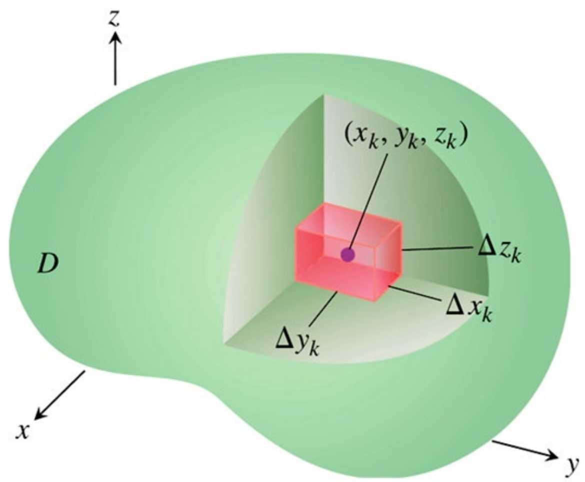
Section 3.4 – Triple Integrals

Triple Integrals

If $F(x,y,z)$ is a function defined on a closed, bounded region *D* in space, such a solid ball or a lump of clay, then the integral of *F* over *D* may be defined in the following way.



$$\Delta V_k = \Delta x_k \Delta y_k \Delta z_k \quad \rightarrow \quad S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta V_k$$

The limit of this summation is the triple integral of *F* over *D*

$$\lim_{n\to\infty} S_n = \iiint_D F(x,y,z)\ dV \quad or \quad \lim_{\|P\|\to} S_n = \iiint_D F(x,y,z)\ dxdydz$$

Volume of a region in Space

Definition

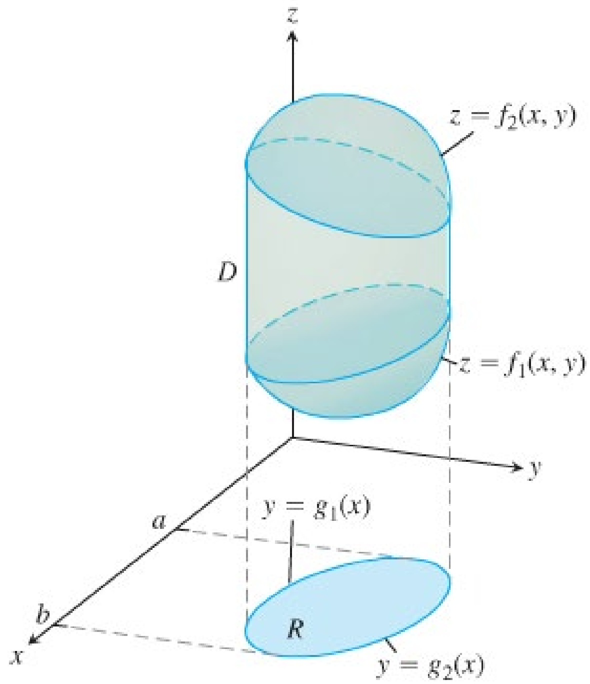
The volume of a closed, bounded region *D* in space is

$$V = \iiint_D dV$$

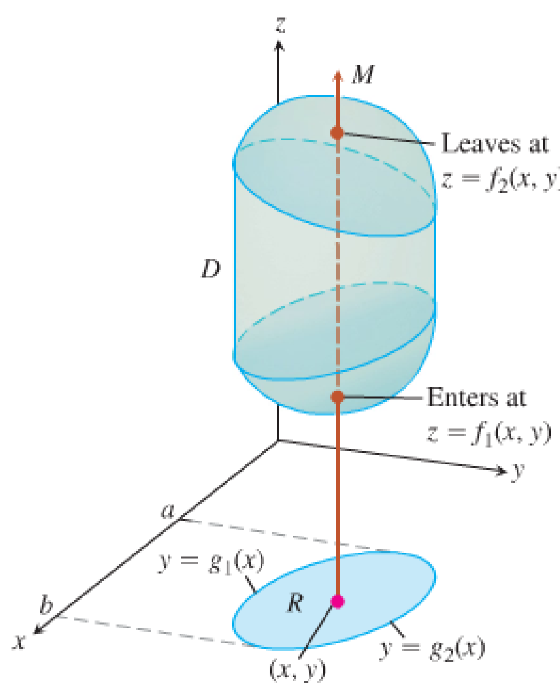
Find Limits of Integration in the Order *dz dy dx*

To evaluate $\iiint\limits_{D} F(x,y,z)\ dV$

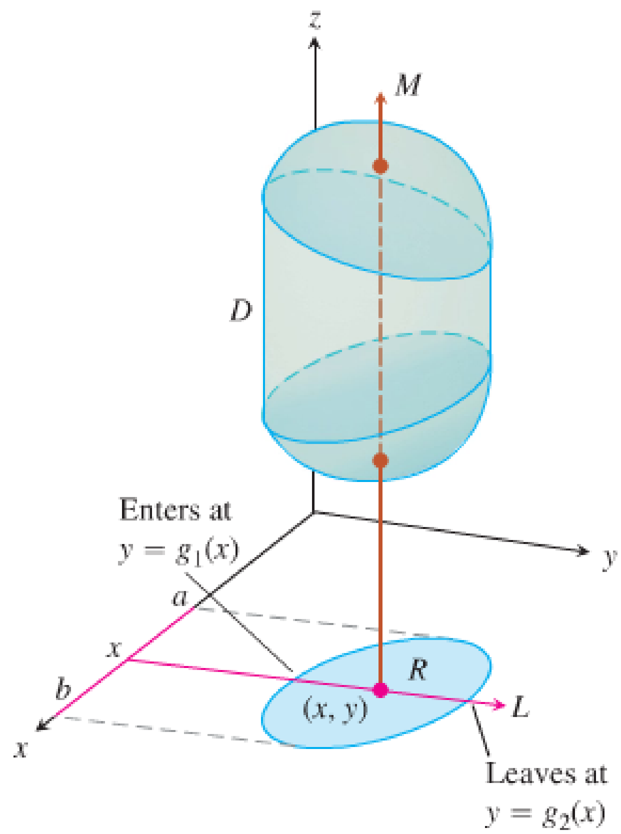
1. ***Sketch***: Sketch the region D along with its "shadow" R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and R.



2. ***Find the z-limits of integration***: Draw a line M passing through (x,y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x,y)$ and leaves at $z = f_2(x,y)$.



3. ***Find the y-limits of integration***: Draw a line L passing through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$.



4. ***Find the x-limits of integration***: Choose x-limits that include all lines through R parallel to the y-axis $(x = a \ \ and \ \ x = b)$.

