Lecture Three

Section 3.1 – Inner Products

Definition

An ***inner product*** on a real vector space *V* is a function that associates a real number $\langle \vec{u},\ \vec{v} \rangle$ with each pair of vectors in *V* in such a way that the following axioms are satisfies for all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in *V* and all scalars *k*.

1. $\langle \vec{u},\ \vec{v} \rangle = \langle \vec{v},\ \vec{u} \rangle$ — ***Symmetry axiom***
2. $\langle \vec{u}+\vec{v},\ \vec{w} \rangle = \langle \vec{u},\ \vec{w} \rangle + \langle \vec{v},\ \vec{w} \rangle$ — ***Additivity axiom***
3. $\langle k\vec{u},\ \vec{v} \rangle = k\langle \vec{u},\ \vec{v} \rangle$ — ***Homogeneity axiom***
4. $\langle \vec{v},\ \vec{v} \rangle \ge 0$ and $\langle \vec{v},\ \vec{v} \rangle = 0$ iff $\vec{v} = 0$ — ***Positivity axiom***

A real vector space with an inner product is called a ***real inner product space***.

$$\boxed{\langle \vec{u},\ \vec{u} \rangle = \vec{u} \bullet \vec{v} = u_1 v_1 + u_2 v_2 + \cdots u_n v_n}$$

This is called the ***Euclidean inner product*** (or the ***standard inner product***)

Definition

If *V* is a real inner product space, then the norm (or length) of a vector $\vec{v}$ in *V* is denoted by $\|\vec{v}\|$ and is defined by

$$\|\vec{v}\| = \sqrt{\langle \vec{v},\ \vec{v} \rangle}$$

And the ***distance*** between two vectors is denoted by $d(\vec{u},\ \vec{v})$ and is defined by

$$d(\vec{u},\ \vec{v}) = \|\vec{u}-\vec{v}\| = \sqrt{\langle \vec{u}-\vec{v},\ \vec{u}-\vec{v} \rangle}$$

A vector of norm 1 is called a ***unit vector***.

Theorem

If ***u*** and ***v*** are vectors in a real inner product space *V*, and if *k* is a scalar, then:

a) $\|\vec{v}\| \ge 0$ with equality *iff* $\vec{v} = 0$

b) $\|k\vec{v}\| = |k|\,\|\vec{v}\|$

c) $d(\vec{u},\ \vec{v}) = d(\vec{v},\ \vec{u})$

d) $d(\vec{u},\ \vec{v}) \ge 0$ with equality *iff* $\vec{u} = \vec{v}$

Although the Euclidean inner product is the most important inner product on $\mathbb{R}^n$, there are various applications in which is desirable to modify it by weighing each term differently. More precisely, if $w_1, w_2, \ldots, w_n$ are positive real numbers, which we will call weighs, and if $\vec{u} = (u_1, u_2, \ldots, u_n)$ and are vectors in $\mathbb{R}^n$, then it can be shown that the formula

$$\langle \vec{u}, \vec{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \ldots + w_n u_n v_n$$

Defines an inner product on $\mathbb{R}^n$ that we call the ***weighted Euclidean inner product*** with weights $\boldsymbol{w_1, w_2, \ldots, w_n}$

Example

Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ be vectors in $\mathbb{R}^2$, verify that the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 3u_1 v_1 + 2u_2 v_2$ satisfies the four inner product axioms.

Solution

Axiom 1: $\langle \vec{u}, \vec{v} \rangle = 3u_1 v_1 + 2u_2 v_2$

$$= 3v_1 u_1 + 2v_2 u_2$$
$$= \langle \vec{v}, \vec{u} \rangle$$

Axiom 2: $\langle \vec{u} + \vec{v}, \ \vec{w} \rangle = 3(u_1 + v_1) w_1 + 2(u_2 + v_2) w_2$

$$= 3(u_1 w_1 + v_1 w_1) + 2(u_2 w_2 + v_2 w_2)$$
$$= 3u_1 w_1 + 3v_1 w_1 + 2u_2 w_2 + 2v_2 w_2$$
$$= (3u_1 w_1 + 2u_2 w_2) + (3v_1 w_1 + 2v_2 w_2)$$
$$= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$$

Axiom 3: $\langle k\vec{u}, \vec{v} \rangle = 3(ku_1) v_1 + 2(ku_2) v_2$

$$= k(3u_1 v_1 + 2u_2 v_2)$$
$$= k\langle \vec{u}, \vec{v} \rangle$$

Axiom 4: $\langle \vec{v}, \vec{v} \rangle = 3v_1 v_1 + 2v_2 v_2$

$$= 3v_1^2 + 2v_2^2 \ge 0$$

$$v_1 = v_2 = 0 \quad \textit{iff} \quad \vec{v} = \vec{0}$$

Exercises *Section* 3.1 – Inner Products

1. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (1, 1)$, $\vec{v} = (3, 2)$, $\vec{w} = (0, -1)$, and $k = 3$. Compute the following.

a) $\langle \vec{u}, \vec{v} \rangle$ *c)* $\langle \vec{u} + \vec{v}, \vec{w} \rangle$ *e)* $d(\vec{u}, \vec{v})$

b) $\langle k\vec{v}, \vec{w} \rangle$ *d)* $\|\vec{v}\|$ *f)* $\|\vec{u} - k\vec{v}\|$

2. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (1, 1)$, $\vec{v} = (3, 2)$, $\vec{w} = (0, -1)$ and $k = 3$. Compute the following for the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 2u_1 v_1 + 3u_2 v_2$.

a) $\langle \vec{u}, \vec{v} \rangle$ *c)* $\langle \vec{u} + \vec{v}, \vec{w} \rangle$ *e)* $d(\vec{u}, \vec{v})$

b) $\langle k\vec{v}, \vec{w} \rangle$ *d)* $\|\vec{v}\|$ *f)* $\|\vec{u} - k\vec{v}\|$

3. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (3, -2)$, $\vec{v} = (4, 5)$, $\vec{w} = (-1, 6)$, and $k = -4$. Verify the following.

a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

d) $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$

e) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$

4. Let $\langle \vec{u}, \vec{v} \rangle$ be the Euclidean inner product on $\mathbb{R}^2$, and let $\vec{u} = (3, -2)$, $\vec{v} = (4, 5)$, $\vec{w} = (-1, 6)$, and $k = -4$. Verify the following for the weighted Euclidean inner product $\langle \vec{u}, \vec{v} \rangle = 4u_1 v_1 + 5u_2 v_2$

a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

b) $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

c) $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

d) $\langle k\vec{u}, \vec{v} \rangle = k\langle \vec{u}, \vec{v} \rangle$

e) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$

5. Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$. Show that the following are inner product on $\mathbb{R}^2$ by verifying that the inner product axioms hold. $\langle \vec{u}, \vec{v} \rangle = 3u_1 v_1 + 5u_2 v_2$

6. Show that the following identity holds for the vectors in any inner product space

$$\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

7. Show that the following identity holds for the vectors in any inner product space

$$\langle \vec{u},\ \vec{v} \rangle = \frac{1}{4}\left(\|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 \right)$$

8. Prove that $\|k\vec{v}\| = |k|\ \|\vec{v}\|$

Section 3.2 – Angle and Orthogonality in Inner Product Spaces

Cosine Formula

If $\vec{u}$ and $\vec{v}$ are nonzero vectors that implies

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\| \ \|\vec{v}\|}$$

$$\theta = \cos^{-1}\left(\frac{\langle \vec{u},\ \vec{v} \rangle}{\|\vec{u}\| \ \|\vec{v}\|}\right)$$

$$-1 \le \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\| \ \|\vec{v}\|} \le 1$$

Example

Let $\mathbb{R}^4$ have the Euclidean inner product. Find the cosine angle θ between the vectors $\vec{u} = (4,\ 3,\ 1,\ -2)$ and $\vec{v} = (-2,\ 1,\ 2,\ 3)$.

Solution

$$\begin{aligned} \|\vec{u}\| &= \sqrt{4^2 + 3^2 + 1^2 + (-2)^2} \\ &= \sqrt{30} \end{aligned}$$

$$\begin{aligned} \|\vec{v}\| &= \sqrt{4+1+4+9} \\ &= \sqrt{18} \\ &= 3\sqrt{2} \end{aligned}$$

$$\begin{aligned} \langle \vec{u},\ \vec{v} \rangle &= 4(-2) + 3(1) + 1(2) - 2(3) \\ &= -9 \end{aligned}$$

$$\begin{aligned} \cos\theta &= -\frac{9}{3\sqrt{30}\sqrt{2}} \\ &= -\frac{3}{\sqrt{60}} \\ &= -\frac{3}{2\sqrt{15}} \end{aligned}$$

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\| \ \|\vec{v}\|}$$

Theorem – Cauchy-Schwarz Inequality

If $\vec{v}$ and $\vec{w}$ are vectors in a real inner product space V, then

$$\left\|\langle \vec{u},\ \vec{v}\rangle\right\| \le \|\vec{u}\|\ \|\vec{v}\|$$

Proof

If either $\vec{u}$ or $\vec{v}$ is equal to zero, then both sides equal to zero

Inequality holds.

