Solution   *Section* 1.4 – Inverse Matrices - Finding A^{-1}

Exercise

Apply Gauss-Jordan method to find the inverse of this triangular "Pascal matrix"

$$\textbf{\textit{Triangular Pascal matrix}} \qquad A=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix}$$

Solution

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\1&1&0&0&0&1&0&0\\1&2&1&0&0&0&1&0\\1&3&3&1&0&0&0&1\end{array}\right]\begin{array}{l}\\R_2-R_1\\R_3-R_1\\R_4-R_1\end{array}$$

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&-1&1&0&0\\0&2&1&0&-1&0&1&0\\0&3&3&1&-1&0&0&1\end{array}\right]\begin{array}{l}\\\\R_3-2R_2\\R_4-3R_2\end{array}$$

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&1&-2&1&0\\0&0&3&1&2&-3&0&1\end{array}\right]\begin{array}{l}\\\\\\R_4-3R_3\end{array}$$

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&1&-2&1&0\\0&0&0&1&-1&3&-3&1\end{array}\right]$$

$$A^{-1}=\begin{pmatrix}1&0&0&0\\-1&1&0&0\\1&-2&1&0\\-1&3&-3&1\end{pmatrix}$$

- The inverse matrix A^{-1} looks like A, except odd-numbered diagonals are multiplied by -1.

Exercise

If A is invertible and $AB = AC$, prove that $B = C$

Solution

$AB = AC$ *Multiply by* A^{-1} *both sides.*

$A^{-1}(AB) = A^{-1}(AC)$ *Multiplication is associative*

$(A^{-1}A)B = (A^{-1}A)C$ $A^{-1}A = I$

$IB = IC$

$\boxed{B = C}$

Exercise

If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find two matrices $B \neq C$ such that $AB = AC$

Solution

Let $B = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ *and* $C = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$\underline{B \neq C \Rightarrow AB = AC}$

Exercise

If A has ***row*** 1 + ***row*** 2 = ***row*** 3, show that A is not invertible

a) Explain why $Ax = (1,\ 0,\ 0)$ can't have a solution.

b) Which right sides $(b_1,\ b_2,\ b_3)$ might allow a solution to $Ax = b$

c) What happens to ***row*** 3 in elimination?

Solution

a) Let $A_1,\ A_2,\ A_3$ be the row vectors of A and x is a solution to $Ax = (1,\ 0,\ 0)$.

Then $A_1.x = 1,\ A_2.x = 0,\ A_3.x = 0$.

Since $\quad A_1 + A_2 = A_3$

Means $\quad A_1.x + A_2.x = A_3.x$

Implies $\quad 1 + 0 = 0$ a contradiction

b) If $Ax = (b_1,\ b_2,\ b_3) \Rightarrow A_1.x = b_1,\ A_2.x = b_2,\ A_3.x = b_3$

Since $\quad A_1 + A_2 = A_3$

$A_1.x + A_2.x = A_3.x$

$\Rightarrow b_1 + b_2 = b_3$

c) In the elimination matrix, the third row will be zero.

Exercise

True or false (with a counterexample if false and a reason if true):

a) A 4 by 4 matrix with a row of zeros is not invertible.

b) A matrix with 1's down the main diagonal is invertible.

c) If A is invertible then A^{-1} is invertible.

d) If A is invertible then A^2 is invertible.

Solution

a) True, because it can have at most 3 pivots.

b) False, if the matrix: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and only has 2 pivots, thus is not invertible.

c) True, If A is invertible then necessarily A^{-1} is invertible.

d) True, $A^2x = 0$ where x is nonzero matrix.

$$A^{-1}A^2x = \left(A^{-1}A\right)Ax = IAx = Ax = 0$$

Since A is invertible, this can only be true if x was zero to begin with. Thus A^2 must also be invertible.

Exercise

Do there exist 2 by 2 matrices A and B with real entries such that $AB - BA = I$, where I is the identity matrix?

Solution

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$= \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$BA = \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}$$

$$AB - BA = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} - \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}$$

$$= \begin{pmatrix} bg-cf & af+bh-be-df \\ ce+dg-ag-ch & cf-bg \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} bg - cf = 1 \\ af + bh - be - df = 0 \\ ce + dg - ag - ch = 0 \\ cf - bg = 1 \end{cases}$$

$$\rightarrow \begin{cases} bg - cf = 1 \\ \underline{cf - bg = 1} \\ 0 = 2 \end{cases}$$

Therefore, $AB - BA \neq I$ for any 2 by 2 matrices.

Exercise

If B is the inverse of A^2, show that AB is the inverse of A.

Solution

Since B is the inverse of A^2 that implies: $\underline{B = \left(A^2\right)^{-1}} = (AA)^{-1} \underline{= A^{-1}A^{-1}}$

Show that AB is the inverse of A

$$(AB)A = \left(A\left(A^{-1}A^{-1}\right)\right)A$$

$$= \left(\left(AA^{-1}\right)A^{-1}\right)A$$

$$= \left(IA^{-1}\right)A$$

$$= A^{-1}A$$

$$\underline{= I}$$

Therefore, AB is the inverse of A.

Exercise

Find and check the inverses (assuming they exist) of these block matrices.

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}$$

Solution

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix}\begin{bmatrix} I & 0 \\ A & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ C+A & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \Rightarrow C + A = 0 \to A = -C$$

$$\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -C & I \end{pmatrix}$$

$$\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}\begin{bmatrix} E & 0 \\ F & G \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} AE & 0 \\ CE+DF & DG \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\Rightarrow \begin{cases} AE = I \\ CE + DF = 0 \\ DG = I \end{cases} \rightarrow \begin{cases} E = A^{-1} \\ \\ G = D^{-1} \end{cases}$$

$$\begin{aligned} CE + DF = 0 \rightarrow CA^{-1} + DF &= 0 \\ DF &= -CA^{-1} \\ D^{-1}DF &= -D^{-1}CA^{-1} \\ IF &= -D^{-1}CA^{-1} \\ F &= -D^{-1}CA^{-1} \end{aligned}$$

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{pmatrix}$$

$$\begin{bmatrix} 0 & I \\ I & D \end{bmatrix}\begin{bmatrix} A & I \\ I & B \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & B \\ A+D & I+DB \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\rightarrow \begin{cases} B = 0 \\ A + D = 0 \\ I + DB = I \end{cases} \Rightarrow \begin{cases} \\ A = -D \\ DB = 0 \end{cases}$$

$$\begin{pmatrix} 0 & I \\ I & D \end{pmatrix}^{-1} = \begin{pmatrix} -D & I \\ I & 0 \end{pmatrix}$$

Exercise

For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

Solution

$c = 0$, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 7 & 0 \end{bmatrix}$ (zero column 2 / row 2)

$c = 2$, $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 8 & 7 & 2 \end{bmatrix}$ (equal rows)

$c = 7$, $A = \begin{bmatrix} 2 & 7 & 7 \\ 7 & 7 & 7 \\ 8 & 7 & 7 \end{bmatrix}$ (equal columns)

Exercise

Find A^{-1} and B^{-1} (if they exist) by elimination.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Solution

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right)\begin{array}{l} \frac{1}{2}R_1 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right)\begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right)\begin{array}{l} \\ \frac{2}{3}R_2 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right)\begin{array}{l} R_1 - \frac{1}{2}R_2 \\ \\ R_3 - \frac{1}{2}R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right)\begin{array}{l} \\ \\ \frac{3}{4}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right) \begin{array}{l} R_1 - \frac{1}{3}R_3 \\ R_2 - \frac{1}{3}R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \frac{1}{2}R_1 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 + R_1 \\ R_3 + R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ R_3 + R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

B^{-1} ***doesn't*** exist, and if we add the columns in B, the result is zero.

