Section 2.3 – Divisibility and Modular Arithmetics

Division

Definition

If a and b are integers with $a \neq 0$, we say that a divides b if there is an integer c such that $b = ac$, or equivalently, if $\frac{b}{a}$ is an integers. When a divides b we say that a is a factor or divisor of b, and that b is multiple of a. The notation $a \mid b$ denotes that a divides b. We write $a \nmid b$ when a does not divide b.

Example

Determine whether $3 \mid 7$ and whether $3 \mid 12$.

Solution

We see that $3 \nmid 7$, because 7/3 is not integer.

$3 \mid 12$ because $12/3 = 4$.

Example

Let n and d be positive integers. How many positive integers not exceeding n are divisible by d?

Solution

The positive integers divisible by d are all the integers of the form dk, where k is a positive integer. Hence, the number of positive integers divisible by d that do not exceed n equals the number of integers k with $0 < k \le n/d$. Therefore, there are $\lfloor n/d \rfloor$ positive integers not exceeding n that are divisible by d.

Theroem

Let a, b, and c integers, where $a \neq 0$. Then

i) If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$;

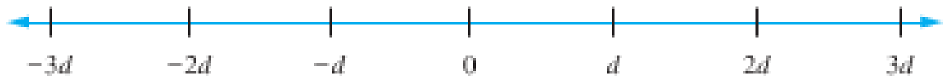
ii) If $a \mid b$, then $a \mid bc$ for all integers c;

iii) If $a \mid b$ and $b \mid c$, then $a \mid c$.

Proof **(*i*)**

Suppose If $a \mid b$ and $a \mid c$. Then, from the definition of divisibility, it follows that there are integers s and t with $b = as$ and $c = at$. Hence,

$$b + c = as + at = a(s + t)$$



Therefore, a divides $b + c$.

Corollary

If a, b, and c integers, where $a \neq 0$, such that $a \mid b$ and $a \mid c$, then $a \mid mb + nc$ whenever m and n are integers.

The Division Algorithm

Theroem

Let a be an integer and d a positive integer. Then there are unique integers q and r, with $0 \le r < d$, such that $a = dq + r$.

Definition

In the equality given in the division algorithm, ***d*** is called the ***divisor***, ***a*** called the ***dividend***, ***q*** is called the ***quotient***, and ***r*** is called the ***remainder***. This notation is used to express the quotient and remainder:

$$q = a \textbf{ div } d, \qquad r = a \textbf{ mod } d$$

Example

What are the quotient and remainder when 101 is divided by 11?

Solution

$101 = 11 \cdot 9 + 2$

Hence, the quotient when 101 is divided by 11 is 9 = 101 ***div*** 11, and the remainder is 2 = 101 ***mod*** 11.

Example

What are the quotient and remainder when -11 is divided by 3?

Solution

$-11 = 3(-4)+1$

Hence, the quotient when -11 is divided by 3 is $-4 = -11$ ***div*** 3,
and the remainder is $1 = -11$ ***mod*** 3.

Modular Arithmetic

Definition

If a and b are integers and m is positive integer, then a is ***congruent*** to b ***modulo*** m if m divides $a-b$. We use the notation $a \equiv b \pmod{m}$ to indicate that a is *congruent* to b *modulo* m. We say that $a \equiv b \pmod{m}$ is a ***congruence*** and that m is its ***modulus*** (plural ***moduli***). If a and b are not congruent modulo m, we write $a \not\equiv b \pmod{m}$

Theorem

Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if
$a \textbf{ mod } m = b \textbf{ mod } m$

Example

Determine whether 17 is congruent to 5 modulo 6 and whether 24 and 14 are congruent modulo 6.

Solution

Because 6 divides $17-5=12$, we see that $17 \equiv 5 \pmod{6}$.

$24-14=10$ is not divisible by 6, we see that $24 \not\equiv 14 \pmod{6}$

Theorem

Let m be a positive integer. The integers a and b are congruent modulo m if and only if there is an integer k such that $a = b + km$.

Proof

If $a \equiv b \pmod{m}$ that implies by the definition of congruence to $m \mid (a-b)$. Which is that there is an integer k such that $a-b=km \quad \Rightarrow \quad a = b + km$.

Conversely, if there is an integer k such that $a = b + km$, then $km = a - b$. Hence, m divides $a - b$, so that $a \equiv b \pmod{m}$

Theorem

Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

$$a + c \equiv b + d \pmod{m} \qquad \text{and} \qquad ac \equiv bd \pmod{m}$$

Proof

Using direct proof. Because $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, by the theorem that are integers s and t with $b = a + sm \quad and \quad d = c + tm$. Hence,

$$b + d = (a + sm) + (c + tm) = (a + c) + m(s + t) \qquad \Rightarrow a + c \equiv b + d \pmod{m}$$

And

$$bd = (a + sm)(c + tm) = ac + m(at + sc + stm) \qquad \Rightarrow ac \equiv bd \pmod{m}$$

Corollary

Let a and b be integers, and let m be a positive integer. Then

$$(a + b) \textbf{ mod } m = ((a \textbf{ mod } m) + (b \textbf{ mod } m)) \textbf{ mod } m$$

and

$$ab \textbf{ mod } m = ((a \textbf{ mod } m)(b \textbf{ mod } m)) \textbf{ mod } m$$

