Solution   *Section* 1.6 – Surface Area

Exercise

Find the lateral (side) surface area of the cone generated by revolving the line segment $y=\frac{x}{2}, \quad 0 \le x \le 4$, about the x-axis. Check your answer with the geometry formula

$$Lateral\ surface\ area = \frac{1}{2} \times base\ circumference \times slant\ height$$

Solution

$$\frac{dy}{dx} = \frac{1}{2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{4}} = \frac{\sqrt{5}}{2}$$

$$S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

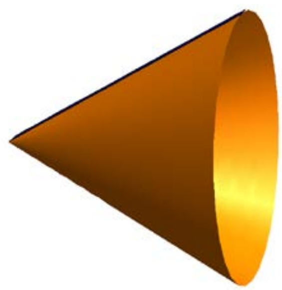
$$= 2\pi \int_0^4 \left(\frac{x}{2}\right)\frac{\sqrt{5}}{2}\,dx$$

$$= \frac{\pi\sqrt{5}}{2}\int_0^4 x\,dx$$

$$= \frac{\pi\sqrt{5}}{2}\frac{1}{2}x^2\Big|_0^4$$

$$= \frac{\pi\sqrt{5}}{4}\left(4^2 - 0\right)$$

$$= 4\pi\sqrt{5}\ \ unit^2$$



$$base\ circumference = 2\pi r = 2\pi(2) = 4\pi$$

$$slant\ height = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$$

$$Lateral\ surface\ area = \frac{1}{2} \times base\ circumference \times slant\ height$$

$$= \frac{1}{2}\times(4\pi)\times\left(2\sqrt{5}\right)$$

$$= 4\pi\sqrt{5}$$

Exercise

Find the lateral surface area of the cone generated by revolving the line segment $y=\frac{x}{2}, \quad 0\le x\le 4$, about the y-axis. Check your answer with the geometry formula

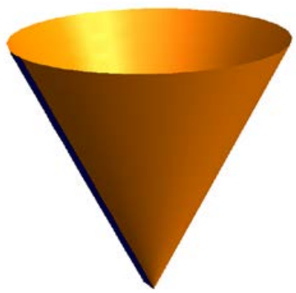
$$\text{Lateral surface area} = \frac{1}{2}\times \text{base circumference}\times \text{slant height}$$

Solution

$$y=\frac{x}{2} \quad \Rightarrow x=2y \to \begin{cases} x=0 & \to y=0 \\ x=4 & \to y=2 \end{cases}$$

$$\frac{dx}{dy}=2$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+4}=\sqrt{5}$$



$$S=2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$=2\pi\int_0^2 2y\sqrt{5}\,dy$$

$$=4\pi\sqrt{5}\int_0^2 y\,dy$$

$$=4\pi\sqrt{5}\,\frac{1}{2}y^2\Big|_0^2$$

$$=2\pi\sqrt{5}(4-0)$$

$$=8\pi\sqrt{5}\ \text{unit}^2$$

$$\text{base circumference}=2\pi(4)=8\pi$$

$$\text{slant height}=\sqrt{4^2+2^2}=\sqrt{20}=2\sqrt{5}$$

$$\text{Lateral surface area}=\frac{1}{2}\times\text{base circumference}\times\text{slant height}$$

$$=\frac{1}{2}\times(8\pi)\times\left(2\sqrt{5}\right)$$

$$=8\pi\sqrt{5}$$

Exercise

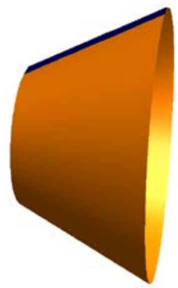
Find the lateral surface area of the cone frustum generated by revolving the line segment $y=\frac{x}{2}+\frac{1}{2}, \quad 1\le x\le 3$, about the *x*-axis. Check your answer with the geometry formula

$$Frustum\ surface\ area=\pi\left(r_1+r_2\right)\times slant\ height$$

Solution

$$\frac{dy}{dx}=\frac{1}{2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{1}{4}}=\frac{\sqrt{5}}{2}$$



$$\begin{aligned}
S&=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\\
&=2\pi\int_1^3\left(\frac{x}{2}+\frac{1}{2}\right)\left(\frac{\sqrt{5}}{2}\right)dx\\
&=\pi\frac{\sqrt{5}}{2}\int_1^3(x+1)\,dx\\
&=\pi\frac{\sqrt{5}}{2}\left[\frac{1}{2}x^2+x\right]_1^3\\
&=\pi\frac{\sqrt{5}}{2}\left[\frac{1}{2}(3)^2+(3)-\left(\frac{1}{2}(1)^2+(1)\right)\right]\\
&=\pi\frac{\sqrt{5}}{2}\left[\frac{9}{2}+3-\frac{3}{2}\right]\\
&=\pi\frac{\sqrt{5}}{2}(6)\\
&=3\pi\sqrt{5}\ unit^2
\end{aligned}$$

$$r_1=\frac{1}{2}+\frac{1}{2}=1 \qquad r_2=\frac{3}{2}+\frac{1}{2}=2$$

$$slant\ height=\sqrt{(2-1)^2+(3-1)^2}=\sqrt{5}$$

$$\begin{aligned}
Frustum\ surface\ area&=\pi\left(r_1+r_2\right)\times slant\ height\\
&=\pi(1+2)\sqrt{5}\\
&=3\pi\sqrt{5}
\end{aligned}$$

Exercise

Find the lateral surface area of the cone frustum generated by revolving the line segment $y = \frac{x}{2} + \frac{1}{2}, \quad 1 \le x \le 3$, about the *y*-axis. Check your answer with the geometry formula

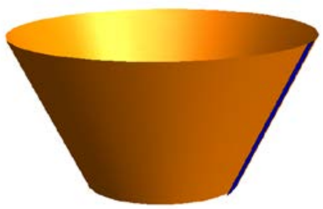
$$Frustum\ surface\ area = \pi\left(r_1 + r_2\right) \times slant\ height$$

Solution

$$y = \frac{x}{2} + \frac{1}{2} \quad \rightarrow 2y = x + 1 \Rightarrow \boxed{x = 2y - 1}$$

$$\frac{dx}{dy} = 2$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1+4} = \sqrt{5}$$



$$S = 2\pi \int_1^2 (2y - 1)\left(\sqrt{5}\right) dy$$

$$= 2\pi\sqrt{5} \int_1^2 (2y-1)\,dy$$

$$= 2\pi\sqrt{5}\left[y^2 - y\right]_1^2$$

$$= 2\pi\sqrt{5}\left[(2)^2 - 2 - \left(1^2 - 1\right)\right]$$

$$= 4\pi\sqrt{5}\ unit^2$$

$$r_1 = 1 \quad r_2 = 3$$

$$slant\ height = \sqrt{(2-1)^2 + (3-1)^2} = \sqrt{5}$$

$$Frustum\ surface\ area = \pi\left(r_1 + r_2\right) \times slant\ height$$

$$= \pi(1+3)\sqrt{5}$$

$$= 4\pi\sqrt{5}$$

Exercise

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve $y = \frac{1}{3}x^3$ about the *x-axis*

Solution

$$y = \frac{1}{3}x^3 \quad \rightarrow \quad y' = x^2$$

$$\sqrt{1+(y')^2} = \sqrt{1+x^4}$$

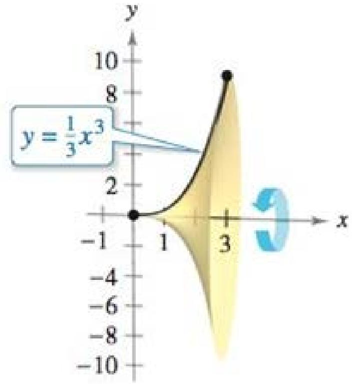
$$S = 2\pi\int_0^3 \frac{1}{3}x^3\sqrt{1+x^4}\ dx \qquad\qquad S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \frac{\pi}{6}\int_0^3 \left(1+x^4\right)^{1/2}\ d\left(1+x^4\right)$$

$$= \frac{\pi}{9}\left(1+x^4\right)^{3/2}\Big|_0^3$$

$$= \frac{\pi}{9}\left((82)^{3/2}-1\right)$$

$$= \frac{\pi}{9}\left(82\sqrt{82}-1\right)$$



Exercise

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve $y = 2\sqrt{x}$ about the *x-axis*

