CHAPTER 14 PARTIAL DERIVATIVES

14.1 FUNCTIONS OF SEVERAL VARIABLES

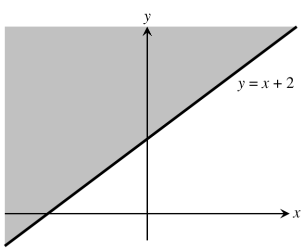
1. (a) $f(0,0)=0$ (b) $f(-1,1)=0$ (c) $f(2,3)=58$
   (d) $f(-3,-2)=33$

2. (a) $f\left(2,\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$ (b) $f\left(-3,\frac{\pi}{12}\right)=-\frac{1}{\sqrt{2}}$ (c) $f\left(\pi,\frac{1}{4}\right)=\frac{1}{\sqrt{2}}$
   (d) $f\left(-\frac{\pi}{2},-7\right)=-1$

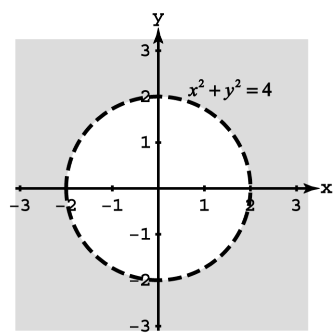
3. (a) $f(3,-1,2)=\frac{4}{5}$ (b) $f\left(1,\frac{1}{2},-\frac{1}{4}\right)=\frac{8}{5}$ (c) $f\left(0,-\frac{1}{3},0\right)=3$
   (d) $f(2,2,100)=0$

4. (a) $f(0,0,0)=7$ (b) $f(2,-3,6)=0$ (c) $f(-1,2,3)=\sqrt{35}$
   (d) $f\left(\frac{4}{\sqrt{2}},\frac{5}{\sqrt{2}},\frac{6}{\sqrt{2}}\right)=\sqrt{\frac{21}{2}}$

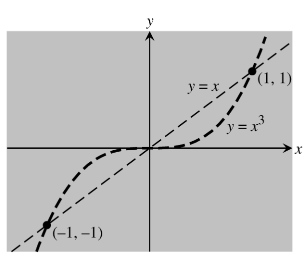
5. Domain: all points (x, y) on or above the line $y = x + 2$



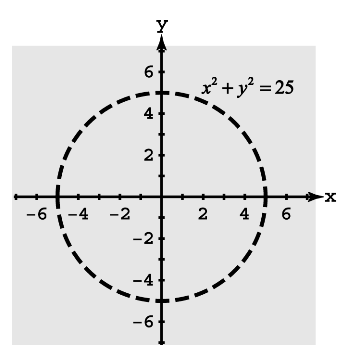
6. Domain: all points (x, y) outside the circle $x^2 + y^2 = 4$



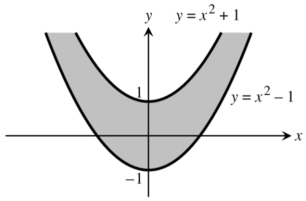
7. Domain: all points (x, y) not liying on the graph of $y = x$ or $y = x^3$



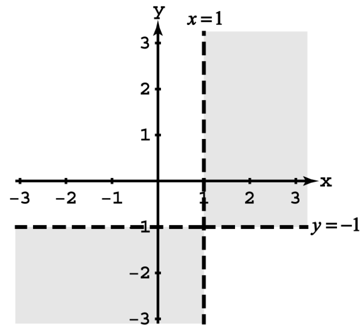
8. Domain: all points (x, y) not liying on the graph of $x^2 + y^2 = 25$



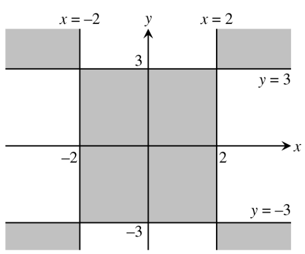
9. Domain: all points (x, y) satisfying $x^2 - 1 \le y \le x^2 + 1$



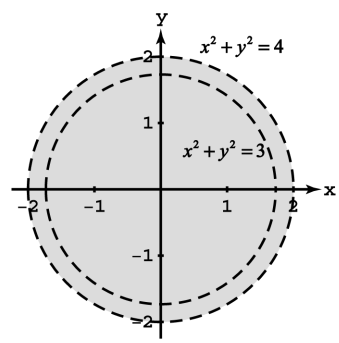
10. Domain: all points (x, y) satisfying $(x - 1)(y + 1) > 0$



11. Domain: all points (x, y) satisfying $(x - 2)(x + 2)(y - 3)(y + 3) \ge 0$



12. Domain: all points (x, y) inside the circle $x^2 + y^2 = 4$ such that $x^2 + y^2 \ne 3$



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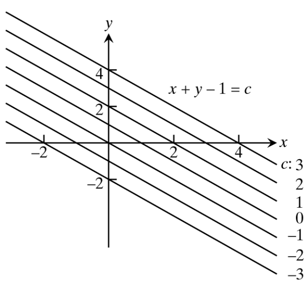

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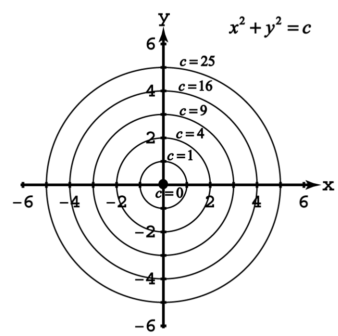

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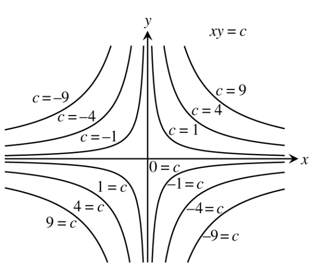

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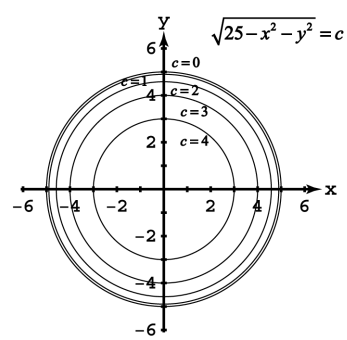


17. (a) Domain: all points in the xy-plane
(b) Range: all real numbers

(c) level curves are straight lines $y - x = c$ parallel to the line $y = x$
(d) no boundary points
(e) both open and closed
(f) unbounded

18. (a) Domain: set of all (x, y) so that $y - x \geq 0 \Rightarrow y \geq x$
(b) Range: $z \geq 0$
(c) level curves are straight lines of the form $y - x = c$ where $c \geq 0$
(d) boundary is $\sqrt{y - x} = 0 \Rightarrow y = x$, a straight line
(e) closed
(f) unbounded

19. (a) Domain: all points in the xy-plane
(b) Range: $z \geq 0$
(c) level curves: for $f(x, y) = 0$, the origin; for $f(x, y) = c > 0$, ellipses with center $(0, 0)$ and major and minor axes along the x- and y-axes, respectively
(d) no boundary points
(e) both open and closed
(f) unbounded

20. (a) Domain: all points in the xy-plane
(b) Range: all real numbers
(c) level curves: for $f(x, y) = 0$, the union of the lines $y = \pm x$; for $f(x, y) = c \neq 0$, hyperbolas centered at $(0, 0)$ with foci on the x-axis if $c > 0$ and on the y-axis if $c < 0$
(d) no boundary points
(e) both open and closed
(f) unbounded

21. (a) Domain: all points in the xy-plane
(b) Range: all real numbers
(c) level curves are hyperbolas with the x- and y-axes as asymptotes when $f(x, y) \neq 0$, and the x- and y-axes when $f(x, y) = 0$
(d) no boundary points
(e) both open and closed
(f) unbounded

22. (a) Domain: all $(x, y) \neq (0, y)$
(b) Range: all real numbers
(c) level curves: for $f(x, y) = 0$, the x-axis minus the origin; for $f(x, y) = c \neq 0$, the parabolas $y = c\,x^2$ minus the origin
(d) boundary is the line $x = 0$
(e) open
(f) unbounded

23. (a) Domain: all (x, y) satisfying $x^2 + y^2 < 16$
(b) Range: $z \geq \frac{1}{4}$
(c) level curves are circles centered at the origin with radii $r < 4$
(d) boundary is the circle $x^2 + y^2 = 16$

(e) open
(f) bounded

24. (a) Domain: all (x, y) satisfying $x^2 + y^2 \leq 9$
(b) Range: $0 \leq z \leq 3$
(c) level curves are circles centered at the origin with radii $r \leq 3$
(d) boundary is the circle $x^2 + y^2 = 9$
(e) closed
(f) bounded

25. (a) Domain: $(x, y) \neq (0, 0)$
(b) Range: all real numbers
(c) level curves are circles with center $(0, 0)$ and radii $r > 0$
(d) boundary is the single point $(0, 0)$
(e) open
(f) unbounded

26. (a) Domain: all points in the xy-plane
(b) Range: $0 < z \leq 1$
(c) level curves are the origin itself and the circles with center $(0, 0)$ and radii $r > 0$
(d) no boundary points
(e) both open and closed
(f) unbounded

27. (a) Domain: all (x, y) satisfying $-1 \leq y - x \leq 1$
(b) Range: $-\frac{\pi}{2} \leq z \leq \frac{\pi}{2}$
(c) level curves are straight lines of the form $y - x = c$ where $-1 \leq c \leq 1$
(d) boundary is the two straight lines $y = 1 + x$ and $y = -1 + x$
(e) closed
(f) unbounded

28. (a) Domain: all (x, y), $x \neq 0$
(b) Range: $-\frac{\pi}{2} < z < \frac{\pi}{2}$
(c) level curves are the straight lines of the form $y = c\,x$, c any real number and $x \neq 0$
(d) boundary is the line $x = 0$
(e) open
(f) unbounded

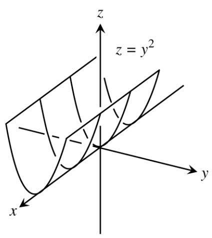
29. (a) Domain: all points (x, y) outside the circle $x^2 + y^2 = 1$
(b) Range: all reals
(c) Circles centered ar the origin with radii $r > 1$
(d) Boundary: the cricle $x^2 + y^2 = 1$
(e) open
(f) unbounded

30. (a) Domain: all points (x, y) inside the circle $x^2 + y^2 = 9$
(b) Range: $z < \ln 9$
(c) Circles centered ar the origin with radii $r < 9$
(d) Boundary: the cricle $x^2 + y^2 = 9$

(e) open
(f) bounded

31. f
32. e
33. a
34. c
35. d
36. b

37. (a)



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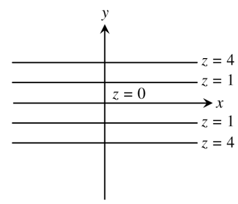

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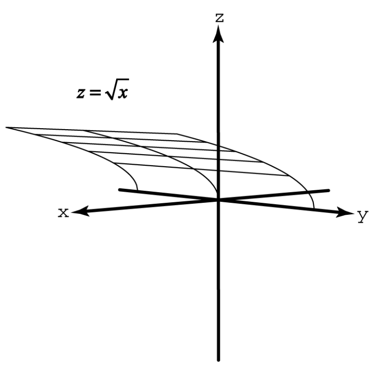

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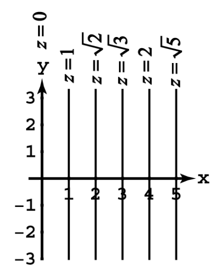
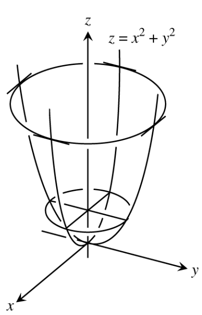


39. (a)



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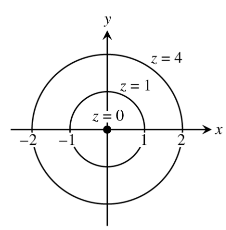
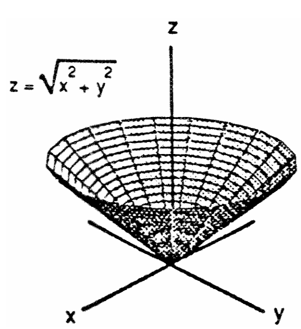


40. (a)



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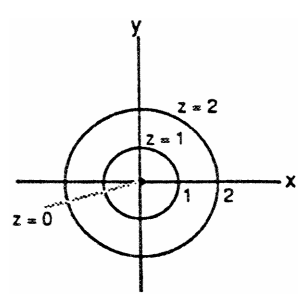

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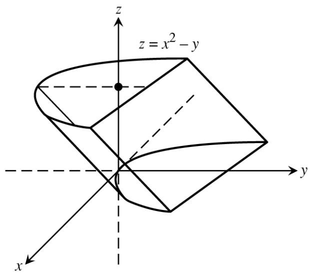

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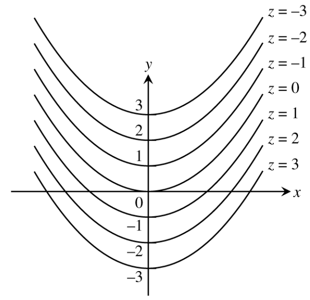

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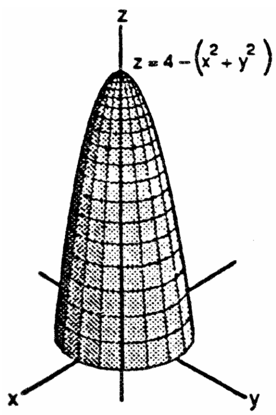

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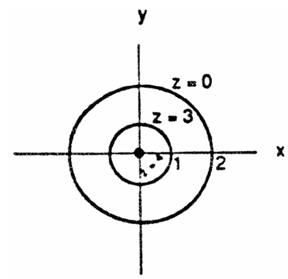
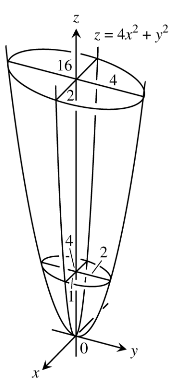


43. (a)



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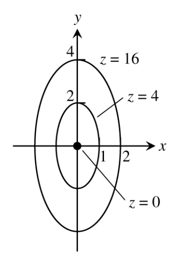

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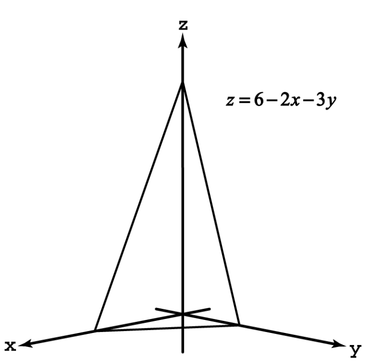

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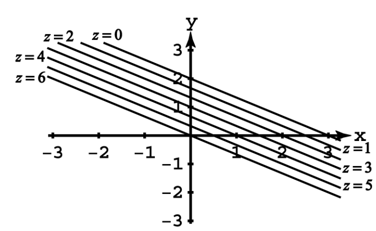

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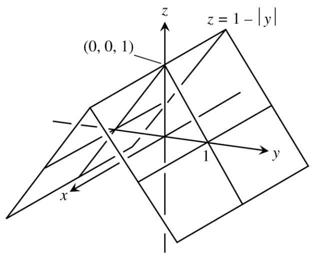

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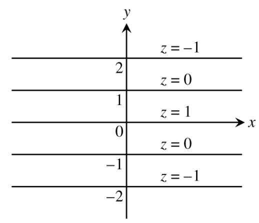

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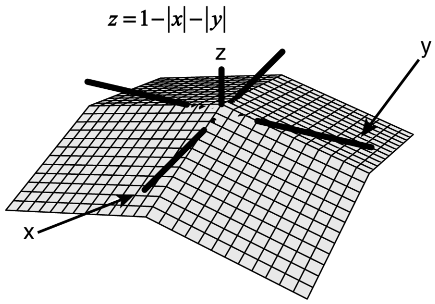

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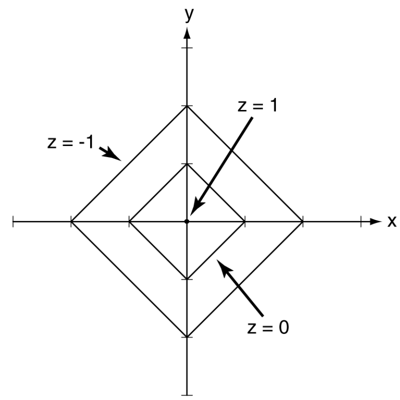

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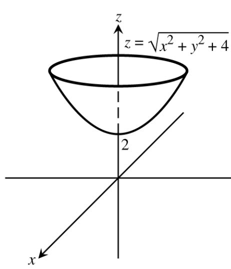
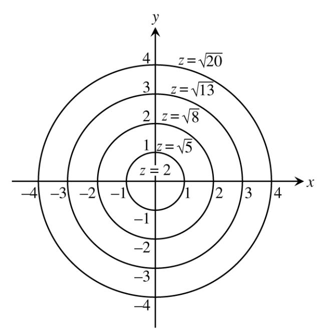
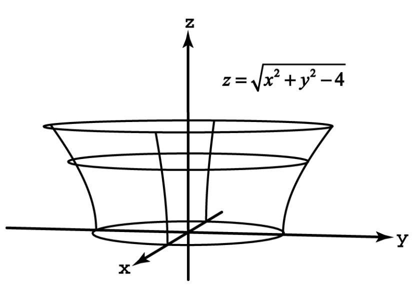


(b)



48. (a)



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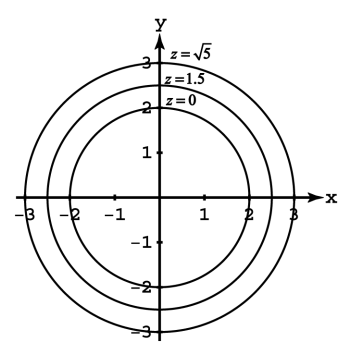


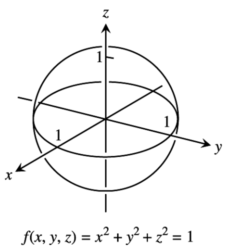
49. $f(x, y) = 16 - x^2 - y^2$ and $\left(2\sqrt{2}, \sqrt{2}\right) \Rightarrow z = 16 - \left(2\sqrt{2}\right)^2 - \left(\sqrt{2}\right)^2 = 6 \Rightarrow 6 = 16 - x^2 - y^2 \Rightarrow x^2 + y^2 = 10$

50. $f(x, y) = \sqrt{x^2 - 1}$ and $(1, 0) \Rightarrow z = \sqrt{1^2 - 1} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = 1$ or $x = -1$

51. $f(x, y) = \sqrt{x + y^2 - 3}$ and $(3, -1) \Rightarrow z = \sqrt{3 + (-1)^2 - 3} = 1 \Rightarrow x + y^2 - 3 = 1 \Rightarrow x + y^2 = 4$

52. $f(x, y) = \frac{2y - x}{x + y + 1}$ and $(-1, 1) \Rightarrow z = \frac{2(1) - (-1)}{(-1) + 1 + 1} = 3 \Rightarrow 3 = \frac{2y - x}{x + y + 1} \Rightarrow y = -4x - 3$

53.



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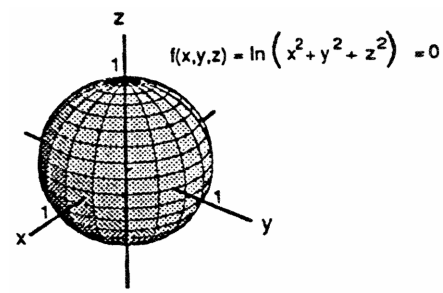

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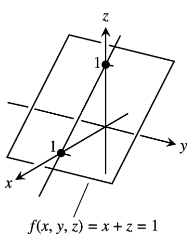

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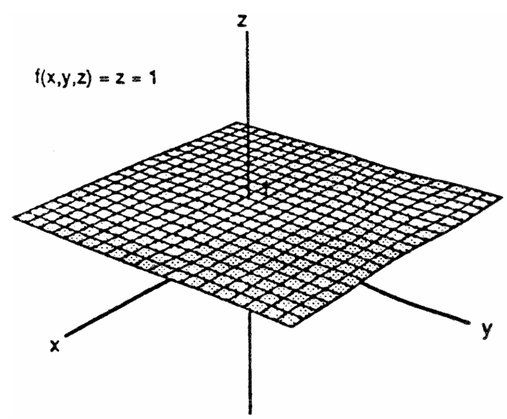

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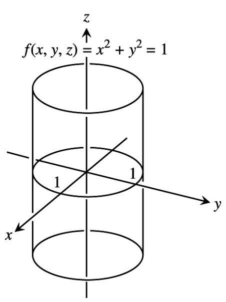

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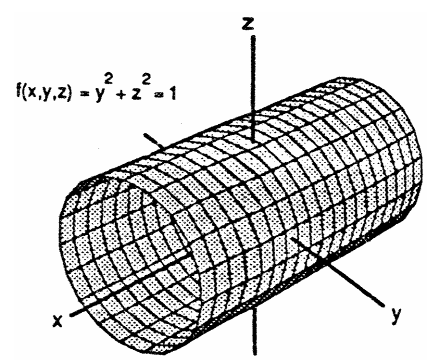

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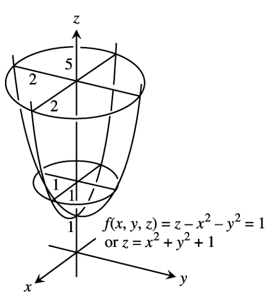

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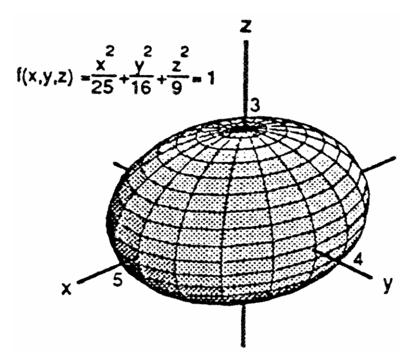


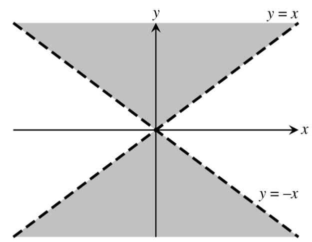
61. $f(x,y,z) = \sqrt{x-y} - \ln z$ at $(3,-1,1) \Rightarrow w = \sqrt{x-y} - \ln z$; at $(3,-1,1) \Rightarrow w = \sqrt{3-(-1)} - \ln 1 = 2$
$\Rightarrow \sqrt{x-y} - \ln z = 2$

62. $f(x,y,z) = \ln(x^2+y+z^2)$ at $(-1,2,1) \Rightarrow w = \ln(x^2+y+z^2)$; at $(-1,2,1) \Rightarrow w = \ln(1+2+1) = \ln 4$
$\Rightarrow \ln 4 = \ln(x^2+y+z^2) \Rightarrow x^2+y+z^2 = 4$

63. $g(x,y,z) = \sqrt{x^2+y^2+z^2}$ at $\left(1,-1,\sqrt{2}\right) \Rightarrow w = \sqrt{x^2+y^2+z^2}$; at $\left(1,-1,\sqrt{2}\right) \Rightarrow w = \sqrt{1^2+(-1)^2+\left(\sqrt{2}\right)^2}$
$= 2 \Rightarrow 2 = \sqrt{x^2+y^2+z^2} \Rightarrow x^2+y^2+z^2 = 4$

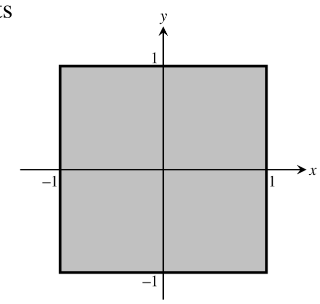
64. $g(x,y,z) = \frac{x-y+z}{2x+y-z}$ at $(1,0,-2) \Rightarrow w = \frac{x-y+z}{2x+y-z}$; at $(1,0,-2) \Rightarrow w = \frac{1-0+(-2)}{2(1)+0-(-2)} = -\frac{1}{4} \Rightarrow -\frac{1}{4} = \frac{x-y+z}{2x+y-z}$
$\Rightarrow 2x - y + z = 0$

65. $f(x,y) = \sum_{n=0}^{\infty} \left(\frac{x}{y}\right)^n = \frac{1}{1-\left(\frac{x}{y}\right)} = \frac{y}{y-x}$ for
$\left|\frac{x}{y}\right| < 1 \Rightarrow$ Domain: all points (x, y) satisfying $|x| < |y|$;
at $(1, 2) \Rightarrow$ since $\left|\frac{1}{2}\right| < 1 \Rightarrow z = \frac{2}{2-1} = 2$
$\Rightarrow \frac{y}{y-x} = 2 \Rightarrow y = 2x$



66. $g(x,y,z) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!\,z^n} = e^{(x+y)/z} \Rightarrow$ Domain: all points (x, y, z) satisfying $z \neq 0$; at $(\ln 4, \ln 9, 2)$
$\Rightarrow w = e^{(\ln 4 + \ln 9)/2} = e^{(\ln 36)/2} = e^{\ln 6} = 6 \Rightarrow 6 = e^{(x+y)/z} \Rightarrow \frac{x+y}{z} = \ln 6$

67. $f(x,y) = \int_x^y \frac{d\theta}{\sqrt{1-\theta^2}} = \sin^{-1}y - \sin^{-1}x \Rightarrow$ Domain: all points (x, y) satisfying $-1 \le x \le 1$ and $-1 \le y \le 1$;
at $(0, 1) \Rightarrow \sin^{-1}1 - \sin^{-1}0 = \frac{\pi}{2} \Rightarrow \sin^{-1}y - \sin^{-1}x = \frac{\pi}{2}$. Since $-\frac{\pi}{2} \le \sin^{-1}y \le \frac{\pi}{2}$ and $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$, in order for $\sin^{-1}y - \sin^{-1}x$ to equal $\frac{\pi}{2}$, $0 \le \sin^{-1}y \le \frac{\pi}{2}$ and $-\frac{\pi}{2} \le \sin^{-1}x \le 0$; that is $0 \le y \le 1$ and $-1 \le x \le 0$. Thus $y = \sin\left(\frac{\pi}{2} + \sin^{-1}x\right) = \sqrt{1-x^2}$, $x \le 0$



68. $g(x,y,z) = \int_x^y \frac{dt}{1+t^2} + \int_0^z \frac{d\theta}{\sqrt{4-\theta^2}} = \tan^{-1}y - \tan^{-1}x + \sin^{-1}\left(\frac{z}{2}\right) \Rightarrow$ Domain: all points (x, y, z) satisfying $-2 \le z \le 2$;
at $\left(0,1,\sqrt{3}\right) \Rightarrow \tan^{-1}1 - \tan^{-1}0 + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{7\pi}{12} \Rightarrow \tan^{-1}y - \tan^{-1}x + \sin^{-1}\left(\frac{z}{2}\right) = \frac{7\pi}{12}$. Since $-\frac{\pi}{2} \le \sin^{-1}\left(\frac{z}{2}\right) \le \frac{\pi}{2}$,
$\frac{\pi}{12} \le \tan^{-1}y - \tan^{-1}x \le \frac{13\pi}{12} \Rightarrow z = 2\sin\left(\frac{7\pi}{12} - \tan^{-1}y + \tan^{-1}x\right)$, $\frac{\pi}{12} \le \tan^{-1}y - \tan^{-1}x \le \frac{13\pi}{12}$

69-72. Example CAS commands:
<u>Maple</u>:

```
with( plots );
f := (x,y) -> x*sin(y/2) + y*sin(2*x);
xdomain := x=0..5*Pi;
ydomain := y=0..5*Pi;
x0,y0 := 3*Pi,3*Pi;
plot3d( f(x,y), xdomain, ydomain, axes=boxed, style=patch, shading=zhue, title="#69(a) (Section 14.1)" );
```

```
plot3d( f(x,y), xdomain, ydomain, grid=[50,50], axes=boxed, shading=zhue, style=patchcontour, orientation=[-90,0],
        title="#69(b) (Section 14.1)" );                                        # (b)
L := evalf( f(x0,y0) );                                 # (c)
plot3d( f(x,y), xdomain, ydomain, grid=[50,50], axes=boxed, shading=zhue, style=patchcontour, contours=[L],
        orientation=[-90,0], title="#45(c) (Section 13.1)" );
```

73-76. Example CAS commands:

Maple:

```
eq := 4*ln(x^2+y^2+z^2)=1;
implicitplot3d( eq, x=-2..2, y=-2..2, z=-2..2, grid=[30,30,30], axes=boxed, title="#73 (Section 14.1)" );
```

77-80. Example CAS commands:

Maple:

```
x := (u,v) -> u*cos(v);
y := (u,v) ->u*sin(v);
z := (u,v) -> u;
plot3d( [x(u,v),y(u,v),z(u,v)], u=0..2, v=0..2*Pi, axes=boxed, style=patchcontour, contours=[($0..4)/2], shading=zhue,
        title="#77 (Section 14.1)" );
```

69-60. Example CAS commands:

Mathematica: (assigned functions and bounds will vary)

For 69 - 72, the command **ContourPlot** draws 2-dimensional contours that are z-level curves of surfaces z = f(x,y).

```
Clear[x, y, f]
f[x_, y_]:= x Sin[y/2] + y Sin[2x]
xmin= 0; xmax= 5π; ymin= 0; ymax= 5π; {x0, y0}={3π, 3π};
cp= ContourPlot[f[x,y], {x, xmin, xmax}, {y, ymin, ymax}, ContourShading → False];
cp0= ContourPlot[[f[x,y], {x, xmin, xmax}, {y, ymin, ymax}, Contours → {f[x0,y0]}, ContourShading → False,
        PlotStyle → {RGBColor[1,0,0]}];
Show[cp, cp0]
```

For 73 - 76, the command **ContourPlot3D** will be used. Write the function f[x, y, z] so that when it is equated to zero, it represents the level surface given.

For 73, the problem associated with Log[0] can be avoided by rewriting the function as x2 + y2 +z2 - e1/4

```
Clear[x, y, z, f]
f[x_, y_, z_]:= x^2 + y^2 + z^2 − Exp[1/4]
ContourPlot3D[f[x, y, z], {x, −5, 5}, {y, −5, 5}, {z, −5, 5}, PlotPoints → {7, 7}];
```

For 77 - 80, the command ParametricPlot3D will be used. To get the z-level curves here, we solve x and y in terms of z and either u or v (v here), create a table of level curves, then plot that table.

```
Clear[x, y, z, u, v]
ParametricPlot3D[{u Cos[v], u Sin[v], u}, {u, 0, 2}, {v, 0, 2p}];
zlevel= Table[{z Cos[v], z sin[v]}, {z, 0, 2, .1}];
ParametricPlot[Evaluate[zlevel],{v, 0, 2π}];
```

14.2 LIMITS AND CONTINUITY IN HIGHER DIMENSIONS

1. $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2-y^2+5}{x^2+y^2+2} = \frac{3(0)^2-0^2+5}{0^2+0^2+2} = \frac{5}{2}$

2. $\lim\limits_{(x,y)\to(0,4)} \frac{x}{\sqrt{y}} = \frac{0}{\sqrt{4}} = 0$

3. $\lim\limits_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1} = \sqrt{3^2+4^2-1} = \sqrt{24} = 2\sqrt{6}$

4. $\lim\limits_{(x,y)\to(2,-3)} \left(\frac{1}{x}+\frac{1}{y}\right)^2 = \left[\frac{1}{2}+\left(\frac{1}{-3}\right)\right]^2 = \left(\frac{1}{6}\right)^2 = \frac{1}{36}$

5. $\lim\limits_{(x,y)\to\left(0,\frac{\pi}{4}\right)} \sec x \tan y = (\sec 0)\left(\tan\frac{\pi}{4}\right) = (1)(1) = 1$

6. $\lim\limits_{(x,y)\to(0,0)} \cos\left(\frac{x^2+y^3}{x+y+1}\right) = \cos\left(\frac{0^2+0^3}{0+0+1}\right) = \cos 0 = 1$

7. $\lim\limits_{(x,y)\to(0,\ln 2)} e^{x-y} = e^{0-\ln 2} = e^{\ln\left(\frac{1}{2}\right)} = \frac{1}{2}$

8. $\lim\limits_{(x,y)\to(1,1)} \ln|1+x^2y^2| = \ln|1+(1)^2(1)^2| = \ln 2$

9. $\lim\limits_{(x,y)\to(0,0)} \frac{e^y \sin x}{x} = \lim\limits_{(x,y)\to(0,0)} (e^y)\left(\frac{\sin x}{x}\right) = e^0 \cdot \lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right) = 1 \cdot 1 = 1$

10. $\lim\limits_{(x,y)\to(1/27,\pi^3)} \cos\sqrt[3]{xy} = \cos\sqrt[3]{\left(\frac{1}{27}\right)\pi^3} = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

11. $\lim\limits_{(x,y)\to(1,\pi/6)} \frac{x\sin y}{x^2+1} = \frac{1\cdot\sin\left(\frac{\pi}{6}\right)}{1^2+1} = \frac{1/2}{2} = \frac{1}{4}$

12. $\lim\limits_{(x,y)\to\left(\frac{\pi}{2},0\right)} \frac{\cos y+1}{y-\sin x} = \frac{(\cos 0)+1}{0-\sin\left(\frac{\pi}{2}\right)} = \frac{1+1}{-1} = -2$

13. $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-2xy+y^2}{x-y} = \lim\limits_{(x,y)\to(1,1)} \frac{(x-y)^2}{x-y} = \lim\limits_{(x,y)\to(1,1)} (x-y) = (1-1) = 0$

14. $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-y^2}{x-y} = \lim\limits_{(x,y)\to(1,1)} \frac{(x+y)(x-y)}{x-y} = \lim\limits_{(x,y)\to(1,1)} (x+y) = (1+1) = 2$

15. $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{xy-y-2x+2}{x-1} = \lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{(x-1)(y-2)}{x-1} = \lim\limits_{(x,y)\to(1,1)} (y-2) = (1-2) = -1$

16. $\lim\limits_{\substack{(x,y)\to(2,-4)\\ y\neq -4,\, x\neq x^2}} \frac{y+4}{x^2y-xy+4x^2-4x} = \lim\limits_{\substack{(x,y)\to(2,-4)\\ y\neq -4,\, x\neq x^2}} \frac{y+4}{x(x-1)(y+4)} = \lim\limits_{\substack{(x,y)\to(2,-4)\\ x\neq x^2}} \frac{1}{x(x-1)} = \frac{1}{2(2-1)} = \frac{1}{2}$

17. $\lim\limits_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}} = \lim\limits_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}+2\right)}{\sqrt{x}-\sqrt{y}} = \lim\limits_{(x,y)\to(0,0)} \left(\sqrt{x}+\sqrt{y}+2\right)$

$= \left(\sqrt{0}+\sqrt{0}+2\right) = 2$

Note: (x,y) must approach $(0,0)$ through the first quadrant only with $x \neq y$.

18. $\lim\limits_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{x+y-4}{\sqrt{x+y}-2} = \lim\limits_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{\left(\sqrt{x+y}+2\right)\left(\sqrt{x+y}-2\right)}{\sqrt{x+y}-2} = \lim\limits_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \left(\sqrt{x+y}+2\right)$

$= \left(\sqrt{2+2}+2\right) = 2+2 = 4$

19. $\lim\limits_{\substack{(x,y)\to(2,0)\\ 2x-y\neq 4}} \frac{\sqrt{2x-y}-2}{2x-y-4} = \lim\limits_{\substack{(x,y)\to(2,0)\\ 2x-y\neq 4}} \frac{\sqrt{2x-y}-2}{\left(\sqrt{2x-y}+2\right)\left(\sqrt{2x-y}-2\right)} = \lim\limits_{(x,y)\to(2,0)} \frac{1}{\sqrt{2x-y}+2}$
$= \frac{1}{\sqrt{(2)(2)-0}+2} = \frac{1}{2+2} = \frac{1}{4}$

20. $\lim\limits_{\substack{(x,y)\to(4,3)\\ x-y\neq 1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1} = \lim\limits_{\substack{(x,y)\to(4,3)\\ x-y\neq 1}} \frac{\sqrt{x}-\sqrt{y+1}}{\left(\sqrt{x}+\sqrt{y+1}\right)\left(\sqrt{x}-\sqrt{y+1}\right)} = \lim\limits_{(x,y)\to(4,3)} \frac{1}{\sqrt{x}+\sqrt{y+1}}$
$= \frac{1}{\sqrt{4}+\sqrt{3+1}} = \frac{1}{2+2} = \frac{1}{4}$

21. $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim\limits_{r\to 0} \frac{\sin(r^2)}{r^2} = \lim\limits_{r\to 0} \frac{2r\cdot\cos(r^2)}{2r} = \lim\limits_{r\to 0} \cos(r^2) = 1$

22. $\lim\limits_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{xy} = \lim\limits_{u\to 0} \frac{1-\cos u}{u} = \lim\limits_{u\to 0} \frac{\sin u}{1} = 0$

23. $\lim\limits_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y} = \lim\limits_{(x,y)\to(1,-1)} \frac{(x+y)(x^2-xy+y^2)}{x+y} = \lim\limits_{(x,y)\to(1,-1)} (x^2-xy+y^2) = \left(1^2-(1)(-1)+(-1)^2\right) = 3$

24. $\lim\limits_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4} = \lim\limits_{(x,y)\to(2,2)} \frac{x-y}{(x+y)(x-y)(x^2+y^2)} = \lim\limits_{(x,y)\to(2,2)} \frac{1}{(x+y)(x^2+y^2)} = \frac{1}{(2+2)(2^2+2^2)} = \frac{1}{32}$

25. $\lim\limits_{P\to(1,3,4)} \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{1}{1}+\frac{1}{3}+\frac{1}{4} = \frac{12+4+3}{12} = \frac{19}{12}$

26. $\lim\limits_{P\to(1,-1,-1)} \frac{2xy+yz}{x^2+z^2} = \frac{2(1)(-1)+(-1)(-1)}{1^2+(-1)^2} = \frac{-2+1}{1+1} = -\frac{1}{2}$

27. $\lim\limits_{P\to(3,3,0)} (\sin^2 x+\cos^2 y+\sec^2 z) = (\sin^2 3+\cos^2 3)+\sec^2 0 = 1+1^2 = 2$

28. $\lim\limits_{P\to\left(-\frac{1}{4},\frac{\pi}{2},2\right)} \tan^{-1}(xyz) = \tan^{-1}\left(-\frac{1}{4}\cdot\frac{\pi}{2}\cdot 2\right) = \tan^{-1}\left(-\frac{\pi}{4}\right)$

29. $\lim\limits_{P\to(\pi,0,3)} ze^{-2y}\cos 2x = 3e^{-2(0)}\cos 2\pi = (3)(1)(1) = 3$

30. $\lim\limits_{P\to(2,-3,6)} \ln\sqrt{x^2+y^2+z^2} = \ln\sqrt{2^2+(-3)^2+6^2} = \ln\sqrt{49} = \ln 7$

31. (a) All (x, y) (b) All (x, y) except $(0, 0)$

32. (a) All (x, y) so that $x \neq y$ (b) All (x, y)

33. (a) All (x, y) except where $x = 0$ or $y = 0$ (b) All (x, y)

34. (a) All (x, y) so that $x^2-3x+2\neq 0 \Rightarrow (x-2)(x-1)\neq 0 \Rightarrow x\neq 2$ and $x\neq 1$
(b) All (x, y) so that $y \neq x^2$

35. (a) All (x, y, z) (b) All (x, y, z) except the interior of the cylinder $x^2+y^2=1$

36. (a) All (x, y, z) so that $xyz > 0$ (b) All (x, y, z)

37. (a) All (x, y, z) with $z \neq 0$ (b) All (x, y, z) with $x^2+z^2\neq 1$

38. (a) All (x, y, z) except $(x, 0, 0)$ (b) All (x, y, z) except $(0, y, 0)$ or $(x, 0, 0)$

39. (a) All (x, y, z) such that $z > x^2 + y^2 + 1$ (b) All (x, y, z) such that $z \neq \sqrt{x^2 + y^2}$

40. (a) All (x, y, z) such that $x^2 + y^2 + z^2 \leq 4$
(b) All (x, y, z) such that $x^2 + y^2 + z^2 \geq 9$ except when $x^2 + y^2 + z^2 = 25$

41. $\lim\limits_{\substack{(x,y) \to (0,0) \\ \text{along } y = x \\ x > 0}} -\frac{x}{\sqrt{x^2+y^2}} = \lim\limits_{x \to 0^+} -\frac{x}{\sqrt{x^2+x^2}} = \lim\limits_{x \to 0^+} -\frac{x}{\sqrt{2}\,|x|} = \lim\limits_{x \to 0^+} -\frac{x}{\sqrt{2}\,x} = \lim\limits_{x \to 0^+} -\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$;
$\lim\limits_{\substack{(x,y) \to (0,0) \\ \text{along } y = x \\ x < 0}} -\frac{x}{\sqrt{x^2+y^2}} = \lim\limits_{x \to 0^-} -\frac{x}{\sqrt{2}\,|x|} = \lim\limits_{x \to 0^-} -\frac{x}{\sqrt{2}(-x)} = \lim\limits_{x \to 0^-} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

42. $\lim\limits_{\substack{(x,y) \to (0,0) \\ \text{along } y = 0}} \frac{x^4}{x^4+y^2} = \lim\limits_{x \to 0} \frac{x^4}{x^4+0^2} = 1$; $\lim\limits_{\substack{(x,y) \to (0,0) \\ \text{along } y = x^2}} \frac{x^4}{x^4+y^2} = \lim\limits_{x \to 0} \frac{x^4}{x^4+(x^2)^2} = \lim\limits_{x \to 0} \frac{x^4}{2x^4} = \frac{1}{2}$

43. $\lim\limits_{\substack{(x,y) \to (0,0) \\ \text{along } y = kx^2}} \frac{x^4-y^2}{x^4+y^2} = \lim\limits_{x \to 0} \frac{x^4-(kx^2)^2}{x^4+(kx^2)^2} = \lim\limits_{x \to 0} \frac{x^4-k^2x^4}{x^4+k^2x^4} = \frac{1-k^2}{1+k^2} \Rightarrow$ different limits for different values of k

44. $\lim\limits_{\substack{(x,y) \to (0,0) \\ \text{along } y = kx \\ k \neq 0}} \frac{xy}{|xy|} = \lim\limits_{x \to 0} \frac{x(kx)}{|x(kx)|} = \lim\limits_{x \to 0} \frac{kx^2}{|kx^2|} = \lim\limits_{x \to 0} \frac{k}{|k|}$; if $k > 0$, the limit is 1; but if $k < 0$, the limit is -1

45. $\lim\limits_{\substack{(x,y) \to (0,0) \\ \text{along } y = kx \\ k \neq -1}} \frac{x-y}{x+y} = \lim\limits_{x \to 0} \frac{x-kx}{x+kx} = \frac{1-k}{1+k} \Rightarrow$ different limits for different values of k, $k \neq -1$

46. $\lim\limits_{\substack{(x,y) \to (0,0) \\ \text{along } y = kx \\ k \neq 1}} \frac{x^2-y}{x-y} = \lim\limits_{x \to 0} \frac{x^2-kx}{x-kx} = \lim\limits_{x \to 0} \frac{x-k}{1-k} = \frac{-k}{1-k} \Rightarrow$ different limits for different values of k, $k \neq 1$

47. $\lim\limits_{\substack{(x,y) \to (0,0) \\ \text{along } y = kx^2 \\ k \neq 0}} \frac{x^2+y}{y} = \lim\limits_{x \to 0} \frac{x^2+kx^2}{kx^2} = \frac{1+k}{k} \Rightarrow$ different limits for different values of k, $k \neq 0$

48. $\lim\limits_{\substack{(x,y) \to (0,0) \\ \text{along } y = kx^2}} \frac{x^2y}{x^4+y^2} = \lim\limits_{x \to 0} \frac{kx^4}{x^4+k^2x^4} = \frac{k}{1+k^2} \Rightarrow$ different limits for different values of k

49. $\lim\limits_{\substack{(x,y) \to (1,1) \\ \text{along } x = 1}} \frac{xy^2-1}{y-1} = \lim\limits_{y \to 1} \frac{y^2-1}{y-1} = \lim\limits_{y \to 1} (y+1) = 2$; $\lim\limits_{\substack{(x,y) \to (1,1) \\ \text{along } y = x}} \frac{xy^2-1}{y-1} = \lim\limits_{y \to 1} \frac{y^3-1}{y-1} = \lim\limits_{y \to 1} (y^2+y+1) = 3$

50. $\lim\limits_{\substack{(x,y) \to (1,-1) \\ \text{along } y = -1}} \frac{xy+1}{x^2-y^2} = \lim\limits_{x \to 1} \frac{-x+1}{x^2-1} = \lim\limits_{x \to 1} \frac{-1}{x+1} = -\frac{1}{2}$; $\lim\limits_{\substack{(x,y) \to (1,-1) \\ \text{along } y = -x^2}} \frac{xy+1}{x^2-y^2} = \lim\limits_{x \to 1} \frac{-x^3+1}{x^2-x^4} = \lim\limits_{x \to 1} \frac{x^2+x+1}{(x+1)(x^2+1)}$
$= \frac{3}{2}$

51. $f(x, y) = \begin{cases} 1 & \text{if } y \geq x^4 \\ 1 & \text{if } y \leq 0 \\ 0 & \text{otherwise} \end{cases}$

(a) $\lim\limits_{(x,y)\to(0,1)} f(x, y) = 1$ since any path through $(0, 1)$ that is close to $(0, 1)$ satisfies $y \geq x^4$

(b) $\lim\limits_{(x,y)\to(2,3)} f(x, y) = 0$ since any path through $(2, 3)$ that is close to $(2, 3)$ does not satisfiy either $y \geq x^4$ or $y \leq 0$

(c) $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } x=0}} f(x, y) = 1$ and $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y=x^2}} f(x, y) = 0 \Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist

52. $f(x, y) = \begin{cases} x^2 & \text{if } x \geq 0 \\ x^3 & \text{if } x < 0 \end{cases}$

(a) $\lim\limits_{(x,y)\to(3,-2)} f(x, y) = 3^2 = 9$ since any path through $(3, -2)$ that is close to $(3, -2)$ satisfies $x \geq 0$

(b) $\lim\limits_{(x,y)\to(-2,1)} f(x, y) = (-2)^3 = -8$ since any path through $(-2, 1)$ that is close to $(-2, 1)$ satisfies $x < 0$

(c) $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$ since the limit is 0 along any path through $(0, 0)$ with $x < 0$ and the limit is also zero along any path through $(0, 0)$ with $x \geq 0$

53. First consider the vertical line $x = 0 \Rightarrow \lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } x=0}} \frac{2x^2y}{x^4+y^2} = \lim\limits_{y\to 0} \frac{2(0)^2 y}{(0)^4+y^2} = \lim\limits_{y\to 0} 0 = 0$. Now consider any nonvertical through $(0, 0)$. The equation of any line through $(0, 0)$ is of the form $y = mx \Rightarrow \lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx}} f(x, y) = \lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx}} \frac{2x^2y}{x^4+y^2}$

$= \lim\limits_{x\to 0} \frac{2x^2(mx)}{x^4+(mx)^2} = \lim\limits_{x\to 0} \frac{2mx^3}{x^4+m^2x^2} = \lim\limits_{x\to 0} \frac{2mx^3}{x^2(x^2+m^2)} = \lim\limits_{x\to 0} \frac{2mx}{(x^2+m^2)} = 0$. Thus $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{any line though } (0,0)}} \frac{2x^2y}{x^4+y^2} = 0$.

54. If f is continuous at (x_0, y_0), then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ must equal $f(x_0, y_0) = 3$. If f is not continuous at (x_0, y_0), the limit could have any value different from 3, and need not even exist.

55. $\lim\limits_{(x,y)\to(0,0)} \left(1 - \frac{x^2y^2}{3}\right) = 1$ and $\lim\limits_{(x,y)\to(0,0)} 1 = 1 \Rightarrow \lim\limits_{(x,y)\to(0,0)} \frac{\tan^{-1} xy}{xy} = 1$, by the Sandwich Theorem

56. If $xy > 0$, $\lim\limits_{(x,y)\to(0,0)} \frac{2|xy| - \left(\frac{x^2y^2}{6}\right)}{|xy|} = \lim\limits_{(x,y)\to(0,0)} \frac{2xy - \left(\frac{x^2y^2}{6}\right)}{xy} = \lim\limits_{(x,y)\to(0,0)} \left(2 - \frac{xy}{6}\right) = 2$ and $\lim\limits_{(x,y)\to(0,0)} \frac{2|xy|}{|xy|} = \lim\limits_{(x,y)\to(0,0)} 2 = 2$; if $xy < 0$, $\lim\limits_{(x,y)\to(0,0)} \frac{2|xy| - \left(\frac{x^2y^2}{6}\right)}{|xy|} = \lim\limits_{(x,y)\to(0,0)} \frac{-2xy - \left(\frac{x^2y^2}{6}\right)}{-xy}$
$= \lim\limits_{(x,y)\to(0,0)} \left(2 + \frac{xy}{6}\right) = 2$ and $\lim\limits_{(x,y)\to(0,0)} \frac{2|xy|}{|xy|} = 2 \Rightarrow \lim\limits_{(x,y)\to(0,0)} \frac{4 - 4\cos\sqrt{|xy|}}{|xy|} = 2$, by the Sandwich Theorem

57. The limit is 0 since $\left|\sin\left(\frac{1}{x}\right)\right| \leq 1 \Rightarrow -1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \Rightarrow -y \leq y\sin\left(\frac{1}{x}\right) \leq y$ for $y \geq 0$, and $-y \geq y\sin\left(\frac{1}{x}\right) \geq y$ for $y \leq 0$. Thus as $(x, y) \to (0, 0)$, both $-y$ and y approach $0 \Rightarrow y\sin\left(\frac{1}{x}\right) \to 0$, by the Sandwich Theorem.

58. The limit is 0 since $\left|\cos\left(\frac{1}{y}\right)\right| \leq 1 \Rightarrow -1 \leq \cos\left(\frac{1}{y}\right) \leq 1 \Rightarrow -x \leq x\cos\left(\frac{1}{y}\right) \leq x$ for $x \geq 0$, and $-x \geq x\cos\left(\frac{1}{y}\right) \geq x$ for $x \leq 0$. Thus as $(x, y) \to (0, 0)$, both $-x$ and x approach $0 \Rightarrow x\cos\left(\frac{1}{y}\right) \to 0$, by the Sandwich Theorem.

59. (a) $f(x, y)\big|_{y=mx} = \frac{2m}{1+m^2} = \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$. The value of $f(x, y) = \sin 2\theta$ varies with θ, which is the line's angle of inclination.

(b) Since $f(x,y)|_{y=mx} = \sin 2\theta$ and since $-1 \le \sin 2\theta \le 1$ for every θ, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ varies from -1 to 1 along $y = mx$.

