# Solution  *Section* 3.3 – Double Integrals in Polar Coordinates

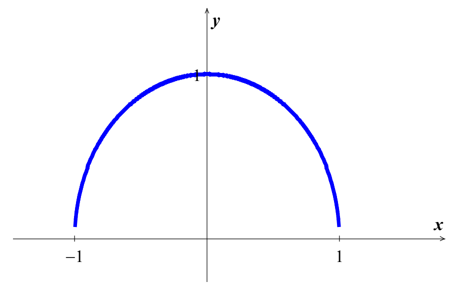
### Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dydx$$

**Solution**

$$y=\sqrt{1-x^2} \quad\Rightarrow\quad y^2=1-x^2 \to x^2+y^2=1=r^2$$




$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dydx=\int_{0}^{\pi}\int_{0}^{1} r\,drd\theta$$

$$=\int_{0}^{\pi}\frac{1}{2}\left(r^2\Big|_{0}^{1}\right.d\theta$$

$$=\frac{1}{2}\int_{0}^{\pi}d\theta$$

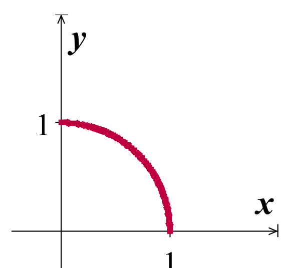
$$=\frac{\pi}{2}$$

### Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}}\left(x^2+y^2\right)dxdy$$

**Solution**




$$\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}}\left(x^2+y^2\right)dxdy=\int_{0}^{\pi/2}d\theta\int_{0}^{1}r^2\,rdr$$

$$=\frac{\pi}{2}\,\frac{1}{4}\left(r^4\Big|_{0}^{1}\right.$$

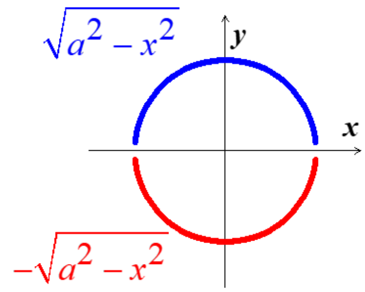
$$=\frac{\pi}{8}$$

### Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx$$

**Solution**




$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx = \int_0^{2\pi} d\theta \int_0^a rdr$$

$$= (2\pi)\ \frac{1}{2}\left(r^2\ \Big|_0^a\right.$$

$$\underline{= \pi a^2}$$

### Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_0^6\int_0^y xdxdy$$

**Solution**

$$x = r\cos\theta, \quad \sin\theta = \frac{6}{r} \to r = \frac{6}{\sin\theta} = 6\csc\theta$$

$$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$$

$$\int_0^6\int_0^y x\,dxdy = \int_{\pi/4}^{\pi/2}\int_0^{6\csc\theta} r^2\cos\theta\ drd\theta$$

$$= \frac{1}{3}\int_{\pi/4}^{\pi/2} \cos\theta\left(r^3\ \Big|_0^{6\csc\theta}\right. d\theta$$

$$= \frac{216}{3}\int_{\pi/4}^{\pi/2} \cos\theta\csc^3\theta\ d\theta$$

$$= 72\int_{\pi/4}^{\pi/2} \cot\theta\ \csc^2\theta\ d\theta \qquad d(\cot\theta) = -\csc^2\theta\ d\theta$$

$$= -72\int_{\pi/4}^{\pi/2} \cot\theta\ d(\cot\theta)$$

$$= -36\left(\cot^2\theta \,\Big|_{\pi/4}^{\pi/2}\right.$$

$$= -36(0-1)$$

$$\underline{= 36}$$

### *Exercise*

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,dydx$$

***Solution***

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,dydx = \int_{\pi}^{3\pi/2}\int_{0}^{1} \frac{2}{1+r}r\;drd\theta$$

$$= 2\int_{\pi}^{3\pi/2} d\theta \int_{0}^{1}\left(1-\frac{1}{1+r}\right)dr$$

$$= 2\left(\frac{3\pi}{2}-\pi\right)\left(r-\ln(1+r)\,\Big|_{0}^{1}\right.$$

$$\underline{= (1-\ln 2)\pi}$$

### *Exercise*

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{0}^{\ln 2}\int_{0}^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\,dxdy$$

***Solution***

$$\int_{0}^{\ln 2}\int_{0}^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\,dxdy = \int_{0}^{\pi/2} d\theta \int_{0}^{\ln 2} e^r r\;dr$$

| | | $\int e^r$ |
|---|---|---|
| + | $r$ | $e^r$ |
| − | 1 | $e^r$ |

$$= \frac{\pi}{2}\left(re^r - e^r\,\Big|_{0}^{\ln 2}\right.$$

$$= \frac{\pi}{2}\left(\ln 2e^{\ln 2} - e^{\ln 2} + 1\right)$$

$$\underline{= \frac{\pi}{2}(2\ln 2 - 1)}$$

### *Exercise*

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dxdy$$

***Solution***

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dxdy = \int_0^{2\pi} d\theta \int_0^1 \ln\left(r^2+1\right)r\,dr$$

$$= (2\pi)\int_0^1 \ln\left(r^2+1\right)\,\frac{1}{2}\,d\left(r^2+1\right) \qquad \int \ln au\ du = u\ln au\ -u$$

$$= \pi\ \left(\left(r^2+1\right)\ln\left(r^2+1\right)-\left(r^2+1\right)\ \Big|_0^1\right.$$

$$= \pi\left(2\ln 2-2+1\right)$$

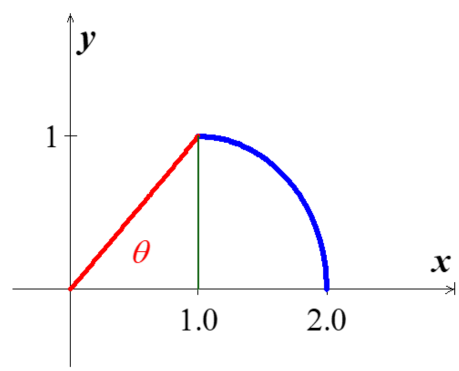
$$\boxed{= \pi\left(\ln 4-1\right)}$$

### *Exercise*

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_1^2\int_0^{\sqrt{2x-x^2}} \frac{1}{\left(x^2+y^2\right)^2}dydx$$

***Solution***




$$y^2 = 2x-x^2 \Rightarrow x^2-2x+1-1+y^2=0 \quad (x-1)^2+y^2=1$$

$$r = \frac{x}{\cos\theta} = \frac{1}{\cos\theta} = \sec\theta$$

$$y=\sqrt{2x-x^2} \quad \to \quad y^2 = 2x-x^2 \Rightarrow x^2+y^2=2x$$

$$r^2 = 2r\cos\theta \to r = 2\cos\theta$$

$$\int_1^2\int_0^{\sqrt{2x-x^2}} \frac{1}{\left(x^2+y^2\right)^2}dydx = \int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta} \frac{1}{r^4}r\,drd\theta$$

$$= \int_0^{\pi/4} \int_{\sec\theta}^{2\cos\theta} r^{-3}\, dr d\theta$$

$$= \int_0^{\pi/4} \left( -\frac{1}{2r^2} \Big|_{\sec\theta}^{2\cos\theta} \right) d\theta$$

$$= \int_0^{\pi/4} \left( -\frac{1}{8\cos^2\theta} + \frac{1}{2\sec^2\theta} \right) d\theta$$

$$= \int_0^{\pi/4} \left( -\frac{1}{8}\sec^2\theta + \frac{1}{2}\cos^2\theta \right) d\theta \qquad \int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$= -\frac{1}{8}\tan\theta + \frac{1}{2}\left( \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right) \Big|_0^{\pi/4}$$

$$= \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta - \frac{1}{8}\tan\theta \Big|_0^{\pi/4}$$

$$= \frac{1}{4}\frac{\pi}{4} + \frac{1}{8} - \frac{1}{8} - (0)$$

$$= \frac{\pi}{16}$$

### *Exercise*

Evaluate the integral by changing to polar coordinates $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2dydx}{\left(1+x^2+y^2\right)^2}$

***Solution***

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2dydx}{\left(1+x^2+y^2\right)^2} = \int_0^{2\pi} d\theta \int_0^1 \frac{2r}{\left(1+r^2\right)^2} dr$$

$$= (2\pi)\int_0^1 \left(1+r^2\right)^{-2} d\left(1+r^2\right)$$

$$= 2\pi \ \frac{-1}{1+r^2} \Big|_0^1$$

$$= 2\pi\left( -\frac{1}{2} + 1 \right)$$

$$= \pi$$

### *Exercise*

Evaluate the integral by changing to polar coordinates $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dxdy$

***Solution***

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dxdy = \int_0^{2\pi} d\theta \int_0^1 \ln\left(r^2+1\right)r\ dr$$

$$= (2\pi)\ \frac{1}{2}\int_0^1 \ln\left(r^2+1\right)\ d\left(r^2+1\right)$$

$$r^2+1=w$$

$$u=\ln w \quad\rightarrow\quad du=\frac{dw}{w} \qquad v=\int dw = w$$

$$\int \ln w\ dw = w\ln w - \int dw$$

$$= w\ln w - w$$

$$= \pi\left(\left(r^2+1\right)\left(\ln\left(r^2+1\right)-1\right)\right)\Big|_0^1$$

$$= \pi\left(2\ln 2 - 2 + 1\right)$$

$$\boxed{= \pi\left(2\ln 2 - 1\right)}$$

### *Exercise*

Evaluate the integral $\displaystyle\int_0^{\infty}\int_0^{\infty} \frac{1}{\left(1+x^2+y^2\right)^2}dxdy$

***Solution***

$$\int_0^{\infty}\int_0^{\infty} \frac{1}{\left(1+x^2+y^2\right)^2}dxdy = \int_0^{\pi/2} d\theta \int_0^{\infty} \frac{1}{\left(1+r^2\right)^2}rdr \qquad d\left(1+r^2\right)=2rdr$$

$$= \frac{\pi}{2}\int_0^{\infty}\left(1+r^2\right)^{-2}\frac{1}{2}d\left(1+r^2\right)$$

$$= \frac{\pi}{4}\left(-\frac{1}{1+r^2}\right|_0^{\infty} \qquad \frac{1}{\infty}=0$$

$$\boxed{= \frac{\pi}{4}}$$

## *Exercise*

Evaluate the integral $\displaystyle\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\ dydx$

***Solution***

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\ dydx = \int_0^{\frac{\pi}{2}} \int_0^3 r\ rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta\ \int_0^3 r^2\ dr$$

$$= \frac{\pi}{2}\left(\frac{1}{3}r^3\ \Big|_0^3\right.$$

$$= \frac{9\pi}{2}$$

## *Exercise*

Evaluate the integral $\displaystyle\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(x^2+y^2\right)^{3/2}\ dydx$

***Solution***

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(x^2+y^2\right)^{3/2}\ dydx = \int_0^{2\pi} \int_0^1 \left(r^2\right)^{3/2}\ rdrd\theta$$

$$= \int_0^{2\pi} d\theta\ \int_0^1 r^4\ dr$$

$$= 2\pi\left(\frac{1}{5}r^5\ \Big|_0^1\right.$$

$$= \frac{2\pi}{5}$$

## *Exercise*

Evaluate the integral $\displaystyle\int_{-4}^4 \int_0^{\sqrt{16-y^2}} \left(16-x^2-y^2\right)\ dxdy$

***Solution***

$$\int_{-4}^{4}\int_{0}^{\sqrt{16-y^2}}\left(16-x^2-y^2\right)\,dxdy=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{4}\left(16-r^2\right)\,rdrd\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_{0}^{4}\left(16r-r^3\right)\,dr$$

$$=\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\left(8r^2-\frac{1}{4}r^4\right|_{0}^{4}$$

$$=\pi\left(128-64\right)$$

$$\underline{=64\pi}$$

### *Exercise*

Evaluate the integral $\displaystyle\int_{0}^{\frac{\pi}{4}}\int_{0}^{\sec\theta} r^3\,drd\theta$

***Solution***

$$\int_{0}^{\frac{\pi}{4}}\int_{0}^{\sec\theta} r^3\,drd\theta=\frac{1}{4}\int_{0}^{\frac{\pi}{4}} r^4\Big|_{0}^{\sec\theta}\,d\theta$$

$$=\frac{1}{4}\int_{0}^{\frac{\pi}{4}}\sec^4\theta\,d\theta$$

$$=\frac{1}{4}\int_{0}^{\frac{\pi}{4}}\sec^2\theta\sec^2\theta\,d\theta$$

$$=\frac{1}{4}\int_{0}^{\frac{\pi}{4}}\left(1+\tan^2\theta\right)\,d\left(\tan\theta\right)$$

$$=\frac{1}{4}\left(\tan\theta+\frac{1}{3}\tan^3\theta\right|_{0}^{\frac{\pi}{4}}$$

$$=\frac{1}{4}\left(1+\frac{1}{3}\right)$$

$$\underline{=\frac{1}{3}}$$

### *Exercise*

Evaluate the integral $\displaystyle\int_0^{\frac{\pi}{2}}\int_1^{\infty}\frac{\cos\theta}{r^3}r\ drd\theta$

