

## ***Solution***      **Section 3.5 – Least Squares Analysis**

### ***Exercise***

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0, 2), (1, 2), (2, 0)\}$$

### ***Solution***

$$\{(0, 2), (1, 2), (2, 0)\}$$

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\text{where } A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

The normal equation formula:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$m = \frac{\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 3 & 3 \end{vmatrix}}$$

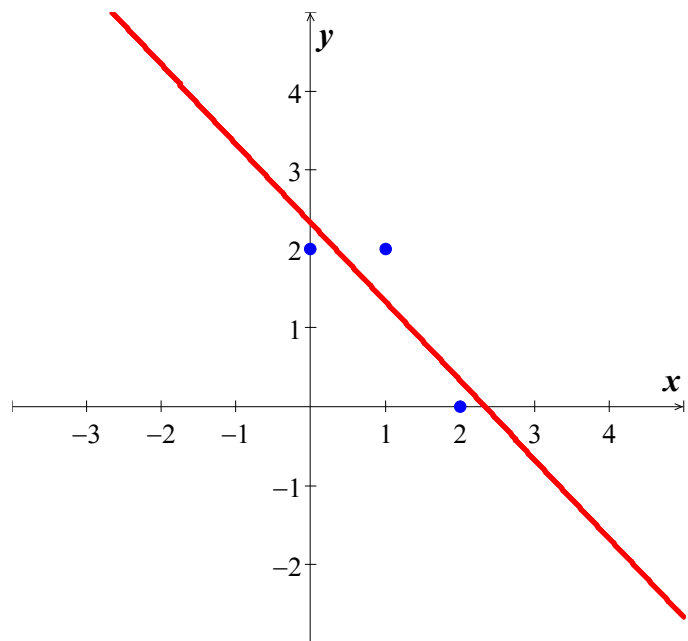
$$= \frac{-6}{6}$$

$$= -1$$

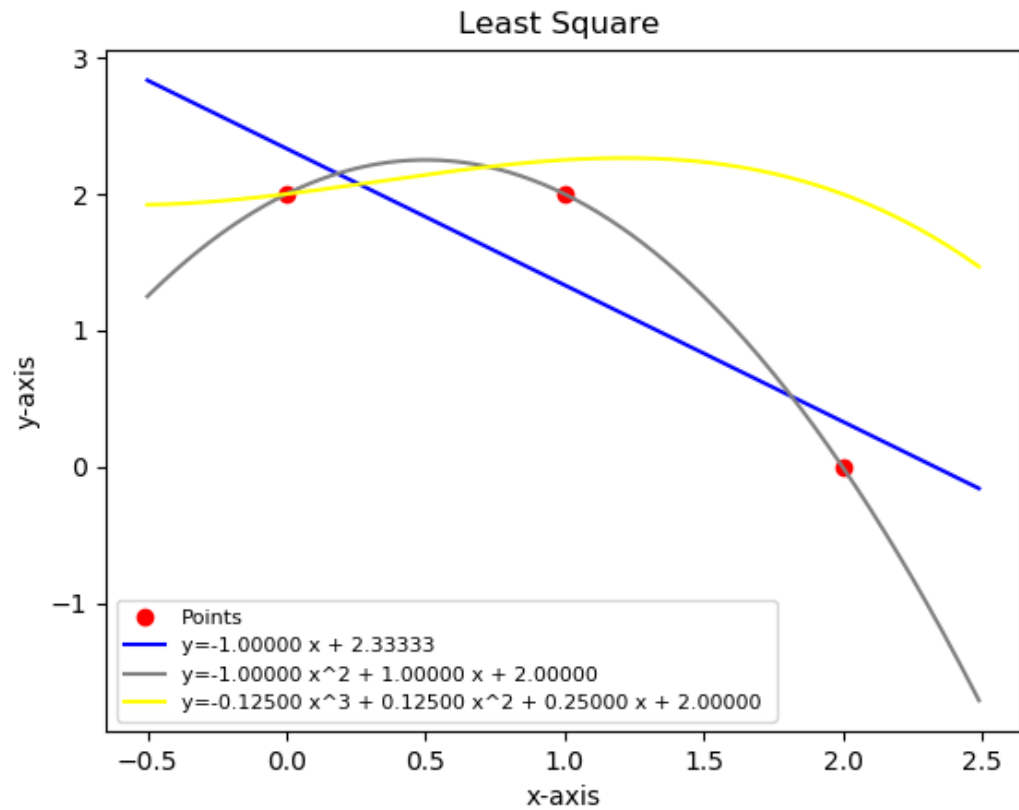
$$b = \frac{\begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix}}{6}$$

$$= \frac{7}{3}$$

$$\text{Thus, } y = -x + \frac{7}{3}$$



$$\begin{aligned}
 A\vec{x} &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ \frac{7}{3} \end{pmatrix} \\
 &= \begin{pmatrix} \frac{7}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix} \\
 \vec{y} - A\vec{x} &= \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{7}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix} \\
 &= \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}
 \end{aligned}$$



**Error:**

$$\begin{aligned}
 \|\vec{y} - A\vec{x}\| &= \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}} \\
 &= \frac{\sqrt{6}}{3} \\
 &\approx 0.8164966
 \end{aligned}$$

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The **second order** equation:

$$y = -x^2 + x + 2$$

**Error** = 0.00000

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The **third order** equation:

$$y = -0.1250x^3 - 0.1250x^2 + 0.25x + 2$$

**Error** = 2.01556

### Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(1, 5), (2, 4), (3, 1), (4, 1), (5, -1)\}$$

### Solution

$$\{(1, 5), (2, 4), (3, 1), (4, 1), (5, -1)\}$$

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{where } A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

The normal equation:  $A^T A \vec{x} = A^T \vec{y}$

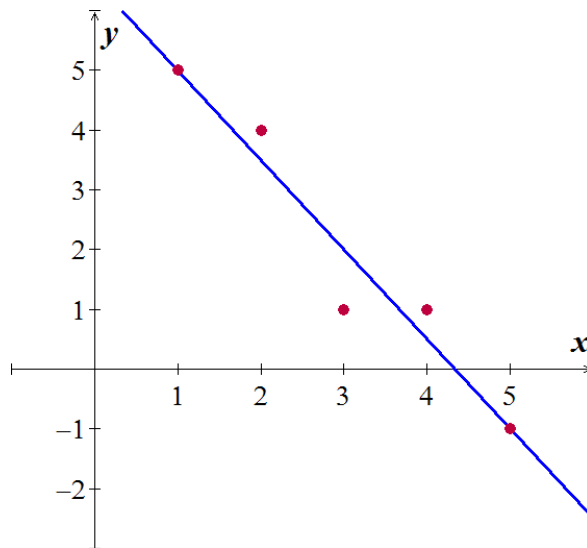
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 55 & 15 \\ 15 & 5 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$$

$$m = \frac{\begin{vmatrix} 15 & 15 \\ 10 & 5 \end{vmatrix}}{\begin{vmatrix} 55 & 15 \\ 15 & 5 \end{vmatrix}} = \frac{-75}{50} = -\frac{3}{2}$$

$$b = \frac{\begin{vmatrix} 55 & 15 \\ 15 & 10 \end{vmatrix}}{50} = \frac{325}{50} = \frac{13}{2}$$

$$\text{Thus, } \underline{y = -\frac{3}{2}x + \frac{13}{2}} \quad \text{or} \quad \underline{y = -1.5x + 6.5}$$

