Lecture Three – Applications of Derivatives

Section 3.1 – Extreme Values of Functions

Definition

Let f be a function with Domain D. Then f has an ***absolute maximum*** value on D at a point c if

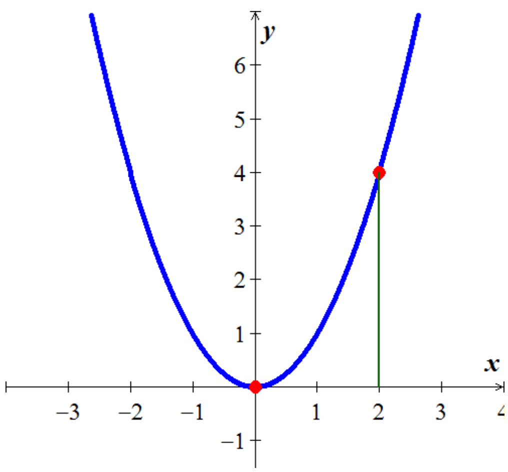
$$f(x) \le f(c) \quad \textit{for all } x \textit{ in } D$$

And an ***absolute maximum*** value on D at a point c if

$$f(x) \ge f(c) \quad \textit{for all } x \textit{ in } D$$

Maximum and minimum values are called ***extreme values*** of the function f .

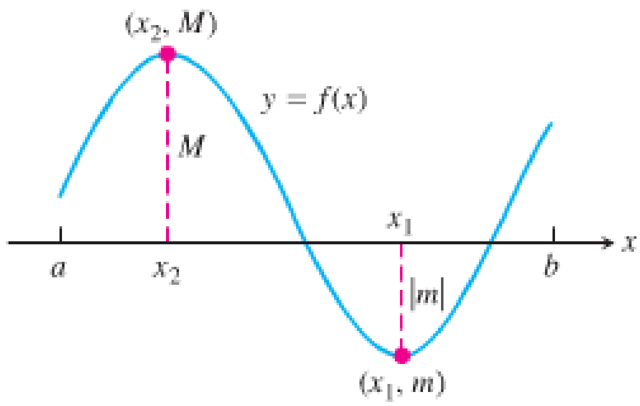
Absolute maxima or minima are also referred to as ***global*** maxima or minima.



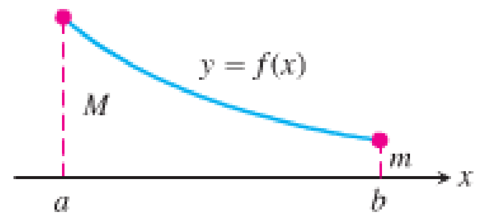
Function rule	***Domain D***	***Absolute extrema on D***
$y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at x = 0.
$y = x^2$	$[0, 2]$	Absolute minimum of 0 at x = 0. Absolute minimum of 4 at x = 2.
$y = x^2$	(0, 2]	No absolute minimum. Absolute minimum of 4 at x = 2.
$y = x^2$	(0, 2)	No absolute extrema.

Theorem – The Extreme Value Theorem

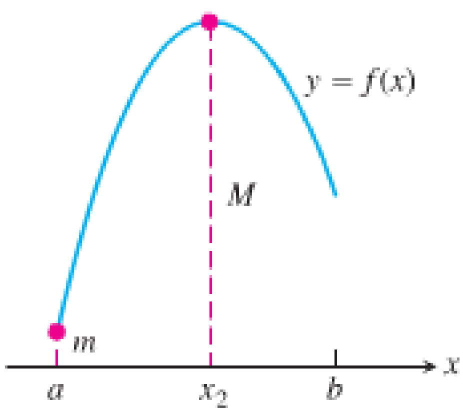
If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other x in $[a, b]$.



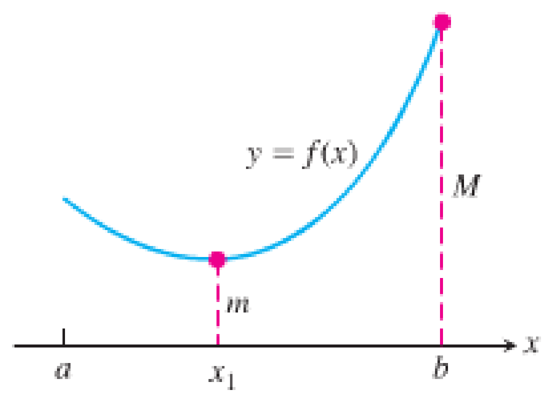
Maximum and minimum
at interior points



Maximum and minimum
at endpoints



Maximum at interior point,
minimum at endpoint

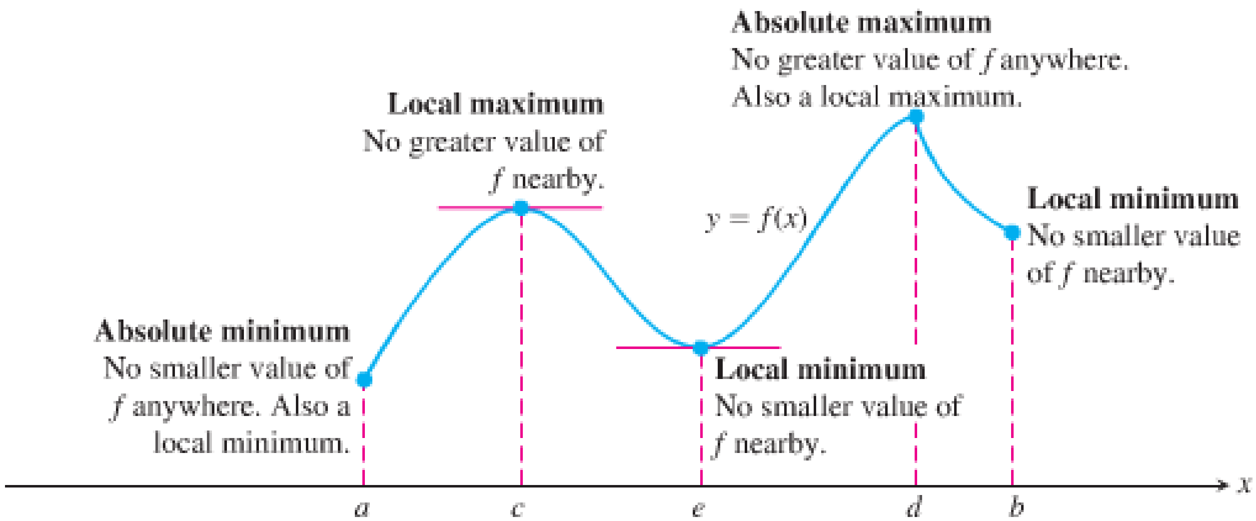


Minimum at interior point,
maximum at endpoint

Definitions

A function f has a ***local maximum*** (***LMAX***) value at a point *c* within its domain *D* if $f(x) \le f(c)$ for all $x \in D$ lying is some open interval containing *c*.

A function f has a ***local minimum*** (***LMIN***) value at a point *c* within its domain *D* if $f(x) \ge f(c)$ for all $x \in D$ lying is some open interval containing *c*.



An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood.

Finding Extrema

Theorem – The First Derivative Theorem for Local Extreme Values

If f has a ***local minimum*** or ***local maximum*** value at a point *c* of its domain *D*, and f' is defined at *c*, then

$$f'(c) = 0$$

Proof

For $f'(c) = 0$ at a local extremum, we need to show that $f'(c)$ can't be positive or negative at *c*.

$$\left.\begin{aligned} f'(c) &= \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \left\{ \frac{f(x) \le f(c)}{x - c > 0} \right\} \le 0 \\ f'(c) &= \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \left\{ \frac{f(x) \le f(c)}{x - c < 0} \right\} \ge 0 \end{aligned}\right. \Rightarrow f'(c) = 0$$

Definition

An interior point of the domain of a function f where f' is zero or undefined is a critical point of f.

How to find the Absolute Extrema of a continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

Example

Find the absolute maximum and minimum values of $f(x) = x^2$ on [–2, 1].

Solution

$$f'(x) = 2x = 0 \quad \Rightarrow \quad x = 0$$

$$f(0) = 0^2 = 0$$

Check: $f(-2) = (-2)^2 = 4$

$$f(1) = (1)^2 = 1$$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$.

Example

Find the absolute maximum and minimum values of $g(t) = 8t - t^4$ on [–2, 1].

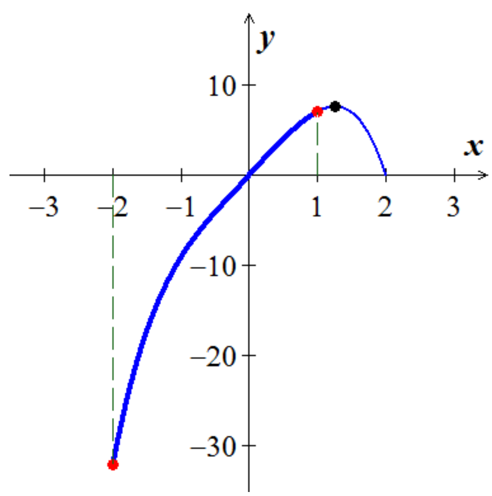
Solution

$$g'(t) = 8 - 4t^3 \quad \Rightarrow \quad t^3 = 2 \rightarrow \boxed{t = \sqrt[3]{2}} > 1$$

Check: $g(-2) = 8(-2) - (-2)^4 = -32$

$$f(1) = 8(1) - (1)^4 = 7$$

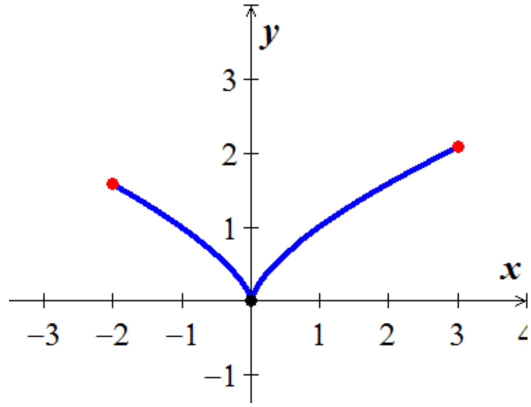
The function has an absolute maximum value of 7 at $x = 1$ and an absolute minimum value of –32 at $x = 0$.



Example

Find the absolute maximum and minimum values of $f(x)=x^{2/3}$ on [–2, 3].

