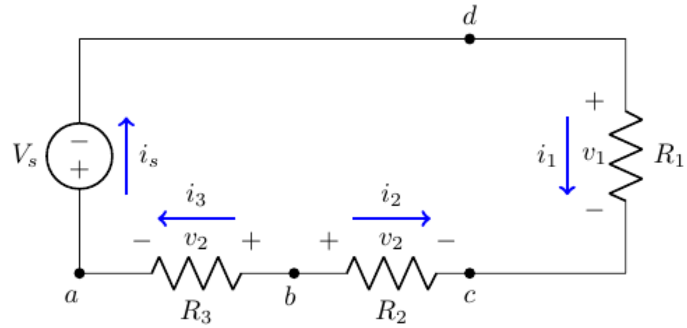
Solution *Section* 1.8 - Basic Electrical Circuit

Exercise

Sum the currents at each node in he circuit



Solution

$$\textit{node } \boldsymbol{a}: \quad i_s - i_1 = 0$$

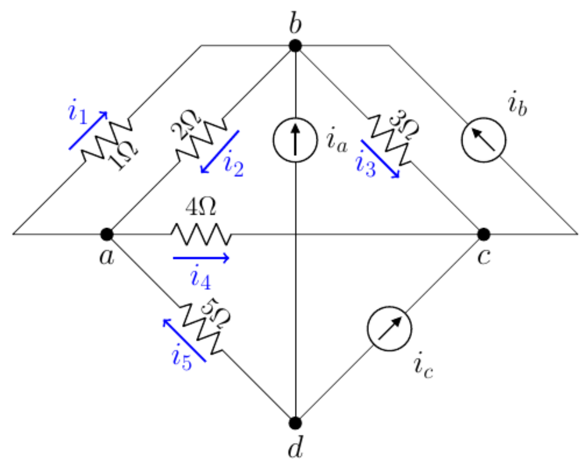
$$\textit{node } \boldsymbol{b}: \quad i_1 + i_c = 0$$

$$\textit{node } \boldsymbol{c}: \quad -i_c - i_1 = 0$$

$$\textit{node } \boldsymbol{d}: \quad i_1 - i_s = 0$$

Exercise

Sum the currents at each node in he circuit



Solution

$$\textit{node } \boldsymbol{a}: \quad i_1 + i_4 - i_2 - i_5 = 0$$

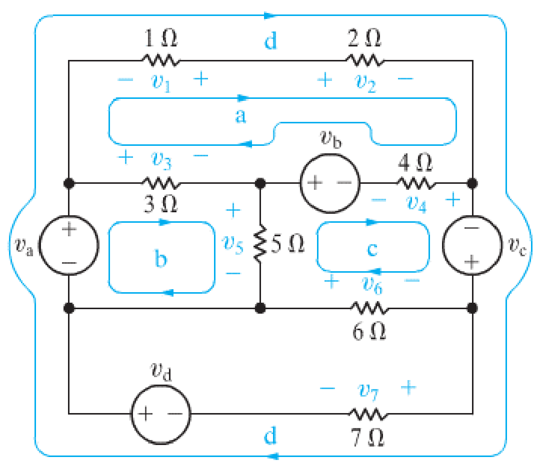
$$\textit{node } \boldsymbol{b}: \quad i_2 + i_3 - i_1 - i_b - i_a = 0$$

$$\textit{node } \boldsymbol{c}: \quad i_b - i_3 - i_4 - i_c = 0$$

$$\textit{node } \boldsymbol{d}: \quad i_5 + i_a + i_c = 0$$

Exercise

Sum the voltges around rach designated path in the circuit



Solution

$$\textit{path } \boldsymbol{a}: \quad -v_1 + v_2 + v_4 - v_b - v_3 = 0$$

$$\textit{path } \boldsymbol{b}: \quad -v_a + v_3 + v_5 = 0$$

$$\textit{path } \boldsymbol{c}: \quad v_b - v_4 - v_c - v_6 - v_5 = 0$$

$$\textit{path } \boldsymbol{d}: \quad -v_a - v_1 + v_2 - v_c + v_7 - v_d = 0$$

Exercise

A resistor $R = 20\ \Omega$ and a capacitor of $C = 0.1\ F$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge on the capacitor at time t for the given $E(t) = 100\sin 2t$

Solution

$$20Q' + \frac{1}{0.1}Q = 100\sin 2t \qquad R\frac{dQ}{dt} + \frac{1}{C}Q = E$$

$$Q' + \frac{1}{2}Q = 5\sin 2t$$

$$e^{\int \frac{1}{2}dt} = e^{t/2}$$

$$\int (5\sin 2t)e^{t/2}dt = \left(-\frac{5}{2}\cos 2t + \frac{5}{8}\sin 2t\right)e^{t/2} - \frac{1}{16}\int (5\sin 2t)e^{t/2}dt$$

$$\frac{17}{16}\int (5\sin 2t)e^{t/2}dt = \frac{5}{8}(-4\cos 2t + \sin 2t)e^{t/2}$$

$$\int (5\sin 2t)e^{t/2}dt = \frac{10}{17}(-4\cos 2t + \sin 2t)e^{t/2}$$

		$\int 5\sin 2t$
+	$e^{t/2}$	$-\frac{5}{2}\cos 2t$
−	$\frac{1}{2}e^{t/2}$	$-\frac{5}{4}\sin 2t$
+	$\frac{1}{4}e^{t/2}$	

$$Q(t) = e^{-t/2}\left(\frac{10}{17}(-4\cos 2t + \sin 2t)e^{t/2} + K\right)$$

$$= \frac{10}{17}(-4\cos 2t + \sin 2t) + Ke^{-t/2}$$

$$Q(0) = 0 \quad \rightarrow \quad 0 = \frac{10}{17}(-4) + K \quad \Rightarrow \quad K = \frac{40}{17}$$

$$Q(t) = \frac{10}{17}(\sin 2t - 4\cos 2t) + \frac{40}{17}e^{-t/2}$$

Exercise

A resistor $R = 20\ \Omega$ and a capacitor of $C = 0.1\ F$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge on the capacitor at time t for the given $E(t) = 100e^{-0.1t}$

Solution

$$20Q' + \frac{1}{0.1}Q = 100e^{-0.1t} \qquad R\frac{dQ}{dt} + \frac{1}{C}Q = E$$

$$Q' + \frac{1}{2}Q = 5e^{-0.1t}$$

$$e^{\int \frac{1}{2}dt} = e^{t/2}$$

$$\int \left(5e^{-0.1t}\right)e^{0.5t}dt = \int 5e^{0.4t}dt = \frac{25}{2}e^{2t/5}$$

$$Q(t)=e^{-t/2}\left(\frac{25}{2}e^{2t/5}+K\right)$$

$$=\frac{25}{2}e^{-t/10}+Ke^{-t/2}$$

$$Q(0)=0 \quad \rightarrow \ 0=\frac{25}{2}+K \ \Rightarrow \ \boxed{K=-\frac{25}{2}}$$

$$\boxed{Q(t)=\frac{25}{2}\left(e^{-t/10}-e^{-t/2}\right)}$$

Exercise

A resistor $R=20\ \Omega$ and a capacitor of $C=0.1\,F$ are joined in series with an electronic force (emf) $E=E(t)$ and no charge on the capacitor at $t=0$. Find the ensuing charge on the capacitor at time t for the given $E(t)=100\left(1-e^{-0.1t}\right)$

