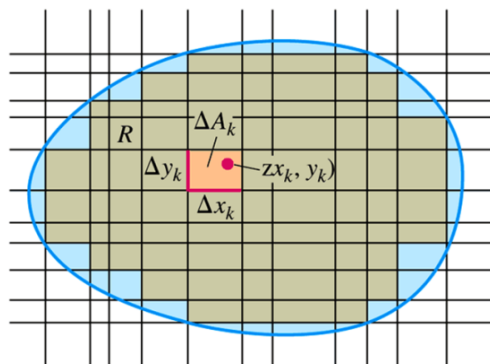


Section 3.2 – Double Integrals over General Regions



Volumes

If $f(x, y)$ is positive and continuous over R , we define the volume of the solid region between R and the

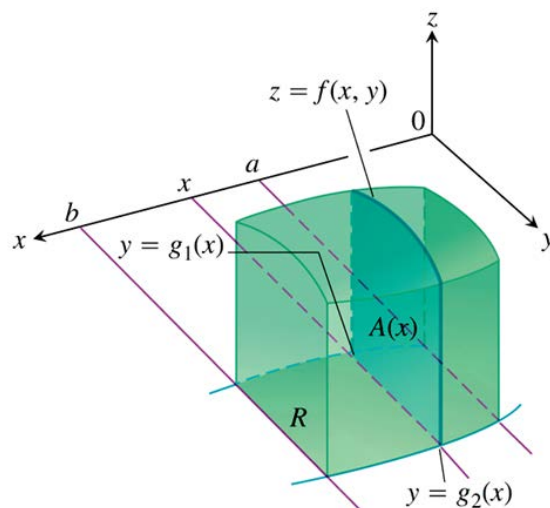
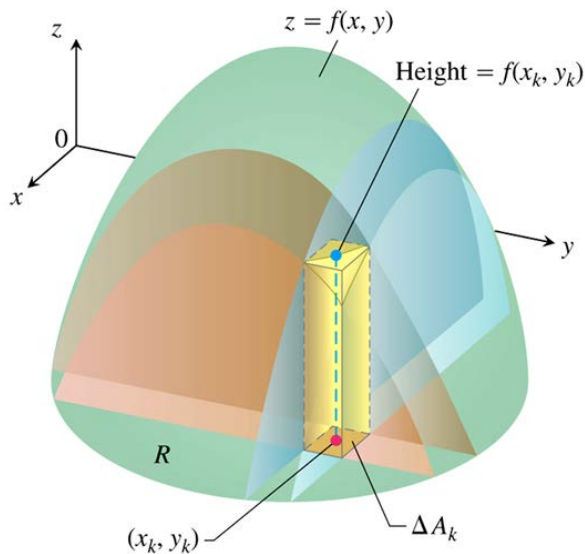
surface $z = f(x, y)$ to be $\iint_R f(x, y) dA$.

If R is a region in the xy -plane, bounded **above** and **below** by the curves $y = g_1(x)$ and $y = g_2(x)$ and on the sides by the lines $x = a$, $x = b$. Calculate the cross-sectional area

$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x, y) dy$$

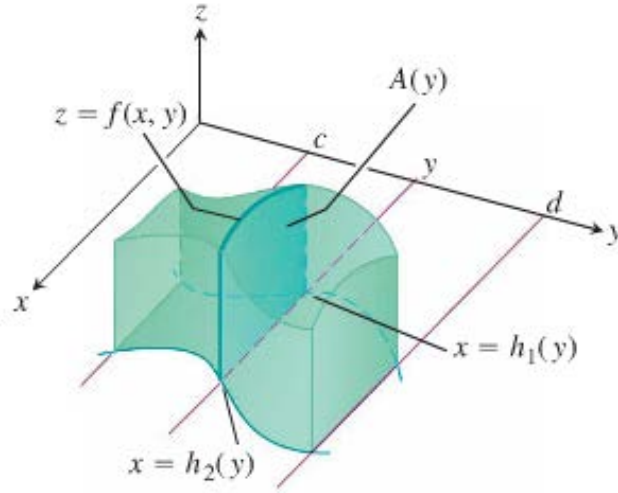
Then integrate $A(x)$ from $x = a$ to $x = b$ to get the volume as an iterated integral

$$V = \int_a^b A(x) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$



Similarly, if R is a region bounded by the curves $x = h_1(y)$ and $x = h_2(y)$ and the lines $y = c$, $y = d$, then the volume calculated by slicing is given by the iterated integral .

