Lecture Three – Multiple Integrals

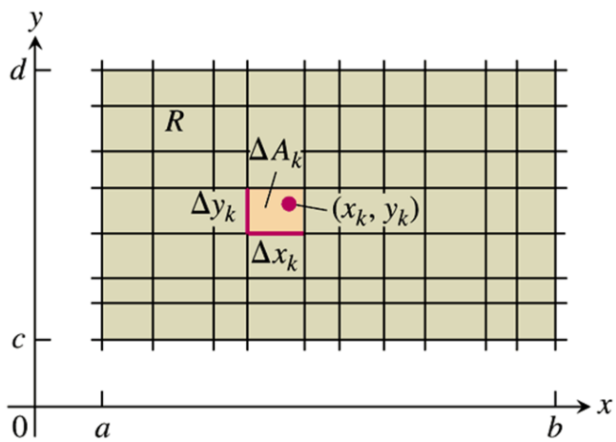
Section 3.1 – Double Integrals over Rectangular Regions

Double Integrals

Consider a function $f(x, y)$ defined on a rectangular region *R*,

$$R: \quad a \le x \le b, \quad c \le y \le d$$

A small rectangular piece of width Δx and height Δy has area $\Delta A = \Delta x \Delta y$.



To form a Riemann sum over *R*, select a point (x_k, y_k) in the kth small rectangle, multiply the value of *f* at that point by the area ΔA_k and add together the products:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

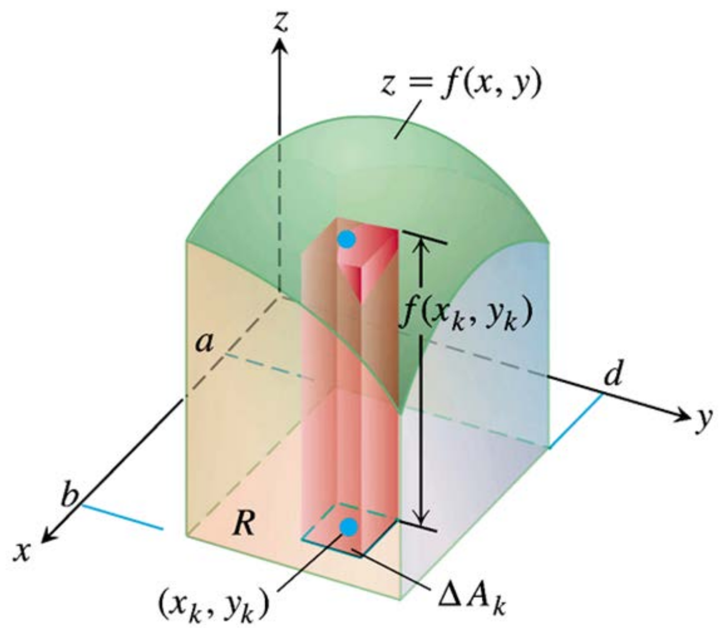
As the rectangles get narrow and short, their number *n* increases, therefore

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

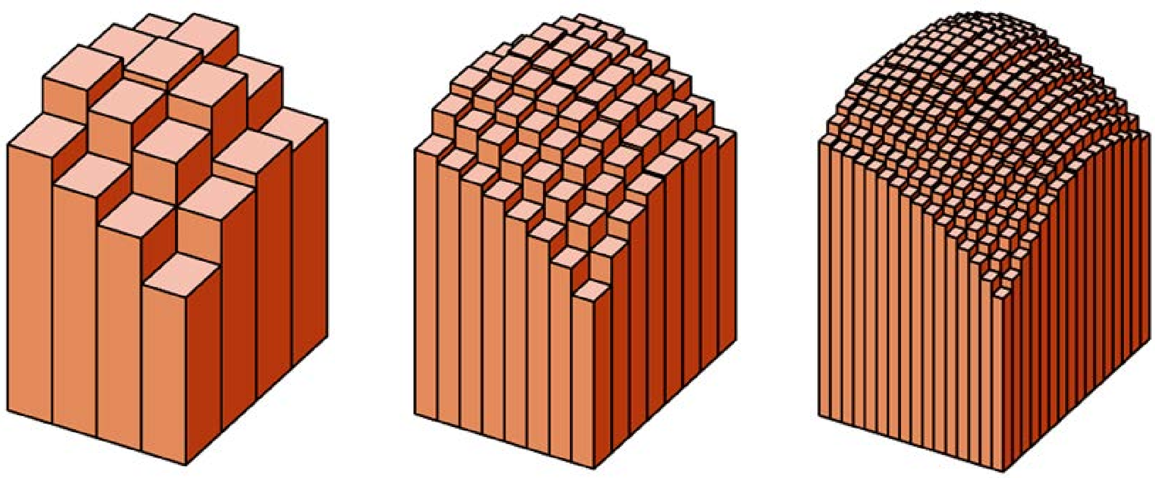
Then the function f is said to be integrable and the limit is called double integral of f over *R*,

$$\iint_R f(x, y)\,dA \quad \text{or} \quad \iint_R f(x, y)\,dxdy$$

Double Integrals as Volumes



$$Volume = \lim_{n \to \infty} S_n = \iint_R f(x,y)\,dA, \;\; where\ \Delta A_k \to 0\ as\ n \to \infty$$



$n = 16$ $\qquad$ $n = 64$ $\qquad$ $n = 256$

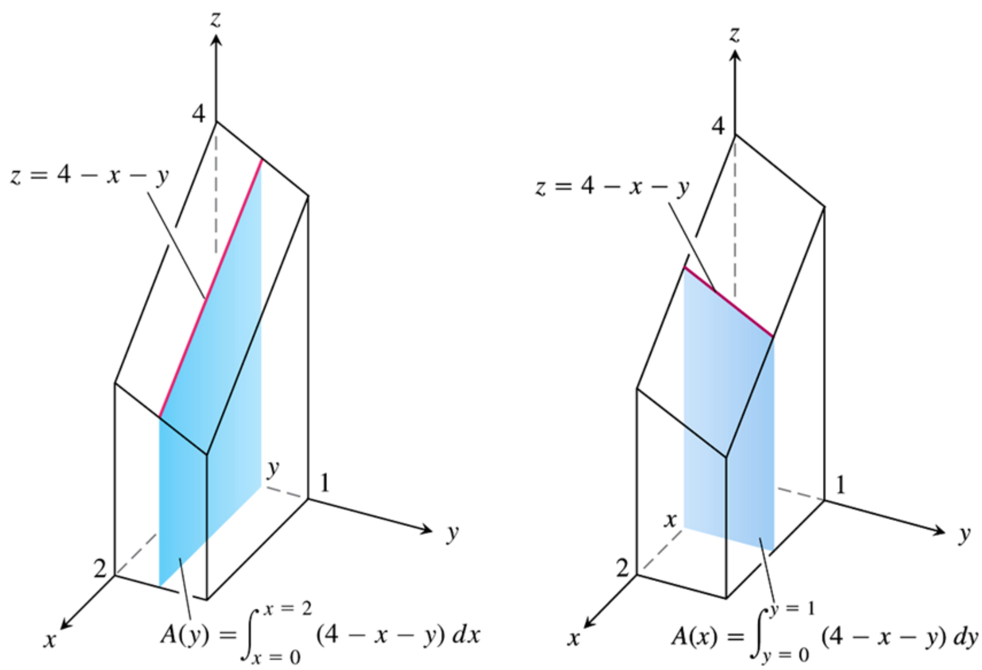
As *n* increases, the ***Riemann sum*** approximations approach the total volume of the solid

Example

Calculate the volume under the plane $z = 4 - x - y$ over the rectangular region $R: \quad 0 \le x \le 2, \quad 0 \le y \le 1$ in the *xy*-plane.

Solution

$$
\begin{aligned}
Volume &= \int_{x=0}^{x=2} A(x)\,dx \\
&= \int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y)\,dy\ dx \\
&= \int_{x=0}^{x=2} \left[4y - xy - \frac{1}{2}y^2 \right]_{y=0}^{y=1} dx \\
&= \int_{x=0}^{x=2} \left(4 - x - \frac{1}{2} \right) dx \\
&= \int_{x=0}^{x=2} \left(\frac{7}{2} - x \right) dx \\
&= \left[\frac{7}{2}x - \frac{1}{2}x^2 \right]_0^2 \\
&= 7 - 2 \\
&= 5 \quad unit^3
\end{aligned}
$$



$$Volume = \int_0^1 \int_0^2 (4 - x - y)\,dx\ dy$$

Theorem – Fubini's Theorem

If $f(x,y)$ is continuous throughout the rectangular region R: $a \le x \le b,\ c \le y \le d$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dxdy = \int_a^b \int_c^d f(x,y)\,dydx$$

Example

Calculate $\iint_R f(x,y)\,dA$ for $f(x,y) = 100 - 6x^2y$ *and* $R:\ 0 \le x \le 2,\quad -1 \le y \le 1$

Solution

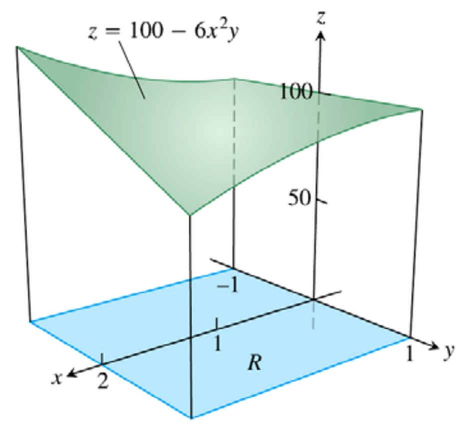
$$\int_{-1}^{1}\int_0^2 \left(100-6x^2y\right)dxdy = \int_{-1}^{1}\left[100x - 2x^3y\right]_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1}(200-16y)\,dy$$

$$= \left[200y - 8y^2\right]_{-1}^{1}$$

$$= 200 - 8 - (-200-8)$$

$$\underline{= 400}$$



Example

Find the volume of the region bounded above the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below the rectangle $R:\ 0 \le x \le 1,\quad 0 \le y \le 2$

Solution

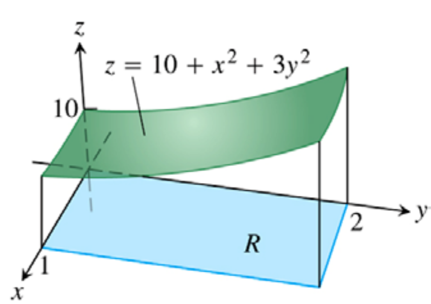
$$Volume = \int_0^1\int_0^2 \left(10 + x^2 + 3y^2\right)dydx$$

$$= \int_0^1 \left[10y + yx^2 + y^3\right]_0^2 dx$$

$$= \int_0^1 \left(2x^2 + 28\right)dx$$

$$= \left[\frac{2}{3}x^3 + 28x\right]_0^1 \quad = \frac{2}{3} + 28$$

$$\underline{= \frac{86}{3}}$$



Exercises *Section* 3.1 – Double Integrals over Rectangular Regions

Evaluate the iterated integral

1. $\displaystyle\int_1^2\int_0^4 2xy\ dydx$

2. $\displaystyle\int_0^2\int_{-1}^1 (x-y)\ dydx$

3. $\displaystyle\int_0^1\int_0^1\left(1-\frac{x^2+y^2}{2}\right)dxdy$

4. $\displaystyle\int_0^3\int_{-2}^0\left(x^2y-2xy\right)dydx$

5. $\displaystyle\int_0^1\int_0^1\frac{y}{1+xy}dxdy$

6. $\displaystyle\int_0^{\ln 2}\int_1^{\ln 5} e^{2x+y}dydx$

7. $\displaystyle\int_0^1\int_1^2 xye^x dydx$

8. $\displaystyle\int_\pi^{2\pi}\int_0^\pi (\sin x+\cos y)dxdy$

Evaluate the double integral over the given region *R*.

9. $\displaystyle\iint_R\left(6y^2-2x\right)dA \quad R:\ \ 0\le x\le 1,\ \ 0\le y\le 2$

10. $\displaystyle\iint_R\left(\frac{\sqrt{x}}{y^2}\right)dA \quad R:\ \ 0\le x\le 4,\ \ 1\le y\le 2$

11. $\displaystyle\iint_R y\sin(x+y)dA \quad R:\ \ -\pi\le x\le 0,\ \ 0\le y\le \pi$

12. $\displaystyle\iint_R e^{x-y}dA \quad R:\ \ 0\le x\le \ln 2,\ \ 0\le y\le \ln 2$

13. $\displaystyle\iint_R \frac{y}{x^2y^2+1}dA \quad R:\ \ 0\le x\le 1,\ \ 0\le y\le 1$

14. Integrate $f(x,y)=\dfrac{1}{xy}$ over the ***square*** $1\le x\le 2,\ \ 1\le y\le 2$

15. Integrate $f(x,y)=y\cos xy$ over the ***rectangle*** $0\le x\le \pi,\ \ 0\le y\le 1$

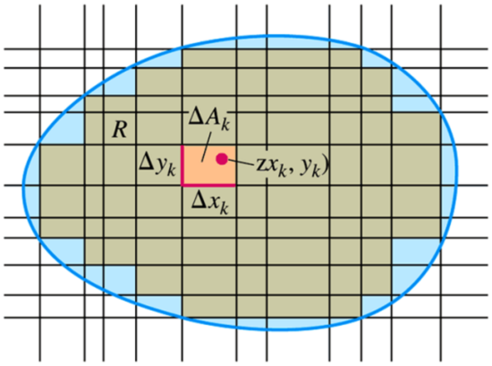
16. Find the volume of the region bounded above the paraboloid $z=x^2+y^2$ and below by the square $R:\ \ -1\le x\le 1,\ \ -1\le y\le 1$

17. Find the volume of the region bounded above the plane $z = \frac{y}{2}$ and below by the rectangle $R: \quad 0 \le x \le 4, \quad 0 \le y \le 2$

18. Find the volume of the region bounded above the surface $z = 4 - y^2$ and below by the rectangle $R: \quad 0 \le x \le 1, \quad 0 \le y \le 2$

19. Find the volume of the region bounded above the ellipitical paraboloid $z = 16 - x^2 - y^2$ and below by the square $R: \quad 0 \le x \le 2, \quad 0 \le y \le 2$

Section 3.2 – Double Integrals over General Regions



Volumes

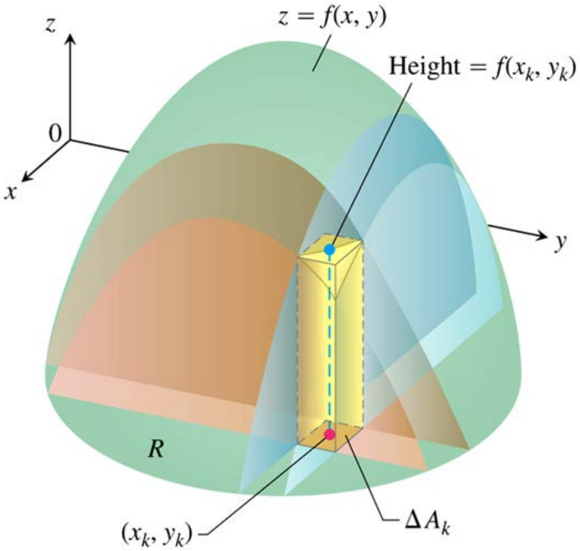
If $f(x,y)$ is positive and continuous over *R*, we define the volume of the solid region between *R* and the surface $z = f(x,y)$ to be $\iint\limits_R f(x,y)\,dA$.

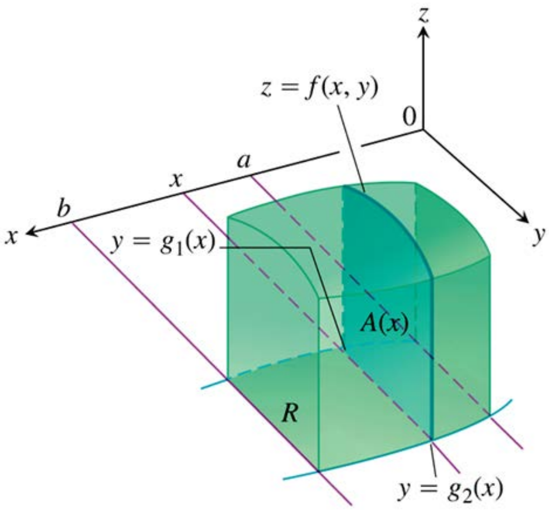
If *R* is a region in the *xy*-plane, bounded ***above*** and ***below*** by the curves $y = g_1(x)$ and $y = g_2(x)$ and on the sides by the lines $x = a,\ x = b$. Calculate the cross-sectional area

$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy$$

Then integrate $A(x)$ from $x = a \ to \ x = b$ to get the volume as an iterated integral

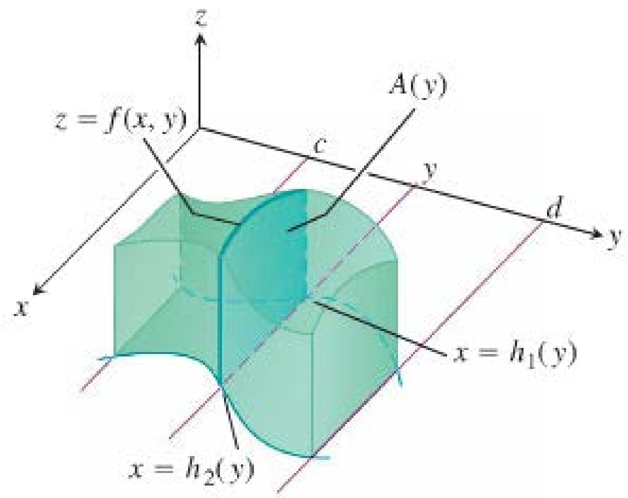
$$V = \int_a^b A(x)\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx$$





Similarly, if R is a region bounded by the curves $x = h_1(y)$ and $x = h_2(y)$ and the lines $y = c,\ y = d$, then the volume calculated by slicing is given by the iterated integral .

$$V = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy$$



$$\int_c^d A(y)\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy$$

$$Volume = \lim \sum f(x_k, y_k)\Delta A_k = \iint_R f(x,y)\,dA$$

Theorem – Fubini's Theorem

Let $f(x,y)$ is continuous on a region R,

1. If R is defined by : $a \le x \le b,\ \ g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on [a, b], then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx$$

2. If R is defined by : $c \le y \le d,\ \ h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on [c, d], then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy$$

Example

Find the volume of the prism whose base is the triangle in the *xy*-plane bounded by the *x*-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = f(x, y) = 3 - x - y$

Solution

$0 \le x \le 1, \quad 0 \le y \le x$

$$V = \int_0^1 \int_0^x (3 - x - y)\,dy\,dx$$

$$= \int_0^1 \left[3y - xy - \tfrac{1}{2}y^2 \right]_0^x dx$$

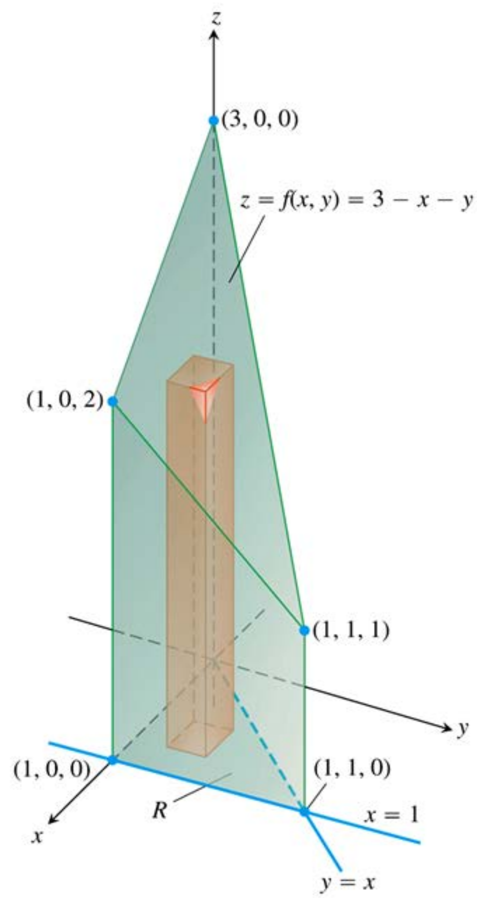
$$= \int_0^1 \left(3x - x^2 - \tfrac{1}{2}x^2 \right) dx$$

$$= \int_0^1 \left(3x - \tfrac{3}{2}x^2 \right) dx$$

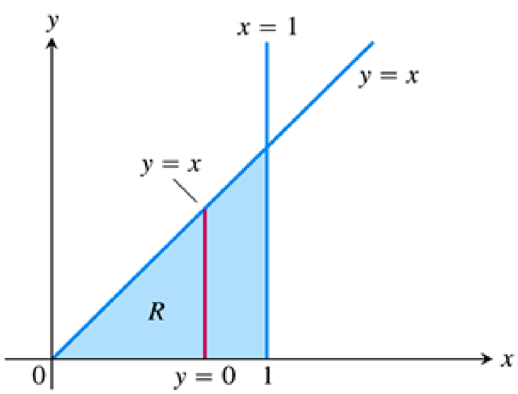
$$= \left[\tfrac{3}{2}x^2 - \tfrac{1}{2}x^3 \right]_0^1$$

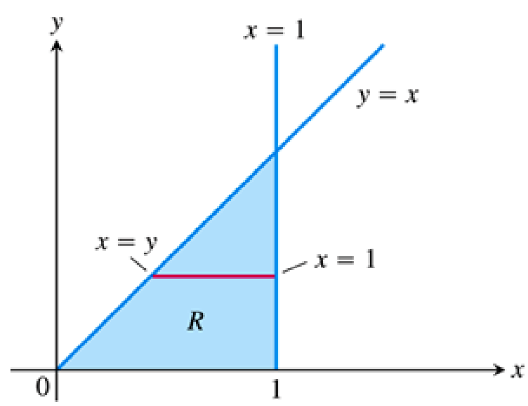
$$= \frac{3}{2} - \frac{1}{2}$$

$$\underline{= 1 \quad unit^3}$$



$$V = \int_0^1 \int_y^1 (3 - x - y)\,dx\,dy = 1$$

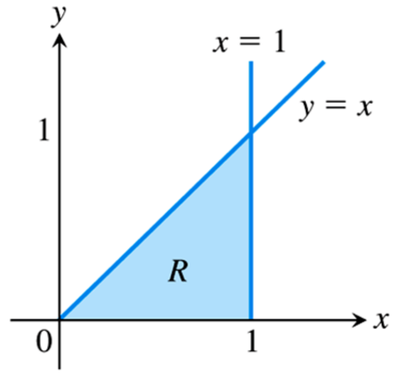




Example

Calculate $\iint\limits_R \frac{\sin x}{x}\, dA$ where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.

