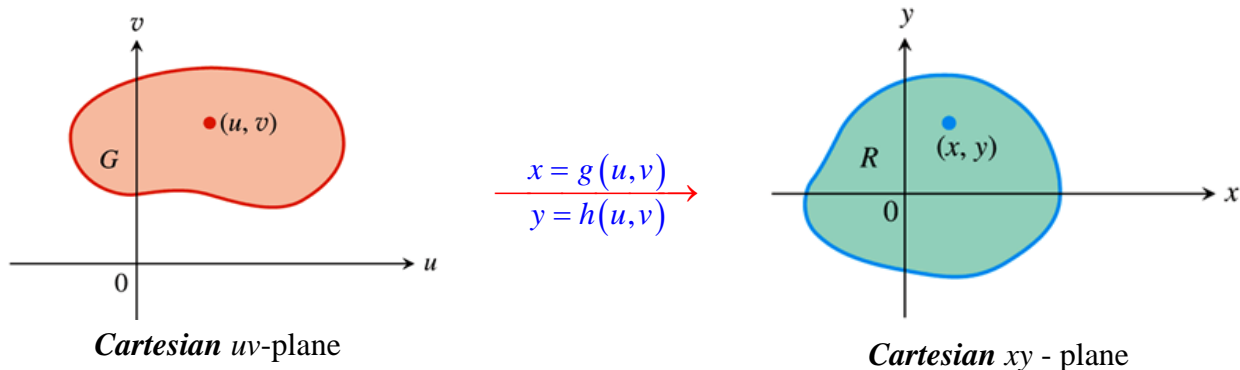# *Section* **3.7 – Change of variables in Multiple Integrals**

Substitution in Double Integrals

Suppose that a region G in the uv-plane is transformed one-to-one into the region R in the xy-plane by equations of the form

$$x = g(u,v), \quad y = h(u,v)$$

Cartesian uv-plane

Cartesian xy - plane

R is the image of G under the transformation, and G the *preimage* of R.

$$\iint_R f(x,y)\,dxdy = \iint_G f(g(u,v),h(u,v))\left|J(u,v)\right|dudv$$

Definition

The ***Jacobian determinant*** or ***Jacobian*** of the coordinate transformation $x = g(u,v), \quad y = h(u,v)$ is
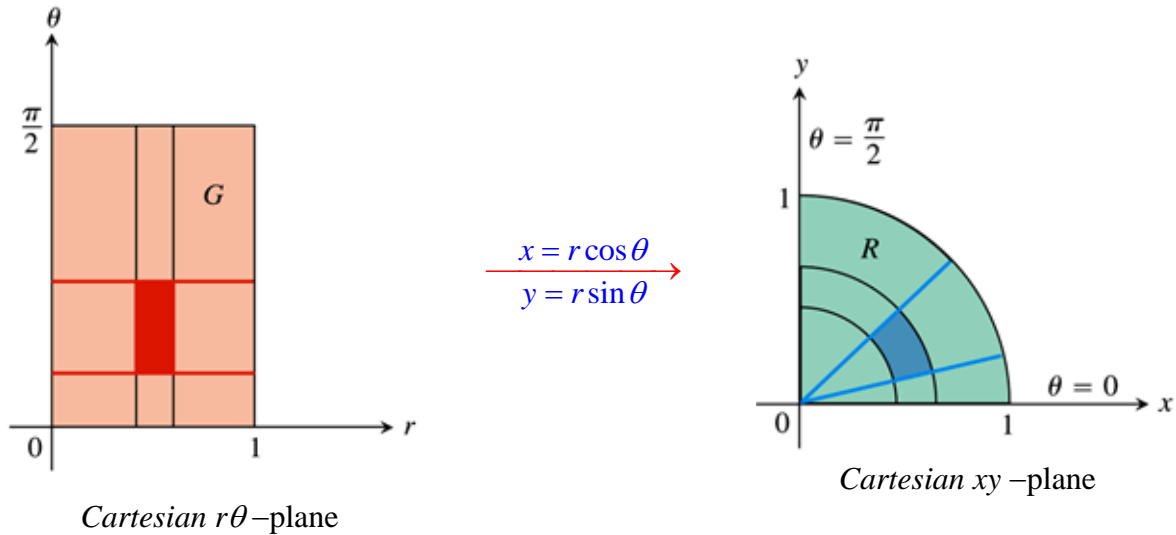
$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

