Solution   Section 3.1 – Definition of the Laplace Transform

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t)=3$

Solution

$$F(s)=\int_0^\infty 3e^{-st}dt$$

$$=\lim_{T\to\infty}\int_0^T 3e^{-st}dt$$

$$=\lim_{T\to\infty}\left(-\frac{3e^{-st}}{s}\right)_{t=0}^{T}$$

$$=\lim_{T\to\infty}\left(-\frac{3}{s}e^{-sT}+\frac{3}{s}\right) \qquad \lim_{T\to\infty}\left(e^{-sT}\right)=0$$

$$\boxed{=\frac{3}{s}}$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t)=e^{-2t}$

Solution

$$F(s)=\int_0^\infty e^{-2t}e^{-st}dt$$

$$=\lim_{T\to\infty}\int_0^T e^{-(s+2)t}dt$$

$$=\lim_{T\to\infty}\left(\frac{-e^{-(s+2)t}}{s+2}\right)_{t=0}^{T}$$

$$=\lim_{T\to\infty}\left(-\frac{e^{-(s+2)T}}{s+2}+\frac{1}{s+2}\right) \qquad \lim_{T\to\infty}\left(e^{-(s+2)T}\right)=0$$

$$\boxed{=\frac{1}{s+2}} \qquad with: s>-2$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t) = te^{-3t}$

Solution

$$F(s) = \int_0^\infty te^{-3t}e^{-st}dt$$

$$= \int_0^\infty te^{-(s+3)t}dt$$

$$u = t \qquad dv = \int e^{-(s+3)t}dt$$

$$du = dt \qquad v = -\frac{1}{s+3}e^{-(s+3)t}$$

$$F(s) = -\frac{1}{s+3}te^{-(s+3)t} - \frac{1}{(s+3)^2}e^{-(s+3)t}$$

$$= \lim_{T\to\infty}\left(-\frac{1}{s+3}te^{-(s+3)t} - \frac{1}{(s+3)^2}e^{-(s+3)t}\right)\Big|_{t=0}^{T}$$

$$= \lim_{T\to\infty}\left(-\frac{1}{s+3}te^{-(s+3)T} - \frac{1}{(s+3)^2}e^{-(s+3)T} + \frac{1}{(s+3)^2}\right) \qquad \lim_{T\to\infty}\left(e^{-\alpha}\right) = 0$$

$$\boxed{= \frac{1}{(s+3)^2}} \qquad \text{with } s > -3$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t) = e^{2t}\cos 3t$

Solution

$$F(s) = \int_0^\infty \left(e^{2t}\cos 3t\right)e^{-st}dt$$

$$= \int_0^\infty e^{-(s-2)t}\cos 3t\ dt \qquad \textit{Integrating by parts}$$

$$\int e^{-(s-2)t}\cos 3t\ dt = \frac{1}{3}\int e^{-(s-2)t}\ d(\sin 3t)$$

$$= \frac{1}{3}\left[e^{-(s-2)t}\sin 3t - \int \sin 3t\ d\left(e^{-(s-2)t}\right)\right]$$

$$= \frac{1}{3}e^{-(s-2)t}\sin 3t + \frac{s-2}{3}\int e^{-(s-2)t}\sin 3t\ dt$$

$$\int e^{-(s-2)t}\sin 3t\ dt$$

$$u = e^{-(s-2)t} \qquad dv = \int \sin 3t\ dt$$

$$du = -(s-2)e^{-(s-2)t}dt \qquad v = -\tfrac{1}{3}\cos 3t$$

$$\int e^{-(s-2)t}\sin 3t\ dt = -\frac{1}{3}e^{-(s-2)t}\cos 3t + \tfrac{1}{3}(s-2)\int e^{-(s-2)t}\cos 3t\ dt$$

$$\int e^{-(s-2)t}\cos 3t\ dt = \tfrac{1}{3}e^{-(s-2)t}\sin 3t + \frac{s-2}{3}\left(-\tfrac{1}{3}e^{-(s-2)t}\cos 3t + \frac{s-2}{3}\int e^{-(s-2)t}\cos 3t\ dt\right)$$

$$= \tfrac{1}{3}e^{-(s-2)t}\sin 3t - \tfrac{1}{9}(s-2)e^{-(s-2)t}\cos 3t - \tfrac{1}{9}(s-2)^2\int e^{-(s-2)t}\cos 3t\ dt$$

$$\left(1+\tfrac{1}{9}(s-2)^2\right)\int e^{-(s-2)t}\cos 3t\ dt = \tfrac{1}{3}e^{-(s-2)t}\sin 3t - \tfrac{1}{9}(s-2)e^{-(s-2)t}\cos 3t$$

$$\left(9+(s-2)^2\right)\int e^{-(s-2)t}\cos 3t\ dt = 3e^{-(s-2)t}\sin 3t - (s-2)e^{-(s-2)t}\cos 3t$$

$$\int e^{-(s-2)t}\cos 3t\ dt = \frac{1}{9+(s-2)^2}\left[3e^{-(s-2)t}\sin 3t - (s-2)e^{-(s-2)t}\cos 3t\right]$$

$$F(s) = \lim_{T\to\infty}\left(\frac{3}{9+(s-2)^2}e^{-(s-2)t}\sin 3t - \frac{s-2}{9+(s-2)^2}e^{-(s-2)t}\cos 3t\right)\Bigg|_{t=0}^{T}$$

$$= \lim_{T\to\infty}\left(\frac{3}{9+(s-2)^2}e^{-(s-2)T}\sin 3T + \frac{s-2}{9+(s-2)^2}e^{-(s-2)T}\cos 3T + \frac{s-2}{9+(s-2)^2}\right)$$

$$\boxed{= \frac{s-2}{9+(s-2)^2}} \qquad s > 2$$

Exercise

Use Definition of Laplace transform to find the Laplace transform of $f(t) = \sin 3t$

Solution

$$F(s) = \int_0^\infty (\sin 3t) e^{-st} dt \qquad\qquad \textit{Integrating by parts}$$

$$u = e^{-st} \qquad dv = \int \sin 3t\ dt$$

$$du = -se^{-st} dt \qquad v = -\frac{1}{3}\cos 3t$$

$$\int \sin 3t\ e^{-st} dt = -\frac{1}{3} e^{-st}\cos 3t - \frac{1}{3}s \int e^{-st}\cos 3t\ dt$$

$$u = e^{-st} \qquad dv = \int \cos 3t\ dt$$

$$du = -se^{-st} dt \qquad v = \frac{1}{3}\sin 3t$$

$$\int \sin 3t\ e^{-st} dt = -\frac{1}{3} e^{-st}\cos 3t - \frac{1}{3}s\left[\frac{1}{3}e^{-st}\sin 3t + \frac{1}{3}s\int e^{-st}\sin 3t\ dt\right]$$

$$\int \sin 3t\ e^{-st} dt + \frac{1}{9}s^2 \int \sin 3t\ e^{-st} dt = -\frac{1}{3}e^{-st}\cos 3t - \frac{1}{9}se^{-st}\sin 3t$$

$$\left(9 + s^2\right)\int \sin 3t\ e^{-st} dt = -(3\cos 3t - s\sin 3t)e^{-st}$$

$$\int \sin 3t\ e^{-st} dt = -\frac{3\cos 3t - s\sin 3t}{s^2+9}e^{-st}$$

$$F(s) = \lim_{T\to\infty}\left(-\frac{3\cos 3t - s\sin 3t}{s^2+9}e^{-st}\right)_{t=0}^{T}$$

$$= -\lim_{T\to\infty}\left[\frac{3\cos 3T - s\sin 3T}{s^2+9}e^{-sT} - \frac{3\cos 3(0) - s\sin 3(0)}{s^2+9}e^{-s(0)}\right]$$

$$= -\left(-\frac{3}{s^2+9}\right)$$

$$= \frac{3}{s^2+9} \qquad s > 0$$

Exercise

Use Definition of Laplace Transform to show the Laplace transform of $f(t)=\cos\omega t$ is

$$F(s)=\frac{s}{s^2+\omega^2}$$

Solution

$$F(s)=\int_0^{\infty}(\cos\omega t)e^{-st}dt$$

Integrating by parts

$$u=e^{-st} \qquad dv=\int\cos\omega t\ dt$$

$$du=-se^{-st}dt \qquad v=\frac{1}{\omega}\sin\omega t$$

$$\int\cos\omega t\ e^{-st}dt=\frac{1}{\omega}e^{-st}\sin\omega t+\frac{s}{\omega}\int e^{-st}\sin\omega t\ dt$$

Integrating the second integral by parts

$$u=e^{-st} \qquad dv=\int\sin\omega t\ dt$$

$$du=se^{-st}dt \qquad v=-\frac{1}{\omega}\cos\omega t$$

$$\int\cos\omega t\ e^{-st}dt=\frac{1}{\omega}e^{-st}\sin\omega t+\frac{s}{\omega}\left(-\frac{1}{\omega}e^{-st}\cos\omega t+\frac{s}{\omega}\int e^{-st}\cos\omega t\ dt\right)$$

$$=\frac{1}{\omega}e^{-st}\sin\omega t-\frac{s}{\omega^2}e^{-st}\cos\omega t+\frac{s^2}{\omega^2}\int e^{-st}\cos\omega t\ dt$$

$$\left(1-\frac{s^2}{\omega^2}\right)\int e^{-st}\cos\omega t\ dt=\frac{1}{\omega}e^{-st}\sin\omega t-\frac{s}{\omega^2}e^{-st}\cos\omega t$$

$$\left(\frac{\omega^2-s^2}{\omega^2}\right)\int e^{-st}\cos\omega t\ dt=\frac{1}{\omega}\left(\sin\omega t-\frac{s}{\omega}\cos\omega t\right)e^{-st}$$

$$\int e^{-st}\cos\omega t\ dt=\frac{\omega^2}{\omega^2-s^2}\frac{1}{\omega^2}(\omega\sin\omega t-s\cos\omega t)e^{-st}$$

$$=\frac{e^{-st}}{\omega^2-s^2}(\omega\sin\omega t-s\cos\omega t)$$

$$F(s)=\lim_{T\to\infty}\frac{e^{-st}}{\omega^2-s^2}(\omega\sin\omega t-s\cos\omega t)\Big|_0^T$$

$$=\lim_{T\to\infty}\left[\frac{e^{-sT}}{\omega^2-s^2}(\omega\sin\omega T-s\cos\omega T)-\frac{1}{\omega^2-s^2}(\omega\sin 0-s\cos 0)\right]$$

$$=0-\frac{1}{\omega^2-s^2}(-s) \qquad \lim_{T\to\infty}e^{-sT}=\lim_{T\to\infty}\frac{1}{e^{-sT}}=0$$

$$\boxed{=\frac{s}{s^2+\omega^2}} \qquad s>0$$

Solution Section 3.2 – Basic Properties of the Laplace Transform

Exercise

Find the Laplace transform and defined the time domain of $y(t) = t^2 + 4t + 5$

Solution

$$\mathcal{L}\left(t^2 + 4t + 5\right)(s) = \mathcal{L}\left(t^2\right)(s) + 4\,\mathcal{L}(4t)(s) + 5\,\mathcal{L}(1)(s)$$

$$= \frac{2!}{s^3} + 4\frac{1}{s^2} + 5\frac{1}{s}$$

$$= \frac{2}{s^3} + \frac{4}{s^2} + \frac{5}{s}$$

$$= \frac{2 + 4s + 5s^2}{s^3} \qquad s > 0$$

Exercise

Find the Laplace transform and defined the time domain of $y(t) = -2\cos t + 4\sin 3t$

Solution

$$\mathcal{L}(-2\cos t + 4\sin 3t)(s) = -2\,\mathcal{L}(\cos t)(s) + 4\,\mathcal{L}(\sin 3t)(s)$$

$$= -2\frac{s}{s^2+1} + 4\frac{3}{s^2+9}$$

$$= \frac{-2s\left(s^2+9\right) + 12\left(s^2+1\right)}{\left(s^2+1\right)\left(s^2+9\right)}$$

$$= \frac{-2s^3 - 18s + 12s^2 + 12}{\left(s^2+1\right)\left(s^2+9\right)}$$

$$= \frac{-2s^3 + 12s^2 - 18s + 12}{\left(s^2+1\right)\left(s^2+9\right)} \qquad s > 0$$

Exercise

Find the Laplace transform and defined the time domain of $y(t) = 2\sin 3t + 3\cos 5t$

Solution

$$\mathcal{L}(2\sin 3t + 3\cos 5t)(s) = 2\,\mathcal{L}(\sin 3t)(s) + 3\,\mathcal{L}(\cos 5t)(s)$$

$$= 2\frac{3}{s^2+9} + 3\frac{s}{s^2+25}$$

$$= \frac{6s^2+150+3s^3+27s}{\left(s^2+9\right)\left(s^2+25\right)}$$

$$= \frac{3s^3+6s^2+27s+150}{\left(s^2+9\right)\left(s^2+25\right)} \quad (s>0)$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

$$y' - 5y = e^{-2t}, \quad \textit{with} \quad y(0) = 1$$

Solution

$$\mathcal{L}(y' - 5y)(s) = \mathcal{L}\left(e^{-2t}\right)(s)$$

$$\mathcal{L}(y')(s) - 5\,\mathcal{L}(y)(s) = \frac{1}{s+2}$$

$$s\mathcal{L}(y)(s) - y(0) - 5\,\mathcal{L}(y)(s) = \frac{1}{s+2}$$

Let $Y(s) = \mathcal{L}(y)(s)$, then

$$sY(s) - 1 - 5Y(s) = \frac{1}{s+2}$$

$$(s-5)Y(s) = \frac{1}{s+2} + 1$$

$$Y(s) = \frac{1}{(s-5)(s+2)} + \frac{1}{(s-5)}$$

$$= \frac{1+s+2}{(s-5)(s+2)}$$

$$= \frac{s+3}{(s-5)(s+2)}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

$$y' - 4y = \cos 2t, \quad \text{with} \quad y(0) = -2$$

Solution

$$\mathcal{L}(y' - 4y)(s) = \mathcal{L}(\cos 2t)(s)$$

$$\mathcal{L}(y')(s) - 4\,\mathcal{L}(y)(s) = \frac{s}{s^2+4}$$

$$s\,\mathcal{L}(y)(s) - y(0) - 4\,\mathcal{L}(y)(s) = \frac{s}{s^2+4}$$

Let $Y(s) = \mathcal{L}(y)(s)$, then

$$sY(s) + 2 - 4Y(s) = \frac{s}{s^2+4}$$

$$(s-4)Y(s) = \frac{s}{s^2+4} - 2$$

$$Y(s) = \frac{s}{(s-4)\left(s^2+4\right)} - \frac{2}{s-4}$$

$$= \frac{s - 2s^2 - 8}{(s-4)\left(s^2+4\right)}$$

$$= \frac{-2s^2 + s - 8}{(s-4)\left(s^2+4\right)}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

$$y'' + 2y' + 2y = \cos 2t; \quad \text{with} \quad y(0) = 1 \;\; \text{and} \;\; y'(0) = 0$$

Solution

$$\mathcal{L}(y'' + 2y' + 2y)(s) = \mathcal{L}(\cos 2t)(s)$$

Let $Y(s) = \mathcal{L}(y)(s)$, then

$$s^2Y(s) - sy(0) - y'(0) + 2\left(sY(s) - y(0)\right) + 2Y(s) = \frac{s}{s^2+4}$$

$$s^2Y(s) - s + 2sY(s) - 2 + 2Y(s) = \frac{s}{s^2+4}$$

$$\left(s^2+2s+2\right)Y(s)=\frac{s}{s^2+4}+s+2$$

$$=\frac{s+s^3+2s^2+4s+8}{s^2+4}$$

$$=\frac{s^3+2s^2+5s+8}{s^2+4}$$

$$\boxed{Y(s)=\frac{s^3+2s^2+5s+8}{\left(s^2+4\right)\left(s^2+2s+2\right)}}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

$$y''+3y'+5y=t+e^{-t}; \quad with \quad y(0)=-1 \;\; and \quad y'(0)=0$$

Solution

$$\mathcal{L}\left(y''+3y'+5y\right)(s)=\mathcal{L}(t)(s)+\mathcal{L}\left(e^{-t}\right)(s)$$

$$s^2Y(s)-sy(0)-y'(0)+3\left(sY(s)-y(0)\right)+5Y(s)=\frac{1}{s^2}+\frac{1}{s+1}$$

$$s^2Y(s)+s+3\left(sY(s)+1\right)+5Y(s)=\frac{s+1+s^2}{s^2(s+1)}$$

$$s^2Y(s)+s+3sY(s)+3+5Y(s)=\frac{s+1+s^2}{s^2(s+1)}$$

$$\left(s^2+3s+5\right)Y(s)=\frac{s+1+s^2}{s^2(s+1)}-s-3$$

$$=\frac{s+1+s^2-s^2(s+1)(s+3)}{s^2(s+1)}$$

$$=\frac{s+1+s^2-s^2\left(s^2+4s+3\right)}{s^2(s+1)}$$

$$=\frac{s+1+s^2-s^4+4s^3+3s^2}{s^2(s+1)}$$

$$=\frac{-s^4+4s^3+4s^2+s+1}{s^2(s+1)}$$

$$\boxed{Y(s)=\frac{-s^4+4s^3+4s^2+s+1}{s^2(s+1)\left(s^2+3s+5\right)}}$$

Exercise

Find the Laplace transform of $y(t)=e^{2t}\cos 2t$

Solution

$$f(t)=\cos 2t \xrightarrow{\mathcal{L}} F(s)=\frac{s}{s^2+4}$$

$$y(t)=e^{2t}\cos 2t \xrightarrow{\mathcal{L}} Y(s)=F(s-2)$$

$$\begin{aligned}Y(s)&=F(s-2)\\&=\frac{s-2}{(s-2)^2+4}\\&=\boxed{\frac{s-2}{s^2-4s+8}}\end{aligned}$$

Exercise

Find the Laplace transform of $y(t)=t\sin 3t$

Solution

$$f(t)=\sin 3t \xrightarrow{\mathcal{L}} F(s)=\frac{3}{s^2+9}$$

$$\begin{aligned}\mathcal{L}\{t\sin 3t\}(s)&=-Y'(s) && \textit{Using Derivative of a Laplace Transform Proposition}\\&=-\frac{3(-2s)}{\left(s^2+9\right)^2}\\&=\boxed{\frac{6s}{\left(s^2+9\right)^2}}\end{aligned}$$

Exercise

Find the Laplace transform of $y(t)=t^2\cos 2t$

Solution

$$f(t)=\cos 2t \xrightarrow{\mathcal{L}} F(s)=\frac{s}{s^2+4}$$

$$\mathcal{L}\left\{t^2\cos 2t\right\}(s)=(-1)^2Y''(s)$$ ***Using Derivative of a Laplace Transform Proposition***

$$=\left(Y'(s)\right)'$$

$$=\frac{d}{ds}\left[\frac{(s^2+4)(1)-s(2s)}{\left(s^2+4\right)^2}\right]$$

$$=\frac{d}{ds}\left[\frac{4-s^2}{\left(s^2+4\right)^2}\right]$$

$$=\frac{-2s\left(s^2+4\right)^2-\left(4-s^2\right)(2)(2s)\left(s^2+4\right)}{\left(s^2+4\right)^4}$$

$$=\left(s^2+4\right)\frac{-2s\left(s^2+4\right)-4s\left(4-s^2\right)}{\left(s^2+4\right)^4}$$

$$=\frac{-2s^3-8s-16s+4s^3}{\left(s^2+4\right)^3}$$

$$=\boxed{\frac{2s^3-24s}{\left(s^2+4\right)^3}}$$

Exercise

Find the Laplace transform of $y(t)=e^{-2t}\left(2t+3\right)$

Solution

$$f(t)=2t+3 \xrightarrow{\mathcal{L}} F(s)=2\frac{1}{s^2}+3\frac{1}{s}$$

$$=\frac{2+3s}{s^2}$$

$$\mathcal{L}\left\{e^{-2t}\left(2t+3\right)\right\}=Y(s+2)$$

$$=\frac{2+3(s+2)}{\left(s+2\right)^2}$$

$$=\frac{3s+8}{\left(s+2\right)^2}$$

Exercise

Find the Laplace transform of $y(t)=t^2e^{2t}$

Solution

$$f(t)=e^{2t} \xrightarrow{\mathcal{L}} F(s)=\frac{1}{s-2}$$

$$\mathcal{L}\left\{t^2e^{2t}\right\}(s)=\left(-1\right)^2Y''(s)$$ ***Using Derivative of a Laplace Transform Proposition***

$$=\frac{d}{ds}\left(\frac{-1}{\left(s-2\right)^2}\right)$$

$$=-\frac{(-1)2(s-2)}{\left(s-2\right)^4}$$

$$=\frac{2}{\left(s-2\right)^3}$$ ***OR Using Laplace Transform** table*

Exercise

Find the Laplace transform of $y(t)=e^{-t}\left(t^2+3t+4\right)$

Solution

$$y(t)=t^2e^{-t}+3te^{-t}+4e^{-t}$$

$$\mathcal{L}\left(t^ne^{-at}\right)(s)=\frac{n!}{(s+a))^{n+1}}$$

$$\begin{aligned}
Y(s)&=\mathcal{L}\left(t^2e^{-t}\right)(s)+3\mathcal{L}\left(te^{-t}\right)(s)+4\mathcal{L}\left(e^{-t}\right)(s)\\
&=\frac{2!}{(s+1)^3}+3\frac{1}{(s+1)^2}+4\frac{1}{s+1}\\
&=\frac{2+3(s+1)+4(s+1)^2}{(s+1)^3}\\
&=\frac{2+3s+3+4s^2+8s+4}{(s+1)^3}\\
&=\boxed{\frac{4s^2+11s+9}{(s+1)^3}}\quad (s>0)
\end{aligned}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

$$y'+2y=t\sin t,\quad with\quad y(0)=1$$

Solution

Let $Y(s)=\mathcal{L}(y)(s)$, then

Left side;

$$\begin{aligned}
\mathcal{L}\left(y'+2y\right)(s)&=s\mathcal{L}(y)(s)-y(0)+2\mathcal{L}(y)(s)\\
&=sY(s)-1+2Y(s)\\
&=(s+2)Y(s)-1
\end{aligned}$$

Right side;

$$f(t)=\sin t\xrightarrow{\mathcal{L}}F(s)=\frac{1}{s^2+1}$$

$$\begin{aligned}
\mathcal{L}\{t\sin t\}(s)&=-F'(s) && \textit{Using Derivative of a Laplace Transform Proposition}\\
&=\frac{2s}{\left(s^2+1\right)^2}
\end{aligned}$$

Therefore,

$$(s+2)Y(s)-1=\frac{2s}{\left(s^2+1\right)^2}$$

$$(s+2)Y(s)=\frac{2s}{\left(s^2+1\right)^2}+1$$

$$\boxed{Y(s)=\frac{2s}{(s+2)\left(s^2+1\right)^2}+\frac{1}{s+2}}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

$$y'-y=t^2e^{-2t}, \quad with \quad y(0)=0$$

Solution

Let $Y(s)=\mathcal{L}(y)(s)$, then

Left side;

$$\mathcal{L}(y'+2y)(s)=s\mathcal{L}(y)(s)-y(0)+2\mathcal{L}(y)(s)$$

$$=sY(s)-Y(s)$$

$$=(s-1)Y(s)$$

Right side;

$$f(t)=e^{2t}\xrightarrow{\mathcal{L}}F(s)=\frac{1}{s-2}$$

$$\mathcal{L}\left\{t^2e^{2t}\right\}(s)=(-1)^2Y''(s)$$ ***Using Laplace Transform table***

$$=\frac{2}{(s-2)^3}$$

$$(s-1)Y(s)=\frac{2}{(s-2)^3}$$

$$\boxed{Y(s)=\frac{2}{(s-1)(s-2)^3}}$$

Exercise

Transform the initial value problem into an algebraic equation involving $\mathcal{L}y)$. Solve the resulting equation for the Laplace transform of y.

$$y'' + y' + 2y = e^{-t}\cos 2t, \quad with \quad y(0) = 1 \ \ and \quad y'(0) = -1$$

Solution

$$\mathcal{L}\left(y'' + y' + 2y\right)(s) = \mathcal{L}\left(e^{-t}\cos 2t\right)$$

$$s^2\mathcal{L}\left(y\right)(s) - sy(0) - y'(0) + s\,\mathcal{L}\left(y\right)(s) - y(0) + 2\,\mathcal{L}\left(y\right)(s) = \frac{s+1}{(s+1)^2+4}$$

$$s^2Y(s) - s + 1 + sY(s) - 1 + 2Y(s) = \frac{s+1}{(s+1)^2+4}$$

$$\left(s^2+s+2\right)Y(s) - s = \frac{s+1}{s^2+2s+1+4}$$

$$\left(s^2+s+2\right)Y(s) = \frac{s+1}{s^2+2s+5} + s$$

$$Y(s) = \frac{s+1}{\left(s^2+2s+5\right)\left(s^2+s+2\right)} + \frac{s}{s^2+s+2}$$

$$= \frac{s+1+s\left(s^2+2s+5\right)}{\left(s^2+2s+5\right)\left(s^2+s+2\right)}$$

$$= \frac{s+1+s^3+2s^2+5s}{\left(s^2+2s+5\right)\left(s^2+s+2\right)}$$

$$= \frac{s^3+2s^2+6s+1}{\left(s^2+2s+5\right)\left(s^2+s+2\right)}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{1}{3s+2}$

Solution

$$Y(s)=\frac{1}{3}\frac{1}{s+2/3} \qquad \textit{Factor}$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{s+2/3}\right\}$$

$$=\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2/3}\right\}$$

$$=\frac{1}{3}e^{-(2/3)t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{2}{3-5s}$

Solution

$$Y(s)=-2\frac{1}{5s-3}$$

$$=-\frac{2}{5}\frac{1}{s-\frac{3}{5}}$$

Thus, by linearity;

$$y(t)=-\frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-\frac{3}{5}}\right\}$$

$$=-\frac{2}{5}e^{(3/5)t}$$

Exercise

Find the inverse Laplace Transform of $Y(s) = \dfrac{1}{s^2+4}$

Solution

$$Y(s) = \frac{1}{2}\frac{2}{s^2+4}$$

$$y(t) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$\boxed{= \frac{1}{2}\sin 2t}$$

Exercise

Find the inverse Laplace Transform of $Y(s) = \dfrac{3}{s^2}$

Solution

$$y(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$\boxed{= 3t}$$

Exercise

Find the inverse Laplace Transform of $Y(s) = \dfrac{3s+2}{s^2+25}$

Solution

$$Y(s) = \frac{3s}{s^2+25} + \frac{2}{s^2+25}$$

$$= 3\frac{s}{s^2+25} + \frac{2}{5}\frac{5}{s^2+25}$$

$$y(t) = 3\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\} + \frac{2}{5}\,\mathcal{L}^{-1}\left\{\frac{5}{s^2+25}\right\}$$

$$\boxed{= 3\cos 5t + \frac{2}{5}\sin 5t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{2-5s}{s^2+9}$

Solution

$$Y(s)=\frac{2}{s^2+9}-\frac{5s}{s^2+25}$$

$$=\frac{2}{3}\frac{3}{s^2+9}-5\frac{s}{s^2+9}$$

$$y(t)=\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}-5\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}$$

$$=\frac{2}{3}\sin 3t-5\cos 3t$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{5}{(s+2)^3}$

Solution

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s+a)^{n+1}}\right\}=t^n e^{-at}$$

$n=2 \quad a=2$

$$Y(s)=\frac{5}{2!}\frac{2!}{(s+2)^3}$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{5}{(s+2)^3}\right\}$$

$$=\frac{5}{2}t^2e^{-2t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{1}{(s-1)^6}$

