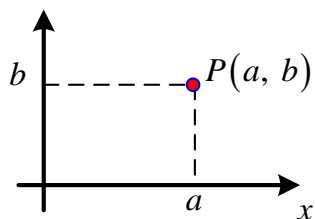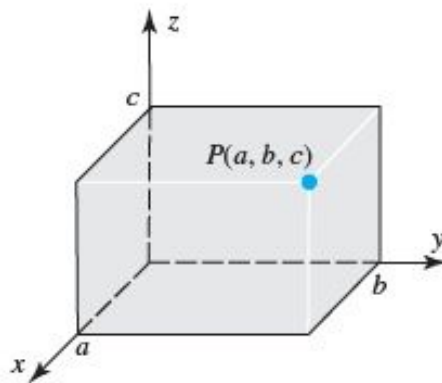# Section 2.7 – Coordinates, Basis and Dimension

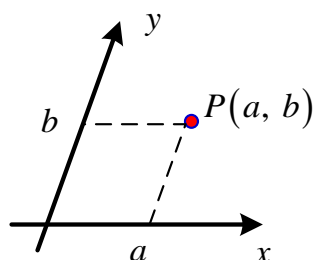## Coordinate Systems in Linear Algebra

In *analytic geometry*, we use rectangular coordinate systems to create a point either in 2-space or 3-space
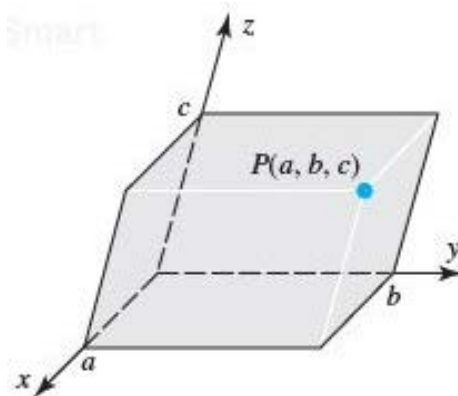


*Coordinates of P in a rectangular coordinate system in 2-space*



*Coordinates of P in a rectangular coordinate system in 3-space*
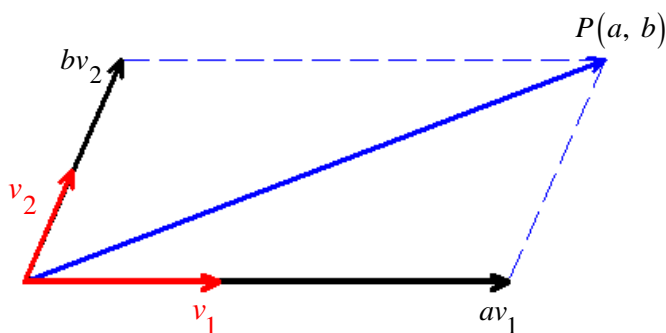


*Coordinates of P in a nonrectangular coordinate system in 2-space*



*Coordinates of P in a nonrectangular coordinate system in 3-space*

In *linear algebra* coordinate systems are commonly specified using vectors rather than coordinate axes.

# Basis

## Definition

If $V$ is any vector space and $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a finite set of vectors in $V$, then $S$ is called a **basis**

for $V$ if the following two conditions hold:

1. $S$ is linearly independent.
2. $S$ spans $V$.

## Example

The columns of $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ produce the "standard basis" for $\mathbb{R}^2$.

### Solution

The basis vectors: $\hat{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $and$ $\hat{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are independent. They span $\mathbb{R}^2$.

## Example

The columns of any invertible $n$ by $n$ matrix give a basis for $\mathbb{R}^n$.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$\qquad$ *Basis* $\qquad\qquad$ *Basis* $\qquad\qquad$ *Not basis*

## Example

The standard unit vectors $\hat{e}_1 = (1, 0, 0, \cdots, 0)$, $\hat{e}_2 = (0, 1, 0, \cdots, 0)$, $\ldots$, $\hat{e}_n = (0, 0, \cdots, 0, 1)$

form a basis in $\mathbb{R}^n$.

