Solution   *Section* 4.4 – Fundamental Theorem of Calculus

Exercise

Evaluate the integral $\displaystyle\int_0^3 (2x+1)dx$

Solution

$$\int_0^3 (2x+1)dx = x^2 + x \Big|_0^3$$

$$= 3^2 + 3 - (0+0)$$

$$= 12$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 x(x-3)dx$

Solution

$$\int_0^2 x(x-3)dx = \int_0^2 \left(x^2 - 3x\right)dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^2$$

$$= \left(\frac{2^3}{3} - \frac{3(2)^2}{2}\right) - \left(\frac{0^3}{3} - \frac{3(2)^2}{2}\right)$$

$$= -\frac{10}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_0^4 \left(3x - \frac{x^3}{4}\right)dx$

Solution

$$\int_0^4 \left(3x - \frac{x^3}{4}\right)dx = \left[3\frac{x^2}{2} - \frac{x^4}{16}\right]_0^4$$

$$= \left(3\frac{(4)^2}{2} - \frac{(4)^4}{16}\right) - 0$$

$$\underline{=8}$$

Exercise

Evaluate the integral $\displaystyle\int_{-2}^{2}\left(x^3 - 2x + 3\right)dx$

Solution

$$\int_{-2}^{2}\left(x^3 - 2x + 3\right)dx = \left[\frac{x^4}{4} - x^2 + 3x\right]_{-2}^{2}$$

$$= \left(\frac{(2)^4}{4} - (2)^2 + 3(2)\right) - \left(\frac{(-2)^4}{4} - (-2)^2 + 3(-2)\right)$$

$$\underline{=12}$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{1}\left(x^2 + \sqrt{x}\right)dx$

Solution

$$\int_{0}^{1}\left(x^2 + \sqrt{x}\right)dx = \left[\frac{x^3}{3} + \frac{2}{3}x^{3/2}\right]_{0}^{1}$$

$$= \left(\frac{(1)^3}{3} + \frac{2}{3}(1)^{3/2}\right) - 0$$

$$\underline{=1}$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{\pi/3} 4\sec u \tan u \; du$

Solution

$$\int_{0}^{\pi/3} 4\sec u \tan u \; du = 4\sec u \Big|_{0}^{\pi/3}$$

$$= 4\left(\sec\frac{\pi}{3} - \sec 0\right)$$

$$= 4(2-1)$$

$$\underline{= 4}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/4}^{3\pi/4} \csc\theta\cot\theta\, d\theta$

Solution

$$\int_{\pi/4}^{3\pi/4} \csc\theta\cot\theta\, d\theta = -\csc\theta\Big|_{\pi/4}^{3\pi/4}$$

$$= -\left(\csc\frac{3\pi}{4} - \csc\frac{\pi}{4}\right)$$

$$= -\left(\sqrt{2} - \sqrt{2}\right)$$

$$\underline{= 0}$$

Exercise

Evaluate the integral $\displaystyle\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt$

Solution

$$\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt = \int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \pi t^{-2}\right) dt$$

$$= \left[4\tan t - \pi t^{-1}\right]_{-\pi/3}^{-\pi/4}$$

$$= \left(4\tan\left(-\frac{\pi}{4}\right) - \pi\left(-\frac{4}{\pi}\right)\right) - \left(4\tan\left(-\frac{\pi}{3}\right) - \pi\left(-\frac{3}{\pi}\right)\right)$$

$$= \left(4(-1) + 4\right) - \left(4\left(-\sqrt{3}\right) + 3\right)$$

$$= -\left(-4\sqrt{3} + 3\right)$$

$$\underline{= 4\sqrt{3} - 3}$$

Exercise

Evaluate the integral $\displaystyle\int_{-3}^{-1} \frac{y^5 - 2y}{y^3}\, dy$

Solution

$$\int_{-3}^{-1} \frac{y^5 - 2y}{y^3}\, dy = \int_{-3}^{-1} \left(\frac{y^5}{y^3} - \frac{2y}{y^3} \right) dy$$

$$= \int_{-3}^{-1} \left(y^2 - 2y^{-2} \right) dy$$

$$= \left[\frac{1}{3} y^3 + 2y^{-1} \right]_{-3}^{-1}$$

$$= \left(\frac{1}{3}(-1)^3 + \frac{2}{-1} \right) - \left(\frac{1}{3}(-3)^3 + \frac{2}{-3} \right)$$

$$= \frac{22}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_{1}^{8} \frac{\left(x^{1/3} + 1\right)\left(2 - x^{2/3}\right)}{x^{1/3}}\, dx$

Solution

$$\int_{1}^{8} \frac{\left(x^{1/3} + 1\right)\left(2 - x^{2/3}\right)}{x^{1/3}}\, dx = \int_{1}^{8} \frac{2x^{1/3} - x + 2 - x^{2/3}}{x^{1/3}}\, dx$$

$$= \int_{1}^{8} \left(2 - x^{2/3} + 2x^{-1/3} - x^{1/3} \right) dx$$

$$= \left[2x - \frac{3}{5} x^{5/3} + 3x^{2/3} - \frac{3}{4} x^{4/3} \right]_{1}^{8}$$

$$= \left(2(8) - \frac{3}{5}(8)^{5/3} + 3(8)^{2/3} - \frac{3}{4}(8)^{4/3} \right) - \left(2(1) - \frac{3}{5}(1)^{5/3} + 3(1)^{2/3} - \frac{3}{4}(1)^{4/3} \right)$$

$$= \left(-\frac{16}{5} \right) - \left(\frac{73}{20} \right)$$

$$= -\frac{137}{20}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x}\,dx$

Solution

$$\begin{aligned}
\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x}\,dx &= \int_{\pi/2}^{\pi} \frac{2\sin x\cos x}{2\sin x}\,dx \\
&= \int_{\pi/2}^{\pi} \cos x dx \\
&= \sin x \Big|_{\pi/2}^{\pi} \\
&= \sin\pi - \sin\frac{\pi}{2} \\
&= -1
\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int_{0}^{\pi/3} (\cos x + \sec x)^2\,dx$

