Lecture One

Section 1.1 – System of Equations

Definition ***Linear equations***

A set of equations is called a system of equations, when a model requires finding the solutions of 2 or more equations.

Types of Solutions

- ✓ Unique One Solution
- ✓ No Solution / Inconsistent
- ✓ Unique Infinite Solutions (*Depend*)

Transformations

An equivalent system is one that has the same solutions as the given system
The following transformations can be applied to a system of equations to get an equivalent system.

1. Exchanging any 2 equations
2. Multiplying both sides of an equation by any nonzero real number
3. Replacing any equation by a nonzero multiple of that equation plus a nonzero multiple by any other equation.

Example

$$2x+3y=12 \quad (1)$$
$$3x-4y=1 \quad (2)$$

<u>*Solution*</u>

$$\begin{aligned} &3(2x+3y=12) \\ &-2(3x-4y=1) \end{aligned} \qquad \begin{array}{r} 6x+9y=36 \\ -6x+8y=-2 \\ \hline 17y=34 \end{array}$$

$$\begin{aligned} &2x+3y=12 \\ &17y=34 \qquad \tfrac{1}{17}R_2 \end{aligned} \qquad \Rightarrow y=\frac{34}{17}=2$$

$$\tfrac{1}{2}R_1 \qquad x+\frac{3}{2}y=6$$

$$x+\frac{3}{2}2=6$$

$$x+3=6 \qquad \Rightarrow x=3$$

Solution: (3, 2)

Example

A restaurant owner orders a replacement set of knives, forks, and spoons. The box arrives containing 40 utensils and weighting 141.3 oz (ignoring the weight of the box). A knife, fork, and spoon weigh 3.9 oz, 3.6 oz, and 3.0 oz, respectively.
How many solutions are there for the number of knives, forks and spoons in the box?

<u>*Solution*</u>

Let: x: Number of knives
y: Number of forks
z: Number of spoons

	Knifes	**Forks**	**Spoons**	**Total**
Number	x	y	z	40
Weight	3.9	3.6	3	141.3

$$\begin{aligned} x+y+z&=40 \\ 3.9x+3.6y+3z&=141.3 \end{aligned}$$

$$3.9R_1 - R_2 \to R_2 \qquad \begin{array}{r} 3.9x+3.9y+3.9z=156 \\ -3.9x-3.6y-3z=-141.3 \\ \hline .3y+.9z=14.7 \end{array}$$

$$.3y+.9z=14.7$$

$$.3y=14.7-.9z$$

$$y=\frac{14.7}{.3}-\frac{.9}{.3}z$$

$$y=49-3z$$

$$\begin{aligned} x+y+z&=40 \\ x&=40-y-z \\ &=40-(49-3z)-z \\ &=40-49+3z-z \\ &=2z-9 \end{aligned}$$

$$(2z-9,\ 49-3z,\ z)$$

$$\begin{aligned} y=49-3z&\ge 0 \\ 49&\ge 3z \\ 3z&\le 49 \\ z&\le 16. \end{aligned} \qquad\qquad \begin{aligned} x=2z-9&\ge 0 \\ 2z&\ge 9 \\ z&\ge 4.5 \end{aligned}$$

Values of ***z***: 5, 6, 7, …., 16

Matrices

$$\begin{array}{cc} & \textit{Column} \\ & \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{array}{c} Row\ 1 \to R_1 \\ Row\ 2 \to R_2 \\ Row\ 3 \to R_3 \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

This is called Matrix (*Matrices*)

Each number in the array is an ***element*** or ***entry***

The matrix is said to be of order $\boldsymbol{m}$ x $\boldsymbol{n}$

$\boldsymbol{m}$: numbers of rows,

$\boldsymbol{n}$: number of columns

When $\boldsymbol{m} = \boldsymbol{n}$, then matrix is said to be ***square***.

Matrices *Size*

a) $\begin{bmatrix} -3 & 5 \\ 2 & 0 \\ 5 & -1 \end{bmatrix}$ 3*x*2 matrix

b) $\begin{bmatrix} 1 & 6 & 5 & -2 & 5 \end{bmatrix}$ 1*x*5 matrix

c) $\begin{bmatrix} 3 \\ -5 \\ 0 \\ 2 \end{bmatrix}$ 5*x*1 matrix

- A matrix containing only 1 row is called a *row matrix* or *row vector*
- A matrix containing only 1 column is called a *column matrix* or *column vector*
- Two matrices are equal if they are the same size and if each pair corresponding elements is equal

Example

a) $\begin{bmatrix} 2 & 1 \\ p & q \end{bmatrix} = \begin{bmatrix} x & y \\ -1 & 0 \end{bmatrix}$ $x = 2,\ y = 1,\ \ p = -1,\ \ q = 0$

b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$ *can't be true*

Addition

The sum of two mxn matrices X and Y is the mxn matrices $X + Y$ in which each element is the sum of the corresponding elements of X and Y.

Example

a) $$\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} 5-4 & -6+6 \\ 8+8 & 9-3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 16 & 6 \end{bmatrix}$$

b) $$\begin{bmatrix} 5 & -8 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -9 & 1 \\ 4 & 2 & -5 \end{bmatrix} = \textit{doesn't exist}$$

Product of a Matrix and a *Scalar*

The product of a scalar k and a matrix X is the matrix kX, each of whose elements is k times the corresponding element of X.

Example

$$(-5)\begin{bmatrix} 3 & 4 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3(-5) & 4(-5) \\ 0(-5) & -1(-5) \end{bmatrix} = \begin{bmatrix} -15 & -20 \\ 0 & 5 \end{bmatrix}$$

Product of Two Matrices

Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix. To find the element in the i^{th} row and j^{th} column of the product matrix AB, multiply each element in the i^{th} row of A by the corresponding element in the j^{th} column of B, and then add these products. The product matrix AB is an $m \times k$ matrix.

Matrix ***A*** *Matrix B*

$m \times n$ $n \times k$

Inner *must be equal*

Outer: Order of AB is $m \times k$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$2x2 \qquad 2x2 \quad \rightarrow \quad 2x2$

$$a_{11} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & \_ \\ \_ & \_ \end{bmatrix}$$

$$a_{12} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & af+bh \\ \_ & \_ \end{bmatrix}$$

$$a_{21} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ ce+dg & \_ \end{bmatrix}$$

$$a_{22} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ \_ & cf+dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Example

Find the product CD of matrices $C = \begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix}$

<u>*Solution*</u>

$$\begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -3(-6)+4(2)+2(3) & -3(4)+4(3)+2(-2) \\ 5(-6)+0(2)+4(3) & 5(4)+0(3)+4(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 32 & -4 \\ -18 & 12 \end{bmatrix}$$

Example – Home Construction

A contractor builds three kinds of houses, models *A*, *B*, and *C*, with a choice of two styles, Spanish and contemporary. Matrix *P* shows the number of each kind of house planned for a new 100-home subdivision. The amounts for each of the exterior materials depend primarily on the style of the house. These amounts are shown in matrix *Q*. (concrete is in cubic yards, lumber in units of 1000 board feet, brick in 1000s, and shingles in units of 100 ft^2.) Matrix *R* gives the cost in dollars for each kind of material.

a) What is the total cost of these materials for each model?
b) How much of each of four kinds of material must be ordered
c) What is the total cost for exterior materials?

<u>Solution</u>

$$\begin{array}{r c} & \begin{array}{cc} \textit{Spanish} & \textit{Contemporary} \end{array} \\ \begin{array}{r} \textit{Model A} \\ \textit{Model B} \\ \textit{Model C} \end{array} & \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} = P \end{array} \qquad \begin{array}{r c} & \begin{array}{cccc} \textit{Concrete} & \textit{Lumber} & \textit{Brick} & \textit{Shingles} \end{array} \\ \begin{array}{r} \textit{Spanish} \\ \textit{Contemporary} \end{array} & \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix} = Q \end{array}$$

$$\begin{array}{r c} & \textit{Cost per unit} \\ \begin{array}{r} \textit{Concrete} \\ \textit{Lumber} \\ \textit{Brick} \\ \textit{Shingles} \end{array} & \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} = R \end{array}$$

a) What is the total cost of these materials for each model?

$$PQ = \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix} = \begin{array}{c c} \begin{array}{cccc} \textit{Concrete} & \textit{Lumber} & \textit{Brick} & \textit{Shingles} \end{array} & \\ \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} & \begin{array}{l} \textit{Model A} \\ \textit{Model B} \\ \textit{Model C} \end{array} \end{array}$$

$$(PQ)R = \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} = \begin{bmatrix} 72{,}900 \\ 54{,}700 \\ 60{,}800 \end{bmatrix} \begin{array}{l} \textit{Model A} \\ \textit{Model B} \\ \textit{Model C} \end{array}$$

The total cost of materials is \$72,900 for model *A*, \$54,700 for model *B*, \$60,800 for model *C*.

b) How much of each of four kinds of material must be ordered

$$\begin{array}{cccc} \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \\ \hline 3800 \quad 130 \quad 1400 \quad 200 \end{array} \qquad T = \begin{bmatrix} 3800 & 130 & 1400 & 200 \end{bmatrix}$$

3800 yd^3 of concrete, 130,000 board feet of lumber, 1,400,000 bricks, and 20,000 ft^2 of shingles are needed.

c) What is the total cost for exterior materials?

$$TR = \begin{bmatrix} 3800 & 130 & 1400 & 200 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} = \begin{bmatrix} 188,400 \end{bmatrix}$$

The total cost for exterior materials is $188,400

The *Multiplicative Identity* Matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \boxed{AI = IA = A}$$

Multiplicative Identity matrix $\boldsymbol{I}$ of order n is unique and given by:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2x2} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3x3} \qquad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplicative *Inverse* of a matrix

Multiplicative inverse of a matrix A_{nxn} and $A^{-1}{}_{nxn}$ if exists, then: $\underline{A \cdot A^{-1} = A^{-1} \cdot A = I}$

Example

Given: $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$ Find A^{-1}

<u>*Solution*</u>

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right] \quad R_2 + R_1$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 5 & 1 & 1 \end{array}\right] \quad \tfrac{1}{5} R_2$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array}\right] \quad R_1 - 3R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & -\frac{3}{5} \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array}\right] \qquad A^{-1} = \begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \Rightarrow \quad A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad-bc=0$, then A^{-1} doesn't exist

$$A=\begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix} \;\rightarrow\; Find \;\; A^{-1}$$

$$\Rightarrow A^{-1}=\frac{1}{(-1)(4)-(-2)(3)}\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}=\frac{1}{2}\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Example

Find A^{-1} *if* $A=\begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

<u>*Solution*</u>

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-3R_1 \end{array}$$

$$\begin{array}{cccccc} 2 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -2 & 0 & 0 \\ \hline 0 & -2 & -3 & -2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 3 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & -3 & 0 & 0 \\ \hline 0 & 0 & -3 & -3 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \; -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \; -\tfrac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \begin{array}{l} R_1-R_3 \\ R_2-\frac{3}{2}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ \hline 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array} \qquad \begin{array}{cccccc} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ \hline 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \qquad A^{-1}=\begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

Solving Systems of Equations with Inverses

To solve the matrix equation $AX = B$.

- X: matrix of the variables
- A: Coefficient matrix
- B: Constant matrix

$$AX = B$$

$$A^{-1}(AX) = A^{-1}B \qquad \textit{Multiply both side by } A^{-1}$$

$$\left(A^{-1}A\right)X = A^{-1}B \qquad \textit{Associate property}$$

$$IX = A^{-1}B \qquad \textit{Multiplicative inverse property}$$

$$X = A^{-1}B \qquad \textit{Identity property}$$

Example

Use the inverse of the coefficient matrix to solve the linear system

$$\begin{aligned} 2x - 3y &= 4 \\ x + 5y &= 2 \end{aligned}$$

<u>*Solution*</u>

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The solution of the system is $(2, 0)$

Example

Three brands of fertilizer are available that provide nitrogen, phosphoric acid, and soluble potash to the soil. One bag of each brand provides the following units of each nutrient.

		Brand		
		Fertifun	***Big Grow***	***Soakem***
Nutrient	*Nitrogen*	1	2	3
	Phosphoric acid	3	1	2
	Potash	2	0	1

For ideal growth, the soil on a Michigan farm needs 18 units of nitrogen, 23 units of phosphoric acid, and 13 units of potash per acre. The corresponding numbers for a California farm are 31, 24, and 11, and for Kansas farm are 20, 19, and 15. How many bags of each brand of fertilizer should be used per acre for ideal growth on each farm?