***Solution***

$$\int_0^{\frac{\pi}{2}}\int_1^{\infty}\frac{\cos\theta}{r^3}r\ drd\theta=\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta\ \int_1^{\infty}\frac{1}{r^2}dr$$

$$=\sin\theta\Big|_0^{\frac{\pi}{2}}\ \left(-\frac{1}{r}\right|_1^{\infty}$$

$$=-(1)(0-1) \qquad\qquad \frac{1}{\infty}=0$$

$$=1$$

### *Exercise*

Find the area of the region cut from the first quadrant by the curve $r=2(2-\sin 2\theta)^{1/2}$

***Solution***

$$\int_0^{\pi/2}\int_0^{2\sqrt{2-\sin 2\theta}}rdrd\theta=\frac{1}{2}\int_0^{\pi/2}\left(r^2\ \Big|_0^{2\sqrt{2-\sin 2\theta}}\right.\ d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}4(2-\sin 2\theta)d\theta$$

$$=2\left(2\theta+\frac{1}{2}\cos 2\theta\ \right|_0^{\pi/2}$$

$$=2\left[\pi-\frac{1}{2}-\left(\frac{1}{2}\right)\right]$$

$$=2(\pi-1)$$

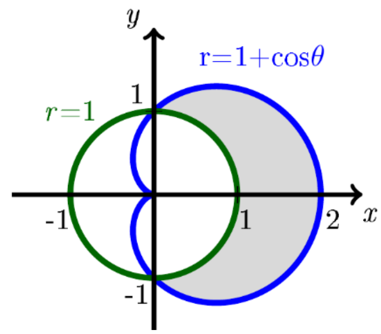
### *Exercise*

Find the area of the region lies inside the cardioid $r=1+\cos\theta$ and outside the circle $r=1$

***Solution***

$$A=2\int_0^{\pi/2}\int_1^{1+\cos\theta}rdrd\theta$$

$$= \int_0^{\pi/2} \left( r^2 \Big|_1^{1+\cos\theta} \right) d\theta$$




$$= \int_0^{\pi/2} \left( (1+\cos\theta)^2 - 1 \right) d\theta$$

$$= \int_0^{\pi/2} \left( 1 + 2\cos\theta + \cos^2\theta - 1 \right) d\theta$$

$$= \int_0^{\pi/2} \left( 2\cos\theta + \cos^2\theta \right) d\theta$$

$$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$= 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \Big|_0^{\pi/2}$$

$$\boxed{= 2 + \frac{\pi}{4}}$$

### *Exercise*

Find the area enclosed by one leaf of the rose $r = 12\cos 3\theta$

***Solution***

$$A = 2\int_0^{\pi/6} \int_0^{12\cos 3\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/6} \left( r^2 \Big|_0^{12\cos 3\theta} \right) d\theta$$

$$= 144 \int_0^{\pi/6} \cos^2 3\theta\,d\theta$$

$$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$= 144 \left( \frac{\theta}{2} + \frac{\sin 6\theta}{12} \Big|_0^{\pi/6} \right.$$

$$= 144\left(\frac{\pi}{12}\right)$$

$$\boxed{= 12\pi}$$

### *Exercise*

Find the area of the region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$

***Solution***

$$A = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left(r^2 \Big|_0^{1-\cos\theta}\right) d\theta$$

$$= 2\int_0^{\pi/2}(1-\cos\theta)^2\, d\theta$$

$$= 2\int_0^{\pi/2}\left(1 - 2\cos\theta + \cos^2\theta\right)d\theta \qquad \int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$= 2\left(\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_0^{\pi/2}$$

$$= 2\left(\frac{\pi}{2} - 2 + \frac{\pi}{4}\right)$$

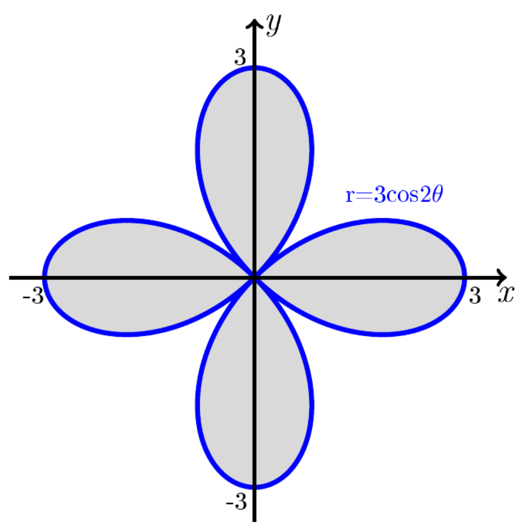
$$= \frac{3\pi}{2} - 4$$

### *Exercise*

Find the area of the region bounded by all leaves of the rose $r = 3\cos 2\theta$

***Solution***

$$A = 4\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{3\cos 2\theta} r\ dr\,d\theta$$




$$= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2 \Big|_0^{3\cos 2\theta} d\theta$$

$$= 18\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta\ d\theta$$

$$= 9\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 + \cos 4\theta)\ d\theta$$

$$= 9\left(\theta + \frac{1}{4}\sin 4\theta\right)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 9\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$= \frac{9\pi}{2}\ \ unit^2$$

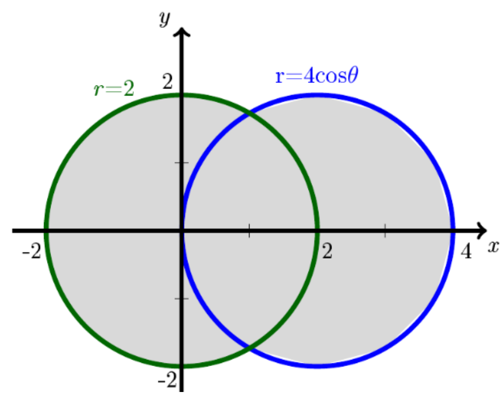
## *Exercise*

Find the area of the region inside both the circles $r = 2$ and $r = 4\cos\theta$

***Solution***

$$r = 4\cos\theta = 2 \quad \to \quad \cos\theta = \frac{1}{2}$$

$$\Rightarrow \ \theta = \frac{\pi}{3},\ \frac{5\pi}{3}$$




$$A = 2\int_0^{\frac{\pi}{3}}\int_0^2 r\ drd\theta\ +\ 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{4\cos\theta} r\ drd\theta$$

$$= \int_0^{\frac{\pi}{3}} d\theta\ \ r^2\Big|_0^2 + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} r^2\Big|_0^{4\cos\theta} d\theta$$

$$= \frac{4\pi}{3} + 16\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2\theta\ d\theta$$

$$= \frac{4\pi}{3} + 8\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1 + \cos 2\theta)\ d\theta$$

$$= \frac{4\pi}{3} + 8\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \frac{4\pi}{3} + 8\left(\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

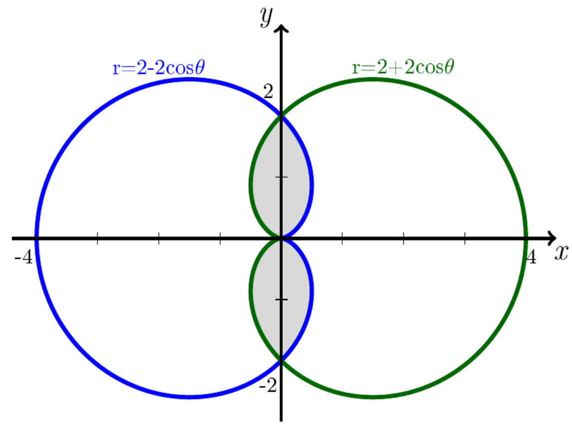
$$= \frac{4\pi}{3} + \frac{4\pi}{3} - 2\sqrt{3}$$

$$= \frac{8\pi}{3} - 2\sqrt{3}\ \ unit^2$$

### *Exercise*

Find the area of the region that lies inside both the cardioids $r = 2 - 2\cos\theta$ and $r = 2 + 2\cos\theta$

***Solution***




$$A = 4\int_0^{\frac{\pi}{2}}\int_0^{2-2\cos\theta} r\ drd\theta$$

$$= 2\int_0^{\frac{\pi}{2}} r^2 \Big|_0^{2-2\cos\theta} d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (2-2\cos\theta)^2\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left(4-8\cos\theta+4\cos^2\theta\right)\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (6-8\cos\theta+2\cos 2\theta)\, d\theta$$

$$= 2(6\theta-8\sin\theta+\sin 2\theta)\Big|_0^{\frac{\pi}{2}}$$

$$= 2(3\pi-8)$$

$$= 6\pi-16\ \ unit^2$$

### *Exercise*

Find the area of the annular region $\{(r,\ \theta):\ \ 1\le r\le 2,\ \ \ 0\le\theta\le 2\pi\}$

***Solution***

$$\int_0^{2\pi}\int_1^2 r\ drd\theta = \int_0^{2\pi} d\theta\ \left(\frac{1}{2}r^2\right. \Big|_1^2$$

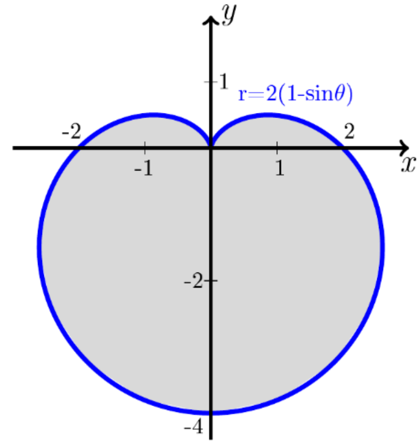
$$= 2\pi\frac{1}{2}(4-1)$$

$$= 3\pi\ \ unit^2$$

### *Exercise*

Find the area of the region bounded by the cardioid $r = 2(1-\sin\theta)$

**Solution**




$$A = \int_0^{2\pi} \int_0^{2(1-\sin\theta)} r \, dr d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2}r^2 \Big|_0^{2(1-\sin\theta)} \right) d\theta$$

$$= 2\int_0^{2\pi} \left(1 - 2\sin\theta + \sin^2\theta\right) d\theta$$

$$= 2\int_0^{2\pi} \left(\frac{3}{2} - 2\sin\theta - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= 2\left(\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta \right)\Big|_0^{2\pi}$$

$$= 2(3\pi + 2 - 2)$$

$$= 6\pi \ \ unit^2$$

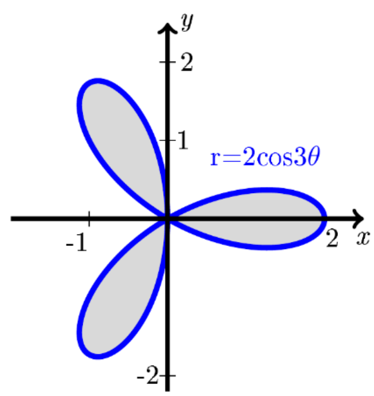
## *Exercise*

Find the area of the region bounded by all leaves of the rose $r = 2\cos 3\theta$

**Solution**

$$r = 2\cos 3\theta = 2 \quad \rightarrow 3\theta = 0 + 2n\pi \quad \Rightarrow \theta = 0, \ \ldots$$

$$r = 2\cos 3\theta = 0 \quad \rightarrow 3\theta = \frac{\pi}{2} + 2n\pi \quad \Rightarrow \theta = \frac{\pi}{6}, \ \ldots$$




$$A = 6\int_0^{\frac{\pi}{6}} \int_0^{2\cos 3\theta} r \, dr d\theta$$

$$= 3\int_0^{\frac{\pi}{6}} \left(r^2 \Big|_0^{2\cos 3\theta} \right) d\theta$$

$$= 12\int_0^{\frac{\pi}{6}} \cos^2 3\theta \, d\theta$$

$$= 6\int_0^{\frac{\pi}{6}} (1 + \cos 6\theta) \, d\theta$$

$$= 6\left(\theta + \frac{1}{6}\sin 6\theta \,\right|_0^{\pi/6}$$

$$= 6\left(\frac{\pi}{6}\right)$$

$$\underline{= \pi \ \ unit^2}$$

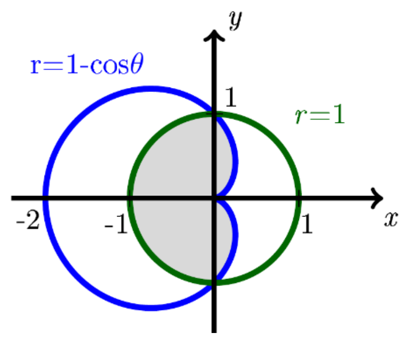
### *Exercise*

Find the area of the region inside both the cardioid $r = 1 - \cos\theta$ and the circle $r = 1$