Suppose that $\vec{u},\ \vec{v} \ne 0$ and if $\vec{w}$ any vector

$\|\vec{w}\| = \vec{w}\ \vec{w} \ge 0$

Let $\vec{w} = \vec{u} - t\vec{v}$, then:

$$\begin{aligned}
0 &\le \vec{w}\vec{w} \\
&= (\vec{u} - t\vec{v})(\vec{u} - t\vec{v}) \\
&= \vec{u} \bullet \vec{u} - t(\vec{u} \bullet \vec{v}) - t(\vec{v} \bullet \vec{u}) + t^2(\vec{v} \bullet \vec{v}) \\
&= \vec{u} \bullet \vec{u} - 2t\,(\vec{u} \bullet \vec{v}) + t^2(\vec{v} \bullet \vec{v}) \qquad \text{Let } t = \frac{\vec{u} \bullet \vec{v}}{\vec{v} \bullet \vec{v}} \\
&= \vec{u} \bullet \vec{u} - 2\left(\frac{\vec{u} \bullet \vec{v}}{\vec{v} \bullet \vec{v}}\right)(\vec{u} \bullet \vec{v}) + \left(\frac{\vec{u} \bullet \vec{v}}{\vec{v} \bullet \vec{v}}\right)^2(\vec{v} \bullet \vec{v}) \\
&= \vec{u} \bullet \vec{u} - 2\frac{(\vec{u} \bullet \vec{v})^2}{\vec{v} \bullet \vec{v}} + \frac{(\vec{u} \bullet \vec{v})^2}{\vec{v} \bullet \vec{v}} \\
&= \vec{u} \bullet \vec{u} - \frac{(\vec{u} \bullet \vec{v})^2}{\vec{v} \bullet \vec{v}} \\
&= \frac{(\vec{u} \bullet \vec{u})(\vec{v} \bullet \vec{v}) - (\vec{u} \bullet \vec{v})^2}{\vec{v} \bullet \vec{v}} \qquad \text{Since } \vec{v} \bullet \vec{v} > 0 \\
&\le (\vec{u} \bullet \vec{u})(\vec{v} \bullet \vec{v}) - (\vec{u} \bullet \vec{v})^2
\end{aligned}$$

$$(\vec{u} \bullet \vec{v})^2 \le (\vec{u} \bullet \vec{u})(\vec{v} \bullet \vec{v})$$

$$\left\|\langle \vec{u},\ \vec{v}\rangle\right\| \le \|\vec{u}\|\ \|\vec{v}\|$$

The following two alternative forms of the Cauchy-Schwarz inequality are useful to know:

$$\langle \vec{u},\ \vec{v}\rangle^2 \le \langle \vec{u},\ \vec{u}\rangle\langle \vec{v},\ \vec{v}\rangle$$

$$\langle \vec{u},\ \vec{v}\rangle^2 \le \|\vec{u}\|^2\ \|\vec{v}\|^2$$

Theorem

If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in a real inner product space V, and if k is any scalar, then

a) $\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$ (***Triangle inequality for vectors***)

b) $d(\vec{u},\ \vec{v}) \le d(\vec{u},\ \vec{w}) + d(\vec{w},\ \vec{v})$ (***Triangle inequality for distances***)

Proof **(*a*)**

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= \langle \vec{u}+\vec{v},\ \vec{u}+\vec{v} \rangle \\
&= \langle \vec{u},\ \vec{u} \rangle + 2\langle \vec{u},\ \vec{v} \rangle + \langle \vec{v},\ \vec{v} \rangle \\
&\le \langle \vec{u},\ \vec{u} \rangle + 2\left|\langle \vec{u},\ \vec{v} \rangle\right| + \langle \vec{v},\ \vec{v} \rangle \\
&\le \langle \vec{u},\ \vec{u} \rangle + 2\|\vec{u}\|\ \|\vec{v}\| + \langle \vec{v},\ \vec{v} \rangle \\
&= \|\vec{u}\|^2 + 2\|\vec{u}\|\ \|\vec{v}\| + \|\vec{v}\|^2 \\
&= \left(\|\vec{u}\| + \|\vec{v}\|\right)^2
\end{aligned}$$

$$\|\vec{u}+\vec{v}\|^2 \le \left(\|\vec{u}\| + \|\vec{v}\|\right)^2$$

$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$$

Definition

Two vectors $\vec{u}$ and $\vec{v}$ in an inner product space are called orthogonal if $\langle \vec{u},\ \vec{v} \rangle = 0$

Example

The vectors $\vec{u} = (1,\ 1)$ and $\vec{v} = (1,\ -1)$ are orthogonal with respect to the Euclidean inner product on $\mathbb{R}^2$, since

$$\begin{aligned}
\vec{u} \bullet \vec{v} &= 1(1) + 1(-1) \\
&= 0
\end{aligned}$$

They are not orthogonal with the respect to the weighted Euclidean inner product $\langle \vec{u},\ \vec{v} \rangle = 3u_1v_1 + 2u_2v_2$, since

$$\begin{aligned}
\langle \vec{u},\ \vec{v} \rangle &= 3(1)(1) + 2(1)(-1) \\
&= 1 \ne 0
\end{aligned}$$

Example

$U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ *and* $V = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ are orthogonal, since

$$\begin{aligned} U \bullet V &= 1(0) + 0(2) + 1(0) + 1(0) \\ &\underline{= 0 \,|} \end{aligned}$$

Definition

If W is a subspace of an inner product space V, then the set of all vectors are orthogonal to every vector in W is called the ***orthogonal complement*** of W and is denoted by the symbol $W^{\perp}$

Theorem

If W is a subspace of an inner product space V, then:

a) $W^{\perp}$ is a subspace of V.

b) $W \cap W^{\perp} = \{0\}$

Proof

a) Let set $W^{\perp}$ contains at least the zero vector, since $\langle \vec{0},\ \vec{w} \rangle = 0$ for every vector $\vec{w}$ in W. We need to show that $W^{\perp}$ is closed under addition and scalar multiplication.

Suppose that $\vec{u}$ and $\vec{v}$ are vectors in $W^{\perp}$, so every vector $\vec{w}$ in W we have $\langle \vec{u},\ \vec{w} \rangle = 0$ and $\langle \vec{v},\ \vec{w} \rangle = 0$

$$\begin{aligned} \langle \vec{u} + \vec{v},\ \vec{w} \rangle &= \langle \vec{u},\ \vec{w} \rangle + \langle \vec{v},\ \vec{w} \rangle \\ &= 0 + 0 \\ &\underline{= 0 \,|} \end{aligned}$$

Closed under addition

$$\begin{aligned} \langle k\vec{u},\ \vec{w} \rangle &= k \langle \vec{u},\ \vec{w} \rangle \\ &= k(0) \\ &\underline{= 0 \,|} \end{aligned}$$

Closed under scalar multiplication

Which proves that $\vec{u} + \vec{w}$ and $k\vec{u}$ are in $W^{\perp}$

b) If $\vec{v}$ is any vector in both W and $W^{\perp}$, then $\vec{v}$ is orthogonal to itself; that is, $\langle \vec{v},\ \vec{v} \rangle = 0$. It follows from the positivity axiom for inner products that $\vec{v} = \vec{0}$

Theorem

If W is a subspace of a finite-dimensional inner product space V, then the orthogonal complement of $W^{\perp}$ is W; that is

$$\left(W^{\perp}\right)^{\perp} = W$$

Example

Let W be the subspace of $\mathbb{R}^6$ spanned by the vectors

$$\vec{w}_1 = (1,\ 3,\ -2,\ 0,\ 2,\ 0), \qquad \vec{w}_2 = (2,\ 6,\ -5,\ -2,\ 4,\ -3)$$
$$\vec{w}_3 = (0,\ 0,\ 5,\ 10,\ 0,\ 15), \qquad \vec{w}_4 = (2,\ 6,\ 0,\ 8,\ 4,\ 18)$$

Find a basis for the orthogonal complement of W.

Solution

The Space W is the same as the row space of the matrix

$$A = \begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 0 & 0 & 4 & 8 & 0 & 18 \end{pmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ R_3 + 5R_2 \\ R_4 + 4R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 \\ 0 & 0 & -1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix} \begin{matrix} \\ -R_2 \\ \\ \frac{1}{6}R_3 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} R_1 - 6R_3 \\ R_2 - 3R_3 \\ \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution

$$\left(x_1, x_2, x_3, x_4, x_5, x_6\right) = \left(-3x_2 - 4x_4 - 2x_5, x_2, -2x_4, x_4, x_5, 0\right)$$

$$= x_2\left(-3, 1, 0, 0, 0, 0\right) + x_4\left(-4, 0, -2, 1, 0, 0\right) + x_5\left(-2, 0, 0, 0, 1, 0\right)$$

$\vec{v}_1 = \left(-3, 1, 0, 0, 0, 0\right)$, $\vec{v}_2 = \left(-4, 0, -2, 1, 0, 0\right)$, $\vec{v}_3 = \left(-2, 0, 0, 0, 1, 0\right)$

Definition

A collection of vectors in $\mathbb{R}^n$ (or inner space) is called orthogonal if any 2 are perpendicular.

$$\vec{v}_i.\vec{v}_j = \vec{v}^T\vec{v} = \begin{cases} 0 & \text{for } i \neq j \quad (\text{orthogonal vectors}) \\ 1 & \text{for } i = j \quad (\text{unit vectors}) \end{cases}$$

Theorem

If $\vec{v}_1, \ldots, \vec{v}_m$ are nonzero orthogonal vectors, then they are linearly independent.

