Section 1.6 – Precise Definition of a Limit

Example

Consider the function $y = 2x - 1$ near $x_0 = 4$. Intuitively it appears that *y* is close to 7 when *x* is close to 4, so $\lim_{x \to 4} (2x - 1) = 7$. However, how close to $x_0 = 4$ does *x* have to be so that $y = 2x - 1$ differs from 7 by, say less than 2 units?

Solution

We need to find the values of *x* for $|y - 7| < 2$.

$|y - 7| = |2x - 1 - 7| = |2x - 8|$

$|2x - 8| < 2$

$-2 < 2x - 8 < 2$

$-2 + 8 < 2x - 8 + 8 < 2 + 8$

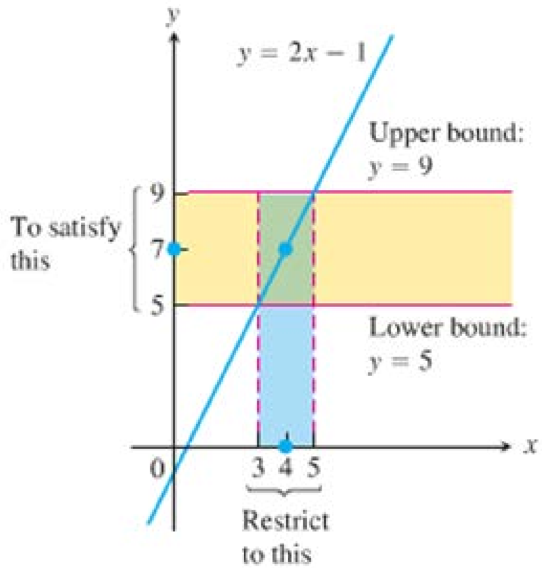
$6 < 2x < 10$

$\frac{6}{2} < \frac{2x}{2} < \frac{10}{2}$

$3 < x < 5$

$3 - 4 < x - 4 < 5 - 4$

$-1 < x - 4 < 1$



Keeping *x* within 1 unit of $x_0 = 4$ will keep *y* within 2 units of $y_0 = 7$

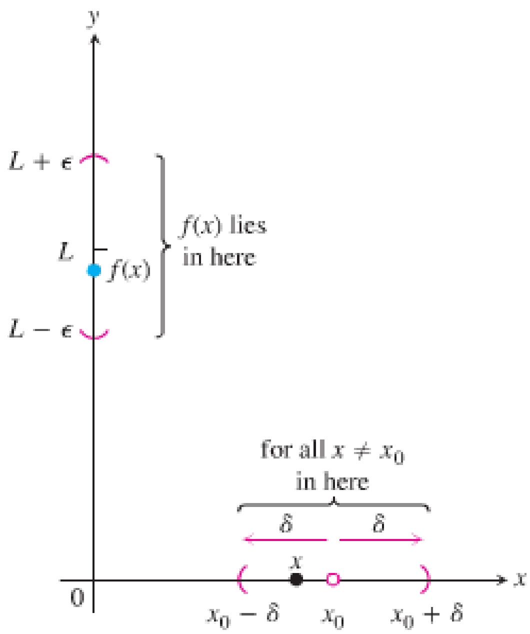
Definition

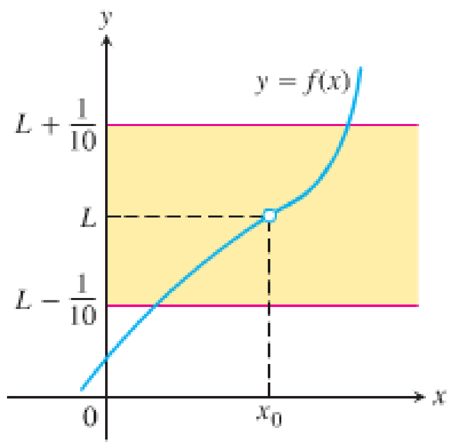
Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that **the limit of** $f(x)$ **as *x* approaches** x_0 **is the number *L***, and write

