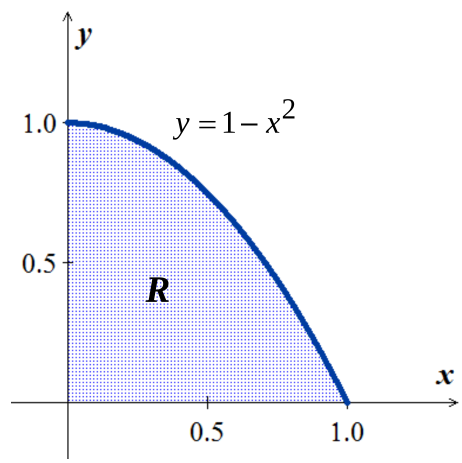
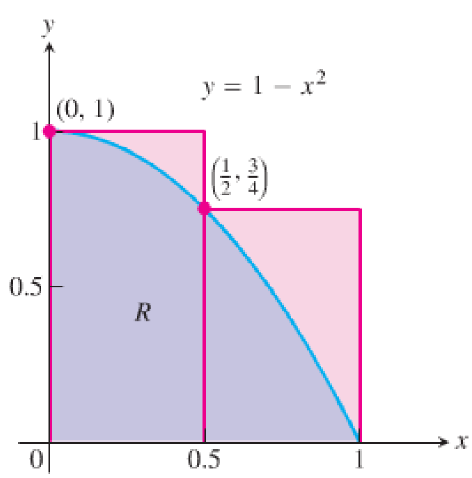
Section 4.2 – Area under Curves

The ***definite integral*** is the key tool in calculus for defining and calculating quantities important to mathematics and science, such as areas, volumes, lengths, and more…

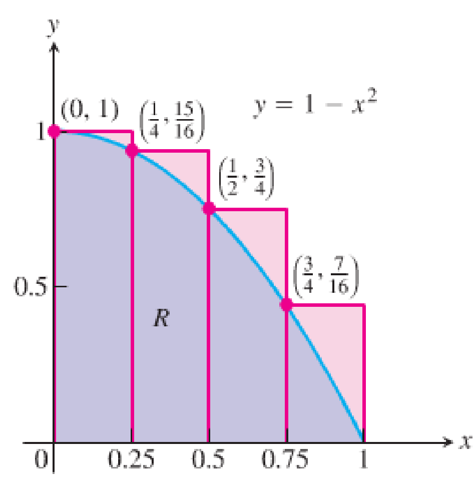
Area

To find the area of the shaded region R that lies above the x-axis, below the graph of $y = 1 - x^2$ and between the vertical lines $x = 0$ and $x = 1$.

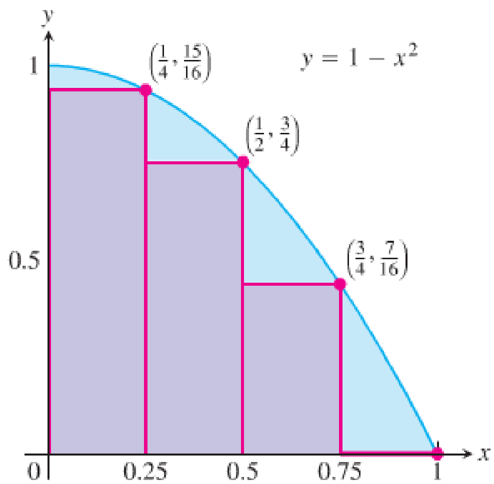




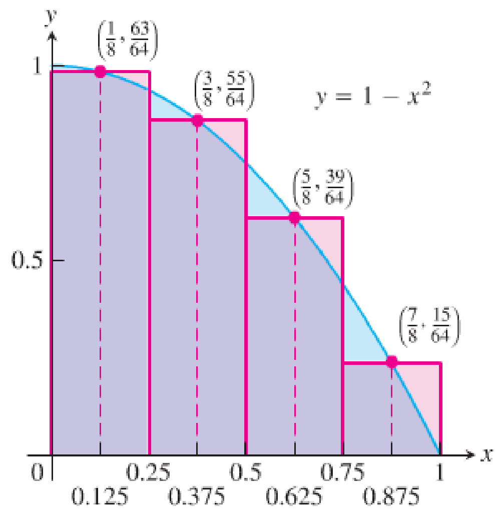
$$Area \approx 1 \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = 0.875$$