Solution

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s+a)^{n+1}}\right\}=t^n e^{-at}$$

$$n=5 \quad a=-1$$

$$Y(s)=\frac{1}{5!}\frac{5!}{(s-1)^6}$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{5!}\frac{5!}{(s-1)^6}\right\}$$

$$\boxed{=\frac{1}{120}t^5e^t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{4(s-1)}{(s-1)^2+4}$

Solution

$$\mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega^2}\right\}=e^{-at}\cos\omega t$$

$$a=-1 \quad \omega=2$$

$$Y(s)=4\frac{s-1}{(s-1)^2+4}$$

$$y(t)=\mathcal{L}^{-1}\left\{4\frac{s-1}{(s-1)^2+4}\right\}$$

$$\boxed{=4e^t\cos 2t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{2s-3}{(s-1)^2+5}$

Solution

$$y(t)=\mathcal{L}^{-1}\left\{\frac{2s-3}{(s-1)^2+5}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{2s-2-1}{(s-1)^2+5}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{2(s-1)}{(s-1)^2+5}-\frac{1}{(s-1)^2+5}\right\}$$

$$=\mathcal{L}^{-1}\left\{2\frac{s-1}{(s-1)^2+5}-\frac{1}{\sqrt{5}}\frac{\sqrt{5}}{(s-1)^2+5}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+a}{(s+a)^2+\omega^2}\right\}=e^{-at}\cos\omega t$$

$$\mathcal{L}^{-1}\left\{\frac{\omega}{(s+a)^2+\omega^2}\right\}=e^{-at}\sin\omega t$$

$$=2e^{t}\cos\sqrt{5}t-\frac{1}{\sqrt{5}}e^{t}\sin\sqrt{5}t$$

$$\underline{=e^{t}\left(2\cos\sqrt{5}t-\frac{\sqrt{5}}{5}\sin\sqrt{5}t\right)}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{2s-1}{(s+1)(s-2)}$

Solution

Use partial fraction $\quad\dfrac{2s-1}{(s+1)(s-2)}=\dfrac{A}{s+1}+\dfrac{B}{s-2}$

$$=\frac{As-2A+Bs+B}{(s+1)(s-2)}$$

$$2s-1=(A+B)s-2A+B$$

$$\begin{cases}A+B=2\\-2A+B=-1\end{cases}\quad\Rightarrow A=B=1$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{2s-1}{(s+1)(s-2)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$\underline{=e^{-t}+e^{2t}}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{2s-2}{(s-4)(s+2)}$

Solution

$$\frac{2s-2}{(s-4)(s+2)}=\frac{A}{s-4}+\frac{B}{s+2}$$

$$=\frac{As+2A+Bs-4B}{(s-4)(s+2)}$$

$$2s-2=(A+B)s+2A-4B$$

$$\begin{cases} A+B=2 \\ 2A-4B=-2 \end{cases} \Rightarrow A=B=1$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{2s-2}{(s-4)(s+2)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s-4}+\frac{1}{s+2}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\boxed{=e^{4t}+e^{-2t}}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{7s^2+3s+16}{(s+1)\left(s^2+4\right)}$

Solution

$$\frac{7s^2+3s+16}{(s+1)\left(s^2+4\right)}=\frac{A}{s+1}+\frac{Bs+C}{s^2+4}$$

$$=\frac{As^2+4A+Bs^2+Bs+Cs+C}{(s+1)\left(s^2+4\right)}$$

$$=\frac{(A+B)s^2+(B+C)s+4A+C}{(s+1)\left(s^2+4\right)}$$

$$7s^2+3s+16=(A+B)s^2+(B+C)s+4A+C$$

$$\begin{cases} A+B=7 \\ B+C=3 \\ 4A+C=16 \end{cases} \quad \left.\begin{matrix} A+B=7 \\ B+C=3 \end{matrix}\right\} \rightarrow A-C=4 \quad 4A+C=16 \quad \Rightarrow 5A=20 \rightarrow A=4 \qquad B=3 \quad C=0$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{7s^2+3s+16}{(s+1)\left(s^2+4\right)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{4}{s+1}+\frac{3s}{s^2+4}\right\}$$

$$=4\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}+3\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$\boxed{=4e^{-t}+3\cos 2t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{1}{(s+2)^2\left(s^2+9\right)}$

Solution

$$\frac{1}{(s+2)^2\left(s^2+9\right)}=\frac{A}{s+2}+\frac{B}{(s+2)^2}+\frac{Cs+D}{s^2+9}$$

$$=\frac{A(s+2)\left(s^2+9\right)+Bs^2+9B+(Cs+D)(s^2+4s+4)}{(s+2)^2\left(s^2+9\right)}$$

$$=\frac{As^3+2As^2+9As+18A+Bs^2+9B+Cs^3+4Cs^2+4Cs+Ds^2+4Ds+4D}{(s+2)^2\left(s^2+9\right)}$$

$$1=\left(A+C\right)s^3+\left(2A+B+4C+D\right)s^2+\left(9A+4C+4D\right)s+18A+9B+4D$$

$$\begin{cases} A+C=0 \\ 2A+B+4C+D=0 \\ 9A+4C+4D=0 \\ 18A+9B+4D=1 \end{cases} \Rightarrow A=\frac{4}{169}\quad B=\frac{1}{13}\quad C=-\frac{4}{169}\quad D=-\frac{5}{169}$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2\left(s^2+9\right)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{4}{169}\frac{1}{s+2}+\frac{1}{13}\frac{1}{(s+2)^2}-\frac{1}{169}\frac{4s+5}{s^2+9}\right\}$$

$$=\frac{4}{169}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}+\frac{1}{13}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}-\frac{4}{169}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}-\frac{5}{169}\mathcal{L}^{-1}\left\{\frac{1}{3}\frac{3}{s^2+9}\right\}$$

$$=\frac{4}{169}e^{-2t}+\frac{1}{13}te^{-2t}-\frac{4}{169}\cos 3t-\frac{5}{507}\sin 3t$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{s}{(s+2)^2\left(s^2+9\right)}$

Solution

$$\frac{s}{(s+2)^2\left(s^2+9\right)}=\frac{A}{s+2}+\frac{B}{(s+2)^2}+\frac{Cs+D}{s^2+9}$$

$$=\frac{As^3+2As^2+9As+18A+Bs^2+9B+Cs^3+4Cs^2+4Cs+Ds^2+4Ds+4D}{(s+1)^2\left(s^2+9\right)}$$

$$s=(A+C)s^3+(2A+B+4C+D)s^2+(9A+4C+4D)s+18A+9B+4D$$

$$\begin{cases} A+C=0 \\ 2A+B+4C+D=0 \\ 9A+4C+4D=1 \\ 18A+9B+4D=0 \end{cases} \Rightarrow A=\frac{5}{169} \quad B=-\frac{2}{13} \quad C=-\frac{5}{169} \quad D=\frac{36}{169}$$

$$y(t)=\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2\left(s^2+9\right)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{5}{169}\frac{1}{s+2}-\frac{2}{13}\frac{1}{(s+2)^2}-\frac{1}{169}\frac{5s+36}{s^2+9}\right\}$$

$$=\frac{5}{169}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}-\frac{2}{13}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}-\frac{5}{169}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}-\frac{36}{169}\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$=\frac{5}{169}e^{-2t}-\frac{2}{13}te^{-2t}-\frac{5}{169}\cos 3t+\frac{12}{169}\sin 3t$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{1}{(s+1)^2\left(s^2-4\right)}$

Solution

$$\frac{1}{(s+1)^2\left(s^2-4\right)}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{Cs+D}{s^2-4}$$

$$=\frac{A(s+1)\left(s^2-4\right)+B\left(s^2-4\right)+(Cs+D)(s+1)^2}{(s+1)^2\left(s^2-4\right)}$$

$$1=As^3-4As+As^2-4A+Bs^2-4B+Cs^3+2Cs^2+Cs+Ds^2+2Ds+D$$
$$=(A+C)s^3+(A+B+2C+D)s^2+(-4A+C+2D)s-4A-4B+D$$

$$\begin{array}{l} s^3 \\ s^2 \\ s^1 \\ s^0 \end{array}\begin{cases} A+C=0 \\ A+B+2C+D=0 \\ -4A+C+2D=0 \\ -4A-4B+D=1 \end{cases} \qquad A=-\frac{2}{15} \quad B=\frac{1}{5} \qquad C=\frac{2}{15} \quad D=-\frac{1}{3}$$

$$Y(s)=-\frac{2}{15}\frac{1}{s+1}+\frac{1}{5}\frac{1}{(s+1)^2}+\frac{\frac{2}{15}s-\frac{1}{3}}{s^2-4}$$
$$=-\frac{2}{15}\frac{1}{s+1}+\frac{1}{5}\frac{1}{(s+1)^2}+\frac{2}{15}\frac{s}{s^2-4}-\frac{1}{3}\frac{1}{s^2-4}$$

$$y(t)=-\frac{2}{15}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}+\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}+\frac{2}{15}\mathcal{L}^{-1}\left\{\frac{s}{s^2-4}\right\}-\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2-4}\right\}$$
$$=-\frac{2}{15}e^{-t}+\frac{1}{5}te^{-t}+\frac{2}{15}\cosh 2t-\frac{1}{6}\sinh 2t$$
$$=-\frac{2}{15}e^{-t}+\frac{1}{5}te^{-t}+\frac{2}{15}\frac{e^{2t}+e^{-2t}}{2}-\frac{1}{6}\frac{e^{2t}-e^{-2t}}{2}$$
$$=-\frac{2}{15}e^{-t}+\frac{1}{5}te^{-t}+\frac{1}{15}e^{2t}+\frac{1}{15}e^{-2t}-\frac{1}{12}e^{2t}+\frac{1}{12}e^{-2t}$$
$$=-\frac{2}{15}e^{-t}+\frac{1}{5}te^{-t}-\frac{1}{60}e^{2t}+\frac{3}{20}e^{-2t}$$

Exercise

Find the inverse Laplace Transform of $Y(s)=\dfrac{7s^2+20s+53}{(s-1)\left(s^2+2s+5\right)}$

Solution

$$\frac{7s^2+20s+53}{(s-1)\left(s^2+2s+5\right)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+2s+5}$$

$$7s^2+20s+53=As^2+2As+5A+Bs^2-Bs+Cs-C$$

$$\begin{matrix} s^2 \\ s^1 \\ s^0 \end{matrix}\begin{cases} A+B=7 \\ 2A-B+c=20 \\ 5A-C=53 \end{cases} \Rightarrow \begin{cases} A=10 \\ B=-3 \\ C=-3 \end{cases}$$

$$\begin{aligned} Y(x) &= \frac{10}{s-1}+\frac{-3s-3}{s^2+2s+5} \\ &= \frac{10}{s-1}-3\frac{s+1}{s^2+2s+5} \end{aligned}$$

$$y(t)=10\mathcal{L}^{-1}\left\{\frac{10}{s-1}\right\}-3\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\}$$

$$\underline{=10e^{t}-3e^{-t}\cos 2t}$$

Exercise

Find the inverse Laplace transform of $F(s)=\dfrac{s^2+1}{s^3-2s^2-8s}$

Solution

$$\frac{s^2+1}{s^3-2s^2-8s}=\frac{A}{s}+\frac{B}{s+2}+\frac{C}{s-4}$$

$$s^2+1=A\left(s^2-2s-8\right)+B\left(s^2-4s\right)+C\left(s^2+2s\right)$$

$$\begin{cases} A+B+C=1 \\ -2A-4B+2C=0 \\ -8A=1 \Rightarrow \underline{A=-\frac{1}{8}} \end{cases} \to \begin{cases} B+C=\frac{9}{8} \\ 2B-C=\frac{1}{8} \end{cases} \to 3B=\frac{5}{4} \Rightarrow \underline{B=\frac{5}{12}} \quad \underline{C=\frac{17}{24}}$$

$$\frac{s^2+1}{s^3-2s^2-8s}=-\frac{1}{8}\frac{1}{s}+\frac{5}{12}\frac{1}{s+2}+\frac{17}{24}\frac{1}{s-4}$$

$$f(t)=\mathcal{L}^{-1}\left\{-\frac{1}{8}\frac{1}{s}+\frac{5}{12}\frac{1}{s+2}+\frac{17}{24}\frac{1}{s-4}\right\}$$

$$\underline{=-\frac{1}{8}+\frac{5}{12}e^{-2t}+\frac{17}{24}e^{4t}}$$

Solution   Section 3.4 – Using Laplace Transform to Solve Differential Equations

Exercise

Solve using the Laplace transform: $y' + 3y = e^{2t}, \quad y(0) = -1$

Solution

$$\mathcal{L}(y' + 3y) = \mathcal{L}\left(e^{2t}\right)$$

$$\mathcal{L}(y') + 3\mathcal{L}(y) = \mathcal{L}\left(e^{2t}\right)$$

$$sY(s) - y(0) + 3Y(s) = \frac{1}{s-2}$$

$$(s+3)Y(s) + 1 = \frac{1}{s-2}$$

$$(s+3)Y(s) = \frac{1}{s-2} - 1$$

$$Y(s) = \frac{1}{(s-2)(s+3)} - \frac{1}{s+3}$$

Using partial fraction:

$$\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$$

$$1 = (A+B)s + 3A - 2B$$

$$\begin{cases} A + B = 0 \\ 3A - 2B = 1 \end{cases} \Rightarrow A = \frac{1}{5} \quad B = -\frac{1}{5}$$

$$Y(s) = \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{1}{s+3} - \frac{1}{s+3}$$

$$y(t) = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{1}{5}e^{2t} - \frac{6}{5}e^{-3t}$$

Exercise

Solve using the Laplace transform: $y' + 9y = e^{-t}, \quad y(0) = 0$

Solution

Let $Y(s) = \mathcal{L}(y)(s)$, then

$$\mathcal{L}(y' + 9y) = \mathcal{L}\left(e^{-t}\right)$$

$$\mathcal{L}(y') + 9\mathcal{L}(y) = \mathcal{L}\left(e^{-t}\right)$$

$$sY(s) - y(0) + 9Y(s) = \frac{1}{s+1}$$

$$(s+9)Y(s) = \frac{1}{s+1}$$

$$Y(s) = \frac{1}{(s+1)(s+9)}$$

Using partial fraction:

$$\frac{1}{(s+1)(s+9)} = \frac{A}{s+1} + \frac{B}{s+9}$$

$$1 = (A+B)s + 9A + B$$

$$\begin{cases} A + B = 0 \\ 9a + B = 1 \end{cases} \Rightarrow A = \frac{1}{8} \quad B = -\frac{1}{8}$$

$$Y(s) = \frac{1}{8}\frac{1}{s+1} - \frac{1}{8}\frac{1}{s+9}$$

$$y(t) = \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+9}\right\}$$

$$= \frac{1}{8}e^{-t} - \frac{1}{8}e^{-9t}$$

Exercise

Solve using the Laplace transform: $y' + 4y = \cos t, \quad y(0) = 0$

Solution

$$\mathcal{L}(y' + 4y) = \mathcal{L}(\cos t)$$

$$sY(s) - y(0) + 4Y(s) = \frac{s}{s^2+1} \qquad\qquad y(0) = 0$$

$$(s+4)Y(s) = \frac{s}{s^2+1}$$

$$Y(s) = \frac{s}{(s+4)\left(s^2+1\right)}$$

Using partial fraction:

$$\frac{s}{(s+4)\left(s^2+1\right)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+1}$$

$$s = As^2 + A + Bs^2 + 4Bs + Cs + 4C$$

$$s = (A+B)s^2 + (4B+C)s + A + 4C$$

$$\begin{cases} A+B=0 \\ 4B+C=1 \Rightarrow A = -\frac{4}{17} \quad B = \frac{4}{17} \quad C = \frac{1}{17} \\ A+4C=0 \end{cases}$$

$$Y(s) = -\frac{4}{17}\frac{1}{s+4} + \frac{4}{17}\frac{s}{s^2+1} + \frac{1}{17}\frac{1}{s^2+1}$$

$$y(t) = -\frac{4}{17}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + \frac{4}{17}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{17}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= -\frac{4}{17}e^{-4t} + \frac{4}{17}e^{-9t}\cos t + \frac{1}{17}\sin t$$

Exercise

Solve using the Laplace transform: $y' + 16y = \sin 3t, \quad y(0) = 1$

Solution

$$\mathcal{L}(y' + 16y) = \mathcal{L}(\sin 3t)$$

$$\mathcal{L}(y') + 16\mathcal{L}(y) = \mathcal{L}(\sin 3t)$$

$$sY(s) - y(0) + 16Y(s) = \frac{3}{s^2+9} \qquad\qquad y(0) = 1$$

$$(s+16)Y(s) - 1 = \frac{3}{s^2+9}$$

$$(s+16)Y(s) = \frac{3}{s^2+9} + 1$$

$$Y(s) = \frac{1}{s+16} + \frac{3}{(s+16)\left(s^2+9\right)}$$

Using partial fraction:

$$\frac{3}{(s+16)\left(s^2+9\right)} = \frac{A}{s+16} + \frac{Bs+C}{s^2+9}$$

$$s = As^2 + 9A + Bs^2 + 16Bs + Cs + 16C$$

$$s = (A+B)s^2 + (16B+C)s + 9A + 16C$$

$$\begin{cases} A + B = 0 \\ 16B + C = 1 \\ 9A + 16C = 0 \end{cases} \Rightarrow A = \frac{3}{265} \quad B = -\frac{3}{265} \quad C = \frac{48}{265}$$

$$Y(s) = \frac{1}{s+16} + \frac{3}{265}\frac{1}{s+16} - \frac{3}{265}\frac{s}{s^2+9} + \frac{48}{265}\frac{1}{s^2+9}$$

$$y(t) = \frac{268}{265}\mathcal{L}^{-1}\left\{\frac{1}{s+16}\right\} - \frac{3}{265}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{48}{265}\mathcal{L}^{-1}\left\{\frac{1}{3}\frac{3}{s^2+9}\right\}$$

$$\boxed{y(t) = \frac{268}{265}e^{-16t} - \frac{3}{265}\cos 3t + \frac{16}{265}\sin 3t}$$

Exercise

Solve using the Laplace transform: $y' + y = te^t, \quad y(0) = -2$

Solution

$$\mathcal{L}(y' + y) = \mathcal{L}\left(te^t\right)$$

$$\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}\left(te^t\right)$$

$$sY(s) - y(0) + Y(s) = \frac{1}{(s-1)^2} \qquad y(0) = -2$$

$$(s+1)Y(s) + 2 = \frac{1}{(s-1)^2}$$

$$(s+1)Y(s) = \frac{1}{(s-1)^2} - 2$$

$$Y(s) = \frac{1}{(s+1)(s-1)^2} - \frac{2}{s+1}$$

Using partial fraction:

$$\frac{1}{(s+1)(s-1)^2} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$= \frac{As^2 - 2As + A + Bs^2 - B + Cs + C}{(s+1)(s-1)^2}$$

$$1 = (A+B)s^2 + (C-2A)s + A - B + C$$

$$\begin{cases} A + B = 0 \\ C - 2A = 0 \\ A - B + C = 1 \end{cases} \Rightarrow A = \frac{1}{4} \quad B = -\frac{1}{4} \quad C = \frac{1}{2}$$

$$Y(s) = \frac{1}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2} - \frac{2}{s+1}$$

$$Y(s) = -\frac{7}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2}$$

$$y(t) = -\frac{7}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$\boxed{= -\frac{7}{4}e^{-t} - \frac{1}{4}e^t + \frac{1}{2}te^t}$$

Exercise

Solve using the Laplace transform: $y' - 4y = t^2 e^{-2t}, \quad y(0) = 1$

Solution

$$\mathcal{L}(y' - 4y) = \mathcal{L}\left(t^2 e^{-2t}\right)$$

$$\mathcal{L}(y') - 4\mathcal{L}(y) = \mathcal{L}\left(t^2 e^{-2t}\right)$$

$$sY(s) - y(0) - 4Y(s) = \frac{2!}{(s+2)^3} \qquad\qquad y(0) = 1$$

$$(s-4)Y(s) - 1 = \frac{2}{(s+2)^3}$$

$$(s-4)Y(s) = 1 + \frac{2}{(s+2)^3}$$

$$Y(s) = \frac{1}{s-4} + \frac{2}{(s-4)(s+2)^3}$$

$$\frac{2}{(s-4)(s+2)^3} = \frac{A}{s-4} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3}$$

$$2 = A\left(s^3 + 6s^2 + 12s + 8A\right) + B\left(s^2 + 4s + 4\right)(s-4) + C(s-4)(s+2) + D(s-4)$$

$$\begin{aligned} 2 = {} & As^3 + 6As^2 + 12As + 8A + Bs^3 + 4Bs^2 + 4Bs - 4Bs^2 - 16Bs - 16B \\ & + Cs^2 - 2Cs - 8C + Ds - 4D \end{aligned}$$

$$2 = (A+B)s^3 + (6A+C)s^2 + (12A - 12B - 2C + D)s + 8A - 16B - 8C - 4D$$

$$\begin{cases} A + B = 0 \\ 6A + C = 0 \\ 12A - 12B - 2C + D = 0 \\ 8A - 16B - 8C - 4D = 2 \end{cases} \Rightarrow \begin{array}{ll} A = \frac{1}{108} & B = -\frac{1}{108} \\ C = -\frac{1}{18} & D = -\frac{1}{3} \end{array}$$

$$\begin{aligned} Y(s) &= \frac{1}{s-4} + \frac{1}{108}\frac{1}{s-4} - \frac{1}{108}\frac{1}{s+2} - \frac{1}{18}\frac{1}{(s+2)^2} - \frac{1}{3}\frac{1}{(s+2)^3} \\ &= \frac{109}{108}\frac{1}{s-4} - \frac{1}{108}\frac{1}{s+2} - \frac{1}{18}\frac{1}{(s+2)^2} - \frac{1}{3}\frac{1}{(s+2)^3} \end{aligned}$$

$$y(t) = \frac{109}{108}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} - \frac{1}{108}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{18}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} - \frac{1}{3}\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2!}{(s+2)^3}\right\}$$

$$\boxed{y(t) = \frac{109}{108}e^{4t} - \frac{1}{108}e^{-2t} - \frac{1}{18}te^{-2t} - \frac{1}{6}t^2 e^{-2t}}$$

Exercise

Solve using the Laplace transform: $y'' - 4y = e^{-t}, \quad y(0) = -1 \quad y'(0) = 0$

Solution

$$\mathcal{L}\left(y'' - 4y\right) = \mathcal{L}\left(e^{-t}\right)$$

$$\mathcal{L}\left(y''\right) - 4\mathcal{L}\left(y\right) = \mathcal{L}\left(e^{-t}\right)$$

$$s^2Y(s) - sy(0) - y'(0) - 4Y(s) = \frac{1}{s+1} \qquad y(0) = -1 \quad y'(0) = 0$$

$$\left(s^2 - 4\right)Y(s) + s = \frac{1}{s+1}$$

$$Y(s) = \frac{1}{(s+1)\left(s^2 - 4\right)} - \frac{s}{s^2 - 4}$$

$$\frac{1}{(s+1)\left(s^2 - 4\right)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s+2}$$

$$1 = As^2 - 4A + Bs^2 + 3Bs + 2B + Cs^2 - Cs - 2C$$

$$1 = (A + B + C)s^2 + (3B - C)s + 2B - 4A - 2C$$

$$\begin{cases} A + B + C = 0 \\ 3B - C = 0 \\ -4A + 2B - 2C = 1 \end{cases} \Rightarrow A = -\frac{1}{3} \quad B = \frac{1}{12} \quad C = \frac{1}{4}$$

$$\frac{s}{s^2 - 4} = \frac{D}{s-2} + \frac{E}{s+2}$$

$$s = (D + E)s + 2D - 2E$$

$$\begin{cases} D + E = 1 \\ 2D - 2E = 0 \end{cases} \Rightarrow D = \frac{1}{2} \quad E = \frac{1}{2}$$

$$Y(s) = -\frac{1}{3}\frac{1}{s+1} + \frac{1}{12}\frac{1}{s-2} + \frac{1}{4}\frac{1}{s+2} - \frac{1}{2}\frac{1}{s-2} - \frac{1}{2}\frac{1}{s+2}$$

$$= -\frac{1}{3}\frac{1}{s+1} - \frac{5}{12}\frac{1}{s-2} - \frac{1}{4}\frac{1}{s+2}$$

$$\boxed{y(t) = -\frac{1}{3}e^{-t} - \frac{5}{12}e^{2t} - \frac{1}{4}e^{-2t}}$$

Exercise

Solve using the Laplace transform: $y'' + 9y = 2\sin 2t, \quad y(0) = 0 \quad y'(0) = -1$

Solution

$$\mathcal{L}\left(y'' + 9y\right) = \mathcal{L}\left(2\sin 2t\right)$$

$$\mathcal{L}\left(y''\right) + 9\mathcal{L} = 2\mathcal{L}\left(\sin 2t\right)$$

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = 2\frac{2}{s^2 + 2^2} \qquad y(0) = 0 \quad y'(0) = -1$$

$$\left(s^2 + 9\right)Y(s) + 1 = \frac{4}{s^2 + 4}$$

$$Y(s) = \frac{4}{\left(s^2 + 9\right)\left(s^2 + 4\right)} - \frac{1}{s^2 + 9}$$

$$\frac{4}{\left(s^2 + 9\right)\left(s^2 + 4\right)} = \frac{A}{s^2 + 9} + \frac{B}{s^2 + 4}$$

$$4 = \left(A + B\right)s^2 + 4A + 9B$$

$$\begin{cases} A + B = 0 \\ 4A + 9B = 4 \end{cases} \Rightarrow A = -\frac{4}{5} \quad B = \frac{4}{5}$$

$$Y(s) = -\frac{4}{5}\frac{1}{s^2 + 9} + \frac{4}{5}\frac{1}{s^2 + 4} - \frac{1}{s^2 + 9}$$

$$= \frac{4}{5}\frac{1}{s^2 + 4} - \frac{9}{5}\frac{1}{s^2 + 9}$$

$$= \frac{4}{5}\frac{1}{2}\frac{2}{s^2 + 4} - \frac{3}{5}\frac{3}{s^2 + 9}$$

$$y(t) = \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{2}{s^2 + 4}\right\} - \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{3}{s^2 + 9}\right\}$$

$$\boxed{y(t) = \frac{2}{5}\sin 2t - \frac{3}{5}\sin 3t}$$

Exercise

Solve using the Laplace transform: $y'' - y = 2t, \quad y(0) = 0 \quad y'(0) = -1$

Solution

$$\mathcal{L}(y'' - y) = \mathcal{L}(2t)$$

$$\mathcal{L}(y'') - \mathcal{L}(y) = 2\mathcal{L}$$

$$s^2 Y(s) - sy(0) - y'(0) - Y(s) = 2\frac{1}{s^2} \qquad\qquad y(0) = 0 \quad y'(0) = -1$$

$$\left(s^2 - 1\right)Y(s) + 1 = \frac{2}{s^2}$$

$$Y(s) = \frac{2}{s^2(s-1)(s+1)} - \frac{1}{(s-1)(s+1)}$$

$$\frac{2}{s^2(s-1)(s+1)} = \frac{A}{s^2} + \frac{B}{s-1} + \frac{C}{s+1}$$

$$2 = As^2 - A + Bs^3 + Bs^2 + Cs^3 - Cs^2$$

$$2 = (B + C)s^3 + (A + B - C)s^2 - A$$

$$\begin{cases} B + C = 0 \\ A + B - C = 0 \\ -A = 2 \end{cases} \Rightarrow A = -2 \quad B = 1 \quad C = -1$$

$$\frac{1}{(s-1)(s+1)} = \frac{D}{s-1} + \frac{E}{s+1}$$

$$\begin{cases} D + E = 0 \\ D - E = 1 \end{cases} \Rightarrow D = \frac{1}{2} \quad E = -\frac{1}{2}$$

$$Y(s) = -\frac{2}{s^2} + \frac{1}{s-1} - \frac{1}{s+1} - \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1}$$

$$= -\frac{2}{s^2} + \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}$$

$$y(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\boxed{y(t) = -2t + \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}}$$

