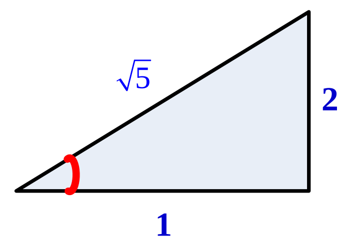
Solution   *Section* 4.2 – Calculus with Parametric Curves

Exercise

Find all the points at which the curve has the given slope. $x = 4\cos t, \quad y = 4\sin t; \quad slope = \frac{1}{2}$

Solution



$$\frac{dy}{dx} = \frac{4\cos t}{-4\sin t} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= -\cot t = \frac{1}{2}$$

$$\cot t = -\frac{1}{2} \quad \Rightarrow \quad t = \cot^{-1}\left(-\frac{1}{2}\right) \quad t \in QII \ \& \ QIV$$

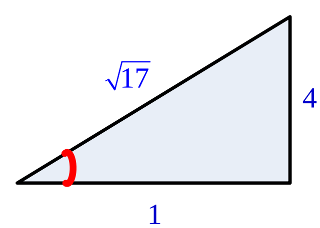
$$x = -\cos\left(\cot^{-1}\frac{1}{2}\right) = -\frac{1}{\sqrt{5}}, \quad y = 4\sin\left(\cot^{-1}\frac{1}{2}\right) = \frac{8}{\sqrt{5}}; \quad \left(-\frac{\sqrt{5}}{5}, \ \frac{8\sqrt{5}}{5}\right)$$

$$x = \cos\left(\cot^{-1}\frac{1}{2}\right) = \frac{1}{\sqrt{5}}, \quad y = -4\sin\left(\cot^{-1}\frac{1}{2}\right) = -\frac{8}{\sqrt{5}}; \quad \left(\frac{\sqrt{5}}{5}, \ -\frac{8\sqrt{5}}{5}\right)$$

Exercise

Find all the points at which the curve has the given slope. $x = 2\cos t, \quad y = 8\sin t; \quad slope = -1$

Solution



$$\frac{dy}{dx} = \frac{8\cos t}{-2\sin t} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= -4\cot t = -1$$

$$\cot t = \frac{1}{4} \quad \Rightarrow \quad t = \cot^{-1}\left(\frac{1}{4}\right) \quad t \in QI \ \& \ QIII$$

$$x = 2\cos\left(\cot^{-1}\frac{1}{4}\right) = \frac{2}{\sqrt{17}}, \quad y = 8\sin\left(\cot^{-1}\frac{1}{4}\right) = \frac{32}{\sqrt{5}}; \quad \left(\frac{2\sqrt{17}}{17}, \ \frac{32\sqrt{17}}{17}\right)$$

$$x = -2\cos\left(\cot^{-1}\frac{1}{4}\right) = -\frac{2}{\sqrt{17}}, \quad y = -8\sin\left(\cot^{-1}\frac{1}{4}\right) = -\frac{32}{\sqrt{5}}; \quad \left(-\frac{2\sqrt{17}}{17}, \ -\frac{32\sqrt{17}}{17}\right)$$

Exercise

Find all the points at which the curve has the given slope. $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}; \quad slope = 1$

Solution

$$\frac{dy}{dx} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$=\frac{t^2+1}{t^2-1}=1$$

$t^2+1\neq 1$ $\therefore$ There are no points on this curve with slope 1.

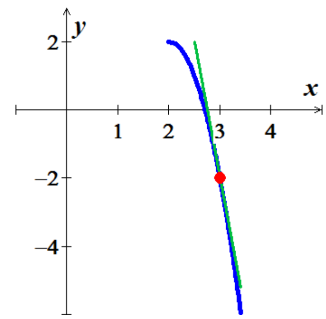
Exercise

Find all the points at which the curve has the given slope. $x=2+\sqrt{t},\quad y=2-4t;\quad slope=-8$

Solution

$$\frac{dy}{dx}=\frac{-4}{\frac{1}{2\sqrt{t}}}=-8\sqrt{t}=-8 \qquad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

$t=1 \quad \rightarrow \quad (3,-2)$



Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of t.

$$x=\sin t,\quad y=\cos t,\quad t=\frac{\pi}{4}$$

Solution

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-\sin t}{\cos t}$$

$$=-\tan t\Big|_{t=\frac{\pi}{4}}$$

$$=-1$$

At $t=\frac{\pi}{4}\Rightarrow x=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2},\quad y=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2} \quad \rightarrow\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$

The equation of the tangent line is $\quad y=-\left(x-\frac{\sqrt{2}}{2}\right)+\frac{\sqrt{2}}{2}=-x+\sqrt{2}$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of t.

$$x=t^2-1,\quad y=t^3+t,\quad t=2$$

Solution

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{3t^2+1}{2t}\Big|_{t=2}=\frac{13}{4}$$

At $t=2\Rightarrow\ x=3,\quad y=10 \quad \rightarrow(3,\ 10)$

The equation of the tangent line is $\quad y=\frac{13}{4}(x-3)+10=\frac{13}{4}x+\frac{1}{4}$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of *t*.

$$x = e^t, \quad y = \ln(t+1), \quad t = 0$$

Solution

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \left.\frac{\frac{1}{t+1}}{e^t}\right|_{t=0} \quad \boxed{=1}$$

At $t = 0 \Rightarrow \ x = 1, \quad y = 0 \quad \to (1,\ 0)$

The equation of the tangent line is $\quad \boxed{y = x - 1}$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of *t*.

$$x = \cos t + t\sin t, \quad y = \sin t - t\cos t, \quad t = \frac{\pi}{4}$$

Solution

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t - \cos t + t\sin t}{-\sin t + \sin t + t\cos t}$$

$$= \tan t\Big|_{t=\frac{\pi}{4}}$$

$$\boxed{=1}$$

At $t = \frac{\pi}{4} \Rightarrow x = \frac{\sqrt{2}}{2} + \frac{\pi}{4}\frac{\sqrt{2}}{2}, \quad y = \frac{\sqrt{2}}{2} - \frac{\pi}{4}\frac{\sqrt{2}}{2} \quad \to \left(\frac{4\sqrt{2}+\pi\sqrt{2}}{8},\ \frac{4\sqrt{2}-\pi\sqrt{2}}{8}\right)$

The equation of the tangent line is $\quad \boxed{y = x - \frac{4\sqrt{2}+\pi\sqrt{2}}{8} + \frac{4\sqrt{2}-\pi\sqrt{2}}{8} = x - \frac{\pi\sqrt{2}}{4}}$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of *t*.

$$x = 6t, \quad y = t^2 + 4, \quad t = 1$$

Solution

$$\frac{dy}{dx} = \frac{2t}{6}\Big|_{t=1} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\boxed{= \frac{1}{3}}$$

At $t = 1 \ \Rightarrow \quad x = 6, \quad y = 5 \quad \to (6,\ 5)$

$$y = \frac{1}{3}(x-6) + 5 \qquad\qquad y = m(x - x_0) + y_0$$

$$\boxed{= \frac{1}{3}x + 3}$$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of t.

$$x = t - 2, \quad y = \frac{1}{t} + 3, \quad t = 1$$

Solution

$$\frac{dy}{dx} = -\frac{1}{t^2}\Big|_{t=1} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\underline{= -1}$$

At $t = 1 \Rightarrow \quad x = -1, \quad y = 4 \quad \to (-1,\ 4)$

$$y = -(x+1) + 4 \qquad\qquad y = m\left(x - x_0\right) + y_0$$

$$\underline{= -x + 3}$$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of t.

$$x = t^2 - t + 2, \quad y = t^3 - 3t, \quad t = -1$$

Solution

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t - 1}\Big|_{t=-1} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\underline{= 0}$$

At $t = -1 \Rightarrow \quad x = 4, \quad y = 2 \quad \to (4,\ 2)$

$$\underline{y = 2} \qquad\qquad y = m\left(x - x_0\right) + y_0$$

Exercise

Find an equation of the line tangent to the curve at the point corresponding to the given value of t.

$$x = -t^2 + 3t, \quad y = 2t^{3/2}, \quad t = \frac{1}{4}$$

Solution

$$\frac{dy}{dx} = \frac{3t^{1/2}}{-2t + 3}\Big|_{t=1/4} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{3}{2}\frac{1}{-\frac{1}{2} + 3}$$

$$\underline{= \frac{3}{5}}$$

At $t = \frac{1}{4} \Rightarrow \quad x = -\frac{1}{16} + \frac{3}{4} = \frac{11}{16}, \quad y = \frac{1}{4} \quad \to \left(\frac{11}{16},\ \frac{1}{4}\right)$

$$y = \frac{3}{5}\left(x - \frac{11}{16}\right) + \frac{1}{4} \qquad\qquad y = m\left(x - x_0\right) + y_0$$

$$\underline{= \frac{3}{5}x - \frac{13}{80}}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = \sin 2\pi t, \quad y = \cos 2\pi t, \quad t = -\frac{1}{6}$

Solution

$$x = \sin 2\pi\left(-\tfrac{1}{6}\right) = -\sin\left(\tfrac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$y = \cos 2\pi\left(-\tfrac{1}{6}\right) = \cos\left(\tfrac{\pi}{3}\right) = \frac{1}{2}$$

The point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$$\frac{dx}{dt} = 2\pi\cos 2\pi t, \quad \frac{dy}{dt} = -2\pi\sin 2\pi t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-2\pi\sin 2\pi t}{2\pi\cos 2\pi t} = -\tan 2\pi t$$

$$\left.\frac{dy}{dx}\right|_{t=-\frac{1}{6}} = -\tan 2\pi\left(-\tfrac{1}{6}\right) = -\tan\left(-\tfrac{\pi}{3}\right) = \sqrt{3}$$

The tangent to the curve at the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is: $\quad y = \sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{1}{2} = \sqrt{3}x + 2$

$$\frac{dy'}{dt} = \frac{d}{dt}(-\tan 2\pi t) = -2\pi\sec^2 2\pi t$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= \frac{-2\pi\sec^2 2\pi t}{2\pi\cos 2\pi t}$$

$$= -\frac{1}{\cos^3 2\pi t}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=-\frac{1}{6}} = -\frac{1}{\cos^3\left(-\frac{\pi}{3}\right)} = -8$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = \cos t, \quad y = \sqrt{3}\cos t, \quad t = \frac{2\pi}{3}$

Solution

$$x = \cos\left(\tfrac{2\pi}{3}\right) = -\tfrac{1}{2}$$

$y=\sqrt{3}\cos\left(\frac{2\pi}{3}\right)=-\frac{\sqrt{3}}{2}$ ***The point*** $\left(-\frac{1}{2},\ -\frac{\sqrt{3}}{2}\right)$

$$\frac{dx}{dt}=-\sin t,\quad \frac{dy}{dt}=-\sqrt{3}\sin t$$

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=\frac{-\sqrt{3}\sin t}{-\sin t}=\sqrt{3}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{2\pi}{3}}=\sqrt{3}$$

The tangent to the curve at the point $\left(-\frac{1}{2},\ -\frac{\sqrt{3}}{2}\right)$ is: $y=\sqrt{3}\left(x+\frac{1}{2}\right)-\frac{\sqrt{3}}{2}=\sqrt{3}x$

$$\frac{dy'}{dt}=\frac{d}{dt}\left(\sqrt{3}\right)=0$$

$$\frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}$$

$$=\frac{0}{-\sin t}$$

$$=0$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{2\pi}{3}}=0$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $x=t,\quad y=\sqrt{t},\quad t=\frac{1}{4}$

Solution

$$x=\frac{1}{4}$$

$y=\sqrt{\frac{1}{4}}=\frac{1}{2}$ ***The point*** $\left(\frac{1}{4},\ \frac{1}{2}\right)$

$$\frac{dx}{dt}=1,\quad \frac{dy}{dt}=\frac{1}{2\sqrt{t}}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=\frac{1}{2\sqrt{t}}\cdot 1=\frac{1}{2\sqrt{t}}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{1}{4}}=\frac{1}{2\sqrt{\frac{1}{4}}}=1$$

The tangent is: $y=\left(x-\frac{1}{4}\right)+\frac{1}{2}=x+\frac{1}{4}$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1}{2\sqrt{t}}\right)$$

$$= -\frac{1}{4}t^{-3/2}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= \frac{-\frac{1}{4}t^{-3/2}}{1}$$

$$= -\frac{1}{4}t^{-3/2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{1}{4}} = -\frac{1}{4}\left(\frac{1}{4}\right)^{-3/2} = \underline{-2}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $x = \sec^2 t - 1$, $y = \tan t$, $t = -\frac{\pi}{4}$

