Section 2.5 – Numerical Integration

Absolute and Relative Error

Definition

Suppose c is a computed numerical solution to a problem having an exact solution x.
There are two common measures of the error in c as an approximation to x:

$$\textbf{\textit{absolute error}} = |c - x| \qquad \& \qquad \textbf{\textit{relative error}} = \frac{|c-x|}{|x|} \quad (if \ x \neq 0)$$

Example

The ancient Greeks used $\frac{22}{7}$ to approximate the value of π. Determine the absolute and relative error in this approximation to π.

Solution

$$absolute\ error = \left|\frac{22}{7} - \pi\right|$$

$$\approx 0.00126$$

$$relative\ error = \frac{\left|\frac{22}{7} - \pi\right|}{\pi}$$

$$\approx .000402$$

$$\approx .04\%$$

Midpoint Rule

Definition

Suppose f is defined and integrable on $[a,\ b]$. The ***midpoint Rule Approximation*** to $\int_a^b f(x)\,dx$ using n equally spaced subintervals on $[a,\ b]$ is

$$M(n) = f(m_1)\Delta x + f(m_2)\Delta x + \cdots + f(m_n)\Delta x$$

$$= \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)\Delta x$$

Where $\Delta x = \dfrac{b-a}{n}$,

$x_0 = a, \qquad x_k = a + k\Delta x$

$m_k = \dfrac{x_{k-1}+x_k}{2}$ is the midpoint of $\left[x_{k-1},\ x_k\right]$, for $k = 1,\ 2,\ \ldots,\ n$.

Example

Approximate $\int_2^4 x^2\,dx$ using the Midpoint Rule with $n = 4$ subinterval

Solution

With $a = 2,\ b = 4 \ \rightarrow\ \Delta x = \dfrac{4-2}{4} = 0.5$

The grid points are:

$x_0 = 2$

$x_1 = 2 + 0.5 \ \underline{= 2.5|}$

$x_2 = 2 + 2(.5) \ \underline{= 3|}$

$x_3 = 2 + 3(0.5) \ \underline{= 3.5|}$

$x_4 = 2 + 4(0.5) \ \underline{= 4|}$

$m_1 = \dfrac{2.5+2}{2} \ \underline{= 2.25|}$

$m_2 = \dfrac{3+2.5}{2} \ \underline{= 2.75|}$

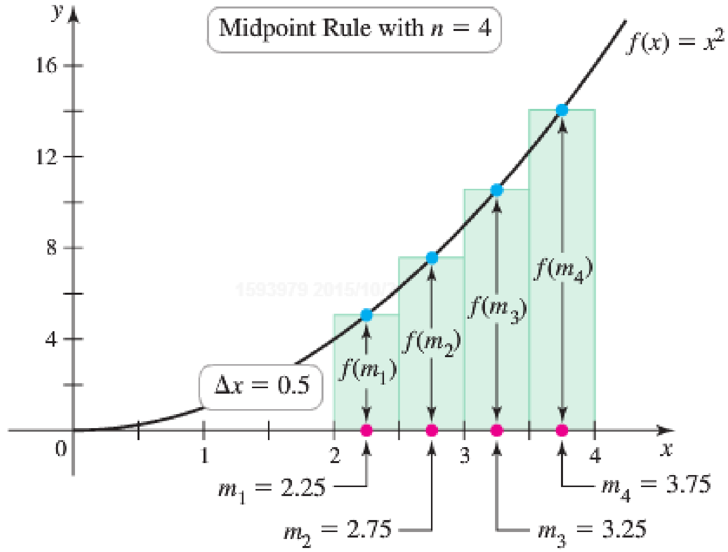
$m_3 = \dfrac{2+3.5}{2} \ \underline{= 3.25|}$