Solution

$$y = 2\sqrt{x} \ \rightarrow\ y' = \frac{1}{\sqrt{x}}$$

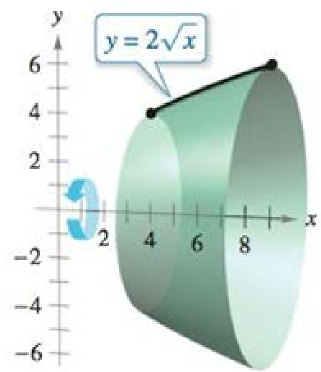
$$\sqrt{1+(y')^2} = \sqrt{1+\frac{1}{x}}$$

$$S = 2\pi\int_4^9 2\sqrt{x}\sqrt{\frac{x+1}{x}}\ dx \qquad\qquad S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= 4\pi\int_4^9 (1+x)^{1/2}\ d(1+x)$$

$$= \frac{8}{3}\pi(1+x)^{3/2}\Big|_4^9$$

$$= \frac{8}{3}\pi\left(10^{3/2}-5^{3/2}\right) \qquad \approx 171.285$$



Exercise

Find the area of the surface generated by $y = \frac{x^3}{9}, \quad 0 \le x \le 2, \quad x - axis$

Solution

$$\frac{dy}{dx} = \frac{1}{3}x^2$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{9}x^4} = \frac{1}{3}\sqrt{9+x^4}$$

$$S = 2\pi\int_0^2 \frac{x^3}{9}\frac{1}{3}\sqrt{9+x^4}\,dx \qquad S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \frac{2\pi}{27}\int_0^2 x^3\sqrt{9+x^4}\,dx \qquad u = 9+x^4 \quad \to du = 4x^3dx \Rightarrow \frac{1}{4}du = x^3dx$$

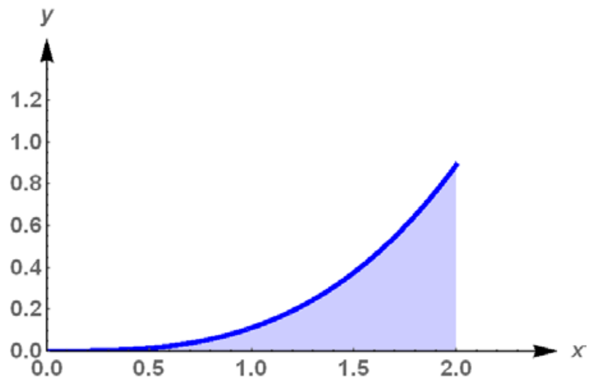
$$= \frac{2\pi}{27}\int_9^{25} u^{1/2}\left(\frac{1}{4}du\right) \qquad \to \begin{cases} x=2 & \to u=25 \\ x=0 & \to u=9 \end{cases}$$

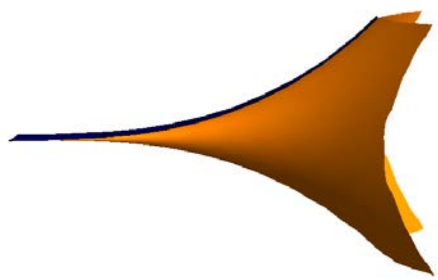
$$= \frac{\pi}{54}\int_9^{25} u^{1/2}du$$

$$= \frac{\pi}{54}\frac{2}{3}u^{3/2}\Big|_9^{25}$$

$$= \frac{\pi}{81}\left(25^{3/2}-9^{3/2}\right)$$

$$\boxed{= \frac{98\pi}{81}\ unit^2}$$





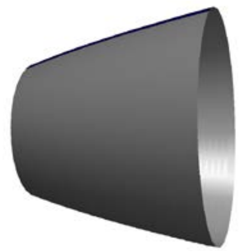
Exercise

Find the area of the surface generated by $y=\sqrt{x+1},\quad 1\le x\le 5,\quad x-axis$

Solution

$$y=\sqrt{x+1}=(x+1)^{1/2}$$

$$\frac{dy}{dx}=\frac{1}{2}(x+1)^{-1/2}$$



$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{1}{4}(x+1)^{-1}}$$

$$=\sqrt{1+\frac{1}{4(x+1)}}$$

$$=\sqrt{\frac{4x+4+1}{4(x+1)}}$$

$$=\frac{1}{2}\sqrt{\frac{4x+5}{x+1}}$$

$$S=2\pi\int_1^5\sqrt{x+1}\,\frac{1}{2}\frac{\sqrt{4x+5}}{\sqrt{x+1}}dx \qquad\qquad S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\pi\int_1^5\sqrt{4x+5}\,dx$$

$$=\frac{\pi}{4}\int_1^5(4x+5)^{1/2}\,d(4x+5)$$

$$=\frac{\pi}{6}(4x+5)^{3/2}\Big|_1^5$$

$$=\frac{\pi}{6}\left(25^{3/2}-9^{3/2}\right)$$

$$=\frac{\pi}{6}(98)$$

$$=\frac{49\pi}{3}\ unit^2$$

Exercise

Find the area of the surface generated by $y=\sqrt{2x-x^2}$, $\quad 0.5\le x\le 1.5, \quad x-axis$

Solution

$$\frac{dy}{dx}=\frac{1}{2}\left(2x-x^2\right)^{-1/2}(2-2x)=(1-x)\left(2x-x^2\right)^{-1/2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+(1-x)^2\left(2x-x^2\right)^{-1}}$$

$$=\sqrt{1+\frac{1-2x+x^2}{2x-x^2}}$$

$$=\sqrt{\frac{2x-x^2+1-2x+x^2}{2x-x^2}}$$

$$=\sqrt{\frac{1}{2x-x^2}}$$

$$=\frac{1}{\sqrt{2x-x^2}}$$

$$S=2\pi\int_{.5}^{1.5}\sqrt{2x-x^2}\,\frac{1}{\sqrt{2x-x^2}}dx \qquad\qquad S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=2\pi\int_{.5}^{1.5}dx$$

$$=2\pi x\Big|_{.5}^{1.5} \qquad =2\pi(1.5-.5)$$

$$=2\pi\ unit^2$$

Exercise

Find the area of the surface generated by $y=3x+4$, $\quad 0\le x\le 6, \quad revolved\ about\ x-axis$

Solution

$$y'=3$$

$$S=2\pi\int_0^6(3x+4)\sqrt{1+9}\,dx \qquad\qquad S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=2\pi\sqrt{10}\left(\frac{3}{2}x^2+4x\right)_0^6$$

$$=2\pi\sqrt{10}(54+24)$$

$$=156\pi\sqrt{10}\ \ unit^2$$

Exercise

Find the area of the surface generated by $y = 12 - 3x, \quad 1 \le x \le 3, \quad$ *revolved about* $x-axis$

Solution

$$y' = -3$$

$$S = 2\pi \int_1^3 (12 - 3x)\sqrt{1+9}\ dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= 2\pi\sqrt{10}\left(12x - \frac{3}{2}x^2\right)\Big|_1^3$$

$$= 2\pi\sqrt{10}\left(36 - \frac{27}{2} - 12 + \frac{3}{2}\right)$$

$$= 24\pi\sqrt{10}\ \ unit^2$$

Exercise

Find the area of the surface generated by $y = 8\sqrt{x}, \quad 9 \le x \le 20, \quad$ *revolved about* $x-axis$

Solution

$$y' = \frac{4}{\sqrt{x}}$$

$$S = 2\pi \int_9^{20} 8\sqrt{x}\sqrt{1+\frac{16}{x}}\ dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= 16\pi \int_9^{20} \sqrt{x}\frac{\sqrt{x+16}}{\sqrt{x}}\ dx$$

$$= 16\pi \int_9^{20} (x+16)^{1/2}\ d(x+16)$$

$$= \frac{32\pi}{3}(x+16)^{3/2}\Big|_9^{20}$$

$$= \frac{32\pi}{3}\left((36)^{3/2} - (25)^{3/2}\right)$$

$$= \frac{32\pi}{3}(216 - 125)$$

$$= \frac{2912\pi}{3}\ \ unit^2$$

Exercise

Find the area of the surface generated by $y = x^3, \quad 0 \le x \le 1, \quad$ *revolved about* $x-axis$

