# *Mathematica* Manual

## Notebook 15: Functions of Several Variables and Partial Derivatives

### ■ Functions of Several Variables

#### ■ Explicit Surfaces

In this section, you are asked to plot surfaces and level curves. For surfaces expressed explicitly as $z = f(x,y)$, we use the command

`Plot3D[f[x,y],{x,xmin,xmax},{y,ymin,ymax}]`

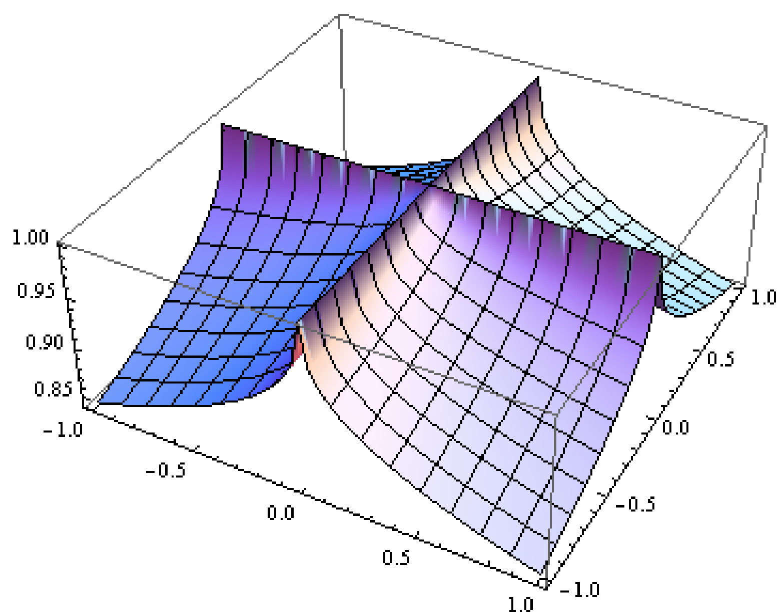
to plot the surface of $f(x, y)$ over the rectangle xmin $\leq x \leq$ xmax, ymin $\leq y \leq$ ymax. The *Mathematica* command

`ContourPlot[f[x,y],{x,xmin,xmax},{y,ymin,ymax},ContourShading →False]`

will plot level curves of $f(x, y)$ in the rectangle xmin $\leq x \leq$ xmax, ymin $\leq y \leq$ ymax. Deleting the option `ContourShading→ False` will result in the level curves being shaded, with lighter shading corresponding to higher parts of the surface of $f(x, y)$. Adding the option `Contours` $\to \{c_1, c_2, \ldots c_n\}$ will instruct *Mathematica* to plot the level curves corresponding to $f(x, y) = c_i$ for $i = 1, 2, \ldots, n$.Using the Plot3D command, some interesting surfaces can be plotted such as the graph of $f(x, y) = \frac{\sin\left(\sqrt[5]{|x\,y|}\right)}{\sqrt[5]{|x\,y|}}$. Change the terms in red for other functions.

```
In[1]:= Plot3D[Sin[Abs[x y]^(1/5)]/Abs[x y]^(1/5), {x, -1, 1}, {y, -1, 1}]
```

Out[1]=




You can see this from different viewpoints by clicking on the surface and moving your mouse.

Here is another example. Change the terms in red for other functions.

**Example:** Consider the function $f(x,\ y) = e^{-x^2-y^2}$ over the rectangle $-2 \le x \le 2$, $-2 \le y \le 2$.

(a) Plot the surface over the given rectangle.

