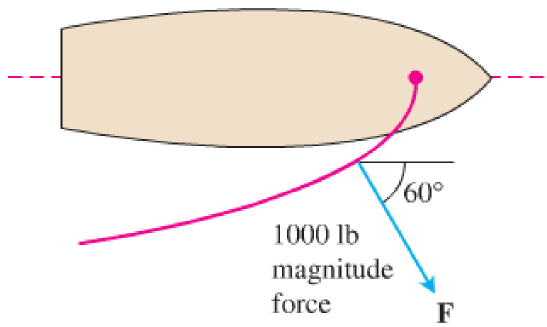
$$W = |F|\left|\overrightarrow{PQ}\right|\cos\theta$$

$$= (200)(20)\cos 30°$$

$$\underline{= 3464.10\ J}$$

Exercise

The wind passing over a boat's sail exerted a 1000-*lb* magnitude force F. How much work did the wind perform in moving the boat forward 1 *mi*? Answer in foot-pounds.



Solution

$$W = |F|\left|\overrightarrow{PQ}\right|\cos\theta$$

$$= (1000N)\left(1\ mi\ \frac{5280\ ft}{1\ mi}\right)\cos 60°$$

$$= 2,640,000\ \ ft\cdot lb$$

Solution Section 1.4 – The Cross Product

Exercise

Find the length and direction of $\boldsymbol{u}\times\boldsymbol{v}$ and $\boldsymbol{v}\times\boldsymbol{u}$: $\quad \boldsymbol{u}=2\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}, \quad \boldsymbol{v}=\boldsymbol{i}-\boldsymbol{k}$

Solution

$$\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 2 & -2 & -1\\ 1 & 0 & -1\end{vmatrix}=2\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$$

Length: $\quad |\boldsymbol{u}\times\boldsymbol{v}|=\sqrt{4+1+4}=\underline{3}$

Direction: $\quad \underline{\frac{1}{3}(2\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k})}$

$$\boldsymbol{v}\times\boldsymbol{u}=-(\boldsymbol{u}\times\boldsymbol{v})=-2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}$$

Length: $\quad |\boldsymbol{v}\times\boldsymbol{u}|=\sqrt{4+1+4}=\underline{3}$

Direction: $\quad \frac{1}{3}(-2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k})$

Exercise

Find the length and direction of $\boldsymbol{u}\times\boldsymbol{v}$ and $\boldsymbol{v}\times\boldsymbol{u}$: $\quad \boldsymbol{u}=\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}, \quad \boldsymbol{v}=0$

Solution

$$\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 1 & -1\\ 0 & 0 & 0\end{vmatrix}=0$$

Length: $\quad \underline{0}$

Direction: No direction

$$\boldsymbol{v}\times\boldsymbol{u}=-(\boldsymbol{u}\times\boldsymbol{v})=0$$

Length: $\quad \underline{0}$

Direction: No direction

Exercise

Find the length and direction of $\boldsymbol{u}\times\boldsymbol{v}$ and $\boldsymbol{v}\times\boldsymbol{u}$: $\quad \boldsymbol{u}=\boldsymbol{i}\times\boldsymbol{j}, \quad \boldsymbol{v}=\boldsymbol{j}\times\boldsymbol{k}$

Solution

$$\boldsymbol{u}\times\boldsymbol{v}=(\boldsymbol{i}\times\boldsymbol{j})\times(\boldsymbol{j}\times\boldsymbol{k})=\boldsymbol{k}\times\boldsymbol{i}=\boldsymbol{j}$$

Length: $\underline{1|}$

Direction: $\underline{\boldsymbol{j}|}$

$$\boldsymbol{v}\times\boldsymbol{u}=-(\boldsymbol{u}\times\boldsymbol{v})=-\boldsymbol{j}$$

Length: $\underline{1|}$

Direction: $\underline{-\boldsymbol{j}|}$

Exercise

Find the length and direction of $\boldsymbol{u}\times\boldsymbol{v}$ and $\boldsymbol{v}\times\boldsymbol{u}$: $\quad \boldsymbol{u}=-8\boldsymbol{i}-2\boldsymbol{j}-4\boldsymbol{k}, \quad \boldsymbol{v}=2\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$

Solution

$$\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ -8 & -2 & -4\\ 2 & 2 & 1\end{vmatrix}=6\boldsymbol{i}-12\boldsymbol{k}$$

Length: $|\boldsymbol{u}\times\boldsymbol{v}|=\sqrt{36+144}=\sqrt{180}=\underline{6\sqrt{5}|}$

Direction: $\dfrac{1}{6\sqrt{5}}(6\boldsymbol{i}-12\boldsymbol{k})=\underline{\dfrac{1}{\sqrt{5}}\boldsymbol{i}-\dfrac{2}{\sqrt{5}}\boldsymbol{k}|}$

$$\boldsymbol{v}\times\boldsymbol{u}=-(\boldsymbol{u}\times\boldsymbol{v})=-6\boldsymbol{i}+12\boldsymbol{k}$$

Length: $|\boldsymbol{v}\times\boldsymbol{u}|==\underline{6\sqrt{5}|}$

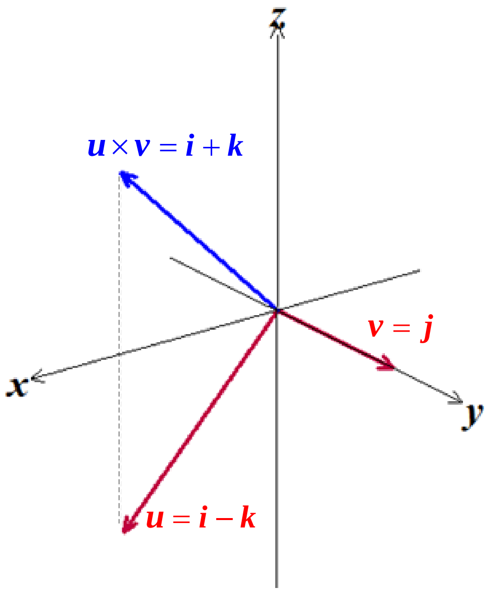
Direction: $\underline{-\dfrac{1}{\sqrt{5}}\boldsymbol{i}+\dfrac{2}{\sqrt{5}}\boldsymbol{k}|}$