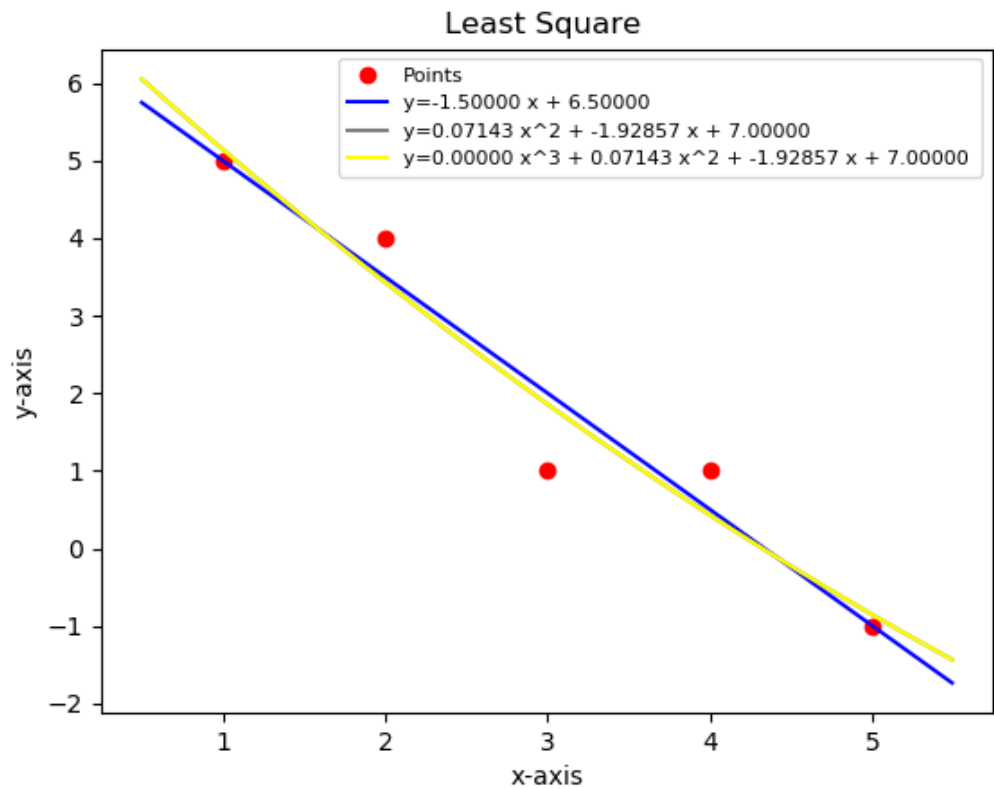


$$A\vec{x} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ \frac{13}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ \frac{7}{2} \\ 2 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 5 \\ 4 \\ 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ \frac{7}{2} \\ 2 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{1}{2} \\ -1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$



**Error:**

$$\begin{aligned} \|\vec{y} - A\vec{x}\| &= \sqrt{\frac{1}{4} + 1 + \frac{1}{4}} \\ &= \frac{\sqrt{6}}{2} \\ &\approx 1.224745 \end{aligned}$$

The **second order** equation:

$$y = 0.07143x^2 - 1.92857x + 7$$

**Error** = 1.19523

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The **third order** equation:

$$y = 0.0x^3 + 0.07143x^2 - 1.92857x + 7$$

**Error** = 1.19523

### Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(0, 1), (1, 3), (2, 4), (3, 4)\}$$

### Solution

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

The normal equation:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix}$$

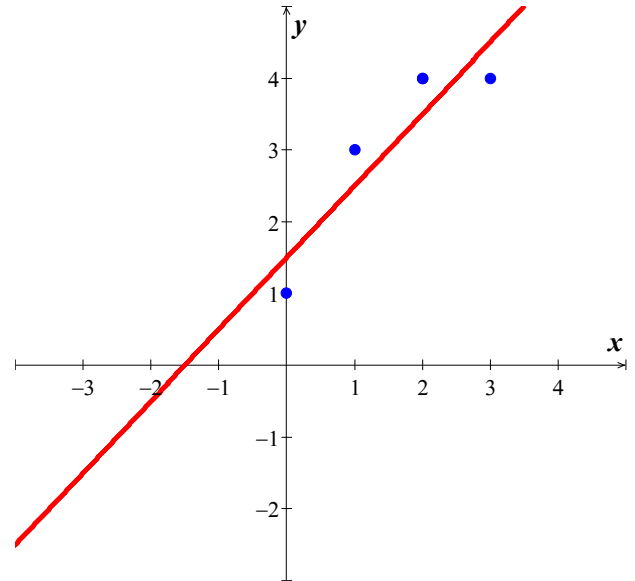
$$\begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 23 \\ 12 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} m \\ b \end{pmatrix} &= \frac{1}{20} \begin{pmatrix} 4 & -6 \\ -6 & 14 \end{pmatrix} \begin{pmatrix} 23 \\ 12 \end{pmatrix} & X = A^{-1}B \\ &= \frac{1}{20} \begin{pmatrix} 20 \\ 30 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix} \end{aligned}$$

We have:  $m = 1$  and  $b = \frac{3}{2}$ .

Thus,  $y = x + \frac{3}{2}$

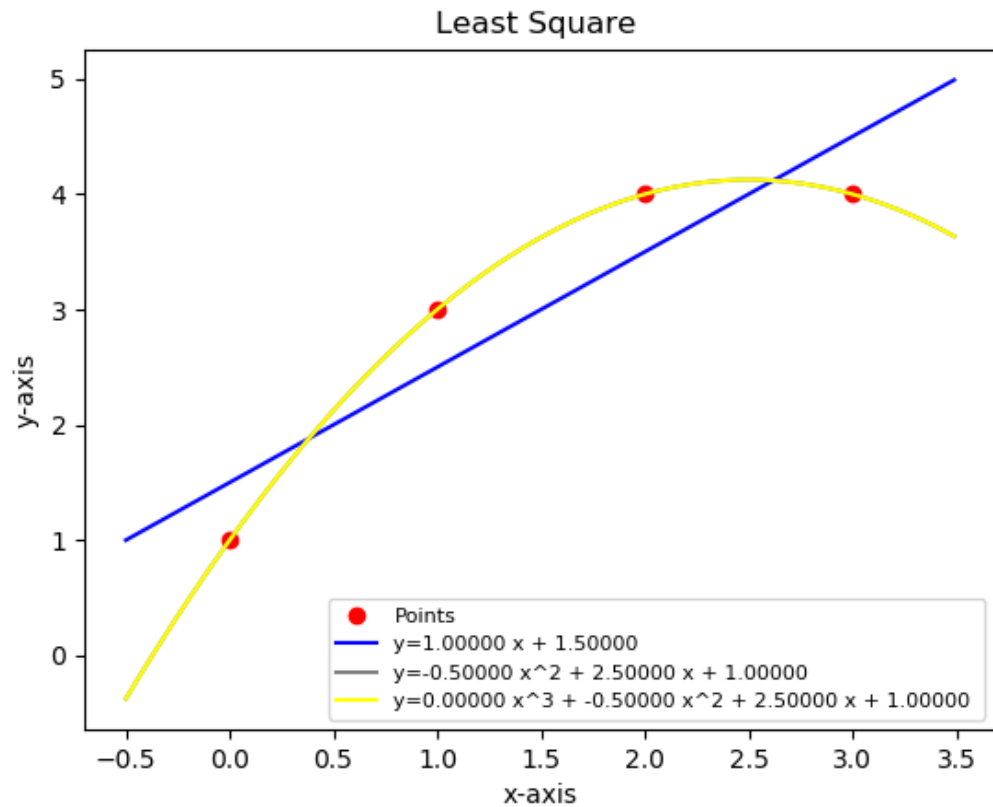
$$A\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$$



$$= \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ \frac{7}{2} \\ \frac{9}{2} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ \frac{7}{2} \\ \frac{9}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$