Solution



$$\int_0^1 \int_0^x \left(\frac{\sin x}{x}\right) dy\ dx = \int_0^1 \left(\frac{\sin x}{x} y\right)\Big|_0^x dx$$

$$= \int_0^1 \sin x dx$$

$$= -\cos x \Big|_0^1$$

$$= -\cos(1) + 1$$

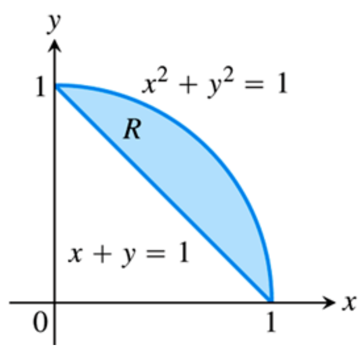
$$\underline{\approx 0.46|}$$

$\int_0^1 \int_y^1 \left(\frac{\sin x}{x}\right) dx\ dy$, we run into a problem because $\int \frac{\sin x}{x} dx$ cannot be expressed in terms of elementary functions.

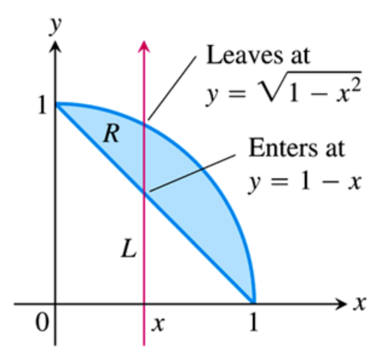
Finding Limits on Intergration

Using Vertical Cross-sections

1. Sketch the region of Integration and label the bounding curves

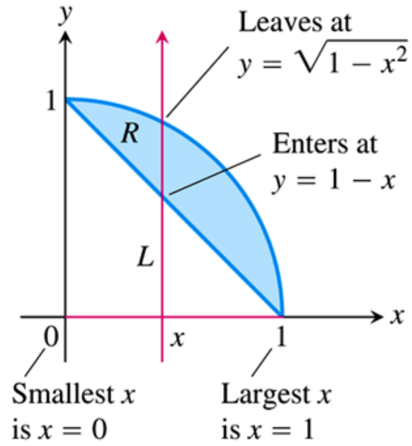


2. *Find the y-limits of integration*. Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants).



3. *Find the x-limits of integration*. Choose x-limits that include all the vertical lines through R. The integral is

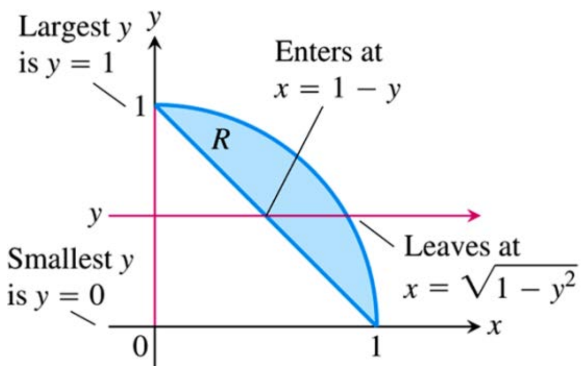
$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=1}\int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx$$



Using Horizontal Cross-sections

To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines.

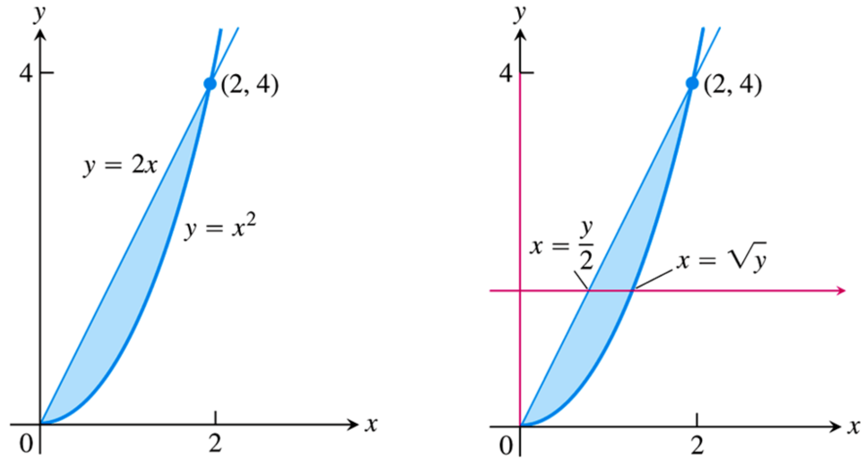
$$\iint_R f(x,y)\,dA = \int_0^1\int_{1-y}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$$



Example

Sketch the region of integration for the integral $\int_0^2 \int_{x^2}^{2x} (4x+2)\,dydx$ and write an equivalent integral with the order of integration reversed.

Solution



The given inequalities are: $x^2 \le y \le 2x \quad and \quad 0 \le x \le 2$

$$\rightarrow \begin{cases} y = x^2 & x = \sqrt{y} \\ y = 2x & x = \dfrac{y}{2} \end{cases} \qquad \rightarrow \begin{cases} x = 0 & y = 0 \\ x = 2 & y = 4 \end{cases}$$

The integral is $\int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,dxdy$

- If $f(x,y)$ and $g(x,y)$ are continuous on the bounded region R, then the following properties hold

1. *Constant Multiple*: $\iint\limits_R cf(x,y)\,dA = c\iint\limits_R f(x,y)\,dA$

2. *Sum and Difference*: $\iint\limits_R \big(f(x,y) \pm g(x,y)\big)\,dA = \iint\limits_R f(x,y)\,dA \pm \iint\limits_R g(x,y)\,dA$

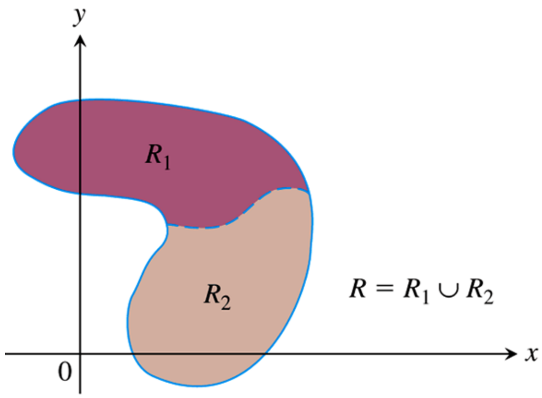
3. *Domination*:

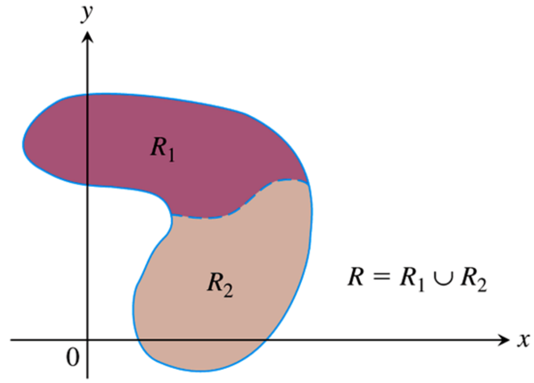
a) $\iint\limits_R f(x,y)\,dA \ge 0 \quad if \quad f(x,y) \ge 0 \ \ on\ R$

b) $\iint\limits_R f(x,y)\,dA \ge \iint\limits_R g(x,y)\,dA \quad if \quad f(x,y) \ge g(x,y) \ \ on\ R$

4. *Additivity*: $\iint\limits_R f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA + \iint\limits_{R_2} f(x,y)\,dA$

If R is the union of two non-overlapping regions R_1 and R_2.



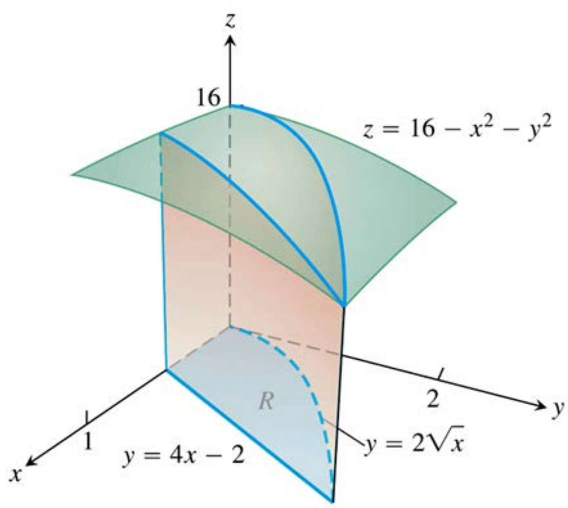


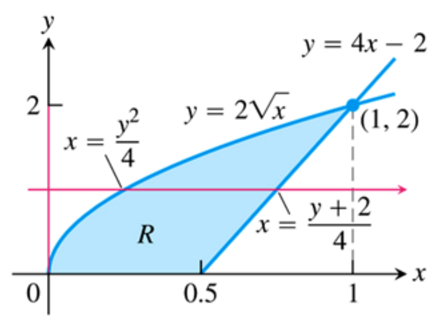
$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

Example

Find the volume of the wedge like solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.

Solution





$y = 2\sqrt{x} \quad \rightarrow x = \dfrac{y^2}{4}$ $\qquad$ $y = 4\dfrac{y^2}{4} - 2 = y^2 - 2 \quad \rightarrow \quad y^2 - y - 2 = 0 \Rightarrow \lfloor y = -1, \ 2 \rfloor$

$y = 4x - 2 \quad \rightarrow x = \dfrac{y+2}{4}$

$$Volume = \int_0^2 \int_{y^2/4}^{(y+2)/4} \left(16 - x^2 - y^2\right) dxdy$$

$$= \int_0^2 \left[16x - \frac{1}{3}x^3 - y^2 x\right]_{y^2/4}^{(y+2)/4} dy$$

$$= \int_0^2 \left[\left(16\frac{y+2}{4} - \frac{1}{3}\left(\frac{y+2}{4}\right)^3 - y^2\frac{y+2}{4} \right) - \left(16\frac{y^2}{4} - \frac{1}{3}\frac{y^6}{64} - \frac{y^4}{4} \right) \right] dy$$

$$= \int_0^2 \left[4y + 8 - \frac{1}{192}\left(y^3 + 6y^2 + 12y + 8\right) - \frac{1}{4}y^3 - \frac{1}{2}y^2 - 4y^2 + \frac{1}{192}y^6 + \frac{1}{4}y^4 \right] dy$$

$$= \int_0^2 \left[4y + 8 - \frac{1}{192}y^3 - \frac{1}{32}y^2 - \frac{1}{16}y - \frac{1}{24} - \frac{1}{4}y^3 - \frac{9}{2}y^2 + \frac{1}{192}y^6 + \frac{1}{4}y^4 \right] dy$$

$$= \int_0^2 \left[\frac{1}{192}y^6 + \frac{1}{4}y^4 - \frac{49}{192}y^3 - \frac{145}{32}y^2 + \frac{63}{16}y + \frac{191}{24} \right] dy$$

$$= \left[\frac{1}{1344}y^7 + \frac{1}{20}y^5 - \frac{49}{768}y^4 - \frac{145}{96}y^3 + \frac{63}{32}y^2 + \frac{191}{24}y \right]_0^2$$

$$\approx 12.4 \quad unit^3$$

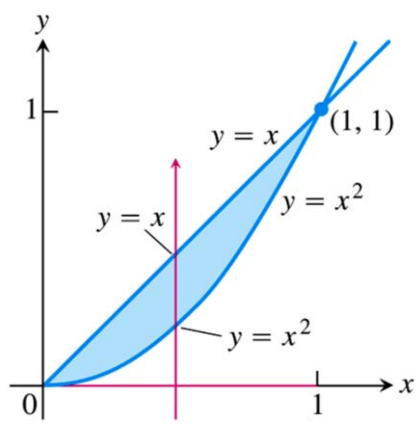
Definition

The area of a closed, bounded plane region R is $\qquad A = \iint\limits_R dA$

Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution



$$y = x = x^2 \quad \to \quad x = 0,\ 1$$

$$A = \int_0^1 \int_{x^2}^{x} dy\,dx$$

$$= \int_0^1 \left[y\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left(x - x^2\right) dx$$

$$= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$$

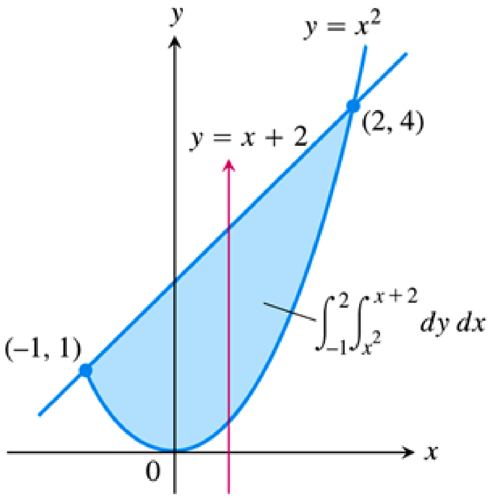
$$= \frac{1}{2} - \frac{1}{3}$$

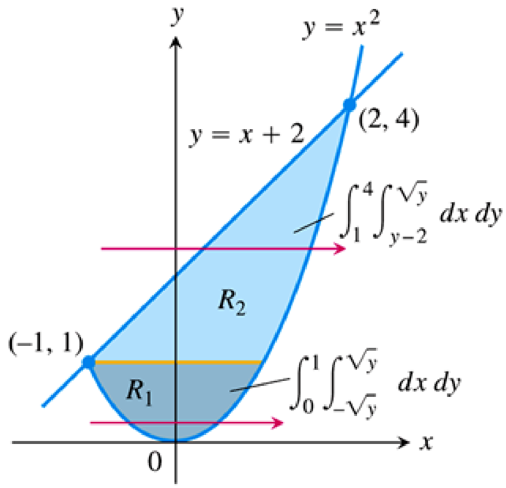
$$= \frac{1}{6} \quad unit^2$$

Example

Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Solution





$y = x^2 = x + 2 \to x^2 - x - 2 = 0 \quad \Rightarrow \quad \boxed{x = -1,\ 2}$

$$A = \int_{-1}^{2}\int_{x^2}^{x+2} dydx$$

$$= \int_{-1}^{2} y\Big|_{x^2}^{x+2} dx$$

$$= \int_{-1}^{2}\left(x + 2 - x^2\right)dx$$

$$= \left[\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3\right]_{-1}^{2}$$

$$= \frac{1}{2}(4) + 2(4) - \frac{1}{3}(8) - \left(\frac{1}{2}(-1)^2 - 2 + \frac{1}{3}\right)$$

$$= \frac{9}{2} \quad unit^2$$

Average values of f over R $= \dfrac{1}{area\ of\ R}\displaystyle\iint_R fdA$

❖ Average value of f over R $= \dfrac{1}{area\ of\ R}\displaystyle\iint_R fdA = \frac{2}{\pi}$

Example

Find the average value of $f(x,y)=x\cos xy$ over the rectangle $R:\ 0\le x\le \pi,\quad 0\le y\le 1.$

Solution

$$\int_0^{\pi}\int_0^{1} x\cos xy\ dydx=\int_0^{\pi}\left[\sin xy\right]_0^1\ dx \qquad\qquad \int x\cos xy\ dy=\sin xy+C$$

$$=\int_0^{\pi}(\sin x-0)\ dx$$

$$=\int_0^{\pi}\sin x\ dx$$

$$=-\cos x\Big|_0^{\pi}$$

$$=1+1$$

$$\underline{=2}$$

Exercises Section 3.2 – Double Integrals over General Regions

Sketch the region of integration and evaluate the integral

1. $\int_0^{\pi}\int_0^{x} x\sin y\,dy\,dx$

2. $\int_0^{\pi}\int_0^{\sin x} y\,dy\,dx$

3. $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dx\,dy$

4. $\int_1^{4}\int_0^{\sqrt{x}} \frac{3}{2}e^{y/\sqrt{x}}\,dy\,dx$

5. Integrate $f(x,y)=\frac{x}{y}$ over the region in the first quadrant bounded by the lines $y=x,\quad y=2x,\quad x=1,\quad and\quad x=2$

6. Integrate $f(x,y)=x^2+y^2$ over the triangular region with vertices $(0,0)$, $(1,0)$ *and* $(0,1)$

7. Integrate $f(s,t)=e^{s}\ln t$ over the region in the first quadrant of the *st*-plane that lies above the curve $s=\ln t$ from $t=1$ to $t=2$.

8. $\int_{-2}^{0}\int_{v}^{-v} 2\,dp\,dv$

9. $\int_{-\pi/3}^{\pi/3}\int_0^{\sec t} 3\cos t\,du\,dt$

Sketch the region of integration, reverse the order of integration, and evaluate the integral

10. $\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\,dy\,dx$

11. $\int_0^{2}\int_x^{2} 2y^2\sin xy\,dy\,dx$

12. $\int_0^{1/16}\int_{y^{1/4}}^{1/2} \cos\left(16\pi x^5\right)dx\,dy$

13. $\int_0^{2}\int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx$

14. Find the volume of the region bounded above the paraboloid $z=x^2+y^2$ and below by the triangle enclosed by the lines $y=x$, $x=0$, and $x+y=2$ in the *xy*-plane

15. Find the volume of the solid that is bounded above the cylinder $z=x^2$ and below by the region enclosed by the parabola $y=2-x^2$ and the line $y=x$ in the *xy*-plane

16. Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$ and the plane $z + y = 3$

17. Find the volume of the solid that is bounded on the front and back by the planes $x = 2$, and $x = 1$, on the sides by the cylinders $y = \pm\frac{1}{x}$ and above and below the planes $z = x + 1$ and $z = 0$.

18. Find the area of the region enclosed by the coordinate axes and the lin $x = 0$ e $x + y = 2$.

19. Find the area of the region enclosed by the lines , $y = 2x$, and $y = 4$

20. Find the area of the region enclosed by the parabola $x = y - y^2$ and the line $y = -x$.

21. Find the area of the region enclosed by the curve $y = e^x$ and the lines $y = 0$, $x = 0$ and $x = \ln 2$

22. Find the area of the region enclosed by the curve $y = \ln x$ and $y = 2\ln x$ and the lines $x = e$ in the first quadrant.

23. Find the area of the region enclosed by the lines $y = x$, $y = \frac{x}{3}$, and $y = 2$

24. Find the area of the region enclosed by the lines $y = x - 2$ and $y = -x$ and the curve $y = \sqrt{x}$

25. Find the area of the region enclosed by the parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$

Find the area of the region

26. $\int_0^6 \int_{y^2/3}^{2y} dxdy$

27. $\int_0^{\pi/4} \int_{\sin x}^{\cos x} dydx$

28. $\int_{-1}^{2} \int_{y^2}^{y+2} dxdy$

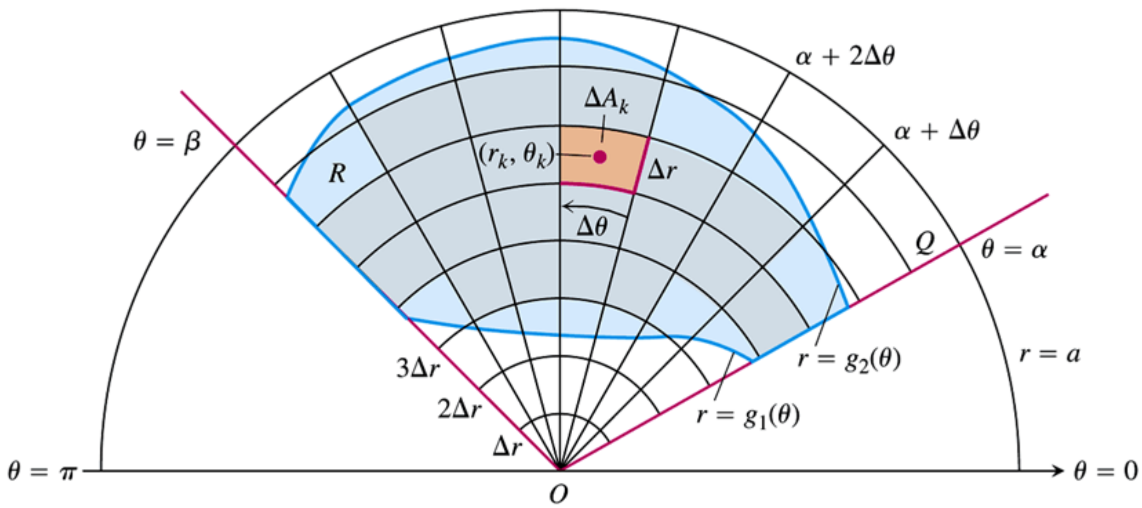
29. $\int_0^2 \int_{x^2-4}^{0} dydx + \int_0^4 \int_0^{\sqrt{x}} dydx$

30. Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2, \quad 0 \le y \le 2$

31. Find the average height of $f(x, y) = \frac{1}{xy}$ over the square $\ln 2 \le x \le 2\ln 2, \quad \ln 2 \le y \le 2\ln 2$

Section 3.3 – Double Integrals in Polar Coordinates

Integrals in Polar Coordinates



$$S_n = \sum_{k=1}^{n} f\left(r_k, \theta_k\right) \Delta A_k$$

If f is continuous throughout R, this sum will approach a limit as Δr and $\Delta\theta$ go to zero. The limit is called the double integral of f over R.

$$\lim_{n \to \infty} S_n = \iint_R f\left(r, \theta\right)\, dA$$

However, the area of a wedge-shaped sector of a circle having radius r and angle θ is

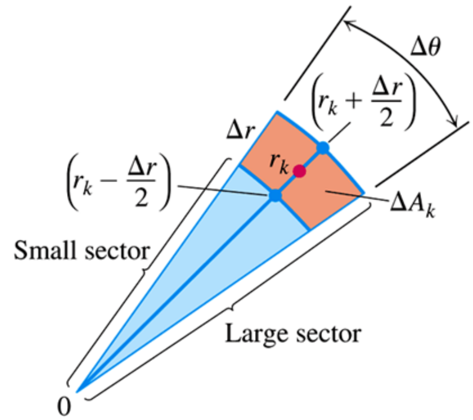
$$A = \frac{1}{2}\theta \cdot r^2$$

Inner radius: $\quad \frac{1}{2}\left(r_k - \frac{\Delta r}{2}\right)^2 \cdot \Delta\theta$

outer radius: $\quad \frac{1}{2}\left(r_k + \frac{\Delta r}{2}\right)^2 \cdot \Delta\theta$

$$\Delta A_k = \begin{pmatrix} \text{area of} \\ \text{large sector} \end{pmatrix} - \begin{pmatrix} \text{area of} \\ \text{small sector} \end{pmatrix}$$

Leads to the formula: $\Delta A_k = r_k \Delta r \Delta\theta$

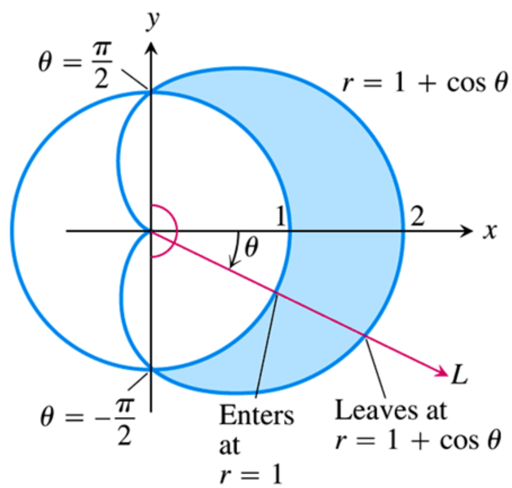


Example

Find the limits of integration for integrating $f(r,\theta)$ over the region *R* that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution

The sketch of the region:



From the graph, we can find the *r - limits of integration*. A typical ray from the origin enters *R* where $r = 1$ and leaves where $r = 1 + \cos\theta$

θ *- limits of integration*: The rays from the origin that intersects *R* run from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$. The integral is

$$\int_{-\pi 2}^{\pi/2} \int_{1}^{1+\cos\theta} f(r,\theta) r \, dr \, d\theta$$

Area in Polar Coordinates

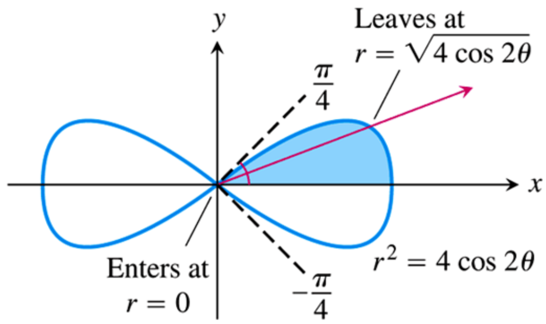
The area of a closed and bounded region *R* in the polar coordinate plane is

$$A = \iint_R r \, dr \, d\theta$$

Example

Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$

Solution



From the graph, we can determine the lemniscate limits of integration, and the total area is 4 times the first-quadrant portion, since it has a form of symmetry.

