Linear 9/10

1.5 Transpose, Diagonal, triangula

A^T D Δ

Symmetry

Transpose

Defn To transpose A is interchanging corresponding Rows & columns

A^T A'

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \qquad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

2×3 3×2

$b : a_{12}$ $b : a^T_{21}$

$$(A^T)_{ij} = A_{ji}$$

Properties

$(A^T)^T = A$ $(A^{-1})^{-1} = A$

$(A \pm B)^T = A^T \pm B^T$

$(kA)^T = kA^T$

$(AB)^T = B^T A^T \leftarrow (AB)^{-1} = B^{-1} A^{-1}$

$\to P$

Theorem

if A is invertible, the A^T is invertible

& $(A^T)^{-1} = (A^{-1})^T$

Proof

$$A^T(A^T)^{-1} = I \quad ?$$

$$A^T(A^T)^{-1} = A^T(A^{-1})^T$$

$$= (A^{-1}A)^T$$

A invertible $AA^{-1} = A^{-1}A = I$

$$= I^T$$

$$= I$$

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$A^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$(A^T)^{-1} = \frac{1}{ad-bc}\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a}{ad-bc} & \frac{-b}{ad-dc} \\ \frac{-c}{ad-bc} & \frac{d}{ad-bc} \end{pmatrix}$$

Trace

$tr(A)$ is defined by the sum of the entries of the main diagonal.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$tr(A) = a_{11} + a_{22} + a_{33}$$

Diagonal

has to be square matrix

$$D = \begin{pmatrix} d_1 & & \emptyset \\ 0 & d_2 & \\ 0 & 0 & 0 \ddots \\ & \emptyset & & d_n \end{pmatrix}$$

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & & \emptyset \\ & \frac{1}{d_2} & \\ \emptyset & & \ddots & \\ & & & \frac{1}{d_n} \end{bmatrix}$$

$$D^k = \begin{bmatrix} d_1^k & & \emptyset \\ & d_2^k & \\ \emptyset & & \ddots & \\ & & & d_n^k \end{bmatrix}$$

Triangular Matrices

$$\begin{bmatrix} a_{11} & & \emptyset \\ & a_{22} & \\ & & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & \\ \emptyset & & a_{33} \end{bmatrix}$$

lower upper

Triangular

Theorem

- Transpose of a lower triangular matrix is upper triangular matrix
- Product of 2 lower triangular matrix is lower Triangular

upper $\to$ upper

- A triangular matrix is invertible iff its diagonal entries are all nonzero (no zero on the main diagonal
- if A is invertible
  A^{-1} of lower (upper) triangular is a lower (upper) Triangular

matrix A can be factorized as lower (L) time Upper (U) triangular

$$A = L\,U$$

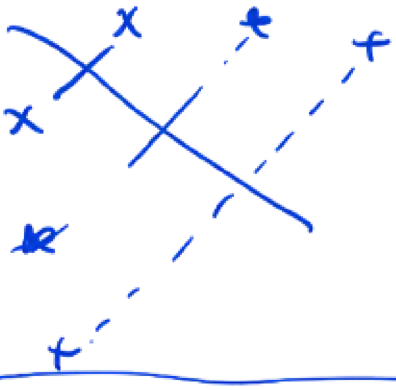
Symmetric Matrices.

Defn A square matrix is said to be symmetric if $A^T = A$

$$a_{ij} = a_{ji}$$

Ex $A = \begin{pmatrix} 1 & -4 \\ -4 & 1 \end{pmatrix} = A^T$

$$A = \begin{pmatrix} 6 & 5 & 1 \\ 5 & 0 & 7 \\ 1 & 7 & -1 \end{pmatrix} = A^T$$



Theorem

If A & B are symmetric matrices w/ same size + if k any scalar:

a) A^T is symmetric

b) $A \pm B$ is "

c) kA is "

if A is invertible & symmetric then A^{-1} is symmetric

Proof

A is invertible $\Rightarrow$ $AA^{-1} = A^{-1}A = I$

A is symmetric $\Rightarrow$ $\underline{A^T = A}$

$A^{-1} = (A^{-1})^T$?

$(A^{-1})^T = (A^T)^{-1}$

$= A^{-1}$ ✓

A is symmetric

$A^T = A$

$\therefore \bar{A}^{-1}$ is symmetric

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① If A is an invertible $\Rightarrow$
$A^T A$ & $A A^T$ are invertible

② A is symmetric, so is A^{-1}

③ A is tri diagonal (only 3 nonzeros diagonal. But A^{-1} is a full matrix

A is symmetric $A = A^T$

$A = LDU$

1.6 Determinant
(square matrix)

$\det(A)$ or $|A|$

$\underline{2\times 2}$ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

① $\det(I_{n\times n}) = 1$

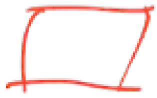
② interchange any 2 rows $\Rightarrow \det = -$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad$$
$$= -(ad - cb)$$
$$= - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

(3) $\begin{vmatrix} ta & tb \\ rc & rd \end{vmatrix} = tr \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

2×2 det = area



n dim = Volume

(4) 2 rows are equals $\Rightarrow$ det $= 0$.

(6) A matrix w/ row of zeros
det $= 0$

* A is triangular or diagonal
then $|A| = \prod_{i=1}^{n} a_{ii}$

$= a_{11} a_{22} a_{33} \cdots a_{nn}$

- A is singular $\Rightarrow \det(A) = 0$
- invertible $\Rightarrow \det(A) \neq 0$
- $|AB| = |BA|$
- $\det(A^T) = \det(A)$

$\det(A+B) \neq \det(A) + \det(B)$

$\det(A+B) \neq \det(A) + \det(B)$

$$\begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix}$$

126

94   114

42

36

$$\begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = 36$$

$$\begin{vmatrix} -8 & 0 & 6 \\ 4 & -5 & 7 \\ -1 & -3 & 5 \end{vmatrix} = -36$$

$x =$

$y =$

$z =$

(200) $(200 + 0) - 48$