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Example

Find the Jacobian for the polar coordinate transformation $x = r\cos\theta, \quad y = r\sin\theta$, write the Cartesian

integral $\displaystyle\iint\limits_R f(x,y)\,dxdy$ as a polar integral.

Solution

Cartesian $r\theta$–plane

Cartesian xy –plane

$x = r\cos\theta, \quad y = r\sin\theta$ transform the rectangle G: $0 \le r \le 1$, $0 \le \theta \le 2\pi$, into the quarter circle R
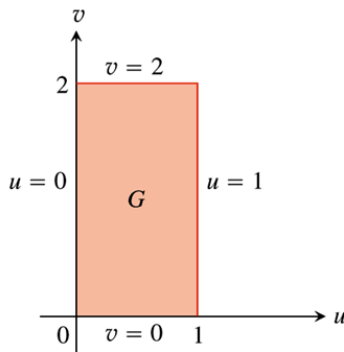
bounded by $x^2 + y^2 = 1$ in QI.

$$J(r,\ \theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r\left(\cos^2\theta + \sin^2\theta\right)$$
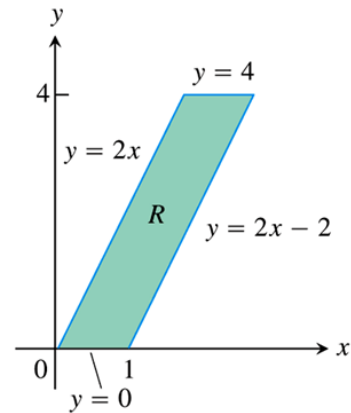
$$\underline{= r}$$

Example

Evaluate $\displaystyle\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\,dxdy$ by applying the transformation $u = \dfrac{2x-y}{2}$, $v = \dfrac{y}{2}$ and

integrating over an appropriate region in the uv-plane.

Solution

$$\rightarrow \underline{y = 2v}, \quad 2u = 2x - y \Rightarrow \underline{x = \frac{2u+y}{2} = \frac{2u+2v}{2} = u + v}$$

$x = u + v$
$y = 2v$

xy-eqns for the boundary of R	Corresponding uv-eqns. for the boundary of G	Simplified uv-eqns.
$x = \dfrac{y}{2}$	$u + v = \dfrac{2v}{2} = v$	$u = 0$
$x = \dfrac{y}{2} + 1$	$u + v = \dfrac{2v}{2} + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial u}(u+v) & \dfrac{\partial}{\partial v}(u+v) \\ \dfrac{\partial}{\partial u}(2v) & \dfrac{\partial}{\partial v}(2v) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}$$

$$\underline{= 2}$$

$$\int_0^4 \int_{y/2}^{(y/2)+1} \left(x - \frac{y}{2}\right)dxdy = \int_0^{v=2} \int_{u=0}^{u=1} u\,|J(u,v)|\,dudv$$

$$= \int_0^{v=2} \int_{u=0}^{u=1} (u)(2)\,dudv$$

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$$= \int_0^{v=2} u^2 \Big|_0^1 \, dv$$

$$= \int_0^{v=2} dv$$

$$= v \Big|_0^2$$

$$\underline{= 2}$$

Example

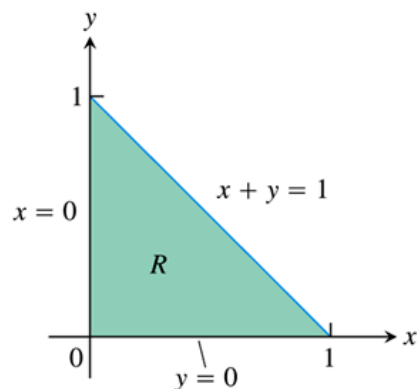Evaluate $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2 \, dy dx$

