Chapter 13. Partial Differential Equations

Section 13.1. Derivation of the Heat Equation

1. The temperature satisfies Dirichlet conditions as given in (1.16). With the data as given the temperature must satisfy

$$\begin{aligned} &u_t(x,t) = ku_{xx}, \quad \text{for } 0 \le x \le L \text{ and } t > 0, \\ &u(0,t) = 5, \quad \text{and} \quad u(L,t) = 25, \quad \text{for } t > 0, \\ &u(x,0) = 15, \quad \text{for } 0 \le x \le L. \end{aligned}$$

2. This follows immediately from the maximum principle.

3. By the equation just before (1.3),

$$\frac{dQ}{dt} = S\int_a^b \frac{\partial}{\partial t}[c\rho u]\,dx.$$

By (1.7)

$$\frac{dQ}{dt} = S\int_a^b \frac{\partial}{\partial x}\left(C\frac{\partial u}{\partial x}\right)dx.$$

Setting these equal, and using the argument leading to (1.9), we get the result.

4. Since the rod is insulated at the endpoints, we have $u_x(0,t) = u_x(L,t) = 0$. Then (1.7) says that $Q_t = 0$, so Q is unchanging.

5. Since α and β are constants, the linearity of differentiation implies that $w_t = \alpha u_t + \beta v_t$ and $w_{xx} = \alpha u_{xx} + \beta v_{xx}$. The result follows immediately.

6. Let $w(x,t) = u(x,t) - v(x,t)$. By the linearity of the heat equation, w is a solution. Since u and v satisfy the same initial and boundary conditions, we have $w(0,t) = 0$ and $w(L,t) = 0$ for $t \ge 0$, and $w(x,0) = 0$ for $0 \le x \le L$. By the maximum principle, $w(x,t) = 0$ for $t \ge 0$ and $0 \le x \le L$. Therefore, $u(x,t) = v(x,t)$ for $t \ge 0$ and $0 \le x \le L$.

7. From Table 1 we see that the thermal diffusivity of aluminum is $k = 0.86$. The boundary conditions are Dirichlet conditions so modifying (1.16) we get

$$\begin{aligned} &u_t(x,t) = 0.86u_{xx}, \quad \text{for } 0 < x < L \text{ and } t > 0, \\ &u(0,t) = 20, \quad \text{and} \quad u(L,t) = 35, \quad \text{for } t > 0, \\ &u(x,0) = 15, \quad \text{for } 0 \le x \le L. \end{aligned}$$

8. From Table 1 we see that the thermal diffusivity of gold is $k = 1.18$. We have a Dirichlet condition at the left-hand endpoint and a Neumann condition at the right-hand endpoint, so the initial/boundary value problem is

$$\begin{aligned} &u_t(x,t) = 1.18u_{xx}, \quad \text{for } 0 < x < L \text{ and } t > 0, \\ &u(0,t) = 20, \quad \text{and} \quad u_x(L,t) = 0, \quad \text{for } t > 0, \\ &u(x,0) = 15, \quad \text{for } 0 \le x \le L. \end{aligned}$$

9. From Table 1 we see that the thermal diffusivity of silver is $k = 1.71$. We now have a Robin condition at the left-hand endpoint.

$$\begin{aligned} &u_t(x,t) = 1.71u_{xx}, \quad \text{for } 0 < x < L \text{ and } t > 0, \\ &u_x(0,t) = 0.0013(u(0,t) - 15), \quad \text{and} \\ &u(L,t) = 35, \quad \text{for } t > 0, \\ &u(x,0) = 15, \quad \text{for } 0 \le x \le L. \end{aligned}$$

Section 13.2. Separation of Variables for the Heat Equation

1. The thermal diffusivity of gold is $k = 1.18\,\text{cm}^2/\text{sec}$. We will let the unit of length be centimeters, so $L = 50$. The boundary conditions are $u(0, t) = 0$ and $u(50, t) = 0$, so the steady-state temperature is 0. Hence the solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin(n\pi x/L),$$

where the coefficients are

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L 100 \sin(n\pi x/L)\, dx \\ &= \frac{200}{n\pi}[1 - \cos(n\pi)] \\ &= \begin{cases} \frac{400}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

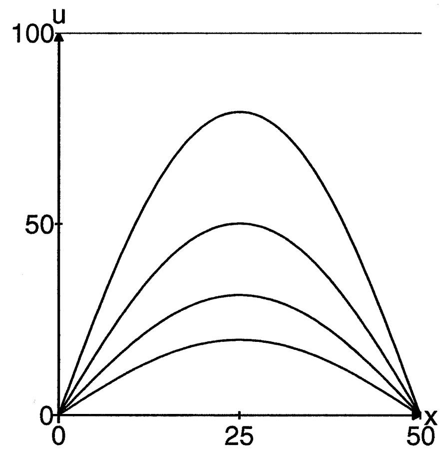
Thus the solution is given by

$$\begin{aligned} u(x, t) = \sum_{p=0}^{\infty} & \frac{400}{(2p+1)\pi} e^{-1.18\times(2p+1)^2\pi^2 t/2500} \\ & \times \sin\left(\frac{n\pi x}{50}\right). \end{aligned}$$

For $t = 100$, the term for $p = 1$ is bounded by 0.65. Hence one term of the series will suffice to estimate the temperature within 1°. Using just this one term, we solve

$$\frac{400}{\pi} e^{-1.18\times\pi^2 t/2500} = 10$$

for $t = 546$ sec. Hence we see that it will take about 546 sec for the temperature to drop below 10°C. The temperature at 100 second intervals is plotted below.



2. The thermal diffusivity of aluminum is $k = 0.84\,\text{cm}^2/\text{sec}$. For $t = 100$, the term for $p = 1$ is bounded by 2.14, while that for $p = 2$ is bounded by 0.006. Hence two terms will suffice to compute the temperature for $t = 100$ sec. Looking at the the term for $p = 0$, we solve

$$\frac{400}{\pi} e^{-0.84\times\pi^2 t/2500} = 10$$

for $t = 767$ sec. For such a time, all of the other terms are extremely small, so we see that it will take about 767 sec for the temperature to drop below 10°C.

The thermal diffusivity of silver is $k = 1.7\,\text{cm}^2/\text{sec}$. For $t = 100$, the term for $p = 1$ is bounded by 0.1.Hence one term will suffice to compute the temperature for $t = 100$ sec. Looking at the the term for $p = 0$, we solve

$$\frac{400}{\pi} e^{-1.7\times\pi^2 t/2500} = 10$$

for $t = 379$ sec. For such a time, all of the other terms are extremely small, so we see that it will take about 379 sec for the temperature to drop below 10°C.

The thermal diffusivity of PVC is $k = 0.0004\,\text{cm}^2/\text{sec}$. For $t = 1$ day $= 86,400$, sec, the term for $p = 2$ is bounded by 0.1.5, while that for $p = 3$ is bounded by 0.07. Hence three terms are needed to compute the temperature for $t = 1$ day. Looking at the the term for $p = 0$, we solve

$$\frac{400}{\pi} e^{-0.0004\times\pi^2 t/2500} = 10$$

for $t = 1.611\times10^6$ sec $= 18.6$ days. For such a time, all of the other terms are extremely small, so we see that it will take about 18, 5 days for the temperature to drop below 10°C.

3. (a) For $0 \le x \le 10$, the steady-state temperature is given by $f(x) = 20 - x$.

   (b) Let $u(x,t)$ be the temperature. It must solve the initial/boundary value problem

$$\begin{aligned} u_t - ku_{xx} &= 0, \quad \text{for } 0 < x < 10 \text{ and } t > 0,\\ u(0,t) &= 20 \quad \text{and} \quad u(10,t) = -10,\\ u(x,0) &= 20 - x. \end{aligned}$$

   This solution will tend to a new steady-state temperature $u_{ss}(x)$ with $u_{ss}(0) = 20$ and $u_{ss}(10) = -10$. Hence $u_{ss}(x) = 20 - 3x$. The difference $v(x,t) = u(x,t) - u_{ss}(x)$ must solve

$$\begin{aligned} v_t - kv_{xx} &= 0, \quad \text{for } 0 < x < 10 \text{ and } t > 0,\\ v(0,t) &= 0 \quad \text{and} \quad v(10,t) = 0,\\ v(x,0) &= (20 - x) - (20 - 3x) = 2x. \end{aligned}$$

   The solution v is given by

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/100} \sin\frac{n\pi x}{10},$$

   where $k = 0.0057\,\text{cm}^2/\text{sec}$ is the thermal diffusivity of brick, and

$$\begin{aligned} b_n &= \frac{2}{10}\int_0^{10} 2x \sin\left(\frac{n\pi x}{10}\right) dx\\ &= (-1)^{n+1}\frac{40}{n\pi}. \end{aligned}$$

   Hence the solution is

$$\begin{aligned} u(x,t) &= (20 - 3x)\\ &\quad + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{40}{n\pi}e^{-0.0057\times n^2\pi^2 t/100}\\ &\quad \times \sin\frac{n\pi x}{10}. \end{aligned}$$

4. The temperature $u(x,t) = u_{ss}(x) + v(x,t)$, where u_{ss} is the steady-state temperature and v is the transient temperature. The steady state temperature satisfies $u_{ss}(0) = 20$ and $u_{ss}(10) = 420$, so $u_{ss}(x) = 20 + 40x$. The transient temperature satisfies

$$\begin{aligned} v_t - kv_{xx} &= 0, \quad \text{for } 0 < x < 10 \text{ and } t > 0,\\ v(0,t) &= 0 \quad \text{and} \quad v(10,t) = 0,\\ v(x,0) &= u(x,0) - u_{ss}(x) = -40x. \end{aligned}$$

   The solution is given by

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/100},$$

   where $k = 5\times10^{-5}\text{cm}^2/\text{sec}$, and

$$\begin{aligned} b_n &= \frac{2}{10}\int_0^{10}(-40x)\sin\left(\frac{n\pi x}{10}\right) dx\\ &= (-1)^n\frac{800}{n\pi}. \end{aligned}$$

   Hence the solution is

$$\begin{aligned} u(x,t) &= (20 + 40x)\\ &\quad + \sum_{n=1}^{\infty}(-1)^n\frac{800}{n\pi}e^{-0.00005\times n^2\pi^2 t/100}. \end{aligned}$$

5. Since $T_0 = T_L = 0$, the steady-state temperature is $0°$. The solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-4n^2\pi^2 t}\sin(n\pi x),$$

   where the coefficients are computed from

$$b_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx.$$

Computing this integral we see that $b_n = 0$ if n is even, and $b_{2n+1} = 4/((2n+1)^3\pi^3)$. Thus the solution is

$$u(x,t) = \frac{4}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}e^{-4(2n+1)^2\pi^2 t} \times \sin((2n+1)\pi x).$$

6. Since $T_0 = T_L = 0$, the steady-state temperature is $0°$. The Fourier series for the initial temperature is given, so we can immediately write down the formula for the temperature

$$u(x,t) = e^{-8t}\sin 2x - e^{-32t}\sin 4x.$$

7. Since $T_0 = T_L = 0$, the steady-state temperature is $0°$. The solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t}\sin(nx),$$

where the coefficients are

$$b_n = \frac{2}{\pi}\int_0^{\pi}\sin^2 x \sin(nx)\,dx = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -8/[\pi n(n^2-4)] & \text{if } n \text{ is odd.}\end{cases}$$

8. Since $T_0 = 0$ and $T_L = 2$, the steady-state temperature is $u_s(x) = 2x$. It remains to find $v(x,t) = u(x,t) - u_s(x)$, which solves the initial/boundary value problem

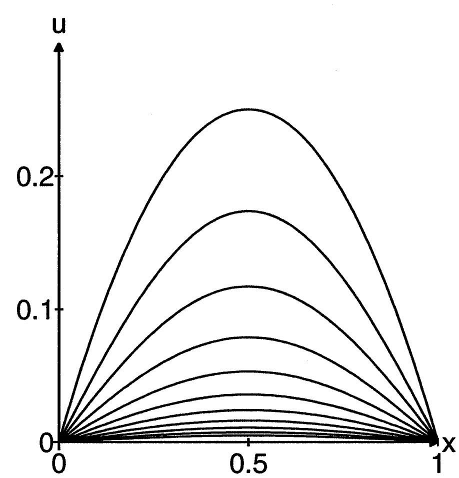
$$\begin{aligned} &v_t(x,t) = v_{xx}(x,t), \quad \text{for } t > 0 \text{ and } 0 \le x \le L, \\ &v(0,t) = 0 \quad \text{and} \quad v(L,t) = 0, \quad \text{for } t > 0, \\ &v(x,0) = -x, \quad \text{for } 0 \le x \le L. \end{aligned}$$

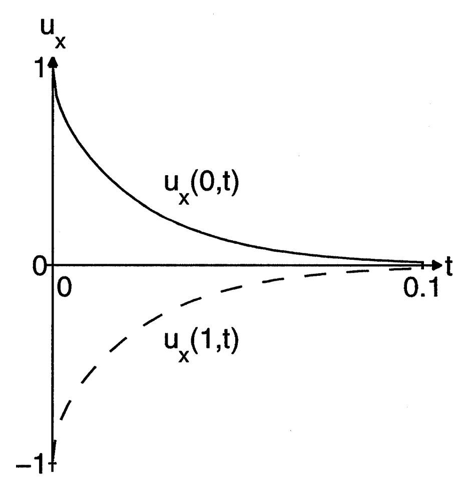
This problem is almost the same as the one in Example 2.17. The solution is

$$v(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 t}\sin n\pi x.$$

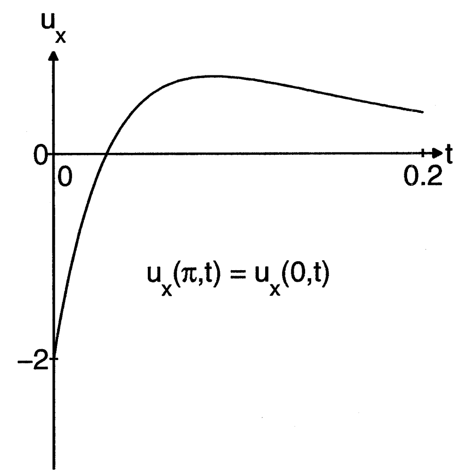
The complete solution is $u(x,t) = u_s(x) + v(x,t)$.

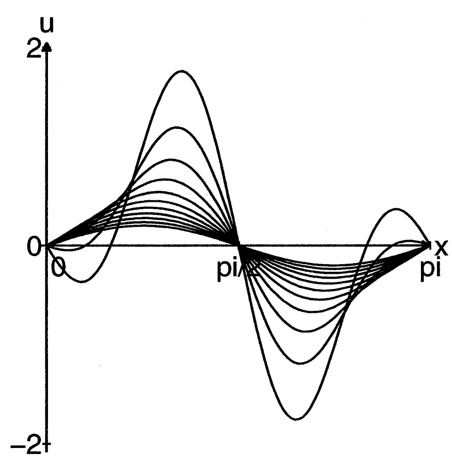
9. This solution is plotted in the first figure for $t = 0$ and then for values of t at intervals of 0.01. The temperature profiles show a steady decrease in temperature as it approaches steady-state. The partial u_x at the endpoints is plotted in the second figure. Notice that $u_x(0,t) > 0$ and $u_x(1,t) < 0$, reflecting the fact that heat is flowing out of the rod at each endpoint, leading to the decrease in the temperature in the rod.



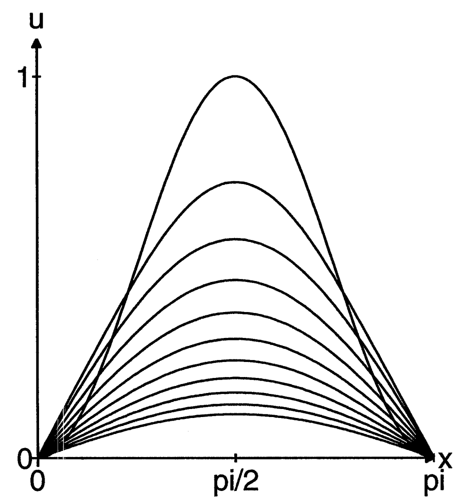


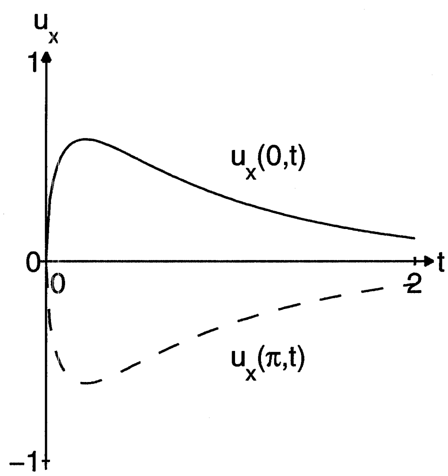
10. The solution is plotted in the first figure for $t = 0$ and then for values of t at intervals of 0.02. The temperature profiles show a steady decrease in in the total heat in the rod as time increases. The temperature decreases from hot spots and increases from cold spots as it approaches steady-state. The partial u_x at the endpoints is plotted in the second figure. Notice that $u_x(0,t) = u_x(\pi,t)$, for all t. As a result, heat flows into the rod at one endpoint as fast as it leaves at the other. The total amount of heat in the rod is constant. Nevertheless the temperature is approaching 0 at each point.



