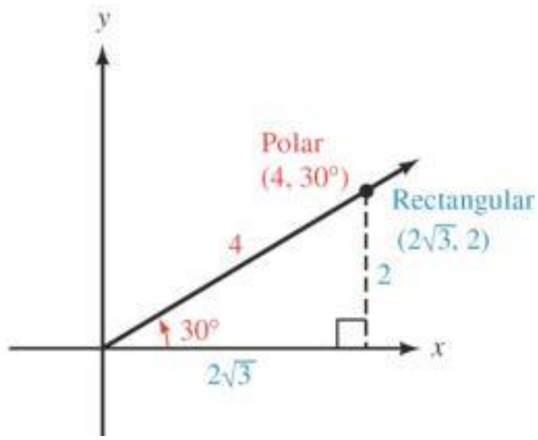
Solution　Section 3.6 – Polar Coordinates

Exercise

Convert to rectangular coordinates. $(4,\ 30°)$

Solution

$$x = r\cos\theta$$
$$= 4\cos 30°$$
$$= 4\left(\frac{\sqrt{3}}{2}\right)$$
$$= 2\sqrt{3}$$



$$y = r\sin\theta$$
$$= 4\sin 30°$$
$$= 4\left(\frac{1}{2}\right)$$
$$= 2$$

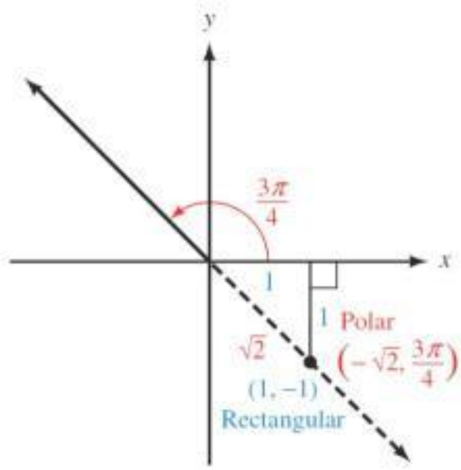
The point $(2\sqrt{3},\ 2)$ in rectangular coordinates is equivalent to $(4,\ 30°)$ in polar coordinates.

Exercise

Convert to rectangular coordinates $\left(-\sqrt{2},\ \frac{3\pi}{4}\right)$.

Solution

$$x = -\sqrt{2}\cos\frac{3\pi}{4}$$
$$= -\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right)$$
$$= 1$$



$$y = -\sqrt{2}\sin\frac{3\pi}{4}$$
$$= -\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)$$
$$= -1$$

The point $(1,\ -1)$ in rectangular coordinates is equivalent to $\left(-\sqrt{2},\ \frac{3\pi}{4}\right)$ in polar coordinates.

Exercise

Convert to rectangular coordinates $(3,\ 270°)$.

Solution

$$\begin{aligned} x &= 3\cos 270° \\ &= 3(0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} y &= 3\sin 270° \\ &= 3(-1) \\ &= -3 \end{aligned}$$

Exercise

Convert to rectangular coordinates $\left(2,\ 60°\right)$

Solution

$$\begin{aligned} x &= 2\cos 60° \\ &= 2\left(\tfrac{1}{2}\right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} y &= 2\sin 60° \\ &= 2\frac{\sqrt{3}}{2} \\ &= \sqrt{3} \end{aligned}$$

$$\boxed{\left(1,\ \sqrt{3}\right)}$$

Exercise

Convert to rectangular coordinates $\left(\sqrt{2},\ -225°\right)$

Solution

$$x = \sqrt{2}\cos\left(-225°\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) = -1$$

$$y = \sqrt{2}\sin\left(-225°\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1$$

$$\boxed{\left(-1,\ 1\right)}$$

Exercise

Convert to rectangular coordinates $\left(4\sqrt{3},\ -\frac{\pi}{6}\right)$

Solution

$$x = 4\sqrt{3}\cos\left(-\frac{\pi}{6}\right) = 4\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = 6$$

$$y = 4\sqrt{3}\sin\left(-\frac{\pi}{6}\right) = 4\sqrt{3}\left(-\frac{1}{2}\right) = -2\sqrt{3} \qquad \Rightarrow \boxed{\left(6,\ -2\sqrt{3}\right)}$$

Exercise

Change the polar coordinates to rectangular coordinates $\left(-2,\ \frac{7\pi}{6}\right)$

Solution

$$x = -2\cos\left(\frac{7\pi}{6}\right) = -2\left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