Solution

$$\begin{cases} u = x+y \\ v = y-2x \end{cases} \rightarrow \begin{cases} x = \dfrac{u}{3} - \dfrac{v}{3} \\ y = \dfrac{2u}{3} + \dfrac{v}{3} \end{cases}$$
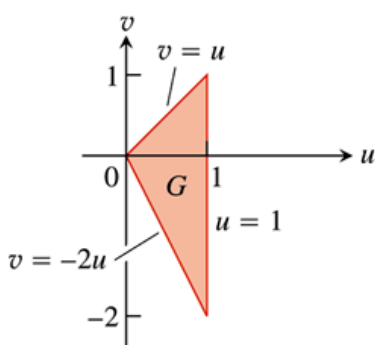
xy-eqns for the boundary of R	Corresponding *uv*-eqns. for the boundary of G	Simplified *uv*-eqns.
$x + y = 1$	$\left(\dfrac{u}{3} - \dfrac{v}{3}\right) + \left(\dfrac{2u}{3} + \dfrac{v}{3}\right) = 1$	$u = 1$
$x = 0$	$\dfrac{u}{3} - \dfrac{v}{3} = 0$	$v = u$
$y = 0$	$\dfrac{2u}{3} + \dfrac{v}{3} = 0$	$v = -2u$

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{3} & -\dfrac{1}{3} \\ \dfrac{2}{3} & \dfrac{1}{3} \end{vmatrix} = \dfrac{1}{3} \Bigg|$$
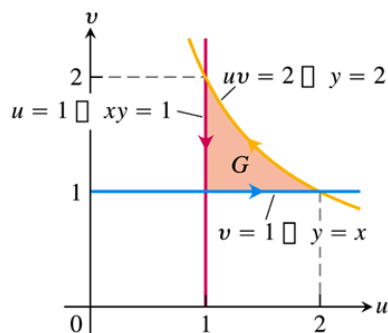
$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dydx = \int_{u=0}^{u=1} \int_{v=-2u}^{v=u} u^{1/2}v^2 \left| J(u,v) \right| dvdu$$

$$= \int_0^1 \int_{-2u}^{u} u^{1/2}v^2 \left(\tfrac{1}{3}\right) dvdu$$

$$= \int_0^1 u^{1/2} \left[\tfrac{1}{9}v^3\right]_{-2u}^{u} du$$

$$= \tfrac{1}{9}\int_0^1 u^{1/2}\left(u^3 + 8u^3\right) du$$

$$= \int_0^1 u^{7/2} du$$

$$= \tfrac{2}{9}u^{9/2}\Big|_0^1$$

$$= \underline{\tfrac{2}{9}}$$

Example

Evaluate $\displaystyle \int_1^2 \int_{1/y}^{y} \sqrt{\dfrac{y}{x}}\, e^{\sqrt{xy}}\, dxdy$

Solution

$$\begin{cases} u = \sqrt{xy} \\ v = \sqrt{\dfrac{y}{x}} \end{cases} \rightarrow \begin{cases} u^2 = xy \\ v^2 = \dfrac{y}{x} \end{cases} \implies x = \dfrac{u}{v}, \quad y = uv$$
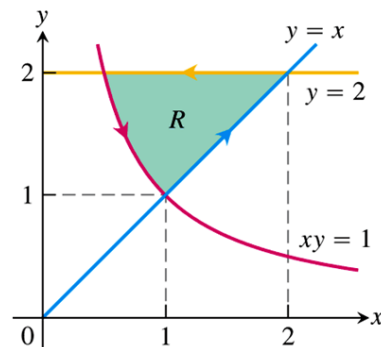
$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{v} & -\dfrac{u}{v^2} \\ v & u \end{vmatrix} = \underline{\dfrac{2u}{v}}$$

xy-eqns for the boundary of R	Corresponding uv-eqns. for the boundary of G	Simplified uv-eqns.
$x = y$	$\dfrac{u}{v} = uv$	$v = 1$
$x = \dfrac{1}{y}$	$\dfrac{u}{v} = \dfrac{1}{uv}$	$u = 1$
$y = 1$	$uv = 1$	
$y = 2$	$uv = 2$	$u = 2 \quad v = \dfrac{2}{u}$