**Error:**

$$\|\vec{y} - A\vec{x}\| = \sqrt{4\left(\frac{1}{4}\right)}$$

$$= 1$$

The **second order** equation:

$$y = -0.50x^2 + 2.5x + 1.0$$

**Error** = 0.00000

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The **third order** equation:

$$y = 0.0x^3 - 0.5x^2 + 2.5x + 1$$

**Error** = 0.00000

### Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-2, 0), (-1, 0), (0, 1), (1, 3), (2, 5)\}$$

### Solution

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

The normal equation:  $A^T A \vec{x} = A^T \vec{y}$

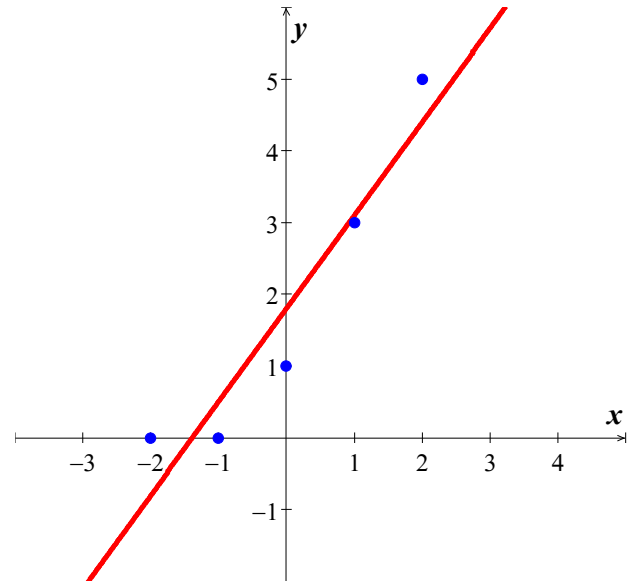
$$\begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{50} \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} 13 \\ 9 \end{pmatrix} = \begin{pmatrix} \frac{13}{10} \\ \frac{9}{5} \end{pmatrix}$$

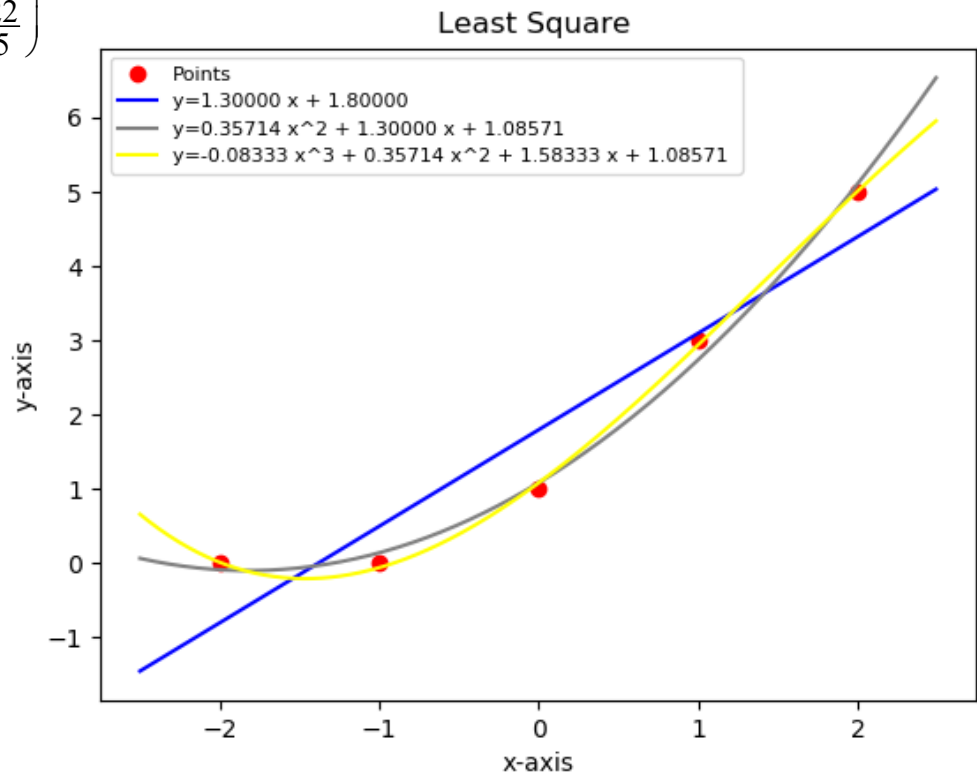
We have:  $m = 1.3$  and  $b = 1.8$

Thus,  $y = \frac{13}{10}x + \frac{9}{5}$



$$A\vec{x} = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{13}{10} \\ \frac{9}{5} \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} \\ \frac{1}{2} \\ \frac{9}{5} \\ \frac{31}{10} \\ \frac{22}{5} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{pmatrix} - \begin{pmatrix} -\frac{4}{5} \\ \frac{1}{2} \\ \frac{9}{5} \\ \frac{31}{10} \\ \frac{22}{5} \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ -\frac{1}{2} \\ -\frac{4}{5} \\ -\frac{1}{10} \\ \frac{3}{5} \end{pmatrix}$$



**Error:**  $\|\vec{y} - A\vec{x}\| = \sqrt{\frac{16}{25} + \frac{1}{4} + \frac{16}{25} + \frac{1}{100} + \frac{9}{25}}$

$$= \sqrt{\frac{41}{25} + \frac{26}{100}}$$

$$= \frac{\sqrt{190}}{10}$$

$$\approx 1.37840$$

The *second order* equation:

$$y = 0.35714x^2 + 1.30x + 1.08571$$

**Error** = 0.33806

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The *third order* equation:

$$y = -0.08333x^3 + 0.35714x^2 + 1.58333x + 1.08571$$

**Error** = 0.11952



### Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(2, 3), (3, 2), (5, 1), (6, 0)\}$$