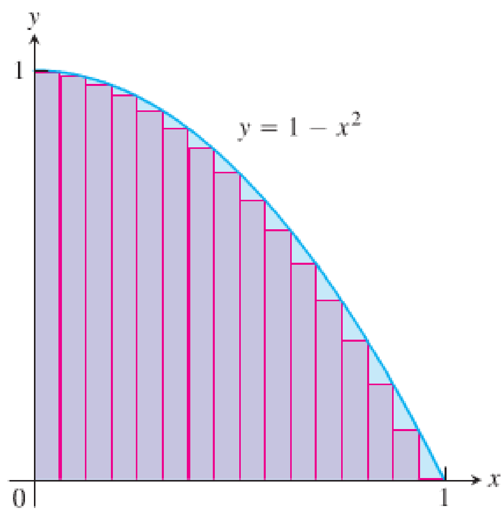
$$Area \approx 1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} = 0.78125$$

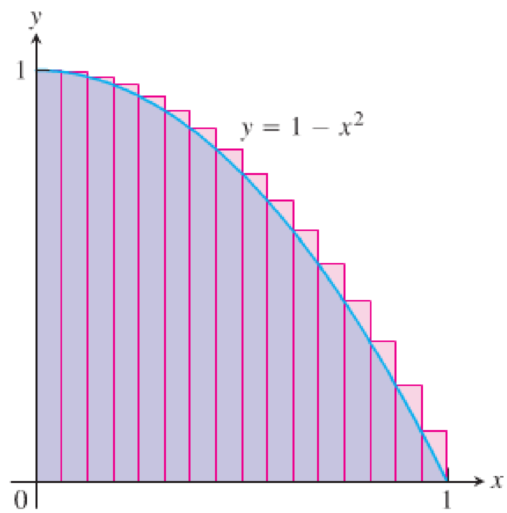


(a)



$$Area \approx \frac{15}{16}\cdot\frac{1}{4}+\frac{3}{4}\cdot\frac{1}{4}+\frac{7}{16}\cdot\frac{1}{4}+0\cdot\frac{1}{4}=0.53125 \qquad Area \approx \frac{63}{64}\cdot\frac{1}{4}+\frac{55}{64}\cdot\frac{1}{4}+\frac{39}{64}\cdot\frac{1}{4}+\frac{15}{64}\cdot\frac{1}{4}=0.671875$$





In each case of the computations, the interval $[a, b]$ over which the function f is defined was subdivided into n equal subintervals (also called ***length***) $\Delta x = \frac{b-a}{n}$, and f was evaluated at a point in each subinterval. The finite sums can be given by the form:

$$f\left(c_1\right)\Delta x + f\left(c_2\right)\Delta x + f\left(c_3\right)\Delta x + \cdots + f\left(c_n\right)\Delta x$$

Distance Traveled

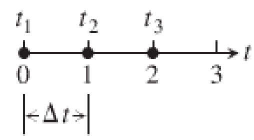
The distance formula is given by: $distance = velocity \times time$

Example

The velocity function of a projectile fired straight up into the air is $f(t) = 160 - 9.8t\ m/\sec$. Use the summation technique to estimate how far the projectile rises during the first 3 *sec*. How close do the sums come to the exact value of 435.9 *m*?

