Section 3.5 – Inverse Trigonometry Functions

Relationships Between f^{-1} *and* f

- $y = f^{-1}(x)$ if and only if $x = f(y)$, where ***x*** is in the domain of f^{-1} and ***y*** is in the domain of f
- Domain of f^{-1} = Range of f
- Range of f^{-1} = Domain of f
- $f\left(f^{-1}(x)\right) = x$ for every x in the domain of f^{-1}
- $f^{-1}\left(f(y)\right) = y$ for every y in the domain of f
- The point (a, b) is on the graph of f ***iff*** the point (b, a) is on the graph of f^{-1}.
- The graphs of f^{-1} and f are reflections of each other through the line $y = x$.

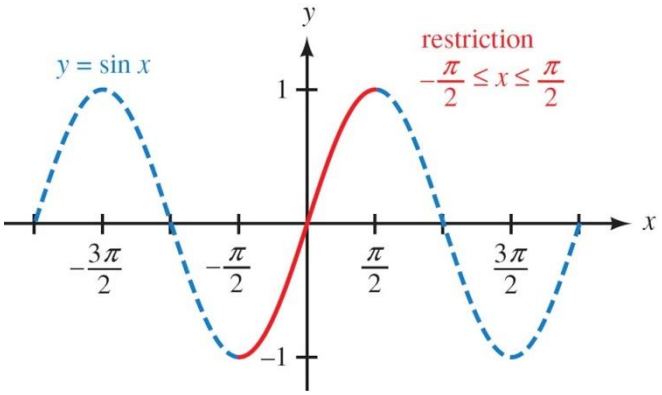
The Inverse *Sine* Function

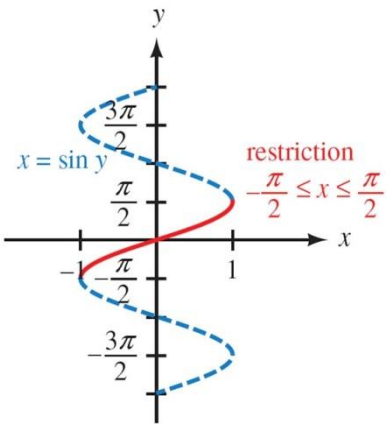
$$y = \sin^{-1} x \quad or \quad y = \arcsin x \quad \textbf{\textit{iff}} \quad x = \sin y \quad for \ -\frac{\pi}{2} \le y \le \frac{\pi}{2} \quad and \quad -1 \le x \le 1$$

Properties of $\sin^{-1}$

$$\sin\left(\sin^{-1} x\right) = \sin\left(\arcsin x\right) = x \quad if \quad -1 \le x \le 1$$

$$\sin^{-1}\left(\sin y\right) = \arcsin\left(\sin y\right) = y \quad if \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$





Example

Find the exact value: $\sin\left(\sin^{-1}\frac{1}{2}\right)$, $\sin^{-1}\left(\sin\frac{\pi}{4}\right)$

Solution

$$\sin\left(\sin^{-1}\frac{1}{2}\right) = \frac{1}{2} \qquad \text{Since } -1 \le \frac{1}{2} \le 1$$

$$\sin^{-1}\left(\sin\frac{\pi}{4}\right) = \frac{\pi}{4} \qquad \text{Since } -\frac{\pi}{2} \le \frac{\pi}{4} \le \frac{\pi}{2}$$

