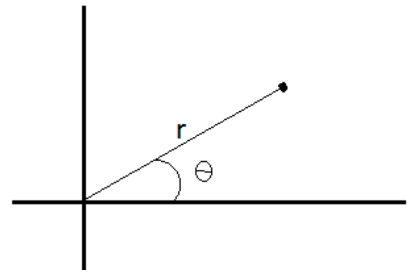
<u>Circular Motion and Applications of Newton's 2nd Law</u>
<u>(Chapter 6 Lecture 1)</u>

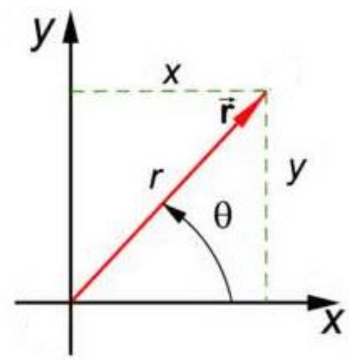
It is easier to describe circular motion in terms of polar coordinates (r, θ) because even though it is a two-dimensional problem in terms of Cartesian coordinates (x,y), it is a one dimensional problem in terms of polar coordinates.

6.1 <u>Polar Coordinate System:</u> In a polar coordinate system a point is identified by its distance from the origin (r) and the angle formed between the position vector of the point and the positive x-axis measure in a counter clockwise direction (negative if measured clockwise)



<u>Relationship between Cartesian and polar coordinates</u>
From the right angled triangle shown, the following relationship can be deduced easily.

$x = r\cos\theta$ $y = r\sin\theta$	$r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right) + (180° \; if \; x < 0)$



<u>Example:</u> Express the polar coordinates $(10, 60°)$ in terms of Cartesian coordinates
<u>Solution</u>

$(r, \theta) = (10,60°)$
$(x, y) = ??$

$r = 10 \qquad \theta = 60°$
$x = r\cos\theta$
$= 10\cos 60 = 5$
$y = 10\sin 60 = 8.7$

$\underline{\therefore (x, y) = (5, 8.7)}$

<u>Polar Unit Vectors:</u>

<u>r-unit vector</u> $(\hat{\boldsymbol{e}}_r)$ has the same direction as the change of the position vector with respect to <u>r</u> while keeping θ constant.

$$\hat{e}_r = \frac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \hat{r}}{\partial r}\right|} \qquad \vec{r} = x\hat{\imath} + y\hat{\jmath} \qquad \text{but } \; x = r\cos\theta$$
$$y = r\sin\theta$$

Thus $\vec{r} = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath}$

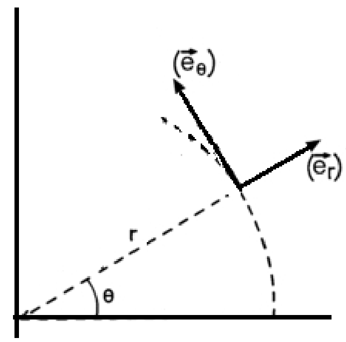
$$\frac{\partial \vec{r}}{\partial r} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$

$$\left|\frac{\partial \vec{r}}{\partial r}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\hat{e}_r = \frac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|} = \frac{\cos\theta\,\hat{\imath} + \sin\theta}{1}$$

$\boxed{\hat{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}}$ $\hat{e}_r$ is usually directed radially outward

***θ*-unit vector** ($\hat{e}_\theta$): has the same direction as the rate of change of the position vector with respect to θ while keeping r constant.



$$\hat{e}_\theta = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|}$$

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath}$$

$$\frac{\partial \vec{r}}{\partial \theta} = -r\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath}$$

$$\left|\frac{\partial \vec{r}}{\partial \theta}\right| = \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2} \quad = r\sqrt{\cos^2\theta + \sin^2\theta} = r$$

$$\therefore \quad \hat{e}_\theta = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|} \frac{= -r\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath}}{r}$$

$$\boxed{\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}}$$

Expressing the position vector in terms of polar unit vectors:

$\vec{r} = x\hat{\imath} + y\hat{\jmath} = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath}$ $\quad$ but $\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} = \hat{e}_r$

