

## ***Solution***      **Section 1.3 – Matrices and Matrix operations**

### ***Exercise***

For the matrices:  $A = \begin{bmatrix} p & 0 \\ q & r \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ , when does  $AB = BA$

### **Solution**

$$\begin{aligned} AB &= \begin{pmatrix} p & 0 \\ q & r \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} p & p \\ q & q+r \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & 0 \\ q & r \end{pmatrix} \\ &= \begin{pmatrix} p+q & r \\ q & r \end{pmatrix} \end{aligned}$$

$$AB = BA$$

$$\begin{pmatrix} p & p \\ q & q+r \end{pmatrix} = \begin{pmatrix} p+q & r \\ q & r \end{pmatrix}$$

$$\begin{cases} p = p+q \\ p = r \\ q+r = r \end{cases} \Rightarrow \begin{cases} q = 0 \\ q = 0 \end{cases}$$

### ***Exercise***

Find values for the variables so that the matrices are equal.  $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

### **Solution**

$$\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$$

$$\Rightarrow \begin{cases} w = 9 & x = 17 \\ y = 8 & z = -12 \end{cases}$$

### Exercise

Find values for the variables so that the matrices are equal.  $\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$

### Solution

$$\begin{cases} \underline{x = 12} \\ y + 3 = 5 \rightarrow \underline{y = 2} \\ 2z = 6 \rightarrow \underline{z = 3} \end{cases}$$

### Exercise

Find values for the variables so that the matrices are equal.  $\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$

### Solution

$$\begin{bmatrix} 5 = y + 3 & x - 4 = 2 & 9 = 9 \\ 2 = z + 4 & -3 = -3 & 8 = 8 \\ 6 = 6 & 0 = 0 & 5 = w \end{bmatrix}$$

$$\rightarrow \begin{cases} y = 2 & z = -2 \\ x = 6 & w = 5 \end{cases}$$

### Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

### Solution

$$\begin{bmatrix} a-5 & 5b & 4c+6 \\ -2d & 7f-6 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

$$\begin{cases} a - 5 = 15 \rightarrow a = 20 \\ 5b = 25 \rightarrow b = 5 \\ 4c + 6 = 6 \rightarrow 4c = 0 \rightarrow c = 0 \\ -2d = -8 \rightarrow d = 4 \\ 7f - 6 = 1 \rightarrow 7f = 7 \rightarrow f = 1 \end{cases}$$

### Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+11 & 12z+1 & 5m \\ 11k & 3 & 1 \end{bmatrix} + \begin{bmatrix} 9a & 9z & 4m \\ 12k & 5 & 3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

### Solution

$$\begin{bmatrix} a+11+9a & 12z+1+9z & 5m+4m \\ 11k+12k & 3+5 & 1+3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 10a+11 & 21z+1 & 9m \\ 23k & 8 & 4 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

$$10a+11=41 \rightarrow 10a=30$$

$$\underline{a=3}$$

$$21z+1=-62 \rightarrow 21z=-63$$

$$\underline{z=-3}$$

$$9m=72 \rightarrow \underline{m=8}$$

$$23k=92 \rightarrow \underline{k=\frac{92}{23}=\underline{4}}$$

### Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} x+2 & 3y+1 & 5z \\ 8w & 2 & 3 \end{bmatrix} + \begin{bmatrix} 3x & 2y & 5z \\ 2w & 5 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$$

### Solution

$$\begin{bmatrix} 4x+2 & 5y+1 & 10z \\ 10w & 7 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$$

$$\begin{cases} 4x+2=10 & \rightarrow \underline{x=2} \\ 5y+1=-14 & \rightarrow \underline{y=-3} \\ 10z=80 & \rightarrow \underline{z=8} \\ 10w=10 & \rightarrow \underline{w=1} \end{cases}$$

### Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} 2x-3 & y-2 & 2z+1 \\ 5 & 2w & 7 \end{bmatrix} + \begin{bmatrix} 3x-3 & y+2 & z-1 \\ -5 & 5w+1 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$$

### Solution

$$\begin{bmatrix} 5x-6 & 2y & 3z \\ 0 & 7w+1 & 10 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$$

$$\begin{cases} 5x-6=20 & \rightarrow x=\frac{26}{5} \\ 2y=8 & \rightarrow y=4 \\ 3z=9 & \rightarrow z=3 \\ 7w+1=8 & \rightarrow w=1 \end{cases}$$

### Exercise

Find a combination  $x_1 w_1 + x_2 w_2 + x_3 w_3$  that gives the zero vector:

$$w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \quad w_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}; \quad w_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are independent or dependent?

