Solution   Section 3.4 – Inverse Matrices

Exercise

Apply Gauss-Jordan method to find the inverse of this triangular "Pascal matrix"

Triangular Pascal matrix $A=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\1&1&0&0&0&1&0&0\\1&2&1&0&0&0&1&0\\1&3&3&1&0&0&0&1\end{array}\right]\begin{array}{l}\\R_2-R_1\\R_3-R_1\\R_4-R_1\end{array}$$

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&-1&1&0&0\\0&2&1&0&-1&0&1&0\\0&3&3&1&-1&0&0&1\end{array}\right]\begin{array}{l}\\\\R_3-2R_2\\R_4-3R_2\end{array}$$

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&1&-2&1&0\\0&0&3&1&2&-3&0&1\end{array}\right]\begin{array}{l}\\\\\\R_4-3R_3\end{array}$$

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&1&-2&1&0\\0&0&0&1&-1&3&-3&1\end{array}\right]$$

$$A^{-1}=\begin{pmatrix}1&0&0&0\\-1&1&0&0\\1&-2&1&0\\-1&3&-3&1\end{pmatrix}$$

- The inverse matrix A^{-1} looks like A, except odd-numbered diagonals are multiplied by -1.

Exercise

If A is invertible and $AB = AC$, prove that $B = C$

Solution

$AB = AC$ ***Multiply by A^{-1} both sides.***

$A^{-1}(AB) = A^{-1}(AC)$ ***Multiplication is associative***

$(A^{-1}A)B = (A^{-1}A)C$ $A^{-1}A = I$

$IB = IC$

$\boxed{B = C}$

Exercise

If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find two matrices $B \neq C$ such that $AB = AC$

Solution

Let $B = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ *and* $C = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$\boxed{B \neq C \Rightarrow AB = AC}$

Exercise

If A has ***row*** 1 + ***row*** 2 = ***row*** 3, show that A is not invertible

a) Explain why $Ax=(1,\ 0,\ 0)$ can't have a solution.

b) Which right sides $\left(b_1,\ b_2,\ b_3\right)$ might allow a solution to $Ax=b$

c) What happens to ***row*** 3 in elimination?

Solution

a) Let $A_1,\ A_2,\ A_3$ be the row vectors of A and x is a solution to $Ax=(1,\ 0,\ 0)$.

Then $A_1.x=1,\ A_2.x=0,\ A_3.x=0$.

Since $\quad A_1+A_2=A_3$

Means $\quad A_1.x+A_2.x=A_3.x$

Implies $\quad 1+0=0$ a contradiction

b) If $Ax=\left(b_1,\ b_2,\ b_3\right)\Rightarrow A_1.x=b_1,\ A_2.x=b_2,\ A_3.x=b_3$

Since $\quad A_1+A_2=A_3$

$A_1.x+A_2.x=A_3.x$

$\Rightarrow b_1+b_2=b_3$

c) In the elimination matrix, the third row will be zero.

Exercise

True or false (with a counterexample if false and a reason if true):

a) A 4 by 4 matrix with a row of zeros is not invertible.

b) A matrix with 1's down the main diagonal is invertible.

c) If A is invertible then A^{-1} is invertible.

d) If A is invertible then A^2 is invertible.

Solution

a) True, because it can have at most 3 pivots.

b) False, if the matrix: $A=\begin{pmatrix}1&0&0\\0&1&1\\0&1&1\end{pmatrix}$ and only has 2 pivots, thus is not invertible.

c) True, If A is invertible then necessarily A^{-1} is invertible.

d) True, $A^2x=0$ where x is nonzero matrix.

$$A^{-1}A^2x=\left(A^{-1}A\right)Ax=IAx=Ax=0$$

Since A is invertible, this can only be true if x was zero to begin with. Thus A^2 must also be invertible.

Exercise

Do there exist 2 by 2 matrices A and B with real entries such that $AB-BA=I$, where I is the identity matrix?