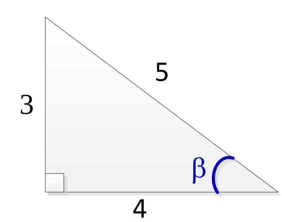
$$y = -2\sin\left(\frac{7\pi}{6}\right) = -2\left(-\frac{1}{2}\right) = 1 \qquad \Rightarrow \boxed{\left(\sqrt{3},\ 1\right)}$$

Exercise

Change the polar coordinates to rectangular coordinates $\left(6,\ \arctan\frac{3}{4}\right)$

Solution

$$\arctan\frac{3}{4} = \beta \quad \Rightarrow \quad \tan\beta = \frac{3}{4}$$



$$x = 2\cos\beta = 2\left(\frac{4}{5}\right) = \frac{8}{5}$$

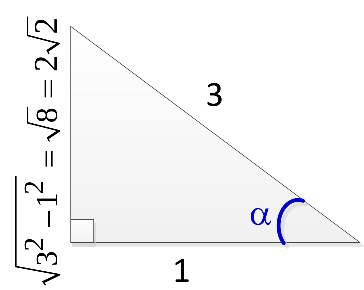
$$y = 2\sin\beta = 2\left(\frac{3}{5}\right) = \frac{6}{5} \qquad \Rightarrow \boxed{\left(\frac{8}{5},\ \frac{6}{5}\right)}$$

Exercise

Change the polar coordinates to rectangular coordinates $\left(10,\ \arccos\left(-\frac{1}{3}\right)\right)$

Solution

$$\arccos\left(-\frac{1}{3}\right) = \alpha \quad \Rightarrow \quad \cos\alpha = -\frac{1}{3} \quad (QII)$$



$$x = 10\cos\alpha = 10\left(-\frac{1}{3}\right) = -\frac{10}{3}$$

$$y = 10\sin\alpha = 10\left(\frac{2\sqrt{2}}{3}\right) = \frac{20\sqrt{2}}{3} \qquad \Rightarrow \boxed{\left(-\frac{10}{3},\ \frac{20\sqrt{2}}{3}\right)}$$

Exercise

Convert to polar coordinates $(3,\ 3)$.

Solution

$$(3,\ 3) \to \begin{cases} r = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2} \\ \hat{\theta} = \tan^{-1}\left(\frac{3}{3}\right) = \tan^{-1}(1) = 45° \end{cases}$$

The angle is in quadrant I; therefore, $\boxed{\theta = 45°}$

$$\boxed{(3,\ 3) \ = \ \left(3\sqrt{2},\ 45°\right)}$$

Exercise

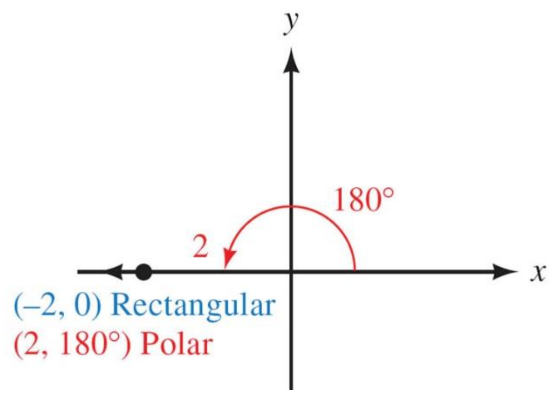
Convert to polar coordinates $(-2,\ 0)$.

Solution

$$r = \pm\sqrt{4+0}$$
$$= \pm\ 2$$

$$\theta = \tan^{-1}\frac{0}{-2}$$
$$= 0°$$

The point $r = 2, \quad \theta = 180°$



Exercise

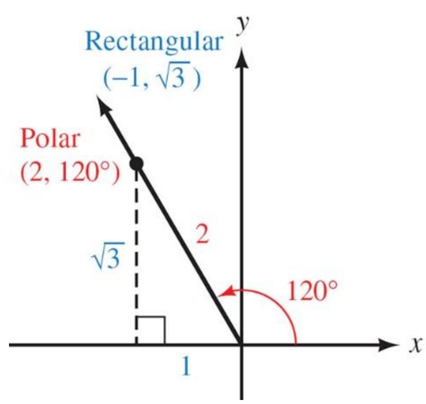
Convert to polar coordinates $(-1,\ \sqrt{3})$.