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\; e^{\sqrt{xy}}\,dxdy = \int_1^2 \int_1^{2/u} 2u\; e^u\,dvdu$$

$$= 2\int_1^2 u\; e^u [v]_1^{2/u}\,du$$

$$= 2\int_1^2 u\; e^u \left(\frac{2}{u} - 1\right)du$$

$$= 2\int_1^2 (2-u)e^u\,du$$

$$= 2\Big[(2-u+1)e^u\Big]_1^2$$

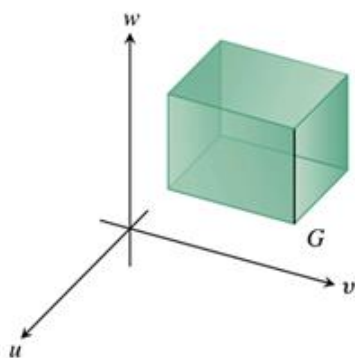$$= 2\Big[(1)e^2 - 2e\Big]$$

$$\underline{= 2e(e-2)}$$

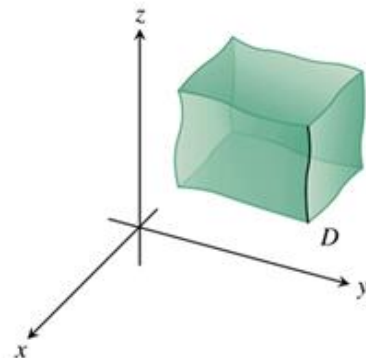|		e^u
(+)	$2-u$	e^u
(−)	-1	e^u
	0	

Substitutions in Triple Integrals

$$x = g(u,v,w), \quad y = h(u,v,w), \quad z = k(u,v,w)$$

Cartesian uvw - plane

Cartesian xyz - plane

$$\iiint\limits_{R} f(x,y,z)\,dxdydz = \iiint\limits_{R} H(u,v,w)\,|J(u,v,w)|\,dudvdw$$

The ***Jacobian determinant*** is

$$J(u,v,w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2mm] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$$

Cube with sides parallel to the axes
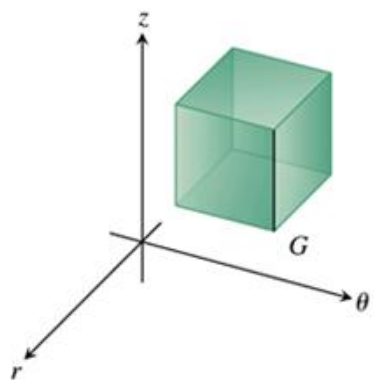
Cube with sides parallel to the axes

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Cartesian rθz - plane
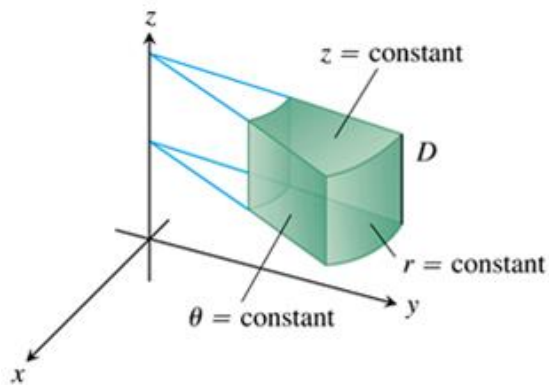
Cartesian xyz - plane

$$J(r,\theta,z) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial z} \\[2mm] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial z} \\[2mm] \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta \underline{= r}$$