Solution

$$x = \sec^2\left(-\frac{\pi}{4}\right) - 1 = 1$$

$$y = \tan\left(-\frac{\pi}{4}\right) = -1$$

The point $(1,\ -1)$

$$\frac{dx}{dt} = 2\sec^2 t\tan t, \quad \frac{dy}{dt} = \sec^2 t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{2\sec^2 t\tan t} = \frac{1}{2\tan t}$$

$$\left.\frac{dy}{dx}\right|_{t=-\frac{\pi}{4}} = \frac{1}{2\tan\left(-\frac{\pi}{4}\right)} = -\frac{1}{2}$$

The tangent is: $y = -\frac{1}{2}(x-1) - 1 = \underline{-\frac{1}{2}x + \frac{1}{2}}$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1}{2}\frac{1}{\tan\theta}\right)$$

$$= \frac{1}{2}\frac{-\sec^2\theta}{\tan^2\theta} = -\frac{1}{2}\frac{\frac{1}{\cos^2\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}$$

$$= -\frac{1}{2}\csc^2\theta$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= \frac{-\frac{1}{2}\csc^2\theta}{2\sec^2 t\tan t} \qquad \frac{-\frac{1}{2}\csc^2\theta}{2\sec^2 t\tan t} = -\frac{1}{4}\frac{\frac{1}{\sin^2 t}}{\frac{1}{\cos^2 t}\frac{\sin t}{\cos t}} = -\frac{1}{4}\frac{\cos^3 t}{\sin^3 t} = -\frac{1}{4}\cot^3 t$$

$$\boxed{= -\frac{1}{4}\cot^3 t}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=-\frac{\pi}{4}} = -\frac{1}{4}\cot^3\left(-\frac{\pi}{4}\right) = \boxed{\frac{1}{4}}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of d^2y/dx^2 at this point $\quad x = \frac{1}{t+1}, \quad y = \frac{t}{t-1}, \quad t = 2$

Solution

$$x = \frac{1}{2+1} = \frac{1}{3} \qquad y = \frac{2}{2-1} = 2 \qquad \textbf{The point } \left(\frac{1}{3},\ 2\right)$$

$$\frac{dx}{dt} = \frac{-1}{(t+1)^2}, \quad \frac{dy}{dt} = \frac{t-1-t}{(t-1)^2} = \frac{-1}{(t-1)^2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{-1}{(t-1)^2}}{\frac{-1}{(t+1)^2}} = \frac{(t+1)^2}{(t-1)^2}$$

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{(2+1)^2}{(2-1)^2} = \boxed{9}$$

The tangent is: $\quad \boxed{y = 9\left(x - \frac{1}{3}\right) + 2 = 9x - 1}$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{t+1}{t-1}\right)^2$$

$$= 2\left(\frac{t+1}{t-1}\right)\left(\frac{t-1-t-1}{(t-1)^2}\right)$$

$$= -4\frac{t+1}{(t-1)^3}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= -4\frac{t+1}{(t-1)^3}\frac{(t+1)^2}{-1}$$

$$\boxed{= 4\frac{(t+1)^3}{(t-1)^3}}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=2} = 4\frac{(2+1)^3}{(2-1)^3} = \boxed{108}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of d^2y/dx^2 at this point $x = t + e^t, \quad y = 1 - e^t, \quad t = 0$

Solution

$x = 0 + e^0 = 1 \qquad y = 1 - e^0 = 0 \qquad$ ***The point*** $(1,\ 0)$

$$\frac{dx}{dt} = 1 + e^t, \quad \frac{dy}{dt} = -e^t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-e^t}{1+e^t}$$

$$\left.\frac{dy}{dx}\right|_{t=0} = -\frac{e^0}{1+e^0} = \boxed{-\frac{1}{2}}$$

The tangent is: $\boxed{y = -\frac{1}{2}(x-1) = -\frac{1}{2}x + \frac{1}{2}}$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{-e^t}{1+e^t}\right)$$

$$= \frac{-e^t\left(1+e^t\right) - e^t\left(-e^t\right)}{\left(1+e^t\right)^2}$$

$$= \frac{-e^t - e^{2t} + e^{2t}}{\left(1+e^t\right)^2}$$

$$= \frac{-e^t}{\left(1+e^t\right)^2}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= \frac{-e^t}{\left(1+e^t\right)^2} \frac{1}{1+e^t}$$

$$\boxed{= \frac{-e^t}{\left(1+e^t\right)^3}}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=0} = \frac{-e^0}{\left(1+e^0\right)^3} = \boxed{-\frac{1}{8}}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = 4t, \quad y = 3t - 2, \quad t = 3$

Solution

$$\frac{dx}{dt} = 4 \quad \frac{dy}{dt} = 3$$

$$\frac{dy}{dx} = \left.\frac{3}{4}\right|_{t=3} = \frac{3}{4} \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$t = 3 \Rightarrow \quad x = 12 \quad y = 7$$

The tangent to the curve at the point $(12, \ 7)$

$$y = \frac{3}{4}(x-12) + 7 \qquad y = m\left(x - x_0\right) + y_0$$

$$\boxed{= \frac{3}{4}x - 2}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{3}{4}\right) = 0$$

$$\frac{d^2y}{dx^2} \boxed{= 0} \qquad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = \sqrt{t}, \quad y = 3t - 1, \quad t = 1$

Solution

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}} \quad \frac{dy}{dt} = 3$$

$$\frac{dy}{dx} = 6\sqrt{t}\Big|_{t=1} \quad \underline{=6|} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$t = 1 \Rightarrow \quad x = 1 \quad y = 2$

The tangent to the curve at the point $(1,\ 2)$

$$y = 6(x-1) + 2 \qquad\qquad y = m\left(x - x_0\right) + y_0$$

$$\underline{= 6x - 4|}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(6\sqrt{t}\right) = \frac{3}{\sqrt{t}}$$

$$\frac{d^2y}{dx^2} = \frac{3}{\sqrt{t}} \cdot 2\sqrt{t} \qquad\qquad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$\underline{= 6|}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\qquad x = t + 1, \quad y = t^2 + 3t, \quad t = -1$

Solution

$$\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = 2t + 3$$

$$\frac{dy}{dx} = 2t + 3\Big|_{t=-1} \quad \underline{=1|} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$t = -1 \Rightarrow \quad x = 2 \quad y = 4$

The tangent to the curve at the point $(2,\ 4)$

$$y = (x-2) + 4 \qquad\qquad y = m\left(x - x_0\right) + y_0$$

$$\underline{= x + 2|}$$

$$\frac{dy'}{dt} = \frac{d}{dt}(2t + 3) = 2$$

$$\frac{d^2y}{dx^2} \ \underline{= 2|} \qquad\qquad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\qquad x = t^2 + 5t + 4, \quad y = 4t, \quad t = 0$

Solution

$$\frac{dx}{dt} = 2t + 5 \quad \frac{dy}{dt} = 4$$

$$\frac{dy}{dx} = \frac{4}{2t+5}\Big|_{t=0} \quad \boxed{= \frac{4}{5}} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$t = 0 \Rightarrow \quad x = 4 \quad y = 0$$

The tangent to the curve at the point $(4,\ 0)$

$$y = \frac{4}{5}(x-4) \qquad\qquad y = m\left(x - x_0\right) + y_0$$

$$\boxed{= \frac{4}{5}x - \frac{16}{5}}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{4}{2t+5}\right)$$

$$= \frac{-8}{(2t+5)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-8}{(2t+5)^2} \cdot \frac{1}{2t+5} \qquad\qquad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= \frac{-8}{(2t+5)^3}\Big|_{t=0}$$

$$\boxed{= -\frac{8}{125}}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\qquad x = 4\cos\theta, \quad y = 4\sin\theta, \quad \theta = \frac{\pi}{4}$

Solution

$$\frac{dx}{d\theta} = -4\sin\theta \quad \frac{dy}{d\theta} = 4\cos\theta$$

$$\frac{dy}{dx} = \frac{4\cos\theta}{-4\sin\theta} \qquad\qquad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= -\cot\theta \Big|_{\theta=\frac{\pi}{4}}$$

$$\boxed{= -1}$$

$$\theta = \frac{\pi}{4} \Rightarrow \quad x = 2\sqrt{2} \quad y = 2\sqrt{2}$$

The tangent to the curve at the point $\left(2\sqrt{2},\ 2\sqrt{2}\right)$:

$$y = -\left(x - 2\sqrt{2}\right) + 2\sqrt{2} \qquad\qquad y = m\left(x - x_0\right) + y_0$$

$$\boxed{= -x + 4\sqrt{2}}$$

$$\frac{dy'}{d\theta} = \frac{d}{d\theta}(-\cot\theta) = \csc^2\theta$$

$$\frac{d^2y}{dx^2} = \frac{\csc^2\theta}{-4\sin\theta} \qquad\qquad \frac{d^2y}{dx^2} = \frac{dy'/d\theta}{dx/d\theta}$$

$$= -\frac{1}{4}\csc^3\theta\Big|_{\theta=\frac{\pi}{4}}$$

$$\boxed{= -\frac{\sqrt{2}}{2}}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = \cos\theta, \quad y = 3\sin\theta, \quad \theta = 0$

Solution

$$\frac{dx}{d\theta} = -\sin\theta \qquad \frac{dy}{d\theta} = 3\cos\theta$$

$$\frac{dy}{dx} = \frac{3\cos\theta}{-\sin\theta} \qquad\qquad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= -3\cot\theta\Big|_{\theta=0}$$

$$\boxed{= \infty}$$

$$\theta = 0 \Rightarrow \quad x = 1 \quad y = 0$$

The tangent to the curve at the point $(1,\ 0)$: $\boxed{x = 1}$

$$\frac{dy'}{d\theta} = \frac{d}{d\theta}(-3\cot\theta) = 3\csc^2\theta$$

$$\frac{d^2y}{dx^2} = \frac{3\csc^2\theta}{-\sin\theta} \qquad\qquad \frac{d^2y}{dx^2} = \frac{dy'/d\theta}{dx/d\theta}$$

$$= -3\csc^3\theta\Big|_{\theta=0}$$

$$\boxed{= \infty} \quad \textit{undefined}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = 2 + \sec\theta, \quad y = 1 + 2\tan\theta, \quad \theta = \frac{\pi}{6}$

Solution

$$\frac{dx}{d\theta} = \sec\theta\tan\theta \qquad \frac{dy}{d\theta} = 2\sec^2\theta$$

$$\frac{dy}{dx} = \frac{2\sec^2\theta}{\sec\theta\tan\theta} \qquad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= 2\csc\theta \Big|_{\theta=\frac{\pi}{6}}$$

$$\underline{= 4}$$

$$\theta = \frac{\pi}{6} \Rightarrow \quad x = 2 + \frac{2}{\sqrt{3}} \quad y = 1 + \frac{2\sqrt{3}}{3}$$

The tangent to the curve at the point $\left(2 + \frac{2\sqrt{3}}{3},\ 1 + \frac{2\sqrt{3}}{3}\right)$:

$$y = 2\left(x - 2 - \frac{2\sqrt{3}}{3}\right) + 1 + \frac{2\sqrt{3}}{3} \qquad y = m\left(x - x_0\right) + y_0$$

$$\underline{= 2x - 3 - \frac{2\sqrt{3}}{3}}$$

$$\frac{dy'}{d\theta} = \frac{d}{d\theta}(2\csc\theta) = -2\csc\theta\cot\theta$$

$$\frac{d^2y}{dx^2} = \frac{-2\csc\theta\cot\theta}{\sec\theta\tan\theta} \qquad \frac{d^2y}{dx^2} = \frac{dy'/d\theta}{dx/d\theta}$$

$$= -2\cot^3\theta \Big|_{\theta=\frac{\pi}{6}}$$

$$\underline{= -6\sqrt{3}}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = \sqrt{t}, \quad y = \sqrt{t-1}, \quad t = 2$

Solution

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}} \quad \frac{dy}{dt} = \frac{1}{2\sqrt{t-1}}$$

$$\frac{dy}{dx} = \frac{2\sqrt{t}}{2\sqrt{t-1}}\Big|_{t=2} \quad \underline{= \sqrt{2}} \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$t = 2 \Rightarrow \quad x = \sqrt{2} \quad y = 1$$