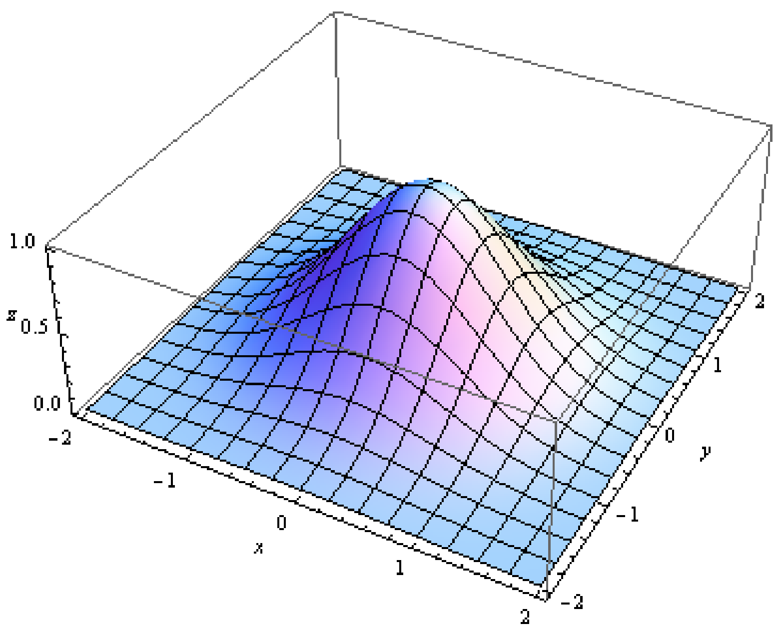
(b) Plot several level curves in the rectangle.

(c) Plot the level curve of $f$ though the point $(1,\ 1)$.

**Part (a)** The graph of the surface is obtained using the `Plot3D` command. The option `PlotPoints->20` increases the resolution of the displayed graph (the default value is `PlotPoints->15`). You may ignore warning messages about possible spelling errors here. Change the terms in red for different functions and different bounds.

In[2]:=
```
Clear[x, y, z]
f[x_, y_] = e^(-x^2-y^2);
xmin = -2; xmax = 2; ymin = -2; ymax = 2;
surface =
 Plot3D[f[x, y], {x, xmin, xmax}, {y, ymin, ymax}, PlotPoints → 20, AxesLabel → {x, y, z}]
```

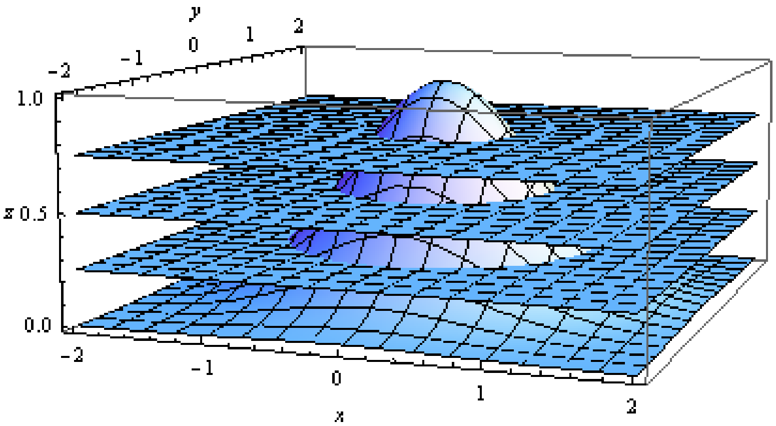
Out[5]=




**Part (b)** Before plotting some level curves of the surface, you might find it helpful to see the three-dimensional images represented by level curves. For example, the following `Plot3D` commands are used to display the level curves corresponding to the horizontal planes

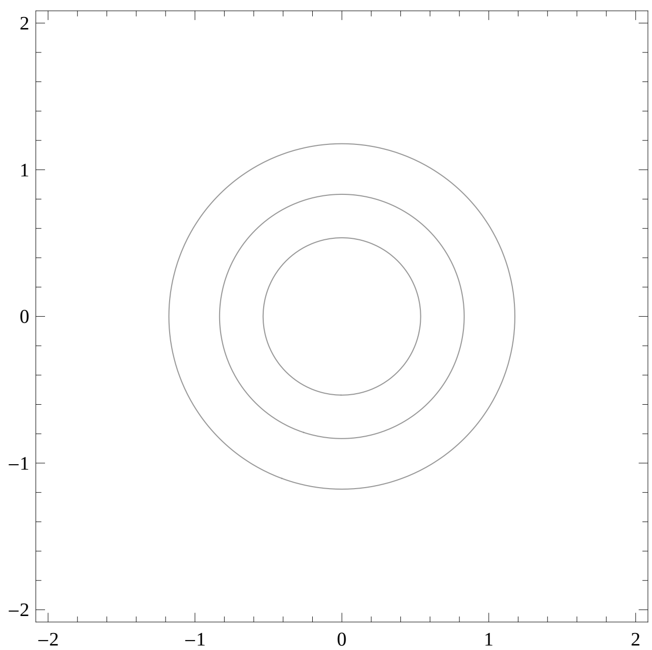
$z = .25$, $z = .5$ and $z = .75$.