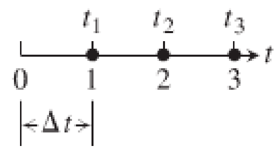
Solution

i. $\Delta t = 1 \text{ sec} \quad \to t = 0, 1, 2$



$$D \approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t$$
$$= f(0)\Delta t + f(1)\Delta t + f(2)\Delta t$$
$$= (160 - 9.8(0))(1) + (160 - 9.8(1))(1) + (160 - 9.8(2))(1)$$
$$= 450.6$$

ii. $\Delta t = 1 \text{ sec} \quad \to t = 1, 2, 3$



$$D \approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t$$
$$= (160 - 9.8(1))(1) + (160 - 9.8(2))(1) + (160 - 9.8(3))(1)$$
$$= 421.2$$

iii. $\Delta t = 0.5 \text{ sec} \quad \to t = 0,\ 0.5,\ 1,\ 1.5,\ 2,\ 2.5$

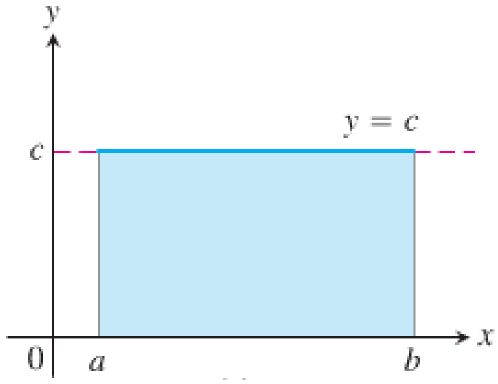
$$D \approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t + f(t_4)\Delta t + f(t_5)\Delta t + f(t_6)\Delta t$$
$$= (160 - 9.8(0))(1) + (160 - 9.8(0.5))(1) + (160 - 9.8(1))(1) + (160 - 9.8(1.5))(1)$$
$$+ (160 - 9.8(2))(1) + (160 - 9.8(2.5))(1)$$
$$\approx 443.25$$

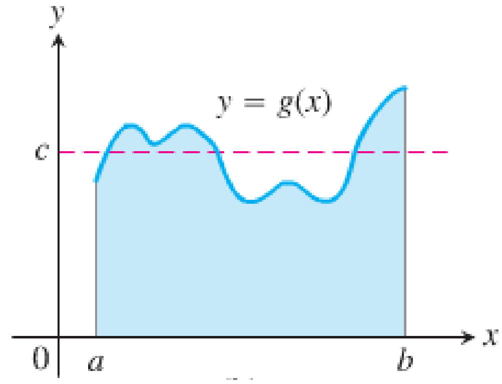
iv. $\Delta t = 0.5 \text{ sec} \quad \to t = 0.5,\ 1,\ 1.5,\ 2,\ 2.5,\ 3$

$$D \approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t + f(t_4)\Delta t + f(t_5)\Delta t + f(t_6)\Delta t$$
$$= (160 - 9.8(0.5))(1) + (160 - 9.8(1))(1) + (160 - 9.8(1.5))(1) + (160 - 9.8(2))(1)$$
$$+ (160 - 9.8(2.5))(1) + (160 - 9.8(3))(1)$$
$$\approx 428.55$$

The true value is 435.9 if you use more subintervals $\Delta t = 0.25 \text{ sec}$, the interval 436.13 & 435.67
The projectile rose about 436 m during the first 3 sec of flight.

Average Value of a Nonnegative Continuous Function



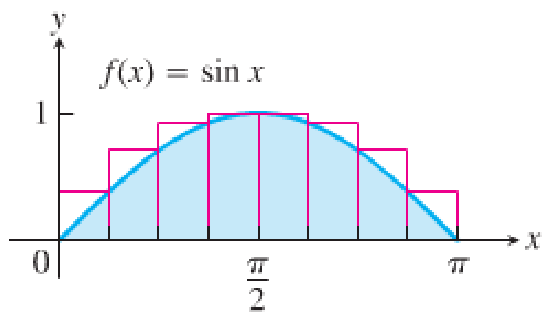


The average value of a collection of *n* numbers $x_1, x_2, \cdots, x_n$ is obtained by adding them together and dividing by *n*.

Example

Estimate the average value of the function $f(x) = \sin x$ on the interval $[0, \pi]$.