For spherical coordinates, ρ, ϕ, and θ take the place of *u, v,* and *w*. The transformation from Cartesian $\rho\phi\theta$-space to Cartesian *xyz* –space is given by

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

The Jacobian of the transformation

$$J(\rho,\phi,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \phi} & \dfrac{\parti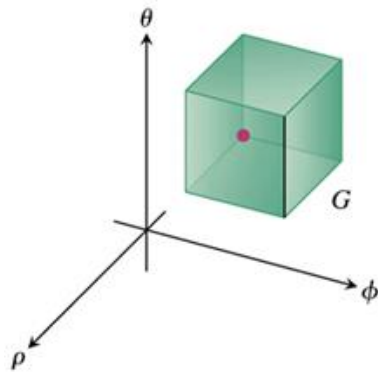al x}{\partial \theta} \\[2mm] \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \theta} \\[2mm] \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \rho^2\cos^2\phi\sin\phi\cos^2\theta + \rho^2\sin^3\phi\sin^2\theta + \rho^2\sin\phi\cos^2\phi\sin^2\theta + \rho^2\sin^3\phi\cos^2\theta$$

$$= \rho^2\cos^2\phi\sin\phi\left(\cos^2\theta + \sin^2\theta\right) + \rho^2\sin^3\phi\left(\sin^2\theta + \cos^2\theta\right)$$

$$= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi$$

$$= \rho^2\sin\phi\left(\cos^2\phi + \sin^2\phi\right)$$

$$\underline{= \rho^2\sin\phi}$$

$$\iiint_D F(x,y,z)\,dxdydz = \iiint_G H(\rho,\phi,\theta)\left|\rho^2\sin\phi\right|d\rho d\phi d\theta$$

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

Cartesian rθz - plane
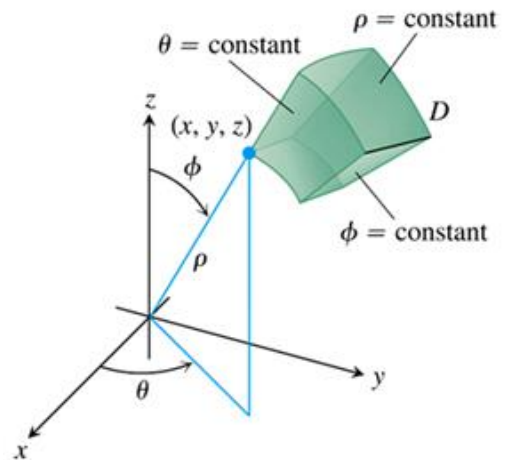
Cartesian xyz - plane

Example

Evaluate $\displaystyle\int_0^3\int_0^4\int_{x=y/2}^{x=(y/2)+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dxdydz$ by applying the transformation

$u=\dfrac{2x-y}{2},\quad v=\dfrac{y}{2},\quad w=\dfrac{z}{3}$ and integrating over an appropriate region in the uvw-plane.

Solution

$$\rightarrow\begin{cases} u=\dfrac{2x-y}{2}\rightarrow x=u+\dfrac{y}{2}=u+v \\[2mm] v=\dfrac{y}{2}\rightarrow y=2v \\[2mm] w=\dfrac{z}{3}\rightarrow z=3w \end{cases}$$