Solution

$$\begin{aligned}
\int_{0}^{\pi/3} (\cos x + \sec x)^2\,dx &= \int_{0}^{\pi/3} \left(\cos^2 x + 2 + \sec^2 x\right)dx \\
&= \int_{0}^{\pi/3} \left(\frac{1}{2} + \frac{1}{2}\cos 2x + 2 + \sec^2 x\right)dx \\
&= \int_{0}^{\pi/3} \left(\frac{5}{2} + \frac{1}{2}\cos 2x + \sec^2 x\right)dx \\
&= \left[\frac{5}{2}x + \frac{1}{4}\sin 2x + \tan x\right]_0^{\pi/3} \\
&= \left(\frac{5}{2}\frac{\pi}{3} + \frac{1}{4}\sin\frac{2\pi}{3} + \tan\frac{\pi}{3}\right) - \left(\frac{5}{2}(0) + \frac{1}{4}\sin(2\cdot 0) + \tan(0)\right) \\
&= \frac{5\pi}{6} + \frac{1}{4}\frac{\sqrt{3}}{2} + \sqrt{3} \\
&= \frac{5\pi}{6} + \frac{9\sqrt{3}}{8}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi} \frac{1}{2}\left(\cos x + |\cos x|\right)dx$

Solution

$$\int_0^{\pi} \frac{1}{2}\left(\cos x + |\cos x|\right)dx = \int_0^{\pi/2} \frac{1}{2}\left(\cos x + \cos x\right)dx + \int_{\pi/2}^{\pi} \frac{1}{2}\left(\cos x - \cos x\right)dx$$

$$= \int_0^{\pi/2} \cos x dx$$

$$= \sin x \Big|_0^{\pi/2}$$

$$\underline{= 1}$$

Exercise

Evaluate the integral $\int_0^{1} 2x\left(4 - x^2\right)dx$

Solution

$$\int_0^{1} 2x\left(4 - x^2\right)dx = \int_0^{1} \left(8x - 2x^3\right)dx$$

$$= 4x^2 - \frac{1}{2}x^4 \Big|_0^1$$

$$= 4 - \frac{1}{2}$$

$$\underline{= \frac{7}{2}}$$

Exercise

Evaluate the integral $\int_0^{4} (8 - 2x)dx$

Solution

$$\int_0^{4} (8 - 2x)dx = 8x - x^2 \Big|_0^4$$

$$= 8(4) - (4)^2 - 0$$

$$\underline{= 16}$$

Exercise

Evaluate the integral $\displaystyle\int_0^4 \frac{1}{\sqrt{16-x^2}}\,dx$

Solution

$$\int_0^4 \frac{1}{\sqrt{16-x^2}}\,dx = \sin^{-1}\frac{x}{4}\Big|_0^4$$

$$= \sin^{-1}\frac{4}{4} - \sin^{-1}0 \qquad \sin^{-1}1 = \frac{\pi}{2}$$

$$= \frac{\pi}{2}$$

Exercise

Evaluate the integral $\displaystyle\int_{-4}^2 (2x+4)\,dx$

Solution

$$\int_{-4}^2 (2x+4)\,dx = x^2 + 4x\Big|_{-4}^2$$

$$= 2^2 + 4(2) - \left(\left(-4^2\right) + 4(-4)\right)$$

$$= 4 + 8 - (16 - 16)$$

$$= 12$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 (1-x)\,dx$

Solution

$$\int_0^2 (1-x)\,dx = x - \frac{1}{2}x^2\Big|_0^2$$

$$= 2 - \frac{1}{2}(2)^2 - 0$$

$$= 0$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 \left(x^2 - 2\right) dx$

Solution

$$\int_0^2 \left(x^2 - 2\right) dx = \frac{1}{3}x^3 - 2x \Big|_0^2$$

$$= \frac{1}{3}(2)^3 - 2(2) - 0$$

$$= \frac{8}{3} - 4$$

$$\boxed{= -\frac{4}{3}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{\pi/2} \cos x \, dx$

Solution

$$\int_0^{\pi/2} \cos x \, dx = \sin x \Big|_0^{\pi/2}$$

$$= \sin\frac{\pi}{2} - \sin 0$$

$$\boxed{= 1}$$

Exercise

Evaluate the integral $\displaystyle\int_1^7 \frac{dx}{x}$

Solution

$$\int_1^7 \frac{dx}{x} = \ln|x| \Big|_1^7$$

$$= \ln 7 - \ln 1$$

$$\boxed{= \ln 7}$$

Exercise

Evaluate the integral $\displaystyle\int_4^9 3\sqrt{x}\, dx$

Solution

$$\int_4^9 3\sqrt{x}\,dx = 2x^{3/2}\Big|_4^9$$

$$= 2\left((9)^{3/2} - (4)^{3/2}\right)$$

$$= 2(27-8)$$

$$\underline{= 38}$$

Exercise

Evaluate the integral $\int_{-2}^{3}\left(x^2 - x - 6\right)dx$

Solution

$$\int_{-2}^{3}\left(x^2 - x - 6\right)dx = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x\Big|_{-2}^{3}$$

$$= \frac{1}{3}3^3 - \frac{1}{2}3^2 - 6(3) - \left(\frac{1}{3}2^3 - \frac{1}{2}2^2 - 6(2)\right)$$

