Section 4.4 – Solution about Singular Points

Solution about Singular Points

The Standard form $\qquad y'' + P(x)y' + Q(x)y = 0$

Definition (*Regular and Irregular Singular Points*)

A singular point $x = x_0$ is said to be a ***regular singular*** point of a differential equation if the functions $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are both analytic at x_0.

A singular point is not regular is said to be an ***irregular singular point*** of the equation.

The singular points are those points where $p(x)$ or $q(x)$ fails to be analytic, when the denominators are zero.

- If $x - x_0$ appears at most to the first power in the denominator of $P(x)$ and at most to the second power in the denominator of $Q(x)$, then $x = x_0$ is a ***regular singular point***.

Example

Determine the singular points for $\left(x^2 - 4\right)^2 y'' + 3(x-2)y' + 5y = 0$

Solution

$$(x-2)^2(x+2)^2 y'' + 3(x-2)y' + 5y = 0$$

$$y'' + 3\frac{x-2}{(x-2)^2(x+2)^2}y' + \frac{5}{(x-2)^2(x+2)^2}y = 0$$

$$P(x) = \frac{3}{(x-2)(x+2)^2} \qquad\qquad Q(x) = \frac{5}{(x-2)^2(x+2)^2}$$

The points are: $x = -2,\ 2$

At $x = -2$

$$p(x) = (x+2)\frac{3}{(x-2)(x+2)^2} = \frac{3}{(x-2)(x+2)}$$

$\boxed{x = -2,\ \ 2}$ $\Rightarrow$ is *not* an analytic at $x = -2$

$$q(x) = (x+2)^2\frac{5}{(x-2)^2(x+2)^2} = \frac{5}{(x-2)^2}$$

$\boxed{x = 2}$ $\Rightarrow$ It is an analytic at $x = 2$

At $x = 2$

$$p(x) = (x-2)\frac{3}{(x-2)(x+2)^2} = \frac{3}{(x+2)}$$

$\boxed{x=-2}$ $\Rightarrow$ It is an analytic at $x=-2$

$$q(x) = (x-2)^2 \frac{5}{(x-2)^2(x+2)^2} = \frac{5}{(x+2)^2}$$

$\boxed{x=-2}$ $\Rightarrow$ It is an analytic at $x=-2$

Frobenius *Theorem*

If $x = x_0$ is a regular singular point of the differential equation. There exists at least one solution of the form

$$y = (x-x_0)^r \sum_{n=0}^{\infty} c_n (x-x_0)^n = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$$

r: constant to be determined.

The series will converge at least on some interval $0 < x - x_0 < R$

The model of Frobenius

The simplest equation, of a second-order linear differential equation near the regular singular point $x=0$, is the constant-coefficient ***equidimensional*** equation

$$x^2 y'' + p_0 x y' + q_0 y = 0$$

If r is a root of the quadratic equation

$$r(r-1) + p_0 r + q_0 = 0$$

Example

Find the exponents in the possible Frobenius series solutions of the equation

$$2x^2(1+x)y'' + 3x(1+x)^3 y' - \left(1-x^2\right)y = 0$$

Solution

$$y'' + \frac{3x(1+x)^3}{2x^2(1+x)}y' - \frac{(1-x)(1+x)}{2x^2(1+x)}y = 0$$

$$y'' + \frac{3}{2}\frac{(1+x)^2}{x}y' - \frac{1}{2}\frac{1-x}{x^2}y = 0$$

Therefore; $p_0 = \frac{3}{2}, \quad q_0 = -\frac{1}{2}$

The indicial equation is $r(r-1) + \frac{3}{2}r - \frac{1}{2} = r^2 + \frac{1}{2}r - \frac{1}{2} = (r+1)\left(r - \frac{1}{2}\right) = 0$

With roots $r_1 = \frac{1}{2}$ *and* $r_2 = -1$

The two possible Frobenius series solutions are then of the forms

$$y_1(x) = x^{1/2} \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = x^{-1} \sum_{n=0}^{\infty} b_n x^n$$

Theorem – Frobenius Series Solutions

Suppose that $x = 0$ is a regular point of the equation $\quad x^2 y'' + p_0 x y' + q_0 y = 0$

Let $\rho > 0$ denote the minimum of the radii of convergence of the power series

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \quad \textit{and} \quad q(x) = \sum_{n=0}^{\infty} q_n x^n$$

Let r_1 and r_2 be the (real) roots, with $r_1 \ge r_2$, of the ***indicial equation*** $I(x) = r(r-1) + p_0 r + q_0 = 0$.

