Section 1.2 – Definitions / Techniques of Limits

Definition of the Limit of a Function

If $f(x)$ becomes arbitrary close to a single number L as x approaches x_0 from either side, then

$$\lim_{x \to x_0} f(x) = L$$

Which is read as "the limit of $f(x)$ as x approaches x_0 is L."

Notation	***Terminology***
$x \to a^-$	***x*** approaches ***a*** from the left (through values ***less*** than a)
$x \to a^+$	***x*** approaches ***a*** from the right (through values ***greater*** than a)

Example

How does the function $f(x) = \dfrac{x^2 - 1}{x - 1}$ behave near $x = 1$?

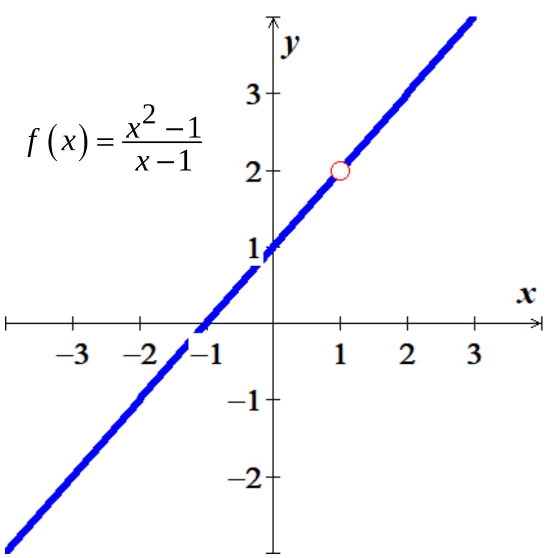
Solution

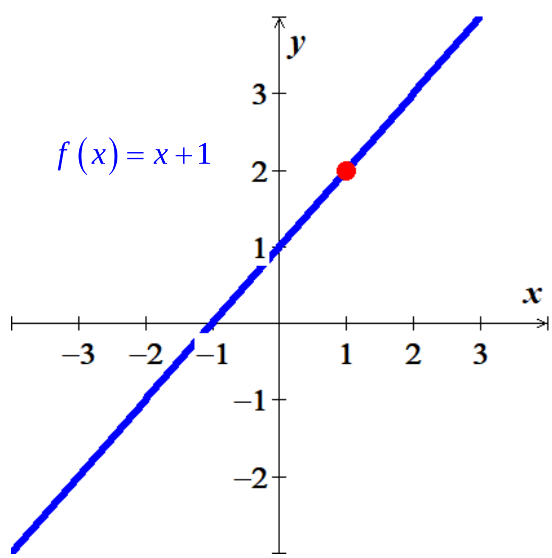
$$f(x) = \frac{(x-1)(x+1)}{x-1}$$

$$= x + 1 \quad \text{for} \quad x \neq 1$$

For $x = 1$:

$$f(x = 1) = 1 + 1 = 2$$



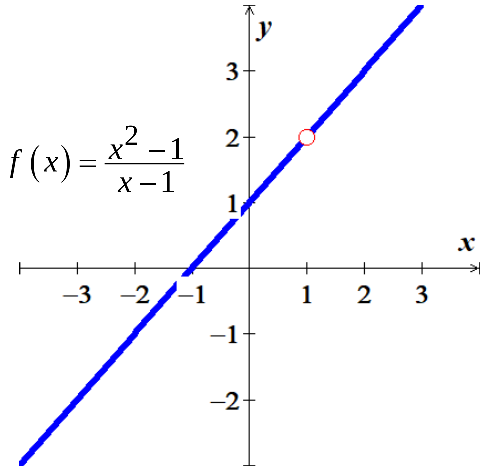


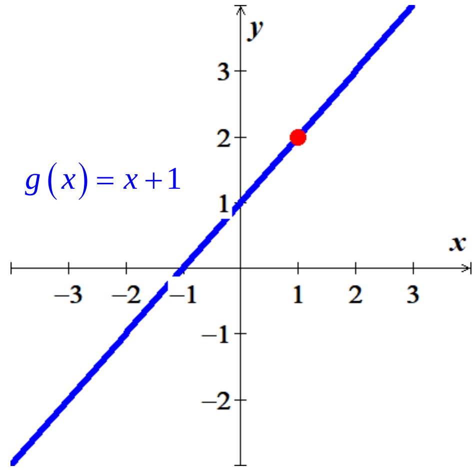
x	.9	.99	.999	1.001	1.01	1.1

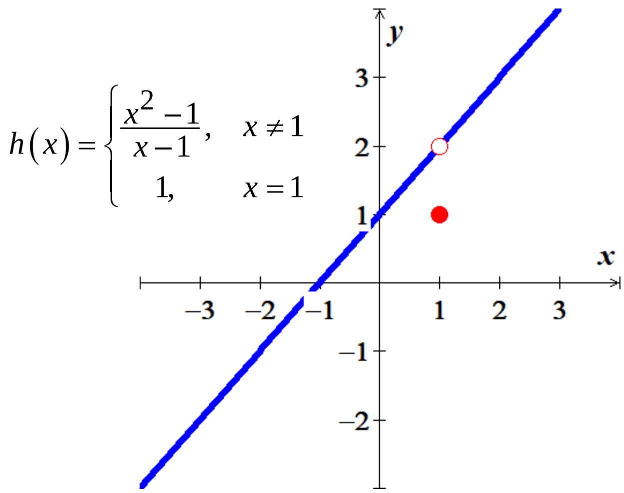
$f(x)$	1.9	1.99	1.999	2.001	2.01	2.1

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1}$$

$$\underline{=2}$$





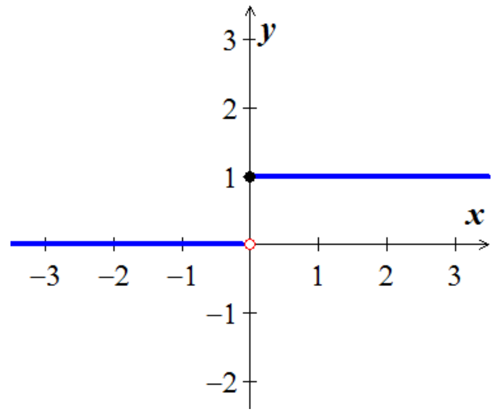


Example

Discuss the behavior of the following function as $x \to 0$.

$$U(x)=\begin{cases}0, & x<0\\ 1, & x\geq 0\end{cases}$$

Solution



The unit step function $U(x)$ has no limit as $x \to 0$, it jumps, because the values jump at $x = 0$.