Solution

$$y' = 3x^2$$

$$S = 2\pi \int_0^1 x^3 \sqrt{1+9x^4}\ dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \frac{\pi}{18}\int_0^1 \left(1+9x^4\right)^{1/2}\ d\left(1+9x^4\right)$$

$$= \frac{\pi}{27}\left(1+9x^4\right)^{3/2}\Big|_0^1$$

$$= \frac{\pi}{27}\left((10)^{3/2}-1\right)$$

$$= \frac{\pi}{27}\left(10\sqrt{10}-1\right)\ unit^2$$

Exercise

Find the area of the surface generated by $y = x^{3/2} - \frac{1}{3}x^{1/2}, \quad 1 \le x \le 2, \quad revolved\ \ about\ \ x-axis$

Solution

$$y' = \frac{3}{2}x^{1/2} - \frac{1}{6}x^{-1/2} = \frac{1}{2}\left(3\sqrt{x} - \frac{1}{3\sqrt{x}}\right) = \frac{9x-1}{6\sqrt{x}}$$

$$S = 2\pi \int_1^2 \left(x^{3/2} - \frac{1}{3}x^{1/2}\right)\sqrt{1+\frac{(9x-1)^2}{36x}}\ dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \frac{2}{3}\pi \int_1^2 \left(3x^{3/2} - x^{1/2}\right)\frac{\sqrt{36x+81x^2-18x+1}}{6\sqrt{x}}\ dx$$

$$= \frac{\pi}{9}\int_1^2 (3x-1)\sqrt{81x^2+18x+1}\ dx$$

$$= \frac{\pi}{9}\int_1^2 (3x-1)\sqrt{(9x+1)^2}\ dx$$

$$= \frac{\pi}{9}\int_1^2 (3x-1)(9x+1)\ dx$$

$$= \frac{\pi}{9}\int_1^2 \left(27x^2-6x-1\right)\ dx$$

$$= \frac{\pi}{9}\left(9x^3-3x^2-x\right)\Big|_1^2 = \frac{\pi}{9}(72-12-2-9+3+1)$$

$$= \frac{53\pi}{9}\ unit^2$$

Exercise

Find the area of the surface generated by $y=\sqrt{4x+6}, \quad 0\le x\le 5, \quad revolved\ about\ x-axis$

Solution

$$y'=\frac{2}{\sqrt{4x+6}}$$

$$S=2\pi\int_0^5 \sqrt{4x+6}\sqrt{1+\frac{4}{4x+6}}\ dx \qquad\qquad S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$=2\pi\int_0^5 \sqrt{4x+6}\sqrt{\frac{4x+6+4}{4x+6}}\ dx$$

$$=2\pi\int_0^5 (4x+10)^{1/2}\ dx$$

$$=\frac{\pi}{2}\int_0^5 (4x+10)^{1/2}\ d(4x+10)$$

$$=\frac{\pi}{3}(4x+10)^{3/2}\Big|_0^5$$

$$=\frac{\pi}{3}\left(30^{3/2}-10^{3/2}\right)$$

$$=\frac{\pi}{3}\left(30\sqrt{30}-10\sqrt{10}\right)$$

$$=\frac{10\pi\sqrt{10}}{3}\left(3\sqrt{3}-1\right)\ \ unit^2$$

Exercise

Find the area of the surface generated by $y=\frac{1}{4}\left(e^{2x}+e^{-2x}\right), \quad -2\le x\le 2, \quad revolved\ about\ x-axis$

Solution

$$y'=\frac{1}{2}\left(e^{2x}-e^{-2x}\right)$$

$$S=2\pi\int_{-2}^{2}\frac{1}{4}\left(e^{2x}+e^{-2x}\right)\sqrt{1+\frac{1}{4}\left(e^{2x}-e^{-2x}\right)^2}\ dx \qquad\qquad S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$=\frac{\pi}{2}\int_{-2}^{2}\left(e^{2x}+e^{-2x}\right)\frac{1}{2}\sqrt{4+e^{4x}-2+e^{-4x}}\ dx$$

$$=\frac{\pi}{4}\int_{-2}^{2}\left(e^{2x}+e^{-2x}\right)\sqrt{e^{4x}+2+e^{-4x}}\ dx$$

$$=\frac{\pi}{4}\int_{-2}^{2}\left(e^{2x}+e^{-2x}\right)\sqrt{\left(e^{2x}+e^{-2x}\right)^2}\ dx$$

$$=\frac{\pi}{4}\int_{-2}^{2}\left(e^{2x}+e^{-2x}\right)^2\ dx$$

$$=\frac{\pi}{4}\int_{-2}^{2}\left(e^{4x}+2+e^{-4x}\right)\ dx$$

$$=\frac{\pi}{4}\left(\frac{1}{4}e^{4x}+2x-\frac{1}{4}e^{-4x}\right)\Big|_{-2}^{2}$$

$$=\frac{\pi}{4}\left(\frac{1}{4}e^{8}+4-\frac{1}{4}e^{-8}-\frac{1}{4}e^{-8}+4+\frac{1}{4}e^{8}\right)$$

$$=\frac{\pi}{4}\left(\frac{1}{2}e^{8}+8-\frac{1}{2}e^{-8}\right)$$

$$\boxed{=\frac{\pi}{8}\left(e^{8}+16-e^{-8}\right)\ \ unit^2}$$

Exercise

Find the area of the surface generated by $y=\frac{1}{8}x^4+\frac{1}{4x^2},\quad 1\le x\le 2,\quad revolved\ about\ x-axis$

Solution

$$y'=\frac{1}{2}x^3-\frac{1}{2x^3}=\frac{1}{2}\frac{x^6-1}{x^3}$$

$$S=2\pi\int_{1}^{2}\left(\frac{1}{8}x^4+\frac{1}{4x^2}\right)\sqrt{1+\left(\frac{x^6-1}{2x^3}\right)^2}\ dx$$

$$S=2\pi\int_{a}^{b}y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$=\frac{\pi}{4}\int_{1}^{2}\left(\frac{x^6+2}{x^2}\right)\sqrt{1+\frac{x^{12}-2x^6+1}{4x^6}}\ dx$$

$$=\frac{\pi}{4}\int_{1}^{2}\left(\frac{x^6+2}{x^2}\right)\sqrt{\frac{x^{12}+2x^6+1}{4x^6}}\ dx$$

$$=\frac{\pi}{4}\int_{1}^{2}\left(\frac{x^6+2}{x^2}\right)\frac{\sqrt{\left(x^6+1\right)^2}}{2x^3}\ dx$$

$$=\frac{\pi}{4}\int_{1}^{2}\frac{\left(x^6+2\right)\left(x^6+1\right)}{2x^5}\ dx$$

$$=\frac{\pi}{4}\int_{1}^{2}\frac{x^{12}+3x^6+2}{2x^5}\ dx$$

$$= \frac{\pi}{8}\int_1^2 \left(x^7 + 3x + 2x^{-5}\right)\, dx$$

$$= \frac{\pi}{8}\left(\frac{1}{8}x^8 + \frac{3}{2}x^2 - \frac{1}{2}x^{-4}\right)\Big|_1^2$$

$$= \frac{\pi}{8}\left(32 + 6 - \frac{1}{32} - \frac{1}{8} - \frac{3}{2} + \frac{1}{2}\right)$$

$$= \frac{\pi}{8}\left(37 - \frac{5}{32}\right)$$

$$= \frac{1179\pi}{256} \quad unit^2$$

Exercise

Find the area of the surface generated by $y = \frac{1}{3}x^3 + \frac{1}{4x}, \quad \frac{1}{2} \le x \le 2, \quad revolved\ about\ x - axis$

