Solution   *Section* 7.7 – Trigonometric Form

Exercise

Write $-\sqrt{3}+i$ in trigonometric form. (Use radian measure)

Solution

$$-\sqrt{3}+i \;\Rightarrow\; \begin{cases} x=-\sqrt{3} \\ y=1 \end{cases}$$

$$r=\sqrt{\left(-\sqrt{3}\right)^2+1^2}$$
$$\underline{=2}$$

$$\hat{\theta}=\tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) \qquad\qquad \theta=\tan^{-1}\frac{y}{x}$$
$$\underline{=\frac{\pi}{6}}$$

The angle is in quadrant *II*, therefore;

$$\theta=\pi-\frac{\pi}{6}$$
$$\underline{=\frac{5\pi}{6}}$$

$$\underline{-\sqrt{3}+i=2\ cis\,\frac{5\pi}{6}}$$

Exercise

Write $3-4i$ in trigonometric form.

Solution

$$3-4i \;\Rightarrow\; \begin{cases} x=3 \\ y=-4 \end{cases}$$

$$r=\sqrt{3^2+(-4)^2}$$
$$\underline{=2}$$

$$\hat{\theta}=\tan^{-1}\left(\frac{4}{3}\right) \qquad\qquad \theta=\tan^{-1}\frac{y}{x}$$
$$\underline{\approx 53^\circ}$$

The angle is in quadrant *II*, therefore;

$$\theta=180^\circ-53^\circ$$
$$\underline{=127^\circ}$$

$$\underline{3-4i=5\ cis127^\circ}$$

Exercise

Write $-21-20i$ in trigonometric form.

Solution

$$-21-20i \quad \Rightarrow \quad \begin{cases} x=-21 \\ y=-20 \end{cases}$$

$$r=\sqrt{(-21)^2+(-20)^2}$$

$$=29$$

$$\hat{\theta}=\tan^{-1}\left(\frac{20}{21}\right) \qquad\qquad \theta=\tan^{-1}\frac{y}{x}$$

$$\approx 43.6°$$

The angle is in quadrant *III*, therefore;

$$\theta=180°+43.6°$$

$$=223.6°$$

$$-21-20i=29\ cis223.6°$$

Exercise

Write $11+2i$ in trigonometric form.

Solution

$$11+2i \quad \Rightarrow \quad \begin{cases} x=11 \\ y=2 \end{cases}$$

$$r=\sqrt{11^2+2^2}$$

$$=\sqrt{125}$$

$$=5\sqrt{5}$$

$$\theta=\tan^{-1}\left(\frac{2}{11}\right) \qquad\qquad \theta=\tan^{-1}\frac{y}{x}$$

$$\approx 10.3°$$

$$11+2i=5\sqrt{5}\ cis10.3°$$

Exercise

Write $\sqrt{3}-i$ in trigonometric form.

Solution

$$\sqrt{3}-i \quad \Rightarrow \quad \begin{cases} x=\sqrt{3} \\ y=-1 \end{cases}$$

$$r=\sqrt{3+1}$$
$$=2$$

$$\hat{\theta}=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \qquad\qquad \theta=\tan^{-1}\frac{y}{x}$$
$$\approx 30^\circ$$

The angle is in quadrant *IV*, therefore;

$$\theta=360^\circ-30^\circ$$
$$=330^\circ$$

$$\sqrt{3}-i=2\ cis330^\circ$$

Exercise

Write $1-\sqrt{3}i$ in trigonometric form.

Solution

$$1-\sqrt{3}i \quad \Rightarrow \quad \begin{cases} x=1 \\ y=-\sqrt{3} \end{cases}$$

$$r=\sqrt{1+3}$$
$$=2$$

$$\hat{\theta}=\tan^{-1}\left(\frac{\sqrt{3}}{1}\right) \qquad\qquad \theta=\tan^{-1}\frac{y}{x}$$
$$=60^\circ$$

The angle is in quadrant *IV*, therefore;

$$\theta=360^\circ-60^\circ$$
$$=300^\circ$$

$$1-\sqrt{3}i=2\ cis300^\circ$$

Exercise

Write $9\sqrt{3}+9i$ in trigonometric form.

