4.7 – Alternating Current Circuits

An alternating current circuit (ac) is a circuit where the voltage and the current vary with time typically like a sine or a cosine. A typical ac signal may be given as

$$v = V\sin(\omega t + \beta)$$

v is the voltage at a given instant of time and is called ***instantaneous*** voltage.
V is the maximum value of the voltage and is called the ***amplitude*** of the voltage.
ω is the number of radians executed per second and is called the ***angular frequency*** of the voltage.

It is related with frequency (f) as $\omega = 2\pi f$ and with period (T) as $\omega = 2\pi / T$.
β is called the phase angle of the voltage.

Its effect is to shift the graph of $\sin(\omega t)$ either to the right (if negative) or to the left (if negative).

Phase angle of signal 2 minus the phase angle of signal 1 is called the ***phase shift*** of signal 2 with respect to signal 1. The one with a bigger phase angle is said to be leading the other and the one with a smaller phase angle is said to be lagging from the other.

If $\quad v_1 = V_1\sin(\omega t + \beta_1)$ *and* $v_2 = V_2\sin(\omega t + \beta_2)$, *then* $\theta = \beta_2 - \beta_1$

Example

An ac voltage varies with time according to the equation $v = 120\sin\left(300t + \frac{\pi}{2}\right)$

a) What is the maximum value of the voltage?
b) How long does it take to make one complete oscillation?
c) What is its phase angle?

Solution

a) $V = 120\text{ V}$

b) $T = \frac{2\pi}{\omega} = \frac{2\pi}{300} \ \underline{= 0.021\ s}$

c) $\underline{\beta = \frac{\pi}{2}}$

Example

Calculate the phase shift between the following pair of signals and indicate which one is leading:

$v_1 = 10\sin(20t + \pi)$ *and* $v_2 = 20\sin\left(20t - \frac{\pi}{2}\right)$

Solution

$\beta_1 = \pi; \quad \beta_2 = -\frac{\pi}{2}$

$\theta = \beta_2 - \beta_1 = -\frac{\pi}{2} - \pi \ \underline{= -\frac{3\pi}{2}}$ $\qquad v_1$ is leading because its phase angle is bigger.

A Resistor Connected to an ac Source

Let's consider a resistor of resistance R connected to an ac source whose voltage varies with time according to the equation $v = V\sin(\omega t)$. Ohm's law applies to ac circuits instantaneously. Therefore the equation $v = iR$ holds where i stands for the instantaneous current. This implies that

$$i = \frac{V}{R}\sin\omega t \quad when \quad v = V\sin(\omega t)$$

This shows that, in a resistor, the phase shift (θ_R) between the voltage and the current is zero. In other words, the voltage and the current are in phase.

$$\theta_R = 0$$

Since $i = I\sin\omega t = \frac{V}{R}\sin\omega t$, Ohm's law also applies to the amplitudes of the voltage and the current.

$$V = IR$$

Example

A resistor of resistance 50 Ω is connected to an ac source whose potential difference varies with time according to the equation $v = 20\sin(100t)$.

a) Calculate the amplitude of the current.
b) Give a formula for the instantaneous current as a function of time.
c) Calculate the instantaneous voltage and current after 10 s.

<u>Solution</u>

Given: $V = 20V; \quad R = 50\Omega; \quad \omega = 100 \ rad/\sec$

a) $\boxed{I = \frac{V}{R} = \frac{20}{50} = 0.4\ A}$

b) $\boxed{i(t) = I\sin\omega t = 0.4\sin 100t}$

c) $i(t=10) = 0.4\sin(100(10)) \ \boxed{= 0.33\ A}$

$v(t=10) = 20\sin(100(10)) \ \boxed{= 16.5\ V}$

A Capacitor Connected to an *ac* Source

Let's consider a capacitor of capacitance C connected to an ac source whose potential difference varies with time according to the equation $v = V\sin(\omega t)$. The instantaneous charge q of the capacitor is related with the instantaneous voltage by $q = vC$. The instantaneous current is equal to the rate of change of the charge or derivative of charge with respect to time in the language of calculus. Therefore, using calculus (because it can't be done algebraically),

