Section 3.5 – Exponential and logarithmic Equations

Exponential Equations

$$b^M = b^N \quad \leftrightarrow \quad M = N \quad \text{for any } b > 0, \neq 1$$

Example

Solve $5^{3x-6} = 125$

Solution

$5^{3x-6} = 5^3$

$3x - 6 = 3$

$3x = 9$

$\boxed{x = 3}$

Example

Solve $8^{x+2} = 4^{x-3}$

Solution

$\left(2^3\right)^{x+2} = \left(2^2\right)^{x-3}$

$2^{3(x+2)} = 2^{2(x-3)}$

$3(x+2) = 2(x-3)$

$3x + 6 = 2x - 6$

$3x - 2x = -6 - 6$

$\boxed{x = -12}$

Using *Natural Logarithms*

1. Isolate the exponential expression
2. Take the natural logarithm on both sides of the equation
3. Simplify using one of the following properties: $\ln b^x = x \ln b$ *or* $\ln e^x = x$
4. Solve for the variable

Example

Solve: $7e^{2x} - 5 = 58$

Solution

$7e^{2x} - 5 = 58$ *Isolate the exponential expression*

$7e^{2x} = 63$ *Divide by 7 both sides*

$e^{2x} = 9$ *Natural logarithm on both sides*

$\ln e^{2x} = \ln 9$ *Use inverse Property*

$2x = \ln 9$

$\boxed{x = \frac{\ln 9}{2} \approx 1.0986}$

Example

Solve: $3^{2x-1} = 7^{x+1}$

Solution

$\ln 3^{2x-1} = \ln 7^{x+1}$ *Natural logarithm on both sides*

$(2x-1)\ln 3 = (x+1)\ln 7$ *Power Rule*

$2x\ln 3 - \ln 3 = x\ln 7 + \ln 7$

$2x\ln 3 - x\ln 7 = \ln 3 + \ln 7$

$x\left(2\ln 3 - \ln 7\right) = \ln 3 + \ln 7$

$\boxed{x = \frac{\ln 3 + \ln 7}{2\ln 3 - \ln 7}}$ $\boxed{\approx 12.1143}$

Logarithmic Equations

1. Express the equation in the form $\log_b M = c$
2. Use the definition of a logarithm to rewrite the equation in exponential form:

