7. The "1-dimensional" wave equation and The d'Alembert Solution

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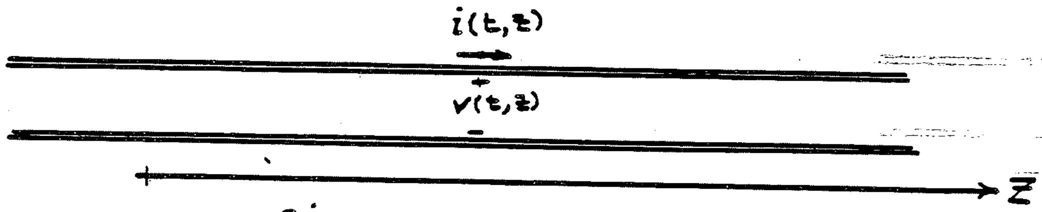
Consider the equations that govern the relationship between voltage and current along a transmission line.



$$-\frac{\partial v}{\partial z} = Ri + L\frac{\partial i}{\partial t}$$

$$-\frac{\partial i}{\partial z} = Gv + C\frac{\partial v}{\partial t}$$

where R is the series resistance
L is the series inductance
G is the shunt conductance
C is the shunt capacitance.
all per unit length.

A very useful approximation that can often be employed is to assume that $R = 0$, and $G = 0$.

Then the equations reduce to

$$\left.\begin{aligned} -\frac{\partial v}{\partial z} &= L\frac{\partial i}{\partial t} \\ -\frac{\partial i}{\partial z} &= C\frac{\partial v}{\partial t}. \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} \frac{\partial^2 v}{\partial z^2} - LC\frac{\partial^2 v}{\partial t^2} &= 0 \\ \frac{\partial^2 i}{\partial z^2} - LC\frac{\partial^2 i}{\partial t^2} &= 0 \end{aligned}\right.$$

The last equation is the one dimensional wave equation and can be solved in a very elegant manner.

Suppose we set $v(t, z) = f(at + bz)$. Then

$$\frac{\partial f}{\partial t} = a f'(at+bz); \quad \frac{\partial f}{\partial z} = b f'(at+bz)$$
$$\frac{\partial^2 f}{\partial t^2} = a^2 f''(at+bz); \quad \frac{\partial^2 f}{\partial z^2} = b^2 f''(at+bz)$$

where $f'(z) = \frac{df(z)}{dz}$, $f''(z) = \frac{d^2 f(z)}{dz^2}$

Substituting this into the wave equation, we find that

$$b^2 f''(at+bz) - LC\, a^2 f''(at+bz) = 0$$
$$b^2 - LC\, a^2 = 0 \implies \boxed{b = \pm\sqrt{LC}\, a}$$

We can choose $a = 1$ and the solutions to the wave equation are

$v(t,z) = f(t + \frac{z}{c})$ and $v(t,z) = f(t - \frac{z}{c})$
where $c = 1/\sqrt{LC}$. The units of c are

$$1/\sqrt{H/m \cdot F/m} = 1/\sqrt{s^2/m^2} = m/s.$$

Since the differential operators are linear, the general solution to the 1-D wave equation is

$$v(t,z) = f_1(t - z/c) + f_2(t + z/c).$$

To solve a physical problem, we must introduce boundary conditions to obtain a unique solution.

Several types of boundary conditions that are also seen in a wider class of problems will be applied to solve some transmission line problems.

