Chapter 6. Numerical Methods

Section 6.1. Euler's Method

1. We have $t_0 = 0$, $y_0 = 1$ and $f(t, y) = t + y$. Thus, the first step of Euler's method is completed as follows.

$$y_1 = y_0 + (t_0 + y_0)\,h = 1 + (0 + 1) \times 0.1 = 1.1$$
$$t_1 = t_0 + h = 0 + 0.1 = 0.1.$$

The second step follows.

$$y_2 = y_1 + (t_1 + y_1)\,h = 1.1 + (0.1 + 1.1) \times 0.1 = 1.22$$
$$t_2 = t_1 + h = 0.1 + 0.1 = 0.2.$$

Continuing in this manner produces the results in the following table.

k	t_k	y_k	$f(t_k, y_k) = y_k$	h	$f(t_k, y_k)h$
0	0.0	1.0000	1.0000	0.1	0.1000
1	0.1	1.1000	1.2000	0.1	0.1200
2	0.2	1.2200	1.4200	0.1	0.1420
3	0.3	1.3620	1.6620	0.1	0.1662
4	0.4	1.5282	1.9282	0.1	0.1928
5	0.5	1.7210	2.2210	0.1	0.2221

2. We have $t_0 = 0$, $y_0 = 1$ and $f(t, y) = y$. Thus, the first step of Euler's method is completed as follows.

$$y_1 = y_0 + y_0\,h = 1 + 1 \times 0.1 = 1.1$$
$$t_1 = t_0 + h = 0 + 0.1 = 0.1.$$

The second step follows.

$$y_2 = y_1 + y_1\,h = 1.1 + 1.1 \times 0.1 = 1.21$$
$$t_2 = t_1 + h = 0.1 + 0.1 = 0.2.$$

Continuing in this manner produces the results in the following table.

k	t_k	y_k	$f(t_k, y_k) = y_k$	h	$f(t_k, y_k)h$
0	0.0	1.0000	1.0000	0.1	0.1000
1	0.1	1.1000	1.1000	0.1	0.1100
2	0.2	1.2100	1.2100	0.1	0.1210
3	0.3	1.3310	1.3310	0.1	0.1331
4	0.4	1.4641	1.4641	0.1	0.1464
5	0.5	1.6105	1.6105	0.1	0.1611

3. We have $t_0 = 0$, $y_0 = 1$ and $f(t, y) = ty$. Thus, the first step of Euler's method is completed as follows.

$$y_1 = y_0 + t_0 y_0 h = 1 + 0 \times 1 \times 0.1 = 1.0$$
$$t_1 = t_0 + h = 0 + 0.1 = 0.1.$$

The second step follows.

$$y_2 = y_1 + t_1 y_1 h = 1.0 + 0.1 \times 1 \times 0.1 = 1.01$$
$$t_2 = t_1 + h = 0.1 + 0.1 = 0.2.$$

Continuing in this manner produces the results in the following table.

k	t_k	y_k	$f(t_k, y_k) = t_k y_k$	h	$f(t_k, y_k)h$
0	0.0	1.0000	0.0000	0.1	0.0000
1	0.1	1.0000	0.1000	0.1	0.0100
2	0.2	1.0100	0.2020	0.1	0.0202
3	0.3	1.0302	0.3091	0.1	0.0309
4	0.4	1.0611	0.4244	0.1	0.0424
5	0.5	1.1036	0.5518	0.1	0.0552

4. We have $x_0 = 0$, $z_0 = 0$ and $f(x, z) = 5 - z$. Thus, the first step of Euler's method is completed as follows.

$$z_1 = z_0 + (5 - z_0)\,h = 0 + (5 - 0) \times 0.1 = 0.5$$
$$x_1 = x_0 + h = 0 + 0.1 = 0.1.$$

The second step follows.

$$z_2 = z_1 + (5 - z_1)\,h = 0.5 + (5 - 0.5) \times 1 \times 0.1 = 0.95$$
$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2.$$

Continuing in this manner produces the results in the following table.

k	x_k	z_k	$f(x_k, z_k) = 5 - z_k$	h	$f(x_k, z_k)h$
0	0.0	0.0000	5.0000	0.1	0.5000
1	0.1	0.5000	4.5000	0.1	0.4500
2	0.2	0.9500	4.0500	0.1	0.4050
3	0.3	1.3550	3.6450	0.1	0.3645
4	0.4	1.7195	3.2805	0.1	0.3281
5	0.5	2.0476	2.9524	0.1	0.2952

5. We have $x_0 = 0$, $z_0 = 1$ and $f(x, z) = x - 2z$. Thus, the first step of Euler's method is completed as follows.

$$z_1 = z_0 + (x_0 - 2z_0)\,h = 1 + (0 - 2 \times 1) \times 0.1 = 0.8$$
$$x_1 = x_0 + h = 0 + 0.1 = 0.1.$$

The second step follows.

$$z_2 = z_1 + (x_1 - 2z_1)\,h = 0.8 + (0.1 - 2 \times 0.8) \times 0.1 = 0.65$$
$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2.$$

Continuing in this manner produces the results in the following table.

k	x_k	z_k	$f(x_k, z_k) = x_k - 2z_k$	h	$f(x_k, z_k)h$
0	0.0	1.0000	-2.0000	0.1	-0.2000
1	0.1	0.8000	-1.5000	0.1	-0.1500
2	0.2	0.6500	-1.1000	0.1	-0.1100
3	0.3	0.5400	-0.7800	0.1	-0.0780
4	0.4	0.4620	-0.5240	0.1	-0.0524
5	0.5	0.4096	-0.3192	0.1	-0.0319

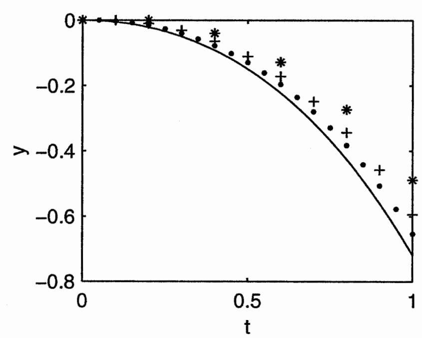
6. An integrating factor for $y' = y - t$ is e^{-t}.

$$\begin{aligned} y' - y &= -t \\ e^{-t}y' - e^{-t}y &= -te^{-t} \\ (e^{-t}y)' &= -te^{-t} \end{aligned}$$

Integrating by parts,

$$\begin{aligned} e^{-t}y &= te^{-t} + e^{-t} + C \\ y &= t + 1 + Ce^{t}. \end{aligned}$$

The initial condition $y(0) = 0$ produces $C = -1$ and $y = t + 1 - e^t$. In the figure, three numerical solutions and the exact solution are pictured. The numerical solutions were calculated using Euler's method and step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$.



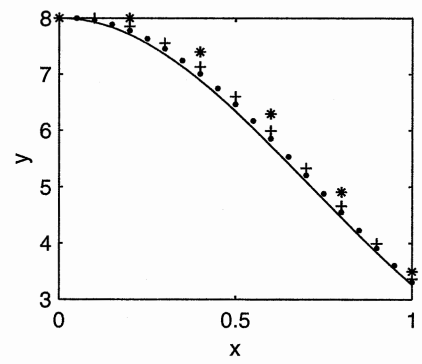
7. An integrating factor for $y' + 2xy = x$ is e^{x^2}.

$$\begin{aligned} e^{x^2}y' + 2xe^{x^2}y &= xe^{x^2} \\ (e^{x^2}y)' &= xe^{x^2} \end{aligned}$$

Integrate.

$$\begin{aligned} e^{x^2}y &= \frac{1}{2}e^{x^2} + C \\ y &= \frac{1}{2} + Ce^{-x^2} \end{aligned}$$

The initial condition $y(0) = 8$ produces $C = 15/2$ and $y = 1/2 + (15/2)e^{-x^2}$. In the figure, three numerical solutions and the exact solution are pictured. The numerical solutions were calculated using Euler's method and step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$.



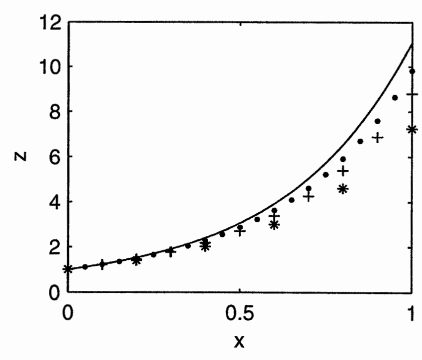
8. An integrating factor for $z - 2z = xe^{2x}$ is e^{-2x}.

$$e^{-2x}z' - 2e^{-2x}z = x$$
$$(e^{-2x}z)' = x$$

Integrate.

$$e^{-2x}z = \frac{1}{2}x^2 + c$$
$$z = \frac{1}{2}x^2e^{2x} + Ce^{2x}$$

The initial condition $z(0) = 1$ produces $C = 1$ and $z = (1/2)x^2e^{2x} + e^{2x}$. In the figure, three numerical solutions and the exact solution are pictured. The numerical solutions were calculated using Euler's method and step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$.



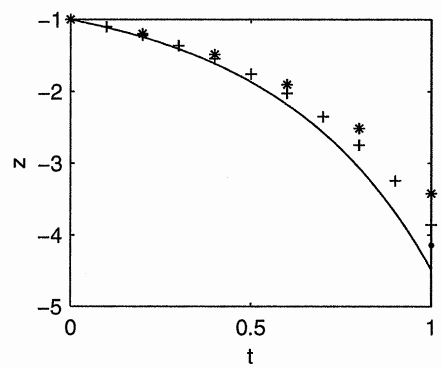
9. The equation $z' = (1+t)z$ is separable.

$$\frac{dz}{z} = (1+t)\,dt$$
$$\ln|z| = t + \frac{1}{2}t^2 + C$$

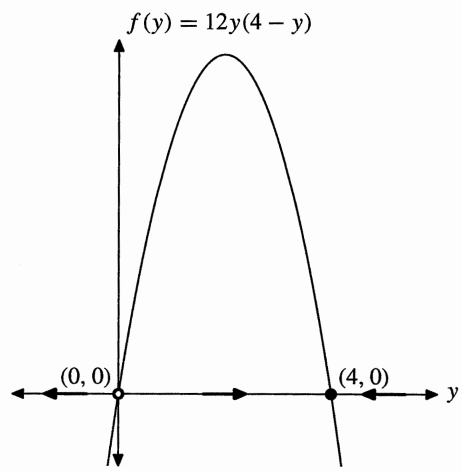
The initial condition $z(0) = -1$ produces $C = 0$ and $\ln|z| = t + (1/2)t^2$. Of course, we choose the negative branch.

$$|z| = e^{t+(1/2)t^2}$$
$$z = -e^{t+(1/2)t^2}$$

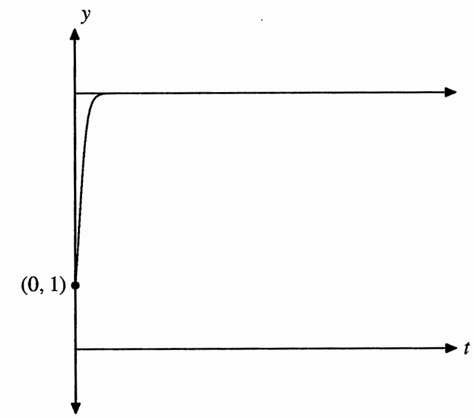
In the figure, three numerical solutions and the exact solution are pictured. The numerical solutions were calculated using Euler's method and step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$.



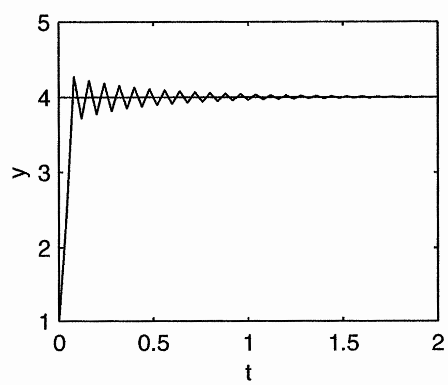
10. (a) Phase plane analysis shows a stable equilibrium point at (4, 0).



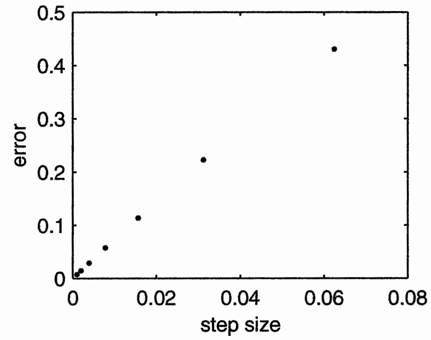
Thus, a solution starting at (0, 1) should approach the stable solution $y = 4$.



(b) The following figure displays a numerical solution of $y' = 12y(4 - y)$, $y(0) = 1$, using Euler's method with step size $h = 0.04$. Note the obvious contradiction of uniqueness as the numerical solution repeatedly crosses the stable solution $y = 4$.



11. (a) Note that a plot of the error versus the step size signifies a linear relationship. Indeed, a line through the origin with the appropriate slope should pass through or close to each data point.



(b) We can estimate the proportionality constant by picking two points from the figure and calculating the slope of the line through the chosen points. Let's use the first and last points.

$$\begin{aligned}\lambda &= \frac{0.0072076631 - 0.4303893417}{0.0009765625 - 0.0625000000}\\ &\approx 6.8784\end{aligned}$$

We can use $E = \lambda h$ to calculate the step size.

$$\begin{aligned}E &= \lambda h\\ h &= \frac{E}{\lambda}\\ h &= \frac{0.001}{6.8784}\\ h &= 1.454 \times 10^{-4}\end{aligned}$$

Since $h = (b - a)/N$,

$$\begin{aligned}N &= \frac{b-a}{h}\\ N &\frac{2-0}{1.454 \times 10^{-4}}\\ N &= 13757.\end{aligned}$$

It will take about 13,757 iterations to achieve the required accuracy.

(c) We ran our Euler routine on a 300 MHz PC using the step size from part (b). The run took approximately 56 seconds and reported an error from the true value of 0.00107417614304.

12. A integrating factor for $y' = t - y$ is e^t. Thus,

$$\begin{aligned}e^t y' + e^t y &= t\\ (e^t y)' &= te^t.\end{aligned}$$

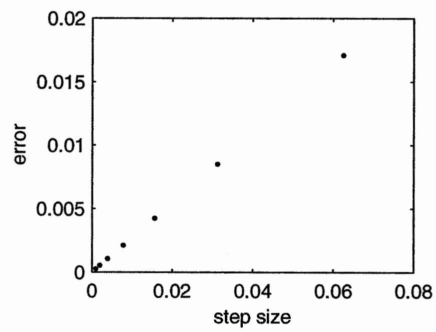
Integrate, using integration by parts.

$$\begin{aligned}e^t y &= te^t - e^t + C\\ y &= t - 1 + Ce^{-t}\end{aligned}$$

The initial condition $y(0) = 1$ provides $C = 2$ and the exact solution $y = t - 1 + 2e^{-t}$. Thus, $y(2) \approx 1.27067$. This value, plus our Euler routine, generated the data in the following table.

Step size h	Euler approx.	True value	Error E_h
0.06250	1.25358	1.27067	0.01709
0.03125	1.26217	1.27067	0.00850
0.01563	1.26643	1.27067	0.00424
0.00781	1.26855	1.27067	0.00212
0.00391	1.26961	1.27067	0.00106
0.00195	1.27014	1.27067	0.00053
0.00098	1.27041	1.27067	0.00026

A plot of the error versus the step size reveals a linear relationship.



Choose two points and use the slope formula to estimate λ.

$$\lambda \approx \frac{0.00026 - 0.01709}{0.00098 - 0.06250}$$
$$\lambda \approx 0.27357$$

Calculate the step size required to maintain error less than 0.01 with the calculation

$$h = \frac{E}{\lambda}$$
$$h \approx \frac{0.01}{0.27357}$$
$$h \approx 0.03655.$$

Calculate the number of iterations with

$$N = \frac{b-a}{h}$$
$$N \approx \frac{2-0}{0.03655}$$
$$N \approx 55.$$

Using a step size $h \approx 0.03655$, our Euler routine on a 300 MHz PC produced an error of 0.00991658799458 in about 0.05 seconds.

13. A integrating factor for $y' = y + \sin t$ is e^{-t}. Thus,

$$\begin{aligned} e^{-t}y' - e^{-t}y &= e^{-t}\sin t \\ (e^{-t}y)' &= e^{-t}\sin t. \end{aligned}$$

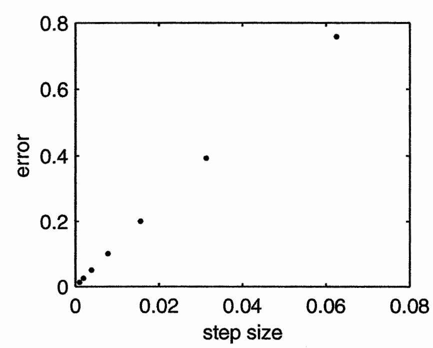
Integrate, using integration by parts.

$$\begin{aligned} e^{-t}y &= -\frac{1}{2}e^{-t}\sin t - \frac{1}{2}e^{-t}\cos t + C \\ y &= -\frac{1}{2}\sin t - \frac{1}{2}\cos t + Ce^{t} \end{aligned}$$

The initial condition $y(0) = 1$ provides $C = 3/2$ and the exact solution $y = -(1/2)\sin t - (1/2)\cos t + (3/2)e^t$. Thus, $y(2) \approx 10.83701$. This value, plus our Euler routine, generated the data in the following table.

Step size h	Euler approx.	True value	Error E_h
0.06250	10.07789	10.83701	0.75912
0.03125	10.44404	10.83701	0.39297
0.01563	10.63700	10.83701	0.20001
0.00781	10.73610	10.83701	0.10091
0.00391	10.78632	10.83701	0.05069
0.00195	10.81161	10.83701	0.02540
0.00098	10.82429	10.83701	0.01271

A plot of the error versus the step size reveals a linear relationship.



Choose two points and use the slope formula to estimate λ.

$$\begin{aligned} \lambda &\approx \frac{0.01271 - 0.75912}{0.00098 - 0.06250} \\ \lambda &\approx 12.1321 \end{aligned}$$

Calculate the step size required to maintain error less than 0.01 with the calculation

$$h = \frac{E}{\lambda}$$
$$h \approx \frac{0.01}{12.1321}$$
$$h \approx 8.2426 \times 10^{-4}.$$

Calculate the number of iterations with

$$N = \frac{b-a}{h}$$
$$N \approx \frac{2-0}{8.2426 \times 10^{-4}}$$
$$N \approx 2426.$$

Using a step size $h \approx 8.2426 \times 10^{-4}$, our Euler routine running on a 300 MHz PC produced an error of 0.01073284032773 in about 3.95 seconds.