Solution



To get the upper sum approximation with 8 rectangles of equal width $\Delta x = \frac{\pi}{8}$.

$$A \approx \left(\sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \sin\frac{3\pi}{8} + \sin\frac{\pi}{2} + \sin\frac{\pi}{2} + \sin\frac{5\pi}{8} + \sin\frac{3\pi}{4} + \sin\frac{7\pi}{8}\right) \cdot \frac{\pi}{8}$$

$$\approx 2.365$$

Finite Sums and Sigma Notation

Sigma notation enables us to write a sum with many terms in the compact form

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

The Greek letter $\sum$ (capital ***sigma***, corresponding to our letter *S*)

The index k ends at $k = n$

The summation symbol $\longrightarrow$ $\displaystyle\sum_{k=1}^{n} a_k$ $\longrightarrow$ a_k is a formula for the kth term

The index k starts at $k = 1$

Example

Sigma Notation	***Written***	***Value of the Sum***
$\sum_{k=1}^{5} k$	$1+2+3+4+5$	15
$\sum_{k=1}^{4} (-1)^k \cdot k$	$(-1)^1 \cdot 1 + (-1)^2 \cdot 2 + (-1)^3 \cdot 3 + (-1)^4 \cdot 4$	$-1+2-3+4 = 2$
$\sum_{k=1}^{3} \frac{k}{k+1}$	$\frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1}$	$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \frac{23}{12}$
$\sum_{k=4}^{5} \frac{k^2}{k-1}$	$\frac{4^2}{4-1} + \frac{5^2}{5-1}$	$\frac{16}{3} + \frac{25}{4} = \frac{139}{12}$

Example

We can write:

$$1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2+10^2=\sum_{k=1}^{10} k^2$$

Example

Express the sum 1+ 3 +5 +7 + 9 in sigma notation.

Solution

Starting with $k = 0$: $1+3+5+7+9=\sum_{k=0}^{4}(2k+1)$

Starting with $k = 1$: $1+3+5+7+9=\sum_{k=1}^{5}(2k-1)$

Theorem on Sums

If $a_1, a_2, a_3, \ldots, a_n, \ldots$ and $b_1, b_2, b_3, \ldots, b_n, \ldots$ are infinite sequences, then for every positive integer ***n***,

(1) $\sum_{k=1}^{n}(a_k+b_k)=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k$

(2) $\sum_{k=1}^{n}(a_k-b_k)=\sum_{k=1}^{n}a_k-\sum_{k=1}^{n}b_k$

(3) $\sum_{k=1}^{n}ca_k=c\left(\sum_{k=1}^{n}a_k\right)$

(4) $\sum_{k=1}^{n}c=n\cdot c$

Proof

$$\sum_{k=1}^{n}\left(a_k+b_k\right)=\left(a_1+b_1\right)+\left(a_2+b_2\right)+\ldots+\left(a_n+b_n\right)$$

$$=\left(a_1+a_2+\ldots+a_n\right)+\left(b_1+b_2+\ldots+b_n\right)$$

$$=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k$$

Example

(1) $\displaystyle\sum_{k=1}^{n}(k+4)=\sum_{k=1}^{n}k+\sum_{k=1}^{n}4=\sum_{k=1}^{n}k+4\cdot n$

(2) $\displaystyle\sum_{k=1}^{n}\left(3k-k^2\right)=3\sum_{k=1}^{n}k-\sum_{k=1}^{n}k^2$

(3) $\displaystyle\sum_{k=1}^{n}\left(-a_k\right)=\sum_{k=1}^{n}(-1)\cdot\left(a_k\right)=(-1)\cdot\sum_{k=1}^{n}\left(a_k\right)=-\sum_{k=1}^{n}a_k$

(4) $\displaystyle\sum_{k=1}^{n}\frac{1}{n}=n\cdot\frac{1}{n}=1$

Example

Show that the sum of the first n integers is $\displaystyle\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$

Solution

The sum of the first 4 integers is: $\displaystyle\sum_{k=1}^{4}k=\frac{4(5)}{2}=10$

To prove the formula in general:

$$\begin{array}{ccccccccc} 1 & + & 2 & + & 3 & + & \cdots & + & n \\ n & + & (n-1) & + & (n-2) & + & \cdots & + & 1 \\ \hline n+1 & + & n+1 & + & n+1 & + & \cdots & + & n+1 \end{array} \rightarrow n(n+1)$$

Since it is twice the desired quantity, the sum of the first n integers is $\dfrac{n(n+1)}{2}$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} = (1+2+3+\cdots+n)^2$$

Limits of Finite Sums

Example

Find the limiting value of lower sum approximations to the area of the region R below the graph of $y = 1 - x^2$ and above the interval [0, 1] on the x-axis using equal-width rectangles whose width approach zero and whose number approaches infinity.

