Solution   *Section* 2.7 – Implicit Differentiation

Exercise

Find $\dfrac{dy}{dx}$: $y^2 + x^2 - 2y - 4x = 4$

Solution

$\dfrac{d}{dx}\left(y^2 + x^2 - 2y - 4x\right) = \dfrac{d}{dx}(4)$

$\dfrac{d}{dx}\left(y^2\right) + \dfrac{d}{dx}\left(x^2\right) - \dfrac{d}{dx}(2y) - \dfrac{d}{dx}(4x) = \dfrac{d}{dx}(4)$

$2y\dfrac{dy}{dx} + 2x - 2\dfrac{dy}{dx} - 4 = 0$

$2(y-1)\dfrac{dy}{dx} = 4 - 2x$

$(y-1)\dfrac{dy}{dx} = 2 - x$

$\dfrac{dy}{dx} = \dfrac{2-x}{y-1}$

Exercise

Find $\dfrac{dy}{dx}$: $x^2y^2 - 2x = 3$

Solution

$2xy^2 + 2x^2yy' - 2 = 0$

$2x^2yy' = 2 - 2xy^2$

$y' = \dfrac{2\left(1 - xy^2\right)}{2x^2y}$

$\dfrac{dy}{dx} = \dfrac{1 - xy^2}{x^2y}$

Exercise

Find $\dfrac{dy}{dx}$: $x + \sqrt{x}\sqrt{y} = y^2$

Solution

$$\frac{d}{dx}\left(x + x^{1/2}y^{1/2}\right) = \frac{d}{dx}y^2$$

$$1 + \frac{d}{dx}\left(x^{1/2}\right)y^{1/2} + x^{1/2}\frac{d}{dx}\left(y^{1/2}\right) = 2y\frac{dy}{dx}$$

$$1 + \frac{1}{2}x^{-1/2}y^{1/2} + x^{1/2}\frac{1}{2}y^{-1/2}\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$1 + \frac{y^{1/2}}{2x^{1/2}} + \frac{x^{1/2}}{2y^{1/2}}\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$1 + \frac{y^{1/2}}{2x^{1/2}} = 2y\frac{dy}{dx} - \frac{x^{1/2}}{2y^{1/2}}\frac{dy}{dx}$$

$$\left(\frac{4y^{3/2} - x^{1/2}}{2y^{1/2}}\right)\frac{dy}{dx} = \frac{2x^{1/2} + y^{1/2}}{2x^{1/2}}$$

$$\frac{dy}{dx} = \frac{2x^{1/2} + y^{1/2}}{2x^{1/2}} \cdot \frac{2y^{1/2}}{4y^{3/2} - x^{1/2}}$$

$$= \frac{4x^{1/2}y^{1/2} + 2y}{8x^{1/2}y^{3/2} - 2x}$$

            Divide every term by 2

$$= \left.\frac{2x^{1/2}y^{1/2} + y}{4x^{1/2}y^{3/2} - x}\right|$$

Exercise

Find $\dfrac{dy}{dx}$: $x^2y + xy^2 = 6$

Solution

$$\left(2xy + x^2\frac{dy}{dx}\right) + \left(y^2 + 2xy\frac{dy}{dx}\right) = 0$$

