Solution   *Section* 1.1 – Velocity and Net Change

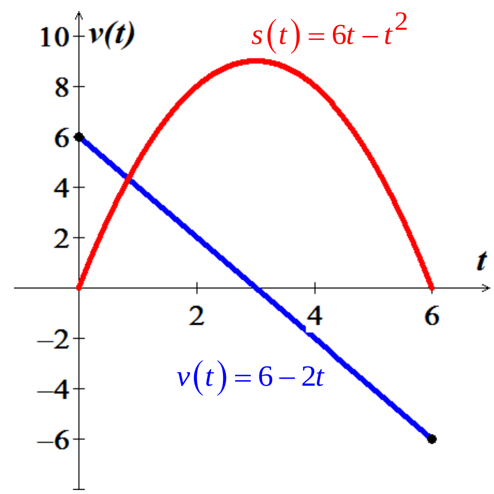
Exercise

Assume t is time measured in seconds and velocities have units of *m/s*. $v(t)=6-2t;\quad 0\le t\le 6$

a) Graph the velocity function over the given interval. Then determine when the motion is in the positive direction.
b) Find the displacement over the given interval.
c) Find the distance traveled over the given interval.

Solution

a) The motion is positive for $0\le t<3$ and negative for $3<t\le 6$



b) Displacement $=\int_0^6 (6-2t)\,dt$

$$=\left[6t-t^2\right]_0^6$$

$$\underline{=0\ \ m}$$

c) Distance traveled is $=\int_0^3 (6-2t)\,dt+\int_3^6 (2t-6)\,dt$

$$=\left[6t-t^2\right]_0^3+\left[t^2-6t\right]_3^6$$

$$=9+9$$

$$\underline{=18\ \ m}$$

Exercise

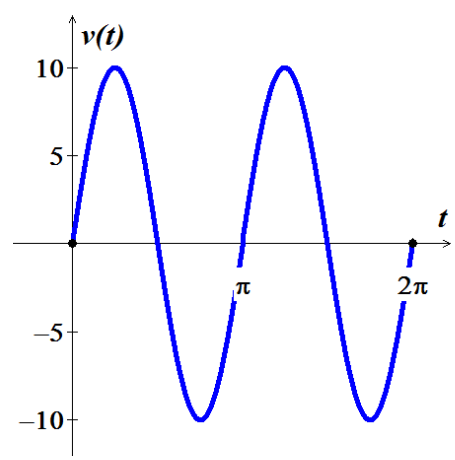
Assume t is time measured in seconds and velocities have units of *m/s*. $v(t)=10\sin 2t;\quad 0\le t\le 2\pi$

a) Graph the velocity function over the given interval. Then determine when the motion is in the positive direction.
b) Find the displacement over the given interval.
c) Find the distance traveled over the given interval.

Solution

a) The motion is positive for $0<t<\frac{\pi}{2}$ & $\pi<t<\frac{3\pi}{2}$

and negative for $\frac{\pi}{2}<t<\pi$ & $\frac{3\pi}{2}<t<2\pi$



b) Displacement $=\int_0^{2\pi} (10\sin 2t)\,dt$

$$= -5\cos 2t \Big|_0^{2\pi}$$

$$\underline{= 0}$$

c) Distance traveled is $= \int_0^{2\pi} (10\sin 2t)\,dt = 4 \cdot \int_0^{\pi/2} (10\sin 2t)\,dt$

$$= -20\cos 2t \Big|_0^{\pi/2}$$

$$= -20(-1-1)$$

$$\underline{= 40 \ \ m}$$

Exercise

Assume t is time measured in seconds and velocities have units of *m/s.* $v(t) = 50e^{-2t}; \quad 0 \le t \le 4$

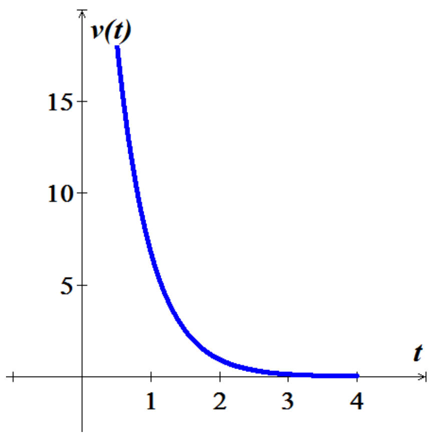
a) Graph the velocity function over the given interval. Then determine when the motion is in the positive direction.

b) Find the displacement over the given interval.

c) Find the distance traveled over the given interval.

Solution

a) The motion is positive for $0 \le t \le 4$



b) Displacement $= \int_0^4 50e^{-2t}\,dt$

$$= -25e^{-2t} \Big|_0^4$$

$$= -25\left(e^{-8} - 1\right)$$

$$\underline{= 25\left(1 - e^{-8}\right) \ \ m} \qquad \underline{\approx 24.992 \ \ m}$$

c) Distance traveled is the same displacement si $\approx 24.992 \ \ m$

Exercise

Consider an object moving along a line with the following velocities and initial positions

$v(t) = 6 - 2t \quad on \ [0, \ 5] \quad s(0) = 0$

a) Graph the velocity function on the given interval. Then determine when the object is moving in the positive direction and when it is moving in the negative direction.

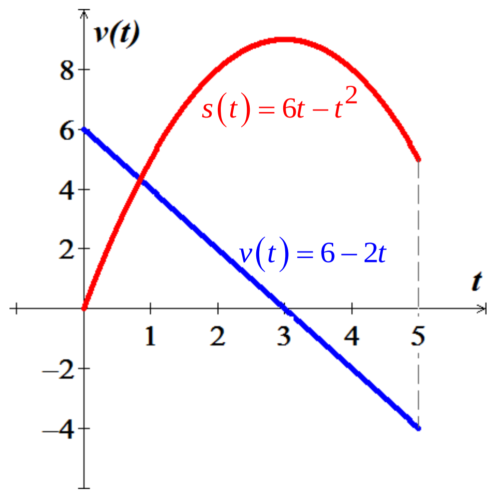
b) Determine the position function for $t \ge 0$ using both the antiderivative method and the Fundamental Theorem of Calculus. Check for agreement between the two methods.

c) Graph the position function on the given interval.

Solution

a) The motion is positive for $0 \le t < 3$ and negative for $3 < t \le 5$

b) $s(t) = \int (6-2t)\,dt = 6t - t^2 + C$



Given: $s(0) = 0 \quad \to \quad \boxed{0 = C}$

$\boxed{s(t) = 6t - t^2}$

Also, $s(t) = s(0) + \int_0^t (6-2x)\,dx$

$= 0 + \left[6x - x^2\right]_0^t$

$\boxed{= 6t - t^2}$

Exercise

Consider an object moving along a line with the following velocities and initial positions

$v(t) = 9 - t^2 \quad on\ [0,\ 4] \quad s(0) = -2$

a) Graph the velocity function on the given interval. Then determine when the object is moving in the positive direction and when it is moving in the negative direction.

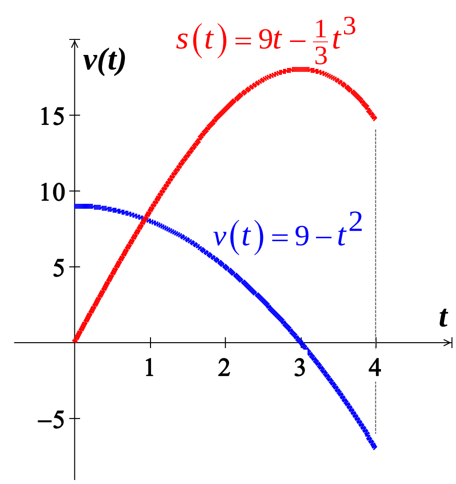
b) Determine the position function for $t \ge 0$ using both the antiderivative method and the Fundamental Theorem of Calculus. Check for agreement between the two methods.

c) Graph the position function on the given interval.

Solution

a) The motion is positive for $0 \le t < 3$ and negative for $3 < t \le 4$

b) $s(t) = \int \left(9 - t^2\right)dt = 9t - \frac{1}{3}t^3 + C$



Given: $s(0) = -2 \quad \to \quad \boxed{-2 = C}$

$\boxed{s(t) = 9t - \frac{1}{3}t^3 - 2}$

Also, $s(t) = s(0) + \int_0^t \left(9 - x^2\right)dx$

$= -2 + \left[9x - \frac{1}{3}x^3\right]_0^t$

$\boxed{= 9t - \frac{1}{3}t^3 - 2}$

Exercise

Find the position and velocity of an object moving along a straight line with the given acceleration, initial velocity, ansd initial position. Assume units of meters and seconds.

$a(t) = -9.8, \quad v(0) = 20, \quad s(0) = 0$

Solution

$$v(t) = \int a(t)\,dt = -\int 9.8\,dt \; \underline{= -9.8t + C_1}$$

Since $v(0) = 20 \quad \rightarrow \quad \underline{20 = C_1}$

$$v(t) \; \underline{= -9.8t + 20}$$

$$s(t) = \int v(t)\,dt = \int (20 - 9.8t)\,dt$$

$$\underline{= 20t - 4.9t^2 + C_2}$$

Since $s(0) = 0 \quad \rightarrow \quad \underline{0 = C_2}$

$$\underline{s(t) = 20t - 4.9t^2}$$

Exercise

Find the position and velocity of an object moving along a straight line with the given acceleration, initial velocity, ansd initial position. Assume units of meters and seconds.

$a(t) = e^{-t}, \quad v(0) = 60, \quad s(0) = 40$

Solution

$$v(t) = \int a(t)\,dt = \int e^{-t}\,dt \; \underline{= -e^{-t} + C_1}$$

Since $v(0) = 60 \quad \rightarrow \quad 20 = -1 + C_1 \Rightarrow \underline{C_1 = 61}$

$$v(t) \; \underline{= -e^{-t} + 61}$$

$$s(t) = \int v(t)\,dt = \int \left(61 - e^{-t}\right)dt$$

$$\underline{= 61t + e^{-t} + C_2}$$

Since $s(0) = 40 \quad \rightarrow \quad 40 = 1 + C_2 \Rightarrow \underline{C_2 = 39}$

$$\underline{s(t) = 61\,t + e^{-t} + 39}$$

Exercise

A mass hanging from a spring is set in motion and its ensuing velocity is given by $v(t)=2\pi\cos\pi t$ for $t\geq 0$. Assume that the position direction is upward and $s(0)=0$.

a) Determine the position function for $t\geq 0$.
b) Graph the position function on the interval [0, 3].
c) At what times does the mass reach its lowest point the first three times?
d) At what times does the mass reach its highest point the first three times?

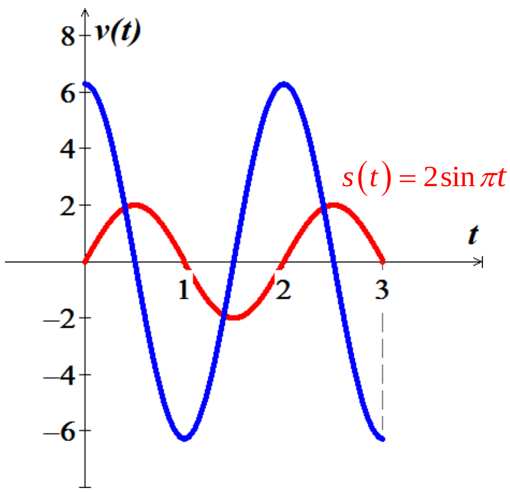
Solution

a) $s(t)=s(0)+\int_0^t (2\pi\cos\pi x)\,dx$

$$=2\sin\pi x\Big|_0^t$$

$$\boxed{=2\sin\pi t}$$

b)



c) The smallest value of Sine is -1, therefore the angle are

$$\frac{4k+3}{2}\pi=\pi t \quad\Rightarrow\quad t=\frac{4k+3}{2}\quad (k=0,1,2)$$

The mass reaches its lowest point at $t=1.5,\ t=3.5,\ and\ t=5.5$

d) The Largest value of Sine is 1, therefore the angle are

$$\frac{4k+1}{2}\pi=\pi t \quad\Rightarrow\quad t=\frac{4k+1}{2}\quad (k=0,1,2)$$

The mass reaches its highest point at $t=0.5,\ t=2.5,\ and\ t=4.5$

Exercise

The velocity of an airplane flying into a headwind is given by $v(t)=30\left(16-t^2\right)$ *mi/hr* for $0\le t\le 3\ hr$.

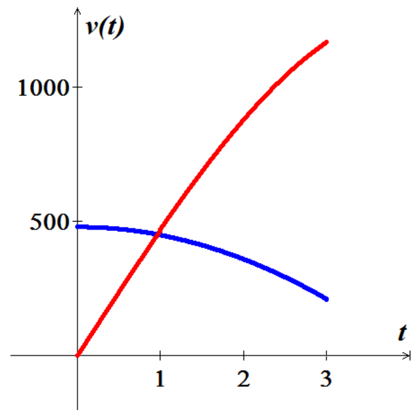
Assume that $s(0)=0$

a) Determine and graph the position function for $0\le t\le 3$.

b) How far does the airplane travel in the first 2 *hr*.?

c) How far has the airplane traveled at the instant its velocity reaches 400 *mi/hr*.?

Solution



a) $s(t)=s(0)+30\int_0^t\left(16-x^2\right)dx$

$=30\left(16x-\frac{1}{3}x^3\right)\Big|_0^t$

$\boxed{=480t-10t^3}$

b) $s(2)=480(2)-10\left(2^3\right)$ $\boxed{=880\ \ miles}$

c) **Given**: $v=400;\ \rightarrow 480-30t^2=400$

$t^2=\frac{8}{3}\ \ \Rightarrow\ \ \boxed{t=\sqrt{\frac{8}{3}}}$

$s\left(\sqrt{\frac{8}{3}}\right)=480\sqrt{\frac{8}{3}}-10\left(\sqrt{\frac{8}{3}}\right)^3$ $\boxed{\approx 740.290\ \ miles}$

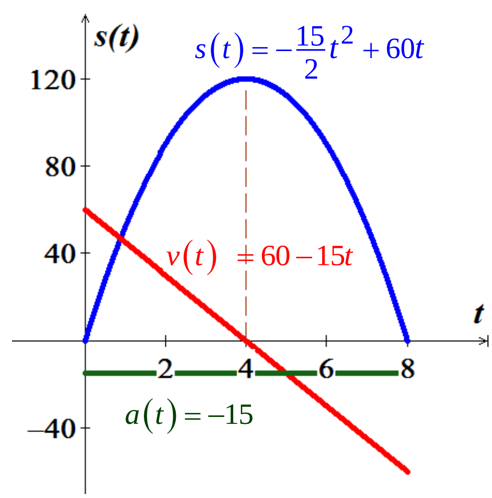
Exercise

A car slows down with an acceleration of $a(t)=-15\ ft/s^2$. Assume that $v(0)=60\ ft/s$ and $s(0)=0$

a) Determine and graph the position function for $t\ge 0$.

b) How far does the car travel in the time it takes to come to rest?

Solution



a) $v(t)=\int a(t)dt=-\int 15\ dt=-15t+C_1$

Given: $v(0)=60\ \ \Rightarrow\ \ \boxed{C_1=60}$

$v(t)\ \boxed{=60-15t}$

$s(t)=\int v(t)dt=\int(60-15t)\ dt$

$=60t-\frac{15}{2}t^2+C_2$

Given: $s(0)=0\ \ \Rightarrow\ \ \boxed{C_2=0}$

$\boxed{s(t)=-\frac{15}{2}t^2+60t}$

b) The car comes to rest when $v(t)=0=60-15t \quad \to \quad \underline{t=4}$

$$s(4)=-\frac{15}{2}(4)^2+60(4)\ \underline{=120\ \ ft}$$

Exercise

The owners of an oil reserve begin extracting oil at $t=0$. Based on estimates of the reserves, suppose the projected extraction rate is given by $Q'(t)=3t^2(40-t)^2$, where $0\le t\le 40$, Q is measured in millions of barrels, and t is measured in years.

a) When does the peak extraction rate occur?
b) How much oil is extracted in the first 10, 20, and 30 years?
c) What is the total amount of oil extracted in 40 year?
d) Is one-fourth of the total oil extracted in the first one-fourth of the extraction period? Explain.

Solution

a)
$$\begin{aligned} Q''(t) &= 6t(40-t)^2-6t^2(40-t) \\ &= 6t(40-t)(40-2t) \\ &= 12t(40-t)(20-t)=0 \end{aligned}$$

0	20	40
+		−

From the table, Q' is maximized at $t=20$; therefore the peak extraction rate is at $t=20\ yrs$

b) In the first **10 years**:

$$\begin{aligned} Q(t) &= \int Q'(t)\,dt=\int_0^{10}\left(3t^4-240t^3+4800t^2\right)dt \\ &= \left[\frac{3}{5}t^5-60t^4+1600t^3\right]_0^{10} \\ &= \frac{3}{5}10^5-6\left(10^5\right)+16\left(10^5\right) \\ &= \frac{53}{5}\cdot 10^5 \\ &\underline{=106\cdot 10^4} \end{aligned}$$

1,060,000 millions of barrels

In the first **20** years

$$\begin{aligned} Q(t) &= \int Q'(t)\,dt=\int_0^{20}\left(3t^4-240t^3+4800t^2\right)dt \\ &= \left[\frac{3}{5}t^5-60t^4+1600t^3\right]_0^{20} \\ &= \frac{3}{5}\cdot 2^5\cdot 10^5-6\cdot 2^4\cdot 10^5+16\cdot 2^3\cdot 10^5 \\ &= 2^5\cdot 10^5\left(\frac{3}{5}-3+4\right) \\ &= 8\cdot 2^6\cdot 10^4 \end{aligned}$$

$$\underline{=2^9\cdot 10^4}\Big|\quad 5{,}120{,}000 \text{ millions of barrels}$$

In the first **30** years

$$Q(t)=\int Q'(t)\,dt=\int_0^{30}\left(3t^4-240t^3+4800t^2\right)dt$$

$$=\left[\frac{3}{5}t^5-60t^4+1600t^3\right]_0^{30}$$

$$=\frac{3}{5}\cdot 3^5\cdot 10^5-6\cdot 3^4\cdot 10^5+16\cdot 3^3\cdot 10^5$$

$$=3^3\cdot 10^5\left(\frac{27}{5}-18+16\right)$$

$$=3^3\cdot 10^5\cdot\frac{17}{5}$$

$$=9{,}180{,}000 \text{ millions of barrels}$$

c) Total amount of oil extracted in 40 year

$$Q(t)=\int Q'(t)\,dt=\int_0^{40}\left(3t^4-240t^3+4800t^2\right)dt$$

$$=\left[\frac{3}{5}t^5-60t^4+1600t^3\right]_0^{40}$$

$$=\frac{3}{5}\cdot 4^5\cdot 10^5-6\cdot 4^4\cdot 10^5+16\cdot 4^3\cdot 10^5$$

$$=4^4\cdot 10^5\left(\frac{12}{5}-6+4\right)$$

$$=4^4\cdot 10^5\cdot\frac{2}{5}$$

$$=10{,}240{,}000 \text{ millions of barrels}$$

d) $\frac{1}{4}(10{,}240{,}000)=2{,}560{,}000\neq 5{,}120{,}000$

No, the amount extracted in the first 10 years is not $\frac{1}{4}$ of the total amount extracted.

Exercise

Starting with an initial value of $P(0)=55$, the population of a prairie dog community grows at a rate of $P'(t)=20-\frac{t}{5}$ (in units of prairie *dogs/month*), for $0\le t\le 200$.

a) What is the population 6 months later?

b) Find the population $P(t)$ for $0\le t\le 200$.

Solution

a) $P(t)=P(0)+\int_0^6\left(20-\frac{t}{5}\right)dt$

$$= 55 + \left[20t - \frac{1}{10}t^2 \right]_0^6$$

$$= 55 + 120 - \frac{18}{5}$$

$$\underline{= \frac{857}{5} \approx 171.4}$$

b) $$P(t) = 55 + \left[20t - \frac{1}{10}t^2 \right]_0^{200}$$

$$= 55 + 4{,}000 - 4{,}000$$

$$\underline{= 55}$$

Exercise

The population of a community of foxes is observed to fluctuate on a 10-year cycle due to variations in the availability of prey. When population measurements began (t = 0 years), the population was 35 foxes. The growth rate in units of *foxes/yr*. was observed to be

$$P'(t) = 5 + 10\sin\left(\frac{\pi t}{5}\right)$$

a) What is the population 15 years later? 35 years later?

b) Find the population $P(t)$ at any time $t \geq 0$.

Solution

a) $$P(t) = P(0) + \int_0^{15} \left(5 + 10\sin\frac{\pi t}{5}\right) dt$$

$$= 35 + \left[5t - \frac{50}{\pi}\cos\frac{\pi t}{5} \right]_0^{15}$$

$$= 35 + 75 + \frac{50}{\pi} + \frac{50}{\pi}$$

$$\underline{= 110 + \frac{100}{\pi} \approx 142 \ \ \textit{foxes}}$$

$$P(t) = 35 + \left[5t - \frac{50}{\pi}\cos\frac{\pi t}{5} \right]_0^{35}$$

$$= 35 + 175 + \frac{50}{\pi} + \frac{50}{\pi}$$

$$\underline{= 210 + \frac{100}{\pi} \approx 242 \ \ \textit{foxes}}$$

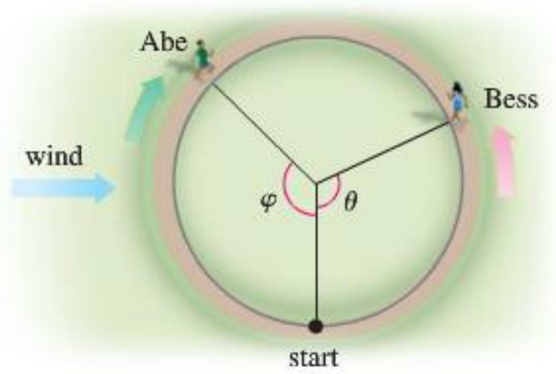
b) $$P(t) = 35 + \int_0^t \left(5 + 10\sin\left(\frac{\pi}{5}x\right)\right) dx$$

$$= 35 + \left[5x - \frac{50}{\pi}\cos\frac{\pi x}{5} \right]_0^t$$

$$\underline{= 35 + 5t - \frac{50}{\pi}\cos\frac{\pi t}{5} + \frac{50}{\pi} \ \ \textit{foxes}}$$

Exercise

A strong west wind blows across a circular running track. Abe and Bess start at the south end of the track and at the same time, Abe starts running clockwise and Bess starts running counterclockwise. Abe runs with a speed (in units of *mi/hr*.) given by $u(\varphi) = 3 - 2\cos\varphi$ and Bess runs with a speed given by $v(\theta) = 3 + 2\cos\theta$, where φ and θ are the central angles of the runners



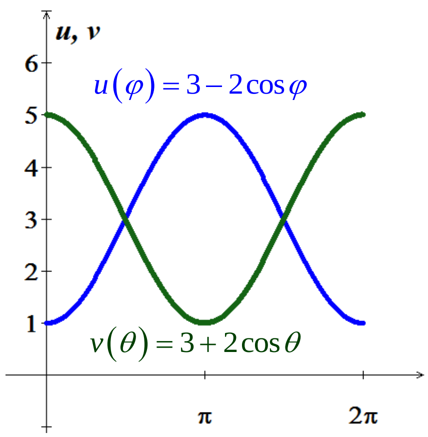
a) Graph the speed functions *u* and *v*, and explain why they describe the runners' speed (in light of the wind).

b) Which runner has the greater average speed for one lap?

c) If the track has a radius of $\frac{1}{10}$ *mi*, how long does it take each runner to complete one lap and who wins the race?