$$= 9 - \frac{9}{2} - 18 - \left(\frac{8}{3} - 2 - 12\right)$$

$$= -\frac{27}{2} + \frac{46}{3}$$

$$\underline{= \frac{11}{6}}$$

Exercise

Evaluate the integral $\int_0^1\left(1 - \sqrt{x}\right)dx$

Solution

$$\int_0^1\left(1 - \sqrt{x}\right)dx = \int_0^1\left(1 - x^{1/2}\right)dx$$

$$= x - \frac{2}{3}x^{3/2}\Big|_0^1$$

$$= 1 - \frac{2}{3}$$

$$\underline{= \frac{1}{3}}$$

Exercise

Evaluate the integral $\int_0^{\pi/4} 2\cos x\, dx$

Solution

$$\int_0^{\pi/4} 2\cos x\, dx = 2\sin x\Big|_0^{\pi/4}$$

$$= 2\left(\sin\frac{\pi}{4} - \sin 0\right)$$

$$\underline{= \sqrt{2}|}$$

Exercise

Evaluate the integral $\int_{-\pi/4}^{7\pi/4} (\sin x + \cos x)\, dx$

Solution

$$\int_{-\pi/4}^{7\pi/4} (\sin x + \cos x)\, dx = -\cos x + \sin x\Big|_{-\pi/4}^{7\pi/4}$$

$$= -\cos\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right) - \left(-\cos\left(\frac{7\pi}{4}\right) + \sin\left(\frac{7\pi}{4}\right)\right)$$

$$= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$$

$$= -\sqrt{2} + \sqrt{2}$$

$$\underline{= 0|} \qquad \textit{or} \quad \text{since } -\frac{\pi}{4} = \frac{7\pi}{4} \quad \int_a^a f(x)\, dx = 0$$

Exercise

Evaluate the integral $\int_0^{\ln 8} e^x dx$

Solution

$$\int_0^{\ln 8} e^x dx = e^x\Big|_0^{\ln 8}$$

$$= e^{\ln 8} - e^0$$

$$= 8 - 1$$

$$\underline{= 7|}$$

Exercise

Evaluate the integral $\displaystyle\int_1^4 \left(\frac{x-1}{x}\right)dx$

Solution

$$\int_1^4 \left(\frac{x-1}{x}\right)dx = \int_1^4 \left(1-\frac{1}{x}\right)dx$$

$$= x - \ln|x| \Big|_1^4$$

$$= 4 - \ln 4 - (1 - \ln 1)$$

$$= 4 - \ln 2^2 - 1$$

$$= 3 - 2\ln 2$$

Exercise

Evaluate the integral $\displaystyle\int_{-2}^{-1} \left(3e^{3x} + \frac{2}{x}\right)dx$

Solution

$$\int_{-2}^{-1} \left(3e^{3x} + \frac{2}{x}\right)dx = e^{3x} + 2\ln|x| \Big|_{-2}^{-1}$$

$$= e^{-3} + 2\ln|-1| - \left(e^{-6} - 2\ln|-2|\right)$$

$$= e^{-3} + 2\ln 1 - e^{-6} + 2\ln 2$$

$$= e^{-3} - e^{-6} + 2\ln 2$$

Exercise

Evaluate the integral $\displaystyle\int_0^2 \frac{dx}{x^2+4}$

Solution

$$\int_0^2 \frac{dx}{x^2+4} = \frac{1}{2}\tan^{-1}\frac{x}{2} \Big|_0^2$$

$$= \frac{1}{2}\left(\tan^{-1}1 - \tan^{-1}0\right)$$

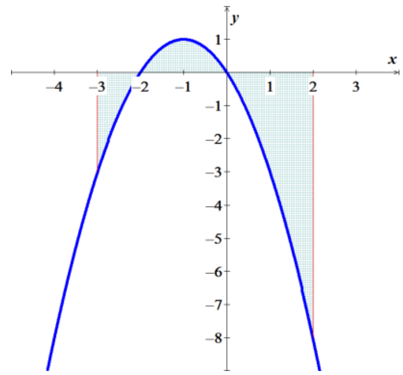
$$= \frac{1}{2}\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{8}$$

Exercise

Find the total area between the region and the x-axis $\qquad y=-x^2-2x, \quad -3\le x\le 2$

Solution



$$-x^2-2x=0 \to -x(x+2)=0 \quad \Rightarrow \boxed{x=-2,0}$$

$$A=-\int_{-3}^{-2}\left(-x^2-2x\right)dx+\int_{-2}^{0}\left(-x^2-2x\right)dx-\int_{0}^{2}\left(-x^2-2x\right)dx$$

$$=-\left[-\frac{1}{3}x^3-x^2\right]_{-3}^{-2}+\left[-\frac{1}{3}x^3-x^2\right]_{-2}^{0}-\left[-\frac{1}{3}x^3-x^2\right]_{0}^{2}$$

$$=-\left[\left(-\tfrac{1}{3}(-2)^3-(-2)^2\right)-\left(-\tfrac{1}{3}(-3)^3-(-3)^2\right)\right]+\left[-\left(-\tfrac{1}{3}(-2)^3-(-2)^2\right)\right]-\left[\left(-\tfrac{1}{3}(2)^3-(2)^2\right)\right]$$

$$=\frac{4}{3}+\frac{4}{3}+\frac{20}{3}$$

$$=\frac{28}{3}\ \ unit^2$$

Exercise

Find the total area between the region and the x-axis $y=x^3-3x^2+2x, \quad 0\le x\le 2$

Solution

$$x^3-3x^2+2x=0 \quad \boxed{x=0,\ 1,\ 2}$$

$$A=\int_{0}^{1}\left(x^3-3x^2+2x\right)dx-\int_{1}^{2}\left(x^3-3x^2+2x\right)dx$$

$$=\left[\frac{1}{4}x^4-x^3+x^2\right]_{0}^{1}-\left[\frac{1}{4}x^4-x^3+x^2\right]_{1}^{2}$$

$$=\frac{1}{4}+\frac{1}{4}$$

$$=\frac{1}{2}\ \ unit^2$$

Exercise

Find the total area between the region and the x-axis $y=x^{1/3}-x, \quad -1\le x\le 8$

