Section 1.4 – Inverse Matrices - Finding A^{-1}

Definition

The matrix A is invertible if there exists a matrix A^{-1} such that:

$$A^{-1}A = AA^{-1} = I$$

where A^{-1} *read as* "A ***inverse***" and A has to be a ***square matrix***.

Not all matrices have inverses.

1. The inverse exists *iff* elimination produces n pivots (row exchanges allow).

2. The matrix A cannot have two different inverses.

3. If A is invertible, the one and only one solution to $Ax = B$ is $x = A^{-1}B$

$$\begin{aligned} AX &= B \\ A^{-1}(AX) &= A^{-1}B && \textit{Multiply both side by } A^{-1} \\ \left(A^{-1}A\right)X &= A^{-1}B && \textit{Associate property} \\ IX &= A^{-1}B && \textit{Multiplicative inverse property} \\ X &= A^{-1}B && \textit{Identity property} \end{aligned}$$

4. Suppose there is a ***nonzero*** vector x such that $Ax = 0$. Then A cannot have an inverse

5. A 2 by 2 matrix is invertible iff $ad - bc$ is not zero.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \textit{Only for 2 by 2 matrices}$$

If $ad - bc = 0$ is the determinant, then A^{-1} doesn't exist

The Inverse of a Product AB

Theorem

If an $n \times n$ matrix has an inverse, that inverse is unique.

Proof

Suppose that A has an inverse A^{-1} and B is a matrix such that $BA = I$

$$\begin{aligned} B &= BI \\ &= B\left(AA^{-1}\right) \\ &= (BA)A^{-1} \\ &= IA^{-1} \\ &= A^{-1} \end{aligned}$$

Therefore, the inverse is unique

Theorem

If A and B are invertible then so is AB. The inverse of a product AB is $(AB)^{-1} = B^{-1}A^{-1}$

Proof

$$\begin{aligned} (AB)\left(B^{-1}A^{-1}\right) &= A\left(BB^{-1}\right)A^{-1} \\ &= (AI)A^{-1} \\ &= AA^{-1} \\ &= \underline{I} \end{aligned}$$

Reverse Order

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

$$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$$

Theorem

If A is invertible and n is a nonnegative integer, then:

a) A^{-1} is *invertible* and $\left(A^{-1}\right)^{-1} = A$

b) A^{n} is *invertible* and $\left(A^{n}\right)^{-1} = A^{-n} = \left(A^{-1}\right)^{n}$

c) kA is *invertible* for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$

Proof

$$
\begin{aligned}
(kA)\left(k^{-1}A^{-1}\right) &= k^{-1}(kA)A^{-1} \\
&= \left(k^{-1}k\right)AA^{-1} \\
&= (1)I \\
&= I
\end{aligned}
$$

$$
\begin{aligned}
\left(k^{-1}A^{-1}\right)(kA) &= k^{-1}\left(kA^{-1}\right)A \\
&= \left(k^{-1}k\right)A^{-1}A \\
&= (1)I \\
&= I
\end{aligned}
$$

Finding A^{-1} using Gauss-Jordan Elimination

$$\left[A \mid I\right] \rightarrow \left[I \mid A^{-1}\right]$$

Find A^{-1} *if* $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccccc} 2 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -2 & 0 & 0 \\ \hline 0 & -2 & -3 & -2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 3 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & -3 & 0 & 0 \\ \hline 0 & 0 & -3 & -3 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \; -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \; -\tfrac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - \frac{3}{2}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ \hline 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array} \qquad \begin{array}{cccccc} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ \hline 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

- ✓ Matrix ***A*** is ***symmetric*** across its main diagonal. So is A^{-1}
- ✓ Matrix ***A*** is ***tridiagonal*** (only three nonzero diagonals). But A^{-1} is a full matrix with no zeros. (another reason we don't compute A^{-1})

Singular *versus* Invertible

A^{-1} exists when A has a full set of n pivots. (Row exchanges allowed)

- With n pivots, elimination solves all the equations $Ax_i = b_i$. The columns x_i go into A^{-1}. Then $AA^{-1} = I$ is at least a ***right-inverse***.
- Elimination is really a sequence of multiplications.

Conclusion

- If A doesn't have n pivots, elimination will lead to a ***zero row***.
- Elimination steps are taken by an invertible M. So a row of MA is zero.
- If $AB = I$ then $MAB = M$. The zero row of MA, times B, gives a zero row of M.
- The invertible matrix M can't have a zero row! A must have n pivots if $AB = I$.