Solution

$$y' = x^2 - \frac{1}{4x^2} = \frac{4x^4 - 1}{4x^2}$$

$$S = 2\pi\int_{1/2}^2 \left(\frac{1}{3}x^3 + \frac{1}{4x}\right)\sqrt{1 + \left(\frac{4x^4 - 1}{4x^2}\right)^2}\, dx \qquad S = 2\pi\int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi\int_{1/2}^2 \left(\frac{4x^4 + 3}{12x}\right)\sqrt{1 + \frac{16x^8 - 8x^4 + 1}{16x^4}}\, dx$$

$$= \frac{\pi}{6}\int_{1/2}^2 \left(\frac{4x^4 + 3}{x}\right)\sqrt{\frac{16x^8 + 8x^4 + 1}{16x^4}}\, dx$$

$$= \frac{\pi}{6}\int_{1/2}^2 \left(\frac{4x^4 + 3}{x}\right)\frac{\sqrt{\left(4x^4 + 1\right)^2}}{4x^2}\, dx$$

$$= \frac{\pi}{24}\int_{1/2}^2 \left(\frac{4x^4 + 3}{x^3}\right)\left(4x^4 + 1\right)\, dx$$

$$= \frac{\pi}{24}\int_{1/2}^2 \left(4x + 3x^{-3}\right)\left(4x^4 + 1\right)\, dx$$

$$= \frac{\pi}{24}\int_{1/2}^2 \left(16x^5 + 16x + 3x^{-3}\right)\, dx$$

$$= \frac{\pi}{24}\left(\frac{8}{3}x^6 + 8x^2 - \frac{3}{2}x^{-2}\right)\Big|_{1/2}^2$$

$$= \frac{\pi}{24}\left(\frac{512}{3} + 32 - \frac{3}{8} - \frac{1}{24} - 2 + 6\right)$$

$$= \frac{\pi}{24}\left(\frac{4086}{24} + 36\right)$$

$$= \frac{\pi}{24}\left(\frac{681}{4} + 36\right)$$

$$= \frac{\pi}{24}\left(\frac{825}{4}\right)$$

$$= \frac{275\pi}{32} \quad unit^2$$

Exercise

Find the area of the surface generated by $y = \sqrt{5x - x^2}, \quad 1 \le x \le 4, \quad revolved\ \ about\ \ x - axis$

Solution

$$y' = \frac{5 - 2x}{2\sqrt{5x - x^2}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(5 - 2x)^2}{4\left(5x - x^2\right)}$$

$$= \frac{20x - 4x^2 + 25 - 20.x + 4x^2}{4\left(5x - x^2\right)}$$

$$= \frac{25}{4\left(5x - x^2\right)}$$

$$S = 2\pi \int_1^4 \sqrt{5x - x^2}\sqrt{\frac{25}{4\left(5x - x^2\right)}}\, dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 5\pi \int_1^4 dx$$

$$= 5\pi x \Big|_1^4$$

$$= 15\pi \ \ unit^2$$

Exercise

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the x-$axis$ $\qquad y = \frac{1}{6}x^3 + \frac{1}{2x}, \quad 1 \le x \le 2$

Solution

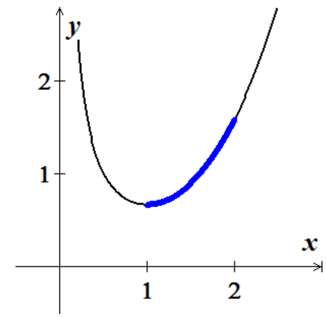
$$y' = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4x^4}}$$

$$= \sqrt{\frac{1}{4}x^4 + \frac{1}{2} + \frac{1}{4x^4}}$$

$$= \sqrt{\left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)^2}$$

$$= \frac{1}{2}x^2 + \frac{1}{2x^2}$$



$$S = 2\pi \int_1^2 \left(\frac{1}{6}x^3 + \frac{1}{2x}\right)\left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)dx \qquad S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_1^2 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3}\right)dx$$

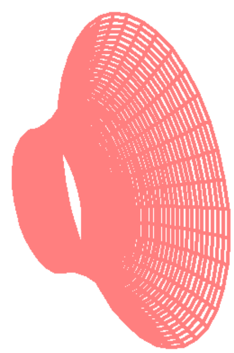
$$= 2\pi \left(\frac{1}{72}x^6 + \frac{1}{6}x^2 - \frac{1}{8x^2}\right)\Big|_1^2$$

$$= 2\pi\left(\frac{64}{72} + \frac{2}{3} - \frac{1}{32} - \frac{1}{72} - \frac{1}{6} + \frac{1}{8}\right)$$

$$= 2\pi\left(\frac{63}{72} + \frac{19}{32}\right)$$

$$= 2\pi\left(\frac{423}{288}\right)$$

$$= \frac{47\pi}{16}$$



Exercise

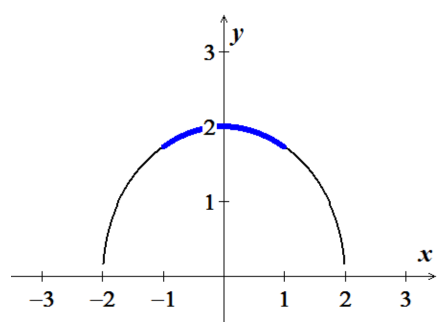
Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the *x-axis* $\qquad y = \sqrt{4-x^2}, \quad -1 \le x \le 1$

Solution

$$y' = \frac{-x}{\sqrt{4-x^2}}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{x^2}{4-x^2}}$$

$$= \sqrt{\frac{4}{4-x^2}}$$



$$S = 2\pi \int_{-1}^{1} \sqrt{4-x^2}\,\frac{2}{\sqrt{4-x^2}}dx \qquad S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 4\pi \int_{-1}^{1} dx$$

$$= 4\pi x \Big|_{-1}^{1}$$

$$\underline{= 8\pi}$$

Exercise

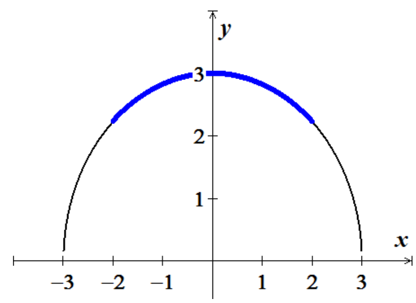
Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the *x-axis* $\quad y = \sqrt{9 - x^2}, \quad -2 \le x \le 2$

Solution

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{9 - x^2}}$$

$$= \sqrt{\frac{9}{9 - x^2}}$$

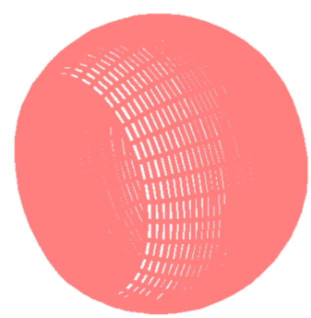


$$S = 2\pi \int_{-2}^{2} \sqrt{9 - x^2} \frac{3}{\sqrt{9 - x^2}} dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 6\pi \int_{-2}^{2} dx$$

$$= 6\pi x \Big|_{-2}^{2}$$

$$\underline{= 24\pi}$$



Exercise

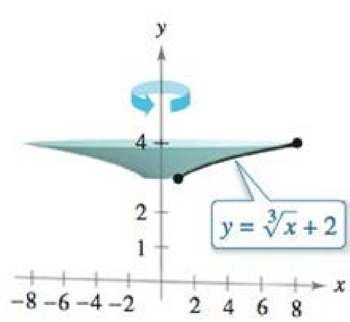
Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the *y-axis*

Solution

$$y' = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{1}{9x^{4/3}}}$$

$$= \frac{\sqrt{9x^{4/3} + 1}}{3x^{2/3}}$$



$$S = 2\pi \int_1^8 x \frac{\sqrt{9x^{4/3}+1}}{3x^{2/3}} dx \qquad S = 2\pi \int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \frac{2}{3}\pi \int_1^8 x^{1/3}\sqrt{9x^{4/3}+1}\ dx$$

$$= \frac{\pi}{18}\int_1^8 \left(9x^{4/3}+1\right)^{1/2}\ d\left(9x^{4/3}+1\right)$$

$$= \frac{\pi}{27}\left(9x^{4/3}+1\right)^{3/2}\Big|_1^8$$

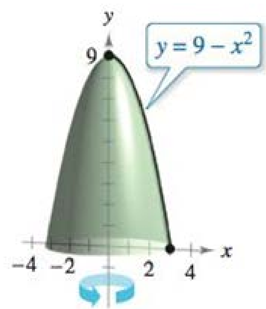
$$= \frac{\pi}{27}\left(\left(72(8)^{1/3}+1\right)^{3/2} - 10^{3/2}\right)$$

$$= \frac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right)$$

Exercise

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the *y-axis*

Solution



$$y' = -2x$$

$$\sqrt{1+(y')^2} = \sqrt{1+4x^2}$$

$$S = 2\pi\int_0^3 x\sqrt{1+4x^2}\ dx \qquad S = 2\pi \int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \frac{\pi}{4}\int_0^3 \left(1+4x^2\right)^{1/2}\ d\left(1+4x^2\right)$$

$$= \frac{\pi}{6}\left(1+4x^2\right)^{3/2}\Big|_0^3$$

$$= \frac{\pi}{6}\left(37\sqrt{37}-1\right)$$

Exercise

Find the area of the surface generated by $y = (3x)^{1/3}; \quad 0 \le x \le \frac{8}{3}$ *about y-axis*