In[6]:=
```
plane1 = Plot3D[.25, {x, xmin, xmax}, {y, ymin, ymax}, PlotPoints -> 2];
plane2 = Plot3D[.5, {x, xmin, xmax}, {y, ymin, ymax}, PlotPoints -> 2];
plane3 = Plot3D[.75, {x, xmin, xmax}, {y, ymin, ymax}, PlotPoints -> 2];
Show[{surface, plane1, plane2, plane3}]
```

Out[9]=



Move the picture around to see it from different angles. Now to show the actual level curves, the following `ContourPlot` command is executed.

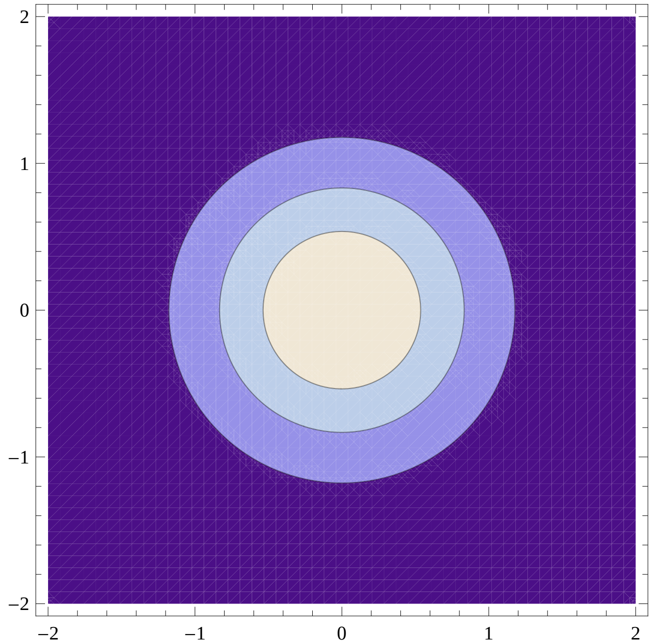
In[10]:=
```
ContourPlot[f[x, y], {x, xmin, xmax}, {y, ymin, ymax},
 PlotPoints -> 50, ContourShading → False, Contours → {.25, .5, .75}]
```

Out[10]=



If we take the default shading, Let's see what we get.

In[11]:=
```
ContourPlot[f[x, y], {x, xmin, xmax},
 {y, ymin, ymax}, PlotPoints -> 50, Contours → {.25, .5, .75}]
```

Out[11]=

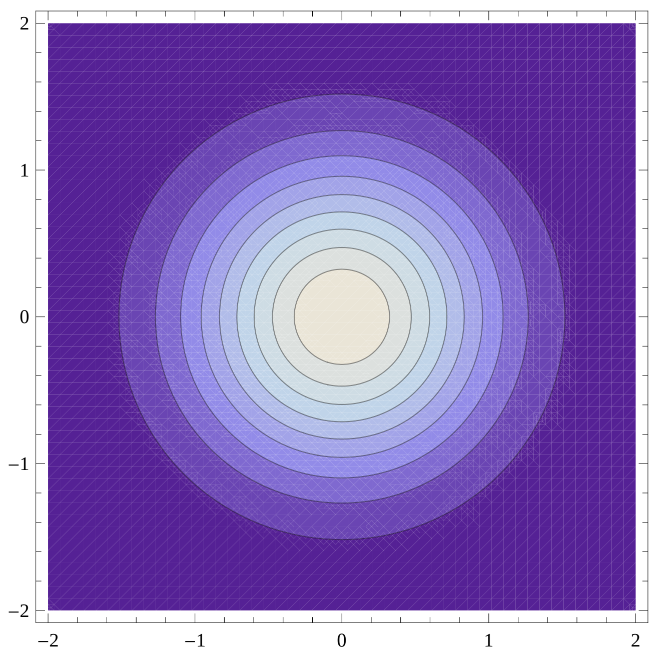


Looking back at your figure, you will see that the lighter shades correspond to the larger values of z.