### Solution

1. $k_1 \hat{e}_1 + k_2 \hat{e}_2 + \ldots + k_n \hat{e}_n = 0 \rightarrow (k_1, k_2, \ldots, k_n) = (0, 0, \ldots, 0)$ it follows that

   $k_1 = k_2 = \ldots = k_n = 0$. That implies they are linearly independent.

2. Every vector $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ can be expressed as $\vec{v} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + \ldots + v_n \hat{e}_n$

   which is linear combination of $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n$. Thus, the standard vector span $\mathbb{R}^n$

Thus, they form a basis for $\mathbb{R}^n$ that we call the **standard basis** for $\mathbb{R}^n$.

# *Example*

Show that the vectors $\vec{v}_1 = (1,\ 2,\ 1)$, $\vec{v}_2 = (2,\ 9,\ 0)$, *and* $\vec{v}_3 = (3,\ 3,\ 4)$ form a basis in $\mathbb{R}^3$

## *Solution*

1. We need to show that the vectors are linearly independent.

$$k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{0}$$

$$k_1(1,\ 2,\ 1) + k_2(2,\ 9,\ 0) + k_3(3,\ 3,\ 4) = (0,\ 0,\ 0)$$

Which yields the homogeneous linear system

$$\rightarrow \begin{cases} k_1 + 2k_2 + 3k_3 = 0 \\ 2k_1 + 9k_2 + 3k_3 = 0 \\ k_1 \qquad\quad + 4k_3 = 0 \end{cases}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} = \underline{-1 \neq 0}$$

$k_1 = 0$, $k_2 = 0$, $k_3 = 0$ has a trivial solution.

The vectors $\vec{v}_1$, $\vec{v}_2$, *and* $\vec{v}_3$ are linearly independent.

2. Every vector can be expressed as $k_1\vec{v}_1 + k_2\vec{v}_2 + k_3\vec{v}_3 = \vec{b}$ which is linear combination. Thus, the standard vector span $\mathbb{R}^3$

That proves that the vectors $\vec{v}_1$, $\vec{v}_2$, *and* $\vec{v}_3$ form a basis in $\mathbb{R}^3$

- The vectors $\vec{v}_1$, $\cdots$, $\vec{v}_n$ are a basis for $\mathbb{R}^n$ exactly when they are the *columns* of an *n* **by** *n* *invertible matrix*. Thus $\mathbb{R}^n$ has infinitely many different bases.

- *The pivots columns of A are a basis for its column space*. The pivot rows of $A$ are a basis for its row space. So are the pivot rows of its echelon form $R$.

## *Example*

Find bases for the column and row spaces of a rank two matrices: $R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

## *Solution*

Columns 1 and 3 are the pivot columns. They are a basis for the column space. It is a subspace of $\mathbb{R}^3$.

Column 2 and 4 are a basis for the same column space.

## *Coordinates Relative to a Basis*

## *Theorem* – **Uniqueness of Basis Representation**

If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for a vector space $V$, then every vector $\vec{v}$ in $V$ can be expressed in the

form $\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_n \vec{v}_n$ in exactly one way.

## *Proof*

Suppose that some vector $\vec{v}$ can be written as

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_n \vec{v}_n$$

Also $\qquad \vec{v} = k_1 \vec{v}_1 + k_2 \vec{v}_2 + \ldots + k_n \vec{v}_n$

Subtracting the second from the first equation

$$0 = \left(c_1 - k_1\right)\vec{v}_1 + \left(c_2 - k_2\right)\vec{v}_2 + \ldots + \left(c_n - k_n\right)\vec{v}_n$$

Since the right side of this equation is a linear combination of vectors in $S$, the linear independence of $S$ implies that

$$c_1 - k_1 = 0, \quad c_2 - k_2 = 0, \quad \ldots, \quad c_n - k_n = 0$$

That implies $\qquad c_1 = k_1, \quad c_2 = k_2, \quad \ldots, \quad c_n = k_n$

Thus, the two expressions for $\vec{v}$ are the same.