Solution

$$9\sqrt{3}+9i \Rightarrow \begin{cases} x=9\sqrt{3} \\ y=9 \end{cases}$$

$$r=9\sqrt{3+1}$$
$$=18$$

$$\theta=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$=30^\circ$$

$$9\sqrt{3}+9i = 18\ cis 30^\circ$$

Exercise

Write $-2+3i$ in trigonometric form.

Solution

$$-2+3i \Rightarrow \begin{cases} x=-2 \\ y=3 \end{cases}$$

$$r=\sqrt{4+9}$$
$$=\sqrt{13}$$

$$\hat{\theta}=\tan^{-1}\left(\frac{3}{2}\right) \qquad \theta=\tan^{-1}\frac{y}{x}$$
$$\approx 56.31^\circ$$

The angle is in quadrant *II*, therefore;

$$\theta=180^\circ-56.31^\circ$$
$$\approx 123.69^\circ$$

$$-2+3i = \sqrt{13}\ cis 123.69^\circ$$

Exercise

Write $4(\cos 30^\circ + i\sin 30^\circ)$ in standard form.

Solution

$$4(\cos 30^\circ + i\sin 30^\circ) = 4\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)$$
$$=2\sqrt{3}+2i$$

Exercise

Write $\sqrt{2}\ cis\frac{7\pi}{4}$ in standard form.

Solution

$$\sqrt{2}\ cis\frac{7\pi}{4}=\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$$

$$=\sqrt{2}\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)$$

$$\underline{=1-i}$$

Exercise

Write $3cis210°$ in standard form.

Solution

$$3cis210°=3\left(\cos 210°+i\sin 210°\right)$$

$$\underline{=-\frac{3\sqrt{3}}{2}-\frac{3}{2}i}$$

Exercise

Write $4\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$ in standard form.

Solution

$$4\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)=4\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)$$

$$\underline{=2\sqrt{2}-2i\sqrt{2}}$$

Exercise

Write $4cis\frac{\pi}{2}$ in standard form.

Solution

$$4cis\frac{\pi}{2}=4\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

$$\underline{=4i}$$

Exercise

Find the quotient $\frac{20cis(75°)}{4cis(40°)}$. Write the result in rectangular form.

Solution

$$\frac{20cis(75°)}{4cis(40°)} = \frac{20}{4}cis(75° - 40°)$$
$$= 5cis(35°)$$
$$= 5(\cos 35° + i\sin 35°)$$
$$\boxed{= 4.1 + 2.87i}$$

Exercise

Divide $z_1 = 1 + i\sqrt{3}$ *by* $z_2 = \sqrt{3} + i$. Write the result in rectangular form.

Solution

$$\frac{z_1}{z_2} = \frac{1 + i\sqrt{3}}{\sqrt{3} + i}$$
$$= \frac{1 + i\sqrt{3}}{\sqrt{3} + i}\,\frac{\sqrt{3} - i}{\sqrt{3} - i}$$
$$= \frac{\sqrt{3} - i + 3i - \sqrt{3}\,i^2}{3 + 1}$$
$$= \frac{2\sqrt{3} + 2i}{4}$$
$$= \frac{2\sqrt{3}}{4} + \frac{2i}{4}$$
$$\boxed{= \frac{\sqrt{3}}{2} + \frac{i}{2}}$$

$\boxed{or}$

$$1 + i\sqrt{3} \quad \rightarrow \quad \begin{cases} r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} \\ \theta = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3} \end{cases}$$

$$\sqrt{3} + i \quad \rightarrow \quad \begin{cases} r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} \\ \theta = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6} \end{cases}$$

$$\frac{z_1}{z_2} = \frac{2cis\frac{\pi}{3}}{2cis\frac{\pi}{6}}$$
$$= \frac{2}{2}cis\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$
$$= \frac{2}{2}cis\left(\frac{\pi}{6}\right)$$

$$= cis\left(\frac{\pi}{6}\right)$$

$$\boxed{= \frac{\sqrt{3}}{2} + \frac{i}{2}}$$

Exercise

Find $(1+i)^8$ and express the result in rectangular form.