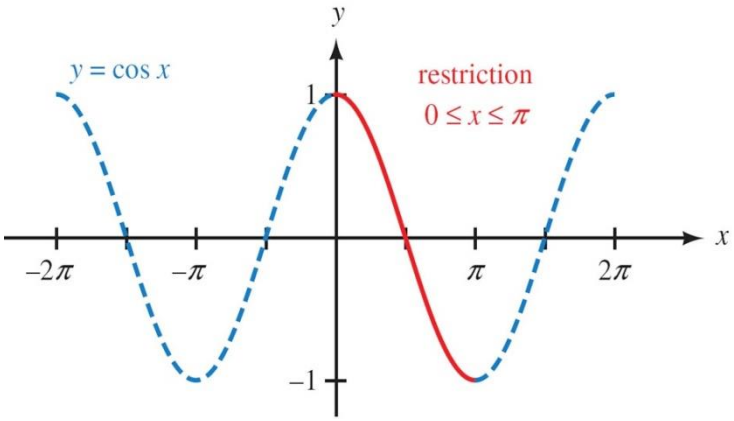
The Inverse *Cosine* Function

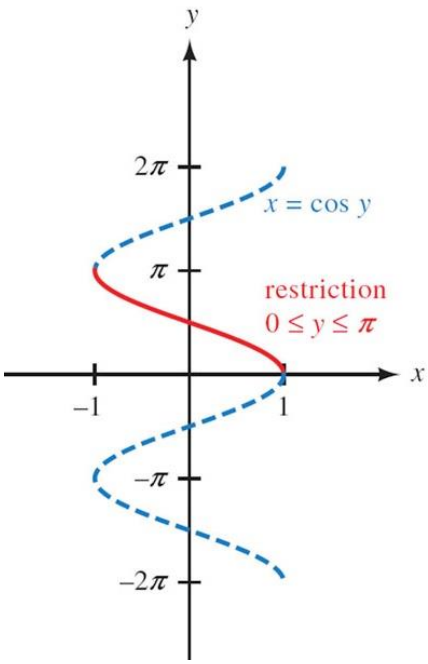
Definition

The inverse cosine function, denoted by $\cos^{-1}$, is defined by

$$y = \cos^{-1} x \quad \textit{iff} \quad x = \cos y \quad \textit{for } 0 \le y \le \pi \quad \textit{and} \quad -1 \le x \le 1$$

Notation	***Meaning***
$y = \cos^{-1} x$ *or* $y = \arccos x$	$x = \cos y$ *and* $0 \le y \le \pi$





Properties of $\cos^{-1}$

$$\cos\left(\cos^{-1} x\right) = \cos\left(\arccos x\right) = x \quad \textit{if} \quad -1 \le x \le 1$$

$$\cos^{-1}\left(\cos y\right) = \arccos\left(\cos y\right) = y \quad \textit{if} \quad 0 \le y \le \pi$$

Example

Find the exact value: $\cos\left(\cos^{-1}(-0.5)\right)$, $\cos^{-1}\left(\cos(3.14)\right)$, $\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)$

Solution

$\cos\left(\cos^{-1}(-0.5)\right) = -0.5$ Since $-1 \le -0.5 \le 1$

$\cos^{-1}\left(\cos(3.14)\right) = 3.14$ Since $0 \le 3.14 \le \pi$

$\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

Example

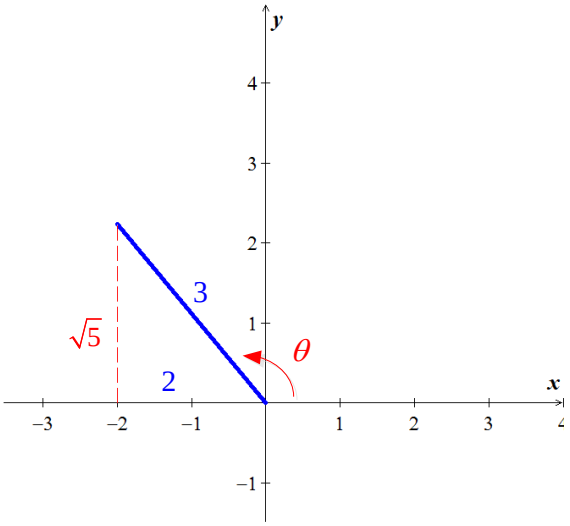
Find the exact value of $\sin\left[\arccos\left(-\frac{2}{3}\right)\right]$