The vectors lie in a \_\_\_\_\_.

The matrix W with those columns is not invertible.

### Solution

$$w_1 - 2w_2 + w_3 = 0; \text{ Therefore those vectors are dependent}$$

The vectors lie in a plane.

### Exercise

The very last words say that the 5 by 5 centered difference matrix is not invertible, Write down the 5 equations  $Cx = b$ . Find a combination of left sides that gives zero. What combination of

$b_1, b_2, b_3, b_4, b_5$  must be zero?

### Solution

The 5 by 5 centered difference matrix is

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

The five equations  $Cx = b$  are:

$$x_2 = b_1, \quad -x_1 + x_3 = b_2, \quad -x_2 + x_4 = b_3, \quad -x_3 + x_5 = b_4, \quad -x_4 = b_5.$$

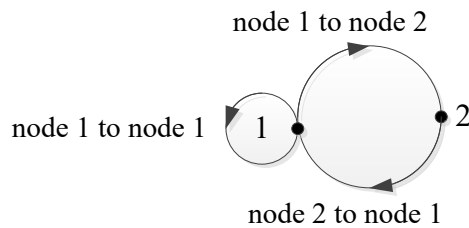
Observe that the sum of the first

$$x_2 - x_2 + x_4 - x_4 = b_1 + b_2 + b_5$$

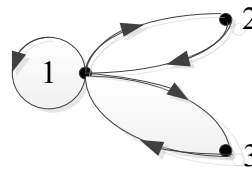
$$0 = b_1 + b_2 + b_5$$

### Exercise

A direct graph starts with  $n$  nodes. There are  $n^2$  possible edges, each edge leaves one of the  $n$  nodes and enters one of the  $n$  nodes (possibly itself). The  $n$  by  $n$  adjacency matrix has  $a_{ij} = 1$  when edge leaves node  $i$  and enter node  $j$ ; if no edge then  $a_{ij} = 0$ . Here are directed graphs and their adjacency matrices:



$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The  $i, j$  entry of  $A^2$  is  $a_{i1}a_{1j} + \dots + a_{in}a_{nj}$ .

Why does that sum count the two-step paths from  $i$  to any node to  $j$ ?

The  $i, j$  entry of  $A^k$  counts  $k$ -steps paths:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{array}{l} \text{counts the paths} \\ \text{with two edges} \end{array} \quad \begin{bmatrix} 1 \text{ to } 2 \text{ to } 1, 1 \text{ to } 1 \text{ to } 1 & 1 \text{ to } 1 \text{ to } 2 \\ 2 \text{ to } 1 \text{ to } 1 & 2 \text{ to } 1 \text{ to } 2 \end{bmatrix}$$

List all 3-step paths between each pair of nodes and compare with  $A^3$ . When  $A^k$  has **no zeros**, that number  $k$  is the diameter of the graph – the number of edges needed to connect the most pair of nodes.

What is the diameter of the second graph?

### Solution

The number  $a_{ik}a_{kj}$  will be “1” if there is an edge from node  $i$  to  $k$  and an edge from  $k$  to  $j$ .

This is a 2-step path. The number  $a_{ik}a_{kj}$  will be “0” if either of those edge (from node  $i$  to  $k$  and from  $k$  to  $j$ ) is missing.

The sum of  $a_{ik}a_{kj}$  is the number of 2-step paths leaving  $i$  and entering  $j$ .

Matrix multiplication is right for this count.

The 3-step paths are counted by  $A^3$ ; we look at paths to node 2:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{array}{l} \text{counts the paths} \\ \text{with three steps} \end{array} \quad \begin{bmatrix} \dots & 1 \text{ to } 1 \text{ to } 1 \text{ to } 2, 1 \text{ to } 2 \text{ to } 1 \text{ to } 2 \\ \dots & 2 \text{ to } 1 \text{ to } 1 \text{ to } 2 \end{bmatrix}$$

The  $A^k$  contain Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, 13, ....

Fibonacci's rule  $F_{k+2} = F_{k+1} + F_k$  show up in  $(A)(A^k) = A^{k+1}$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} = \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix} = A^{k+1}$$

There are **13 six-step** paths from node one to node 1.