$$V = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$



$$\int_c^d A(y) dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

$$Volume = \lim \sum f(x_k, y_k) \Delta A_k = \iint_R f(x, y) dA$$

Theorem – Fubini's Theorem

Let $f(x, y)$ is continuous on a region R ,

1. If R is defined by : $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

2. If R is defined by : $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

Example

Find the volume of the prism whose base is the triangle in the xy -plane bounded by the x -axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = f(x, y) = 3 - x - y$

Solution

$$0 \leq x \leq 1, \quad 0 \leq y \leq x$$

$$V = \int_0^1 \int_0^x (3 - x - y) dy dx$$

$$= \int_0^1 \left[3y - xy - \frac{1}{2}y^2 \right]_0^x dx$$

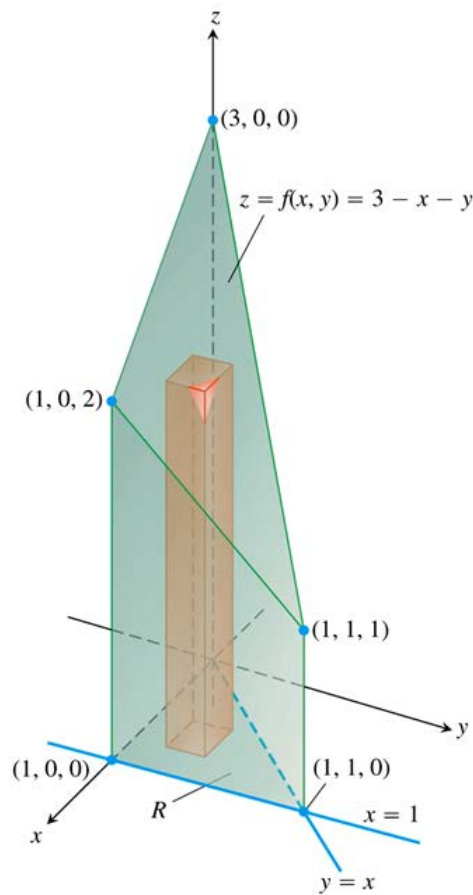
$$= \int_0^1 \left(3x - x^2 - \frac{1}{2}x^2 \right) dx$$

$$= \int_0^1 \left(3x - \frac{3}{2}x^2 \right) dx$$

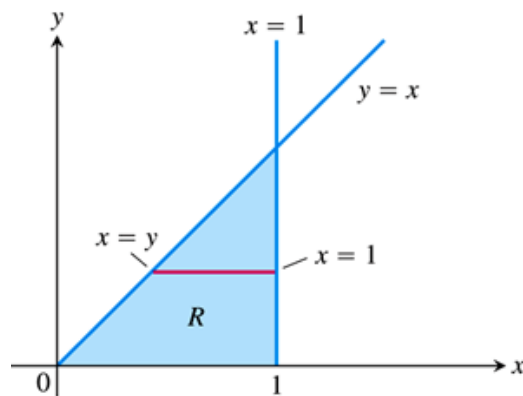
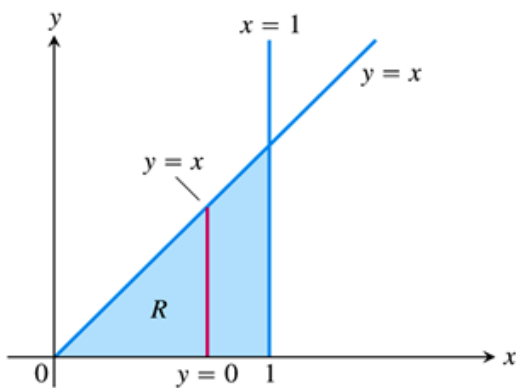
$$= \left[\frac{3}{2}x^2 - \frac{1}{2}x^3 \right]_0^1$$

$$= \frac{3}{2} - \frac{1}{2}$$

$$= \underline{1 \text{ unit}^3}$$



$$V = \int_0^1 \int_y^1 (3 - x - y) dx dy = 1$$

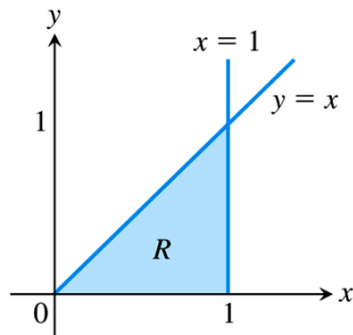


Example

Calculate $\iint_R \frac{\sin x}{x} dA$ where R is the triangle in the xy -plane bounded by the x -axis, the line $y = x$, and the line $x = 1$.