Solution



$$f'(x)=\frac{2}{3}x^{-1/3}=\frac{2}{3\sqrt[3]{3}}=0 \quad \Rightarrow \quad \textit{Undefined}$$

Critical point: $\quad f(0)=0$

Endpoint values: $\quad f(-2)=(-2)^{2/3}=\sqrt[3]{2^2}=\sqrt[3]{4}$

$$f(3)=(3)^{2/3}=\sqrt[3]{3^2}=\sqrt[3]{9}$$

Absolute MIN: $\quad (0, 0)$

Absolute MAX: $\quad \left(3,\ \sqrt[3]{9}\right)$

Critical Points ***(CP)*** or ***Critical Numbers***

The critical points for a function f are those numbers c in the domain of f for which $f'(c)=0$ or $f'(c)$ doesn't exist. A critical point is a point whose *x*-coordinate is the critical point c, and whose *y*-coordinate is $f(c)$

$f(x)=x^2$

$\Rightarrow f'(x)=2x=0$

$\rightarrow \boxed{x=0}$ is a critical point.

If $f'(x)=0$ undefined

Exercises Section 3.1 – Extreme Values of Functions

Find the absolute maximum and minimum values of each function. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

1. $f(x)=\frac{2}{3}x-5 \qquad -2\le x\le 3$

2. $f(x)=x^2-1 \qquad -1\le x\le 2$

3. $f(x)=-\frac{1}{x^2} \qquad 0.5\le x\le 2$

4. $f(x)=\sqrt{4-x^2} \qquad -2\le x\le 1$

5. $f(\theta)=\sin\theta \qquad -\frac{\pi}{2}\le\theta\le\frac{5\pi}{6}$

6. $g(x)=\sec x \qquad -\frac{\pi}{3}\le x\le\frac{\pi}{6}$

Find the absolute maximum and minimum values of each function

7. $f(x)=x^{4/3}, \quad -1\le x\le 8$

8. $f(\theta)=\theta^{3/5}, \quad -32\le\theta\le 1$

Determine all critical points of each function

9. $y=x^2-6x+7$

10. $g(x)=(x-1)^2(x-3)^2$

11. $f(x)=\frac{x^2}{x-2}$

12. $g(x)=x^2-32\sqrt{x}$

Find the extreme values (absolute and local) of the function and where they occur

13. $y=x^3-2x+4$

14. $y=\sqrt{x^2-1}$

15. $y=\frac{1}{\sqrt[3]{1-x^2}}$

16. $y=\frac{x+1}{x^2+2x+2}$

Find the critical points, domain endpoints, and local extreme values (absolute and local) for each function.

17. $y=x^{2/3}(x+2)$

18. $y=x^2\sqrt{3-x}$

19. $y=x\sqrt{4-x^2}$

20. Let $f(x)=(x-2)^{2/3}$

a) Does $f'(2)$ exist?

b) Show the only local extreme value of f occurs at $x = 2$.

c) Does the result in part (*b*) contradict the Extreme Value Theorem?

21. Find the absolute extrema of $f(x)=x^{8/3}-16x^{2/3}$ on the interval $[-1,\ 8]$.

22. Find the minimum and maximum values of $f(x)=x^2-8x+10$ on the interval [0, 7].

23. Find the absolute extrema of $f(x) = x^{8/3} - 16x^{2/3}$ on the interval $[-1,\ 8]$.

24. Find the absolute extrema of the function on the closed interval $f(x) = 2(3 - x),\ \ [-1,\ 2]$

25. Find the absolute extrema of the function on the closed interval $f(x) = x^3 - 3x^2,\ \ [0,\ 4]$

26. Find the absolute extrema of the function on the closed interval
$f(x) = \frac{1}{3}x^3 - 2x^2 + 3x - 4,\ \ [-2,\ 5]$

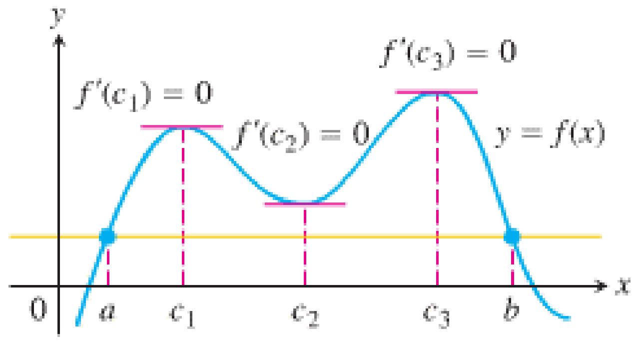
27. Find the absolute extrema of the function on the closed interval $f(x) = \dfrac{1}{x+2},\ \ [-4,\ 1]$

28. Find the absolute extrema of the function on the closed interval $f(x) = \left(x^2 + 4\right)^{2/3},\ \ [-2,\ 2]$

Section 3.2 – The Mean Value Theorem

Rolle's Theorem

Suppose that $y = f(x)$ is continuous at every point of the closed interval [a, b] and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least on number c in (a, b) at which $f'(c) = 0$



Proof

Being continuous, f assumes absolute maximum and minimum values on [a, b]. These can occur only

1. At interior points where f' is zero,
2. At interior points where f' does not exist,
3. At the endpoints of the function's domain, in this case a and b.

By hypothesis, f has a derivative at every interior point. That rules out possibility (2), leaving us with interior points where $f' = 0$ and with the two endpoints a and b.
If either maximum or the minimum occurs at a point c between a and b. then $f' = 0$.
If both the absolute maximum and the absolute minimum occur at the endpoints, then because $f(a) = f(b)$ it must be the case that f is a constant function with $f(x) = f(a) = f(b)$ for every $x \in [a, b]$. Therefore $f'(x) = 0$ and the point c can be taken anywhere in the interior (a, b).

Example

Show that the equation $x^3 + 3x + 1 = 0$ has exactly one real solution.

Solution

$f(x) = x^3 + 3x + 1$

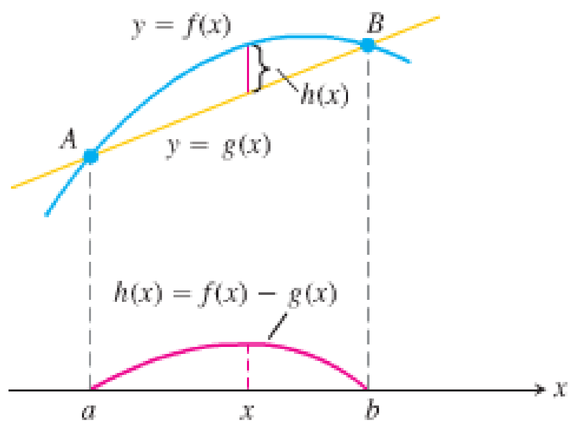
$f(-1) = -3$ and $f(0) = 1$, the Intermediate Value Theorem the equation has one real solution in the open interval (−1, 0).

$f'(x) = 3x^2 + 3 > 0$ (***always positive***). Rolle's Theorem would guarantees the existence of a point $x = c$ in between them where f' was zero. Therefore, f has no more than one zero.

The Mean Value *Theorem*

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



Example

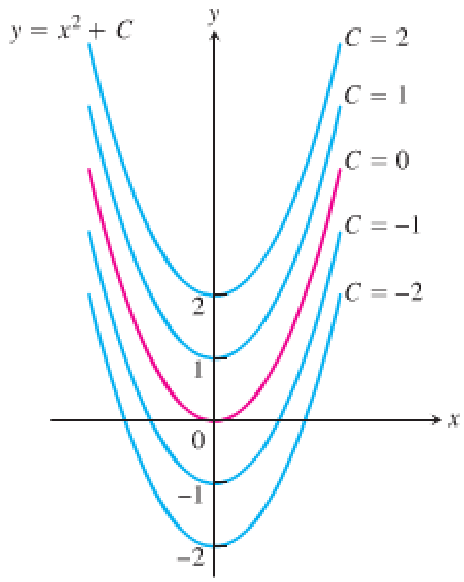
The function $f(x) = x^2$ is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $\frac{4-0}{2-0} = 2$. In this case we can identify c by solving the equation $2c = 2$ to get $c = 1$. However, it is not always easy to find c algebraically, even though we know it always exists.

Corollary

If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

Corollary

If $f'(x) = g'(x)$ at each point x of an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).



Section 3.3 – Monotonic Functions and the First Derivative Test

Increasing and Decreasing Functions

Corollary

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b)
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b)
3. If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b)

Example

Find the open intervals on which the function $f(x) = x^3 - 12x - 5$ is increasing or decreasing

Solution

$$f'(x) = 3x^2 - 12$$

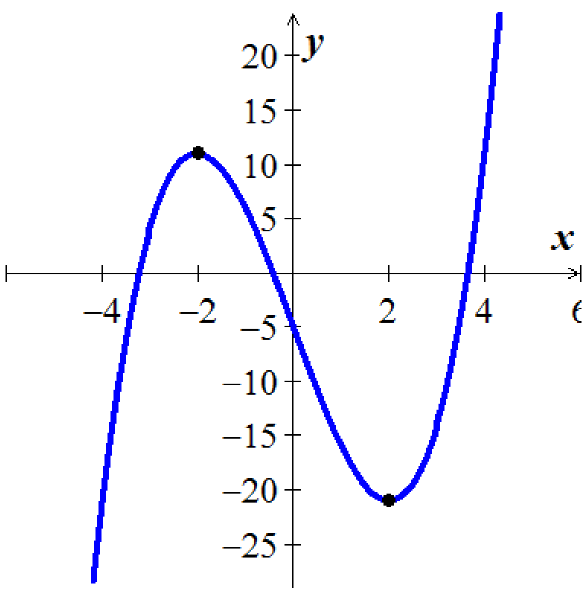
$$3\left(x^2 - 4\right) = 0 \Rightarrow \boxed{x = \pm 2} \quad (\textbf{\textit{CP}})$$

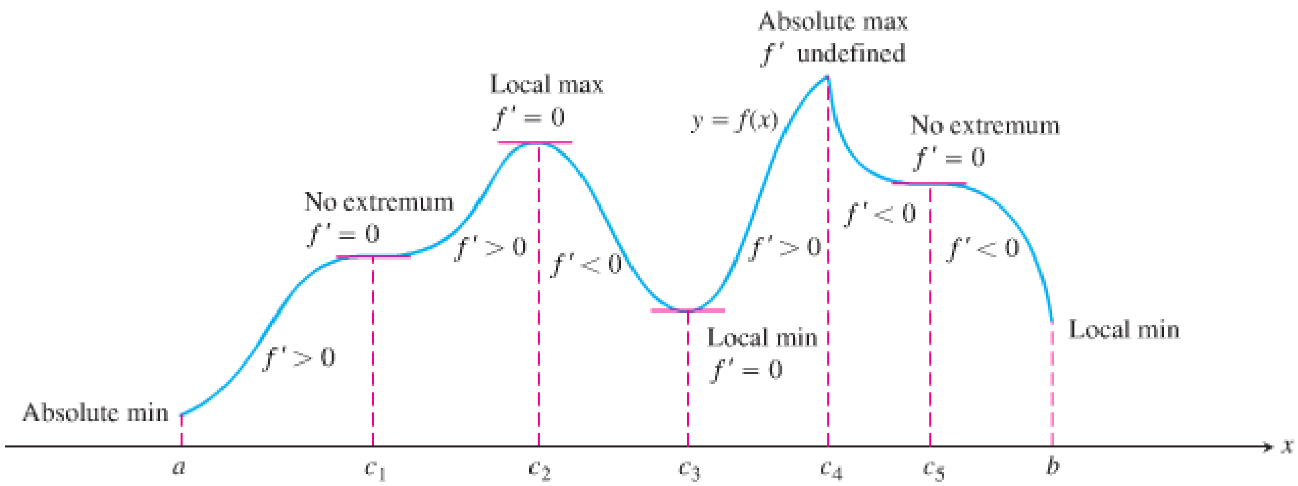
$-\infty$ $\quad -2 \quad$ 2 $\quad \infty$