The tangent to the curve at the point $\left(\sqrt{2},\ 1\right)$

$$y = \sqrt{2}\left(x - \sqrt{2}\right) + 1 \qquad y = m\left(x - x_0\right) + y_0$$

$$\underline{= \sqrt{2}x - 1}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{\sqrt{t}}{\sqrt{t-1}}\right) \qquad \left(U^nV^m\right)' = U^{n-1}V^{m-1}\left(nU'V + mUV'\right)$$

$$=\frac{\frac{1}{2}t-\frac{1}{2}-\frac{1}{2}t}{(t-1)^{3/2}\sqrt{t}}$$

$$=-\frac{1}{2}\frac{1}{(t-1)^{3/2}\sqrt{t}}$$

$$\frac{d^2y}{dx^2}=-\frac{1}{2}\frac{1}{(t-1)^{3/2}\sqrt{t}}\cdot 2\sqrt{t} \qquad \frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}$$

$$=-\frac{1}{(t-1)^{3/2}}\Bigg|_{t=2}$$

$$\boxed{=-1}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x=\cos^3\theta,\quad y=\sin^3\theta,\quad \theta=\frac{\pi}{4}$

Solution

$$\frac{dx}{d\theta}=-3\sin\theta\cos^2\theta \qquad \frac{dy}{d\theta}=3\cos\theta\sin^2\theta$$

$$\frac{dy}{dx}=\frac{3\cos\theta\sin^2\theta}{-3\sin\theta\cos^2\theta} \qquad \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}$$

$$=-\tan\theta\Big|_{\theta=\frac{\pi}{4}}$$

$$\boxed{=-1}$$

$$\theta=\frac{\pi}{4}\Rightarrow \quad x=\frac{\sqrt{2}}{4} \quad y=\frac{\sqrt{2}}{4}$$

The tangent to the curve at the point $\left(\frac{\sqrt{2}}{4},\ \frac{\sqrt{2}}{4}\right)$:

$$y=-\left(x-\frac{\sqrt{2}}{4}\right)+\frac{\sqrt{2}}{4} \qquad y=m\left(x-x_0\right)+y_0$$

$$\boxed{=-x+\frac{\sqrt{2}}{2}}$$

$$\frac{dy'}{d\theta}=\frac{d}{d\theta}(-\tan\theta)=-\sec^2\theta$$

$$\frac{d^2y}{dx^2}=\frac{-\sec^2\theta}{-3\sin\theta\cos^2\theta} \qquad \frac{d^2y}{dx^2}=\frac{dy'/d\theta}{dx/d\theta}$$

$$= \left.\frac{1}{3\sin\theta\cos^4\theta}\right|_{\theta=\frac{\pi}{4}}$$

$$\boxed{= \frac{4\sqrt{2}}{3}}$$

Exercise

Find the tangent to the curve at the point defined by the given value of t. Also find the value of $\frac{d^2y}{dx^2}$ at this point $\quad x = \theta - \sin\theta, \quad y = 1 - \cos\theta, \quad \theta = \pi$

Solution

$$\frac{dx}{d\theta} = 1 - \cos\theta \qquad \frac{dy}{d\theta} = \sin\theta$$

$$\frac{dy}{dx} = \left.\frac{\sin\theta}{1-\cos\theta}\right|_{\theta=\pi} \qquad\qquad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$\boxed{= 0}$$

$$\theta = \pi \Rightarrow \quad x = \pi \quad y = 2$$

The tangent to the curve at the point $(\pi,\ 2)$:

$$\boxed{y = 2} \qquad\qquad y = m\left(x - x_0\right) + y_0$$

$$\frac{dy'}{d\theta} = \frac{d}{d\theta}\left(\frac{\sin\theta}{1-\cos\theta}\right)$$

$$= \frac{\cos\theta - \cos^2\theta - \sin^2\theta}{(1-\cos\theta)^2}$$

$$= \frac{\cos\theta - 1}{(1-\cos\theta)^2}$$

$$= \frac{-1}{1-\cos\theta}$$

$$\frac{d^2y}{dx^2} = \left.\left(\frac{-1}{1-\cos\theta}\right)\frac{1}{1-\cos\theta}\right|_{\theta=\pi} \qquad\qquad \frac{d^2y}{dx^2} = \frac{dy'/d\theta}{dx/d\theta}$$

$$= \left.\frac{-1}{(1-\cos\theta)^2}\right|_{\theta=\pi}$$

$$\boxed{= -\frac{1}{4}}$$

Exercise

Find the equations of the tangent lines at the point where the curve crosses itself

$$x = 2\sin 2t, \quad y = 3\sin t$$

Solution

$x = y \to \quad 2\sin 2t = 3\sin t \quad \Rightarrow \underline{t = 0,\ \pi}$

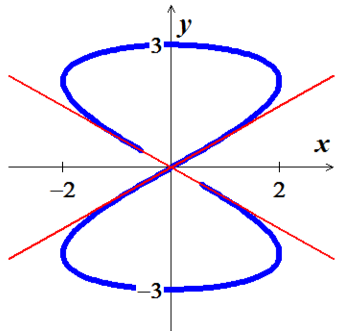
$\frac{dx}{dt} = 4\cos 2t, \quad \frac{dy}{dt} = 3\cos t$

$\frac{dy}{dx} = \frac{3\cos t}{4\cos 2t}$

At $t = 0 \quad \frac{dy}{dx} = \frac{3}{4}$

The point at $t = 0$ is $(0,\ 0)$

The tangent line: $\underline{y = \frac{3}{4}x}$



At $t = \pi \quad \frac{dy}{dx} = -\frac{3}{4}$

The point at $t = 0$ is $(0,\ 0)$

The tangent line: $\underline{y = -\frac{3}{4}x}$

Exercise

Find the equations of the tangent lines at the point where the curve crosses itself

$$x = 2 - \pi\cos t, \quad y = 2t - \pi\sin t$$

Solution

The graph crosses itself at the point $(2,\ 0)$

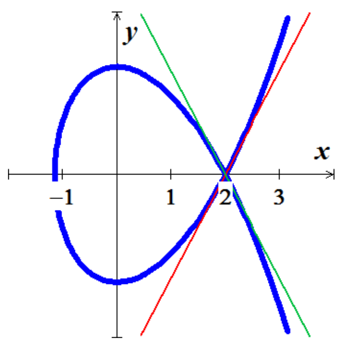
$x = 2 - \pi\cos t = 2 \ \to \ \cos t = 0 \quad \Rightarrow \underline{t = \pm\frac{\pi}{2}}$

$\frac{dx}{dt} = \pi\sin t, \quad \frac{dy}{dt} = 2 - \pi\cos t$

$\frac{dy}{dx} = \frac{2 - \pi\cos t}{\pi\sin t}$

At $t = \frac{\pi}{2} \quad \frac{dy}{dx} = \frac{2}{\pi}$

The tangent line: $y = \frac{2}{\pi}(x - 2) \ \underline{= \frac{2}{\pi}x - \frac{4}{\pi}}$



At $t = -\frac{\pi}{2} \quad \frac{dy}{dx} = -\frac{2}{\pi}$

The tangent line: $y = -\frac{2}{\pi}(x - 2) \ \underline{= -\frac{2}{\pi}x + \frac{4}{\pi}}$

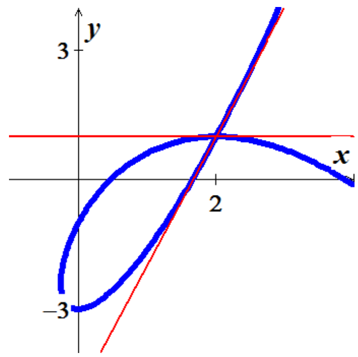
Exercise

Find the equations of the tangent lines at the point where the curve crosses itself

$$x = t^2 - t, \quad y = t^3 - 3t - 1$$

Solution

The graph crosses itself at the point $(2,\ 1)$



$x = t^2 - t = 2 \ \rightarrow\ t^2 - t - 2 = 0 \quad \Rightarrow\ \underline{t = -1,\ 2}$

$\dfrac{dx}{dt} = 2t - 1, \quad \dfrac{dy}{dt} = 3t^2 - 3$

$\dfrac{dy}{dx} = \dfrac{3t^2 - 3}{2t - 1}$

At $t = -1 \quad \dfrac{dy}{dx} = 0$

The tangent line: $y \underline{= 1}$

At $t = 2 \quad \dfrac{dy}{dx} = 3$

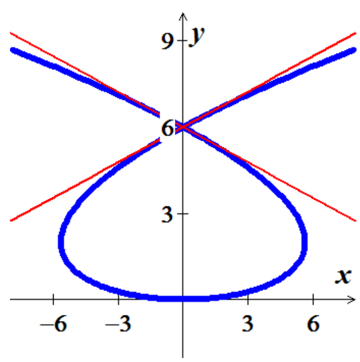
The tangent line: $y = 3(x - 2) + 1 \underline{= 3x - 5}$

Exercise

Find the equations of the tangent lines at the point where the curve crosses itself

$$x = t^3 - 6t, \quad y = t^2$$

Solution



The graph crosses itself at the point $(0,\ 6)$

$y = t^2 = 6 \quad \Rightarrow\ \underline{t = \pm\sqrt{6}}$

$\dfrac{dx}{dt} = 3t^2 - 6, \quad \dfrac{dy}{dt} = 2t$

$\dfrac{dy}{dx} = \dfrac{2t}{3t^2 - 6}$

At $t = -\sqrt{6} \quad \dfrac{dy}{dx} = \dfrac{-2\sqrt{6}}{12} = -\dfrac{\sqrt{6}}{6}$

The tangent line: $\underline{y = -\dfrac{\sqrt{6}}{6}x + 6}$

At $t = \sqrt{6} \quad \dfrac{dy}{dx} = \dfrac{2\sqrt{6}}{12} = \dfrac{\sqrt{6}}{6}$

The tangent line: $\underline{y = \dfrac{\sqrt{6}}{6}x + 6}$

Exercise

Find the slope of the curve $x=f(t),\ y=g(t)$ at the given value of *t*. Define *x* and *y* as differentiable functions. $x^3+2t^2=9,\quad 2y^3-3t^2=4,\quad t=2$

Solution

$$x^3+2(2)^2=9 \Rightarrow x^3=9-8=1 \to \boxed{x=1}$$

$$2y^3-3(2)^2=4 \quad\Rightarrow\quad 2y^3=4+12=16 \quad\to y^3=8 \ \Rightarrow \boxed{y=2}$$

$$x^3+2t^2=9 \quad\Rightarrow\quad 3x^2\frac{dx}{dt}+4t=0$$

$$3x^2\frac{dx}{dt}=-4t$$

$$\frac{dx}{dt}=-\frac{4t}{3x^2}$$

$$2y^3-3t^2=4 \quad\Rightarrow\quad 6y^2\frac{dy}{dt}-6t=0$$

$$y^2\frac{dy}{dt}=t$$

$$\frac{dy}{dt}=\frac{t}{y^2}$$

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\frac{t}{y^2}}{-\frac{4t}{3x^2}}=-\frac{3x^2}{4y^2}$$

$$\left.\frac{dy}{dx}\right|_{t=2}=-\frac{3(1)^2}{4(2)^2}=\underline{\left.-\frac{3}{16}\right|}$$

Exercise

Find the slope of the curve $x=f(t),\ y=g(t)$ at the given value of *t*. Define *x* and *y* as differentiable functions. $\quad x+2x^{3/2}=t^2+t,\quad y\sqrt{t+1}+2t\sqrt{y}=4,\quad t=0$

Solution

$$x+2x^{3/2}=0^2+0 \quad\Rightarrow\quad x\left(1+2x^{1/2}\right)=0 \qquad \to \boxed{x=0} \quad \cancel{x^{1/2}=-\frac{1}{2}} \ (False)$$

$$y\sqrt{0+1}+2(0)\sqrt{y}=4 \quad\Rightarrow\quad \boxed{y=4}$$

$$x+2x^{3/2}=t^2+t \quad\Rightarrow\quad \frac{dx}{dt}+3x^{1/2}\frac{dx}{dt}=2t+1$$

$$\frac{dx}{dt}\left(1+3x^{1/2}\right)=2t+1$$

$$\frac{dx}{dt} = \frac{2t+1}{1+3x^{1/2}}$$

$$y\sqrt{t+1} + 2t\sqrt{y} = 4 \quad \Rightarrow \quad \frac{dy}{dt}\sqrt{t+1} + \frac{1}{2}y(t+1)^{-1/2} + 2\sqrt{y} + 2t\left(\frac{1}{2}y^{-1/2}\right)\frac{dy}{dt} = 0$$

$$\frac{dy}{dt}\left(\sqrt{t+1} + \frac{t}{\sqrt{y}}\right) = -\frac{y}{2\sqrt{t+1}} - 2\sqrt{y}$$

$$\frac{dy}{dt}\left(\frac{\sqrt{t+1}\sqrt{y}+t}{\sqrt{y}}\right) = \frac{-y-4\sqrt{t+1}\sqrt{y}}{2\sqrt{t+1}}$$

$$\frac{dy}{dt} = \frac{-y-4\sqrt{t+1}\sqrt{y}}{2\sqrt{t+1}} \cdot \frac{\sqrt{y}}{\sqrt{t+1}\sqrt{y}+t}$$

$$\frac{dy}{dt} = \frac{-y\sqrt{y}-4y\sqrt{t+1}}{2(t+1)\sqrt{y}+2t\sqrt{t+1}}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-y\sqrt{y}-4y\sqrt{t+1}}{2(t+1)\sqrt{y}+2t\sqrt{t+1}} \cdot \frac{1+3\sqrt{x}}{2t+1}$$

