

Section 2.8 – Applications

Motion with Resistance Proportional to Velocity

An object with a mass m is moving along a coordinate line with position function s and velocity v at time t .

From *Newton's second law* of motion, the resistance force opposing the motion is

$$\begin{aligned}\text{Force} &= \text{mass} \times \text{acceleration} \\ &= m \frac{dv}{dt}\end{aligned}$$

If the resisting force is proportional to velocity, we have

$$m \frac{dv}{dt} = -kv \quad \text{or} \quad \frac{dv}{dt} = -\frac{k}{m}v \quad (k > 0)$$

This is a separable differential equation, the solution with initial condition $v = v_0$ at $t = 0$ is

$$v = v_0 e^{-(k/m)t}$$

In the English system, where the weight is measured in pounds, mass is measured in slugs. Thus,

$$\text{Pounds} = \text{slugs} \times 32$$

Example

For a 192-lb ice skater, the k is about $\frac{1}{3}$ slug/sec and $m = \frac{192}{32} = 6$ slugs. How long will it take the skater to coast from 11 ft/sec (7.5 mph) to 1 ft/sec? How far will the skater coast before coming to a complete stop?

Solution

$$v = v_0 e^{-(k/m)t} \qquad \frac{k}{m} = \frac{\frac{1}{3}}{6} = \frac{1}{18}$$

$$1 = 11e^{-(1/18)t}$$

$$e^{-(1/18)t} = \frac{1}{11}$$

$$-\frac{1}{18}t = \ln\left(\frac{1}{11}\right) = -\ln 11$$

$$t = 18 \ln 11 \approx \underline{43 \text{ sec}}$$

$$\text{Distance} = \frac{v_0 m}{k} = \frac{11(6)}{1/3} = \underline{198 \text{ ft}}$$

Mixtures Problems

The physical representation of the rate of change:

$$\frac{dy}{dt} = \text{rate of change} = \text{rate in} - \text{rate out}$$

This is referred to as a **balance law**.

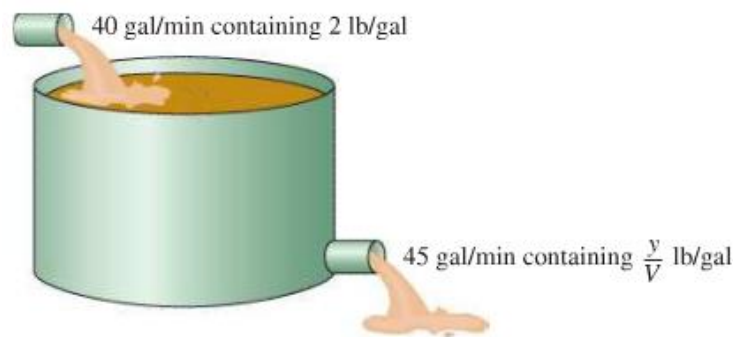
$$\text{Rate} = \text{Volume Rate (gal/min)} \times \text{Concentration (lb/gal)}$$

If $y(t)$ is the amount of chemical in the container at time t and $V(t)$ is the total volume of liquid in the container at time t , then the departure rate of the chemical at time t is

$$\begin{aligned} \text{Departure Rate} &= \frac{y(t)}{V(t)} \bullet (\text{outflow rate}) \\ &= \left(\frac{\text{concentration in}}{\text{container at } t} \right) \bullet (\text{outflow rate}) \end{aligned}$$

Example

In an oil refinery, a storage tank contains 2000 gal of gasoline that initially has 100 lb. of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 lb. of additive per gallon is pumped into the tank at a rate of 40 gal/min. The well-mixed solution is pumped out at a rate of 45 gal/min. How much of the additive is in the tank 20 min after the pumping process begins?