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z) \; dz \; dy \; dx$$

Example

Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

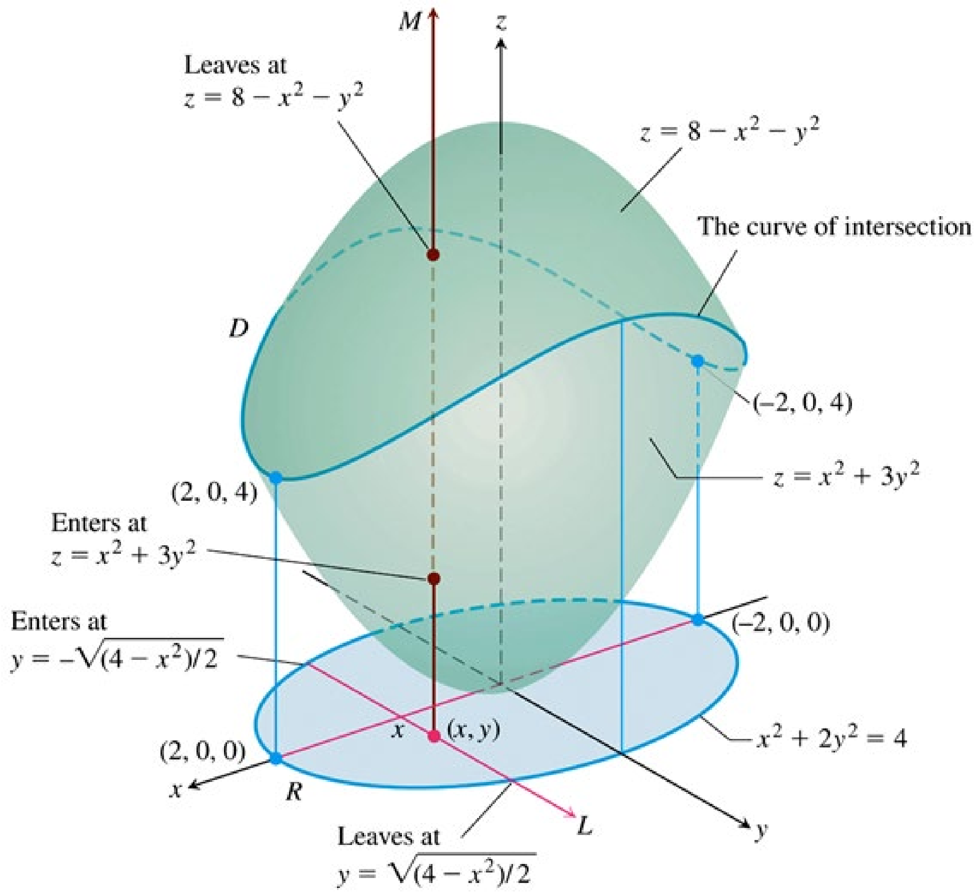
Solution

z-limits: $x^2 + 3y^2 \le z \le 8 - x^2 - y^2$

y-limits: $z = x^2 + 3y^2 = 8 - x^2 - y^2 \quad \to \quad 2x^2 + 4y^2 = 8 \Rightarrow x^2 + 2y^2 = 4$

$$y^2 = \frac{4 - x^2}{2} \quad \Rightarrow y = \pm\sqrt{\frac{4 - x^2}{2}} \quad \to -\sqrt{\frac{4 - x^2}{2}} \le y \le \sqrt{\frac{4 - x^2}{2}}$$

x-limits: $x^2 + 2y^2 = 4 \quad (y = 0) \quad \to x = \pm 2$



$$V = \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dzdydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} z \Big|_{x^2+3y^2}^{8-x^2-y^2} dydx$$

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \left(8 - x^2 - y^2 - x^2 - 3y^2\right)\, dydx$$

$$= \int_{-2}^{2} \left(\left(8-2x^2\right)y - \frac{4}{3}y^3 \;\Big|_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\right) dx$$

$$= \int_{-2}^{2} \left[\left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}} - \frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2} + \left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}} - \frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right] dx$$

$$= \int_{-2}^{2} \left[2\left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right] dx$$

$$= \int_{-2}^{2} \left[2\left(\frac{2}{2}\right)(2)\left(4-x^2\right)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right] dx$$

$$= \int_{-2}^{2} \left[8\left(\frac{4-x^2}{2}\right)\left(\frac{4-x^2}{2}\right)^{1/2} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right] dx$$

$$= \int_{-2}^{2} \left[8\left(\frac{4-x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right] dx$$

$$= \frac{16}{3}\int_{-2}^{2} \left(\frac{4-x^2}{2}\right)^{3/2} dx$$

$$= \frac{16}{3(2)^{3/2}}\int_{-2}^{2} \left(4-x^2\right)^{3/2} dx \qquad \frac{16}{3(2)^{3/2}}\frac{2^{1/2}}{2^{1/2}} = \frac{16\sqrt{2}}{3\cdot 4} = \frac{4\sqrt{2}}{3}$$

$$x = 2\sin u \quad dx = 2\cos u\,du \quad \left(4 - x^2 = 4 - 4\sin^2 u = 4\cos^2 u\right)$$

$$\begin{cases} x = 2 & \to u = \sin^{-1}\frac{x}{2} = \sin^{-1} 1 = \frac{\pi}{2} \\ x = -2 & \to u = \sin^{-1}(-1) = -\frac{\pi}{2} \end{cases}$$

$$= \frac{4\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} \left(4\cos^2 u\right)^{3/2} \left(2\cos u\; du\right)$$

$$= \frac{4\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} 16(\cos u)^3 (\cos u)\, du$$

$$= \frac{64\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} \cos^4 u\ du$$

$$= \frac{64\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos 2u}{2}\right)^2 du$$

$$= \frac{16\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} \left(1+2\cos 2u+\cos^2 2u\right) du$$

$$= \frac{16\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} \left(1+2\cos 2u+\frac{1}{2}+\frac{1}{2}\cos 4u\right) du$$

$$= \frac{16\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2} \left(\frac{3}{2}+2\cos 2u+\frac{1}{2}\cos 4u\right) du$$

$$= \frac{16\sqrt{2}}{3}\ \left(\frac{3}{2}u+\sin 2u+\frac{1}{8}\sin 4u\ \right|_{-\pi/2}^{\pi/2}$$

$$= \frac{16\sqrt{2}}{3}\left[\frac{3\pi}{4}+\sin\pi+\frac{1}{8}\sin 2\pi-\left(-\frac{3\pi}{4}-\sin\pi-\frac{1}{8}\sin 2\pi\right)\right]$$

$$= \frac{16\sqrt{2}}{3}\left(\frac{3\pi}{2}\right)$$

$$\underline{= 8\pi\sqrt{2}\quad unit^3}$$

Example

Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D with vertices (0, 0, 0), (1, 1, 0), (0, 1, 0), and (0, 1, 1). Use the order of integration $dydzdx$.