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{-4\sqrt{4}-4(4)\sqrt{0+1}}{2(0+1)\sqrt{4}+2(0)\sqrt{0+1}} \cdot \frac{1+3\sqrt{0}}{2(0)+1} = \underline{-6}|$$

Exercise

Find the slope of the curve $x = f(t)$, $y = g(t)$ at the given value of t. Define x and y as differentiable functions. $t = \ln(x-t)$, $y = te^t$, $t = 0$

Solution

$$0 = \ln(x-0) \quad \Rightarrow \quad \ln x = 0 \ \rightarrow \ \boxed{x=1}$$

$$y = (0)e^0 \quad \Rightarrow \quad \boxed{y=0}$$

$$t = \ln(x-t) \quad \Rightarrow \quad 1 = \frac{\frac{dx}{dt}-1}{x-t}$$

$$\frac{dx}{dt} - 1 = x - t$$

$$\frac{dx}{dt} = x - t + 1$$

$$y = te^t \quad \Rightarrow \quad \frac{dy}{dt} = e^t + te^t = e^t(1+t)$$

$$\frac{dy}{dx} = \frac{e^t(1+t)}{x-t+1} \qquad\qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{e^0(1+0)}{1-0+1} = \underline{\frac{1}{2}}|$$

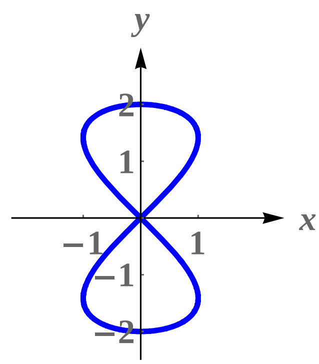
Exercise

Consider Lissajous curve, estimate the coordinates of the points on the curve at which there is

$$x = \sin 2t, \quad y = 2\sin t; \quad 0 \le t \le 2\pi$$

a) A horizontal tangent line

b) A vertical tangent line.



Solution

a) $\dfrac{dy}{dx} = \dfrac{2\cos t}{2\cos 2t} = 0 \qquad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

$\cos t = 0 \quad \to \quad t = \dfrac{\pi}{2}, \ \dfrac{3\pi}{2}$

$t = \dfrac{\pi}{2} \to \quad x = \sin \pi = 0 \quad y = 2\sin\dfrac{\pi}{2} = 2 \quad \boxed{(0,\ 2)}$

$t = \dfrac{3\pi}{2} \to \quad x = \sin 3\pi = 0 \quad y = 2\sin\dfrac{3\pi}{2} = -2 \quad \boxed{(0,\ -2)}$

b) Vertical tangent line: $\cos 2t = 0 \quad \cos t \ne 0$

$\cos 2t = 0 \quad \to \quad 2t = \dfrac{\pi}{2}, \ \dfrac{3\pi}{2}, \ \dfrac{5\pi}{2}. \ \dfrac{7\pi}{2} \quad \Rightarrow \boxed{t = \dfrac{\pi}{4}, \ \dfrac{3\pi}{4}, \ \dfrac{5\pi}{4}. \ \dfrac{7\pi}{4}}$

$t = \dfrac{\pi}{4} \to \quad x = 1 \quad y = \sqrt{2} \quad \boxed{\left(1,\ \sqrt{2}\right)}$

$t = \dfrac{3\pi}{4} \to \quad x = -1 \quad y = \sqrt{2} \quad \boxed{\left(-1,\ \sqrt{2}\right)}$

$t = \dfrac{5\pi}{4} \to \quad x = -1 \quad y = -\sqrt{2} \quad \boxed{\left(-1,\ -\sqrt{2}\right)}$

$t = \dfrac{7\pi}{4} \to \quad x = 1 \quad y = -\sqrt{2} \quad \boxed{\left(1,\ -\sqrt{2}\right)}$

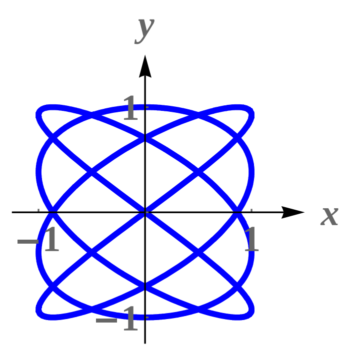
Exercise

Consider Lissajous curve, estimate the coordinates of the points on the curve at which there is

$$x = \sin 4t, \quad y = \sin 3t; \quad 0 \le t \le 2\pi$$

a) A horizontal tangent line

b) A vertical tangent line.



Solution

a) $\dfrac{dy}{dx} = \dfrac{3\cos 3t}{4\cos 4t} = 0 \qquad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

$\cos 3t = 0 \quad \to \quad 3t = \dfrac{\pi}{2}, \ \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ \dfrac{7\pi}{2}, \dfrac{9\pi}{2}, \ \dfrac{11\pi}{2}$

$\boxed{t = \dfrac{\pi}{6}, \ \dfrac{\pi}{2}, \dfrac{5\pi}{6}, \ \dfrac{7\pi}{6}, \dfrac{3\pi}{2}, \ \dfrac{11\pi}{6}}$

$t = \dfrac{\pi}{6} \to \quad x = -\dfrac{\sqrt{3}}{2} \quad y = 1 \quad \boxed{\left(-\dfrac{\sqrt{3}}{2},\ 1\right)}$

$t=\frac{\pi}{2} \to \quad x=0 \quad y=-1 \quad \boxed{(0,\ -1)}$

$t=\frac{5\pi}{6} \to \quad x=\frac{\sqrt{3}}{2} \quad y=-1 \quad \boxed{\left(\frac{\sqrt{3}}{2},\ -1\right)}$

$t=\frac{7\pi}{6} \to \quad x=-\frac{\sqrt{3}}{2} \quad y=1 \quad \boxed{\left(-\frac{\sqrt{3}}{2},\ -1\right)}$

$t=\frac{3\pi}{2} \to \quad x=0 \quad y=1 \quad \boxed{(0,\ 1)}$

$t=\frac{11\pi}{6} \to \quad x=\frac{\sqrt{3}}{2} \quad y=1 \quad \boxed{\left(\frac{\sqrt{3}}{2},\ 1\right)}$

b) Vertical tangent line: $\cos 4t=0 \quad \cos 3t \neq 0$

$\cos 4t=0 \quad \to \quad 4t=\frac{(n+1)\pi}{2} \quad \Rightarrow \boxed{t=\frac{(n+1)\pi}{8}}$

$t=\frac{(n+1)\pi}{8} \to \quad x=\pm 1 \quad y=\pm\sin\frac{3\pi}{8} \quad \boxed{\left(\pm 1,\ \pm\sin\frac{3\pi}{8}\right)}$

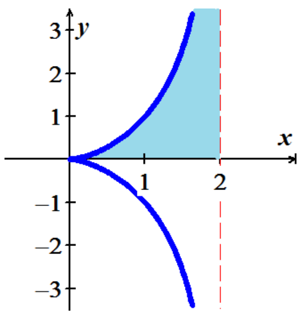
Exercise

Find the area of the region $\quad x=2\sin^2\theta, \quad y=2\sin^2\theta\tan\theta, \quad 0\le\theta<\frac{\pi}{2}$

Solution

$dx=4\sin\theta\cos\theta\ d\theta$

$$A=\int_0^{\pi/2} 2\sin^2\theta\tan\theta(4\sin\theta\cos\theta)d\theta \qquad A=\int_a^b y\,dx$$



$$=8\int_0^{\pi/2}\sin^4\theta\ d\theta$$

$$=2\int_0^{\pi/2}(1-\cos 2\theta)^2\ d\theta$$

$$=2\int_0^{\pi/2}\left(1-2\cos 2\theta+\cos^2 2\theta\right)\ d\theta$$

$$=2\int_0^{\pi/2}\left(1-2\cos 2\theta+\frac{1}{2}+\frac{1}{2}\cos 4\theta\right)\ d\theta$$

$$=2\left(\frac{3}{2}\theta-\sin 2\theta+\frac{1}{8}\sin 4\theta\right)\Big|_0^{\pi/2}$$

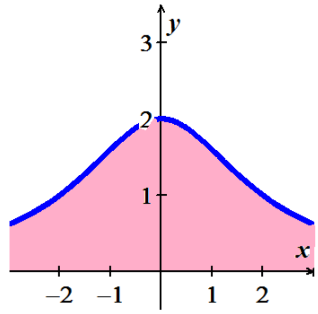
$$=2\left(\frac{3\pi}{4}\right)$$

$$\boxed{=\frac{3\pi}{2}}$$

Exercise

Find the area of the region $\quad x = 2\cot\theta, \quad y = 2\sin^2\theta, \quad 0 \le \theta < \pi$

Solution



$$dx = -2\csc^2\theta \, d\theta$$

$$A = -4\int_0^{\pi} \sin^2\theta \csc^2\theta \, d\theta \qquad\qquad A = \int_a^b y dx$$

$$= -4\int_0^{\pi} d\theta \qquad = -4\int_{\pi/2}^{0} d\theta$$

$$= \left|-4\theta\right|_0^{\pi}$$

$$= 4\pi$$

Exercise

Find the area under one arch of the cycloid $x = a(t - \sin t), \quad y = a(1 - \cos t)$

Solution

$$A = \int_0^{2\pi} y dx$$

$$= \int_0^{2\pi} a(1-\cos t) d\left[a(t-\sin t)\right] \qquad\qquad d\left[a(t-\sin t)\right] = a(1-\cos t)dt$$

$$= \int_0^{2\pi} a^2(1-\cos t)^2 dt$$

$$= a^2\int_0^{2\pi} \left(1 - 2\cos t + \cos^2 t\right) dt$$

$$= a^2\int_0^{2\pi} \left(1 - 2\cos t + \frac{1+\cos 2t}{2}\right) dt$$

$$= a^2\int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{1}{2}\cos 2t\right) dt$$

$$= a^2\left[\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t\right]_0^{2\pi}$$

$$= a^2\left[\frac{3}{2}(2\pi) - 2\sin(2\pi) + \frac{1}{4}\sin 2(2\pi) - 0\right]$$

$$= 3\pi a^2$$

Exercise

Find the area enclosed by the y–axis and the curve $\quad x = t - t^2, \quad y = 1 + e^{-t}$

Solution

$$x = t - t^2 = 0 \quad \Rightarrow \quad \boxed{t = 0,\ 1}$$

$$A = \int_0^1 x\,dy$$

$$= \int_0^1 \left(t - t^2\right) d\left(1 + e^{-t}\right)$$

$$= \int_0^1 \left(t - t^2\right)\left(-e^{-t}\right) dt$$

$$= -\int_0^1 \left(t - t^2\right) e^{-t}\,dt$$

		$\int e^{-t}$
$t - t^2$	$(+)$	$-e^{-t}$
$1 - 2t$	$(-)$	e^{-t}
-2	$(+)$	$-e^{-t}$

$$= -\left[\left(t - t^2\right)\left(-e^{-t}\right) - (1 - 2t)\left(e^{-t}\right) + (-2)\left(-e^{-t}\right)\right]_0^1$$

$$= -\left[e^{-t}\left(t^2 - t\right) - e^{-t}(1 - 2t) + 2e^{-t}\right]_0^1$$

$$= -\left[e^{-1}\left(1^2 - 1\right) - e^{-1}(1 - 2(1)) + 2e^{-1} - \left(e^{-0}\left(0^2 - 0\right) - e^{-0}(1 - 2(0)) + 2e^{-0}\right)\right]$$

$$= -\left[e^{-1} + 2e^{-1} - (-1 + 2)\right]$$

$$= -\left(3e^{-1} - 1\right)$$

$$= 1 - 3e^{-1}$$

$$\boxed{= 1 - \frac{3}{e}}$$

Exercise

Find the area enclosed by the ellipse $\quad x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi$