$f'(-3) > 0$	$f'(0) < 0$	$f'(3) > 0$
Increasing	***Decreasing***	***Increasing***

Increasing: $(-\infty, -2)$ and $(2, \infty)$

Decreasing: $(-2, 2)$

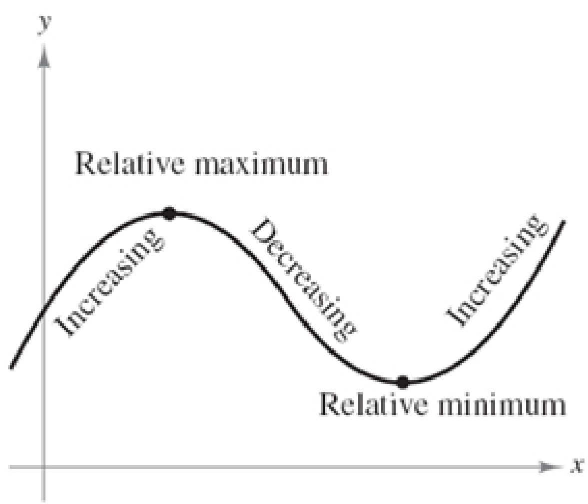




First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f .

1. If f' changes from negative to positive at c, then f has a local minimum (***LMIN***).
2. If f' changes from positive to negative at c, then f has a local maximum (***LMAX***).
3. If f' doesn't change sign at c, then f has no local extremum at c.



Example

Find the open intervals on which the function $f(x) = x^{1/3}(x-4)$ is increasing or decreasing

Solution

$$f(x) = x^{4/3} - 4x^{1/3}$$

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}\left(x^{1/3} - x^{-2/3}\right)\frac{x^{2/3}}{x^{2/3}}$$

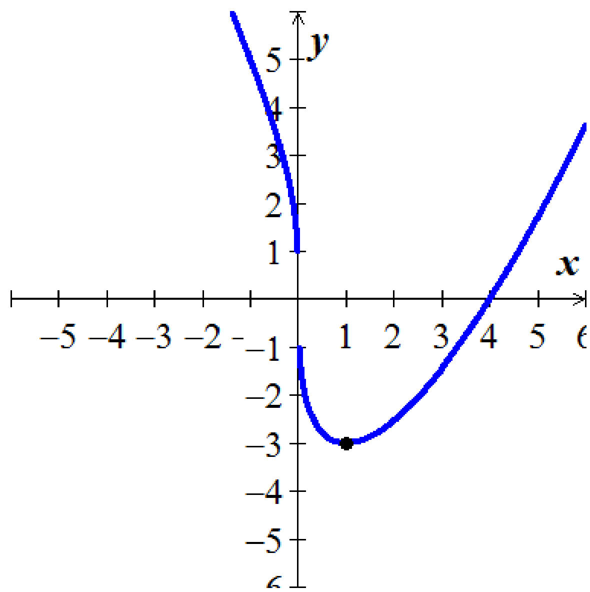
$$= \frac{4}{3}\frac{x-1}{x^{2/3}} = 0$$

$$\Rightarrow \begin{cases} x = 1 \\ x \neq 0 \end{cases} \quad (\boldsymbol{CP})$$

$-\infty$	0	1	∞
$f'(-1) < 0$	$f'(0.5) < 0$	$f'(2) > 0$	
Decreasing	***Decreasing***	***Increasing***	

Increasing: $(1, \infty)$

Decreasing: $(-\infty, 1)$



Exercises *Section* 3.3 – Monotonic Functions and the First Derivative Test

Find the critical numbers and the open intervals on which the function is increasing or decreasing.

1. $f(x) = x^3 + 3x^2 - 9x + 4$

2. $f(x) = (x-1)^{2/3}$

3. $f(x) = x\sqrt{x+1}$

4. $f(x) = \dfrac{x}{x^2+4}$

5. $f(x) = \dfrac{x}{x^2+1}$

6. $f(x) = x\sqrt{x+1}$

7. $f(x) = x^3 - 12x$

8. $f(x) = x^{2/3}$

Find the open intervals on which the function is increasing and decreasing. Then, identify the function's local and absolute extreme values, if any, saying where they occur.

9. $g(t) = -t^2 - 3t + 3$

10. $h(x) = 2x^3 - 18x$

11. $f(\theta) = 3\theta^2 - 4\theta^3$

12. $g(x) = x^4 - 4x^3 + 4x^2$

13. $f(x) = x - 6\sqrt{x-1}$

14. $f(x) = \dfrac{x^3}{3x^2+1}$

15. $f(x) = x^{1/3}(x+8)$

Find all relative Extrema as well as where the function is increasing and decreasing

16. $f(x) = 2x^3 - 6x + 1$

17. $f(x) = 6x^{2/3} - 4x$

18. $f(x) = x^4 - 4x^3$

19. $f(x) = 3x^{2/3} - 2x$

20. $y = \sqrt{4 - x^2}$

21. $f(x) = x\sqrt{x+1}$

22. $f(x) = \dfrac{x}{x^2+1}$

23. $f(x) = x^4 - 8x^2 + 9$

Find the local extrema of each function on the given interval, and say where they occur

24. $f(x) = \sin 2x \quad 0 \le x \le \pi$

25. $f(x) = \sqrt{3}\cos x + \sin x \quad 0 \le x \le 2\pi$

26. $f(x) = \frac{x}{2} - 2\sin\frac{x}{2} \quad 0 \le x \le 2\pi$

27. $f(x) = \sec^2 x - 2\tan x \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

28. A county realty group estimates that the number of housing starts per year over the next three years will be

$$H(r) = \frac{300}{1 + 0.03r^2}$$

Where r is the mortgage rate (in percent).

a) Where is $H(r)$ increasing?

b) Where is $H(r)$ decreasing?

29. Suppose the total cost $C(x)$ to manufacture a quantity x of insecticide (in hundreds of liters) is given by $C(x) = x^3 - 27x^2 + 240x + 750$. Where is $C(x)$ decreasing?

30. A manufacturer sells telephones with cost function $C(x) = 6.14x - 0.0002x^2, \quad 0 \le x \le 950$ and revenue function $R(x) = 9.2x - 0.002x^2, \quad 0 \le x \le 950$. Determine the interval(s) on which the profit function is increasing.

31. The cost of a computer system increases with increased processor speeds. The cost C of a system as a function of processor speed is estimated as $C(x) = 14x^2 - 4x + 1200$, where x is the processor speed in MHz. Determine the intervals where the cost function $C(x)$ is decreasing.

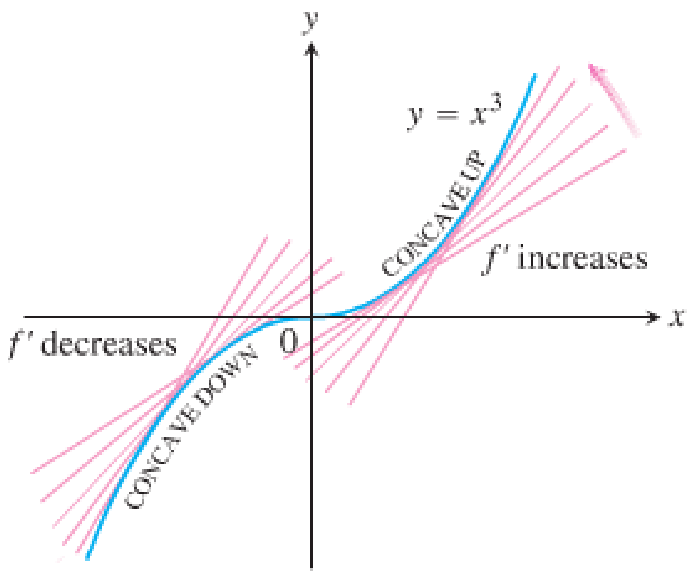
32. The percent of concentration of a drug in the bloodstream t hours after the drug is administered is given by $K(t) = \dfrac{t}{t^2 + 36}$. On what time interval is the concentration of the drug increasing?

33. Coughing forces the trachea to contract, this in turn affects the velocity of the air through the trachea. The velocity of the air during coughing can be modeled by: $v = k(R - r)r^2, \quad 0 \le r < R$ where k is a constant, R is the normal radius of the trachea (also a constant) and r is the radius of the trachea during coughing. What radius r will produce the maximum air velocity?

34. $P(x) = -x^3 + 15x^2 - 48x + 450, \quad x \ge 3$ is an approximation to the total profit (in thousands of dollars) from the sale of x hundred thousand tires. Find the number of hundred thousands of tires that must be sold to maximize profit.

35. $P(x) = -x^3 + 3x^2 + 360x + 5000; \quad 6 \le x \le 20$ is an approximation to the number of salmon swimming upstream to spawn, where x represents the water temperature in degrees Celsius. Find the temperature that produces the maximum number of salmon.

Section 3.4 – Concavity and Curve Sketching

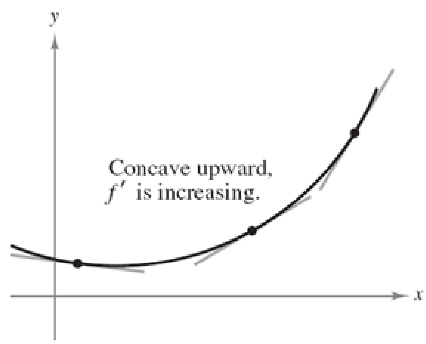


Concavity

Definition

Let f be differentiable on an open interval I . The graph of f is

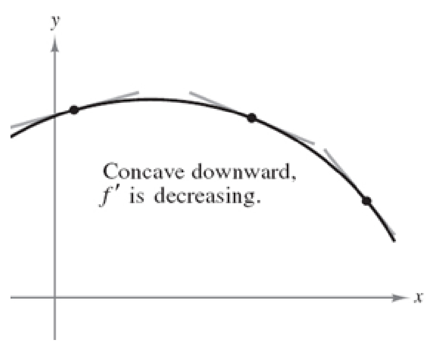
1. **Concave *upward*** on I if f' is increasing on the interval.
2. **Concave *downward*** on I if f' is decreasing on the interval.



