Solution   *Section* 4.1 – Matrix Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

Exercise

Find the standard matrix for the transformation defined by the equations

a) $\begin{cases} w_1 = 2x_1 - 3x_2 + 4x_4 \\ w_2 = 3x_1 + 5x_2 - x_4 \end{cases}$

b) $\begin{cases} w_1 = 7x_1 + 2x_2 - 8x_3 \\ w_2 = \quad\quad - x_2 + 5x_3 \\ w_3 = 4x_1 + 7x_2 - x_3 \end{cases}$

c) $\begin{cases} w_1 = x_1 \\ w_2 = x_1 + x_2 \\ w_3 = x_1 + x_2 + x_3 \\ w_4 = x_1 + x_2 + x_3 + x_4 \end{cases}$

Solution

a) $$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 & 4 \\ 3 & 5 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The standard matrix is $\begin{bmatrix} 2 & -3 & 0 & 4 \\ 3 & 5 & 0 & -1 \end{bmatrix}$

b) $$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The standard matrix is $\begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix}$

c) $$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The standard matrix is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Exercise

Find the standard matrix for the operator *T* defined by the formula

$$T\left(x_1, x_2\right) = \left(2x_1 - x_2, x_1 + x_2\right)$$

Solution

$$T\left(x_1, x_2\right) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The standard matrix is $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

Exercise

Find the standard matrix for the operator *T* defined by the formula

$$T\left(x_1, x_2, x_3\right) = \left(x_1 + 2x_2 + x_3,\ x_1 + 5x_2,\ x_3\right)$$

Solution

$$T\left(x_1, x_2, x_3\right) = \begin{bmatrix} x_1 + 2x_2 + x_3 \\ x_1 + 5x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The standard matrix is $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Exercise

Find the standard matrix for the operator T defined by the formula

$$T\left(x_1, x_2, x_3\right) = \left(4x_1,\ 7x_2,\ -8x_3\right)$$

Solution

$$T\left(x_1, x_2, x_3\right) = \begin{pmatrix} 4x_1 \\ 7x_2 \\ -8x_3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The standard matrix is $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -8 \end{pmatrix}$

Exercise

Find the standard matrix for the operator T defined by the formula

$$T\left(x_1, x_2\right) = \left(x_2,\ -x_1,\ x_1 + 3x_2,\ x_1 - x_2\right)$$

Solution

$$T\left(x_1, x_2\right) = \begin{pmatrix} x_2 \\ -x_1 \\ x_1 + 3x_2 \\ x_1 - x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The matrix is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{pmatrix}$

Exercise

Find the standard matrix for the operator T defined by the formula

$$T\left(x_1, x_2, x_3\right) = \left(0,\ \ 0,\ 0,\ 0,\ 0\right)$$

Solution

$$T\left(x_1, x_2, x_3\right) = \left(0,\ \ 0,\ 0,\ 0,\ 0\right)$$

The matrix is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Exercise

Find the standard matrix for the operator T defined by the formula

$$T\left(x_1, x_2, x_3, x_4\right) = \left(x_4,\ x_1,\ x_3,\ x_2,\ x_1 - x_3\right)$$

Solution

$$T\left(x_1, x_2, x_3, x_4\right) = \left(x_4,\ x_1,\ x_3,\ x_2,\ x_1 - x_3\right)$$

$$T\left(x_1, x_2, x_3, x_4\right) = \begin{pmatrix} x_4 \\ x_1 \\ x_3 \\ x_2 \\ x_1 - x_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

The matrix is $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}$

Exercise

Find the standard matrix for the operator T defined by the formula

$$T\left(x_1, x_2, x_3, x_4\right) = \left(7x_1 + 2x_2 - x_3 + x_4,\ x_2 + x_3,\ -x_1\right)$$

Solution

$$T\left(x_1, x_2, x_3, x_4\right) = \begin{pmatrix} 7x_1 + 2x_2 - x_3 + x_4 \\ x_2 + x_3 \\ -x_1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