Deleting the `Contours` option allows *Mathematica* to choose the level curves to be plotted.

In[12]:=
```
ContourPlot[f[x, y], {x, xmin, xmax}, {y, ymin, ymax}, PlotPoints -> 50]
```

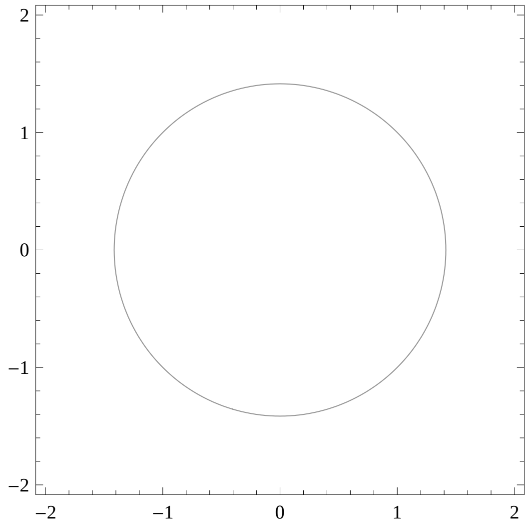
Out[12]=



**Part (c)** The following graph plots the level curve passing through (1, 1). Why?

In[13]:=
```
ContourPlot[f[x, y], {x, xmin, xmax}, {y, ymin, ymax},
 PlotPoints -> 50, ContourShading → False, Contours → {f[1, 1]}]
```

Out[13]=




## ■ Implicit Surfaces

We will use the command:

`ContourPlot3D[f[x,y,z],{x, xmin, xmax},{y, ymin,  ymax},{z,zmin,zmax},Contours->{c)]`

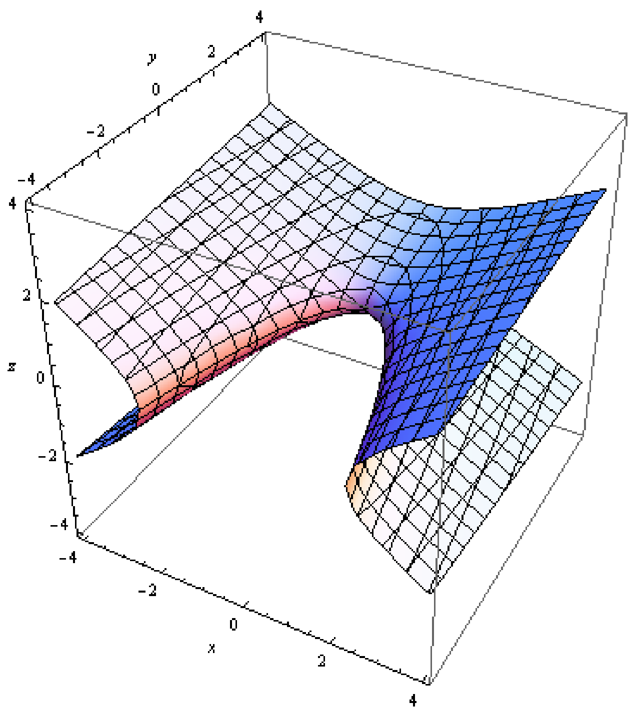
which will plot the level surface $f(x, y, z) = c$.

**Example:** Plot the level surface $x^2 + y - 3z^2 = 1$.

The surface is plotted where the option and `AxesLabel->{x,y,z}` is added so that the $x$, $y$ and $z$ axes are labeled.