<u>Solution</u>

$$\begin{aligned} x+2y+3z&=b_1 \\ 3x+y+2z&=b_2 \\ 2x \qquad +z&=b_3 \end{aligned} \qquad A=\begin{bmatrix}1&2&3\\3&1&2\\2&0&1\end{bmatrix} \qquad \Rightarrow \quad A^{-1}=\begin{bmatrix}-\frac{1}{3}&\frac{2}{3}&-\frac{1}{3}\\-\frac{1}{3}&\frac{5}{3}&-\frac{7}{3}\\\frac{2}{3}&-\frac{4}{3}&\frac{5}{3}\end{bmatrix}$$

For Michigan farm: $B=\begin{bmatrix}18\\23\\13\end{bmatrix} \quad \Rightarrow \quad X=A^{-1}B=\begin{bmatrix}-\frac{1}{3}&\frac{2}{3}&-\frac{1}{3}\\-\frac{1}{3}&\frac{5}{3}&-\frac{7}{3}\\\frac{2}{3}&-\frac{4}{3}&\frac{5}{3}\end{bmatrix}\begin{bmatrix}18\\23\\13\end{bmatrix}=\begin{bmatrix}5\\2\\3\end{bmatrix}$

For California farm: $B=\begin{bmatrix}31\\24\\11\end{bmatrix} \quad \Rightarrow \quad X=\begin{bmatrix}-\frac{1}{3}&\frac{2}{3}&-\frac{1}{3}\\-\frac{1}{3}&\frac{5}{3}&-\frac{7}{3}\\\frac{2}{3}&-\frac{4}{3}&\frac{5}{3}\end{bmatrix}\begin{bmatrix}31\\24\\11\end{bmatrix}=\begin{bmatrix}2\\4\\7\end{bmatrix}$

For Kansas farm: $B=\begin{bmatrix}20\\19\\15\end{bmatrix} \quad \Rightarrow \quad X=\begin{bmatrix}-\frac{1}{3}&\frac{2}{3}&-\frac{1}{3}\\-\frac{1}{3}&\frac{5}{3}&-\frac{7}{3}\\\frac{2}{3}&-\frac{4}{3}&\frac{5}{3}\end{bmatrix}\begin{bmatrix}20\\19\\15\end{bmatrix}=\begin{bmatrix}1\\-10\\13\end{bmatrix}$

(-10) *is impossible to have a negative bags*.

Exercises *Section* 1.1 – System of Equations

1. Solve the system $\begin{cases} 2x + y - z = 2 & (1) \\ x + 3y + 2z = 1 & (2) \\ x + y + z = 2 & (3) \end{cases}$

2. Solve the system $\begin{cases} 3x_1 + x_2 - 2x_3 = 2 \\ x_1 - 2x_2 + x_3 = 3 \\ 2x_1 - x_2 - 3x_3 = 3 \end{cases}$

3. Solve the system: $\begin{cases} 2x_1 - 2x_2 + x_3 = 3 \\ 3x_1 + x_2 - x_3 = 7 \\ x_1 - 3x_2 + 2x_3 = 0 \end{cases}$

4. Katherine invests \$10,000 received from her grandmother in three ways. With one part, she buys U.S saving bonds at an interest rate of 2.5% per year. She uses the second part, which amounts to twice the first, to buy mutual funds that offer a return of 6% per year. She puts the rest of the money into a money market account paying 4.5% annual interest. The first year her investments bring a return of \$470. How much did she invest in each way?

5. A Company that rents small moving trucks wants to purchase 25 trucks with a combined capacity of 28,000 cubic ft. Three different types of trucks are available: a 10-foot truck with a capacity of 350 cubic feet, a 14-foot truck with a capacity of 700 cubic feet, and a 24-foot truck with a capacity of 1,400 cubic feet. How many truck of each type of truck should the company purchase?

6. A flight leaves New York at 8 PM and arrives in Paris at 9 AM (Paris time). This 13-hour difference includes the flight time plus the change in time zones. The return leaves Paris at 1 PM and arrives in New York at 3 PM (NY time). This 2-hour difference includes the flight time minus time zones, plus an extra hour due to the fact that flying westward is against the wind. Find the actual flight time eastward and the difference in time zones.

7. Find the variables, if possible: $\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$

8. Find the variables, if possible: $\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$

9. Evaluate: $\begin{bmatrix} 2 & 4 & 5 & -7 \\ 6 & -3 & 12 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 0 & -10 & 1 \\ -2 & 8 & -9 & 11 \end{bmatrix}$

10. Evaluate: $\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix}$

11. Evaluate: $\begin{bmatrix} 5 & 4 \\ -3 & 7 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix}$

12. Find: $-4\begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix} + 5\begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix}$

13. Find: $\begin{bmatrix} 2 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ -1 & 4 \\ 0 & 2 \end{bmatrix}$

14. Find: $\begin{bmatrix} 2 \\ -9 \\ 12 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$

15. Find: $\begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$

16. Find: $\begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$

17. Given: $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ $\quad F = \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}$ Find: $3F + 2A$

18. Given: $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix}$ Find:

a) $A - B$

b) $3A + 2B$

19. Given: $A = \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}$ Find: AB *and* BA

20. Sal's Shoes and Fred's Footwear both have outlets in California and Arizona. Sal's sells shoes for \$80, sandals for \$40, and boots for \$120. Fred's prices are \$60, \$30, and \$150 for shoes, sandals and boots, respectively. Half of all sales in California stores are shoes, 1/4 are sandals, and 1/4 are boots. In Arizona the fractions are 1/5 shoes, 1/5 are sandals, and 3/5 are boots.

a) Write a 2 *x* 3 matrix called P representing prices for the two stores and three types of footwear.

b) Write a 2 *x* 3 matrix called F representing fraction of each type of footwear sold in each state.

c) Only one of the two products PF and FP is meaningful. Determine which one it is, calculate the product, and describe what the entries represent.

21. Use the inverse of the coefficient matrix to solve the linear system

$$\begin{aligned} 2x+5y&=15 \\ x+4y&=9 \end{aligned}$$

22. Find the inverse of: $A=\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

23. Find the inverse of: $A=\begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

24. Find the inverse of: $M=\begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

25. Use the inverse of the coefficient matrix to solve the linear system: $\begin{cases} 3x_1 - x_2 + x_3 = 3 \\ -x_1 + x_2 = -3 \\ x_1 + x_3 = 2 \end{cases}$

26. An investment advisor currently has two types of investment available for clients: a conservative investment A that pays 8% per year and an investment B paying 24% per year. Clients may divide their investments between the two to achieve any total return desired between 8% and 24%. However, the higher the desired return, the higher the risk. How should each client listed in the table invest to achieve the indicated return?

Section 1.2 – Graphing Linear Inequalities

A linear inequalities in two variables has the form

$$ax + by \le c$$
$$ax + by < c$$
$$ax + by \ge c$$
$$ax + by > c$$

1. Determine Line type
   a. A ***solid*** line if the original inequality $\le or \ge$
   b. A ***dashed*** line if the original inequality $< or >$
2. Choose a test point from on the half-planes (*Choose (0, 0) if the graph is not pass thru*)
3. If the result is True statement, shade the half-plane, otherwise, the other half

Example

Graph the linear inequality: $2x - 3y \le 12$

<u>*Solution*</u>

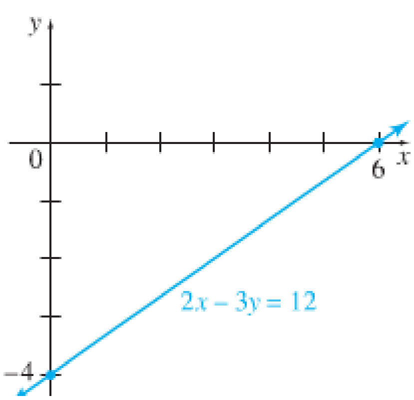
$2x - 3y = 12$

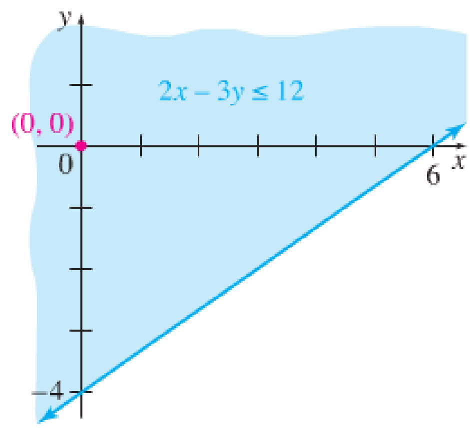
x	y
0	-4
6	0

Test: using the point (0, 0)

$2(0) - 3(0) \le 12$

$0 \le 12 \quad \to \ True$





Example

Graph the system: $\begin{cases} y < -3x + 12 \\ x < 2y \end{cases}$

<u>*Solution*</u>

$y = -3x + 12$

x	y
0	12
4	0

$x = 2y$

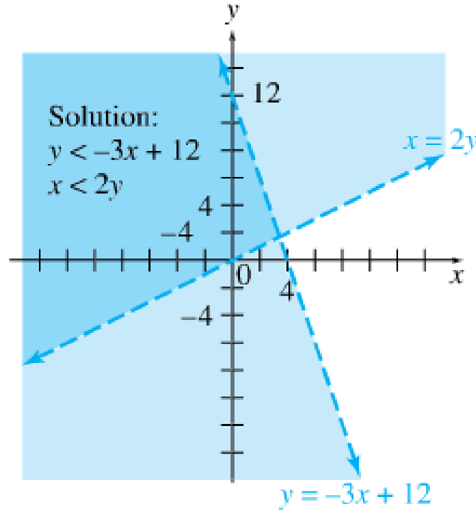
x	y
0	0
2	1

Test: (0, 0)

$0 < -3(0) + 12 \ \to T$

(1,0)

$1 < 2(0) \ \to F$



Definition

A region consisting of the overlapping parts of two or more graphs of inequalities (Heavily shaded region or the intersection) is called the ***region of feasible solution*** or the ***feasible region***.

A solution region of a system of linear inequalities

Bounded: if it can be enclosed within circle

Unbounded: ∞

Corner point is a point in the feasible region where the boundary lines of two constraints cross.

Example

Graph the feasible region for the system $\begin{cases} y \le -2x + 8 \\ -2 \le x \le 1 \end{cases}$

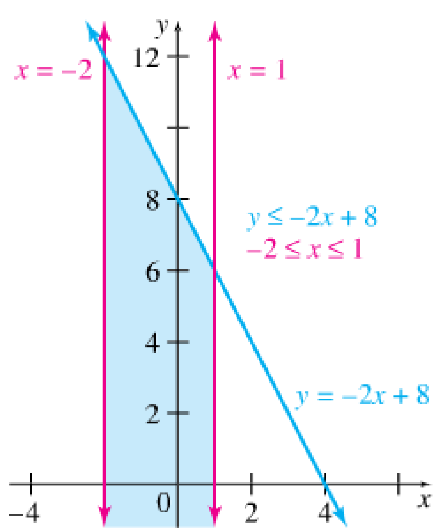
<u>*Solution*</u>

$y = -2x + 8$

x	y
0	8
4	0

$0 \le -2(0) + 8$

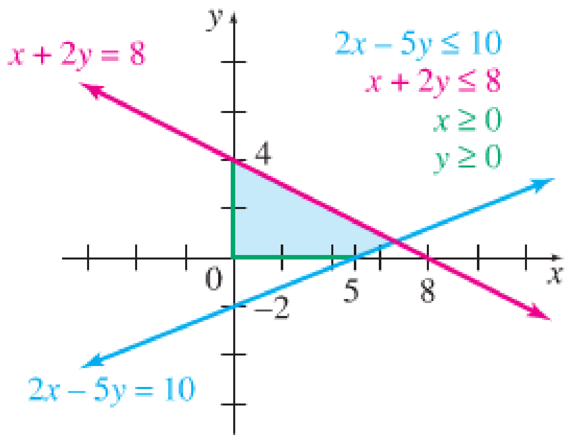
$0 \le 8 \quad \to \ True$



Example

Graph the feasible region for the system $\begin{cases} 2x-5y \le 10 \\ x+2y \le 8 \\ x, y \ge 0 \end{cases}$

<u>*Solution*</u>



Example

Happy Ice Cream Cone Company makes cake cones and sugar cones, both of which must be processed in the mixing department and the baking department. Manufacturing one batch of cake cones requires 1 hour in the mixing department and 2 hours in the baking department, and producing one batch of sugar cones requires 2 hours in the mixing department and 1 hour in the baking department. Each department is operated for at most 12 hours per day.

a) Write a system of inequalities that expresses these restrictions.

b) Graph the feasible region

c) Using the graph from part (b), can 3 batches of cake cones and 2 batches of sugar cones be manufactured in one day

d) can 4 batches of cake cones and 6 batches of sugar cones be manufactured in one day