***Solution***

$r = 1 - \cos\theta = 1 \quad \rightarrow \quad \cos\theta = 0$

$\theta = \frac{\pi}{2}, \ \frac{3\pi}{2}$




$$A = \left(area\ of\ \frac{1}{2}\ circle\right) + 2\int_0^{\pi/2}\int_0^{1-\cos\theta} r\ drd\theta$$

$$= \frac{\pi}{2} + \int_0^{\pi/2} r^2 \Big|_0^{1-\cos\theta} d\theta$$

$$= \frac{\pi}{2} + \int_0^{\pi/2} (1-\cos\theta)^2\, d\theta$$

$$= \frac{\pi}{2} + \int_0^{\pi/2} \left(1 - 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= \frac{\pi}{2} + \int_0^{\pi/2} \left(\frac{3}{2} - 2\cos\theta + \cos 2\theta\right) d\theta$$

$$= \frac{\pi}{2} + \left(\frac{3}{2}\theta - 2\sin\theta + \frac{1}{2}\sin 2\theta \,\right|_0^{\pi/2}$$

$$= \frac{\pi}{2} + \frac{3\pi}{4} - 2$$

$$\underline{= \frac{5\pi}{4} - 2 \ \ unit^2}$$

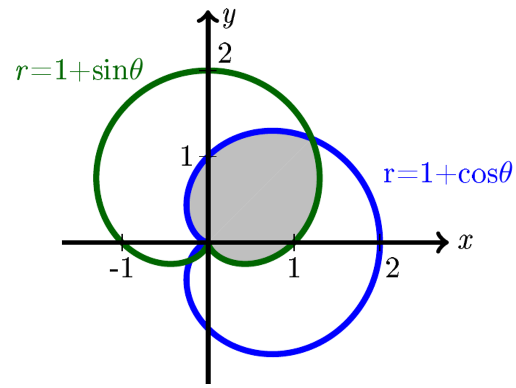
### *Exercise*

Find the area of the region inside both the cardioid $r = 1 + \sin\theta$ and the cardioid $r = 1 + \cos\theta$

***Solution***

$$r = 1 + \sin\theta = 1 + \cos\theta \quad \to \quad \sin\theta = \cos\theta$$

$\theta = \frac{\pi}{4}, \ \frac{5\pi}{4}$, and due to the symmetry;




$$A = 2\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \int_{0}^{1+\cos\theta} r\ dr d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} r^2 \Big|_{0}^{1+\cos\theta} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1+\cos\theta)^2\ d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left(1 + 2\cos\theta + \cos^2\theta\right)\ d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left(\frac{3}{2} + 2\cos\theta + \cos 2\theta\right)\ d\theta$$

$$= \frac{3}{2}\theta + 2\sin\theta + \frac{1}{2}\sin 2\theta \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \frac{15\pi}{8} - \sqrt{2} + \frac{1}{2} - \frac{3\pi}{8} - \sqrt{2} - \frac{1}{2}$$

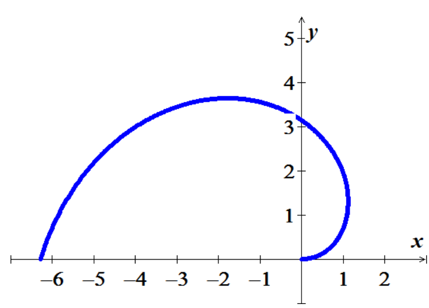
$$= \frac{3\pi}{2} - 2\sqrt{2}\ \ unit^2$$

### *Exercise*

Find the area of the region bounded by the spiral $r = 2\theta$, for $0 \le \theta \le \pi$, and the $x$-axis.

***Solution***

$$A = \int_0^{\pi} \int_0^{2\theta} r\ dr d\theta$$



$$= \frac{1}{2} \int_0^{\pi} r^2 \Big|_0^{2\theta} \, d\theta$$

$$= 2 \int_0^{\pi} \theta^2 \, d\theta$$

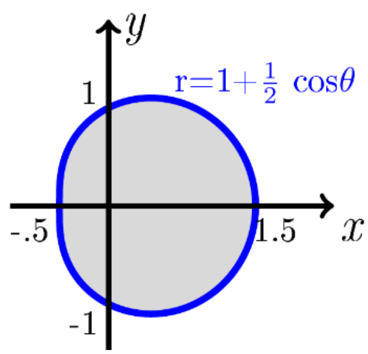
$$= \frac{2}{3} \theta^3 \Big|_0^{\pi}$$

$$= \frac{2\pi^3}{3} \ unit^2$$

## *Exercise*

Find the area of the region inside the limaçon $r = 1 + \frac{1}{2}\cos\theta$

***Solution***




$$A = \int_0^{2\pi} \int_0^{1+\frac{1}{2}\cos\theta} r \, dr d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} r^2 \Big|_0^{1+\frac{1}{2}\cos\theta} \, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(1 + \frac{1}{2}\cos\theta\right)^2 \, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(1 + \cos\theta + \frac{1}{4}\cos^2\theta\right) d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\frac{9}{8} + \cos\theta + \frac{1}{8}\cos 2\theta\right) d\theta$$

$$= \frac{1}{2} \left(\frac{9}{8}\theta + \sin\theta + \frac{1}{16}\sin 2\theta \Big|_0^{2\pi}\right.$$

$$= \frac{9\pi}{8} \ unit^2$$

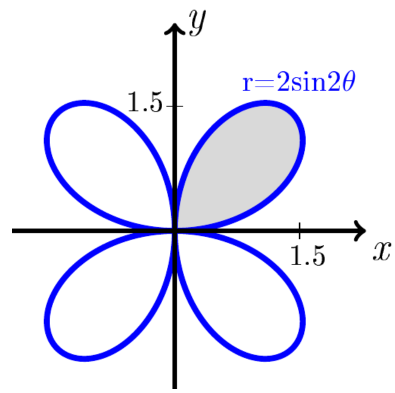
### *Exercise*

Find the area of the region bounded by $r = 2\sin 2\theta$ in $QI$.

***Solution***

$$r = 2\sin 2\theta = 0$$

$$2\theta = n\pi \quad \Rightarrow \quad \boxed{\theta = 0, \ \frac{\pi}{2}}$$




$$A = \int_0^{\frac{\pi}{2}} \int_0^{2\sin 2\theta} r \ dr d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} r^2 \Big|_0^{2\sin 2\theta} \ d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^2 2\theta \ d\theta$$

$$= \int_0^{\frac{\pi}{2}} (1 - \cos 4\theta) \ d\theta$$

$$= \theta - \frac{1}{4}\sin 4\theta \Big|_0^{\frac{\pi}{2}}$$

$$\boxed{= \frac{\pi}{2} \ \ unit^2}$$

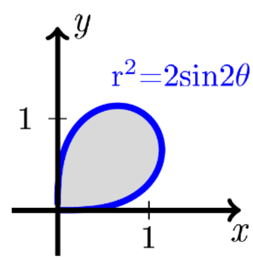
### *Exercise*

Find the area of the region bounded by $r^2 = 2\sin 2\theta$ in $QI$.

***Solution***

$$r^2 = 2\sin 2\theta = 0$$

$$2\theta = n\pi \quad \Rightarrow \quad \boxed{\theta = 0, \ \frac{\pi}{2}}$$




$$A = \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2\sin 2\theta}} r \ dr d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} r^2 \Big|_0^{\sqrt{2\sin 2\theta}} \ d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin 2\theta \, d\theta$$

$$= -\frac{1}{2}\cos 2\theta \Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{2}(-1-1)$$

$$\boxed{= 1 \ \ unit^2}$$

## *Exercise*

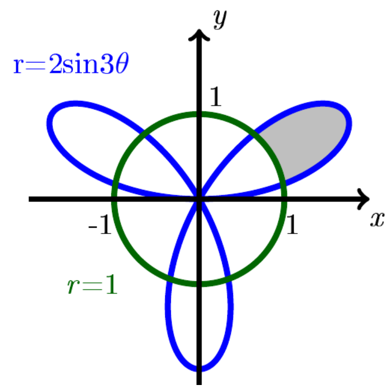
Find the area of the region outside the circle $r = 1$ and inside the rose $r = 2\sin 3\theta$ in $Q$I.

***Solution***

$$r = 2\sin 3\theta = 1$$

$$3\theta = \frac{\pi}{6}, \ \frac{5\pi}{6} \quad \Rightarrow \quad \boxed{\theta = \frac{\pi}{18}, \ \frac{5\pi}{18}}$$

$$A = \int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} \int_1^{2\sin 3\theta} r \, dr d\theta$$




$$= \frac{1}{2}\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} r^2 \Big|_1^{2\sin 3\theta} d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} \left(4\sin^2 3\theta - 1\right) d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} (1 - 2\cos 6\theta) \, d\theta$$

$$= \frac{1}{2}\left(\theta - \frac{1}{3}\cos 6\theta \right. \Big|_{\frac{\pi}{18}}^{\frac{5\pi}{18}}$$

$$= \frac{1}{2}\left(\frac{5\pi}{18} - \frac{1}{3}\cos\frac{5\pi}{3} - \frac{\pi}{18} + \frac{1}{3}\cos\frac{\pi}{3}\right)$$

$$= \frac{1}{2}\left(\frac{2\pi}{9} - \frac{1}{6} + \frac{1}{6}\right)$$

$$\boxed{= \frac{\pi}{9} \ \ unit^2}$$

### *Exercise*

Find the area of the region outside the circle $r=\frac{1}{2}$ and inside the circle $r=1+\cos\theta$

***Solution***

$$A=2\int_0^{\pi/2}\int_{\frac{1}{2}}^{1+\cos\theta} r\,dr\,d\theta$$

$$=\int_0^{\pi/2}\left(r^2\Big|_{\frac{1}{2}}^{1+\cos\theta}\right)d\theta$$

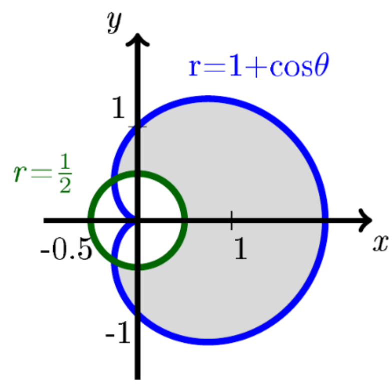
$$=\int_0^{\pi/2}\left[(1+\cos\theta)^2-\frac{1}{4}\right]d\theta$$

$$=\int_0^{\pi/2}\left(\frac{3}{4}+2\cos\theta+\cos^2\theta\right)d\theta$$

$$=\int_0^{\pi/2}\left(\frac{5}{4}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$=\frac{5}{4}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\Big|_0^{\pi/2}$$

$$=\frac{5\pi}{8}+2\ \ unit^2$$




### *Exercise*

Integrate $f(x,y)=\dfrac{\ln\left(x^2+y^2\right)}{\sqrt{x^2+y^2}}$ over the region $1\le x^2+y^2\le e$

***Solution***

$$\int_0^{2\pi}\int_1^{\sqrt{e}}\left(\frac{\ln r^2}{r}\right)r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_1^{\sqrt{e}}2\ln r\,dr$$

$$=2\pi\ \left(r\ln r-r\Big|_1^{\sqrt{e}}\right.$$

$$=2\pi\left(\sqrt{e}\ln e^{1/2}-\sqrt{e}-(0-1)\right)$$

$$=2\pi\left(2-\sqrt{e}\right)$$

## *Exercise*

The region enclosed by the lemniscates $r^2 = 2\cos 2\theta$ is the base of a solid right cylinder whose top is bounded by the sphere $z = \sqrt{2-r^2}$. Find the cylinder's volume.