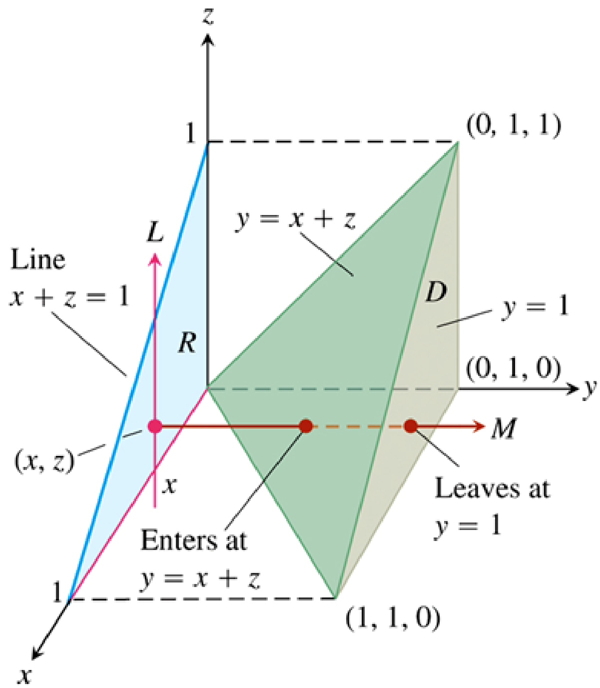
Solution

From the sketch, the upper (right-hand) bounding surface of D lies in the plane $y = 1$.

The lower (left-hand) bounding surface lies in the plane $y = x + z$.

The upper boundary of R is the line $z = 1 - x$.

The lower boundary is the line $z = 0$.



***y*-limits**: The line through $(x,\ z)$ in R parallel to the y-axis enters D at $y = x + z$ and leaves at $y = 1$.

***z*-limits**: The line through $(x,\ z)$ in R parallel to the z-axis enters R at $z = 0$ and leaves at $z = 1 - x$.

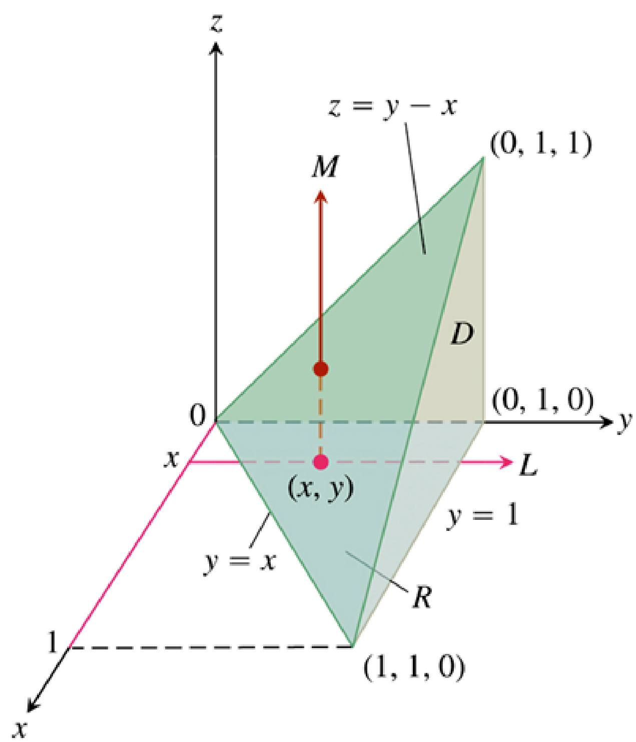
***x*-limits**: $0 \le x \le 1$

$$\int_0^1 \int_0^{1-x} \int_{x+z}^1 F(x, y, z)\ dydzdx$$

Example

Integrate $F(x,y,z)=1$ over the tetrahedron D in the previous example in the order $dz\ dy\ dx$, and then integrate in the order $dy\ dz\ dx$.

Solution



z-limits of integration: A line M parallel to the z-axis through a typical point (x, y) in the xy-plane "shadow" enters the tetrahedron at $z = 0$ and exists through the upper plane where $z = y - x$. $0 \le z \le y - x$

Line is given by: $ax + by + cz = 0$ passes through the 2 points:

$(1,\ 1,\ 0) \to a + b = 0 \quad \Rightarrow \quad a = -b$

and $(0,\ 1,\ 1) \to b + c = 0 \quad \Rightarrow \quad c = -b$

$\to -bx + by - bz = 0$

$-x + y - z = 0 \quad \Rightarrow \quad z = y - x$

y-limits of integration: On the xy-plane, where $z = 0$, the sloped side of the tetrahedron crosses the plane along the line $y = x$. A line L through (x, y) parallel to the y-axis enters the shadow in the xy-plane at $y = x$ and exists at $y = 1$. $x \le y \le 1$

x-limits of integration: A line L parallel to the y-axis through a typical point (x, y) in the xy-plane sweeps out the shadow, where $0 \le x \le 1$ at the point $(1,\ 1,\ 0)$

The integral is: $$\int_0^1 \int_x^1 \int_0^{y-x} F(x,y,z)\ dzdydx$$

$$V=\int_0^1\int_x^1\int_0^{y-x} dzdydx$$

$$=\int_0^1\int_x^1 z\Big|_0^{y-x} dydx$$

$$=\int_0^1\int_x^1 (y-x)\,dydx$$

$$=\int_0^1\left(\tfrac{1}{2}y^2-xy\ \Big|_x^1\ dx\right.$$

$$=\int_0^1 \left[\tfrac{1}{2}-x-\left(\tfrac{1}{2}x^2-x^2\right)\right] dx$$

$$=\int_0^1\left(\tfrac{1}{2}-x+\tfrac{1}{2}x^2\right)dx$$

$$=\left(\frac{1}{2}x-\frac{1}{2}x^2+\frac{1}{6}x^3\ \right|_0^1$$

$$=\frac{1}{2}-\frac{1}{2}+\frac{1}{6}$$

$$\boxed{=\frac{1}{6}\quad unit^3}$$

$$V=\int_0^1\int_0^{1-x}\int_{x+z}^1 dydzdx$$

$$=\int_0^1\int_0^{1-x} y\Big|_{x+z}^1 dzdx$$

$$=\int_0^1\int_0^{1-x}(1-x-z)\,dzdx$$

$$=\int_0^1\left(z-xz-\frac{1}{2}z^2\ \right|_0^{1-x} dx$$

$$=\int_0^1\left(1-x-x(1-x)-\tfrac{1}{2}(1-x)^2\right) dx$$

$$=\int_0^1\left((1-x)(1-x)-\tfrac{1}{2}(1-x)^2\right) dx$$

$$= \int_0^1 \left((1-x)^2 - \frac{1}{2}(1-x)^2 \right) dx$$

$$= \frac{1}{2} \int_0^1 (1-x)^2 \, dx$$

$$= -\frac{1}{6}(1-x)^3 \Big|_0^1$$

$$= \frac{1}{6} \quad unit^3$$

Example

Evaluate the integral $\displaystyle\int_0^1 \int_x^{x^2} \int_{xy}^{x^2y^3} xy \, dzdydx$