Exercise

Solve using the Laplace transform: $y'' + 3y' = -3t, \quad y(0) = -1 \quad y'(0) = 1$

Solution

$$\mathcal{L}(y'' + 3y') = \mathcal{L}(-3t)$$

$$\mathcal{L}(y'') + 3\mathcal{L}(y') = -3\mathcal{L}(t)$$

$$s^2Y(s) - sy(0) - y'(0) + 3\left(sY(s) - y(0)\right) = -3\frac{1}{s^2} \qquad y(0) = -1 \quad y'(0) = 1$$

$$s^2Y(s) + s - 1 + 3sY(s) + 3 = -\frac{3}{s^2}$$

$$\left(s^2 + 3s\right)Y(s) = -\frac{3}{s^2} - s - 2$$

$$Y(s) = -\frac{3}{s^3(s+3)} - \frac{s+2}{s(s+3)}$$

$$\frac{3}{s^3(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+3}$$

$$3 = As^2(s+3) + Bs(s+3) + C(s+3) + Ds^3$$

$$3 = (A+D)s^3 + (3A+B)s^2 + (3B+C) + 3C$$

$$\begin{cases} A + D = 0 \\ 3A + B = 0 \\ 3B + C = 0 \\ 3C = 3 \end{cases} \Rightarrow \begin{matrix} A = \frac{1}{9} & B = -\frac{1}{3} \\ C = 1 & D = -\frac{1}{9} \end{matrix}$$

$$\frac{s+2}{s(s+3)} = \frac{E}{s} + \frac{F}{s+3}$$

$$\begin{cases} E + F = 1 \\ 3E = 2 \end{cases} \Rightarrow E = \frac{2}{3} \quad F = \frac{1}{3}$$

$$Y(s) = -\left(\frac{1}{9}\frac{1}{s} - \frac{1}{3}\frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{9}\frac{1}{s+3}\right) - \left(\frac{2}{3}\frac{1}{s} + \frac{1}{3}\frac{1}{s+3}\right)$$

$$= -\frac{1}{9}\frac{1}{s} + \frac{1}{3}\frac{1}{s^2} - \frac{1}{s^3} + \frac{1}{9}\frac{1}{s+3} - \frac{2}{3}\frac{1}{s} - \frac{1}{3}\frac{1}{s+3}$$

$$= -\frac{7}{9}\frac{1}{s} + \frac{1}{3}\frac{1}{s^2} - \frac{1}{s^3} - \frac{2}{9}\frac{1}{s+3}$$

$$y(t) = -\frac{7}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{2!}\frac{2!}{s^3}\right\} - \frac{2}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$\boxed{y(t) = -\frac{7}{9} + \frac{1}{3}t - \frac{1}{2}t^2 - \frac{2}{9}e^{-3t}}$$

Exercise

Solve using the Laplace transform: $y'' - y' - 2y = e^{2t}, \quad y(0) = -1 \quad y'(0) = 0$

Solution

$$\mathcal{L}(y'' - y' - 2y) = \mathcal{L}\left(e^{2t}\right)$$

$$s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) - 2Y(s) = \frac{1}{s-2} \qquad y(0) = -1 \quad y'(0) = 0$$

$$s^2Y(s) + s - sY(s) - 1 - 2Y(s) = \frac{1}{s-2}$$

$$\left(s^2 - s - 2\right)Y(s) = \frac{1}{s-2} - s + 1$$

$$(s+1)(s-2)(Y(s) = \frac{1}{s-2} - s + 1$$

$$Y(s) = \frac{1}{(s+1)(s-2)^2} - \frac{s-1}{(s+1)(s-2)}$$

$$= \frac{1-(s-1)(s-2)}{(s+1)(s-2)^2}$$

$$= \frac{-s^2+3s-1}{(s+1)(s-2)^2}$$

$$= \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$-s^2 + 3s - 1 = As^2 - 4As + 4A + Bs^2 - Bs - 2B + Cs + C$$

$$-s^2 + 3s - 1 = (A+B)s^2 + (-4A - B + C)s + 4A - 2B + C$$

$$\begin{cases} A + B = -1 \\ -4A - B + C = 3 \\ 4A - 2B + C = -1 \end{cases} \Rightarrow A = -\frac{5}{9} \quad B = -\frac{4}{9} \quad C = \frac{1}{3}$$

$$Y(s) = -\frac{5}{9}\frac{1}{s+1} - \frac{4}{9}\frac{1}{s-2} + \frac{1}{3}\frac{1}{(s-2)^2}$$

$$y(t) = -\frac{5}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{4}{9}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$\boxed{y(t) = -\frac{5}{9}e^{-t} - \frac{4}{9}e^{2t} + \frac{1}{3}te^{2t}}$$

Exercise

Solve using the Laplace transform: $y'' + y = -2\cos 2t, \quad y(0) = 1 \quad y'(0) = -1$

Solution

$$\mathcal{L}\left(y'' + y\right)(s) = \mathcal{L}\left(-2\cos 2t\right)(s)$$

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = -2\frac{s}{s^2+4} \qquad\qquad y(0) = 1 \quad y'(0) = -1$$

$$s^2 Y(s) - s + 1 + Y(s) = -2\frac{s}{s^2+4}$$

$$\left(s^2+1\right)Y(s) = \frac{-2s}{s^2+4} + s - 1$$

$$Y(s) = \frac{-2s}{\left(s^2+4\right)\left(s^2+1\right)} + \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

$$= \frac{As}{s^2+4} + \frac{Bs}{s^2+1} + \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

$$\frac{-2s}{\left(s^2+4\right)\left(s^2+1\right)} = \frac{As}{s^2+4} + \frac{Bs}{s^2+1}$$

$$\Rightarrow \quad -2s = As\left(s^2+1\right) + Bs\left(s^2+4\right)$$

$$-2s = As^3 + As + Bs^3 + 4Bs$$

$$-2s = (A+B)s^3 + (A+4B)s$$

$$\begin{cases} A + B = 0 \\ A + 4B = -2 \end{cases} \Rightarrow \boxed{A = \frac{2}{3}} \quad \boxed{B = -\frac{2}{3}}$$

$$= \frac{2}{3}\frac{s}{s^2+4} - \frac{2}{3}\frac{s}{s^2+1} + \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

$$= \frac{2}{3}\frac{s}{s^2+4} + \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{s^2+1}$$

$$\boxed{y(t) = \frac{2}{3}\cos 2t + \frac{1}{3}\cos t - \sin t}$$

Exercise

Solve using the Laplace transform: $y'' + 9y = 3\sin 2t, \quad y(0) = 0 \quad y'(0) = -1$

Solution

$$\mathcal{L}(y'' + 9y)(s) = \mathcal{L}(3\sin 2t)(s)$$

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = 3\frac{1}{s^2+4} \qquad\qquad y(0) = 0 \quad y'(0) = -1$$

$$s^2Y(s) + 1 + Y(s) = \frac{3}{s^2+4}$$

$$\left(s^2+1\right)Y(s) = \frac{3}{s^2+4} - 1$$

$$\left(s^2+1\right)Y(s) = \frac{-s^2-1}{s^2+4}$$

$$Y(s) = \frac{-s^2-1}{\left(s^2+4\right)\left(s^2+1\right)}$$

$$= \frac{A}{s^2+4} + \frac{B}{s^2+1}$$

$$\frac{A}{s^2+4} + \frac{B}{s^2+1} = \frac{As^2 + A + Bs^2 + 4B}{\left(s^2+4\right)\left(s^2+1\right)}$$

$$\begin{matrix} s^2 \\ s^0 \end{matrix} \begin{cases} A + B = -1 \\ A + 4B = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{2}{3} \\ B = -\frac{5}{3} \end{cases}$$

$$Y(s) = \frac{2}{3}\frac{1}{s^2+4} - \frac{5}{3}\frac{1}{s^2+1}$$

$$y(t) = \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} - \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{2}{3}\sin 2t - \frac{5}{3}\sin t$$

Exercise

Use Laplace transform to find the solution to the initial value problem

$$x'' - x' - 6x = 0; \quad x(0) = 2, \quad x'(0) = -1$$

Solution

$$\mathcal{L}\left\{x'' - x' - 6x\right\} = 0$$

$$\left(s^2X(s) - sx(0) - x'(0)\right) - \left(sX(s) - x(0)\right) - 6X(s) = 0$$

$$\left(s^2 - s - 6\right)X(s) - 2s + 3 = 0$$

$$\left(s^2-s-6\right)X(s)=2s-3$$

$$X(s)=\frac{2s-3}{(s-3)(s+2)}=\frac{A}{s-3}+\frac{B}{s+2}$$

$$2s-3=A(s+2)+B(s-3)$$

$$\begin{cases} A+B=2 \\ 2A-3B=-3 \end{cases} \rightarrow \quad A=\frac{3}{5} \quad B=\frac{7}{5}$$

$$X(s)=\frac{3}{5}\frac{1}{s-3}+\frac{7}{5}\frac{1}{s+2} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}=e^{at}$$

$$x(t)=\frac{3}{5}e^{3t}+\frac{7}{5}e^{-2t}$$

Exercise

Solve the initial value problem $x''+4x'+4x=t^2;\quad x(0)=x'(0)=0$

Solution

$$\mathcal{L}\left\{x''+4x'+4x\right\}=\mathcal{L}\left\{t^2\right\}$$

$$s^2X(s)-sx(0)-x'(0)+4\left[sX(s)-x(0)\right]+4X(s)=\frac{2}{s^3}$$

$$\left(s^2+4s+4\right)X(s)=\frac{2}{s^3}$$

$$X(s)=\frac{2}{s^3(s+2)^2}=\frac{A}{s^3}+\frac{B}{s^2}+\frac{C}{s}+\frac{D}{(s+2)^2}+\frac{E}{s+2}$$

$$2=A\left(s^2+4s+4\right)+Bs\left(s^2+4s+4\right)+Cs^2\left(s^2+4s+4\right)+Ds^3+Es^3(s+2)$$

$$2=As^2+4As+4A+Bs^3+4Bs^2+4Bs+Cs^4+4Cs^3+4Cs^2+Ds^3+Es^4+2Es^3$$

$$\begin{cases} C+E=0 \\ B+4C+D+2E=0 \\ A+4B+4C=0 \\ 4A+4B=0 \\ 4A=2 \rightarrow A=\frac{1}{2} \end{cases} \rightarrow \quad \begin{aligned} &E=-C=-\frac{3}{8} \\ &D=-2C-B=-\frac{1}{4} \\ &C=-\frac{3}{4},\ B=\frac{3}{8} \\ &B=-\frac{1}{2} \end{aligned}$$

$$X(s)=\frac{\frac{1}{2}}{s^3}-\frac{\frac{1}{2}}{s^2}+\frac{\frac{3}{8}}{s}-\frac{\frac{1}{4}}{(s+2)^2}-\frac{\frac{3}{8}}{s+2}$$

$$x(t)=\frac{1}{2}t^2-\frac{1}{2}t+\frac{3}{8}-\frac{1}{4}te^{-2t}-\frac{3}{8}e^{-2t}$$

Exercise

Solve the initial value problem

$$y^{(4)}+2y''+y=4te^t;\quad y(0)=y'(0)=y''(0)=y^{(3)}(0)=0$$

Solution

$$\mathcal{L}\left\{y^{(4)}+2y''+y\right\}=\mathcal{L}\left\{4te^t\right\}$$

$$s^4Y(s)+2s^2Y(s)+Y(s)=\frac{4}{(s-1)^2}\qquad \mathcal{L}\left\{te^t\right\}=\frac{1}{(s-1)^2}$$

$$\left(s^4+2s^2+1\right)Y(s)=\frac{4}{(s-1)^2}\qquad \left(s^4+2s^2+1\right)=\left(s^2+1\right)^2$$

$$Y(s)=\frac{4}{\left(s^2+1\right)^2(s-1)^2}=\frac{As+B}{s^2+1}+\frac{Cs+D}{\left(s^2+1\right)^2}+\frac{E}{s-1}+\frac{F}{(s-1)^2}$$

$$(As+B)\left(s^2+1\right)(s-1)^2+(Cs+D)(s-1)^2+E(s-1)\left(s^2+1\right)^2+F\left(s^2+1\right)^2=4$$

$$if\ \ s=1\ \Rightarrow\ \boxed{F=1}$$

$$(As+B)\left(s^4-s^3+2s^2-2s+1\right)+Cs^3-2Cs^2+Cs+Ds^2-2Ds+D+Es^5+2Es^3$$
$$+Es-Es^4-2Es^2-E+s^4+2s^2+1=4$$

$$\begin{aligned}
&A+E=0\\
&-2A+B-E+1=0\\
&2A-2B+C+2E=0\\
&-2A+2B-2C+D-2E+2=0\\
&A-2B+C-2D+E=0\\
&B+D-E+1=4
\end{aligned}
\quad\rightarrow\quad
\begin{cases}
A=2 & B=1\\
C=2 & D=0\\
E=-2 & F=1
\end{cases}$$

$$Y(s)=\frac{2s+1}{s^2+1}+\frac{2s}{\left(s^2+1\right)^2}-\frac{2}{s-1}+\frac{1}{(s-1)^2}\qquad \mathcal{L}^{-1}\frac{s}{\left(s^2+k^2\right)^2}=\frac{1}{2k}t\sin kt$$

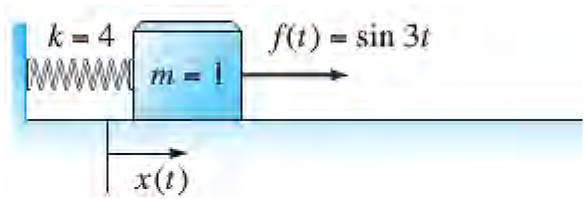
$$y(t)=2\cos t+\sin t+2\frac{1}{2}t\sin t-2e^t+te^t$$

$$\underline{=2\cos t+(1+t)\sin t+(t-2)e^t}$$

Exercise

Solve the initial value problem $x'' + 4x = \sin 3t; \quad x(0) = x'(0) = 0.$

Such problem arises in the motion of a mass-and-spring system with external force as shown below.



Solution

$$\mathcal{L}\{x'' + 4x\} = \mathcal{L}\{\sin 3t\}$$

$$\mathcal{L}\{x''\} - \mathcal{L}\{4x\} = \mathcal{L}\{\sin 3t\}$$

$$\left(s^2 X(s) - sx(0) - x'(0)\right) + 4X(x) = \frac{3}{s^2+9}$$

$$\left(s^2+4\right) X(x) = \frac{3}{s^2+9}$$

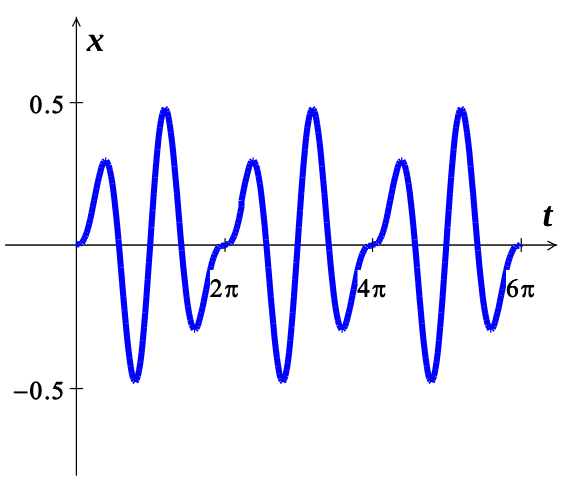
$$X(x) = \frac{3}{\left(s^2+4\right)\left(s^2+9\right)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$3 = (As+B)\left(s^2+9\right) + (Cs+D)\left(s^2+4\right)$$

$$3 = (A+C)s^3 + (B+D)s^2 + (9A+4C)s + 9B + 4D$$

$$\begin{cases} A+C=0 \\ B+D=0 \\ 9A+4C=0 \\ 9B+4D=3 \end{cases} \Rightarrow \begin{cases} A=C=0 \\ B=\frac{3}{5};\ D=-\frac{3}{5} \end{cases}$$

$$X(x) = \frac{3}{5}\frac{1}{s^2+4} - \frac{3}{5}\frac{1}{s^2+9} = \frac{3}{10}\frac{2}{s^2+4} - \frac{1}{5}\frac{3}{s^2+9}$$



$$x(t) = \mathcal{L}^{-1}[X(s)]$$

$$= \frac{3}{10}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

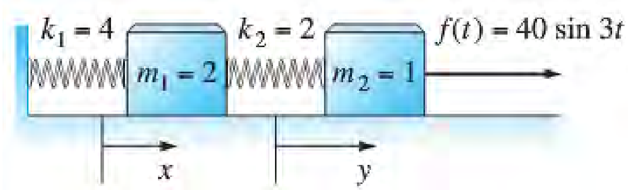
$$\boxed{= \frac{3}{10}\sin 2t - \frac{1}{5}\sin 3t}$$

Exercise

Solve the system $\begin{cases} 2x'' = -6x + 2y \\ y'' = 2x - 2y + 40\sin 3t \end{cases}$

Subject to the initial conditions $x(0) = x'(0) = y(0) = y'(0) = 0$

Thus the force $f(t) = 40\sin 3t$ is applied to the second mass as shown below, beginning at time $t = 0$ when the system is at rest in its equilibrium position.



Solution

Let $X(s) = \mathcal{L}\{x(t)\}$ *and* $Y(s) = \mathcal{L}\{y(t)\}$

$$\begin{cases} \mathcal{L}\{2x''\} = \mathcal{L}\{-6x + 2y\} \\ \mathcal{L}\{y''\} = \mathcal{L}\{2x - 2y + 40\sin 3t\} \end{cases}$$

$$\begin{cases} 2\left(s^2 X(s) - sx(0) - x'(0)\right) = -6X(x) + 2Y(s) \\ s^2 Y(s) - sy(0) - y'(0) = 2X(s) - 2Y(s) + 40\dfrac{3}{s^2+9} \end{cases} \rightarrow \begin{cases} \left(s^2+3\right)X(s) - Y(s) = 0 \\ -2X(s) + \left(s^2+2\right)Y(s) = \dfrac{120}{s^2+9} \end{cases}$$

Solve the system using Cramer's rule

$$D = \begin{vmatrix} s^2+3 & -1 \\ -2 & s^2+2 \end{vmatrix} = s^4 + 5s^2 + 4 = \left(s^2+1\right)\left(s^2+4\right)$$

$$D_X = \begin{vmatrix} 0 & -1 \\ \frac{120}{s^2+9} & s^2+2 \end{vmatrix} = \frac{120}{s^2+9} \quad \Rightarrow \quad X(s) = \frac{120}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)}$$

$$D_Y = \begin{vmatrix} s^2+3 & 0 \\ -2 & \frac{120}{s^2+9} \end{vmatrix} = \frac{120\left(s^2+3\right)}{s^2+9} \quad \Rightarrow \quad Y(s) = \frac{120\left(s^2+3\right)}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)}$$

$$X(s) = \frac{120}{\left(s^2+1\right)\left(s^2+4\right)\left(s^2+9\right)} = \frac{A}{s^2+1} + \frac{B}{s^2+4} + \frac{C}{s^2+9}$$

(*Nothing that the denominator factors are linear in* s^2)

$$120 = A\left(s^2+4\right)\left(s^2+9\right) + B\left(s^2+1\right)\left(s^2+9\right) + C\left(s^2+1\right)\left(s^2+4\right)$$

$$= (A+B+C)s^4 + (13A+10B+5C)s^2 + 36A + 9B + 4C$$

$$\begin{cases} A+B+C=0 \\ 13A+10B+5C=0 \\ 36A+9B+4C=120 \end{cases} \Rightarrow \begin{matrix} A=5 \\ B=-8 \\ C=3 \end{matrix}$$

$$X(s)=\frac{5}{s^2+1}-4\frac{2}{s^2+4}+\frac{3}{s^2+9} \quad \Rightarrow \quad \underline{x(t)=5\sin t-4\sin 2t+\sin 3t}$$

$$Y(s)=\frac{120s^2+360}{\left(s^2+9\right)\left(s^2+1\right)\left(s^2+4\right)}=\frac{A}{s^2+1}+\frac{B}{s^2+4}+\frac{C}{s^2+9}$$

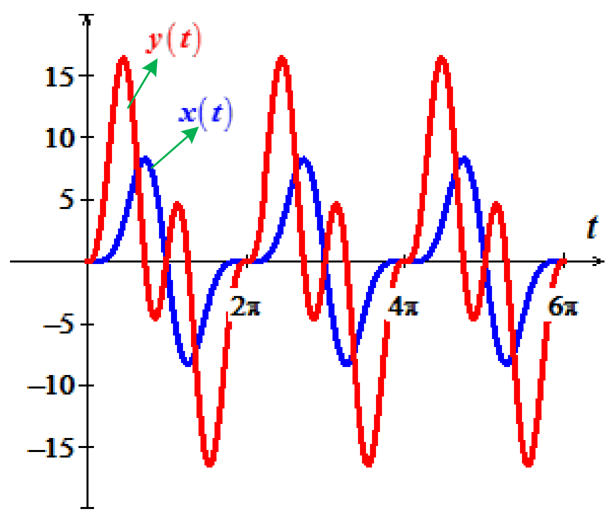
$$120s^2+360=A\left(s^2+4\right)\left(s^2+9\right)+B\left(s^2+1\right)\left(s^2+9\right)+C\left(s^2+1\right)\left(s^2+4\right)$$

$$=(A+B+C)s^4+(13A+10B+5C)s^2+36A+9B+4C$$

$$\begin{cases} A+B+C=0 \\ 13A+10B+5C=120 \\ 36A+9B+4C=360 \end{cases} \Rightarrow \begin{matrix} A=10 \\ B=8 \\ C=-18 \end{matrix}$$

$$Y(s)=10\frac{1}{s^2+1}+4\frac{1}{s^2+4}-6\frac{3}{s^2+9}$$

$$\Rightarrow \quad \underline{y(t)=10\sin t+4\sin 2t-6\sin 3t}$$



Exercise

Given: $y'' - 4y' + 3y = 0, \quad y(0) = 1 \quad y'(0) = -1$

a) Show that the general solution is: $y(t) = C_1 e^{3t} + C_2 e^t$ and find C_1 *and* C_2

b) Use Laplace transform to solve the system

Solution

a) $\lambda^2 - 4\lambda + 3 = 0 \quad \Rightarrow \quad \boxed{\lambda = 3,\ 1}$

That implies to the general solution: $y = C_1 e^{3t} + C_2 e^t$

$$1 = C_1 e^{3(0)} + C_2 e^{(0)}$$

$$1 = C_1 + C_2$$

$$y' = 3C_1 e^{3t} + C_2 e^t$$

$$-1 = 3C_1 e^{3(0)} + C_2 e^{(0)}$$

$$-1 = 3C_1 + C_2$$

$$\begin{cases} C_1 + C_2 = 1 \\ 3C_1 + C_2 = -1 \end{cases} \Rightarrow \boxed{C_1 = -1} \quad \boxed{C_2 = 2}$$

Therefore; the general solution is: $\boxed{y = -e^{3t} + 2e^t}$

b) $\mathcal{L}\left(y'' - 4y' + 3y\right)(s) = 0$

$$s^2 Y(s) - sy(0) - y'(0) - 4\left(sY(s) - y(0)\right) + 3Y(s) = 0$$

$$s^2 Y(s) - s + 1 - 4\left(sY(s) - 1\right) + 3Y(s) = 0$$

$$s^2 Y(s) - s + 1 - 4sY(s) + 4 + 3Y(s) = 0$$

$$\left(s^2 - 4s + 3\right) Y(s) = s - 5$$

$$Y(s) = \frac{s-5}{s^2 - 4s + 3}$$

$$= \frac{s-5}{(s-1)(s-3)}$$

$$= \frac{A}{s-1} + \frac{B}{s-3} = \frac{(A+B)s - 3A - B}{(s-1)(s-3)}$$

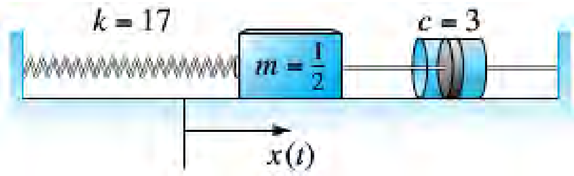
$$\begin{cases} A + B = 1 \\ -3A - B = -5 \end{cases} \Rightarrow \boxed{A = 2} \quad \boxed{B = -1}$$

$$= \frac{2}{s-1} - \frac{1}{s-3}$$

That implies: $\boxed{y(t) = 2e^t - e^{3t}}$

Exercise

Consider a mass-spring system with $m = \frac{1}{2}, \quad k = 17, \quad and \ c = 3$.



Let $x(t)$ be the displacement of the mass *m* from its equilibrium position. If the mass is set in motion with $x(0) = 3 \ and \ x'(0) = 1$, find $x(t)$ for the resulting damped free oscillations.

Solution

The differential equation of $mx'' + cx' + k = 0$ is $\frac{1}{2}x'' + 3x' + 17 = 0$

$x'' + 6x' + 34 = 0; \quad x(0) = 3 \ and \ x'(0) = 1$

$\mathcal{L}\{x'' + 6x' + 34\} = 0$

$s^2 X(s) - sx(0) - x'(0) + 6[sX(s) - x(0)] + 34X(s) = 0$

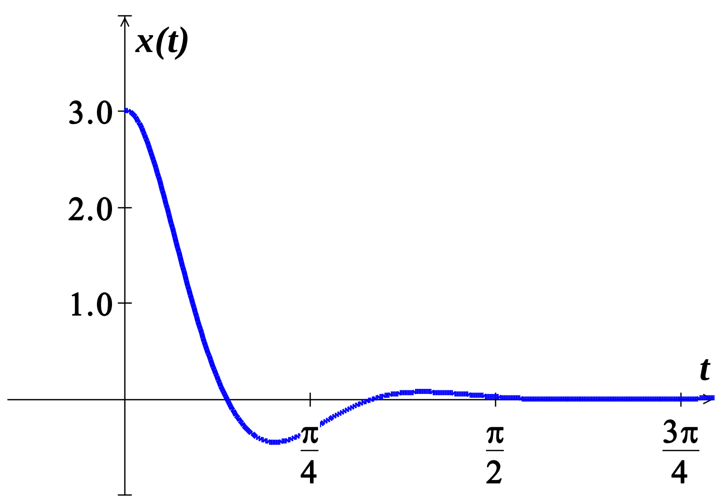
$(s^2 + 6s + 34)X(s) = 3s + 1 + 18$

$$X(s) = \frac{3s+19}{s^2+6s+34}$$

$$= \frac{3s+19}{(s+3)^2+25} \qquad\qquad 3s + 19 = 3(s+3) + 10 = 3(s+3) + 2\cdot 5$$

$$= 3\frac{s+3}{(s+3)^2+25} + 2\frac{5}{(s+3)^2+25}$$

$\boxed{x(t) = 3e^{-3t}\cos 5t + 2e^{-3t}\sin 5t}$



Exercise

Consider a mass-spring-dashpot system with $m=\frac{1}{2}, \quad k=17, \quad c=3, \quad and\ f(t)=15\sin 2t$ with initial conditions $x(0)=x'(0)=0$. Let $x(t)$ be the displacement of the mass *m* from its equilibrium position. Find the resulting transient motion and steady periodic motion of the mass..

