Solution *Section* 4.2 – General Linear Transformations

Exercise

The matrix $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ gives a shearing transformation $T(x, y) = (x,\ 3x + y)$.

What happens to (1, 0) and (2, 0) on the x-axis.

What happens to the points on the vertical lines $x = 0$ and $x = a$?.

Solution

The points (1, 0) and (2, 0) on the x-axis transform by T to (1, 3) and (2, 6). The horizontal x-axis transforms to the straight line with slope 3 (going through (0, 0) of course). The points on the y-axis are not moved because $T(0, y) – (0, y)$. The y -axis is the line of eigenvectors of T with $\lambda = 1$.

The vertical line $x = a$ is moved up by $3a$, since $3a$ is added to the y component. This is ***shearing***. Vertical lines slide higher as you go from left to right.

Exercise

A nonlinear transformation T is invertible if every $\vec{b}$ in the output space comes from exactly one x in the input space. $T(\vec{x}) = \vec{b}$ always has exactly one solution. Which of these transformation (on real numbers $\vec{x}$ is invertible and what is T^{-1}? None are linear, not even T_3. When you solve $T(\vec{x}) = \vec{b}$, you are inverting T:

$$T_1(\vec{x}) = x^2 \quad T_2(\vec{x}) = x^3 \quad T_3(\vec{x}) = x + 9 \quad T_4(\vec{x}) = e^x \quad T_5(\vec{x}) = \frac{1}{x} \quad \textit{for nonzero } x's$$

Solution

T_1 is not invertible because

$x^2 = 1 \rightarrow x = \pm 1$ and $x^2 = -1$ has no solution.

T_4 is not invertible because

$e^x = -1$ has no solution.

T_2 is invertible.

The solutions to $x^3 = b \rightarrow x = b^{1/3} = T_2^{-1}(b)$

T_3 is invertible.

The solutions to $x + 9 = b \rightarrow x = b - 9 = T_3^{-1}(b)$

T_5 is invertible.

The solutions to $\frac{1}{x} = b \rightarrow x = \frac{1}{b} = T_5^{-1}(b)$

Exercise

If S and T are linear transformations, is $S(T(\vec{v}))$ linear or quadratic?

a) If $S(\vec{v})=\vec{v}$ and $T(\vec{v})=\vec{v}$, then $S(T(\vec{v}))=\vec{v}$ or $\vec{v}^2$?

b) $S(\vec{w}_1+\vec{w}_2)=S(\vec{w}_1)+S(\vec{w}_2)$ and $T(\vec{v}_1+\vec{v}_2)=T(\vec{v}_1)+T(\vec{v}_2)$ combine into

$$S(T(\vec{v}_1+\vec{v}_2))=S(\_\_\_\_\_)=\_\_\_\_+\_\_\_\_\_\_$$

Solution

a) $S(T(\vec{v}))=S(\vec{v})$ $\qquad T(\vec{v})=\vec{v}$

$=\vec{v}$ $\qquad S(\vec{v})=\vec{v}$

b) $S(T(\vec{v}_1+\vec{v}_2))=S(T(\vec{v}_1)+T(\vec{v}_2))$

$=S(T(\vec{v}_1))+S(T(\vec{v}_2))$

It is quadratic.

Exercise

Find the range and kernel (like the column space and nullspace) of T:

a) $T(v_1, v_2)=(v_2, v_1)$

b) $T(v_1, v_2, v_3)=(v_1, v_2)$

c) $T(v_1, v_2)=(0, 0)$

d) $T(v_1, v_2)=(v_1, v_1)$

Solution

a) Range is the line $y = 0$, Kernel is the line $x = y$ in the xy plane.

b) Range is the xy plane, Kernel is the complementary line in $\mathbb{R}^3$.

c) Range is the point (0, 0), Kernel is plane

d) Range is the line $x = y$ in the xy plane, Kernel is the line $x = 0$.

Exercise

M is any 2 by 2 matrix and $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. The transformation T is defined by $T(M) = AM$. What rules of matrix multiplication show that T is linear?

Solution

The distribution law and the association law for multiplication give the linearity

$$\begin{aligned} A(cM + dN) &= A(cM) + A(dN) \\ &= (Ac)M + (Ad)N \\ &= cA(M) + dA(N) \end{aligned}$$

Exercise

Which of these transformations satisfy $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ and which satisfy $T(c\vec{v}) = cT(\vec{v})$?

a) $T(\vec{v}) = \dfrac{\vec{v}}{\|\vec{v}\|}$

b) $T(\vec{v}) = v_1 + v_2 + v_3$

c) $T(\vec{v}) = (v_1,\ 2v_2,\ 3v_3)$

d) $T(\vec{v}) =$ largest component of $\vec{v}$.

Solution

a) This is scaling the vector into a normal vector. This it is impossible that we get additivity, because the sums of normal vectors don't have to be normal. For example $T(0, 1)$ and $T(1, 0)$ for instance. However, true to its name this does have the scaling property. For $\boldsymbol{c}$ value, this value will be canceled from $\vec{v}$ and $\|\vec{v}\|$.

b) This satisfies both. One immediate way to see that it is matrix multiplication by [1, 1, 1], which is a linear operation and thus satisfies both properties.

c) This satisfies both. This a matrix multiplication by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

d) Doesn't satisfy additivity [(0, 1) and (1, 0) still work]. Scaling doesn't work either, if we scale by -1 we now pick out the negative of the smallest component, which doesn't have to be related in any way to the largest component.