$= r[\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}]$

$$\therefore \quad \boxed{\vec{r} = r\vec{e}_r}$$

Rate of Change of $\hat{e}_r$ with respect to θ

$$\frac{\partial \vec{e}_r}{\partial \theta} = \frac{d}{d\theta}[\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}]$$

$$= -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} = \hat{e}_\theta$$

$$\therefore \quad \boxed{\frac{\partial \hat{e}_r}{\partial \theta} = \vec{e}_\theta}$$

Rate of Change of $\vec{e}_\theta$ with respect to θ

$$\frac{\partial \hat{e}_\theta}{\partial \theta} = \frac{d}{d\theta}[-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}]$$

$$= -\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath}$$

$$= -[\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}] = -\hat{e}_r$$

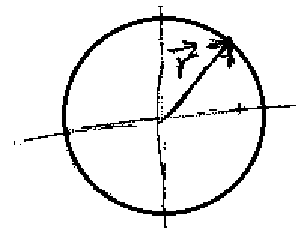
$$\boxed{\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r}$$

Example: A particle is located at the point (3,4) m. Express its position vector in terms of polar unit vectors

Solution

$$(x, y) = (3,4)m \quad \Rightarrow x = 3m\,;y = 4m$$
$$\vec{r} = r\hat{e}_r \quad \text{but} \quad r = \sqrt{(3m)^2 + (4m)^2} = 5m$$

6.2 Circular Motion in terms of polar coordinates



For a circular motion, if the origin is the center of the circular path, r remains constant.

Motion Variables of Circular Motion in terms of polar unit vectors

Position Vector: $\vec{r} = r\hat{e}_r$

Velocity Vector: $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e}_r) = r\frac{d}{dt}\hat{e}_r$

$$= r\frac{d}{dt}[\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}]$$
$$\vec{v} = r\frac{d}{dt}[\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}]$$
$$= r\left[\frac{d}{d\theta}\cos\theta\frac{d\theta}{dt}\hat{\imath} + \frac{d}{d\theta}\sin\theta\frac{d\theta}{dt}\hat{\jmath}\right]$$
$$= r\left[-\sin\theta\frac{d\theta}{dt}\hat{\imath} + \cos\theta\frac{d\theta}{dt}\hat{\jmath}\right]$$
$$= r\frac{d\theta}{dt}[-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}]$$
$$\vec{v} = r\frac{d\theta}{dt}\hat{e}_\theta$$

$\frac{d\theta}{dt}$ which is the rate of change of angular displacement (θ) with time is called angular speed and denoted by ω. $\omega = \frac{d\theta}{dt}$

$$\boxed{\vec{v} = r\omega\hat{e}_\theta}$$

Relationship between linear speed (v) and angular speed (ω)

If $\underline{\theta}$ is in radians & is a central angle that subtends an arc length $\underline{s}$ in a circle of radius $\underline{r}$, then

$$\boxed{\theta = s/r}$$

$\Delta\theta = \frac{\Delta s}{r}$ dividing by Δt $\frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$	or	$\Delta s = r\,\Delta\theta$ But $\frac{\Delta s}{\Delta t} = v$ & $\frac{\Delta\theta}{\Delta t} = \omega$ $\therefore \boxed{v = r\omega}$

$\therefore$ the velocity of a particle executing circular motion may be written as

$$\boxed{\vec{v} = r\omega\vec{e}_\theta = v\hat{e}_\theta}$$

Acceleration ($\vec{a}$)

$\vec{a} = \frac{d\vec{v}}{dt}$ by $\vec{v} = v\hat{e}_\theta$

$$\vec{a} = \frac{d}{dt}(v\hat{e}_\theta) = \frac{dv}{dt}\hat{e}_\theta + v\frac{d\hat{e}_\theta}{dt}$$

$$\frac{d\hat{e}_\theta}{dt} = \frac{d}{dt}[-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}]$$

$$= -\cos\theta\frac{d\theta}{dt}\hat{\imath} + \left(-\sin\theta\frac{d\theta}{dt}\hat{\jmath}\right)$$

$$= -\frac{d\theta}{dt}[\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}]$$