In[14]:=
```
ContourPlot3D[x^2 + y - 3 z^2, {x, -4, 4}, {y, -4, 4},
 {z, -4, 4}, Contours -> {1}, AxesLabel -> {x, y, z}]
```

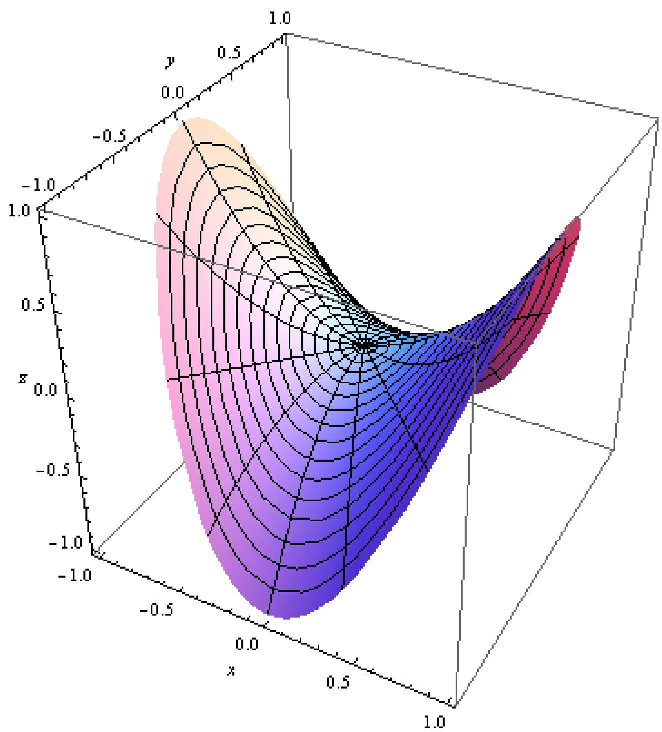
Out[14]=



**Parametrized Surfaces**

The *Mathematica* command
`ParametricPlot3D[{f[u, v], g[u, v], h[u, v]}, {u, umin, umax}, {v, umin, umax}]`
can be used to plot a surface described by the parametric equations
$x = f(u, v)$, $y = g(u, v)$ and $z = h(u, v)$. Consider the following example.

In[15]:=
```
ParametricPlot3D[{u Cos[v], u Sin[v], u^2 (Cos[v]^2 - Sin[v]^2)},
  {u, 0, 1}, {v, 0, 2 π}, AxesLabel → {x, y, z}]
```

Out[15]=



## Extreme Values and Saddle Points

### Computing Partial Derivatives

The command $\partial_x$ `f[x, y]` or `D[f[x,y],x]` can be used to compute $f_x(x, y)$ and the command $\partial_{x,y}$ `f[x, y]` or `D[f[x,y],x,y]` can be used to find $f_{xy}(x, y)$.

In[16]:=
```
Clear[x, y]
∂x Sin[x^2 y]
```

Out[17]= `2 x y Cos[x^2 y]`

In[18]:= `∂x,y Sin[x^2 y]`

Out[18]= `2 x Cos[x^2 y] - 2 x^3 y Sin[x^2 y]`

In[19]:= `D[Sin[x^2 y], x, y]`

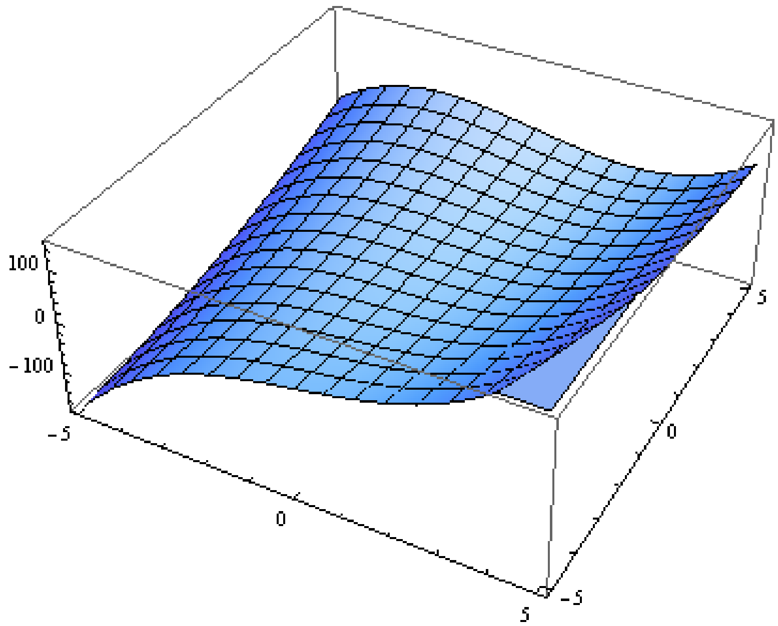
Out[19]= `2 x Cos[x^2 y] - 2 x^3 y Sin[x^2 y]`

**Example:** Consider the function $f(x, y) = x^3 + y^2 - 3xy$ on $-5 \leq x \leq 5$, $-5 \leq y \leq 5$. Plot the function over the given rectangle and then find and classify the critical points.