Solution

$$x^{1/3}-x=0 \to x^{1/3}\left(1-x^{2/3}\right)=0 \quad \boxed{x=0,\ \pm 1}$$

$$A = -\int_{-1}^{0}\left(x^{1/3} - x\right)dx + \int_{0}^{1}\left(x^{1/3} - x\right)dx - \int_{1}^{8}\left(x^{1/3} - x\right)dx$$

$$= -\left[\frac{3}{4}x^{4/3} - \frac{1}{2}x^2\right]_{-1}^{0} + \left[\frac{3}{4}x^{4/3} - \frac{1}{2}x^2\right]_{0}^{1} - \left[\frac{3}{4}x^{4/3} - \frac{1}{2}x^2\right]_{1}^{8}$$

$$= \left(\frac{3}{4} - \frac{1}{2}\right) + \left(\frac{3}{4} - \frac{1}{2}\right) - \left[(12 - 32) - \left(\frac{3}{4} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{81}{4}$$

$$= \frac{83}{4}\ \ unit^2$$

Exercise

Find the total area between the region and the x-axis $f(x) = x^2 + 1, \quad 2 \le x \le 3$

Solution

$$Area = \int_{2}^{3}\left(x^2 + 1\right)dx$$

$$= \left[\frac{1}{3}x^3 + x\right]_{2}^{3}$$

$$= \left(\frac{1}{3}3^3 + 3\right) - \left(\frac{1}{3}2^3 + 2\right)$$

$$= (9 + 3) - \left(\frac{8}{3} + 2\right)$$

$$= 12 - \left(\frac{14}{3}\right)$$

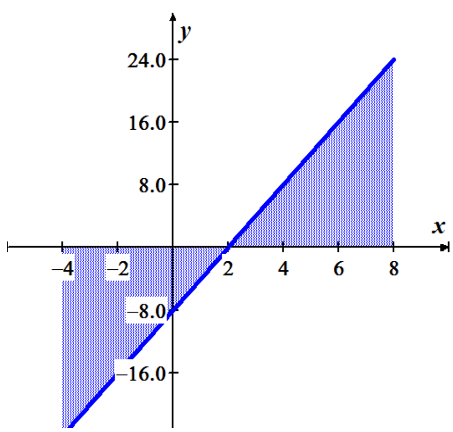
$$= \frac{22}{3}\ \ unit^2$$

Exercise

Find the area of the region between the graph of $y = 4x - 8$ and the x-axis, for $-4 \le x \le 8$

Solution

$y = 4x - 8 = 0 \quad \Rightarrow \quad \underline{x = 2}$



$$Area = \int_{-4}^{2}(4x - 8)\,dx + \int_{2}^{8}(4x - 8)\,dx$$

$$= \left|2x^2 - 8x\Big|_{-4}^{2}\right| + 2x^2 - 8x\Big|_{2}^{8}$$

$$= \left|8 - 16 - (32 + 32)\right| + \left(128 - 64 - (8 - 16)\right)$$

$$= |-72| + 72$$

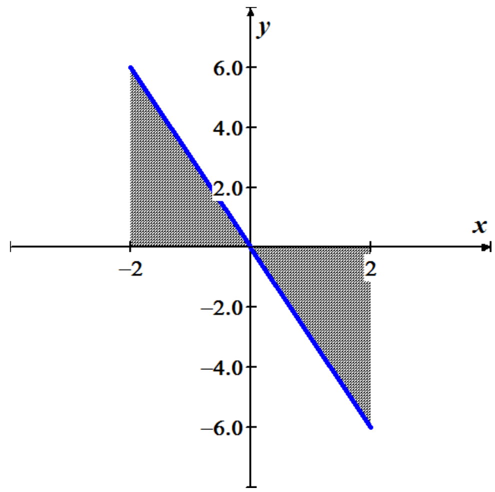
$$= 144 \text{ unit}^2$$

Exercise

Find the area of the region between the graph of $y = -3x$ and the x-*axis*, for $-2 \le x \le 2$

Solution

$$y = -3x = 0 \to \underline{x = 0}$$



$$Area = \int_{-2}^{0} (-3x)\,dx + \int_{0}^{2} (-3x)\,dx$$

$$= 2\int_{-2}^{0} (-3x)\,dx$$

$$= 3\left(-x^2\right)\Big|_{-2}^{0}$$

$$= 3(0+4)$$

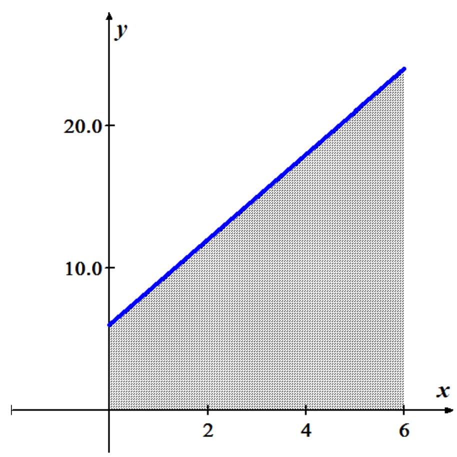
$$= 12 \text{ unit}^2$$

Exercise

Find the area of the region between the graph of $y = 3x + 6$ and the x-*axis*, for $0 \le x \le 6$

Solution

$$y = 3x + 6 = 0 \to \underline{x = -2}$$



$$Area = \int_{0}^{6} (3x+6)\,dx$$

$$= \frac{3}{2}x^2 + 6x\Big|_{0}^{6}$$

$$= \frac{3}{2}(36) + 36 - 0$$

$$= 90 \text{ unit}^2$$

Exercise

Find the area of the region between the graph of $y = 1 - |x|$ and the x-*axis*, for $-2 \le x \le 2$