$$\lim_{x \to x_0} f(x) = L$$

If, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all *x*,

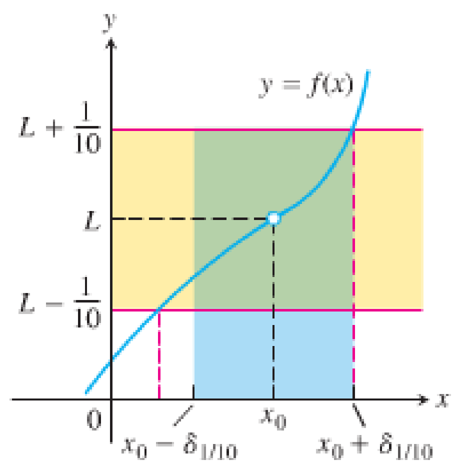
$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$





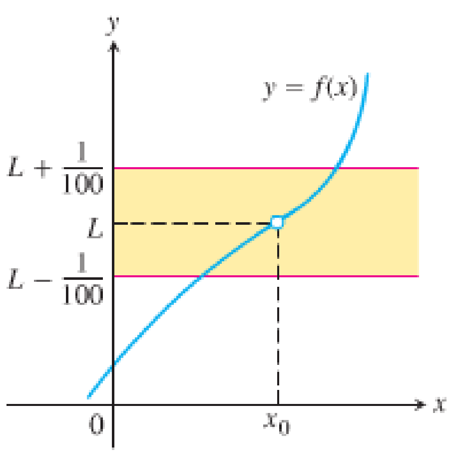
The challenge:

Make $\left|f(x) - L\right| < \epsilon = \frac{1}{10}$



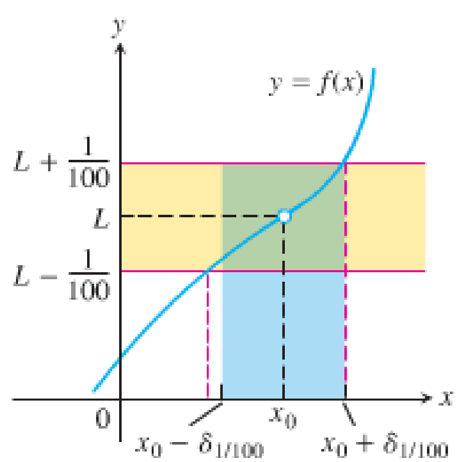
Response:

$\left|x - x_0\right| < \delta_{1/10}$ (a number)



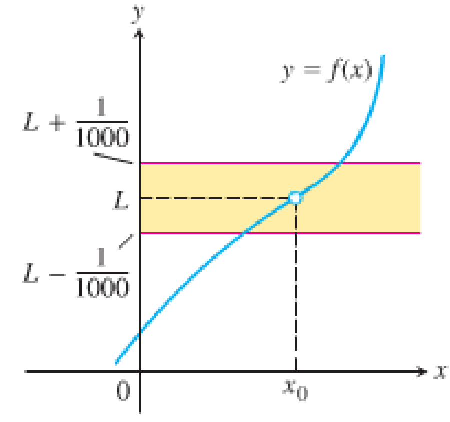
New challenge:

Make $\left|f(x) - L\right| < \epsilon = \frac{1}{100}$



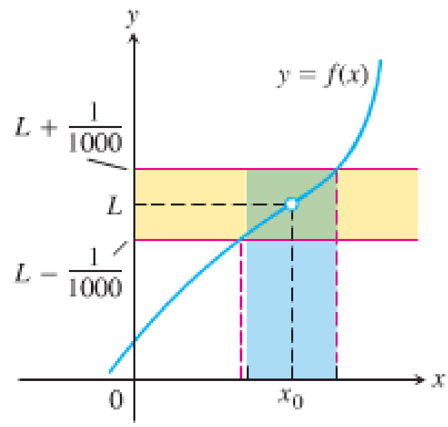
Response:

$\left|x - x_0\right| < \delta_{1/100}$



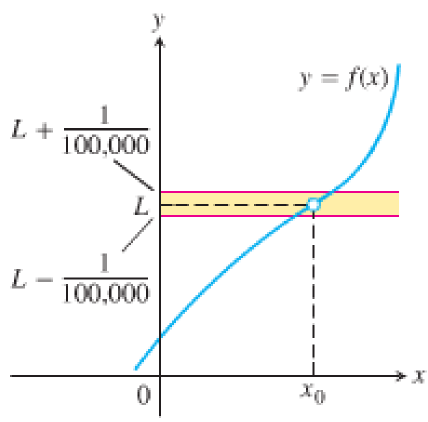
New challenge:

$\epsilon = \frac{1}{1000}$



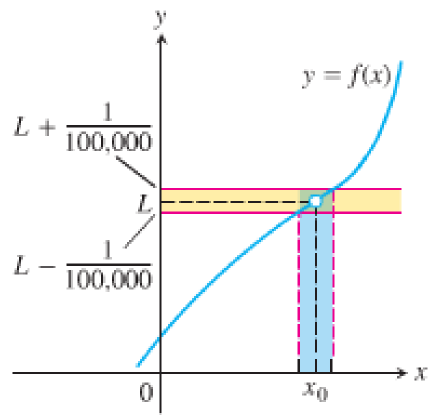
Response:

$|x - x_0| < \delta_{1/1000}$



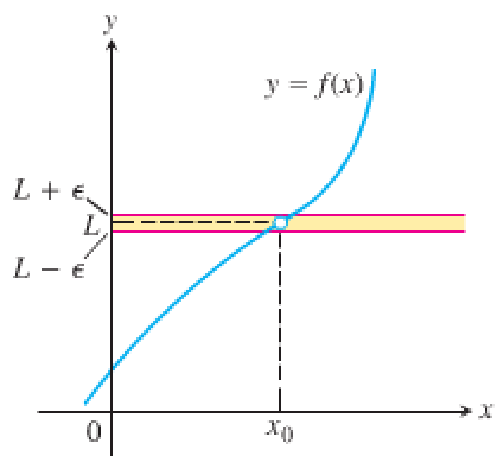
New challenge:

$\epsilon = \frac{1}{100{,}000}$



Response:

$|x - x_0| < \delta_{1/100{,}000}$



New challenge:

$\epsilon = \cdots$

Example

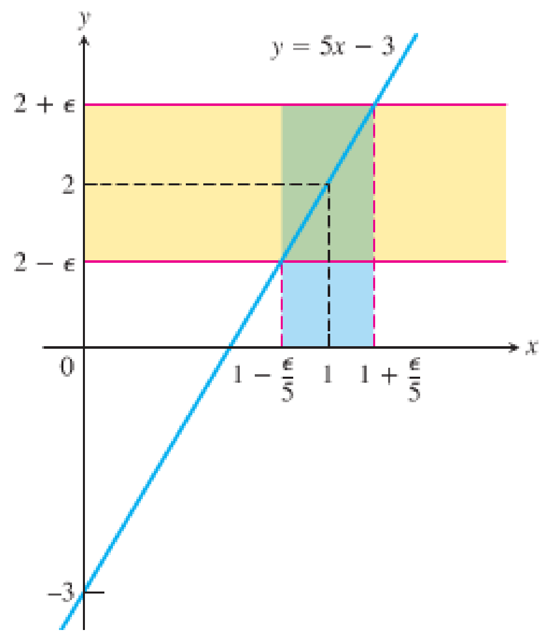
Show that $\lim_{x \to 1} (5x-3) = 2$

Solution

Let $x_0 = 1, \quad f(x) = 5x-3, \quad and \quad L = 2$.