$$x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -2xy - y^2$$

$$\left(x^2 + 2xy\right)\frac{dy}{dx} = -2xy - y^2$$

$$\frac{dy}{dx} = \left.\frac{-2xy - y^2}{x^2 + 2xy}\right|$$

Exercise

Find $\dfrac{dy}{dx}$: $x^3 - xy + y^3 = 1$

Solution

$$3x^2 - \left(y + x\dfrac{dy}{dx}\right) + 3y^2\dfrac{dy}{dx} = 0$$

$$3x^2 - y - x\dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx} = 0$$

$$\left(3y^2 - x\right)\dfrac{dy}{dx} = y - 3x^2$$

$$\boxed{\dfrac{dy}{dx} = \dfrac{y - 3x^2}{3y^2 - x}}$$

Exercise

Find $\dfrac{dy}{dx}$: $y^2 = \dfrac{x-1}{x+1}$

Solution

$$2yy' = \dfrac{1(x+1) - (1)(x-1)}{(x+1)^2}$$

$$2yy' = \dfrac{x+1-x+1}{(x+1)^2}$$

$$y' = \dfrac{2}{2y(x+1)^2}$$

$$\boxed{\dfrac{dy}{dx} = \dfrac{1}{y(x+1)^2}}$$

Exercise

Find $\dfrac{dy}{dx}$: $(3xy + 7)^2 = 6y$

Solution

$$2(3xy + 7)(3y + 3xy') = 6y'$$

$$6(3xy + 7)(y + xy') = 6y'$$ *Divide by 6 both sides*

$$(3xy + 7)(y + xy') = y'$$

$$3xy^2 + 3x^2yy' + 7y + 7xy' = y'$$

$$3x^2yy' + 7xy' - y' = -3xy^2 - 7y$$

$$\left(3x^2y + 7x - 1\right)y' = -\left(3xy^2 + 7y\right)$$

$$\dfrac{dy}{dx} = -\dfrac{3xy^2 + 7y}{3x^2y + 7x - 1}$$

Exercise

Find $\dfrac{dy}{dx}:$ $xy = \cot(xy)$

Solution

$$y + xy' = -\csc^2(xy)\ (y + xy')$$

$$y + xy' = -y\csc^2(xy) - x\csc^2(xy)y'$$

$$x\csc^2(xy)y' + xy' = -y\csc^2(xy) - y$$

$$x\left(\csc^2(xy) + 1\right)y' = -y\left(\csc^2(xy) + 1\right)$$

$$y' = -\dfrac{y\left(\csc^2(xy) + 1\right)}{x\left(\csc^2(xy) + 1\right)}$$

$$\dfrac{dy}{dx} = -\dfrac{y}{x}$$

Exercise

Find $\dfrac{dy}{dx}:$ $x + \tan(xy) = 0$

Solution

$$1 + \sec^2(xy)(y + xy') = 0$$

$$1 + y\sec^2(xy) + x\sec^2(xy)y' = 0$$

$$x\sec^2(xy)y' = -y\sec^2(xy) - 1$$

$$y' = -\dfrac{y\sec^2(xy)}{x\sec^2(xy)} - \dfrac{1}{x\sec^2(xy)}$$

$$\dfrac{dy}{dx} = -\dfrac{y}{x} - \dfrac{\cos^2 x}{x}$$

$$= \dfrac{-y - \cos^2 x}{x}$$

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Exercise

Find $\dfrac{dy}{dx}$: $\quad x\cos(2x+3y) = y\sin x$

Solution

$$\cos(2x+3y) - \sin(2x+3y)(2x+3y') = y'\sin x + y\cos x$$

$$\cos(2x+3y) - 2x\sin(2x+3y) - 3\sin(2x+3y)y' = y'\sin x + y\cos x$$

$$\cos(2x+3y) - 2x\sin(2x+3y) - y\cos x = y'\sin x + 3\sin(2x+3y)y'$$

$$\cos(2x+3y) - 2x\sin(2x+3y) - y\cos x = y'\left(\sin x + 3\sin(2x+3y)\right)$$

$$\boxed{y' = \frac{\cos(2x+3y) - 2x\sin(2x+3y) - y\cos x}{\sin x + 3\sin(2x+3y)}}$$

Exercise

Find $\dfrac{dy}{dx}$: $\quad y = \dfrac{e^y}{1+\sin x}$

Solution

$$y(1+\sin x) = e^y$$

$$y'(1+\sin x) + y\cos x = y'e^y$$

$$y'\left(e^y - 1 - \sin x\right) = y\cos x$$

$$\boxed{\frac{dy}{dx} = \frac{y\cos x}{e^y - 1 - \sin x}}$$

Exercise

Find $\dfrac{dy}{dx}$: $\quad \sin x\cos(y-1) = \dfrac{1}{2}$

Solution

$$\cos x\cos(y-1) - y'\sin x\sin(y-1) = 0$$

$$y'\sin x\sin(y-1) = \cos x\cos(y-1)$$

$$y' = \frac{\cos x\cos(y-1)}{\sin x\sin(y-1)}$$

$$\boxed{\frac{dy}{dx} = \cot x\cot(y-1)}$$

Exercise

Find $\dfrac{dy}{dx}$: $y\sqrt{x^2 + y^2} = 15$

Solution

$$y'\sqrt{x^2 + y^2} + \frac{1}{2}y\left(2x + 2yy'\right)\left(x^2 + y^2\right)^{-1/2} = 0 \qquad \times \sqrt{x^2 + y^2}$$

