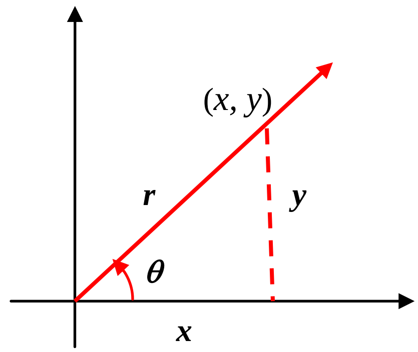
Section 1.2 – Trigonometric Functions

Let (x, y) be a point on the terminal side of an angle θ in standard position

The distance from the point to the origin is given by: $r=\sqrt{x^2+y^2}$



Six Trigonometry Functions

$$\sin\theta=\frac{Opposite}{Hypotenuse}=\frac{opp}{hyp}=\frac{y}{r}$$

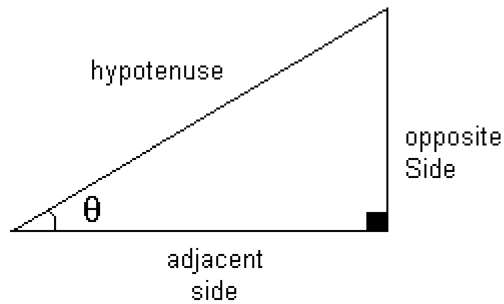
$$\cos\theta=\frac{Adjacent}{Hypotenuse}=\frac{adj}{hyp}=\frac{x}{r}$$

$$\tan\theta=\frac{opp}{adj}=\frac{\sin\theta}{\cos\theta}=\frac{y}{x}$$

$$\cot\theta=\frac{adj}{opp}=\frac{\cos\theta}{\sin\theta}=\frac{1}{\tan\theta}=\frac{x}{y}$$

$$\sec\theta=\frac{hyp}{adj}=\frac{1}{\cos\theta}=\frac{r}{x}$$

$$\csc\theta=\frac{hyp}{opp}=\frac{1}{\sin\theta}=\frac{r}{y}$$



Undefined Function Values

If the terminal side of a quadrantal angle lies along the ***y*-axis**, then the ***tangent*** and ***secant*** functions are undefined.

If the terminal side of a quadrantal angle lies along the ***x*-axis**, then the ***cotangent*** and ***cosecant*** functions are undefined.

Example

Find the six trigonometry functions of θ if θ is in the standard position and the point $(8, 15)$ is on the terminal side of θ.

Solution

$$\Rightarrow r=\sqrt{x^2+y^2}=\sqrt{8^2+15^2}=17$$

$$\sin\theta=\frac{y}{r}=\frac{15}{17}\qquad \cos\theta=\frac{x}{r}=\frac{8}{17}\qquad \tan\theta=\frac{y}{x}=\frac{15}{8}$$

$$\csc\theta=\frac{r}{y}=\frac{17}{15}\qquad \sec\theta=\frac{r}{x}=\frac{17}{8}\qquad \cot\theta=\frac{x}{y}=\frac{8}{15}$$

Example

Find the sine and cosine of 45° at the convenient point (1, 1)

<u>*Solution*</u>

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

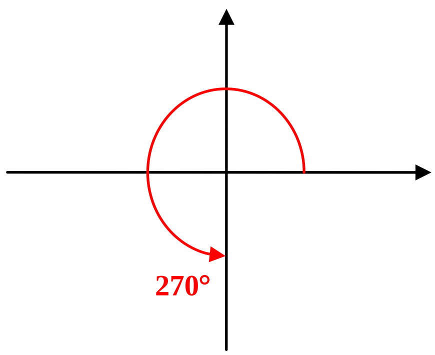
$$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}} \qquad \cos\theta = \frac{x}{r} = \frac{1}{\sqrt{2}}$$

Example

Find the six trigonometry functions of 270°

<u>*Solution*</u>

The convenient point (0, -1)



$$\Rightarrow r = \sqrt{0^2 + (-1)^2}$$

$$= \sqrt{1}$$

$$= 1$$

$$\sin 270^\circ = \frac{y}{r} = -1 \qquad \cot 270^\circ = \frac{x}{y} = \frac{0}{-1} = 0$$

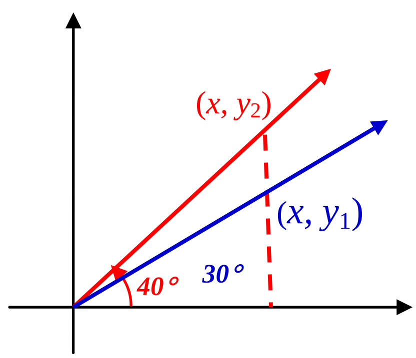
$$\cos 270^\circ = \frac{x}{r} = \frac{0}{1} = 0 \qquad \sec 270^\circ = \frac{r}{x} = \frac{1}{0} = \infty$$

$$\tan 270^\circ = \frac{y}{x} = \frac{-1}{0} = \text{undefined} = -\infty \qquad \csc 270^\circ = \frac{r}{y} = \frac{1}{-1} = -1$$

Example

Which will be greater, tan 30° or tan 40°?