Definition

A vector $\vec{v}$ is called normal if $\|\vec{v}\| = 1$

A collection of vectors $\vec{v}_1, \ldots, \vec{v}_m$ is called orthonormal if they are orthogonal and each $\|\vec{v}_i\| = 1$.

An orthonormal basis is a basis made up of orthonormal vectors.

Example

Q rotates every vector in the plane through the angle θ.

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$Q^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= Q^T$$

The dot product $\left(\cos\theta\sin\theta - \sin\theta\cos\theta = 0\right)$, the columns are orthogonal.

They are unit vectors because $\cos^2\theta + \sin^2\theta = 1$. Those columns give an orthonormal basis for the plane $\mathbb{R}^2$.

We have: $QQ^T = I = Q^T Q$ (This type is called ***rotation***)

Exercises *Section* 3.2 – Angle and Orthogonality in Inner Product Spaces

1. Which of the following form orthonormal sets?

a) $(1, 0), (0, 2)$ in $\mathbb{R}^2$

b) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^2$

c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^2$

d) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ in $\mathbb{R}^3$

e) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ in $\mathbb{R}^3$

f) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ in $\mathbb{R}^3$

2. Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

a) $\vec{u}=(1, -3), \quad \vec{v}=(2, 4)$

b) $\vec{u}=(-1, 0), \quad \vec{v}=(3, 8)$

c) $\vec{u}=(-1, 5, 2), \quad \vec{v}=(2, 4, -9)$

d) $\vec{u}=(4, 1, 8), \quad \vec{v}=(1, 0, -3)$

e) $\vec{u}=(1, 0, 1, 0), \quad \vec{v}=(-3, -3, -3, -3)$

f) $\vec{u}=(2, 1, 7, -1), \quad \vec{v}=(4, 0, 0, 0)$

g) $\vec{u}=(1, 3, -5, 4), \quad \vec{v}=(2, -4, 4, 1)$

h) $\vec{u}=(1, 2, 3, 4), \quad \vec{v}=(-1, -2, -3, -4)$

3. Find the cosine of the angle between ***A*** and ***B***.

a) $A=\begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B=\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$

b) $A=\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B=\begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$

c) $A=\begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

d) $A=\begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B=\begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$

4. Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

a) $\vec{u}=(-1, 3, 2), \quad \vec{v}=(4, 2, -1)$

b) $\vec{u}=(a, b), \quad \vec{v}=(-b, a)$

c) $\vec{u}=(-2, -2, -2), \quad \vec{v}=(1, 1, 1)$

d) $\vec{u}=(-4, 6, -10, 1), \quad \vec{v}=(2, 1, -2, 9)$

e) $\vec{u}=(-4, 6, -10, 1), \quad \vec{v}=(2, 1, -2, 9)$

5. Do there exist scalars *k* and *l* such that the vectors $\vec{u}=(2, k, 6), \quad \vec{v}=(l, 5, 3), \quad \textit{and} \quad \vec{w}=(1, 2, 3)$ are mutually orthogonal with respect to the Euclidean inner product?

6. Let $\mathbb{R}^3$ have the Euclidean inner product. For which values of k are $\vec{u}$ and $\vec{v}$ orthogonal?

a) $\vec{u} = (2, 1, 3), \quad \vec{v} = (1, 7, k)$ *b)* $\vec{u} = (k, k, 1), \quad \vec{v} = (k, 5, 6)$

7. Let V be an inner product space. Show that if $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors in V, then $\|\vec{u} - \vec{v}\| = \sqrt{2}$

8. Let $\boldsymbol{S}$ be a subspace of $\mathbb{R}^n$. Explain what $\left(\boldsymbol{S}^{\perp}\right)^{\perp} = \boldsymbol{S}$ means and why it is true.

9. The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ – (***note*** that all six edges have length $\sqrt{2}$, so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ to the vertices?

10. Determine if the given vectors are orthogonal.

$\vec{x}_1 = (1, 0, 1, 0), \quad \vec{x}_2 = (0, 1, 0, 1), \quad \vec{x}_3 = (1, 0, -1, 0), \quad \vec{x}_4 = (1, 1, -1, -1)$

11. Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

a) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \ \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

b) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \ \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) \ \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

12. Consider vectors $\vec{u} = (2, 3, 5) \quad \vec{v} = (1, -4, 3) \quad in \quad \mathbb{R}^3$

a) $\langle \vec{u}, \vec{v} \rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine between $\vec{u}$ *and* $\vec{v}$

13. Consider vectors $\vec{u} = (1, 1, 1) \quad \vec{v} = (1, 2, -3) \quad in \quad \mathbb{R}^3$

a) $\langle \vec{u}, \vec{v} \rangle$ *b)* $\|\vec{u}\|$ *c)* $\|\vec{v}\|$ *d)* Cosine θ between $\vec{u}$ *and* $\vec{v}$

14. Consider vectors $\vec{u} = (1, 2, 5) \quad \vec{v} = (2, -3, 5) \quad \vec{w} = (4, 2, -3) \quad in \quad \mathbb{R}^3$

a) $\langle \vec{u}, \vec{v} \rangle$ *d)* $\|\vec{u}\|$ *g)* Cosine α between $\vec{u}$ *and* $\vec{v}$

b) $\langle \vec{u}, \vec{w} \rangle$ *e)* $\|\vec{v}\|$ *h)* Cosine β between $\vec{u}$ *and* $\vec{w}$

c) $\langle \vec{v}, \vec{w} \rangle$ *f)* $\|\vec{w}\|$ *i)* Cosine θ between $\vec{v}$ *and* $\vec{w}$

j) $(\vec{u} + \vec{v}) \bullet \vec{w}$

15. Consider polynomial $f(t) = 3t - 5; \quad g(t) = t^2 \quad in \quad \mathbb{P}(t)$

a) $\langle f, g \rangle$ *b)* $\|f\|$ *c)* $\|g\|$ *d)* Cosine between $f \ and \ g$

16. Consider polynomial $f(t) = t + 2\ ; \quad g(t) = 3t - 2\ ; \quad h(t) = t^2 - 2t - 3 \quad in \quad \mathbb{P}(t)$

a) $\langle f, g \rangle$ *d)* $\|f\|$ *g)* Cosine α between $f \ and \ g$

b) $\langle f, h \rangle$ *e)* $\|g\|$ *h)* Cosine β between $f \ and \ h$

c) $\langle g, h \rangle$ *f)* $\|h\|$ *i)* Cosine θ between $g \ and \ h$

17. Suppose $\langle \vec{u}, \vec{v} \rangle = 3 + 2\,i$ in a complex inner product space *V*. Find:

a) $\langle (2 - 4\,i)\vec{u}, \vec{v} \rangle$ *b)* $\langle \vec{u}, (4 + 3\,i)\vec{v} \rangle$ *c)* $\langle (3 - 6\,i)\vec{u}, (5 - 2\,i)\vec{v} \rangle$ *d)* $\|\vec{u}, \vec{v}\|$

18. Find the Fourier coefficient *c* and the projection $c\vec{v}$ of $\vec{u} = (3 + 4\,i,\ 2 - 3\,i)$ along $\vec{v} = (5 + i,\ 2i)$ in $\mathbb{C}^2$

19. Suppose $\vec{v} = (1,\ 3,\ 5,\ 7)$. Find the projection of $\vec{v}$ onto *W* or find $\vec{w} \in W$ that minimizes $\|\vec{v} - \vec{w}\|$, where *W* is the subspace of $\mathbb{R}^4$ spanned by:

a) $\vec{u}_1 = (1,\ 1,\ 1,\ 1) \quad and \quad \vec{u}_2 = (1,\ -3,\ 4,\ -2)$

b) $\vec{v}_1 = (1,\ 1,\ 1,\ 1) \quad and \quad \vec{v}_2 = (1,\ 2,\ 3,\ 2)$

20. Suppose $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ is an orthogonal set of vectors. Prove that (*Pythagoras*)

$$\|\vec{u}_1 + \vec{u}_2 + \ \ldots + \vec{u}_n\|^2 = \|\vec{u}_1\|^2 + \|\vec{u}_2\|^2 + \ \ldots + \|\vec{u}_n\|^2$$

21. Suppose *A* is an orthogonal matrix. Show that $\langle \vec{u}A, \vec{v}A \rangle = \langle \vec{u}, \vec{v} \rangle$ for any $\vec{u}, \vec{v} \in V$

22. Suppose *A* is an orthogonal matrix. Show that $\|\vec{u}A\| = \|\vec{u}\|$ for every $\vec{u} \in V$

23. Let *V* be an inner product space over $\mathbb{R}$ or $\mathbb{C}$. Show that

$$\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$

If and only if

$$\|s\vec{u} + t\vec{v}\| = s\|\vec{u}\| + t\|\vec{v}\| \quad for\ all \quad s,\ t \ge 0$$

24. Let *V* be an inner product vector space over $\mathbb{R}$.

a) If e_1, e_2, e_3 are three vectors in *V* with pairwise product negative, that is,

$$\langle e_1, e_2 \rangle < 0, \quad i, j = 1,\ 2,\ 3, \quad i \ne j$$

Show that e_1, e_2, e_3 are linearly independent.

b) Is it possible for three vectors on the *xy*–plane to have pairwise negative products?

c) Does part (a) remain valid when the word "negative: is replaced with positive?

d) Suppose $\vec{u},\ \vec{v},\ and\ \vec{w}$ are three unit vectors in the *xy*–plane. What are the maximum and minimum values that

$$\langle \vec{u},\ \vec{v} \rangle + \langle \vec{v},\ \vec{w} \rangle + \langle \vec{w},\ \vec{u} \rangle$$

Can attain? And when?