Solution

$$\begin{aligned}\int_0^1 \int_0^x \left(\frac{\sin x}{x} \right) dy \, dx &= \int_0^1 \left(\frac{\sin x}{x} y \right)_0^x dx \\&= \int_0^1 \sin x \, dx \\&= -\cos x \Big|_0^1 \\&= -\cos(1) + 1 \\&= \underline{1 - \cos 1} \quad \underline{\approx 0.46}\end{aligned}$$

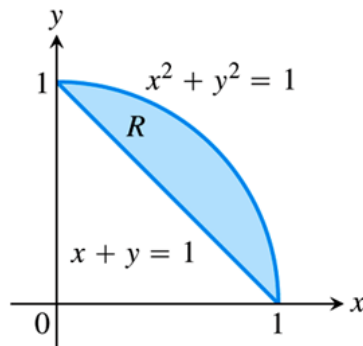


$\int_0^1 \int_y^1 \left(\frac{\sin x}{x} \right) dx \, dy$, we run into a problem because $\int \frac{\sin x}{x} dx$ cannot be expressed in terms of elementary functions.

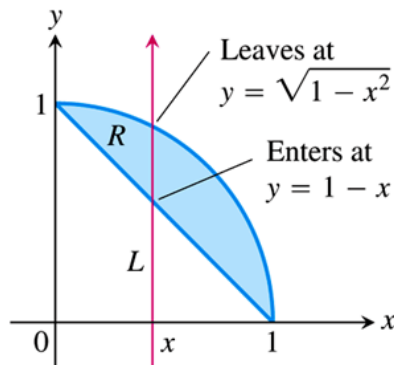
Finding Limits on Intergration

Using Vertical Cross-sections

1. Sketch the region of Integration and label the bounding curves

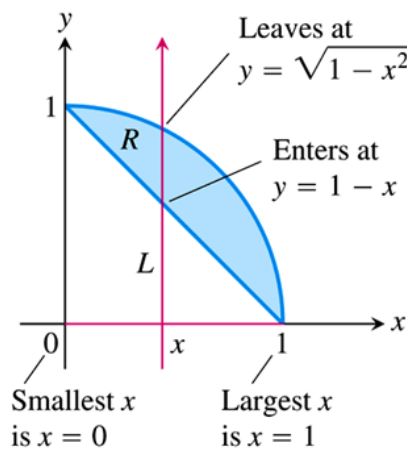


2. Find the y-limits of integration. Imagine a vertical line L cutting through R in the direction of increasing y . Mark the y -values where L enters and leaves. These are the y -limits of integration and are usually functions of x (instead of constants).



3. Find the x -limits of integration. Choose x -limits that include all the vertical lines through R . The integral is

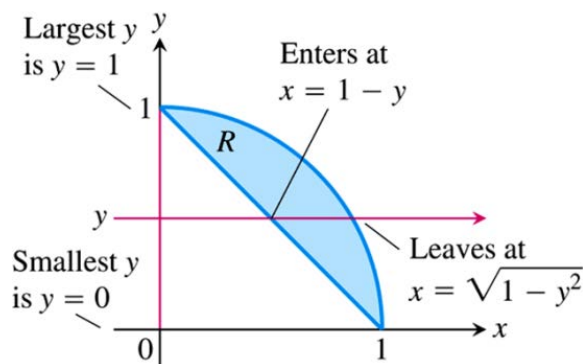
$$\iint_R f(x, y) dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x, y) dy dx$$



Using Horizontal Cross-sections

To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines.