Solution

$$\int_0^1 \int_x^{x^2} \int_{xy}^{x^2y^3} xy \, dzdydx = \int_0^1 \int_x^{x^2} xyz \Big|_{xy}^{x^2y^3} dydx$$

$$= \int_0^1 \int_x^{x^2} xy\left(x^2y^3 - xy\right) dydx$$

$$= \int_0^1 \int_x^{x^2} \left(x^3y^4 - x^2y^2\right) dydx$$

$$= \int_0^1 \left(\frac{1}{5}x^3y^5 - \frac{1}{3}x^2y^3 \right) \Big|_x^{x^2} dx$$

$$= \int_0^1 \left(\frac{1}{5}x^{13} - \frac{1}{3}x^8 - \frac{1}{5}x^8 + \frac{1}{3}x^5 \right) dx$$

$$= \int_0^1 \left(\frac{1}{5}x^{13} - \frac{8}{15}x^8 + \frac{1}{3}x^5 \right) dx$$

$$= \left(\frac{1}{70}x^{14} - \frac{8}{135}x^9 + \frac{1}{18}x^6 \right. \Big|_0^1$$

$$= \frac{1}{70} - \frac{8}{135} + \frac{1}{18}$$

$$= \frac{2}{189}$$

Example

Evaluate the integral $\int_0^a \int_0^{a-z} \int_0^{a-y-z} yz\ dxdydz$

Solution

$$\int_0^a \int_0^{a-z} \int_0^{a-y-z} yz\ dxdydz = \int_0^a \int_0^{a-z} yzx \Big|_0^{a-y-z} dydz$$

$$= \int_0^a \int_0^{a-z} \left(ayz - y^2z - yz^2\right) dydz$$

$$= \int_0^a \left(\frac{1}{2}azy^2 - \frac{1}{3}zy^3 - \frac{1}{2}z^2y^2 \Big|_0^{a-z} dz\right.$$

$$= \int_0^a \left(\frac{1}{2}\left(az - z^2\right)(a-z)^2 - \frac{1}{3}z(a-z)^3\right) dz$$

$$= \frac{1}{6}\int_0^a (a-z)^2 \left(3\left(az - z^2\right) - 2z(a-z)\right) dz$$

$$= \frac{1}{6}\int_0^a (a-z)^2 \left(3az - 3z^2 - 2az + 2z^2\right) dz$$

$$= \frac{1}{6}\int_0^a (a-z)^2 \left(az - z^2\right) dz$$

$$= \frac{1}{6}\int_0^a z(a-z)^3\ dz$$

$$= \frac{1}{6}\int_0^a \left(a^3z - 3a^2z^2 + 3az^3 - z^4\right) dz$$

$$= \frac{1}{6}\left(\frac{1}{2}a^3z^2 - a^2z^3 + \frac{3}{4}az^4 - \frac{1}{5}z^5 \Big|_0^a\right.$$

$$= \frac{1}{6}\left(\frac{1}{2}a^5 - a^5 + \frac{3}{4}a^5 - \frac{1}{5}a^5\right)$$

$$= \frac{a^5}{6}\left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right)$$

$$= \frac{a^5}{6}\left(\frac{1}{20}\right)$$

$$\boxed{= \frac{a^5}{120}}$$

Example

Find the volume bounded by the cylinder $z = \frac{4}{y^2+1}$, bounded by the planes $y = x, \quad y = 3, \quad x = 0, \quad z = 0$

Solution

$$0 \le z \le \frac{4}{y^2+1}$$

$$0 \le x \le y$$

$$0 \le y \le 3$$

$$V = \int_0^3 \int_0^y \int_0^{\frac{4}{y^2+1}} dz dx dy$$

$$= \int_0^3 \int_0^y z \Big|_0^{\frac{4}{y^2+1}} dx dy$$

$$= \int_0^3 \int_0^y \frac{4}{y^2+1} dx dy$$

$$= \int_0^3 \frac{4}{y^2+1} x \Big|_0^y dy$$

$$= \int_0^3 \frac{4y}{y^2+1} dy$$

$$= 2 \int_0^3 \frac{1}{y^2+1} d\left(y^2+1\right)$$

$$= 2 \ln\left(y^2+1\right) \Big|_0^3$$

$$\boxed{= 2 \ln(10) \ \ unit^3}$$

Average Value of a Function in Space

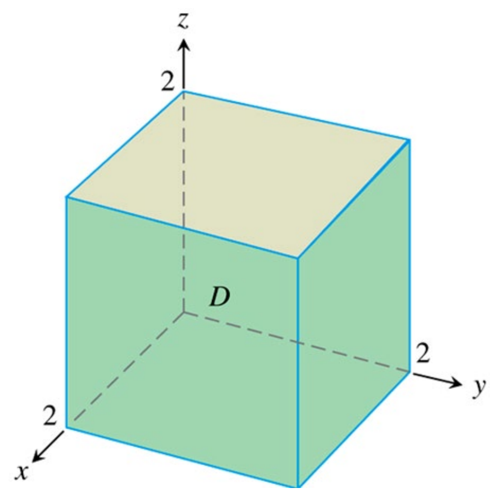
The average value of a function F over a region D in space is defined by the formula

$$\textbf{Average value} \text{ of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint_D F\,dV$$

Example

Find the average of $F(x,y,z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x=2$, $y=2$, and $z=2$ in the first octant.