60. $|xy\,(x^2-y^2)| = |xy|\,|x^2-y^2| \le |x|\,|y|\,|x^2+y^2| = \sqrt{x^2}\,\sqrt{y^2}\,|x^2+y^2| \le \sqrt{x^2+y^2}\,\sqrt{x^2+y^2}\,|x^2+y^2|$
$= (x^2+y^2)^2 \Rightarrow \left|\frac{xy\,(x^2-y^2)}{x^2+y^2}\right| \le \frac{(x^2+y^2)^2}{x^2+y^2} = x^2+y^2 \Rightarrow -(x^2+y^2) \le \frac{xy\,(x^2-y^2)}{x^2+y^2} \le (x^2+y^2)$
$\Rightarrow \lim\limits_{(x,y)\to(0,0)} \left(xy\,\frac{x^2-y^2}{x^2+y^2}\right) = 0$ by the Sandwich Theorem, since $\lim\limits_{(x,y)\to(0,0)} \pm(x^2+y^2) = 0$; thus, define $f(0,0) = 0$

61. $\lim\limits_{(x,y)\to(0,0)} \frac{x^3-xy^2}{x^2+y^2} = \lim\limits_{r\to 0} \frac{r^3\cos^3\theta - (r\cos\theta)(r^2\sin^2\theta)}{r^2\cos^2\theta + r^2\sin^2\theta} = \lim\limits_{r\to 0} \frac{r(\cos^3\theta - \cos\theta\sin^2\theta)}{1} = 0$

62. $\lim\limits_{(x,y)\to(0,0)} \cos\left(\frac{x^3-y^3}{x^2+y^2}\right) = \lim\limits_{r\to 0} \cos\left(\frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\right) = \lim\limits_{r\to 0} \cos\left[\frac{r(\cos^3\theta - \sin^3\theta)}{1}\right] = \cos 0 = 1$

63. $\lim\limits_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2} = \lim\limits_{r\to 0} \frac{r^2\sin^2\theta}{r^2} = \lim\limits_{r\to 0} (\sin^2\theta) = \sin^2\theta$; the limit does not exist since $\sin^2\theta$ is between 0 and 1 depending on θ

64. $\lim\limits_{(x,y)\to(0,0)} \frac{2x}{x^2+x+y^2} = \lim\limits_{r\to 0} \frac{2r\cos\theta}{r^2+r\cos\theta} = \lim\limits_{r\to 0} \frac{2\cos\theta}{r+\cos\theta} = \frac{2\cos\theta}{\cos\theta}$; the limit does not exist for $\cos\theta = 0$

65. $\lim\limits_{(x,y)\to(0,0)} \tan^{-1}\left[\frac{|x|+|y|}{x^2+y^2}\right] = \lim\limits_{r\to 0} \tan^{-1}\left[\frac{|r\cos\theta|+|r\sin\theta|}{r^2}\right] = \lim\limits_{r\to 0} \tan^{-1}\left[\frac{|r|\,(|\cos\theta|+|\sin\theta|)}{r^2}\right]$;
if $r \to 0^+$, then $\lim\limits_{r\to 0^+} \tan^{-1}\left[\frac{|r|\,(|\cos\theta|+|\sin\theta|)}{r^2}\right] = \lim\limits_{r\to 0^+} \tan^{-1}\left[\frac{|\cos\theta|+|\sin\theta|}{r}\right] = \frac{\pi}{2}$; if $r \to 0^-$, then
$\lim\limits_{r\to 0^-} \tan^{-1}\left[\frac{|r|\,(|\cos\theta|+|\sin\theta|)}{r^2}\right] = \lim\limits_{r\to 0^-} \tan^{-1}\left(\frac{|\cos\theta|+|\sin\theta|}{-r}\right) = \frac{\pi}{2} \Rightarrow$ the limit is $\frac{\pi}{2}$

66. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim\limits_{r\to 0} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2} = \lim\limits_{r\to 0} (\cos^2\theta - \sin^2\theta) = \lim\limits_{r\to 0} (\cos 2\theta)$ which ranges between -1 and 1 depending on $\theta \Rightarrow$ the limit does not exist

67. $\lim\limits_{(x,y)\to(0,0)} \ln\left(\frac{3x^2-x^2y^2+3y^2}{x^2+y^2}\right) = \lim\limits_{r\to 0} \ln\left(\frac{3r^2\cos^2\theta - r^4\cos^2\theta\sin^2\theta + 3r^2\sin^2\theta}{r^2}\right)$
$= \lim\limits_{r\to 0} \ln(3 - r^2\cos^2\theta\sin^2\theta) = \ln 3 \Rightarrow$ define $f(0,0) = \ln 3$

68. $\lim\limits_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^2} = \lim\limits_{r\to 0} \frac{(3r\cos\theta)(r^2\sin^2\theta)}{r^2} = \lim\limits_{r\to 0} 3r\cos\theta\sin^2\theta = 0 \Rightarrow$ define $f(0,0) = 0$

69. Let $\delta = 0.1$. Then $\sqrt{x^2+y^2} < \delta \Rightarrow \sqrt{x^2+y^2} < 0.1 \Rightarrow x^2+y^2 < 0.01 \Rightarrow |x^2+y^2-0| < 0.01$
$\Rightarrow |f(x,y) - f(0,0)| < 0.01 = \epsilon$.

70. Let $\delta = 0.05$. Then $|x| < \delta$ and $|y| < \delta \Rightarrow |f(x,y) - f(0,0)| = \left|\frac{y}{x^2+1} - 0\right| = \left|\frac{y}{x^2+1}\right| \le |y| < 0.05 = \epsilon$.

71. Let $\delta = 0.005$. Then $|x| < \delta$ and $|y| < \delta \Rightarrow |f(x,y) - f(0,0)| = \left|\frac{x+y}{x^2+1} - 0\right| = \left|\frac{x+y}{x^2+1}\right| \le |x+y| < |x|+|y|$
$< 0.005 + 0.005 = 0.01 = \epsilon$.

72. Let $\delta = 0.01$. Since $-1 \le \cos x \le 1 \Rightarrow 1 \le 2+\cos x \le 3 \Rightarrow \frac{1}{3} \le \frac{1}{2+\cos x} \le 1 \Rightarrow \frac{|x+y|}{3} \le \left|\frac{x+y}{2+\cos x}\right| \le |x+y|$
$\le |x|+|y|$. Then $|x| < \delta$ and $|y| < \delta \Rightarrow |f(x,y) - f(0,0)| = \left|\frac{x+y}{2+\cos x} - 0\right| = \left|\frac{x+y}{2+\cos x}\right| \le |x|+|y| < 0.01 + 0.01$
$= 0.02 = \epsilon$.

73. Let $\delta = 0.04$. Since $y^2 \le x^2 + y^2 \Rightarrow \frac{y^2}{x^2+y^2} \le 1 \Rightarrow \frac{|x|y^2}{x^2+y^2} \le |x| = \sqrt{x^2} \le \sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y) - f(0,0)|$ $= \left|\frac{xy^2}{x^2+y^2} - 0\right| < 0.04 = \epsilon$.

74. Let $\delta = 0.01$. If $|y| \le 1$, then $y^2 \le |y| = \sqrt{y^2} \le \sqrt{x^2+y^2}$, so $|x| = \sqrt{x^2} \le \sqrt{x^2+y^2} \Rightarrow |x| + y^2 \le 2\sqrt{x^2+y^2}$. Since $x^2 \le x^2 + y^2 \Rightarrow \frac{x^2}{x^2+y^2} \le 1$ and $y^2 \le x^2 + y^2 \Rightarrow \frac{y^2}{x^2+y^2} \le 1$. Then $\frac{|x^3+y^4|}{x^2+y^2} \le \frac{x^2}{x^2+y^2}|x| + \frac{y^2}{x^2+y^2}y^2 \le |x| + y^2 < 2\delta$ $\Rightarrow |f(x,y) - f(0,0)| = \left|\frac{x^3+y^4}{x^2+y^2} - 0\right| < 2(0.01) = 0.002 = \epsilon$.

75. Let $\delta = \sqrt{0.015}$. Then $\sqrt{x^2+y^2+z^2} < \delta \Rightarrow |f(x,y,z) - f(0,0,0)| = |x^2+y^2+z^2 - 0| = |x^2+y^2+z^2|$ $= \left(\sqrt{x^2+t^2+x^2}\right)^2 < \left(\sqrt{0.015}\right)^2 = 0.015 = \epsilon$.

76. Let $\delta = 0.2$. Then $|x| < \delta$, $|y| < \delta$, and $|z| < \delta \Rightarrow |f(x,y,z) - f(0,0,0)| = |xyz - 0| = |xyz| = |x|\,|y|\,|z| < (0.2)^3$ $= 0.008 = \epsilon$.

77. Let $\delta = 0.005$. Then $|x| < \delta$, $|y| < \delta$, and $|z| < \delta \Rightarrow |f(x,y,z) - f(0,0,0)| = \left|\frac{x+y+z}{x^2+y^2+z^2+1} - 0\right|$ $= \left|\frac{x+y+z}{x^2+y^2+z^2+1}\right| \le |x+y+z| \le |x| + |y| + |z| < 0.005 + 0.005 + 0.005 = 0.015 = \epsilon$.

78. Let $\delta = \tan^{-1}(0.1)$. Then $|x| < \delta$, $|y| < \delta$, and $|z| < \delta \Rightarrow |f(x,y,z) - f(0,0,0)| = |\tan^2 x + \tan^2 y + \tan^2 z|$ $\le |\tan^2 x| + |\tan^2 y| + |\tan^2 z| = \tan^2 x + \tan^2 y + \tan^2 z < \tan^2\delta + \tan^2\delta + \tan^2\delta = 0.01 + 0.01 + 0.01 = 0.03 = \epsilon$.

79. $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = \lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} (x+y-z) = x_0 + y_0 - z_0 = f(x_0,y_0,z_0) \Rightarrow$ f is continuous at every (x_0,y_0,z_0)

80. $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = \lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} (x^2+y^2+z^2) = x_0^2 + y_0^2 + z_0^2 = f(x_0,y_0,z_0) \Rightarrow$ f is continuous at every point (x_0,y_0,z_0)

14.3 PARTIAL DERIVATIVES

1. $\frac{\partial f}{\partial x} = 4x$, $\frac{\partial f}{\partial y} = -3$

2. $\frac{\partial f}{\partial x} = 2x - y$, $\frac{\partial f}{\partial y} = -x + 2y$

3. $\frac{\partial f}{\partial x} = 2x(y+2)$, $\frac{\partial f}{\partial y} = x^2 - 1$

4. $\frac{\partial f}{\partial x} = 5y - 14x + 3$, $\frac{\partial f}{\partial y} = 5x - 2y - 6$

5. $\frac{\partial f}{\partial x} = 2y(xy-1)$, $\frac{\partial f}{\partial y} = 2x(xy-1)$

6. $\frac{\partial f}{\partial x} = 6(2x-3y)^2$, $\frac{\partial f}{\partial y} = -9(2x-3y)^2$

7. $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$, $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$

8. $\frac{\partial f}{\partial x} = \frac{2x^2}{\sqrt[3]{x^3+\left(\frac{y}{2}\right)}}$, $\frac{\partial f}{\partial y} = \frac{1}{3\sqrt[3]{x^3+\left(\frac{y}{2}\right)}}$

9. $\frac{\partial f}{\partial x} = -\frac{1}{(x+y)^2} \cdot \frac{\partial}{\partial x}(x+y) = -\frac{1}{(x+y)^2}$, $\frac{\partial f}{\partial y} = -\frac{1}{(x+y)^2} \cdot \frac{\partial}{\partial y}(x+y) = -\frac{1}{(x+y)^2}$

10. $\frac{\partial f}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$, $\frac{\partial f}{\partial y} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$

11. $\frac{\partial f}{\partial x} = = \frac{(xy-1)(1) - (x+y)(y)}{(xy-1)^2} = \frac{-y^2-1}{(xy-1)^2}$, $\frac{\partial f}{\partial y} = \frac{(xy-1)(1) - (x+y)(x)}{(xy-1)^2} = \frac{-x^2-1}{(xy-1)^2}$

12. $\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = -\frac{y}{x^2\left[1+\left(\frac{y}{x}\right)^2\right]} = -\frac{y}{x^2+y^2}$, $\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{x\left[1+\left(\frac{y}{x}\right)^2\right]} = \frac{x}{x^2+y^2}$

13. $\frac{\partial f}{\partial x} = e^{(x+y+1)} \cdot \frac{\partial}{\partial x}(x+y+1) = e^{(x+y+1)}$, $\frac{\partial f}{\partial y} = e^{(x+y+1)} \cdot \frac{\partial}{\partial y}(x+y+1) = e^{(x+y+1)}$

14. $\frac{\partial f}{\partial x} = -e^{-x}\sin(x+y) + e^{-x}\cos(x+y)$, $\frac{\partial f}{\partial y} = e^{-x}\cos(x+y)$

15. $\frac{\partial f}{\partial x} = \frac{1}{x+y} \cdot \frac{\partial}{\partial x}(x+y) = \frac{1}{x+y}$, $\frac{\partial f}{\partial y} = \frac{1}{x+y} \cdot \frac{\partial}{\partial y}(x+y) = \frac{1}{x+y}$

16. $\frac{\partial f}{\partial x} = e^{xy} \cdot \frac{\partial}{\partial x}(xy) \cdot \ln y = ye^{xy}\ln y$, $\frac{\partial f}{\partial y} = e^{xy} \cdot \frac{\partial}{\partial y}(xy) \cdot \ln y + e^{xy} \cdot \frac{1}{y} = xe^{xy}\ln y + \frac{e^{xy}}{y}$

17. $\frac{\partial f}{\partial x} = 2\sin(x-3y) \cdot \frac{\partial}{\partial x}\sin(x-3y) = 2\sin(x-3y)\cos(x-3y) \cdot \frac{\partial}{\partial x}(x-3y) = 2\sin(x-3y)\cos(x-3y)$,
$\frac{\partial f}{\partial y} = 2\sin(x-3y) \cdot \frac{\partial}{\partial y}\sin(x-3y) = 2\sin(x-3y)\cos(x-3y) \cdot \frac{\partial}{\partial y}(x-3y) = -6\sin(x-3y)\cos(x-3y)$

18. $\frac{\partial f}{\partial x} = 2\cos(3x-y^2) \cdot \frac{\partial}{\partial x}\cos(3x-y^2) = -2\cos(3x-y^2)\sin(3x-y^2) \cdot \frac{\partial}{\partial x}(3x-y^2)$
$= -6\cos(3x-y^2)\sin(3x-y^2)$,
$\frac{\partial f}{\partial y} = 2\cos(3x-y^2) \cdot \frac{\partial}{\partial y}\cos(3x-y^2) = -2\cos(3x-y^2)\sin(3x-y^2) \cdot \frac{\partial}{\partial y}(3x-y^2)$
$= 4y\cos(3x-y^2)\sin(3x-y^2)$

19. $\frac{\partial f}{\partial x} = yx^{y-1}$, $\frac{\partial f}{\partial y} = x^y \ln x$

20. $f(x,y) = \frac{\ln x}{\ln y} \Rightarrow \frac{\partial f}{\partial x} = \frac{1}{x\ln y}$ and $\frac{\partial f}{\partial y} = \frac{-\ln x}{y(\ln y)^2}$

21. $\frac{\partial f}{\partial x} = -g(x)$, $\frac{\partial f}{\partial y} = g(y)$

22. $f(x,y) = \sum_{n=0}^{\infty}(xy)^n$, $|xy| < 1 \Rightarrow f(x,y) = \frac{1}{1-xy} \Rightarrow \frac{\partial f}{\partial x} = -\frac{1}{(1-xy)^2} \cdot \frac{\partial}{\partial x}(1-xy) = \frac{y}{(1-xy)^2}$ and
$\frac{\partial f}{\partial y} = -\frac{1}{(1-xy)^2} \cdot \frac{\partial}{\partial y}(1-xy) = \frac{x}{(1-xy)^2}$

23. $f_x = y^2$, $f_y = 2xy$, $f_z = -4z$

24. $f_x = y+z$, $f_y = x+z$, $f_z = y+x$

25. $f_x = 1$, $f_y = -\frac{y}{\sqrt{y^2+z^2}}$, $f_z = -\frac{z}{\sqrt{y^2+z^2}}$

26. $f_x = -x\left(x^2+y^2+z^2\right)^{-3/2}$, $f_y = -y\left(x^2+y^2+z^2\right)^{-3/2}$, $f_z = -z\left(x^2+y^2+z^2\right)^{-3/2}$

27. $f_x = \frac{yz}{\sqrt{1-x^2y^2z^2}}$, $f_y = \frac{xz}{\sqrt{1-x^2y^2z^2}}$, $f_z = \frac{xy}{\sqrt{1-x^2y^2z^2}}$

28. $f_x = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}$, $f_y = \frac{z}{|x+yz|\sqrt{(x+yz)^2-1}}$, $f_z = \frac{y}{|x+yz|\sqrt{(x+yz)^2-1}}$

29. $f_x = \frac{1}{x+2y+3z}$, $f_y = \frac{2}{x+2y+3z}$, $f_z = \frac{3}{x+2y+3z}$

30. $f_x = yz \cdot \frac{1}{xy} \cdot \frac{\partial}{\partial x}(xy) = \frac{(yz)(y)}{xy} = \frac{yz}{x}$, $f_y = z\ln(xy) + yz \cdot \frac{\partial}{\partial y}\ln(xy) = z\ln(xy) + \frac{yz}{xy} \cdot \frac{\partial}{\partial y}(xy) = z\ln(xy) + z$,
$f_z = y\ln(xy) + yz \cdot \frac{\partial}{\partial z}\ln(xy) = y\ln(xy)$

31. $f_x = -2xe^{-(x^2+y^2+z^2)}$, $f_y = -2ye^{-(x^2+y^2+z^2)}$, $f_z = -2ze^{-(x^2+y^2+z^2)}$

32. $f_x = -yze^{-xyz}$, $f_y = -xze^{-xyz}$, $f_z = -xye^{-xyz}$

33. $f_x = \operatorname{sech}^2(x + 2y + 3z)$, $f_y = 2\operatorname{sech}^2(x + 2y + 3z)$, $f_z = 3\operatorname{sech}^2(x + 2y + 3z)$

34. $f_x = y\cosh(xy - z^2)$, $f_y = x\cosh(xy - z^2)$, $f_z = -2z\cosh(xy - z^2)$

35. $\frac{\partial f}{\partial t} = -2\pi\sin(2\pi t - \alpha)$, $\frac{\partial f}{\partial \alpha} = \sin(2\pi t - \alpha)$

36. $\frac{\partial g}{\partial u} = v^2 e^{(2u/v)} \cdot \frac{\partial}{\partial u}\left(\frac{2u}{v}\right) = 2ve^{(2u/v)}$, $\frac{\partial g}{\partial v} = 2ve^{(2u/v)} + v^2 e^{(2u/v)} \cdot \frac{\partial}{\partial v}\left(\frac{2u}{v}\right) = 2ve^{(2u/v)} - 2ue^{(2u/v)}$

37. $\frac{\partial h}{\partial \rho} = \sin\phi\cos\theta$, $\frac{\partial h}{\partial \phi} = \rho\cos\phi\cos\theta$, $\frac{\partial h}{\partial \theta} = -\rho\sin\phi\sin\theta$

38. $\frac{\partial g}{\partial r} = 1 - \cos\theta$, $\frac{\partial g}{\partial \theta} = r\sin\theta$, $\frac{\partial g}{\partial z} = -1$

39. $W_p = V$, $W_v = P + \frac{\delta v^2}{2g}$, $W_\delta = \frac{Vv^2}{2g}$, $W_v = \frac{2V\delta v}{2g} = \frac{V\delta v}{g}$, $W_g = -\frac{V\delta v^2}{2g^2}$

40. $\frac{\partial A}{\partial c} = m$, $\frac{\partial A}{\partial h} = \frac{q}{2}$, $\frac{\partial A}{\partial k} = \frac{m}{q}$, $\frac{\partial A}{\partial m} = \frac{k}{q} + c$, $\frac{\partial A}{\partial q} = -\frac{km}{q^2} + \frac{h}{2}$

41. $\frac{\partial f}{\partial x} = 1 + y$, $\frac{\partial f}{\partial y} = 1 + x$, $\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial y^2} = 0$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 1$

42. $\frac{\partial f}{\partial x} = y\cos xy$, $\frac{\partial f}{\partial y} = x\cos xy$, $\frac{\partial^2 f}{\partial x^2} = -y^2\sin xy$, $\frac{\partial^2 f}{\partial y^2} = -x^2\sin xy$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \cos xy - xy\sin xy$

43. $\frac{\partial g}{\partial x} = 2xy + y\cos x$, $\frac{\partial g}{\partial y} = x^2 - \sin y + \sin x$, $\frac{\partial^2 g}{\partial x^2} = 2y - y\sin x$, $\frac{\partial^2 g}{\partial y^2} = -\cos y$, $\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial^2 g}{\partial x \partial y} = 2x + \cos x$

44. $\frac{\partial h}{\partial x} = e^y$, $\frac{\partial h}{\partial y} = xe^y + 1$, $\frac{\partial^2 h}{\partial x^2} = 0$, $\frac{\partial^2 h}{\partial y^2} = xe^y$, $\frac{\partial^2 h}{\partial y \partial x} = \frac{\partial^2 h}{\partial x \partial y} = e^y$

45. $\frac{\partial r}{\partial x} = \frac{1}{x+y}$, $\frac{\partial r}{\partial y} = \frac{1}{x+y}$, $\frac{\partial^2 r}{\partial x^2} = \frac{-1}{(x+y)^2}$, $\frac{\partial^2 r}{\partial y^2} = \frac{-1}{(x+y)^2}$, $\frac{\partial^2 r}{\partial y \partial x} = \frac{\partial^2 r}{\partial x \partial y} = \frac{-1}{(x+y)^2}$

46. $\frac{\partial s}{\partial x} = \left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \left(-\frac{y}{x^2}\right)\left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] = \frac{-y}{x^2+y^2}$, $\frac{\partial s}{\partial y} = \left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] = \frac{x}{x^2+y^2}$,
$\frac{\partial^2 s}{\partial x^2} = \frac{y(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$, $\frac{\partial^2 s}{\partial y^2} = \frac{-x(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$, $\frac{\partial^2 s}{\partial y \partial x} = \frac{\partial^2 s}{\partial x \partial y} = \frac{(x^2+y^2)(-1)+y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

47. $\frac{\partial w}{\partial x} = 2x\tan(xy) + x^2\sec^2(xy) \cdot y = 2x\tan(xy) + x^2 y\sec^2(xy)$, $\frac{\partial w}{\partial y} = x^2\sec^2(xy) \cdot x = x^3\sec^2(xy)$,
$\frac{\partial^2 w}{\partial x^2} = 2\tan(xy) + 2x\sec^2(xy) \cdot y + 2xy\sec^2(xy) + x^2 y\,(2\sec(xy)\sec(xy)\tan(xy) \cdot y)$
$= 2\tan(xy) + 4xy\sec^2(xy) + 2x^2y^2\sec^2(xy)\tan(xy)$, $\frac{\partial^2 w}{\partial y^2} = x^3(2\sec(xy)\sec(xy)\tan(xy) \cdot x) = 2x^4\sec^2(xy)\tan(xy)$
$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = 3x^2\sec^2(xy) + x^3(2\sec(xy)\sec(xy)\tan(xy) \cdot y) = 3x^2\sec^2(xy) + x^3 y\sec^2(xy)\tan(xy)$

48. $\frac{\partial w}{\partial x} = ye^{x^2-y} \cdot 2x = 2xy\,e^{x^2-y}$, $\frac{\partial w}{\partial y} = (1)e^{x^2-y} + ye^{x^2-y} \cdot (-1) = e^{x^2-y}(1-y)$,
$\frac{\partial^2 w}{\partial x^2} = 2y\,e^{x^2-y} + 2xy\left(e^{x^2-y} \cdot 2x\right) = 2ye^{x^2-y}(1+2x^2)$, $\frac{\partial^2 w}{\partial y^2} = \left(e^{x^2-y} \cdot (-1)\right)(1-y) + e^{x^2-y}(-1)$
$= e^{x^2-y}(y-2)$, $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = \left(e^{x^2-y} \cdot 2x\right)(1-y) = 2x\,e^{x^2-y}(1-y)$

49. $\frac{\partial w}{\partial x} = \sin(x^2y) + x\cos(x^2y) \cdot 2xy = \sin(x^2y) + 2x^2y\cos(x^2y)$, $\frac{\partial w}{\partial y} = x\cos(x^2y) \cdot x^2 = x^3\cos(x^2y)$,
$\frac{\partial^2 w}{\partial x^2} = \cos(x^2y) \cdot 2xy + 4xy\cos(x^2y) - 2x^2y\sin(x^2y) \cdot 2xy = 6xy\cos(x^2y) - 4x^3y^2\sin(x^2y)$,
$\frac{\partial^2 w}{\partial y^2} = -x^3\sin(x^2y) \cdot x^2 = -x^5\sin(x^2y)$, $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = 3x^2\cos(x^2y) - x^3\sin(x^2y) \cdot 2xy = 3x^2\cos(x^2y) - 2x^4y\sin(x^2y)$

50. $\frac{\partial w}{\partial x} = \frac{(x^2+y)-(x-y)(2x)}{(x^2+y)^2} = \frac{-x^2+2xy+y}{(x^2+y)^2}$, $\frac{\partial w}{\partial y} = \frac{(x^2+y)(-1)-(x-y)}{(x^2+y)^2} = \frac{-x^2-x}{(x^2+y)^2}$,
$\frac{\partial^2 w}{\partial x^2} = \frac{(x^2+y)^2(-2x+2y)-(-x^2+2xy+y)2(x^2+y)(2x)}{\left[(x^2+y)^2\right]^2} = \frac{2(x^3-3x^2y-3xy+y^2)}{(x^2+y)^3}$,
$\frac{\partial^2 w}{\partial y^2} = \frac{(x^2+y)^2\cdot 0-(-x^2-x)2(x^2+y)\cdot 1}{\left[(x^2+y)^2\right]^2} = \frac{2x^2+2x}{(x^2+y)^3}$, $\frac{\partial^2 w}{\partial y\partial x} = \frac{\partial^2 w}{\partial x\partial y} = \frac{(x^2+y)^2(2x+1)-(-x^2+2xy+y)2(x^2+y)\cdot 1}{\left[(x^2+y)^2\right]^2}$
$= \frac{2x^3+3x^2-2xy-y}{(x^2+y)^3}$

51. $\frac{\partial w}{\partial x} = \frac{2}{2x+3y}$, $\frac{\partial w}{\partial y} = \frac{3}{2x+3y}$, $\frac{\partial^2 w}{\partial y\partial x} = \frac{-6}{(2x+3y)^2}$, and $\frac{\partial^2 w}{\partial x\partial y} = \frac{-6}{(2x+3y)^2}$

52. $\frac{\partial w}{\partial x} = e^x + \ln y + \frac{y}{x}$, $\frac{\partial w}{\partial y} = \frac{x}{y} + \ln x$, $\frac{\partial^2 w}{\partial y\partial x} = \; = \frac{1}{y} + \frac{1}{x}$, and $\frac{\partial^2 w}{\partial x\partial y} = \frac{1}{y} + \frac{1}{x}$

53. $\frac{\partial w}{\partial x} = y^2 + 2xy^3 + 3x^2y^4$, $\frac{\partial w}{\partial y} = 2xy + 3x^2y^2 + 4x^3y^3$, $\frac{\partial^2 w}{\partial y\partial x} = 2y + 6xy^2 + 12x^2y^3$, and $\frac{\partial^2 w}{\partial x\partial y} = 2y + 6xy^2 + 12x^2y^3$

54. $\frac{\partial w}{\partial x} = \sin y + y\cos x + y$, $\frac{\partial w}{\partial y} = x\cos y + \sin x + x$, $\frac{\partial^2 w}{\partial y\partial x} = \cos y + \cos x + 1$, and $\frac{\partial^2 w}{\partial x\partial y} = \cos y + \cos x + 1$

55. (a) x first (b) y first (c) x first (d) x first (e) y first (f) y first

56. (a) y first three times (b) y first three times (c) y first twice (d) x first twice

57. $f_x(1,2) = \lim_{h\to 0} \frac{f(1+h,2)-f(1,2)}{h} = \lim_{h\to 0} \frac{[1-(1+h)+2-6(1+h)^2]-(2-6)}{h} = \lim_{h\to 0} \frac{-h-6(1+2h+h^2)+6}{h}$
$= \lim_{h\to 0} \frac{-13h-6h^2}{h} = \lim_{h\to 0} (-13-6h) = -13$,
$f_y(1,2) = \lim_{h\to 0} \frac{f(1,2+h)-f(1,2)}{h} = \lim_{h\to 0} \frac{[1-1+(2+h)-3(2+h)]-(2-6)}{h} = \lim_{h\to 0} \frac{(2-6-2h)-(2-6)}{h}$
$= \lim_{h\to 0} (-2) = -2$

58. $f_x(-2,1) = \lim_{h\to 0} \frac{f(-2+h,1)-f(-2,1)}{h} = \lim_{h\to 0} \frac{[4+2(-2+h)-3-(-2+h)]-(-3+2)}{h}$
$= \lim_{h\to 0} \frac{(2h-1-h)+1}{h} = \lim_{h\to 0} 1 = 1$,
$f_y(-2,1) = \lim_{h\to 0} \frac{f(-2,1+h)-f(-2,1)}{h} = \lim_{h\to 0} \frac{\left[4-4-3(1+h)+2(1+h)^2\right]-(-3+2)}{h}$
$= \lim_{h\to 0} \frac{(-3-3h+2+4h+2h^2)+1}{h} = \lim_{h\to 0} \frac{h+2h^2}{h} = \lim_{h\to 0} (1+2h) = 1$

59. $f_x(-2,3) = \lim_{h\to 0} \frac{f(-2+h,3)-f(-2,3)}{h} = \lim_{h\to 0} \frac{\sqrt{2(-2+h)+9-1}-\sqrt{-4+9-1}}{h}$
$= \lim_{h\to 0} \frac{\sqrt{2h+4}-2}{h} = \lim_{h\to 0} \left(\frac{\sqrt{2h+4}-2}{h} \cdot \frac{\sqrt{2h+4}+2}{\sqrt{2h+4}+2}\right) = \lim_{h\to 0} \frac{2}{\sqrt{2h+4}+2} = \frac{1}{2}$,
$f_y(-2,3) = \lim_{h\to 0} \frac{f(-2,3+h)-f(-2,3)}{h} = \lim_{h\to 0} \frac{\sqrt{-4+3(3+h)-1}-\sqrt{-4+9-1}}{h}$
$= \lim_{h\to 0} \frac{\sqrt{3h+4}-2}{h} = \lim_{h\to 0} \left(\frac{\sqrt{3h+4}-2}{h} \cdot \frac{\sqrt{3h+4}+2}{\sqrt{3h+4}+2}\right) = \lim_{h\to 0} \frac{3}{\sqrt{2h+4}+2} = \frac{3}{4}$

60. $f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{\sin(h^3+0)}{h^2+0}-0}{h} = \lim_{h\to 0} \frac{\sin h^3}{h^3} = 1$
$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{\sin(0+h^4)}{0+h^2}-0}{h} = \lim_{h\to 0} \frac{\sin h^4}{h^3} = \lim_{h\to 0} \left(h \cdot \frac{\sin h^4}{h^4}\right) = 0\cdot 1 = 0$

61. (a) In the plane $x = 2 \Rightarrow f_y(x,y) = 3 \Rightarrow f_y(2,-1) = 3 \Rightarrow m = 3$
(b) In the plane $y = -1 \Rightarrow f_x(x,y) = 2 \Rightarrow f_y(2,-1) = 2 \Rightarrow m = 2$

62. (a) In the plane $x = -1 \Rightarrow f_y(x, y) = 3y^2 \Rightarrow f_y(-1, 1) = 3(1)^2 = 3 \Rightarrow m = 3$
(b) In the plane $y = 1 \Rightarrow f_x(x, y) = 2x \Rightarrow f_y(-1, 1) = 2(-1) = -2 \Rightarrow m = -2$

63. $f_z(x_0, y_0, z_0) = \lim\limits_{h \to 0} \frac{f(x_0, y_0, z_0 + h) - f(x_0, y_0, z_0)}{h}$;
$f_z(1, 2, 3) = \lim\limits_{h \to 0} \frac{f(1, 2, 3+h) - f(1, 2, 3)}{h} = \lim\limits_{h \to 0} \frac{2(3+h)^2 - 2(9)}{h} = \lim\limits_{h \to 0} \frac{12h + 2h^2}{h} = \lim\limits_{h \to 0} (12 + 2h) = 12$

64. $f_y(x_0, y_0, z_0) = \lim\limits_{h \to 0} \frac{f(x_0, y_0 + h, z_0) - f(x_0, y_0, z_0)}{h}$;
$f_y(-1, 0, 3) = \lim\limits_{h \to 0} \frac{f(-1, h, 3) - f(-1, 0, 3)}{h} = \lim\limits_{h \to 0} \frac{(2h^2 + 9h) - 0}{h} = \lim\limits_{h \to 0} (2h + 9) = 9$

65. $y + \left(3z^2 \frac{\partial z}{\partial x}\right) x + z^3 - 2y \frac{\partial z}{\partial x} = 0 \Rightarrow (3xz^2 - 2y) \frac{\partial z}{\partial x} = -y - z^3 \Rightarrow$ at $(1, 1, 1)$ we have $(3 - 2) \frac{\partial z}{\partial x} = -1 - 1$ or $\frac{\partial z}{\partial x} = -2$

66. $\left(\frac{\partial x}{\partial z}\right) z + x + \left(\frac{y}{x}\right) \frac{\partial x}{\partial z} - 2x \frac{\partial x}{\partial z} = 0 \Rightarrow \left(z + \frac{y}{x} - 2x\right) \frac{\partial x}{\partial z} = -x \Rightarrow$ at $(1, -1, -3)$ we have $(-3 - 1 - 2) \frac{\partial x}{\partial z} = -1$ or $\frac{\partial x}{\partial z} = \frac{1}{6}$

67. $a^2 = b^2 + c^2 - 2bc \cos A \Rightarrow 2a = (2bc \sin A) \frac{\partial A}{\partial a} \Rightarrow \frac{\partial A}{\partial a} = \frac{a}{bc \sin A}$; also $0 = 2b - 2c \cos A + (2bc \sin A) \frac{\partial A}{\partial b}$
$\Rightarrow 2c \cos A - 2b = (2bc \sin A) \frac{\partial A}{\partial b} \Rightarrow \frac{\partial A}{\partial b} = \frac{c \cos A - b}{bc \sin A}$

68. $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{(\sin A) \frac{\partial a}{\partial A} - a \cos A}{\sin^2 A} = 0 \Rightarrow (\sin A) \frac{\partial a}{\partial x} - a \cos A = 0 \Rightarrow \frac{\partial a}{\partial A} = \frac{a \cos A}{\sin A}$; also
$\left(\frac{1}{\sin A}\right) \frac{\partial a}{\partial B} = b(-\csc B \cot B) \Rightarrow \frac{\partial a}{\partial B} = -b \csc B \cot B \sin A$

69. Differentiating each equation implicitly gives $1 = v_x \ln u + \left(\frac{v}{u}\right) u_x$ and $0 = u_x \ln v + \left(\frac{u}{v}\right) v_x$ or

$$\left.\begin{aligned} (\ln u)\, v_x + \left(\tfrac{v}{u}\right) u_x &= 1 \\ \left(\tfrac{u}{v}\right) v_x + (\ln v)\, u_x &= 0 \end{aligned}\right\} \Rightarrow v_x = \frac{\begin{vmatrix} 1 & \frac{v}{u} \\ 0 & \ln v \end{vmatrix}}{\begin{vmatrix} \ln u & \frac{v}{u} \\ \frac{u}{v} & \ln v \end{vmatrix}} = \frac{\ln v}{(\ln u)(\ln v) - 1}$$

70. Differentiating each equation implicitly gives $1 = (2x)x_u - (2y)y_u$ and $0 = (2x)x_u - y_u$ or

$$\left.\begin{aligned} (2x)x_u - (2y)y_u &= 1 \\ (2x)x_u - \quad y_u &= 0 \end{aligned}\right\} \Rightarrow x_u = \frac{\begin{vmatrix} 1 & -2y \\ 0 & -1 \end{vmatrix}}{\begin{vmatrix} 2x & -2y \\ 2x & -1 \end{vmatrix}} = \frac{-1}{-2x + 4xy} = \frac{1}{2x - 4xy} \text{ and}$$

$y_u = \frac{\begin{vmatrix} 2x & 1 \\ 2x & 0 \end{vmatrix}}{-2x + 4xy} = \frac{-2x}{-2x + 4xy} = \frac{2x}{2x - 4xy} = \frac{1}{1 - 2y}$; next $s = x^2 + y^2 \Rightarrow \frac{\partial s}{\partial u} = 2x \frac{\partial x}{\partial u} + 2y \frac{\partial y}{\partial u}$
$= 2x \left(\frac{1}{2x - 4xy}\right) + 2y \left(\frac{1}{1 - 2y}\right) = \frac{1}{1 - 2y} + \frac{2y}{1 - 2y} = \frac{1 + 2y}{1 - 2y}$

71. $f_x(x, y) = \begin{cases} 0 & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases} \Rightarrow f_x(x, y) = 0$ for all points (x, y); at $y = 0$, $f_y(x, 0) = \lim\limits_{h \to 0} \frac{f(x, 0+h) - f(x, 0)}{h} = \lim\limits_{h \to 0} \frac{f(x, h) - 0}{h}$
$= \lim\limits_{h \to 0} \frac{f(x, h)}{h} = 0$ because $\lim\limits_{h \to 0^-} \frac{f(x, h)}{h} = \lim\limits_{h \to 0^-} \frac{h^3}{h} = 0$ and $\lim\limits_{h \to 0^+} \frac{f(x, h)}{h} = \lim\limits_{h \to 0^+} \frac{h^2}{h} = 0 \Rightarrow f_y(x, y) = \begin{cases} 3y^2 & \text{if } y \geq 0 \\ -2y & \text{if } y < 0 \end{cases}$;
$f_{yx}(x, y) = f_{xy}(x, y) = 0$ for all points (x, y)

72. At $x = 0$, $f_x(0, y) = \lim\limits_{h \to 0} \frac{f(0+h, y) - f(0, y)}{h} = \lim\limits_{h \to 0} \frac{f(h, y) - 0}{h} = \lim\limits_{h \to 0} \frac{f(h, y)}{h}$ which does not exist because $\lim\limits_{h \to 0^-} \frac{f(h, y)}{h}$
$= \lim\limits_{h \to 0^-} \frac{h^2}{h} = 0$ and $\lim\limits_{h \to 0^+} \frac{f(h, y)}{h} = \lim\limits_{h \to 0^+} \frac{\sqrt{h}}{h} = \lim\limits_{h \to 0^+} \frac{1}{\sqrt{h}} = +\infty \Rightarrow f_x(x, y) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } x > 0 \\ 2x & \text{if } x < 0 \end{cases}$;
$f_y(x, y) = \begin{cases} 0 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \Rightarrow f_y(x, y) = 0$ for all points (x, y); $f_{yx}(x, y) = 0$ for all points (x, y), while $f_{xy}(x, y) = 0$ for all points (x, y) such that $x \neq 0$.

73. $\frac{\partial f}{\partial x} = 2x,\ \frac{\partial f}{\partial y} = 2y,\ \frac{\partial f}{\partial z} = -4z \Rightarrow \frac{\partial^2 f}{\partial x^2} = 2,\ \frac{\partial^2 f}{\partial y^2} = 2,\ \frac{\partial^2 f}{\partial z^2} = -4 \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 2 + 2 + (-4) = 0$

74. $\frac{\partial f}{\partial x} = -6xz,\ \frac{\partial f}{\partial y} = -6yz,\ \frac{\partial f}{\partial z} = 6z^2 - 3\left(x^2 + y^2\right),\ \frac{\partial^2 f}{\partial x^2} = -6z,\ \frac{\partial^2 f}{\partial y^2} = -6z,\ \frac{\partial^2 f}{\partial z^2} = 12z \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
$= -6z - 6z + 12z = 0$

75. $\frac{\partial f}{\partial x} = -2e^{-2y} \sin 2x,\ \frac{\partial f}{\partial y} = -2e^{-2y} \cos 2x,\ \frac{\partial^2 f}{\partial x^2} = -4e^{-2y} \cos 2x,\ \frac{\partial^2 f}{\partial y^2} = 4e^{-2y} \cos 2x \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$
$= -4e^{-2y} \cos 2x + 4e^{-2y} \cos 2x = 0$

76. $\frac{\partial f}{\partial x} = \frac{x}{x^2 + y^2},\ \frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2},\ \frac{\partial^2 f}{\partial x^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},\ \frac{\partial^2 f}{\partial y^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = 0$

77. $\frac{\partial f}{\partial x} = 3,\ \frac{\partial f}{\partial y} = 2,\ \frac{\partial^2 f}{\partial x^2} = 0,\ \frac{\partial^2 f}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 + 0 = 0$

78. $\frac{\partial f}{\partial x} = \frac{1/y}{1 + \left(\frac{x}{y}\right)^2} = \frac{y}{y^2 + x^2},\ \frac{\partial f}{\partial y} = \frac{-x/y^2}{1 + \left(\frac{x}{y}\right)^2} = \frac{-x}{y^2 + x^2},\ \frac{\partial^2 f}{\partial x^2} = \frac{(y^2 + x^2) \cdot 0 - y \cdot 2x}{(y^2 + x^2)^2} = \frac{-2xy}{(y^2 + x^2)^2},\ \frac{\partial^2 f}{\partial y^2} = \frac{(y^2 + x^2) \cdot 0 - (-x) \cdot 2y}{(y^2 + x^2)^2} = \frac{2xy}{(y^2 + x^2)^2}$
$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{-2xy}{(y^2 + x^2)^2} + \frac{2xy}{(y^2 + x^2)^2} = 0$

79. $\frac{\partial f}{\partial x} = -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}(2x) = -x\left(x^2 + y^2 + z^2\right)^{-3/2},\ \frac{\partial f}{\partial y} = -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}(2y)$
$= -y\left(x^2 + y^2 + z^2\right)^{-3/2},\ \frac{\partial f}{\partial z} = -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}(2z) = -z\left(x^2 + y^2 + z^2\right)^{-3/2};$
$\frac{\partial^2 f}{\partial x^2} = -\left(x^2 + y^2 + z^2\right)^{-3/2} + 3x^2\left(x^2 + y^2 + z^2\right)^{-5/2},\ \frac{\partial^2 f}{\partial y^2} = -\left(x^2 + y^2 + z^2\right)^{-3/2} + 3y^2\left(x^2 + y^2 + z^2\right)^{-5/2},$
$\frac{\partial^2 f}{\partial z^2} = -\left(x^2 + y^2 + z^2\right)^{-3/2} + 3z^2\left(x^2 + y^2 + z^2\right)^{-5/2} \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
$= \left[-\left(x^2 + y^2 + z^2\right)^{-3/2} + 3x^2\left(x^2 + y^2 + z^2\right)^{-5/2}\right] + \left[-\left(x^2 + y^2 + z^2\right)^{-3/2} + 3y^2\left(x^2 + y^2 + z^2\right)^{-5/2}\right]$
$+ \left[-\left(x^2 + y^2 + z^2\right)^{-3/2} + 3z^2\left(x^2 + y^2 + z^2\right)^{-5/2}\right] = -3\left(x^2 + y^2 + z^2\right)^{-3/2} + \left(3x^2 + 3y^2 + 3z^2\right)\left(x^2 + y^2 + z^2\right)^{-5/2} = 0$

80. $\frac{\partial f}{\partial x} = 3e^{3x+4y} \cos 5z,\ \frac{\partial f}{\partial y} = 4e^{3x+4y} \cos 5z,\ \frac{\partial f}{\partial z} = -5e^{3x+4y} \sin 5z;\ \frac{\partial^2 f}{\partial x^2} = 9e^{3x+4y} \cos 5z,\ \frac{\partial^2 f}{\partial y^2} = 16e^{3x+4y} \cos 5z,$
$\frac{\partial^2 f}{\partial z^2} = -25e^{3x+4y} \cos 5z \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 9e^{3x+4y} \cos 5z + 16e^{3x+4y} \cos 5z - 25e^{3x+4y} \cos 5z = 0$

81. $\frac{\partial w}{\partial x} = \cos(x + ct),\ \frac{\partial w}{\partial t} = c\cos(x + ct);\ \frac{\partial^2 w}{\partial x^2} = -\sin(x + ct),\ \frac{\partial^2 w}{\partial t^2} = -c^2 \sin(x + ct) \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2\left[-\sin(x + ct)\right] = c^2\ \frac{\partial^2 w}{\partial x^2}$

82. $\frac{\partial w}{\partial x} = -2\sin(2x + 2ct),\ \frac{\partial w}{\partial t} = -2c\sin(2x + 2ct);\ \frac{\partial^2 w}{\partial x^2} = -4\cos(2x + 2ct),\ \frac{\partial^2 w}{\partial t^2} = -4c^2 \cos(2x + 2ct)$
$\Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2[-4\cos(2x + 2ct)] = c^2\ \frac{\partial^2 w}{\partial x^2}$

83. $\frac{\partial w}{\partial x} = \cos(x + ct) - 2\sin(2x + 2ct),\ \frac{\partial w}{\partial t} = c\cos(x + ct) - 2c\sin(2x + 2ct);$
$\frac{\partial^2 w}{\partial x^2} = -\sin(x + ct) - 4\cos(2x + 2ct),\ \frac{\partial^2 w}{\partial t^2} = -c^2 \sin(x + ct) - 4c^2 \cos(2x + 2ct)$
$\Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2[-\sin(x + ct) - 4\cos(2x + 2ct)] = c^2\ \frac{\partial^2 w}{\partial x^2}$

84. $\frac{\partial w}{\partial x} = \frac{1}{x + ct},\ \frac{\partial w}{\partial t} = \frac{c}{x + ct};\ \frac{\partial^2 w}{\partial x^2} = \frac{-1}{(x + ct)^2},\ \frac{\partial^2 w}{\partial t^2} = \frac{-c^2}{(x + ct)^2} \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2\left[\frac{-1}{(x + ct)^2}\right] = c^2\ \frac{\partial^2 w}{\partial x^2}$

85. $\frac{\partial w}{\partial x} = 2\sec^2(2x - 2ct),\ \frac{\partial w}{\partial t} = -2c\sec^2(2x - 2ct);\ \frac{\partial^2 w}{\partial x^2} = 8\sec^2(2x - 2ct)\tan(2x - 2ct),$
$\frac{\partial^2 w}{\partial t^2} = 8c^2 \sec^2(2x - 2ct)\tan(2x - 2ct) \Rightarrow ux\frac{\partial^2 w}{\partial t^2} = c^2[8\sec^2(2x - 2ct)\tan(2x - 2ct)] = c^2\ \frac{\partial^2 w}{\partial x^2}$

86. $\frac{\partial w}{\partial x} = -15\sin(3x+3ct) + e^{x+ct}$, $\frac{\partial w}{\partial t} = -15c\sin(3x+3ct) + ce^{x+ct}$; $\frac{\partial^2 w}{\partial x^2} = -45\cos(3x+3ct) + e^{x+ct}$, $\frac{\partial^2 w}{\partial t^2} = -45c^2\cos(3x+3ct) + c^2e^{x+ct} \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2\left[-45\cos(3x+3ct) + e^{x+ct}\right] = c^2\frac{\partial^2 w}{\partial x^2}$

87. $\frac{\partial w}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} = \frac{\partial f}{\partial u}(ac) \Rightarrow \frac{\partial^2 w}{\partial t^2} = (ac)\left(\frac{\partial^2 f}{\partial u^2}\right)(ac) = a^2c^2\frac{\partial^2 f}{\partial u^2}$; $\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial f}{\partial u}\cdot a \Rightarrow \frac{\partial^2 w}{\partial x^2} = \left(a\frac{\partial^2 f}{\partial u^2}\right)\cdot a$ $= a^2\frac{\partial^2 f}{\partial u^2} \Rightarrow \frac{\partial^2 w}{\partial t^2} = a^2c^2\frac{\partial^2 f}{\partial u^2} = c^2\left(a^2\frac{\partial^2 f}{\partial u^2}\right) = c^2\frac{\partial^2 w}{\partial x^2}$

88. If the first partial derivatives are continuous throughout an open region R, then by Theorem 3 in this section of the text, $f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$, where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$. Then as $(x,y) \to (x_0,y_0)$, $\Delta x \to 0$ and $\Delta y \to 0 \Rightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0) \Rightarrow$ f is continuous at every point (x_0,y_0) in R.

89. Yes, since f_{xx}, f_{yy}, f_{xy}, and f_{yx} are all continuous on R, use the same reasoning as in Exercise 76 with $f_x(x,y) = f_x(x_0,y_0) + f_{xx}(x_0,y_0)\,\Delta x + f_{xy}(x_0,y_0)\,\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$ and $f_y(x,y) = f_y(x_0,y_0) + f_{yx}(x_0,y_0)\,\Delta x + f_{yy}(x_0,y_0)\,\Delta y + \hat{\epsilon}_1\Delta x + \hat{\epsilon}_2\Delta y$. Then $\lim_{(x,y)\to(x_0,y_0)} f_x(x,y) = f_x(x_0,y_0)$ and $\lim_{(x,y)\to(x_0,y_0)} f_y(x,y) = f_y(x_0,y_0)$.

90. To find α and β so that $u_t = u_{xx} \Rightarrow u_t = -\beta\sin(\alpha x)e^{-\beta t}$ and $u_x = \alpha\cos(\alpha x)e^{-\beta t} \Rightarrow u_{xx} = -\alpha^2\sin(\alpha x)e^{-\beta t}$; then $u_t = u_{xx} \Rightarrow -\beta\sin(\alpha x)e^{-\beta t} = -\alpha^2\sin(\alpha x)e^{-\beta t}$, thus $u_t = u_{xx}$ only if $\beta = \alpha^2$

91. $f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{h\cdot 0^2}{h^2+0^4}-0}{h} = \lim_{h\to 0}\frac{0}{h} = 0$; $f_y(0,0) = \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{0\cdot h^2}{0^2+h^4}-0}{h} = \lim_{h\to 0}\frac{0}{h} = 0$; $\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x=ky^2}} f(x,y) = \lim_{y\to 0}\frac{(ky^2)y^2}{(ky^2)^2+y^4} = \lim_{y\to 0}\frac{ky^4}{k^2y^4+y^4} = \lim_{y\to 0}\frac{k}{k^2+1} = \frac{k}{k^2+1} \Rightarrow$ different limits for different values of $k \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist $\Rightarrow f(x,y)$ is not continuous at $(0,0) \Rightarrow$ by Theorem 4, $f(x,y)$ is not differentiable at $(0,0)$.

92. $f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{f(h,0)-1}{h} = \lim_{h\to 0}\frac{1-1}{h} = 0$; $f_y(0,0) = \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0}\frac{f(0,h)-1}{h} = \lim_{h\to 0}\frac{1-1}{h} = 0$; $\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=x^2}} f(x,y) = \lim_{y\to 0} 0 = 0$ but $\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=1.5x^2}} f(x,y) = \lim_{y\to 0} 1 = 1 \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist $\Rightarrow f(x,y)$ is not continuous at $(0,0) \Rightarrow$ by Theorem 4, $f(x,y)$ is not differentiable at $(0,0)$.