$$\begin{aligned} i &= \frac{dq}{dt} \\ &= C\frac{dv}{dt} \\ &= C\frac{d}{dt}\{V\sin(\omega t)\} \\ &= CV\omega\cos(\omega t) \qquad \textit{but} \quad \cos\omega t = \sin\left(\omega t + \frac{\pi}{2}\right) \\ &= CV\omega\sin\left(\omega t + \frac{\pi}{2}\right) \qquad \text{when } v = V\sin(\omega t) \end{aligned}$$

This implies that for a capacitor, the voltage lags from the voltage by $\frac{\pi}{2}$ or 90°.

$$\theta_C = -\frac{\pi}{2}$$

where θ_C is the phase shift of the voltage with respect to current. Since

$$i = I\sin\left(\omega t + \frac{\pi}{2}\right) = CV\omega\sin\left(\omega t + \frac{\pi}{2}\right) \quad \Rightarrow \quad I = CV\omega$$

Where I is the amplitude of the current. The ratio between the amplitude of the voltage across a capacitor and the amplitude of the current across a capacitor is defined to be the ***capacitive reactance*** (X_C) of the capacitor.

$$X_C = \frac{V}{I}$$

The unit of measurement for capacitive reactance is ohm. Replacing I by $VC\omega$, it follows that the capacitive reactance of a capacitor is inversely proportional to the frequency of the voltage (current).

$$X_C = \frac{1}{C\omega}$$

Example

A capacitor of capacitance 20 μF to an ac source whose potential difference varies with time according to the equation $v = 12\sin(500t)$.

a) Calculate its capacitive reactance.
b) Calculate the amplitude of the current
c) Give a formula for the current as a function of time

Solution

Given: $V = 12V$; $C = 20\mu F = 2\times10^{-5}$; $\omega = 500\ \ rad/\sec$

a) $X_C = \dfrac{1}{\omega C} = \dfrac{1}{500\times2\times10^{-5}} = 100\ \Omega$

b) $I = \dfrac{V}{X_C} = \dfrac{12}{100} = 0.12\ A$

c) For a capacitor, the current leads the voltage by $\pi/2$. Therefore, since the phase angle of the voltage is zero, the phase angle of the current must be $\theta_C = -\frac{\pi}{2}\ \pi/2$.

$$i(t) = I\sin\left(\omega t + \frac{\pi}{2}\right) = 0.12\sin\left(500t + \frac{\pi}{2}\right)\ \ A$$

An Inductor Connected to an ac Source

Let's consider an inductor of inductance L connected to an ac source where the current in the circuit varies with time according to the equation $i = I \sin(\omega t)$. From Kirchhoff's loop rule, the voltage of the source is the negative of the self-induced voltage (so that they add up to zero):

$$v = -E_{self}$$

But $E_{self} = -L\dfrac{di}{dt}$ (*using calculus*)

Hence $v = L\dfrac{di}{dt} = L\dfrac{d}{dt}\{I\sin(\omega t)\}$

$$= LI\omega\cos\omega t \qquad but \quad \cos\omega t = \sin\left(\omega t + \frac{\pi}{2}\right)$$

$$= LI\omega\sin\left(\omega t + \frac{\pi}{2}\right) \qquad \text{when } i = I\sin(\omega t)$$

This means, for an inductor, the voltage leads the current by $\frac{\pi}{2}$ or 90°.

$$\theta_L = \frac{\pi}{2}$$

where $\theta_L = \beta_f - \beta_i$ is the phase shift of the voltage with respect to current. Since

$$v = V\sin\left(\omega t + \frac{\pi}{2}\right) = LI\omega\sin\left(\omega t + \frac{\pi}{2}\right) \Rightarrow V = L\omega I$$

The ratio between the amplitude of the voltage across an inductor and the current across the inductor is called the *inductive reactance* (X_L) of the inductor.

$$X_L = \frac{V}{I}$$

The unit of measurement for inductive reactance is the ohm. Replacing V by $L\omega I$, It is seen that the inductive reactance of an inductor is directly proportional to the frequency of the signal.

$$X_L = \omega L$$

Example

An inductor of inductance 15 *m*H is connected to an ac source whose potential difference varies with time according to the equation $v = 40\sin(250t)$.

a) Calculate its inductive reactance.
b) Calculate the amplitude of the current.
c) Give a formula for the current as a function of time.

