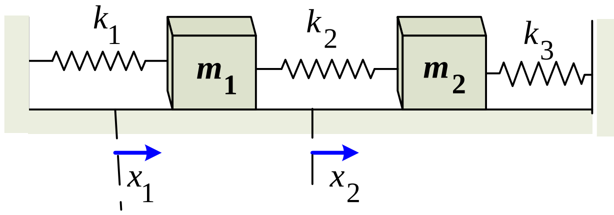
Exercise

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

$$m_1 = m_2 = 1; \quad k_1 = 0,\ k_2 = 2,\ k_3 = 0 \quad (\textit{no walls})$$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 \end{cases} \Rightarrow \begin{cases} x_1'' = -2x_1 + 2x_2 \\ x_2'' = 2x_1 - 2x_2 \end{cases}$$

$$x'' = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\vec{x} \quad \to \quad A = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -2-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} \\ &= (-2-\lambda)^2 - 4 \\ &= \lambda^2 + 4\lambda = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = 0, \quad \lambda_2 = -4$

The natural frequencies: $\omega_1 = 0 \quad \textit{and} \quad \omega_2 = \sqrt{-(-4)} = 2$

For $\lambda_1 = 0 \quad \Rightarrow \quad (A - 0I)V_1 = 0$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = b \ \to\ V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \vec{x}_1(t) = (a_1 + b_1 t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -4 \quad \Rightarrow \quad (A + 4I)V_2 = 0$

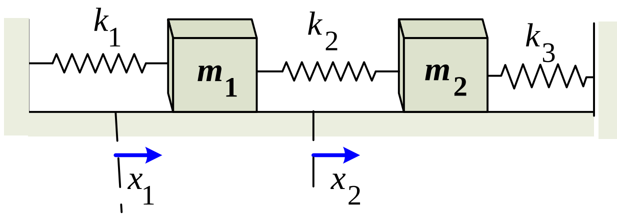
$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = -b \quad \to\ V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \Rightarrow \vec{x}_2(t) = (a_2 \cos 2t + b_2 \sin 2t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1 + b_1 t + a_2 \cos 2t + b_2 \sin 2t \\ \vec{x}_2(t) = a_1 + b_1 t - a_2 \cos 2t - b_2 \sin 2t \end{cases}$$

In the degenerate natural mode with frequency $\omega_1 = 0$ the 2 masses move by translation without oscillating. At frequency $\omega_2 = 2$ they oscillate in opposite directions with equal amplitudes.

Exercise

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

$$m_1 = m_2 = 1; \quad k_1 = 1,\ k_2 = 2,\ k_3 = 1$$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 \end{cases} \Rightarrow \begin{cases} x_1'' = -3x_1 + 2x_2 \\ x_2'' = 2x_1 - 3x_2 \end{cases}$$

$$x'' = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}\vec{x} \quad \to \quad A = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -3-\lambda & 2 \\ 2 & -3-\lambda \end{vmatrix}$$

$$= (-3-\lambda)^2 - 4$$

$$= \lambda^2 + 4\lambda + 5 = 0$$

The eigenvalues are: $\lambda_1 = -1, \quad \lambda_2 = -5$

The natural frequencies: $\omega_1 = 1 \quad and \quad \omega_2 = \sqrt{5}$

For $\lambda_1 = -1 \quad \Rightarrow \quad (A + I)V_1 = 0$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = b \quad \to V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1(t) = (a_1 \cos t + b_1 \sin t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -5 \quad \Rightarrow \quad (A + 5I)V_2 = 0$

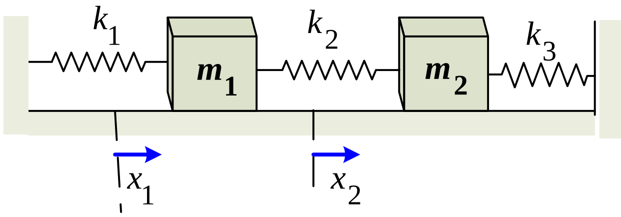
$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = -b \quad \to V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \vec{x}_2(t) = (a_2 \cos t\sqrt{5} + b_2 \sin t\sqrt{5})\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1 \cos t + b_1 \sin t + a_2 \cos t\sqrt{5} + b_2 \sin t\sqrt{5} \\ \vec{x}_2(t) = a_1 \cos t + b_1 \sin t - a_2 \cos t\sqrt{5} - b_2 \sin t\sqrt{5} \end{cases}$$

In the degenerate natural mode with frequency $\omega_1 = 1$ the 2 masses move in the same direction with equal amplitudes of oscillation. At frequency $\omega_2 = \sqrt{5}$ they oscillate in opposite directions with equal amplitudes.

Exercise

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

$$m_1 = m_2 = 1; \quad k_1 = 2,\ k_2 = 1,\ k_3 = 2$$

Solution

$$\begin{cases} m_1 x_1'' = -\left(k_1 + k_2\right)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - \left(k_2 + k_3\right)x_2 \end{cases} \Rightarrow \begin{cases} x_1'' = -3x_1 + x_2 \\ x_2'' = x_1 - 3x_2 \end{cases}$$

$$x'' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\vec{x} \quad \rightarrow \quad A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$$

$$\begin{aligned} \left|A - \lambda I\right| &= \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} \\ &= \left(-3-\lambda\right)^2 - 1 \\ &= \lambda^2 + 4\lambda + 8 = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = -2, \quad \lambda_2 = -4$

The natural frequencies: $\omega_1 = \sqrt{2} \quad$ *and* $\quad \omega_2 = 2$

For $\lambda_1 = -2 \quad \Rightarrow \quad \left(A + 2I\right)V_1 = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = b \quad \rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1(t) = \left(a_1 \cos t\sqrt{2} + b_1 \sin t\sqrt{2}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -4 \quad \Rightarrow \quad \left(A + 4I\right)V_2 = 0$

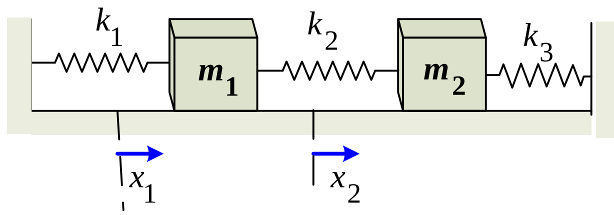
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = -b \quad \rightarrow V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \vec{x}_2(t) = \left(a_2 \cos 2t + b_2 \sin 2t\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1 \cos t\sqrt{2} + b_1 \sin t\sqrt{2} + a_2 \cos 2t + b_2 \sin 2t \\ \vec{x}_2(t) = a_1 \cos t\sqrt{2} + b_1 \sin t\sqrt{2} - a_2 \cos 2t - b_2 \sin 2t \end{cases}$$

In the degenerate natural mode with frequency $\omega_1 = \sqrt{2}$ the 2 masses move in the same direction with equal amplitudes of oscillation. At frequency $\omega_2 = 2$ they oscillate in opposite directions with equal amplitudes.