### Solution

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

The normal equation:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 74 & 16 \\ 16 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 17 \\ 6 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 74 & 16 \\ 16 & 4 \end{vmatrix} = 40 \quad \Delta_m = \begin{vmatrix} 17 & 16 \\ 6 & 4 \end{vmatrix} = -28 \quad \Delta_b = \begin{vmatrix} 74 & 17 \\ 16 & 6 \end{vmatrix} = 172$$

$$m = -\frac{28}{40} = -\frac{7}{10}$$

$$b = \frac{172}{40} = \frac{43}{10}$$

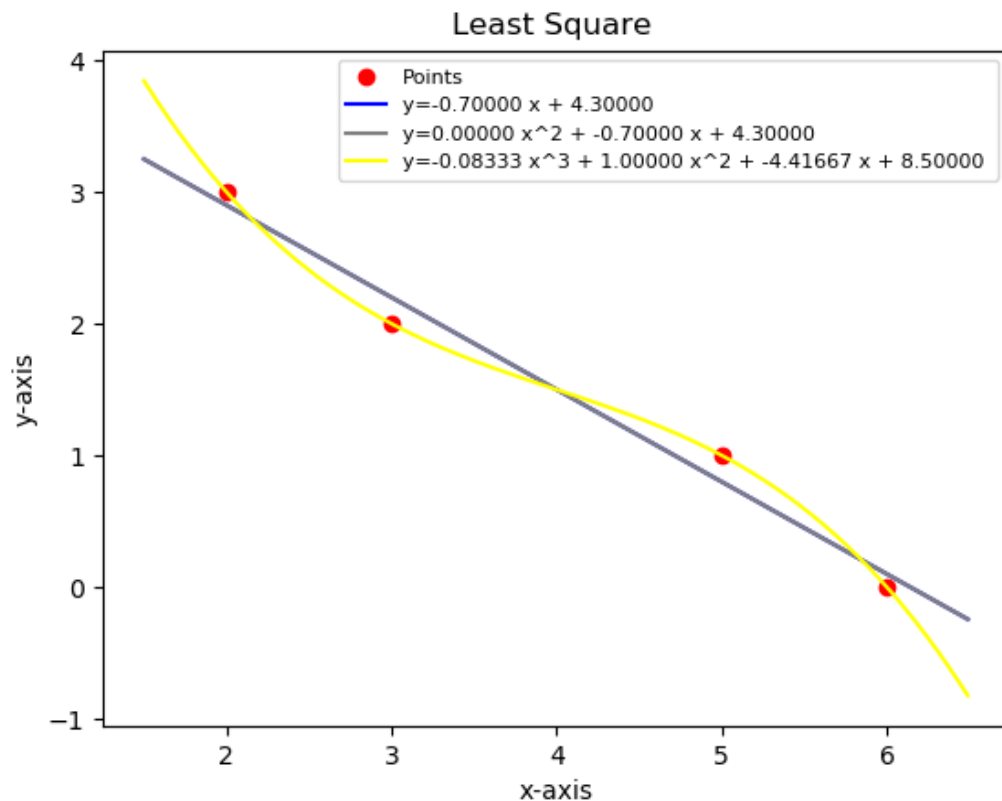
$$\text{Thus, } y = -\frac{7}{10}x + \frac{43}{10}$$

$$A\vec{x} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} -\frac{7}{10} \\ \frac{43}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{29}{10} \\ \frac{22}{10} \\ \frac{8}{10} \\ \frac{1}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{29}{10} \\ \frac{22}{10} \\ \frac{8}{10} \\ \frac{1}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{10} \\ -\frac{1}{5} \\ \frac{1}{5} \\ -\frac{1}{10} \end{pmatrix}$$



**Error:**

$$\|\vec{y} - A\vec{x}\| = \sqrt{2\left(\frac{1}{100} + \frac{1}{25}\right)}$$

$$= \frac{\sqrt{10}}{10}$$

$$= 0.31623$$

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The **second order** equation:

$$y = 0.0x^2 - 0.7x + 4.3$$

$$\text{Error} = 0.31623$$

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The **third order** equation:

$$y = -0.08333x^3 + x^2 - 4.41667x + 8.5$$

$$\text{Error} = 0.00000$$

### Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(-1, 0), (0, 1), (1, 2), (2, 4)\}$$

### Solution

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

The normal equation:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 10 \\ 7 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 6 & 2 \\ 2 & 4 \end{vmatrix} = 20 \quad \Delta_m = \begin{vmatrix} 10 & 2 \\ 7 & 4 \end{vmatrix} = 26 \quad \Delta_b = \begin{vmatrix} 6 & 10 \\ 2 & 7 \end{vmatrix} = 22$$

$$m = \frac{26}{20} = \frac{13}{10}$$

$$b = \frac{22}{20} = \frac{11}{10}$$

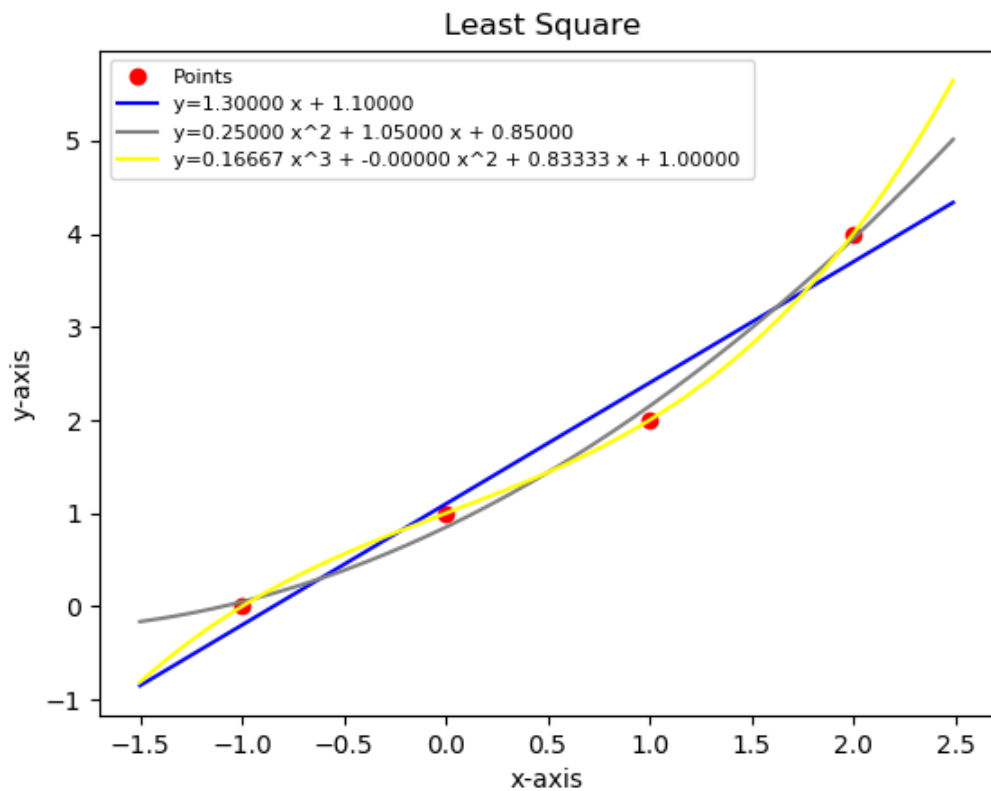
$$\text{Thus, } y = \frac{13}{10}x + \frac{11}{10}$$

$$A\vec{x} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{13}{10} \\ \frac{11}{10} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{5} \\ \frac{11}{10} \\ \frac{12}{5} \\ \frac{37}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} -\frac{1}{5} \\ \frac{11}{10} \\ \frac{12}{5} \\ \frac{37}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} \\ -\frac{1}{10} \\ -\frac{2}{5} \\ \frac{3}{10} \end{pmatrix}$$



**Error:**

$$\|\vec{y} - A\vec{x}\| = \sqrt{\frac{1}{25} + \frac{1}{100} + \frac{4}{25} + \frac{9}{100}}$$

$$= \sqrt{\frac{4+1+16+9}{100}}$$

$$= \frac{\sqrt{30}}{10}$$

$$= 0.54772$$

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The **second order** equation:

$$y = 0.25x^2 + 1.05x + 0.85$$

$$\text{Error} = 0.22361$$

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The **third order** equation:

$$y = 0.16667x^3 + 0.82222x + 1$$

$$\text{Error} = 0.00000$$

### Exercise

Find the equation of the line that best fits the given points in the least-squares sense and find the error.