To the left of zero $\left(\textit{negative value } \mathbf{0^-}\right)$ $U(x) = 0$. For the positive values of x close to zero $\left(\mathbf{0^+}\right)$ $U(x) = 1$

One-Sided Limits

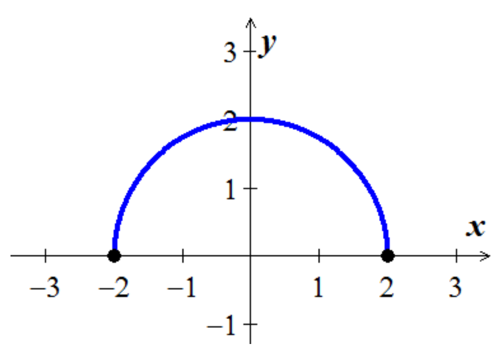
To have a limit L as x approaches c, a function f must be defined on ***both sides*** of c and its values $f(x)$ must approach L as x approaches c from either side. Because of this, ordinary limits are called ***two-sided***. If f fails to have two-sided limit at c, it may still have one-sided limit.

If the approach is from the *right*, the limit is a ***right-hand limit***. $\lim\limits_{x \to c^+} f(x) = L$

If the approach is from the *left*, the limit is a ***left-hand limit***. $\lim\limits_{x \to c^-} f(x) = M$

Example

The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2,\ 2]$; its graph is the semicircle.



We have: $\lim\limits_{x \to -2^+} \sqrt{4 - x^2} = 0$ *and* $\lim\limits_{x \to 2^-} \sqrt{4 - x^2} = 0$

The function doesn’t have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$.

It does not have ordinary two-sided limits at either –2 or 2.

Theorem

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \textit{and} \quad \lim_{x\to c^+} f(x) = L$$

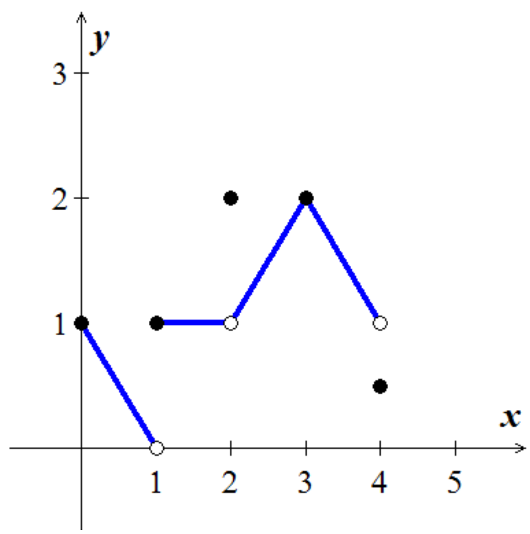
Properties of Limits

Constant function $(f(x)=k)$: $\quad \lim_{x\to x_0} f(x) = \lim_{x\to x_0} k = k$

Identity function $(f(x)=x)$: $\quad \lim_{x\to x_0} f(x) = \lim_{x\to x_0} x = x_0$

Example

Given the function graphed:



At $x = 0$: $\quad \lim_{x\to 0^+} f(x) = 1$

$\lim_{x\to 0^-} f(x)$ *and* $\lim_{x\to 0} f(x)$ don’t exist. The function is not defined to the left of $x = 0$

At $x = 1$: $\quad \lim_{x\to 1^-} f(x) = 0 \qquad \lim_{x\to 1^+} f(x) = 1$

$\lim_{x\to 1} f(x)$ doesn’t exist. The right-hand and left-hand limits are not equal.

At $x = 2$: $\quad \lim_{x\to 2^-} f(x) = 1 \qquad \lim_{x\to 2^+} f(x) = 1$

$\lim_{x\to 2} f(x) = 2$ even though $f(2) = 2$

At $x = 3$: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) \boxed{= 2}$

At $x = 4$: $\lim_{x\to 4^-} f(x) = 1$ even though $f(4) \neq 1$

$\lim_{x\to 4^+} f(x)$ *and* $\lim_{x\to 4} f(x)$ do not exist.

The function is not defined to the right of $x = 4$

Definitions

We say that $f(x)$ has right-hand limit L at x_0 and $\lim_{x\to x_0^+} f(x) = L$

If for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \varepsilon$$

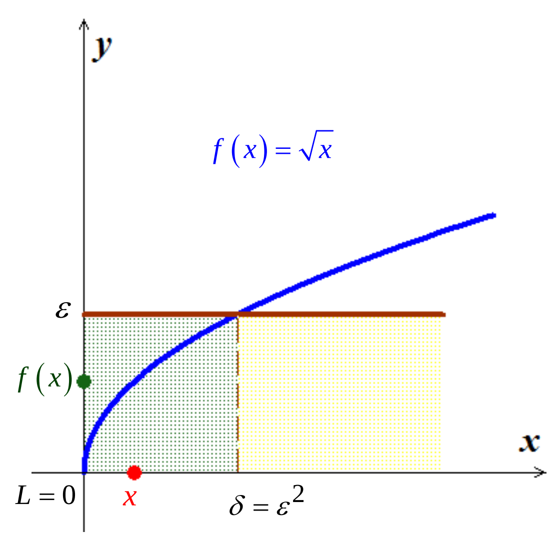
We say that $f(x)$ has left-hand limit L at x_0 and $\lim_{x\to x_0^-} f(x) = L$

If for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon$$

Example

Prove that $\lim_{x\to 0^+} \sqrt{x} = 0$



Solution

Let $\varepsilon > 0$ be given. $x_0 = 0$, $L = 0$, *Find* $\delta > 0 \ni \forall x$

$$0 < x < \delta \Rightarrow \left|\sqrt{x} - 0\right| < \varepsilon$$

or $\quad 0 < x < \delta \Rightarrow \sqrt{x} < \varepsilon$

$(\sqrt{x})^2 < \varepsilon^2$

$\Rightarrow x < \varepsilon^2$ *if* $0 < x < \delta$

If we choose $\delta = \varepsilon^2$, we have

$$0 < x < \delta = \varepsilon^2 \quad \Rightarrow \quad \sqrt{x} < \varepsilon$$