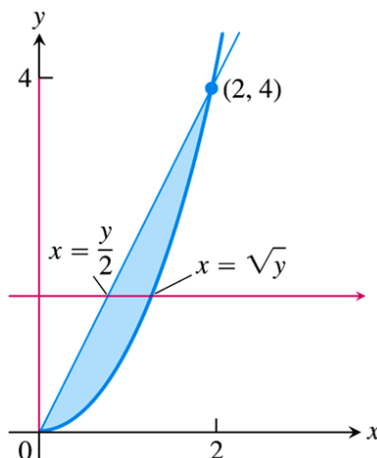
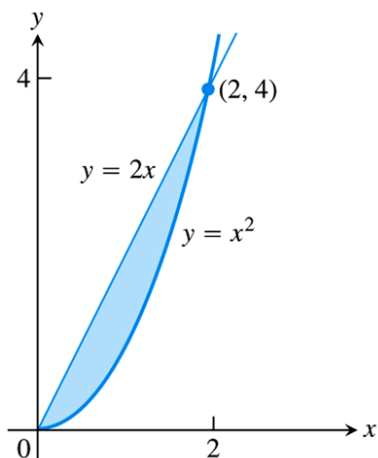
$$\iint_R f(x, y) dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y) dx dy$$



Example

Sketch the region of integration for the integral $\int_0^2 \int_{x^2}^{2x} (4x+2) dy dx$ and write an equivalent integral with the order of integration reversed.

Solution



The given inequalities are: $x^2 \leq y \leq 2x$ and $0 \leq x \leq 2$

$$\rightarrow \begin{cases} y = x^2 & x = \sqrt{y} \\ y = 2x & x = \frac{y}{2} \end{cases} \quad \rightarrow \begin{cases} x = 0 & y = 0 \\ x = 2 & y = 4 \end{cases}$$

The integral is $\int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2) dx dy$

✚ If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R , then the following properties hold

1. *Constant Multiple:*
$$\iint_R cf(x, y) dA = c \iint_R f(x, y) dA$$

2. *Sum and Difference:*
$$\iint_R (f(x, y) \pm g(x, y)) dA = \iint_R f(x, y) dA \pm \iint_R g(x, y) dA$$

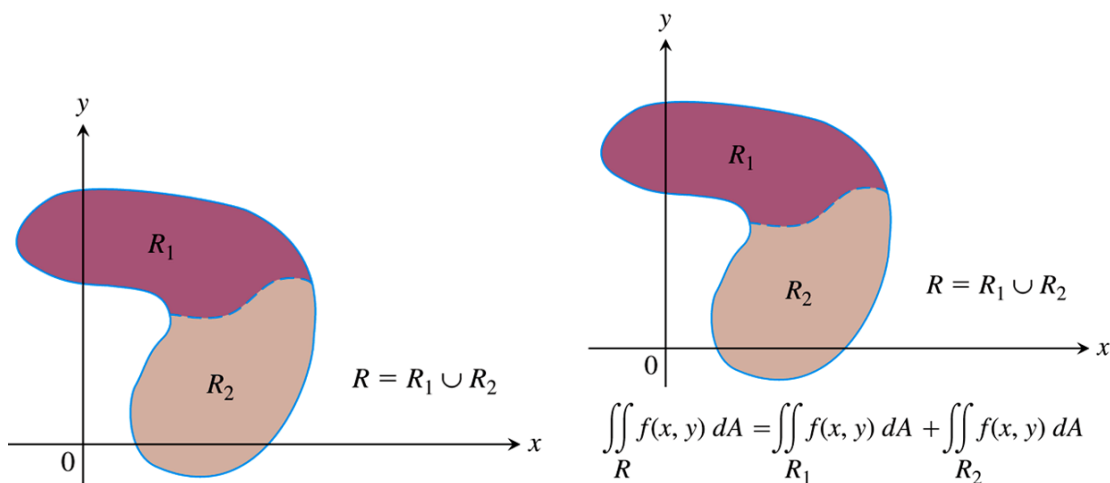
3. *Domination:*

a)
$$\iint_R f(x, y) dA \geq 0 \quad \text{if} \quad f(x, y) \geq 0 \quad \text{on} \quad R$$

b)
$$\iint_R f(x, y) dA \geq \iint_R g(x, y) dA \quad \text{if} \quad f(x, y) \geq g(x, y) \quad \text{on} \quad R$$