How large could $\tan\theta$ be?



<u>*Solution*</u>

$$\tan 30^\circ = \frac{y_1}{x}$$

$$\tan 40^\circ = \frac{y_2}{x}$$

$$Ratio: \frac{y_2}{x} > \frac{y_1}{x}$$

→tan 40°> tan 30°

No limit as to how large $\tan\theta$ can be

Function	I	II	III	IV
y = sin x	+	+	-	-
y = cosx	+	-	-	+
y = tan x	+	-	+	-
y = cot x	+	-	+	-
y = cscx	+	+	-	-
y = sec x	+	-	-	+

Example

If $\cos\theta = \frac{\sqrt{3}}{2}$, and θ is QIV, find $\sin\theta$ and $\tan\theta$.

<u>*Solution*</u>

$$\cos\theta = \frac{\sqrt{3}}{2} = \frac{x}{r} \quad \to x = \sqrt{3}, \quad r = 2$$

$$r^2 = x^2 + y^2$$

$$\Rightarrow y^2 = r^2 - x^2$$

$$y = \sqrt{r^2 - x^2}$$

$$y = \sqrt{2^2 - \left(\sqrt{3}\right)^2}$$

$$= \sqrt{4-3}$$

$$\underline{=1|} \qquad \text{Since } \theta \text{ is Q IV} \Rightarrow \boxed{y = -1}$$

$$\sin\theta = \frac{y}{r} = -\frac{1}{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta = \frac{1}{\csc\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Ratio Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$x^2 + y^2 = r^2$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1 \quad \Rightarrow \quad \boxed{\cos^2\theta + \sin^2\theta = 1}$$

Solving for $\cos\theta$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$$

Solving for $\sin\theta$

$$\sin^2\theta = 1 - \cos^2\theta \quad \Rightarrow \quad \boxed{\sin\theta = \pm\sqrt{1 - \cos^2\theta}}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\left(\frac{\cos\theta}{\cos\theta}\right)^2 + \left(\frac{\sin\theta}{\cos\theta}\right)^2 = \left(\frac{1}{\cos\theta}\right)^2$$

$$\boxed{1 + \tan^2\theta = \sec^2\theta}$$

$\cos^2\theta + \sin^2\theta = 1$
$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$
$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$
$1 + \tan^2\theta = \sec^2\theta$
$1 + \cot^2\theta = \csc^2\theta$

Example

Prove $\sin\theta\cot\theta = \cos\theta$

<u>*Solution*</u>

$$\begin{aligned}\sin\theta\cot\theta &= \sin\theta\frac{\cos\theta}{\sin\theta}\\ &= \cos\theta\end{aligned}$$

Example

Write $\tan\theta$ in terms of $\sin\theta$.

<u>*Solution*</u>

$$\begin{aligned}\tan\theta &= \frac{\sin\theta}{\cos\theta}\\ &= \frac{\sin\theta}{\pm\sqrt{1-\sin^2\theta}}\\ &= \pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\end{aligned}$$

Example

If $\cos\theta = \frac{1}{2}$ and θ terminated in QIV, find the remaining trigonometric ratios for θ.

<u>*Solution*</u>

$$\begin{aligned}\sin\theta &= -\sqrt{1-\cos^2\theta}\\ &= -\sqrt{1-\left(\frac{1}{2}\right)^2}\\ &= -\sqrt{1-\frac{1}{4}}\\ &= -\sqrt{\frac{3}{4}}\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{1/2} = 2$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\sqrt{3}/2} = -\frac{2}{\sqrt{3}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$$

$$\cot\theta = -\frac{1}{\sqrt{3}}$$

Example

Find $\sin\theta$ and $\cos\theta$, given that $\tan\theta = \frac{4}{3}$ and θ is in QIII.