Solution

a) Abe starts out running into a headwind
Bess starts out running with a tailwind.



b) Abe's average speed $= \frac{1}{2\pi}\int_0^{2\pi} (3 - 2\cos\varphi)\,d\varphi$

$$= \frac{1}{2\pi}\left[3\varphi - 2\sin\varphi\right]_0^{2\pi}$$

$$\boxed{= 3 \ \ mph}$$

Bess' average speed $= \frac{1}{2\pi}\int_0^{2\pi} (3 + 2\cos\theta)\,d\theta$

$$= \frac{1}{2\pi}\left[3\theta + 2\sin\theta\right]_0^{2\pi}$$

$$\boxed{= 3 \ \ mph}$$

They have the same average speed.

c) The track is $\frac{1}{10}$ *mi* in radius $\Rightarrow s = \frac{1}{10}\varphi$ $\boxed{s = r\theta}$

We have $u = \frac{ds}{dt} = \frac{1}{10}\frac{d\varphi}{dt} = 3 - 2\cos\varphi$

$$dt = \frac{1}{10(3 - 2\cos\varphi)}d\varphi$$

Abe's time:

$$T=\int_0^T dt=\int_0^{2\pi}\frac{1}{10(3-2\cos\varphi)}d\varphi$$

$$=\frac{2}{10\sqrt{5}}\tan^{-1}\left(\sqrt{5}\ \tan\frac{\varphi}{2}\right)\Bigg|_0^{2\pi}$$

$$=\frac{1}{5\sqrt{5}}[\pi-0]$$

$$\boxed{=\frac{\pi\sqrt{5}}{25}}$$

Let $\varphi=2\tan^{-1}u \quad\Rightarrow\quad u=\tan\frac{\varphi}{2}$

$$d\varphi=\frac{2}{1+u^2}du$$

$$\cos\varphi=\frac{1-\tan^2\frac{\varphi}{2}}{1+\tan^2\frac{\varphi}{2}}=\frac{1-u^2}{1+u^2}$$

$$\int\frac{d\varphi}{3-2\cos\varphi}=\int\frac{2}{1+u^2}\frac{1}{3-2\frac{1-u^2}{1+u^2}}du$$

$$=\int\frac{2}{1+5u^2}du$$

$$=\frac{2}{\sqrt{5}}\int\frac{1}{1+\left(\sqrt{5}u\right)^2}d\left(\sqrt{5}u\right)$$

$$=\frac{2}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}u\right)$$

$$=\frac{2}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\ \tan\frac{\varphi}{2}\right)$$

Bess' time:

$$T=\int_0^T dt=\int_0^{2\pi}\frac{1}{10(3+2\cos\theta)}d\theta$$

$$=\frac{1}{10}\int_0^{2\pi}\frac{2}{1+u^2}\frac{1}{3+2\frac{1-u^2}{1+u^2}}du$$

$$=\frac{1}{5}\int_0^{2\pi}\frac{du}{5+u^2}$$

$$=\frac{1}{5\sqrt{5}}\tan^{-1}\left(\frac{1}{\sqrt{5}}\ \tan u\right)\Bigg|_0^{2\pi}$$

$$=\frac{1}{5\sqrt{5}}\tan^{-1}\left(\frac{1}{\sqrt{5}}\ \tan\frac{\theta}{2}\right)\Bigg|_0^{2\pi}$$

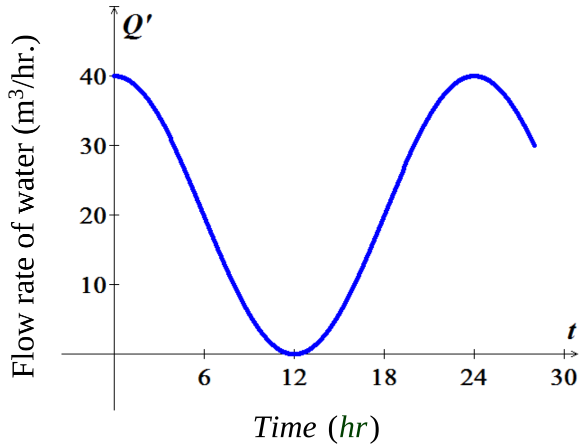
$$\boxed{=\frac{\pi\sqrt{5}}{25}}$$

They tie the race, both have average speed $\frac{2\pi}{10T}=\frac{2\pi}{10\frac{\pi\sqrt{5}}{25}}=\frac{1}{\frac{\sqrt{5}}{5}}\ \boxed{=\sqrt{5}}$

Exercise

A reservoir with a capacity of 2500 m^3 is filled with a single inflow pipe. The reservoir is empty and the inflow pipe is opened at $t = 0$. Letting $Q(t)$ be the amount of water in the reservoir at time t, the flow rate of water into reservoir (in m^3 / hr) oscillates on 24-*hr* cycle and is given by

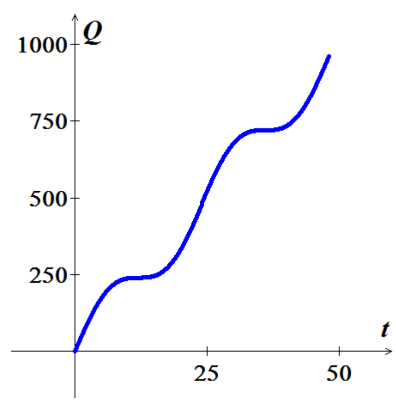
$$Q'(t) = 20\left[1 + \cos\frac{\pi t}{12}\right]$$



a) How much water flows into the reservoir in the first 2 hr.?

b) Find and graph the function that gives the amount of water in the reservoir over the interval $[0,\ t]$ where $t \ge 0$.

c) When is the reservoir full?



Solution

a) $Q(t) = \int_0^2 20\left(1 + \cos\frac{\pi t}{12}\right)dt$

$= \left(20t + \frac{240}{\pi}\sin\frac{\pi t}{12}\right)\Big|_0^2$

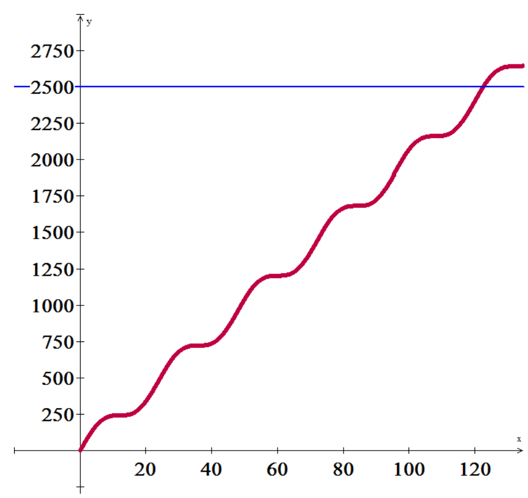
$= 400 + \frac{240}{\pi}\cdot\frac{1}{2}$

$\boxed{= 400 + \frac{120}{\pi} \quad \approx 78.197\ m^3}$

b) $Q(t) = \int_0^t 20\left(1 + \cos\frac{\pi x}{12}\right)dx$

$= \left(20x + \frac{240}{\pi}\sin\frac{\pi x}{12}\right)\Big|_0^t$

$\boxed{= 20t + \frac{240}{\pi}\sin\frac{\pi t}{12} \quad m^3}$



c) The reservoir is full when $20t + \frac{240}{\pi}\sin\frac{\pi t}{12} = 2500$

Using program: $\boxed{T \approx 122.6\ \ hrs}$

Solution *Section* 1.2 – Region between Curves

Exercise

Find the area of the region bounded by the graphs of $y = 2x - x^2$ and $y = -3$

Solution

$$y = -3 \quad \rightarrow \quad 2x - x^2 = -3 \Rightarrow x^2 - 2x - 3 = 0 \quad \boxed{x = -1,\ 3}$$

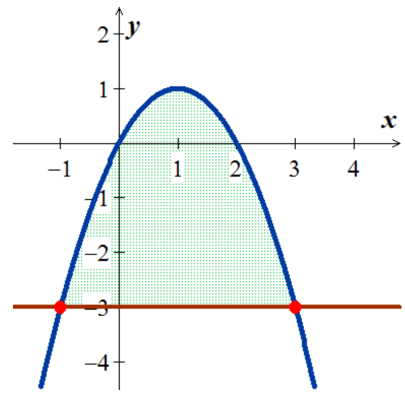
$$A = \int_{-1}^{3} \left[2x - x^2 - (-3) \right] dx$$

$$= \left[x^2 - \frac{x^3}{3} + 3x \right]_{-1}^{3}$$

$$= \left((3)^2 - \frac{(3)^3}{3} + 3(3) \right) - \left((-1)^2 - \frac{(-1)^3}{3} + 3(-1) \right)$$

$$= (9 - 9 + 9) - \left(1 + \frac{1}{3} - 3 \right)$$

$$= \frac{32}{3} \ \ unit^2$$



Exercise

Find the area of the region bounded by the graphs of $y = 7 - 2x^2$ and $y = x^2 + 4$

Solution

$$7 - 2x^2 = x^2 + 4$$

$$-3x^2 = -3 \quad \rightarrow \quad x^2 = 1 \Rightarrow \boxed{x = \pm 1}$$

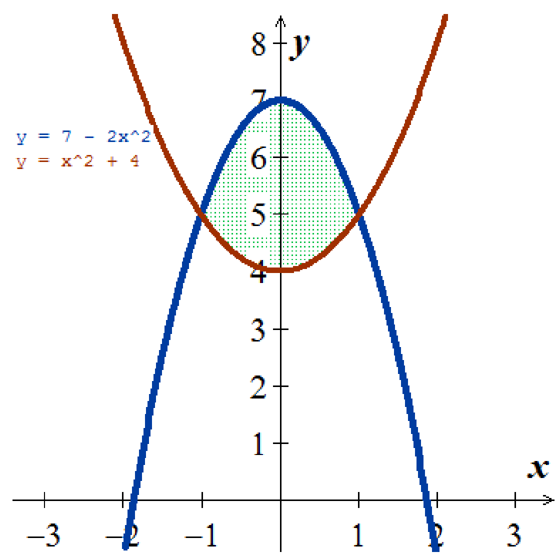
$$A = \int_{-1}^{1} \left[\left(7 - 2x^2 \right) - \left(x^2 + 4 \right) \right] dx$$

$$= \int_{-1}^{1} \left(3 - 3x^2 \right) dx$$

$$= \left[3x - 3\frac{x^3}{3} \right]_{-1}^{1}$$

$$= \left(3(1) - (1)^3 \right) - \left(3(-1) - (-1)^3 \right)$$

$$= 4 \ \ unit^2$$



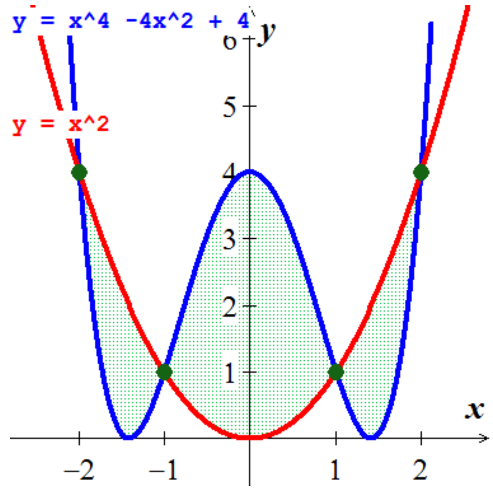
Exercise

Find the area of the region bounded by the graphs of $y = x^4 - 4x^2 + 4$ and $y = x^2$

Solution

$$x^4 - 4x^2 + 4 = x^2$$

$$x^4 - 5x^2 + 4 = 0 \quad \rightarrow \quad \boxed{x = \pm 1, \ \pm 2}$$



$$A = \int_{-2}^{-1} \left(x^2 - \left(x^4 - 4x^2 + 4 \right) \right) dx + \int_{-1}^{1} \left(x^4 - 4x^2 + 4 - \left(x^2 \right) \right) dx + \int_{1}^{2} \left(x^2 - \left(x^4 - 4x^2 + 4 \right) \right) dx$$

$$= \int_{-2}^{-1} \left(-x^4 + 5x^2 - 4 \right) dx + \int_{-1}^{1} \left(x^4 - 5x^2 + 4 \right) dx + \int_{1}^{2} \left(-x^4 + 5x^2 - 4 \right) dx$$

$$= \left[-\frac{x^5}{5} + \frac{5}{3}x^3 - 4x \right]_{-2}^{-1} + \left[\frac{x^5}{5} - \frac{5}{3}x^3 + 4x \right]_{-1}^{1} + \left[-\frac{x^5}{5} + \frac{5}{3}x^3 - 4x \right]_{1}^{2}$$

$$= \left[\left(-\frac{(-1)^5}{5} + \frac{5}{3}(-1)^3 - 4(-1) \right) - \left(-\frac{(-2)^5}{5} + \frac{5}{3}(-2)^3 - 4(-2) \right) \right]$$

$$+ \left[\left(\frac{(1)^5}{5} - \frac{5}{3}(1)^3 + 4(1) \right) - \left(\frac{(-1)^5}{5} - \frac{5}{3}(-1)^3 + 4(-1) \right) \right]$$

$$+ \left[\left(-\frac{(2)^5}{5} + \frac{5}{3}(2)^3 - 4(2) \right) - \left(-\frac{(1)^5}{5} + \frac{5}{3}(1)^3 - 4(1) \right) \right]$$

$$= \left(\tfrac{1}{5} - \tfrac{5}{3} + 4 \right) - \left(\tfrac{32}{5} - \tfrac{40}{3} + 8 \right) + \left(\tfrac{1}{5} - \tfrac{5}{3} + 4 \right) - \left(-\tfrac{1}{5} + \tfrac{5}{3} - 4 \right) + \left(-\tfrac{32}{5} + \tfrac{40}{3} - 8 \right) - \left(-\tfrac{1}{5} + \tfrac{5}{3} - 4 \right)$$

$$= 8 \ \ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $x = 2y^2, \quad x = 0, \quad and \quad y = 3$

Solution

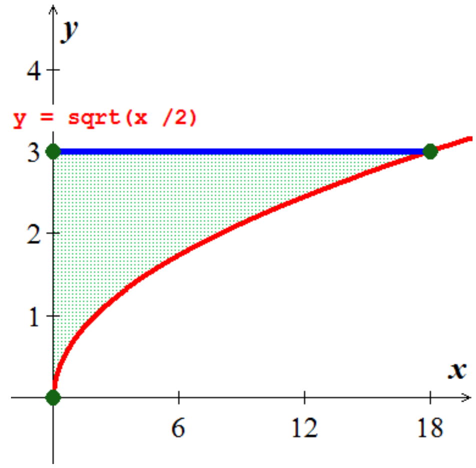
$$y = 3 \quad \to \quad \boxed{x = 2y^2 = 18}$$

$$A = \int_0^3 2y^2 dy$$

$$= \frac{2}{3}\left[y^3\right]_0^3$$

$$= \frac{2}{3}\left(3^3 - 0\right)$$

$$\boxed{= 18 \ \ unit^2}$$



Exercise

Find the area of the region bounded by the graphs of $x = y^3 - y^2 \quad and \quad x = 2y$

Solution

$$y^3 - y^2 = 2y$$

$$y^3 - y^2 - 2y = 0$$

$$y\left(y^2 - y - 2\right) = 0 \quad \to \quad \boxed{y = 0, \ -1, \ 2}$$

$$A = \int_{-1}^{0}\left[y^3 - y^2 - (2y)\right]dy + \int_0^2\left[2y - \left(y^3 - y^2\right)\right]dy$$

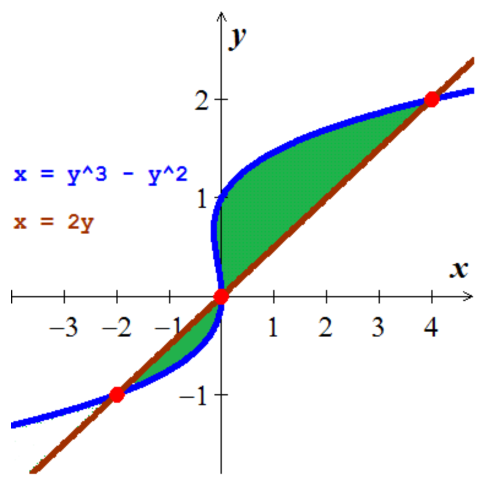
$$= \int_{-1}^{0}\left(y^3 - y^2 - 2y\right)dy + \int_0^2\left(2y - y^3 + y^2\right)dy$$

$$= \left[\frac{y^4}{4} - \frac{y^3}{3} - y^2\right]_{-1}^{0} + \left[y^2 - \frac{y^4}{4} + \frac{y^3}{3}\right]_0^2$$

$$= \left[0 - \left(\frac{1}{4} + \frac{1}{3} - 1\right)\right] + \left[\left(4 - 4 + \frac{8}{3}\right) - 0\right]$$

$$= \frac{5}{12} + \frac{8}{3}$$

$$\boxed{= \frac{37}{12} \ \ unit^2}$$

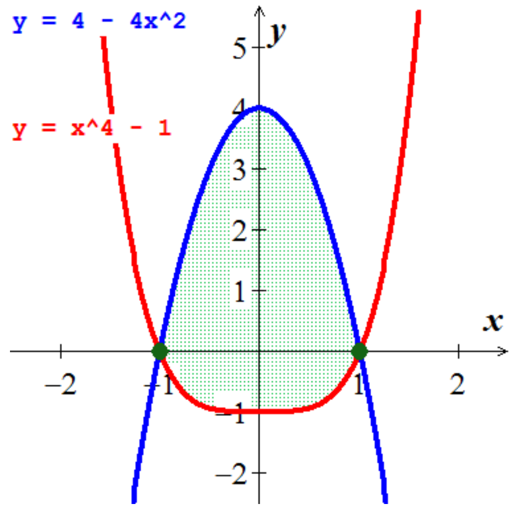


Exercise

Find the area of the region bounded by the graphs of $4x^2 + y = 4$ *and* $x^4 - y = 1$

Solution

$$4x^2 + y = 4 \quad \to \quad y = 4 - 4x^2$$
$$x^4 - y = 1 \quad and \quad y = x^4 - 1$$



$$A = \int_{-1}^{1} \left[4 - 4x^2 - \left(x^4 - 1 \right) \right] dx$$
$$= \int_{-1}^{1} \left(x^4 - 4x^2 + 5 \right) dx$$
$$= \left[\frac{x^5}{5} - 4\frac{x^3}{3} + 5x \right]_{-1}^{1}$$
$$= \left(\frac{1}{5} - \frac{4}{3} + 5 \right) - \left(-\frac{1}{5} + \frac{4}{3} - 5 \right)$$
$$= \frac{105}{15} \ \ unit^2$$

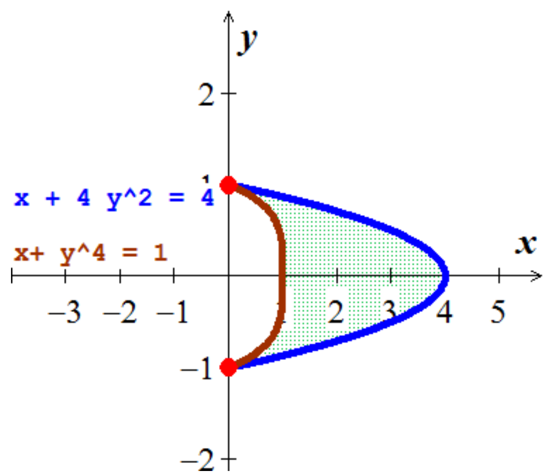
Exercise

Find the area of the region bounded by the graphs of $x + 4y^2 = 4$ *and* $x + y^4 = 1,$ *for* $x \ge 0$

Solution

$$x = 4 - 4y^2 \quad x = 1 - y^4 \quad \to 4 - 4y^2 = 1 - y^4$$
$$y^4 - 4y^2 + 3 = 0 \quad \to \quad y^2 = 1,\ 3 \Rightarrow y = \pm 1,\ \pm\sqrt{3}$$

$$\begin{cases} y = \pm 1 & \to \underline{x = 1 - (\pm 1)^4 = 0} \\ y = \pm\sqrt{3} & \to x = 1 - \left(\pm\sqrt{3} \right)^4 = -8 < 0 \times \end{cases}$$



$$A = \int_{-1}^{1} \left[4 - 4y^2 - \left(1 - y^4 \right) \right] dy$$
$$= \int_{-1}^{1} \left(3 - 4y^2 + y^4 \right) dy$$
$$= \left[3y - 4\frac{y^3}{3} + \frac{y^5}{5} \right]_{-1}^{1}$$
$$= \left(3 - \frac{4}{3} + \frac{1}{5} \right) - \left(-3 + \frac{4}{3} - \frac{1}{5} \right)$$
$$= \frac{56}{15} \ \ unit^2$$

Exercise

Find the area of the region bounded by the graphs of $y = 2\sin x,\quad and \quad y = \sin 2x,\quad 0 \le x \le \pi$

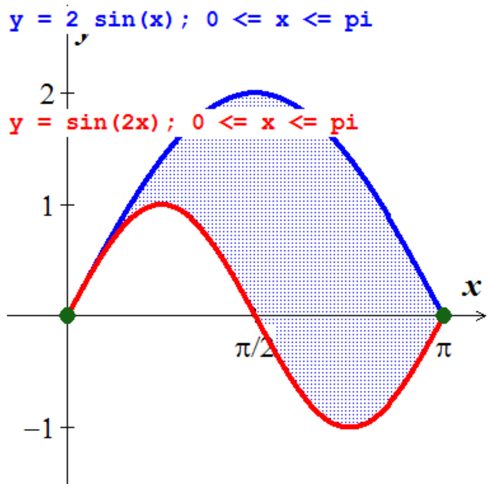
Solution

$$y = 2\sin x = \sin 2x$$

$$2\sin x = 2\sin x\cos x$$

$$2\sin x - 2\sin x\cos x = 0$$

$$2\sin x(1-\cos x) = 0 \quad \to \quad \begin{cases} \sin x = 0 & x = 0, \pi \\ \cos x = 1 & x = 0 \end{cases}$$



$$A = \int_0^{\pi} (2\sin x - \sin 2x)\,dx$$

$$= \left[-2\cos x + \tfrac{1}{2}\cos 2x\right]_0^{\pi}$$

$$= \left(-2(-1) + \tfrac{1}{2}(1)\right) - \left(-2 + \tfrac{1}{2}\right)$$

$$\boxed{= 4 \ \ unit^2}$$

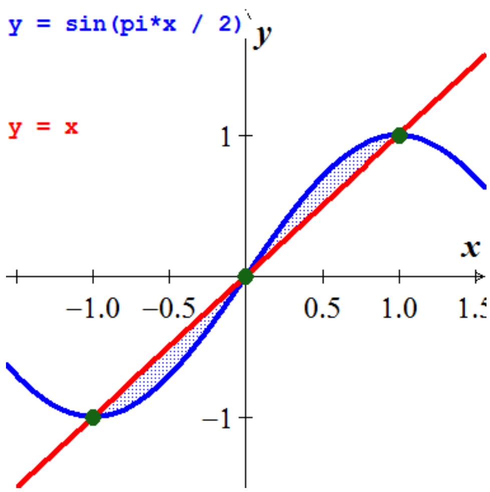
Exercise

Find the area of the region bounded by the graphs of $y = \sin\frac{\pi x}{2} \quad and \quad y = x$

Solution

$$y = \sin\frac{\pi x}{2} = x \quad \to \quad \boxed{x = -1,\ 1}$$

$$A = \int_{-1}^{0}\left(\sin\tfrac{\pi x}{2} - x\right)dx + \int_0^1 \left(\sin\tfrac{\pi x}{2} - x\right)dx$$



$$= 2\int_0^1 \left(\sin\tfrac{\pi x}{2} - x\right)dx$$

$$= 2\left[-\frac{2}{\pi}\cos\frac{\pi x}{2} - \frac{x^2}{2}\right]_0^1$$

$$= 2\left[\left(0 - \tfrac{1}{2}\right) - \left(-\tfrac{2}{\pi} - 0\right)\right]$$

$$= 2\left(-\tfrac{1}{2} + \tfrac{2}{\pi}\right)$$

$$= 2\left(\frac{-\pi + 4}{2\pi}\right)$$

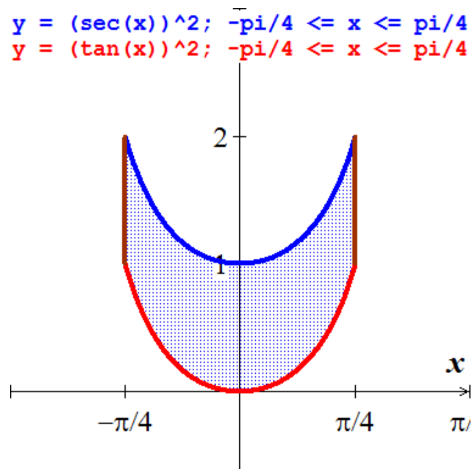
$$\boxed{= \frac{4-\pi}{\pi} \ \ unit^2}$$

Exercise

Find the area of the region bounded by the graphs of $y = \sec^2 x, \quad y = \tan^2 x, \quad x = -\frac{\pi}{4}, \quad \text{and} \quad x = \frac{\pi}{4}$

Solution

$$
\begin{aligned}
A &= \int_{-\pi/4}^{\pi/4} \left(\sec^2 x - \tan^2 x\right) dx \\
&= \int_{-\pi/4}^{\pi/4} \left(\sec^2 x - \left(\sec^2 x - 1\right)\right) dx \\
&= \int_{-\pi/4}^{\pi/4} dx \\
&= x \Big|_{-\pi/4}^{\pi/4} \\
&= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) \\
&= \frac{\pi}{2} \ \ unit^2
\end{aligned}
$$

