Similar Matrices

B is similar A

$$B = P^{-1} A P \quad \text{or} \quad A = P B P^{-1}$$

Ex

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} cd & d^2 \\ -c^2 & -cd \end{pmatrix}$$

$B = \mathbf{0}$

$tr(A) = 0 = tr(B)$

$|A| = |D| = 0$

#34

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$$

$|A| = 1 \neq |B| = -2$

Not similar

$|A| = |B| \Rightarrow A, B$ are similar

$|A| \neq |B| \quad A + B$ are is similar.

Fibonacci numbers

$0, 1, 1, 2, 3, 5, 8, \ldots$

$$F_{k+2} = F_{k+1} + F_k$$

F_{100}

$$\vec{u}_k = \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} \qquad \left\{ \begin{array}{l} F_{k+2} = F_{k+1} + F_k \\ F_{k+1} = F_{k+1} \end{array} \right\}$$

$$\vec{u}_{k+1} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad F_{100} = \begin{pmatrix} F_{101} \\ F_{100} \end{pmatrix}$$

$$\vec{u}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{u}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \vec{u}_3 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \vec{u}_4 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$\vec{u}_{100} = \begin{pmatrix} F_{101} \\ F_{100} \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2 - \lambda - 1 = 0$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

For $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $(A - \lambda_1 I) V_1 = 0$

$$\rightarrow \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{5}}{2} & 1 \\ 1 & -\frac{1}{2} - \frac{\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x = \lambda_1 y_1$$

$$V_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$$

For $\lambda_2 \Rightarrow \quad (A - \lambda_2 I) V_2 = 0$

$$\begin{pmatrix} 1-\lambda_2 & 1 \\ 1 & -\lambda_2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x - \lambda_2 y_2$$

$$V_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

$$S = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$$

$$S^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$$

$$\vec{u}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} - \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

$$\vec{u}_0 = \frac{1}{\lambda_1 - \lambda_2} (\vec{V}_1 - \vec{V}_2)$$

$$\vec{u}_0 = \frac{\vec{v}_1 - \vec{v}_2}{\lambda_1 - \lambda_2}$$

$$\vec{u}_{100} = \frac{(\lambda_1)^{100}\vec{v}_1 - (\lambda_2)^{10}\vec{v}_2}{\lambda_1 - \lambda_2}$$

$$\vec{F}_{100} = \underbrace{\frac{1}{\lambda_1 - \lambda_2}\left((\lambda_1)^{100} - (\lambda_2)^{100}\right)}_{\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{100} - \left(\frac{1-\sqrt{5}}{2}\right)^{100}\right)}$$

Jordan Form

$$J_i = \begin{bmatrix} \lambda_1 & 1 & 0 & \cdots & 0 \\ & \lambda_2 & 1 & \ddots & 0 \\ & & \ddots & \ddots & 1 \\ & 0 & & & \lambda_n \end{bmatrix}$$

4.6 Orthogonal Diagonalization

Defn $A_{(n \times n)}$ is called orthogonally diagonalizable $\exists$ orthogonal P

$$P^{-1}AP = D$$

Theorem
A is orthogonally diagonalizable
A has an orthonormal set eigenvectors
A is symmetric

Ex

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

orthogonal
P?
diagonalizes

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{vmatrix}$$

$$= (4-\lambda)^3 + 16 - 12(4-\lambda)$$

$$= 64 - 48\lambda + 12\lambda^2 - \lambda^3 + 16 - 48 + 12\lambda$$

$$= -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

$$\underline{\lambda_{1,2,3} = 2, 2, 8}$$

$$\begin{array}{c|cccc} 2 & -1 & 12 & -36 & 32 \\ & & -2 & 20 & -32 \\ \hline & -1 & 10 & -16 & \boxed{0} \end{array}$$

For $\lambda_1 = 2 \Rightarrow (A - \lambda_1 I) V_1 = 0$

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$2x_1 + 2y_1 + 2z_1 = 0$

$$x_1 + y_1 + z_1 = 0$$

if $z_1 = 0 \Rightarrow x_1 = -y_1$ $\qquad V_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

if $y_1 = 0 \Rightarrow x_1 = -z_1$ $\qquad V_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

