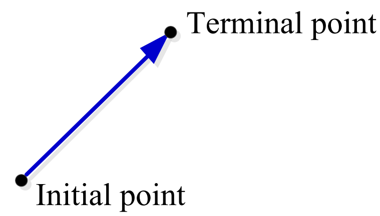
Lecture 1 – Vectors (*Review*)

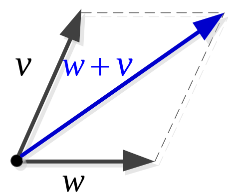
The direction of the arrowhead specifies the ***direction*** of the vector and the ***length*** of the arrow specifies the *magnitude*.

The tail of the arrow is called the ***initial point*** of the vector and the tip the ***terminal point***.



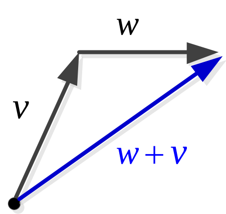
Parallelogram Rule for Vector Addition

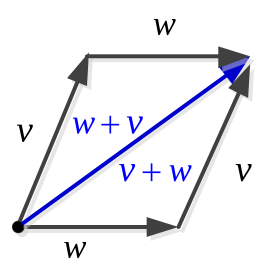
If $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors that are positioned so their initial points coincides, then the vectors form adjacent sides of a parallelogram, and then the sum $\boldsymbol{v} + \boldsymbol{w}$ is the vector represented by the arrow from the common initial point of $\boldsymbol{v}$ and $\boldsymbol{w}$ to the opposite vertex of the parallelogram.



Triangle Rule for Vector Addition

If $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors that are positioned so the initial point of w is at the terminal point of $\boldsymbol{v}$, then the sum $\boldsymbol{v} + \boldsymbol{w}$ is represented by the arrow from the initial point of $\boldsymbol{v}$ to the terminal point of $\boldsymbol{w}$.



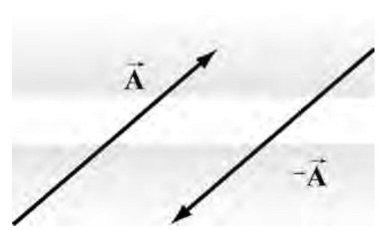


$$\boxed{\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}}$$

Inverse element for Vector Addition

For every vector $\boldsymbol{A}$, there is a unique inverse vector

$$(-1)A \equiv -A \quad \textit{such that} \quad A + (A) = 0$$

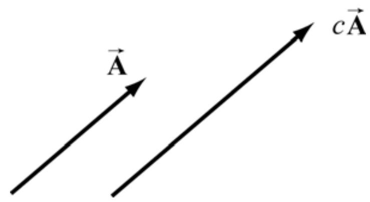


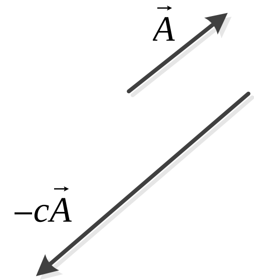
This means that the vector $-A$ has the same magnitude as A,

$$|A| = |-A| = A$$

Scalar Field

A scalar function of one or more variables;
e.g. The distribution of locally averaged family incomes over the United States.
Or the electric charge distribution of the surface of a piece of metal.

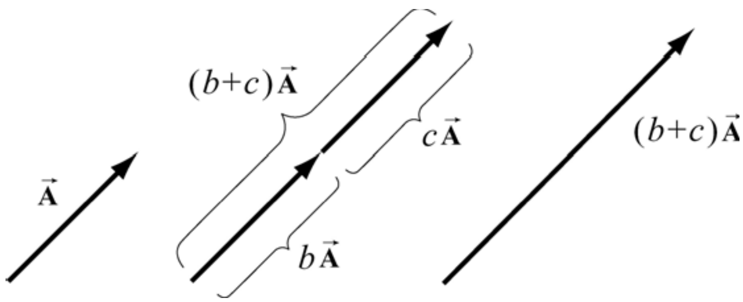




Distributive Law for Scalar Addition

The multiplication operation also satisfies a distributive law for the addition of numbers.
Let b and c be real numbers. Then