14.4 THE CHAIN RULE

1. (a) $\frac{\partial w}{\partial x} = 2x$, $\frac{\partial w}{\partial y} = 2y$, $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t \Rightarrow \frac{dw}{dt} = -2x\sin t + 2y\cos t = -2\cos t\sin t + 2\sin t\cos t$ $= 0$; $w = x^2+y^2 = \cos^2 t + \sin^2 t = 1 \Rightarrow \frac{dw}{dt} = 0$
   (b) $\frac{dw}{dt}(\pi) = 0$

2. (a) $\frac{\partial w}{\partial x} = 2x$, $\frac{\partial w}{\partial y} = 2y$, $\frac{dx}{dt} = -\sin t + \cos t$, $\frac{dy}{dt} = -\sin t - \cos t \Rightarrow \frac{dw}{dt}$
   $= (2x)(-\sin t + \cos t) + (2y)(-\sin t - \cos t)$
   $= 2(\cos t + \sin t)(\cos t - \sin t) - 2(\cos t - \sin t)(\sin t + \cos t) = (2\cos^2 t - 2\sin^2 t) - (2\cos^2 t - 2\sin^2 t)$
   $= 0$; $w = x^2+y^2 = (\cos t + \sin t)^2 + (\cos t - \sin t)^2 = 2\cos^2 t + 2\sin^2 t = 2 \Rightarrow \frac{dw}{dt} = 0$
   (b) $\frac{dw}{dt}(0) = 0$

3. (a) $\frac{\partial w}{\partial x} = \frac{1}{z}$, $\frac{\partial w}{\partial y} = \frac{1}{z}$, $\frac{\partial w}{\partial z} = \frac{-(x+y)}{z^2}$, $\frac{dx}{dt} = -2\cos t \sin t$, $\frac{dy}{dt} = 2\sin t \cos t$, $\frac{dz}{dt} = -\frac{1}{t^2}$

$\Rightarrow \frac{dw}{dt} = -\frac{2}{z}\cos t \sin t + \frac{2}{z}\sin t \cos t + \frac{x+y}{z^2t^2} = \frac{\cos^2 t + \sin^2 t}{\left(\frac{1}{t^2}\right)(t^2)} = 1$; $w = \frac{x}{z} + \frac{y}{z} = \frac{\cos^2 t}{\left(\frac{1}{t}\right)} + \frac{\sin^2 t}{\left(\frac{1}{t}\right)} = t \Rightarrow \frac{dw}{dt} = 1$

(b) $\frac{dw}{dt}(3) = 1$

4. (a) $\frac{\partial w}{\partial x} = \frac{2x}{x^2+y^2+z^2}$, $\frac{\partial w}{\partial y} = \frac{2y}{x^2+y^2+z^2}$, $\frac{\partial w}{\partial z} = \frac{2z}{x^2+y^2+z^2}$, $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$, $\frac{dz}{dt} = 2t^{-1/2}$

$\Rightarrow \frac{dw}{dt} = \frac{-2x\sin t}{x^2+y^2+z^2} + \frac{2y\cos t}{x^2+y^2+z^2} + \frac{4zt^{-1/2}}{x^2+y^2+z^2} = \frac{-2\cos t \sin t + 2\sin t \cos t + 4\left(4t^{1/2}\right)t^{-1/2}}{\cos^2 t + \sin^2 t + 16t}$

$= \frac{16}{1+16t}$; $w = \ln(x^2+y^2+z^2) = \ln(\cos^2 t + \sin^2 t + 16t) = \ln(1+16t) \Rightarrow \frac{dw}{dt} = \frac{16}{1+16t}$

(b) $\frac{dw}{dt}(3) = \frac{16}{49}$

5. (a) $\frac{\partial w}{\partial x} = 2ye^x$, $\frac{\partial w}{\partial y} = 2e^x$, $\frac{\partial w}{\partial z} = -\frac{1}{z}$, $\frac{dx}{dt} = \frac{2t}{t^2+1}$, $\frac{dy}{dt} = \frac{1}{t^2+1}$, $\frac{dz}{dt} = e^t \Rightarrow \frac{dw}{dt} = \frac{4yte^x}{t^2+1} + \frac{2e^x}{t^2+1} - \frac{e^t}{z}$

$= \frac{(4t)(\tan^{-1} t)(t^2+1)}{t^2+1} + \frac{2(t^2+1)}{t^2+1} - \frac{e^t}{e^t} = 4t\tan^{-1} t + 1$; $w = 2ye^x - \ln z = (2\tan^{-1} t)(t^2+1) - t$

$\Rightarrow \frac{dw}{dt} = \left(\frac{2}{t^2+1}\right)(t^2+1) + (2\tan^{-1} t)(2t) - 1 = 4t\tan^{-1} t + 1$

(b) $\frac{dw}{dt}(1) = (4)(1)\left(\frac{\pi}{4}\right) + 1 = \pi + 1$

6. (a) $\frac{\partial w}{\partial x} = -y\cos xy$, $\frac{\partial w}{\partial y} = -x\cos xy$, $\frac{\partial w}{\partial z} = 1$, $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = \frac{1}{t}$, $\frac{dz}{dt} = e^{t-1} \Rightarrow \frac{dw}{dt} = -y\cos xy - \frac{x\cos xy}{t} + e^{t-1}$

$= -(\ln t)[\cos(t\ln t)] - \frac{t\cos(t\ln t)}{t} + e^{t-1} = -(\ln t)[\cos(t\ln t)] - \cos(t\ln t) + e^{t-1}$; $w = z - \sin xy$

$= e^{t-1} - \sin(t\ln t) \Rightarrow \frac{dw}{dt} = e^{t-1} - [\cos(t\ln t)]\left[\ln t + t\left(\frac{1}{t}\right)\right] = e^{t-1} - (1+\ln t)\cos(t\ln t)$

(b) $\frac{dw}{dt}(1) = 1 - (1+0)(1) = 0$

7. (a) $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (4e^x \ln y)\left(\frac{\cos v}{u\cos v}\right) + \left(\frac{4e^x}{y}\right)(\sin v) = \frac{4e^x \ln y}{u} + \frac{4e^x \sin v}{y}$

$= \frac{4(u\cos v)\ln(u\sin v)}{u} + \frac{4(u\cos v)(\sin v)}{u\sin v} = (4\cos v)\ln(u\sin v) + 4\cos v$;

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (4e^x \ln y)\left(\frac{-u\sin v}{u\cos v}\right) + \left(\frac{4e^x}{y}\right)(u\cos v) = -(4e^x\ln y)(\tan v) + \frac{4e^x u\cos v}{y}$

$= [-4(u\cos v)\ln(u\sin v)](\tan v) + \frac{4(u\cos v)(u\cos v)}{u\sin v} = (-4u\sin v)\ln(u\sin v) + \frac{4u\cos^2 v}{\sin v}$;

$z = 4e^x \ln y = 4(u\cos v)\ln(u\sin v) \Rightarrow \frac{\partial z}{\partial u} = (4\cos v)\ln(u\sin v) + 4(u\cos v)\left(\frac{\sin v}{u\sin v}\right)$

$= (4\cos v)\ln(u\sin v) + 4\cos v$; also $\frac{\partial z}{\partial v} = (-4u\sin v)\ln(u\sin v) + 4(u\cos v)\left(\frac{u\cos v}{u\sin v}\right)$

$= (-4u\sin v)\ln(u\sin v) + \frac{4u\cos^2 v}{\sin v}$

(b) At $\left(2, \frac{\pi}{4}\right)$: $\frac{\partial z}{\partial u} = 4\cos\frac{\pi}{4}\ln\left(2\sin\frac{\pi}{4}\right) + 4\cos\frac{\pi}{4} = 2\sqrt{2}\ln\sqrt{2} + 2\sqrt{2} = \sqrt{2}(\ln 2 + 2)$;

$\frac{\partial z}{\partial v} = (-4)(2)\sin\frac{\pi}{4}\ln\left(2\sin\frac{\pi}{4}\right) + \frac{(4)(2)\left(\cos^2\frac{\pi}{4}\right)}{\left(\sin\frac{\pi}{4}\right)} = -4\sqrt{2}\ln\sqrt{2} + 4\sqrt{2} = -2\sqrt{2}\ln 2 + 4\sqrt{2}$

8. (a) $\frac{\partial z}{\partial u} = \left[\frac{\left(\frac{1}{y}\right)}{\left(\frac{x}{y}\right)^2+1}\right]\cos v + \left[\frac{\left(\frac{-x}{y^2}\right)}{\left(\frac{x}{y}\right)^2+1}\right]\sin v = \frac{y\cos v}{x^2+y^2} - \frac{x\sin v}{x^2+y^2} = \frac{(u\sin v)(\cos v) - (u\cos v)(\sin v)}{u^2} = 0$;

$\frac{\partial z}{\partial v} = \left[\frac{\left(\frac{1}{y}\right)}{\left(\frac{x}{y}\right)^2+1}\right](-u\sin v) + \left[\frac{\left(\frac{-x}{y^2}\right)}{\left(\frac{x}{y}\right)^2+1}\right]u\cos v = -\frac{yu\sin v}{x^2+y^2} - \frac{xu\cos v}{x^2+y^2} = \frac{-(u\sin v)(u\sin v) - (u\cos v)(u\cos v)}{u^2}$

$= -\sin^2 v - \cos^2 v = -1$; $z = \tan^{-1}\left(\frac{x}{y}\right) = \tan^{-1}(\cot v) \Rightarrow \frac{\partial z}{\partial u} = 0$ and $\frac{\partial z}{\partial v} = \left(\frac{1}{1+\cot^2 v}\right)(-\csc^2 v)$

$= \frac{-1}{\sin^2 v + \cos^2 v} = -1$

(b) At $\left(1.3, \frac{\pi}{6}\right)$: $\frac{\partial z}{\partial u} = 0$ and $\frac{\partial z}{\partial v} = -1$

9. (a) $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} = (y+z)(1) + (x+z)(1) + (y+x)(v) = x + y + 2z + v(y+x)$
$= (u+v) + (u-v) + 2uv + v(2u) = 2u + 4uv$; $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$
$= (y+z)(1) + (x+z)(-1) + (y+x)(u) = y - x + (y+x)u = -2v + (2u)u = -2v + 2u^2$;
$w = xy + yz + xz = (u^2 - v^2) + (u^2 v - uv^2) + (u^2 v + uv^2) = u^2 - v^2 + 2u^2 v \Rightarrow \frac{\partial w}{\partial u} = 2u + 4uv$ and
$\frac{\partial w}{\partial v} = -2v + 2u^2$

(b) At $\left(\frac{1}{2}, 1\right)$: $\frac{\partial w}{\partial u} = 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right)(1) = 3$ and $\frac{\partial w}{\partial v} = -2(1) + 2\left(\frac{1}{2}\right)^2 = -\frac{3}{2}$

10. (a) $\frac{\partial w}{\partial u} = \left(\frac{2x}{x^2+y^2+z^2}\right)(e^v \sin u + ue^v \cos u) + \left(\frac{2y}{x^2+y^2+z^2}\right)(e^v \cos u - ue^v \sin u) + \left(\frac{2z}{x^2+y^2+z^2}\right)(e^v)$
$= \left(\frac{2ue^v \sin u}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\right)(e^v \sin u + ue^v \cos u)$
$+ \left(\frac{2ue^v \cos u}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\right)(e^v \cos u - ue^v \sin u)$
$+ \left(\frac{2ue^v}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\right)(e^v) = \frac{2}{u}$;
$\frac{\partial w}{\partial v} = \left(\frac{2x}{x^2+y^2+z^2}\right)(ue^v \sin u) + \left(\frac{2y}{x^2+y^2+z^2}\right)(ue^v \cos u) + \left(\frac{2z}{x^2+y^2+z^2}\right)(ue^v)$
$= \left(\frac{2ue^v \sin u}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\right)(ue^v \sin u)$
$+ \left(\frac{2ue^v \cos u}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\right)(ue^v \cos u)$
$+ \left(\frac{2ue^v}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\right)(ue^v) = 2$; $w = \ln(u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}) = \ln(2u^2e^{2v})$
$= \ln 2 + 2\ln u + 2v \Rightarrow \frac{\partial w}{\partial u} = \frac{2}{u}$ and $\frac{\partial w}{\partial v} = 2$

(b) At $(-2, 0)$: $\frac{\partial w}{\partial u} = \frac{2}{-2} = -1$ and $\frac{\partial w}{\partial v} = 2$

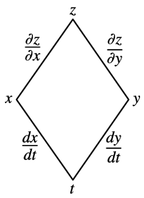
11. (a) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = \frac{1}{q-r} + \frac{r-p}{(q-r)^2} + \frac{p-q}{(q-r)^2} = \frac{q-r+r-p+p-q}{(q-r)^2} = 0$;
$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} = \frac{1}{q-r} - \frac{r-p}{(q-r)^2} + \frac{p-q}{(q-r)^2} = \frac{q-r-r+p+p-q}{(q-r)^2} = \frac{2p-2r}{(q-r)^2}$
$= \frac{(2x+2y+2z)-(2x+2y-2z)}{(2z-2y)^2} = \frac{z}{(z-y)^2}$; $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$
$= \frac{1}{q-r} + \frac{r-p}{(q-r)^2} - \frac{p-q}{(q-r)^2} = \frac{q-r+r-p-p+q}{(q-r)^2} = \frac{2q-2p}{(q-r)^2} = \frac{-4y}{(2z-2y)^2} = -\frac{y}{(z-y)^2}$;
$u = \frac{p-q}{q-r} = \frac{2y}{2z-2y} = \frac{y}{z-y} \Rightarrow \frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = \frac{(z-y)-y(-1)}{(z-y)^2} = \frac{z}{(z-y)^2}$, and $\frac{\partial u}{\partial z} = \frac{(z-y)(0)-y(1)}{(z-y)^2}$
$= -\frac{y}{(z-y)^2}$

(b) At $\left(\sqrt{3}, 2, 1\right)$: $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = \frac{1}{(1-2)^2} = 1$, and $\frac{\partial u}{\partial z} = \frac{-2}{(1-2)^2} = -2$

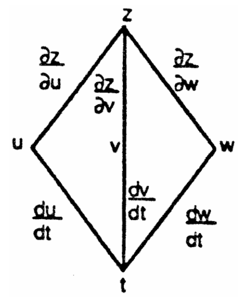
12. (a) $\frac{\partial u}{\partial x} = \frac{e^{qr}}{\sqrt{1-p^2}}(\cos x) + (re^{qr}\sin^{-1} p)(0) + (qe^{qr}\sin^{-1} p)(0) = \frac{e^{qr}\cos x}{\sqrt{1-p^2}} = \frac{e^{z\ln y}\cos x}{\sqrt{1-\sin^2 x}} = y^z$ if $-\frac{\pi}{2} < x < \frac{\pi}{2}$;
$\frac{\partial u}{\partial y} = \frac{e^{qr}}{\sqrt{1-p^2}}(0) + (re^{qr}\sin^{-1} p)\left(\frac{z^2}{y}\right) + (qe^{qr}\sin^{-1} p)(0) = \frac{z^2\, re^{qr}\sin^{-1} p}{y} = \frac{z^2\left(\frac{1}{z}\right)y^z x}{y} = xzy^{z-1}$;
$\frac{\partial u}{\partial z} = \frac{e^{qr}}{\sqrt{1-p^2}}(0) + (re^{qr}\sin^{-1} p)(2z\ln y) + (qe^{qr}\sin^{-1} p)\left(-\frac{1}{z^2}\right) = (2zre^{qr}\sin^{-1} p)(\ln y) - \frac{qe^{qr}\sin^{-1} p}{z^2}$
$= (2z)\left(\frac{1}{z}\right)(y^z x\ln y) - \frac{(z^2\ln y)(y^z)x}{z^2} = xy^z\ln y$; $u = e^{z\ln y}\sin^{-1}(\sin x) = xy^z$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2} \Rightarrow \frac{\partial u}{\partial x} = y^z$,
$\frac{\partial u}{\partial y} = xzy^{z-1}$, and $\frac{\partial u}{\partial z} = \; = xy^z\ln y$ from direct calculations

(b) At $\left(\frac{\pi}{4}, \frac{1}{2}, -\frac{1}{2}\right)$: $\frac{\partial u}{\partial x} = \left(\frac{1}{2}\right)^{-1/2} = \sqrt{2}$, $\frac{\partial u}{\partial y} = \left(\frac{\pi}{4}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)^{(-1/2)-1} = -\frac{\pi\sqrt{2}}{4}$, $\frac{\partial u}{\partial z} = \left(\frac{\pi}{4}\right)\left(\frac{1}{2}\right)^{-1/2}\ln\left(\frac{1}{2}\right) = -\frac{\pi\sqrt{2}\ln 2}{4}$

13. $\frac{dz}{dt} = \frac{\partial z}{\partial x}\,\frac{dx}{dt} + \frac{\partial z}{\partial y}\,\frac{dy}{dt}$

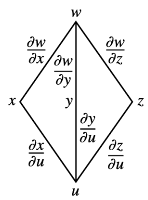


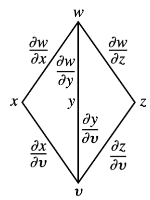
14. $\frac{dz}{dt} = \frac{\partial z}{\partial u}\,\frac{du}{dt} + \frac{\partial z}{\partial v}\,\frac{dv}{dt} + \frac{\partial x}{\partial w}\,\frac{dw}{dt}$



15. $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\,\frac{\partial z}{\partial u}$

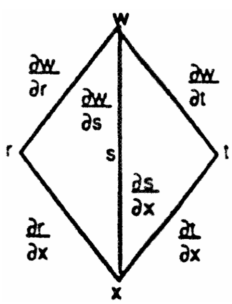
$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\,\frac{\partial z}{\partial v}$

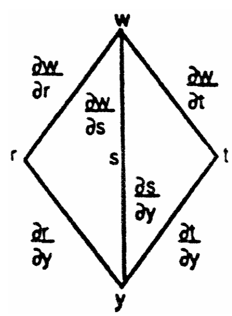




16. $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\,\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\,\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\,\frac{\partial t}{\partial x}$

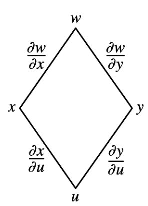
$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\,\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\,\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\,\frac{\partial t}{\partial y}$

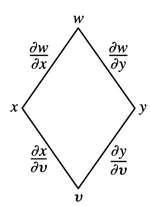




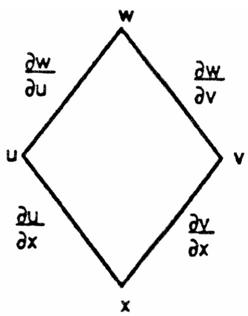
17. $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial u}$

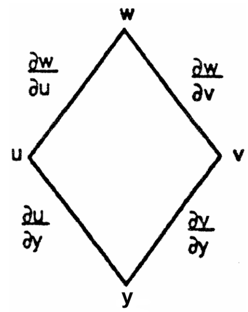
$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\,\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\,\frac{\partial y}{\partial v}$



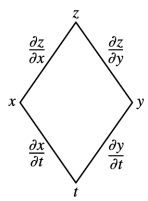


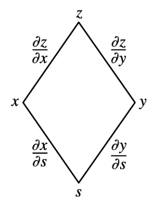
18. $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$ $\qquad$ $\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$



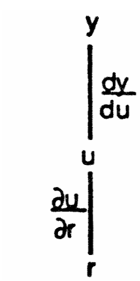


19. $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$ $\qquad$ $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$

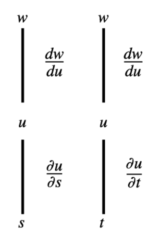




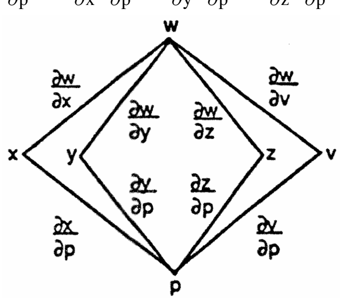
20. $\frac{\partial y}{\partial r} = \frac{dy}{du}\frac{\partial u}{\partial r}$



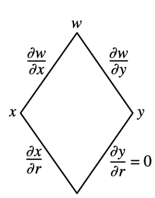
21. $\frac{\partial w}{\partial s} = \frac{dw}{du}\frac{\partial u}{\partial s}$ $\qquad$ $\frac{\partial w}{\partial t} = \frac{dw}{du}\frac{\partial u}{\partial t}$

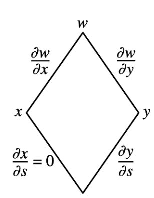


22. $\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial p} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}$

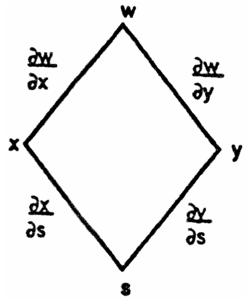


23. $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{dx}{dr} + \frac{\partial w}{\partial y}\frac{dy}{dr} = \frac{\partial w}{\partial x}\frac{dx}{dr}$ since $\frac{dy}{dr} = 0$ $\qquad$ $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{dy}{ds} = \frac{\partial w}{\partial y}\frac{dy}{ds}$ since $\frac{dx}{ds} = 0$





24. $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$



25. Let $F(x,y) = x^3 - 2y^2 + xy = 0 \Rightarrow F_x(x,y) = 3x^2 + y$ and $F_y(x,y) = -4y + x \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2+y}{(-4y+x)}$ $\Rightarrow \frac{dy}{dx}(1,1) = \frac{4}{3}$

26. Let $F(x,y) = xy + y^2 - 3x - 3 = 0 \Rightarrow F_x(x,y) = y - 3$ and $F_y(x,y) = x + 2y \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y-3}{x+2y}$ $\Rightarrow \frac{dy}{dx}(-1,1) = 2$

27. Let $F(x,y) = x^2 + xy + y^2 - 7 = 0 \Rightarrow F_x(x,y) = 2x + y$ and $F_y(x,y) = x + 2y \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x+y}{x+2y}$ $\Rightarrow \frac{dy}{dx}(1,2) = -\frac{4}{5}$

28. Let $F(x,y) = xe^y + \sin xy + y - \ln 2 = 0 \Rightarrow F_x(x,y) = e^y + y\cos xy$ and $F_y(x,y) = xe^y + x\sin xy + 1$ $\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{e^y + y\cos xy}{xe^y + x\sin xy + 1} \Rightarrow \frac{dy}{dx}(0, \ln 2) = -(2 + \ln 2)$

29. Let $F(x,y,z) = z^3 - xy + yz + y^3 - 2 = 0 \Rightarrow F_x(x,y,z) = -y, F_y(x,y,z) = -x + z + 3y^2, F_z(x,y,z) = 3z^2 + y$ $\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-y}{3z^2+y} = \frac{y}{3z^2+y} \Rightarrow \frac{\partial z}{\partial x}(1,1,1) = \frac{1}{4}$; $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-x+z+3y^2}{3z^2+y} = \frac{x-z-3y^2}{3z^2+y}$ $\Rightarrow \frac{\partial z}{\partial y}(1,1,1) = -\frac{3}{4}$

30. Let $F(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0 \Rightarrow F_x(x,y,z) = -\frac{1}{x^2}, F_y(x,y,z) = -\frac{1}{y^2}, F_z(x,y,z) = -\frac{1}{z^2}$ $\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{z^2}\right)} = -\frac{z^2}{x^2} \Rightarrow \frac{\partial z}{\partial x}(2,3,6) = -9$; $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\left(-\frac{1}{y^2}\right)}{\left(-\frac{1}{z^2}\right)} = -\frac{z^2}{y^2} \Rightarrow \frac{\partial z}{\partial y}(2,3,6) = -4$

31. Let $F(x,y,z) = \sin(x+y) + \sin(y+z) + \sin(x+z) = 0 \Rightarrow F_x(x,y,z) = \cos(x+y) + \cos(x+z)$, $F_y(x,y,z) = \cos(x+y) + \cos(y+z), F_z(x,y,z) = \cos(y+z) + \cos(x+z) \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ $= -\frac{\cos(x+y)+\cos(x+z)}{\cos(y+z)+\cos(x+z)} \Rightarrow \frac{\partial z}{\partial x}(\pi,\pi,\pi) = -1$; $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\cos(x+y)+\cos(y+z)}{\cos(y+z)+\cos(x+z)} \Rightarrow \frac{\partial z}{\partial y}(\pi,\pi,\pi) = -1$

32. Let $F(x,y,z) = xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0 \Rightarrow F_x(x,y,z) = e^y + \frac{2}{x}, F_y(x,y,z) = xe^y + e^z, F_z(x,y,z) = ye^z$ $\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\left(e^y + \frac{2}{x}\right)}{ye^z} \Rightarrow \frac{\partial z}{\partial x}(1, \ln 2, \ln 3) = -\frac{4}{3\ln 2}$; $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xe^y+e^z}{ye^z} \Rightarrow \frac{\partial z}{\partial y}(1, \ln 2, \ln 3) = -\frac{5}{3\ln 2}$

33. $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = 2(x+y+z)(1) + 2(x+y+z)[-\sin(r+s)] + 2(x+y+z)[\cos(r+s)]$ $= 2(x+y+z)[1 - \sin(r+s) + \cos(r+s)] = 2[r - s + \cos(r+s) + \sin(r+s)][1 - \sin(r+s) + \cos(r+s)]$ $\Rightarrow \left.\frac{\partial w}{\partial r}\right|_{r=1,s=-1} = 2(3)(2) = 12$

34. $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} = y\left(\frac{2v}{u}\right) + x(1) + \left(\frac{1}{z}\right)(0) = (u+v)\left(\frac{2v}{u}\right) + \frac{v^2}{u} \Rightarrow \left.\frac{\partial w}{\partial v}\right|_{u=-1,v=2} = (1)\left(\frac{4}{-1}\right) + \left(\frac{4}{-1}\right) = -8$

35. $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} = \left(2x - \frac{y}{x^2}\right)(-2) + \left(\frac{1}{x}\right)(1) = \left[2(u - 2v + 1) - \frac{2u+v-2}{(u-2v+1)^2}\right](-2) + \frac{1}{u-2v+1}$ $\Rightarrow \left.\frac{\partial w}{\partial v}\right|_{u=0,v=0} = -7$

36. $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (y\cos xy + \sin y)(2u) + (x\cos xy + x\cos y)(v)$
$= [uv\cos(u^3v + uv^3) + \sin uv](2u) + [(u^2+v^2)\cos(u^3v+uv^3) + (u^2+v^2)\cos uv](v)$
$\Rightarrow \left.\frac{\partial z}{\partial u}\right|_{u=0,v=1} = 0 + (\cos 0 + \cos 0)(1) = 2$

37. $\frac{\partial z}{\partial u} = \frac{dz}{dx}\frac{\partial x}{\partial u} = \left(\frac{5}{1+x^2}\right)e^u = \left[\frac{5}{1+(e^u+\ln v)^2}\right]e^u \Rightarrow \left.\frac{\partial z}{\partial u}\right|_{u=\ln 2,v=1} = \left[\frac{5}{1+(2)^2}\right](2) = 2;$
$\frac{\partial z}{\partial v} = \frac{dz}{dx}\frac{\partial x}{\partial v} = \left(\frac{5}{1+x^2}\right)\left(\frac{1}{v}\right) = \left[\frac{5}{1+(e^u+\ln v)^2}\right]\left(\frac{1}{v}\right) \Rightarrow \left.\frac{\partial z}{\partial v}\right|_{u=\ln 2,v=1} = \left[\frac{5}{1+(2)^2}\right](1) = 1$

38. $\frac{\partial z}{\partial u} = \frac{dz}{dq}\frac{\partial q}{\partial u} = \left(\frac{1}{q}\right)\left(\frac{\sqrt{v+3}}{1+u^2}\right) = \left(\frac{1}{\sqrt{v+3}\tan^{-1}u}\right)\left(\frac{\sqrt{v+3}}{1+u^2}\right) = \frac{1}{(\tan^{-1}u)(1+u^2)} \Rightarrow \left.\frac{\partial z}{\partial u}\right|_{u=1,v=-2} = \frac{1}{(\tan^{-1}1)(1+1^2)} = \frac{2}{\pi};$
$\frac{\partial z}{\partial v} = \frac{dz}{dq}\frac{\partial q}{\partial v} = \left(\frac{1}{q}\right)\left(\frac{\tan^{-1}u}{2\sqrt{v+3}}\right) = \left(\frac{1}{\sqrt{v+3}\tan^{-1}u}\right)\left(\frac{\tan^{-1}u}{2\sqrt{v+3}}\right) = \frac{1}{2(v+3)} \Rightarrow \left.\frac{\partial z}{\partial v}\right|_{u=1,v=-2} = \frac{1}{2}$

39. Let $x = s^3 + t^2 \Rightarrow w = f(s^3+t^2) = f(x) \Rightarrow \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s} = f'(x)\cdot 3s^2 = 3s^2e^{s^3+t^2}$, $\frac{\partial w}{\partial t} = \frac{dw}{dx}\frac{\partial x}{\partial t} = f'(x)\cdot 2t = 2t\,e^{s^3+t^2}$

40. Let $x = ts^2$ and $y = \frac{s}{t} \Rightarrow w = f\left(ts^2, \frac{s}{t}\right) = f(x,y) \Rightarrow \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = f_x(x,y)\cdot 2ts + f_y(x,y)\cdot\frac{1}{t}$
$= (ts^2)\left(\frac{s}{t}\right)\cdot 2ts + \frac{(ts^2)^2}{2}\cdot\frac{1}{t} = 2s^4t + \frac{s^4t}{2} = \frac{5s^4t}{2}$; $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} = f_x(x,y)\cdot s^2 + f_y(x,y)\cdot\frac{-s}{t^2}$
$= (ts^2)\left(\frac{s}{t}\right)\cdot s^2 + \frac{(ts^2)^2}{2}\cdot\left(-\frac{s}{t^2}\right) = s^5 - \frac{s^5}{2} = \frac{s^5}{2}$

41. $V = IR \Rightarrow \frac{\partial V}{\partial I} = R$ and $\frac{\partial V}{\partial R} = I$; $\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt} = R\frac{dI}{dt} + I\frac{dR}{dt} \Rightarrow -0.01$ volts/sec
$= (600 \text{ ohms})\frac{dI}{dt} + (0.04 \text{ amps})(0.5 \text{ ohms/sec}) \Rightarrow \frac{dI}{dt} = -0.00005$ amps/sec

42. $V = abc \Rightarrow \frac{dV}{dt} = \frac{\partial V}{\partial a}\frac{da}{dt} + \frac{\partial V}{\partial b}\frac{db}{dt} + \frac{\partial V}{\partial c}\frac{dc}{dt} = (bc)\frac{da}{dt} + (ac)\frac{db}{dt} + (ab)\frac{dc}{dt}$
$\Rightarrow \left.\frac{dV}{dt}\right|_{a=1,b=2,c=3} = (2\text{ m})(3\text{ m})(1\text{ m/sec}) + (1\text{ m})(3\text{ m})(1\text{ m/sec}) + (1\text{ m})(2\text{ m})(-3\text{ m/sec}) = 3\text{ m}^3\text{/sec}$
and the volume is increasing; $S = 2ab + 2ac + 2bc \Rightarrow \frac{dS}{dt} = \frac{\partial S}{\partial a}\frac{da}{dt} + \frac{\partial S}{\partial b}\frac{db}{dt} + \frac{\partial S}{\partial c}\frac{dc}{dt}$
$= 2(b+c)\frac{da}{dt} + 2(a+c)\frac{db}{dt} + 2(a+b)\frac{dc}{dt} \Rightarrow \left.\frac{dS}{dt}\right|_{a=1,b=2,c=3}$
$= 2(5\text{ m})(1\text{ m/sec}) + 2(4\text{ m})(1\text{ m/sec}) + 2(3\text{ m})(-3\text{ m/sec}) = 0\text{ m}^2\text{/sec}$ and the surface area is not changing;
$D = \sqrt{a^2+b^2+c^2} \Rightarrow \frac{dD}{dt} = \frac{\partial D}{\partial a}\frac{da}{dt} + \frac{\partial D}{\partial b}\frac{db}{dt} + \frac{\partial D}{\partial c}\frac{dc}{dt} = \frac{1}{\sqrt{a^2+b^2+c^2}}\left(a\frac{da}{dt} + b\frac{db}{dt} + c\frac{dc}{dt}\right) \Rightarrow \left.\frac{dD}{dt}\right|_{a=1,b=2,c=3}$
$= \left(\frac{1}{\sqrt{14}\text{ m}}\right)[(1\text{ m})(1\text{ m/sec}) + (2\text{ m})(1\text{ m/sec}) + (3\text{ m})(-3\text{ m/sec})] = -\frac{6}{\sqrt{14}}\text{ m/sec} < 0 \Rightarrow$ the diagonals are decreasing in length

43. $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}(1) + \frac{\partial f}{\partial v}(0) + \frac{\partial f}{\partial w}(-1) = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}$,
$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = \frac{\partial f}{\partial u}(-1) + \frac{\partial f}{\partial v}(1) + \frac{\partial f}{\partial w}(0) = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$, and
$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} = \frac{\partial f}{\partial u}(0) + \frac{\partial f}{\partial v}(-1) + \frac{\partial f}{\partial w}(1) = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w} \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$

44. (a) $\frac{\partial w}{\partial r} = f_x\frac{\partial x}{\partial r} + f_y\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta$ and $\frac{\partial w}{\partial \theta} = f_x(-r\sin\theta) + f_y(r\cos\theta) \Rightarrow \frac{1}{r}\frac{\partial w}{\partial \theta} = -f_x\sin\theta + f_y\cos\theta$

(b) $\frac{\partial w}{\partial r}\sin\theta = f_x\sin\theta\cos\theta + f_y\sin^2\theta$ and $\left(\frac{\cos\theta}{r}\right)\frac{\partial w}{\partial \theta} = -f_x\sin\theta\cos\theta + f_y\cos^2\theta$
$\Rightarrow f_y = (\sin\theta)\frac{\partial w}{\partial r} + \left(\frac{\cos\theta}{r}\right)\frac{\partial w}{\partial \theta}$; then $\frac{\partial w}{\partial r} = f_x\cos\theta + \left[(\sin\theta)\frac{\partial w}{\partial r} + \left(\frac{\cos\theta}{r}\right)\frac{\partial w}{\partial \theta}\right](\sin\theta) \Rightarrow f_x\cos\theta$
$= \frac{\partial w}{\partial r} - (\sin^2\theta)\frac{\partial w}{\partial r} - \left(\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial w}{\partial \theta} = (1-\sin^2\theta)\frac{\partial w}{\partial r} - \left(\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial w}{\partial \theta} \Rightarrow f_x = (\cos\theta)\frac{\partial w}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial w}{\partial \theta}$

(c) $(f_x)^2 = (\cos^2\theta)\left(\frac{\partial w}{\partial r}\right)^2 - \left(\frac{2\sin\theta\cos\theta}{r}\right)\left(\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}\right) + \left(\frac{\sin^2\theta}{r^2}\right)\left(\frac{\partial w}{\partial \theta}\right)^2$ and
$(f_y)^2 = (\sin^2\theta)\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{2\sin\theta\cos\theta}{r}\right)\left(\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}\right) + \left(\frac{\cos^2\theta}{r^2}\right)\left(\frac{\partial w}{\partial \theta}\right)^2 \Rightarrow (f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$

45. $w_x = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = x\frac{\partial w}{\partial u} + y\frac{\partial w}{\partial v} \Rightarrow w_{xx} = \frac{\partial w}{\partial u} + x\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right) + y\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial v}\right)$
$= \frac{\partial w}{\partial u} + x\left(\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v \partial u}\frac{\partial v}{\partial x}\right) + y\left(\frac{\partial^2 w}{\partial u \partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial x}\right) = \frac{\partial w}{\partial u} + x\left(x\frac{\partial^2 w}{\partial u^2} + y\frac{\partial^2 w}{\partial v \partial u}\right) + y\left(x\frac{\partial^2 w}{\partial u \partial v} + y\frac{\partial^2 w}{\partial v^2}\right)$
$= \frac{\partial w}{\partial u} + x^2\frac{\partial^2 w}{\partial u^2} + 2xy\frac{\partial^2 w}{\partial v \partial u} + y^2\frac{\partial^2 w}{\partial v^2}$; $w_y = \frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = -y\frac{\partial w}{\partial u} + x\frac{\partial w}{\partial v}$
$\Rightarrow w_{yy} = -\frac{\partial w}{\partial u} - y\left(\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v \partial u}\frac{\partial v}{\partial y}\right) + x\left(\frac{\partial^2 w}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial y}\right)$
$= -\frac{\partial w}{\partial u} - y\left(-y\frac{\partial^2 w}{\partial u^2} + x\frac{\partial^2 w}{\partial v \partial u}\right) + x\left(-y\frac{\partial^2 w}{\partial u \partial v} + x\frac{\partial^2 w}{\partial v^2}\right) = -\frac{\partial w}{\partial u} + y^2\frac{\partial^2 w}{\partial u^2} - 2xy\frac{\partial^2 w}{\partial v \partial u} + x^2\frac{\partial^2 w}{\partial v^2}$; thus
$w_{xx} + w_{yy} = (x^2+y^2)\frac{\partial^2 w}{\partial u^2} + (x^2+y^2)\frac{\partial^2 w}{\partial v^2} = (x^2+y^2)(w_{uu}+w_{vv}) = 0$, since $w_{uu} + w_{vv} = 0$

46. $\frac{\partial w}{\partial x} = f'(u)(1) + g'(v)(1) = f'(u) + g'(v) \Rightarrow w_{xx} = f''(u)(1) + g''(v)(1) = f''(u) + g''(v)$;
$\frac{\partial w}{\partial y} = f'(u)(i) + g'(v)(-i) \Rightarrow w_{yy} = f''(u)(i^2) + g''(v)(i^2) = -f''(u) - g''(v) \Rightarrow w_{xx} + w_{yy} = 0$

47. $f_x(x,y,z) = \cos t$, $f_y(x,y,z) = \sin t$, and $f_z(x,y,z) = t^2 + t - 2 \Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$
$= (\cos t)(-\sin t) + (\sin t)(\cos t) + (t^2+t-2)(1) = t^2 + t - 2$; $\frac{df}{dt} = 0 \Rightarrow t^2 + t - 2 = 0 \Rightarrow t = -2$
or $t = 1$; $t = -2 \Rightarrow x = \cos(-2)$, $y = \sin(-2)$, $z = -2$ for the point $(\cos(-2), \sin(-2), -2)$; $t = 1 \Rightarrow x = \cos 1$, $y = \sin 1$, $z = 1$ for the point $(\cos 1, \sin 1, 1)$

48. $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = (2xe^{2y}\cos 3z)(-\sin t) + (2x^2e^{2y}\cos 3z)\left(\frac{1}{t+2}\right) + (-3x^2e^{2y}\sin 3z)(1)$
$= -2xe^{2y}\cos 3z \sin t + \frac{2x^2e^{2y}\cos 3z}{t+2} - 3x^2e^{2y}\sin 3z$; at the point on the curve $z = 0 \Rightarrow t = z = 0$
$\Rightarrow \left.\frac{dw}{dt}\right|_{(1,\ln 2,0)} = 0 + \frac{2(1)^2(4)(1)}{2} - 0 = 4$

49. (a) $\frac{\partial T}{\partial x} = 8x - 4y$ and $\frac{\partial T}{\partial y} = 8y - 4x \Rightarrow \frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = (8x-4y)(-\sin t) + (8y-4x)(\cos t)$
$= (8\cos t - 4\sin t)(-\sin t) + (8\sin t - 4\cos t)(\cos t) = 4\sin^2 t - 4\cos^2 t \Rightarrow \frac{d^2T}{dt^2} = 16\sin t\cos t$;
$\frac{dT}{dt} = 0 \Rightarrow 4\sin^2 t - 4\cos^2 t = 0 \Rightarrow \sin^2 t = \cos^2 t \Rightarrow \sin t = \cos t$ or $\sin t = -\cos t \Rightarrow t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$ on the interval $0 \le t \le 2\pi$;
$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{\pi}{4}} = 16\sin\frac{\pi}{4}\cos\frac{\pi}{4} > 0 \Rightarrow$ T has a minimum at $(x,y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$;
$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{3\pi}{4}} = 16\sin\frac{3\pi}{4}\cos\frac{3\pi}{4} < 0 \Rightarrow$ T has a maximum at $(x,y) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$;
$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{5\pi}{4}} = 16\sin\frac{5\pi}{4}\cos\frac{5\pi}{4} > 0 \Rightarrow$ T has a minimum at $(x,y) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$;
$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{7\pi}{4}} = 16\sin\frac{7\pi}{4}\cos\frac{7\pi}{4} < 0 \Rightarrow$ T has a maximum at $(x,y) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

(b) $T = 4x^2 - 4xy + 4y^2 \Rightarrow \frac{\partial T}{\partial x} = 8x - 4y$, and $\frac{\partial T}{\partial y} = 8y - 4x$ so the extreme values occur at the four points found in part (a): $T\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = T\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 4\left(\frac{1}{2}\right) - 4\left(-\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) = 6$, the maximum and
$T\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = T\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 4\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) = 2$, the minimum

50. (a) $\frac{\partial T}{\partial x} = y$ and $\frac{\partial T}{\partial y} = x \Rightarrow \frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = y\left(-2\sqrt{2}\sin t\right) + x\left(\sqrt{2}\cos t\right)$
$= \left(\sqrt{2}\sin t\right)\left(-2\sqrt{2}\sin t\right) + \left(2\sqrt{2}\cos t\right)\left(\sqrt{2}\cos t\right) = -4\sin^2 t + 4\cos^2 t = -4\sin^2 t + 4(1 - \sin^2 t)$
$= 4 - 8\sin^2 t \Rightarrow \frac{d^2T}{dt^2} = -16\sin t\cos t$; $\frac{dT}{dt} = 0 \Rightarrow 4 - 8\sin^2 t = 0 \Rightarrow \sin^2 t = \frac{1}{2} \Rightarrow \sin t = \pm\frac{1}{\sqrt{2}} \Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ on the interval $0 \le t \le 2\pi$;
$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{\pi}{4}} = -8\sin 2\left(\frac{\pi}{4}\right) = -8 \Rightarrow$ T has a maximum at $(x,y) = (2,1)$;
$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{3\pi}{4}} = -8\sin 2\left(\frac{3\pi}{4}\right) = 8 \Rightarrow$ T has a minimum at $(x,y) = (-2,1)$;

$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{5\pi}{4}} = -8\sin 2\left(\frac{5\pi}{4}\right) = -8 \Rightarrow$ T has a maximum at $(x, y) = (-2, -1)$;

$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{7\pi}{4}} = -8\sin 2\left(\frac{7\pi}{4}\right) = 8 \Rightarrow$ T has a minimum at $(x, y) = (2, -1)$

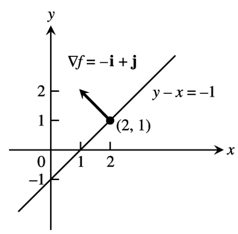
(b) $T = xy - 2 \Rightarrow \frac{\partial T}{\partial x} = y$ and $\frac{\partial T}{\partial y} = x$ so the extreme values occur at the four points found in part (a): $T(2, 1) = T(-2, -1) = 0$, the maximum and $T(-2, 1) = T(2, -1) = -4$, the minimum

51. $G(u, x) = \int_a^u g(t, x)\,dt$ where $u = f(x) \Rightarrow \frac{dG}{dx} = \frac{\partial G}{\partial u}\frac{du}{dx} + \frac{\partial G}{\partial x}\frac{dx}{dx} = g(u, x)f'(x) + \int_a^u g_x(t, x)\,dt$; thus $F(x) = \int_0^{x^2}\sqrt{t^4 + x^3}\,dt \Rightarrow F'(x) = \sqrt{(x^2)^4 + x^3}\,(2x) + \int_0^{x^2}\frac{\partial}{\partial x}\sqrt{t^4 + x^3}\,dt = 2x\sqrt{x^8 + x^3} + \int_0^{x^2}\frac{3x^2}{2\sqrt{t^4 + x^3}}\,dt$

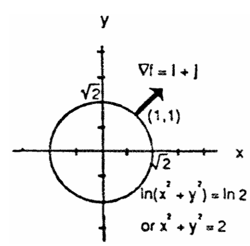
52. Using the result in Exercise 51, $F(x) = \int_{x^2}^1\sqrt{t^3 + x^2}\,dt = -\int_1^{x^2}\sqrt{t^3 + x^2}\,dt \Rightarrow F'(x)$ $= \left[-\sqrt{(x^2)^3 + x^2}\,x^2 - \int_1^{x^2}\frac{\partial}{\partial x}\sqrt{t^3 + x^2}\,dt\right] = -x^2\sqrt{x^6 + x^2} + \int_{x^2}^1\frac{x}{\sqrt{t^3 + x^2}}\,dt$

14.5 DIRECTIONAL DERIVATIVES AND GRADIENT VECTORS

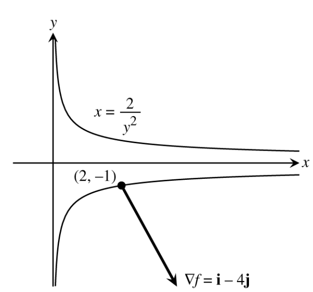
1. $\frac{\partial f}{\partial x} = -1, \frac{\partial f}{\partial y} = 1 \Rightarrow \nabla f = -\mathbf{i} + \mathbf{j}$; $f(2, 1) = -1$ $\Rightarrow -1 = y - x$ is the level curve



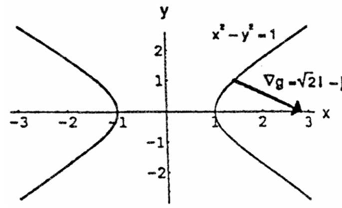
2. $\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2} \Rightarrow \frac{\partial f}{\partial x}(1, 1) = 1$; $\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2}$ $\Rightarrow \frac{\partial f}{\partial y}(1, 1) = 1 \Rightarrow \nabla f = \mathbf{i} + \mathbf{j}$; $f(1, 1) = \ln 2 \Rightarrow \ln 2$ $= \ln(x^2 + y^2) \Rightarrow 2 = x^2 + y^2$ is the level curve



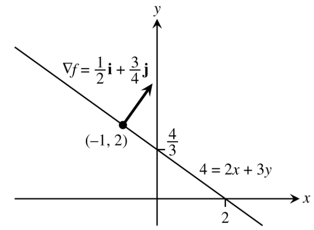
3. $\frac{\partial g}{\partial x} = y^2 \Rightarrow \frac{\partial g}{\partial x}(2, -1) = 1$; $\frac{\partial g}{\partial y} = 2xy \Rightarrow \frac{\partial g}{\partial x}(2, -1) = -4$; $\Rightarrow \nabla g = \mathbf{i} - 4\mathbf{j}$; $g(2, -1) = 2 \Rightarrow x = \frac{2}{y^2}$ is the level curve



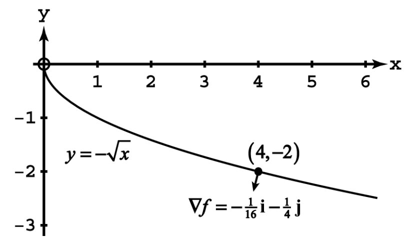
4. $\frac{\partial g}{\partial x} = x \Rightarrow \frac{\partial g}{\partial x}\left(\sqrt{2}, 1\right) = \sqrt{2}$; $\frac{\partial g}{\partial y} = -y$ $\Rightarrow \frac{\partial g}{\partial y}\left(\sqrt{2}, 1\right) = -1 \Rightarrow \nabla g = \sqrt{2}\,\mathbf{i} - \mathbf{j}$; $g\left(\sqrt{2}, 1\right) = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{x^2}{2} - \frac{y^2}{2}$ or $1 = x^2 - y^2$ is the level curve



5. $\frac{\partial f}{\partial x} = \frac{1}{\sqrt{2x+3y}} \Rightarrow \frac{\partial f}{\partial x}(-1,2) = \frac{1}{2}$; $\frac{\partial f}{\partial y} = \frac{3}{2\sqrt{2x+3y}}$
$\Rightarrow \frac{\partial f}{\partial x}(-1,2) = \frac{3}{4}$; $\Rightarrow \nabla f = \frac{1}{2}\mathbf{i} + \frac{3}{4}\mathbf{j}$; $f(-1,2) = 2$
$\Rightarrow 4 = 2x + 3y$ is the level curve



6. $\frac{\partial f}{\partial x} = \frac{y}{2y^2\sqrt{x} + 2x^{3/2}} \Rightarrow \frac{\partial f}{\partial x}(4,-2) = -\frac{1}{16}$;
$\frac{\partial f}{\partial y} = -\frac{\sqrt{x}}{2y^2+x} \Rightarrow \frac{\partial f}{\partial y}(4,-2) = -\frac{1}{4} \Rightarrow \nabla f = -\frac{1}{16}\mathbf{i} - \frac{1}{4}\mathbf{j}$;
$f(4,-2) = -\frac{\pi}{4} \Rightarrow y = -\sqrt{x}$ is the level curve



7. $\frac{\partial f}{\partial x} = 2x + \frac{z}{x} \Rightarrow \frac{\partial f}{\partial x}(1,1,1) = 3$; $\frac{\partial f}{\partial y} = 2y \Rightarrow \frac{\partial f}{\partial y}(1,1,1) = 2$; $\frac{\partial f}{\partial z} = -4z + \ln x \Rightarrow \frac{\partial f}{\partial z}(1,1,1) = -4$;
thus $\nabla f = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

8. $\frac{\partial f}{\partial x} = -6xz + \frac{z}{x^2z^2+1} \Rightarrow \frac{\partial f}{\partial x}(1,1,1) = -\frac{11}{2}$; $\frac{\partial f}{\partial y} = -6yz \Rightarrow \frac{\partial f}{\partial y}(1,1,1) = -6$; $\frac{\partial f}{\partial z} = 6z^2 - 3\left(x^2+y^2\right) + \frac{x}{x^2z^2+1}$
$\Rightarrow \frac{\partial f}{\partial z}(1,1,1) = \frac{1}{2}$; thus $\nabla f = -\frac{11}{2}\mathbf{i} - 6\mathbf{j} + \frac{1}{2}\mathbf{k}$

9. $\frac{\partial f}{\partial x} = -\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}} + \frac{1}{x} \Rightarrow \frac{\partial f}{\partial x}(-1,2,-2) = -\frac{26}{27}$; $\frac{\partial f}{\partial y} = -\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}} + \frac{1}{y} \Rightarrow \frac{\partial f}{\partial y}(-1,2,-2) = \frac{23}{54}$;
$\frac{\partial f}{\partial z} = -\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}} + \frac{1}{z} \Rightarrow \frac{\partial f}{\partial z}(-1,2,-2) = -\frac{23}{54}$; thus $\nabla f = -\frac{26}{27}\mathbf{i} + \frac{23}{54}\mathbf{j} - \frac{23}{54}\mathbf{k}$

10. $\frac{\partial f}{\partial x} = e^{x+y}\cos z + \frac{y+1}{\sqrt{1-x^2}} \Rightarrow \frac{\partial f}{\partial x}\left(0,0,\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + 1$; $\frac{\partial f}{\partial y} = e^{x+y}\cos z + \sin^{-1} x \Rightarrow \frac{\partial f}{\partial y}\left(0,0,\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$;
$\frac{\partial f}{\partial z} = -e^{x+y}\sin z \Rightarrow \frac{\partial f}{\partial z}\left(0,0,\frac{\pi}{6}\right) = -\frac{1}{2}$; thus $\nabla f = \left(\frac{\sqrt{3}+2}{2}\right)\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$

11. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{4\mathbf{i}+3\mathbf{j}}{\sqrt{4^2+3^2}} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$; $f_x(x,y) = 2y \Rightarrow f_x(5,5) = 10$; $f_y(x,y) = 2x - 6y \Rightarrow f_y(5,5) = -20$
$\Rightarrow \nabla f = 10\mathbf{i} - 20\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = 10\left(\frac{4}{5}\right) - 20\left(\frac{3}{5}\right) = -4$

12. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{3\mathbf{i}-4\mathbf{j}}{\sqrt{3^2+(-4)^2}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$; $f_x(x,y) = 4x \Rightarrow f_x(-1,1) = -4$; $f_y(x,y) = 2y \Rightarrow f_y(-1,1) = 2$
$\Rightarrow \nabla f = -4\mathbf{i} + 2\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = -\frac{12}{5} - \frac{8}{5} = -4$

13. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{12\mathbf{i}+5\mathbf{j}}{\sqrt{12^2+5^2}} = \frac{12}{13}\mathbf{i} + \frac{5}{13}\mathbf{j}$; $g_x(x,y) = \frac{y^2+2}{(xy+2)^2} \Rightarrow g_x(1,-1) = 3$; $g_y(x,y) = -\frac{x^2+2}{(xy+2)^2} \Rightarrow g_y(1,-1) = -3$
$\Rightarrow \nabla g = 3\mathbf{i} - 3\mathbf{j} \Rightarrow (D_{\mathbf{u}}g)_{P_0} = \nabla g \cdot \mathbf{u} = \frac{36}{13} - \frac{15}{13} = \frac{21}{13}$

14. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{3\mathbf{i}-2\mathbf{j}}{\sqrt{3^2+(-2)^2}} = \frac{3}{\sqrt{13}}\mathbf{i} - \frac{2}{\sqrt{13}}\mathbf{j}$; $h_x(x,y) = \frac{\left(\frac{-y}{x^2}\right)}{\left(\frac{y}{x}\right)^2+1} + \frac{\left(\frac{y}{2}\right)\sqrt{3}}{\sqrt{1-\left(\frac{x^2y^2}{4}\right)}} \Rightarrow h_x(1,1) = \frac{1}{2}$;
$h_y(x,y) = \frac{\left(\frac{1}{x}\right)}{\left(\frac{y}{x}\right)^2+1} + \frac{\left(\frac{x}{2}\right)\sqrt{3}}{\sqrt{1-\left(\frac{x^2y^2}{4}\right)}} \Rightarrow h_y(1,1) = \frac{3}{2} \Rightarrow \nabla h = \frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} \Rightarrow (D_{\mathbf{u}}h)_{P_0} = \nabla h \cdot \mathbf{u} = \frac{3}{2\sqrt{13}} - \frac{6}{2\sqrt{13}}$
$= -\frac{3}{2\sqrt{13}}$

15. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{3\mathbf{i}+6\mathbf{j}-2\mathbf{k}}{\sqrt{3^2+6^2+(-2)^2}} = \frac{3}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{2}{7}\mathbf{k}$; $f_x(x,y,z) = y+z \Rightarrow f_x(1,-1,2) = 1$; $f_y(x,y,z) = x+z$ $\Rightarrow f_y(1,-1,2) = 3$; $f_z(x,y,z) = y+x \Rightarrow f_z(1,-1,2) = 0 \Rightarrow \nabla f = \mathbf{i} + 3\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = \frac{3}{7} + \frac{18}{7} = 3$

16. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$; $f_x(x,y,z) = 2x \Rightarrow f_x(1,1,1) = 2$; $f_y(x,y,z) = 4y$ $\Rightarrow f_y(1,1,1) = 4$; $f_z(x,y,z) = -6z \Rightarrow f_z(1,1,1) = -6 \Rightarrow \nabla f = 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u}$ $= 2\left(\frac{1}{\sqrt{3}}\right) + 4\left(\frac{1}{\sqrt{3}}\right) - 6\left(\frac{1}{\sqrt{3}}\right) = 0$

17. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{2\mathbf{i}+\mathbf{j}-2\mathbf{k}}{\sqrt{2^2+1^2+(-2)^2}} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$; $g_x(x,y,z) = 3e^x \cos yz \Rightarrow g_x(0,0,0) = 3$; $g_y(x,y,z) = -3ze^x \sin yz$ $\Rightarrow g_y(0,0,0) = 0$; $g_z(x,y,z) = -3ye^x \sin yz \Rightarrow g_z(0,0,0) = 0 \Rightarrow \nabla g = 3\mathbf{i} \Rightarrow (D_{\mathbf{u}}g)_{P_0} = \nabla g \cdot \mathbf{u} = 2$

18. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{\sqrt{1^2+2^2+2^2}} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$; $h_x(x,y,z) = -y \sin xy + \frac{1}{x} \Rightarrow h_x\left(1,0,\frac{1}{2}\right) = 1$; $h_y(x,y,z) = -x \sin xy + ze^{yz} \Rightarrow h_y\left(1,0,\frac{1}{2}\right) = \frac{1}{2}$; $h_z(x,y,z) = ye^{yz} + \frac{1}{z} \Rightarrow h_z\left(1,0,\frac{1}{2}\right) = 2 \Rightarrow \nabla h = \mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}$ $\Rightarrow (D_{\mathbf{u}}h)_{P_0} = \nabla h \cdot \mathbf{u} = \frac{1}{3} + \frac{1}{3} + \frac{4}{3} = 2$

19. $\nabla f = (2x+y)\mathbf{i} + (x+2y)\mathbf{j} \Rightarrow \nabla f(-1,1) = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{-\mathbf{i}+\mathbf{j}}{\sqrt{(-1)^2+1^2}} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$; f increases most rapidly in the direction $\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$; $(D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = \sqrt{2}$ and $(D_{-\mathbf{u}}f)_{P_0} = -\sqrt{2}$

20. $\nabla f = (2xy + ye^{xy} \sin y)\mathbf{i} + (x^2 + xe^{xy} \sin y + e^{xy} \cos y)\mathbf{j} \Rightarrow \nabla f(1,0) = 2\mathbf{j} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \mathbf{j}$; f increases most rapidly in the direction $\mathbf{u} = \mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = -\mathbf{j}$; $(D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f|$ $= 2$ and $(D_{-\mathbf{u}}f)_{P_0} = -2$