$m_4 = \dfrac{3.5+4}{2} \ \underline{= 3.75|}$

$$M(4) = f(m_1)\Delta x + f(m_2)\Delta x + f(m_3)\Delta x + f(m_4)\Delta x$$

$$= \left(2.25^2 + 2.75^2 + 3.25^2 + 3.75^2\right)(0.5)$$

$$= 18.625$$



$$Exact = \int_2^4 x^2 dx$$

$$= \frac{1}{3}x^3 \Big|_2^4$$

$$= \frac{56}{3}$$

$$absolute\ error = \left|18.625 - \frac{56}{3}\right|$$

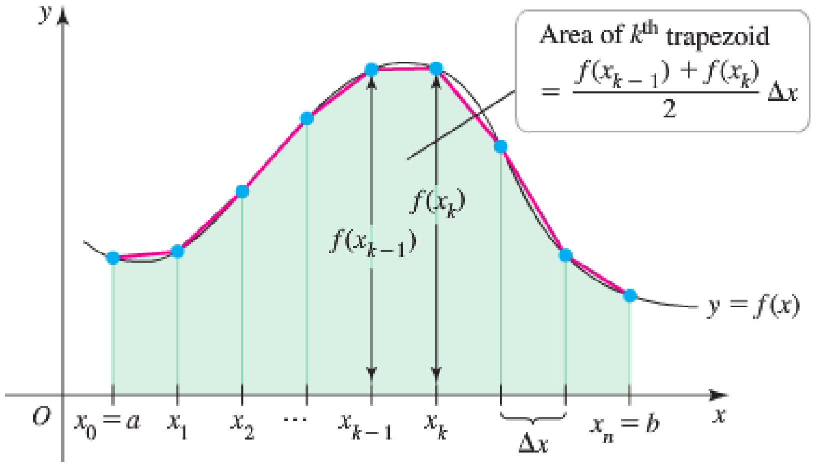
$$\approx 0.0417$$

$$relative\ error = \frac{\left|18.625 - \frac{56}{3}\right|}{\frac{56}{3}}$$

$$\approx .00223 \ = .223\%$$

Trapezoid Approximations

The ***Trapezoid Rule*** for the value of a definite integral is based on approximating the region between a curve and the x-axis with trapezoids instead of rectangles.



The length of each subinterval is $\Delta x = \dfrac{b-a}{n}$ is called the ***step size*** or ***mesh size***.

The area of a trapezoid: $\Delta x \cdot \left(\dfrac{y_{i-1} + y_i}{2} \right)$

The area is the approximation by adding the areas of all trapezoids:

$$\begin{aligned} T &= \tfrac{1}{2}\left(y_0 + y_1\right)\Delta x + \tfrac{1}{2}\left(y_1 + y_2\right)\Delta x + \cdots + \tfrac{1}{2}\left(y_{n-2} + y_{n-1}\right)\Delta x + \tfrac{1}{2}\left(y_{n-1} + y_n\right)\Delta x \\ &= \tfrac{1}{2}\Delta x\left(y_0 + y_1 + y_1 + y_2 + \cdots + y_{n-2} + y_{n-1} + y_{n-1} + y_n\right) \\ &= \tfrac{1}{2}\Delta x\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-2} + 2y_{n-1} + y_n\right) \end{aligned}$$

The Trapezoid Rule

If f is continuous on $[a, b]$ and if a regular partition of $[a, b]$ is determined by the numbers $a = x_0,\ x_1,\ \ldots,\ x_n = b$, then

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f\left(x_0\right) + 2f\left(x_1\right) + 2f\left(x_2\right) + \ldots + 2f\left(x_{n-1}\right) + f\left(x_n\right) \right]$$

$$T(n) = \left(\frac{1}{2} f\left(x_0\right) + \sum_{k=1}^{n-1} f\left(x_k\right) + \frac{1}{2} f\left(x_n\right) \right) \Delta x$$

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \ \ldots, \quad x_{n-1} = a + (n-1)\Delta x, \quad x_n = b$$

Where $\Delta x = \dfrac{b-a}{n}$ and $x_0 = a, \quad x_k = a + k\Delta x$

Error Estimate for the Trapezoidal Rule

If M is a positive real number such that $\left|f''(x)\right| \le M$ for all x in $[a, b]$, then the error involved in using the Trapezoidal Rule is not greater than $\dfrac{M(b-a)^3}{12n^2}$

Example

Use the Trapezoid Rule with $n = 4$ to estimate $\int_1^2 x^2 dx$. Compare the estimate with the exact value.