Solution

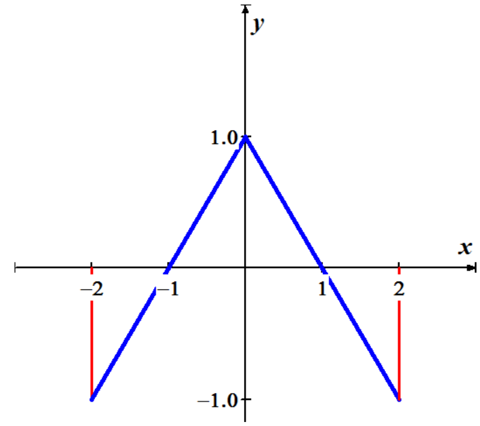
$$y = 1 - x = 0 \to \underline{x = 1}$$

$$Area = 2\int_0^1 (1-x)\,dx + 2\int_1^2 (1-x)\,dx$$

$$= 2\left(x - \frac{1}{2}x^2\right)\Big|_0^1 + 2\left|\left(x - \frac{1}{2}x^2\right)\Big|_1^2\right|$$

$$= 4\left(1 - \frac{1}{2}\right)$$

$$= 2 \ \ unit^2$$

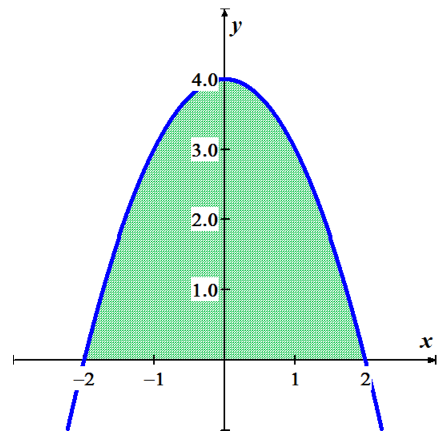


Exercise

Find the area of the region above the *x-axis* bounded by $y = 4 - x^2$

Solution

$$y = 4 - x^2 = 0 \ \Rightarrow \ x = \pm 2$$



$$Area = \int_{-2}^{2} \left(4 - x^2\right) dx$$

$$= 4x - \frac{1}{3}x^3 \Big|_{-2}^{2}$$

$$= 8 - \frac{8}{3} - \left(-8 + \frac{8}{3}\right)$$

$$= 2\left(\frac{16}{3}\right)$$

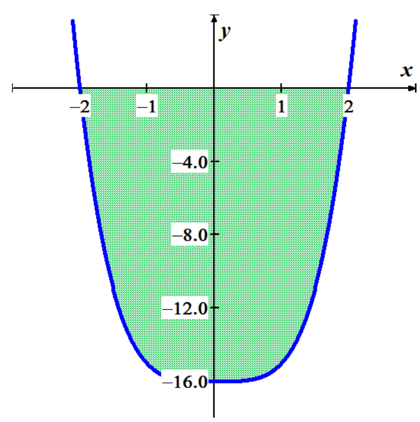
$$= \frac{32}{3} \ \ unit^2$$

Exercise

Find the area of the region above the *x-axis* bounded by $y = x^4 - 16$

Solution

$$y = x^4 - 16 = 0 \ \Rightarrow \ x = \pm 2$$



$$Area = \left|\int_{-2}^{2} \left(x^4 - 16\right) dx\right| = -\int_{2}^{2} \left(x^4 - 16\right) dx$$

$$= -\frac{1}{5}x^5 + 16x \Big|_{-2}^{2}$$

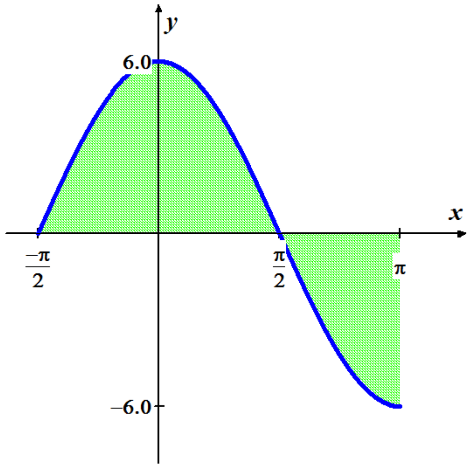
$$= -\frac{32}{5} + 32 - \left(\frac{32}{5} - 32\right)$$

$$= \frac{256}{5} \ \ unit^2$$

Exercise

Find the area of the region between the graph of $y = 6\cos x$ and the *x-axis*, for $-\frac{\pi}{2} \le x \le \pi$

Solution



$$y = 6\cos x = 0 \ \Rightarrow \ \underline{x = \pm\frac{\pi}{2}}$$

$$Area = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (6\cos x)\,dx + \int_{\frac{\pi}{2}}^{\pi} (-6\cos x)\,dx$$

$$= 6\sin x \Big|_{-\pi/2}^{\pi/2} - 6\sin x \Big|_{\pi/2}^{\pi}$$

$$= 6\left(\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right) - 6\left(\sin\pi - \sin\left(\frac{\pi}{2}\right)\right)$$

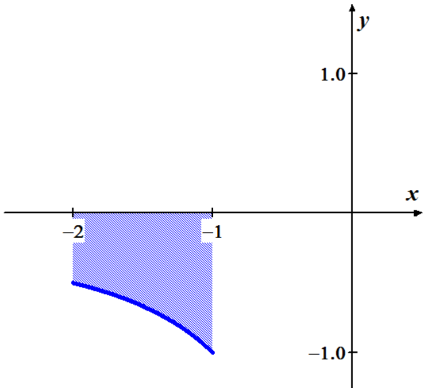
$$= 6(1+1) - 6(0-1)$$

$$\underline{= 18 \ \ unit^2}$$

Exercise

Find the area of the region between the graph of $f(x) = \frac{1}{x}$ and the *x-axis*, for $-2 \le x \le -1$