Solution

$$20Q'+\frac{1}{0.1}Q=100\left(1-e^{-0.1t}\right) \qquad\qquad R\frac{dQ}{dt}+\frac{1}{C}Q=E$$

$$Q'+\frac{1}{2}Q=5\left(1-e^{-0.1t}\right)$$

$$e^{\int \frac{1}{2}dt}=e^{t/2}$$

$$\int 5\left(1-e^{-0.1t}\right)e^{0.5t}dt=5\int\left(e^{t/2}-e^{0.4t}\right)dt=5\left(2e^{t/2}-\frac{5}{2}e^{2t/5}\right)$$

$$Q(t)=e^{-t/2}\left(10e^{t/2}-\frac{25}{2}e^{2t/5}+K\right)$$

$$=10-\frac{25}{2}e^{-t/10}+Ke^{-t/2}$$

$$Q(0)=0 \quad \rightarrow \ 0=10-\frac{25}{2}+K \ \Rightarrow \ \boxed{K=\frac{5}{2}}$$

$$\boxed{Q(t)=10-\frac{25}{2}e^{-t/10}+\frac{5}{2}e^{-t/2}}$$

Exercise

A resistor $R=20\ \Omega$ and a capacitor of $C=0.1\,F$ are joined in series with an electronic force (emf) $E=E(t)$ and no charge on the capacitor at $t=0$. Find the ensuing charge on the capacitor at time t for the given $E(t)=100\cos 3t$

Solution

$$20Q'+\frac{1}{0.1}Q=100\cos 3t \qquad\qquad R\frac{dQ}{dt}+\frac{1}{C}Q=E$$

$$Q'+\frac{1}{2}Q=5\cos 3t$$

$$e^{\int \frac{1}{2}dt}=e^{t/2}$$

$$\int (5\cos 3t)e^{t/2}dt = \left(\frac{5}{3}\sin 3t + \frac{5}{18}\cos 3t\right)e^{t/2} - \frac{1}{36}\int (5\cos 3t)e^{t/2}dt$$

$$\frac{37}{36}\int (5\cos 3t)e^{t/2}dt = \frac{5}{18}(6\sin 3t + \cos 3t)e^{t/2}$$

$$\int (5\cos 3t)e^{t/2}dt = \frac{10}{37}(6\sin 3t + \cos 3t)e^{t/2}$$

		$\int 5\cos 3t$
+	$e^{t/2}$	$\frac{5}{3}\sin 3t$
−	$\frac{1}{2}e^{t/2}$	$-\frac{5}{9}\cos 3t$
+	$\frac{1}{4}e^{t/2}$	

$$Q(t) = e^{-t/2}\left(\frac{10}{37}(6\sin 3t + \cos 3t)e^{t/2} + K\right)$$

$$= \frac{10}{37}(6\sin 3t + \cos 3t) + Ke^{-t/2}$$

$$Q(0) = 0 \quad \rightarrow \quad 0 = \frac{10}{37}(1) + K \;\Rightarrow\; \boxed{K = -\frac{10}{37}}$$

$$\boxed{Q(t) = \frac{10}{37}\left(6\sin 3t + \cos 3t - e^{-t/2}\right)}$$

Exercise

An inductor $(L = 1\ H)$ and a resistor $(R = 0.1\ \Omega)$ are joined in series with an electronic force (emf) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing charge current in the current at time t for the given $E(t) = 10 - 2t$

Solution

$$\frac{dI}{dt} + 0.1I = 10 - 2t \qquad\qquad L\frac{dI}{dt} + RI = E(t)$$

$$e^{\int .1dt} = e^{t/10}$$

$$\int (10-2t)e^{t/10}dt = (100 - 20t + 200)e^{t/10} = (300 - 20t)e^{t/10}$$

$$I(t) = e^{-10t}\left((300-20t)e^{t/10} + K\right)$$

$$= 300 - 20t + Ke^{-10t}$$

$$I(0) = 0 \quad \rightarrow \quad \boxed{K = -300}$$

$$\boxed{I(t) = 300 - 20t - 300e^{-t/10}}$$

Exercise

An inductor $(L = 1\ H)$ and a resistor $(R = 0.1\ \Omega)$ are joined in series with an electronic force (*emf*) $E = E(t)$ and no charge on the capacitor at $t = 0$. Find the ensuing current in the current at time t for the given $E(t) = 4\cos 3t$

Solution

$$\frac{dI}{dt}+0.1I=4\cos 3t \qquad\qquad L\frac{dI}{dt}+RI=E(t)$$

$$e^{\int .1dt}=e^{t/10}$$

$$\int (4\cos 3t)e^{t/10}=\left(\frac{4}{3}\sin 3t+\frac{2}{45}\cos 3t\right)e^{t/10}-\frac{1}{900}\int (4\cos 3t)e^{t/10}$$

$$\frac{901}{900}\int (4\cos 3t)e^{t/10}=\frac{2}{45}(30\sin 3t+\cos 3t)e^{t/10}$$

$$\int (4\cos 3t)e^{t/10}=\frac{40}{901}(30\sin 3t+\cos 3t)e^{t/10}$$

		$\int 4\cos 3t$
+	$e^{t/10}$	$\frac{4}{3}\sin 3t$
−	$\frac{1}{10}e^{t/10}$	$-\frac{4}{9}\cos 3t$
+	$\frac{1}{100}e^{t/10}$	

$$I(t)=e^{-t/10}\left(\frac{40}{901}(30\sin 3t+\cos 3t)e^{t/10}+K\right)$$

$$=\frac{40}{901}(30\sin 3t+\cos 3t)+Ke^{-t/10}$$

$$I(0)=0 \quad\rightarrow\quad \boxed{K=-\frac{40}{901}}$$

$$\boxed{I(t)=\frac{40}{901}\left(30\sin 3t+\cos 3t-e^{-t/10}\right)}$$

Exercise

An inductor $(L=1\ H)$ and a resistor $(R=0.1\ \Omega)$ are joined in series with an electronic force (*emf*) $E=E(t)$ and no charge on the capacitor at $t=0$. Find the ensuing current in the current at time *t* for the given $E(t)=4\sin 2\pi t$

Solution

$$\frac{dI}{dt}+0.1I=4\sin 2\pi t \qquad\qquad L\frac{dI}{dt}+RI=E(t)$$

$$e^{\int .1dt}=e^{t/10}$$

$$\int (4\sin 2\pi t)e^{t/10}=\left(-\frac{2}{\pi}\cos 2\pi t+\frac{1}{10\pi^2}\sin 2\pi t\right)e^{t/10}-\frac{1}{400\pi^2}\int (4\sin 2\pi t)e^{t/10}$$

$$\frac{1+400\pi^2}{400\pi^2}\int (4\sin 2\pi t)e^{t/10}=\frac{1}{10\pi^2}(-20\pi\cos 2\pi t+\sin 2\pi t)e^{t/10}$$

$$\int (4\sin 2\pi t)e^{t/10}=\frac{40}{1+400\pi^2}(-20\pi\cos 2\pi t+\sin 2\pi t)e^{t/10}$$

		$\int 4\sin 2\pi t$
+	$e^{t/10}$	$-\frac{2}{\pi}\cos 2\pi t$
−	$\frac{1}{10}e^{t/10}$	$-\frac{1}{\pi^2}\sin 2\pi t$
+	$\frac{1}{100}e^{t/10}$	

$$I(t)=e^{-t/10}\left(\frac{40}{1+400\pi^2}(-20\pi\cos 2\pi t+\sin 2\pi t)e^{t/10}+K\right)$$

$$=\frac{40}{1+400\pi^2}(-20\pi\cos 2\pi t+\sin 2\pi t)+Ke^{-t/10}$$

$$I(0)=0 \quad\rightarrow 0=\frac{40}{1+400\pi^2}(-20\pi)+K \Rightarrow \boxed{K=\frac{800\pi}{1+25\pi^2}}$$

$$I(t)=\frac{40}{1+400\pi^2}(-20\pi\cos 2\pi t+\sin 2\pi t)+\frac{800}{1+400\pi^2}e^{-t/10}$$

$$=\boxed{\frac{40}{1+400\pi^2}\left(\sin 2\pi t-20\pi\cos 2\pi t+20\pi e^{-t/10}\right)}$$

Exercise

An *RL* circuit with a $1-\Omega$ resistor and a 0.1-H inductor is driven by a voltage $E(t)=\sin 100t\ V$. If the initial inductor current is zero, determine the subsequence resistor and inductor current and the voltages.