Solution

Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$$AB=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}=\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$BA=\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}$$

$$\begin{aligned} AB-BA &= \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}-\begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix} \\ &= \begin{pmatrix} bg-cf & af+bh-be-df \\ ce+dg-ag-ch & cf-bg \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{cases} bg-cf=1 \\ af+bh-be-df=0 \\ ce+dg-ag-ch=0 \\ cf-bg=1 \end{cases} \rightarrow \begin{cases} bg-cf=1 \\ \underline{cf-bg=1} \\ 0=2 \end{cases}$$

Therefore, $AB-BA\neq I$ for any 2 by 2 matrices.

Exercise

If B is the inverse of A^2, show that AB is the inverse of A.

Solution

Since B is the inverse of A^2 that implies: $\underline{B = \left(A^2\right)^{-1} = (AA)^{-1} = A^{-1}A^{-1}}$

Show that AB is the inverse of A

$$\begin{aligned}(AB)A &= \left(A\left(A^{-1}A^{-1}\right)\right)A \\ &= \left(\left(AA^{-1}\right)A^{-1}\right)A \\ &= \left(IA^{-1}\right)A \\ &= A^{-1}A \\ &\underline{= I}\end{aligned}$$

Therefore, AB is the inverse of A.

Exercise

Find and check the inverses (assuming they exist) of these block matrices.

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}$$

Solution

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix}\begin{bmatrix} I & 0 \\ A & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ C+A & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \Rightarrow C + A = 0 \to A = -C$$

$$\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -C & I \end{pmatrix}$$

$$\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}\begin{bmatrix} E & 0 \\ F & G \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} AE & 0 \\ CE+DF & DG \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\Rightarrow \begin{cases} AE = I \\ CE + DF = 0 \\ DG = I \end{cases} \rightarrow \begin{cases} E = A^{-1} \\ \\ G = D^{-1} \end{cases}$$

$$\begin{aligned} CE + DF = 0 \rightarrow CA^{-1} + DF &= 0 \\ DF &= -CA^{-1} \\ D^{-1}DF &= -D^{-1}CA^{-1} \\ IF &= -D^{-1}CA^{-1} \\ F &= -D^{-1}CA^{-1} \end{aligned}$$

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{pmatrix}$$

$$\begin{bmatrix} 0 & I \\ I & D \end{bmatrix}\begin{bmatrix} A & I \\ I & B \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & B \\ A+D & I+DB \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\rightarrow \begin{cases} B = 0 \\ A + D = 0 \\ I + DB = I \end{cases} \Rightarrow \begin{cases} A = -D \\ DB = 0 \end{cases}$$

$$\begin{pmatrix} 0 & I \\ I & D \end{pmatrix}^{-1} = \begin{pmatrix} -D & I \\ I & 0 \end{pmatrix}$$

Exercise

For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

<u>Solution</u>

$c = 0$, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 7 & 0 \end{bmatrix}$ (zero column 2 / row 2)

$$c=2,\ A=\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 8 & 7 & 2 \end{bmatrix} \text{ (equal rows)}$$

$$c=7,\ A=\begin{bmatrix} 2 & 7 & 7 \\ 7 & 7 & 7 \\ 8 & 7 & 7 \end{bmatrix} \text{ (equal columns)}$$

Exercise

Find A^{-1} and B^{-1} (if they exist) by elimination.

$$A=\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad B=\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Solution

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \frac{1}{2}R_1 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2-R_1 \\ R_3-R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right) \begin{array}{l} \\ \frac{2}{3}R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right) \begin{array}{l} R_1-\frac{1}{2}R_2 \\ \\ R_3-\frac{1}{2}R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right) \begin{array}{l} \\ \\ \frac{3}{4}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right)\begin{array}{l} R_1 - \frac{1}{3}R_3 \\ R_2 - \frac{1}{3}R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right)\begin{array}{l} \frac{1}{2}R_1 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right)\begin{array}{l} \\ R_2 + R_1 \\ R_3 + R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right)\begin{array}{l} \\ \\ R_3 + R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

B^{-1} doesn't exist, and if we add the columns in B, the result is zero.