Exercise

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

$$m_1 = 1,\ m_2 = 2;\quad k_1 = 2,\ k_2 = k_3 = 4$$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1+k_2)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - (k_2+k_3)x_2 \end{cases} \Rightarrow \begin{cases} x_1'' = -6x_1 + 4x_2 \\ 2x_2'' = 4x_1 - 8x_2 \end{cases} \to \begin{cases} x_1'' = -6x_1 + 4x_2 \\ x_2'' = 2x_1 - 4x_2 \end{cases}$$

$$x'' = \begin{pmatrix} -6 & 4 \\ 2 & -4 \end{pmatrix}\vec{x} \quad \to \quad A = \begin{pmatrix} -6 & 4 \\ 2 & -4 \end{pmatrix}$$

$$\begin{aligned} |A-\lambda I| &= \begin{vmatrix} -6-\lambda & 4 \\ 2 & -4-\lambda \end{vmatrix} \\ &= (-6-\lambda)(-4-\lambda) - 8 \\ &= \lambda^2 + 10\lambda + 16 = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = -2,\quad \lambda_2 = -8$

The natural frequencies: $\omega_1 = \sqrt{2}$ *and* $\omega_2 = 2\sqrt{2}$

For $\lambda_1 = -2 \ \Rightarrow\ (A+2I)V_1 = 0$

$$\begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow\ a = b \quad \to V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1(t) = \left(a_1 \cos t\sqrt{2} + b_1 \sin t\sqrt{2}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -8 \ \Rightarrow\ (A+8I)V_2 = 0$

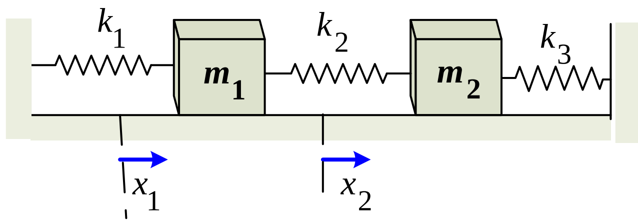
$$\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow\ a = -2b \to V_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow \vec{x}_2(t) = \left(a_2 \cos t\sqrt{8} + b_2 \sin t\sqrt{8}\right)\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1 \cos t\sqrt{2} + b_1 \sin t\sqrt{2} + 2a_2 \cos t\sqrt{8} + 2b_2 \sin t\sqrt{8} \\ \vec{x}_2(t) = a_1 \cos t\sqrt{2} + b_1 \sin t\sqrt{2} - a_2 \cos t\sqrt{8} - b_2 \sin t\sqrt{8} \end{cases}$$

In the degenerate natural mode with frequency $\omega_1 = \sqrt{2}$ the 2 masses move in the same direction with equal amplitudes of oscillation. At frequency $\omega_2 = \sqrt{8}$ they oscillate in opposite directions with amplitude of oscillation of m_1 twice that of m_2.

Exercise

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



The mass-and-spring system is set in motion from rest $x_1'(0) = x_2'(0) = 0$ in its equilibrium position $x_1(0) = x_2(0) = 0$.

Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

For the given external forces $F_1(t)$ and $F_2(t)$ acting on the masses m_1 and m_2, respectively.

Find the resulting motion of the system and describe it as a superposition of oscillations at three different frequencies.

$$m_1 = m_2 = 1; \quad k_1 = 1,\ k_2 = 4,\ k_3 = 1 \quad F_1(t) = 96\cos 5t, \quad F_2(t) = 0$$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 + 96\cos 5t \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 \end{cases} \Rightarrow \begin{cases} x_1'' = -5x_1 + 4x_2 + 96\cos 5t \\ x_2'' = 4x_1 - 5x_2 \end{cases}$$

$$A = \begin{pmatrix} -5 & 4 \\ 4 & -5 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -5-\lambda & 4 \\ 4 & -5-\lambda \end{vmatrix}$$

$$= (-5-\lambda)^2 - 16$$

$$= \lambda^2 + 10\lambda + 9 = 0$$

The eigenvalues are: $\lambda_1 = -1, \quad \lambda_2 = -9$

The natural frequencies: $\omega_1 = 1 \quad \omega_2 = 3 \quad \omega_3 = 5$

For $\lambda_1 = -1 \quad \Rightarrow \quad (A + I)V_1 = 0$

$$\begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = b \quad \rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \vec{x}_1(t) = (a_1 \cos t + b_1 \sin t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -9 \quad \Rightarrow \quad (A + 9I)V_2 = 0$

$$\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = -b \quad \rightarrow V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \Rightarrow \vec{x}_2(t) = (a_2 \cos 3t + b_2 \sin 3t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1 \cos t + b_1 \sin t + a_2 \cos 3t + b_2 \sin 3t + c_1 \cos 5t \\ \vec{x}_2(t) = a_1 \cos t + b_1 \sin t - a_2 \cos 3t - b_2 \sin 3t + c_2 \cos 5t \end{cases}$$

$$\begin{cases} \vec{x}_1''(t) = -a_1 \cos t - b_1 \sin t - 9a_2 \cos 3t - 9b_2 \sin 3t - 25c_1 \cos 5t \\ \vec{x}_2''(t) = -a_1 \cos t - b_1 \sin t + 9a_2 \cos 3t + 9b_2 \sin 3t - 25c_2 \cos 5t \end{cases}$$

$$x_1'' = -5x_1 + 4x_2 + 96\cos 5t$$

$$-a_1 \cos t - b_1 \sin t - 9a_2 \cos 3t - 9b_2 \sin 3t - 25c_1 \cos 5t =$$

$$-5a_1 \cos t - 5b_1 \sin t - 5a_2 \cos 3t - 5b_2 \sin 3t - 5c_1 \cos 5t$$

$$+ 4a_1 \cos t + 4b_1 \sin t - 4a_2 \cos 3t - 4b_2 \sin 3t + 4c_2 \cos 5t + 96\cos 5t$$

$$-25c_1 \cos 5t = -5c_1 \cos 5t + 4c_2 \cos 5t + 96\cos 5t$$

$$-20c_1 - 4c_2 = 96 \rightarrow \boxed{5c_1 + c_2 = -24}$$

$$x_2'' = 4x_1 - 5x_2$$

$$-25c_2 \cos 5t = 4c_1 \cos 5t - 5c_2 \cos 5t \qquad \rightarrow \boxed{c_1 = -5c_2}$$

$$5(-5c_2) + c_2 = -24 \quad \Rightarrow \quad \boxed{c_2 = 1,\ c_1 = -5}$$

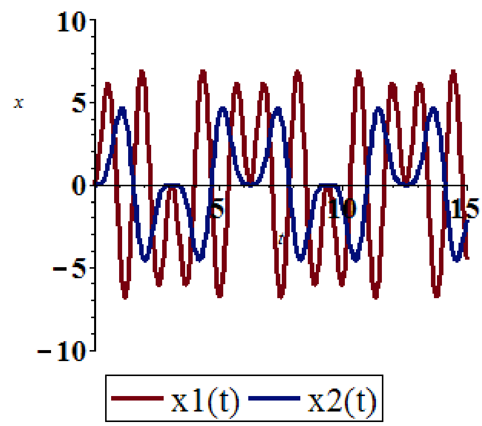
$$\begin{cases} \vec{x}_1(t) = a_1 \cos t + b_1 \sin t + a_2 \cos 3t + b_2 \sin 3t - 5\cos 5t \\ \vec{x}_2(t) = a_1 \cos t + b_1 \sin t - a_2 \cos 3t - b_2 \sin 3t + \cos 5t \end{cases}$$