Solution

$$0.1\frac{dI}{dt}+I=\sin 100t \qquad\qquad L\frac{dI}{dt}+RI=E(t)$$

$$\frac{dI}{dt}+10I=10\sin 100t$$

$$e^{\int 10dt}=e^{10t}$$

$$\int(10\sin 100t)e^{10t}dt=\left(-\frac{1}{10}\cos 100t+\frac{1}{100}\sin 100t\right)e^{10t}-\int\left(\frac{1}{10}\sin 100t\right)e^{10t}dt$$

$$\left(10+\frac{1}{10}\right)\int(\sin 100t)e^{10t}dt=\frac{1}{100}(-10\cos 100t+\sin 100t)e^{10t}$$

$$\frac{101}{10}\int(10\sin 100t)e^{10t}dt=\frac{1}{100}(-10\cos 100t+\sin 100t)e^{10t}$$

$$\int(10\sin 100t)e^{10t}dt=\frac{1}{1010}(-10\cos 100t+\sin 100t)e^{10t}$$

		$\int 10\sin 100t$
+	e^{10t}	$-\frac{1}{10}\cos 100t$
−	$10e^{10t}$	$-\frac{1}{1000}\sin 100t$
+	$100e^{10t}$	

$$I(t)=e^{-10t}\left(\frac{1}{1010}(-10\cos 100t+\sin 100t)e^{10t}+K\right)$$

$$=\boxed{\frac{1}{1010}(-10\cos 100t+\sin 100t)+Ke^{-10t}}$$

$$I(0)=0\quad\rightarrow\quad 0=\frac{1}{1010}(-10)+K\quad\Rightarrow\quad \boxed{K=\frac{1}{101}}$$

$$\boxed{I(t)=\frac{1}{1010}(-10\cos 100t+\sin 100t)+\frac{1}{101}e^{-10t}}$$

The voltage at the resistor:

$$\boxed{E_R(t)}=RI=\boxed{\frac{1}{1010}(-10\cos 100t+\sin 100t)+\frac{1}{101}e^{-10t}}$$

The voltage at the inductor:

$$E_L(t)=L\frac{dI}{dt}=0.1\left(\frac{1}{1010}\left(10^3\sin 100t+100\cos 100t\right)-\frac{10}{101}e^{-10t}\right)$$

$$=\boxed{\frac{10}{101}\sin 100t+\frac{1}{101}\sin 100t-\frac{1}{101}e^{-10t}}$$

Exercise

An RL circuit with a $1-\Omega$ resistor and a 0.01-H inductor is driven by a voltage $E(t)=\sin 100t\ V$. If the initial inductor current is zero, determine the subsequence resistor and inductor current and the voltages.

Solution

$$0.01\frac{dI}{dt}+I=\sin 100t \qquad L\frac{dI}{dt}+RI=E(t)$$

$$\frac{dI}{dt}+100I=100\sin 100t$$

$$e^{\int 100dt}=e^{100t}$$

$$\int(100\sin 100t)e^{100t}dt=(-\cos 100t+\sin 100t)e^{100t}-\int(100\sin 100t)e^{100t}dt$$

$$2\int(100\sin 100t)e^{100t}dt=(-\cos 100t+\sin 100t)e^{100t}$$

$$\int(100\sin 100t)e^{100t}dt=\frac{1}{2}(-\cos 100t+\sin 100t)e^{100t}$$

		$\int 100\sin 100t$
+	e^{100t}	$-\cos 100t$
−	$100e^{100t}$	$-\frac{1}{100}\sin 100t$
+	$10000e^{100t}$	

$$I(t)=e^{-100t}\left(\frac{1}{2}(-\cos 100t+\sin 100t)e^{100t}+K\right)$$

$$=\frac{1}{2}(-\cos 100t+\sin 100t)+Ke^{-100t}$$

$$I(0)=0 \quad\to\quad 0=\frac{1}{2}(-1)+K \ \Rightarrow\ K=\frac{1}{2}$$

$$I(t)=\frac{1}{2}(-\cos 100t+\sin 100t)+\frac{1}{2}e^{-100t}$$

The voltage at the resistor:

$$E_R(t)=RI=\frac{1}{2}\left(\sin 100t-\cos 100t+e^{-100t}\right)$$

The voltage at the inductor:

$$E_L(t)=L\frac{dI}{dt}=(0.01)\frac{1}{2}\left(100\cos 100t+100\sin 100t-100e^{-100t}\right)$$

$$=\frac{1}{2}\left(\cos 100t+\sin 100t+e^{-10t}\right)$$

Exercise

An RL circuit with a $5-\Omega$ resistor and a 0.05-H inductor is driven by a voltage $E(t)=5\cos 120t\ V$. If the initial inductor current is 1 A, determine the subsequence resistor and inductor current and the voltages.

Solution

$$0.05\frac{dI}{dt}+5I=5\cos 120t \qquad L\frac{dI}{dt}+RI=E(t)$$

$$\frac{dI}{dt}+100I=100\cos 120t$$

$$e^{\int 100dt} = e^{100t}$$

$$\int (100\cos 120t)e^{100t}dt = \left(\frac{5}{6}\sin 120t + \frac{25}{36}\cos 120t\right)e^{100t} - \frac{25}{36}\int (100\cos 120t)e^{100t}dt$$

$$\left(1+\frac{25}{36}\right)\int (100\cos 120t)e^{100t}dt = \left(\frac{5}{6}\sin 120t + \frac{25}{36}\cos 120t\right)e^{100t}$$

$$\frac{61}{36}\int (100\cos 120t)e^{100t}dt = \frac{5}{36}(6\sin 120t + 5\cos 120t)e^{100t}$$

$$\int (100\cos 120t)e^{100t}dt = \frac{5}{61}(6\sin 120t + 5\cos 120t)e^{100t}$$

		$\int 100\cos 120t$
+	e^{100t}	$\frac{5}{6}\sin 120t$
−	$100e^{100t}$	$-\frac{1}{144}\cos 120t$
+	$10^4 e^{100t}$	

$$I(t) = e^{-100t}\left(\frac{5}{61}(6\sin 120t + 5\cos 120t)e^{100t} + K\right)$$

$$= \frac{5}{61}(6\sin 120t + 5\cos 120t) + Ke^{-100t}$$

$$I(0)=1 \quad \to 1 = \frac{5}{61}(5) + K \;\Rightarrow\; K = \frac{36}{61}$$

$$I(t) = \frac{5}{61}(6\sin 120t + 5\cos 120t) + \frac{36}{61}e^{-100t}$$

The voltage at the resistor:

$$E_R(t) = RI = \frac{25}{61}(6\sin 120t + 5\cos 120t) + \frac{180}{61}e^{-100t}$$

The voltage at the inductor:

$$E_L(t) = L\frac{dI}{dt} = (0.05)\left(\frac{25}{61}(720\cos 120t - 600\sin 120t) - \frac{18000}{61}e^{-100t}\right)$$

$$= 14.754\cos 120t - 12.295\sin 120t - 14.754e^{-100t}$$

Exercise

An *RC* circuit with a $1-\Omega$ resistor and a 10^{-6}-F capacitor is driven by a voltage $E(t) = \sin 100t\ V$. If the initial capacitor current is zero, determine the subsequence resistor and capacitor current and the voltages.