Solution

$$A = \int_0^{2\pi} y\,dx = 2\left|\int_0^{\pi} y\,dx\right|$$

$$= 2\int_0^{\pi} b\sin t\ d\left[a\cos t\right]$$

$$= 2\int_0^{\pi} b\sin t\left(-a\sin t\right)\ dt$$

$$= -2ab\int_0^{\pi} \sin^2 t \ dt$$

$$= -2ab\int_0^{\pi} \left(\frac{1-\cos 2t}{2}\right) dt$$

$$= -ab\left[t - \frac{1}{2}\sin 2t\right]_0^{\pi}$$

$$= -ab\left(\pi - \frac{1}{2}\sin 2\pi - 0\right)$$

$$= \left|-\pi ab\right|$$

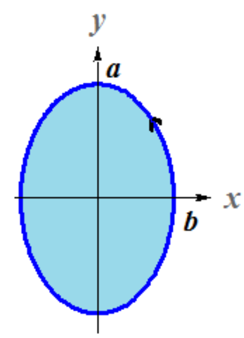
$$= \pi ab$$

Exercise

Find the area of the closed curve *Ellipse* $\begin{cases} x = b\cos t \\ y = a\sin t \end{cases}$ $0 \le t \le 2\pi$

Solution

$$A = \int_0^{2\pi} ydx = 2\left|\int_0^{\pi} ydx\right|$$



$$= 2\int_0^{\pi} a\sin t \ d\left(b\cos t\right)$$

$$= -2ab\int_0^{\pi} \sin^2 t \ dt$$

$$= -2ab\int_0^{\pi} \left(\frac{1-\cos 2t}{2}\right) dt$$

$$= -ab\left[t - \frac{1}{2}\sin 2t\right]_0^{\pi}$$

$$= -ab\left(\pi - \frac{1}{2}\sin 2\pi - 0\right)$$

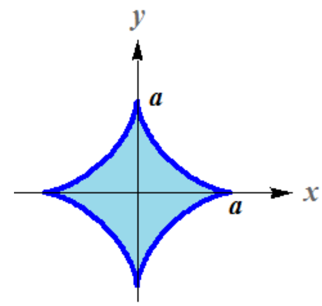
$$= \left|-\pi ab\right|$$

$$= \pi ab$$

Exercise

Find the area of the closed curve *Astroid* $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}$ $0 \le t \le 2\pi$

Solution



$$A = \int_0^{2\pi} y\,dx$$

$$= 4\int_0^{\pi/2} a\sin^3 t \left| d\left(a\cos^3 t\right)\right|$$

$$= 12a^2 \int_0^{\pi/2} \sin^4 t \cos^2 t \; dt$$

$$= 12a^2 \int_0^{\pi/2} \left(\frac{1-\cos 2t}{2}\right)^2 \left(\frac{1+\cos 2t}{2}\right)\; dt$$

$$= \frac{3}{2}a^2 \int_0^{\pi/2} \left(1 - 2\cos 2t + \cos^2 2t\right)\left(1+\cos 2t\right)\; dt$$

$$= \frac{3}{2}a^2 \int_0^{\pi/2} \left(1 - \cos 2t - \cos^2 2t + \cos^3 2t\right)\; dt$$

$$= \frac{3}{2}a^2 \int_0^{\pi/2} \left(\frac{1}{2} - \cos 2t - \cos 4t\right)\; dt + \frac{3}{2}a^2 \int_0^{\pi/2} \cos^2 2t \cos 2t \; dt$$

$$= \frac{3}{2}a^2 \left(\frac{1}{2}t - \frac{1}{2}\sin 2t - \frac{1}{4}\sin 4t\right)\Big|_0^{\pi/2} + \frac{3}{4}a^2 \int_0^{\pi/2} \left(1 - \sin^2 2t\right)\; d\left(\sin 2t\right)$$

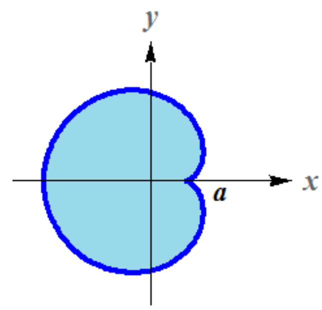
$$= \frac{3}{2}a^2 \left(\frac{\pi}{4}\right) + \frac{3}{4}a^2 \left(\sin 2t - \frac{1}{3}\sin^3 2t\right)\Big|_0^{\pi/2}$$

$$= \frac{3}{8}\pi a^2$$

Exercise

Find the area of the closed curve *Cardioid* $\begin{cases} x = 2a\cos t - a\cos 2t \\ y = 2a\sin t - a\sin 2t \end{cases}$ $0 \le t \le 2\pi$

Solution



$$A = \int_0^{2\pi} y\,dx$$

$$= 2\left|\int_0^{\pi} \left(2a\sin t - a\sin 2t\right)\; d\left(2a\cos t - a\cos 2t\right)\right|$$

$$= 2\left|\int_0^{\pi}(2a\sin t - a\sin 2t)(-2a\sin t + 2a\sin 2t)\ dt\right|$$

$$= 4a^2\left|\int_0^{\pi}(2\sin t - \sin 2t)(-\sin t + \sin 2t)\ dt\right|$$

$$= 4a^2\left|\int_0^{\pi}\left(-2\sin^2 t + 3\sin t\sin 2t - \sin^2 2t\right)\ dt\right| \qquad \sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$= 4a^2\left|\int_0^{\pi}\left(-1 + \cos 2t + \frac{3}{2}\cos t - \frac{3}{2}\cos 3t - \frac{1}{2} + \frac{1}{2}\cos 4t\right)\ dt\right|$$

$$= 4a^2\left|\int_0^{\pi}\left(-\frac{3}{2} + \cos 2t + \frac{3}{2}\cos t - \frac{3}{2}\cos 3t + \frac{1}{2}\cos 4t\right)\ dt\right|$$

$$= 4a^2\left|\left(-\frac{3}{2}t + \frac{1}{2}\sin 2t + \frac{3}{2}\sin t - \frac{1}{2}\sin 3t + \frac{1}{8}\sin 4t\right)\Big|_0^{\pi}\right|$$

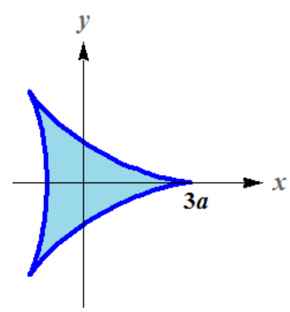
$$= 4a^2\left|\left(-\frac{3\pi}{2}\right)\right|$$

$$\underline{= 6\pi a^2}$$

Exercise

Find the area of the closed curve *Deltoid* $\begin{cases} x = 2a\cos t + a\cos 2t \\ y = 2a\sin t - a\sin 2t \end{cases}$ $0 \le t \le 2\pi$

Solution



$$A = \int_0^{2\pi} y dx$$

$$= 2\left|\int_0^{\pi}(2a\sin t - a\sin 2t)\ d(2a\cos t + a\cos 2t)\right|$$

$$= 2\left|\int_0^{\pi}(2a\sin t - a\sin 2t)(-2a\sin t - 2a\sin 2t)\ dt\right|$$

$$= 4a^2\left|\int_0^{\pi}(2\sin t - \sin 2t)(\sin t + \sin 2t)\ dt\right|$$

$$= 4a^2\left|\int_0^{\pi}\left(2\sin^2 t + \sin t\sin 2t - \sin^2 2t\right)\ dt\right| \qquad \sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$= 4a^2\left|\int_0^{\pi}\left(1-\cos 2t+\frac{1}{2}\cos t-\frac{1}{2}\cos 3t-\frac{1}{2}+\frac{1}{2}\cos 4t\right)\,dt\right|$$

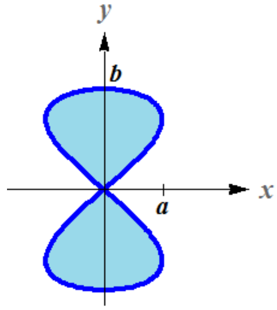
$$= 4a^2\left|\int_0^{\pi}\left(\frac{1}{2}-\cos 2t+\frac{1}{2}\cos t-\frac{1}{2}\cos 3t+\frac{1}{2}\cos 4t\right)\,dt\right|$$

$$= 4a^2\left|\left(\frac{1}{2}t-\frac{1}{2}\sin 2t+\frac{1}{2}\sin t-\frac{1}{6}\sin 3t+\frac{1}{8}\sin 4t\right)\Big|_0^{\pi}\right|$$

$$\underline{= 2\pi a^2}$$

Exercise

Find the area of the closed curve *Hourglass* $\begin{cases} x = a\sin 2t \\ y = b\sin t \end{cases}$ $0 \le t \le 2\pi$



Solution

$$A = \int_0^{2\pi} y\,dx$$

$$= 2\left|\int_0^{\pi}(b\sin t)\,d(a\sin 2t)\right|$$

$$= 4ab\left|\int_0^{\pi}(\sin t\cos 2t)\,dt\right| \qquad \sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$= 2ab\left|\int_0^{\pi}(\sin 3t+\sin(-t))\,dt\right|$$

$$= 2ab\left|\int_0^{\pi}(\sin 3t-\sin t)\,dt\right|$$

$$= 2ab\left|\left(-\frac{1}{3}\cos 3t+\cos t\right)\Big|_0^{\pi}\right|$$

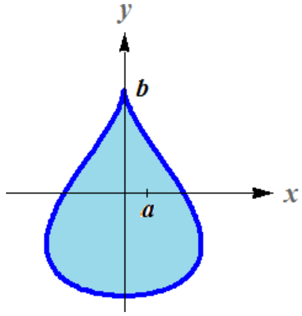
$$= 2ab\left|\frac{1}{3}-1+\frac{1}{3}-1\right|$$

$$\underline{= \frac{8}{3}ab}$$

Exercise

Find the area of the closed curve *Teardrop* $\begin{cases} x = 2a\cos t - a\sin 2t \\ y = b\sin t \end{cases}$ $0 \le t \le 2\pi$

Solution



$$A = \int_0^{2\pi} y\,dx$$

$$= 2\left|\int_{-\pi/2}^{\pi/2} (b\sin t)\ d(2a\cos t - a\sin 2t)\right|$$

$$= 2\left|\int_{-\pi/2}^{\pi/2} (b\sin t)(-2a\sin t - 2a\cos 2t)\ dt\right|$$

$$= 4ab\left|\int_{-\pi/2}^{\pi/2} \left(\sin^2 t + \sin t\cos 2t\right)\ dt\right| \qquad \sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$= 2ab\left|\int_{-\pi/2}^{\pi/2} (1 - \cos 2t + \sin 3t - \sin t)\ dt\right|$$

$$= 2ab\left|\left(t - \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \cos t\right)\Big|_{-\pi/2}^{\pi/2}\right|$$

$$= 2ab\left|\left(\frac{\pi}{2} + \frac{\pi}{2}\right)\right|$$

$$= 2\pi ab$$

Exercise

Find the lengths of the curves $x = \cos t, \quad y = t + \sin t, \quad 0 \le t \le \pi$

Solution

$$x = \cos t \quad \Rightarrow \quad \frac{dx}{dt} = -\sin t$$

$$y = t + \sin t \quad \Rightarrow \quad \frac{dy}{dt} = 1 + \cos t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\sin^2 t + (1+\cos t)^2}$$

$$= \sqrt{\sin^2 t + 1 + 2\cos t + \cos^2 t}$$

$$= \sqrt{2 + 2\cos t}$$

$$L = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$L = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= \int_0^{\pi} \sqrt{2+2\cos t}\; dt$$

$$= \sqrt{2}\int_0^{\pi} \sqrt{(1+\cos t)\frac{1-\cos t}{1-\cos t}}\; dt$$

$$= \sqrt{2}\int_0^{\pi} \sqrt{\frac{1-\cos^2 t}{1-\cos t}}\; dt$$

$$= \sqrt{2}\int_0^{\pi} \sqrt{\frac{\sin^2 t}{1-\cos t}}\; dt$$

$$= \sqrt{2}\int_0^{\pi} \frac{\sin t}{\sqrt{1-\cos t}}\; dt \qquad d(1-\cos t) = \sin t\,dt$$

$$= \sqrt{2}\int_0^{\pi} \frac{d(1-\cos t)}{\sqrt{1-\cos t}}$$

$$= \sqrt{2}\Big[2\sqrt{1-\cos t}\,\Big]_0^{\pi}$$

$$= 2\sqrt{2}\left(\sqrt{1-\cos\pi} - \sqrt{1-\cos 0}\right)$$

$$= 2\sqrt{2}\left(\sqrt{2}-0\right)$$

$$\underline{= 4|}$$

$$= \int_0^{\pi} \sqrt{4\sin^2\frac{t}{2}}\; dt \qquad 2\sin^2\frac{t}{2} = 1+\cos t$$

$$= 2\int_0^{\pi} \sin\frac{t}{2}\; dt$$

$$= -4\cos\frac{t}{2}\Big|_0^{\pi}$$

$$= -4(0-1)$$

$$\underline{= 4|}$$

Exercise

Find the lengths of the curves $\qquad x = t^3, \quad y = \frac{3}{2}t^2, \quad 0 \le t \le \sqrt{3}$