**Solution**

$$V = 4\int_0^{\pi/4}\int_0^{\sqrt{2\cos 2\theta}} r\sqrt{2-r^2}\, dr d\theta \qquad d\left(2-r^2\right) = -2rdr$$

$$= -2\int_0^{\pi/4}\int_0^{\sqrt{2\cos 2\theta}} \left(2-r^2\right)^{1/2} d\left(2-r^2\right) d\theta$$

$$= -2\int_0^{\pi/4} \tfrac{2}{3}\left(2-r^2\right)^{3/2} \Big|_0^{\sqrt{2\cos 2\theta}} d\theta$$

$$= -\tfrac{4}{3}\int_0^{\pi/4} \left[(2-2\cos 2\theta)^{3/2} - 2^{3/2}\right] d\theta$$

$$= -\tfrac{4}{3}\int_0^{\pi/4} \left[2^{3/2}(1-\cos 2\theta)^{3/2}\right] d\theta + \tfrac{4}{3}\int_0^{\pi/4} 2^{3/2}\, d\theta$$

$$= -\tfrac{4}{3}2\sqrt{2}\int_0^{\pi/4} \left(2\sin^2\theta\right)^{3/2} d\theta + \tfrac{4}{3}2\sqrt{2}\left(\tfrac{\pi}{4}\right)$$

$$= -\frac{8\sqrt{2}}{3}\int_0^{\pi/4} 2\sqrt{2}\, \sin^3\theta\, d\theta + \frac{2\pi}{3}\sqrt{2}$$

$$= -\frac{32}{3}\int_0^{\pi/4} \sin^2\theta\, \sin\theta\, d\theta + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\int_0^{\pi/4} \left(1-\cos^2\theta\right) d(\cos\theta) + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\left(\cos\theta - \tfrac{1}{3}\cos^3\theta\right. \Big|_0^{\pi/4} + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\left[\frac{\sqrt{2}}{2} - \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3 - \left(1-\frac{1}{3}\right)\right] + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12} - \frac{2}{3}\right) + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\left(\frac{5\sqrt{2}-8}{12}\right) + \frac{2\pi\sqrt{2}}{3}$$

$$= 8\left(\frac{5\sqrt{2}-8}{9}\right)+\frac{2\pi\sqrt{2}}{3}$$

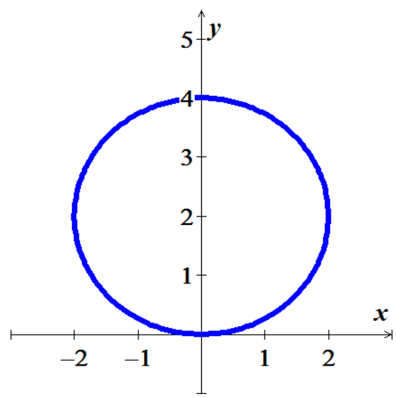
$$\boxed{= \frac{40\sqrt{2}-64+6\pi\sqrt{2}}{9}\ \ unit^3}$$

## Exercise

Evaluate $\iint\limits_R (x+y)\,dA$; $R$ is the disk bounded by circle $r = 4\sin\theta$

**Solution**

$$\iint\limits_R (x+y)\,dA = \int_0^{\pi}\int_0^{4\sin\theta}(r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$$




$$= \int_0^{\pi}\int_0^{4\sin\theta}(\cos\theta+\sin\theta)r^2\ dr\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi}(\cos\theta+\sin\theta)\ r^3\Big|_0^{4\sin\theta}\ d\theta$$

$$= \frac{64}{3}\int_0^{\pi}(\cos\theta+\sin\theta)\sin^3\theta\ d\theta$$

$$= \frac{64}{3}\int_0^{\pi}\cos\theta\sin^3\theta\ d\theta + \frac{64}{3}\int_0^{\pi}\sin^4\theta\ d\theta$$

$$= \frac{64}{3}\int_0^{\pi}\sin^3\theta\ d(\sin\theta) + \frac{64}{3}\int_0^{\pi}\frac{1}{4}(1-\cos 2\theta)^2\ d\theta$$

$$= \frac{16}{3}\sin^4\theta\Big|_0^{\pi} + \frac{64}{3}\int_0^{\pi}\frac{1}{4}\left(1-2\cos 2\theta+\cos^2 2\theta\right)\ d\theta$$

$$= \frac{16}{3}\int_0^{\pi}\left(\frac{3}{2}-2\cos 2\theta+\cos 4\theta\right)\ d\theta$$

$$= \frac{16}{3}\left(\frac{3}{2}\theta-\sin 2\theta+\frac{1}{4}\sin 4\theta\right)\Big|_0^{\pi}$$

$$\boxed{= 8\pi}$$

## Exercise

Find the volume of the solid bounded above by the paraboloid $z=2-x^2-y^2$ and below by the plane $z=1$

**Solution**

$$z=2-x^2-y^2-1 \quad \to \quad x^2+y^2=1$$

$$0\le r\le 1 \quad \& \quad 0\le\theta\le 2\pi$$

$$V=\iint_R \left(2-x^2-y^2-1\right)dA$$

$$=\int_0^{2\pi} d\theta \int_0^1 \left(1-r^2\right)r\,dr$$

$$=2\pi \int_0^1 \left(r-r^3\right)dr$$

$$=2\pi\left(\frac{1}{2}r^2-\frac{1}{4}r^4\right.\Big|_0^1$$

$$=2\pi\left(\frac{1}{2}-\frac{1}{4}\right)$$

$$=\frac{\pi}{2}\ unit^3$$

## Exercise

Find the volume of the solid bounded above by the paraboloid $z=8-x^2-3y^2$ and below by the hyperbolic paraboloid $z=x^2-y^2$

**Solution**

$$z=8-x^2-3y^2=x^2-y^2 \quad \to \quad x^2+y^2=4$$

$$0\le r\le 2 \quad \& \quad 0\le\theta\le 2\pi$$

$$V=\iint_R \left(8-x^2-3y^2-x^2+y^2\right)dA$$

$$=\iint_R \left(8-2\left(x^2+y^2\right)\right)dA$$

$$=2\int_0^{2\pi} d\theta \int_0^2 \left(4-r^2\right)r\,dr$$

$$= 4\pi \int_0^2 \left(4r - r^3\right)\, dr$$

$$= 4\pi\left(2r^2 - \frac{1}{4}r^4 \right|_0^2$$

$$= 4\pi(8-4)$$

$$\underline{= 16\pi \ \ unit^3}$$

### Exercise

Evaluate the integral over $R$ using polar coordinates

$$\iint\limits_R \left(x^2 + y^2\right) dA; \quad R = \{(r,\ \theta):\ \ 0 \le r \le 4, \quad 0 \le \theta \le 2\pi\}$$

***Solution***

$$\iint\limits_R \left(x^2 + y^2\right) dA = \int_0^{2\pi} \int_0^4 \left(r^2\right) r\ drd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^4 r^3\ dr$$

$$= 2\pi\left(\frac{1}{4}r^4 \right|_0^4$$

$$\underline{= 128\pi}$$

### Exercise

Evaluate the integral over $R$ using polar coordinates

$$\iint\limits_R 2xydA; \quad R = \left\{(r,\ \theta):\ \ 1 \le r \le 3, \quad 0 \le \theta \le \frac{\pi}{2}\right\}$$

***Solution***

$$\iint\limits_R (2xy)\, dA = \int_0^{\frac{\pi}{2}} \int_1^3 2(r\cos\theta)(r\sin\theta) r\ drd\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin 2\theta\ d\theta \int_1^3 r^3\ dr$$

$$= -\frac{1}{2}\cos 2\theta \Big|_0^{\frac{\pi}{2}} \left(\frac{1}{4}r^4\Big|_1^3\right)$$

$$= -\frac{1}{8}(-1-1)\ (81-1)$$

$$\underline{= 20}\ |$$

## Exercise

Evaluate the integral over $R$ using polar coordinates

$$\iint_R 2xy\ dA; \quad R = \left\{(x,\ y):\ \ x^2 + y^2 \le 9, \quad y \ge 0\right\}$$

***Solution***

$$x^2 + y^2 = 9 \quad \to \quad 0 \le r \le 3$$

$$y \ge 0 \quad \to \quad 0 \le \theta \le \pi$$

$$\iint_R (2xy)\,dA = \int_0^\pi \int_0^3 2(r\cos\theta)(r\sin\theta)r\ drd\theta$$

$$= \int_0^\pi \sin 2\theta\ d\theta \ \int_0^3 r^3\ dr$$

$$= -\frac{1}{2}\cos 2\theta \Big|_0^{\pi} \left(\frac{1}{4}r^4\Big|_0^3\right)$$

$$= -\frac{1}{8}(-1+1)\ (81-1)$$

$$\underline{= 0}\ |$$

## Exercise

Evaluate the integral over $R$ using polar coordinates

$$\iint_R \frac{dA}{1+x^2+y^2}; \quad R = \left\{(r,\ \theta):\ \ 1 \le r \le 2, \quad 0 \le \theta \le \pi\right\}$$

***Solution***

$$\iint_R \frac{dA}{1+x^2+y^2} = \int_0^\pi \int_1^2 \frac{1}{1+r^2} r\ drd\theta$$

$$= \frac{1}{2} \int_0^{\pi} d\theta \int_1^2 \frac{1}{1+r^2} \, d\left(1+r^2\right)$$

$$= \frac{\pi}{2} \ln\left(1+r^2\right) \Big|_1^2$$

$$= \frac{\pi}{2} (\ln 5 - \ln 2)$$

$$= \frac{\pi}{2} \ln \frac{5}{2}$$

### Exercise

Evaluate the integral over $R$ using polar coordinates

$$\iint_R \frac{dA}{\sqrt{16 - x^2 - y^2}}; \quad R = \left\{(x, y): \; x^2 + y^2 \le 4, \quad y \ge 0\right\}$$

**Solution**

$x^2 + y^2 = 4 \quad \to \quad 0 \le r \le 2$

$y \ge 0 \quad \to \quad 0 \le \theta \le \pi$

$$\iint_R \frac{dA}{\sqrt{16 - x^2 - y^2}} = \int_0^{\pi} \int_0^2 \frac{1}{\sqrt{16 - r^2}} r \, dr d\theta$$

$$= -\frac{1}{2} \int_0^{\pi} d\theta \int_0^2 \left(16 - r^2\right)^{-1/2} \, d\left(16 - r^2\right)$$

$$= -\pi \left(16 - r^2\right)^{1/2} \Big|_0^2$$

$$= -\pi \left(2\sqrt{3} - 4\right)$$

$$= 2\pi \left(2 - \sqrt{3}\right)$$

### Exercise

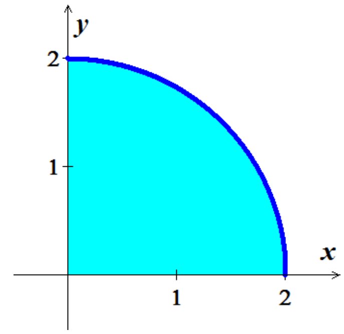
Evaluate the integral over $R$ using polar coordinates

$$\iint_R \frac{dA}{\sqrt{16 - x^2 - y^2}}; \quad R = \left\{(x, y): \; x^2 + y^2 \le 4, \quad x, \, y \ge 0\right\}$$

**Solution**

$x^2 + y^2 = 4 \quad \to \quad 0 \le r \le 2$

$x,\ y \geq 0 \quad \to \quad 0 \leq \theta \leq \frac{\pi}{2}$




$$\iint_R \frac{dA}{\sqrt{16 - x^2 - y^2}} = \int_0^{\frac{\pi}{2}} \int_0^2 \frac{1}{\sqrt{16 - r^2}}\, r\ dr d\theta$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta \int_0^2 \left(16 - r^2\right)^{-1/2} d\left(16 - r^2\right)$$

$$= -\frac{\pi}{2} \left(16 - r^2\right)^{1/2} \Big|_0^2$$

$$= -\frac{\pi}{2} \left(2\sqrt{3} - 4\right)$$

$$= \pi\left(2 - \sqrt{3}\right)$$

## *Exercise*

Evaluate the integral over $R$ using polar coordinates

$$\iint_R e^{-x^2 - y^2} dA; \quad R = \left\{(x,\ y):\ \ x^2 + y^2 \leq 9\right\}$$

***Solution***

$$\iint_R e^{-x^2 - y^2} dA = \int_0^{2\pi} \int_0^3 e^{-r^2} r\ dr d\theta$$

$$= -\frac{1}{2} \int_0^{2\pi} d\theta \int_0^3 e^{-r^2} d\left(-r^2\right)$$

$$= -\pi e^{-r^2} \Big|_0^3$$

$$= -\pi\left(e^{-9} - 1\right)$$

$$= \pi\left(1 - e^{-9}\right)$$

## *Exercise*

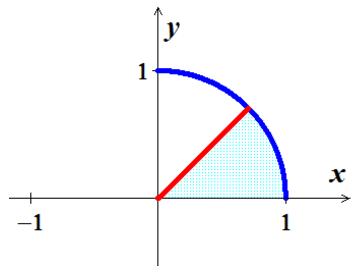
Evaluate the integral over $R$ using polar coordinates

$$\iint_R \sqrt{x^2+y^2}\, dA; \quad R=\left\{(x,\ y):\ \ y\le x\le 1,\quad 0\le y\le 1\right\}$$

**Solution**

$$y=x \quad \to \quad \cos\theta=\sin\theta \;\Rightarrow\; \theta=\frac{\pi}{4}$$

$$y=r\sin\theta\le 1 \quad \to \quad r\le\sec\theta$$




$$\iint_R \sqrt{x^2+y^2}\, dA=\int_0^{\frac{\pi}{4}}\int_0^{\sec\theta} r^2\, drd\theta$$

$$=\frac{1}{3}\int_0^{\frac{\pi}{4}} r^3\Big|_0^{\sec\theta} d\theta$$

$$=\frac{1}{3}\int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta$$

$$=\frac{1}{6}\left(\sec\theta\tan\theta+\ln\left|\sec\theta+\tan\theta\right|\right)\Big|_0^{\frac{\pi}{4}}$$