Test for Concavity

Let f be function whose second derivative exists on an open interval *I*.

1. If $f''(x) > 0$ for all x in I, then f is ***concave up*** on I.
2. If $f''(x) < 0$ for all x in I, then f is ***concave down*** on I.



i. Locate the x values @ which $f''(x) = 0$ or undefined

ii. Use these test x-value to determine the test intervals

iii. Test the sign of $f''(x)$ in each interval

Example

Determine the intervals on which the graph of the function is concave upward or concave downward.

$$f(x) = x^4 - 8x^3 + 18x^2$$

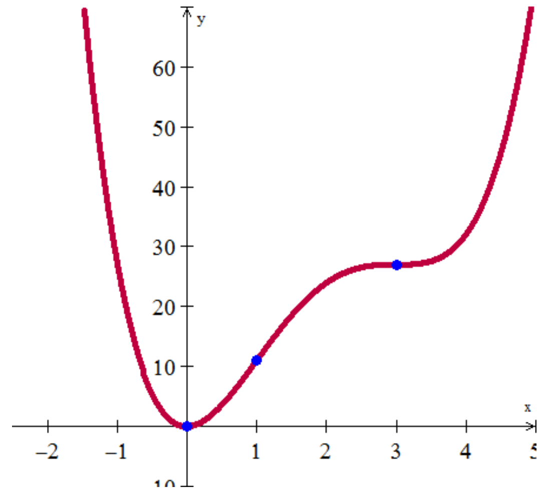
Solution

$f'(x) = 4x^3 - 24x^2 + 36x$

$f''(x) = 12x^2 - 48x + 36$

Solve for x: $x = 1 \quad x = 3$

$-\infty$	1	3	∞
$f''(0) > 0$		$f''(2) < 0$	$f''(4) > 0$
upward		***downward***	***upward***



f is concave upward on $(-\infty,\ 1)$ *and* $(3,\ \infty)$

f is concave downward on $(1,\ 3)$

Example

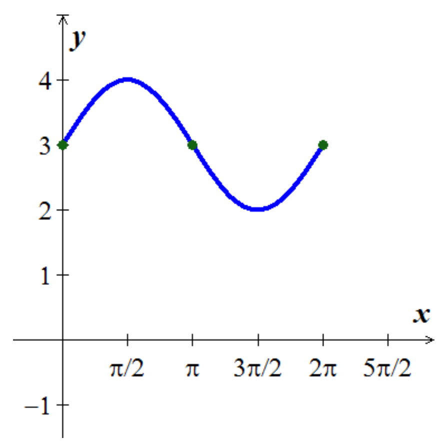
Determine the concavity of $y = 3 + \sin x$ *on* $[0,\ 2\pi]$

Solution

$y' = \cos x$

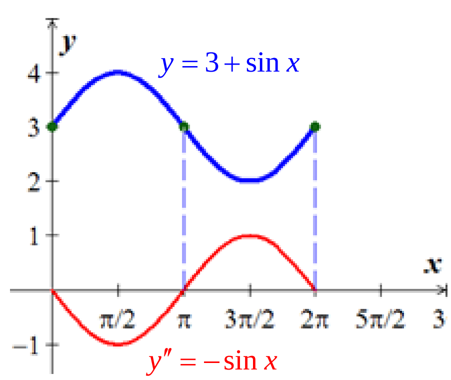
$y'' = -\sin x = 0 \quad \Rightarrow \quad \boxed{x = 0,\ \pi,\ 2\pi}$

0	π	2π
$f''\left(\frac{\pi}{2}\right) < 0$		$f''\left(\frac{3\pi}{2}\right) > 0$
downward		***upward***

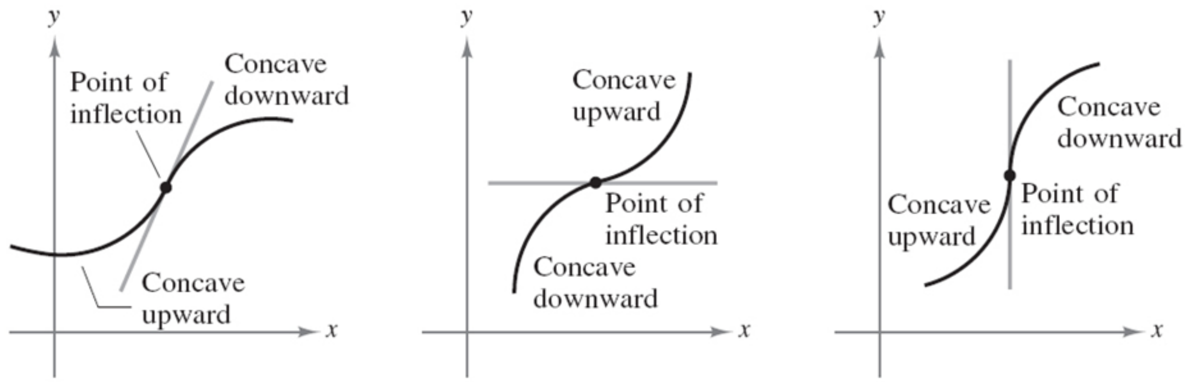


The graph y is concave down on $(0,\ \pi)$

The graph y is concave up on $(\pi,\ 2\pi)$



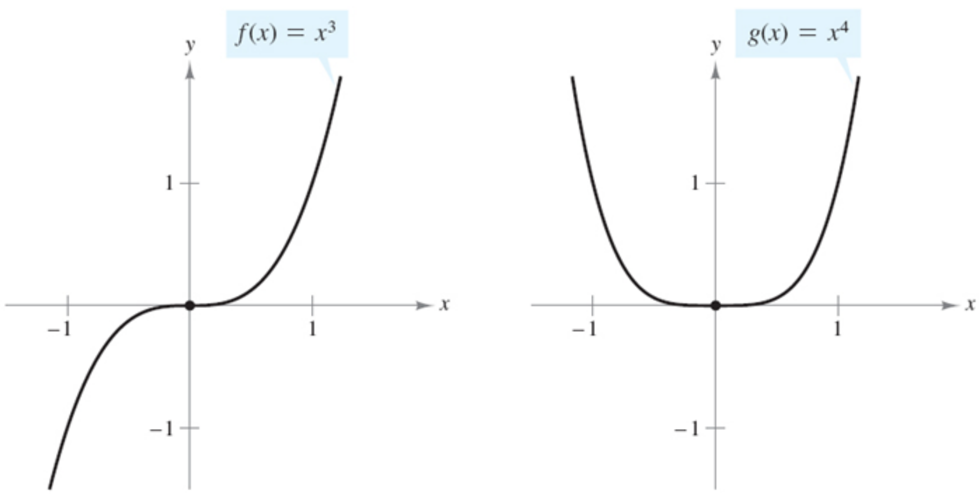
Points of Inflection



Definition

If the graph of a continuous function has a tangent line @ a point where its concavity changes from upward to downward (or down to upward) then the point is a ***point of inflection***.

At a point of inflection $(c,\ f(c))$, either $f''(c)=0$ *or* $f''(c)$ fails to exist.



$f''(0) = 0$, and (0, 0) is a point of inflection.

$g''(0) = 0$, but (0, 0) is not a point of inflection.

Example

A particle is moving along a horizontal coordinate line (positive to the right) with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \quad t \ge 0$$

Find the velocity and acceleration, and describe the motion of the particle.

Solution

The velocity is: $v(t) = s'(t) = 6t^2 - 28t + 22 = 0 \quad \Rightarrow \quad \boxed{t = 1, \ \frac{11}{3}}$

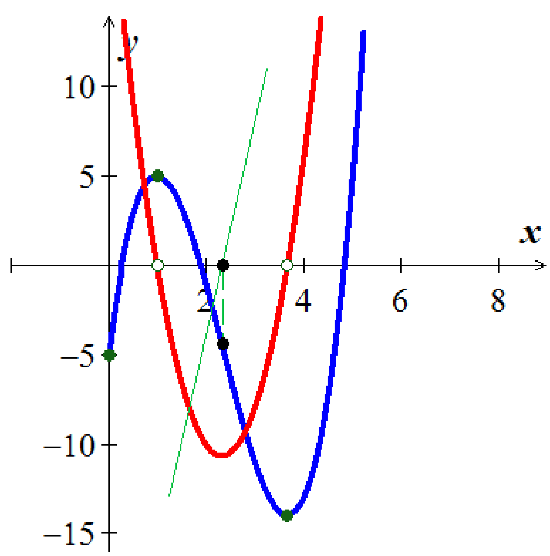
The acceleration is: $a(t) = v'(t) = 12t - 28 = 0 \quad \Rightarrow \quad \boxed{t = \frac{7}{3}}$

$0 \qquad 1 \qquad \frac{7}{3} \qquad \frac{11}{3}$

$f'(.5) > 0$	$f'(2) < 0$	$f'(4) > 0$
Increasing	***Decreasing***	***Increasing***
right	***left***	***right***

$f''(1) < 0$	$f''(4) > 0$
Concave down	***Concave up***

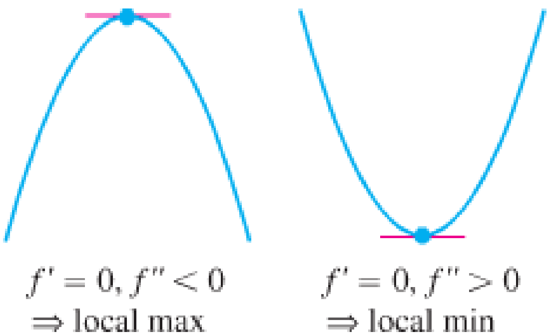
The particle starts moving to the right while slowing down, and then reverses by moving to the left at $t = 1$ under the influence of the leftward acceleration over the time interval $\left[0, \ \frac{7}{3}\right)$. The acceleration then changes direction changes direction at $t = \frac{7}{3}$ but the particles continues moving leftward, while slowing down under the rightward acceleration. At $t = \frac{11}{3}$ the particle reverses direction again; moving to the right in the same direction as the acceleration.



Second Derivative Test for local Extrema

Let $f'(c)=0$ and let f'' exist ($\exists$)

1. If $f'(c)=0$ *and* $f''(c)>0 \Rightarrow f$ is a local Minimum at $x=c$
2. If $f'(c)=0$ *and* $f''(c)<0 \Rightarrow f$ is a local Maximum at $x=c$
3. If $f'(c)=0$ *and* $f''(c)=0 \Rightarrow$ Test fails $\rightarrow$ use f' to determine Max, Min.