But $\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} = \hat{e}_r$

$$= \frac{d\hat{e}_\theta}{dt} = -\frac{d\theta}{dt}\,\hat{e}_r \quad \& \quad \frac{d\theta}{dt} = \omega$$

$$\frac{d\vec{e}_\theta}{dt} = -\omega\hat{e}_r$$

$$\vec{a} = \frac{dv}{dt}\hat{e}_\theta + v\frac{d\vec{e}_\theta}{dt}$$

$$= \frac{dv}{dt}\vec{e}_\theta - v\omega\hat{e}_r \quad \text{but} \quad \omega = \frac{v}{r}$$

$$\therefore \boxed{\vec{a} = \frac{dv}{dt}\vec{e}_\theta - \frac{v^2}{r}\hat{e}_r}$$

The first term of the acceleration is acceleration due to change of speed and is called tangential acceleration (a_t)

$$\boxed{\vec{a}_t = \frac{dv}{dt}\vec{e}_\theta}$$

$$\& \text{ with } \boxed{\frac{dv}{dt} = a_t} \quad \boxed{\vec{a}_t = a_t\hat{e}_\theta}$$

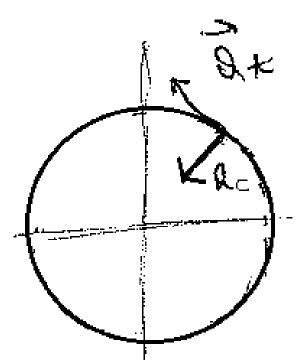
The direction of tangential acceleration is always tangent to the circular path.

The second term is acceleration due to change of direction & is called centripetal or radial acceleration (a_c)

$$\boxed{\begin{array}{c}\vec{a} = -\dfrac{v^2}{r}\hat{e}_r \\ a_c = \dfrac{v^2}{r}\end{array}}$$

The direction of centripetal acceleration is always towards the center of the circle.

Since tangential and centripetal acceleration ($or\ \hat{e}_\theta\ \&\ \hat{e}_r$) are perpendicular to each other, the magnitude of the acceleration of a circular motion can be obtained from Pythagorean theorem.



$$\boxed{a = \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{r}\right)^2} = \sqrt{a_t^2 + a_c^2}}$$

Force:

From Newton's 2nd Law

$$\vec{F} = m\vec{a}$$

but for a circular motion $\vec{a} = \frac{dv}{dt}\vec{e}_\theta - \frac{v^2}{r}\vec{e}_r$

$$\therefore \quad \boxed{\vec{F} = m\frac{dv}{dt}\vec{e}_\theta - \frac{mv^2}{r}\vec{e}_r}$$

Example: Obtain expressions for the acceleration & force of a particle undergoing a uniform circular motion.

Solution

Uniform Circular motion ⇒ $v = constant$

$\therefore \quad \frac{dv}{dt} = 0$

$$\vec{a} = \frac{dv}{dt}\vec{e}_\theta - \frac{v^2}{dr}\vec{e}_r \quad \text{with } \frac{dv}{dt} = 0$$

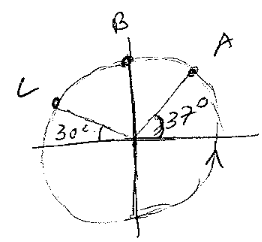
which reduces to

$$\vec{a} = -\frac{v^2}{r}\vec{e}_r$$

Similarly $\vec{F} = m\frac{dv}{dt}\vec{e}_\theta - \frac{mv^2}{r}\vec{e}_r$ reduces to

$$\vec{F} = -\frac{mv^2}{r}\vec{e}_r$$

Example: A particle is revolving in a circular path of radius 10 m with a uniform speed of 5 m/s



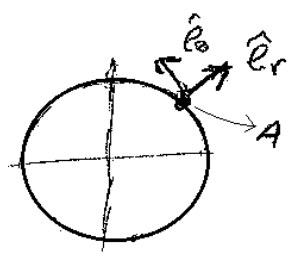
a) At point A show the polar unit vectors graphically

Solution

$\hat{e}_r$ unit vector is directed radially outward

$\hat{e}_\theta$ is tangent to the circular trajectory

b) Express the polar unit vectors at point A in the $\hat{\imath} - \hat{\jmath}$ notation