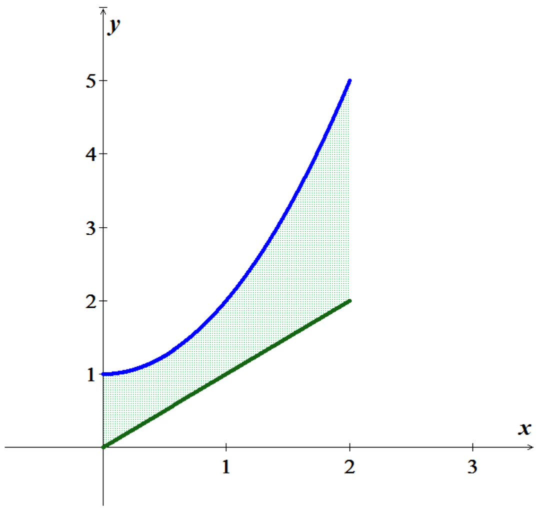


Exercise

Find the area of the region bounded by the graphs of $y = x^2 + 1$ and $y = x$ for $0 \le x \le 2$

Solution

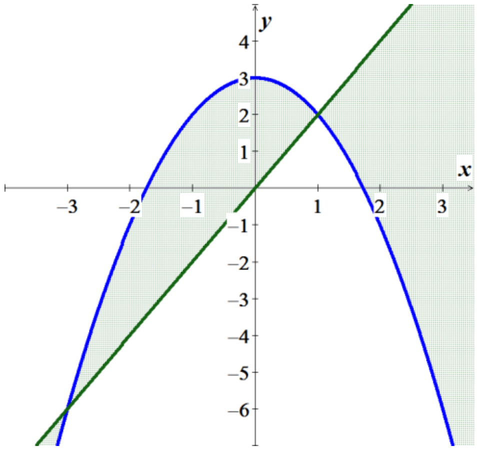
$$
\begin{aligned}
A &= \int_0^2 [(x^2 + 1) - x] dx \\
&= \int_0^2 \left(x^2 - x + 1\right) dx \\
&= \frac{x^3}{3} - \frac{x^2}{2} + 1x \Big|_0^2 \\
&= \frac{2^3}{3} - \frac{2^2}{2} + 1(2) - 0 \\
&= \frac{8}{3} \ \ unit^2
\end{aligned}
$$



Exercise

Find the area of the region bounded by the graphs of $y = 3 - x^2$ and $y = 2x$

Solution



$$x^2 + 2x - 3 = 0 \quad \rightarrow \boxed{x = 1, -3}$$

$$A = \int_{-3}^{1} \left(\left(3 - x^2\right) - 2x \right) dx$$

$$= \int_{-3}^{1} \left(-x^2 - 2x + 3 \right) dx$$

$$= -\frac{x^3}{3} - 2\frac{x^2}{2} + 3x \Big|_{-3}^{1}$$

$$= -\frac{1^3}{3} - 1^2 + 3(1) - \left[-\frac{(-3)^3}{3} - (-3)^2 + 3(-3) \right] = -\frac{1}{3} - 1 + 3 - [9 - 9 - 9]$$

$$= 11 - \frac{1}{3}$$

$$= \frac{32}{3} \;\; unit^2$$

Exercise

Find the area of the region bounded by the graphs of $y = x^2 - x - 2$ and x-axis

Solution

The intersection points: $x^2 - x - 2 = 0 \;\Rightarrow \boxed{x = -1, 2}$

$$A = \int_{-1}^{2} [0 - (x^2 - x - 2)] dx$$

$$= -\frac{x^3}{3} + \frac{x^2}{2} + 2x \Big|_{-1}^{2}$$

$$= -\frac{2^3}{3} + \frac{2^2}{2} + 2(2) - \left[-\frac{(-1)^3}{3} + \frac{(-1)^2}{2} + 2(-1) \right]$$

$$= -\frac{8}{3} + 2 + 4 - \left[\frac{1}{3} + \frac{1}{2} - 2 \right]$$

$$= \frac{10}{3} + \frac{7}{6}$$

$$= \frac{9}{2} \;\; unit^2$$

Exercise

Find the area of the region bounded by the graphs of $f(x) = x^3 + 2x^2 - 3x$ and $g(x) = x^2 + 3x$

Solution

$$x^3 + 2x^2 - 3x = x^2 + 3x \quad \to \quad x^3 + x^2 - 6x = 0$$

$$x(x^2 + x - 6) = 0 \Rightarrow \begin{cases} x = 0 \\ x^2 + x - 6 = 0 \end{cases} \to \boxed{x = -3,\ 0,\ 2}$$

$$A = \int_{-3}^{0} (f - g)dx + \int_{0}^{2} (g - f)dx$$

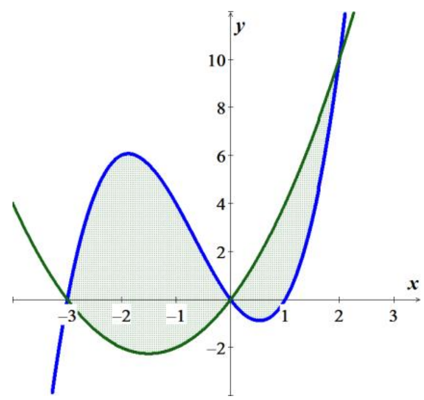
$$= \int_{-3}^{0} (x^3 + 2x^2 - 3x - (x^2 + 3x))dx + \int_{0}^{2} (x^2 + 3x - (x^3 + 2x^2 - 3x))dx$$

$$= \int_{-3}^{0} (x^3 + x^2 - 6x)dx + \int_{0}^{2} (-x^3 - x^2 + 6x)dx$$

$$= \frac{x^4}{4} + \frac{x^3}{3} - 3x^2 \Big|_{-3}^{0} + \left[-\frac{x^4}{4} - \frac{x^3}{3} + 3x^2 \right]_{0}^{2}$$

$$= 0 - \left(\frac{(-3)^4}{4} + \frac{(-3)^3}{3} - 3(-3)^2 \right) + \left[\left(-\frac{2^4}{4} - \frac{2^3}{3} + 32^2 \right) - 0 \right]$$

$$= \frac{253}{12}\ unit^2 \qquad \approx 21.083$$



Exercise

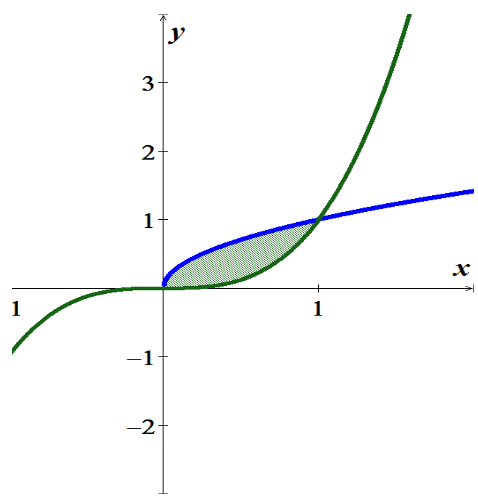
Find the area between the curves $y = x^{1/2}$ *and* $y = x^3$

Solution

$$x^3 = x^{1/2} \quad \textit{Square both sides} \quad \to x^6 = x$$

$$x^6 - x = 0$$

$$x\left(x^5 - 1\right) = 0 \qquad \to x = 0 \quad x^5 - 1 = 0 \Rightarrow x = 1$$



$$A = \int_0^1 \left(x^{1/2} - x^3\right)dx$$

$$= \frac{2}{3}x^{3/2} - \frac{1}{4}x^4 \Big|_0^1$$

$$= \frac{2}{3}1^{3/2} - \frac{1}{4}1^4 - 0$$

$$= \frac{2}{3} - \frac{1}{4}$$

$$= \frac{8-3}{12}$$

$$= \frac{5}{12}\ unit^2$$

Exercise

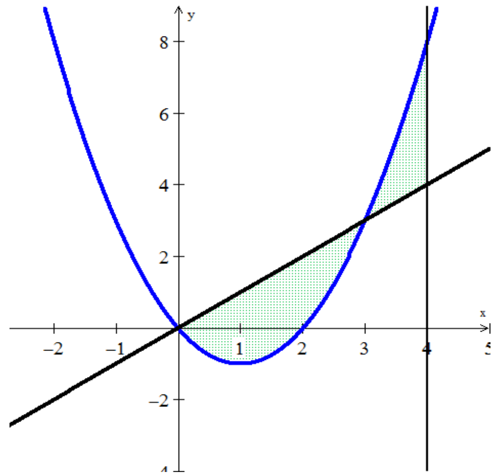
Find the area of the region bounded by the graphs of $y = x^2 - 2x$ and $y = x$ on [0, 4].

Solution

$$x^2 - 2x = x$$

$$x^2 - 3x = 0$$

$$x(x-3) = 0 \Rightarrow \boxed{x = 0, 3}$$



$$A = \int_0^3 \left(x - \left(x^2 - 2x\right)\right)dx + \int_3^4 \left(x^2 - 2x - x\right)dx$$

$$= \int_0^3 \left(-x^2 + 3x\right)dx + \int_3^4 \left(x^2 - 3x\right)dx$$

$$= \left(-\frac{1}{3}x^3 + \frac{3}{2}x^2\right)\Big|_0^3 + \left(\frac{1}{3}x^3 - \frac{3}{2}x^2\right)\Big|_3^4$$

$$= \left(-\frac{1}{3}3^3 + \frac{3}{2}3^2\right) + \left[\left(\frac{1}{3}4^3 - \frac{3}{2}4^2\right) - \left(\frac{1}{3}3^3 - \frac{3}{2}3^2\right)\right]$$

$$= \left(\frac{9}{2}\right) + \left[\left(-\frac{8}{3}\right) - \left(-\frac{9}{2}\right)\right]$$

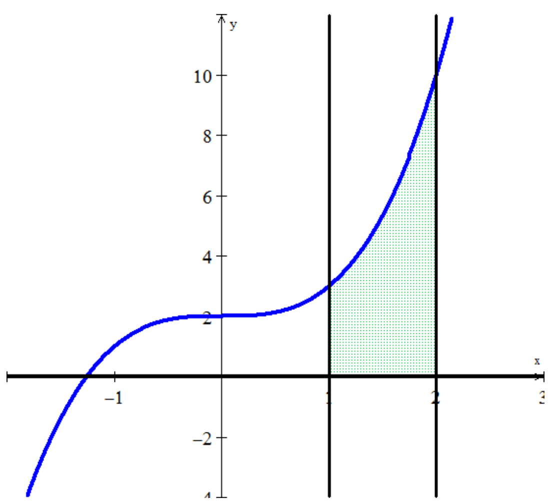
$$= \frac{9}{2} - \frac{8}{3} + \frac{9}{2}$$

$$= \frac{19}{3} \ \ unit^2$$

Exercise

Find the area between the curves $x = 1, \quad x = 2, \quad y = x^3 + 2, \quad y = 0$

Solution



$$A = \int_1^2 \left(x^3 + 2 - 0\right)dx$$

$$= \frac{1}{4}x^4 + 2x \Big|_1^2$$

$$= \left(\frac{1}{4}2^4 + 2(2)\right) - \left(\frac{1}{4}1^4 + 2(1)\right)$$

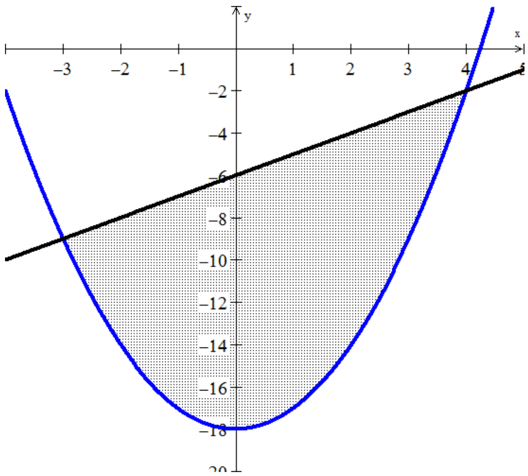
$$= (8) - \left(\frac{9}{4}\right)$$

$$= \frac{23}{4} \ \ unit^2$$

Exercise

Find the area between the curves $y = x^2 - 18, \quad y = x - 6$

Solution



$$x^2 - 18 = x - 6$$

$$x^2 - x - 12 = 0 \to \boxed{x = -3, 4}$$

$$A = \int_{-3}^{4} \left(x^2 - 18 - (x - 6)\right) dx$$

$$= \int_{-3}^{4} \left(x^2 - x - 12\right) dx$$

$$= \frac{1}{3}x^3 - \frac{1}{2}x^2 - 12x \Big|_{-3}^{4}$$

$$= \left(\frac{1}{3}4^3 - \frac{1}{2}4^2 - 12(4)\right) - \left(\frac{1}{3}(-3)^3 - \frac{1}{2}(-3)^2 - 12(-3)\right)$$

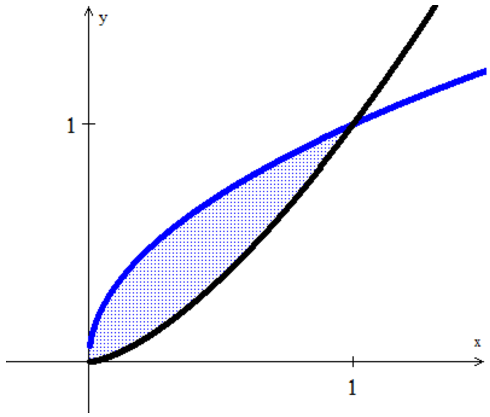
$$= \left(-\frac{104}{3}\right) - \left(\frac{45}{2}\right)$$

$$= \frac{343}{6} \ unit^2$$

Exercise

Find the area between the curves $y = \sqrt{x}, \quad y = x\sqrt{x}$

Solution



$$x\sqrt{x} = \sqrt{x} \quad \Rightarrow \quad \left(x\sqrt{x}\right)^2 = \left(\sqrt{x}\right)^2$$

$$x^2 x = x \quad \to \quad x\left(x^2 - 1\right) = 0$$

$$\boxed{x = 0} \quad x^2 - 1 = 0 \Rightarrow x = \pm 1 (no\ negative) \quad \boxed{x = 1}$$

$$A = \int_{0}^{1} \left(\sqrt{x} - x\sqrt{x}\right) dx$$

$$= \int_{0}^{1} \left(x^{1/2} - x^{3/2}\right) dx$$

$$= \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} \Big|_{0}^{1}$$

$$= \left(\frac{2}{3}1^{3/2} - \frac{2}{5}1^{5/2}\right) - \left(\frac{2}{3}0^{3/2} - \frac{2}{5}0^{5/2}\right)$$

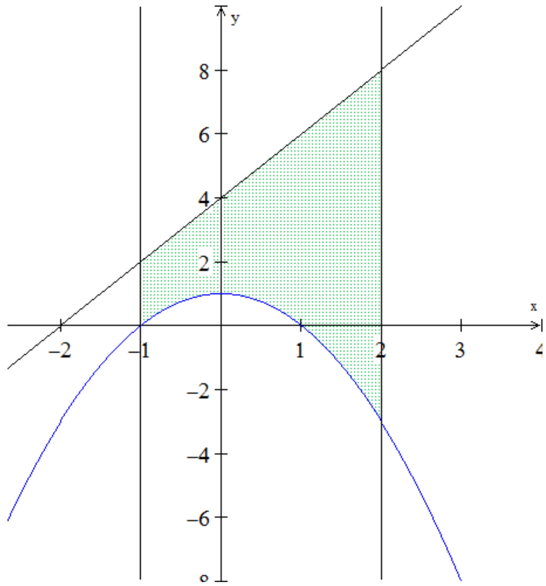
$$= \left(\frac{2}{3} - \frac{2}{5}\right) - 0$$

$$= \frac{4}{15} \ unit^2$$

Exercise

Find the area bounded by $f(x) = -x^2 + 1$, $g(x) = 2x + 4$, $x = -1$, and $x = 2$

Solution



$$f \cap g \Rightarrow \quad -x^2 + 1 = 2x + 4$$

$$x^2 + 2x + 3 = 0$$

$$\Rightarrow x = -1 \pm i\sqrt{2}$$

$$A = \int_{-1}^{2} (g - f)dx$$

$$= \int_{-1}^{2} \left(2x + 4 - \left(-x^2 + 1\right)\right)dx$$

$$= \int_{-1}^{2} \left(x^2 + 2x + 3\right)dx$$

$$= \frac{1}{3}x^3 + x^2 + 3x \Big|_{-1}^{2}$$

$$= \left(\frac{1}{3}(2)^3 + (2)^2 + 3(2)\right) - \left(\frac{1}{3}(-1)^3 + (-1)^2 + 3(-1)\right)$$

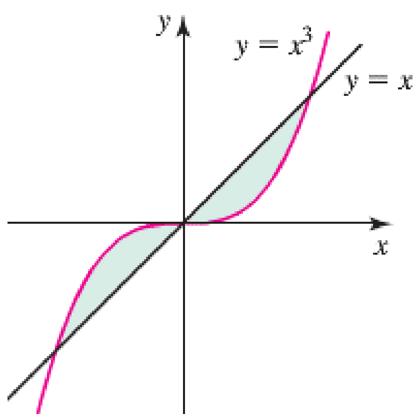
$$= \left(\frac{8}{3} + 4 + 6\right) - \left(-\frac{1}{3} + 1 - 3\right)$$

$$= \frac{8}{3} + 10 + \frac{1}{3} + 2$$

$$= 15 \ \ unit^2$$

Exercise

Determine the area of the shaded region in



Solution

$$y = x^3 = x \quad \rightarrow \quad x\left(x^2 - 1\right) = 0$$

$$\therefore x = 0, \ \pm 1$$

$$Area = \int_{-1}^{0} \left(x^3 - x\right)dx + \int_{0}^{1} \left(x - x^3\right)dx$$

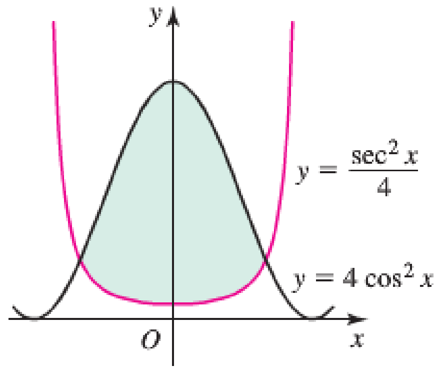
$$= \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{-1}^{0} + \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_{0}^{1}$$

$$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4}$$

$$= \frac{1}{2} \ \ unit^2$$

Exercise

Determine the area of the shaded region in



Solution

$$y = \frac{\sec^2 x}{4} = 4\cos^2 x \quad \to \quad \cos^4 x = \frac{1}{16}$$

$$\cos x = \pm\frac{1}{2} \quad \to \quad \boxed{x = \pm\frac{\pi}{3}}$$

By the symmetry;

$$Area = 2\int_0^{\pi/3} \left(4\cos^2 x - \frac{1}{4}\sec^2 x\right) dx$$

$$= 2\int_0^{\pi/3} \left(2 + 2\cos 2x - \frac{1}{4}\sec^2 x\right) dx$$

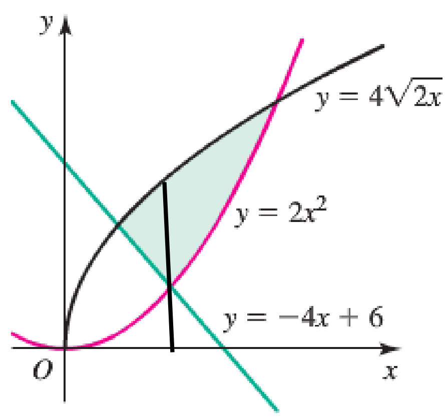
$$= 2\left[2x + \sin 2x - \frac{1}{4}\tan x\right]_0^{\pi/3}$$

$$= 2\left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}\right)$$

$$\boxed{= \frac{4\pi}{3} + \frac{\sqrt{3}}{2} \quad unit^2}$$

Exercise

Determine the area of the shaded region in



Solution

$$y = 4\sqrt{2x} = -4x + 6 \quad \to \quad \left(4\sqrt{2x}\right)^2 = (-4x+6)^2$$

$$32x = 16x^2 - 48x + 36$$

$$16x^2 - 80x + 36 = 0 \quad \to \quad x = \frac{1}{2}, \ \cancel{\frac{9}{2}}$$

$$y = 4\sqrt{2x} = 2x^2 \quad \to \quad \left(4\sqrt{2x}\right)^2 = \left(2x^2\right)^2$$

$$32x = 4x^4 \quad \to \quad 4x\left(x^3 - 8\right) = 0 \quad \to \quad x = 2, \ \cancel{0}$$

$$y = 2x^2 = -4x + 6 \quad \to \quad x^2 + 2x - 3 = 0 \quad \to \quad x = 1, \ \cancel{-3}$$

$$Area = \int_{1/2}^{1} \left(4\sqrt{2x} - (-4x+6)\right) dx + \int_1^2 \left(4\sqrt{2x} - 2x^2\right) dx$$

$$= \left(\frac{8\sqrt{2}}{3}x^{3/2} + 2x^2 - 6x\right)\Big|_{1/2}^{1} + \left(\frac{8\sqrt{2}}{3}x^{3/2} - \frac{2}{3}x^3\right)\Big|_1^2$$

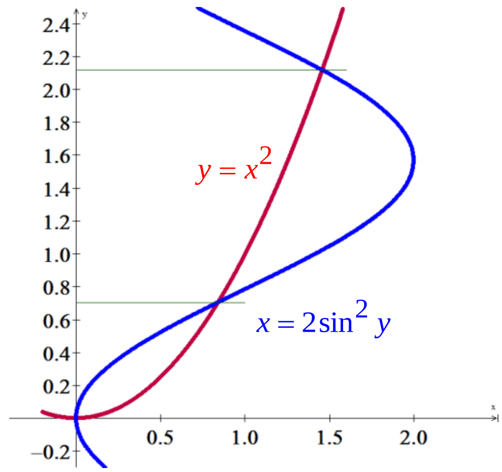
$$=\left(\frac{8\sqrt{2}}{3}+2-6-\frac{8\sqrt{2}}{3}\frac{1}{2\sqrt{2}}-\frac{1}{2}+3\right)+\left(\frac{32}{3}-\frac{16}{3}-\frac{8\sqrt{2}}{3}+\frac{2}{3}\right)$$

$$=-1-\frac{4}{3}-\frac{1}{2}+6$$

$$\underline{=\frac{19}{6}\ unit^2}$$

Exercise

Determine the area of the shaded region in



Solution

From the graph the intersection are: $y=0,\ \ y\approx .705,\ \ y\approx 2.12$

$$A=\int_0^{.705}\left(\sqrt{y}-2\sin^2 y\right)dy+\int_{.705}^{2.12}\left(2\sin^2 y-\sqrt{y}\right)dy$$

$$=\int_0^{.705}\left(y^{1/2}-1+\cos 2y\right)dy+\int_{.705}^{2.12}\left(1-\cos 2y-y^{1/2}\right)dy$$

$$=\left(\frac{2}{3}y^{3/2}-y+\frac{1}{2}\sin 2y\right)\Big|_0^{.705}+\left(y-\frac{1}{2}\sin 2y-\frac{2}{3}y^{3/2}\right)\Big|_{.705}^{2.12}$$

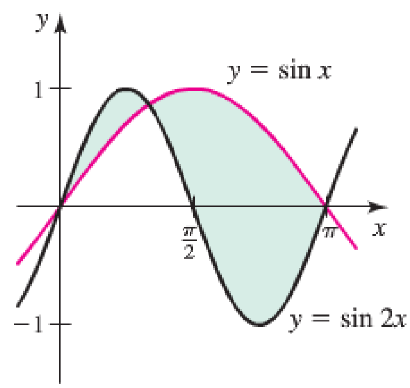
$$=\frac{2}{3}(.705)^{3/2}-0.705+\frac{1}{2}\sin(1.41)+2.12-\frac{1}{2}\sin(4.24)-\frac{2}{3}(2.12)^{3/2}-.705+\frac{1}{2}\sin(1.41)+\frac{2}{3}(.705)^{3/2}$$

$$\underline{\approx .8738\ \ unit^2}$$

Exercise

Determine the area of the shaded regions between $y=\sin x$ *and* $y=\sin 2x$, *for* $0\le x\le \pi$

Solution



$$y=\sin x=\sin 2x$$

$$\sin x=2\sin x\cos x\quad\to\quad \sin x(2\cos x-1)=0$$

$$\sin x=0\quad\to \underline{x=0,\ \pi}$$

$$\cos x=\frac{1}{2}\quad\to \underline{x=\frac{\pi}{3}}$$

$$A=\int_0^{\pi/3}(\sin 2x-\sin x)dx+\int_{\pi/3}^{\pi}(\sin x-\sin 2x)dx$$

$$=\left[-\frac{1}{2}\cos 2x+\cos x\right]_0^{\pi/3}+\left[-\cos x+\frac{1}{2}\cos 2x\right]_{\pi/3}^{\pi}$$

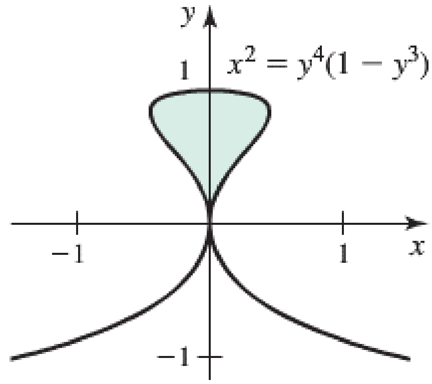
$$=\left(\frac{1}{4}+\frac{1}{2}+\frac{1}{2}-1\right)+\left(1+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}\right)$$

$$\underline{=\frac{5}{2}\ \ unit^2}$$

Exercise

Determine the area of the shaded region bounded by the curve $x^2 = y^4\left(1-y^3\right)$

Solution



$$x^2 = y^4\left(1-y^3\right) \ \rightarrow \ x = y^2\sqrt{1-y^3}$$

Since it is symmetric about *y*-axis, then

$$A = 2\int_0^1 y^2\sqrt{1-y^3}\ dy$$

$$= -\frac{2}{3}\int_0^1 \left(1-y^3\right)^{1/2}\ d\left(1-y^3\right)$$

$$= -\frac{4}{9}\left(1-y^3\right)^{3/2}\Big|_0^1$$

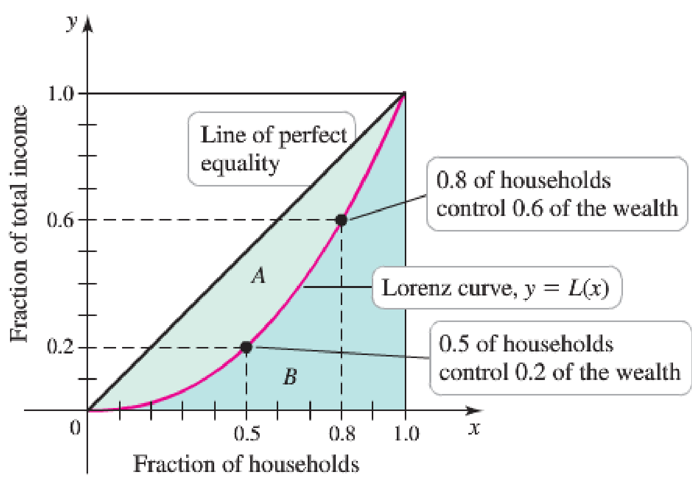
$$\boxed{= \frac{4}{9}\ unit^2}$$

Exercise

A Lorenz curve is given by $y = L(x)$, where $0 \le x \le 1$ represents the lowest fraction of the population of a society in terms of wealth and $0 \le y \le 1$ represents the fraction of the total wealth that is owned by that fraction of the society. For example, the Lorenz curve in the figure shows that $L(0.5) = 0.2$, which means that the lowest 0.5 (50%) of the society owns 0.2 (20%) of the wealth.

a) A Lorenz curve $y = L(x)$ is accompanied by the line $y = x$, called the ***line of perfect equality***. Explain why this line is given the name.

b) Explain why a Lorenz curve satisfies the conditions $L(0) = 0,\ L(1) = 1,\ and\ L'(x) \ge 0 \quad on\ [0,\ 1]$



c) Graph the Lorenz curves $L(x) = x^p$ corresponding to *p* = 1.1, 1.5, 2, 3, 4. Which value of *p* corresponds to the ***most*** equitable distribution of wealth (closest to the line of perfect equality)? Which value of *p* corresponds to the ***least*** equitable distribution of wealth? Explain.

d) The information in the Lorenz curve is often summarized in a single measure called the ***Gini index***, which is defined as follows. Let *A* be the area of the region between $y=x$ and $y=L(x)$ and Let *B* be the area of the region between $y=L(x)$ and the *x*-axis. Then the Gini index is $G=\dfrac{A}{A+B}$.