Given initial values: $x_1'(0) = x_2'(0) = 0$ and $x_1(0) = x_2(0) = 0$.

$$\begin{cases} \vec{x}_1(0) = a_1 + a_2 - 5 = 0 \\ \vec{x}_2(0) = a_1 - a_2 + 1 = 0 \end{cases} \rightarrow \boxed{a_1 = 2,\ a_2 = 3}$$

$$\begin{cases} \vec{x}_1'(0) = b_1 + 3b_2 = 0 \\ \vec{x}_2'(0) = b_1 - 3b_2 = 0 \end{cases} \rightarrow \boxed{b_1 = b_2 = 0}$$

$$\begin{cases} \vec{x}_1(t) = 2\cos t + 3\cos 3t - 5\cos 5t \\ \vec{x}_2(t) = 2\cos t - 3\cos 3t + \cos 5t \end{cases}$$



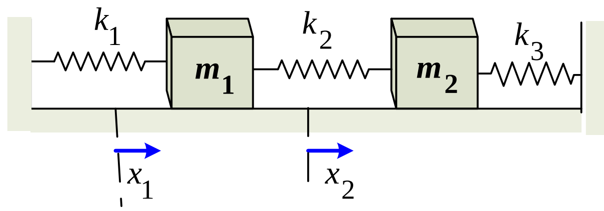
At frequency $\omega_1 = 1$ the 2 masses move in the same direction with equal amplitudes of oscillation.

At frequency $\omega_2 = 3$ the 2 masses move in the opposite direction with equal amplitudes of oscillation.

At frequency $\omega_3 = 5$ they oscillate in opposite directions with amplitude of oscillation of m_1 **5** times that of m_2.

Exercise

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



The mass-and-spring system is set in motion from rest $x_1'(0) = x_2'(0) = 0$ in its equilibrium position $x_1(0) = x_2(0) = 0$.

Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

For the given external forces $F_1(t)$ and $F_2(t)$ acting on the masses m_1 and m_2, respectively.

Find the resulting motion of the system and describe it as a superposition of oscillations at three different frequencies.

$$m_1 = 1,\ m_2 = 2; \quad k_1 = 1,\ k_2 = k_3 = 2; \quad F_1(t) = 0,\ F_2(t) = 120\cos 3t$$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 + 120\cos 3t \end{cases} \Rightarrow \begin{cases} x_1'' = -3x_1 + 2x_2 \\ 2x_2'' = 2x_1 - 4x_2 + 120\cos 3t \end{cases}$$

$$\rightarrow \begin{cases} x_1'' = -3x_1 + 2x_2 \\ x_2'' = x_1 - 2x_2 + 60\cos 3t \end{cases} \qquad A = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} \\ &= (-3-\lambda)(-2-\lambda) - 2 \\ &= \lambda^2 + 5\lambda + 4 = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = -1, \quad \lambda_2 = -4$

The natural frequencies: $\omega_1 = 1 \quad \omega_2 = 2 \quad \omega_3 = 3$

For $\lambda_1 = -1 \ \Rightarrow \ (A + I)V_1 = 0$

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \Rightarrow \ a = b \ \rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ \Rightarrow \vec{x}_1(t) = (a_1 \cos t + b_1 \sin t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -4 \ \Rightarrow \ (A + 4I)V_2 = 0$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \Rightarrow \ a = -2b \ \rightarrow V_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \ \Rightarrow \vec{x}_2(t) = (a_2 \cos 2t + b_2 \sin 2t)\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1\cos t + b_1\sin t + 2a_2\cos 2t + 2b_2\sin 2t + c_1\cos 3t \\ \vec{x}_2(t) = a_1\cos t + b_1\sin t - a_2\cos 2t - b_2\sin 2t + c_2\cos 3t \end{cases}$$

$$\begin{cases} \vec{x}''_{1p} = -9c_1\cos 3t \\ \vec{x}''_{2p} = -9c_2\cos 3t \end{cases}$$

$$x_1'' = -3x_1 + 2x_2$$

$$-9c_1\cos 3t = -3c_1\cos 3t + 2c_2\cos 3t \quad \Rightarrow \quad -6c_1 = 2c_2 \quad \to \quad \boxed{-3c_1 = c_2}$$

$$x_2'' = x_1 - 2x_2 + 60\cos 3t$$

$$-9c_2\cos 3t = c_1\cos 3t - 2c_2\cos 3t + 60\cos 3t \quad \Rightarrow \quad \boxed{c_1 + 7c_2 = -60}$$

$$c_1 + 7(-3c_1) = -60 \quad \Rightarrow \quad \boxed{c_1 = 3, \ c_2 = -9}$$

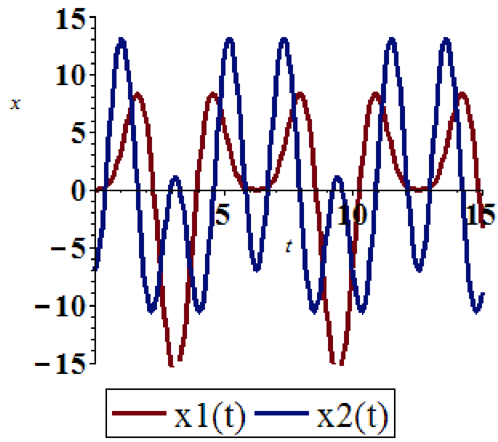
$$\begin{cases} \vec{x}_1(t) = a_1\cos t + b_1\sin t + 2a_2\cos 2t + 2b_2\sin 2t + 3\cos 3t \\ \vec{x}_2(t) = a_1\cos t + b_1\sin t - a_2\cos 2t - b_2\sin 2t - 9\cos 3t \end{cases}$$

Given initial values: $x_1'(0) = x_2'(0) = 0$ and $x_1(0) = x_2(0) = 0$.

$$\begin{cases} \vec{x}_1(0) = a_1 + 2a_2 + 3 = 0 \\ \vec{x}_2(0) = a_1 - a_2 - 9 = 0 \end{cases} \to \quad \boxed{a_1 = 5, \ a_2 = -4}$$

$$\begin{cases} \vec{x}'_1(0) = b_1 + 4b_2 = 0 \\ \vec{x}'_2(0) = b_1 - 2b_2 = 0 \end{cases} \to \quad \boxed{b_1 = b_2 = 0}$$

$$\begin{cases} \vec{x}_1(t) = 5\cos t - 8\cos 2t + 3\cos 3t \\ \vec{x}_2(t) = 5\cos t + 4\cos 2t - 9\cos 3t \end{cases}$$



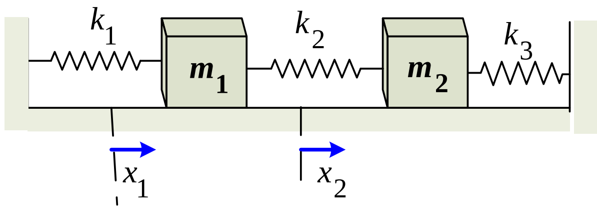
At frequency $\omega_1 = 1$ the 2 masses oscillate in the same direction with equal amplitudes.