Solution



$$Area = -\int_{-2}^{-1} \frac{1}{x}\,dx$$

$$= -\ln|x| \Big|_{-2}^{-1}$$

$$= -\ln|-1| + \ln|-2|$$

$$= -\ln 1 + \ln 2$$

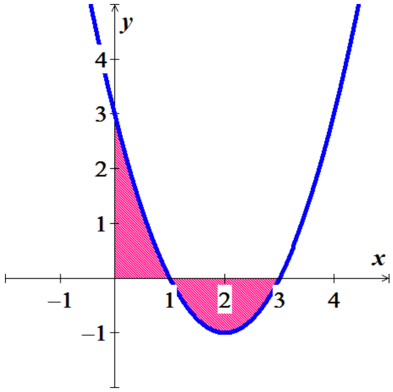
$$\underline{= \ln 2 \ \ unit^2}$$

Exercise

Find the area of the region bounded by the graph of

$f(x) = x^2 - 4x + 3 \quad x\text{-axis} \quad on \ 0 \le x \le 3$

Solution



$$f(x) = x^2 - 4x + 3 = 0 \quad \rightarrow \quad \underline{x = 1, \ 3}$$

$$A = \int_0^1 \left(x^2 - 4x + 3\right)dx - \int_1^3 \left(x^2 - 4x + 3\right)dx$$

$$= \frac{1}{3}x^3 - 2x^2 + 3x\Big|_0^1 - \left(\frac{1}{3}x^3 - 2x^2 + 3x\right)\Big|_1^3$$

$$= \frac{1}{3} - 2 + 3 - \left(9 - 18 + 9 - \frac{1}{3} + 2 - 3\right)$$

$$\boxed{= \frac{8}{3}\ \ unit^2}$$

Exercise

Find the area of the region bounded by the graph of $f(x) = x^2 + 4x + 3$ *x-axis on* $-3 \le x \le 0$

Solution

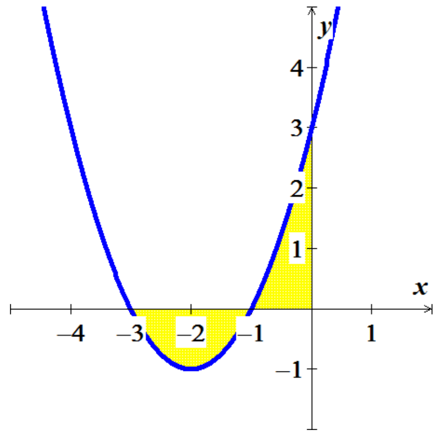
$$f(x) = x^2 + 4x + 3 = 0 \quad \to \quad \boxed{x = -1,\ -3}$$

$$A = -\int_{-3}^{-1}\left(x^2 + 4x + 3\right)dx + \int_{-1}^{0}\left(x^2 + 4x + 3\right)dx$$

$$= -\left(\frac{1}{3}x^3 + 2x^2 + 3x\right)\Big|_{-3}^{-1} + \left(\frac{1}{3}x^3 + 2x^2 + 3x\right)\Big|_{-1}^{0}$$

$$= -\left(-\frac{1}{3} + 2 - 3 + 9 - 18 + 9\right) + \frac{1}{3} - 2 + 3$$

$$\boxed{= \frac{8}{3}\ \ unit^2}$$



Exercise

Find the area of the region bounded by the graph of $f(x) = x^2 - 3x + 2$ *x-axis on* $0 \le x \le 2$

Solution

$$f(x) = x^2 - 3x + 2 = 0 \quad \to \quad \boxed{x = 1,\ 2}$$

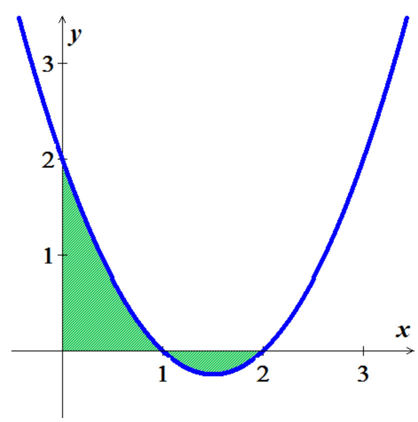
$$A = \int_{0}^{1}\left(x^2 - 3x + 2\right)dx - \int_{1}^{2}\left(x^2 - 3x + 2\right)dx$$

$$= \left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right)\Big|_0^1 - \left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right)\Big|_1^2$$

$$= \frac{1}{3} - \frac{3}{2} + 2 - \left(\frac{8}{3} - 6 + 4 - \frac{1}{3} + \frac{3}{2} - 2\right)$$

$$= -\frac{7}{6} + 2 - \left(\frac{8}{3} - 2 + \frac{7}{6} - 2\right)$$

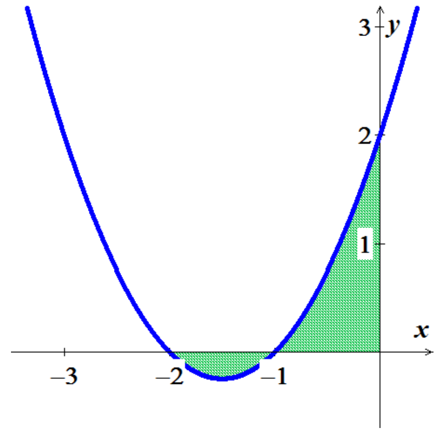
$$\boxed{= 1\ \ unit^2}$$



Exercise

Find the area of the region bounded by the graph of $f(x)=x^2+3x+2$ *x-axis on* $-2\le x\le 0$