4. *Additivity:*
$$\iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA$$

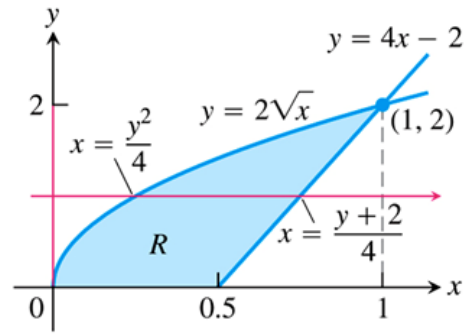
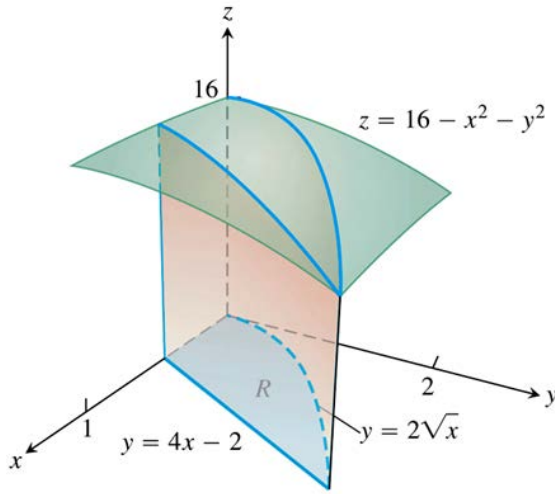
If R is the union of two non-overlapping regions R_1 and R_2 .



Example

Find the volume of the wedge like solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x -axis.

Solution



$$y = 2\sqrt{x} \rightarrow x = \frac{y^2}{4}$$

$$y = 4x - 2 \rightarrow x = \frac{y+2}{4}$$

$$y = 4\frac{y^2}{4} - 2 = y^2 - 2 \rightarrow y^2 - y - 2 = 0 \Rightarrow \underline{y = -1, 2}$$

$$\begin{aligned} \text{Volume} &= \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2) dx dy \\ &= \int_0^2 \left(16x - \frac{1}{3}x^3 - y^2x \right) \Big|_{y^2/4}^{(y+2)/4} dy \\ &= \int_0^2 \left[\left(16\frac{y+2}{4} - \frac{1}{3}\left(\frac{y+2}{4}\right)^3 - y^2\frac{y+2}{4} \right) - \left(16\frac{y^2}{4} - \frac{1}{3}\frac{y^6}{64} - \frac{y^4}{4} \right) \right] dy \\ &= \int_0^2 \left[4y + 8 - \frac{1}{192}(y^3 + 6y^2 + 12y + 8) - \frac{1}{4}y^3 - \frac{1}{2}y^2 - 4y^2 + \frac{1}{192}y^6 + \frac{1}{4}y^4 \right] dy \\ &= \int_0^2 \left(4y + 8 - \frac{1}{192}y^3 - \frac{1}{32}y^2 - \frac{1}{16}y - \frac{1}{24} - \frac{1}{4}y^3 - \frac{9}{2}y^2 + \frac{1}{192}y^6 + \frac{1}{4}y^4 \right) dy \\ &= \int_0^2 \left(\frac{1}{192}y^6 + \frac{1}{4}y^4 - \frac{49}{192}y^3 - \frac{145}{32}y^2 + \frac{63}{16}y + \frac{191}{24} \right) dy \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1344} y^7 + \frac{1}{20} y^5 - \frac{49}{768} y^4 - \frac{145}{96} y^3 + \frac{63}{32} y^2 + \frac{191}{24} y \Big|_0^2 \\
&= \frac{2}{21} + \frac{8}{5} - \frac{49}{48} - \frac{145}{12} + \frac{63}{8} + \frac{191}{12} \\
&= \frac{178}{105} + \frac{513}{48} \\
&= \frac{62,409}{5,040} \text{ unit}^3 \quad \approx 12.4 \text{ unit}^3
\end{aligned}$$

Definition

The area of a closed, bounded plane region R is $A = \iint_R dA$

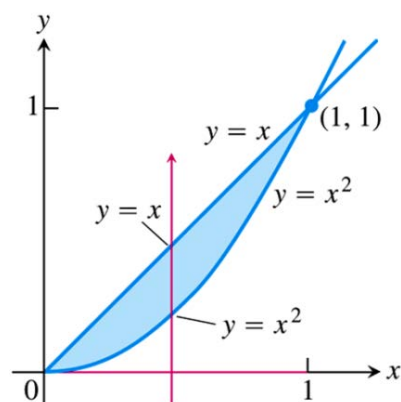
Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution

$$y = x = x^2 \rightarrow x = 0, 1$$

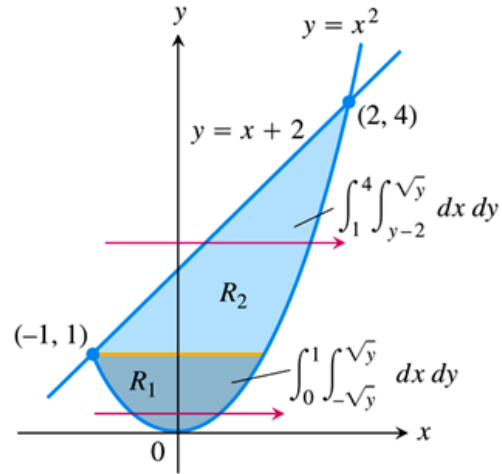
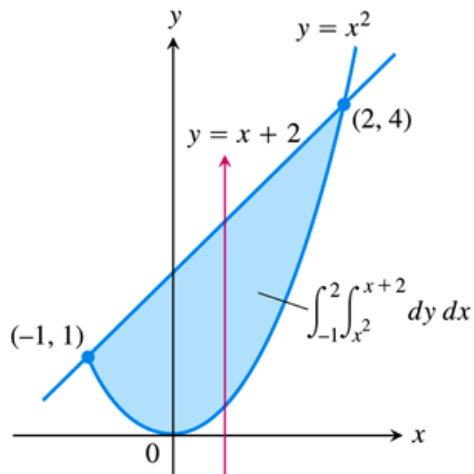
$$\begin{aligned}
A &= \int_0^1 \int_{x^2}^x dy dx \\
&= \int_0^1 y \Big|_{x^2}^x dx \\
&= \int_0^1 (x - x^2) dx \\
&= \frac{1}{2} x^2 - \frac{1}{3} x^3 \Big|_0^1 \\
&= \frac{1}{2} - \frac{1}{3} \\
&= \frac{1}{6} \text{ unit}^2
\end{aligned}$$



Example

Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Solution



$$y = x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$\underline{x = -1, 2}$$

$$A = \int_{-1}^2 \int_{x^2}^{x+2} dy dx$$

$$= \int_{-1}^2 y \Big|_{x^2}^{x+2} dx$$

$$= \int_{-1}^2 (x + 2 - x^2) dx$$

$$= \frac{1}{2}x^2 + 2x - \frac{1}{3}x^3 \Big|_{-1}^2$$

$$= \frac{1}{2}(4) + 2(2) - \frac{1}{3}(8) - \left(\frac{1}{2}(-1)^2 - 2 + \frac{1}{3} \right)$$

$$\underline{= \frac{9}{2} \text{ unit}^2}$$

$$\text{Average values of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f dA$$

$$\diamond \text{ Average value of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f dA = \underline{\frac{2}{\pi}}$$

Example

Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R: 0 \leq x \leq \pi, 0 \leq y \leq 1$.

Solution

$$\begin{aligned} \int_0^\pi \int_0^1 x \cos xy \, dy dx &= \int_0^\pi \sin xy \Big|_0^1 dx & \int x \cos xy \, dy &= \sin xy + C \\ &= \int_0^\pi (\sin x - 0) \, dx \\ &= \int_0^\pi \sin x \, dx \\ &= -\cos x \Big|_0^\pi \\ &= 1 + 1 \\ &= \underline{2} \end{aligned}$$

Exercises Section 3.2 – Double Integrals over General Regions

(1 – 4) Sketch the region of integration and evaluate the integral

1. $\int_0^{\pi} \int_0^x x \sin y \, dy \, dx$

3. $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y} \, dx \, dy$

2. $\int_0^{\pi} \int_0^{\sin x} y \, dy \, dx$

4. $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}} \, dy \, dx$

5. Integrate $f(x, y) = \frac{x}{y}$ over the region in the first quadrant bounded by the lines

$$y = x, \quad y = 2x, \quad x = 1, \quad \text{and} \quad x = 2$$

6. Integrate $f(x, y) = x^2 + y^2$ over the triangular region with vertices $(0,0)$, $(1,0)$ and $(0,1)$

7. Integrate $f(s, t) = e^s \ln t$ over the region in the first quadrant of the st -plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$.

