Ex Vol? $z = 0,\ y = x,\ x = 1$

$$z = f(x,y) = 3 - x - y$$

$$\boxed{0 \le y \le x \qquad 0 \le x \le 1} \quad (1)$$

$$\boxed{0 \le y \le 1 \qquad y \le x \le 1} \quad (2)$$

$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy$$

$$= \int_0^1 \left(3x - \tfrac{1}{2}x^2 - yx \Big|_y^1\right) dy$$

$$= \int_0^1 \left(3 - \tfrac{1}{2} - y - 3y + \tfrac{1}{2}y^2 + y^2\right) dy$$

$$= \int_0^1 \left(\tfrac{5}{2} - 4y + \tfrac{3}{2}y^2\right) dy$$

$$= \tfrac{5}{2}y - 2y^2 + \tfrac{1}{2}y^3 \Big|_0^1$$

$$= \tfrac{5}{2} - 2 + \tfrac{1}{2}$$

$$= 1 \ \text{unit}^3$$

[illegible]

$$\iint_R \frac{\sin x}{x}\,dx\,dy = \int_0^1 \int_0^x \frac{\sin x}{x}\,dy\,dx$$

$$= \int_0^1 \frac{\sin x}{x}\, y \Big|_0^x \,dx$$

$$= \int_0^1 \sin x\,dx$$

$$= -\cos x \Big|_0^1$$

$$= -\cos 1 + \cos 0$$

$$= \boxed{1 - \cos 1}$$

Ex $z = 16 - x^2 - y^2$ $\begin{cases} y = 2\sqrt{x} \\ y = 4x - 2 \\ y = 0 \end{cases}$

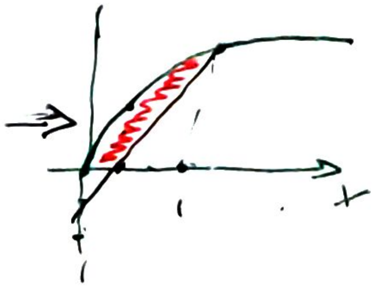
V?

$y = 2\sqrt{x} = 4x - 2$

$(\sqrt{x})^2 = (2x - 1)^2$

$x = 4x^2 - 4x + 1$

$4x^2 - 5x + 1 = 0$



$y = 2\sqrt{x} \Rightarrow x = \frac{y^2}{4}$ $\left(x = 1, \frac{1}{4}\right) \rightarrow \#$

Testing.

$4x = y + 2$

$$V = \int_0^2 \int_{y^2/4}^{\frac{y+2}{4}} (16 - x^2 - y^2)\,dx\,dy$$

$$= \int_0^2 16x - \frac{1}{3}x^3 - y^2 x \Big|_{y^2/4}^{\frac{y+2}{4}} dy$$

$$= \int_0^2 \Big(4y + 8 - \overset{\frac{1}{192}}{\frac{1}{3}\frac{1}{64}}(y^3 + 6y^2 + 12y + 8) - \frac{1}{4}y^3 - \frac{1}{2}y^2 - 4y^2 + \frac{1}{192}y^6 + \frac{1}{4}y^4 \Big) dy$$

$$= \int_0^2 \left(\frac{1}{192}y^6 + \frac{1}{4}y^4 - \frac{49}{192}y^3 - \frac{145}{32}y^2 + \frac{63}{16}y + \frac{191}{24} \right) dy$$

$$= \frac{1}{1344}y^7 + \frac{1}{20}y^5 - \frac{49}{768}y^4 - \frac{145}{96}y^3 + \frac{63}{32}y^2 + \frac{191}{24}y \Big|_0^2$$

$$= \frac{2^7}{1344} + \frac{2^5}{20} - \frac{49 \times 2^4}{768} - \frac{145}{12} + \frac{63}{8} + \frac{191}{12}$$

