Section 4.4 – Eigenvalues and Eigenvectors

In many problems in science and mathematics, linear equations $A\vec{x} = \vec{b}$ come from steady state problems. Eigenvalues have their greatest importance in dynamic problems. The solution of $A\vec{x} = \lambda\vec{x}$ *or* $\frac{d\vec{x}}{dt} = A\vec{x}$ (is changing with time) has nonzero solutions. (***All matrices are square***)

Definition

Suppose A is an n x n matrix and

$$\boldsymbol{\lambda}\vec{x} = \boldsymbol{A}\vec{x}$$

The values of λ are called eigenvalues of the matrix $\boldsymbol{A}$ and the nonzero vectors $\vec{x}$ in $\mathbb{R}^n$ are called the eigenvectors corresponding to that eigenvalue (λ).

λ is the eigenvalue associated with or corresponding to the eigenvector $\vec{x}$.

- One of the meanings of the word "***eigen***" in German is "***proper***"; eigenvalues are also called ***proper values, characteristic values,*** or ***latent roots***.

Example

The vector $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 3$ since

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 6 \end{bmatrix} \\ &= 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= 3\vec{x} \end{aligned}$$

Eigenvalues and eigenvectors have a useful geometric interpretation in $\mathbb{R}^2$ and $\mathbb{R}^3$.

The equation for the *eigenvalues*

Let's rewrite the equation $\lambda\vec{x} = A\vec{x}$.

$A\vec{x} - \lambda\vec{x} = 0$ $\qquad\qquad$ λ: are the eigenvalues and not a vector

$A\vec{x} - \lambda I\vec{x} = 0$

$(A - \lambda I)\vec{x} = 0$

The matrix $A - \lambda I$ times the eigenvectors $\vec{x}$ is the zero vector.

The eigenvectors make up the nullspace of $A - \lambda I$.

Definition

The number λ is an eigenvalue of $\boldsymbol{A}$ if and only if $A - \lambda I$ is singular:

$$\boxed{\det(A - \lambda I) = 0}$$

This equation $\det(A - \lambda I) = 0$ is called ***characteristic equation*** of $\boldsymbol{A}$; the scalars satisfying this equation are the eigenvalues of $\boldsymbol{A}$. when expanding the determinant $\det(A - \lambda I)$ is a polynomial in λ of degree *n*, called the ***characteristic polynomial*** of $\boldsymbol{A}$.

Example

Find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

Solution

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix}$$

$$= (3-\lambda)(-\lambda) + 2$$

$$= \lambda^2 - 3\lambda + 2$$

The characteristic equation of $\boldsymbol{A}$ is:

$$\lambda^2 - 3\lambda + 2 = 0$$

The eigenvalues of $\boldsymbol{A}$ are $\lambda_1 = 1, \ \lambda_2 = 2$

Theorem

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A.

Example

Find the eigenvalues of the lower triangular matrix

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{3}{2} & 0 \\ 5 & -8 & -\frac{1}{4} \end{pmatrix}$$

Solution

The eigenvalues are: $\lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{3}{2}, \quad and \quad \lambda_3 = -\frac{1}{4}$

Theorem

If A is an $n \times n$ matrix, the following are equivalent.

a) λ is an eigenvalue of A.

b) The system of equations $(A - \lambda I)\vec{x} = \vec{0}$ has nontrivial solutions.

c) There is a nonzero vector $\vec{x}$ in $\mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$.

d) λ is a real solution of the characteristic equation $\det(A - \lambda I) = 0$

Eigenvectors

To find the eigenvector $\vec{x}$, for each eigenvalue λ solve $(A-\lambda I)\vec{x}=0$ *or* $A\vec{x}=\lambda\vec{x}$

From the eigenvalues, the eigenvectors, in the form $\boldsymbol{V}_i=\begin{pmatrix} x_i \\ y_i \end{pmatrix}$, of the system can be determined by letting:

$$(A-\lambda_1 I)V_1=0 \qquad \text{and}\,(A-\lambda_2 I)V_2=0$$

Example

Find the eigenvalues and the eigenvectors of the matrix $A=\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