Exercise

Find A^{-1} using the Gauss-Jordan method, which has a remarkable inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 + R_2 \\ \\ \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ R_2 + R_3 \\ \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ R_3 + R_4 \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise

Find the inverse, if exists of $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$

Solution

$$\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}^{-1} = \frac{1}{12+12}\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{12} & \frac{1}{6} \\ \frac{1}{8} & \frac{1}{4} \end{bmatrix}$$

Exercise

Find the inverse, if exists of $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}^{-1} = \frac{1}{7-8}\begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix}$$

$$= -\begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$$

Exercise

Find the inverse, if exists of $\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$

Solution

$$\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}^{-1} = \frac{1}{-15-24}\begin{bmatrix} 5 & -6 \\ -4 & -3 \end{bmatrix}$$

$$= -\frac{1}{39}\begin{bmatrix} 5 & -6 \\ -4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{39} & \frac{2}{13} \\ \frac{4}{39} & \frac{1}{13} \end{bmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{-4+6}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ -\frac{1}{3} & 1 \end{bmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{10-10}\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

$$= \frac{1}{0}\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$$

∴ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$

Solution

$$\left[\begin{array}{cc|cc} -2 & 3 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{array}\right] \quad -\frac{1}{2}R_1$$

$$\begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \end{array}$$

$$\begin{array}{cccc} -3 & 4 & 0 & 1 \\ 3 & -\frac{9}{2} & -\frac{3}{2} & 0 \\ \hline 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ -3 & 4 & 0 & 1 \end{array}\right] \quad R_2 + 3R_1$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right] \quad -2R_2$$

$$\begin{array}{cccc} 0 & 1 & 3 & -2 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 3 & -2 \end{array}\right] \quad R_1 + \frac{3}{2}R_2 \qquad \begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & \frac{9}{2} & -3 \\ \hline 1 & 0 & 4 & -3 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 4 & -3 \\ 0 & 1 & 3 & -2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} a & b \\ 3 & 3 \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{3a - 3b}\begin{bmatrix} 3 & -b \\ -3 & a \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{3(a-b)} & \frac{-b}{3(a-b)} \\ \frac{-3}{3(a-b)} & \frac{a}{3(a-b)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{a-b} & \frac{-b}{3(a-b)} \\ \frac{-1}{a-b} & \frac{a}{3(a-b)} \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} -2 & a \\ 4 & a \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{-2a - 4a}\begin{bmatrix} a & -a \\ -4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{-6a} & \frac{-a}{-6a} \\ \frac{-4}{-6a} & \frac{-2}{-6a} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{6} & \frac{1}{6} \\ \frac{2}{3a} & \frac{1}{3a} \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} 4 & 4 \\ b & a \end{bmatrix}$

Solution

$$A^{-1} = \frac{1}{4a-4b}\begin{bmatrix} a & -4 \\ -b & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{4(a-b)} & \frac{-4}{4(a-b)} \\ \frac{-b}{4(a-b)} & \frac{4}{4(a-b)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{4(a-b)} & \frac{-1}{a-b} \\ \frac{-b}{4(a-b)} & \frac{1}{a-b} \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{-1+4}\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{3}\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{14}\begin{pmatrix} -1 & -4 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{14} & \frac{2}{7} \\ \frac{3}{14} & -\frac{1}{7} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{5}\begin{pmatrix} -1 & -3 \\ -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{11}\begin{pmatrix} 5 & -3 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{11} & -\frac{3}{11} \\ \frac{2}{11} & \frac{1}{11} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{0}\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} -6 & 9 \\ 2 & -3 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{18-18}\begin{pmatrix} & \\ & \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} -2 & 7 \\ 0 & 2 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -7 \\ 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{7}{4} \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$

Solution

$$A = \frac{1}{-16+16}\begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Solution

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 1 \\ a & a \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{a}\begin{pmatrix} a & -1 \\ -a & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\frac{1}{a} \\ -1 & \frac{2}{a} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} b & 3 \\ b & 2 \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{b}\begin{pmatrix} 2 & -3 \\ -b & b \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{b} & \frac{3}{b} \\ 1 & -1 \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & a \\ 3 & a \end{pmatrix}$

Solution

$$A^{-1} = -\frac{1}{2a}\begin{pmatrix} a & -a \\ -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2a} & -\frac{1}{2a} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{a^2 - 4}\begin{pmatrix} a & -2 \\ -2 & a \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a}{a^2-4} & \frac{-2}{a^2-4} \\ \frac{-2}{a^2-4} & \frac{a}{a^2-4} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{0}\begin{pmatrix} & \\ & \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} -3 & \frac{1}{2} \\ 6 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{0}\begin{pmatrix} & \\ & \end{pmatrix}$$

$\therefore$ Inverse ***doesn't exist***

Exercise

Find A^{-1} *if* $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccccc} 2 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -2 & 0 & 0 \\ \hline 0 & -2 & -3 & -2 & 1 & 0 \end{array} \qquad \begin{array}{cccccc} 3 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & -3 & 0 & 0 \\ \hline 0 & 0 & -3 & -3 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \; -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \; -\tfrac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - \frac{3}{2}R_3 \\ \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ \hline 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array} \qquad \begin{array}{cccccc} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ \hline 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