Solution

$$1+i \quad \Rightarrow \quad \begin{cases} x = 1 \\ y = 1 \end{cases}$$

$$r = \sqrt{1^2 + 1^2}$$
$$\boxed{= \sqrt{2}}$$

$$\theta = \tan^{-1}(1)$$
$$\boxed{= \frac{\pi}{4}}$$

$$\boxed{1+i = \sqrt{2}cis\frac{\pi}{4}}$$

$$(1+i)^8 = \left(\sqrt{2}cis\frac{\pi}{4}\right)^8$$
$$= \left(\sqrt{2}\right)^8 cis\left[8\left(\frac{\pi}{4}\right)\right]$$
$$= 16cis2\pi$$
$$= 16(\cos 2\pi + i\sin 2\pi)$$
$$= 16(1 + i0)$$
$$\boxed{= 16}$$

Exercise

Find $(1+i)^{10}$ and express the result in rectangular form.

Solution

$$1+i \quad \Rightarrow \quad \begin{cases} x = 1 \\ y = 1 \end{cases}$$

$$r = \sqrt{1^2 + 1^2}$$
$$\boxed{= \sqrt{2}}$$

$$\theta = \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

$$1 + i = \sqrt{2} cis \frac{\pi}{4}$$

$$(1+i)^{10} = \left(\sqrt{2} cis \frac{\pi}{4}\right)^{10}$$

$$= \left(\sqrt{2}\right)^{10} cis\left[10\left(\frac{\pi}{4}\right)\right]$$

$$= 32 cis \frac{5\pi}{2}$$

$$= 32 cis \frac{\pi}{2}$$

$$= 32\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)$$

$$= 32(0 + i)$$

$$= 32i$$

Exercise

Find and express the result in rectangular form $(1-i)^5$

Solution

$$1 - i \quad \Rightarrow \quad \begin{cases} x = 1 \\ y = -1 \end{cases}$$

$$r = \sqrt{1^2 + 1^2}$$

$$= \sqrt{2}$$

$$\hat{\theta} = \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

The angle is in quadrant *IV*, therefore;

$$\theta = 2\pi - \frac{\pi}{4}$$

$$= \frac{7\pi}{4}$$

$$1 - i = \sqrt{2}\ cis \frac{7\pi}{4}$$

$$(1-i)^5 = \left(\sqrt{2}\ cis \frac{7\pi}{4}\right)^5$$

$$= 4\sqrt{2}\left(\ cis\left(5 \times \frac{7\pi}{4}\right)\right)$$

$$= 4\sqrt{2}\left(\ cis\frac{35\pi}{4}\right) \qquad 35\pi - 8\times 2\pi = 3\pi$$

$$= 4\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$\underline{= -4 + 4i}$$

Exercise

Find and express the result in rectangular form $\left(1-\sqrt{5}i\right)^8$

Solution

$$1-\sqrt{5}i \;\Rightarrow\; \begin{cases} x = 1 \\ y = -\sqrt{5} \end{cases}$$

$$r = \sqrt{1+5}$$

$$\underline{= \sqrt{6}}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{\sqrt{5}}{1}\right)$$

$$\underline{\approx 66^\circ}$$

The angle is in quadrant *IV*, therefore;