Section 3.3 – Gram-Schmidt Process

Definition

A set of two or more vectors in a real inner product space is said to be ***orthogonal*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be ***orthonormal***.

Theorem

1. If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an orthogonal basis for an inner product space V, and if $\vec{u}$ is any vector in V, then

$$\vec{u} = \frac{\langle \vec{u}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{\langle \vec{u}, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 + \cdots + \frac{\langle \vec{u}, \vec{v}_n \rangle}{\|\vec{v}_n\|^2}\vec{v}_n$$

2. If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an orthonormal basis for an inner product space V, and if $\vec{u}$ is any vector in V, then

$$\vec{u} = \langle \vec{u}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{u}, \vec{v}_2 \rangle \vec{v}_2 + \cdots + \langle \vec{u}, \vec{v}_n \rangle \vec{v}_n$$

Proof

1. Since $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for V, every vector $\vec{u}$ in V can be expressed in the form

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

Let show that $c_i = \dfrac{\langle \vec{u}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2}$ *for* $i = 1, 2, \ldots, n$

$$\begin{aligned}\langle \vec{u}, \vec{v}_i \rangle &= \langle c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n, \ \vec{v}_i \rangle \\ &= c_1 \langle \vec{v}_1, \vec{v}_i \rangle + c_2 \langle \vec{v}_2, \vec{v}_i \rangle + \cdots + c_n \langle \vec{v}_n, \vec{v}_i \rangle\end{aligned}$$

Since S is an orthogonal set, all of the inner products in the last equality are zero except the i^{th}, so we have

$$\begin{aligned}\langle \vec{u}, \vec{v}_i \rangle &= c_i \langle \vec{v}_i, \vec{v}_i \rangle \\ &= c_i \|\vec{v}_i\|^2\end{aligned}$$

The Gram-Schmidt Process

To convert a basis $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r\}$ into an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$, perform the following computations:

***Step* 1**: $\vec{v}_1 = \vec{u}_1$

***Step* 2**: $\vec{v}_2 = \vec{u}_2 - \dfrac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$

***Step* 3**: $\vec{v}_3 = \vec{u}_3 - \dfrac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \dfrac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$

***Step* 4**: $\vec{v}_4 = \vec{u}_4 - \dfrac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \dfrac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \dfrac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$

To convert the orthogonal basis into an orthonormal basis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$, normalize the orthogonal basis vectors. $\boxed{\vec{q}_i = \dfrac{\vec{v}_i}{\|\vec{v}_i\|}}$

Example

Assume that the vector space $\mathbb{R}$ has the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors

$$\vec{u}_1 = (1,\ 1,\ 1) \quad \vec{u}_2 = (0,\ 1,\ 1) \quad \vec{u}_3 = (0,\ 0,\ 1)$$

Into the orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, and then normalize the ***orthogonal*** basis vectors to obtain an orthonormal basis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$

<u>Solution</u>

$$\vec{v}_1 = \vec{u}_1$$
$$= (1,\ 1,\ 1)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (0,\ 1,\ 1) - \frac{0+1+1}{1^2+1^2+1^2}(1,\ 1,\ 1)$$

$$= (0,\ 1,\ 1) - \frac{2}{3}(1,\ 1,\ 1)$$

$$\boxed{= \left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}$$

$$\vec{v}_3 = \vec{u}_3 - proj_{\vec{v}_2}\vec{u}_3$$

$$= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (0,\ 0,\ 1) - \frac{0+0+1}{1^2+1^2+1^2}(1,\ 1,\ 1) - \frac{0+0+\frac{1}{3}}{\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$= (0,\ 0,\ 1) - \frac{1}{3}(1,\ 1,\ 1) - \frac{\frac{1}{3}}{\frac{2}{3}}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$= (0,\ 0,\ 1) - \frac{1}{3}(1,\ 1,\ 1) - \frac{1}{2}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$\boxed{= \left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$\boxed{= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}{\sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}}$$

$$= \frac{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}{\frac{\sqrt{6}}{3}}$$

$$\boxed{= \left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)}$$

$$\begin{aligned}
\vec{q}_3 &= \frac{\vec{v}_3}{\|\vec{v}_3\|} \\
&= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\sqrt{0^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}} \\
&= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\frac{\sqrt{2}}{2}} \\
&= \left(0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)
\end{aligned}$$

Gram-Schmidt Process (*Orthonormal*)

Suppose $\vec{v}_1, \ldots, \vec{v}_n$ linearly independent in $\mathbb{R}^n$, construct *n* ***orthonormal*** $\vec{u}_1, \ldots, \vec{u}_n$ that span the same space: span $\{\vec{u}_1, \ldots, \vec{u}_k\}$ = span $\{\vec{v}_1, \ldots, \vec{v}_k\}$

***Step* 1**: Since $\vec{v}_i$ are linearly independent ($\neq 0$), so $\|\vec{v}_1\| \neq 0$ (to create a normal vector)

Let $\vec{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \vec{q}_1$, then $\|\vec{u}_1\| = 1$ since $\vec{u}_1$ is orthonormal and span $\{\vec{u}_1\} = span\{\vec{v}_1\}$

$\vec{w}_1 = \vec{v}_1 \Rightarrow \vec{v}_1 = \|\vec{w}_1\|\vec{u}_1$

***Step* 2**: $\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{q}_1)\vec{q}_1$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{u}_1}{\|\vec{v}_1\|}\vec{v}_1 \qquad (\vec{w}_2 \perp \vec{u}_1)$$

$$\vec{v}_2 = \|\vec{w}_2\|\vec{u}_2 + (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 \qquad \vec{w}_2 = \|\vec{w}_2\|\vec{u}_2$$

$$\boxed{\vec{q}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}}$$

***Step* 3**: $\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{q}_1)\vec{q}_1 - (\vec{v}_3 \cdot \vec{q}_2)\vec{q}_2$

$$\boxed{\vec{q}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}}$$

	$\vec{q}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$
$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{q}_1)\vec{q}_1$	$\vec{q}_2 = \dfrac{\vec{w}_2}{\|\vec{w}_2\|}$
$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{q}_1)\vec{q}_1 - (\vec{v} \cdot \vec{q}_2)\vec{q}_2$	$\vec{q}_3 = \dfrac{\vec{w}_3}{\|\vec{w}_3\|}$
$\vec{w}_n = \vec{v}_n - (\vec{v}_n \cdot \vec{q}_1)\vec{q}_1 - (\vec{v}_n \cdot \vec{q}_2)\vec{q}_2 - \ldots - (\vec{v}_n \cdot \vec{q}_{n-1})\vec{q}_{n-1}$	$\vec{q}_n = \dfrac{\vec{w}_n}{\|\vec{w}_n\|}$

Example

Use the Gram-Schmidt process to find an ***orthonormal*** basis for the subspaces of

$$\vec{v}_1 = (1,\ 1,\ 0,\ 0) \quad \vec{v}_2 = (0,\ 1,\ 1,\ 0) \quad \vec{v}_3 = (1,\ 0,\ 1,\ 1)$$

Solution

Step 1: $\vec{q}_1 = \dfrac{(1,\ 1,\ 0,\ 0)}{\sqrt{1^2+1^2+0+0}}$ $\qquad \vec{q}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$

$$= \frac{(1,\ 1,\ 0,\ 0)}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$$