Elementary Matrices

Definition

Let ***e*** be an elementary row operation. Then the $n \times n$ ***elementary matrix*** E associated with ***e*** is the matrix obtained by applying ***e*** to the $n \times n$ identity matrix. Thus

$$E = eI$$

Example

a) $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \rightarrow$ *Multiply* R_2 *of* I *by* -3

b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow$ *Multiply the third row by* -5

c) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ *Interchange the first and second rows*

d) $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ *Add* -3 *times* R_1 *to* R_2

Theorem

Let e be an elementary operation and let E be the corresponding elementary matrix $E = e(I)$. Then for every $m \times n$ matrix A

$$e(A) = EA$$

That is, an elementary row operation can be performed on A by multiplying A on the left by the corresponding elementary matrix.

Example $m \times m$

Let $A = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$ $M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

$$MA = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 & 6 & 2 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$

This result can be obtained from A by multiplying the first row by 2.

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & 3 & 1 \\ -6 & 2 & 1 & 5 \\ 1 & 0 & 2 & 4 \end{bmatrix}$$

This result can be obtained from A by interchanging rows 2 and 3.

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ 0 & -4 & 10 & 8 \end{bmatrix}$$

This result can be obtained from A by adding 3 times row 1 to row 3.

Uniqueness of Echelon Form

Two matrices *A* and *B* are row-equivalent if and only if they have the same reduced echelon form.

Proof

If *A* and *B* have the same reduced echelon form *E*, then *A* is row-equivalent to *E* and *E* is row-equivalent to *B*. It follows that *A* is row-equivalent to *B*.
Now Suppose *A* and *B* are row-equivalent. Let E_1 be a reduced echelon form of *A* and E_2 be a reduced echelon form of *B*. Then E_1 and E_2 are row equivalent.
Suppose $E_1 = IF_1 \quad and \quad E_2 = IF_2$. Since E_1 and E_2 are row equivalent, $E_2 = CE_1$ for some matrix *C*. This means $I = CI$ and $F_2 = CF_1$. But then $C = I \quad and \quad F_2 = F_1$.

Example

Show that the two matrices are row equivalent

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

Solution

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{matrix} R_1 + R_2 \\ R_2 - 2R_1 \end{matrix}$$

$$= \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

$$\underline{= B}$$

Definition

A relationship ~ (equivalent) between elements of a set is called an equivalence relation if

- ✓ *A* ~ *A* is always true,
- ✓ *A* ~ *B* always implies *B* ~ *A*,
- ✓ *A* ~ *B* and *B* ~ *C* always implies *A* ~ *C*.

Exercises *Section* 1.4 – Inverse Matrices - Finding A^{-1}

1. Apply Gauss-Jordan method to find the inverse of this triangular "Pascal matrix"

Triangular Pascal matrix $$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

2. If A is invertible and $AB = AC$, prove that $B = C$

3. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find two matrices $B \neq C$ such that $AB = AC$

4. If A has ***row*** 1 + ***row*** 2 = ***row*** 3, show that A is not invertible

a) Explain why $Ax = (1,\ 0,\ 0)$ can't have a solution.

b) Which right sides $(b_1,\ b_2,\ b_3)$ might allow a solution to $Ax = b$

c) What happens to ***row*** 3 in elimination?

5. True or false (with a counterexample if false and a reason if true):

a) A 4 by 4 matrix with a row of zeros is not invertible.
b) A matrix with 1's down the main diagonal is invertible.
c) If A is invertible then A^{-1} is invertible.
d) If A is invertible then A^2 is invertible.

6. Do there exist 2 by 2 matrices A and B with real entries such that $AB - BA = I$, where I is the identity matrix?

7. If B is the inverse of A^2, show that AB is the inverse of A.

8. Find and check the inverses (assuming they exist) of these block matrices.

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}$$

9. For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

10. Find A^{-1} and B^{-1} (if they exist) by elimination.

$$A=\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}\quad B=\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}$$