## *Definition*

If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for a vector space $V$, and $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$

is the expression for a vector $\vec{v}$ in terms of the basis $S$, then the scalars $c_1, c_2, \ldots, c_n$ are called

coordinates of $\vec{v}$ relative to the basis $S$. The vector $(c_1, c_2, \ldots, c_n)$ in $\mathbb{R}^n$ constructed from these

coordinates is called ***coordinate vector of v relative to S***; it is denoted by

$$(\vec{v})_S = (c_1, c_2, \ldots, c_n)$$

**A *one-to-one correspondence***
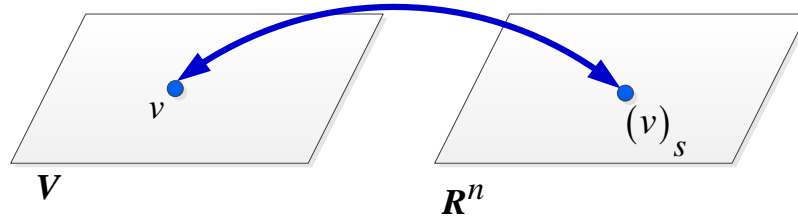


## *Example*

*a)* Given the vectors $\vec{v}_1 = (1, 2, 1)$, $\vec{v}_2 = (2, 9, 0)$, *and* $\vec{v}_3 = (3, 3, 4)$ form a basis for $\mathbb{R}^3$. Find the

coordinate vector of $\vec{v} = (5, -1, 9)$ relative to the basis $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

*b)* Find the coordinate vector of $\vec{v}$ in $\mathbb{R}^3$ whose coordinate relative to $S$ is $(\vec{v})_S = (-1, 3, 2)$.

## *Solution*

*a)* To find $(\vec{v})_S$ we must first express $\vec{v}$ as a linear combination of the vectors in $S$;

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$$

$$(5, -1, 9) = c_1(1, 2, 1) + c_2(2, 9, 0) + c_3(3, 3, 4)$$

Which gives: 
$$\begin{cases} c_1 + 2c_2 + 3c_3 = 5 \\ 2c_1 + 9c_2 + 3c_3 = -1 \\ c_1 \quad\quad + 4c_3 = 9 \end{cases}$$

Solving this system, we obtain $c_1 = 1, \quad c_2 = -1, \quad c_3 = 2$.

Therefore $(\vec{v})_S = (1, -1, 2)$

*b)* $\vec{v} = (-1)\vec{v}_1 + 3\vec{v}_2 + 2\vec{v}_3$

$= (-1)(1, 2, 1) + 3(2, 9, 0) + 2(3, 3, 4)$

$\underline{= (11, 31, 7)}$

# Dimension

If $\vec{v}_1$, ..., $\vec{v}_m$ and $\vec{w}_1$, ..., $\vec{w}_n$ are both bases for the same vector space, then $m = n$.

## Note

*V* may have many different bases, but they all must have the same number of elements.

## Proof

Let **S** and **W** be bases of **V** can be written as a linear combination of vectors in **S**.

$$\vec{w}_1 = a_{11}\vec{s}_1 + ... + a_{1m}s_m$$
$$\vec{w}_2 = a_{21}\vec{s}_1 + ... + a_{2m}\vec{s}_m$$
$$\vdots$$
$$\vec{w}_n = a_{n1}\vec{s}_1 + ... + a_{nm}\vec{s}_m$$

But since **V** is a basis, $c_1\vec{w}_1 + ... + c_n\vec{w}_n = 0 \Leftrightarrow c_i = 0$ (to be linearly independent, otherwise to be linearly dependent with at least 1 of t $c_i \neq 0$)

$$c_1\left(a_{11}s_1 + ... + a_{1m}s_m\right) + c_2\left(a_{21}s_1 + ... + a_{2m}s_m\right) + ... + c_n\left(a_{n1}s_1 + ... + a_{nm}s_m\right) = 0$$

$$\left(c_1 a_{11} + c_2 a_{21} + ... + c_n a_{n1}\right)s_1 + ... + \left(c_1 a_{1m} + c_2 a_{2m} + ... + c_n a_{nm}\right)s_m = 0$$

$$c_1 a_{11} + c_2 a_{21} + ... + c_n a_{n1} = 0$$
$$\Leftrightarrow$$
$$c_1 a_{1m} + c_2 a_{2m} + ... + c_n a_{nm} = 0$$

$\therefore$ $S_i$'s linear independent.