Solution

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$$

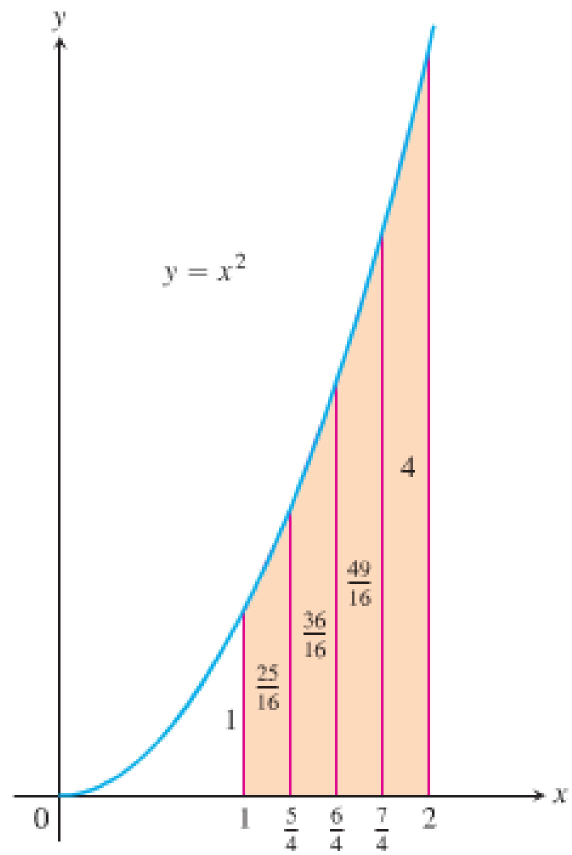
$$x_0 = 1$$

$$x_1 = 1 + \frac{1}{4} = \frac{5}{4}$$

$$x_2 = 1 + 2\left(\frac{1}{4}\right) = \frac{6}{4}$$

$$x_3 = 1 + 3\left(\frac{1}{4}\right) = \frac{7}{4}$$

$$x_4 = 2$$



$$T = \frac{1}{2}\Delta x\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right)$$

$$= \frac{1}{2}\cdot\frac{1}{4}\left(1^2 + 2\left(\frac{5}{4}\right)^2 + 2\left(\frac{6}{4}\right)^2 + 2\left(\frac{7}{4}\right)^2 + 2^2\right)$$

