

## ***Solution***      **Section 3.1 – Inner Products**

### ***Exercise***

Let  $\langle \vec{u}, \vec{v} \rangle$  be the Euclidean inner product on  $\mathbb{R}^2$ , and let  $\vec{u} = (1, 1)$ ,  $\vec{v} = (3, 2)$ ,  $\vec{w} = (0, -1)$ , and  $k = 3$ . Compute the following.

- |  |   |                             |
|--|---|-----------------------------|
| a) $\langle \vec{u}, \vec{v} \rangle$  | c) $\langle \vec{u} + \vec{v}, \vec{w} \rangle$ | e) $d(\vec{u}, \vec{v})$    |
| b) $\langle k\vec{v}, \vec{w} \rangle$ | d) $\ \vec{v}\ $                                | f) $\ \vec{u} - k\vec{v}\ $ |

### ***Solution***

$$\begin{aligned} a) \quad \langle \vec{u}, \vec{v} \rangle &= 1(3) + 1(2) \\ &= 5 \end{aligned}$$

$$\begin{aligned} b) \quad \langle k\vec{v}, \vec{w} \rangle &= \langle 3\vec{v}, \vec{w} \rangle \\ &= 9 \cdot 0 + 6 \cdot (-1) \\ &= -6 \end{aligned}$$

$$\begin{aligned} c) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \\ &= 1 \cdot 0 + 1 \cdot (-1) + 3 \cdot 0 + 2 \cdot (-1) \\ &= -3 \end{aligned}$$

$$\begin{aligned} d) \quad \|\vec{v}\| &= \sqrt{\langle \vec{v}, \vec{v} \rangle} \\ &= \sqrt{3^2 + 2^2} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} e) \quad d(\vec{u}, \vec{v}) &= \|\vec{u} - \vec{v}\| \\ &= \|(-2, -1)\| \\ &= \sqrt{(-2)^2 + (-1)^2} \\ &= \sqrt{5} \end{aligned}$$

$$\begin{aligned} f) \quad \|\vec{u} - k\vec{v}\| &= \|(1, 1) - 3(3, 2)\| \\ &= \|(-8, -5)\| \\ &= \sqrt{(-8)^2 + (-5)^2} \\ &= \sqrt{89} \end{aligned}$$

### Exercise

Let  $\langle \vec{u}, \vec{v} \rangle$  be the Euclidean inner product on  $\mathbb{R}^2$ , and let  $\vec{u} = (1, 1)$ ,  $\vec{v} = (3, 2)$ ,  $\vec{w} = (0, -1)$  and  $k = 3$ .

Compute the following for the weighted Euclidean inner product  $\langle \vec{u}, \vec{v} \rangle = 2u_1v_1 + 3u_2v_2$ .

a)  $\langle \vec{u}, \vec{v} \rangle$

c)  $\langle \vec{u} + \vec{v}, \vec{w} \rangle$

e)  $d(\vec{u}, \vec{v})$

b)  $\langle k\vec{v}, \vec{w} \rangle$

d)  $\|\vec{v}\|$

f)  $\|\vec{u} - k\vec{v}\|$

### Solution

a)  $\langle \vec{u}, \vec{v} \rangle = 2(1)(3) + 3(1)(2)$

$= 12$

b)  $\langle k\vec{v}, \vec{w} \rangle = 2(3 \cdot 3)(0) + 3(3 \cdot 2)(-1)$

$= -18$

c)  $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

$= 1 \cdot 0 + 1 \cdot (-1) + 3 \cdot 0 + 2 \cdot (-1)$

$= -3$

d)  $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

$= \sqrt{2(3)(3) + 3(2)(2)}$

$= \sqrt{30}$

e)  $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$

$= \|\langle (-2, -1) \rangle\|$

$= \sqrt{2(-2)(-2) + 3(-1)(-1)}$

$= \sqrt{11}$

f)  $\|\vec{u} - k\vec{v}\| = \|(1, 1) - 3(3, 2)\|$

$= \|\langle (-8, -5) \rangle\|$

$= \sqrt{2(-8)^2 + 3(-5)^2}$

$= \sqrt{203}$

### Exercise

Let  $\langle \vec{u}, \vec{v} \rangle$  be the Euclidean inner product on  $\mathbb{R}^2$ , and let  $\vec{u} = (3, -2)$ ,  $\vec{v} = (4, 5)$ ,  $\vec{w} = (-1, 6)$ , and  $k = -4$ . Verify the following.