Exercise

Find A^{-1}, where $A=\begin{bmatrix}1&2&-1\\3&5&3\\2&4&3\end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc}1&2&-1&1&0&0\\3&5&3&0&1&0\\2&4&3&0&0&1\end{array}\right]\begin{array}{l}\\R_2-3R_1\\R_3+2R_1\end{array}$$

$$\begin{array}{rrrrrr}3&5&3&0&1&0\\-3&-6&3&-3&0&0\\\hline 0&-1&6&-3&1&0\end{array}\qquad\begin{array}{rrrrrr}2&4&3&0&0&1\\-2&-4&2&-2&0&0\\\hline 0&0&5&-2&0&1\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&2&-1&1&0&0\\0&-1&6&-3&1&0\\0&0&5&-2&0&1\end{array}\right]\;-R_2$$

$$\left[\begin{array}{ccc|ccc}1&2&-1&1&0&0\\0&1&-6&3&-1&0\\0&0&5&-2&0&1\end{array}\right]\begin{array}{l}R_1-2R_2\\\\\\\end{array}$$

$$\begin{array}{rrrrrr}1&2&-1&1&0&0\\0&-2&12&-6&2&0\\\hline 1&0&11&-5&2&0\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&11&-5&2&0\\0&1&-6&3&-1&0\\0&0&5&-2&0&1\end{array}\right]\begin{array}{l}\\\\\frac{1}{5}R_3\end{array}$$

$$\begin{array}{rrrrrr}0&0&1&-\frac{2}{5}&0&\frac{1}{5}\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&11&-5&2&0\\0&1&-6&3&-1&0\\0&0&1&-\frac{2}{5}&0&\frac{1}{5}\end{array}\right]\begin{array}{l}R_1-11R_3\\R_2+6R_3\\\\\end{array}$$

$$\begin{array}{rrrrrr}0&1&-6&3&-1&0\\0&0&6&-\frac{12}{5}&0&\frac{6}{5}\\\hline 0&1&0&\frac{3}{5}&-1&\frac{6}{5}\end{array}\qquad\begin{array}{rrrrrr}1&0&11&-5&2&0\\0&0&-11&\frac{22}{5}&0&-\frac{11}{5}\\\hline 1&0&0&-\frac{3}{5}&2&-\frac{11}{5}\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&0&-\frac{3}{5}&2&-\frac{11}{5}\\0&1&0&\frac{3}{5}&-1&\frac{6}{5}\\0&0&1&-\frac{2}{5}&0&\frac{1}{5}\end{array}\right]$$

$$A^{-1}=\begin{bmatrix}-\frac{3}{5}&2&-\frac{11}{5}\\\frac{3}{5}&-1&\frac{6}{5}\\-\frac{2}{5}&0&\frac{1}{5}\end{bmatrix}$$

Exercise

Find A^{-1}, where $A=\begin{bmatrix}1 & 2 & -1\\ -2 & 0 & 1\\ 1 & -1 & 0\end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ -2 & 0 & 1 & 0 & 1 & 0\\ 1 & -1 & 0 & 0 & 0 & 1\end{array}\right] \begin{array}{l} R_2+2R_1\\ R_3-R_1\end{array}$$

$$\begin{array}{cccccc} -2 & 0 & 1 & 0 & 1 & 0\\ 2 & 4 & -2 & 2 & 0 & 0\\ \hline 0 & 4 & -1 & 2 & 1 & 0\end{array} \qquad \begin{array}{cccccc} 1 & -1 & 0 & 0 & 0 & 1\\ -1 & -2 & 1 & -1 & 0 & 0\\ \hline 0 & -3 & 1 & -1 & 0 & 1\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & 4 & -1 & 2 & 1 & 0\\ 0 & -3 & 1 & -1 & 0 & 1\end{array}\right] \tfrac{1}{4}R_2$$

$$\begin{array}{cccccc} 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0\\ 0 & -3 & 1 & -1 & 0 & 1\end{array}\right] \begin{array}{l} R_1-2R_2\\ \\ R_3+3R_2\end{array}$$

$$\begin{array}{cccccc} 0 & -3 & 1 & -1 & 0 & 1\\ 0 & 3 & -\frac{3}{4} & \frac{3}{2} & \frac{3}{4} & 0\\ \hline 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1\end{array} \qquad \begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0\\ 0 & -2 & \frac{1}{2} & -1 & -\frac{1}{2} & 0\\ \hline 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0\\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0\\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1\end{array}\right] 4R_3$$

$$\begin{array}{cccccc} 0 & 0 & 1 & 2 & 3 & 4\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0\\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0\\ 0 & 0 & 1 & 2 & 3 & 4\end{array}\right] \begin{array}{l} R_1+\frac{1}{2}R_3\\ R_2+\frac{1}{4}R_3\end{array}$$

$$\begin{array}{cccccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0\\ 0 & 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2\\ \hline 1 & 0 & 0 & 1 & 1 & 2\end{array} \qquad \begin{array}{cccccc} 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0\\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1\\ \hline 0 & 1 & 0 & 1 & 1 & 1\end{array}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 1 & 2\\ 0 & 1 & 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 2 & 3 & 4\end{array}\right]$$

$$A^{-1}=\begin{bmatrix}1 & 1 & 2\\ 1 & 1 & 1\\ 2 & 3 & 4\end{bmatrix}$$

Exercise

Find A^{-1}, where $A=\begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \qquad \frac{1}{-2}R_1 \qquad 1 \quad -\frac{5}{2} \quad -\frac{3}{2} \quad -\frac{1}{2} \quad 0 \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \qquad R_2-4R_1 \qquad \begin{array}{cccccc} 4 & -1 & 3 & 0 & 1 & 0 \\ -4 & 10 & 6 & 2 & 0 & 0 \\ \hline 0 & 9 & 9 & 2 & 1 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \qquad R_3-7R_1 \qquad \begin{array}{cccccc} 7 & -2 & 5 & 0 & 0 & 1 \\ -7 & \frac{35}{2} & \frac{21}{2} & \frac{7}{2} & 0 & 0 \\ \hline 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}\right] \qquad \frac{1}{9}R_2 \qquad 0 \quad 1 \quad 1 \quad \frac{2}{9} \quad \frac{1}{9} \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}\right] \qquad R_3-\frac{31}{2}R_2 \qquad \begin{array}{cccccc} 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \\ 0 & -\frac{31}{2} & -\frac{31}{2} & -\frac{31}{9} & -\frac{31}{18} & 0 \\ \hline 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array}\right]$$

$\therefore$ The inverse matrix ***doesn't exist***

Exercise

Find the inverse, if exists, of $\quad A=\begin{pmatrix}1 & 1 & 0\\ -1 & 3 & 4\\ 0 & 4 & 3\end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 0 & 0\\ -1 & 3 & 4 & 0 & 1 & 0\\ 0 & 4 & 3 & 0 & 0 & 1\end{array}\right)\quad R_2+R_1 \qquad \begin{array}{cccccc}1 & 1 & 0 & 1 & 0 & 0\\ -1 & 3 & 4 & 0 & 1 & 0\\ \hline 0 & 4 & 4 & 1 & 1 & 0\end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 0 & 0\\ 0 & 4 & 4 & 1 & 1 & 0\\ 0 & 4 & 3 & 0 & 0 & 1\end{array}\right)\quad \tfrac{1}{4}R_2$$