Solution

Given: $V = 40V$; $L = 15mH = 0.05$; $\omega = 250 \ rad/\sec$

a) $X_L = \omega L = (250)(0.015) = 3.75\ \Omega$

b) $\boxed{I = \frac{V}{X_C} = \frac{40}{3.75} = 10.7\ A}$

c) For an inductor, the current lags from the voltage by $\pi/2$ and the phase angle of the voltage is zero, the phase angle of the current must be -$\pi/2$.

$$\boxed{i(t) = I\sin\left(\omega t - \frac{\pi}{2}\right) = 10.7\sin\left(250t - \frac{\pi}{2}\right)\ A}$$

Series Combination of a Resistor, an Inductor and a Capacitor Connected to an ac Source

Let's consider a series combination of a resistor of resistance *R*, an inductor of inductance *L* and a capacitor of capacitance *C* connected to an ac source where the current in the circuit varies with time according the equation $i = I\sin(\omega t)$. The currents through the resistor (i_R), the inductor (i_L) and the capacitor (i_C) are equal and they are equal to the current in the circuit.

$$i_R = i_L = i_C = i = I\sin(\omega t)$$

The net instantaneous potential difference (v) is equal to the sum of the instantaneous potential differences across the resistor (v_R), the inductor (v_L) and capacitor (v_C).

$$v = v_R + v_L + v_C$$

In a resistor, the voltage and the current are in phase; that is, their phase angles are equal. Therefore, since $i = I\sin(\omega t)$, the instantaneous potential difference across the resistor is given by

$$v_R = V_R \sin(\omega t)$$

Where $V_R = IR$.

In an inductor, the voltage leads the current by $\pi/2$; that is, the phase angle of the voltage is $\pi/2$ more than the phase angle of the current. Since the current is given as $i = I\sin(\omega t)$, the instantaneous potential difference across the inductor is given by

$$v_L = V_L \sin\left(\omega t + \frac{\pi}{2}\right)$$

Where $V_L = IX_L$.

In a capacitor, the potential difference lags from the current by $\pi/2$; that is the phase angle of the voltage is $\pi/2$ less than the phase angle of the current. Since the current is given as $i = I\sin(\omega t)$, the instantaneous voltage across the capacitor is given by

$$v_C = V_C \sin\left(\omega t - \frac{\pi}{2}\right)$$

where $V_C = IX_C$.

The net instantaneous voltage is the sum of the instantaneous voltages across the resistor, inductor and capacitor:

$$v = V_R \sin(\omega t) + V_L \sin\left(\omega t + \frac{\pi}{2}\right) + V_C \sin\left(\omega t - \frac{\pi}{2}\right)$$

But $\quad \sin\left(\omega t + \frac{\pi}{2}\right) = \cos\omega t \quad$ and $\quad \sin\left(\omega t - \frac{\pi}{2}\right) = -\cos\omega t$

Therefore,

$$v = V_R \sin(\omega t) + V_L \cos(\omega t) - V_C \cos(\omega t)$$

$$\boxed{v = V_R \sin(\omega t) + (V_L - V_C)\cos(\omega t)}$$

Also, if the phase shift of the net voltage with respect to the current is θ, then

$$v = V\sin(\omega t + \theta)$$

This expression of v can be expressed in terms of $cos(\omega t)$ and $sin(\omega t)$ by expanding the sine:

$$v = V\cos\theta\sin\omega t + V\sin\theta\cos\omega t$$

The coefficients of $\cos(\omega t)$ and $\sin(\omega t)$ of both expressions of v can be equated because $\cos(\omega t)$ and $\sin(\omega t)$ are independent functions.