$$x=u+v$$
$$y=2v$$
$$z=3w$$

Rear plane:
$x=\dfrac{y}{2}$, or $y=2x$

Front plane:
$x=\dfrac{y}{2}+1$, or $y=2x-2$
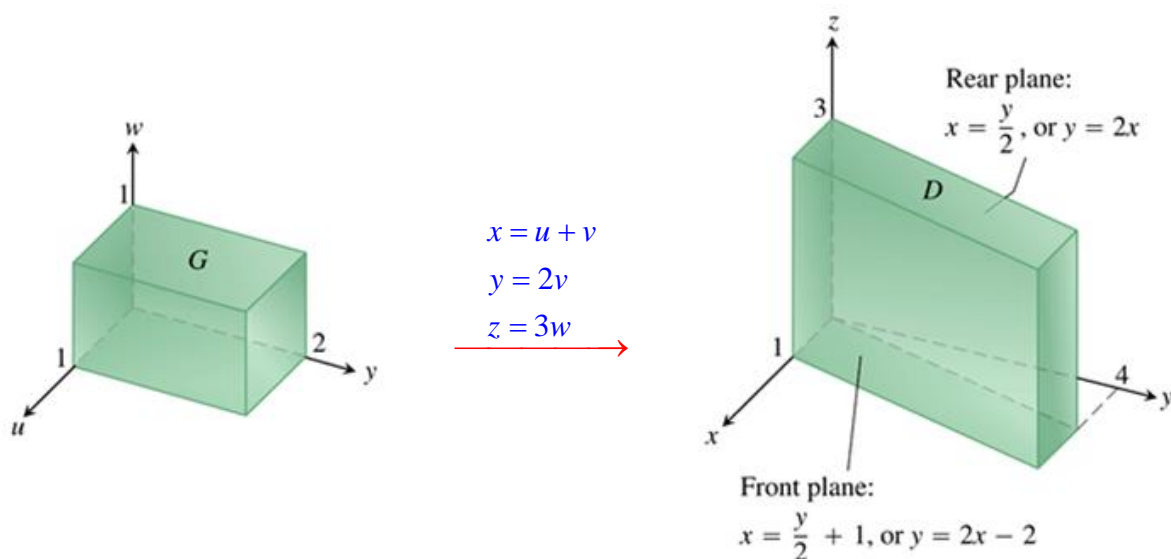
xyz-eqns for the boundary of *D*	Corresponding *uvw*- eqns. for the boundary of *G*	Simplified *uvw*- eqns.
$x=\dfrac{y}{2}$	$u+v=v$	$u=0$
$x=\dfrac{y}{2}+1$	$u+v=v+1$	$u=1$
$y=0$	$2v=0$	$v=0$
$y=4$	$2v=4$	$v=2$
$z=0$	$3w=0$	$w=0$
$z=3$	$3w=3$	$w=1$

$$J(u,v,w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2mm] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$$\underline{= 6}$$

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx\,dy\,dz = \int_0^1 \int_0^2 \int_0^1 (u+w)\left| J(u,v,w) \right|\, du\,dv\,dw$$

$$= 6 \int_0^1 \int_0^2 \int_0^1 (u+w)\,du\,dv\,dw$$

$$= 6 \int_0^1 \int_0^2 \left[\frac{u^2}{2} + wu \right]_0^1 dv\,dw$$

$$= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + w \right) dv\,dw$$

$$= 6 \int_0^1 \left[\frac{1}{2}v + wv \right]_0^2 dw$$

$$= 6 \int_0^1 (1 + 2w)\,dw$$

$$= 6 \left[w + w^2 \right]_0^1$$

$$= 6(1 + 1)$$

$$\underline{= 12}$$

Exercises   *Section* 3.7 – **Change of Variables in Multiple Integrals**

Let $S = \{0 \le u \le 1, \ 0 \le v \le 1\}$ be a unit square in the *uv*-plane. Find the image of *S* in the *xy*-plane under the following transformations.

1. $T: \quad x = v, \ y = u$

2. $T: \quad x = -v, \ y = u$

3. $T: \quad x = \dfrac{u+v}{2}, \ y = \dfrac{u-v}{2}$

4. $T: \quad x = u, \ y = 2v + 2$

5.   a) Solve the system $u = x - y, \quad v = 2x + y$ for *x* and *y* in terms of *u* and *v*. Then find the value of the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$

b) Find the image under the transformation $u = x - y, \ v = 2x + y$ of the triangular region with vertices (0, 0), (1, 1), and (1, −2) in the *xy*-plane. Sketch the transformed region in the *uv*-plane.