Show that $G=2A=1-2\int_0^1 L(x)\,dx$.

e) Compute the Gini index for the cases $L(x)=x^p$ and *p* = 1.1, 1.5, 2, 3, 4.

f) What is the smallest interval [*a*, *b*] on which values of the Gini index lie, for $L(x)=x^p$ with $p\ge 1$? Which endpoints of [*a*, *b*] correspond to the least and most equitable distribution of wealth?

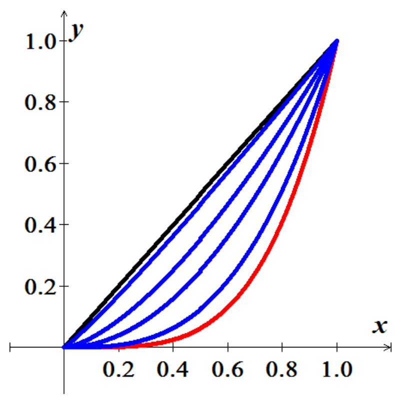
g) Consider the Lorenz curve described by $L(x)=\dfrac{5x^2}{6}+\dfrac{x}{6}$. Show that it satisfies the conditions $L(0)=0,\ L(1)=1,\ and\ L'(x)\ge 0\quad on\ [0,\ 1]$. Find the Gini index for this function.

Solution

a) Let the point $N=(a,\ a)$ on the curve $y=x$ would represent the notion that the lowest *p*% of the society owns *p*% of the wealth, which would represent a form of equality.

b) The function must be increasing and concave up because the poorest *p*% cannot own more than *p*% of the wealth.

c) $y=x^{1.1}$ is closet to $y=x$, and $y=x^4$ is furthest from $y=x$



d) Since, $B=\int_0^1 L(x)\,dx$ and $A+B=\frac{1}{2}$

Then $A=\frac{1}{2}-B=\frac{1}{2}-\int_0^1 L(x)\,dx$

$$G=\frac{A}{A+B}=\frac{A}{\frac{1}{2}}=2A=1-2\int_0^1 L(x)\,dx \quad \checkmark$$

e) For $L(x)=x^p$

$$\begin{aligned} G&=1-2\int_0^1 x^p\,dx \\ &=1-\frac{2}{p+1}\left(x^{p+1}\right)\Big|_0^1 \\ &=1-\frac{2}{p+1} \\ &=\frac{p-1}{p+1} \end{aligned}$$

P	1.1	1.5	2	3	4
G	$\frac{1}{21}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{5}$

f) For $p=1 \quad \rightarrow \quad G=\dfrac{p-1}{p+1}=0$

$\lim\limits_{p\to\infty} \frac{p-1}{p+1} \boxed{=1}$, the largest value of G approaches 1.

g) $L(x) = \frac{5x^2}{6} + \frac{x}{6} \qquad \to L(0) = 0, \quad L(0) = 1$

$L'(x) = \frac{5}{3}x + \frac{1}{6} > 0 \quad x \in [0,\ 1]$

$L''(x) = \frac{5}{3} > 0$

The Gini index is:

$$G = 1 - 2\int_0^1 \left(\frac{5x^2}{6} + \frac{x}{6} \right) dx$$

$$= 1 - 2\left(\frac{5x^3}{18} + \frac{x^2}{12} \right) \Big|_0^1$$

$$= 1 - 2\left(\frac{5}{18} + \frac{1}{12} \right)$$

$$= 1 - \frac{5}{9} - \frac{1}{6}$$

$$\boxed{= \frac{5}{18}}$$

Solution   Section 1.3 – Volumes by Slicing

Exercise

The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$. Find the volume of the solid.

Solution

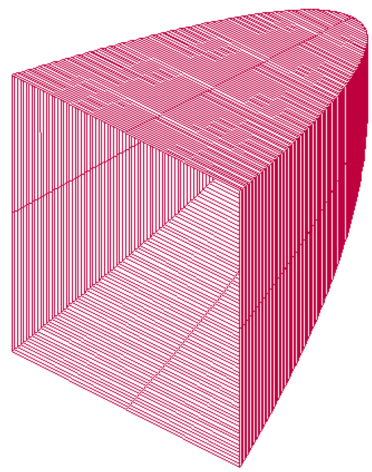
$$A(x) = \tfrac{1}{2}(diagonal)^2$$

$$= \tfrac{1}{2}\left(\sqrt{x} - \left(-\sqrt{x}\right)\right)^2$$

$$= \tfrac{1}{2}\left(2\sqrt{x}\right)^2$$

$$= \tfrac{1}{2}(4x)$$

$$= 2x \ \ unit^2 \qquad a = 0, \quad b = 4;$$



$$V = \int_a^b A(x)\,dx$$

$$= \int_0^4 2x\,dx$$

$$= \left[x^2\right]_0^4$$

$$= 4^2 - 0$$

$$= 16 \ \ unit^3$$

Exercise

The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$. Find the volume of the solid.

Solution

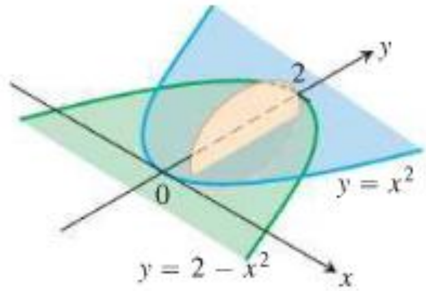
$$y = 2 - x^2 = x^2 \quad \Rightarrow \quad 2x^2 = 2 \to x^2 = 1 \ \Rightarrow \boxed{x = \pm 1}$$

$$A(x) = \tfrac{\pi}{4}(diameter)^2 = \tfrac{\pi}{4}\left(2 - x^2 - x^2\right)^2$$

$$= \frac{\pi}{4}\left(2\left(1-x^2\right)\right)^2$$

$$= \frac{\pi}{4}4\left(1-2x^2+x^4\right)$$

$$\boxed{= \pi\left(1-2x^2+x^4\right) \ \ unit^2} \qquad a=-1, \quad b=1;$$



$$V = \int_a^b A(x)dx$$

$$= \int_{-1}^{1} \pi\left(1-2x^2+x^4\right)dx$$

$$= \pi\left[x-\frac{2}{3}x^3+\frac{1}{5}x^5\right]_{-1}^{1} \qquad = \pi\left[\left(1-\frac{2}{3}(1)^3+\frac{1}{5}(1)^5\right)-\left(-1-\frac{2}{3}(-1)^3+\frac{1}{5}(-1)^5\right)\right]$$

$$= \pi\left[\left(1-\frac{2}{3}+\frac{1}{5}\right)-\left(-1+\frac{2}{3}-\frac{1}{5}\right)\right]$$

$$= 2\pi\left(1-\frac{2}{3}+\frac{1}{5}\right)$$

$$\boxed{= \frac{16\pi}{15} \ \ unit^3}$$

Exercise

The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the axis between these planes are squares whose bases run from the semicircle $y=-\sqrt{1-x^2}$ to the semicircle $y=\sqrt{1-x^2}$. Find the volume of the solid.

Solution

$$A(x) = \frac{1}{2}(diagonal)^2$$

$$= \frac{1}{2}\left(\sqrt{1-x^2}-\left(-\sqrt{1-x^2}\right)\right)^2$$

$$= \frac{1}{2}\left(2\sqrt{1-x^2}\right)^2$$

$$\boxed{= 2\left(1-x^2\right) \ \ unit^2} \qquad a=-1, \quad b=1;$$

$$V = \int_a^b A(x)dx$$

$$= \int_{-1}^{1} 2\left(1-x^2\right)dx$$

$$= 2\left[x - \frac{1}{3}x^3\right]_{-1}^{1}$$

$$= 2\left[\left(1 - \frac{1}{3}1^3\right) - \left((-1) - \frac{1}{3}(-1)^3\right)\right]$$

$$= 4\left(1 - \frac{1}{3}\right)$$

$$= \frac{8}{3}\ unit^3$$

Exercise

The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval [0, π] on the *x*-axis. The cross-sections perpendicular to the *x*-axis are
Find the volume of the solid.

a) Equilateral triangles with bases running from the *x*-axis to the curve as shown

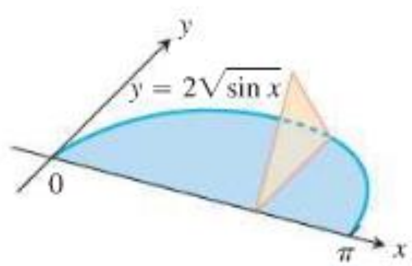
b) Squares with bases running from the *x*-axis to the curve.

Solution

a) $A(x) = \frac{1}{2}(side)(side)\cdot\sin\frac{\pi}{3}$ ***Equilateral triangle*** $\theta = \frac{\pi}{3}$

$$= \frac{1}{2}\left(2\sqrt{\sin x}\right)\left(2\sqrt{\sin x}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \sqrt{3}\sin x\ \ unit^2 \qquad a = 0, \quad b = \pi;$$



$$V = \sqrt{3}\int_0^{\pi} \sin x dx$$

$$= \sqrt{3}\left[-\cos x\right]_0^{\pi}$$

$$= -\sqrt{3}\left[\cos\pi - \cos 0\right]$$

$$= 2\sqrt{3}\ \ unit^3$$

b) $A(x) = (side)^2 = \left(2\sqrt{\sin x}\right)^2 = 4\sin x\ \ unit^2 \qquad a = 0, \quad b = \pi;$

$$V = 4\int_0^{\pi} \sin x dx$$

$$= 4\left[-\cos x\right]_0^{\pi}$$

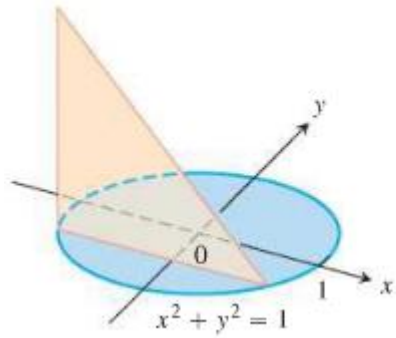
$$= -4\left[\cos\pi - \cos 0\right]$$

$$= 8\ \ unit^3$$

Exercise

The base of the solid is the disk $x^2+y^2\le 1$. The cross-sections by planes perpendicular to the y-axis between $y=-1$ and $y=1$ are isosceles right triangles with one leg in the disk.

Solution



$$x^2+y^2=1 \quad\to\quad x^2=1-y^2 \Rightarrow \boxed{x=\pm\sqrt{1-y^2}}$$

$$A(y)=\tfrac{1}{2}(leg)(leg)=\tfrac{1}{2}\left[\sqrt{1-y^2}-\left(-\sqrt{1-y^2}\right)\right]^2$$

$$=\frac{1}{2}\left[2\sqrt{1-y^2}\right]^2$$

$$=2\left(1-y^2\right)\ unit^2 \qquad c=-1,\quad d=1;$$

$$V=\int_c^d A(y)dy$$

$$=\int_{-1}^{1}2\left(1-y^2\right)dy$$

$$=2\left[y-\tfrac{1}{3}y^3\right]_{-1}^{1}$$

$$=2\left[\left(1-\tfrac{1}{3}1^3\right)-\left((-1)-\tfrac{1}{3}(-1)^3\right)\right]$$

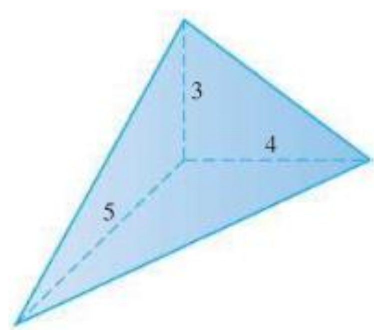
$$=4\left(1-\tfrac{1}{3}\right)$$

$$=\frac{8}{3}\ unit^3$$

Exercise

Find the volume of the given tetrahedron. (***Hint***: Consider slices perpendicular to one of the labeled edges)

Solution



Let consider the slices perpendicular to edge labeled 5 are triangles.

By similar triangles, we have: $\frac{height}{base}=\frac{h}{b}=\frac{3}{4} \quad\Rightarrow\quad h=\frac{3}{4}b$

The equation of the line through (5, 0) and (0, 4) is:

$$y=\tfrac{4-0}{0-5}(x-0)+4 \quad\to\quad y=-\tfrac{4}{5}x+4$$

Therefore, the length of the base: $b=-\frac{4}{5}x+4$

$$h=\tfrac{3}{4}b=\tfrac{3}{4}\left(-\tfrac{4}{5}x+4\right)=-\tfrac{3}{5}x+3$$

$$A(x)=\frac{1}{2}(base)\cdot(height)=\frac{1}{2}\left(-\frac{4}{5}x+4\right)\left(-\frac{3}{5}x+3\right)$$
$$=\frac{1}{2}\left(\frac{12}{25}x^2-\frac{24}{5}x+12\right)$$
$$=\frac{6}{25}x^2-\frac{12}{5}x+5 \quad unit^2$$

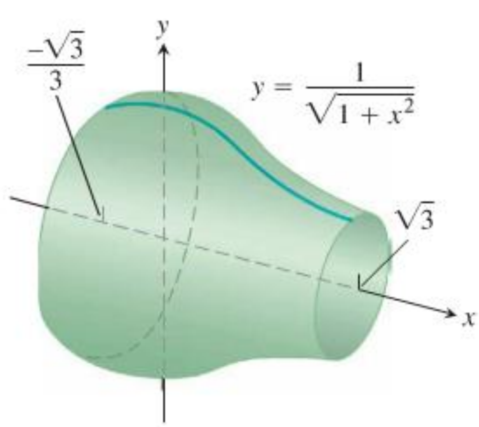
$$V=\int_a^b A(x)dx$$
$$=\int_0^5\left(\frac{6}{25}x^2-\frac{12}{5}x+5\right)dx$$
$$=\left[\frac{2}{25}x^3-\frac{6}{5}x^2+5x\right]_0^5$$
$$=\left[\frac{2}{25}(5)^3-\frac{6}{5}(5)^2+5(5)\right]$$
$$=10 \ \ unit^3$$

Exercise

Find the volume of the solid of revolution

Solution

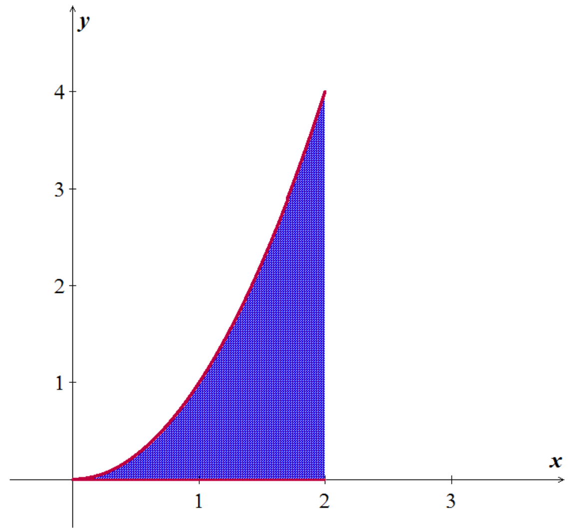
$$V=\pi\int_{-\sqrt{3}/3}^{\sqrt{3}}\left(\frac{1}{\sqrt{1+x^2}}\right)^2 dx$$
$$=\pi\int_{-\sqrt{3}/3}^{\sqrt{3}}\frac{1}{1+x^2}dx$$
$$=\pi\left[\tan^{-1}x\right]_{-\sqrt{3}/3}^{\sqrt{3}}$$
$$=\pi\left(\tan^{-1}\sqrt{3}-\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right)$$
$$=\pi\left(\frac{\pi}{3}-\left(-\frac{\pi}{6}\right)\right)$$
$$=\pi\left(\frac{\pi}{3}+\frac{\pi}{6}\right)$$
$$=\pi\left(\frac{\pi}{2}\right)$$
$$=\frac{\pi^2}{2} \quad unit^3$$



Exercise

Find the volume of the solid generated by revolving the region bounded by $y = x^2$ and the lines $y = 0$, $x = 2$ about the x-axis.

Solution



$$R(x) = x^2$$

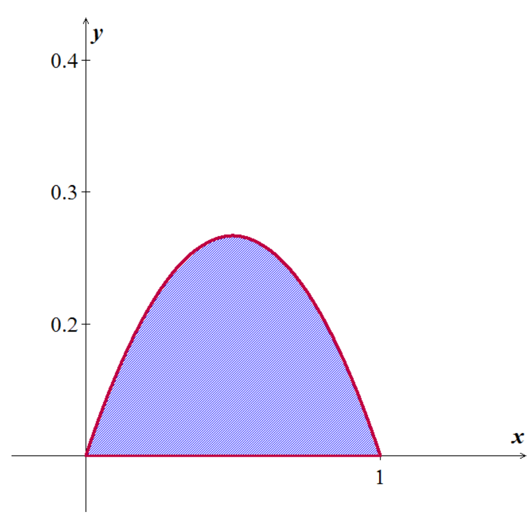
$$\begin{aligned} V &= \int_0^2 \pi\left[R(x)\right]^2 dx \\ &= \pi \int_0^2 \left(x^2\right)^2 dx \\ &= \pi \int_0^2 x^4 dx \\ &= \pi\left[\frac{1}{5}x^5\right]_0^2 \\ &= \pi\left(\frac{1}{5}(2)^5 - 0\right) \\ &= \boxed{\frac{32\pi}{5}\ \ unit^3} \end{aligned}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y = x - x^2$ and the line $y = 0$ about the x-axis.

Solution

$R(x) = x - x^2 \qquad\qquad x - x^2 = 0 \ \rightarrow\ x = 0,\ 1$

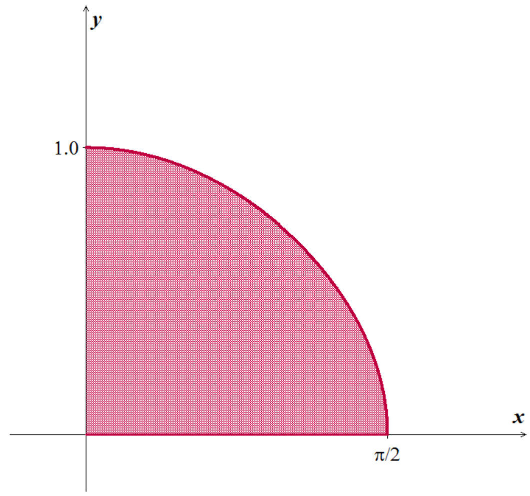


$$\begin{aligned} V &= \int_0^1 \pi\left[x - x^2\right]^2 dx \\ &= \pi \int_0^1 \left(x^2 - 2x^3 + x^4\right) dx \\ &= \pi\left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_0^1 \\ &= \pi\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) \\ &= \boxed{\frac{\pi}{30}\ \ unit^3} \end{aligned}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{\cos x}$ and the lines $0 \le x \le \frac{\pi}{2}$, $\quad y=0, \quad x=0$ about the *x*-axis.

Solution



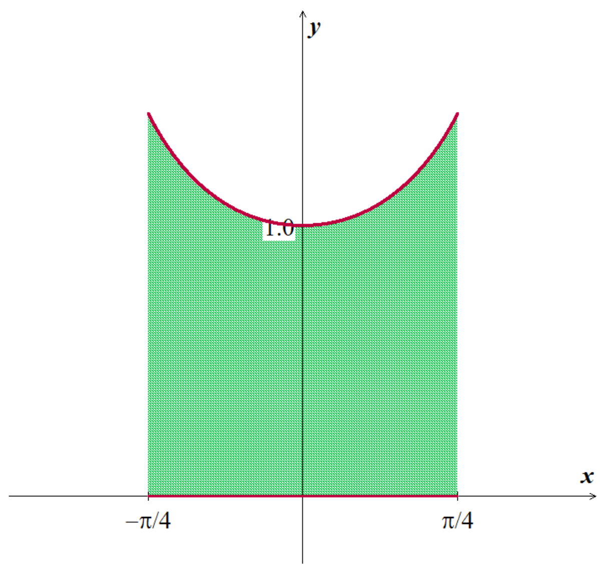
$$R(x)=\sqrt{\cos x}$$

$$\begin{aligned}
V &= \int_0^{\pi/2} \pi\left[\sqrt{\cos x}\right]^2 dx \\
&= \pi\int_0^{\pi/2} \cos x dx \\
&= \pi\left[\sin x\right]_0^{\pi/2} \\
&= \pi(1-0) \\
&= \pi \ \ unit^3
\end{aligned}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y=\sec x$ and the lines $y=0, \quad x=-\frac{\pi}{4}, \quad x=\frac{\pi}{4}$ about the *x*-axis.

Solution



$$R(x)=\sec x$$

$$\begin{aligned}
V &= \pi\int_{-\pi/4}^{\pi/4} \sec^2 x dx \\
&= \pi\left[\tan x\right]_{-\pi/4}^{\pi/4} \\
&= \pi\left(\tan\left(\frac{\pi}{4}\right)-\tan\left(-\frac{\pi}{4}\right)\right) \\
&= \pi(1-(-1)) \\
&= 2\pi \ \ unit^3
\end{aligned}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $x=\sqrt{5}\,y^2$ and the lines $x=0, \quad y=-1, \quad y=1$ about the *y*-axis.