Solution

$$Q' + 10^6 Q = \sin 100t \qquad\qquad R\frac{dQ}{dt} + \frac{1}{C}Q = E$$

$$e^{\int 10^6 dt} = e^{10^6 t}$$

$$\int (\sin 100t)e^{10^6 t}dt = \left(-\frac{1}{100}\cos 100t + 100\sin 100t\right)e^{10^6 t} - 10^8\int (\sin 100t)e^{10^6 t}dt$$

$$\left(10^8+1\right)\int (\sin 100t)e^{10^6 t}dt = \frac{1}{100}\left(-\cos 100t + 10^4\sin 100t\right)e^{10^6 t}$$

$$\int (\sin 100t) e^{10^6 t} dt = \frac{1}{100\left(10^8+1\right)}\left(-\cos 100t + 10^4 \sin 100t\right) e^{10^6 t}$$

		$\int \sin 100t$
+	$e^{10^6 t}$	$-\frac{1}{100}\cos 100t$
−	$10^6 e^{10^6 t}$	$-10^{-4}\sin 100t$
+	$10^{12} e^{10^6 t}$	

$$Q(t) = e^{-10^6 t}\left(\frac{1}{10^{10}+100}\left(-\cos 100t + 10^4 \sin 100t\right) e^{10^6 t} + K\right)$$

$$= \frac{1}{10^{10}+100}\left(-\cos 100t + 10^4 \sin 100t\right) + Ke^{-10^6 t}$$

$$Q(0) = 0 \quad \rightarrow \; 0 = \frac{1}{10^{10}+100}(-1) + K \;\Rightarrow\; \boxed{K = \frac{1}{10^{10}+100}}$$

$$\boxed{Q(t) = \frac{1}{10^{10}+100}\left(-\cos 100t + 10^4 \sin 100t + e^{-10^6 t}\right)}$$

The voltage across the capacitor is:

$$\boxed{V_C = \frac{Q}{C} = \frac{10^6}{10^{10}+100}\left(-\cos 100t + 10^4 \sin 100t + e^{-10^6 t}\right)}$$

The current is:

$$I(t) = \frac{dQ}{dt} = \frac{1}{10^{10}+100}\left(100 \sin 100t + 10^6 \cos 100t - 10^6 e^{-10^6 t}\right)$$

$$\boxed{= \frac{1}{10^8+1}\left(\sin 100t + 10^4 \cos 100t - 10^4 e^{-10^6 t}\right)}$$

The voltage across the resistor is:

$$\boxed{V_R = RI = \frac{1}{10^8+1}\left(\sin 100t + 10^4 \cos 100t - 10^4 e^{-10^6 t}\right)}$$

Exercise

Solve the general initial value problem modeling the *RC* circuit $R\frac{dQ}{dt} + \frac{1}{C}Q = E, \quad Q(0) = 0$

Where E is a constant source of *emf*

Solution

$$\frac{dQ}{dt} + \frac{1}{RC}Q = \frac{E}{R}$$

$$e^{\int \frac{1}{RC} dt} = e^{t/RC}$$

$$\int \frac{E}{R} e^{t/RC} dt = \frac{E}{R} RCe^{t/RC} = ECe^{t/RC}$$

$$Q(t) = \frac{1}{e^{t/RC}}\left(ECe^{t/RC} - K\right) = EC - Ke^{-t/RC}$$

$$Q(t=0) = EC - K$$

$$0 = EC - K \quad \Rightarrow \quad K = EC$$

$$Q(t) = \boxed{EC\left(1 - e^{-t/RC}\right)}$$

Exercise

Solve the general initial value problem modeling the *LR* circuit $L\frac{dI}{dt} + RI = E, \quad I(0) = I_0$

Where E is a constant source of emf

Solution

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}$$

$$e^{\int \frac{R}{L}dt} = e^{(R/L)t}$$

$$\int \frac{E}{L}e^{(R/L)t}dt = \frac{E}{L}\frac{L}{R}e^{(R/L)t} = \frac{E}{R}e^{(R/L)t}$$

$$I(t) = \frac{1}{e^{(R/L)t}}\left(\frac{E}{R}e^{(R/L)t} - K\right)$$

$$= \frac{E}{R} - Ke^{-(R/L)t}$$

$$I(t=0) = \frac{E}{R} - K$$

$$I_0 = \frac{E}{R} - K \quad \Rightarrow \quad K = \frac{E}{R} - I_0$$

$$I(t) = \frac{E}{R} - \left(\frac{E}{R} - I_0\right)e^{-Rt/L}$$

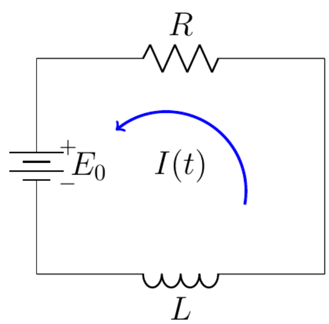
$$I(t) = \boxed{\frac{1}{R}\left(E + \left(RI_0 - E\right)e^{-Rt/L}\right)}$$

Exercise

For the given *RL*−circuit

Where E_0 is a constant source of *emf* at time $t = 0$.

Find the current $I(t)$ flowing in the circuit.



Solution

$$L\frac{dI}{dt} + RI = E_0$$

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E_0}{L}$$

$$e^{\int \frac{R}{L}dt} = e^{(R/L)t}$$

$$\int \frac{E_0}{L}e^{(R/L)t}dt = \frac{E_0}{L}\frac{L}{R}e^{(R/L)t} = \frac{E_0}{R}e^{(R/L)t}$$

$$I(t)=\frac{1}{e^{(R/L)t}}\left(\frac{E_0}{R}e^{(R/L)t}-K\right)$$

$$=\frac{E_0}{R}-Ke^{-(R/L)t}$$

$$I(t=0)=\frac{E_0}{R}-K=I_0$$

$$\boxed{K=\frac{E_0}{R}-I_0}$$

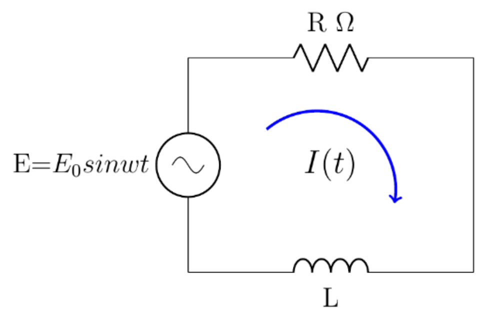
$$\boxed{I(t)=\frac{E_0}{R}-\left(\frac{E_0}{R}-I_0\right)e^{-(R/L)t}}$$

Exercise

For the given $RL-$circuit

Where $E=E_0\sin\omega t$ is the impressed voltage.

Find the current $I(t)$ flowing in the circuit.



Solution

$$L\frac{dI}{dt}+RI=E_0\sin\omega t$$

$$\frac{dI}{dt}+\frac{R}{L}I=\frac{E_0}{L}\sin\omega t$$

$$e^{\int\frac{R}{L}dt}=e^{(R/L)t}$$

$$\int\frac{E_0}{L}(\sin\omega t)e^{(R/L)t}dt=\frac{E_0}{L}\left(\sin\omega t-\frac{L}{R}\omega\cos\omega t\right)e^{(R/L)t}-\omega^2\frac{E_0}{L}\frac{L^2}{R^2}\int(\sin\omega t)e^{(R/L)t}dt$$

$$\frac{E_0}{L}\left(1+\frac{LE_0\omega^2}{R^2}\right)\int(\sin\omega t)e^{(R/L)t}dt=\frac{E_0}{L}\left(\sin\omega t-\frac{L}{R}\omega\cos\omega t\right)e^{(R/L)t}$$

$$\int\frac{E_0}{L}(\sin\omega t)e^{(R/L)t}dt=\frac{R^2}{1+LE_0\omega^2}\frac{E_0}{L}\left(\sin\omega t-\frac{L}{R}\omega\cos\omega t\right)e^{(R/L)t}$$

$$I(t)=\frac{1}{e^{(R/L)t}}\left(\frac{R^2E_0}{L+E_0L^2\omega^2}\left(\sin\omega t-\frac{L}{R}\omega\cos\omega t\right)e^{(R/L)t}+K\right)$$

$$=\frac{R^2E_0}{L+E_0L^2\omega^2}\left(\sin\omega t-\frac{L\omega}{R}\cos\omega t\right)+Ke^{-(R/L)t}$$

$$I(t=0)=-\frac{L\omega}{R}\frac{R^2E_0}{L+E_0L^2\omega^2}+K=I_0$$

$$\boxed{K = I_0 + \frac{R\omega E_0}{1 + E_0 L\omega^2}}$$

$$I(t) = \frac{R^2 E_0}{L + E_0 L^2 \omega^2}\left(\sin \omega t - \frac{L\omega}{R}\cos \omega t\right) + \left(I_0 + \frac{R\omega E_0}{1 + E_0 L\omega^2}\right)e^{-(R/L)t}$$