a) Radiation condition - infinite transmission line

-=-

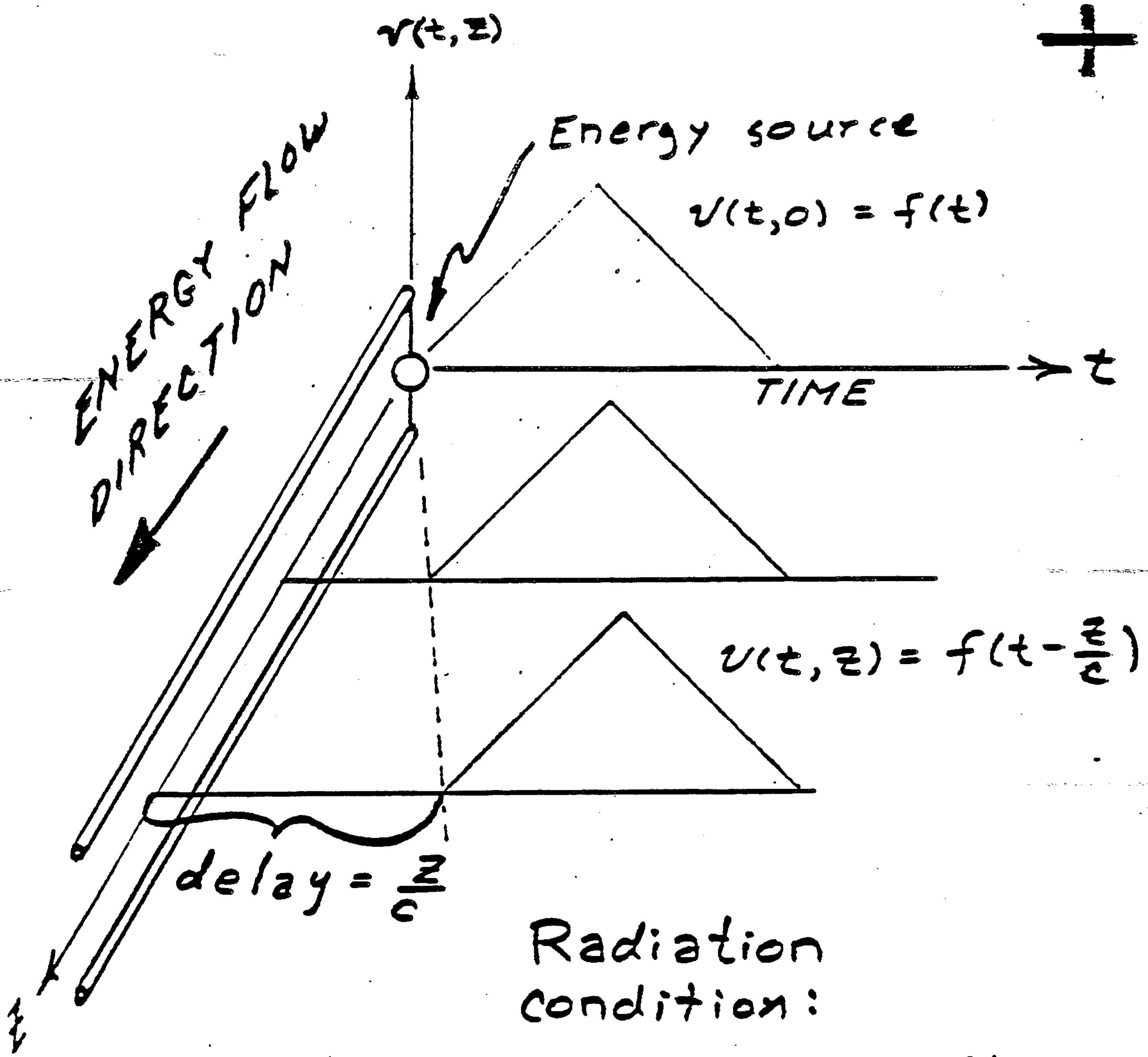
Physically what does the solution $v(t,z) = f(t-z/c)$ represent?

This represents a wave that is traveling along the transmission line in the positive z direction.

It is an undistorted wave because the response at a distance z is the same as that at distance 0 except that it is DELAYED by z/c where c is the wave speed.

Similarly, a solution of the form $v(t,z) = f(t+z/c)$ is a wave traveling in the negative z direction.

Consider the uniform transmission with a source of $v(t,0) = f(t)$ shown in figure 1.