Solution

$$R(y)=\sqrt{5}y^2$$

$$V=\pi\int_{-1}^{1}\left[\sqrt{5}y^2\right]^2 dy$$

$$=\pi\int_{-1}^{1} 5y^4 dy$$

$$=\pi\left(y^5\right)\Big|_{-1}^{1}$$

$$=\pi\left(1-(-1)\right)$$

$$\underline{=2\pi \ \ unit^3}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y=2\sqrt{x}$ and the lines $y=2, \quad x=0$ about the *x*-axis.

Solution

$$r(x)=2\sqrt{x} \quad and \quad R(x)=2$$

$$V=\pi\int_{0}^{1}\left(\left[R(x)\right]^2-\left[r(x)\right]^2\right)dx$$

$$=\pi\int_{0}^{1}\left((2)^2-\left(2\sqrt{x}\right)^2\right)dx$$

$$=\pi\int_{0}^{1}(4-4x)\,dx$$

$$=4\pi\left(x-\tfrac{1}{2}x^2\right)\Big|_{0}^{1}$$

$$=4\pi\left[\left(1-\tfrac{1}{2}\right)-0\right]$$

$$\underline{=2\pi \ \ unit^3}$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $y = \sec x, \quad y = \tan x$ and the lines $x = 0, \quad x = 1$ about the *x*-axis.

Solution

$$r(x) = \tan x \quad \text{and} \quad R(x) = \sec x$$

$$V = \pi \int_0^1 \left([R(x)]^2 - [r(x)]^2 \right) dx$$

$$= \pi \int_0^1 \left(\sec^2 x - \tan^2 x \right) dx$$

$$= \pi \int_0^1 (1)\, dx$$

$$= \pi (x)\Big|_0^1$$

$$= \pi \ \ unit^3$$

Exercise

Find the volume of the solid generated by revolving the region bounded by $x = \sqrt{2\sin 2y}$ and the lines $0 \le y \le \frac{\pi}{2}, \quad x = 0$ about the *y*-axis.

Solution

$$V = \pi \int_0^{\pi/2} \left(\sqrt{2\sin 2y} \right)^2 dy$$

$$= 2\pi \int_0^{\pi/2} (\sin 2y)\, dy$$

$$= -\pi \cos 2y \Big|_0^{\pi/2}$$

$$= -\pi(-1-1)$$

$$= 2\pi \ \ unit^3$$

Exercise

Find the volume of a solid ball having radius *a*.

Solution

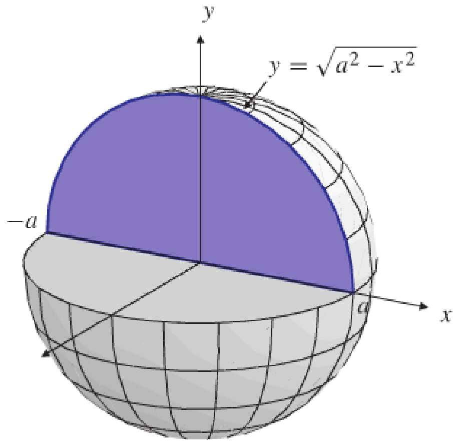
$$V = \pi\int_{-a}^{a}\left(\sqrt{a^2-x^2}\right)^2 dx$$

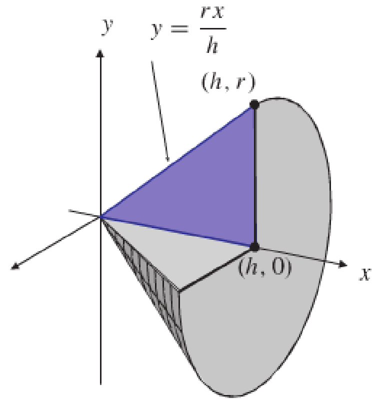
$$= 2\pi\int_{0}^{a}\left(a^2-x^2\right)dx$$

$$= 2\pi\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$= 2\pi\left(a^3-\frac{a^3}{3}\right)$$

$$= \frac{4}{3}\pi a^3 \quad unit^3$$





Exercise

You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentation at home persuades you that you can get one that hold about 3 *L* if you make it 9 *cm* deep and give the sphere a radius of 16 *cm*. To be sure, you picture the wok as a solid of revolution, and calculate its volume with an integral. To the nearest cubic centimeter, what volume do you really get? $\left(1\ L = 1{,}000\ cm^3\right)$

Solution

$$x^2+y^2=256 \quad\Rightarrow\quad x^2=256-y^2$$

$$R(y)=\sqrt{256-y^2}$$

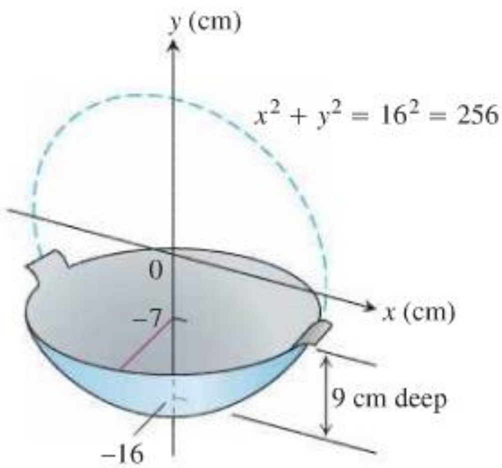
$$V=\pi\int_{-16}^{-7}\left[\sqrt{256-y^2}\right]^2 dy$$

$$=\pi\int_{-16}^{-7}\left(256-y^2\right)dy$$

$$=\pi\left[256y-\frac{1}{3}y^3\right]_{-16}^{-7}$$

$$=\pi\left[\left(256(-7)-\frac{1}{3}(-7)^3\right)-\left(256(-16)-\frac{1}{3}(-16)^3\right)\right]$$

$$=1053\pi\ \ cm^3 \quad \approx 3308\ \ cm^3$$



Exercise

A curved wedge is cut from a circular cylinder of radius 3 by two planes. On plane is perpendicular to the axis of the cylinder. The second place crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

Solution

The base of the cylinder is a circle $x^2 + y^2 = 9$.

Since the second plane cut the base at the cylinder at the center, therefore, the base of the wedge is semi-circle. $y = \pm\sqrt{9-x^2} = radius$.

When we slice the wedge by a plane perpendicular to the axis of the cylinder, we obtained a cross-section at *x* which is a rectangle of height *x*.

The area of this cross-section is: $A(x) = height \times width$

$$= x\left(2\sqrt{9-x^2}\right)$$

$$= 2x\sqrt{9-x^2}$$

The rectangles run from $x = 0$ to $x = 3$, so

$$V = \int_0^3 A(x)\,dx$$

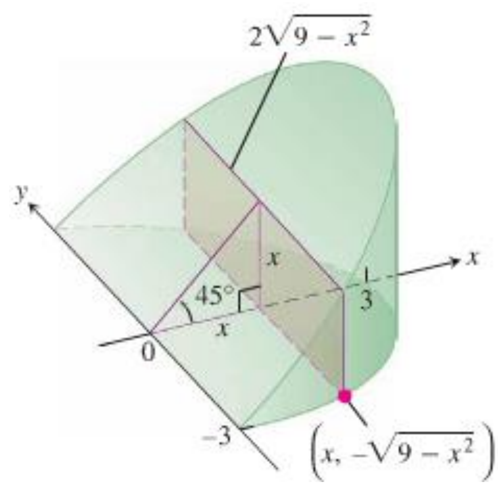
$$= \int_0^3 2x\sqrt{9-x^2}\,dx \qquad or \;\; u = 9 - x^2 \;\; \rightarrow \;\; du = -2x\,dx$$

$$= -\int_0^3 \left(9-x^2\right)^{1/2} d\left(9-x^2\right)$$

$$= -\frac{2}{3}\left[\left(9-x^2\right)^{3/2}\right]_0^3$$

$$= -\frac{2}{3}\left[\left(9-3^2\right)^{3/2} - \left(9-0^2\right)^{3/2}\right]$$

$$\underline{= 18 \;\; unit^3}$$



Exercise

Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

Solution

$$V=\int_1^4 \pi\left[R(x)\right]^2 dx$$

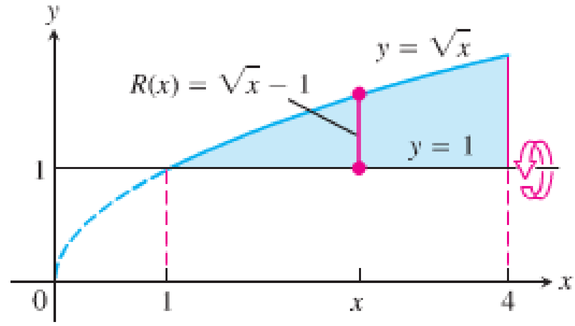
$$=\pi\int_1^4 \left[\sqrt{x}-1\right]^2 dx$$

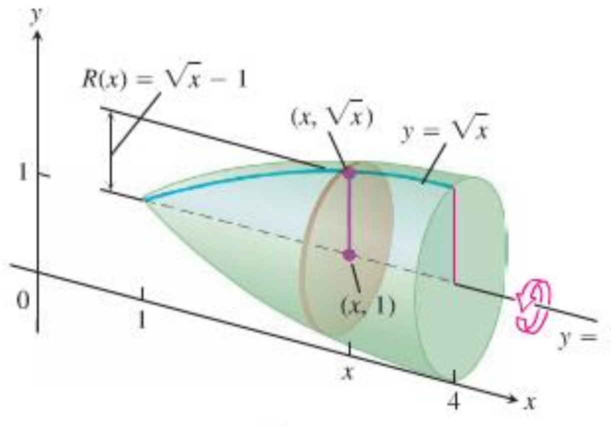
$$=\pi\int_1^4 \left(x-2\sqrt{x}+1\right)dx$$

$$=\pi\left[\frac{x^2}{2}-2\frac{2}{3}x^{3/2}+x\right]_1^4$$

$$=\pi\left[\left(\frac{4^2}{2}-\frac{4}{3}4^{3/2}+4\right)-\left(\frac{1^2}{2}-\frac{4}{3}1^{3/2}+1\right)\right]$$

$$\underline{=\frac{7\pi}{6}\ \ unit^3}$$





Exercise

The solid with a circular base of radius 5 whose cross sections perpendicular to the base and parallel to the *x*-axis are equilateral triangles. Use the general slicing method to find the volume of the solid.

Solution

$$x^2+y^2=5^2 \quad\rightarrow\quad y=\pm\sqrt{25-x^2}$$

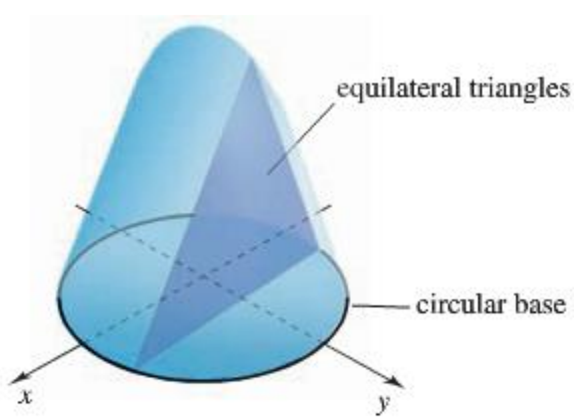
$$A(x)=\left(2\sqrt{25-x^2}\right)^2 \ \ \underline{=100-4x^2}$$

$$V=\int_0^5\left(100-4x^2\right)dx$$

$$=\left(100x-\frac{4}{3}x^3\right)\Big|_0^5$$

$$=500-\frac{500}{3}$$

$$\underline{=\frac{1000}{3}\ \ unit^3}$$



Exercise

The solid whose base is the region bounded by $y = x^2$ and the line $y = 1$ and whose cross sections perpendicular to the base and parallel to the *x*-axis squares. Use the general slicing method to find the volume of the solid.

Solution

$$y = x^2 \quad \Rightarrow \quad x = \sqrt{y}$$

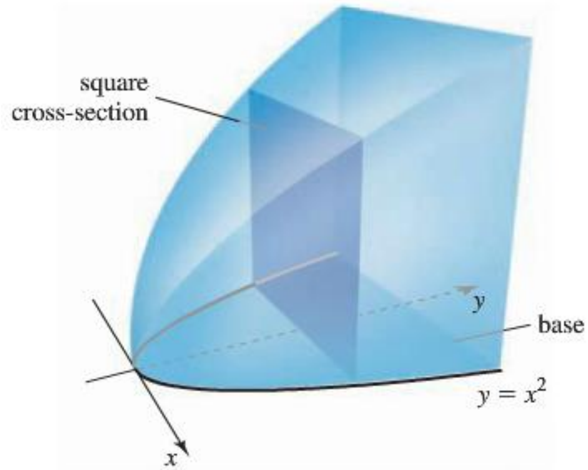
$$V = \int_0^1 A(y)\,dy$$

$$= \int_0^1 \left(2\sqrt{y}\right)^2 dy$$

$$= \int_0^1 (4y)\,dy$$

$$= 2y^2 \Big|_0^1$$

$$= 2 \ \ unit^3$$



Exercise

A circular cylinder of radius *r* and height *h* whose curved surface is at an angle of $\frac{\pi}{4}$ *rad* . Use the general slicing method to find the volume of the solid

Solution

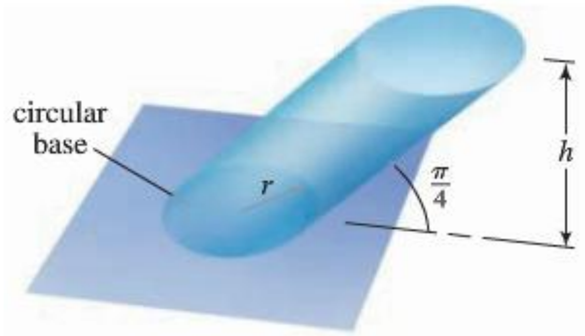
The cross sections are all circles with area πr^2

$$V = \int_0^h \left(\pi r^2\right) dz$$

$$= \pi r^2 z \Big|_0^h$$

$$= \pi r^2 h \ \ unit^3$$

∴ The 45° angle does not affect the volume.



Exercise

Let R be the region bounded by $y = e^{-x}$, $y = 0$, $x = 0$, $x = \ln 4$. Use the disk method to find the volume of the solid generated when R is revolved about x-axis.

Solution

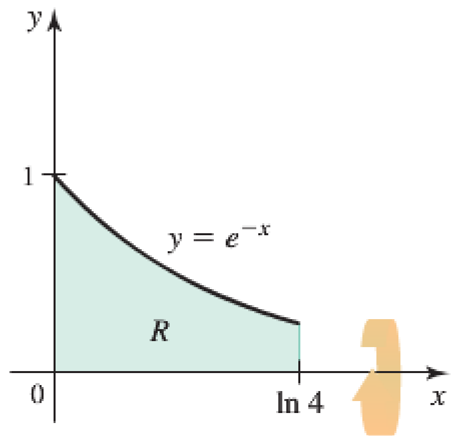
$$V = \pi \int_0^{\ln 4} e^{-2x} dx$$

$$= -\frac{\pi}{2} e^{-2x} \Big|_0^{\ln 4}$$

$$= -\frac{\pi}{2}\left(e^{-2\ln 4} - 1\right)$$

$$= -\frac{\pi}{2}\left(\frac{1}{16} - 1\right)$$

$$= \frac{15\pi}{32} \ unit^3$$



Exercise

Let R be the region bounded by $y = \cos x$, $y = 0$, $x = 0$. Use the disk method to find the volume of the solid generated when R is revolved about x-axis,

Solution

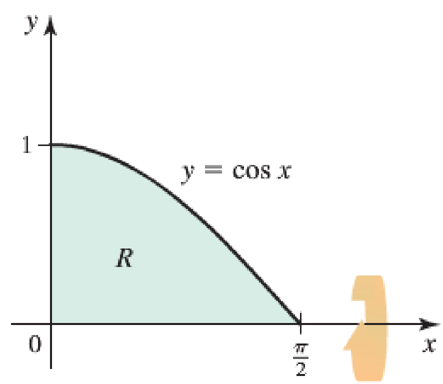
$$V = \pi \int_0^{\pi/2} \cos^2 x \ dx$$

$$= \frac{\pi}{2} \int_0^{\pi/2} (1 + \cos 2x) \ dx$$

$$= \frac{\pi}{2}\left(x + \frac{1}{2}\sin 2x\right)\Big|_0^{\pi/2}$$

$$= \frac{\pi}{2}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi^2}{4} \ unit^3$$



Exercise

Let R be the region bounded by $y = e^{x/2}$, $y = e^{-x}$, $x = \ln 2$, $x = \ln 8$. Use the disk method to find the volume of the solid generated when R is revolved about x-axis.

Solution

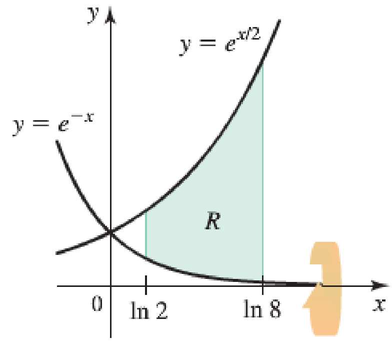
$$V = \pi \int_{\ln 2}^{\ln 8} \left(\left(e^{x/2}\right)^2 - \left(e^{-x}\right)^2 \right) dx$$

$$= \pi \int_{\ln 2}^{\ln 8} \left(e^{x} - e^{-2x}\right) dx$$

$$= \pi \left(e^{x} + \frac{1}{2} e^{-2x}\right) \Big|_{\ln 2}^{\ln 8}$$

$$= \pi \left(8 + \frac{1}{2}\frac{1}{64} - 2 - \frac{1}{8}\right)$$

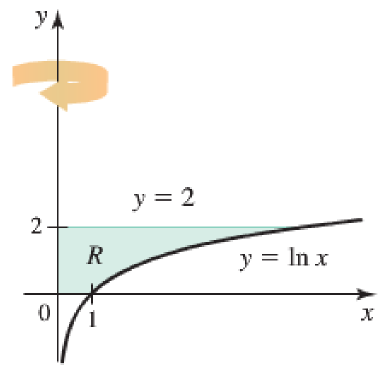
$$= \frac{753\pi}{128} \ \ unit^3$$



Exercise

Let R be the region bounded by $y = 0, \quad y = \ln x, \quad y = 2, \quad x = 0$. Use the disk method to find the volume of the solid generated when R is revolved about y-axis.

Solution



$$y = \ln x \quad \rightarrow \quad x = e^{y}$$

$$V = \pi \int_{0}^{2} e^{2y} dy$$

$$= \frac{\pi}{2} e^{2y} \Big|_{0}^{2}$$

$$= \frac{\pi}{2}\left(e^{4} - 1\right) \ \ unit^3$$

Exercise

Let R be the region bounded by $y = \sin^{-1} x, \quad x = 0, \quad y = \frac{\pi}{4}$. Use the disk method to find the volume of the solid generated when R is revolved about y-axis.

Solution

$$y = \sin^{-1} x \quad \rightarrow \quad x = \sin y$$

$$V = \pi \int_{0}^{\pi/4} \sin^2 y \, dy$$

$$= \frac{\pi}{2} \int_{0}^{\pi/4} (1 - \cos 2y) \, dy$$

$$= \frac{\pi}{2}\left(y - \frac{1}{2} \sin 2y\right) \Big|_{0}^{\pi/4}$$

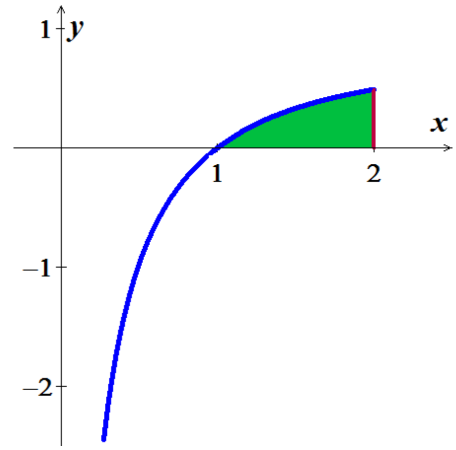
$$= \frac{\pi}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

$$= \frac{\pi(\pi - 2)}{8} \ \ unit^3$$

Exercise

Find the volume of the solid of revolution bounded by $y = \frac{\ln x}{\sqrt{x}}$, $y = 0$, *and* $x = 2$ revolved about the x-axis. Sketch the region

Solution



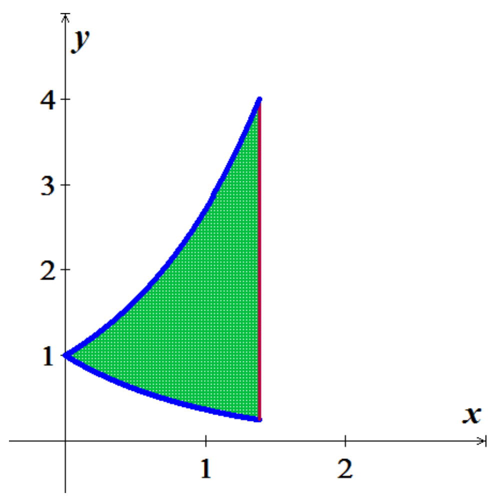
$$y = \frac{\ln x}{\sqrt{x}} = 0 \ \rightarrow \ \underline{x = 1}$$

$$\begin{aligned} V &= \pi \int_1^2 \left(\frac{\ln x}{\sqrt{x}}\right)^2 dx \\ &= \pi \int_1^2 \frac{\ln^2 x}{x} dx \\ &= \pi \int_1^2 \ln^2 x \, d(\ln x) \\ &= \frac{\pi}{3}\left[\ln^3 x\right]_1^2 \\ &= \frac{\pi \ln^3 2}{3} \quad unit^3 \end{aligned}$$

Exercise

Find the volume of the solid of revolution bounded by $y = e^{-x}$, $y = e^{x}$, $x = 0$, $x = \ln 4$ revolved about the x-axis. Sketch the region

Solution



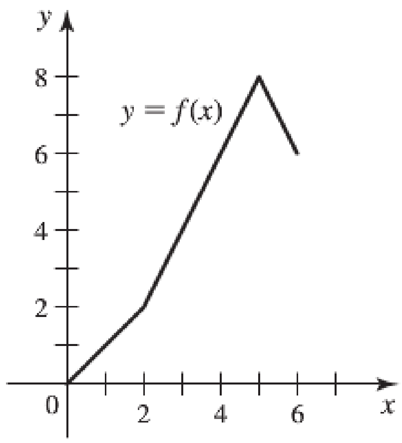
$$\begin{aligned} V &= \pi \int_0^{\ln 4} \left(e^{2x} - e^{-2x}\right) dx \\ &= \frac{\pi}{2}\left(e^{2x} + e^{-2x}\right)\Big|_0^{\ln 4} \\ &= \frac{\pi}{2}\left(16 + \frac{1}{16}\right) \\ &= \frac{225\pi}{32} \quad unit^3 \end{aligned}$$

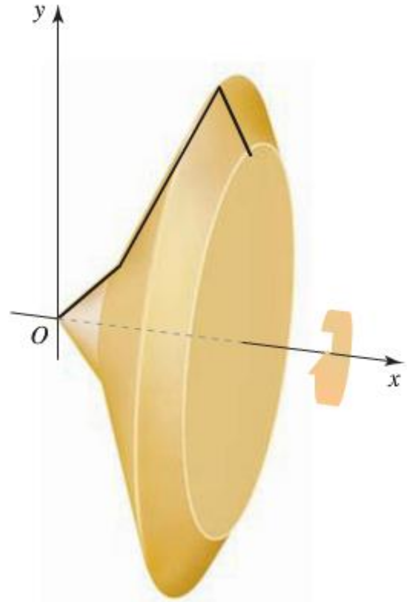
$e^{2\ln 4} = e^{\ln 4^2} = 16$

Exercise

Let $f(x)=\begin{cases} x & if \quad 0\le x\le 2 \\ 2x-2 & if \quad 2<x\le 5 \\ -2x+18 & if \quad 5<x\le 6 \end{cases}$

Find the volume of the solid formed when the region bounded by the graph of *f*, the *x*-axis, and the line $x=6$ is revolved about the *x*-axis





Solution

$$V=\pi\int_0^2 x^2dx+\pi\int_2^5(2x-2)^2\,dx+\pi\int_5^6(18-2x)^2\,dx$$

$$=\pi\int_0^2 x^2dx+\frac{\pi}{2}\int_2^5(2x-2)^2\,d(2x-2)-\frac{\pi}{2}\int_5^6(18-2x)^2\,d(18-2x)$$

$$=\frac{\pi}{3}x^3\Big|_0^2+\frac{\pi}{6}(2x-2)^3\Big|_2^5-\frac{\pi}{6}(18-2x)^3\Big|_5^6$$

$$=\pi\left[\frac{8}{3}+\frac{1}{6}(512-8)-\frac{1}{6}(216-512)\right]$$

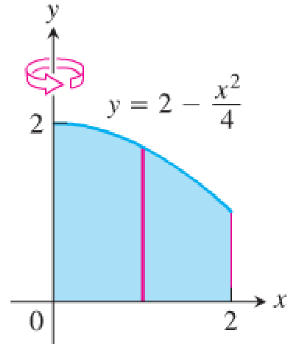
$$=\pi\left(\frac{8}{3}+\frac{252}{3}+\frac{148}{3}\right)$$

$$\underline{=136\pi\ \ unit^3}$$

Solution   *Section* 1.4 – Volumes by Shells

Exercise

Use the shell method to find the volume of the solid generated by revolving the shaded region about the indicated axis



Solution

$$V = \int_a^b 2\pi \begin{pmatrix} shell \\ radius \end{pmatrix} \begin{pmatrix} shell \\ height \end{pmatrix} dx$$

$$= \int_0^2 2\pi(x)\left(2 - \frac{x^2}{4}\right) dx$$

$$= 2\pi \int_0^2 \left(2x - \frac{x^3}{4}\right) dx$$

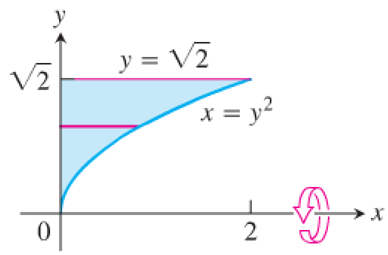
$$= 2\pi \left(x^2 - \frac{x^4}{16} \right)_0^2$$

$$= 2\pi \left[\left(2^2 - \frac{2^4}{16} \right) - 0 \right]$$

$$= 6\pi \ \ unit^3$$

Exercise

Use the shell method to find the volume of the solid generated by revolving the shaded region about the indicated axis



Solution

$$V = \int_0^{\sqrt{2}} 2\pi(y)\left(y^2\right) dy$$

$$= 2\pi \int_0^{\sqrt{2}} y^3 dy$$

$$= 2\pi \left(\frac{y^4}{4} \right)_0^{\sqrt{2}}$$

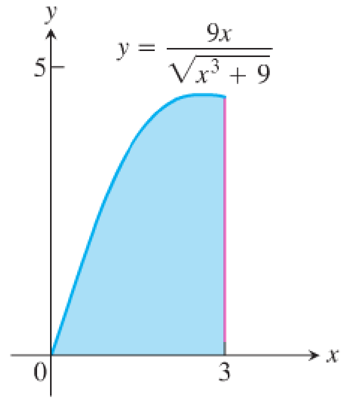
$$= 2\pi \frac{\left(\sqrt{2}\right)^4}{4}$$

$$= 2\pi \ \ unit^3$$

Exercise

Use the shell method to find the volume of the solid generated by revolving the shaded region about the *y*-axis

Solution



$$V = \int_a^b 2\pi \binom{shell}{radius}\binom{shell}{height} dx$$

$$= \int_0^3 2\pi(x)\left(\frac{9x}{\sqrt{x^3+9}}\right) dx$$

$$= 2\pi \int_0^3 \left(\frac{9x^2}{\sqrt{x^3+9}}\right) dx$$

$$= 2\pi \int_0^3 3\left(x^3+9\right)^{1/2} d\left(x^3+9\right) \qquad d\left(x^3+9\right) = 3x^2 dx$$

$$= 6\pi \left[2\left(x^3+9\right)^{1/2} \right]_0^3$$

$$= 12\pi \left[\left(3^3+9\right)^{1/2} - (0+9)^{1/2} \right]$$

$$= 12\pi[6-3]$$

$$\underline{= 36\pi \ \ unit^3}$$

Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = x^2, \quad y = 2 - x, \quad x = 0, \quad for\ x \ge 0$ about the *y*-axis.

