Exercise

Find the length of the curve $y=\frac{1}{3}\left(x^2+2\right)^{3/2}$ from $x = 0$ to $x = 3$.

Solution

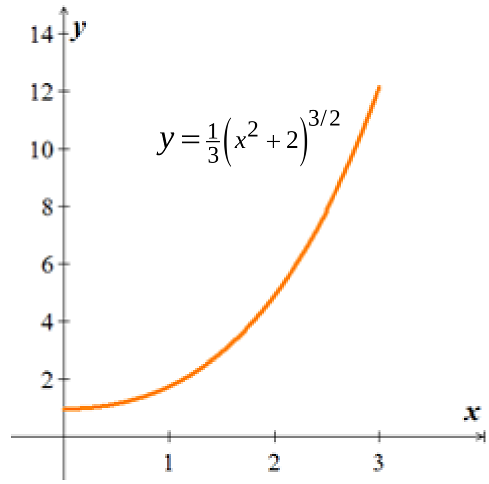
$$\frac{dy}{dx}=\frac{1}{3}\frac{3}{2}\left(x^2+2\right)^{1/2}(2x)=x\left(x^2+2\right)^{1/2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+x^2\left(x^2+2\right)}$$

$$=\sqrt{1+x^4+2x^2}$$

$$=\sqrt{\left(x^2+1\right)^2}$$

$$=x^2+1$$



$$L=\int_0^3\left(x^2+1\right)dx$$

$$=\left[\frac{1}{3}x^3+x\right]_0^3$$

$$=\frac{1}{3}(3)^3+(3)-0$$

$$\underline{=12\ \ unit}$$

Exercise

Find the length of the curve $y=(x)^{3/2}$ from $x = 0$ to $x = 4$.

Solution

$$\frac{dy}{dx}=\frac{3}{2}x^{1/2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{9}{4}x}$$

$$=\sqrt{\frac{4+9x}{4}}$$

$$=\frac{1}{2}\sqrt{4+9x}$$

$$L=\int_0^4\frac{1}{2}(4+9x)^{1/2}dx \qquad u=4+9x \quad \Rightarrow du=9dx \quad \rightarrow \frac{1}{9}du=dx \quad \begin{cases} x=4 & \rightarrow u=40 \\ x=0 & \rightarrow u=4 \end{cases}$$

$$= \frac{1}{2}\int_4^{40} \frac{1}{9}u^{1/2}du$$

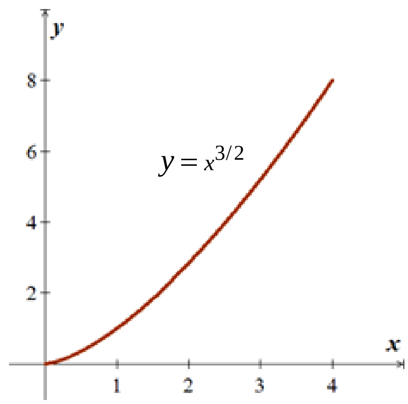
$$= \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_4^{40}$$

$$= \frac{1}{27}\left(40^{3/2} - 4^{3/2}\right)$$

$$= \frac{1}{27}\left(\sqrt{40^3} - \sqrt{4^3}\right)$$

$$= \frac{1}{27}\left(80\sqrt{10} - 8\right)$$

$$\boxed{= \frac{8}{27}\left(10\sqrt{10} - 1\right) \ \ unit}$$



Exercise

Find the length of the curve $x = \dfrac{y^{3/2}}{3} - y^{1/2}$ from $y = 1$ to $y = 9$.

Solution

$$\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} = \frac{1}{2}\left(y^{1/2} - \frac{1}{y^{1/2}}\right)$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\frac{1}{4}\left(y^{1/2} - \frac{1}{y^{1/2}}\right)^2}$$

$$= \sqrt{1+\frac{1}{4}\left(y - 2 + \frac{1}{y}\right)}$$

$$= \sqrt{1+\frac{1}{4}y - \frac{1}{2} + \frac{1}{4y}}$$

$$= \sqrt{\frac{1}{4}y + \frac{1}{2} + \frac{1}{4y}}$$

$$= \sqrt{\frac{1}{4}\left(y + 2 + \frac{1}{y}\right)}$$

$$= \frac{1}{2}\sqrt{\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2}$$

$$= \frac{1}{2}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)$$

$$L = \frac{1}{2}\int_1^9 \left(y^{1/2} + y^{-1/2}\right)dy$$

$a = \frac{1}{3}, \quad m = \frac{3}{2}, \quad b = -1, \quad n = \frac{1}{2}$

1. $m + n = \frac{3}{2} + \frac{1}{2} = 2$ ✓
2. $abmn = \frac{1}{3}(-1)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(\frac{1}{3}y^{3/2} + y^{1/2}\right)_1^9$$

$$= 9 + 3 - \frac{4}{3}$$

$$\boxed{= \frac{32}{3} \ \ unit}$$

$$=\frac{1}{2}\left[\frac{2}{3}y^{3/2}+2y^{1/2}\right]_1^9$$

$$=\left[\frac{1}{3}y^{3/2}+y^{1/2}\right]_1^9$$

$$=\left[\frac{1}{3}9^{3/2}+9^{1/2}-\left(\frac{1}{3}1^{3/2}+1^{1/2}\right)\right]$$

$$=\frac{1}{3}3^3+3-\left(\frac{1}{3}+1\right)$$

$$=9+3-\frac{4}{3}$$

$$=\frac{32}{3}\ \ unit$$

Exercise

Find the length of the curve $x=\dfrac{y^3}{6}+\dfrac{1}{2y}$ from $y = 2$ to $y = 3$.