$$V_R = V\cos(\theta) \; . \; . \; . \; . \; . \; (1)$$

$$V_L - V_C = V\sin(\theta) \; . \; . \; . \; . \; . \; (2)$$

An expression for the phase shift of the voltage with respect to the current can be obtained by dividing equation (2) by equation (1):

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{V_L - V_C}{V_R} = \frac{IX_L - IX_C}{IR} = \frac{X_L - X_C}{R}$$

Thus θ is given as follows:

$$\theta = \arctan\left(\frac{X_L - X_C}{R}\right) = \arctan\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$$

An expression for the amplitude of the net voltage can be obtained by squaring equations (**1**) and (**2**) and adding:

$$V_R^2 + (V_L - V_C)^2 = (V\cos\theta)^2 + (V\sin\theta)^2 = V^2$$

Therefore, the amplitude of the net voltage is related with the amplitudes of the voltages across the resistor, inductor and capacitor as follows:

$$V = \sqrt{V_R^2 + (V_L - V_C)^2}$$

The total ***impedance*** (***Z***) of the series combination is defined to be the ratio between the between the amplitude of the net voltage and the amplitude of the current in the circuit.

$$Z = \frac{V}{I}$$

An expression for Z in terms of R, L, and C can be obtained by expressing the voltages in terms of the current:

$$Z = \frac{V}{I} = \frac{\sqrt{V_R^2 + (V_L - V_C)^2}}{I} = \frac{\sqrt{(IR)^2 + (IX_L - IX_C)^2}}{I}$$

Hence,

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

Example

A 200 Ω resistor, a 20 H inductor and a 5 *m*F capacitor are connected in series and then connected to an ac source whose potential difference varies with time according to the equation $v = 120\sin(15t)$

a) Calculate the impedance of the circuit.
b) Calculate the amplitude of the current.
c) Calculate the amplitudes of the voltages across the resistor, inductor and capacitor.
d) Calculate the phase shift of the voltage with respect to the current. Which one is leading?
e) Obtain a formula for the current as a function of time.
f) Obtain formulas for the voltages across the resistor, inductor and capacitor as a function of time.

Solution

Given: $R = 200\ \Omega;\quad L = 20\ H;\quad C = 5mF = 5\times10^{-3} F;\quad \omega = 15\ rad/s;\quad V = 120V$

a) $$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$= \sqrt{200^2 + \left((15)(20) - \frac{1}{(15)\left(5\times10^{-3}\right)}\right)^2}$$

$$\boxed{= 349.5\ \Omega}$$

b) $$I = \frac{V}{Z} = \frac{120}{349.5}\ \boxed{= 0.343\ A}$$

c) $$V_R = IR = 0.34 * 200\ \boxed{= 68.6\ V}$$

$$V_L = I\omega L = 0.34 * 15 * 20\ \boxed{= 102.9\ V}$$

$$V_C = \frac{I}{\omega C} = \frac{0.34}{15 * 5\times10^{-3}}\ \boxed{= 4.6\ V}$$

d) $$\theta = \arctan\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) = \arctan\left(\frac{15*20 - \frac{1}{15*5\times10^{-3}}}{200}\right)\ \boxed{= 55^\circ}$$

Since the phase shift of voltage with respect to current $(\theta = \beta_v - \beta_i)$ is positive, the voltage is leading.