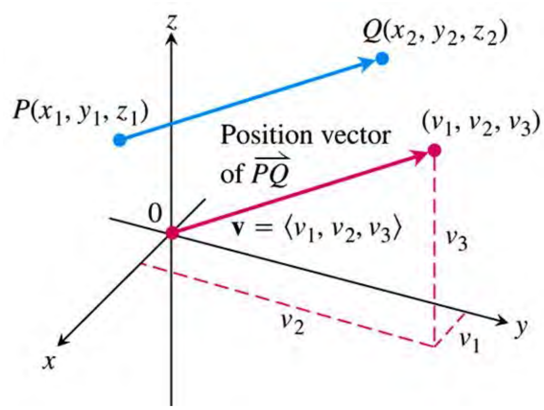
$$(b+c)\vec{A} = b\vec{A} + c\vec{A}$$



Identity Element for Scalar Multiplication

The number 1 acts as an identity element for multiplication

$$1\vec{A} = \vec{A}$$



Definition

The vector represented by the direected line segment $\overrightarrow{PQ}$ has initial point P and terminal point Q and its length is denoted by $\left|\overrightarrow{PQ}\right|$

Magnitude

The vector components $\vec{A} = \left(A_x, A_y, A_z\right)$

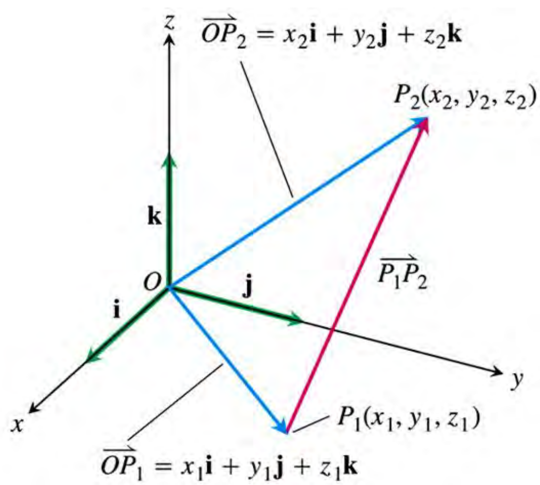
$$\left|\vec{A}\right| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

The magnitude or length of the vector $v = \overrightarrow{PQ}$ is the nonegative number

$$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$$

Unit vectors

The idea of multiplication by real numbers allows us to define a set of unit vectors at each point in space. We associate to each point P in space, a set of three unit vectors $\left(\hat{i}, \hat{j}, \hat{k}\right)$. A unit vector means that the magnitude is one: $\left|\hat{i}\right| = 1, \quad \left|\hat{j}\right| = 1, \quad and \ \left|\hat{k}\right| = 1$



A vector v of length 1 is called a ***unit vector***. The ***standard unit vectors*** are

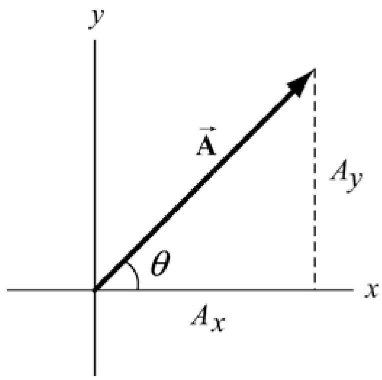
$$\hat{i} = \langle 1,\ 0,\ 0\rangle, \quad \hat{j} = \langle 0,\ 1,\ 0\rangle, \quad and \quad \hat{k} = \langle 0,\ 0,\ 1\rangle$$

Any vector $v = \left\langle v_1, v_2, v_3 \right\rangle$ can be written as a linear combination of the standard unit vectors as follows:

$$\begin{aligned} v &= \left\langle v_1, v_2, v_3 \right\rangle \\ &= \left\langle v_1, 0, 0 \right\rangle + \left\langle 0, v_2, 0 \right\rangle + \left\langle 0, 0, v_3 \right\rangle \\ &= v_1 \langle 1, 0, 0\rangle + v_2 \langle 0, 1, 0\rangle + v_3 \langle 0, 0, 1\rangle \\ &= v_1\hat{i} + v_2\hat{j} + v_3\hat{k} \end{aligned}$$

Direction

Let's consider a vector $\vec{\boldsymbol{A}} = \left(A_x, A_y, 0\right)$. Since the z-component is zero, the vector $\vec{\boldsymbol{A}}$ lies in the x-y plane. Let θ denote the angle that the vector $\vec{\boldsymbol{A}}$ makes in the counterclockwise direction with the positive x-axis.