$$= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right)$$

$$= \frac{75}{32}$$

$$\approx 2.34375$$

$$\int_1^2 x^2 dx = \frac{1}{3}x^3 \Big|_1^2$$

$$= \frac{1}{3}\left(2^3 - 1^3\right)$$

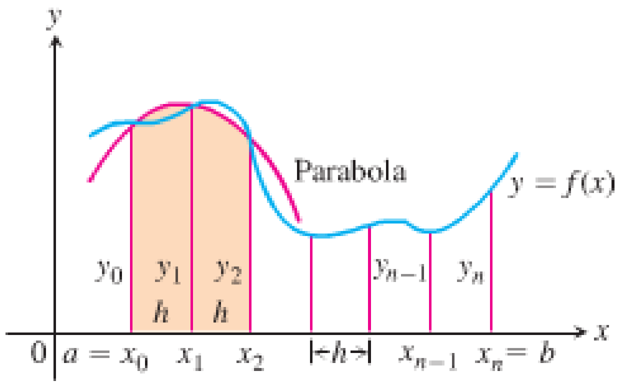
$$= \frac{7}{3} \approx 2.3333$$

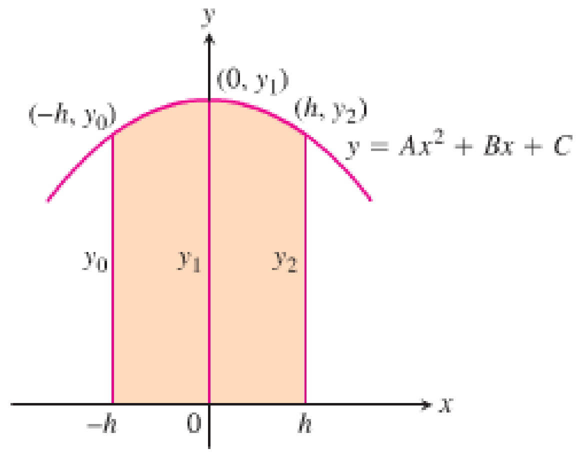
The difference: $2.34375 - 2.3333 \approx 0.01042$

The percentage error: $\dfrac{2.34375 - 2.33333}{2.33333} \approx 0.004466 \quad .446\%$

Simpson's Rule: Approximations Using Parabolas

We partition the interval $[a, b]$ into n subintervals of equal length $h = \Delta x = \dfrac{b-a}{n}$ *n : even number*





The parabola has an equation of the form: $y = Ax^2 + Bx + C$

So the area under it from $x = -h$ to $x = h$ is

$$\begin{aligned}
A_p &= \int_{-h}^{h} \left(Ax^2 + Bx + C\right) dx \\
&= \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx \Big]_{-h}^{h} \\
&= \frac{A}{3}h^3 + \frac{B}{2}h^2 + Ch - \left(\frac{A}{3}(-h)^3 + \frac{B}{2}(-h)^2 + C(-h)\right) \\
&= \frac{A}{3}h^3 + \frac{B}{2}h^2 + Ch + \frac{A}{3}h^3 - \frac{B}{2}h^2 + Ch \\
&= \frac{2}{3}Ah^3 + 2Ch \\
&= \frac{h}{3}\left(2Ah^2 + 6C\right)
\end{aligned}$$

Since the curve passes through the three points $\left(-h, y_0\right)$, $\left(0, y_1\right)$, *and* $\left(h, y_2\right)$

$$y_0 = Ah^2 - Bh + C \quad y_1 = C \quad y_2 = Ah^2 + Bh + C$$

$$C = y_1, \quad Ah^2 - Bh = y_0 - y_1$$

$$\underline{Ah^2 + Bh = y_2 - y_1}$$

$$2Ah^2 = y_0 - 2y_1 + y_2$$

$$\begin{aligned}
A_p &= \frac{h}{3}\left(2Ah^2 + 6C\right) \\
&= \frac{h}{3}\left(y_0 - 2y_1 + y_2 + 6y_1\right) \\
&= \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)
\end{aligned}$$

Computing the areas under all the parabolas and adding the results gives the approximation

$$\int_a^b f(x)dx \approx \frac{h}{3}\left(y_0+4y_1+y_2\right)+\frac{h}{3}\left(y_2+4y_3+y_4\right)+\cdots+\frac{h}{3}\left(y_{n-2}+4y_{n-1}+y_n\right)$$

$$=\frac{h}{3}\left(y_0+4y_1+2y_2+4y_3+2y_4+\cdots+2y_{n-2}+4y_{n-1}+y_n\right)$$

Simpson's Rule

To approximate $\int_a^b f(x)dx$, use $S=\frac{\Delta x}{3}\left(y_0+4y_1+2y_2+4y_3+2y_4+\cdots+2y_{n-2}+4y_{n-1}+y_n\right)$