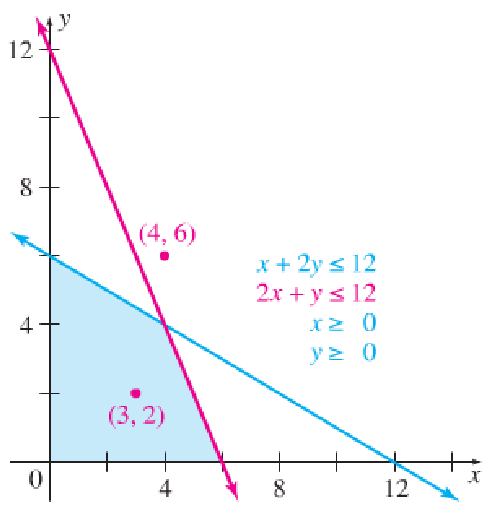
<u>*Solution*</u>

a) Let x: *number of cake cones*
y: *number of sugar cones*

	Cake	***Sugar***		***Total***
	x	y		
Hours in Mixing Dept.	1	2	$\le$	12
Hours in Baking Dept.	2	1	$\le$	12

$$\begin{cases} x+2y\le 12 \\ 2x+y\le 12 \end{cases}$$

b)



c) 3 batches of cake cones and 2 batches of sugar cones correspond to the point (3, 2). It is possible to manufacture these since the point (3, 2) is in the feasible region

d) 4 batches of cake cones and 6 batches of sugar cones correspond to the point (4, 6). It is *not* possible to manufacture these since the point (4, 6) is *not* in the feasible region

Exercises *Section* 1.2 – Graphing Linear Inequalities

Graph the feasible region for the system

1. $\begin{cases} 3x - 2y \ge 6 \\ x + y \le -5 \\ y \le 4 \end{cases}$

2. $\begin{cases} x + y \ge 6 \\ 2x - y \ge 0 \end{cases}$

3. $\begin{cases} 3x + y \le 21 \\ x - 2y \le 0 \end{cases}$

4. $\begin{cases} 5x + y \ge 20 \\ x + y \ge 12 \\ x + 3y \ge 18 \\ x \ge 0 \\ y \ge 0 \end{cases}$

5. A manufacturing plant makes two types of inflatable boats, a two-person boat and a four-person boat. Each two-person boat requires 0.9 labor-hour in the cutting department and 0.8 labor-hour in the assembly department. Each four-person boat requires 1.8 labor-hours in the cutting department and 1.2 labor-hours in the assembly department. The maximum labor-hours available each month in the cutting and assembly departments are 864 and 672, respectively.

a) Summarize the information in a table

b) If x two-person boat and y four-person boats are manufactured each month, write a system of linear inequalities that reflect the conditions indicated. Find the set of feasible solutions graphically

Section 1.3 – Solving Linear Programming and Applications

These approaches help management to make a decision

1) Graph the region (feasible region).
2) Identify the corner points.
3) Evaluate the objective function at each corner point.
4) interpret the optimal solution {maximum or minimum and where it occurs}

A solution region of a system of linear inequalities

Bounded: if it can be enclosed within circle

Unbounded: ∞

Corner point is a point in the feasible region where the boundary lines are intersected.

If an optimum value (either maximum or minimum) of the objective function exists, it will occur at one or more of the corner points of the feasible region.

At least = Min. is $\geq$

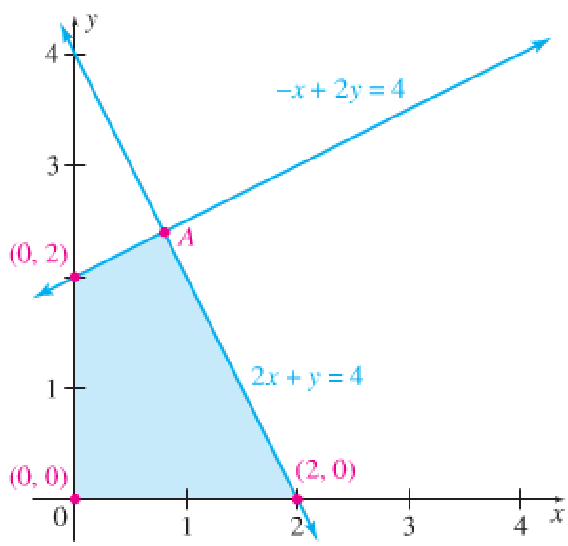
At the most = Max $\leq$

Maximization

Example

Find the maximum value of the objective function $z = 3x + 4y$ subject to the following constraints

$$\begin{cases} 2x + y \leq 4 & (1) \\ -x + 2y \leq 4 & (2) \\ x, y \geq 0 & (3) \end{cases}$$



$(1) \cap (2) \quad \rightarrow \quad A = \left(\frac{4}{5},\ \frac{12}{5}\right)$

$(1) \cap x-axis \quad \rightarrow \quad (2,\ 0)$

$(2) \cap y-axis \quad \rightarrow \quad (0,\ 2)$

$$A = \left(\frac{4}{5},\ \frac{12}{5}\right) \Rightarrow z = 3x + 4y$$
$$= 3\frac{4}{5} + 4\frac{12}{5}$$
$$= 12$$

The corner point A leads to the largest value of *z*.

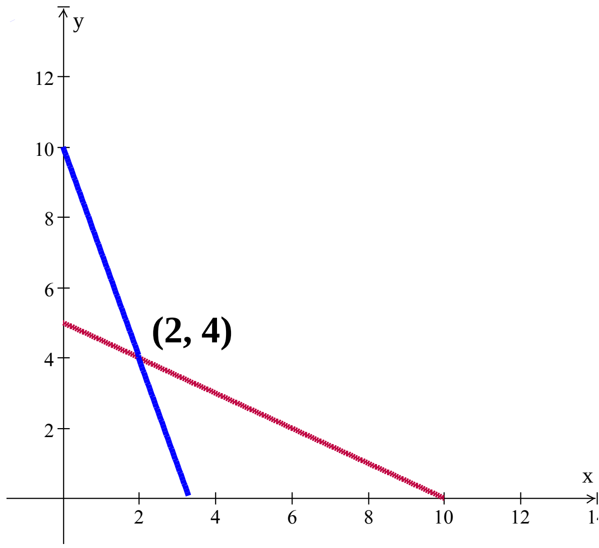
The feasible region is bounded

Minimization

Example

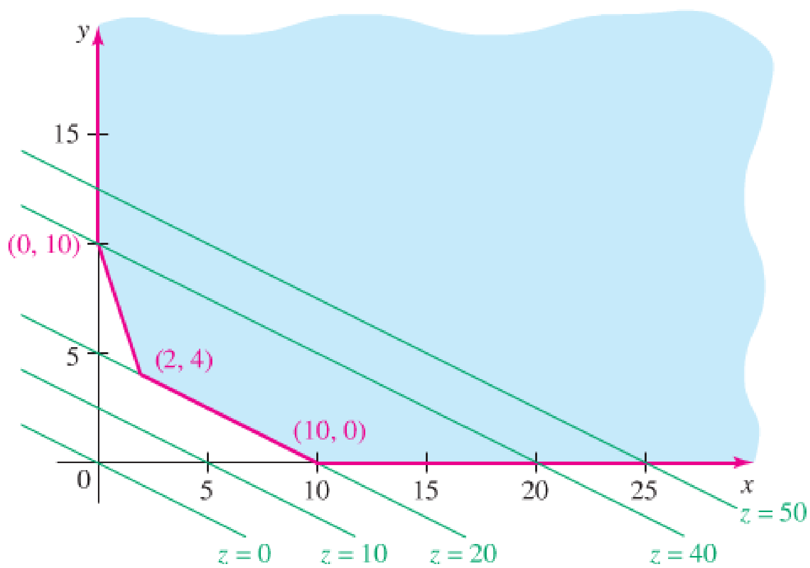
Solve the following linear programming problem

$$\begin{aligned} &\textit{Minimize} && z = 2x + 4y \\ &\textit{Subject to} && x + 2y \ge 10 \\ & && 3x + y \ge 10 \\ & && x, y \ge 0 \end{aligned}$$



The minimum value: $z = 2x + 4y$

$$= 2(2) + 4(4)$$
$$= 20$$



The feasible region is unbounded

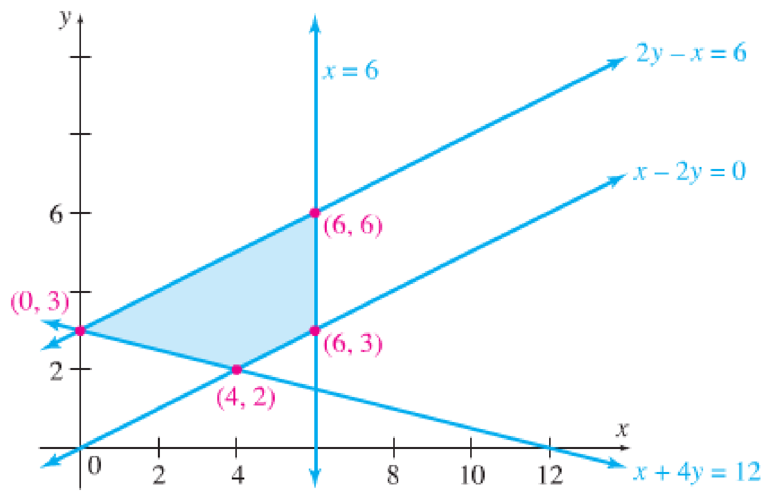
Maximization and Minimization

Example

Solve the following linear programming problem

$$z = x + 10y$$

$$\textit{Subject to} \quad \begin{aligned} x + 4y &\ge 12 \\ x - 2y &\le 0 \\ -x + 2y &\le 6 \\ x &\le 6 \end{aligned}$$



At $(0,\ 3) \;\Rightarrow\; z = 0 + 10(3) = 30$

At $(4,\ 2) \;\Rightarrow\; z = 4 + 10(2) = 24$

At $(6,\ 3) \;\Rightarrow\; z = 6 + 10(3) = 36$

At $(6,\ 6) \;\Rightarrow\; z = 6 + 10(6) = 66$

The minimum value of $\boldsymbol{z}$ is 24 at the corner point (4, 2).

The maximum value of $\boldsymbol{z}$ is 66 at the corner point (6, 6).

Example

Mr. Trenga plans to start a new business called River Explorers, which will rent canoes and kayaks to people to travel 10 miles down the Clarion River in Cook Forest State Park. He has $45,000 to purchase new boats. He can buy the canoes for $600 each and the kayaks for $750 each. His facility can hold up to 65 boats. The canoes will rent for $25 a day and the kayaks will rent for $30 a day. How many canoes and how many kayaks should he buy to earn the most revenue?

Solution

x: represents the number of canoes

y : represents the number of kayaks

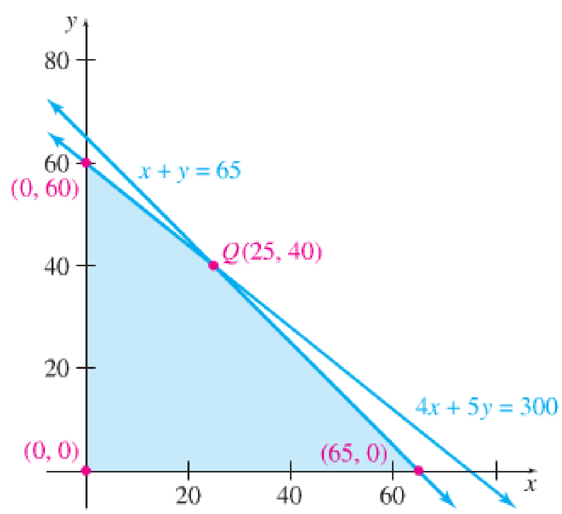
	Canoes	***Kayaks***		***Total***
	x	***y***	$\le$	65
Cost of Each	600	750	$\le$	45,000
Revenue	25	30		

$$\begin{cases} x+y\le 65 \\ 600x+750y\le 45,000 \end{cases} \quad \textit{divide both sides by } 150$$

$$\begin{cases} x+y\le 65 \\ 4x+5y\le 300 \end{cases}$$

The mathematical model for this problem for the given linear programming problem is as follows

$$\begin{array}{lll} \textit{Maximize} & z=25x+30y & (1) \\ \textit{Subject to} & x+y\le 65 & (2) \\ & 4x+5y\le 300 & (3) \\ & x, y\ge 0 & \end{array}$$



The corner points are

$$(0, 0)$$

$$(2)\cap(3) \rightarrow (25,\ 40)$$

$$(3)\cap x-axis \rightarrow (65,\ 0)$$

$$(2)\cap y-axis \rightarrow (0,\ 60)$$

Corner points	***z = 25x +30y***
(0, 0)	z = 25(0) +30(0) **= 0**
(25, 40)	z = 25(25) +30(40) = **1825**
(65, 0)	z = 25(65) +30(0) = **1625**
(0, 60)	z = 25(0) +30(60) = **1800**

The objective function, which represents revenue, is maximized when x = 25 and y = 40. He should buy 25 canoes and 40 kayaks.