6.   Let *R* be the region in the first quadrant of the *xy*-plane bounded by the hyperbolas $xy = 1, \ xy = 9$ and the lines $y = x, \ y = 4x$. Use the transformation $x = \dfrac{u}{v}, \ y = uv$ with $u > 0, \ and \ v > 0$ to rewrite

$$\iint\limits_{R} \left(\sqrt{\frac{y}{x}} + \sqrt{xy} \right) dx\, dy$$

As an integral over an appropriate region *G* in the uv-plane. Then evaluate the *uv*-integral over *G*.

7.   The area πab of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ can be found by integrating the function $f(x, y) = 1$ over the region bounded by the ellipse in the *xy*-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x = au, \ y = bv$ and evaluate the transformed integral over the disk *G*: $u^2 + v^2 \le 1$ in the *uv*-plane. Find the area this way.

8.   Use the transformation $x = u + \dfrac{1}{2}v, \ y = v$ to evaluate the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3 (2x - y) e^{(2x-y)^2} dx\, dy$$

By first writing it as an integral over a region *G* in the *uv*-plane.

9.   Use the transformation $x = \dfrac{u}{v}, \ y = uv$ to evaluate the integral

$$\int_1^2 \int_{1/y}^{y} \left(x^2 + y^2\right) dx dy + \int_2^4 \int_{y/4}^{4/y} \left(x^2 + y^2\right) dx dy$$

10. Find the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$ of the transformation

   a) $x = u \cos v, \quad y = u \sin v$

   b) $x = u \sin v, \quad y = u \cos v$

11. Find the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$ of the transformation

   a) $x = u \cos v, \quad y = u \sin v, \quad z = w$

   b) $x = 2u - 1, \quad y = 3v - 4, \quad z = \frac{1}{2}(w - 4)$

12. Evaluate the appropriate determinant to show that the Jacobian of the transformation from Cartesian $\rho\phi\theta$–space to Cartesian xyz–space is $\rho^2 \sin \phi$

13. How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.

14. Find the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

   (**Hint**: Let $x = au, \ y = bv, \ and \ z = cw$. Then find the volume of an appropriate region in uvw-space)

15. Use the transformation $x = u^2 - v^2, \ y = 2uv$ to evaluate the integral

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2} \ dy dx$$

   (**Hint**: Show that the image of the triangular region **G** with vertices (0, 0), (1, 0), (1, 1) in the uv-plane is the region of integration **R** in the xy-plane defined by the limits of integration.)

16. Evaluate $\displaystyle\iint\limits_R y^4 dA$; R is the region bounded by the hyperbolas $xy = 1$ and $xy = 4$ and the lines

   $\dfrac{y}{x} = 1$, and $\dfrac{y}{x} = 3$

17. Evaluate $\displaystyle\iint\limits_{R} \left(y^2 + xy - 2x^2\right) dA$; R is the region bounded by the lines $y = x$, $y = x - 3$,

$y = -2x + 3$, and $y = -2x - 3$

18. Evaluate $\displaystyle\iint\limits_{D} x\, dV$; R is bounded by the planes $y - 2x = 0$, $y - 2x = 1$, $z - 3y = 0$, $z - 3y = 1$,

$z - 4x = 0$ and $z - 4x = 3$

19. Let R be the region bounded by the lines $x + y = 1$; $x + y = 4$; $x - 2y = 0$; $x - 2y = -4$

Evaluate the integral $\displaystyle\iint\limits_{R} 3xy\, dA$

20. Let R be the region bounded by the square with vertices (0, 1), (1, 2), (2, 1), & (1, 0).

Evaluate the integral $\displaystyle\iint\limits_{R} (x + y)^2 \sin^2(x - y)\, dA$

21. Evaluate $\displaystyle\iiint\limits_{D} yz\, dV$ D is bounded by the planes: $x + 2y = 1$, $x + 2y = 2$, $x - z = 0$, $x - z = 2$,

$2y - z = 0$, and $2y - z = 3$

22. Evaluate $\displaystyle\iint\limits_{R} xy\, dA$; R is the square with vertices $(0,\ 0)$, $(1,\ 1)$, $(2,\ 0)$, and $(1,\ -1)$