According to the definition, this shows that $\lim\limits_{x \to 0^+} \sqrt{x} = 0$

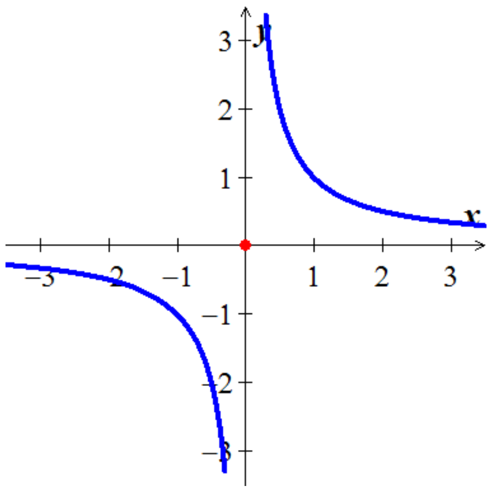
Example

Discuss the behavior of the following function as $x \to 0$.

$a)\quad g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \qquad\qquad b)\quad f(x) = \begin{cases} 0, & x \leq 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$

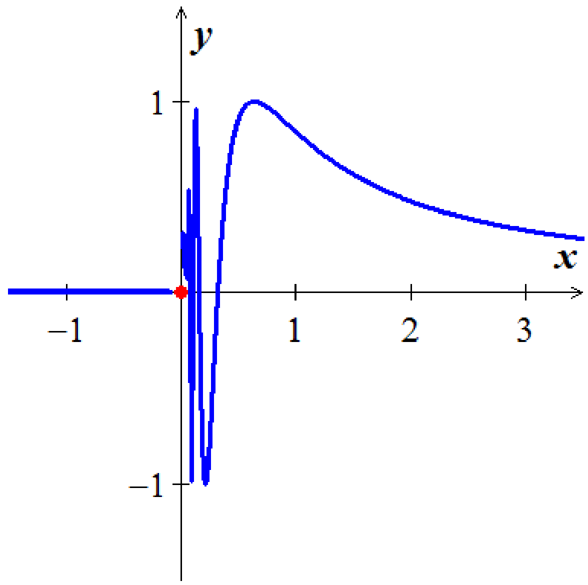
Solution

a)



$g(x)$ has *no limit* as $x \to 0$ because the values of $g(x)$ grow arbitrary large (negative and positive) value as $x \to 0$ and do not stay close.

b)



$f(x)$ has *no limit* as $x \to 0$ because the function's values oscillate between −1 and +1 in every open interval containing 0. The values do not stay close to any one number as $x \to 0$.

Limit Laws

If $\lim\limits_{x \to c} f(x) = L$ *and* $\lim\limits_{x \to c} g(x) = M$

Constant Multiple Rule: $\lim\limits_{x \to c} [bf(x)] = b \lim\limits_{x \to c} f(x) \; \underline{= bL}$

Sum and Difference Rules: $\lim\limits_{x \to c} [f(x) \pm g(x)] = \lim\limits_{x \to c} f(x) \pm \lim\limits_{x \to c} g(x) \; \underline{= L \pm M}$

Product Rule: $\lim\limits_{x \to c} [f(x) \cdot g(x)] = \lim\limits_{x \to c} f(x) \cdot \lim\limits_{x \to c} g(x) \; \underline{= L.M}$

Quotient Rule: $\lim\limits_{x \to c} \left(\dfrac{f(x)}{g(x)} \right) = \dfrac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)} \; \underline{= \dfrac{L}{M}}$ $\quad M \neq 0$

Power Rule: $\lim\limits_{x \to c} (f(x))^n = \left[\lim\limits_{x \to c} f(x) \right]^n \; \underline{= L^n}$

Root Rule: $\lim\limits_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to c} f(x)} \; \underline{= \sqrt[n]{L}}$ $\quad n > 0, \quad L > 0,$ *n is even*

Example

Find the following limits:

$a)\ \lim_{x\to c}\left(x^3+4x^2-3\right)$ $\qquad b)\ \lim_{x\to c}\dfrac{x^4+x^2-1}{x^2+5}$ $\qquad c)\ \lim_{x\to -2}\sqrt{4x^2-3}$

Solution

a) $\lim_{x\to c}\left(x^3+4x^2-3\right)=\lim_{x\to c}x^3+\lim_{x\to c}4x^2-\lim_{x\to c}(3)$ ***Sum and Difference Rules***

$$=c^3+4c^2-3$$

b) $\lim_{x\to c}\dfrac{x^4+x^2-1}{x^2+5}=\dfrac{\lim_{x\to c}\left(x^4+x^2-1\right)}{\lim_{x\to c}\left(x^2+5\right)}$ ***Quotient Rule***

$$=\frac{\lim_{x\to c}x^4+\lim_{x\to c}x^2-\lim_{x\to c}1}{\lim_{x\to c}x^2+\lim_{x\to c}5}$$ ***Sum and Difference Rules***

$$=\frac{c^4+c^2-1}{c^2+5}$$

c) $\lim_{x\to -2}\sqrt{4x^2-3}=\sqrt{\lim_{x\to -2}\left(4x^2-3\right)}$ ***Root Rule***

$$=\sqrt{\lim_{x\to -2}4x^2-\lim_{x\to -2}3}$$ ***Difference Rule***

$$=\sqrt{4(-2)^2-3}$$

$$=\sqrt{16-3}$$

$$=\sqrt{13}$$

Theorem – Limits of Polynomials

If $P(x)=a_n x^n + a_{n-1}x^{n-1}+\cdots+a_1 x + a_0$, then $\lim\limits_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1}+\cdots+a_1 c + a_0$

Theorem – Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c)\neq 0$, then $\lim\limits_{x\to c}\dfrac{P(x)}{Q(x)}=\dfrac{P(c)}{Q(c)}$