Solution

Let y be the amount (in lb.) of additive in the tank at time t and $y(0) = 100$

$$\begin{aligned} V(t) &= 2000 + \left(40 \frac{\text{gal}}{\text{min}} - 45 \frac{\text{gal}}{\text{min}} \right) (t \text{ min}) \\ &= 2000 - 5t \end{aligned}$$

$$\begin{aligned} \text{Rate out} &= \frac{y(t)}{V(t)} \bullet \text{outflow rate} \\ &= \left(\frac{y}{2000 - 5t} \right) (45) \\ &= \frac{45y}{2000 - 5t} \frac{\text{lb}}{\text{min}} \end{aligned}$$

$$\text{Rate in} = \left(2 \frac{\text{lb}}{\text{gal}} \right) \left(40 \frac{\text{gal}}{\text{min}} \right) = \underline{80 \frac{\text{lb}}{\text{min}}}$$

$$\frac{dy}{dt} = 80 - \frac{45y}{2000-5t}$$

$$\frac{dy}{dt} + \frac{45}{2000-5t} y = 80$$

$$\begin{aligned} v(t) &= e^{\int p dt} = e^{\int \frac{45}{2000-5t} dt} & d(2000-5t) &= -5dt \Rightarrow dt = -\frac{1}{5} d(2000-5t) \\ &= e^{\int \frac{45}{2000-5t} \left(-\frac{1}{5} d(2000-5t) \right)} \\ &= e^{-\int \frac{9}{2000-5t} d(2000-5t)} \\ &= e^{-9 \ln(2000-5t)} \\ &= (2000-5t)^{-9} \end{aligned}$$

$$\begin{aligned} y &= \frac{1}{(2000-5t)^{-9}} \left(\int 80 \cdot (2000-5t)^{-9} dt + C \right) \\ &= (2000-5t)^9 \left(80 \int (2000-5t)^{-9} \left(-\frac{1}{5} d(2000-5t) \right) + C \right) \\ &= (2000-5t)^9 \left(-16 \frac{(2000-5t)^{-8}}{-8} + C \right) \\ &= (2000-5t)^9 \left(2(2000-5t)^{-8} + C \right) \\ &= 2(2000-5t) + C(2000-5t)^9 \end{aligned}$$

$$y(\textcolor{red}{0}) = 2(2000-5(\textcolor{red}{0})) + C(2000-5(\textcolor{red}{0}))^9$$

$$100 = 4000 + C(2000)^9$$

$$C(2000)^9 = -3900$$

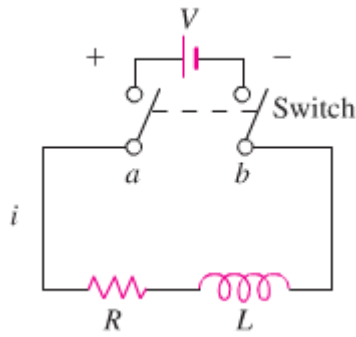
$$C = -\frac{3900}{(2000)^9}$$

$$y = 2(2000-5t) - \frac{3900}{(2000)^9} (2000-5t)^9$$

$$y(20) = 2(2000-5(\textcolor{red}{20})) - \frac{3900}{(2000)^9} (2000-5(\textcolor{red}{20}))^9$$

$$\approx \underline{\textcolor{blue}{1342} \text{ lb.}}$$

RL Circuits



The diagram represents an electrical circuit whose total resistance is a constant R ohms and whose self-inductance is coil, is L henries.

Ohm's Law: $V = Ri$

$$L \frac{di}{dt} + Ri = V$$

i : current in amperes

t : time in seconds

Example

The switch in the RL circuit is closed at time $t = 0$. How will the current flow as a function of time?