Solution

$$3x = y^3 \to x = \frac{1}{3}y^3 \quad \Rightarrow x' = y^2 \qquad \begin{cases} x = 0 & \to y = 0 \\ x = \frac{8}{3} & \to y = \left(3\frac{8}{3}\right)^{1/3} = 2 \end{cases}$$

$$S = 2\pi \int_0^2 \frac{1}{3} y^3 \sqrt{1+y^4}\ dy \qquad\qquad S = 2\pi \int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

$$= \frac{\pi}{6} \int_0^2 \left(1+y^4\right)^{1/2}\ d\left(1+y^4\right)$$

$$= \frac{\pi}{9} \left(1+y^4\right)^{3/2} \Big|_0^2$$

$$= \frac{\pi}{9} \left((17)^{3/2} - 1\right)$$

$$= \frac{\pi}{9} \left(17\sqrt{17} - 1\right)$$

Exercise

Find the area of the surface generated of the curve $y = 4x - 1$ between the points $(1,\ 3)$ and $(4,\ 15)$ *about y-axis*

Solution

$$y = 4x - 1 \rightarrow x = \frac{1}{4}(y+1) \quad \Rightarrow x' = \frac{1}{4}$$

$$S = 2\pi \int_3^{15} \frac{1}{4}(y+1)\sqrt{1+\frac{1}{16}}\ dy \qquad\qquad S = 2\pi \int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

$$= \frac{\pi}{2} \int_3^{15} (y+1)\sqrt{\frac{17}{16}}\ dy$$

$$= \frac{\pi\sqrt{17}}{8} \left(\frac{1}{2}y^2 + y\right) \Big|_3^{15}$$

$$= \frac{\pi\sqrt{17}}{8} \left(\frac{225}{2} + 15 - \frac{9}{2} - 3\right)$$

$$= \frac{\pi\sqrt{17}}{8}(120)$$

$$= 15\pi\sqrt{17}$$

Exercise

Find the area of the surface generated of the curve $y = \frac{1}{2}\ln\left(2x + \sqrt{4x^2 - 1}\right)$ between the points $\left(\frac{1}{2},\ 0\right)$ and $\left(\frac{17}{16},\ \ln 2\right)$ *about y-axis*

Solution

$$2y = \ln\left(2x + \sqrt{4x^2 - 1}\right) \rightarrow \left(2x + \sqrt{4x^2 - 1}\right)^2 = \left(e^{2y}\right)^2$$

$$4x^2 + 4x\sqrt{4x^2 - 1} + 4x^2 - 1 = e^{4y}$$

$$4x\left(2x + \sqrt{4x^2 - 1}\right) = e^{4y} + 1 \qquad\qquad 2x + \sqrt{4x^2 - 1} = e^{2y}$$

$$4x\left(e^{2y}\right) = e^{4y} + 1$$

$$x = \frac{e^{4y} + 1}{4e^{2y}} = \frac{1}{4}\left(e^{2y} + e^{-2y}\right)$$

$$x' = \frac{1}{2}\left(e^{2y} - e^{-2y}\right)$$

$$S = 2\pi\int_0^{\ln 2} \frac{1}{4}\left(e^{2y} + e^{-2y}\right)\sqrt{1 + \frac{1}{4}\left(e^{2y} - e^{-2y}\right)^2}\; dy \qquad\qquad S = 2\pi\int_c^d x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\; dy$$

$$= \frac{\pi}{4}\int_0^{\ln 2} \left(e^{2y} + e^{-2y}\right)\sqrt{4 + e^{4y} - 2 + e^{-4y}}\; dy$$

$$= \frac{\pi}{4}\int_0^{\ln 2} \left(e^{2y} + e^{-2y}\right)\sqrt{\left(e^{2y} + e^{-2y}\right)^2}\; dy$$

$$= \frac{\pi}{4}\int_0^{\ln 2} \left(e^{2y} + e^{-2y}\right)^2\; dy$$

$$= \frac{\pi}{4}\int_0^{\ln 2} \left(e^{4y} + 2 + e^{-4y}\right)\; dy$$

$$= \frac{\pi}{4}\left(\frac{1}{4}e^{4y} + 2y - \frac{1}{4}e^{-4y}\right)\Bigg|_0^{\ln 2}$$

$$= \frac{\pi}{4}\left(\frac{1}{4}e^{4\ln 2} + 2\ln 2 - \frac{1}{4}e^{-4\ln 2} - \frac{1}{4} + \frac{1}{4}\right)$$

$$= \frac{\pi}{4}\left(\frac{1}{4}e^{\ln 2^4} + 2\ln 2 - \frac{1}{4}e^{\ln 2^{-4}}\right)$$

$$= \frac{\pi}{4}\left(\frac{1}{4}2^4 + 2\ln 2 - \frac{1}{4}2^{-4}\right)$$

$$= \frac{\pi}{4}\left(4 + 2\ln 2 - \frac{1}{64}\right)$$

$$= \frac{\pi}{4}\left(\frac{255}{64} + 2\ln 2\right)\; unit^2$$

Exercise

Find the area of the surface generated by $x=\sqrt{12y-y^2};\quad 2\le y\le 10\quad about\ y\text{-}axis$

Solution

$$x'=\frac{6-y}{\sqrt{12y-y^2}}$$

$$S=2\pi\int_2^{10}\sqrt{12y-y^2}\sqrt{1+\frac{(6-y)^2}{12y-y^2}}\,dy \qquad S=2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$=2\pi\int_2^{10}\sqrt{12y-y^2+36-12y+y^2}\,dy$$

$$=12\pi\int_2^{10}dy$$

$$=12\pi y\Big|_2^{10}$$

$$=96\pi\ \ unit^2$$

Exercise

Find the area of the surface generated by $x=4y^{3/2}-\frac{1}{12}y^{1/2};\quad 1\le y\le 4\quad about\ y\text{-}axis$

Solution

$$x'=6y^{1/2}-\frac{1}{24\sqrt{y}}=\frac{144y-1}{24\sqrt{y}}$$

$$S=2\pi\int_1^4\left(4y^{3/2}-\frac{1}{12}y^{1/2}\right)\sqrt{1+\frac{(144y-1)^2}{576y}}\,dy \qquad S=2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$=2\pi\int_1^4\left(4y^{3/2}-\frac{1}{12}y^{1/2}\right)\sqrt{\frac{576y+(144y)^2-288y+1}{576y}}\,dy$$

$$=\frac{\pi}{12}\int_1^4\left(4y^{3/2}-\frac{1}{12}y^{1/2}\right)\frac{1}{\sqrt{y}}\sqrt{(144y+1)^2}\,dy$$

$$=\frac{\pi}{144}\int_1^4(48y-1)(144y+1)\,dy$$

$$=\frac{\pi}{144}\int_1^4\left(6{,}912y^2-96y-1\right)dy$$

$$=\frac{\pi}{144}\left(2304y^3-48y^2-y\right)\Big|_1^4$$

$$= \frac{\pi}{144}(147,456 - 768 - 4 - 2304 + 48 + 1)$$

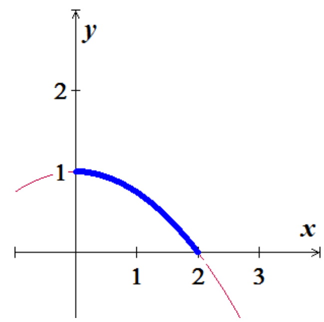
$$= \frac{144,429\pi}{144}$$

$$= \frac{48,143\ \pi}{48}\ \ unit^2$$

Exercise

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the *y-axis* $\quad y = 1 - \frac{1}{4}x^2, \quad 0 \le x \le 2$

Solution



$$y' = -\frac{1}{2}x$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{4}}$$

$$= \frac{1}{2}\sqrt{4 + x^2}$$

$$S = 2\pi \int_0^2 x \frac{\sqrt{4 + x^2}}{2}\ dx \qquad S = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \frac{\pi}{2}\int_0^2 \left(4 + x^2\right)^{1/2}\ d\left(4 + x^2\right)$$

$$= \frac{\pi}{3}\left(4 + x^2\right)^{3/2}\Big|_0^2$$

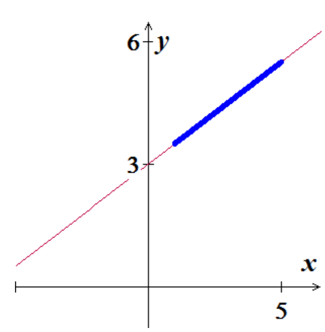
$$= \frac{\pi}{3}\left(8^{3/2} - 4^{3/2}\right)$$

$$= \frac{\pi}{3}\left(16\sqrt{2} - 8\right) \quad \approx 15.318\ \ unit^2$$

Exercise

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the *y-axis* $\quad y = \frac{1}{2}x + 3, \quad 1 \le x \le 5$

Solution



$$y' = \frac{1}{2}$$

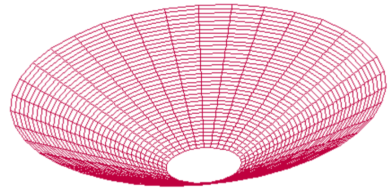
$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$$

$$S = \pi\sqrt{5}\int_1^5 x\ dx \qquad S = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \pi\sqrt{5}\left(\frac{1}{2}x^2\right)\Big|_1^5$$

$$= \frac{\sqrt{5}}{2}\pi(25-1)$$

$$= 12\pi\sqrt{5} \quad unit^2$$



Exercise

A right circular cone is generated by revolving the region bounded by $y = \frac{3}{4}x$, $y = 3$, and $x = 0$ about the y-*axis*. Find the lateral surface area of the cone.