(red corrections: $\frac{8}{5}$ over $\frac{2^5}{20}$, 48 under 768)

$$= \frac{62,409}{5,040} \text{ unit}^3$$

Ex. A? $y=x$ $y=x^2$ Q I

$$y = x = x^2 \Rightarrow \underline{x = 0, 1}$$

$$A = \int_0^1 \int_{x^2}^{x} dy\, dx$$

$$= \int_0^1 y \Big|_{x^2}^{x} dx$$

$$= \int_0^1 (x - x^2)\, dx$$

$$= \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6} \text{ unit}^2$$

Ex A? $y = x^2$ $y = x+2$

$$x^2 = x+2$$

$$x^2 - x - 2 = 0 \Rightarrow \underline{x = -1,\ 2}$$

$$A = \int_{-1}^{2} \int_{x^2}^{x-2} dy\, dx$$

$$\int_a^b d\theta = b - a$$

$$= \int_{-1}^{2} y \Big|_{x^2}^{x+2} dx$$

$$= \int_{-1}^{2} (x+2-x^2)\, dx$$

$$= \frac{1}{2}x^2 + 2x - \frac{1}{3}x^3 \Big|_{-1}^{2}$$

$$= 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$= \underline{\frac{9}{2}\ \text{unit}^2}$$

Ex A? $y = x^2$ $y^2 = x \to y = \sqrt{x}$

$$y^2 = x = x^4 \to x = 0, 1$$

$$\text{Area} = \int_0^1 \int_{x^2}^{\sqrt{x}} dy\, dx$$

$$= \int_0^1 (-x^2 + x^{1/2})\, dx$$

$$= -\frac{1}{3}x^3 + \frac{2}{3}x^{3/2} \Big|_0^1$$

$$= -\frac{1}{3} + \frac{2}{3}$$

$$= \frac{1}{3} \text{ unit}^2$$

3.3

$$A = \frac{1}{2} r^2 \theta$$

Polar $\to$ $r\,dr\,d\theta = dx\,dy$

$$A = \iint_R r\,dr\,d\theta = \iint_R dx\,dy$$

Ex A? $r^2 = 4\cos 2\theta$

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta$$

$$= \frac{4}{2}\int_0^{\pi/4} r^2 \Big|_0^{\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\frac{\cancel{2}}{\cancel{2}}\int_0^{\pi/4} \cos 2\theta \; d(2\theta)$$

$$= 4\sin 2\theta \Big|_0^{\pi/4}$$

$$\boxed{= 4 \ \text{unit}^2}$$

$\iint_R e^{x^2+y^2}\,dy\,dx$ $\quad R:\ y = 0 \quad y = \sqrt{1 - x^2}$ ($1 = r^2$)

$$\iint_R e^{x^2+y^2}\,dy\,dx = \int_0^{\pi} d\theta \int_0^1 r e^{r^2}\,dr$$

$$= \frac{\pi}{2}\int_0^1 e^{r^2}\,d(r^2)$$

$$= \frac{\pi}{2}\, e^{r^2}\Big|_0^1$$

$$\boxed{= \frac{\pi}{2}(e - 1)}$$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$$

$$= \int_0^{\pi/2} d\theta \int_0^1 \overbrace{r^2 r}^{r^3}\,dr$$

$$= \frac{\pi}{2} \frac{1}{4} r^4 \Big|_0^1$$

$$= \frac{\pi}{8}$$

Ex V? $z = 9 - x^2 - y^2$ $x^2+y^2=1$

$$V = \int_0^{2\pi} d\theta \int_0^1 (9-r^2)\, r\,dr$$

$$= 2\pi \int_0^1 (9r - r^3)\,dr$$

$$= 2\pi \left(\frac{9}{2} r^2 - \frac{1}{4} r^4 \Big|_0^1 \right.$$

$$= 2\pi \left(\frac{9}{2} - \frac{1}{4} \right)$$

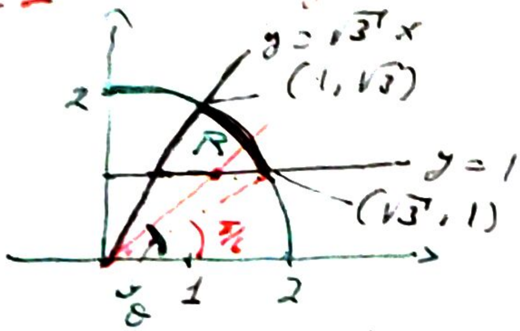
$$= \frac{17\pi}{2} \text{ unit}^3$$

Ex. A? $x^2+y^2=4$

$\uparrow y=1$ $\qquad$ $\downarrow y=\sqrt{3}\,x$

$x = r\cos\theta$

$y = r\sin\theta$



$\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$

$? \le r \le 2$

$y = r\sin\theta = 1 \Rightarrow r = \frac{1}{\sin\theta} = \csc\theta$

$\csc\theta \le r \le 2$

$$A = \int_{\pi/6}^{\pi/3} \int_{\csc\theta}^{2} r\,dr\,d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/3} r^2 \Big|_{\csc\theta}^{2} d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/3} (4-\csc^2\theta)\,d\theta$$