## Exercise

$A$  is 3 by 5,  $B$  is 5 by 3,  $C$  is 5 by 1, and  $D$  is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

a)  $AB$

c)  $ABD$

e)  $ABC$

g)  $A(B+C)$

b)  $BA$

d)  $DBA$

f)  $ABCD$

## Solution

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

a)  $AB: (3 \times 5)(5 \times 3) = (3 \times 3)$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{pmatrix}$$

b)  $BA: (5 \times 3)(3 \times 5) = (5 \times 5)$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}$$

c)  $ABD : (3 \times 5)(5 \times 3)(3 \times 1) = (3 \times 1)$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ 15 \\ 15 \end{pmatrix}$$

d)  $DBA : (3 \times 1)(5 \times 3)(3 \times 5) = NA$

e)  $ABC : (3 \times 5)(5 \times 3)(5 \times 1) = NA$

f)  $ABCD : (3 \times 5)(5 \times 3)(5 \times 1)(3 \times 1) = NA$

g)  $A(B + C) : (3 \times 5)((5 \times 3) + (5 \times 1)) = NA$

Matrices  $B$  and  $C$  are not the same size.

## Exercise

What rows or columns or matrices do you multiply to find.

- The third column of  $AB$ ?
- The second column of  $AB$ ?
- The first row of  $AB$ ?
- The second row of  $AB$ ?
- The entry in row 3, column 4 of  $AB$ ?
- The entry in row 2, column 3 of  $AB$ ?

## Solution

- $A$  (column 3 of  $B$ )
- $A$  (column 2 of  $B$ )
- (Row 1 of  $A$ )  $B$
- (Row 2 of  $A$ )  $B$
- (Row 3 of  $A$ ) (Column 4 of  $B$ )
- (Row 2 of  $A$ ) (Column 3 of  $B$ )

### Exercise

Add  $AB$  to  $AC$  and compare with  $A(B + C)$ :

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

### Solution

$$AB = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 0 & 7 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$$

$$\begin{aligned} A(B + C) &= \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \left( \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} AB + AC &= \begin{bmatrix} 0 & 7 \\ 0 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix} \end{aligned}$$

$$\underline{A(B + C) = AB + AC}$$

### Exercise

True or False

- a) If  $A^2$  is defined then  $A$  is necessarily square.
- b) If  $AB$  and  $BA$  are defined then  $A$  and  $B$  are square.
- c) If  $AB$  and  $BA$  are defined then  $AB$  and  $BA$  are square.
- d) If  $AB = B$ , then  $A = I$

### Solution

- a) True
- b) False, if  $A$  has an order  $m$  by  $n$  and  $B$   $n$  by  $m$ :  $AB : m \times m$      $BA : n \times n$
- c) True;  $AB : m \times m$      $BA : n \times n$
- d) False, if  $B$  is the matrix of all zeros.



### Exercise

a) Find a nonzero matrix  $A$  such that  $A^2 = 0$

b) Find a matrix that has  $A^2 \neq 0$  but  $A^3 = 0$

### Solution

a) A nonzero matrix  $A$  such that  $A^2 = 0$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

b) A matrix that has  $A^2 \neq 0$  but  $A^3 = 0$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = A^2 A$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

### Exercise

Suppose you solve  $Ax = b$  for three special right sides  $b$ :

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If the three solutions  $x_1, x_2, x_3$  are the columns of a matrix  $X$ , what is  $A$  times  $X$ ?

### Solution

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Therefore,  $Ax = I$

### ***Exercise***

Show that  $(A + B)^2$  is different from  $A^2 + 2AB + B^2$ , when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

Write down the correct rule for  $(A + B)(A + B) = A^2 + \underline{\hspace{2cm}} + B^2$

### ***Solution***

$$\begin{aligned} A + B &= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} (A + B)^2 &= \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix} \end{aligned}$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}$$

$$2AB = 2 \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} A^2 + 2AB + B^2 &= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 14 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix} \neq \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$$

$$\boxed{(A+B)^2 \neq A^2 + 2AB + B^2}$$

$$\begin{aligned} BA &= \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A^2 + AB + BA + B^2 &= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 4 \\ 5 & 6 \end{bmatrix} \end{aligned}$$

$$\boxed{(A+B)(A+B) = A^2 + \textcolor{teal}{-}AB + \textcolor{red}{BA}\textcolor{teal}{-} + B^2}$$

### Exercise

Find the product of the 2 matrices by rows or by columns:  $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

### Solution

$$\begin{aligned} \text{By rows: } \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} &= \begin{pmatrix} (2 \quad 3) \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ (5 \quad 1) \begin{pmatrix} 4 \\ 2 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} \textcolor{blue}{14} \\ \textcolor{blue}{22} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{By columns: } \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} &= 4 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \textcolor{blue}{14} \\ \textcolor{blue}{22} \end{pmatrix} \end{aligned}$$

### ***Exercise***

Find the product of the 2 matrices by rows or by columns:  $\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