Solution

$$mx''+cx'+k=f(t)$$

$$\frac{1}{2}x''+3x'+17=15\sin 2t$$

$$x''+6x'+34=30\sin 2t; \quad x(0)=x'(0)=0$$

$$\mathcal{L}\{x''+6x'+34\}=\mathcal{L}\{30\sin 2t\}$$

$$\left(s^2+6s+34\right)X(s)=\frac{60}{s^2+4}$$

$$X(s)=\frac{60}{\left(s^2+4\right)\left((s+3)^2+25\right)}=\frac{As+B}{s^2+4}+\frac{Cs+D}{(s+3)^2+25}$$

$$60=(As+B)\left(s^2+6s+34\right)+(Cs+D)\left(s^2+4\right)$$

$$=As^3+6As^2+34As+Bs^2+6Bs+34B+Cs^3+4Cs+Ds^2+4D$$

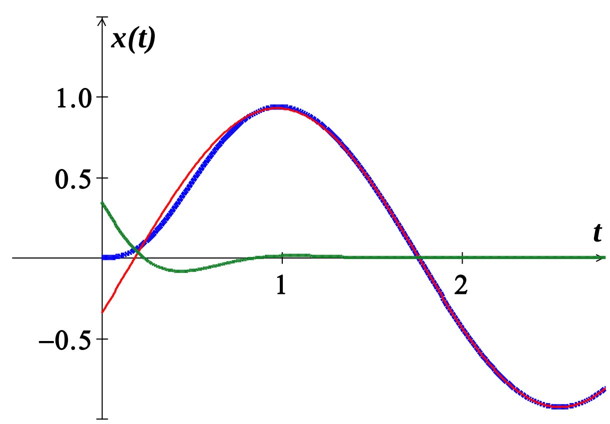
$$\begin{cases} A+C=0 \\ 6A+B+D=0 \\ 34A+6B+4C=0 \\ 34B+4D=60 \end{cases} \xrightarrow{rref} \begin{matrix} A=-\frac{10}{29} & B=\frac{50}{29} \\ C=\frac{10}{29} & D=\frac{10}{29} \end{matrix}$$

$$X(s)=\frac{1}{29}\frac{-10s+50}{s^2+4}+\frac{1}{29}\frac{10s+10}{(s+3)^2+25}$$

$$=\frac{1}{29}\left(\frac{-10s+25\cdot 2}{s^2+4}+\frac{10(s+3)-4\cdot 5}{(s+3)^2+25}\right)$$

$$x(t)=\mathcal{L}^{-1}\{X(s)\}$$

$$x(t)=\underbrace{\frac{5}{29}(5\sin 2t-2\cos 2t)}_{steady\ periodic}+\underbrace{\frac{7}{29}e^{-3t}(5\cos 5t-2\sin 5t)}_{transient}$$



The terms of circular frequency 2 constitute the steady state periodic forced oscillation of the mass, whereas the exponentially damped terms of circular frequency 5 constitute its transient motion, which disappears very rapidly.

Solution — Section 3.5 - Basic Electrical Circuits

Exercise

A resistor $R = 20\ \Omega$ and a capacitor of $C = 0.1\ F$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge on the capacitor at time t for the given $E(t) = 100\sin 2t$

Solution

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E$$

$$20Q' + \frac{1}{0.1}Q = 100\sin 2t$$

$$\mathcal{L}\left(Q' + 0.5Q\right) = 5\mathcal{L}\left(\sin 2t\right)$$

$$sQ(s) - Q(0) + 0.5Q(s) = 5\frac{2}{s^2+4} \qquad Q(0) = 0$$

$$\left(s + 0.5\right)Q(s) = 10\frac{1}{s^2+4}$$

$$Q(s) = 10\frac{1}{\left(s+0.5\right)\left(s^2+4\right)}$$

$$\frac{1}{\left(s+0.5\right)\left(s^2+4\right)} = \frac{A}{s+0.5} + \frac{Bs+C}{s^2+4}$$

$$1 = As^2 + 4A + Bs^2 + 0.5Bs + Cs + 0.5C$$

$$1 = \left(A+B\right)s^2 + \left(0.5B + C\right)s + 4A + 0.5C$$

$$\begin{cases} A + B = 0 \\ 0.5B + C = 0 \\ 4A + 0.5C = 1 \end{cases} \Rightarrow A = \frac{4}{17} \quad B = -\frac{4}{17} \quad C = \frac{2}{17}$$

$$Q(s) = 10\left(\frac{4}{17}\frac{1}{s+\frac{1}{2}} - \frac{4}{17}\frac{s}{s^2+4} + \frac{2}{17}\frac{1}{s^2+4}\right)$$

$$= \frac{1}{17}\left(40\frac{1}{s+\frac{1}{2}} - 40\frac{s}{s^2+4} + 10\frac{2}{s^2+4}\right)$$

$$Q(t) = \frac{1}{17}\left(40\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\} - 40\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 10\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\right)$$

$$\boxed{= \frac{1}{17}\left(40e^{-t/2} - 40\cos 2t + 10\sin 2t\right)}$$

Exercise

A resistor $R = 20\ \Omega$ and a capacitor of $C = 0.1\ F$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge on the capacitor at time t for the given $E(t) = 100e^{-0.1t}$

Solution

$$20Q' + \frac{1}{0.1}Q = 100e^{-0.1t}$$

$$\mathcal{L}(Q' + 0.5Q) = 5\mathcal{L}\left(e^{-0.1t}\right)$$

$$sQ(s) - Q(0) + 0.5Q(s) = 5\frac{1}{s+0.1} \qquad Q(0) = 0$$

$$(s+0.5)Q(s) = 5\frac{1}{s+0.1}$$

$$Q(s) = 5\frac{1}{(s+0.5)(s+0.1)}$$

$$\frac{1}{(s+0.5)(s+0.1)} = \frac{A}{s+0.5} + \frac{B}{s+0.1}$$

$$1 = (A+B)s + 0.1A + 0.5B$$

$$\begin{cases} A + B = 0 \\ 0.1A + 0.5B = 1 \end{cases} \Rightarrow A = -\frac{5}{2} \quad B = \frac{5}{2}$$

$$Q(s) = 5\left(-\frac{5}{2}\frac{1}{s+\frac{1}{2}} + \frac{5}{2}\frac{1}{s+0.1}\right)$$

$$= \frac{25}{2}\left(-\frac{1}{s+\frac{1}{2}} + \frac{1}{s+\frac{1}{10}}\right)$$

$$Q(t) = \frac{25}{2}\left(\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{10}}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\}\right)$$

$$\boxed{= \frac{25}{2}\left(e^{-t/10} - e^{-t/2}\right)}$$

Exercise

A resistor $R = 20\ \Omega$ and a capacitor of $C = 0.1\ F$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge on the capacitor at time t for the given $E(t) = 100\left(1 - e^{-0.1t}\right)$

Solution

$$20Q' + \frac{1}{0.1}Q = 100\left(1 - e^{-0.1t}\right)$$

$$\mathcal{L}\left(Q' + 0.5Q\right) = 5\mathcal{L}\left(1 - e^{-0.1t}\right)$$

$$sQ(s) - Q(0) + 0.5Q(s) = 5\left(\frac{1}{s} - \frac{1}{s+0.1}\right) \qquad Q(0) = 0$$

$$(s + 0.5)Q(s) = 5\left(\frac{1}{s} - \frac{1}{s+0.1}\right)$$

$$Q(s) = 5\frac{1}{s+0.5}\left(\frac{s+0.1-s}{s(s+0.1)}\right)$$

$$Q(s) = 0.5\frac{1}{s(s+0.5)(s+0.1)}$$

$$\frac{1}{s(s+0.5)(s+0.1)} = \frac{A}{s} + \frac{B}{s+0.5} + \frac{C}{s+0.1}$$

$$1 = A\left(s^2 + 0.6s + 0.05\right) + B\left(s^2 + 0.1s\right) + C\left(s^2 + 0.5s\right)$$

$$1 = (A + B + C)s^2 + (0.6A + 0.1B + 0.5C)s + 0.05A$$

$$\begin{cases} A + B + C = 0 \\ 0.6A + 0.1B + 0.5C = 0 \\ 0.05A = 1 \end{cases} \Rightarrow A = \frac{1}{0.05} = 20 \quad B = 5 \quad C = -25$$

$$Q(s) = \frac{1}{2}\left(20\frac{1}{s} + 5\frac{1}{s+0.5} - 25\frac{1}{s+0.1}\right)$$

$$= 10\frac{1}{s} + \frac{5}{2}\frac{1}{s+\frac{1}{2}} - \frac{25}{2}\frac{1}{s+\frac{1}{10}}$$

$$Q(t) = \left(10\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\} - \frac{25}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{10}}\right\}\right)$$

$$\boxed{= 10 + \frac{5}{2}e^{-t/2} - \frac{25}{2}e^{-t/10}}$$

Exercise

A resistor $R = 20\ \Omega$ and a capacitor of $C = 0.1\ F$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge on the capacitor at time t for the given $E(t) = 100\cos 3t$

Solution

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E$$

$$20Q' + \frac{1}{0.1}Q = 100\cos 3t$$

$$Q' + \frac{1}{0.1(20)}Q = \frac{100}{20}\cos 3t$$

$$\mathcal{L}(Q' + 0.5Q) = 5\mathcal{L}(\cos 3t)$$

$$sQ(s) - Q(0) + 0.5Q(s) = 5\frac{s}{s^2 + 9} \qquad Q(0) = 0$$

$$(s + 0.5)Q(s) = 5\frac{s}{s^2 + 9}$$

$$Q(s) = 5\frac{s}{(s + 0.5)(s^2 + 9)}$$

$$\frac{s}{(s + 0.5)(s^2 + 9)} = \frac{A}{s + 0.5} + \frac{Bs + C}{s^2 + 9}$$

$$s = As^2 + 9A + Bs^2 + 0.5Bs + Cs + 0.5C$$

$$s = (A + B)s^2 + (0.5B + C)s + 9A + 0.5C$$

$$\begin{cases} A + B = 0 \\ 0.5B + C = 1 \\ 9A + 0.5C = 0 \end{cases} \Rightarrow A = -\frac{2}{37} \quad B = \frac{2}{37} \quad C = \frac{36}{37}$$

$$Q(s) = 5\left(-\frac{2}{37}\frac{1}{s + \frac{1}{2}} + \frac{2}{37}\frac{s}{s^2 + 9} + \frac{36}{37}\frac{1}{s^2 + 9} \right)$$

$$= \frac{1}{37}\left(-10\frac{1}{s + \frac{1}{2}} + 10\frac{s}{s^2 + 9} + \frac{5(36)}{3}\frac{3}{s^2 + 9} \right)$$

$$Q(t) = \frac{1}{37}\left(-10\mathcal{L}^{-1}\left\{\frac{1}{s + \frac{1}{2}}\right\} + 10\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 9}\right\} + 60\mathcal{L}^{-1}\left\{\frac{3}{s^2 + 9}\right\} \right)$$

$$\boxed{= \frac{1}{37}\left(-10e^{-t/2} + 10\cos 3t + 60\sin 3t\right)}$$

Exercise

An inductor $(L = 1\ H)$ and a resistor $(R = 0.1\ \Omega)$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge current in the current at time t for the given $E(t) = 10 - 2t$

Solution

$$L\frac{dI}{dt} + RI = E(t)$$

$$\frac{dI}{dt} + 0.1I = 10 - 2t$$

$$\mathcal{L}(I' + 0.1I) = \mathcal{L}(10 - 2t)$$

$$sI(s) - I(0) + 0.1I(s) = \frac{10}{s} - \frac{2}{s^2} \qquad I(0) = 0$$

$$(s + 0.1)I(s) = \frac{10s - 2}{s^2}$$

$$I(s) = \frac{10s - 2}{s^2(s + 0.1)}$$

$$\frac{10s - 2}{s^2(s + 0.1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + 0.1}$$

$$10s - 2 = As(s + 0.1) + B(s + 0.1) + Cs^2$$

$$10s - 2 = (A + C)s^2 + (B + 0.1A)s + 0.1B$$

$$\begin{cases} A + C = 0 \\ 0.1A + B = 10 \\ 0.1B = -2 \end{cases} \Rightarrow A = 300 \quad B = -20 \quad C = -300$$

$$I(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + 0.1}$$

$$= 300\frac{1}{s} - 20\frac{1}{s^2} - 300\frac{1}{s + \frac{1}{10}}$$

$$I(t) = \left(3000\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 20\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 300\mathcal{L}^{-1}\left\{\frac{1}{s + \frac{1}{10}}\right\}\right)$$

$$\underline{= 300 - 20t - 300e^{-t/10}}$$

Exercise

An inductor $(L = 1\ H)$ and a resistor $(R = 0.1\ \Omega)$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing current in the current at time t for the given $E(t) = 4\cos 3t$

Solution

$$L\frac{dI}{dt} + RI = E(t)$$

$$\frac{dI}{dt} + 0.1I = 4\cos 3t$$

$$\mathcal{L}\left(I' + 0.1I\right) = 4\mathcal{L}\left(\cos 3t\right)$$

$$sI(s) - I(0) + 0.1I(s) = 4\frac{s}{s^2 + 9} \qquad I(0) = 0$$

$$\left(s + 0.1\right)I(s) = 4\frac{s}{s^2 + 9}$$

$$I(s) = 4\frac{s}{\left(s + 0.1\right)(s^2 + 9)}$$

$$\frac{s}{\left(s + 0.1\right)(s^2 + 9)} = \frac{A}{s + 0.1} + \frac{Bs + C}{s^2 + 9}$$

$$s = As^2 + 9A + Bs^2 + 0.1Bs + Cs + 0.1C$$

$$s = \left(A + B\right)s^2 + \left(0.1B + C\right)s + 9A + 0.1C$$

$$\begin{cases} A + B = 0 \\ 0.1B + C = 1 \\ 9A + 0.1C = 0 \end{cases} \Rightarrow A = -\frac{10}{901} \quad B = \frac{10}{901} \quad C = \frac{900}{901}$$

$$I(s) = 4\left(\frac{A}{s + 0.1} + \frac{Bs}{s^2 + 9} + \frac{C}{s^2 + 9}\right)$$

$$= 4\left(-\frac{10}{901}\frac{1}{s + 0.1} + \frac{10}{901}\frac{s}{s^2 + 9} + \frac{900}{901}\frac{1}{3}\frac{3}{s^2 + 9}\right)$$

$$I(t) = \frac{1}{901}\left(-40\mathcal{L}^{-1}\left\{\frac{1}{s + 0.1}\right\} + 40\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 9}\right\} + 1200\mathcal{L}^{-1}\left\{\frac{3}{s^2 + 9}\right\}\right)$$

$$= \frac{1}{901}\left(-40e^{-t/10} + 40\cos 3t + 1200\sin 3t\right)$$

Exercise

An inductor $(L = 1\ H)$ and a resistor $(R = 0.1\ \Omega)$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing current in the current at time t for the given $E(t) = 4\sin 2\pi t$

Solution

$$L\frac{dI}{dt} + RI = E(t)$$

$$\frac{dI}{dt} + 0.1I = 4\sin 2\pi t$$

$$\mathcal{L}\left(I' + 0.1I\right) = 4\mathcal{L}\left(\sin 2\pi t\right)$$

$$sI(s) - I(0) + 0.1I(s) = 4\frac{2\pi}{s^2 + 4\pi^2} \qquad I(0) = 0$$

$$(s + 0.1)I(s) = 8\pi\frac{1}{s^2 + 4\pi^2}$$

$$I(s) = 8\pi\frac{1}{(s + 0.1)\left(s^2 + 4\pi^2\right)}$$

$$\frac{1}{(s + 0.1)\left(s^2 + 4\pi^2\right)} = \frac{A}{s + 0.1} + \frac{Bs + C}{s^2 + 4\pi^2}$$

$$s = As^2 + 4\pi^2 A + Bs^2 + 0.1Bs + Cs + 0.1C$$

$$s = (A + B)s^2 + (0.1B + C)s + 4\pi^2 A + 0.1C$$

$$\begin{cases} A + B = 0 \\ 0.1B + C = 0 \\ 4\pi^2 A + 0.1C = 1 \end{cases} \Rightarrow A = \frac{100}{1 + 400\pi^2} \quad B = -\frac{100}{1 + 400\pi^2} \quad C = \frac{10}{1 + 400\pi^2}$$

$$I(s) = 8\pi\left(\frac{A}{s + 0.1} + \frac{Bs}{s^2 + 4\pi^2} + \frac{C}{s^2 + 4\pi^2}\right)$$

$$= 8\pi\left(\frac{100}{1 + 400\pi^2}\frac{1}{s + 0.1} - \frac{100}{1 + 400\pi^2}\frac{s}{s^2 + 4\pi^2} + \frac{10}{1 + 400\pi^2}\frac{1}{s^2 + 4\pi^2}\right)$$

$$I(t) = \frac{8}{1 + 400\pi^2}\left(100\pi\mathcal{L}^{-1}\left\{\frac{1}{s + 0.1}\right\} - 100\pi\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 4\pi^2}\right\} + 10\pi\frac{1}{2\pi}\mathcal{L}^{-1}\left\{\frac{2\pi}{s^2 + 4\pi^2}\right\}\right)$$

$$\boxed{= \frac{8}{1 + 400\pi^2}\left(100\pi e^{-t/10} - 100\pi\cos 2\pi t + 5\sin 2\pi t\right)}$$

Exercise

Solve the general initial value problem modeling the *RC* circuit

$$R\frac{dQ}{dt}+\frac{1}{C}Q=E, \quad Q(0)=0$$

Where E is a constant source of emf

Solution

$$\frac{dQ}{dt}+\frac{1}{RC}Q=\frac{E}{R}$$

$$e^{\int \frac{1}{RC}dt}=e^{t/RC}$$

$$\int \frac{E}{R}e^{t/RC}dt=\frac{E}{R}RCe^{t/RC}=ECe^{t/RC}$$

$$Q(t)=\frac{1}{e^{t/RC}}\left(ECe^{t/RC}-K\right)=EC-Ke^{-t/RC}$$

$$Q(t=0)=EC-K$$

$$0=EC-K \quad \Rightarrow \quad K=EC$$

$$\boxed{Q(t)=EC\left(1-e^{-t/RC}\right)}$$

OR

$$\mathcal{L}\left(\frac{dQ}{dt}+\frac{1}{RC}Q\right)=\mathcal{L}\left(\frac{E}{R}\right)$$

$$sQ(s)-Q(0)+\frac{1}{RC}Q(s)=\frac{E}{R}\frac{1}{s} \qquad Q(0)=0$$

$$\left(s+\frac{1}{RC}\right)Q(s)=\frac{E}{R}\frac{1}{s}$$

$$Q(s)=\frac{E}{R}\frac{1}{s\left(s+\frac{1}{RC}\right)}$$

$$\frac{1}{s\left(s+\frac{1}{RC}\right)}=\frac{A}{s}+\frac{B}{s+\frac{1}{RC}}=\frac{As+\frac{1}{RC}A+Bs}{s\left(s+\frac{1}{RC}\right)}$$

$$\begin{cases} A+B=0 \\ \frac{1}{RC}A=1 \end{cases} \rightarrow \quad A=RC \quad B=-RC$$

$$Q(t)=\frac{E}{R}\left(RC\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-RC\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{RC}}\right\}\right)$$

$$=\frac{E}{R}\left(RC-RCe^{-t/RC}\right)$$

$$=EC\left(1-e^{-t/RC}\right)$$

Exercise

Solve the general initial value problem modeling the *LR* circuit $L\frac{dI}{dt}+RI=E, \quad I(0)=I_0$

Where E is a constant source of emf

Solution

$$\frac{dI}{dt}+\frac{R}{L}I=\frac{E}{L}$$

$$e^{\int \frac{R}{L}dt}=e^{(R/L)t}$$

$$\int \frac{E}{L}e^{(R/L)t}dt=\frac{E}{L}\frac{L}{R}e^{(R/L)t}=\frac{E}{R}e^{(R/L)t}$$

$$I(t)=\frac{1}{e^{(R/L)t}}\left(\frac{E}{R}e^{(R/L)t}-K\right)=\frac{E}{R}-Ke^{-(R/L)t}$$

$$I(t=0)=\frac{E}{R}-K$$

$$I_0=\frac{E}{R}-K \quad \Rightarrow \quad K=\frac{E}{R}-I_0$$

$$I(t)=\frac{E}{R}-\left(\frac{E}{R}-I_0\right)e^{-Rt/L}$$

$$\boxed{I(t)=\frac{1}{R}\left(E+\left(RI_0-E\right)e^{-Rt/L}\right)}$$

OR

$$\mathcal{L}\left(\frac{dI}{dt}+\frac{R}{L}I\right)=\mathcal{L}\left(\frac{E}{L}\right)$$

$$sI(s)-I(0)+\frac{R}{L}I(s)=\frac{E}{L}\frac{1}{s} \qquad\qquad I(0)=I_0$$

$$\left(s+\frac{R}{L}\right)I(s)=\frac{E}{L}\frac{1}{s}+I_0$$

$$I(s)=\frac{E}{L}\frac{1}{s\left(s+\frac{R}{L}\right)}+I_0\frac{1}{s+\frac{R}{L}}$$

$$\frac{1}{s\left(s+\frac{R}{L}\right)}=\frac{A}{s}+\frac{B}{s+\frac{R}{L}}=\frac{As+\frac{R}{L}A+Bs}{s\left(s+\frac{R}{L}\right)}$$

$$\begin{cases}A+B=0\\ \frac{R}{L}A=1\end{cases} \quad\rightarrow\quad A=\frac{L}{R} \quad B=-\frac{L}{R}$$

$$I(t)=\frac{E}{L}\left(\frac{L}{R}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\frac{L}{R}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{R}{L}}\right\}\right)+I_0\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{R}{L}}\right\}$$

$$=\frac{E}{L}\left(\frac{L}{R}-\frac{L}{R}e^{-Rt/L}\right)+I_0e^{-Rt/L}$$

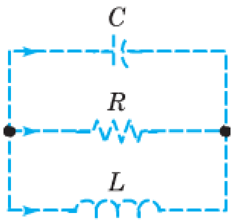
$$=\frac{E}{R}-\frac{E}{R}e^{-Rt/L}+I_0e^{-Rt/L}$$

$$= \frac{1}{R}\left(E - Ee^{-Rt/L} + RI_0 e^{-Rt/L}\right)$$

$$\boxed{= \frac{1}{R}\left(E + \left(RI_0 - E\right)e^{-Rt/L}\right)}$$

Exercise

Consider the circuit shown and let I_1, I_2, and I_3 be the currents through the capacitor, resistor, and inductor, respectively. Let V_1, V_2, and V_3 be the corresponding voltage drops. The arrows denote the arbitrary chosen directions in which currents and voltage drops will be taken to be positive.



a) Applying Kirchhoff's second law to the upper loop in the circuit, show that

$$V_1 - V_2 = 0 \quad and \quad V_2 - V_3 = 0$$

b) Applying Kirchhoff's first law to either node in the circuit, show that

$$I_1 + I_2 + I_3 = 0$$

c) Use the current-voltage relation through each element in the circuit to obtain the equations

$$CV_1' = I_1, \quad V_2 = RI_2, \quad LI_3' = V_3$$

d) Eliminate V_2, V_3, I_1 and I_2 to obtain

$$CV_1' = -I_3 - \frac{V_1}{R}, \quad LI_3' = V_1$$

Solution

a) Taking the clockwise loop around each paths, it is easy to see that voltage drops are given by

$$V_1 - V_2 = 0 \quad and \quad V_2 - V_3 = 0$$

b) Consider the right node. The current is given by $I_1 + I_2$. The current leaving the node is $-I_3$.

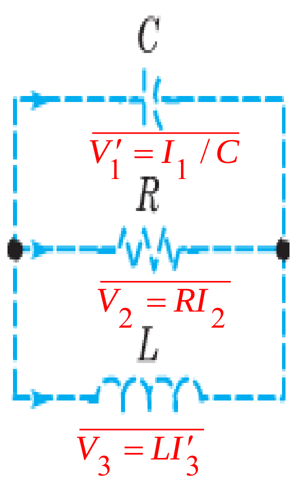
Hence the cursing through the node is $I_1 + I_2 - \left(-I_3\right)$.

Based on Kirchhoff's first law, $I_1 + I_2 + I_3 = 0$

c) In the capacitor $\quad CV_1' = I_1$

In the resistor $\quad V_2 = RI_2$

In the inductor $\quad LI_3' = V_3$



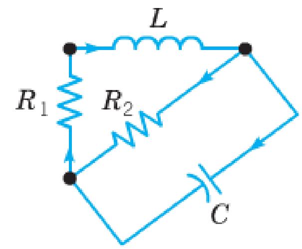
d) Based on part (a), $V_1 = V_2 = V_3$. Based on part (*b*),

$$CV_1' + \frac{1}{R}V_2 + I_3 = 0$$

It follows: $CV_1' = -I_3 - \frac{V_1}{R}, \quad LI_3' = V_1$

Exercise

Consider the circuit. Use the method outlined to show that the current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations.



$$L\frac{dI}{dt}=-R_1I-V,\quad C\frac{dV}{dt}=I-\frac{V}{R_2}$$

Solution

let I_1, I_2, I_3 and I_4 be the current through the resistors, inductor, and capacitor, respectively.

Assign V_1, V_2, V_3 and V_4 to be the corresponding voltage drops.

Based on Kirchhoff's second law, the net voltage drops, around each loop, satisfy

$$V_1+V_3+V_4=0,\quad V_1+V_3+V_2=0\quad and\quad V_4-V_2=0$$

Applying Kirchhoff's first law:

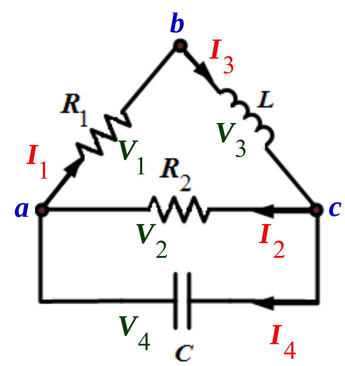
Node ***a***: $\quad I_1-\left(I_2+I_4\right)=0$

Node ***b***: $\quad I_1-I_3=0\ \rightarrow\ I_1=I_3$

Node ***c***: $\quad I_2+I_4-I_1=0\ \rightarrow\ I_2+I_4=I_1$

$$I_2+I_4=I_3\ \Rightarrow\ \boxed{I_2+I_4-I_3=0}$$

Using the current-voltage relations:



$$V_1=R_1I_1=R_1I_3\qquad V_2=R_2I_2$$

$$LI_3'=V_3\qquad CV_4'=I_4$$

$$V_1+V_3+V_4=0\ \Rightarrow\ \boxed{R_1I_3+LI_3'+V_4=0}$$

$$I_4=I_3-I_2\ \Rightarrow\ \boxed{CV_4'=I_3-\frac{V_2}{R_2}}$$

Let $I_3=I\quad and\quad V_4=V$

$$R_1I+LI'+V=0\ \Rightarrow\ \boxed{LI'=-R_1I-V}$$

$$CV_4'=I_3-\frac{V_2}{R_2}\ \Rightarrow\ \boxed{CV'=I-\frac{V}{R_2}}$$

Exercise

Consider an electric circuit containing a capacitor, resistor, and battery.
The charge $Q(t)$ on the capacitor satisfies the equation

$$R\frac{dQ}{dt}+\frac{Q}{C}=V$$

Where *R* is the resistance, *C* is the capacitance, and *V* is the constant voltage supplied by the battery.