$$= \frac{1}{2}\left(4\theta + \cot\theta\right)\Big|_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left(\frac{4\pi}{3} + \frac{1}{\sqrt{3}} - \frac{2\pi}{3} - \sqrt{3}\right) \qquad \frac{\sqrt{3}}{3} - \sqrt{3}$$

$$= \frac{1}{2}\left(\frac{2\pi}{3} - \frac{2\sqrt{3}}{3}\right)$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{3} \quad \text{unit}^2$$

Ex $\iint e^{-x^2-y^2}\,dA$ QI $x^2+y^2=a^2$

$0 \le \theta \le \frac{\pi}{2}$ $0 \le r \le a$

$$\iint e^{-(x^2+y^2)}\,dA = \int_0^{\pi/2} d\theta \int_0^a r e^{-r^2}\,dr$$

$$= -\frac{\pi}{4} \int_0^a e^{-r^2}\,d(-r^2)$$

$$= -\frac{\pi}{4}\, e^{-r^2}\Big|_0^a$$

$$= \frac{-\pi}{4}\left(e^{-a^2}-1\right)$$

$$= \frac{\pi}{4}\left(1-\frac{1}{e^{a^2}}\right)$$

10 3.3

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\,dx\,dy$$

$$= \int_{0}^{2\pi}\int_{0}^{1} \ln(r^2+1)(r\,dr\,d\theta)$$

$$= \frac{1}{2}\int_{0}^{2\pi} d\theta \int_{0}^{1} \ln(r^2+1)\,d(r^2+1)$$

$$= \pi\,(r^2+1)\left(\ln(r^2+1)-1\right)\Big|_0^1 \qquad \int \ln x\,dx = x\ln x - x,$$

$$= \pi\left[2(\ln 2 - 1) + 1\right]$$

$$= \pi\,(2\ln 2 - 1)$$

$$\boxed{= \pi\,(\ln 4 - 1)}$$

$u = \ln x \qquad v = \int dx$

$du = \frac{1}{x}dx \qquad = x$

$$\int \ln x\,dx = x\ln x - \int dx$$

$$= x\ln x - x$$

$y = \ln x$

$x = e^y$

$dx = e^y\,dy$

$$\int \ln x\,dx = \int y\,e^y\,dy$$

$$= (y-1)e^y$$

$$= (\ln x - 1)x$$

A11 $\displaystyle\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{dy\,dx}{(x^2+y^2)^2}$

$0 \le y \le \sqrt{2x-x^2}$ $\qquad$ $1 \le x \le 2$

$y^2 = 2x - x^2$

$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$

$0 \le \theta \le \frac{\pi}{4}$

$x = r\cos\theta = 1$

$\left[r = \frac{1}{\cos\theta} = \sec\theta \right]$

$x^2 + y^2 = 2x$

$r^2 = 2r\cos\theta \to r = 2\cos\theta$

$$\int_0^{\pi/4} \int_{\sec\theta}^{2\cos\theta} \frac{r\,dr\,d\theta}{r^4}$$

$$= \int_0^{\pi/4} \int_{\sec\theta}^{2\cos\theta} r^{-3}\,dr\,d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/4} \frac{1}{r^2}\Big|_{\sec\theta}^{2\cos\theta} d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/4} \left(\frac{1}{4\cos^2\theta} - \frac{1}{\sec^2\theta} \right) d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/4} \left(\frac{1}{4}\sec^2\theta - \cos^2\theta \right) d\theta$$

$$= -\frac{1}{2}\left[\frac{1}{4}\tan\theta\Big|_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4}(1+\cos 2\theta)\,d\theta\right]$$

$$= -\frac{1}{2}\left(\frac{1}{4} - \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\Big|_0^{\pi/4}\right.\right.$$

$$= -\frac{1}{2}\left(\frac{1}{4} - \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right)\right)$$

$$= -\frac{1}{2}\left(-\frac{\pi}{8}\right)$$

$$\boxed{= \frac{\pi}{16}}$$