Example

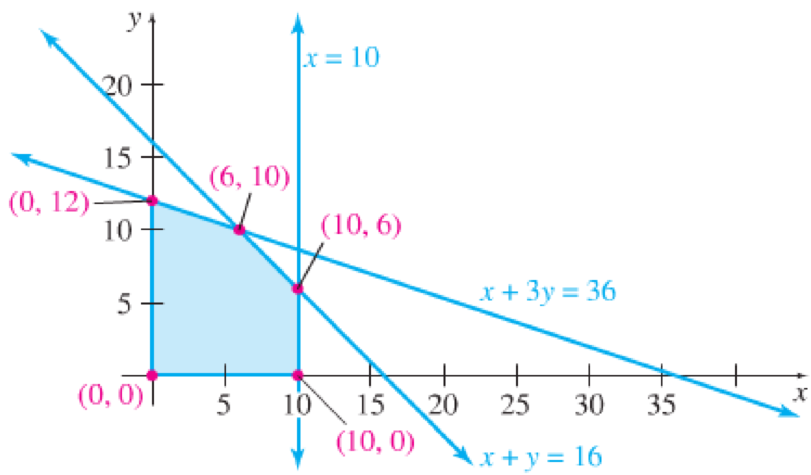
A 4-H member raises only goats and pigs. She wants to raise no more than 16 animals, including no more than 10 goats. She spends $25 to raise a goat and $75 to raise a pig, and she has $900 available for the project. The 4-H member wishes to maximize her profits. Each goat produces $12 in profit and each pig $40 in profit

Solution

	Goats	***Pigs***		***Total***
Raised	***x***	***y***	≤	16
Goat Limit	1		≤	10
Cost	$25	$75	≤	$900
Profit	$12	$40		

$$25x + 75y \le 900 \quad \Rightarrow \quad x + 3y \le 36$$

$$\begin{array}{lll} \textit{Maximize} & z = 12x + 40y & (1) \\ \textit{Subject to} & x + y \le 16 & (2) \\ & x \le 10 & (3) \\ & x + 3y \le 36 & (4) \\ & x, y \ge 0 & \end{array}$$



The corner points are

$$\begin{array}{lll} & & (0, 0) \\ (2) \cap (3) & \to & (10,\ 6) \\ (2) \cap (4) & \to & (6,\ 10) \\ (2) \cap x-axis & \to & (10,\ 0) \\ (4) \cap y-axis & \to & (0,\ 12) \end{array}$$

Corner points	***z = 12x + 40y***
(0, 0)	z = 12(0) + 40(0) **= 0**
(10, 6)	z = 12(10) + 40(6) = **360**
(6, 10)	z = 12(6) + 40(10) = **472**
(10, 0)	z = 12(10) + 40(0) = **120**
(0, 12)	z = 12(0) + 40(12) = **480**

Therefore, 12 pigs and no goats will produce a maximum profit of $480.

Example

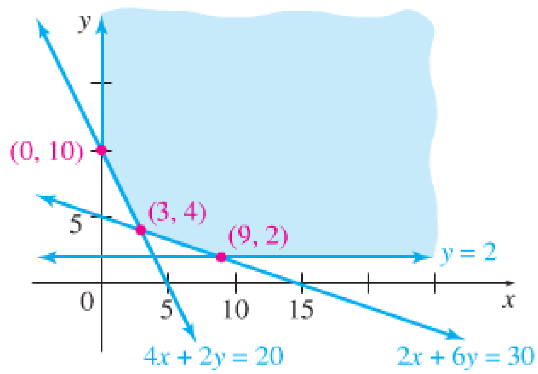
Certain animals in a rescue shelter must have at least 30 *g* of protein and at least 20 *g* of fat per feeding period. These nutrients come from food *A*, which costs 18 cents per unit and supplies 2 *g* of protein and 4 *g* of fat; and food *B*, which costs 12 cents per unit and supplies 6 *g* of protein and 2 *g* of fat. Food *B* is bought under a long-term contract requiring that at least 2 units of *B* be used per serving.

How much of each food must be bought to produce the minimum cost per serving?

<u>Solution</u>

	Food A	***Food B***		***Total***
	x	***y***		
Proteins	2	6	$\geq$	30
Fat	4	2	$\geq$	20
Long-Term Contract		1	$\geq$	2
Cost	18	12		

$$\begin{aligned} &\textit{Minimize} & z &= 0.18x + 0.12y && (1)\\ &\textit{Subject to} & 2x+6y &\geq 30 && (2)\\ && 4x+2y &\geq 20 && (3)\\ && y &\geq 2 && (4)\\ && x &\geq 0 && (5) \end{aligned}$$



The corner points are

$(2)\bigcap(3) \quad \rightarrow \quad (3,\ 4)$

$(2)\bigcap(4) \quad \rightarrow \quad (9,\ 2)$

$(3)\bigcap y-axis \quad \rightarrow \quad (0,\ 10)$

Corner points	***z = 0.18x + 0.12y***
(3, 4)	z = 0.18(3) + 0.12(4) = **1.02**
(9, 2)	z = 0.18(9) + 0.12(2) = **1.86**
(0, 10)	z = 0.18(0) + 0.12(10) = **1.20**

Therefore, 3 units of food *A* and 4 units of food *B* will produce the minimum cost of $1.02 per serving.

Exercises *Section* 1.3 – Solving Linear Programming and Applications

1. Maximize and minimize $z = 4x + 2y$ subject to the constraints

$$\begin{cases} 2x + y \le 20 & (1) \\ 10x + y \ge 36 & (2) \\ 2x + 5y \ge 36 & (3) \\ x, y \ge 0 \end{cases}$$

2. A manufacturing plant makes two types of inflatable boats, a two-person boat and a four-person boat. Each two-person boat requires 0.9 labor-hour in the cutting department and 0.8 labor-hour in the assembly department. Each four-person boat requires 1.8 labor-hours in the cutting department and 1.2 labor-hours in the assembly department. The maximum labor-hours available each month in the cutting and assembly departments are 864 and 672, respectively. The company makes a profit of $25 on each two-person boat and $40 on each four-person boat

a) Identify the decision variables
b) Summarize the relevant material in a table
c) Write the objective function P.
d) Write the problem constraints and the nonnegative constraints
e) Determine how many boats should be manufactured each month to maximize the profit. What is the maximum profit?

3. A chicken farmer can buy a special food mix A at 20¢ per pound and a special food mix B at 40¢ per pound. Each pound of mix A contains 3,000 units of nutrient N_1 and 1,000 units of nutrient N_2, and Each pound of mix B contains 4,000 units of nutrient N_1 and 4,000 units of nutrient N_2. If the minimum daily requirements for the chickens collectively are 36,000 units of nutrient N_1 and 20,000 units of nutrient N_2, how many pounds of each food mix should be used each day to minimize daily food costs while meeting (or exceeding) the minimum daily nutrient requirements? What is the minimum daily cost? Construct a mathematical model and solve using the geometric method.

4. A company produces small engines for several manufacturers. The company receives orders from two assembly plants for their Top-flight engine. Plant I needs at least 45 engines, and plant II needs at least 32 engines. The company can send at most 90 engines to these two assembly plants. It costs $30 per engine to ship to plant I and $40 per engine to ship to plant II. Plant I gives the company $20 in rebates toward its products for each engine they buy, while plant II gives similar $15 rebates. The company estimates that they need at least $1200 in rebates to cover products they plan to buy from the two plants. How many engines should be shipped to each plant to minimize shipping costs? What is the minimum cost?

5. The Muro Manufacturing Company makes two kinds of plasma screen TV sets. It produces the Flexscan set that sells for $350 profit and the Panoramic I that sells for $500 profit. On the assembly line, the Flexscan requires 5 hours, and the Panoramic I takes 7 hours. The cabinet shop spends 1 hour on the cabinet for the Flexscan and 2 hours on the cabinet for the Panoramic I. Both sets require 4 hours for testing and packing. On a particular production run, the Muro Company has available 3600 work-hours on the assembly line, 900 work-hours in the cabinet shop, and 2600 work-hours in the testing and packing department. How many sets of each type should it produce to make a maximum profit? What is the maximum profit?

6. The manufacturing process requires that oil refineries must manufacture at least 2 gal of gasoline for every gallon of fuel oil. To meet the winter demand for fuel oil, at least 3 million gal a day must be produced. The demand for gasoline is no more than 6.4 million gal per day. It takes 0.25 hour to ship each million gal of gasoline and 1 hour to ship each million gal of fuel oil out of the warehouse. No more than 4.65 hours are available for shipping. If the refinery sells gasoline for $2.50 per gal and fuel oil for $2 per gal, how many of each should be produced to maximize revenue? Find the maximum revenue.

7. A small country can grow only two crops for export, coffee and cocoa. The country has 500,000 hectares of land available for the crops. Long-term contracts require that at least 100,000 hectares be devoted to coffee and at least 200,000 hectares to cocoa. Cocoa must be processed locally, and production bottlenecks limit cocoa to 270,000 hectares. Coffee requires two workers per hectare, with cocoa requiring five. No more than 1,750,000 people are available for working with these corps. Coffee produces a profit of $220 per hectares and cocoa a profit of 4550 per hectare. How many hectares should the country devote to each crop in order to maximize profit? Find the maximum profit.

8. A pension fund manager decides to invest a total of at most $39 million in U.S. treasury bonds paying 4% annual interest and in mutual funds paying 8% annual interest. He plans to invest at least $5 million in bonds and at least $10 million in mutual funds. Bonds have an initial fee of $100 per million dollars, while the fee for mutual funds is $200 per million. The fund manager is allowed to spend no more than $5000 on fees. How much should be invested in each to maximize annual interest? What is the maximum annual interest?

9. Mark, who is ill, takes vitamin pills. Each day he must have at least 16 units of vitamin A, 5 units of vitamin B, and 20 units of vitamin C. he can choose between pill 1, which contains 8 units of A, 1 of B, and 2 of C; and pill 2, which contains 2 units of A, 1 of B, and 7 of C. Pill 1 costs 15¢, and pill 2 costs 30¢.

a) How many of each pill should be buy in order to minimize his cost?

b) What is the minimum cost?

c) For the solution in part a, Mark is receiving more than he needs of at least one vitamin. Identify that vitamin, and tell how much surplus he is receiving. Is there any ways he can avoid receiving that surplus while still meeting the other constraints and minimizing the cost?

10. A certain predator requires at least 10 units of protein and 8 units of fat per day. One prey of species I provides 5 units of protein and 2 units of fat; one prey of species II provides 3 units of protein and 4 units of fat. Capturing and digesting each species-II prey requires 3 units of energy, and capturing and digesting each species-I prey requires 2 units of energy. How many of each prey would meet the predator's daily food requirements with the least expenditure of energy?

11. A dietician is planning a snack package of fruit and nuts. Each ounce of fruit will supply zero units of protein, 2 units of carbohydrates, and 1 unit of fat, and will contain 20 calories. Each ounce of nuts will supply 3 units of protein, 1 unit of carbohydrates, and 2 units of fat, and will contain 30 calories. Every package must provide at least 6 units of protein, at least 10 units of carbohydrates, and no more than 9 units of fat. Find the number of ounces of fruit and number of ounces of nuts that will meet the requirement with the least number of calories. What is the least number of calories?

12. An anthropology article presents a hypothetical situation that could be described by a linear programming model. Suppose a population gathers plants and animals for survival. They need at least 360 units of energy, 300 units of protein, and 8 hides during some time period. One unit of plants provides 30 units of energy, 10 units of protein, and no hides. One animal provides 20 units of energy, 25 units of protein, and 1 hide.

13. In a small town in South Carolina, zoning rules require that the window space (in square feet) in a house be at least one-sixth of the space used up by solid walls. The cost to build windows is $10 per ft^2, while the cost to build solid walls is $20 per ft^2. The total amount available for building walls and windows is no more than $12,000. The estimated monthly cost to heat the house is $0,32 for each square foot of windows and $0.20 for each square foot of solid walls. Find the maximum total area (windows plus walls) if no more than $160 per month is available to pay for heat.

14. A manufacturing company makes two types of water skis, a trick ski and a slalom ski. The trick ski requires 9 labor-hours for fabricating and 1 labor-hour for finishing. The slalom ski requires 5 labor-hours for fabricating and 1 labor-hour for finishing. The maximum labor-hours available per day for fabricating and finishing are 135 and 20 respectively. If x is the number of trick skis and y is the number of slalom skis produced per day, write a system of linear inequalities that indicates appropriate restraints on x and y. Find the set of feasible solutions graphically for the number of each type of ski that can be produced.