Now all bases of **V** have some number of elements, we can define the dimension (is # of vectors in a basis)

## *Definition*

The dimension of a finite-dimensional vector space $V$ is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for $V$. in addition, the zero-vector space is defined to have dimension zero.

1. $\text{Dim}(\mathbf{V})$ = # elements in basis. If $\mathbf{V}$ is finite.

2. If $V = \{\vec{0}\}$, then $\text{Dim}(\mathbf{V}) = 0$, even though there is no basis.

3. $\text{Dim}(\mathbf{V})$ may be infinite.

➤ $\dim\left(\mathbb{R}^n\right) = n$      **The standard basis has $n$ vectors.**

➤ $\dim\left(P_n\right) = n+1$      **The standard basis has $n + 1$ vectors.**

➤ $\dim\left(M_{mn}\right) = mn$      **The standard basis has $mn$ vectors.**

## **Bases for Matrix Spaces** *and Function Spaces*

Independence, basis, and dimension are not all restricted to column vectors.

- The dimension of the whole $n$ by $n$ space is $n^2$

- The dimension of the subspace of upper triangular matrices is $\frac{1}{2}n^2 + \frac{1}{2}n$

- The dimension of the subspace of diagonal matrices is $n$

- The dimension of the subspace of symmetric matrices is $\frac{1}{2}n^2 + \frac{1}{2}n$

## *Function Spaces*

The equations:

| | | |
|---|---|---|
| $y'' = 0$ | is solved by any linear function | $y = cx + d$ |
| $y'' = -y$ | is solved by any combination | $y = c\sin x + d\cos x$ |
| $y'' = y$ | is solved by any combination | $y = ce^x + de^{-x}$ |

## *Example*

Find a basis for and the dimension of the solution space of the homogeneous system

$$x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad = 0$$
$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0$$
$$5x_3 - 10x_4 \qquad\quad + 15x_6 = 0$$
$$2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 + 18x_6 = 0$$

## *Solution*

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & -10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{pmatrix} \quad \begin{matrix} \\ R_2 - 2R_1 \\ \\ R_4 - 2R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 5 & -10 & 0 & 15 & 0 \\ 0 & 0 & 4 & 8 & 0 & 18 & 0 \end{pmatrix} \quad \begin{matrix} \\ -R_2 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 5 & -10 & 0 & 15 & 0 \\ 0 & 0 & 4 & 8 & 0 & 18 & 0 \end{pmatrix} \quad \begin{matrix} R_1 + 2R_2 \\ \\ R_3 - 5R_2 \\ R_4 - 4R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & -20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \end{pmatrix} \quad \begin{matrix} \\ \\ -\dfrac{1}{20}R_3 \\ \dfrac{1}{6}R_4 \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 6 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{matrix} R_1 - 4R_3 \\ R_2 - 2R_3 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{matrix} R_1 - 6R_4 \\ R_2 - 3R_4 \\ \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 3 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad x_1 = -3x_2 - 2x_5$$

The solution $\left(x_1, x_2, x_3, x_4, x_5, x_6\right) = \left(-3x_2 - 2x_5, x_2, 0, 0, x_5, 0\right)$

$$= x_2\left(-3, 1, 0, 0, 0, 0\right) + x_5\left(-2, 0, 0, 0, 1, 0\right)$$

The solution space has dimension 2.

## Plus/Minus Theorem

### *Theorem*

Let $S$ be a nonempty set of vector space $V$.

a) If $S$ is a linearly independent set, and if $\vec{v}$ is a vector in $V$ that is outside of span(S), the set $S \cup \{\vec{v}\}$ that results by inserting $\vec{v}$ into $S$ is still linearly independent.

b) If $\vec{v}$ is a vector in $S$ that is expressible as a linear combination of other vectors in $S$, and if $S - \{\vec{v}\}$ denotes the set obtained by removing $\vec{v}$ from $S$, then $S$ and $S - \{\vec{v}\}$ span the same space; that is,

$$span(S) = span(S - \{\mathbf{v}\})$$
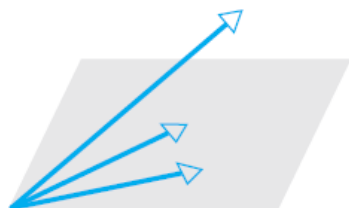


| The vector outside the plane can be adjoined to the other two without affecting their linear independence | Any of the vectors can be removed, and the remaining two still span the plane | Either of the collinear vectors can be removed, and the remaining two will still span the plane |