Solution

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}$$

$$e^{\int p dt} = e^{\int \left(\frac{R}{L}\right) dt} = e^{\frac{R}{L}t}$$

$$i = \frac{1}{e^{\int p dt}} \left(\int f \cdot e^{\int p dt} dt + C \right)$$

$$= \frac{1}{e^{\frac{R}{L}t}} \left(\int \frac{V}{L} \cdot e^{\frac{R}{L}t} dt + C \right)$$

$$= e^{-\frac{R}{L}t} \left(\frac{V}{L} \frac{L}{R} e^{\frac{R}{L}t} + C \right)$$

$$= e^{-\frac{R}{L}t} \left(\frac{V}{R} e^{\frac{R}{L}t} + C \right)$$

$$t = 0 \Rightarrow i = 0 \quad i = e^{-\frac{R}{L}(0)} \left(\frac{V}{R} e^{\frac{R}{L}(0)} + C \right)$$

$$0 = \frac{V}{R} + C$$

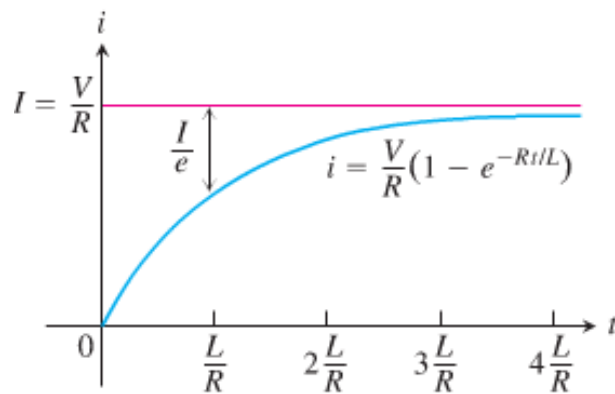
$$\boxed{C = -\frac{V}{R}}$$

$$i(t) = e^{-\frac{R}{L}t} \left(\frac{V}{R} e^{\frac{R}{L}t} - \frac{V}{R} \right)$$

$$\boxed{= \frac{V}{R} - \frac{V}{R} e^{-\frac{R}{L}t}}$$

$$\lim_{t \rightarrow \infty} i = \lim_{t \rightarrow \infty} \left(\frac{V}{R} - \frac{V}{R} e^{-\frac{R}{L}t} \right) = \frac{V}{R} - \frac{V}{R} \cdot 0 = \underline{\frac{V}{R}}$$

The current approaches the *steady-state value* $\frac{V}{R}$.



Exercises Section 2.8 – Applications

1. A 66-kg cyclist on a 7-kg bicycle starts coasting on level ground at 9 m/sec. The $k \approx 3.9 \text{ kg / sec}$
 - a) About how far will the cyclist coast before reaching a complete stop?
 - b) How long will it take the cyclist's speed to drop to 1 m/sec?
2. Suppose that an Iowa class battleship has mass 51,000 metric tons (51,000,000 kg) and $k \approx 59,000 \text{ kg / sec}$. Assume that the ship loses power when it is moving at a speed of 9 m/sec.
 - a) About how far will the ship coast before it is dead in the water?
 - b) About how long will it take the ship's speed to drop to 1 m/sec?
3. A 200-gal tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 lb./gal of concentrate enters the tank at the rate of 5 gal/min, and the well-stirred mixture is withdrawn at the rate of 3 gal/min.
 - a) At what time will the tank be full?
 - b) At the time the tank is full, how many pounds of concentrate will it contain?
4. A tank contains 100 gal of fresh water. A solution containing 1 lb./gal of soluble lawn fertilizer runs into the tank at the rate of 1 gal/min, and the mixture is pumped out of the tank at a rate of 3 gal/min. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.
5. An Executive conference room of a corporation contains 4500 ft^3 of air initially free of carbon monoxide. Starting at time $t = 0$, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 ft^3 / min . A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 ft^3 / min . Find the time when the concentration of carbon monoxide in the room reaches 0.01%.
6. Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the substance B at time $t = 0$, and if x is the amount of product at time t , then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt} = k(a-x)(b-x) \quad \text{or} \quad \frac{1}{(a-x)(b-x)} \frac{dx}{dt} = k$$

Where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t .

a) If $a = b$

b) If $a \neq b$

Assume in each case that $x = 0$ when $t = 0$

7. The tank initially holds 100 *gal* of pure water. At time $t = 0$, a solution containing 2 *lb* of salt per gallon begins to enter the tank at the rate of 3 gallons per minute. At the same time a drain is opened at the bottom of the tank so that the volume of solution in the tank remains constant.