Solution



$$f(x)=x^2+3x+2=0 \quad\rightarrow\quad \underline{x=-1,\ -2}$$

$$A=-\int_{-2}^{-1}\left(x^2+3x+2\right)dx+\int_{-1}^{0}\left(x^2+3x+2\right)dx$$

$$=-\left(\frac{1}{3}x^3+\frac{3}{2}x^2+2x\right)\Big|_{-2}^{-1}+\left(\frac{1}{3}x^3+\frac{3}{2}x^2+2x\right)\Big|_{-1}^{0}$$

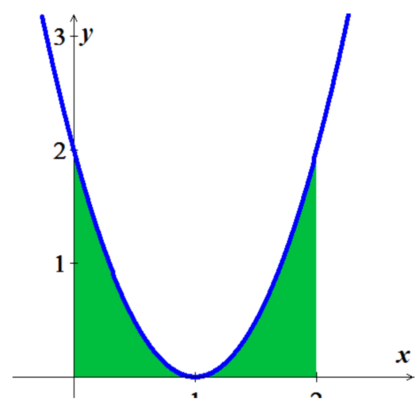
$$=-\left(\frac{8}{3}-6+4-\frac{1}{3}+\frac{3}{2}-2\right)+\frac{1}{3}-\frac{3}{2}+2$$

$$=1\ \ unit^2$$

Exercise

Find the area of the region bounded by the graph of $f(x)=2x^2-4x+2$ *x-axis on* $0\le x\le 2$

Solution



$$f(x)=2x^2-4x+2=0 \quad\rightarrow\quad \underline{x=1}$$

$$A=\int_{0}^{1}\left(2x^2-4x+2\right)dx+\int_{1}^{2}\left(2x^2-4x+2\right)dx$$

$$=\left(\frac{2}{3}x^3-2x^2+2x\right)\Big|_{0}^{1}+\left(\frac{2}{3}x^3-2x^2+2x\right)\Big|_{1}^{2}$$

$$=\frac{2}{3}-2+2+\frac{16}{3}-8+4-\frac{2}{3}+2-2$$

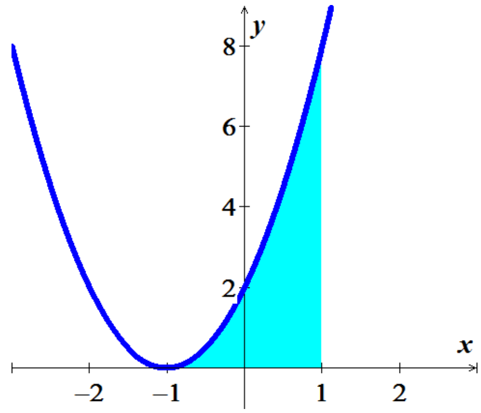
$$=\frac{4}{3}\ \ unit^2$$

Exercise

Find the area of the region bounded by the graph of $f(x)=2x^2+4x+2$ *x-axis on* $-1\le x\le 1$

Solution

$$f(x)=2x^2+4x+2=0 \quad\rightarrow\quad \underline{x=-1}$$



$$A=\int_{-1}^{1}\left(2x^2+4x+2\right)dx$$

$$=\left(\frac{2}{3}x^3+2x^2+2x\right)\Big|_{-1}^{1}$$

$$=\frac{2}{3}+2+2+\frac{2}{3}-2+2$$

$$=\frac{16}{3}\ \ unit^2$$

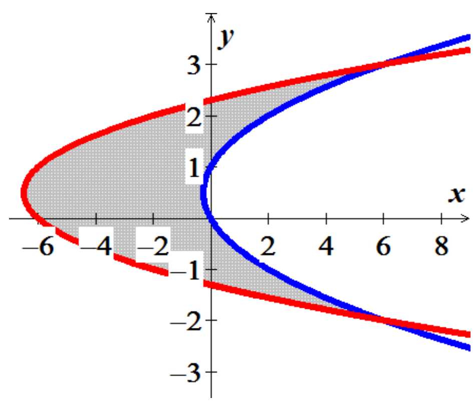
Exercise

Find the area of the region bounded by the graphs of $x=y^2-y$ and $x=2y^2-2y-6$

Solution

$$x=2y^2-2y-6=y^2-y$$

$$y^2-y-6=0 \quad\rightarrow\quad \underline{y=-2,\ 3}$$



$$A=\int_{-2}^{3}\left(y^2-y-2y^2+2y+6\right)dy$$

$$=\int_{-2}^{3}\left(-y^2+y+6\right)dy$$

$$=-\frac{1}{3}y^3+\frac{1}{2}y^2+6y\Big|_{-2}^{3}$$

$$=-9+\frac{9}{2}+18-\frac{8}{3}-2+12$$

$$=19+\frac{11}{6}$$

$$=\frac{125}{6}\ \ unit^2$$

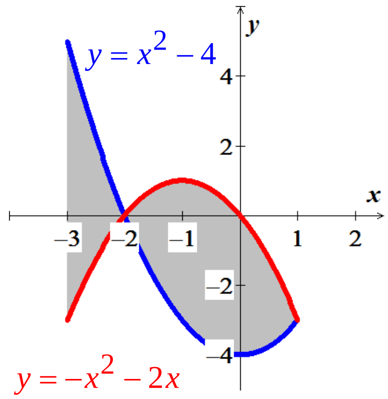
Exercise

Find the area of the region bounded by the graphs of $y = x^2 - 4$ & $y = -x^2 - 2x$ on $-3 \le x \le 1$