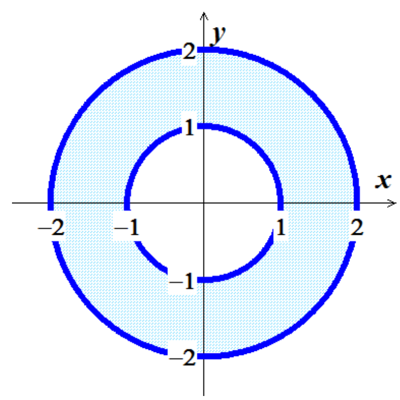
$$=\frac{1}{6}\left(\sqrt{2}+\ln\left(\sqrt{2}+1\right)\right)$$

## *Exercise*

Evaluate the integral over $R$ using polar coordinates

$$\iint_R \sqrt{x^2+y^2}\, dA; \quad R=\left\{(x,\ y):\ \ 1\le x^2+y^2\le 2\right\}$$

**Solution**




$$\iint_R \sqrt{x^2+y^2}\, dA=\int_0^{2\pi}\int_1^{2} r^2\, drd\theta$$

$$=\frac{1}{3}\int_0^{2\pi} d\theta\ \ r^3\Big|_1^2$$

$$=\frac{2\pi}{3}(8-1)$$

$$=\frac{14\pi}{3}$$

### *Exercise*

Evaluate the integral over $R$ using polar coordinates

$$\iint_R \frac{dA}{\left(x^2+y^2\right)^{5/2}}; \quad R=\left\{(r,\ \theta):\ 1\le r\le\infty,\quad 0\le\theta\le 2\pi\right\}$$

***Solution***

$$\iint_R \frac{dA}{\left(x^2+y^2\right)^{5/2}} = \int_0^{2\pi}\int_1^{\infty}\frac{1}{r^5}\ rdrd\theta$$

$$= \int_0^{2\pi} d\theta \int_1^{\infty} r^{-4}\ dr$$

$$= 2\pi\left(-\frac{1}{3}\frac{1}{r^3}\right)\Big|_1^{\infty}$$

$$= -\frac{2\pi}{3}(0-1)$$

$$\boxed{=\frac{2\pi}{3}}$$

### *Exercise*

Evaluate the integral over $R$ using polar coordinates

$$\iint_R e^{-x^2-y^2}\,dA; \quad R=\left\{(r,\ \theta):\ 0\le r\le\infty,\quad 0\le\theta\le\frac{\pi}{2}\right\}$$

***Solution***

$$\iint_R e^{-x^2-y^2}\,dA = \int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}\ rdrd\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} e^{-r^2}\,d\left(-r^2\right)$$

$$= -\frac{\pi}{4}e^{-r^2}\Big|_0^{\infty}$$

$$= -\frac{\pi}{4}(0-1)$$

$$\boxed{=\frac{\pi}{4}}$$

### Exercise

Evaluate the integral over $R$ using polar coordinates

$$\iint_R \frac{dA}{\left(1+x^2+y^2\right)^2}; \quad R \in QI$$

***Solution***

$$\iint_R \frac{dA}{\left(1+x^2+y^2\right)^2} = \int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{1}{\left(1+r^2\right)^2} r\,drd\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} \frac{1}{\left(1+r^2\right)^2} d\left(1+r^2\right)$$

$$= \frac{1}{2}\frac{\pi}{2}\left(-\frac{1}{1+r^2}\right)\Big|_0^{\infty}$$

$$= \frac{\pi}{4}(-0+1)\Big|_0^{\infty}$$

$$= \frac{\pi}{4}$$

### Exercise

Find the volume of a bowl holds water if it is filled to a depth of four units?

*a)* The paraboloid $z = x^2 + y^2$, for $0 \le z \le 4$

*b)* The cone $z = \sqrt{x^2 + y^2}$, for $0 \le z \le 4$

*c)* The hyperboloid $z = \sqrt{1 + x^2 + y^2}$, for $1 \le z \le 5$

*d)* Which bowl holds more water?

*e)* To what weight (above the bottom of the bowl) must the cone and paraboloid bowls be filled to hold the same volume of water as the hyperboloid bowl filled to a depth of 4 units $(1 \le z \le 5)$

***Solution***

***a)*** $$V = \iint_R \left(4 - \left(x^2 + y^2\right)\right) dA$$

$$= \int_0^{2\pi} \int_0^{2} \left(4 - r^2\right) r\,drd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^2 \left(4r - r^3\right) dr$$

$$= 2\pi \left(2r^2 - \frac{1}{4}r^4\right)\Big|_0^2$$

$$= 2\pi(8-4)$$

$$= 8\pi \ \ unit^3$$

***b)*** $0 \le z = \sqrt{x^2 + y^2} \le 4$

$0 \le x^2 + y^2 \le 16$

$$V = \iint\limits_R \left(4 - \sqrt{x^2 + y^2}\right) dA$$

$$= \int_0^{2\pi} \int_0^4 (4-r) r \ drd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^4 \left(4r - r^2\right) dr$$

$$= 2\pi \left(2r^2 - \frac{1}{3}r^3\right)\Big|_0^4$$

$$= 2\pi\left(32 - \frac{64}{3}\right)$$

$$= \frac{64\pi}{3} \ unit^3$$

***c)*** $1 \le z = \sqrt{1 + x^2 + y^2} \le 5$

$1 \le 1 + x^2 + y^2 \le 25 \quad \rightarrow \quad 0 \le x^2 + y^2 \le 24$

$$V = \iint\limits_R \left(5 - \sqrt{1 + x^2 + y^2}\right) dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{24}} \left(5 - \sqrt{1 + r^2}\right) r \ drd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{24}} \left(5r - r\sqrt{1 + r^2}\right) dr$$

$$= 2\pi \int_0^{\sqrt{24}} 5r \ dr - \pi \int_0^{\sqrt{24}} \left(1 + r^2\right)^{1/2} \ d\left(1 + r^2\right)$$

$$= 5\pi r^2 \Big|_0^{\sqrt{24}} - \frac{2\pi}{3}\left(1+r^2\right)^{3/2} \Big|_0^{\sqrt{24}}$$

$$= 5\pi(24) - \frac{2}{3}\pi(125-1)$$

$$= \frac{112\pi}{3} \ unit^3$$

***d)*** The hyperboloid bowl holds most water of $\frac{112\pi}{3}$ $unit^3$.

***e)*** Let the height $= h$

*Paraboloid*: $z = x^2 + y^2 = h$

$$V = \int_0^{2\pi} \int_0^{\sqrt{h}} \left(r^2\right) r \ dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{h}} r^3 \ dr$$

$$= \frac{\pi}{2} r^4 \Big|_0^{\sqrt{h}}$$

$$= \frac{\pi}{2} h^2 = \frac{112\pi}{3}$$

$$h^2 = \frac{224}{3} \quad \rightarrow \quad h = \sqrt{\frac{224}{3}} \ units$$

*Cone*: $z = \sqrt{x^2 + y^2} = h$

$$V = \int_0^{2\pi} \int_0^{h} r^2 \ dr d\theta$$

$$= \int_0^{2\pi} d\theta \ \frac{1}{3} r^3 \Big|_0^{h}$$

$$= \frac{2\pi}{3} h^3 = \frac{112\pi}{3}$$

$$h^3 = 56 \quad \rightarrow \quad h = \sqrt[3]{56} \ units$$

## *Exercise*

Consider the surface $z = x^2 - y^2$

*a)* Find the region in the *xy*-plane in polar coordinates for which $z \ge 0$.

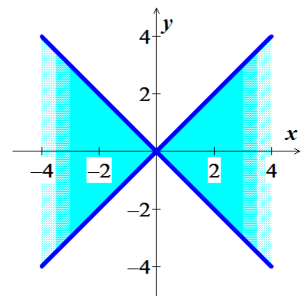
*b)* Let $R = \left\{(r, \theta): \ 0 \le r \le a, \ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}\right\}$, which is a sector of a circle of radius a. Find the volume of the region below the hyperbolic paraboloid and above the region *R*.

***Solution***

***a)*** $z = x^2 - y^2 \ge 0 \quad \to \quad x^2 \ge y^2$

$-|y| \le x \le |y|$

$R = \left\{ (r,\ \theta):\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4},\ \ \frac{3\pi}{4} \le \theta \le \frac{5\pi}{4} \right\}$




***b)*** $V = \iint\limits_R \left(x^2 - y^2\right) dA$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^a \left(r^2 \cos^2\theta - r^2 \sin^2\theta\right) r\ dr d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\cos^2\theta - \sin^2\theta\right) d\theta \ \int_0^a r^3\ dr$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos 2\theta)\, d\theta \ \left(\frac{1}{4} r^4\right)\Big|_0^a$$

$$= \frac{1}{2} \sin 2\theta \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \ \left(\frac{1}{4} a^4\right)$$

$$= \frac{1}{2}(1+1)\ \left(\frac{1}{4} a^4\right)$$

$$= \frac{1}{4} a^4$$

## ***Exercise***

A cake is shaped like a hemisphere of radius 4 with its base on the *xy*-plane. A wedge of the cake is removed by making two slices from the center of the cake outward, perpendicular to the *xy*-plane and separated by an angle of $\varphi$.

*a)* Use a double integral to find the volume of the slice for $\varphi = \frac{\pi}{4}$.

*b)* Suppose the cake is sliced by a plane perpendicular to the *xy*-plane at $x = a > 0$. Let *D* be the smaller of the two pieces produced. For what value of *a* is the volume of *D* equal to the volume in part (*a*)?

***Solution***

***a)*** $V = \iint\limits_R \left(4^2 - \left(x^2 + y^2\right)\right) dA$

$$= \int_0^{\frac{\pi}{4}} \int_0^4 \sqrt{16 - r^2}\ r\ dr d\theta$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{4}} d\theta \int_0^4 \left(16 - r^2\right)^{1/2}\ d\left(16 - r^2\right)$$

$$= -\frac{1}{2}\left(\frac{\pi}{4}\right)\frac{2}{3}\left(16 - r^2\right)^{3/2} \Big|_0^4$$

$$= -\frac{\pi}{12}(-64)$$

$$= \frac{16\pi}{3}\ unit^3$$

Geometrically, this slice is $\frac{1}{8}$ of the hemispherical cake.

The formula for the volume of a sphere is $\frac{4\pi}{3}$ , them the volume of the slice is

$$V = \frac{1}{8}\frac{1}{2}\frac{4\pi}{3} = \frac{16\pi}{3}\ unit^3 \quad \checkmark$$

***b)*** $V = \iint_R \left(16 - \left(x^2 + y^2\right)\right) dA$

$$= \int_0^{\varphi} d\theta \int_0^4 \sqrt{16 - r^2}\ r\ dr$$

$$= -\frac{\varphi}{2} \int_0^4 \left(16 - r^2\right)^{1/2}\ d\left(16 - r^2\right)$$

$$= -\frac{\varphi}{3}\left(16 - r^2\right)^{3/2} \Big|_0^4$$

$$= \frac{64\pi}{3}\ unit^3$$

## *Exercise*

Suppose the density of a thin plate represented by the region $R$ is $\rho(r,\ \theta)$ (in units of mass per area). The mass of the plate is $\iint_R \rho(r,\ \theta)\, dA$. Find the mass of the thin half annulus

$R = \{(r,\ \theta):\ \ 1 \le r \le 4,\quad 0 \le \theta \le \pi\}$ with a density $\rho(r,\ \theta) = 4 + r\sin\theta$

***Solution***

$$\iint\limits_R \rho(r,\ \theta)\,dA = \int_0^{\pi}\int_1^4 (4+r\sin\theta)\,r\ drd\theta$$

$$= \int_0^{\pi}\int_1^4 \left(4r + r^2\sin\theta\right)\ drd\theta$$

$$= \int_0^{\pi} \left(2r^2 + \frac{1}{3}r^3\sin\theta\right)\Big|_1^4\ d\theta$$

$$= \int_0^{\pi} \left(32 + \frac{64}{3}\sin\theta - 2 - \frac{1}{3}\sin\theta\right)\ d\theta$$

$$= \int_0^{\pi} (30 + 21\sin\theta)\ d\theta$$

$$= (30\theta - 21\cos\theta)\Big|_0^{\pi}$$

$$= 30\pi + 21 + 21$$

$$= 30\pi + 42$$

### *Exercise*

An important integral in statistics associated with the normal distribution is $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. It is evaluated in the following steps.

*a)* Assume that $I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dxdy$

Where we have chosen the variables of integration to be $x$ and $y$ and then written the product as an iterated integral. Evaluate this integral in polar coordinates and show that $I = \sqrt{\pi}$. Why is the solution $I = -\sqrt{\pi}$ rejected?

*b)* Evaluate $\int_0^{\infty} e^{-x^2}\,dx$, $\int_0^{\infty} xe^{-x^2}\,dx$, and $\int_0^{\infty} x^2e^{-x^2}\,dx$.