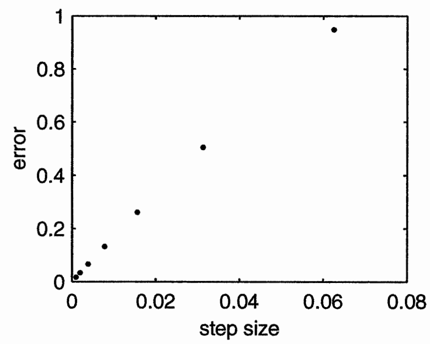
14. The equation $y' = ty$ is separable. Thus,

$$\frac{dy}{y} = t\,dt$$
$$\ln y = \frac{1}{2}t^2 + C.$$

The initial condition $y(0) = 1$ provides $C = 0$ and the exact solution $y = e^{(1/2)t^2}$. Thus, $y(2) \approx 7.38906$. This value, plus our Euler routine, generated the data in the following table.

Step size h	Euler approx.	True value	Error E_h
0.06250	6.44037	7.38906	0.94868
0.03125	6.88449	7.38906	0.50456
0.01563	7.12849	7.38906	0.26056
0.00781	7.25660	7.38906	0.13245
0.00391	7.32227	7.38906	0.06678
0.00195	7.35552	7.38906	0.03353
0.00098	7.37225	7.38906	0.01680

A plot of the error versus the step size reveals a linear relationship.



Choose two points and use the slope formula to estimate λ.

$$\lambda \approx \frac{0.01680 - 0.94868}{0.00098 - 0.06250}$$
$$\lambda \approx 15.1467$$

Calculate the step size required to maintain error less than 0.01 with the calculation

$$h = \frac{E}{\lambda}$$
$$h \approx \frac{0.01}{15.1467}$$
$$h \approx 6.60207 \times 10^{-4}.$$

Calculate the number of iterations with

$$N = \frac{b-a}{h}$$
$$N \approx \frac{2-0}{6.60207 \times 10^{-4}}$$
$$N \approx 3029.$$

Using a step size $h \approx 6.60207 \times 10^{-4}$, our Euler routine running on a 300 MHz PC produced an error of 0.01136462652632 in about 5.39 seconds.

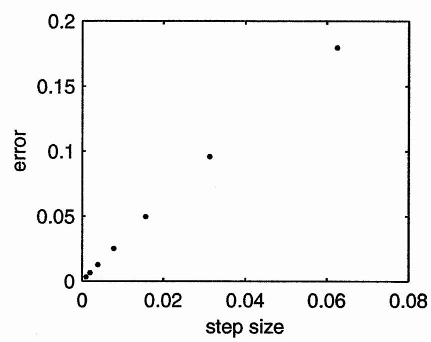
15. The equation $y' = -(1/3)y^2$ is separable. Thus,

$$\frac{dy}{y^2} = -\frac{1}{3}\,dt$$
$$-y^{-1} = -\frac{1}{3}t + C$$
$$y = \frac{1}{(1/3)t + C}$$
$$y = \frac{3}{t + 3C}.$$

The initial condition $y(0) = 1$ provides $C = -1$ and the exact solution $y = 3/(t-3)$. Thus, $y(2) = -3$. This value, plus our Euler routine, generated the data in the following table.

Step size h	Euler approx.	True value	Error E_h
0.06250	-2.82040	-3.00000	0.17960
0.03125	-2.90412	-3.00000	0.09588
0.01563	-2.95035	-3.00000	0.04965
0.00781	-2.97472	-3.00000	0.02528
0.00391	-2.98725	-3.00000	0.01275
0.00195	-2.99359	-3.00000	0.00641
0.00098	-2.99679	-3.00000	0.00321

A plot of the error versus the step size reveals a linear relationship.



Choose two points and use the slope formula to estimate λ.

$$\lambda \approx \frac{0.00321 - 0.17960}{0.00098 - 0.06250}$$
$$\lambda \approx 2.8670$$

Calculate the step size required to maintain error less than 0.01 with the calculation

$$h = \frac{E}{\lambda}$$
$$h \approx \frac{0.01}{2.8670}$$
$$h \approx 0.003487.$$

Calculate the number of iterations with

$$N = \frac{b-a}{h}$$
$$N \approx \frac{2-0}{0.003487}$$
$$N \approx 573.$$

Using a step size $h \approx 0.003487$, our Euler routine on a 300 MHz PC produced an error of 0.01139175100243 in about 0.66 seconds.

16. Because $E_h \approx \lambda h$, halving the step size should halve the error. A simple calculation will show this.

$$E \approx \lambda\left(\frac{1}{2}h\right) \approx \frac{1}{2}(\lambda h) \approx \frac{1}{2}E_h$$

Because the number of iterations with step size h is $N_h = (b-a)/h$, halving the step size should double the number of iterations. A simple calculation shows this.

$$N \approx \frac{b-a}{\frac{1}{2}h} \approx 2\left(\frac{b-a}{h}\right) \approx 2N_h$$

17. We have $t_0 = 0$, $x_0 = 1$, and $y_0 = 0$. We also have $f(t,x,y) = y$ and $g(t,x,y) = -x$. The first step of Euler's method is completed as follows.

$$\begin{aligned} x_1 &= x_0 + y_0 h = 1 + 0 \times 0.1 = 1 \\ y_1 &= y_0 - x_0 h = 0 - 1 \times 0.1 = -0.1 \\ t_1 &= t_0 + h = 0 + 0.1 = 0.1 \end{aligned}$$

We iterate a second time as follows.

$$\begin{aligned} x_2 &= x_1 + y_1 h = 1 - 1 \times 0.1 = 0.99 \\ y_2 &= y_1 - x_1 h = -0.1 - 1 \times 0.1 = -0.2 \\ t_2 &= t_1 + h = 0.1 + 0.1 = 0.2 \end{aligned}$$

Continuing in this manner produces the results in the following table.

t_k	x_k	y_k	$f(t_k, x_k, y_k)h = y_k h$	$g(t_k, x_k, y_k)h = -x_k h$
0.0	1.0000	0.0000	0.0000	-0.1000
0.1	1.0000	-0.1000	-0.0100	-0.1000
0.2	0.9900	-0.2000	-0.0200	-0.0990
0.3	0.9700	-0.2990	-0.0299	-0.0970
0.4	0.9401	-0.3960	-0.0396	-0.0940
0.5	0.9005	-0.4900	-0.0490	-0.0901

18. We have $t_0 = 0$, $x_0 = 1$, and $y_0 = -1$. We also have $f(t,x,y) = y$ and $g(t,x,y) = -x$. The first step of Euler's method is completed as follows.

$$\begin{aligned} x_1 &= x_0 + y_0 h = 1 - 1 \times 0.1 = 0.9 \\ y_1 &= y_0 - x_0 h = -1 - 1 \times 0.1 = -1.1 \\ t_1 &= t_0 + h = 0 + 0.1 = 0.1 \end{aligned}$$

We iterate a second time as follows.

$$\begin{aligned} x_2 &= x_1 + y_1 h = 0.9 - 1.1 \times 0.1 = 0.79 \\ y_2 &= y_1 - x_1 h = -1.1 - 0.9 \times 0.1 = -1.19 \\ t_2 &= t_1 + h = 0.1 + 0.1 = 0.2 \end{aligned}$$

Continuing in this manner produces the results in the following table.

t_k	x_k	y_k	$f(t_k, x_k, y_k)h = y_k h$	$g(t_k, x_k, y_k)h = -x_k h$
0.0	1.0000	-1.0000	-0.1000	-0.1000
0.1	0.9000	-1.1000	-0.1100	-0.0900
0.2	0.7900	-1.1900	-0.1190	-0.0790
0.3	0.6710	-1.2690	-0.1269	-0.0671
0.4	0.5441	-1.3361	-0.1336	-0.0544
0.5	0.4105	-1.3905	-0.1391	-0.0410

19. We have $t_0 = 0$, $x_0 = 0$, and $y_0 = -1$. We also have $f(t, x, y) = -2y$ and $g(t, x, y) = x$. The first step of Euler's method is completed as follows.

$$\begin{aligned} x_1 &= x_0 - 2y_0 h = 0 - 2 \times -1 \times 0.1 = 0.2 \\ y_1 &= y_0 + x_0 h = -1 + 0 \times 0.1 = -1 \\ t_1 &= t_0 + h = 0 + 0.1 = 0.1 \end{aligned}$$

We iterate a second time as follows.

$$\begin{aligned} x_2 &= x_1 - 2y_1 h = 0.2 - 2 \times -1 \times 0.1 = 0.4 \\ y_2 &= y_1 + x_1 h = -1 + 0.2 \times 0.1 = -0.98 \\ t_2 &= t_1 + h = 0.1 + 0.1 = 0.2 \end{aligned}$$

Continuing in this manner produces the results in the following table.

t_k	x_k	y_k	$f(t_k, x_k, y_k)h = -2y_k h$	$g(t_k, x_k, y_k)h = x_k h$
0.0	0.0000	-1.0000	-0.1000	0.0000
0.1	0.2000	-1.0000	-0.1000	-0.0200
0.2	0.4000	-0.9800	-0.0980	-0.0400
0.3	0.5960	-0.9400	-0.0940	-0.0596
0.4	0.7840	-0.8804	-0.0880	-0.0784
0.5	0.9601	-0.8020	-0.0802	-0.0960

20. We have $t_0 = 0$, $x_0 = 1$, and $y_0 = 1$. We also have $f(t, x, y) = x + y$ and $g(t, x, y) = x - y$. The first step of Euler's method is completed as follows.

$$\begin{aligned} x_1 &= x_0 + (x_0 + y_0)h = 1 + (1 + 1) \times 0.1 = 1.2 \\ y_1 &= y_0 + (x_0 - y_0)h = 1 + (1 - 1) \times 0.1 = 1 \\ t_1 &= t_0 + h = 0 + 0.1 = 0.1 \end{aligned}$$

We iterate a second time as follows.

$$\begin{aligned} x_2 &= x_1 + (x_1 + y_1)h = 1.2 + (1.2 + 1) \times 0.1 = 1.42 \\ y_2 &= y_1 + (x_1 - y_1)h = 1 + (1.2 - 1) \times 0.1 = 1.02 \\ t_2 &= t_1 + h = 0.1 + 0.1 = 0.2 \end{aligned}$$

Continuing in this manner produces the results in the following table.

t_k	x_k	y_k	$f(t_k, x_k, y_k)h = (x_k + y_k)h$	$g(t_k, x_k, y_k)h = (x_k - y_k)h$
0.0	1.0000	1.0000	0.2000	0.0000
0.1	1.2000	1.0000	0.2200	0.0200
0.2	1.4200	1.0200	0.2440	0.0400
0.3	1.6640	1.0600	0.2724	0.0604
0.4	1.9364	1.1204	0.3057	0.0816
0.5	2.2421	1.2020	0.3444	0.1040

21. We have $t_0 = 0$, $x_0 = 1$, and $y_0 = -1$. We also have $f(t, x, y) = -y$ and $g(t, x, y) = x + y$. The first step of Euler's method is completed as follows.

$$x_1 = x_0 - y_0 h = 1 + 1 \times 0.1 = 1.1$$
$$y_1 = y_0 + (x_0 + y_0)h = -1 + (1 - 1) \times 0.1 = -1$$
$$t_1 = t_0 + h = 0 + 0.1 = 0.1$$

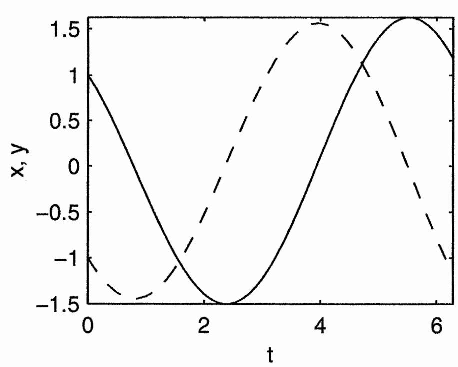
We iterate a second time as follows.

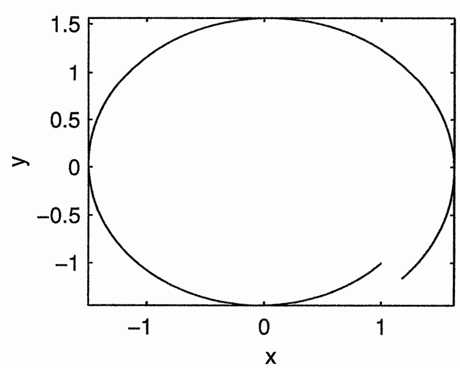
$$x_2 = x_1 - y_1 h = 1.1 + 1 \times 0.1 = 1.2$$
$$y_2 = y_1 + (x_1 + y_1)h = -1 + (1.1 - 1) \times 0.1 = -0.99$$
$$t_2 = t_1 + h = 0.1 + 0.1 = 0.2$$

Continuing in this manner produces the results in the following table.

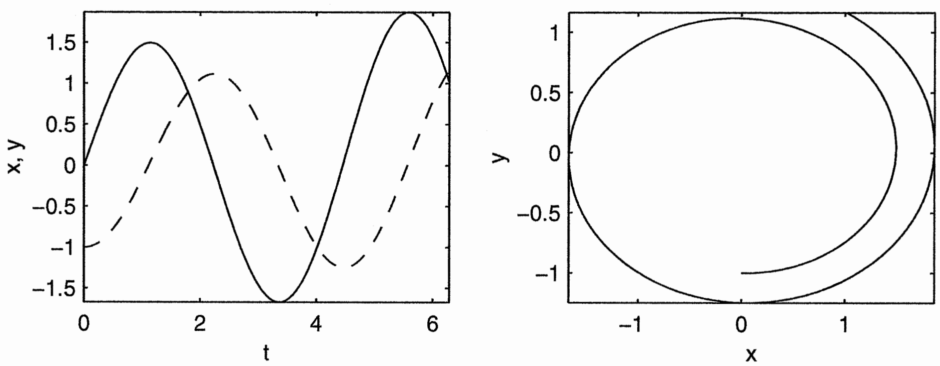
t_k	x_k	y_k	$f(t_k, x_k, y_k)h = -y_k h$	$g(t_k, x_k, y_k)h = (x_k + y_k)h$
0.0	1.0000	-1.0000	0.1000	0.0000
0.1	1.1000	-1.0000	0.1000	0.0100
0.2	1.2000	-0.9900	0.0990	0.0210
0.3	1.2990	-0.9690	0.0969	0.0330
0.4	1.3959	-0.9360	0.0936	0.0460
0.5	1.4895	-0.8900	0.0890	0.0599

22. Pictured below are two plots. The first shows the plots of x versus t (solid line) and y versus t (dashed line) on the time interval $[0, 2\pi]$. The second shows the plot of y versus x. The plots were generated with Euler's method, using a step size $h = 0.05$.

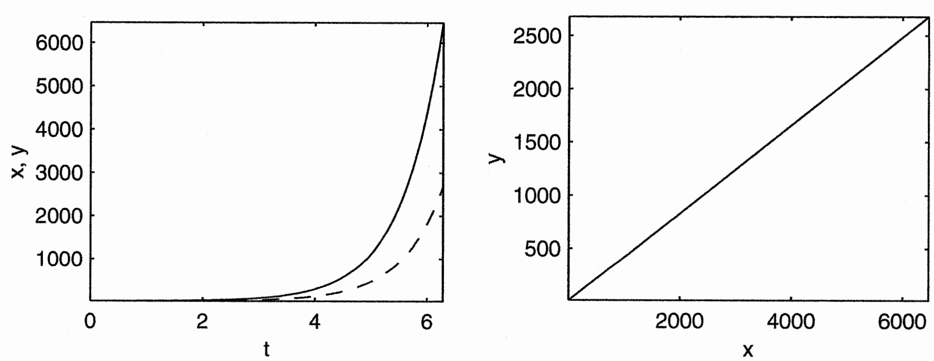




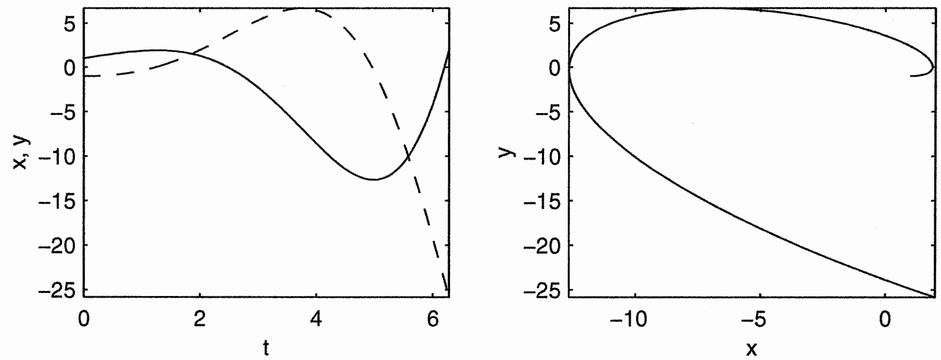
23. Pictured below are two plots. The first shows the plots of x versus t (solid line) and y versus t (dashed line) on the time interval $[0, 2\pi]$. The second shows the plot of y versus x. The plots were generated with Euler's method, using a step size $h = 0.05$.



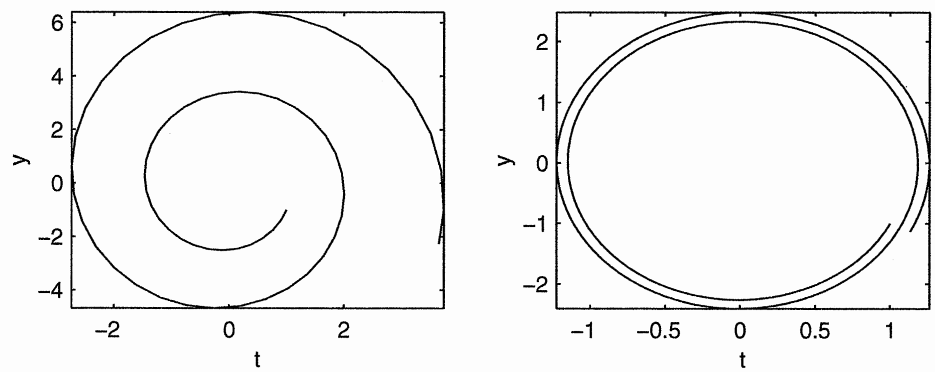
24. Pictured below are two plots. The first shows the plots of x versus t (solid line) and y versus t (dashed line) on the time interval $[0, 2\pi]$. The second shows the plot of y versus x. The plots were generated with Euler's method, using a step size $h = 0.05$.