$$A_x = A\cos\theta, \qquad A_y = A\sin\theta$$

We can write a vector in the x-y plane as

$$\vec{\boldsymbol{A}} = A\cos\theta\,\hat{\boldsymbol{i}} + A\sin\theta\,\hat{\boldsymbol{j}}$$

$$\tan\theta = \frac{A_y}{A_x}$$

Proporties of Vector Operations

Let ***u, v, w*** be vectors and *a, b* be scalars

1. $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$
2. $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$
3. $\boldsymbol{u}+\boldsymbol{0}=\boldsymbol{u}$
4. $\boldsymbol{u}+(-\boldsymbol{u})=\boldsymbol{0}$
5. $0\boldsymbol{u}=\boldsymbol{0}$
6. $1\boldsymbol{u}=\boldsymbol{u}$
7. $a(b\boldsymbol{u})=(ab)\boldsymbol{u}$
8. $(a+b)\boldsymbol{u}=a\boldsymbol{u}+b\boldsymbol{u}$
9. $a(\boldsymbol{u}+\boldsymbol{v})=a\boldsymbol{u}+a\boldsymbol{v}$

Vector Field

A vector function of one or more variables; e.g. the velocity of a projectile fired from a cannon as a function of time.
Or the electric field as a function of time and space.

Why vectors?

Some quantities of physical interest cannot be characterized by a single number.

When we apply vectors to physical quantities it's nice to keep in the back of our minds all these formal properties. However from the physicist's point of view, we are interested in representing physical quantities such as displacement, velocity, acceleration, force, impulse, momentum, torque, and angular momentum as vectors. We can't add force to velocity or subtract momentum from torque. We must always understand the physical context for the vector quantity. Thus, instead of approaching vectors as formal mathematical objects we shall instead consider the following essential properties that enable us to represent physical quantities as vectors.

In order to fully characterize a vector field, $\vec{F}$, which represents, say force, you must

1. Tell the point in space and time at which you are interested in determining *F*.
2. Tell the ***direction*** in which *F* acts.
3. Tell how ***much*** force is being applied $\left(\left|\vec{F}\right|\right)$.

How is a vector represented?

1. ***Graphically***: This type of representation is useful for illustrative purposes for vector fields and can be useful computationally for two-dimensional vector fields.
2. ***Resolution into components***: This is most commonly how we represent a vector.
   For example, in rectangular components, we represent a vector by specifying its projections along the *x*, *y*, and *z* axes.

The ***Base*** vectors for rectangular components are vectors with ***unit*** magnitude and with a direction along the *x*, *y*, and *z* axes.

The most common notations are

NOTATION	***Base Vector Direction***		
	x	y	z
	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
	$\bullet 1$	$\bullet 2$	$\bullet 3$
	$\hat{x}$	$\hat{y}$	$\hat{z}$

In general, a circumflex, "^." Over a symbol will be used to indicate that symbol represents a unit vector.

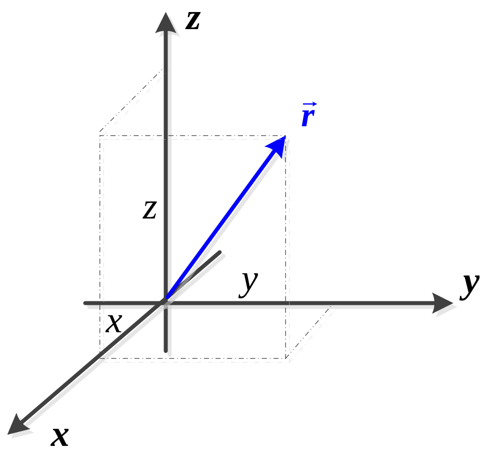
Thus, in our notation, a vector $\vec{F}$, would be written as

$$\vec{F} = F_x \hat{x} + F_y \hat{y} + F_z \hat{z}$$

In rectangular coordinates.

Rather than writing out "*x, y, z*" every time, it is preferable to define the position vector,

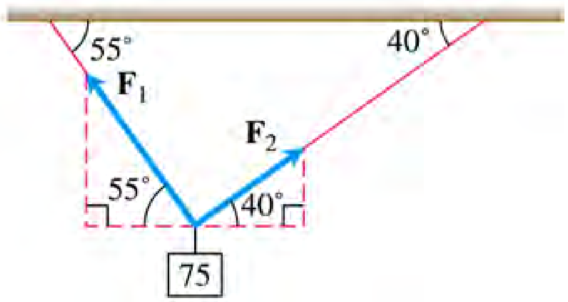
$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$



Thus, we write $\vec{F} = \vec{F}(\vec{r},\ t)$

Example

A 75-N weight is suspended by two wires.