Solution

$$V = \int_a^b 2\pi \binom{shell}{radius}\binom{shell}{height} dx$$

$$= \int_0^1 2\pi(x)\left((2-x) - x^2\right) dx$$

$$= 2\pi \int_0^1 x\left(2 - x - x^2\right) dx$$

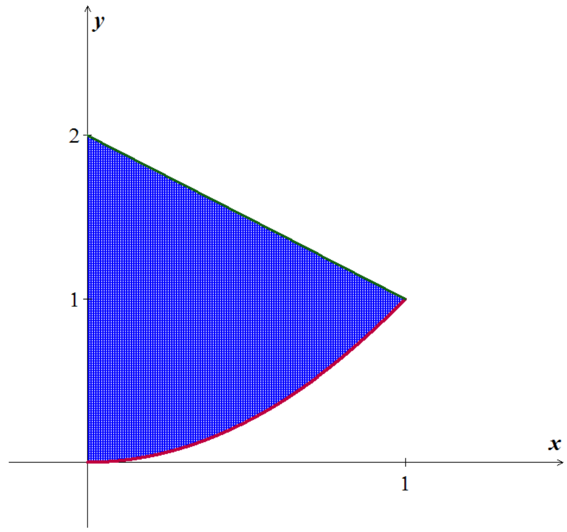
$$= 2\pi \int_0^1 \left(2x - x^2 - x^3\right) dx$$

$$= 2\pi \left[x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^1$$

$$= 2\pi \left(1 - \frac{1}{3} - \frac{1}{4}\right)$$

$$= 12\pi \left(\frac{5}{12}\right)$$

$$\underline{= \frac{5\pi}{6} \ \ unit^3}$$



Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = 2 - x^2, \quad y = x^2, \quad x = 0$ about the *y*-axis.

Solution

$$y = 2 - x^2 = x^2 \quad \rightarrow \quad 2x^2 = 2 \Rightarrow x^2 = 1 \rightarrow \boxed{x = \pm 1}$$

Since *about* $y - axis, \quad a = x = 0 \quad b = 1$

$$V = \int_0^1 2\pi(x)\left(\left(2 - x^2\right) - x^2\right) dx$$

$$= 2\pi \int_0^1 x\left(2 - 2x^2\right) dx$$

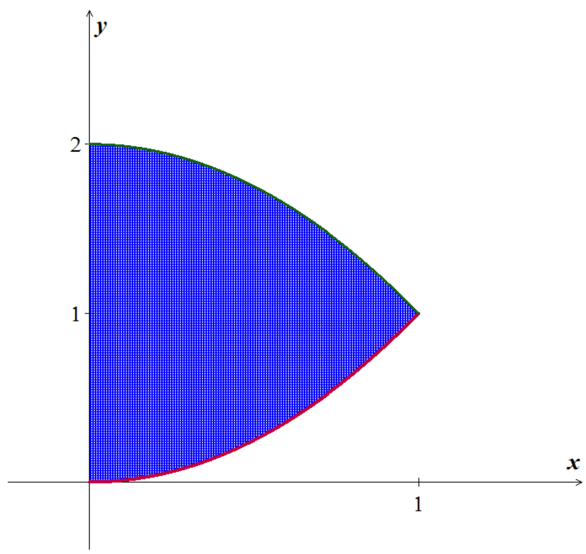
$$= 4\pi \int_0^1 \left(x - x^3\right) dx$$

$$= 4\pi \left[\frac{1}{2}x^2 - \frac{1}{4}x^4 \right]_0^1$$

$$= 4\pi \left[\frac{1}{2} - \frac{1}{4} \right]_0^1$$

$$= 4\pi \left(\frac{1}{4}\right)$$

$$\underline{= \pi \ \ unit^3}$$



Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y=\frac{3}{2\sqrt{x}}$, $\quad y=0$, $\quad x=1$, $\quad x=4$ about the *y*-axis.

Solution

$$V=\int_1^4 2\pi(x)\left(\frac{3}{2\sqrt{x}}-0\right)dx$$

$$=\pi\int_1^4 x\left(3x^{-1/2}\right)dx$$

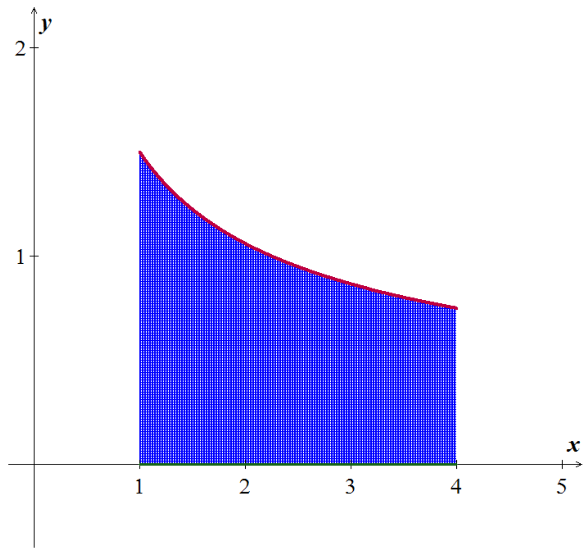
$$=3\pi\int_1^4 x^{1/2}dx$$

$$=3\pi\left[\frac{2}{3}x^{3/2}\right]_1^4$$

$$=2\pi\left[4^{3/2}-1^{3/2}\right]$$

$$=2\pi(7)$$

$$\boxed{=14\pi \ \ unit^3}$$



Exercise

Let $g(x)=\begin{cases}\dfrac{(\tan x)^2}{x} & 0<x\le\frac{\pi}{4}\\ 0 & x=0\end{cases}$

a) Show that $x\cdot g(x)=(\tan x)^2,\quad 0\le x\le\frac{\pi}{4}$

b) Find the volume of the solid generated by revolving the shaded region about the *y*-axis.

Solution

a) $x\cdot g(x)=\begin{cases}x\cdot\dfrac{(\tan x)^2}{x} & 0<x\le\frac{\pi}{4}\\ x\cdot 0 & x=0\end{cases} \quad\Rightarrow x\cdot g(x)=\begin{cases}\tan^2 x & 0<x\le\frac{\pi}{4}\\ 0 & x=0\end{cases}$

Since $x=0 \quad\to\quad \tan x=0 \qquad\qquad \Rightarrow x\cdot g(x)=\begin{cases}\tan^2 x & 0<x\le\frac{\pi}{4}\\ \tan^2 x & x=0\end{cases}$

$$\Rightarrow \boxed{x\cdot g(x)=\tan^2 x \qquad 0\le x\le\frac{\pi}{4}}$$

b) $V = \int_a^b 2\pi \begin{pmatrix} shell \\ radius \end{pmatrix} \begin{pmatrix} shell \\ height \end{pmatrix} dx$

$= 2\pi \int_0^{\pi/4} x \cdot g(x)\, dx$

$= 2\pi \int_0^{\pi/4} \tan^2 x\ dx$

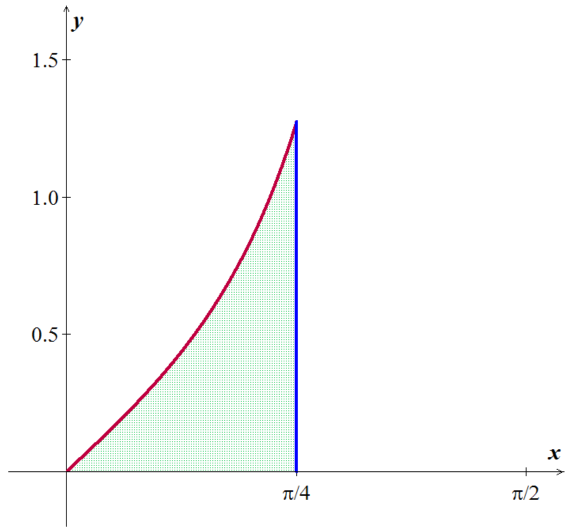
$= 2\pi \left[\tan x - x\right]_0^{\pi/4}$

$= 2\pi \left[\left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - \left(\tan 0 - 0\right)\right]$

$= 2\pi \left(1 - \frac{\pi}{4}\right)$

$= 2\pi \left(\frac{4-\pi}{4}\right)$

$= \frac{4\pi - \pi^2}{2}\ \ unit^3$



Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $x = \sqrt{y}, \quad x = -y, \quad y = 2$ about the x-axis.

Solution

$x = \sqrt{y} = -y \quad \to y = 0 = c$

$V = \int_0^2 2\pi(y)\left(\sqrt{y} - (-y)\right) dy$

$= 2\pi \int_0^2 \left(y^{3/2} + y^2\right) dy$

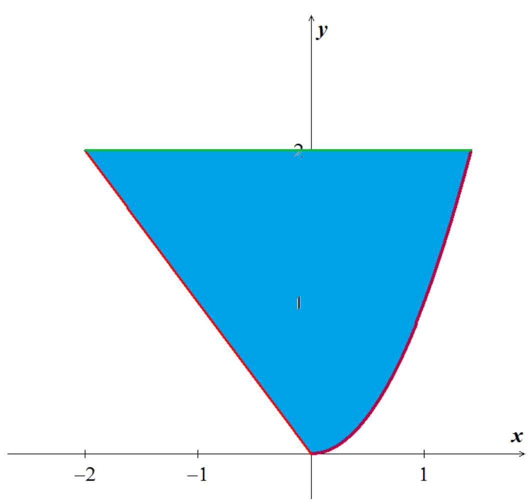
$= 2\pi \left[\frac{2}{5} y^{5/2} + \frac{1}{3} y^3\right]_0^2$

$= 2\pi \left[\frac{2}{5}(2)^{5/2} + \frac{1}{3}(2)^3\right]$

$= 2\pi \left[\frac{8\sqrt{2}}{5} + \frac{8}{3}\right]$

$= 16\pi \left(\frac{3\sqrt{2}+5}{15}\right)$

$= \frac{16}{15}\pi\left(3\sqrt{2}+5\right)\ \ unit^3$

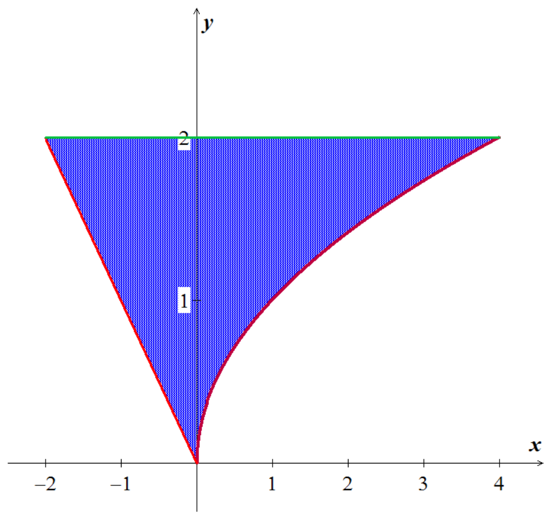


Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $x = y^2, \quad x = -y, \quad y = 2, \quad y \ge 0$ about the *x*-axis.

Solution

$$x = y^2 = -y \quad \to y = 0 = c \quad d = 2$$



$$V = \int_0^2 2\pi(y)\left(y^2 - (-y)\right)dy$$

$$= 2\pi \int_0^2 \left(y^3 + y^2\right)dy$$

$$= 2\pi \left[\frac{1}{4}y^4 + \frac{1}{3}y^3\right]_0^2$$

$$= 2\pi \left(\frac{1}{4}(2)^4 + \frac{1}{3}(2)^3\right)$$

$$= 2\pi \left(4 + \frac{8}{3}\right)$$

$$= 2\pi \left(\frac{20}{3}\right)$$

$$= \frac{40\pi}{3} \quad unit^3$$

Exercise

Use the shell method to find the volumes of the solids generated by revolving the shaded regions about the *indicated* axes.

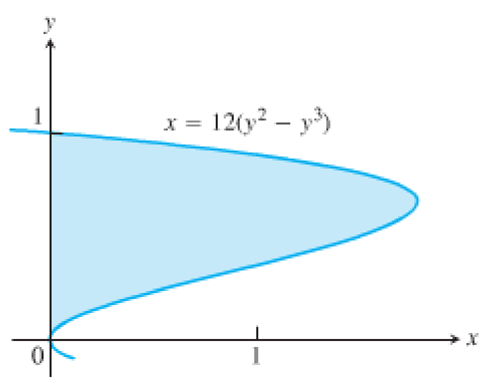
a) The *x*-axis

b) The line $y = 1$

c) The line $y = \frac{8}{5}$

d) The line $y = -\frac{2}{5}$

Solution



a) $V = \int_0^1 2\pi(y) \cdot \left[12\left(y^2 - y^3\right)\right]dy$

$$= 24\pi \int_0^1 \left(y^3 - y^4\right)dy$$

$$= 24\pi \left[\frac{y^4}{4} - \frac{y^5}{5}\right]_0^1$$

$$= 24\pi\left(\frac{1}{4} - \frac{1}{5}\right)$$

$$= 24\pi\left(\frac{1}{20}\right)$$

$$\boxed{= \frac{6\pi}{5} \quad unit^3}$$

b) $$V = \int_0^1 2\pi(1-y)\cdot\left[12\left(y^2 - y^3\right)\right]dy$$

$$= 24\pi\int_0^1 \left(y^2 - y^3 - y^3 + y^4\right)dy$$

$$= 24\pi\int_0^1 \left(y^2 - 2y^3 + y^4\right)dy$$

$$= 24\pi\left[\frac{y^3}{3} - \frac{y^4}{2} + \frac{y^5}{5}\right]_0^1$$

$$= 24\pi\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right)$$

$$= 24\pi\left(\frac{1}{30}\right)$$

$$\boxed{= \frac{4\pi}{5} \quad unit^3}$$

c) $$V = \int_c^d 2\pi\begin{pmatrix} shell \\ radius \end{pmatrix}\begin{pmatrix} shell \\ height \end{pmatrix}dy$$

$$= 2\pi\int_0^1 \left(\frac{8}{5} - y\right)\cdot\left[12\left(y^2 - y^3\right)\right]dy$$

$$= 24\pi\int_0^1 \left(\frac{8}{5}y^2 - \frac{8}{5}y^3 - y^3 + y^4\right)dy$$

$$= 24\pi\int_0^1 \left(\frac{8}{5}y^2 - \frac{13}{5}y^3 + y^4\right)dy$$

$$= 24\pi\left[\frac{8}{15}y^3 - \frac{13}{20}y^4 + \frac{y^5}{5}\right]_0^1$$

$$= 24\pi\left(\frac{8}{15} - \frac{13}{20} + \frac{1}{5}\right)$$

$$= 24\pi\left(\frac{5}{60}\right)$$

$$\boxed{= 2\pi \quad unit^3}$$

d) $V = \int_0^1 2\pi\left(y+\frac{2}{5}\right)\cdot\left[12\left(y^2-y^3\right)\right]dy$

$$= 24\pi\int_0^1 \left(y^3-y^4+\frac{2}{5}y^2-\frac{2}{5}y^3\right)dy$$

$$= 24\pi\int_0^1 \left(\frac{3}{5}y^3-y^4+\frac{2}{5}y^2\right)dy$$

$$= 24\pi\left[\frac{3}{20}y^4-\frac{1}{5}y^4+\frac{2}{15}y^3\right]_0^1$$

$$= 24\pi\left(\frac{3}{20}-\frac{1}{5}+\frac{2}{15}\right)$$

$$= 24\pi\left(\frac{5}{60}\right)$$

$$= 2\pi \ \ unit^3$$

Exercise

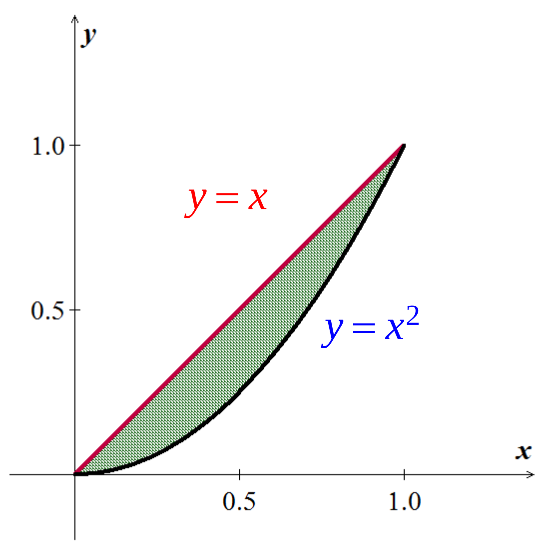
Compute the volume of the solid generated by revolving the region bounded by the lines $y = x \quad and \quad y = x^2$ about each coordinate axis using

a) The *shell* method

b) The *washer* method

Solution

$$y = x = x^2 \Rightarrow x^2 - x = 0 \to \boxed{x = 0,\ 1}$$



a) x-axis

$$V = \int_c^d 2\pi\begin{pmatrix} shell \\ radius\end{pmatrix}\begin{pmatrix} shell \\ height\end{pmatrix}dy$$

$$= \int_0^1 2\pi(y)\cdot\left[\sqrt{y}-y\right]dy$$

$$= 2\pi\int_0^1 \left(y^{3/2}-y^2\right)dy$$

$$= 2\pi\left[\frac{2}{5}y^{5/2}-\frac{1}{3}y^3\right]_0^1$$

$$= 2\pi\left(\frac{2}{5}-\frac{1}{3}\right)$$

$$= \frac{2\pi}{15} \ \ unit^3$$

y-axis

$$V=\int_a^b 2\pi\begin{pmatrix}shell\\radius\end{pmatrix}\begin{pmatrix}shell\\height\end{pmatrix}dx$$

$$=2\pi\int_0^1 (x)\left(x-x^2\right)dx$$

$$=2\pi\int_0^1 \left(x^2-x^3\right)dx$$

$$=2\pi\left[\tfrac{1}{3}x^3-\tfrac{1}{4}x^4\right]_0^1$$

$$=2\pi\left(\tfrac{1}{3}-\tfrac{1}{4}\right)$$

$$=\frac{\pi}{6}\ \ unit^3$$

b) x-axis $\quad R(x)=x \quad and \quad r(x)=x^2$

$$V=\int_a^b \pi\left[R(x)^2-r(x)^2\right]dx$$

$$=\pi\int_0^1 \left(x^2-x^4\right)dx$$

$$=\pi\left[\tfrac{1}{3}x^3-\tfrac{1}{5}x^5\right]_0^1$$

$$=\pi\left(\tfrac{1}{3}-\tfrac{1}{5}\right)$$

$$=\frac{2\pi}{15}\ \ unit^3$$

y-axis $\quad R(y)=\sqrt{y} \quad and \quad r(y)=y$

$$V=\int_c^d \pi\left[R(y)^2-r(y)^2\right]dy$$

$$=\pi\int_0^1 \left(y-y^2\right)dy$$

$$=\pi\left[\tfrac{1}{2}y^2-\tfrac{1}{3}y^3\right]_0^1$$

$$=\pi\left(\tfrac{1}{2}-\tfrac{1}{3}\right)$$

$$=\frac{\pi}{6}\ \ unit^3$$

Exercise

Use the *washer* method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = \sqrt{x}, \quad y = 2, \quad x = 0$ about

a) the *x*-axis

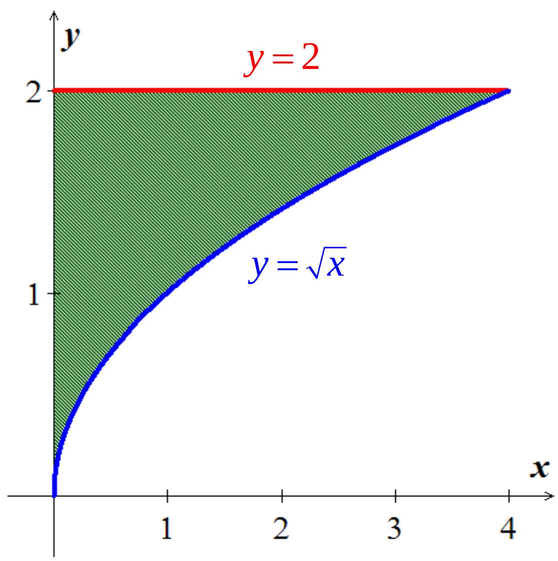
b) the *y*-axis

c) the line $x = 4$

d) the line $y = 1$

Solution

a) x-axis



$$V = \int_c^d 2\pi \binom{shell}{radius}\binom{shell}{height} dy$$

$$= \int_0^2 2\pi (y)\cdot\left(y^2 - 0\right)dy$$

$$= 2\pi \int_0^2 y^3 dy$$

$$= \frac{1}{2}\pi y^4 \Big|_0^2$$

$$= \frac{1}{2}\pi (2)^4$$

$$\underline{= 8\pi \ \ unit^3}$$

b) y-axis

$$V = \int_a^b 2\pi \binom{shell}{radius}\binom{shell}{height} dx$$

$$= 2\pi \int_0^4 (x)\cdot\left(2 - \sqrt{x}\right)dx$$

$$= 2\pi \int_0^4 \left(2x - x^{3/2}\right)dx$$

$$= 2\pi \left[x^2 - \frac{2}{5}x^{5/2}\right]_0^4$$

$$= 2\pi \left(16 - \frac{64}{5}\right)$$

$$\underline{= \frac{32\pi}{5} \ \ unit^3}$$

c) the line $x = 4$

$$V = \int_a^b 2\pi \begin{pmatrix} shell \\ radius \end{pmatrix} \begin{pmatrix} shell \\ height \end{pmatrix} dx$$

$$= \int_0^4 2\pi (4-x)\left(2-\sqrt{x}\right) dx$$

$$= 2\pi \int_0^4 \left(8 - 4x^{1/2} - 2x - x^{3/2}\right) dx$$

$$= 2\pi \left[8x - \frac{8}{3}x^{3/2} - x^2 - \frac{2}{5}x^{5/2} \right]_0^4$$

$$= 2\pi \left[8(4) - \frac{8}{3}(4)^{3/2} - (4)^2 - \frac{2}{5}(4)^{5/2} - 0 \right]$$

$$= 2\pi \left(32 - \frac{64}{3} - 16 + \frac{64}{5} \right)$$

$$\boxed{= \frac{224\pi}{15} \quad unit^3}$$

d) the line $y = 1$

$$V = 2\pi \int_0^2 (2-y)\left(y^2\right) dy$$

$$= 2\pi \int_0^2 \left(2y^2 - y^3\right) dy$$

$$= 2\pi \left[\frac{2}{3}y^3 - \frac{1}{4}y^4 \right]_0^2$$

$$= 2\pi \left[\frac{2}{3}(2)^3 - \frac{1}{4}(2)^4 \right]$$

$$= 2\pi \left(\frac{16}{3} - \frac{16}{4} \right)$$

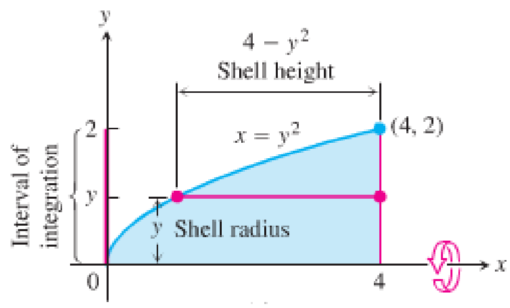
$$= \frac{32\pi}{12}$$

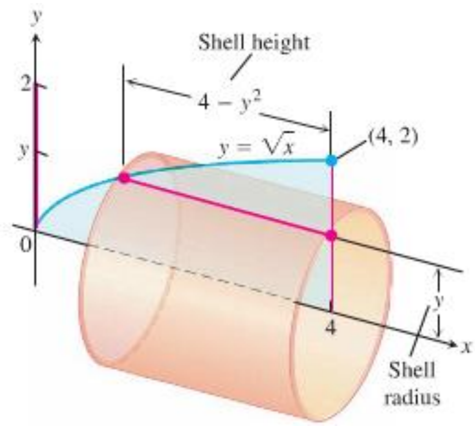
$$\boxed{= \frac{8\pi}{3} \quad unit^3}$$

Exercise

The region bounded by the curve $y=\sqrt{x}$, the *x*-axis, and the line *x* = 4 is revolved about the *x*-axis to generate a solid. Find the volume of the solid.