Solution

$$\frac{dx}{dy}=\frac{1}{2}y^2-\frac{1}{2y^2}=\frac{1}{2}\left(y^2-y^{-2}\right)$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2}=\sqrt{1+\frac{1}{4}\left(y^2-y^{-2}\right)^2}$$

$$=\frac{1}{2}\sqrt{4+\left(y^4-2+y^{-4}\right)}$$

$$=\frac{1}{2}\sqrt{y^4+2+y^{-4}}$$

$$=\frac{1}{2}\sqrt{\left(y^2+y^{-2}\right)^2}$$

$$=\frac{1}{2}\left(y^2+y^{-2}\right)$$

$$L=\frac{1}{2}\int_2^3\left(y^2+y^{-2}\right)dy$$

$$=\frac{1}{2}\left[\frac{1}{3}y^3-y^{-1}\right]_2^3$$

$$=\frac{1}{2}\left[\left(\frac{1}{3}3^3-3^{-1}\right)-\left(\frac{1}{3}2^3-2^{-1}\right)\right]$$

$$=\frac{1}{2}\left[9-\frac{1}{3}-\left(\frac{8}{3}-\frac{1}{2}\right)\right]$$

$$=\frac{1}{2}\left(\frac{26}{3}-\frac{13}{6}\right)$$

$$=\frac{13}{4}\ \ unit$$

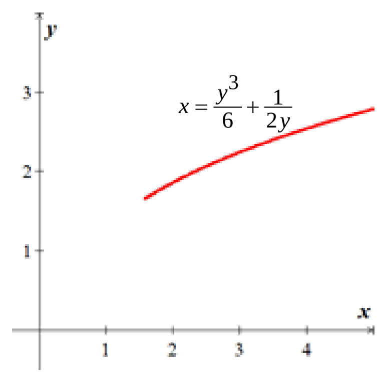
$a=\frac{1}{6},\quad m=3,\quad b=\frac{1}{2},\quad n=-1$

1. $m+n=3-1=2$ ✓
2. $abmn=\frac{1}{6}\left(\frac{1}{2}\right)(3)(-1)=-\frac{1}{4}$ ✓

$$L=\left(\frac{y^3}{6}-\frac{1}{2y}\right)_1^9$$

$$=\frac{1}{2}\left[9-\frac{1}{3}-\left(\frac{8}{3}-\frac{1}{2}\right)\right]$$

$$=\frac{13}{4}\ \ unit$$



Exercise

Find the length of the curve $f(x)=x^3+\dfrac{1}{12x}$ *for* $\dfrac{1}{2}\le x\le 2$

Solution

$a=1,\quad m=3,\quad b=\dfrac{1}{12},\quad n=-1$

1. $m+n=2$ ✓ **2.** $abmn=-\frac{1}{4}$ ✓

$$L=\left(x^3-\frac{1}{12x}\right)\Big|_{1/2}^{2}$$

$$=8-\frac{1}{24}-\frac{1}{8}+\frac{1}{6}$$

$$=8\ \ unit$$

Exercise

Find the length of the curve of $f(x)=\frac{1}{5}x^5+\dfrac{1}{12x^3}\quad 1\le x\le 2$

Solution

$a=\frac{1}{5},\quad m=5,\quad b=\frac{1}{12},\quad n=-3$

1. $m+n=5-3=2$ ✓ **2.** $abmn=\frac{1}{5}\left(\frac{1}{12}\right)(5)(-3)=-\frac{1}{4}$ ✓

$$L=\frac{1}{5}x^5-\frac{1}{12x^3}\Big|_{1}^{2}$$

$$=\frac{32}{5}-\frac{1}{96}-\frac{1}{5}+\frac{1}{12}$$

$$=\frac{31}{5}+\frac{7}{96}$$

$$=\frac{3011}{480}$$

Exercise

Find the length of the curve of $y=\frac{1}{3}x^{1/2}-x^{3/2},\quad 0\le x\le\frac{1}{3}$

Solution

$a=\frac{1}{3},\quad m=\frac{1}{2},\quad b=-1,\quad n=\frac{3}{2}$

3. $m+n=\dfrac{1}{2}+\dfrac{3}{2}=2$ ✓ **4.** $abmn=\frac{1}{3}(-1)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)=-\frac{1}{4}$ ✓

$$L=\frac{1}{3}x^{1/2}+x^{3/2}\Big|_{0}^{1/3}$$

$$= \frac{1}{3\sqrt{3}} + \frac{1}{3\sqrt{3}}$$

$$= \frac{2}{3\sqrt{3}} \qquad \qquad = \frac{2\sqrt{3}}{9}$$

Exercise

Find the length of the curve of $\qquad y = \frac{1}{3}x^3 + \frac{1}{4x}, \quad 1 \le x \le 2$