Example

Find all relative extrema for $f(x)=4x^3+7x^2-10x+8$

Solution

$f'(x)=12x^2+14x-10=0 \quad \rightarrow x=-\frac{5}{3} \quad x=\frac{1}{2}$

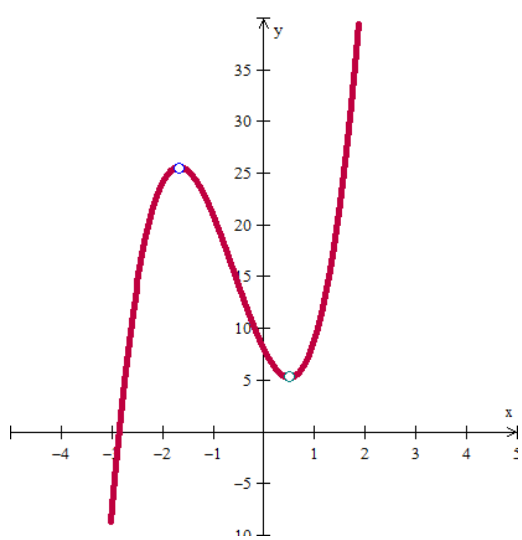
$f''(x)=24x+14$

$f''\left(-\frac{5}{3}\right)=24\left(-\frac{5}{3}\right)+14=-26<0$ Leads to local maximum

$f''\left(\frac{1}{2}\right)=24\left(\frac{1}{2}\right)+14=26>0$ Leads to local minimum

RMAX: $\left(-\frac{5}{3}, \frac{691}{27}\right)$

RMIN: $\left(\frac{1}{2}, \frac{21}{4}\right)$



Example

Sketch a graph of the function $f(x) = x^4 - 4x^3 + 10$ using the following steps

a) Identify where the extrema of f occur

b) Find the intervals on which f is increasing and decreasing

c) Find where the graph of f is concave up and down

d) Sketch the general shape of the graph for f

e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

Solution

$$f'(x) = 4x^3 - 12x^2$$

$$= 4x^2(x-3) = 0 \quad \Rightarrow \quad \boxed{x = 0,\ 0} \quad \boxed{x = 3} \ (CN)$$

$-\infty \qquad 0 \qquad 3 \qquad \infty$

$f'(-1) < 0$ ***decreasing***	$f'(1) < 0$ ***decreasing***	$f'(4) > 0$ ***increasing***

a) $x = 3 \Rightarrow \lfloor y = 3^4 - 4(3)^3 + 10 = -17 \rfloor$

A local minimum at $(3, -17)$

b) f is ***decreasing***: $(-\infty,\ 0] \cup [0,\ 3)$

f is ***increasing***: $(3,\ \infty)$

c) $f''(x) = 12x^2 - 24x = 12x(x-2) = 0$

$$\Rightarrow \begin{cases} \boxed{x = 0} & \rightarrow f(0) = 10 \\ \boxed{x = 2} & \rightarrow f(2) = -6 \end{cases}$$

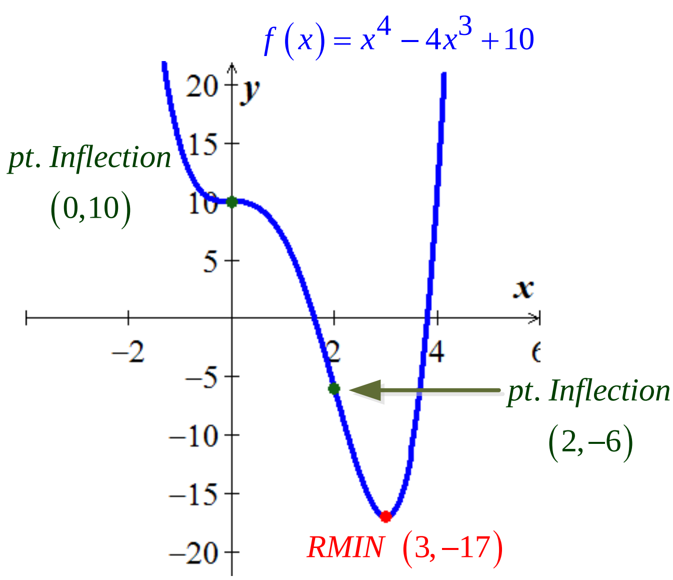
$-\infty \qquad 0 \qquad 2 \qquad \infty$

$f''(-1) > 0$ ***Concave up***	$f''(1) < 0$ ***Concave down***	$f''(3) > 0$ ***Concave up***

f is ***concave up***: $(-\infty,\ 0) \cup (2,\ \infty)$

f is ***concave down***: $(0,\ 2)$

d) $f(x = 0) = 0^4 - 4(0)^3 + 10 = 10$



Example

Sketch the graph of $f(x)=\dfrac{(x+1)^2}{1+x^2}$

Solution

Domain of f is $(-\infty,\ \infty)$

$\Rightarrow$ Horizontal Asymptotes $y=1$

$$f'(x)=\frac{2(x+1)\left(1+x^2\right)-2x(x+1)^2}{\left(1+x^2\right)^2}$$

$u=(x+1)^2 \quad v=1+x^2$

$u'=2(x+1) \quad v'=2x$

$$=\frac{2(x+1)\left[\left(1+x^2\right)-x(x+1)\right]}{\left(1+x^2\right)^2}$$

$$=\frac{2(x+1)\left[1+x^2-x^2-x\right]}{\left(1+x^2\right)^2}$$

$$=\frac{2(x+1)(1-x)}{\left(1+x^2\right)^2}$$

$$=2\frac{1-x^2}{\left(1+x^2\right)^2}$$

$$(x+1)(1-x)=0 \quad \Rightarrow \quad \boxed{x=\pm 1}\ (\boldsymbol{CN})$$

$$f''(x)=2\frac{(-2x)\left(1+x^2\right)^2-2\left(1+x^2\right)(2x)\left(1-x^2\right)}{\left(1+x^2\right)^4}$$

$$=2\frac{-2x\left(1+x^2\right)-4x\left(1-x^2\right)}{\left(1+x^2\right)^4}\left(1+x^2\right)$$

$$=2\frac{-2x-2x^3-4x+4x^3}{\left(1+x^2\right)^3}$$

$$=2\frac{2x^3-6x}{\left(1+x^2\right)^3}$$

$$= \frac{4x\left(x^2-3\right)}{\left(1+x^2\right)^3} = 0 \qquad \rightarrow \boxed{x=0} \quad \boxed{x=\pm\sqrt{3}}$$ ***Point of inflections***

$-\infty$	$-\sqrt{3}$	-1	$\mathbf{0}$	1	$\sqrt{3}$	∞
$f'(-2)<0$ ***Decreasing***			$f'(.5)>0$ ***Increasing***		$f'(2)<0$ ***Decreasing***	
$f''(-4)<0$ ***Concave down***		$f''(-1)>0$ ***Concave up***		$f''(1)<0$ ***Concave down***		$f''(4)>0$ ***Concave up***

RMAX: $(1,\ 2)$

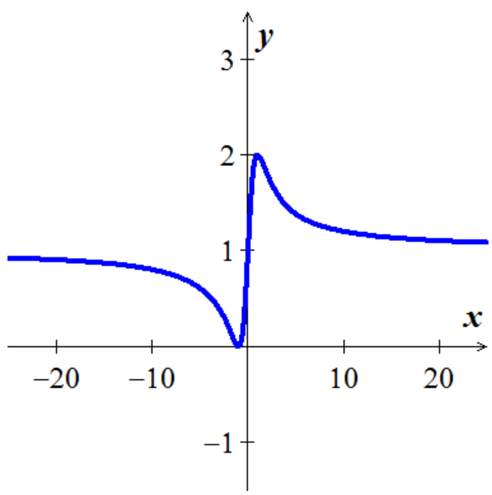
RMIN: $(-1,\ 0)$

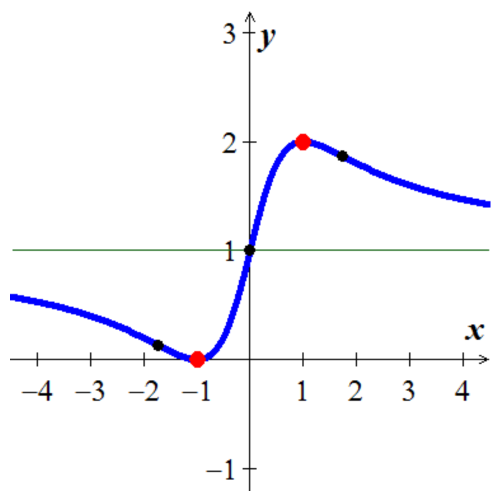
Decreasing: $(-\infty,\ -1)\cup(1,\ \infty)$

Increasing: $(-1,\ 1)$

Concave down: $\left(-\infty,\ -\sqrt{3}\right)$ *and* $\left(0,\ \sqrt{3}\right)$

Concave up: $\left(-\sqrt{3},\ 0\right)$ *and* $\left(\sqrt{3},\ \infty\right)$





Example

Sketch the graph of $f(x)=\dfrac{x^2+4}{2x}$

Solution

$$f(x)=\frac{x^2+4}{2x}$$

$$=\frac{x^2}{2x}+\frac{4}{2x}$$

$$=\frac{x}{2}+\frac{2}{x} \qquad \boxed{y=\frac{x}{2}} \text{ Oblique Asymptote}$$

$$f'(x)=\frac{1}{2}-\frac{2}{x^2}$$

$$=\frac{x^2-4}{2x^2}=0 \qquad \Rightarrow \quad \boxed{x=\pm 2} \ (CN)$$

$$f'(x)=\frac{1}{2}-\frac{2}{x^2} \quad \rightarrow \quad f''(x)=\frac{4}{x^3}$$

No point of inflection and when $\begin{cases} x>0 & \rightarrow f''>0 \\ x<0 & \rightarrow f''<0 \end{cases}$

$-\infty \qquad -2 \qquad 0 \qquad 2 \qquad \infty$

$f'(-3)>0$ ***Increasing***	$f'(0)<0$ ***Decreasing***	$f'(3)>0$ ***Increasing***

$f''(-1)<0$ ***Concave down***	$f''(1)>0$ ***Concave up***

RMIN: $(2,\ 2)$

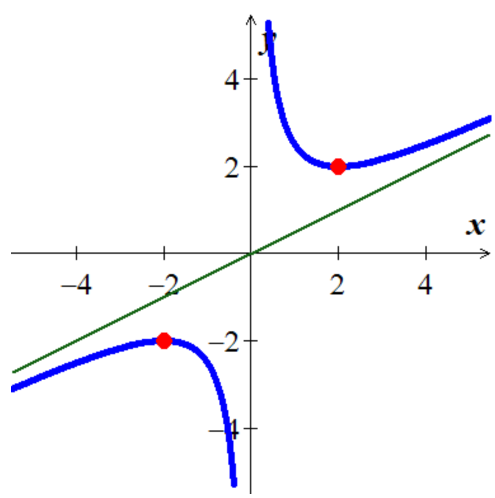
RMAX: $(-2,\ -2)$

Decreasing: $(-2,\ 2)$

Increasing: $(-\infty,\ -2)\cup(2,\ \infty)$

Concave down: $(-\infty,\ 0)$

Concave up: $(0,\ \infty)$



Exercises Section 3.4 – Concavity and Curve Sketching

Determine the intervals on which the graph of the function is concave upward or concave downward.