### **Solution**

$$\begin{aligned} \text{By rows: } \begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} &= \begin{pmatrix} (3 \ 6)(2 \ -1) \\ (6 \ 12)(2 \ -1) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{By columns: } \begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} &= 2 \begin{pmatrix} 3 \\ 6 \end{pmatrix} - 1 \begin{pmatrix} 6 \\ 12 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

### ***Exercise***

Find the product of the 2 matrices by rows or by columns:  $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

### **Solution**

$$\begin{aligned} \text{By rows: } \begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} (1 \ 2 \ 4)(3 \ 1 \ 1) \\ (2 \ 0 \ 1)(3 \ 1 \ 1) \end{pmatrix} \\ &= \begin{pmatrix} 1(3) + 2(1) + 4(1) \\ 2(3) + 0(1) + 1(1) \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ 7 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{By columns: } \begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} &= 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ 7 \end{pmatrix} \end{aligned}$$

### Exercise

Find the product of the 2 matrices by rows or by columns:  $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

### Solution

$$\begin{aligned} \text{By rows: } \begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} &= \begin{pmatrix} (1 \ 2 \ 4)(2 \ 2 \ 3) \\ (-2 \ 3 \ 1)(2 \ 2 \ 3) \\ (-4 \ 1 \ 2)(2 \ 2 \ 3) \end{pmatrix} \\ &= \begin{pmatrix} 18 \\ 5 \\ 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{By columns: } \begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} &= 2 \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 18 \\ 5 \\ 0 \end{pmatrix} \end{aligned}$$

### Exercise

Given  $A = \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$   $B = \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix}$  Find  $A + B$ ,  $2A$ , and  $-B$

### Solution

$$\begin{aligned} A + B &= \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix} + \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -2 \\ 8 & -2 & 0 \end{bmatrix} \end{aligned}$$

$$2A = 2 \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 2 & 6 \\ 6 & -2 & -4 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\begin{aligned} -B &= - \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 2 & 3 \\ 1 & 0 & 0 \\ -8 & 2 & 4 \end{bmatrix} \end{aligned}$$

### ***Exercise***

Given  $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$  and  $B = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}$ . Find  $AB$  and  $BA$ .

### **Solution**

$$AB = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 13 \\ 4 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} -16 & 29 \\ -4 & 10 \end{bmatrix}$$

**Note:**  $AB \neq BA$

### ***Exercise***

Given  $A = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$   $B = \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

### **Solution**

$$\begin{aligned} AB &= \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix} \\ &= \begin{pmatrix} -6 & 17 \\ 6 & -8 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 14 \\ 1 & -20 \end{pmatrix} \end{aligned}$$

### ***Exercise***

Given  $A = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$   $B = \begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

### **Solution**

$$\begin{aligned} AB &= \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -3 & -11 \\ 4 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -7 & 1 \\ 16 & 4 \end{pmatrix} \end{aligned}$$

### ***Exercise***

Given  $A = \begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}$   $B = \begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

### **Solution**

$$\begin{aligned} AB &= \begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 10 & 6 \\ 14 & -7 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 14 & -1 \\ 0 & -11 \end{pmatrix} \end{aligned}$$

### ***Exercise***

Given  $A = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}$   $B = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

### **Solution**

$$\begin{aligned} AB &= \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 2 \\ 4 & -4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
 BA &= \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix} \\
 &= \begin{pmatrix} 4 & -4 \\ 14 & -12 \end{pmatrix}
 \end{aligned}$$

### ***Exercise***

Given  $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}$   $B = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

### **Solution**

$$\begin{aligned}
 AB &= \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & -4 & 5 \\ 6 & 0 & 3 \\ -3 & -2 & 1 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 BA &= \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix} \\
 &= \begin{pmatrix} 5 & -13 \\ 3 & -4 \end{pmatrix}
 \end{aligned}$$

### ***Exercise***

Given  $A = \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}$   $B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

### **Solution**

$$\begin{aligned}
 AB &= \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \\
 &= \begin{pmatrix} -1 & 5 & 4 \\ 2 & -3 & 8 \\ -3 & 8 & -5 \end{pmatrix}
 \end{aligned}$$



$$BA = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -14 & 7 \\ -4 & 5 \end{pmatrix}$$

### ***Exercise***

Given  $A = \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix}$   $B = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

### **Solution**

$$AB = \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 24 & 4 \\ 2 & -6 & -2 \\ -13 & 12 & 10 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 8 \\ -10 & 10 \end{pmatrix}$$