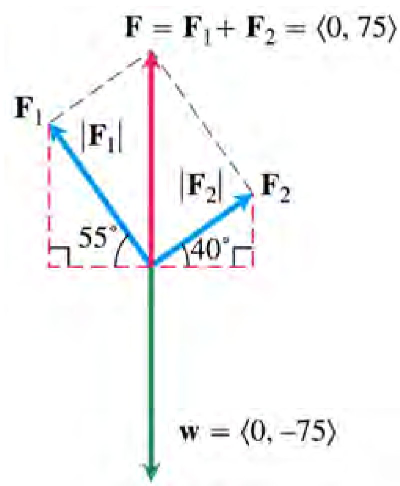


Find the forces F_1 and F_2 acting both wires

Solution

$$F_1 = \left\langle -\left|F_1\right|\cos 55°,\ \left|F_1\right|\sin 55°\right\rangle$$

$$F_2 = \left\langle \left|F_2\right|\cos 40°,\ \left|F_2\right|\sin 40°\right\rangle$$



$$F_1 + F_2 = \langle 0,\ 75\rangle$$

$$-\left|F_1\right|\cos 55° + \left|F_2\right|\cos 40° = 0 \quad\Rightarrow\quad \left|F_2\right| = \left|F_1\right|\frac{\cos 55°}{\cos 40°}$$

$$\left|F_1\right|\sin 55° + \left|F_2\right|\sin 40° = 75$$

$$\left|F_1\right|\sin 55° + \left|F_1\right|\frac{\cos 55°}{\cos 40°}\sin 40° = 75$$

$$\left|F_1\right|\left(\sin 55° + \cos 55° \tan 40°\right) = 75$$

$$\left|F_1\right| = \frac{75}{\sin 55° + \cos 55° \tan 40°} \approx 57.67\ N$$

$$\left|F_2\right| = 57.67\frac{\cos 55°}{\cos 40°} \approx 43.18\ N$$

The force vectors are then:

$$F_1 = \left\langle -\left|F_1\right|\cos 55°,\ \left|F_1\right|\sin 55°\right\rangle$$

$$= \langle -57.67\cos 55°,\ 57.67\sin 55°\rangle$$

$$= \langle -33.08,\ 47.24\rangle$$

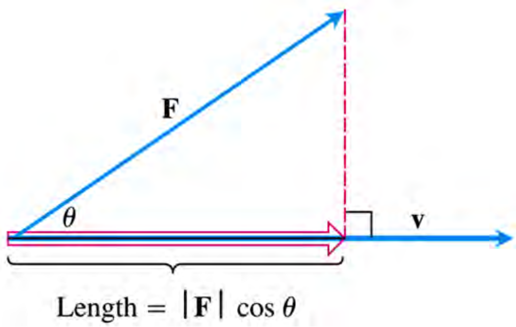
$$F_2 = \left\langle \left|F_2\right|\cos 40°,\ \left|F_2\right|\sin 40°\right\rangle$$

$$= \langle 43.18\cos 40°,\ 43.18\sin 40°\rangle$$

$$= \langle 33.08,\ 27.76\rangle$$

The Dot Product

If a force $\boldsymbol{F}$ is applied to a particle moving along a path, we often need to know the magnitude of the force and the direction of motion.



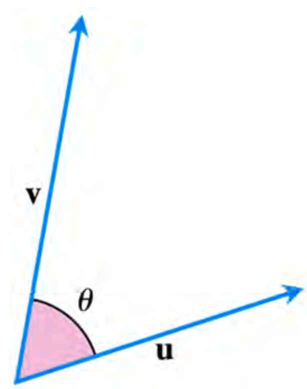
To calculate the angle between two vectors directly from their component, called the ***dot product***, also called *inner* or *scalar* products.

Angle between Vectors

Theorem

The angle θ between two nonzero vectors $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\boxed{\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\boldsymbol{u}||\boldsymbol{v}|}\right)}$$



Definition

The dot product $\boldsymbol{u} \cdot \boldsymbol{v}$ of vector $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1v_1 + u_2v_2 + u_3v_3$$

Example

Find the dot product:

a) $\langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle$

b) $\left(\frac{1}{2}\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}\right) \cdot (4\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

Solution

a) $\langle 1,-2,-1\rangle \cdot \langle -6,2,-3\rangle = 1(-6)+(-2)(2)+(-1)(-3)$

$$= -7$$

b) $\left(\frac{1}{2}\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}\right)\cdot(4\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}) = \frac{1}{2}(4)+3(-1)+1(2)$

$$= 1$$

Example

Find the angle between $\boldsymbol{u} = \boldsymbol{i}-2\boldsymbol{j}-2\boldsymbol{k}$ *and* $\boldsymbol{v} = 6\boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k}$

