

## ***Solution***      **Section 1.7 – Length of Curves**

### ***Exercise***

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = (2 \cos t) \hat{i} + (2 \sin t) \hat{j} + \sqrt{5}t \hat{k}; \quad 0 \leq t \leq \pi$$

### **Solution**

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -(2 \sin t) \hat{i} + (2 \cos t) \hat{j} + \sqrt{5} \hat{k}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{4 \sin^2 t + 4 \cos^2 t + 5} \\ &= \sqrt{4 + 5} \\ &= 3 \end{aligned}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = -\frac{2 \sin t}{3} \hat{i} + \frac{2 \cos t}{3} \hat{j} + \frac{\sqrt{5}}{3} \hat{k}$$

$$\begin{aligned} \text{Length: } s &= \int_0^{\pi} |\vec{v}(t)| \, dt \\ &= \int_0^{\pi} 3 \, dt \\ &= 3t \Big|_0^{\pi} \\ &= 3\pi \end{aligned}$$

### ***Exercise***

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = t\hat{i} + \frac{2}{3}t^{3/2} \hat{k}; \quad 0 \leq t \leq 8$$

### **Solution**

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \hat{i} + t^{1/2} \hat{k}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{1+t}$$

$$\vec{T} = \frac{1}{\sqrt{1+t}} \hat{i} + \frac{t^{1/2}}{\sqrt{1+t}} \hat{k}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\text{Length: } s = \int_0^8 |\vec{v}(t)| \, dt$$

$$\begin{aligned}
&= \int_0^8 (1+t)^{1/2} dt &= \int_0^8 (1+t)^{1/2} d(1+t) \\
&= \frac{2}{3}(1+t)^{3/2} \Big|_0^8 \\
&= \frac{2}{3} \left[ (9)^{3/2} - 1 \right] \\
&= \frac{2}{3}(27-1) \\
&= \frac{52}{3}
\end{aligned}$$

### Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = (2+t)\hat{i} - (t+1)\hat{j} + t\hat{k}; \quad 0 \leq t \leq 3$$

### Solution

$$\vec{v}(t) = \hat{i} - \hat{j} + \hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{1+1+1} = \sqrt{3}$$

$$\vec{T} = \frac{1}{\sqrt{3}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\begin{aligned}
\text{Length: } s &= \int_0^3 |\vec{v}(t)| dt \\
&= \int_0^3 \sqrt{3} dt \\
&= \sqrt{3}t \Big|_0^3 \\
&= 3\sqrt{3}
\end{aligned}$$

### Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = (\cos^3 t)\hat{i} + (\sin^3 t)\hat{k}; \quad 0 \leq t \leq \frac{\pi}{2}$$

### Solution

$$\vec{v}(t) = -\left(3\cos^2 t \sin t\right)\hat{i} + \left(3\sin^2 t \cos t\right)\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned}
|\vec{v}| &= \sqrt{9 \cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t} \\
&= 3 \sqrt{\cos^2 t \sin^2 t (\cos^2 t + \sin^2 t)} \\
&= 3 \sqrt{\cos^2 t \sin^2 t} \\
&= 3 |\cos t \sin t|
\end{aligned}$$

$$\begin{aligned}
\vec{T} &= -\left(\frac{3 \cos^2 t \sin t}{3 |\cos t \sin t|}\right) \hat{i} + \left(\frac{3 \sin^2 t \cos t}{3 |\cos t \sin t|}\right) \hat{k} & \vec{T} &= \frac{\vec{v}}{|\vec{v}|} \\
&= -(\cos t) \hat{i} + (\sin t) \hat{k}
\end{aligned}$$

$$\begin{aligned}
L &= \int_0^{\pi/2} 3 \cos t \sin t \, dt \\
&= \frac{3}{2} \int_0^{\pi/2} \sin 2t \, dt \\
&= \frac{3}{2} \left[ -\frac{1}{2} \cos 2t \right]_0^{\pi/2} \\
&= -\frac{3}{4} (\cos \pi - \cos 0) \\
&= -\frac{3}{4} (-2) \\
&= \frac{3}{2}
\end{aligned}$$

$$\begin{aligned}
L &= \int_a^b |\vec{v}(t)| \, dt \\
\sin 2t &= 2 \cos t \sin t
\end{aligned}$$

## Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = (t \cos t) \hat{i} + (t \sin t) \hat{j} + \left(\frac{2\sqrt{2}}{3} t^{3/2}\right) \hat{k}; \quad 0 \leq t \leq \pi$$

## Solution

$$\vec{v}(t) = (\cos t - t \sin t) \hat{i} + (\sin t + t \cos t) \hat{j} + \left(\sqrt{2} t^{1/2}\right) \hat{k} \quad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned}
|\vec{v}(t)| &= \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + 2t} \\
&= \sqrt{\cos^2 t - 2t \cos t + \sin^2 t + \sin^2 t + 2t \cos t + \cos^2 t + 2t} \\
&= \sqrt{2 + 2t}
\end{aligned}$$

$$\vec{T} = \frac{1}{\sqrt{2}} \left( \frac{\cos t - t \sin t}{\sqrt{1+t}} \right) \hat{i} + \frac{1}{\sqrt{2}} \left( \frac{\sin t + t \cos t}{\sqrt{1+t}} \right) \hat{j} + \left( \sqrt{\frac{t}{1+t}} t^{1/2} \right) \hat{k} \quad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\begin{aligned}
L &= \int_0^{\pi} \sqrt{2}\sqrt{1+t} \, dt & L &= \int_a^b |\vec{v}(t)| \, dt \\
&= \sqrt{2} \int_0^{\pi} \sqrt{1+t} \, d(1+t) \\
&= \sqrt{2} \int_0^{\pi} \sqrt{1+t} \, d(1+t) \\
&= \sqrt{2} (\sqrt{1+\pi} - 1)
\end{aligned}$$

### Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\mathbf{r}(t) = (t \sin t + \cos t) \mathbf{i} + (t \cos t - \sin t) \mathbf{j}; \quad \sqrt{2} \leq t \leq 2$$

### Solution

$$\begin{aligned}
\vec{v}(t) &= (\sin t + t \cos t - \sin t) \hat{i} + (\cos t - t \sin t - \cos t) \hat{j} & \vec{v}(t) &= \frac{d\vec{r}}{dt} \\
&= (t \cos t) \hat{i} - (t \sin t) \hat{j} \\
|\vec{v}| &= \sqrt{t^2 \cos^2 t + t^2 \sin^2 t} \\
&= \sqrt{t^2 (\cos^2 t + \sin^2 t)} \\
&= \sqrt{t^2} \\
&= |t| \\
&= t \quad \text{because } \sqrt{2} \leq t \leq 2
\end{aligned}$$

$$\begin{aligned}
\mathbf{T} &= \frac{\mathbf{v}}{|\mathbf{v}|} = \left( \frac{t \cos t}{t} \right) \mathbf{i} - \left( \frac{t \sin t}{t} \right) \mathbf{j} \\
&= (\cos t) \mathbf{i} - (\sin t) \mathbf{j}
\end{aligned}$$

$$\begin{aligned}
L &= \int_{\sqrt{2}}^2 t \, dt & L &= \int_a^b |\vec{v}(t)| \, dt \\
&= \frac{1}{2} t^2 \Big|_{\sqrt{2}}^2 \\
&= \frac{1}{2} (4 - 2) \\
&= 1
\end{aligned}$$

### Exercise

Find the point on the curve  $\vec{r}(t) = (5 \sin t)\hat{i} + (5 \cos t)\hat{j} + 12t\hat{k}$  at a distance  $26\pi$  units along the curve from the point  $(0, 5, 0)$  in the direction of increasing arc length.

### Solution

$$\vec{v} = (5 \cos t)\hat{i} - (5 \sin t)\hat{j} + 12\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{25 \cos^2 t + 25 \sin^2 t + 144} \\ &= \sqrt{25(\cos^2 t + \sin^2 t) + 144} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

$$s = \int_0^{t_0} 13 \, dt \qquad s = \int_0^{t_0} |\vec{v}(t)| \, dt$$

$$= 13t_0$$

$$s = 26\pi = 13t_0$$

$$\underline{t_0 = 2\pi}$$

$$\begin{aligned} \vec{r}(t = 2\pi) &= (5 \sin 2\pi)\hat{i} + (5 \cos 2\pi)\hat{j} + 12(2\pi)\hat{k} \\ &= 0\hat{i} + 5\hat{j} + 24\pi\hat{k} \end{aligned}$$