Example

Find the limit: $\lim\limits_{x\to -1}\dfrac{x^3+4x^2-3}{x^2+5}$

Solution

$$\begin{aligned}\lim_{x\to -1}\frac{x^3+4x^2-3}{x^2+5} &= \frac{(-1)^3+4(-1)^2-3}{(-1)^2+5}\\ &= \frac{0}{6}\\ &= 0\end{aligned}$$

Eliminating Zero Denominators Algebraically

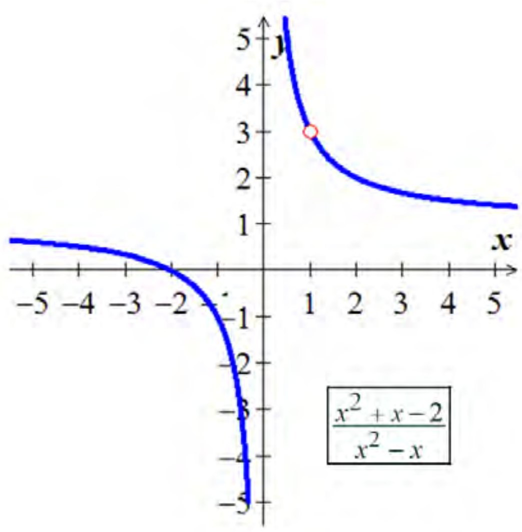
Example

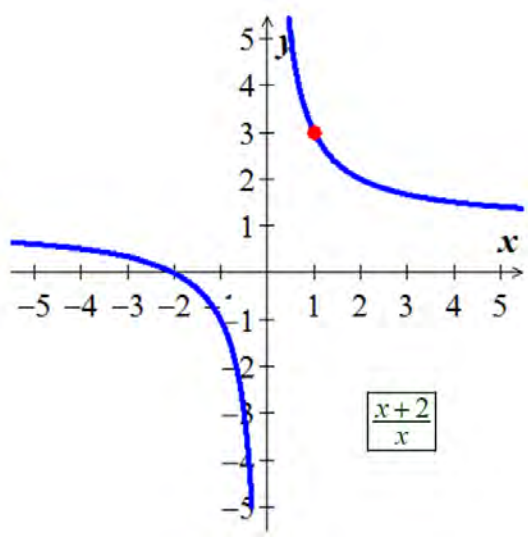
Evaluate: $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{x^2-x}$

Solution

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x} = \frac{1^2+1-2}{1^2-1} = \frac{0}{0}$$

$$\begin{aligned}\lim_{x\to 1}\frac{x^2+x-2}{x^2-x} &= \lim_{x\to 1}\frac{(x-1)(x+2)}{x(x-1)}\\ &= \lim_{x\to 1}\frac{(x+2)}{x}\\ &= \frac{1+2}{1}\\ &= 3\end{aligned}$$





Example

Evaluate: $\lim\limits_{x\to 0} \dfrac{\sqrt{x^2+100}-10}{x^2}$

Solution

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} = \frac{\sqrt{0+100}-10}{0} = \frac{0}{0}$$

$$\frac{\sqrt{x^2+100}-10}{x^2} = \frac{\sqrt{x^2+100}-10}{x^2} \bullet \frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$$

$(a-b)(a+b) = a^2 - b^2; \quad \left(\sqrt{a}\right)^2 = a$

$$= \frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$= \frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$= \frac{1}{\sqrt{x^2+100}+10}$$

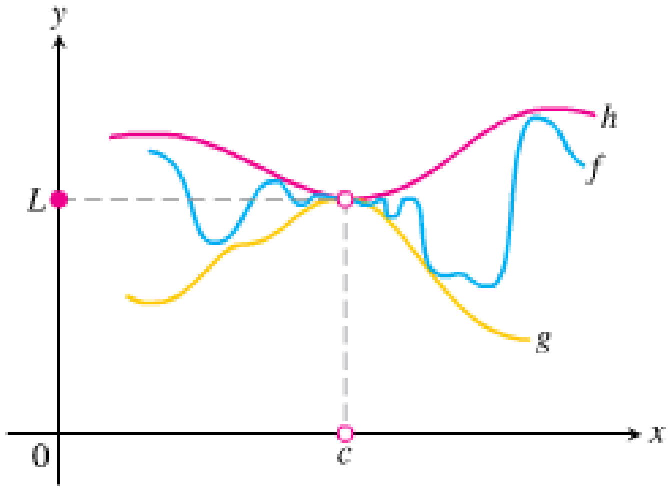
$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x\to 0} \frac{1}{\sqrt{x^2+100}+10}$$

$$= \frac{1}{\sqrt{0+100}+10}$$

$$= \frac{1}{10+10}$$

$$= \frac{1}{20}$$

The Sandwich (Squeeze) Theorem



Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L \quad \textit{then} \quad \lim_{x\to c} f(x) = L$$

Example

Given that $1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2}$ $\quad$ *for all* $x \ne 0$, find the $\lim_{x\to 0} u(x)$, no matter how complicated u is.

Solution

$$\lim_{x\to 0}\left(1 - \frac{x^2}{4}\right) = 1 - \frac{0}{4}$$

$$= 1$$

$$\lim_{x\to 0}\left(1 + \frac{x^2}{2}\right) = 1$$

The Sandwich theorem implies that $\lim_{x\to 0} u(x) = 1$

Theorem

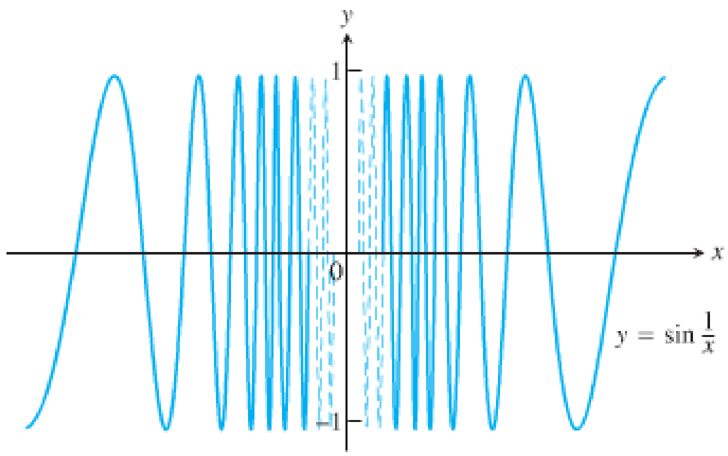
Suppose that $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x)$$

Example

Show that $y = \sin\left(\frac{1}{x}\right)$ has no limit as x approaches zero from either side.