Solution

$$r = \pm\sqrt{1+3}$$
$$= \pm\ 2$$

$$\theta = \tan^{-1}\frac{\sqrt{3}}{-1}$$
$$= 120°$$

The point $r = 2, \quad \theta = 120°$



Exercise

Convert to polar coordinates $\left(-3,\ -3\right)$ $r \ge 0$ $0° \le \theta < 360°$

Solution

$$\left(-3,\ -3\right) \to \begin{cases} r = \sqrt{(-3)^2 + (-3)^2} = 3\sqrt{2} \\ \hat{\theta} = \tan^{-1}\left(\frac{3}{3}\right) = \tan^{-1}(1) = 45° \end{cases}$$

The angle is in quadrant III; therefore, $\theta = 180° + 45° = 225°$

$$\boxed{\left(-3,\ -3\right) = \left(3\sqrt{2},\ 225°\right)}$$

Exercise

Convert to polar coordinates $\left(2,\ -2\sqrt{3}\right)$ $r \ge 0$ $0° \le \theta < 360°$

Solution

$$\left(2,\ -2\sqrt{3}\right) \to \begin{cases} r = \sqrt{2^2 + \left(-2\sqrt{3}\right)^2} = 4 \\ \hat{\theta} = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \tan^{-1}\left(\sqrt{3}\right) = 60° \end{cases}$$

The angle is in quadrant IV; therefore, $\theta = 360° - 60° = 300°$

$$\boxed{\left(2,\ -2\sqrt{3}\right) = \left(4,\ 300°\right)}$$

Exercise

Convert to polar coordinates $\left(-2,\ 0\right)$ $r \ge 0$ $0 \le \theta < 2\pi$

Solution

$$\left(-2,\ 0\right) \to \begin{cases} r = \sqrt{(-2)^2 + 0^2} = 2 \\ \hat{\theta} = \tan^{-1}\left(\frac{0}{2}\right) = 0 \Rightarrow \theta = \pi \end{cases}$$

$$\boxed{\left(-2,\ 0\right) = \left(2,\ \pi\right)}$$

Exercise

Convert to polar coordinates $\left(-1,\ -\sqrt{3}\right)$ $\quad r \ge 0 \quad 0 \le \theta < 2\pi$

Solution

$$\left(-1,\ -\sqrt{3}\right) \to \begin{cases} r = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = 2 \\ \hat{\theta} = \tan^{-1}\left(\dfrac{\sqrt{3}}{1}\right) = \dfrac{\pi}{3} \end{cases}$$

The angle is in quadrant III; therefore, $\theta = \pi + \dfrac{\pi}{3} = \dfrac{4\pi}{3}$

$$\boxed{\left(-1,\ -\sqrt{3}\right) = \left(2,\ \dfrac{4\pi}{3}\right)}$$

Exercise

Change the rectangular coordinates to polar coordinates $\left(7,\ -7\sqrt{3}\right)$ $\quad r > 0 \quad 0 \le \theta < 2\pi$

Solution

$$\left(7,\ -7\sqrt{3}\right) \to \begin{cases} r = \sqrt{(7)^2 + \left(-7\sqrt{3}\right)^2} = \sqrt{196} = 14 \\ \hat{\theta} = \tan^{-1}\left(\dfrac{7\sqrt{3}}{7}\right) = \dfrac{\pi}{3} \end{cases}$$

The angle is in quadrant IV; therefore, $\theta = 2\pi - \dfrac{\pi}{3} = \dfrac{5\pi}{3}$

$$\boxed{\left(-7,\ -7\sqrt{3}\right) = \left(14,\ \dfrac{5\pi}{3}\right)}$$

Exercise

Change the rectangular coordinates to polar coordinates $\left(-2\sqrt{2},\ -2\sqrt{2}\right)$ $\quad r > 0 \quad 0 \le \theta < 2\pi$

Solution

$$\left(-2\sqrt{2},\ -2\sqrt{2}\right) \to \begin{cases} r = \sqrt{\left(-2\sqrt{2}\right)^2 + \left(-2\sqrt{2}\right)^2} = 4 \\ \hat{\theta} = \tan^{-1}\left(\dfrac{-2\sqrt{2}}{-2\sqrt{2}}\right) = \dfrac{\pi}{4} \end{cases}$$

The angle is in quadrant III; therefore, $\theta = \pi + \dfrac{\pi}{4} = \dfrac{5\pi}{4}$

$$\boxed{\left(-2\sqrt{2},\ -2\sqrt{2}\right) = \left(4,\ \dfrac{5\pi}{4}\right)}$$

Exercise

The point $(0,\ -3)$ in rectangular coordinates is equivalent to $(3,\ 270°)$ in polar coordinates.