Solution

$$y = x^2 - 4 = -x^2 - 2x$$

$$2x^2 + 2x - 4 = 0 \quad \to \quad \underline{x = 1, \ -2}$$



$$A = \int_{-3}^{-2} \left(x^2 - 4 + x^2 + 2x\right)dx + \int_{-2}^{1} \left(-x^2 - 2x - x^2 + 4\right)dx$$

$$= \int_{-3}^{-2} \left(2x^2 + 2x - 4\right)dx + \int_{-2}^{1} \left(-2x^2 - 2x + 4\right)dx$$

$$= \left(\frac{2}{3}x^3 + x^2 - 4x\right)\Big|_{-3}^{-2} + \left(-\frac{2}{3}x^3 - x^2 + 4x\right)\Big|_{-2}^{1}$$

$$= -\frac{16}{3} + 12 + 18 - 21 + \left(-\frac{2}{3} + 3\right) - \frac{16}{3} + 12$$

$$= -\frac{34}{3} + 24$$

$$= \frac{38}{3} \ \ unit^2$$

Exercise

Compute the area of the region bounded by the graph of f and the x-*axis* on the given interval.

$$f(x) = \frac{1}{x^2 + 1} \quad on \ \left[-1, \ \sqrt{3}\right]$$

Solution

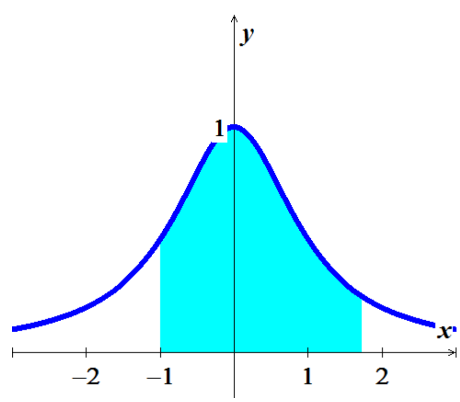
$$A = \int_{-1}^{\sqrt{3}} \frac{1}{x^2 + 1}\, dx \qquad \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \tan^{-1} x \Big|_{-1}^{\sqrt{3}}$$

$$= \tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(1)$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12} \ \ unit^2$$

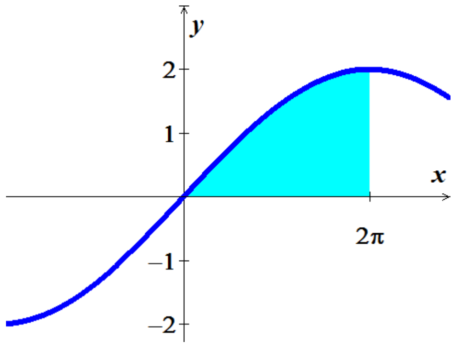


Exercise

Compute the area of the region bounded by the graph of f and the x-*axis* on the given interval.

$$f(x) = 2\sin\frac{x}{4} \quad on \quad [0,\ 2\pi]$$

Solution



$$A = \int_0^{2\pi} 2\sin\frac{x}{4}\, dx$$

$$= -8\cos\frac{x}{4}\Big|_0^{2\pi}$$

$$= -8(0-1)$$

$$= 8\ \ unit^2$$

Exercise

Archimedes, inventor, military engineer, physicist, and the greatest mathematician of classical times in the Western world, discovered that the area under a parabolic arch is two-thirds the base times the height. Sketch the parabolic arch $y = h - \left(\frac{4h}{b^2}\right)x^2 \quad -\frac{b}{2} \le x \le \frac{b}{2}$, assuming that h and b are positive. Then use calculus to find the area of the region enclosed between the arch and the x-axis

Solution

$$A = \int_{-b/2}^{b/2} \left(h - \left(\frac{4h}{b^2}\right)x^2\right)dx$$

$$= \left[hx - \frac{4h}{b^2}\frac{x^3}{3}\right]_{-b/2}^{b/2}$$

$$= \left(\frac{hb}{2} - \frac{4h}{3b^2}\frac{b^3}{8}\right) - \left(-\frac{hb}{2} + \frac{4h}{3b^2}\frac{b^3}{8}\right)$$

$$= \left(\frac{hb}{2} - \frac{hb}{6}\right) - \left(-\frac{hb}{2} + \frac{hb}{6}\right)$$

$$= \frac{hb}{3} + \frac{hb}{3}$$

$$= \frac{2}{3}bh\ \ unit^2$$

Exercise

Suppose that a company's marginal revenue from the manufacture and sale of eggbeaters is

$$\frac{dr}{dx} = 2 - \frac{2}{(x+1)^2}$$

Where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand eggbeaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

Solution

$$r = \int_0^3 \left(2 - 2(x+1)^{-2}\right)dx \qquad\qquad d(x+1) = dx$$

$$= \int_0^3 2dx - \int_0^3 2(x+1)^{-2}\, d(x+1)$$

$$= 2x\Big|_0^3 + 2(x+1)^{-1}\Big|_0^3$$

$$= 6 + 2(4)^{-1} - 2$$

$$\underline{= 4.5} \quad \rightarrow \quad \underline{\$4500.00}$$

Exercise

The height H (*feet*) of a palm tree after growing for t years is given by

$$H = \sqrt{t+1} + 5t^{1/3} \quad \textit{for} \quad 0 \le t \le 8$$

a) Find the tree's height when $t = 0$, $t = 4$, and $t = 8$.

b) Find the tree's average height for $0 \le t \le 8$

Solution

a) $t = 0 \ \Rightarrow\ H = 1\, ft$

$t = 4 \ \Rightarrow\ H = 10.17\, ft$

$t = 8 \ \Rightarrow\ H = 13\, ft$

b) Average height $= \frac{1}{8-0}\int_0^8 \left(\sqrt{t+1} + 5t^{1/3}\right)dt \qquad\qquad d(t+1) = dt$

$$= \frac{1}{8}\int_0^8 (t+1)^{1/2}\, d(t+1) + \frac{5}{8}\int_0^8 t^{1/3}dt$$

$$= \left[\frac{1}{12}(t+1)^{3/2} + \frac{15}{32}t^{4/3}\right]_0^8$$

$$= \frac{1}{12}(9)^{3/2} + \frac{15}{32}(8)^{4/3} - \frac{1}{12}$$

$$\underline{\approx 9.67\ ft}$$