Then

- ✓ For $x > 0$, there exist a solution of the form

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \quad (a_0 \ne 0) \text{ corresponding to the larger root } r_1.$$

$$y_2(x) = y_1(x)\ln x + x^{r_1+1} \sum_{n=0}^{\infty} b_n x^n$$

- ✓ If $r_1 - r_2 = N$, a positive integer, then the equation has two solutions y_1 and y_2 of the form

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \quad \textit{and} \quad y_2(x) = C y_1(x) + x^{r_2} \sum_{n=0}^{\infty} b_n x^n \quad (a_0, b_0 \ne 0)$$

The radii of convergence of the power series of this theorem are all at least ρ. The coefficients in these series (and the constant C) may be determined by direct substitution of the series.

Example

Find the general solution to the equation $2xy'' + y' - 4y = 0$ near the point $x_0 = 0$

Solution

$$\left(\frac{x}{2}\right)2xy'' + \left(\frac{x}{2}\right)y' - \left(\frac{x}{2}\right)4y = 0$$

$$x^2y'' + \frac{1}{2}xy' - 2xy = 0$$

That implies to $p(x) = \frac{1}{2}$ and $q(x) = -2x$, both are analytic. Hence, $x_0 = 0$ is a regular point

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$2x^2y'' + xy' - 4xy = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - 4x \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - 4x \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+r)(n+r-1) + (n+r)\right] a_n x^{n+r} - 4x \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-2) + (n+r)\right] a_n x^{n+r} - 4 \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$x^r \left(\sum_{n=0}^{\infty} \left[(n+r)(2n+2r-1)\right] a_n x^{n} - 4 \sum_{n=0}^{\infty} a_n x^{n+1} \right) = 0$$

$$x^r \left(r(2r-1)a_0 + \underbrace{\sum_{n=1}^{\infty} (n+r)(2n+2r-1) a_n x^{n-1}}_{k=n} - 4 \underbrace{\sum_{n=0}^{\infty} a_n x^{n+1}}_{k=n+1} \right) = 0$$

$$x^r\left(r(2r-1)a_0+\sum_{k=1}^{\infty}(k+r)(2k+2r-1)a_k x^k-4\sum_{k=1}^{\infty}a_{k-1}x^k\right)=0$$

$$x^r\left(r(2r-1)a_0+\sum_{k=1}^{\infty}\left[(k+r)(2k+2r-1)a_k-4a_{k-1}\right]x^k\right)=0$$

$$\begin{cases} r(2r-1)a_0=0 & \Rightarrow \boxed{r=0} \quad \boxed{r=\frac{1}{2}} \\ (k+r)(2k+2r-1)a_k-4a_{k-1}=0 & \Rightarrow \boxed{a_k=\dfrac{4}{(k+r)(2k+2r-1)}a_{k-1}} \end{cases}$$

$r=0$

$$a_k=\frac{4}{k(2k-1)}a_{k-1}$$

$$a_1=\frac{4}{1}a_0$$

$$a_2=\frac{4}{2\cdot 3}a_1=\frac{4^2}{1\cdot 2\cdot 3}a_0$$

$$a_3=\frac{4}{3\cdot 5}a_2=\frac{4^3}{1\cdot 2\cdot 3\cdot 3\cdot 5}a_0$$

$$a_4=\frac{4}{4\cdot 7}a_3=\frac{4^3}{4!(1\cdot 3\cdot 5\cdot 7)}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_n=\frac{4^n}{n!\ 1\cdot 3\cdot 5\cdots(2n-1)}a_0$$