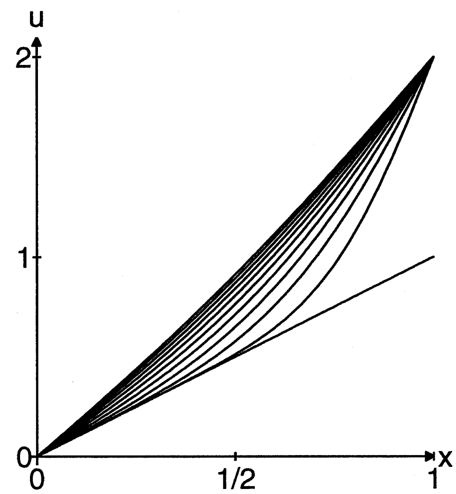


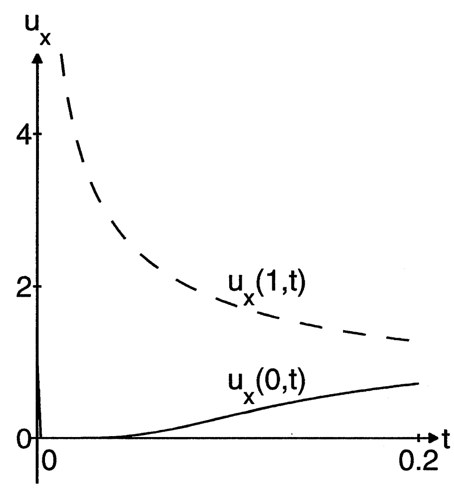
11. This solution is plotted in the first figure for $t = 0$ and then for values of t at intervals of 0.2. The temperature profiles show a steady decrease in temperature as it approaches steady-state. The partial u_x at the endpoints is plotted in the second figure. Notice that $u_x(0,t) > 0$ and $u_x(1,t) < 0$, reflecting the fact that heat is flowing out of the rod at each endpoint, leading to the decrease in the temperature in the rod. However, notice that there is an increase in the temperature near the endpoints for a short period of time. This is because heat flow faster from th center of the rod into these areas than it flows out through the endpoints.





12. This solution is plotted in the first figure for $t = 0$ and then for values of t at intervals of 0.02. The temperature profiles show a steady increase in temperature as it approaches steady-state. The partial u_x at the endpoints is plotted in the second figure. Notice that $u_x(0, t) > 0$ and $u_x(1, t) > 0$, reflecting the fact that heat is flowing out of the rod at $x = 0$, but into the rod at $x = 1$. Since the temperature at $x = 0$ is smaller than that at $x = 1$, heat flows from $x = 1$ to $x = 0$.





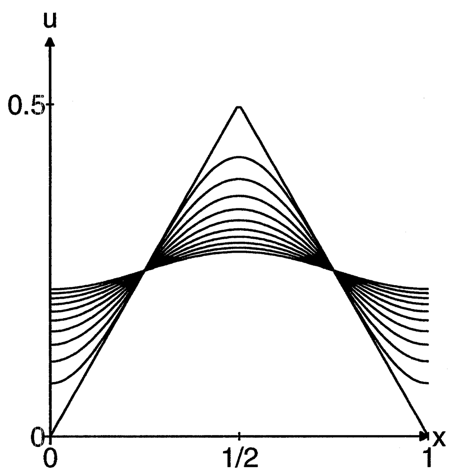
13. The solution is

$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t} \cos(n\pi x),$$

where the coefficients a_n are the Fourier cosine coefficients of f. After computing the coefficients, we get the series

$$u(x,t) = \frac{1}{4} - \frac{8}{\pi^2}\sum_{n=0}^{\infty} \frac{1}{(4n+2)^2} e^{-(4n+2)^2\pi^2 t}$$
$$\times \cos((4n+2)\pi x).$$

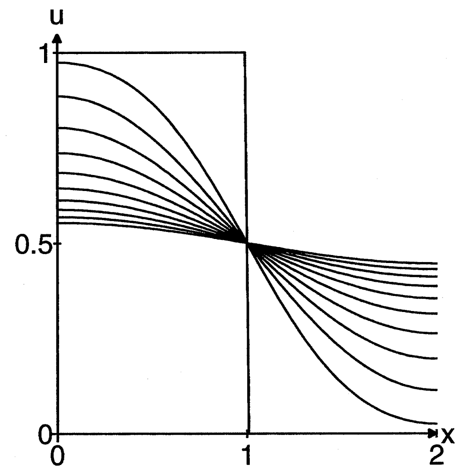
The solution is plotted in the next figure at $t = 0$ and then at intervals of 0.005. The temperature approaches the steady-state temperature of 1/4, which is the average temperature in the rod initially.



14. The solution is

$$u(x,t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{n} e^{-(2n+1)^2\pi^2 t/4}$$
$$\times \cos\left(\frac{(2n+1)\pi x}{2}\right).$$

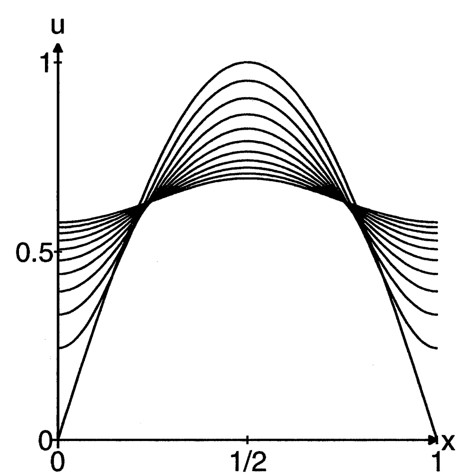
The solution is plotted in the next figure at $t = 0$ and then at intervals of 0.1. The temperature approaches the steady-state temperature of 1/2, which is the average temperature in the rod initially.



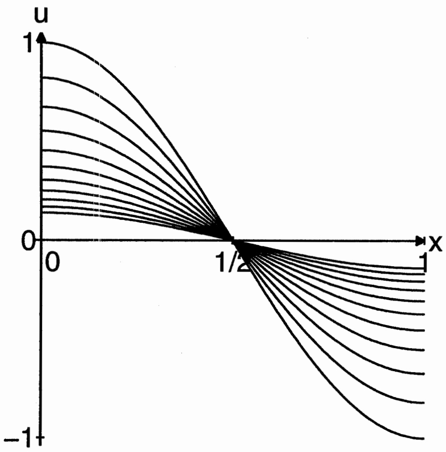
15. The solution is

$$u(x,t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1} e^{-4n^2\pi^2 t} \cos(2n\pi x).$$

The solution is plotted in the next figure at $t = 0$ and then at intervals of 0.1. The temperature approaches the steady-state temperature of $2/\pi$, which is the average temperature in the rod initially.



16. The solution is $u(x,t) = e^{-\pi^2 t}\cos(\pi x)$. The solution is plotted in the next figure at $t = 0$ and then at intervals of 0.02. The temperature approaches the steady-state temperature of 0, which is the average temperature in the rod initially.



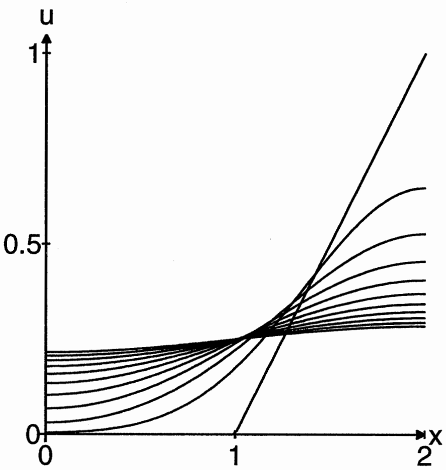
17. The solution is

$$u(x,t) = \frac{1}{4} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t/4}\cos\left(\frac{n\pi x}{2}\right),$$

where for $n \geq 1$,

$$a_n = \frac{4}{n^2\pi^2} \times \begin{cases} 0, & \text{if } n = 4k, \\ -1, & \text{if } n = 4k \pm 1, \\ 2, & \text{if } n = 4k+2. \end{cases}$$

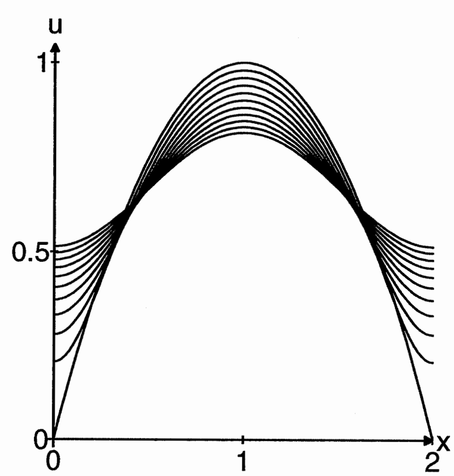
The solution is plotted in the next figure at $t = 0$ and then at intervals of 0.1. The temperature approaches the steady-state temperature of 1/4, which is the average temperature in the rod initially.



18. The solution is

$$u(x,t) = \frac{2}{3} - \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{4n^2} e^{-n^2\pi^2 t}\cos(n\pi x).$$

The solution is plotted in the next figure at $t = 0$ and then at intervals of 0.01. The temperature approaches the steady-state temperature of 2/3, which is the average temperature in the rod initially.



19. Since v does not depend on t, the partial differential equation reduces to $-kv'' = p/c\rho$, which is equivalent to what is to be shown. If $u(x,t) = u_h(x,t) + v(x)$, then

$$\frac{\partial u}{\partial t} = \frac{\partial u_h}{\partial t} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u_h}{\partial x^2} + v'' = \frac{\partial^2 u_h}{\partial x^2} - \frac{P}{k}.$$

Hence

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u_h}{\partial t} - k\left(\frac{\partial^2 u_h}{\partial x^2} - \frac{P}{k}\right) = P.$$

In addition, $u(x,0) = u_h(x,0) + v(x) = f(x)$. Finally, $u(0,t) = u_h(0,t) + v(0) = A$ and $u(L,t) = u_h(L,t) + v(L) = B$.

20. Since $v'' = -P(x)/k = -6x$, we have $v(x) = -x^2 + \alpha x + \beta$, where α and β are constants of integration. The boundary conditions say that $0 = \beta$ and $1 = -1 + \alpha + \beta$. Hence $\beta = 0$ and $\alpha = 2$, so $v(x) = 2x - x^3$.

We have to solve the initial/boundary value problem for the heat equation with initial value $u_h(x,0) = f(x) - v(x) = \sin \pi x - 2x + x^3$. The Fourier sine series for v is

$$v(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x,$$

where

$$a_n = 2\int_0^1 v(x)\sin n\pi x\,dx = 2\int_0^1 (2x - x^3)\sin n\pi x\,dx.$$

After integrating by parts three times, we find that

$$a_n = (-1)^{n+1}\left[\frac{2}{n\pi} + \frac{12}{n^3\pi^3}\right].$$

Therefore

$$\begin{aligned} u_h(x,t) &= e^{-\pi t}\sin \pi^2 x \\ &\quad + \sum_{n=1}^{\infty}(-1)^n e^{-n^2\pi^2 t}\left[\frac{2}{n\pi} + \frac{12}{n^3\pi^3}\right]\sin n\pi x. \end{aligned}$$

Finally,

$$\begin{aligned} u(x,t) &= u_h(x,t) + v(x) \\ &= e^{-\pi t}\sin \pi^2 x \\ &\quad + \sum_{n=1}^{\infty}(-1)^n e^{-n^2\pi^2 t}\left[\frac{2}{n\pi} + \frac{12}{n^3\pi^3}\right]\sin n\pi x \\ &\quad + 2x - x^3. \end{aligned}$$

21. Since $v'' = -P(x)/k = -e^{-x}$, we have $v(x) = -e^{-x} + \alpha x + \beta$, where α and β are constants of integration. The boundary conditions say that $1 = \beta$ and $-1/e = -1/e + \alpha + \beta$. Hence $\beta = 1$ and $\alpha = -1$, so $v(x) = 1 - x - e^{-x}$.

We have to solve the initial/boundary value problem for the heat equation with initial value $u_h(x,0) = f(x) - v(x) = \sin \pi x + e^{-x} - 1 + x$. The Fourier sine series for v is

$$v(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x,$$

where

$$\begin{aligned} a_n &= 2\int_0^1 v(x)\sin n\pi x\,dx \\ &= 2\int_0^1 (1 - x - e^{-x})\sin n\pi x\,dx \\ &= \frac{2}{n\pi} - \frac{2n\pi[1 - (-1)^n/e]}{1 + n^2\pi^2}. \end{aligned}$$

Therefore

$$u_h(x,t) = e^{-\pi^2 t}\sin \pi x - \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t}\sin n\pi x.$$

Finally,

$$\begin{aligned} u(x,t) &= u_h(x,t) + v(x) \\ &= 1 - x - e^{-x} + e^{-\pi^2 t}\sin \pi x \\ &\quad - \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t}\sin n\pi x. \end{aligned}$$

22. Suppose that $\lambda < 0$, and write $\lambda = -\mu^2$, where $\mu > 0$. The differential equation in (2.27) becomes $X'' - \mu^2 X = 0$, which has general soluton $X(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$. Since $X'(x) = \mu[C_1 e^{\mu x} - C_2 e^{-\mu x}]$, the first boundary condition says that $0 = X'(0) = C_1 - C_2$. Hence $C_2 = C_1$ and $X(x) = C_1(e^{\mu x} + e^{-\mu x})$. The second boundary condition then says that $0 = X'(L) = C_1(e^{\mu L} - e^{\mu L})$. For all $L > 0$, $e^{\mu L} > 0 > e^{-\mu L}$, so we conclude that $C_1 = 0$. Hence the only solution to the boundary value problem is $X(x) = 0$, so $\lambda = -\mu^2$ is not an eigenvalue.

Section 13.3. The Wave Equation

1. The Fourier sine series for f is $\sum_{n=1}^{\infty} a_n \sin n\pi x$, where

$$\begin{aligned} a_n &= 2\int_0^1 f(x)\sin n\pi x\,dx \\ &= \frac{1}{2}\int_0^1 x(1-x)\sin n\pi x\,dx \\ &= \begin{cases} 0, & \text{if } n \text{ is even,} \\ \dfrac{2}{n^3\pi^3}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

From (3.5), we conclude that

$$\sum_{k=0}^{\infty} \frac{2\sin((2k+1)\pi x)\cos((2k+1)\pi t)}{(2k+1)^3\pi^3}.$$

2. The Fourier sine series for f is $\sum_{n=1}^{\infty} a_n \sin n\pi x$, where

$$a_n = \frac{1}{5}\int_0^{10} f(x)\sin\frac{n\pi x}{10}\,dx.$$

Since $f(10-x) = f(x)$, and $\sin(n\pi - \alpha) = -\cos n\pi \sin\alpha$, it follows that

$$\begin{aligned} &\int_0^{10} f(x)\sin\frac{n\pi x}{10}\,dx \\ &= \int_0^5 f(x)\sin\frac{n\pi x}{10}\,dx \\ &\quad + \int_5^{10} f(x)\sin\frac{n\pi x}{10}\,dx \\ &= [1-(-1)^n]\int_0^5 f(x)\sin\frac{n\pi x}{10}\,dx. \end{aligned}$$

Hence $a_n = 0$ if n is even, and

$$\begin{aligned} a_{2k+1} &= \frac{1}{50}\int_0^5 x\sin\frac{(2k+1)\pi x}{10}\,dx \\ &= (-1)^k\frac{4}{(2k+1)^2\pi^2}. \end{aligned}$$

From (3.5), we conclude that

$$\begin{aligned} &u(x,t) \\ &= 4\sum_{k=0}^{\infty} \frac{(-1)^k\sin((2k+1)\pi x)\cos((2k+1)\pi t)}{(2k+1)^2\pi^2}. \end{aligned}$$

3. The Fourier sine series for g is $\sum_{n=1}^{\infty} b_n \sin n\pi x$, where

$$\begin{aligned} b_n &= 2\int_0^1 \sin n\pi x\,dx \\ &= \frac{2}{n\pi}[1-\cos n\pi] \\ &= \begin{cases} 0 & n \text{ even} \\ 4/n\pi & n \text{ odd.} \end{cases} \end{aligned}$$

From (3.5) and (3.7), we conclude that

$$u(x,t) = 4\sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi x \cdot \sin(2n+1)\pi t}{(2n+1)^2\pi^2}.$$

4. The Fourier sine series for g is $\sum_{n=1}^{\infty} b_n \sin n\pi x$,

where

$$\begin{aligned} b_n &= 2\int_0^1 g(x)\sin n\pi x\,dx \\ &= -\int_0^{1/2} \sin n\pi x\,dx + \int_{1/2}^1 \sin n\pi x\,dx \\ &= \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - 1 - \cos n\pi + \cos\frac{n\pi}{2}\right]. \end{aligned}$$

Thus $b_n = 0$ if n is odd, $b_{4n} = 0$ and $b_{4n+2} = -4/n\pi$. From (3.5) and (3.7), we conclude that

$$\begin{aligned} &u(x,t) \\ &= -4\sum_{k=0}^{\infty} \frac{\sin(4k+2)\pi x \cdot \sin\sin(4k+2)\pi t}{(4n+2)^2\pi^2}. \end{aligned}$$

5. The Fourier sine series for g is $\sum_{n=1}^{\infty} b_n \sin n\pi x$, where

$$\begin{aligned} b_n &= \frac{2}{3}\int_0^3 g(x)\sin\frac{n\pi x}{3}\,dx \\ &= \frac{2}{3}\int_1^2 \sin\frac{n\pi x}{3}\,dx \\ &= \frac{2[\cos(n\pi/3) - \cos(2n\pi/3)]}{n\pi}. \end{aligned}$$

From (3.5) and (3.7), we conclude that

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/3)\sin(n\pi t/3),$$

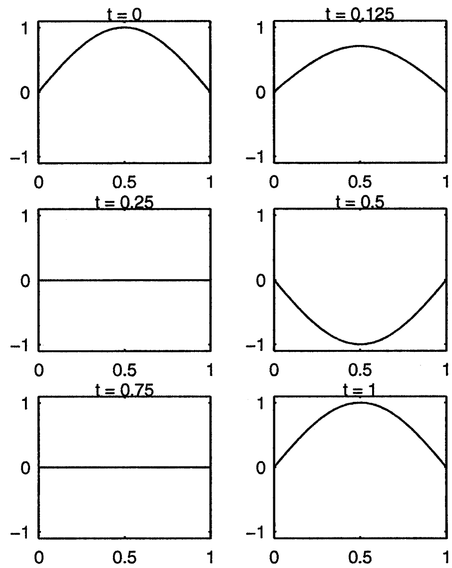
where

$$B_n = \frac{3b_n}{n\pi} = \frac{6[\cos(n\pi/3) - \cos(2n\pi/3)]}{n^2\pi^2}.$$

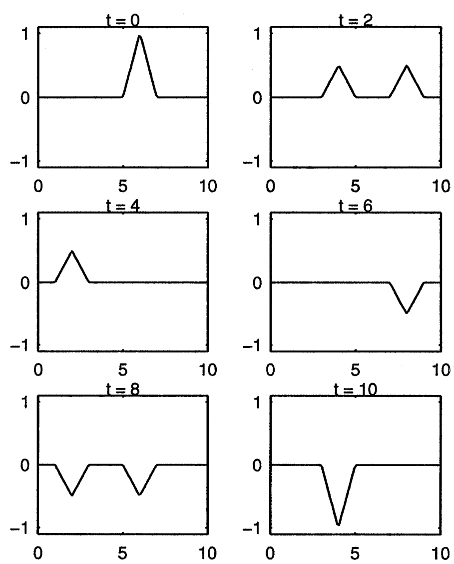
6. Let u_1 be the solution with all of the same data, except that $g = 0$, and let u_2 be the solution with all the same data except that $f = 0$. We found u_1 in Exercise 1 and we found $-u_2$ in Exercise 3. By the linearity of the wave equation, $u = u_1 + u_2$.