Solution

$\hat{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$ & $\theta = 37°$

$= \cos 37°\,\hat{\imath} + \sin 37°\,\hat{\jmath}$

$= 0.8\,\hat{\imath} + 0.6\hat{\jmath}$

$\vec{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$

$= -\sin 37°\,\hat{\imath} + \cos 37°\,\hat{\jmath}$

$-0.6\hat{\imath} + 0.8\hat{\jmath}$

c) At point B, express the velocity vector in polar & in Cartesian $(\hat{\imath} - \hat{\jmath})$ notation

Solution

Polar: $\vec{v} = v\vec{e}_\theta$ but $v = 5$ m/s

$\therefore \quad \vec{v} = 5$ m/s $\hat{e}_\theta$

Cartesian: $\vec{v} = 5$ m/s $\vec{e}_\theta$

But $\vec{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$ At point B $\theta = 90°$

$\hat{e}_\theta = -\sin 90\,\hat{\imath} + \cos 90\,\hat{\jmath} = -\hat{\imath}$

$$\vec{v} = 5 \text{ m/s } \vec{e}_\theta = -5 \text{ m/s } \hat{\imath}$$

At point C, express the acceleration vector in terms of polar & Cartesian unit vectors

Solution

At point C $\theta = 180° - 30° = 150°$ (remember angles are measured wrt the positive x-axis)

Polar: $\vec{a} = \frac{dv}{dt}\vec{e}_\theta - \frac{v^2}{r}\hat{e}_r$ $v = 5$ m/s ; $r = 10\ m$ & $\frac{dv}{dt} = 0$ because $v =$ constant ∴ $\vec{a} = -\frac{5^2}{10}\vec{e}_r$ m/s^2 = -2.5 m/s^2 $\hat{e}_r$	Cartesian: $\vec{a} = -2.5$ m/s^2 $\hat{e}_r$ but $\hat{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$ $= \cos 150\,\hat{\imath} + \sin 150\,\hat{\jmath}$ $-0.87\hat{\imath} + 0.5\hat{\jmath}$ ∴ $\vec{a} = -2.5$ m/$s^2[-0.87\hat{\imath} + 0.5\hat{\jmath}]$ $= [2.175\hat{\imath} - 1.25\hat{\jmath}]$ m/s^2

Example: A particle of mass 3 kg travelling in a circular path of radius 4m increased its speed uniformly from 2 m/s to 12 m/s in 5 seconds. Its speed by the time it passes point P is 10 m/s

a) Find its acceleration at point P in polar and Cartesian coordinates

Solution

$\theta = 180 - 45 = 135°$ $r = 4m$ $v_P = 10$ m/s	$v_i = 2$ m/s $v_f = 12$ m/s $t = 5\ s$

Polar: $\vec{a} = \frac{dv}{dt}\hat{e}_\theta - \frac{v^2}{r}\hat{e}_r$ $\frac{dv}{dt} = a_t = \frac{v_f - v_i}{t} = \frac{12-2}{5} = 2$ m/s^2 ∴ $\vec{a} = 2$ m/s^2 $\hat{e}_\theta - \frac{10^2}{4}$ m/s^2 $\hat{e}_r$ $= 2$ m/s^2 $\hat{e}_\theta -$ 25 m/s^2 $\hat{e}_r$	Cartesian: $\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$ $= -\sin 135°\,\hat{\imath} + \cos 135°\,\hat{\jmath}$ $= -0.7\hat{\imath} - 0.7\hat{\jmath}$ $\hat{e}_r = cos\theta\hat{\imath} + sin\theta\hat{\jmath}$ $= \cos 135°\,\hat{\imath} + \sin 135°\,\hat{\jmath}$ $= -0.7\hat{\imath} + 0.7\hat{\jmath}$ $\vec{a} = 2$ m/$s^2\vec{e}_\theta - 25$ m/s^2 $\hat{e}_r$ $= 2\frac{m}{s^2}[-0.7\hat{\imath} - 0.7\hat{\jmath}] - 25\frac{m}{s^2}[-0.7\hat{\imath} + 0.7\hat{\jmath}]$ $= -1.4\frac{m}{s^2}\hat{\imath} - 1.4\frac{m}{s^2} + 17.5\frac{m}{s^2} - 17.5\frac{m}{s^2}\hat{\jmath}$ Collecting similar terms $\vec{a} = 16.1\frac{m}{s^2} - 18.1\frac{m}{s^2}$

b) Find the force acting on the particle at point P in polar & Cartesian unit vectors