Solution

$$y' = \frac{3}{4}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{9}{16}} = \frac{5}{4}$$

$$y = 3 = \frac{3}{4}x \;\Rightarrow\; x = 4$$

$$S = \frac{5\pi}{2}\int_0^4 x\, dx \qquad\qquad S = 2\pi\int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \frac{5\pi}{4}x^2\Big|_0^4$$

$$= 20\pi \quad unit^2$$

Exercise

A right circular cone is generated by revolving the region bounded by $y = \frac{h}{r}x$, $y = h$, and $x = 0$ about the y-*axis*. Verify that the lateral surface area of the cone is $S = \pi r\sqrt{r^2+h^2}$

Solution

$$y' = \frac{h}{r}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{h^2}{r^2}}$$

$$= \frac{\sqrt{r^2+h^2}}{r}$$

$$y = h = \frac{h}{r}x \;\Rightarrow\; x = r$$

$$S = 2\pi \int_0^r x \frac{\sqrt{r^2 + h^2}}{r}\, dx \qquad\qquad S = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \frac{\pi\sqrt{r^2 + h^2}}{r}\left(x^2\right)\Big|_0^r$$

$$\boxed{= \pi r\sqrt{r^2 + h^2}}$$

Exercise

Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{9 - x^2}$, $0 \le x \le 2$, about the *y-axis*

Solution

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{9 - x^2}} = \frac{3}{\sqrt{9 - x^2}}$$

$$S = 2\pi \int_0^2 x \frac{3}{\sqrt{9 - x^2}}\, dx \qquad\qquad S = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= -3\pi \int_0^2 \left(9 - x^2\right)^{-1/2}\, d\left(9 - x^2\right)$$

$$= -6\pi\left(9 - x^2\right)^{1/2}\Big|_0^2$$

$$= -6\pi\left(\sqrt{5} - 3\right)$$

$$\boxed{= 6\pi\left(3 - \sqrt{5}\right)}$$

Exercise

Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{r^2 - x^2}$, $0 \le x \le a$, about the *y-axis*. Assume that $a < r$.

Solution

$$y' = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{r^2 - x^2}} = \frac{r}{\sqrt{r^2 - x^2}}$$

$$S = 2\pi \int_0^a x \frac{r}{\sqrt{r^2 - x^2}}\, dx \qquad S = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= -\pi r \int_0^a \left(r^2 - x^2\right)^{-1/2}\, d\left(r^2 - x^2\right)$$

$$= -2\pi r\sqrt{r^2 - x^2}\Big|_0^a$$

$$= -2\pi r\left(\sqrt{r^2 - a^2} - r\right)$$

$$= 2\pi r\left(r - \sqrt{r^2 - a^2}\right)$$

Exercise

Find the area of the surface generated by the curve $y = 1 + \sqrt{1 - x^2}$ between the points $(1,\ 1)$ and $\left(\frac{\sqrt{3}}{1},\ \frac{3}{2}\right)$ *about y-axis*

Solution

$$\left(\sqrt{1 - x^2}\right)^2 = (y - 1)^2 \quad \Rightarrow \quad 1 - x^2 = y^2 - 2y + 1$$

$$x = \sqrt{2y - y^2} \ \rightarrow\ x' = \frac{1 - y}{\sqrt{2y - y^2}}$$

$$S = 2\pi \int_1^{3/2} \sqrt{2y - y^2}\sqrt{1 + \frac{(1 - y)^2}{2y - y^2}}\, dy \qquad S = 2\pi \int_c^d x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 2\pi \int_1^{3/2} \sqrt{2y - y^2 + 1 - 2y + y^2}\, dy$$

$$= 2\pi \int_1^{3/2} dy$$

$$= 2\pi y\Big|_1^{3/2}$$

$$= \pi \ \ unit^2$$

Exercise

Find the area of the surface generated by $x = 2\sqrt{4-y} \quad 0 \le y \le \frac{15}{4}, \quad y - axis$

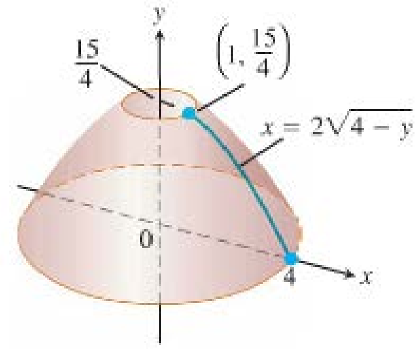
Solution

$$\frac{dy}{dx} = 2\frac{1}{2}(4-y)^{-1/2}(-1) = \frac{-1}{\sqrt{4-y}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{4-y}}$$

$$= \sqrt{\frac{4-y+1}{4-y}}$$

$$= \sqrt{\frac{5-y}{4-y}}$$



$$S = 2\pi \int_0^{15/4} 2\sqrt{4-y}\,\frac{\sqrt{5-y}}{\sqrt{4-y}}\,dy$$

$$= 4\pi \int_0^{15/4} \sqrt{5-y}\;dy \qquad d(5-y) = -dy$$

$$= 4\pi \int_0^{15/4} (5-y)^{1/2}\;\left(-d(5-y)\right)$$

$$= -4\pi\frac{2}{3}(5-y)^{3/2}\Big|_0^{15/4}$$

$$= -\frac{8\pi}{3}\left[\left(5-\frac{15}{4}\right)^{3/2} - (5-0)^{3/2}\right]$$

$$= -\frac{8\pi}{3}\left[\left(\frac{5}{4}\right)^{3/2} - 5^{3/2}\right]$$

$$= -\frac{8\pi}{3}\left[\frac{5\sqrt{5}}{8} - 5\sqrt{5}\right]$$

$$= -\frac{8\pi}{3}5\sqrt{5}\left(\frac{1}{8}-1\right)$$

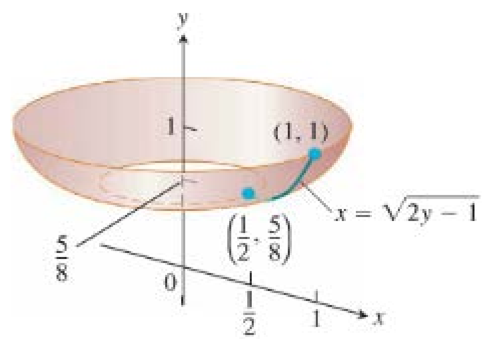
$$= -\frac{8\pi}{3}5\sqrt{5}\left(-\frac{7}{8}\right)$$

$$\boxed{= \frac{35\pi\sqrt{5}}{3}\ unit^2}$$

Exercise

Find the area of the surface generated by $x=\sqrt{2y-1}\quad \frac{5}{8}\le y\le 1,\quad y-axis$

Solution



$$\frac{dy}{dx}=\frac{1}{2}(2y-1)^{-1/2}(2)=\frac{1}{\sqrt{2y-1}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{1}{2y-1}}$$

$$=\sqrt{\frac{2y}{2y-1}}$$

$$S=2\pi\int_{5/8}^{1}\sqrt{2y-1}\frac{\sqrt{2y}}{\sqrt{2y-1}}dy$$

$$=2\pi\int_{5/8}^{1}\sqrt{2y}\ dy \qquad\qquad u=2y\to du=2dy$$

$$=2\pi\int_{5/8}^{1}u^{1/2}\ \left(\tfrac{1}{2}du\right)$$

$$=\pi\int_{5/8}^{1}u^{1/2}\ du$$

$$=\frac{2\pi}{3}(2y)^{3/2}\Big|_{5/8}^{1}$$

$$=\frac{2\pi}{3}\left((2)^{3/2}-\left(\tfrac{5}{4}\right)^{3/2}\right)$$

$$=\frac{2\pi}{3}\left(2\sqrt{2}-\frac{5\sqrt{5}}{8}\right)$$

$$=\frac{2\pi}{3}\left(\frac{16\sqrt{2}-5\sqrt{5}}{8}\right)$$

$$\boxed{=\frac{\pi}{12}\left(16\sqrt{2}-5\sqrt{5}\right)\ unit^2}$$

Exercise

$y=\frac{1}{3}\left(x^2+2\right)^{3/2},\quad 0\le x\le\sqrt{2};\quad y-axis$ (Hint: Express $ds=\sqrt{dx^2+dy^2}$ in terms of dx, and evaluate the integral $S=\int 2\pi x\ ds$ with appropriate limits.)