Solution

$$x = t^3 \quad \Rightarrow \quad \frac{dx}{dt} = 3t^2$$

$$y = \frac{3}{2}t^2 \quad \Rightarrow \quad \frac{dy}{dt} = 3t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9t^4 + 9t^2}$$

$$= 3t\sqrt{t^2+1}$$

$$L = \int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

$$= \int_0^{\sqrt{3}} 3t\sqrt{t^2+1}\; dt \qquad u = t^2+1 \quad \Rightarrow \quad du = 2t\,dt \quad \rightarrow \begin{cases} t = \sqrt{3} \Rightarrow u = 4 \\ t = 0 \Rightarrow u = 1 \end{cases}$$

$$= \frac{3}{2}\int_1^4 \sqrt{u}\ du$$

$$= \frac{3}{2}\frac{2}{3}\left[u^{3/2}\right]_1^4$$

$$= 4^{3/2} - 1$$

$$\underline{= 7}$$

Exercise

Find the lengths of the curves $x = 8\cos t + 8t\sin t, \quad y = 8\sin t - 8t\cos t, \quad 0 \le t \le \frac{\pi}{2}$

Solution

$$x = 8\cos t + 8t\sin t \quad \Rightarrow \quad \frac{dx}{dt} = -8\sin t + 8\sin t + 8t\cos t = \underline{8t\cos t}$$

$$y = 8\sin t - 8t\cos t \quad \Rightarrow \quad \frac{dy}{dt} = 8\cos t - 8\cos t + 8t\sin t = \underline{8t\sin t}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(8t\cos t)^2 + (8t\sin t)^2}$$

$$= \sqrt{(8t)^2\cos^2 t + (8t)^2\sin^2 t}$$

$$= 8t\sqrt{\cos^2 t + \sin^2 t} \qquad \cos^2 t + \sin^2 t = 1$$

$$= 8t$$

$$L = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= \int_0^{\pi/2} 8t\ dt$$

$$= 4t^2\Big|_0^{\pi/2}$$

$$= 4\left(\frac{\pi^2}{4} - 0\right)$$

$$\underline{= \pi^2}$$

Exercise

Find the lengths of the curves $x = \ln(\sec t + \tan t) - \sin t, \quad y = \cos t, \quad 0 \le t \le \frac{\pi}{3}$

Solution

$$x = \ln(\sec t + \tan t) - \sin t \quad \Rightarrow \quad \frac{dx}{dt} = \frac{\sec t\tan t + \sec^2 t}{\sec t + \tan t} - \cos t$$

$$= \frac{\sec t(\tan t + \sec t)}{\sec t + \tan t} - \cos t$$

$$= \underline{\sec t - \cos t}|$$

$$y = \cos t \quad \Rightarrow \quad \frac{dy}{dt} = \underline{-\sin t}|$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(\sec t - \cos t)^2 + (-\sin t)^2}$$

$$= \sqrt{\sec^2 t - 2\sec t \cos t + \cos^2 t + \sin^2 t}$$

$$= \sqrt{\sec^2 t - 2\frac{1}{\cos t}\cos t + 1}$$

$$= \sqrt{\sec^2 t - 2 + 1}$$

$$= \sqrt{\sec^2 t - 1}$$

$$= \sqrt{\tan^2 t}$$

$$= \tan t$$

$$L = \int_0^{\pi/3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi/3} \tan t\; dt \qquad \int_0^{\pi/3} \frac{\sin t}{\cos t}\, dt = \int_0^{\pi/3} -\frac{d(\cos t)}{\cos t}$$

$$= -\ln|\cos t|\Big|_0^{\pi/3}$$

$$= -\ln\cos\frac{\pi}{3} + \ln\cos 0$$

$$= -\ln\frac{1}{2} + \ln 1$$

$$= \underline{\ln 2}|$$

Exercise

Find the arc length of the Hypocycloid perimeter curve: $x = a\cos\theta, \quad y = a\sin\theta$

Solution

$$x = a\cos\theta \quad \to \quad \frac{dx}{d\theta} = -a\sin\theta$$

$$y = a\sin\theta \quad \to \quad \frac{dy}{d\theta} = a\cos\theta$$

$$L = 4\int_0^{\pi/2} \sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\; d\theta \qquad L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 4a\int_0^{\pi/2} d\theta$$

$$= 4a\theta\Big|_0^{\pi/2}$$

$$\underline{= 2\pi a}$$

Exercise

Find the arc length of the circle circumference: $x = a\cos^3\theta, \quad y = a\sin^3\theta$

Solution

$$\frac{dx}{d\theta} = -3a\sin\theta\cos^2\theta \quad \frac{dy}{d\theta} = 3a\cos\theta\sin^2\theta$$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{9a^2\sin^2\theta\cos^4\theta + 9a^2\cos^2\theta\sin^4\theta}$$

$$= 3a\sin\theta\cos\theta\sqrt{\cos^2\theta + \sin^2\theta}$$

$$= 3a\sin\theta\cos\theta$$

$$L = 4\int_0^{\pi/2} 3a\sin\theta\cos\theta\ d\theta \qquad L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= 6a\int_0^{\pi/2} \sin 2\theta\ d\theta$$

$$= -3a\cos 2\theta\Big|_0^{\pi/2}$$

$$= -3a(-1-1)$$

$$\underline{= 6a}$$

Exercise

Find the arc length of the Cycloid arch: $x = a(\theta - \sin\theta), \quad y = a(1-\cos\theta)$

Solution

$$x = a(\theta - \sin\theta) \ \rightarrow \ \frac{dx}{d\theta} = a(1-\cos\theta)$$

$$y = a(1-\cos\theta) \ \rightarrow \ \frac{dy}{d\theta} = a\sin\theta$$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2\left(1-2\cos\theta+\cos^2\theta\right) + a^2\sin^2\theta}$$

$$= a\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}$$

$$= a\sqrt{2-2\cos\theta}$$

$$L = 2a\sqrt{2}\int_0^{\pi} \sqrt{1-\cos\theta}\ d\theta \qquad\qquad L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= 2a\sqrt{2}\int_0^{\pi} \sqrt{1-\cos\theta}\ \frac{\sqrt{1+\cos\theta}}{\sqrt{1+\cos\theta}}\ d\theta$$

$$= 2a\sqrt{2}\int_0^{\pi} \frac{\sin\theta}{\sqrt{1+\cos\theta}}\ d\theta$$

$$= -2a\sqrt{2}\int_0^{\pi} (1+\cos\theta)^{-1/2}\ d(1+\cos\theta)$$

$$= -4a\sqrt{2}\sqrt{1+\cos\theta}\Big|_0^{\pi}$$

$$= -4a\sqrt{2}\left(0-\sqrt{2}\right)$$

$$\underline{= 8a|}$$

Exercise

Find the arc length of the involute of a circle: $\qquad x = \cos\theta + \theta\sin\theta, \quad y = \sin\theta - \theta\cos\theta$

Solution

$$x = \cos\theta + \theta\sin\theta \quad \to \quad \frac{dx}{d\theta} = -\sin\theta + \sin\theta + \theta\cos\theta = \theta\cos\theta$$

$$y = \sin\theta - \theta\cos\theta \quad \to \quad \frac{dy}{d\theta} = \theta\sin\theta$$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{\theta^2\cos^2\theta + \theta^2\sin^2\theta}$$

$$= \theta\sqrt{\cos^2\theta + \sin^2\theta}$$

$$= \theta$$

$$L = \int_0^{2\pi} \theta\ d\theta \qquad\qquad L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= \frac{1}{2}\theta^2\Big|_0^{2\pi}$$

$$\underline{= 2\pi^2|}$$

Exercise

Find the area of the surface generated by revolving the curve about each given axis.

$$x = \frac{1}{3}t^3, \quad y = t+1, \quad 1 \le t \le 2, \quad y\text{-axis}$$

Solution

$$\frac{dx}{dt} = t^2, \quad \frac{dy}{dt} = 1$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{t^4 + 1}$$

$$S = 2\pi \int_1^2 \frac{1}{3} t^3 \sqrt{t^4 + 1}\, dt \qquad\qquad S = 2\pi \int_a^b y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \frac{1}{6}\pi \int_1^2 \sqrt{t^4 + 1}\, d\left(t^4 + 1\right)$$

$$= \frac{\pi}{9}\left(t^4 + 1\right)^{3/2} \Big|_1^2$$

$$= \frac{\pi}{9}\left(17^{3/2} - 2^{3/2}\right)$$

$$\boxed{= \frac{\pi}{9}\left(17\sqrt{17} - 2\sqrt{2}\right)}$$

Exercise

Find the areas of the surfaces generated by revolving the curves

$$x = \frac{2}{3} t^{3/2}, \quad y = 2\sqrt{t}, \quad 0 \le t \le \sqrt{3}; \quad x - axis$$

Solution

$$x = \frac{2}{3} t^{3/2} \quad \Rightarrow \quad \frac{dx}{dt} = \boxed{t^{1/2}}$$

$$y = 2\sqrt{t} \quad \Rightarrow \quad \frac{dy}{dt} = \boxed{t^{-1/2}}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(t^{1/2}\right)^2 + \left(t^{-1/2}\right)^2}$$

$$= \sqrt{t + t^{-1}}$$

$$= \sqrt{\frac{t^2 + 1}{t}}$$

$$A = 2\pi \int_0^{\sqrt{3}} x ds$$

$$= 2\pi \int_0^{\sqrt{3}} \frac{2}{3} t^{3/2} \sqrt{\frac{t^2 + 1}{t}} dt$$

$$= \frac{4\pi}{3} \int_0^{\sqrt{3}} t\sqrt{t^2 + 1} dt \qquad u = t^2 + 1 \quad \Rightarrow \quad du = 2tdt \quad \rightarrow \begin{cases} t = \sqrt{3} \Rightarrow u = 4 \\ t = 0 \Rightarrow u = 1 \end{cases}$$

$$= \frac{2\pi}{3}\int_1^4 \sqrt{u}\ du$$

$$= \frac{2\pi}{3}\left[\frac{2}{3}u^{3/2}\right]_1^4$$

$$= \frac{4\pi}{9}\left(4^{3/2}-1\right)$$

$$= \frac{28\pi}{9}$$

Exercise

Find the areas of the surfaces generated by revolving the curves

$$x = t+\sqrt{2}, \quad y = \frac{t^2}{2}+\sqrt{2}t, \quad -\sqrt{2} \le t \le \sqrt{2}; \quad y-axis$$

Solution

$$x = t+\sqrt{2} \quad \Rightarrow \quad \frac{dx}{dt} = 1$$

$$y = \frac{t^2}{2}+\sqrt{2}t \quad \Rightarrow \quad \frac{dy}{dt} = t+\sqrt{2}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{1^2+\left(t+\sqrt{2}\right)^2}$$

$$= \sqrt{1+t^2+2\sqrt{2}t+2}$$

$$= \sqrt{t^2+2\sqrt{2}\ t+3}$$

$$A = 2\pi\int_{-\sqrt{2}}^{\sqrt{2}} xds$$

$$= 2\pi\int_{-\sqrt{2}}^{\sqrt{2}} \left(t+\sqrt{2}\right)\sqrt{t^2+2\sqrt{2}\ t+3}\ dt \qquad u = t^2+2\sqrt{2}\ t+3 \quad \Rightarrow \quad du = \left(2t+2\sqrt{2}\right)d = 2\left(t+\sqrt{2}\right)dt$$

$$\rightarrow \begin{cases} t = \sqrt{2} \Rightarrow u = 9 \\ t = -\sqrt{2} \Rightarrow u = 1 \end{cases}$$

$$= \pi\int_1^9 \sqrt{u}\ du$$

$$= \pi\left[\frac{2}{3}u^{3/2}\right]_1^9$$

$$= \frac{2\pi}{3}\left(9^{3/2}-1\right)$$

$$= \frac{52\pi}{9}$$

Exercise

Find the areas of the surfaces generated by revolving the curves $x = 2t, \quad y = 3t; \quad 0 \le t \le 3 \quad x\text{-}axis$