Section 1.4 – Slack Variables and the Pivot

http:/www.zweigmedia.com/RealWorld/simplex.html

http://people.richland.edu/james/ictcm/2006/pivot.html

A linear programming problem is in standard maximum form if the following conditions are satisfied

1. The objective function is to be maximized
2. All variables are nonnegative $(x_i \ge 0)$
3. All remaining constraints are stated in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \le b \qquad \text{with } b \ge 0$$

$$\begin{array}{ll} \textit{Maximize} & z = 3x_1 + 2x_2 + x_3 \\ \textit{subject to} & 2x_1 + x_2 + x_3 \le 150 \\ & x_1 + 2x_2 + 8x_3 \le 200 \\ & 2x_1 + 3x_2 + x_3 \le 320 \\ \textit{with} & x_1, x_2, x_3 \ge 0 \end{array}$$

Slack Variables

Example

$$\begin{array}{ll} \textit{Maximize} & z = 3x_1 + 2x_2 + x_3 \\ \textit{subject to} & 2x_1 + x_2 + x_3 + s_1 \qquad\qquad = 150 \\ & x_1 + 2x_2 + 8x_3 \quad + s_2 \quad\; = 200 \\ & 2x_1 + 3x_2 + x_3 \qquad\quad + s_3 = 320 \\ \textit{with} & x_1, x_2, x_3, s_1, s_2, s_3 \ge 0 \end{array}$$

The variables s_1, s_2, and s_3 are called ***slack variables*** because each makes up the difference between the left or right sides of an inequality system (takes up any slack).

The objective function may be rewritten as: $z - 3x_1 - 2x_2 - x_3 = 0$

The equations can be written as the following augmented matrix

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 2 & 1 & 1 & 1 & 0 & 0 & 0 & 150 \\ 1 & 2 & 8 & 0 & 1 & 0 & 0 & 200 \\ 2 & 3 & 1 & 0 & 0 & 1 & 0 & 320 \\ \hline -3 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \rightarrow \textit{Indicators}$$

This matrix is called the initial ***simplex tableau***. The numbers in the bottom row are called ***indicators***.

Standard Maximization Problems in standard Form

Maximize: $z = 50x_1 + 80x_2$

Subject to

$$x_1 + 2x_2 \le 32$$
$$3x_1 + 4x_2 \le 84$$
$$x_1, x_2 \ge 0$$

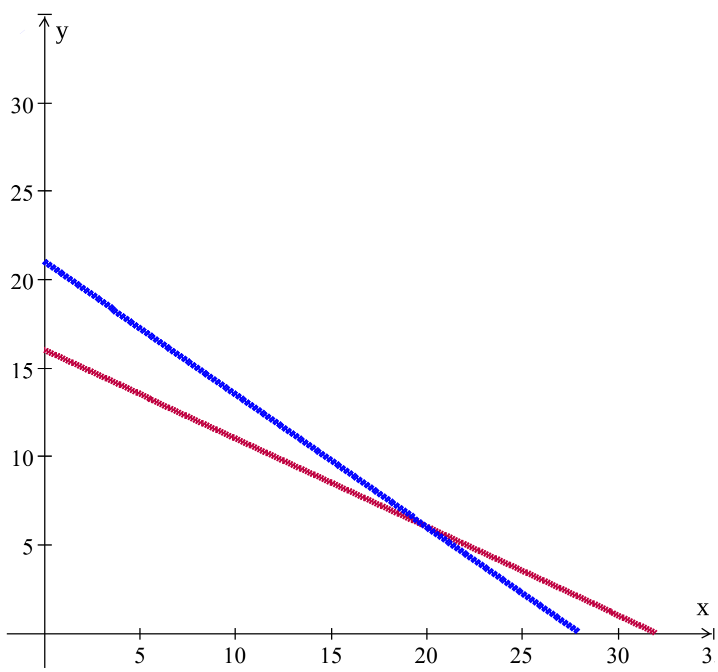
Slack variables

Convert the constraint inequalities to linear equations by using a simple device called a slack variable.

$$x_1 + 2x_2 + s_1 \quad = 32$$
$$3x_1 + 4x_2 \quad + s_2 = 84$$

The variables s_1 and s_2 are called ***slack variables*** because each makes up the difference between the left or right sides of an inequality system.

x_1	x_2	s_1	s_2	Intersection Point	**Feasible**
0	0	32	84	(0, 0)	Yes
0	16	0	20	(0, 16)	Yes
0	21	-10	0	(0, 21)	No
32	0	0	-12	(32, 0)	No
28	0	4	0	(28, 0)	Yes
20	6	0	0	(20, 6)	Yes



The Initial Simplex Tableau {*A tableau is just a special augmented matrix*}

When creating the Initial tableau, enter the coefficients of the variables just like we do with Gauss-Jordan method. Label the columns with the appropriate variables. To identify the basic variables, look for the columns that have a 1 and the rest of the entries are 0. Beside the row that has the 1.

Initial Simplex Tableau

$$\begin{array}{c} \begin{array}{ccccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & & \end{array} \\ \left[\begin{array}{rrrrrrr|r} 2 & 1 & 1 & 1 & 0 & 0 & 0 & 150 \\ 1 & 2 & 8 & 0 & 1 & 0 & 0 & 200 \\ 2 & 3 & 1 & 0 & 0 & 1 & 0 & 320 \\ \hline -3 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Basic Variables

s_1, s_2, s_3, and P are ***basic*** variables.

That means that x_1, x_2, and x_3 are ***nonbasic*** (at this juncture) $\Rightarrow$ x_1, x_2, and $x_3 = 0$.

If you write the corresponding system from the tableau with x_1, x_2, and $x_3 = 0$, you get the following.

$$s_1 = 150 \quad s_2 = 200 \quad s_3 = 320 \quad P = 0$$

So the ***basic feasible solution*** at this point is:

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ s_1 = 150,\ s_2 = 200,\ s_3 = 320,\ P = 0$$

Determining the pivot element

Pivot Column:

Look at the elements on the bottom row to the left of the P column. The value that is most negative identifies the pivot column.

$$\begin{array}{c} \begin{array}{ccccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P & & \end{array} \\ \left[\begin{array}{rrrrrrr|r} 2 & 1 & 1 & 1 & 0 & 0 & 0 & 150 \\ 1 & 2 & 8 & 0 & 1 & 0 & 0 & 200 \\ 2 & 3 & 1 & 0 & 0 & 1 & 0 & 320 \\ \hline -3 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

(-3) ***column* 1** is the pivot column.

Pivot Row:

Divide each positive number above the most negative indicator into the corresponding constants at the far right. The smallest quotient indicates the pivot row.

$$\begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P & \end{array} \\ \left[\begin{array}{ccccccc|c} 2 & 1 & 1 & 1 & 0 & 0 & 0 & 150 \\ 1 & 2 & 8 & 0 & 1 & 0 & 0 & 200 \\ 2 & 3 & 1 & 0 & 0 & 1 & 0 & 320 \\ \hline -3 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} \frac{150}{2} = 75 \leftarrow \\ \frac{200}{1} = 200 \\ \frac{320}{2} = 160 \end{array}$$

***Row* 1** is the pivot row

The Pivoting Process:

After identified the pivot element, the pivot process can begin.
First, make the pivot element become a 1 by using the appropriate row operation, then use the 1 to eliminate all other elements in its column using Gauss-Jordan elimination.
We are pivoting on the column 1, row 1.

First Pivot

$$\begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P & \end{array} \\ \left[\begin{array}{ccccccc|c} 2 & 1 & 1 & 1 & 0 & 0 & 0 & 150 \\ 1 & 2 & 8 & 0 & 1 & 0 & 0 & 200 \\ 2 & 3 & 1 & 0 & 0 & 1 & 0 & 320 \\ \hline -3 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} \frac{1}{2}R_1 \\ \\ \\ \\ \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P & \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & .5 & .5 & .5 & 0 & 0 & 0 & 75 \\ 1 & 2 & 8 & 0 & 1 & 0 & 0 & 200 \\ 2 & 3 & 1 & 0 & 0 & 1 & 0 & 320 \\ \hline -3 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 + 3R_1 \end{array}$$

$$\begin{array}{c} \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P & \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & .5 & .5 & .5 & 0 & 0 & 0 & 75 \\ 0 & 1.5 & 7.5 & -.5 & 1 & 0 & 0 & 125 \\ 0 & 2 & 0 & -1 & 0 & 1 & 0 & 170 \\ \hline 0 & -.5 & .5 & 1.5 & 0 & 0 & 1 & 225 \end{array}\right] \end{array}$$

So the basic feasible solution at this point is:

$x_1 = 75,\ x_2 = 0,\ x_3 = 0,\ s_1 = 0,\ s_2 = 125,\ s_3 = 170,\ P = 225$

Second Pivot

$$\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P \end{array} \\
\left[\begin{array}{ccccccc|c}
1 & .5 & .5 & .5 & 0 & 0 & 0 & 75 \\
0 & 1.5 & 7.5 & -.5 & 1 & 0 & 0 & 125 \\
0 & 2 & 0 & -1 & 0 & 1 & 0 & 170 \\ \hline
0 & -.5 & .5 & 1.5 & 0 & 0 & 1 & 225
\end{array}\right]
\end{array}
\quad
\begin{array}{l}
\frac{75}{.5}=150 \\
\frac{125}{1.5}=83.3 \\
\frac{170}{2}=85
\end{array}$$

$$\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P \end{array} \\
\left[\begin{array}{ccccccc|c}
1 & .5 & .5 & .5 & 0 & 0 & 0 & 75 \\
0 & 1 & 5 & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & \frac{250}{3} \\
0 & 2 & 0 & -1 & 0 & 1 & 0 & 170 \\ \hline
0 & -.5 & .5 & 1.5 & 0 & 0 & 1 & 225
\end{array}\right]
\end{array}
\quad
\begin{array}{l}
R_1 - .5R_2 \\
\\
R_3 - 2R_2 \\
R_4 + .5R_2
\end{array}$$

$$\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & P \end{array} \\
\left[\begin{array}{ccccccc|c}
1 & 0 & -2 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & \frac{100}{3} \\
0 & 1 & 5 & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & \frac{250}{3} \\
0 & 0 & -10 & -\frac{1}{3} & -\frac{4}{3} & 1 & 0 & \frac{10}{3} \\ \hline
0 & 0 & 3 & \frac{4}{3} & \frac{1}{3} & 0 & 1 & \frac{800}{3}
\end{array}\right]
\end{array}$$

Exercises *Section* 1.4 – Slack Variables and the Pivot

1. Write the initial simplex tableau for each linear programing problem

a) *Maximized* : $z = 7x_1 + x_2$

subject to : $4x_1 + 2x_2 \le 5$

$x_1 + 2x_2 \le 4$

$x_1, x_2 \ge 0$

b) *Maximized* : $z = x_1 + 3x_2$

subject to : $2x_1 + 3x_2 \le 100$

$5x_1 + 4x_2 \le 200$

$x_1, x_2 \ge 0$

c) *Maximized* : $z = x_1 + 3x_2$

subject to : $x_1 + x_2 \le 10$

$5x_1 + 2x_2 \le 4$

$x_1 + 2x_2 \le 36$

$x_1, x_2 \ge 0$

d) *Maximized* : $z = 5x_1 + 3x_2$

subject to : $x_1 + x_2 \le 25$

$4x_1 + 3x_2 \le 48$

$x_1, x_2 \ge 0$

2. Pivot once as indicated in each simplex tableau. Read the solution from the result

a)

$$\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\
\left[\begin{array}{cccccc|c}
1 & 2 & 4 & 1 & 0 & 0 & 56 \\
2 & \{2\} & 1 & 0 & 1 & 0 & 40 \\ \hline
-1 & -3 & -2 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}$$

b)

$$\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\
\left[\begin{array}{cccccc|c}
2 & 3 & 4 & 1 & 0 & 0 & 18 \\
6 & \{3\} & 2 & 0 & 1 & 0 & 15 \\ \hline
-1 & -6 & -2 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}$$

c)

$$\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array} \\
\left[\begin{array}{ccccccc|c}
2 & 2 & \{1\} & 1 & 0 & 0 & 0 & 12 \\
1 & 2 & 3 & 0 & 1 & 0 & 0 & 45 \\
3 & 1 & 1 & 0 & 0 & 1 & 0 & 20 \\ \hline
-2 & -1 & -3 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}$$

d)

$$\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array} \\
\left[\begin{array}{ccccccc|c}
2 & \{2\} & 3 & 1 & 0 & 0 & 0 & 500 \\
4 & 1 & 1 & 0 & 1 & 0 & 0 & 300 \\
7 & 2 & 4 & 0 & 0 & 1 & 0 & 700 \\ \hline
-3 & -4 & -2 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}$$

3. The authors of a best-selling textbook in finite mathematics are told that, for the next edition of their book, each simple figure would cost the project \$20, each figure with additions would cost \$35, and each computer-drawn sketch would cost \$60. They are limited to 400 figures, for which they are allowed to spend up to \$2200. The number of computer-drawn sketches must be no more than the number of the other two types combined, and there must be at least twice as many simple figures as there are figures with additions. If each simple figure increases the royalties by \$95, each figure with additions increases royalties by \$200, and each computer-drawn figure increases royalties by \$325, how many of each type of figure should be included to maximize royalties, assuming that all art costs are borne by the publisher?

4. A manufacturer of bicycles builds racing, touring, and mountain models. The bicycles are made of both aluminum and steel. The company has available 91,800 units of steel and 42,000 units of aluminum. The racing, touring, and mountain models need 17, 27, and 34 units steel, and 12, 21, and 15 units of aluminum respectively. How many of each type of bicycle should be made in order to maximize profit if the company makes \$8 per racing bike, \$12 per touring bike, and \$22 per mountain bike? What is the maximum possible profit?