Solution

$$r = \sqrt{0 + (-3)^2} = 3$$

$$\hat{\theta} = \tan^{-1}\frac{0}{3} = 90°$$

The point $\boxed{(3,\ 270°)}$

Exercise

The point $(1,\ -1)$ in rectangular coordinates is equivalent to $\left(-\sqrt{2},\ \frac{3\pi}{4}\right)$ in polar coordinates.

Solution

$$r = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$$

$$\hat{\theta} = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$\theta \in QIV \quad \to \quad \theta = \frac{7\pi}{4}$$

$$\left(\sqrt{2},\ \frac{7\pi}{4}\right) \quad \Leftrightarrow \quad \left(-\sqrt{2},\ \frac{3\pi}{4}\right)$$

Exercise

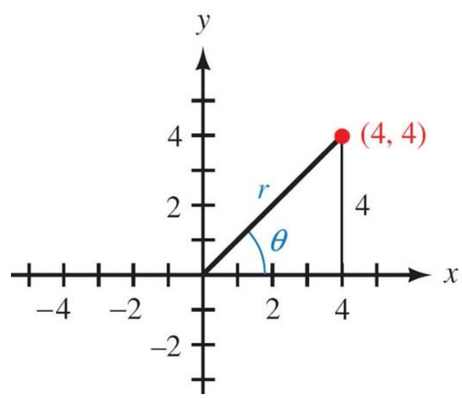
A point lies at (4, 4) on a rectangular coordinate system. Give its address in polar coordinates $(r,\ \theta)$

Solution

$$r = \sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{4}{4}\right) = \tan^{-1}(1) = 45°$$

$$\boxed{\left(4\sqrt{2},\ 45°\right)}$$



Exercise

Write the equation in rectangular coordinates $r^2 = 4$

Solution

$$r^2 = 4$$

$$\boxed{x^2 + y^2 = 4}$$

Exercise

Write the equation in rectangular coordinates $r = 6\cos\theta$

Solution

$$r = 6\cos\theta$$

$$r = 6\frac{x}{r}$$

$$r^2 = 6x$$

$$\boxed{x^2 + y^2 = 6x}$$

Exercise

Write the equation in rectangular coordinates $r^2 = 4\cos 2\theta$

Solution

$$r^2 = 4\left(\cos^2\theta - \sin^2\theta\right) \qquad \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r}$$

$$= 4\left(\left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2\right)$$

$$= 4\left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right)$$

$$= 4\left(\frac{x^2 - y^2}{r^2}\right)$$

$$r^4 = 4\left(x^2 - y^2\right) \qquad r^2 = x^2 + y^2$$

$$\boxed{\left(x^2 + y^2\right)^4 = 4x^2 - 4y^2}$$

Exercise

Write the equation in rectangular coordinates $r\left(\cos\theta - \sin\theta\right) = 2$

Solution

$$r\left(\cos\theta - \sin\theta\right) = 2 \qquad \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r}$$

$$r\left(\frac{x}{r} - \frac{y}{r}\right) = 2$$

$$r\left(\frac{x - y}{r}\right) = 2$$

$$\boxed{x - y = 2}$$

Exercise

Write the equation in rectangular coordinates $r^2 = 4\sin 2\theta$

Solution

$$r^2 = 4\sin 2\theta \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= 4(2\sin\theta\cos\theta) \qquad \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r}$$

$$= 8\left(\frac{y}{r}\right)\left(\frac{x}{r}\right)$$

$$= 8\frac{xy}{r^2}$$

$$r^4 = 8xy \qquad r^2 = x^2 + y^2$$

$$\boxed{\left(x^2 + y^2\right)^2 = 8xy}$$

Exercise

Find an equation in x and y that has the same graph as polar equation. $r\sin\theta = -2$

Solution

$$r\sin\theta = -2 \qquad y = r\sin\theta$$

$$\boxed{y = -2}$$

Exercise

Find an equation in x and y that has the same graph as polar equation. $\theta = \frac{\pi}{4}$

Solution

$$\tan\theta = \tan\frac{\pi}{4}$$

$$\frac{y}{x} = 1$$

$$\boxed{y = x}$$

Exercise

Find an equation in x and y that has the same graph as polar $r^2\left(4\sin^2\theta - 9\cos^2\theta\right) = 36$