Solution



$$Volume = 2 \cdot 2 \cdot 2$$

$$= 8 \ \ unit^3$$

The value of the integral of F over the cube is

$$V = \int_0^2 \int_0^2 \int_0^2 xyz\ dx\ dy\ dz$$

$$= \int_0^2 zdz \int_0^2 ydy \int_0^2 xdx$$

$$= \left(\frac{1}{2}z^2\right|_0^2 \left(\frac{1}{2}y^2\right|_0^2 \left(\frac{1}{2}x^2\right|_0^2$$

$$= \frac{1}{8}(4)(4)(4)$$

$$= 8 \ \ unit^3$$

$$\textbf{Average value} \text{ of } xyz \text{ over } cube = \frac{1}{\text{volume of } D} \iiint_{cube} xyz\ dV$$

$$= \left(\frac{1}{8}\right)(8)$$

$$= 1$$

Exercises Section 3.4 – Triple Integrals

(1 – 31) Evaluate the integral

1. $\int_0^1 \int_0^1 \int_0^1 \left(x^2+y^2+z^2\right) dzdydx$

2. $\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dzdxdy$

3. $\int_0^{\pi/6} \int_0^1 \int_{-2}^3 y\sin z \ dxdydz$

4. $\int_{-1}^1 \int_0^1 \int_0^2 (x+y+z)\, dydxdz$

5. $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dzdydx$

6. $\int_0^1 \int_0^{1-x^2} \int_0^{4-x^2-y} x dzdydx$

7. $\int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \cos(u+v+w)\, dudvdw$

8. $\int_0^{\pi/4} \int_0^{\ln\sec v} \int_{-\infty}^{2t} e^x dxdtdv$

9. $\int_0^1 \int_{-z}^{z} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dydxdz$

10. $\int_0^{\pi} \int_0^{y} \int_0^{\sin x} dzdxdy$

11. $\int_0^9 \int_0^1 \int_{2y}^2 \frac{4\sin x^2}{\sqrt{z}} dxdydz$

12. $\int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \cos(x+y+z)\, dxdydz$

13. $\int_1^e \int_1^x \int_0^z \frac{2y}{z^3} dydzdx$

14. $\int_{\ln 6}^{\ln 7} \int_0^{\ln 2} \int_{\ln 4}^{\ln 5} e^{(x+y+z)} dzdydx$

15. $\int_0^1 \int_0^{x^2} \int_0^{x+y} (2x-y-z)\, dzdydx$

16. $\int_{-2}^2 \int_3^6 \int_0^2 dxdydz$

17. $\int_{-1}^1 \int_{-1}^2 \int_0^1 6xyz \ dydxdz$

18. $\int_{-2}^2 \int_1^2 \int_1^e \frac{xy^2}{z} dzdxdy$

19. $\int_0^{\ln 4} \int_0^{\ln 3} \int_0^{\ln 2} e^{-x+y+z} dxdydz$

20. $\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\frac{\pi}{2}} \sin \pi x \cos y \sin 2z \ dydxdz$

21. $\int_0^2 \int_1^2 \int_0^1 yze^x dxdzdy$

22. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dzdydx$

23. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 2xz \ dzdydx$

24. $\int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \int_0^{16-\frac{1}{4}x^2-y^2} dzdxdy$

25. $\displaystyle\int_1^6 \int_0^{4-\frac{2}{3}y} \int_0^{12-2y-3z} \frac{1}{y}\, dxdzdy$

26. $\displaystyle\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{1+x^2+z^2}} dydxdz$

27. $\displaystyle\int_0^{\pi} \int_0^{\pi} \int_0^{\sin x} \sin y\, dzdxdy$

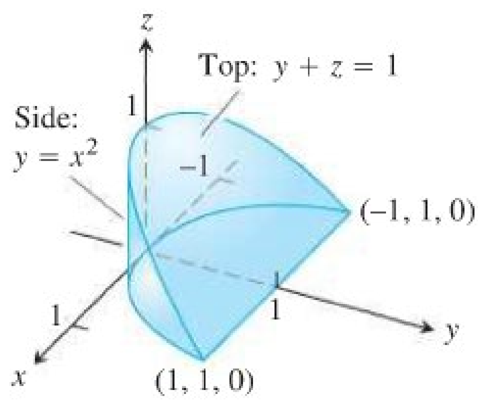
28. $\displaystyle\int_0^{\ln 8} \int_1^{\sqrt{z}} \int_{\ln y}^{\ln 2y} e^{x+y^2-z}\, dxdydz$

29. $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{2-x} 4yz\, dzdydx$

30. $\displaystyle\int_0^2 \int_0^4 \int_{y^2}^4 \sqrt{x}\, dzdxdy$

31. $\displaystyle\int_0^1 \int_y^{2-y} \int_0^{2-x-y} xy\, dzdxdy$

32. Here is the region of integration of the integral $\displaystyle\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dzdydx$



a) *dydzdx* ***b***) *dydxdz* ***c***) *dxdydz* ***d***) *dxdzdy* ***e***) *dzdxdy*

(**33 – 37**) Use another order to evaluate

33. $\displaystyle\int_0^5 \int_{-1}^0 \int_0^{4x+4} dydxdz$

34. $\displaystyle\int_0^1 \int_{-2}^2 \int_0^{\sqrt{4-y^2}} dzdydx$

35. $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dydzdx$

36. $\displaystyle\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-z^2}} dydzdx$

37. $\displaystyle\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin\sqrt{yz}}{x^{3/2}}\, dydxdz$

(**38 – 39**) Evaluate

38. $\displaystyle\iiint_D (xy + xz + yz)\,dV$; $\qquad D = \{(x,\ y,\ z):\ -1 \le x \le 1,\quad -2 \le y \le 2,\quad -3 \le z \le 3\}$

39. $\displaystyle\iiint_D xyze^{-x^2-y^2}\,dV$; $D = \{(x,\ y,\ z):\ 0 \le x \le \sqrt{\ln 2},\quad 0 \le y \le \sqrt{\ln 4},\quad 0 \le z \le 1\}$

40. Let $D = \{(x, y, z):\ 0 \le x \le y^2,\ 0 \le y \le z^3,\ 0 \le z \le 2\}$

a) Use a triple integral to find the volume of *D*.

b) In theory, how many other possible orderings of the variables (besides the one used in part (*a*)) can be used to find the volume of *D*? Verify the result of part (*a*) using one of these other ordering.

c) What is the volume of the region $D = \{(x, y, z):\ 0 \le x \le y^p,\ 0 \le y \le z^q,\ 0 \le z \le 2\}$, where *p* and *q* are positive real numbers?

41. Find the volume the parallelepiped (slanted box) with vertices $(0,\ 0,\ 0)$, $(1,\ 0,\ 0)$, $(0,\ 1,\ 0)$, $(1,\ 1,\ 0)$, $(0,\ 1,\ 1)$, $(1,\ 1,\ 1)$, $(0,\ 2,\ 1)$, $(1,\ 2,\ 1)$

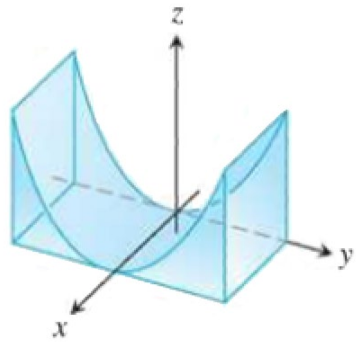
42. Find the volume the larger of two solids formed when the parallelepiped with vertices $(0,\ 0,\ 0)$, $(2,\ 0,\ 0)$, $(0,\ 2,\ 0)$, $(2,\ 2,\ 0)$, $(0,\ 1,\ 1)$, $(2,\ 1,\ 1)$, $(0,\ 3,\ 1)$, $(2,\ 3,\ 1)$ is sliced by the plane $y = 2$.