Radiation condition:

Energy must flow __away__ *from the sourc*

- The solution $f(t+\frac{z}{c})$ is __excluded__ by the radiation condition.

The boundary conditions are
$\left.\begin{array}{l} v(t,o) = f(t) \\ \text{Radiation condition} \end{array}\right\}$

FIG 1

b) Loaded transmission lines

To consider a loaded line, we must find how v and i are related.

Consider first a wave propagating in the +z direction. We have seen that $v(t,z) = f(t-z/c)$.

Since the current must satisfy exactly the same differential equation, $i(t,z) = g(t-z/c)$.

Using the relation,

$$-\frac{\partial v}{\partial z} = L\frac{\partial i}{\partial t}, \quad \Rightarrow \quad -\frac{\partial}{\partial z} f(t-z/c) = L\frac{\partial}{\partial t} g(t-z/c)$$

We conclude that

$$\frac{1}{c} f'(t-z/c) = Lg'(t-z/c) \quad \text{or} \quad f'(u) = Lc\,g'(u)$$

which implies that $f(u) = (cL)\,g(u) + K$ where K is some constant. That constant represents a DC voltage between the two parallel wires in the line.

It is of little physical interest and is taken to be zero.

The coefficient, cL, is

$$Z_0 = cL = \frac{L}{\sqrt{LC}} = \sqrt{\frac{L}{C}}$$

and is called the characteristic impedance or wave impedance of the transmission line. Its reciprocal, Y_0, is called the characteristic admittance.

Therefore

$$v(t,z) = Z_0\, i(t,z) \quad \text{or} \quad i(t,z) = Y_0\, v(t,z)$$

for waves propagating in the +z direction.

Similarly, for a wave propagating in the −z direction, the current and voltage are related by

$$v(t,z) = -Z_0\, i(t,z) \quad \text{or} \quad i(t,z) = -Y_0\, v(t,z)$$

Therefore, in general,

$$v(t,z) = f_1(t - z/c) + f_2(t + z/c)$$
$$i(t,z) = Y_0 f_1(t - z/c) - Y_0 f_2(t + z/c).$$

(1) Shorted transmission line

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The boundary conditions for a transmission line excited by a voltage source at $z=0$ of $f(t)$ and loaded at $z=z_0$ with a short are:

$$v(t,0) = f(t)$$

$$v(t,z_0) = 0.$$

We must assume both waves going in the + and − directions:

$$v(t,z) = f_1(t - z/c) + f_2(t + z/c)$$

Applying the boundary conditions, we find that

$$f_2(t + z_0/c) = -f_1(t - z_0/c) \implies$$

$$f_2(t) = -f_1(t - 2z_0/c),$$

and

$$f_1(t) = f(t) - f_2(t)$$
$$= f(t) + f_1(t - 2z_0/c).$$

Therefore, we have that if $f(t) = 0$ for $t < 0$, then
$f_1(t) = 0$ also for $t < 0$.
Therefore, $f_1(t - 2z_0/c) = 0$ for $t < 2z_0/c$.

Thus, for $t < 2z_0/c$,
$f_1(t) = f(t)$.