Solution

$$r^2\left(4\sin^2\theta - 9\cos^2\theta\right) = 36 \qquad \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r}$$

$$r^2\left(4\frac{y^2}{r^2} - 9\frac{x^2}{r^2}\right) = 36$$

$$r^2\left(\frac{4y^2-9x^2}{r^2}\right)=36$$

$$\boxed{4y^2-9x^2=36}$$

Exercise

Find an equation in x and y that has the same graph as polar $r^2\left(\cos^2\theta+4\sin^2\theta\right)=16$

Solution

$$r^2\left(\cos^2\theta+4\sin^2\theta\right)=16 \qquad \cos\theta=\frac{x}{r}\quad \sin\theta=\frac{y}{r}$$

$$r^2\left(\frac{x^2}{r^2}+4\frac{y^2}{r^2}\right)=16$$

$$r^2\left(\frac{x^2+4y^2}{r^2}\right)=16$$

$$\boxed{x^2+4y^2=16}$$

Exercise

Find an equation in x and y that has the same graph as polar $r\left(\sin\theta-2\cos\theta\right)=6$

Solution

$$r\left(\sin\theta-2\cos\theta\right)=6 \qquad \cos\theta=\frac{x}{r}\quad \sin\theta=\frac{y}{r}$$

$$r\left(\frac{y}{r}-2\frac{x}{r}\right)=6$$

$$r\left(\frac{y-2x}{r}\right)=6$$

$$\boxed{y-2x=6}$$

Exercise

Find an equation in x and y that has the same graph as polar $r\left(\sin\theta+r\cos^2\theta\right)=1$

Solution

$$r\left(\sin\theta+r\cos^2\theta\right)=1 \qquad \cos\theta=\frac{x}{r}\quad \sin\theta=\frac{y}{r}$$

$$r\left(\frac{y}{r}+r\frac{x^2}{r^2}\right)=1$$

$$r\left(\frac{y}{r}+\frac{x^2}{r}\right)=1$$

$$r\left(\frac{y+x^2}{r}\right)=1$$

$$\boxed{y+x^2=1}$$

Exercise

Find an equation in x and y that has the same graph as polar $\quad r=8\sin\theta-2\cos\theta$

Solution

$$r=8\sin\theta-2\cos\theta \qquad \cos\theta=\frac{x}{r} \quad \sin\theta=\frac{y}{r}$$

$$r=8\frac{y}{r}-2\frac{x}{r}$$

$$r^2=8y-2x \qquad r^2=x^2+y^2$$

$$\boxed{x^2+y^2=8y-2x}$$

Exercise

Find an equation in x and y that has the same graph as polar $\quad r=\tan\theta$

Solution

$$r=\tan\theta$$

$$x^2+y^2=\frac{y^2}{x^2}$$

$$x^4+x^2y^2=y^2$$

$$\boxed{\sqrt{x^2+y^2}=\frac{y}{x}}$$

Exercise

Find a polar equation that has the same graph as the equation in x and y. $\quad y^2=6x$

Solution

$$y^2=6x \qquad x=r\cos\theta \quad y=r\sin\theta$$

$$(r\sin\theta)^2=6(r\cos\theta)$$

$$r^2\sin^2\theta=6r\cos\theta$$

$$\boxed{r=6\frac{\cos\theta}{\sin^2\theta}}$$

Exercise

Find a polar equation that has the same graph as the equation in x and y. $xy = 8$

Solution

$$xy = 8 \qquad x = r\cos\theta \quad y = r\sin\theta$$

$$(r\cos\theta)(r\sin\theta) = 8$$

$$\boxed{r^2 = \frac{8}{\cos\theta\sin\theta}}$$

Exercise

Find a polar equation that has the same graph as the equation in x and y. $(x+2)^2 + (y-3)^2 = 13$

Solution

$$x^2 + 4x + 4 + y^2 - 6y + 9 = 13$$

$$x^2 + 4x + y^2 - 6y = 13 - 9 - 4$$

$$x^2 + 4x + y^2 - 6y = 0$$

$$x^2 + y^2 = 6y - 4x \qquad x = r\cos\theta \quad y = r\sin\theta$$

$$r^2 = 6r\sin\theta - 4r\cos\theta$$

$$r^2 = r(6\sin\theta - 4\cos\theta) \qquad \textit{Divide by } r$$

$$\boxed{r = 6\sin\theta - 4\cos\theta}$$

Exercise

Find a polar equation that has the same graph as the equation in x and y. $y^2 - x^2 = 4$