43. Find the volume of the pyramid with vertices $(0,\ 0,\ 0)$, $(2,\ 0,\ 0)$, $(2,\ 2,\ 0)$, $(0,\ 2,\ 0)$, $(0,\ 0,\ 4)$

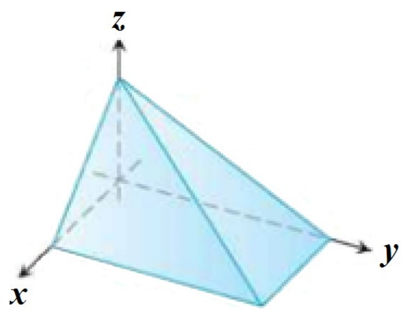
44. Two different tetrahedrons fill the region in the first octant bounded by the coordinate planes and the plane $x + y + z = 4$. Both solids have densities that vary in the *z*-direction between $\rho = 4$ and $\rho = 8$, according to the functions $\rho_1 = 8 - z$ and $\rho_2 = 4 + z$. Find the mass of each solid

45. Suppose a wedge of cheese fills the region in the first octant bounded by the planes $y = z$, $y = 4$ and $x = 4$. You could divide the wedge into two equal pieces (by volume) if you sliced the wedge with the plane $x = 2$. Instead find *a* with $0 < a < 1$ such that slicing the wedge with the plane $y = a$ divides the wedge into two pieces of equal volume

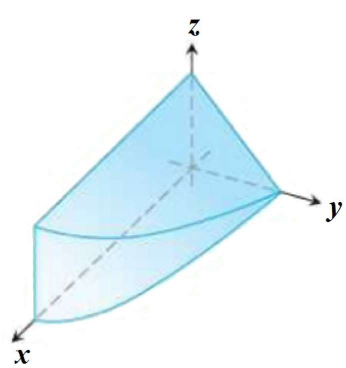
46. Find the volumes of the region between the cylinder $z = y^2$ and the xy-plane that is bounded by the planes $x = 0,\quad x = 1,\quad y = -1,\quad y = 1$



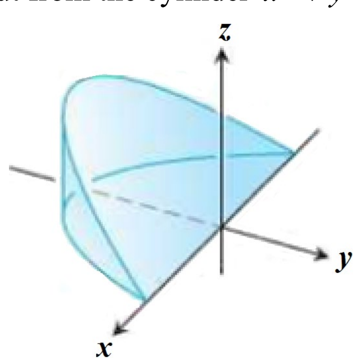
47. Find the volumes of the region in the first octant bounded by the coordinate planes and the planes $x + z = 1,\quad y + 2z = 2$



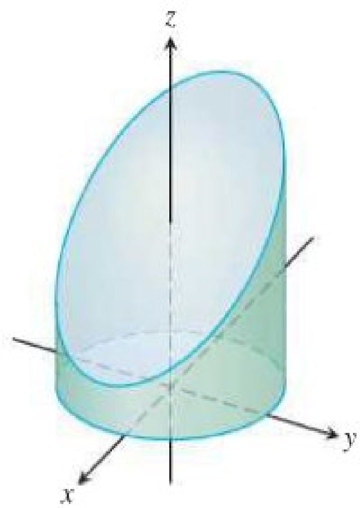
48. Find the volumes of the region in the first octant bounded by the coordinate planes and the plane $y + z = 2$, and the cylinder $x = 4 - y^2$



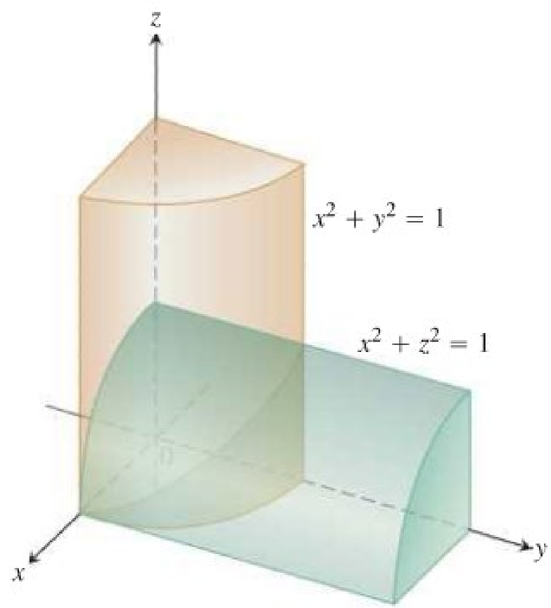
49. Find the volumes of the wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y,\quad z = 0$



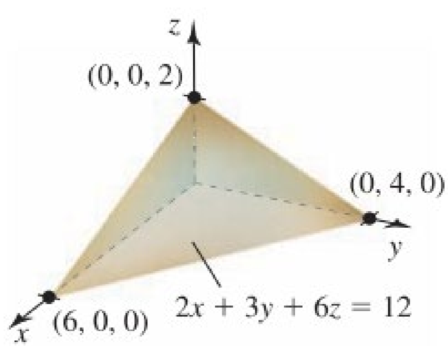
50. Find the volumes of the region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$



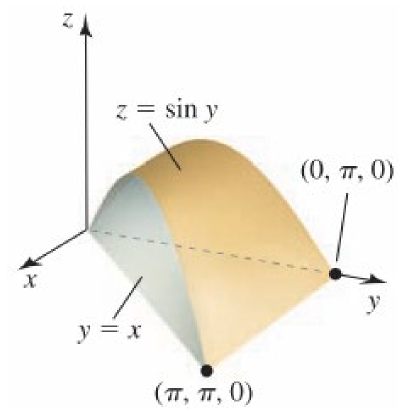
51. Find the volumes of the region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown below



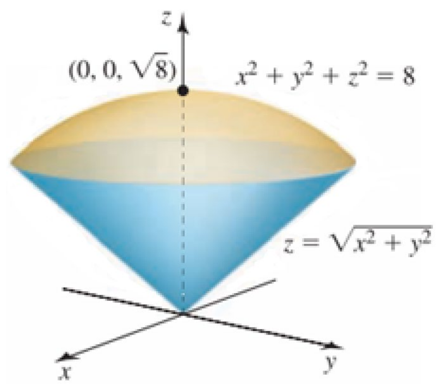
52. Find the volume of the solid in the first octant bounded by the plane $2x + 3y + 6z = 12$ and the coordinate planes