Solution

$$dy = \frac{1}{3}\frac{3}{2}\left(x^2+2\right)^{1/2}(2x)dx = x\sqrt{x^2+2}\ dx$$

$$ds = \sqrt{dx^2 + \left(x\sqrt{x^2+2}\ dx\right)^2}$$

$$= \sqrt{dx^2 + x^2\left(x^2+2\right)dx^2}$$

$$= \sqrt{1+x^4+2x^2}\,dx$$

$$= \sqrt{\left(1+x^2\right)^2}\,dx$$

$$= \left(1+x^2\right)dx$$

$$S = \int 2\pi x\ ds$$

$$= 2\pi\int_0^{\sqrt{2}} x\left(1+x^2\right)\ dx \qquad\qquad d\left(1+x^2\right) = 2xdx$$

$$= \pi\int_0^{\sqrt{2}} \left(1+x^2\right)\ d\left(1+x^2\right)$$

$$= \pi\frac{1}{2}u^2\Big|_1^3$$

$$= \frac{\pi}{2}\left(3^2-1^2\right)$$

$$= \frac{\pi}{2}(8)$$

$$= 4\pi\ unit^2$$

Exercise

Find the area of the surface generated by revolving the curve $x = \frac{1}{2}\left(e^y+e^{-y}\right), \quad 0 \le y \le \ln 2,$ about *y-axis*

Solution

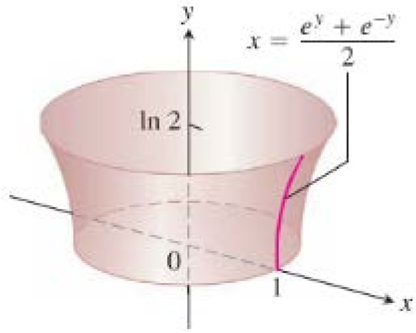
$$S = 2\pi\int_0^{\ln 2} \frac{1}{2}\left(e^y+e^{-y}\right)\sqrt{1+\left(\frac{e^y-e^{-y}}{2}\right)^2}dy$$

$$= \pi\int_0^{\ln 2} \left(e^y+e^{-y}\right)\sqrt{1+\frac{e^{2y}+e^{-2y}-2}{4}}dy$$

$$= \pi \int_0^{\ln 2} \left(e^{y} + e^{-y}\right)\sqrt{\frac{4 + e^{2y} + e^{-2y} - 2}{4}}\, dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^{y} + e^{-y}\right)\sqrt{e^{2y} + e^{-2y} + 2}\, dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^{y} + e^{-y}\right)\sqrt{\left(e^{y} + e^{-y}\right)^2}\, dy$$



$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^{y} + e^{-y}\right)^2 dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^{2y} + e^{-2y} + 2\right) dy$$

$$= \frac{\pi}{2}\left[\frac{1}{2}e^{2y} - \frac{1}{2}e^{-2y} + 2y\right]_0^{\ln 2}$$

$$= \frac{\pi}{2}\left[\left(\frac{1}{2}e^{2\ln 2} - \frac{1}{2}e^{-2\ln 2} + 2\ln 2\right) - \left(\frac{1}{2}e^{0} - \frac{1}{2}e^{0} + 0\right)\right]$$

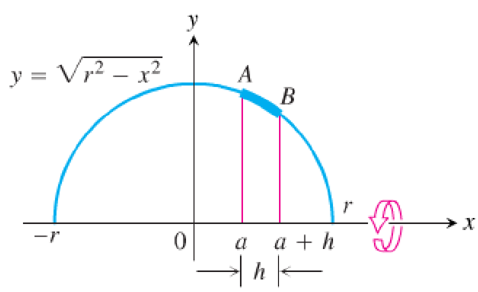
$$= \frac{\pi}{2}\left(\frac{1}{2}\cdot 4 - \frac{1}{2}\cdot\frac{1}{4} + 2\ln 2\right)$$

$$= \frac{\pi}{2}\left(\frac{15}{8} + 2\ln 2\right)\ unit^2$$

Exercise

Did you know that if you can cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle $y = \sqrt{r^2 - x^2}$ shown here is revolved about the x-axis to generate a sphere. Let $\boldsymbol{AB}$ be an arc of the semicircle that lies above an interval of length $\boldsymbol{h}$ on the x-axis. Show that the area swept out by $\boldsymbol{AB}$ does not depend on the location of the interval. (It does depend on the length of the interval.)

Solution



$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{r^2 - x^2}} = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{r^2 - x^2}}$$

$$= \sqrt{\frac{r^2}{r^2 - x^2}}$$

$$S = 2\pi \int_a^{a+h} \sqrt{r^2 - x^2}\, \frac{r}{\sqrt{r^2 - x^2}}\, dx$$

$$= 2\pi r \int_a^{a+h} dx$$

$$= 2\pi r x \Big|_a^{a+h}$$

$$= 2\pi r (a + h - a)$$

$$\underline{= 2\pi r h \ \ unit^2}$$

Example

The curved surface of a funnel is generated by revolving the graph of $y = f(x) = x^3 + \frac{1}{12x}$ on the interval $[1,\ 2]$ about the x-*axis*. Approximately what volume of paint is needed to cover the outside of the funnel with a layer of paint 0.05 *cm* thick? Assume that *x* and *y* measured in centimeters.

Solution

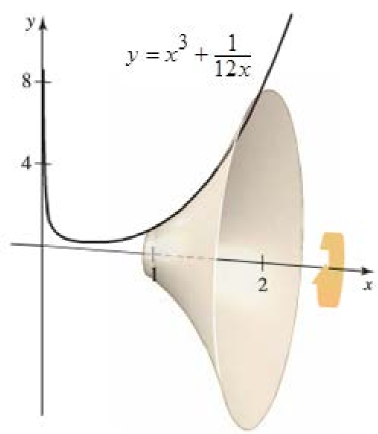
$$f'(x) = 3x^2 - \frac{1}{12x^2}$$

$$1 + f'(x)^2 = 1 + \left(3x^2 - \frac{1}{12x^2}\right)^2$$

5 $$= 1 + 9x^4 - \frac{1}{2} + \frac{1}{144x^4}$$

$$= 9x^4 + \frac{1}{2} + \frac{1}{144x^4}$$

$$= \left(3x^2 + \frac{1}{12x^2}\right)^2$$



$$S = 2\pi \int_1^2 \left(x^3 + \frac{1}{12x}\right) \sqrt{\left(3x^2 + \frac{1}{12x^2}\right)^2}\, dx$$

$$= 2\pi \int_1^2 \left(x^3 + \frac{1}{12x}\right)\left(3x^2 + \frac{1}{12x^2}\right) dx$$

$$= 2\pi \int_1^2 \left(3x^5 + \frac{x}{3} + \frac{1}{144} x^{-3}\right) dx$$

$$= 2\pi \left(\frac{1}{2}x^6 + \frac{1}{6}x^2 - \frac{1}{288}x^{-2}\right)\Big|_1^2$$

$$= 2\pi\left(32 + \frac{2}{3} - \frac{1}{1152} - \frac{1}{2} - \frac{1}{6} + \frac{1}{288}\right)$$

$$= 2\pi\left(\frac{36864 + 768 - 1 - 576 - 192 + 4}{1152}\right)$$

$$= \frac{12,289}{192}\pi \ \ cm^2$$

Because the paint layer is 0.05 *cm* thick, the approximate volume of paint needed is

$$= \left(\frac{12,289}{192}\pi \ cm^2\right)(0.05 \ \ cm) \ \approx 10.1 \ \ cm^3$$

Exercise

When the circle $x^2 + (y-a)^2 = r^2$ on the interval $[-r, \ r]$ is revolved about the *x-axis*, the result is the surface of a torus, where $0 < r < a$. Show that the surface area of the torus is $S = 4\pi^2 ar$.