Exercise

Find A^{-1} using the Gauss-Jordan method, which has a remarkable inverse.

$$A=\begin{bmatrix}1 & -1 & 1 & -1\\ 0 & 1 & -1 & 1\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Solution

$$\left(\begin{array}{cccc|cccc}1 & -1 & 1 & -1 & 1 & 0 & 0 & 0\\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right) R_1+R_2$$

$$\left(\begin{array}{cccc|cccc}1 & 0 & 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right) R_2+R_3$$

$$\left(\begin{array}{cccc|cccc}1 & 0 & 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right) R_3+R_4$$

$$\left(\begin{array}{cccc|cccc}1 & 0 & 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right)$$

$$A^{-1}=\begin{pmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}$$

Exercise

Find the inverse.

a) $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$

c) $\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

e) $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$

f) $\begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

g) $\begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}$

Solution

a) $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{12} & \frac{1}{6} \\ \frac{1}{8} & \frac{1}{4} \end{bmatrix}$

b) $A^{-1} = \frac{1}{7-8}\begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix}$

$= -\begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$

c) $A^{-1} = \frac{1}{-15-24}\begin{bmatrix} 5 & -6 \\ -4 & -3 \end{bmatrix}$

$= -\frac{1}{39}\begin{bmatrix} 5 & -6 \\ -4 & -3 \end{bmatrix}$

$= \begin{bmatrix} -\frac{5}{39} & \frac{2}{13} \\ \frac{4}{39} & \frac{1}{13} \end{bmatrix}$

d) $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_3 - R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \quad R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \quad -\tfrac{1}{2}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \quad \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

e) $$\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}^{-1} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{pmatrix}$$

f) $$\begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{pmatrix} \frac{\sqrt{2}}{26} & -\frac{3\sqrt{2}}{26} & 0 \\ \frac{2\sqrt{2}}{13} & \frac{\sqrt{2}}{26} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

g) $$\begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}^{-1} = \textit{doesn't exist}$$ This matrix is ***singular***

Exercise

Show that A is not invertible for any values of the entries

$$A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

Solution

Since the matrix A had zero's on its diagonals, therefore A is not invertible.

Exercise

Prove that if A is an invertible matrix and B is row equivalent to A, then B is also invertible.

Solution

Since B is row equivalent to A, there exist some elementary matrices $E_1, E_2, \ldots, E_n$ such that $B = E_n \cdots E_1 A$. Because $E_1, E_2, \ldots, E_n$ and A are invertible, then B is also invertible.

Exercise

Determine if the given matrix has an inverse, and find the inverse if it exists. Check your answer by multiplying $A \cdot A^{-1} = I$

a) $\begin{bmatrix} 2 & 3 \\ -3 & -5 \end{bmatrix}$ *b)* $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}$

Solution

a) $2(-5) - 3(-3) = -10 + 9 = -1$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} -5 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ -3 & -2 \end{bmatrix}$$

$$AA^{-1} = \begin{pmatrix} 2 & 3 \\ -3 & -5 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

b) $$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \quad R_3 - 2R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 3 & 3 & -2 & 0 & 1 \end{array}\right] \quad R_3 - 3R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & * & * & * \end{array}\right]$$

The inverse matrix doesn't exist

Exercise

Show that the inverse of $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is $\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$

Solution

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\theta)\cos(-\theta) - (\sin\theta)\sin(-\theta) & (\cos\theta)\sin(-\theta) - (\sin\theta)\cos(-\theta) \\ (-\sin\theta)\cos(-\theta) - (\cos\theta)\sin(-\theta) & (-\sin\theta)\sin(-\theta) + (\cos\theta)\cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\theta + \sin\theta\sin\theta & -\cos\theta\sin\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin\theta\sin\theta + \cos\theta\cos\theta \end{bmatrix} \qquad \begin{cases} \cos(-\theta) = \cos\theta & (even) \\ \sin(-\theta) = -\sin\theta & (odd) \end{cases}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$