Exercise

Sketch the coordinate axes and then include the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{u}\times\boldsymbol{v}$ as vectors starting origin for $\boldsymbol{u}=\boldsymbol{i}-\boldsymbol{k}, \quad \boldsymbol{v}=\boldsymbol{j}$

Solution

$$\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 0 & -1\\ 0 & 1 & 0\end{vmatrix}=\boldsymbol{i}+\boldsymbol{k}$$

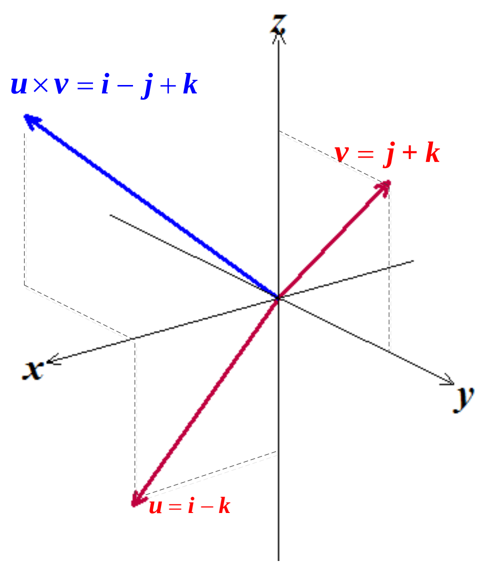


Exercise

Sketch the coordinate axes and then include the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{u}\times\boldsymbol{v}$ as vectors starting origin for $\boldsymbol{u}=\boldsymbol{i}-\boldsymbol{k}, \quad \boldsymbol{v}=\boldsymbol{j}+\boldsymbol{k}$

Solution

$$\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 0 & -1\\ 0 & 1 & 1\end{vmatrix}=\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}$$



Exercise

Find the area of the triangle determined by the points P, Q, and R, and then find a unit vector perpendicular to plane PQR. $P(1,\ -1,\ 2),\quad Q(2,\ 0,\ -1),\quad and\quad R(0,\ 2,\ 1)$

Solution

$$\overrightarrow{PQ}=(2-1)\boldsymbol{i}+(0+1)\boldsymbol{j}+(-1-2)\boldsymbol{k}=\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k}$$

$$\overrightarrow{PR}=(0-1)\boldsymbol{i}+(2+1)\boldsymbol{j}+(1-2)\boldsymbol{k}=-\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}$$

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 1 & -3\\ -1 & 3 & -1\end{vmatrix}=8\boldsymbol{i}+4\boldsymbol{j}+4\boldsymbol{k}$$

$$\begin{aligned}Area&=\frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|\\&=\frac{1}{2}\sqrt{8^2+4^2+4^2}\\&=\frac{1}{2}\sqrt{96}\\&=2\sqrt{6}\end{aligned}$$

$$\begin{aligned}\boldsymbol{u}&=\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|}\\&=\frac{1}{4\sqrt{6}}(8\boldsymbol{i}+4\boldsymbol{j}+4\boldsymbol{k})\\&=\frac{1}{\sqrt{6}}(2\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})\end{aligned}$$

Exercise

Find the area of the triangle determined by the points P, Q, and R, and then find a unit vector perpendicular to plane PQR. $P(1,\ 1,\ 1),\quad Q(2,\ 1,\ 3),\quad and\quad R(3,\ -1,\ 1)$

Solution

$$\overrightarrow{PQ}=(2-1)\boldsymbol{i}+(1-1)\boldsymbol{j}+(3-1)\boldsymbol{k}=\boldsymbol{i}+2\boldsymbol{k}$$

$$\overrightarrow{PR}=(3-1)\boldsymbol{i}+(-1-1)\boldsymbol{j}+(1-1)\boldsymbol{k}=2\boldsymbol{i}-2\boldsymbol{j}$$

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 0 & 2\\ 2 & -2 & 0\end{vmatrix}=4\boldsymbol{i}+4\boldsymbol{j}-2\boldsymbol{k}$$

$$Area = \frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|$$

$$= \frac{1}{2}\sqrt{16+16+4}$$

$$= \frac{1}{2}\sqrt{36}$$

$$\boxed{= 3}$$

$$\boldsymbol{u} = \frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|}$$

$$= \frac{1}{6}\left(4\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}\right)$$

$$\boxed{= \frac{1}{3}\left(2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}\right)}$$

Exercise

Find the area of the triangle determined by the points P, Q, and R, and then find a unit vector perpendicular to plane $PQ\,R$. $P(-2,\ 2,\ 0),\quad Q(0,\ 1,\ -1),\quad and\ \ R(-1,\ 2,\ -2)$

Solution

$$\overrightarrow{PQ} = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$$

$$\overrightarrow{PR} = \boldsymbol{i} - 2\boldsymbol{k}$$

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -1 & -1 \\ 1 & 0 & -2 \end{vmatrix} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$$

$$Area = \frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|$$

$$= \frac{1}{2}\sqrt{4+9+1}$$

$$\boxed{= \frac{\sqrt{14}}{2}}$$

$$\boldsymbol{u} = \frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|}$$

$$\boxed{= \frac{1}{\sqrt{14}}\left(2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}\right)}$$