Solution

$$\det(A-\lambda I)=\det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix}$$

$$=(1-\lambda)(4-\lambda)-4$$

$$=\lambda^2-5\lambda+4-4$$

$$=\lambda^2-5\lambda$$

$$=\lambda(\lambda-5)=\mathbf{0}$$

The eigenvalues of $\boldsymbol{A}$ are: $\lambda_1=0 \quad \lambda_2=5$

For $\lambda_1=0$, we have:

$$(A-\lambda_1 I)V_1=0$$

$$\begin{pmatrix} 1-0 & 2 \\ 2 & 4-0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x+2y=0 \\ \end{cases}$$

$\underline{x=-2y}\,|$

If $y=-1 \Rightarrow x=2$

Therefore, the eigenvector $\boxed{V_1=\begin{pmatrix} 2 \\ -1 \end{pmatrix}}$

Or $\begin{pmatrix} -2y \\ y \end{pmatrix}=y\begin{pmatrix} -2 \\ 1 \end{pmatrix} \Rightarrow V_1=\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ *or* $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

For $\lambda_2 = 5 \quad \Rightarrow \quad \left(A - \lambda_2 I\right)V_2 = 0$:

$$\begin{pmatrix} 1-5 & 2 \\ 2 & 4-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \ 2x - y = 0$$

$\underline{2x = y|}$

Therefore, the eigenvector $V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Power of a Matrix

Theorem

If k is a positive integer, λ is an eigenvalue of a matrix A, and $\vec{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\vec{x}$ is a corresponding eigenvector.

Example

Find the eigenvalues of A^7 for $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

Solution

$$\det\left(A - \lambda I\right) = \begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}$$

$$= \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

The eigenvalues of A: $\lambda_1 = 1 \quad and \quad \lambda_2 = 2$

The eigenvalues of A^7 are:

$\lambda_1 = 1^7 = 1 \quad and \quad \lambda_2 = 2^7 = 128$

Theorem

A square matrix A is invertible *iff* $\lambda = 0$ is not an eigenvalue of A.

Summary

To solve the eigenvalue problem for an n by n matrix:

1. Compute the determinant of $A-\lambda I$. With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It is a polynomial in λ of degree n.
2. Find the roots of this polynomial, by solving $\det(A-\lambda I)=0$. The n roots are the n eigenvalues of A. They make $A-\lambda I$ singular.
3. For each eigenvalue λ, solve $(A-\lambda I)\vec{x}=\vec{0}$ ***to find an eigenvector x***.

Imaginary Eigenvalues

Example

Find the eigenvalues and the eigenvectors of the matrix $A=\begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}$

Solution

$$\det(A-\lambda I)=\begin{vmatrix} -2-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix}$$

$$=(-2-\lambda)(2-\lambda)+5$$

$$=\lambda^2+1=0$$

$$\Rightarrow \lambda^2=-1$$

The eigenvalues are: $\boxed{\lambda_{1,2}=\pm i}$

For $\lambda_1=i$: $(A-\lambda_1 I)V_1=0$

$$\begin{pmatrix} -2-i & -1 \\ 5 & 2-i \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad (-2-i)x_1-y_1=0$$

$$\Rightarrow \boxed{(2+i)x_1=-y_1}$$

Therefore, the eigenvector $V_1=\begin{pmatrix} -1 \\ 2+i \end{pmatrix}$

$\lambda_1=-i: (A-\lambda_2 I)V_2=0$

$$\begin{pmatrix} -2+i & -1 \\ 5 & 2+i \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad (-2+i)x_2-y_2=0$$

$$\Rightarrow \boxed{(-2+i)x_2=y_2}$$

Therefore, the eigenvector $\boxed{V_2=\begin{pmatrix} -1 \\ 2-i \end{pmatrix}}$

Example

Find the eigenvalues and the eigenvectors of the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix}$$

$$= \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 = -1$$

The eigenvalues are: $\boxed{\lambda_{1,2} = \pm i}$

The matrix ***A*** is a 90° rotation which has no real eigenvalues or eigenvectors.
No vector $A\vec{x}$ stays in the same direction as $\vec{x}$ (except the zero vector which is useless).
If we add the eigenvalues together the result is zero which is the trace of ***A***.