At frequency $\omega_2 = 2$ the 2 masses oscillate in opposite directions with equal amplitudes of m_1 ***twice*** that of m_2.

At frequency $\omega_3 = 3$ they oscillate in opposite directions with amplitude of oscillation of m_1 **3** times that of m_2.

Exercise

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



The mass-and-spring system is set in motion from rest $x_1'(0)=x_2'(0)=0$ in its equilibrium position $x_1(0)=x_2(0)=0$.

Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

For the given external forces $F_1(t)$ and $F_2(t)$ acting on the masses m_1 and m_2, respectively.

Find the resulting motion of the system and describe it as a superposition of oscillations at three different frequencies.

$$m_1=m_2=1; \quad k_1=4,\ k_2=6,\ k_3=4; \quad F_1(t)=30\cos t,\ F_2(t)=60\cos t$$

Solution

$$\begin{cases} m_1x_1''=-(k_1+k_2)x_1+k_2x_2+30\cos t \\ m_2x_2''=k_2x_1-(k_2+k_3)x_2+60\cos t \end{cases} \Rightarrow \begin{cases} x_1''=-10x_1+6x_2+30\cos t \\ x_2''=6x_1-10x_2+60\cos t \end{cases}$$

$$A=\begin{pmatrix} -10 & 6 \\ 6 & -10 \end{pmatrix}$$

$$\begin{aligned} |A-\lambda I| &= \begin{vmatrix} -10-\lambda & 6 \\ 6 & -10-\lambda \end{vmatrix} \\ &= (-10-\lambda)^2-36 \\ &= \lambda^2+20\lambda+64=0 \end{aligned}$$

The eigenvalues are: $\lambda_1=-4, \quad \lambda_2=-16$

The natural frequencies: $\omega_1=2 \quad \omega_2=4 \quad \omega_3=1$

For $\lambda_1=-4 \quad \Rightarrow \quad (A+4I)V_1=0$

$$\begin{pmatrix} -6 & 6 \\ 6 & -6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a=b \quad \to V_1=\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \vec{x}_1(t)=\left(a_1\cos 2t+b_1\sin 2t\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2=-16 \quad \Rightarrow \quad (A+16I)V_2=0$

$$\begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2a=-3b$$

$$\to V_2=\begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad \Rightarrow \vec{x}_2(t)=\left(a_2\cos 4t+b_2\sin 4t\right)\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1 \cos 2t + b_1 \sin 2t + 3a_2 \cos 3t + 3b_2 \sin 3t + c_1 \cos t \\ \vec{x}_2(t) = a_1 \cos 2t + b_1 \sin 2t - 2a_2 \cos 3t - 2b_2 \sin 3t + c_2 \cos t \end{cases}$$

$$\begin{cases} \vec{x}''_{1p} = -c_1 \cos t \\ \vec{x}''_{2p} = -c_2 \cos t \end{cases}$$

$$x''_1 = -10x_1 + 6x_2 + 30\cos t$$

$$-c_1 \cos t = -10c_1 \cos t + 6c_2 \cos t + 30\cos t \quad \Rightarrow 9c_1 - 6c_2 = 30 \quad \Rightarrow \boxed{3c_1 - 2c_2 = 10}$$

$$x''_2 = 6x_1 - 10x_2 + 60\cos t$$

$$-c_2 \cos t = 6c_1 \cos t - 10c_2 \cos t + 60\cos t \quad \Rightarrow -6c_1 + 9c_2 = 60 \quad \Rightarrow \boxed{-2c_1 + 3c_2 = 20}$$

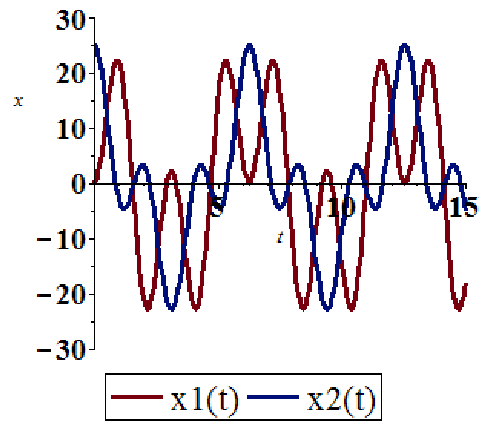
$$5c_1 = 70 \quad \Rightarrow \boxed{c_1 = 14, \ \ c_2 = 16}$$

$$\begin{cases} \vec{x}_1(t) = a_1 \cos 2t + b_1 \sin 2t + 3a_2 \cos 3t + 3b_2 \sin 3t + 14\cos t \\ \vec{x}_2(t) = a_1 \cos 2t + b_1 \sin 2t - 2a_2 \cos 3t - 2b_2 \sin 3t + 16\cos t \end{cases}$$

Given initial values: $x'_1(0) = x'_2(0) = 0$ and $x_1(0) = x_2(0) = 0$.

$$\begin{cases} \vec{x}_1(0) = a_1 + 3a_2 + 14 = 0 \\ \vec{x}_2(0) = a_1 - 2a_2 + 16 = 0 \end{cases} \rightarrow \boxed{a_1 = 1, \ a_2 = -5}$$

$$\begin{cases} \vec{x}'_1(0) = 2b_1 + 9b_2 = 0 \\ \vec{x}'_2(0) = 2b_1 - 6b_2 = 0 \end{cases} \rightarrow \boxed{b_1 = b_2 = 0}$$



$$\begin{cases} \vec{x}_1(t) = \cos 2t - 15\cos 3t + 14\cos t \\ \vec{x}_2(t) = \cos 2t + 10\cos 3t + 16\cos t \end{cases}$$

At frequency $\omega_1 = 2$ the 2 masses oscillate in the same direction of m_1 ***twice*** that of m_2.

At frequency $\omega_2 = 3$ the 2 masses oscillate in opposite directions with equal amplitudes of m_1 **3** *times* that of m_2.

At frequency $\omega_3 = 1$ they oscillate in the same direction with equal amplitudes of oscillation.