For $2z_0/c < t < 4z_0/c$,

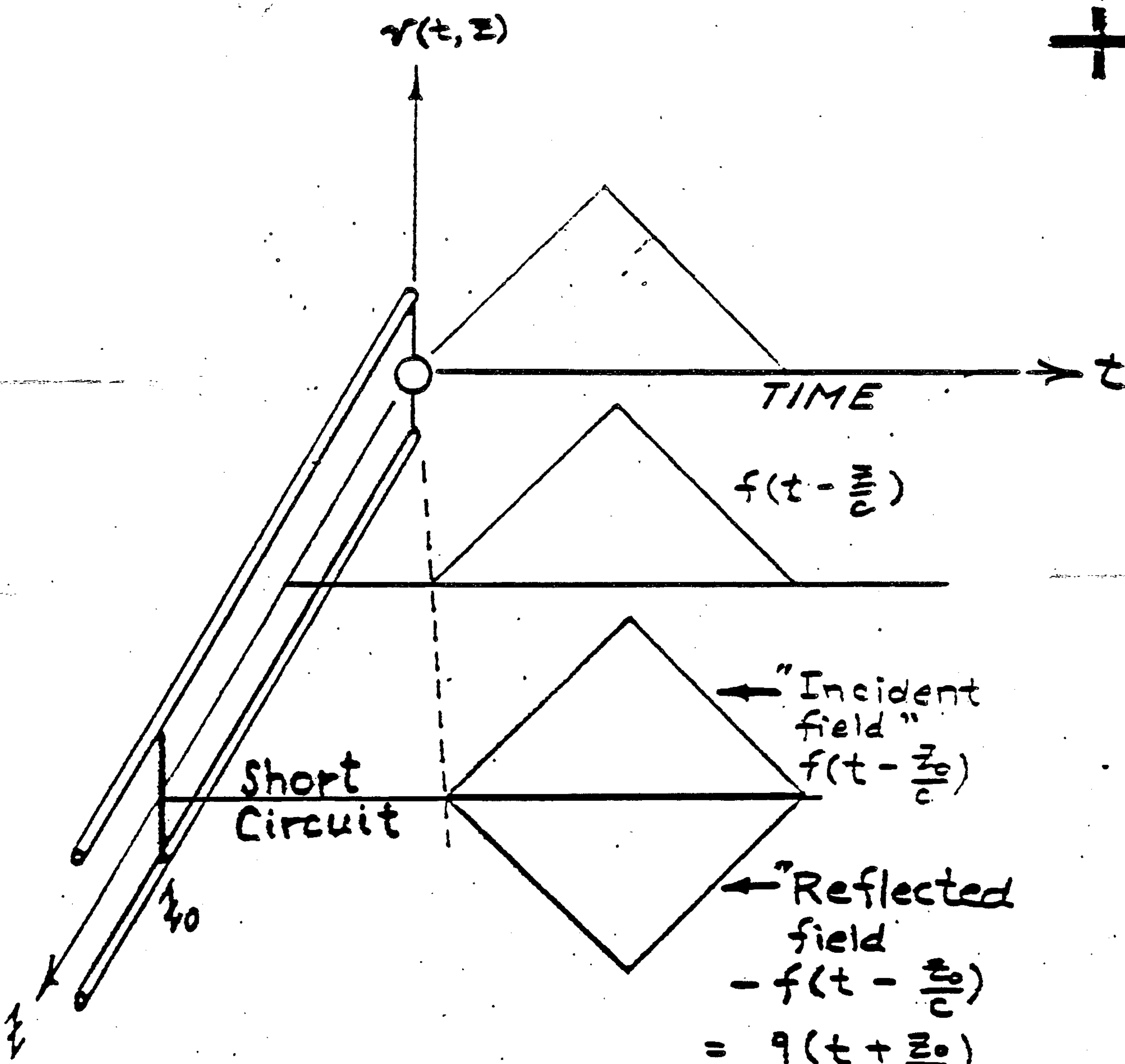
$$\begin{aligned} f_1(t) &= f(t) + f_1(t - 2z_0/c) \\ &= f(t) + f(t - 2z_0/c). \end{aligned}$$

This process can be carried out infinitely many times with the result that

$$\begin{aligned} f_1(t) &= f(t) + f(t - 2z_0/c) + f(t - 4z_0/c) \\ &\quad + f(t - 6z_0/c) + \cdots \\ &= \sum_{k=0}^{\infty} f(t - 2kz_0/c) \end{aligned}$$

$$f_2(t) = -\sum_{k=0}^{\infty} f(t - 2(k+1)z_0/c).$$

All of these analytical results can be constructed very easily by graphical methods. (Figs. 2-3a)



Any obstruction will reflect energy.