Solution

$$x^2 + (y-a)^2 = r^2 \quad \Rightarrow \quad (y-a)^2 = r^2 - x^2$$

$$y = a \pm \sqrt{r^2 - x^2}$$

$$f(x) = a + \sqrt{r^2 - x^2}$$

$$1 + f'(x)^2 = 1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2$$

$$= 1 + \frac{x^2}{r^2 - x^2}$$

$$= \frac{r^2}{r^2 - x^2}$$

$$S_1 = 2\pi \int_{-r}^{r} \left(a + \sqrt{r^2 - x^2}\right) \frac{r}{\sqrt{r^2 - x^2}} \ dx$$

$$= 4\pi \int_0^r \left(\frac{ar}{\sqrt{r^2 - x^2}} + r\right) dx$$

$$= 4\pi \left[ar \sin^{-1}\left(\frac{x}{r}\right) + rx \right]_0^r$$

$$= 4\pi \left(ar\frac{\pi}{2} + r^2 \right)$$

$$= 2\pi^2 ar + 4\pi r^2$$

$$S_2 = 2\pi \int_{-r}^{r} \left(a - \sqrt{r^2 - x^2}\right) \frac{r}{\sqrt{r^2 - x^2}}\, dx$$

$$= 4\pi \int_0^r \left(\frac{ar}{\sqrt{r^2 - x^2}} - r \right) dx$$

$$= 4\pi \left[ar \sin^{-1}\left(\frac{x}{r}\right) - rx \right]_0^r$$

$$= 4\pi \left(ar\frac{\pi}{2} - r^2 \right)$$

$$= 2\pi^2 ar - 4\pi r^2$$

$$S = 2\pi^2 ar + 4\pi r^2 + 2\pi^2 ar - 4\pi r^2 \; = 4\pi^2 ar \;\; unit^2$$

Exercise

A 1.5–*mm* layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the curve $y = \sqrt{8x - x^2}$ on the interval $[1,\ 7]$ is revolved about the *x-axis* Assume *x* and *y* are in *meters*.

Solution

$$y' = \frac{4 - x}{\sqrt{8x - x^2}}$$

$$S = 2\pi \int_1^7 \sqrt{8x - x^2} \sqrt{1 + \frac{(4 - x)^2}{8x - x^2}}\, dx \qquad\qquad S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_1^7 \sqrt{8x - x^2} \frac{\sqrt{8x - x^2 + 16 - 8x + x^2}}{\sqrt{8x - x^2}}\, dx$$

$$= 2\pi \int_1^7 \sqrt{16}\, dx$$

$$= 8\pi x \Big|_1^7$$

$$= 48\pi \;\; m^2$$

The volume of paint required to cover the surface to a thickness 0.0015 *m* is

$$V = 48\pi(0.0015) \; \approx 0.226195 \;\; m^3 \qquad\qquad 1\, m^3 = 264.172052\ gal$$

$$= 0.226195 \times 264.172052 \; \approx 59.75 \;\; gal$$

Exercise

A 1.5–*mm* layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the upper portion of the circle $x^2 + y^2 = 100$ on the interval $[-8,\ 8]$ is revolved about the x-axis. Assume *x* and *y* are in *meters*.

Solution

$$y = \sqrt{100 - x^2} \quad \Rightarrow \quad y' = \frac{-x}{\sqrt{100 - x^2}}$$

$$S = 2\pi \int_{-8}^{8} \sqrt{100 - x^2} \sqrt{1 + \frac{x^2}{100 - x^2}}\ dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$= 2\pi \int_{-8}^{8} \sqrt{100 - x^2}\ \frac{\sqrt{100 - x^2 + x^2}}{\sqrt{100 - x^2}}\ dx$$

$$= 20\pi \int_{-8}^{8} dx$$

$$= 20\pi x \Big|_{-8}^{8}$$

$$= 320\pi\ \ m^2$$

The volume of paint required to cover the surface to a thickness 0.0015 *m* is

$$V = 320\pi(0.0015) \approx 1.507965\ \ m^3 \qquad\qquad 1\ m^3 = 264.172052\ gal$$

$$= 1.507965 \times 264.172052 \approx 398.36\ \ gal$$

Exercise

Suppose a sphere of radius *r* is sliced by two horizontal planes *h* units apart. Show that the surface area of the resulting zone on the sphere is $2\pi h$, independent of the location of the cutting planes.

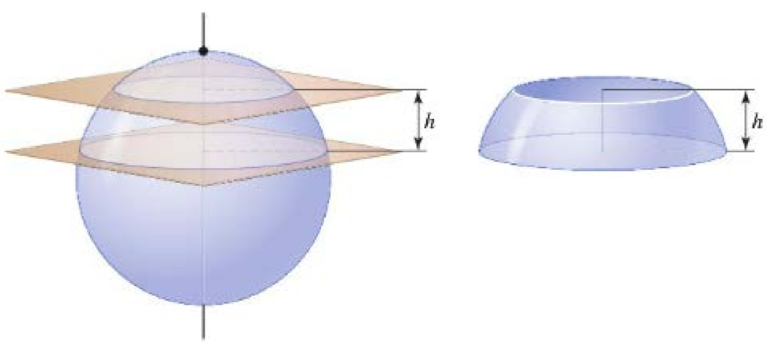
Solution

$$f(x) = \sqrt{r^2 - x^2}$$

$$1 + f'(x)^2 = 1 + \left(\frac{x}{\sqrt{r^2 - x^2}}\right)^2$$

$$= 1 + \frac{x^2}{r^2 - x^2}$$

$$= \frac{r^2}{r^2 - x^2}$$



$$S = 2\pi \int_a^{a+h} \sqrt{r^2 - x^2} \frac{r}{\sqrt{r^2 - x^2}} \, dx$$

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2} dx$$

$$= 2\pi r x \Big|_a^{a+h}$$

$$= 2\pi r (a + h - a)$$

$$= 2\pi rh \ \ unit^2$$

Exercise

An ornamental light bulb is designed by revolving the graph of $y = \frac{1}{3}x^{1/2} - x^{3/2}, \quad 0 \le x \le \frac{1}{3}$ about the x-*axis* , where x and y are mesured in *feet*. Find the surface area of the bulb and use the result to approximate the amount of glass needed to make the bulb.
(Assume that the glass is 0.015 *inch* thick)

Solution

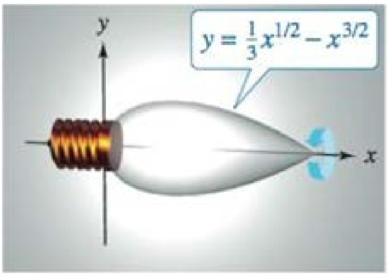
$$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{1}{36}x^{-1} - \frac{1}{2} + \frac{9}{4}x}$$

$$= \frac{1}{6}\sqrt{x^{-1} + 18 + 81x}$$

$$= \frac{1}{6}\sqrt{\left(x^{-1/2} + 9x^{1/2}\right)^2}$$

$$= \frac{1}{6}\left(x^{-1/2} + 9x^{1/2}\right)$$



$$S = 2\pi \frac{1}{6} \int_0^{1/3} \left(\frac{1}{3}x^{1/2} - x^{3/2}\right)\left(x^{-1/2} + 9x^{1/2}\right) \, dx$$

$$= \frac{\pi}{3} \int_0^{1/3} \left(\frac{1}{3} + 2x - 9x^2\right) \, dx$$

$$= \frac{\pi}{3}\left(\frac{1}{3}x + x^2 - 3x^3\right)\Big|_0^{1/3}$$

$$= \frac{\pi}{3}\left(\frac{1}{9} + \frac{1}{9} - \frac{1}{9}\right)$$

$$= \frac{\pi}{27} \quad \approx 0.1164 \ ft^2 \quad \approx 16.8 \ in^2$$

Amount of glass needed: $V = \frac{\pi}{2}\left(\frac{0.015}{12}\right) \approx 0.00015 \ ft^3 \ \ \approx 0.25 \ in^3$