21. $\nabla f = \frac{1}{y}\mathbf{i} - \left(\frac{x}{y^2} + z\right)\mathbf{j} - y\mathbf{k} \Rightarrow \nabla f(4,1,1) = \mathbf{i} - 5\mathbf{j} - \mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{\mathbf{i}-5\mathbf{j}-\mathbf{k}}{\sqrt{1^2+(-5)^2+(-1)^2}} = \frac{1}{3\sqrt{3}}\mathbf{i} - \frac{5}{3\sqrt{3}}\mathbf{j} - \frac{1}{3\sqrt{3}}\mathbf{k}$; f increases most rapidly in the direction of $\mathbf{u} = \frac{1}{3\sqrt{3}}\mathbf{i} - \frac{5}{3\sqrt{3}}\mathbf{j} - \frac{1}{3\sqrt{3}}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{1}{3\sqrt{3}}\mathbf{i} + \frac{5}{3\sqrt{3}}\mathbf{j} + \frac{1}{3\sqrt{3}}\mathbf{k}$; $(D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = 3\sqrt{3}$ and $(D_{-\mathbf{u}}f)_{P_0} = -3\sqrt{3}$

22. $\nabla g = e^y\mathbf{i} + xe^y\mathbf{j} + 2z\mathbf{k} \Rightarrow \nabla g\left(1, \ln 2, \frac{1}{2}\right) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla g}{|\nabla g|} = \frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{\sqrt{2^2+2^2+1^2}} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$; g increases most rapidly in the direction $\mathbf{u} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$; $(D_{\mathbf{u}}g)_{P_0} = \nabla g \cdot \mathbf{u} = |\nabla g| = 3$ and $(D_{-\mathbf{u}}g)_{P_0} = -3$

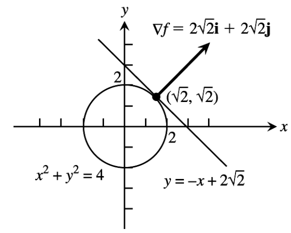
23. $\nabla f = \left(\frac{1}{x} + \frac{1}{x}\right)\mathbf{i} + \left(\frac{1}{y} + \frac{1}{y}\right)\mathbf{j} + \left(\frac{1}{z} + \frac{1}{z}\right)\mathbf{k} \Rightarrow \nabla f(1,1,1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$; f increases most rapidly in the direction $\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$; $(D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = 2\sqrt{3}$ and $(D_{-\mathbf{u}}f)_{P_0} = -2\sqrt{3}$

24. $\nabla h = \left(\frac{2x}{x^2+y^2-1}\right)\mathbf{i} + \left(\frac{2y}{x^2+y^2-1} + 1\right)\mathbf{j} + 6\mathbf{k} \Rightarrow \nabla h(1,1,0) = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla h}{|\nabla h|} = \frac{2\mathbf{i}+3\mathbf{j}+6\mathbf{k}}{\sqrt{2^2+3^2+6^2}}$ $= \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$; h increases most rapidly in the direction $\mathbf{u} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} - \frac{6}{7}\mathbf{k}$; $(D_{\mathbf{u}}h)_{P_0} = \nabla h \cdot \mathbf{u} = |\nabla h| = 7$ and $(D_{-\mathbf{u}}h)_{P_0} = -7$

25. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} \Rightarrow \nabla f\left(\sqrt{2}, \sqrt{2}\right) = 2\sqrt{2}\,\mathbf{i} + 2\sqrt{2}\,\mathbf{j}$

$\Rightarrow$ Tangent line: $2\sqrt{2}\left(x - \sqrt{2}\right) + 2\sqrt{2}\left(y - \sqrt{2}\right) = 0$

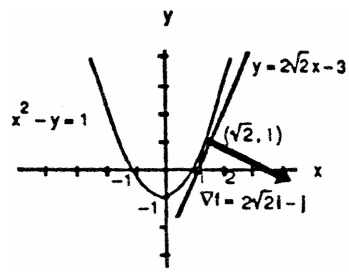
$\Rightarrow \sqrt{2}x + \sqrt{2}y = 4$



26. $\nabla f = 2x\mathbf{i} - \mathbf{j} \Rightarrow \nabla f\left(\sqrt{2}, 1\right) = 2\sqrt{2}\,\mathbf{i} - \mathbf{j}$

$\Rightarrow$ Tangent line: $2\sqrt{2}\left(x - \sqrt{2}\right) - (y - 1) = 0$

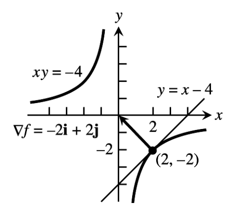
$\Rightarrow y = 2\sqrt{2}x - 3$



27. $\nabla f = y\mathbf{i} + x\mathbf{j} \Rightarrow \nabla f(2, -2) = -2\mathbf{i} + 2\mathbf{j}$

$\Rightarrow$ Tangent line: $-2(x - 2) + 2(y + 2) = 0$

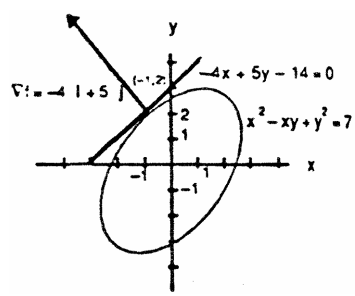
$\Rightarrow y = x - 4$



28. $\nabla f = (2x - y)\mathbf{i} + (2y - x)\mathbf{j} \Rightarrow \nabla f(-1, 2) = -4\mathbf{i} + 5\mathbf{j}$

$\Rightarrow$ Tangent line: $-4(x + 1) + 5(y - 2) = 0$

$\Rightarrow -4x + 5y - 14 = 0$



29. $\nabla f = (2x - y)\mathbf{i} + (-x + 2y - 1)\mathbf{j}$

(a) $\nabla f(1, -1) = 3\mathbf{i} - 4\mathbf{j} \Rightarrow |\nabla f(1, -1)| = 5 \Rightarrow D_{\mathbf{u}}f(1, -1) = 5$ in the direction of $\mathbf{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$

(b) $-\nabla f(1, -1) = -3\mathbf{i} + 4\mathbf{j} \Rightarrow |\nabla f(1, -1)| = 5 \Rightarrow D_{\mathbf{u}}f(1, -1) = -5$ in the direction of $\mathbf{u} = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$

(c) $D_{\mathbf{u}}f(1, -1) = 0$ in the direction of $\mathbf{u} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$ or $\mathbf{u} = -\frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$

(d) Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} \Rightarrow |\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1$; $D_{\mathbf{u}}f(1, -1) = \nabla f(1, -1) \cdot \mathbf{u} = (3\mathbf{i} - 4\mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j})$
$= 3u_1 - 4u_2 = 4 \Rightarrow u_2 = \frac{3}{4}u_1 - 1 \Rightarrow u_1^2 + \left(\frac{3}{4}u_1 - 1\right)^2 = 1 \Rightarrow \frac{25}{16}u_1^2 - \frac{3}{2}u_1 = 0 \Rightarrow u_1 = 0$ or $u_1 = \frac{24}{25}$;
$u_1 = 0 \Rightarrow u_2 = -1 \Rightarrow \mathbf{u} = -\mathbf{j}$, or $u_1 = \frac{24}{25} \Rightarrow u_2 = -\frac{7}{25} \Rightarrow \mathbf{u} = \frac{24}{25}\mathbf{i} - \frac{7}{25}\mathbf{j}$

(e) Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} \Rightarrow |\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1$; $D_{\mathbf{u}}f(1, -1) = \nabla f(1, -1) \cdot \mathbf{u} = (3\mathbf{i} - 4\mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j})$
$= 3u_1 - 4u_2 = -3 \Rightarrow u_1 = \frac{4}{3}u_2 - 1 \Rightarrow \left(\frac{4}{3}u_2 - 1\right)^2 + u_2^2 = 1 \Rightarrow \frac{25}{9}u_2^2 - \frac{8}{3}u_2 = 0 \Rightarrow u_2 = 0$ or $u_2 = \frac{24}{25}$;
$u_2 = 0 \Rightarrow u_1 = -1 \Rightarrow \mathbf{u} = -\mathbf{i}$, or $u_2 = \frac{24}{25} \Rightarrow u_2 = \frac{7}{25} \Rightarrow \mathbf{u} = \frac{7}{25}\mathbf{i} + \frac{24}{25}\mathbf{j}$

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30. $\nabla f = \frac{2y}{(x+y)^2}\mathbf{i} - \frac{2x}{(x+y)^2}\mathbf{j}$

(a) $\nabla f\left(-\frac{1}{2}, \frac{3}{2}\right) = 3\mathbf{i} + \mathbf{j} \Rightarrow \left|\nabla f\left(-\frac{1}{2}, \frac{3}{2}\right)\right| = \sqrt{10} \Rightarrow D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = \sqrt{10}$ in the direction of $\mathbf{u} = \frac{3}{\sqrt{10}}\mathbf{i} + \frac{1}{\sqrt{10}}\mathbf{j}$

(b) $-\nabla f\left(-\frac{1}{2}, \frac{3}{2}\right) = -3\mathbf{i} - \mathbf{j} \Rightarrow \left|\nabla f\left(-\frac{1}{2}, \frac{3}{2}\right)\right| = \sqrt{10} \Rightarrow D_{\mathbf{u}}f(1, -1) = -\sqrt{10}$ in the direction of $\mathbf{u} = -\frac{3}{\sqrt{10}}\mathbf{i} - \frac{1}{\sqrt{10}}\mathbf{j}$

(c) $D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = 0$ in the direction of $\mathbf{u} = \frac{1}{\sqrt{10}}\mathbf{i} - \frac{3}{\sqrt{10}}\mathbf{j}$ or $\mathbf{u} = -\frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$

(d) Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} \Rightarrow |\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1$; $D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = \nabla f\left(-\frac{1}{2}, \frac{3}{2}\right) \cdot \mathbf{u} = (3\mathbf{i} + \mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j})$
$= 3u_1 + u_2 = -2 \Rightarrow u_2 = -3u_1 - 2 \Rightarrow u_1^2 + (-3u_1 - 2)^2 = 1 \Rightarrow 10u_1^2 + 12u_1 + 3 = 0 \Rightarrow u_1 = \frac{-6 \pm \sqrt{6}}{10}$
$u_1 = \frac{-6+\sqrt{6}}{10} \Rightarrow u_2 = \frac{-2-3\sqrt{6}}{10} \Rightarrow \mathbf{u} = \frac{-6+\sqrt{6}}{10}\mathbf{i} + \frac{-2-3\sqrt{6}}{10}\mathbf{j}$, or $u_1 = \frac{-6-\sqrt{6}}{10} \Rightarrow u_2 = \frac{-2+3\sqrt{6}}{10}$
$\Rightarrow \mathbf{u} = \frac{-6-\sqrt{6}}{10}\mathbf{i} + \frac{-2+3\sqrt{6}}{10}\mathbf{j}$

(e) Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} \Rightarrow |\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1$; $D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = \nabla f\left(-\frac{1}{2}, \frac{3}{2}\right) \cdot \mathbf{u} = (3\mathbf{i} + \mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j})$
$= 3u_1 + u_2 = 1 \Rightarrow u_2 = 1 - 3u_1 \Rightarrow u_1^2 + (1 - 3u_1)^2 = 1 \Rightarrow 10u_1^2 - 6u_1 = 0 \Rightarrow u_1 = 0$ or $u_1 = \frac{3}{5}$;
$u_1 = 0 \Rightarrow u_2 = 1 \Rightarrow \mathbf{u} = \mathbf{j}$, or $u_1 = \frac{3}{5} \Rightarrow u_2 = -\frac{4}{5} \Rightarrow \mathbf{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$

31. $\nabla f = y\mathbf{i} + (x + 2y)\mathbf{j} \Rightarrow \nabla f(3, 2) = 2\mathbf{i} + 7\mathbf{j}$; a vector orthogonal to ∇f is $\mathbf{v} = 7\mathbf{i} - 2\mathbf{j} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{7\mathbf{i} - 2\mathbf{j}}{\sqrt{7^2 + (-2)^2}}$
$= \frac{7}{\sqrt{53}}\mathbf{i} - \frac{2}{\sqrt{53}}\mathbf{j}$ and $-\mathbf{u} = -\frac{7}{\sqrt{53}}\mathbf{i} + \frac{2}{\sqrt{53}}\mathbf{j}$ are the directions where the derivative is zero

32. $\nabla f = \frac{4xy^2}{(x^2+y^2)^2}\mathbf{i} - \frac{4x^2y}{(x^2+y^2)^2}\mathbf{j} \Rightarrow \nabla f(1, 1) = \mathbf{i} - \mathbf{j}$; a vector orthogonal to ∇f is $\mathbf{v} = \mathbf{i} + \mathbf{j}$
$\Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i}+\mathbf{j}}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ and $-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ are the directions where the derivative is zero

33. $\nabla f = (2x - 3y)\mathbf{i} + (-3x + 8y)\mathbf{j} \Rightarrow \nabla f(1, 2) = -4\mathbf{i} + 13\mathbf{j} \Rightarrow |\nabla f(1, 2)| = \sqrt{(-4)^2 + (13)^2} = \sqrt{185}$; no, the maximum rate of change is $\sqrt{185} < 14$

34. $\nabla T = 2y\mathbf{i} + (2x - z)\mathbf{j} - y\mathbf{k} \Rightarrow \nabla T(1, -1, 1) = -2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow |\nabla T(1, -1, 1)| = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6}$; no, the minimum rate of change is $-\sqrt{6} > -3$

35. $\nabla f = f_x(1, 2)\mathbf{i} + f_y(1, 2)\mathbf{j}$ and $\mathbf{u}_1 = \frac{\mathbf{i}+\mathbf{j}}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \Rightarrow (D_{\mathbf{u}_1}f)(1, 2) = f_x(1, 2)\left(\frac{1}{\sqrt{2}}\right) + f_y(1, 2)\left(\frac{1}{\sqrt{2}}\right)$
$= 2\sqrt{2} \Rightarrow f_x(1, 2) + f_y(1, 2) = 4$; $\mathbf{u}_2 = -\mathbf{j} \Rightarrow (D_{\mathbf{u}_2}f)(1, 2) = f_x(1, 2)(0) + f_y(1, 2)(-1) = -3 \Rightarrow -f_y(1, 2) = -3$
$\Rightarrow f_y(1, 2) = 3$; then $f_x(1, 2) + 3 = 4 \Rightarrow f_x(1, 2) = 1$; thus $\nabla f(1, 2) = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-\mathbf{i} - 2\mathbf{j}}{\sqrt{(-1)^2 + (-2)^2}}$
$= -\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = -\frac{1}{\sqrt{5}} - \frac{6}{\sqrt{5}} = -\frac{7}{\sqrt{5}}$

36. (a) $(D_{\mathbf{u}}f)_P = 2\sqrt{3} \Rightarrow |\nabla f| = 2\sqrt{3}$; $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i}+\mathbf{j}-\mathbf{k}}{\sqrt{1^2+1^2+(-1)^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$; thus $\mathbf{u} = \frac{\nabla f}{|\nabla f|}$
$\Rightarrow \nabla f = |\nabla f|\mathbf{u} \Rightarrow \nabla f = 2\sqrt{3}\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right) = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

(b) $\mathbf{v} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i}+\mathbf{j}}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = 2\left(\frac{1}{\sqrt{2}}\right) + 2\left(\frac{1}{\sqrt{2}}\right) - 2(0) = 2\sqrt{2}$

37. The directional derivative is the scalar component. With ∇f evaluated at P_0, the scalar component of ∇f in the direction of $\mathbf{u}$ is $\nabla f \cdot \mathbf{u} = (D_{\mathbf{u}}f)_{P_0}$.

38. $D_{\mathbf{i}}f = \nabla f \cdot \mathbf{i} = (f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}) \cdot \mathbf{i} = f_x$; similarly, $D_{\mathbf{j}}f = \nabla f \cdot \mathbf{j} = f_y$ and $D_{\mathbf{k}}f = \nabla f \cdot \mathbf{k} = f_z$

39. If (x, y) is a point on the line, then $\mathbf{T}(x, y) = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j}$ is a vector parallel to the line $\Rightarrow \mathbf{T} \cdot \mathbf{N} = 0$
$\Rightarrow A(x - x_0) + B(y - y_0) = 0$, as claimed.

40. (a) $\nabla(kf) = \frac{\partial(kf)}{\partial x}\mathbf{i} + \frac{\partial(kf)}{\partial y}\mathbf{j} + \frac{\partial(kf)}{\partial z}\mathbf{k} = k\left(\frac{\partial f}{\partial x}\right)\mathbf{i} + k\left(\frac{\partial f}{\partial y}\right)\mathbf{j} + k\left(\frac{\partial f}{\partial z}\right)\mathbf{k} = k\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) = k\nabla f$

(b) $\nabla(f+g) = \frac{\partial(f+g)}{\partial x}\mathbf{i} + \frac{\partial(f+g)}{\partial y}\mathbf{j} + \frac{\partial(f+g)}{\partial z}\mathbf{k} = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}\right)\mathbf{j} + \left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial z}\right)\mathbf{k}$

$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} + \frac{\partial g}{\partial z}\mathbf{k} = \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) + \left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right) = \nabla f + \nabla g$

(c) $\nabla(f-g) = \nabla f - \nabla g$ (Substitute $-g$ for g in part (b) above)

(d) $\nabla(fg) = \frac{\partial(fg)}{\partial x}\mathbf{i} + \frac{\partial(fg)}{\partial y}\mathbf{j} + \frac{\partial(fg)}{\partial z}\mathbf{k} = \left(\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial x}f\right)\mathbf{i} + \left(\frac{\partial f}{\partial y}g + \frac{\partial g}{\partial y}f\right)\mathbf{j} + \left(\frac{\partial f}{\partial z}g + \frac{\partial g}{\partial z}f\right)\mathbf{k}$

$= \left(\frac{\partial f}{\partial x}g\right)\mathbf{i} + \left(\frac{\partial g}{\partial x}f\right)\mathbf{i} + \left(\frac{\partial f}{\partial y}g\right)\mathbf{j} + \left(\frac{\partial g}{\partial y}f\right)\mathbf{j} + \left(\frac{\partial f}{\partial z}g\right)\mathbf{k} + \left(\frac{\partial g}{\partial z}f\right)\mathbf{k}$

$= f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right) + g\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) = f\nabla g + g\nabla f$

(e) $\nabla\left(\frac{f}{g}\right) = \frac{\partial\left(\frac{f}{g}\right)}{\partial x}\mathbf{i} + \frac{\partial\left(\frac{f}{g}\right)}{\partial y}\mathbf{j} + \frac{\partial\left(\frac{f}{g}\right)}{\partial z}\mathbf{k} = \left(\frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}\right)\mathbf{i} + \left(\frac{g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}}{g^2}\right)\mathbf{j} + \left(\frac{g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}}{g^2}\right)\mathbf{k}$

$= \left(\frac{g\frac{\partial f}{\partial x}\mathbf{i} + g\frac{\partial f}{\partial y}\mathbf{j} + g\frac{\partial f}{\partial z}\mathbf{k}}{g^2}\right) - \left(\frac{f\frac{\partial g}{\partial x}\mathbf{i} + f\frac{\partial g}{\partial y}\mathbf{j} + f\frac{\partial g}{\partial z}\mathbf{k}}{g^2}\right) = \frac{g\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)}{g^2} - \frac{f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right)}{g^2}$

$= \frac{g\nabla f}{g^2} - \frac{f\nabla g}{g^2} = \frac{g\nabla f - f\nabla g}{g^2}$

14.6 TANGENT PLANES AND DIFFERENTIALS

1. (a) $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow \nabla f(1,1,1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow$ Tangent plane: $2(x-1) + 2(y-1) + 2(z-1) = 0$ $\Rightarrow x + y + z = 3$;
   (b) Normal line: $x = 1 + 2t,\ y = 1 + 2t,\ z = 1 + 2t$

2. (a) $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k} \Rightarrow \nabla f(3,5,-4) = 6\mathbf{i} + 10\mathbf{j} + 8\mathbf{k} \Rightarrow$ Tangent plane: $6(x-3) + 10(y-5) + 8(z+4) = 0$ $\Rightarrow 3x + 5y + 4z = 18$;
   (b) Normal line: $x = 3 + 6t,\ y = 5 + 10t,\ z = -4 + 8t$

3. (a) $\nabla f = -2x\mathbf{i} + 2\mathbf{k} \Rightarrow \nabla f(2,0,2) = -4\mathbf{i} + 2\mathbf{k} \Rightarrow$ Tangent plane: $-4(x-2) + 2(z-2) = 0$ $\Rightarrow -4x + 2z + 4 = 0 \Rightarrow -2x + z + 2 = 0$;
   (b) Normal line: $x = 2 - 4t,\ y = 0,\ z = 2 + 2t$

4. (a) $\nabla f = (2x+2y)\mathbf{i} + (2x-2y)\mathbf{j} + 2z\mathbf{k} \Rightarrow \nabla f(1,-1,3) = 4\mathbf{j} + 6\mathbf{k} \Rightarrow$ Tangent plane: $4(y+1) + 6(z-3) = 0$ $\Rightarrow 2y + 3z = 7$;
   (b) Normal line: $x = 1,\ y = -1 + 4t,\ z = 3 + 6t$

5. (a) $\nabla f = (-\pi\sin\pi x - 2xy + ze^{xz})\,\mathbf{i} + (-x^2 + z)\,\mathbf{j} + (xe^{xz} + y)\,\mathbf{k} \Rightarrow \nabla f(0,1,2) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow$ Tangent plane: $2(x-0) + 2(y-1) + 1(z-2) = 0 \Rightarrow 2x + 2y + z - 4 = 0$;
   (b) Normal line: $x = 2t,\ y = 1 + 2t,\ z = 2 + t$

6. (a) $\nabla f = (2x-y)\mathbf{i} - (x+2y)\mathbf{j} - \mathbf{k} \Rightarrow \nabla f(1,1,-1) = \mathbf{i} - 3\mathbf{j} - \mathbf{k} \Rightarrow$ Tangent plane: $1(x-1) - 3(y-1) - 1(z+1) = 0 \Rightarrow x - 3y - z = -1$;
   (b) Normal line: $x = 1 + t,\ y = 1 - 3t,\ z = -1 - t$

7. (a) $\nabla f = \mathbf{i} + \mathbf{j} + \mathbf{k}$ for all points $\Rightarrow \nabla f(0,1,0) = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow$ Tangent plane: $1(x-0) + 1(y-1) + 1(z-0) = 0$ $\Rightarrow x + y + z - 1 = 0$;
   (b) Normal line: $x = t,\ y = 1 + t,\ z = t$

8. (a) $\nabla f = (2x - 2y - 1)\mathbf{i} + (2y - 2x + 3)\mathbf{j} - \mathbf{k} \Rightarrow \nabla f(2,-3,18) = 9\mathbf{i} - 7\mathbf{j} - \mathbf{k} \Rightarrow$ Tangent plane: $9(x-2) - 7(y+3) - 1(z-18) = 0 \Rightarrow 9x - 7y - z = 21$;
   (b) Normal line: $x = 2 + 9t,\ y = -3 - 7t,\ z = 18 - t$

9. $z = f(x,y) = \ln(x^2+y^2) \Rightarrow f_x(x,y) = \frac{2x}{x^2+y^2}$ and $f_y(x,y) = \frac{2y}{x^2+y^2} \Rightarrow f_x(1,0) = 2$ and $f_y(1,0) = 0 \Rightarrow$ from Eq. (4) the tangent plane at $(1,0,0)$ is $2(x-1) - z = 0$ or $2x - z - 2 = 0$

10. $z = f(x,y) = e^{-(x^2+y^2)} \Rightarrow f_x(x,y) = -2xe^{-(x^2+y^2)}$ and $f_y(x,y) = -2ye^{-(x^2+y^2)} \Rightarrow f_x(0,0) = 0$ and $f_y(0,0) = 0$ $\Rightarrow$ from Eq. (4) the tangent plane at $(0,0,1)$ is $z - 1 = 0$ or $z = 1$

11. $z = f(x,y) = \sqrt{y-x} \Rightarrow f_x(x,y) = -\frac{1}{2}(y-x)^{-1/2}$ and $f_y(x,y) = \frac{1}{2}(y-x)^{-1/2} \Rightarrow f_x(1,2) = -\frac{1}{2}$ and $f_y(1,2) = \frac{1}{2}$ $\Rightarrow$ from Eq. (4) the tangent plane at $(1,2,1)$ is $-\frac{1}{2}(x-1) + \frac{1}{2}(y-2) - (z-1) = 0 \Rightarrow x - y + 2z - 1 = 0$

12. $z = f(x,y) = 4x^2 + y^2 \Rightarrow f_x(x,y) = 8x$ and $f_y(x,y) = 2y \Rightarrow f_x(1,1) = 8$ and $f_y(1,1) = 2 \Rightarrow$ from Eq. (4) the tangent plane at $(1,1,5)$ is $8(x-1) + 2(y-1) - (z-5) = 0$ or $8x + 2y - z - 5 = 0$

13. $\nabla f = \mathbf{i} + 2y\mathbf{j} + 2\mathbf{k} \Rightarrow \nabla f(1,1,1) = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\nabla g = \mathbf{i}$ for all points; $\mathbf{v} = \nabla f \times \nabla g$

$$\Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 2\mathbf{j} - 2\mathbf{k} \Rightarrow \text{Tangent line: } x = 1,\ y = 1 + 2t,\ z = 1 - 2t$$

14. $\nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} \Rightarrow \nabla f(1,1,1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$; $\nabla g = 2x\mathbf{i} + 4y\mathbf{j} + 6z\mathbf{k} \Rightarrow \nabla g(1,1,1) = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$;

$$\Rightarrow \mathbf{v} = \nabla f \times \nabla g \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 4 & 6 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} \Rightarrow \text{Tangent line: } x = 1 + 2t,\ y = 1 - 4t,\ z = 1 + 2t$$

15. $\nabla f = 2x\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow \nabla f\left(1,1,\frac{1}{2}\right) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\nabla g = \mathbf{j}$ for all points; $\mathbf{v} = \nabla f \times \nabla g$

$$\Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{k} \Rightarrow \text{Tangent line: } x = 1 - 2t,\ y = 1,\ z = \tfrac{1}{2} + 2t$$

16. $\nabla f = \mathbf{i} + 2y\mathbf{j} + \mathbf{k} \Rightarrow \nabla f\left(\frac{1}{2},1,\frac{1}{2}\right) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\nabla g = \mathbf{j}$ for all points; $\mathbf{v} = \nabla f \times \nabla g$

$$\Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{k} \Rightarrow \text{Tangent line: } x = \tfrac{1}{2} - t,\ y = 1,\ z = \tfrac{1}{2} + t$$

17. $\nabla f = (3x^2 + 6xy^2 + 4y)\mathbf{i} + (6x^2y + 3y^2 + 4x)\mathbf{j} - 2z\mathbf{k} \Rightarrow \nabla f(1,1,3) = 13\mathbf{i} + 13\mathbf{j} - 6\mathbf{k}$; $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$$\Rightarrow \nabla g(1,1,3) = 2\mathbf{i} + 2\mathbf{j} + 6\mathbf{k};\ \mathbf{v} = \nabla f \times \nabla g \Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 13 & 13 & -6 \\ 2 & 2 & 6 \end{vmatrix} = 90\mathbf{i} - 90\mathbf{j} \Rightarrow \text{Tangent line:}$$

$x = 1 + 90t,\ y = 1 - 90t,\ z = 3$

18. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} \Rightarrow \nabla f\left(\sqrt{2},\sqrt{2},4\right) = 2\sqrt{2}\,\mathbf{i} + 2\sqrt{2}\,\mathbf{j}$; $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow \nabla g\left(\sqrt{2},\sqrt{2},4\right)$

$$= 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} - \mathbf{k};\ \mathbf{v} = \nabla f \times \nabla g \Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\sqrt{2} & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 2\sqrt{2} & -1 \end{vmatrix} = -2\sqrt{2}\,\mathbf{i} + 2\sqrt{2}\,\mathbf{j} \Rightarrow \text{Tangent line:}$$

$x = \sqrt{2} - 2\sqrt{2}\,t,\ y = \sqrt{2} + 2\sqrt{2}\,t,\ z = 4$

19. $\nabla f = \left(\frac{x}{x^2+y^2+z^2}\right)\mathbf{i} + \left(\frac{y}{x^2+y^2+z^2}\right)\mathbf{j} + \left(\frac{z}{x^2+y^2+z^2}\right)\mathbf{k} \Rightarrow \nabla f(3,4,12) = \frac{3}{169}\mathbf{i} + \frac{4}{169}\mathbf{j} + \frac{12}{169}\mathbf{k}$;
$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}}{\sqrt{3^2 + 6^2 + (-2)^2}} = \frac{3}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{2}{7}\mathbf{k} \Rightarrow \nabla f \cdot \mathbf{u} = \frac{9}{1183}$ and $df = (\nabla f \cdot \mathbf{u})\,ds = \left(\frac{9}{1183}\right)(0.1) \approx 0.0008$

20. $\nabla f = (e^x \cos yz)\,\mathbf{i} - (ze^x \sin yz)\,\mathbf{j} - (ye^x \sin yz)\,\mathbf{k} \Rightarrow \nabla f(0,0,0) = \mathbf{i}\,;\ \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2\mathbf{i}+2\mathbf{j}-2\mathbf{k}}{\sqrt{2^2+2^2+(-2)^2}}$
$= \frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k} \Rightarrow \nabla f \cdot \mathbf{u} = \frac{1}{\sqrt{3}}$ and $df = (\nabla f \cdot \mathbf{u})\, ds = \frac{1}{\sqrt{3}}(0.1) \approx 0.0577$

21. $\nabla g = (1+\cos z)\mathbf{i} + (1-\sin z)\mathbf{j} + (-x\sin z - y\cos z)\mathbf{k} \Rightarrow \nabla g(2,-1,0) = 2\mathbf{i} + \mathbf{j} + \mathbf{k};\ \mathbf{A} = \overrightarrow{P_0P_1} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
$\Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-2\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{\sqrt{(-2)^2+2^2+2^2}} = -\frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k} \Rightarrow \nabla g \cdot \mathbf{u} = 0$ and $dg = (\nabla g \cdot \mathbf{u})\, ds = (0)(0.2) = 0$

22. $\nabla h = [-\pi y \sin(\pi xy) + z^2]\,\mathbf{i} - [\pi x \sin(\pi xy)]\,\mathbf{j} + 2xz\mathbf{k} \Rightarrow \nabla h(-1,-1,-1) = (\pi \sin \pi + 1)\mathbf{i} + (\pi \sin \pi)\mathbf{j} + 2\mathbf{k}$
$= \mathbf{i} + 2\mathbf{k}\,;\ \mathbf{v} = \overrightarrow{P_0P_1} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ where $P_1 = (0,0,0) \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k}$
$\Rightarrow \nabla h \cdot \mathbf{u} = \frac{3}{\sqrt{3}} = \sqrt{3}$ and $dh = (\nabla h \cdot \mathbf{u})\, ds = \sqrt{3}(0.1) \approx 0.1732$

23. (a) The unit tangent vector at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ in the direction of motion is $\mathbf{u} = \frac{\sqrt{3}}{2}\,\mathbf{i} - \frac{1}{2}\,\mathbf{j}\,;$
$\nabla T = (\sin 2y)\mathbf{i} + (2x\cos 2y)\mathbf{j} \Rightarrow \nabla T\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \left(\sin\sqrt{3}\right)\mathbf{i} + \left(\cos\sqrt{3}\right)\mathbf{j} \Rightarrow D_{\mathbf{u}}T\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \nabla T \cdot \mathbf{u}$
$= \frac{\sqrt{3}}{2}\sin\sqrt{3} - \frac{1}{2}\cos\sqrt{3} \approx 0.935°$ C/ft
(b) $\mathbf{r}(t) = (\sin 2t)\mathbf{i} + (\cos 2t)\mathbf{j} \Rightarrow \mathbf{v}(t) = (2\cos 2t)\mathbf{i} - (2\sin 2t)\mathbf{j}$ and $|\mathbf{v}| = 2;\ \frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$
$= \nabla T \cdot \mathbf{v} = \left(\nabla T \cdot \frac{\mathbf{v}}{|\mathbf{v}|}\right)|\mathbf{v}| = (D_{\mathbf{u}}T)\,|\mathbf{v}|$, where $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$; at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ we have $\mathbf{u} = \frac{\sqrt{3}}{2}\,\mathbf{i} - \frac{1}{2}\,\mathbf{j}$ from part (a)
$\Rightarrow \frac{dT}{dt} = \left(\frac{\sqrt{3}}{2}\sin\sqrt{3} - \frac{1}{2}\cos\sqrt{3}\right)\cdot 2 = \sqrt{3}\sin\sqrt{3} - \cos\sqrt{3} \approx 1.87°$ C/sec

24. (a) $\nabla T = (4x - yz)\mathbf{i} - xz\mathbf{j} - xy\mathbf{k} \Rightarrow \nabla T(8,6,-4) = 56\mathbf{i} + 32\mathbf{j} - 48\mathbf{k}\,;\ \mathbf{r}(t) = 2t^2\mathbf{i} + 3t\mathbf{j} - t^2\mathbf{k} \Rightarrow$ the particle is at the point $P(8,6,-4)$ when $t = 2$; $\mathbf{v}(t) = 4t\mathbf{i} + 3\mathbf{j} - 2t\mathbf{k} \Rightarrow \mathbf{v}(2) = 8\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \frac{8}{\sqrt{89}}\,\mathbf{i} + \frac{3}{\sqrt{89}}\,\mathbf{j} - \frac{4}{\sqrt{89}}\,\mathbf{k} \Rightarrow D_{\mathbf{u}}T(8,6,-4) = \nabla T \cdot \mathbf{u} = \frac{1}{\sqrt{89}}[56\cdot 8 + 32\cdot 3 - 48\cdot(-4)] = \frac{736}{\sqrt{89}}\,°$ C/m
(b) $\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = \nabla T \cdot \mathbf{v} = (\nabla T \cdot \mathbf{u})\,|\mathbf{v}| \Rightarrow$ at $t = 2$, $\frac{dT}{dt} = D_{\mathbf{u}}T\big|_{t=2}\mathbf{v}(2) = \left(\frac{736}{\sqrt{89}}\right)\sqrt{89} = 736°$ C/sec

25. (a) $f(0,0) = 1,\ f_x(x,y) = 2x \Rightarrow f_x(0,0) = 0,\ f_y(x,y) = 2y \Rightarrow f_y(0,0) = 0 \Rightarrow L(x,y) = 1 + 0(x-0) + 0(y-0) = 1$
(b) $f(1,1) = 3,\ f_x(1,1) = 2,\ f_y(1,1) = 2 \Rightarrow L(x,y) = 3 + 2(x-1) + 2(y-1) = 2x + 2y - 1$

26. (a) $f(0,0) = 4,\ f_x(x,y) = 2(x+y+2) \Rightarrow f_x(0,0) = 4,\ f_y(x,y) = 2(x+y+2) \Rightarrow f_y(0,0) = 4$
$\Rightarrow L(x,y) = 4 + 4(x-0) + 4(y-0) = 4x + 4y + 4$
(b) $f(1,2) = 25,\ f_x(1,2) = 10,\ f_y(1,2) = 10 \Rightarrow L(x,y) = 25 + 10(x-1) + 10(y-2) = 10x + 10y - 5$

27. (a) $f(0,0) = 5,\ f_x(x,y) = 3$ for all (x,y), $f_y(x,y) = -4$ for all $(x,y) \Rightarrow L(x,y) = 5 + 3(x-0) - 4(y-0) = 3x - 4y + 5$
(b) $f(1,1) = 4,\ f_x(1,1) = 3,\ f_y(1,1) = -4 \Rightarrow L(x,y) = 4 + 3(x-1) - 4(y-1) = 3x - 4y + 5$

28. (a) $f(1,1) = 1,\ f_x(x,y) = 3x^2y^4 \Rightarrow f_x(1,1) = 3,\ f_y(x,y) = 4x^3y^3 \Rightarrow f_y(1,1) = 4$
$\Rightarrow L(x,y) = 1 + 3(x-1) + 4(y-1) = 3x + 4y - 6$
(b) $f(0,0) = 0,\ f_x(0,0) = 0,\ f_y(0,0) = 0 \Rightarrow L(x,y) = 0$

29. (a) $f(0,0) = 1,\ f_x(x,y) = e^x\cos y \Rightarrow f_x(0,0) = 1,\ f_y(x,y) = -e^x\sin y \Rightarrow f_y(0,0) = 0$
$\Rightarrow L(x,y) = 1 + 1(x-0) + 0(y-0) = x + 1$
(b) $f\left(0,\frac{\pi}{2}\right) = 0,\ f_x\left(0,\frac{\pi}{2}\right) = 0,\ f_y\left(0,\frac{\pi}{2}\right) = -1 \Rightarrow L(x,y) = 0 + 0(x-0) - 1\left(y - \frac{\pi}{2}\right) = -y + \frac{\pi}{2}$

30. (a) $f(0,0)=1$, $f_x(x,y)=-e^{2y-x} \Rightarrow f_x(0,0)=-1$, $f_y(x,y)=2e^{2y-x} \Rightarrow f_y(0,0)=2$
$\Rightarrow L(x,y)=1-1(x-0)+2(y-0)=-x+2y+1$
(b) $f(1,2)=e^3$, $f_x(1,2)=-e^3$, $f_y(1,2)=2e^3 \Rightarrow L(x,y)=e^3-e^3(x-1)+2e^3(y-2)=-e^3x+2e^3y-2e^3$

31. (a) $W(20,25)=11°F$; $W(30,-10)=-39°F$; $W(15,15)=0°F$
(b) $W(10,-40)=-65.5°F$; $W(50,-40)=-88°F$; $W(60,30)=10.2°F$;
(c) $W(25,5)=-17.4088°F$; $\frac{\partial W}{\partial V}=-\frac{5.72}{V^{0.84}}+\frac{0.0684t}{V^{0.84}} \Rightarrow \frac{\partial W}{\partial V}(25,5)=-0.36$; $\frac{\partial W}{\partial T}=0.6215+0.4275V^{0.16}$
$\Rightarrow \frac{\partial W}{\partial T}(25,5)=1.3370 \Rightarrow L(V,T)=-17.4088-0.36(V-25)+1.337(T-5)=1.337T-0.36V-15.0938$
(d) i) $W(24,6)\approx L(24,6)=-15.7118\approx-15.7°F$
ii) $W(27,2)\approx L(27,2)=-22.1398\approx-22.1°F$
ii) $W(5,-10)\approx L(5,-10)=-30.2638\approx-30.2°F$ This value is very different because the point $(5,-10)$ is not close to the point $(25,5)$.

32. $W(50,-20)=-59.5298°F$; $\frac{\partial W}{\partial V}=-\frac{5.72}{V^{0.84}}+\frac{0.0684t}{V^{0.84}} \Rightarrow \frac{\partial W}{\partial V}(50,-20)=-0.2651$; $\frac{\partial W}{\partial T}=0.6215+0.4275V^{0.16}$
$\Rightarrow \frac{\partial W}{\partial T}(50,-20)=1.4209 \Rightarrow L(V,T)=-59.5298-0.2651(V-50)+1.4209(T+20)$
$=1.4209T-0.2651V-17.8568$
(a) $W(49,-22)\approx L(49,-22)=-62.1065\approx-62.1°F$
(b) $W(53,-19)\approx L(53,-19)=-58.9042\approx-58.9°F$
(c) $W(60,-30)\approx L(60,-30)=-76.3898\approx-76.4°F$

33. $f(2,1)=3$, $f_x(x,y)=2x-3y \Rightarrow f_x(2,1)=1$, $f_y(x,y)=-3x \Rightarrow f_y(2,1)=-6 \Rightarrow L(x,y)=3+1(x-2)-6(y-1)$
$=7+x-6y$; $f_{xx}(x,y)=2$, $f_{yy}(x,y)=0$, $f_{xy}(x,y)=-3 \Rightarrow M=3$; thus $|E(x,y)|\le\left(\frac{1}{2}\right)(3)\left(|x-2|+|y-1|\right)^2$
$\le\left(\frac{3}{2}\right)(0.1+0.1)^2=0.06$

34. $f(2,2)=11$, $f_x(x,y)=x+y+3 \Rightarrow f_x(2,2)=7$, $f_y(x,y)=x+\frac{y}{2}-3 \Rightarrow f_y(2,2)=0$
$\Rightarrow L(x,y)=11+7(x-2)+0(y-2)=7x-3$; $f_{xx}(x,y)=1$, $f_{yy}(x,y)=\frac{1}{2}$, $f_{xy}(x,y)=1$
$\Rightarrow M=1$; thus $|E(x,y)|\le\left(\frac{1}{2}\right)(1)\left(|x-2|+|y-2|\right)^2\le\left(\frac{1}{2}\right)(0.1+0.1)^2=0.02$

35. $f(0,0)=1$, $f_x(x,y)=\cos y \Rightarrow f_x(0,0)=1$, $f_y(x,y)=1-x\sin y \Rightarrow f_y(0,0)=1$
$\Rightarrow L(x,y)=1+1(x-0)+1(y-0)=x+y+1$; $f_{xx}(x,y)=0$, $f_{yy}(x,y)=-x\cos y$, $f_{xy}(x,y)=-\sin y \Rightarrow M=1$;
thus $|E(x,y)|\le\left(\frac{1}{2}\right)(1)\left(|x|+|y|\right)^2\le\left(\frac{1}{2}\right)(0.2+0.2)^2=0.08$

36. $f(1,2)=6$, $f_x(x,y)=y^2-y\sin(x-1) \Rightarrow f_x(1,2)=4$, $f_y(x,y)=2xy+\cos(x-1) \Rightarrow f_y(1,2)=5$
$\Rightarrow L(x,y)=6+4(x-1)+5(y-2)=4x+5y-8$; $f_{xx}(x,y)=-y\cos(x-1)$, $f_{yy}(x,y)=2x$,
$f_{xy}(x,y)=2y-\sin(x-1)$; $|x-1|\le0.1 \Rightarrow 0.9\le x\le1.1$ and $|y-2|\le0.1 \Rightarrow 1.9\le y\le2.1$; thus the max of $|f_{xx}(x,y)|$ on R is 2.1, the max of $|f_{yy}(x,y)|$ on R is 2.2, and the max of $|f_{xy}(x,y)|$ on R is $2(2.1)-\sin(0.9-1)$
$\le4.3 \Rightarrow M=4.3$; thus $|E(x,y)|\le\left(\frac{1}{2}\right)(4.3)\left(|x-1|+|y-2|\right)^2\le(2.15)(0.1+0.1)^2=0.086$

37. $f(0,0)=1$, $f_x(x,y)=e^x\cos y \Rightarrow f_x(0,0)=1$, $f_y(x,y)=-e^x\sin y \Rightarrow f_y(0,0)=0$
$\Rightarrow L(x,y)=1+1(x-0)+0(y-0)=1+x$; $f_{xx}(x,y)=e^x\cos y$, $f_{yy}(x,y)=-e^x\cos y$, $f_{xy}(x,y)=-e^x\sin y$;
$|x|\le0.1 \Rightarrow -0.1\le x\le0.1$ and $|y|\le0.1 \Rightarrow -0.1\le y\le0.1$; thus the max of $|f_{xx}(x,y)|$ on R is $e^{0.1}\cos(0.1)$
≤1.11, the max of $|f_{yy}(x,y)|$ on R is $e^{0.1}\cos(0.1)\le1.11$, and the max of $|f_{xy}(x,y)|$ on R is $e^{0.1}\sin(0.1)$
$\le0.12 \Rightarrow M=1.11$; thus $|E(x,y)|\le\left(\frac{1}{2}\right)(1.11)\left(|x|+|y|\right)^2\le(0.555)(0.1+0.1)^2=0.0222$

38. $f(1,1)=0$, $f_x(x,y)=\frac{1}{x} \Rightarrow f_x(1,1)=1$, $f_y(x,y)=\frac{1}{y} \Rightarrow f_y(1,1)=1 \Rightarrow L(x,y)=0+1(x-1)+1(y-1)$ $= x+y-2$; $f_{xx}(x,y)=-\frac{1}{x^2}$, $f_{yy}(x,y)=-\frac{1}{y^2}$, $f_{xy}(x,y)=0$; $|x-1|\le 0.2 \Rightarrow 0.98\le x\le 1.2$ so the max of $|f_{xx}(x,y)|$ on R is $\frac{1}{(0.98)^2}\le 1.04$; $|y-1|\le 0.2 \Rightarrow 0.98\le y\le 1.2$ so the max of $|f_{yy}(x,y)|$ on R is $\frac{1}{(0.98)^2}\le 1.04 \Rightarrow M=1.04$; thus $|E(x,y)|\le\left(\frac{1}{2}\right)(1.04)\left(|x-1|+|y-1|\right)^2\le(0.52)(0.2+0.2)^2=0.0832$

39. (a) $f(1,1,1)=3$, $f_x(1,1,1)=y+z|_{(1,1,1)}=2$, $f_y(1,1,1)=x+z|_{(1,1,1)}=2$, $f_z(1,1,1)=y+x|_{(1,1,1)}=2$ $\Rightarrow L(x,y,z)=3+2(x-1)+2(y-1)+2(z-1)=2x+2y+2z-3$
(b) $f(1,0,0)=0$, $f_x(1,0,0)=0$, $f_y(1,0,0)=1$, $f_z(1,0,0)=1 \Rightarrow L(x,y,z)=0+0(x-1)+(y-0)+(z-0)=y+z$
(c) $f(0,0,0)=0$, $f_x(0,0,0)=0$, $f_y(0,0,0)=0$, $f_z(0,0,0)=0 \Rightarrow L(x,y,z)=0$

40. (a) $f(1,1,1)=3$, $f_x(1,1,1)=2x|_{(1,1,1)}=2$, $f_y(1,1,1)=2y|_{(1,1,1)}=2$, $f_z(1,1,1)=2z|_{(1,1,1)}=2$ $\Rightarrow L(x,y,z)=3+2(x-1)+2(y-1)+2(z-1)=2x+2y+2z-3$
(b) $f(0,1,0)=1$, $f_x(0,1,0)=0$, $f_y(0,1,0)=2$, $f_z(0,1,0)=0 \Rightarrow L(x,y,z)=1+0(x-0)+2(y-1)+0(z-0)$ $=2y-1$
(c) $f(1,0,0)=1$, $f_x(1,0,0)=2$, $f_y(1,0,0)=0$, $f_z(1,0,0)=0 \Rightarrow L(x,y,z)=1+2(x-1)+0(y-0)+0(z-0)$ $=2x-1$

41. (a) $f(1,0,0)=1$, $f_x(1,0,0)=\frac{x}{\sqrt{x^2+y^2+z^2}}\Big|_{(1,0,0)}=1$, $f_y(1,0,0)=\frac{y}{\sqrt{x^2+y^2+z^2}}\Big|_{(1,0,0)}=0$, $f_z(1,0,0)=\frac{z}{\sqrt{x^2+y^2+z^2}}\Big|_{(1,0,0)}=0 \Rightarrow L(x,y,z)=1+1(x-1)+0(y-0)+0(z-0)=x$
(b) $f(1,1,0)=\sqrt{2}$, $f_x(1,1,0)=\frac{1}{\sqrt{2}}$, $f_y(1,1,0)=\frac{1}{\sqrt{2}}$, $f_z(1,1,0)=0$ $\Rightarrow L(x,y,z)=\sqrt{2}+\frac{1}{\sqrt{2}}(x-1)+\frac{1}{\sqrt{2}}(y-1)+0(z-0)=\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y$
(c) $f(1,2,2)=3$, $f_x(1,2,2)=\frac{1}{3}$, $f_y(1,2,2)=\frac{2}{3}$, $f_z(1,2,2)=\frac{2}{3} \Rightarrow L(x,y,z)=3+\frac{1}{3}(x-1)+\frac{2}{3}(y-2)+\frac{2}{3}(z-2)$ $=\frac{1}{3}x+\frac{2}{3}y+\frac{2}{3}z$

42. (a) $f\left(\frac{\pi}{2},1,1\right)=1$, $f_x\left(\frac{\pi}{2},1,1\right)=\frac{y\cos xy}{z}\Big|_{\left(\frac{\pi}{2},1,1\right)}=0$, $f_y\left(\frac{\pi}{2},1,1\right)=\frac{x\cos xy}{z}\Big|_{\left(\frac{\pi}{2},1,1\right)}=0$, $f_z\left(\frac{\pi}{2},1,1\right)=\frac{-\sin xy}{z^2}\Big|_{\left(\frac{\pi}{2},1,1\right)}=-1 \Rightarrow L(x,y,z)=1+0\left(x-\frac{\pi}{2}\right)+0(y-1)-1(z-1)=2-z$
(b) $f(2,0,1)=0$, $f_x(2,0,1)=0$, $f_y(2,0,1)=2$, $f_z(2,0,1)=0 \Rightarrow L(x,y,z)=0+0(x-2)+2(y-0)+0(z-1)=2y$

43. (a) $f(0,0,0)=2$, $f_x(0,0,0)=e^x|_{(0,0,0)}=1$, $f_y(0,0,0)=-\sin(y+z)|_{(0,0,0)}=0$, $f_z(0,0,0)=-\sin(y+z)|_{(0,0,0)}=0 \Rightarrow L(x,y,z)=2+1(x-0)+0(y-0)+0(z-0)=2+x$
(b) $f\left(0,\frac{\pi}{2},0\right)=1$, $f_x\left(0,\frac{\pi}{2},0\right)=1$, $f_y\left(0,\frac{\pi}{2},0\right)=-1$, $f_z\left(0,\frac{\pi}{2},0\right)=-1 \Rightarrow L(x,y,z)$ $=1+1(x-0)-1\left(y-\frac{\pi}{2}\right)-1(z-0)=x-y-z+\frac{\pi}{2}+1$
(c) $f\left(0,\frac{\pi}{4},\frac{\pi}{4}\right)=1$, $f_x\left(0,\frac{\pi}{4},\frac{\pi}{4}\right)=1$, $f_y\left(0,\frac{\pi}{4},\frac{\pi}{4}\right)=-1$, $f_z\left(0,\frac{\pi}{4},\frac{\pi}{4}\right)=-1 \Rightarrow L(x,y,z)$ $=1+1(x-0)-1\left(y-\frac{\pi}{4}\right)-1\left(z-\frac{\pi}{4}\right)=x-y-z+\frac{\pi}{2}+1$

44. (a) $f(1,0,0)=0$, $f_x(1,0,0)=\frac{yz}{(xyz)^2+1}\Big|_{(1,0,0)}=0$, $f_y(1,0,0)=\frac{xz}{(xyz)^2+1}\Big|_{(1,0,0)}=0$, $f_z(1,0,0)=\frac{xy}{(xyz)^2+1}\Big|_{(1,0,0)}=0 \Rightarrow L(x,y,z)=0$
(b) $f(1,1,0)=0$, $f_x(1,1,0)=0$, $f_y(1,1,0)=0$, $f_z(1,1,0)=1 \Rightarrow L(x,y,z)=0+0(x-1)+0(y-1)+1(z-0)=z$
(c) $f(1,1,1)=\frac{\pi}{4}$, $f_x(1,1,1)=\frac{1}{2}$, $f_y(1,1,1)=\frac{1}{2}$, $f_z(1,1,1)=\frac{1}{2} \Rightarrow L(x,y,z)=\frac{\pi}{4}+\frac{1}{2}(x-1)+\frac{1}{2}(y-1)+\frac{1}{2}(z-1)$ $=\frac{1}{2}x+\frac{1}{2}y+\frac{1}{2}z+\frac{\pi}{4}-\frac{3}{2}$

45. $f(x, y, z) = xz - 3yz + 2$ at $P_0(1, 1, 2) \Rightarrow f(1, 1, 2) = -2$; $f_x = z$, $f_y = -3z$, $f_z = x - 3y \Rightarrow L(x, y, z)$
$= -2 + 2(x - 1) - 6(y - 1) - 2(z - 2) = 2x - 6y - 2z + 6$; $f_{xx} = 0$, $f_{yy} = 0$, $f_{zz} = 0$, $f_{xy} = 0$, $f_{yz} = -3$
$\Rightarrow M = 3$; thus, $|E(x, y, z)| \le \left(\frac{1}{2}\right)(3)(0.01 + 0.01 + 0.02)^2 = 0.0024$

46. $f(x, y, z) = x^2 + xy + yz + \frac{1}{4} z^2$ at $P_0(1, 1, 2) \Rightarrow f(1, 1, 2) = 5$; $f_x = 2x + y$, $f_y = x + z$, $f_z = y + \frac{1}{2} z$
$\Rightarrow L(x, y, z) = 5 + 3(x - 1) + 3(y - 1) + 2(z - 2) = 3x + 3y + 2z - 5$; $f_{xx} = 2$, $f_{yy} = 0$, $f_{zz} = \frac{1}{2}$, $f_{xy} = 1$, $f_{xz} = 0$,
$f_{yz} = 1 \Rightarrow M = 2$; thus $|E(x, y, z)| \le \left(\frac{1}{2}\right)(2)(0.01 + 0.01 + 0.08)^2 = 0.01$

47. $f(x, y, z) = xy + 2yz - 3xz$ at $P_0(1, 1, 0) \Rightarrow f(1, 1, 0) = 1$; $f_x = y - 3z$, $f_y = x + 2z$, $f_z = 2y - 3x$
$\Rightarrow L(x, y, z) = 1 + (x - 1) + (y - 1) - (z - 0) = x + y - z - 1$; $f_{xx} = 0$, $f_{yy} = 0$, $f_{zz} = 0$, $f_{xy} = 1$, $f_{xz} = -3$,
$f_{yz} = 2 \Rightarrow M = 3$; thus $|E(x, y, z)| \le \left(\frac{1}{2}\right)(3)(0.01 + 0.01 + 0.01)^2 = 0.00135$

48. $f(x, y, z) = \sqrt{2} \cos x \sin(y + z)$ at $P_0\left(0, 0, \frac{\pi}{4}\right) \Rightarrow f\left(0, 0, \frac{\pi}{4}\right) = 1$; $f_x = -\sqrt{2} \sin x \sin(y + z)$,
$f_y = \sqrt{2} \cos x \cos(y + z)$, $f_z = \sqrt{2} \cos x \cos(y + z) \Rightarrow L(x, y, z) = 1 - 0(x - 0) + (y - 0) + \left(z - \frac{\pi}{4}\right)$
$= y + z - \frac{\pi}{4} + 1$; $f_{xx} = -\sqrt{2} \cos x \sin(y + z)$, $f_{yy} = -\sqrt{2} \cos x \sin(y + z)$, $f_{zz} = -\sqrt{2} \cos x \sin(y + z)$,
$f_{xy} = -\sqrt{2} \sin x \cos(y + z)$, $f_{xz} = -\sqrt{2} \sin x \cos(y + z)$, $f_{yz} = -\sqrt{2} \cos x \sin(y + z)$. The absolute value of each of these second partial derivatives is bounded above by $\sqrt{2} \Rightarrow M = \sqrt{2}$; thus $|E(x, y, z)|$
$\le \left(\frac{1}{2}\right)\left(\sqrt{2}\right)(0.01 + 0.01 + 0.01)^2 = 0.000636.$

49. $T_x(x, y) = e^y + e^{-y}$ and $T_y(x, y) = x\left(e^y - e^{-y}\right) \Rightarrow dT = T_x(x, y)\,dx + T_y(x, y)\,dy$
$= \left(e^y + e^{-y}\right) dx + x\left(e^y - e^{-y}\right) dy \Rightarrow dT|_{(2, \ln 2)} = 2.5\,dx + 3.0\,dy$. If $|dx| \le 0.1$ and $|dy| \le 0.02$, then the maximum possible error in the computed value of T is $(2.5)(0.1) + (3.0)(0.02) = 0.31$ in magnitude.