### *Theorem*

If $W$ is a subspace of a finite-dimensional vector space $V$, then

a) $W$ is finite-dimensional

b) $\dim(W) \le \dim(V)$

c) $W = V$ if and only if $\dim(W) = \dim(V)$

# *Exercises*    *Section* **2.7 – Coordinates, Basis and Dimension**

1.  Suppose $\vec{v}_1, \ldots, \vec{v}_n$ is a basis for $\mathbb{R}^n$ and the $n$ by $n$ matrix $A$ is invertible. Show that $A\vec{v}_1, \ldots, A\vec{v}_n$ is also a basis for $\mathbb{R}^n$.

2.  Consider the matrix $A = \begin{pmatrix} 1 & 0 & a \\ 2 & -1 & b \\ 1 & 1 & c \\ -2 & 1 & d \end{pmatrix}$

   *a)*  Which vectors $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ will make the columns of $A$ linearly dependent?

   *b)*  Which vectors $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ will make the columns of $A$ a basis for $\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : y + w = 0 \right\}$?

   *c)*  For $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$, compute a basis for the four subspaces.

3.  Find a basis for $x - 2y + 3z = 0$ in $\mathbb{R}^3$.

   Find a basis for the intersection of that plane with $xy$ plane. Then find a basis for all vectors perpendicular to the plane.

4.  **U** comes from **A** by subtracting row 1 from row 3:
   $$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad and \quad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   Find the bases for the two column spaces. Find the bases for the two row spaces. Find bases for the two nullspaces.

**5.** Write a 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives $c_1 P_1 + \ldots + c_5 P_5 = 0$, and check entries to prove $c_i$ is zero.) The five permutation matrices are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

**6.** Choose three independent columns of $A = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix}$. Then choose a different three independent columns. Explain whether either of these choices forms a basis for $C(A)$.

**7.** Which of the following sets of vectors are bases for $\mathbb{R}^2$?

    *a)* $\{(2, 1), (3, 0)\}$                        *b)* $\{(0, 0), (1, 3)\}$

**8.** Which of the following sets of vectors are bases for $\mathbb{R}^3$?

    *a)* $\{(1, 0, 0), (2, 2, 0), (3, 3, 3)\}$         *c)* $\{(2, -3, 1), (4, 1, 1), (0, -7, 1)\}$

    *b)* $\{(3, 1, -4), (2, 5, 6), (1, 4, 8)\}$         *d)* $\{(1, 6, 4), (2, 4, -1), (-1, 2, 5)\}$

**9.** Let $V$ be the space spanned by $\vec{v}_1 = \cos^2 x, \quad \vec{v}_2 = \sin^2 x, \quad \vec{v}_3 = \cos 2x$

    *a)* Show that $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is not a basis for $V$.

    *b)* Find a basis for $V$.

**10.** Find the coordinate vector of $\vec{w}$ relative to the basis $S = \{\vec{u}_1, \vec{u}_2\}$ for $\mathbb{R}^2$

    *a)* $\vec{u}_1 = (1, 0), \quad \vec{u}_2 = (0, 1), \quad \vec{w} = (3, -7)$     *d)* $\vec{u}_1 = (1, -1), \quad \vec{u}_2 = (1, 1), \quad \vec{w} = (0, 1)$

    *b)* $\vec{u}_1 = (2, -4), \quad \vec{u}_2 = (3, 8), \quad \vec{w} = (1, 1)$     *e)* $\vec{u}_1 = (1, -1), \quad \vec{u}_2 = (1, 1), \quad \vec{w} = (1, 1)$

    *c)* $\vec{u}_1 = (1, 1), \quad \vec{u}_2 = (0, 2), \quad \vec{w} = (a, b)$

**11.** Find the coordinate vector of $\vec{v}$ relative to the basis $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

    *a)* $\vec{v} = (2, -1, 3), \quad \vec{v}_1 = (1, 0, 0), \quad \vec{v}_2 = (2, 2, 0), \quad \vec{v}_3 = (3, 3, 3)$