The point is:  $\underline{(0, 5, 24\pi)}$

### Exercise

Find the arc length parameter along the curve from the point where  $t = 0$ . Also, find the length of the indicated portion of the curve.  $\vec{r}(t) = (4 \cos t)\hat{i} + (4 \sin t)\hat{j} + 3t\hat{k}; \quad 0 \leq t \leq \frac{\pi}{2}$

### Solution

$$\vec{v} = -(4 \sin t)\hat{i} + (4 \cos t)\hat{j} + 3\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{16 \sin^2 t + 16 \cos^2 t + 9} \\ &= \sqrt{16 + 9} \\ &= 5 \end{aligned}$$

$$\begin{aligned}
 s &= \int_0^t 5 \, dt \\
 &= 5t \Big| \\
 s\left(\frac{\pi}{2}\right) &= \frac{5\pi}{2}
 \end{aligned}
 \qquad
 s = \int_0^t |\vec{v}(\tau)| \, d\tau$$

### Exercise

Find the arc length parameter along the curve from the point where  $t = 0$ . Also, find the length of the indicated portion of the curve.  $\vec{r}(t) = (e^t \cos t)\hat{i} + (e^t \sin t)\hat{j} + e^t \hat{k}; \quad -\ln 4 \leq t \leq 0$

### Solution

$$\vec{v}(t) = (e^t \cos t - e^t \sin t)\hat{i} + (e^t \sin t + e^t \cos t)\hat{j} + e^t \hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned}
 |\vec{v}| &= \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2 + e^{2t}} \\
 &= \sqrt{e^{2t} \cos^2 t - e^{2t} \cos t \sin t + e^{2t} \sin^2 t + e^{2t} \sin^2 t + e^{2t} \cos^2 t + e^{2t} \cos t \sin t + e^{2t} \cos^2 t + e^{2t}} \\
 &= \sqrt{2e^{2t} (\cos^2 t + \sin^2 t) + e^{2t}} \\
 &= \sqrt{3e^{2t}} \\
 &= \sqrt{3} e^t
 \end{aligned}$$

$$\begin{aligned}
 s(t) &= \int_0^t \sqrt{3} e^\tau \, d\tau \qquad s = \int_0^t |\vec{v}(\tau)| \, d\tau \\
 &= \sqrt{3} e^\tau \Big|_0^t \\
 &= \sqrt{3} (e^t - 1)
 \end{aligned}$$

$$s(0) = \sqrt{3} (e^0 - 1) = 0$$

$$\begin{aligned}
 s(-\ln 4) &= \sqrt{3} (e^{-\ln 4} - 1) \\
 &= \sqrt{3} \left( e^{\ln \frac{1}{4}} - 1 \right) \\
 &= \sqrt{3} \left( \frac{1}{4} - 1 \right)
 \end{aligned}$$

$$\begin{aligned}
&= \sqrt{3} \left( -\frac{3}{4} \right) \\
&= \frac{3\sqrt{3}}{4} \\
s(-\ln 4) - s(0) &= \underline{\underline{\frac{3\sqrt{3}}{4}}}
\end{aligned}$$

### Exercise

Find the arc length parameter along the curve from the point where  $t = 0$ . Also, find the length of the indicated portion of the curve.  $\vec{r}(t) = (1 + 2t)\hat{i} + (1 + 3t)\hat{j} + (6 - 6t)\hat{k}$ ;  $-1 \leq t \leq 0$

### Solution

$$\vec{v} = 2\hat{i} + 3\hat{j} - 6\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{4 + 9 + 36} = 7$$

$$\begin{aligned}
s(t) &= \int_0^t 7 \, d\tau \\
&= 7t
\end{aligned}
\qquad
s = \int_0^t |\vec{v}(\tau)| \, d\tau$$

$$\text{Length: } s(0) - s(-1) = 0 - (-7) = \underline{\underline{7}}$$

### Exercise

Find the arc length of the curve  $\vec{r}(t) = \langle 2t^{9/2}, t^3 \rangle$  for  $0 \leq t \leq 2$