Exercise

Consider the basis $S = \left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\}$ for R^3, where $\vec{v}_1 = (1, 1, 1)$ $\vec{v}_2 = (1, 1, 0)$ $\vec{v}_3 = (1, 0, 0)$ and let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation for which

$$T\left(\vec{v}_1\right) = (2, -1, 4), \quad T\left(\vec{v}_2\right) = (3, 0, 1), \quad T\left(\vec{v}_3\right) = (-1, 5, 1)$$

Find a formula for $T\left(\vec{x}_1, \vec{x}_2, \vec{x}_3\right)$, and then use that formula to compute $T(2, 4, -1)$

Solution

Assume: $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

$$\begin{aligned}\left(\vec{x}_1, \vec{x}_2, \vec{x}_3\right) &= c_1(1, 1, 1) + c_2(1, 1, 0) + c_3(1, 0, 0)\\ &= \left(c_1 + c_2 + c_3, c_1 + c_2, c_1\right)\end{aligned}$$

$$\begin{cases} c_1 + c_2 + c_3 = x_1 \\ c_1 + c_2 \qquad = x_2 \\ c_1 \qquad\qquad = x_3 \end{cases}$$

$$\begin{cases} c_3 = x_1 - x_2 \\ c_2 = x_2 - x_3 \\ c_1 = x_3 \end{cases}$$

$$\begin{aligned}T\left(\vec{x}_1, \vec{x}_2, \vec{x}_3\right) &= x_3 T\left(\vec{v}_1\right) + \left(x_2 - x_3\right)T\left(\vec{v}_2\right) + \left(x_1 - x_2\right)T\left(\vec{v}_3\right)\\ &= x_3(2, -1, 4) + \left(x_2 - x_3\right)(3, 0, 1) + \left(x_1 - x_2\right)(-1, 5, 1)\\ &= \left(2x_3 + 3x_2 - 3x_3 - x_1 + x_2,\ -x_3 + 5x_1 - 5x_2,\ 4x_3 + x_2 - x_3 + x_1 - x_2\right)\\ &= \left(-x_1 + 4x_2 - x_3,\ 5x_1 - 5x_2 - x_3,\ x_1 + 3x_3\right)\end{aligned}$$

$$\begin{aligned}T(2, 4, -1) &= (-2 + 16 + 1,\ 10 - 20 + 1,\ 2 - 3)\\ &\underline{= (15, -9, -1)}\end{aligned}$$

Exercise

Consider the basis $S=\left\{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\right\}$ for R^3, where $\vec{v}_1=(1,\ 2,\ 1)$ $\vec{v}_2=(2,\ 9,\ 0)$ $\vec{v}_3=(3,\ 3,\ 4)$ and let $T:\ \mathbb{R}^3\to\mathbb{R}^3$ be the linear transformation for which

$$T\left(\vec{v}_1\right)=(1,\ 0),\quad T\left(\vec{v}_2\right)=(-1,\ 1),\quad T\left(\vec{v}_3\right)=(0,\ 1)$$

Find a formula for $T\left(\vec{x}_1,\ \vec{x}_2,\ \vec{x}_3\right)$, and then use that formula to compute $T(7,\ 13,\ 7)$

Solution

Assume: $\vec{u}=c_1\vec{v}_1+\ c_2\vec{v}_2+\ c_3\vec{v}_3$

$$\left(\vec{x}_1,\ \vec{x}_2,\ \vec{x}_3\right)=c_1\,(1,\ 2,\ 1)+\ c_2\,(2,\ 9,\ 0)+\ c_3\,(3,\ 3,\ 4)$$

$$=\left(c_1+2c_2+3c_3,\ 2c_1+9c_2+3c_3,\ c_1+4c_3\right)$$

$$\begin{cases} c_1+2c_2+3c_3=x_1 \\ 2c_1+9c_2+3c_3=x_2 \\ c_1 \qquad\quad +4c_3=x_3 \end{cases}$$

$$\begin{cases} c_1+7c_2=x_2-x_1 \\ \\ c_1+4c_3=x_3 \end{cases}$$

$$2c_1+\frac{9}{7}x_2-\frac{9}{7}x_1-\frac{9}{7}c_1+\frac{3}{4}x_3-\frac{3}{4}c_1=x_2$$

$$2c_1-\frac{9}{7}c_1-\frac{3}{4}c_1=x_2-\frac{9}{7}x_2+\frac{9}{7}x_1-\frac{3}{4}x_3$$

$$-\frac{1}{28}c_1=\frac{9}{7}x_1-\frac{2}{7}x_2-\frac{3}{4}x_3$$

$$\boxed{c_1=-36x_1+8x_2+21x_3}$$

$$c_3=\frac{1}{4}x_3-\frac{1}{4}c_1$$

$$\boxed{c_3=9x_1-2x_2-5x_3}$$

$$c_2=\frac{1}{7}x_2-\frac{1}{7}x_1-\frac{1}{7}c_1$$

$$c_2=\frac{1}{7}x_2-\frac{1}{7}x_1+\frac{36}{7}x_1-\frac{8}{7}x_2-3x_3$$

$$\boxed{c_2=5x_1-x_2-3x_3}$$

$$T\left(\vec{x}_1, \vec{x}_2, \vec{x}_3\right) = \left(-36x_1 + 8x_2 + 21x_3\right)T\left(\vec{v}_1\right) + \left(5x_1 - x_2 - 3x_3\right)T\left(\vec{v}_2\right)$$
$$+\left(9x_1 - 2x_2 - 5x_3\right)T\left(\vec{v}_3\right)$$
$$= \left(-36x_1 + 8x_2 + 21x_3\right)(1,\ 0) + \left(5x_1 - x_2 - 3x_3\right)(-1,\ 1) + \left(9x_1 - 2x_2 - 5x_3\right)(0,\ 1)$$
$$= \left(36x_1 - 8x_2 + 21x_3 - 5x_1 + x_2 + 3x_3,\ 5x_1 - x_2 - 3x_3 + 9x_1 - 2x_2 - 5x_3\right)$$
$$= \left(41x_1 + 9x_2 + 24x_3,\ 14x_1 - 3x_2 - 8x_3\right)$$

$$T(7,\ 13,\ 7) = \left(37(7) - 13(13) + 24(7),\ \ 8(7) + 3(13) - 8(7)\right)$$
$$\underline{= (-2,\ 3)}$$