1. $f(x)=\dfrac{x^2-1}{2x+1}$

2. $f(x)=-4x^3-8x^2+32$

3. $f(x)=\dfrac{12}{x^2+4}$

4. Find the points of inflection. $f(x)=x^3-9x^2+24x-18$

5. Find the second derivative of $f(x)=-2\sqrt{x}$ and discuss the concavity of the graph

6. Find the extrema using the second derivative test $f(x)=\dfrac{4}{x^2+1}$

7. Discuss the concavity of the graph of *f* and find its points of inflection. $f(x)=x^4-2x^3+1$

8. Find all relative extrema of $f(x)=x^4-4x^3+1$

Sketch the graph

9. $y=x^3-3x+3$

10. $y=-x^4+6x^2-4$

11. $y=x\left(\frac{x}{2}-5\right)^4$

12. $y=x+\sin x \quad 0\le x\le 2\pi$

13. $y=\cos x+\sqrt{3}\sin x \quad 0\le x\le 2\pi$

14. $y=\dfrac{x}{\sqrt{x^2+1}}$

15. $y=x^2+\dfrac{2}{x}$

16. $y=\dfrac{x^2-3}{x-2}$

17. $y=\dfrac{5}{x^4+5}$

18. $y=\dfrac{x^2-49}{x^2+5x-14}$

19. $y=\dfrac{x^4+1}{x^2}$

20. $y=\dfrac{x^2-4}{x^2-1}$

21. $y=-\dfrac{x^2-x+1}{x-1}$

22. $y=\dfrac{x^3-3x^2+3x-1}{x^2+x-2}$

23. $y=\dfrac{4x}{x^2+4}$

24. The revenue *R* generated from sales of a certain product is related to the amount *x* spent on advertising by

$$R(x)=\frac{1}{15{,}000}\left(600x^2-x^3\right), \qquad 0\le x\le 600$$

Where *x* and *R* are in thousands of dollars.
Is there a point of diminishing returns for this function?

25. Find the point of diminishing returns (x, y) for the function

$$R(x) = -x^3 + 45x^2 + 400x + 8000, \quad 0 \le x \le 20$$

where $R(x)$ represents revenue in thousands of dollars and x represents the amount spent on advertising in tens of thousands of dollars.

Section 3.5 – Applied Optimization

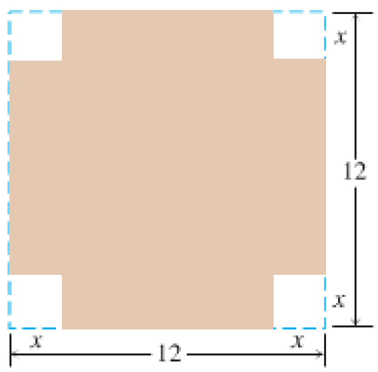
Solving Applied Optimization Problems

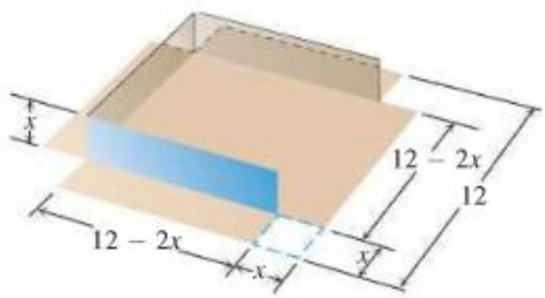
1. Read the problem
2. Draw a picture
3. Introduce variables
4. Write an equation for the unknown quantity
5. Test the critical points and endpoints in the domain of the unknown

Example

An open-top box is to be made by cutting small congruent squares from the corners of a 12-in. by 12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Solution





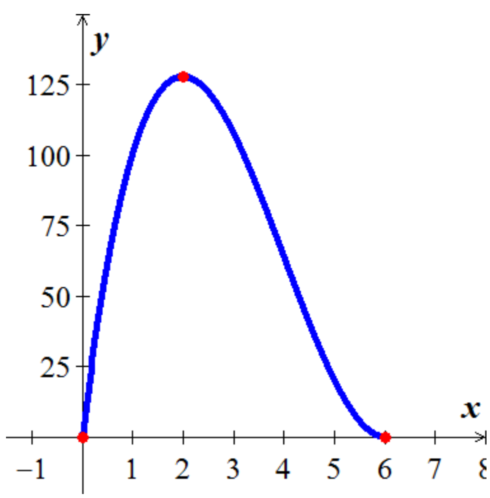
$$\begin{aligned} V(x) &= hlw \\ &= x(12-2x)^2 \\ &= x\left(144-48x+4x^2\right) \\ &= 4x^3-48x^2+144x \qquad 0 \le x \le 6 \end{aligned}$$

$$\frac{dV}{dx} = 12x^2 - 96x + 144 = 0$$

$$\boxed{x = 2,\ 6} \quad (\boldsymbol{CP})$$

$$V(2) = 4(2)^3 - 48(2)^2 + 144(2) = \underline{128}$$

$$V(6) = 4(6)^3 - 48(6)^2 + 144(6) = \underline{0}$$

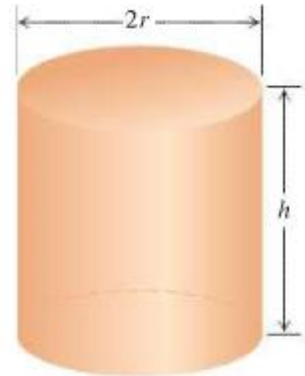


The maximum volume is 128 in^3. The cutout squares is 2 in.

Example

You have been asked to design a one-liter can shaped like a right circular cylinder. What dimensions will use the least material?

Solution



Volume of can:

$$V = \pi r^2 h = 1 \ liter = 1000 \ cm^3$$

Surface area of can:

$$A = 2\pi r^2 + 2\pi rh$$

$$\pi r^2 h = 1000 \quad \Rightarrow \quad h = \frac{1000}{\pi r^2}$$

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2000}{r}$$

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2} = 0 \qquad \textit{Solve for r}$$

$$4\pi r = \frac{2000}{r^2}$$

$$r^3 = \frac{500}{\pi}$$

$$\underline{\lfloor r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \rfloor}$$

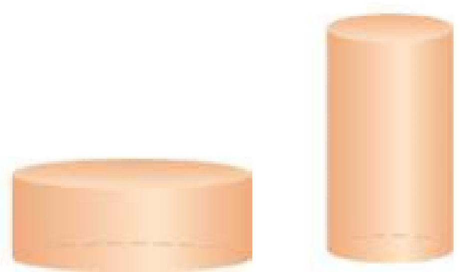
0	5.42
$A'(1) < 0$ *decreasing*	$A'(6) > 0$ *increasing*

$$\underline{\lfloor h = \frac{1000}{\pi r^2} = \frac{1000}{\pi (5.42)^2} \approx 10.84 \rfloor} \quad (h = 2r)$$

The graph is concave up on its domain.

The on2-liter can that uses the least material has height equal to twice its radius.

$$\boxed{r \approx 5.42 \ cm \quad h \approx 10.84 \ cm}$$



Short and wide Tall and thin

Example

A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area of the rectangle can have, and what are its dimensions?

Solution

The equation of the circle is given by the equation: $x^2 + y^2 = 4$

Therefore, the semicircle is: $y = \sqrt{4 - x^2}$

The dimensions of semicircle: $\quad length : 2x \quad height : \sqrt{4 - x^2}$

Area: $A(x) = lh = 2x\sqrt{4 - x^2}$

$$A'(x) = 2\sqrt{4 - x^2} - \frac{2x^2}{\sqrt{4 - x^2}} \qquad \begin{aligned} u &= 2x & v &= \sqrt{4 - x^2} \\ u' &= 2 & v' &= \frac{-x}{\sqrt{4 - x^2}} \end{aligned}$$

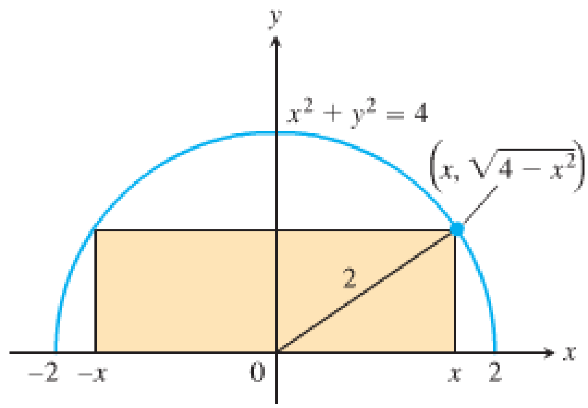
$$= \frac{8 - 2x^2 - 2x^2}{\sqrt{4 - x^2}}$$

$$= \frac{8 - 4x^2}{\sqrt{4 - x^2}} = 0 \qquad \textbf{\textit{Solve for x}}$$

$$8 - 4x^2 = 0 \quad \Rightarrow \quad x^2 = 2 \rightarrow \boxed{x = \pm\sqrt{2}}$$

$$A(\sqrt{2}) = 2(\sqrt{2})\sqrt{4 - (\sqrt{2})^2} = 4$$

$$A(2) = 2(2)\sqrt{4 - (2)^2} = 0$$



The area has a maximum value of 4 when the height is $x = \sqrt{2}$ and length $2x = 2\sqrt{2}$

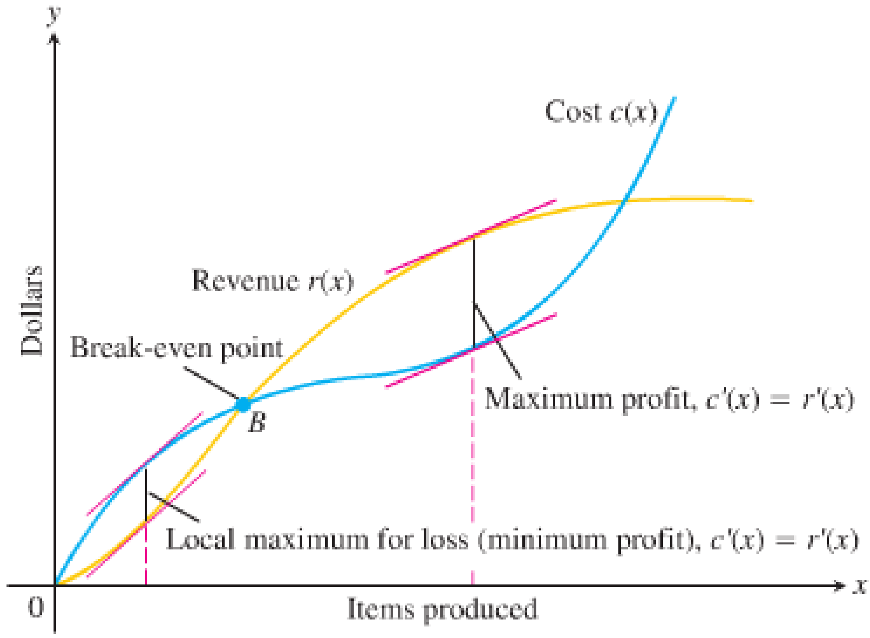
Example from *Economics*

$r(x) =$ Revenue from selling x items

$c(x) =$ Cost of producing the x items

$p(x) = r(x) - c(x)$ Profit from producing and selling x items

At a production level yielding maximum profit, marginal revenue equals to marginal cost.