Solution

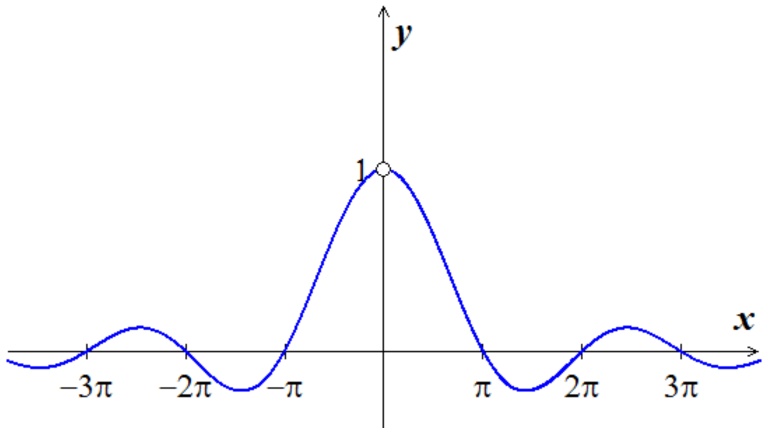


As x approaches zero, its reciprocal, $\frac{1}{x}$, grows without bound and the values of $\sin\left(\frac{1}{x}\right)$ cycle repeatedly from -1 to 1.

There is no single number L that the function's values stay increasingly close to as x approaches zero.

The function has neither a right-hand limit nor a left-hand limit at $x = 0$.

Limit Involving $\frac{\sin\theta}{\theta}$



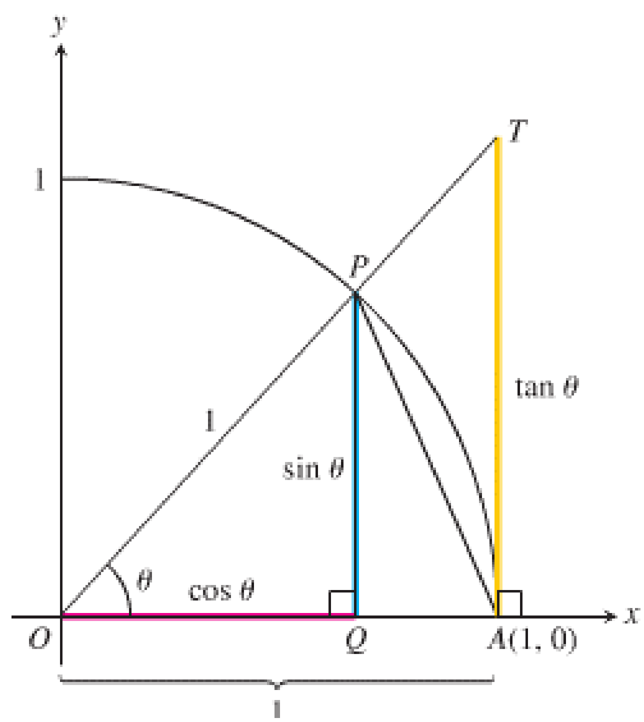
A central fact about $\frac{\sin\theta}{\theta}$ is that in radian measure it limit as $\theta \to 0$ is **1**.

Theorem

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \quad (\theta \textit{ in rad.})$$

Proof

We need to show that the right-hand limit is 1, $\theta < \frac{\pi}{2}$



Notice that:

Area ΔOAP < *Area Sector* OAP < *Area* ΔOAT

$$Area\ \Delta OAP = \tfrac{1}{2} base \times height = \tfrac{1}{2}(1)(\sin\theta)$$

$$Area\ Sector\ \Delta OAP = \tfrac{1}{2} r^2 \times \theta = \tfrac{1}{2}(1)^2(\theta) = \frac{\theta}{2}$$

$$Area\ \Delta OAP = \tfrac{1}{2} base \times height = \tfrac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta$$

$$\Rightarrow \frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

$$\frac{2}{\sin\theta}\frac{1}{2}\sin\theta < \frac{1}{2}\theta\frac{2}{\sin\theta} < \frac{1}{2}\frac{\sin\theta}{\cos\theta}\frac{2}{\sin\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$ ***Taking reciprocals reverses the inequalities***

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Since $\lim\limits_{\theta\to 0^+} \cos\theta = 1$, then

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta}$$

So $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$

Example

Show that $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x} = 0$

Solution

Using the half-angle formula: $\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right)$

$$\begin{aligned}
\lim_{x\to 0}\frac{\cos x-1}{x} &= \lim_{x\to 0}\frac{1-2\sin^2\left(\frac{x}{2}\right)-1}{x} \\
&= \lim_{x\to 0}\frac{-2\sin^2\left(\frac{x}{2}\right)}{x} \qquad \textit{Let } \theta = \frac{x}{2} \\
&= -\lim_{\theta\to 0}\frac{2\sin^2(\theta)}{2\theta} \\
&= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta \\
&= -(1)(0) \\
&= 0
\end{aligned}$$

Example

Show that $\lim_{x\to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$

Solution

$$\lim_{x\to 0} \frac{\sin 2x}{5x} = \lim_{x\to 0} \frac{\left(\frac{2}{5}\right)\sin 2x}{\left(\frac{2}{5}\right)5x}$$

Since we need 2x in the denominator

$$= \frac{2}{5} \lim_{x\to 0} \frac{\sin 2x}{2x}$$

$$= \frac{2}{5}(1)$$

$$= \frac{2}{5}$$

Example

Show that $\lim_{x\to 0} \frac{\tan x \ \sec 2x}{3x} = \frac{1}{3}$

Solution

$$\lim_{x\to 0} \frac{\tan x \ \sec 2x}{3x} = \frac{1}{3} \lim_{x\to 0} \frac{1}{x} \cdot \frac{\sin x}{\cos x} \cdot \frac{1}{\cos 2x}$$

$$= \frac{1}{3} \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} \cdot \frac{1}{\cos 2x}$$