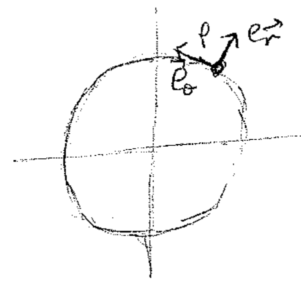
Solution

m= 3kg.

Polar $\vec{a} = 2$ m/$s^2\vec{e}_\theta - 25$ m/s^2 $\hat{e}_r$ $\vec{F} = m\vec{a} = 3kg[2$ m/$s^2\vec{e}_\theta - 25$ m/s^2 $\hat{e}_r]$ $6N\,\vec{e}_\theta - 75N\vec{e}_r$	Cartesian: $\vec{a} = 16.1\frac{m}{s^2} - 18.1\frac{m}{s^2}$ $\vec{F} = m\vec{a} = 3kg\left[16.1\frac{m}{s^2} - 18.1\frac{m}{s^2}\right]$ $= 48.3\ N\,\hat{\imath} - 54.3\ N\,\hat{\jmath}$

Examples of Applications of Newton's 2nd Law to circular motion
(Chapter 6 Lecture 2)

1) A particle of mass 4 kg. is revolving in a circular path of radius 2 m with a uniform speed of 5m/s. At an arbitrary point P in the circular trajectory:



a) Calculate its acceleration

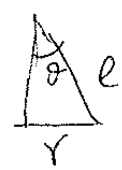
$$\vec{a} = \frac{dv}{dt}\vec{e}_\theta - \frac{v^2}{r}\hat{e}_r$$
$$= (0)\vec{e}_\theta - \frac{5^2}{2}\hat{e}_r$$
$$= -12.5 \text{ m}/s^2$$

$m = 4kg$
$r = 2m$
$v = 5$ m/s
$\frac{dv}{dt} = 0$ (b/c the speed is constant)

b) Calculate the force acting on it

$$\vec{F} = m\vec{a}$$
$$= 4kg\left(-12.5\frac{m}{s^2}\hat{e}_r\right)$$
$$\vec{F} = -50N\ \hat{e}_r$$

2) A pendulum of length l and inclination of θ is revolving in a horizontal circle as shown.



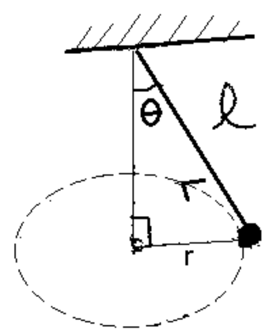
a) Express the radius of the circle in terms of l and θ

Solution

From the right angled triangle shown

$\sin\theta = \frac{r}{l}$

$\boxed{r = l\sin\theta}$



b) Express the speed by which it is revolving in terms of l and θ

Solution

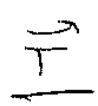
Forces acting are

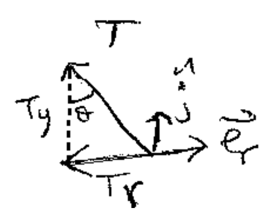
1. Tension in the string $(\vec{T})$
2. Weight of the object $(\vec{w})$

These forces can be decomposed into a radial component $(\hat{e}_r)$ and a vertical component $\hat{j}$

$$\vec{T} = T\sin\theta\,(-\hat{e}_r) + T\cos\theta\,\hat{j}$$
$$\Rightarrow T_r = -\sin\theta\,; T_y = T\cos\theta$$