		$\int e^{(R/L)t}$
+	$\sin \omega t$	$\frac{L}{R}e^{(R/L)t}$
−	$\omega \cos \omega t$	$\left(\frac{L}{R}\right)^2 e^{(R/L)t}$
+	$-\omega^2 \sin \omega t$	

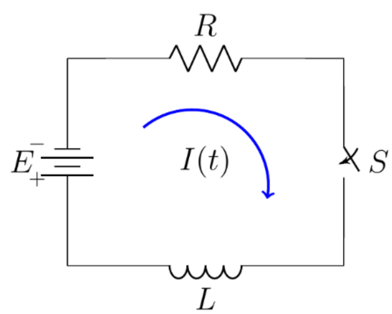
Exercise

For the given $RL-$circuit

Which has a constant impressed voltage E, a resistor of resistance R, and a coil of impedance L.

Find the current $I(t)$ flowing in the circuit.

Solution



$$L\frac{dI}{dt} + RI = E$$

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}$$

$$e^{\int \frac{R}{L}dt} = e^{(R/L)t}$$

$$\int \frac{E}{L}e^{(R/L)t}dt = \frac{E}{L}\frac{L}{R}e^{(R/L)t} = \frac{E}{R}e^{(R/L)t}$$

$$I(t) = \frac{1}{e^{(R/L)t}}\left(\frac{E}{R}e^{(R/L)t} - K\right)$$

$$= \frac{E}{R} - Ke^{-(R/L)t}$$

$$I(t = 0) = \frac{E}{R} - K = 0$$

$$\boxed{K = \frac{E}{R}}$$

$$\boxed{I(t) = \frac{E}{R} - \frac{E}{R}e^{-(R/L)t}}$$

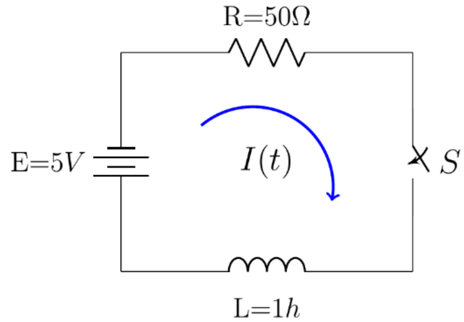
$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty}\left(\frac{E}{R} - \frac{E}{R}e^{-(R/L)t}\right) = \boxed{\frac{E}{R}}$$

Exercise

For the given $RL-$circuit

Which has a constant impressed voltage E, a resistor of resistance R, and a coil of impedance L. Find the current $I(t)$ flowing in the circuit.

Solution



$$L\frac{dI}{dt}+RI=E$$

$$\frac{dI}{dt}+50I=5$$

$$e^{\int 50dt}=e^{50t}$$

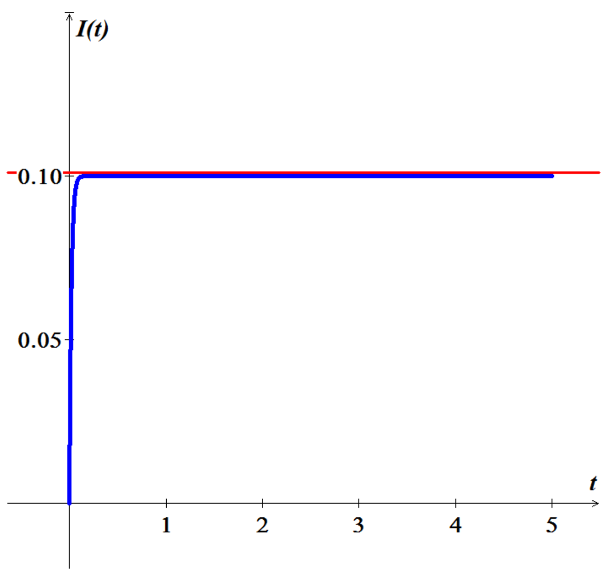
$$\int 5e^{50t}dt=\frac{1}{10}e^{50t}$$

$$I(t)=\frac{1}{e^{50t}}\left(\frac{1}{10}e^{50t}+K\right)$$

$$=\frac{1}{10}+Ke^{-50t}$$

$$I(t=0)=\frac{1}{10}+K=0 \quad\rightarrow\quad \boxed{K=-\frac{1}{10}}$$

$$\boxed{I(t)=\frac{1}{10}\left(1-e^{-50t}\right)}$$



Exercise

Consider the circuit shown and let I_1, I_2, and I_3 be the currents through the capacitor, resistor, and inductor, respectively. Let V_1, V_2, and V_3 be the corresponding voltage drops. The arrows denote the arbitrary chosen directions in which currents and voltage drops will be taken to be positive.

a) Applying Kirchhoff's second law to the upper loop in the circuit, show that

$$V_1 - V_2 = 0 \quad \textit{and} \quad V_2 - V_3 = 0$$

b) Applying Kirchhoff's first law to either node in the circuit, show that $I_1 + I_2 + I_3 = 0$

c) Use the current-voltage relation through each element in the circuit to obtain the equations

$$CV_1' = I_1, \quad V_2 = RI_2, \quad LI_3' = V_3$$

d) Eliminate V_2, V_3, I_1 and I_2 to obtain $\quad CV_1' = -I_3 - \frac{V_1}{R}, \quad LI_3' = V_1$

Solution

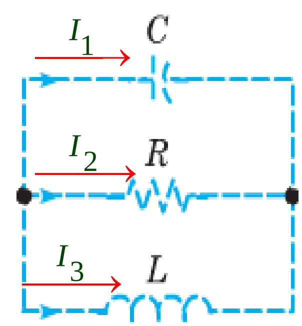
a) Taking the clockwise loop around each paths, it is easy to see that voltage drops are given by

$$V_1 - V_2 = 0 \quad \textit{and} \quad V_2 - V_3 = 0$$

b) Consider the right node. The current is given by $I_1 + I_2$. The current leaving the node is $-I_3$.

Hence the cursing through the node is $I_1 + I_2 - \left(-I_3\right)$.

Based on Kirchhoff's first law, $I_1 + I_2 + I_3 = 0$



c) In the capacitor $\quad CV_1' = I_1$

In the resistor $\quad V_2 = RI_2$

In the inductor $\quad LI_3' = V_3$

d) Based on part (a), $V_1 = V_2 = V_3$. Based on part (*b*),

$$CV_1' + \frac{1}{R}V_2 + I_3 = 0$$

It follows: $CV_1' = -I_3 - \frac{V_1}{R}, \quad LI_3' = V_1$

Exercise

Consider the circuit. Use the method outlined to show that the current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations.

$$L\frac{dI}{dt} = -R_1 I - V, \quad C\frac{dV}{dt} = I - \frac{V}{R_2}$$

Solution

let I_1, I_2, I_3 and I_4 be the current through the resistors, inductor, and capacitor, respectively.

Assign V_1, V_2, V_3 and V_4 to be the corresponding voltage drops.