8. Evaluate $\int_{-2}^0 \int_v^{-v} 2 \, dp \, dv$

9. Evaluate $\int_{-\pi/3}^{\pi/3} \int_0^{\sec t} 3 \cos t \, du \, dt$

(10 – 13) Sketch the region of integration, reverse the order of integration, and evaluate the integral

10. $\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} \, dy \, dx$

12. $\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos(16\pi x^5) \, dx \, dy$

11. $\int_0^2 \int_x^2 2y^2 \sin xy \, dy \, dx$

13. $\int_0^2 \int_0^{4-x^2} \frac{x e^{2y}}{4-y} \, dy \, dx$

14. Find the volume of the region bounded above the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy -plane

15. Find the volume of the solid that is bounded above the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy -plane

16. Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$ and the plane $z + y = 3$
17. Find the volume of the solid that is bounded on the front and back by the planes $x = 2$, and $x = 1$, on the sides by the cylinders $y = \pm \frac{1}{x}$ and above and below the planes $z = x + 1$ and $z = 0$.
18. Find the volume under the parabolic cylinder $z = x^2$ above the region enclosed by the parabola $y = 6 - x^2$ and the line $y = x$ in the xy -plane
19. Find the area of the region enclosed by the line $y = 2x + 4$ and the parabola $y = 4 - x^2$ in the xy -plane.
20. Find the area of the region enclosed by the coordinate axes and the line $x + y = 2$.
21. Find the area of the region enclosed by the lines $y = 2x$, and $y = 4$
22. Find the area of the region enclosed by the parabola $x = y - y^2$ and the line $y = -x$.
23. Find the area of the region enclosed by the curve $y = e^x$ and the lines $y = 0$, $x = 0$ and $x = \ln 2$
24. Find the area of the region enclosed by the curve $y = \ln x$ and $y = 2 \ln x$ and the lines $x = e$ in the first quadrant.
25. Find the area of the region enclosed by the lines $y = x$, $y = \frac{x}{3}$, and $y = 2$
26. Find the area of the region enclosed by the lines $y = x - 2$ and $y = -x$ and the curve $y = \sqrt{x}$
27. Find the area of the region enclosed by the parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$
28. Find the area of the region bounded by the lines $y = -x - 4$, $y = x$, and $y = 2x - 4$. Make a sketch of the region.
29. Find the area of the region bounded by the lines $y = |x|$ and $y = 20 - x^2$. Make a sketch of the region.
30. Find the area of the region bounded by the lines $y = x^2$ and $y = 1 + x - x^2$. Make a sketch of the region.

(31 – 34) Find the area of the region

$$31. \int_0^6 \int_{y^2/3}^{2y} dx dy$$

$$33. \int_{-1}^2 \int_{y^2}^{y+2} dx dy$$

$$32. \int_0^{\pi/4} \int_{\sin x}^{\cos x} dy dx$$

$$34. \int_0^2 \int_{x^2-4}^0 dy dx + \int_0^4 \int_0^{\sqrt{x}} dy dx$$

35. Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \leq x \leq 2$, $0 \leq y \leq 2$

36. Find the average height of $f(x, y) = \frac{1}{xy}$ over the square $\ln 2 \leq x \leq 2\ln 2$, $\ln 2 \leq y \leq 2\ln 2$

(37 – 40) Evaluate the integral over the given region

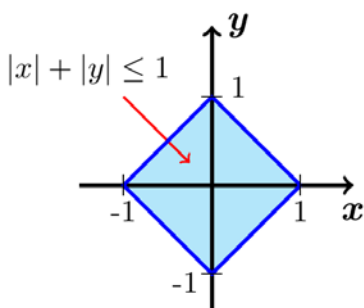
37. $\iint_R y dA$ $R = \left\{ (x, y) : 0 \leq x \leq \frac{\pi}{3}, 0 \leq y \leq \sec x \right\}$

38. $\iint_R (x + y) dA$ R is the region bounded by $y = \frac{1}{x}$ and $y = \frac{5}{2} - x$

39. $\iint_R \frac{xy}{1 + x^2 + y^2} dA$ $R = \left\{ (x, y) : 0 \leq y \leq x, 0 \leq x \leq 2 \right\}$

40. $\iint_R x \sec^2 y dA$ $R = \left\{ (x, y) : 0 \leq y \leq x^2, 0 \leq x \leq \frac{\sqrt{\pi}}{2} \right\}$

41. Consider the region $R = \left\{ (x, y) : |x| + |y| \leq 1 \right\}$



a) Use a double integral to show that the area of R is 2.

b) Find the volume of the square column whose base is R and whose upper surface is $z = 12 - 3x - 4y$.

c) Find the volume of the solid above R and beneath the cylinder $x^2 + z^2 = 1$.

d) Find the volume of the pyramid whose base is R and whose vertex is on the z -axis at $(0, 0, 6)$