For any given $\varepsilon > 0$, there exists a $\delta > 0$ so that $x \neq 1$ and x is within distance δ of $x_0 = 1$, that is

$$0 < |x-1| < \delta \quad \Rightarrow \quad |f(x) - 2| < \varepsilon$$



$$|(5x-3)-2| < \varepsilon$$

$$|5x-5| < \varepsilon$$

$$5|x-1| < \varepsilon$$

$$|x-1| < \frac{\varepsilon}{5}$$

Thus, we can take: $\delta = \frac{\varepsilon}{5}$

If $0 < |x-1| < \delta = \frac{\varepsilon}{5}$

$$\begin{aligned} |(5x-3)-2| &= |5x-5| \\ &= 5|x-1| \\ &= 5\frac{\varepsilon}{5} \\ &= \varepsilon \end{aligned}$$

Which proves that $\lim_{x \to 1} (5x-3) = 2$

Example

Prove the results presented graphically $\lim_{x \to x_0} x = x_0$

Solution

Let $\varepsilon > 0$ be given, we must find $\delta > 0$ such that for all x

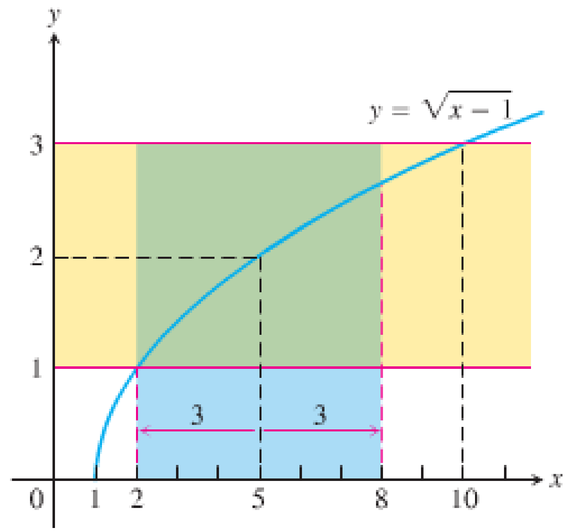
$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |x - x_0| < \varepsilon$$

This implication will hold if $\delta = \varepsilon$ or any smaller number.

Example

For the limit $\lim\limits_{x\to 5}\sqrt{x-1}=2$, find a δ > 0 that works for ε = 1. That is, find a δ > 0 such that for all *x*:

$$0<|x-5|<\delta \quad \Rightarrow \quad \left|\sqrt{x-1}-2\right|<1$$



Solution

$$\left|\sqrt{x-1}-2\right|<1$$

$$-1<\sqrt{x-1}-2<1$$

$$-1{\color{red}+2}<\sqrt{x-1}-2{\color{red}+2}<1{\color{red}+2}$$

$$1<\sqrt{x-1}<3 \qquad \textit{Square all sides}$$

$$1<x-1<9$$

$$1{\color{red}+1}<x-1{\color{red}+1}<9{\color{red}+1}$$

$$2<x<10$$

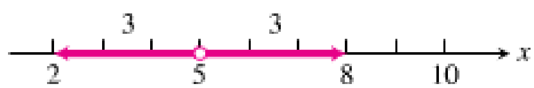
The inequality holds for all *x* in the open interval $(2,\ 10)$.

So it holds for all $x\neq 5$ in the interval as well.

Finding δ value.

$$5-\delta<x<5+\delta$$

Centered at $x_0=5$ inside the interval $(2,\ 10)$



$$\begin{cases} 5-\delta=2 \\ 5+\delta<10 \end{cases} \rightarrow \delta=3 \ (\textit{to be centered})$$

$$0<|x-5|<3 \quad \Rightarrow \quad \left|\sqrt{x-1}-2\right|<1$$

How to Find *Algebraically* a δ for a Given *f*, *L*, x_0, and $\varepsilon > 0$

The process of finding a $\delta > 0$ such that for all *x*:

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

Can be accomplished in two steps

1. Solve the inequality $|f(x) - L| < \varepsilon$ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.
2. Find a value of $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \varepsilon$ will hold for all $x \neq x_0$ in this δ–interval.