Solution

$$a = \frac{1}{3}, \quad m = 3, \quad b = \frac{1}{4}, \quad n = -1$$

5. $m + n = 3 - 1 = 2$ ✓ $\qquad$ **6.** $abmn = \frac{1}{3}\left(\frac{1}{4}\right)(3)(-1) = -\frac{1}{4}$ ✓

$$L = \frac{1}{3}x^3 - \frac{1}{4x}\Big|_1^2$$

$$= \frac{8}{3} - \frac{1}{8} - \frac{1}{3} + \frac{1}{4} \qquad \qquad = \frac{7}{3} + \frac{1}{8}$$

$$= \frac{59}{24}$$

Exercise

Find the length of the curve of $\qquad y = 2e^x + \frac{1}{8}e^{-x} \quad 0 \le x \le \ln 2$

Solution

$$a = 2, \quad m = 1, \quad b = \frac{1}{8}, \quad n = -1$$

7. $m = -n = 1$ ✓ $\qquad$ **8.** $abmn = 2\left(\frac{1}{8}\right)(1)(-1) = -\frac{1}{4}$ ✓

$$L = 2e^x - \frac{1}{8}e^{-x}\Big|_0^{\ln 2}$$

$$= 2e^{\ln 2} - \frac{1}{8}e^{-\ln 2} - 2 + \frac{1}{8}$$

$$= 4 - \frac{1}{16} - \frac{15}{8}$$

$$= \frac{33}{16}$$

Exercise

Find the length of the curve of $\qquad y = e^{2x} + \frac{1}{16}e^{-2x}, \quad 0 \le x \le \ln 3$

Solution

$$a = 1, \quad m = 2, \quad b = \frac{1}{16}, \quad n = -2$$

9. $m = -n = 2$ ✓

10. $abmn = 1\left(\frac{1}{16}\right)(2)(-2) = -\frac{1}{4}$ ✓

$$L = e^{2x} - \frac{1}{16}e^{-2x}\Big|_0^{\ln 3}$$

$$= e^{2\ln 3} - \frac{1}{16}e^{-2\ln 3} - 1 + \frac{1}{16}$$

$$= 9 - \frac{1}{16}\left(\frac{1}{9}\right) - \frac{15}{16}$$

$$= \frac{1{,}160}{144}$$

$$= \frac{145}{18}$$

Exercise

Find the length of the curve $y = \ln(\cos x) \quad 0 \le x \le \frac{\pi}{4}$

Solution

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$$

$$L = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\ dx$$

$$= \int_0^{\pi/4} \sqrt{\sec^2 x}\ dx$$

$$= \int_0^{\pi/4} \sec x\ dx$$

$$= \left[\ln|\sec x + \tan x|\right]_0^{\pi/4}$$

$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0|$$

$$= \ln\left|\sqrt{2} + 1\right| - \ln|1 + 0|$$

$$= \ln\left|\sqrt{2} + 1\right| - 0$$

$$= \ln\left(\sqrt{2} + 1\right) \ \ unit$$

Exercise

Find the length of the curve $f(y) = 2e^{\sqrt{2}y} + \frac{1}{16}e^{-\sqrt{2}y} \quad for \quad 0 \le y \le \frac{\ln 2}{\sqrt{2}}$

Solution

$$a = 2, \quad m = \sqrt{2}, \quad b = \frac{1}{16}, \quad n = -\sqrt{2}$$

1. $m=-n$ ✓

2. $abmn = 2\left(\sqrt{2}\right)\left(\frac{1}{16}\right)\left(-\sqrt{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(2e^{\sqrt{2}y} + \frac{1}{16}e^{-\sqrt{2}y}\right)\Bigg|_0^{\ln 2/\sqrt{2}}$$

$$= 2e^{\ln 2} + \frac{1}{16}e^{-\ln 2} - 2 - \frac{1}{16}$$

$$= 4 + \frac{1}{32} - \frac{33}{16}$$

$$= \frac{63}{32} \ unit$$

Exercise

Find the length of the curve $y = \frac{x^3}{3} + x^2 + x + 1 + \frac{1}{4x+4} \quad 0 \le x \le 2$

Solution

$$\frac{dy}{dx} = x^2 + 2x + 1 - \frac{1}{4}\frac{1}{(x+1)^2} = (x+1)^2 - \frac{1}{4}\frac{1}{(x+1)^2}$$