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4}(2\cos 2\theta)\,d\theta$$

$$= 4\int_0^{\pi/4}\cos 2\theta\ d(2\theta)$$

$$= 4\sin 2\theta\Big|_0^{\pi/4}$$

$$= 4\sin\frac{\pi}{2}$$

$$\boxed{= 4 \quad unit^2}$$

Changing Cartesian Integrals into Polar Integrals

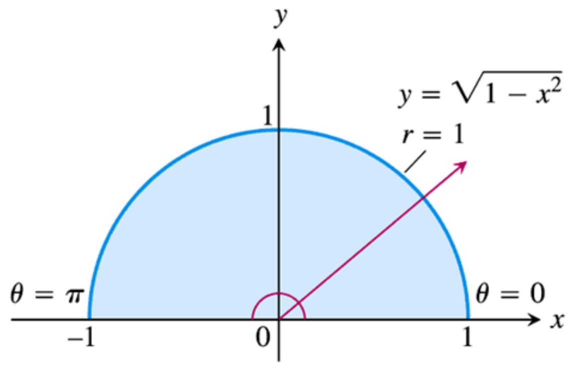
$$\iint_R f(x,y)\,dxdy = \iint_G f(r\cos\theta, r\sin\theta)\,r\ drd\theta$$

Example

Evaluate $\iint_R e^{x^2+y^2}\,dydx$

Where R is the semicircular region bounded by the x-axis and the curve $y=\sqrt{1-x^2}$

Solution



$$\iint_R e^{x^2+y^2}\,dydx = \int_0^\pi \int_0^1 e^{r^2}\,r\ drd\theta \qquad d\left(r^2\right)=2rdr$$

$$= \frac{1}{2}\int_0^\pi \int_0^1 e^{r^2}\,d\left(r^2\right)d\theta$$

$$= \frac{1}{2}\int_0^\pi \left[e^{r^2}\right]_0^1 d\theta$$

$$= \frac{1}{2}\int_0^\pi (e-1)\,d\theta$$

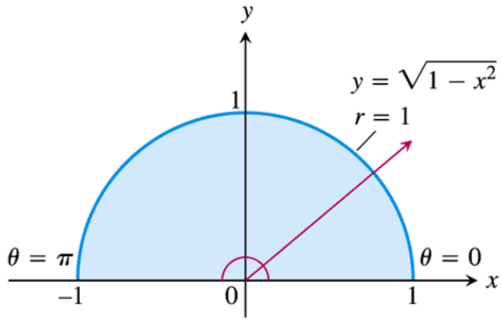
$$= \frac{1}{2}(e-1)\theta\Big|_0^\pi$$

$$\boxed{= \frac{\pi}{2}(e-1)}$$

Example

Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \left(x^2+y^2\right) dydx$

Solution



Since: $0 \le x \le 1 \quad \rightarrow \textit{ interior of } \; x^2+y^2=1 \quad \textit{and in QI}$

Let: $r^2 = x^2+y^2 \quad \textit{with} \quad 0 \le r \le 1$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \left(x^2+y^2\right) dydx = \int_0^{\pi/2} \int_0^1 \left(r^2\right) rdrd\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{4}r^4\right]_0^1 d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} d\theta$$

$$= \frac{1}{4}\theta \Big|_0^{\pi/2}$$

$$= \frac{\pi}{8}$$

- Or we can use the integral table to solve it

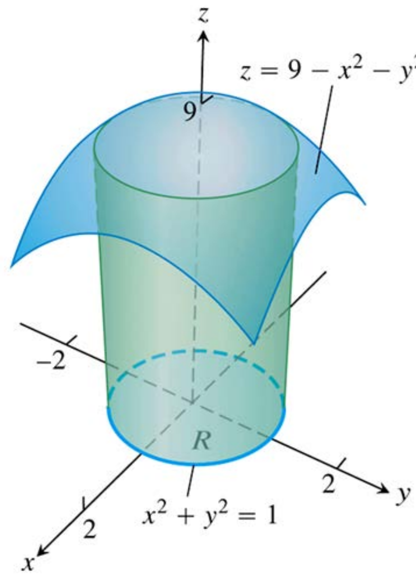
$$\int_0^1 \int_0^{\sqrt{1-x^2}} \left(x^2+y^2\right) dydx = \int_0^1 \left[x^2\sqrt{1-x^2} + \frac{1}{3}\left(1-x^2\right)^3\right] dx$$

Example

Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the *xy*-plane.

Solution

The region of integration R is the unit circle: $x^2 + y^2 = 1 \quad \rightarrow \quad r = 1, \quad 0 \le \theta \le 2\pi$



$$Volume = \int_0^{2\pi}\int_0^1 \left(9 - r^2\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(9r - r^3\right) dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{9}{2}r^2 - \frac{1}{4}r^4\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{9}{2} - \frac{1}{4}\right) d\theta$$

$$= \frac{17}{4}\int_0^{2\pi} d\theta$$

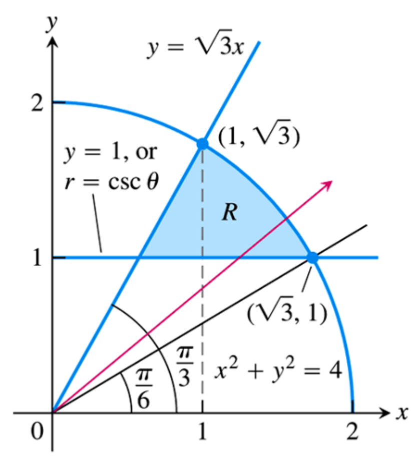
$$= \frac{17}{4}\theta\Big|_0^{2\pi}$$

$$= \frac{17\pi}{2} \quad unit^3$$

Example

Using the polar integration, find the area of the region R in the *xy*-plane enclosed by the circle $x^2 + y^2 = 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.

Solution



The $y = \sqrt{3}x$ has a slope of $\sqrt{3} = \tan\theta \Rightarrow \theta = \frac{\pi}{3}$

Line $y = 1$ intersects $x^2 + y^2 = 4$ when $x^2 + 1 = 4 \rightarrow x = \sqrt{3}$.

A line from origin to $\left(\sqrt{3}, 1\right)$ has a slope of

$$\frac{1}{\sqrt{3}} = \tan\theta \rightarrow \theta = \frac{\pi}{6}$$

$$\therefore \quad \boxed{\frac{\pi}{6} \le \theta \le \frac{\pi}{3}}$$

The polar coordinate r varies from the horizontal line $y = 1$ to the circle $x^2 + y^2 = 4$.

Substituting $r\sin\theta$ for y: $y=1 \to r\sin\theta = 1 \Rightarrow \underline{r = \frac{1}{\sin\theta} = \csc\theta}$ and the radius of the circle is 2.

$\therefore \quad \boxed{\csc\theta \le r \le 2}$

$$Area = \int_{\pi/6}^{\pi/3}\int_{\csc\theta}^{2} r\,dr\,d\theta$$

$$= \int_{\pi/6}^{\pi/3}\left[\frac{1}{2}r^2\right]_{\csc\theta}^{2} d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/3}\left(4-\csc^2\theta\right)d\theta$$

$$= \frac{1}{2}\left[4\theta+\cot\theta\right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left[\frac{4\pi}{3}+\frac{1}{\sqrt{3}}-\left(\frac{4\pi}{6}+\sqrt{3}\right)\right]$$

$$= \frac{1}{2}\left(\frac{2\pi}{3}+\frac{\sqrt{3}}{3}-\sqrt{3}\right)$$

$$= \frac{1}{2}\left(\frac{2\pi-2\sqrt{3}}{3}\right)$$

$$= \frac{\pi-\sqrt{3}}{3} \quad unit^2$$

Exercises *Section* 3.3 – Double Integrals in Polar Coordinates

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

1. $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dydx$

2. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} \left(x^2+y^2\right)dxdy$

3. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx$

4. $\displaystyle\int_{0}^{6}\int_{0}^{y} xdxdy$

5. $\displaystyle\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}dydx$

6. $\displaystyle\int_{0}^{\ln 2}\int_{0}^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}dxdy$

7. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dxdy$

8. $\displaystyle\int_{1}^{2}\int_{0}^{\sqrt{2x-x^2}} \frac{1}{\left(x^2+y^2\right)^2}dydx$

9. Find the area of the region cut from the first quadrant by the curve $r=2(2-\sin 2\theta)^{1/2}$

10. Find the area of the region lies inside the cardioid $r=1+\cos\theta$ and outside the circle $r=1$

11. Find the area enclosed by one leaf of the rose $r=12\cos 3\theta$

12. Find the area of the region common to the interiors of the cardioids $r=1+\cos\theta$ and $r=1-\cos\theta$

13. Integrate $f(x,y)=\dfrac{\ln\left(x^2+y^2\right)}{\sqrt{x^2+y^2}}$ over the region $1\le x^2+y^2\le e$

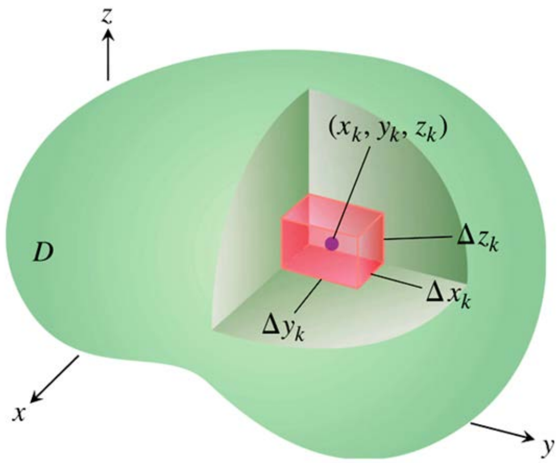
14. Evaluate the integral $\displaystyle\int_{0}^{\infty}\int_{0}^{\infty} \frac{1}{\left(1+x^2+y^2\right)^2}dxdy$

15. The region enclosed by the lemniscates $r^2=2\cos 2\theta$ is the base of a solid right cylinder whose top is bounded by the sphere $z=\sqrt{2-r^2}$. Find the cylinder's volume.

Section 3.4 – Triple Integrals

Triple Integrals

If $F(x,y,z)$ is a function defined on a closed, bounded region D in space, such a solid ball or a lump of clay, then the integral of F over D may be defined in the following way.



$$\Delta V_k = \Delta x_k \Delta y_k \Delta z_k \quad \to \quad S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta V_k$$

The limit of this summation is the triple integral of F over D

$$\lim_{n\to\infty} S_n = \iiint\limits_D F(x,y,z)\,dV \quad or \quad \lim_{\|P\|\to} S_n = \iiint\limits_D F(x,y,z)\,dxdydz$$

Volume of a region in Space

Definition

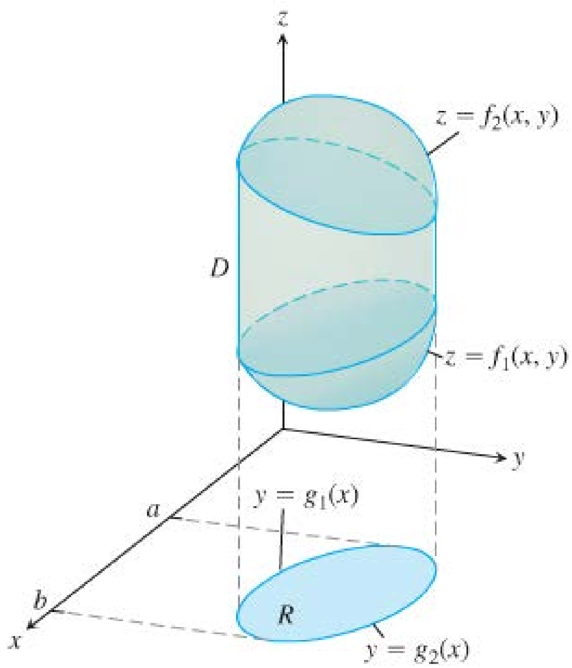
The volume of a closed, bounded region D in space is

$$V = \iiint\limits_D dV$$

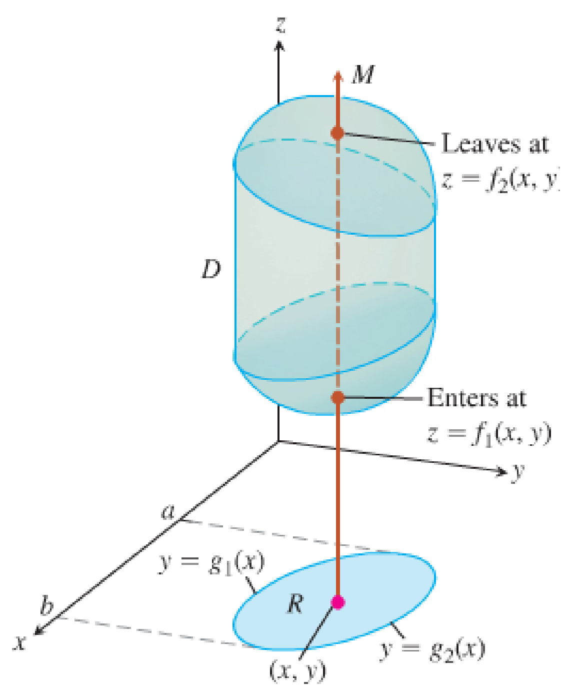
Find Limits of Integration in the Order *dz dy dx*

To evaluate $\iiint_D F(x,y,z)\,dV$

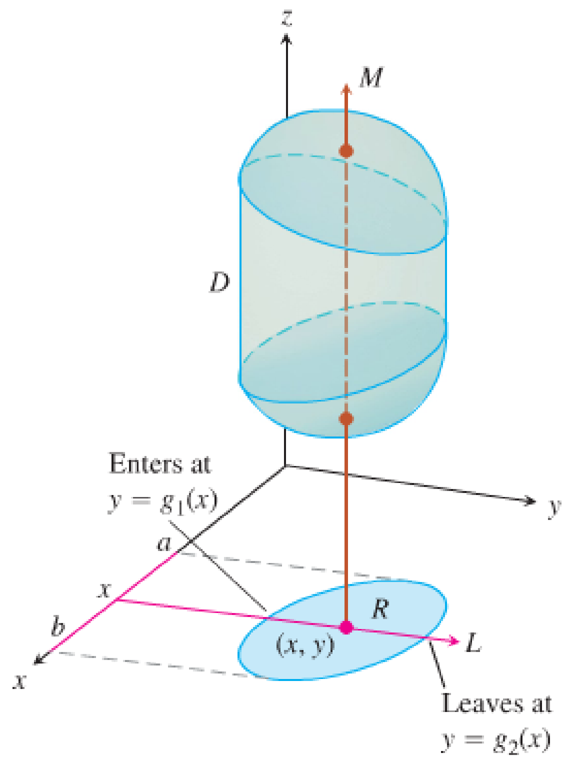
1. ***Sketch***: Sketch the region D along with its "shadow" R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and R.



2. ***Find the z-limits of integration***: Draw a line M passing through (x,y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x,y)$ and leaves at $z = f_2(x,y)$.



3. ***Find the y-limits of integration***: Draw a line L passing through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$.



4. ***Find the x-limits of integration***: Choose x-limits that include all lines through R parallel to the y-axis $(x = a \;\; and \;\; x = b)$.

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dzdydx$$

Example

Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

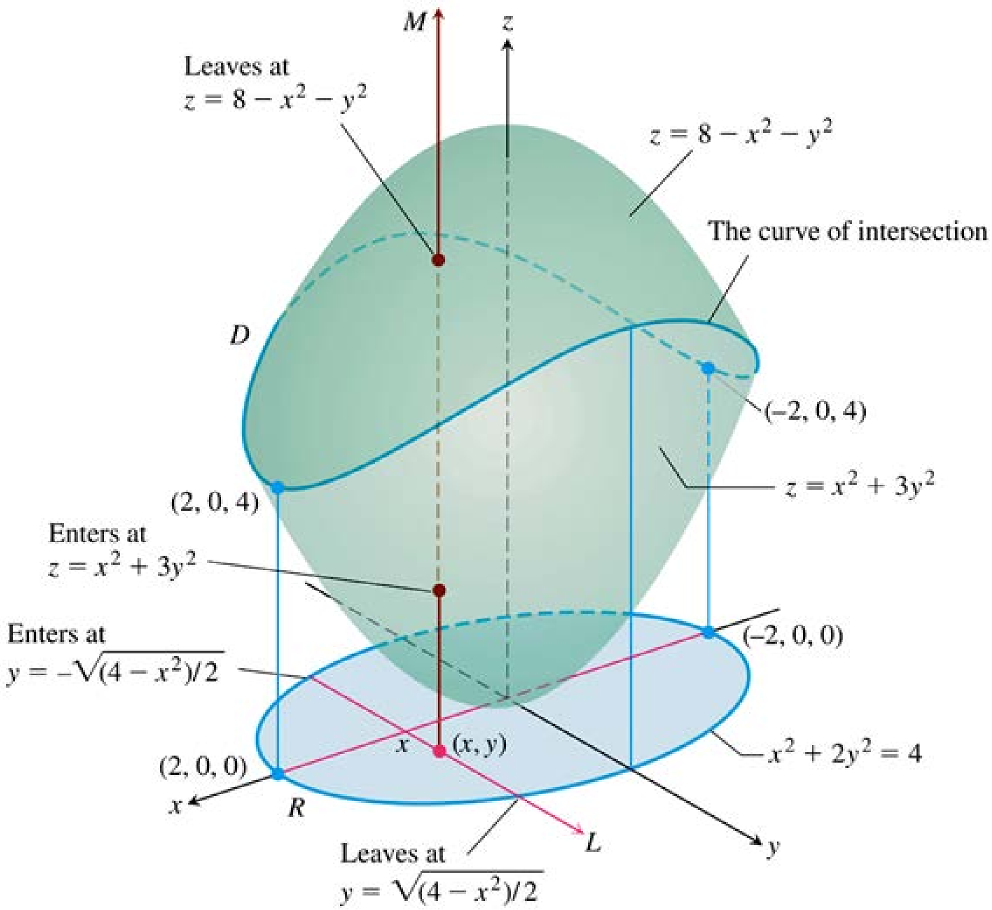
Solution

z-limits: $x^2 + 3y^2 \le z \le 8 - x^2 - y^2$

y-limits: $z = x^2 + 3y^2 = 8 - x^2 - y^2 \quad \rightarrow \quad 2x^2 + 4y^2 = 8 \;\Rightarrow\; x^2 + 2y^2 = 4$

$$y^2 = \frac{4 - x^2}{2} \quad \Rightarrow y = \pm\sqrt{\frac{4 - x^2}{2}} \quad \rightarrow -\sqrt{\frac{4 - x^2}{2}} \le y \le \sqrt{\frac{4 - x^2}{2}}$$

x-limits: $x^2 + 2y^2 = 4 \quad (y = 0) \quad \rightarrow x = \pm 2$



$$V=\int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dzdydx$$

$$=\int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\left[z\right]_{x^2+3y^2}^{8-x^2-y^2} dydx$$

$$=\int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\left(8-x^2-y^2-x^2-3y^2\right)dydx$$

$$=\int_{-2}^{2}\left[\left(8-2x^2\right)y-\frac{4}{3}y^3\right]_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}dx$$

$$=\int_{-2}^{2}\left[\left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}}-\frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2}+\left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}}-\frac{4}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right]dx$$

$$=\int_{-2}^{2}\left[2\left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right]dx$$

$$=\int_{-2}^{2}\left[2\left(\frac{2}{2}\right)(2)\left(4-x^2\right)\sqrt{\frac{4-x^2}{2}}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right]dx$$

$$=\int_{-2}^{2}\left[8\left(\frac{4-x^2}{2}\right)\left(\frac{4-x^2}{2}\right)^{1/2}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right]dx$$

$$=\int_{-2}^{2}\left[8\left(\frac{4-x^2}{2}\right)^{3/2}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right]dx$$

$$=\int_{-2}^{2}\left[\frac{16}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right]dx$$

$$=\frac{16}{3(2)^{3/2}}\int_{-2}^{2}\left(4-x^2\right)^{3/2}dx \qquad \frac{16}{3(2)^{3/2}}\frac{2^{1/2}}{2^{1/2}}=\frac{16\sqrt{2}}{3\cdot 4}=\frac{4\sqrt{2}}{3}$$

$$x=2\sin u \quad dx=2\cos u\,du \quad \left(4-x^2=4-4\sin^2 u=4\cos^2 u\right)$$

$$\begin{cases} x=2 & \to u=\sin^{-1}\frac{x}{2}=\sin^{-1}1=\frac{\pi}{2} \\ x=-2 & \to u=\sin^{-1}(-1)=-\frac{\pi}{2} \end{cases}$$

$$=\frac{4\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}\left(4\cos^2 u\right)^{3/2}\left(2\cos u\ du\right)$$

$$=\frac{4\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}16(\cos u)^3(\cos u)du$$

$$=\frac{64\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}\cos^4 u\ du$$

$$=\frac{64\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2u}{2}\right)^2 du$$

$$=\frac{16\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}\left(1+2\cos 2u+\cos^2 2u\right)du$$

$$=\frac{16\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}\left(1+2\cos 2u+\frac{1}{2}+\frac{1}{2}\cos 4u\right)du$$

$$=\frac{16\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}\left(\frac{3}{2}+2\cos 2u+\frac{1}{2}\cos 4u\right)du$$

$$=\frac{16\sqrt{2}}{3}\left[\frac{3}{2}u+\sin 2u+\frac{1}{8}\sin 4u\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{16\sqrt{2}}{3}\left[\frac{3\pi}{4} + \sin\pi + \frac{1}{8}\sin 2\pi - \left(-\frac{3\pi}{4} - \sin\pi - \frac{1}{8}\sin 2\pi\right)\right]$$

$$= \frac{16\sqrt{2}}{3}\left(\frac{3\pi}{2}\right)$$

$$\underline{= 8\pi\sqrt{2} \quad unit^3}$$

Example

Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D with vertices (0, 0, 0), (1, 1, 0), (0, 1, 0), and (0, 1, 1). Use the order of integration $dydzdx$.