The boundary conditions are:

$$\begin{cases} v(t,0) = f(t) \\ v(t,z_0) = 0 \end{cases}$$

FIG 2

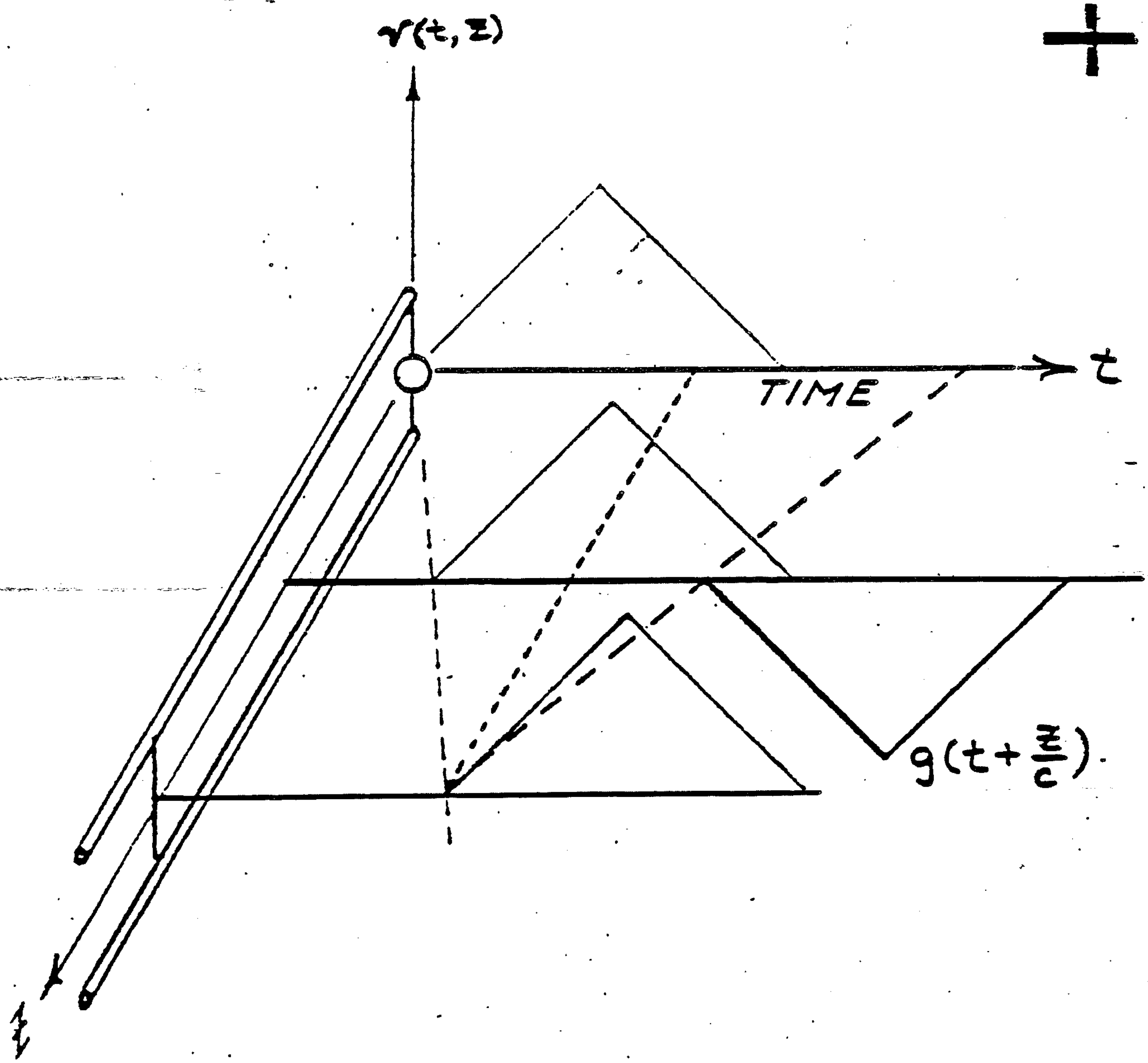


FIG. 3

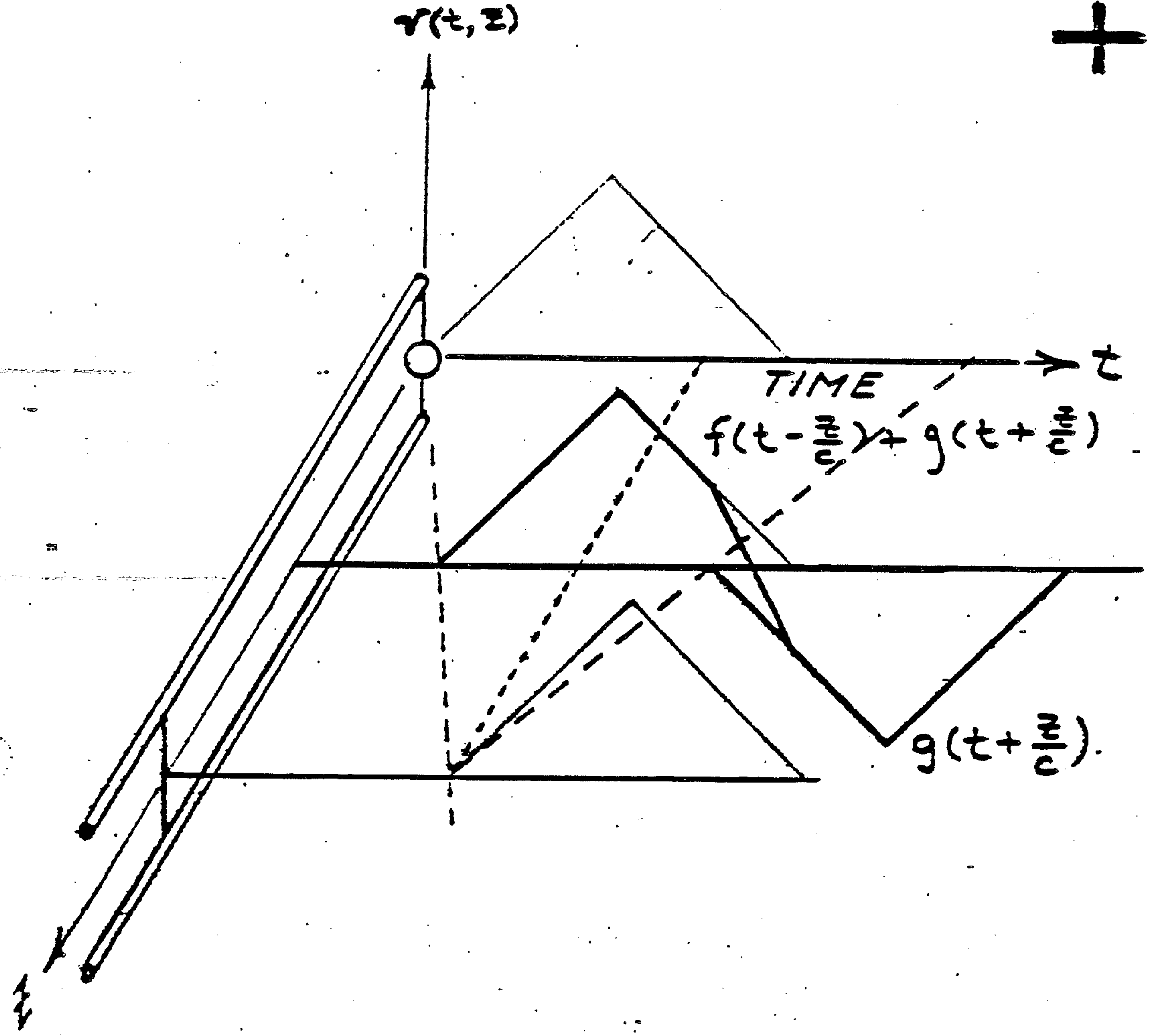


FIG. 3a

(2) Opened transmission line

The boundary conditions for a transmission line excited by a voltage source at $z=0$ of $f(t)$ and loaded at $z=z_0$ with an open are:

$$v(t,0) = f(t)$$

$$i(t,z_0) = 0.$$

Again we must assume both positive and negative going waves. Applying the boundary condition on the current, we find that $f_1(t-z_0/c) = f_2(t+z_0/c)$ and hence the incident and reflected voltages add at the open.