How much salt is in the tank after 60 *min*?

What will be the eventual salt content in the tank?

8. The 600-*gal* tank is filled with 300 *gal* of pure water. A spigot is opened above the tank and a salt solution containing 1.5 *lb.* of salt per gallon of solution begins flowing into the tank at the rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave tank at a rate of 1 *gal/min*. What will be the salt content in the tank at the precise moment that the volume of solution in the tank is equal to the tank's capacity (600 *gal*)?

9. The amount of drug in the blood of a patient (in *mg*) due to an intravenous line is governed by the initial value problem

$$y'(t) = -0.02y + 3, \quad y(0) = 0 \text{ for } t \geq 0$$

Where t is measured in hours

- Find and graph the solution of the initial value problem.
 - What is the steady-state level of the drug?
 - When does the drug level reach 90% of the steady-state value?
10. A fish hatchery has 500 *fish* at time $t = 0$, when harvesting begins at a rate of b *fish/yr.* where $b > 0$. The fish population is modeled by the initial value problem.

$$y'(t) = 0.1y - b, \quad y(0) = 500 \quad \text{for } t \geq 0$$

Where t is measured in years.

- Find the fish population for $t \geq 0$ in terms of the harvesting rate b .
 - Graph the solution in the case that $b = 40$ *fish / yr*. Describe the solution.
 - Graph the solution in the case that $b = 60$ *fish / yr*. Describe the solution.
11. A community of hares on an island has a population of 50 when observations begin at $t = 0$. The population for $t \geq 0$ is modeled by the initial value problem.

$$\frac{dP}{dt} = 0.08P \left(1 - \frac{P}{200} \right), \quad P(0) = 50$$

- Find the solution of the initial value problem.
 - What is the steady-state population?
12. When an infected person is introduced into a closed and otherwise healthy community, the number of people who become infected with the disease (in the absence of any intervention) may be modeled by the logistic equation

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{A} \right), \quad P(0) = P_0$$

Where k is a positive infection rate, A is the number of people in the community, and P_0 is the number of infected people at $t = 0$. The model assumes no recovery or intervention.

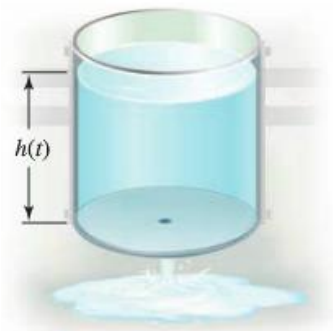
- a) Find the solution of the initial value problem in terms of k , A , and P_0 .
- b) Graph the solution in the case that $k = 0.025$, $A = 300$, and $P_0 = 1$.
- c) For fixed values of k and A , describe the long-term behavior of the solutions for any P_0 with $0 < P_0 < A$

- 13.** An object in free fall may be modeled by assuming that the only forces at work are the gravitational force and resistance (friction due to the medium in which the objects falls). By Newton's second law (mass \times acceleration = the sum of the external forces), the velocity of the object satisfies the differential equation

$$\underbrace{m}_{\text{mass}} \cdot \underbrace{v'(t)}_{\text{acceleration}} = \underbrace{mg + f(v)}_{\text{external force}}$$

Where f is a function that models the resistance and the positive direction is downward. One common assumption (often used for motion in air) is that $f(v) = -kv^2$, where $k > 0$ is a drag coefficient.