Exercise

Verify that $(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}=(\boldsymbol{v}\times\boldsymbol{w})\bullet\boldsymbol{u}=(\boldsymbol{w}\times\boldsymbol{u})\bullet\boldsymbol{v}$ and find the volume of the parallelepiped determined by $\boldsymbol{u}=2\boldsymbol{i},\quad \boldsymbol{v}=2\boldsymbol{j},\quad \textit{and}\quad \boldsymbol{w}=2\boldsymbol{k}$

Solution

Let $\boldsymbol{u}=\langle u_1, u_2, u_3\rangle,\quad \boldsymbol{v}=\langle v_1, v_2, v_3\rangle,\quad \boldsymbol{w}=\langle w_1, w_2, w_3\rangle$

$$(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}=\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\qquad (\boldsymbol{v}\times\boldsymbol{w})\bullet\boldsymbol{u}=\begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix}\qquad (\boldsymbol{w}\times\boldsymbol{u})\bullet\boldsymbol{v}=\begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Which all have the same absolute value, by interchanging the rows the determinant does not change its absolute value.

$$\textit{Volume}=(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}=\begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}=\underline{8}$$

Exercise

Find the volume of the parallelepiped determined by

$$\boldsymbol{u}=\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k},\quad \boldsymbol{v}=2\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k},\quad \textit{and}\quad \boldsymbol{w}=-\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$$

Solution

$$\textit{Volume}=(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}=\textit{abs}\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 2 & -1 \end{vmatrix}=\underline{3}$$

Exercise

Find the volume of the parallelepiped determined by

$$\boldsymbol{u}=\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k},\quad \boldsymbol{v}=-\boldsymbol{i}-\boldsymbol{k},\quad \textit{and}\quad \boldsymbol{w}=2\boldsymbol{i}+4\boldsymbol{j}-2\boldsymbol{k}$$

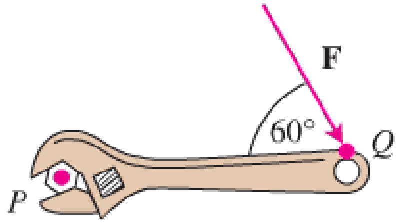
Solution

$$\textit{Volume}=(\boldsymbol{u}\times\boldsymbol{v})\bullet\boldsymbol{w}=\textit{abs}\begin{vmatrix} 1 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix}=\underline{8}$$

Exercise

Find the magnitude of the torque force exerted by $\boldsymbol{F}$ on the bolt at P if $\left|\overrightarrow{PQ}\right| = 8\ in.$ and $\left|\boldsymbol{F}\right| = 30\ lb.$

Solution



$$\left|\overrightarrow{PQ}\times\boldsymbol{F}\right| = \left|\overrightarrow{PQ}\right|\left|\boldsymbol{F}\right|\sin 60°$$

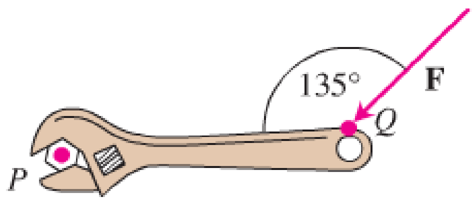
$$= \frac{8}{12}(30)\frac{\sqrt{3}}{2}$$

$$= 10\sqrt{3}\ \ ft.lb$$

Exercise

Find the magnitude of the torque force exerted by $\boldsymbol{F}$ on the bolt at P if $\left|\overrightarrow{PQ}\right| = 8\ in.$ and $\left|\boldsymbol{F}\right| = 30\ lb.$

Solution



$$\left|\overrightarrow{PQ}\times\boldsymbol{F}\right| = \left|\overrightarrow{PQ}\right|\left|\boldsymbol{F}\right|\sin 135°$$

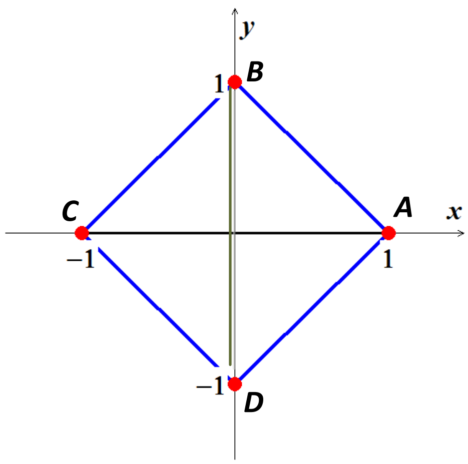
$$= \frac{8}{12}(30)\frac{\sqrt{2}}{2}$$

$$= 10\sqrt{2}\ \ ft.lb$$

Exercise

Find the area of the parallelogram whose vertices are: $A(1,\ 0),\quad B(0,\ 1),\quad C(-1,\ 0),\quad D(0,\ -1)$

Solution



$$\overrightarrow{AB} = -\boldsymbol{i} + \boldsymbol{j} \quad \overrightarrow{AD} = -\boldsymbol{i} - \boldsymbol{j}$$

$$Area(\triangle ABD) = Area(\triangle CBD)$$

$$Area = \left|\overrightarrow{AB}\times\overrightarrow{AD}\right|$$

$$= abs\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 1 & 0 \\ -1 & -1 & 0 \end{vmatrix}$$

$$= abs\left|2\boldsymbol{k}\right|$$

$$= 2$$

Exercise

Find the area of the parallelogram whose vertices are: $A(0,\ 0),\quad B(7,\ 3),\quad C(9,\ 8),\quad D(2,\ 5)$

Solution

$\overrightarrow{AB} = 7\boldsymbol{i} + 3\boldsymbol{j} \quad \overrightarrow{AC} = 9\boldsymbol{i} + 8\boldsymbol{j}$