$(-\vec{e}_r$ means directed towards center)





$$T = -m|g|\hat{j} \Rightarrow T_r = 0 \quad T_y = -m|g|$$
$$\vec{F}_{net} = \vec{T} + \vec{w} = m\vec{a}$$

$$\sum F_y = T_y + w_y = ma_y$$

$a_y = 0$ because the particle is not moving upward

$$T\cos\theta - m|g| = 0$$

$$\boxed{T\cos\theta = m|g|}\ldots\ldots\ldots(1)$$

$$\sum F_r = T_r + w_r = m\vec{a}_r$$

$$-T\sin\theta + 0 = -\frac{mv^2}{r}$$

$$T\sin\theta = \frac{mv^2}{r} \quad \text{but} \quad r = l\sin\theta$$

$$\boxed{T\sin\theta = \frac{mv^2}{l\sin\theta}}\ldots\ldots(2)$$

$$\vec{a}_r = -\frac{v^2}{r}\vec{e}_r$$

$$a_r = -\frac{v^2}{r}$$

Dividing Equation (2) by (1)

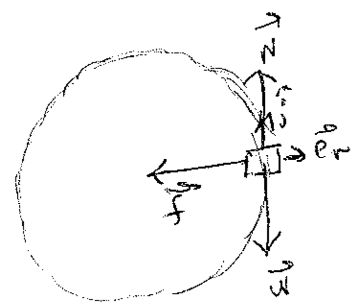
$$\frac{T\sin\theta}{T\cos\theta} = \frac{\frac{mv^2}{l\sin\theta}}{m|g|} \quad \text{but} \quad \frac{sin\theta}{\cos\theta} = \tan\theta$$

$$\tan\theta = \frac{v^2}{l|g|\sin\theta} \implies v^2 = l|g|\tan\theta\sin\theta$$

$$\boxed{v = \sqrt{l|g|\tan\theta\sin\theta}}$$

3) A car is revolving in a circular path of radius <u>r,</u> The coefficient of friction between the tires & the ground is μ_o. Find the expression for the maximum speed by which the car can make it without skidding.

<u>Solution</u>



Forces acting on the car

1. Force of friction directed towards the center of the circle ($\vec{f}$)
2. Weight of the car ($\vec{w}$)
3. Upward normal force exerted by the ground ($\vec{N}$)

These vectors can be decomposed in radial (along $\hat{e}_r$) and upward ($\hat{\jmath}$) vectors.

<u>$\vec{f}$</u>: $\quad \vec{f} = f(-\hat{e}_r)$

$\Rightarrow f_r = -f \quad f_y = 0$

<u>$\vec{N}$:</u> $\quad \vec{N} = N\hat{\jmath}$

$\Rightarrow N_r = 0 \quad N_y = N$

<u>$\vec{w}$:</u> $\quad \vec{w} = m|g|(-\hat{\jmath})$

$w_r = 0 \quad w_y = -m|g|$

$$\sum F_y = f_y + N_y + w_y = ma_y = 0$$

$$0 + N - m|g| = 0 \qquad N = m|g|$$

$$\sum F_r = f_r + N_r + w_r = ma_r$$

$$-f + 0 + 0 = -\frac{mv^2}{r}$$

$$\vec{a} = -\frac{v^2}{r}\hat{e}_r$$

$$a_r = -\frac{v^2}{r}$$

$$f = \frac{mv^2}{r}$$

but $f = \mu N = \mu m|g|$

$$\therefore \ \mu\, m|g| = \frac{mv^2}{r}$$

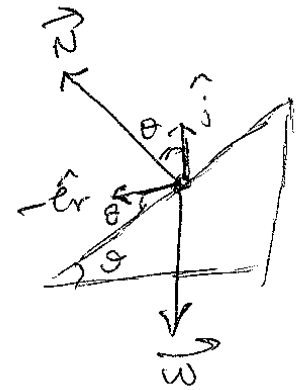
$$v^2 = r\mu|g|$$

$$\boxed{v = \sqrt{r\mu|g|}}$$

4) It is possible for a car to turn on a circular path in a frictionless road if the road is banked, because the component of the normal force directed towards the center of curvature will contribute to the centripetal force. What should the banking angle of the road be if the car is to turn with a speed <u>s</u> in a

curve of radius of curvature $\underline{r}$ (assuming there is no friction)?

Solution



Forces Acting

1. The weight of the car
2. Normal force exerted by the surface of the road

The direction of the centripetal acceleration towards the center of the curvature (i.e. in the direction of $-\hat{e}_r$ as shown).