$$\{(1, 0), (2, 1), (4, 2), (5, 3)\}$$

### Solution

Let  $y = mx + b$  be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

The normal equation:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 46 & 12 \\ 12 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 25 \\ 6 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 46 & 12 \\ 12 & 4 \end{vmatrix} = 40 \quad \Delta_m = \begin{vmatrix} 25 & 12 \\ 6 & 4 \end{vmatrix} = 28 \quad \Delta_b = \begin{vmatrix} 46 & 25 \\ 12 & 6 \end{vmatrix} = -24$$

$$m = \frac{28}{40} = \frac{7}{10}$$

$$b = -\frac{24}{40} = -\frac{3}{5}$$

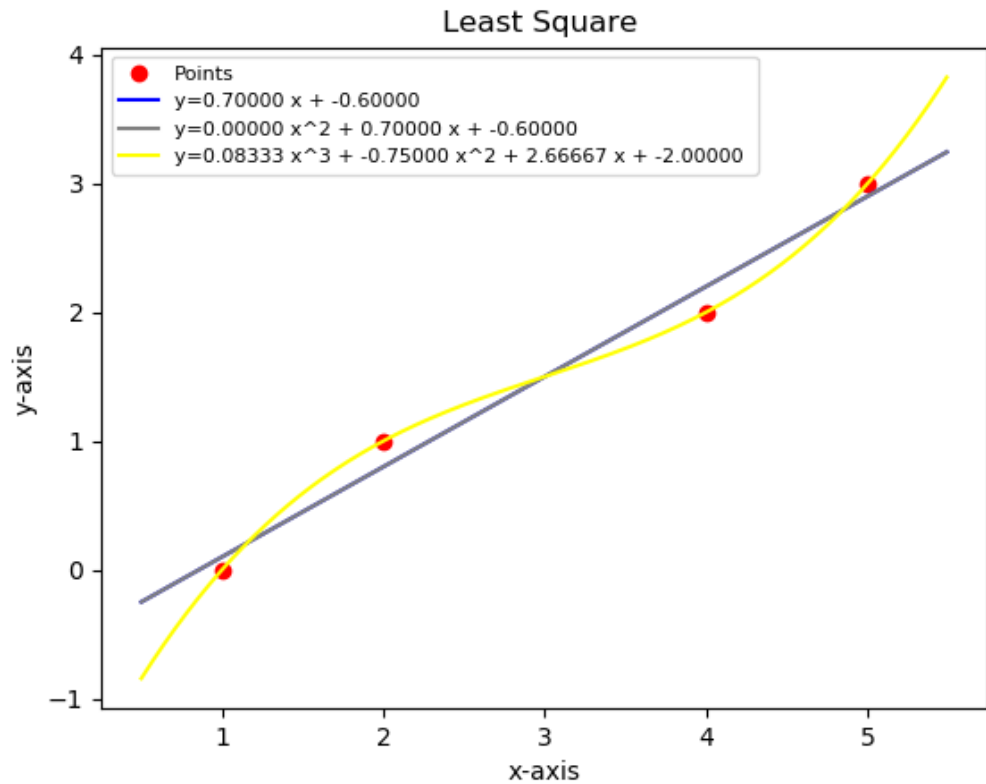
$$\text{Thus, } y = \frac{7}{10}x - \frac{3}{5}$$

$$A\vec{x} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} \frac{7}{10} \\ -\frac{3}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{10} \\ \frac{4}{5} \\ \frac{11}{5} \\ \frac{29}{10} \end{pmatrix}$$

$$\vec{y} - A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} \frac{1}{10} \\ \frac{4}{5} \\ \frac{11}{5} \\ \frac{29}{10} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{10} \\ \frac{1}{5} \\ -\frac{1}{5} \\ \frac{1}{10} \end{pmatrix}$$



**Error:**

$$\|\vec{y} - A\vec{x}\| = \sqrt{2\left(\frac{1}{100} + \frac{1}{25}\right)}$$

$$= \frac{\sqrt{10}}{10}$$

$$= 0.31623$$

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The **second order** equation:

$$y = 0.0x^2 + 0.7x - .6$$

$$\text{Error} = 0.31623$$

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The **third order** equation:

$$y = 0.08333x^3 - 0.75x^2 + 2.66667x - 2$$

$$\text{Error} = 0.00000$$

### Exercise

Find the orthogonal projection of the vector  $\vec{u}$  on the subspace of  $\mathbb{R}^4$  spanned by the vectors

$$\vec{u} = (-3, -3, 8, 9); \quad \vec{v}_1 = (3, 1, 0, 1), \quad \vec{v}_2 = (1, 2, 1, 1), \quad \vec{v}_3 = (-1, 0, 2, -1)$$

### Solution

$$\text{Let } A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{pmatrix}$$

$$A^T \vec{u} = \begin{pmatrix} 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -3 \\ 8 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 8 \\ 10 \end{pmatrix}$$

The normal solution is  $A^T A \vec{x} = A^T \vec{u}$

$$\begin{pmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ 10 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 11 & 6 & -4 \\ 6 & 7 & 0 \\ -4 & 0 & 6 \end{vmatrix} = 134$$

$$\Delta_1 = \begin{vmatrix} -3 & 6 & -4 \\ 8 & 7 & 0 \\ 10 & 0 & 6 \end{vmatrix} = -134$$

$$\Delta_2 = \begin{vmatrix} 11 & -3 & -4 \\ 6 & 8 & 0 \\ -4 & 10 & 6 \end{vmatrix} = 268$$

$$\underline{x_1 = \frac{-134}{134} = -1} \quad \underline{x_2 = \frac{268}{134} = 2} \quad \underline{x_3 = \frac{134}{134} = 1}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

So  $\text{proj}_W \vec{u} = A\vec{x}$

$$= \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 3 \\ 4 \\ 0 \end{pmatrix}$$

$$\underline{\text{proj}_W \vec{u} = (-2, 3, 4, 0)}$$

### Exercise

Find the orthogonal projection of the vector  $\vec{u}$  on the subspace of  $\mathbb{R}^4$  spanned by the vectors

$$\vec{u} = (6, 3, 9, 6); \quad \vec{v}_1 = (2, 1, 1, 1), \quad \vec{v}_2 = (1, 0, 1, 1), \quad \vec{v}_3 = (-2, -1, 0, -1)$$

### Solution

$$\text{Let } A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix}$$

$$A^T \vec{u} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 9 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 30 \\ 21 \\ -21 \end{pmatrix}$$