Step 2: $\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{q}_1)\vec{q}_1$

$$= (0,\ 1,\ 1,\ 0) - \left[(0,\ 1,\ 1,\ 0) \cdot \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)\right]\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$$

$$= (0,\ 1,\ 1,\ 0) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$$

$$= (0,\ 1,\ 1,\ 0) - \left(\frac{1}{2},\ \frac{1}{2},\ 0,\ 0\right)$$

$$= \left(-\frac{1}{2},\ \frac{1}{2},\ 1,\ 0\right)$$

$$\|\vec{w}_2\| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1}$$

$$= \frac{\sqrt{3}}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{6}}{2}$$

$$\vec{q}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{\left(-\frac{1}{2},\ \frac{1}{2},\ 1,\ 0\right)}{\frac{\sqrt{6}}{2}}$$

$$= \frac{2}{\sqrt{6}}\left(-\frac{1}{2},\ \frac{1}{2},\ 1,\ 0\right)$$

$$= \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right)$$

***Step* 3**: $\vec{v}_3 \bullet \vec{q}_1 = (1,\ 0,\ 1,\ 1) \bullet \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$

$$= \frac{1}{\sqrt{2}}$$

$$\vec{v}_3 \bullet \vec{q}_2 = (1,\ 0,\ 1,\ 1) \bullet \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right)$$

$$= -\frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}}$$

$$\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{q}_1\right)\vec{q}_1 - \left(\vec{v} \bullet \vec{q}_2\right)\vec{q}_2$$

$$= (1,\ 0,\ 1,\ 1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right) - \frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right)$$

$$= (1,\ 0,\ 1,\ 1) - \left(\frac{1}{2},\ \frac{1}{2},\ 0,\ 0\right) - \left(-\frac{1}{6},\ \frac{1}{6},\ \frac{2}{6},\ 0\right)$$

$$= \left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$\vec{q}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$

$$= \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 1^2}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \frac{1}{\sqrt{\frac{21}{9}}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \frac{3}{\sqrt{21}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \left(\frac{2}{\sqrt{21}},\ -\frac{2}{\sqrt{21}},\ \frac{2}{\sqrt{21}},\ \frac{3}{\sqrt{21}}\right)$$

The ***orthonormal*** basis:

$$\left\{ \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right),\ \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right),\ \left(\frac{2}{\sqrt{21}},\ -\frac{2}{\sqrt{21}},\ \frac{2}{\sqrt{21}},\ \frac{3}{\sqrt{21}}\right) \right\}$$

QR–Decomposition

Problem

If A is an $m \times n$ matrix with linearly independent column vectors, and if Q is the matrix that results by applying the Gram-Schmidt process to the column vectors of A, what relationship, if any, exists between A and Q?

To solve this problem, suppose that the column vectors of A are $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ and the orthonormal column vectors of Q are $\vec{q}_1, \vec{q}_2, \dots, \vec{q}_n$.

$$\begin{array}{l} \vec{u}_1 = \langle \vec{u}_1, \vec{q}_1 \rangle \vec{q}_1 + \langle \vec{u}_1, \vec{q}_2 \rangle \vec{q}_2 + \cdots + \langle \vec{u}_1, \vec{q}_n \rangle \vec{q}_n \\ \vec{u}_2 = \langle \vec{u}_2, \vec{q}_1 \rangle \vec{q}_1 + \langle \vec{u}_2, \vec{q}_2 \rangle \vec{q}_2 + \cdots + \langle \vec{u}_2, \vec{q}_n \rangle \vec{q}_n \\ \ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \vec{u}_n = \langle \vec{u}_n, \vec{q}_1 \rangle \vec{q}_1 + \langle \vec{u}_n, \vec{q}_2 \rangle \vec{q}_2 + \cdots + \langle \vec{u}_n, \vec{q}_n \rangle \vec{q}_n \end{array}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \cdots & \langle \vec{u}_n, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \cdots & \langle \vec{u}_n, \vec{q}_2 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \langle \vec{u}_n, \vec{q}_n \rangle \end{bmatrix}$$

The equation $A = QR$ is a factorization of A into the product of a matrix Q with orthonormal column vectors and an invertible upper triangular matrix R. We call it the ***QR-decomposition of A***.

Theorem

If A is an $m \times n$ matrix with linearly independent column vectors, then A can be factored as

$$A = QR$$

Where Q is an $m \times n$ matrix with orthonormal column vectors, and R is an $n \times n$ invertible upper triangular matrix.

Example

Find the QR-decomposition of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Solution

The column vectors of are

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

From the previous example

$$\vec{q}_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \qquad \vec{q}_2 = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \qquad \vec{q}_3 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \langle \vec{u}_3, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \langle \vec{u}_3, \vec{q}_2 \rangle \\ 0 & 0 & \langle \vec{u}_3, \vec{q}_3 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} & 0 + (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} & 0 + 0 + (1)\frac{1}{\sqrt{3}} \\ 0 & 0\left(\frac{-2}{\sqrt{6}}\right) + (1)\frac{1}{\sqrt{6}} + (1)\frac{1}{\sqrt{6}} & 0\left(\frac{-2}{\sqrt{6}}\right) + 0\frac{1}{\sqrt{6}} + (1)\frac{1}{\sqrt{6}} \\ 0 & 0 & 0 + (0)\frac{-1}{\sqrt{2}} + (1)\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{Q} \cdot \underbrace{\begin{pmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}}_{R}$$

Calculus: Applying the Gram-Schmidt Process

We can apply the Gram-Schmidt orthogonalization procedure to generate some polynomials that are orthonormal on the interval $x \in [-1,\ 1]$ with inner product

$$\langle p,\ q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

Example

Apply the Gram-Schmidt orthonormalization process to the basis $B = \{1,\ x,\ x^2\}$ in $\mathbb{P}_2$ using the inner product

Solution

$B = \{1,\ x,\ x^2\}$

Let $\vec{u}_1 = 1,\quad \vec{u}_2 = x,\quad \vec{u}_3 = x^2$

$\vec{v}_1 = \vec{u}_1 = 1$

$$\langle \vec{v}_1,\ \vec{v}_1\rangle = \int_{-1}^{1} dx$$

$$= x \Big|_{-1}^{1}$$

$$= 2$$

$$\langle \vec{u}_2,\ \vec{v}_1\rangle = \int_{-1}^{1} x\ dx$$

$$= \frac{1}{2}x^2 \Big|_{-1}^{1}$$

$$= 0$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= x - \frac{0}{2}(1)$$

$$= x$$

$$\langle \vec{v}_2,\ \vec{v}_2\rangle = \int_{-1}^{1} x^2\ dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$= \frac{2}{3}$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \int_{-1}^{1} x^2 \, dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$= \frac{2}{3}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \int_{-1}^{1} x^3 \, dx$$

$$= \frac{1}{4}x^4 \Big|_{-1}^{1}$$

$$= 0$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2} \vec{v}_1 - \frac{\left\langle \vec{u}_3, \vec{v}_2 \right\rangle}{\left\| \vec{v}_2 \right\|^2} \vec{v}_2$$

$$= x^2 - \frac{3}{2}(x)(0) - \frac{1}{2}\frac{2}{3}$$

$$= x^2 - \frac{1}{3}$$

$$\left\langle \vec{v}_3, \vec{v}_3 \right\rangle = \int_{-1}^{1} \left(x^2 - \frac{1}{3} \right)^2 dx$$

$$= \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9} \right) dx$$

$$= \left(\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x \right) \Big|_{-1}^{1}$$

$$= \frac{1}{5} - \frac{2}{9} + \frac{1}{9} + \frac{1}{5} - \frac{2}{9} + \frac{1}{9}$$

$$= \frac{8}{45}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$= \frac{1}{\sqrt{2}}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{x}{\sqrt{2/3}}$$

$$\boxed{= \frac{\sqrt{3}}{\sqrt{2}}\, x}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$

$$\boxed{= \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \frac{1}{3}\right)}$$

The ***orthonormal*** basis is $\left\{ \frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}\, x,\ \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right) \right\}$

Exercises Section 3.3 – Gram-Schmidt Process

(1 – 14) Use the Gram-Schmidt process to find an ***orthonormal*** basis for the subspaces of $\mathbb{R}^m$.

1. $\vec{u}_1 = (1,\ -3),\quad \vec{u}_2 = (2,\ 2)$

2. $\vec{u}_1 = (1,\ 0),\quad \vec{u}_2 = (3,\ -5)$

3. $\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ (1,\ 2,\ 1)\}$

4. $\{(1,\ 1,\ 1),\ (0,\ 1,\ 1),\ (0,\ 0,\ 1)\}$

5. $\{(1,\ 1,\ 1),\ (0,\ 2,\ 1),\ (1,\ 0,\ 3)\}$

6. $\{(2,\ 2,\ 2),\ (1,\ 0,\ -1),\ (0,\ 3,\ 1)\}$

7. $\{(1,\ -1,\ 0),\ (0,\ 1,\ 0),\ (2,\ 3,\ 1)\}$

8. $\{(3,\ 0,\ 4),\ (-1,\ 0,\ 7),\ (2,\ 9,\ 11)\}$

9. $\{(1,\ 1,\ 1,\ 1),\ (1,\ 2,\ 1,\ 0),\ (1,\ 3,\ 0,\ 0)\}$

10. $\{(0,\ 2,\ -1,\ 1),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 1,\ 1,\ -1)\}$

11. $\vec{u}_1 = (1,\ 0,\ 0),\quad \vec{u}_2 = (3,\ 7,\ -2),\quad \vec{u}_3 = (0,\ 4,\ 1)$

12. $\vec{u}_1 = (1, 1, 1, 1),\quad \vec{u}_2 = (1,\ 2,\ 4,\ 5),\quad \vec{u}_3 = (1,\ -3,\ -4,\ -2)$