$\lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \lim_{x\to 0} \frac{1}{\cos x} = 1, \quad \lim_{x\to 0} \frac{1}{\cos 2x} = 1$

$$= \frac{1}{3}(1)(1)(1)$$

$$= \frac{1}{3}$$

Exercises *Section* 1.2 – Definitions / Techniques of Limits

(**1 – 121**) Find the limit:

1. $\lim\limits_{x\to 3} (-1)$

2. $\lim\limits_{x\to -1} 3$

3. $\lim\limits_{x\to 1000} 18\pi^2$

4. $\lim\limits_{x\to 1} \sqrt{5x+6}$

5. $\lim\limits_{x\to 9} \sqrt{x}$

6. $\lim\limits_{x\to -3} \left(x^2+3x\right)$

7. $\lim\limits_{x\to -4} |x-4|$

8. $\lim\limits_{x\to 4} (x+2)$

9. $\lim\limits_{x\to 4} (x-4)$

10. $\lim\limits_{x\to 2} (5x-6)^{3/2}$

11. $\lim\limits_{x\to 9} \dfrac{x-9}{\sqrt{x}-3}$

12. $\lim\limits_{x\to 1} (2x+4)$

13. $\lim\limits_{x\to 1} \dfrac{x^2-4}{x-2}$

14. $\lim\limits_{x\to 2} \dfrac{x^2+4}{x-2}$

15. $\lim\limits_{x\to 0} \dfrac{|x|}{x}$

16. $\lim\limits_{x\to 3} \dfrac{x^2-x-1}{\sqrt{x+1}}$

17. $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x-2}$

18. $\lim\limits_{x\to 0} (3x-2)$

19. $\lim\limits_{x\to 1} (2x^2-x+4)$

20. $\lim\limits_{x\to -2} \left(x^3-2x^2+4x+8\right)$

21. $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$

22. $\lim\limits_{x\to 2} \dfrac{x^3-8}{x-2}$

23. $\lim\limits_{x\to 3} \dfrac{x^2+x-12}{x-3}$

24. $\lim\limits_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$

25. $\lim\limits_{x\to -2} \dfrac{5}{x+2}$

26. $\lim\limits_{x\to 0} \dfrac{3}{\sqrt{3x+1}+1}$

27. $\lim\limits_{x\to 3} \dfrac{\sqrt{x+1}-1}{x}$

28. $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$

29. $\lim\limits_{x\to -2} \dfrac{|x+2|}{x+2}$

30. $\lim\limits_{x\to 0} (2z-8)^{1/3}$

31. $\lim\limits_{x\to 2} \dfrac{x^2-7x+10}{x-2}$

32. $\lim\limits_{x\to 0} \dfrac{5x^3+8x^2}{3x^4-16x^2}$

33. $\lim\limits_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1}$

34. $\lim\limits_{u\to 1} \dfrac{u^4-1}{u^3-1}$

35. $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x+3}-2}$

36. $\lim\limits_{x\to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$

37. $\lim\limits_{x\to -3} \dfrac{2-\sqrt{x^2-5}}{x+3}$

38. $\lim\limits_{x\to 0} (2\sin x-1)$

39. $\lim\limits_{x\to 0} \sin^2 x$

40. $\lim\limits_{x\to 0} \sec x$

41. $\lim\limits_{x\to 0} \dfrac{1+x+\sin x}{3\cos x}$

42. $\lim\limits_{x\to -\pi} \sqrt{x+4}\cos(x+\pi)$

43. $\lim\limits_{x\to -0.5^-} \sqrt{\dfrac{x+2}{x+1}}$

44. $\lim\limits_{x\to 1^+} \sqrt{\dfrac{x-1}{x+2}}$

45. $\lim\limits_{x\to -2^+} \left(\dfrac{x}{x+1}\right)\left(\dfrac{2x+5}{x^2+x}\right)$

46. $\lim\limits_{x\to 0^+} \dfrac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}$

47. $\lim\limits_{x\to -2^+} (x+3)\dfrac{|x+2|}{x+2}$

48. $\lim\limits_{x\to 1^+} \dfrac{\sqrt{2x}(x-1)}{|x-1|}$

49. $\lim\limits_{x\to 0^-} \dfrac{x}{\sin 3x}$

50. $\lim\limits_{\theta\to 0} \dfrac{\sin\sqrt{2}.\theta}{\sqrt{2}.\theta}$

51. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{4x}$

52. $\lim\limits_{x\to 0} \dfrac{\tan 2x}{x}$

53. $\lim\limits_{x\to 0} 6x^2(\cot x)(\csc 2x)$

54. $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\sin 2\theta}$

55. $\lim\limits_{h\to 0} \dfrac{\sin(\sin h)}{\sin h}$

56. $\lim\limits_{\theta\to 0} \dfrac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}$

57. $\lim\limits_{\theta\to \pi/4} \dfrac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta}$

58. $\lim\limits_{x\to \pi/2} \dfrac{\dfrac{1}{\sqrt{\sin x}}-1}{x+\dfrac{\pi}{2}}$

59. $\lim\limits_{x\to 1} \dfrac{x^3-7x^2+12x}{4-x}$

60. $\lim\limits_{x\to 4} \dfrac{x^3-7x^2+12x}{4-x}$

61. $\lim\limits_{x\to 1} \dfrac{1-x^2}{x^2-8x+7}$

62. $\lim\limits_{x\to 3} \dfrac{\sqrt{3x+16}-5}{x-3}$

63. $\lim\limits_{x\to 3} \dfrac{1}{x-3}\left(\dfrac{1}{\sqrt{x+1}}-\dfrac{1}{2}\right)$