Exercise

let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be vectors in a vector space V, and let $T: V \to R^3$ be the linear transformation for which

$$T\left(\vec{v}_1\right) = (1,\ -1,\ 2),\quad T\left(\vec{v}_2\right) = (0,\ 3,\ 2),\quad T\left(\vec{v}_3\right) = (-3,\ 1,\ 2).$$

Find $T\left(2\vec{v}_1 - 3\vec{v}_2 + 4\vec{v}_3\right)$

Solution

$$T\left(2\vec{v}_1 - 3\vec{v}_2 + 4\vec{v}_3\right) = 2T\left(\vec{v}_1\right) - 3T\left(\vec{v}_2\right) + 4T\left(\vec{v}_3\right)$$
$$= 2(1,\ -1,\ 2) - 3(0,\ 3,\ 2) + 4(-3,\ 1,\ 2)$$
$$= (2,\ -2,\ 4) - (0,\ 9,\ 6) + (-12,\ 4,\ 8)$$
$$\underline{= (-10,\ -7,\ 6)}$$

Exercise

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operation given by the formula $T(x,\ y) = (2x - y,\ -8x + 4y)$

Which of the following vectors are in $R(T)$

a) $(1,\ -4)$ ***b***) $(5,\ 0)$ ***c***) $(-3,\ 12)$

Solution

a) $T(x, y) = (2x - y,\ -8x + 4y) = (1,\ -4)$

$$\begin{cases} 2x - y = 1 \\ -8x + 4y = -4 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ -8 & 4 & -4 \end{array}\right] \quad R_2 + 4R_1$$

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \quad \frac{1}{2}R_1$$

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]$$

This is a consistent system, therefore (1, −4) is in $R(T)$

b) $T(x,y) = (2x - y,\ -8x + 4y) = (5,\ 0)$

$$\begin{cases} 2x - y = 5 \\ -8x + 4y = 0 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & -1 & 5 \\ -8 & 4 & 0 \end{array}\right] \quad R_2 + 4R_1$$

$$\left[\begin{array}{cc|c} 2 & -1 & 5 \\ 0 & 0 & 20 \end{array}\right] \quad \rightarrow\ 0 \neq 20$$

This is an inconsistent system, therefore (5, 0) is not in $R(T)$

c) $T(x,y) = (2x - y,\ -8x + 4y) = (-3,\ 12)$

$$\begin{cases} 2x - y = -3 \\ -8x + 4y = 12 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & -1 & -3 \\ -8 & 4 & 12 \end{array}\right] \quad R_2 + 4R_1$$

$$\left[\begin{array}{cc|c} 2 & -1 & -3 \\ 0 & 0 & 0 \end{array}\right] \quad \frac{1}{2}R_1$$

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \end{array}\right]$$

This is a consistent system, therefore (−3, 12) is in $R(T)$

Exercise

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operation given by the formula $T(x, y) = (2x - y, \ -8x + 4y)$

Which of the following vectors are in $\ker(T)$

a) $(5,\ 10)$ ***b***) $(3,\ 2)$ ***c***) $(1,\ 1)$

Solution

a) $T(5,\ 10) = (10 - 10,\ -40 + 40)$

$= (0,\ 0)$

Therefore (5, 10) is in $\ker(T)$

b) $T(3,\ 2) = (6 - 2,\ -24 + 8)$

$= (4,\ -16)$

Therefore (3, 2) is not in $\ker(T)$

c) $T(1,\ 1) = (2 - 1,\ -8 + 4)$

$= (1,\ -4)$

Therefore (1, 1) is not in $\ker(T)$

Exercise

Let $T: \mathbb{R}^4 \to \mathbb{R}^3$ be the linear operation given by the formula

$$T\left(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\right) = \left(4x_1 + x_2 - 2x_3 - 3x_4,\ \ 2x_1 + x_2 + x_3 - 4x_4,\ \ 6x_1 - 9x_3 + 9x_4\right)$$

Which of the following vectors are in $R(T)$

a) $(0,\ 0,\ 6)$ ***b***) $(1,\ 3,\ 0)$ ***c***) $(2,\ 4,\ 1)$

Solution

a) $T\left(x_1, x_2, x_3, x_4\right) = \left(4x_1 + x_2 - 2x_3 - 3x_4,\ 2x_1 + x_2 + x_3 - 4x_4,\ 6x_1 - 9x_3 + 9x_4\right) = (0,\ 0,\ 6)$

$$\begin{cases} 4x_1 + x_2 - 2x_3 - 3x_4 = 0 \\ 2x_1 + x_2 + x_3 - 4x_4 = 0 \\ 6x_1 \qquad\ \ - 9x_3 + 9x_4 = 6 \end{cases}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 0 \\ 2 & 1 & 1 & -4 & 0 \\ 6 & 0 & -9 & 9 & 6 \end{array}\right] \quad \begin{array}{c} \\ 2R_2 - R_1 \\ 2R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & -3 & -12 & 27 & 12 \end{array}\right] \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 2 & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & 0 & 0 & 12 & 12 \end{array}\right] \quad \begin{array}{l} \\ \\ \frac{1}{12}R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 2 & 0 \\ 0 & 1 & 4 & -5 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 + 5R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 0 & -2 \\ 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} \frac{1}{4}R_1 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -\frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