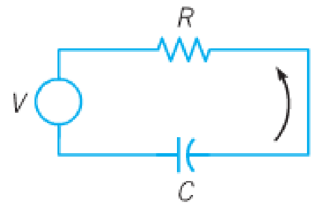
a) If $Q(0)=0$, find $Q(t)$ at time *t*.

b) Find the limiting value Q_L that $Q(t)$ approaches after a long time.

c) Suppose that $Q(t_1)=Q(t)$ and that at time $t=t_1$ the battery is removed and the circuit is closed again. Find $Q(t)$ for $t>t_1$.

Solution

a) $R\frac{dQ}{dt}+\frac{Q}{C}=V \quad \Rightarrow \quad R\frac{dQ}{dt}=V-\frac{Q}{C}$



$$R\frac{dQ}{dt}=\frac{CV-Q}{C}$$

$$\frac{dQ}{dt}=\frac{CV-Q}{RC}$$

$$\frac{dQ}{Q-CV}=-\frac{1}{RC}dt$$

$$\int\frac{dQ}{Q-CV}=-\int\frac{1}{RC}dt$$

$$\ln|Q-CV|=-\frac{t}{CR}+A$$

$$Q-CV=e^{-t/CR+A}$$

$$Q=De^{-t/CR}+CV$$

$$Q(0)=0$$

$$0=De^{-0}+CV$$

$$D=-CV$$

$$Q=-CVe^{-t/CR}+CV$$

$$Q(t)=CV\left(1-e^{-t/CR}\right)$$

b) $\lim\limits_{t\to\infty}Q(t)=CV\lim\limits_{t\to\infty}\left(1-e^{-t/CR}\right)=CV(1-0)=CV$

$$=CV(1-0)$$

$$\lim_{t\to\infty}e^{-t/CR}=\lim_{t\to\infty}\frac{1}{e^{t/CR}}\to\frac{1}{\infty}=0$$

$$\underline{=CV}|$$

c) In this case, $R\frac{dQ}{dt}+\frac{Q}{C}=0$, $Q(t_1)=CV$

$$\frac{dQ}{dt} = -\frac{Q}{CR}$$

$$\int \frac{dQ}{Q} = -\frac{1}{CR}\int dt$$

$$\ln|Q| = -\frac{t}{CR} + A$$

The solution is $Q = Ee^{-t/CR}$

So, $Q(t_1) = Ee^{-t_1/CR} = CV$

$$E = CVe^{t_1/CR}$$

$$Q(t) = CVe^{t_1/CR}e^{-t/CR}$$

$$= CVe^{(t_1-t)/CR}$$

$$\underline{= CVe^{-(t-t_1)/CR}} \quad for \quad t \geq t_1$$

Solutions   Section 3.7 – Basic Theory of Linear Systems

Exercise

Determine if the system is linear, and if so determine which is homogeneous? or inhomogeneous?

$$\begin{cases} x_1' = -2x_1 + x_1 x_2 \\ x_2' = -3x_1 - x_2 \end{cases}$$

Solution

The system is nonlinear because of the term $x_1 x_2$

Exercise

Determine if the system is linear, and if so determine which is homogeneous? or inhomogeneous?

$$\begin{cases} x_1' = -x_2 \\ x_2' = \sin x_1 \end{cases}$$

Solution

The system is nonlinear because of the term $\sin x_1$

Exercise

Determine if the system is linear, and if so determine which is homogeneous? or inhomogeneous?

$$\begin{cases} x_1' = x_1 + (\sin t) x_2 \\ x_2' = 2tx_1 - x_2 \end{cases}$$

Solution

The system is linear and homogeneous, because $f_1(t) = f_2(t) = 0$

$$x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + f_1(t)$$
$$x_2' = a_{21}(t)x_1 + a_{2n}(t)x_2 + f_2(t)$$

Exercise

Write the given system of equations in matrix-form then show that the given vector is a solution to the system

$$\begin{cases} x_1' = -3x_1 + x_2 \\ x_2' = -2x_1 \end{cases} \qquad v = \left(-e^{-2t} + e^{-t}, -e^{-2t} + 2e^{-t}\right)^T$$

Solution

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_1 = -e^{-2t} + e^{-t} \qquad x_2 = -e^{-2t} + 2e^{-t}$$

$$x_1' = -3x_1 + x_2$$

$$2e^{-2t} - e^{-t} = -3\left(-e^{-2t} + e^{-t}\right) + \left(-e^{-2t} + 2e^{-t}\right)$$

$$2e^{-2t} - e^{-t} = 3e^{-2t} - 3e^{-t} - e^{-2t} + 2e^{-t}$$

$$2e^{-2t} - e^{-t} = 2e^{-2t} - e^{-t}$$

$$x_2' = \left(-e^{-2t} + 2e^{-t}\right)'$$

$$= 2e^{-2t} - 2e^{-t}$$

$$= -2\left(-e^{-2t} + e^{-t}\right)$$

$$= -2x_1$$

Exercise

Write the given system of equations in matrix-form then show that the given vector is a solution to the system

$$\begin{cases} x_1' = -x_1 + 4x_2 \\ x_2' = \quad\quad\ 3x_2 \end{cases} \qquad v = \left(e^{3t} - e^{-t}, e^{3t}\right)^T$$

Solution

$$x_1 = e^{3t} - e^{-t} \qquad\qquad x_2 = e^{3t}$$

$$\begin{aligned} x_1' &= 3e^{3t} + e^{-t} \\ &= 4e^{3t} - e^{3t} + e^{-t} \\ &= -\left(e^{3t} - e^{-t}\right) + 4e^{3t} \\ &= -x_1 + 4x_2 \end{aligned}$$

$$x_2' = 3e^{3t} = 3x_2$$

Exercise

Verify by substitution that $x_1(t)$ and $x_2(t)$ are solutions of the given homogenous equation. Show also that the solutions $x_1(t)$ and $x_2(t)$ are linearly independent. Find the solution of the given homogeneous equation with the initial condition $x(0) = x_0$

$$x_1(t) = \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix}, \qquad x_2(t) = \begin{pmatrix} -e^{-2t} \\ e^{-2t} \end{pmatrix}$$

$$x' = \begin{pmatrix} -6 & -4 \\ 8 & 6 \end{pmatrix} x \qquad x(0) = \begin{pmatrix} -5 \\ 8 \end{pmatrix}$$

Solution

$$x_1'(t) = \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix}' = \begin{pmatrix} -2e^{2t} \\ 4e^{2t} \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} -6 & -4 \\ 8 & 6 \end{pmatrix} x_1 &= \begin{pmatrix} -6 & -4 \\ 8 & 6 \end{pmatrix} \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} -2e^{2t} \\ 4e^{2t} \end{pmatrix} \qquad \Rightarrow \text{Therefore, } x_1 \text{ is a solution.} \end{aligned}$$

$$x_2'(t) = \begin{pmatrix} -e^{-2t} \\ e^{-2t} \end{pmatrix}' = \begin{pmatrix} 2e^{2t} \\ -2e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} -6 & -4 \\ 8 & 6 \end{pmatrix} x_2 = \begin{pmatrix} -6 & -4 \\ 8 & 6 \end{pmatrix} \begin{pmatrix} -e^{2t} \\ e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} -2e^{2t} \\ 2e^{2t} \end{pmatrix} \quad \Rightarrow x_2 \text{ is also a solution.}$$

$$x_1(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \qquad x_2(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$x(t) = C_1 \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} -e^{-2t} \\ e^{-2t} \end{pmatrix}$$

$$x(0) = C_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -5 \\ 8 \end{pmatrix} = \begin{pmatrix} -C_1 - C_2 \\ 2C_1 + C_2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -C_1 - C_2 = -5 \\ 2C_1 + C_2 = 8 \end{cases} \rightarrow \qquad \boxed{C_1 = 3} \quad \boxed{C_2 = 2}$$

$$x(t) = 3\begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix} + 2\begin{pmatrix} -e^{-2t} \\ e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} -3e^{2t} - 2e^{-2t} \\ 6e^{2t} + 2e^{-2t} \end{pmatrix}$$

Exercise

Verify by substitution that $x_1(t)$ and $x_2(t)$ are solutions of the given homogenous equation. Show also that the solutions $x_1(t)$ and $x_2(t)$ are linearly independent. Find the solution of the given homogeneous equation with the initial condition $x(0) = x_0$

$$x_1(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}, \qquad x_2(t) = \begin{pmatrix} e^{2t}(t+2) \\ e^{2t}(t+1) \end{pmatrix}$$

$$x' = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} x \qquad\qquad x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solution

$$x_1'(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}' = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} x_1(t) = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix} = x_1'$$

$$x_2'(t) = \begin{pmatrix} e^{2t}(t+2) \\ e^{2t}(t+1) \end{pmatrix}'$$

$$= \begin{pmatrix} 2e^{2t}(t+2) + e^{2t} \\ 2e^{2t}(t+1) + e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2te^{2t} + 4e^{2t} + e^{2t} \\ 2te^{2t} + 2e^{2t} + e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2te^{2t} + 5e^{2t} \\ 2te^{2t} + 3e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} x_2(t) = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} te^{2t} + 2e^{2t} \\ te^{2t} + e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 2te^{2t} + 5e^{2t} \\ 2te^{2t} + 3e^{2t} \end{pmatrix} = x_2'$$

$$x_1(0)=\begin{pmatrix}1\\1\end{pmatrix}\qquad x_2(0)=\begin{pmatrix}2\\1\end{pmatrix}$$

The vectors are independent (x_1 is not a multiple of x_2), so the x_1 and x_2 are independent. Therefore, the general solution is:

$$x(t)=C_1\begin{pmatrix}e^{2t}\\e^{2t}\end{pmatrix}+C_2\begin{pmatrix}e^{2t}(t+2)\\e^{2t}(t+1)\end{pmatrix}$$

$$x(0)=C_1\begin{pmatrix}1\\1\end{pmatrix}+C_2\begin{pmatrix}2\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}C_1+2C_2\\C_1+C_2\end{pmatrix}\qquad \begin{cases}C_1+2C_2=0\\C_1+C_2=1\end{cases}\rightarrow\qquad \boxed{C_1=2}\qquad \boxed{C_2=-1}$$

$$\begin{aligned}x(t)&=2\begin{pmatrix}e^{2t}\\e^{2t}\end{pmatrix}-\begin{pmatrix}te^{2t}+2e^{2t}\\te^{2t}+e^{2t}\end{pmatrix}\\&=\begin{pmatrix}-te^{2t}\\-te^{2t}+e^{2t}\end{pmatrix}\end{aligned}$$

Exercise

Verify by substitution that $x_1(t)$ and $x_2(t)$ are solutions of the given homogenous equation. Show also that the solutions $x_1(t)$ and $x_2(t)$ are linearly independent. Find the solution of the given homogeneous equation with the initial condition $x(0)=x_0$

$$x_1(t)=\begin{pmatrix}\frac{1}{2}\cos t-\frac{1}{2}\sin t\\\cos t\end{pmatrix},\qquad x_2(t)=\begin{pmatrix}\frac{1}{2}\cos t+\frac{1}{2}\sin t\\\sin t\end{pmatrix}$$

$$x'=\begin{pmatrix}1&-1\\2&-1\end{pmatrix}x\qquad\qquad x(0)=\begin{pmatrix}1\\0\end{pmatrix}$$

Solution

$$\begin{aligned}x_1'(t)&=\begin{pmatrix}\frac{1}{2}\cos t-\frac{1}{2}\sin t\\\cos t\end{pmatrix}'\\&=\begin{pmatrix}-\frac{1}{2}\sin t-\frac{1}{2}\cos t\\-\sin t\end{pmatrix}\end{aligned}$$

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} x_1 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \cos t \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2}\cos t - \frac{1}{2}\sin t \\ -\sin t \end{pmatrix}$$

$$x_2'(t) = \begin{pmatrix} \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \sin t \end{pmatrix}'$$

$$= \begin{pmatrix} -\frac{1}{2}\sin t + \frac{1}{2}\cos t \\ \cos t \end{pmatrix}'$$

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} x_2 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \sin t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \cos t \end{pmatrix}$$

$$x_1(0) = \begin{pmatrix} \frac{1}{2}\cos(0) - \frac{1}{2}\sin(0) \\ \cos(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

$$x_2(0) = \begin{pmatrix} \frac{1}{2}\cos(0) + \frac{1}{2}\sin(0) \\ \sin(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

The vectors are independent (x_1 is not a multiple of x_2), so the x_1 and x_2 are independent. Therefore, the general solution is:

$$x(t) = C_1 \begin{pmatrix} \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \cos t \end{pmatrix} + C_2 \begin{pmatrix} \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \sin t \end{pmatrix}$$

$$x(0) = C_1 \begin{pmatrix} \frac{1}{2}\cos(0) - \frac{1}{2}\sin(0) \\ \cos(0) \end{pmatrix} + C_2 \begin{pmatrix} \frac{1}{2}\cos(0) + \frac{1}{2}\sin(0) \\ \sin(0) \end{pmatrix}$$

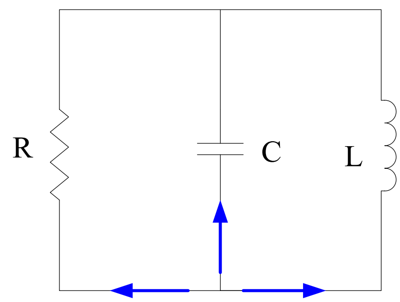
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}C_1 + \frac{1}{2}C_2 \\ C_1 \end{pmatrix} \Rightarrow \quad \boxed{C_1 = 0} \quad \boxed{C_2 = 2}$$

$$x(t) = 2\begin{pmatrix} \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \sin t \end{pmatrix} = \begin{pmatrix} \cos t + \sin t \\ 2\sin t \end{pmatrix}$$

Exercise

Consider the RLC parallel circuit below. Let V represent the voltage drop across the capacitor and I represent the current across the inductor.



Show that:

$$V' = -\frac{V}{RC} - \frac{1}{C}$$
$$I' = \frac{V}{L}$$

Solution

Using Kirchhoff's current law: $I_1 + I_2 + I_3 = 0$

In the RC loop: $V_1 - V_2 = 0$

In the LC loop: $V_2 - V_3 = 0$

$V_2 = RI_2, \quad CV_1' = I_1, \quad LI_3' = V_3$

Since the circuit elements are in parallel, therefore $V_1 = V_2 = V_3 = V$

$$LI_3' = V_1 \Rightarrow \boxed{I_3' = \frac{V_1}{L}}$$

$$CV_1' = I_1$$
$$= -I_2 - I_3 \qquad\qquad V_2 = RI_2$$
$$= -\frac{V_2}{R} - I_3 \qquad\qquad V_2 = V_1$$
$$= -\frac{V_1}{R} - I_3$$

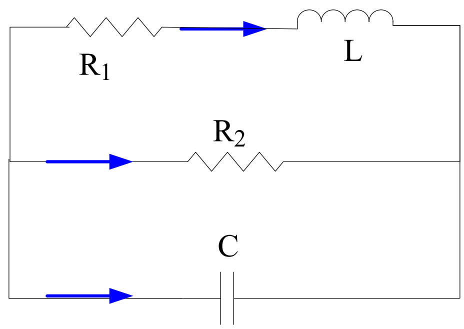
$$V_1' = -\frac{V_1}{CR} - \frac{I_3}{C}$$

Since $V_1 = V$ *and* $I_3 = I$

$$\Rightarrow \begin{cases} I' = \dfrac{V}{L} \\ V' = -\dfrac{V}{CR} - \dfrac{I}{C} \end{cases}$$

Exercise

Consider the RLC parallel circuit below. Let V represent the voltage drop across the capacitor and I represent the current across the inductor.



Show that:

$$CV' = -I - \frac{V}{R_2}$$

$$LI' = -R_1 I + V$$

Solution

Using Kirchhoff's current law: $I + I_2 + I_3 = 0$ **(1)**

In the $R_1 L R_2$ loop: $R_1 I + LI' - R_2 I_2 = 0$ **(2)**

In the $R_2 C$ loop: $R_2 I_2 - V = 0$ **(3)**

From (3): $V = R_2 I_2 \Rightarrow I_2 = \frac{V}{R_2}$

From (2): $LI' = -R_1 I + R_2 I_2$ $\qquad V = R_2 I_2$

$$= -R_1 I + V$$

From (1): $I_2 = -I - I_3$

$$\frac{V}{R_2} = -I - I_3$$

However, the voltage drop across the capacitor is: $V = \frac{q}{C}$

$$\Rightarrow CV = q$$

$$CV' = q' \qquad I_3 = q'$$

$$CV' = I_3$$

$$\frac{V}{R_2} = -I - CV'$$

$$CV' = -I - \frac{V}{R_2}$$

Exercise

Let I_1 and I_2 represent the current flow across the indicators L_1 and L_2 respectively. Show that the circuit is modeled by the system

$$\begin{cases} L_1 I_1' = -R_1 I_1 - R_1 I_2 + E \\ L_2 I_2' = -R_1 I_1 - \left(R_1 + R_2\right) I_2 + E \end{cases}$$

Solution

By Kirchhoff's second law:

$$I = I_1 + I_2$$

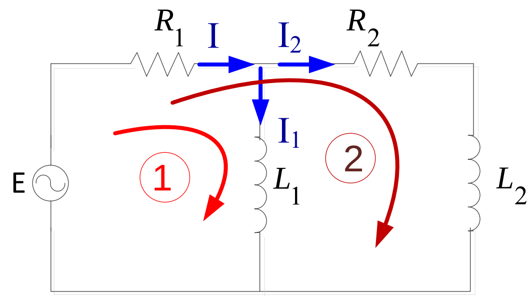
From loop **1**:

$$-E + R_1 I + L_1 I_1' = 0$$

$$L_1 I_1' = E - R_1 I$$

$$= E - R_1\left(I_1 + I_2\right)$$

$$\underline{= -R_1 I_1 - R_1 I_2 + E}$$



From loop **2**:

$$-E + R_1 I + R_2 I_2 + L_2 I_2' = 0$$

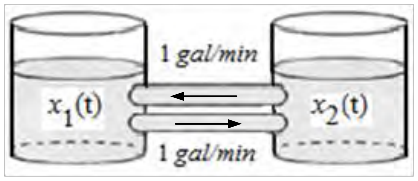
$$L_2 I_2' = -R_1 I - R_2 I_2 + E$$

$$= -R_1\left(I_1 + I_2\right) - R_2 I_2 + E$$

$$\underline{= -R_1 I_1 - R_1 I_2 - R_2 I_2 + E}$$

Exercise

Two tanks are connected by two pipes. Each tank contains 500 gallons of a salt solution. Through on pipe solution is pumped from the first tank to the second at 1 *gal/min*. Through the other pipe, solution is pumped at the same rate from the second to the first tank. Show the salt content in each tank varies with time.



Solution

$x_1(t)$ and $x_2(t)$ represent the salt content.

Rate out $= 1\ gal\ /\ min \times \dfrac{x_1}{500} lb\ /\ gal = \dfrac{x_1}{500} lb\ /\ min$

Rate in $= 1\ gal\ /\ min \times \dfrac{x_2}{500} lb\ /\ gal = \dfrac{x_2}{500} lb\ /\ min$

$$\frac{dx_1}{dt} = \text{Rate out - Rate in}$$

$$\frac{dx_1}{dt} = \frac{x_2}{500} - \frac{x_1}{500}$$

$$\frac{dx_1}{dt} = -\frac{x_1}{500} + \frac{x_2}{500}$$

And

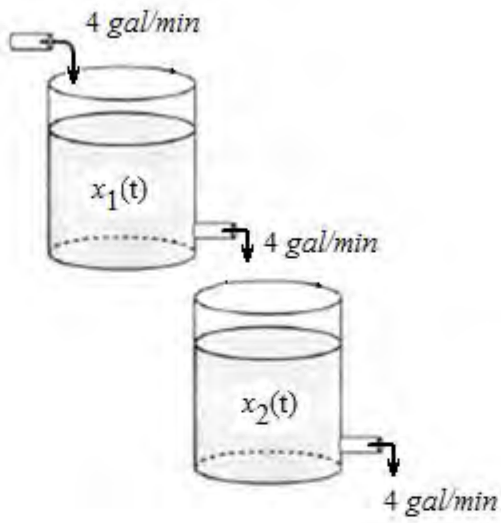
$$\frac{dx_2}{dt} = \frac{x_1}{500} - \frac{x_2}{500}$$

$$x' = Ax$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -\frac{1}{500} & \frac{1}{500} \\ \frac{1}{500} & -\frac{1}{500} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Exercise

Each tank contains 100 gallons of a salt solution. Pure water flows into the upper tank at a rate of 4 *gal/min*. Salt solution drains from the upper tank into the lower tank at a rate of 4 *gal/min*. Finally, salt solution drains from the lower tank at a rate of 4 *gal/min*, effectively keeping the volume of solution in each tank at a constant 100 *gal*. If the initial salt content of the upper and lower tanks is 10 and 20 pounds, respectively. Set up an initial value problem that models the amount of salt in each tank over time (do not solve).Write the model in matrix-vector form. Is the system homogeneous or inhomogeneous?



Solution

For the first tank: Rate out $= 4\ gal\ /\ min \times \dfrac{x_1}{100} lb\ /\ gal$

$$= \frac{x_1}{25} lb\ /\ min$$

$$\frac{dx_1}{dt} = \text{Rate out - Rate in}$$

$$\boxed{\frac{dx_1}{dt} = -\frac{x_1}{25}}$$

For the second tank: Rate out $= 4\ gal\ /\ min \times \dfrac{x_2}{100} lb\ /\ gal$

$$= \frac{x_2}{25} lb\ /\ min$$

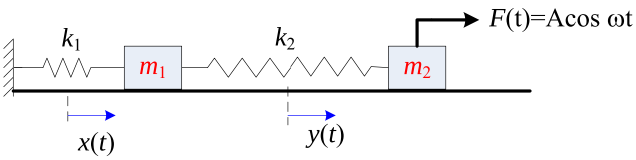
$$\frac{dx_2}{dt} = \text{Rate out - Rate in}$$

$$\boxed{\frac{dx_2}{dt} = \frac{x_1}{25} - \frac{x_2}{25}}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -\frac{1}{25} & 0 \\ \frac{1}{25} & -\frac{1}{25} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Exercise

Two masses on a frictionless tabletop are connected with a spring having spring constant k_2. The first mass is connected to a vertical support with a spring having spring constant k_1. The second mass is shaken harmonically via a force equaling $F = A\cos\omega t$. Let $x(t)$ and $y(t)$ measure the displacements of the masses m_1 and m_2, respectively, from their equilibrium positions as a function of time. If both masses start from rest at their equilibrium positions at time $t = 0$.



Set up an initial value problem that models the position of the masses over time (do not solve).Write the model in matrix-vector form. Is the system homogeneous or inhomogeneous?

Solution

By Newton's Law; the first mass:

$$m_1 x'' = -k_1 x + k_2 (y - x)$$

$$x'' = -\frac{k_1}{m_1} x + \frac{k_2}{m_1}(y - x)$$

The second mass:

$$m_2 y'' = -k_2 (y - x) + A\cos\omega t$$

$$y'' = -\frac{k_2}{m_2}(y - x) + \frac{A}{m_2}\cos\omega t$$

Let assume: $x_1 = x, \quad x_2 = x', \quad x_3 = y, \quad x_4 = y'$

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{k_1}{m_1} x_1 + \frac{k_2}{m_1}(x_3 - x_1) \\ x_3' = x_4 \\ x_4' = -\frac{k_2}{m_2}(x_3 - x_1) + \frac{A}{m_2}\cos\omega t \end{cases} \Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = -\left(\frac{k_1}{m_1} + \frac{k_2}{m_1}\right) x_1 + \frac{k_2}{m_1} x_3 \\ x_3' = x_4 \\ x_4' = \frac{k_2}{m_2} x_1 - \frac{k_2}{m_2} x_3 + \frac{A}{m_2}\cos\omega t \end{cases}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1 + k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{A}{m_2}\cos\omega t \end{pmatrix}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices. $A=\begin{pmatrix} 12 & 14 \\ -7 & -9 \end{pmatrix}$

Solution

$$\begin{aligned}\det(A-\lambda I)&=\begin{vmatrix} 12-\lambda & 14 \\ -7 & -9-\lambda \end{vmatrix}\\ &=(12-\lambda)(-9-\lambda)-(14)(-7)\\ &=-108-12\lambda+9\lambda+\lambda^2+98\\ &=\lambda^2-3\lambda-10=0\end{aligned}$$

Thus, the eigenvalues are: $\lambda_1=-2$ *and* $\lambda_2=5$

For $\lambda_1=-2$, we have

$$\left(A-\lambda_1 I\right)V_1=0$$

$$\begin{pmatrix} 12+2 & 14 \\ -7 & -9+2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 14 \\ -7 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 14x+14y=0 \\ -7x-7y=0 \end{cases} \Rightarrow x=-y$$

$$\Rightarrow \boxed{V_1=\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

For $\lambda_2=5$, we have

$$\left(A-\lambda_2 I\right)V_2=0$$

$$\begin{pmatrix} 7 & 14 \\ -7 & -14 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 7x+14y=0 \\ -7x-14y=0 \end{cases} \Rightarrow x=-2y$$

$$\Rightarrow \boxed{V_2=\begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices. $A=\begin{pmatrix}-4 & 1\\ -2 & 1\end{pmatrix}$

Solution

$$\det(A-\lambda I)=\begin{vmatrix}-4-\lambda & 1\\ -2 & 1-\lambda\end{vmatrix}$$
$$=(-4-\lambda)(1-\lambda)+2$$
$$=\lambda^2+3\lambda-2=0$$

Thus, the eigenvalues are: $\lambda_1=\frac{-3-\sqrt{17}}{2}$ *and* $\lambda_2=\frac{-3+\sqrt{17}}{2}$

For $\lambda_1=\frac{-3-\sqrt{17}}{2}$, we have: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix}-4-\frac{-3-\sqrt{17}}{2} & 1\\ -2 & 1-\frac{-3-\sqrt{17}}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}\frac{-5+\sqrt{17}}{2} & 1\\ -2 & \frac{5+\sqrt{17}}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{cases}\frac{-5+\sqrt{17}}{2}x+y=0\\ -2x+\frac{5+\sqrt{17}}{2}y=0\end{cases}\Rightarrow x=\left(\frac{5+\sqrt{17}}{4}\right)y$$

$$\Rightarrow V_1=\begin{pmatrix}\frac{5+\sqrt{17}}{4}\\ 1\end{pmatrix}$$

For $\lambda_2=\frac{-3+\sqrt{17}}{2}$, we have: $(A-\lambda_2 I)V_2=0$

$$\begin{pmatrix}-4-\frac{-3+\sqrt{17}}{2} & 1\\ -2 & 1-\frac{-3+\sqrt{17}}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}\frac{-5-\sqrt{17}}{2} & 1\\ -2 & \frac{5-\sqrt{17}}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{cases}\frac{-5-\sqrt{17}}{2}x+y=0\\ -2x+\frac{5-\sqrt{17}}{2}y=0\end{cases}\Rightarrow x=\left(\frac{5-\sqrt{17}}{4}\right)y$$

$$\Rightarrow V_2=\begin{pmatrix}\frac{5-\sqrt{17}}{4}\\ 1\end{pmatrix}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices. $A=\begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$