### ***Exercise***

Given  $A = \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix}$   $B = \begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$  Find  $AB$  and  $BA$  if possible

### **Solution**

$$AB = \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3(3) + 2(0) - 3(1) & 3(-4) + 2(1) - 3(0) \\ 0(3) + 1(0) + 0(1) & 0(-4) + 1(1) + 0(0) \end{bmatrix}$$

$$\begin{aligned}
&= \begin{bmatrix} 6 & -10 \\ 0 & 1 \end{bmatrix} \\
BA &= \begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 3(3) - 4(0) & 3(2) - 4(1) & 3(-3) - 4(0) \\ 0(3) + 1(0) & 0(2) + 1(1) & 0(-3) + 1(0) \\ 1(3) + 0(0) & 1(2) + 0(1) & 1(-3) + 0(0) \end{bmatrix} \\
&= \begin{bmatrix} 9 & 2 & -9 \\ 0 & 1 & 0 \\ 3 & 2 & -3 \end{bmatrix}
\end{aligned}$$

### ***Exercise***

Given  $A = \begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix}$   $B = \begin{bmatrix} 4 & -2 \\ -2 & 0 \\ 9 & 1 \end{bmatrix}$  Find  $AB$  and  $BA$  if possible

### **Solution**

$AB = \text{Undefined}$

$$\begin{aligned}
BA &= \begin{bmatrix} 4 & -2 \\ -2 & 0 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 22 & 12 \\ -10 & -6 \\ 44 & 27 \end{bmatrix}
\end{aligned}$$

### ***Exercise***

Given  $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$   $B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$  Find  $AB$  and  $BA$  if possible

### **Solution**

$$AB = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}$$

$$BA = \text{Undefined}$$

### Exercise

Given  $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$   $B = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

### Solution

$$\begin{aligned} AB &= \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 11 & -2 & -6 \\ 0 & -1 & 2 \\ 5 & -3 & -4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 6 & -3 & 9 \\ 2 & -3 & 4 \\ 4 & -6 & 3 \end{pmatrix} \end{aligned}$$

### Exercise

Given  $A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix}$   $B = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

### Solution

$$\begin{aligned} AB &= \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 8 & -2 \\ 3 & -8 & 4 \\ -2 & 13 & -5 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
 BA &= \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix} \\
 &= \begin{pmatrix} -4 & 5 & -1 \\ 11 & -5 & 6 \\ -8 & 7 & -4 \end{pmatrix}
 \end{aligned}$$

### Exercise

Given  $A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}$   $B = \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}$ . Find  $AB$  and  $BA$ .

### Solution

$$\begin{aligned}
 AB &= \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix} \\
 &= \begin{pmatrix} -5 & -7 & 2 \\ -6 & 2 & 2 \\ -8 & -6 & 0 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 BA &= \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix} \\
 &= \begin{pmatrix} -1 & 6 & 1 \\ 7 & 0 & 5 \\ 4 & -4 & -2 \end{pmatrix}
 \end{aligned}$$

### Exercise

Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Compute the following (where possible):

$$\text{a) } D + E \quad \text{b) } D - E \quad \text{c) } 5A \quad \text{d) } -7C \quad \text{e) } 2B - C \quad \text{f) } -3(D + 2E)$$

### Solution

$$\begin{aligned}
 a) \quad D + E &= \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} 7 & 6 & 5 \\ -2 & 1 & 3 \\ 7 & 3 & 7 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 b) \quad D - E &= \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 c) \quad 5A &= 5 \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 15 & 0 \\ -5 & 10 \\ 5 & 5 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 d) \quad -7C &= -7 \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} -7 & -28 & -14 \\ -21 & -7 & -35 \end{bmatrix}
 \end{aligned}$$

e) Since  $B$  and  $C$  are not the same size

$2B - C$ : *can't be calculated*

$$\begin{aligned}
 f) \quad -3(D + 2E) &= -3 \left( \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 12 & 2 & 6 \\ -2 & 2 & 4 \\ 8 & 2 & 6 \end{bmatrix} \right) \\
 &= -3 \begin{bmatrix} 13 & 7 & 8 \\ -3 & 2 & 5 \\ 11 & 4 & 10 \end{bmatrix} \\
 &= \begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix}
 \end{aligned}$$

## Exercise

Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} -3 & -1 \\ 2 & 1 \\ 4 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$$

Compute the following (where possible):

- a)  $A + B$       b)  $A + C$       c)  $AB$       d)  $BA$       e)  $CD$       f)  $DC$   
g)  $BD$       h)  $DB$       i)  $A^2$       j)  $B^2$       k)  $D^2$

## Solution

- a) Since  $A$  and  $B$  are not the same size, then

$$A + B = \text{can't be calculated}$$

$$\begin{aligned} \text{b) } A + C &= \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -3 & -1 \\ 2 & 1 \\ 4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 3 \\ 5 & 4 \end{bmatrix} \end{aligned}$$

- c)  $A: 3 \times 2$      $B: 3 \times 3$

$AB$  *can't be calculated*, since the inner are not equal.