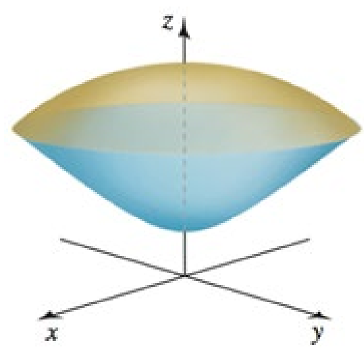
53. Find the volume of the solid in the first octant formed when the cylinder $z = \sin y$, for $0 \le y \le \pi$, is sliced by the planes $y = x$ and $x = 0$



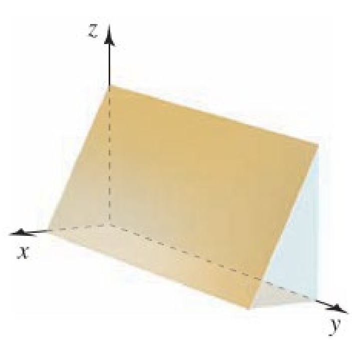
54. Find the volume of the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and bounded above the sphere $x^2 + y^2 + z^2 = 8$



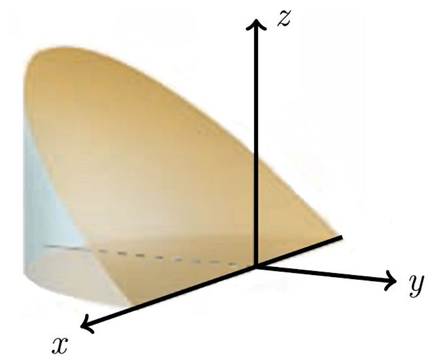
55. The solid between the sphere $x^2 + y^2 + z^2 = 19$ and the hyperboloid $z^2 - x^2 - y^2 = 1$, for $z > 0$



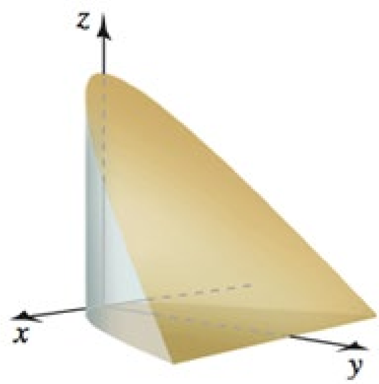
56. Find the volume of the prism in the first octant bounded below by $z = 2 - 4x$ and $y = 8$



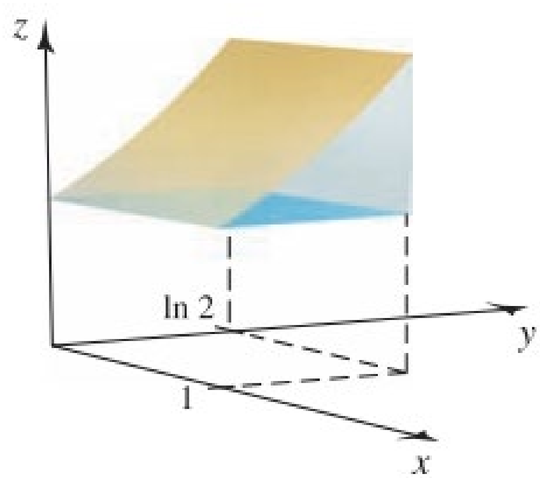
57. Find the volume of the wedge above the *xy*-plane formed when the cylinder $x^2 + y^2 = 4$ is cut by the planes $z = 0$ and $y = -z$



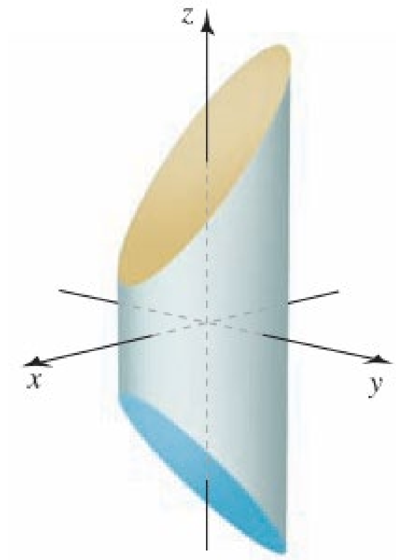
58. The wedge bounded by the parabolic cylinder $y = x^2$ and the planes $z = 3 - y$ and $z = 0$



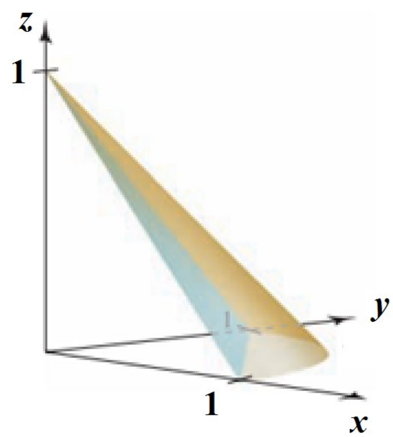
59. Find the volume of the solid bounded by the surfaces $z = e^y$ and $z = 1$ over the rectangle $\{(x, y):\ 0 \le x \le 1,\ 0 \le y \le \ln 2\}$



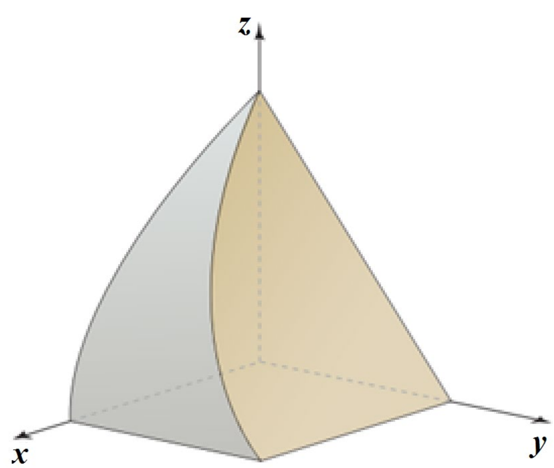
60. Find the volume of the wedge of the cylinder $x^2 + 4y^2 = 4$ created by the planes $z = 3 - x$ and $z = x - 3$