11. Find A^{-1} using the Gauss-Jordan method, which has a remarkable inverse.

$$A=\begin{bmatrix}1&-1&1&-1\\0&1&-1&1\\0&0&1&-1\\0&0&0&1\end{bmatrix}$$

Find the inverse, if exists, of

12. $\begin{bmatrix}6&-4\\-3&2\end{bmatrix}$

13. $\begin{bmatrix}1&4\\2&7\end{bmatrix}$

14. $\begin{bmatrix}-3&6\\4&5\end{bmatrix}$

15. $A=\begin{bmatrix}2&-6\\1&-2\end{bmatrix}$

16. $A=\begin{bmatrix}10&-2\\-5&1\end{bmatrix}$

17. $A=\begin{bmatrix}-2&3\\-3&4\end{bmatrix}$

18. $A=\begin{bmatrix}a&b\\3&3\end{bmatrix}$

19. $A=\begin{bmatrix}-2&a\\4&a\end{bmatrix}$

20. $A=\begin{bmatrix}4&4\\b&a\end{bmatrix}$

21. $A=\begin{pmatrix}-1&2\\-2&1\end{pmatrix}$

22. $A=\begin{pmatrix}1&-2\\2&-1\end{pmatrix}$

23. $A=\begin{pmatrix}2&4\\3&-1\end{pmatrix}$

24. $A=\begin{pmatrix}-1&3\\2&-1\end{pmatrix}$

25. $A=\begin{pmatrix}1&3\\-2&5\end{pmatrix}$

26. $A=\begin{pmatrix}4&6\\2&3\end{pmatrix}$

27. $A=\begin{pmatrix}-6&9\\2&-3\end{pmatrix}$

28. $A=\begin{pmatrix}-2&7\\0&2\end{pmatrix}$

29. $A=\begin{pmatrix}4&-16\\1&-4\end{pmatrix}$

30. $A=\begin{pmatrix}2&1\\1&1\end{pmatrix}$

31. $A=\begin{pmatrix}2&1\\a&a\end{pmatrix}$

32. $A=\begin{pmatrix}b&3\\b&2\end{pmatrix}$

33. $A=\begin{pmatrix}1&a\\3&a\end{pmatrix}$

34. $A=\begin{pmatrix}a&2\\2&a\end{pmatrix}$

35. $A=\begin{pmatrix}4&-2\\2&-1\end{pmatrix}$

36. $A=\begin{pmatrix}-3&\frac{1}{2}\\6&-1\end{pmatrix}$

37. $A=\begin{bmatrix}1&0&1\\2&-2&-1\\3&0&0\end{bmatrix}$

38. $A=\begin{bmatrix}1&2&-1\\3&5&3\\2&4&3\end{bmatrix}$

39. $A=\begin{bmatrix}1&2&-1\\-2&0&1\\1&-1&0\end{bmatrix}$

40. $A=\begin{bmatrix}-2&5&3\\4&-1&3\\7&-2&5\end{bmatrix}$

41. $A=\begin{pmatrix}1&1&0\\-1&3&4\\0&4&3\end{pmatrix}$

42. $A=\begin{pmatrix}1 & -1 & 1\\ 0 & -2 & 1\\ -2 & -3 & 0\end{pmatrix}$

43. $A=\begin{pmatrix}1 & 0 & 2\\ -1 & 2 & 3\\ 1 & -1 & 0\end{pmatrix}$

44. $A=\begin{pmatrix}1 & 1 & 1\\ 3 & 2 & -1\\ 3 & 1 & 2\end{pmatrix}$

45. $A=\begin{pmatrix}3 & 3 & 1\\ 1 & 2 & 1\\ 2 & -1 & 1\end{pmatrix}$

46. $A=\begin{pmatrix}-3 & 1 & -1\\ 1 & -4 & -7\\ 1 & 2 & 5\end{pmatrix}$

47. $A=\begin{pmatrix}1 & 1 & -3\\ 2 & -4 & 1\\ -5 & 7 & 1\end{pmatrix}$

48. $A=\begin{pmatrix}1 & 2 & -1\\ 3 & 5 & 3\\ 2 & 4 & 3\end{pmatrix}$

49. $A=\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 0\end{pmatrix}$

50. $A=\begin{pmatrix}\frac{1}{5} & \frac{1}{5} & -\frac{2}{5}\\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10}\\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10}\end{pmatrix}$

51. $A=\begin{pmatrix}\sqrt{2} & 3\sqrt{2} & 0\\ -4\sqrt{2} & \sqrt{2} & 0\\ 0 & 0 & 1\end{pmatrix}$

52. $A=\begin{pmatrix}-8 & 17 & 2 & \frac{1}{3}\\ 4 & 0 & \frac{2}{5} & -9\\ 0 & 0 & 0 & 0\\ -1 & 13 & 4 & 2\end{pmatrix}$

53. $A=\begin{bmatrix}-2 & -3 & 4 & 1\\ 0 & 1 & 1 & 0\\ 0 & 4 & -6 & 1\\ -2 & -2 & 5 & 1\end{bmatrix}$

54. $A=\begin{bmatrix}1 & -14 & 7 & 38\\ -1 & 2 & 1 & -2\\ 1 & 2 & -1 & -6\\ 1 & -2 & 3 & 6\end{bmatrix}$

55. $A=\begin{bmatrix}10 & 20 & -30 & 15\\ 3 & -7 & 14 & -8\\ -7 & -2 & -1 & 2\\ 4 & 4 & -3 & 1\end{bmatrix}$