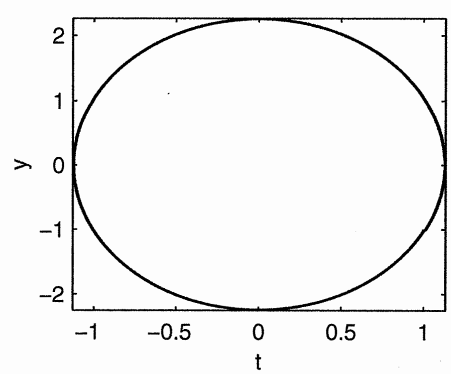
25. Pictured below are two plots. The first shows the plots of x versus t (solid line) and y versus t (dashed line) on the time interval $[0, 2\pi]$. The second shows the plot of y versus x. The plots were generated with Euler's method, using a step size $h = 0.05$.



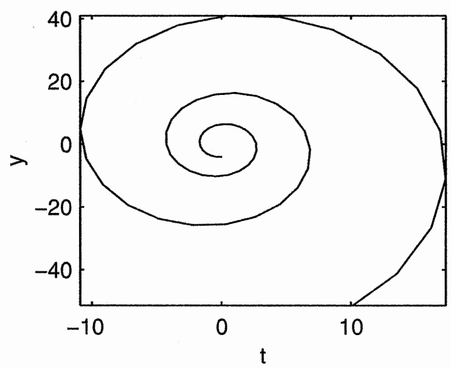
26. The first plot uses step size $h = 0.1$, the second uses $h = 0.01$

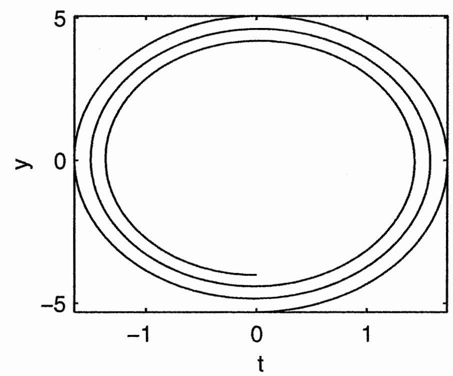


The third plot uses step size $h = 0.001$.

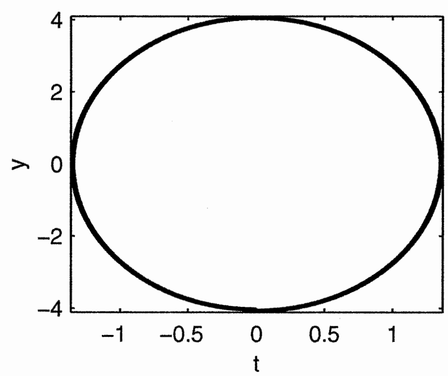


27. The first plot uses step size $h = 0.1$, the second uses $h = 0.01$

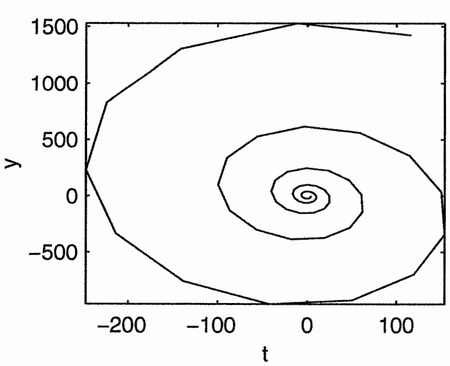


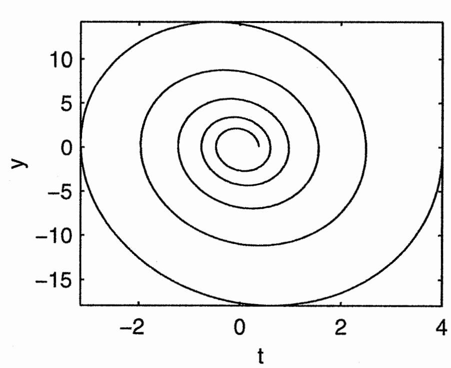


The third plot uses step size $h = 0.001$.

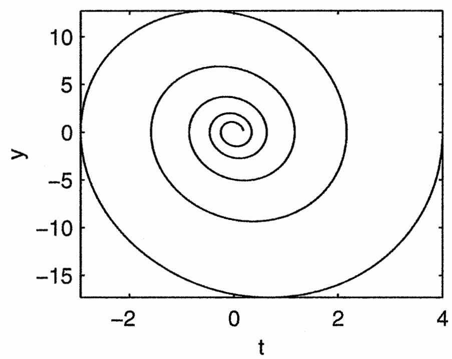


28. The first plot uses step size $h = 0.1$, the second uses $h = 0.01$

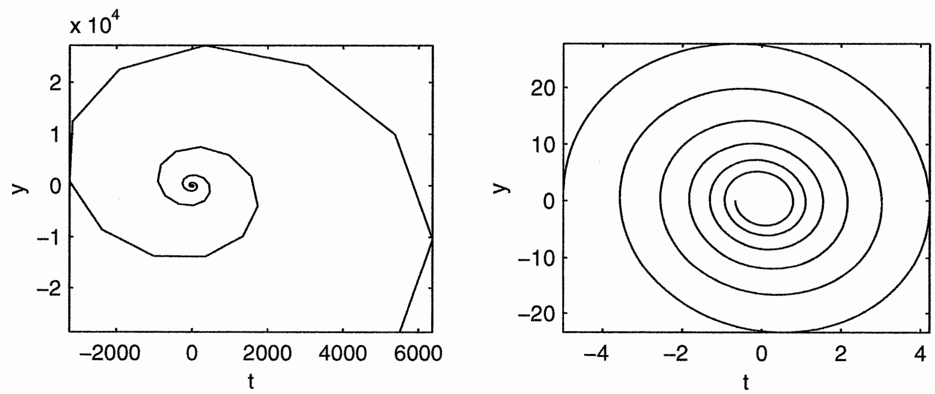




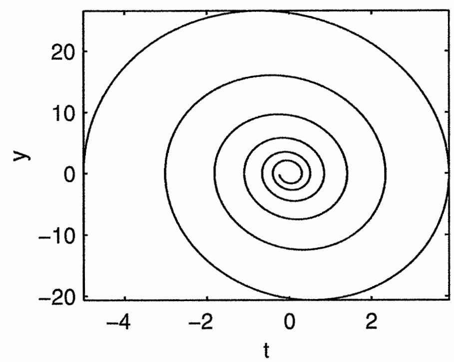
The third plot uses step size $h = 0.001$.



29. The first plot uses step size $h = 0.1$, the second uses $h = 0.01$



The third plot uses step size $h = 0.001$.



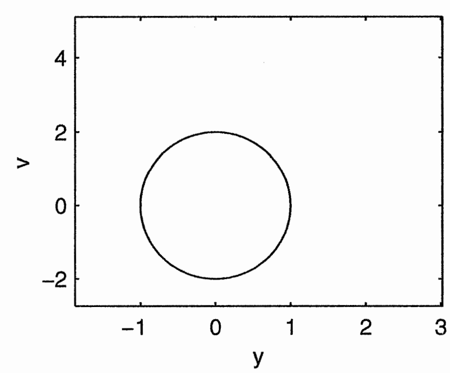
30. (a) Substitute $y(t) = \cos 2t$ and $v(t) = -2\sin 2t$ into the first equation, $y' = v$.

$$\begin{aligned} y' &= v \\ (\cos 2t)' &= -2\sin 2t \\ -2\sin 2t &= -2\sin 2t \end{aligned}$$

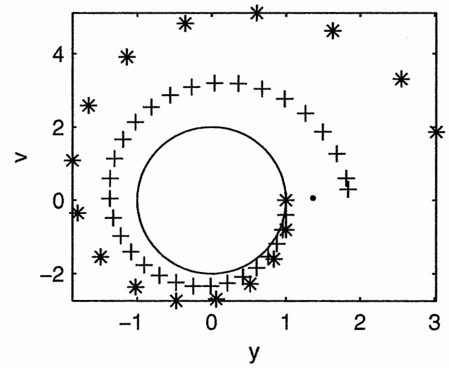
Thus, the first equation is satisfied. Substitute the proposed solutions into the second equation.

$$\begin{aligned} v' &= -4y \\ (-2\sin 2t)' &= -4\cos 2t \\ -4\cos 2t &= -4\cos 2t \end{aligned}$$

The second equation is also satisfied. Therefore, $y(t) = \cos 2t$, $v(t) = -2\sin 2t$ is a solution of the system. A plot of v versus y on the interval $[0, \pi]$ follows.



(b) In the following plot, we've superimposed Euler solutions on the exact solution from part (a). Note the improvement in the numerical solution as the step size decreases from $h = 0.2$ to $h = 0.1$ to $h = 0.05$.



Section 6.2. Runge-Kutta Methods

1. We start with initial condition $t_0 = 0$ and $y_0 = 1$. Also, $f(t, y) = t + y$. The first step of the Runge-Kutta 2 algorithm follows. First we compute the slopes.

$$s_1 = f(t_0, y_0) = f(0, 1) = 0 + 1 = 1$$
$$s_2 = f(t_0 + h, y_0 + hs_1) = f(0.1, 1.1) = 0.1 + 1.1 = 1.2$$

You can now update y and t.

$$y_1 = y_0 + h\frac{s_1 + s_2}{2} = 1 + 0.1\frac{1 + 1.2}{2} = 1.1$$
$$t_1 = t_0 + h = 0 + 0.1 = 0.1$$

The second iteration begins with computing the slopes.

$$s_1 = f(t_1, y_1) = f(0.1, 1.11) = 0.1 + 1.11 = 1.21$$
$$s_2 = f(t_1 + h, y_1 + hs_1) = f(0.2, 1.231) = 0.2 + 1.231 = 1.431$$

You can now update y and t.

$$y_2 = y_1 + h\frac{s_1 + s_2}{2} = 1.11 + 0.1\frac{1.21 + 1.431}{2} = 1.24205$$
$$t_2 = t_1 + h = 0.1 + 0.1 = 0.2$$

Continuing in this manner, we can complete the table.

k	t_k	y_k	s_1	s_2	h	$h(s_1+s_2)/2$
0	0.0	1.0000	1.0000	1.2000	0.1	0.1100
1	0.1	1.1100	1.2100	1.4310	0.1	0.1321
2	0.2	1.2421	1.4421	1.6863	0.1	0.1564
3	0.3	1.3985	1.6985	1.9683	0.1	0.1833
4	0.4	1.5818	1.9818	2.2800	0.1	0.2131
5	0.5	1.7949	2.2949	2.6244	0.1	0.2460

2. We begin with $t_0 = 0$, $y_0 = 1$ and $f(t, y) = y$. First, the slopes.

$$s_1 = f(t_0, y_0) = f(0, 1) = 1$$
$$s_2 = f(t_0 + h, y_0 + hs_1) = f(0.1, 1.1) = 1.1$$

Update t and y.

$$y_1 = y_0 + h\frac{s_1 + s_2}{2} = 1 + 0.1\frac{1 + 1.1}{2} = 1.105$$
$$t_1 = t_0 + h = 0 + 0.1 = 0.1$$

Continuing in this manner, we arrive at the following table.

k	t_k	y_k	s_1	s_2	h	$h(s_1+s_2)/2$
0	0.0	1.0000	1.0000	1.1000	0.1	0.1050
1	0.1	1.1050	1.1050	1.2155	0.1	0.1160
2	0.2	1.2210	1.2210	1.3431	0.1	0.1282
3	0.3	1.3492	1.3492	1.4842	0.1	0.1417
4	0.4	1.4909	1.4909	1.6400	0.1	0.1565
5	0.5	1.6474	1.6474	1.8122	0.1	0.1730

3. We begin with $t_0 = 0$, $y_0 = 1$ and $f(t, y) = ty$. First, the slopes.

$$s_1 = f(t_0, y_0) = f(0, 1) = 0 \times 1 = 0$$
$$s_2 = f(t_0 + h, y_0 + hs_1) = f(0.1, 1) = 0.1 \times 1 = 0.1$$

Update t and y.

$$y_1 = y_0 + h\frac{s_1 + s_2}{2} = 1 + 0.1\frac{0 + 0.1}{2} = 1.005$$
$$t_1 = t_0 + h = 0 + 0.1 = 0.1$$

Continuing in this manner, we arrive at the following table.

k	t_k	y_k	s_1	s_2	h	$h(s_1+s_2)/2$
0	0.0	1.0000	0.0000	0.1000	0.1	0.0050
1	0.1	1.0050	0.1005	0.2030	0.1	0.0152
2	0.2	1.0202	0.2040	0.3122	0.1	0.0258
3	0.3	1.0460	0.3138	0.4309	0.1	0.0372
4	0.4	1.0832	0.4333	0.5633	0.1	0.0498
5	0.5	1.1331	0.5665	0.7138	0.1	0.0640

4. We begin with $x_0 = 0$, $z_0 = 0$ and $f(x, z) = 5 - z$. First, the slopes.

$$s_1 = f(x_0, z_0) = f(0, 0) = 5 - 0 = 5$$
$$s_2 = f(x_0 + h, z_0 + hs_1) = f(0.1, 0.5) = 5 - 0.5 = 4.5$$

Update x and z.

$$z_1 = z_0 + h\frac{s_1 + s_2}{2} = 0 + 0.1\frac{5 + 4.5}{2} = 0.475$$
$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

Continuing in this manner, we arrive at the following table.

k	x_k	z_k	s_1	s_2	h	$h(s_1 + s_2)/2$
0	0.0	0.0000	5.0000	4.5000	0.1	0.4750
1	0.1	0.4750	4.5250	4.0725	0.1	0.4299
2	0.2	0.9049	4.0951	3.6856	0.1	0.3890
3	0.3	1.2939	3.7061	3.3355	0.1	0.3521
4	0.4	1.6460	3.3540	3.0186	0.1	0.3186
5	0.5	1.9646	3.0354	2.7318	0.1	0.2884

5. We begin with $x_0 = 0$, $z_0 = 1$ and $f(x, z) = x - 2z$. First, the slopes.

$$s_1 = f(x_0, z_0) = f(0, 1) = 0 - 2(1) = -2$$
$$s_2 = f(x_0 + h, z_0 + hs_1) = f(0.1, 0.8) = 0.1 - 2(0.8) = -1.5$$

Update x and z.

$$z_1 = z_0 + h\frac{s_1 + s_2}{2} = 1 + 0.1\frac{-2 - 1.5}{2} = 0.825$$
$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

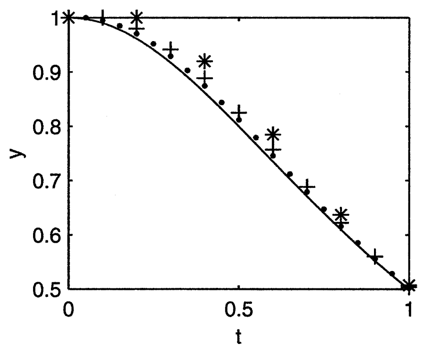
Continuing in this manner, we arrive at the following table.

k	x_k	z_k	s_1	s_2	h	$h(s_1 + s_2)/2$
0	0.0	1.0000	-2.0000	-1.5000	0.1	-0.1750
1	0.1	0.8250	-1.5500	-1.1400	0.1	-0.1345
2	0.2	0.6905	-1.1810	-0.8448	0.1	-0.1013
3	0.3	0.5892	-0.8784	-0.6027	0.1	-0.0741
4	0.4	0.5152	-0.6303	-0.4042	0.1	-0.0517
5	0.5	0.4634	-0.4268	-0.2415	0.1	-0.0334

6. The equation is separable.

$$\frac{dy}{y^2} = -2t\,dt$$
$$-y^{-1} = -t^2 + C$$
$$y = \frac{1}{t^2 + C}$$

The initial condition $y(0) = 1$ provides $C = -1$ and the exact solution $y = 1/(t^2 + 1)$. In the image that follows, the exact solution is plotted on the interval [0, 1]. Further, Euler's method is used to superimpose three additional numeric solutions, using step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$, respectively.



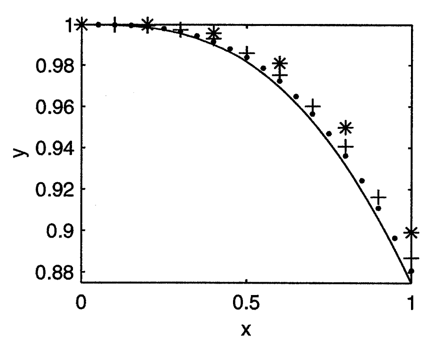
7. An integrating factor is e^x.

$$\begin{aligned} e^x z' + e^x z &= e^x \cos x \\ (e^z)' &= e^x \cos x \end{aligned}$$

Integration is by parts.

$$\begin{aligned} e^x z &= \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + C \\ z &= \frac{1}{2} \sin x + \frac{1}{2} \cos x + C e^{-x} \end{aligned}$$

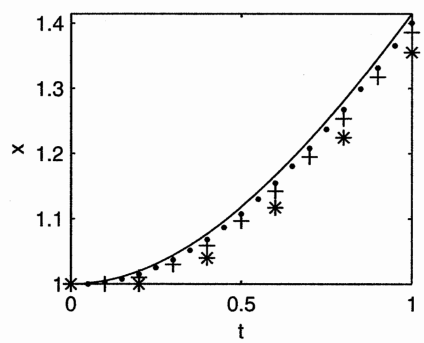
The initial condition $z(0) = 1$ provides $C = 1/2$ and the exact solution $z = (1/2)\sin x + (1/2)\cos x + (1/2)e^{-x}$. In the image that follows, the exact solution is plotted on the interval [0, 1]. Further, Euler's method is used to superimpose three additional numeric solutions, using step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$, respectively.



8. The equation is separable.

$$\begin{aligned} x\,dx &= t\,dt \\ \frac{1}{2}x^2 &= \frac{1}{2}t^2 + C \\ x^2 &= t^2 + 2C \end{aligned}$$

The initial condition $x(0) = 1$ provides $C = 1/2$ and the implicit solution $x^2 = t^2 + 1$. Of course, because we want $x(0) = 1$, we take the right-hand branch, $x = \sqrt{t^2 + 1}$. In the image that follows, the exact solution is plotted on the interval [0, 1]. Further, Euler's method is used to superimpose three additional numeric solutions, using step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$, respectively.