***Solution***

***a)*** $I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)}dxdy$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,rdrd\theta$$

$$= -\frac{1}{2}\int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2} d\left(-r^2\right)$$

$$= -\frac{1}{2}(2\pi)(0-1)$$

$$\boxed{= \pi}$$

The integrand is positive everywhere, so the integral of a positive function is positive.

***b)*** $\displaystyle\int_0^{\infty} e^{-x^2} dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2} dx$

$$\boxed{= \frac{\sqrt{\pi}}{2}}$$

$$\int_0^{\infty} xe^{-x^2} dx = -\frac{1}{2}\int_0^{\infty} e^{-x^2} d\left(-x^2\right)$$

$$= -\frac{1}{2}e^{-x^2}\Big|_0^{\infty}$$

$$= -\frac{1}{2}(0-1)$$

$$\boxed{= \frac{1}{2}}$$

$u = x \qquad dv = xe^{-x^2} dx$

$du = dx \qquad v = -\frac{1}{2}e^{-x^2}$

$$\int_0^{\infty} x^2 e^{-x^2} dx = -\frac{1}{2}xe^{-x^2}\Big|_0^{\infty} + \frac{1}{2}\int_0^{\infty} e^{-x^2} dx$$

$$= 0 + \frac{1}{2}\frac{\sqrt{\pi}}{2}$$

$$\boxed{= \frac{\sqrt{\pi}}{4}}$$

## *Exercise*

For what values of $p$ does the integral $\displaystyle\iint_R \frac{k}{\left(x^2+y^2\right)^p} dA$ exist in the following cases?

*a)* $R = \left\{(r,\ \theta):\ \ 1 \le r \le \infty,\quad 0 \le \theta \le 2\pi\right\}$

*b)* $R = \left\{(r,\ \theta):\ \ 0 \le r \le 1,\quad 0 \le \theta \le 2\pi\right\}$

***Solution***

***a)*** $$\iint_R \frac{k}{\left(x^2+y^2\right)^p}\,dA = \int_0^{2\pi}\int_1^{\infty} \frac{k}{r^{2p}}\; r\; dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_1^{\infty} k r^{1-2p}\; dr$$

$$= \frac{k\pi}{1-p}\left(r^{2-2p}\right)\Big|_1^{\infty}$$

$$= \frac{k\pi}{1-p}\left(r^{-2(p-1)}\right)\Big|_1^{\infty}$$

If $p-1<0 \quad \to \quad p<1$ the integral diverges.

If $p-1>0 \quad \to \quad p>1$ the integral converges.

$$\iint_R \frac{k}{\left(x^2+y^2\right)^p}\,dA = \frac{k\pi}{1-p}\left(r^{-2(p-1)}\right)\Big|_1^{\infty}$$

$$= \frac{k\pi}{1-p}(0-1)$$

$$= \frac{k\pi}{p-1}$$

***b)*** $$\iint_R \frac{k}{\left(x^2+y^2\right)^p}\,dA = \int_0^{2\pi}\int_0^{1} \frac{k}{r^{2p}}\; r\; dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{1} k r^{1-2p}\; dr$$

$$= \frac{k\pi}{1-p}\left(\frac{1}{r^{2(p-1)}}\right)\Big|_0^{1}$$

$$= \frac{k\pi}{1-p}\left(1-\frac{1}{0}\right)$$

If $p-1>0 \quad \to \quad p>1$ the integral diverges.

If $p-1<0 \quad \to \quad p<1$ the integral converges.

$$\iint_R \frac{k}{\left(x^2+y^2\right)^p}\,dA = \frac{k\pi}{1-p}(1-0)$$

$$= \frac{k\pi}{1-p}$$

## *Exercise*

Consider the integral $\displaystyle\iint\limits_R \frac{1}{\left(1+x^2+y^2\right)^2}\,dA$ where $R=\{(x,\ y):\ \ 0\le x\le 1,\ \ \ 0\le y\le a\}$

*a)* Evaluate $I$ for $a=1$.

*b)* Evaluate $I$ for arbitrary $a>0$.

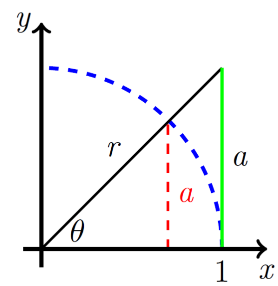
*c)* Let $a\to\infty$ in part (*b*) to find $I$ over the infinite strip $R=\{(x,\ y):\ \ 0\le x\le 1,\ \ \ 0\le y\le \infty\}$

***Solution***

$0\le x=r\cos\theta\le 1 \quad\to\quad 0\le r\le \sec\theta$

$0\le y=r\sin\theta\le a \quad\to\quad 0\le r\le a\csc\theta$

$\tan\theta=\frac{a}{1} \quad\to\quad \theta=\tan^{-1}a$




***a)*** $\theta=\tan^{-1}1=\frac{\pi}{4}$

$$\iint\limits_R \frac{1}{\left(1+x^2+y^2\right)^2}\,dA=\int_0^1\int_0^1 \frac{1}{\left(1+x^2+y^2\right)^2}\,dydx$$

$$=\int_0^{\frac{\pi}{4}}\int_0^{\sec\theta}\frac{1}{\left(1+r^2\right)^2}\,rdrd\theta+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\csc\theta}\frac{1}{\left(1+r^2\right)^2}\,rdrd\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}\int_0^{\sec\theta}\frac{d\left(1+r^2\right)}{\left(1+r^2\right)^2}\,d\theta+\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\csc\theta}\frac{d\left(1+r^2\right)}{\left(1+r^2\right)^2}\,d\theta$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{1}{1+r^2}\Big|_0^{\sec\theta}\,d\theta-\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1}{1+r^2}\Big|_0^{\csc\theta}\,d\theta$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(\frac{1}{1+\sec^2\theta}-1\right)d\theta-\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\frac{1}{1+\csc^2\theta}-1\right)d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{\sec^2\theta}{2+\tan^2\theta}\,d\theta+\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\csc^2\theta}{2+\cot^2\theta}\,d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{d(\tan\theta)}{2+\tan^2\theta}-\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{d(\cot\theta)}{2+\cot^2\theta}$$

$$\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1} \frac{\tan\theta}{\sqrt{2}} \Bigg|_0^{\frac{\pi}{4}} - \frac{1}{2\sqrt{2}} \tan^{-1} \frac{\cot\theta}{\sqrt{2}} \Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2\sqrt{2}} \left( \tan^{-1} \frac{1}{\sqrt{2}} + \tan^{-1} \frac{1}{\sqrt{2}} \right)$$

$$\boxed{= \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{1}{\sqrt{2}} \right)}$$

***b)*** $$\iint_R \frac{1}{\left(1+x^2+y^2\right)^2} dA = \int_0^1 \int_0^a \frac{1}{\left(1+x^2+y^2\right)^2} dydx$$

$$= \int_0^{\tan^{-1} a} \int_0^{\sec\theta} \frac{1}{\left(1+r^2\right)^2} rdrd\theta + \int_{\tan^{-1} a}^{\frac{\pi}{2}} \int_0^{a\csc\theta} \frac{1}{\left(1+r^2\right)^2} rdrd\theta$$

$$= \frac{1}{2} \int_0^{\tan^{-1} a} \int_0^{\sec\theta} \frac{d\left(1+r^2\right)}{\left(1+r^2\right)^2} rd\theta + \frac{1}{2} \int_{\tan^{-1} a}^{\frac{\pi}{2}} \int_0^{a\csc\theta} \frac{d\left(1+r^2\right)}{\left(1+r^2\right)^2} d\theta$$

$$= -\frac{1}{2} \int_0^{\tan^{-1} a} \frac{1}{1+r^2} \Bigg|_0^{\sec\theta} d\theta - \frac{1}{2} \int_{\tan^{-1} a}^{\frac{\pi}{2}} \frac{1}{1+r^2} \Bigg|_0^{a\csc\theta} d\theta$$

$$= -\frac{1}{2} \int_0^{\tan^{-1} a} \left( \frac{1}{1+\sec^2\theta} - 1 \right) d\theta - \frac{1}{2} \int_{\tan^{-1} a}^{\frac{\pi}{2}} \left( \frac{1}{1+a^2\csc^2\theta} - 1 \right) d\theta$$

$$= \frac{1}{2} \int_0^{\tan^{-1} a} \frac{\sec^2\theta}{2+\tan^2\theta} d\theta + \frac{1}{2} \int_{\tan^{-1} a}^{\frac{\pi}{2}} \frac{a^2\csc^2\theta}{1+a^2\csc^2\theta} d\theta$$

$$= \frac{1}{2} \int_0^{\tan^{-1} a} \frac{\sec^2\theta}{2+\tan^2\theta} d\theta + \frac{1}{2} \int_{\tan^{-1} a}^{\frac{\pi}{2}} \frac{\csc^2\theta}{\frac{1}{a^2}+\csc^2\theta} d\theta$$

$$= \frac{1}{2} \int_0^{\tan^{-1} a} \frac{\sec^2\theta}{2+\tan^2\theta} d\theta + \frac{1}{2} \int_{\tan^{-1} a}^{\frac{\pi}{2}} \frac{\csc^2\theta}{\frac{1}{a^2}+1+\cot^2\theta} d\theta$$

$$= \frac{1}{2} \int_0^{\tan^{-1} a} \frac{d(\tan\theta)}{2+\tan^2\theta} - \frac{1}{2} \int_{\tan^{-1} a}^{\frac{\pi}{2}} \frac{d(\cot\theta)}{\frac{1+a^2}{a^2}+\cot^2\theta}$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1} \frac{\tan\theta}{\sqrt{2}} \Bigg|_0^{\tan^{-1} a} - \frac{a}{2\sqrt{1+a^2}} \tan^{-1} \left( \frac{a}{\sqrt{1+a^2}} \cot\theta \right) \Bigg|_{\tan^{-1} a}^{\frac{\pi}{2}}$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{a}{\sqrt{2}}\right) + \frac{a}{2\sqrt{1+a^2}}\tan^{-1}\left(\frac{a}{\sqrt{1+a^2}}\cot\left(\tan^{-1}a\right)\right)$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{a}{\sqrt{2}}\right) + \frac{a}{2\sqrt{1+a^2}}\tan^{-1}\left(\frac{a}{\sqrt{1+a^2}}\frac{1}{a}\right)$$

$$\boxed{= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{a}{\sqrt{2}}\right) + \frac{a}{2\sqrt{1+a^2}}\tan^{-1}\left(\frac{1}{\sqrt{1+a^2}}\right)}$$

***c)*** $$\lim_{a\to\infty}\left(\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{a}{\sqrt{2}}\right) + \frac{a}{2\sqrt{1+a^2}}\tan^{-1}\left(\frac{1}{\sqrt{1+a^2}}\right)\right) = \frac{1}{2\sqrt{2}}\tan^{-1}(\infty) + \frac{a}{2\sqrt{1+a^2}}\tan^{-1}(0)$$

$$= \frac{1}{2\sqrt{2}}\frac{\pi}{2} - 0$$

$$\boxed{= \frac{\pi\sqrt{2}}{8}}$$

## *Exercise*

In polar coordinates an equation of an ellipse with eccentricity $0 < e < 1$ and semimajor axis $a$ is

$$r = \frac{a\left(1-e^2\right)}{1+e\cos\theta}$$

*a)* Write the integral that gives the area of the ellipse.