56. Show that A is not invertible for any values of the entries

$$A=\begin{bmatrix}0 & a & 0 & 0 & 0\\ b & 0 & c & 0 & 0\\ 0 & d & 0 & e & 0\\ 0 & 0 & f & 0 & g\\ 0 & 0 & 0 & h & 0\end{bmatrix}$$

57. Prove that if A is an invertible matrix and B is row equivalent to A, then B is also invertible.

58. Determine if the given matrix has an inverse, and find the inverse if it exists. Check your answer by multiplying $A \cdot A^{-1} = I$

a) $\begin{bmatrix} 2 & 3 \\ -3 & -5 \end{bmatrix}$ *b)* $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}$

59. Show that the inverse of $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is $\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$

60. If the product $C = AB$ is invertible (and *A* & *B* are square matrices), find a formula for A^{-1} that involves C^{-1} and *B*.

Hence, it is not possible to multiply a non-invertible matrix by another matric and obtain an invertible matrix as a result.

61. Prove that if *A* is an $m \times n$ matrix, there is an invertible matrix *C* such that CA is in reduced row-echelon form.

62. Prove that 2 $m \times n$ matrices *A* and *B* are row equivalent if and only if there exists a nonsingular matrix *P* such that $B = PA$

63. Let *A* and *B* be 2 $m \times n$ matrices. Suppose *A* is row equivalent to *B*. Prove that *A* is nonsingular if and only if *B* is nonsingular.

64. Show that if *A* and *B* are two $n \times n$ invertible matrices then *A* is row equivalent to *B*.

65. Prove that a square matrix *A* is nonsingular if and only if *A* is a product of elementary matrices.

66. Show that if $A \sim B$ (that is, if they are row equivalent), then $EA = B$ for some matrix *E* which is a product of elementary matrices.

67. Show that if $EA = B$ for some matrix *E* which is a product of elementary matrices, then $AC \sim BC$ for every $n \times n$ matrix *C*.

68. Let $A\vec{x} = 0$ be a homogeneous system of *n* linear equations in *n* unknowns that has only the trivial solution. Show that of *k* is any positive integer, then the system $A^k \vec{x} = 0$ also has only trivial solution.

69. Let $A\vec{x} = 0$ be a homogeneous system of *n* linear equations in *n* unknowns, and let *Q* be an invertible $n \times n$ matrix. Show that $A\vec{x} = 0$ has just trivial solution if and only if $(QA)\vec{x} = 0$ has just trivial solution.

70. Let $A\vec{x}=b$ be any consistent system of linear equations, and let $\vec{x}_1$ be a fixed solution. Show that every solution to the system can be written in the form $\vec{x}=\vec{x}_1+\vec{x}_0$ where $\vec{x}_0$ is a solution to $A\vec{x}=0$. Show also that every matrix of this form is a solution.

71. If *A* and *B* are $n\times n$ matrices satisfying $A^2=B^2=(AB)^2=I_n$. Prove that $AB=BA$.

72. Let $A=\begin{pmatrix}0&1&0\\0&0&1\\5&0&0\end{pmatrix}$. Verify that $A^3=5I$, then find A^{-1} in term of *A*.

73. Consider $B(A,\ I)=(BA,\ B)$, thus if *B* is the inverse of *A*, then $(BA,\ B)$ becomes $\left(I,\ A^{-1}\right)$. On the other hand *B* is a product of elementary matrices since it is invertible. This indicates that the inverse of *A* can be obtained by applying elementary row operations to $(A,\ I)$ to get $\left(I,\ A^{-1}\right)$.

Find the inverses of

a) $A=\begin{pmatrix}1&0&0\\0&1&0\\a&b&1\end{pmatrix}$

b) $B=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\a&b&c&d\end{pmatrix}$

74. Let $A,\ B,\ C,\ X,\ Y,\ Z\in M_n(\mathbb{C})$, *A* and *C* are invertible. Find

a) $\begin{pmatrix}A&B\\0&C\end{pmatrix}^{-1}$

b) $\begin{pmatrix}I&X&Y\\0&I&Z\\0&0&I\end{pmatrix}^{-1}$

75. Suppose that *A*, *B*, and *A* – *B* are invertible $n\times n$ matrices. Show that

$$(A-B)^{-1}=A^{-1}+A^{-1}\left(B^{-1}-A^{-1}\right)^{-1}A^{-1}$$

76. Suppose *P* is invertible and $A=PBP^{-1}$. Solve for *B* in terms of *A*.

77. Suppose $(A-B)C=0$, where *A* and *B* are $m\times n$ matrices and *C* is invertible. Show that $A=B$.