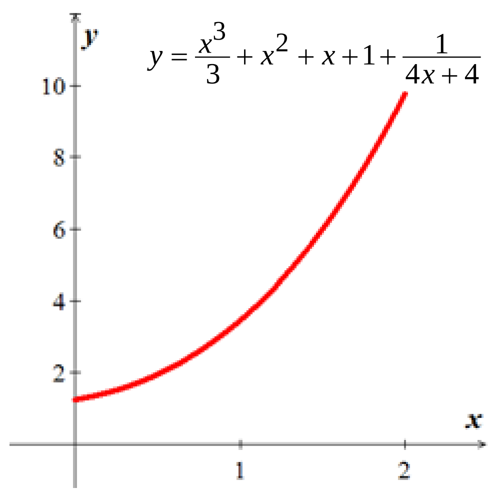
$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left((x+1)^2 - \frac{1}{4}\frac{1}{(x+1)^2}\right)^2}$$

$$= \sqrt{1+(x+1)^4 - \frac{1}{2} - \frac{1}{16}\frac{1}{(x+1)^4}}$$

$$= \sqrt{(x+1)^4 + \frac{1}{2} - \frac{1}{16}\frac{1}{(x+1)^4}}$$

$$= \sqrt{\left((x+1)^2 + \frac{1}{4}\frac{1}{(x+1)^2}\right)^2}$$

$$= (x+1)^2 + \frac{1}{4}(x+1)^{-2}$$



$$L = \int_0^2 \left((x+1)^2 + \frac{1}{4}(x+1)^{-2}\right)dx \qquad u = x+1 \quad \Rightarrow du = dx \quad \begin{cases} x=2 & \rightarrow u=3 \\ x=0 & \rightarrow u=1 \end{cases}$$

$$= \int_1^3 \left(u^2 + \frac{1}{4}u^{-2}\right)du$$

$$= \left[\frac{1}{3}u^3 - \frac{1}{4}u^{-1}\right]_1^3$$

$$= 9 - \frac{1}{12} - \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{53}{6} \ unit$$

Exercise

Find the length of the curve $y=\ln\left(e^x-1\right)-\ln\left(e^x+1\right) \quad \ln 2 \le x \le \ln 3$

Solution

$$y=\ln\left(e^x-1\right)-\ln\left(e^x+1\right) \quad \Rightarrow \frac{dy}{dx}=\frac{e^x}{e^x-1}-\frac{e^x}{e^x+1}$$

$$=\frac{e^{2x}+e^x-e^{2x}+e^x}{e^{2x}-1}$$

$$=\frac{2e^x}{e^{2x}-1}$$

$$L=\int_{\ln 2}^{\ln 3}\sqrt{1+\left(\frac{2e^x}{e^{2x}-1}\right)^2}\,dx$$

$$=\int_{\ln 2}^{\ln 3}\sqrt{1+\frac{4e^{2x}}{e^{4x}-2e^{2x}+1}}\,dx$$

$$=\int_{\ln 2}^{\ln 3}\sqrt{\frac{e^{4x}-2e^{2x}+1+4e^{2x}}{\left(e^{2x}-1\right)^2}}\,dx$$

$$=\int_{\ln 2}^{\ln 3}\sqrt{\frac{e^{4x}+2e^{2x}+1}{\left(e^{2x}-1\right)^2}}\,dx$$

$$=\int_{\ln 2}^{\ln 3}\sqrt{\frac{\left(e^{2x}+1\right)^2}{\left(e^{2x}-1\right)^2}}\,dx$$

$$=\int_{\ln 2}^{\ln 3}\frac{e^{2x}+1}{e^{2x}-1}\,dx$$

$$=\int_{\ln 2}^{\ln 3}\frac{\frac{e^{2x}+1}{e^x}}{\frac{e^{2x}-1}{e^x}}\,dx$$

$$=\int_{\ln 2}^{\ln 3}\frac{\frac{e^{2x}}{e^x}+\frac{1}{e^x}}{\frac{e^{2x}}{e^x}-\frac{1}{e^x}}\,dx$$

$$=\int_{\ln 2}^{\ln 3}\frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$$

or Let $u=e^x-e^{-x} \quad \Rightarrow du=\left(e^x+e^{-x}\right)dx$

$$= \int_{\ln 2}^{\ln 3} \frac{1}{e^x - e^{-x}}\, d\left(e^x - e^{-x}\right) \qquad\qquad d\left(e^x - e^{-x}\right) = \left(e^x + e^{-x}\right)dx$$

$$= \left[\ln\left|e^x - e^{-x}\right|\right]_{\ln 2}^{\ln 3}$$

$$= \ln\left(3 - \tfrac{1}{3}\right) - \ln\left(2 - \tfrac{1}{2}\right)$$

$$= \ln\left(\tfrac{8}{3}\right) - \ln\left(\tfrac{3}{2}\right)$$

$$= \ln\left(\tfrac{16}{9}\right) \ \ unit$$

Exercise

Find the length of the curve $f(x) = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2} \quad 1 \le x \le 4$

Solution

$$a = \tfrac{2}{3}, \quad m = \tfrac{3}{2}, \quad b = -\tfrac{1}{2}, \quad n = \tfrac{1}{2}$$