$$y'\left(x^2 + y^2\right) + y\left(x + yy'\right) = 0$$

$$y'\left(x^2 + y^2\right) + y^2 y' = -xy$$

$$y'\left(x^2 + 2y^2\right) = -xy$$

$$\underline{\dfrac{dy}{dx} = -\dfrac{xy}{x^2 + 2y^2}}$$

Exercise

Find $\dfrac{dr}{d\theta}$ $r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4}$

Solution

$$r - 2\theta^{1/2} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4}$$

$$\frac{dr}{d\theta} - 2\frac{1}{2}\theta^{-1/2} = \frac{3}{2}\frac{2}{3}\theta^{-1/3} + \frac{4}{3}\frac{3}{4}\theta^{-1/4}$$

$$\underline{\dfrac{dr}{d\theta} = \theta^{-1/3} + \theta^{-1/4} + \theta^{-1/2}}$$

Exercise

Find $\dfrac{dr}{d\theta}$ $\sin(r\theta) = \frac{1}{2}$

Solution

$$\cos(r\theta)\left(\theta\frac{dr}{d\theta} + r\right) = 0$$

$$\theta\frac{dr}{d\theta} + r = 0 \qquad \cos(r\theta) \neq 0$$

$$\underline{\dfrac{dr}{d\theta} = -\dfrac{r}{\theta}} \qquad \cos(r\theta) \neq 0$$

Exercise

Find $\dfrac{d^2y}{dx^2}$ $x^{2/3} + y^{2/3} = 1$

$\dfrac{2}{3}x^{-1/3} + \dfrac{2}{3}y^{-1/3}y' = 0$ *Multiply all terms by* $\dfrac{3}{2}$

$x^{-1/3} + y^{-1/3}y' = 0$

$y^{-1/3}y' = -x^{-1/3}$

$y' = -\dfrac{x^{-1/3}}{y^{-1/3}}$

$\quad = -\dfrac{y^{1/3}}{x^{1/3}} = -\left(\dfrac{y}{x}\right)^{1/3}$

$y'' = -\dfrac{1}{3}\left(\dfrac{y}{x}\right)^{-2/3}\left(\dfrac{xy' - y}{x^2}\right)$

$\quad = -\dfrac{1}{3}\left(\dfrac{x}{y}\right)^{2/3}\left(\dfrac{-x\left(\dfrac{y}{x}\right)^{1/3} - y}{x^2}\right) = \dfrac{1}{3}\left(\dfrac{x^{4/3}y^{1/3}}{y^{2/3}x^2} + \dfrac{x^{2/3}y}{y^{2/3}x^2}\right)$

$\quad = \dfrac{1}{3}\left(\dfrac{x}{y}\right)^{2/3}\left(\dfrac{x\dfrac{y^{1/3}}{x^{1/3}} + y}{x^2}\right)$

$\quad = \dfrac{1}{3}\dfrac{x^{2/3}}{y^{2/3}}\dfrac{x^{2/3}y^{1/3} + y}{x^2}$

$\quad = \dfrac{1}{3}\left(\dfrac{1}{y^{1/3}x^{2/3}} + \dfrac{y^{1/3}}{x^{4/3}}\right)$