Solution

$$\boldsymbol{u}\cdot\boldsymbol{v} = 1(6)+(-2)(3)+(-2)(2) = -4$$

$$\boldsymbol{u} = \sqrt{1^2+(-2)^2+(-2)^2} = 3$$

$$\boldsymbol{v} = \sqrt{6^2+3^2+2^2} = 7$$

$$\theta = \cos^{-1}\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$$

$$= \cos^{-1}\left(\frac{-4}{(3)(7)}\right)$$

$$\approx 1.76\ rad$$

Definition **Perpendicular (*Orthogonal*) Vectors**

Vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal (or perpendicular) iff $\boldsymbol{u}\cdot\boldsymbol{v} = 0$

Dot Product Properties and Vector Projection

If $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are any vectors and c is a scalar, then

a) $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{u}$

b) $\boldsymbol{u}\cdot\boldsymbol{u} = |\boldsymbol{u}|^2$

c) $\boldsymbol{u}\cdot(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{u}\cdot\boldsymbol{v}+\boldsymbol{u}\cdot\boldsymbol{w}$

d) $\boldsymbol{u}\cdot(\boldsymbol{v}-\boldsymbol{w}) = \boldsymbol{u}\cdot\boldsymbol{v}-\boldsymbol{u}\cdot\boldsymbol{w}$

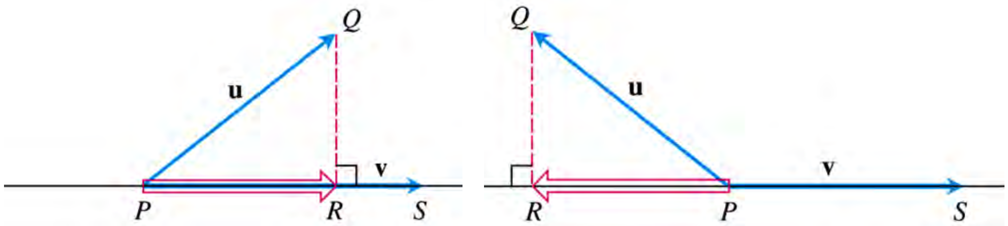
e) $(\boldsymbol{u}+\boldsymbol{v})\cdot\boldsymbol{w} = \boldsymbol{u}\cdot\boldsymbol{w}+\boldsymbol{v}\cdot\boldsymbol{w}$

f) $(\boldsymbol{u}-\boldsymbol{v})\cdot\boldsymbol{w} = \boldsymbol{u}\cdot\boldsymbol{w}-\boldsymbol{v}\cdot\boldsymbol{w}$

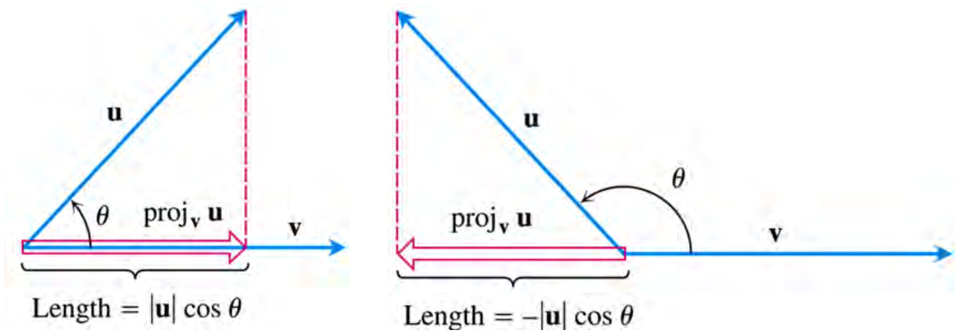
g) $c(\boldsymbol{u}\cdot\boldsymbol{v}) = (c\boldsymbol{u})\cdot\boldsymbol{v} = \boldsymbol{u}\cdot(c\boldsymbol{v})$

h) $0\cdot\boldsymbol{v} = \boldsymbol{v}\cdot 0 = 0$

The vector projection of $\boldsymbol{u}=\overrightarrow{PQ}$ onto a nonzero vector $\boldsymbol{v}=\overrightarrow{PS}$ is the vector $\overrightarrow{PR}$ determined by dropping a perpendicular from Q to the line PS.



The notation for this vector is $\quad proj_{\boldsymbol{v}}\boldsymbol{u} \quad$ (The vector projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$)



$$proj_{\boldsymbol{v}}\boldsymbol{u}=\left(|\boldsymbol{u}|\cos\theta\right)\frac{\boldsymbol{u}}{|\boldsymbol{v}|}=\left(\frac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\boldsymbol{v}$$

The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$ is the scalar: $\quad |\boldsymbol{u}|\cos\theta=\frac{\boldsymbol{u}.\boldsymbol{v}}{|\boldsymbol{v}|}=\boldsymbol{u}.\frac{\boldsymbol{v}}{|\boldsymbol{v}|}$

Example

Find the vector projection of $\boldsymbol{u}=6\boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k}$ *onto* $\boldsymbol{v}=\boldsymbol{i}-2\boldsymbol{j}-2\boldsymbol{k}$ and the scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$.