e) Since the voltage leads the current by θ and the phase angle of the voltage is zero , the phase angle of the current should be $-\theta$

$$\theta = 55^\circ = 55^\circ\frac{\pi}{180^\circ}\ \boxed{= 0.96\ rad}$$

$$\beta_v = 0\quad (because\ v = V\sin\omega t)$$

$$\theta = \beta_v - \beta_i \quad\Rightarrow\quad \boxed{\beta_i = \beta_v - \theta = 0 - 0.96 = -0.96}$$

$$i(t) = I\sin\left(\omega t + \beta_i\right) \boxed{= 0.343\sin(15t - 0.96)\ A}$$

f) $\theta_R = 0;\quad \theta_L = \frac{\pi}{2};\quad \theta_C = -\frac{\pi}{2}$

$$\theta_R = \beta_{Rv} - \beta_i \quad \Rightarrow \quad \boxed{\beta_{Rv}} = \theta_R + \beta_i = 0 + (-0.96) \boxed{= -0.96}$$

$$v_R = V_R \sin\left(\omega t + \beta_{Rv}\right) \boxed{= 68.6\ \sin(15t - 0.96)\ V}$$

$$\theta_L = \beta_{Lv} - \beta_i \quad \Rightarrow \quad \boxed{\beta_{Lv}} = \theta_L + \beta_i \boxed{= \frac{\pi}{2} - 0.96}$$

$$v_L = V_L \sin\left(\omega t + \beta_{Lv}\right) \boxed{= 102.9\ \sin\left(15t + \frac{\pi}{2} - 0.96\right)\ V}$$

$$\theta_C = \beta_{Cv} - \beta_i \quad \Rightarrow \quad \boxed{\beta_{Cv}} = \theta_C + \beta_i \boxed{= -\frac{\pi}{2} - 0.96}$$

$$v_C = V_C \sin\left(\omega t + \beta_{Cv}\right) \boxed{= 102.9\ \sin\left(15t - \frac{\pi}{2} - 0.96\right)\ V}$$

Resonant Frequency

The Amplitude of the current through a series connection of a resistor, an inductor and capacitor depends on the frequency of the source, because the impedance of the combination depends on frequency. The frequency for which the amplitude of the current is the maximum for a given amplitude of the voltage is called resonant frequency of the circuit. Since $I = \frac{V}{Z}$, I is maximum when Z is minimum. Since $Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$, Z is minimum when $\omega L - \frac{1}{\omega C} = 0$. Therefore the resonant angular frequency $\left(\omega_0\right)$ is the frequency that makes this expression zero.

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

The resonant frequency is obtained by dividing the resonant angular frequency by 2π; that is,

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

Resonant frequency has a very important application in the tuning circuits of radios, televisions and others. A tuning circuit may have a variable capacitor. The capacitance of the variable capacitor can be dialed so that the resonant frequency of the tuning circuit is equal to the frequency of the signal to be picked. When signals with different frequencies arrive on the device, only the signal whose frequency is equal to the resonant frequency will be received with a significant current.

Example

The tuning circuit of a radio consists of a series combination of a 1000 Ω resistor, 0.004 H and a variable capacitor. To what capacitance should the capacitor be dialed, if the radio is to pick a signal whose frequency is $2 \times 10^6 Hz$.

Solution

The capacitance of the capacitor should be dialed to a value that makes the resonant frequency of the tuning circuit equal to the frequency of the signal to be picked.

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$f_0^2 = \frac{1}{4\pi^2 LC}$$

$$C = \frac{1}{4\pi^2 L f_0^2} = \frac{1}{4(3.14)^2(0.004)\left(2\times10^6\right)} = 1.6\times10^{-12}\ F$$

Root Mean Square Value

The ***root mean square (*****RMS*****) value*** of an ac signal is defined to be the square root of the average of the square of the signal. Let's consider an ac voltage that varies with time according to the equation $v = V\sin\omega t$.

Squaring $v^2 = V^2 \sin^2(\omega t)$.

The square of the sine can be expanded using the double angle formula:

$$v^2 = \frac{1}{2}V^2 - \frac{1}{2}V^2\cos(2\omega t)$$

The average of the first term is itself $\frac{1}{2}V^2$ because it is a constant.