The normal solution is  $A^T A\vec{x} = A^T \vec{u}$



$$\begin{pmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 30 \\ 21 \\ -21 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 7 & 4 & -6 \\ 4 & 3 & -3 \\ -6 & -3 & 6 \end{vmatrix} = 3$$

$$\Delta_1 = \begin{vmatrix} 30 & 4 & -6 \\ 21 & 3 & -3 \\ -21 & -3 & 6 \end{vmatrix} = 18$$

$$\Delta_2 = \begin{vmatrix} 7 & 30 & -6 \\ 4 & 21 & -3 \\ -6 & -21 & 6 \end{vmatrix} = 9$$

$$\underline{x_1 = \frac{18}{3} = 6 \quad x_2 = \frac{9}{3} = 3 \quad x_3 = 4}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

$$\text{So } \text{proj}_W \vec{u} = A\vec{x}$$

$$= \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 2 \\ 9 \\ 5 \end{pmatrix}$$

$$\underline{\text{proj}_W \vec{v} = (7, 2, 9, 5)}$$

### ***Exercise***

Find the orthogonal projection of the vector  $\vec{u}$  on the subspace of  $\mathbb{R}^4$  spanned by the vectors

$$\vec{u} = (-2, 0, 2, 4); \quad \vec{v}_1 = (1, 1, 3, 0), \quad \vec{v}_2 = (-2, -1, -2, 1), \quad \vec{v}_3 = (-3, -1, 1, 3)$$

### ***Solution***

$$\text{Let } A = \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{pmatrix}$$

$$A^T \vec{u} = \begin{pmatrix} 1 & 1 & 3 & 0 \\ -2 & -1 & -2 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix}$$

The normal solution is  $A^T A \vec{x} = A^T \vec{u}$

$$\begin{pmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 11 & -9 & -1 \\ -9 & 10 & 8 \\ -1 & 8 & 20 \end{vmatrix} = 10 \quad \Delta_1 = \begin{vmatrix} 4 & -9 & -1 \\ 4 & 10 & 8 \\ 20 & 8 & 20 \end{vmatrix} = -8 \quad \Delta_2 = \begin{vmatrix} 11 & 4 & -1 \\ -9 & 4 & 8 \\ -1 & 20 & 20 \end{vmatrix} = -16$$

$$\underline{x_1 = \frac{-8}{10} = -\frac{4}{5}} \quad \underline{x_2 = \frac{-16}{10} = -\frac{8}{5}} \quad \underline{x_3 = \frac{8}{5}}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} \\ -\frac{8}{5} \\ \frac{8}{5} \end{pmatrix}$$

$$proj_W \vec{u} = A \vec{x}$$

$$= \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} \\ -\frac{8}{5} \\ \frac{8}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{12}{5} \\ -\frac{4}{5} \\ \frac{12}{5} \\ \frac{16}{5} \end{pmatrix}$$

$$\underline{\text{proj}_W \vec{u} = \left( -\frac{12}{5}, -\frac{4}{5}, \frac{12}{5}, \frac{16}{5} \right)}$$

### Exercise

Find the standard matrix for the orthogonal projection  $P$  of  $\mathbb{R}^2$  on the line passes through the origin and makes an angle  $\theta$  with the positive  $x$ -axis.

#### Solution

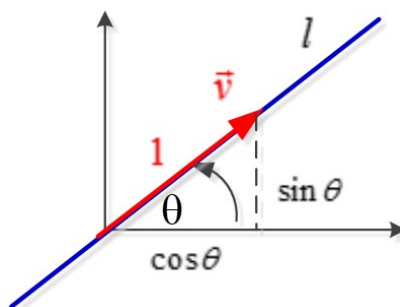
Since the line  $l$  is 2-dimensional, then we can take  $\vec{v} = (\cos \theta, \sin \theta)$  as a basis for this subspace

$$A = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

$$[P] = A^T A$$

$$= [\cos \theta \quad \sin \theta] \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

$$= \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}$$



### Exercise

Hooke's law in physics states that the length  $x$  of a uniform spring is a linear function of the force  $y$  applied to it. If we express the relationship as  $y = mx + b$ , then the coefficient  $m$  is called the spring constant.

Suppose a particular unstretched spring has a measured length of 6.1 inches.(i.e.,  $x = 6.1$  when  $y = 0$ ).

Forces of 2 pounds, 4 pounds, and 6 pounds are then applied to the spring, and the corresponding lengths are found to be 7.6 inches, 8.7 inches, and 10.4 inches. Find the spring constant.

#### Solution

$$M = \begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

The normal equation:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6.1 & 1 \\ 7.6 & 1 \\ 8.7 & 1 \\ 10.4 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 6.1 & 7.6 & 8.7 & 10.4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix}$$

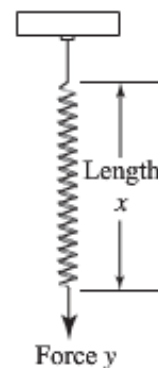
$$\begin{pmatrix} 278.82 & 32.8 \\ 32.8 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 112.4 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{39.44} \begin{pmatrix} 4 & -32.8 \\ -32.8 & 278.2 \end{pmatrix} \begin{pmatrix} 112.4 \\ 12 \end{pmatrix}$$

$$= \frac{1}{39.44} \begin{pmatrix} 56 \\ -348.32 \end{pmatrix}$$

$$= \begin{pmatrix} 1.4 \\ -8.8 \end{pmatrix}$$

Thus, the estimated value of the spring constant is  $\approx 1.4$  pounds



## Exercise

Prove:

If  $A$  has a linearly independent column vectors, and if  $\vec{b}$  is orthogonal to the column space of  $A$ , then the least squares solution of  $A\vec{x} = \vec{b}$  is  $\vec{x} = \vec{0}$ .

## Solution

If  $A$  has linearly independent column vectors, then  $A^T A$  is invertible and the least squares solution of  $A\vec{x} = \vec{b}$  is the solution of  $A^T A \vec{x} = A^T \vec{b}$ , but since  $\vec{b}$  is orthogonal to the column space of  $A$ .

$A^T \vec{b} = 0$ , so  $\vec{x}$  is a solution of  $A^T A \vec{x} = 0$ .

Thus  $\vec{x} = \vec{0}$  since  $A^T A$  is invertible.

## Exercise

Let  $A$  be an  $m \times n$  matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of  $\mathbb{R}^n$  onto the row space of  $A$ .

## Solution

$A^T$  will have linearly independent column vectors, and the column space  $A^T$  is the row space of  $A$ .

Thus, the standard matrix for the orthogonal projection of  $\mathbb{R}^n$  onto the row space of  $A$  is

$$\begin{aligned}
 [P] &= A^T \left[ \begin{pmatrix} A^T \\ A^T \end{pmatrix} \right]^{-1} \begin{pmatrix} A^T \end{pmatrix}^T \\
 &= A^T (AA^T)^{-1} A
 \end{aligned}$$

### Exercise

Let  $W$  be the line with parametric equations  $x = 2t, \quad y = -t, \quad z = 4t$

- Find a basis for  $W$ .
- Find the standard matrix for the orthogonal projection on  $W$ .
- Use the matrix in part (b) to find the orthogonal projection of a point  $P_0(x_0, y_0, z_0)$  on  $W$ .
- Find the distance between the point  $P_0(2, 1, -3)$  and the line  $W$ .

### Solution

**a)**  $W = \text{span}\{(2, -1, 4)\}$

So that the vector  $(2, -1, 4)$  forms a basis for  $W$  (linear independence)

**b)** Let  $A = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$

$$[P] = A(A^T A)^{-1} A^T$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \left( \begin{bmatrix} 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} [21]^{-1} \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$= \frac{1}{21} \begin{bmatrix} 4 & -2 & 8 \\ -2 & 1 & -4 \\ 8 & -4 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix}$$

$$c) \begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \frac{4}{21}x_0 - \frac{2}{21}y_0 + \frac{8}{21}z_0 \\ -\frac{2}{21}x_0 + y_0 - \frac{4}{21}z_0 \\ \frac{8}{21}x_0 - \frac{4}{21}y_0 + \frac{16}{21}z_0 \end{bmatrix}$$

$$d) \begin{bmatrix} \frac{4}{21} & -\frac{2}{21} & \frac{8}{21} \\ -\frac{2}{21} & 1 & -\frac{4}{21} \\ \frac{8}{21} & -\frac{4}{21} & \frac{16}{21} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{6}{7} \\ \frac{3}{7} \\ -\frac{12}{7} \end{bmatrix}$$

The distance between  $P_0$  and  $W$  equals to the distance between  $P_0$  and its projection on  $W$ .