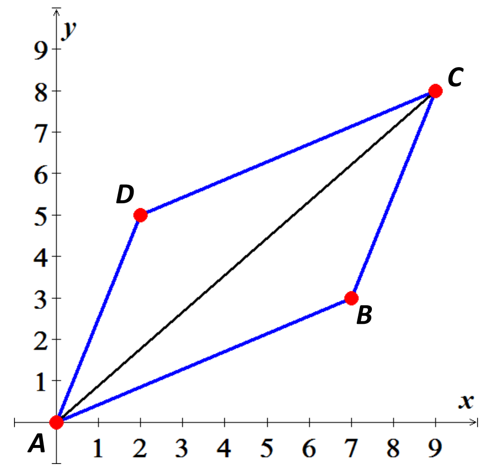
$Area(\triangle ABC) = Area(\triangle ACD)$

$$Area = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= abs\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 7 & 3 & 0 \\ 9 & 8 & 0 \end{vmatrix}$$

$$= abs|29\boldsymbol{k}|$$

$$\underline{= 29|}$$



Exercise

Find the area of the parallelogram whose vertices are: $A(-1,\ 2),\quad B(2,\ 0),\quad C(7,\ 1),\quad D(4,\ 3)$

Solution

$\overrightarrow{AB} = 3\boldsymbol{i} - 2\boldsymbol{j} \quad \overrightarrow{AC} = 8\boldsymbol{i} - \boldsymbol{j}$

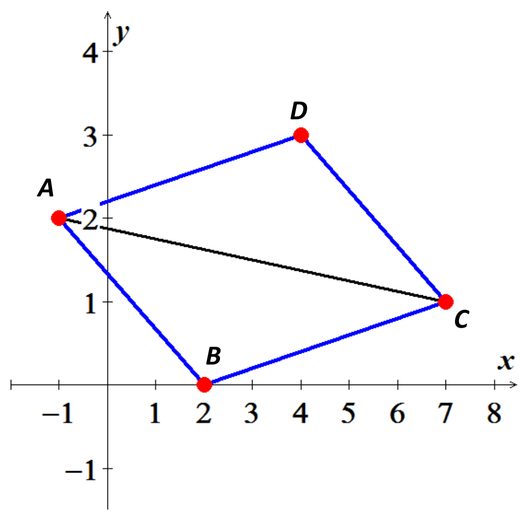
$Area(\triangle ABC) = Area(\triangle ACD)$

$$Area = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= abs\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & -2 & 0 \\ 8 & -1 & 0 \end{vmatrix}$$

$$= abs|13\boldsymbol{k}|$$

$$\underline{= 13|}$$



Exercise

Find the area of the parallelogram whose vertices are:
$A(0,\ 0,\ 0),\quad B(3,\ 2,\ 4),\quad C(5,\ 1,\ 4),\quad D(2,\ -1,\ 0)$

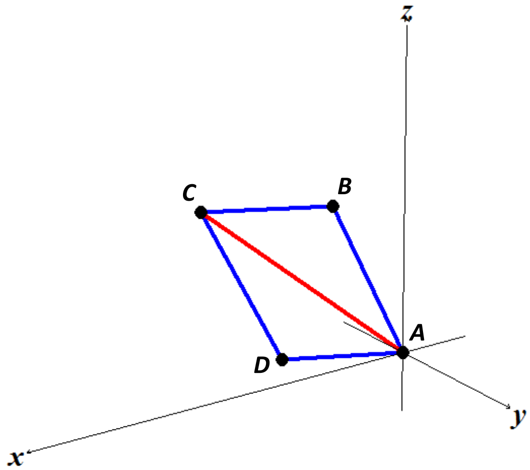
Solution

$\overrightarrow{AB} = 3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k} \quad \overrightarrow{DC} = 3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$

$\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$

$\overrightarrow{AD} = 2\boldsymbol{i} - \boldsymbol{j} \quad \overrightarrow{BC} = 2\boldsymbol{i} - \boldsymbol{j}$

$\overrightarrow{AD}$ is parallel to $\overrightarrow{BC}$



$$Area = \left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= abs\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & 4 \\ 2 & -1 & 0 \end{vmatrix}$$

$$= abs\left|4\boldsymbol{i} + 8\boldsymbol{j} - 7\boldsymbol{k}\right|$$

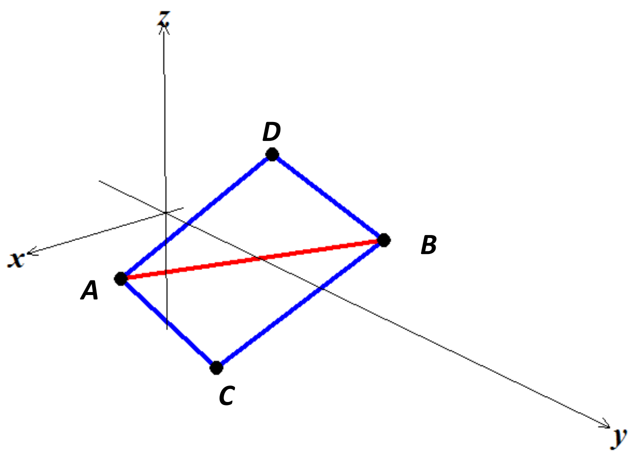
$$\underline{= \sqrt{129}}$$

Exercise

Find the area of the parallelogram whose vertices are:
$A(1,\ 0,\ -1),\quad B(1,\ 7,\ 2),\quad C(2,\ 4,\ -1),\quad D(0,\ 3,\ 2)$

Solution

$\overrightarrow{AC} = \boldsymbol{i} + 4\boldsymbol{j} \quad \overrightarrow{CB} = -\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}$



$$Area = \left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= abs\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 4 & 0 \\ -1 & 3 & 3 \end{vmatrix}$$

$$= abs\left|12\boldsymbol{i} - 3\boldsymbol{j} + 7\boldsymbol{k}\right|$$

$$\sqrt{144 + 9 + 49}$$

$$\underline{= \sqrt{202}}$$

Exercise

Find the area of the triangle whose vertices are: $A(0,\ 0),\quad B(-2,\ 3),\quad C(3,\ 1)$