Exercise

Find $\dfrac{d^2y}{dx^2}$ $2\sqrt{y} = x - y$

$2\dfrac{1}{2}y^{-1/2}y' = 1 - y'$

$$2\frac{1}{2}y^{-1/2}y' + y' = 1$$

$$\left(y^{-1/2} + 1\right)y' = 1$$

$$y' = \frac{1}{y^{-1/2}+1} = \frac{1}{\frac{1}{\sqrt{y}}+1} = \frac{\sqrt{y}}{1+\sqrt{y}}$$

$$\left(y^{-1/2}+1\right)y'' + \left(-\frac{1}{2}y^{-3/2}y'\right)y' = 0$$

$$\left(y^{-1/2}+1\right)y'' - \frac{1}{2}y^{-3/2}(y')^2 = 0$$

$$\left(y^{-1/2}+1\right)y'' = \frac{1}{2}y^{-3/2}\left(\frac{1}{y^{-1/2}+1}\right)^2$$

$$y'' = \frac{1}{2}y^{-3/2}\frac{1}{\left(y^{-1/2}+1\right)^2}\frac{1}{y^{-1/2}+1}$$

$$= \frac{1}{2}y^{-3/2}\frac{1}{\left(\frac{1}{\sqrt{y}}+1\right)^3}$$

$$= \frac{1}{2}y^{-3/2}\frac{1}{\left(\frac{1+\sqrt{y}}{\sqrt{y}}\right)^3}$$

$$= \frac{1}{2}y^{-3/2}\frac{1}{\frac{\left(1+\sqrt{y}\right)^3}{\left(y^{1/2}\right)^3}}$$

$$= \frac{1}{2}y^{-3/2}\frac{y^{3/2}}{\left(1+\sqrt{y}\right)^3}$$

$$= \frac{1}{2\left(1+\sqrt{y}\right)^3}$$

Exercise

If $x^3 + y^3 = 16$, find the value of $\dfrac{d^2y}{dx^2}$ at the point (2, 2).

Solution

$$3x^2 + 3y^2y' = 0$$

$$3y^2y' = -3x^2$$

$$y^2 y' = -x^2$$

$$2yy'y' + y^2 y'' = -2x$$

$$y^2 y'' = -2x - 2y(y')^2$$

$$y^2 y'' = -2x - 2y\left(\frac{-x^2}{y^2}\right)^2$$

$$y^2 y'' = -2x - 2\frac{x^4}{y^3}$$

$$y'' = -2\frac{x}{y^2} - 2\frac{x^4}{y^5}$$

$$= \frac{-2xy^3 - 2x^4}{y^5}$$

$$y''\Big|_{(2,2)} = \frac{-2(2)2^3 - 2(2)^4}{2^5}$$

$$= \frac{-2^5 - 2^5}{2^5}$$

$$= -2$$

Exercise

Find dy/dx: $x^2 - xy + y^2 = 4$ and evaluate the derivative at the given point $(0,-2)$

Solution

$$2x - (y + xy') + 2yy' = 0$$

$$-y - xy' + 2yy' = -2x$$

$$(2y - x)y' = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}\Bigg|$$

$$@(0,-2) \rightarrow \frac{dy}{dx} = \frac{-2 - 2(0)}{2(-2) - (0)}$$

$$= \frac{-2}{-4}$$

$$= \frac{1}{2}$$

Exercise

Find the slope of the curve $\left(x^2 + y^2\right)^2 = (x - y)^2$ at the point $(-2, 1)$ and $(-2, -1)$

Solution

1 and -1

Exercise

Find the slope of the tangent line to the circle $x^2 - 9y^2 = 16$ at the point $(5, 1)$