Solution

$$y^2 - x^2 = 4 \qquad x = r\cos\theta \quad y = r\sin\theta$$

$$r^2\sin^2\theta - r^2\cos^2\theta = 4$$

$$r^2\left(\sin^2\theta - \cos^2\theta\right) = 4 \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$r^2(-\cos 2\theta) = 4$$

$$\boxed{r^2 = -\frac{4}{\cos 2\theta}}$$

Exercise

Write the equation in polar coordinates $x + y = 5$

Solution

$$r\cos\theta + r\sin\theta = 5 \qquad\qquad x = r\cos\theta \quad y = r\sin\theta$$

$$r(\cos\theta + \sin\theta) = 5$$

$$\boxed{r = \frac{5}{\cos\theta + \sin\theta}}$$

Exercise

Write the equation in polar coordinates $x^2 + y^2 = 9$

Solution

$$x^2 + y^2 = 9 \qquad\qquad r^2 = x^2 + y^2$$

$$\boxed{r^2 = 9}$$

Exercise

Write the equation in polar coordinates $x^2 + y^2 = 4x$

Solution

$$r^2 = 4r\cos\theta$$

$$\frac{r^2}{r} = \frac{4r\cos\theta}{r}$$

$$\boxed{r = 4\cos\theta}$$

Exercise

Write the equation in polar coordinates $y = -x$

Solution

$$y = -x \qquad\qquad x = r\cos\theta \quad y = r\sin\theta$$

$$r\sin\theta = -r\cos\theta$$

$$\boxed{\sin\theta = -\cos\theta}$$

Exercise

Write the equation in polar coordinates $x + y = 4$

Solution

$$r\cos\theta + r\sin\theta = 4 \qquad\qquad x = r\cos\theta \quad y = r\sin\theta$$