The average of the second term is zero because cosine alternates between equal (numerically) positives and negatives periodically. Therefore the RMS value $\left(V_{RMS}\right)$ of an ac signal is equal to the square root of $\frac{1}{2}V^2$; that is, the RMS value of an ac signal is obtained by dividing the amplitude by $\sqrt{2}$.

$$V_{RMS} = \frac{V}{\sqrt{2}}$$

Similarly the RMS value of the current is given as $I_{RMS} = \frac{I}{\sqrt{2}}$. ac voltmeters and ammeters are designed to measure RMS values.

Example

An ac voltmeter connected to an ac voltage reads 10 *V*. What is the amplitude of the voltage?

Solution

The reading of the voltmeter is equal to the RMS value of the voltage.

$$V = \sqrt{2}V_{RMS} = \sqrt{2}(10) = 14.1\ V$$

Average Power

The instantaneous power dissipated in ac circuits is given as the product of the instantaneous voltage and instantaneous current. Suppose the current varies with time in the form $i = I \sin \omega t$ and the phase shift of the voltage with respect to the current is θ. Then, the voltage is given as $v = V \sin(\omega t)$ and the instantaneous power varies with time as

$$VI \sin(\omega t)\sin(\omega t + \theta) = VI \cos(\theta)\sin^2(\omega t) + VI \sin(\theta)\sin(\omega t)\cos(\omega t)$$

The last expression is obtained by expanding $\sin(\omega t + \theta)$.

The average of the first term is $\frac{1}{2}VI \cos(\omega t)$ because the average of $\sin^2(\omega t)$ is equal to $\frac{1}{2}$ as obtained in the previous section. The average of the second term is zero because

$$\sin(\omega t)\cos(\omega t) = \frac{1}{2}\sin(2\omega t)$$

whose average is zero because sine alternates between equal (numerically) positives and negatives periodically. Therefore, the average power (P_{av}) of an ac circuit is given as follows:

$$P_{av} = \frac{1}{2}VI \cos\theta = I_{RMS} V_{RMS} \cos\theta$$

The value $\cos\theta$ is called the power factor of the circuit.

The circuit yields no power if the phase shift of voltage with respect to current is $\pm\frac{\pi}{2}$ (because $\cos\left(\pm\frac{\pi}{2}\right) = 0$. For example, no power can be extracted from an inductor or a capacitor because their phase shifts are $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ respectively. Only a resistor yields power because for a resistor the phase shift is zero and $\cos(0) = 1$. Actually, it can be said that the power dissipated in a series connection of a resistor, inductor and capacitor, the power dissipated in the circuit is equal to the power dissipated in the resistor.

Since $\cos\theta = \frac{V_R}{V} = \frac{IR}{V}$

The average power can also be written as

$$P_{av} = \frac{1}{2}I^2 R = I_{RMS}^2 R$$

Example

A 500 Ω resistor, a 60 H inductor and a 0.006 F capacitor are connected in series and then connected to a potential difference that varies with time according to the equation $v = 20\sin(20t)$.

a) Calculate the average power dissipated in the circuit

b) Calculate the average power dissipated across the resistor, inductor and capacitor

Solution

Given: $V = 20\ V;\quad \omega = 20\ rad/s;\quad R = 500\ \Omega;\quad L = 20\ H;\quad C = 0.006\ F$

a) $I = \frac{V}{Z}$

$$= \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$= \frac{20}{\sqrt{500^2 + \left(20*20 - \frac{1}{20*0.06}\right)^2}}$$

$$\underline{= 0.015\ A}|$$

$$P_{av} = \frac{1}{2}I^2R$$

$$= \frac{1}{2}(0.015)^2(500)$$

$$\underline{= 0.06\ W}|$$

b) The power dissipated across the resistor is equal to the power dissipated in the circuit which is 0.06 W. The power dissipated across the inductor is zero because the phase shift of voltage with respect to current is $\frac{\pi}{2}$. The average power dissipated across the capacitor is zero because the phase shift of the voltage with respect to current is $-\frac{\pi}{2}$.