Decomposing both forces in to component along $-\hat{e}_r$ & $\hat{\jmath}$

$\vec{N} = N\sin\theta\,(-\hat{e}_r) + N\cos\theta\,\hat{\jmath}$ $N_r = -N\sin\theta \qquad N_y = N\cos\theta$	$\vec{w} = -m\lvert g\rvert\hat{\jmath}$ $w_y = -m\lvert g\rvert \qquad w_r = 0$

$$\sum F_y = N_y + w_y = ma_y$$

but $a_y = 0$ because no acceleration in the $\hat{\jmath}$ direction

$$\therefore\ N_y + w_y = 0$$

$$N\cos\theta - m|g| = 0$$

$$N = \frac{m|g|}{\cos\theta} \ldots\ldots (1)$$

$$\sum F_r = N_r + w_r = ma_r$$

$$-N\sin\theta + 0 = -\frac{mv^2}{r}$$

$$N\sin\theta = \frac{mv^2}{r} \ldots\ldots (2)$$

Substituting for N in (2) for (1)

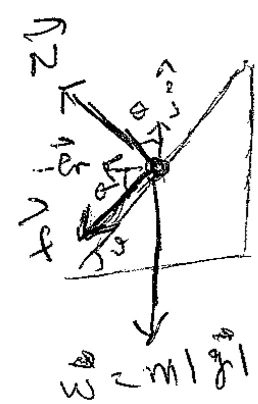
$$\frac{m|g|}{\cos\theta}\sin\theta = \frac{mv^2}{r}$$

$$|g|\tan\theta = \frac{v^2}{r}$$

$$\Rightarrow \tan\theta = \frac{v^2}{r|g|}$$

$$\boxed{\theta = \tan^{-1}\left(\frac{v^2}{r|g|}\right)}$$

5) A car is turning in a banked road with an angle of inclination θ & radius of curvature r. If the coefficient of friction between the road and the tires is μ, find an expression for the maximum speed by which it can make it without sliding.



Solution

Forces acting are:

1. Its weight $(\vec{w})$
2. Normal Force $(\vec{N})$
3. Friction $(\vec{f})$

The direction of the acceleration (centripetal) is towards the center of curvature (along $-\hat{e}_r$)

$$\vec{a} = -\frac{v^2}{r}\hat{e}_r \quad \Rightarrow \quad a_r = -\frac{v^2}{r} \qquad a_y = 0$$

Decomposing the forces into radial (along $-\hat{e}_r$) & vertical (along $\hat{\jmath}$) components

$$\vec{w} = -m|g|\hat{\jmath}$$

$$w_r = 0 \quad w_y = -m|g|$$

$$\vec{N} = N\cos\theta\,\hat{\jmath} + N\sin\theta\,(-\hat{e}_r)$$

$$\Rightarrow\ N_y = N\cos\theta \qquad N_r = -N\sin\theta$$

$$\vec{f} = f\cos\theta\,(-\hat{e}_r) + f\sin\theta\,(-\hat{\jmath})$$

$f_r = -f\cos\theta \quad f_y = -f\sin\theta$

$$\sum F_y = w_y + N_y + f_y = ma_y$$

but $a_y = 0$

$$\Rightarrow\ -m|g| + N\cos\theta - f\sin\theta = 0$$

$$-m|g| + N\cos\theta - \mu N\sin\theta = 0$$

$$N(\cos\theta - \mu\sin\theta) = m|g|$$

$$N = \frac{m|g|}{\cos\theta - \mu\sin\theta} \ldots\ldots (1)$$

$$\sum F_r = w_r + N_r + f_r = ma_r$$

but $a_r = -\frac{v^2}{r}$

$$0 - N\sin\theta + (-f\cos\theta) = -\frac{mv^2}{r}$$

but $f = \mu N$

$$N\sin\theta + \mu N\cos\theta = \frac{mv^2}{r}$$

$$N(\sin\theta + \mu\cos\theta) = \frac{mv^2}{r} \ldots\ldots (2)$$

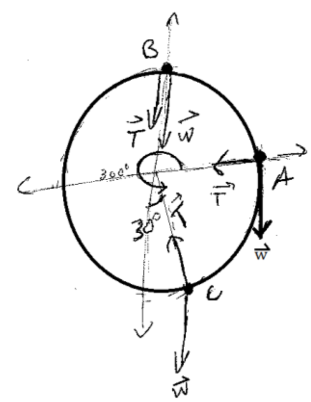
Substituting for N in (2) from (1)

$$\frac{m|g|}{\cos\theta - \mu\sin\theta} * (\sin\theta + \mu\cos\theta) = \frac{mv^2}{r}$$

$$v^2 = r|g|\left[\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right]$$

$$\boxed{v = \sqrt{r|g|\left[\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right]}}$$