Solution

$$\det(A-\lambda I)=\begin{vmatrix} 5-\lambda & 3 \\ -6 & -4-\lambda \end{vmatrix}$$

$$=(-4-\lambda)(5-\lambda)+18$$

$$=\lambda^2-\lambda-2=0$$

Thus, the eigenvalues are: $\lambda_1=-1$ *and* $\lambda_2=2$

For $\lambda_1=-1$, we have: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} 6 & 3 \\ -6 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 6x+3y=0 \\ -6x-3y=0 \end{cases} \Rightarrow y=-2x$$

$$\Rightarrow \underline{V_1=\begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

For $\lambda_2=2 \quad \Rightarrow (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix} 3 & 3 \\ -6 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3x+3y=0 \\ -6x-6y=0 \end{cases} \Rightarrow y=-x$$

$$\Rightarrow \underline{V_2=\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices.

$$A = \begin{pmatrix} -2 & 3 \\ 0 & -5 \end{pmatrix}$$

Solution

$$\det(A-\lambda I) = \begin{vmatrix} -2-\lambda & 3 \\ 0 & -5-\lambda \end{vmatrix}$$

$$= (-2-\lambda)(-5-\lambda) - 0$$

$$= (2+\lambda)(5+\lambda) = 0$$

Thus, the eigenvalues are: $\lambda_1 = -5$ *and* $\lambda_2 = -2$

For $\lambda_1 = -5$, we have: $(A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3x + 3y = 0 \Rightarrow y = -x$$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

For $\lambda_2 = -2 \Rightarrow (A-\lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 3 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3y = 0 \\ -3y = 0 \end{cases} \Rightarrow y = 0$$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices.

$$A = \begin{pmatrix} 6 & 10 \\ -5 & -9 \end{pmatrix}$$

Solution

$$\det(A-\lambda I) = \begin{vmatrix} 6-\lambda & 10 \\ -5 & -9-\lambda \end{vmatrix}$$

$$= (6-\lambda)(-9-\lambda) + 50$$

$$= -54 + 3\lambda + \lambda^2 + 50$$
$$= \lambda^2 + 3\lambda - 4 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -4$ *and* $\lambda_2 = 1$

For $\lambda_1 = -4$, we have: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 10 & 10 \\ -5 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 10x + 10y = 0 \\ -5x - 5y = 0 \end{cases} \Rightarrow y = -x$$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

For $\lambda_2 = 1 \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 5 & 10 \\ -5 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 5x + 10y = 0 \\ -5x - 10y = 0 \end{cases} \Rightarrow x = -2y$$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices.

$$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix}$$
$$= (3-\lambda)(-1-\lambda) - 0$$
$$= \lambda^2 - 2\lambda - 3$$

The characteristic equation: $\lambda^2 - 2\lambda - 3$

$$\lambda^2 - 2\lambda - 3 = 0$$

The eigenvalues are $\lambda_1 = -1$ *and* $\lambda_2 = 3$

$$\lambda_1 = -1 \rightarrow (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 4 & 0 \\ 8 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} 4x = 0 \\ 8x = 0 \end{cases} \Rightarrow x = 0$$

Therefore the eigenvector $\underline{V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

$$\lambda_2 = 3 \quad \to \quad (A - \lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} 0 & 0 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} 0 = 0 \\ 8x - 4y = 0 \end{cases} \Rightarrow 8x = 4y \to \boxed{2x = y}$$

Therefore the eigenvector $\underline{V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices.

$$A = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 10 - \lambda & -9 \\ 4 & -2 - \lambda \end{vmatrix}$$

$$= (10 - \lambda)(-2 - \lambda) + 36$$

$$= \lambda^2 - 8\lambda + 16$$

$\Rightarrow$ The characteristic equation: $\lambda^2 - 8\lambda + 16$

$$\lambda^2 - 8\lambda + 16 = 0$$

$\Rightarrow$ The eigenvalues are $\lambda_{1,2} = 4$

$$\lambda_1 = 4 \quad \to \quad (A - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} 6x - 9y = 0 \\ 4x - 6y = 0 \end{cases} \Rightarrow \boxed{2x = 3y}$$

Therefore the eigenvector $\underline{V_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}}$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices.

$$A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

Solution

$$\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix}$$

$$= (4-\lambda)(1-\lambda)(1-\lambda) + 2(1-\lambda)$$

$$= (1-\lambda)\left[(4-\lambda)(1-\lambda) + 2\right]$$

$$= (1-\lambda)\left(\lambda^2 - 5\lambda + 6\right)$$

$\Rightarrow$ The characteristic equation: $-\lambda^3 + 6\lambda^2 - 11\lambda + 6$

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

The eigenvalues are $\boxed{\lambda = 1,\ 2,\ 3}$

$$\lambda_1 = 1 \quad \rightarrow \quad \begin{pmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} 3x_1 + x_3 = 0 \\ \quad x_1 = 0 \end{cases} \Rightarrow x_1 = x_3 = 0$$

Therefore the eigenvector $V_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\lambda_2 = 2 \quad \rightarrow \quad \begin{pmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} 2x_1 + x_3 = 0 \\ -2x_1 - x_2 = 0 \\ -2x_1 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = -2x_1 \\ x_2 = -2x_1 \end{cases}$$

Therefore the eigenvector $\boxed{V_2 = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}}$

$$\lambda_3 = 3 \quad \rightarrow \quad \begin{pmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} x_1 + x_3 = 0 \\ -2x_1 - 2x_2 = 0 \\ -2x_1 - 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = -x_1 \\ x_2 = -x_1 \end{cases}$$

Therefore the eigenvector $\boxed{V_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{pmatrix}$$

Solution

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 4 & 3-\lambda & 2 \\ -8 & -4 & -3-\lambda \end{vmatrix}$$

$$= (1-\lambda)(3-\lambda)(-3-\lambda) + 8(1-\lambda)$$

$$= -9 + 9\lambda + \lambda^2 - \lambda^3 + 8 - 8\lambda$$

$$= -\lambda^3 + \lambda^2 + \lambda - 1 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1$ *and* $\lambda_{2,3} = 1$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 4 & 2 & 2 \\ -8 & -4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 0=0 \\ 4x+2y+2z=0 \\ -8x-4y-4z=0 \end{cases} \Rightarrow \begin{cases} \\ 2x=-y-z \\ 2x=-y-z \end{cases} \rightarrow \begin{matrix} If \ \boxed{x=0} \\ \boxed{y=-z} \end{matrix}$$

$$\Rightarrow \underline{\left| V_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right|}$$

For $\lambda_{2,3} = -1 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 2 & 0 & 0 \\ 4 & 4 & 2 \\ -8 & -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x=0 \\ 4x+4y+2z=0 \\ -8x-4y-2z=0 \end{cases} \Rightarrow \begin{cases} \boxed{x=0} \\ 4y=-2z \rightarrow \boxed{z=-2y} \\ 4y=-2z \end{cases}$$

$$\Rightarrow \underline{\left| V_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right|}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices.

$$A = \begin{pmatrix} -1 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & -12 & 2 \end{pmatrix}$$

Solution

$$\left|A - \lambda I\right| = \begin{vmatrix} -1-\lambda & -4 & -2 \\ 0 & 1-\lambda & 1 \\ -6 & -12 & 2-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(1-\lambda)(2-\lambda) + 24 - 12(1-\lambda) + 12(-1-\lambda)$$

$$= -\lambda^3 + 2\lambda^2 + \lambda - 2 + 24 - 12 + 12\lambda - 12 - 12\lambda$$

$$= -\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -1 \quad \lambda_2 = 1 \quad and \quad \lambda_3 = 2$

For $\lambda_1 = -1 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 0 & -4 & -2 \\ 0 & 2 & 1 \\ -6 & -12 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -4y - 2z = 0 \\ 2y + z = 0 \\ -6x - 12y + 3z = 0 \end{cases} \Rightarrow \begin{cases} -4y = 2z \\ 2y = -z \\ -6x = 12y - 3z \end{cases} \rightarrow \begin{array}{l} \boxed{y = -\frac{1}{2}z} \\ x = \frac{-9z}{-6} = \frac{3}{2}z \end{array}$$

$$\Rightarrow V_1 = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -2 & -4 & -2 \\ 0 & 0 & 1 \\ -6 & -12 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2x - 4y - 2z = 0 \\ \boxed{z = 0} \\ -6x - 12y + 2z = 0 \end{cases} \Rightarrow \begin{cases} -2x - 4y = 0 \\ \\ -6x - 12y = 0 \end{cases} \rightarrow \boxed{x = -2y}$$

$$\Rightarrow V_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_3 = 2 \quad \Rightarrow (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -3 & -4 & -2 \\ 0 & -1 & 1 \\ -6 & -12 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -3x - 4y - 2z = 0 \\ -y + z = 0 \\ -6x - 12y = 0 \end{cases} \Rightarrow \boxed{y = z} \rightarrow \begin{cases} -3x = 6z \\ \\ -6x = 12z \end{cases} \rightarrow \begin{cases} \\ \\ \boxed{x = -2z} \end{cases}$$

$$\Rightarrow V_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices.

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$$

Solution

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(4-\lambda)(-1-\lambda) - 4 - 8 + 4(4-\lambda) + 4(3-\lambda) + 2\lambda + 2$$

$$= -\lambda^3 + 6\lambda^2 - 5\lambda - 12 - 12 + 16 - 4\lambda + 12 - 4\lambda + 2\lambda + 2$$

$$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 1 \quad \lambda_2 = 2 \quad and \quad \lambda_3 = 3$

For $\lambda_1 = 1 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2x + 2y + 2z = 0 \\ x + 3y + z = 0 \\ -2x - 4y - 2z = 0 \end{cases} \Rightarrow (1)\ \&\ (3) \to \boxed{y = 0} \to \begin{cases} 2x + 2z = 0 \\ x + z = 0 \end{cases} \Rightarrow \boxed{x = -z}$$

$$\Rightarrow V_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x + 2y + 2z = 0 \\ x + 2y + z = 0 \\ -2x - 4y - 3z = 0 \end{cases} \Rightarrow \begin{cases} 2x + 4y + 2z = 0 \\ -2x - 4y - 3z = 0 \end{cases} \to \boxed{z = 0} \to \boxed{x = -2y}$$

$$\Rightarrow V_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_3 = 3 \quad \Rightarrow \left(A - \lambda_3 I\right)V_3 = 0$

$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ -2 & -4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2y + 2z = 0 \\ x + y + z = 0 \\ -2x - 4y - 4z = 0 \end{cases} \Rightarrow \boxed{y = -z} \rightarrow \boxed{x = 0}$$

$$\Rightarrow V_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices.

$$A = \begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}$$

Solution

$$\left|A - \lambda I\right| = \begin{vmatrix} -6-\lambda & 4 & 4 \\ -4 & 2-\lambda & 4 \\ -10 & 8 & 4-\lambda \end{vmatrix}$$

$$= (-6-\lambda)(2-\lambda)(4-\lambda) - 160 - 128 + 40(2-\lambda) + 32(6+\lambda) + 16(4-\lambda)$$

$$= -\lambda^3 + 4\lambda = 0$$

Thus, the eigenvalues are: $\lambda_1 = 0 \quad \lambda_2 = -2 \quad and \quad \lambda_3 = 2$

For $\lambda_1 = 0 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -6x + 4y + 4z = 0 \\ -4x + 2y + 4z = 0 \\ -10x + 8y + 4z = 0 \end{cases} \Rightarrow 2x - 2y = 0 \rightarrow \boxed{x = y} \rightarrow \begin{matrix} -2x + 4z = 0 \\ -2x + 4z = 0 \end{matrix} \Rightarrow \boxed{x = 2z = y}$$

$$\Rightarrow \boxed{V_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = -2 \quad \Rightarrow \left(A - \lambda_2 I\right) V_2 = 0$

$$\begin{pmatrix} -4 & 4 & 4 \\ -4 & 4 & 4 \\ -10 & 8 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -4x + 4y + 4z = 0 \\ -4x + 4y + 4z = 0 \\ -10x + 8y + 6z = 0 \end{cases} \Rightarrow \begin{matrix} -x + y + z = 0 \\ -5x + 4y + 3z = 0 \end{matrix} \Rightarrow \begin{matrix} 4x - 4y = 4z \\ -5x + 4y = -3z \end{matrix} \rightarrow \begin{matrix} \boxed{x = -z} \\ \boxed{y = -2z} \end{matrix}$$

$$\Rightarrow \boxed{V_2 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}}$$

For $\lambda_3 = 2 \quad \Rightarrow \left(A - \lambda_3 I\right) V_3 = 0$

$$\begin{pmatrix} -8 & 4 & 4 \\ -4 & 0 & 4 \\ -10 & 8 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -8x + 4y + 4z = 0 \\ -4x + 4z = 0 \\ -10x + 8y + 2z = 0 \end{cases} \Rightarrow \boxed{x = z} \Rightarrow \begin{matrix} 4y - 4z = 0 \\ \\ 8y - 8z = 0 \end{matrix} \Rightarrow \boxed{y = z}$$

$$\Rightarrow \boxed{V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

Exercise

Find the eigenvalues and the eigenvectors for each of the matrices.

$$A = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution

$$\det(A-\lambda I) = \begin{pmatrix} -\lambda & 0 & 2 & 0 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -2-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & 0 & 2 \\ 1 & -\lambda & 1 \\ 0 & 1 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left(\lambda^2(-2-\lambda)+2+\lambda\right)$$

$$= (1-\lambda)\left(-\lambda^3-2\lambda^2+\lambda+2\right)$$

$$= \lambda^4+\lambda^3-3\lambda^2-\lambda+2$$

$\Rightarrow$ The characteristic equation: $\lambda^4+\lambda^3-3\lambda^2-\lambda+2$

$\lambda^4+\lambda^3-3\lambda^2-\lambda+2=0 \quad \Rightarrow \quad$ The eigenvalues are $\boxed{\lambda=-2,-1,1,1}$

$$\lambda_1=-2 \quad \to \quad \begin{pmatrix} 2 & 0 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} 2x_1+2x_3=0 \\ x_1+2x_2+x_3=0 \\ x_2=0 \\ x_4=0 \end{cases}$$

$$\to \begin{cases} x_1=-x_3 \\ x_1=-x_3 \\ x_2=0 \\ x_4=0 \end{cases}$$

Therefore the eigenvector $V_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$\lambda_2=-1 \quad \to \quad \begin{pmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} x_1+2x_3=0 \\ x_1+x_2+x_3=0 \\ x_2-x_3=0 \\ x_4=0 \end{cases}$$

$$\rightarrow \begin{cases} x_1 = -2x_3 \\ x_1 = -x_2 - x_3 \\ x_2 = x_3 \end{cases}$$

Therefore the eigenvector $V_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

$$\lambda_3 = 1 \quad \rightarrow \quad \begin{pmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \begin{cases} -x_1 + 2x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_2 - 3x_3 = 0 \end{cases}$$

$$\rightarrow \begin{cases} x_1 = 2x_3 \\ x_1 = x_2 - x_3 \\ x_2 = 3x_3 \end{cases}$$

Therefore the eigenvector $V_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_4 = 1 \quad \rightarrow$ Therefore the eigenvector $V_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Exercise

Find a fundamental set of solutions for the system $x' = Ax$, where A is the given matrices.

$$A = \begin{pmatrix} 2 & 0 \\ -4 & -2 \end{pmatrix}$$

Solution

$$\left|A - \lambda I\right| = \begin{vmatrix} 2-\lambda & 0 \\ -4 & -2-\lambda \end{vmatrix}$$

$$= (2-\lambda)(-2-\lambda) - 0$$

$$= \lambda^2 - 4 \; = 0$$

Thus, the eigenvalues are: $\lambda_1 = -2 \;\; and \;\; \lambda_2 = 2$

For $\lambda_1 = -2 \quad \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} 4 & 0 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 4x = 0 \\ -4x = 0 \end{cases} \Rightarrow \boxed{x = 0} \quad \boxed{y = 1}$$

The eigenvector is: $V_1 = (0,\ 1)^T$

The solution is: $x_1(t) = e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

For $\lambda_2 = 2 \quad \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} 0 & 0 \\ -4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 0 = 0 \\ -4x - 4y = 0 \end{cases} \Rightarrow \boxed{x = -y}$$

The eigenvector is: $V_2 = (-1,\ 1)^T$

The solution is: $\boxed{x_2(t) = e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

Since the vectors V_1 and V_2 are independent, the solutions $x_1(t)$ and $x_2(t)$ are independent for all t and for a fundamental set of solutions.

Exercise

Find a fundamental set of solutions for the system $x' = Ax$, where A is the given matrices.

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -6 \\ -2 & 3 & 4 \end{pmatrix}$$

Solution

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 0 & 0 \\ 2 & -5-\lambda & -6 \\ -2 & 3 & 4-\lambda \end{vmatrix}$$

$$= \lambda^3 + 2\lambda^2 - \lambda - 2 = 0$$

Thus, the eigenvalues are: $\lambda_1 = -2 \quad \lambda_2 = -1 \quad$ *and* $\quad \lambda_3 = 1$

For $\lambda_1 = -2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & -6 \\ -2 & 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \boxed{x = 0} \\ 2x - 3y - 6z = 0 \Rightarrow 3y = -6z \rightarrow \boxed{y = -2z} \\ -2x + 3y + 6z = 0 \end{cases}$$

$$\Rightarrow V_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -1 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & -4 & -6 \\ -2 & 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 0 = 0 \\ 2x - 4y - 6z = 0 \Rightarrow -4y = 6z \rightarrow \boxed{y = -z} \rightarrow 2x = 4y + 6z = 2z \Rightarrow \boxed{x = z} \\ -2x + 3y + 5z = 0 \qquad 3y = -5z \end{cases}$$

$$\Rightarrow V_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 1 \quad \Rightarrow (A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -2 & 0 & 0 \\ 2 & -6 & -6 \\ -2 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2x = 0 & \boxed{x=0} \\ 2x-6y-6z=0 \Rightarrow -6y=6z \to \boxed{y=-z} \\ -2x+3y+3z=0 \quad 3y=-3z \end{cases}$$

$$\Rightarrow V_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

The vectors are given by: $V = \begin{pmatrix} 0 & 1 & 0 \\ -2 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

$$\det(V) = \begin{vmatrix} 0 & 1 & 0 \\ -2 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$$

The solutions are independent for all t and form a fundamental set of solutions.

Exercise

Find the general solution of the system $y' = Ay$

$$A = \begin{pmatrix} -1 & 6 \\ -3 & 8 \end{pmatrix} \qquad y(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

<u>***Solution***</u>

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -1-\lambda & 6 \\ -3 & 8-\lambda \end{vmatrix} \\ &= (-1-\lambda)(8-\lambda) + 18 \\ &= -8 + 7\lambda + \lambda^2 + 18 \\ &= \lambda^2 + 7\lambda + 10 \end{aligned}$$

Thus, the eigenvalues are: $\lambda_1 = 2$ *and* $\lambda_2 = 5$

For $\lambda_1 = 2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -3x + 6y = 0 \\ -3x + 6y = 0 \end{cases} \Rightarrow \boxed{x = 2y}$$

The eigenvector is: $V_1 = (2,\ 1)^T$

The solution is: $y_1(t) = e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

For $\lambda_2 = 5 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -6x + 6y = 0 \\ -3x + 3y = 0 \end{cases} \Rightarrow \boxed{x = y}$$

The eigenvector is: $V_2 = (1,\ 1)^T$

The solution is: $y_2(t) = e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The general solution is given by: $y(t) = C_1 y_1(t) + C_2 y_2(t)$

$$y(t) = C_1 e^{2t}\begin{pmatrix}2\\1\end{pmatrix} + C_2 e^{5t}\begin{pmatrix}1\\1\end{pmatrix}$$

$$y(0) = C_1\begin{pmatrix}2\\1\end{pmatrix} + C_2\begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}2C_1 + C_2\\C_1 + C_2\end{pmatrix}$$

$$\begin{cases}2C_1 + C_2 = 1\\C_1 + C_2 = -2\end{cases} \rightarrow \boxed{C_1 = 3} \qquad \boxed{C_2 = -5}$$

The particular solution is:

$$\boxed{y(t) = 3e^{2t}\begin{pmatrix}2\\1\end{pmatrix} - 5e^{5t}\begin{pmatrix}1\\1\end{pmatrix}}$$

Exercise

Find the general solution of the system $y' = Ay$

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \qquad y(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Solution

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} \\ &= (1-\lambda)(4-\lambda) + 2 \\ &= 4 - 5\lambda + \lambda^2 + 2 \\ &= \lambda^2 - 5\lambda + 6 \end{aligned}$$

Thus, the eigenvalues are: $\lambda_1 = 2$ *and* $\lambda_2 = 3$

For $\lambda_1 = 2 \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x + 2y = 0 \\ -x + 2y = 0 \end{cases} \Rightarrow \boxed{x = 2y}$$

The eigenvector is: $V_1 = (2,\ 1)^T$

The solution is: $y_1(t) = e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

For $\lambda_2 = 3 \quad \Rightarrow (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2x + 2y = 0 \\ -x + y = 0 \end{cases} \Rightarrow \boxed{x = y}$$

The eigenvector is: $V_2 = (1,\ 1)^T$

The solution is: $y_2(t) = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The general solution is given by: $y(t) = C_1 y_1(t) + C_2 y_2(t)$

$$y(t) = C_1 e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$y(0) = C_1\begin{pmatrix}2\\1\end{pmatrix} + C_2\begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}2C_1 + C_2\\C_1 + C_2\end{pmatrix}$$

$$\begin{cases}2C_1 + C_2 = 3\\C_1 + C_2 = 2\end{cases} \rightarrow \boxed{C_1 = 1} \qquad \boxed{C_2 = 1}$$

The particular solution is:

$$\boxed{y(t) = e^{2t}\begin{pmatrix}2\\1\end{pmatrix} + e^{3t}\begin{pmatrix}1\\1\end{pmatrix}}$$

Exercise

Find the general solution of the system $y' = Ay$

$$A = \begin{pmatrix} -4 & -8 \\ 4 & 4 \end{pmatrix} \qquad y(0) = \begin{pmatrix} 0 & 2 \end{pmatrix}^T$$

Solution

$$|A - \lambda I| = \begin{vmatrix} -4-\lambda & -8 \\ 4 & 4-\lambda \end{vmatrix}$$

$$= (-4-\lambda)(4-\lambda) + 32$$

$$= -16 + \lambda^2 + 32$$

$$= \lambda^2 + 16 \underline{= 0}$$

$$\lambda^2 = -16 \Rightarrow \lambda = \pm 4i$$

Thus, the eigenvalues are: $\lambda_1 = -4i$ *and* $\lambda_2 = 4i$

For $\lambda = 4i \quad \Rightarrow (A - \lambda I)V = 0$

$$\begin{pmatrix} -4-4i & -8 \\ 4 & 4-4i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} (-4-4i)x - 8y = 0 \\ 4x + (4-4i)y = 0 \end{cases} \Rightarrow \textit{divide by } 4 \begin{cases} -x - ix - 2y = 0 \\ x + y - iy = 0 \end{cases}$$

$$-ix - y - iy = 0$$

$$ix = (-1-i)y \Rightarrow \underline{x = \frac{-1-i}{i} y \frac{i}{i}} = \frac{-i+1}{-1} y = \underline{(-1+i)y}$$

The eigenvector is: $V = (-1+i,\ 1)^T$

$$z(t) = e^{4it}\begin{pmatrix} -1+i \\ 1 \end{pmatrix}$$

$$= (\cos 4t + i\sin 4t)\left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$$

$$= \cos 4t\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \sin 4t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\left(\sin 4t\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \cos 4t\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$= \begin{pmatrix} -\cos 4t - \sin 4t \\ \cos 4t \end{pmatrix} + i\begin{pmatrix} -\sin 4t + \cos 4t \\ \sin 4t \end{pmatrix}$$

$$y_1(t) = \begin{pmatrix} -\cos 4t - \sin 4t \\ \cos 4t \end{pmatrix} \ \& \ y_2(t) = \begin{pmatrix} -\sin 4t + \cos 4t \\ \sin 4t \end{pmatrix}$$

The general solution is given by: $y(t) = C_1 y_1(t) + C_2 y_2(t)$

$$y(t) = C_1 \begin{pmatrix} -\cos 4t - \sin 4t \\ \cos 4t \end{pmatrix} + C_2 \begin{pmatrix} \cos 4t - \sin 4t \\ \sin 4t \end{pmatrix}$$

$$y(0) = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -C_1 + C_2 \\ C_1 \end{pmatrix}$$

$$\Rightarrow \boxed{C_1 = 2} \quad \boxed{C_2 = 2}$$

$$y(t) = 2 \begin{pmatrix} -\cos 4t - \sin 4t \\ \cos 4t \end{pmatrix} + 2 \begin{pmatrix} \cos 4t - \sin 4t \\ \sin 4t \end{pmatrix}$$

$$= \begin{pmatrix} -4\sin 4t \\ 2\cos 4t + 2\sin 4t \end{pmatrix}$$

Exercise

Find the general solution of the system $y' = Ay$

$$A = \begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix} \qquad y(0) = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$$

Solution

$$\begin{aligned}
|A-\lambda I| &= \begin{vmatrix} -1-\lambda & -2 \\ 4 & 3-\lambda \end{vmatrix} \\
&= (-1-\lambda)(3-\lambda)+8 \\
&= -3-2\lambda+\lambda^2+8 \\
&= \lambda^2-2\lambda+5 \underline{=0}
\end{aligned}$$

$$\Rightarrow \lambda = 1 \pm 2i$$

Thus, the eigenvalues are: $\lambda_1 = 1+2i$ *and* $\lambda_2 = 1-2i$

For $\lambda = 1+2i \quad \Rightarrow (A-\lambda I)V = 0$

$$\begin{pmatrix} -2-2i & -2 \\ 4 & 2-2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} (-2-2i)x-2y=0 \\ 4x+(2-2i)y=0 \end{cases} \Rightarrow \textit{divide by } 2 \begin{cases} -x-ix-y=0 \\ 2x+y-iy=0 \end{cases}$$

$$x-ix-iy=0$$

$$(1-i)x = iy \Rightarrow \frac{i}{i}\frac{1-i}{i}x = y$$

$$\Rightarrow y = -(i+1)x$$

The eigenvector is: $V = \begin{pmatrix} 1, & -1-i \end{pmatrix}^T$

$$\begin{aligned}
z(t) &= e^{(1+2i)t}\begin{pmatrix} 1 \\ -1-i \end{pmatrix} \\
&= e^t e^{2it}\begin{pmatrix} 1 \\ -1-i \end{pmatrix} \\
&= e^t\left(\cos 2t + i\sin 2t\right)\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) \\
&= e^t\left[\cos 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\cos 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix} + i\sin 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] \\
&= e^t\left[\cos 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] + ie^t\left[\left(\cos 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)\right] \\
&= e^t\begin{pmatrix} \cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix} + ie^t\begin{pmatrix} \sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix}
\end{aligned}$$

$$y_1(t) = e^t \begin{pmatrix} \cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix} \qquad \& \qquad y_2(t) = e^t \begin{pmatrix} \sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix}$$

Form a fundamental equation:

$$y(t) = C_1 e^t \begin{pmatrix} \cos 2t \\ -\cos 2t + \sin 2t \end{pmatrix} + C_2 e^t \begin{pmatrix} \sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix}$$

$$y(0) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 \\ -C_1 - C_2 \end{pmatrix} \quad \Rightarrow \quad \boxed{C_1 = 0} \quad \boxed{C_2 = -1}$$

$$\boxed{y(t) = -e^t \begin{pmatrix} \sin 2t \\ -\cos 2t - \sin 2t \end{pmatrix}}$$

Exercise

Find the general solution of the system $y' = Ay$

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \qquad y(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Solution