Solution

$$\overrightarrow{AB} = -2\boldsymbol{i} + 3\boldsymbol{j} \quad \overrightarrow{AC} = 3\boldsymbol{i} + \boldsymbol{j}$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= \left(\frac{1}{2}\right) abs \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2 & 3 & 0 \\ 3 & 1 & 0 \end{vmatrix}$$

$$= \left(\frac{1}{2}\right) abs \left|-11\boldsymbol{k}\right|$$

$$= \frac{11}{2}$$

Exercise

Find the area of the triangle whose vertices are: $A(-1,\ -1),\quad B(3,\ 3),\quad C(2,\ 1)$

Solution

$$\overrightarrow{AB} = 4\boldsymbol{i} + 4\boldsymbol{j} \quad \overrightarrow{AC} = 3\boldsymbol{i} + 2\boldsymbol{j}$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= \frac{1}{2}\left\| \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 4 & 4 & 0 \\ 3 & 2 & 0 \end{vmatrix} \right\|$$

$$= \frac{1}{2}\left\|-4\boldsymbol{k}\right\|$$

$$= 2$$

Exercise

Find the area of the triangle whose vertices are: $A(1,\ 0,\ 0),\quad B(0,\ 0,\ 2),\quad C(0,\ 0,\ -1)$

Solution

$$\overrightarrow{AB} = -\boldsymbol{i} + 2\boldsymbol{k} \quad \overrightarrow{AC} = -\boldsymbol{i} - \boldsymbol{k}$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= \frac{1}{2}\left\|\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 0 & 2 \\ -1 & 0 & -1 \end{vmatrix}\right\|$$

$$= \frac{1}{2}\|3\boldsymbol{k}\|$$

$$\underline{= \frac{3}{2}|}$$

Exercise

Find the area of the triangle whose vertices are: $A(0,\ 0,\ 0),\quad B(-1,\ 1,\ -1),\quad C(3,\ 0,\ 3)$

Solution

$$\overrightarrow{AB} = -\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} \quad \overrightarrow{AC} = 3\boldsymbol{i} + 3\boldsymbol{k}$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= \frac{1}{2}\left\|\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 1 & -1 \\ 3 & 0 & 3 \end{vmatrix}\right\|$$

$$= \frac{1}{2}\|3\boldsymbol{i} - 3\boldsymbol{k}\|$$

$$= \frac{1}{2}\sqrt{9+9}$$

$$\underline{= \frac{3\sqrt{2}}{2}|}$$

Exercise

Find the volume of the parallelepiped if four of its eight vertices are:
$A(0,\ 0,\ 0),\quad B(1,\ 2,\ 0),\quad C(0,\ -3,\ 2),\quad D(3,\ -4,\ 5)$

Solution

$$\overrightarrow{AB} = \boldsymbol{i} + 2\boldsymbol{j} \quad \overrightarrow{AC} = -3\boldsymbol{j} + 2\boldsymbol{k} \quad \overrightarrow{AD} = 3\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$$

$$\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) \bullet \overrightarrow{AD} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 3 & -4 & 5 \end{vmatrix} = 5$$

$$Volume = \left|\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) \bullet \overrightarrow{AD}\right| = \underline{5|}$$

Solution Section 1.5 – Lines and Planes in Space

Exercise

Find the parametric equation for the line through the point $P(3,-4,-1)$ parallel to the vector $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$

Solution

$x=3+t, \quad y=-4+t, \quad z=-1+t$

Exercise

Find the parametric equation for the line through the points $P(1,2,-1)$ and $Q(-1,0,1)$

Solution

The direction: $\overrightarrow{PQ}=-2\boldsymbol{i}-2\boldsymbol{j}+2\boldsymbol{k}$ and $P(1,2,-1)$

$x=1-2t, \quad y=2-2t, \quad z=-1+2t$

Exercise

Find the parametric equation for the line through the points $P(-2,0,3)$ and $Q(3,5,-2)$

Solution

The direction: $\overrightarrow{PQ}=5\boldsymbol{i}+5\boldsymbol{j}-5\boldsymbol{k}$ and $P(-2,0,3)$

$x=-2+5t, \quad y=5t, \quad z=3-5t$

Exercise

Find the parametric equation for the line through the origin parallel to the vector $2\boldsymbol{j}+\boldsymbol{k}$

Solution

The direction: $2\boldsymbol{j}+\boldsymbol{k}$ and $P(0,0,0)$

$x=0, \quad y=2t, \quad z=t$

Exercise

Find the parametric equation for the line through the point $P(3,-2,1)$ parallel to the line

$x=1+2t, \quad y=2-t, \quad z=3t$

Solution

The direction: $2\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$ and $P(3,-2,1)$

$x=3+2t, \quad y=-2-t, \quad z=1+3t$

Exercise

Find the parametric equation for the line through $(2,4,5)$ perpendicular to the plane $3x+7y-5z=21$

Solution

The direction: $3\boldsymbol{i}+7\boldsymbol{j}-5\boldsymbol{k}$ and $(2,4,5)$

$x=2+3t, \quad y=4+7t, \quad z=5-5t$

Exercise

Find the parametric equation for the line through $(2,3,0)$ perpendicular to the vectors $\boldsymbol{u}=\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}$ and $\boldsymbol{v}=3\boldsymbol{i}+4\boldsymbol{j}+5\boldsymbol{k}$

Solution

The direction: $\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 2 & 3\\ 3 & 4 & 5\end{vmatrix}=-2\boldsymbol{i}+4\boldsymbol{j}-2\boldsymbol{k}$ and $(2,3,0)$

$x=2-2t, \quad y=3+4t, \quad z=-2t$

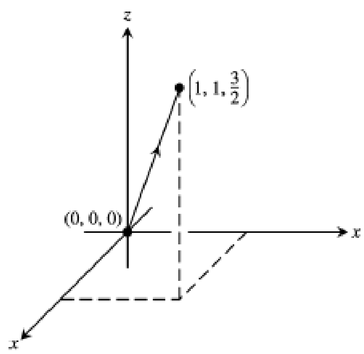
Exercise

Find the parameterization for the line segment joining the points $(0,0,0)$, $\left(1,1,\frac{3}{2}\right)$. Draw coordinate axes and sketch the segment, indicate the direction on increasing t for the parametrization.