$\lambda_1 = i$: $\quad (A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -ix + y = 0 \\ \end{cases}$$

$$\Rightarrow \boxed{x = -iy}$$

Therefore, the eigenvector $\boxed{V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}}$

$\lambda_2 = -i$: $\quad (A - \lambda_2 I)V_2 = 0$

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow ix + y = 0$$

$$\Rightarrow \boxed{y = -ix}$$

Therefore, the eigenvector $\boxed{V_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}}$

Exercises *Section* 4.4 – Eigenvalues and Eigenvectors

1. Find the eigenvalues and eigenvectors of $A,\ A^2,\ A^{-1},\ and\ \ A+4I$:

$$A=\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad and \quad A^2=\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

Check the trace $\lambda_1+\lambda_2$ and the determinant $\lambda_1\lambda_2$ for A and also A^2.

2. Show directly that the given vectors are eigenvectors of the given matrix. What are the corresponding eigenvalues

$$\vec{v}_1=\begin{bmatrix} 1 \\ -3 \end{bmatrix},\quad \vec{v}_2=\begin{bmatrix} 2 \\ 1 \end{bmatrix},\quad A=\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$$

3. For which real numbers c does this matrix A have

$$A=\begin{pmatrix} 2 & -c \\ -1 & 2 \end{pmatrix}$$

a) Two real eigenvalues and eigenvectors.

b) A repeated eigenvalue with only one eigenvector

c) Two complex eigenvalues and eigenvectors.

4. Find the eigenvalues of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{AB}$, and $\boldsymbol{BA}$:

$$A=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad AB=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad BA=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

a) The eigenvalues of $\boldsymbol{AB}$ (are equal to) (are not equal to) eigenvalues of $\boldsymbol{A}$ times eigenvalues of $\boldsymbol{B}$.

b) The eigenvalues of $\boldsymbol{AB}$ (are equal to) (are not equal to) eigenvalues of $\boldsymbol{BA}$.

5. When $a+b=c+d$ show that (1, 1) is an eigenvector and find both eigenvalues of

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

6. The eigenvalues of A equal to the eigenvalues of A^T. This is because $\det(A-\lambda I)$ equals $\det\left(A^T-\lambda I\right)$. That is true because _____. Show by an example that the eigenvectors of A and A^T are not the same.

7. Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. Compute the eigenvalues and eigenvectors of A.

8. Let $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

a) What is the characteristic polynomial for A (i.e. compute $\det(A - \lambda I)$?

b) Verify that 1 is an eigenvalue of A. What is a corresponding eigenvector?

c) What are the other eigenvalues of A?

(9 – 58) For the following matrices:

i. Find the characteristic equation.

ii. Find the eigenvalues.

iii. Find the eigenvectors.

9. $\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

10. $\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

11. $\begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}$

12. $\begin{pmatrix} -2 & -7 \\ 1 & 2 \end{pmatrix}$

13. $\begin{pmatrix} 12 & 14 \\ -7 & -9 \end{pmatrix}$

14. $\begin{pmatrix} -4 & 1 \\ -2 & 1 \end{pmatrix}$

15. $\begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$

16. $\begin{pmatrix} -2 & 3 \\ 0 & -5 \end{pmatrix}$

17. $\begin{pmatrix} 2 & 0 \\ -4 & -2 \end{pmatrix}$

18. $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

19. $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$

20. $\begin{pmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{pmatrix}$

21. $\begin{pmatrix} -\frac{5}{2} & 2 \\ \frac{3}{4} & -2 \end{pmatrix}$

22. $\begin{pmatrix} 3 & -1 \\ 9 & -3 \end{pmatrix}$

23. $\begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}$

24. $\begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix}$

25. $\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$

26. $\begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}$

27. $\begin{pmatrix} 4 & 5 \\ -2 & 6 \end{pmatrix}$

28. $\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$

29. $\begin{pmatrix} 6 & -6 \\ 4 & -4 \end{pmatrix}$

30. $\begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix}$

31. $\begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix}$

32. $\begin{pmatrix} 6 & -10 \\ 2 & -3 \end{pmatrix}$

33. $\begin{pmatrix} 11 & -15 \\ 6 & -8 \end{pmatrix}$

34. $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

35. $\begin{pmatrix} 9 & 2 \\ 2 & 6 \end{pmatrix}$

36. $\begin{pmatrix} 13 & 4 \\ 4 & 7 \end{pmatrix}$

37. $\begin{pmatrix} 5 & -1 \\ 3 & -1 \end{pmatrix}$

38. $\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$

39. $\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$

40. $\begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -6 \\ -2 & 3 & 4 \end{pmatrix}$