Based on Kirchhoff's second law, the net voltage drops, around each loop, satisfy

$$V_1 + V_3 + V_4 = 0, \quad V_1 + V_3 + V_2 = 0 \quad \text{and} \quad V_4 - V_2 = 0$$

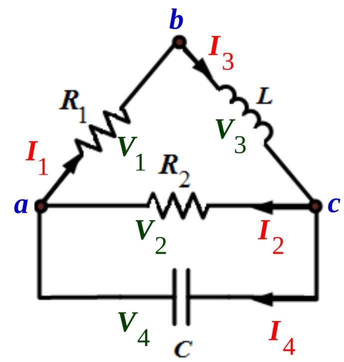
Applying Kirchhoff's first law:

Node ***a***: $\quad I_1 - \left(I_2 + I_4\right) = 0$

Node ***b***: $\quad I_1 - I_3 = 0 \quad \rightarrow \quad I_1 = I_3$

Node ***c***: $\quad I_2 + I_4 - I_1 = 0 \quad \rightarrow \quad I_2 + I_4 = I_1$

$$I_2 + I_4 = I_3 \quad \Rightarrow \quad \boxed{I_2 + I_4 - I_3 = 0}$$



Using the current-voltage relations:

$$V_1 = R_1 I_1 = R_1 I_3 \qquad V_2 = R_2 I_2$$

$$L I_3' = V_3 \qquad C V_4' = I_4$$

$$V_1 + V_3 + V_4 = 0 \quad \Rightarrow \quad \boxed{R_1 I_3 + L I_3' + V_4 = 0}$$

$$I_4 = I_3 - I_2 \quad \Rightarrow \quad \boxed{C V_4' = I_3 - \frac{V_2}{R_2}}$$

Let $I_3 = I$ *and* $V_4 = V$

$$R_1 I + L I' + V = 0 \quad \Rightarrow \quad \boxed{L I' = -R_1 I - V}$$

$$C V_4' = I_3 - \frac{V_2}{R_2} \quad \Rightarrow \quad \boxed{C V' = I - \frac{V}{R_2}}$$

Exercise

Consider an electric circuit containing a capacitor, resistor, and battery.

The charge $Q(t)$ on the capacitor satisfies the equation

$$R\frac{dQ}{dt} + \frac{Q}{C} = V$$

Where R is the resistance, C is the capacitance, and V is the constant voltage supplied by the battery.

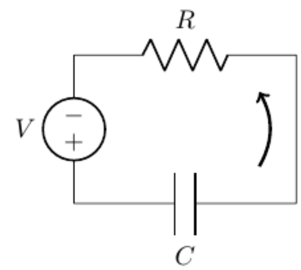
a) If $Q(0) = 0$, find $Q(t)$ at time *t*.

b) Find the limiting value Q_L that $Q(t)$ approaches after a long time.

c) Suppose that $Q\left(t_1\right) = Q(t)$ and that at time $t = t_1$ the battery is removed and the circuit is closed again. Find $Q(t)$ for $t > t_1$.

Solution

a) $R\frac{dQ}{dt}+\frac{Q}{C}=V \quad \Rightarrow \quad R\frac{dQ}{dt}=V-\frac{Q}{C}$



$$R\frac{dQ}{dt}=\frac{CV-Q}{C}$$

$$\frac{dQ}{dt}=\frac{CV-Q}{RC}$$

$$\frac{dQ}{Q-CV}=-\frac{1}{RC}dt$$

$$\int\frac{dQ}{Q-CV}=-\int\frac{1}{RC}dt$$

$$\ln|Q-CV|=-\frac{t}{CR}+A$$

$$Q-CV=e^{-t/CR+A}$$

$$Q=De^{-t/CR}+CV$$

$$Q(0)=0$$

$$0=De^{-0}+CV$$

$$D=-CV$$

$$Q=-CVe^{-t/CR}+CV$$

$$\boxed{Q(t)=CV\left(1-e^{-t/CR}\right)}$$

b) $\lim_{t\to\infty}Q(t)=CV\lim_{t\to\infty}\left(1-e^{-t/CR}\right)$

$$=CV(1-0)$$

$$\lim_{t\to\infty}e^{-t/CR}=\lim_{t\to\infty}\frac{1}{e^{t/CR}}\to\frac{1}{\infty}=0$$

$$\boxed{=CV}$$

c) In this case, $R\frac{dQ}{dt}+\frac{Q}{C}=0$, $Q(t_1)=CV$

$$\frac{dQ}{dt}=-\frac{Q}{CR}$$

$$\int\frac{dQ}{Q}=-\frac{1}{CR}\int dt$$

$$\ln|Q|=-\frac{t}{CR}+A$$

The solution is $Q=Ee^{-t/CR}$

So, $Q(t_1)=Ee^{-t_1/CR}=CV$

$$E=CVe^{t_1/CR}$$

$$Q(t)=CVe^{t_1/CR}e^{-t/CR}$$

$$=CVe^{(t_1-t)/CR}$$

$$\underline{= CVe^{-(t-t_1)/CR}} \quad for \quad t \ge t_1$$

Exercise

A circuit containing an electromotive force, a capacitor with a capacitance of C farads (F), and a resistor with a resistance of R ohms (Ω). The voltage drop across the capacitor is $\frac{Q}{C}$, where Q is the charge (in coulombs), so in this case ***Kirchhoff's Law*** gives

$$RI + \frac{Q}{C} = E(t)$$

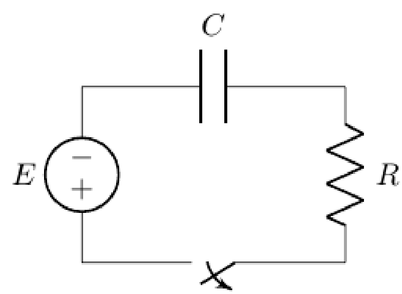
But $I = \frac{dQ}{dt}$, so we have
$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

Find the charge and the current at time t

a) Suppose the resistance is $5\ \Omega$, the capacitance is 0.05 F, a battery gives voltage of 60 V and initial charge is $Q(0) = 0\ C$

b) Suppose the resistance is $2\ \Omega$, the capacitance is 0.01 F, $E(t) = 10\sin 60t$ and initial charge is $Q(0) = 0\ C$

Solution



a) $5\frac{dQ}{dt} + \frac{1}{.05}Q = 60 \quad \rightarrow \quad \frac{dQ}{dt} + 4Q = 12$

$$e^{\int 4dt} = e^{4t}$$

$$\int 12e^{4t}dt = 3e^{4t}$$

$$Q(t) = \frac{1}{e^{4t}}\left(3e^{4t} + C\right) \ \underline{= 3 + Ce^{-4t}}$$

$$Q(0) = 3 + C = 0 \Rightarrow \underline{C = -3}$$

$$\underline{Q(t) = 3\left(1 - e^{-4t}\right)}$$

$$I = \frac{dQ}{dt} \ \underline{= 12e^{-4t}}$$

b) $2\frac{dQ}{dt} + \frac{1}{.01}Q = 10\sin 60t \quad \rightarrow \quad \frac{dQ}{dt} + 50Q = 5\sin 60t$

$$e^{\int 50dt} = e^{50t}$$

$$5\int e^{50t}(\sin 60t)\,dt =$$

$$\int e^{50t}(\sin 60t)\,dt = \left(-\frac{1}{60}\cos 60t + \frac{1}{72}\sin 60t\right)e^{50t} - \frac{25}{36}\int e^{50t}(\sin 60t)\,dt$$

$$\frac{61}{36}\int e^{50t}(\sin 60t)\,dt = \left(-\frac{1}{60}\cos 60t + \frac{1}{72}\sin 60t\right)e^{50t}$$

$$\int e^{50t}(\sin 60t)\,dt = \frac{36}{21{,}960}(-6\cos 60t + 5\sin 60t)e^{50t}$$

$$5\int e^{50t}(\sin 60t)\,dt = \frac{1}{122}(-6\cos 60t + 5\sin 60t)e^{50t}$$

		$\int \sin 60t$
+	e^{50t}	$-\frac{1}{60}\cos 60t$
−	$50e^{50t}$	$-\frac{1}{3600}\sin 60t$
+	$2500e^{50t}$	$-\frac{1}{3600}\int \sin 60t$

$$Q(t) = \frac{1}{e^{50t}}\left(\frac{1}{122}(-6\cos 60t + 5\sin 60t)e^{50t} + C\right)$$