The distance between  $(2, 1, -3)$  and  $\left(-\frac{6}{7}, \frac{3}{7}, -\frac{12}{7}\right)$  is

$$\begin{aligned} d &= \sqrt{\left(2 + \frac{6}{7}\right)^2 + \left(1 - \frac{3}{7}\right)^2 + \left(-3 + \frac{12}{7}\right)^2} \\ &= \sqrt{\frac{400}{49} + \frac{16}{49} + \frac{81}{49}} \\ &= \frac{\sqrt{497}}{7} \end{aligned}$$

### Exercise

In  $R^3$ , consider the line  $l$  given by the equations  $x = t, y = t, z = t$

And the line  $m$  given by the equations  $x = s, y = 2s - 1, z = 1$

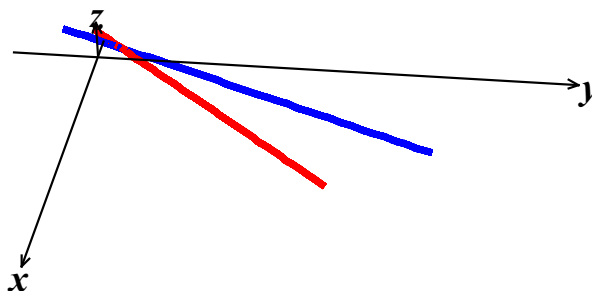
Let  $P$  be the point on  $l$ , and let  $Q$  be a point on  $m$ . Find the values of  $t$  and  $s$  that minimize the distance between the lines by minimizing the squared distance  $\|P - Q\|^2$

### Solution

When  $t = 1 \Rightarrow$  Let  $P = (1, 1, 1)$  is on line  $l$

When  $s = 1 \Rightarrow$  Let  $Q = (1, 1, 1)$  is on line  $m$

$$\|P - Q\| = \sqrt{(1-1)^2 + (1-1)^2 + (1-1)^2} = 0 \geq 0$$



Thus, these are the values  $P = (1, 1, 1)$  and  $Q = (1, 1, 1)$  are the values for  $s = t = 1$  that minimize the distance between the lines.

### Exercise

Determine whether the statement is true or false,

- If  $A$  is an  $m \times n$  matrix, then  $A^T A$  is a square matrix.
- If  $A^T A$  is invertible, then  $A$  is invertible.
- If  $A$  is invertible, then  $A^T A$  is invertible.
- If  $A\vec{x} = \vec{b}$  is a consistent linear system, then  $A^T A\vec{x} = A^T \vec{b}$  is also consistent.
- If  $A\vec{x} = \vec{b}$  is an inconsistent linear system, then  $A^T A\vec{x} = A^T \vec{b}$  is also inconsistent.
- Every linear system has a least squares solution.
- Every linear system has a unique least squares solution.
- If  $A$  is an  $m \times n$  matrix with linearly independent columns and  $\vec{b}$  is in  $R^m$ , then  $A\vec{x} = \vec{b}$  has a unique least squares solution.

### Solution

- True;**  $A^T A$  is an  $n \times n$  matrix
- False;** only square matrix has inverses, but  $A^T A$  can be invertible when  $A$  is not square matrix.
- True;** if  $A$  is invertible, so is  $A^T$ , so the product  $A^T A$  is also invertible
- True**
- False;** the system  $A^T A\vec{x} = A^T \vec{b}$  may be consistent
- True**
- False;** the least squares solution may involve a parameter
- True;** if  $A$  has linearly independent column vectors; then  $A^T A$  is invertible, so  $A^T A\vec{x} = A^T \vec{b}$  has a unique solution

### Exercise

A certain experiment produces the data  $\{(1, 1.8), (2, 2.7), (3, 3.4), (4, 3.8), (5, 3.9)\}$ .

Find the function that it will fit these data in the form of  $y = \beta_1 x + \beta_2 x^2$

### Solution

**Given:** the equation  $y = \beta_1 x + \beta_2 x^2$  that best fits the given points. Then

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$$

The normal equation formula:  $A^T A \vec{x} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix} \begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix}$$

$$\begin{pmatrix} 55 & 225 \\ 225 & 979 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 52.1 \\ 201.5 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 55 & 225 \\ 225 & 979 \end{vmatrix} = 3,220 \quad \Delta\beta_1 = \begin{vmatrix} 52.1 & 225 \\ 201.5 & 979 \end{vmatrix} = 5,668.4 \quad \Delta\beta_2 = \begin{vmatrix} 55 & 52.1 \\ 225 & 201.5 \end{vmatrix} = -640$$

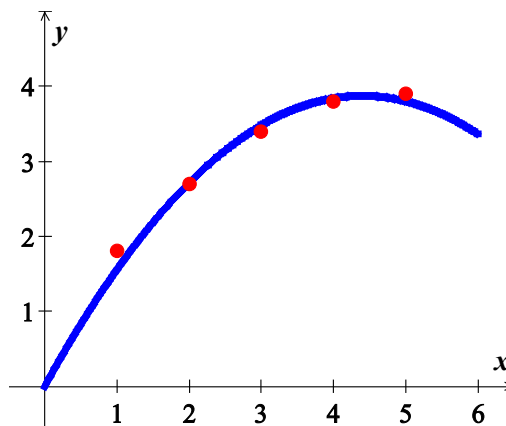
$$\beta_1 = \frac{5,668.4}{3,220}$$

$$\approx 1.76$$

$$\beta_2 = -\frac{640}{3,220}$$

$$\approx -0.199$$

$$y = 1.76x - .2x^2$$





### Exercise

According to Kepler's first law, a comet should have an ellipse, parabolic, or hyperbolic orbit (with gravitational attractions from the planets ignored). In suitable polar coordinates, the position  $(r, \nu)$  of a comet satisfies an equation of the form

$$r = \beta + e(r \cdot \cos \nu)$$

Where  $\beta$  is a constant and  $e$  is the eccentricity of the orbit, with  $0 \leq e < 1$  for an ellipse,  $e = 1$  for a parabolic, and  $e > 1$  for a hyperbola.

Suppose observations of a newly discovered comet provide the data below.

$\nu$	.88	1.10	1.42	1.77	2.14
$r$	3.00	2.30	1.65	1.25	1.01

Determine the type of orbit, and predict where the orbit will be when  $\nu = 4.6$  (*radians*)?