$$\theta = 360^\circ - 66^\circ$$

$$\underline{= 294^\circ}$$

$$\underline{1-\sqrt{5}i = \sqrt{6}\ cis294^\circ}$$

$$\left(1-\sqrt{5}i\right)^8 = \left(\sqrt{6}cis294^\circ\right)^8$$

$$= \left(\sqrt{6}\right)^8\left(cis2352^\circ\right) \qquad 2352^\circ - 6\times 360^\circ = 192^\circ$$

$$= 1296\left(\cos 192^\circ + i\sin 192^\circ\right)$$

$$= 1296\left(-.978 - 0.208i\right)$$

$$\underline{= -1267.488 - 269.568\ i}$$

Exercise

Find and express the result in rectangular form $\left(3cis80°\right)^3$

Solution

$$\left(3cis80°\right)^3 = 3^3\left(cis240°\right)$$

$$= 27\left(\cos 240° + i\sin 240°\right)$$

$$= 27\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= -\frac{27}{2} - i\frac{27\sqrt{3}}{2}$$

Exercise

Find and express the result in rectangular form $\left(\sqrt{3}cis10°\right)^6$

Solution

$$\left(\sqrt{3}cis10°\right)^6 = 27\left(cis60°\right)$$

$$= 27\left(\cos 60° + i\sin 60°\right)$$

$$= \frac{27}{2} + i\frac{27\sqrt{3}}{2}$$

Exercise

Find and express the result in rectangular form $\left(\sqrt{2} - i\right)^6$

Solution

$$\sqrt{2} - i \quad \Rightarrow \quad \begin{cases} x = \sqrt{2} \\ y = -1 \end{cases}$$

$$r = \sqrt{2+1}$$

$$= \sqrt{3}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\approx 35.26°$$

The angle is in quadrant *IV*, therefore;

$$\theta = 360° - 35.26°$$

$$= 324.74°$$

$$\sqrt{2} - i = \sqrt{3}\ cis324.74°$$

$$\left(\sqrt{2}-i\right)^6 = \left(\sqrt{3}\ cis 324.74°\right)^6$$
$$= 27\left(cis 1948.44°\right) \qquad 1948.44° - 5\times 360° = 148.44°$$
$$= 27\left(\cos 148.44° + i\sin 148.44°\right)$$
$$\underline{= -23 + 14.142i}$$

Exercise

Find and express the result in rectangular form $\left(4cis 40°\right)^6$

Solution

$$\left(4cis 40°\right)^6 = 4^6\left(cis\left(6\times 40°\right)\right)$$
$$= 4^6\left(\cos 240° + i\sin 240°\right)$$
$$= 4096\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$\underline{= -2048 + 2048\ i\sqrt{3}}$$

Exercise

Find and express the result in rectangular form $\left(2cis 30°\right)^5$

Solution

$$\left(2cis 30°\right)^5 = 2^5 cis\left(5(30°)\right)$$
$$= 32\left(\cos 150° + i\sin 150°\right)$$
$$= 32\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$
$$\underline{= -16\sqrt{3} + 16i}$$

Exercise

Find and express the result in rectangular form $\left(\frac{1}{2}cis 72°\right)^5$

Solution

$$\left(\frac{1}{2}cis 72°\right)^5 = \frac{1}{2^5}cis\left(5\times 72°\right)$$
$$= \frac{1}{32}cis\left(\cos 360° + i\sin 360°\right)$$
$$\underline{= \frac{1}{32}}$$

Exercise

Find ***fifth*** roots of $z = 1 + i\sqrt{3}$ and express the result in rectangular form.