Here is a plot of the function.

In[20]:=
```
Clear[f];
f[x_, y_] = x^3 + y^2 - 3 x y;
Plot3D[f[x, y], {x, -5, 5}, {y, -5, 5}, PlotPoints → 25]
```

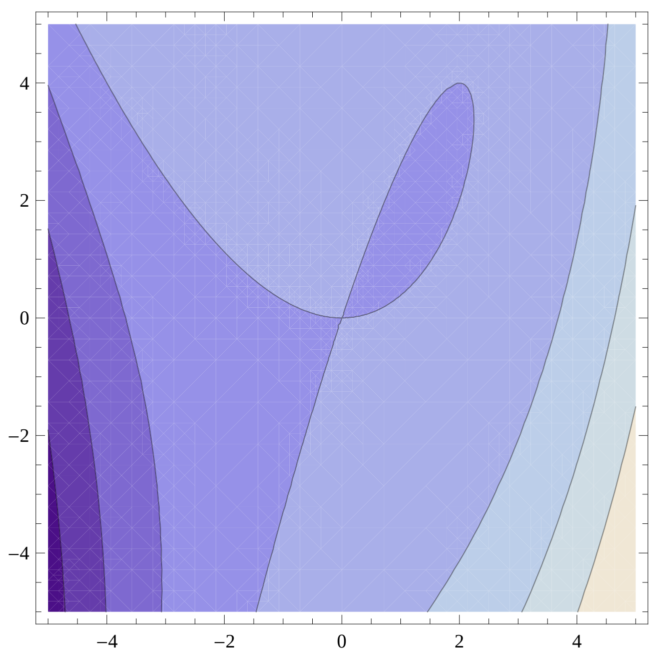
Out[22]=




We will plot some contours; recall that lighter shades represent regions where the function is larger.

In[23]:=
```
ContourPlot[f[x, y], {x, -5, 5}, {y, -5, 5}]
```

Out[23]=



Now the first partial derivatives are found.

```
In[24]:= xparder = ∂x f[x, y] // Simplify
         yparder = ∂y f[x, y] // Simplify

Out[24]= 3 (x^2 - y)

Out[25]= -3 x + 2 y
```

The command `Solve[{m[x,y]==0,n[x,y]==0},{x,y}]` can be used to simultaneously solve $m(x,\ y) = 0$ and $n(x,\ y) = 0$.

```
In[26]:= sol = Solve[{xparder == 0, yparder == 0}, {x, y}]

Out[26]= {{y → 0, x → 0}, {y → 9/4, x → 3/2}}
```

Now the second partial derivatives of the function are computed.

```
In[27]:= xxparder = ∂x,x f[x, y] // Simplify;
         yyparder = ∂x,y f[x, y] // Simplify;
         xyparder = ∂x,y f[x, y] // Simplify;
         discrim = xxparder yyparder - xyparder^2;
```

We will now evaluate the second partial with respect to x and the discriminant at the critical points identified in our above solution.

```
In[31]:= xxparder /. sol[[1]]
         discrim /. sol[[1]]

Out[31]= 0

Out[32]= -9

In[33]:= xxparder /. sol[[2]]
         discrim /. sol[[2]]

Out[33]= 9

Out[34]= -36
```

From the above, you can conclude that both critical points are saddle points.