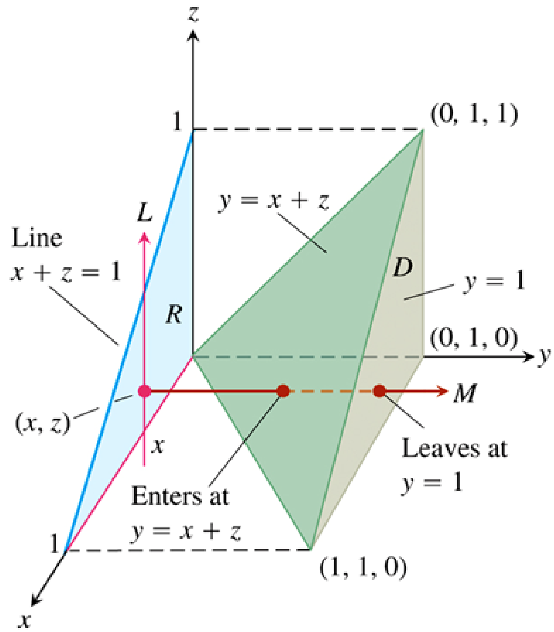
Solution

From the sketch, the upper (right-hand) bounding surface of D lies in the plane $y = 1$.

The lower (left-hand) bounding surface lies in the plane $y = x + z$.

The upper boundary of R is the line $z = 1 - x$.

The lower boundary is the line $z = 0$.



y-limits: The line through $(x,\ z)$ in R parallel to the y-axis enters D at $y = x + z$ and leaves at $y = 1$.

z-limits: The line through $(x,\ z)$ in R parallel to the z-axis enters R at $z = 0$ and leaves at $z = 1 - x$.

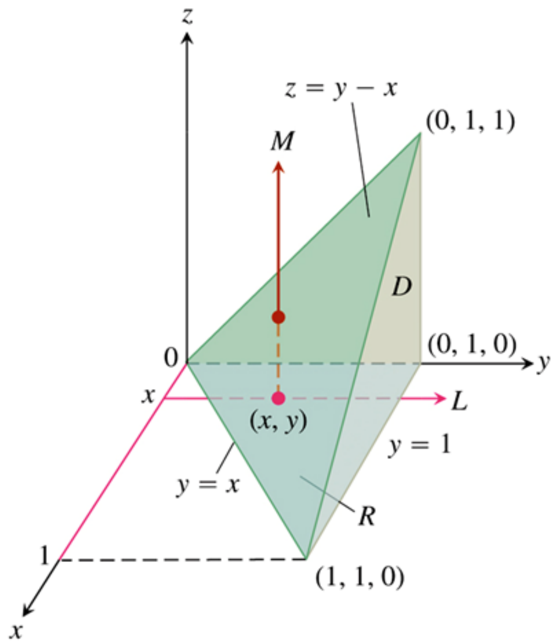
x-limits: $0 \le x \le 1$

$$\int_0^1 \int_0^{1-x} \int_{x+z}^{1} F(x, y, z)\, dydzdx$$

Example

Integrate $F(x,y,z)=1$ over the tetrahedron D in the previous example in the order $dz\ dy\ dx$, and then integrate in the order $dy\ dz\ dx$.

Solution



z-limits of integration: A line M parallel to the z-axis through a typical point (x, y) in the xy-plane "shadow" enters the tetrahedron at $z = 0$ and exists through the upper plane where $z = y - x$. $0 \le z \le y - x$

Line is given by: $ax + by + cz = 0$ passes through the 2 points:

$(1,1,0) \to a + b = 0 \quad \Rightarrow \quad a = -b$

and $(0,1,1) \to b + c = 0 \quad \Rightarrow \quad c = -b$

$\to -bx + by - bz = 0$

$-x + y - z = 0 \quad \Rightarrow \quad z = y - x$

y-limits of integration: On the xy-plane, where $z = 0$, the sloped side of the tetrahedron crosses the plane along the line $y = x$. A line L through (x, y) parallel to the y-axis enters the shadow in the xy-plane at $y = x$ and exists at $y = 1$. $x \le y \le 1$

x-limits of integration: A line L parallel to the y-axis through a typical point (x, y) in the xy-plane sweeps out the shadow, where $0 \le x \le 1$ at the point $(1,1,0)$

The integral is: $\displaystyle\int_0^1 \int_x^1 \int_0^{y-x} F(x,y,z)\,dzdydx$

$$V=\int_0^1\int_x^1\int_0^{y-x} dzdydx$$
$$=\int_0^1\int_x^1 [z]_0^{y-x}\, dydx$$
$$=\int_0^1\int_x^1 (y-x)\,dydx$$
$$=\int_0^1\left[\tfrac{1}{2}y^2-xy\right]_x^1 dx$$
$$=\int_0^1\left[\tfrac{1}{2}-x-\left(\tfrac{1}{2}x^2-x^2\right)\right]dx$$
$$=\int_0^1\left(\tfrac{1}{2}-x+\tfrac{1}{2}x^2\right)dx$$
$$=\left[\frac{1}{2}x-\frac{1}{2}x^2+\frac{1}{6}x^3\right]_0^1$$
$$=\frac{1}{2}-\frac{1}{2}+\frac{1}{6}$$
$$=\frac{1}{6}\quad unit^3$$

$$V=\int_0^1\int_0^{1-x}\int_{x+z}^{1} dydzdx$$
$$=\int_0^1\int_0^{1-x} [y]_{x+z}^{1}\, dzdx$$
$$=\int_0^1\int_0^{1-x} (1-x-z)\,dzdx$$
$$=\int_0^1\left[z-xz-\tfrac{1}{2}z^2\right]_0^{1-x} dx$$
$$=\int_0^1\left(1-x-x(1-x)-\tfrac{1}{2}(1-x)^2\right)dx$$
$$=\int_0^1\left((1-x)(1-x)-\tfrac{1}{2}(1-x)^2\right)dx$$

$$= \int_0^1 \left((1-x)^2 - \frac{1}{2}(1-x)^2 \right) dx$$

$$= \int_0^1 \frac{1}{2}(1-x)^2 \, dx$$

$$= -\frac{1}{6}(1-x)^3 \Big|_0^1$$

$$= \frac{1}{6} \quad unit^3$$

Average Value of a Function in Space

The average value of a function F over a region D in space is defined by the formula

$$\textbf{\textit{Average value}} \text{ of } F \text{ over } D = \frac{1}{\text{volume of } D}\iiint\limits_D F\,dV$$

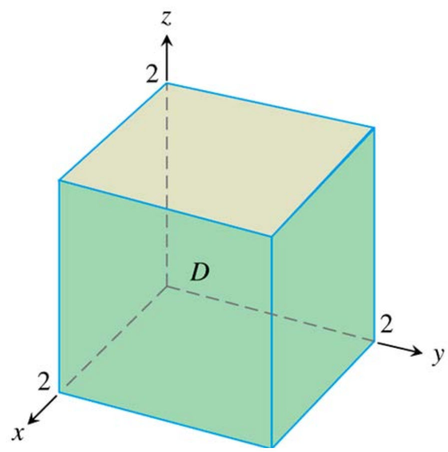
Example

Find the average of $F(x,y,z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x=2$, $y=2$, and $z=2$ in the first octant.

Solution

$Volume = 2\cdot 2\cdot 2 = 8$

The value of the integral of F over the cube is



$$V = \int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz$$

$$= \int_0^2 z\,dz \int_0^2 y\,dy \int_0^2 x\,dx$$

$$= \left[\frac{1}{2}z^2\right]_0^2 \left[\frac{1}{2}y^2\right]_0^2 \left[\frac{1}{2}x^2\right]_0^2$$

$$= \frac{1}{8}(4)(4)(4)$$

$$= 8 \quad unit^3$$

$$\textbf{\textit{Average value}} \text{ of } xyz \text{ over } cube = \frac{1}{\text{volume of } D}\iiint\limits_{cube} xyz\,dV$$

$$= \left(\frac{1}{8}\right)(8)$$

$$= 1$$

Exercises Section 3.4 – Triple Integrals

Evaluate the integral

1. $\displaystyle\int_0^1\int_0^1\int_0^1 \left(x^2+y^2+z^2\right)dzdydx$

2. $\displaystyle\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dzdxdy$

3. $\displaystyle\int_0^{\pi/6}\int_0^1\int_{-2}^3 y\sin z\ dxdydz$

4. $\displaystyle\int_{-1}^1\int_0^1\int_0^2 (x+y+z)dydxdz$

5. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} dzdydx$

6. $\displaystyle\int_0^1\int_0^{1-x^2}\int_0^{4-x^2-y} xdzdydx$

7. $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \cos(u+v+w)dudvdw$

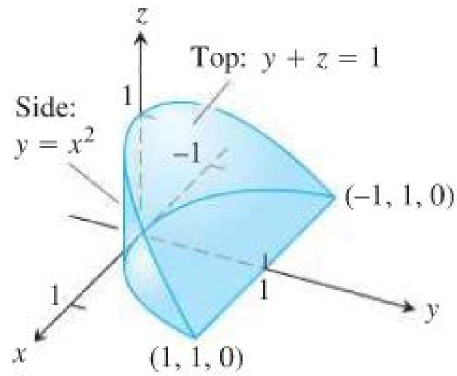
8. $\displaystyle\int_0^{\pi/4}\int_0^{\ln\sec v}\int_{-\infty}^{2t} e^x dxdtdv$

9. Here is the region of integration of the integral

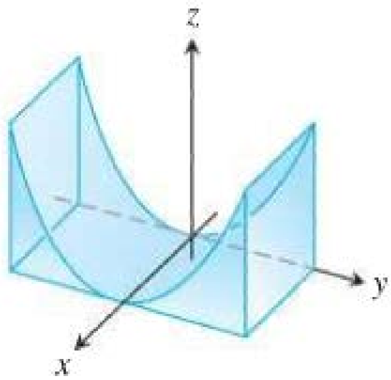
$$\int_{-1}^1\int_{x^2}^1\int_0^{1-y} dzdydx$$

a) *dydzdx* *b*) *dydxdz* *c*) *dxdydz*

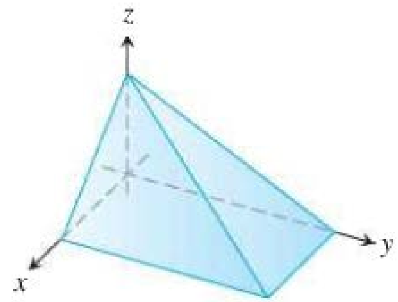
d) *dxdzdy* *e*) *dzdxdy*



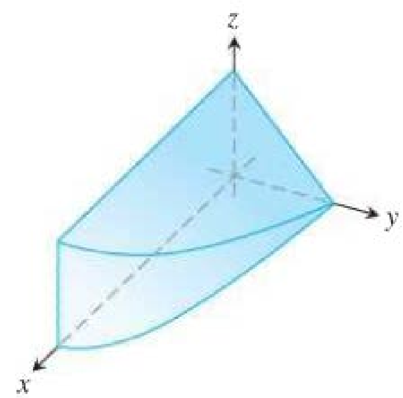
10. Find the volumes of the region between the cylinder $z=y^2$ and the *xy*-plane that is bounded by the planes $x=0,\quad x=1,\quad y=-1,\quad y=1$



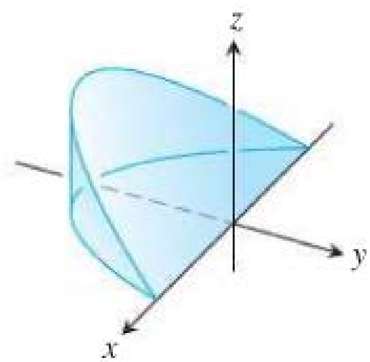
11. Find the volumes of the region in the first octant bounded by the coordinate planes and the planes $x + z = 1, \quad y + 2z = 2$



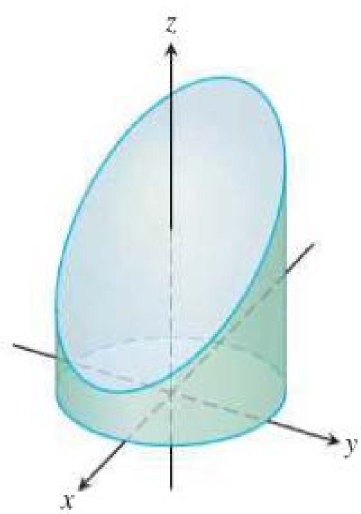
12. Find the volumes of the region in the first octant bounded by the coordinate planes and the plane $y + z = 2$, and the cylinder $x = 4 - y^2$



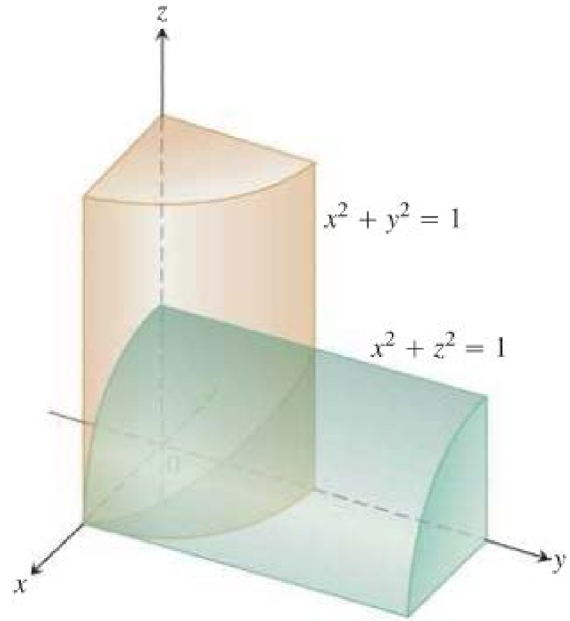
13. Find the volumes of the wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y, \quad z = 0$



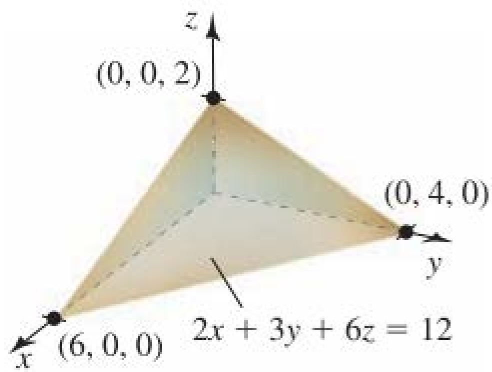
14. Find the volumes of the region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$



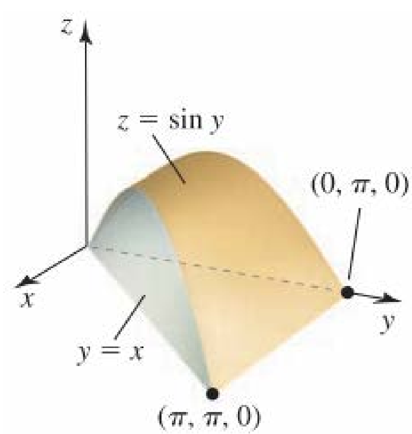
15. Find the volumes of the region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown below



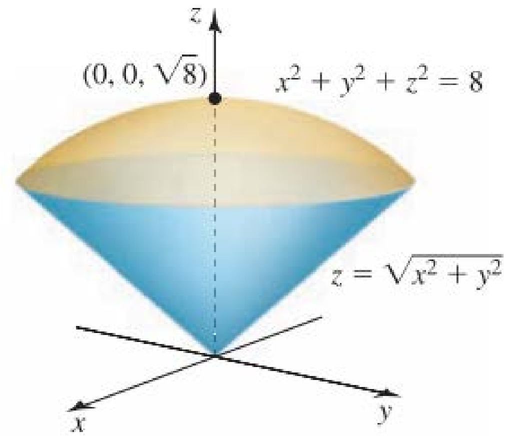
16. Find the volume of the solid in the first octant bounded by the plane $2x + 3y + 6z = 12$ and the coordinate planes



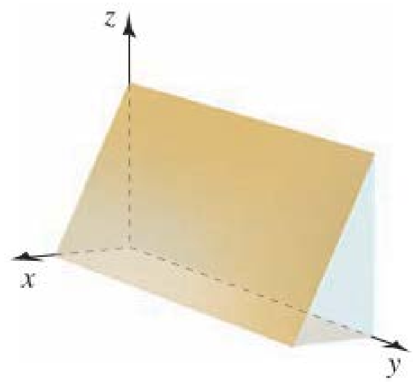
17. Find the volume of the solid in the first octant formed when the cylinder $z = \sin y$, for $0 \le y \le \pi$, is sliced by the planes $y = x$ and $x = 0$



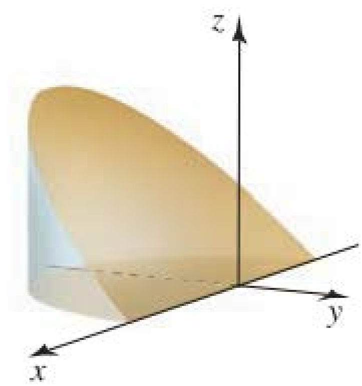
18. Find the volume of the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and bounded above the sphere $x^2 + y^2 + z^2 = 8$



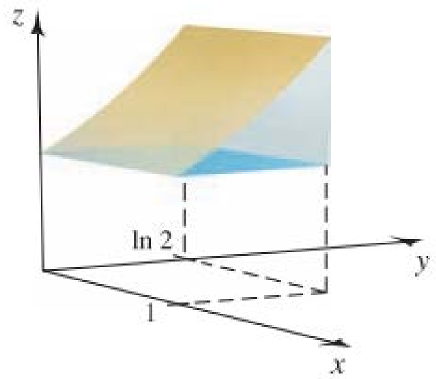
19. Find the volume of the prism in the first octant bounded below by $z = 2 - 4x$ and $y = 8$



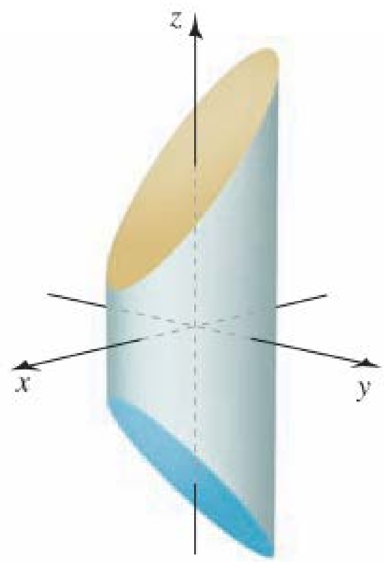
20. Find the volume of the wedge above the *xy*-plane formed when the cylinder $x^2 + y^2 = 4$ is cut by the planes $z = 0$ and $y = -z$



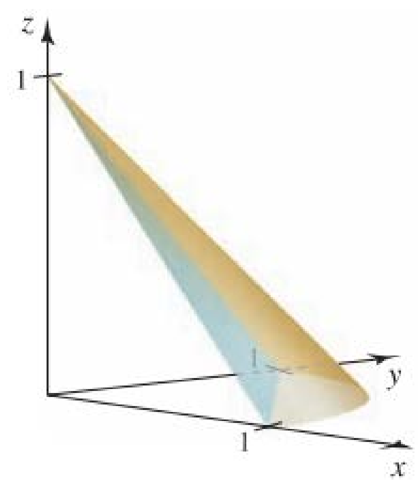
21. Find the volume of the solid bounded by the surfaces $z = e^y$ and $z = 1$ over the rectangle $\{(x, y): 0 \le x \le 1, 0 \le y \le \ln 2\}$



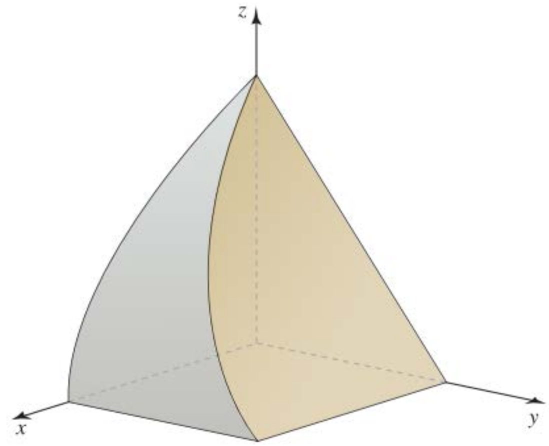
22. Find the volume of the wedge of the cylinder $x^2 + 4y^2 = 4$ created by the planes $z = 3 - x$ and $z = x - 3$



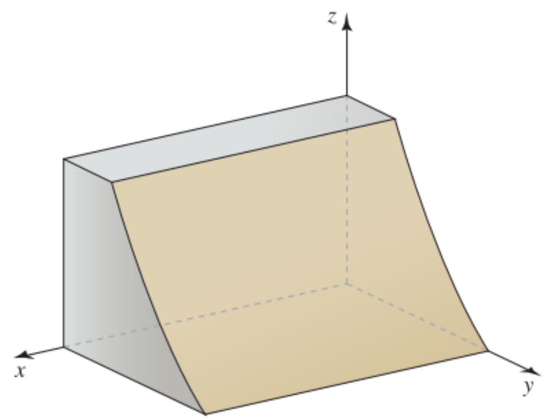
23. Find the volume of the solid in the first octant bounded by the cone $z = 1 - \sqrt{x^2 + y^2}$ and the plane $x + y + z = 1$