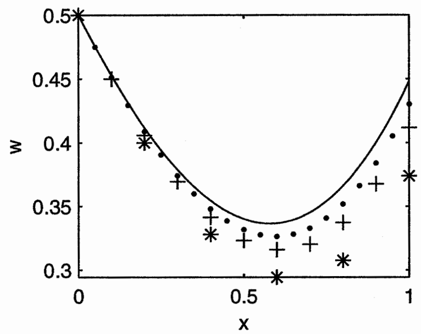
9. An integrating factor is e^x.

$$\begin{aligned} e^x w' + e^x w &= x^2 e^x \\ (e^x w)' &= x^2 e^x \end{aligned}$$

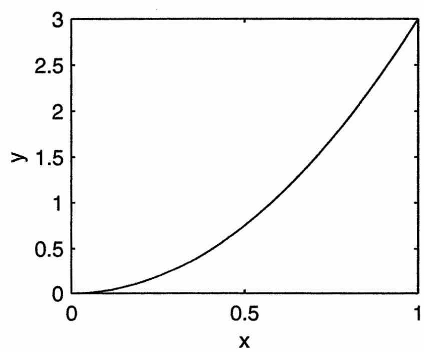
Integrate by parts.

$$\begin{aligned} e^x w &= x^2 e^x - 2xe^x + 2e^x + C \\ w &= x^2 - 2x + 2 + Ce^{-x} \end{aligned}$$

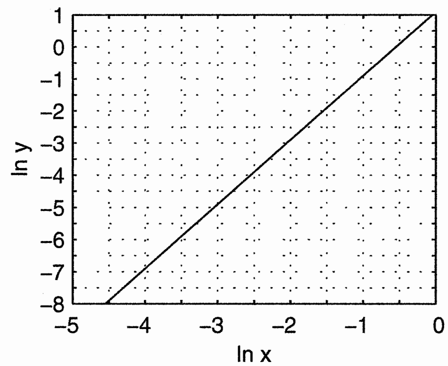
The initial condition $w(0) = 1/2$ provides $C = -3/2$ and the exact solution $w = x^2 - 2x + 2 - (3/2)e^{-x}$. In the image that follows, the exact solution is plotted on the interval [0, 1]. Further, Euler's method is used to superimpose three additional numeric solutions, using step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$, respectively.



10. The plot of the power function $y = 3x^2$ on [0, 1].



The difficulty most encounter is in circumventing the logarithm of zero. We actually made our plot on [0.01, 1] (close enough).

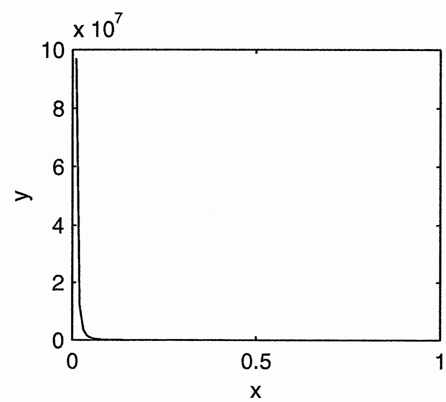


Pick any two points on the line and use the slope formula. We estimate $(-4, -6.8)$ and $(-1, -0.8)$ from the graph, giving a slope of

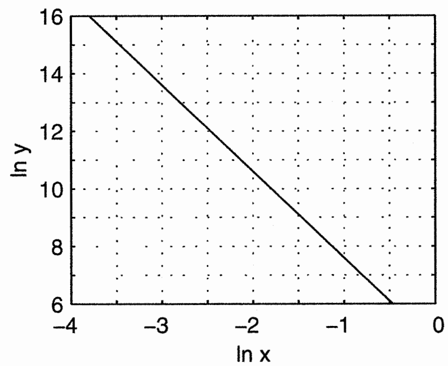
$$m \approx \frac{-6.8 - (-4)}{-0.8 - (-1)} \approx 2.$$

Note that the slope of the line is identical to the exponent of the power function $y = 3x^2$.

11. The plot of the power function $y = 100x^{-3}$ on $[0, 1]$.



The difficulty most encounter is in circumventing the logarithm of zero. We actually made our plot on $[0.01, 1]$ (close enough).

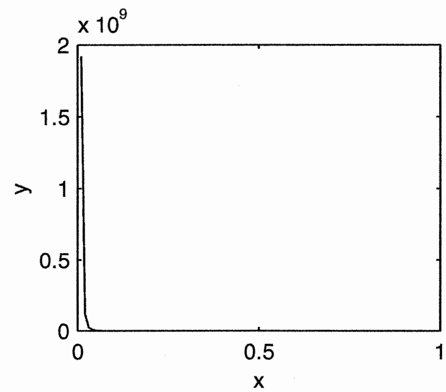


Pick any two points on the line and use the slope formula. We estimate $(-3, 13.6)$ and $(-1, 7.6)$ from the graph, giving a slope of

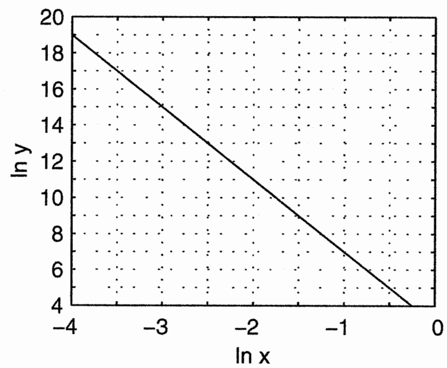
$$m \approx \frac{7.6 - 13.6}{-1 - (-3))} \approx -3.$$

Note that the slope of the line is identical to the exponent of the power function $y = 100x^{-3}$.

12. The plot of the power function $y = 20x^{-4}$ on $[0, 1]$.



The difficulty most encounter is in circumventing the logarithm of zero. We actually made our plot on $[0.01, 1]$ (close enough).

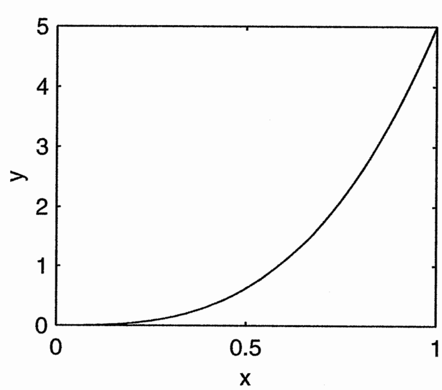


Pick any two points on the line and use the slope formula. We estimate $(-3, 15)$ and $(-1, 7)$ from the graph, giving a slope of

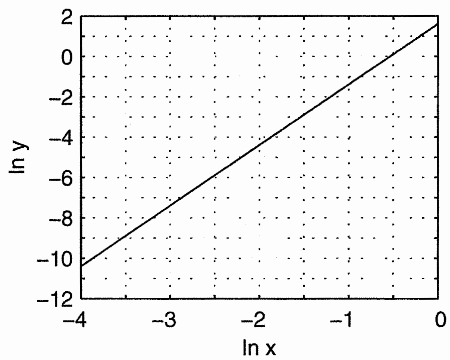
$$m \approx \frac{7 - 15}{-1 - (-3)} \approx -4.$$

Note that the slope of the line is identical to the exponent of the power function $y = 20x^{-4}$.

13. The plot of the power function $y = 5x^3$ on $[0, 1]$.



The difficulty most encounter is in circumventing the logarithm of zero. We actually made our plot on $[0.01, 1]$ (close enough).

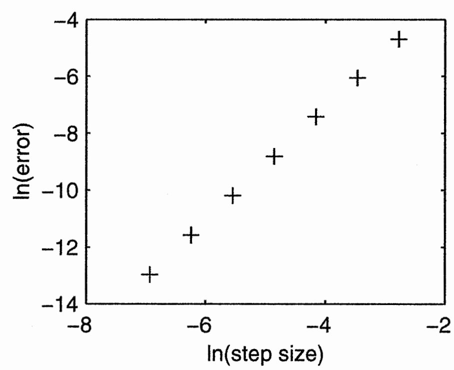


Pick any two points on the line and use the slope formula. We estimate $(-3, -7.4)$ and $(-1, -1.4)$ from the graph, giving a slope of

$$m \approx \frac{-1.4 - (-7.4)}{-1 - (-3))} \approx 3.$$

Note that the slope of the line is identical to the exponent of the power function $y = 5x^3$.

14. (a) Prepare a plot of $\ln(E_h)$ (column 4) versus the logarithm of the step size (column 1).



Note the linear relationship between $\ln(E_h)$ and $\ln h$.

(b) Using a calculator, we fit a linear model to the data, having form

$$\ln E_h = 1.9901 \ln h + 0.8397.$$

(c) Solve the equation in part (b) for E_h. Exponentiate both sides.

$$E_h = e^{1.9901 \ln h + 0.8397}$$
$$E_h = e^{0.8397} e^{\ln h^{1.9901}}$$
$$E_h = 2.316 h^{1.9901}$$

Note that the exponent is near 2. This agrees with our claim that RK2 is a second order method.

(d) Let $E_h = 0.001$ in the formula developed in part (c).

$$.001 = 2.316h^{1.9901}$$
$$h = \left(\frac{0.001}{2.316}\right)^{1/1.9901}$$
$$h \approx 0.0203$$

The number of iterations is calculated with

$$N = \frac{b-a}{h}$$
$$N = \frac{2}{0.0203}$$
$$N \approx 99$$

It will take approximately 99 iterations of the RK2 method to achieve the desired accuracy.

(e) We ran the RK2 algorithm on a computer with step size $h = 0.0203$. A 300 MHz PC reported an error of 0.0009957 in an elapsed time of about 0.17 seconds.

15. An integrating factor is e^t.

$$e^t y' + e^t y = e^t \sin t$$
$$(e^t y)' = e^t \sin t$$

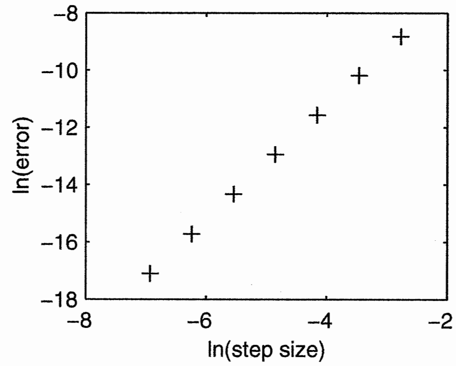
Integrate by parts.

$$e^t y = \frac{1}{2}e^t \sin t - \frac{1}{2}e^t \cos t + C$$
$$y = \frac{1}{2}\sin t - \frac{1}{2}\cos t + Ce^{-t}$$

The initial condition $y(0) = 1$ gives us $C = 3/2$ and the exact solution $y = (1/2)\sin t - (1/2)\cos t + (3/2)e^{-t}$. Thus, the true solution at $t = 2$ is $y(2) \approx .8657250565$. This value allows us to use RK2 and a computer to construct the following table.

Step size h	RK2 approx.	True value	Error E_h
0.06250000	0.86557660	0.86572506	0.00014846
0.03125000	0.86568735	0.86572506	0.00003771
0.01562500	0.86571556	0.86572506	0.00000950
0.00781250	0.86572267	0.86572506	0.00000238
0.00390625	0.86572446	0.86572506	0.00000060
0.00195313	0.86572491	0.86572506	0.00000015
0.00097656	0.86572502	0.86572506	0.00000004

A logarithmic plot of the error in column 4 versus the step size in column 1 reveals the linear relationship between $\ln E_h$ and $\ln h$.



A computer was used to find a linear fit.

$$\ln E_h = 1.9938 \ln h - 3.278$$
$$E_h = e^{-3.278} e^{\ln h^{1.9938}}$$
$$E_h = 0.03770 h^{1.9938}$$

Note that the exponent is approximately 2, which is in keeping with the fact that RK2 is a second order method.

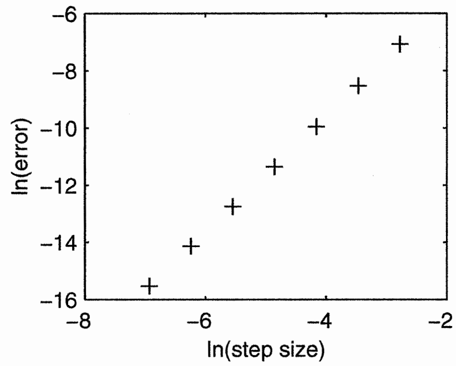
16. The equation is separable.

$$\frac{dy}{y} = -t^2\,dt$$
$$\ln y = -\frac{1}{3}t^3 + C$$

The initial condition $y(0) = 1$ gives $C = 0$. Thus, $y = e^{-(1/3)t^3}$ and $y(2) \approx 0.06948345$. This value allows us to use RK2 and a computer to construct the following table.

Step size h	RK2 approx.	True value	Error E_h
0.06250000	0.07032847	0.06948345	0.00084502
0.03125000	0.06968051	0.06948345	0.00019706
0.01562500	0.06953107	0.06948345	0.00004761
0.00781250	0.06949516	0.06948345	0.00001170
0.00390625	0.06948635	0.06948345	0.00000290
0.00195313	0.06948417	0.06948345	0.00000072
0.00097656	0.06948363	0.06948345	0.00000018

A logarithmic plot of the error in column 4 versus the step size in column 1 reveals the linear relationship between $\ln E_h$ and $\ln h$.



A computer was used to find a linear fit.

$$\ln E_h = 2.029 \ln h - 1.490$$
$$E_h = e^{-1.490} e^{\ln h^{2.029}}$$
$$E_h = 0.2254 h^{2.029}$$

Note that the exponent is approximately 2, which is in keeping with the fact that RK2 is a second order method.

17. The equation is separable.

$$y^2\,dy = t\,dt$$
$$\frac{1}{3}y^2 = \frac{1}{2}t^2 + C$$

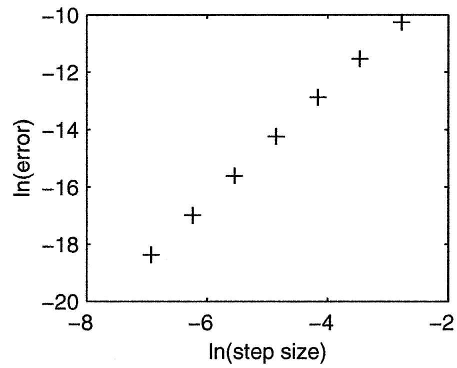
The initial condition $y(0) = 1$ gives $C = 1/3$. Thus,

$$\frac{1}{3}y^3 = \frac{1}{2}t^2 + \frac{1}{3}$$
$$y = \sqrt[3]{(3/2)t^2 + 1}$$

Thus, $y(2) \approx 1.91293118$. This value allows us to use RK2 and a computer to construct the following table.

Step size h	RK2 approx.	True value	Error E_h
0.06250000	1.91289601	1.91293118	0.00003517
0.03125000	1.91292139	1.91293118	0.00000979
0.01562500	1.91292861	1.91293118	0.00000257
0.00781250	1.91293053	1.91293118	0.00000066
0.00390625	1.91293102	1.91293118	0.00000017
0.00195313	1.91293114	1.91293118	0.00000004
0.00097656	1.91293117	1.91293118	0.00000001

A logarithmic plot of the error in column 4 versus the step size in column 1 reveals the linear relationship between $\ln E_h$ and $\ln h$.



A computer was used to find a linear fit.

$$
\begin{aligned}
\ln E_h &= 1.958 \ln h - 4.765 \\
E_h &= e^{-4.765} e^{\ln h^{1.958}} \\
E_h &= 0.008522 h^{1.958}
\end{aligned}
$$

Note that the exponent is approximately 2, which is in keeping with the fact that RK2 is a second order method.

18. An integrating factor is e^{3t}.

$$
\begin{aligned}
e^{3t} y' + 3e^{3t} y &= t e^{3t} \\
(e^{3t} y)' &= t e^{3t}
\end{aligned}
$$

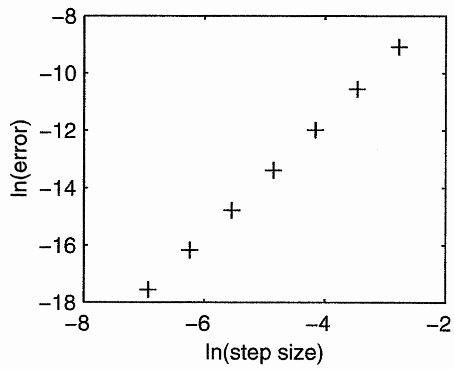
Integration by parts.

$$
\begin{aligned}
e^{3t} y &= \frac{1}{3} t e^{3t} - \frac{1}{9} e^{3t} + C \\
y &= \frac{1}{3} t - \frac{1}{9} + C e^{-3t}
\end{aligned}
$$

The initial condition $y(0) = 1$ gives us $C = 10/9$ and the exact solution $y = (1/3)t - 1/9 + (10/9)e^{-3t}$. Thus, $y(2) \approx 0.55830972$. This value allows us to use RK2 and a computer to construct the following table.

Step size h	RK2 approx.	True value	Error E_h
0.06250000	0.55842346	0.55830972	0.00011373
0.03125000	0.55833582	0.55830972	0.00002610
0.01562500	0.55831600	0.55830972	0.00000628
0.00781250	0.55831126	0.55830972	0.00000154
0.00390625	0.55831011	0.55830972	0.00000038
0.00195313	0.55830982	0.55830972	0.00000009
0.00097656	0.55830975	0.55830972	0.00000002

A logarithmic plot of the error in column 4 versus the step size in column 1 reveals the linear relationship between $\ln E_h$ and $\ln h$.



A computer was used to find a linear fit.

$$\begin{aligned}
\ln E_h &= 2.033 \ln h - 3.492 \\
E_h &= e^{-3.492} e^{\ln h^{2.033}} \\
E_h &= 0.03043 h^{2.033}
\end{aligned}$$

Note that the exponent is approximately 2, which is in keeping with the fact that RK2 is a second order method.

19. We've seen that the error made by RK2 is related to the step size via $E_h = \lambda h^2$. So, if h leads to an error of E_h, then the error associated with step size $h/\sqrt{2}$ is

$$\begin{aligned}
E &= \lambda \left(\frac{h}{\sqrt{2}}\right)^2 \\
E &= \lambda \frac{h^2}{2} \\
E &= \frac{1}{2} \lambda h^2 \\
E &= \frac{1}{2} E_h.
\end{aligned}$$

Thus, dividing the step size by the square root of 2 halves the error. Of course, if N_h is the number of iterations associated with step size h, then

$$N_h = \frac{b-a}{h}.$$

If we reduce the step size to $h/\sqrt{2}$, then the number of iterations is

$$\begin{aligned}
N &= \frac{b-a}{h/\sqrt{2}} \\
N &= \sqrt{2} \frac{b-a}{h} \\
N &= \sqrt{2} N_h.
\end{aligned}$$

Thus, the number of iterations is increased by a factor of $\sqrt{2}$.

This exercise may be better understood in reverse. That is, what happens when we halve the step size? If E_h is the error associated with step size h, then the error associated with step size $h/2$ is

$$E = \lambda\left(\frac{h}{2}\right)^2$$
$$E = \frac{1}{4}\lambda h^2$$
$$E = \frac{1}{4}E_h.$$

Thus, halving the step size quarters the error, which is considerably better than the performance given by Euler's method.