Solution

The lower sum approximation using n rectangles of equal width: $\Delta x = \frac{1-0}{n} = \frac{1}{n}$

By subdividing the interval [0, 1] into n equal width subintervals:

$$[0,\ 1] = \left[0,\ \frac{1}{n}\right], \left[\frac{1}{n},\ \frac{2}{n}\right], \left[\frac{2}{n},\ \frac{3}{n}\right],\ \ldots,\ \left[\frac{n-1}{n},\ \frac{n}{n}\right] = \left[\frac{k-1}{n},\ \frac{k}{n}\right]$$

$$f\left(\frac{k}{n}\right) = 1 - \left(\frac{k}{n}\right)^2$$

$$\left[f\left(\frac{1}{n}\right)\right]\cdot\left(\frac{1}{n}\right) + \left[f\left(\frac{2}{n}\right)\right]\cdot\left(\frac{1}{n}\right) + \cdots + \left[f\left(\frac{n}{n}\right)\right]\cdot\left(\frac{1}{n}\right)$$

We can write this in sigma notation:

$$\begin{aligned}
\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\left(\frac{1}{n}\right) &= \sum_{k=1}^{n}\left[1 - \left(\frac{k}{n}\right)^2\right]\cdot\left(\frac{1}{n}\right) \\
&= \sum_{k=1}^{n}\left(\frac{1}{n} - \frac{k^2}{n^3}\right) \\
&= \sum_{k=1}^{n}\frac{1}{n} - \sum_{k=1}^{n}\frac{k^2}{n^3} \\
&= n\cdot\frac{1}{n} - \frac{1}{n^3}\sum_{k=1}^{n} k^2 \\
&= 1 - \frac{1}{n^3}\cdot\frac{n\cdot(n+1)\cdot(2n+1)}{6}
\end{aligned}$$

$$=1-\frac{2n^3+3n^2+n}{6n^3}$$

$$=\frac{6n^3-2n^3-3n^2-n}{6n^3}$$

$$=\frac{4n^3-3n^2-n}{6n^3}$$

$$\lim_{x\to\infty}\left(\frac{4n^3-3n^2-n}{6n^3}\right)=\frac{4}{6}=\underline{\frac{2}{3}|}$$

The lower sum approximation converge to $\frac{2}{3}$

The upper sum approximation also converge to $\frac{2}{3}$

Review

Definition of Arithmetic Sequence

A sequence $a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$ is an arithmetic sequence if there is a real number $\boldsymbol{d}$ such that for every positive integer $\boldsymbol{k}$,

$$a_{k+1}=a_k+d$$

The number $d=a_{k+1}-a_k$ is called the ***common difference*** of the sequence.

The nth Term of an Arithmetic Sequence: $\boxed{a_n=a_1+(n-1)d}$

Example

Express the sum in terms of summation notation: $4+11+18+25+32$. (Answers are not unique)

Solution

Number of terms: $n = 5$

Difference in terms: $d = 11 - 4 = 7$

$a_n=a_1+(n-1)d$

$\underline{|a_n}=4+(n-1)7=4+7n-7\ \underline{=7n-3|}$

$$\sum_{n=1}^{5}(7n-3)$$

Theorem

Formulas for S_n

If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an arithmetic sequence with common difference ***d***, then the *n*th partial sum S_n (that is, the sum of the first ***n*** terms) is given by either

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right] \quad \text{or} \quad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

Definition of *Geometric* Sequence

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a geometric sequence if $a_1 \neq 0$ and if there is a real number $r \neq 0$ such that for every positive integer k.

$$a_{k+1} = a_k r$$

The number $r = \dfrac{a_{k+1}}{a_k}$ is called the ***common ratio*** of the sequence.