$$\left(\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 0 & 0\\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0\\ 0 & 4 & 3 & 0 & 0 & 1\end{array}\right)\quad \begin{array}{l}R_1-R_2\\ \\ R_3-4R_2\end{array} \qquad \begin{array}{cccccc}0 & 4 & 3 & 0 & 0 & 1\\ 0 & -4 & -4 & -1 & -1 & 0\\ \hline 0 & 0 & -1 & -1 & -1 & 1\end{array} \qquad \begin{array}{cccccc}1 & 1 & 0 & 1 & 0 & 0\\ 0 & -1 & -1 & -\frac{1}{4} & -\frac{1}{4} & 0\\ \hline 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0\end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0\\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0\\ 0 & 0 & -1 & -1 & -1 & 1\end{array}\right)\quad -R_3$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0\\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0\\ 0 & 0 & 1 & 1 & 1 & -1\end{array}\right)\quad \begin{array}{l}R_1+R_3\\ R_2-R_3\end{array} \qquad \begin{array}{cccccc}1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0\\ 0 & 0 & 1 & 1 & 1 & -1\\ \hline 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1\end{array} \qquad \begin{array}{cccccc}0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0\\ 0 & 0 & -1 & -1 & -1 & 1\\ \hline 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1\end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1\\ 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1\\ 0 & 0 & 1 & 1 & 1 & -1\end{array}\right)$$

$$A^{-1}=\begin{pmatrix}\frac{7}{4} & \frac{3}{4} & -1\\ -\frac{3}{4} & -\frac{3}{4} & 1\\ 1 & 1 & -1\end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix}1 & -1 & 1\\ 0 & -2 & 1\\ -2 & -3 & 0\end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc}1 & -1 & 1 & 1 & 0 & 0\\ 0 & -2 & 1 & 0 & 1 & 0\\ -2 & -3 & 0 & 0 & 0 & 1\end{array}\right) \quad R_3+2R_1 \qquad \begin{array}{cccccc}-2 & -3 & 0 & 0 & 0 & 1\\ 2 & -2 & 2 & 2 & 0 & 0\\ \hline 0 & -5 & 2 & 2 & 0 & 1\end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & -1 & 1 & 1 & 0 & 0\\ 0 & -2 & 1 & 0 & 1 & 0\\ 0 & -5 & 2 & 2 & 0 & 1\end{array}\right) \quad -\tfrac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc}1 & -1 & 1 & 1 & 0 & 0\\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0\\ 0 & -5 & 2 & 2 & 0 & 1\end{array}\right) \quad \begin{array}{l}R_1+R_2\\ \\ R_3+5R_2\end{array} \qquad \begin{array}{cccccc}1 & -1 & 1 & 1 & 0 & 0\\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0\\ \hline 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0\end{array} \qquad \begin{array}{cccccc}0 & -5 & 2 & 2 & 0 & 1\\ 0 & 5 & -\frac{5}{2} & 0 & -\frac{5}{2} & 0\\ \hline 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1\end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0\\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0\\ 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1\end{array}\right) \quad -2R_3$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0\\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0\\ 0 & 0 & 1 & -4 & 5 & -2\end{array}\right) \quad \begin{array}{l}R_1-\frac{1}{2}R_3\\ R_2+\frac{1}{2}R_3\end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & 0 & 3 & -3 & 1\\ 0 & 1 & 0 & -2 & 2 & -1\\ 0 & 0 & 1 & -4 & 5 & -2\end{array}\right)$$

$$A^{-1}=\begin{pmatrix}3 & -3 & 1\\ -2 & 2 & -1\\ -4 & 5 & -2\end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right) \frac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right) R_3 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right) 2R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right) \begin{array}{l} R_1 - 2R_3 \\ R_2 - \frac{5}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix}1 & 1 & 1\\ 3 & 2 & -1\\ 3 & 1 & 2\end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 3 & 2 & -1 & 0 & 1 & 0\\ 3 & 1 & 2 & 0 & 0 & 1\end{array}\right)\begin{array}{l} \\ R_2-3R_1\\ R_3-3R_1\end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & -1 & -4 & -3 & 1 & 0\\ 0 & -2 & -1 & -3 & 0 & 1\end{array}\right)\begin{array}{l} \\ -R_2\\ \end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 1 & 4 & 3 & -1 & 0\\ 0 & -2 & -1 & -3 & 0 & 1\end{array}\right)\begin{array}{l}R_1-R_2\\ \\ R_3+2R_2\end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & -3 & -2 & 1 & 0\\ 0 & 1 & 4 & 3 & -1 & 0\\ 0 & 0 & 7 & 3 & -2 & 1\end{array}\right)\begin{array}{l} \\ \\ \frac{1}{7}R_3\end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & -3 & -2 & 1 & 0\\ 0 & 1 & 4 & 3 & -1 & 0\\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7}\end{array}\right)\begin{array}{l}R_1+3R_3\\ R_2-4R_3\\ \end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & 0 & -\frac{5}{7} & \frac{1}{7} & \frac{3}{7}\\ 0 & 1 & 0 & \frac{9}{7} & \frac{1}{7} & -\frac{4}{7}\\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7}\end{array}\right)$$

$$A^{-1}=\begin{pmatrix}-\frac{5}{7} & \frac{1}{7} & \frac{3}{7}\\ \frac{9}{7} & \frac{1}{7} & -\frac{4}{7}\\ \frac{3}{7} & -\frac{2}{7} & \frac{1}{7}\end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix}3&3&1\\1&2&1\\2&-1&1\end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc}3&3&1&1&0&0\\1&2&1&0&1&0\\2&-1&1&0&0&1\end{array}\right)\begin{array}{l}\frac{1}{3}R_1\\ \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc}1&1&\frac{1}{3}&\frac{1}{3}&0&0\\1&2&1&0&1&0\\2&-1&1&0&0&1\end{array}\right)\begin{array}{l}\\R_2-R_1\\R_3-2R_1\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&1&\frac{1}{3}&\frac{1}{3}&0&0\\0&1&\frac{2}{3}&-\frac{1}{3}&1&0\\0&-3&\frac{1}{3}&-\frac{2}{3}&0&1\end{array}\right)\begin{array}{l}R_1-R_2\\ \\R_3+3R_2\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&-\frac{1}{3}&\frac{2}{3}&-1&0\\0&1&\frac{2}{3}&-\frac{1}{3}&1&0\\0&0&\frac{7}{3}&-\frac{5}{3}&3&1\end{array}\right)\begin{array}{l}\\ \\ \frac{3}{7}R_3\end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&-\frac{1}{3}&\frac{2}{3}&-1&0\\0&1&\frac{2}{3}&-\frac{1}{3}&1&0\\0&0&1&-\frac{5}{7}&\frac{9}{7}&\frac{3}{7}\end{array}\right)\begin{array}{l}R_1+\frac{1}{3}R_3\\R_2-\frac{2}{3}R_3\\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc}1&0&0&\frac{3}{7}&-\frac{4}{7}&\frac{1}{7}\\0&1&0&\frac{1}{7}&\frac{1}{7}&-\frac{2}{7}\\0&0&1&-\frac{5}{7}&\frac{9}{7}&\frac{3}{7}\end{array}\right)$$

$$A^{-1}=\begin{pmatrix}\frac{3}{7}&-\frac{4}{7}&\frac{1}{7}\\\frac{1}{7}&\frac{1}{7}&-\frac{2}{7}\\-\frac{5}{7}&\frac{9}{7}&\frac{3}{7}\end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix} -3 & 1 & -1 \\ 1 & -4 & -7 \\ 1 & 2 & 5 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} -3 & 1 & -1 & 1 & 0 & 0 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right) \quad -\frac{1}{3}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{11}{3} & -\frac{22}{3} & \frac{1}{3} & 1 & 0 \\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1 \end{array}\right) \quad -\frac{3}{11}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0 \\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1 \end{array}\right) \quad R_3 - \frac{7}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0 \\ 0 & 0 & 0 & & & \end{array}\right)$$