$x_0=a,\ x_1=a+\Delta x,\ x_2=a+2\Delta x,\ \ldots,\ x_{n-1}=a+(n-1)\Delta x,\ x_n=b$

Where $\Delta x=\frac{b-a}{n}$

$$\int_a^b f(x)dx \approx \frac{b-a}{3n}\left[f\left(x_0\right)+4f\left(x_1\right)+2f\left(x_2\right)+4f\left(x_3\right)+\ldots+2f\left(x_{n-2}\right)+4f\left(x_{n-1}\right)+f\left(x_n\right)\right]$$

Error Estimate for the Trapezoidal Rule

If M is a positive real number such that $\left|f^{(4)}(x)\right|\le M$ for all x in $[a, b]$, then the error involved in using the Simpson's Rule is not greater than $\dfrac{M(b-a)^5}{180n^4}$

Example

Use Simpson's Rule with $n=4$ to approximate $\int_0^2 5x^4dx$

Solution

$$\Delta x=\frac{b-a}{n}=\frac{2-0}{4}=\frac{1}{2}$$

$$x_0=0,\quad x_1=\frac{1}{2},\quad x_2=0+2\frac{1}{2}=1,\quad x_3=0+3\frac{1}{2}=\frac{3}{2},\quad x_4=2$$

$$S=\frac{\Delta x}{3}\left(y_0+4y_1+2y_2+4y_3+y_4\right)$$

$$=\frac{1}{3}\frac{1}{2}\left(5(0)^4+4(5)\left(\frac{1}{2}\right)^4+2(5)(1)^4+4(5)\left(\frac{3}{2}\right)^4+5(2)^4\right)$$

$=\frac{1}{6}\left(0+\frac{5}{4}+10+\frac{405}{4}+80\right)$

$=\frac{1}{6}\left(\frac{385}{2}\right)$

$=\frac{385}{12}$

≈ 32.08333 *The exact value is* 32.

Example

The table lists rates of change $s'(t)$ in global sea level $s(t)$ in various years from 1995 $(t=0)$ to 2011 $(t=16)$, with rates of change reported in *mm/yr*.

Years	1995	1998	2000	2002	2003	2007	2009	2011
t	0	3	5	7	8	12	14	16
$s'(t)$ *(mm/yr)*	0.51	5.19	4.39	2.21	5.24	0.63	4.19	2.38

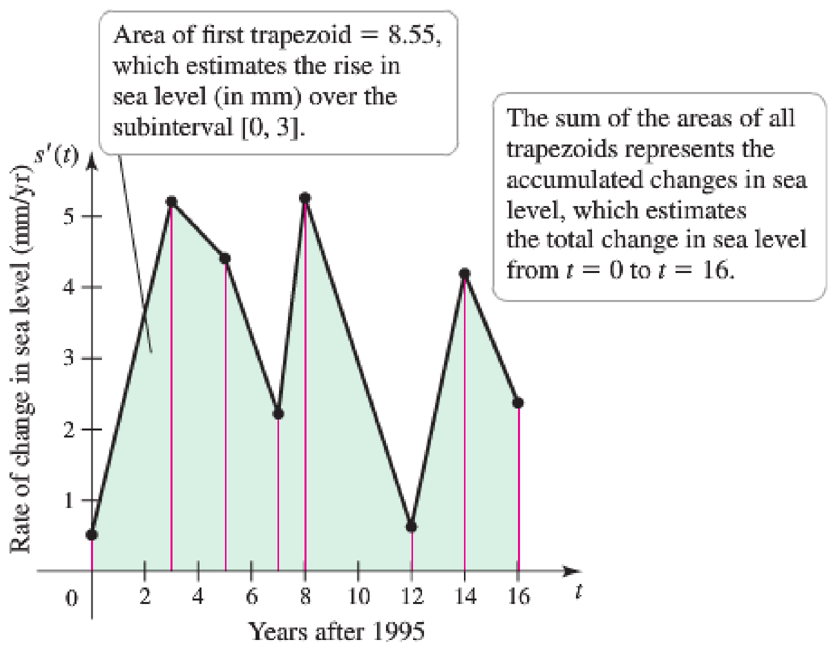
a) Assuming $s'(t)$ is continuous on $[0, 16]$, explain how a definite integral can be used to find the net change in sea level from 1995 to 2011; then write the definite integral.

b) Use the data in the table and generalize the trapezoid Rule to estimate the value of the integral from part (*a*).