41. $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$

42. $\begin{pmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{pmatrix}$

43. $\begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

44. $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

45. $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}$

46. $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$

47. $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

48. $\begin{pmatrix} -\frac{5}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 \\ 1 & 1 & -\frac{5}{2} \end{pmatrix}$

49. . $\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

50. $\begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$

51. $\begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}$

52. $\begin{pmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{pmatrix}$

53. $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 2 \\ -8 & -4 & -3 \end{pmatrix}$

54. $\begin{pmatrix} -1 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & -12 & 2 \end{pmatrix}$

55. $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$

56. $\begin{pmatrix} -6 & 4 & 4 \\ -4 & 2 & 4 \\ -10 & 8 & 4 \end{pmatrix}$

57. $\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

58. $\begin{pmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

59. Find the eigenvalues of A^9 for $A = \begin{pmatrix} 1 & 3 & 7 & 11 \\ 0 & \frac{1}{2} & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

60. Given: $A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$. Compute A^{11}

61. Find the eigenvalues of the matrices

$$A = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0.61 & 0.52 \\ 0.39 & 0.48 \end{pmatrix}, \quad A^\infty = \begin{pmatrix} 0.57143 & 0.57143 \\ 0.42857 & 0.42857 \end{pmatrix}, \quad and \ B = \begin{pmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{pmatrix}$$

62. Given the matrix $\begin{bmatrix} -1 & -3 \\ -3 & 7 \end{bmatrix}$

a) Find the characteristic polynomial.
b) Find the eigenvalues
c) Find the bases for its eigenspaces
d) Graph the eigenspaces
e) Verify directly that $A\vec{v} = \lambda\vec{v}$, for all associated eigenvectors and eigenvalues.

63. Given the matrix $\begin{bmatrix} 5 & 0 & -4 \\ 0 & -3 & 0 \\ -4 & 0 & -1 \end{bmatrix}$

a) Find the characteristic polynomial.
b) Find the eigenvalues
c) Find the bases for its eigenspaces
d) Graph the eigenspaces
e) Verify directly that $A\vec{v} = \lambda\vec{v}$, for all associated eigenvectors and eigenvalues.

64. Explain why a 2×2 matrix can have at most two distinct eigenvalues. Explain why an $n \times n$ matrix can have at most *n* distinct eigenvalues.

65. Construct an example of a 2×2 matrix with only one distinct eigenvalue.

66. Let λ be an eigenvalue of an invertible matrix *A*. Show that λ^{-1} is an eigenvalue of A^{-1}.

67. Show that if A^2 is the zero matrix, then the only eigenvalue of *A* is 0.

68. Show that λ is an eigenvalue of *A* if and only if λ is an eigenvalue of A^T.

69. For $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, find one eigenvalue, without calculation. Justify your answer.

70. For $A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$, find one eigenvalue, and two linearly independent eigenvectors, without calculation. Justify your answer.

71. Consider an $n \times n$ matrix *A* with the property that the row sums all equal the same number ***s***. Show that ***s*** is an eigenvalue of *A*.

72. Consider an $n \times n$ matrix *A* with the property that the column sums all equal the same number ***s***. Show that ***s*** is an eigenvalue of *A*.

73. Let A be the matrix of the linear transformation T on $\mathbb{R}^2$

T: reflects points across some line through the origin.

Without writing A, find an eigenvalue of A and describe the eigenspace.

74. Let A be the matrix of the linear transformation T on $\mathbb{R}^2$

T: reflects points about some line through the origin.

Without writing A, find an eigenvalue of A and describe the eigenspace.

75. Show that if $\vec{v}$ is an eigenvector of the matrix product AB and $B\vec{v} \neq \vec{0}$, then $B\vec{v}$ is an eigenvector of BA

76. Explain and demonstrate that the eigenspace of a matrix A corresponding to some eigenvalue λ is a subspace.

77. If λ is an eigenvalue of the matrix A, prove that λ^2 is an eigenvalue of A^2.