$\therefore$ Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix}1 & 1 & -3\\ 2 & -4 & 1\\ -5 & 7 & 1\end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc}1 & 1 & -3 & 1 & 0 & 0\\ 2 & -4 & 1 & 0 & 1 & 0\\ -5 & 7 & 1 & 0 & 0 & 1\end{array}\right)\begin{matrix}\\ R_2-2R_1\\ R_3+5R_1\end{matrix}$$

$$\left(\begin{array}{ccc|ccc}1 & 1 & -3 & 1 & 0 & 0\\ 0 & -6 & 7 & -2 & 1 & 0\\ 0 & 12 & -14 & 5 & 0 & 1\end{array}\right)\ -\tfrac{1}{6}R_2$$

$$\left(\begin{array}{ccc|ccc}1 & 1 & -3 & 1 & 0 & 0\\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0\\ 0 & 12 & -14 & 5 & 0 & 1\end{array}\right)\begin{matrix}R_1-R_2\\ \\ R_3-12R_2\end{matrix}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & -3 & 1 & 0 & 0\\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0\\ 0 & 0 & 0 & & & \end{array}\right)$$

$\therefore$ Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $A=\begin{pmatrix}1 & 2 & -1\\ 3 & 5 & 3\\ 2 & 4 & 3\end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 3 & 5 & 3 & 0 & 1 & 0\\ 2 & 4 & 3 & 0 & 0 & 1\end{array}\right)\begin{matrix}\\ R_3-3R_1\\ R_3-2R_1\end{matrix}$$

$$\left(\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & -1 & 6 & -3 & 1 & 0\\ 0 & 0 & 5 & -2 & 0 & 1\end{array}\right)\ -R_2$$

$$\left(\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 0 & 1 & -6 & 3 & -1 & 0\\ 0 & 0 & 5 & -2 & 0 & 1\end{array}\right)\begin{matrix}R_1-2R_2\\ \\ \\ \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}\right) \quad \tfrac{1}{5}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right) \quad \begin{array}{l} R_1 - 11R_3 \\ R_2 + 6R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{pmatrix}$$

Exercise

Find the inverse, if exists of $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_3 - R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \quad R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \quad -\tfrac{1}{2}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right] \quad \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Exercise

Find the inverse, if exists of $A = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{pmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & 1 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} & 0 & 1 & 0 \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} & 0 & 0 & 1 \end{array}\right] \begin{array}{l} 5R_1 \\ 10R_2 \\ 10R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 2 & 2 & 1 & 0 & 10 & 0 \\ 2 & -8 & 1 & 0 & 0 & 10 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 0 & 5 & -10 & 10 & 0 \\ 0 & -10 & 5 & -10 & 0 & 10 \end{array}\right] \begin{array}{l} \\ R_2 - \frac{1}{10}R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & \frac{9}{2} & -9 & 10 & -1 \\ 0 & -10 & 5 & -10 & 0 & 10 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 10R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{13}{2} & 14 & -10 & 1 \\ 0 & 1 & \frac{9}{2} & -9 & 10 & -1 \\ 0 & 0 & 50 & -100 & 100 & 0 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{50}R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{13}{2} & 14 & -10 & 1 \\ 0 & 1 & \frac{9}{2} & -9 & 10 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right] \begin{array}{l} R_1 + \frac{13}{2} R_3 \\ R_2 - \frac{9}{2} R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{bmatrix}$$

Exercise

Find the inverse, if exists of $A = \begin{pmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} \sqrt{2} & 3\sqrt{2} & 0 & 1 & 0 & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_2 + 4R_1$$

$$\left[\begin{array}{ccc|ccc} \sqrt{2} & 3\sqrt{2} & 0 & 1 & 0 & 0 \\ 0 & 13\sqrt{2} & 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} 13R_1 - 3R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 13\sqrt{2} & 0 & 0 & 1 & -3 & 0 \\ 0 & 13\sqrt{2} & 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \frac{1}{13\sqrt{2}} R_1 \\ \frac{1}{13\sqrt{2}} R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{13\sqrt{2}} & -\frac{3}{13\sqrt{2}} & 0 \\ 0 & 1 & 0 & \frac{4}{13\sqrt{2}} & \frac{1}{13\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{pmatrix} \frac{\sqrt{2}}{26} & -\frac{3\sqrt{2}}{26} & 0 \\ \frac{2\sqrt{2}}{13} & \frac{\sqrt{2}}{26} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise

Find the inverse, if exists of $A = \begin{pmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{pmatrix}$

Solution

$$\begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}^{-1} = doesn't\ exist$$

Since this matrix is ***singular***, row 3 all zeros.

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -6 & 1 \\ -2 & -2 & 5 & 1 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} -2 & -3 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad -\frac{1}{2}R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_4 + 2R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_4 - R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