Solution





$$V = 2\pi\int_c^d \left(\begin{matrix}shell\\radius\end{matrix}\right)\left(\begin{matrix}shell\\height\end{matrix}\right)dy$$

$$= 2\pi\int_0^2 (y)\left(4-y^2\right)dy$$

$$= 2\pi\int_0^2 \left(4y-y^3\right)dy$$

$$= 2\pi\left[2y^2-\frac{y^4}{4}\right]_0^2$$

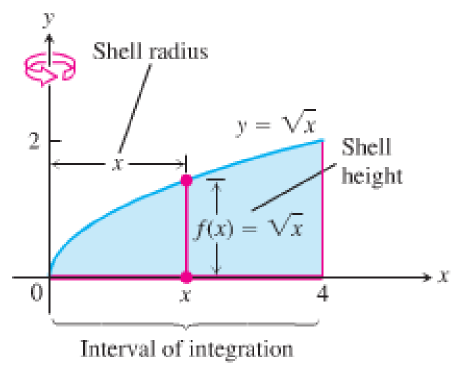
$$= 2\pi\left[2(2)^2-\frac{(2)^4}{4}\right]$$

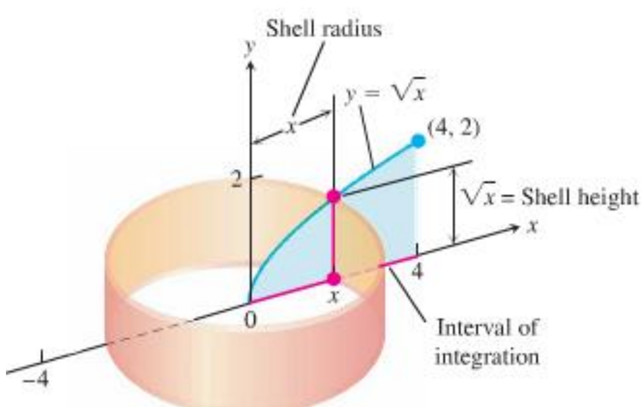
$$\underline{= 8\pi\ \ unit^3}$$

Exercise

The region bounded by the curve $y=\sqrt{x}$, the *x*-axis, and the line *x* = 4 is revolved about the *y*-axis to generate a solid. Find the volume of the solid.

Solution





$$V = \int_a^b 2\pi\left(\begin{matrix}shell\\radius\end{matrix}\right)\left(\begin{matrix}shell\\height\end{matrix}\right)dx$$

$$= 2\pi\int_0^4 (x)\left(\sqrt{x}\right)dx$$

$$= 2\pi\int_0^4 x^{3/2}dx$$

$$= 2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4$$

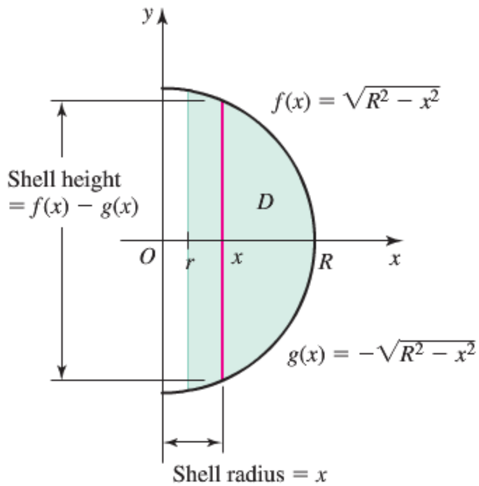
$$= \frac{4}{5}\pi\left[4^{5/2}\right]$$

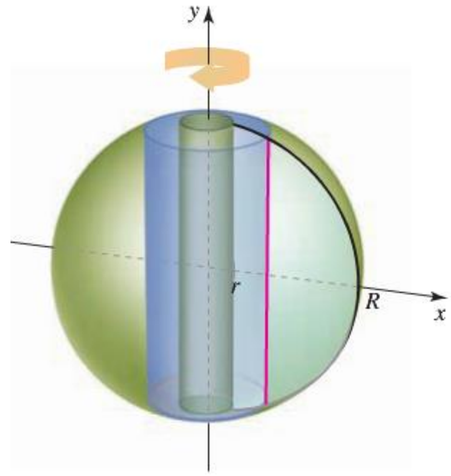
$$\underline{= \frac{128\pi}{5}\ \ unit^3}$$

Exercise

A cylinder hole with radius r is drilled symmetrically through the center of a sphere with radius R, where $r \le R$. What is the volume of the remaining material?

Solution





Let D be the region in the xy-plane bounded above by $f(x) = \sqrt{R^2 - x^2}$, the upper half of the circle of radius R, and bounded below by $g(x) = -\sqrt{R^2 - x^2}$, the lower half of the circle of radius R, for $r \le x \le R$.

The radius of a typical shell is x. Height is $f(x) - g(x) = 2\sqrt{R^2 - x^2}$

$$V = 2\pi \int_r^R x\left(2\sqrt{R^2 - x^2}\right)dx$$

$$= -2\pi \int_r^R \left(R^2 - x^2\right)^{1/2} d\left(R^2 - x^2\right)$$

$$= -\frac{4}{3}\pi\left(R^2 - x^2\right)^{3/2}\Big|_r^R$$

$$\boxed{= \frac{4}{3}\pi\left(R^2 - r^2\right)^{3/2} \quad unit^3}$$

Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = \sqrt{x}, \quad y = 2 - x, \quad y = 0$ about the x-axis.

Solution

$x = y^2$

$y = 2 - x^2 = 2 - y^2 \ \Rightarrow\ y^2 + y - 2 = 0 \to y = \cancel{-2}, 1$

Given: $y = 0$

$$V = 2\pi \int_0^1 y\left(2 - y - y^2\right) dy$$

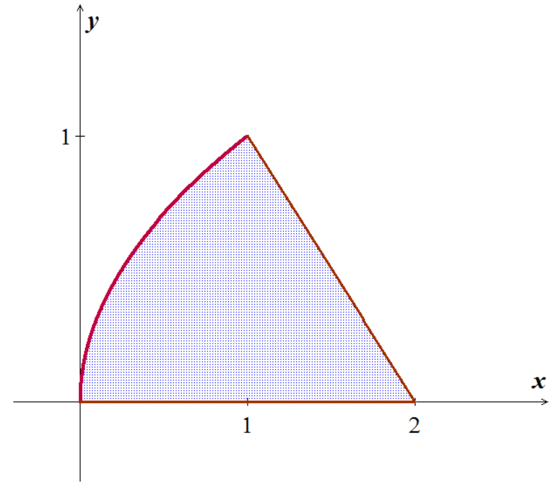
$$= 2\pi \int_0^1 \left(2y - y^2 - y^3\right) dy$$

$$= 2\pi \left(y^2 - \frac{1}{3}y^3 - \frac{1}{4}y^4\right)\Big|_0^1$$

$$= 2\pi \left(1 - \frac{1}{3} - \frac{1}{4}\right)$$

$$= 2\pi \left(\frac{5}{12}\right)$$

$$= \frac{5\pi}{6} \quad unit^3$$



Exercise

Find the volume of the region bounded by $y = \frac{\ln x}{x^2}$, $y = 0$, $x = 1$, *and* $x = 3$ revolved about the *y-axis*

Solution

$$V = 2\pi \int_1^3 x \frac{\ln x}{x^2} dx$$

$$= 2\pi \int_1^3 \ln x \, d(\ln x)$$

$$= \pi (\ln x)^2 \Big|_1^3$$

$$= \pi (\ln 3)^2 \quad unit^3$$

Exercise

Find the volume of the region bounded by $y = \frac{e^x}{x}$, $y = 0$, $x = 1$, *and* $x = 2$ revolved about the *y-axis*

Solution

$$V = 2\pi \int_1^2 x \frac{e^x}{x} dx$$

$$= 2\pi \int_1^2 e^x dx$$

$$= 2\pi e^x \Big|_1^2$$

$$\boxed{= 2\pi\left(e^2 - e\right)\ \ unit^3}$$

Exercise

Find the volume of the region bounded by $y^2 = \ln x,\quad y^2 = \ln x^3,\quad and\ y = 2$ revolved about the *x-axis*

Solution

$$\begin{cases} y^2 = \ln x & \rightarrow \quad x = e^{y^2} \\ y^2 = \ln x^3 & \rightarrow \quad x = e^{y^2/3} \end{cases}$$

$$V = 2\pi \int_0^2 y\left(e^{y^2} - e^{y^2/3}\right)dy$$

$$= \pi \int_0^2 \left(e^{y^2} - e^{y^2/3}\right)d\left(y^2\right)$$

$$= \pi\left(e^{y^2} - 3e^{y^2/3}\right)\Big|_0^2$$

$$= \pi\left(e^4 - 3e^{4/3} - 1 + 3\right)$$

$$\boxed{= \pi\left(2 + e^4 - 3e^{4/3}\right)\ \ unit^3}$$

Exercise

Find the volume using both the disk/washer and shell methods of

$$y = (x-2)^3 - 2, \quad x = 0, \quad y = 25; \text{ revolved about the } y\text{-}axis$$

Solution

Using ***washers***:

$$(x-2)^3 = y + 2 \quad \rightarrow \quad x = 2 + \sqrt[3]{y+2}$$

$$x = 0 \ \Rightarrow\ y = (-2)^3 - 2 = -10$$

$$V = \pi \int_{-10}^{25} \left(2 + \sqrt[3]{y+2}\right)^2 dy \qquad\qquad V = \pi \int_c^d f(y)^2\, dy$$

$$= \pi \int_{-10}^{25} \left(4 + 4(y+2)^{1/3} + (y+2)^{2/3}\right)d(y+2)$$

$$= \pi\left(4(y+2)+3(y+2)^{4/3}+\frac{3}{5}(y+2)^{5/3}\right)\Big|_{-10}^{25}$$

$$= \pi\left(108+3(27)^{4/3}+\frac{3}{5}(27)^{5/3}-\left(-32+3(-8)^{4/3}+\frac{3}{5}(-8)^{5/3}\right)\right)$$

$$= \pi\left(108+243+\frac{729}{5}+32-48+\frac{96}{5}\right)$$

$$= \pi(335+165)$$

$$= 500\pi \ \ unit^3$$

Using ***Shells***:

$$y = 25 \quad \to \; x = 2+\sqrt[3]{27} = 5$$

$$V = 2\pi\int_0^5 x\left(25-(x-2)^3+2\right)dx \qquad\qquad V = 2\pi\int_a^b x\left(f(x)-g(x)\right)dx$$

$$= 2\pi\int_0^5 x\left(27-x^3+6x^2-12x+8\right)dx$$

$$= 2\pi\int_0^5 \left(-x^4+6x^3-12x^2+35x\right)dx$$

$$= 2\pi\left(-\frac{1}{5}x^5+\frac{3}{2}x^4-4x^3+\frac{35}{2}x^2\right)\Big|_0^5$$

$$= 2\pi\left(-5^4+\frac{3}{2}5^4-4(5)^3+\frac{35}{2}(5)^2\right)$$

$$= 2\pi\left(-625+\frac{1875}{2}-500+\frac{875}{2}\right)$$

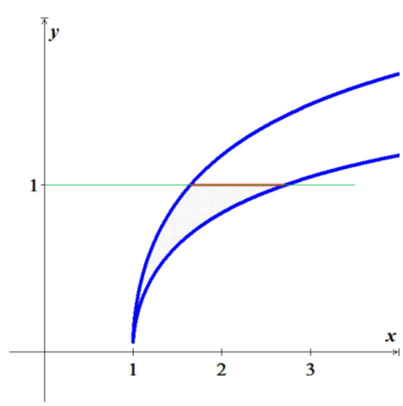
$$= 2\pi(250)$$

$$= 500\pi \ \ unit^3$$

Exercise

Find the volume using both the disk/washer and shell methods of

$$y = \sqrt{\ln x}, \quad y = \sqrt{\ln x^2}, \quad y = 1; \text{ revolved about the } x\text{-}axis$$



Solution

Using ***washers***:

$$y = \sqrt{\ln x} = \sqrt{\ln x^2} \quad \to \quad \ln x = \ln x^2$$

$$x = x^2 \;\Rightarrow\; x = 0,\ 1$$

$$y = 1 = \sqrt{\ln x} \quad \Rightarrow \quad x = e$$

$$y = 1 = \sqrt{\ln x^2} \quad \Rightarrow \quad x^2 = e \;\to\; x = \sqrt{e}$$

$$V=\pi\int_1^{\sqrt{e}}\left(\ln x^2-\ln x\right)dx+\pi\int_{\sqrt{e}}^{e}(1-\ln x)\,dx \qquad V=\pi\int_a^b\left(f(x)^2-g(x)^2\right)dx$$

$$=\pi\int_1^{\sqrt{e}}(2\ln x-\ln x)\,dx+\pi\int_{\sqrt{e}}^{e}(1-\ln x)\,dx$$

$$=\pi\int_1^{\sqrt{e}}(\ln x)\,dx+\pi\int_{\sqrt{e}}^{e}(1-\ln x)\,dx \qquad \int \ln x\,dx=x\ln x-x$$

$$=\pi(x\ln x-x)\Big|_1^{\sqrt{e}}+\pi(2x-x\ln x)\Big|_{\sqrt{e}}^{e}$$

$$=\pi\left(\tfrac{1}{2}\sqrt{e}-\sqrt{e}+1\right)+\pi\left(2e-e-2\sqrt{e}+\tfrac{1}{2}\sqrt{e}\right)$$

$$=\pi\left(-\tfrac{1}{2}\sqrt{e}+1+e-2\sqrt{e}+\tfrac{1}{2}\sqrt{e}\right)$$

$$=\pi\left(e-2\sqrt{e}+1\right)$$

$$\boxed{=\pi\left(\sqrt{e}-1\right)^2\ \ unit^3}$$

Using ***Shells***:

$$y=\sqrt{\ln x}\ \Rightarrow\ \boxed{x=e^{y^2}}$$

$$y=\sqrt{\ln x^2}\ \Rightarrow\ 2\ln x=y^2\ \to\ \boxed{x=e^{y^2/2}}$$

$$V=2\pi\int_0^1 y\left(e^{y^2}-e^{y^2/2}\right)dy \qquad V=2\pi\int_c^d y\left(p(y)-q(y)\right)dy$$

$$=\pi\int_0^1 e^{y^2}d\left(y^2\right)-2\pi\int_0^1 e^{y^2/2}d\left(\tfrac{1}{2}y^2\right)$$

$$=\pi\left(e^{y^2}-2e^{y^2/2}\right)\Big|_0^1$$

$$=\pi\left(e-2e^{1/2}-1+2\right)$$

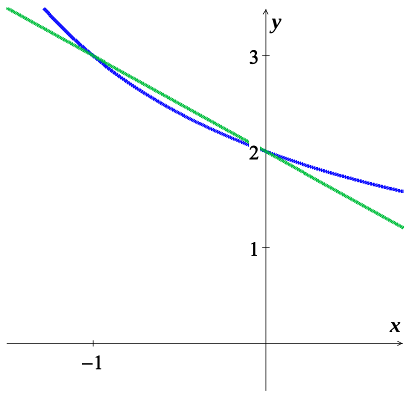
$$=\pi\left(e-2\sqrt{e}+1\right)$$

$$\boxed{=\pi\left(\sqrt{e}-1\right)^2\ \ unit^3}$$

Exercise

Find the volume using both the disk/washer and shell methods of

$$y = \frac{6}{x+3}, \quad y = 2 - x; \text{ revolved about the } x\text{-axis}$$



Solution

Using ***washers***:

$$y = \frac{6}{x+3} = 2 - x$$

$$-x^2 - x + 6 = 6$$

$$x(x+1) = 0 \Rightarrow \underline{x = -1,\ 0}|$$

$$V = \pi\int_{-1}^{0}\left((2-x)^2 - \frac{36}{(x+3)^2}\right)dx \qquad V = \pi\int_a^b\left(f(x)^2 - g(x)^2\right)dx$$

$$= \pi\int_{-1}^{0} -(2-x)^2\, d(2-x) - \pi\int_{-1}^{0}\frac{36}{(x+3)^2}\, d(x+3)$$

$$= \pi\left(-\frac{1}{3}(2-x)^3 + \frac{36}{x+3}\right)_{-1}^{0}$$

$$= \pi\left(-\frac{8}{3} + 12 + 9 - 18\right)$$

$$= \frac{\pi}{3}\ \ unit^3$$

Using ***Shells***:

$$y = \frac{6}{x+3} \ \rightarrow\ x = \frac{6}{y} - 3$$

$$y = 2 - x \ \rightarrow\ x = 2 - y$$

$$V = 2\pi\int_2^3 y\left(2 - y - \frac{6}{y} + 3\right)dy \qquad V = 2\pi\int_c^d y\left(p(y) - q(y)\right)dy$$

$$= 2\pi\int_2^3\left(5y - y^2 - 6\right)dy$$

$$= 2\pi\left(\frac{5}{2}y^2 - \frac{1}{3}y^3 - 6y\right)_2^3$$

$$= 2\pi\left(\frac{45}{2} - 9 - 18 - 10 + \frac{8}{3} + 12\right)$$

$$= 2\pi\left(\frac{151}{6} - 25\right)$$

$$= \frac{\pi}{3}\ \ unit^3$$

Solution Section 1.5 – Length of Curves

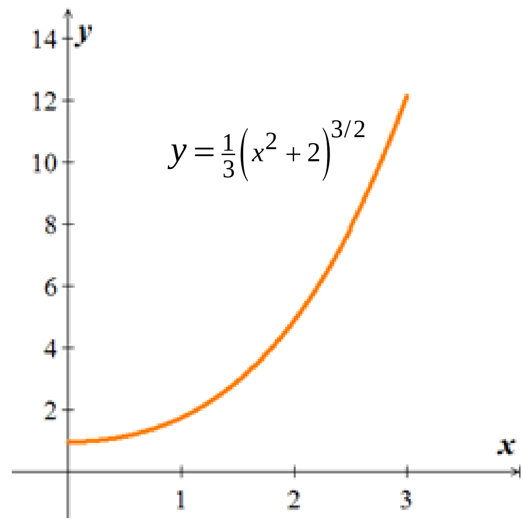
Exercise

Find the length of the curve $y=\frac{1}{3}\left(x^2+2\right)^{3/2}$ from $x = 0$ to $x = 3$.

Solution

$$\frac{dy}{dx}=\frac{1}{3}\frac{3}{2}\left(x^2+2\right)^{1/2}(2x)=x\left(x^2+2\right)^{1/2}$$

$$\begin{aligned}\sqrt{1+\left(\frac{dy}{dx}\right)^2}&=\sqrt{1+x^2\left(x^2+2\right)}\\&=\sqrt{1+x^4+2x^2}\\&=\sqrt{\left(x^2+1\right)^2}\\&=x^2+1\end{aligned}$$



$$\begin{aligned}L&=\int_0^3\left(x^2+1\right)\,dx\\&=\left[\frac{1}{3}x^3+x\right]_0^3\\&=\frac{1}{3}(3)^3+(3)-0\\&=12\ \ unit\end{aligned}$$

Exercise

Find the length of the curve $y=(x)^{3/2}$ from $x = 0$ to $x = 4$.

Solution

$$\frac{dy}{dx}=\frac{3}{2}x^{1/2}$$

$$\begin{aligned}\sqrt{1+\left(\frac{dy}{dx}\right)^2}&=\sqrt{1+\frac{9}{4}x}\\&=\sqrt{\frac{4+9x}{4}}\\&=\frac{1}{2}\sqrt{4+9x}\end{aligned}$$

$$L=\int_0^4\frac{1}{2}(4+9x)^{1/2}\,dx \qquad u=4+9x\ \Rightarrow du=9dx\ \rightarrow\frac{1}{9}du=dx\quad\begin{cases}x=4 & \rightarrow u=40\\ x=0 & \rightarrow u=4\end{cases}$$

$$= \frac{1}{2}\int_4^{40} \frac{1}{9}u^{1/2}du$$

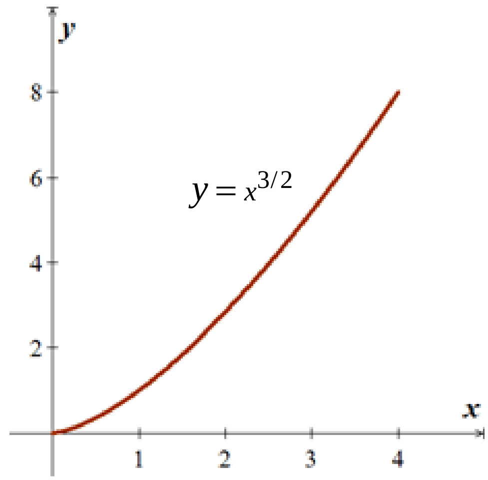
$$= \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_4^{40}$$

$$= \frac{1}{27}\left(40^{3/2} - 4^{3/2}\right)$$

$$= \frac{1}{27}\left(\sqrt{40^3} - \sqrt{4^3}\right)$$

$$= \frac{1}{27}\left(80\sqrt{10} - 8\right)$$

$$= \frac{8}{27}\left(10\sqrt{10} - 1\right) \ \ unit$$



Exercise

Find the length of the curve $x = \frac{y^{3/2}}{3} - y^{1/2}$ from $y = 1$ to $y = 9$.