Solution

$$1 + i\sqrt{3} \Rightarrow \begin{cases} x = 1 \\ y = \sqrt{3} \end{cases}$$

$$r = \sqrt{1+3}$$
$$\underline{= 2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$
$$\underline{= 60^\circ}$$

$$\underline{1 + i\sqrt{3} = 2\ cis 60^\circ}$$

$$\left(1 + i\sqrt{3}\right)^{1/5} = \left(2\ cis 60^\circ\right)^{1/5}$$
$$= \sqrt[5]{2}\left(cis \frac{60^\circ}{5} + \frac{360^\circ k}{5}\right)$$
$$\underline{= \sqrt[5]{2}\ cis\left(12^\circ + 72^\circ k\right)} \qquad k = 0,\ 1,\ 2,\ 3,\ 4$$

For $k = 0$

$$\sqrt[5]{2}\ cis\left(12^\circ + 72^\circ 0\right) \ \underline{= \sqrt[5]{2}\ cis 12^\circ}$$

For $k = 1$

$$\sqrt[5]{2}\ cis\left(12^\circ + 72^\circ.(1)\right) \ \underline{= \sqrt[5]{2}\ cis 84^\circ}$$

For $k = 2$

$$\sqrt[5]{2}\ cis\left(12^\circ + 72^\circ.(2)\right) \ \underline{= \sqrt[5]{2}\ cis 156^\circ}$$

For $k = 3$

$$\sqrt[5]{2}\ cis\left(12^\circ + 72^\circ.(3)\right) \ \underline{= \sqrt[5]{2}\ cis 228^\circ}$$

For $k = 4$

$$\sqrt[5]{2}\ cis\left(12^\circ + 72^\circ.(4)\right) \ \underline{= \sqrt[5]{2}\ cis 300^\circ}$$

Exercise

Find the ***fourth*** roots of $z = 16cis60°$

Solution

$$\sqrt[4]{z} = \sqrt[4]{16}\ cis\left(\frac{60°}{4} + \frac{360°}{4}k\right)$$

$$= 2cis\left(15° + 90°k\right) \qquad k = 0,\ 1,\ 2,\ 3$$

For $k = 0$

$$2\ cis\left(15° + 90°(0)\right) = 2cis15°$$

For $k = 1$

$$2\ cis\left(15° + 90°(1)\right) = 2cis105°$$

For $k = 2$

$$2\ cis\left(15° + 90°(2)\right) = 2cis195°$$

For $k = 3$

$$2\ cis\left(15° + 90°(3)\right) = 2cis285°$$

Exercise

Find the ***fourth*** roots of $\sqrt{3} - i$

Solution

$$\sqrt{3} - i \quad \Rightarrow \quad \begin{cases} x = \sqrt{3} \\ y = -1 \end{cases}$$

$$r = \sqrt{3+1}$$

$$= 2$$

$$\hat{\theta} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{6}$$

The angle is in quadrant *IV*, therefore;

$$\theta = 2\pi - \frac{\pi}{6}$$

$$= \frac{11\pi}{6}$$

$$\sqrt{3} - i = 2\ cis\frac{11\pi}{6}$$

$$\sqrt[4]{\sqrt{3} - i} = \sqrt[4]{2\ cis\frac{11\pi}{6}}$$

$$= \sqrt[4]{2}\ cis\left(\frac{1}{4}\frac{11\pi}{6} + \frac{2\pi k}{4}\right)$$

$$= \boxed{\sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + \frac{\pi k}{2}\right)} \qquad k = 0,\ 1,\ 2,\ 3$$

For $k = 0$

$$\sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + 0\right) = \boxed{\sqrt[4]{2}\ cis\frac{11\pi}{24}}$$

For $k = 1$

$$\sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + \frac{\pi}{2}\right) = \boxed{\sqrt[4]{2}\ cis\frac{23\pi}{24}}$$

For $k = 2$

$$\sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + \pi\right) = \boxed{\sqrt[4]{2}\ cis\frac{35\pi}{24}}$$

For $k = 3$

$$\sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + \frac{3\pi}{2}\right) = \boxed{\sqrt[4]{2}\ cis\frac{47\pi}{24}}$$