### Solution

**Given:** the equation in the form  $r = \beta + e(r \cdot \cos \nu)$

$$3 = \beta + e(3 \cdot \cos(.88)) = \beta + 1.911e$$

$$2.3 = \beta + e(2.3 \cos(1.1)) = \beta + 1.043e$$

$$1.65 = \beta + e(1.65 \cos(1.42)) = \beta + .248e$$

$$1.25 = \beta + e(1.25 \cos(1.77)) = \beta - .247e$$

$$1.01 = \beta + e(1.01 \cos(2.14)) = \beta - .544e$$

$$\begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix} \begin{pmatrix} \beta \\ e \end{pmatrix} = \begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$$

$$\text{where } A = \begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} \beta \\ e \end{pmatrix} \quad \vec{r} = \begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$$

The normal equation formula:  $A^T A \vec{v} = A^T \vec{r}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1.911 & 1.043 & .248 & -.247 & -.544 \end{pmatrix} \begin{pmatrix} 1 & 1.911 \\ 1 & 1.043 \\ 1 & .248 \\ 1 & -.247 \\ 1 & -.544 \end{pmatrix} \begin{pmatrix} \beta \\ e \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1.911 & 1.043 & .248 & -.247 & -.544 \end{pmatrix} \begin{pmatrix} 3 \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2.411 \\ 2.411 & 5.158 \end{pmatrix} \begin{pmatrix} \beta \\ e \end{pmatrix} = \begin{pmatrix} 9.21 \\ 7.683 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 5 & 2.411 \\ 2.411 & 5.158 \end{vmatrix} = 19.98 \quad \Delta_{\beta} = \begin{vmatrix} 9.21 & 2.411 \\ 7.683 & 5.158 \end{vmatrix} = 28.98 \quad \Delta_e = \begin{vmatrix} 5 & 9.21 \\ 2.411 & 7.683 \end{vmatrix} = 16.21$$

$$\beta = \frac{28.98}{19.98}$$

$$\approx 1.45$$

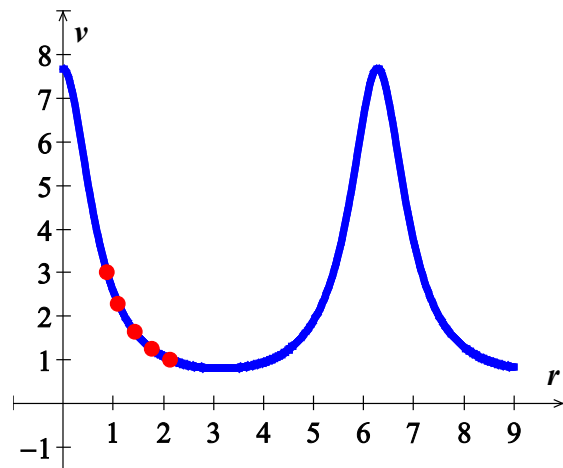
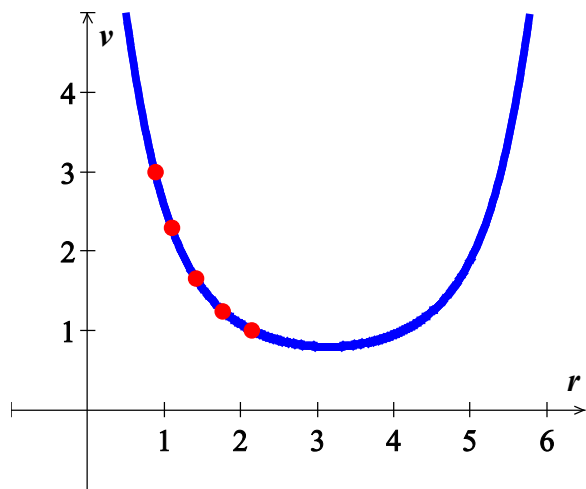
$$e = \frac{16.21}{19.98}$$

$$\approx 0.811 < 1$$

Therefore, the orbit is an *ellipse* type since  $e \approx 0.811 < 1$

Since  $r = \beta + e(r \cdot \cos v)$

$$\text{Then, } r(v) = \frac{1.45}{1 - 0.811 \cdot \cos v}$$



$$r(4.6) = \frac{1.45}{1 - 0.811 \cdot \cos 4.6}$$

$$\approx 1.329$$

### Exercise

To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second, from  $t = 0$  to  $t = 12$

The position (in *feet*) were:

0, 8.8, 29.9, 62.0, 104.7, 159.1, 222.0, 294.5, 380.4, 471.1, 571.7, 686.8, and 809.2

- a) Find the least square cubic curve  $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$  for these data.
- b) Estimate the velocity of the plane when  $t = 4.5$  *sec*, using the result from part (a).

### Solution

**Given:** the equation is in form  $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \\ 1 & 6 & 36 & 216 \\ 1 & 7 & 49 & 343 \\ 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \\ 1 & 12 & 144 & 1728 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8.8 \\ 29.9 \\ 62 \\ 104.7 \\ 159.1 \\ 222.0 \\ 294.5 \\ 380.4 \\ 471.1 \\ 571.7 \\ 686.8 \\ 809.2 \end{pmatrix}$$

$\mathbf{A} \quad \vec{t} = \vec{y}$

The normal equation formula:  $A^T A \vec{t} = A^T \vec{y}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \\ 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 & 100 & 121 & 144 & \\ 0 & 1 & 8 & 27 & 64 & 125 & 216 & 343 & 512 & 729 & 1000 & 1331 & 1728 & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \\ 1 & 6 & 36 & 216 \\ 1 & 7 & 49 & 343 \\ 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \\ 1 & 12 & 144 & 1728 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \\ 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 & 100 & 121 & 144 & \\ 0 & 1 & 8 & 27 & 64 & 125 & 216 & 343 & 512 & 729 & 1000 & 1331 & 1728 & \end{pmatrix} \begin{pmatrix} 0 \\ 8.8 \\ 29.9 \\ 62 \\ 104.7 \\ 159.1 \\ 222.0 \\ 294.5 \\ 380.4 \\ 471.1 \\ 571.7 \\ 686.8 \\ 809.2 \end{pmatrix}$$

$$\begin{pmatrix} 13 & 78 & 650 & 6,084 \\ 78 & 650 & 6,084 & 60,710 \\ 650 & 6,084 & 60,710 & 630,708 \\ 6,084 & 60,710 & 630,708 & 6,735,950 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 3,800.2 \\ 35,127.7 \\ 348,063.9 \\ 3,599,800.9 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} 13 & 78 & 650 & 6,084 \\ 78 & 650 & 6,084 & 60,710 \\ 650 & 6,084 & 60,710 & 630,708 \\ 6,084 & 60,710 & 630,708 & 6,735,950 \end{vmatrix} = 97,538,785,344$$

$$\Delta_0 = \begin{vmatrix} 3800.2 & 78 & 650 & 6,084 \\ 35,127.7 & 650 & 6,084 & 60,710 \\ 348,063.9 & 6,084 & 60,710 & 630,708 \\ 3,599,800.9 & 60,710 & 630,708 & 6,735,950 \end{vmatrix} = -83,470,691,303.8916$$

Or I use my program to find the values

```
rref = (Matrix([
  [1, 0, 0, 0, -0.855769230765803],
  [0, 1, 0, 0, 4.70248501498163],
  [0, 0, 1, 0, 5.55536963037029],
  [0, 0, 0, 1, -0.0273601398601744]]))
```

$$\beta_0 \approx -0.855769$$

$$\beta_1 \approx 4.702485$$

$$\beta_2 \approx 5.55537$$

$$\beta_3 \approx -0.02736$$

$$y(t) = -0.855769 + 4.702485t + 5.55537t^2 - 0.02736t^3$$

$$\text{Error} = 3.9734$$