64. $\lim\limits_{x\to 1/3} \dfrac{x-\dfrac{1}{3}}{(3x-1)^2}$

65. $\lim\limits_{x\to 3} \dfrac{x^4-81}{x-3}$

66. $\lim\limits_{x\to 1} \dfrac{x^5-1}{x-1}$

67. $\lim\limits_{x\to 81} \dfrac{\sqrt[4]{x}-3}{x-81}$

68. $\lim\limits_{x\to 1} \dfrac{\sqrt[3]{x}-1}{x-1}$

69. $\lim\limits_{x\to 2} \dfrac{x^5-32}{x-2}$

70. $\lim\limits_{x\to 1} \dfrac{x^6-1}{x-1}$

71. $\lim\limits_{x\to -1} \dfrac{x^7+1}{x+1}$

72. $\lim\limits_{x\to a} \dfrac{x^5-a^5}{x-a}$

73. $\lim\limits_{x\to a} \dfrac{x^n-a^n}{x-a} \quad n\in\mathbb{Z}^+$

74. $\lim\limits_{h\to 0} \dfrac{100}{(10h-1)^{11}+2}$

75. $\lim\limits_{h\to 0} \dfrac{(5+h)^2-25}{h}$

76. $\lim\limits_{x\to 3} \dfrac{\dfrac{1}{x^2+2x}-\dfrac{1}{15}}{x-3}$

77. $\lim\limits_{x\to 1} \dfrac{\sqrt{10x-9}-1}{x-1}$

78. $\lim\limits_{x\to 2} \left(\dfrac{1}{x-2}-\dfrac{2}{x^2-2x}\right)$

79. $\lim\limits_{x\to c} \dfrac{x^2-2cx+c^2}{x-c}$

80. $\lim\limits_{x\to -c} \dfrac{x^2+5cx+4c^2}{x^2+cx}$

81. $\lim\limits_{x\to 16} \dfrac{\sqrt[4]{x}-2}{x-16}$

82. $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$

83. $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{4x+5}-3}$

84. $\lim\limits_{x\to 4} \dfrac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}$

85. $\lim\limits_{x\to 0} \dfrac{x}{\sqrt{ax+1}-1} \quad (a\neq 0)$

86. $\lim\limits_{x\to \pi} \dfrac{\cos^2 x+3\cos x+2}{\cos x+1}$

87. $\lim\limits_{x\to \frac{3\pi}{2}} \dfrac{\sin^2 x+6\sin x+5}{\sin^2 x-1}$

88. $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\sin x-1}{\sqrt{\sin x}-1}$

89. $\lim\limits_{x\to 0} \dfrac{\dfrac{1}{2+\sin x}-\dfrac{1}{2}}{\sin x}$

90. $\lim\limits_{x\to 0} \dfrac{e^{2x}-1}{e^x-1}$

91. $\lim\limits_{x\to \frac{\pi}{4}} \csc x$

92. $\lim\limits_{x\to 4} \dfrac{x-5}{(x^2-10x+24)^2}$

93. $\lim\limits_{x\to 0} \dfrac{\cos x-1}{\sin^2 x}$

94. $\lim\limits_{x\to 0} \dfrac{1-\cos^2 x}{\sin x}$

95. $\lim\limits_{x\to 0} \dfrac{x^3-5x^2}{x^2}$

96. $\lim\limits_{x\to 5} \dfrac{4x^2-100}{x-5}$

97. $\lim\limits_{x\to 3} \dfrac{\sqrt{9-6x+x^2}}{x-3}$

98. $\lim\limits_{x\to 3} \dfrac{\sqrt{9+6x+x^2}}{x-3}$

99. $\lim\limits_{x\to 3} \dfrac{\sqrt{x^2-9}}{x-3}$

100. $\lim\limits_{x\to \frac{4\pi}{3}} \sin x$

101. $\lim\limits_{x\to \frac{2\pi}{3}} \cos x$

102. $\lim\limits_{x\to \frac{7\pi}{4}} \sin x$

103. $\lim\limits_{x\to 1} \dfrac{\sin\sqrt{1-x}}{\sqrt{1-x^2}}$

104. $\lim\limits_{x\to 2} \dfrac{\sin\sqrt{2-x}}{\sqrt{4-x^2}}$

105. $\lim\limits_{x\to 0} \dfrac{\sin\left(\sqrt{5}\,x\right)}{\sin\left(\sqrt{3}\,x\right)}$

106. $\lim\limits_{x\to 0} \dfrac{\sin\left(\sqrt{15}\,x\right)}{\sin\left(\sqrt{3}\,x\right)}$

107. $\lim\limits_{x\to 0^+} \dfrac{x-\sqrt{x}}{\sqrt{\sin x}}$

108. $\lim\limits_{x\to 1} \dfrac{x-\sqrt{x}}{\sqrt{\sin x}}$

109. $\lim\limits_{x\to \pi} \dfrac{x-\sqrt{x}}{\sqrt{\sin x}}$

110. $\lim\limits_{x\to 0} e^{x^3}$

111. $\lim\limits_{x\to 1} e^{x^2}$

112. $\lim\limits_{x\to 1} e^{x^3-1}$

113. $\lim\limits_{x\to -1} e^{x^3-1}$

114. $\lim\limits_{x\to 2} \left(e^{x^2}-\ln x\right)$

115. $\lim\limits_{x\to 1} \left(e^{x^2}-\ln x\right)$

116. $\lim\limits_{x\to e} \ln x$

117. $\lim\limits_{x\to e} \ln x^2$

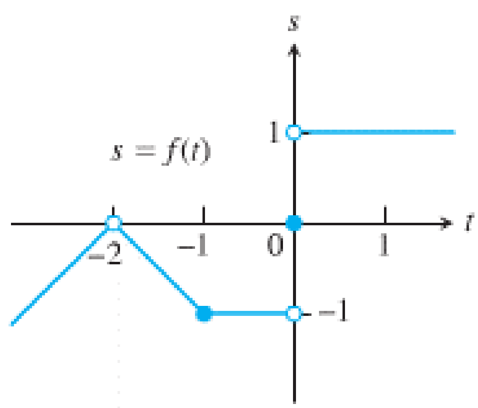
118. $\lim\limits_{x\to 0^+} \ln x$

119. $\lim\limits_{x\to 1} \dfrac{1}{\ln x}$

120. $\lim\limits_{x\to e} \ln e^{2x}$

121. $\lim\limits_{x\to 1} \ln e^{x^2}$

122. For the function $f(t)$ graphed, find the following limits or explain why they do not exist.



a) $\lim\limits_{t\to -2} f(t)$ b) $\lim\limits_{t\to -1} f(t)$ c) $\lim\limits_{t\to 0} f(t)$ d) $\lim\limits_{t\to -0.5} f(t)$