Solution

$$\theta = \arccos\left(-\frac{2}{3}\right) \Rightarrow \cos\theta = -\frac{2}{3} \qquad 0 \le \theta \le \pi$$

$$y = \sqrt{3^2 - 2^2} = \sqrt{5}$$

$$\sin\left[\arccos\left(-\frac{2}{3}\right)\right] = \sin\theta = \frac{\sqrt{5}}{3}$$



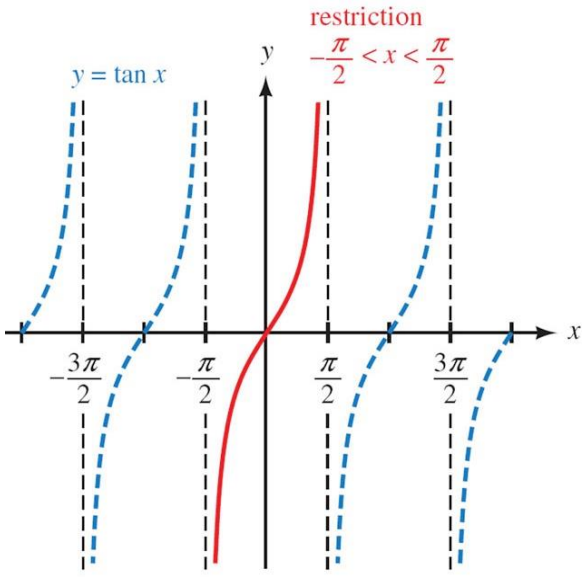
The Inverse *Tangent* Function

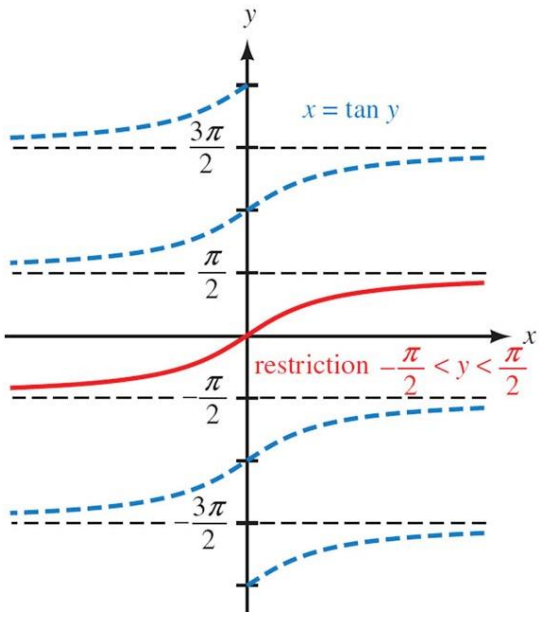
Definition

The inverse cosine function, denoted by $\tan^{-1}$, is defined by

$y = \tan^{-1} x$ *iff* $x = \tan y$ for any real number x and for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$y = \tan^{-1} x$ *or* $y = \arctan x$





Properties of $\tan^{-1}$

$$\tan\left(\tan^{-1} x\right) = \tan\left(\arctan x\right) = x \quad \textit{for} \quad \textit{every } x$$

$$\tan^{-1}\left(\tan y\right) = \arctan\left(\tan y\right) = y \quad \textit{if} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

Example

Find the exact value: $\tan\left(\tan^{-1}(1000)\right)$, $\tan^{-1}\left(\tan\frac{\pi}{4}\right)$, $\arctan(\tan\pi)$

Solution

$\tan\left(\tan^{-1}1000\right)=1000$

$\tan^{-1}\left(\tan\frac{\pi}{4}\right)=\frac{\pi}{4}$ Since $-\frac{\pi}{2}\le\frac{\pi}{4}\le\frac{\pi}{2}$

$\arctan(\tan\pi)=\arctan(0)=0$ $\therefore \pi>\frac{\pi}{2}$

Example

Evaluate in radians without using a calculator or tables.