This is a consistent system, therefore (0, 0, 6) is in $R(T)$

b) $T(x_1, x_2, x_3, x_4) = (4x_1 + x_2 - 2x_3 - 3x_4,\ 2x_1 + x_2 + x_3 - 4x_4,\ 6x_1 - 9x_3 + 9x_4) = (1,\ 3,\ 0)$

$$\begin{cases} 4x_1 + x_2 - 2x_3 - 3x_4 = 1 \\ 2x_1 + x_2 + x_3 - 4x_4 = 3 \\ 6x_1 \qquad - 9x_3 + 9x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 1 \\ 2 & 1 & 1 & -4 & 3 \\ 6 & 0 & -9 & 9 & 0 \end{array}\right] \quad \begin{array}{l} \\ 2R_2 - R_1 \\ 2R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 1 \\ 0 & 1 & 4 & -5 & 5 \\ 0 & -3 & -12 & 27 & -3 \end{array}\right] \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 2 & -4 \\ 0 & 1 & 4 & -5 & 5 \\ 0 & 0 & 0 & 12 & 12 \end{array}\right] \quad \begin{array}{l} \\ \\ \frac{1}{12}R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 2 & -4 \\ 0 & 1 & 4 & -5 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 + 5R_3 \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 0 & -6 \\ 0 & 1 & 4 & 0 & 10 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad \tfrac{1}{4}R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -\frac{3}{2} & 0 & -\frac{3}{2} \\ 0 & 1 & 4 & 0 & 10 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

This is a consistent system, therefore (1, 3, 0) is in $R(T)$

c) $T\left(x_1, x_2, x_3, x_4\right) = \left(4x_1 + x_2 - 2x_3 - 3x_4,\ 2x_1 + x_2 + x_3 - 4x_4,\ 6x_1 - 9x_3 + 9x_4\right) = (2,\ 4,\ 1)$

$$\begin{cases} 4x_1 + x_2 - 2x_3 - 3x_4 = 2 \\ 2x_1 + x_2 + x_3 - 4x_4 = 4 \\ 6x_1 \qquad - 9x_3 + 9x_4 = 1 \end{cases}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 2 \\ 2 & 1 & 1 & -4 & 4 \\ 6 & 0 & -9 & 9 & 1 \end{array}\right] \quad \begin{array}{l} \\ 2R_2 - R_1 \\ 2R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 2 \\ 0 & 1 & 4 & -5 & 6 \\ 0 & -3 & -12 & 27 & -4 \end{array}\right] \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 2 & -4 \\ 0 & 1 & 4 & -5 & 6 \\ 0 & 0 & 0 & 12 & 14 \end{array}\right] \quad \begin{array}{l} \\ \\ \tfrac{1}{12}R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 2 & -4 \\ 0 & 1 & 4 & -5 & 6 \\ 0 & 0 & 0 & 1 & \frac{7}{6} \end{array}\right] \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 + 5R_3 \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 4 & 0 & -6 & 0 & -\frac{19}{3} \\ 0 & 1 & 4 & 0 & \frac{71}{6} \\ 0 & 0 & 0 & 1 & \frac{7}{6} \end{array}\right] \qquad \frac{1}{4}R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -\frac{3}{2} & 0 & -\frac{19}{12} \\ 0 & 1 & 4 & 0 & \frac{71}{6} \\ 0 & 0 & 0 & 1 & \frac{7}{6} \end{array}\right]$$

This is a consistent system, therefore (2, 4, 1) is in $R(T)$

Exercise

Let $T: \mathbb{R}^4 \to \mathbb{R}^3$ be the linear operation given by the formula

$$T\left(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\right) = \left(4x_1 + x_2 - 2x_3 - 3x_4,\ \ 2x_1 + x_2 + x_3 - 4x_4,\ \ 6x_1 - 9x_3 + 9x_4\right)$$

Which of the following vectors are in $\ker(T)$

a) $(3,\ -8,\ 2,\ 0)$ ***b***) $(0,\ 0,\ 0,\ 1)$ ***c***) $(0,\ -4,\ 1,\ 0)$

Solution

a) $T(3,\ -8,\ 2,\ 0) = (12-8-4,\ 6-8+2,\ 18-18)$

$$= (0,\ 0,\ 0)$$

Therefore, (3, –8, 2, 0) is in $\ker(T)$

b) $T(0,\ 0,\ 0,\ 1) = (-3,\ -4,\ 9)$

Therefore, (0, 0, 0, 1) is ***not*** in $\ker(T)$

c) $T(0,\ -4,\ 1,\ 0) = (-4-2,\ -4+1,\ -9)$

$$= (-6,\ -3,\ -9)$$

Therefore, (0, –4, 1, 0) is ***not*** in $\ker(T)$

Exercise

Determine if the given function T is a linear transformation

$$T: M_{22} \to M_{22} \quad by \quad T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2ab & 3cd \\ 0 & 0 \end{bmatrix}$$

Solution

Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$$T(A+B) = T\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2(a_1 + a_2)(b_1 + b_2) & 3(c_1 + c_2)(d_1 + d_2) \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2a_1b_1 + 2a_1b_2 + 2a_2b_1 + 2a_2b_2 & 3c_1d_1 + 3c_1d_2 + 3c_2d_1 + 3c_2d_2 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2a_1b_1 & 3c_1d_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2a_2b_2 & 3c_2d_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2a_1b_2 + 2a_2b_1 & 3c_1d_2 + 3c_2d_1 \\ 0 & 0 \end{bmatrix}$$

$$= T(A) + T(B) + \begin{bmatrix} 2a_1b_2 + 2a_2b_1 & 3c_1d_2 + 3c_2d_1 \\ 0 & 0 \end{bmatrix}$$

$$\underline{\neq T(A) + T(B)}$$

Function T is NOT a linear transformation.