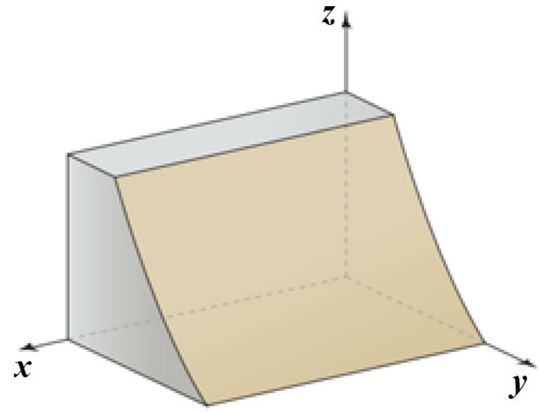
61. Find the volume of the solid in the first octant bounded by the cone $z = 1 - \sqrt{x^2 + y^2}$ and the plane $x + y + z = 1$



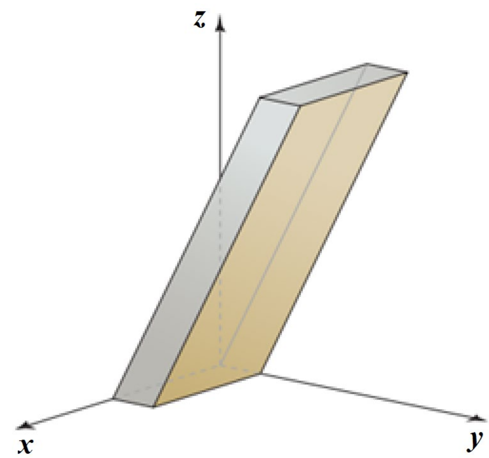
62. Find the volume of the solid bounded by $x = 0$, $x = 1 - z^2$, $y = 0$, $z = 0$, and $z = 1 - y$



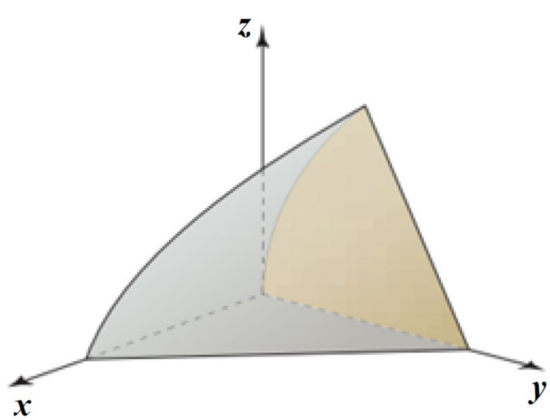
63. Find the volume of the solid bounded by $x=0$, $x=2$, $y=0$, $y=e^{-z}$, $z=0$, and $z=1$



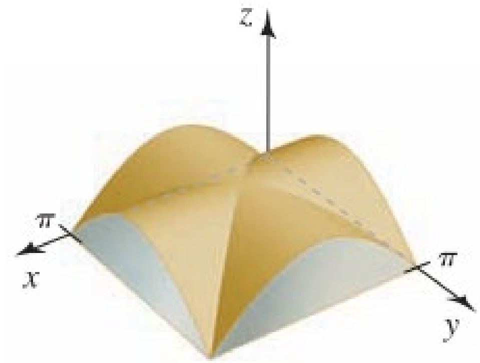
64. Find the volume of the solid bounded by $x=0$, $x=2$, $y=z$, $y=z+1$, $z=0$, and $z=4$



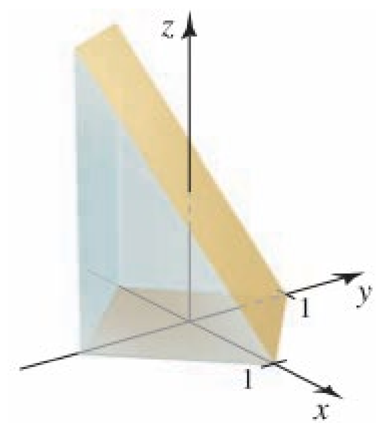
65. Find the volume of the solid bounded by $x=0$, $y=z^2$, $z=0$, and $z=2-x-y$



66. Find the volume of the solid common to the cylinders $z = \sin x$ and $z = \sin y$ over the square $R = \{(x, y):\ 0 \le x \le \pi,\ 0 \le y \le \pi\}$



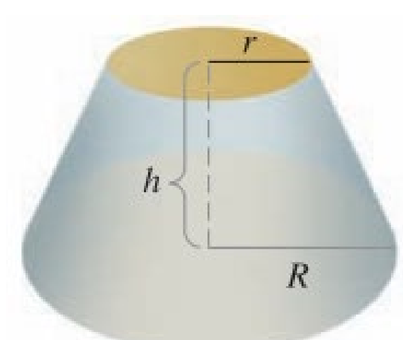
67. Find the volume of the wedge of the square column $|x| + |y| = 1$ created by the planes $z = 0$ and $x + y + z = 1$



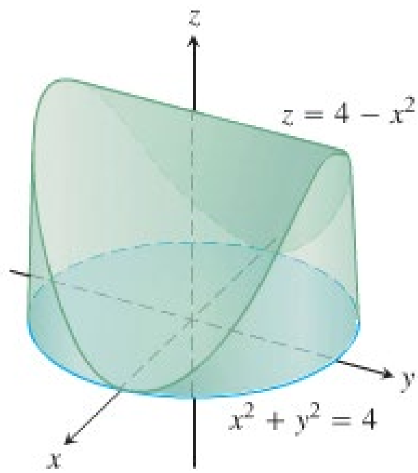
68. Find the volume of a right circular cone with height h and base radius r.

69. Find the volume of a tetrahedron whose vertices are located at $(0,\ 0,\ 0)$, $(a,\ 0,\ 0)$, $(0,\ b,\ 0)$, and $(0,\ 0,\ c)$

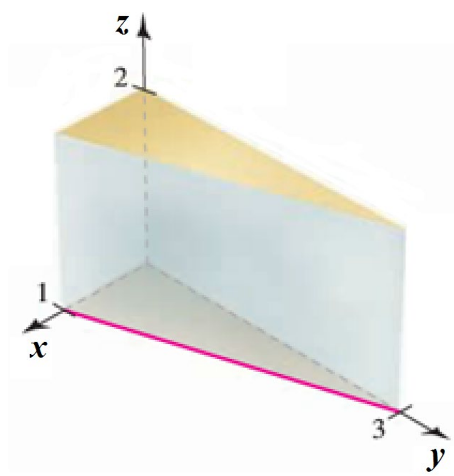
70. Find the volume of a truncated cone of height h whose ends have radii r and R.



71. Find the volume of the solid that is bounded above by the cylinder $z = 4 - x^2$, on the sides by the cylinder $x^2 + y^2 = 4$, and below by the *xy*-plane.



72. Find the volume of the prism in the first octant bounded by the planes $y = 3 - 3x$ *and* $z = 2$



73. Find the volume of the prism in the first octant bounded by the planes

$x^2 + y^2 = 4$ *and* $x^2 + z^2 = 4$