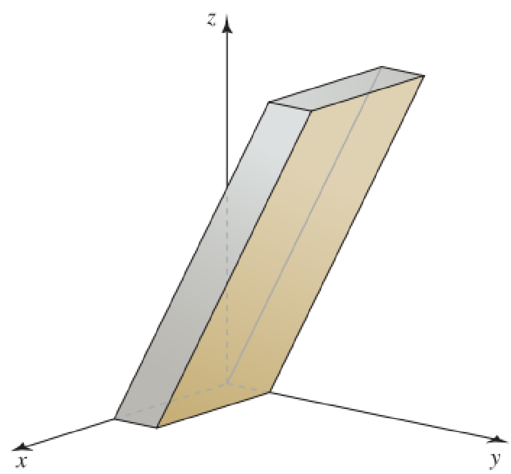
24. Find the volume of the solid bounded by $x = 0$, $x = 1 - z^2$, $y = 0$, $z = 0$, and $z = 1 - y$



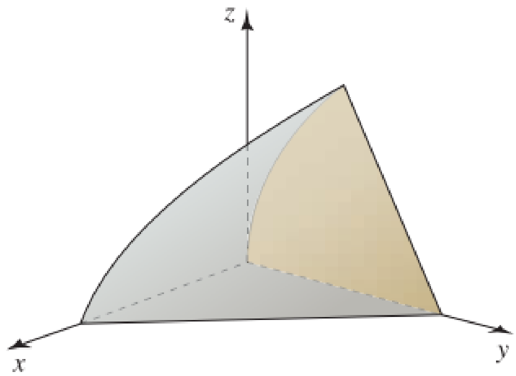
25. Find the volume of the solid bounded by $x = 0$, $x = 2$, $y = 0$, $y = e^{-z}$, $z = 0$, and $z = 1$



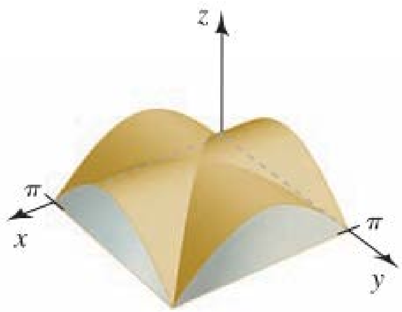
26. Find the volume of the solid bounded by $x = 0$, $x = 2$, $y = z$, $y = z + 1$, $z = 0$, and $z = 4$



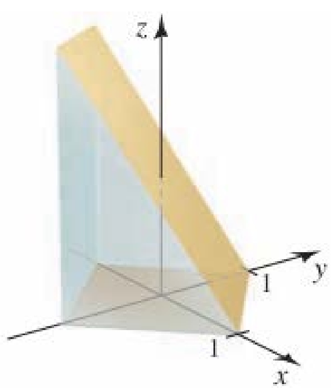
27. Find the volume of the solid bounded by $x = 0$, $y = z^2$, $z = 0$, and $z = 2 - x - y$



28. Find the volume of the solid common to the cylinders $z = \sin x$ and $z = \sin y$ over the square $R = \{(x, y):\ 0 \le x \le \pi,\ 0 \le y \le \pi\}$



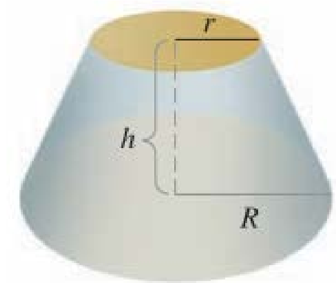
29. Find the volume of the wedge of the square column $|x| + |y| = 1$ created by the planes $z = 0$ and $x + y + z = 1$



30. Find the volume of a right circular cone with height h and base radius r.

31. Find the volume of a tetrahedron whose vertices are located at $(0,\ 0,\ 0)$, $(a,\ 0,\ 0)$, $(0,\ b,\ 0)$, and $(0,\ 0,\ c)$

32. Find the volume of a truncated cone of height h whose ends have radii r and R.



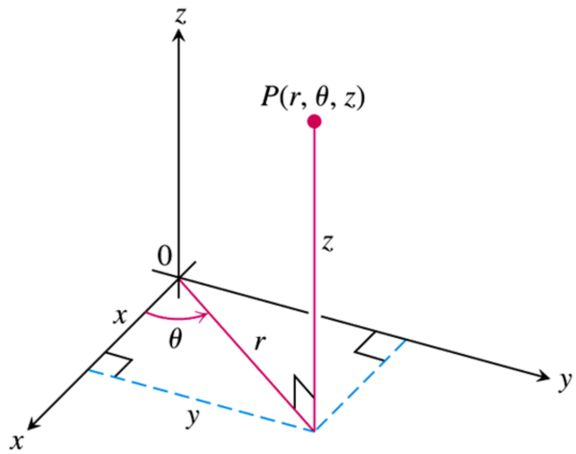
Section 3.5 – Triple Integrals in Cylindrical and Spherical Coordinates

Integration in Cylindrical Coordinates

Definition

Cylindrical coordinates represents a point P in space by ordered triples (r,θ,z) in which

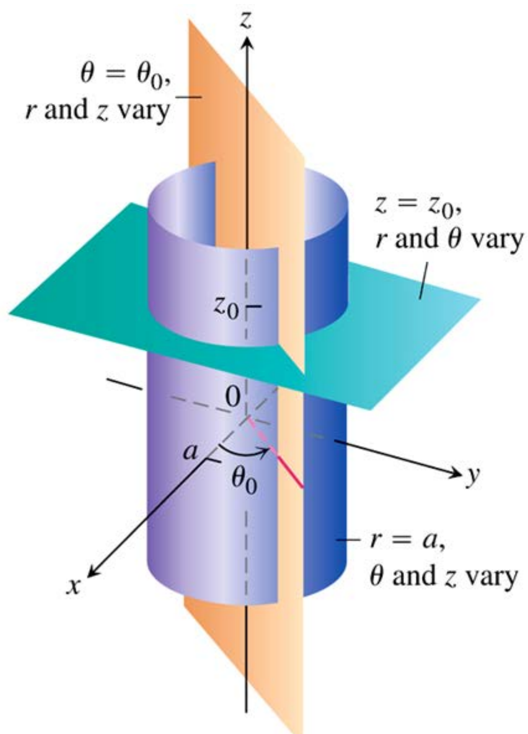
1. r *and* θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.



Equations Reating Rectangular (x,y,z) and Cylindrical (r,θ,z) Coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$



The triple integral of a function f over D is obtained by taking a limit of such Riemann sums with partitions whose norms approach zero:

$$\lim_{n\to\infty} S_n = \iiint_D f dV = \iiint_D f \ dz \ r dr d\theta$$

Example

Find the limits of integration in cylindrical coordinates for integrating a function $f(r,\theta,z)$ over the region *D* bounded below by the plane $z=0$, laterally by the circular cylinder $x^2+(y-1)^2=1$, and above by the paraboloid $z=x^2+y^2$.

Solution

Base of *D* is the region's projection *R* on the *xy*-plane.

The boundary of *R* is the circle $x^2+(y-1)^2=1$.

The polar coordinate equation is

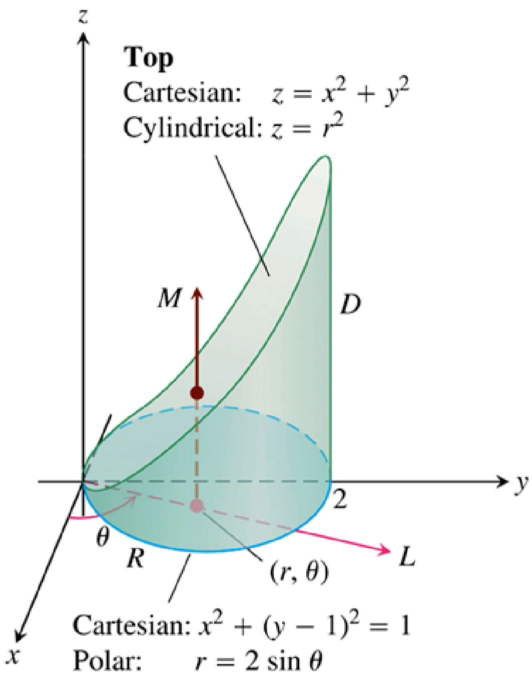
$$x^2+(y-1)^2=1$$

$$x^2+y^2-2y+1=1$$

$$r^2-2r\sin\theta=0$$

$$r(r-2\sin\theta)=0$$

$$r=2\sin\theta$$



z-limits: A line *M* through a typical point $(r,\ \theta)$ in *R* // *z*-axis enters *D* at $z=0$ and leaves at $z=x^2+y^2=r^2$

r-limits: starts at $r=0$ and ends at $r=2\sin\theta$

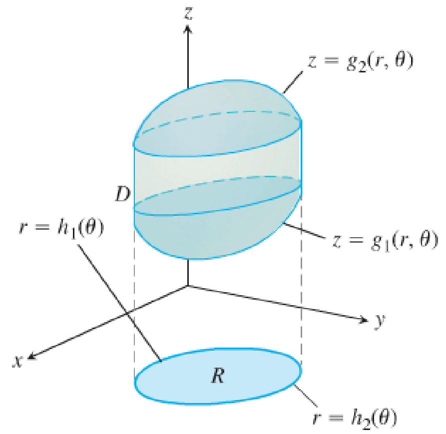
θ-limits: From $\theta=0$ to $\theta=\pi$

$$\iiint_D f\ dz\ r dr d\theta = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} f(r,\theta,z)\,dz\ r\ dr d\theta$$

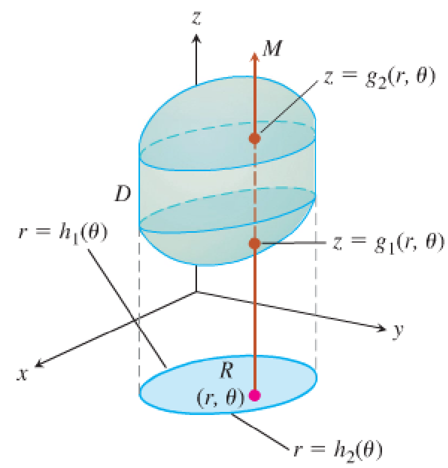
How to integrate in Cylindrical Coordinates

To evaluate $$\iiint\limits_{D} F(r,\ \theta,\ z)\,dV$$

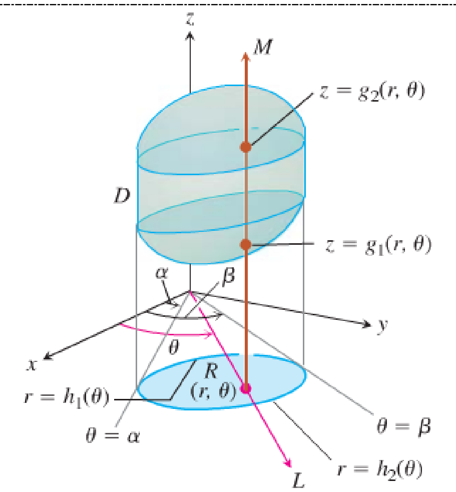
1. ***Sketch***: Sketch the region D along with its projection R on the xy-plane. Label the upper and lower bounding surfaces of D and R.



2. ***Find the z-limits of integration***: Draw a line M passing through (r,θ) in R // z-axis. As z increases, M enters D at $z = g_1(r,\theta)$ to $z = g_2(r,\theta)$.



3. ***Find the r-limits of integration***: Draw a line L passing through (r,θ) from the origin. From $r = h_1(\theta)$ to $r = h_2(\theta)$.



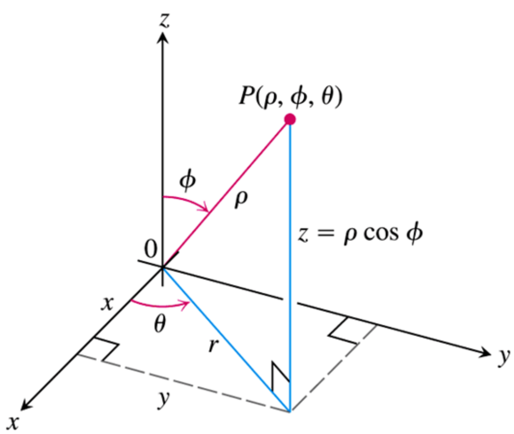
4. ***Find the θ-limits of integration***: As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ *and* $\theta = \beta$.

$$\iiint\limits_{D} F(r,\ \theta,\ z)\,dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)}\int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} F(r,\theta,z)\,dz\ r\,dr\,d\theta$$

Definition

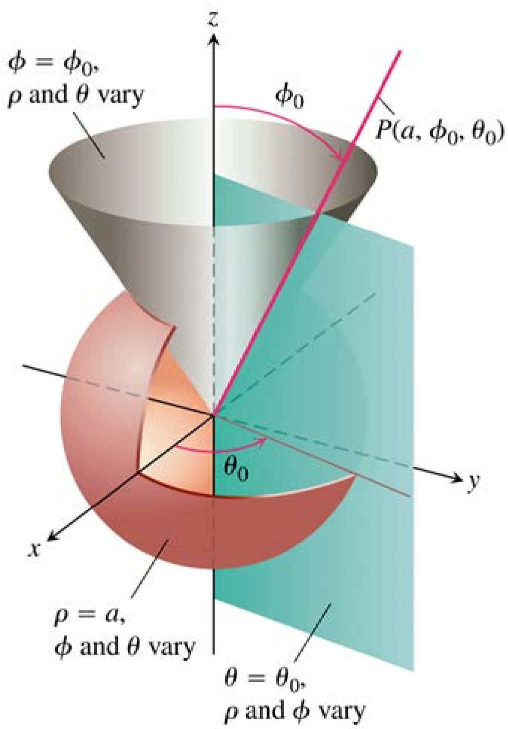
Spherical coordinates represent a point P in space by ordered triple $(\rho,\ \phi,\ \theta)$ in which

1. ρ is the distance from P to the origin.
2. ϕ is the angle $\overrightarrow{OP}$ makes with positive z-axis $(0 \le \phi \le \pi)$.
3. θ is the angle from the cylindrical coordinates $(0 \le \theta \le 2\pi)$



Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$r = \rho\sin\phi, \quad x = r\cos\theta = \rho\sin\phi\cos\theta,$$
$$z = \rho\cos\phi, \quad y = r\sin\theta = \rho\sin\phi\sin\theta,$$
$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$$



Example

Find a spherical coordinate equation for the sphere $x^2+y^2+(z-1)^2=1$

Solution

$$x^2+y^2+(z-1)^2=1$$

$$\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta+(\rho\cos\phi-1)^2=1$$

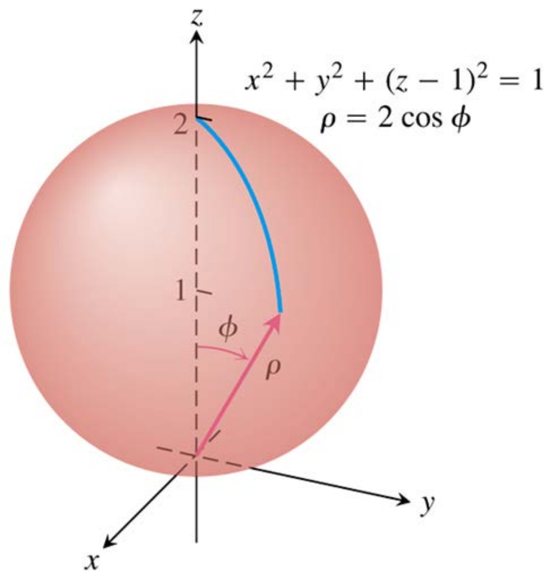
$$\rho^2\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right)+\rho^2\cos^2\phi-2\rho\cos\phi+1=1 \qquad \cos^2\theta+\sin^2\theta=1$$

$$\rho^2\left(\sin^2\phi+\cos^2\phi\right)-2\rho\cos\phi=0$$

$$\rho^2-2\rho\cos\phi=0$$

$$\rho(\rho-2\cos\phi)=0 \qquad \rho>0$$

$$\boxed{\rho=2\cos\phi}$$



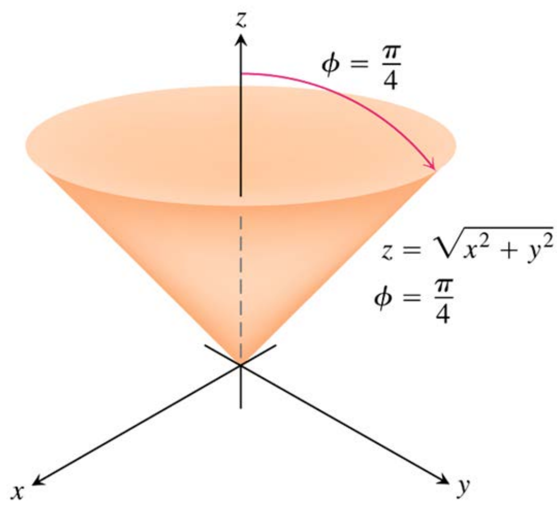
The angle ϕ varies from 0 to the north pole of the sphere to $\frac{\pi}{2}$ at the south pole; the angle θ doesn't appear in the expression for ρ, reflecting the symmetry about the *z*-axis.

Example

Find a spherical coordinate equation for the sphere $z=\sqrt{x^2+y^2}$

Solution

The cone is symmetric with respect to the *z*-axis and cuts the first quadrant of the yz-plane along the line $z=y$. The angle between the cone and the positive *z*-axis is therefore $\frac{\pi}{4}$ rad. The cone consists of the points whose spherical coordinates have $\phi=\frac{\pi}{4}$.



$$z=\sqrt{x^2+y^2}$$

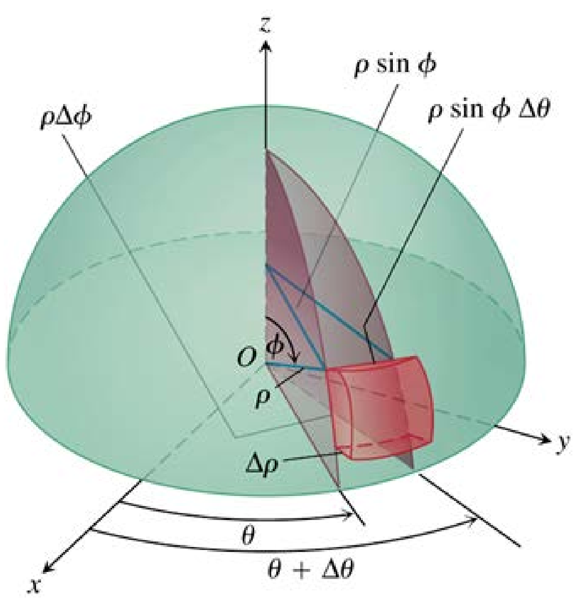
$$\rho\cos\phi=\sqrt{\rho^2\sin^2\phi}$$

$$\rho\cos\phi=\rho\sin\phi$$

$$\cos\phi=\sin\phi \quad \rightarrow \quad \boxed{\phi=\frac{\pi}{4}}$$

Volume Differential in Spherical Coordinates

$$dV=\rho^2\sin\phi\; d\rho\; d\phi\; d\theta$$



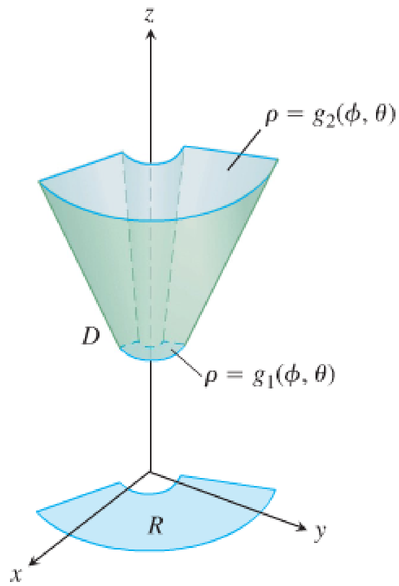
$$dV = d\rho\cdot\rho d\phi\cdot\rho\sin\phi d\theta=\rho^2\sin\phi\; d\rho\; d\phi\; d\theta$$

How to integrate in Spherical Coordinates

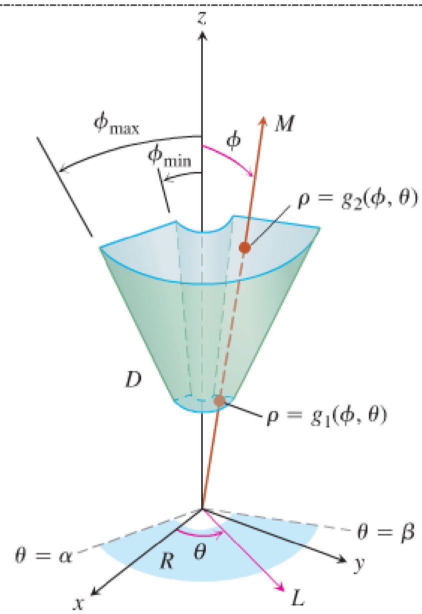
To evaluate $$\iiint_D F(\rho, \phi, \theta)\,dV$$

1. ***Sketch***: Sketch the region D along its projection R on the xy-plane. Label the surface that bound of D.

2. ***Find the ρ-limits of integration***: Draw a ray M from the origin through D making an angle ϕ with the positive z-axis. Also draw the projection of M on the xy-plane (call the projection L). The ray L makes an angle θ with the positive x-axis. As ρ increases, M enters D at $\rho = g_1(\phi,\theta)$ to $\rho = g_2(\phi,\theta)$.



3. ***Find the ϕ-limits of integration***: For the given θ, the angle ϕ that M makes with the z-axis runs $\phi = \phi_{\min}$ to $\phi = \phi_{\max}$.