∴ Inverse ***does not exist***

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 1 & -14 & 7 & 38 \\ -1 & 2 & 1 & -2 \\ 1 & 2 & -1 & -6 \\ 1 & -2 & 3 & 6 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & -6 & 0 & 0 & 1 & 0 \\ 1 & -2 & 3 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 + R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & -12 & 8 & 36 & 1 & 1 & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array}\right] \quad -\frac{1}{12}R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 + 14R_2 \\ \\ R_3 - 16R_2 \\ R_4 - 12R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & \frac{8}{3} & 4 & \frac{1}{3} & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array}\right] \quad \frac{3}{8}R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + \frac{7}{3}R_3 \\ R_2 + \frac{2}{3}R_3 \\ \\ R_4 - 4R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & -2 & -\frac{1}{2} & -1 & -\frac{3}{2} & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ -\frac{1}{2}R_4 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array}\right] \begin{array}{l} R_1 + \frac{1}{2}R_4 \\ R_2 + 2R_4 \\ R_3 - \frac{3}{2}R_4 \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ 0 & 0 & 1 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{bmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 10 & 20 & -30 & 15 \\ 3 & -7 & 14 & -8 \\ -7 & -2 & -1 & 2 \\ 4 & 4 & -3 & 1 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} 10 & 20 & -30 & 15 & 1 & 0 & 0 & 0 \\ 3 & -7 & 14 & -8 & 0 & 1 & 0 & 0 \\ -7 & -2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & -3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad \frac{1}{10}R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 3 & -7 & 14 & -8 & 0 & 1 & 0 & 0 \\ -7 & -2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & -3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 + 7R_1 \\ R_4 - 4R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 0 & -13 & 23 & -\frac{25}{2} & -\frac{3}{10} & 1 & 0 & 0 \\ 0 & 12 & -22 & \frac{25}{2} & \frac{7}{10} & 0 & 1 & 0 \\ 0 & -4 & 9 & -5 & -\frac{2}{5} & 0 & 0 & 1 \end{array}\right] \quad -\frac{1}{13}R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 12 & -22 & \frac{25}{2} & \frac{7}{10} & 0 & 1 & 0 \\ 0 & -4 & 9 & -5 & -\frac{2}{5} & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 - 12R_2 \\ R_4 + 4R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 0 & -\frac{10}{13} & \frac{25}{26} & \frac{11}{26} & \frac{12}{13} & 1 & 0 \\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1 \end{array}\right] \quad -\frac{13}{10}R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - \frac{7}{13}R_3 \\ R_2 + \frac{23}{13}R_3 \\ \\ R_4 - \frac{25}{13}R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{19}{20} & -\frac{11}{5} & -\frac{23}{10} & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array}\right] \begin{array}{l} \\ R_2 + R_4 \\ R_3 + R_4 \\ \\ \end{array}.$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ \frac{4}{5}R_4 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array}\right] \begin{array}{l} R_1 - \frac{1}{4}R_4 \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{bmatrix}$$

Exercise

Show that A is not invertible for any values of the entries

$$A=\begin{bmatrix}0&a&0&0&0\\ b&0&c&0&0\\ 0&d&0&e&0\\ 0&0&f&0&g\\ 0&0&0&h&0\end{bmatrix}$$

Solution

Since the matrix A had zero's on its diagonals, therefore A is not invertible.

Exercise

Prove that if A is an invertible matrix and B is row equivalent to A, then B is also invertible.

Solution

Since B is row equivalent to A, there exist some elementary matrices $E_1, E_2, \ldots, E_n$ such that $B = E_n \ \ldots\ E_1 A$. Because $E_1, E_2, \ldots, E_n$ and A are invertible, then B is also invertible.

Exercise

Determine if the given matrix has an inverse, and find the inverse if it exists. Check your answer by multiplying $A \cdot A^{-1} = I$

a) $\begin{bmatrix}2&3\\ -3&-5\end{bmatrix}$ *b)* $\begin{bmatrix}1&0&1\\ 0&1&1\\ 2&3&5\end{bmatrix}$

Solution

a) $2(-5)-3(-3)=-10+9=-1$

$$A^{-1}=\frac{1}{-1}\begin{bmatrix}-5&-3\\ 3&2\end{bmatrix}=\begin{bmatrix}5&3\\ -3&-2\end{bmatrix}$$

$$AA^{-1}=\begin{pmatrix}2&3\\ -3&-5\end{pmatrix}\begin{pmatrix}5&3\\ -3&-2\end{pmatrix}=\begin{pmatrix}1&0\\ 0&1\end{pmatrix}=I$$

b) $\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\ 0&1&1&0&1&0\\ 2&3&5&0&0&1\end{array}\right]$ R_3-2R_1

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\ 0&1&1&0&1&0\\ 0&3&3&-2&0&1\end{array}\right] \quad R_3-3R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & * & * & * \end{array}\right]$$

The inverse matrix doesn't exist

Exercise

Show that the inverse of $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is $\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$

Solution

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\theta)\cos(-\theta) - (\sin\theta)\sin(-\theta) & (\cos\theta)\sin(-\theta) - (\sin\theta)\cos(-\theta) \\ (-\sin\theta)\cos(-\theta) - (\cos\theta)\sin(-\theta) & (-\sin\theta)\sin(-\theta) + (\cos\theta)\cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\theta + \sin\theta\sin\theta & -\cos\theta\sin\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin\theta\sin\theta + \cos\theta\cos\theta \end{bmatrix} \qquad \begin{cases} \cos(-\theta) = \cos\theta & (even) \\ \sin(-\theta) = -\sin\theta & (odd) \end{cases}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

Exercise

If the product $C = AB$ is invertible (and A & B are square matrices), find a formula for A^{-1} that involves C^{-1} and B.

Hence, it is not possible to multiply a non-invertible matrix by another matric and obtain an invertible matrix as a result.

Solution

Since $C = AB$ is invertible, the $CC^{-1} = C^{-1}C = I$

$CC^{-1} = I$

$(AB)C^{-1} = I$

$A\left(BC^{-1}\right) = I$

$A^{-1}A\left(BC^{-1}\right) = A^{-1}I$

$I\left(BC^{-1}\right) = A^{-1}$

$\underline{BC^{-1} = A^{-1}}\,|$ ✓

Exercise

Prove that if A is an $m \times n$ matrix, there is an invertible matrix C such that CA is in reduced row-echelon form.

Solution

The reduced row-echelon form of A can be written in the form $E_n \ldots E_2 E_1 A$. where $E_1,\ E_2,\ \ldots,\ E_n$ are elementary matrices.

Let $C = E_n \cdots E_2 E_1$, then C is invertible since $E_1,\ E_2,\ \ldots,\ E_n$ are invertible.

Hence, there exists such a matrix C.

Exercise

Prove that 2 $m \times n$ matrices A and B are row equivalent if and only if there exists a nonsingular matrix P such that $B = PA$

Solution

Suppose that $A \sim B$, then there exist elementary matrices $E_1,\ E_2,\ \ldots,\ E_n$ such that $B = E_n \cdots E_2 E_1 A$.

Let $P = E_n \cdots E_2 E_1 \Rightarrow$ by the theorem, P is nonsingular.

Suppose that $B = PA$, for some nonsingular matrix P. By the theorem, P is row equivalent to I_k.

That is, $I_k = E_n \ldots E_2 E_1 P$.

Thus, $B = E_1^{-1} E_2^{-1} \ldots E_n^{-1} A$ and this implies that A is row equivalent to B.

Exercise

Let A and B be 2 $m \times n$ matrices. Suppose A is row equivalent to B. Prove that A is nonsingular if and only if B is nonsingular.

Solution

Suppose that A is row equivalent to B. Then, there exists a nonsingular matrix P such that $B = PA$. If A is nonsingular then B is nonsingular.

Conversely, if B is nonsingular then $A = P^{-1}B$ is nonsingular.