1. $m + n = \frac{3}{2} + \frac{1}{2} = 2$ ✓ **2.** $abmn = \frac{2}{3}\left(\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(\tfrac{2}{3}x^{3/2} + \tfrac{1}{2}x^{1/2}\right)\Big|_1^4$$

$$= \tfrac{2}{3}4^{3/2} + 1 - \tfrac{2}{3} - \tfrac{1}{2}$$

$$= \tfrac{16}{3} - \tfrac{2}{3} + \tfrac{1}{2}$$

$$= \tfrac{31}{6} \ \ unit$$

Exercise

Find the length of the curve $f(x) = x^3 + \frac{1}{12x} \quad 1 \le x \le 4$

Solution

$$a = 1, \quad m = 3, \quad b = \tfrac{1}{12}, \quad n = -1$$

1. $m + n = 3 - 1 = 2$ ✓ **2.** $abmn = (1)\left(\frac{1}{12}\right)(3)(-1) = -\frac{1}{4}$ ✓

$$L = \left(x^3 - \tfrac{1}{12x}\right)\Big|_1^4$$

$$= 4^3 - \tfrac{1}{48} - 1 + \tfrac{1}{12}$$

$$= 63 + \tfrac{3}{48}$$

$$= \frac{3,027}{48} \ unit$$

Exercise

Find the length of the curve $f(x) = \frac{1}{8}x^4 + \frac{1}{4x^2} \quad 1 \le x \le 10$

Solution

$$a = \frac{1}{8}, \quad m = 4, \quad b = \frac{1}{4}, \quad n = -2$$

1. $m + n = 4 - 2 = 2$ ✓ **2.** $abmn = \left(\frac{1}{8}\right)\left(\frac{1}{4}\right)(4)(-2) = -\frac{1}{4}$ ✓

$$L = \left(\frac{1}{8}x^4 - \frac{1}{4x^2}\right)\Big|_1^{10}$$

$$= \frac{10^4}{8} - \frac{1}{400} - \frac{1}{8} + \frac{1}{4}$$

$$= \frac{9,999}{8} + \frac{99}{400}$$

$$= \frac{9}{8}\left(1111 + \frac{11}{50}\right)$$

$$= \frac{9}{8}\left(\frac{55,561}{50}\right)$$

$$= \frac{500,049}{400} \ unit$$

Exercise

Find the length of the curve $f(x) = \frac{1}{4}x^4 + \frac{1}{8x^2} \quad 3 \le x \le 8$

Solution

$$a = \frac{1}{4}, \quad m = 4, \quad b = \frac{1}{8}, \quad n = -2$$

1. $m + n = 4 - 2 = 2$ ✓ **2.** $abmn = \left(\frac{1}{4}\right)\left(\frac{1}{8}\right)(4)(-2) = -\frac{1}{4}$ ✓

$$L = \left(\frac{1}{4}x^4 - \frac{1}{8x^2}\right)\Big|_3^{8}$$

$$= \frac{8^4}{4} - \frac{1}{8^3} - \frac{81}{4} + \frac{1}{72}$$

$$= \frac{4,015}{4} - \frac{1}{512} + \frac{1}{72}$$

$$= \frac{1}{4}\left(4,015 - \frac{1}{128} + \frac{1}{18}\right)$$

$$= \frac{1}{4}\left(4,015 + \frac{55}{1152}\right)$$

$$\underline{= \frac{4,625,335}{4,608} \ \ unit}$$

Exercise

Find the length of the curve $f(x) = \frac{1}{10}x^5 + \frac{1}{6x^3} \quad 1 \le x \le 7$

Solution

$$a = \frac{1}{10}, \quad m = 5, \quad b = \frac{1}{6}, \quad n = -3$$

1. $m + n = 5 - 3 = 2$ ✓ **2.** $abmn = \left(\frac{1}{10}\right)\left(\frac{1}{6}\right)(5)(-3) = -\frac{1}{4}$ ✓

$$L = \left(\frac{1}{10}x^5 - \frac{1}{6x^3}\right)\Big|_1^7$$

$$= \frac{7^5}{10} - \frac{1}{2058} - \frac{1}{10} + \frac{1}{6}$$

$$= \frac{8403}{5} + \frac{57}{343}$$

$$\underline{= \frac{2,882,514}{1,715} \ \ unit}$$

Exercise

Find the length of the curve $f(x) = \frac{3}{10}x^{1/3} - \frac{3}{2}x^{5/3} \quad 0 \le x \le 12$ b

Solution

$$a = \frac{1}{10}, \quad m = 5, \quad b = \frac{1}{6}, \quad n = -3$$

1. $m + n = 5 - 3 = 2$ ✓ **2.** $abmn = \left(\frac{1}{10}\right)\left(\frac{1}{6}\right)(5)(-3) = -\frac{1}{4}$ ✓

$$L = \left(\frac{3}{10}x^{1/3} + \frac{3}{2}x^{5/3}\right)\Big|_0^{12}$$

$$= \frac{3}{10}\sqrt[3]{12} + \frac{3}{2}12\sqrt[3]{144}$$

$$\underline{= \frac{3}{10}\sqrt[3]{12} + 18\sqrt[3]{144} \ \ unit} \qquad = \frac{3}{10}\sqrt[3]{12}\left(1 + 600\sqrt[3]{12}\right)$$

Exercise

Find the length of the curve $f(x) = x^{1/2} - \frac{1}{3}x^{3/2} \quad 2 \le x \le 9$