a. $\sin^{-1}\frac{1}{2}$

$$-\frac{\pi}{2}\le angle\le\frac{\pi}{2}\Rightarrow\sin\frac{\pi}{6}=\frac{1}{2}$$

$$\sin^{-1}\frac{1}{2}=\frac{\pi}{6}$$

b. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

$$0<angle<\pi\Rightarrow\cos\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}$$

$$\arccos\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{6}$$

c. $\tan^{-1}(-1)$

$$-\frac{\pi}{2}<angle<\frac{\pi}{2}\Rightarrow\tan\left(-\frac{\pi}{4}\right)=-1$$

$$\tan^{-1}(-1)=-\frac{\pi}{4}$$

Example

Use a calculator to evaluate each expression to the nearest tenth of a degree

a. $\arcsin(0.5075)$

$$\arcsin(0.5075) = 30.5°$$

b. $\arcsin(-0.5075)$

$$\arcsin(-0.5075) = -30.5°$$

c. $\cos^{-1}(0.6428)$

$$\cos^{-1}(0.6428) = 50.0°$$

d. $\cos^{-1}(-0.6428)$

$$\cos^{-1}(-0.6428) = 130.0°$$

e. $\arctan(4.474)$

$$\arctan(4.474) = 77.4°$$

f. $\arctan(-4.474)$

$$\arctan(-4.474) = -77.4°$$

Example

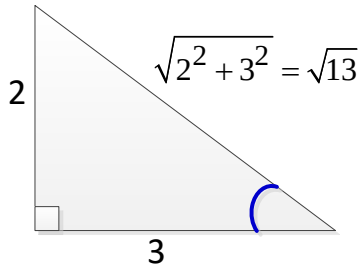
Find the exact value: $\sec\left(\arctan\frac{2}{3}\right)$

Solution

$$\alpha = \arctan\frac{2}{3} \rightarrow \tan\alpha = \frac{2}{3}$$

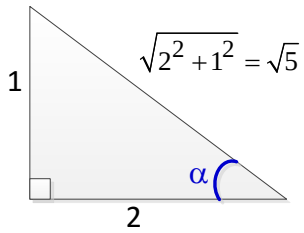
$$\sec\left(\arctan\frac{2}{3}\right) = \sec\alpha$$

$$= \frac{\sqrt{13}}{3}$$



Example

Find the exact value: $\sin\left(\arctan\frac{1}{2}-\arccos\frac{4}{5}\right)$



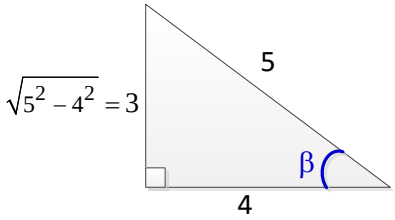
Solution

$\alpha=\arctan\frac{1}{2}$ $\qquad$ $\beta=\arccos\frac{4}{5}$

$\tan\alpha=\frac{1}{2}$ $\qquad$ $\cos\beta=\frac{4}{5}$

$\sin\alpha=\frac{1}{\sqrt{5}}$ $\qquad$ $\sin\beta=\frac{3}{5}$

$\cos\alpha=\frac{2}{\sqrt{5}}$



$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$$=\frac{1}{\sqrt{5}}\frac{4}{5}-\frac{2}{\sqrt{5}}\frac{3}{5}$$

$$=\frac{4}{5\sqrt{5}}-\frac{6}{5\sqrt{5}}$$

$$=-\frac{2}{5\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}}$$

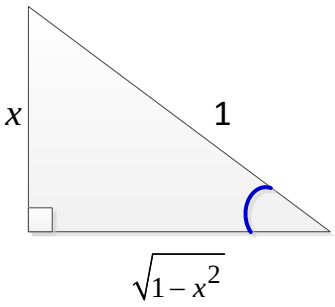
$$=-\frac{2\sqrt{5}}{25}$$

Example

If $-1\le x\le 1$, rewrite $\cos\left(\sin^{-1}x\right)$ as an algebraic expression in x.