50. $V_r = 2\pi rh$ and $V_h = \pi r^2 \Rightarrow dV = V_r\,dr + V_h\,dh \Rightarrow \frac{dV}{V} = \frac{2\pi rh\,dr + \pi r^2\,dh}{\pi r^2 h} = \frac{2}{r}\,dr + \frac{1}{h}\,dh$; now $\left|\frac{dr}{r} \cdot 100\right| \le 1$ and
$\left|\frac{dh}{h} \cdot 100\right| \le 1 \Rightarrow \left|\frac{dV}{V} \cdot 100\right| \le \left|\left(2\frac{dr}{r}\right)(100) + \left(\frac{dh}{h}\right)(100)\right| \le 2\left|\frac{dr}{r} \cdot 100\right| + \left|\frac{dh}{h} \cdot 100\right| \le 2(1) + 1 = 3 \Rightarrow 3\%$

51. $\frac{dx}{x} \le 0.02$, $\frac{dy}{y} \le 0.03$

(a) $S = 2x^2 + 4xy \Rightarrow dS = (4x + 4y)dx + 4x\,dy = (4x^2 + 4xy)\frac{dx}{x} + 4xy\frac{dy}{y} \le (4x^2 + 4xy)(0.02) + (4xy)(0.03)$
$= 0.04(2x^2) + 0.05(4xy) \le 0.05(2x^2) + 0.05(4xy) = (0.05)(2x^2 + 4xy) = 0.05S$

(b) $V = x^2 y \Rightarrow dV = 2xy\,dx + x^2 dy = 2x^2 y\frac{dx}{x} + x^2 y\frac{dy}{y} \le (2x^2 y)(0.02) + (x^2 y)(0.03) = 0.07(x^2 y) = 0.07V$

52. $V = \frac{4\pi}{3} r^3 + \pi r^2 h \Rightarrow dV = (4\pi r^2 + 2\pi rh)dr + \pi r^2 dh$; $r = 10$, $h = 15$, $dr = \frac{1}{2}$ and $dh = 0 \Rightarrow$
$dV = \left(4\pi(10)^2 + 2\pi(10)(15)\right)\left(\frac{1}{2}\right) + \pi(10)^2(0) = 350\pi \text{ cm}^3$

53. $V_r = 2\pi rh$ and $V_h = \pi r^2 \Rightarrow dV = V_r\,dr + V_h\,dh \Rightarrow dV = 2\pi rh\,dr + \pi r^2\,dh \Rightarrow dV|_{(5,12)} = 120\pi\,dr + 25\pi\,dh$;
$|dr| \le 0.1$ cm and $|dh| \le 0.1$ cm $\Rightarrow dV \le (120\pi)(0.1) + (25\pi)(0.1) = 14.5\pi \text{ cm}^3$; $V(5, 12) = 300\pi \text{ cm}^3$
$\Rightarrow$ maximum percentage error is $\pm\frac{14.5\pi}{300\pi} \times 100 = \pm 4.83\%$

54. (a) $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow -\frac{1}{R^2}\,dR = -\frac{1}{R_1^2}\,dR_1 - \frac{1}{R_2^2}\,dR_2 \Rightarrow dR = \left(\frac{R}{R_1}\right)^2 dR_1 + \left(\frac{R}{R_2}\right)^2 dR_2$

(b) $dR = R^2\left[\left(\frac{1}{R_1^2}\right)dR_1 + \left(\frac{1}{R_2^2}\right)dR_2\right] \Rightarrow dR|_{(100,400)} = R^2\left[\frac{1}{(100)^2}\,dR_1 + \frac{1}{(400)^2}\,dR_2\right] \Rightarrow$ R will be more sensitive to a variation in R_1 since $\frac{1}{(100)^2} > \frac{1}{(400)^2}$

(c) From part (a), $dR = \left(\frac{R}{R_1}\right)^2 dR_1 + \left(\frac{R}{R_2}\right)^2 dR_2$ so that R_1 changing from 20 to 20.1 ohms $\Rightarrow$ $dR_1 = 0.1$ ohm and R_2 changing from 25 to 24.9 ohms $\Rightarrow$ $dR_2 = -0.1$ ohms; $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ $\Rightarrow$ $R = \frac{100}{9}$ ohms $\Rightarrow$ $dR\big|_{(20,25)} = \frac{\left(\frac{100}{9}\right)^2}{(20)^2}(0.1) + \frac{\left(\frac{100}{9}\right)^2}{(25)^2}(-0.1) \approx 0.011$ ohms $\Rightarrow$ percentage change is $\frac{dR}{R}\big|_{(20,25)} \times 100 = \frac{0.011}{\left(\frac{100}{9}\right)} \times 100 \approx 0.1\%$

55. $A = xy \Rightarrow dA = x\,dy + y\,dx$; if $x > y$ then a 1-unit change in y gives a greater change in dA than a 1-unit change in x. Thus, pay more attention to y which is the smaller of the two dimensions.

56. (a) $f_x(x, y) = 2x(y + 1) \Rightarrow f_x(1, 0) = 2$ and $f_y(x, y) = x^2 \Rightarrow f_y(1, 0) = 1 \Rightarrow df = 2\,dx + 1\,dy \Rightarrow$ df is more sensitive to changes in x

(b) $df = 0 \Rightarrow 2\,dx + dy = 0 \Rightarrow 2\,\frac{dx}{dy} + 1 = 0 \Rightarrow \frac{dx}{dy} = -\frac{1}{2}$

57. (a) $r^2 = x^2 + y^2 \Rightarrow 2r\,dr = 2x\,dx + 2y\,dy \Rightarrow dr = \frac{x}{r}\,dx + \frac{y}{r}\,dy \Rightarrow dr\big|_{(3,4)} = \left(\frac{3}{5}\right)(\pm 0.01) + \left(\frac{4}{5}\right)(\pm 0.01) = \pm\frac{0.07}{5} = \pm 0.014 \Rightarrow \left|\frac{dr}{r} \times 100\right| = \left|\pm\frac{0.014}{5} \times 100\right| = 0.28\%$; $d\theta = \frac{\left(-\frac{y}{x^2}\right)}{\left(\frac{y}{x}\right)^2 + 1}\,dx + \frac{\left(\frac{1}{x}\right)}{\left(\frac{y}{x}\right)^2 + 1}\,dy = \frac{-y}{y^2 + x^2}\,dx + \frac{x}{y^2 + x^2}\,dy \Rightarrow d\theta\big|_{(3,4)} = \left(\frac{-4}{25}\right)(\pm 0.01) + \left(\frac{3}{25}\right)(\pm 0.01) = \frac{\mp 0.04}{25} + \frac{\pm 0.03}{25}$
$\Rightarrow$ maximum change in $d\theta$ occurs when dx and dy have opposite signs ($dx = 0.01$ and $dy = -0.01$ or vice versa) $\Rightarrow d\theta = \frac{\pm 0.07}{25} \approx \pm 0.0028$; $\theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 0.927255218 \Rightarrow \left|\frac{d\theta}{\theta} \times 100\right| = \left|\frac{\pm 0.0028}{0.927255218} \times 100\right| \approx 0.30\%$

(b) the radius r is more sensitive to changes in y, and the angle θ is more sensitive to changes in x

58. (a) $V = \pi r^2 h \Rightarrow dV = 2\pi rh\,dr + \pi r^2\,dh \Rightarrow$ at $r = 1$ and $h = 5$ we have $dV = 10\pi\,dr + \pi\,dh \Rightarrow$ the volume is about 10 times more sensitive to a change in r

(b) $dV = 0 \Rightarrow 0 = 2\pi rh\,dr + \pi r^2\,dh = 2h\,dr + r\,dh = 10\,dr + dh \Rightarrow dr = -\frac{1}{10}\,dh$; choose $dh = 1.5$ $\Rightarrow dr = -0.15 \Rightarrow h = 6.5$ in. and $r = 0.85$ in. is one solution for $\Delta V \approx dV = 0$

59. $f(a, b, c, d) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \Rightarrow f_a = d, f_b = -c, f_c = -b, f_d = a \Rightarrow df = d\,da - c\,db - b\,dc + a\,dd$; since $|a|$ is much greater than $|b|$, $|c|$, and $|d|$, the function f is most sensitive to a change in d.

60. $u_x = e^y$, $u_y = xe^y + \sin z$, $u_z = y\cos z \Rightarrow du = e^y\,dx + (xe^y + \sin z)\,dy + (y\cos z)\,dz$
$\Rightarrow du\big|_{(2,\ln 3,\frac{\pi}{2})} = 3\,dx + 7\,dy + 0\,dz = 3\,dx + 7\,dy \Rightarrow$ magnitude of the maximum possible error $\le 3(0.2) + 7(0.6) = 4.8$

61. $Q_K = \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left(\frac{2M}{h}\right)$, $Q_M = \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left(\frac{2K}{h}\right)$, and $Q_h = \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left(\frac{-2KM}{h^2}\right)$
$\Rightarrow dQ = \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left(\frac{2M}{h}\right)dK + \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left(\frac{2K}{h}\right)dM + \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left(\frac{-2KM}{h^2}\right)dh$
$= \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left[\frac{2M}{h}\,dK + \frac{2K}{h}\,dM - \frac{2KM}{h^2}\,dh\right] \Rightarrow dQ\big|_{(2,20,0.0.05)}$
$= \frac{1}{2}\left[\frac{(2)(2)(20)}{0.05}\right]^{-1/2}\left[\frac{(2)(20)}{0.05}\,dK + \frac{(2)(2)}{0.05}\,dM - \frac{(2)(2)(20)}{(0.05)^2}\,dh\right] = (0.0125)(800\,dK + 80\,dM - 32{,}000\,dh)$
$\Rightarrow$ Q is most sensitive to changes in h

62. $A = \frac{1}{2}ab\sin C \Rightarrow A_a = \frac{1}{2}b\sin C$, $A_b = \frac{1}{2}a\sin C$, $A_c = \frac{1}{2}ab\cos C$
$\Rightarrow dA = \left(\frac{1}{2}b\sin C\right)da + \left(\frac{1}{2}a\sin C\right)db + \left(\frac{1}{2}ab\cos C\right)dC$; $dC = |2°| = |0.0349|$ radians, $da = |0.5|$ ft, $db = |0.5|$ ft; at $a = 150$ ft, $b = 200$ ft, and $C = 60°$, we see that the change is approximately
$dA = \frac{1}{2}(200)(\sin 60°)|0.5| + \frac{1}{2}(150)(\sin 60°)|0.5| + \frac{1}{2}(200)(150)(\cos 60°)|0.0349| = \pm 338\ \text{ft}^2$

63. $z = f(x, y) \Rightarrow g(x, y, z) = f(x, y) - z = 0 \Rightarrow g_x(x, y, z) = f_x(x, y), g_y(x, y, z) = f_y(x, y)$ and $g_z(x, y, z) = -1$ $\Rightarrow g_x(x_0, y_0, f(x_0, y_0)) = f_x(x_0, y_0), g_y(x_0, y_0, f(x_0, y_0)) = f_y(x_0, y_0)$ and $g_z(x_0, y_0, f(x_0, y_0)) = -1 \Rightarrow$ the tangent plane at the point P_0 is $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - [z - f(x_0, y_0)] = 0$ or $z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$

64. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} = 2(\cos t + t \sin t)\mathbf{i} + 2(\sin t - t\cos t)\mathbf{j}$ and $\mathbf{v} = (t\cos t)\mathbf{i} + (t \sin t)\mathbf{j} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$ $= \frac{(t\cos t)\mathbf{i} + (t\sin t)\mathbf{j}}{\sqrt{(t\cos t)^2 + (t\sin t)^2}} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ since $t > 0 \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u}$ $= 2(\cos t + t\sin t)(\cos t) + 2(\sin t - t\cos t)(\sin t) = 2$

65. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + 2t\mathbf{k}$ and $\mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$ $= \frac{(-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}}{\sqrt{(\sin t)^2 + (\cos t)^2 + 1^2}} = \left(\frac{-\sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\cos t}{\sqrt{2}}\right)\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u}$ $= (2\cos t)\left(\frac{-\sin t}{\sqrt{2}}\right) + (2\sin t)\left(\frac{\cos t}{\sqrt{2}}\right) + (2t)\left(\frac{1}{\sqrt{2}}\right) = \frac{2t}{\sqrt{2}} \Rightarrow (D_{\mathbf{u}}f)\left(\frac{-\pi}{4}\right) = \frac{-\pi}{2\sqrt{2}}$, $(D_{\mathbf{u}}f)(0) = 0$ and $(D_{\mathbf{u}}f)\left(\frac{\pi}{4}\right) = \frac{\pi}{2\sqrt{2}}$

66. $\mathbf{r} = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} - \frac{1}{4}(t + 3)\mathbf{k} \Rightarrow \mathbf{v} = \frac{1}{2}t^{-1/2}\mathbf{i} + \frac{1}{2}t^{-1/2}\mathbf{j} - \frac{1}{4}\mathbf{k}$; $t = 1 \Rightarrow x = 1, y = 1, z = -1 \Rightarrow P_0 = (1, 1, -1)$ and $\mathbf{v}(1) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \frac{1}{4}\mathbf{k}$; $f(x, y, z) = x^2 + y^2 - z - 3 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$ $\Rightarrow \nabla f(1, 1, -1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$; therefore $\mathbf{v} = \frac{1}{4}(\nabla f) \Rightarrow$ the curve is normal to the surface

67. $\mathbf{r} = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} + (2t - 1)\mathbf{k} \Rightarrow \mathbf{v} = \frac{1}{2}t^{-1/2}\mathbf{i} + \frac{1}{2}t^{-1/2}\mathbf{j} + 2\mathbf{k}$; $t = 1 \Rightarrow x = 1, y = 1, z = 1 \Rightarrow P_0 = (1, 1, 1)$ and $\mathbf{v}(1) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}$; $f(x, y, z) = x^2 + y^2 - z - 1 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow \nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$; now $\mathbf{v}(1) \cdot \nabla f(1, 1, 1) = 0$, thus the curve is tangent to the surface when $t = 1$

14.7 EXTREME VALUES AND SADDLE POINTS

1. $f_x(x, y) = 2x + y + 3 = 0$ and $f_y(x, y) = x + 2y - 3 = 0 \Rightarrow x = -3$ and $y = 3 \Rightarrow$ critical point is $(-3, 3)$; $f_{xx}(-3, 3) = 2, f_{yy}(-3, 3) = 2, f_{xy}(-3, 3) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(-3, 3) = -5$

2. $f_x(x, y) = 2y - 10x + 4 = 0$ and $f_y(x, y) = 2x - 4y + 4 = 0 \Rightarrow x = \frac{2}{3}$ and $y = \frac{4}{3} \Rightarrow$ critical point is $\left(\frac{2}{3}, \frac{4}{3}\right)$; $f_{xx}\left(\frac{2}{3}, \frac{4}{3}\right) = -10, f_{yy}\left(\frac{2}{3}, \frac{4}{3}\right) = -4, f_{xy}\left(\frac{2}{3}, \frac{4}{3}\right) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(\frac{2}{3}, \frac{4}{3}\right) = 0$

3. $f_x(x, y) = 2x + y + 3 = 0$ and $f_y(x, y) = x + 2 = 0 \Rightarrow x = -2$ and $y = 1 \Rightarrow$ critical point is $(-2, 1)$; $f_{xx}(-2, 1) = 2, f_{yy}(-2, 1) = 0, f_{xy}(-2, 1) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -1 < 0 \Rightarrow$ saddle point

4. $f_x(x, y) = 5y - 14x + 3 = 0$ and $f_y(x, y) = 5x - 6 = 0 \Rightarrow x = \frac{6}{5}$ and $y = \frac{69}{25} \Rightarrow$ critical point is $\left(\frac{6}{5}, \frac{69}{25}\right)$; $f_{xx}\left(\frac{6}{5}, \frac{69}{25}\right) = -14, f_{yy}\left(\frac{6}{5}, \frac{69}{25}\right) = 0, f_{xy}\left(\frac{6}{5}, \frac{69}{25}\right) = 5 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -25 < 0 \Rightarrow$ saddle point

5. $f_x(x, y) = 2y - 2x + 3 = 0$ and $f_y(x, y) = 2x - 4y = 0 \Rightarrow x = 3$ and $y = \frac{3}{2} \Rightarrow$ critical point is $\left(3, \frac{3}{2}\right)$; $f_{xx}\left(3, \frac{3}{2}\right) = -2, f_{yy}\left(3, \frac{3}{2}\right) = -4, f_{xy}\left(3, \frac{3}{2}\right) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(3, \frac{3}{2}\right) = \frac{17}{2}$

6. $f_x(x, y) = 2x - 4y = 0$ and $f_y(x, y) = -4x + 2y + 6 = 0 \Rightarrow x = 2$ and $y = 1 \Rightarrow$ critical point is $(2, 1)$; $f_{xx}(2, 1) = 2, f_{yy}(2, 1) = 2, f_{xy}(2, 1) = -4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -12 < 0 \Rightarrow$ saddle point

7. $f_x(x,y) = 4x + 3y - 5 = 0$ and $f_y(x,y) = 3x + 8y + 2 = 0 \Rightarrow x = 2$ and $y = -1 \Rightarrow$ critical point is $(2,-1)$; $f_{xx}(2,-1) = 4$, $f_{yy}(2,-1) = 8$, $f_{xy}(2,-1) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 23 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(2,-1) = -6$

8. $f_x(x,y) = 2x - 2y - 2 = 0$ and $f_y(x,y) = -2x + 4y + 2 = 0 \Rightarrow x = 1$ and $y = 0 \Rightarrow$ critical point is $(1,0)$; $f_{xx}(1,0) = 2$, $f_{yy}(1,0) = 4$, $f_{xy}(1,0) = -2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(1,0) = 0$

9. $f_x(x,y) = 2x - 2 = 0$ and $f_y(x,y) = -2y + 4 = 0 \Rightarrow x = 1$ and $y = 2 \Rightarrow$ critical point is $(1,2)$; $f_{xx}(1,2) = 2$, $f_{yy}(1,2) = -2$, $f_{xy}(1,2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0 \Rightarrow$ saddle point

10. $f_x(x,y) = 2x + 2y = 0$ and $f_y(x,y) = 2x = 0 \Rightarrow x = 0$ and $y = 0 \Rightarrow$ critical point is $(0,0)$; $f_{xx}(0,0) = 2$, $f_{yy}(0,0) = 0$, $f_{xy}(0,0) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0 \Rightarrow$ saddle point

11. $f_x(x,y) = \frac{112x - 8x}{\sqrt{56x^2 - 8y^2 - 16x - 31}} - 8 = 0$ and $f_y(x,y) = \frac{-8y}{\sqrt{56x^2 - 8y^2 - 16x - 31}} = 0 \Rightarrow$ critical point is $\left(\frac{16}{7},0\right)$; $f_{xx}\left(\frac{16}{7},0\right) = -\frac{8}{15}$, $f_{yy}\left(\frac{16}{7},0\right) = -\frac{8}{15}$, $f_{xy}\left(\frac{16}{7},0\right) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = \frac{64}{225} > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(\frac{16}{7},0\right) = -\frac{16}{7}$

12. $f_x(x,y) = \frac{-2x}{3(x^2+y^2)^{2/3}} = 0$ and $f_y(x,y) = \frac{-2y}{3(x^2+y^2)^{2/3}} = 0 \Rightarrow$ there are no solutions to the system $f_x(x,y) = 0$ and $f_y(x,y) = 0$, however, we must also consider where the partials are undefined, and this occurs when $x = 0$ and $y = 0$ $\Rightarrow$ critical point is $(0,0)$. Note that the partial derivatives are defined at every other point other than $(0,0)$. We cannot use the second derivative test, but this is the only possible local maximum, local minimum, or saddle point. $f(x,y)$ has a local maximum of $f(0,0) = 1$ at $(0,0)$ since $f(x,y) = 1 - \sqrt[3]{x^2+y^2} \le 1$ for all (x,y) other than $(0,0)$.

13. $f_x(x,y) = 3x^2 - 2y = 0$ and $f_y(x,y) = -3y^2 - 2x = 0 \Rightarrow x = 0$ and $y = 0$, or $x = -\frac{2}{3}$ and $y = \frac{2}{3} \Rightarrow$ critical points are $(0,0)$ and $\left(-\frac{2}{3},\frac{2}{3}\right)$; for $(0,0)$: $f_{xx}(0,0) = 6x|_{(0,0)} = 0$, $f_{yy}(0,0) = -6y|_{(0,0)} = 0$, $f_{xy}(0,0) = -2$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0 \Rightarrow$ saddle point; for $\left(-\frac{2}{3},\frac{2}{3}\right)$: $f_{xx}\left(-\frac{2}{3},\frac{2}{3}\right) = -4$, $f_{yy}\left(-\frac{2}{3},\frac{2}{3}\right) = -4$, $f_{xy}\left(-\frac{2}{3},\frac{2}{3}\right) = -2$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 12 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(-\frac{2}{3},\frac{2}{3}\right) = \frac{170}{27}$

14. $f_x(x,y) = 3x^2 + 3y = 0$ and $f_y(x,y) = 3x + 3y^2 = 0 \Rightarrow x = 0$ and $y = 0$, or $x = -1$ and $y = -1 \Rightarrow$ critical points are $(0,0)$ and $(-1,-1)$; for $(0,0)$: $f_{xx}(0,0) = 6x|_{(0,0)} = 0$, $f_{yy}(0,0) = 6y|_{(0,0)} = 0$, $f_{xy}(0,0) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2$ $= -9 < 0 \Rightarrow$ saddle point; for $(-1,-1)$: $f_{xx}(-1,-1) = -6$, $f_{yy}(-1,-1) = -6$, $f_{xy}(-1,-1) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2$ $= 27 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-1,-1) = 1$

15. $f_x(x,y) = 12x - 6x^2 + 6y = 0$ and $f_y(x,y) = 6y + 6x = 0 \Rightarrow x = 0$ and $y = 0$, or $x = 1$ and $y = -1 \Rightarrow$ critical points are $(0,0)$ and $(1,-1)$; for $(0,0)$: $f_{xx}(0,0) = 12 - 12x|_{(0,0)} = 12$, $f_{yy}(0,0) = 6$, $f_{xy}(0,0) = 6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2$ $= 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(0,0) = 0$; for $(1,-1)$: $f_{xx}(1,-1) = 0$, $f_{yy}(1,-1) = 6$, $f_{xy}(1,-1) = 6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point

16. $f_x(x,y) = 3x^2 + 6x = 0 \Rightarrow x = 0$ or $x = -2$; $f_y(x,y) = 3y^2 - 6y = 0 \Rightarrow y = 0$ or $y = 2 \Rightarrow$ the critical points are $(0,0)$, $(0,2)$, $(-2,0)$, and $(-2,2)$; for $(0,0)$: $f_{xx}(0,0) = 6x + 6|_{(0,0)} = 6$, $f_{yy}(0,0) = 6y - 6|_{(0,0)} = -6$, $f_{xy}(0,0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point; for $(0,2)$: $f_{xx}(0,2) = 6$, $f_{yy}(0,2) = 6$, $f_{xy}(0,2) = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(0,2) = -12$; for $(-2,0)$: $f_{xx}(-2,0) = -6$, $f_{yy}(-2,0) = -6$, $f_{xy}(-2,0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-2,0) = -4$; for $(-2,2)$: $f_{xx}(-2,2) = -6$, $f_{yy}(-2,2) = 6$, $f_{xy}(-2,2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point

17. $f_x(x,y) = 3x^2 + 3y^2 - 15 = 0$ and $f_y(x,y) = 6xy + 3y^2 - 15 = 0 \Rightarrow$ critical points are $(2,1)$, $(-2,-1)$, $\left(0, \sqrt{5}\right)$, and $\left(0, -\sqrt{5}\right)$; for $(2,1)$: $f_{xx}(2,1) = 6x|_{(2,1)} = 12$, $f_{yy}(2,1) = (6x+6y)|_{(2,1)} = 18$, $f_{xy}(2,1) = 6y|_{(2,1)} = 6$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 180 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(2,1) = -30$; for $(-2,-1)$: $f_{xx}(-2,-1) = 6x|_{(-2,-1)}$ $= -12$, $f_{yy}(-2,-1) = (6x+6y)|_{(-2,-1)} = -18$, $f_{xy}(-2,-1) = 6y|_{(-2,-1)} = -6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 180 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-2,-1) = 30$; for $\left(0, \sqrt{5}\right)$: $f_{xx}\left(0, \sqrt{5}\right) = 6x\big|_{\left(0,\sqrt{5}\right)} = 0$, $f_{yy}\left(0, \sqrt{5}\right)$ $= (6x+6y)|_{\left(0,\sqrt{5}\right)} = 6\sqrt{5}$, $f_{xy}\left(0, \sqrt{5}\right) = 6y\big|_{\left(0,\sqrt{5}\right)} = 6\sqrt{5} \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -180 < 0 \Rightarrow$ saddle point; for $\left(0, -\sqrt{5}\right)$: $f_{xx}\left(0, -\sqrt{5}\right) = 6x\big|_{\left(0,-\sqrt{5}\right)} = 0$, $f_{yy}\left(0, -\sqrt{5}\right) = (6x+6y)|_{\left(0,-\sqrt{5}\right)} = -6\sqrt{5}$, $f_{xy}\left(0, -\sqrt{5}\right) = 6y\big|_{\left(0,-\sqrt{5}\right)} = -6\sqrt{5} \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -180 < 0 \Rightarrow$ saddle point.

18. $f_x(x,y) = 6x^2 - 18x = 0 \Rightarrow 6x(x-3) = 0 \Rightarrow x = 0$ or $x = 3$; $f_y(x,y) = 6y^2 + 6y - 12 = 0 \Rightarrow 6(y+2)(y-1) = 0$ $\Rightarrow y = -2$ or $y = 1 \Rightarrow$ the critical points are $(0,-2)$, $(0,1)$, $(3,-2)$, and $(3,1)$; $f_{xx}(x,y) = 12x - 18$, $f_{yy}(x,y) = 12y + 6$, and $f_{xy}(x,y) = 0$; for $(0,-2)$: $f_{xx}(0,-2) = -18$, $f_{yy}(0,-2) = -18$, $f_{xy}(0,-2) = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 324 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(0,-2) = 20$; for $(0,1)$: $f_{xx}(0,1) = -18$, $f_{yy}(0,1) = 18$, $f_{xy}(0,1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -324 < 0 \Rightarrow$ saddle point; for $(3,-2)$: $f_{xx}(3,-2) = 18$, $f_{yy}(3,-2) = -18$, $f_{xy}(3,-2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -324 < 0 \Rightarrow$ saddle point; for $(3,1)$: $f_{xx}(3,1) = 18$, $f_{yy}(3,1) = 18$, $f_{xy}(3,1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 324 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(3,1) = -34$

19. $f_x(x,y) = 4y - 4x^3 = 0$ and $f_y(x,y) = 4x - 4y^3 = 0 \Rightarrow x = y \Rightarrow x\left(1 - x^2\right) = 0 \Rightarrow x = 0, 1, -1 \Rightarrow$ the critical points are $(0,0)$, $(1,1)$, and $(-1,-1)$; for $(0,0)$: $f_{xx}(0,0) = -12x^2|_{(0,0)} = 0$, $f_{yy}(0,0) = -12y^2|_{(0,0)} = 0$, $f_{xy}(0,0) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -16 < 0 \Rightarrow$ saddle point; for $(1,1)$: $f_{xx}(1,1) = -12$, $f_{yy}(1,1) = -12$, $f_{xy}(1,1) = 4$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(1,1) = 2$; for $(-1,-1)$: $f_{xx}(-1,-1) = -12$, $f_{yy}(-1,-1) = -12$, $f_{xy}(-1,-1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-1,-1) = 2$

20. $f_x(x,y) = 4x^3 + 4y = 0$ and $f_y(x,y) = 4y^3 + 4x = 0 \Rightarrow x = -y \Rightarrow -x^3 + x = 0 \Rightarrow x\left(1 - x^2\right) = 0 \Rightarrow x = 0, 1, -1$ $\Rightarrow$ the critical points are $(0,0)$, $(1,-1)$, and $(-1,1)$; $f_{xx}(x,y) = 12x^2$, $f_{yy}(x,y) = 12y^2$, and $f_{xy}(x,y) = 4$; for $(0,0)$: $f_{xx}(0,0) = 0$, $f_{yy}(0,0) = 0$, $f_{xy}(0,0) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -16 < 0 \Rightarrow$ saddle point; for $(1,-1)$: $f_{xx}(1,-1) = 12$, $f_{yy}(1,-1) = 12$, $f_{xy}(1,-1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(1,-1) = -2$; for $(-1,1)$: $f_{xx}(-1,1) = 12$, $f_{yy}(-1,1) = 12$, $f_{xy}(-1,1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(-1,1) = -2$

21. $f_x(x,y) = \frac{-2x}{\left(x^2+y^2-1\right)^2} = 0$ and $f_y(x,y) = \frac{-2y}{\left(x^2+y^2-1\right)^2} = 0 \Rightarrow x = 0$ and $y = 0 \Rightarrow$ the critical point is $(0,0)$; $f_{xx} = \frac{4x^2-2y^2+2}{\left(x^2+y^2-1\right)^3}$, $f_{yy} = \frac{-2x^2+4y^2+2}{\left(x^2+y^2-1\right)^3}$, $f_{xy} = \frac{8xy}{\left(x^2+y^2-1\right)^3}$; $f_{xx}(0,0) = -2$, $f_{yy}(0,0) = -2$, $f_{xy}(0,0) = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(0,0) = -1$

22. $f_x(x,y) = -\frac{1}{x^2} + y = 0$ and $f_y(x,y) = x - \frac{1}{y^2} = 0 \Rightarrow x = 1$ and $y = 1 \Rightarrow$ the critical point is $(1,1)$; $f_{xx} = \frac{2}{x^3}$, $f_{yy} = \frac{2}{y^3}$, $f_{xy} = 1$; $f_{xx}(1,1) = 2$, $f_{yy}(1,1) = 2$, $f_{xy}(1,1) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} > 2 \Rightarrow$ local minimum of $f(1,1) = 3$

23. $f_x(x,y) = y\cos x = 0$ and $f_y(x,y) = \sin x = 0 \Rightarrow x = n\pi$, n an integer, and $y = 0 \Rightarrow$ the critical points are $(n\pi, 0)$, n an integer (Note: $\cos x$ and $\sin x$ cannot both be 0 for the same x, so $\sin x$ must be 0 and $y = 0$); $f_{xx} = -y\sin x$, $f_{yy} = 0$, $f_{xy} = \cos x$; $f_{xx}(n\pi, 0) = 0$, $f_{yy}(n\pi, 0) = 0$, $f_{xy}(n\pi, 0) = 1$ if n is even and $f_{xy}(n\pi, 0) = -1$ if n is odd $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -1 < 0 \Rightarrow$ saddle point.

24. $f_x(x,y) = 2e^{2x}\cos y = 0$ and $f_y(x,y) = -e^{2x}\sin y = 0 \Rightarrow$ no solution since $e^{2x} \neq 0$ for any x and the functions cos y and sin y cannot equal 0 for the same y $\Rightarrow$ no critical points $\Rightarrow$ no extrema and no saddle points

25. $f_x(x,y) = (2x-4)e^{x^2+y^2-4x} = 0$ and $f_y(x,y) = 2ye^{x^2+y^2-4x} = 0 \Rightarrow$ critical point is $(2,0)$; $f_{xx}(2,0) = \frac{2}{e^4}$, $f_{xy}(2,0) = 0$, $f_{yy}(2,0) = \frac{2}{e^4} \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = \frac{4}{e^8} > 0$ and $f_{xx} > 0 \Rightarrow$ local mimimum of $f(2,0) = \frac{1}{e^4}$

26. $f_x(x,y) = -ye^x = 0$ and $f_y(x,y) = e^y - e^x = 0 \Rightarrow$ critical point is $(0,0)$; $f_{xx}(2,0) = 0$, $f_{xy}(2,0) = -1$, $f_{yy}(2,0) = 1$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -1 < 0 \Rightarrow$ saddle point

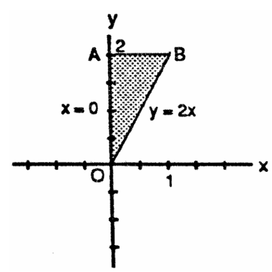
27. $f_x(x,y) = 2xe^{-y} = 0$ and $f_y(x,y) = 2ye^{-y} - e^{-y}(x^2+y^2) = 0 \Rightarrow$ critical points are $(0,0)$ and $(0,2)$; for $(0,0)$: $f_{xx}(0,0) = 2e^{-y}|_{(0,0)} = 2$, $f_{yy}(0,0) = (2e^{-y} - 4ye^{-y} + e^{-y}(x^2+y^2))|_{(0,0)} = 2$, $f_{xy}(0,0) = -2xe^{-y}|_{(0,0)} = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} > 0 \Rightarrow$ local mimimum of $f(0,0) = 0$; for $(0,2)$: $f_{xx}(0,2) = 2e^{-y}|_{(0,2)} = \frac{2}{e^2}$, $f_{yy}(0,2) = (2e^{-y} - 4ye^{-y} + e^{-y}(x^2+y^2))|_{(0,2)} = -\frac{2}{e^2}$, $f_{xy}(0,2) = -2xe^{-y}|_{(0,2)} = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -\frac{4}{e^4} < 0$ $\Rightarrow$ saddle point

28. $f_x(x,y) = e^x(x^2 - 2x + y^2) = 0$ and $f_y(x,y) = -2ye^x = 0 \Rightarrow$ critical points are $(0,0)$ and $(-2,0)$; for $(0,0)$: $f_{xx}(0,0) = e^x(x^2+4x+2-y^2)|_{(0,0)} = 2$, $f_{yy}(0,0) = -2e^x|_{(0,0)} = -2$, $f_{xy}(0,0) = -2ye^x|_{(0,0)} = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0$ and $f_{xx} > 0 \Rightarrow$ saddle point; for $(-2,0)$: $f_{xx}(-2,0) = e^x(x^2+4x+2-y^2)|_{(-2,0)} = -\frac{2}{e^2}$, $f_{yy}(-2,0) = -2e^x|_{(-2,0)} = -\frac{2}{e^2}$, $f_{xy}(-2,0) = -2ye^x|_{(-2,0)} = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = \frac{4}{e^4} > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-2,0) = \frac{4}{e^2}$

29. $f_x(x,y) = -4 + \frac{2}{x} = 0$ and $f_y(x,y) = -1 + \frac{1}{y} = 0 \Rightarrow$ critical point is $\left(\frac{1}{2}, 1\right)$; $f_{xx}\left(\frac{1}{2}, 1\right) = -8$, $f_{yy}\left(\frac{1}{2}, 1\right) = -1$, $f_{xy}\left(\frac{1}{2}, 1\right) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 8 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(\frac{1}{2}, 1\right) = -3 - 2\ln 2$

30. $f_x(x,y) = 2x + \frac{1}{x+y} = 0$ and $f_y(x,y) = -1 + \frac{1}{x+y} = 0 \Rightarrow$ critical point is $\left(-\frac{1}{2}, \frac{3}{2}\right)$; $f_{xx}\left(-\frac{1}{2}, \frac{3}{2}\right) = 1$, $f_{yy}\left(-\frac{1}{2}, \frac{3}{2}\right) = -1$, $f_{xy}\left(-\frac{1}{2}, \frac{3}{2}\right) = -1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -2 < 0 \Rightarrow$ saddle point

31. (i) On OA, $f(x,y) = f(0,y) = y^2 - 4y + 1$ on $0 \le y \le 2$;
$f'(0,y) = 2y - 4 = 0 \Rightarrow y = 2$;
$f(0,0) = 1$ and $f(0,2) = -3$

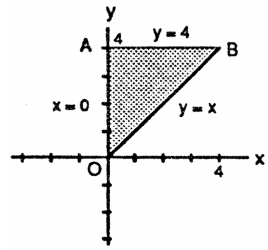


(ii) On AB, $f(x,y) = f(x,2) = 2x^2 - 4x - 3$ on $0 \le x \le 1$;
$f'(x,2) = 4x - 4 = 0 \Rightarrow x = 1$;
$f(0,2) = -3$ and $f(1,2) = -5$

(iii) On OB, $f(x,y) = f(x,2x) = 6x^2 - 12x + 1$ on $0 \le x \le 1$; endpoint values have been found above;
$f'(x,2x) = 12x - 12 = 0 \Rightarrow x = 1$ and $y = 2$, but $(1,2)$ is not an interior point of OB

(iv) For interior points of the triangular region, $f_x(x,y) = 4x - 4 = 0$ and $f_y(x,y) = 2y - 4 = 0$ $\Rightarrow x = 1$ and $y = 2$, but $(1,2)$ is not an interior point of the region. Therefore, the absolute maximum is 1 at $(0,0)$ and the absolute minimum is -5 at $(1,2)$.

32. (i) On OA, $D(x, y) = D(0, y) = y^2 + 1$ on $0 \le y \le 4$;
$D'(0, y) = 2y = 0 \Rightarrow y = 0$; $D(0, 0) = 1$ and
$D(0, 4) = 17$

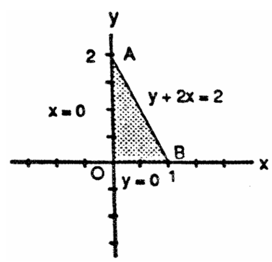


(ii) On AB, $D(x, y) = D(x, 4) = x^2 - 4x + 17$ on
$0 \le x \le 4$; $D'(x, 4) = 2x - 4 = 0 \Rightarrow x = 2$ and $(2, 4)$
is an interior point of AB; $D(2, 4) = 13$ and
$D(4, 4) = D(0, 4) = 17$

(iii) On OB, $D(x, y) = D(x, x) = x^2 + 1$ on $0 \le x \le 4$;
$D'(x, x) = 2x = 0 \Rightarrow x = 0$ and $y = 0$, which is not an interior point of OB; endpoint values have been found above

(iv) For interior points of the triangular region, $f_x(x, y) = 2x - y = 0$ and $f_y(x, y) = -x + 2y = 0 \Rightarrow x = 0$ and $y = 0$, which is not an interior point of the region. Therefore, the absolute maximum is 17 at $(0, 4)$ and $(4, 4)$, and the absolute minimum is 1 at $(0, 0)$.

33. (i) On OA, $f(x, y) = f(0, y) = y^2$ on $0 \le y \le 2$;
$f'(0, y) = 2y = 0 \Rightarrow y = 0$ and $x = 0$; $f(0, 0) = 0$ and
$f(0, 2) = 4$

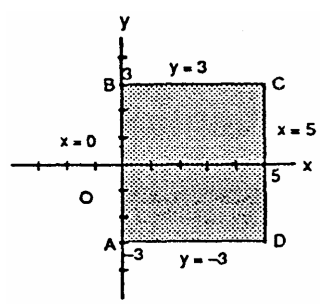


(ii) On OB, $f(x, y) = f(x, 0) = x^2$ on $0 \le x \le 1$;
$f'(x, 0) = 2x = 0 \Rightarrow x = 0$ and $y = 0$; $f(0, 0) = 0$ and
$f(1, 0) = 1$

(iii) On AB, $f(x, y) = f(x, -2x + 2) = 5x^2 - 8x + 4$ on
$0 \le x \le 1$; $f'(x, -2x + 2) = 10x - 8 = 0 \Rightarrow x = \frac{4}{5}$
and $y = \frac{2}{5}$; $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{4}{5}$; endpoint values have been found above.

(iv) For interior points of the triangular region, $f_x(x, y) = 2x = 0$ and $f_y(x, y) = 2y = 0 \Rightarrow x = 0$ and $y = 0$, but $(0, 0)$ is not an interior point of the region. Therefore the absolute maximum is 4 at $(0, 2)$ and the absolute minimum is 0 at $(0, 0)$.

34. (i) On AB, $T(x, y) = T(0, y) = y^2$ on $-3 \le y \le 3$;
$T'(0, y) = 2y = 0 \Rightarrow y = 0$ and $x = 0$; $T(0, 0) = 0$,
$T(0, -3) = 9$, and $T(0, 3) = 9$



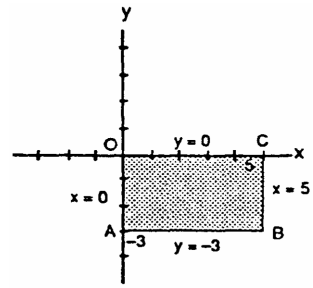
(ii) On BC, $T(x, y) = T(x, 3) = x^2 - 3x + 9$ on $0 \le x \le 5$;
$T'(x, 3) = 2x - 3 = 0 \Rightarrow x = \frac{3}{2}$ and $y = 3$;
$T\left(\frac{3}{2}, 3\right) = \frac{27}{4}$ and $T(5, 3) = 19$

(iii) On CD, $T(x, y) = T(5, y) = y^2 + 5y - 5$ on
$-3 \le y \le 3$; $T'(5, y) = 2y + 5 = 0 \Rightarrow y = -\frac{5}{2}$ and
$x = 5$; $T\left(5, -\frac{5}{2}\right) = -\frac{45}{4}$, $T(5, -3) = -11$ and $T(5, 3) = 19$

(iv) On AD, $T(x, y) = T(x, -3) = x^2 - 9x + 9$ on $0 \le x \le 5$; $T'(x, -3) = 2x - 9 = 0 \Rightarrow x = \frac{9}{2}$ and $y = -3$;
$T\left(\frac{9}{2}, -3\right) = -\frac{45}{4}$, $T(0, -3) = 9$ and $T(5, -3) = -11$

(v) For interior points of the rectangular region, $T_x(x, y) = 2x + y - 6 = 0$ and $T_y(x, y) = x + 2y = 0 \Rightarrow x = 4$ and $y = -2 \Rightarrow (4, -2)$ is an interior critical point with $T(4, -2) = -12$. Therefore the absolute maximum is 19 at $(5, 3)$ and the absolute minimum is -12 at $(4, -2)$.

35. (i) On OC, $T(x, y) = T(x, 0) = x^2 - 6x + 2$ on $0 \le x \le 5$; $T'(x, 0) = 2x - 6 = 0 \Rightarrow x = 3$ and $y = 0$; $T(3, 0) = -7$, $T(0, 0) = 2$, and $T(5, 0) = -3$



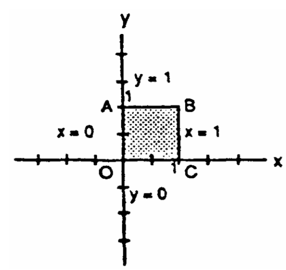
(ii) On CB, $T(x, y) = T(5, y) = y^2 + 5y - 3$ on $-3 \le y \le 0$; $T'(5, y) = 2y + 5 = 0 \Rightarrow y = -\frac{5}{2}$ and $x = 5$; $T\left(5, -\frac{5}{2}\right) = -\frac{37}{4}$ and $T(5, -3) = -9$

(iii) On AB, $T(x, y) = T(x, -3) = x^2 - 9x + 11$ on $0 \le x \le 5$; $T'(x, -3) = 2x - 9 = 0 \Rightarrow x = \frac{9}{2}$ and $y = -3$; $T\left(\frac{9}{2}, -3\right) = -\frac{37}{4}$ and $T(0, -3) = 11$

(iv) On AO, $T(x, y) = T(0, y) = y^2 + 2$ on $-3 \le y \le 0$; $T'(0, y) = 2y = 0 \Rightarrow y = 0$ and $x = 0$, but $(0, 0)$ is not an interior point of AO

(v) For interior points of the rectangular region, $T_x(x, y) = 2x + y - 6 = 0$ and $T_y(x, y) = x + 2y = 0 \Rightarrow x = 4$ and $y = -2$, an interior critical point with $T(4, -2) = -10$. Therefore the absolute maximum is 11 at $(0, -3)$ and the absolute minimum is -10 at $(4, -2)$.

36. (i) On OA, $f(x, y) = f(0, y) = -24y^2$ on $0 \le y \le 1$; $f'(0, y) = -48y = 0 \Rightarrow y = 0$ and $x = 0$, but $(0, 0)$ is not an interior point of OA; $f(0, 0) = 0$ and $f(0, 1) = -24$



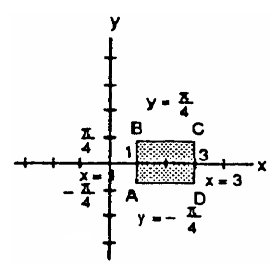
(ii) On AB, $f(x, y) = f(x, 1) = 48x - 32x^3 - 24$ on $0 \le x \le 1$; $f'(x, 1) = 48 - 96x^2 = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$ and $y = 1$, or $x = -\frac{1}{\sqrt{2}}$ and $y = 1$, but $\left(-\frac{1}{\sqrt{2}}, 1\right)$ is not in the interior of AB; $f\left(\frac{1}{\sqrt{2}}, 1\right) = 16\sqrt{2} - 24$ and $f(1, 1) = -8$

(iii) On BC, $f(x, y) = f(1, y) = 48y - 32 - 24y^2$ on $0 \le y \le 1$; $f'(1, y) = 48 - 48y = 0 \Rightarrow y = 1$ and $x = 1$, but $(1, 1)$ is not an interior point of BC; $f(1, 0) = -32$ and $f(1, 1) = -8$

(iv) On OC, $f(x, y) = f(x, 0) = -32x^3$ on $0 \le x \le 1$; $f'(x, 0) = -96x^2 = 0 \Rightarrow x = 0$ and $y = 0$, but $(0, 0)$ is not an interior point of OC; $f(0, 0) = 0$ and $f(1, 0) = -32$

(v) For interior points of the rectangular region, $f_x(x, y) = 48y - 96x^2 = 0$ and $f_y(x, y) = 48x - 48y = 0$ $\Rightarrow x = 0$ and $y = 0$, or $x = \frac{1}{2}$ and $y = \frac{1}{2}$, but $(0, 0)$ is not an interior point of the region; $f\left(\frac{1}{2}, \frac{1}{2}\right) = 2$. Therefore the absolute maximum is 2 at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and the absolute minimum is -32 at $(1, 0)$.

37. (i) On AB, $f(x, y) = f(1, y) = 3\cos y$ on $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$; $f'(1, y) = -3\sin y = 0 \Rightarrow y = 0$ and $x = 1$; $f(1, 0) = 3$, $f\left(1, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$, and $f\left(1, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$



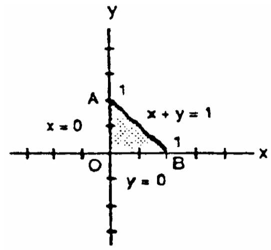
(ii) On CD, $f(x, y) = f(3, y) = 3\cos y$ on $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$; $f'(3, y) = -3\sin y = 0 \Rightarrow y = 0$ and $x = 3$; $f(3, 0) = 3$, $f\left(3, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$ and $f\left(3, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$

(iii) On BC, $f(x, y) = f\left(x, \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(4x - x^2)$ on $1 \le x \le 3$; $f'\left(x, \frac{\pi}{4}\right) = \sqrt{2}(2 - x) = 0 \Rightarrow x = 2$ and $y = \frac{\pi}{4}$; $f\left(2, \frac{\pi}{4}\right) = 2\sqrt{2}$, $f\left(1, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$, and $f\left(3, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$

(iv) On AD, $f(x, y) = f\left(x, -\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(4x - x^2)$ on $1 \le x \le 3$; $f'\left(x, -\frac{\pi}{4}\right) = \sqrt{2}(2 - x) = 0 \Rightarrow x = 2$ and $y = -\frac{\pi}{4}$; $f\left(2, -\frac{\pi}{4}\right) = 2\sqrt{2}$, $f\left(1, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$, and $f\left(3, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$

(v) For interior points of the region, $f_x(x,y) = (4-2x)\cos y = 0$ and $f_y(x,y) = -(4x-x^2)\sin y = 0 \Rightarrow x = 2$ and $y = 0$, which is an interior critical point with $f(2,0) = 4$. Therefore the absolute maximum is 4 at $(2,0)$ and the absolute minimum is $\frac{3\sqrt{2}}{2}$ at $\left(3,-\frac{\pi}{4}\right)$, $\left(3,\frac{\pi}{4}\right)$, $\left(1,-\frac{\pi}{4}\right)$, and $\left(1,\frac{\pi}{4}\right)$.

38. (i) On OA, $f(x,y) = f(0,y) = 2y+1$ on $0 \le y \le 1$; $f'(0,y) = 2 \Rightarrow$ no interior critical points; $f(0,0) = 1$ and $f(0,1) = 3$



(ii) On OB, $f(x,y) = f(x,0) = 4x+1$ on $0 \le x \le 1$; $f'(x,0) = 4 \Rightarrow$ no interior critical points; $f(1,0) = 5$

(iii) On AB, $f(x,y) = f(x,-x+1) = 8x^2-6x+3$ on $0 \le x \le 1$; $f'(x,-x+1) = 16x-6 = 0 \Rightarrow x = \frac{3}{8}$ and $y = \frac{5}{8}$; $f\left(\frac{3}{8},\frac{5}{8}\right) = \frac{15}{8}$, $f(0,1) = 3$, and $f(1,0) = 5$

(iv) For interior points of the triangular region, $f_x(x,y) = 4-8y = 0$ and $f_y(x,y) = -8x+2 = 0$ $\Rightarrow y = \frac{1}{2}$ and $x = \frac{1}{4}$ which is an interior critical point with $f\left(\frac{1}{4},\frac{1}{2}\right) = 2$. Therefore the absolute maximum is 5 at $(1,0)$ and the absolute minimum is 1 at $(0,0)$.

39. Let $F(a,b) = \int_a^b (6-x-x^2)\,dx$ where $a \le b$. The boundary of the domain of F is the line $a = b$ in the ab-plane, and $F(a,a) = 0$, so F is identically 0 on the boundary of its domain. For interior critical points we have: $\frac{\partial F}{\partial a} = -(6-a-a^2) = 0 \Rightarrow a = -3, 2$ and $\frac{\partial F}{\partial b} = (6-b-b^2) = 0 \Rightarrow b = -3, 2$. Since $a \le b$, there is only one interior critical point $(-3,2)$ and $F(-3,2) = \int_{-3}^{2} (6-x-x^2)\,dx$ gives the area under the parabola $y = 6-x-x^2$ that is above the x-axis. Therefore, $a = -3$ and $b = 2$.

40. Let $F(a,b) = \int_a^b (24-2x-x^2)^{1/3}\,dx$ where $a \le b$. The boundary of the domain of F is the line $a = b$ and on this line F is identically 0. For interior critical points we have: $\frac{\partial F}{\partial a} = -(24-2a-a^2)^{1/3} = 0 \Rightarrow a = 4, -6$ and $\frac{\partial F}{\partial b} = (24-2b-b^2)^{1/3} = 0 \Rightarrow b = 4, -6$. Since $a \le b$, there is only one critical point $(-6,4)$ and $F(-6,4) = \int_{-6}^{4} (24-2x-x^2)\,dx$ gives the area under the curve $y = (24-2x-x^2)^{1/3}$ that is above the x-axis. Therefore, $a = -6$ and $b = 4$.