Solution

$$a = 1, \quad m = \frac{1}{2}, \quad b = -\frac{1}{3}, \quad n = \frac{3}{2}$$

1. $m + n = \frac{1}{2} + \frac{3}{2} = 2$ ✓ **2.** $abmn = (1)\left(-\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(x^{1/2} + \frac{1}{3}x^{3/2}\right)\Big|_2^9$$

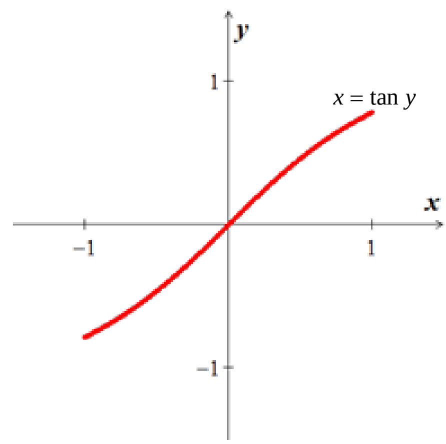
$$= 3 + 9 - \sqrt{2} - \frac{2\sqrt{2}}{3}$$

$$\underline{= \frac{1}{3}\left(36 - 5\sqrt{2}\right) \ \ unit} \qquad = \frac{3}{10}\sqrt[3]{12}\left(1 + 600\sqrt[3]{12}\ \right)$$

Exercise

Find the length of the curve $x = \int_0^y \sqrt{\sec^4 t - 1}\, dt \quad -\frac{\pi}{4} \le y \le \frac{\pi}{4}$

Solution



$$\frac{dx}{dy} = \sqrt{\sec^4 y - 1}$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \sec^4 y - 1}$$

$$= \sqrt{\sec^4 y}$$

$$= \sec^2 y$$

$$L = \int_{-\pi/4}^{\pi/4} \sec^2 y \ \ dy$$

$$= \tan y \Big|_{-\pi/4}^{\pi/4}$$

$$= 1 - (-1)$$

$$\underline{= 2 \ \ unit}$$

Exercise

Find the length of the curve $y = 3 - 2x \quad 0 \le x \le 2$. Check your answer by finding the length of the segment as the hypotenuse of a right triangle.

Solution

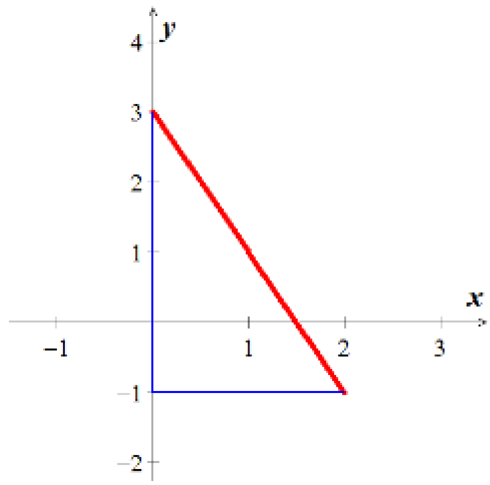
$$\frac{dy}{dx} = -2$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+4} = \sqrt{5}$$

$$L = \int_0^2 \sqrt{5}\ dx$$

$$= \sqrt{5}x \Big|_0^2$$

$$= 2\sqrt{5}\ \ unit$$



$$\begin{cases} x = 0 & \rightarrow y = 3 \\ x = 2 & \rightarrow y = -1 \end{cases}$$

$$d = \sqrt{(2-0)^2 + (3+1)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}\ \ unit$$

Exercise

Find a curve through the origin in the *xy*-plane whose length from $x = 0$ to $x = 1$ is $L = \int_0^1 \sqrt{1+\frac{1}{4}e^x}\ dx$

Solution

$$L = \int_0^1 \sqrt{1+\frac{1}{4}e^x}\ dx \quad \Rightarrow \frac{dy}{dx} = \frac{e^{x/2}}{2} \qquad \rightarrow dy = \frac{e^{x/2}}{2}dx$$

$$y = \int \frac{e^{x/2}}{2}dx = e^{x/2} + C$$

$$0 = e^{0/2} + C$$

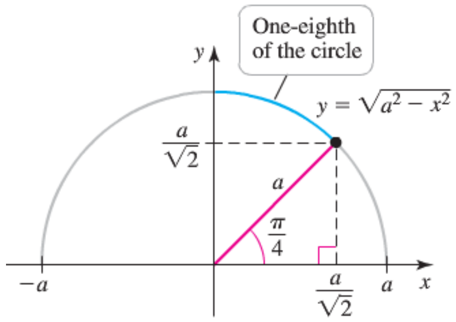
$$0 = 1 + C \qquad \Rightarrow \ C = -1$$

$$y = e^{x/2} - 1$$

Exercise

Confirm that the circumference of a circle of radius a is $2\pi a$.