For $\lambda_3 = 8 \Rightarrow (A - \lambda_3 I) V_3 = 0$

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2x_3 + y_3 + z_3 = 0 \\ x_3 - 2y_3 + z_3 = 0 & (1) \\ x_3 + y_3 - 2z_3 = 0 & (2) \end{cases}$$

$\boxed{D = 0}$

$(1) - (2) \Rightarrow -3y_3 + 3z_3 = 0 \Rightarrow y_3 = z_3$

$$-2x_3 = -2y_3$$

$$x_3 = y_3$$

$V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$V_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad V_2 = \begin{pmatrix} -1 \\ 0 \\ +1 \end{pmatrix} \quad V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{u}_1 = \frac{(-1, 1, 0)}{\sqrt{2}} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\,\vec{u}_1$$

$$= (-1, 0, 1) - \left((-1, 0, 1) \cdot \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)\right)\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= (-1, 0, 1) - \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= \left(-\frac{1}{2}, -\frac{1}{2}, 1\right)$$

$$\vec{u}_2 = \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4} + 1}}\left(-\frac{1}{2}, -\frac{1}{2}, 1\right)$$

$$= \frac{2}{\sqrt{6}}\left(-\frac{1}{2}, -\frac{1}{2}, 1\right)$$

$$= \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$

$$= \begin{pmatrix}1\\1\\1\end{pmatrix} - \underbrace{\left(\begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}-\frac{1}{\sqrt{2}}\\ 1/\sqrt{2}\\ 0\end{pmatrix}\right)}_{=0} \vec{u}_1$$

$$- \underbrace{\left(\begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}-\frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}}\\ 2/\sqrt{6}\end{pmatrix}\right)}_{=0} \vec{u}_2$$

$$\vec{w}_3 = (1, 1, 1)$$

$$\vec{u}_3 = \frac{(1, 1, 1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$ orthogonal

$$P^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$P^{-1} A P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

Matrix P to be orthogonal $PP^T = I$

If you found P (orthogonal)

$$P^{-1} = P^T$$

$$A = \lambda_1 \vec{u}_1 \vec{u}_1^T + \lambda_2 \vec{u}_2 \vec{u}_2^T + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T$$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}$$

$$= \lambda^2 + \lambda - 6 = 0$$

$$\lambda_{1,2} = -3, 2$$

For $\lambda_1 = -3$ $\quad (A - \lambda_1 I) V_1 = 0$

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow 2x_1 = -y_1$$

$$V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

For $\lambda_2 = 2 \Rightarrow (A - \lambda_2 I) V_2 = 0$

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x_2 = 2y_2$$

$$V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix}1\\-2\end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix}2\\1\end{pmatrix}$$

$$\vec{u}_1 = \frac{(1,-2)}{\sqrt{1+4}} = \left(\frac{1}{\sqrt5}, \frac{-2}{\sqrt5}\right)$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\,\vec{u}_1$$

$$= \begin{pmatrix}2\\1\end{pmatrix} - \left(\begin{pmatrix}2\\1\end{pmatrix} \cdot \begin{pmatrix}\frac{1}{\sqrt5}\\ \frac{-2}{\sqrt5}\end{pmatrix}\right)\begin{pmatrix}\frac{1}{\sqrt5}\\ \frac{-2}{\sqrt5}\end{pmatrix}$$