Solution

a) The net charge in any quantity Q over the interval $[a, b]$ is $Q(b)-Q(a)$

Net change in $s(t) = S(b)-S(a)=\int_0^{16} s'(t)\,dt$



b) From the figure the values accompanied by 7 trapezoids whose area approximates $\int_0^{16} s'(t)\,dt$

Area of the ***first*** trapezoid:

$$T_1 = \frac{1}{2}\left(s'(0)+s'(3)\right)\cdot 3$$
$$= \frac{1}{2}(0.51+5.19)\cdot 3$$
$$\underline{= 8.55|}$$

$$T_2 = \frac{1}{2}\left(s'(3)+s'(5)\right)\cdot 2$$
$$= \frac{1}{2}(5.19+4.39)\cdot 2$$
$$\underline{= 9.58|}$$

$$T_3 = \frac{1}{2}\left(s'(5)+s'(7)\right)\cdot 2$$
$$= \frac{1}{2}(4.39+2.21)\cdot 2$$
$$\underline{= 6.6|}$$

$$T_4 = \frac{1}{2}\left(s'(7)+s'(8)\right)\cdot 1$$
$$= \frac{1}{2}(2.21+5.24)$$
$$\underline{= 3.725|}$$

$$T_5 = \frac{1}{2}\left(s'(8)+s'(12)\right)\cdot 4$$
$$= \frac{1}{2}(5.24+0.63)\cdot 4$$
$$\underline{= 11.74|}$$

$$T_6 = \frac{1}{2}\left(s'(12)+s'(14)\right)\cdot 2$$
$$= \frac{1}{2}(0.63+4.19)\cdot 2$$
$$\underline{= 4.82|}$$

$$T_7 = \frac{1}{2}\left(s'(14)+s'(16)\right)\cdot 2$$
$$= \frac{1}{2}(4.19+2.38)\cdot 2$$
$$\underline{= 6.57|}$$

$$T(7) = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7$$
$$\underline{\approx 51.585 \;\; mm\;|}$$

Exercises Section 2.5 – Numerical Integration

Find the *Midpoint* Rule approximations to

1. $\int_0^1 \sin \pi x \, dx \quad n = 6 \ \textit{subintervals}$

2. $\int_0^\pi x^2 \sin x \, dx \quad n = 8 \ \textit{subintervals}$

3. $\int_0^1 e^{-\sqrt{x}} dx \quad n = 6 \ \textit{subintervals}$

4. $\int_0^1 e^{-x} dx \quad n = 8 \ \textit{subintervals}$

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of 10^{-4} by (***a***) the *Trapezoid* Rule and (***b***) *Simpson's* Rule.

5. $\int_1^3 (2x-1) dx$

6. $\int_{-1}^1 \left(x^2+1\right) dx$

7. $\int_2^4 \frac{1}{(s-1)^2} ds$

Find the *Trapezoid* & *Simpson's* Rule approximations and error to

8. $\int_0^1 \sin \pi x \, dx \quad n = 6 \ \textit{subintervals}$

9. $\int_0^1 e^{-x} dx \quad n = 8 \ \textit{subintervals}$

10. $\int_1^5 \left(3x^2 - 2x\right) dx \quad n = 8 \ \textit{subintervals}$

11. $\int_0^{\pi/4} 3 \sin 2x \, dx \quad n = 8 \ \textit{subintervals}$

12. $\int_0^8 e^{-2x} \, dx \quad n = 8 \ \textit{subintervals}$

13. $\int_{-1}^1 \sqrt{x^2+1} \, dx \quad n = 8 \ \textit{subintervals}$

14. $\int_0^{1/2} \sin\left(x^2\right) dx \quad n = 4 \ \textit{subintervals}$

15. $\int_{\pi/2}^\pi \frac{\sin x}{x} \, dx \quad n = 6 \ \textit{subintervals}$