- a) Show that the equation can be written in the form $v'(t) = g - av^2$ where $a = \frac{k}{m}$
 - b) For what (positive) value of v is $v'(t) = 0$? (This equilibrium solution is called the **terminal velocity**.)
 - c) Find the solution of this separable equation assuming $v(0) = 0$ and $0 < v(t)^2 < \frac{g}{a}$ for $t \geq 0$
 - d) Graph the solution found in part (c) with $g = 9.8 \text{ m/s}^2$, $m = 1 \text{ kg}$, and $k = 0.1 \text{ kg/m}$, and verify the terminal velocity agrees with the value found in part (b).
- 14.** An open cylindrical tank initially filled with water drains through a hole in the bottom of the tank according to Torricelli's Law. If $h(t)$ is the depth of water in the tank for $t \geq 0$, then Torricelli's Law implies $h'(t) = -k\sqrt{h}$, where k is a constant that includes the acceleration due to gravity, the radius of the tank, and the radius of the drain. Assume that the initial depth of the water is $h(0) = H$



- a) Find the solution of the initial value problem.
- b) Find the solution in the case that $k = 0.1$ and $H = 0.5 \text{ m}$.
- c) In general, how long does it take the tank to drain in terms of k and H ?

15. The reaction of chemical compounds can often be modeled by differential equations. Let $y(t)$ be the concentration of a substance in reaction for $t \geq 0$ (typical units of y are *moles/L*). The change in the concentration of a substance, under appropriate conditions, is $\frac{dy}{dt} = -ky^n$, where $k > 0$ is a rate constant and the positive integer n is the order of the reaction.
- Show that for a first-order reaction ($n = 1$), the concentration obeys an exponential decay law.
 - Solve the initial value problem for a second-order reaction ($n = 2$) assuming $y(0) = y_0$
 - Graph and compare the concentration for a first-order and second-order reaction with $k = 0.1$ and $y_0 = 1$

16. The growth of cancer tumors may be modeled by the Gomperts growth equation. Let $M(t)$ be the mass of the tumor for $t \geq 0$. The relevant initial value problem is

$$\frac{dM}{dt} = -aM \ln \frac{M}{K}, \quad M(0) = M_0$$

Where a and K are positive constants and $0 < M_0 < K$

- Graph the growth rate function $R(M) = -aM \ln \frac{M}{K}$ assuming $a = 1$ and $K = 4$. For what values of M is the growth rate positive? For what values of M is maximum?
 - Solve the initial value problem and graph the solution for $a = 1$, $K = 4$, and $M_0 = 1$. Describe the growth pattern of the tumor. Is the growth unbounded? If not, what is the limiting size of the tumor?
 - In the general equation, what is the meaning of K ?
17. An endowment is an investment account in which the balance ideally remains constant and withdrawals are made on the interest earned by the account. Such an account may be modeled by the initial value problem $B'(t) = aB - m$ for $t \geq 0$, with $B(0) = B_0$. The constant a reflects the annual interest rate, m is the annual rate of withdrawal, and B_0 is the initial balance in the account.
- Solve the initial value problem with $a = 0.05$, $m = \$1000 / \text{yr.}$ and $B_0 = \$15,000$. Does the balance in the account increase or decrease?
 - If $a = 0.05$ and $B_0 = \$50,000$, what is the annual withdrawal rate m that ensures a constant balance in the account? What is the constant balance?
18. The halibut fishery has been modeled by the differential equation $\frac{dy}{dt} = ky \left(1 - \frac{y}{M}\right)$
- Where $y(t)$ is the biomass (the total mass of the members of the population) in kilograms at time t (measured in years), the carrying capacity is estimated to be $M = 8 \times 10^7 \text{ kg}$ and $k = 0.71 \text{ per year}$.
- If $y(0) = 2 \times 10^7 \text{ kg}$, find the biomass a year later.
 - How long will it take for the biomass to reach $4 \times 10^7 \text{ kg}$.

19. Suppose a population $P(t)$ satisfies $\frac{dP}{dt} = 0.4P - 0.001P^2$, $P(0) = 50$ where t is measured in years.

- What is the carrying capacity?
- What is $P'(0)$?
- When will the population reach 50% of the carrying capacity?