$$= \frac{1}{122}(-6\cos 60t + 5\sin 60t) + Ce^{-50t}$$

$$Q(0) = -\frac{6}{122} + C = 0 \;\Rightarrow\; \boxed{C = \frac{3}{61}}$$

$$\boxed{Q(t) = \frac{1}{122}\left(-5\cos 60t + 6\sin 60t + 6e^{-50t}\right)}$$

$$I = \frac{dQ}{dt} = \frac{1}{122}\left(300\sin 60t + 360\cos 60t - 300e^{-50t}\right)$$

$$\boxed{= \frac{30}{61}\left(5\sin 60t + 6\cos 60t - 5e^{-50t}\right)}$$

Exercise

A heart pacemaker consists of a switch, a battery voltage E_0, a capacitor with constant capacitance *C*, and the heart as a resistor with constant resistance *R*. When the switch is closed, the capacitor charges; when the switch is open, the capacitor discharges, sending an electrical stimulus to the heart. During the time the heart is being stimulated, the voltage *E* across the heart satisfies the linear differential equation

$$\frac{dE}{dt} = -\frac{1}{RC}E$$

Solve the *DE*, subject to $E(4) = E_0$

Solution

$$\int \frac{dE}{E} = -\frac{1}{RC}\int dt$$

$$\ln E = -\frac{1}{RC}t + C$$

$$E = e^{-\frac{1}{RC}t + C} = Ae^{-\frac{1}{RC}t}$$

$$E(4) = E_0 \quad\to\quad E_0 = Ae^{-\frac{4}{RC}} \Rightarrow \boxed{A = E_0 e^{\frac{4}{RC}}}$$

$$E(t) = E_0 e^{\frac{4}{RC}} e^{-\frac{1}{RC}t} \boxed{= E_0 e^{\frac{4-t}{RC}}}$$

Exercise

A 30–volt electromotive force is applied to an *LR*-series circuit in which the inductance is 0.1 *henry* and the resistance is 50 *ohms*.

a) Find the current $i(t)$ if $i(0)=0$

b) Determine the current as $t\to\infty$

c) Solve the equation when $E(t)=E_0\sin\omega t$ and $i(0)=i_0$

Solution

a) $0.1\frac{di}{dt}+50i=30$ $\qquad L\frac{di}{dt}+Ri=E(t)$

$$\frac{di}{dt}+500i=300$$

$$e^{\int 500dt}=e^{500t}$$

$$\int 300e^{500t}dt=\frac{3}{5}e^{500t}$$

$$i(t)=e^{-500t}\left(\frac{3}{5}e^{500t}+C\right)\qquad i(0)=0$$

$$0=\frac{3}{5}+C\ \rightarrow\ \boxed{C=-\frac{3}{5}}$$

$$\boxed{i(t)=\frac{3}{5}-\frac{3}{5}e^{-500t}}$$

b) $\lim\limits_{t\to\infty} i(t)=\lim\limits_{t\to\infty}\left(\frac{3}{5}-\frac{3}{5}e^{-500t}\right)\boxed{=\frac{3}{5}}$

c) $\frac{di}{dt}+500i=10E_0\sin\omega t$

		$\int\sin\omega t$
+	e^{500t}	$-\frac{1}{\omega}\cos\omega t$
−	$500e^{500t}$	$-\frac{1}{\omega^2}\sin\omega t$
+	$25\times10^4e^{500t}$	$-\int\frac{1}{\omega^2}\sin\omega t$

$$\int 10E_0(\sin\omega t)e^{500t}dt=$$

$$\int(\sin\omega t)e^{500t}dt=\left(-\frac{1}{\omega}\cos\omega t+\frac{500}{\omega^2}\sin\omega t\right)e^{500t}-\frac{25\times10^4}{\omega^2}\int(\sin\omega t)e^{500t}dt$$

$$\left(\frac{\omega^2+25\times10^4}{\omega^2}\right)\int(\sin\omega t)e^{500t}dt=\frac{1}{\omega^2}(-\omega\cos\omega t+500\sin\omega t)e^{500t}$$

$$\int 10E_0(\sin\omega t)e^{500t}dt=\frac{10E_0}{\omega^2+25\times10^4}(-\omega\cos\omega t+500\sin\omega t)e^{500t}$$

$$i(t)=e^{-500t}\left(\frac{10E_0}{\omega^2+25\times10^4}(-\omega\cos\omega t+500\sin\omega t)e^{500t}+C\right)\qquad i(0)=i_0$$

$$0=-\frac{10\omega tE_0}{\omega^2+25\times10^4}+C\ \rightarrow\ \boxed{C=\frac{10\omega tE_0}{\omega^2+25\times10^4}}$$

$$\boxed{i(t)=\frac{10E_0}{\omega^2+25\times10^4}\left(-\omega\cos\omega t+500\sin\omega t-\omega e^{-500t}\right)}$$

Exercise

A 100–volt electromotive force is applied to an *RC*-series circuit in which the resistance is 200 *ohms* and the capacitance is 10^{-4} *farad*.

a) Find the charge $q(t)$ if $q(0)=0$

b) Find the current as $i(t)$

Solution

Given: $R=200\ \Omega,\quad C=10^{-4}\ F\quad E(t)=100\ V$

a) $200\frac{dq}{dt}+10^4 q=100$ $\qquad R\frac{dq}{dt}+\frac{1}{C}q=E(t)$

$$\frac{dq}{dt}+50q=\frac{1}{2}$$

$$e^{\int 50dt}=e^{50t}$$

$$\int \frac{1}{2}e^{50t}dt=\frac{1}{100}e^{50t}$$

$$q(t)=\frac{1}{e^{50t}}\left(\frac{1}{100}e^{50t}+C\right)$$

$$=\frac{1}{100}+Ce^{-50t}$$

$$q(0)=0\quad \to 0=\frac{1}{100}+C\quad \Rightarrow\ \boxed{C=-\frac{1}{100}}$$

$$\boxed{q(t)=\frac{1}{100}-\frac{1}{100}e^{-50t}}$$

b) $\boxed{i(t)=\frac{1}{2}e^{-50t}}$ $\qquad i(t)=\frac{dq}{dt}$

Exercise

A 200–volt electromotive force is applied to an *RC*-series circuit in which the resistance is 1000 *ohms* and the capacitance is 5×10^{-6} *farad*.

a) Find the charge $q(t)$ if $i(0)=0.4$

b) Determine the charge as $t\to\infty$

Solution

Given: $R=1000\ \Omega,\quad C=5\times10^{-6}\ F\quad E(t)=200\ V$

a) $1000\frac{dq}{dt}+\frac{1}{5}10^6 q=200$ $\qquad R\frac{dq}{dt}+\frac{1}{C}q=E(t)$

$$\frac{dq}{dt}+200q=\frac{1}{5}$$

$$e^{\int 200dt}=e^{200t}$$

$$\int \frac{1}{5}e^{200t}dt = \frac{1}{1000}e^{200t}$$

$$q(t) = \frac{1}{e^{200t}}\left(\frac{1}{1000}e^{200t} + C\right)$$

$$= \frac{1}{1000} + Ce^{-200t}$$

$$i(t) = -200Ce^{-200t} \qquad i(t) = \frac{dq}{dt}$$

$$i(0) = 0.4 \quad \to 0.4 = -200C \quad \Rightarrow \boxed{C = -\frac{1}{500}}$$

$$\boxed{q(t) = \frac{1}{1000} - \frac{1}{500}e^{-200t}}$$

b) $$\lim_{t\to\infty} q(t) = \lim_{t\to\infty}\left(\frac{1}{1000} - \frac{1}{500}e^{-200t}\right) \qquad \lim_{t\to\infty}\left(e^{-200t}\right) = \lim_{t\to\infty}\left(\frac{1}{e^{200t}}\right) = 0$$

$$\boxed{= \frac{1}{1000}}$$

Exercise

An electromotive force

$$E(t) = \begin{cases} 120, & 0 \le t \le 20 \\ 0, & t > 20 \end{cases}$$

Is applied to an *LR*-series circuit in which the inductance is 20 *henries* and resistance is 2 *ohms*. Find the current $i(t)$ if $i(0) = 0$