<u>Solution</u>

$$proj_{\boldsymbol{v}}\boldsymbol{u}=\frac{\boldsymbol{u}\bullet\boldsymbol{v}}{|\boldsymbol{v}|^2}\boldsymbol{v}=\frac{6(1)+3(-2)+2(-2)}{1^2+(-2)^2+(-2)^2}(\boldsymbol{i}-2\boldsymbol{j}-2\boldsymbol{k})=\frac{-4}{9}(\boldsymbol{i}-2\boldsymbol{j}-2\boldsymbol{k})$$

$$\boxed{=-\frac{4}{9}\boldsymbol{i}+\frac{8}{9}\boldsymbol{j}+\frac{8}{9}\boldsymbol{k}}$$

$$\boldsymbol{u}\cos\theta=\boldsymbol{u}\bullet\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=(6\boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k})\bullet\frac{\boldsymbol{i}-2\boldsymbol{j}-2\boldsymbol{k}}{\sqrt{1^2+(-2)^2+(-2)^2}}$$

$$=(6\boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k})\bullet\left(\frac{1}{3}\boldsymbol{i}-\frac{2}{3}\boldsymbol{j}-\frac{2}{3}\boldsymbol{k}\right)$$

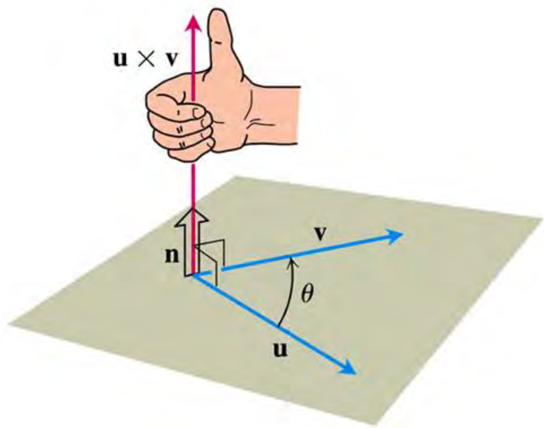
$$=6\left(\frac{1}{3}\right)+3\left(-\frac{2}{3}\right)+2\left(-\frac{2}{3}\right)$$

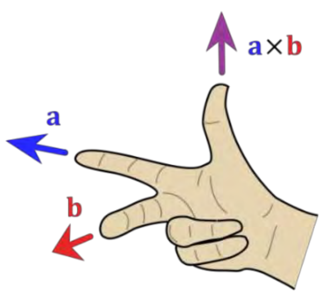
$$\boxed{=-\frac{4}{3}}$$

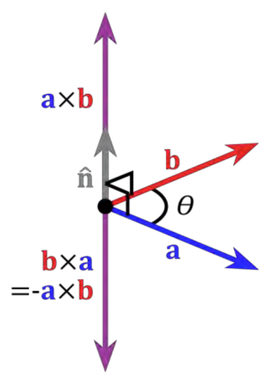
The *Cross Product*

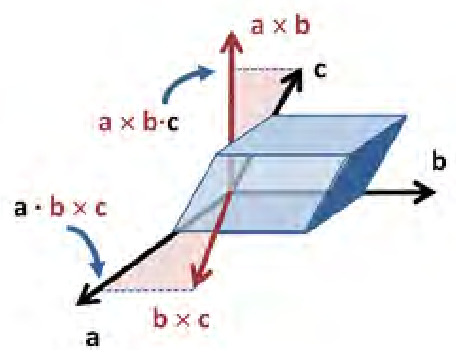
To find a vector in 3-space that is perpendicular to two vectors; the type of vector multiplication that facilities this construction is the cross product.

We start with two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in space. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are no parallel, they determine a plane. We select a unit vector $\boldsymbol{n}$ perpendicular to the plane by the ***right-hand rule***. Then the cross product $\boldsymbol{u}\times\boldsymbol{v}$ ("$\boldsymbol{u}$ cross $\boldsymbol{v}$") is the vector defined as follows









Definition

$$\boldsymbol{u}\times\boldsymbol{v} = \left(|\boldsymbol{u}||\boldsymbol{v}|\sin\theta\right)\boldsymbol{n}$$

Parallel Vectors

Nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel iff $\boldsymbol{u}\times\boldsymbol{v} = \boldsymbol{0}$