20. Let $P(t)$ be the performance level of someone learning a skill as a function of the training time t . The graph of P is called a **learning curve**. We proposed the differential equation

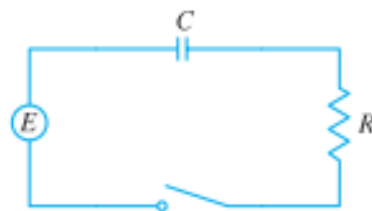
$$\frac{dP}{dt} = k(M - P(t))$$

As a reasonable model for learning, where k is a positive constant. Solve it as a linear differential equation and use your solution to graph the learning curve.

21. A circuit containing an electromotive force, a capacitor with a capacitance of C farads (F), and a resistor with a resistance of R ohms (Ω). The voltage drop across the capacitor is $\frac{Q}{C}$, where Q is the charge (in coulombs), so in this case **Kirchhoff's Law** gives

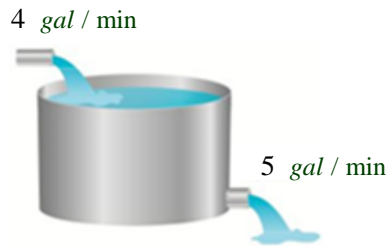
$$RI + \frac{Q}{C} = E(t)$$

But $I = \frac{dQ}{dt}$, so we have $R \frac{dQ}{dt} + \frac{1}{C}Q = E(t)$



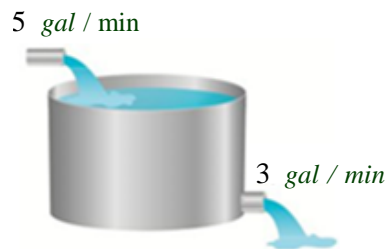
Find the charge and the current at time t

- Suppose the resistance is 5Ω , the capacitance is $0.05 F$, a battery gives voltage of $60 V$ and initial charge is $Q(0) = 0 C$
 - Suppose the resistance is 2Ω , the capacitance is $0.01 F$, $E(t) = 10\sin 60t$ and initial charge is $Q(0) = 0 C$
22. A 30-volt electromotive force is applied to an LR -series circuit in which the inductance is 0.1 henry and the resistance is 50 ohms.
- Find the current $i(t)$ if $i(0) = 0$
 - Determine the current as $t \rightarrow \infty$
 - Solve the equation when $E(t) = E_0 \sin \omega t$ and $i(0) = i_0$
23. A tank contains 50 gallons of a solution composed of 90% water and 10% alcohol. A second solution containing 50% water and 50% alcohol is added to the tank at the rate of 4 gal / min. As the second solution is being added, the tank is being drained at a rate of 5 gal / min. The solution in the tank is stirred constantly. How much alcohol is in the tank after 10 minutes?



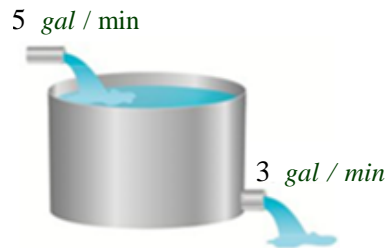
24. A 200-gallon tank is half full of distilled water. At time $t = 0$, a concentrate solution containing 0.5 lb/gal enters the tank at the rate of 5 gal / min , and well-stirred mixture is withdrawn at the rate of 3 gal / min .

- At what time will the tank be full?
- At the time the tank is full, how many pounds of concentrate will it contain?



25. A 200-gallon tank is half full of distilled water. At time $t = 0$, a concentrate solution containing 1 lb/gal enters the tank at the rate of 5 gal / min , and well-stirred mixture is withdrawn at the rate of 3 gal / min .

- At what time will the tank be full?
- At the time the tank is full, how many pounds of concentrate will it contain?



26. A 200-gallon tank is full of a concentrate solution containing 25 lb . Starting at time $t = 0$, distilled water is admitted to the tank at the rate of 10 gal / min , and well-stirred mixture is withdrawn at the same rate.

- Find the amount of concentrate in the solution as a function of t .
- Find the time at which the amount of concentrate in the tank reaches 15 pounds.
- Find the quantity of the concentrate in the solution as $t \rightarrow \infty$.