The formula for the nth Term of a Geometric Sequence: $\boxed{a_n = a_1 r^{n-1}}$

The common ratio for: $6, \ -12, \ 24, \ -48, \ \ldots, \ (-2)^{n-1}(6), \ \ldots$ is $= \frac{-12}{6} = -2$

Example

Express the sum in terms of summation notation (Answers are not unique.)

$$\frac{1}{4} - \frac{1}{12} + \frac{1}{36} - \frac{1}{108}$$

Solution

$$\frac{1}{4} - \frac{1}{12} + \frac{1}{36} - \frac{1}{108} = \frac{1}{4} - \frac{1}{4}\frac{1}{3^1} + \frac{1}{4}\frac{1}{3^2} - \frac{1}{4}\frac{1}{3^3}$$

$$= \sum_{n=1}^{4} (-1)^{n+1} \frac{1}{4}\left(\frac{1}{3}\right)^{n-1}$$

Theorem: Formula for S_n

The nth partial sum S_n of a geometric sequence with first term a_1 and common ratio $r \neq 1$ is

$$\boxed{S_n = a_1 \frac{1 - r^n}{1 - r}}$$

Riemann Sums

The theory of limits of finite approximations was made precise by the German mathematician ***Bernhard Riemann***.

We introduce the notion of a *Riemann sum*, which underlies the theory of the definite integral.

Let a closed interval [*a*, *b*] be partitioned by points $a < x_1 < x_2 < \cdots < x_{n-1} < b$

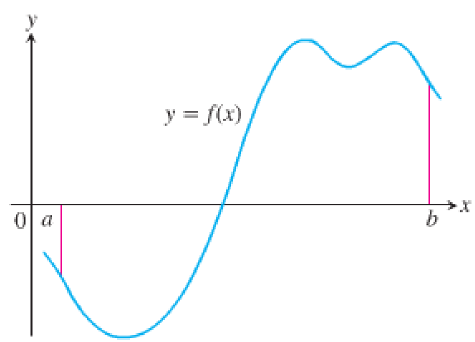
To make the notation consistent, so that

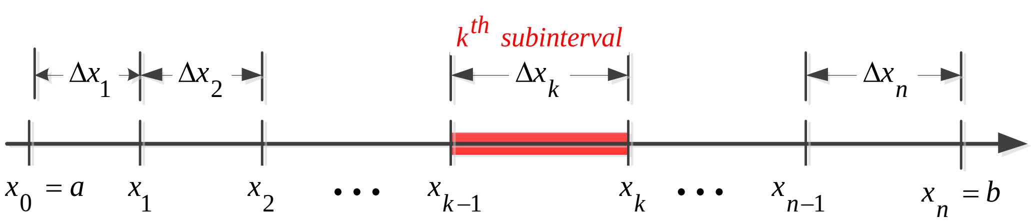
$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

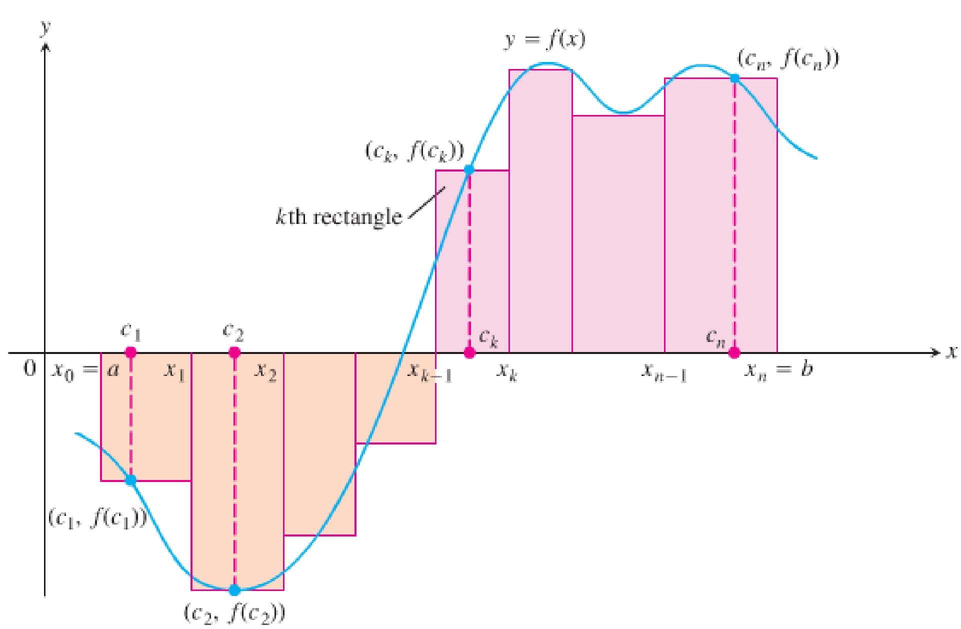
The set: $P = \{x_0, x_1, x_2, \cdots, x_{n-1}, x_n\}$ is called a partition of [*a*, *b*].