Exercise

Show that if A and B are two $n \times n$ invertible matrices then A is row equivalent to B.

Solution

Since A is invertible, then A is a row equivalent to I_n. That is, there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that $I_n = E_k E_{k-1} \cdots E_1 A$.

Similarly, there exist elementary matrices $F_1, F_2, \ldots, F_k$ such that $I_n = F_i F_{i-1} \cdots F_1 B$.

Hence, $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n$

$$= E_1^{-1} E_2^{-1} \cdots E_k^{-1} \left(F_i F_{i-1} \cdots F_1 B \right)$$

$$= \left(E_1^{-1} E_2^{-1} \cdots E_k^{-1} F_i F_{i-1} \cdots F_1 B \right)$$

That is, A row equivalent to B.

Exercise

Prove that a square matrix A is nonsingular if and only if A is a product of elementary matrices.

Solution

Suppose that A is nonsingular. Then A is row equivalent to I_n. That is, there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that $I_n = E_k E_{k-1} \cdots E_1 A \quad \to \quad A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n$.

But each E_i^{-1} is an elementary matrix.

Conversely, suppose that $A = E_1 E_2 \ \ldots E_k$, then $\left(E_1 E_2 \ \ldots E_k \right)^{-1} A = I_n$

That is, A is nonsingular.

Exercise

Show that if $A \sim B$ (that is, if they are row equivalent), then $EA = B$ for some matrix E which is a product of elementary matrices.

Solution

If $A \sim B$, there is some sequence of elementary row operations which, when performed on A, produce B.

Further, multiplying on the left by the corresponding elementary matrix is the same as performing that row operation. So we have

$A \sim E_1 A$

$\quad \sim E_2 E_1 A$

$\sim E_k E_{k-1} \dots E_2 E_1 A$

$\underline{= B|}$

Thus, if $E = E_k \dots E_1$, then we have $EA = B$

Exercise

Show that if $EA = B$ for some matrix *E* which is a product of elementary matrices, then $AC \sim BC$ for every $n \times n$ matrix *C*.

Solution

Let $E = E_k E_{k-1} \dots E_1$, where each E_i is an elementary matrix.

$$\begin{aligned} AC &\sim E_1 AC \\ &\sim E_2 E_1 AC \\ &\sim E_k E_{k-1} \dots E_2 E_1 AC \\ &= EAC \qquad \text{since } EA = B \\ &= BC \end{aligned}$$

Therefore; $\underline{AC \sim BC\ |}$

Exercise

Let $A\vec{x} = 0$ be a homogeneous system of *n* linear equations in *n* unknowns that has only the trivial solution. Show that of *k* is any positive integer, then the system $A^k \vec{x} = 0$ also has only trivial solution.

Solution

Since *A* is a square matrix, thus *A* has only the trivial solution. That implies that *A* is invertible.

But A^k is also invertible so $A^k \vec{x} = 0$ has only trivial solution.

Exercise

Let $A\vec{x} = 0$ be a homogeneous system of *n* linear equations in *n* unknowns, and let *Q* be an invertible $n \times n$ matrix. Show that $A\vec{x} = 0$ has just trivial solution if and only if $(QA)\vec{x} = 0$ has just trivial solution.

Solution

A is a square $(n \times n)$ matrix. If $A\vec{x} = 0$ has just a trivial solution, then *A* is invertible. Since *Q* is an invertible $n \times n$ matrix, then QA is also invertible.

Thus, $(QA)\vec{x} = 0$ has trivial solution.

On the other hand, if $(QA)\vec{x} = 0$ has trivial solution, then QA is also invertible.

Since Q is invertible, then Q^{-1} is also invertible.

Thus, $A = Q^{-1}QA$ is invertible, i.e $A\vec{x} = 0$ has just trivial solution, equivalent $A\vec{x} = 0$ has just trivial solution if and only if $(QA)\vec{x} = 0$ has just trivial solution.

Exercise

Let $A\vec{x} = b$ be any consistent system of linear equations, and let $\vec{x}_1$ be a fixed solution. Show that every solution to the system can be written in the form $\vec{x} = \vec{x}_1 + \vec{x}_0$ where $\vec{x}_0$ is a solution to $A\vec{x} = 0$. Show also that every matrix of this form is a solution.

Solution

Since $\vec{x}_0$ is a solution to $A\vec{x} = 0$, we have $A\vec{x}_0 = 0$.

Adding $A\vec{x}_0 = 0$ to $A\vec{x} = b$, then

$$A\vec{x} + Ax_0 = b + 0$$

$$A\left(\vec{x} + \vec{x}_0\right) = b$$

As adding an equation to the original equation does not affect the solution.

If we let $\vec{x}_1$ be a fixed solution, then every solution to $A\vec{x} = b$ is $\vec{x} = \vec{x}_1 + \vec{x}_0$.

Besides,

$$\begin{aligned} A\left(\vec{x} + \vec{x}_0\right) &= A\vec{x} + Ax_0 \\ &= b + 0 \\ &= b \end{aligned}$$

So, every matrix (vector) in the form $\vec{x}_1 + \vec{x}_0$ is a solution to $A\vec{x} = b$

Exercise

If A and B are $n \times n$ matrices satisfying $A^2 = B^2 = (AB)^2 = I_n$. Prove that $AB = BA$.

Solution

Since $A^2 = B^2 = (AB)^2 = I_n$, then A, B, AB are nonsingular.

$$A^2 = I \;\rightarrow\; A = A^{-1}$$

$$B^2 = I \;\rightarrow\; B = B^{-1}$$

$$(AB)^2 = I \;\rightarrow\; AB = (AB)^{-1}$$

$$AB = (AB)^{-1}$$
$$= B^{-1}A^{-1}$$
$$\underline{= BA} \quad \checkmark$$

Exercise

Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix}$. Verify that $A^3 = 5I$, then find A^{-1} in term of A.

Solution

$$A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 1 \\ 5 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}$$

$$A^3 = AA^2$$
$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 5 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$
$$= 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\underline{= 5I}$$

Since $A^3 = AA^2 = 5I$

$$\frac{1}{5}\left(AA^2\right) = I$$
$$A\left(\frac{1}{5}A^2\right) = I$$
$$\underline{A^{-1} = \frac{1}{5}A^2}$$

Exercise

Consider $B(A,\ I)=(BA,\ B)$, thus if B is the inverse of A, then $(BA,\ B)$ becomes $\left(I,\ A^{-1}\right)$. On the other hand B is a product of elementary matrices since it is invertible. This indicates that the inverse of A can be obtained by applying elementary row operations to $(A,\ I)$ to get $\left(I,\ A^{-1}\right)$.