Exercise

Consider a mass-and-spring system containing two masses $m_1 = m_2 = 1$ whose displacement functions $x(t)$ and $y(t)$ satisfy the differential equations

$$x'' = -40x + 8y$$
$$y'' = 12x - 60y$$

a) Describe the two fundamental modes of free oscillation of the system.

b) Assume that the two masses start in motion with the initial conditions

$x(0) = 19,\ x'(0) = 12 \quad and \quad y(0) = 3,\ y'(0) = 6$

And are acted on by the same force, $F_1(t) = F_2(t) = -195\cos 7t$. Describe the resulting motion as a superposition of oscillations at three different frequencies.

Solution

a) $A = \begin{pmatrix} -40 & 8 \\ 12 & -60 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -40-\lambda & 8 \\ 12 & -60-\lambda \end{vmatrix}$$
$$= (-40-\lambda)(-60-\lambda) - 96$$
$$= \lambda^2 + 100\lambda + 144 = 0$$

The eigenvalues are: $\lambda_1 = -36, \quad \lambda_2 = -64$

The natural frequencies: $\omega_1 = 6 \quad \omega_2 = 8$

For $\lambda_1 = -36 \quad \Rightarrow \quad (A + 36I)V_1 = 0$

$$\begin{pmatrix} -4 & 8 \\ 12 & -24 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = 2b \quad \to V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \Rightarrow \vec{x}_1(t) = (a_1\cos 6t + b_1\sin 6t)\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -64 \quad \Rightarrow \quad (A + 64I)V_2 = 0$

$$\begin{pmatrix} 24 & 8 \\ 12 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 3a = -b \quad \to V_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad \Rightarrow \vec{x}_2(t) = (a_2\cos 8t + b_2\sin 8t)\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\begin{cases} \vec{x}(t) = 2a_1\cos 6t + 2b_1\sin 6t + a_2\cos 8t + b_2\sin 8t \\ \vec{y}(t) = a_1\cos 6t + b_1\sin 6t - 3a_2\cos 8t - 3b_2\sin 8t \end{cases}$$

In mode 1: At frequency $\omega_1 = 6,$ the 2 masses oscillate in the same direction of m_1 ***twice*** of m_2.

In mode 2: At frequency $\omega_2 = 8,$ the 2 masses oscillate in opposite directions of oscillation of m_1 ***3 times*** that of m_2.

b) ***Given*** $x(0) = 19,\ x'(0) = 12 \quad y(0) = 3,\ y'(0) = 6$ and $F_1(t) = F_2(t) = -195\cos 7t$

$$x'' = -40x + 8y - 195\cos 7t$$
$$y'' = 12x - 60y - 195\cos 7t$$

$$\begin{cases}\vec{x}(t)=2a_1\cos 6t+2b_1\sin 6t+a_2\cos 8t+b_2\sin 8t+c_1\cos 7t\\ \vec{y}(t)=a_1\cos 6t+b_1\sin 6t-3a_2\cos 8t-3b_2\sin 8t+c_2\cos 7t\end{cases}$$

$$\begin{cases}x''_p=-49c_1\cos 7t\\ y''_p=-49c_2\cos 7t\end{cases}$$

$$x''=-40x+8y-195\cos 7t$$

$$-49c_1\cos 7t=-40c_1\cos 7t+8c_2\cos 7t-195\cos 7t \quad\Rightarrow\quad \boxed{9c_1+8c_2=195}$$

$$y''=12x-60y-195\cos 7t$$

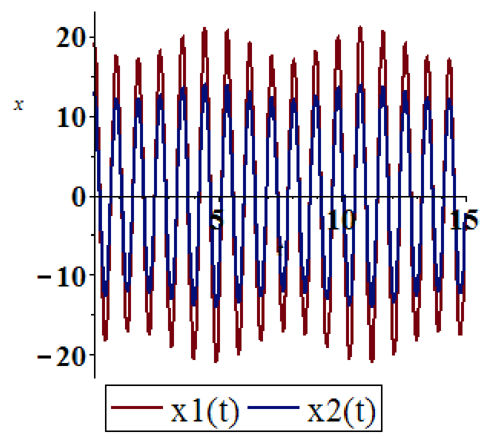
$$-49c_2\cos 7t=12c_1\cos 7t-60c_2\cos 7t-195\cos 7t \quad\Rightarrow\quad \boxed{12c_1-11c_2=195}$$

$$\Rightarrow\quad \boxed{c_1=19,\ \ c_2=3}$$

$$\begin{cases}\vec{x}(t)=2a_1\cos 6t+2b_1\sin 6t+a_2\cos 8t+b_2\sin 8t+19\cos 7t\\ \vec{y}(t)=a_1\cos 6t+b_1\sin 6t-3a_2\cos 8t-3b_2\sin 8t+3\cos 7t\end{cases}$$

$$\begin{cases}x(0)=2a_1+a_2+19=19\\ y(0)=a_1-3a_2+3=3\end{cases} \rightarrow \begin{cases}2a_1+a_2=0\\ a_1-3a_2=0\end{cases} \Rightarrow\quad \boxed{a_1=0\ ,a_2=0}$$

$$\Rightarrow\begin{cases}x(t)=2b_1\sin 6t+b_2\sin 8t+19\cos 7t\\ y(t)=b_1\sin 6t-3b_2\sin 8t+3\cos 7t\end{cases}$$



$$\begin{cases}x'(0)=12b_1+8b_2=12\\ y'(0)=6b_1-24b_2=6\end{cases} \Rightarrow\quad \boxed{b_1=1\ ,b_2=0}$$

$$\Rightarrow\begin{cases}x(t)=2\sin 6t+19\cos 7t\\ y(t)=\ \sin 6t+3\cos 7t\end{cases}$$

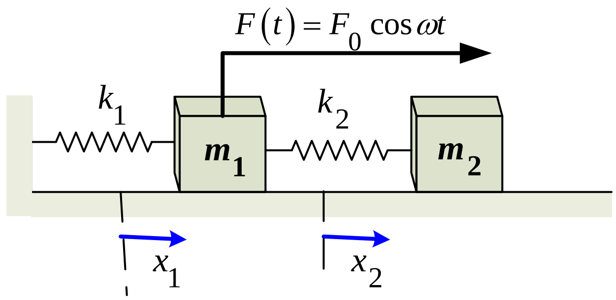
At frequency $\omega_1=6$, the 2 masses oscillate in the same direction with amplitude of motion of m_1 ***twice*** that of m_2.

At frequency $\omega_3=7$, the 2 masses oscillate in the same direction with amplitude of motion of m_1 being $\frac{19}{3}$ *times* that of m_2.

At frequency $\omega_2=8$, the expected oscillation is missing.

Exercise

Consider a mass-and-spring system shown below. Assume that $m_1 = 1;$ $k_1 = 50,$ $k_2 = 10;$ $F_0 = 5$ in mks units, and that $\omega = 10$. Then find m_2 so that in the resulting steady periodic oscillations, the mass m_1 will remain at rest (!).