7. The odd periodic extension of $f(x) = \sin\pi x$ is $f_{op}(x) = \sin\pi x$. Hence the solution is

$$\begin{aligned} u(x,t) &= [f_{op}(x+ct) + f_{op}(x-ct)]/2 \\ &= [\sin\pi(c+2t) + \sin\pi(x-2t)]/2 \\ &= \sin\pi x\cos 2\pi t. \end{aligned}$$



8. The solution is $u(x,t) = [f_{op}(x+ct) + f_{op}(x-ct)]/2$. It is plotted in the figure.



9. The Fourier sine series converges to $f_{op}(x)$ for all x, so

$$f_{op}(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x, \quad \text{for all } x.$$

Using the trigonometric identity, we get

$$\begin{aligned} u_1(x,t) &= \sum_{n=1}^{\infty} a_n \sin n\pi x \cos n\pi t \\ &= \frac{1}{2}\sum_{n=1}^{\infty} a_n \sin n\pi(x+t) \\ &\quad + \frac{1}{2}\sum_{n=1}^{\infty} a_n \sin n\pi(x-t) \\ &= \frac{1}{2}\left[f_{op}(x+t) + f_{op}(x-t)\right] \\ &= u_2(x,t). \end{aligned}$$

10. We look for product solutions of the form $u(x,t) = X(x)T(t)$. Substituting into the differential equaiton, it becomes $[T'' + T' + T]X = X''T$. Dividing by $u = XT$, we get

$$\frac{T'' + T' + T}{T} = \frac{X''}{X}.$$

Since the left-hand side is a function of t and the right-hand side is a function of x, there must be a constant λ such that both are equal to $-\lambda$. Hence we have

$$T'' + T' + T = -\lambda T \quad \text{and} \quad -X'' = \lambda X.$$

Since u satisfies the boundary conditions, we also have $X(0) = X(1) = 0$. The Sturm Liouville problem for X has solutions

$$\lambda_n = n^2\pi^2 \quad \text{with} \quad X_n(x) = \sin n\pi x.$$

The equation for T becomes

$$T'' + T' + (n^2\pi^2 + 1)T = 0.$$

This is a second order equation with constant coefficients. If we set

$$\omega_n = \sqrt{n^2\pi^2 + 3/4},$$

the equation has the fundamental set of solutions

$$e^{-t/2}\sin\omega_n t \quad \text{and} \quad e^{-t/2}\cos\omega_n t.$$

Hence we have the general solution

$$u(x,t) = e^{-t/2}\sum_{n=1}^{\infty}\left[a_n\cos\omega_n t + b_n \sin\omega_n t\right] \times \sin n\pi x.$$

The initial conditions become

$$\begin{aligned} f(x) = u(x,0) &= \sum_{n=1}^{\infty} a_n \sin n\pi x, \\ 0 = u_t(0,x) &= \sum_{n=1}^{\infty}\left[\omega_n b_n - \frac{a_n}{2}\right]\sin n\pi x. \end{aligned}$$

The first series is the Fourier sine series for f, so

$$a_n = 2\int_0^1 f(x)\sin n\pi x\,dx.$$

From the second series we see that $b_n = a_n/(2\omega_n)$. hence the solution is

$$u(x,t) = e^{-t/2}\sum_{n=1}^{\infty} a_n\left[\cos\omega_n t + \frac{1}{2\omega_n}\sin\omega_n t\right] \times \sin n\pi x.$$

11. We start with the d'Alembert solution

$$u(x,t) = F(x+ct) + G(x-ct).$$

The initial conditions are

$$\begin{aligned} 0 &= F(x) + G(x) \\ g(x) &= c[F'(x) - G'(x)]. \end{aligned}$$

By the first equation, $G = -F$, and then by the second equation

$$F'(x) = \frac{1}{2c}g(x) \quad \text{for } 0 \le x \le L.$$

The first boundary condition is

$$0 = u(0,t) = F(ct) - F(-ct),$$

from which we conclude that F is an even function. The second boundary condition is

$$0 = u(L,t) = F(L+ct) - F(L-ct).$$

If we set $ct = y + L$, and remember that F is even, this equation implies that

$$F(y+2L) = F(-y) = F(y).$$

Hence F is even and $2L$-periodic. Therefore, the derivative F' is odd and periodic. Since $F'(x) = g(x)/2c$ for $0 \le x \le L$, we set $F'(x) = g_{op}(x)$ for all x. Then by the fundamental theorem of calculus,

$$\begin{aligned} u(x,t) &= F(x+ct) - F(x-ct) \\ &= \int_{x-ct}^{x+ct} F'(s)\,ds \\ &= \frac{1}{2c}\int_{x-ct}^{x+ct} g_{op}(s)\,ds. \end{aligned}$$

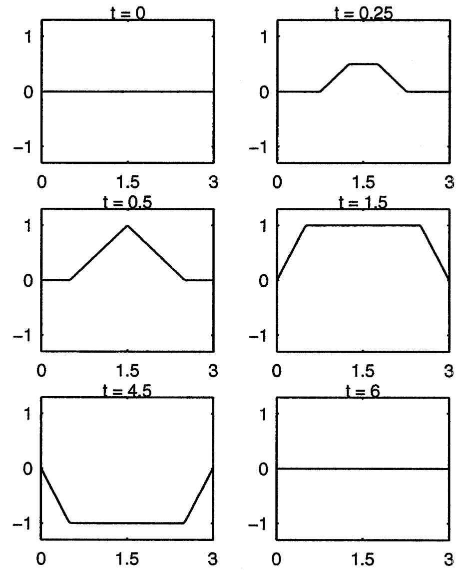
12. When we integrate the function g, we get

$$\begin{aligned} F(x) &= \int_0^x g(x)\,dx \\ &= \begin{cases} 0, & \text{for } 0 \le x \le 1, \\ 2x-2, & \text{for } 1 < x \le 2, \\ 2, & \text{for } 2 < x \le 3. \end{cases} \end{aligned}$$

The function F must be even and periodic with period 6. Then

$$u(x,t) = \frac{1}{2}[F(x+t) - F(x-t)].$$

The solution is plotted below.



13. Let u_1 be the solution with the same data, but with $g = 0$. According to (3.18)

$$u_1(x,t) = \frac{1}{2}\left[f_{op}(x+ct) - f_{op}(x-ct)\right].$$

Let u_2 be the solution with the same data, but with $f = 0$. According to Exercise 11

$$u_2(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g_{op}(s)\,ds.$$

Since the wave equation is linear, the function $u = u_1 + u_2$ is a solution, and satisfies the needed boundary and initial conditions. Hence

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[f_{op}(x+ct) - f_{op}(x-ct)\right] \\ &\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} g_{op}(s)\,ds. \end{aligned}$$

14. (a) If we substitute $v(x)$ into (3.20) we get $0 = c^2 v_{xx} - g$. This is an ordinary differential equation for v, and its general solution is $v(x) = gx^2/(2c^2) + Ax + B$. From the boundary conditions we get $0 = v(0) = B$, and then $0 =$

$v(L) = gL^2/(2c^2) - AL$, or $A = -gL^{(}2c^2)$. Hence

$$v(x) = \frac{g}{2c^2}L(L-x).$$

The function $w = u - v$ satisfies $w_{tt} = u_{tt} = c^2u_{xx} - g$. and $w_{xx} = u_{xx} - v_{xx} = u_{xx} - g/c^2$. Hence $w_{tt} = c^2 w_{xx}$, so w satisfies the wave equation.

(b) The function $w(x,t) = u(x,t) - v(x)$ satisfies the wave equation, the Dirichlet boundary conditions, and the initial conditions $w(x,0) = u(x,0) - v(x) = -v(x) = -gL(L-x)/(2c^2)$ and $w_t(x,0) = u_t(x,0) = 0$. Hence

$$w(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{cn\pi t}{L}\right),$$

where

$$a_n = -\frac{2}{L}\int_0^L \frac{g}{2c^2}L(L-x)\sin\left(\frac{n\pi x}{L}\right)dx$$
$$= \begin{cases} 0 & \text{if } n \text{ is even,} \\ -4gL^2/(c^2\pi^3 n^3) & \text{if } n \text{ is odd.} \end{cases}$$

Hence

$$w(x,t) = -\frac{4gL^2}{c^2\pi^3}\sum_{n\,\text{odd}} \frac{1}{n^3}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{cn\pi t}{L}\right),$$

and our solution is

$$u(x,t) = v(x) + w(x,t).$$

15. Proceeding according to the hint, we get

$$E'(t) = \frac{1}{2}\int_0^L \left[\rho u_t^2 + T u_x^2\right]_t dx$$
$$= \int_0^L [\rho u_t u_{tt} + T u_x u_{xt}]\, dx$$
$$= T\int_0^L [u_t u_{xx} + u_x u_{xt}]\, dx,$$

since $u_{tt} = c^2 u_{xx} = T u_{xx}/\rho$. It then follows that

$$E'(t) = T\int_0^L [u_t u_x]_x\, dx$$
$$= 0 \quad \text{since } u_t(0,t) = 0 = u_t(L,t).$$

For the last step we notice that $u_t(0,t) = [u(0,t)]_t = 0$, and a similar calculation for the other endpoint.

16. By fingering the string precisely in the middle, you are halving the length of the string. As a result the fundamental frequency $\omega_1 = c\pi/L$ is doubled.

17. (a) When we substitute $u(x,t) = X(x)T(t)$ into equation (3.22), and separate variables, we get

$$\frac{T'' + 2kT'}{c^2 T} = \frac{X''}{X}.$$

Since the left-hand side is a function of t and the right-hand side is a function of x, both must be equal to a constant, which we designate as $-\lambda$. Thus X must satisfy $X'' = -\lambda X$, with $X(0) = X(L) = 0$. This is the same boundary value problem as discussed in the text. Hence the solutions are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{and} \quad X_n(x) = \sin\frac{n\pi x}{L},$$

for $n = 1, 2, \ldots$.
The equation for T becomes

$$T'' + 2kT' + [n^2\pi^2c^2/L^2]T = 0.$$

Since $k < \pi c/L$, the functions $e^{-kt}\sin(\mu_n)$ and $e^{-kt}\cos(\mu_n)$, where

$$\mu_n = \sqrt{(n^2\pi^2c^2/L^2) - k^2},$$

form a fundamental set of solutions. Hence we have product solutions

$$u_n(x,t) = e^{-kt}\cos(\mu_n t)\sin\left(\frac{n\pi x}{L}\right) \quad \text{and}$$
$$v_n(x,t) = e^{-kt}\sin(\mu_n t)\sin\left(\frac{n\pi x}{L}\right)$$

for $n = 1, 2, \ldots$.

(b) The general solution to equation (3.22) and the boundary conditions is the infinite series

$$u(x,t) = e^{-kt}\sum_{n=1}^{\infty}[a_n u_n(x,t) + b_n v_n(x,t)].$$

To satisfy the initial conditions we must have

$$f(x) = u(x, 0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{and}$$

$$g(x) = u_t(x, 0) = -\sum_{n=1}^{\infty} \mu_n b_n \sin\left(\frac{n\pi x}{L}\right).$$

Hence we must have

$$a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad \text{and}$$

$$b_n = -\frac{2}{L\mu_n}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Section 13.4. Laplace's Equation

1. Using Figure 1, we have $f(x) = 10$, and $g = h = k = 0$. According to (4.12), (4.13), and (4.14), the solution is given by

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sinh[n\pi(y-2)] \sin(n\pi x),$$

where

$$b_n = -\frac{2}{\sinh(2n\pi)}\int_0^1 10 \sin(n\pi x)\, dx = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -40/n\pi \sinh(2n\pi) & \text{if } n \text{ is odd.} \end{cases}$$

Hence

$$u(x, y) = \sum_{k=0}^{\infty} \frac{40}{(2k+1)\pi} \frac{\sinh[(2k+1)\pi(2-y)]}{\sinh[(4k+2)\pi]} \sin(n\pi x).$$

2. Using Figure 1, we have $f(x) = 100x$, $h(x) = 20$, and $g = k = 0$. According to (4.12), (4.13), and (4.14), the solution is

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi y}{10}\right) \sin\left(\frac{n\pi x}{10}\right) + \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi(y-25)}{10}\right) \sin\left(\frac{n\pi x}{10}\right),$$

where

$$a_n = \frac{2}{10\sinh(2.5n\pi)}\int_0^{10} 20 \sin\left(\frac{n\pi x}{10}\right) dx = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 80/[n\pi \sinh(2.5n\pi)] & \text{if } n \text{ is odd.} \end{cases}$$

and

$$b_n = \frac{-2}{10\sinh(2.5n\pi)}\int_0^{10} 100x \sin\left(\frac{n\pi x}{10}\right) dx = (-1)^n \frac{200}{n\pi \sinh(2.5n\pi)}.$$

Hence,

$$\begin{aligned} u(x, y) =& 80\sum_{k=0}^{\infty} \frac{1}{(2k+1)\pi} \frac{\sinh((2k+1)\pi y/10)}{\sinh(2.5(2k+1)\pi)} \sin\left(\frac{(2k+1)\pi x}{10}\right) \\ &+ 200\sum_{n=1}^{\infty} (-1)^n \frac{\sinh(n\pi(y-25)/10}{\sinh(2.5n\pi)} \sin\left(\frac{(2k+1)\pi x}{10}\right). \end{aligned}$$

3. The derivation is just like that in the text. In outline, we use separation of variables to find the product solutions

$$\sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b} \quad\text{and}\quad \sinh\frac{n\pi(x-a)}{b}\sin\frac{n\pi y}{b}.$$

The general solution therefore has the form

$$u(x,y)=\sum_{n=1}^{\infty}\left[a_n\sinh\frac{n\pi x}{b}+b_n\sinh\frac{n\pi(x-a)}{b}\right]\sin\frac{n\pi y}{b},$$

The constants are determined from the boundary conditions. At $x=0$, we have

$$g(y)=u(0,y)=\sum_{n=1}^{\infty}-b_n\sinh\frac{n\pi a}{b}\sin\frac{n\pi y}{b},$$

and at $x=a$,

$$k(y)=u(a,y)=\sum_{n=1}^{\infty}a_n\sinh\frac{n\pi a}{b}\sin\frac{n\pi y}{b}.$$

These are the Fourier sine series for g and k, so the result follows.

4. For $j=1,2,3,4$, let u_j be the temperature where the jth side is kept at 100°, while the other three sides are at 0°. By symmetry, all four temperatures are equal at the center. On the other hand, the sum $u_1+u_2+u_3+u_4$ is a solution to Laplace's equation and is equal to 100° on the entire boundary. Hence at the center, $100=u_1+u_2+u_3+u_4=4u_1$. Hence the temperature at the center is 25°.

