Section 1.4 – Slack Variables and the Pivot

http:/www.zweigmedia.com/RealWorld/simplex.html

http://people.richland.edu/james/ictcm/2006/pivot.html

A linear programming problem is in standard maximum form if the following conditions are satisfied

1. The objective function is to be maximized
2. All variables are nonnegative $\left(x_i \ge 0\right)$
3. All remaining constraints are stated in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \le b \qquad \text{with } b \ge 0$$

$$\begin{array}{ll}
\textit{Maximize} & z = 3x_1 + 2x_2 + x_3 \\
\textit{subject to} & 2x_1 + x_2 + x_3 \le 150 \\
 & x_1 + 2x_2 + 8x_3 \le 200 \\
 & 2x_1 + 3x_2 + x_3 \le 320 \\
\textit{with} & x_1, x_2, x_3 \ge 0
\end{array}$$

Slack Variables

Example

$$\begin{array}{ll}
\textit{Maximize} & z = 3x_1 + 2x_2 + x_3 \\
\textit{subject to} & 2x_1 + x_2 + x_3 + s_1 = 150 \\
 & x_1 + 2x_2 + 8x_3 + s_2 = 200 \\
 & 2x_1 + 3x_2 + x_3 + s_3 = 320 \\
\textit{with} & x_1, x_2, x_3, s_1, s_2, s_3 \ge 0
\end{array}$$

The variables s_1 , s_2, and s_3 are called ***slack variables*** because each makes up the difference between the left or right sides of an inequality system (takes up any slack).

The objective function may be rewritten as: $z - 3x_1 - 2x_2 - x_3 = 0$

The equations can be written as the following augmented matrix

$$\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array} \\
\left[\begin{array}{ccccccc|c}
2 & 1 & 1 & 1 & 0 & 0 & 0 & 150 \\
1 & 2 & 8 & 0 & 1 & 0 & 0 & 200 \\
2 & 3 & 1 & 0 & 0 & 1 & 0 & 320 \\
\hline
-3 & -2 & -1 & 0 & 0 & 0 & 1 & 0
\end{array}\right] \rightarrow \textit{Indicators}
\end{array}$$

This matrix is called the initial ***simplex tableau***. The numbers in the bottom row are called ***indicators***.

Standard Maximization Problems in standard Form

Maximize: $z = 50x_1 + 80x_2$

Subject to
$$x_1 + 2x_2 \le 32$$
$$3x_1 + 4x_2 \le 84$$
$$x_1, x_2 \ge 0$$

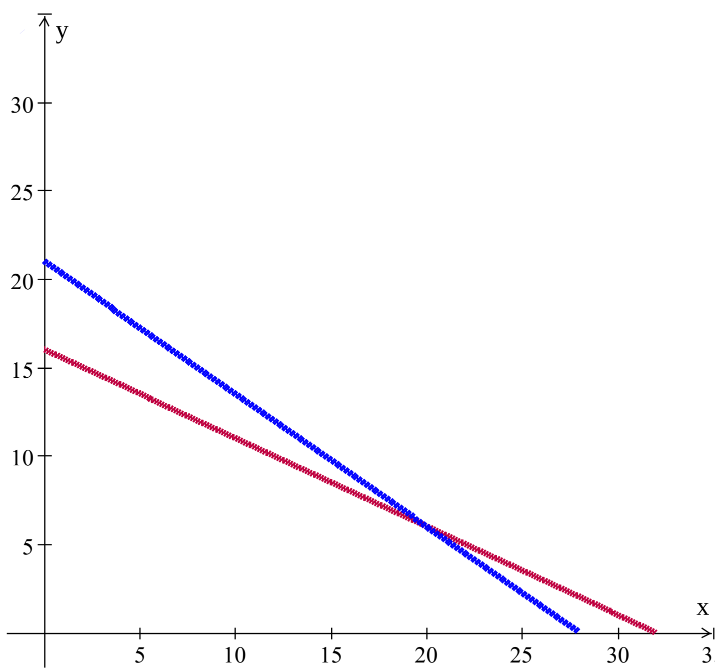
Slack variables

Convert the constraint inequalities to linear equations by using a simple device called a slack variable.

$$x_1 + 2x_2 + s_1 \quad = 32$$
$$3x_1 + 4x_2 \quad + s_2 = 84$$

The variables s_1 and s_2 are called ***slack variables*** because each makes up the difference between the left or right sides of an inequality system.

x_1	x_2	s_1	s_2	Intersection Point	**Feasible**
0	0	32	84	(0, 0)	Yes
0	16	0	20	(0, 16)	Yes
0	21	-10	0	(0, 21)	No
32	0	0	-12	(32, 0)	No
28	0	4	0	(28, 0)	Yes
20	6	0	0	(20, 6)	Yes



The Initial Simplex Tableau {*A tableau is just a special augmented matrix*}

When creating the Initial tableau, enter the coefficients of the variables just like we do with Gauss-Jordan method. Label the columns with the appropriate variables. To identify the basic variables, look for the columns that have a 1 and the rest of the entries are 0. Beside the row that has the 1.