Thus the effect of the second mass-and-spring pair will be to neutralize the effect of the force on the first mass. This is an example of a dynamic damper. It has an electrical analogy that some cable companies use to prevent your reception of certain cable channels.

Solution

$$F(t) = F_0 \cos \omega t = 5\cos 10t$$

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 + 5\cos 10t \\ m_2 x_2'' = -k_2(x_2 - x_1) \end{cases} \Rightarrow \begin{cases} x_1'' = -60x_1 + 10x_2 + 5\cos 10t \\ m_2 x_2'' = 10x_1 - 10x_2 \end{cases}$$

$$\begin{cases} x_{1p} = c_1 \cos 10t \\ x_{2p} = c_2 \cos 10t \end{cases} \rightarrow \begin{cases} x_{1p}'' = -100c_1 \cos 10t \\ x_{2p}'' = -100c_2 \cos 10t \end{cases}$$

$$x_1'' = -60x_1 + 10x_2 + 5\cos 10t$$

$$-100c_1 \cos 10t = -60c_1 \cos 10t + 10c_2 \cos 10t + 5\cos 10t \quad \rightarrow \quad \boxed{-40c_1 - 10c_2 = 5}$$

$$m_2 x_2'' = 10x_1 - 10x_2$$

$$-100m_2 c_2 \cos 10t = 10c_1 \cos 10t - 10c_2 \cos 10t \quad \rightarrow \quad \boxed{c_1 - (1 - 10m_2)c_2 = 0}$$

$$-40(1 - 10m_2)c_2 - 10c_2 = 5 \qquad\qquad c_1 = (1 - 10m_2)c_2$$

$$390m_2 c_2 = 45 \quad \Rightarrow \quad \boxed{c_2 = \frac{3}{26m_2}} \quad \rightarrow c_1 = (1 - 10m_2)\frac{3}{26m_2} = \boxed{\frac{3}{26m_2} - \frac{15}{13}}$$

$$-40\left(\frac{3}{26m_2} - \frac{15}{13}\right) - 10\frac{3}{26m_2} = 5$$

$$-4.615 + 46.154m_2 - 1.154 = 5m_2$$

$$41.154m_2 = 5.769$$

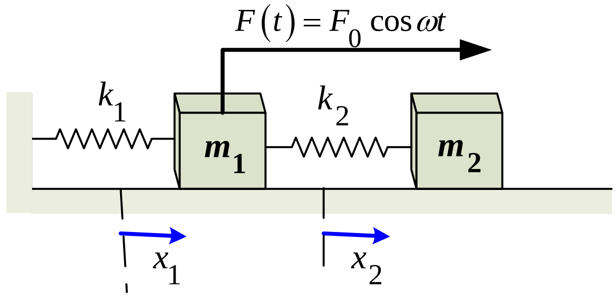
$$\boxed{m_2 \approx 0.1 \ \ slug} \qquad \Rightarrow \quad c_1 = \frac{3}{26m_2} - \frac{15}{13} \boxed{\approx 0} \quad c_2 = \frac{3}{26m_2} \boxed{\approx 1.15}$$

Since $c_1 = 0$, so the mass m_1 remains at rest.

Exercise

Consider a mass-and-spring system shown below. Assume that

$$m_1 = 2,\ m_2 = \frac{1}{2};\quad k_1 = 75,\ k_2 = 25;\quad F_0 = 100\quad \textit{and}\ \ \omega = 10\ \text{(in mks units)}.$$



Find the solution of the system $M\vec{x}'' = K\vec{x} + F$ that satisfies the initial conditions $\vec{x}(0) = \vec{x}'(0) = \mathbf{0}$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 + 100\cos 10t \\ m_2 x_2'' = -k_2(x_2 - x_1) \end{cases} \Rightarrow \begin{cases} 2x_1'' = -100x_1 + 25x_2 + 100\cos 10t \\ \frac{1}{2}x_2'' = 25x_1 - 25x_2 \end{cases}$$

$$\begin{cases} x_1'' = -50x_1 + \frac{25}{2}x_2 + 50\cos 10t \\ x_2'' = 50x_1 - 50x_2 \end{cases} \quad \to A = \begin{bmatrix} -50 & \frac{25}{2} \\ 50 & -50 \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -50-\lambda & \frac{25}{2} \\ 50 & -50-\lambda \end{vmatrix} \\ &= (-50-\lambda)^2 - 625 \\ &= \lambda^2 + 100\lambda - 1875 = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = -25,\quad \lambda_2 = -75$

The natural frequencies: $\omega_1 = 5\quad \omega_2 = 5\sqrt{3}$

For $\lambda_1 = -25 \Rightarrow (A + 25I)V_1 = 0$

$$\begin{pmatrix} -25 & \frac{25}{2} \\ 50 & -25 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2a = b \quad \to V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \vec{x}_1(t) = (a_1\cos 5t + b_1\sin 5t)\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = -75 \Rightarrow (A + 75I)V_2 = 0$

$$\begin{pmatrix} 25 & \frac{25}{2} \\ 50 & 25 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2a = -b \quad \to V_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow \vec{x}_2(t) = (a_2\cos 5t\sqrt{3} + b_2\sin 5t\sqrt{3})\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\begin{cases} x_1(t) = a_1\cos 5t + b_1\sin 5t + a_2\cos 5t\sqrt{3} + b_2\sin 5t\sqrt{3} \\ x_2(t) = 2a_1\cos 5t + 2b_1\sin 5t - 2a_2\cos 5t\sqrt{3} - 2b_2\sin 5t\sqrt{3} \end{cases}$$

$$\begin{cases} x_{1p} = c_1 \cos 10t \\ x_{2p} = c_2 \cos 10t \end{cases} \to \begin{cases} x''_{1p} = -100c_1 \cos 10t \\ x''_{2p} = -100c_2 \cos 10t \end{cases}$$

$$x''_1 = -50x_1 + \frac{25}{2}x_2 + 50\cos 10t$$

$$-100c_1 \cos 10t = -50c_1 \cos 10t + \frac{25}{2}c_2 \cos 10t + 50\cos 10t$$

$$\Rightarrow\ 50c_1 + \frac{25}{2}c_2 = -50 \quad \Rightarrow\ \underline{4c_1 + c_2 = -4}$$

$$x''_2 = 50x_1 - 50x_2$$

$$-100c_2 = 50c_1 - 50c_2 \quad \Rightarrow\ \underline{c_1 + c_2 = 0}$$

$$\underline{c_1 = -\frac{4}{3},\ c_2 = \frac{4}{3}}$$

$$\begin{cases} x_1(t) = a_1 \cos 5t + b_1 \sin 5t + a_2 \cos 5t\sqrt{3} + b_2 \sin 5t\sqrt{3} - \frac{4}{3}\cos 10t \\ x_2(t) = 2a_1 \cos 5t + 2b_1 \sin 5t - 2a_2 \cos 5t\sqrt{3} - 2b_2 \sin 5t\sqrt{3} + \frac{4}{3}\cos 10t \end{cases}$$