Example

Prove that $\lim\limits_{x \to 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

Solution

We need to show that given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all *x*:

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |f(x) - 4| < \varepsilon$$

1. *Solve the inequality* $|f(x) - 4| < \varepsilon$ *to find an open interval containing* $x_0 = 2$ *on which the inequality holds for all* $x \neq x_0$.

For $x \neq x_0 = 2$, $f(x) = x^2$, and the inequality to solve is $|x^2 - 4| < \varepsilon$:

$|x^2 - 4| < \varepsilon$

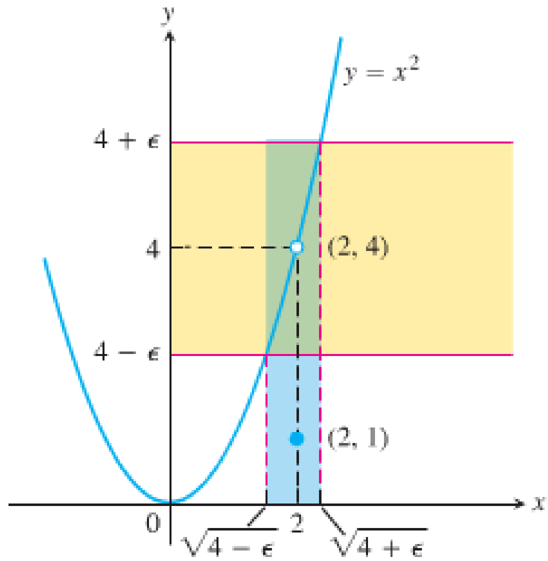
$-\varepsilon < x^2 - 4 < \varepsilon$ ***Add 4 to all sides***

$4 - \varepsilon < x^2 < 4 + \varepsilon$ ***Square root***

$\sqrt{4 - \varepsilon} < |x| < \sqrt{4 + \varepsilon}$ ***Assume $\varepsilon < 4$***

$\sqrt{4 - \varepsilon} < x < \sqrt{4 + \varepsilon}$

The inequality $|f(x) - 4| < \varepsilon$ holds for all $x \neq 2$ in the open interval $\left(\sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon}\right)$



2. *Find a value of $\delta > 0$ that places the open interval $(2-\delta,\ 2+\delta)$ inside the interval $\left(\sqrt{4-\varepsilon},\ \sqrt{4+\varepsilon}\right)$.*

Take δ to be the distance from $x_0 = 2$ to the nearer endpoint of $\left(\sqrt{4-\varepsilon},\ \sqrt{4+\varepsilon}\right)$.

$\Rightarrow \delta = \min\left(2-\sqrt{4-\varepsilon},\ \sqrt{4+\varepsilon}-2\right)$.

$$0 < |x-2| < \delta$$

$$-\left(2-\sqrt{4-\varepsilon}\right) < x-2 < \sqrt{4+\varepsilon}-2$$

$$-2+\sqrt{4-\varepsilon} < x-2 < \sqrt{4+\varepsilon}-2$$

$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$$

$$\therefore\ \ 0 < |x-2| < \delta \quad \Rightarrow \quad |f(x)-4| < \varepsilon$$

Example

Given that $\lim\limits_{x\to c} f(x) = L$ *and* $\lim\limits_{x\to c} g(x) = M$, prove that $\lim\limits_{x\to c}\left(f(x)+g(x)\right) = L+M$

Solution

We need to show that given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x:

$$0 < |x-c| < \delta \quad \Rightarrow \quad \left|f(x)+g(x)-(L+M)\right| < \varepsilon$$

$$\left|f(x)+g(x)-(L+M)\right| = \left|f(x)+g(x)-L-M\right|$$

$$= \left| \left(f(x) - L \right) + \left(g(x) - M \right) \right| \qquad \textit{Triangle Inequality } |a+b| \le |a| + |b|$$

$$\le \left| \left(f(x) - L \right) \right| + \left| \left(g(x) - M \right) \right|$$