$$\log_b M = c \quad \Rightarrow \quad b^c = M$$

3. Solve for the variable
4. Check proposed solution in the original equation. Include only the set for M > 0

Example

Solve: $\log(x) + \log(x-3) = 1$

Solution

$\log\left(x(x-3)\right) = 1$ ***Product Rule***

$x(x-3) = 10^1$ ***Convert to exponential form***

$x^2 - 3x = 10$

$x^2 - 3x - 10 = 0$ ***Solve for x***

$x = -2,\ 5$

Check: $x = -2 \Rightarrow \cancel{\log(-2) + \log(x-3) = 1}$

$x = 5 \Rightarrow \log(5) + \log(5-3) = 1$

$\therefore$ ***Solution***: $\boxed{x = 5}$

Example

Solve: $\log_6(3x+2) + \log_6(x-1) = 1$

Solution

$\log_6\left[(3x+2)(x-1)\right] = 1$ ***Product Rule***

$(3x+2)(x-1) = 6^1$ ***Convert to exponential form***

$3x^2 - x - 2 = 6$

$3x^2 - x - 8 = 0$ ***Solve for x***

$x = \frac{1-\sqrt{97}}{6} < 0$ $\quad$ $x = \frac{1+\sqrt{97}}{6} > 1$

$\therefore$ ***Solution***: $\boxed{x = \frac{1+\sqrt{97}}{6}}$

Property of Logarithmic Equality

For any $M > 0$, $N > 0$, $b > 0, \neq 1$

$$\log_b M = \log_b N \quad \Rightarrow \quad M = N$$

Example

Solve: $\ln(x-3) = \ln(7x-23) - \ln(x+1)$

Solution

$\ln(x-3) = \ln\left(\dfrac{7x-23}{x+1}\right)$ ***Quotient Rule***

$x-3 = \dfrac{7x-23}{x+1}$

$(x-3)(x+1) = 7x-23$

$x^2 - 2x - 3 = 7x - 23$

$x^2 - 9x + 20 = 0$

$\underline{x = 4,\ 5|}$

Check: $x = 4 \Rightarrow \ln(4-3) = \ln(7(4)-23) - \ln(4+1)$

$x = 5 \Rightarrow \ln(5-3) = \ln(7(5)-23) - \ln(5+1)$

$\therefore$ ***Solution***: $\underline{x = 4,\ 5|}$

Example

Solve: $\log(x+6) - \log(x+2) = \log x$

Solution

$\log\dfrac{x+6}{x+2} = \log x$ ***Quotient Rule***

$\dfrac{x+6}{x+2} = x$ ***Multiply by x + 2***

$x + 6 = x(x+2)$

$x + 6 = x^2 + 2x$

$x^2 + x - 6 = 0$ ***Solve for x***

$\underline{x = -3,\ 2|}$

Check: $x = -3 \to \log(-3+6) - \log(-3+2) = \log(-3)$

$x = 2 \to \log(2+6) - \log(2+2) = \log(2)$

Or ***Domain***

$\therefore$ ***Solution***: $\underline{x = 2|}$

Exercises *Section* 3.5 – Exponential and logarithmic Equations

(**1 – 105**) Solve the equations

1. $2^x = 128$

2. $3^x = 243$

3. $5^x = 70$

4. $6^x = 50$

5. $5^x = 134$

6. $7^x = 12$

7. $9^x = \frac{1}{\sqrt[3]{3}}$

8. $49^x = \frac{1}{343}$

9. $2^{5x+3} = \frac{1}{16}$

10. $\left(\frac{2}{5}\right)^x = \frac{8}{125}$

11. $2^{3x-7} = 32$

12. $4^{2x-1} = 64$

13. $3^{1-x} = \frac{1}{27}$

14. $2^{-x^2} = 5$

15. $2^{-x} = 8$

16. $\left(\frac{1}{3}\right)^x = 81$

17. $3^{-x} = 120$

18. $27 = 3^{5x}\ 9^{x^2}$

19. $4^{x+3} = 3^{-x}$

20. $2^{x+4} = 8^{x-6}$

21. $8^{x+2} = 4^{x-3}$

22. $7^x = 12$

23. $5^{x+4} = 4^{x+5}$

24. $5^{x+2} = 4^{1-x}$

25. $3^{2x-1} = 0.4^{x+2}$

26. $4^{3x-5} = 16$

27. $4^{x+3} = 3^{-x}$

28. $7^{2x+1} = 3^{x+2}$

29. $3^{x-1} = 7^{2x+5}$

30. $4^{x-2} = 2^{3x+3}$