- d)  $B: 3 \times 3$      $A: 3 \times 2$

$$\begin{aligned} BA &= \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 12 \\ -1 & 2 \\ -10 & 5 \end{bmatrix} \end{aligned}$$

- e)  $C: 3 \times 2$      $D: 2 \times 2$

$$\begin{aligned} CD &= \begin{bmatrix} -3 & -1 \\ 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -14 & 3 \\ 10 & -2 \\ 22 & -4 \end{bmatrix} \end{aligned}$$

f)  $D: 2 \times 2$   $C: 3 \times 2$

$DC$  *can't be calculated*, since the inner are not equal.

g)  $B: 3 \times 3$   $D: 2 \times 2$

$BD$  *can't be calculated*, since the inner are not equal.

h)  $D: 2 \times 2$   $B: 3 \times 3$

$DB$  *can't be calculated*, since the inner are not equal.

i)  $A^2$  *can't be calculated*, since  $A$  is not square matrix.

$$j) \quad B^2 = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 12 & 8 \\ -2 & -4 & -2 \\ -17 & -16 & 1 \end{bmatrix}$$

$$k) \quad D^2 = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -4 \\ 8 & -2 \end{bmatrix}$$

### Exercise

Let  $B = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$ , show that  $B^4 = \begin{pmatrix} a^4 & 0 \\ a^3 + a^2b + ab^2 + b^3 & b^4 \end{pmatrix}$

### Solution

$$B^2 = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$

$$= \begin{pmatrix} a^2 & 0 \\ a+b & b^2 \end{pmatrix}$$

$$B^4 = B^2 B^2 = \begin{pmatrix} a^2 & 0 \\ a+b & b^2 \end{pmatrix} \begin{pmatrix} a^2 & 0 \\ a+b & b^2 \end{pmatrix}$$

$$= \begin{pmatrix} a^4 & 0 \\ a^3 + a^2b + ab^2 + b^3 & b^4 \end{pmatrix} \quad \checkmark$$

### Exercise

Let  $B = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$ , show that  $B^n = \begin{pmatrix} a^n & 0 \\ \sum_{k=0}^{n-1} a^{n-1-k} b^k & b^n \end{pmatrix}$

### Solution

$$\begin{aligned} n=2 \rightarrow B^2 &= \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} \\ &= \begin{pmatrix} a^2 & 0 \\ a+b & b^2 \end{pmatrix} \quad \checkmark \end{aligned}$$

Let assume  $B^n = \begin{pmatrix} a^n & 0 \\ \sum_{k=0}^{n-1} a^{n-1-k} b^k & b^n \end{pmatrix}$  is true

We need to also prove that it is true for  $B^{n+1} = \begin{pmatrix} a^{n+1} & 0 \\ \sum_{k=0}^n a^{n-k} b^k & b^{n+1} \end{pmatrix}$

$$\begin{aligned} B^{n+1} &= B^n B = \begin{pmatrix} a^n & 0 \\ \sum_{k=0}^{n-1} a^{n-1-k} b^k & b^n \end{pmatrix} \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} \\ &= \begin{pmatrix} a^{n+1} & 0 \\ b^n + a \sum_{k=0}^{n-1} a^{n-1-k} b^k & b^{n+1} \end{pmatrix} \\ &= \begin{pmatrix} a^{n+1} & 0 \\ b^n + \sum_{k=0}^n a^{n-k} b^k & b^{n+1} \end{pmatrix} \\ &= \begin{pmatrix} a^{n+1} & 0 \\ \sum_{k=0}^n a^{n-k} b^k & b^{n+1} \end{pmatrix} \quad \checkmark \end{aligned}$$



### Exercise

Let  $A = \begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix}$ . Prove that  $A^n = \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$  if  $n \geq 1$

### Solution

Using the principle of mathematical induction.