Exercise

Determine if the given function T is a linear transformation

$$T: M_{22} \to M_{22} \quad by \quad T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+d & 0 \\ 0 & b+c \end{bmatrix}$$

Solution

Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$$T(A+B) = T\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1+a_2+d_1+d_2 & 0 \\ 0 & b_1+b_2+c_1+c_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1+d_1 & 0 \\ 0 & b_1+c_1 \end{bmatrix} + \begin{bmatrix} a_2+d_2 & 0 \\ 0 & b_2+c_2 \end{bmatrix}$$

$$= T(A)+T(B) \quad \surd$$

$$T(kA) = T\left(k\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} ka_1+kd_1 & 0 \\ 0 & kb_1+kc_1 \end{bmatrix}$$

$$= \begin{bmatrix} k(a_1+d_1) & 0 \\ 0 & k(b_1+c_1) \end{bmatrix}$$

$$= k\begin{bmatrix} a_1+d_1 & 0 \\ 0 & b_1+c_1 \end{bmatrix}$$

$$= kT(A) \quad \surd$$

Since $T(A+B)=T(A)+T(B)$ and $T(kA)=kT(A)$, then function T is a linear transformation.

Exercise

Determine if the given function T is a linear transformation where A is fixed 2×3 matrix

$$T: M_{22} \to M_{23} \quad by \quad T(B)=BA$$

Solution

$$T(B+C) = (B+C)A$$

$$= BA+CA$$

$$= T(B)+T(C)$$

$$T(rB) = (rB)A$$

$$= r(BA)$$

$$= rT(B)$$

Function T is a linear transformation

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T = f_A$

$$T(x, y) = \left(x^2, y\right)$$

Solution

Let $\vec{u} = \left(x_1, y_1\right)$ *and* $\vec{v} = \left(x_2, y_2\right)$

$$\begin{aligned} T(\vec{u}+\vec{v}) &= T\left(x_1 + x_2,\ y_1 + y_2\right) \\ &= \left(\left(x_1 + x_2\right)^2,\ y_1 + y_2\right) \\ &= \left(x_1^2 + x_2^2 + 2x_1x_2,\ y_1 + y_2\right) \\ &= \left(x_1^2,\ y_1\right) + \left(x_2^2,\ y_2\right) + \left(2x_1x_2,\ 0\right) \\ &= T(\vec{u}) + T(\vec{v}) + \left(2x_1x_2,\ 0\right) \\ &\neq T(\vec{u}) + T(\vec{v}) \end{aligned}$$

The function T is ***not*** a linear transformation.

Domain: $T:\ \mathbb{R}^2 \to \mathbb{R}^2$

Exercise

Determine if the given function T is a linear transformation. Also, give the domain and range of T; if T is linear, find the A such $T = f_A$.

$$T(x, y, z) = (2x + y,\ x - y + z)$$

Solution

Let $\vec{u} = \left(x_1, y_1, z_1\right)$ *and* $\vec{v} = \left(x_2, y_2, z_2\right)$

$$\begin{aligned} T(\vec{u}+\vec{v}) &= T\left(x_1 + x_2, y_1 + y_2, z_1 + z_2\right) \\ &= \left(2\left(x_1 + x_2\right) + y_1 + y_2,\ \ x_1 + x_2 - \left(y_1 + y_2\right) + z_1 + z_2\right) \\ &= \left(2x_1 + y_1 + 2x_2 + y_2,\ \ x_1 - y_1 + z_1 + x_2 - y_2 + z_2\right) \\ &= \left(2x_1 + y_1,\ x_1 - y_1 + z_1\right) + \left(2x_2 + y_2,\ x_2 - y_2 + z_2\right) \\ &= T\left(x_1, y_1, z_1\right) + T\left(x_2, y_2, z_2\right) \end{aligned}$$

$$= T(\vec{u}) + T(\vec{v})$$

$$T(r\vec{u}) = T(rx_1,\ ry_1,\ rz_1)$$

$$= (2rx_1 + ry_1,\ rx_1 - ry_1 + rz_1)$$

$$= r(2x_1 + y_1,\ x_1 - y_1 + z_1)$$

$$= rT(\vec{u})$$

Since $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{u}) = rT(\vec{u})$, then function T is a linear transformation.