5. Since $f=0$ and $a=b=1$, we see from (4.12) that the solution is given by

$$u(x,y)=\sum_{n=1}^{\infty}a_n\sinh(n\pi y)\sin(n\pi x),$$

where

$$a_n=\frac{2}{\sinh(n\pi)}\int_0^1 h(x)\sin(n\pi x)\,dx.$$

We find that $a_n=0$ if n is even, and

$$a_{2n+1}=\frac{4}{\sinh((2n+1)\pi)}\cdot\frac{(-1)^n}{(2n+1)^2\pi^2}.$$

Hence the temperature is

$$u(x,y)=4\sum_{n=0}^{\infty}\frac{(-1)^n\sinh((2n+1)\pi y)\sin((2n+1)\pi x)}{(2n+1)^2\pi^2\sinh((2n+1)\pi)}.$$

6. Notice that the odd periodic extension of h is the square wave which we considered it Example 2.8 in Section 12.2. We found that its Fourier sine series is

$$h(x)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin((2n+1)\pi x).$$

Therefore the temperature is

$$u(x, y) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sinh((2n+1)\pi y) \cdot \sin((2n+1)\pi x)}{(2n+1)\sinh((2n+1)\pi)}.$$

7. The function $f(x) = \sin(2\pi x)$ is expressed in terms of its Fourier sine series, so we can immediately write down the temperature:

$$u(x, y) = \frac{-\sinh(2\pi(y-1))\sin(2\pi x)}{\sinh(2\pi)}.$$

8. Since $h = 0$ and $a = 1$, we see from (4.12) that the solution is given by

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sinh(n\pi(y-2))\sin(n\pi x),$$

where

$$b_n = \frac{-2}{\sinh(2n\pi)} \int_0^1 f(x)\sin(n\pi x)\,dx.$$

We have $f(x) = \sin^2(\pi x) = [1 - \cos(2\pi x)]/2$. Using the identity $\cos(a)\sin(b) = [\sin(b+a) + \sin(b-a)]/2$, we have

$$\begin{aligned}
\int_0^1 f(x)\sin(n\pi x)\,dx &= \int_0^1 \sin^2(\pi x)\sin(n\pi x)\,dx \\
&= \int_0^1 \left[\frac{1}{2}\sin(n\pi x) - \frac{1}{4}\sin((n+2)\pi x) - \frac{1}{4}\sin((n-2)\pi x)\right] dx \\
&= \begin{cases} 0, & \text{if } n \text{ is even,} \\ \dfrac{-4}{\pi n(n^2-4)}, & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Therefore,

$$u(x, y) = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{\sinh(n\pi(y-2))\sin(n\pi x)}{(2n+1)[(2n+1)^2-4]\sinh(2n\pi)}.$$

9. We see from (4.12) that the solution is given by

$$u(x, y) = \sum_{n=1}^{\infty} [a_n \sinh(n\pi y) + b_n \sinh(n\pi(y-2)] \sin(n\pi x),$$

where

$$\begin{aligned}
a_n = b_n &= \frac{2}{\sinh(2n\pi)} \int_0^1 \sin(n\pi x)\,dx \\
&= \frac{2}{\sinh(2n\pi)} \begin{cases} 0, & \text{if } n \text{ is even,} \\ \dfrac{2}{n\pi}, & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Therefore the temperature is

$$u(x, y) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sinh((2n+1)\pi y) + \sinh((2n+1)\pi(y-2))}{(2n+1)\sinh((4n+2)\pi)} \sin((2n+1)\pi x).$$

10. Let u_1 be the temperature with all of the data the same, except that $f = 0$, and let u_2 be the temperature with all of the data the same, except that $h = 0$. From Exercise 5 we see that

$$u_1(x, y) = 4 \sum_{n=0}^{\infty} \frac{(-1)^n \sinh((2n+1)\pi y)\sin((2n+1)\pi x)}{(2n+1)^2\pi^2 \sinh((2n+1)\pi)}.$$

From Exercise 7 we see that

$$u_2(x, y) = \frac{-\sinh(2\pi(y-1))\sin(2\pi x)}{\sinh(2\pi)}.$$

By the linearity of the problem, $u = u_1 + u_2$.

11. (a) Using (4.12), we see that the temperature is given by

$$u(x, y) = -\sum_{n=0}^{\infty} B_n \frac{\sinh(n\pi(y-L))}{\sinh(n\pi L)} \sin(n\pi x).$$

(b) In part (a), the dependence on L comes from the factor

$$\frac{-\sinh(n\pi(y-L))}{\sinh(n\pi L)} = \frac{e^{n\pi(L-y)} - e^{n\pi(y-L)}}{e^{n\pi L} - e^{-n\pi L}}.$$

If we factor $e^{n\pi L}$ from both numerator and denominator, we get

$$\frac{e^{-n\pi y} - e^{n\pi(y-2L)}}{1 - e^{-2n\pi L}}.$$

When we let L increase to ∞, this converges to $e^{-n\pi y}$. Hence the solution over the finite rectangle converges to

$$u(x, y) = \sum_{n=1}^{\infty} B_n e^{-n\pi y} \sin(n\pi x).$$

12. Since Laplace's equation is linear, the function $w(x, y) = u(x, y) - v(x, y)$ satisfies Laplace's equation. By assumption, $w(x, y) = 0$ at very point $(x, y) \in \partial D$. Since, by the maximum principle, w must achieve both its maximum and minimum on ∂D, we must have $w(x, y) = 0$ at every point $(x, y) \in D$. Therefore, $u(x, y) = v(x, y)$ at every point $(x, y) \in D$.

Section 13.5. Laplace's Equation on a Disk

1. The computations of u_{xx} and u_{yy} proceed just as they do in polar coordinates, since the equations for x and y are are the same. Since the coordinate z is unchanged, u_{zz} is also unchanged. Hence

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

2. Differentiating $r^2 = x^2 + y^2 + z^2$ with respect to x, y, and z, we get

$$r_x = \frac{x}{r} = \cos\theta \sin\phi, \quad r_y = \frac{y}{r} = \sin\theta\sin\phi, \quad \text{and} \quad r_z = \frac{z}{r} = \cos\phi.$$

From $\tan\theta = y/x$, we see that $1 + \tan^2\theta = (x^2+y^2)/x^2$. Notice also that $x^2 + y^2 = r^2\sin^2\phi$. Differentiating $\tan\theta = y/x$ with respect to x, and using these formulas, we get $(1+\tan^2\theta)\theta_x = -y/x^2$. Thus $\theta_x = -y/(x^2+y^2)$. Doing the same for y, we see that

$$\theta_x = \frac{-y}{x^2+y^2} = \frac{-\sin\theta}{r\sin\phi}, \quad \theta_y = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r\sin\phi}, \quad \text{and} \quad \theta_z = 0.$$

Since $\tan\phi = \sqrt{x^2+y^2}/z$, $1 + \tan^2\phi = r^2/z^2$. Differentiating $\tan^2\phi = (x^2+y^2)/z^2$ we then compute that

$$\phi_x = \frac{\cos\theta\cos\phi}{r}, \quad \phi_y = \frac{\sin\theta\cos\phi}{r}, \quad \text{and} \quad \phi_z = \frac{-\sin\phi}{r}.$$

By the chain rule,

$$u_x = u_r r_x + u_\theta\theta_x + u_\phi\phi_x = u_r\cdot\cos\theta\sin\phi + u_\theta\cdot\frac{-\sin\theta}{r\sin\phi} + u_\phi\cdot\frac{\cos\theta\cos\phi}{r}.$$

Differentiating once again we get

$$\begin{aligned} u_{xx} = u_{rr}\cdot\cos^2\theta\sin^2\phi &+ \left(\frac{u_\theta}{r}\right)_r\cdot\frac{-\sin\theta\cos\phi}{r} + \left(\frac{u_\phi}{r}\right)_r\cdot\frac{\cos^2\theta\sin\phi\cos\phi}{r} \\ &+ (u_r\cos\theta)_\theta\cdot\frac{-\sin\theta}{r} + (u_\theta\sin\theta)_\theta\cdot\frac{\sin\theta}{r^2\sin^2\phi} + (u_\phi\cos\theta)_\theta\cdot\frac{-\sin\theta\cos\phi}{r^2\sin\phi} \\ &+ (u_r\sin\phi)_\phi\cdot\frac{\cos^2\theta\cos\phi}{r} + \left(\frac{u_\theta}{-\sin\phi}\right)_\phi\cdot\frac{\sin\theta\cos\theta\cos\phi}{r^2} + (u_\phi\cos\phi)_\phi\cdot\frac{\cos^2\theta\cos\phi}{r^2}. \end{aligned}$$

Similarly,

$$\begin{aligned} u_{yy} = u_{rr}\cdot\sin^2\theta\sin^2\phi &+ \left(\frac{u_\theta}{r}\right)_r\cdot\frac{\sin\theta\cos\phi}{r} + + \left(\frac{u_\phi}{r}\right)_r\cdot\frac{\sin^2\theta\sin\phi\cos\phi}{r} \\ &+ (u_r\sin\theta)_\theta\cdot\frac{\cos\theta}{r} + (u_\theta\cos\theta)_\theta\cdot\frac{\cos\theta}{r\sin^2\phi} + (u_\phi\sin\theta)_\theta\cdot\frac{\cos\theta\cos\phi}{r^2\sin\phi} \\ &+ (u_r\sin\phi)_\phi\cdot\frac{\sin^2\theta\cos\phi}{r} + \left(\frac{u_\theta}{\sin\phi}\right)_\phi\cdot\frac{\sin\theta\cos\theta\cos\phi}{r^2} + (u_\phi\cos\phi)_\phi\cdot\frac{\sin^2\theta\cos\phi}{r^2}. \end{aligned}$$

Finally,

$$u_{zz} = u_{rr} \cdot \cos^2\phi + \left(\frac{u_\phi}{r}\right)_r \cdot (-\sin\phi\cos\phi) + (u_r \cos\phi)_\phi \cdot \frac{-\sin\phi}{r} + (u_\phi \sin\phi)_\phi \cdot \frac{\sin\phi}{r^2}.$$

The last term is simpler because $\theta_z = 0$.

Since $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$ we have to combine these terms. This is made easier if you systematically combine terms by type. A term is of type (a, b) if it contains a factor of $(u_a \cdot *)_b$, where $*$ stands for an arbitrary expression. Of course, a and b stand for r, θ, or ϕ. For example, the (r, r) term is

$$u_{rr} \cdot \left(\cos^2\theta \sin^2\phi + \sin^2\theta \sin^2\phi + \cos^2\phi\right) = u_{rr}.$$

The (θ, r) term is

$$\left(\frac{u_\theta}{r}\right)_r \cdot \frac{-\sin\theta\cos\phi}{r} + \left(\frac{u_\theta}{r}\right)_r \cdot \frac{\sin\theta\cos\phi}{r} = 0.$$

The (r, θ) term is

$$(u_r \cos\theta)_\theta \cdot \frac{-\sin\theta}{r} + (u_r \sin\theta)_\theta \cdot \frac{\cos\theta}{r} = \frac{u_r}{r}.$$

Proceeding in this manner through all 9 terms we get

$$\nabla^2 u = u_{rr} + \frac{2u_r}{r} + \frac{u_{\theta\theta}}{r^2 \sin\phi} + \frac{u_\phi \cos\phi}{r^2 \sin\phi} + \frac{u_{\phi\phi}}{r^2}.$$

The first two terms and the last two terms can be combined to get

$$\nabla^2 u = \frac{1}{r^2}(r^2 u_r)_r + \frac{u_{\theta\theta}}{r^2 \sin\phi} + \frac{1}{r^2 \sin\phi}(u_\phi \cdot \sin\phi)_\phi.$$

3. The function $u(x, y) = 1 + y$ is a solution to Laplace's equation, and is equal to $f(x, y)$ on the surface of the can. Hence the temperature is $u(x, y) = 1 + y$.

4. The effect of the rays of the sun should be proportional to the normal component of the ray. This is $\sin\phi$ times the full force of the ray. A little geometry shows that $\phi = \theta$, the polar angle. Hence assuming the boundary temperature is

$$f(\theta) = \begin{cases} \sin\theta, & \text{for } 0 \le \theta \le \pi, \\ 0, & \text{for } \pi < \theta \le 2\pi. \end{cases}$$

The Fourier series for f is

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos n\theta + B_n \sin n\theta],$$

where $A_0 = (1/\pi) \int_0^\pi \sin\theta \, d\theta = 2/\pi$, and

$$A_n = \frac{1}{\pi} \int_0^\pi \sin\theta \cos n\theta \, d\theta = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ \dfrac{-2}{\pi(n^2-1)}, & \text{if } n \text{ is even}, \end{cases}$$

and

$$B_n = \frac{1}{\pi} \int_0^\pi \sin\theta \sin n\theta = \begin{cases} 1/2, & \text{for } n = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Hence,

$$u(r,\theta) = \frac{1}{\pi} + \frac{r\sin\theta}{2} - 2\sum_{n=1}^{\infty} \frac{r^{2n}\cos 2n\theta}{\pi(4n^2-1)}.$$

5. Since the Fourier series for f is $f(\theta) = \sin^2\theta = [1-\cos 2\theta]/2$, the temperature is $u(r,\theta) = [1-r^2\cos 2\theta]/2$.

6. Since the Fourier series for f is $f(\theta) = \cos^2\theta = [1+\cos 2\theta]/2$, the temperature is $u(r,\theta) = [1+r^2\cos 2\theta]/2$.

7. The Fourier series for f is

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty}[A_n\cos n\theta + B_n\sin n\theta].$$

First, we have

$$A_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta = \frac{1}{\pi}\int_0^{2\pi}\theta(2\pi-\theta)\,d\theta = \frac{4\pi^2}{3},$$

Next, for $n \geq 1$, we integrate by parts twice to get

$$A_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta = \frac{1}{\pi}\int_0^{2\pi}\pi(2\pi-\theta)\cos n\theta\,d\theta = \frac{-4}{n^2}.$$

Finally, since the periodic extension of f is even, the sine coefficients are $B_n = 0$. Thus the temperature is

$$u(r,\theta) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{r^n\cos n\theta}{n^2}.$$

8. The Fourier series for f is $f(\theta) = \sin\theta\cos\theta = (1/2)\sin 2\theta$, so the temperature is $u(r,\theta) = (r/2)\sin 2\theta$.

9. Let $u(r)$ denote the temperature at distance r from the center of the circles. Then by (5.2), $\nabla^2 u = u_{rr} + u_r/r = 0$. Thus u satisfies the Euler equation $ru_{rr} + u_r = 0$, and has the form $u(r) = A + B\ln r$. Using the boundary conditions to solve for the coefficients A and B we find that

$$u(r) = T_1 + (T_2 - T_1)\frac{\ln(r/a)}{\ln(b/a)}.$$

10. As in Exercise 9, we know that $u(r) = A + B\ln r$. Using the boundary conditions we see that $B = 0$ and $A = T$. Thus $u(r) = T$, and the temperature is constant throughout the plate.

11. Since the origin is not in the ring-shaped plate, we cannot eliminate the second solution to Euler's equation in (5.9). Hence the general series solution has the form

$$u(r,\theta) = \frac{A_0 + C_0\ln r}{2} + \sum_{n=1}^{\infty}\left[(A_nr^n + C_nr^{-n})\cos n\theta + (B_nr^n + D_nr^{-n}\sin n\theta\right]$$

Since $u(a,\theta) = 0$, all of the coefficients of the trigonometric functions must equal 0 at $r = a$. Using this to evaluate A_n and B_n we get

$$u(r,\theta) = \frac{C_0\ln(r/a)}{2} + \sum_{n=1}^{\infty}\left[C_n(r^{-n} - a^{-2n}r^n)\cos n\theta + D_n(r^{-n} - a^{-2n}r^n)\sin n\theta\right]$$

We can then use $u(b,\theta) = f(\theta)$ to compute the upper case coefficients, since these reduce to the Fourier coefficients of f. We get

$$\begin{aligned}
C_0 \ln(b/a) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta \\
C_n(b^{-n} - a^{-2n}b^n) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta \\
D_n(b^{-n} - a^{-2n}b^n) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta
\end{aligned}$$

12. Separation of variables in polar coordinates proceeds as before, but instead of periodic boundary conditions, we now have that $u(r,0) = u(r,\pi) = 0$. Hence the Sturm-Liouville problem is

$$T'' + \lambda T = 0 \quad \text{with} \quad T(0) = T(\pi) = 0.$$

The eigenvalues and eigenfunctions are

$$\lambda_n = n^2 \quad \text{and} \quad T_n(\theta) = \sin n\theta \quad \text{for } n \geq 1.$$

Hence the product solutions are of the form $r^n \sin n\theta$, and the general series solution is

$$u(r,\theta) = \sum_{n=1}^{\infty} B_n r^n \sin n\theta.$$

At $r = a$ we have $u(a,\theta) = f(\theta)$ so we get the Fourier sine series for f. Hence the coefficients are given by

$$B_n = \frac{2}{\pi a^n}\int_0^{\pi} f(\theta)\sin n\theta\,d\theta.$$

13. Separation of variables in polar coordinates proceeds as before, but we now have that $u(r,0) = u(r,\theta_0) = 0$. Hence the Sturm-Liouville problem is

$$T'' + \lambda T = 0 \quad \text{with} \quad T(0) = T(\theta_0) = 0.$$

The eigenvalues and eigenfunctions are

$$\lambda_n = n^2/\theta_0^2 \quad \text{and} \quad T_n(\theta) = \sin(n\theta/\theta_0) \quad \text{for } n \geq 1.$$

Hence the product solutions are of the form $r^{n\pi/\theta_0}\sin(n\theta/\theta_0)$, and the general series solution is

$$u(r,\theta) = \sum_{n=1}^{\infty} B_n r^{n\pi/\theta_0}\sin(n\theta/\theta_0).$$

At $r = a$ we have $u(a,\theta) = f(\theta)$ so we get the Fourier sine series for f. Hence the coefficients are given by

$$B_n = \frac{2}{\theta_0 a^n}\int_0^{\theta_0} f(\theta)\sin(n\theta/\theta_0)\,d\theta.$$

Section 13.6. Sturm-Liouville Problems

1. The operator in (b) is $L\phi = x\phi'' + \phi' = (x\phi')'$. The operator in (e) is $L\phi = \sin x\phi'' + \cos x\phi' = (\sin x\phi')'$. The operator in (f) is $L\phi = (1-x^2)\phi'' - 2x\phi' = [(1-x^2)\phi']'$. These are therefore formally self-adjoint. The others are not.