20. We have $t_0 = 0$, $y_0 = 1$ and $f(t, y) = y$. First compute four slopes.

$$\begin{aligned}
s_1 &= f(t_0, y_0) = f(0, 1) = 1\\
s_2 &= f(t_0 + h/2, y_0 + hs_1/2) = f(0.05, 1.05) = 1.05\\
s_3 &= f(t_0 + h/2, y_0 + hs_2/2) = f(0.05, 1.0525) = 1.0525\\
s_4 &= f(t_0 + h, y_0 + hs_3) = f(0.1, 1.10525) = 1.10525
\end{aligned}$$

Update t and y.

$$\begin{aligned}
y_1 &= y_0 + h\frac{s_1 + 2s_2 + 2s_3 + s_4}{6}\\
&= 1 + 0.1\frac{1 + 2(1.05) + 2(1.0525) + 1.10525}{6} = 1.05170833\\
t_1 &= t_0 + h = 0 + 0.1 = 0.1
\end{aligned}$$

Continuing in this manner produces the following table.

t_k	y_k	s_1	s_2	s_3	s_4	$h\frac{s_1+2s_2+2s_3+s_4}{6}$
0.0	1.0000	1.0000	1.0500	1.0525	1.1053	0.1052
0.1	1.1052	1.1052	1.1604	1.1632	1.2215	0.1162
0.2	1.2214	1.2214	1.2825	1.2855	1.3500	0.1285
0.3	1.3499					

21. We have $t_0 = 0$, $y_0 = 1$ and $f(t, y) = t + y$. First compute four slopes.

$$\begin{aligned}
s_1 &= f(t_0, y_0) = f(0, 1) = 1\\
s_2 &= f(t_0 + h/2, y_0 + hs_1/2) = f(0.05, 1.05) = 1.10\\
s_3 &= f(t_0 + h/2, y_0 + hs_2/2) = f(0.05, 1.055) = 1.105\\
s_4 &= f(t_0 + h, y_0 + hs_3) = f(0.1, 1.1105) = 1.2105
\end{aligned}$$

Update t and y.

$$\begin{aligned}
y_1 &= y_0 + h\frac{s_1 + 2s_2 + 2s_3 + s_4}{6}\\
&= 1 + 0.1\frac{1 + 2(1.10) + 2(1.105) + 1.2105}{6} = 1.110341667\\
t_1 &= t_0 + h = 0 + 0.1 = 0.1
\end{aligned}$$

Continuing in this manner produces the following table.

t_k	y_k	s_1	s_2	s_3	s_4	$h\frac{s_1+2s_2+2s_3+s_4}{6}$
0.0	1.0000	1.0000	1.1000	1.1050	1.2105	0.1103
0.1	1.1103	1.2103	1.3209	1.3264	1.4430	0.1325
0.2	1.2428	1.4428	1.5649	1.5711	1.6999	0.1569
0.3	1.3997					

22. We have $x_0 = 0$, $z_0 = 0$ and $f(x, z) = 5 - z$. First compute four slopes.

$$\begin{aligned}
s_1 &= f(x_0, z_0) = f(0, 0) = 5 \\
s_2 &= f(x_0 + h/2, z_0 + hs_1/2) = f(0.05, 0.25) = 4.75 \\
s_3 &= f(x_0 + h/2, z_0 + hs_2/2) = f(0.05, 0.2375) = 4.7625 \\
s_4 &= f(x_0 + h, z_0 + hs_3) = f(0.1, 0.47625) = 4.52375
\end{aligned}$$

Update x and z.

$$\begin{aligned}
z_1 &= z_0 + h\frac{s_1 + 2s_2 + 2s_3 + s_4}{6} \\
&= 0 + 0.1\frac{5 + 2(4.75) + 2(4.7625) + 4.52375}{6} = 0.4758125 \\
x_1 &= x_0 + h = 0 + 0.1 = 0.1
\end{aligned}$$

Continuing in this manner produces the following table.

s x_k	z_k	s_1	s_2	s_3	s_4	$h\frac{s_1+2s_2+2s_3+s_4}{6}$
0.0	0.0000	5.0000	4.7500	4.7625	4.5237	0.4758
0.1	0.4758	4.5242	4.2980	4.3093	4.0933	0.4305
0.2	0.9063	4.0937	3.8890	3.8992	3.7037	0.3896
0.3	1.2959					

23. We have $x_0 = 0$, $z_0 = 1$ and $f(x, z) = x - 2z$. First compute four slopes.

$$\begin{aligned}
s_1 &= f(x_0, z_0) = f(0, 1) = -2 \\
s_2 &= f(x_0 + h/2, z_0 + hs_1/2) = f(0.05, 0.9) = -1.75 \\
s_3 &= f(x_0 + h/2, z_0 + hs_2/2) = f(0.05, 0.9125) = -1.775 \\
s_4 &= f(x_0 + h, z_0 + hs_3) = f(0.1, 0.0.8225) = -1.545
\end{aligned}$$

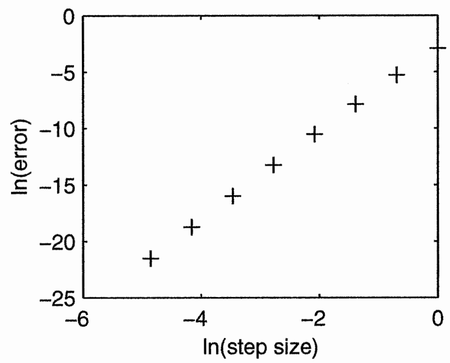
Update x and z.

$$\begin{aligned}
z_1 &= z_0 + h\frac{s_1 + 2s_2 + 2s_3 + s_4}{6} \\
&= 1 + 0.1\frac{-2 + 2(-1.75) + 2(-1.775) + (-1, 545)}{6} = 0.8234166667 \\
x_1 &= x_0 + h = 0 + 0.1 = 0.1
\end{aligned}$$

Continuing in this manner produces the following table.

s x_k	z_k	s_1	s_2	s_3	s_4	$h\frac{s_1+2s_2+2s_3+s_4}{6}$
0.0	1.0000	-2.0000	-1.7500	-1.7750	-1.5450	-0.1766
0.1	0.8234	-1.5468	-1.3422	-1.3626	-1.1743	-0.1355
0.2	0.6879	-1.1758	-1.0082	-1.0250	-0.8708	-0.1019
0.3	0.5860					

24. (a) Prepare a plot of $\ln(E_h)$ (column 4) versus the logarithm of the step size (column 1).



Note the linear relationship between $\ln(E_h)$ and $\ln h$.

(b) Using a calculator, we fit a linear model to the data, having form

$$\ln E_h = 3.8569 \ln h - 2.647.$$

(c) Solve the equation in part (b) for E_h. Exponentiate both sides.

$$E_h = e^{3.8569 \ln h - 2.647}$$
$$E_h = e^{-2.647} e^{\ln h^{3.8569}}$$
$$E_h = 0.07086 h^{3.8569}$$

Note that the exponent is near 4. This agrees with our claim that RK4 is a fourth order method.

(d) Let $E_h = 0.000001$ in the formula developed in part (c).

$$.000001 = 0.07086 h^{3.8569}$$
$$h = \left(\frac{0.000001}{0.07086}\right)^{1/3.8569}$$
$$h \approx 0.0552$$

The number of iterations is calculated with

$$N = \frac{b-a}{h}$$
$$N = \frac{2}{0.0552}$$
$$N \approx 36$$

It will take approximately 36 iterations of the RK4 method to achieve the desired accuracy.

(e) We ran the RK4 algorithm on a computer with step size $h = 0.0552$. A 300 MHz PC reported an error of 1.085×10^{-6} in an elapsed time of about 0.11 seconds.

25. An integrating factor is e^t.

$$e^t y' + e^t y = e^t \sin t$$
$$(e^t y)' = e^t \sin t$$

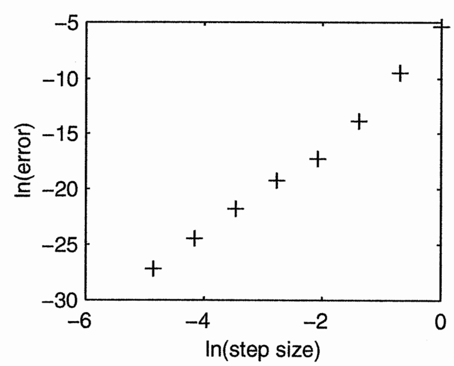
Integrate by parts.

$$e^t y = \frac{1}{2} e^t \sin t - \frac{1}{2} e^t \cos t + C$$
$$y = \frac{1}{2} \sin t - \frac{1}{2} \cos t + Ce^{-t}$$

The initial condition $y(0) = 1$ gives us $C = 3/2$ and the exact solution $y = (1/2)\sin t - (1/2)\cos t + (3/2)e^{-t}$. Thus, the true solution at $t = 2$ is $y(2) \approx .8657250565$. This value allows us to use RK4 and a computer to construct the following table.

Step size h	RK4 approx.	True value	Error E_h
1.0000000	0.87036095021	0.86572505654	0.00463589367
0.5000000	0.86580167967	0.86572505654	0.00007662313
0.2500000	0.86572599485	0.86572505654	0.00000093831
0.1250000	0.86572502504	0.86572505654	0.00000003150
0.0625000	0.86572505215	0.86572505654	0.00000000439
0.0312500	0.86572505620	0.86572505654	0.00000000034
0.0156250	0.86572505652	0.86572505654	0.00000000002
0.0078125	0.86572505654	0.86572505654	0.00000000000

A logarithmic plot of the error in column 4 versus the step size in column 1 reveals the linear relationship between $\ln E_h$ and $\ln h$.



A computer was used to find a linear fit.

$$\ln E_h = 4.352 \ln h - 6.778$$
$$E_h = e^{-6.778} e^{\ln h^{4.352}}$$
$$E_h = 0.001138 h^{4.352}$$

Note that the exponent is approximately 4, which is in keeping with the fact that RK4 is a fourth order method.

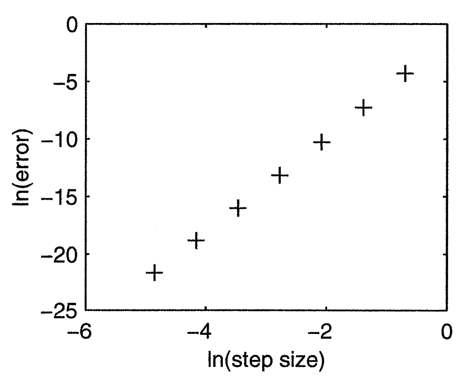
26. The equation is separable.

$$\frac{dy}{y} = -t^2\,dt$$
$$\ln y = -\frac{1}{3}t^3 + C$$

The initial condition $y(0) = 1$ gives $C = 0$. Thus, $y = e^{-(1/3)t^3}$ and $y(2) \approx 0.06948345$. This value allows us to use RK4 and a computer to construct the following table.

Step size h	RK4 approx.	True value	Error E_h
0.50000000	0.08308652	0.06948345	0.01360307
0.25000000	0.07018576	0.06948345	0.00070231
0.12500000	0.06951834	0.06948345	0.00003489
0.06250000	0.06948535	0.06948345	0.00000190
0.03125000	0.06948356	0.06948345	0.00000011
0.01562500	0.06948346	0.06948345	0.00000001
0.00781250	0.06948345	0.06948345	0.00000000

A logarithmic plot of the error in column 4 versus the step size in column 1 reveals the linear relationship between $\ln E_h$ and $\ln h$.



A computer was used to find a linear fit.

$$\ln E_h = 4.165 \ln h - 1.516$$
$$E_h = e^{-1.516} e^{\ln h^{4.165}}$$
$$E_h = 0.2195 h^{4.165}$$

Note that the exponent is approximately 4, which is in keeping with the fact that RK4 is a fourth order method.

27. The equation is separable.

$$y^2\,dy = t\,dt$$

$$\frac{1}{3}y^2 = \frac{1}{2}t^2 + C$$

The initial condition $y(0) = 1$ gives $C = 1/3$. Thus,

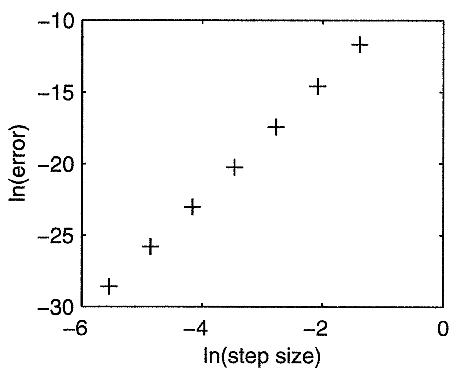
$$\frac{1}{3}y^3 = \frac{1}{2}t^2 + \frac{1}{3}$$

$$y = \sqrt[3]{(3/2)t^2 + 1}$$

Thus, $y(2) \approx 1.91293118$. This value allows us to use RK4 and a computer to construct the following table.

Step size h	RK4 approx.	True value	Error E_h
0.25000000	1.91293958	1.91293118	0.00000840
0.12500000	1.91293164	1.91293118	0.00000046
0.06250000	1.91293121	1.91293118	0.00000003
0.03125000	1.91293118	1.91293118	0.00000000
0.01562500	1.91293118	1.91293118	0.00000000
0.00781250	1.91293118	1.91293118	0.00000000
0.00390625	1.91293118	1.91293118	0.00000000

A logarithmic plot of the error in column 4 versus the step size in column 1 reveals the linear relationship between $\ln E_h$ and $\ln h$.



A computer was used to find a linear fit.

$$\ln E_h = 4.055 \ln h - 6.139$$

$$E_h = e^{-6.139} e^{\ln h^{4.055}}$$

$$E_h = 0.002157 h^{4.055}$$

Note that the exponent is approximately 4, which is in keeping with the fact that RK4 is a fourth order method.

28. An integrating factor is e^{3t}.

$$e^{3t}y' + 3e^{3t}y = te^{3t}$$
$$(e^{3t}y)' = te^{3t}$$

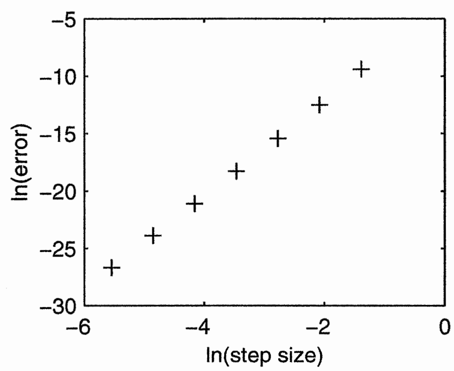
Integration by parts.

$$e^{3t}y = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + C$$
$$y = \frac{1}{3}t - \frac{1}{9} + Ce^{-3t}$$

The initial condition $y(0) = 1$ gives us $C = 10/9$ and the exact solution $y = (1/3)t - 1/9 + (10/9)e^{-3t}$. Thus, $y(2) \approx 0.55830972$. This value allows us to use RK4 and a computer to construct the following table.

Step size h	RK2 approx.	True value	Error E_h
0.25000000	0.55839264	0.55830972	0.00008291
0.12500000	0.55831345	0.55830972	0.00000373
0.06250000	0.55830992	0.55830972	0.00000020
0.03125000	0.55830974	0.55830972	0.00000001
0.01562500	0.55830973	0.55830972	0.00000000
0.00781250	0.55830972	0.55830972	0.00000000
0.00390625	0.55830972	0.55830972	0.00000000

A logarithmic plot of the error in column 4 versus the step size in column 1 reveals the linear relationship between $\ln E_h$ and $\ln h$.



A computer was used to find a linear fit.

$$\ln E_h = 4.134 \ln h - 3.850$$
$$E_h = e^{-3.850} e^{\ln h^{4.134}}$$
$$E_h = 0.02127h^{4.134}$$

Note that the exponent is approximately 4, which is in keeping with the fact that RK4 is a fourth order method.

29. Let $E_h = \lambda h^4$ be the error associated with the step size h. Now, what will be the error if we halve the step size?

$$E = \lambda \left(\frac{h}{2}\right)^4$$
$$E = \frac{1}{16}\lambda h^4$$
$$E = \frac{1}{16}E_h$$

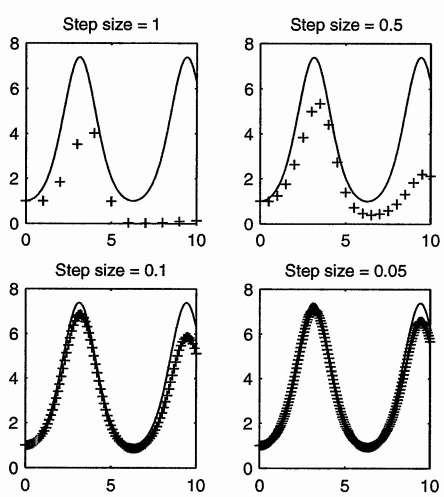
Thus, having the step size reduces the error to 1/16 of its former size.

Section 6.3. Numerical Error Comparisons

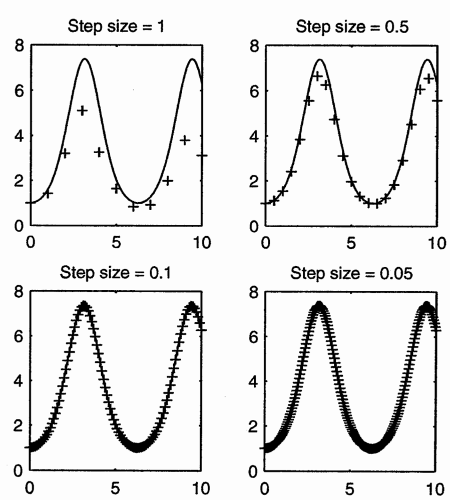
1. The equation is separable.

$$\frac{dx}{x} = \sin t \, dt$$
$$\ln x = -\cos t + C$$

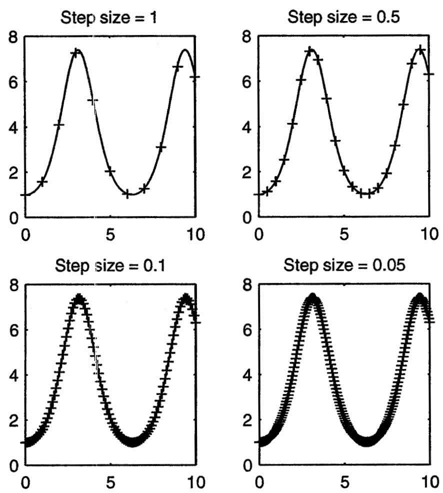
The initial condition $x(0) = 1$ gives us $C = 1$ and $\ln x = 1 - \cos t$. Solving for x, $x = e^{1-\cos t}$. Euler's method, with step sizes $h = 1$, 0.5, 0.1, and 0.05, produces the following result.