Solution

The direction: $\overrightarrow{PQ}=\boldsymbol{i}+\boldsymbol{j}+\frac{3}{2}\boldsymbol{k}$ and $(0,0,0)$

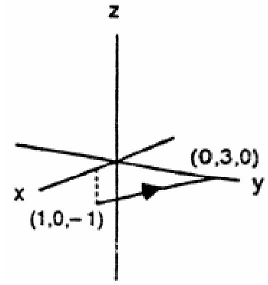
$x=t, \quad y=t, \quad z=\frac{3}{2}t, \quad 0\le t\le 1$



Exercise

Find the parameterization for the line segment joining the points $(1,0,-1)$, $(0,3,0)$. Draw coordinate axes and sketch the segment, indicate the direction on increasing t for the parametrization.

Solution



The direction: $\overrightarrow{PQ}=-\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$ and $(1,0,-1)$

$$x=1-t, \quad y=3t, \quad z=-1+t, \quad 0\le t\le 1$$

Exercise

Find equation for the plane through $P_0(0,2,-1)$ normal to $\boldsymbol{n}=3\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}$

Solution

$$3(x-0)-2(y-2)-(z+1)=0$$

$$3x-2y+4-z-1=0$$

$$\boxed{3x-2y-z=-3}$$

Exercise

Find equation for the plane through $(1,-1,3)$ parallel to the plane $3x+y+z=7$

Solution

$$3(x-1)+(y+1)+(z-3)=0$$

$$3x-3+y+1+z-3=0$$

$$\boxed{3x+y+z=5}$$

Exercise

Find equation for the plane through $(1,1,-1)$, $(2,0,2)$ and $(0,-2,1)$

Solution

$$\overrightarrow{PQ}=\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k} \quad \overrightarrow{PS}=-\boldsymbol{i}-3\boldsymbol{j}+2\boldsymbol{k}$$

$$\overrightarrow{PQ}\times\overrightarrow{PS}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & -1 & 3\\ -1 & -3 & 2\end{vmatrix}=7\boldsymbol{i}-5\boldsymbol{j}-4\boldsymbol{k} \text{ is normal to the plane.}$$

$$7(x-2)-5(y+0)-4(z-2)=0$$

$$7x-14-5y-4z+8=0$$

$$\boxed{7x-5y-4z=6}$$

Exercise

Find equation for the plane through $P_0(2,4,5)$ perpendicular to the line $x=5+t, \quad y=1+3t, \quad z=4t$

Solution

$$x=5+t, \quad y=1+3t, \quad z=4t \qquad \Rightarrow \quad \boldsymbol{n}=\boldsymbol{i}+3\boldsymbol{j}+4\boldsymbol{k}$$

$$1(x-2)+3(y-4)+4(z-5)=0$$

$$x-2+3y-12+4z-20=0$$

$$\boxed{x+3y+4z=34}$$

Exercise

Find equation for the plane through $A(1,-2,1)$ perpendicular to the vector from the origin to A.

Solution

$$\Rightarrow \quad \boldsymbol{n}=\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}$$

$$1(x-1)-2(y+2)+1(z-1)=0$$

$$x-1-2y-4+z-1=0$$

$$\boxed{x-2y+z=6}$$

Exercise

Find the point of intersection of the lines $x=2t+1, \quad y=3t+2, \quad z=4t+3$ and $x=s+2, \quad y=2s+4, \quad z=-4s-1$, and find the plane determined by these lines.

Solution

$$\begin{cases} x=2t+1=s+2 \\ y=3t+2=2s+4 \\ z=4t+3=-4s-1 \end{cases} \Rightarrow \begin{cases} 2t-s=1 \\ 3t-2s=2 \end{cases} \to \boxed{t=0} \quad \boxed{s=-1}$$

$$z=4t+3=-4s-1 \quad \Rightarrow \quad 4(0)+3=-4(-1)-1 \to 3=3 \ \surd \text{ (satisfied)}$$

The lines intersect when $t=0$ and $s=-1 \Rightarrow$ The point of intersection $x=1, \quad y=2, \quad z=3$

Therefore; the point is $\boxed{P(1,\ 2,\ 3)}$

The normal vectors: $\boldsymbol{n}_1=2\boldsymbol{i}+3\boldsymbol{j}+4\boldsymbol{k} \quad \boldsymbol{n}_2=\boldsymbol{i}+2\boldsymbol{j}-4\boldsymbol{k}$

$$\boldsymbol{n}_1\times\boldsymbol{n}_2=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix}=-20\boldsymbol{i}+12\boldsymbol{j}+\boldsymbol{k}$$ $\boldsymbol{n}_1$ *and* $\boldsymbol{n}_2$ are directions of the lines.