The partition *P* divides [*a*, *b*] into *n* closed subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$$







These products are:

$$S_P = \sum_{k=1}^{n} f(c_k)\Delta x_k$$

The sum S_P is called a ***Riemann sum*** *for f on the interval* [*a*, *b*], and c_k in the subintervals.

If we choose *n* subintervals all having equal width $\Delta x = \frac{b-a}{n}$ to partition [*a*, *b*], then choose the point c_k to be the right-hand endpoints of each subintervals when forming the Riemann sum. This choice leads to the Riemann sum formula

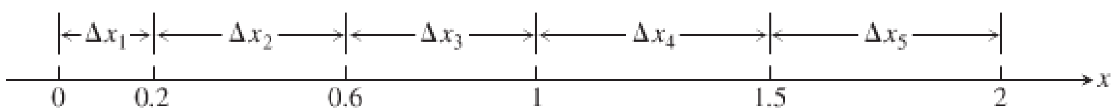
$$S_n = \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right) \cdot \left(\frac{b-a}{n}\right)$$

Example

The set $P = \{0,\ 0.2,\ 0.6,\ 1,\ 1.5,\ 2\}$ is a partition of [0, 2]

Solution

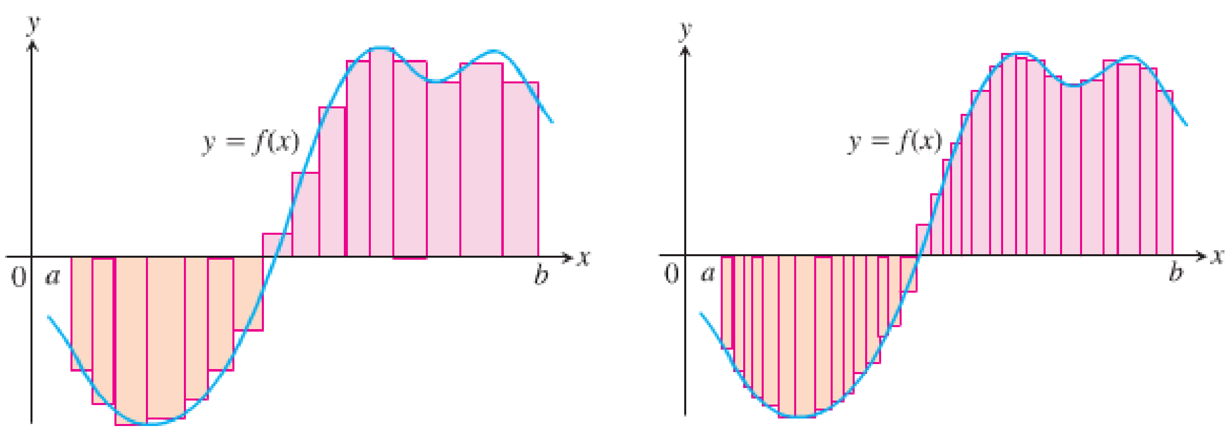
There are five subintervals of *P*: [0, 0.2], [0.2, 0.6], [0.6, 1], [1, 1.5], and [1.5, 2]