31. $3^{5x-8} = 9^{x+2}$

32. $3^{x+4} = 2^{1-3x}$

33. $3^{2-3x} = 4^{2x+1}$

34. $4^{x+3} = 3^{-x}$

35. $7^{x+6} = 7^{3x-4}$

36. $2^{-100x} = (0.5)^{x-4}$

37. $4^x\left(\frac{1}{2}\right)^{3-2x} = 8.\left(2^x\right)^2$

38. $5^x + 125\left(5^{-x}\right) = 30$

39. $4^x - 3\left(4^{-x}\right) = 8$

40. $5^{3x-6} = 125$

41. $e^x = 15$

42. $e^{x+1} = 20$

43. $9e^x = 107$

44. $e^{x\ln 3} = 27$

45. $e^{x^2} = e^{7x-12}$

46. $f(x) = xe^{x} + e^{x}$

47. $f(x) = x^{3}\left(4e^{4x}\right) + 3x^{2}e^{4x}$

48. $e^{2x} - 2e^{x} - 3 = 0$

49. $e^{0.08t} = 2500$

50. $e^{x^{2}} = 200$

51. $e^{2x+1} \cdot e^{-4x} = 3e$

52. $e^{2x} - 8e^{x} + 7 = 0$

53. $e^{2x} + 2e^{x} - 15 = 0$

54. $e^{x} + e^{-x} - 6 = 0$

55. $e^{1-3x} \cdot e^{5x} = 2e$

56. $6\ln(2x) = 30$

57. $\log_{5}(x-7) = 2$

58. $\log_{4}(5+x) = 3$

59. $\log(4x-18) = 1$

60. $\log_{3} x = -2$

61. $\log\left(x^{2}+19\right) = 2$

62. $\ln\left(x^{2}-12\right) = \ln x$

63. $\log\left(2x^{2}+3x\right) = \log(10x+30)$

64. $\log_{5}(2x+3) = \log_{5} 11 + \log_{5} 3$

65. $\log_{3} x - \log_{9}(x+42) = 0$

66. $\log_{5} x + \log_{5}(4x-1) = 1$

67. $\log x - \log(x+3) = 1$

68. $\log x + \log(x-9) = 1$

69. $\log_{2}(x+1) + \log_{2}(x-1) = 3$

70. $\log_{8}(x+1) - \log_{8} x = 2$

71. $\ln(x+8) + \ln(x-1) = 2\ln x$

72. $\ln(4x+6) - \ln(x+5) = \ln x$

73. $\ln(5+4x) - \ln(x+3) = \ln 3$

74. $\ln\sqrt[4]{x} = \sqrt{\ln x}$

75. $\sqrt{\ln x} = \ln\sqrt{x}$

76. $\log x^{2} = (\log x)^{2}$

77. $\log x^{3} = (\log x)^{2}$

78. $\log(\log x) = 1$

79. $\log(\log x) = 2$

80. $\ln(\ln x) = 2$

81. $\ln\left(e^{x^{2}}\right) = 64$

82. $e^{\ln(x-1)} = 4$

83. $10^{\log(2x+5)} = 9$

84. $\log\sqrt{x^{3}-9} = 2$

85. $\log\sqrt{x^{3}-17} = \frac{1}{2}$

86. $\log_{4} x = \log_{4}(8-x)$

87. $\log_{7}(x-5) = \log_{7}(6x)$

88. $\ln x^{2} = \ln(12-x)$

89. $\log_{2}(x+7) + \log_{2} x = 3$

90. $\ln x = 1 - \ln(x+2)$

91. $\ln x = 1 + \ln(x+1)$

92. $\log_{6}(2x-3) = \log_{6} 12 - \log_{6} 3$

93. $\log(3x+2) + \log(x-1) = 1$

94. $\log_{5}(x+2) + \log_{5}(x-2) = 1$

95. $\log_2 x + \log_2 (x-4) = 2$

96. $\log_3 x + \log_3 (x+6) = 3$

97. $\log_3 (x+3) + \log_3 (x+5) = 1$

98. $\ln x = \frac{1}{2}\ln\left(2x + \frac{5}{2}\right) + \frac{1}{2}\ln 2$

99. $\ln(-4-x) + \ln 3 = \ln(2-x)$

100. $\log_4 x + \log_4 (x-2) = \log_4 (15)$

101. $\ln(x-5) - \ln(x+4) = \ln(x-1) - \ln(x+2)$

102. $\ln(4-x) = \ln(x+8) + \ln(2x+13)$

103. $\log\left(x^2+4\right) - \log(x+2) = 2 + \log(x-2)$

104. $\log_3 (x-2) = \log_3 27 - \log_3 (x-4) - 5^{\log_5 1}$

105. $\log_2 (x+3) = \log_2 (x-3) + \log_3 9 + 4^{\log_4 3}$

106. Solve for t using logarithms with base a: $2a^{t/3} = 5$

107. Solve for t using logarithms with base a: $K = H - Ca^t$