Find the inverses of

a) $A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix}$ *b)* $B=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{pmatrix}$

Solution

a)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix} R_3 - aR_1 \qquad E_{31}=-a$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{pmatrix} R_3 - bR_2 \qquad E_{32}=-b$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$A^{-1}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & -b & 1 \end{pmatrix}$$

b) First, we have move row 4 to row 1, for the calculation

$$\begin{pmatrix} a & b & c & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{a}R_1 \qquad E_{11}=\frac{1}{a}$$

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{c}{a} & \frac{d}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} R_1 - \frac{b}{a}R_2 \qquad E_{12}=-\frac{b}{a}$$

$$\begin{pmatrix} 1 & 0 & \frac{c}{a} & \frac{d}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_1 - \frac{c}{a}R_3 \qquad E_{13} = -\frac{c}{a}$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{d}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_1 - \frac{d}{a}R_4 \qquad E_{14} = -\frac{d}{a}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$E = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} & -\frac{c}{a} & -\frac{d}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since we move Row 4 to Row 1, we must move Column 1 to Column 4 to get the inverse matrix.

$$B^{-1} = \begin{pmatrix} -\frac{b}{a} & -\frac{c}{a} & -\frac{d}{a} & \frac{1}{a} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Exercise

Let $A, B, C, X, Y, Z \in M_n(\mathbb{C})$, A and C are invertible. Find

a) $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}^{-1}$ *b)* $\begin{pmatrix} I & X & Y \\ 0 & I & Z \\ 0 & 0 & I \end{pmatrix}^{-1}$

Solution

a) $\left(\begin{array}{cc|cc} A & B & I & 0 \\ 0 & C & 0 & I \end{array}\right) \quad A^{-1}R_1$

$$\left(\begin{array}{cc|cc} A^{-1}A & A^{-1}B & A^{-1}I & 0 \\ 0 & C & 0 & I \end{array}\right)$$

$$\left(\begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ 0 & C & 0 & I \end{array}\right) \quad C^{-1}R_2$$

$$\left(\begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ 0 & C^{-1}C & 0 & C^{-1}I \end{array}\right)$$

$$\left(\begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ 0 & I & 0 & C^{-1} \end{array}\right) \quad R_1 - A^{-1}BR_2$$

$$\left(\begin{array}{cc|cc} I & A^{-1}B - A^{-1}BI & A^{-1} & -A^{-1}BC^{-1} \\ 0 & I & 0 & C^{-1} \end{array}\right)$$

$$\left(\begin{array}{cc|cc} I & A^{-1}B - A^{-1}B & A^{-1} & -A^{-1}BC^{-1} \\ 0 & I & 0 & C^{-1} \end{array}\right)$$

$$\left(\begin{array}{cc|cc} I & 0 & A^{-1} & -A^{-1}BC^{-1} \\ 0 & I & 0 & C^{-1} \end{array}\right)$$

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{pmatrix}$$

b)
$$\left(\begin{array}{ccc|ccc} I & X & Y & I & 0 & 0 \\ 0 & I & Z & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 & I \end{array}\right) \quad R_1 - XR_2$$

$$\left(\begin{array}{ccc|ccc} I & 0 & Y - XZ & I & -X & 0 \\ 0 & I & Z & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 & I \end{array}\right) \quad \begin{array}{c} R_1 - (Y - XZ)R_3 \\ R_2 - ZR_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} I & 0 & 0 & I & -X & XZ - Y \\ 0 & I & 0 & 0 & I & -Z \\ 0 & 0 & I & 0 & 0 & I \end{array}\right)$$

$$\begin{pmatrix} I & X & Y \\ 0 & I & Z \\ 0 & 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & -X & XZ - Y \\ 0 & I & -Z \\ 0 & 0 & I \end{pmatrix}$$

Exercise

Suppose that *A, B*, and $A-B$ are invertible $n \times n$ matrices. Show that

$$(A-B)^{-1} = A^{-1} + A^{-1}\left(B^{-1} - A^{-1}\right)^{-1} A^{-1}$$

Solution

A, B, and $A-B$ are invertible Then

$AA^{-1} = A^{-1}A = I \quad BB^{-1} = B^{-1}B = I$

$(A-B)(A-B)^{-1} = (A-B)^{-1}(A-B) = I$

Let:

$(A-B)^{-1}(A-B) = I$

Then, we need to prove that

$$\left(A^{-1} + A^{-1}\left(B^{-1} - A^{-1}\right)^{-1} A^{-1}\right)(A-B) \overset{?}{=} I$$

$$\left(A^{-1} + A^{-1}\left(B^{-1} - A^{-1}\right)^{-1} A^{-1}\right)(A-B) = \left(A^{-1} + A^{-1}\left(A\left(B^{-1} - A^{-1}\right)\right)^{-1}\right)(A-B)$$

$$\left(A\left(B^{-1} - A^{-1}\right)\right)^{-1} = \left(B^{-1} - A^{-1}\right)^{-1} A^{-1}$$

$$= \left(A^{-1} + A^{-1}\left(AB^{-1} - AA^{-1}\right)^{-1}\right)(A-B)$$

$$= \left(A^{-1} + A^{-1}\left(AB^{-1} - I\right)^{-1}\right)(A-B)$$

$$= \left(A^{-1} + A^{-1}\left(AB^{-1} - BB^{-1}\right)^{-1}\right)(A-B)$$

$$= \left(A^{-1} + A^{-1}\left((A-B)B^{-1}\right)^{-1}\right)(A-B)$$

$$\left((A-B)B^{-1}\right)^{-1} = B(A-B)^{-1}$$

$$= \left(A^{-1} + A^{-1}\left(B(A-B)^{-1}\right)\right)(A-B)$$

$$= \left(A^{-1} + A^{-1}B(A-B)^{-1}\right)(A-B)$$

$$= A^{-1}(A-B) + A^{-1}B(A-B)^{-1}(A-B)$$

$$= A^{-1}A - A^{-1}B + A^{-1}BI$$

$$= I - A^{-1}B + A^{-1}B$$

$= I$ ✓

Therefore; $(A-B)^{-1} = A^{-1} + A^{-1}\left(B^{-1} - A^{-1}\right)^{-1} A^{-1}$

Exercise

Suppose P is invertible and $A = PBP^{-1}$. Solve for B in terms of A.

Solution

Since P is invertible, then $PP^{-1} = P^{-1}P = I$

$A = PBP^{-1}$

$P^{-1}AP = P^{-1}PBP^{-1}P \qquad PP^{-1} = P^{-1}P = I$

$P^{-1}AP = IBI \qquad BI = B$

$\underline{P^{-1}AP = B}$

Exercise

Suppose $(A-B)C = 0$, where A and B are $m \times n$ matrices and C is invertible. Show that $A = B$.

Solution

Since C is invertible, then $CC^{-1} = C^{-1}C = I$

$(A-B)C = 0$

$(A-B)CC^{-1} = 0C^{-1}$

$(A-B)I = 0$

$A - B = 0$

$A - B + B = 0 + B$

$\underline{A = B}$ ✓