Domain: $T: \ \mathbb{R}^3 \to \mathbb{R}^2$

$$T(x,\ y,\ z) = (2x + y,\ \ x - y + z)$$

$$= \begin{pmatrix} 2x + y \\ x - y + z \end{pmatrix}$$

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ A = & \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \end{array}$$

Exercise

Determine if the given function T is a linear transformation. Also, give the domain and range of T; if T is linear, find the A such $T = f_A$.

$$T(x,\ y,\ z) = (z - x,\ z - y)$$

Solution

Let $\vec{u} = (x_1,\ y_1,\ z_1)$ *and* $\vec{v} = (x_2,\ y_2,\ z_2)$

$$T(\vec{u}+\vec{v}) = T(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$$

$$= \left(z_1 + z_2 - (x_1 + x_2),\ z_1 + z_2 - (y_1 + y_2)\right)$$

$$= (z_1 + z_2 - x_1 - x_2,\ \ z_1 + z_2 - y_1 - y_2)$$

$$= (z_1 - x_1,\ \ z_1 - y_1) + (z_2 - x_2,\ \ z_2 - y_2)$$

$$= T(x_1,\ y_1,\ z_1) + T(x_2,\ y_2,\ z_2)$$

$$= T(\vec{u}) + T(\vec{v})$$

$$T(r\vec{u}) = T(rx_1,\ ry_1,\ rz_1)$$

$$= \left(rz_1 - rx_1,\ rz_1 - ry_1\right)$$

$$= r\left(z_1 - x_1,\ z_1 - y_1\right)$$

$$= rT(\vec{u})$$

Since $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{u}) = rT(\vec{u})$, then function T is a linear transformation.

Domain: $T:\ \mathbb{R}^3 \to \mathbb{R}^2$

$$T(x, y, z) = (z - x, z - y)$$

$$= \begin{pmatrix} -x + z \\ -y + z \end{pmatrix}$$

$$\begin{array}{cc} & \begin{matrix} x & y & z \end{matrix} \\ A = & \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T = f_A$

$$T\left(x_1, x_2, x_3\right) = \left(2x_1 - x_2 + x_3,\ \ x_2 - 4x_3\right)$$

Solution

Let $\vec{u} = \left(x_1, x_2, x_3\right)$ *and* $\vec{v} = \left(y_1, y_2, y_3\right)$

$$T(\vec{u}+\vec{v}) = T\left(x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3\right)$$

$$= \left(2\left(x_1 + y_1\right) - x_2 - y_2 + x_3 + y_3\ ,\ x_2 + y_2 - 4\left(x_3 + y_3\right)\right)$$

$$= \left(2x_1 + 2y_1 - x_2 - y_2 + x_3 + y_3\ ,\ x_2 + y_2 - 4x_3 - 4y_3\right)$$

$$= \left(2x_1 - x_2 + x_3\ , x_2 - 4x_3\right) + \left(2y_1 - y_2 + y_3\ , y_2 - 4y_3\right)$$

$$= T(\vec{u}) + T(\vec{v})$$

$$T(r\vec{u}) = T\left(rx_1, rx_2, rx_3\right)$$

$$= \left(2rx_1 - rx_2 + rx_3\ , rx_2 - 4rx_3\right)$$

$$= r\left(2x_1 - x_2 + x_3\ , x_2 - 4x_3\right)$$

$$= rT(\vec{u})$$

Since $T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v})$ and $T(r\vec{u})=rT(\vec{u})$, then function T is a linear transformation.

Domain: $T:\ \mathbb{R}^3 \to \mathbb{R}^2$

$$T\left(x_1, x_2, x_3\right)=\left(2x_1-x_2+x_3,\ \ x_2-4x_3\right)$$
$$=\begin{pmatrix} 2x_1-x_2+x_3 \\ x_2-4x_3 \end{pmatrix}$$

$$A=\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 2 & -1 & 1 \\ 0 & -1 & -4 \end{pmatrix} \end{array}$$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T=f_A$

$$T\left(x_1, x_2\right)=\left(2x_1-x_2,\ -3x_1+x_2,\ 2x_1-3x_2\right)$$

Solution

Let $\vec{u}=\left(u_1, u_2\right)$ *and* $\vec{v}=\left(v_1, v_2\right)$

$$T(\vec{u}+\vec{v})=T\left(u_1+v_1,\ u_2+v_2\right)$$
$$=\left(2\left(u_1+v_1\right)-\left(u_2+v_2\right),\ -3\left(u_1+v_1\right)+\left(u_2+v_2\right),\ 2\left(u_1+v_1\right)-3\left(u_2+v_2\right)\right)$$
$$=\left(2u_1+2v_1-u_2-v_2,\ -3u_1-3v_1+u_2+v_2,\ 2u_1+2v_1-3u_2-3v_2\right)$$
$$=\left(\left(2u_1-u_2\right)+\left(2v_1-v_2\right),\ \left(-3u_1+u_2\right)+\left(-3v_1+v_2\right),\ \left(2u_1-3u_2\right)+\left(2v_1-3v_2\right)\right)$$
$$=\left(2u_1-u_2,\ -3u_1+u_2,\ 2u_1-3u_2\right)+\left(2v_1-v_2,\ -3v_1+v_2,\ 2v_1-3v_2\right)$$
$$=T(\vec{u})+T(\vec{v})$$

$$T(r\vec{u})=T\left(r\left(u_1, u_2\right)\right)$$
$$=T\left(ru_1, ru_2\right)$$
$$=\left(2ru_1-ru_2,\ -3ru_1+ru_2,\ 2ru_1-3ru_2\right)$$
$$=r\left(2u_1-u_2,\ -3u_1+u_2,\ 2u_1-3u_2\right)$$
$$=rT(\vec{u})$$

Since $T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v})$ and $T(r\vec{u})=rT(\vec{u})$, then function T is a linear transformation.