5. ***Find the θ-limits of integration***: As L sweeps over R as θ runs from α to β.

$$\iiint_D f(\rho, \phi, \theta)\,dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}}\int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho, \phi, \theta)\rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

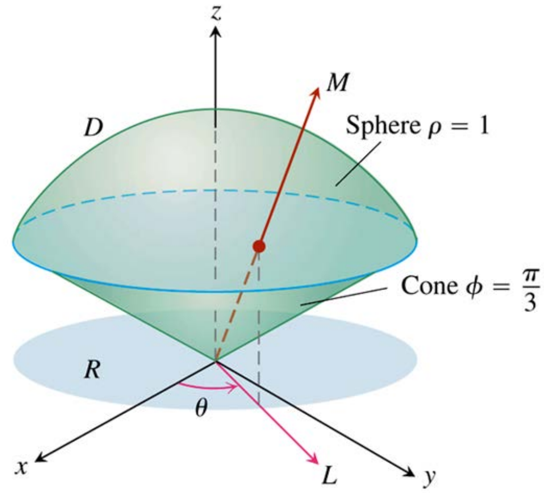
Example

Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \le 1$ by the cone $\phi = \frac{\pi}{3}$

Solution

$$f(\rho,\ \phi,\ \theta) = 1$$

$$V = \iiint\limits_D \rho^2 \sin\phi \ d\rho d\phi d\theta$$



ρ-limits: Draw a ray M from the origin through D making an angle ϕ with the positive z-axis. And L, the projection of M on the xy-plane, along with the angle θ that L makes with the positive x-axis. Ray M enters D form $\rho = 0$ to $\rho = 1$

ϕ-limits: The cone $\phi = \frac{\pi}{3}$ makes with the positive z-axis. $0 \le \phi \le \frac{\pi}{3}$

θ-limits: $0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \ d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{1}{3}\rho^3\right]_0^1 \sin\phi \ d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3}\sin\phi \ d\phi d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi} \left[\cos\phi\right]_0^{\pi/3} d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi} \left(\frac{1}{2} - 1\right) d\theta$$

$$= \frac{1}{6}\int_0^{2\pi} d\theta$$

$$= \frac{1}{6}\theta \Big|_0^{2\pi}$$

$$= \frac{1}{6}(2\pi - 0)$$

$$= \frac{\pi}{3} \quad unit^3$$

Coordinate Conversion Formulas

Cylindrical to Rectangular	*Spherical to Rectangular*	*Spherical to Cylindrical*
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\ dy\ dz \\ &= dz\ r dr\ d\theta \\ &= \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta \end{aligned}$$

Exercises *Section* 3.5 – Triple Integrals in Cylindrical and Spherical Coordinates

Evaluate the cylindrical coordinate integral

1. $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\, rdr\, d\theta$

2. $\displaystyle\int_0^{2\pi}\int_0^{\theta/(2\pi)}\int_r^{3+24r^2} dz\, rdr\, d\theta$

3. $\displaystyle\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} zdz\, rdr\, d\theta$

4. $\displaystyle\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}\left(r^2\sin^2\theta+z^2\right)dz\, rdr\, d\theta$

Evaluate the integral

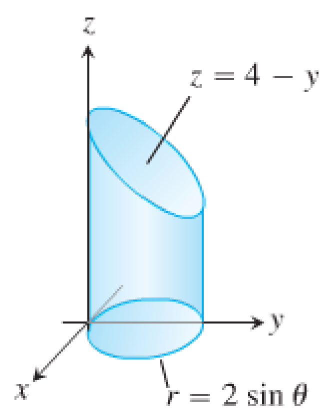
5. $\displaystyle\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3 dr\, dz\, d\theta$

6. $\displaystyle\int_0^1\int_0^{\sqrt{z}}\int_0^{2\pi}\left(r^2\cos^2\theta+z^2\right)r\, d\theta\, drdz$

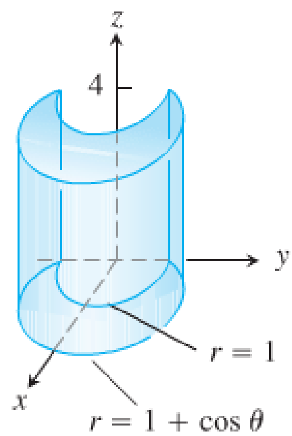
7. $\displaystyle\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi}(r\sin\theta+1)r\, d\theta\, dz\, dr$

8. Convert the integral $\displaystyle\int_{-1}^1\int_0^{\sqrt{1-y^2}}\int_0^x\left(x^2+y^2\right)dzdxdy$ to an equivalent integral in cylindrical coordinates and evaluate the result.

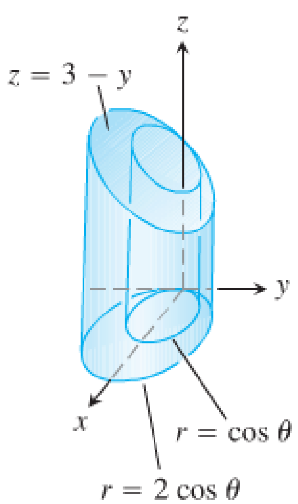
9. Set up the iterated integral for evaluating $\displaystyle\iiint_D f(r,\theta,z)\,dzdrd\theta$ over the region D that is the right circular cylinder whose base is the circle $r=2\sin\theta$ in the *xy*-plane and whose top lies in the plane $z=4-y$



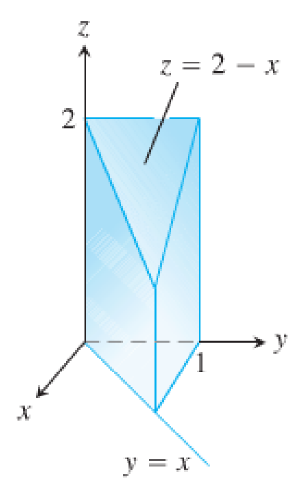
10. Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,dr\,d\theta$ over the region D which is the solid right cylinder whose base is the region in the xy-plane that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$



11. Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,dr\,d\theta$ over the region D which is the solid right cylinder whose base is the region between the circles $r = \cos\theta$ and $r = 2\cos\theta$ and whose top lies in the plane $z = 3 - y$



12. Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,dr\,d\theta$ over the region D which is the prism whose base is the triangle in the xy-plane bounded by the y-axis and the lines $y = x$ and $y = 1$ and whose top lies in the plane $z = 2 - x$



13. Evaluate the spherical coordinate integral $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

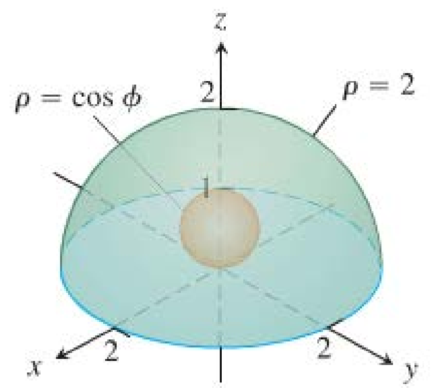
14. Evaluate the spherical coordinate integral $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} (\rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

15. Evaluate the spherical coordinate integral $\displaystyle\int_0^{3\pi/2}\int_0^{\pi}\int_0^{1} 5\rho^3 \sin^3\phi \, d\rho \, d\phi \, d\theta$

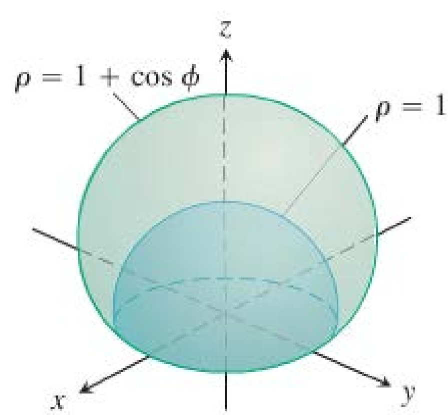
16. Evaluate the integral $\displaystyle\int_0^{2}\int_{-\pi}^{0}\int_{\pi/4}^{\pi/2} \rho^3 \sin 2\phi \, d\phi \, d\theta \, d\rho$

17. Evaluate the integral $\displaystyle\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2} 5\rho^4 \sin^3\phi \, d\rho \, d\theta \, d\phi$

18. Find the volume of the solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2, \;\; z \ge 0$

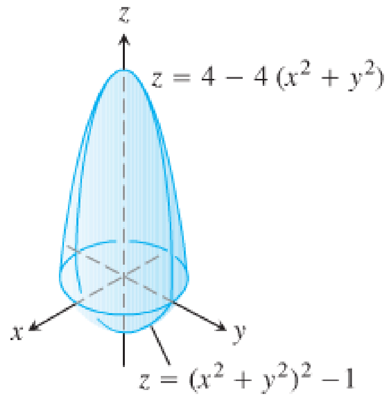


19. Find the volume of the solid bounded below by the hemisphere $\rho = 1, \;\; z \ge 0$, and above the cardioid of revolution $\rho = 1 + \cos\phi$

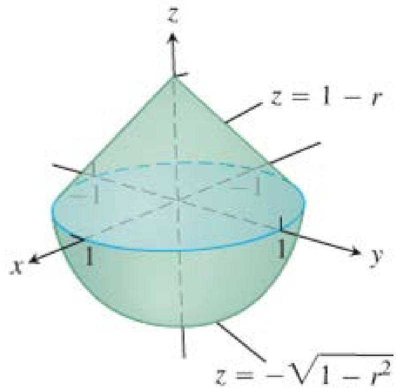


20. Find the volume of the solid

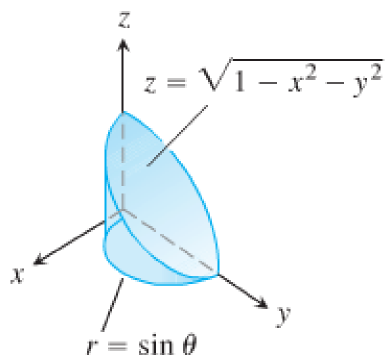
a)



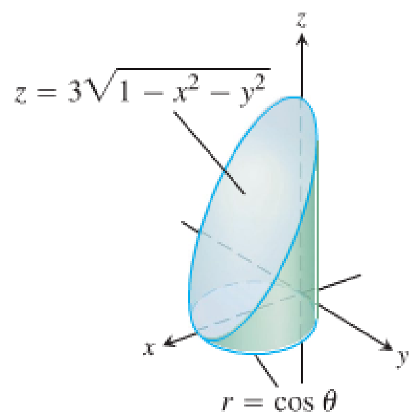
b)



c)



d)

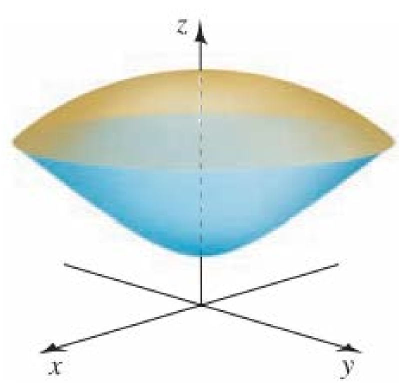


21. Find the volume of the smaller region cut from the solid sphere $\rho \le 2$ by the plane $z = 1$

22. Find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2 + 1$

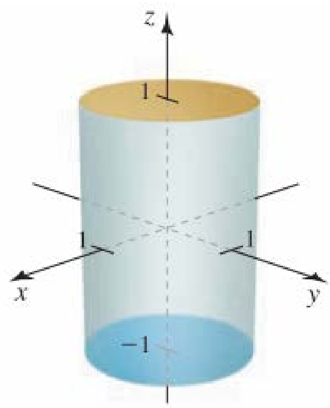
23. Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$

24. Find the volume of the solid between the sphere $x^2 + y^2 + z^2 = 19$ and the hyperboloid $z^2 - x^2 - y^2 = 1$ for $z > 0$



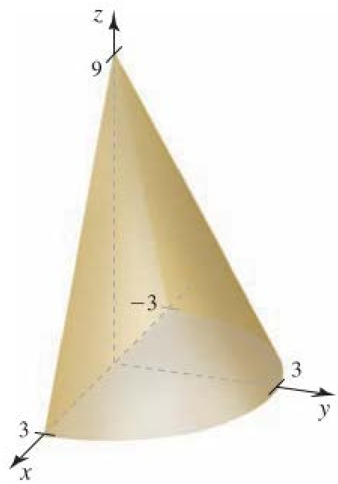
25. Evaluate the integral in cylindrical coordinates

$$\int_0^{2\pi}\int_0^1\int_{-1}^1 r\, dzdrd\theta$$



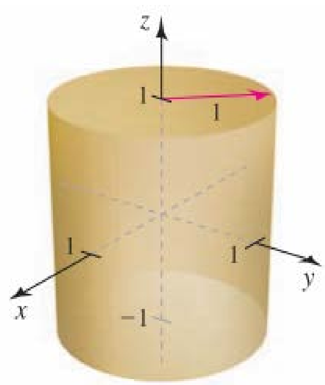
26. Evaluate the integral in cylindrical coordinates

$$\int_0^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_0^{9-3\sqrt{x^2+y^2}} dzdxdy$$



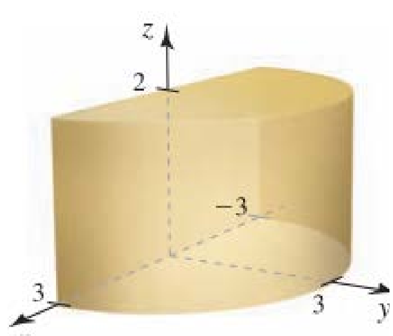
27. Evaluate the integral in cylindrical coordinates

$$\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{-1}^1 \left(x^2+y^2\right)^{3/2}\, dzdxdy$$

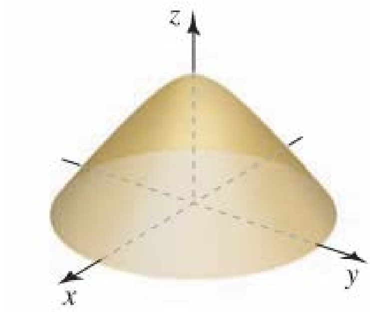


28. Evaluate the integral in cylindrical coordinates

$$\int_{-3}^3\int_0^{\sqrt{9-x^2}}\int_0^2 \frac{1}{1+x^2+y^2}\, dzdydx$$

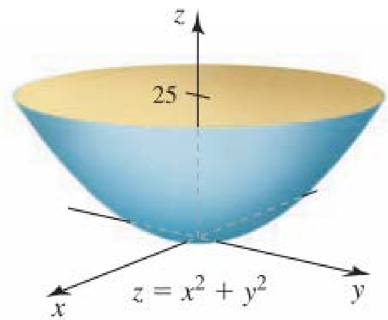


29. Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the plane $z = 0$ and the hyperboloid $z = \sqrt{17} - \sqrt{1 + x^2 + y^2}$

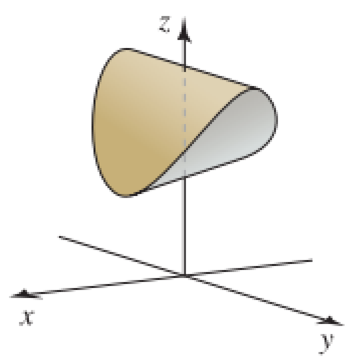


$z = \sqrt{17} - \sqrt{1 + x^2 + y^2}$

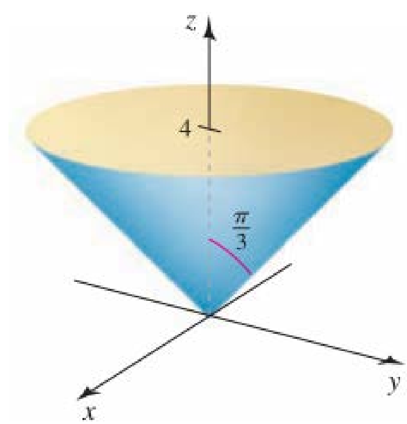
30. Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the plane $z = 25$ and the paraboloid $z = x^2 + y^2$



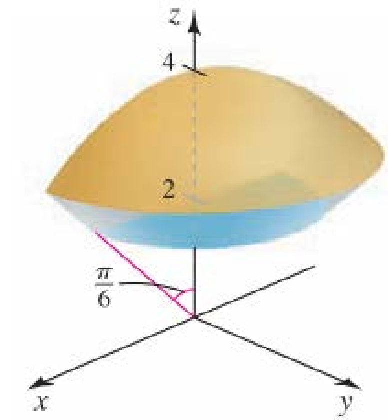
31. Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the parabolic cylinders $z = y^2 + 1$ and $z = 2 - x^2$



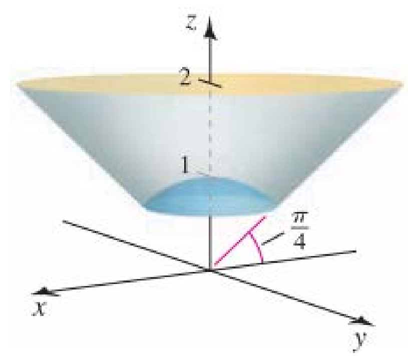
32. Evaluate the integral $\int_0^{2\pi} \int_0^{\pi/3} \int_0^{4\sec\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$



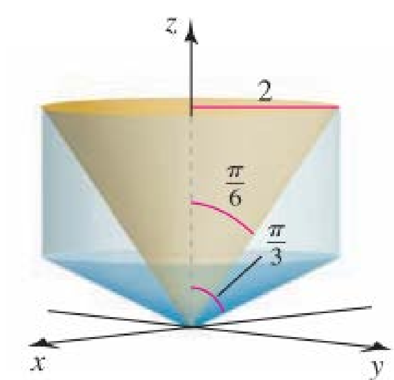
33. Evaluate the integral $\int_0^{\pi}\int_0^{\pi/6}\int_{2\sec\varphi}^{4} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$



34. Evaluate the integral $\int_0^{2\pi}\int_0^{\pi/4}\int_1^{2\sec\varphi} \left(\rho^{-3}\right)\rho^2 \sin\varphi \, d\rho d\varphi d\theta$

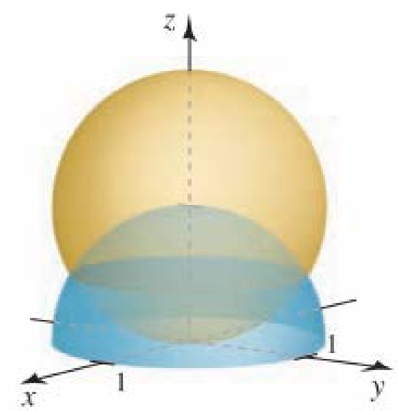


35. Evaluate the integral $\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^{2\csc\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$



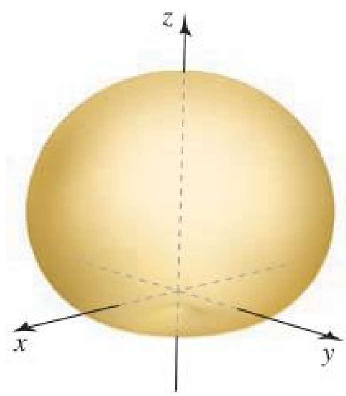
36. Use the spherical coordinates to find the volume of a ball of radius $a > 0$

37. Use the spherical coordinates to find the volume of the solid bounded by the sphere $\rho = 2\cos\varphi$ and the hemisphere $\rho = 1,\ z \geq 0$

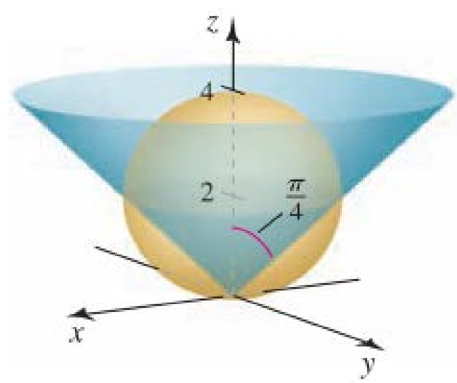


38. Use the spherical coordinates to find the volume of the solid of revolution

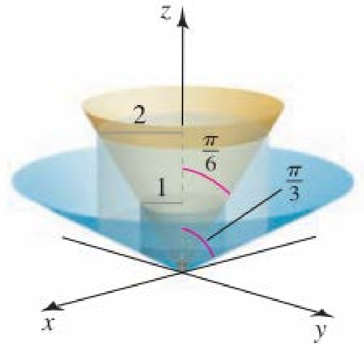
$D = \{(\rho, \varphi, \theta):\ 0 \leq \rho \leq 1 + \cos\varphi,\ 0 \leq \varphi \leq \pi,\ 0 \leq \theta \leq 2\pi\}$



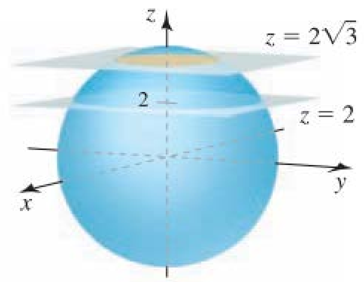
39. Use the spherical coordinates to find the volume of the solid outside the cone $\varphi = \frac{\pi}{4}$ and inside the sphere $\rho = 4\cos\varphi$



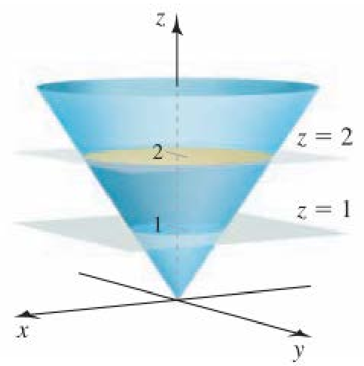
40. Use the spherical coordinates to find the volume of the solid bounded by the cylinders $r = 1$ and $r = 2$, and the cone $\varphi = \frac{\pi}{6}$ and $\varphi = \frac{\pi}{3}$



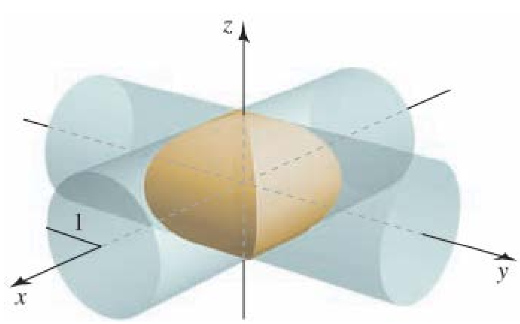
41. Use the spherical coordinates to find the volume of the ball $\rho \le 4$ that lies between the planes $z = 2$ and $z = 2\sqrt{3}$



42. Use the spherical coordinates to find the volume of the solid inside the cone $z = \left(x^2 + y^2\right)^{1/2}$ that lies between the planes $z = 1$ and $z = 2$



43. The *x*- and *y*-axes from the axes of two right circular cylinders with radius 1.



Find the volume of the solid that is common to the two cylinders.