Exercise

Find the ***fourth*** roots of $4 - 4\sqrt{3}i$

Solution

$$4 - 4\sqrt{3}i \Rightarrow \begin{cases} x = 4 \\ y = -4\sqrt{3} \end{cases}$$

$$r = 4\sqrt{3+1}$$

$$= \boxed{8}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{4\sqrt{3}}{4}\right)$$

$$= \tan^{-1}\left(\sqrt{3}\right)$$

$$= \boxed{\frac{\pi}{3}}$$

The angle is in quadrant *IV*, therefore;

$$\theta = 2\pi - \frac{\pi}{3}$$

$$= \boxed{\frac{5\pi}{3}}$$

$$\boxed{4 - 4\sqrt{3}i = 8\ cis\frac{5\pi}{3}}$$

$$\sqrt[4]{4 - 4\sqrt{3}i} = \sqrt[4]{8\ cis\frac{5\pi}{3}}$$

$$=\sqrt[4]{8}\ cis\left(\frac{5\pi}{12}+\frac{\pi k}{2}\right) \qquad k=0,\ 1,\ 2,\ 3$$

For $k=0$

$$\sqrt[4]{8}\ cis\left(\frac{5\pi}{12}+0\right)=\sqrt[4]{8}\ cis\frac{5\pi}{12}$$

For $k=1$

$$\sqrt[4]{8}\ cis\left(\frac{5\pi}{12}+\frac{\pi}{2}\right)=\sqrt[4]{8}\ cis\frac{11\pi}{12}$$

For $k=2$

$$\sqrt[4]{8}\ cis\left(\frac{5\pi}{12}+\pi\right)=\sqrt[4]{8}\ cis\frac{17\pi}{12}$$

For $k=3$

$$\sqrt[4]{8}\ cis\left(\frac{5\pi}{12}+\frac{3\pi}{2}\right)=\sqrt[4]{8}\ cis\frac{23\pi}{12}$$

Exercise

Find the ***fourth*** roots of $-16i$

Solution

$$-16i \Rightarrow \begin{cases} x=0 \\ y=-16 \end{cases}$$

$$r=16$$

$$\theta=\frac{3\pi}{2}$$

$$-16i=16\ cis\frac{3\pi}{2}$$

$$\sqrt[4]{-16i}=\sqrt[4]{16\ cis\frac{3\pi}{2}}$$

$$=2\ cis\left(\frac{3\pi}{8}+\frac{\pi k}{2}\right) \qquad k=0,\ 1,\ 2,\ 3$$

For $k=0$

$$2\ cis\left(\frac{3\pi}{8}+0\right)=2\ cis\frac{3\pi}{8}$$

For $k=1$

$$2\ cis\left(\frac{3\pi}{8}+\frac{\pi}{2}\right)=2\ cis\frac{7\pi}{8}$$

For $k=2$

$$2\ cis\left(\frac{3\pi}{8}+\pi\right)=2\ cis\frac{11\pi}{8}$$

For $k=3$

$$2\ cis\left(\frac{3\pi}{8}+\frac{3\pi}{2}\right) \underline{= 2\ cis\frac{15\pi}{8}}$$

Exercise

Find the ***cube*** roots of 27.

Solution

$$\sqrt[3]{27}=\left(27cis0°\right)^{1/3}$$

$$=\sqrt[3]{27}\ cis\left(\frac{0°}{3}+\frac{360°}{3}k\right)$$

$$\underline{=3\ cis\left(0°+120°k\right)} \qquad k=0,\ 1,\ 2$$

For $k=0$

$$z=3\ cis\left(0°+120°(0)\right) \underline{=2\ cis0°}$$

For $k=1$

$$z=3\ cis\left(0°+120°(1)\right) \underline{=2\ cis120°}$$

For $k=2$

$$z=3\ cis\left(0°+120°(2)\right) \underline{=2\ cis240°}$$

Exercise

Find the ***cube*** roots of $8-8i$

Solution

$$8-8i \Rightarrow \begin{cases} x=8 \\ y=-8 \end{cases}$$

$$r=8\sqrt{1+1}$$

$$\underline{=8\sqrt{2}}$$

$$\hat{\theta}=\tan^{-1}\left(\frac{8}{8}\right)$$

$$=\tan^{-1}(1)$$

$$\underline{=\frac{\pi}{4}}$$

The angle is in quadrant *IV*, therefore;