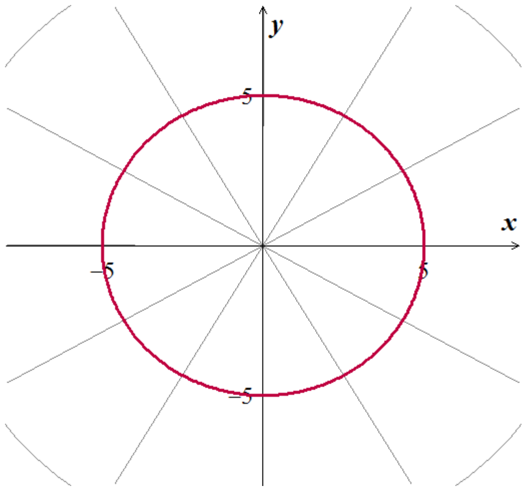
$$r(\cos\theta + \sin\theta) = 4$$

$$\boxed{r = \frac{4}{\cos\theta + \sin\theta}}$$

Exercise

Sketch the graph of the polar equation $r = 5$

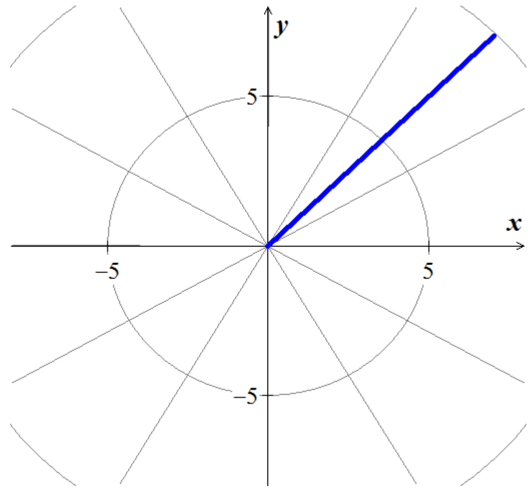
Solution



Exercise

Sketch the graph of the polar equation $\theta = \frac{\pi}{4}$

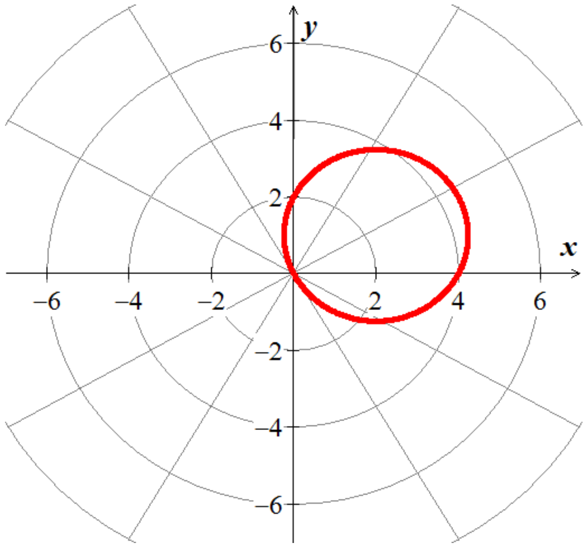
Solution



Exercise

Sketch graph $r = 4\cos\theta + 2\sin\theta$

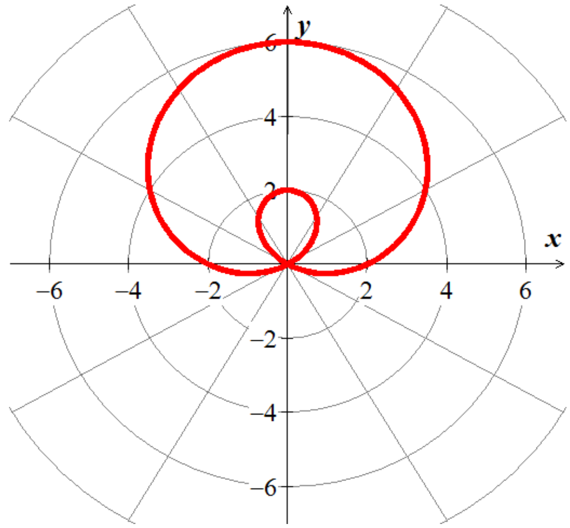
Solution



Exercise

Sketch the graph of the polar $r = 2 + 4\sin\theta$

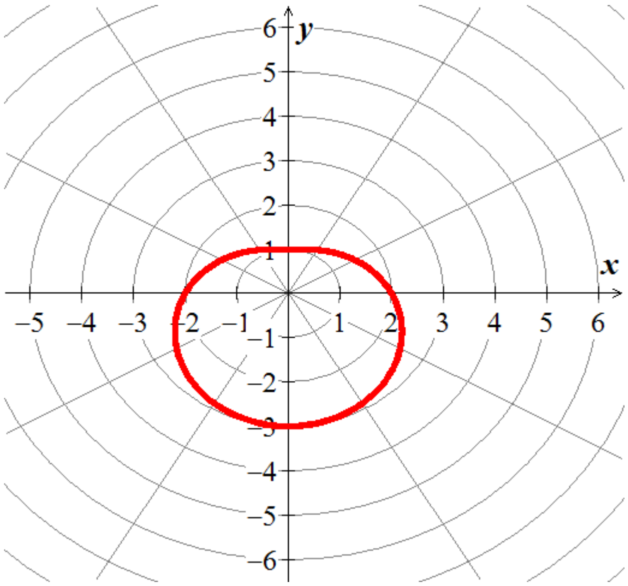
Solution



Exercise

Sketch the graph $r = 2 - \cos\theta$

Solution

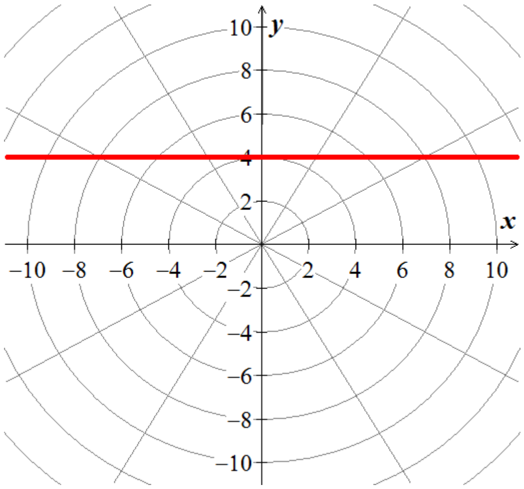


Exercise

Sketch the graph $r = 4\csc\theta$

Solution

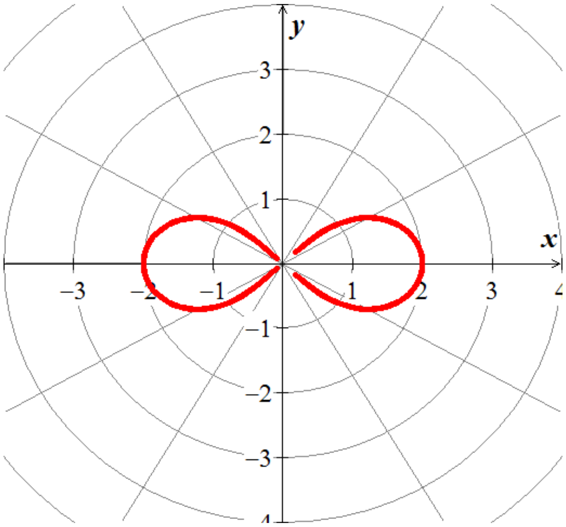
$$r = 4\csc\theta = \frac{4}{\sin\theta} \Rightarrow r\sin\theta = 4 = y$$