### Solution

$$\begin{aligned}
\vec{v}(t) &= \langle 9t^{7/2}, 3t^2 \rangle & \vec{v}(t) &= \frac{d\vec{r}}{dt} \\
L &= \int_0^2 \sqrt{(9t^{7/2})^2 + (3t^2)^2} \, dt & L &= \int_0^2 \left| \frac{d\vec{r}}{dt} \right| \, dt \\
&= \int_0^2 \sqrt{81t^7 + 9t^4} \, dt \\
&= \int_0^2 3t^2 \sqrt{9t^3 + 1} \, dt \\
&= \frac{1}{9} \int_0^2 (9t^3 + 1)^{1/2} d(9t^3 + 1)
\end{aligned}$$

$$\begin{aligned}
&= \frac{2}{27} \left( 9t^3 + 1 \right)^{3/2} \Big|_0^2 \\
&= \frac{2}{27} \left( 73\sqrt{73} - 1 \right) \quad \text{unit}
\end{aligned}$$

### Exercise

Find the arc length of the curve  $\vec{r}(t) = \left\langle t^2, \frac{4\sqrt{2}}{3}t^{3/2}, 2t \right\rangle$  for  $1 \leq t \leq 3$

### Solution

$$\begin{aligned}
\frac{d\vec{r}}{dt} &= \left\langle 2t, 2\sqrt{2} t^{1/2}, 2 \right\rangle \\
L &= \int_1^3 \sqrt{4t^2 + 8t + 4} \, dt & L &= \int_a^b |\vec{r}'(t)| \, dt \\
&= 2 \int_1^3 \sqrt{(t+1)^2} \, dt \\
&= 2 \int_1^3 (t+1) \, dt \\
&= 2 \left( \frac{1}{2}t^2 + t \right) \Big|_1^3 \\
&= 2 \left( \frac{9}{2} + 1 - \frac{1}{2} - 1 \right) \\
&= 12 \quad \text{unit}
\end{aligned}$$

### Exercise

Find the arc length of the curve  $\vec{r}(t) = \left\langle t, \ln \sec t, \ln(\sec t + \tan t) \right\rangle$  for  $0 \leq t \leq \frac{\pi}{4}$

### Solution

$$\begin{aligned}
\frac{d\vec{r}}{dt} &= \left\langle 1, \frac{\tan t \sec t}{\sec t}, \frac{\tan t \sec t + \sec^2 t}{\sec t + \tan t} \right\rangle \\
&= \left\langle 1, \tan t, \sec t \right\rangle \\
L &= \int_0^{\pi/4} \sqrt{1 + \tan^2 t + \sec^2 t} \, dt & L &= \int_a^b |\vec{r}'(t)| \, dt
\end{aligned}$$



$$\begin{aligned}
&= \int_0^{\pi/4} \sqrt{2 \sec^2 t} \, dt \\
&= \sqrt{2} \int_0^{\pi/4} \sec t \, dt \\
&= \sqrt{2} \ln(\sec t + \tan t) \Big|_0^{\pi/4} \\
&= \sqrt{2} \ln(\sqrt{2} + 1)
\end{aligned}$$

### Exercise

Find the lengths of the curves  $\vec{r}(t) = (2 \cos t)\hat{i} + (2 \sin t)\hat{j} + t^2 \hat{k}; \quad 0 \leq t \leq \frac{\pi}{4}$

### Solution

$$\begin{aligned}
\frac{d\vec{r}}{dt} &= (-2 \sin t)\hat{i} + (2 \cos t)\hat{j} + 2t \hat{k} \\
\left| \frac{d\vec{r}}{dt} \right| &= \sqrt{4 \sin^2 t + 4 \cos^2 t + 4t^2} \\
&= \sqrt{4 + 4t^2} \\
&= 2\sqrt{1 + t^2} \\
L &= 2 \int_0^{\pi/4} \sqrt{1 + t^2} \, dt \qquad L = \int_a^b |\vec{r}'(t)| \, dt \\
&= t\sqrt{1 + t^2} + \ln\left(t + \sqrt{1 + t^2}\right) \Big|_0^{\pi/4} \\
&= \frac{\pi}{4} \sqrt{1 + \frac{\pi^2}{16}} + \ln\left(\frac{\pi}{4} + \sqrt{1 + \frac{\pi^2}{16}}\right) - 0 - \ln 1 \\
&= \frac{\pi}{4} \sqrt{1 + \frac{\pi^2}{16}} + \ln\left(\frac{\pi}{4} + \sqrt{1 + \frac{\pi^2}{16}}\right)
\end{aligned}$$