Properties of the Cross Product

If ***u***, ***v*** and ***w*** are any vectors and *r, s* are scalars, then

a) $\boldsymbol{u}\times\boldsymbol{v} = -(\boldsymbol{v}\times\boldsymbol{u})$

b) $\boldsymbol{u}\times(\boldsymbol{v}+\boldsymbol{w}) = (\boldsymbol{u}\times\boldsymbol{v})+(\boldsymbol{u}\times\boldsymbol{w})$

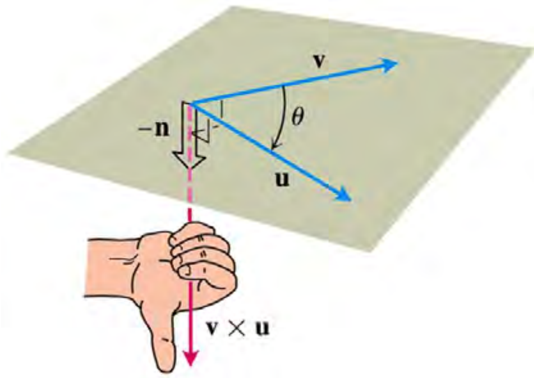
c) $(\boldsymbol{u}+\boldsymbol{v})\times\boldsymbol{w} = (\boldsymbol{u}\times\boldsymbol{w})+(\boldsymbol{v}\times\boldsymbol{w})$

d) $r(\boldsymbol{u}\times\boldsymbol{v}) = (r\boldsymbol{u})\times\boldsymbol{v} = \boldsymbol{u}\times(r\boldsymbol{v})$

e) $(r\boldsymbol{u})\times(s\boldsymbol{v}) = (rs)(\boldsymbol{u}\times\boldsymbol{v})$

f) $\boldsymbol{u}\times 0 = 0\times\boldsymbol{u} = 0$

g) $\boldsymbol{u}\times\boldsymbol{u} = 0$



Note:

✓ $\boldsymbol{i}\times\boldsymbol{i} = \boldsymbol{j}\times\boldsymbol{j} = \boldsymbol{k}\times\boldsymbol{k} = \mathbf{0}$

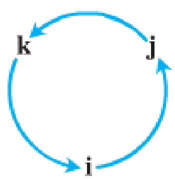
✓ $\boldsymbol{i}\times\boldsymbol{j} = \boldsymbol{k}, \quad \boldsymbol{j}\times\boldsymbol{k} = \boldsymbol{i}, \quad \boldsymbol{k}\times\boldsymbol{i} = \boldsymbol{j}$

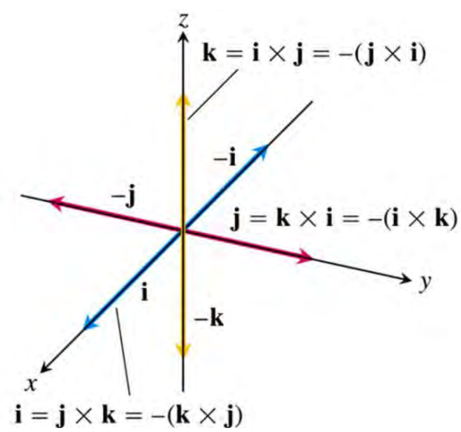
✓ $\boldsymbol{j}\times\boldsymbol{i} = -\boldsymbol{k}, \quad \boldsymbol{k}\times\boldsymbol{j} = -\boldsymbol{i}, \quad \boldsymbol{i}\times\boldsymbol{k} = -\boldsymbol{j}$

✓ $\boldsymbol{i}\times\boldsymbol{j} = -(\boldsymbol{j}\times\boldsymbol{i}) = \boldsymbol{k}$

✓ $\boldsymbol{j}\times\boldsymbol{k} = -(\boldsymbol{k}\times\boldsymbol{j}) = \boldsymbol{i}$

✓ $\boldsymbol{k}\times\boldsymbol{i} = -(\boldsymbol{i}\times\boldsymbol{k}) = \boldsymbol{j}$

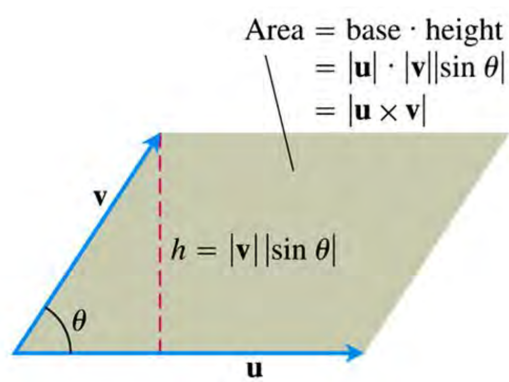




$|\boldsymbol{u}\times\boldsymbol{v}|$ Is the *Area* of the Parallelogram

Because ***n*** is a unit vector, the magnitude of $\boldsymbol{u}\times\boldsymbol{v}$ is

$$|\boldsymbol{u}\times\boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}||\sin\theta||n| = |\boldsymbol{u}||\boldsymbol{v}||\sin\theta|$$