Section 1.5 – Maximization Problems *with constraints of the form* ≤

Example

A farmer has 100 acres of available land on which he wishes to plant a mixture of potatoes, corn, and cabbage. It costs him \$400 to produce an acre of potatoes, \$160 to produce an acre of corn, and \$280 to produce an acre of cabbage. He has a maximum of \$20,000 to spend. He makes a profit of \$120 per acre of potatoes, \$40 per acre of corn, and \$60 per acre of cabbage. How many acres of each crop should he plant to maximize his profit?

Solution

	Potatoes	*Corn*	*Cabbage*		*Total*
Number of acres	x_1	x_2	x_3	≤	100
Cost (per acre)	\$400	\$160	\$280	≤	\$20,000
Profit (per acre)	\$120	\$40	\$60		

Maximize $P = 120x_1 + 40x_2 + 60x_3$ (1)

subject to $x_1 + x_2 + x_3 \le 100$ (2)

$400x_1 + 160x_2 + 280x_3 \le 20,000$ (3)

with $x_1, x_2, x_3 \ge 0$

Divide **(3)** by 40: $\frac{1}{40}(3) \Rightarrow 10x_1 + 4x_2 + 7x_3 \le 500$

$$\begin{aligned} P - 120x_1 - 40x_2 - 60x_3 &= 0 && (1)\\ x_1 + x_2 + x_3 + s_1 &= 100 && (2)\\ 10x_1 + 4x_2 + 7x_3 + s_2 &= 500 && (3)\\ x_1, x_2, x_3, s_1, s_2 &\ge 0 \end{aligned}$$

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & P \end{array}$$
$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 100\\ 10 & 4 & 7 & 0 & 1 & 0 & 500\\ \hline -120 & -40 & -60 & 0 & 0 & 1 & 0 \end{array}\right]$$

So the ***basic feasible solution*** at this point is:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ s_1 = 100,\ s_2 = 500,\ P = 0$

First Pivot

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & P & & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 100 \\ 10 & 4 & 7 & 0 & 1 & 0 & 500 \\ \hline -120 & -40 & -60 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} \frac{100}{1} = 100 \\ \frac{500}{10} = 50 \leftarrow \\ \end{array}$$

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & P & & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 100 \\ 10 & 4 & 7 & 0 & 1 & 0 & 500 \\ \hline -120 & -40 & -60 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \frac{1}{10} R_2$$

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & P & & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 100 \\ 1 & .4 & .7 & 0 & .1 & 0 & 50 \\ \hline -120 & -40 & -60 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 120R_2 \end{array}$$

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & P & & \end{array} \\ \left[\begin{array}{cccccc|c} 0 & .6 & .3 & 1 & -.1 & 0 & 50 \\ 1 & .4 & .7 & 0 & 1 & 0 & 50 \\ \hline 0 & 8 & 24 & 0 & 12 & 1 & 6000 \end{array}\right] \end{array}$$

So the ***basic feasible solution*** at this point is:

$$x_1 = 50, \; x_2 = 0, \; x_3 = 0, \; s_1 = 50, \; s_2 = 0, \; P = 6000$$

The farmer will make a maximum of $6,000 by planning 50 acres of potatoes, no acres of corn, and no acres of cabbage.

Simplex Method

1. Determine the objective constraints
2. Write all the necessary constraints
3. Convert each constraint into an equation by adding a slack variable in each.
4. Set up the initial simplex tableau.
5. Locate the most negative indicator. If there are two such indicators, choose the one farther to the left.
6. Form the necessary quotients to find the pivot. Disregard any quotients with 0 or a negative number in the denominator. The smallest nonnegative quotient gives the location of the pivot. If all quotients must be disregarded, no maximum solution exists. If two quotients are both equal and smallest, choose the pivot in the row nearest the top of the matrix.
7. Use the row operations to change all other numbers in the pivot column to zero by adding a suitable multiple of the pivot row to a positive multiple of each row.
8. If the indicators are all positive or 0, this is the final tableau.
9. Read the solution from the final tableau

Example

Solve the simplex method:

$$\begin{aligned} &\textit{Maximize}: && P = 25x_1 + 30x_2 \\ &\textit{subject to}: && x_1 + x_2 \le 65 \\ & && 4x_1 + 5x_2 \le 300 \\ &\textit{with} && x_1, x_2 \ge 0 \end{aligned}$$

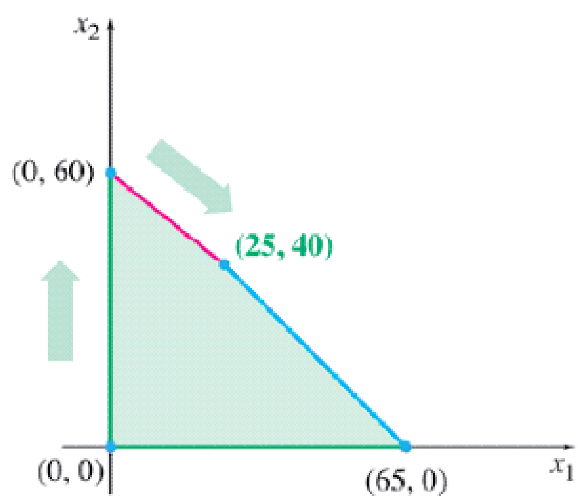
The initial tableau

$$\begin{array}{cl} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} & \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 65 \\ 4 & 5 & 0 & 1 & 0 & 300 \\ \hline -25 & -30 & 0 & 0 & 1 & 0 \end{array}\right] & \begin{array}{l} \frac{65}{1}=65 \\ \frac{300}{5}=60 \\ \end{array} \end{array}$$

$$\begin{array}{cl} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} & \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 65 \\ 4 & 5 & 0 & 1 & 0 & 300 \\ \hline -25 & -30 & 0 & 0 & 1 & 0 \end{array}\right] & \begin{array}{l} \\ \frac{1}{5}R_2 \\ \end{array} \end{array}$$

$$\begin{array}{cl} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} & \\ \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 65 \\ .8 & 1 & 0 & .2 & 0 & 60 \\ \hline -25 & -30 & 0 & 0 & 1 & 0 \end{array}\right] & \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 30R_2 \end{array} \end{array}$$

$$\begin{array}{cl} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} & \\ \left[\begin{array}{ccccc|c} .2 & 0 & 1 & -.2 & 0 & 5 \\ .8 & 1 & 0 & .2 & 0 & 60 \\ \hline -1 & 0 & 0 & 6 & 1 & 1800 \end{array}\right] & \begin{array}{l} \frac{1}{.2}R_1 \\ \\ \end{array} \end{array}$$



$$\begin{array}{cl} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} & \\ \left[\begin{array}{ccccc|c} 1 & 0 & 5 & -1 & 0 & 25 \\ .8 & 1 & 0 & .2 & 0 & 60 \\ \hline -1 & 0 & 0 & 6 & 1 & 1800 \end{array}\right] & \begin{array}{l} \\ R_2 - .8R_1 \\ R_3 + R_1 \end{array} \end{array}$$

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & P & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 5 & -1 & 0 & 25 \\ 0 & 1 & -4 & 1 & 0 & 40 \\ \hline 0 & 0 & 5 & 5 & 1 & 1825 \end{array}\right] \end{array}$$

The solution is $x_1 = 25$ and $x_2 = 40$ with $P = 1825$

Example

Find the pivot for the following initial simplex tableau

$$\begin{array}{cccccc}
x_1 & x_2 & s_1 & s_2 & s_3 & z
\end{array}$$

$$\left[\begin{array}{cccccc|c}
1 & -2 & 1 & 0 & 0 & 0 & 100 \\
3 & 4 & 0 & 1 & 0 & 0 & 200 \\
5 & 0 & 0 & 0 & 1 & 0 & 150 \\ \hline
-10 & -25 & 0 & 0 & 0 & 1 & 0
\end{array}\right]$$

The most negative indicator is -25, the pivot column is 2

Pivot row: $\frac{100}{-2}, \ \frac{200}{4}, \ \frac{150}{0}$

Variables have to be nonnegative, and no zero in denominator, the only usable quotient is $\frac{200}{4}$

The pivot is (4) column 2 , row 2

Exercises Section 1.5 – Maximization Problems *with constraints of the form* $\le$

1. Solve the simplex method:

Maximize: $P = 50x_1 + 80x_2$

Subject to
$$\begin{aligned} x_1 + 2x_2 &\le 32 \\ 3x_1 + 4x_2 &\le 84 \\ x_1, x_2 &\ge 0 \end{aligned}$$

2. Solve the simplex method:

Maximize: $P = 2x_1 + 3x_2$

Subject to: $\begin{cases} -3x_1 + 4x_2 \le 12 \\ x_2 \le 2 \\ x_1, x_2 \ge 0 \end{cases}$

3. Solve the simplex method:

Maximize: $P = 2x_1 + x_2$

Subject to: $\begin{cases} 5x_1 + x_2 \le 9 \\ x_1 + x_2 \le 5 \\ x_1, x_2 \ge 0 \end{cases}$

4. The initial tableau of a linear programing is given. Use the simplex method to solve it.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 2 & 4 & 1 & 0 & 0 & 8 \\ 5 & 8 & 1 & 0 & 1 & 0 & 10 \\ \hline -3 & -24 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

5. Carrie is working to raise money for the homeless by sending information letters and making follow-up calls to local labor organizations and church groups. She discovers that each church group requires 2 hours of letter writing and 1 hour of follow-up, while for each labor union she needs 2 hours of letter writing and 3 hours of follow-up. Carrie can raise \$100 from each church group and \$200 from each union local, and she has a maximum of 16 hours of letter-writing time and a maximum of 12 hours of follow-up time available per month. Determine the most profitable mixture of groups she should contact and the most money she can raise in a month.

6. The Texas Poker Company assembles three different poker sets. Each Royal Flush poker set contains 1000 poker chips, 4 decks of cards, 10 dice, and 2 dealer buttons. Each Deluxe Diamond poker set contains 600 poker chips, 2 decks of cards, 5 dice, and one dealer button. The full House poker set contains 300 poker chips, 2 decks of cards, 5 dice, and one dealer button. The Texas Poker Company has 2,800,000 poker chips, 10,000 decks of cards, 25,000 dice, and 6000 dealer buttons in stock. They earn a profit of $38 for each Royal Flush poker set, $22 for each Deluxe Diamond poker set, and $12 for each Full House poker set.

a) How many of each type of poker set should they assemble to maximize profit? What is the maximum profit?

b) Find the values of any nonzero slack variables and describe what they tell you about any unused components.

7. The Muro Manufacturing Company makes two kinds of plasma screen TV sets. It produces the Flexscan set that sells for $350 profit and the Panoramic I that sells for $500 profit. On the assembly line, the Flexscan requires 5 hours, and the Panoramic I takes 7 hours. The cabinet shop spends 1 hour on the cabinet for the Flexscan and 2 hours on the cabinet for the Panoramic I. Both sets require 4 hours for testing and packing. On a particular production run, the Muro Company has available 3600 work-hours on the assembly line, 900 work-hours in the cabinet shop, and 2600 work-hours in the testing and packing department.

a) How many sets of each type should it produce to make a maximum profit? What is the maximum profit?

b) Find the values of any nonzero slack variables and describe what they tell you about unused time.

8. A baker has 150 units of flour, 90 of sugar, and 150 of raisins. A loaf of raisin bread requires 1 unit of flour, 1 of sugar, and 2 of raisins, while a raisin cake needs 5, 2, and 1 units, respectively.

a) If raisin bread sells for $1.75 a loaf and raisin cake for $4.00 each, how many of each should baked so that gross income is maximized?

b) What is the maximum gross income?

c) Does it require all of the available units of flour, sugar, and raisins to produce the number the maximum profit? If not, how much of each ingredient is left over? Compare any leftover to the value of the relevant slack variable.

9. A farmer owns a 100 acre farm and plans to plant at most three crops. The seed for crops A, B, and C costs $24, $40, and $30 per acre, respectively. A maximum of $3,600 can be spent on seed. Crops A, B, and C require 1, 2, and 2 workdays per acre, respectively, and three are a maximum of 160 workdays available. If the farmer can make a profit of $140 per acre on crop A, $200 per acre on crop B, and $160 per acre on crop C, how many acres of each crop that should be planted to maximize the profit?

10. A candy company makes three types of candy, solid, fruit, and cream filled, and packages these candies in three different assortments. A box of assortment I contains 4 solid, 4 fruit, and 12 cream and sells for $9.40. A box of assortment II contains 12 solid, 4 fruit, and 4 cream and sells for $7.60. A box of assortment III contains 8 solid, 8 fruit, and 8 cream and sells for $11.00. The manufacturing costs per piece of candy are $0.20 for solid, $0.25 for fruit, and $0.30 for cream. The company can manufacture up to 4800 solid, 4000 fruit, and 5600 cream candies weekly. How many boxes of each type should the company produce in order to maximize profit? What is their maximum profit?