Solution

$2x - 18y\dfrac{dy}{dx} = 0$

$-18y\dfrac{dy}{dx} = -2x$

$\dfrac{dy}{dx} = \dfrac{-2x}{-18y} = \dfrac{x}{9y}$

@ $(5, 1)$ → $\dfrac{dy}{dx} = \dfrac{5}{9(1)}$

$\qquad\qquad = \dfrac{5}{9}$

Exercise

Find the slope of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$

Solution

$\dfrac{d}{dx}\left(x^2 + y^2\right) = \dfrac{d}{dx}(25)$

$2x + 2y\dfrac{dy}{dx} = 0$

$2y\dfrac{dy}{dx} = -2x$

$\dfrac{dy}{dx} = -\dfrac{x}{y}$

Slope: $\dfrac{dy}{dx} = -\dfrac{3}{-4}$

$\qquad\quad = \dfrac{3}{4}$

Exercise

Find an equation of the line tangent to the following curves at the given point

$$y = 3x^3 + \sin x; \quad (0, 0)$$

Solution

$$m = y' = 9x^2 + \cos x \Big|_{(0, 0)}$$

$$= 1$$

$$\underline{y = x}$$

$$y = m(x - x_1) + y_1$$

Exercise

Find an equation of the line tangent to the following curves at the given point

$$y = \frac{4x}{x^2 + 3}; \quad (3, 1)$$

Solution

$$m = y' = \frac{4x^2 + 12 - 8x^2}{\left(x^2 + 3\right)^2}$$

$$= \frac{12 - 4x^2}{\left(x^2 + 3\right)^2}\Bigg|_{(3, 1)}$$

$$= \frac{-24}{144}$$

$$= -\frac{1}{6}$$

$$y = -\frac{1}{6}(x - 3) + 1 \qquad\qquad y = m(x - x_1) + y_1$$

$$= -\frac{1}{6}x + \frac{3}{2}$$

Exercise

Find an equation of the line tangent to the following curves at the given point

$$y + \sqrt{xy} = 6; \quad (1, 4)$$

Solution

$$y' + \frac{1}{2}(y + xy')\frac{1}{\sqrt{xy}} = 0 \Bigg|_{(1, 4)}$$

$$y' + \frac{1}{2}(4 + y')\frac{1}{2} = 0$$

$$y' + \frac{1}{4}y' = -1$$

$$\frac{5}{4}y' = -1$$

$$\underline{m = y' = -\frac{4}{5}}$$

$$y = -\frac{4}{5}(x-1)+4 \qquad\qquad y = m(x - x_1) + y_1$$

$$\underline{= -\frac{4}{5}x + \frac{24}{5}}$$

Exercise

Find an equation of the line tangent to the following curves at the given point

$$x^2 y + y^3 = 75; \quad (4, \ 3)$$

Solution

$$2xy + x^2 y' + 3y^2 y' = 0 \ \Big|_{(4, \ 3)}$$

$$(16 + 27)y' = -24$$

$$\underline{y' = -\frac{24}{43} = m}$$

$$y = -\frac{24}{43}(x-4)+3 \qquad\qquad y = m(x - x_1) + y_1$$

$$\underline{= -\frac{24}{43}x + \frac{225}{43}}$$

Exercise

Find the equation of the tangent line to the circle $x^3 + y^3 = 9xy$ at the point (2, 4)

Solution

$$3x^2 + 3y^2 y' = 9y + 9xy'$$

$$3y^2 y' - 9xy' = 9y - 3x^2$$

$$\left(3y^2 - 9x\right)y' = 9y - 3x^2$$

$$y' = \frac{3\left(3y - x^2\right)}{3\left(y^2 - 3x\right)}$$

$$= \frac{3y - x^2}{y^2 - 3x}$$

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$$m\Big|_{(2,4)} = \frac{3(4)-2^2}{4^2-3(2)}$$

$$= \frac{8}{10}$$

$$\underline{= \frac{4}{5}}\Big|$$

$$y = \frac{4}{5}(x-2)+4 \qquad\qquad\qquad y = m(x-x_1)+y_1$$

$$= \frac{4}{5}x - \frac{8}{5}+4$$

$$\underline{y = \frac{4}{5}x + \frac{12}{5}}\Big|$$

Exercise

Find the lines that are (**a**) tangent and (**b**) normal to the curve $x^2 + xy - y^2 = 1$ at the point $(2, 3)$.