Figure 4 shows the graphically constructed solution to this problem.

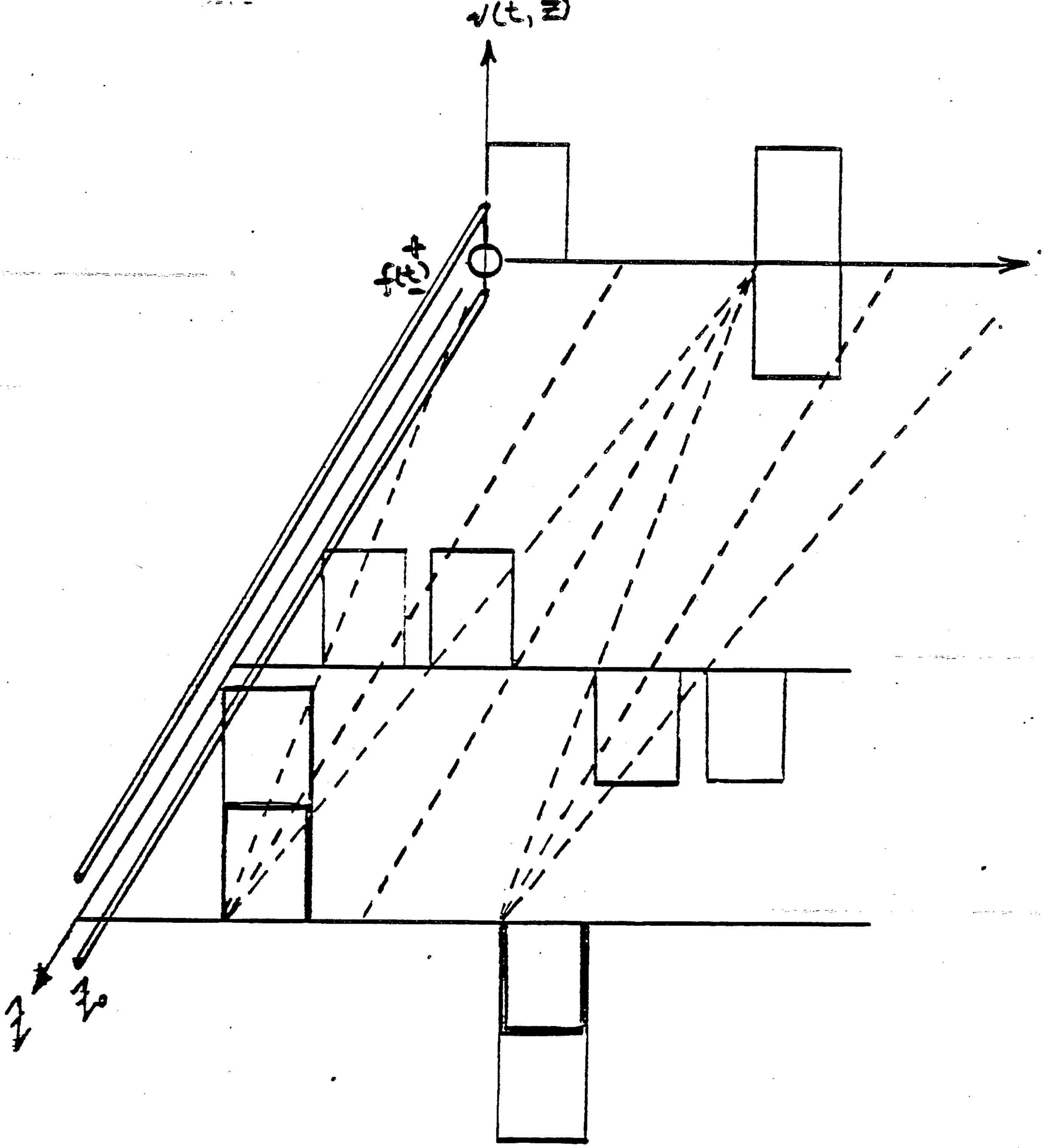
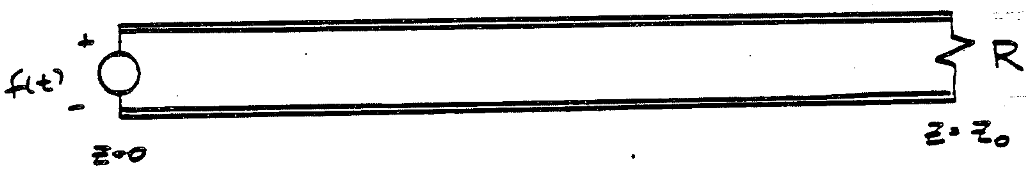