*b)* Show that the area of an ellipse is $\pi ab$, where $b^2 = a^2\left(1-e^2\right)$

***Solution***

***a)*** $$A = \iint_R 1\,dA$$

$$= \int_0^{2\pi}\int_0^{\frac{a\left(1-e^2\right)}{1+e\cos\theta}} r\ dr d\theta$$

***b)*** $$A = \int_0^{2\pi}\int_0^{\frac{a\left(1-e^2\right)}{1+e\cos\theta}} r\ dr d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} r^2 \Big|_0^{a\left(1-e^2\right)/(1+e\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{a^2\left(1-e^2\right)^2}{\left(1+e\cos\theta\right)^2}\,d\theta$$

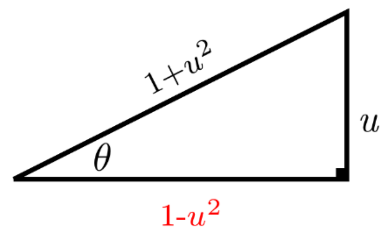
$$= a^2\left(1-e^2\right)^2 \int_0^{\pi} \frac{1}{\left(1+e\cos\theta\right)^2}\,d\theta$$

$$\tan^2\alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha}$$

$$\tan^2\alpha + \tan^2\alpha\cos 2\alpha = 1-\cos 2\alpha$$

$$\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$

$$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$




$$\left(1+e\cos\theta\right)^2 = \left(1+e\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)^2$$

$$= \frac{1}{\left(1+\tan^2\frac{\theta}{2}\right)^2}\left(1+e+(1-e)\tan^2\frac{\theta}{2}\right)^2 \qquad \tan\frac{\theta}{2} = u$$

$$= \frac{1}{\left(1+u^2\right)^2}\left(1+e+(1-e)u^2\right)^2$$

$$\tan\frac{\theta}{2} = u \quad \rightarrow \quad \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta = du$$

$$d\theta = 2\cos^2\frac{\theta}{2}\;du$$

$$= \frac{2}{1+u^2}\,du$$

$$= a^2\left(1-e^2\right)^2 \int_0^{\pi} \frac{\left(1+u^2\right)^2}{\left(1+e+(1-e)u^2\right)^2}\frac{2}{1+u^2}\,du$$

$$= 2a^2\left(1-e^2\right)^2 \int_0^{\pi} \frac{1+u^2}{\left(1+e+(1-e)u^2\right)^2}\,du$$

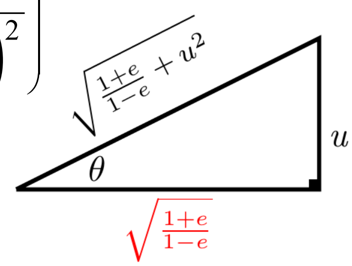
$$\frac{1+u^2}{\left(1+e+(1-e)u^2\right)^2} = \frac{Au+B}{1+e+(1-e)u^2} + \frac{Cu+D}{\left(1+e+(1-e)u^2\right)^2}$$

$$1+u^2=(1+e)Au+(1-e)Au^3+(1+e)B+(1-e)Bu^2+Cu+D$$

$$\begin{array}{lll}
u^3 & (1-e)A=0 & \to A=0 \\
u^2 & (1-e)B=1 & \to B=\dfrac{1}{1-e} \\
u & (1+e)A+C=0 & \to C=0 \\
1 & (1+e)B+D=1 & \to D=1-\dfrac{1+e}{1-e}=-\dfrac{2e}{1-e}
\end{array}$$

$$=\frac{2a^2}{1-e}\left(1-e^2\right)^2\int_0^\pi\frac{du}{1+e+(1-e)u^2}-\frac{4ea^2}{1-e}\left(1-e^2\right)^2\int_0^\pi\frac{du}{\left(1+e+(1-e)u^2\right)^2}$$

$$=\frac{2a^2}{1-e}\left(1-e^2\right)^2\left(\frac{1}{1-e}\int_0^\pi\frac{du}{\frac{1+e}{1-e}+u^2}-\frac{2e}{(1-e)^2}\int_0^\pi\frac{du}{\left(\frac{1+e}{1-e}+u^2\right)^2}\right)$$




$$u=\sqrt{\frac{1+e}{1-e}}\tan\alpha\quad\to du=\sqrt{\frac{1+e}{1-e}}\sec^2\alpha\,d\alpha$$

$$\frac{1+e}{1-e}+u^2=\frac{1+e}{1-e}\sec^2\alpha$$

$$=\frac{2a^2}{1-e}\left(1-e^2\right)^2\left(\frac{1}{1-e}\sqrt{\frac{1-e}{1+e}}\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\ \tan\frac{\theta}{2}\right)\Bigg|_0^\pi-\frac{2e}{(1-e)^2}\int_0^\pi\sqrt{\frac{1+e}{1-e}}\frac{\sec^2\alpha\,d\alpha}{\left(\frac{1+e}{1-e}\right)^2\sec^4\alpha}\right)$$

$$=\frac{2a^2}{1-e}\left(1-e^2\right)^2\left(\frac{1}{1-e}\sqrt{\frac{1-e}{1+e}\cdot\frac{1-e}{1-e}}\tan^{-1}(\infty)-\frac{2e}{(1-e)^{1/2}(1+e)^{3/2}}\int_0^\pi\frac{1}{\sec^2\alpha}d\alpha\right)$$

$$=2a^2\frac{\left(1-e^2\right)^2}{1-e}\left(\frac{\pi}{2\sqrt{1-e^2}}-\frac{2e}{(1+e)\sqrt{1-e^2}}\int_0^\pi\cos^2\alpha\ d\alpha\right)$$

$$=2a^2\frac{\left(1-e^2\right)^2}{(1-e)\sqrt{1-e^2}}\left(\frac{\pi}{2}-\frac{e}{(1+e)}\int_0^\pi(1+\cos2\alpha)\ d\alpha\right)$$

$$=2a^2\frac{\left(1-e^2\right)^2}{(1-e)\sqrt{1-e^2}}\left(\frac{\pi}{2}-\frac{e}{(1+e)}\left(\alpha+\frac{1}{2}\sin2\alpha\right)\Bigg|_0^\pi\right)$$

$$\frac{1}{2}\sin2\alpha=\sin\alpha\cos\alpha$$

$$=\frac{u}{\sqrt{\frac{1+e}{1-e}+u^2}}\frac{\sqrt{\frac{1+e}{1-e}}}{\sqrt{\frac{1+e}{1-e}+u^2}}\qquad\qquad u=\tan\frac{\theta}{2}$$

$$= \sqrt{\frac{1+e}{1-e}} \frac{\tan\frac{\theta}{2}}{\frac{1+e}{1-e} + \tan^2\frac{\theta}{2}}$$

$$= 2a^2 \frac{\left(1-e^2\right)^2}{(1-e)\sqrt{1-e^2}} \left( \frac{\pi}{2} - \frac{e}{(1+e)} \left( \arctan\left( \sqrt{\frac{1-e}{1-e}} \tan\frac{\theta}{2} \right) + \sqrt{\frac{1+e}{1-e}} \frac{\tan\frac{\theta}{2}}{\frac{1+e}{1-e} + \tan^2\frac{\theta}{2}} \right) \Bigg|_0^{\pi} \right)$$

$$\lim_{\theta\to\pi} \frac{\tan\frac{\theta}{2}}{\frac{1+e}{1-e} + \tan^2\frac{\theta}{2}} = \frac{\infty}{\infty}$$

$$= \lim_{\theta\to\pi} \frac{\tan\frac{\theta}{2}}{\tan^2\frac{\theta}{2}}$$

$$= \lim_{\theta\to\pi} \frac{1}{\tan\frac{\theta}{2}}$$

$$= 0$$

$$= 2a^2 \frac{\left(1-e^2\right)^{3/2}}{1-e} \left( \frac{\pi}{2} - \frac{e\pi}{2(1+e)} \right)$$

$$= \pi a^2 \frac{\left(1-e^2\right)^{3/2}}{1-e} \left( \frac{1+e-e}{1+e} \right)$$

$$= \pi a^2 \frac{\left(1-e^2\right)^{3/2}}{1-e^2}$$

$$= \pi a^2 \left(1-e^2\right)^{1/2} \qquad b^2 = a^2\left(1-e^2\right)$$

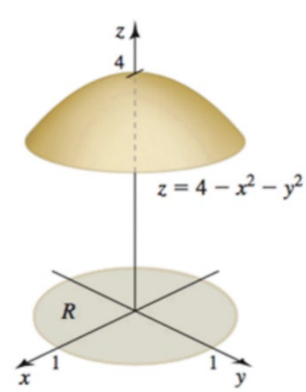
$$= \pi a \sqrt{a\left(1-e^2\right)}$$

$$= \pi ab$$

### *Exercise*

Find the volume of the solid below the paraboloid $z = 4 - x^2 - y^2$ and above

$$R = \{(r,\ \theta):\quad 0 \le r \le 1,\quad 0 \le \theta \le 2\pi\}$$




***Solution***

$$V = \iint\limits_R \left(4 - x^2 - y^2\right) dA$$

$$= \int_0^{2\pi} \int_0^1 \left(4 - r^2\right) r \, dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 \left(4r - r^3\right) dr$$

$$= 2\pi \left(2r^2 - \frac{1}{4} r^4\right) \Big|_0^1$$

$$= 2\pi \left(2 - \frac{1}{4}\right)$$

$$= \frac{7\pi}{2}$$

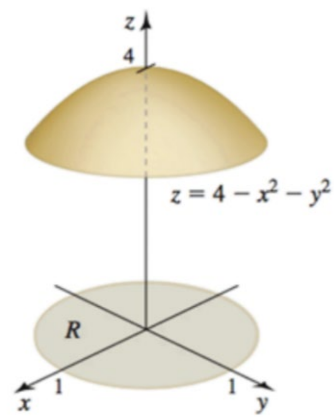
### *Exercise*

Find the volume of the solid below the paraboloid $z = 4 - x^2 - y^2$ and above

$$R = \{(r, \theta): \quad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi\}$$

***Solution***

$$V = \iint_R \left(4 - x^2 - y^2\right) dA$$




$$= \int_0^{2\pi} \int_0^2 \left(4 - r^2\right) r \, dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^2 \left(4r - r^3\right) dr$$

$$= 2\pi \left(2r^2 - \frac{1}{4} r^4\right) \Big|_0^2$$

$$= 2\pi (8 - 4)$$

$$= 8\pi$$

### *Exercise*

Find the volume of the solid below the paraboloid $z = 4 - x^2 - y^2$ and above

$$R = \{(r, \theta): \quad 1 \le r \le 2, \quad 0 \le \theta \le 2\pi\}$$

***Solution***

$$V = \iint_R \left(4 - x^2 - y^2\right) dA$$

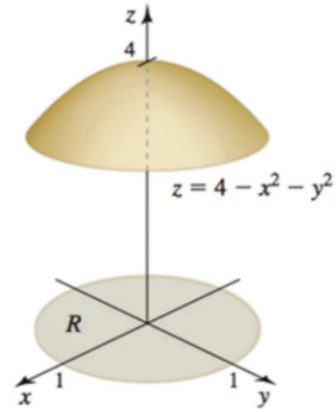
$$= \int_0^{2\pi} \int_1^2 \left(4 - r^2\right) r \, dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_1^2 \left(4r - r^3\right) dr$$

$$= 2\pi \left(2r^2 - \frac{1}{4} r^4\right) \Big|_1^2$$

$$= 2\pi \left(8 - 4 - 2 + \frac{1}{4}\right)$$

$$= \frac{9\pi}{2}$$




### *Exercise*

Find the volume of the solid below the paraboloid $z = 4 - x^2 - y^2$ and above

$$R = \left\{ (r,\ \theta) : \ \ 1 \le r \le 2, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \right\}$$

***Solution***

$$V = \iint_R \left(4 - x^2 - y^2\right) dA$$

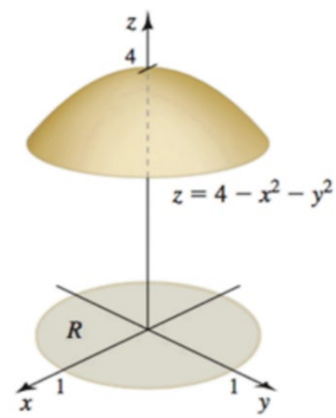
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^2 \left(4 - r^2\right) r \, dr d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_1^2 \left(4r - r^3\right) dr$$

$$= \pi \left(2r^2 - \frac{1}{4} r^4\right) \Big|_1^2$$

$$= \pi \left(8 - 4 - 2 + \frac{1}{4}\right)$$

$$= \frac{9\pi}{4}$$

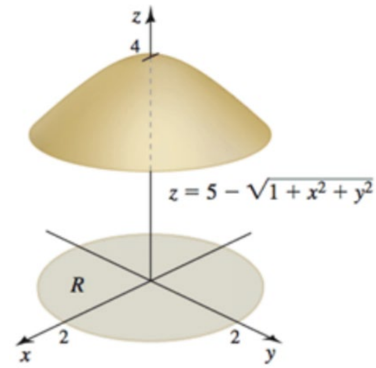


## Exercise

Find the volume of the solid below the hyperboloid $z = 5-\sqrt{1+x^2+y^2}$ and above

$$R = \{(r,\ \theta):\ \ 0 \le r \le 2,\ \ 0 \le \theta \le 2\pi\}$$

***Solution***




$$V = \iint_R \left(5-\sqrt{1+x^2+y^2}\right) dA$$

$$= \int_0^{2\pi}\int_0^2 \left(5-\sqrt{1+r^2}\right) r\ drd\theta$$

$$= \int_0^{2\pi} d\theta\ \int_0^2 \left(5r - r\sqrt{1+r^2}\right)\ dr$$

$$= 2\pi\int_0^2 5r\ dr - 2\pi\int_0^2 r\sqrt{1+r^2}\ dr$$

$$= 5\pi\left(r^2\right)\Big|_0^2 - \pi\int_0^2 \left(1+r^2\right)^{1/2}\ d\left(1+r^2\right)$$

$$= 20\pi - \frac{2\pi}{3}\left(1+r^2\right)^{3/2}\Big|_0^2$$

$$= 20\pi - \frac{2\pi}{3}\left(5\sqrt{5}-1\right)$$

$$= 20\pi - \frac{10\pi\sqrt{5}}{3} + \frac{2\pi}{3}$$

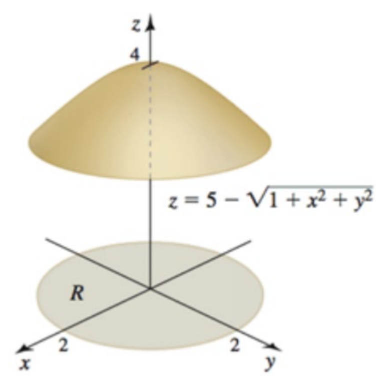
$$= \frac{\pi}{3}\left(62-10\sqrt{5}\right)$$