2. This problem was solved in Section 2. According to (2.28), the eigenvalues and eigenfunctions are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{and}$$
$$\phi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$$

for $n = 0, 1, 2, 3, \ldots$.

3. We use Proposition 6.24 to conclude that all of the eigenvalues are positive. Setting $\lambda = \omega^2$, where $\omega > 0$, the differential equation becomes $\phi'' + \omega^2\phi = 0$. The general solution is $\phi(x) = A\cos\omega x + B\sin\omega x$. By the first boundary condition, $0 = \phi'(0) = \omega B$. Hence $B = 0$. Then, by the second boundary condition, $0 = \phi(1) = A\cos\omega$. Hence $\omega = \pi/2 + n\pi$ for a nonnegative integer n. We conclude that the eigenvalues and eigenfunctions are

$$\lambda_n = (\pi/2 + n\pi)^2 \quad \text{and}$$
$$\phi_n(x) = \cos(\pi/2 + n\pi)x$$

for $n = 0, 1, 2, 3, \ldots$.

4. We use Proposition 6.24 to conclude that all of the eigenvalues are positive. Setting $\lambda = \omega^2$, where $\omega > 0$, the differential equation becomes $\phi'' + \omega^2\phi = 0$. The general solution is $\phi(x) = A\cos\omega x + B\sin\omega x$. By the first boundary condition, $0 = \phi'(0) = \omega B$. Hence $B = 0$. Then, by the second boundary condition, $0 = \phi'(1) + \phi(1) = A[-\omega\sin\omega + \cos\omega]$. Hence, we are looking for solutions to the equation $\tan\omega = 1/\omega$. For every $n \geq 1$, there is a solution ω_n satisfying $(n-1)\pi < \omega_n < (n-1)\pi + \pi/2$. We conclude that the eigenvalues and eigenfunctions are

$$la_n = \omega_n^2 \quad \text{and}$$
$$\phi_n(x) = \cos\omega_n x$$

for $n = 1, 2, 3, \ldots$.

5. We use Proposition 6.24 to conclude that all of the eigenvalues are positive. Setting $\lambda = \omega^2$, where $\omega > 0$, the differential equation becomes $\phi'' + \omega^2\phi = 0$. The general solution is $\phi(x) = A\cos\omega x + B\sin\omega x$. By the first boundary condition, $0 = \phi'(0) - \phi(0) = \omega B - A$. Hence $A = \omega B$. so the solution becomes $\phi(x) = B(\omega\cos\omega x + \sin\omega x)$. Then, by the second boundary condition, $0 = \phi(1) = B[\omega\cos\omega + \sin\omega]$. Hence, we are looking for solutions to the equation $\tan\omega = -1/\omega$. For every $n \geq 1$, there is a solution ω_n satisfying $n\pi - \pi/2 < \omega_n < n\pi$. We conclude that the eigenvalues and eigenfunctions are

$$\lambda_n = \omega_n^2 \quad \text{and}$$
$$\phi_n(x) = \omega_n\cos\omega_n x + \sin\omega_n x,$$

for $n = 1, 2, 3, \ldots$.

6. We need to find functions p, q, and μ such that

$$\begin{aligned}\mu[\phi'' + 4\phi' + \lambda\phi] &= (p\phi')' + \lambda q\phi \\ &= p\phi'' + p'\phi' + \lambda q\phi.\end{aligned}$$

Equating the coefficients of the derivatives of ϕ, we see that $p = \mu$, $p' = 4\mu$, and $q = \mu$. From the first two equations we get the equation $p' = 4p$, which has solution $p(x) = e^{4x}$. Since $q = \mu = p$, we see that the equation becomes $-(e^{4x}\phi')' = \lambda e^{4x}\phi$.

7. We need to find functions p, q, and μ such that

$$\begin{aligned}\mu[2x\phi'' + \lambda\phi] &= (p\phi')' + \lambda q\phi \\ &= p\phi'' + p'\phi' + \lambda q\phi.\end{aligned}$$

Equating the coefficients of the derivatives of ϕ, we see that $p = 2x\mu$, $p' = 0$, and $q = \mu$. From the second equation we see that p is a constant, so we set $p = 1$. Then $q = \mu = 1/2x$, so the equation becomes $-\phi'' = \lambda(1/2x)\phi$.

8. We need to find functions p, q, and μ such that

$$\begin{aligned}&\mu[x(x-1)\phi'' + 2x\phi' + \lambda\phi] \\ &\quad = (p\phi')' + \lambda q\phi \\ &\quad = p\phi'' + p'\phi' + \lambda q\phi.\end{aligned}$$

Equating the coefficients of the derivatives of ϕ, we see that $p = x(x-1)\mu$, $p' = 2x\mu$, and

$q = \mu$. From the first two equations we get the equation $p' = (2/(x-1))p$, which has solution $p(x) = (x-1)^2$. Hence $\mu = (x-1)/x$, and $q = \mu = (x-1)/x$, so the equation becomes $-[(x-1)^2\phi']' = \lambda[(x-1)/x]\phi$.

9. We need to find functions p, q, and μ such that

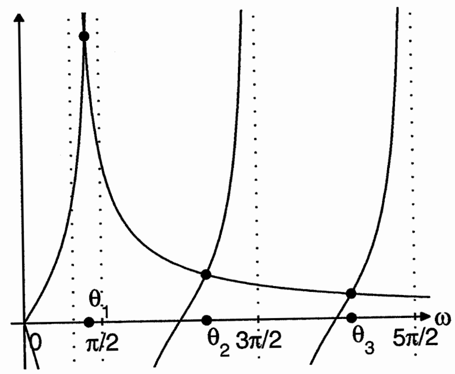
$$\begin{aligned}\mu[x^2\phi'' - 2x\phi' + \lambda\phi] &= (p\phi')' + \lambda q\phi \\ &= p\phi'' + p'\phi' + \lambda q\phi.\end{aligned}$$

Equating the coefficients of the derivatives of ϕ, we see that $p = x^2\mu$, $p' = -2x\mu$, and $q = \mu$. From the first two equations we get the equation $p' = -2x^{-1}p$, which has solution $p(x) = x^{-2}$. Hence $\mu = x^{-4}$, and $q = \mu = x^{-4}$, so the equation becomes $-(x^{-2}\phi')' = \lambda x^{-4}\phi$.

10. Notice that the boundary conditions imply that $\phi(0)\phi'(0) = \phi(0)^2$, and $\phi(1)\phi'(1) = -\phi(1)^2$. Hence $\phi\phi'|_0^1 = -[\phi(0)^2 + \phi(1)^2] \le 0$. Therefore Proposition 6.24 implies that the eigenvalues are nonnegative. Furthermore, if $\lambda = 0$ is an eigenvalue, then the eigenfunction ϕ is a constant. Since $\phi' = 0$, the boundary conditions imply that $\phi(0) = \phi(1) = 0$, so $\phi = 0$. Thus all of the eigenvalues are positive. Setting $\lambda = \omega^2$, where $\omega > 0$, the differential equation becomes $\phi'' + \omega^2\phi = 0$. The general solution is $\phi(x) = A\cos\omega x + B\sin\omega x$. By the first boundary condition, $0 = \phi'(0) - \phi(0) = \omega B - A$. Hence $A = \omega B$, so the solution becomes $\phi(x) = B(\omega\cos\omega x + \sin\omega x)$. Then, by the second boundary condition, $0 = \phi'(1) + \phi(1) = B[(1-\omega^2)\sin\omega + 2\omega\cos\omega]$. Hence, we are looking for solutions to the equation $\tan\omega = 2\omega/(\omega^2 - 1)$. The functions $\tan\omega$ and $2\omega/(\omega^2-1)$ are plotted in the accompanying figure. The points of intersection determine the solutions ω_n. Notice that for $0 < \omega < 1$, $\tan\omega > 0$, while $2\omega/(\omega^2 - 1) < 0$. Hence the first root satisfies $1 < \omega_1 < \pi/2$. For $n \ge 2$ we have $(n-1)\pi < \omega_n < (n-1)\pi + \pi/2$. We conclude that the eigenvalues and eigenfunctions are

$$\begin{aligned}\lambda_n &= \omega_n^2 \quad \text{and} \\ \phi_n(x) &= \omega_n\cos\omega_n x + \sin\omega_n x,\end{aligned}$$

for $n = 1, 2, 3, \ldots$.



11. (a) If $\phi \ne 0$ satisfies the boundary conditions we have $p\phi\phi'|_0^1 = \phi(1)\phi'(1) = a\phi(1)^2 \ge 0$.

(b) First notice that when $\lambda = 0$, the equation is $\phi'' = 0$, which has solution $\phi(x) = Ax + B$. The first boundary condition implies that $B = \phi(0) = 0$, so $\phi(x) = Ax$. The second condition then implies that $A(1-a) = 0$. We get a nontrivial solution when $A \ne 0$, and therefore only when $a = 1$. Thus 0 is an eigenvalue only when $a = 1$. Next set $\lambda = -\nu^2$ where $\nu > 0$. The differential equation becomes $\phi'' = \nu^2\phi$, and has solution $\phi(x) = Ae^{\nu x} + Be^{-\nu x}$. The first boundary condition is $0 = \phi(0) = A + B$. Hence $B = -A$, and $\phi(x) = A(e^{\nu x} - e^{-\nu x}) = 2A\sinh\nu x$. The second boundary condition can be written as $\phi'(1) = a\phi(1)$, which becomes $2\nu A\cosh\nu = aA\sinh\nu$. This can be written as $\nu = a\tanh\nu$. To examine solutions, we graph the two functions ν and $a\tanh\nu$. Since $\tanh'(0) = 1$, we see that the slope of $a\tanh\nu$ at $\nu = 0$ is a. If $a < 1$, the graph of $a\tanh\nu$ lies below the graph of ν, so there is no solution, and therefore no negative eigenvalue. On the other hand if $a > 1$, the two graphs intersect precisely once, and the value of ν for which this occurs is the lone negative eigenvalue.

12. Using the definition of L, we have

$$\begin{aligned}\int_a^b Lf \cdot g\,dx &= \int_a^b [-(pf')' + qf](x)g(x)\,dx \\ &= -\int_a^b (pf')'g\,dx \\ &\quad + \int_a^b qfg\,dx \\ &= -pf'g\big|_a^b + \int_a^b f'(pg')\,dx \\ &\quad + \int_a^b qfg\,dx \\ &= -pf'g\big|_a^b + pfg'\big|_a^b \\ &\quad - \int_a^b f(pg')'\,dx + \int_a^b qfg\,dx \\ &= \int_a^b f \cdot Lg\,dx + p(fg' - f'g)\big|_a^b.\end{aligned}$$

13. (a) Proposition 6.12 says that if L is a formally self-adjoint operator, then

$$\int_a^b Lf \cdot g\,dx = \int_a^b f \cdot Lg\,dx + p(fg' - f'g)\big|_a^b,$$

for any two functions f and g which have two continuous derivatives. If both f and g vanish at the endpoints, then $p(fg' - f'g)|_a^b = 0$, so the result follows.

(b) Direct computation shows that

$$\begin{aligned}\int_0^1 Lf \cdot g\,dx &= -\int_0^1 [x^2 + x^3 - 2x^4]\,dx \\ &= \frac{-13}{30},\end{aligned}$$

while

$$\begin{aligned}\int_a^b f \cdot Lg\,dx &= \int_0^1 [2x - 6x^2 + x^3 + 3x^4]\,dx \\ &= \frac{-3}{20}.\end{aligned}$$

14. The differential equation to be solved is

$$-\phi'' = \lambda\phi \quad \text{or} \quad \phi'' + \lambda\phi = 0.$$

By Proposition 6.24 there are no negative eigenvalues. For $\lambda = 0$, the only possible eigenfunction is a constant function. Checking shows that constants are eigenfunctions. Hence we set $\lambda_0 = 0$ and it is simplest to take $c_0(x) = 1$.

If $\lambda > 0$, we can set $\lambda = \omega^2$, where $\omega > 0$. Then the differential equation is $\phi'' + \omega^2\phi = 0$. This equation has general solution $\phi(x) = C_1 \cos\omega x + C_2 \sin\omega x$. Using the fact that sin is odd and cos is even, the periodic boundary conditions yield

$$\begin{aligned}0 &= B_1\phi = \phi(\pi) - \phi(-\pi) = 2C_2 \sin\omega\pi \\ 0 &= B_2\phi = \phi'(\pi) - \phi'(-\pi) = 2C_1 \sin\omega\pi\end{aligned}$$

If $\sin\omega\pi \neq 0$ we have $C_1 = C_2 = 0$. The only solution is $\phi = 0$, so $\lambda = \omega^2$ is not an eigenvalue. However, if $\sin\omega\pi = 0$, the constants C_1 and C_2 are both arbitrary, and any solution to the differential equation is an eigenfunction. Of course $\sin\omega\pi = 0$ if and only if ω is an integer, and since $\omega > 0$ it must be a positive integer. Thus for each positive integer we get the eigenvalue $\lambda = n^2$, and for each such eigenvalue there are two linearly independent eigenfunctions $\cos nx$ and $\sin nx$. We will set $\lambda_n = n^2$, with corresponding eigenfunctions $c_n(x) = \cos nx$ and $s_n(x) = \sin nx$.

15. Substituting $u(x,t) = X(x)T(t)$ into the differential equation, we get $XT_t = k[X_{xx} - qX]T$. Dividing by kXT, we get

$$\frac{T_t}{kT} = \frac{X_{xx} - qX}{X}.$$

Since the left-hand side is a function of t and the right-hand side is a function of x, both must be constant. Therefore, there is a constant λ such that

$$T' = \lambda T \quad \text{and} \quad -X'' + qX = \lambda X.$$

At $x = 0$ we have $0 = u(0,t) = X(x)T(t)$. Hence $X(0) = 0$. At $x = L$ the boundary condition is $0 = u_t(L,t) = X'(L)T(t)$, so $X'(L) = 0$.

16. Suppose that the boundary condition at $x = a$ is unmixed, so that

$$B_1\phi = \alpha_1\phi'(a) + \beta_1\phi(a) = 0.$$

The eigenfunctions ϕ_1 and ϕ_2 both satisfy this boundary condition. Consequently α_1 and β_1 satisfy the system of equations

$$\begin{pmatrix} \phi_1'(a) & \phi_1(a) \\ \phi_2'(a) & \phi_2(a) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $(\alpha_1, \beta_1)^T$ is a nonzero vector, the matrix must have determinant equal to 0. Hence $\phi_1'(a)\phi_2(a) - \phi_1(a)\phi_2'(a) = 0$. Thus the Wronskian of ϕ_1 and ϕ_2, defined by $W = \phi_1\phi_2' - \phi_1'\phi_2$ satisfies $W(a) = 0$. Since both ϕ_1 and ϕ_2 are solutions to the differential equation (6.18), Proposition 1.25 of section 4.1 implies that $W(x) = 0$ for all x. According to Proposition 1.26 of section 4.1, this implies that the functions ϕ_1 and ϕ_2 are linearly dependent. Therefore two linearly independent eigenfunctions cannot exist.