16. $\int_0^{\pi/4} x \tan x \, dx \quad n = 6 \ \textit{subintervals}$

17. $\int_0^1 e^{-x^2} \, dx \quad n = 10 \ \textit{subintervals}$

18. $\int_0^2 \frac{1}{\sqrt{1+x^2}} \, dx \quad n = 10 \ \textit{subintervals}$

19. $\int_0^{1/2} \sin\left(e^{x/2}\right) dx \quad n = 8 \ \textit{subintervals}$

20. $\int_2^3 \frac{1}{\ln x} \, dx \quad n = 10 \ \textit{subintervals}$

21. $\int_1^2 e^{1/x} \, dx \quad n = 4 \ \textit{subintervals}$

22. $\displaystyle\int_0^1 \ln\left(1+e^x\right)\, dx \quad n=8 \;\; subintervals$

23. $\displaystyle\int_0^1 x^5 e^x\, dx \quad n=10 \;\; subintervals$

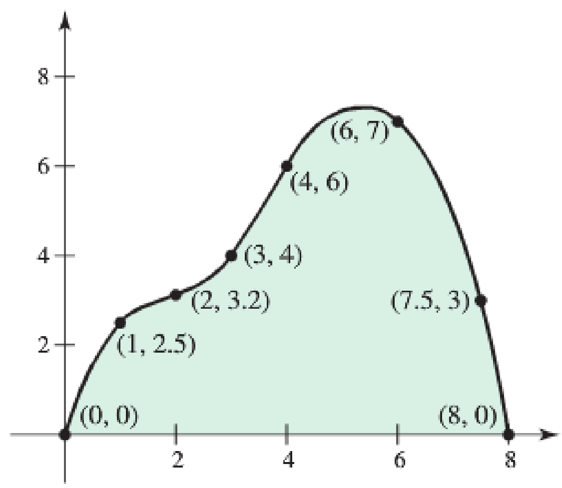
24. $\displaystyle\int_0^4 \sqrt{x}\, \sin x\, dx \quad n=8 \;\; subintervals$

25. $\displaystyle\int_0^3 \frac{1}{1+x^5}\, dx \quad n=6 \;\; subintervals$

26. $\displaystyle\int_1^4 \frac{e^x}{x}\, dx \quad n=10 \;\; subintervals$

27. $\displaystyle\int_1^2 \frac{dx}{x} \quad n=10 \;\; subintervals$

28. A piece of wood paneling must be cut in the shape shown below. The coordinates of several point on its curved surface are also shown (with units of inches).



a) Estimate the surface area of the paneling using the Trapezoid Rule
b) Estimate the surface area of the paneling using a left Riemann sum.
c) Could two identical pieces be cut from a 9-*in* by 9-*in* piece of wood?

29. The region bounded by the curves $y=\dfrac{1}{1+e^{-x}}$, $x=0$ and $x=10$ is rotated about $x-axis$. Use Simpson's Rule with $n=10$ to estimate the volume of the resulting solid.

30. A pendulum with length *L* that makes a maximum angle θ_0 with the vertical. Using Newton's Second Law it can be shown that the period *T* (the time for one complete swing) is given by

$$T=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2 x}}$$

Where $k=\sin\left(\frac{1}{2}\theta_0\right)$ and *g* is the acceleration due to gravity. If $L=1\ m$ and $\theta_0=42°$, use Simpson's Rule with $n=10$ to find the period.