$$\theta=2\pi-\frac{\pi}{4}$$

$$\underline{=\frac{7\pi}{4}}$$

$$\sqrt[3]{8-8i} = \sqrt[3]{8\sqrt{2}\ cis\frac{7\pi}{4}}$$

$$= 2\sqrt[3]{2}\ cis\left(\frac{7\pi}{12}+\frac{2\pi k}{3}\right) \qquad k = 0,\ 1,\ 2$$

For $k = 0$

$$z = 2\sqrt[3]{2}\ cis\left(\frac{7\pi}{12}+0\right) = 2\sqrt[3]{2}\ cis\frac{7\pi}{12}$$

For $k = 1$

$$z = 2\sqrt[3]{2}\ cis\left(\frac{7\pi}{12}+\frac{2\pi}{3}\right) = 2\sqrt[3]{2}\ cis\frac{15\pi}{12}$$

For $k = 2$

$$z = 2\sqrt[3]{2}\ cis\left(\frac{7\pi}{12}+\frac{4\pi}{3}\right) = 2\sqrt[3]{2}\ cis\frac{23\pi}{12}$$

Exercise

Find the ***cube*** roots of -8

Solution

$r = 8$

$\theta = \frac{3\pi}{2}$

$$\sqrt[3]{-8} = \sqrt[3]{8\ cis\frac{3\pi}{2}}$$

$$= 2\ cis\left(\frac{\pi}{2}+\frac{2\pi k}{3}\right) \qquad k = 0,\ 1,\ 2$$

For $k = 0$

$$z = 2\ cis\left(\frac{\pi}{2}+0\right) = 2\ cis\frac{\pi}{2}$$

For $k = 1$

$$z = 2\ cis\left(\frac{\pi}{2}+\frac{2\pi}{3}\right) = 2\ cis\frac{7\pi}{6}$$

For $k = 2$

$$z = 2\ cis\left(\frac{\pi}{2}+\frac{4\pi}{3}\right) = 2\ cis\frac{11\pi}{6}$$

Exercise

Find all complex number solutions of $x^3 + 1 = 0$.

Solution

$$x^3 + 1 = 0$$

$$x^3 = -1$$

$$\underline{r = 1}$$

$$\theta = \tan^{-1}\left(\frac{0}{-1}\right)$$

$$\underline{= \pi}$$

$$x^3 = 1\ cis(\pi)$$

$$x = (1\ cis\pi)^{1/3}$$

$$= (1)^{1/3}\ cis\left(\frac{\pi}{3} + \frac{2\pi}{3}k\right)$$

$$\underline{= cis\left(\frac{\pi}{3} + \frac{2\pi}{3}k\right)} \qquad k = 0,\ 1,\ 2$$

For $k = 0$

$$x = cis\left(\frac{\pi}{3} + \frac{2\pi}{3}(0)\right)$$

$$= cis\frac{\pi}{3}$$

$$= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$\underline{= \frac{1}{2} + i\frac{\sqrt{3}}{2}}$$

For $k = 1$

$$x = cis\left(\frac{\pi}{3} + \frac{2\pi}{3}(1)\right)$$

$$= cis\left(\frac{3\pi}{3}\right)$$

$$= cis\pi$$

$$= \cos\pi + i\sin\pi$$

$$\underline{= -1}$$

For $k = 2$

$$x = cis\left(\frac{\pi}{3} + \frac{2\pi}{3}(2)\right)$$

$$= cis\frac{5\pi}{3}$$

$$= \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$

$$= \frac{1}{2} - i\frac{\sqrt{3}}{2}$$