Solution



$$f(x) = \sqrt{a^2 - x^2} \quad \text{for} \quad -a \le x \le a$$

$$f'(x) = -\frac{x}{\sqrt{a^2 - x^2}} \qquad \text{but } \underline{x \ne \pm a}$$

$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2}}$$

$$= \frac{a}{\sqrt{a^2 - x^2}}$$

Let's compute the length of $\frac{1}{8}$ of the circle on $\left[0, \frac{a}{\sqrt{2}}\right]$

$$L = 8a \int_0^{a/\sqrt{2}} \frac{dx}{\sqrt{a^2 - x^2}}$$

$$= 8a \sin^{-1}\left(\frac{x}{a}\right)\Big|_0^{a/\sqrt{2}}$$

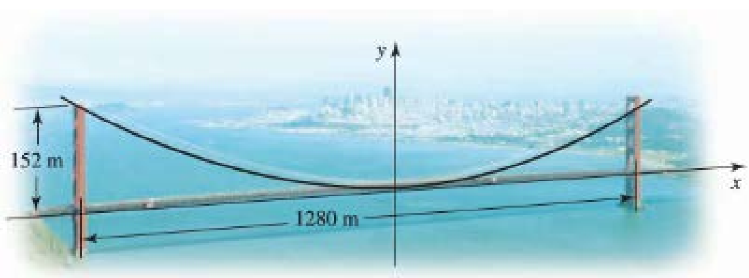
$$= 8a \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= 8a\left(\frac{\pi}{4}\right)$$

$$\underline{= 2\pi a \ \text{ unit}}$$

Exercise

The profile of the cables on a suspension bridge may be modeled by a parabola. The central span of the Golden Gate Bridge is 1280 *m* long and 152 *m* high. The parabola $y = 0.00037x^2$ gives a good fit to the shape of the cables, where $|x| \le 640$, and x and y are measured in *meters*. Approximate the length of the cables that stretch between the tops of the two towers.



Solution

$$y' = 0.00074x$$

$$L = \int_{-640}^{640} \sqrt{1+(.00074x)^2}\ dx \qquad \int \sqrt{a^2+x^2}\,dx = \tfrac{x}{2}\sqrt{a^2+x^2} + \tfrac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|$$

$$= \left(\tfrac{x}{2}\sqrt{1+x^2} + \tfrac{1}{2}\ln\left|x+\sqrt{1+x^2}\right|\right)\Bigg|_{-640}^{640}$$

$$= 320\sqrt{1+640^2} + \tfrac{1}{2}\ln\left|640+\sqrt{1+640^2}\right| + 320\sqrt{1+x^2} - \tfrac{1}{2}\ln\left|-640+\sqrt{1+640^2}\right|$$

$$\underline{\approx 1326.4\ m}$$

Exercise

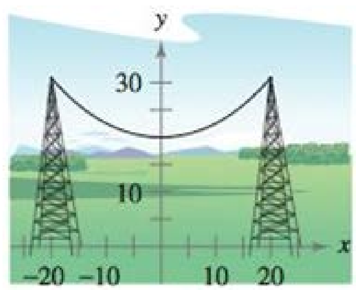
Electrical wires suspended between two towers form a caternary modeled by the equation

$$y = 20\cosh\frac{x}{20}, \quad -20 \le x \le 20$$

Where x and y are measured in *meters*. The towers are 40 *meters* apart. Find the length of the suspended cable.

Solution

$$y = 20\cosh\frac{x}{20} \quad \to y' = \sinh\frac{x}{20}$$



$$\sqrt{1+(y')^2} = \sqrt{1+\sinh^2\frac{x}{20}}$$

$$= \sqrt{\cosh^2\frac{x}{20}}$$

$$= \cosh\frac{x}{20}$$

$$L = \int_{-20}^{20} \cosh\frac{x}{20}\,dx$$

$$= 2(20)\sinh\frac{x}{20}\Bigg|_{0}^{20}$$

$$= 40(\sinh 1 - \sinh 0)$$

$$\underline{= 40\sinh 1}$$

Exercise

A barn is 100 *feet* long and 40 *feet* wide. A cross section of the roof is the inverted caternary $y = 31 - 10\left(e^{x/20} + e^{-x/20}\right)$. Find the number of ***square feet*** of roofing on the barn.

Solution

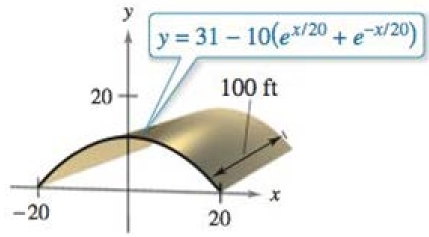
$$a = 10, \quad m = \frac{1}{20}, \quad b = 10, \quad n = -\frac{1}{20}$$

1. $m = -n$ ✓ **2.** $abmn = 10(10)\left(\frac{1}{20}\right)\left(-\frac{1}{20}\right) = -\frac{1}{4}$ ✓

$$L = 10\left(e^{x/20} - e^{-x/20}\right)\Big|_{-20}^{20}$$

$$= 10\left(e - \frac{1}{e} - \frac{1}{e} + e\right)$$

$$= 20\left(e - \frac{1}{e}\right) \approx 47\ ft$$



$\therefore$ There are $100(47) = 4,700\ ft^2$ of roofing on the barn

Exercise

A cable for a suspension bridge has the shape of a parabola with equation $y = kx^2$. Let h represent the height of the cable from it lowest point to its highest point and let $2w$ represent the total span of the bridge.