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} \\ &= (3-\lambda)(1-\lambda)+1 \\ &= 3-4\lambda+\lambda^2+1 \\ &= \lambda^2-4\lambda+4 = 0 \end{aligned}$$

Thus, the eigenvalues are: $\lambda_1 = \lambda_2 = 2$

For $\lambda = 2 \quad \Rightarrow (A-2I)V_1 = 0$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x - y = 0 \\ x - y = 0 \end{cases} \Rightarrow \boxed{x = y}$$

The eigenvector is: $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The solution is: $y_1(t) = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For the second eigenvector $V_2 \quad \Rightarrow (A-2I)V_2 = V_1$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{cases} x - y = 1 \\ x - y = 1 \end{cases} \Rightarrow \text{if } y = 0 \Rightarrow \boxed{x = 1}$$

The eigenvector is: $V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The solution is: $y_2(t) = e^{2t}\left(V_2 + tV_1\right)$

$$= e^{2t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

Therefore, the final solution can be written as: $y(t) = C_1 y_1(t) + C_2 y_2(t)$

$$y(t) = C_1 e^{2t}\begin{pmatrix}1\\1\end{pmatrix} + C_2 e^{2t}\left(\begin{pmatrix}1\\0\end{pmatrix} + t\begin{pmatrix}1\\1\end{pmatrix}\right)$$

$$= e^{2t}\left(C_1\begin{pmatrix}1\\1\end{pmatrix} + C_2\begin{pmatrix}1\\0\end{pmatrix} + C_2 t\begin{pmatrix}1\\1\end{pmatrix}\right)$$

$$y(0) = C_1\begin{pmatrix}1\\1\end{pmatrix} + C_2\begin{pmatrix}1\\0\end{pmatrix}$$

$$\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}C_1 + C_2\\C_1\end{pmatrix}$$

$$\to \begin{cases} C_1 + C_2 = 2 \\ \boxed{C_1 = -1} \end{cases} \Rightarrow \underline{C_2 = 2 - C_1 = 3}$$

$$y(t) = e^{2t}\left(-\begin{pmatrix}1\\1\end{pmatrix} + 3\begin{pmatrix}1\\0\end{pmatrix} + 3t\begin{pmatrix}1\\1\end{pmatrix}\right)$$

$$\underline{= e^{2t}\begin{pmatrix}2+3t\\-1+3t\end{pmatrix}}$$

Exercise

Find the general solution of the system $y' = Ay$

$$A = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \qquad y(0) = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

Solution

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} \\ &= (-3-\lambda)(-1-\lambda) + 1 \\ &= 3 + 4\lambda + \lambda^2 + 1 \\ &= \lambda^2 + 4\lambda + 4 \ \underline{= 0} \end{aligned}$$

Thus, the eigenvalues are: $\lambda_1 = \lambda_2 = -2$

For $\lambda = -2 \quad \Rightarrow (A + 2I)V_1 = 0$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x + y = 0 \\ -x + y = 0 \end{cases} \Rightarrow \boxed{x = y}$$

The eigenvector is: $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The solution is: $y_1(t) = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For the second eigenvector $V_2 \quad \Rightarrow (A + 2I)V_2 = V_1$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{cases} -x + y = 1 \\ -x + y = 1 \end{cases} \Rightarrow \textit{if} \ \ y = 0 \Rightarrow \boxed{x = -1}$$

The eigenvector is: $V_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

The solution is: $y_2(t) = e^{-2t}\left(V_2 + tV_1\right)$

$$= e^{-2t}\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$

Therefore, the final solution can be written as: $y(t) = C_1 y_1(t) + C_2 y_2(t)$

$$y(t) = C_1 e^{-2t}\begin{pmatrix}1\\1\end{pmatrix} + C_2 e^{-2t}\left(\begin{pmatrix}-1\\0\end{pmatrix} + t\begin{pmatrix}1\\1\end{pmatrix}\right)$$

$$= e^{-2t}\left(C_1\begin{pmatrix}1\\1\end{pmatrix} + C_2\begin{pmatrix}-1\\0\end{pmatrix} + C_2 t\begin{pmatrix}1\\1\end{pmatrix}\right)$$

$$y(0) = C_1\begin{pmatrix}1\\1\end{pmatrix} + C_2\begin{pmatrix}-1\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\-3\end{pmatrix} = \begin{pmatrix}C_1 - C_2\\C_1\end{pmatrix}$$

$$\rightarrow \begin{cases} C_1 - C_2 = 0 \\ \boxed{C_1 = -3} \end{cases} \Rightarrow \underline{C_2 = C_1 = -3}$$

$$y(t) = e^{-2t}\left(-3\begin{pmatrix}1\\1\end{pmatrix} - 3\begin{pmatrix}-1\\0\end{pmatrix} - 3t\begin{pmatrix}1\\1\end{pmatrix}\right)$$

$$\underline{= e^{-2t}\begin{pmatrix}-3t\\-3-3t\end{pmatrix}}$$

Exercise

Find the general solution of the system $y' = Ay$

$$A = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} \qquad y(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Solution

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{vmatrix}$$

$$= (2-\lambda)(6-\lambda) + 4$$

$$= 12 - 2\lambda - 6\lambda + \lambda^2 + 4$$

$$= \lambda^2 - 8\lambda + 16 = 0$$

Thus, the eigenvalues are: $\lambda_1 = \lambda_2 = 4$

For $\lambda = 4 \quad \Rightarrow (A - 4I)V_1 = 0$

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2x + 4y = 0 \\ -x + 2y = 0 \end{cases} \Rightarrow \boxed{x = 2y}$$

The eigenvector is: $V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The solution is: $y_1(t) = e^{4t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

For the second eigenvector $V_2 \quad \Rightarrow (A - 4I)V_2 = V_1$

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{cases} -2x + 4y = 2 \\ -x + 2y = 1 \end{cases} \Rightarrow \text{if } y = 0 \Rightarrow \boxed{x = -1}$$

The eigenvector is: $V_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

The solution is: $y_2(t) = e^{4t}\left(V_2 + tV_1\right)$

$$= e^{4t}\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$$

Therefore, the final solution can be written as: $y(t) = C_1 y_1(t) + C_2 y_2(t)$

$$y(t) = C_1 e^{-2t}\begin{pmatrix}1\\1\end{pmatrix} + C_2 e^{-2t}\left(\begin{pmatrix}-1\\0\end{pmatrix} + t\begin{pmatrix}1\\1\end{pmatrix}\right)$$

$$= e^{4t}\left(C_1\begin{pmatrix}2\\1\end{pmatrix} + C_2\begin{pmatrix}-1\\0\end{pmatrix} + C_2 t\begin{pmatrix}2\\1\end{pmatrix}\right)$$

$$y(0) = C_1\begin{pmatrix}2\\1\end{pmatrix} + C_2\begin{pmatrix}-1\\0\end{pmatrix}$$

$$\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}2C_1 - C_2\\ C_1\end{pmatrix}$$

$$\to \begin{cases} 2C_1 - C_2 = 3 \\ \boxed{C_1 = 1} \end{cases} \Rightarrow \boxed{C_2 = 2C_1 - 3 = -1}$$

$$y(t) = e^{4t}\left(\begin{pmatrix}2\\1\end{pmatrix} - \begin{pmatrix}-1\\0\end{pmatrix} - t\begin{pmatrix}2\\1\end{pmatrix}\right)$$

$$\boxed{= e^{4t}\begin{pmatrix}3-2t\\ 1-t\end{pmatrix}}$$

Exercise

Find the general solution of the system $A = \begin{pmatrix} -8 & -10 \\ 5 & 7 \end{pmatrix}$ $y(0) = \begin{pmatrix} 3 & 1 \end{pmatrix}^T$

Solution

$$y' = Ay \quad |A - \lambda I| = \begin{vmatrix} -8-\lambda & -10 \\ 5 & 7-\lambda \end{vmatrix}$$

$$= (-8-\lambda)(7-\lambda) + 50$$

$$= -56 + 8\lambda - 7\lambda + \lambda^2 + 50$$

$$= \lambda^2 + \lambda - 6 \boxed{= 0}$$

Thus, the eigenvalues are: $\lambda_1 = -3$ *and* $\lambda_2 = 2$

For $\lambda_1 = -3 \quad \Rightarrow (A + 3I)V_1 = 0$

$$\begin{pmatrix} -5 & -10 \\ 5 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -5x - 10y = 0 \\ 5x + 10y = 0 \end{cases} \Rightarrow 5x = -10y \to \boxed{x = -2y}$$

The eigenvector is: $V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

The solution is: $y_1(t) = e^{-3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

For $\lambda_2 = 2 \quad \Rightarrow (A - 2I)V_2 = 0$

$$\begin{pmatrix} -10 & -10 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -10x - 10y = 0 \\ 5x + 5y = 0 \end{cases} \Rightarrow 5x = -5y \to \boxed{x = -y}$$

The eigenvector is: $V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

The solution is: $y_2(t) = e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Therefore, the final solution can be written as: $y(t) = C_1 y_1(t) + C_2 y_2(t)$

$$y(t) = C_1 e^{-3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$y(0) = C_1\begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2C_1 - C_2 \\ C_1 + C_2 \end{pmatrix}$$

$$\begin{cases} -2C_1 - C_2 = 3 \\ C_1 + C_2 = 1 \end{cases} \rightarrow \boxed{C_1 = -4} \Rightarrow \boxed{C_2 = 1 - C_1 = 5}$$

$$-C_1 = 4$$

$$y(t) = -4e^{-3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + 5e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 8e^{-3t} - 5e^{2t} \\ -4e^{-3t} + 5e^{2t} \end{pmatrix}}$$

Exercise

Find the real and imaginary part of $z(t) = e^{2it}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$

Solution

$$z(t) = (\cos 2t + i\sin 2t)\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2t + i\sin 2t \\ \cos 2t + i\sin 2t + i\cos 2t - \sin 2t \end{pmatrix}$$

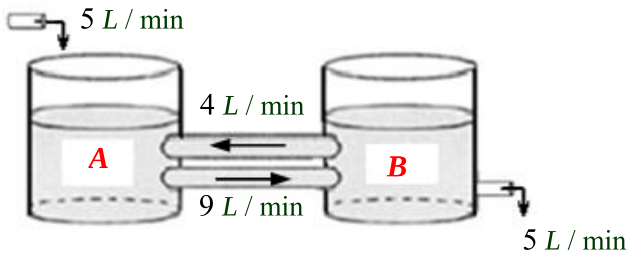
$$= \begin{pmatrix} \cos 2t + i\sin 2t \\ \cos 2t - \sin 2t + i(\sin 2t + \cos 2t) \end{pmatrix}$$

The real part is: $\boxed{(\cos 2t,\ \cos 2t - \sin 2t)^T}$

The imaginary part is: $\boxed{(\sin 2t,\ \sin 2t + \cos 2t)^T}$

Exercise

Two tanks, each containing 360 liters of a salt solution. Pure water pours into tank *A* at a rate of 5 *L/min*. There are two pipes connecting tank *A* to tank *B*. The first pumps salt solution from tank *B* into tank *A* at a rate of 4 *L/min*. The second pumps salt solution from tank *A* into tank *B* at a rate of 9 *L/min*. Finally, there is a drain on tank *B* from which salt solution drains at a rate of 5 *L/min*. Thus, each tank maintains a constant volume of 360 *liters* of salt solution. Initially, there are 60 *kg* of salt present in tank *A*, but tank *B* contains pure water.



a) Set up, in matrix-vector form, an initial value problem that models the salt content in each tank over time.

b) Find the eigenvalues and eigenvectors of the coefficient matrix in part (*a*), then find the general solution in vector form. Find the solution that satisfies the initial conditions posed in part (*a*).

c) Plot each component of your solution in part (*b*) over a period of four time constants $\left\lfloor 0,\ 4T_c \right\rfloor$.

What is the eventual salt content in each tank? Give both a physical and a mathematical reason for your answer.

Solution

a) Let $x_A(t)$ and $x_A(t)$ represent the number of pounds of salt as a function of time.

Tank *A*:

$$\text{Rate in } = (5+4)\ \frac{L}{\min}\frac{x_A}{360}\frac{kg}{L} = \frac{x_A}{40}\, kg\,/\,min$$

$$\text{Rate out } = 4\ \frac{L}{\min}\frac{x_B}{360}\frac{kg}{L} = \frac{x_B}{90}\ kg\,/\,min$$

$$\frac{dx_A}{dt} = \text{Rate in} - \text{Rate out} = -\frac{x_A}{40} + \frac{x_B}{90}$$

Tank *B*:

$$\text{Rate in } = 9\ \frac{L}{\min}\frac{x_A}{360}\frac{kg}{L} = \frac{x_A}{40}\, kg\,/\,min$$

$$\text{Rate out } = (5+4)\ \frac{L}{\min}\frac{x_B}{360}\frac{kg}{L} = \frac{x_B}{40}\ kg\,/\,min$$

$$\frac{dx_B}{dt} = \text{Rate in} - \text{Rate out} = \frac{x_A}{40} - \frac{x_B}{40}$$

$$\begin{cases} x'_A = -\dfrac{x_A}{40} + \dfrac{x_B}{90} \\ x'_B = \dfrac{x_A}{40} - \dfrac{x_B}{40} \end{cases}$$

The system is: $x' = Ax(t)$

$$\begin{pmatrix} x_A \\ x_B \end{pmatrix}' = \begin{pmatrix} -\frac{1}{40} & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{40} \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix}$$

With initial 60 kg of salt in tank A; $\begin{pmatrix} x_A(0) \\ x_B(0) \end{pmatrix} = \begin{pmatrix} 60 \\ 0 \end{pmatrix}$

b) $\det(A - \lambda I) = \begin{vmatrix} -\frac{1}{40} - \lambda & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{40} - \lambda \end{vmatrix}$

$$= \left(-\frac{1}{40} - \lambda\right)\left(-\frac{1}{40} - \lambda\right) - \frac{1}{90}\frac{1}{40}$$

$$= \frac{1}{1600} + \frac{1}{20}\lambda + \lambda^2 - \frac{1}{3600}$$

$$= \lambda^2 + \frac{1}{20}\lambda + \frac{5}{14400}$$

$\therefore$ The eigenvalues are: $\boxed{\lambda_1 = -\frac{1}{120}}$ *and* $\boxed{\lambda_2 = -\frac{1}{24}}$

For $\lambda_1 = -\frac{1}{120} \quad \Rightarrow \quad (A - \lambda_1 I)V_1 = 0$, we have

$$\begin{pmatrix} -\frac{1}{40} + \frac{1}{120} & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{40} + \frac{1}{120} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{60} & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{60} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{60} & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{60} \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad x - \frac{2}{3}y = 0$$

$$V_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \to \quad x_1(t) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-t/120}$$

For $\lambda_2 = -\frac{1}{24} \quad \Rightarrow \quad (A - \lambda_2 I)V_2 = 0$, we have

$$\begin{pmatrix} -\frac{1}{40}+\frac{1}{24} & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{40}+\frac{1}{24} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{60} & \frac{1}{90} \\ \frac{1}{40} & \frac{1}{60} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{60} & \frac{1}{90} \\ \frac{1}{40} & \frac{1}{60} \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & \frac{2}{3} \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad x+\frac{2}{3}y=0$$

$$V_2=\begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad \to \quad \underline{x_2(t)=\begin{pmatrix} -2 \\ 3 \end{pmatrix}e^{-t/24}}\Big|$$

$$x(t)=C_1\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{-t/120}+C_2\begin{pmatrix} -2 \\ 3 \end{pmatrix}e^{-t/24}$$

Given $\begin{pmatrix} x_A(0) \\ x_B(0) \end{pmatrix}=\begin{pmatrix} 60 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 60 \\ 0 \end{pmatrix}=C_1\begin{pmatrix} 2 \\ 3 \end{pmatrix}+C_2\begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 60 \\ 0 \end{pmatrix}=\begin{pmatrix} 2C_1-2C_2 \\ 3C_1+3C_2 \end{pmatrix}$$

$$\begin{cases} 2C_1-2C_2=60 \\ 3C_1+3C_2=0 \end{cases} \quad \to \quad C_1=15 \quad C_2=-15$$

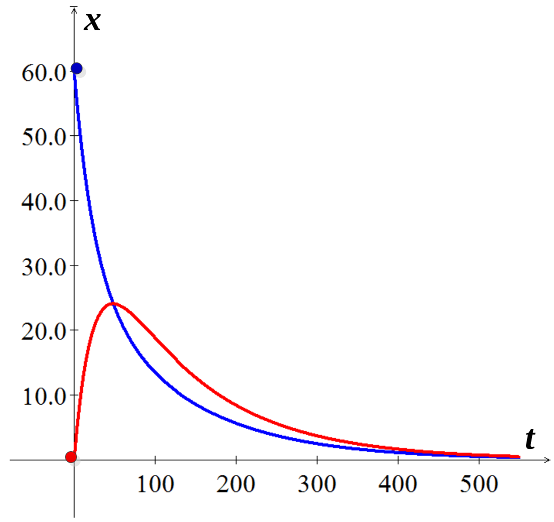
$$\underline{x(t)=15\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{-t/120}-15\begin{pmatrix} -2 \\ 3 \end{pmatrix}e^{-t/24}}\Big|$$

c) $x(t)=\begin{pmatrix} 30 & 30 \\ 45 & -45 \end{pmatrix}\begin{pmatrix} e^{-t/120} \\ e^{-t/24} \end{pmatrix}$

$$\begin{pmatrix} x_A \\ x_B \end{pmatrix}=\begin{pmatrix} 30e^{-t/120}+30e^{-t/24} \\ 45e^{-t/120}-45e^{-t/24} \end{pmatrix}$$

The time constant on $e^{-t/120}$ is $T_c=120$

The time constant on $e^{-t/24}$ is $T_c=24$



If we choose the larger of these two time constants over a period of four time constants $\left\lfloor 0,\ 4T_c \right\rfloor = [0,\ 480]$.

This allows enough time to show both components decaying to zero.

Physically, if we keep pouring pure water into the tank *B*, eventually the system will purge itself of all salt content.

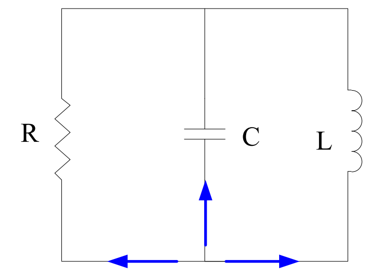
Mathematically:

$$\begin{cases} 30e^{-t/120} + 30e^{-t/24} \xrightarrow[t\to\infty]{} 0 \\ 45e^{-t/120} - 45e^{-t/24} \xrightarrow[t\to\infty]{} 0 \end{cases}$$

Exercise

Consider the *RLC* parallel circuit below. Let *V* represent the voltage drop across the capacitor and *I* represent the current across the inductor that satisfied the system.

$$\begin{cases} V' = -\dfrac{V}{RC} - \dfrac{1}{C} \\ I' = \dfrac{V}{L} \end{cases}$$



Suppose that the resistance is $R = \frac{1}{2}\Omega$, the capacitor is $C = 1\ farad$, and the inductance is $L = \frac{1}{2}\ henry$. If the initial voltage across the capacitor is $V(0) = 10\ volts$ and there is no initial current across the inductor, solve the system to determine the voltage and current as a function of time. Plot the voltage and current as a function of time. Assume current flows in the directions indicated.

Solution

$$\begin{cases} V' = -\dfrac{V}{\frac{1}{2}(1)} - \dfrac{1}{1} \\ I' = \dfrac{V}{\frac{1}{2}} \end{cases} \quad \to \quad \begin{cases} V' = -2V - 1 \\ I' = 2V \end{cases}$$

$$\begin{pmatrix} V \\ I \end{pmatrix}' = \begin{pmatrix} -2 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & -1 \\ 2 & -\lambda \end{vmatrix}$$

$$= (-2-\lambda)(-\lambda) + 2$$

$$= \lambda^2 + 2\lambda + 2 = 0$$

$\therefore$ The eigenvalues are: $\lambda = -1 \pm i$

For $\lambda_1 = -1 + i \quad \Rightarrow (A - \lambda_1 I)V = 0$

$$\begin{pmatrix} -1-i & -1 \\ 2 & 1-i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad \begin{cases} -x - y - ix = 0 \\ 2x + y - iy = 0 \end{cases} \to 2x = (-1+i)y$$

$$V = \begin{pmatrix} -1+i \\ 2 \end{pmatrix} \quad \to \quad z(t) = \begin{pmatrix} -1+i \\ 2 \end{pmatrix} e^{(-1+i)t}$$

$$z(t) = \begin{pmatrix} -1+i \\ 2 \end{pmatrix} e^{-t} e^{it}$$

$$= e^{-t}(\cos t + i \sin t)\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

$$= e^{t}\left[\cos t\begin{pmatrix}-1\\2\end{pmatrix}-\sin t\begin{pmatrix}1\\0\end{pmatrix}\right]+ie^{t}\left[\left(\sin t\begin{pmatrix}-1\\2\end{pmatrix}+\cos\begin{pmatrix}1\\0\end{pmatrix}\right)\right]$$

$$= e^{-t}\begin{pmatrix}-\cos t-\sin t\\2\cos t\end{pmatrix}+ie^{-t}\begin{pmatrix}\cos t-\sin t\\2\sin t\end{pmatrix}$$

$$\boxed{x(t)=C_1e^{-t}\begin{pmatrix}-\cos t-\sin t\\2\cos t\end{pmatrix}+C_2e^{-t}\begin{pmatrix}\cos t-\sin t\\2\sin t\end{pmatrix}}$$

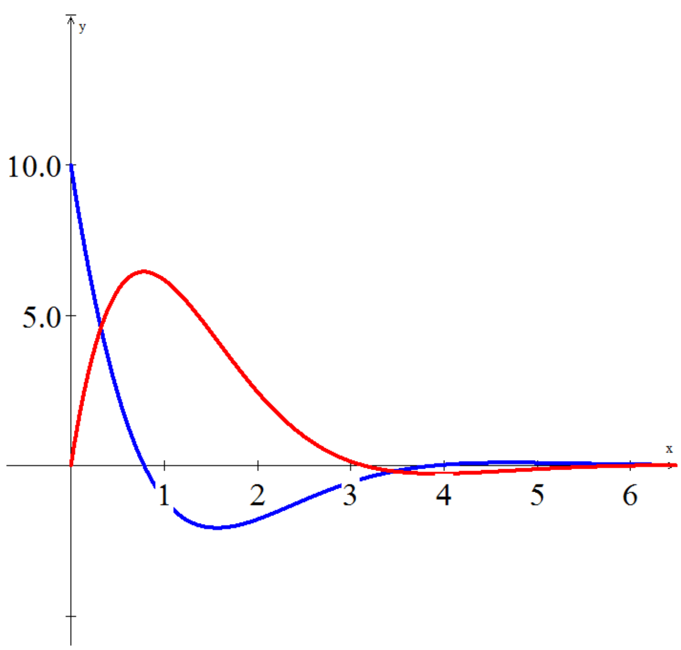
$$x(0)=(1)\begin{pmatrix}-1-0\\2(1)\end{pmatrix}+i(1)\begin{pmatrix}1-0\\0\end{pmatrix}$$

$$\begin{pmatrix}10\\0\end{pmatrix}=C_1\begin{pmatrix}-1\\2\end{pmatrix}+C_2\begin{pmatrix}1\\0\end{pmatrix}$$

$$\begin{pmatrix}10\\0\end{pmatrix}=\begin{pmatrix}-1C_1+C_2\\2C_1\end{pmatrix}\quad\Rightarrow\quad C_1=0\quad C_2=10$$

$$\boxed{x(t)=10e^{-t}\begin{pmatrix}\cos t-\sin t\\2\sin t\end{pmatrix}}$$

$$\begin{pmatrix}V(t)\\I(t)\end{pmatrix}=x(t)=\boxed{\begin{pmatrix}10e^{-t}(\cos t-\sin t)\\20e^{-t}\sin t\end{pmatrix}}$$



Exercise

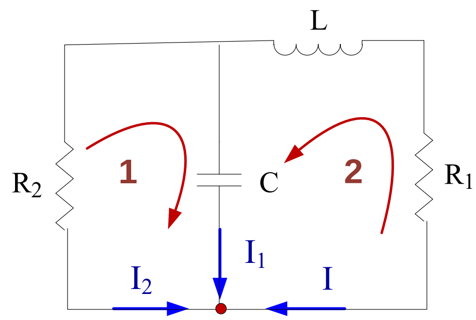
Show that the voltage V across the capacitor and the current I through the inductor satisfy the system

$$\begin{cases} I' = -\frac{R_1}{L} I + \frac{1}{L} V \\ V' = -\frac{1}{C} I - \frac{1}{R_2 C} V \end{cases}$$

Suppose that the capacitance is $C = 1$ *farad*, the inductance is $L = 1$ *henry*, the leftmost resistor has resistance $R_2 = 1\ \Omega$, and the rightmost resistor has resistance $R_1 = 5\ \Omega$. If the initial voltage across the capacitor is 12 volts and the initial current through the inductor is zero, determine the voltage V across the capacitor and the current I through the inductor as functions of time. Plot the voltage and current as functions of time. Assume current flows in the directions indicated.