13. $\vec{u}_1 = (1, 1, 1, 1),\quad \vec{u}_2 = (1,\ 1,\ 2,\ 4),\quad \vec{u}_3 = (1,\ 2,\ -4,\ -3)$

14. $\vec{u}_1 = (0,\ 2,\ 1,\ 0),\quad \vec{u}_2 = (1,\ -1,\ 0,\ 0),\quad \vec{u}_3 = (1,\ 2,\ 0,\ -1),\quad \vec{u}_4 = (1,\ 0,\ 0,\ 1)$

(15 – 26) Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

15. $\{(1,\ 1,\ 0),\ (1,\ 2,\ 0),\ (0,\ 1,\ 2)\}$

16. $\{(1,\ -2,\ 2),\ (2,\ 2,\ 1),\ (2,\ -1,\ -2)\}$

17. $\{(1,\ 0,\ 0),\ (1,\ 1,\ 1),\ (1,\ 1,\ -1)\}$

18. $\{(4,\ -3,\ 0),\ (1,\ 2,\ 0),\ (0,\ 0,\ 4)\}$

19. $\{(0,\ 1,\ 2),\ (2,\ 0,\ 0),\ (1,\ 1,\ 1)\}$

20. $\{(0,\ 1,\ 1),\ (1,\ 1,\ 0),\ (1,\ 0,\ 1)\}$

21. $\{(1,\ 2,\ -2),\ (1,\ 0,\ -4),\ (5,\ 2,\ 0),\ (1,\ 1,\ -1)\}$

22. $\{(-3,\ 1,\ 2),\ (1,\ 1,\ 1),\ (2,\ 0,\ -1),\ (1,\ -3,\ 2)\}$

23. $\{(2,\ 1,\ 1),\ (0,\ 3,\ -1),\ (3,\ -4,\ -2),\ (-1,\ -1,\ 3)\}$

24. $\vec{u}_1 = (1, 1, 0, -1),\quad \vec{u}_2 = (1, 3, 0, 1),\quad \vec{u}_3 = (4, 2, 2, 0)$

25. $\vec{u}_1 = (1, 1, 1, 1),\quad \vec{u}_2 = (1, 1, 2, 4),\quad \vec{u}_3 = (1,\ 2,\ -4,\ -3)$

26. $\{(3,\ 4,\ 0,\ 0),\ (-1,\ 1,\ 0,\ 0),\ (2,\ 1,\ 0,\ -1),\ (0,\ 1,\ 1,\ 0)\}$

27. Find the ***QR***-decomposition of

a) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

28. Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product

$\vec{u} = (0,\ -2,\ 2,\ 1),\quad \vec{v} = (-1,\ -1,\ 1,\ 1)$

(**29 – 33**) Apply the Gram-Schmidt ***orthonormalization*** process in $\mathbb{C}^0\ [-1,\ 1]$ spanned by the functions, using the inner product

29. $f_1(x) = x + 2,\quad f_2(x) = x^2 - 3x + 4$

30. $f_1(x) = x,\quad f_2(x) = x^3,\quad f_3(x) = x^5$

31. $f_1(x) = 1,\quad f_2(x) = x,\quad f_3(x) = \frac{1}{2}\left(3x^2 - 1\right)$

32. $f_1(x) = 1,\quad f_2(x) = \sin \pi x,\quad f_3(x) = \cos \pi x$

33. $f_1(x) = \sin \pi x,\quad f_2(x) = \sin 2\pi x,\quad f_3(x) = \sin 3\pi x$

34. For $\mathbb{P}_3[x]$, define the inner product over $\mathbb{R}$ as

$$\langle f,\ g\rangle = \int_{-1}^{1} f(x)g(x)\ dx$$

a) If $f(x) = 1$ is a unit vector in $\mathbb{P}_3[x]$?

b) Find an orthonormal basis for the subspace spanned by x *and* x^2.

c) Complete the basis in part (*b*) to an orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product.

d) Is

$$[f,\ g] = \int_{0}^{1} f(x)g(x)\ dx$$

Also, an inner product for $\mathbb{P}_3[x]$

e) Find a pair of vectors $\vec{v}$ and $\vec{w}$ such that

$$\langle \vec{v},\ \vec{w} \rangle = 0 \quad but \quad [\vec{v},\ \vec{w}] \neq 0$$

f) Is the basis found in part (*c*) are orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product in part (*d*)?

Section 3.4 – Orthogonal Matrices

Definition

A square matrix A is said to be orthogonal if its transpose is the same as its inverse, that is, if

$$A^{-1} = A^T$$

or, equivalently, if

$$AA^T = A^T A = I$$

Example

The matrix $A = \begin{pmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{pmatrix}$

Solution

$$A^T A = \begin{pmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{pmatrix} \begin{pmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example

The matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Solution

$$A^T A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem

The following are equivalent for $n \times n$ matrix A.

a) A is orthogonal.

b) The row vectors of A form an orthonormal set in R^n with the Euclidean inner product.

c) The column vectors of A form an orthonormal set in R^n with the Euclidean inner product.

Theorem

a) The inverse of an orthogonal matrix is orthogonal

b) A product of orthogonal matrices is orthogonal

c) If A is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$

Theorem

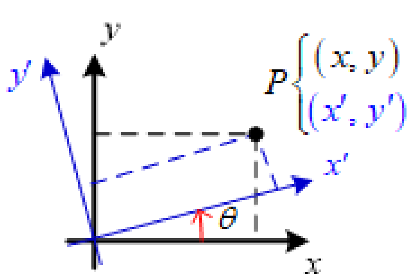
If A is an $n \times n$ matrix, then the following are equivalent

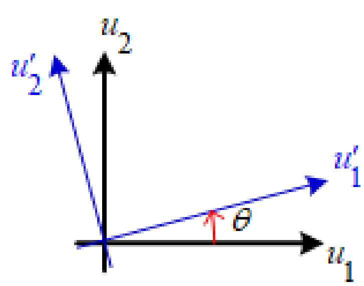
a) A is orthogonal.

b) $\|A\vec{x}\| = \|\vec{x}\|$ for all $\boldsymbol{x}$ in R^n.

c) $A\vec{x} \bullet A\vec{y} = \vec{x} \bullet \vec{y}$ for all $\vec{x}$ and $\vec{y}$ in R^n.

Let $\vec{u}_1$ and $\vec{u}_2$ be the unit vectors along the x- and y-axes and unit vectors $\vec{u}'_1$ and $\vec{u}'_2$ along the x'- and y'-axes.



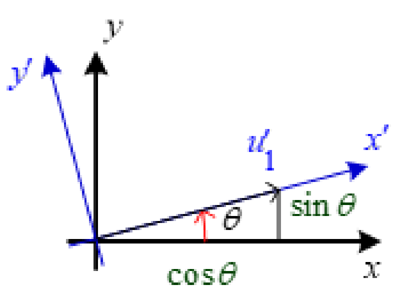


The new coordinates (x', y') and the old coordinates (x, y) of a point P will be related by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

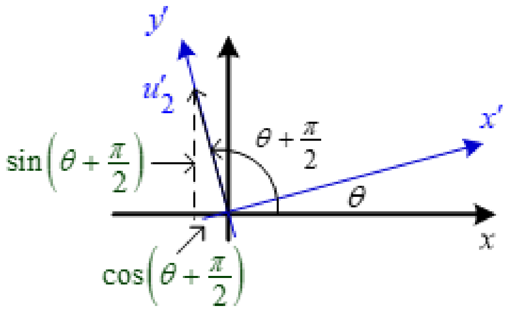
$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$P^{-1} = P^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$



$$\begin{bmatrix} \vec{x}' \\ \vec{y}' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix}$$

$$\rightarrow \begin{cases} x' = x\cos\theta + y\sin\theta \\ y' = -x\sin\theta + y\cos\theta \end{cases}$$



These are sometimes called the ***rotation equations***.

Example

Use the form $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ to find the new coordinates of the point $Q(2,\ 1)$ if the coordinate axes of a rectangular coordinate system are rotated through an angle of $\theta = \frac{\pi}{4}$.

Solution

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

The new coordinates of Q are $(x',\ y') = \left(\frac{3}{\sqrt{2}},\ -\frac{1}{\sqrt{2}} \right)$

Exercises *Section* 3.4 – Orthogonal Matrices

(1 – 2) Show that the matrix is orthogonal

1. $A=\begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$

2. $A=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

(3 – 12) Determine if the matrix is orthogonal. For those that is orthogonal find the inverse.

3. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

5. $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

6. $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

7. $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$

8. $\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

9. $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

10. $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$

11. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$

13. Find a last column so that the resulting matrix is orthogonal

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \cdots \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & \cdots \end{bmatrix}$$

14. Determine if the given matrix is orthogonal. If it is, find its inverse

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{5} & \frac{3}{7} \\ \frac{4}{9} & \frac{3}{5} & -\frac{2}{7} \\ \frac{8}{9} & -\frac{2}{5} & \frac{3}{7} \end{bmatrix}$$

15. Prove that if A is orthogonal, then A^T is orthogonal.

16. Prove that if A is orthogonal, then A^{-1} is orthogonal.

17. Prove that if A and B are orthogonal, then AB is orthogonal.

18. Let Q be an $n \times n$ orthogonal matrix, and let A be an $n \times n$ matrix. Show that $\det\left(QAQ^T\right) = \det(A)$

19. Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{pmatrix}$

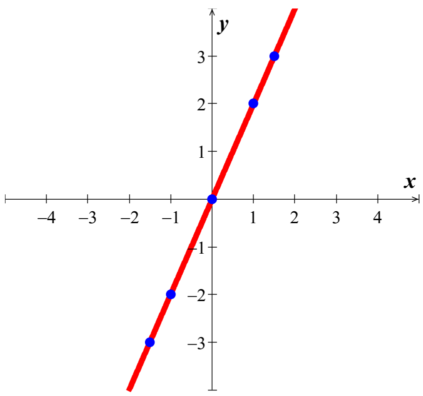
a) Is matrix A an orthogonal matrix?
b) Let B be the matrix obtained by normalizing each row of A, find B.
c) Is B an orthogonal matrix?
d) Are the columns of B orthogonal?

Section 3.5 – Least Squares Analysis

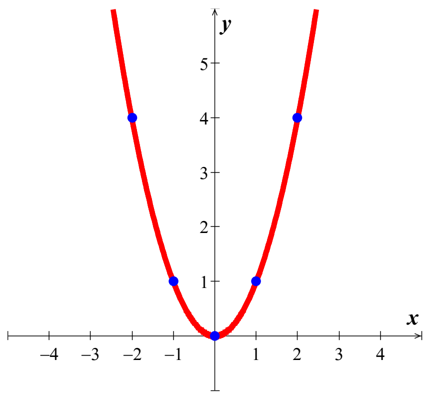
The use to ***best*** fit data, we will use results about orthogonal projections in inner product spaces to obtain a technique for fitting a line or other polynomial.

Fitting a Curve to Data

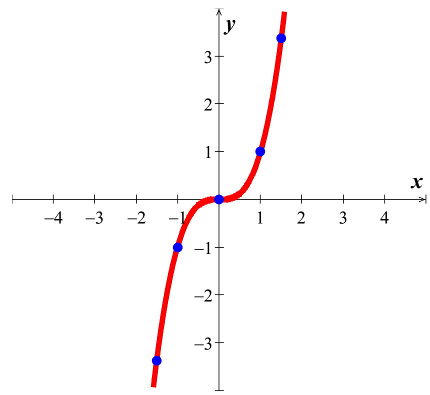
The common problem is to obtain a mathematical relationship between 2 variables x and y by ***fitting*** a curve to points in the xy-plane.
Some possibility of fitting the data



$y = ax + b$



$y = ax^2 + bx + c$



$y = ax^3 + bx^2 + cx + d$

Least Squares Fit of a Straight Line

Recall that a system of equations $A\vec{x} = \vec{y}$ is called inconsistent if it does not have a solution. Suppose we want to fit a straight line $y = mx + b$ to the determined points $(x_1, y_1), \ \ldots, \ (x_n, y_n)$

If the data points were collinear, the line would pass through all n points and the unknown coefficients m and b would satisfy the equations

$$\begin{matrix} y_1 = mx_1 + b \\ y_2 = mx_2 + b \\ \vdots \quad \vdots \quad \vdots \\ y_n = mx_n + b \end{matrix} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\vec{y}}$$

The problem is to find m and b that minimize the errors is some sense.

Least Square Problem

Given a linear system $A\vec{x} = \vec{y}$ of *m* equations in *n* unknowns, find a vector $\vec{x}$ that minimizes $\|\vec{y} - A\vec{x}\|$ with respect to the Euclidean inner product on $\mathbb{R}^m$. We call such as $\vec{x}$ a least squares solution of the system, we call $\vec{y} - A\vec{x}$ the least squares error vectors, and we call $\|\vec{y} - A\vec{x}\|$ the least squares error.

$$A\boldsymbol{x} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix}$$

The term "***least square solution***" results from the fact the minimizing $\|\vec{y} - A\vec{x}\| = e_1^2 + e_2^2 + \ldots + e_m^2$

Example

Find the sums of squares of the errors of (2, 4), (4, 8), (6, 6)

Solution

$$4 = 2m + b \quad \Rightarrow \quad 4 - 2m - b = e_1$$
$$8 = 4m + b \quad \Rightarrow \quad 8 - 4m - b = e_2$$
$$6 = 6m + b \quad \Rightarrow \quad 6 - 6m - b = e_3$$

$$e_1^2 + e_2^2 + \ldots + e_m^2 = (4 - 2m - b)^2 + (8 - 4m - b)^2 + (6 - 6m - b)^2$$

The least squares problem for this example to find the values *m* and *b* for which $e_1^2 + e_2^2 + \ldots + e_m^2$ is a minimum.

Theorem

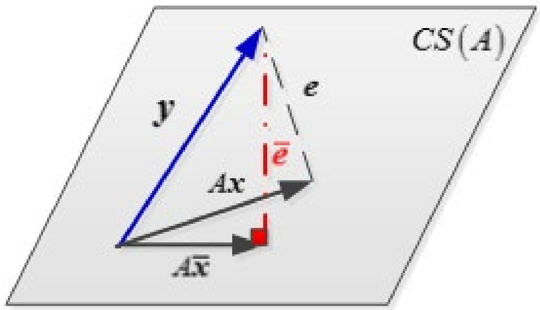
If A is an $m \times n$ matrix, the equation $A\vec{x} = \vec{y}$ has a solution if and only if $\vec{y}$ is in the column space of A.

$$\vec{y} - A\vec{x} = \vec{e}$$

$A\vec{x}$ is a vector that is in the column space of A. For this A the column space is a plane is $\mathbb{R}^m$

$\vec{y}$ is a vector, not in the column space of A (otherwise $A\vec{x} = \vec{y}$ has an exact solution)

$\vec{e}$ is the error vector, the difference between $\vec{y}$ and $A\vec{x}$



The length $\|\vec{e}\|$ is a *minimum* exactly when $\vec{e} \perp CS(A)$

Best Approximation *Theorem*

If $CS(A)$ is a finite dimensional subspace of an inner product space, and if $\vec{y}$ is a vector in ***V***, then $proj_{CS(A)}\vec{y}$ is the best approximation to ***y*** from $CS(A)$ is the sense that

$$\left\| \vec{y} - proj_{CS(A)}\vec{y} \right\| < \| \vec{y} - CS(A) \|$$

For every vector $\vec{w}$ in $CS(A)$ that is different from $proj_{CS(A)}\vec{y}$

Theorem

For every linear system $A\vec{x} = \vec{y}$, the associated normal system

$$A^T A\vec{x} = A^T \vec{y}$$

Is consistent, and all solutions are least squares solutions of $A\vec{x} = \vec{y}$

If the columns of A are linearly independent, then $A^T A$ is invertible so has a unique solution $\overline{x}$.
This solution is often expressed theoretically as

$$\left(A^T A\right)^{-1} A^T A\overline{x} = \left(A^T A\right)^{-1} A^T \vec{y}$$

$$\bar{x} = \left(A^T A\right)^{-1} A^T \vec{y}$$

Proof

Let the vector $\bar{\boldsymbol{x}}$ is a least squares solution to $A\vec{x} = \vec{y} \quad \Leftrightarrow \quad \left(\vec{y} - A\bar{x}\right) \perp CS(A)$

$$\left(\vec{y} - A\bar{x}\right) \cdot \vec{z} = 0 \qquad \vec{z} \text{ in } CS(A) \quad \& \quad \vec{z} = A\vec{w}$$

$$\left(\vec{y} - A\bar{x}\right) \cdot A\vec{w} = 0 \qquad \vec{w} \text{ in } \mathbb{R}^n$$

$$A^T\left(\vec{y} - A\bar{x}\right) \cdot \vec{w} = 0$$

$$A^T\left(\vec{y} - A\bar{\boldsymbol{x}}\right) = 0$$

$$A^T\vec{y} - A^T A\bar{\boldsymbol{x}} = 0$$

$$A^T\vec{y} = A^T A\bar{x}$$

Theorem

If A is an $m \times n$ matrix, then the following are equivalent

a) A has linearly independent column vectors.

b) $A^T A$ is invertible.

Example

Find the equation of the line that best fits the given points in the least-squares sense.