Marginal Analysis

Profit = P Revenue = R Cost = C $P = R - C$

The derivatives of these quantities are called ***Marginal***

$$\frac{dP}{dx} = \text{Marginal Profit}$$

$$\frac{dR}{dx} = \text{Marginal Revenue}$$

$$\frac{dC}{dx} = \text{Marginal Cost}$$

Example

Suppose that $r(x)=9x$ and $c(x)=x^3-6x^2+15x$, where *x* represents millions of MP3 players produced. Is there a production level that maximizes profit? If so, what is it?

Solution

$r'(x)=9 \qquad c'(x)=3x^2-12x+15$

To find the intersection between the 2 derivatives, set $r'(x)=c'(x)$

$3x^2-12x+15=9$

$3x^2-12x+6=0 \quad \rightarrow \quad \begin{cases} x=2-\sqrt{2}\approx .586 \\ x=2+\sqrt{2}\approx 3.414 \end{cases}$

The possible productions are $x\approx 0.586$ *or* $x\approx 3.414$ million.

$p(x)=r(x)-c(x)=9x-x^3+6x^2-15x$

$p(x)=-x^3+6x^2-6x$

$p'(x)=-3x^2+12x-6$

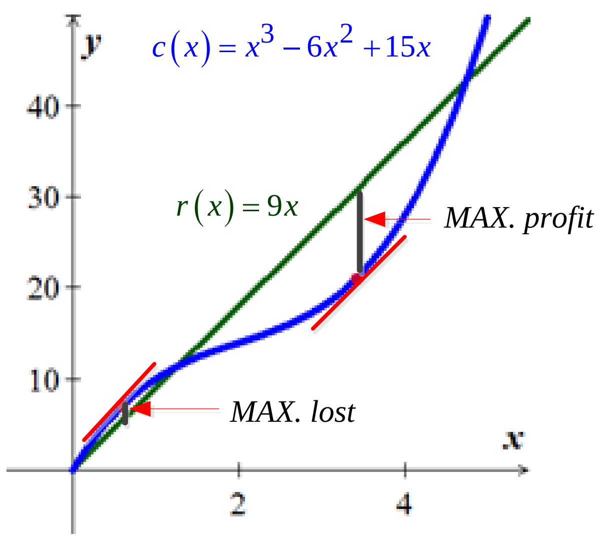
$p''(x)=-6x+12=0 \quad \rightarrow \quad \boxed{x=2}$

Concave down: $(0,\ 2)$

Concave up: $(2,\ \infty)$

The maximum profit is about $x\approx 3.414$ *million*

The maximum lost is about $x\approx .586$ *million*



Exercises Section 3.5 – Applied Optimization

1. Find two nonnegative numbers x and y for which $2x + y = 30$, such that xy^2 is maximized.

2. A rectangular page will contain 54 in^2 of print. The margins at the top and bottom of the page are 1.5 *inches* wide. The margins on each side are 1 *inch* wide. What should the dimensions of the page be to minimize the amount of paper used?

3. The product of two numbers is 72. Minimize the sum of the second number and twice the first number

4. Verify the function $V = 27x - \frac{1}{4}x^3$ has an absolute maximum when $x = 6$. What is the maximum volume?

5. A net enclosure for golf practice is open at one end. The volume of the enclosure is $83\frac{1}{3}$ cubic meters. Find the dimensions that require the least amount of netting.

6. Find two numbers x and y such that their sum is 480 and $x^2 y$ is maximized.

7. If the price charged for a candy bar is $p(x)$ cents, then x thousand candy bars will be sold in a certain city, where $p(x) = 82 - \dfrac{x}{20}$. How many candy bars must be sold to maximize revenue?

8. $S(x) = -x^3 + 6x^2 + 288x + 4000;\ \ 4 \le x \le 20$ is an approximation to the number of salmon swimming upstream to spawn, where x represents the water temperature in degrees Celsius. Find the temperature that produces the maximum number of salmon.

9. A company wishes to manufacture a box with a volume of 52 cubic feet that is open on top and is twice as long as it is wide. Find the width of the box that can be produced using the minimum amount of material.

10. A rectangular field is to be enclosed on four sides with a fence. Fencing costs \$3 per foot for two opposite sides, and \$4 per foot for the other two sides. Find the dimensions of the field of area 730 square feet that would be the cheapest to enclose.

11. A page is to contain 30 square inches of print. The margins at the top and bottom of the page are 2 inches wide. The margins on the sides are 1 inch wide. What dimensions will minimize the amount of paper used?

12. Find the points of $y = 4 - x^2$ that are closet to (0, 3)

13. A manufacturer wants to design an open box that has a square base and a surface area of 108 in^2. What dimensions will produce a box with a maximum volume?

14. A company wants to manufacture cylinder aluminum can with a volume $1000cm^3$. What should the radius and height of the can be to minimize the amount of aluminum used?

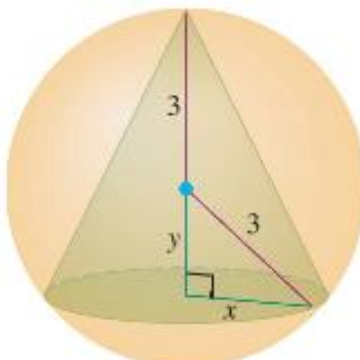
15. What is the smallest perimeter possible for a rectangle whose area is 16 in^2, and what are its dimensions?

16. A rectangle has its base on the x-axis and its upper vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

17. You are planning to make an open rectangular box from an 8-in. by 15-in. piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way, and what is its volume?

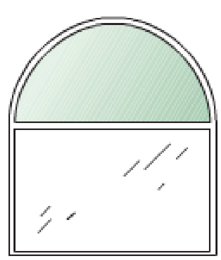
18. A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?

19. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.



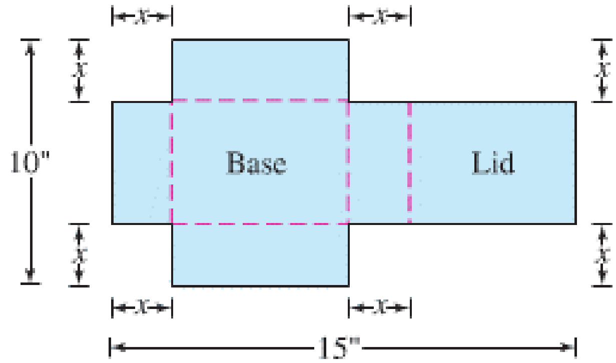
20. What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cm^3?

21. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.



22. The cost per hour for fuel to run a train is $\frac{v^2}{4}$ dollars, where v is the speed of the train in miles per hour. (Note that the cost goes up as the square of the speed.) Other costs, including labor are \$300 per hour. How fast should the train travel on a 360-mile trip to minimize the total cost for the trip?

23. A piece of cardboard measures 10-in. by 15-in. Two equal squares are removed from the corners of 10-in. side. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.

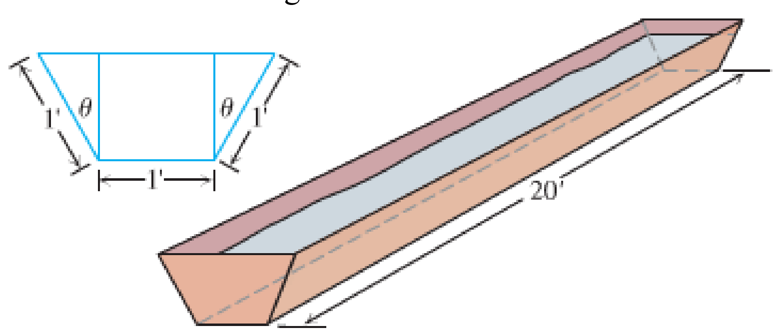


a) Write a formula $V(x)$ for the volume of the box

b) Find the domain of V for the problem situation and graph V over this domain

c) Use the graphical or analytically method to find the maximum volume and the value of x that gives it.

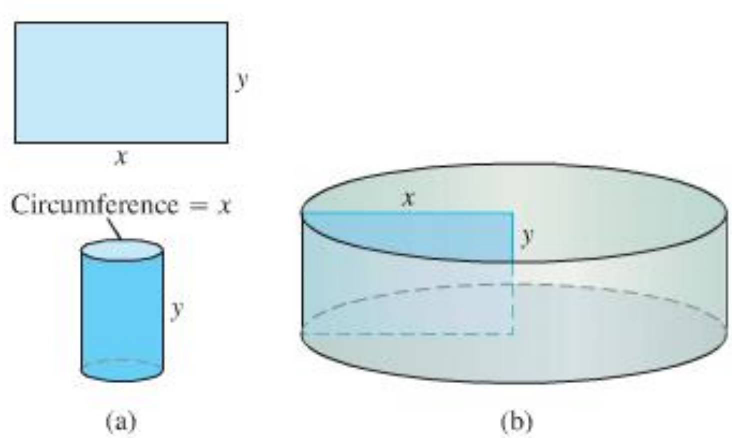
24. The trough in the figure is to be made to the dimensions shown. Only the angle θ can be varied. What value of θ will maximize the trough's volume?



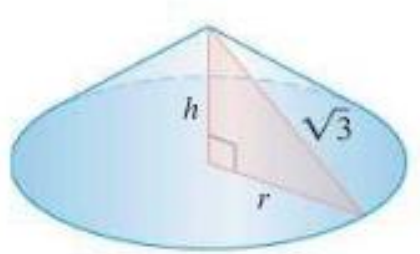
25. Compare the answers to the following two construction problems.

a) A rectangular sheet of perimeter 36 *cm* and dimensions x *cm* and y *cm* is to be rolled into a cylinder as shown in part (a) of the figure. What values of x and y give the largest volume?

b) The same sheet is to be revolved about one of the sides of length y to sweep out the cylinder as shown in part (b) of the figure. What values of x and y give the largest volume?