$r=\frac{1}{2}$

$$a_k=\frac{4}{\left(k+\frac{1}{2}\right)\left(2k+2\frac{1}{2}-1\right)}a_{k-1}=\frac{4}{k(2k+1)}a_{k-1}$$

$$a_1=\frac{4}{1\cdot 3}a_0$$

$$a_2=\frac{4}{2\cdot 5}a_1=\frac{4^2}{1\cdot 2\cdot 3\cdot 5}a_0$$

$$a_3=\frac{4}{3\cdot 7}a_2=\frac{4^3}{3!(3\cdot 5\cdot 7)}a_0$$

$$a_4=\frac{4}{4\cdot 9}a_3=\frac{4^3}{4!(3\cdot 5\cdot 7\cdot 9)}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_n=\frac{4^n}{n!\ 3\cdot 5\cdots(2n+1)}a_0$$

$$y_1(x)=x^0\left(a_0+\sum_{n=0}^{\infty}\frac{4}{n!\ 1\cdot 3\cdot 5\cdots(2n-1)}a_0x^n\right)=a_0\left(1+\sum_{n=0}^{\infty}\frac{4}{n!\ 1\cdot 3\cdot 5\cdots(2n-1)}x^n\right)$$

$$y_2(x)=x^{1/2}\left(a_0+\sum_{n=0}^{\infty}\frac{4^n}{n!\ 3\cdot 5\cdots(2n+1)}a_0x^n\right)=a_0x^{1/2}\left(1+\sum_{n=0}^{\infty}\frac{4^n}{n!\ 3\cdot 5\cdots(2n+1)}x^n\right)$$

$$y(x)=C_1y_1(x)+C_2y_2(x)$$

$$=C_1\left(1+\sum_{n=0}^{\infty}\frac{4}{n!\ 1\cdot 3\cdot 5\cdots(2n-1)}x^n\right)+C_2x^{1/2}\left(1+\sum_{n=0}^{\infty}\frac{4^n}{n!\ 3\cdot 5\cdots(2n+1)}x^n\right)$$

Example

Find the general solution to the equation $3xy'' + y' - y = 0$

Solution

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

$$3xy'' + y' - y = 0$$

$$3x \sum_{n=0}^{\infty} (n+r+2)(n+r+1) c_{n+2} x^{n+r} + \sum_{n=0}^{\infty} (n+r+1) c_{n+1} x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r+2)(n+r+1) c_{n+2} x^{n+r+1} + \sum_{n=0}^{\infty} \left[(n+r+1) c_{n+1} - c_n\right] x^{n+r} = 0$$

$$3\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} c_n (n+r)(3n+3r-3+1) x^{n-1} x^r - \sum_{n=0}^{\infty} c_n x^n x^r = 0$$

$$x^r \left(\sum_{n=0}^{\infty} c_n (n+r)(3n+3r-2) x^{n-1} - \sum_{n=0}^{\infty} c_n x^n \right) = 0$$

$$x^r \left(c_0 r(3r-2) x^{-1} + \underbrace{\sum_{n=1}^{\infty} c_n (n+r)(3n+3r-2) x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \right) = 0$$

$$x^r \left(c_0 r(3r-2) x^{-1} + \sum_{k=0}^{\infty} c_{k+1} (k+r+1)(3k+3r+1) x^k - \sum_{k=0}^{\infty} c_k x^k \right) = 0$$

$$x^r\left(c_0 r(3r-2)x^{-1}+\sum_{k=0}^{\infty}\left[c_{k+1}(k+r+1)(3k+3r+1)-c_k\right]x^k\right)=0$$

$$\begin{cases} c_0 r(3r-2)=0 & \Rightarrow \boxed{r=0} \quad \boxed{r=\frac{2}{3}} \\ c_{k+1}(k+r+1)(3k+3r+1)-c_k=0 & \Rightarrow \boxed{c_{k+1}=\dfrac{c_k}{(k+r+1)(3k+3r+1)}} \end{cases}$$

$r=0$

$$c_{k+1}=\frac{c_k}{(k+1)(3k+1)}$$

$$c_1=c_0$$

$$c_2=\frac{c_1}{2\cdot 4}=\frac{c_0}{(2)(4)}$$

$$c_3=\frac{c_2}{3\cdot 7}=\frac{c_0}{2\cdot 3(4\cdot 7)}$$

$$c_4=\frac{c_3}{4\cdot 10}=\frac{c_0}{2\cdot 3\cdot 4(4\cdot 7\cdot 10)}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$c_n=\frac{c_0}{n!\ 4\cdot 7\cdot 10\cdots(3n-2)}$$