123. Suppose $\lim\limits_{x\to c} f(x)=5$ *and* $\lim\limits_{x\to c} g(x)=-2$. Find

a) $\lim\limits_{x\to c} f(x)g(x)$

b) $\lim\limits_{x\to c} 2f(x)g(x)$

c) $\lim\limits_{x\to c} \left(f(x)+3g(x)\right)$

d) $\lim\limits_{x\to c} \dfrac{f(x)}{f(x)-g(x)}$

124. Explain why the limits do not exist for $\lim\limits_{x\to 0} \dfrac{x}{|x|}$

(**125 – 126**) Evaluate the limit using the form $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ for

125. $f(x)=x^2, \quad x=1$

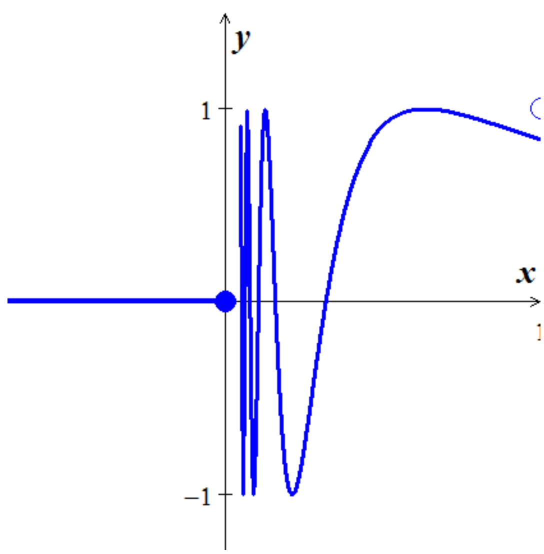
126. $f(x)=\sqrt{3x+1}, \quad x=0$

127. If $\lim\limits_{x\to 4} \dfrac{f(x)-5}{x-2}=1$, find $\lim\limits_{x\to 4} f(x)$

128. If $\lim\limits_{x\to 0} \dfrac{f(x)}{x^2}=1$, find $\lim\limits_{x\to 0} f(x)$ *and* $\lim\limits_{x\to 0} \dfrac{f(x)}{x}$

129. If $x^4 \le f(x) \le x^2 \quad -1 \le x \le 1$ and $x^2 \le f(x) \le x^4 \quad x<-1$ *and* $x>1$. At what points c do you automatically know $\lim\limits_{x\to c} f(x)$? What can you say about the value of the limits at these points?

130. Let $f(x)=\begin{cases} 0, & x\le 0 \\ \sin\dfrac{1}{x}, & x>0 \end{cases}$

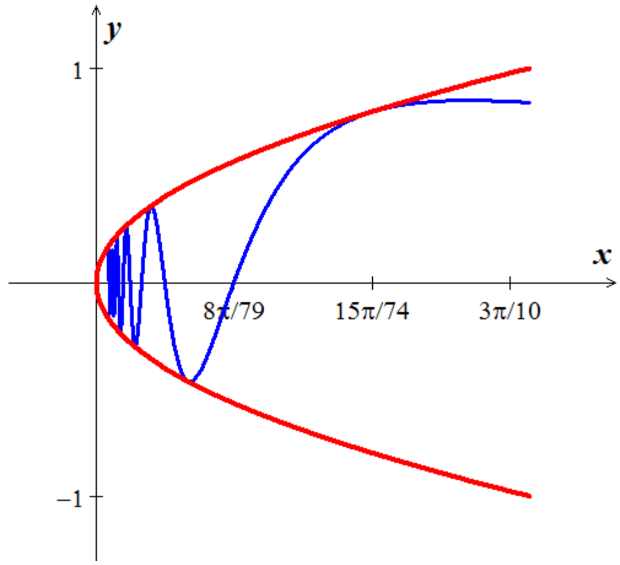


a) Does $\lim\limits_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

131. Let $g(x) = \sqrt{x}\ \sin\frac{1}{x}$

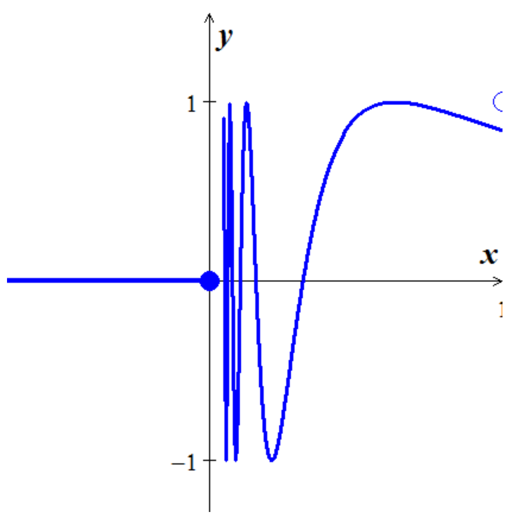


a) Does $\lim\limits_{x \to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x \to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x \to 0} g(x)$ exist? If so, what is it? If not, why not?

132. Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$



d) Does $\lim\limits_{x \to 0^+} f(x)$ exist? If so, what is it? If not, why not?

e) Does $\lim\limits_{x \to 0^-} f(x)$ exist? If so, what is it? If not, why not?

f) Does $\lim_{x \to 0} f(x)$ exist? If so, what is it? If not, why not?

133. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

a) $\lim_{x \to -1^+} f(x) = 1$

b) $\lim_{x \to 0^-} f(x) = 0$

c) $\lim_{x \to 0^-} f(x) = 1$

d) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

e) $\lim_{x \to 0} f(x)$ *exists*

f) $\lim_{x \to 0} f(x) = 0$

g) $\lim_{x \to 0} f(x) = 1$

h) $\lim_{x \to 1} f(x) = 1$

i) $\lim_{x \to 1} f(x) = 0$

j) $\lim_{x \to 2^-} f(x) = 2$

k) $\lim_{x \to -1^-} f(x) = 0$ *does not exist*

l) $\lim_{x \to 2^+} f(x) = 0$