41. $T_x(x,y) = 2x-1 = 0$ and $T_y(x,y) = 4y = 0 \Rightarrow x = \frac{1}{2}$ and $y = 0$ with $T\left(\frac{1}{2},0\right) = -\frac{1}{4}$; on the boundary $x^2+y^2 = 1$: $T(x,y) = -x^2-x+2$ for $-1 \le x \le 1 \Rightarrow T'(x,y) = -2x-1 = 0 \Rightarrow x = -\frac{1}{2}$ and $y = \pm\frac{\sqrt{3}}{2}$; $T\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right) = \frac{9}{4}$, $T\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right) = \frac{9}{4}$, $T(-1,0) = 2$, and $T(1,0) = 0 \Rightarrow$ the hottest is $2\frac{1}{4}{}^\circ$ at $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$; the coldest is $-\frac{1}{4}{}^\circ$ at $\left(\frac{1}{2},0\right)$.

42. $f_x(x,y) = y+2-\frac{2}{x} = 0$ and $f_y(x,y) = x-\frac{1}{y} = 0 \Rightarrow x = \frac{1}{2}$ and $y = 2$; $f_{xx}\left(\frac{1}{2},2\right) = \frac{2}{x^2}\Big|_{\left(\frac{1}{2},2\right)} = 8$, $f_{yy}\left(\frac{1}{2},2\right) = \frac{1}{y^2}\Big|_{\left(\frac{1}{2},2\right)} = \frac{1}{4}$, $f_{xy}\left(\frac{1}{2},2\right) = 1 \Rightarrow f_{xx}f_{yy}-f_{xy}^2 = 1 > 0$ and $f_{xx} > 0 \Rightarrow$ a local minimum of $f\left(\frac{1}{2},2\right)$ $= 2-\ln\frac{1}{2} = 2+\ln 2$

43. (a) $f_x(x,y) = 2x-4y = 0$ and $f_y(x,y) = 2y-4x = 0 \Rightarrow x = 0$ and $y = 0$; $f_{xx}(0,0) = 2$, $f_{yy}(0,0) = 2$, $f_{xy}(0,0) = -4 \Rightarrow f_{xx}f_{yy}-f_{xy}^2 = -12 < 0 \Rightarrow$ saddle point at $(0,0)$

(b) $f_x(x,y) = 2x-2 = 0$ and $f_y(x,y) = 2y-4 = 0 \Rightarrow x = 1$ and $y = 2$; $f_{xx}(1,2) = 2$, $f_{yy}(1,2) = 2$, $f_{xy}(1,2) = 0 \Rightarrow f_{xx}f_{yy}-f_{xy}^2 = 4 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum at $(1,2)$

(c) $f_x(x,y) = 9x^2 - 9 = 0$ and $f_y(x,y) = 2y + 4 = 0 \Rightarrow x = \pm 1$ and $y = -2$; $f_{xx}(1,-2) = 18x\big|_{(1,-2)} = 18$, $f_{yy}(1,-2) = 2$, $f_{xy}(1,-2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum at $(1,-2)$; $f_{xx}(-1,-2) = -18$, $f_{yy}(-1,-2) = 2$, $f_{xy}(-1,-2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point at $(-1,-2)$

44. (a) Minimum at $(0,0)$ since $f(x,y) > 0$ for all other (x,y)
(b) Maximum of 1 at $(0,0)$ since $f(x,y) < 1$ for all other (x,y)
(c) Neither since $f(x,y) < 0$ for $x < 0$ and $f(x,y) > 0$ for $x > 0$
(d) Neither since $f(x,y) < 0$ for $x < 0$ and $f(x,y) > 0$ for $x > 0$
(e) Neither since $f(x,y) < 0$ for $x < 0$ and $y > 0$, but $f(x,y) > 0$ for $x > 0$ and $y > 0$
(f) Minimum at $(0,0)$ since $f(x,y) > 0$ for all other (x,y)

45. If $k = 0$, then $f(x,y) = x^2 + y^2 \Rightarrow f_x(x,y) = 2x = 0$ and $f_y(x,y) = 2y = 0 \Rightarrow x = 0$ and $y = 0 \Rightarrow (0,0)$ is the only critical point. If $k \neq 0$, $f_x(x,y) = 2x + ky = 0 \Rightarrow y = -\frac{2}{k}x$; $f_y(x,y) = kx + 2y = 0 \Rightarrow kx + 2\left(-\frac{2}{k}x\right) = 0$ $\Rightarrow kx - \frac{4x}{k} = 0 \Rightarrow \left(k - \frac{4}{k}\right)x = 0 \Rightarrow x = 0$ or $k = \pm 2 \Rightarrow y = \left(-\frac{2}{k}\right)(0) = 0$ or $y = \pm x$; in any case $(0,0)$ is a critical point.

46. (See Exercise 45 above): $f_{xx}(x,y) = 2$, $f_{yy}(x,y) = 2$, and $f_{xy}(x,y) = k \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 - k^2$; f will have a saddle point at $(0,0)$ if $4 - k^2 < 0 \Rightarrow k > 2$ or $k < -2$; f will have a local minimum at $(0,0)$ if $4 - k^2 > 0 \Rightarrow -2 < k < 2$; the test is inconclusive if $4 - k^2 = 0 \Rightarrow k = \pm 2$.

47. No; for example $f(x,y) = xy$ has a saddle point at $(a,b) = (0,0)$ where $f_x = f_y = 0$.

48. If $f_{xx}(a,b)$ and $f_{yy}(a,b)$ differ in sign, then $f_{xx}(a,b)\,f_{yy}(a,b) < 0$ so $f_{xx}f_{yy} - f_{xy}^2 < 0$. The surface must therefore have a saddle point at (a,b) by the second derivative test.

49. We want the point on $z = 10 - x^2 - y^2$ where the tangent plane is parallel to the plane $x + 2y + 3z = 0$. To find a normal vector to $z = 10 - x^2 - y^2$ let $w = z + x^2 + y^2 - 10$. Then $\nabla w = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$ is normal to $z = 10 - x^2 - y^2$ at (x,y). The vector ∇w is parallel to $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ which is normal to the plane $x + 2y + 3z = 0$ if $6x\mathbf{i} + 6y\mathbf{j} + 3\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ or $x = \frac{1}{6}$ and $y = \frac{1}{3}$. Thus the point is $\left(\frac{1}{6}, \frac{1}{3}, 10 - \frac{1}{36} - \frac{1}{9}\right)$ or $\left(\frac{1}{6}, \frac{1}{3}, \frac{355}{36}\right)$.

50. We want the point on $z = x^2 + y^2 + 10$ where the tangent plane is parallel to the plane $x + 2y - z = 0$. Let $w = z - x^2 - y^2 - 10$, then $\nabla w = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$ is normal to $z = x^2 + y^2 + 10$ at (x,y). The vector ∇w is parallel to $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ which is normal to the plane if $x = \frac{1}{2}$ and $y = 1$. Thus the point $\left(\frac{1}{2}, 1, \frac{1}{4} + 1 + 10\right)$ or $\left(\frac{1}{2}, 1, \frac{45}{4}\right)$ is the point on the surface $z = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$.

51. $d(x,y,z) = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} \Rightarrow$ we can minimize $d(x,y,z)$ by minimizing $D(x,y,z) = x^2 + y^2 + z^2$; $3x + 2y + z = 6 \Rightarrow z = 6 - 3x - 2y \Rightarrow D(x,y) = x^2 + y^2 + (6 - 3x - 2y)^2 \Rightarrow D_x(x,y) = 2x - 6(6 - 3x - 2y) = 0$ and $D_y(x,y) = 2y - 4(6 - 3x - 2y) = 0 \Rightarrow$ critical point is $\left(\frac{9}{7}, \frac{6}{7}\right) \Rightarrow z = \frac{3}{7}$; $D_{xx}\left(\frac{9}{7}, \frac{6}{7}\right) = 20$, $D_{yy}\left(\frac{1}{2}, 1\right) = 10$, $D_{xy}\left(\frac{1}{2}, 1\right) = 12 \Rightarrow D_{xx}D_{yy} - D_{xy}^2 = 56 > 0$ and $D_{xx} > 0 \Rightarrow$ local minimum of $d\left(\frac{9}{7}, \frac{6}{7}, \frac{3}{7}\right) = \frac{3\sqrt{14}}{7}$

52. $d(x,y,z) = \sqrt{(x-2)^2 + (y+1)^2 + (z-1)^2} \Rightarrow$ we can minimize $d(x,y,z)$ by minimizing $D(x,y,z) = (x-2)^2 + (y+1)^2 + (z-1)^2$; $x + y - z = 2 \Rightarrow z = x + y - 2$ $\Rightarrow D(x,y) = (x-2)^2 + (y+1)^2 + (x+y-3)^2 \Rightarrow D_x(x,y) = 2(x-2) + 2(x+y-3) = 0$ and $D_y(x,y) = 2(y+1) + 2(x+y-3) = 0 \Rightarrow$ critical point is $\left(\frac{8}{3}, -\frac{1}{3}\right) \Rightarrow z = \frac{1}{3}$; $D_{xx}\left(\frac{8}{3}, -\frac{1}{3}\right) = 4$, $D_{yy}\left(\frac{8}{3}, -\frac{1}{3}\right) = 4$, $D_{xy}\left(\frac{8}{3}, -\frac{1}{3}\right) = 2 \Rightarrow D_{xx}D_{yy} - D_{xy}^2 = 12 > 0$ and $D_{xx} > 0 \Rightarrow$ local minimum of $d\left(\frac{8}{3}, -\frac{1}{3}, \frac{1}{3}\right) = \frac{2}{\sqrt{3}}$

53. $s(x, y, z) = x^2 + y^2 + z^2$; $x + y + z = 9 \Rightarrow z = 9 - x - y \Rightarrow s(x, y) = x^2 + y^2 + (9 - x - y)^2$
$\Rightarrow s_x(x, y) = 2x - 2(9 - x - y) = 0$ and $s_y(x, y) = 2y - 2(9 - x - y) = 0 \Rightarrow$ critical point is $(3, 3) \Rightarrow z = 3$;
$s_{xx}(3, 3) = 4$, $s_{yy}(3, 3) = 4$, $s_{xy}(3, 3) = 2 \Rightarrow s_{xx}s_{yy} - s_{xy}^2 = 12 > 0$ and $s_{xx} > 0 \Rightarrow$ local minimum of $s(3, 3, 3) = 27$

54. $p(x, y, z) = xyz$; $x + y + z = 3 \Rightarrow z = 3 - x - y \Rightarrow p(x, y) = xy(3 - x - y) = 3xy - x^2y - xy^2$
$\Rightarrow p_x(x, y) = 3y - 2xy - y^2 = 0$ and $p_y(x, y) = 3x - x^2 - 2xy = 0 \Rightarrow$ critical points are $(0, 0)$, $(0, 3)$, $(3, 0)$, and $(1, 1)$; for $(0, 0) \Rightarrow z = 3$; $p_{xx}(0, 0) = 0$, $p_{yy}(0, 0) = 0$, $p_{xy}(0, 0) = 3 \Rightarrow p_{xx}p_{yy} - p_{xy}^2 = -9 < 0 \Rightarrow$ saddle point;
for $(0, 3) \Rightarrow z = 0$; $p_{xx}(0, 3) = -6$, $p_{yy}(0, 3) = 0$, $p_{xy}(0, 3) = -3 \Rightarrow p_{xx}p_{yy} - p_{xy}^2 = -9 < 0 \Rightarrow$ saddle point;
for $(3, 0) \Rightarrow z = 0$; $p_{xx}(3, 0) = 0$, $p_{yy}(3, 0) = -6$, $p_{xy}(3, 0) = -3 \Rightarrow p_{xx}p_{yy} - p_{xy}^2 = -9 < 0 \Rightarrow$ saddle point;
for $(1, 1) \Rightarrow z = 1$; $p_{xx}(1, 1) = -2$, $p_{yy}(1, 1) = -2$, $p_{xy}(1, 1) = -1 \Rightarrow p_{xx}p_{yy} - p_{xy}^2 = 3 > 0$ and $p_{xx} < 0 \Rightarrow$ local maximum of $p(1, 1, 1) = 1$

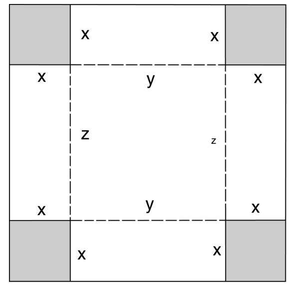
55. $s(x, y, z) = xy + yz + xz$; $x + y + z = 6 \Rightarrow z = 6 - x - y \Rightarrow s(x, y) = xy + y(6 - x - y) + x(6 - x - y)$
$= 6x + 6y - xy - x^2 - y^2 \Rightarrow s_x(x, y) = 6 - 2x - y = 0$ and $s_y(x, y) = 6 - x - 2y = 0 \Rightarrow$ critical point is $(2, 2)$
$\Rightarrow z = 2$; $s_{xx}(2, 2) = -2$, $s_{yy}(2, 2) = -2$, $s_{xy}(2, 2) = -1 \Rightarrow s_{xx}s_{yy} - s_{xy}^2 = 3 > 0$ and $s_{xx} < 0 \Rightarrow$ local maximum of $s(2, 2, 2) = 12$

56. $d(x, y, z) = \sqrt{(x + 6)^2 + (y - 4)^2 + (z - 0)^2} \Rightarrow$ we can minimize $d(x, y, z)$ by minimizing
$D(x, y, z) = (x + 6)^2 + (y - 4)^2 + z^2$; $z = \sqrt{x^2 + y^2} \Rightarrow D(x, y) = (x + 6)^2 + (y - 4)^2 + x^2 + y^2$
$= 2x^2 + 2y^2 + 12x - 8y + 52 \Rightarrow D_x(x, y) = 4x + 12 = 0$ and $D_y(x, y) = 4y - 8 = 0 \Rightarrow$ critical point is $(-3, 2)$
$\Rightarrow z = \sqrt{13}$; $D_{xx}(-3, 2) = 4$, $D_{yy}(-3, 2) = 4$, $D_{xy}(-3, 2) = 0 \Rightarrow D_{xx}D_{yy} - D_{xy}^2 = 16 > 0$ and $D_{xx} > 0 \Rightarrow$ local minimum of $d\left(-3, 2, \sqrt{13}\right) = \sqrt{26}$

57. $V(x, y, z) = (2x)(2y)(2z) = 8xyz$; $x^2 + y^2 + z^2 = 4 \Rightarrow z = \sqrt{4 - x^2 - y^2} \Rightarrow V(x, y) = 8xy\sqrt{4 - x^2 - y^2}$,
$x \ge 0$ and $y \ge 0 \Rightarrow V_x(x, y) = \frac{32y - 16x^2y - 8y^3}{\sqrt{4 - x^2 - y^2}} = 0$ and $V_y(x, y) = \frac{32x - 16xy^2 - 8x^3}{\sqrt{4 - x^2 - y^2}} = 0 \Rightarrow$ critical points are
$(0, 0)$, $\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$, $\left(\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$, $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$, and $\left(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$. Only $(0, 0)$ and $\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ satisfy $x \ge 0$ and $y \ge 0$
$V(0, 0) = 0$ and $V\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = \frac{64}{3\sqrt{3}}$; On $x = 0$, $0 \le y \le 2 \Rightarrow V(0, y) = 8(0)y\sqrt{4 - 0^2 - y^2} = 0$, no critical points,
$V(0, 0) = 0$, $V(0, 2) = 0$; On $y = 0$, $0 \le x \le 2 \Rightarrow V(x, 0) = 8x(0)\sqrt{4 - x^2 - 0^2} = 0$, no critical points, $V(0, 0) = 0$,
$V(0, 2) = 0$; On $y = \sqrt{4 - x^2}$, $0 \le x \le 2 \Rightarrow V\left(x, \sqrt{4 - x^2}\right) = 8x\sqrt{4 - x^2}\sqrt{4 - x^2 - \left(\sqrt{4 - x^2}\right)^2} = 0$
no critical points, $V(0, 2) = 0$, $V(2, 0) = 0$. Thus, there is a maximum volume of $\frac{64}{3\sqrt{3}}$ if the box is $\frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}}$.

58. $S(x, y, z) = 2xy + 2yz + 2xz$; $xyz = 27 \Rightarrow z = \frac{27}{xy} \Rightarrow S(x, y, z) = 2xy + 2y\left(\frac{27}{xy}\right) + 2x\left(\frac{27}{xy}\right) = 2xy + \frac{54}{x} + \frac{54}{y}$, $x > 0$,
$y > 0$; $S_x(x, y) = 2y - \frac{54}{x^2} = 0$ and $S_y(x, y) = 2x - \frac{54}{y^2} = 0 \Rightarrow$ Critical point is $(3, 3) \Rightarrow z = 3$; $S_{xx}(3, 3) = 4$,
$S_{yy}(3, 3) = 4$, $D_{xy}(3, 3) = 2 \Rightarrow D_{xx}D_{yy} - D_{xy}^2 = 12 > 0$ and $D_{xx} > 0 \Rightarrow$ local minimum of $S(3, 3, 3) = 54$

59. Let x = height of the box, y = width, and z = length, cut out squares of length x from corner of the material See diagram at right. Fold along the dashed lines to form the box. From the diagram we see that the length of the material is $2x+y$ and the width is $2x+z$. Thus $(2x+y)(2x+z)=12$ $\Rightarrow z=\frac{2(6-2x^2+xy)}{2x+y}$. Since $V(x,y,z)=xyz$ $\Rightarrow V(x,y)=\frac{2xy(6-2x^2+xy)}{2x+y}$, where $x>0$, $y>0$.



$V_x(x,y)=\frac{4(3y^2-4x^3y-4x^2y^2-xy^3)}{(2x+y)^2}=0$ and

$V_y(x,y)=\frac{2(12x^2-4x^4-4x^3y-x^2y^2)}{(2x+y)^2}=0 \Rightarrow$ critical points are $\left(\sqrt{3},0\right)$, $\left(-\sqrt{3},0\right)$, $\left(\frac{1}{\sqrt{3}},\frac{4}{\sqrt{3}}\right)$, and $\left(-\frac{1}{\sqrt{3}},-\frac{4}{\sqrt{3}}\right)$. Only $\left(\sqrt{3},0\right)$ and $\left(\frac{1}{\sqrt{3}},\frac{4}{\sqrt{3}}\right)$ satisfy $x>0$ and $y>0$. For $\left(\sqrt{3},0\right)$: $z=0$; $V_{xx}\left(\sqrt{3},0\right)=0$, $V_{yy}\left(\sqrt{3},0\right)=-2\sqrt{3}$, $V_{xy}\left(\sqrt{3},0\right)=-4\sqrt{3} \Rightarrow V_{xx}V_{yy}-V_{xy}^2=-48<0 \Rightarrow$ saddle point. For $\left(\frac{1}{\sqrt{3}},\frac{4}{\sqrt{3}}\right)$: $z=\frac{4}{\sqrt{3}}$; $V_{xx}\left(\frac{1}{\sqrt{3}},\frac{4}{\sqrt{3}}\right)=-\frac{80}{3\sqrt{3}}$, $V_{yy}\left(\frac{1}{\sqrt{3}},\frac{4}{\sqrt{3}}\right)=-\frac{2}{3\sqrt{3}}$, $V_{xy}\left(\frac{1}{\sqrt{3}},\frac{4}{\sqrt{3}}\right)=-\frac{4}{3\sqrt{3}} \Rightarrow V_{xx}V_{yy}-V_{xy}^2=\frac{16}{3}>0$ and $V_{xx}<0 \Rightarrow$ local maximum of $V\left(\frac{1}{\sqrt{3}},\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}}\right)=\frac{16}{3\sqrt{3}}$

60. (a) (i) On $x=0$, $f(x,y)=f(0,y)=y^2-y+1$ for $0\le y\le 1$; $f'(0,y)=2y-1=0 \Rightarrow y=\frac{1}{2}$ and $x=0$; $f\left(0,\frac{1}{2}\right)=\frac{3}{4}$, $f(0,0)=1$, and $f(0,1)=1$

(ii) On $y=1$, $f(x,y)=f(x,1)=x^2+x+1$ for $0\le x\le 1$; $f'(x,1)=2x+1=0 \Rightarrow x=-\frac{1}{2}$ and $y=1$, but $\left(-\frac{1}{2},1\right)$ is outside the domain; $f(0,1)=1$ and $f(1,1)=3$

(iii) On $x=1$, $f(x,y)=f(1,y)=y^2+y+1$ for $0\le y\le 1$; $f'(1,y)=2y+1=0 \Rightarrow y=-\frac{1}{2}$ and $x=1$, but $\left(1,-\frac{1}{2}\right)$ is outside the domain; $f(1,0)=1$ and $f(1,1)=3$

(iv) On $y=0$, $f(x,y)=f(x,0)=x^2-x+1$ for $0\le x\le 1$; $f'(x,0)=2x-1=0 \Rightarrow x=\frac{1}{2}$ and $y=0$; $f\left(\frac{1}{2},0\right)=\frac{3}{4}$; $f(0,0)=1$, and $f(1,0)=1$

(v) On the interior of the square, $f_x(x,y)=2x+2y-1=0$ and $f_y(x,y)=2y+2x-1=0 \Rightarrow 2x+2y=1$ $\Rightarrow (x+y)=\frac{1}{2}$. Then $f(x,y)=x^2+y^2+2xy-x-y+1=(x+y)^2-(x+y)+1=\frac{3}{4}$ is the absolute minimum value when $2x+2y=1$.

(b) The absolute maximum is $f(1,1)=3$.

61. (a) $\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=\frac{dx}{dt}+\frac{dy}{dt}=-2\sin t+2\cos t=0 \Rightarrow \cos t=\sin t \Rightarrow x=y$

(i) On the semicircle $x^2+y^2=4$, $y\ge 0$, we have $t=\frac{\pi}{4}$ and $x=y=\sqrt{2} \Rightarrow f\left(\sqrt{2},\sqrt{2}\right)=2\sqrt{2}$. At the endpoints, $f(-2,0)=-2$ and $f(2,0)=2$. Therefore the absolute minimum is $f(-2,0)=-2$ when $t=\pi$; the absolute maximum is $f\left(\sqrt{2},\sqrt{2}\right)=2\sqrt{2}$ when $t=\frac{\pi}{4}$.

(ii) On the quartercircle $x^2+y^2=4$, $x\ge 0$ and $y\ge 0$, the endpoints give $f(0,2)=2$ and $f(2,0)=2$. Therefore the absolute minimum is $f(2,0)=2$ and $f(0,2)=2$ when $t=0,\frac{\pi}{2}$ respectively; the absolute maximum is $f\left(\sqrt{2},\sqrt{2}\right)=2\sqrt{2}$ when $t=\frac{\pi}{4}$.

(b) $\frac{dg}{dt}=\frac{\partial g}{\partial x}\frac{dx}{dt}+\frac{\partial g}{\partial y}\frac{dy}{dt}=y\frac{dx}{dt}+x\frac{dy}{dt}=-4\sin^2 t+4\cos^2 t=0 \Rightarrow \cos t=\pm\sin t \Rightarrow x=\pm y$.

(i) On the semicircle $x^2+y^2=4$, $y\ge 0$, we obtain $x=y=\sqrt{2}$ at $t=\frac{\pi}{4}$ and $x=-\sqrt{2}$, $y=\sqrt{2}$ at $t=\frac{3\pi}{4}$. Then $g\left(\sqrt{2},\sqrt{2}\right)=2$ and $g\left(-\sqrt{2},\sqrt{2}\right)=-2$. At the endpoints, $g(-2,0)=g(2,0)=0$. Therefore the absolute minimum is $g\left(-\sqrt{2},\sqrt{2}\right)=-2$ when $t=\frac{3\pi}{4}$; the absolute maximum is $g\left(\sqrt{2},\sqrt{2}\right)=2$ when $t=\frac{\pi}{4}$.

(ii) On the quartercircle $x^2 + y^2 = 4$, $x \ge 0$ and $y \ge 0$, the endpoints give $g(0, 2) = 0$ and $g(2, 0) = 0$. Therefore the absolute minimum is $g(2, 0) = 0$ and $g(0, 2) = 0$ when $t = 0, \frac{\pi}{2}$ respectively; the absolute maximum is $g\left(\sqrt{2}, \sqrt{2}\right) = 2$ when $t = \frac{\pi}{4}$.

(c) $\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt} = 4x\frac{dx}{dt} + 2y\frac{dy}{dt} = (8 \cos t)(-2 \sin t) + (4 \sin t)(2 \cos t) = -8 \cos t \sin t = 0$
$\Rightarrow\ t = 0, \frac{\pi}{2}, \pi$ yielding the points $(2, 0)$, $(0, 2)$ for $0 \le t \le \pi$.

(i) On the semicircle $x^2 + y^2 = 4$, $y \ge 0$ we have $h(2, 0) = 8$, $h(0, 2) = 4$, and $h(-2, 0) = 8$. Therefore, the absolute minimum is $h(0, 2) = 4$ when $t = \frac{\pi}{2}$; the absolute maximum is $h(2, 0) = 8$ and $h(-2, 0) = 8$ when $t = 0, \pi$ respectively.

(ii) On the quartercircle $x^2 + y^2 = 4$, $x \ge 0$ and $y \ge 0$ the absolute minimum is $h(0, 2) = 4$ when $t = \frac{\pi}{2}$; the absolute maximum is $h(2, 0) = 8$ when $t = 0$.

62. (a) $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 2\frac{dx}{dt} + 3\frac{dy}{dt} = -6 \sin t + 6 \cos t = 0 \ \Rightarrow\ \sin t = \cos t \ \Rightarrow\ t = \frac{\pi}{4}$ for $0 \le t \le \pi$.

(i) On the semi-ellipse, $\frac{x^2}{9} + \frac{y^2}{4} = 1$, $y \ge 0$, $f(x, y) = 2x + 3y = 6 \cos t + 6 \sin t = 6\left(\frac{\sqrt{2}}{2}\right) + 6\left(\frac{\sqrt{2}}{2}\right) = 6\sqrt{2}$ at $t = \frac{\pi}{4}$. At the endpoints, $f(-3, 0) = -6$ and $f(3, 0) = 6$. The absolute minimum is $f(-3, 0) = -6$ when $t = \pi$; the absolute maximum is $f\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 6\sqrt{2}$ when $t = \frac{\pi}{4}$.

(ii) On the quarter ellipse, at the endpoints $f(0, 2) = 6$ and $f(3, 0) = 6$. The absolute minimum is $f(3, 0) = 6$ and $f(0, 2) = 6$ when $t = 0, \frac{\pi}{2}$ respectively; the absolute maximum is $f\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 6\sqrt{2}$ when $t = \frac{\pi}{4}$.

(b) $\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt} = (2 \sin t)(-3 \sin t) + (3 \cos t)(2 \cos t) = 6\left(\cos^2 t - \sin^2 t\right) = 6 \cos 2t = 0$
$\Rightarrow\ t = \frac{\pi}{4}, \frac{3\pi}{4}$ for $0 \le t \le \pi$.

(i) On the semi-ellipse, $g(x, y) = xy = 6 \sin t \cos t$. Then $g\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 3$ when $t = \frac{\pi}{4}$, and $g\left(-\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = -3$ when $t = \frac{3\pi}{4}$. At the endpoints, $g(-3, 0) = g(3, 0) = 0$. The absolute minimum is $g\left(-\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = -3$ when $t = \frac{3\pi}{4}$; the absolute maximum is $g\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 3$ when $t = \frac{\pi}{4}$.

(ii) On the quarter ellipse, at the endpoints $g(0, 2) = 0$ and $g(3, 0) = 0$. The absolute minimum is $g(3, 0) = 0$ and $g(0, 2) = 0$ at $t = 0, \frac{\pi}{2}$ respectively; the absolute maximum is $g\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 3$ when $t = \frac{\pi}{4}$.

(c) $\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt} = 2x\frac{dx}{dt} + 6y\frac{dy}{dt} = (6 \cos t)(-3 \sin t) + (12 \sin t)(2 \cos t) = 6 \sin t \cos t = 0$
$\Rightarrow\ t = 0, \frac{\pi}{2}, \pi$ for $0 \le t \le \pi$, yielding the points $(3, 0)$, $(0, 2)$, and $(-3, 0)$.

(i) On the semi-ellipse, $y \ge 0$ so that $h(3, 0) = 9$, $h(0, 2) = 12$, and $h(-3, 0) = 9$. The absolute minimum is $h(3, 0) = 9$ and $h(-3, 0) = 9$ when $t = 0, \pi$ respectively; the absolute maximum is $h(0, 2) = 12$ when $t = \frac{\pi}{2}$.

(ii) On the quarter ellipse, the absolute minimum is $h(3, 0) = 9$ when $t = 0$; the absolute maximum is $h(0, 2) = 12$ when $t = \frac{\pi}{2}$.

63. $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}$

(i) $x = 2t$ and $y = t + 1 \ \Rightarrow\ \frac{df}{dt} = (t + 1)(2) + (2t)(1) = 4t + 2 = 0 \ \Rightarrow\ t = -\frac{1}{2} \ \Rightarrow\ x = -1$ and $y = \frac{1}{2}$ with $f\left(-1, \frac{1}{2}\right) = -\frac{1}{2}$. The absolute minimum is $f\left(-1, \frac{1}{2}\right) = -\frac{1}{2}$ when $t = -\frac{1}{2}$; there is no absolute maximum.

(ii) For the endpoints: $t = -1 \ \Rightarrow\ x = -2$ and $y = 0$ with $f(-2, 0) = 0$; $t = 0 \ \Rightarrow\ x = 0$ and $y = 1$ with $f(0, 1) = 0$. The absolute minimum is $f\left(-1, \frac{1}{2}\right) = -\frac{1}{2}$ when $t = -\frac{1}{2}$; the absolute maximum is $f(0, 1) = 0$ and $f(-2, 0) = 0$ when $t = -1, 0$ respectively.

(iii) There are no interior critical points. For the endpoints: $t = 0 \ \Rightarrow\ x = 0$ and $y = 1$ with $f(0, 1) = 0$; $t = 1 \ \Rightarrow\ x = 2$ and $y = 2$ with $f(2, 2) = 4$. The absolute minimum is $f(0, 1) = 0$ when $t = 0$; the absolute maximum is $f(2, 2) = 4$ when $t = 1$.

64. (a) $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

(i) $x = t$ and $y = 2 - 2t \Rightarrow \frac{df}{dt} = (2t)(1) + 2(2-2t)(-2) = 10t - 8 = 0 \Rightarrow t = \frac{4}{5} \Rightarrow x = \frac{4}{5}$ and $y = \frac{2}{5}$ with $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{16}{25} + \frac{4}{25} = \frac{4}{5}$. The absolute minimum is $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{4}{5}$ when $t = \frac{4}{5}$; there is no absolute maximum along the line.

(ii) For the endpoints: $t = 0 \Rightarrow x = 0$ and $y = 2$ with $f(0,2) = 4$; $t = 1 \Rightarrow x = 1$ and $y = 0$ with $f(1,0) = 1$. The absolute minimum is $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{4}{5}$ at the interior critical point when $t = \frac{4}{5}$; the absolute maximum is $f(0,2) = 4$ at the endpoint when $t = 0$.

(b) $\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} = \left[\frac{-2x}{(x^2+y^2)^2}\right]\frac{dx}{dt} + \left[\frac{-2y}{(x^2+y^2)^2}\right]\frac{dy}{dt}$

(i) $x = t$ and $y = 2 - 2t \Rightarrow x^2 + y^2 = 5t^2 - 8t + 4 \Rightarrow \frac{dg}{dt} = -\left(5t^2 - 8t + 4\right)^{-2}[(-2t)(1) + (-2)(2-2t)(-2)]$ $= -\left(5t^2 - 8t + 4\right)^{-2}(-10t + 8) = 0 \Rightarrow t = \frac{4}{5} \Rightarrow x = \frac{4}{5}$ and $y = \frac{2}{5}$ with $g\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{1}{\left(\frac{4}{5}\right)} = \frac{5}{4}$. The absolute maximum is $g\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{5}{4}$ when $t = \frac{4}{5}$; there is no absolute minimum along the line since x and y can be as large as we please.

(ii) For the endpoints: $t = 0 \Rightarrow x = 0$ and $y = 2$ with $g(0,2) = \frac{1}{4}$; $t = 1 \Rightarrow x = 1$ and $y = 0$ with $g(1,0) = 1$. The absolute minimum is $g(0,2) = \frac{1}{4}$ when $t = 0$; the absolute maximum is $g\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{5}{4}$ when $t = \frac{4}{5}$.

65. $w = (mx_1 + b - y_1)^2 + (mx_2 + b - y_2)^2 + \cdots + (mx_n + b - y_n)^2$

$\Rightarrow \frac{\partial w}{\partial m} = 2(mx_1 + b - y_1)(x_1) + 2(mx_2 + b - y_2)(x_2) + \cdots + 2(mx_n + b - y_n)(x_n)$

$\Rightarrow \frac{\partial w}{\partial b} = 2(mx_1 + b - y_1)(1) + 2(mx_2 + b - y_2)(1) + \cdots + 2(mx_n + b - y_n)(1)$

$\frac{\partial w}{\partial m} = 0 \Rightarrow 2\left[(mx_1 + b - y_1)(x_1) + (mx_2 + b - y_2)(x_2) + \cdots + (mx_n + b - y_n)(x_n)\right] = 0$

$\Rightarrow mx_1^2 + bx_1 - x_1y_1 + mx_2^2 + bx_2 - x_2y_2 + \cdots + mx_n^2 + bx_n - x_ny_n = 0$

$\Rightarrow m(x_1^2 + x_2^2 + \cdots + x_n^2) + b(x_1 + x_2 + \cdots + x_n) - (x_1y_1 + x_2y_2 + \cdots + x_ny_n) = 0$

$\Rightarrow m\sum_{k=1}^{n}(x_k^2) + b\sum_{k=1}^{n}x_k - \sum_{k=1}^{n}(x_ky_k) = 0$

$\frac{\partial w}{\partial b} = 0 \Rightarrow 2\left[(mx_1 + b - y_1) + (mx_2 + b - y_2) + \cdots + (mx_n + b - y_n)\right] = 0$

$\Rightarrow mx_1 + b - y_1 + mx_2 + b - y_2 + \cdots + mx_n + b - y_n = 0$

$\Rightarrow m(x_1 + x_2 + \cdots + x_n) + (b + b + \cdots + b) - (y_1 + y_2 + \cdots + y_n) = 0$

$\Rightarrow m\sum_{k=1}^{n}x_k + b\sum_{k=1}^{n}1 - \sum_{k=1}^{n}y_k = 0 \Rightarrow m\sum_{k=1}^{n}x_k + bn - \sum_{k=1}^{n}y_k = 0 \Rightarrow b = \frac{1}{n}\left(\sum_{k=1}^{n}y_k - m\sum_{k=1}^{n}x_k\right).$

Substituting for b in the equation obtained for $\frac{\partial w}{\partial m}$ we get $m\sum_{k=1}^{n}(x_k^2) + \frac{1}{n}\left(\sum_{k=1}^{n}y_k - m\sum_{k=1}^{n}x_k\right)\sum_{k=1}^{n}x_k - \sum_{k=1}^{n}(x_ky_k) = 0.$

Multiply both sides by n to obtain $mn\sum_{k=1}^{n}(x_k^2) + \left(\sum_{k=1}^{n}y_k - m\sum_{k=1}^{n}x_k\right)\sum_{k=1}^{n}x_k - n\sum_{k=1}^{n}(x_ky_k) = 0$

$\Rightarrow mn\sum_{k=1}^{n}(x_k^2) + \left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}y_k\right) - m\left(\sum_{k=1}^{n}x_k\right)^2 - n\sum_{k=1}^{n}(x_ky_k) = 0$

$\Rightarrow mn\sum_{k=1}^{n}(x_k^2) - m\left(\sum_{k=1}^{n}x_k\right)^2 = n\sum_{k=1}^{n}(x_ky_k) - \left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}y_k\right)$

$\Rightarrow m\left[n\sum_{k=1}^{n}(x_k^2) - \left(\sum_{k=1}^{n}x_k\right)^2\right] = n\sum_{k=1}^{n}(x_ky_k) - \left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}y_k\right)$

$$\Rightarrow m = \frac{n\sum_{k=1}^{n}(x_ky_k) - \left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}y_k\right)}{n\sum_{k=1}^{n}(x_k^2) - \left(\sum_{k=1}^{n}x_k\right)^2} = \frac{\left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}y_k\right) - n\sum_{k=1}^{n}(x_ky_k)}{\left(\sum_{k=1}^{n}x_k\right)^2 - n\sum_{k=1}^{n}(x_k^2)}$$

To show that these values for m and b minimize the sum of the squares of the distances, use second derivative test.

$\frac{\partial^2 w}{\partial m^2} = 2x_1^2 + 2x_2^2 + \cdots + 2x_n^2 = 2\sum_{k=1}^{n}(x_k^2)$; $\frac{\partial^2 w}{\partial m\,\partial b} = 2x_1 + 2x_2 + \cdots + 2x_n = 2\sum_{k=1}^{n}x_k$; $\frac{\partial^2 w}{\partial b^2} = 2 + 2 + \cdots + 2 = 2n$

The discriminant is: $\left(\frac{\partial^2 w}{\partial m^2}\right)\left(\frac{\partial^2 w}{\partial b^2}\right) - \left(\frac{\partial^2 w}{\partial m\,\partial b}\right)^2 = \left[2\sum_{k=1}^{n}(x_k^2)\right](2n) - \left[2\sum_{k=1}^{n}x_k\right]^2 = 4\left[n\sum_{k=1}^{n}(x_k^2) - \left(\sum_{k=1}^{n}x_k\right)^2\right].$

Now, $n\sum_{k=1}^{n}(x_k^2) - \left(\sum_{k=1}^{n}x_k\right)^2 = n(x_1^2 + x_2^2 + \cdots + x_n^2) - (x_1 + x_2 + \cdots + x_n)(x_1 + x_2 + \cdots + x_n)$

$= n\,x_1^2 + n\,x_2^2 + \cdots + n\,x_n^2 - x_1^2 - x_1x_2 - \cdots - x_1x_n - x_2x_1 - x_2^2 - \cdots - x_2x_n - x_nx_1 - x_nx_2 - \cdots - x_n^2$

$= (n-1)\,x_1^2 + (n-1)\,x_2^2 + \cdots + (n-1)\,x_n^2 - 2\,x_1x_2 - 2\,x_1x_3 - \cdots - 2\,x_1x_n - 2\,x_2x_3 - \cdots - 2\,x_2x_n - \cdots - 2\,x_{n-1}x_n$

$= (x_1^2 - 2\,x_1x_2 + x_2^2) + (x_1^2 - 2\,x_1x_3 + x_3^2) + \cdots + (x_1^2 - 2\,x_1x_n + x_n^2) + (x_2^2 - 2\,x_2x_3 + x_3^2) + \cdots + (x_2^2 - 2\,x_2x_n + x_n^2)$
$\quad + \cdots + (x_{n-1}^2 - 2\,x_{n-1}x_n + x_n^2)$

$= (x_1 - x_2)^2 + (x_1 - x_3)^2 + \cdots + (x_1 - x_n)^2 + (x_2 - x_3)^2 + \cdots + (x_2 - x_n)^2 + \cdots + (x_{n-1} - x_n)^2 \ge 0.$

Thus we have : $\left(\frac{\partial^2 w}{\partial m^2}\right)\left(\frac{\partial^2 w}{\partial b^2}\right) - \left(\frac{\partial^2 w}{\partial m\,\partial b}\right)^2 = 4\left[n\sum_{k=1}^{n}(x_k^2) - \left(\sum_{k=1}^{n}x_k\right)^2\right] \ge 4(0) = 0$. If $x_1 = x_2 = \cdots = x_n$ then

$\left(\frac{\partial^2 w}{\partial m^2}\right)\left(\frac{\partial^2 w}{\partial b^2}\right) - \left(\frac{\partial^2 w}{\partial m\,\partial b}\right)^2 = 0$. Also, $\frac{\partial^2 w}{\partial m^2} = 2\sum_{k=1}^{n}(x_k^2) \ge 0$. If $x_1 = x_2 = \cdots = x_n = 0$, then $\frac{\partial^2 w}{\partial m^2} = 0$.

Provided that at least one x_i is nonzero and different from the rest of x_j, $j \ne i$, then $\left(\frac{\partial^2 w}{\partial m^2}\right)\left(\frac{\partial^2 w}{\partial b^2}\right) - \left(\frac{\partial^2 w}{\partial m\,\partial b}\right)^2 > 0$ and $\frac{\partial^2 w}{\partial m^2} > 0 \Rightarrow$ the values given above for m and b minimize w.

66. $m = \frac{(0)(5) - 3(6)}{(0)^2 - 3(8)} = \frac{3}{4}$ and
$b = \frac{1}{3}\left[5 - \frac{3}{4}(0)\right] = \frac{5}{3}$
$\Rightarrow\ y = \frac{3}{4}x + \frac{5}{3}$; $y\big|_{x=4} = \frac{14}{3}$

k	x_k	y_k	x_k^2	x_ky_k
1	−2	0	4	0
2	0	2	0	0
3	2	3	4	6
Σ	0	5	8	6

67. $m = \frac{(2)(-1) - 3(-14)}{(2)^2 - 3(10)} = -\frac{20}{13}$ and
$b = \frac{1}{3}\left[-1 - \left(-\frac{20}{13}\right)(2)\right] = \frac{9}{13}$
$\Rightarrow\ y = -\frac{20}{13}x + \frac{9}{13}$; $y\big|_{x=4} = -\frac{71}{13}$

k	x_k	y_k	x_k^2	x_ky_k
1	−1	2	1	−2
2	0	1	0	0
3	3	−4	9	−12
Σ	2	−1	10	−14

68. $m = \frac{(3)(5) - 3(8)}{(3)^2 - 3(5)} = \frac{3}{2}$ and
$b = \frac{1}{3}\left[5 - \frac{3}{2}(3)\right] = \frac{1}{6}$
$\Rightarrow\ y = \frac{3}{2}x + \frac{1}{6}$; $y\big|_{x=4} = \frac{37}{6}$

k	x_k	y_k	x_k^2	x_ky_k
1	0	0	0	0
2	1	2	1	2
3	2	3	4	6
Σ	3	5	5	8

69-74. Example CAS commands:

Maple:

```
f := (x,y) -> x^2+y^3-3*x*y;
x0,x1 := -5,5;
y0,y1 := -5,5;
plot3d( f(x,y), x=x0..x1, y=y0..y1, axes=boxed, shading=zhue, title="#69(a) (Section 14.7)" );
plot3d( f(x,y), x=x0..x1, y=y0..y1, grid=[40,40], axes=boxed, shading=zhue, style=patchcontour, title="#69(b)
    (Section 14.7)" );
fx := D[1](f);                                    # (c)
fy := D[2](f);
crit_pts := solve( {fx(x,y)=0,fy(x,y)=0}, {x,y} );
fxx := D[1](fx);                                  # (d)
fxy := D[2](fx);
```

```
fyy := D[2](fy);
discr := unapply( fxx(x,y)*fyy(x,y)-fxy(x,y)^2, (x,y) );
for CP in {crit_pts} do                                    # (e)
  eval( [x,y,fxx(x,y),discr(x,y)], CP );
end do;
# (0,0) is a saddle point
# ( 9/4,  3/2) is a local minimum
```

Mathematica: (assigned functions and bounds will vary)

```
Clear[x,y,f]
f[x_,y_]:= x^2 + y^3 − 3x y
xmin= −5;  xmax= 5;  ymin= −5;  ymax= 5;
Plot3D[f[x,y], {x, xmin, xmax}, {y, ymin, ymax}, AxesLabel → {x, y, z}]
ContourPlot[f[x,y], {x, xmin, xmax}, {y, ymin, ymax}, ContourShading → False, Contours → 40]
fx= D[f[x,y], x];
fy= D[f[x,y], y];
critical=Solve[{fx==0, fy==0},{x, y}]
fxx= D[fx, x];
fxy= D[fx, y];
fyy= D[fy, y];
discriminant= fxx fyy − fxy^2
{{x, y}, f[x, y], discriminant, fxx} /.critical
```

14.8 LAGRANGE MULTIPLIERS

1. $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 4y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 4y\mathbf{j}) \Rightarrow y = 2x\lambda$ and $x = 4y\lambda$ $\Rightarrow x = 8x\lambda^2 \Rightarrow \lambda = \pm\frac{\sqrt{2}}{4}$ or $x = 0$.
CASE 1: If $x = 0$, then $y = 0$. But $(0, 0)$ is not on the ellipse so $x \neq 0$.
CASE 2: $x \neq 0 \Rightarrow \lambda = \pm\frac{\sqrt{2}}{4} \Rightarrow x = \pm\sqrt{2}y \Rightarrow \left(\pm\sqrt{2}y\right)^2 + 2y^2 = 1 \Rightarrow y = \pm\frac{1}{2}$.
Therefore f takes on its extreme values at $\left(\pm\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$ and $\left(\pm\frac{\sqrt{2}}{2}, -\frac{1}{2}\right)$. The extreme values of f on the ellipse are $\pm\frac{\sqrt{2}}{2}$.

2. $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow y = 2x\lambda$ and $x = 2y\lambda$ $\Rightarrow x = 4x\lambda^2 \Rightarrow x = 0$ or $\lambda = \pm\frac{1}{2}$.
CASE 1: If $x = 0$, then $y = 0$. But $(0, 0)$ is not on the circle $x^2 + y^2 - 10 = 0$ so $x \neq 0$.
CASE 2: $x \neq 0 \Rightarrow \lambda = \pm\frac{1}{2} \Rightarrow y = 2x\left(\pm\frac{1}{2}\right) = \pm x \Rightarrow x^2 + (\pm x)^2 - 10 = 0 \Rightarrow x = \pm\sqrt{5} \Rightarrow y = \pm\sqrt{5}$.
Therefore f takes on its extreme values at $\left(\pm\sqrt{5}, \sqrt{5}\right)$ and $\left(\pm\sqrt{5}, -\sqrt{5}\right)$. The extreme values of f on the circle are 5 and -5.

3. $\nabla f = -2x\mathbf{i} - 2y\mathbf{j}$ and $\nabla g = \mathbf{i} + 3\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow -2x\mathbf{i} - 2y\mathbf{j} = \lambda(\mathbf{i} + 3\mathbf{j}) \Rightarrow x = -\frac{\lambda}{2}$ and $y = -\frac{3\lambda}{2}$ $\Rightarrow \left(-\frac{\lambda}{2}\right) + 3\left(-\frac{3\lambda}{2}\right) = 10 \Rightarrow \lambda = -2 \Rightarrow x = 1$ and $y = 3 \Rightarrow$ f takes on its extreme value at $(1, 3)$ on the line. The extreme value is $f(1, 3) = 49 - 1 - 9 = 39$.

4. $\nabla f = 2xy\mathbf{i} + x^2\mathbf{j}$ and $\nabla g = \mathbf{i} + \mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2xy\mathbf{i} + x^2\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j}) \Rightarrow 2xy = \lambda$ and $x^2 = \lambda$ $\Rightarrow 2xy = x^2 \Rightarrow x = 0$ or $2y = x$.
CASE 1: If $x = 0$, then $x + y = 3 \Rightarrow y = 3$.

CASE 2: If $x \neq 0$, then $2y = x$ so that $x + y = 3 \Rightarrow 2y + y = 3 \Rightarrow y = 1 \Rightarrow x = 2$.
Therefore f takes on its extreme values at $(0, 3)$ and $(2, 1)$. The extreme values of f are $f(0, 3) = 0$ and $f(2, 1) = 4$.

5. We optimize $f(x, y) = x^2 + y^2$, the square of the distance to the origin, subject to the constraint $g(x, y) = xy^2 - 54 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = y^2\mathbf{i} + 2xy\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} = \lambda\left(y^2\mathbf{i} + 2xy\mathbf{j}\right) \Rightarrow 2x = \lambda y^2$ and $2y = 2\lambda xy$.
CASE 1: If $y = 0$, then $x = 0$. But $(0, 0)$ does not satisfy the constraint $xy^2 = 54$ so $y \neq 0$.
CASE 2: If $y \neq 0$, then $2 = 2\lambda x \Rightarrow x = \frac{1}{\lambda} \Rightarrow 2\left(\frac{1}{\lambda}\right) = \lambda y^2 \Rightarrow y^2 = \frac{2}{\lambda^2}$. Then $xy^2 = 54 \Rightarrow \left(\frac{1}{\lambda}\right)\left(\frac{2}{\lambda^2}\right) = 54$
$\Rightarrow \lambda^3 = \frac{1}{27} \Rightarrow \lambda = \frac{1}{3} \Rightarrow x = 3$ and $y^2 = 18 \Rightarrow x = 3$ and $y = \pm 3\sqrt{2}$.
Therefore $\left(3, \pm 3\sqrt{2}\right)$ are the points on the curve $xy^2 = 54$ nearest the origin (since $xy^2 = 54$ has points increasingly far away as y gets close to 0, no points are farthest away).

6. We optimize $f(x, y) = x^2 + y^2$, the square of the distance to the origin subject to the constraint $g(x, y) = x^2y - 2 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = 2xy\mathbf{i} + x^2\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = 2xy\lambda$ and $2y = x^2\lambda \Rightarrow \lambda = \frac{2y}{x^2}$, since $x = 0 \Rightarrow y = 0$ (but $g(0, 0) \neq 0$). Thus $x \neq 0$ and $2x = 2xy\left(\frac{2y}{x^2}\right) \Rightarrow x^2 = 2y^2 \Rightarrow \left(2y^2\right)y - 2 = 0 \Rightarrow y = 1$ (since $y > 0$) $\Rightarrow x = \pm\sqrt{2}$. Therefore $\left(\pm\sqrt{2}, 1\right)$ are the points on the curve $x^2y = 2$ nearest the origin (since $x^2y = 2$ has points increasingly far away as x gets close to 0, no points are farthest away).

7. (a) $\nabla f = \mathbf{i} + \mathbf{j}$ and $\nabla g = y\mathbf{i} + x\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} + \mathbf{j} = \lambda(y\mathbf{i} + x\mathbf{j}) \Rightarrow 1 = \lambda y$ and $1 = \lambda x \Rightarrow y = \frac{1}{\lambda}$ and $x = \frac{1}{\lambda} \Rightarrow \frac{1}{\lambda^2} = 16 \Rightarrow \lambda = \pm\frac{1}{4}$. Use $\lambda = \frac{1}{4}$ since $x > 0$ and $y > 0$. Then $x = 4$ and $y = 4 \Rightarrow$ the minimum value is 8 at the point $(4, 4)$. Now, $xy = 16$, $x > 0$, $y > 0$ is a branch of a hyperbola in the first quadrant with the x-and y-axes as asymptotes. The equations $x + y = c$ give a family of parallel lines with $m = -1$. As these lines move away from the origin, the number c increases. Thus the minimum value of c occurs where $x + y = c$ is tangent to the hyperbola's branch.
(b) $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = \mathbf{i} + \mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j}) \Rightarrow y = \lambda = x\ y + y = 16 \Rightarrow y = 8$ $\Rightarrow x = 8 \Rightarrow f(8, 8) = 64$ is the maximum value. The equations $xy = c$ ($x > 0$ and $y > 0$ or $x < 0$ and $y < 0$ to get a maximum value) give a family of hyperbolas in the first and third quadrants with the x- and y-axes as asymptotes. The maximum value of c occurs where the hyperbola $xy = c$ is tangent to the line $x + y = 16$.

8. Let $f(x, y) = x^2 + y^2$ be the square of the distance from the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = (2x + y)\mathbf{i} + (2y + x)\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = \lambda(2x + y)$ and $2y = \lambda(2y + x) \Rightarrow \frac{2y}{2y+x} = \lambda$
$\Rightarrow 2x = \left(\frac{2y}{2y+x}\right)(2x + y) \Rightarrow x(2y + x) = y(2x + y) \Rightarrow x^2 = y^2 \Rightarrow y = \pm x$.
CASE 1: $y = x \Rightarrow x^2 + x(x) + x^2 - 1 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$ and $y = x$.
CASE 2: $y = -x \Rightarrow x^2 + x(-x) + (-x)^2 - 1 = 0 \Rightarrow x = \pm 1$ and $y = -x$. Thus $f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{3}$
$= f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ and $f(1, -1) = 2 = f(-1, 1)$.
Therefore the points $(1, -1)$ and $(-1, 1)$ are the farthest away; $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ are the closest points to the origin.

9. $V = \pi r^2 h \Rightarrow 16\pi = \pi r^2 h \Rightarrow 16 = r^2 h \Rightarrow g(r, h) = r^2 h - 16$; $S = 2\pi rh + 2\pi r^2 \Rightarrow \nabla S = (2\pi h + 4\pi r)\mathbf{i} + 2\pi r\mathbf{j}$ and $\nabla g = 2rh\mathbf{i} + r^2\mathbf{j}$ so that $\nabla S = \lambda \nabla g \Rightarrow (2\pi rh + 4\pi r)\mathbf{i} + 2\pi r\mathbf{j} = \lambda\left(2rh\mathbf{i} + r^2\mathbf{j}\right) \Rightarrow 2\pi rh + 4\pi r = 2rh\lambda$ and $2\pi r = \lambda r^2$ $\Rightarrow r = 0$ or $\lambda = \frac{2\pi}{r}$. But $r = 0$ gives no physical can, so $r \neq 0 \Rightarrow \lambda = \frac{2\pi}{r} \Rightarrow 2\pi h + 4\pi r = 2rh\left(\frac{2\pi}{r}\right) \Rightarrow 2r = h$ $\Rightarrow 16 = r^2(2r) \Rightarrow r = 2 \Rightarrow h = 4$; thus $r = 2$ cm and $h = 4$ cm give the only extreme surface area of $24\pi\ \text{cm}^2$. Since $r = 4$ cm and $h = 1$ cm $\Rightarrow V = 16\pi\ \text{cm}^3$ and $S = 40\pi\ \text{cm}^2$, which is a larger surface area, then $24\pi\ \text{cm}^2$ must be the minimum surface area.