$r=\frac{2}{3}$

$$c_{k+1}=\frac{c_k}{\left(k+\frac{5}{3}\right)(3k+3)}=\frac{c_k}{(3k+5)(k+1)}$$

$$c_1=\frac{c_0}{5\cdot 1}$$

$$c_2=\frac{c_1}{8\cdot 2}=\frac{c_0}{5\cdot 8\cdot 1\cdot 2}$$

$$c_3=\frac{c_2}{11\cdot 3}=\frac{c_0}{(5\cdot 8\cdot 11)(1\cdot 2\cdot 3)}$$

$$c_4=\frac{c_3}{14\cdot 4}=\frac{c_0}{(5\cdot 8\cdot 11\cdot 14)(1\cdot 2\cdot 3\cdot 4)}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$c_n=\frac{c_0}{n!\ 5\cdot 8\cdot 11\cdots(3n+2)}$$

$$y_1(x)=x^0\left(c_0+\sum_{n=0}^{\infty}\frac{c_0}{n!\ 4\cdot 7\cdot 10\cdots(3n-2)}x^n\right)$$

$$=c_0\left(1+\sum_{n=0}^{\infty}\frac{1}{n!\ 4\cdot 7\cdot 10\cdots(3n-2)}x^n\right)$$

$$y_2(x)=x^{2/3}\left(c_0+\sum_{n=0}^{\infty}\frac{c_0}{n!\ 5\cdot 8\cdots(3n+2)}x^n\right)$$

$$=c_0x^{2/3}\left(1+\sum_{n=0}^{\infty}\frac{1}{n!\ 5\cdot 8\cdots(3n+2)}x^n\right)$$

$$y(x)=C_1y_1(x)+C_2y_2(x)$$

$$=C_1\left(1+\sum_{n=0}^{\infty}\frac{1}{n!\ 4\cdot 7\cdot 10\cdots(3n-2)}x^n\right)+C_2x^{2/3}\left(1+\sum_{n=0}^{\infty}\frac{1}{n!\ 5\cdot 8\cdots(3n+2)}x^n\right)$$

OR

$$y''+\frac{1}{3x}y'-\frac{1}{3x}y=0$$

$$p(x)=(x-x_0)P(x)=x\frac{1}{3x}=\frac{1}{3} \qquad p(x)=a_0+a_1x+\cdots$$

$$q(x)=\left(x-x_0\right)^2 Q(x)=x^2\left(-\tfrac{1}{3x}\right)=-\tfrac{1}{3}x \qquad\qquad q(x)=b_0+b_1x+\cdots$$

$$r(r-1)+a_0r+b_0=0$$

$$r(r-1)+\tfrac{1}{3}r+0=0$$

$$r^2-r+\tfrac{1}{3}r=0$$

$$3r^2-2r=0$$

$$\underline{r(3r-2)=0}$$

Theorem The Extended Theorem and Procedure of ***Frobenius***

The *ODE* is given by: $x^2y''+xp(x)y'+q(x)y=0$

Has a regular singular point at $x=0$. The extended Method of ***Frobenius*** produce *two* independent solutions of the *ODE* if the indicial roots are real.

- Find the indicial roots r_1 and r_2 of the indicial polynomial $f(r)=r^2+\left(p_0-1\right)r+q_0$

  Verify that they are real; index them such that $r_2\le r_1$

- Construct the solution $y_1(x)=x^{r_1}\sum_{n=0}^{\infty}a_nx^n \quad \left(a_0=1\right)$ by the method of Frobenius. The recursion

  formula is $f\left(r_1+n\right)a_n=\sum_{k=0}^{n-1}\left[\left(k+r_1\right)p_{n-k}+q_{n-k}\right]a_k$

- If $r_1=r_2 \quad\Rightarrow\quad y_2(x)=y_1(x)\ln x+x^{r_1}\sum_{n=0}^{\infty}c_nx^n \qquad (x>0)$

- If r_1-r_2 is a positive integer, then a second independent solution has the form

$$y_2(x)=\alpha y_1(x)\ln x+x^{r_2}\left(1+\sum_{n=1}^{\infty}d_nx^n\right)$$

Exercises   *Section* 4.4 – Solution about Singular Points

Determine all the singular points of the given differential equation and classify each singular point as regular or irregular