Solution

$$x = 2t \quad \to \quad \frac{dx}{dt} = 2$$

$$y = 3t \quad \to \quad \frac{dy}{dt} = 3$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{4+9} = \sqrt{13}$$

$$S = 2\pi \int_0^3 (3t)\sqrt{13}\ dt \qquad\qquad S = 2\pi \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= 3\pi\sqrt{13}\left[t^2\right]_0^3$$

$$= 27\pi\sqrt{13}\ \ unit^2$$

Exercise

Find the areas of the surfaces generated by revolving the curves $x = 2t, \quad y = 3t; \quad 0 \le t \le 3 \quad y\text{-}axis$

Solution

$$x = 2t \quad \to \quad \frac{dx}{dt} = 2$$

$$y = 3t \quad \to \quad \frac{dy}{dt} = 3$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{13}$$

$$S = 2\pi \int_0^3 (2t)\sqrt{13}\ dt \qquad\qquad S = 2\pi \int_a^b x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= 2\pi\sqrt{13}\left[t^2\right]_0^3$$

$$= 18\pi\sqrt{13}\ \ unit^2$$

Exercise

Find the areas of the surfaces generated by revolving the curves $x = t, \quad y = 4 - 2t; \quad 0 \le t \le 2 \quad x\text{-}axis$

Solution

$$\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = -2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\sqrt{5}$$

$$S=2\pi\int_0^2 (4-2t)\sqrt{5}\ dt \qquad\qquad S=2\pi\int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

$$=2\pi\sqrt{5}\left[4t-t^2\right]_0^2$$

$$=8\pi\sqrt{5}\ \ unit^2$$

Exercise

Find the areas of the surfaces generated by revolving the curves $x=t,\quad y=4-2t;\quad 0\le t\le 2\quad y\text{-}axis$

Solution

$$\frac{dx}{dt}=1 \qquad \frac{dy}{dt}=-2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\sqrt{5}$$

$$S=2\pi\int_0^2 (t)\sqrt{5}\ dt \qquad\qquad S=2\pi\int_a^b x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

$$=\pi\sqrt{5}\left[t^2\right]_0^2$$

$$=4\pi\sqrt{5}\ \ unit^2$$

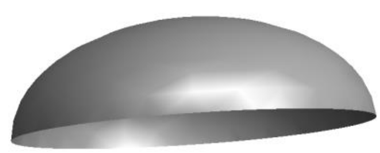
Exercise

Find the area of the surface generated by revolving the curve about each given axis.

$$x=5\cos\theta,\quad y=5\sin\theta,\quad 0\le\theta\le\frac{\pi}{2},\quad y\text{-}axis$$

Solution

$$\frac{dx}{d\theta}=-5\sin\theta \qquad \frac{dy}{d\theta}=5\cos\theta$$



$$\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}=\sqrt{25\sin^2\theta+25\cos^2\theta}\ =5$$

$$S=2\pi\int_0^{\pi/2} 5\cos\theta(5)\ d\theta \qquad\qquad S=2\pi\int_a^b x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

$$=50\pi\sin\theta\Big|_0^{\pi/2}$$

$$=50\pi$$

Exercise

Find the area of the surface generated by revolving the curve about each given axis.

$$x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad 0 \le \theta \le \pi, \quad x\text{-}axis$$

Solution

$$x = a\cos^3\theta \quad \to \quad \frac{dx}{d\theta} = -3a\sin\theta\cos^2\theta$$

$$y = a\sin^3\theta \quad \to \quad \frac{dy}{d\theta} = 3a\cos\theta\sin^2\theta$$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{9a^2\sin^2\theta\cos^4\theta + 9a^2\cos^2\theta\sin^4\theta}$$

$$= 3a\sin\theta\cos\theta$$

$$S = 2\pi\int_0^{\pi/2} a\sin^3\theta\left(3a\sin\theta\cos\theta\right)\, d\theta \qquad\qquad S = 2\pi\int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 12a^2\pi\int_0^{\pi/2} \sin^4\theta\; d\left(\sin\theta\right)$$

$$= \frac{12a^2\pi}{5}\sin^5\theta \Big|_0^{\pi/2}$$

$$= \frac{12}{5}\pi a^2$$

Exercise

Find the area of the surface generated by revolving the curve about each given axis.

$$x = a\cos\theta, \quad y = b\sin\theta, \quad 0 \le \theta \le 2\pi$$

a) $x\text{-}axis$ *b)* $y\text{-}axis$

Solution

$$x = a\cos\theta \quad \to \quad \frac{dx}{d\theta} = -a\sin\theta$$

$$y = b\sin\theta \quad \to \quad \frac{dy}{d\theta} = b\cos\theta$$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$

a) $$S = 4\pi\int_0^{\pi/2} b\sin\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\; d\theta \qquad\qquad S = \pi\int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 4\pi\int_0^{\pi/2} b\sin\theta\sqrt{a^2\left(1-\cos^2\theta\right) + b^2\cos^2\theta}\; d\theta$$

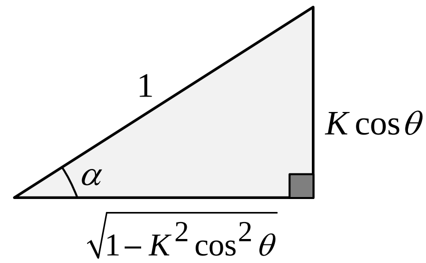
$$= 4\pi \int_0^{\pi/2} b\sin\theta \sqrt{a^2 + \left(b^2 - a^2\right)\cos^2\theta}\; d\theta$$

$$= 4\pi \int_0^{\pi/2} ab\sin\theta \sqrt{1 - \left(\frac{a^2 - b^2}{a^2}\right)\cos^2\theta}\; d\theta \qquad Let: \; K^2 = \frac{a^2 - b^2}{a^2}$$

$$= 4\pi \int_0^{\pi/2} ab\sin\theta \sqrt{1 - K^2\cos^2\theta}\; d\theta$$

$$K\cos\theta = \sin\alpha \qquad \sqrt{1 - K^2\cos^2\theta} = \cos\alpha$$

$$-K\sin\theta d\theta = \cos\alpha d\alpha$$



$$= -\frac{4ab\pi}{K} \int_0^{\pi/2} \cos^2\alpha \; d\alpha$$

$$= -\frac{2ab\pi}{K} \int_0^{\pi/2} (1 + \cos 2\alpha)\; d(\alpha)$$

$$= -\frac{2ab\pi}{K}\left(\alpha + \frac{1}{2}\sin 2\alpha\right)\Big|_0^{\pi/2}$$

$$= -\frac{2ab\pi}{K}\left(\arcsin(K\cos\theta) + K\cos\theta\sqrt{1 - K^2\cos^2\theta}\right)\Big|_0^{\pi/2}$$

$$= -\frac{2a^2b\pi}{\sqrt{a^2 - b^2}}\left(-\arcsin\left(\frac{\sqrt{a^2 - b^2}}{a}\right) - \frac{\sqrt{a^2 - b^2}}{a}\right) \qquad e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{c}{a}: \; \textit{eccentricity}$$

$$\boxed{= \frac{2ab\pi}{e}\left(e + \arcsin(e)\right)} \qquad c = \sqrt{a^2 - b^2}$$

b) $$S = 4\pi \int_0^{\pi/2} a\cos\theta \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\; d\theta \qquad S = \pi\int_a^b x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

$$= 4a\pi \int_0^{\pi/2} \cos\theta \sqrt{\left(a^2 - b^2\right)\sin^2\theta + b^2}\; d\theta$$

$$= 4a\pi \int_0^{\pi/2} \cos\theta \sqrt{c^2\sin^2\theta + b^2}\; d\theta$$

$$c\sin\theta = b\tan\alpha \qquad \sqrt{c^2\sin^2\theta + b^2} = b\sec\alpha$$

$$c\cos\theta d\theta = b\sec^2\alpha \; d\alpha$$

$$= 4a\pi \int_0^{\pi/2} \frac{b^2}{c}\sec^3\alpha \; d\alpha \qquad \int \sec^3 x\, dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right|$$

$$= \frac{2ab^2\pi}{c^2}\left(\sec\alpha\tan\alpha + \ln\left|\sec\alpha + \tan\alpha\right|\right)\Big|_0^{\pi/2}$$

$$= \frac{2ab^2\pi}{c}\left(\frac{c\sin\theta\sqrt{c^2\sin^2\theta + b^2}}{b^2} + \ln\left|\frac{c\sin\theta + \sqrt{c^2\sin^2\theta + b^2}}{b}\right|\right)\Big|_0^{\pi/2}$$

$$= \frac{2ab^2\pi}{c}\left(\frac{c\sqrt{c^2 + b^2}}{b^2} + \ln\left|\frac{c + \sqrt{c^2 + b^2}}{b}\right|\right)$$

$$= 2a\pi\sqrt{a^2-b^2+b^2} + \frac{2ab^2\pi}{c}\ln\left|\frac{\sqrt{a^2-b^2}+\sqrt{a^2-b^2+b^2}}{b}\right|$$

$$= 2a^2\pi + \frac{2ab^2\pi}{\sqrt{a^2-b^2}}\ln\left|\frac{\sqrt{a^2-b^2}+a}{b}\right|$$

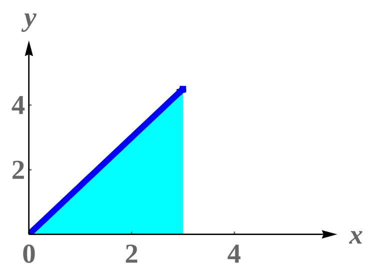
$$\underline{= 2a^2\pi + \frac{2b^2\pi}{e}\ln\left|\frac{a(e+1)}{b}\right|} \qquad = 2a^2\pi + \frac{b^2\pi}{e}\ln\left|\frac{1+e}{1-e}\right|$$

Exercise

Find the area of the surface generated by revolving the curve about each given axis.

$x = 2t, \quad y = 3t, \quad 0 \le t \le 3$

a) *x-axis* *b)* *y-axis*



Solution

$x = 2t \quad \to \quad \frac{dx}{dt} = 2$

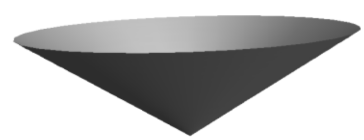
$y = 3t \quad \to \quad \frac{dy}{dt} = 3$

a) $S = 2\pi\int_0^3 3t\sqrt{4+9}\ dt \qquad S = 2\pi\int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$

$$= 3\pi\sqrt{13}\ t^2\Big|_0^3$$

$$\underline{= 27\pi\sqrt{13}}$$

b) $S = 2\pi\int_0^3 2t\sqrt{13}\ dt \qquad S = 2\pi\int_a^b x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$



$$= 2\pi\sqrt{13}\ t^2\Big|_0^3$$

$$\underline{= 18\pi\sqrt{13}}$$

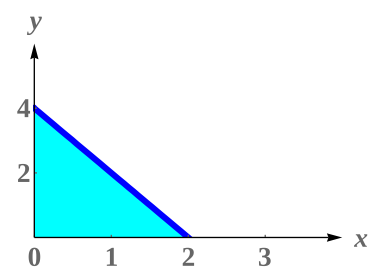
Exercise

Find the area of the surface generated by revolving the curve about each given axis.