For  $n=1 \rightarrow A = \begin{bmatrix} 1+6 & 4 \\ -9 & 1-6 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix}$  ✓  $P_1$  is true

Assume that  $P_n$  is true,  $A^n = \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$

We need to prove that  $P_{n+1}$ :

$$\begin{aligned} A^{n+1} &= \begin{bmatrix} 1+6(n+1) & 4(n+1) \\ -9(n+1) & 1-6(n+1) \end{bmatrix} \\ &= \begin{bmatrix} 7+6n & 4n+4 \\ -9n-9 & -6n-5 \end{bmatrix} \quad \text{is also true.} \end{aligned}$$

$$\begin{aligned} A^{n+1} &= AA^n \\ &= \begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix} \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix} \\ &= \begin{bmatrix} 7+42n-36n & 28n+4-24n \\ -9-54n+45n & -36n-5+30n \end{bmatrix} \\ &= \begin{bmatrix} 7+6n & 4n+4 \\ -9n-9 & -6n-5 \end{bmatrix} \quad \text{✓ } P_{n+1} \text{ is also true} \end{aligned}$$

$\therefore$  By mathematical induction, the proof of  $A^n = \begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$  is completed.

### Exercise

Let  $A = \begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix}$ . Prove that  $A^n = \begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix}$  if  $n \geq 1$

### Solution

Using the principle of mathematical induction.

For  $n=1 \rightarrow A^1 = \begin{bmatrix} (1+1)a & -a^2 \\ 1a^0 & (1-1)a \end{bmatrix} = \begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix}$  ✓  $P_1$  is true

Assume that  $P_n$  is true,  $A^n = \begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix}$

We need to prove that  $P_{n+1}$  :

$$A^{n+1} = \begin{bmatrix} (n+2)a^{n+1} & -(n+1)a^{n+2} \\ (n+1)a^n & -na^{n+1} \end{bmatrix} \text{ is also } \textcolor{red}{true}.$$

$$\begin{aligned} A^{n+1} &= AA^n \\ &= \begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix} \\ &= \begin{bmatrix} 2(n+1)a^{n+1} - na^{n+1} & -2na^{n+2} - (1-n)a^{n+2} \\ (n+1)a^n & -na^{n+1} \end{bmatrix} \\ &= \begin{bmatrix} (2n+2-n)a^{n+1} & -(2n+1-n)a^{n+2} \\ (n+1)a^n & -na^{n+1} \end{bmatrix} \\ &= \begin{bmatrix} (n+2)a^{n+1} & -(n+1)a^{n+2} \\ (n+1)a^n & -na^{n+1} \end{bmatrix} \checkmark P_{n+1} \text{ is also true} \end{aligned}$$

$\therefore$  By mathematical induction, the proof of  $A^n = \begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix}$  is completed.

### ***Exercise***

The following system of recurrence relations holds for all  $n \geq 0$

$$\begin{cases} x_{n+1} = 7x_n + 4y_n \\ y_{n+1} = -9x_n - 5y_n \end{cases}$$

Solve the system for  $x_n$  and  $y_n$  in terms of  $x_0$  and  $y_0$

### **Solution**

$$\begin{cases} x_{n+1} = 7x_n + 4y_n \\ y_{n+1} = -9x_n - 5y_n \end{cases}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$X_{n+1} = AX_n$$

$$A = \begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix} \quad X_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$X_1 = AX_0$$

$$X_2 = AX_1 = A(AX_0) = A^2X_0$$

$$X_3 = AX_2 = A(A^2X_0) = A^3X_0$$

$$\vdots \quad \vdots$$

$$X_n = A^nX_0$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix}^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Since, when  $\begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix}$  that implies  $A^n = \begin{pmatrix} 1+6n & 4n \\ -9n & 1-6n \end{pmatrix}$  (from previous prove).

$$\begin{aligned} \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= \begin{pmatrix} 1+6n & 4n \\ -9n & 1-6n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \begin{pmatrix} (1+6n)x_0 + 4ny_0 \\ -9nx_0 + (1-6n)y_0 \end{pmatrix} \end{aligned}$$

$$\therefore \begin{cases} x_n = (1+6n)x_0 + 4ny_0 \\ y_n = -9nx_0 + (1-6n)y_0 \end{cases}$$

### Exercise

If  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , prove that  $A^2 - (a+d)A + (ad-bc)I_{2 \times 2} = 0$

### Solution

$$\begin{aligned} A^2 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} \end{aligned}$$

$$A^2 - (a+d)A + (ad-bc)I = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} - (a+d) \begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} - \begin{pmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{pmatrix} + \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\
&= \begin{pmatrix} a^2 + bc - a^2 - ad + ad - bc & ab + bd - ab - bd \\ ac + cd - ac - cd & bc + d^2 - ad - d^2 + ad - bc \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \underline{0} \quad \checkmark
\end{aligned}$$