FIG 4

(3) Resistively loaded transmission lines

The boundary conditions in this case are:

$$v(t,0) = f(t)$$
$$v(t,z_0) = R\,i(t,z_0),$$



$$v(t,z) = f_1(t - z/c) + f_2(t + z/c)$$
$$i(t,z) = Y_0 f_1(t - z/c) - Y_0 f_2(t + z/c)$$

where R is the load resistance.

Applying the boundary condition at the load,

$$f_1(t - z_0/c) + f_2(t + z_0/c) = RY_0 f_1(t - z_0/c) - RY_0 f_2(t + z_0/c)$$

$$\Rightarrow \quad \frac{f_2(t + z_0/c)}{f_1(t - z_0/c)} = -\frac{1 - RY_0}{1 + RY_0} \equiv \Gamma.$$

The ratio of $f_2(t+z_0/c)$ to $f_1(t-z_0/c)$ is called the "reflection coefficient" since it is the fraction of the incident voltage that is reflected back toward the source.

Applying the boundary condition at the source, we obtain

$$f_1(t) + f_2(t) = f(t)$$

But $f_2(t + z_0/c) = \Gamma f_1(t - z_0/c) \Rightarrow$

$$f_2(t) = \Gamma f_1(t - 2z_0/c).$$

If $f(t) = 0$ for $t < 0$, then

$$\begin{aligned} f_1(t) &= f(t) - \Gamma f_1(t - 2z_0/c) \\ &= f(t) - \Gamma f(t - 2z_0/c) \quad \text{for } 0 \leq t \leq 4z_0/c \\ &= f(t) - \Gamma f(t - 2z_0/c) + \Gamma^2 f(t - 4z_0/c) \\ &\quad + \cdots \\ &= \sum_{k=0}^{\infty} f(t - 2kz_0/c)(-\Gamma)^k \end{aligned}$$

$$\begin{aligned} f_2(t) &= \Gamma \sum_{k=0}^{\infty} f(t - 2z_0/c - 2kz_0/c)(-\Gamma)^k \\ &= -\sum_{k=0}^{\infty} f\left[t - \frac{2(k+1)z_0}{c}\right](-\Gamma)^{k+1} \\ &= -\sum_{k=1}^{\infty} f\left[t - \frac{2kz_0}{c}\right](-\Gamma)^k \end{aligned}$$

Therefore, $$\begin{aligned} v(t, z) &= f_1(t - z/c) + f_2(t + z/c) \\ &= f(t - z/c) + \sum_{k=1}^{\infty} (-\Gamma)^k \cdot \\ &\quad \left\{ f\left[t - \frac{(z + 2kz_0)}{c}\right] - f\left[t + \frac{(z - 2kz_0)}{c}\right] \right. \end{aligned}$$