Section 13.7. Orthogonality and Generalized Fourier Series

1. The eigenfunctions are $\phi_n(x) = \sin((2n+1)\pi x/2)$. We compute

$$(\phi_n, \phi_n) = \int_0^1 \sin^2 \frac{(2n+1)\pi x}{2}\, dx = \frac{1}{2}, \quad \text{and}$$
$$(f, \phi_n) = \int_0^1 \sin \frac{(2n+1)\pi x}{2}\, dx = \frac{2}{(2n+1)\pi}.$$

Consequently, the coefficients are

$$c_n = \frac{4}{(2n+1)\pi},$$

and the generalized Fourier series is

$$1 = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin \frac{(2n+1)\pi x}{2}.$$

2. From Exercise 1 we know that $(\phi_n, \phi_n) = 1/2$. We compute that

$$\begin{aligned}(f, \phi_n) &= \int_0^1 \sin \pi x \cdot \sin \frac{(2n+1)\pi x}{2}\, dx \\ &= \frac{1}{2}\int_0^1 \cos \frac{(2n-1)\pi x}{2} \\ &\quad - \frac{1}{2}\int_0^1 \cos \frac{(2n+3)\pi x}{2},\, dx \\ &= (-1)^{n+1}\left[\frac{1}{\pi(2n-1)} - \frac{1}{\pi(2n+3)}\right] \\ &= (-1)^{n+1}\frac{4}{\pi(2n-1)(2n+3)}.\end{aligned}$$

The generalized Fourier series is

$$\begin{aligned} f(x) &= \sin \pi x \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}8}{\pi(2n-1)(2n+3)} \sin \frac{(2n+1)\pi x}{2} \end{aligned}$$

3. From Exercise 1 we know that $(\phi_n, \phi_n) = 1/2$. Using integration by parts, we compute that

$$\begin{aligned}(f, \phi_n) &= \int_0^1 (1-x)\cdot \sin \frac{(2n+1)\pi x}{2}\, dx \\ &= -\frac{2(1-x)}{(2n+1)\pi} \cos \frac{(2n+1)\pi x}{2}\Bigg|_0^1 \\ &\quad - \frac{2}{(2n+1)\pi}\int_0^1 \cos \frac{(2n+1)\pi x}{2}\, dx \\ &= \frac{2}{(2n+1)\pi} + (-1)^{n+1}\frac{4}{(2n+1)^2\pi^2}.\end{aligned}$$

The generalized Fourier series is

$$\begin{aligned} f(x) &= 1 - x \\ &= \sum_{n=0}^{\infty}\left[\frac{4}{(2n+1)\pi} + (-1)^{n+1}\frac{8}{(2n+1)^2\pi^2}\right] \\ &\quad \times \sin \frac{(2n+1)\pi x}{2}. \end{aligned}$$

4. From Exercise 1 we know that $(\phi_n, \phi_n) = 1/2$. Since

$\sin^2 x = [1 - \cos 2x]/2$, we have

$$(f, \phi_n) = \int_0^1 \sin^2 \pi x \cdot \sin \frac{(2n+1)\pi x}{2}\,dx$$
$$= \frac{1}{2}\int_0^1 \sin \frac{(2n+1)\pi x}{2}\,dx$$
$$- \frac{1}{2}\int_0^1 \cos 2\pi x \cdot \sin \frac{(2n+1)\pi x}{2}\,dx$$

For the first integral we have

$$\int_0^1 \sin \frac{(2n+1)\pi x}{2}\,dx = \frac{2}{(2n+1)\pi}.$$

For the second we use the identity $\sin\alpha \cdot \cos\beta = [\sin(\alpha+\beta) + \sin(\alpha-\beta)]/2$. The second integral becomes

$$\int_0^1 \cos 2\pi x \cdot \sin \frac{(2n+1)\pi x}{2}\,dx$$
$$= \frac{1}{2}\int_0^1 \sin \frac{(2n+5)\pi x}{2}\,dx$$
$$+ \frac{1}{2}\int_0^1 \sin \frac{(2n-3)\pi x}{2}\,dx$$
$$= \frac{1}{2}\cdot\frac{2}{(2n+5)\pi} + \frac{1}{2}\cdot\frac{2}{(2n-3)\pi}$$
$$= \frac{4n+2}{\pi(2n-3)(2n+5)}.$$

Thus

$$(f, \phi_n) = \frac{-16}{\pi(2n-3)(2n+1)(2n+5)},$$

and

$$\sin^2 \pi x = \sum_{n=1}^{\infty} \frac{-32}{\pi(2n-3)(2n+1)(2n+5)} \times \sin \frac{(2n+1)\pi x}{2}.$$

5. The eigenfunctions are $\phi_n(x) = \sin(\theta_n x)$, where θ_n is the nth root of the equation $\tan\theta = -\theta$. This equation can also be written as $\sin\theta = -\theta\cos\theta$. We compute

$$(\phi_n, \phi_n) = \int_0^1 \sin^2 \theta_n x\,dx$$
$$= \frac{1}{2}\int_0^1 [1 - \cos 2\theta_n x]\,dx$$
$$= \frac{1}{2} - \frac{1}{4\theta_n}\sin 2\theta_n$$
$$= \frac{1}{2}\left[1 - \frac{1}{\theta_n}\sin\theta_n\cos\theta_n\right]$$
$$= \frac{1}{2}[1 + \cos^2\theta_n].$$

We also compute

$$(f, \phi_n) = \int_0^1 \sin\theta_n x\,dx = \frac{1-\cos\theta_n}{\theta_n}.$$

Thus the coefficients are

$$c_n = \frac{2(1-\cos\theta_n)}{\theta_n(1+\cos^2\theta_n)}.$$

We have

$$1 = f(x) = \sum_{n=1}^{\infty} \frac{2(1-\cos\theta_n)}{\theta_n(1+\cos^2\theta_n)}\sin(\theta_n x).$$

6. We compute

$$(f, \phi_n) = \int_0^1 \sin\pi x \cdot \sin\theta_n x\,dx$$
$$= \frac{1}{2}\int_0^1 [\cos(\theta_n - \pi)x - \cos(\theta_n + \pi)x]\,dx$$
$$= \frac{\pi\sin\theta_n}{\pi^2 - \theta_n^2}.$$

Therefore the coefficients are

$$c_n = \frac{2\pi\sin\theta_n}{(\pi^2 - \theta_n^2)(1+\cos^2\theta_n)},$$

and

$$\sin\pi x = \sum_{n=1}^{\infty} \frac{2\pi\sin\theta_n \cdot \sin\theta_n x}{(\pi^2 - \theta_n^2)(1+\cos^2\theta_n)}.$$

7. Integrating by parts, we get

$$
\begin{aligned}
(f, \phi_n) &= \int_0^1 (1-x)\sin\theta_n x\,dx \\
&= -\frac{1}{\theta_n}(1-x)\cos\theta_n x\Big|_0^1 \\
&\quad + \frac{1}{\theta_n}\int_0^1 \cos\theta_n x\,dx \\
&= \frac{\theta_n - \sin\theta_n}{\theta_n^2}.
\end{aligned}
$$

Using the fact that $\sin\theta_n = -\theta_n\cos\theta_n$, we see that the coefficients are

$$c_n = \frac{2(\theta_n - \sin\theta_n)}{\theta_n^2(1+\cos^2\theta_n)} = \frac{2(1+\cos\theta_n)}{\theta_n(1+\cos^2\theta_n)},$$

and

$$1 - x = \sum_{n=1}^{\infty} \frac{2(1+\cos\theta_n)}{\theta_n(1+\cos^2\theta_n)}\sin\theta_n x.$$

8. Since $\sin^2\alpha = [1-\cos 2\alpha]/2$, and $\cos\alpha\cdot\sin\beta = [\sin(\beta+\alpha)+\sin(\beta-\alpha)]/2$,

$$
\begin{aligned}
(f, \phi_n) &= \frac{1}{2}\int_0^1 \sin^2\pi x\cdot\sin\theta_n x\,dx \\
&= \frac{1}{2}\int_0^1 \sin\theta_n x\,dx \\
&\quad - \frac{1}{4}\int_0^1 \sin(\theta_n + 2\pi)x\,dx \\
&\quad - \frac{1}{4}\int_0^1 \sin(\theta_n - 2\pi)x\,dx \\
&= \frac{1-\cos\theta_n}{2}\left[\frac{1}{\theta_n} - \frac{1}{2(\theta_n+2\pi)} - \frac{1}{2(\theta_n - 2\pi)}\right] \\
&= \frac{2\pi^2(1-\cos\theta_n)}{\theta_n(4\pi^2 - \theta_n^2)}.
\end{aligned}
$$

Hence

$$\sin^2\pi x = \sum_{n=1}^{\infty} \frac{4\pi^2(1-\cos\theta_n)}{\theta_n(4\pi^2-\theta_n^2)(1+\cos^2\theta_n)}\sin\theta_n x.$$

9. The solution in general is given in 7.22. For $f(x) = 1$ we computed the generalized Fourier coefficients in Exercise 1. The solution is

$$u(t,x) = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} e^{-(2n+1)^2\pi^2 t/4} \times \sin\frac{(2n+1)\pi x}{2}.$$

---

10. The solution in general is given in 7.22. For $f(x) = \sin\pi s$ we computed the generalized Fourier coefficients in Exercise 2. The solution is

$$u(t,x) = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{8}{\pi(2n-1)(2n+3)} e^{-(2n+1)^2\pi^2 t/4}\sin\frac{(2n+1)\pi x}{2}.$$

11. The solution in general is given in 7.22. For $f(x) = 1 - x$ we computed the generalized Fourier coefficients in Exercise 3. The solution is

$$u(t,x) = \sum_{n=0}^{\infty}\left[\frac{4}{(2n+1)\pi} + (-1)^{n+1}\frac{8}{(2n+1)^2\pi^2}\right] e^{-(2n+1)^2\pi^2 t/4}\sin\frac{(2n+1)\pi x}{2}.$$

12. The solution in general is given in 7.22. For $f(x) = \sin^2 \pi x$ we computed the generalized Fourier coefficients in Exercise 4. The solution is

$$u(t, x) = \sum_{n=0}^{\infty} \frac{-32}{\pi(2n-3)(2n+1)(2n+5)} e^{-(2n+1)^2\pi^2 t/4} \sin \frac{(2n+1)\pi x}{2}.$$

13. The solution to the initial/boundary value problem is

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \sin \frac{\theta_n x}{L},$$

where $\lambda_n = \theta_n^2$. For $f(x) = 1$ we computed the generalized Fourier coefficients in Exercise 5. The solution is

$$u(t, x) = \sum_{n=1}^{\infty} \frac{2(1-\cos\theta_n)}{\theta_n(1+\cos^2\theta_n)} e^{-\theta_n^2 t} \sin \theta_n x.$$

14. The solution to the initial/boundary value problem is

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \sin \frac{\theta_n x}{L},$$

where $\lambda_n = \theta_n^2$. For $f(x) = \sin \pi x$ we computed the generalized Fourier coefficients in Exercise 6. The solution is

$$u(t, x) = \sum_{n=1}^{\infty} \frac{2\pi \sin\theta_n}{(\pi^2 - \theta_n^2)(1+\cos^2\theta_n)} e^{-\theta_n^2 t} \sin \theta_n x.$$

15. The solution to the initial/boundary value problem is

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \sin \frac{\theta_n x}{L},$$

where $\lambda_n = \theta_n^2$. For $f(x) = 1 - x$ we computed the generalized Fourier coefficients in Exercise 7. The solution is

$$u(t, x) = \sum_{n=1}^{\infty} \frac{2(1+\cos\theta_n)}{\theta_n(1+\cos^2\theta_n)} e^{-\theta_n^2 t} \sin \theta_n x.$$

16. The solution to the initial/boundary value problem is

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \sin \frac{\theta_n x}{L},$$

where $\lambda_n = \theta_n^2$. For $f(x) = \sin^2 \pi x$ we computed the generalized Fourier coefficients in Exercise 8. The solution is

$$u(t, x) = \sum_{n=1}^{\infty} \frac{4\pi^2(1-\cos\theta_n)}{\theta_n(4\pi^2 - \theta_n^2)(1+\cos^2\theta_n)} e^{-\theta_n^2 t} \sin \theta_n x.$$

17. We look for product solutions of the form $u(x, y) = X(x)Y(y)$ which satisfy the boundary conditions in the variable x. Separating the variables in the differential equation

$$\nabla^2 u = X''Y + XY'' = 0,$$

we get the ordinary differential equations

$$-X'' = \lambda X \quad \text{and} \quad Y'' - \lambda Y = 0,$$

where λ is a constant. Invoking the boundary conditions, we see that X must be a solution to the Sturm Liouville problem

$$-X'' = \lambda X \quad \text{with} \quad X(0) = X'(1) = 0.$$

This is the Sturm Liouville problem in Example 6.26. The solutions are

$$\lambda_n = \frac{(2n+1)^2\pi^2}{4} \quad \text{and} \quad \phi_n(x) = \sin\frac{(2n+1)\pi x}{2} \quad \text{for } n = 0, 1, 2, \ldots.$$

In analogy with equation 4.11, for each positive integer n, we get two product solutions

$$u_n(x, y) = \sinh\left(\frac{(2n+1)\pi y}{2}\right)\sin\left(\frac{(2n+1)\pi x}{2}\right) \quad \text{and}$$
$$v_n(x, y) = \sinh\left(\frac{(2n+1)\pi(y-1)}{2}\right)\sin\left(\frac{(2n+1)\pi x}{2}\right)$$

Thus we get the general solution

$$\begin{aligned} u(x, y) &= \sum_{n=1}^{\infty} a_n u_n(x, y) + \sum_{n=1}^{\infty} b_n v_n(x, y) \\ &= \sum_{n=1}^{\infty} a_n \sinh\left(\frac{(2n+1)\pi y}{2}\right)\sin\left(\frac{(2n+1)\pi x}{2}\right) \\ &\quad + \sum_{n=1}^{\infty} b_n \sinh\left(\frac{(2n+1)\pi(y-1)}{2}\right)\sin\left(\frac{(2n+1)\pi x}{2}\right), \end{aligned}$$

For the boundary condition at $y = 0$ we have the generalized Fourier series

$$T_1 = u(x, 0) = -\sum_{n=1}^{\infty} b_n \sinh\left(\frac{(2n+1)\pi}{2}\right)\sin\left(\frac{(2n+1)\pi x}{2}\right).$$

The coefficients are given by

$$-b_n \sinh\left(\frac{(2n+1)\pi}{2}\right) = \frac{(T_1, X_n)}{(X_n, X_n)} = \frac{4T_1}{(2n+1)\pi},$$

or

$$b_n = \frac{-4T_1}{(2n+1)\pi\,\sinh((2n+1)\pi/2)}.$$

Similarly

$$a_n = \frac{4T_2}{(2n+1)\pi \sinh((2n+1)\pi/2)}.$$

Hence the temperature is

$$u(x, y) = \sum_{n=1}^{\infty} \frac{4}{(2n+1)\pi \sinh((2n+1)\pi/2)} \sin\left(\frac{(2n+1)\pi x}{2}\right)$$
$$\cdot \left[T_2 \sinh\left(\frac{(2n+1)\pi y}{2}\right) - T_1 \sinh\left(\frac{(2n+1)\pi(y-1)}{2}\right)\right].$$

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Section 13.8. Temperatures in a Ball—Legendre Polynomials

1. We have the boundary temperature $f(z) = T$. If a is the radius of the sphere, we have $f(as) = T = TP_0(s)$. Hence $u(r, phi) = TP_0(\cos\phi) = T$.

2. We have the boundary temperature $f(z) = 1 - z$. Since $f(s) = 1 - s = P_0(s) - P_1(s)$, the temperature in the sphere is $u(r, \phi) = P_0(\cos\phi) + rP_1(\cos\phi) = 1 - r\cos\phi$. Since $z = r\cos\phi$, the formula for the temperature in cartesian coordinates is $u(x, y, z) = 1 - z$.

3. We have $f(s) = s^3 = [2P_3(s) + 3P_1(s)]/5$. Therefore the temperature is

$$u(r, \phi) = \frac{2}{5}r^3 P_3(\cos\phi) + \frac{3}{5}rP_1(\cos\phi) = \frac{1}{5}r^3\left[5\cos^3\phi - 3\cos\phi\right] + \frac{3}{5}r\cos\phi.$$

When put into cartesian coordinates, this becomes

$$u(x, y, z) = z^3 + \frac{3}{5}z(1 - x^2 - y^2 - z^2).$$

4. We have $f(s) = s^4 = [8P_4(s) + 20P_2(s) + 7P_0(s)]/35$. Therefore the temperature is

$$u(r, \phi) = \frac{1}{35}\left[8r^4 P_4(\cos\phi) + 20r^2 P_2(\cos\phi) + 7\right]$$
$$= \frac{1}{35}\left[r^4(35\cos^4\phi - 30\cos^2\phi + 3) + 10r^2(3\cos^2\phi - 1) + 7\right].$$

When put into cartesian coordinates, this becomes

$$u(x, y, z) = z^4 + \frac{1}{35}\left[30z^2(1 - x^2 - y^2 - z^2) - 7(x^2 - y^2 - z^2) + 7\right].$$

5. We have

$$f(s) = \begin{cases} 10, & \text{for } s > 0, \text{ and} \\ 0, \text{ for } s \le 0. \end{cases}$$

The Legendre coefficients for f are given by

$$c_n = \frac{2n+1}{2} \int_0^1 P_n(s)\, ds.$$

Evaluating explicitly we get

$$c_0 = \frac{1}{2}\int_0^1 10\, ds = 5, \quad c_1 = \frac{3}{2}\int_0^1 10s\, ds = \frac{15}{2},$$
$$c_2 = \frac{5}{2}\int_0^1 10\frac{3s^2-1}{2}\, ds = 0, \quad c_3 = \frac{7}{2}\int_0^1 10\frac{5s^3-3s}{2}\, ds = -\frac{35}{8}.$$

Hence the temperature is given approximately by

$$\begin{aligned} u(r,\phi) &= \frac{1}{2} + \frac{3}{4} r\cos\phi - \frac{7}{16} r^3 \frac{5\cos^3\phi - 3\cos\phi}{2} + \cdots \\ &= \frac{1}{2} + \frac{3}{4} z - \frac{35}{32} z^3 + \frac{21}{32} z(x^2+y^2+z^2) + \cdots . \end{aligned}$$

6. According to (8.9), the solution is $u(r,\phi) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\phi$, where the coefficients are given by (8.11). Using the hint and the fact that $P_n(1) = 1$, we get

$$\begin{aligned} n\int_0^1 P_n(s)\, ds &= \int_0^1 sP_n'(s)\, ds - \int_0^1 P_{n-1}'(s)\, ds \\ &= 1 - \int_0^1 P_n(s)\, ds - [1 - P_{n-1}(0)]. \end{aligned}$$

Collecting the terms involving the integral of P_n on the left, we see that

$$(n+1)\int_0^1 P_n(s)\, ds = P_{n-1}(0).$$

We get

$$\int_0^1 P_{2n}(s)\, ds = 0 \quad \text{for } n \ge 1, \text{ and}$$
$$\int_0^1 P_{2n+1}(s)\, ds = (-1)^n \frac{(2n)!}{2^{2n+2}(n+1)(n!)^2} \quad \text{for } n \ge 0.$$

Then we have $c_0 = 10/2 = 5$, $c_{2n} = 0$ for $n \ge 1$, and

$$c_{2n+1} = (-1)^n \frac{10\cdot(4n+3)(2n)!}{2^{2n+2}(n+1)(n!)^2} \quad \text{for } n \ge 0.$$

Therefore

$$u(r,\phi) = 5 + 10\sum_{n=0}^{\infty} (-1)^n \frac{(4n+3)(2n)!}{2^{2n+2}(n+1)(n!)^2} r^{2n+1} P_{2n+1}(\cos\phi).$$