Initial Simplex Tableau

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 2 & 1 & 1 & 1 & 0 & 0 & 0 & 150 \\ 1 & 2 & 8 & 0 & 1 & 0 & 0 & 200 \\ 2 & 3 & 1 & 0 & 0 & 1 & 0 & 320 \\ \hline -3 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Basic Variables

s_1, s_2, s_3, and P are ***basic*** variables.

That means that x_1, x_2, and x_3 are ***nonbasic*** (at this juncture) $\Rightarrow$ x_1, x_2, and $x_3 = 0$.

If you write the corresponding system from the tableau with x_1, x_2, and $x_3 = 0$, you get the following.

$$s_1 = 150 \quad s_2 = 200 \quad s_3 = 320 \quad P = 0$$

So the ***basic feasible solution*** at this point is:

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ s_1 = 150,\ s_2 = 200,\ s_3 = 320,\ P = 0$$

Determining the pivot element

Pivot Column:

Look at the elements on the bottom row to the left of the P column. The value that is most negative identifies the pivot column.

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P \end{array} \\ \left[\begin{array}{ccccccc|c} 2 & 1 & 1 & 1 & 0 & 0 & 0 & 150 \\ 1 & 2 & 8 & 0 & 1 & 0 & 0 & 200 \\ 2 & 3 & 1 & 0 & 0 & 1 & 0 & 320 \\ \hline -3 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

(-3) ***column* 1** is the pivot column.

Pivot Row:

Divide each positive number above the most negative indicator into the corresponding constants at the far right. The smallest quotient indicates the pivot row.

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P & \end{array} \\ \left[\begin{array}{ccccccc|c} 2 & 1 & 1 & 1 & 0 & 0 & 0 & 150 \\ 1 & 2 & 8 & 0 & 1 & 0 & 0 & 200 \\ 2 & 3 & 1 & 0 & 0 & 1 & 0 & 320 \\ \hline -3 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} \frac{150}{2} = 75 \leftarrow \\ \frac{200}{1} = 200 \\ \frac{320}{2} = 160 \end{array}$$

***Row* 1** is the pivot row

The Pivoting Process:

After identified the pivot element, the pivot process can begin.
First, make the pivot element become a 1 by using the appropriate row operation, then use the 1 to eliminate all other elements in its column using Gauss-Jordan elimination.
We are pivoting on the column 1, row 1.

First Pivot

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P & \end{array} \\ \left[\begin{array}{ccccccc|c} 2 & 1 & 1 & 1 & 0 & 0 & 0 & 150 \\ 1 & 2 & 8 & 0 & 1 & 0 & 0 & 200 \\ 2 & 3 & 1 & 0 & 0 & 1 & 0 & 320 \\ \hline -3 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \frac{1}{2}R_1$$

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P & \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & .5 & .5 & .5 & 0 & 0 & 0 & 75 \\ 1 & 2 & 8 & 0 & 1 & 0 & 0 & 200 \\ 2 & 3 & 1 & 0 & 0 & 1 & 0 & 320 \\ \hline -3 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 + 3R_1 \end{array}$$

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P & \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & .5 & .5 & .5 & 0 & 0 & 0 & 75 \\ 0 & 1.5 & 7.5 & -.5 & 1 & 0 & 0 & 125 \\ 0 & 2 & 0 & -1 & 0 & 1 & 0 & 170 \\ \hline 0 & -.5 & .5 & 1.5 & 0 & 0 & 1 & 225 \end{array}\right] \end{array}$$

So the basic feasible solution at this point is:

$x_1 = 75,\ x_2 = 0,\ x_3 = 0,\ s_1 = 0,\ s_2 = 125,\ s_3 = 170,\ P = 225$

Second Pivot

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & .5 & .5 & .5 & 0 & 0 & 0 & 75 \\ 0 & 1.5 & 7.5 & -.5 & 1 & 0 & 0 & 125 \\ 0 & 2 & 0 & -1 & 0 & 1 & 0 & 170 \\ \hline 0 & -.5 & .5 & 1.5 & 0 & 0 & 1 & 225 \end{array}\right] \end{array} \quad \begin{array}{l} \frac{75}{.5}=150 \\ \frac{125}{1.5}={\color{red}83.3} \\ \frac{170}{2}=85 \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & .5 & .5 & .5 & 0 & 0 & 0 & 75 \\ 0 & 1 & 5 & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & \frac{250}{3} \\ 0 & 2 & 0 & -1 & 0 & 1 & 0 & 170 \\ \hline 0 & -.5 & .5 & 1.5 & 0 & 0 & 1 & 225 \end{array}\right] \end{array} \quad \begin{array}{l} R_1 - .5R_2 \\ \\ R_3 - 2R_2 \\ R_4 + .5R_2 \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 0 & -2 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & \frac{100}{3} \\ 0 & 1 & 5 & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & \frac{250}{3} \\ 0 & 0 & -10 & -\frac{1}{3} & -\frac{4}{3} & 1 & 0 & \frac{10}{3} \\ \hline 0 & 0 & 3 & \frac{4}{3} & \frac{1}{3} & 0 & 1 & \frac{800}{3} \end{array}\right] \end{array}$$

Exercises Section 1.4 – Slack Variables and the Pivot

1. Write the initial simplex tableau for each linear programing problem

a) $\text{Maximized}: \quad z = 7x_1 + x_2$

$\text{subject to}: \quad 4x_1 + 2x_2 \le 5$

$x_1 + 2x_2 \le 4$

$x_1, x_2 \ge 0$

b) $\text{Maximized}: \quad z = x_1 + 3x_2$

$\text{subject to}: \quad 2x_1 + 3x_2 \le 100$

$5x_1 + 4x_2 \le 200$

$x_1, x_2 \ge 0$

c) $\text{Maximized}: \quad z = x_1 + 3x_2$

$\text{subject to}: \quad x_1 + x_2 \le 10$

$5x_1 + 2x_2 \le 4$

$x_1 + 2x_2 \le 36$

$x_1, x_2 \ge 0$

d) $\text{Maximized}: \quad z = 5x_1 + 3x_2$

$\text{subject to}: \quad x_1 + x_2 \le 25$

$4x_1 + 3x_2 \le 48$

$x_1, x_2 \ge 0$

2. Pivot once as indicated in each simplex tableau. Read the solution from the result

a)
$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & z & \end{matrix} \\ \left[\begin{array}{cccccc|c} 1 & 2 & 4 & 1 & 0 & 0 & 56 \\ 2 & \{2\} & 1 & 0 & 1 & 0 & 40 \\ \hline -1 & -3 & -2 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

b)
$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & z & \end{matrix} \\ \left[\begin{array}{cccccc|c} 2 & 3 & 4 & 1 & 0 & 0 & 18 \\ 6 & \{3\} & 2 & 0 & 1 & 0 & 15 \\ \hline -1 & -6 & -2 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

c)
$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \end{matrix} \\ \left[\begin{array}{ccccccc|c} 2 & 2 & \{1\} & 1 & 0 & 0 & 0 & 12 \\ 1 & 2 & 3 & 0 & 1 & 0 & 0 & 45 \\ 3 & 1 & 1 & 0 & 0 & 1 & 0 & 20 \\ \hline -2 & -1 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

d)
$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \end{matrix} \\ \left[\begin{array}{ccccccc|c} 2 & \{2\} & 3 & 1 & 0 & 0 & 0 & 500 \\ 4 & 1 & 1 & 0 & 1 & 0 & 0 & 300 \\ 7 & 2 & 4 & 0 & 0 & 1 & 0 & 700 \\ \hline -3 & -4 & -2 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

3. The authors of a best-selling textbook in finite mathematics are told that, for the next edition of their book, each simple figure would cost the project \$20, each figure with additions would cost \$35, and each computer-drawn sketch would cost \$60. They are limited to 400 figures, for which they are allowed to spend up to \$2200. The number of computer-drawn sketches must be no more than the number of the other two types combined, and there must be at least twice as many simple figures as there are figures with additions. If each simple figure increases the royalties by \$95, each figure with additions increases royalties by \$200, and each computer-drawn figure increases royalties by \$325, how many of each type of figure should be included to maximize royalties, assuming that all art costs are borne by the publisher?

4. A manufacturer of bicycles builds racing, touring, and mountain models. The bicycles are made of both aluminum and steel. The company has available 91,800 units of steel and 42,000 units of aluminum. The racing, touring, and mountain models need 17, 27, and 34 units steel, and 12, 21, and 15 units of aluminum respectively. How many of each type of bicycle should be made in order to maximize profit if the company makes \$8 per racing bike, \$12 per touring bike, and \$22 per mountain bike? What is the maximum possible profit?