Solution

$$\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} = \frac{1}{2}\left(y^{1/2} - \frac{1}{y^{1/2}}\right)$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\frac{1}{4}\left(y^{1/2} - \frac{1}{y^{1/2}}\right)^2}$$

$$= \sqrt{1+\frac{1}{4}\left(y - 2 + \frac{1}{y}\right)}$$

$$= \sqrt{1+\frac{1}{4}y - \frac{1}{2} + \frac{1}{4y}}$$

$$= \sqrt{\frac{1}{4}y + \frac{1}{2} + \frac{1}{4y}}$$

$$= \sqrt{\frac{1}{4}\left(y + 2 + \frac{1}{y}\right)}$$

$$= \frac{1}{2}\sqrt{\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2}$$

$$= \frac{1}{2}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)$$

$$L = \frac{1}{2}\int_1^9 \left(y^{1/2} + y^{-1/2}\right)dy$$

$a = \frac{1}{3}, \quad m = \frac{3}{2}, \quad b = -1, \quad n = \frac{1}{2}$

1. $m + n = \frac{3}{2} + \frac{1}{2} = 2$ ✓
2. $abmn = \frac{1}{3}(-1)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(\frac{1}{3}y^{3/2} + y^{1/2}\right)_1^9$$

$$= 9 + 3 - \frac{4}{3}$$

$$= \frac{32}{3} \ \ unit$$

$$= \frac{1}{2}\left[\frac{2}{3}y^{3/2} + 2y^{1/2}\right]_1^9$$

$$= \left[\frac{1}{3}y^{3/2} + y^{1/2}\right]_1^9$$

$$= \left[\frac{1}{3}9^{3/2} + 9^{1/2} - \left(\frac{1}{3}1^{3/2} + 1^{1/2}\right)\right]$$

$$= \frac{1}{3}3^3 + 3 - \left(\frac{1}{3} + 1\right)$$

$$= 9 + 3 - \frac{4}{3}$$

$$= \frac{32}{3} \quad unit$$

Exercise

Find the length of the curve $x = \frac{y^3}{6} + \frac{1}{2y}$ from $y = 2$ to $y = 3$.

Solution

$$\frac{dx}{dy} = \frac{1}{2}y^2 - \frac{1}{2y^2} = \frac{1}{2}\left(y^2 - y^{-2}\right)$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \frac{1}{4}\left(y^2 - y^{-2}\right)^2}$$

$$= \frac{1}{2}\sqrt{4 + \left(y^4 - 2 + y^{-4}\right)}$$

$$= \frac{1}{2}\sqrt{y^4 + 2 + y^{-4}}$$

$$= \frac{1}{2}\sqrt{\left(y^2 + y^{-2}\right)^2}$$

$$= \frac{1}{2}\left(y^2 + y^{-2}\right)$$

$$L = \frac{1}{2}\int_2^3 \left(y^2 + y^{-2}\right)dy$$

$$= \frac{1}{2}\left[\frac{1}{3}y^3 - y^{-1}\right]_2^3$$

$$= \frac{1}{2}\left[\left(\frac{1}{3}3^3 - 3^{-1}\right) - \left(\frac{1}{3}2^3 - 2^{-1}\right)\right]$$

$$= \frac{1}{2}\left[9 - \frac{1}{3} - \left(\frac{8}{3} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left(\frac{26}{3} - \frac{13}{6}\right)$$

$$= \frac{13}{4} \quad unit$$

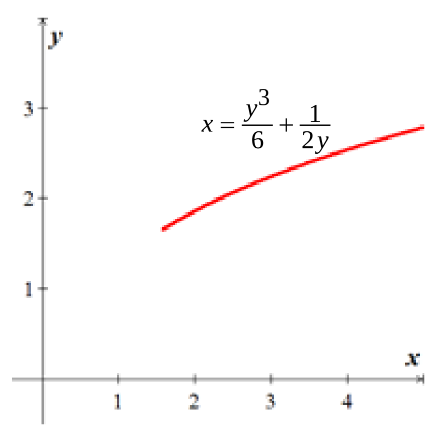
$a = \frac{1}{6}, \quad m = 3, \quad b = \frac{1}{2}, \quad n = -1$

1. $m + n = 3 - 1 = 2$ ✓
2. $abmn = \frac{1}{6}\left(\frac{1}{2}\right)(3)(-1) = -\frac{1}{4}$ ✓

$$L = \left(\frac{y^3}{6} - \frac{1}{2y}\right)_1^9$$

$$= \frac{1}{2}\left[9 - \frac{1}{3} - \left(\frac{8}{3} - \frac{1}{2}\right)\right]$$

$$= \frac{13}{4} \quad unit$$



Exercise

Find the length of the curve $f(x)=x^3+\frac{1}{12x}$ *for* $\frac{1}{2}\le x\le 2$

Solution

$$a=1,\quad m=3,\quad b=\frac{1}{12},\quad n=-1$$

1. $m+n=2$ ✓ **2.** $abmn=-\frac{1}{4}$ ✓

$$L=\left(x^3-\frac{1}{12x}\right)\Big|_{1/2}^{2}$$

$$=8-\frac{1}{24}-\frac{1}{8}+\frac{1}{6}$$

$$\underline{=8\ \textit{unit}}$$

Exercise

Find the length of the curve $y=\ln(\cos x)\quad 0\le x\le\frac{\pi}{4}$

Solution

$$\frac{dy}{dx}=\frac{-\sin x}{\cos x}=-\tan x$$

$$L=\int_0^{\pi/4}\sqrt{1+\tan^2 x}\ dx$$

$$=\int_0^{\pi/4}\sqrt{\sec^2 x}\ dx$$

$$=\int_0^{\pi/4}\sec x\ dx$$

$$=\Big[\ln\left|\sec x+\tan x\right|\Big]_0^{\pi/4}$$

$$=\ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right|-\ln\left|\sec 0+\tan 0\right|$$

$$=\ln\left|\sqrt{2}+1\right|-\ln\left|1+0\right|$$

$$=\ln\left|\sqrt{2}+1\right|-0$$

$$\underline{=\ln\left(\sqrt{2}+1\right)\ \textit{unit}}$$

Exercise

Find the length of the curve $y = 2e^{\sqrt{2}y} + \frac{1}{16}e^{-\sqrt{2}y}$ *for* $0 \le y \le \frac{\ln 2}{\sqrt{2}}$

Solution

$a = 2, \quad m = \sqrt{2}, \quad b = \frac{1}{16}, \quad n = -\sqrt{2}$

1. $m = -n$ ✓ **2.** $abmn = 2\left(\sqrt{2}\right)\left(\frac{1}{16}\right)\left(-\sqrt{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(2e^{\sqrt{2}y} + \frac{1}{16}e^{-\sqrt{2}y}\right)\Big|_0^{\ln 2/\sqrt{2}}$$

$$= 2e^{\ln 2} + \frac{1}{16}e^{-\ln 2} - 2 - \frac{1}{16}$$

$$= 4 + \frac{1}{32} - \frac{33}{16}$$

$$= \frac{63}{32} \ \textit{unit}$$

Exercise

Find the length of the curve $y = \frac{x^3}{3} + x^2 + x + 1 + \frac{1}{4x+4} \quad 0 \le x \le 2$

Solution

$$\frac{dy}{dx} = x^2 + 2x + 1 - \frac{1}{4}\frac{1}{(x+1)^2} = (x+1)^2 - \frac{1}{4}\frac{1}{(x+1)^2}$$

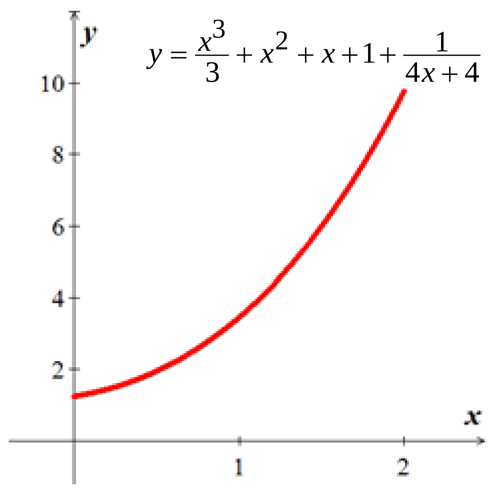
$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left((x+1)^2 - \frac{1}{4}\frac{1}{(x+1)^2}\right)^2}$$

$$= \sqrt{1 + (x+1)^4 - \frac{1}{2} - \frac{1}{16}\frac{1}{(x+1)^4}}$$

$$= \sqrt{(x+1)^4 + \frac{1}{2} - \frac{1}{16}\frac{1}{(x+1)^4}}$$

$$= \sqrt{\left((x+1)^2 + \frac{1}{4}\frac{1}{(x+1)^2}\right)^2}$$

$$= (x+1)^2 + \frac{1}{4}(x+1)^{-2}$$



$$L = \int_0^2 \left((x+1)^2 + \frac{1}{4}(x+1)^{-2}\right)dx \qquad u = x+1 \ \Rightarrow du = dx \quad \begin{cases} x = 2 & \to u = 3 \\ x = 0 & \to u = 1 \end{cases}$$

$$= \int_1^3 \left(u^2 + \frac{1}{4}u^{-2}\right)du$$

$$=\left[\frac{1}{3}u^3-\frac{1}{4}u^{-1}\right]_1^3$$

$$=9-\frac{1}{12}-\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$=9-\frac{1}{12}-\frac{1}{12}$$

$$=\frac{53}{6}\ \ unit$$

Exercise

Find the length of the curve $y=\ln\left(e^x-1\right)-\ln\left(e^x+1\right)\quad \ln 2\le x\le \ln 3$

Solution

$$y=\ln\left(e^x-1\right)-\ln\left(e^x+1\right)\ \Rightarrow \frac{dy}{dx}=\frac{e^x}{e^x-1}-\frac{e^x}{e^x+1}$$

$$=\frac{e^{2x}+e^x-e^{2x}+e^x}{e^{2x}-1}$$

$$=\frac{2e^x}{e^{2x}-1}$$

$$L=\int_{\ln 2}^{\ln 3}\sqrt{1+\left(\frac{2e^x}{e^{2x}-1}\right)^2}\ dx$$

$$=\int_{\ln 2}^{\ln 3}\sqrt{1+\frac{4e^{2x}}{e^{4x}-2e^{2x}+1}}\ dx$$

$$=\int_{\ln 2}^{\ln 3}\sqrt{\frac{e^{4x}-2e^{2x}+1+4e^{2x}}{\left(e^{2x}-1\right)^2}}\ dx$$

$$=\int_{\ln 2}^{\ln 3}\sqrt{\frac{e^{4x}+2e^{2x}+1}{\left(e^{2x}-1\right)^2}}\ dx$$

$$=\int_{\ln 2}^{\ln 3}\sqrt{\frac{\left(e^{2x}+1\right)^2}{\left(e^{2x}-1\right)^2}}\ dx$$

$$=\int_{\ln 2}^{\ln 3}\frac{e^{2x}+1}{e^{2x}-1}\ dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\frac{e^{2x}+1}{e^x}}{\frac{e^{2x}-1}{e^x}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\frac{e^{2x}}{e^x}+\frac{1}{e^x}}{\frac{e^{2x}}{e^x}-\frac{1}{e^x}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{e^x+e^{-x}}{e^x-e^{-x}}\, dx \qquad \textit{or} \;\; \text{Let } u = e^x - e^{-x} \;\; \Rightarrow du = \left(e^x+e^{-x}\right)dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{1}{e^x-e^{-x}}\, d\left(e^x-e^{-x}\right) \qquad d\left(e^x-e^{-x}\right) = \left(e^x+e^{-x}\right)dx$$

$$= \left[\ln\left|e^x-e^{-x}\right|\right]_{\ln 2}^{\ln 3}$$

$$= \ln\left(3-\tfrac{1}{3}\right) - \ln\left(2-\tfrac{1}{2}\right)$$

$$= \ln\left(\tfrac{8}{3}\right) - \ln\left(\tfrac{3}{2}\right)$$

$$= \ln\left(\tfrac{16}{9}\right) \;\; \textit{unit}$$

Exercise

Find the length of the curve $f(x) = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2} \quad 1 \le x \le 4$

Solution

$a = \frac{2}{3}, \quad m = \frac{3}{2}, \quad b = -\frac{1}{2}, \quad n = \frac{1}{2}$

1. $m + n = \frac{3}{2} + \frac{1}{2} = 2$ ✓ **2.** $abmn = \frac{2}{3}\left(\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(\frac{2}{3}x^{3/2} + \frac{1}{2}x^{1/2}\right)\Big|_1^4$$

$$= \frac{2}{3}4^{3/2} + 1 - \frac{2}{3} - \frac{1}{2}$$

$$= \frac{16}{3} - \frac{2}{3} + \frac{1}{2}$$

$$= \frac{31}{6} \;\; \textit{unit}$$

Exercise

Find the length of the curve $f(x)=x^3+\frac{1}{12x}\quad 1\le x\le 4$

Solution

$a=1,\quad m=3,\quad b=\frac{1}{12},\quad n=-1$

1. $m+n=3-1=2$ ✓ **2.** $abmn=(1)\left(\frac{1}{12}\right)(3)(-1)=-\frac{1}{4}$ ✓

$$L=\left(x^3-\frac{1}{12x}\right)\Big|_1^4$$

$$=4^3-\frac{1}{48}-1+\frac{1}{12}$$

$$=63+\frac{3}{48}$$

$$=\frac{3,027}{48}\ \text{unit}$$

Exercise

Find the length of the curve $f(x)=\frac{1}{8}x^4+\frac{1}{4x^2}\quad 1\le x\le 10$

Solution

$a=\frac{1}{8},\quad m=4,\quad b=\frac{1}{4},\quad n=-2$

1. $m+n=4-2=2$ ✓ **2.** $abmn=\left(\frac{1}{8}\right)\left(\frac{1}{4}\right)(4)(-2)=-\frac{1}{4}$ ✓

$$L=\left(\frac{1}{8}x^4-\frac{1}{4x^2}\right)\Big|_1^{10}$$

$$=\frac{10^4}{8}-\frac{1}{400}-\frac{1}{8}+\frac{1}{4}$$

$$=\frac{9,999}{8}+\frac{99}{400}$$

$$=\frac{9}{8}\left(1111+\frac{11}{50}\right)$$

$$=\frac{9}{8}\left(\frac{55,561}{50}\right)$$

$$=\frac{500,049}{400}\ \text{unit}$$

Exercise

Find the length of the curve $f(x)=\frac{1}{4}x^4+\frac{1}{8x^2} \quad 3\le x\le 8$

Solution

$a=\frac{1}{4}, \quad m=4, \quad b=\frac{1}{8}, \quad n=-2$

1. $m+n=4-2=2$ ✓ **2.** $abmn=\left(\frac{1}{4}\right)\left(\frac{1}{8}\right)(4)(-2)=-\frac{1}{4}$ ✓

$$L=\left(\frac{1}{4}x^4-\frac{1}{8x^2}\right)\Big|_3^8$$

$$=\frac{8^4}{4}-\frac{1}{8^3}-\frac{81}{4}+\frac{1}{72}$$

$$=\frac{4,015}{4}-\frac{1}{512}+\frac{1}{72}$$

$$=\frac{1}{4}\left(4,015-\frac{1}{128}+\frac{1}{18}\right)$$

$$=\frac{1}{4}\left(4,015+\frac{55}{1152}\right)$$

$$=\frac{4,625,335}{4,608} \quad unit$$

Exercise

Find the length of the curve $f(x)=\frac{1}{10}x^5+\frac{1}{6x^3} \quad 1\le x\le 7$

Solution

$a=\frac{1}{10}, \quad m=5, \quad b=\frac{1}{6}, \quad n=-3$

1. $m+n=5-3=2$ ✓ **2.** $abmn=\left(\frac{1}{10}\right)\left(\frac{1}{6}\right)(5)(-3)=-\frac{1}{4}$ ✓

$$L=\left(\frac{1}{10}x^5-\frac{1}{6x^3}\right)\Big|_1^7$$

$$=\frac{7^5}{10}-\frac{1}{2058}-\frac{1}{10}+\frac{1}{6}$$

$$=\frac{8403}{5}+\frac{57}{343}$$

$$=\frac{2,882,514}{1,715} \quad unit$$

Exercise

Find the length of the curve $f(x)=\frac{3}{10}x^{1/3}-\frac{3}{2}x^{5/3} \quad 0\le x\le 12$

Solution

$$a=\frac{1}{10}, \quad m=5, \quad b=\frac{1}{6}, \quad n=-3$$

1. $m+n=5-3=2$ ✓ **2.** $abmn=\left(\frac{1}{10}\right)\left(\frac{1}{6}\right)(5)(-3)=-\frac{1}{4}$ ✓

$$L=\left(\frac{3}{10}x^{1/3}+\frac{3}{2}x^{5/3}\right)\Big|_0^{12}$$

$$=\frac{3}{10}\sqrt[3]{12}+\frac{3}{2}12\sqrt[3]{144}$$

$$\boxed{=\frac{3}{10}\sqrt[3]{12}+18\sqrt[3]{144} \quad unit} \qquad =\frac{3}{10}\sqrt[3]{12}\left(1+600\sqrt[3]{12}\right)$$

Exercise

Find the length of the curve $f(x)=x^{1/2}-\frac{1}{3}x^{3/2} \quad 2\le x\le 9$

Solution

$$a=1, \quad m=\frac{1}{2}, \quad b=-\frac{1}{3}, \quad n=\frac{3}{2}$$

1. $m+n=\frac{1}{2}+\frac{3}{2}=2$ ✓ **2.** $abmn=(1)\left(-\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)=-\frac{1}{4}$ ✓

$$L=\left(x^{1/2}+\frac{1}{3}x^{3/2}\right)\Big|_2^{9}$$

$$=3+9-\sqrt{2}-\frac{2\sqrt{2}}{3}$$

$$\boxed{=\frac{1}{3}\left(36-5\sqrt{2}\right) \quad unit} \qquad =\frac{3}{10}\sqrt[3]{12}\left(1+600\sqrt[3]{12}\right)$$

Exercise

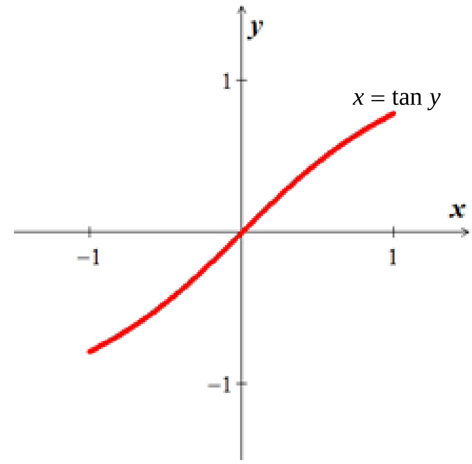
Find the length of the curve $x=\int_0^y \sqrt{\sec^4 t-1}\,dt \quad -\frac{\pi}{4}\le y\le\frac{\pi}{4}$

Solution

$$\frac{dx}{dy}=\sqrt{\sec^4 y-1}$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2}=\sqrt{1+\sec^4 y-1}$$

$$= \sqrt{\sec^4 y}$$
$$= \sec^2 y$$



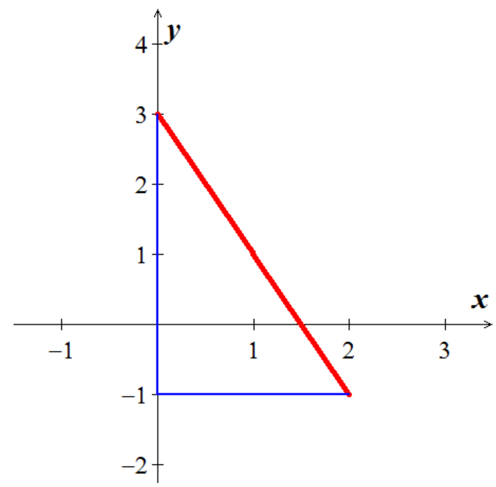
$$L = \int_{-\pi/4}^{\pi/4} \sec^2 y \; dy$$
$$= \tan y \Big|_{-\pi/4}^{\pi/4}$$
$$= 1 - (-1)$$
$$\underline{= 2 \ \ unit}$$

Exercise

Find the length of the curve $y = 3 - 2x \quad 0 \le x \le 2$. Check your answer by finding the length of the segment as the hypotenuse of a right triangle.

Solution

$$\frac{dy}{dx} = -2$$
$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1+4} = \sqrt{5}$$
$$L = \int_0^2 \sqrt{5} \; dx$$
$$= \sqrt{5}x \Big|_0^2$$
$$\underline{= 2\sqrt{5} \ \ unit}$$



$$\begin{cases} x = 0 & \rightarrow y = 3 \\ x = 2 & \rightarrow y = -1 \end{cases}$$

$$d = \sqrt{(2-0)^2 + (3+1)^2} = \sqrt{4+16} = \sqrt{20} \ \underline{= 2\sqrt{5} \ \ unit}$$

Exercise

Find a curve through the origin in the *xy*-plane whose length from $x = 0$ to $x = 1$ is $L = \int_0^1 \sqrt{1+\frac{1}{4}e^x}\, dx$

Solution

$$L = \int_0^1 \sqrt{1+\frac{1}{4}e^x}\, dx \quad \Rightarrow \frac{dy}{dx} = \frac{e^{x/2}}{2} \qquad \to dy = \frac{e^{x/2}}{2}dx$$

$$y = \int \frac{e^{x/2}}{2}dx = e^{x/2} + C$$

$$0 = e^{0/2} + C$$

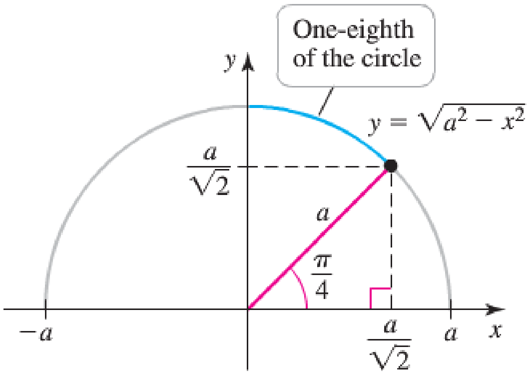
$$0 = 1 + C \qquad \Rightarrow \underline{C = -1}$$

$$\underline{y = e^{x/2} - 1}$$

Exercise

Confirm that the circumference of a circle of radius a is $2\pi a$.

Solution



$$f(x) = \sqrt{a^2 - x^2} \quad for \quad -a \le x \le a$$

$$f'(x) = -\frac{x}{\sqrt{a^2 - x^2}} \qquad \text{but } \underline{x \ne \pm a}$$

$$\sqrt{1+f'(x)^2} = \sqrt{1+\frac{x^2}{a^2 - x^2}}$$

$$= \frac{a}{\sqrt{a^2 - x^2}}$$

Let's compute the length of $\frac{1}{8}$ of the circle on $\left[0, \frac{a}{\sqrt{2}}\right]$

$$L = 8a\int_0^{a/\sqrt{2}} \frac{dx}{\sqrt{a^2 - x^2}}$$

$$= 8a\sin^{-1}\left(\frac{x}{a}\right)\Big|_0^{a/\sqrt{2}}$$

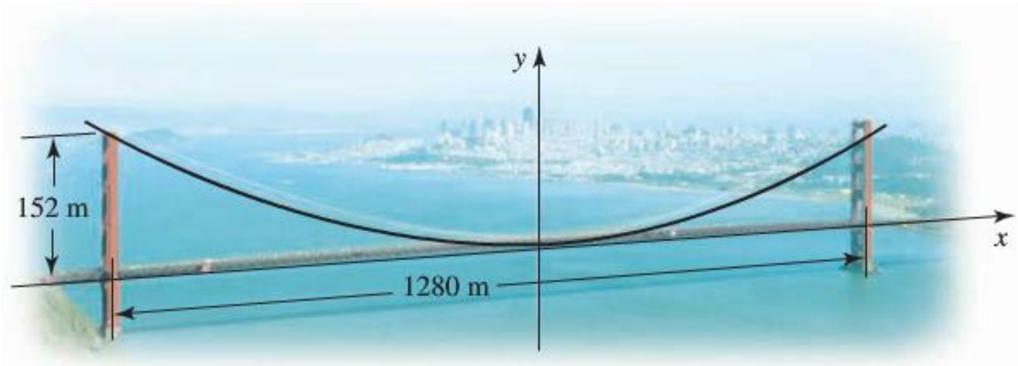
$$= 8a\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= 8a\left(\frac{\pi}{4}\right)$$

$$\underline{= 2\pi a \;\; unit}$$

Exercise

The profile of the cables on a suspension bridge may be modeled by a parabola. The central span of the Golden Gate Bridge is 1280 *m* long and 152 *m* high. The parabola $y = 0.00037x^2$ gives a good fit to the shape of the cables, where $|x| \le 640$, and x and y are measured in meters. Approximate the length of the cables that stretch between the tops of the two towers.



Solution

$$y' = 0.00074x$$

$$L = \int_{-640}^{640} \sqrt{1 + (.00074x)^2}\, dx \qquad\qquad \int \sqrt{a^2 + x^2}\,dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\ln\left|x + \sqrt{a^2 + x^2}\right|$$

$$= \left(\frac{x}{2}\sqrt{1 + x^2} + \frac{1}{2}\ln\left|x + \sqrt{1 + x^2}\right|\right)\Bigg|_{-640}^{640}$$

$$= 320\sqrt{1 + 640^2} + \frac{1}{2}\ln\left|640 + \sqrt{1 + 640^2}\right| + 320\sqrt{1 + x^2} - \frac{1}{2}\ln\left|-640 + \sqrt{1 + 640^2}\right|$$

$$\approx 1326.4\ m$$