    *b)* $\vec{v} = (5, -12, 3), \quad \vec{v}_1 = (1, 2, 3), \quad \vec{v}_2 = (-4, 5, 6), \quad \vec{v}_3 = (7, -8, 9)$

**12.** Show that $\{A_1, A_2, A_3, A_4\}$ is a basis for $M_{22}$, and express $A$ as a linear combination of the basis vectors

a) $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $A = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$

b) $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

c) $A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$, $A_1 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

**13.** Consider the eight vectors

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

a) List all of the one-element. Linearly dependent sets formed from these.

b) What are the two-element, linearly dependent sets?

c) Find a three-element set spanning a subspace of dimension three, and dimension of two? One? Zero?

d) Which four-element sets are linearly dependent? Explain why.

**(14 – 18)** Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space

**14.** $\begin{cases} x_1 + x_2 - x_3 = 0 \\ -2x_1 - x_2 + 2x_3 = 0 \\ -x_1 \quad\quad + x_3 = 0 \end{cases}$

**17.** $\begin{cases} x + y + z = 0 \\ 3x + 2y - 2z = 0 \\ 4x + 3y - z = 0 \\ 6x + 5y + z = 0 \end{cases}$

**15.** $\begin{cases} 3x_1 + x_2 + x_3 + x_4 = 0 \\ 5x_1 - x_2 + x_3 - x_4 = 0 \end{cases}$

**18.** $\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 \quad\quad + 5x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$

**16.** $\begin{cases} x_1 - 3x_2 + x_3 = 0 \\ 2x_1 - 6x_2 + 2x_3 = 0 \\ 3x_1 - 9x_2 + 3x_3 = 0 \end{cases}$

**19.** If $AS = SA$ for the shift matrix $S$. Show that $A$ must have this special form:

$$\text{If } \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\text{then } A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

"The subspace of matrices that commute with the shift $S$ has dimension _____."

**20.** Find bases for the following subspaces of $\mathbb{R}^3$

    *a)* All vectors of the form $(a, b, c, 0)$

    *b)* All vectors of the form $(a, b, c, d)$, where $d = a + b$ and $c = a - b$.

    *c)* All vectors of the form $(a, b, c, d)$, where $a = b = c = d$.

**21.** Find a basis for the null space of $A$.

    *a)* $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

    *b)* $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

    *c)* $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

**22.** Find a basis for the subspace of $\mathbb{R}^4$ spanned by the given vectors

    *a)* $(1, 1, -4, -3)$, $(2, 0, 2, -2)$, $(2, -1, 3, 2)$

    *b)* $(-1, 1, -2, 0)$, $(3, 3, 6, 0)$, $(9, 0, 0, 3)$

**23.** Determine whether the given vectors form a basis for the given vector space

    *a)* $\vec{v}_1(3, -2, 1)$, $\vec{v}_2(2, 3, 1)$, $\vec{v}_3(2, 1, -3)$, *in* $\mathbb{R}^3$

    *b)* $\vec{v}_1 = (1, 1, 0, 0)$, $\vec{v}_2 = (0, 1, 1, 0)$, $\vec{v}_3 = (0, 0, 1, 1)$, $\vec{v}_4 = (1, 0, 0, 1)$, *for* $\mathbb{R}^4$

    *c)* $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $M_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $M_{22}$

**24.** Find a basis for, and the dimension of, the null space of the given matrix $\begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & -2 & -2 & 1 \end{bmatrix}$

**25.** Let $\mathbb{R}$ be the set of all real numbers and let $\mathbb{R}^+$ be the set of all positive real numbers. Show that $\mathbb{R}^+$ is a vector space over $\mathbb{R}$ under the addition

$$\alpha \oplus \beta = \alpha\beta \quad \alpha, \ \beta \in \mathbb{R}^+$$

And the scalar multiplication

$$a \odot \alpha = \alpha^a \quad \alpha \in \mathbb{R}^+, \ a \in \mathbb{R}$$

Find the dimension of the vector space. Is $\mathbb{R}^+$ also a vector space if the scalar multiplication is instead defined as

$$a \otimes \alpha = a^\alpha \quad \alpha \in \mathbb{R}^+, \ a \in \mathbb{R}?$$