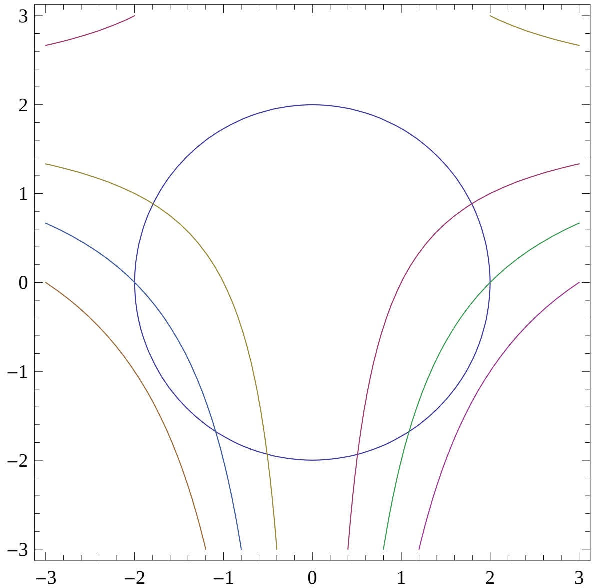
## ■ Lagrange Multipliers

### ■ A Function of Two Variables

Following is an example. You may change the terms in red for other functions.
**Example:** Maximize and minimize the function $f(x,\ y) = 2\,x - x\,y$ subject to the constraint $x^2 + y^2 - 4 = 0$. We will begin by defining our functions and also the gradient function, and plotting the constraint together with some contours of the function to be minimized.

In[35]:=
```
Clear[x, y, f, g, λ]
f[x_, y_] := 2 x - x y
g[x_, y_] := x^2 + y^2 - 4
gradient[f_] := {D[f, x], D[f, y]}
ContourPlot[{g[x, y] == 0, f[x, y] == 2, f[x, y] == -2,
  f[x, y] == 4, f[x, y] == -4, f[x, y] == 6, f[x, y] == -6}, {x, -3, 3}, {y, -3, 3}]
```

Out[39]=




Next, we solve the equations that will yield the desired results.

In[40]:=
```
solution = Solve[{gradient[f[x, y]] == λ gradient[g[x, y]],
          g[x, y] == 0},
  {x, y, λ}]
```

Out[40]= $\left\{\{\lambda \to 0,\ x \to 0,\ y \to 2\},\ \{\lambda \to 0,\ x \to 0,\ y \to 2\},\ \left\{\lambda \to -\frac{\sqrt{3}}{2},\ x \to -\sqrt{3},\ y \to -1\right\},\ \left\{\lambda \to \frac{\sqrt{3}}{2},\ x \to \sqrt{3},\ y \to -1\right\}\right\}$

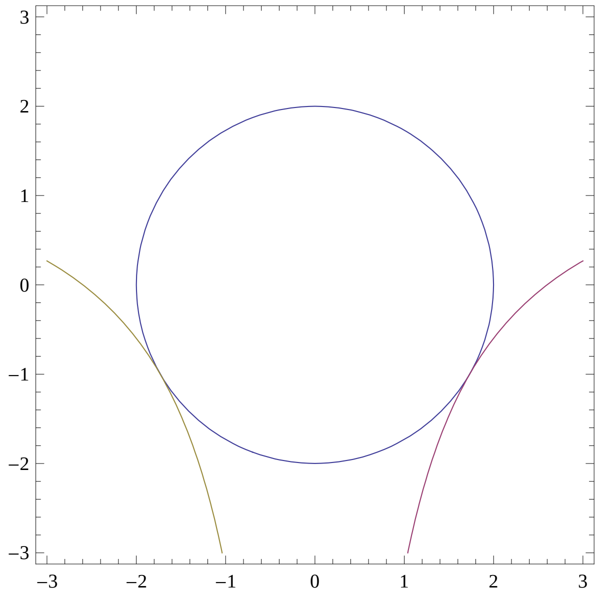
In[41]:=
```
f[x, y] /. solution
```

Out[41]= $\left\{0,\ 0,\ -3\sqrt{3},\ 3\sqrt{3}\right\}$

Look at the plot of the constraint and the contours representing the function at the largest and smallest values identified by the solution.

In[42]:=
```
ContourPlot[{g[x, y] == 0, f[x, y] == 3 √3, f[x, y] == -3 √3}, {x, -3, 3}, {y, -3, 3}]
```

Out[42]=




Notice how these contours are tangent to the contstraint curve.