### Exercise

If  $A = \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}$ , use the fact  $A^2 = 4A - 3I$  and mathematical induction, to prove that

$$A^n = \frac{3^n - 1}{2} A + \frac{3 - 3^n}{2} I \quad \text{if } n \geq 1$$

### Solution

$$\begin{aligned}
A^2 &= \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 13 & -12 \\ 4 & -3 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
4A - 3I &= 4 \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 16 & -12 \\ 4 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 13 & -12 \\ 4 & -3 \end{pmatrix} \\
&= \underline{A^2}
\end{aligned}$$

Using mathematical induction model

$$\text{For } n=1 \rightarrow A^1 = \frac{3^1 - 1}{2} A + \frac{3 - 3^1}{2} I$$

$$A = A + 0 = A \quad \checkmark \quad \text{is true for } P_1$$

$$\text{Assume is true for } P_k \rightarrow A^k = \frac{3^k - 1}{2} A + \frac{3 - 3^k}{2} I$$

$$\text{We need to prove that is also true for } P_{k+1} \rightarrow A^{k+1} = \frac{3^{k+1} - 1}{2} A + \frac{3 - 3^{k+1}}{2} I$$

$$A^{k+1} = A A^k$$

$$\begin{aligned}
&= A \left( \frac{3^k - 1}{2} A + \frac{3 - 3^k}{2} I \right) \\
&= \frac{3^k - 1}{2} A^2 + \frac{3 - 3^k}{2} (AI) \quad A^2 = 4A - 3I \\
&= \frac{3^k - 1}{2} (4A - 3I) + \frac{3 - 3^k}{2} A \\
&= 2(3^k - 1)A - \frac{3(3^k - 1)}{2} I + \frac{3 - 3^k}{2} A \\
&= \left( 2 \cdot 3^k - 2 + \frac{3 - 3^k}{2} \right) A - \frac{3^{k+1} - 3}{2} I \\
&= \left( \frac{4 \cdot 3^k - 4 + 3 - 3^k}{2} \right) A - \frac{3^{k+1} - 3}{2} I \\
&= \left( \frac{3 \cdot 3^k - 1}{2} \right) A - \frac{3^{k+1} - 3}{2} I \\
&= \frac{3^{k+1} - 1}{2} A + \frac{3 - 3^{k+1}}{2} I \quad \checkmark \text{ is also true for } P_{k+1}
\end{aligned}$$

By mathematical induction, the proof that  $A^n = \frac{3^n - 1}{2} A + \frac{3 - 3^n}{2} I$  if  $n \geq 1$  is completed.

### Exercise

A sequence of numbers  $x_1, x_2, \dots, x_n, \dots$  satisfies the recurrence relation  $x_{n+1} = ax_n + bx_{n-1}$  for  $n \geq 1$ , where  $a$  and  $b$  are constants. Prove that

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = A \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

Where  $A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$  and hence express  $\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$  in terms of  $\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$ .

If  $a = 4$  and  $b = -3$ , use the previous question to find a formula for  $x_n$  in terms  $x_1$  and  $x_0$

### Solution

$$\begin{aligned}
x_{n+1} &= ax_n + bx_{n-1} \\
&= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}
\end{aligned}$$

$$x_n = x_n$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

$$= A \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

$$x_{n+1} = ax_n + bx_{n-1}$$

$$= 4x_n - 3x_{n-1} \quad |$$

$$n=1 \rightarrow x_2 = 4x_1 - 3x_0$$

$$n=2 \rightarrow x_3 = 4x_2 - 3x_1$$

$$= 4(4x_1 - 3x_0) - 3x_1 = (4^2 - 3)x_1 - 3x_0$$

$$= 13x_1 - 12x_0$$

$$n=3 \rightarrow x_4 = 4x_3 - 3x_2$$

$$= 4(13x_1 - 12x_0) - 3(4x_1 - 3x_0)$$

$$= 40x_1 - 39x_0$$

$$n=4 \rightarrow x_5 = 4x_4 - 3x_3$$

$$= 4(40x_1 - 39x_0) - 3(13x_1 - 12x_0)$$

$$= 121x_1 - 120x_0$$

	$x_1$	$x_0$
$n=2 \rightarrow$	4	-3
$n=3 \rightarrow$	13	-12
$n=4 \rightarrow$	40	-39
$n=5 \rightarrow$	121	-120

$$x_n = \frac{3^n - 1}{2}x_1 + \frac{3 - 3^n}{2}x_0 \quad |$$