RK2 does a little better.



RK4 is the most accurate.

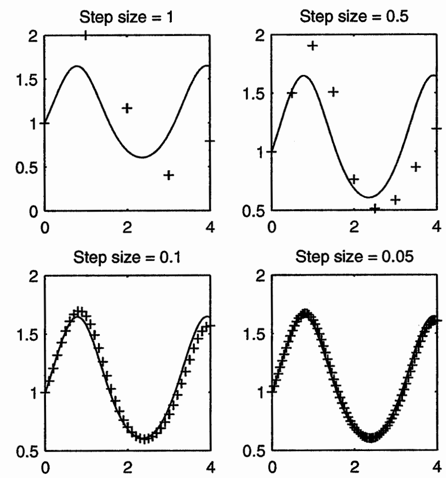


2. The equation is separable.

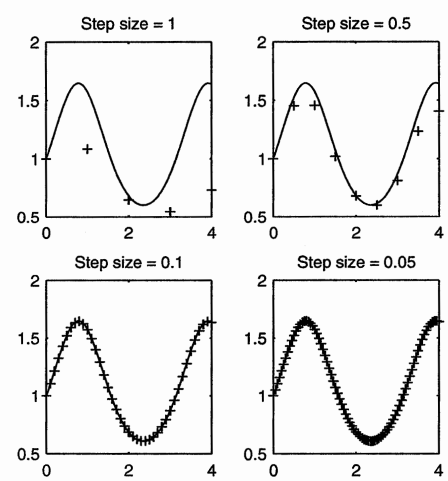
$$\frac{dx}{x} = \cos 2t\, dt$$

$$\ln x = \frac{1}{2} \sin 2t + C$$

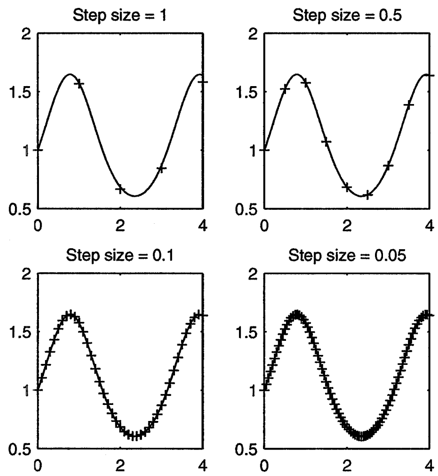
The initial condition $x(0) = 1$ gives us $C = 0$ and $\ln x = (1/2)\sin 2t$. Solving for x, $x = e^{(1/2)\sin 2t}$. Euler's method, with step sizes $h = 1, 0.5, 0.1$, and 0.05, produces the following result.



RK2 does a little better.



RK4 is the most accurate.



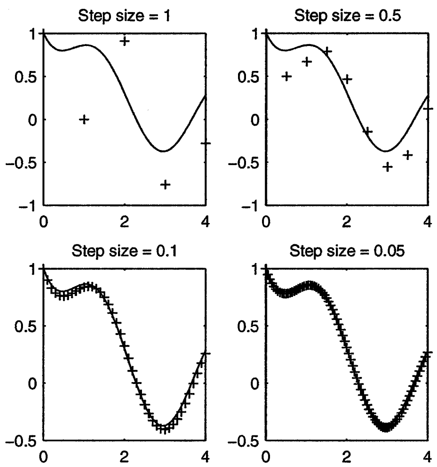
3. An integrating factor is e^t.

$$e^t x' + e^t x = e^t \sin 2t$$
$$(e^t x)' = e^t \sin 2t$$

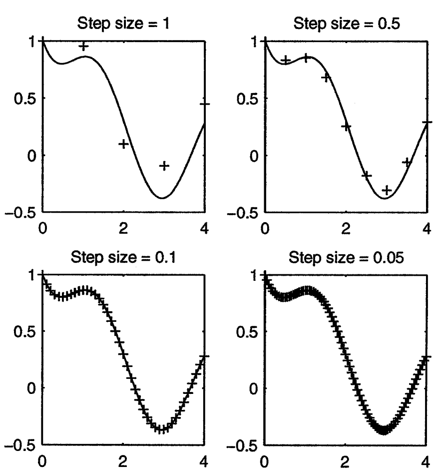
Integration is by parts.

$$e^t x = \frac{1}{5} e^t \sin 2t - \frac{2}{5} e^t \cos 2t + C$$
$$x = \frac{1}{5} \sin 2t - \frac{2}{5} \cos 2t + ce^{-t}$$

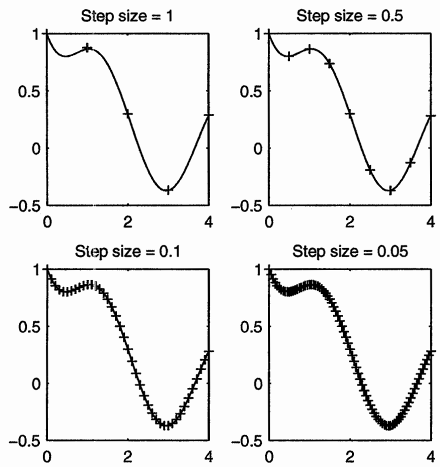
The initial condition $x(0) = 1$ gives us $C = 7/5$ and $x = (1/5)\sin 2t - (2/5)\cos 2t + (7/5)e^{-t}$. Euler's method, with step sizes $h = 1$, 0.5, 0.1, and 0.05, produces the following result.



RK2 does a little better.



RK4 is the most accurate.



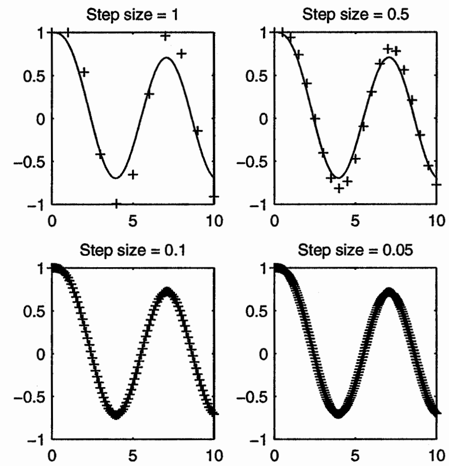
4. An integrating factor is e^t.

$$\begin{aligned} e^t x' + e^t x &= e^t \cos t \\ (e^t x)' &= e^t \cos t \end{aligned}$$

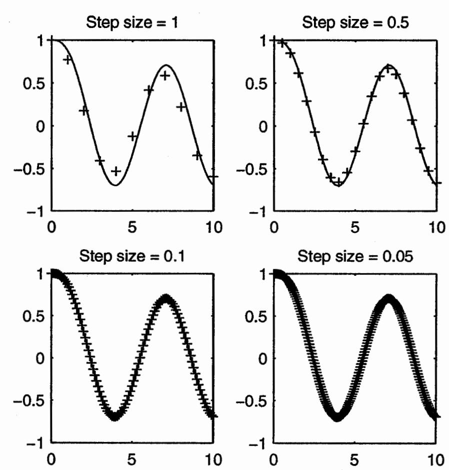
Integration is by parts.

$$\begin{aligned} e^t x &= \frac{1}{2} e^t \cos t + \frac{1}{2} e^t \sin t + C \\ x &= \frac{1}{2} \cos t + \frac{1}{2} \sin t + C e^{-t} \end{aligned}$$

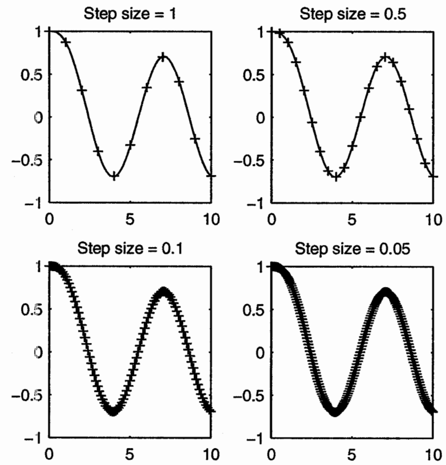
The initial condition $x(0) = 1$ gives us $C = 1/2$ and $x = (1/2)\cos t + (1/2)\sin t + (1/2)e^{-t}$. Euler's method, with step sizes $h = 1$, 0.5, 0.1, and 0.05, produces the following result.



RK2 does a little better.



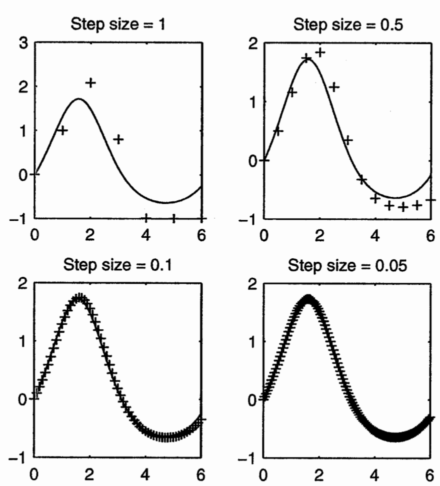
RK4 is the most accurate.



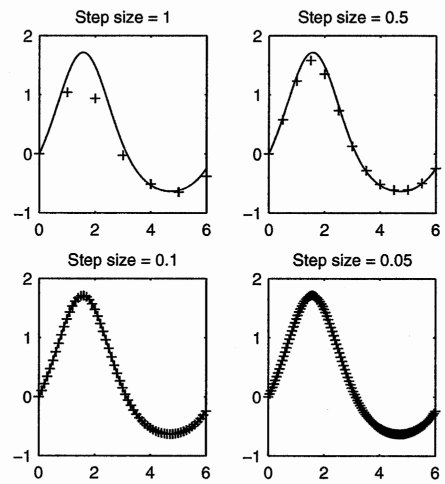
5. The equation is separable.

$$\frac{dx}{1+x} = \cos t\, dt$$
$$\ln(1+x) = \sin t + C$$

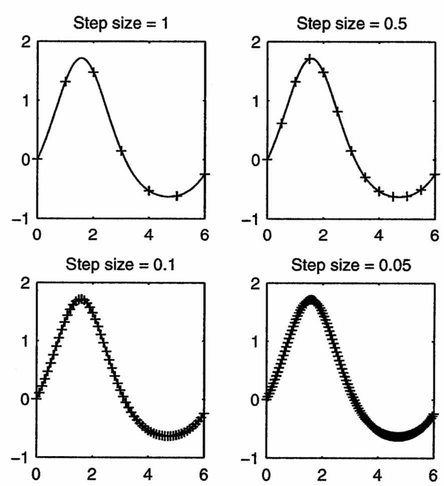
The initial condition $x(0) = 0$ gives us $C = 0$ and $\ln(1+x) = \sin t$. Solving for x, $x = -1 + e^{\sin t}$. Euler's method, with step sizes $h = 1, 0.5, 0.1$, and 0.05, produces the following result.



RK2 does a little better.



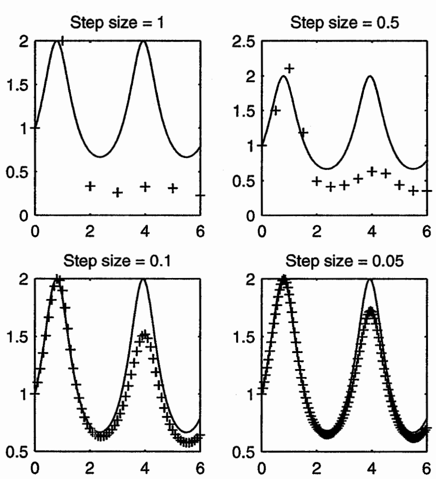
RK4 is the most accurate.



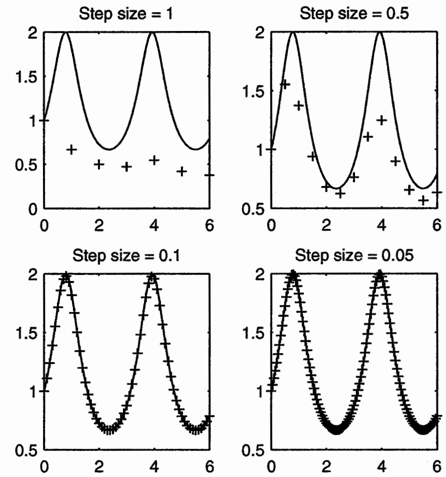
6. The equation is separable.

$$\begin{aligned} \frac{dx}{x^2} &= \cos 2t\, dt \\ -\frac{1}{x} &= \frac{1}{2}\sin 2t + C \\ x &= \frac{-2}{\sin 2t + 2C} \end{aligned}$$

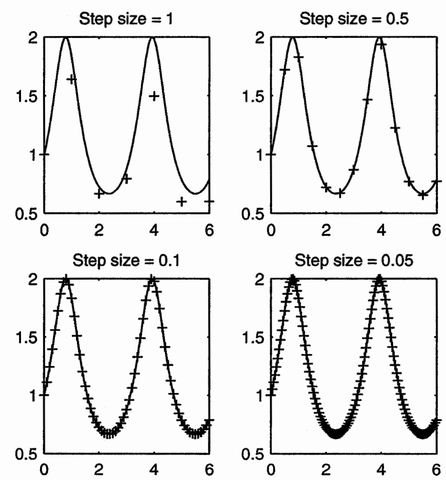
The initial condition $x(0) = 1$ gives us $C = -1$ and $x = 2/(2 - \sin 2t)$. Euler's method, with step sizes $h = 1$, 0.5, 0.1, and 0.05, produces the following result.



RK2 does a little better.



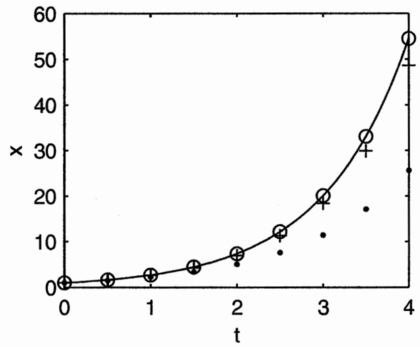
RK4 is the most accurate.



7. The equation is separable.

$$\begin{aligned} \frac{dx}{x} &= dt \\ \ln x &= t + C \end{aligned}$$

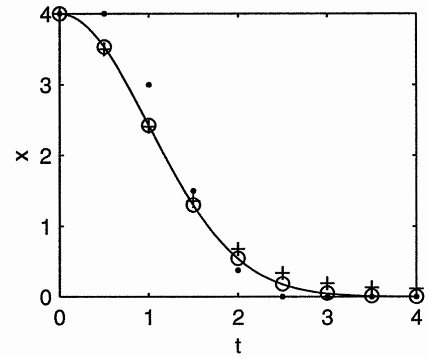
The initial condition $x(0) = 1$ gives $C = 0$ and $x = e^t$. The figure that follows shows three numerical solutions on the interval [0, 4] with step size $h = 0.5$. The exact solution is the solid line, the Euler solution is plotted with discrete dots, the RK2 solution is plotted with discrete plus signs, and the RK4 solution is plotted with discrete circles.



8. The equation is separable. Thus,

$$\begin{aligned}\frac{dx}{x} &= -t\,dt\\ \ln x &= -\frac{1}{2}t^2 + C\\ x &= De^{-(1/2)t^2},\end{aligned}$$

where $D = e^C$. The initial condition $x(0) = 4$ gives $D = 4$ and $x = 4e^{-(1/2)t^2}$. The figure that follows shows three numerical solutions on the interval [0, 4] with step size $h = 0.5$. The exact solution is the solid line, the Euler solution is plotted with discrete dots, the RK2 solution is plotted with discrete plus signs, and the RK4 solution is plotted with discrete circles.



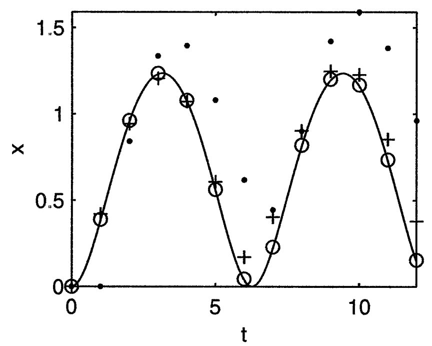
9. The equation is separable.

$$\begin{aligned}(x+1)\,dx &= \sin t\,dt\\ \frac{1}{2}x^2 + x &= -\cos t + C\end{aligned}$$

The initial condition $x(0) = 0$ gives $C = 1$ and

$$\begin{aligned}\frac{1}{2}x^2 + x &= 1 - \cos t\\ x^2 + 2x &= 2 - 2\cos t\\ (x+1)^2 &= 3 - 2\cos t\\ x &= -1 \pm \sqrt{3 - 2\cos t}\end{aligned}$$

Because $x(0) = 0$, choose $x = -1 + \sqrt{3 - 2\cos t}$. The figure that follows shows three numerical solutions on the interval [0, 12] with step size $h = 1$. The exact solution is the solid line, the Euler solution is plotted with discrete dots, the RK2 solution is plotted with discrete plus signs, and the RK4 solution is plotted with discrete circles.



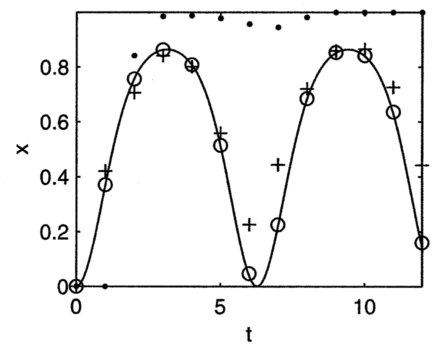
10. The equation is separable.

$$\begin{aligned}\frac{dx}{1-x} &= \sin t\,dt \\ -\ln|1-x| &= -\cos t + C \\ \ln|1-x| &= \cos t - C\end{aligned}$$

The initial condition $x(0) = 0$ gives $C = 1$ and

$$\begin{aligned}\ln|1-x| &= \cos t - 1 \\ |1-x| &= e^{-1+\cos t} \\ 1-x &= \pm e^{-1+\cos t} \\ x &= 1 \pm e^{-1+\cos t}.\end{aligned}$$

Because $x(0) = 0$, choose $x = 1 - e^{-1+\cos t}$. The figure that follows shows three numerical solutions on the interval [0, 12] with step size $h = 1$. The exact solution is the solid line, the Euler solution is plotted with discrete dots, the RK2 solution is plotted with discrete plus signs, and the RK4 solution is plotted with discrete circles.



11. (a) First, we complete the entries in the table using a computer.