11. A small company manufactures three different electronic components for computers. Component A requires 2 hours of fabrication and 1 hour of assembly; component B requires 3 hours of fabrication and 1 hour of assembly; and component C requires 2 hours of fabrication and 2 hours of assembly. The company has up to 1,000 labor-hours of fabrication time and 800 labor-hours of assembly time available per week. The profit on each component, A, B, and C is $7, $8, and $10, respectively. How many components of each week in order to maximize its profit (assuming that all components that it manufactures can be sold)? What is the maximum profit?

12. An investor has at most $100,000 to invest in government bonds, mutual funds, and money market funds. The average yields for government bonds, mutual funds, and money market funds are 8%, 13%,, and 15%, respectively. The investor's policy requires that the total amount invested in mutual and money market funds not exceed the amount invested in government bonds. How much should be invested in each type of investment in order to maximize the return? What is the maximum return?

13. A department store chain up to $20,000 to spend on television advertising for a sale. All ads will be placed with one television station, where 30-second as costs $1,000 on daytime TV and is viewed by 14,000 potential customers, $2000 on prime-time TV and is viewed by 24,000 potential customer, and $1,500 on late-night TV and is viewed by 18,000 potential customers. The television station will not accept a total of more than 15 ads in all three time periods. How many ads should be placed in each time period in order to maximize the number of potential customers who will see the ads? How many potential customers will see the ads?

14. A political scientist has received a grant to find a research project involving voting trends. The budget of the grant includes $3,200 for conducting door-to-door interviews the day before an election. Undergraduate students, graduate students, and faculty members will be hired to conduct the interviews. Each undergraduate student will conduct 18 interviews and be paid $100. Each graduate student will conduct 25 interviews and be paid $150. Each faculty members will conduct 30 interviews and be paid $200. Due to limited transportation facilities, no more than 20 interviews can be hired. How many undergraduate students, graduate students, and faculty members should be hired in order to maximize the number of interviews that will be conducted? What is the maximum number of interviews?

Section 1.6 – Minimization Problems $\geq$ (*Duality*)

The simplex method can only be used if the problem is as a standard maximization problem (problem constraints of the form ≤). With standard minimization problems, you must be minimizing the objective function, subject to constraints that are ≥ any real number. {You don't have the nonnegative restriction for the constants as you did with standard maximization.}

Formation of the Dual Problem

$$Minimize: \ C = 16x_1 + 45x_2$$

$$Subject\ to: \begin{cases} 2x_1 + 5x_2 \geq 50 \\ x_1 + 3x_2 \geq 27 \\ x_1, x_2 \geq 0 \end{cases}$$

The Coefficient Matrix

$$A = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & \end{array} \\ \left[\begin{array}{cc|c} 2 & 5 & 50 \\ 1 & 3 & 27 \\ \hline 16 & 45 & 1 \end{array}\right] \end{array}$$

Transpose

$$A^T = \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & \end{array} \\ \left[\begin{array}{cc|c} 2 & 1 & 16 \\ 5 & 3 & 45 \\ \hline 50 & 27 & 1 \end{array}\right] \end{array}$$

$$\begin{aligned} 2y_1 + y_2 &\leq 16 \\ 5y_1 + 3y_2 &\leq 45 \\ 50y_1 + 27y_2 &= P \end{aligned}$$

Dual Problem

$$\begin{array}{ll} Maximize & P = 50y_1 + 27y_2 \\ Subject\ to & 2y_1 + y_2 \leq 16 \\ & 5y_1 + 3y_2 \leq 45 \\ & y_1, y_2 \geq 0 \end{array}$$

Formation of the Dual Problem

Given a minimization problem with $\geq$ problem constraints,

- *Use the coefficients and constants in the problem constraints and the objective function to form a matrix A with the coefficients of the objective function in the last row*

- *Interchange the rows and columns of matrix A to form the matrix* A^T *, the transpose of A.*

- *Use the rows of* A^T *to form a maximization problem with* $\leq$ *problem constraints.*

Example

Solve the following minimization problem by maximizing the dual:

$$\begin{aligned} &Minimize: \ C = 16x_1 + 9x_2 + 21x_3 \\ &Subject\ to: \ x_1 + x_2 + 3x_3 \geq 12 \\ &\qquad 2x_1 + x_2 + x_3 \geq 16 \\ &\qquad x_1, x_2, x_3 \geq 0 \end{aligned}$$

<u>*Solution*</u>

$$A = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 12 \\ 2 & 1 & 1 & 16 \\ \hline 16 & 9 & 21 & 1 \end{array}\right] \qquad A^T = \left[\begin{array}{cc|c} 1 & 2 & 16 \\ 1 & 1 & 9 \\ 3 & 1 & 21 \\ \hline 12 & 16 & 1 \end{array}\right]$$

The dual problem is:

$$Maximize: \ P = 12y_1 + 16y_2$$

$$Subject\ to: \begin{cases} y_1 + 2y_2 \leq 16 \\ y_1 + y_2 \leq 9 \\ 3y_1 + y_2 \leq 21 \\ y_1, y_2 \geq 0 \end{cases}$$

$$\begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & x_3 & P \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 2 & 1 & 0 & 0 & 0 & 16 \\ 1 & 1 & 0 & 1 & 0 & 0 & 9 \\ 3 & 1 & 0 & 0 & 1 & 0 & 21 \\ \hline -12 & \langle -16 \rangle & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \frac{1}{2}R_1$$

$$
\begin{array}{c}
\begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & x_3 & P \end{array} \\
\left[\begin{array}{cccccc|c}
.5 & 1 & .5 & 0 & 0 & 0 & 8 \\
1 & 1 & 0 & 1 & 0 & 0 & 9 \\
3 & 1 & 0 & 0 & 1 & 0 & 21 \\
\hline
-12 & -16 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
\quad R_2 - R_1
\qquad
\begin{array}{ccccccc}
1 & 1 & 0 & 1 & 0 & 0 & 9 \\
-.5 & -1 & -.5 & 0 & 0 & 0 & -8 \\
\hline
.5 & 0 & -.5 & 1 & 0 & 0 & 1
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & x_3 & P \end{array} \\
\left[\begin{array}{cccccc|c}
.5 & 1 & .5 & 0 & 0 & 0 & 8 \\
.5 & 0 & -.5 & 1 & 0 & 0 & 1 \\
3 & 1 & 0 & 0 & 1 & 0 & 21 \\
\hline
-12 & -16 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
\quad R_3 - R_1
\qquad
\begin{array}{ccccccc}
3 & 1 & 0 & 0 & 1 & 0 & 21 \\
-.5 & -1 & -.5 & 0 & 0 & 0 & -8 \\
\hline
2.5 & 0 & -.5 & 0 & 1 & 0 & 13
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & x_3 & P \end{array} \\
\left[\begin{array}{cccccc|c}
.5 & 1 & .5 & 0 & 0 & 0 & 8 \\
.5 & 0 & -.5 & 1 & 0 & 0 & 1 \\
3 & 1 & 0 & 0 & 1 & 0 & 21 \\
\hline
-12 & -16 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
\quad R_4 + 16R_1
\qquad
\begin{array}{ccccccc}
-12 & -16 & 0 & 0 & 0 & 1 & 0 \\
8 & 16 & 8 & 0 & 0 & 0 & 128 \\
\hline
-4 & 0 & 8 & 0 & 0 & 1 & 128
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & x_3 & P \end{array} \\
\left[\begin{array}{cccccc|c}
.5 & 1 & .5 & 0 & 0 & 0 & 8 \\
\langle .5 \rangle & 0 & -.5 & 1 & 0 & 0 & 1 \\
2.5 & 0 & -.5 & 0 & 1 & 0 & 13 \\
\hline
-4 & 0 & 8 & 0 & 0 & 1 & 128
\end{array}\right]
\end{array}
\quad \frac{1}{.5}R_2
$$

$$
\begin{array}{c}
\begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & x_3 & P \end{array} \\
\left[\begin{array}{cccccc|c}
.5 & 1 & .5 & 0 & 0 & 0 & 8 \\
1 & 0 & -1 & 2 & 0 & 0 & 2 \\
2.5 & 0 & -.5 & 0 & 1 & 0 & 13 \\
\hline
-4 & 0 & 8 & 0 & 0 & 1 & 128
\end{array}\right]
\end{array}
\quad
\begin{array}{l}
R_1 - .5R_2 \\
\\
R_3 - 2.5R_2 \\
R_4 + 4R_2
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & x_3 & P \end{array} \\
\left[\begin{array}{cccccc|c}
0 & 1 & 1 & -1 & 0 & 0 & 7 \\
1 & 0 & -1 & 2 & 0 & 0 & 2 \\
0 & 0 & 2 & -5 & 1 & 0 & 8 \\
\hline
0 & 0 & 4 & 8 & 0 & 1 & 136
\end{array}\right]
\end{array}
$$

Min $C = 136$ @ $x_1 = 4,\ x_2 = 8,\ x_3 = 0$

Example

Solve the following minimization problem by maximizing the dual:

$Minimize: \ C = 2x_1 + 3x_2$

$$Subject\ to: \begin{cases} x_1 - 2x_2 \ge 2 \\ -x_1 + x_2 \ge 1 \\ x_1, x_2 \ge 0 \end{cases}$$

<u>Solution</u>

Coefficient Matrix $\quad$ ***Transpose***

$$A = \left[\begin{array}{cc|c} 1 & -2 & 2 \\ -1 & 1 & 1 \\ \hline 2 & 3 & 1 \end{array}\right] \qquad A^T = \left[\begin{array}{cc|c} 1 & -1 & 2 \\ -2 & 1 & 3 \\ \hline 2 & 1 & 1 \end{array}\right]$$

Dual Problem

$Maximize: \ P = 2y_1 + y_2$

$$Subject\ to: \begin{cases} y_1 - y_2 \le 2 \\ -2y_1 + y_2 \le 3 \\ y_1, y_2 \ge 0 \end{cases}$$

$$\begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & x_1 & x_2 & P \end{array} \\ \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 0 & 0 & 2 \\ (-2) & 1 & 0 & 1 & 0 & 3 \\ \hline \langle -2 \rangle & -1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad -\tfrac{1}{2}R_2$$

$$\begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & x_1 & x_2 & P \end{array} \\ \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 0 & 0 & 2 \\ 1 & -.5 & 0 & -.5 & 0 & -1.5 \\ \hline -2 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad R_1 - R_2$$

$$\begin{array}{cccccc} 1 & -1 & 1 & 0 & 0 & 2 \\ -1 & .5 & 0 & .5 & 0 & 1.5 \\ \hline 0 & -.5 & 1 & .5 & 0 & 3.5 \end{array}$$

$$\begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & x_1 & x_2 & P \end{array} \\ \left[\begin{array}{ccccc|c} 0 & -.5 & 1 & .5 & 0 & 3.5 \\ 1 & -.5 & 0 & -.5 & 0 & -1.5 \\ \hline -2 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad R_3 + 2R_2$$

$$\begin{array}{cccccc} -2 & -1 & 0 & 0 & 1 & 0 \\ 2 & -1 & 0 & -1 & 0 & -3 \\ \hline 0 & -2 & 0 & -1 & 1 & -3 \end{array}$$

$$\begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & x_1 & x_2 & P \end{array} \\ \left[\begin{array}{ccccc|c} 0 & -.5 & 1 & .5 & 0 & 3.5 \\ 1 & -.5 & 0 & -.5 & 0 & -1.5 \\ \hline 0 & -2 & 0 & -1 & 1 & -3 \end{array}\right] \end{array} \quad 2R_1$$

$$\begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & P & \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & -2 & -1 & 0 & -7 \\ 1 & -.5 & 0 & -.5 & 0 & -1.5 \\ \hline 0 & -2 & 0 & -1 & 1 & -3 \end{array}\right] \begin{array}{l} \\ R_2 + .5R_1 \\ R_3 + 2R_1 \end{array} \end{array}$$

$$\begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & x_1 & x_2 & P & \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & -2 & -1 & 0 & -7 \\ 1 & 0 & -1 & -1 & 0 & -5 \\ \hline 0 & 0 & -4 & -3 & 1 & -17 \end{array}\right] \end{array}$$

No optimal solution

Caution: For the row associated with the objective function, copy the coefficients exactly as written. You derive the dual problem from the transpose matrix in reverse order, from the way you got the coefficient matrix.

Caution: When writing the dual problem, be sure to do the following:

1) Change Minimize to Maximize and change C to P.
2) Change the x-variables to y-variables.
3) Change $\leq$ constraints to $\geq$ constraints.
4) Don't forget the nonnegative constraints.