$$\begin{cases} x_1(0) = a_1 + a_2 - \frac{4}{3} = 0 \\ x_2(0) = 2a_1 - 2a_2 + \frac{4}{3} = 0 \end{cases} \quad \begin{cases} a_1 + a_2 = \frac{4}{3} \\ 2a_1 - 2a_2 = -\frac{4}{3} \end{cases} \quad \underline{a_1 = \frac{1}{3},\ a_2 = 1}$$

$$\begin{cases} x'_1(t) = -5a_1 \sin 5t + 5b_1 \cos 5t - 5a_2\sqrt{3}\sin 5t\sqrt{3} + 5b_2\sqrt{3}\cos 5t\sqrt{3} + \frac{40}{3}\sin 10t \\ x'_2(t) = -10a_1 \sin 5t + 10b_1 \cos 5t + 10a_2\sqrt{3}\sin 5t\sqrt{3} - 10b_2\sqrt{3}\cos 5t\sqrt{3} - \frac{40}{3}\sin 10t \end{cases}$$

$$\begin{cases} x'_1(0) = 5b_1 + 5\sqrt{3}b_2 = 0 \\ x'_2(0) = 10b_1 - 10\sqrt{3}b_2 = 0 \end{cases} \Rightarrow\ \underline{b_1 = b_2 = 0}$$

$$\begin{cases} x_1(t) = \frac{1}{3}\cos 5t + \cos 5t\sqrt{3} - \frac{4}{3}\cos 10t \\ x_2(t) = \frac{2}{3}\cos 5t - 2\cos 5t\sqrt{3} + \frac{4}{3}\cos 10t \end{cases}$$

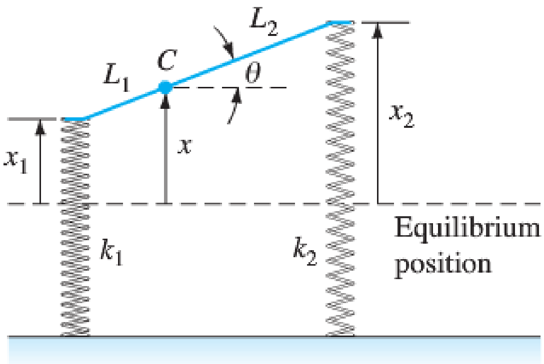
At frequency $\omega_1 = 5,$ the 2 masses oscillate in the same direction with amplitude of motion of m_1 ***half*** that of m_2.

At frequency $\omega_2 = 5\sqrt{3},$ the 2 masses oscillate in opposite directions with amplitude of motion of m_1 being ***half*** that of m_2.

At frequency $\omega_3 = 10$ the 2 masses oscillate in opposite directions with equal amplitudes.

Exercise

A car with two axles and with separate front and rear suspension systems.



We assume that the car body acts as would a solid bar of mass m and length $L = L_1 + L_2$. It has moment of inertia I about its center of mass *C*, which is at distance L_1 from the front of the car. The car has front and back suspension springs with Hooke's constants k_1 and k_2, respectively. When the car is in motion, let $x(t)$ denote the vertical displacement of the center of mass of the car from equilibrium; let $\theta(t)$ denote its angular displacement (in radians) from the horizontal. Then Newton's laws of motion for linear and angular acceleration can be used to derive the equations.

$$mx'' = -(k_1 + k_2)x + (k_1L_1 - k_2L_2)\theta$$

$$I\theta'' = (k_1L_1 - k_2L_2)x - \left(k_1L_1^2 + k_2L_2^2\right)\theta$$

Suppose that $m = 75$ *slugs* (the car weighs 2400 *lb*), $L_1 = 7\, ft$, $L_2 = 3\, ft$ (it's a rear engine car), $k_1 = k_2 = 2000 \; lb/ft$, and $I = 1000 \; ft.lb.s^2$.

a) Find the two natural frequencies ω_1 and ω_2 of the car.

b) Now suppose that the car is driven at a speed of $v \; ft/\sec$ along a washboard surface shaped like a sine curve with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1$ *or* $\omega = \omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

Solution

a) $\begin{cases} 75x'' = -4000x + 8000\theta \\ 1000\theta'' = 8000x - (98000 + 18000)\theta \end{cases}$

$$\begin{cases} x'' = -\frac{160}{3}x + \frac{320}{3}\theta \\ \theta'' = 8x - 116\theta \end{cases} \quad \rightarrow A = \begin{bmatrix} -\frac{160}{3} & \frac{320}{3} \\ 8 & -116 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\frac{160}{3} - \lambda & \frac{320}{3} \\ 8 & -116 - \lambda \end{vmatrix}$$

$$= \left(-\frac{160}{3} - \lambda\right)(-116 - \lambda) - \frac{2560}{3}$$

$$=\lambda^2+\frac{508}{3}\lambda-\frac{48640}{3}=0$$

The eigenvalues are: $\lambda_1 \approx -41.8285, \quad \lambda_2 \approx -127.5049$

The natural frequencies: $\omega_1 \approx \underline{6.4675\ \ rad/\sec|} \quad \omega_2 \approx \underline{11.2918\ \ rad/\sec|}$

$$\omega_1=\frac{6.4675}{2\pi}\approx \underline{1.0293\ \ Hz|} \quad \omega_2=\frac{11.2918}{2\pi}\approx \underline{1.7971\ \ Hz|}$$

b) $\omega=\frac{\pi}{20}v \quad \Rightarrow \quad v=\frac{20}{\pi}\omega$

$$v_1=\frac{20}{\pi}\omega_1=\frac{(20)(6.4675)}{\pi} \underline{\approx 41\ \ ft/\sec|} \qquad (41)(0.681818) \underline{\approx 28\ \ mph|}$$

$$v_2=\frac{20}{\pi}\omega_2=\frac{(20)(11.2918)}{\pi} \underline{\approx 72\ \ ft/\sec|} \qquad (72)(0.681818) \underline{\approx 49\ \ mph|}$$

Exercise

The system is taken as a model for an undamped car with the given parameters in *fps* units.