1. $x^2y'' + 3y' - xy = 0$

2. $\left(x^2 + x\right)y'' + 3y' - 6xy = 0$

3. $\left(x^2 - 1\right)y'' + (1 - x)y' + \left(x^2 - 2x + 1\right)y = 0$

4. $e^x y'' - \left(x^2 - 1\right)y' + 2xy = 0$

5. $\ln(x-1)y'' + (\sin 2x)y' - e^x y = 0$

6. $xy'' + x(1-x)^{-1}y' + (\sin x)y = 0$

7. $x^3y'' + 4x^2y' + 3y = 0$

8. $x(x+3)^2 y'' - y = 0$

9. $\left(x^2 - 9\right)^2 y'' + (x+3)y' + 2y = 0$

10. $y'' - \frac{1}{x}y' + \frac{1}{(x-1)^3}y = 0$

11. $\left(x^3 + 4x\right)y'' - 2xy' + 6y = 0$

12. $x^2(x-5)^2 y'' + 4xy' + \left(x^2 - 25\right)y = 0$

13. $\left(x^2 + x - 6\right)y'' + (x+3)y' + (x-2)y = 0$

14. $x\left(x^2 + 1\right)^2 y'' + y = 0$

15. $x^3\left(x^2 - 25\right)(x-2)^2 y'' + 3x(x-2)y' + 7(x+5)y = 0$

16. $\left(x^3 - 2x^2 - 3x\right)^2 y'' + x(x-3)^2 y' - (x+1)y = 0$

17. $\left(1 - x^2\right)y'' + (\tan x)y' + x^{5/3}y = 0$

18. $x(x-1)^2(x+2)y'' + x^2y' - \left(x^3 + 2x - 1\right)y = 0$

19. $x^4\left(x^2 + 1\right)(x-1)^2 y'' + 4x^3(x-1)y' + (x+1)y = 0$

Determine whether $x = 0$ is an ordinary point, singular point, or irregular singular point of the given differential equation

20. $xy'' + (1 - \cos x)\,y' + x^2 y = 0$

21. $\left(e^x - 1 - x\right) y'' + xy = 0$

Find the Frobenius series solutions near the point $x = 0$

22. $2x^2 y'' + 3xy' - \left(1 + x^2\right) y = 0$

23. $2x^2 y'' - xy' + \left(1 + x^2\right) y = 0$

24. $2xy'' + (1 + x)y' + y = 0$

25. $xy'' + 2y' + xy = 0$

26. $2xy'' - y' + 2y = 0$

27. $2xy'' + 5y' + xy = 0$

28. $4xy'' + \frac{1}{2} y' + y = 0$

29. $2x^2 y'' - xy' + \left(x^2 + 1\right) y = 0$

30. $2xy'' - (3 + 2x)\,y' + y = 0$

31. $3xy'' + (2 - x)\,y' - y = 0$

32. $xy'' + (x - 6)\,y' - 3y = 0$

33. $x(x - 1)\,y'' + 3y' - 2y = 0$

34. $x^2 y'' - \left(x - \frac{2}{9}\right) y = 0$

35. $x^2 y'' + x(3 + x)\,y' - 3y = 0$

36. $x^2 y'' + \left(x^2 - 2x\right) xy' + 2y = 0$

37. $x^2 y'' + \left(x^2 + 2x\right) y' - 2y = 0$

38. $2xy'' + 3y' - y = 0$

39. $2xy'' - y' - y = 0$

40. $2xy'' + (1 + x)\,y' + y = 0$

41. $2xy'' + \left(1 - 2x^2\right) y' - 4xy = 0$

42. $2x^2 y'' + xy' - \left(1 + 2x^2\right) y = 0$

43. $2x^2 y'' + xy' - \left(3 - 2x^2\right) y = 0$

44. $3xy'' + 2y' + 2y = 0$

45. $3x^2 y'' + 2xy' + x^2 y = 0$

46. $3x^2 y'' - xy' + y = 0$

47. $4xy'' + 2y' + y = 0$

48. $6x^2 y'' + 7xy' - \left(x^2 + 2\right) y = 0$

49. $xy'' + y' + 2y = 0$

50. $2x(1 - x)\,y'' + (1 + x)\,y' - y = 0$

51. $x^2 y'' + \left(x^2 + \frac{1}{2}x\right) y' + xy = 0$

52. $18x^2 y'' + 3x(x + 5)\,y' - (10x + 1)\,y = 0$

53. $2x^2 y'' + 7x(x + 1)\,y' - 3y = 0$

54. Find the Frobenius series solutions:

$$x(1 - x)\,y'' + \left[c - (a + b + 1)\,x\right] y' - aby = 0 \quad (\textit{Gauss' Hypergeometric})$$