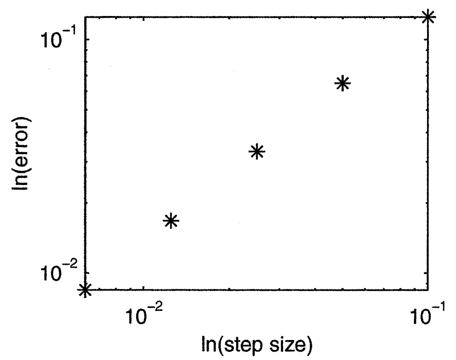
h	Euler	RK2	RK4
0.10000	2.59374	2.71408	2.71828
0.05000	2.65330	2.71719	2.71828
0.02500	2.68506	2.71800	2.71828
0.01250	2.70148	2.71821	2.71828
0.00625	2.70984	2.71826	2.71828

Next, since we will be plotting the magnitude of the error, we craft a new table with the errors at each step size. The error is calculated by taking the magnitude of the difference of $x(1) = e$ and the solution at $t = 1$ given by the numerical routine.

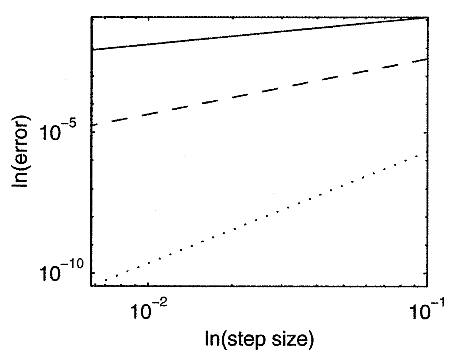
h	Eul error	RK2 error	RK4 error
0.10000	0.12454	0.00420	0.00000
0.05000	0.06498	0.00109	0.00000
0.02500	0.03322	0.00028	0.00000
0.01250	0.01680	0.00007	0.00000
0.00625	0.00845	0.00002	0.00000

We will work with the internal precision of the numbers in the computer, not the numbers displayed in the above tables. Here is a loglog graph of the Euler error versus the step size.

Note the logarithmic scale on each axis.



(b) What follows is a plot of the logarithm of the error versus the logarithm of the step size for each numerical method. Note the logarithmic scale on each axis. Euler (solid line), RK2 (dashed), RK4 (dotted).



(c) We used a computer to find the slope of each line in part (b). For Euler's method, the slope was reported as 0.9716, which is close to 1, consistent with the fact that the Euler routine is a first order algorithm. The slope of the second line came in at 1.9755, which is close to 2, consistent with the fact that RK2 is a second order method. Finally, the slope of the third line was estimated as 3.9730, which is close to 4, consistent with the fact that RK4 is a fourth order method.

12. The equation is separable.

$$\frac{dx}{x} = -tx$$

$$\ln x = -\frac{1}{2}t^2 + C$$

The initial condition $x(0) = 4$ gives $C = \ln 4$. Thus,

$$\ln x = -\frac{1}{2}t^2 + \ln 4$$

$$x = e^{-(1/2)t^2 + \ln 4}$$

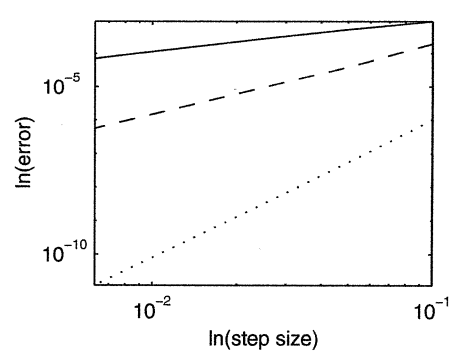
$$x = e^{\ln 4} e^{-(1/2)t^2}$$

$$x = 4e^{-(1/2)t^2}.$$

A table is constructed to evaluate the error made at each step size. The error is the magnitude of the difference between $x(4) = 4e^{-8}$ and the approximation predicted by the particular method (Eul, RK2, RK4) at $t = 4$. The step size used in constructing the table is $h = 0.1$.

h	Eul error	RK2 error	RK4 error
0.10000	0.00089	0.00019	0.00000
0.05000	0.00051	0.00004	0.00000
0.02500	0.00027	0.00001	0.00000
0.01250	0.00014	0.00000	0.00000
0.00625	0.00007	0.00000	0.00000

Again, we use the internal precision of each number to continue with the analysis. A loglog plot of the error versus step size follows for each numerical method. Euler (solid line), RK2 (dashed), RK4 (dotted).



Finally, we used a computer routine to find the slope of each line. Euler: 0.9145, RK2: 2.0978, RK4: 4.0919. Note that each of these numbers is consistent with the order of the particular method.

13. The equation is separable.

$$\begin{aligned}\frac{dx}{x^2} &= -t\,dt \\ -\frac{1}{x} &= -\frac{1}{2}t^2 + C \\ \frac{1}{x} &= \frac{1}{2}t^2 - C\end{aligned}$$

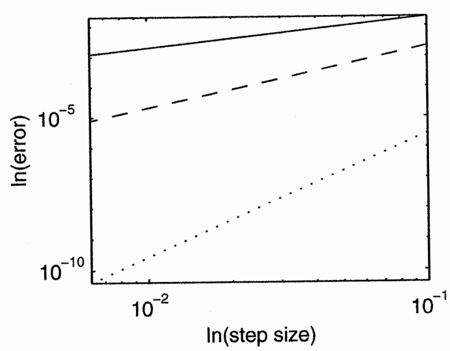
The initial condition $x(0) = 3$ gives $C = -(1/3)$. Thus,

$$\begin{aligned}\frac{1}{x} &= \frac{1}{2}t^2 + \frac{1}{3} \\ x &= \frac{6}{3t^2+2}.\end{aligned}$$

A table is constructed to evaluate the error made at each step size. The error is the magnitude of the difference between $x(2) = 3/7$ and the approximation predicted by the particular method (Eul, RK2, RK4) at $t = 2$. The step size used in constructing the table is $h = 0.1$.

h	Eul error	RK2 error	RK4 error
0.10000	0.02045	0.00220	0.00000
0.05000	0.01001	0.00052	0.00000
0.02500	0.00495	0.00013	0.00000
0.01250	0.00246	0.00003	0.00000
0.00625	0.00123	0.00001	0.00000

Again, we use the internal precision of each number to continue with the analysis. A loglog plot of the error versus step size follows for each numerical method. Euler (solid line), RK2 (dashed), RK4 (dotted).



Finally, we used a computer routine to find the slope of each line. Euler: 1.0135, RK2: 2.0303, RK4: 4.0256. Note that each of these numbers is consistent with the order of the particular method.

14. (a) Substitute $x = \cos t$ and $y = -\sin t$ into the first equation.

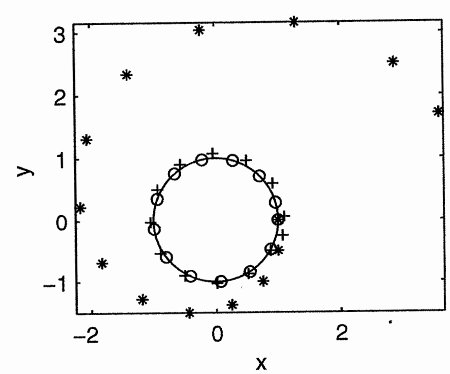
$$\begin{aligned}x' &= y \\ (\cos t)' &= -\sin t \\ -\sin t &= -\sin t\end{aligned}$$

Therefore, the first equation is satisfied. Substitute into the second equation.

$$\begin{aligned}y' &= -x \\ (-\sin t)' &= -\cos t \\ -\cos t &= -\cos t\end{aligned}$$

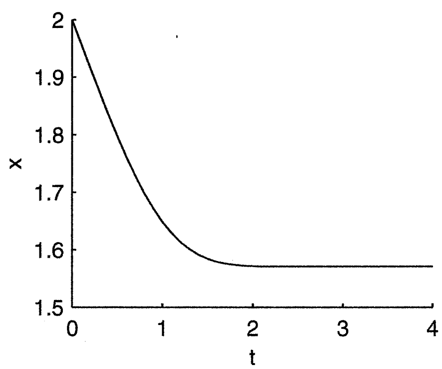
Therefore, the equations satisfy the second equation. Thus, $x = \cos t$ and $y = -\sin t$ is a solution of the system. Furthermore, $x(0) = \cos 0 = 1$ and $y(0) = -\sin 0 = 0$, so the initial conditions are also satisfied.

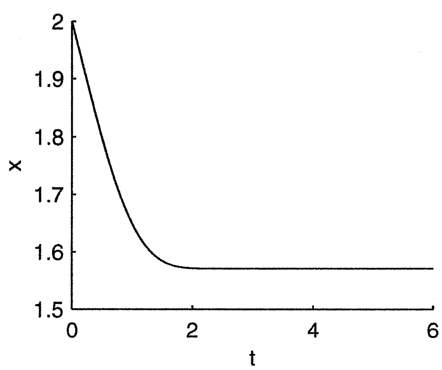
(b) The exact solution plus three numerical solutions are shown in the next figure. In each case, a step size of $h = 0.5$ was used. Note that the RK4 is again the most accurate. Exact (solid), Euler (asterisk), RK2 (plus), RK4 (circles).



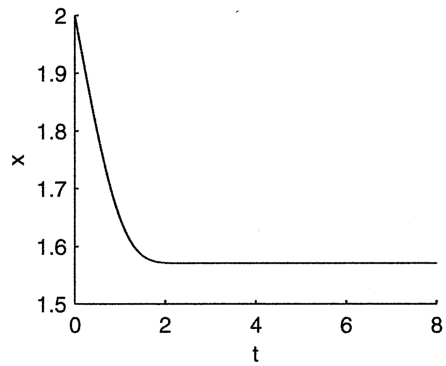
Section 6.4. Practical Use of Solvers

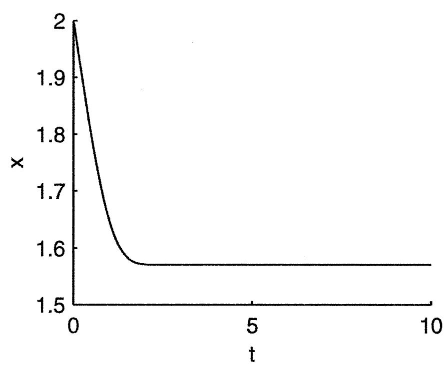
1. The following figure shows the solution on $[0, 4]$, with step-size $h = 0.1$, and elapsed time $t \approx 0.11$ s. The next figure shows the solution on $[0, 6]$, with step-size $h = 0.01$, and elapsed time $t \approx 1.43$ s. The run was done on a P166 Pentium machine. Answers will vary on different systems.



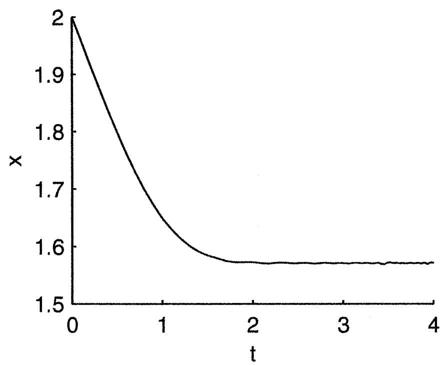


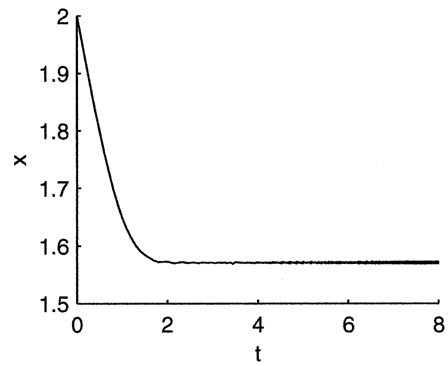
The following figure shows the solution on $[0, 8]$, with step-size $h = 0.001$, and elapsed time $t \approx 19.12$ s. The next figure shows the solution on $[0, 10]$, with step-size $h = 0.0001$, and elapsed time $t \approx 238.6$ s. The run was also done on a P166 machine. Answers will vary on different systems.

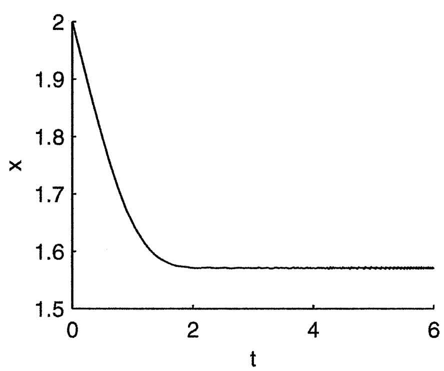


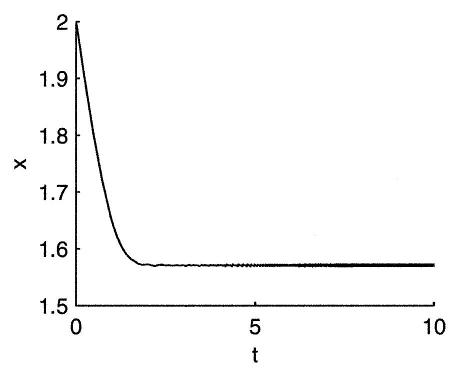


2. The following figure shows the solution on $[0, 4]$. The elapsed time was $t \approx 0.55$ s, the minimum step-size was 0.006, and the maximum step-size was 0.1. The next figure shows the solution on $[0, 6]$. The elapsed time was $t \approx 1.27$ s, the minimum step-size was 0.0018, and the maximum step size was 0.15.



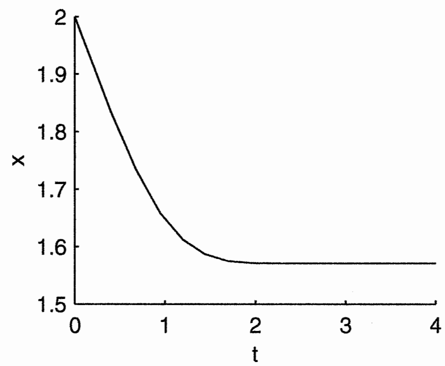


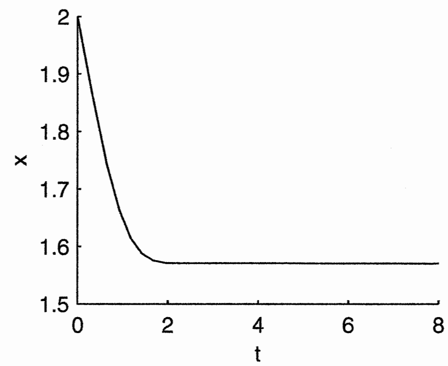


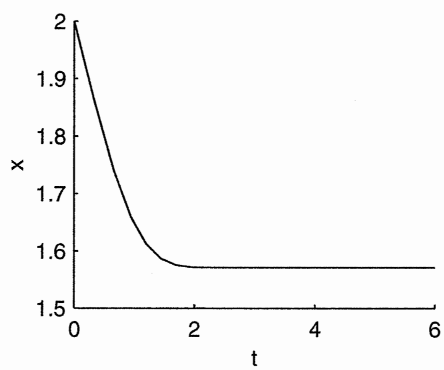


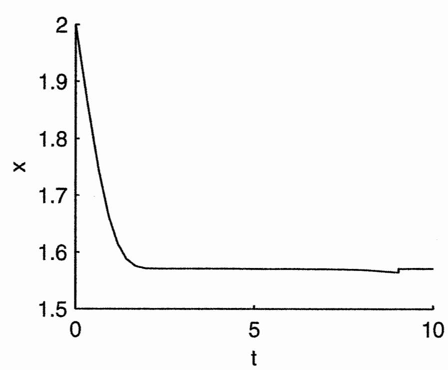
The following figure shows the solution on [0, 8]. The elapsed time was $t \approx 8.46$ s, the minimum step-size was 1.6×10^{-4}, and the maximum step-size was 0.2. The next figure shows the solution on [0, 10]. The elapsed times was $t \approx 66.19$ s, the minimum step-size was 2.1×10^{-5}, and the maximum step-size was 0.24. All runs were on a 166 MHz machine. Answers will vary on other systems.

3. The following figure shows the solution on the interval [0, 4]. The elapsed time was $t \approx 0.72$ s, the minimum step-size was 0.2420, and the maximum step-size was 0.4232. The next figure shows the solution on [0, 6]. The elapsed times was $t \approx 0.22$ s, the minimum step-size was 0.2422, and the maximum step-size was 0.6304.



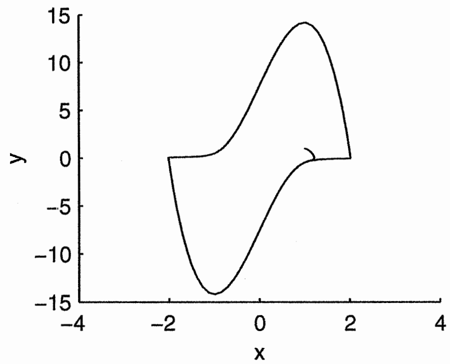


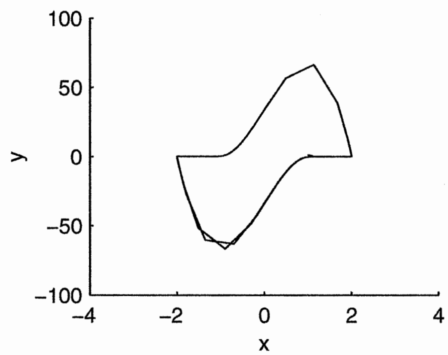




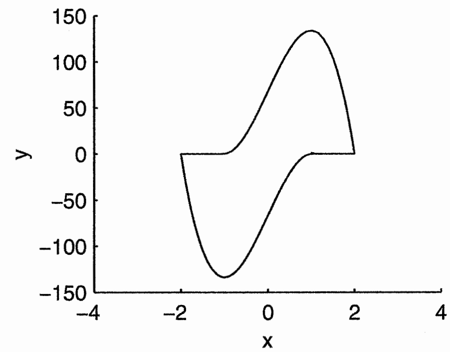
The following figure shows the solution on the interval [0, 8]. The elapsed times was $t \approx 0.28$ s. The minimum step-size was 0.1571, and the maximum step-size was 0.8. The next figure shows the solution on [0, 10]. The elapsed time was $t \approx 0.44$ s, the minimum step-size was 4.3×10^{-4}, and the maximum step-size was 1. Results were run on a 166 MHz system. Answers will vary on other systems.

4. The following figure shows the solution for $\mu = 10$ on the interval $I = [0, 20]$. The elapsed time was $t \approx 8.02$ s, using step size $h = 0.01$. The next figure shows the solution for $\mu = 50$ on the interval $I = [0, 100]$. The elapsed time was $t \approx 43.89$, using step-size $h = 0.01$.