$m=100; \quad I=800; \quad L_1=L_2=5; \quad k_1=k_2=2000$

a) Find the two natural frequencies ω_1 and ω_2 of the car (in hertz).

b) Assume that his car is driven along a sinusoidal washboard surface with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega=\frac{2\pi}{40}v=\frac{\pi}{20}v$. Resonance occurs when $\omega=\omega_1$ *or* $\omega=\omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

Solution

a)

$$\begin{cases} mx''=-(k_1+k_2)x+(k_1L_1-k_2L_2)\theta \\ I\theta''=(k_1L_1-k_2L_2)x-\left(k_1L_1^2+k_2L_2^2\right)\theta \end{cases} \rightarrow \begin{cases} 100x''=-4000x \\ 800\theta''=-100{,}000\theta \end{cases}$$

$$\begin{cases} x''=-40x \\ \theta''=-125\theta \end{cases} \rightarrow A=\begin{bmatrix} -40 & 0 \\ 0 & -125 \end{bmatrix}$$

$$|A-\lambda I|=\begin{vmatrix} -40-\lambda & 0 \\ 0 & -125-\lambda \end{vmatrix}=(-40-\lambda)(-125-\lambda)=0$$

The eigenvalues are: $\lambda_1=-40, \quad \lambda_2=-125$

The natural frequencies: $\omega_1=\sqrt{40}\approx \underline{6.325\ \ rad/\sec|} \quad \omega_2=\sqrt{125}\ \underline{\approx 11.180\ \ rad/\sec|}$

$$\omega_1=\frac{6.325}{2\pi}\approx \underline{1.0067\ \ Hz|} \quad \omega_2=\frac{11.180}{2\pi}\approx \underline{1.779\ \ Hz|}$$

b)

$$v_1=\frac{20}{\pi}\omega_1=\frac{(20)(6.325)}{\pi} \underline{\approx 40.26\ \ ft/\sec|} \qquad (40.26)(0.681818) \underline{\approx 27\ \ mph|}$$

$$v_2=\frac{20}{\pi}\omega_2=\frac{(20)(11.180)}{\pi} \underline{\approx 71.18\ \ ft/\sec|} \qquad (71.18)(0.681818) \underline{\approx 49\ \ mph|}$$

Exercise

The system is taken as a model for an undamped car with the given parameters in *fps* units.

$m = 100;\quad I = 1000;\quad L_1 = 6,\ L_2 = 4;\quad k_1 = k_2 = 2000$

a) Find the two natural frequencies ω_1 and ω_2 of the car (in hertz).

b) Assume that his car is driven along a sinusoidal washboard surface with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1$ *or* $\omega = \omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

Solution

a)
$$\begin{cases} mx'' = -\left(k_1 + k_2\right)x + \left(k_1L_1 - k_2L_2\right)\theta \\ I\theta'' = \left(k_1L_1 - k_2L_2\right)x - \left(k_1L_1^2 + k_2L_2^2\right)\theta \end{cases} \rightarrow \begin{cases} 100x'' = -4000x + 4000\theta \\ 1000\theta'' = 4000x - 104,000\theta \end{cases}$$

$$\begin{cases} x'' = -40x + 40\theta \\ \theta'' = 4x - 104\theta \end{cases} \rightarrow A = \begin{bmatrix} -40 & 40 \\ 4 & -104 \end{bmatrix}$$

$$\begin{aligned} \left|A - \lambda I\right| &= \begin{vmatrix} -40-\lambda & 40 \\ 4 & -104-\lambda \end{vmatrix} \\ &= \left(-40-\lambda\right)\left(-104-\lambda\right) - 160 \\ &= \lambda^2 + 144\lambda + 4000 = 0 \end{aligned} \qquad \lambda_{1,2} = -72 \pm 4\sqrt{74}$$

The eigenvalues are: $\lambda_1 \approx -37.591,\quad \lambda_2 \approx -106.409$

The natural frequencies: $\omega_1 = \sqrt{37.591} \approx \underline{6.131\ \ rad/\sec}\quad \omega_2 = \sqrt{106.409}\ \underline{\approx 10.315\ \ rad/\sec}$

$$\omega_1 = \frac{6.131}{2\pi} \underline{\approx .9758\ \ Hz} \quad \omega_2 = \frac{10.315}{2\pi} \underline{\approx 1.6417\ \ Hz}$$

b)
$$v_1 = \frac{20}{\pi}\omega_1 = \frac{(20)(6.131)}{\pi}\ \underline{\approx 39.03\ \ ft/\sec}\quad (39.03)(0.681818)\ \underline{\approx 27\ \ mph}$$

$$v_2 = \frac{20}{\pi}\omega_2 = \frac{(20)(10.315)}{\pi}\ \underline{\approx 65.67\ \ ft/\sec}\quad (65.67)(0.681818)\ \underline{\approx 45\ \ mph}$$

Exercise

The system is taken as a model for an undamped car with the given parameters in *fps* units.

$m = 100;\quad I = 800;\quad L_1 = L_2 = 5;\quad k_1 = 1000,\ k_2 = 2000$

a) Find the two natural frequencies ω_1 and ω_2 of the car (in hertz).

b) Assume that his car is driven along a sinusoidal washboard surface with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1$ *or* $\omega = \omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

Solution

a) $$\begin{cases} mx'' = -\left(k_1 + k_2\right)x + \left(k_1L_1 - k_2L_2\right)\theta \\ I\theta'' = \left(k_1L_1 - k_2L_2\right)x - \left(k_1L_1^2 + k_2L_2^2\right)\theta \end{cases} \quad \to \begin{cases} 100x'' = -3000x - 5000\theta \\ 800\theta'' = -5000x - 75{,}000\theta \end{cases}$$

$$\begin{cases} x'' = -30x - 50\theta \\ \theta'' = -\frac{25}{4}x - \frac{375}{4}\theta \end{cases} \quad \to A = \begin{bmatrix} -30 & -50 \\ -\frac{25}{4} & -\frac{375}{4} \end{bmatrix}$$

$$\begin{aligned} \left|A - \lambda I\right| &= \begin{vmatrix} -30-\lambda & -50 \\ -\frac{25}{4} & -\frac{375}{4} - \lambda \end{vmatrix} \\ &= (-30-\lambda)\left(-\frac{375}{4} - \lambda\right) - \frac{625}{2} \\ &= \lambda^2 + \frac{495}{4}\lambda + 2500 = 0 \end{aligned} \qquad \lambda_{1,2} = \frac{-495 \pm 5\sqrt{3401}}{8}$$

The eigenvalues are: $\lambda_1 \approx -25.426, \quad \lambda_2 \approx -98.234$

The natural frequencies: $\omega_1 = \sqrt{25.426} \approx 5.0424 \ \ rad/\sec$ $\quad \omega_2 = \sqrt{98.234} \approx 9.9158 \ \ rad/\sec$

$$\omega_1 = \frac{5.0424}{2\pi} \approx .8025 \ \ Hz \qquad \omega_2 = \frac{9.9158}{2\pi} \approx 1.5781 \ \ Hz$$

b) $$v_1 = \frac{20}{\pi}\omega_1 = \frac{(20)(5.0424)}{\pi} \approx 32.10 \ \ ft/\sec \qquad (32.1)(0.681818) \approx 22 \ \ mph$$

$$v_2 = \frac{20}{\pi}\omega_2 = \frac{(20)(9.9158)}{\pi} \approx 63.13 \ \ ft/\sec \qquad (63.13)(0.681818) \approx 43 \ \ mph$$