7. The Legendre coefficients for f are given by

$$c_n = \frac{2n+1}{2}\int_0^1 s P_n(s)\,ds.$$

Evaluating explicitly we get

$$c_0 = \frac{1}{2}\int_0^1 s\,ds = \frac{1}{4}, \quad c_1 = \frac{3}{2}\int_0^1 s\cdot s\,ds = \frac{1}{2},$$
$$c_2 = \frac{5}{2}\int_0^1 s\cdot\frac{3s^2-1}{2}\,ds = \frac{5}{16}, \quad c_3 = \frac{7}{2}\int_0^1 s\cdot\frac{5s^3-3s}{2}\,ds = 0.$$

8. According to (8.9), the solution is $u(r,\phi) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\phi$, where the coefficients are given by (8.11). Using the hint and the fact that $P_n(1) = 1$, we get

$$\begin{aligned} n\int_0^1 s P_n(s)\,ds &= \int_0^1 s^2 P_n'(s)\,ds - \int_0^1 s P_{n-1}'(s)\,ds \\ &= 1 - 2\int_0^1 s P_n(s)\,ds - 1 + \int_0^1 P_{n-1}(s)\,ds. \end{aligned}$$

Collecting the terms involving the integral of P_n on the left, we see that

$$(n+2)\int_0^1 P_n(s)\,ds = \int_0^1 P_{n-1}(s)\,ds.$$

This last integral was computed in Exercise 6, and we get

$$\int_0^1 P_n(s)\,ds = \frac{1}{n(n+2)}P_{n-2}(0) \quad \text{for } n \ge 2.$$

We get

$$\int_0^1 s P_{2n+1}(s)\,ds = 0 \quad \text{for } n \ge 1, \text{ and}$$
$$\int_0^1 s P_{2n}(s)\,ds = (-1)^{n-1}\frac{(2n-2)!}{2^{2n}n(n+1)[(n-1)!]^2} \quad \text{for } n \ge 1.$$

Then we have $c_0 = 1/4$, $c_1 = 1/2$, $c_{2n+1} = 0$ for $n \ge 1$, and

$$c_{2n} = (-1)^{n-1}\frac{(4n+1)(2n-2)!}{2^{2n+1}n(n+1)[(n-1)!]^2} \quad \text{for } n \ge 1.$$

Therefore

$$u(r,\phi) = \frac{1}{4} + \frac{1}{2}r P_1(\cos\phi) + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(4n+1)(2n-2)!}{2^{2n+1}n(n+1)[(n-1)!]^2}r^{2n}P_{2n}(\cos\phi).$$

9. Consider the temperature v of the ball with the botton half at 10° and the top half at 0°. Then $u + v = 100$ on the boundary, so $u + v = 100$ throughout the ball. In addition, by symmetry, $u = v$ at the center of the ball. Therefore $u = 5°$ at the center.

10. Since the boundary temperature is independent of the angle variables, we can assume that the temperature is a function of the radius only. Hence $u = u(r)$. Laplace's equation for such a function is

$$0 = \nabla^2 u = \frac{1}{r^2}(r^2 u')' = 0.$$

We conclude that $u(r) = Ar^{-1} + B$, for suitable constants A and B. The boundary conditions reduce to $A + B = 0$ and $A/2 + B = 10$. Solving, we find that

$$u(r) = 20\left(1 - \frac{1}{r}\right).$$

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Section 13.9. The Heat and Wave Equations in Higher Dimension

1. If $u(t, x, y) = T(t)\phi(x, y)$, then the equation $u_t = \nabla^2 u$ becomes $T_t\phi = T\nabla^2\phi$. Separating variables we get

$$\frac{T_t}{T} = \frac{\nabla^2\phi}{\phi}.$$

Since the left-hand side depends only on t, while the right-hand side depends on (x, y), we conclude that both are equal to the same constant, which we write as $-\lambda$. Both differential equations follow. Since $0 = u(t, x, 0) = T(t)\phi(x, 0)$, we conclude that $\phi(x, 0) = 0$. The other boundary conditions follow in the same way.

2. We look for product solutions of the form $\phi(x, y) = X(x)Y(y)$. The differential equation becomes $-X_{xx}Y - XY_{yy} = \lambda XY$. Dividing by XY we get

$$-\frac{X_{xx}}{X} - \frac{Y_{yy}}{Y} = \lambda.$$

Each summand must be constant, so we conclude that there are constants μ and ν such that

$$-X_{xx} = \mu X \quad \text{and} \quad -Y_{yy} = \nu Y, \quad \text{with} \quad \mu + \nu = \lambda.$$

The boundary conditions for ϕ translate into

$$X(0) = X'(1) = 0 \quad \text{and} \quad Y(0) = Y(0) = 0.$$

Thus we have two Sturm Liouville problems. The problem for Y we solved in Section 2. The solutions are

$$\nu_q = q^2\pi^2 \quad \text{with} \quad Y_q(y) = \sin q\pi y, \quad \text{for } q \geq 1.$$

The problem for X we solved in Section 6. The solutions are

$$\mu_p = \frac{(2p+1)^2\pi^2}{4} \quad \text{with} \quad X_p(x) = \sin\left(\frac{(2p+1)\pi x}{2}\right), \quad \text{for } p \geq 0.$$

Thus the eigenvalues are $\lambda_{p,q} = \mu_p + \nu_q$, and the eigenfunctions are $\phi_{p,q}(x, y) = X_p(x)Y_q(y)$.

3.
$$\begin{aligned}
&\int_D \phi_{p,q}(x, y)\, \phi_{p',q'}(x, y)\, dx\, dy \\
&= \int_0^1 \sin\left(\frac{(2p+1)\pi x}{2}\right) \sin\left(\frac{(2p'+1)\pi x}{2}\right) dx \int_0^1 \sin(q\pi y)\sin(q'\pi y)\, dy
\end{aligned}$$
The first integral is equal to 0 if $p' \neq p$ and equal to 1/2 if $p' = p$. The second integral is equal to 0 if $q' \neq q$ and and equal to 1/2 if $q' = q$. The result follows from this.

4. With $\lambda = \lambda_{p,q}$, the equation for T in Exercise 1 has solution $T_{p,q}(t) = e^{-\lambda_{p,q}t}$. Hence we have the product solutions $u_{p,q}(t, x, y) = T_{p,q}(t)\phi_{p.q}(x, y)$ to the heat equation that also satisfy the Dirichlet boundary conditions. Hence, the series
$$u(t, x, y) = \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} c_{p,q} e^{\lambda_{p,q}kt} \phi_{p,q}(x, y),$$
is a solution, provided that it converges.
The initial condition is
$$f(x, y) = u(0, x, y) = \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} c_{p,q}\phi_{p,q}(x, y).$$
If this is true, then multiplying f by $\phi_{p',q'}$ and integrating over D, we get, using the orthogonality relations in Exercise 3,
$$\begin{aligned}
\int_D f\phi_{p',q'}\, dx\, dy &= \int_D \phi_{p',q'} \left(\sum_{p=0}^{\infty}\sum_{q=1}^{\infty} c_{p,q}\phi_{p,q}\right) dx\, dy \\
&= \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} c_{p,q} \int_D \phi_{p',q'}\phi_{p,q}\, dx\, dy \\
&= c_{p',q'} \cdot \frac{1}{4}.
\end{aligned}$$
The result follows.

5. It is necessary to start by finding a steady-state temperatue u_s that solves the boundary value problem
$$\begin{aligned}
\nabla^2 u_s &= 0, \quad \text{in } D, \\
u_s(x, 0) = u_s(x, 1) = T_1, &\quad \text{and} \quad u_s(0, y) = (u_s)_x(1, y) = 0.
\end{aligned}$$
Then the function $v(t, x, y) = u(t, x, y) - u_s(x, y)$ must be found to solve the initial/boundary value problem
$$\begin{aligned}
v_t &= \nabla^2 v, \\
v(t, x, 0) = v(t, x, 1) = T_1, &\quad \text{and} \quad v(t, 0, y) = v_x(t, 1, y) = 0 \\
v(0, x, y) &= f(x, y) - u_s(x, y),
\end{aligned}$$
which is a variant of the problem in these exercises. Then the solution is $u(t, x, y) = u_s(x, y) + v(t, x, y)$

6. In this case the eigenvalue problem for the Laplacian is modified to $-\nabla^2\phi = \lambda\phi$ with the boundary conditions
$$\phi(x, 0) = \phi(x, 1) = 0, \quad \text{and} \quad \phi_x(0, y) = \phi_x(1, y) = 0.$$

When we separate variables on this problem we get the two Sturm Liouville problems

$$\begin{aligned} -X'' &= \mu X \quad \text{with} \quad X'(0) = X'(1) = 0, \quad \text{and} \\ -Y'' &= \nu Y \quad \text{with} \quad Y(0) = Y(1) = 0. \end{aligned}$$

We solved both of these in Section 2, getting

$$\begin{aligned} \mu_p &= p^2\pi^2 \quad \text{with} \quad X_p(x) = \cos p\pi x, \quad \text{for } p \ge 0, \text{ and} \\ \nu_q &= q^2\pi^2 \quad \text{with} \quad Y_q(y) = \sin q\pi y, \quad \text{for } q \ge 1. \end{aligned}$$

Thus the solutions to the eigenvalue problem are

$$\lambda_{p,q} = (p^2 + q^2)\pi^2 \quad \text{with} \quad \phi_{p,q}(x, y) = \cos(p\pi x)\sin(q\pi y), \quad \text{for } p \ge 0 \text{ and } q \ge 1.$$

Everything else is the same.

7. (a) We check the four cases. Remember that $0 < x < \pi$ and $0 < y < \pi$.

$$0 < x < \min\{y, \pi - x, \pi - y\} \Rightarrow \begin{cases} -\pi < x - y < 0 & \text{so } F(x - y) = \pi - y + x \\ 0 < x < \pi/2 & \\ 0 < x + y < \pi & \text{so } F(x + y) = \pi - x - y \end{cases}$$

Therefore, $F(x - y) - F(x + y) = 2x$.

$$0 < y < \min\{x, \pi - x, \pi - y\} \Rightarrow \begin{cases} 0 < x - y < \pi & \text{so } F(x - y) = \pi - x + y \\ 0 < x + y < \pi & \text{so } F(x + y) = \pi - x - y \\ 0 < y < \pi/2 & \end{cases}$$

Therefore, $F(x - y) - F(x + y) = 2y$.

$$0 < \pi - x < \min\{x, y, \pi - y\} \Rightarrow \begin{cases} x > \pi/2 & \\ x + y > \pi & \\ -\pi < x - y < 0 & \text{so } F(x - y) = \pi - y + x \end{cases}$$

The second inequality and the periodicity of F implies that $F(x+y) = F(x+y-2\pi) = \pi - (2\pi - x - y) = x + y - \pi$. Therefore, $F(x - y) - F(x + y) = 2(\pi - x)$.

$$0 < \pi - y < \min\{x, y, \pi - x\} \Rightarrow \begin{cases} x + y > \pi & \\ y > \pi/2 & \\ -\pi < x - y < 0 & \text{so } F(x - y) = \pi - y + x \end{cases}$$

The first inequality and the periodicity of F implies that $F(x + y) = F(x + y - 2\pi) = \pi - (2\pi - x - y) = x + y - \pi$. Therefore, $F(x - y) - F(x + y) = 2(\pi - y)$.

(b) F is an even function, so its Fourier sereis in the cosine series

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz,$$

where the coefficients are $a_0 = \pi$, and

$$a_n = \frac{2}{\pi}\int_0^\pi (\pi - z)\cos nz\,dz = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 4/(n^2\pi), & \text{if } n \text{ is odd.} \end{cases}$$

Hence

$$F(z) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)z}{(2n+1)^2}.$$

(c) The addition formula for the cosine is $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$. Hence

$$F(x-y) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x\cdot\cos(2n+1)y + \sin(2n+1)x\cdot\sin(2n+1)y}{(2n+1)^2}, \quad \text{and}$$

$$F(x+y) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x\cdot\cos(2n+1)y - \sin(2n+1)x\cdot\sin(2n+1)y}{(2n+1)^2}.$$

Hence

$$\begin{aligned} f(x,y) &= [F(x-y) - F(x+y)]/2 \\ &= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)x\cdot\sin(2n+1)y}{(2n+1)^2}. \end{aligned}$$

(d) Notice that only the diagonal terms $p = q = 2n+1$ occur. The eigenvalues are $\lambda_{2n+1,2n+1} = 2(2n+1)^2$. Hence,

$$u(t,x,y) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)x\cdot\sin(2n+1)y\cdot\cos\sqrt{2}c(2n+1)t}{(2n+1)^2}.$$

The time frequencies are $(2n+1)\cdot\sqrt{2}ct$. Since they are all integer multiples of the fundamental frequency $\sqrt{2}ct$, the vibration is periodic in time with period $2\pi/[\sqrt{2}ct]$.

8. By computation with the chain rule, we see that

$$\begin{aligned} u_{xx}(t,x,y) &= u_{yy}(t,x,y) \\ &= \frac{1}{4}[F''(x-y+\sqrt{2}ct) + F''(x-y-\sqrt{2}ct) \\ &\qquad - F''(x+y+\sqrt{2}ct) - F''(x+y-\sqrt{2}ct)], \end{aligned}$$

while

$$\begin{aligned} u_{tt}(t,x,y) = 2c^2\cdot\frac{1}{4}[&F''(x-y+\sqrt{2}ct) + F''(x-y-\sqrt{2}ct) \\ &- F''(x+y+\sqrt{2}ct) - F''(x+y-\sqrt{2}ct)]. \end{aligned}$$

Hence $u_{tt} = c^2[u_{xx} + u_{yy}]$. Setting $t = 0$, we see that

$$\begin{aligned} u(0,x,y) &= \frac{1}{4}[F(x-y) + F(x-y) - F(x+y) - F(x+y)] \\ &= f(x,y). \end{aligned}$$

Differentiating u with respect to t, and evaluating at $t = 0$, we get

$$\begin{aligned} u_t(0, x, y) &= \frac{\sqrt{2}c}{4}[F'(x-y) - F'(x-y) - F'(x+y) + F'(x+y)] \\ &= 0. \end{aligned}$$

For the boundary conditions, first setting $x = 0$,

$$\begin{aligned} u(t, 0, y) &= \frac{1}{4}[F(-y+\sqrt{2}ct) + F(-y-\sqrt{2}ct) - F(y+\sqrt{2}ct) - F(y-\sqrt{2}ct)] \\ &= 0, \end{aligned}$$

since F is an even function. Next setting $x = \pi$,

$$\begin{aligned} u(t, \pi, y) &= \frac{1}{4}[F(\pi - y + \sqrt{2}ct) + F(\pi - y - \sqrt{2}ct) \\ &\qquad - F(\pi + y + \sqrt{2}ct) - F(\pi + y - \sqrt{2}ct)] \\ &= \frac{1}{4}[F(\pi - y + \sqrt{2}ct) + F(\pi - y - \sqrt{2}ct) \\ &\qquad - F(-\pi + y + \sqrt{2}ct) - F(-\pi + y - \sqrt{2}ct)] \\ &= 0, \end{aligned}$$

since F is 2π-periodic and even. The other boundary conditions are verified in a similar way.

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Section 13.10. Domains with Circular Symmetry—Bessel Functions

1. Using polar coordinates, we have

$$\begin{aligned} \int_D \phi_{n,k}\phi_{n',k'}\,dx\,dy &= \int_D \cos n\theta \cdot J_n(\alpha_{n,k}r/a) \cdot \cos n'\theta \cdot J_{n'}(\alpha_{n',k'}r/a)\,dx\,dy \\ &= \int_0^{2\pi} \cos n\theta \cdot \cos n'\theta\,d\theta \cdot \int_0^a J_n(\alpha_{n,k}r/a) \cdot J_{n'}(\alpha_{n',k'}r/a)\,r\,dr. \end{aligned}$$

If $n \neq n'$, the first integral is equal to 0. If $k \neq k'$, the second integral is equal to 0. If $n = n'$, the first integral is equal to π, and if $k = k'$ the second is equal to $a^2 J_{n+1}^2(\alpha_{n,k})/2$, so the first orthogonality relation is verified. Similarly,

$$\begin{aligned} \int_D \psi_{n,k}\psi_{n',k'}\,dx\,dy &= \int_D \sin n\theta \cdot J_n(\alpha_{n,k}r/a) \cdot \sin n'\theta \cdot J_{n'}(\alpha_{n',k'}r/a)\,dx\,dy \\ &= \int_0^{2\pi} \sin n\theta \cdot \sin n'\theta\,d\theta \cdot \int_0^a J_n(\alpha_{n,k}r/a) \cdot J_{n'}(\alpha_{n',k'}r/a)\,r\,dr. \end{aligned}$$

Again, if $n \neq n'$, the first integral is equal to 0, and if $k \neq k'$, the second integral is equal to 0. If $n = n'$, the first integral is equal to π, and if $k = k'$ the second is equal to $a^2 J_{n+1}^2(\alpha_{n,k})/2$, so the second orthogonality relation is verified. Finally

$$\begin{aligned}\int_D \phi_{n,k}\psi_{n',k'}\,dx\,dy &= \int_D \cos n\theta \cdot J_n(\alpha_{n,k}r/a) \cdot \sin n'\theta \cdot J_{n'}(\alpha_{n',k'}r/a)\,dx\,dy \\ &= \int_0^{2\pi} \cos n\theta \cdot \sin n'\theta\,d\theta \cdot \int_0^a J_n(\alpha_{n,k}r/a) \cdot J_{n'}(\alpha_{n',k'}r/a)\,r\,dr.\end{aligned}$$

The first integral is equal to 0, so we are through.