Solution

$$2x + y + xy' - 2yy' = 0$$

$$(x-2y)y' = -2x - y$$

$$y' = \frac{-2x-y}{x-2y} = \frac{2x+y}{2y-x}$$

a) *tangent slope* $= y'\Big|_{(2,\,3)}$

$$= \frac{2(2)+3}{2(3)-2}$$

$$\underline{= \frac{7}{4}}\Big|$$

$$y = \frac{7}{4}(x-2)+3 \qquad\qquad y = m(x-x_1)+y_1$$

$$= \frac{7}{4}x - \frac{7}{2}+3$$

$$\underline{= \frac{7}{4}x - \frac{1}{2}}\Big|$$

b) *normal slope* $= -\frac{4}{7}$

$$y = -\frac{4}{7}(x-2)+3 \qquad\qquad y = m(x-x_1)+y_1$$

$$y = \frac{4}{7}x - \frac{8}{7}+3$$

$$\underline{y = -\frac{4}{7}x + \frac{29}{7}}\Big|$$

Exercise

Find the lines that are (***a***) tangent and (***b***) normal to the curve $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$ at the point $(-1, 0)$.

Solution

$$12x + 3y + 3xy' + 4yy' + 17y' = 0$$

$$(3x + 4y + 17)y' = -12x - 3y$$

$$y' = \frac{-12x - 3y}{3x + 4y + 17}$$

a) tangent slope $= y'\Big|_{(-1, 0)}$

$$= \frac{-12(-1) - 3(0)}{3(-1) + 4(0) + 17}$$

$$= \underline{\frac{6}{7}}$$

$$y = \frac{6}{7}(x + 1) \qquad\qquad\qquad y = m(x - x_1) + y_1$$

$$\underline{= \frac{6}{7}x + \frac{6}{7}}$$

b) normal slope $= -\frac{7}{6}$

$$y = -\frac{7}{6}(x + 1) \;\Rightarrow\; \underline{y = -\frac{7}{6}x - \frac{7}{6}} \qquad\qquad y = m(x - x_1) + y_1$$

Exercise

Find the lines that are (***a***) tangent and (***b***) normal to the curve $x^2 \cos^2 y - \sin y = 0$ at the point $(0, \pi)$.

Solution

$$2x\cos^2 y + x^2\left(2\cos y(-\sin y)y'\right) - (\cos y)y' = 0$$

$$\left(-2x^2 \cos y \sin y - \cos y\right)y' = -2x\cos^2 y$$

$$y' = \frac{-2x\cos^2 y}{-\left(2x^2 \sin y + 1\right)\cos y} = \frac{2x\cos y}{2x^2 \sin y + 1}$$

a) tangent slope $= y'\Big|_{(0, \pi)}$

$$= \frac{2(0)\cos(\pi)}{2(0)^2 \sin(\pi) + 1}$$

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$$= 0 \rfloor$$

$$y - \pi = 0(x - 0)$$

$$\underline{y = \pi} \rfloor$$

b) *normal slope* $= 0$

$$\Rightarrow \underline{x = 0} \rfloor$$

Exercise

Suppose that x and y are both functions of t, which can be considered to represent time, and that x and y are related by the equation $xy^2 + y = x^2 + 17$

Suppose further that when $x = 2$ and $y = 3$, then $\dfrac{dx}{dt} = 13$. Find the value of the $\dfrac{dy}{dt}$ at that moment.

Solution

$$y^2 \frac{dx}{dt} + 2xy \frac{dy}{dt} + \frac{dy}{dt} = 2x \frac{dx}{dt}$$

$$3^2(13) + 2(2)(3)\frac{dy}{dt} + \frac{dy}{dt} = 2(2)(13)$$

$$117 + 12\frac{dy}{dt} + \frac{dy}{dt} = 52$$

$$13\frac{dy}{dt} = -65$$

$$\frac{dy}{dt} = \frac{-65}{13}$$

$$\underline{= -5} \rfloor$$

Exercise

A cone-shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of 0.2 *cm* per hour, while the length is increasing at a rate of 0.8 *cm* per *hour*. If the icicle is currently 4 *cm* in radius and 20 *cm* long, is the volume of the icicle increasing or decreasing and at what rate?

Solution

The volume of the cone is given by the formula: $V = \frac{1}{3}\pi r^2 h$.

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$$

Given the values:

$$\frac{dr}{dt} = -0.2 \qquad \frac{dh}{dt} = 0.8 \qquad r = 4 \qquad h = 20$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(2(4)(20)(-0.2) + 4^2(0.8)\right)$$

$$= -20$$

The volume is decreasing at a rate of 20 cm^3 per *hour*.