(40, 482), (45, 467), (50, 452), (55, 432), (60, 421)

Solution

Let $y = mx + b$ be the equation of the line that best fits the given points. Then

$$\begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$$

Where $A = \begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix} \quad \boldsymbol{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \boldsymbol{y} = \begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$

Using the normal equation formula: $A^T Ax = A^T y$

$$\begin{pmatrix} 40 & 45 & 50 & 55 & 60 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 40 & 1 \\ 45 & 1 \\ 50 & 1 \\ 55 & 1 \\ 60 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 40 & 45 & 50 & 55 & 60 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 482 \\ 467 \\ 452 \\ 432 \\ 421 \end{pmatrix}$$

$$\begin{pmatrix} 12,750 & 250 \\ 250 & 5 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 111,970 \\ 2,255 \end{pmatrix}$$

$X = A^{-1}B$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{1250} \begin{pmatrix} 5 & -250 \\ -250 & 12,750 \end{pmatrix} \begin{pmatrix} 111,970 \\ 2,255 \end{pmatrix}$$

$$= \begin{pmatrix} -3.12 \\ 607 \end{pmatrix}$$

Or

$$m = \frac{\begin{vmatrix} 111,970 & 250 \\ 2,255 & 5 \end{vmatrix}}{\begin{vmatrix} 12,750 & 250 \\ 250 & 5 \end{vmatrix}}$$

$$= \frac{-3,900}{1,250}$$

$$\boxed{= -\frac{78}{25}}$$

$$b = \frac{\begin{vmatrix} 12,750 & 111,970 \\ 250 & 2,255 \end{vmatrix}}{1,250}$$

$$= \frac{758,750}{1,250}$$

$$\boxed{= 607}$$

Thus, $\boxed{y = -\frac{78}{25}x + 607}$ *or* $\boxed{y = -3.12x + 607}$

Example

Given the system equation: $\begin{cases} x_1 - x_2 = 4 \\ 3x_1 + 2x_2 = 1 \\ -2x_1 + 4x_2 = 3 \end{cases}$

a) Find the least-squares solution of the linear system $A\vec{x} = \vec{y}$

b) Find the orthogonal projection of $\vec{y}$ on the column space of A

c) Find the ***error vector*** and the ***error***

Solution

a) $A = \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} m \\ b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$

$$A^T A\vec{x} = A^T \vec{y}$$

$$\begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 14 & -3 \\ -3 & 21 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{285}\begin{pmatrix} 21 & 3 \\ 3 & 14 \end{pmatrix}\begin{pmatrix} 1 \\ 10 \end{pmatrix} \qquad X = A^{-1}B$$

$$= \begin{pmatrix} \frac{51}{285} \\ \frac{143}{285} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{17}{95} \\ \frac{143}{285} \end{pmatrix}$$

Thus $\underline{y = \frac{17}{95}x + \frac{143}{285}}$ *or* $\underline{y = 0.1789x + 0.5018}$

b) The orthogonal projection of $\vec{y}$ on the column space of A

$$A\vec{x} = \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} \frac{17}{95} \\ \frac{143}{285} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{pmatrix}$$

c) $\vec{y} - A\vec{x} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -\frac{92}{285} \\ \frac{439}{285} \\ \frac{94}{57} \end{pmatrix}$

$$= \begin{pmatrix} \frac{1232}{285} \\ -\frac{154}{285} \\ \frac{4}{3} \end{pmatrix}$$

The ***error***: $\|\vec{y} - A\vec{x}\| = \sqrt{\left(\frac{1232}{285}\right)^2 + \left(-\frac{154}{285}\right)^2 + \left(\frac{4}{3}\right)^2}$

$$\approx 4.556$$

Exercises *Section* 3.5 – Least Squares Analysis

(1 – 7) Find the equation of the line that best fits the given points in the least-squares sense and find the error.

1. $\{(0,\ 2),\ (1,\ 2),\ (2,\ 0)\}$

2. $\{(1,\ 5),\ (2,\ 4),\ (3,\ 1),\ (4,\ 1),\ (5,\ -1)\}$

3. $\{(0,\ 1),\ (1,\ 3),\ (2,\ 4),\ (3,\ 4)\}$

4. $\{(-2,\ 0),\ (-1,\ 0),\ (0,\ 1),\ (1,\ 3),\ (2,\ 5)\}$

5. $\{(2,\ 3),\ (3,\ 2),\ (5,\ 1),\ (6,\ 0)\}$

6. $\{(-1,\ 0),\ (0,\ 1),\ (1,\ 2),\ (2,\ 4)\}$

7. $\{(1,\ 0),\ (2,\ 1),\ (4,\ 2),\ (5,\ 3)\}$

(8 – 10) Find the orthogonal projection of the vector $\vec{u}$ on the subspace of $\mathbb{R}^4$ spanned by the vectors

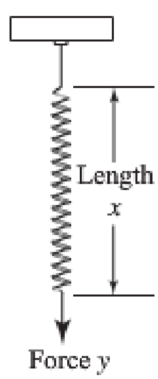
8. $\vec{u}=(-3,\ -3,\ 8,\ 9);\quad \vec{v}_1=(3,\ 1,\ 0,\ 1),\quad \vec{v}_2=(1,\ 2,\ 1,\ 1),\quad \vec{v}_3=(-1,\ 0,\ 2,\ -1)$

9. $\vec{u}=(6,\ 3,\ 9,\ 6);\quad \vec{v}_1=(2,\ 1,\ 1,\ 1),\quad \vec{v}_2=(1,\ 0,\ 1,\ 1),\quad \vec{v}_3=(-2,\ -1,\ 0,\ -1)$

10. $\vec{u}=(-2,\ 0,\ 2,\ 4);\quad v_1=(1,\ 1,\ 3,\ 0),\quad \vec{v}_2=(-2,\ -1,\ -2,\ 1),\quad \vec{v}_3=(-3,\ -1,\ 1,\ 3)$

11. Find the standard matrix for the orthogonal projection P of $\mathbb{R}^2$ on the line passes through the origin and makes an angle θ with the positive x-axis.

12. Hooke's law in physics states that the length x of a uniform spring is a linear function of the force y applied to it. If we express the relationship as $y = mx + b$, then the coefficient m is called the spring constant.



Suppose a particular unstretched spring has a measured length of 6.1 *inches*.(i.e., $x = 6.1$ when $y = 0$). Forces of 2 pounds, 4 pounds, and 6 pounds are then applied to the spring, and the corresponding lengths are found to be 7.6 inches, 8.7 inches, and 10.4 inches. Find the spring constant.

13. Prove: If A has a linearly independent column vectors, and if $\boldsymbol{b}$ is orthogonal to the column space of A, then the least squares solution of $A\vec{x}=\vec{b}$ is $\vec{x}=\vec{0}$.

14. Let A be an $m\times n$ matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of $\mathbb{R}^n$ onto the row space of A.

15. Let W be the line with parametric equations $x=2t, \quad y=-t, \quad z=4t$

a) Find a basis for W.
b) Find the standard matrix for the orthogonal projection on W.
c) Use the matrix in part (*b*) to find the orthogonal projection of a point $P_0\left(x_0, y_0, z_0\right)$ on W.
d) Find the distance between the point $P_0\left(2, 1, -3\right)$ and the line W.

16. In $\mathbb{R}^3$, consider the line l given by the equations $x=t, \quad y=t, \quad z=t$

And the line m given by the equations $x=s, \quad y=2s-1, \quad z=1$

Let P be the point on l, and let Q be a point on m.

Find the values of t and s that minimize the distance between the lines by minimizing the squared distance $\|P-Q\|^2$

17. Determine whether the statement is true or false,

a) If A is an $m\times n$ matrix, then A^TA is a square matrix.
b) If A^TA is invertible, then A is invertible.
c) If A is invertible, then A^TA is invertible.
d) If $A\vec{x}=\vec{b}$ is a consistent linear system, then $A^TA\vec{x}=A^T\vec{b}$ is also consistent.
e) If $A\vec{x}=\vec{b}$ is an inconsistent linear system, then $A^TA\vec{x}=A^T\vec{b}$ is also inconsistent.
f) Every linear system has a least squares solution.
g) Every linear system has a unique least squares solution.
h) If A is an $m\times n$ matrix with linearly independent columns and $\vec{b}$ is in R^m, then $A\vec{x}=\vec{b}$ has a unique least squares solution.

18. A certain experiment produces the data $\{(1, 1.8), (2, 2.7), (3, 3.4), (4, 3.8), (5, 3.9)\}$.

Find the function that it will fit these data in the form of $y=\beta_1x+\beta_2x^2$

19. According to Kepler's first law, a comet should have an ellipse, parabolic, or hyperbolic orbit (with gravitational attractions from the planets ignored). In suitable polar coordinates, the position $(r,\ \upsilon)$ of a comet satisfies an equation of the form

$$r = \beta + e(r \cdot \cos\upsilon)$$

Where β is a constant and e is the eccentricity of the orbit, with $0 \le e < 1$ for an ellipse, $e = 1$ for a parabolic, and $e > 1$ for a hyperbola.

Suppose observations of a newly discovered comet provide the data below.

υ	.88	1.10	1.42	1.77	2.14
r	3.00	2.30	1.65	1.25	1.01

Determine the type of orbit, and predict where the orbit will be when $\upsilon = 4.6$ $(\textit{radians})$?

20. To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second, from $t = 0$ to $t = 12$

The position (in *feet*) were:

$0,\ 8.8,\ 29.9,\ 62.0,\ 104.7,\ 159.1,\ 222.0,\ 294.5,\ 380.4,\ 471.1,\ 571.7,\ 686.8,\ \textit{and}\ 809.2$

a) Find the least square cubic curve $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$ for these data.

b) Estimate the velocity of the plane when $t = 4.5$ *sec*, using the result from part (*a*).