Show that the length C of the cable is given by $C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\, dx$

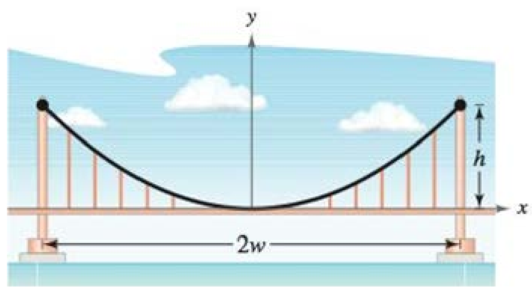
Solution

$$y' = 2kx$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + 4k^2x^2}$$

At $(w,\ h) \to h = kw^2 \quad \Rightarrow\ k = \frac{h}{w^2}$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{4h^2}{w^4}x^2}$$



$\therefore$ By symmetry: $C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\, dx$

Exercise

Find the total length of the graph of the astroid $x^{2/3} + y^{2/3} = 4$

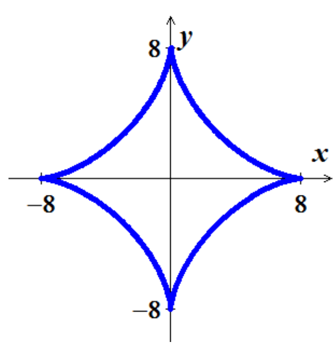
Solution

$$x^{2/3} + y^{2/3} = 4 \quad \Rightarrow \quad y = \left(4 - x^{2/3}\right)^{3/2}$$

$$y' = \frac{3}{2}\left(-\frac{2}{3}x^{-1/3}\right)\left(4 - x^{2/3}\right)^{1/2}$$

$$= -\frac{1}{x^{1/3}}\left(4 - x^{2/3}\right)^{1/2}$$

$$1 + (y')^2 = 1 + \frac{1}{x^{2/3}}\left(4 - x^{2/3}\right)$$



$$= \frac{4}{x^{2/3}}$$

$$y = 0 \quad \to x^{2/3} = 4 \;\Rightarrow\; \underline{x = 4^{3/2} = 8}$$

$$L = 4\int_0^8 \sqrt{\frac{4}{x^{2/3}}}\, dx$$

$$= 8\int_0^8 x^{-1/3}\, dx$$

$$= 12x^{2/3}\Big|_0^8$$

$$= 12(4-0)$$

$$\underline{= 48}$$

Exercise

Find the arc length from $(0,\ 3)$ clockwise to $\left(2,\ \sqrt{5}\right)$ along the circle $x^2 + y^2 = 9$

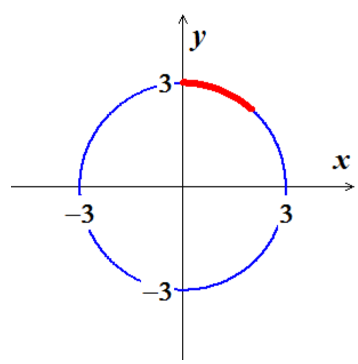
Solution

$$y = \sqrt{9-x^2} \quad \Rightarrow \quad y' = -\frac{x}{\sqrt{9-x^2}}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{x^2}{9-x^2}}$$

$$= \sqrt{\frac{9}{9-x^2}}$$

$$= \frac{3}{\sqrt{9-x^2}}$$



$$L = \int_0^2 \frac{3}{\sqrt{9-x^2}}\, dx$$

$$= 3\arcsin\frac{x}{3}\Big|_0^2$$

$$\underline{= 3\arcsin\frac{2}{3}} \quad \underline{\approx 2.1892}$$

Exercise

Find the arc length from $(-3,\ 4)$ clockwise to $(4,\ 3)$ along the circle $x^2 + y^2 = 25$. Show that the result is one-fourth the circumference of the circle.

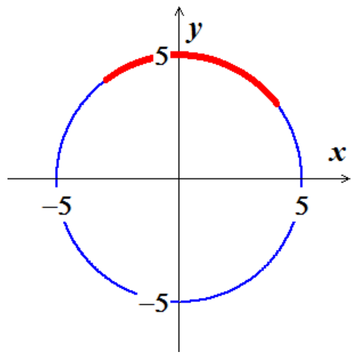
Solution

$$y = \sqrt{25 - x^2} \quad \Rightarrow \quad y' = -\frac{x}{\sqrt{25 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{25 - x^2}}$$

$$= \sqrt{\frac{25}{25 - x^2}}$$

$$= \frac{5}{\sqrt{25 - x^2}}$$



$$L = \int_{-3}^{4} \frac{5}{\sqrt{25 - x^2}}\, dx$$

$$= 5 \arcsin \frac{x}{5} \Big|_{-3}^{4}$$

$$= 5\left(\arcsin \frac{4}{5} + \arcsin \frac{3}{5}\right) \quad \approx 7.854$$