## Exercise

Find the volume of the solid below the hyperboloid $z = 5-\sqrt{1+x^2+y^2}$ and above

$$R = \{(r,\ \theta):\ \ 0 \le r \le 1,\ \ 0 \le \theta \le \pi\}$$

***Solution***




$$V = \iint_R \left(5-\sqrt{1+x^2+y^2}\right) dA$$

$$= \int_0^{\pi}\int_0^1 \left(5-\sqrt{1+r^2}\right) r\ drd\theta$$

$$= \int_0^{\pi} d\theta\ \int_0^1 \left(5r - r\sqrt{1+r^2}\right)\ dr$$

$$= \pi\int_0^1 5r\,dr - \pi\int_0^1 r\sqrt{1+r^2}\,dr$$

$$= \frac{5}{2}\pi\left(r^2\right)\Big|_0^1 - \frac{\pi}{2}\int_0^1 \left(1+r^2\right)^{1/2}\,d\left(1+r^2\right)$$

$$= \frac{5\pi}{2} - \frac{\pi}{3}\left(1+r^2\right)^{3/2}\Big|_0^1$$

$$= \frac{5\pi}{2} - \frac{\pi}{3}\left(2\sqrt{2}-1\right)$$

$$= \frac{5\pi}{2} - \frac{2\pi\sqrt{2}}{3} + \frac{\pi}{3}$$

$$= \frac{\pi}{6}\left(17-4\sqrt{2}\right)$$

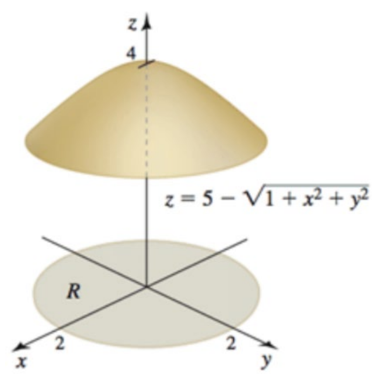
### *Exercise*

Find the volume of the solid below the hyperboloid $z = 5-\sqrt{1+x^2+y^2}$ and above

$$R = \left\{(r,\ \theta):\ \ \sqrt{3} \le r \le 2\sqrt{2},\quad 0 \le \theta \le 2\pi\right\}$$

***Solution***

$$V = \iint\limits_R \left(5-\sqrt{1+x^2+y^2}\right)dA$$




$$= \int_0^{2\pi}\int_{\sqrt{3}}^{2\sqrt{2}} \left(5-\sqrt{1+r^2}\right)r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_{\sqrt{3}}^{2\sqrt{2}} \left(5r - r\sqrt{1+r^2}\right)dr$$

$$= 2\pi\int_{\sqrt{3}}^{2\sqrt{2}} 5r\,dr - 2\pi\int_{\sqrt{3}}^{2\sqrt{2}} r\sqrt{1+r^2}\,dr$$

$$= 5\pi\left(r^2\right)\Big|_{\sqrt{3}}^{2\sqrt{2}} - \pi\int_{\sqrt{3}}^{2\sqrt{2}} \left(1+r^2\right)^{1/2}\,d\left(1+r^2\right)$$

$$= 5\pi(8-3) - \frac{2\pi}{3}\left(1+r^2\right)^{3/2}\Big|_{\sqrt{3}}^{2\sqrt{2}}$$

$$= 25\pi - \frac{2\pi}{3}(27-8)$$

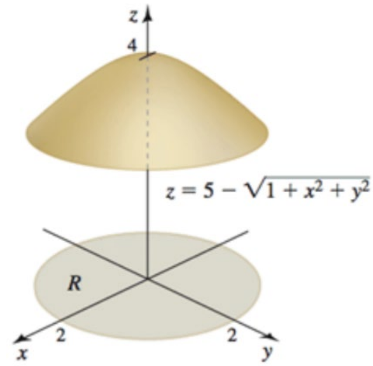
$$= \frac{37\pi}{3}$$

## *Exercise*

Find the volume of the solid below the hyperboloid $z = 5 - \sqrt{1 + x^2 + y^2}$ and above

$$R = \left\{ (r,\ \theta):\ \ \sqrt{3} \le r \le \sqrt{15},\ \ -\frac{\pi}{2} \le \theta \le \pi \right\}$$

***Solution***




$$V = \iint\limits_R \left(5 - \sqrt{1 + x^2 + y^2}\right) dA$$

$$= \int_{-\frac{\pi}{2}}^{\pi} \int_{\sqrt{3}}^{\sqrt{15}} \left(5 - \sqrt{1 + r^2}\right) r\ dr d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi} d\theta \ \int_{\sqrt{3}}^{\sqrt{15}} \left(5r - r\sqrt{1 + r^2}\right)\ dr$$

$$= \left(\pi + \frac{\pi}{2}\right) \int_{\sqrt{3}}^{\sqrt{15}} 5r\ dr - \frac{3\pi}{2} \int_{\sqrt{3}}^{\sqrt{15}} r\sqrt{1 + r^2}\ dr$$

$$= \frac{15\pi}{4}\left(r^2\right)\Big|_{\sqrt{3}}^{\sqrt{15}} - \frac{3\pi}{4} \int_{\sqrt{3}}^{\sqrt{15}} \left(1 + r^2\right)^{1/2}\ d\left(1 + r^2\right)$$

$$= \frac{15\pi}{4}(12) - \frac{\pi}{2}\left(1 + r^2\right)^{3/2}\Big|_{\sqrt{3}}^{\sqrt{15}}$$

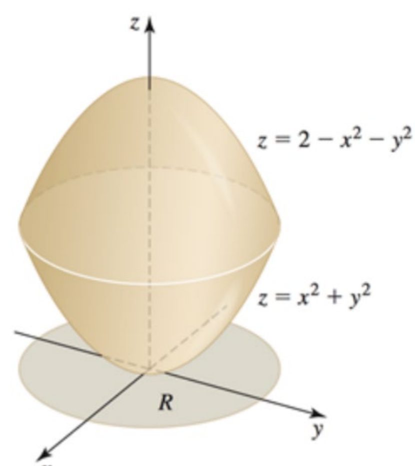
$$= 45\pi - \frac{\pi}{2}(64 - 8)$$

$$\underline{= 17\pi}\ |$$

## *Exercise*

Find the volume of the solid between the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$

***Solution***




$$z = x^2 + y^2 = 2 - x^2 - y^2$$

$$2x^2 + 2y^2 = 2 \quad \rightarrow \quad x^2 + y^2 = 1$$

$$0 \le r \le 1 \quad \& \quad 0 \le \theta \le 2\pi$$

$$V = \iint\limits_R \left(\left(2 - x^2 - y^2\right) - \left(x^2 + y^2\right)\right) dA$$

$$= \int_0^{2\pi} \int_0^1 \left(2 - r^2 - r^2\right) r\ dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 \left(2r - 2r^3\right) dr$$

$$= 2\pi \left(r^2 - \frac{1}{2} r^4\right) \Big|_0^1$$

$$= 2\pi \left(1 - \frac{1}{2}\right)$$

$$\underline{= \pi}$$

## *Exercise*

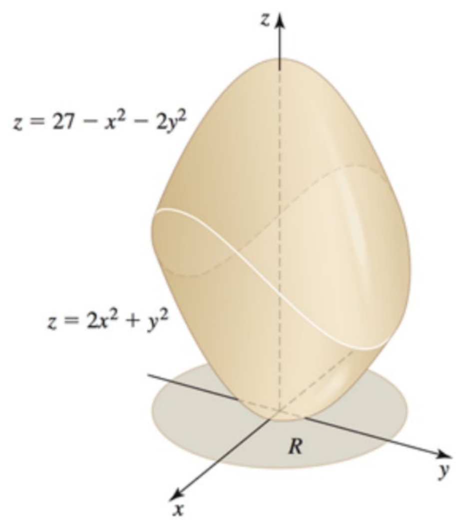
Find the volume of the solid between the paraboloids $z = 2x^2 + y^2$ and $z = 27 - x^2 - 2y^2$

***Solution***

$$z = 2x^2 + y^2 = 27 - x^2 - 2y^2$$

$$3x^2 + 3y^2 = 27 \quad \rightarrow \quad x^2 + y^2 = 9$$

$$0 \le r \le 3 \quad \& \quad 0 \le \theta \le 2\pi$$




$$V = \iint_R \left(\left(27 - x^2 - 2y^2\right) - \left(2x^2 + y^2\right)\right) dA$$

$$= \iint_R \left(27 - 3\left(x^2 + y^2\right)\right) dA$$

$$= 3 \int_0^{2\pi} \int_0^3 \left(9 - r^2\right) r \, dr d\theta$$

$$= 3 \int_0^{2\pi} d\theta \int_0^3 \left(9r - r^3\right) dr$$

$$= 6\pi \left(\frac{9}{2} r^2 - \frac{1}{4} r^4\right) \Big|_0^3$$

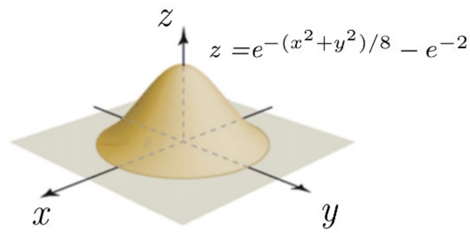
$$= 6\pi \left(\frac{81}{2} - \frac{81}{4}\right)$$

$$\underline{= \frac{243\pi}{2}}$$

### *Exercise*

Find the volume of island $z = e^{-\left(x^2+y^2\right)/8} - e^{-2}$

**Solution**




$$z = e^{-\left(x^2+y^2\right)/8} - e^{-2} = 0$$

$$e^{-\left(x^2+y^2\right)/8} = e^{-2}$$

$$-\frac{x^2+y^2}{8} = -2 \quad \to \quad x^2+y^2 = 16$$

$$V = \int_0^{2\pi} \int_0^4 \left(e^{-r^2/8} - e^{-2}\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^4 \left(re^{-r^2/8} - re^{-2}\right) dr$$

$$= -8\pi \int_0^4 e^{-r^2/8}\, d\left(-\frac{1}{8}r^2\right) - 2\pi \int_0^4 e^{-2} r\, dr$$

$$= -8\pi e^{-r^2/8}\Big|_0^4 - \pi e^{-2} r^2 \Big|_0^4$$

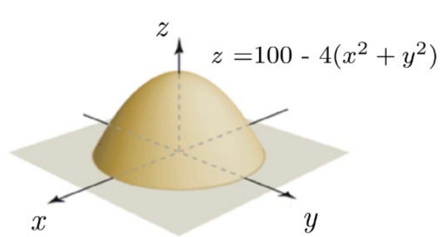
$$= -8\pi\left(e^{-2} - 1\right) - 16\pi e^{-2}$$

$$= 8\pi - 24\pi e^{-2}$$

### *Exercise*

Find the volume of island $z = 100 - 4\left(x^2+y^2\right)$

**Solution**




$$z = 100 - 4\left(x^2+y^2\right) = 0 \quad \to \quad x^2+y^2 = 25$$

$$V = \int_0^{2\pi} \int_0^5 \left(100 - 4r^2\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^5 \left(100r - 4r^3\right) dr$$

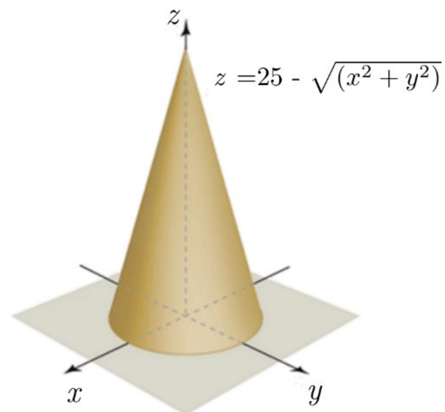
$$= 2\pi\left(50r^2 - r^4\right)\Big|_0^5$$

$$= 2\pi(1250 - 625)$$

$$= 1,250\pi$$

### ***Exercise***

Find the volume of island $z = 25 - \sqrt{x^2 + y^2}$




***Solution***

$$z = 25 - \sqrt{x^2 + y^2} \quad \rightarrow \quad x^2 + y^2 \le 25^2$$

$$V = \int_0^{2\pi} \int_0^{25} (25 - r) r \, dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{25} \left(25r - r^2\right) dr$$

$$= 2\pi \left( \frac{25}{2} r^2 - \frac{1}{3} r^3 \right) \Big|_0^{25}$$

$$= 2\pi (15{,}625) \left( \frac{1}{2} - \frac{1}{3} \right)$$

$$= \frac{15{,}625\pi}{3}$$