Example

A computer manufacturing company has two assembly plants, plant A and plant B, and two distribution outlets, outlet **I** and outlet **II**. Plant A can assemble at most 700 computers a month, and plant B can assemble at most 900 computers a month. Outlet I must have at least 500 computers a month, and outlet II must have at least 1,000 computers a month. Transportation costs for shipping one computer from each plant to each outlet are as follows: \$7 from plant A to outlet I, \$5 from plant A to outlet II, \$4 from plant B to outlet I, \$3 from plant B to outlet II. Find a shipping schedule that will minimize the total cost of shipping the computers from the assembly plants to the distribution outlets. What is this minimum cost?

Solution

	I	**II**		***Assembly***
Plant A	\$7	\$5		
			$\leq$	700
Plant B	\$4	\$3		
			$\leq$	900
Min	500	1000		

Number Shipped: From Plant A: $x_1 + x_2 \le 700$

From Plant B: $x_3 + x_4 \le 900$

From outlet I: $x_1 + x_3 \ge 500$

From outlet II: $x_2 + x_4 \ge 1000$

Total Shipping Cost: $C = 7x_1 + 5x_2 + 4x_3 + 3x_4$

$$\textit{Minimize}: \quad C = 7x_1 + 5x_2 + 4x_3 + 3x_4$$

$$\textit{Subject to} \quad \begin{cases} x_1 + x_2 \le 700 \\ x_3 + x_4 \le 900 \\ x_1 + x_3 \ge 500 \\ x_2 + x_4 \ge 1000 \\ x_1, x_2, x_3, x_4 \ge 0 \end{cases}$$

$$\textit{Minimize}: \quad C = 7x_1 + 5x_2 + 4x_3 + 3x_4$$

$$\textit{Subject to} \quad \begin{cases} -x_1 - x_2 \ge -700 \\ -x_3 - x_4 \ge -900 \\ x_1 + x_3 \ge 500 \\ x_2 + x_4 \ge 1000 \\ x_1, x_2, x_3, x_4 \ge 0 \end{cases}$$

$$A = \left[\begin{array}{cccc|c} -1 & -1 & 0 & 0 & -700 \\ 0 & 0 & -1 & -1 & -900 \\ 1 & 0 & 1 & 0 & 500 \\ 0 & 1 & 0 & 1 & 1,000 \\ \hline 7 & 5 & 4 & 3 & 1 \end{array}\right] \qquad A^T = \left[\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 7 \\ -1 & 0 & 0 & 1 & 5 \\ 0 & -1 & 1 & 0 & 4 \\ 0 & -1 & 0 & 1 & 3 \\ \hline -700 & -900 & 500 & 1000 & 1 \end{array}\right]$$

The dual problem is:

$$\textit{Maximize}: \quad C = -700y_1 - 900y_2 + 500y_3 + 1000y_4$$

$$\textit{Subject to} \quad \begin{cases} -y_1 + y_3 \le 7 \\ -y_1 - y_4 \le 5 \\ -y_2 + y_3 \le 4 \\ -y_2 + y_4 \le 3 \\ y_1, y_2, y_3, y_4 \ge 0 \end{cases}$$

$$\begin{array}{cc}
 & \begin{array}{ccccccccc|c} y_1 & y_2 & y_3 & y_4 & x_1 & x_2 & x_3 & x_4 & P & \end{array} \\
\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ P \end{array} & \left[\begin{array}{ccccccccc|c}
-1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 7 \\
-1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 5 \\
0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 4 \\
0 & -1 & 0 & (1) & 0 & 0 & 0 & 1 & 0 & 3 \\ \hline
700 & 900 & -500 & \langle -1000 \rangle & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
\begin{array}{l} \\ R_2 - R_4 \\ \\ \\ \\ R_5 + 1000R_4 \end{array}$$

$$\begin{array}{cc}
 & \begin{array}{ccccccccc|c} y_1 & y_2 & y_3 & y_4 & x_1 & x_2 & x_3 & x_4 & P & \end{array} \\
\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ y_4 \\ P \end{array} & \left[\begin{array}{ccccccccc|c}
-1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 7 \\
-1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 2 \\
0 & -1 & (1) & 0 & 0 & 0 & 1 & 0 & 0 & 4 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 3 \\ \hline
700 & -100 & \langle -500 \rangle & 0 & 0 & 0 & 0 & 1000 & 1 & 3000
\end{array}\right]
\end{array}
\begin{array}{l} R_1 - R_3 \\ \\ \\ \\ \\ R_5 + 500R_3 \end{array}$$

$$\begin{array}{cc}
 & \begin{array}{ccccccccc|c} y_1 & y_2 & y_3 & y_4 & x_1 & x_2 & x_3 & x_4 & P & \end{array} \\
\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ y_4 \\ P \end{array} & \left[\begin{array}{ccccccccc|c}
-1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 3 \\
-1 & (1) & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 2 \\
0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 4 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 3 \\ \hline
700 & \langle -600 \rangle & 0 & 0 & 0 & 0 & 500 & 1000 & 1 & 5000
\end{array}\right]
\end{array}
\begin{array}{l} R_1 - R_2 \\ \\ \\ R_3 + R_2 \\ R_4 + R_2 \\ R_5 + 600R_2 \end{array}$$

$$\begin{array}{cc}
 & \begin{array}{ccccccccc|c} y_1 & y_2 & y_3 & y_4 & x_1 & x_2 & x_3 & x_4 & P & \end{array} \\
\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ y_4 \\ P \end{array} & \left[\begin{array}{ccccccccc|c}
0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 1 \\
-1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3 \\
-1 & 0 & 1 & 0 & 0 & 1 & 1 & -1 & 0 & 6 \\
-1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 5 \\ \hline
100 & 0 & 0 & 0 & 0 & 600 & 500 & 400 & 1 & 6200
\end{array}\right]
\end{array}$$

$Min\ C = \$6,200 \quad x_1 = 0,\ x_2 = 600,\ x_3 = 500,\ x_4 = 400$

Plant *A*:	outlet I = 0	outlet II = 600
Plant *B*:	outlet I = 500	outlet II = 400

Rewriting a linear programming problem so that it is in standard maximization or standard minimization form.

When a problem is not in one of standard forms, you may be able to rewrite it by multiplying all of the terms of the inequality by –1. {This reverses the direction of the inequality}

Example:

$$\begin{aligned} &\textit{Maximize}: \; P = 3x_1 + 5x_2 \\ &\qquad\qquad 2x_1 - x_2 \ge -10 \\ &\textit{Subject To}: \; x_1 + 2x_2 \le 8 \\ &\qquad\qquad x_1, x_2 \ge 0 \end{aligned}$$

Rewritten:

$$\begin{aligned} &\textit{Maximize}: \; P = 3x_1 + 5x_2 \\ &\qquad\qquad -2x_1 + x_2 \le 10 \\ &\textit{Subject To}: \; x_1 + 2x_2 \le 8 \\ &\qquad\qquad x_1, x_2 \ge 0 \end{aligned}$$

Example:

$$\begin{aligned} &\textit{Minimize}: \; C = 21x_1 + 50x_2 \\ &\qquad\qquad x_1 - 5x_2 \le 2 \\ &\textit{Subject To}: \; 3x_1 + 7x_2 \ge 17 \\ &\qquad\qquad x_1, x_2 \ge 0 \end{aligned}$$

Rewritten:

$$\begin{aligned} &\textit{Minimize}: \; C = 21x_1 + 50x_2 \\ &\qquad\qquad -x_1 + 5x_2 \ge -2 \\ &\textit{Subject To}: \; 3x_1 + 7x_2 \ge 17 \\ &\qquad\qquad x_1, x_2 \ge 0 \end{aligned}$$

The following problem can't be "fixed".

$$\begin{aligned} &\textit{Maximize}: \; P = 3x_1 + 5x_2 \\ &\qquad\qquad 2x_1 - x_2 \ge 10 \\ &\textit{Subject To}: \; x_1 + 2x_2 \le 8 \\ &\qquad\qquad x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} &\textit{Maximize}: \; P = 3x_1 + 5x_2 \\ &\qquad\qquad -2x_1 + x_2 \le -10 \\ &\textit{Subject To}: \; x_1 + 2x_2 \le 8 \\ &\qquad\qquad x_1, x_2 \ge 0 \end{aligned}$$

Even though you can fix the inequality; in standard max problems the constants can't be negative.
This example is what we call a mixed constraint problem.

Exercises   *Section* 1.6 – Minimization Problems ≥ (*Duality*)

1. Find the transpose of the matrix

$a)\ \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 4 & 4 & 4 \end{bmatrix}$ $b)\ \begin{bmatrix} -3 & -8 & 1 \\ 5 & -2 & 5 \\ 9 & 6 & -2 \\ 4 & 5 & 8 \end{bmatrix}$

2. Solve the following minimization problem by maximizing the dual:

$$Maximize:\ P = 12y_1 + 17y_2$$

$$Subject\ To:\ \begin{cases} 2y_1 + 3y_2 \le 21 \\ 5y_1 + 7y_2 \le 50 \\ y_1, y_2 \ge 0 \end{cases}$$

3. Solve the following minimization problem by maximizing the dual:

$$Minimize:\ C = 16x_1 + 8x_2 + 4x_3$$

$$Subject\ to\ \begin{cases} 3x_1 + 2x_2 + 2x_3 \ge 16 \\ 4x_1 + 3x_2 + x_3 \ge 14 \\ 5x_1 + 3x_2 + x_3 \ge 12 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

4. Customers buy 14 units of regular beer and 20 units of light beer monthly. The brewery decides to produce extra beer, beyond that needed to satisfy the customers. The cost per unit of regular beer is \$33,000 and the cost per unit of light beer is \$44,000. Every unit of regular beer brings in \$200,000 in revenue, while every unit of light beer brings in \$400,000 in revenue. The brewery wants at least \$16,000,000 in revenue. At least 18 additional units of beer can be sold. How much of each beer type should be made so as to minimize total production costs? What is the minimum cost?

5. Acme Micros markets computers with single-sided and double-sided drives. The disk driers are supplied by two other companies, Associated Electronics and Digital Drives. Associated Electronics charges \$250 for a single-sided disk drive and \$350 for a double-sided disk drive. Digital Drives charges \$290 for a single-sided disk drive and \$320 for a double-sided disk drive. Associated Electronics can supply at most 1,000 disk drives in any combination of single-sided and double-sided drives. The combined monthly total supplied by Digital Drives cannot exceed 2,000 disk drives. Acme Micros needs at least 1,200 single-sided drives and at least 1,600 double-sided drives each month. How many disk drives of each type should Acme Micros order from each supplier in order to meet its monthly demand and minimize the purchase cost? What is the minimum purchase cost?

6. A farmer can buy three types of plant food, mix A, mix B, and mix C. Each cubic yard of mix A contains 20 pounds of phosphoric acid, 10 pounds of nitrogen, and 10 pound of potash. Each cubic yard of mix B contains 10 pounds of phosphoric acid, 10 pounds of nitrogen, and 15 pound of potash. . Each cubic yard of mix C contains 20 pounds of phosphoric acid, 20 pounds of nitrogen, and 5 pound of potash. The minimum monthly requirements are 480 pounds of phosphoric acid, 320 pounds of nitrogen, and 225 pound of potash. If mix A costs $30 per cubic yard, nix B costs $36 per cubic yard, and mix C $39 per cubic yard, how many cubic yards of each mix should the farmer blend to meet the minimum monthly requirements at a minimal cost? What is the minimum cost?

7. Mark, who is ill, takes vitamin pills. Each day he must have at least 16 units of vitamin A, 5 units of vitamin B, and 20 units of vitamin C. he can choose between pill #1, which contains 8 units of A, 1 of B, and 2 of C; and pill #2, which contains 2 units of A, 1 of B, and 7 of C. Pill 1 costs 15¢, and pill 2 costs 30¢.

a) How many of each pill should be buy in order to minimize his cost?
b) What is the minimum cost?
c) For the solution in part a, Mark is receiving more than he needs of at least one vitamin. Identify that vitamin, and tell how much surplus he is receiving. Is there any ways he can avoid receiving that surplus while still meeting the other constraints and minimizing the cost?

8. One gram of soybean meal provides at least 2.5 units of vitamins and 5 calories. One gram of meat byproducts provides at least 4.5 units of vitamins and 3 calories. One gram of grain provides at least 5 units of vitamins and 10 calories. If a gram of soybean meal costs 6 cents, a gram of meat byproducts 8 cents, and a gram of grain 9 cents, what mixture of these three ingredients will provide at least 54 units of vitamins and 60 calories per serving at minimum cost? What will be the minimum cost?

9. A metropolitan school district has two high-schools that are overcrowded and two that are underenrolled. In order to balance the enrollment, the school board has decided to bus students from the crowded schools to the underenrolled schools. North Division High School has 300 more students than it should have, and South Division High School has 500 more students than it should have. Central High School can accommodate 400 additional students and Washington High School can accommodate 500 additional students. The weekly cost of busing a student from North Division to Central is $5, from North Division to Washington is $2, from South Division to Central is $3, and from South Division to Washington is $4. Determine the number of students that should be bused from each of the overcrowded schools to each of the underenrolled schools in order to balance the enrollment and minimize the cost of busing the students. What is the minimum cost?