The plane containing the lines is represented by

$$-20(x-1)+12(y-2)+1(z-3)=0 \quad \Rightarrow \quad \boxed{-20x+12y+z=7}$$

Exercise

Find the plane determined by the intersecting lines:

$$L_1: x = -1 + t, \quad y = 2 + t, \quad z = 1 - t; \quad -\infty < t < \infty$$

$$L_2: x = 1 - 4s, \quad y = 1 + 2s, \quad z = 2 - 2s; \quad -\infty < s < \infty$$

Solution

The normal vectors: $\boldsymbol{n}_1 = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} \quad \boldsymbol{n}_2 = -4\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$

$$\boldsymbol{n} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{vmatrix} = 6\boldsymbol{j} + 6\boldsymbol{k}$$

Let $t = 0$ $L_1: x = -1, \quad y = 2, \quad z = 1; \quad \Rightarrow \boxed{P(-1, 2, 1)}$

Therefore; the desired plane is:

$$0(x+1) + 6(y-2) + 6(z-1) = 0$$

$$6y - 12 + 6z - 6 = 0$$

$$6y + 6z = 18 \quad \Rightarrow \quad \boxed{y + z = 3}$$

Exercise

Find a plane through $P_0(2, 1, -1)$ and perpendicular to the line of intersection of the planes $2x + y - z = 3, \quad x + 2y + z = 2$

Solution

The normal vectors: $\boldsymbol{n}_1 = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} \quad \boldsymbol{n}_2 = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

$\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\boldsymbol{j} - 3\boldsymbol{j} + 3\boldsymbol{k}$ is the vector in the direction of the line of intersection of the planes.

$$\Rightarrow 3(x-2) - 3(y-1) + 3(z+1) = 0$$

$$3x - 3y + 3z = 0$$

$\boxed{x - y + z = 0}$ is the desired plane containing $P_0(2, 1, -1)$

Exercise

Find the distance from the point to the plane $(0,0,12), \quad x=4t, \quad y=-2t, \quad z=2t$

Solution

At $t=0 \Rightarrow \boxed{P(0,0,0)}$ and let $S(0,0,12)$

$\overrightarrow{PS}=12\boldsymbol{k}$ and $\boldsymbol{v}=4\boldsymbol{i}-2\boldsymbol{j}+2\boldsymbol{k}$

$$\overrightarrow{PS}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 0 & 0 & 12\\ 4 & -2 & 2\end{vmatrix}=24\boldsymbol{i}+48\boldsymbol{j}$$

$$\begin{aligned}d&=\frac{\left|\overrightarrow{PS}\times\boldsymbol{v}\right|}{|\boldsymbol{v}|}\\&=\frac{\sqrt{24^2+48^2}}{\sqrt{16+4+4}}\\&=\frac{24\sqrt{5}}{\sqrt{24}}\\&=\sqrt{5}\sqrt{24}\\&=2\sqrt{30}\end{aligned}$$

Exercise

Find the distance from the point to the plane $(2,1,-1), \quad x=2t, \quad y=1+2t, \quad z=2t$

Solution

At $t=0 \Rightarrow \boxed{P(0,1,0)}$ and let $S(2,1,-1)$

$\overrightarrow{PS}=2\boldsymbol{i}-\boldsymbol{k}$ and $\boldsymbol{v}=2\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k}$

$$\overrightarrow{PS}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 2 & 0 & -1\\ 2 & 2 & 2\end{vmatrix}=2\boldsymbol{i}-6\boldsymbol{j}+4\boldsymbol{k}$$

$$\begin{aligned}d&=\frac{\left|\overrightarrow{PS}\times\boldsymbol{v}\right|}{|\boldsymbol{v}|}\\&=\frac{\sqrt{4+36+16}}{\sqrt{4+4+4}}\\&=\frac{\sqrt{56}}{\sqrt{12}}\end{aligned}$$

$$= \frac{2\sqrt{14}}{2\sqrt{3}}$$

$$\boxed{= \sqrt{\frac{14}{3}} \quad unit}$$

Exercise

Find the distance from the point to the plane $(3,-1,4), \quad x = 4 - t, \quad y = 3 + 2t, \quad z = -5 + 3t$

Solution

At $t = 0 \Rightarrow \boxed{P(4,3,-5)}$ and let $S(3,-1,4)$

$\overrightarrow{PS} = -\boldsymbol{i} - 4\boldsymbol{j} + 9\boldsymbol{k}$ and $\boldsymbol{v} = -\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$

$$\overrightarrow{PS} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 2 & 3 \\ -1 & 2 & 3 \end{vmatrix} = -30\boldsymbol{i} - 6\boldsymbol{j} - 6\boldsymbol{k}$$

$$d = \frac{\left|\overrightarrow{PS} \times \boldsymbol{v}\right|}{|\boldsymbol{v}|}$$

$$= \frac{\sqrt{900 + 36 + 36}}{\sqrt{1 + 4 + 9}}$$

$$= \sqrt{\frac{972}{14}}$$

$$= \sqrt{\frac{486}{7}}$$

$$= \frac{9\sqrt{6}}{\sqrt{7}} \frac{\sqrt{7}}{\sqrt{7}}$$

$$\boxed{= \frac{9\sqrt{42}}{7} \quad unit}$$

Exercise

Find the distance from the point to the plane $(2,-3,4), \quad x + 2y + 2z = 13$

Solution

$\Rightarrow \boxed{P(13,0,0)}$ and let $S(2,-3,4)$

$\overrightarrow{PS} = -11\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{n} = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