### Exercise

Find the lengths of the curves  $\vec{r}(t) = (3 \cos t)\hat{i} + (3 \sin t)\hat{j} + 2t^{3/2} \hat{k}; \quad 0 \leq t \leq 3$

### Solution

$$\frac{d\vec{r}}{dt} = (-3 \sin t) \hat{i} + (3 \cos t) \hat{j} + 3t^{1/2} \hat{k}$$

$$\begin{aligned} \left| \frac{d\vec{r}}{dt} \right| &= \sqrt{9 \sin^2 t + 9 \cos^2 t + 9t} \\ &= 3\sqrt{1+t^2} \end{aligned}$$

$$L = 3 \int_0^3 \sqrt{1+t} \, dt$$

$$= 3 \int_0^3 (1+t)^{1/2} \, d(1+t)$$

$$= 2(1+t)^{3/2} \Big|_0^3$$

$$= 2(4^{3/2} - 1)$$

$$= 2(8 - 1)$$

$$= 14$$

$$L = \int_a^b |\vec{r}'(t)| \, dt$$

### Exercise

The acceleration of a wayward firework is given by  $\vec{a}(t) = \sqrt{2}\hat{j} + 2t\hat{k}$  for  $0 \leq t \leq 3$ . Suppose the initial velocity of the firework is  $\vec{v}(0) = 1$ .

- Find the velocity of the firework, for  $0 \leq t \leq 3$ .
- Find the length of the trajectory of the firework over the interval  $0 \leq t \leq 3$

### Solution

$$a) \quad \vec{v} = \int \langle 0, \sqrt{2}, 2t \rangle \, dt$$

$$= \langle 0, \sqrt{2}t, t^2 \rangle + \vec{C}$$

$$\vec{v}(0) = 1 = \langle 1, 0, 0 \rangle$$

$$\langle 1, 0, 0 \rangle = \langle 0, 0, 0 \rangle + \vec{C}$$

$$\vec{C} = \langle 1, 0, 0 \rangle$$

$$\vec{v}(t) = \langle 0, \sqrt{2}t, t^2 \rangle + \langle 1, 0, 0 \rangle$$

$$= \langle 1, \sqrt{2}t, t^2 \rangle$$

$$\begin{aligned}
 b) \quad L &= \int_0^3 \sqrt{1+2t^2+t^4} \, dt \\
 &= \int_0^3 \sqrt{(1+t^2)^2} \, dt \\
 &= \int_0^3 (1+t^2) \, dt \\
 &= t + \frac{1}{3}t^3 \Big|_0^3 \\
 &= 3 + 9 \\
 &= \underline{12} \text{ unit}
 \end{aligned}$$

$$L = \int_a^b |\vec{r}'(t)| \, dt$$

### Exercise

If a string wound around a fixed circle in unwound while held taut in the plane of the circle, its end  $P$  traces an involute of the circle. The circle in question is the circle  $x^2 + y^2 = 1$  and the tracing point starts at  $(1, 0)$ . The unwound portion of the string is tangent to the circle at  $Q$ , and  $t$  is the radian measure of the angle from the position  $x$ -axis to segment  $OQ$ . Derive the parametric equations

$$x = \cos t + t \sin t, \quad y = \sin t - t \cos t, \quad t > 0 \text{ of the point } P(x, y) \text{ for the involute.}$$

### Solution

$$\angle PQB = \angle QOB = t$$

$$PQ = \text{arc}(AQ) = t$$

$$PQ = \text{Length of the unwound string}$$

$$\text{arc}(AQ)$$

$$\Delta PDQ: \begin{cases} \sin t = \frac{DP}{PQ} = \frac{DP}{t} \rightarrow \underline{DP = t \sin t} \\ \cos t = \frac{QD}{PQ} = \frac{QD}{t} \rightarrow \underline{QD = t \cos t} \end{cases}$$

$$\begin{aligned}
 x &= OB + BC \\
 &= OB + DP \\
 &= \underline{\cos t + t \sin t}
 \end{aligned}$$

$$\begin{aligned}
 y &= PC \\
 &= QB - QD \\
 &= \underline{\sin t - t \cos t}
 \end{aligned}$$