Exercise

Sketch the graph $r^2 = 4\cos 2\theta$

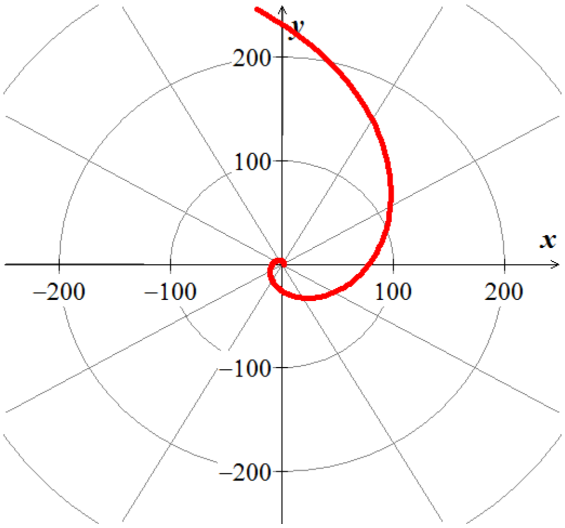
Solution



Exercise

Sketch the graph $r = 2^{\theta} \quad \theta \ge 0$

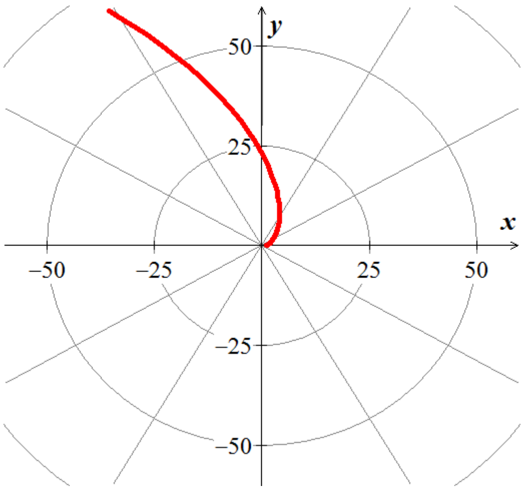
Solution



Exercise

Sketch the graph of the polar equation $r = e^{2\theta} \quad \theta \ge 0$

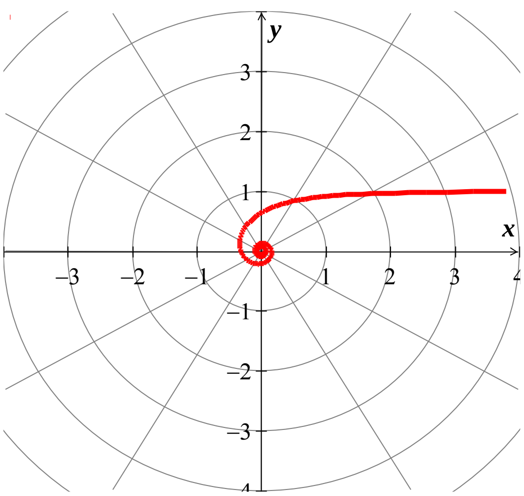
Solution

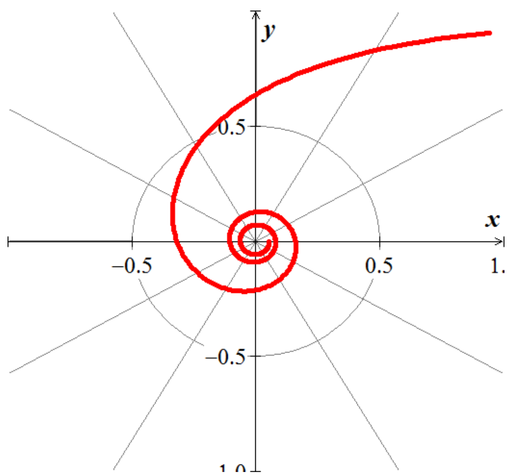


Exercise

Sketch the graph of the polar equation $r\theta = 1 \quad \theta > 0$

Solution





Exercise

Sketch the graph of the polar equation $r = 2 + 2\sec\theta$

Solution