2. If we multiply the series for f_0 by $\phi_{p,q}$ and integrate term by term, we get

$$\begin{aligned}&\int_D f_0(x, y)\phi_{p,q}(x, y)\,dx\,dy \\ &= \sum_{k=1}^{\infty} a_{0,k} \int_0^a \int_0^{2\pi} J_0\left(\frac{\alpha_{0,k}r}{a}\right) \cdot \cos p\theta \cdot J_p(\alpha_{p,q}r/a)\;d\theta\,r\,dr \\ &+ \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \int_0^a \int_0^{2\pi} J_n\left(\frac{\alpha_{n,k}r}{a}\right)\left[a_{n,k}\cos n\theta + b_{n,k}\sin n\theta\right] \cdot \cos p\theta \cdot J_p(\alpha_{p,q}r/a)\;d\theta\,r\,dr.\end{aligned}$$

By the orthogonality relations, each summand is equal to 0, except when $n = p$ and $k = q$, and we get

$$\int_D f_0(x, y)\phi_{p,q}(x, y)\,dx\,dy = a_{p,q}\frac{\pi a^2 J_{p+1}^2(\alpha_{p,q})}{2}.$$

Thus

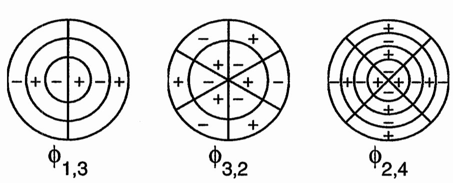
$$a_{p,q} = \frac{2}{\pi a^2 J_{p+1}^2(\alpha_{p,q})} \int_D f_0(x, y)\phi_{p,q}(x, y)\,dx\,dy.$$

The argument for $b_{p,q}$ is similar. Differentiating u with respect to t, and evaluating at $t = 0$, we get

$$f_1(r, \theta) = \frac{c\alpha_{n,k}}{a}\left\{\sum_{k=1}^{\infty} c_{0,k} J_0\left(\frac{\alpha_{0,k}r}{a}\right) + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} J_n\left(\frac{\alpha_{n,k}r}{a}\right)\left[c_{n,k}\cos n\theta + d_{n,k}\sin n\theta\right]\right\}.$$

Then the same argument as before yields the formulas for $c_{p,q}$ and $d_{p,q}$.

3.



4. The coefficients as given in (10.15) reduce to

$$\begin{aligned} a_{n,k} &= \frac{2}{\pi a^2 J_{n+1}(\alpha_{n,k})} \int_0^a \int_0^{2\pi} f(r) J_n\left(\frac{\alpha_{n,k} r}{a}\right) \cos n\theta \, r \, dr \, d\theta \\ &= \int_0^a f(r)\, J_n\left(\frac{\alpha_{n,k} r}{a}\right) r \, dr \cdot \int_0^{2\pi} \cos n\theta \, d\theta. \end{aligned}$$

The last integral is equal to 0 if $n \geq 1$, so $a_{n,k} = 0$ for $n \geq 0$. Similarly, $b_{n,k} = 0$ for $n \geq 0$. Thus the series for u reduces to

$$u(t, r, \theta) = \sum_{k=1}^{\infty} J_0\left(\frac{\alpha_{0,k} r}{a}\right) \left[a_{0,k} \cos \frac{c\alpha_{0,k} t}{a} + b_{0,k} \sin \frac{c\alpha_{0,k} t}{a} \right].$$

5. (a) The Laplacian of u in polar coordinates is $\nabla^2 u = u_{rr} + u_r/r + u_{\theta\theta}/r^2$. Since u does not depend on θ, this reduces to $\nabla^2 u = u_{rr} + u_r/r$.

(b) If we substitute $u(t, r) = T(t)R(r)$ into the heat equation, we get $RT_t = [R_{rr} + R_r/r]T$. When we separate variables, we get the two equations

$$T' = -\lambda k T \quad \text{and} \quad -[R_{rr} + \frac{1}{r} R_r] = \lambda R.$$

The first equation has solution $T(t) = e^{-\lambda k t}$. If we multiply the second equation by r it becomes $-(rR')' = \lambda r R$. Thus we are left with the Sturm Liouville problem in (10.7) with $n = 0$. The solutions are given in (10.8):

$$\lambda_p = \frac{\alpha_{0,p}^2}{a^2} \quad \text{and} \quad R_p(r) = J_0(\alpha_{0,p} r/a), \quad \text{for } p = 1, 2, \cdots.$$

Hence the product solutions are

$$e^{-k\alpha_{0,p}^2 t/a^2} J_0(\alpha_{0,p} r/a), \quad \text{for } p = 1, 2, \cdots.$$

(c) The solution is

$$u(t, r) = \sum_{p=1}^{\infty} c_p e^{-kt\alpha_{0,p}^2 t/a^2} J_0(\alpha_{0,p} r/a),$$

where

$$f(r) = u(0, r) = \sum_{p=1}^{\infty} c_p J_0(\alpha_{0,p} r/a)$$

is the Bessel series for f. The coefficients are computed in (10.10):

$$c_p = \frac{2}{a^2 J_1^2(\alpha_{0,p})} \int_0^a f(r) J_0(\alpha_{0,p} r/a)\, r \, dr.$$

6. We follow the procedure in text, and look for product solutions of the form $u(t, x, y) = T(t)\phi(x, y)$. When we substitute into the heat equation and separate variables, we see that there is a constant λ such that $T' = -\lambda k T$,

and ϕ is a solution to the eigenvalue problem in (10.1). The solutions are in (10.11). Since $T(t) = e^{-k\lambda t}$, where λ is the appropriate eigenvalue, we see that the solution is

$$u(t, r, \theta) = \sum_{p=1}^{\infty} a_{0,p} e^{-k\alpha_{0,p}^2 t/a^2} J_0\left(\frac{\alpha_{0,p} r}{a}\right)$$
$$+ \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \left[a_{n,p}\cos n\theta + b_{n,p}\sin n\theta\right] e^{-k\alpha_{n,p}^2 t/a^2} J_n\left(\frac{\alpha_{n,p} r}{a}\right).$$

The coefficients are found by using the initial condition:

$$f(r,\theta) = u(0, r, \theta) = \sum_{p=1}^{\infty} a_{0,p} J_0\left(\frac{\alpha_{0,p} r}{a}\right) \quad + \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \left[a_{n,p}\cos n\theta + b_{n,p}\sin n\theta\right] J_n\left(\frac{\alpha_{n,p} r}{a}\right).$$

Using the orthogonality relations in the usual way, we get

$$a_{n,p} = \frac{2}{\pi a^2 J_{n+1}(\alpha_{n,p})} \int_0^a \int_0^{2\pi} f(r,\theta) J_n\left(\frac{\alpha_{n,p} r}{a}\right) \cos n\theta \, r \, dr \, d\theta, \quad \text{and}$$
$$b_{n,p} = \frac{2}{\pi a^2 J_{n+1}^2(\alpha_{n,p})} \int_0^a \int_0^{2\pi} f(r,\theta) J_n\left(\frac{\alpha_{n,p} r}{a}\right) \sin n\theta \, r \, dr \, d\theta.$$

7. (a) We look for product solutions $u(t, x, y) = T(t)\phi(x, y)$. When we substitute into the heat equation and separate variables, we see that there is a constant λ such that $T' = -k\lambda T$, and $-\nabla^2\phi = \lambda\phi$. Since

$$\frac{\partial u}{\partial \mathbf{n}}(t, x, y)T(t) = T(t)\frac{\partial \phi}{\partial \mathbf{n}}(x, y),$$

we conclude that ϕ must satisfy a Neumann condition. Hence the eigenvalue problem is

$$-\nabla^2\phi(x, y) = \lambda\phi(x, y), \quad \text{for } (x, y) \in D, \text{ and}$$
$$\frac{\partial \phi}{\partial \mathbf{n}}(x, y) = 0, \quad \text{for } (x, y) \in \partial D.$$

(b) Since we know the Laplacian in polar coordinates, it is only necessary to understand what $\partial\phi/\partial\mathbf{n}$ iis in polar coordinates. Since the normal direction to the boundary at a point on the circular boundary of the disk points in the direction of increasing r, we see that $\partial\phi/\partial\mathbf{n} = \phi_r$. Hence the problem becomes

$$-\left[\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta}\right](r,\theta) = \lambda\phi(r,\theta), \quad \text{for } r < 1,$$
$$\phi_r(a, \theta) = 0 \quad \text{for } 0 \le \theta \le 2\pi.$$

(c) We look for product solutions $\phi(r,\theta) = R(r)U(\theta)$. Substituting and separating variables, we get the equations in refcdrum3. The Sturm Liouville problem for U is the same and has the solutions in (10.4). The Sturm Liouville problem for R, however, is different. It becomes

$$-(rR')' + \frac{n^2}{r}R = \lambda r R \quad \text{for } 0 < r < 1,$$
$$R \text{ and } R' \text{ are continuous at } r = 0,$$
$$R'(a) = 0.$$

Proposition 6.24 implies that all of the eigenvalues are nonnegative. If $\lambda = 0$ is an eigenvalue, the corresponding eigenfunction must be a constant. Notice that this occurs when $n = 0$. Thus $\lambda_{0,0} = 0$ as an eigenvalue, with eigenfunction $R_0 = 1$. In other cases the eigenvalues are positive, so we set $\lambda = \nu^2$. As before, the differential equation is transformed into Bessel's equation and has solutions $J_n(\nu r)$ which are bounded at $r = 0$. To satisfy the second boundary condition, we must have $J_n'(\nu) = 0$. Let $\beta_{n,k}$ be the sequence of zeros of J_n'. Thus the eigenvalues and eigenfunctions are

$$\lambda_k = \beta_{n,k}^2 \quad \text{and} \quad R_k(r) = J_n(\beta_{n,k} r/a) \quad \text{for } k = 1, 2, \cdots,$$

for $k \geq 1$, with the addition of $\lambda_{0,0} = 0$ and $R_0 = 1$ when $n = 0$. Consequently, the eigenvalues for the eigenvalue problem on the disk are $\lambda_{0,0} = 0$ and $\lambda_{n,k} = \beta_{n,k}^2$ for $n \geq 0$ and $k \geq 1$.

8. (a) We have $\nabla^2 u = \nabla^2 \phi \cdot Z + \phi \cdot Z_{zz} = 0$. Dividing by ϕZ, we get

$$\frac{\nabla^2 \phi}{\phi} = -\frac{Z''}{Z} = -\lambda,$$

where λ is a constant, since the two sides of the first equality depend on different variables. Hence

$$-\nabla^2 \phi = \lambda \phi, \quad \text{and} \quad Z'' = \lambda Z.$$

(b) λ and ϕ must solve the eigenvalue problem for the disk in (10.2). The solutions are in (10.11). If we set $\omega_{n,k} = \sqrt{\lambda_{n,k}}$, then the equation for Z becomes $Z + \omega_{n,k}^2 Z = 0$. As we found in Section 4, a convenient fundamental set of solutions is

$$\sinh(\omega_{n,k} z) = \frac{e^{\omega_{n,k} z} - e^{-\omega_{n,k} z}}{2} \quad \text{and} \quad \sinh(\omega_{n,k}(z - L)) = \frac{e^{\omega_{n,k}(z-L)} - e^{-\omega_{n,k}(z-L)}}{2}.$$

Therefore the product solutions in the cylinder are:

$$\begin{gathered} \sinh(\omega_{n,k} z) \cdot \phi_{n,k}(r,\theta), \quad \sinh(\omega_{n,k} z) \cdot \psi_{n,k}(r,\theta), \\ \sinh(\omega_{n,k}(z - L)) \cdot \phi_{n,k}(r,\theta), \quad \text{and} \quad \sinh(\omega_{n,k}(z - L)) \cdot \psi_{n,k}(r,\theta), \end{gathered}$$

(c) By linearity, any function of the form

$$\begin{aligned} u(r,\theta,z) = {} & \sum_{k=1}^{\infty} J_0\left(\frac{\alpha_{0,k} r}{a}\right)\left[a_{0,k} \sinh(\omega_{n,k} z) + c_{0,k} \sinh(\omega_{n,k}(z - L))\right] \\ & + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} J_n\left(\frac{\alpha_{n,k} r}{a}\right)\left[a_{n,k} \cos n\theta + b_{n,k} \sin n\theta\right] \sinh(\omega_{n,k} z) \\ & + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} J_n\left(\frac{\alpha_{n,k} r}{a}\right)\left[c_{n,k} \cos n\theta + d_{n,k} \sin n\theta\right] \sinh(\omega_{n,k}(z - L)) \end{aligned}$$

is formally a solution to Laplace's equation on the cylinder that vanishes on the curved portion of the boundary. To satisfy the boundary conditions on the flat portions, we first set

$$\begin{aligned} 0 = u(r,\theta,L) = {} & \sum_{k=1}^{\infty} a_{0,k} J_0\left(\frac{\alpha_{0,k} r}{a}\right) \sinh(\omega_{n,k} L) \\ & + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} J_n\left(\frac{\alpha_{n,k} r}{a}\right)\left[a_{n,k} \cos n\theta + b_{n,k} \sin n\theta\right] \sinh(\omega_{n,k} L) \end{aligned}$$

Thus $a_{n,k} = b_{n,k} = 0$. At the other end we have

$$f(r,\theta) = u(r,\theta,0) = -\sum_{k=1}^{\infty} c_{0,k} J_0\left(\frac{\alpha_{0,k} r}{a}\right) \sinh(\omega_{n,k} L)$$
$$- \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} J_n\left(\frac{\alpha_{n,k} r}{a}\right)\left[c_{n,k}\cos n\theta + d_{n,k}\sin n\theta\right]\sinh(\omega_{n,k} L)$$

From this expansion for f we can evaluate the coefficients using the orthogonality relations in (10.12). We get

$$c_{n,k} = \frac{-2}{\pi a^2 J_{n+1}(\alpha_{n,k})\sinh(\omega_{n,k} L)} \int_0^a \int_0^{2\pi} f(r,\theta) J_n\left(\frac{\alpha_{n,k} r}{a}\right) \cos n\theta \, r\, dr\, d\theta, \quad \text{and}$$
$$d_{n,k} = \frac{-2}{\pi a^2 J_{n+1}^2(\alpha_{n,k})\sinh(\omega_{n,k} L)} \int_0^a \int_0^{2\pi} f(r,\theta) J_n\left(\frac{\alpha_{n,k} r}{a}\right) \sin n\theta \, r\, dr\, d\theta.$$

9. (a) We have $-\nabla^2 u = -\nabla^2 A \cdot B - A \cdot B_{zz} = \lambda AB$. If we divides by $u = AB$, this becomes

$$\frac{-\nabla^2 A}{A} + \frac{-B''}{B} = \lambda.$$

Since each of the summands depend on different variables, they must both be constant. Hence there exist constants μ and ν such that $\mu + \nu = \lambda$ for which

$$-\nabla^2 A = \mu A \quad \text{and} \quad -B'' = \nu B.$$

Since $u = AB$ vanishes on the boundary, we must have

$$A(a,\theta) = 0 \quad \text{for } (r,\theta) \in \partial D, \qquad \text{and} \quad B(0) = B(L) = 0.$$

(b) The eigenvalue problem for A is the same as that in (10.1). The solutions are in (10.11):

$$\mu_{0,k} = \frac{\alpha_{0,k}^2}{a^2} \quad \text{with} \quad \phi_{0,k}(r,\theta) = J_0(\alpha_{0,k} r/a)$$
$$\text{for } n = 0 \text{ and } k = 1,\ 2,\ 3,\ \ldots$$
$$\mu_{n,k} = \frac{\alpha_{n,k}^2}{a^2} \quad \text{with} \quad \begin{cases} \phi_{n,k}(r,\theta) = \cos n\theta \cdot J_n(\alpha_{n,k} r/a) & \text{and} \\ \psi_{n,k}(r,\theta) = \sin n\theta \cdot J_n(\alpha_{n,k} r/a) \end{cases}$$
$$\text{for } n = 1,\ 2,\ 3,\ \ldots \text{ and } k = 1,\ 2,\ 3,\ \ldots.$$

We are well familiar with the Sturm Liouville problem for B. The solutions are

$$\nu_l = \frac{l^2\pi^2}{L^2} \quad \text{with} \quad Z_l(z) = \sin\frac{l\pi z}{L}.$$

Consequently, the eigenvalues for the cylinder are

$$\lambda_{n,k,l} = \mu_{nk} + \nu_l = \frac{\alpha_{n,k}^2}{a^2} + \frac{l^2\pi^2}{L^2},$$

for $n = 0, 1, 2, \ldots, k = 1, 2, \ldots,$ and $l = 1, 2, \ldots.$ The associated eigenfunctions are

$$u_{0,k,l} = \phi_{0,k} \cdot Z_l = J_0(\alpha_{0,k} r/a) \cdot \sin \frac{l\pi z}{L},$$

for $n = 0$, and

$$u_{n,k,l} = \phi_{n,k} \cdot Z_l = \cos n\theta \cdot J_n(\alpha_{n,k} r/a) \cdot \sin \frac{l\pi z}{L},$$

$$v_{n,k,l} = \psi_{n,k} \cdot Z_l = \sin n\theta \cdot J_n(\alpha_{n,k} r/a) \cdot \sin \frac{l\pi z}{L},$$

for $n \geq 1$.