Solution

Using the identity $1+\tan^2\theta = \sec^2\theta$

$$\sec^2\theta = 1+\left(\frac{4}{3}\right)^2$$

$$= 1+\frac{16}{9}$$

$$= \frac{25}{9}$$

$$\sec\theta = -\sqrt{\frac{25}{9}} = -\frac{5}{3} \qquad \theta \in QIII \Rightarrow \cos\theta < 0 \rightarrow \sec\theta < 0$$

$$\cos\theta = -\frac{3}{5}$$

$$\sin^2\theta = 1-\cos^2\theta$$

$$= 1-\left(-\frac{3}{5}\right)^2$$

$$= 1-\frac{9}{25}$$

$$= \frac{16}{25}$$

$$\sin\theta = -\frac{4}{5} \quad (\theta \in QIII)$$

Example

Show that the following statement is true by transforming the left side into the right side.

$$\cos\theta\tan\theta = \sin\theta$$

Solution

$$\cos\theta\tan\theta = \cos\theta\frac{\sin\theta}{\cos\theta}$$

$$= \sin\theta$$

Example

Simplify the expression $\sqrt{x^2+9}$ as much as possible after substituting $3\tan\theta$ for x

Solution

$$x = 3\tan\theta$$

$$\sqrt{x^2+9} = \sqrt{(3\tan\theta)^2+9}$$

$$= \sqrt{9\tan^2\theta+9}$$

$$= \sqrt{9\left(\tan^2\theta+1\right)}$$

$$= 3\sqrt{\sec^2\theta}$$

$$= 3\sec\theta$$

Exercise *Section* 1.2 – Trigonometric Functions

1. Find the six trigonometry functions of θ if θ is in the standard position and the point $(-2, 3)$ is on the terminal side of θ.

2. Find the six trigonometry functions of θ if θ is in the standard position and the point $(-3, -4)$ is on the terminal side of θ.

3. Find the six trigonometry functions of θ in standard position with terminal side through the point $(-3,\ 0)$.

4. Find the six trigonometry functions of θ if θ is in the standard position and the point $(12, -5)$ is on the terminal side of θ.

5. Find the values of the six trigonometric functions for an angle of $90°$.

6. Indicate the two quadrants θ could terminate in if $\cos\theta = \frac{1}{2}$

7. Indicate the two quadrants θ could terminate in if $\csc\theta = -2.45$

8. Find the remaining trigonometric function of θ if $\sin\theta = \frac{12}{13}$ and θ terminates in QI.

9. Find the remaining trigonometric function of θ if $\cot\theta = -2$ and θ terminates in QII.

10. Find the remaining trigonometric function of θ if $\tan\theta = \frac{3}{4}$ and θ terminates in QIII.

11. Find the remaining trigonometric function of θ if $\cos\theta = \frac{24}{25}$ and θ terminates in QIV.

12. Find the remaining trigonometric functions of θ if $\cos\theta = \frac{\sqrt{3}}{2}$ and θ is terminates in QIV.

13. Find the remaining trigonometric functions of θ if $\tan\theta = -\frac{1}{2}$ and $\cos\theta > 0$.

14. If $\sin\theta = -\frac{5}{13}$, and θ is QIII, find $\cos\theta$ and $\tan\theta$.

15. If $\cos\theta = \frac{3}{5}$, and θ is QIV, find $\sin\theta$ and $\tan\theta$.

16. Use the reciprocal identities if $\cos\theta = \frac{\sqrt{3}}{2}$ find $\sec\theta$

17. Find $\cos\theta$, given that $\sec\theta = \frac{5}{3}$

18. Find $\sin\theta$, given that $\csc\theta = -\frac{\sqrt{12}}{2}$

19. Use a ratio identity to find $\tan\theta$ if $\sin\theta = \frac{3}{5}$ and $\cos\theta = -\frac{4}{5}$

20. If $\cos\theta = -\frac{1}{2}$ and θ terminates in QII, find $\sin\theta$

21. If $\sin\theta = \frac{3}{5}$ and θ terminated in QII, find $\cos\theta$ and $\tan\theta$.

22. Find $\tan\theta$ if $\sin\theta = \frac{1}{3}$ and θ terminates in QI

23. Find the remaining trigonometric ratios of θ, if $\sec\theta = -3$ and $\theta \in QIII$

24. Using the calculator and rounding your answer to the nearest hundredth, find the remaining trigonometric ratios of θ if $\csc\theta = -2.45$ and $\theta \in QIII$

25. Write $\frac{\sec\theta}{\csc\theta}$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

26. Write $\cot\theta - \csc\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

27. Write $\frac{\sin\theta}{\cos\theta} + \frac{1}{\sin\theta}$ in terms of $\sin\theta$ and/or $\cos\theta$, and then simplify if possible.

28. Write $\sin\theta\cot\theta + \cos\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify if possible.

29. Multiply $(1-\cos\theta)(1+\cos\theta)$

30. Multiply $(\sin\theta + 2)(\sin\theta - 5)$

31. Simplify the expression $\sqrt{25 - x^2}$ as much as possible after substituting $5\sin\theta$ for x.

32. Simplify the expression $\sqrt{4x^2 + 16}$ as much as possible after substituting $2\tan\theta$ for x