Section 3.6 – Integrals for Mass Calculations

Mass and Moment Calculations

We treat coil springs and wires as masses distributed along smooth curves in space. The distribution is described by a continuous density function $\delta(x,y,z)$ representing mass per unit length. When a curve C is parametrized by $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}, \quad a \le t \le b$, the density is the function $\delta(x(t), y(t), z(t))$, and then the arc length differential is given by

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

The formula of mass is

$$M = m = \int_a^b \delta(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

Mass and moment formulas for coil springs, wires, and thin rods lying along a smooth curve C in space

Mass: $m = \int_C \delta ds \qquad \delta = \delta(x,y,z)$ *is the density at* (x,y,z)

First moments about the coordinates planes:

$$M_{yz} = \int_C x\delta ds, \quad M_{xz} = \int_C y\delta ds, \quad M_{xy} = \int_C z\delta ds$$

Coordinates of the center of mass:

$$\overline{x} = \frac{M_{yz}}{M}, \quad \overline{y} = \frac{M_{xz}}{M}, \quad \overline{z} = \frac{M_{xy}}{M}$$

Moments of inertia about axes and other lines:

$$I_x = \int_C \left(y^2 + z^2\right)\delta ds, \quad I_y = \int_C \left(x^2 + z^2\right)\delta ds, \quad I_z = \int_C \left(x^2 + y^2\right)\delta ds$$

$$I_L = \int_C r^2 \delta ds \qquad r(x,y,z) = \text{distance from the point } (x,y,z) \text{ to the line } L$$

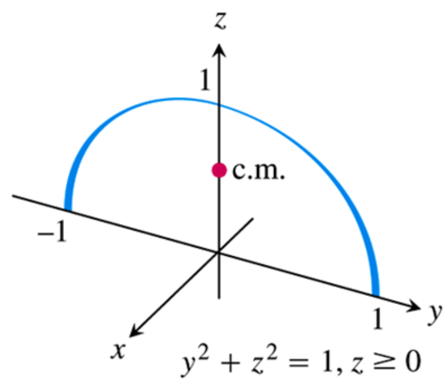
Example

A slender metal arch, denser at the bottom than top, lies along the semicircle $z^2+y^2=1,\ \ z\ge 0$, in the *yz*-plane. Find the center of the arch's mass if the density at the point (x,y,z) on the arch is $\delta(x,y,z)=2-z$

Solution

$\bar{x}=0$ *and* $\bar{y}=0$, because the arch lies in the *yz*-plane with its mass distributed symmetrically about the *z*-axis.

$$\boldsymbol{r}(t)=(\cos t)\,\boldsymbol{j}+(\sin t)\,\boldsymbol{k},\quad 0\le t\le\pi$$



$$\begin{aligned}|v(t)|&=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\\&=\sqrt{(0)^2+(-\sin t)^2+(\cos t)^2}\\&=\sqrt{\sin^2 t+\cos^2 t}\\&=1\end{aligned}$$

$$\Rightarrow ds=|v|\,dt=dt$$

$$\begin{aligned}m&=\int_0^\pi (2-z)\ dt\\&=\int_0^\pi (2-\sin t)\ dt\\&=\left[2t+\cos t\right]_0^\pi\\&=2\pi+\cos\pi-\cos 0\\&=2\pi-2\end{aligned}$$

$$\begin{aligned}M_{xy}&=\int_C z\delta ds\\&=\int_C z(2-z)ds\\&=\int_0^\pi(\sin t)(2-\sin t)\ dt\\&=\int_0^\pi\left(2\sin t-\sin^2 t\right)\ dt\\&=\left[-2\cos t-\frac{t}{2}+\frac{\sin 2t}{4}\right]_0^\pi\end{aligned}$$

$$= -2(-1) - \frac{\pi}{2} + 2$$

$$= 4 - \frac{\pi}{2}$$

$$\boxed{= \frac{8-\pi}{2}}$$

$$\bar{z} = \frac{M_{xy}}{m}$$

$$= \frac{8-\pi}{2} \cdot \frac{1}{2\pi - 2}$$

$$= \frac{8-\pi}{4\pi - 4}$$

$$\boxed{\approx 0.57}$$

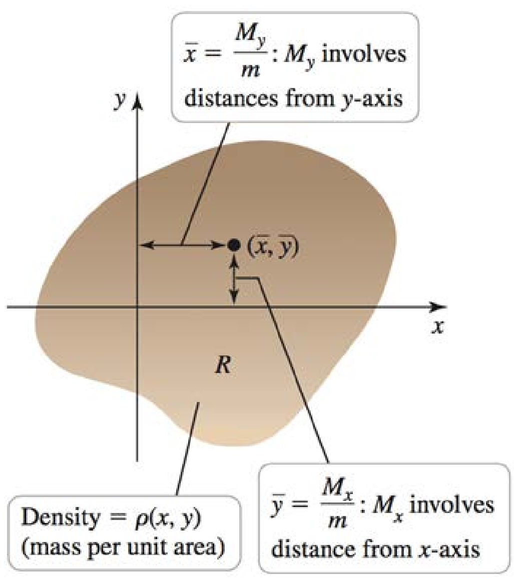
The center mass is $(0,\ 0,\ 0.57)$

Two-Dimensional Objects

Definition

Let ρ be an integrable area density function defined over a closed bounded region $\boldsymbol{R}$ in $\mathbb{R}^2$. The coordinates of the center of mass of the object represented by $\boldsymbol{R}$ are:

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_R x\rho(x,y)\,dA \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_R y\rho(x,y)\,dA$$



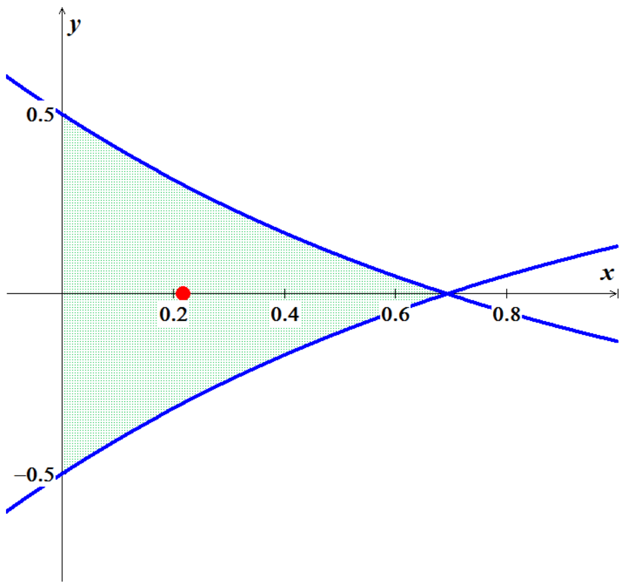
Where $m = \iint_R \rho(x,y)\,dA$ is the mass, and M_y and M_x are the moments with respect to the y-axis and x-axis, respectively. If ρ is constant, the center of mass is called the ***centroid*** and is independent of the density,

Example

Find the centroid (center of mass) of the constant density, dart-shaped region bounded by the y-axis and the curves $y = e^{-x} - \frac{1}{2}$ *and* $y = \frac{1}{2} - e^{-x}$

Solution

Assume: $\rho = 1$



$$m = \int_0^{\ln 2} \int_{\frac{1}{2}-e^{-x}}^{e^{-x}-\frac{1}{2}} 1\, dy\, dx$$

$$= \int_0^{\ln 2} y \Big|_{\frac{1}{2}-e^{-x}}^{e^{-x}-\frac{1}{2}} dx$$

$$= \int_0^{\ln 2} \left(2e^{-x} - 1\right) dx$$

$$= \left[-2e^{-x} - x\right]_0^{\ln 2}$$

$$= -2e^{-\ln 2} - \ln 2 + 2$$

$$= -2\left(\frac{1}{2}\right) - \ln 2 + 2$$

$$= 1 - \ln 2 \quad \approx 0.307$$

$$M_y = \int_0^{\ln 2} \int_{\frac{1}{2}-e^{-x}}^{e^{-x}-\frac{1}{2}} x\, dy\, dx$$

$$= \int_0^{\ln 2} xy \Big|_{\frac{1}{2}-e^{-x}}^{e^{-x}-\frac{1}{2}} dx$$

$$= \int_0^{\ln 2} x\left(e^{-x} - \frac{1}{2} - \frac{1}{2} + e^{-x}\right) dx$$

$$= \int_0^{\ln 2} x\left(2e^{-x} - 1\right) dx$$

$$\int xe^{ax}dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right)$$

		$\int e^{-x}$
+	x	$-e^{-x}$
–	1	e^{-x}

$$= \left[-2xe^{-x} - 2e^{-x} - \frac{1}{2}x^2\right]_0^{\ln 2}$$

$$= -2(\ln 2)\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right) - \frac{1}{2}(\ln 2)^2 + 0 + 2 + 0$$

$$= 1 - \ln 2 - \frac{1}{2}(\ln 2)^2 \quad \approx 0.067$$

$$\bar{x} = \frac{M_y}{m} = \frac{0.067}{0.307} \approx 0.217$$

The center of mass is located approximately at $(0.217,\ 0)$

Three-Dimensional Objects

Definition

Let ρ be an integrable area density function defined over a closed bounded region $\boldsymbol{D}$ in $\mathbb{R}^3$. The coordinates of the center of mass of the object represented by $\boldsymbol{D}$ are:

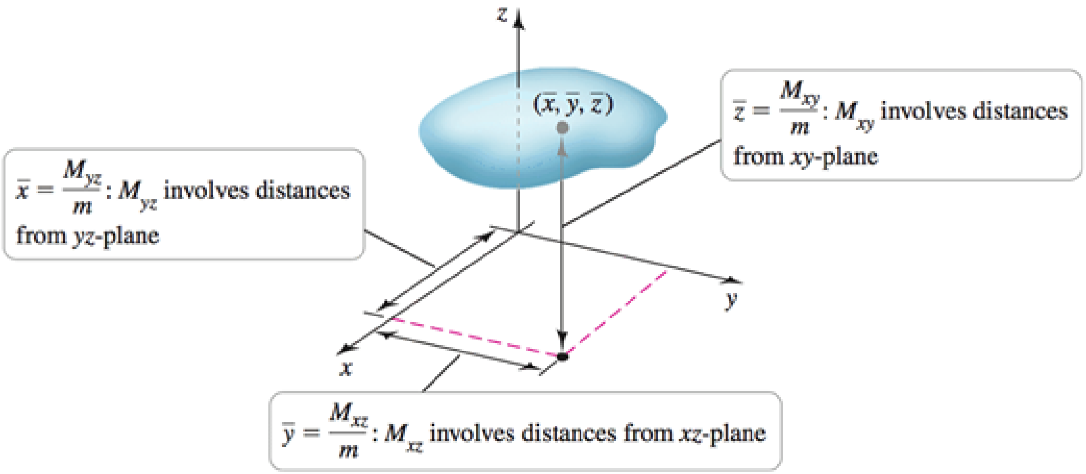
$$\overline{x} = \frac{M_{yz}}{m} = \frac{1}{m} \iiint_D x\, \rho(x,y,z)\, dV$$

$$\overline{y} = \frac{M_{xz}}{m} = \frac{1}{m} \iiint_D y\, \rho(x,y,z)\, dV$$

$$\overline{z} = \frac{M_{xy}}{m} = \frac{1}{m} \iiint_D z\, \rho(x,y,z)\, dV$$

Where $m = \iiint_D \rho(x,y,z)\, dV$ is the mass.

$M_{yz},\ M_{xz},\ and\ M_{xy}$ are the moments with respect to the coordinates planes.



Example

Find the center of mass of the constant density solid cone $\boldsymbol{D}$ bounded by the surface

$$z = 4 - \sqrt{x^2 + y^2} \quad and \quad z = 0$$

Solution

The one is symmetric about the z-axis and has uniform density, the center of mass lies on the z-axis, that is, $\overline{x} = 0$ *and* $\overline{y} = 0$.

The disk has a radius of 4 and centered at the origin. Therefore, the cone has height 4 and radius 4; by the volume formula is $\frac{1}{3}\pi hr^2 = \frac{1}{3}\pi 4\left(4^2\right) = \frac{64\pi}{3}$.

The cone has a constant density, so we assume that $\rho = 1$ and its mass is $\boxed{m = \frac{64\pi}{3}}$

$$z = 4 - \sqrt{x^2 + y^2} = 4 - r$$

$$\begin{aligned}
M_{xy} &= \int_0^{2\pi}\int_0^4\int_0^{4-r} z\, dz\, rdrd\theta \\
&= \int_0^{2\pi}\int_0^4 r\left[\frac{1}{2}z^2\right]_0^{4-r} drd\theta \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^4 r(4-r)^2\, drd\theta \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^4 \left(16r - 8r^2 + r^3\right) drd\theta \\
&= \frac{1}{2}\left[8r^2 - \frac{8}{3}r^3 + \frac{1}{4}r^4\right]_0^4 \left[\theta\right]_0^{2\pi} \\
&= \frac{1}{2}\left(128 - \frac{512}{3} + 64\right)(2\pi) \\
&= \boxed{\frac{64\pi}{3}}
\end{aligned}$$

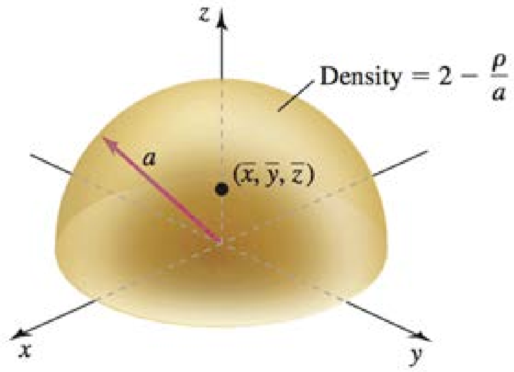
$$\overline{z} = \frac{M_{xy}}{m} = \frac{64\pi/3}{64\pi/3} \boxed{= 1}$$

$\therefore$ The center of mass is located at $(0,\ 0,\ 1)$

Example

Find the center of mass of the interior of the hemisphere $\boldsymbol{D}$ of a radius a with its base on the *xy*-plane. The density of the objects is $f(\rho,\ \phi,\ \theta)=2-\frac{\rho}{a}$ (heavy near the center and light near the outer surface.)

Solution



The one is symmetric about the *z*-axis and has uniform density, the center of mass lies on the *z*-axis, that is, $\bar{x}=0$ *and* $\bar{y}=0$.

$$m=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a}\left(2-\frac{\rho}{a}\right)\rho^2\sin\phi\ d\rho\ d\phi\ d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin\phi d\phi\int_0^{a}\left(2\rho^2-\frac{1}{a}\rho^3\right)\ d\rho$$

$$=\theta\Big|_0^{2\pi}\ -\cos\phi\Big|_0^{\pi/2}\ \left[\frac{2}{3}\rho^3-\frac{1}{4a}\rho^4\right]_0^a$$

$$=(2\pi)(1)\left(\frac{2}{3}a^3-\frac{1}{4}a^3\right)$$

$$\boxed{=\frac{5\pi}{6}a^3}$$

In spherical coordinates $z=\rho\cos\phi$.

$$M_{xy}=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{a}\rho\cos\phi\left(2-\frac{\rho}{a}\right)\rho^2\sin\phi\ d\rho\ d\phi\ d\theta$$

$$M_{xy}=\iiint_D z\ \rho(x,y,z)dV$$

$$=\int_0^{2\pi}d\theta\ \int_0^{\pi/2}\frac{1}{2}\sin 2\phi d\phi\ \int_0^{a}\left(2\rho^3-\frac{1}{a}\rho^4\right)d\rho$$

$$2\sin\phi\cos\phi=\sin 2\phi$$

$$=\theta\Big|_0^{2\pi}\ \left[-\frac{1}{4}\cos 2\phi\Big|_0^{\pi/2}\right.\ \left[\frac{1}{2}\rho^4-\frac{1}{5a}\rho^5\right]_0^a$$

$$=-\frac{1}{4}(2\pi)(-2)\left(\frac{1}{2}a^4-\frac{1}{5}a^4\right)$$

$$\boxed{=\frac{3\pi}{10}a^4}$$

$$\bar{z}=\frac{M_{xy}}{m}=\frac{\frac{3\pi a^4}{10}}{\frac{5\pi a^3}{6}}\ \boxed{=\frac{9a}{25}}\quad \boxed{=0.36a}$$

However, the center of mass of a uniform-density hemisphere solid of radius a is $\frac{3a}{8}=0.375a$ units above the base. In this particular case, the variable density shifts the center of mass.

Exercises Section 3.6 – Integrals for Mass Calculations

Find the mass and center of mass of the thin rods with the following density functions.

1. $\rho(x)=1+\sin x \quad for \ \ 0\le x\le \pi$

2. $\rho(x)=1+x^3 \quad for \ \ 0\le x\le 1$

3. $\rho(x)=2-\frac{x^2}{16} \quad for \ \ 0\le x\le 4$

4. $\rho(x)=2+\cos x \quad for \ \ 0\le x\le \pi$

5. $\rho(x)=\begin{cases} x^2 & if \ \ 0\le x\le 1 \\ 2x-x^2 & if \ \ 1\le x\le 2 \end{cases}$

Find the mass and centroid (center of mass) of the following thin plates, assuming a constant density. Sketch the region corresponding to the plate and indicate the location of the center of mass. Use symmetry when possible to simplify your work.

6. The region bounded by $y=\sin x$ and $y=1-\sin x$ between $x=\frac{\pi}{4} \quad and \quad x=\frac{3\pi}{4}$

7. The region bounded by $y=1-|x|$ and the x-axis

8. The region bounded by $y=e^x, \quad y=e^{-x}, \quad x=0, \quad and \quad x=\ln 2$

9. The region bounded by $y=\ln x, \quad x-axis, \quad and \quad x=e$

10. The region bounded by $x^2+y^2=1 \quad and \quad x^2+y^2=9, \quad for \ \ y\ge 0$

Find the coordinates of the center of mass of the following plane regions with variable density. Describe the distribution of mass in the region

11. $R=\{(x,\ y):\ 0\le x\le 4,\ 0\le y\le 2\}; \quad \rho(x,\ y)=1+\frac{x}{2}$

12. The triangular plate in the first quadrant bounded by $x+y=4$ with $\rho(x,\ y)=1+x+y$

13. The upper half $(y\ge 0)$ of the disk bounded by the circle $x^2+y^2=4$ with $\rho(x,\ y)=1+\frac{y}{2}$

14. The upper half $(y\ge 0)$ of the disk bounded by the ellipse $x^2+9y^2=9$ with $\rho(x,\ y)=1+y$

Find the center of mass of the following solids, assuming a constant density of 1. Sketch the region and indicate the location of the centroid. Use symmetry when possible and choose a convenient coordinate system.

15. The upper half of the ball $x^2+y^2+z^2\le 16 \quad (for\ z\ge 0)$

16. The region bounded by the paraboloid $z=x^2+y^2$ and the plane $z=25$

17. The tetrahedron in the first octant bounded by $z=1-x-y$ and the coordinate planes

18. The solid bounded by the cone $z=16-r$ and the plane $z=0$

19. Consider the thin constant-density plate $\{(r,\ \theta):\ a \le r \le 1,\ 0 \le \theta \le \pi\}$ bounded by two semicircles and the *x*-axis.

a) Find the graph the *y*-coordinate of the center of mass of the plate as a function of a.

b) For what value of a is the center of mass on the edge of the plate?

20. Consider the thin constant-density plate $\left\{(\rho,\ \phi,\ \theta):\ 0 < a \le \rho \le 1,\ 0 \le \phi \le \frac{\pi}{2},\ 0 \le \theta \le 2\pi\right\}$ bounded by two hemispheres and the *xy*-axis.

a) Find the graph the *z*-coordinate of the center of mass of the plate as a function of a.

b) For what value of a is the center of mass on the edge of the solid?

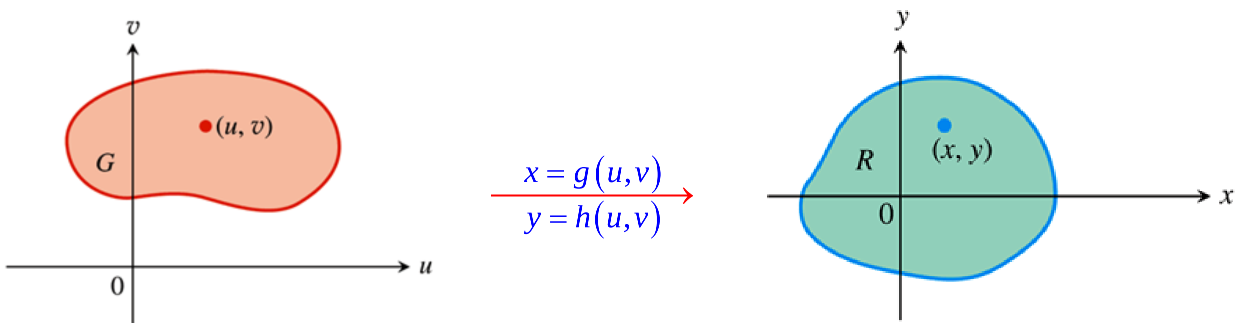
21. A cylindrical soda can has a radius of 4 *cm* and a height of 12 *cm*. When the can is full of soda, the center of mass of the contents of the can is 6 cm above the base on the axis of the can (halfway along the axis of the can). As the can is drained, the center of mass descends for a while. However, when the can is empty (filled only with air), the center of mass is once again 6 *cm* above the base on the axis of the can. Find the depth of soda in the can for which the center of mass is at its lowest point. Neglect the mass of the can, and assume the density of the soda is $1\ g / cm^3$ and the density of air is $0.001\ g / cm^3$.

Section 3.7 – Change of variables in Multiple Integrals

Substitution in Double Integrals

Suppose that a region G in the uv-plane is transformed one-to-one into the region R in the xy-plane by equations of the form

$$x = g(u,v), \quad y = h(u,v)$$



Cartesian uv-plane

Cartesian xy - plane

R is the image of G under the transformation, and G the ***preimage*** of R.

$$\iint_R f(x,y)\,dxdy = \iint_G f\big(g(u,v),h(u,v)\big)\left|J(u,v)\right|dudv$$

Definition

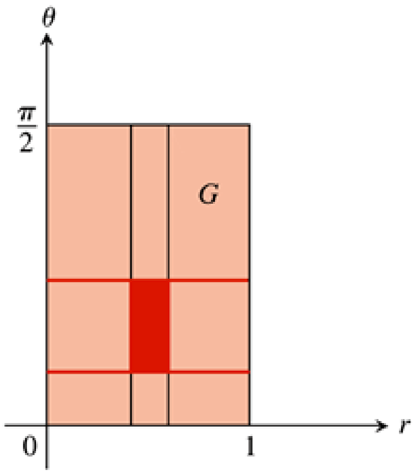
The ***Jacobian determinant*** or ***Jacobian*** of the coordinate transformation $x = g(u,v), \quad y = h(u,v)$ is

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

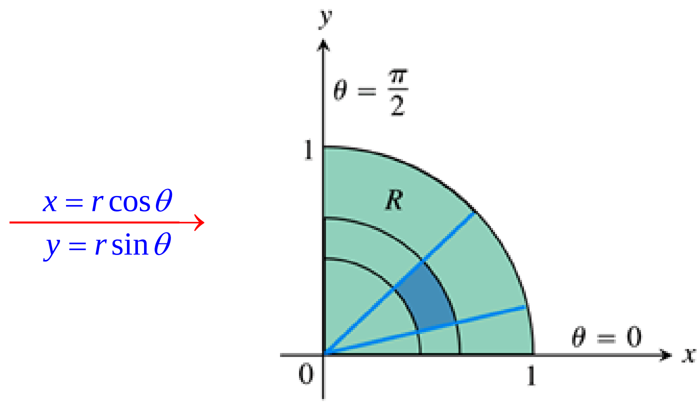
Example

Find the Jacobian for the polar coordinate transformation $x = r\cos\theta, \quad y = r\sin\theta$, write the Cartesian integral $\iint\limits_R f(x, y)\,dxdy$ as a polar integral.