The matrix is $\begin{pmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$

Exercise

Plot $\vec{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and their images under the given transformation T

$$T(\vec{x}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

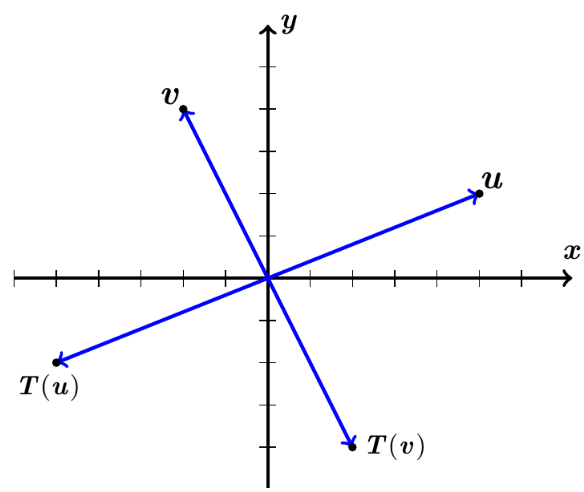
Solution

$$T(\vec{u}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ -2 \end{pmatrix}$$

$$T(\vec{v}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$



$\therefore$ *Reflection* through the origin $(180°)$

Exercise

Plot $\vec{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and their images under the given transformation T

$$T(\vec{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Solution

$$T(\vec{u}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

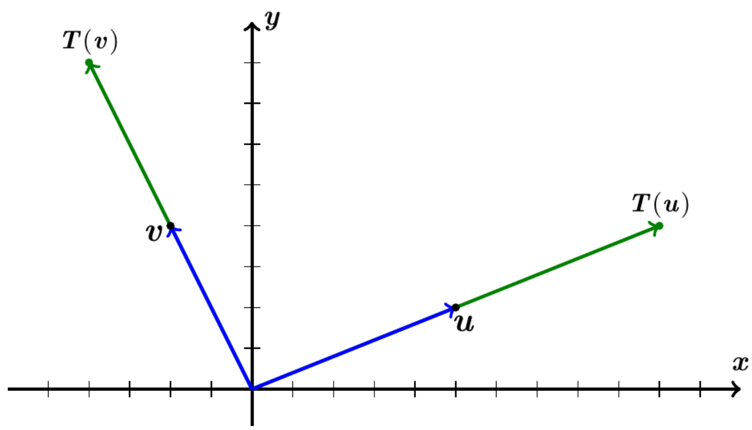
$$= \begin{pmatrix} 10 \\ 4 \end{pmatrix}$$

$$T(\vec{v}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} 10 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$T(\vec{u}) = \vec{u}$$



$\therefore$ *Dilation* with factor $(k = 2)$ on $\mathbb{R}^2$

Exercise

Plot $\vec{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and their images under the given transformation T

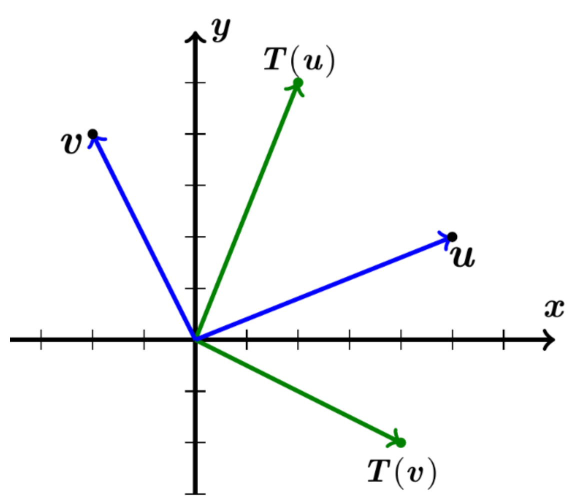
$$T(\vec{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Solution

$$T(\vec{u}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$T(\vec{v}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$



$$= \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\vec{x}_1 = \vec{x}_2$$