6) An object of mass 3kg is revolving in a vertical circle of radius 2m by means of a rod (as shown) with a uniform speed of 10 m/s. Calculate the tension in the rod:



a) At point A

Forces acting are

1. Tension in the rod ($\vec{T}$)
2. Weight of the object

The acceleration is given by $\vec{a} = \frac{dv}{dt}\vec{e}_\theta - \frac{v^2}{r}\vec{e}_r$

but $\frac{dv}{dt} = 0$ because the speed is constant

$$\therefore\ \vec{a} = -\frac{v^2}{r}\vec{e}_r$$

$\vec{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$ & $\theta = 0$ at point A

$$= \cos 0\,\hat{\imath} + \sin 0\,\hat{\jmath}$$

$$\vec{e}_r = \hat{\imath}$$

$$\vec{a} = -\frac{v^2}{r}\vec{e}_r = \frac{-(10\,m/s)^2}{2m}\hat{\imath}$$

$$= -50\ m/s^2\,\hat{\imath}$$

Decomposing the vectors $\vec{w}$ along $\hat{\imath}$ & $\hat{\jmath}$

$$\vec{w} = -m|g|\hat{\jmath} = -3(10)N\,\hat{\jmath}$$

$$= -30N\hat{\jmath}$$

$$\therefore \sum \vec{F} = \vec{T} + \vec{w} = m\vec{a}$$

$$\vec{T} = m\vec{a} - \vec{w}$$

$$\vec{T} = 3kg\left(-50\frac{m}{s^2}\right)\hat{\imath} - (-30N\hat{\jmath})$$

$$\underline{\vec{T} = -150N\hat{\imath} + 30N\hat{\jmath}}$$

b) At point A

Forces acting are

1. Tension in the rod ($\vec{T}$)
2. Weight of the object

$$\vec{a} = -\frac{v^2}{r}\vec{e}_r$$

$\vec{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$ & $\theta = 90°$ at point B

$$= \cos 90°\,\hat{\imath} + \sin 90°\,\hat{\jmath}$$

$$\vec{e}_r = \hat{\jmath}$$

$$\therefore \vec{a} = -50N\,\hat{j}$$

And $\vec{w} = -m|g|\hat{j} = -30N\hat{j}$

$$\sum \vec{F} = \vec{T} + \vec{w} = m\vec{a}$$

$$\vec{T} = m\vec{a} - \vec{w}$$

$$= 3(-50\hat{j}) - (-30N\hat{j})$$

$$\underline{\vec{T} = -120N\,\hat{j}}$$

c) At point A

Forces acting are

1. Tension in the rod ($\vec{T}$)
2. Weight of the object

$$\vec{a} = -\frac{v^2}{r}\vec{e}_r$$

$\vec{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{j}$ &

$\theta = 270 + 30 = 300°$ at point C

$= \cos 300°\ \hat{\imath} + \sin 300°\,\hat{j}$

$\vec{e}_r = 0.5\hat{\imath} - 0.87\hat{j}$

$$\therefore \quad \vec{a} = -50\,m/s^2\,(0.5\hat{\imath} - 0.87\hat{j})$$

And $\vec{w} = -m|g|\hat{j} = -30N\hat{j}$

$$\sum \vec{F} = \vec{T} + \vec{w} = m\vec{a}$$

$$\vec{T} = m\vec{a} - \vec{w}$$

$$= 3(-25\hat{\imath} + 43.5\hat{j}) - (-30N\hat{j})$$

$$\underline{\vec{T} = -75\hat{\imath} + 160.5\hat{j}}$$