Domain: $T: \mathbb{R}^2 \to \mathbb{R}^3$

$$T(x_1, x_2) = (2x_1 - x_2,\ -3x_1 + x_2,\ 2x_1 - 3x_2)$$

$$= \begin{pmatrix} 2x_1 - x_2 \\ -3x_1 + x_2 \\ 2x_1 - 3x_2 \end{pmatrix}$$

$$\begin{array}{cc} x_1 & x_2 \end{array}$$
$$A = \begin{pmatrix} 2 & -1 \\ -3 & 1 \\ 2 & -3 \end{pmatrix}$$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T = f_A$

$$T(x_1, x_2) = (x_1 + 4x_2,\ 0,\ x_1 - 3x_2,\ x_1)$$

Solution

Let $\vec{u} = (u_1, u_2)$ *and* $\vec{v} = (v_1, v_2)$

$$T(\vec{u} + \vec{v}) = T(u_1 + v_1,\ u_2 + v_2)$$

$$= \big((u_1 + v_1) + 4(u_2 + v_2),\ 0,\ (u_1 + v_1) - 3(u_2 + v_2),\ (u_1 + v_1)\big)$$

$$= (u_1 + v_1 + 4u_2 + 4v_2,\ 0,\ u_1 + v_1 - 3u_2 - 3v_2,\ u_1 + v_1)$$

$$= \big((u_1 + 4u_2) + (v_1 + 4v_2),\ 0,\ (u_1 - 3u_2) + (v_1 - 3v_2),\ u_1 + v_1\big)$$

$$= (u_1 + 4u_2,\ 0,\ u_1 - 3u_2,\ u_1) + (v_1 + 4v_2,\ 0,\ v_1 - 3v_2,\ v_1)$$

$$= T(\vec{u}) + T(\vec{v})$$

$$T(r\vec{u}) = T\big(r(u_1, u_2)\big)$$

$$= T(ru_1, ru_2)$$

$$= (ru_1 + 4ru_2,\ 0,\ ru_1 - 3ru_2,\ ru_1)$$

$$= r(u_1 + 4u_2,\ 0,\ u_1 - 3u_2,\ u_1)$$

$$= rT(\vec{u})$$

Since $T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v})$ and $T(r\vec{u})=rT(\vec{u})$, then function T is a linear transformation.

Domain: $T:\ \mathbb{R}^2 \to \mathbb{R}^4$

$$T(x_1,\ x_2)=(x_1+4x_2,\ 0,\ x_1-3x_2,\ x_1)$$

$$=\begin{pmatrix} x_1+4x_2 \\ 0 \\ x_1-3x_2 \\ x_1 \end{pmatrix}$$

$$\begin{array}{cc} & \begin{array}{cc} x_1 & x_2 \end{array} \\ A= & \begin{pmatrix} 1 & 4 \\ 0 & 0 \\ 1 & -3 \\ 1 & 0 \end{pmatrix} \end{array}$$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T=f_A$

$$T(x_1,\ x_2,\ x_3)=(x_1-5x_2+4x_3,\quad x_2-6x_3)$$

Solution

Let $\vec{u}=(u_1,\ u_2,\ u_3)$ *and* $\vec{v}=(v_1,\ v_2,\ v_3)$

$$\begin{aligned}T(\vec{u}+\vec{v})&=T(u_1+v_1,\ u_2+v_2,\ u_3+v_3)\\ &=\big((u_1+v_1)-5(u_2+v_2)+4(u_3+v_3),\ \ (u_2+v_2)-6(u_3+v_3)\big)\\ &=(u_1+v_1-5u_2-5v_2+4u_3+4v_3,\ u_2+v_2-6u_3-6v_3)\\ &=\big((u_1-5u_2+4u_3)+(v_1-5v_2+4v_3),\ \ (u_2-6u_3)+(v_2-6v_3)\big)\\ &=(u_1-5u_2+4u_3,\ u_2-6u_3)+(v_1-5v_2+4v_3,\ v_2-6v_3)\\ &=T(\vec{u})+T(\vec{v})\end{aligned}$$

$$\begin{aligned}T(r\vec{u})&=T\big(r(u_1,\ u_2,\ u_3)\big)\\ &=T(ru_1,\ ru_2,\ ru_3)\\ &=(ru_1-5ru_2+4ru_3,\ ru_2-6ru_3)\end{aligned}$$

$$= r\left(u_1 - 5u_2 + 4u_3,\ u_2 - 6u_3\right)$$
$$= rT(\vec{u})$$

Since $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{u}) = rT(\vec{u})$, then function T is a linear transformation.

Domain: $T:\ \mathbb{R}^3 \to \mathbb{R}^2$

$$T\left(x_1, x_2, x_3\right) = \left(x_1 - 5x_2 + 4x_3,\ \ x_2 - 6x_3\right)$$
$$= \begin{pmatrix} x_1 - 5x_2 + 4x_3 \\ x_2 - 6x_3 \end{pmatrix}$$

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ A = & \begin{pmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{pmatrix} \end{array}$$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T = f_A$

$$T\left(x_1, x_2, x_3, x_4\right) = \left(x_1 + 2x_2,\ 0,\ 2x_2 + x_4,\ x_2 - x_4\right)$$