Arithmetic Modulo m

We define addition by: $a +_m b = (a + b) \textbf{ mod } m$ and multiplication by $a \cdot_m b = (a \cdot b) \textbf{ mod } m$

Exercises *Section* 2.3 – Divisibility and Modular Arithmetics

1. Does 17 divide each of these numbers?

a) 68 ***b***) 84 ***c***) 35 ***d***) 1001

2. Prove that if a is an integer other than 0, then

a) $1 \text{ } divides \text{ } a$ ***b***) $a \text{ } divides \text{ } 0$

3. Show that if $a \mid b$ and $b \mid a$, where a and b are integers, then $a = b$ or $a = -b$.

4. Show that if a, b, and c are integers, where $a \neq 0$ and $c \neq 0$, such that $ac \mid bc$, then $a \mid b$

5. What are the quotient and remainder when

a) 19 is divided by 7?
b) -111 is divided by 11?
c) 789 is divided by 23?
d) 1001 is divided by 13?
e) 0 is divided by 19?
f) 3 is divided by 5?
g) -1 is divided by 3?
h) 4 is divided by 1?

6. What time does a 12-hour clock read

a) 80 hours after it reads 11:00?
b) 40 hours before it reads 12:00?
c) 100 hours after it reads 6:00?

7. What time does a 24-hour clock read

a) 100 hours after it reads 2:00?
b) 45 hours before it reads 12:00?
c) 168 hours after it reads 19:00?

8. Suppose a and b are integers, $a \equiv 4 \pmod{13}$, and $b \equiv 9 \pmod{13}$. Find the integer c with $0 \le c \le 12$ such that

a) $c \equiv 9a \pmod{13}$
b) $c \equiv 11b \pmod{13}$
c) $c \equiv a + b \pmod{13}$
d) $c \equiv 2a + 3b \pmod{13}$
e) $c \equiv a^2 + b^2 \pmod{13}$
f) $c \equiv a^3 - b^3 \pmod{13}$

9. Suppose a and b are integers, $a \equiv 11 \; (\boldsymbol{mod}\ 19)$, and $b \equiv 3 \; (\boldsymbol{mod}\ 19)$. Find the integer c with $0 \le c \le 10$ such that

a) $c \equiv a - b \; (\boldsymbol{mod}\ 19)$

b) $c \equiv 7a + 3b \; (\boldsymbol{mod}\ 19)$

c) $c \equiv 2a^2 + 3b^2 \; (\boldsymbol{mod}\ 19)$

d) $c \equiv a^3 + 4b^3 \; (\boldsymbol{mod}\ 19)$

10. Let m be a positive integer. Show that $a \equiv b (\text{mod } m)$ if a **mod** $m = b$ **mod** m

11. Show that if n and k are positive integers, then $\left[n/k \right] = \left[\frac{n-1}{k} \right] + 1$

12. Evaluate these quantities

a) -17 ***mod*** 2
b) 144 ***mod*** 7
c) -101 ***mod*** 13
d) 199 ***mod*** 19
e) 13 ***mod*** 3
f) -97 ***mod*** 11

13. Find a **div** m and a **mod** m when

a) $a = 228$, $m = 119$
b) $a = 9009$, $m = 223$
c) $a = -10101$, $m = 333$
d) $a = -765432$, $m = 38271$

14. Find the integer a such that

a) $a \equiv -15 (\boldsymbol{mod}\ 27)$ *and* $-26 \le a \le 0$

b) $a \equiv 24 (\boldsymbol{mod}\ 31)$ *and* $-15 \le a \le 15$

c) $a \equiv 99 (\boldsymbol{mod}\ 41)$ *and* $100 \le a \le 140$

d) $a \equiv 43 (\boldsymbol{mod}\ 23)$ *and* $-22 \le a \le 0$

e) $a \equiv 17 (\boldsymbol{mod}\ 29)$ *and* $-14 \le a \le 14$

15. Decide whether each of these integers is congruent to 5 modulo 17.

a) 37   *b*) 66   *c*) −17   *d*) −67

16. Find each of these values.

a) $(-133 \textbf{ mod } 23 + 261 \textbf{ mod } 23) \textbf{ mod } 23$

b) $\left(457 \ \mathbf{mod}\ 23 \cdot 182 \ \mathbf{mod}\ 23\right) \ \mathbf{mod}\ 23$

c) $\left(177 \ \mathbf{mod}\ 31 + 270 \ \mathbf{mod}\ 31\right) \ \mathbf{mod}\ 31$

d) $\left(19^2 \ \boldsymbol{mod}\ 41\right) \boldsymbol{mod}\ 9$

e) $\left(32^3 \ \boldsymbol{mod}\ 13\right)^2 \ \boldsymbol{mod}\ 11$

f) $\left(99^2 \ \boldsymbol{mod}\ 32\right)^3 \ \boldsymbol{mod}\ 15$

g) $\left(3^4 \ \boldsymbol{mod}\ 17\right)^2 \ \boldsymbol{mod}\ 11$

h) $\left(19^3 \ \boldsymbol{mod}\ 23\right)^2 \ \boldsymbol{mod}\ 31$

i) $\left(89^3 \ \boldsymbol{mod}\ 79\right)^4 \ \boldsymbol{mod}\ 26$