Section 3.4 – Comparison Tests

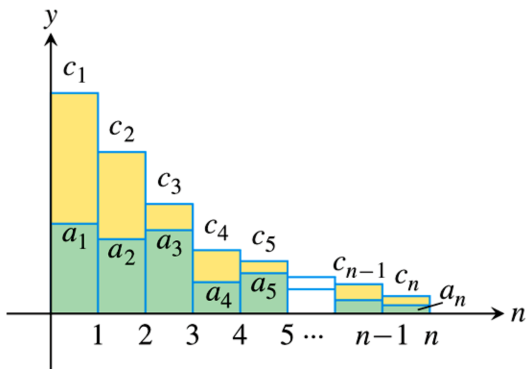
Theorem

Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with nonnegative terms. Suppose that for some integer N.

$$d_n \le a_n \le c_n \quad \text{for all} \quad n > N$$

a) If $\sum c_n$ converges, then $\sum a_n$ also converges.

b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges.



Example

Use the comparison Test to determine if $\displaystyle\sum_{n=1}^{\infty} \frac{5}{5n-1}$ converges or diverges.

Solution

$$\frac{5}{5n-1} = \frac{1}{n-\frac{1}{5}} > \frac{1}{n}$$

The series diverges because its nth term is greater than the nth term of the divergent harmonic series.

Example

Use the comparison Test to determine if $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ converges or diverges.

Solution

$$\sum_{n=0}^{\infty} \frac{1}{n!} < 1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{1-\frac{1}{2}} = 3$$ The series converges.

Theorem – Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \ge N$ (N an integer)

1. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge

2. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges

3. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges

Example

Does the series $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots$ converge or diverge?

Solution

$$\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$$

Let $a_n = \frac{2n+1}{n^2+2n+1} \to \frac{2n}{n^2} = \frac{2}{n}$

$\frac{2}{n} > b_n = \frac{1}{n}$

Since $\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ *diverges*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n+1}{n^2+2n+1} \cdot \frac{n}{1} \underline{= 2} \quad \Rightarrow \sum a_n \text{ diverges}$$

Example

Does the series $\frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots = \sum\limits_{n=1}^{\infty} \frac{1}{2^n - 1}$ converge or diverge?

Solution

Let $a_n = \frac{1}{2^n - 1} \quad \to \quad b_n = \frac{1}{2^n}$

Since $\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty} \frac{1}{2^n}$ *converges*

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2^n}{2^n-1}$$

$$= \lim_{n\to\infty}\frac{1}{1-\frac{1}{2^n}}$$

$$\underline{=1}$$

$\Rightarrow \sum a_n$ converges by the Limit Comparison Test.

Example

Does the series $\frac{1+2\ln 2}{9}+\frac{1+3\ln 3}{14}+\frac{1+4\ln 4}{21}+\cdots=\sum_{n=2}^{\infty}\frac{1+n\ln n}{n^2+5}$ converge or diverge?

Solution

Let $a_n = \frac{1+n\ln n}{n^2+5} \quad \to \quad b_n = \frac{n\ln n}{n^2} = \frac{\ln n}{n} > \frac{1}{n}$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n}$ *diverges*

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1+n\ln n}{n^2+5}\cdot\frac{n}{1}$$

$$= \lim_{n\to\infty}\frac{n+n^2\ln n}{n^2+5}$$

$$\underline{=\infty}$$

$\Rightarrow \sum a_n$ diverges by the Limit Comparison Test.

Example

Does the series $\sum_{n=2}^{\infty}\frac{\ln n}{n^{3/2}}$ converge?

Solution

Let $a_n = \frac{\ln n}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}} = b_n$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\ln n}{n^{3/2}}\cdot\frac{n^{5/4}}{1}$$

$$= \lim_{n\to\infty} \frac{\ln n}{n^{1/4}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{4}n^{-3/4}}$$

L'hôpital Rule

$$= \lim_{n\to\infty} \frac{4}{n^{1/4}}$$

$$= 0$$

$\Rightarrow \sum a_n$ converges by the Limit Comparison Test.

Exercises *Section* 3.4 – Comparison Tests

Use the Comparison Test to determine if the series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+30}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$

3. $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{3n+1}{n^3+1}$

8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$

11. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

12. $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{5^n+3}$

13. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n+1}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

15. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$

17. $\displaystyle\sum_{n=0}^{\infty} e^{-n^2}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n-1}$

Use the Limit Comparison Test to determine if the series converges or diverges.

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{n^3-n^2+3}$

20. $\displaystyle\sum_{n=2}^{\infty} \frac{n(n+1)}{\left(n^2+1\right)(n-1)}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\,4^n}$

23. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$

24. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{4^n+1}$

29. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{5^n+1}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2(n+3)}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

Use any method to determine if the series converges or diverges

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+\sqrt[3]{n}}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$

37. $\displaystyle\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$

38. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$

39. $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin n}{n^2}$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$

43. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{an+b}$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$

46. $\displaystyle\sum_{n=0}^{\infty} 5\left(-\frac{4}{3}\right)^n$

47. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n+1}$

48. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3-8}$

49. $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n-2}$

50. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n+1}-\frac{1}{n+2}\right)$

51. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\left(n^2+1\right)^2}$

52. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

53. $\displaystyle\sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1}$