Determinant Formula for $\boldsymbol{u}\times\boldsymbol{v}$

Definition

The cross product of $\boldsymbol{u}=\left(u_1,\ u_2,\ u_3\right)$ and $\boldsymbol{v}=\left(v_1,\ v_2,\ v_3\right)$ is the vector

$$\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$$

$$=\begin{vmatrix}u_2 & v_2\\ u_3 & v_3\end{vmatrix}\boldsymbol{i}-\begin{vmatrix}u_1 & v_1\\ u_3 & v_3\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}u_1 & v_2\\ u_2 & v_1\end{vmatrix}\boldsymbol{k}$$

$$=\left(u_2v_3-u_3v_2\right)\boldsymbol{i}-\left(u_1v_3-u_3v_1\right)\boldsymbol{j}+\left(u_1v_2-u_2v_1\right)\boldsymbol{k}$$

$$\boxed{=\left(u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1\right)}$$

Example

Find $\boldsymbol{u}\times\boldsymbol{v}$ and $\boldsymbol{v}\times\boldsymbol{u}$ if $\boldsymbol{u}=2\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{v}=-4\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$

<u>Solution</u>

$$\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 2 & 1 & 1\\ -4 & 3 & 1\end{vmatrix}$$

$$=\begin{vmatrix}1 & 1\\ 3 & 1\end{vmatrix}\boldsymbol{i}-\begin{vmatrix}2 & 1\\ -4 & 1\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}2 & 1\\ -4 & 3\end{vmatrix}\boldsymbol{k}$$

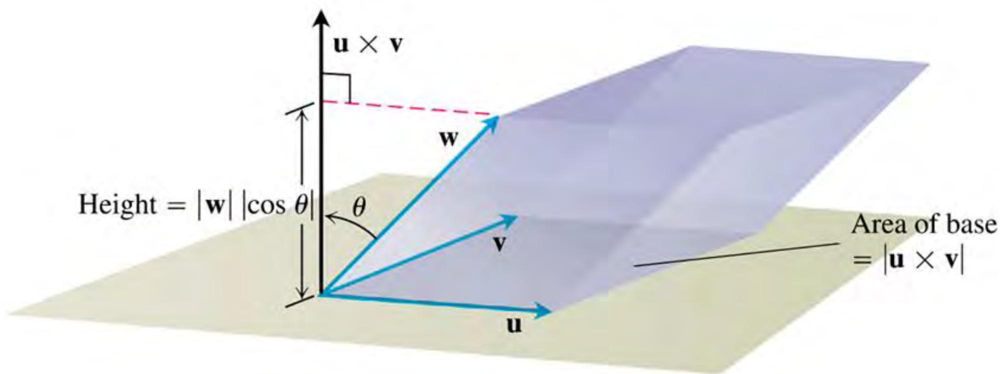
$$=-2\boldsymbol{i}-6\boldsymbol{j}+10\boldsymbol{k}$$

$$\boldsymbol{v}\times\boldsymbol{u}=-\boldsymbol{u}\times\boldsymbol{v}=2\boldsymbol{i}+6\boldsymbol{j}-10\boldsymbol{k}$$

Triple Scalar or Box Product

The product $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$ is called the triple scalar product of ***u, v***, and ***w*** (in that order).

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = |\boldsymbol{u} \times \boldsymbol{v}||\boldsymbol{w}||\cos\theta|$$



Volume

The Volume of the Parallelepiped is

$$\begin{aligned} V &= (area\ of\ \ base).(height) \\ &= |\boldsymbol{u} \times \boldsymbol{v}||\boldsymbol{w}||\cos\theta| \\ &= |\boldsymbol{u} \times \boldsymbol{v}|\frac{|(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}|}{|\boldsymbol{u} \times \boldsymbol{v}|} \\ &= |(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}| \end{aligned}$$

$$V = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = \left| \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \right|$$

Example

Find the volume of the box (parallelepiped) determined by $\boldsymbol{u} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k},\ \boldsymbol{v} = -2\boldsymbol{i} + 3\boldsymbol{k},$ and $\boldsymbol{w} = 7\boldsymbol{j} - 4\boldsymbol{k}$.

<u>Solution</u>

$$V = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = \left| \det \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{bmatrix} \right| = |-23| = 23$$

The volume is 23 units cubed.