$\therefore$ *reflection* about the line $\vec{x}_1 = \vec{x}_2$

Exercise

Plot $\vec{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and their images under the given transformation T

$$T(\vec{x}) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

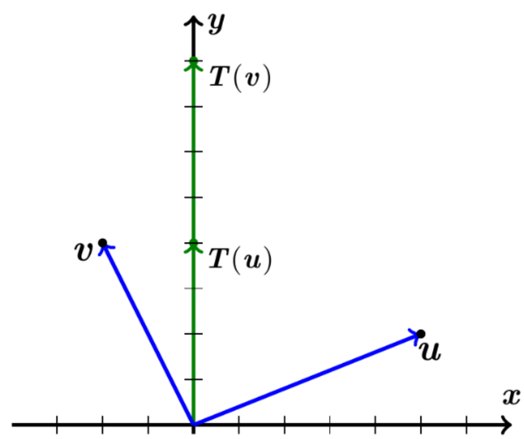
Solution

$$T(\vec{u}) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$T(\vec{v}) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 8 \end{pmatrix}$$



$$\underset{T(\vec{u})}{\begin{pmatrix} 0 \\ 4 \end{pmatrix}} = 2 \underset{\vec{u}}{\begin{pmatrix} * \\ 2 \end{pmatrix}}$$

$$\underset{T(\vec{v})}{\begin{pmatrix} 0 \\ 8 \end{pmatrix}} = 2 \underset{\vec{v}}{\begin{pmatrix} * \\ 4 \end{pmatrix}}$$

$\therefore$ Orthogonal projection on the y-axis and with *Dilation* with factor $(k = 2)$

Exercise

Plot $\vec{u}=\begin{pmatrix}5\\2\end{pmatrix},\quad \vec{v}=\begin{pmatrix}-2\\4\end{pmatrix}$ and their images under the given transformation T

$$T(\vec{x})=\begin{pmatrix}1&0\\1&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

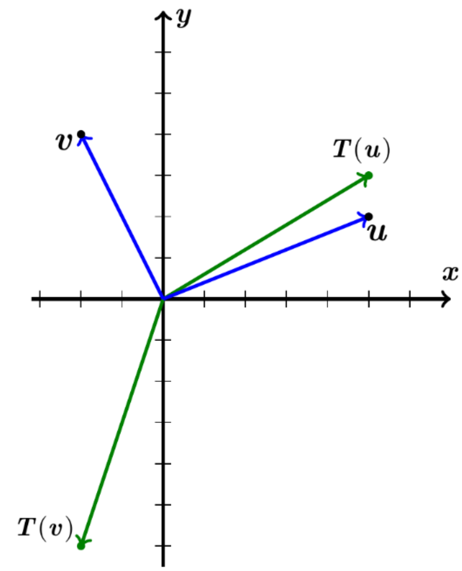
Solution

$$T(\vec{u})=\begin{pmatrix}1&0\\1&-1\end{pmatrix}\begin{pmatrix}5\\2\end{pmatrix}$$

$$=\begin{pmatrix}5\\3\end{pmatrix}$$

$$T(\vec{v})=\begin{pmatrix}1&0\\1&-1\end{pmatrix}\begin{pmatrix}-2\\4\end{pmatrix}$$

$$=\begin{pmatrix}-2\\-6\end{pmatrix}$$



$$\underset{T(\vec{u})}{\begin{pmatrix}5\\3\end{pmatrix}}=\underset{\vec{u}}{\begin{pmatrix}5\\2\end{pmatrix}}$$

$$\underset{T(\vec{v})}{\begin{pmatrix}-2\\-6\end{pmatrix}}=\underset{\vec{v}}{\begin{pmatrix}-2\\4\end{pmatrix}}$$

$\therefore$ *Expansion* of $\vec{u}$ in the y–direction with factor $k=\frac{3}{2}$

$\therefore$ *Expansion* of $\vec{v}$ in the y–direction with factor $k=\frac{3}{2}$ and reflection about x–axis.