Solution

$\alpha=\sin^{-1}x\rightarrow\sin\alpha=x=\frac{x}{1}$



$\cos\left(\sin^{-1}x\right)=\cos\alpha=\frac{\sqrt{1-x^2}}{1}=\sqrt{1-x^2}$

Exercises *Section* 3.5 – Inverse Trigonometric Functions

Find the exact value of the expression whenever it is defined

1. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

2. $\arccos\left(\frac{\sqrt{2}}{2}\right)$

3. $\arctan\left(-\frac{\sqrt{3}}{3}\right)$

4. $\sin\left[\arcsin\left(-\frac{3}{10}\right)\right]$

5. $\tan\left[\arctan(14)\right]$

6. $\sin\left[\sin^{-1}\left(\frac{2}{3}\right)\right]$

7. $\cos^{-1}\left[\cos\left(\frac{5\pi}{6}\right)\right]$

8. $\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]$

9. $\arcsin\left[\sin\left(-\frac{\pi}{2}\right)\right]$

10. $\arccos\left[\cos(0)\right]$

11. $\arctan\left[\tan\left(-\frac{\pi}{4}\right)\right]$

12. $\sin\left[\arcsin\left(\frac{1}{2}\right)+\arccos 0\right]$

13. $\cos\left[\arctan\left(-\frac{3}{4}\right)-\arcsin\frac{4}{5}\right]$

14. $\tan\left[\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)\right]$

15. $\sin\left[2\arccos\left(-\frac{3}{5}\right)\right]$

16. $\cos\left[2\sin^{-1}\left(\frac{15}{17}\right)\right]$

17. $\tan\left[2\tan^{-1}\left(\frac{3}{4}\right)\right]$

18. $\cos\left[\frac{1}{2}\tan^{-1}\left(\frac{8}{15}\right)\right]$

Evaluate without using a calculator

19. $\cos\left(\cos^{-1}\frac{3}{5}\right)$

20. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

21. $\tan\left(\cos^{-1}\frac{3}{5}\right)$

22. $\sin\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$

23. $\cos\left(\sin^{-1}\frac{1}{2}\right)$

24. $\sin\left(\sin^{-1}\frac{3}{5}\right)$

25. $\cos\left(\tan^{-1}\frac{3}{4}\right)$

26. $\tan\left(\sin^{-1}\frac{3}{5}\right)$

27. $\sec\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$

28. $\cot\left(\tan^{-1}\frac{1}{2}\right)$

Write an equivalent expression that involves x only for

29. $\cos\left(\cos^{-1}x\right)$

30. $\tan\left(\cos^{-1}x\right)$

31. $\csc\left(\sin^{-1}\frac{1}{x}\right)$

32. $\sin\left(\tan^{-1}x\right);\quad x>0$

33. $\sec\left(\sin^{-1}\frac{x}{\sqrt{x^2+4}}\right)\quad x>0$

34. $\cot\left(\sin^{-1}\frac{\sqrt{x^2-9}}{x}\right)\quad x>0$

35. $\sin\left(2\sin^{-1}x\right)\quad x>0$

36. $\cos\left(2\tan^{-1}x\right),\quad x>0$

37. $\cos\left(\frac{1}{2}\arccos x\right),\quad x>0$

38. $\tan\left(\frac{1}{2}\cos^{-1}\frac{1}{x}\right),\quad x>0$

39. $\sec\left(\tan^{-1}\frac{2}{\sqrt{x^2-4}}\right)\quad x>0$

40. $\sec\left(\sin^{-1}\frac{\sqrt{x^2-25}}{x}\right)\quad x>0$

41. $\sin\left(\cos^{-1}\frac{x}{\sqrt{x^2+4}}\right)\quad x>0$

Sketch the graph of the equation:

42. $y = \sin^{-1} 2x$ **43.** $y = \sin^{-1}(x-2) + \frac{\pi}{2}$ **44.** $y = \cos^{-1} \frac{1}{2} x$

45. Evaluate $\sin\left(\tan^{-1} \frac{3}{4}\right)$ without using a calculator

46. Evaluate $\sin(\cos^{-1} x)$ as an equivalent expression in x only