Solution

For $0 \le t \le 20$

$$20\frac{di}{dt} + 2i = 120 \qquad L\frac{di}{dt} + Ri = E(t)$$

$$\frac{di}{dt} + \frac{1}{10}i = 6$$

$$e^{\int \frac{1}{10}dt} = e^{t/10}$$

$$\int 6e^{t/10}dt = 60e^{t/10}$$

$$i(t) = e^{-t/10}\left(60e^{t/10} + C\right)$$

$$= 60 + Ce^{-t/10}$$

$$i(0) = 0 \quad \to 0 = 60 + C \quad \Rightarrow \boxed{C = -60}$$

$$\boxed{i(t) = 60 - 60e^{-t/10}}$$

For $t > 20$

$$20\frac{di}{dt}+2i=0 \qquad\qquad L\frac{di}{dt}+Ri=E(t)$$

$$\frac{di}{dt}=-\frac{1}{10}i$$

$$\int\frac{di}{i}=-\int\frac{1}{10}dt$$

$$\ln i=-\frac{1}{10}t+C$$

$$i(t)=C_1e^{-t/10}$$

$$\boxed{i(t=20)=60-60e^{-2}}$$

$$i(20)=60-60e^{-2} \quad \to 60-60e^{-2}=Ce^{-2} \quad \Rightarrow \boxed{C=60\left(e^2-1\right)}$$

$$\boxed{i(t)=60\left(e^2-1\right)e^{-t/10}}$$

$$i(t)=\begin{cases} 60-60e^{-t/10} & 0\le t\le 20 \\ 60\left(e^2-1\right)e^{-t/10} & t>20 \end{cases}$$

Exercise

Suppose an *RC*-series circuit has a variable resistor. If the resistance at time *t* is given by $R=k_1+k_2t$, where k_1 and k_2 are known positive constants, then

$$\left(k_1+k_2t\right)\frac{dq}{dt}+\frac{1}{C}q=E(t)$$

If $E(t)=E_0$ and $q(0)=q_0$, where E_0 and q_0 are constants, show that

$$q(t)=E_0C+\left(q_0-E_0C\right)\left(\frac{k_1}{k_1+k_2t}\right)^{1/Ck_2}$$

Solution

$$\left(k_1+k_2t\right)\frac{dq}{dt}=E_0-\frac{1}{C}q$$

$$\int\frac{dq}{E_0-\frac{1}{C}q}=\int\frac{dt}{k_1+k_2t}$$

$$-C\ln\left(E_0-\frac{1}{C}q\right)=\frac{1}{k_2}\ln\left(k_1+k_2t\right)+\ln A$$

$$\ln\left(E_0-\frac{1}{C}q\right)^{-C}=\ln\left(k_1+k_2t\right)^{1/k_2}+\ln A$$

$$\left(\frac{CE_0-q}{C}\right)^{-C}=A\left(k_1+k_2t\right)^{1/k_2}$$

$$q(0)=q_0 \quad \to \quad \left(\frac{C}{CE_0-q_0}\right)^C = A\left(k_1\right)^{1/k_2}$$

$$A=\left(\frac{C}{CE_0-q_0}\right)^C \left(k_1\right)^{-1/k_2}$$

$$\left(\frac{C}{CE_0-q}\right)^C = \left(\frac{C}{CE_0-q_0}\right)^C \left(\frac{1}{k_1}\right)^{1/k_2} \left(k_1+k_2t\right)^{1/k_2}$$

$$\left(\frac{1}{CE_0-q}\right)^C = \left(\frac{1}{CE_0-q_0}\right)^C \left(\frac{k_1+k_2t}{k_1}\right)^{1/k_2}$$

$$\frac{1}{CE_0-q} = \frac{1}{CE_0-q_0}\left(\frac{k_1+k_2t}{k_1}\right)^{1/Ck_2}$$

$$CE_0-q=\left(CE_0-q_0\right)\left(\frac{k_1}{k_1+k_2t}\right)^{1/Ck_2}$$

$$q(t)=E_0C-\left(CE_0-q_0\right)\left(\frac{k_1}{k_1+k_2t}\right)^{1/Ck_2}$$

$$\boxed{q(t)=E_0C+\left(q_0-E_0C\right)\left(\frac{k_1}{k_1+k_2t}\right)^{1/Ck_2}}$$

Exercise

A heart pacemaker, consists of a switch, a battery, a capacitor, and the heart as a resistor.

When the switch S is at P, the capacitor charges; when S is at Q, the capacitor discharges, sending an electrical stimulus to the heart. The electrical stimulus is being applied to the heart, the voltage E across the heart satisfies the linear DE.

$$\frac{dE}{dt}=-\frac{1}{RC}E$$

a) Let assume that over the time interval of length t_1, $0<t<t_1$, the switch S is at position P and the capacitor is being charges. When the switch is moved to position Q at time t_1 the capacitor discharges, sending an impulse to the heart over the time interval of length t_2 : $t_1 \le t < t_1+t_2$. Thus over the initial charging/discharging interval $0<t<t_1+t_2$ the voltage to the heart is actually modeled by the piecewise-defined differential equation

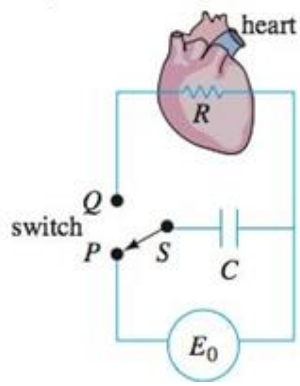
$$\frac{dE}{dt}=\begin{cases} 0, & 0<t<t_1 \\ -\frac{1}{RC}E, & t_1 \le t < t_1+t_2 \end{cases}$$

By moving S between P and Q, the charging and discharging over time intervals of lengths t_1 and t_2 is repeated indefinitely. Suppose $t_1 = 4\ s$, $t_2 = 2\ s$. $E_0 = 12\ V$, and $E(0) = 0$, $E(4) = 12$, $E(6) = 0$, $E(10) = 12$, $E(12) = 0$, and so on.

Solve for $E(t)$ for $0 \le t \le 24$

b) Suppose for the sake of illustration that $R = C = 1$. Graph the solution in part (*a*) for $0 \le t \le 24$

Solution



a) $\dfrac{dE}{dt} = -\dfrac{1}{RC}E \quad \to \quad \dfrac{dE}{E} = -\dfrac{1}{RC}dt$

$$\int \frac{dE}{E} = -\int \frac{1}{RC}dt$$

$$\ln E = -\frac{1}{RC}t + C$$

$$E(t) = e^{-\frac{1}{RC}t + A_1} = Ae^{-\frac{1}{RC}t}$$

For $0 \le t < 4$, $6 \le t < 10$, and $12 \le t < 16$ $\to$ $\underline{E(t) = 0}$

For $E(4) = E(10) = E(16) = 12$

$$Ae^{-\frac{1}{RC}t} = 12 \Rightarrow \underline{A = 12e^{\frac{1}{RC}t}}$$

$E(4) \ \to \ A = 12e^{4/RC}$

$E(10) \ \to \ A = 12e^{10/RC}$

$E(16) \ \to \ A = 12e^{16/RC}$

$$E(t) = \begin{cases} 0 & 0 \le t < 4,\ \ 6 \le t < 10,\ \ 12 \le t < 16 \\ 12e^{\frac{4-t}{RC}} & 4 \le t < 6 \\ 12e^{\frac{10-t}{RC}} & 10 \le t < 12 \\ 12e^{\frac{16-t}{RC}} & 16 \le t < 18 \\ 12e^{\frac{22-t}{RC}} & 22 \le t < 24 \end{cases}$$

b)