26. A right triangle whose hypotenuse is $\sqrt{3}\ m$ long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.



27. The height above the ground of an object moving vertically is given by

$$s = -16t^2 + 96t + 112$$

With s in feet and t in seconds. Find

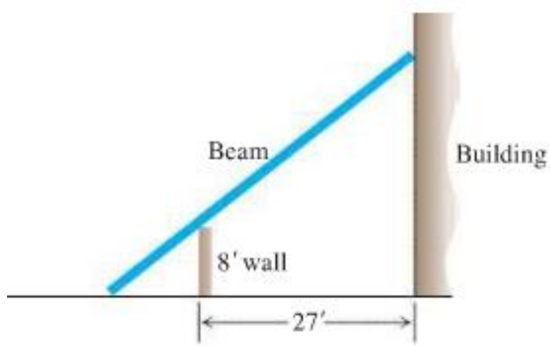
a) The object's velocity when $t = 0$

b) Its maximum height and when it occurs

c) Its velocity when $s = 0$

28. Jane is 2 mi offshore in a boat and wishes to reach a coastal village 6 mi down a straight shoreline from the point nearest the boat. She can row 2 mph and can walk 5 mph. Where should she land her boat to reach the village in the least amount of time?

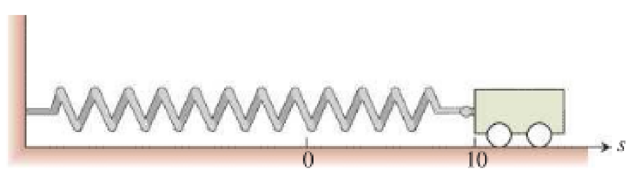
29. The 8-ft wall stands 27 ft from the building. Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.



30. A small frictionless cart, attached to the wall by a spring, is pulled 10 cm from its rest position and released at time $t = 0$ to roll back and forth for 4 sec. Its position at time t is $s = 10\cos \pi t$

a) What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?

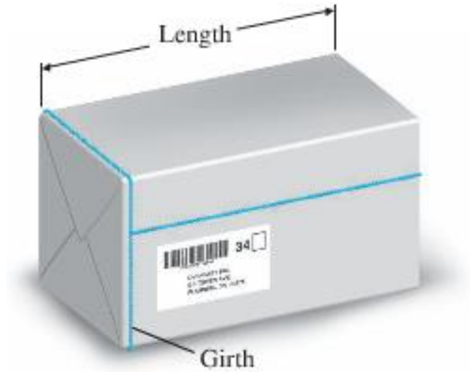
b) Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?



31. A parcel delivery service will deliver a package only if the length plus girth (distance around) does not exceed 108 inches.

a) Find the dimensions of a rectangular box with square ends that satisfies the delivery service's restriction and has maximum volume. What is the maximum volume?

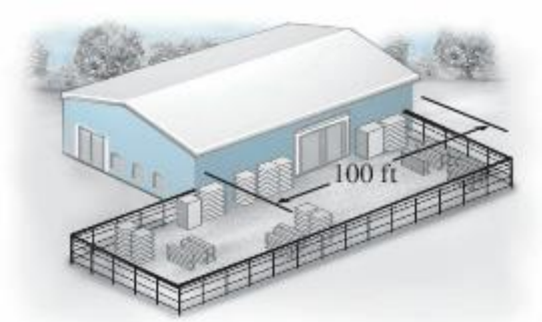
b) Find the dimensions (radius and height) of a cylinder container that meets the delivery service's requirement and has maximum volume. What is the maximum volume?



32. The owner of a retail lumber store wants to construct a fence an outdoor storage are adjacent to the store, using all of the store as part of one side of the area. Find the dimensions that will enclose the largest area if

a) 240 feet fencing material are used.

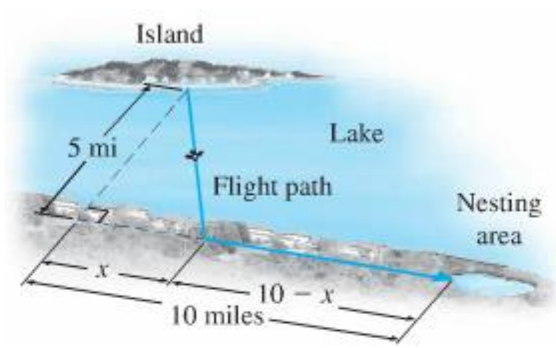
b) 400 feet fencing material are used.



33. Some birds tend to avoid flights over large bodies of water during daylight hours. Suppose that an adult bird with this tendency is taken from its nesting area on the edge of a large lake to an island 5 miles offshore and is then release.

a) If it takes only 1.4 times as much energy to fly over water as land, how far up the shore (x, in miles) should the bird head to minimize the total energy expended in returning to the nesting area?

b) If it takes only 1.1 times as much energy to fly over water as land, how far up the shore should the bird head to minimize the total energy expended in returning to the nesting area?



34. A pharmacy has a uniform annual demand for 200 bottles of a certain antibiotic. It costs \$10 to store one bottle for one year and \$40 to place an order. How many times during the year should the pharmacy order the antibiotic in order to minimize the total storage and reorder costs?

35. A 300-room hotel in Las Vegas is filled to capacity every night at \$80 a room. For each \$1 increase in rent, 3 fewer rooms are rented. If each rented room costs \$10 to service per day, how much should be the management charge for each room to maximize gross profit? What is the maximum gross profit?

36. A multimedia company anticipates that there will be a demand for 20,000 copies of a certain DVD during the next year. It costs the company \$0.50 to store a DVD for one year. Each time it must make additional DVDs, it costs \$200 to set up the equipment. How many DVDs should the company make during each production run to minimize its total storage and setup costs?

37. A university student center sells 1,600 cups of coffee per day at a price of \$2.40.

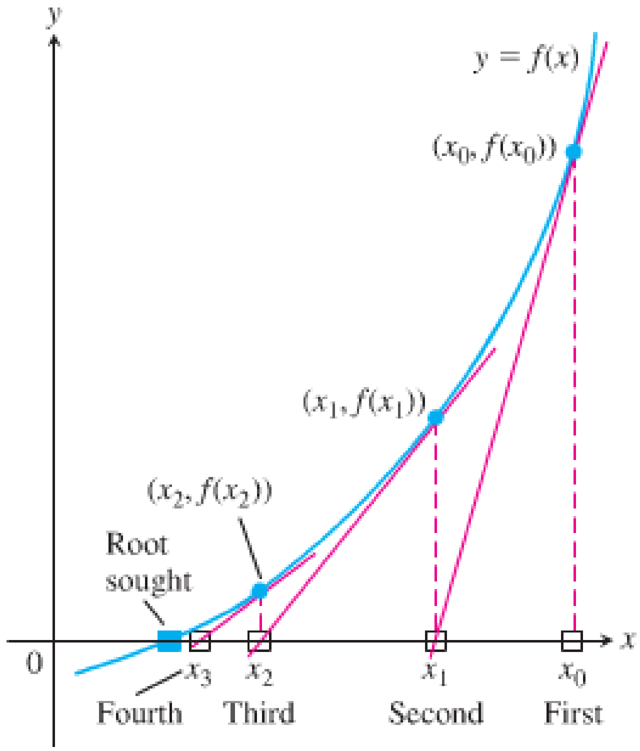
a) A market survey shows that for every \$0.05 reduction in price, 50 more cups of coffee will be sold. How much should be the student center charge for a cup of coffee in order to maximize revenue?

b) A different market survey shows that for every \$0.10 reduction in the original \$2.40 price, 60 more cups of coffee will be sold. Now how much should the student center charge for a cup of coffee in order to maximize revenue?

Section 3.6 – Newton's Method

Procedure for *Newton's Method*

The goal of Newton's method, also called the ***Newton-Raphson*** method, for estimating a solution of an equation $f(x)=0$ is to produce a sequence of approximations that approach the solution.



We begin with the first number x_0 of the sequence. Then the function is approximated by its tangent line, and one computes the *x*-intercept of this tangent line. At each step the method approximates a zero of *f* with a zero of one of its linearization.

> Initial estimates, x_0, the method than uses the tangent curve $y=f(x)$ @ $(x_0, f(x_0))$ to approximate the curve, calling the point x_1 where the tangent meets the *x*-axis. The number x_1 usually a better approximation to the solution that is x_0. The point x_2 where the tangent to the curve at $(x_1, f(x_1))$ crosses the *x*-axis is the next approximation in the sequence. We continue on using each approximation to generate the next, until we are close enough to the root to stop.

The point-slope equation for the tangent to the curve at $(x_n, f(x_n))$ is

$$y = f(x_n) + f'(x_n)\cdot(x - x_n)$$

We can find where it crosses the *x*-axis by setting $y = 0$.

$$0 = f\left(x_n\right) + f'\left(x_n\right)\cdot\left(x - x_n\right) \quad \Rightarrow \quad x - x_n = -\frac{f\left(x_n\right)}{f'\left(x_n\right)}$$

$$\boxed{x = x_n - \frac{f\left(x_n\right)}{f'\left(x_n\right)}}$$

Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f\left(x_n\right)}{f'\left(x_n\right)} \quad if\ f'\left(x_n\right) \neq 0$$

Example

Find the positive root of the equation $f(x) = x^2 - 2 = 0$

Solution

$f'(x) = 2x$

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n^2 - 2}{2x_n} \\ &= x_n - \frac{x_n}{2} + \frac{1}{x_n} \\ &= \frac{x_n}{2} + \frac{1}{x_n} \end{aligned}$$

Example

Find the x-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

Solution

$$x^3 - x = 1$$

$$x^3 - x - 1 = 0$$

$$f(x) = x^3 - x - 1$$

$$\begin{cases} f(1) = -1 \\ f(2) = 5 \end{cases}$$

$$f'(x) = 3x^2 - 1$$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	1	–1	2	1.5
1	1.5	0.875	5.75	1.347826087
2	1.347826087	0.100682173	4.449905482	1.325200399
3	1.325200399	0.002058362	4.268468292	1.324718174
4	1.324718174	0.000000924	4.264634722	1.324717957
5	1.324717957	–1.8672 E–13	4.264632999	1.324717957

The result: $\boxed{x = 1.324717957}$

Exercises *Section* 3.6 – Newton's Method

1. Use Newton's method to estimate the on real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find x_2

2. Use Newton's method to estimate the on real solution of $x^4 + x - 3 = 0$. Start with $x_0 = -1$ for the left-hand zero and with $x_0 = 1$ for the zero on the right. Then, in each case, find x_2

3. Use Newton's method to estimate the on real solution of $2x - x^2 + 1 = 0$. Start with $x_0 = 0$ for the left-hand zero and with $x_0 = 2$ for the zero on the right. Then, in each case, find x_2

4. Use Newton's method to estimate the on real solution of $x^4 - 2 = 0$. Start with $x_0 = 1$ and then find x_2