The lengths of the subintervals are:

$$\Delta x_1 = 0.2 \quad \Delta x_2 = 0.4 \quad \Delta x_3 = 0.4 \quad \Delta x_4 = 0.5 \quad \Delta x_5 = 0.5$$

The longest subinterval length is 0.5, so the norm of the partition is $\|P\| = 0.5$



Exercises Section 4.2 – Area under Curves

Use finite approximations to estimate the area under the graph of the function using

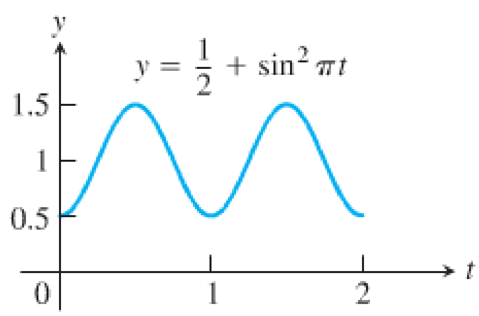
a) A lower sum with two rectangles of equal width
b) A lower sum with four rectangles of equal width
c) A upper sum with two rectangles of equal width
d) A upper sum with four rectangles of equal width

1. $f(x)=\frac{1}{x}$ *between* $x=1$ *and* $x=5$

2. $f(x)=4-x^2$ *between* $x=-2$ *and* $x=2$

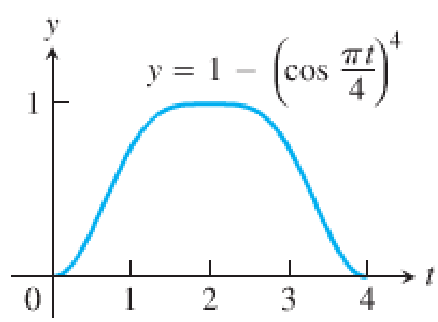
3. Use finite approximations to estimate the average value of *f* on the given interval by partitioning the interval into four subintervals of equal length and evaluating *f* at the subinterval midpoints.

$$f(t)=\frac{1}{2}+\sin^2 \pi t \quad on \quad [0,\ 2]$$



4. Use finite approximations to estimate the average value of *f* on the given interval by partitioning the interval into four subintervals of equal length and evaluating *f* at the subinterval midpoints.

$$f(t)=1-\left(\cos\frac{\pi t}{4}\right)^4 \quad on \quad [0,\ 4]$$



Write the sums without sigma notation. Then evaluate them:

5. $\sum_{k=1}^{2}\frac{6k}{k+1}$ **6.** $\sum_{k=1}^{3}\frac{k-1}{k}$ **7.** $\sum_{k=1}^{5}\sin k\pi$ **8.** $\sum_{k=1}^{4}(-1)^k\cos k\pi$

9. Write the following expression 1 + 2 + 4 + 8 + 16 + 32 in sigma notation

10. Write the following expression 1 – 2 + 4 – 8 + 16 – 32 in sigma notation

11. Write the following expression $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ in sigma notation

12. Write the following expression $-\frac{1}{5}+\frac{2}{5}-\frac{3}{5}+\frac{4}{5}-\frac{5}{5}$ in sigma notation

13. Suppose that $\sum_{k=1}^{n} a_k = -5$ *and* $\sum_{k=1}^{n} b_k = 6$. Find the value of $\sum_{k=1}^{n}\left(b_k - 2a_k\right)$

Evaluate the sums

14. $\sum_{k=1}^{10} k^3$

15. $\sum_{k=1}^{7} (-2k)$

16. $\sum_{k=1}^{5} \frac{\pi k}{15}$

17. $\sum_{k=1}^{5} k(3k+5)$

18. $\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$

19. $\sum_{k=1}^{500} 7$

20. $\sum_{k=18}^{71} k(k-1)$

21. $\sum_{k=1}^{n} \left(\frac{1}{n}+2n\right)$

22. Graph the function $f(x) = x^2 - 1$ over the given interval [0, 2]. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\sum_{k=1}^{4} f\left(c_k\right)\Delta x_k$, given c_k is the

a) Left-hand endpoint

b) Right-hand endpoint

c) Midpoint of kth subinterval.

(Make a separate sketch for each set of rectangles.)