Solution

The current coming into the node at a must equal the current coming out,



$$I + I_1 + I_2 = 0$$

$$-R_2 I_2 + V = 0$$

$$-R_2\left(-I - I_1\right) + V = 0$$

$$R_2 I + R_2 I_1 = -V$$

The voltage across the capacitor follows the law

$V = \frac{1}{C} q_1$, where q_1 is the charge in the capacitor.

$$CV = q_1$$

$$(CV)' = (q_1)'$$

$$CV' = q_1' = I_1$$

$$R_2 I + R_2 I_1 = -V \quad \rightarrow \quad R_2 I + R_2 (CV') = -V$$

$$R_2 CV' = -V - R_2 I$$

$$\boxed{V' = -\frac{1}{R_2 C} V - \frac{1}{C} I}$$

Loop **2**:

$$-V + LI' + R_1 I = 0$$

$$LI' = V - R_1 I$$

$$\boxed{I' = \frac{1}{L} V - \frac{R_1}{L} I}$$

$$\begin{pmatrix} V \\ I \end{pmatrix}' = \begin{pmatrix} -\frac{1}{R_2 C} & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R_1}{L} \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{(1)(1)} & -\frac{1}{1} \\ \frac{1}{1} & -\frac{5}{1} \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -1 \\ 1 & -5-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(-5-\lambda) + 1$$

$$= \lambda^2 + 6\lambda + 6 = 0$$

$\therefore$ The eigenvalues are: $\lambda = -3 \pm \sqrt{3}$

For $\lambda_1 = -3 + \sqrt{3} \quad \Rightarrow (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2-\sqrt{3} & -1 \\ 1 & -2-\sqrt{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{cases} (2-\sqrt{3})x - y = 0 \\ x - (2+\sqrt{3})y = 0 \end{cases}$$

$$V_1 = \begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} \quad \rightarrow \quad \underline{x_1(t) = \begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} e^{(-3+\sqrt{3})t}}$$

For $\lambda_2 = -3 - \sqrt{3}$

$$V_2 = \begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix} \quad \rightarrow \quad \underline{x_2(t) = \begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix} e^{(-3-\sqrt{3})t}}$$

$$\mathbf{x}(t) = C_1 e^{(-3+\sqrt{3})t} \begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} + C_2 e^{(-3-\sqrt{3})t} \begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix}$$

Given: $\quad V_0 = 12\ V \quad I_0 = 0A$

$$\begin{pmatrix} 12 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix}$$

$$\begin{cases} (2+\sqrt{3})C_1 + (2-\sqrt{3})C_2 = 12 \\ C_1 + C_2 = 0 \end{cases} \quad \Rightarrow \quad C_1 = 2\sqrt{3},\ C_2 = -2\sqrt{3}$$

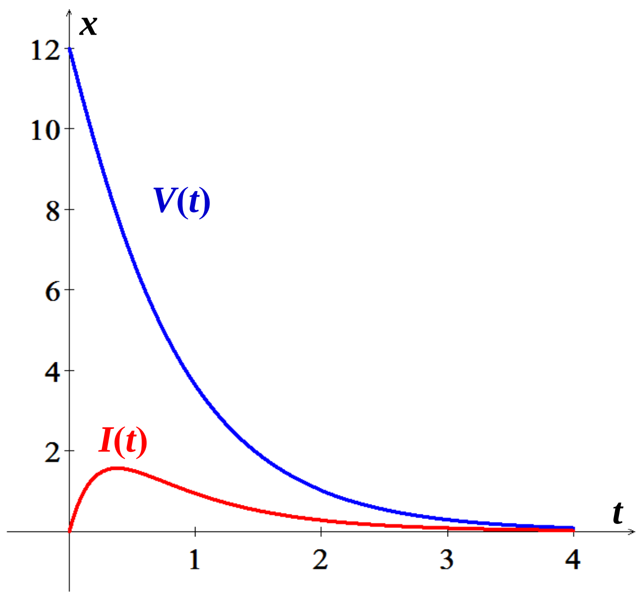
$$\mathbf{x}(t) = 2\sqrt{3} e^{(-3+\sqrt{3})t} \begin{pmatrix} 2+\sqrt{3} \\ 1 \end{pmatrix} - 2\sqrt{3} e^{(-3-\sqrt{3})t} \begin{pmatrix} 2-\sqrt{3} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} \left(4\sqrt{3}+6\right)e^{\left(-3+\sqrt{3}\right)t} - \left(4\sqrt{3}-6\right)e^{\left(-3-\sqrt{3}\right)t} \\ 2\sqrt{3}e^{\left(-3+\sqrt{3}\right)t} - 2\sqrt{3}e^{\left(-3-\sqrt{3}\right)t} \end{pmatrix}$$

Which leads to the solutions

$$V(t) = \left(4\sqrt{3}+6\right)e^{\left(-3+\sqrt{3}\right)t} - \left(4\sqrt{3}-6\right)e^{\left(-3-\sqrt{3}\right)t}$$

$$I(t) = 2\sqrt{3}e^{\left(-3+\sqrt{3}\right)t} - 2\sqrt{3}e^{\left(-3-\sqrt{3}\right)t}$$



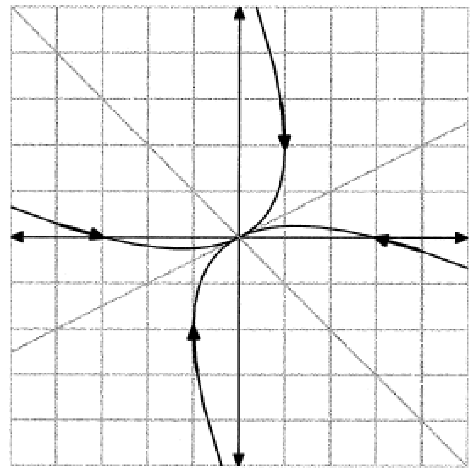
Solution   Section 3.10 – Phase Plane Portraits

Exercise

Sketch a rough approximation of a solution in each region determined by the half-line solutions. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

$$y(t) = C_1 e^{-t}\begin{pmatrix}2\\1\end{pmatrix} + C_2 e^{-2t}\begin{pmatrix}-1\\1\end{pmatrix}$$

Solution



Both eigenvalues are negative, so the equilibrium point at the origin is a sink.

Solutions dive toward the origin to the slow exponential solution, $e^{-t}\begin{pmatrix}2\\1\end{pmatrix}$.

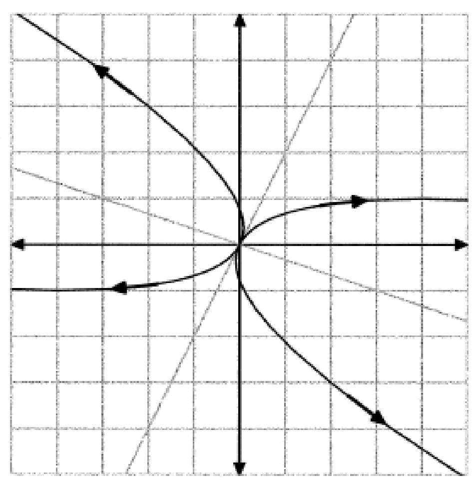
Solutions dive toward the origin to the fast exponential solution, $e^{-2t}\begin{pmatrix}-1\\1\end{pmatrix}$.

Exercise

Sketch a rough approximation of a solution in each region determined by the half-line solutions. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

$$y(t) = C_1 e^{t}\begin{pmatrix}-1\\-2\end{pmatrix} + C_2 e^{2t}\begin{pmatrix}3\\-1\end{pmatrix}$$

Solution



Both eigenvalues are positive, so the equilibrium point at the origin is a source.

Solutions emanate from the origin tangent to the slow exponential solution, $e^{t}(-1,\ -2)^T$.

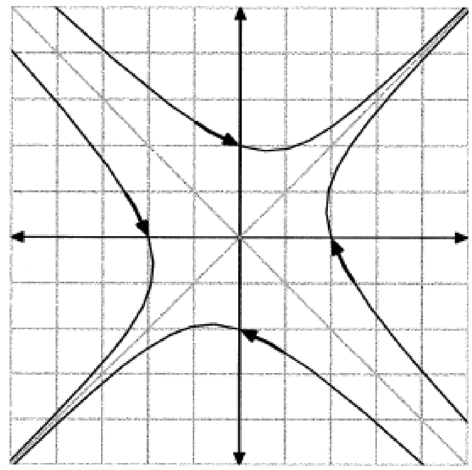
Solutions emanate from the origin to the fast exponential solution, $e^{2t}(3,\ -1)^T$.

Exercise

Sketch a rough approximation of a solution in each region determined by the half-line solutions. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

$$y(t) = C_1 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Solution



One eigenvalue is negative and the other positive. So the equilibrium point on the origin is a saddle.

As $t \to +\infty$, solutions parallel the exponential solution $e^{t}(1,\ 1)^T$

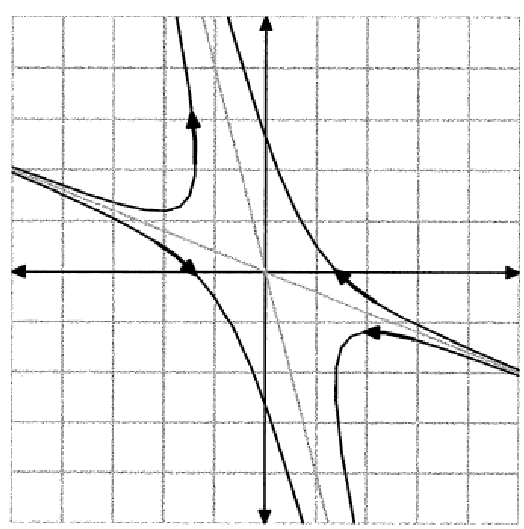
As $t \to -\infty$, solutions parallel the exponential solution $e^{-2t}(1,\ -1)^T$

Exercise

Sketch a rough approximation of a solution in each region determined by the half-line solutions. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

$$y(t) = C_1 e^{-t} \begin{pmatrix} -5 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

Solution



One eigenvalue is negative and the other positive. So the equilibrium point on the origin is a saddle.

As $t \to +\infty$, solutions parallel the exponential solution $e^{2t}(-1,\ 4)^T$

As $t \to -\infty$, solutions parallel the exponential solution $e^{-t}(-5,\ 2)^T$

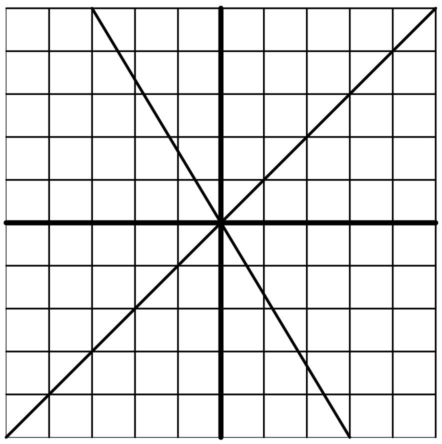
Exercise

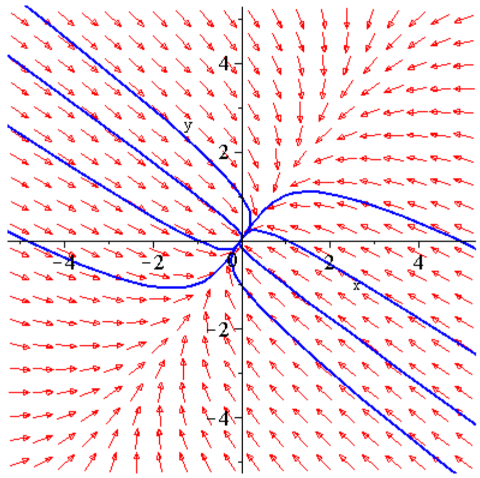
Sketch a rough approximation of the given system. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

$$y' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} y$$

Solution

Asymptotically stable sink at the center





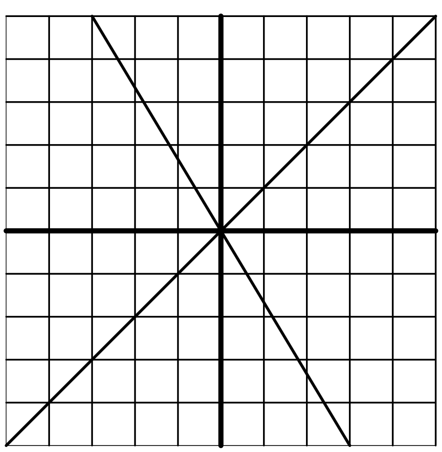
Exercise

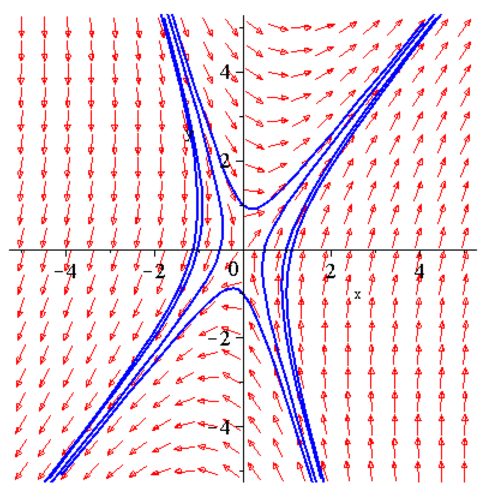
Sketch a rough approximation of the given system. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

$$y' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} y$$

Solution

Saddle point at (0, 0); semi-stable





Exercise

Calculate the eigenvalues to determine the behavior of the system whether the equilibrium point at the origin is the center, a spiral sink or a source. Calculate and sketch the vector generated by the right-hand side of the system at the point (1, 0). Use this to help sketch the solution trajectory for the system passing through the point (1, 0).

$$y' = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} y$$

Solution

Equilibrium point at the origin is the center

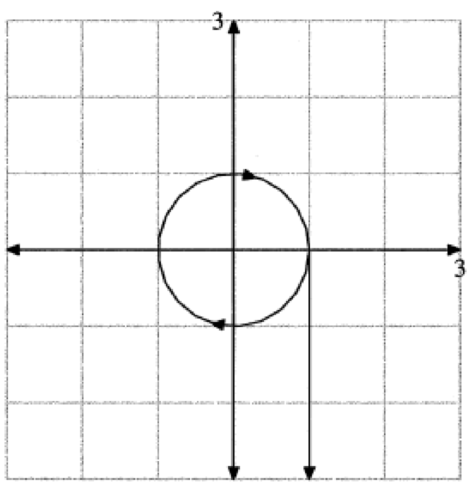
$A = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$ has a trace $T = 0$ and determinant $D = 9$.

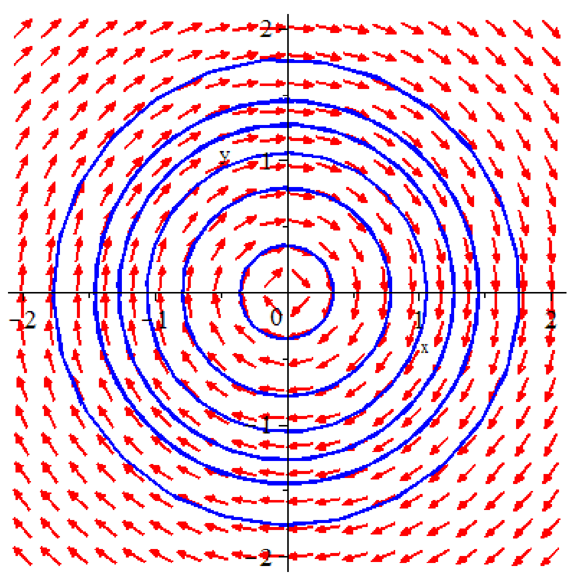
$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -\lambda & 3 \\ -3 & -\lambda \end{vmatrix} \\ &= \lambda^2 + 9 = 0 \end{aligned}$$

Therefore; the eigenvalues are: $\lambda_1 = 3i \quad and \quad \lambda_2 = -3i$

$$\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

$\therefore$ The motion is clockwise.





Exercise

Calculate the eigenvalues to determine the behavior of the system whether the equilibrium point at the origin is the center, a spiral sink or a source. Calculate and sketch the vector generated by the right-hand side of the system at the point (1, 0). Use this to help sketch the solution trajectory for the system passing through the point (1, 0).

$$y' = \begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix} y$$

Solution

$$A = \begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix}$$

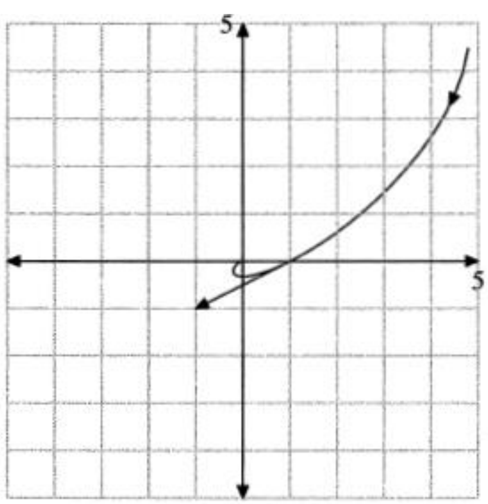
$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -2-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} \\ &= (-2-\lambda)(-\lambda) + 2 \\ &= \lambda^2 + 2\lambda + 2 = 0 \end{aligned}$$

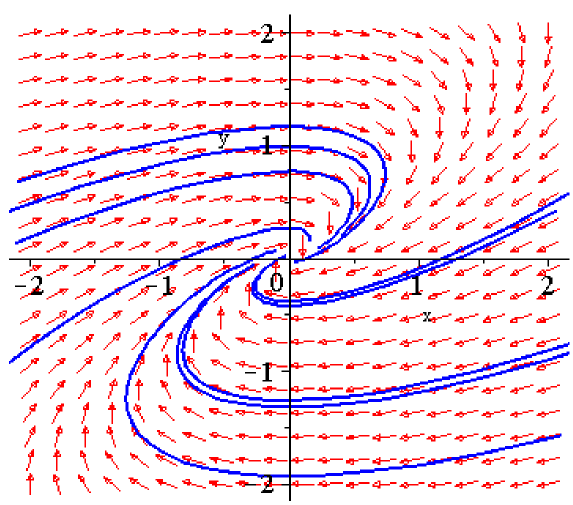
Therefore; the eigenvalues are: $\lambda_1 = -1 + i$ *and* $\lambda_2 = -1 - i$

Because both the real part of the eigenvalues is negative, the equilibrium point at the origin is a spiral sink

$$\begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$\therefore$ The motion is clockwise.





Exercise

Calculate the eigenvalues to determine the behavior of the system whether the equilibrium point at the origin is the center, a spiral sink or a source. Calculate and sketch the vector generated by the right-hand side of the system at the point (1, 0). Use this to help sketch the solution trajectory for the system passing through the point (1, 0).

$$y' = \begin{pmatrix} 7 & -10 \\ 4 & -5 \end{pmatrix} y$$

Solution

$$A = \begin{pmatrix} 7 & -10 \\ 4 & -5 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 7-\lambda & -10 \\ 4 & -5-\lambda \end{vmatrix}$$

$$= (7-\lambda)(-5-\lambda) + 40$$

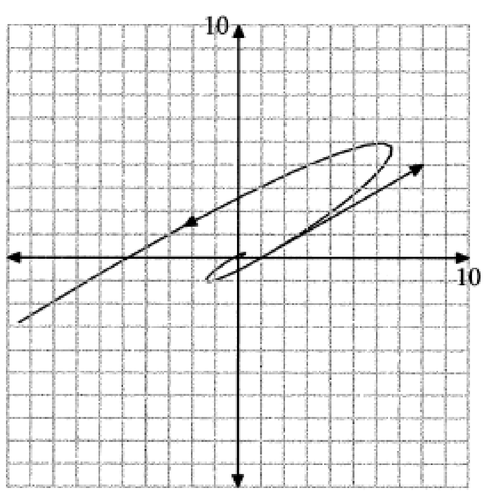
$$= \lambda^2 - 2\lambda + 5 = 0$$

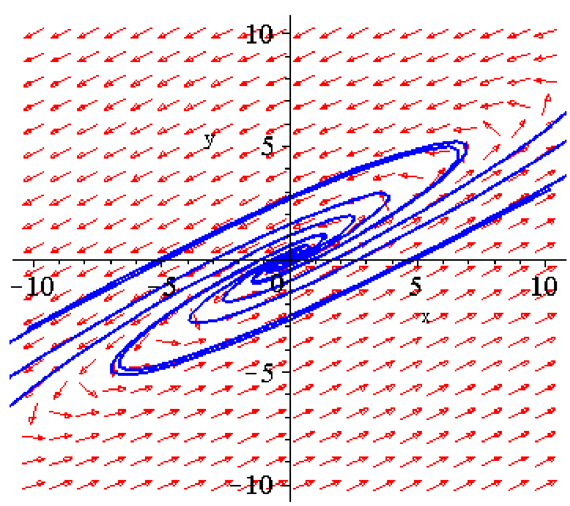
Therefore; the eigenvalues are: $\lambda_1 = 1 + 2i \quad and \quad \lambda_2 = 1 - 2i$

Because both the real part of the eigenvalues is positive, the equilibrium point at the origin is a spiral source.

$$\begin{pmatrix} 7 & -10 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

$\therefore$ The motion is counterclockwise.





Exercise

Calculate the eigenvalues to determine the behavior of the system whether the equilibrium point at the origin is the center, a spiral sink or a source. Calculate and sketch the vector generated by the right-hand side of the system at the point (1, 0). Use this to help sketch the solution trajectory for the system passing through the point (1, 0).

$$y' = \begin{pmatrix} -4 & 8 \\ -4 & 4 \end{pmatrix} y$$

Solution

$$A = \begin{pmatrix} -4 & 8 \\ -4 & 4 \end{pmatrix}$$

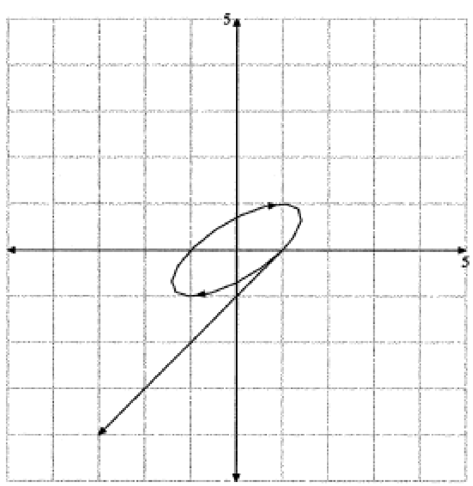
$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -4-\lambda & 8 \\ -4 & 4-\lambda \end{vmatrix} \\ &= (4-\lambda)(-4-\lambda) + 32 \\ &= \lambda^2 + 16 = 0 \end{aligned}$$

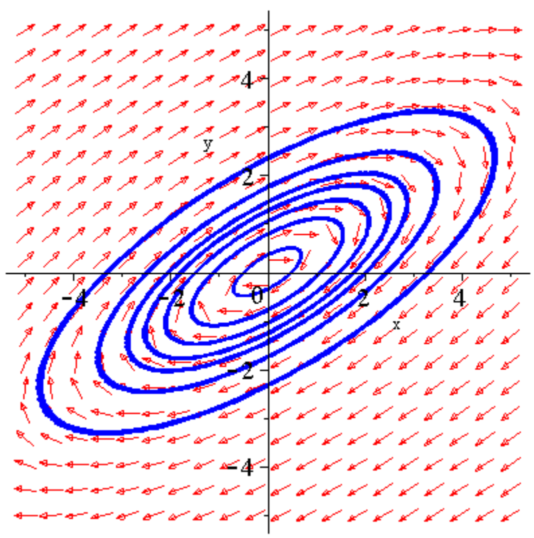
Therefore; the eigenvalues are: $\lambda_1 = -4i \quad and \quad \lambda_2 = 4i$

Because both the real part of the eigenvalues is zero, the equilibrium point at the origin is a center.

$$\begin{pmatrix} -4 & 8 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$$

$\therefore$ The motion is clockwise.





Exercise

Calculate the eigenvalues to determine the behavior of the system whether the equilibrium point at the origin is the center, a spiral sink or a source. Calculate and sketch the vector generated by the right-hand side of the system at the point (1, 0). Use this to help sketch the solution trajectory for the system passing through the point (1, 0).

$$y' = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} y$$

Solution

$$A = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix}$$

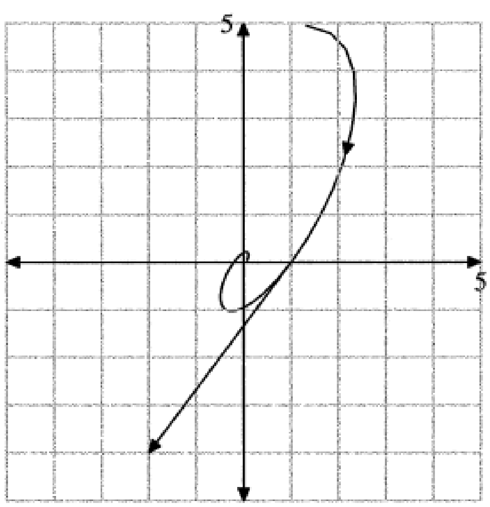
$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -3-\lambda & 2 \\ -4 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)(-3-\lambda) + 8 \\ &= \lambda^2 + 2\lambda + 5 = 0 \end{aligned}$$

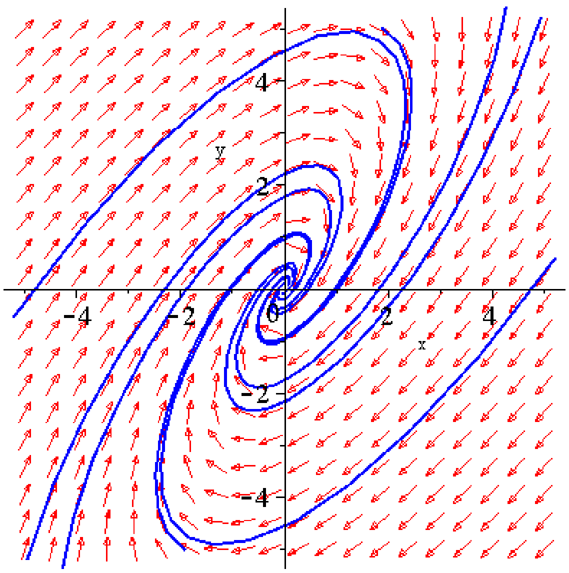
Therefore; the eigenvalues are: $\lambda_1 = -1 + 2i$ *and* $\lambda_2 = -1 - 2i$

Because both the real part of the eigenvalues is negative, the equilibrium point at the origin is a spiral sink.

$$\begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$$

$\therefore$ The motion is clockwise.





Exercise

For the given system $y' = \begin{pmatrix} -1 & 6 \\ -3 & 8 \end{pmatrix} y$

a) Sketch a rough approximation of the given system. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

b) Find the solution of the initial-value problem $y(0) = (0,\ 1)^T$

Solution

a) $|A - \lambda I| = \begin{vmatrix} -1-\lambda & 6 \\ -3 & 8-\lambda \end{vmatrix}$

$$= (-1-\lambda)(8-\lambda) + 18$$

$$= -8 + \lambda - 8\lambda + \lambda^2 + 18$$

$$= \lambda^2 - 7\lambda + 10 = 0$$

Thus, the eigenvalues are: $\lambda_1 = 2$ *and* $\lambda_2 = 5$

For $\lambda_1 = 2 \quad \Rightarrow (A - 2I)V_1 = 0$

$$\begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -3x + 6y = 0 \\ -3x + 6y = 0 \end{cases} \Rightarrow -3x = -6y \to \boxed{x = 2y}$$

The eigenvector is: $V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The solution is: $y_1(t) = e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

For $\lambda_2 = 5 \quad \Rightarrow (A - 5I)V_2 = 0$

$$\begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -6x + 6y = 0 \\ -3x + 3y = 0 \end{cases} \Rightarrow -6x = -6y \to \boxed{x = y}$$

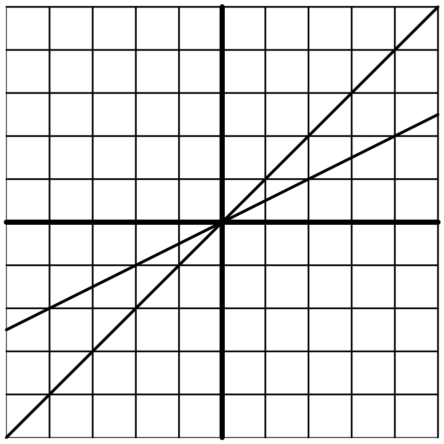
The eigenvector is: $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

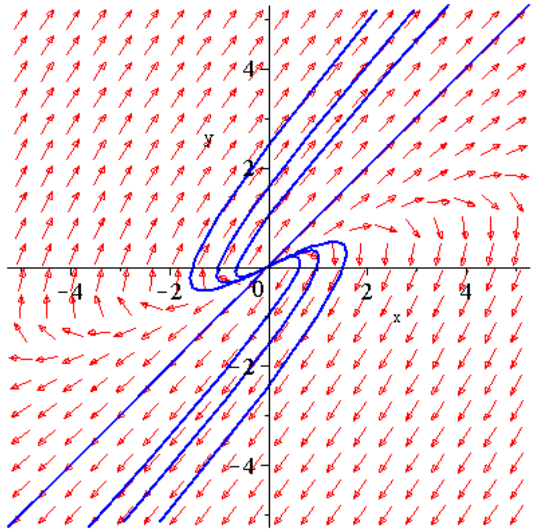
The solution is: $y_2(t) = e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Therefore, the final solution can be written as:

$$y(t) = C_1 e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Unstable at the center (source)





b) $y(0) = C_1 e^{2(0)} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{5(0)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ -2 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2C_1 + C_2 \\ C_1 + C_2 \end{pmatrix}$

$\rightarrow \begin{cases} 2C_1 + C_2 = 1 \\ C_1 + C_2 = -2 \end{cases} \xrightarrow{\quad rref \quad} C_1 = 3 \quad C_2 = -5$

$\boxed{y(t) = 3e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 5e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$