$$a) \quad \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$$

$$d) \quad \langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle$$

$$b) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$$

$$e) \quad \langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$$

$$c) \quad \langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$$

### Solution

$$a) \quad \langle \vec{u}, \vec{v} \rangle = 3 \cdot 4 + (-2) \cdot (5)$$

$$= 2$$

$$\langle \vec{v}, \vec{u} \rangle = 4 \cdot 3 + (5) \cdot (-2)$$

$$= 2$$

$$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \quad \checkmark$$

$$b) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle (7, 3), (-1, 6) \rangle$$

$$= 7(-1) + 3(6)$$

$$= 11$$

$$\langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle = (3)(-1) + (-2)(6) + (4)(-1) + (5)(6)$$

$$= 11$$

$$\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \quad \checkmark$$

$$c) \quad \langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle (3, -2), (3, 11) \rangle$$

$$= 3(3) + (-2)(11)$$

$$= -13$$

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle = (3)(4) + (-2)(5) + (3)(-1) + (-2)(6)$$

$$= -13$$

$$\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle \quad \checkmark$$

$$d) \quad \langle k\vec{u}, \vec{v} \rangle = (-4 \cdot 3) \cdot 4 + ((-4)(-2)) \cdot (5)$$

$$= -8$$

$$k \langle \vec{u}, \vec{v} \rangle = (-4)(3 \cdot 4 + (-2) \cdot (5))$$

$$= -8$$

$$\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle \quad \checkmark$$

$$\begin{aligned}
 e) \quad \langle \vec{0}, \vec{v} \rangle &= 0 \cdot 4 + 0 \cdot (5) \\
 &= 0 \\
 \langle \vec{v}, \vec{0} \rangle &= 4 \cdot 0 + (5) \cdot (0) \\
 &= 0 \\
 \langle \vec{0}, \vec{v} \rangle &= \langle \vec{v}, \vec{0} \rangle = 0 \quad \checkmark
 \end{aligned}$$

### Exercise

Let  $\langle \vec{u}, \vec{v} \rangle$  be the Euclidean inner product on  $\mathbb{R}^2$ , and let  $\vec{u} = (3, -2)$ ,  $\vec{v} = (4, 5)$ ,  $\vec{w} = (-1, 6)$ , and  $k = -4$ . Verify the following for the weighted Euclidean inner product  $\langle \vec{u}, \vec{v} \rangle = 4u_1v_1 + 5u_2v_2$ .

$$\begin{aligned}
 a) \quad \langle \vec{u}, \vec{v} \rangle &= \langle \vec{v}, \vec{u} \rangle & d) \quad \langle k\vec{u}, \vec{v} \rangle &= k \langle \vec{u}, \vec{v} \rangle \\
 b) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle & e) \quad \langle \vec{0}, \vec{v} \rangle &= \langle \vec{v}, \vec{0} \rangle = 0 \\
 c) \quad \langle \vec{u}, \vec{v} + \vec{w} \rangle &= \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle
 \end{aligned}$$

### Solution

$$\begin{aligned}
 a) \quad \langle \vec{u}, \vec{v} \rangle &= 4 \cdot 3 \cdot 4 + 5 \cdot (-2) \cdot (5) \\
 &= -2 \\
 \langle \vec{v}, \vec{u} \rangle &= 4 \cdot 4 \cdot 3 + 5 \cdot (5) \cdot (-2) \\
 &= -2 \\
 \langle \vec{u}, \vec{v} \rangle &= \langle \vec{v}, \vec{u} \rangle \quad \checkmark \\
 b) \quad \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle (7, 3), (-1, 6) \rangle \\
 &= 4 \cdot 7 \cdot (-1) + 5 \cdot 3 \cdot (6) \\
 &= 62 \\
 \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle &= 4 \cdot (3) \cdot (-1) + 5 \cdot (-2) \cdot (6) + 4 \cdot (4) \cdot (-1) + 5 \cdot (5) \cdot (6) \\
 &= 62 \\
 \langle \vec{u} + \vec{v}, \vec{w} \rangle &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \quad \checkmark \\
 c) \quad \langle \vec{u}, \vec{v} + \vec{w} \rangle &= \langle (3, -2), (3, 11) \rangle \\
 &= 4 \cdot 3 \cdot (3) + 5 \cdot (-2) \cdot (11) \\
 &= -74 \\
 \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle &= 4 \cdot (3) \cdot (4) + 5 \cdot (-2) \cdot (5) + 4 \cdot (3) \cdot (-1) + 5 \cdot (-2) \cdot (6)
 \end{aligned}$$

$$\underline{= -74}$$

$$\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle \quad \checkmark$$

$$d) \quad \langle k\vec{u}, \vec{v} \rangle = 4 \cdot (-4 \cdot 3) \cdot 4 + 5 \cdot ((-4)(-2)) \cdot (5) \\ \underline{= 8}$$