$|\boldsymbol{n}| = \sqrt{1 + 4 + 4} = \underline{3}|$

$$d = \left| \overrightarrow{PS} \bullet \frac{\boldsymbol{n}}{|\boldsymbol{n}|} \right|$$

$$= \left| (-11\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}) \bullet \left(\frac{1}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + \frac{2}{3}\boldsymbol{k} \right) \right|$$

$$= \left| -\frac{11}{3} - \frac{6}{3} + \frac{8}{3} \right|$$

$$= 3 \quad unit$$

Exercise

Find the distance from the point to the plane $(0,0,0), \quad 3x + 2y + 6z = 6$

Solution

$$3x + 2y + 6z = 6, \quad 3x + 2(0) + 6(0) = 6 \to \boxed{x = 2}$$

$$\Rightarrow \boxed{P(2,0,0)} \text{ and let } S(0,0,0)$$

$$\overrightarrow{PS} = -2\boldsymbol{i} \text{ and } \boldsymbol{n} = 3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k} \qquad \to |\boldsymbol{n}| = \sqrt{9 + 4 + 36} = 7$$

$$d = \left| \overrightarrow{PS} \bullet \frac{\boldsymbol{n}}{|\boldsymbol{n}|} \right|$$

$$= \left| (-2\boldsymbol{i}) \bullet \left(\frac{3}{7}\boldsymbol{i} + \frac{2}{7}\boldsymbol{j} + \frac{6}{7}\boldsymbol{k} \right) \right|$$

$$= \frac{6}{7} \quad unit$$

Exercise

Find the distance from the point to the plane $(0,1,1), \quad 4y + 3z = -12$

Solution

$$\Rightarrow \boxed{P(0,-3,0)} \text{ and let } S(0,1,1)$$

$$\overrightarrow{PS} = 4\boldsymbol{j} + \boldsymbol{k} \text{ and } \boldsymbol{n} = 4\boldsymbol{j} + 3\boldsymbol{k} \qquad \to |\boldsymbol{n}| = \sqrt{16 + 9} = 5$$

$$d = \left| \overrightarrow{PS} \bullet \frac{\boldsymbol{n}}{|\boldsymbol{n}|} \right|$$

$$= \left| (4\boldsymbol{j} + \boldsymbol{k}) \bullet \left(\frac{4}{5}\boldsymbol{j} + \frac{3}{5}\boldsymbol{k} \right) \right|$$

$$= \left| \frac{16}{5} + \frac{3}{5} \right|$$

$$= \frac{19}{5} \quad unit$$

Exercise

Find the distance from the plane $x+2y+6z=1$ to the plane $x+2y+6z=10$

Solution

$$x+2y+6z=1 \quad \Rightarrow \quad P(1,0,0)$$

$$x+2y+6z=10 \quad \Rightarrow \quad S(10,0,0)$$

$$\overrightarrow{PS}=9\boldsymbol{i} \text{ and } \boldsymbol{n}=\boldsymbol{i}+2\boldsymbol{j}+6\boldsymbol{k} \qquad \rightarrow |\boldsymbol{n}|=\sqrt{1+4+36}=\boxed{\sqrt{41}}$$

$$\begin{aligned} d &= \left|\overrightarrow{PS}\cdot\frac{\boldsymbol{n}}{|\boldsymbol{n}|}\right| \\ &= \left|(9\boldsymbol{i})\cdot\frac{1}{\sqrt{41}}(\boldsymbol{i}+2\boldsymbol{j}+6\boldsymbol{k})\right| \\ &= \frac{1}{\sqrt{41}}|9| \\ &= \boxed{\frac{9}{\sqrt{41}}} \end{aligned}$$

Exercise

Find the angle between the planes $x+y=1, \quad 2x+y-2z=2$

Solution

The vectors: $\quad \boldsymbol{n}_1=\boldsymbol{i}+\boldsymbol{j}, \quad \boldsymbol{n}_2=2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}$ are normal to the planes.

The angle between them is:

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{\boldsymbol{n}_1\cdot\boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}\right) \\ &= \cos^{-1}\left(\frac{2+1}{\sqrt{1+1}\sqrt{4+1+4}}\right) \\ &= \cos^{-1}\left(\frac{3}{3\sqrt{2}}\right) \\ &= \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ &= \boxed{\frac{\pi}{4}} \end{aligned}$$

Exercise

Find the angle between the planes $5x + y - z = 10, \quad x - 2y + 3z = -1$

Solution

The vectors: $\boldsymbol{n}_1 = 5\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}, \quad \boldsymbol{n}_2 = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ are normal to the planes.

The angle between them is:

$$\begin{aligned}
\theta &= \cos^{-1}\left(\frac{\boldsymbol{n}_1 \bullet \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}\right) \\
&= \cos^{-1}\left(\frac{5-2-3}{\sqrt{25+1+1}\sqrt{1+4+9}}\right) \\
&= \cos^{-1}(0) \\
&= \frac{\pi}{2}
\end{aligned}$$

Exercise

Find the point in which the line meets the plane $x = 1 - t, \quad y = 3t, \quad z = 1 + t; \quad 2x - y + 3z = 6$

Solution

$$2(1-t) - 3t + 3(1+t) = 6$$

$$2 - 2t - 3t + 3 + 3t = 6$$

$$-2t = 1$$

$$\boxed{t = -\frac{1}{2}}$$

$$x = 1 - \left(-\frac{1}{2}\right) = \frac{3}{2}, \quad y = 3\left(-\frac{1}{2}\right) = -\frac{3}{2}, \quad z = 1 + \left(-\frac{1}{2}\right) = \frac{1}{2}$$

$$\Rightarrow \boxed{P\left(\frac{3}{2},\ -\frac{3}{2},\ \frac{1}{2}\right)}$$

Exercise

Find the point in which the line meets the plane $x = 2, \quad y = 3 + 2t, \quad z = -2 - 2t; \quad 6x + 3y - 4z = -12$

Solution

$$12 + 3(3 + 2t) - 4(-2 - 2t) = -12$$

$$12 + 9 + 6t + 8 + 8t = -12$$

$$14t = -41$$

$$\boxed{t = -\frac{41}{14}}$$

$$x = 2, \quad y = 3 + 2\left(-\frac{41}{14}\right) = -\frac{20}{7}, \quad z = -2 - 2\left(-\frac{41}{14}\right) = \frac{27}{7}$$

$$\Rightarrow \boxed{P\left(2,\ -\frac{20}{7},\ \frac{27}{7}\right)}$$