$$= \begin{pmatrix}2\\1\end{pmatrix}$$

$$\vec{u}_2 = \left(\frac{2}{\sqrt5}, \frac{1}{\sqrt5}\right)$$

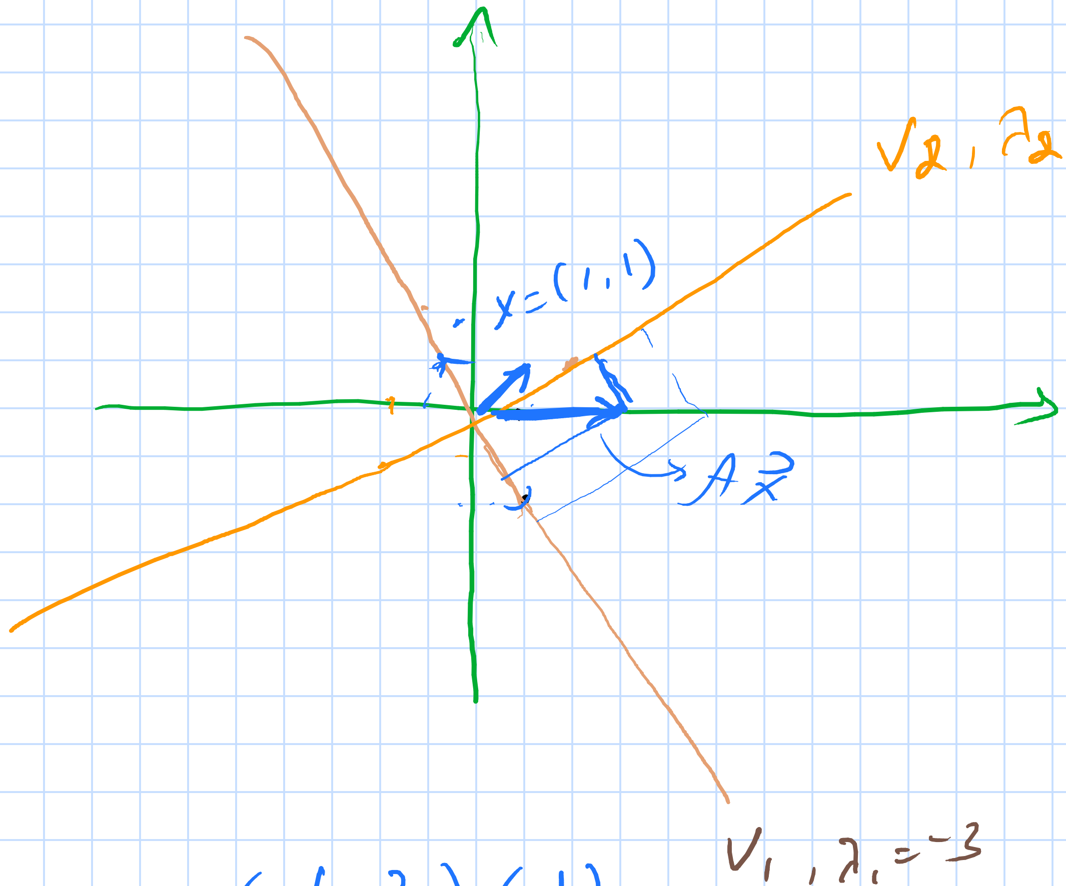
$$\begin{array}{cc} 7 & 6 \\ \multicolumn{2}{c}{\begin{pmatrix}\frac{1}{5} & \frac{-2}{5}\\ \frac{-2}{5} & \frac{4}{5}\end{pmatrix}} \end{array}$$

$$\begin{pmatrix}1 & 2\\2 & -2\end{pmatrix} = -3\begin{pmatrix}\frac{1}{\sqrt5}\\ \frac{-2}{\sqrt5}\end{pmatrix}\left[\frac{1}{\sqrt5} \quad \frac{-2}{\sqrt5}\right]$$

$$+\, 2\begin{pmatrix}\frac{2}{\sqrt5}\\ \frac{1}{\sqrt5}\end{pmatrix}\left[\frac{2}{\sqrt5} \quad \frac{1}{\sqrt5}\right]$$

$$= \begin{pmatrix}-\frac{3}{5} & \frac{6}{5}\\ \frac{6}{5} & \frac{-12}{5}\end{pmatrix} + \begin{pmatrix}\frac{8}{5} & \frac{4}{5}\\ \frac{4}{5} & \frac{2}{5}\end{pmatrix}$$

$$= \begin{pmatrix}1 & 2\\2 & -2\end{pmatrix} \checkmark$$



$$A\vec{x} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$