23. Evaluate $\displaystyle\iint\limits_{R} x^2 y\, dA$; $R = \{(x,\ y):\ 0 \le x \le 2,\ \ x \le y \le x + 4\}$

24. Evaluate $\displaystyle\iint\limits_{R} x^2\sqrt{x + 2y}\, dA$; $R = \left\{(x,\ y):\ 0 \le x \le 2,\ \ -\frac{x}{2} \le y \le 1 - x\right\}$

25. Evaluate $\displaystyle\iint\limits_{R} xy\, dA$; where R is bounded by the ellipse $9x^2 + 4y^2 = 36$.

26. Evaluate $\displaystyle\int_{0}^{1}\int_{y}^{y+2} \sqrt{x - y}\, dxdy$

27. Evaluate $\displaystyle\iint\limits_{R} \sqrt{y^2 - x^2}\ dA$; where R is the diamond bounded by $y - x = 0,\ \ y - x = 2,\ \ y + x = 0,$

and $y + x = 2$

28. Evaluate $\displaystyle\iint\limits_{R} \left(\frac{y - x}{y + 2x + 1}\right)^4 dA$; where R is the parallelogram bounded by $y - x = 1,\ \ y - x = 2,$

$y + 2x = 0$, and $y + 2x = 4$

29. Evaluate $\displaystyle\iint\limits_{R} e^{xy}\ dA$; where R is the region bounded by $xy = 1,\ \ xy = 4,\ \dfrac{y}{x} = 1$, and $\dfrac{y}{x} = 3$

30. Evaluate $\displaystyle\iint\limits_{R} xy\ dA$; where R is the region bounded by the hyperbolas $xy = 1,\ \ xy = 4,\ \ y = 1$, and

$y = 3$

31. Evaluate $\displaystyle\iint\limits_{R} (x - y)\sqrt{x - 2y}\ dA$; where R is the triangular region bounded by $y = 0,\ \ x - 2y = 0,$

and $x - y = 1$

32. Evaluate $\displaystyle\iiint\limits_{D} xy\ dV : D$ is bounded by the planes: $y - x = 0,\ \ y - x = 2,\ \ z - y = 0,\ \ z - y = 2,$

$z = 0$, and $z = 3$

33. Evaluate $\displaystyle\iiint\limits_{D} dV : D$ is bounded by the planes: $y - 2x = 0,\ \ y - 2x = 1,\ \ z - 3y = 0,\ \ z - 3y = 1,$

$z - 4x = 0$, and $z - 4x = 3$

34. Evaluate $\displaystyle\iiint\limits_{D} z\ dV : D$ is bounded by the paraboloid $z = 16 - x^2 - 4y^2$ and the xy-plane.

35. Evaluate $\displaystyle\iiint\limits_{D} dV : D$ is bounded by the upper half of the ellipsoid $\dfrac{x^2}{9} + \dfrac{y^2}{4} + z^2 = 1$ and the xy-

plane.

36. Evaluate $\displaystyle\iiint\limits_{D} xz\ dV : D$ is bounded by the planes: $y = x,\ \ y = x + 2,\ \ x - z = 0,\ \ z = x + 3,\ \ z = 0,$

and $z = 4$

(37 − 41) Let R be the region bounded by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$ are real numbers.

37. Find the area of R.

38. Evaluates $\displaystyle\iint_R |xy|\, dA$

39. Find the center of mass of the upper half of R $(y \geq 0)$ assuming it has a constant density.

40. Find the average square of the distance between points of R and the origin.

41. Find the average distance between points in the upper half of R and the x-axis.

(42 − 45) Let D be the region bounded by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, where $a > 0$, $b > 0$ and $c > 0$ are real numbers.

42. Find the Volume of D.

43. Evaluates $\displaystyle\iint_D |xyz|\, dV$

44. Find the center of mass of the upper half of D $(z \geq 0)$ assuming it has a constant density.

45. Find the average square of the distance between points of D and the origin.