Solution

Let $\vec{u} = \left(u_1, u_2, u_3, u_4\right)$ *and* $\vec{v} = \left(v_1, v_2, v_3, v_4\right)$

$$T(\vec{u}+\vec{v}) = T\left(u_1 + v_1,\ u_2 + v_2,\ u_3 + v_3,\ u_4 + v_4\right)$$
$$= \left(\left(u_1 + v_1\right) + 2\left(u_2 + v_2\right),\ 0,\ 2\left(u_2 + v_2\right) - 3\left(u_4 + v_4\right),\ \left(u_2 + v_2\right) - \left(u_4 + v_4\right)\right)$$
$$= \left(u_1 + v_1 + 2u_2 + 2v_2,\ 0,\ 2u_2 + 2v_2 - 3u_4 - 3v_4,\ u_2 + v_2 - u_4 - v_4\right)$$
$$= \left(\left(u_1 + 2u_2\right) + \left(v_1 + 2v_2\right),\ 0,\ \left(2u_2 - 3u_4\right) + \left(2v_2 - 3v_4\right),\ \left(u_2 - u_4\right) + \left(v_2 - v_4\right)\right)$$
$$= \left(u_1 + 2u_2,\ 0,\ 2u_2 - 3u_4,\ u_2 - u_4\right) + \left(v_1 + 2v_2,\ 0,\ 2v_2 - 3v_4,\ v_2 - v_4\right)$$
$$= T(\vec{u}) + T(\vec{v})$$

$$T(r\vec{u}) = T\left(r\left(u_1, u_2, u_3, u_4\right)\right)$$
$$= T\left(ru_1, ru_2, ru_3, ru_4\right)$$
$$= \left(ru_1 + 2ru_2,\ 0,\ 2ru_2 - 3ru_4,\ ru_2 - ru_4\right)$$

$$= r\left(u_1 + 2u_2,\ \ 0,\ \ 2u_2 - 3u_4,\ \ u_2 - u_4\right)$$
$$= rT\left(\vec{u}\right)$$

Since $T\left(\vec{u}+\vec{v}\right) = T\left(\vec{u}\right) + T\left(\vec{v}\right)$ and $T\left(r\vec{u}\right) = rT\left(\vec{u}\right)$, then function T is a linear transformation.

Domain: $T:\ \mathbb{R}^4 \to \mathbb{R}^4$

$$T\left(x_1,\ x_2,\ x_3,\ x_4\right) = \left(x_1 + 2x_2,\ \ 0,\ \ 2x_2 + x_4,\ \ x_2 - x_4\right)$$
$$= \begin{pmatrix} x_1 + 2x_2 \\ 0 \\ 2x_2 + x_4 \\ x_2 - x_4 \end{pmatrix}$$

$$\begin{array}{cc} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ A = & \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \end{array}$$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T = f_A$

$$T\left(x_1,\ x_2,\ x_3,\ x_4\right) = 3x_1 + 4x_3 - 2x_4$$

Solution

Let $\vec{u} = \left(u_1,\ u_2,\ u_3,\ u_4\right)$ *and* $\vec{v} = \left(v_1,\ v_2,\ v_3,\ v_4\right)$

$$T\left(\vec{u}+\vec{v}\right) = T\left(u_1 + v_1,\ \ u_2 + v_2,\ \ u_3 + v_3,\ \ u_4 + v_4\right)$$
$$= 3\left(u_1 + v_1\right) + 4\left(u_3 + v_3\right) - 2\left(u_4 + v_4\right)$$
$$= 3u_1 + 3v_1 + 4u_3 + 4v_3 - 2u_4 - 2v_4$$
$$= \left(3u_1 + 4u_3 - 2u_4\right) + \left(3v_1 + 4v_3 - 2v_4\right)$$
$$= T\left(\vec{u}\right) + T\left(\vec{v}\right)$$

$$T\left(r\vec{u}\right) = T\left(r\left(u_1,\ u_2,\ u_3,\ u_4\right)\right)$$
$$= T\left(ru_1,\ ru_2,\ ru_3,\ ru_4\right)$$

$$= 3ru_1 + 4ru_3 - 2ru_4$$
$$= r\left(3u_1 + 4u_3 - 2u_4\right)$$
$$= rT(\vec{u})$$

Since $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(r\vec{u}) = rT(\vec{u})$, then function T is a linear transformation.

Domain: $T: \ \mathbb{R}^4 \to \mathbb{R}^1$

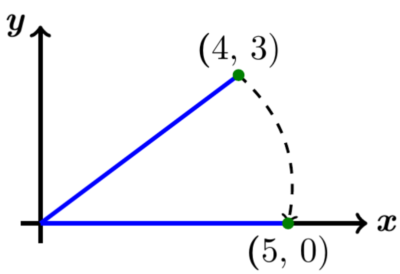
$$T\left(x_1, x_2, x_3, x_4\right) = 3x_1 + 4x_3 - 2x_4$$
$$= \left(3x_1 + 4x_3 - 2x_4\right)$$

$$\begin{array}{cc} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ A = & \begin{pmatrix} 3 & 0 & 4 & -2 \end{pmatrix} \end{array}$$

Exercise

A Givens rotation is a linear transformation from $\mathbb{R}^n$ *to* $\mathbb{R}^n$ used in computer to create a zero entry in a vector (usually a column of a matrix). The standard matrix of a Givens rotation in $\mathbb{R}^2$ has the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}; \qquad a^2 + b^2 = 1$$



A Givens rotation in $\mathbb{R}^2$

Find a and b that $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is rotated into $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$.

Solution

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$$\begin{cases} 4a - 3b = 5 & (1) \\ 4b + 3a = 0 & \rightarrow \quad b = -\frac{3a}{4} \end{cases}$$

$$(1) \quad \rightarrow \quad 4a - 3\left(-\frac{3a}{4}\right) = 5$$

$$4a + \frac{9a}{4} = 5$$

$$\left(4 + \frac{9}{4}\right)a = 5$$

$$\frac{25}{4}a = 5$$

$$a = \frac{4}{5}$$

$$b = -\frac{3}{4}\frac{4}{5}$$

$$b = -\frac{3}{5}$$

$$A = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}$$