$$k \langle \vec{u}, \vec{v} \rangle = (-4)(4 \cdot 3 \cdot 4 + 5 \cdot (-2) \cdot (5)) \\ \underline{= 8}$$

$$\langle k\vec{u}, \vec{v} \rangle = k \langle \vec{u}, \vec{v} \rangle \quad \checkmark$$

$$e) \quad \langle \vec{0}, \vec{v} \rangle = 4 \cdot 0 \cdot 4 + 5 \cdot 0 \cdot (5) \\ \underline{= 0}$$

$$\langle \vec{v}, \vec{0} \rangle = 4 \cdot 4 \cdot 0 + 5 \cdot (5) \cdot (0) \\ \underline{= 0}$$

$$\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0 \quad \checkmark$$

### Exercise

Let  $\vec{u} = (u_1, u_2)$  and  $\vec{v} = (v_1, v_2)$ . Show that the following are inner product on  $\mathbb{R}^2$  by verifying that the inner product axioms hold.  $\langle \vec{u}, \vec{v} \rangle = 3u_1v_1 + 5u_2v_2$

### Solution

$$\text{Axiom 1: } \langle \vec{u}, \vec{v} \rangle = 3u_1v_1 + 5u_2v_2 \\ = 3v_1u_1 + 5v_2u_2 \\ = \langle \vec{v}, \vec{u} \rangle \quad \checkmark$$

$$\text{Axiom 2: } \langle \vec{u} + \vec{v}, \vec{w} \rangle = 3(u_1 + v_1)w_1 + 5(u_2 + v_2)w_2 \\ = 3(u_1w_1 + v_1w_1) + 5(u_2w_2 + v_2w_2) \\ = 3u_1w_1 + 3v_1w_1 + 5u_2w_2 + 5v_2w_2 \\ = (3u_1w_1 + 5u_2w_2) + (3v_1w_1 + 5v_2w_2) \\ = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \quad \checkmark$$

$$\text{Axiom 3: } \langle k\vec{u}, \vec{v} \rangle = 3(ku_1)v_1 + 5(ku_2)v_2 \\ = k(3u_1v_1 + 5u_2v_2)$$

$$= k \langle \vec{u}, \vec{v} \rangle \quad \checkmark$$

$$\text{Axiom 4: } \langle \vec{v}, \vec{v} \rangle = 3v_1v_1 + 5v_2v_2$$

$$= 3v_1^2 + 5v_2^2 \geq 0$$

$$v_1 = v_2 = 0 \quad \text{iff} \quad \vec{v} = \vec{0} \quad \checkmark$$

### Exercise

Show that the following identity holds for the vectors in any inner product space

$$\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

### Solution

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 &= \langle \vec{u} + \vec{v}, \vec{u} + \vec{v} \rangle + \langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle \\ &= \langle \vec{u}, \vec{u} + \vec{v} \rangle + \langle \vec{v}, \vec{u} + \vec{v} \rangle + \langle \vec{u}, \vec{u} - \vec{v} \rangle - \langle \vec{v}, \vec{u} - \vec{v} \rangle \\ &= \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle + \langle \vec{u}, \vec{u} \rangle - \langle \vec{u}, \vec{v} \rangle - \langle \vec{v}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle \\ &= 2\langle \vec{u}, \vec{u} \rangle + 2\langle \vec{v}, \vec{v} \rangle \\ &= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2 \quad \checkmark \end{aligned}$$

### Exercise

Show that the following identity holds for the vectors in any inner product space

$$\langle \vec{u}, \vec{v} \rangle = \frac{1}{4} \left( \|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 \right)$$

### Solution

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= \langle \vec{u} + \vec{v}, \vec{u} + \vec{v} \rangle = \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 \\ \|\vec{u} - \vec{v}\|^2 &= \langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle = \|\vec{u}\|^2 - 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 \\ \|\vec{u} + \vec{v}\|^2 &= \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 \\ - \|\vec{u} - \vec{v}\|^2 &= \|\vec{u}\|^2 - 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 \\ \hline \|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 &= 4\langle \vec{u}, \vec{v} \rangle \\ \langle \vec{u}, \vec{v} \rangle &= \frac{1}{4} \left( \|\vec{u} + \vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 \right) \quad \checkmark \end{aligned}$$

### ***Exercise***

Prove that  $\|k\vec{v}\| = |k| \|\vec{v}\|$

### **Solution**

$$\begin{aligned}\|k\vec{v}\|^2 &= \langle k\vec{v}, \vec{v} \rangle \\ &= k^2 \langle \vec{v}, \vec{v} \rangle \\ &= k^2 \|\vec{v}\|^2\end{aligned}$$

$$\|k\vec{v}\| = |k| \|\vec{v}\| \quad \checkmark$$