### ▪ A Function of Three Variables

**Example:** Minimize the function $f(x, y, z) = xz + xy$ subject to the constraints $x^2 + z^2 - 4 = 0$ and $x^2 + y^2 - 5 = 0$.

**Step one:** First the functions $f$, $g_1$, $g_2$ and $h$ are defined as shown in the following input cell.

In[43]:=
```
Clear[f, h, x, y, z];
f = x z + x y;
g1 = x^2 + z^2 - 4;
g2 = x^2 + y^2 - 5;
h = f - λ1 g1 - λ2 g2;
```

**Step two:** The partial derivatives are then computed and placed in equations which all have zero on the right side.

In[48]:=
```
eq1 = ∂x h == 0;
eq2 = ∂y h == 0;
eq3 = ∂z h == 0;
eq4 = ∂λ1 h == 0;
eq5 = ∂λ2 h == 0;
```

**Step Three:** The `Solve` command is used to solve the system of equations formed in step two.

In[53]:=
```
sol = Solve[{eq1, eq2, eq3, eq4, eq5}, {x, y, z, λ1, λ2}]
```

Out[53]=
$$\left\{\left\{\lambda_1 \to -\frac{\sqrt{5}}{4},\ \lambda_2 \to -\frac{1}{\sqrt{5}},\ y \to -\frac{5}{3},\ z \to -\frac{4}{3},\ x \to \frac{2\sqrt{5}}{3}\right\},\right.$$
$$\left\{\lambda_1 \to -\frac{\sqrt{5}}{4},\ \lambda_2 \to -\frac{1}{\sqrt{5}},\ y \to \frac{5}{3},\ z \to \frac{4}{3},\ x \to -\frac{2\sqrt{5}}{3}\right\},$$
$$\left\{\lambda_1 \to \frac{\sqrt{5}}{4},\ \lambda_2 \to \frac{1}{\sqrt{5}},\ y \to -\frac{5}{3},\ z \to -\frac{4}{3},\ x \to -\frac{2\sqrt{5}}{3}\right\},$$
$$\left.\left\{\lambda_1 \to \frac{\sqrt{5}}{4},\ \lambda_2 \to \frac{1}{\sqrt{5}},\ y \to \frac{5}{3},\ z \to \frac{4}{3},\ x \to \frac{2\sqrt{5}}{3}\right\}\right\}$$

**Step Four:** The function $f$ is then evaluated at each solution to determine the extreme values subject to the constraints asked for in the exercise. What are the extreme values based upon the following work?

In[54]:=
```
{{x, y, z}, f} /. sol[[1]]
{{x, y, z}, f} /. sol[[2]]
{{x, y, z}, f} /. sol[[3]]
{{x, y, z}, f} /. sol[[4]]
```

Out[54]= $\left\{\left\{\frac{2\sqrt{5}}{3},\ -\frac{5}{3},\ -\frac{4}{3}\right\},\ -2\sqrt{5}\right\}$

Out[55]= $\left\{\left\{-\frac{2\sqrt{5}}{3},\ \frac{5}{3},\ \frac{4}{3}\right\},\ -2\sqrt{5}\right\}$

Out[56]= $\left\{\left\{-\frac{2\sqrt{5}}{3},\ -\frac{5}{3},\ -\frac{4}{3}\right\},\ 2\sqrt{5}\right\}$

Out[57]= $\left\{\left\{\frac{2\sqrt{5}}{3},\ \frac{5}{3},\ \frac{4}{3}\right\},\ 2\sqrt{5}\right\}$

We can look at the results in decimal form also.

In[58]:=
```
{{x, y, z}, f} /. sol[[1]] // N
{{x, y, z}, f} /. sol[[2]] // N
{{x, y, z}, f} /. sol[[3]] // N
{{x, y, z}, f} /. sol[[4]] // N
```

Out[58]= {{1.49071, -1.66667, -1.33333}, -4.47214}

Out[59]= {{-1.49071, 1.66667, 1.33333}, -4.47214}

Out[60]= {{-1.49071, -1.66667, -1.33333}, 4.47214}

Out[61]= {{1.49071, 1.66667, 1.33333}, 4.47214}