Solution



Cartesian $r\theta$ –plane



Cartesian xy –plane

$x = r\cos\theta, \quad y = r\sin\theta$ transform the rectangle G: $0 \le r \le 1, \ 0 \le \theta \le 2\pi$, into the quarter circle R bounded by $x^2 + y^2 = 1$ in QI.

$$J(r, \theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r\left(\cos^2\theta + \sin^2\theta\right)$$

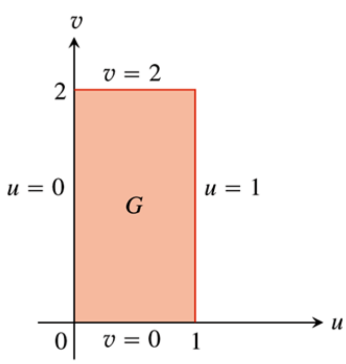
$$\underline{= r|}$$

Example

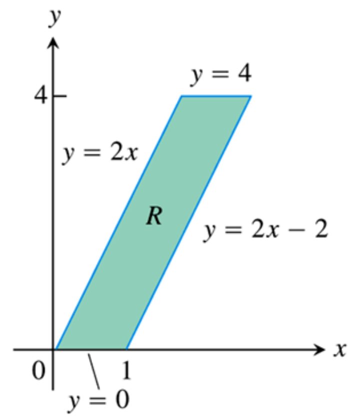
Evaluate $\displaystyle\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\,dxdy$ by applying the transformation $u=\dfrac{2x-y}{2}$, $v=\dfrac{y}{2}$ and integrating over an appropriate region in the *uv*-plane.

Solution

$$\rightarrow \underline{y=2v}\big|,\quad 2u=2x-y \Rightarrow \big|\underline{x=\frac{2u+y}{2}=\frac{2u+2v}{2}=u+v}\big|$$



$$\xrightarrow[y=2v]{x=u+v}$$



xy*-eqns for the boundary of *R	**Corresponding *uv*-eqns. for the boundary of *G***	**Simplified *uv*-eqns.**
$x=\dfrac{y}{2}$	$u+v=\dfrac{2v}{2}=v$	$u=0$
$x=\dfrac{y}{2}+1$	$u+v=\dfrac{2v}{2}+1=v+1$	$u=1$
$y=0$	$2v=0$	$v=0$
$y=4$	$2v=4$	$v=2$

$$J(u,v)=\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial u}(u+v) & \dfrac{\partial}{\partial v}(u+v) \\ \dfrac{\partial}{\partial u}(2v) & \dfrac{\partial}{\partial v}(2v) \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}$$

$$\underline{=2}\big|$$

$$\int_0^4 \int_{y/2}^{(y/2)+1} \left(x-\frac{y}{2}\right) dxdy = \int_0^{v=2}\int_{u=0}^{u=1} u\left|J(u,v)\right| dudv$$

$$= \int_0^{v=2}\int_{u=0}^{u=1} (u)(2)\,dudv$$

$$= \int_0^{v=2} u^2 \Big|_0^1 \, dv$$

$$= \int_0^{v=2} dv$$

$$= v \Big|_0^2$$

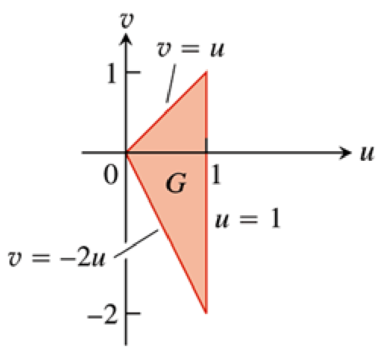
$$\underline{= 2}$$

Example

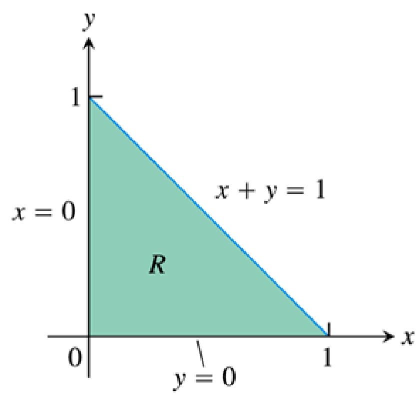
Evaluate $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\, dydx$

Solution

$$\begin{cases} u = x + y \\ v = y - 2x \end{cases} \to \begin{cases} x = \frac{u}{3} - \frac{v}{3} \\ y = \frac{2u}{3} + \frac{v}{3} \end{cases}$$



$$\xrightarrow[y = \frac{2u}{3} + \frac{v}{3}]{x = \frac{u}{3} - \frac{v}{3}}$$



xy*-eqns for the boundary of *R	**Corresponding *uv*-eqns. for the boundary of *G***	**Simplified *uv*-eqns.**
$x + y = 1$	$\left(\frac{u}{3} - \frac{v}{3}\right) + \left(\frac{2u}{3} + \frac{v}{3}\right) = 1$	$u = 1$
$x = 0$	$\frac{u}{3} - \frac{v}{3} = 0$	$v = u$
$y = 0$	$\frac{2u}{3} + \frac{v}{3} = 0$	$v = -2u$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \underline{\frac{1}{3}}$$

$$\int_0^1\int_0^{1-x}\sqrt{x+y}\,(y-2x)^2\,dydx = \int_{u=0}^{u=1}\int_{v=-2u}^{v=u} u^{1/2}v^2\left|J(u,v)\right|dvdu$$

$$= \int_0^1\int_{-2u}^{u} u^{1/2}v^2\left(\frac{1}{3}\right)dvdu$$

$$= \int_0^1 u^{1/2}\left[\frac{1}{9}v^3\right]_{-2u}^{u} du$$

$$= \frac{1}{9}\int_0^1 u^{1/2}\left(u^3+8u^3\right)du$$

$$= \int_0^1 u^{7/2}du$$

$$= \frac{2}{9}u^{9/2}\Big|_0^1$$

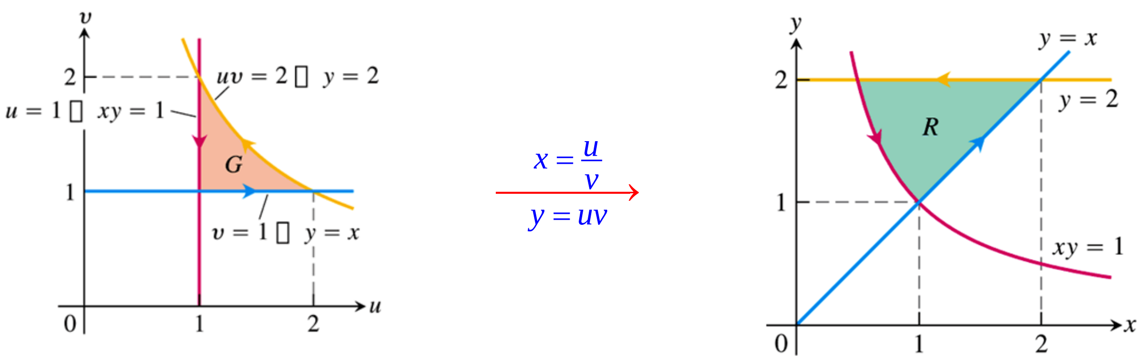
$$\underline{= \frac{2}{9}}$$

Example

Evaluate $\displaystyle\int_1^2\int_{1/y}^{y}\sqrt{\frac{y}{x}}\,e^{\sqrt{xy}}\,dxdy$

Solution

$$\begin{cases} u=\sqrt{xy} \\ v=\sqrt{\dfrac{y}{x}} \end{cases} \to \begin{cases} u^2 = xy \\ v^2 = \dfrac{y}{x} \end{cases} \quad \Rightarrow \quad x=\frac{u}{v}, \quad y=uv$$



$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{v} & -\dfrac{u}{v^2} \\ v & u \end{vmatrix} = \underline{\frac{2u}{v}}$$

xy-eqns for the boundary of ***R***	Corresponding *uv*-eqns. for the boundary of ***G***	Simplified *uv*-eqns.
$x=y$	$\frac{u}{v}=uv$	$v=1$
$x=\frac{1}{y}$	$\frac{u}{v}=\frac{1}{uv}$	$u=1$
$y=1$	$uv=1$	
$y=2$	$uv=2$	$u=2 \quad v=\frac{2}{u}$

$$\int_1^2\int_{1/y}^{y}\sqrt{\frac{y}{x}}\,e^{\sqrt{xy}}\,dxdy=\int_1^2\int_1^{2/u}2u\;e^u\,dvdu$$

$$=2\int_1^2 u\;e^u\left[v\right]_1^{2/u}du$$

$$=2\int_1^2 u\;e^u\left(\frac{2}{u}-1\right)du$$

$$=2\int_1^2(2-u)e^u\,du$$

$$=2\left[(2-u+1)e^u\right]_1^2$$

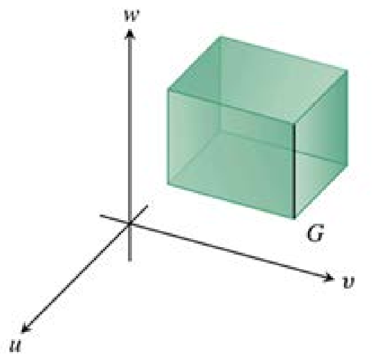
$$=2\left[(1)e^2-2e\right]$$

$$=2e(e-2)$$

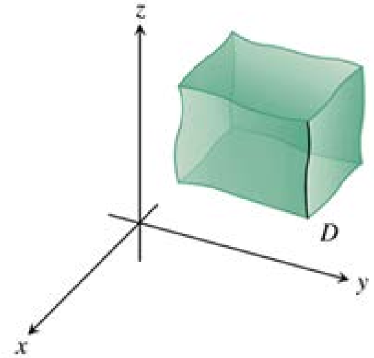
		e^u
$(+)$	$2-u$	e^u
$(-)$	-1	e^u
	0	

Substitutions in Triple Integrals

$$x = g(u,v,w), \quad y = h(u,v,w), \quad z = k(u,v,w)$$



$$\xrightarrow[]{\substack{x=g(u,v,w)\\ y=h(u,v,w)\\ z=k(u,v,w)}}$$



Cartesian *uvw* - plane

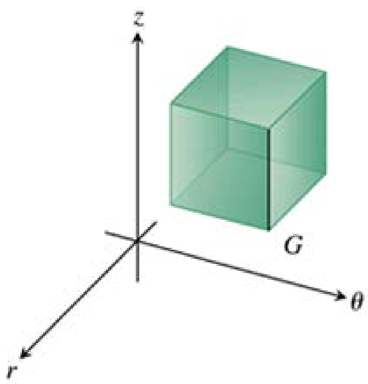
Cartesian *xyz* - plane

$$\iiint\limits_R f(x,y)\,dxdy = \iiint\limits_R H(u,v,w)\left|J(u,v,w)\right| dudvdw$$

The ***Jacobian determinant*** is

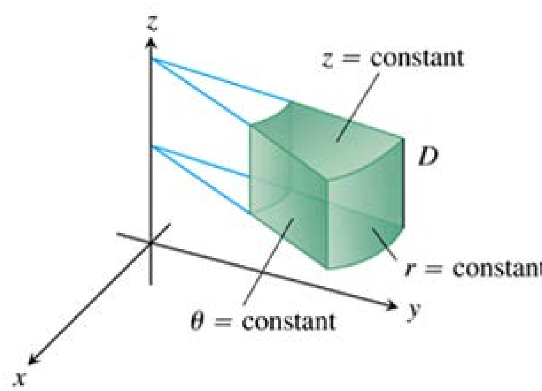
$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

Cube with sides parallel to the axes



$$\xrightarrow[]{\substack{x=r\cos\theta\\ y=r\sin\theta\\ z=z}}$$

Cube with sides parallel to the axes



Cartesian $r\theta z$ - plane

Cartesian *xyz* - plane

$$J(r,\theta,z) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta \underline{= r}$$

For spherical coordinates, ρ, ϕ, and θ take the place of u, v, and w. The transformation from Cartesian $\rho\phi\theta$-space to Cartesian xyz –space is given by

$$x=\rho\sin\phi\cos\theta,\quad y=\rho\sin\phi\sin\theta,\quad z=\rho\cos\phi$$

The Jacobian of the transformation

$$J(\rho,\phi,\theta)=\begin{vmatrix}\frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta}\\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta}\end{vmatrix}$$

$$=\begin{vmatrix}\sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta\\ \cos\phi & -\rho\sin\phi & 0\end{vmatrix}$$

$$=\rho^2\cos^2\phi\sin\phi\cos^2\theta+\rho^2\sin^3\phi\sin^2\theta+\rho^2\sin\phi\cos^2\phi\sin^2\theta+\rho^2\sin^3\phi\cos^2\theta$$

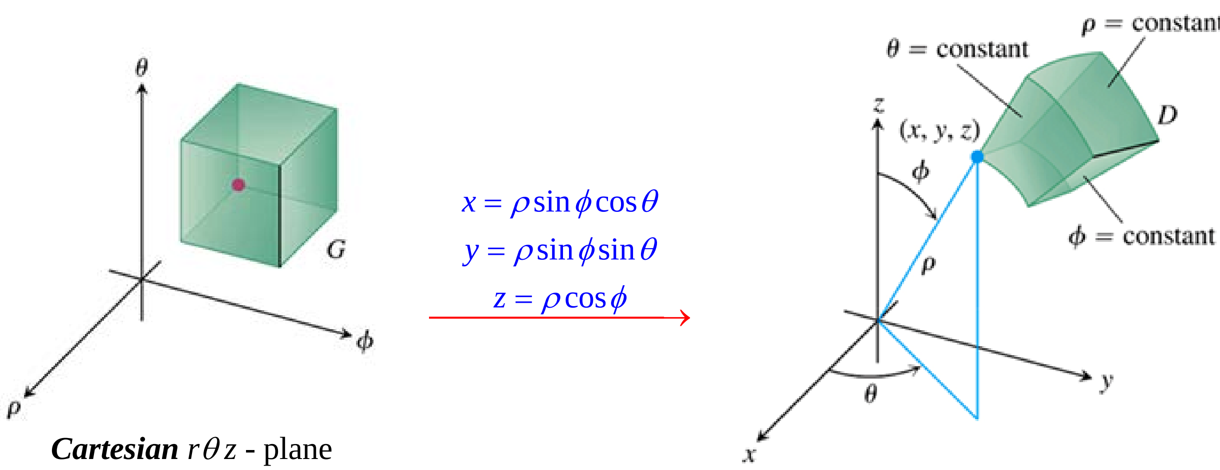
$$=\rho^2\cos^2\phi\sin\phi\left(\cos^2\theta+\sin^2\theta\right)+\rho^2\sin^3\phi\left(\sin^2\theta+\cos^2\theta\right)$$

$$=\rho^2\cos^2\phi\sin\phi+\rho^2\sin^3\phi$$

$$=\rho^2\sin\phi\left(\cos^2\phi+\sin^2\phi\right)$$

$$\underline{=\rho^2\sin\phi}$$

$$\iiint\limits_D F(x,y,z)\,dxdydz=\iiint\limits_G H(\rho,\phi,\theta)\left|\rho^2\sin\phi\right|d\rho\, d\phi\, d\theta$$



Cartesian $r\theta z$ - plane

Cartesian xyz - plane

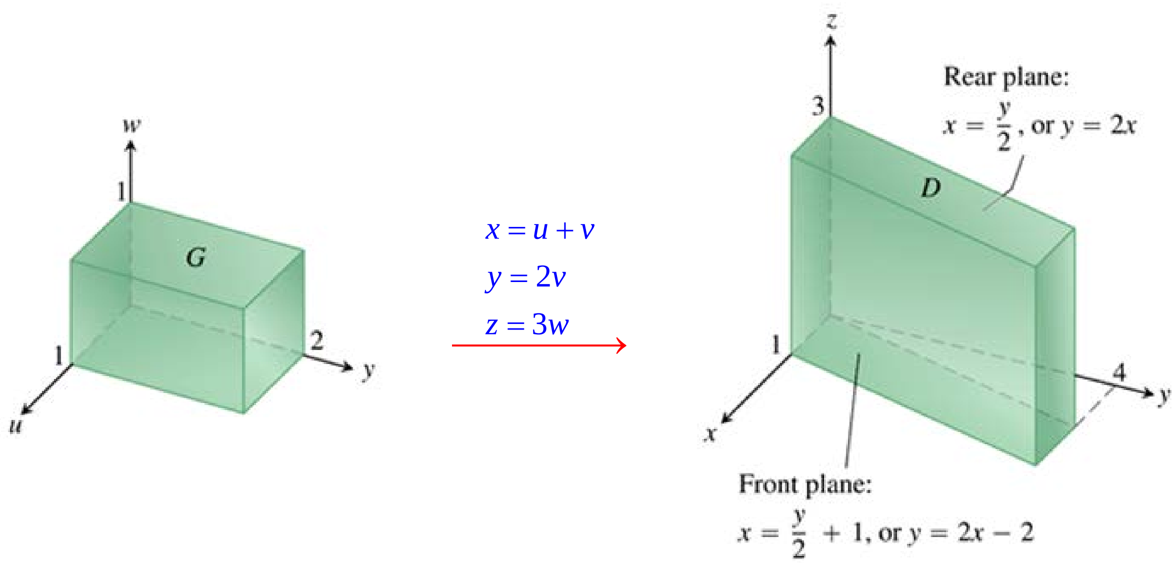
Example

Evaluate $\displaystyle\int_0^3\int_0^4\int_{x=y/2}^{x=(y/2)+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dxdydz$ by applying the transformation

$u=\dfrac{2x-y}{2},\quad v=\dfrac{y}{2},\quad w=\dfrac{z}{3}$ and integrating over an appropriate region in the *uvw*-plane.

Solution

$$\rightarrow\begin{cases} u=\dfrac{2x-y}{2}\rightarrow x=u+\dfrac{y}{2}=u+v \\ \qquad v=\dfrac{y}{2}\rightarrow y=2v \\ \qquad w=\dfrac{z}{3}\rightarrow z=3w \end{cases}$$



xyz*-*eqns* for the boundary of *D	**Corresponding *uvw*- *eqns*. for the boundary of *G***	**Simplified *uvw*- *eqns*.**
$x=\dfrac{y}{2}$	$u+v=v$	$u=0$
$x=\dfrac{y}{2}+1$	$u+v=v+1$	$u=1$
$y=0$	$2v=0$	$v=0$
$y=4$	$2v=4$	$v=2$
$z=0$	$3w=0$	$w=0$
$z=3$	$3w=3$	$w=1$

$$J(u,v,w)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$$\underline{=6}$$

$$\int_0^3\int_0^4\int_{x=y/2}^{x=(y/2)+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dxdydz=\int_0^1\int_0^2\int_0^1(u+w)\left|J(u,v,w)\right|\,dudvdw$$

$$=6\int_0^1\int_0^2\int_0^1(u+w)dudvdw$$

$$=6\int_0^1\int_0^2\left[\frac{u^2}{2}+wu\right]_0^1 dvdw$$

$$=6\int_0^1\int_0^2\left(\frac{1}{2}+w\right)dvdw$$

$$=6\int_0^1\left[\frac{1}{2}v+wv\right]_0^2 dw$$

$$=6\int_0^1(1+2w)dw$$

$$=6\left[w+w^2\right]_0^1$$

$$=6(1+1)$$

$$\underline{=12}$$

Exercises *Section* 3.7 – Change of Variables in Multiple Integrals

1. *a*) Solve the system $u = x - y, \quad v = 2x + y$ for *x* and *y* in terms of *u* and *v*. Then find the value of the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$

b) Find the image under the transformation $u = x - y, \ \ v = 2x + y$ of the triangular region with vertices (0, 0), (1, 1), and (1, –2) in the *xy*-plane. Sketch the transformed region in the *uv*-plane.

2. Let *R* be the region in the first quadrant of the *xy*-plane bounded by the hyperbolas $xy = 1, \ xy = 9$ and the lines $y = x, \ y = 4x$. Use the transformation $x = \frac{u}{v}, \ y = uv$ with $u > 0, \ and \ v > 0$ to rewrite

$$\iint\limits_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy$$

As an integral over an appropriate region *G* in the uv-plane. Then evaluate the *uv*-integral over *G*.

3. The area πab of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be found by integrating the function $f(x,y) = 1$ over the region bounded by the ellipse in the *xy*-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x = au, \ y = bv$ and evaluate the transformed integral over the disk *G*: $u^2 + v^2 \le 1$ in the *uv*-plane. Find the area this way.

4. Use the transformation $x = u + \frac{1}{2}v, \ \ y = v$ to evaluate the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3 (2x - y) e^{(2x-y)^2} dxdy$$

By first writing it as an integral over a region *G* in the *uv*-plane.

5. Use the transformation $x = \frac{u}{v}, \ \ y = uv$ to evaluate the integral

$$\int_1^2 \int_{1/y}^{y} \left(x^2 + y^2\right) dxdy + \int_2^4 \int_{y/4}^{4/y} \left(x^2 + y^2\right) dxdy$$

6. Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ of the transformation

a) $x = u\cos v, \quad y = u\sin v$

b) $x = u\sin v, \quad y = u\cos v$

7. Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ of the transformation

a) $x = u\cos v, \quad y = u\sin v, \quad z = w$

b) $x = 2u - 1, \quad y = 3v - 4, \quad z = \frac{1}{2}(w - 4)$

8. Evaluate the appropriate determinant to show that the Jacobian of the transformation from Cartesian $\rho\phi\theta$–space to Cartesian *xyz*–space is $\rho^2 \sin\phi$

9. How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.

10. Find the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

(***Hint***: Let $x = au,\ y = bv,\ and\ z = cw$. Then find the volume of an appropriate region in *uvw*-space)

11. Use the transformation $x = u^2 - v^2,\ \ y = 2uv$ to evaluate the integral

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2}\ dydx$$

(***Hint***: Show that the image of the triangular region ***G*** with vertices (0, 0), (1, 0), (1, 1) in the *uv*-plane is the region of integration ***R*** in the *xy*-plane defined by the limits of integration.)