10. For a cylinder of radius r and height h we want to maximize the surface area $S = 2\pi rh$ subject to the constraint $g(r,h) = r^2 + \left(\frac{h}{2}\right)^2 - a^2 = 0$. Thus $\nabla S = 2\pi h\mathbf{i} + 2\pi r\mathbf{j}$ and $\nabla g = 2r\mathbf{i} + \frac{h}{2}\mathbf{j}$ so that $\nabla S = \lambda \nabla g \Rightarrow 2\pi h = 2\lambda r$ and $2\pi r = \frac{\lambda h}{2} \Rightarrow \frac{\pi h}{r} = \lambda$ and $2\pi r = \left(\frac{\pi h}{r}\right)\left(\frac{h}{2}\right) \Rightarrow 4r^2 = h^2 \Rightarrow h = 2r \Rightarrow r^2 + \frac{4r^2}{4} = a^2 \Rightarrow 2r^2 = a^2 \Rightarrow r = \frac{a}{\sqrt{2}}$ $\Rightarrow h = a\sqrt{2} \Rightarrow S = 2\pi\left(\frac{a}{\sqrt{2}}\right)\left(a\sqrt{2}\right) = 2\pi a^2$.

11. $A = (2x)(2y) = 4xy$ subject to $g(x,y) = \frac{x^2}{16} + \frac{y^2}{9} - 1 = 0$; $\nabla A = 4y\mathbf{i} + 4x\mathbf{j}$ and $\nabla g = \frac{x}{8}\mathbf{i} + \frac{2y}{9}\mathbf{j}$ so that $\nabla A = \lambda \nabla g \Rightarrow 4y\mathbf{i} + 4x\mathbf{j} = \lambda\left(\frac{x}{8}\mathbf{i} + \frac{2y}{9}\mathbf{j}\right) \Rightarrow 4y = \left(\frac{x}{8}\right)\lambda$ and $4x = \left(\frac{2y}{9}\right)\lambda \Rightarrow \lambda = \frac{32y}{x}$ and $4x = \left(\frac{2y}{9}\right)\left(\frac{32y}{x}\right)$ $\Rightarrow y = \pm\frac{3}{4}x \Rightarrow \frac{x^2}{16} + \frac{\left(\frac{\pm 3}{4}x\right)^2}{9} = 1 \Rightarrow x^2 = 8 \Rightarrow x = \pm 2\sqrt{2}$. We use $x = 2\sqrt{2}$ since x represents distance. Then $y = \frac{3}{4}\left(2\sqrt{2}\right) = \frac{3\sqrt{2}}{2}$, so the length is $2x = 4\sqrt{2}$ and the width is $2y = 3\sqrt{2}$.

12. $P = 4x + 4y$ subject to $g(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$; $\nabla P = 4\mathbf{i} + 4\mathbf{j}$ and $\nabla g = \frac{2x}{a^2}\mathbf{i} + \frac{2y}{b^2}\mathbf{j}$ so that $\nabla P = \lambda \nabla g$ $\Rightarrow 4 = \left(\frac{2x}{a^2}\right)\lambda$ and $4 = \left(\frac{2y}{b^2}\right)\lambda \Rightarrow \lambda = \frac{2a^2}{x}$ and $4 = \left(\frac{2y}{b^2}\right)\left(\frac{2a^2}{x}\right) \Rightarrow y = \left(\frac{b^2}{a^2}\right)x \Rightarrow \frac{x^2}{a^2} + \frac{\left(\frac{b^2}{a^2}\right)^2 x^2}{b^2} = 1 \Rightarrow \frac{x^2}{a^2} + \frac{b^2x^2}{a^4}$ $= 1 \Rightarrow \left(a^2 + b^2\right)x^2 = a^4 \Rightarrow x = \frac{a^2}{\sqrt{a^2+b^2}}$, since $x > 0 \Rightarrow y = \left(\frac{b^2}{a^2}\right)x = \frac{b^2}{\sqrt{a^2+b^2}} \Rightarrow$ width $= 2x = \frac{2a^2}{\sqrt{a^2+b^2}}$ and height $= 2y = \frac{2b^2}{\sqrt{a^2+b^2}} \Rightarrow$ perimeter is $P = 4x + 4y = \frac{4a^2+4b^2}{\sqrt{a^2+b^2}} = 4\sqrt{a^2+b^2}$

13. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = (2x-2)\mathbf{i} + (2y-4)\mathbf{j}$ so that $\nabla f = \lambda \nabla g = 2x\mathbf{i} + 2y\mathbf{j} = \lambda[(2x-2)\mathbf{i} + (2y-4)\mathbf{j}]$ $\Rightarrow 2x = \lambda(2x-2)$ and $2y = \lambda(2y-4) \Rightarrow x = \frac{\lambda}{\lambda-1}$ and $y = \frac{2\lambda}{\lambda-1}$, $\lambda \neq 1 \Rightarrow y = 2x \Rightarrow x^2 - 2x + (2x)^2 - 4(2x) = 0$ $\Rightarrow x = 0$ and $y = 0$, or $x = 2$ and $y = 4$. Therefore $f(0,0) = 0$ is the minimum value and $f(2,4) = 20$ is the maximum value. (Note that $\lambda = 1$ gives $2x = 2x - 2$ or $0 = -2$, which is impossible.)

14. $\nabla f = 3\mathbf{i} - \mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 3 = 2\lambda x$ and $-1 = 2\lambda y \Rightarrow \lambda = \frac{3}{2x}$ and $-1 = 2\left(\frac{3}{2x}\right)y$ $\Rightarrow y = -\frac{x}{3} \Rightarrow x^2 + \left(-\frac{x}{3}\right)^2 = 4 \Rightarrow 10x^2 = 36 \Rightarrow x = \pm\frac{6}{\sqrt{10}} \Rightarrow x = \frac{6}{\sqrt{10}}$ and $y = -\frac{2}{\sqrt{10}}$, or $x = -\frac{6}{\sqrt{10}}$ and $y = \frac{2}{\sqrt{10}}$. Therefore $f\left(\frac{6}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) = \frac{20}{\sqrt{10}} + 6 = 2\sqrt{10} + 6 \approx 12.325$ is the maximum value, and $f\left(-\frac{6}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right)$ $= -2\sqrt{10} + 6 \approx -0.325$ is the minimum value.

15. $\nabla T = (8x - 4y)\mathbf{i} + (-4x + 2y)\mathbf{j}$ and $g(x,y) = x^2 + y^2 - 25 = 0 \Rightarrow \nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla T = \lambda \nabla g$ $\Rightarrow (8x - 4y)\mathbf{i} + (-4x + 2y)\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 8x - 4y = 2\lambda x$ and $-4x + 2y = 2\lambda y \Rightarrow y = \frac{-2x}{\lambda-1}$, $\lambda \neq 1$ $\Rightarrow 8x - 4\left(\frac{-2x}{\lambda-1}\right) = 2\lambda x \Rightarrow x = 0$, or $\lambda = 0$, or $\lambda = 5$.

CASE 1: $x = 0 \Rightarrow y = 0$; but $(0,0)$ is not on $x^2 + y^2 = 25$ so $x \neq 0$.

CASE 2: $\lambda = 0 \Rightarrow y = 2x \Rightarrow x^2 + (2x)^2 = 25 \Rightarrow x = \pm\sqrt{5}$ and $y = 2x$.

CASE 3: $\lambda = 5 \Rightarrow y = \frac{-2x}{4} = -\frac{x}{2} \Rightarrow x^2 + \left(-\frac{x}{2}\right)^2 = 25 \Rightarrow x = \pm 2\sqrt{5} \Rightarrow x = 2\sqrt{5}$ and $y = -\sqrt{5}$, or $x = -2\sqrt{5}$ and $y = \sqrt{5}$.

Therefore $T\left(\sqrt{5}, 2\sqrt{5}\right) = 0° = T\left(-\sqrt{5}, -2\sqrt{5}\right)$ is the minimum value and $T\left(2\sqrt{5}, -\sqrt{5}\right) = 125°$ $= T\left(-2\sqrt{5}, \sqrt{5}\right)$ is the maximum value. (Note: $\lambda = 1 \Rightarrow x = 0$ from the equation $-4x + 2y = 2\lambda y$; but we found $x \neq 0$ in CASE 1.)

16. The surface area is given by $S = 4\pi r^2 + 2\pi rh$ subject to the constraint $V(r,h) = \frac{4}{3}\pi r^3 + \pi r^2 h = 8000$. Thus $\nabla S = (8\pi r + 2\pi h)\mathbf{i} + 2\pi r\mathbf{j}$ and $\nabla V = \left(4\pi r^2 + 2\pi rh\right)\mathbf{i} + \pi r^2\mathbf{j}$ so that $\nabla S = \lambda \nabla V = (8\pi r + 2\pi h)\mathbf{i} + 2\pi r\mathbf{j}$ $= \lambda\left[\left(4\pi r^2 + 2\pi rh\right)\mathbf{i} + \pi r^2\mathbf{j}\right] \Rightarrow 8\pi r + 2\pi h = \lambda\left(4\pi r^2 + 2\pi rh\right)$ and $2\pi r = \lambda\pi r^2 \Rightarrow r = 0$ or $2 = r\lambda$. But $r \neq 0$

so $2 = r\lambda \Rightarrow \lambda = \frac{2}{r} \Rightarrow 4r + h = \frac{2}{r}(2r^2 + rh) \Rightarrow h = 0 \Rightarrow$ the tank is a sphere (there is no cylindrical part) and $\frac{4}{3}\pi r^3 = 8000 \Rightarrow r = 10\left(\frac{6}{\pi}\right)^{1/3}$.

17. Let $f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2$ be the square of the distance from $(1, 1, 1)$. Then $\nabla f = 2(x-1)\mathbf{i} + 2(y-1)\mathbf{j} + 2(z-1)\mathbf{k}$ and $\nabla g = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ so that $\nabla f = \lambda \nabla g$ $\Rightarrow 2(x-1)\mathbf{i} + 2(y-1)\mathbf{j} + 2(z-1)\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \Rightarrow 2(x-1) = \lambda, 2(y-1) = 2\lambda, 2(z-1) = 3\lambda$ $\Rightarrow 2(y-1) = 2[2(x-1)]$ and $2(z-1) = 3[2(x-1)] \Rightarrow x = \frac{y+1}{2} \Rightarrow z + 2 = 3\left(\frac{y+1}{2}\right)$ or $z = \frac{3y-1}{2}$; thus $\frac{y+1}{2} + 2y + 3\left(\frac{3y-1}{2}\right) - 13 = 0 \Rightarrow y = 2 \Rightarrow x = \frac{3}{2}$ and $z = \frac{5}{2}$. Therefore the point $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$ is closest (since no point on the plane is farthest from the point $(1, 1, 1)$).

18. Let $f(x, y, z) = (x-1)^2 + (y+1)^2 + (z-1)^2$ be the square of the distance from $(1, -1, 1)$. Then $\nabla f = 2(x-1)\mathbf{i} + 2(y+1)\mathbf{j} + 2(z-1)\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow x - 1 = \lambda x, y + 1 = \lambda y$ and $z - 1 = \lambda z \Rightarrow x = \frac{1}{1-\lambda}, y = -\frac{1}{1-\lambda}$, and $z = \frac{1}{1-\lambda}$ for $\lambda \neq 1 \Rightarrow \left(\frac{1}{1-\lambda}\right)^2 + \left(\frac{-1}{1-\lambda}\right)^2 + \left(\frac{1}{1-\lambda}\right)^2 = 4$ $\Rightarrow \frac{1}{1-\lambda} = \pm\frac{2}{\sqrt{3}} \Rightarrow x = \frac{2}{\sqrt{3}}, y = -\frac{2}{\sqrt{3}}, z = \frac{2}{\sqrt{3}}$ or $x = -\frac{2}{\sqrt{3}}, y = \frac{2}{\sqrt{3}}, z = -\frac{2}{\sqrt{3}}$. The largest value of f occurs where $x < 0, y > 0$, and $z < 0$ or at the point $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$ on the sphere.

19. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance from the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = 2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}) \Rightarrow 2x = 2x\lambda, 2y = 2y\lambda$, and $2z = -2z\lambda \Rightarrow x = 0$ or $\lambda = 1$.
CASE 1: $\lambda = 1 \Rightarrow 2y = -2y \Rightarrow y = 0; 2z = -2z \Rightarrow z = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$ and $y = z = 0$.
CASE 2: $x = 0 \Rightarrow y^2 - z^2 = 1$, which has no solution.
Therefore the points on the unit circle $x^2 + y^2 = 1$, are the points on the surface $x^2 + y^2 - z^2 = 1$ closest to the origin. The minimum distance is 1.

20. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = y\mathbf{i} + x\mathbf{j} - \mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(y\mathbf{i} + x\mathbf{j} - \mathbf{k}) \Rightarrow 2x = \lambda y, 2y = \lambda x$, and $2z = -\lambda$ $\Rightarrow x = \frac{\lambda y}{2} \Rightarrow 2y = \lambda\left(\frac{\lambda y}{2}\right) \Rightarrow y = 0$ or $\lambda = \pm 2$.
CASE 1: $y = 0 \Rightarrow x = 0 \Rightarrow -z + 1 = 0 \Rightarrow z = 1$.
CASE 2: $\lambda = 2 \Rightarrow x = y$ and $z = -1 \Rightarrow x^2 - (-1) + 1 = 0 \Rightarrow x^2 + 2 = 0$, so no solution.
CASE 3: $\lambda = -2 \Rightarrow x = -y$ and $z = 1 \Rightarrow (-y)y - 1 + 1 = 0 \Rightarrow y = 0$, again.
Therefore $(0, 0, 1)$ is the point on the surface closest to the origin since this point gives the only extreme value and there is no maximum distance from the surface to the origin.

21. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = -y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(-y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}) \Rightarrow 2x = -y\lambda, 2y = -x\lambda$, and $2z = 2z\lambda \Rightarrow \lambda = 1$ or $z = 0$.
CASE 1: $\lambda = 1 \Rightarrow 2x = -y$ and $2y = -x \Rightarrow y = 0$ and $x = 0 \Rightarrow z^2 - 4 = 0 \Rightarrow z = \pm 2$ and $x = y = 0$.
CASE 2: $z = 0 \Rightarrow -xy - 4 = 0 \Rightarrow y = -\frac{4}{x}$. Then $2x = \frac{4}{x}\lambda \Rightarrow \lambda = \frac{x^2}{2}$, and $-\frac{8}{x} = -x\lambda \Rightarrow -\frac{8}{x} = -x\left(\frac{x^2}{2}\right)$ $\Rightarrow x^4 = 16 \Rightarrow x = \pm 2$. Thus, $x = 2$ and $y = -2$, or $x = -2$ and $y = 2$.
Therefore we get four points: $(2, -2, 0), (-2, 2, 0), (0, 0, 2)$ and $(0, 0, -2)$. But the points $(0, 0, 2)$ and $(0, 0, -2)$ are closest to the origin since they are 2 units away and the others are $2\sqrt{2}$ units away.

22. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = \lambda yz, 2y = \lambda xz$, and $2z = \lambda xy \Rightarrow 2x^2 = \lambda xyz$ and $2y^2 = \lambda yxz$ $\Rightarrow x^2 = y^2 \Rightarrow y = \pm x \Rightarrow z = \pm x \Rightarrow x(\pm x)(\pm x) = 1 \Rightarrow x = \pm 1 \Rightarrow$ the points are $(1, 1, 1), (1, -1, -1)$, $(-1, -1, 1)$, and $(-1, 1, -1)$.

23. $\nabla f = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} - 2\mathbf{j} + 5\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 1 = 2x\lambda$, $-2 = 2y\lambda$, and $5 = 2z\lambda \Rightarrow x = \frac{1}{2\lambda}$, $y = -\frac{1}{\lambda} = -2x$, and $z = \frac{5}{2\lambda} = 5x \Rightarrow x^2 + (-2x)^2 + (5x)^2 = 30 \Rightarrow x = \pm 1$. Thus, $x = 1, y = -2, z = 5$ or $x = -1, y = 2, z = -5$. Therefore $f(1, -2, 5) = 30$ is the maximum value and $f(-1, 2, -5) = -30$ is the minimum value.

24. $\nabla f = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 1 = 2x\lambda$, $2 = 2y\lambda$, and $3 = 2z\lambda \Rightarrow x = \frac{1}{2\lambda}$, $y = \frac{1}{\lambda} = 2x$, and $z = \frac{3}{2\lambda} = 3x \Rightarrow x^2 + (2x)^2 + (3x)^2 = 25 \Rightarrow x = \pm \frac{5}{\sqrt{14}}$. Thus, $x = \frac{5}{\sqrt{14}}$, $y = \frac{10}{\sqrt{14}}$, $z = \frac{15}{\sqrt{14}}$ or $x = -\frac{5}{\sqrt{14}}$, $y = -\frac{10}{\sqrt{14}}$, $z = -\frac{15}{\sqrt{14}}$. Therefore $f\left(\frac{5}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{15}{\sqrt{14}}\right) = 5\sqrt{14}$ is the maximum value and $f\left(-\frac{5}{\sqrt{14}}, -\frac{10}{\sqrt{14}}, -\frac{15}{\sqrt{14}}\right) = -5\sqrt{14}$ is the minimum value.

25. $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = x + y + z - 9 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = \mathbf{i} + \mathbf{j} + \mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) \Rightarrow 2x = \lambda$, $2y = \lambda$, and $2z = \lambda \Rightarrow x = y = z \Rightarrow x + x + x - 9 = 0 \Rightarrow x = 3$, $y = 3$, and $z = 3$.

26. $f(x, y, z) = xyz$ and $g(x, y, z) = x + y + z^2 - 16 = 0 \Rightarrow \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $\nabla g = \mathbf{i} + \mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + 2z\mathbf{k}) \Rightarrow yz = \lambda$, $xz = \lambda$, and $xy = 2z\lambda \Rightarrow yz = xz \Rightarrow z = 0$ or $y = x$. But $z > 0$ so that $y = x \Rightarrow x^2 = 2z\lambda$ and $xz = \lambda$. Then $x^2 = 2z(xz) \Rightarrow x = 0$ or $x = 2z^2$. But $x > 0$ so that $x = 2z^2 \Rightarrow y = 2z^2 \Rightarrow 2z^2 + 2z^2 + z^2 = 16 \Rightarrow z = \pm \frac{4}{\sqrt{5}}$. We use $z = \frac{4}{\sqrt{5}}$ since $z > 0$. Then $x = \frac{32}{5}$ and $y = \frac{32}{5}$ which yields $f\left(\frac{32}{5}, \frac{32}{5}, \frac{4}{\sqrt{5}}\right) = \frac{4096}{25\sqrt{5}}$.

27. $V = xyz$ and $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0 \Rightarrow \nabla V = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla V = \lambda \nabla g \Rightarrow yz = \lambda x$, $xz = \lambda y$, and $xy = \lambda z \Rightarrow xyz = \lambda x^2$ and $xyz = \lambda y^2 \Rightarrow y = \pm x \Rightarrow z = \pm x \Rightarrow x^2 + x^2 + x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{3}}$ since $x > 0 \Rightarrow$ the dimensions of the box are $\frac{1}{\sqrt{3}}$ by $\frac{1}{\sqrt{3}}$ by $\frac{1}{\sqrt{3}}$ for maximum volume. (Note that there is no minimum volume since the box could be made arbitrarily thin.)

28. $V = xyz$ with x, y, z all positive and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$; thus $V = xyz$ and $g(x, y, z) = bcx + acy + abz - abc = 0 \Rightarrow \nabla V = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $\nabla g = bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}$ so that $\nabla V = \lambda \nabla g \Rightarrow yz = \lambda bc$, $xz = \lambda ac$, and $xy = \lambda ab \Rightarrow xyz = \lambda bcx$, $xyz = \lambda acy$, and $xyz = \lambda abz \Rightarrow \lambda \neq 0$. Also, $\lambda bcx = \lambda acy = \lambda abz \Rightarrow bx = ay$, $cy = bz$, and $cx = az \Rightarrow y = \frac{b}{a}x$ and $z = \frac{c}{a}x$. Then $\frac{x}{a} + \frac{y}{b} + \frac{c}{z} = 1 \Rightarrow \frac{x}{a} + \frac{1}{b}\left(\frac{b}{a}x\right) + \frac{1}{c}\left(\frac{c}{a}x\right) = 1 \Rightarrow \frac{3x}{a} = 1 \Rightarrow x = \frac{a}{3} \Rightarrow y = \left(\frac{b}{a}\right)\left(\frac{a}{3}\right) = \frac{b}{3}$ and $z = \left(\frac{c}{a}\right)\left(\frac{a}{3}\right) = \frac{c}{3} \Rightarrow V = xyz = \left(\frac{a}{3}\right)\left(\frac{b}{3}\right)\left(\frac{c}{3}\right) = \frac{abc}{27}$ is the maximum volume. (Note that there is no minimum volume since the box could be made arbitrarily thin.)

29. $\nabla T = 16x\mathbf{i} + 4z\mathbf{j} + (4y - 16)\mathbf{k}$ and $\nabla g = 8x\mathbf{i} + 2y\mathbf{j} + 8z\mathbf{k}$ so that $\nabla T = \lambda \nabla g \Rightarrow 16x\mathbf{i} + 4z\mathbf{j} + (4y - 16)\mathbf{k} = \lambda(8x\mathbf{i} + 2y\mathbf{j} + 8z\mathbf{k}) \Rightarrow 16x = 8x\lambda$, $4z = 2y\lambda$, and $4y - 16 = 8z\lambda \Rightarrow \lambda = 2$ or $x = 0$.

CASE 1: $\lambda = 2 \Rightarrow 4z = 2y(2) \Rightarrow z = y$. Then $4z - 16 = 16z \Rightarrow z = -\frac{4}{3} \Rightarrow y = -\frac{4}{3}$. Then

$$4x^2 + \left(-\frac{4}{3}\right)^2 + 4\left(-\frac{4}{3}\right)^2 = 16 \Rightarrow x = \pm \frac{4}{3}.$$

CASE 2: $x = 0 \Rightarrow \lambda = \frac{2z}{y} \Rightarrow 4y - 16 = 8z\left(\frac{2z}{y}\right) \Rightarrow y^2 - 4y = 4z^2 \Rightarrow 4(0)^2 + y^2 + (y^2 - 4y) - 16 = 0$

$\Rightarrow y^2 - 2y - 8 = 0 \Rightarrow (y - 4)(y + 2) = 0 \Rightarrow y = 4$ or $y = -2$. Now $y = 4 \Rightarrow 4z^2 = 4^2 - 4(4)$

$\Rightarrow z = 0$ and $y = -2 \Rightarrow 4z^2 = (-2)^2 - 4(-2) \Rightarrow z = \pm\sqrt{3}$.

The temperatures are $T\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right) = 642\frac{2}{3}^\circ$, $T(0, 4, 0) = 600^\circ$, $T\left(0, -2, \sqrt{3}\right) = \left(600 - 24\sqrt{3}\right)^\circ$, and $T\left(0, -2, -\sqrt{3}\right) = \left(600 + 24\sqrt{3}\right)^\circ \approx 641.6^\circ$. Therefore $\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$ are the hottest points on the space probe.

30. $\nabla T = 400yz^2\mathbf{i} + 400xz^2\mathbf{j} + 800xyz\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla T = \lambda \nabla g$ $\Rightarrow 400yz^2\mathbf{i} + 400xz^2\mathbf{j} + 800xyz\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 400yz^2 = 2x\lambda$, $400xz^2 = 2y\lambda$, and $800xyz = 2z\lambda$. Solving this system yields the points $(0, \pm 1, 0)$, $(\pm 1, 0, 0)$, and $\left(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{\sqrt{2}}{2}\right)$. The corresponding temperatures are $T(0, \pm 1, 0) = 0$, $T(\pm 1, 0, 0) = 0$, and $T\left(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{\sqrt{2}}{2}\right) = \pm 50$. Therefore 50 is the maximum temperature at $\left(\frac{1}{2}, \frac{1}{2}, \pm \frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{2}, \pm \frac{\sqrt{2}}{2}\right)$; -50 is the minimum temperature at $\left(\frac{1}{2}, -\frac{1}{2}, \pm \frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}, \pm \frac{\sqrt{2}}{2}\right)$.

31. $\nabla U = (y+2)\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2\mathbf{i} + \mathbf{j}$ so that $\nabla U = \lambda \nabla g \Rightarrow (y+2)\mathbf{i} + x\mathbf{j} = \lambda(2\mathbf{i} + \mathbf{j}) \Rightarrow y + 2 = 2\lambda$ and $x = \lambda \Rightarrow y + 2 = 2x \Rightarrow y = 2x - 2 \Rightarrow 2x + (2x - 2) = 30 \Rightarrow x = 8$ and $y = 14$. Therefore $U(8, 14) = \$128$ is the maximum value of U under the constraint.

32. $\nabla M = (6+z)\mathbf{i} - 2y\mathbf{j} + x\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla M = \lambda \nabla g \Rightarrow (6+z)\mathbf{i} - 2y\mathbf{j} + x\mathbf{k}$ $= \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 6 + z = 2x\lambda$, $-2y = 2y\lambda$, $x = 2z\lambda \Rightarrow \lambda = -1$ or $y = 0$.

CASE 1: $\lambda = -1 \Rightarrow 6 + z = -2x$ and $x = -2z \Rightarrow 6 + z = -2(-2z) \Rightarrow z = 2$ and $x = -4$. Then $(-4)^2 + y^2 + 2^2 - 36 = 0 \Rightarrow y = \pm 4$.

CASE 2: $y = 0$, $6 + z = 2x\lambda$, and $x = 2z\lambda \Rightarrow \lambda = \frac{x}{2z} \Rightarrow 6 + z = 2x\left(\frac{x}{2z}\right) \Rightarrow 6z + z^2 = x^2$ $\Rightarrow (6z + z^2) + 0^2 + z^2 = 36 \Rightarrow z = -6$ or $z = 3$. Now $z = -6 \Rightarrow x^2 = 0 \Rightarrow x = 0$; $z = 3$ $\Rightarrow x^2 = 27 \Rightarrow x = \pm 3\sqrt{3}$.

Therefore we have the points $\left(\pm 3\sqrt{3}, 0, 3\right)$, $(0, 0, -6)$, and $(-4, \pm 4, 2)$. Then $M\left(3\sqrt{3}, 0, 3\right) = 27\sqrt{3} + 60$ ≈ 106.8, $M\left(-3\sqrt{3}, 0, 3\right) = 60 - 27\sqrt{3} \approx 13.2$, $M(0, 0, -6) = 60$, and $M(-4, 4, 2) = 12 = M(-4, -4, 2)$. Therefore, the weakest field is at $(-4, \pm 4, 2)$.

33. Let $g_1(x, y, z) = 2x - y = 0$ and $g_2(x, y, z) = y + z = 0 \Rightarrow \nabla g_1 = 2\mathbf{i} - \mathbf{j}$, $\nabla g_2 = \mathbf{j} + \mathbf{k}$, and $\nabla f = 2x\mathbf{i} + 2\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x\mathbf{i} + 2\mathbf{j} - 2z\mathbf{k} = \lambda(2\mathbf{i} - \mathbf{j}) + \mu(\mathbf{j} + \mathbf{k}) \Rightarrow 2x\mathbf{i} + 2\mathbf{j} - 2z\mathbf{k} = 2\lambda\mathbf{i} + (\mu - \lambda)\mathbf{j} + \mu\mathbf{k}$ $\Rightarrow 2x = 2\lambda$, $2 = \mu - \lambda$, and $-2z = \mu \Rightarrow x = \lambda$. Then $2 = -2z - x \Rightarrow x = -2z - 2$ so that $2x - y = 0$ $\Rightarrow 2(-2z - 2) - y = 0 \Rightarrow -4z - 4 - y = 0$. This equation coupled with $y + z = 0$ implies $z = -\frac{4}{3}$ and $y = \frac{4}{3}$. Then $x = \frac{2}{3}$ so that $\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right)$ is the point that gives the maximum value $f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \left(\frac{2}{3}\right)^2 + 2\left(\frac{4}{3}\right) - \left(-\frac{4}{3}\right)^2 = \frac{4}{3}$.

34. Let $g_1(x, y, z) = x + 2y + 3z - 6 = 0$ and $g_2(x, y, z) = x + 3y + 9z - 9 = 0 \Rightarrow \nabla g_1 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\nabla g_2 = \mathbf{i} + 3\mathbf{j} + 9\mathbf{k}$, and $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ $= \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \mu(\mathbf{i} + 3\mathbf{j} + 9\mathbf{k}) \Rightarrow 2x = \lambda + \mu$, $2y = 2\lambda + 3\mu$, and $2z = 3\lambda + 9\mu$. Then $0 = x + 2y + 3z - 6$ $= \frac{1}{2}(\lambda + \mu) + (2\lambda + 3\mu) + \left(\frac{9}{2}\lambda + \frac{27}{2}\mu\right) - 6 \Rightarrow 7\lambda + 17\mu = 6$; $0 = x + 3y + 9z - 9$ $\Rightarrow \frac{1}{2}(\lambda + \mu) + \left(3\lambda + \frac{9}{2}\mu\right) + \left(\frac{27}{2}\lambda + \frac{81}{2}\mu\right) - 9 \Rightarrow 34\lambda + 91\mu = 18$. Solving these two equations for λ and μ gives $\lambda = \frac{240}{59}$ and $\mu = -\frac{78}{59} \Rightarrow x = \frac{\lambda + \mu}{2} = \frac{81}{59}$, $y = \frac{2\lambda + 3\mu}{2} = \frac{123}{59}$, and $z = \frac{3\lambda + 9\mu}{2} = \frac{9}{59}$. The minimum value is $f\left(\frac{81}{59}, \frac{123}{59}, \frac{9}{59}\right) = \frac{21{,}771}{59^2} = \frac{369}{59}$. (Note that there is no <u>maximum</u> value of f subject to the constraints because at least one of the variables x, y, or z can be made arbitrary and assume a value as large as we please.)

35. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance from the origin. We want to minimize $f(x, y, z)$ subject to the constraints $g_1(x, y, z) = y + 2z - 12 = 0$ and $g_2(x, y, z) = x + y - 6 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\nabla g_1 = \mathbf{j} + 2\mathbf{k}$, and $\nabla g_2 = \mathbf{i} + \mathbf{j}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x = \mu$, $2y = \lambda + \mu$, and $2z = 2\lambda$. Then $0 = y + 2z - 12$ $= \left(\frac{\lambda}{2} + \frac{\mu}{2}\right) + 2\lambda - 12 \Rightarrow \frac{5}{2}\lambda + \frac{1}{2}\mu = 12 \Rightarrow 5\lambda + \mu = 24$; $0 = x + y - 6 = \frac{\mu}{2} + \left(\frac{\lambda}{2} + \frac{\mu}{2}\right) - 6 \Rightarrow \frac{1}{2}\lambda + \mu = 6$ $\Rightarrow \lambda + 2\mu = 12$. Solving these two equations for λ and μ gives $\lambda = 4$ and $\mu = 4 \Rightarrow x = \frac{\mu}{2} = 2$, $y = \frac{\lambda + \mu}{2} = 4$, and $z = \lambda = 4$. The point $(2, 4, 4)$ on the line of intersection is closest to the origin. (There is no maximum distance from the origin since points on the line can be arbitrarily far away.)

36. The maximum value is $f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \frac{4}{3}$ from Exercise 33 above.

37. Let $g_1(x, y, z) = z - 1 = 0$ and $g_2(x, y, z) = x^2 + y^2 + z^2 - 10 = 0 \Rightarrow \nabla g_1 = \mathbf{k}$, $\nabla g_2 = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, and $\nabla f = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k} = \lambda(\mathbf{k}) + \mu(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$ $\Rightarrow 2xyz = 2x\mu$, $x^2z = 2y\mu$, and $x^2y = 2z\mu + \lambda \Rightarrow xyz = x\mu \Rightarrow x = 0$ or $yz = \mu \Rightarrow \mu = y$ since $z = 1$.
CASE 1: $x = 0$ and $z = 1 \Rightarrow y^2 - 9 = 0$ (from g_2) $\Rightarrow y = \pm 3$ yielding the points $(0, \pm 3, 1)$.
CASE 2: $\mu = y \Rightarrow x^2z = 2y^2 \Rightarrow x^2 = 2y^2$ (since $z = 1$) $\Rightarrow 2y^2 + y^2 + 1 - 10 = 0$ (from g_2) $\Rightarrow 3y^2 - 9 = 0$ $\Rightarrow y = \pm\sqrt{3} \Rightarrow x^2 = 2\left(\pm\sqrt{3}\right)^2 \Rightarrow x = \pm\sqrt{6}$ yielding the points $\left(\pm\sqrt{6}, \pm\sqrt{3}, 1\right)$.
Now $f(0, \pm 3, 1) = 1$ and $f\left(\pm\sqrt{6}, \pm\sqrt{3}, 1\right) = 6\left(\pm\sqrt{3}\right) + 1 = 1 \pm 6\sqrt{3}$. Therefore the maximum of f is $1 + 6\sqrt{3}$ at $\left(\pm\sqrt{6}, \sqrt{3}, 1\right)$, and the minimum of f is $1 - 6\sqrt{3}$ at $\left(\pm\sqrt{6}, -\sqrt{3}, 1\right)$.

38. (a) Let $g_1(x, y, z) = x + y + z - 40 = 0$ and $g_2(x, y, z) = x + y - z = 0 \Rightarrow \nabla g_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\nabla g_2 = \mathbf{i} + \mathbf{j} - \mathbf{k}$, and $\nabla w = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ so that $\nabla w = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$ $\Rightarrow yz = \lambda + \mu$, $xz = \lambda + \mu$, and $xy = \lambda - \mu \Rightarrow yz = xz \Rightarrow z = 0$ or $y = x$.
CASE 1: $z = 0 \Rightarrow x + y = 40$ and $x + y = 0 \Rightarrow$ no solution.
CASE 2: $x = y \Rightarrow 2x + z - 40 = 0$ and $2x - z = 0 \Rightarrow z = 20 \Rightarrow x = 10$ and $y = 10 \Rightarrow w = (10)(10)(20)$ $= 2000$

(b) $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j}$ is parallel to the line of intersection $\Rightarrow$ the line is $x = -2t + 10$, $y = 2t + 10$, $z = 20$. Since $z = 20$, we see that $w = xyz = (-2t + 10)(2t + 10)(20) = (-4t^2 + 100)(20)$ which has its maximum when $t = 0 \Rightarrow x = 10$, $y = 10$, and $z = 20$.

39. Let $g_1(x, y, z) = y - x = 0$ and $g_2(x, y, z) = x^2 + y^2 + z^2 - 4 = 0$. Then $\nabla f = y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$, $\nabla g_1 = -\mathbf{i} + \mathbf{j}$, and $\nabla g_2 = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k} = \lambda(-\mathbf{i} + \mathbf{j}) + \mu(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$ $\Rightarrow y = -\lambda + 2x\mu$, $x = \lambda + 2y\mu$, and $2z = 2z\mu \Rightarrow z = 0$ or $\mu = 1$.
CASE 1: $z = 0 \Rightarrow x^2 + y^2 - 4 = 0 \Rightarrow 2x^2 - 4 = 0$ (since $x = y$) $\Rightarrow x = \pm\sqrt{2}$ and $y = \pm\sqrt{2}$ yielding the points $\left(\pm\sqrt{2}, \pm\sqrt{2}, 0\right)$.
CASE 2: $\mu = 1 \Rightarrow y = -\lambda + 2x$ and $x = \lambda + 2y \Rightarrow x + y = 2(x + y) \Rightarrow 2x = 2(2x)$ since $x = y \Rightarrow x = 0 \Rightarrow y = 0$ $\Rightarrow z^2 - 4 = 0 \Rightarrow z = \pm 2$ yielding the points $(0, 0, \pm 2)$.
Now, $f(0, 0, \pm 2) = 4$ and $f\left(\pm\sqrt{2}, \pm\sqrt{2}, 0\right) = 2$. Therefore the maximum value of f is 4 at $(0, 0, \pm 2)$ and the minimum value of f is 2 at $\left(\pm\sqrt{2}, \pm\sqrt{2}, 0\right)$.

40. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance from the origin. We want to minimize $f(x, y, z)$ subject to the constraints $g_1(x, y, z) = 2y + 4z - 5 = 0$ and $g_2(x, y, z) = 4x^2 + 4y^2 - z^2 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\nabla g_1 = 2\mathbf{j} + 4\mathbf{k}$, and $\nabla g_2 = 8x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ $= \lambda(2\mathbf{j} + 4\mathbf{k}) + \mu(8x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k}) \Rightarrow 2x = 8x\mu$, $2y = 2\lambda + 8y\mu$, and $2z = 4\lambda - 2z\mu \Rightarrow x = 0$ or $\mu = \frac{1}{4}$.
CASE 1: $x = 0 \Rightarrow 4(0)^2 + 4y^2 - z^2 = 0 \Rightarrow z = \pm 2y \Rightarrow 2y + 4(2y) - 5 = 0 \Rightarrow y = \frac{1}{2}$, or $2y + 4(-2y) - 5 = 0$ $\Rightarrow y = -\frac{5}{6}$ yielding the points $\left(0, \frac{1}{2}, 1\right)$ and $\left(0, -\frac{5}{6}, \frac{5}{3}\right)$.
CASE 2: $\mu = \frac{1}{4} \Rightarrow y = \lambda + y \Rightarrow \lambda = 0 \Rightarrow 2z = 4(0) - 2z\left(\frac{1}{4}\right) \Rightarrow z = 0 \Rightarrow 2y + 4(0) = 5 \Rightarrow y = \frac{5}{2}$ and $(0)^2 = 4x^2 + 4\left(\frac{5}{2}\right)^2 \Rightarrow$ no solution.
Then $f\left(0, \frac{1}{2}, 1\right) = \frac{5}{4}$ and $f\left(0, -\frac{5}{6}, \frac{5}{3}\right) = 25\left(\frac{1}{36} + \frac{1}{9}\right) = \frac{125}{36} \Rightarrow$ the point $\left(0, \frac{1}{2}, 1\right)$ is closest to the origin.

41. $\nabla f = \mathbf{i} + \mathbf{j}$ and $\nabla g = y\mathbf{i} + x\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} + \mathbf{j} = \lambda(y\mathbf{i} + x\mathbf{j}) \Rightarrow 1 = y\lambda$ and $1 = x\lambda \Rightarrow y = x$ $\Rightarrow y^2 = 16 \Rightarrow y = \pm 4 \Rightarrow (4,4)$ and $(-4,-4)$ are candidates for the location of extreme values. But as $x \to \infty$, $y \to \infty$ and $f(x,y) \to \infty$; as $x \to -\infty$, $y \to 0$ and $f(x,y) \to -\infty$. Therefore no maximum or minimum value exists subject to the constraint.

42. Let $f(A,B,C) = \sum_{k=1}^{4} (Ax_k + By_k + C - z_k)^2 = C^2 + (B + C - 1)^2 + (A + B + C - 1)^2 + (A + C + 1)^2$. We want to minimize f. Then $f_A(A,B,C) = 4A + 2B + 4C$, $f_B(A,B,C) = 2A + 4B + 4C - 4$, and $f_C(A,B,C) = 4A + 4B + 8C - 2$. Set each partial derivative equal to 0 and solve the system to get $A = -\frac{1}{2}$, $B = \frac{3}{2}$, and $C = -\frac{1}{4}$ or the critical point of f is $\left(-\frac{1}{2}, \frac{3}{2}, -\frac{1}{4}\right)$.

43. (a) Maximize $f(a,b,c) = a^2b^2c^2$ subject to $a^2 + b^2 + c^2 = r^2$. Thus $\nabla f = 2ab^2c^2\mathbf{i} + 2a^2bc^2\mathbf{j} + 2a^2b^2c\mathbf{k}$ and $\nabla g = 2a\mathbf{i} + 2b\mathbf{j} + 2c\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2ab^2c^2 = 2a\lambda$, $2a^2bc^2 = 2b\lambda$, and $2a^2b^2c = 2c\lambda$ $\Rightarrow 2a^2b^2c^2 = 2a^2\lambda = 2b^2\lambda = 2c^2\lambda \Rightarrow \lambda = 0$ or $a^2 = b^2 = c^2$.

CASE 1: $\lambda = 0 \Rightarrow a^2b^2c^2 = 0$.

CASE 2: $a^2 = b^2 = c^2 \Rightarrow f(a,b,c) = a^2a^2a^2$ and $3a^2 = r^2 \Rightarrow f(a,b,c) = \left(\frac{r^2}{3}\right)^3$ is the maximum value.

(b) The point $\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)$ is on the sphere if $a + b + c = r^2$. Moreover, by part (a), $abc = f\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)$ $\le \left(\frac{r^2}{3}\right)^3 \Rightarrow (abc)^{1/3} \le \frac{r^2}{3} = \frac{a+b+c}{3}$, as claimed.

44. Let $f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_ix_i = a_1x_1 + a_2x_2 + \ldots + a_nx_n$ and $g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_n^2 - 1$. Then we want $\nabla f = \lambda \nabla g \Rightarrow a_1 = \lambda(2x_1)$, $a_2 = \lambda(2x_2)$, $\ldots$, $a_n = \lambda(2x_n)$, $\lambda \ne 0 \Rightarrow x_i = \frac{a_i}{2\lambda} \Rightarrow \frac{a_1^2}{4\lambda^2} + \frac{a_2^2}{4\lambda^2} + \ldots + \frac{a_n^2}{4\lambda^2} = 1$ $\Rightarrow 4\lambda^2 = \sum_{i=1}^{n} a_i^2 \Rightarrow 2\lambda = \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \Rightarrow f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_ix_i = \sum_{i=1}^{n} a_i\left(\frac{a_i}{2\lambda}\right) = \frac{1}{2\lambda}\sum_{i=1}^{n} a_i^2 = \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$ is the maximum value.

45-50. Example CAS commands:

Maple:

```
f := (x,y,z) -> x*y+y*z;
g1 := (x,y,z) -> x^2+y^2-2;
g2 := (x,y,z) -> x^2+z^2-2;
h := unapply( f(x,y,z)-lambda[1]*g1(x,y,z)-lambda[2]*g2(x,y,z), (x,y,z,lambda[1],lambda[2]) );    # (a)
hx := diff( h(x,y,z,lambda[1],lambda[2]), x );                                                     #(b)
hy := diff( h(x,y,z,lambda[1],lambda[2]), y );
hz := diff( h(x,y,z,lambda[1],lambda[2]), z );
hl1 := diff( h(x,y,z,lambda[1],lambda[2]), lambda[1] );
hl2 := diff( h(x,y,z,lambda[1],lambda[2]), lambda[2] );
sys := { hx=0, hy=0, hz=0, hl1=0, hl2=0 };
q1 := solve( sys, {x,y,z,lambda[1],lambda[2]} );                                                  # (c)
q2 := map(allvalues,{q1});
for p in q2 do                                                                                     # (d)
  eval( [x,y,z,f(x,y,z)], p );
  ``=evalf(eval( [x,y,z,f(x,y,z)], p ));
end do;
```

Mathematica: (assigned functions will vary)

```
Clear[x, y, z, lambda1, lambda2]
f[x_,y_,z_]:= x y + y z
g1[x_,y_,z_]:= x^2 + y^2 - 2
g2[x_,y_,z_]:= x^2 + z^2 - 2
h = f[x, y, z] - lambda1 g1[x, y, z] - lambda2 g2[x, y, z];
hx= D[h, x];  hy= D[h, y];  hz= D[h,z]; hL1=D[h, lambda1];  hL2= D[h, lambda2];
critical=Solve[{hx==0, hy==0, hz==0, hL1==0, hL2==0, g1[x,y,z]==0, g2[x,y,z]==0},
{x, y, z, lambda1, lambda2}]//N
{{x, y, z}, f[x, y, z]}/.critical
```

14.9 TAYLOR'S FORMULA FOR TWO VARIABLES

1. $f(x,y) = xe^y \Rightarrow f_x = e^y, f_y = xe^y, f_{xx} = 0, f_{xy} = e^y, f_{yy} = xe^y$
$\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2f_{yy}(0,0)\right]$
$= 0 + x \cdot 1 + y \cdot 0 + \frac{1}{2}(x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0) = x + xy$ quadratic approximation;
$f_{xxx} = 0, f_{xxy} = 0, f_{xyy} = e^y, f_{yyy} = xe^y$
$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3f_{xxx}(0,0) + 3x^2yf_{xxy}(0,0) + 3xy^2f_{xyy}(0,0) + y^3f_{yyy}(0,0)\right]$
$= x + xy + \frac{1}{6}(x^3 \cdot 0 + 3x^2y \cdot 0 + 3xy^2 \cdot 1 + y^3 \cdot 0) = x + xy + \frac{1}{2}xy^2$, cubic approximation

2. $f(x,y) = e^x \cos y \Rightarrow f_x = e^x \cos y, f_y = -e^x \sin y, f_{xx} = e^x \cos y, f_{xy} = -e^x \sin y, f_{yy} = -e^x \cos y$
$\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2f_{yy}(0,0)\right]$
$= 1 + x \cdot 1 + y \cdot 0 + \frac{1}{2}\left[x^2 \cdot 1 + 2xy \cdot 0 + y^2 \cdot (-1)\right] = 1 + x + \frac{1}{2}(x^2 - y^2)$, quadratic approximation;
$f_{xxx} = e^x \cos y, f_{xxy} = -e^x \sin y, f_{xyy} = -e^x \cos y, f_{yyy} = e^x \sin y$
$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3f_{xxx}(0,0) + 3x^2yf_{xxy}(0,0) + 3xy^2f_{xyy}(0,0) + y^3f_{yyy}(0,0)\right]$
$= 1 + x + \frac{1}{2}(x^2 - y^2) + \frac{1}{6}\left[x^3 \cdot 1 + 3x^2y \cdot 0 + 3xy^2 \cdot (-1) + y^3 \cdot 0\right]$
$= 1 + x + \frac{1}{2}(x^2 - y^2) + \frac{1}{6}(x^3 - 3xy^2)$, cubic approximation

3. $f(x,y) = y \sin x \Rightarrow f_x = y \cos x, f_y = \sin x, f_{xx} = -y \sin x, f_{xy} = \cos x, f_{yy} = 0$
$\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2f_{yy}(0,0)\right]$
$= 0 + x \cdot 0 + y \cdot 0 + \frac{1}{2}(x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0) = xy$, quadratic approximation;
$f_{xxx} = -y \cos x, f_{xxy} = -\sin x, f_{xyy} = 0, f_{yyy} = 0$
$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3f_{xxx}(0,0) + 3x^2yf_{xxy}(0,0) + 3xy^2f_{xyy}(0,0) + y^3f_{yyy}(0,0)\right]$
$= xy + \frac{1}{6}(x^3 \cdot 0 + 3x^2y \cdot 0 + 3xy^2 \cdot 0 + y^3 \cdot 0) = xy$, cubic approximation

4. $f(x,y) = \sin x \cos y \Rightarrow f_x = \cos x \cos y, f_y = -\sin x \sin y, f_{xx} = -\sin x \cos y, f_{xy} = -\cos x \sin y,$
$f_{yy} = -\sin x \cos y \Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2f_{yy}(0,0)\right]$
$= 0 + x \cdot 1 + y \cdot 0 + \frac{1}{2}(x^2 \cdot 0 + 2xy \cdot 0 + y^2 \cdot 0) = x$, quadratic approximation;
$f_{xxx} = -\cos x \cos y, f_{xxy} = \sin x \sin y, f_{xyy} = -\cos x \cos y, f_{yyy} = \sin x \sin y$
$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3f_{xxx}(0,0) + 3x^2yf_{xxy}(0,0) + 3xy^2f_{xyy}(0,0) + y^3f_{yyy}(0,0)\right]$
$= x + \frac{1}{6}\left[x^3 \cdot (-1) + 3x^2y \cdot 0 + 3xy^2 \cdot (-1) + y^3 \cdot 0\right] = x - \frac{1}{6}(x^3 + 3xy^2)$, cubic approximation

5. $f(x,y) = e^x \ln(1+y) \Rightarrow f_x = e^x \ln(1+y), f_y = \frac{e^x}{1+y}, f_{xx} = e^x \ln(1+y), f_{xy} = \frac{e^x}{1+y}, f_{yy} = -\frac{e^x}{(1+y)^2}$
$\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2f_{yy}(0,0)\right]$
$= 0 + x \cdot 0 + y \cdot 1 + \frac{1}{2}\left[x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot (-1)\right] = y + \frac{1}{2}(2xy - y^2)$, quadratic approximation;
$f_{xxx} = e^x \ln(1+y), f_{xxy} = \frac{e^x}{1+y}, f_{xyy} = -\frac{e^x}{(1+y)^2}, f_{yyy} = \frac{2e^x}{(1+y)^3}$

$\Rightarrow\ f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= y + \frac{1}{2}\left(2xy - y^2\right) + \frac{1}{6}\left[x^3 \cdot 0 + 3x^2 y \cdot 1 + 3xy^2 \cdot (-1) + y^3 \cdot 2\right]$
$= y + \frac{1}{2}\left(2xy - y^2\right) + \frac{1}{6}\left(3x^2 y - 3xy^2 + 2y^3\right)$, cubic approximation

6. $f(x,y) = \ln(2x + y + 1) \Rightarrow f_x = \frac{2}{2x+y+1}$, $f_y = \frac{1}{2x+y+1}$, $f_{xx} = \frac{-4}{(2x+y+1)^2}$, $f_{xy} = \frac{-2}{(2x+y+1)^2}$,
$f_{yy} = \frac{-1}{(2x+y+1)^2} \Rightarrow f(x,y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 0 + x \cdot 2 + y \cdot 1 + \frac{1}{2}\left[x^2 \cdot (-4) + 2xy \cdot (-2) + y^2 \cdot (-1)\right] = 2x + y + \frac{1}{2}\left(-4x^2 - 4xy - y^2\right)$
$= (2x + y) - \frac{1}{2}(2x + y)^2$, quadratic approximation;
$f_{xxx} = \frac{16}{(2x+y+1)^3}$, $f_{xxy} = \frac{8}{(2x+y+1)^3}$, $f_{xyy} = \frac{4}{(2x+y+1)^3}$, $f_{yyy} = \frac{2}{(2x+y+1)^3}$
$\Rightarrow\ f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= (2x + y) - \frac{1}{2}(2x + y)^2 + \frac{1}{6}\left(x^3 \cdot 16 + 3x^2 y \cdot 8 + 3xy^2 \cdot 4 + y^3 \cdot 2\right)$
$= (2x + y) - \frac{1}{2}(2x + y)^2 + \frac{1}{3}\left(8x^3 + 12x^2 y + 6xy^2 + y^2\right)$
$= (2x + y) - \frac{1}{2}(2x + y)^2 + \frac{1}{3}(2x + y)^3$, cubic approximation

7. $f(x,y) = \sin(x^2 + y^2) \Rightarrow f_x = 2x\cos(x^2 + y^2)$, $f_y = 2y\cos(x^2 + y^2)$, $f_{xx} = 2\cos(x^2 + y^2) - 4x^2\sin(x^2 + y^2)$,
$f_{xy} = -4xy\sin(x^2 + y^2)$, $f_{yy} = 2\cos(x^2 + y^2) - 4y^2\sin(x^2 + y^2)$
$\Rightarrow\ f(x,y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 0 + x \cdot 0 + y \cdot 0 + \frac{1}{2}\left(x^2 \cdot 2 + 2xy \cdot 0 + y^2 \cdot 2\right) = x^2 + y^2$, quadratic approximation;
$f_{xxx} = -12x\sin(x^2 + y^2) - 8x^3\cos(x^2 + y^2)$, $f_{xxy} = -4y\sin(x^2 + y^2) - 8x^2 y\cos(x^2 + y^2)$,
$f_{xyy} = -4x\sin(x^2 + y^2) - 8xy^2\cos(x^2 + y^2)$, $f_{yyy} = -12y\sin(x^2 + y^2) - 8y^3\cos(x^2 + y^2)$
$\Rightarrow\ f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= x^2 + y^2 + \frac{1}{6}\left(x^3 \cdot 0 + 3x^2 y \cdot 0 + 3xy^2 \cdot 0 + y^3 \cdot 0\right) = x^2 + y^2$, cubic approximation

8. $f(x,y) = \cos(x^2 + y^2) \Rightarrow f_x = -2x\sin(x^2 + y^2)$, $f_y = -2y\sin(x^2 + y^2)$,
$f_{xx} = -2\sin(x^2 + y^2) - 4x^2\cos(x^2 + y^2)$, $f_{xy} = -4xy\cos(x^2 + y^2)$, $f_{yy} = -2\sin(x^2 + y^2) - 4y^2\cos(x^2 + y^2)$
$\Rightarrow\ f(x,y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 1 + x \cdot 0 + y \cdot 0 + \frac{1}{2}\left[x^2 \cdot 0 + 2xy \cdot 0 + y^2 \cdot 0\right] = 1$, quadratic approximation;
$f_{xxx} = -12x\cos(x^2 + y^2) + 8x^3\sin(x^2 + y^2)$, $f_{xxy} = -4y\cos(x^2 + y^2) + 8x^2 y\sin(x^2 + y^2)$,
$f_{xyy} = -4x\cos(x^2 + y^2) + 8xy^2\sin(x^2 + y^2)$, $f_{yyy} = -12y\cos(x^2 + y^2) + 8y^3\sin(x^2 + y^2)$
$\Rightarrow\ f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= 1 + \frac{1}{6}\left(x^3 \cdot 0 + 3x^2 y \cdot 0 + 3xy^2 \cdot 0 + y^3 \cdot 0\right) = 1$, cubic approximation

9. $f(x,y) = \frac{1}{1-x-y} \Rightarrow f_x = \frac{1}{(1-x-y)^2} = f_y$, $f_{xx} = \frac{2}{(1-x-y)^3} = f_{xy} = f_{yy}$
$\Rightarrow\ f(x,y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 1 + x \cdot 1 + y \cdot 1 + \frac{1}{2}\left(x^2 \cdot 2 + 2xy \cdot 2 + y^2 \cdot 2\right) = 1 + (x + y) + \left(x^2 + 2xy + y^2\right)$
$= 1 + (x + y) + (x + y)^2$, quadratic approximation; $f_{xxx} = \frac{6}{(1-x-y)^4} = f_{xxy} = f_{xyy} = f_{yyy}$
$\Rightarrow\ f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= 1 + (x + y) + (x + y)^2 + \frac{1}{6}\left(x^3 \cdot 6 + 3x^2 y \cdot 6 + 3xy^2 \cdot 6 + y^3 \cdot 6\right)$
$= 1 + (x + y) + (x + y)^2 + \left(x^3 + 3x^2 y + 3xy^2 + y^3\right) = 1 + (x + y) + (x + y)^2 + (x + y)^3$, cubic approximation

10. $f(x,y) = \frac{1}{1-x-y+xy} \Rightarrow f_x = \frac{1-y}{(1-x-y+xy)^2}$, $f_y = \frac{1-x}{(1-x-y+xy)^2}$, $f_{xx} = \frac{2(1-y)^2}{(1-x-y+xy)^3}$,
$f_{xy} = \frac{1}{(1-x-y+xy)^2}$, $f_{yy} = \frac{2(1-x)^2}{(1-x-y+xy)^3}$
$\Rightarrow\ f(x,y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 1 + x \cdot 1 + y \cdot 1 + \frac{1}{2}\left(x^2 \cdot 2 + 2xy \cdot 1 + y^2 \cdot 2\right) = 1 + x + y + x^2 + xy + y^2$, quadratic approximation;

$f_{xxx} = \frac{6(1-y)^3}{(1-x-y+xy)^4}$, $f_{xxy} = \frac{[-4(1-x-y+xy)+6(1-y)(1-x)](1-y)}{(1-x-y+xy)^4}$,
$f_{xyy} = \frac{[-4(1-x-y+xy)+6(1-x)(1-y)](1-x)}{(1-x-y+xy)^4}$, $f_{yyy} = \frac{6(1-x)^3}{(1-x-y+xy)^4}$
$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= 1 + x + y + x^2 + xy + y^2 + \frac{1}{6}\left(x^3 \cdot 6 + 3x^2y \cdot 2 + 3xy^2 \cdot 2 + y^3 \cdot 6\right)$
$= 1 + x + y + x^2 + xy + y^2 + x^3 + x^2y + xy^2 + y^3$, cubic approximation

11. $f(x,y) = \cos x \cos y \Rightarrow f_x = -\sin x \cos y$, $f_y = -\cos x \sin y$, $f_{xx} = -\cos x \cos y$, $f_{xy} = \sin x \sin y$, $f_{yy} = -\cos x \cos y \Rightarrow f(x,y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 1 + x \cdot 0 + y \cdot 0 + \frac{1}{2}\left[x^2 \cdot (-1) + 2xy \cdot 0 + y^2 \cdot (-1)\right] = 1 - \frac{x^2}{2} - \frac{y^2}{2}$, quadratic approximation. Since all partial derivatives of f are products of sines and cosines, the absolute value of these derivatives is less than or equal to 1 $\Rightarrow E(x,y) \le \frac{1}{6}\left[(0.1)^3 + 3(0.1)^3 + 3(0.1)^3 + 0.1)^3\right] \le 0.00134$.