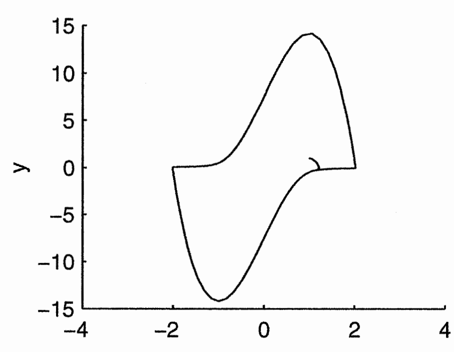


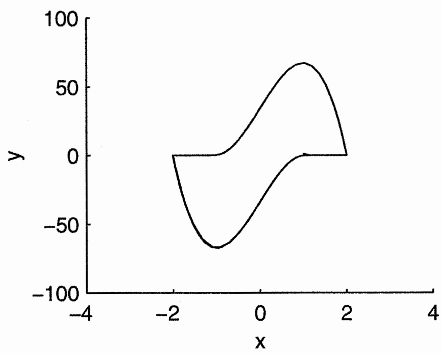


The following figure shows the solution for $\mu = 100$ on the interval $I = [0, 200]$. The elapsed time was $t \approx 844.53$ s, using step-size $h = 0.001$. The runs were made on a 166 MHz machine. Answers will vary on different systems.

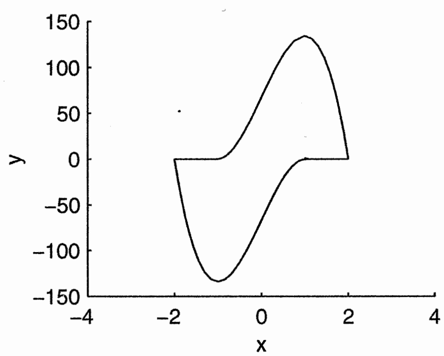


5. Te following figure shows the solution for $\mu = 10$ on $I = [0, 20]$. The elapsed time was $t \approx 2.03$ s, the minimum step-size was 0.0051, and the maximum step-size was 0.0889. The next figure shows the solution for $\mu = 50$ on $I = [0, 100]$. The elapsed time was $t \approx 39.27$ s, the minimum step-size was 0.0010, and the maximum step-size was 0.0585.

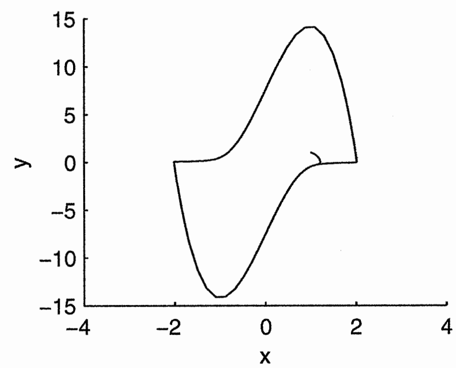


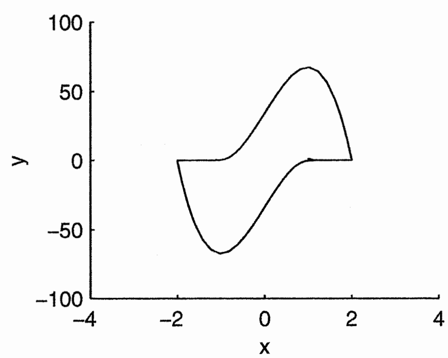


The following figure shows the solution for $\mu = 50$ on $I = [0, 200]$. The elapsed time was $t \approx 187.9$ s, the minimum step-size was 5.3×10^{-4}, and the maximum step-size was 0.0442. Answers will vary on different systems.

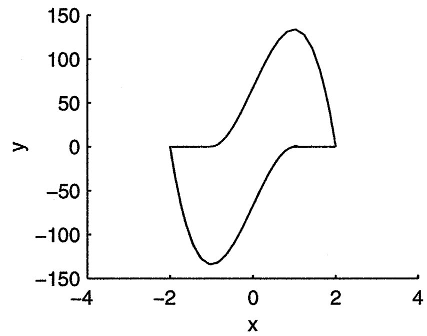


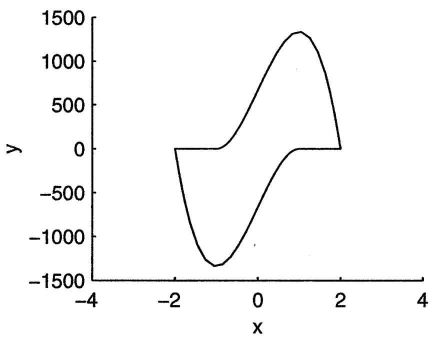
6. The following figure shows the solution for $\mu = 10$ on $I = [0, 20]$. The elapsed times was $t \approx 3.79$ s, the minimum step-size was 0.0035, and the maximum step-size was 0.4918. The next figure shows the solution for $\mu = 50$ on $I = [0, 100]$. The elapsed time was $t \approx 7.69$ s, the minimum step-size was 7.4×10^{-4}, and the maximum step-size was 2.1179.





The following figure shows the solution for $\mu = 100$ on $I = [0, 200]$. The elapsed time was $t \approx 9.67$ s, the minimum step-size was 4.2×10^{-4}, and the maximum step-size was 4.1960. The next figure shows the solution for $\mu = 1000$ on the interval $I = [0, 2000]$. The elapsed time was $t \approx 11.42$ s.





7. (a) If we substitute $x(t) = 2e^{-t} + \sin t$ into the left-hand side of the first equation,

$$x' = (2e^{-t} + \sin t)' = -2e^{-t} + \cos t.$$

If we substitute $x(t)$ and $y(t) = 2e^{-t} + \cos t$ into the right-hand side of the first equation,

$$\begin{aligned} &-2x + y + 2\sin t \\ &\quad = -2(2e^{-t} + \sin t) \\ &\qquad + (2e^{-t} + \cos t) + 2\sin t \\ &\quad = -2e^{-t} + \cos t. \end{aligned}$$

Thus, the solution satisfies the first equation. Substituting into the left-hand side of the second equation,

$$y' = (2e^{-t} + \cos t)' = -2e^{-t} - \sin t.$$

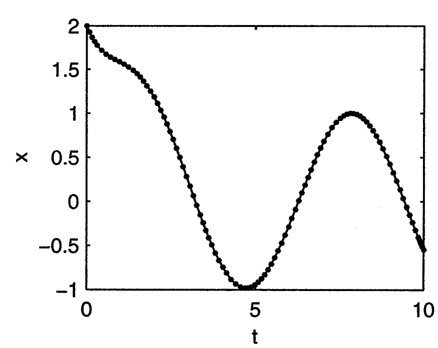
Substituting into the right-hand side of the second equation,

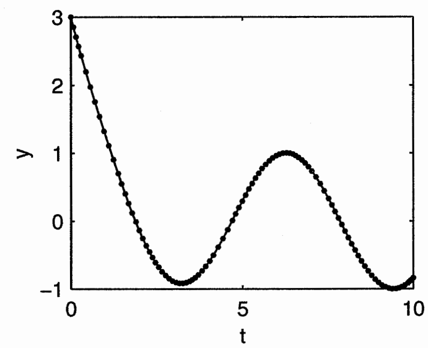
$$\begin{aligned} &x - 2y + 2(\cos t - \sin t) \\ &\quad = 2e^{-t} + \sin t - 2(2e^{-t} + \cos t) \\ &\qquad + 2\cos t - 2\sin t \\ &\quad = -2e^{-t} - \sin t. \end{aligned}$$

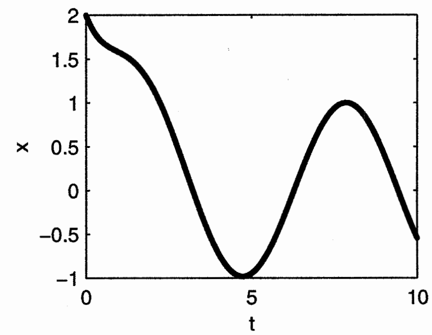
Thus, the solutions satisfy the second equation. Finally, $x(0) = 2e^0 + \sin 0 = 2$ and $y(0) = 2e^0 + \cos 0 = 3$.

Matlab's `ode45`, a variable-step solver, produced a numerical solution in about 0.22 seconds on a 300 MHz PC. The calculation required about 15,353 floating point operations, which was calculated with Matlab's `flops` command.

Two images follow. The first contains the exact and numerical solution of x versus t. The second contains that of y versus t.







(b) First, we'll show that $x(t) = 2e^{-t} + \sin t$ and $y(t) = 2e^{-t} + \cos t$ are also solutions of the second system. Obviously, because the first equation of the second system is identical to the first equation of the first system, both $x(t)$ and $y(t)$ still satisfy this equation. Now for the second equation. Subbing $y(t)$ into the left-hand side of the second equation,

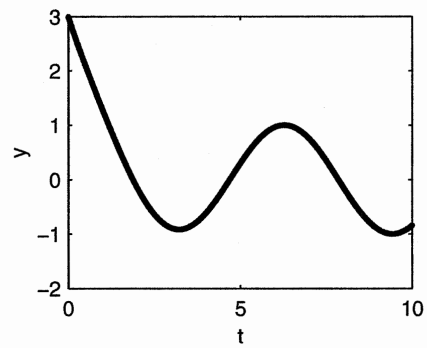
$$y' = (2e^{-t} + \cos t)' = -2e^{-t} - \sin t.$$

Substituting $x(t)$ and $y(t)$ into the right-hand side of the second equation,

$$\begin{aligned} &-2x + y + 2\sin t \\ &\quad = 998(2e^{-t} + \sin t) - 999(2e^{-t} + \cos t) \\ &\qquad + 999\cos t - 999\sin t \\ &\quad = -2e^{-t} - \sin t. \end{aligned}$$

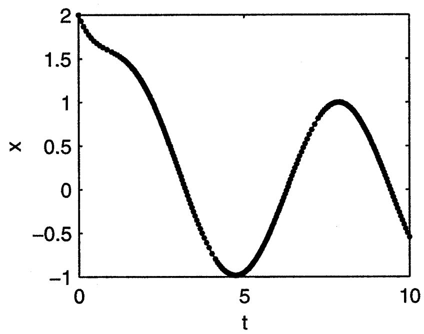
Thus, the equations satisfy the second system. Finally, both $x(t)$ and $y(t)$ still satisfy the same initial conditions.

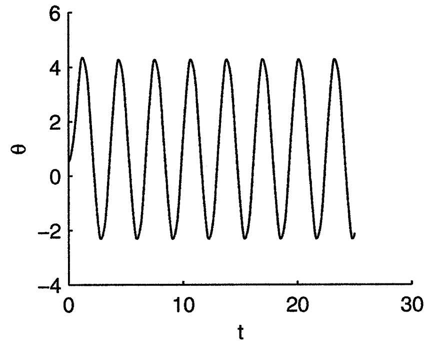
The same solver, `ode45`, was used to produce a numerical solution of the second system, same initial conditions, same interval, and, of course, same exact solutions.

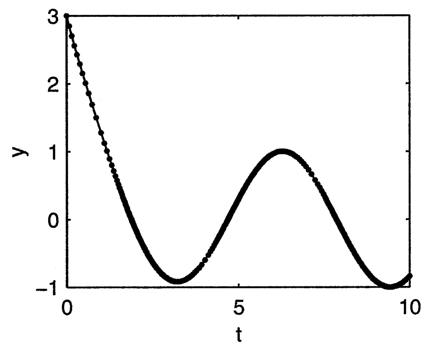


However, the time required elevated to about 12 seconds and the floating point operations required increased to 1,778,814! Two things are apparent. First, the second system is stiff. Secondly, stiffness is not a property of the solution, as both systems in this case have precisely the same solution. Rather, stiffness is some inherent part of the differential equations.

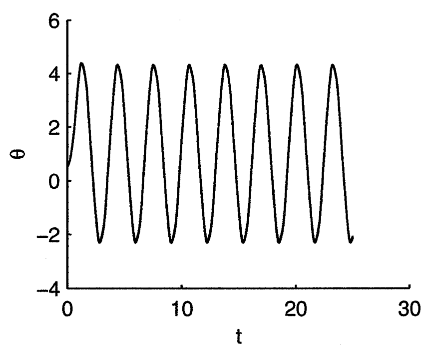
(c) This time we solved the second system with Matlab's `ode23s`, a stiff solver. The performance improved dramatically during the run, which lasted about 2.31 seconds, using approximately 84,703 floating point operations. This considerable savings in resources is evident is the less dense plots of the second system that follow.





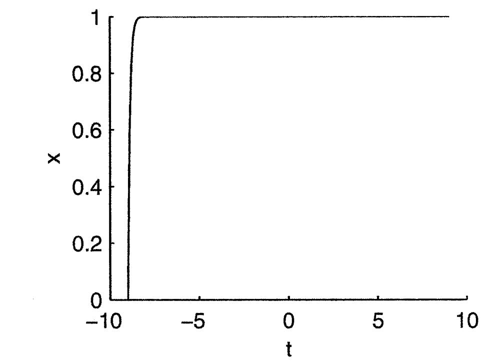


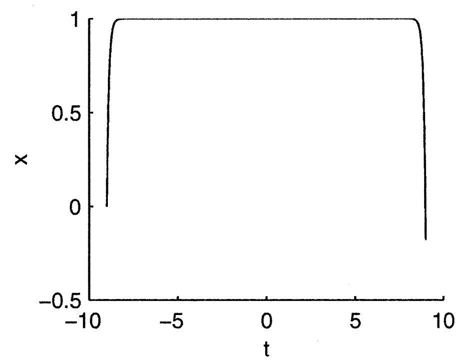
8. Using RK4, we found nice agreement on [0, 25] using a step-size $h = 0.1$, but even better agreement using $h = 0.05$, as the figures that follow show.

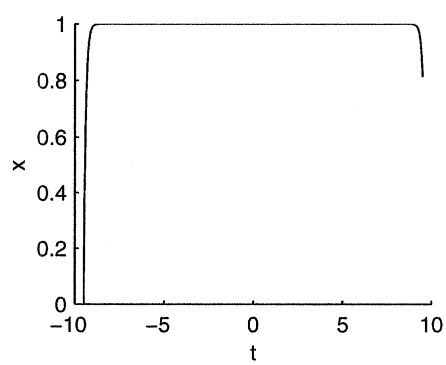


In a similar manner, on [0, 50] with $h = 0.05$ some of the color of the first graph bled through the color of the second graph, but reducing the step size to $h = 0.025$ rectified the situation, producing excellent agreement between the two solutions. On [0, 75], some color bled through at $h = 0.025$, but halving the step size to $h = 0.0125$ rectified the situation. Halving this again produced agreement on [0, 100], but the computation was agonizingly slow on a 166 MHz machine. One would be inclined to estimate that one would have to have the step-size 4 more times to $h \approx 3.9 \times 10^{-4}$ get good agreement on [0, 200].

9. The initial value problem $x' = t(x-1), x(-10) = 0$, has solution $x(t) = 1 - e^{(t^2-100)/2}$. Note that $x(10) = 0$. We used Matlab and an RK4 routine to produce numerical solutions at step sizes $h = 0.1$, 0.01, and 0.001. Note that none of these solutions give the correct solution (namely, zero) at $t = 10$.



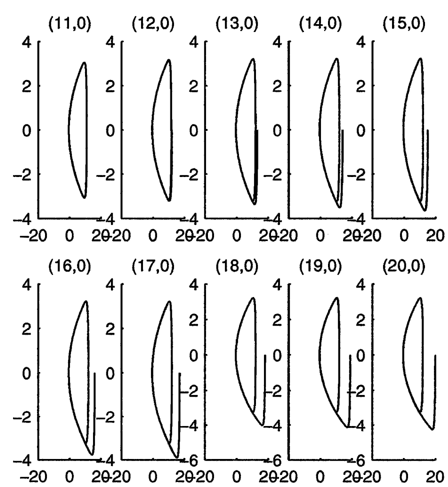




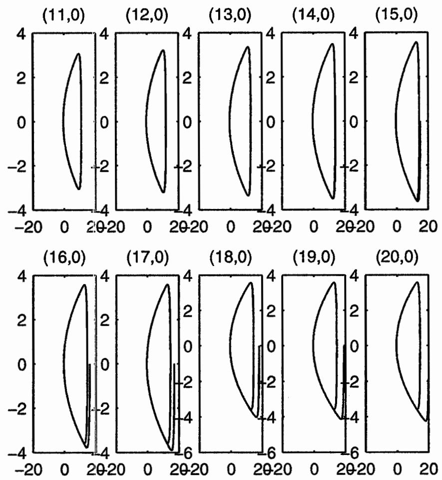
The run time and floating point operations at each step size are summarized in the following table.

Step size	Run Time	Flops
0.1	0.17	6,250
0.01	2.64	62,050
0.001	105.9	620,050

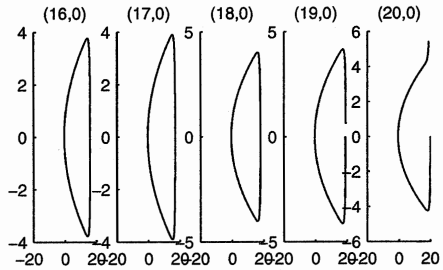
10. We decided to attack this experiment with the RK4 algorithm. Fixing a step size, $h = 0.1$, setting the time interval, $[0, 18]$, then using initial conditions $(k, 0)$, $k = 11, 11, \ldots, 20$, we found an image that helped gain insight into the task at hand. Each solution in the following image is marked with its initial condition.



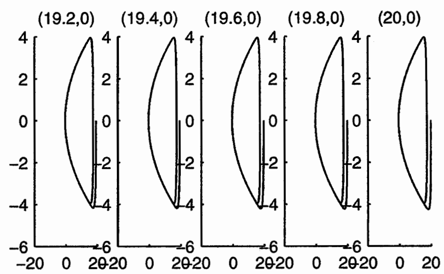
The solutions with initial conditions $(11, 0)$ and $(12, 0)$ appear to be closed curves. The others do not. We decreased the step size to $h = 0.05$.



The first five appear to be closed curves, the others do not. So, now we are thinking that problems might occur on our system in the last five images; that is, starting with initial condition $(k, 0)$, where $16 \le k \le 20$. Again, we decrease the step size, but concentrate on the last five initial conditions. We use step size $h = 0.01$ for our next image.



Some sort of strange behavior occurs between initial conditions $(19, 0)$ and $(20, 0)$. We increased the time interval to $[0, 20]$, then looked at initial conditions $(k, 0)$, with $k = 19.2, 19.4, \ldots, 20$. We also try a step size of $h = 0.005$.



At this point, we will stop and leave the remainder of the exploration to you. Perhaps a reduction is step size will allow the solution with initial condition $(20, 0)$ to close, perhaps we can move beyond $(20, 0)$. But the machine time is quickly becoming prohibitive, at least on our system.