Since $\lim_{x \to c} f(x) = L$, there exists a number $\delta_1 > 0$ such that for all *x*:

$$0 < |x - c| < \delta_1 \quad \Rightarrow \quad |f(x) - L| < \frac{\varepsilon}{2}$$

Similarly, since $\lim_{x \to c} g(x) = M$, there exists a number $\delta_2 > 0$ such that for all *x*:

$$0 < |x - c| < \delta_2 \quad \Rightarrow \quad |g(x) - M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$, the smaller of δ_1 and δ_2. If $0 < |x - c| < \delta$ then $0 < |x - c| < \delta_1$, so $|f(x) - L| < \frac{\varepsilon}{2}$ and $|x - c| < \delta_2$, so $|g(x) - M| < \frac{\varepsilon}{2}$. Therefore

$$|f(x) + g(x) - (L + M)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This show that $\lim_{x \to c} \left(f(x) + g(x) \right) = L + M$

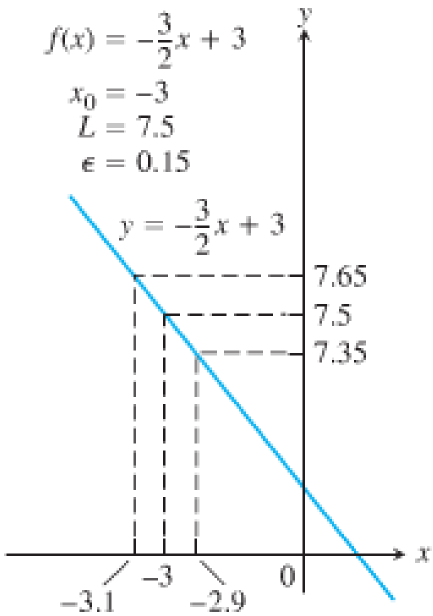
Exercises *Section* 1.6 – Precise Definition of Limits

Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \;\Rightarrow\; a < x < b$ for

1. $a = 1, \quad b = 7, \quad x_0 = 5$

2. $a = -\frac{7}{2}, \quad b = -\frac{1}{2}, \quad x_0 = -\frac{3}{2}$

3. Use the graph to find a $\delta > 0$ such that for all x $0 < |x - x_0| < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon$



(4 – 11) Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

4. $f(x) = x + 1, \quad L = 5, \quad x_0 = 4, \quad \varepsilon = 0.01$

5. $f(x) = 2x - 1, \quad L = 3, \quad x_0 = 2, \quad \varepsilon = 0.1$

6. $f(x) = x + 2, \quad L = 3, \quad x_0 = 1, \quad \varepsilon = 0.001$

7. $f(x) = 3x + 2, \quad L = 2, \quad x_0 = 0, \quad \varepsilon = 0.1$

8. $f(x) = \sqrt{x + 1}, \quad L = 1, \quad x_0 = 0, \quad \varepsilon = 0.1$

9. $f(x) = \sqrt{x - 7}, \quad L = 4, \quad x_0 = 23, \quad \varepsilon = 1$

10. $f(x) = x^2, \quad L = 3, \quad x_0 = \sqrt{3}, \quad \varepsilon = 0.1$

11. $f(x) = \frac{120}{x}, \quad L = 5, \quad x_0 = 24, \quad \varepsilon = 1$

(12 – 17) Give a formal proof that

12. $\lim\limits_{x\to 4}(9-x)=5$

13. $\lim\limits_{x\to 1}\dfrac{1}{x}=1$

14. $\lim\limits_{x\to 5}\dfrac{x^2-25}{x-5}=10$

15. $\lim\limits_{x\to 0} f(x)=0$ *if* $f(x)=\begin{cases}2x, & x<0\\ \dfrac{x}{2}, & x\ge 0\end{cases}$

16. $\lim\limits_{x\to 1}(5x-2)=3$

17. $\lim\limits_{x\to 2}\dfrac{1}{(x-2)^4}=\infty$

18. Prove that $\lim\limits_{x\to 0} x\sin\left(\dfrac{1}{x}\right)=0$