$x = t, \quad y = 4-2t, \quad 0 \le t \le 2$

a) *x-axis* *b)* *y-axis*

Solution



$$x = t \quad \to \quad \frac{dx}{dt} = 1$$

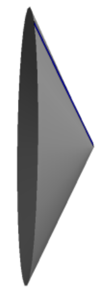
$$y = 4 - 2t \quad \to \quad \frac{dy}{dt} = -2$$

a) $S = 2\pi \int_0^2 (4 - 2t)\sqrt{1+4}\ dt$

$$S = 2\pi \int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= 2\pi\sqrt{5}\left(4t - t^2\right)\Big|_0^2$$

$$\underline{= 8\pi\sqrt{5}}$$

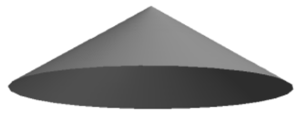


b) $S = 2\pi \int_0^2 t\sqrt{5}\ dt$

$$S = 2\pi \int_a^b x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= \pi\sqrt{5}\ t^2\Big|_0^2$$

$$\underline{= 4\pi\sqrt{5}}$$



Exercise

Use the parametric equations $x = t^2\sqrt{3}$ *and* $y = 3t - \frac{1}{3}t^3$ to

a) Graph the curve on the interval $-3 \le t \le 3$.

b) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

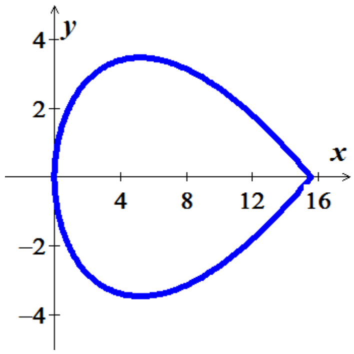
c) Find the equation of the tangent line at the point $\left(\sqrt{3},\ \frac{8}{3}\right)$

d) Find the length of the curve

e) Find the surface area generated by revolving the curve about the *x-axis*

Solution

a)



b) $\boxed{\frac{dy}{dx} = \frac{3 - t^2}{2t\sqrt{3}}}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy'}{dt} = \frac{1}{2\sqrt{3}}\frac{-2t^2 - 3 + t^2}{t^2} = -\frac{t^2 + 3}{2\sqrt{3}t^2}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = -\frac{t^2+3}{2\sqrt{3}t^2} \cdot \frac{1}{2t\sqrt{3}}$$

$$= -\frac{t^2+3}{12t^3}$$

c) $\left(\sqrt{3},\ \frac{8}{3}\right) \to x = t^2\sqrt{3} = \sqrt{3} \quad \Rightarrow t=1$

$$m = \left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{3-t^2}{2t\sqrt{3}}\right|_{t=1} = \frac{1}{\sqrt{3}}$$

$$y = \frac{\sqrt{3}}{3}\left(x-\sqrt{3}\right) + \frac{8}{3}$$

$$= \frac{\sqrt{3}}{3}x + \frac{5}{3}$$

d) $\frac{dx}{dt} = 2t\sqrt{3} \qquad \frac{dy}{dt} = 3 - t^2$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$L = \int_{-3}^{3} \sqrt{12t^2 + 9 - 6t^2 + t^4}\ dt$$

$$= \int_{-3}^{3} \sqrt{\left(t^2+3\right)^2}\ dt$$

$$= \int_{-3}^{3} \left(t^2+3\right)\ dt$$

$$= \left.\frac{1}{3}t^3 + 3t\right|_{-3}^{3}$$

$$= 9+9+9+9$$

$$= 36\ \ unit$$

e) $S = 2\pi\int_0^3 \left(3t - \frac{1}{3}t^3\right)\left(t^2+3\right)dt$

$$S = 2\pi\int_a^b y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= 2\pi\int_0^3 \left(2t^3 - \frac{1}{3}t^5 + 9t\right)dt$$

$$= 2\pi\left[\frac{1}{2}t^4 - \frac{1}{18}t^6 + \frac{9}{2}t^2\right]_0^3$$

$$= 2\pi\left(\frac{81}{2} - \frac{81}{2} + \frac{81}{2}\right)$$

$$= 81\pi\ \ unit^2$$

Exercise

Use the parametric equations $\quad x = a(\theta - \sin\theta) \quad$ *and* $\quad y = a(1 - \cos\theta) \quad a > 0$

a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$

b) Find the equation of the tangent line at the point where $\theta = \dfrac{\pi}{6}$

c) Find all points (if any) of horizontal tangency.

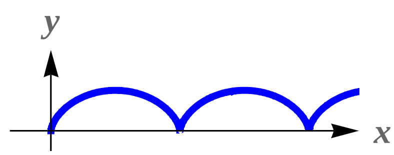
d) Determine where the curve is concave upward or concave downward.

e) Find the length of one arc of the curve

Solution

a) $\dfrac{dx}{d\theta} = a(1 - \cos\theta) \qquad \dfrac{dy}{d\theta} = a\sin\theta$

$$\boxed{\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta}} \qquad\qquad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$



$$\begin{aligned}
\frac{dy'}{d\theta} &= \frac{d}{d\theta}\left(\frac{\sin\theta}{1 - \cos\theta}\right) \\
&= \frac{\cos\theta - \cos^2\theta - \sin^2\theta}{(1 - \cos\theta)^2} \\
&= \frac{\cos\theta - 1}{(1 - \cos\theta)^2} \\
&= \frac{-1}{1 - \cos\theta}
\end{aligned}$$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \left(\frac{-1}{1 - \cos\theta}\right)\frac{1}{a(1 - \cos\theta)} \qquad\qquad \frac{d^2y}{dx^2} = \frac{dy'/d\theta}{dx/d\theta} \\
&\boxed{= \frac{-1}{a(1 - \cos\theta)^2}}
\end{aligned}$$

b) At $\theta = \dfrac{\pi}{6} \qquad x = a\left(\dfrac{\pi}{6} - \dfrac{1}{2}\right) \quad y = a\left(1 - \dfrac{\sqrt{3}}{2}\right)$

$$\begin{aligned}
m = \frac{dy}{dx} &= \frac{\sin\theta}{1 - \cos\theta}\bigg|_{\theta = \frac{\pi}{6}} \\
&= \frac{\frac{1}{2}}{1 - \frac{\sqrt{3}}{2}} \\
&\boxed{= \frac{1}{2 - \sqrt{3}}}
\end{aligned}$$

Tangent Line:

$$y = \frac{1}{2 - \sqrt{3}}\left(x - \frac{\pi a}{6} + \frac{a}{2}\right) + a - \frac{a\sqrt{3}}{2} \qquad\qquad y = m\left(x - x_0\right) + y_0$$

$$\boxed{= \left(2+\sqrt{3}\right)\left(x-\frac{\pi a}{6}+\frac{a}{2}\right)+a-\frac{a\sqrt{3}}{2}}$$

c) $\dfrac{dy}{dx}=\dfrac{\sin\theta}{1-\cos\theta}=0 \quad\Rightarrow\quad \sin\theta=0, \quad \boxed{\theta=(2n+1)\pi}$

$1-\cos\theta\neq 0 \quad\to\theta=2\pi n$

$x=a(2n+1)\pi, \quad y=2a$

Points of horizontal tangency: $\boxed{(x,\ y)=\left(a(2n+1)\pi,\ 2a\right)}$

d) Concave downward on all open θ–intervals

$\ldots,\ (-2\pi,\ 0),\ (0,\ 2\pi),\ (2\pi,\ 4\pi),\ \ldots$

e) $\dfrac{dx}{d\theta}=a(1-\cos\theta) \quad \dfrac{dy}{d\theta}=a\sin\theta$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}=\sqrt{a^2\left(1-2\cos\theta+\cos^2\theta\right)+a^2\sin^2\theta}$$

$$=a\sqrt{1-2\cos\theta+\cos^2\theta+\sin^2\theta}$$

$$=a\sqrt{2-2\cos\theta}$$

$$L=2a\sqrt{2}\int_0^{\pi}\sqrt{1-\cos\theta}\ d\theta \qquad\qquad L=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

$$=2a\sqrt{2}\int_0^{\pi}\sqrt{1-\cos\theta}\ \frac{\sqrt{1+\cos\theta}}{\sqrt{1+\cos\theta}}\ d\theta$$

$$=2a\sqrt{2}\int_0^{\pi}\frac{\sin\theta}{\sqrt{1+\cos\theta}}\ d\theta$$

$$=-2a\sqrt{2}\int_0^{\pi}(1+\cos\theta)^{-1/2}\ d(1+\cos\theta)$$

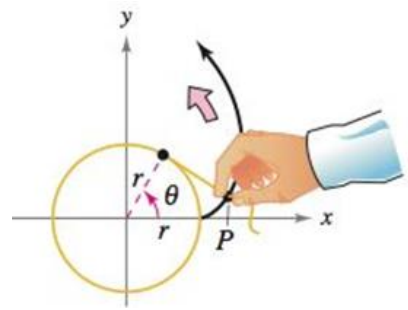
$$=-4a\sqrt{2}\sqrt{1+\cos\theta}\Big|_0^{\pi}$$

$$=-4a\sqrt{2}\left(0-\sqrt{2}\right)$$

$$\boxed{=8a}$$

Exercise

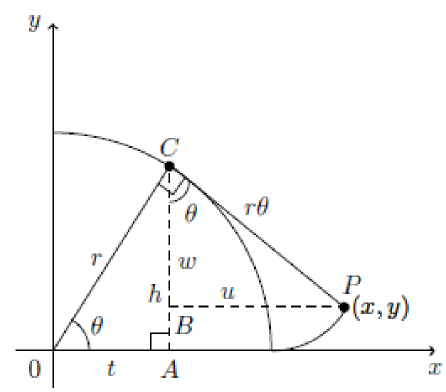
The involute of a circle is described by the endpoint P of a string that is held taut as it is unwound from a spool that does not turn.



Show that a parametric representation of the involute is

$$x = r(\cos\theta + \theta\sin\theta) \quad \text{and} \quad y = r(\sin\theta - \theta\cos\theta)$$

Solution



$\Delta OAC: \quad \cos\theta = \dfrac{t}{r} \quad \sin\theta = \dfrac{h}{r}$

$\Delta PBC: \quad \cos\theta = \dfrac{w}{r\theta} \quad \sin\theta = \dfrac{u}{r\theta}$

$x = t + u = r\cos\theta + r\theta\sin\theta$

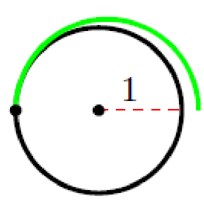
$\quad \boxed{= r(\cos\theta + \theta\sin\theta)}$

$y = h - w = r\sin\theta - r\theta\cos\theta$

$\quad \boxed{= r(\sin\theta - \theta\cos\theta)}$

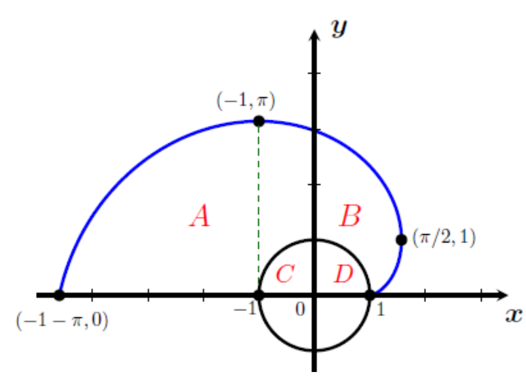
Exercise

The figure shows a piece of string tied to a circle with a radius of one unit. The string is just long enough to reach the opposite side if the circle.



Find the area that is covered when the string is unwounded counterclockwise.

Solution



From previous exercise, we have

$$x = \cos\theta + \theta\sin\theta \quad \text{and} \quad y = \sin\theta - \theta\cos\theta$$

At $(-1, \pi)$, the string is fully extended and has length x.

The area of region $\boldsymbol{A}$ is: $\frac{1}{4}\pi r^2 \boxed{= \frac{1}{4}\pi^3}$

The area of region $\boldsymbol{C + D}$ is: $\frac{1}{2}\pi r^2 \boxed{= \frac{\pi}{2}}$

$$\frac{dx}{d\theta} = -\sin\theta + \sin\theta + \theta\cos\theta \underline{= \theta\cos\theta}$$

$$\frac{dx}{d\theta} = \theta\cos\theta = 0 \quad \rightarrow \quad \theta = \frac{\pi}{2} \quad (\theta = 0 \ \ is\ cusp)$$

The area of the region $B + C + D$ is given by

$$\int_{\pi}^{\pi/2} ydx - \int_{0}^{\pi/2} ydx = \int_{\pi}^{0} ydx$$

$$A_2 = \int_{\pi}^{0} (\sin\theta - \theta\cos\theta)\theta\cos\theta \ d\theta$$

$$= \int_{\pi}^{0} \left(\theta\cos\theta\sin\theta - \theta^2\cos^2\theta\right) \ d\theta$$

$$= \int_{\pi}^{0} \left(\frac{1}{2}\theta\sin 2\theta - \frac{1}{2}\theta^2 - \frac{1}{2}\theta^2\cos 2\theta\right) \ d\theta$$

		$\int \sin 2\theta$
+	θ	$-\frac{1}{2}\cos 2\theta$
–	1	$-\frac{1}{4}\sin 2\theta$

$$= -\frac{1}{4}\theta\cos 2\theta + \frac{1}{8}\sin 2\theta - \frac{1}{6}\theta^3 - \frac{1}{4}\theta^2\sin 2\theta - \frac{1}{4}\theta\cos 2\theta + \frac{1}{8}\sin 2\theta \Big|_{\pi}^{0}$$

$$= \frac{\pi}{4} + \frac{\pi^3}{6} + \frac{\pi}{4}$$

$$= \boxed{\frac{\pi^3}{6} + \frac{\pi}{2}}$$

		$\int \cos 2\theta$
+	$\frac{1}{2}\theta^2$	$\frac{1}{2}\sin 2\theta$
–	θ	$-\frac{1}{4}\cos 2\theta$
+	1	$-\frac{1}{8}\sin 2\theta$

Total area covered $= 2\left(\frac{\pi^3}{4} + \frac{\pi^3}{6} + \frac{\pi}{2} - \frac{\pi}{2}\right)$

$$\boxed{= \frac{5\pi^3}{6}}$$