12. $f(x,y) = e^x \sin y \Rightarrow f_x = e^x \sin y$, $f_y = e^x \cos y$, $f_{xx} = e^x \sin y$, $f_{xy} = e^x \cos y$, $f_{yy} = -e^x \sin y$
$\Rightarrow f(x,y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 0 + x \cdot 0 + y \cdot 1 + \frac{1}{2}\left(x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0\right) = y + xy$, quadratic approximation. Now, $f_{xxx} = e^x \sin y$, $f_{xxy} = e^x \cos y$, $f_{xyy} = -e^x \sin y$, and $f_{yyy} = -e^x \cos y$. Since $|x| \le 0.1$, $|e^x \sin y| \le |e^{0.1} \sin 0.1| \approx 0.11$ and $|e^x \cos y| \le |e^{0.1} \cos 0.1| \approx 1.11$. Therefore,
$E(x,y) \le \frac{1}{6}\left[(0.11)(0.1)^3 + 3(1.11)(0.1)^3 + 3(0.11)(0.1)^3 + (1.11)(0.1)^3\right] \le 0.000814$.

14.10 PARTIAL DERIVATIVES WITH CONSTRAINED VARIABLES

1. $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$:

(a) $\begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} x = x(y,z) \\ y = y \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_z = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$; $\frac{\partial z}{\partial y} = 0$ and $\frac{\partial z}{\partial y} = 2x\frac{\partial x}{\partial y} + 2y\frac{\partial y}{\partial y}$
$= 2x\frac{\partial x}{\partial y} + 2y \Rightarrow 0 = 2x\frac{\partial x}{\partial y} + 2y \Rightarrow \frac{\partial x}{\partial y} = -\frac{y}{x} \Rightarrow \left(\frac{\partial w}{\partial y}\right)_z = (2x)\left(-\frac{y}{x}\right) + (2y)(1) + (2z)(0) = -2y + 2y = 0$

(b) $\begin{pmatrix} x \\ z \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y(x,z) \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_x = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z}$; $\frac{\partial x}{\partial z} = 0$ and $\frac{\partial z}{\partial z} = 2x\frac{\partial x}{\partial z} + 2y\frac{\partial y}{\partial z}$
$\Rightarrow 1 = 2y\frac{\partial y}{\partial z} \Rightarrow \frac{\partial y}{\partial z} = \frac{1}{2y} \Rightarrow \left(\frac{\partial w}{\partial z}\right)_x = (2x)(0) + (2y)\left(\frac{1}{2y}\right) + (2z)(1) = 1 + 2z$

(c) $\begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} x = x(y,z) \\ y = y \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_y = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z}$; $\frac{\partial y}{\partial z} = 0$ and $\frac{\partial z}{\partial z} = 2x\frac{\partial x}{\partial z} + 2y\frac{\partial y}{\partial z}$
$\Rightarrow 1 = 2x\frac{\partial x}{\partial z} \Rightarrow \frac{\partial x}{\partial z} = \frac{1}{2x} \Rightarrow \left(\frac{\partial w}{\partial z}\right)_y = (2x)\left(\frac{1}{2x}\right) + (2y)(0) + (2z)(1) = 1 + 2z$

2. $w = x^2 + y - z + \sin t$ and $x + y = t$:

(a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y \\ z = z \\ t = x + y \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_{x,z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y}$; $\frac{\partial x}{\partial y} = 0$, $\frac{\partial z}{\partial y} = 0$, and
$\frac{\partial t}{\partial y} = 1 \Rightarrow \left(\frac{\partial w}{\partial y}\right)_{x,t} = (2x)(0) + (1)(1) + (-1)(0) + (\cos t)(1) = 1 + \cos t = 1 + \cos(x+y)$

(b) $\begin{pmatrix} y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} x = t - y \\ y = y \\ z = z \\ t = t \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_{z,t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y}$; $\frac{\partial z}{\partial y} = 0$ and $\frac{\partial t}{\partial y} = 0$
$\Rightarrow \frac{\partial x}{\partial y} = \frac{\partial t}{\partial y} - \frac{\partial y}{\partial y} = -1 \Rightarrow \left(\frac{\partial w}{\partial y}\right)_{z,t} = (2x)(-1) + (1)(1) + (-1)(0) + (\cos t)(0) = 1 - 2(t-y) = 1 + 2y - 2t$

(c) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y \\ z = z \\ t = x + y \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_{x,y} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial z};\ \frac{\partial x}{\partial z} = 0$ and $\frac{\partial y}{\partial z} = 0$

$\Rightarrow \left(\frac{\partial w}{\partial z}\right)_{x,y} = (2x)(0) + (1)(0) + (-1)(1) + (\cos t)(0) = -1$

(d) $\begin{pmatrix} y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} x = t - y \\ y = y \\ z = z \\ t = t \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_{y,t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial z};\ \frac{\partial y}{\partial z} = 0$ and $\frac{\partial t}{\partial z} = 0$

$\Rightarrow \left(\frac{\partial w}{\partial z}\right)_{y,t} = (2x)(0) + (1)(0) + (-1)(1) + (\cos t)(0) = -1$

(e) $\begin{pmatrix} x \\ z \\ t \end{pmatrix} \to \begin{pmatrix} x = x \\ y = t - x \\ z = z \\ t = t \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial t}\right)_{x,z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial t};\ \frac{\partial x}{\partial t} = 0$ and $\frac{\partial z}{\partial t} = 0$

$\Rightarrow \left(\frac{\partial w}{\partial t}\right)_{x,z} = (2x)(0) + (1)(1) + (-1)(0) + (\cos t)(1) = 1 + \cos t$

(f) $\begin{pmatrix} y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} x = t - y \\ y = y \\ z = z \\ t = t \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial t}\right)_{y,z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial t};\ \frac{\partial y}{\partial t} = 0$ and $\frac{\partial z}{\partial t} = 0$

$\Rightarrow \left(\frac{\partial w}{\partial t}\right)_{y,z} = (2x)(1) + (1)(0) + (-1)(0) + (\cos t)(1) = \cos t + 2x = \cos t + 2(t - y)$

3. $U = f(P, V, T)$ and $PV = nRT$

(a) $\begin{pmatrix} P \\ V \end{pmatrix} \to \begin{pmatrix} P = P \\ V = V \\ T = \frac{PV}{nR} \end{pmatrix} \to U \Rightarrow \left(\frac{\partial U}{\partial P}\right)_V = \frac{\partial U}{\partial P}\frac{\partial P}{\partial P} + \frac{\partial U}{\partial V}\frac{\partial V}{\partial P} + \frac{\partial U}{\partial T}\frac{\partial T}{\partial P} = \frac{\partial U}{\partial P} + \left(\frac{\partial U}{\partial V}\right)(0) + \left(\frac{\partial U}{\partial T}\right)\left(\frac{V}{nR}\right)$

$= \frac{\partial U}{\partial P} + \left(\frac{\partial U}{\partial T}\right)\left(\frac{V}{nR}\right)$

(b) $\begin{pmatrix} V \\ T \end{pmatrix} \to \begin{pmatrix} P = \frac{nRT}{V} \\ V = V \\ T = T \end{pmatrix} \to U \Rightarrow \left(\frac{\partial U}{\partial T}\right)_V = \frac{\partial U}{\partial P}\frac{\partial P}{\partial T} + \frac{\partial U}{\partial V}\frac{\partial V}{\partial T} + \frac{\partial U}{\partial T}\frac{\partial T}{\partial T} = \left(\frac{\partial U}{\partial P}\right)\left(\frac{nR}{V}\right) + \left(\frac{\partial U}{\partial V}\right)(0) + \frac{\partial U}{\partial T}$

$= \left(\frac{\partial U}{\partial P}\right)\left(\frac{nR}{V}\right) + \frac{\partial U}{\partial T}$

4. $w = x^2 + y^2 + z^2$ and $y \sin z + z \sin x = 0$

(a) $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y \\ z = z(x, y) \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial x}\right)_y = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x};\ \frac{\partial y}{\partial x} = 0$ and

$(y \cos z)\frac{\partial z}{\partial x} + (\sin x)\frac{\partial z}{\partial x} + z \cos x = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-z \cos x}{y \cos z + \sin x}$. At $(0, 1, \pi)$, $\frac{\partial z}{\partial x} = \frac{-\pi}{-1} = \pi$

$\Rightarrow \left(\frac{\partial w}{\partial x}\right)_{y|(0,1,\pi)} = (2x)(1) + (2y)(0) + (2z)(\pi)\big|_{(0,1,\pi)} = 2\pi^2$

(b) $\begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} x = x(y, z) \\ y = y \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_y = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z} = (2x)\frac{\partial x}{\partial z} + (2y)(0) + (2z)(1)$

$= (2x)\frac{\partial x}{\partial z} + 2z$. Now $(\sin z)\frac{\partial y}{\partial z} + y \cos z + \sin x + (z \cos x)\frac{\partial x}{\partial z} = 0$ and $\frac{\partial y}{\partial z} = 0$

$\Rightarrow y \cos z + \sin x + (z \cos x)\frac{\partial x}{\partial z} = 0 \Rightarrow \frac{\partial x}{\partial z} = \frac{-y \cos z - \sin x}{z \cos x}$. At $(0, 1, \pi)$, $\frac{\partial x}{\partial z} = \frac{1 - 0}{(\pi)(1)} = \frac{1}{\pi}$

$\Rightarrow \left(\frac{\partial w}{\partial z}\right)_{y|(0,1,\pi)} = 2(0)\left(\frac{1}{\pi}\right) + 2\pi = 2\pi$

5. $w = x^2y^2 + yz - z^3$ and $x^2 + y^2 + z^2 = 6$

(a) $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y \\ z = z(x, y) \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_x = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$

$= (2xy^2)(0) + (2x^2y + z)(1) + (y - 3z^2)\frac{\partial z}{\partial y} = 2x^2y + z + (y - 3z^2)\frac{\partial z}{\partial y}$. Now $(2x)\frac{\partial x}{\partial y} + 2y + (2z)\frac{\partial z}{\partial y} = 0$ and $\frac{\partial x}{\partial y} = 0 \Rightarrow 2y + (2z)\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{y}{z}$. At $(w, x, y, z) = (4, 2, 1, -1)$, $\frac{\partial z}{\partial y} = -\frac{1}{-1} = 1 \Rightarrow \left(\frac{\partial w}{\partial y}\right)_x\Big|_{(4,2,1,-1)}$ $= [(2)(2)^2(1) + (-1)] + [1 - 3(-1)^2](1) = 5$

(b) $\begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} x = x(y,z) \\ y = y \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_z = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$

$= (2xy^2)\frac{\partial x}{\partial y} + (2x^2y + z)(1) + (y - 3z^2)(0) = (2x^2y)\frac{\partial x}{\partial y} + 2x^2y + z$. Now $(2x)\frac{\partial x}{\partial y} + 2y + (2z)\frac{\partial z}{\partial y} = 0$ and $\frac{\partial z}{\partial y} = 0 \Rightarrow (2x)\frac{\partial x}{\partial y} + 2y = 0 \Rightarrow \frac{\partial x}{\partial y} = -\frac{y}{x}$. At $(w, x, y, z) = (4, 2, 1, -1)$, $\frac{\partial x}{\partial y} = -\frac{1}{2} \Rightarrow \left(\frac{\partial w}{\partial y}\right)_z\Big|_{(4,2,1,-1)}$ $= (2)(2)(1)^2\left(-\frac{1}{2}\right) + (2)(2)^2(1) + (-1) = 5$

6. $y = uv \Rightarrow 1 = v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}$; $x = u^2 + v^2$ and $\frac{\partial x}{\partial y} = 0 \Rightarrow 0 = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} \Rightarrow \frac{\partial v}{\partial y} = \left(-\frac{u}{v}\right)\frac{\partial u}{\partial y} \Rightarrow 1$ $= v\frac{\partial u}{\partial y} + u\left(-\frac{u}{v}\frac{\partial u}{\partial y}\right) = \left(\frac{v^2 - u^2}{v}\right)\frac{\partial u}{\partial y} \Rightarrow \frac{\partial u}{\partial y} = \frac{v}{v^2 - u^2}$. At $(u, v) = \left(\sqrt{2}, 1\right)$, $\frac{\partial u}{\partial y} = \frac{1}{1^2 - \left(\sqrt{2}\right)^2} = -1$ $\Rightarrow \left(\frac{\partial u}{\partial y}\right)_x = -1$

7. $\begin{pmatrix} r \\ \theta \end{pmatrix} \to \begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix} \Rightarrow \left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$; $x^2 + y^2 = r^2 \Rightarrow 2x + 2y\frac{\partial y}{\partial x} = 2r\frac{\partial r}{\partial x}$ and $\frac{\partial y}{\partial x} = 0 \Rightarrow 2x = 2r\frac{\partial r}{\partial x}$ $\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \Rightarrow \left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{\sqrt{x^2 + y^2}}$

8. If x, y, and z are independent, then $\left(\frac{\partial w}{\partial x}\right)_{y,z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x}$ $= (2x)(1) + (-2y)(0) + (4)(0) + (1)\left(\frac{\partial t}{\partial x}\right) = 2x + \frac{\partial t}{\partial x}$. Thus $x + 2z + t = 25 \Rightarrow 1 + 0 + \frac{\partial t}{\partial x} = 0 \Rightarrow \frac{\partial t}{\partial x} = -1$ $\Rightarrow \left(\frac{\partial w}{\partial x}\right)_{y,z} = 2x - 1$. On the other hand, if x, y, and t are independent, then $\left(\frac{\partial w}{\partial x}\right)_{y,t}$ $= \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x} = (2x)(1) + (-2y)(0) + 4\frac{\partial z}{\partial x} + (1)(0) = 2x + 4\frac{\partial z}{\partial x}$. Thus, $x + 2z + t = 25$ $\Rightarrow 1 + 2\frac{\partial z}{\partial x} + 0 = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{1}{2} \Rightarrow \left(\frac{\partial w}{\partial x}\right)_{y,t} = 2x + 4\left(-\frac{1}{2}\right) = 2x - 2$.

9. If x is a differentiable function of y and z, then $f(x, y, z) = 0 \Rightarrow \frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} = 0$ $\Rightarrow \left(\frac{\partial x}{\partial y}\right)_z = -\frac{\partial f/\partial y}{\partial f/\partial z}$. Similarly, if y is a differentiable function of x and z, $\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\partial f/\partial z}{\partial f/\partial x}$ and if z is a differentiable function of x and y, $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\partial f/\partial x}{\partial f/\partial y}$. Then $\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y$ $= \left(-\frac{\partial f/\partial y}{\partial f/\partial z}\right)\left(-\frac{\partial f/\partial z}{\partial f/\partial x}\right)\left(-\frac{\partial f/\partial x}{\partial f/\partial y}\right) = -1$.

10. $z = z + f(u)$ and $u = xy \Rightarrow \frac{\partial z}{\partial x} = 1 + \frac{df}{du}\frac{\partial u}{\partial x} = 1 + y\frac{df}{du}$; also $\frac{\partial z}{\partial y} = 0 + \frac{df}{du}\frac{\partial u}{\partial y} = x\frac{df}{du}$ so that $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$ $= x\left(1 + y\frac{df}{du}\right) - y\left(x\frac{df}{du}\right) = x$

11. If x and y are independent, then $g(x, y, z) = 0 \Rightarrow \frac{\partial g}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial y} = 0$ and $\frac{\partial x}{\partial y} = 0 \Rightarrow \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial y} = 0$ $\Rightarrow \left(\frac{\partial z}{\partial y}\right)_x = -\frac{\partial g/\partial y}{\partial g/\partial z}$, as claimed.

12. Let x and y be independent. Then $f(x, y, z, w) = 0$, $g(x, y, z, w) = 0$ and $\frac{\partial y}{\partial x} = 0$ $\Rightarrow \frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = 0$ and $\frac{\partial g}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial x} = 0$ imply

$$\begin{cases} \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = -\frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial x} = -\frac{\partial g}{\partial x} \end{cases} \Rightarrow \left(\frac{\partial z}{\partial x}\right)_y = \frac{\begin{vmatrix} -\frac{\partial f}{\partial x} & \frac{\partial f}{\partial w} \\ -\frac{\partial g}{\partial x} & \frac{\partial g}{\partial w} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{vmatrix}} = \frac{-\frac{\partial f}{\partial x}\frac{\partial g}{\partial w} + \frac{\partial g}{\partial x}\frac{\partial f}{\partial w}}{\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial g}{\partial z}\frac{\partial f}{\partial w}} = -\frac{\frac{\partial f}{\partial x}\frac{\partial g}{\partial w} - \frac{\partial f}{\partial w}\frac{\partial g}{\partial x}}{\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial f}{\partial w}\frac{\partial g}{\partial z}}, \text{ as claimed.}$$

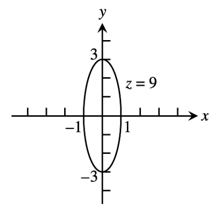
Likewise, $f(x, y, z, w) = 0$, $g(x, y, z, w) = 0$ and $\frac{\partial x}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$ $= \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = 0$ and (similarly) $\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial y} = 0$ imply

$$\begin{cases} \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = -\frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial y} = -\frac{\partial g}{\partial y} \end{cases} \Rightarrow \left(\frac{\partial w}{\partial y}\right)_x = \frac{\begin{vmatrix} \frac{\partial f}{\partial z} & -\frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial z} & -\frac{\partial g}{\partial y} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{vmatrix}} = \frac{-\frac{\partial f}{\partial z}\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial g}{\partial z}\frac{\partial f}{\partial w}} = -\frac{\frac{\partial f}{\partial z}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial z}}{\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial f}{\partial w}\frac{\partial g}{\partial z}}, \text{ as claimed.}$$

CHAPTER 14 PRACTICE EXERCISES

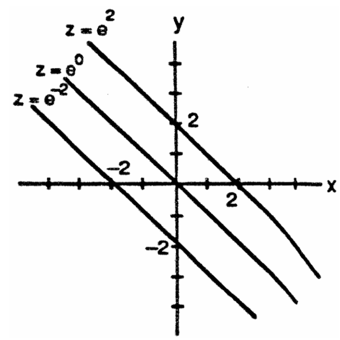
1. Domain: All points in the xy-plane
   Range: $z \geq 0$

   Level curves are ellipses with major axis along the y-axis and minor axis along the x-axis.



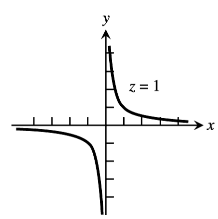
2. Domain: All points in the xy-plane
   Range: $0 < z < \infty$

   Level curves are the straight lines $x + y = \ln z$ with slope -1, and $z > 0$.



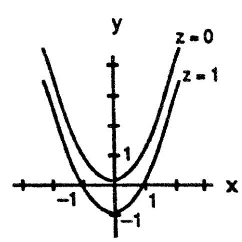
3. Domain: All (x, y) such that $x \neq 0$ and $y \neq 0$
   Range: $z \neq 0$

   Level curves are hyperbolas with the x- and y-axes as asymptotes.



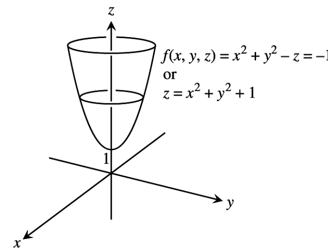
4. Domain: All (x, y) so that $x^2 - y \geq 0$
   Range: $z \geq 0$

   Level curves are the parabolas $y = x^2 - c$, $c \geq 0$.



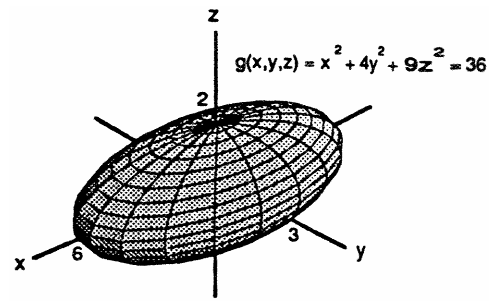
5. Domain: All points (x, y, z) in space
Range: All real numbers

Level surfaces are paraboloids of revolution with the z-axis as axis.



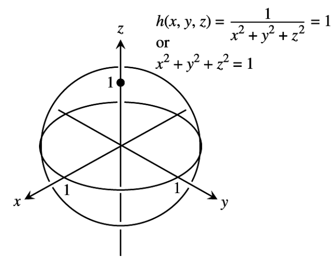
6. Domain: All points (x, y, z) in space
Range: Nonnegative real numbers

Level surfaces are ellipsoids with center (0, 0, 0).



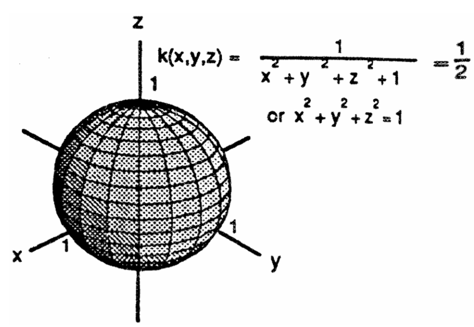
7. Domain: All (x, y, z) such that $(x, y, z) \neq (0, 0, 0)$
Range: Positive real numbers

Level surfaces are spheres with center (0, 0, 0) and radius $r > 0$.



8. Domain: All points (x, y, z) in space
Range: (0, 1]

Level surfaces are spheres with center (0, 0, 0) and radius $r > 0$.



9. $\lim\limits_{(x, y) \to (\pi, \ln 2)} e^y \cos x = e^{\ln 2} \cos \pi = (2)(-1) = -2$

10. $\lim\limits_{(x, y) \to (0, 0)} \frac{2+y}{x+\cos y} = \frac{2+0}{0+\cos 0} = 2$

11. $\lim\limits_{\substack{(x, y) \to (1, 1) \\ x \neq \pm y}} \frac{x-y}{x^2-y^2} = \lim\limits_{\substack{(x, y) \to (1, 1) \\ x \neq \pm y}} \frac{x-y}{(x-y)(x+y)} = \lim\limits_{(x, y) \to (1, 1)} \frac{1}{x+y} = \frac{1}{1+1} = \frac{1}{2}$

12. $\lim\limits_{(x, y) \to (1, 1)} \frac{x^3y^3-1}{xy-1} = \lim\limits_{(x, y) \to (1, 1)} \frac{(xy-1)(x^2y^2+xy+1)}{xy-1} = \lim\limits_{(x, y) \to (1, 1)} (x^2y^2+xy+1) = 1^2 \cdot 1^2 + 1 \cdot 1 + 1 = 3$

13. $\lim\limits_{P \to (1, -1, e)} \ln|x+y+z| = \ln|1+(-1)+e| = \ln e = 1$

14. $\lim\limits_{P \to (1, -1, -1)} \tan^{-1}(x+y+z) = \tan^{-1}(1+(-1)+(-1)) = \tan^{-1}(-1) = -\frac{\pi}{4}$

15. Let $y = kx^2$, $k \neq 1$. Then $\lim\limits_{\substack{(x,y)\to(0,0)\\ y\neq x^2}} \frac{y}{x^2-y} = \lim\limits_{(x,kx^2)\to(0,0)} \frac{kx^2}{x^2-kx^2} = \frac{k}{1-k^2}$ which gives different limits for different values of k $\Rightarrow$ the limit does not exist.

16. Let $y = kx$, $k \neq 0$. Then $\lim\limits_{\substack{(x,y)\to(0,0)\\ xy\neq 0}} \frac{x^2+y^2}{xy} = \lim\limits_{(x,kx)\to(0,0)} \frac{x^2+(kx)^2}{x(kx)} = \frac{1+k^2}{k}$ which gives different limits for different values of k $\Rightarrow$ the limit does not exist.

17. Let $y = kx$. Then $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \frac{x^2-k^2x^2}{x^2+k^2x^2} = \frac{1-k^2}{1+k^2}$ which gives different limits for different values of k $\Rightarrow$ the limit does not exist so $f(0,0)$ cannot be defined in a way that makes f continuous at the origin.

18. Along the x-axis, $y = 0$ and $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x-y)}{|x|+|y|} = \lim\limits_{x\to 0} \frac{\sin x}{|x|} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$, so the limit fails to exist $\Rightarrow$ f is not continuous at $(0,0)$.

19. $\frac{\partial g}{\partial r} = \cos\theta + \sin\theta$, $\frac{\partial g}{\partial \theta} = -r\sin\theta + r\cos\theta$

20. $\frac{\partial f}{\partial x} = \frac{1}{2}\left(\frac{2x}{x^2+y^2}\right) + \frac{\left(-\frac{y}{x^2}\right)}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} = \frac{x-y}{x^2+y^2}$, $\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{2y}{x^2+y^2}\right) + \frac{\left(\frac{1}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = \frac{y}{x^2+y^2} + \frac{x}{x^2+y^2} = \frac{x+y}{x^2+y^2}$

21. $\frac{\partial f}{\partial R_1} = -\frac{1}{R_1^2}$, $\frac{\partial f}{\partial R_2} = -\frac{1}{R_2^2}$, $\frac{\partial f}{\partial R_3} = -\frac{1}{R_3^2}$

22. $h_x(x,y,z) = 2\pi\cos(2\pi x + y - 3z)$, $h_y(x,y,z) = \cos(2\pi x + y - 3z)$, $h_z(x,y,z) = -3\cos(2\pi x + y - 3z)$

23. $\frac{\partial P}{\partial n} = \frac{RT}{V}$, $\frac{\partial P}{\partial R} = \frac{nT}{V}$, $\frac{\partial P}{\partial T} = \frac{nR}{V}$, $\frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$

24. $f_r(r,\ell,T,w) = -\frac{1}{2r^2\ell}\sqrt{\frac{T}{\pi w}}$, $f_\ell(r,\ell,T,w) = -\frac{1}{2r\ell^2}\sqrt{\frac{T}{\pi w}}$, $f_T(r,\ell,T,w) = \left(\frac{1}{2r\ell}\right)\left(\frac{1}{\sqrt{\pi w}}\right)\left(\frac{1}{2\sqrt{T}}\right)$
$= \frac{1}{4r\ell}\sqrt{\frac{1}{T\pi w}} = \frac{1}{4r\ell T}\sqrt{\frac{T}{\pi w}}$, $f_w(r,\ell,T,w) = \left(\frac{1}{2r\ell}\right)\sqrt{\frac{T}{\pi}}\left(-\frac{1}{2}w^{-3/2}\right) = -\frac{1}{4r\ell w}\sqrt{\frac{T}{\pi w}}$

25. $\frac{\partial g}{\partial x} = \frac{1}{y}$, $\frac{\partial g}{\partial y} = 1 - \frac{x}{y^2} \Rightarrow \frac{\partial^2 g}{\partial x^2} = 0$, $\frac{\partial^2 g}{\partial y^2} = \frac{2x}{y^3}$, $\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial^2 g}{\partial x \partial y} = -\frac{1}{y^2}$

26. $g_x(x,y) = e^x + y\cos x$, $g_y(x,y) = \sin x \Rightarrow g_{xx}(x,y) = e^x - y\sin x$, $g_{yy}(x,y) = 0$, $g_{xy}(x,y) = g_{yx}(x,y) = \cos x$

27. $\frac{\partial f}{\partial x} = 1 + y - 15x^2 + \frac{2x}{x^2+1}$, $\frac{\partial f}{\partial y} = x \Rightarrow \frac{\partial^2 f}{\partial x^2} = -30x + \frac{2-2x^2}{(x^2+1)^2}$, $\frac{\partial^2 f}{\partial y^2} = 0$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 1$

28. $f_x(x,y) = -3y$, $f_y(x,y) = 2y - 3x - \sin y + 7e^y \Rightarrow f_{xx}(x,y) = 0$, $f_{yy}(x,y) = 2 - \cos y + 7e^y$, $f_{xy}(x,y) = f_{yx}(x,y) = -3$

29. $\frac{\partial w}{\partial x} = y\cos(xy+\pi)$, $\frac{\partial w}{\partial y} = x\cos(xy+\pi)$, $\frac{dx}{dt} = e^t$, $\frac{dy}{dt} = \frac{1}{t+1}$
$\Rightarrow \frac{dw}{dt} = [y\cos(xy+\pi)]e^t + [x\cos(xy+\pi)]\left(\frac{1}{t+1}\right)$; $t = 0 \Rightarrow x = 1$ and $y = 0$
$\Rightarrow \left.\frac{dw}{dt}\right|_{t=0} = 0 \cdot 1 + [1 \cdot (-1)]\left(\frac{1}{0+1}\right) = -1$

30. $\frac{\partial w}{\partial x} = e^y$, $\frac{\partial w}{\partial y} = xe^y + \sin z$, $\frac{\partial w}{\partial z} = y\cos z + \sin z$, $\frac{dx}{dt} = t^{-1/2}$, $\frac{dy}{dt} = 1 + \frac{1}{t}$, $\frac{dz}{dt} = \pi$
$\Rightarrow \frac{dw}{dt} = e^y t^{-1/2} + (xe^y + \sin z)\left(1 + \frac{1}{t}\right) + (y\cos z + \sin z)\pi$; $t = 1 \Rightarrow x = 2$, $y = 0$, and $z = \pi$
$\Rightarrow \left.\frac{dw}{dt}\right|_{t=1} = 1 \cdot 1 + (2 \cdot 1 - 0)(2) + (0+0)\pi = 5$

31. $\frac{\partial w}{\partial x} = 2\cos(2x - y)$, $\frac{\partial w}{\partial y} = -\cos(2x - y)$, $\frac{\partial x}{\partial r} = 1$, $\frac{\partial x}{\partial s} = \cos s$, $\frac{\partial y}{\partial r} = s$, $\frac{\partial y}{\partial s} = r$
$\Rightarrow \frac{\partial w}{\partial r} = [2\cos(2x - y)](1) + [-\cos(2x - y)](s)$; $r = \pi$ and $s = 0 \Rightarrow x = \pi$ and $y = 0$
$\Rightarrow \left.\frac{\partial w}{\partial r}\right|_{(\pi,0)} = (2\cos 2\pi) - (\cos 2\pi)(0) = 2$; $\frac{\partial w}{\partial s} = [2\cos(2x - y)](\cos s) + [-\cos(2x - y)](r)$
$\Rightarrow \left.\frac{\partial w}{\partial s}\right|_{(\pi,0)} = (2\cos 2\pi)(\cos 0) - (\cos 2\pi)(\pi) = 2 - \pi$

32. $\frac{\partial w}{\partial u} = \frac{dw}{dx}\frac{\partial x}{\partial u} = \left(\frac{x}{1+x^2} - \frac{1}{x^2+1}\right)(2e^u \cos v)$; $u = v = 0 \Rightarrow x = 2 \Rightarrow \left.\frac{\partial w}{\partial u}\right|_{(0,0)} = \left(\frac{2}{5} - \frac{1}{5}\right)(2) = \frac{2}{5}$;
$\frac{\partial w}{\partial v} = \frac{dw}{dx}\frac{\partial x}{\partial v} = \left(\frac{x}{1+x^2} - \frac{1}{x^2+1}\right)(-2e^u \sin v) \Rightarrow \left.\frac{\partial w}{\partial v}\right|_{(0,0)} = \left(\frac{2}{5} - \frac{1}{5}\right)(0) = 0$

33. $\frac{\partial f}{\partial x} = y + z$, $\frac{\partial f}{\partial y} = x + z$, $\frac{\partial f}{\partial z} = y + x$, $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$, $\frac{dz}{dt} = -2\sin 2t$
$\Rightarrow \frac{df}{dt} = -(y + z)(\sin t) + (x + z)(\cos t) - 2(y + x)(\sin 2t)$; $t = 1 \Rightarrow x = \cos 1$, $y = \sin 1$, and $z = \cos 2$
$\Rightarrow \left.\frac{df}{dt}\right|_{t=1} = -(\sin 1 + \cos 2)(\sin 1) + (\cos 1 + \cos 2)(\cos 1) - 2(\sin 1 + \cos 1)(\sin 2)$

34. $\frac{\partial w}{\partial x} = \frac{dw}{ds}\frac{\partial s}{\partial x} = (5)\frac{dw}{ds}$ and $\frac{\partial w}{\partial y} = \frac{dw}{ds}\frac{\partial s}{\partial y} = (1)\frac{dw}{ds} = \frac{dw}{ds} \Rightarrow \frac{\partial w}{\partial x} - 5\frac{\partial w}{\partial y} = 5\frac{dw}{ds} - 5\frac{dw}{ds} = 0$

35. $F(x, y) = 1 - x - y^2 - \sin xy \Rightarrow F_x = -1 - y\cos xy$ and $F_y = -2y - x\cos xy \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-1 - y\cos xy}{-2y - x\cos xy}$
$= \frac{1 + y\cos xy}{-2y - x\cos xy} \Rightarrow$ at $(x, y) = (0, 1)$ we have $\left.\frac{dy}{dx}\right|_{(0,1)} = \frac{1+1}{-2} = -1$

36. $F(x, y) = 2xy + e^{x+y} - 2 \Rightarrow F_x = 2y + e^{x+y}$ and $F_y = 2x + e^{x+y} \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2y + e^{x+y}}{2x + e^{x+y}}$
$\Rightarrow$ at $(x, y) = (0, \ln 2)$ we have $\left.\frac{dy}{dx}\right|_{(0,\ln 2)} = -\frac{2\ln 2 + 2}{0 + 2} = -(\ln 2 + 1)$

37. $\nabla f = (-\sin x \cos y)\mathbf{i} - (\cos x \sin y)\mathbf{j} \Rightarrow \nabla f|_{(\frac{\pi}{4},\frac{\pi}{4})} = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} \Rightarrow |\nabla f| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$;
$\mathbf{u} = \frac{\nabla f}{|\nabla f|} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} \Rightarrow$ f increases most rapidly in the direction $\mathbf{u} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$; $(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = \frac{\sqrt{2}}{2}$ and $(D_{-\mathbf{u}}f)_{P_0} = -\frac{\sqrt{2}}{2}$;
$\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} + 4\mathbf{j}}{\sqrt{3^2 + 4^2}} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} \Rightarrow (D_{\mathbf{u}_1}f)_{P_0} = \nabla f \cdot \mathbf{u}_1 = \left(-\frac{1}{2}\right)\left(\frac{3}{5}\right) + \left(-\frac{1}{2}\right)\left(\frac{4}{5}\right) = -\frac{7}{10}$

38. $\nabla f = 2xe^{-2y}\mathbf{i} - 2x^2e^{-2y}\mathbf{j} \Rightarrow \nabla f|_{(1,0)} = 2\mathbf{i} - 2\mathbf{j} \Rightarrow |\nabla f| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$; $\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$
$\Rightarrow$ f increases most rapidly in the direction $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$; $(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = 2\sqrt{2}$ and $(D_{-\mathbf{u}}f)_{P_0} = -2\sqrt{2}$; $\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$
$\Rightarrow (D_{\mathbf{u}_1}f)_{P_0} = \nabla f \cdot \mathbf{u}_1 = (2)\left(\frac{1}{\sqrt{2}}\right) + (-2)\left(\frac{1}{\sqrt{2}}\right) = 0$

39. $\nabla f = \left(\frac{2}{2x + 3y + 6z}\right)\mathbf{i} + \left(\frac{3}{2x + 3y + 6z}\right)\mathbf{j} + \left(\frac{6}{2x + 3y + 6z}\right)\mathbf{k} \Rightarrow \nabla f|_{(-1,-1,1)} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$;
$\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \Rightarrow$ f increases most rapidly in the direction $\mathbf{u} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} - \frac{6}{7}\mathbf{k}$; $(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = 7$, $(D_{-\mathbf{u}}f)_{P_0} = -7$;
$\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \Rightarrow (D_{\mathbf{u}_1}f)_{P_0} = (D_{\mathbf{u}}f)_{P_0} = 7$

40. $\nabla f = (2x + 3y)\mathbf{i} + (3x + 2)\mathbf{j} + (1 - 2z)\mathbf{k} \Rightarrow \nabla f|_{(0,0,0)} = 2\mathbf{j} + \mathbf{k}$; $\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{2}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k} \Rightarrow$ f increases most rapidly in the direction $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{\sqrt{5}}\mathbf{j} - \frac{1}{\sqrt{5}}\mathbf{k}$;
$(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = \sqrt{5}$ and $(D_{-\mathbf{u}}f)_{P_0} = -\sqrt{5}$; $\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$
$\Rightarrow (D_{\mathbf{u}_1}f)_{P_0} = \nabla f \cdot \mathbf{u}_1 = (0)\left(\frac{1}{\sqrt{3}}\right) + (2)\left(\frac{1}{\sqrt{3}}\right) + (1)\left(\frac{1}{\sqrt{3}}\right) = \frac{3}{\sqrt{3}} = \sqrt{3}$

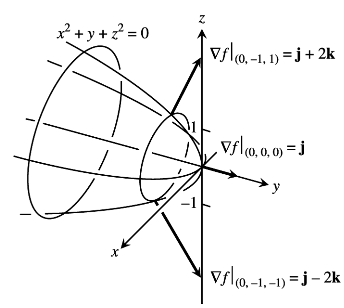
41. $\mathbf{r} = (\cos 3t)\mathbf{i} + (\sin 3t)\mathbf{j} + 3t\mathbf{k} \Rightarrow \mathbf{v}(t) = (-3 \sin 3t)\mathbf{i} + (3 \cos 3t)\mathbf{j} + 3\mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{3}\right) = -3\mathbf{j} + 3\mathbf{k}$
$\Rightarrow \mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$; $f(x, y, z) = xyz \Rightarrow \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$; $t = \frac{\pi}{3}$ yields the point on the helix $(-1, 0, \pi)$
$\Rightarrow \nabla f|_{(-1,0,\pi)} = -\pi\mathbf{j} \Rightarrow \nabla f \cdot \mathbf{u} = (-\pi\mathbf{j}) \cdot \left(-\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}\right) = \frac{\pi}{\sqrt{2}}$

42. $f(x, y, z) = xyz \Rightarrow \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$; at $(1, 1, 1)$ we get $\nabla f = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow$ the maximum value of $D_{\mathbf{u}}f|_{(1,1,1)} = |\nabla f| = \sqrt{3}$

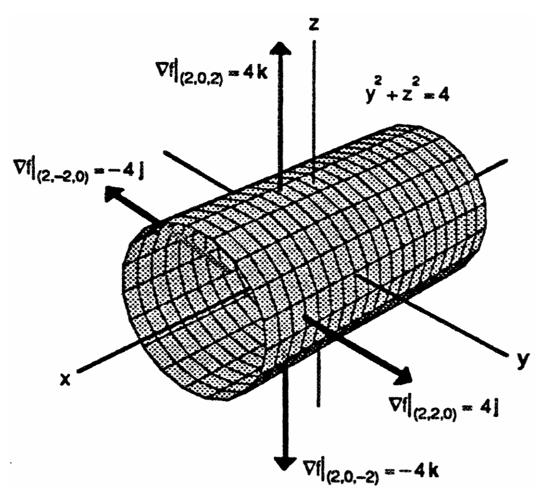
43. (a) Let $\nabla f = a\mathbf{i} + b\mathbf{j}$ at $(1, 2)$. The direction toward $(2, 2)$ is determined by $\mathbf{v}_1 = (2-1)\mathbf{i} + (2-2)\mathbf{j} = \mathbf{i} = \mathbf{u}$ so that $\nabla f \cdot \mathbf{u} = 2 \Rightarrow a = 2$. The direction toward $(1, 1)$ is determined by $\mathbf{v}_2 = (1-1)\mathbf{i} + (1-2)\mathbf{j} = -\mathbf{j} = \mathbf{u}$ so that $\nabla f \cdot \mathbf{u} = -2 \Rightarrow -b = -2 \Rightarrow b = 2$. Therefore $\nabla f = 2\mathbf{i} + 2\mathbf{j}$; $f_x(1, 2) = f_y(1, 2) = 2$.
(b) The direction toward $(4, 6)$ is determined by $\mathbf{v}_3 = (4-1)\mathbf{i} + (6-2)\mathbf{j} = 3\mathbf{i} + 4\mathbf{j} \Rightarrow \mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$
$\Rightarrow \nabla f \cdot \mathbf{u} = \frac{14}{5}$.

44. (a) True (b) False (c) True (d) True

45. $\nabla f = 2x\mathbf{i} + \mathbf{j} + 2z\mathbf{k} \Rightarrow$
$\nabla f|_{(0,-1,-1)} = \mathbf{j} - 2\mathbf{k}$,
$\nabla f|_{(0,0,0)} = \mathbf{j}$,
$\nabla f|_{(0,-1,1)} = \mathbf{j} + 2\mathbf{k}$



46. $\nabla f = 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow$
$\nabla f|_{(2,2,0)} = 4\mathbf{j}$,
$\nabla f|_{(2,-2,0)} = -4\mathbf{j}$,
$\nabla f|_{(2,0,2)} = 4\mathbf{k}$,
$\nabla f|_{(2,0,-2)} = -4\mathbf{k}$



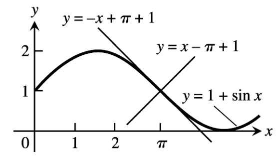
47. $\nabla f = 2x\mathbf{i} - \mathbf{j} - 5\mathbf{k} \Rightarrow \nabla f|_{(2,-1,1)} = 4\mathbf{i} - \mathbf{j} - 5\mathbf{k} \Rightarrow$ Tangent Plane: $4(x-2) - (y+1) - 5(z-1) = 0$
$\Rightarrow 4x - y - 5z = 4$; Normal Line: $x = 2 + 4t$, $y = -1 - t$, $z = 1 - 5t$

48. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k} \Rightarrow \nabla f|_{(1,1,2)} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow$ Tangent Plane: $2(x-1) + 2(y-1) + (z-2) = 0$
$\Rightarrow 2x + 2y + z - 6 = 0$; Normal Line: $x = 1 + 2t$, $y = 1 + 2t$, $z = 2 + t$

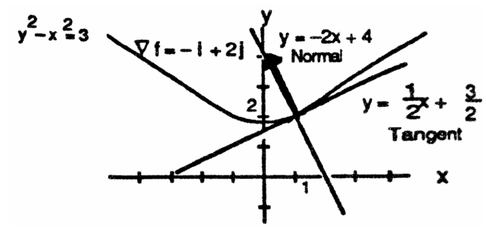
49. $\frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2} \Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(0,1,0)} = 0$ and $\frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2} \Rightarrow \left.\frac{\partial z}{\partial y}\right|_{(0,1,0)} = 2$; thus the tangent plane is
$2(y-1) - (z-0) = 0$ or $2y - z - 2 = 0$

50. $\frac{\partial z}{\partial x} = -2x\left(x^2+y^2\right)^{-2} \Rightarrow \left.\frac{\partial z}{\partial x}\right|_{\left(1,1,\frac{1}{2}\right)} = -\frac{1}{2}$ and $\frac{\partial z}{\partial y} = -2y\left(x^2+y^2\right)^{-2} \Rightarrow \left.\frac{\partial z}{\partial y}\right|_{\left(1,1,\frac{1}{2}\right)} = -\frac{1}{2}$; thus the tangent plane is $-\frac{1}{2}(x-1) - \frac{1}{2}(y-1) - \left(z-\frac{1}{2}\right) = 0$ or $x+y+2z-3=0$

51. $\nabla f = (-\cos x)\mathbf{i} + \mathbf{j} \Rightarrow \left.\nabla f\right|_{(\pi,1)} = \mathbf{i}+\mathbf{j} \Rightarrow$ the tangent line is $(x-\pi)+(y-1)=0 \Rightarrow x+y=\pi+1$; the normal line is $y-1 = 1(x-\pi) \Rightarrow y = x-\pi+1$



52. $\nabla f = -x\mathbf{i} + y\mathbf{j} \Rightarrow \left.\nabla f\right|_{(1,2)} = -\mathbf{i}+2\mathbf{j} \Rightarrow$ the tangent line is $-(x-1)+2(y-2)=0 \Rightarrow y = \frac{1}{2}x + \frac{3}{2}$; the normal line is $y-2 = -2(x-1) \Rightarrow y = -2x+4$



53. Let $f(x,y,z) = x^2+2y+2z-4$ and $g(x,y,z) = y-1$. Then $\nabla f = \left.2x\mathbf{i}+2\mathbf{j}+2\mathbf{k}\right|_{\left(1,1,\frac{1}{2}\right)} = 2\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ and $\nabla g = \mathbf{j} \Rightarrow \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\mathbf{i}+2\mathbf{k} \Rightarrow$ the line is $x = 1-2t$, $y=1$, $z = \frac{1}{2}+2t$

54. Let $f(x,y,z) = x+y^2+z-2$ and $g(x,y,z) = y-1$. Then $\nabla f = \left.\mathbf{i}+2y\mathbf{j}+\mathbf{k}\right|_{\left(\frac{1}{2},1,\frac{1}{2}\right)} = \mathbf{i}+2\mathbf{j}+\mathbf{k}$ and $\nabla g = \mathbf{j} \Rightarrow \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i}+\mathbf{k} \Rightarrow$ the line is $x = \frac{1}{2}-t$, $y=1$, $z=\frac{1}{2}+t$

55. $f\left(\frac{\pi}{4},\frac{\pi}{4}\right) = \frac{1}{2}$, $f_x\left(\frac{\pi}{4},\frac{\pi}{4}\right) = \left.\cos x \cos y\right|_{(\pi/4,\pi/4)} = \frac{1}{2}$, $f_y\left(\frac{\pi}{4},\frac{\pi}{4}\right) = \left.-\sin x \sin y\right|_{(\pi/4,\pi/4)} = -\frac{1}{2}$ $\Rightarrow L(x,y) = \frac{1}{2}+\frac{1}{2}\left(x-\frac{\pi}{4}\right) - \frac{1}{2}\left(y-\frac{\pi}{4}\right) = \frac{1}{2}+\frac{1}{2}x-\frac{1}{2}y$; $f_{xx}(x,y) = -\sin x \cos y$, $f_{yy}(x,y) = -\sin x\cos y$, and $f_{xy}(x,y) = -\cos x \sin y$. Thus an upper bound for E depends on the bound M used for $|f_{xx}|$, $|f_{xy}|$, and $|f_{yy}|$. With $M = \frac{\sqrt{2}}{2}$ we have $|E(x,y)| \le \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)\left(\left|x-\frac{\pi}{4}\right| + \left|y-\frac{\pi}{4}\right|\right)^2 \le \frac{\sqrt{2}}{4}(0.2)^2 \le 0.0142$; with $M=1$, $|E(x,y)| \le \frac{1}{2}(1)\left(\left|x-\frac{\pi}{4}\right|+\left|y-\frac{\pi}{4}\right|\right)^2 = \frac{1}{2}(0.2)^2 = 0.02$.

56. $f(1,1) = 0$, $f_x(1,1) = \left.y\right|_{(1,1)} = 1$, $f_y(1,1) = \left.x-6y\right|_{(1,1)} = -5 \Rightarrow L(x,y) = (x-1)-5(y-1) = x-5y+4$; $f_{xx}(x,y) = 0$, $f_{yy}(x,y) = -6$, and $f_{xy}(x,y) = 1 \Rightarrow$ maximum of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ is 6 $\Rightarrow M = 6$ $\Rightarrow |E(x,y)| \le \frac{1}{2}(6)\left(|x-1|+|y-1|\right)^2 = \frac{1}{2}(6)(0.1+0.2)^2 = 0.27$

57. $f(1,0,0) = 0$, $f_x(1,0,0) = \left.y-3z\right|_{(1,0,0)} = 0$, $f_y(1,0,0) = \left.x+2z\right|_{(1,0,0)} = 1$, $f_z(1,0,0) = \left.2y-3x\right|_{(1,0,0)} = -3$ $\Rightarrow L(x,y,z) = 0(x-1)+(y-0)-3(z-0) = y-3z$; $f(1,1,0) = 1$, $f_x(1,1,0) = 1$, $f_y(1,1,0) = 1$, $f_z(1,1,0) = -1$ $\Rightarrow L(x,y,z) = 1+(x-1)+(y-1)-1(z-0) = x+y-z-1$

58. $f\left(0,0,\frac{\pi}{4}\right) = 1$, $f_x\left(0,0,\frac{\pi}{4}\right) = \left.-\sqrt{2}\sin x \sin(y+z)\right|_{\left(0,0,\frac{\pi}{4}\right)} = 0$, $f_y\left(0,0,\frac{\pi}{4}\right) = \left.\sqrt{2}\cos x\cos(y+z)\right|_{\left(0,0,\frac{\pi}{4}\right)} = 1$, $f_z\left(0,0,\frac{\pi}{4}\right) = \left.\sqrt{2}\cos x\cos(y+z)\right|_{\left(0,0,\frac{\pi}{4}\right)} = 1 \Rightarrow L(x,y,z) = 1+1(y-0)+1\left(z-\frac{\pi}{4}\right) = 1+y+z-\frac{\pi}{4}$; $f\left(\frac{\pi}{4},\frac{\pi}{4},0\right) = \frac{\sqrt{2}}{2}$, $f_x\left(\frac{\pi}{4},\frac{\pi}{4},0\right) = -\frac{\sqrt{2}}{2}$, $f_y\left(\frac{\pi}{4},\frac{\pi}{4},0\right) = \frac{\sqrt{2}}{2}$, $f_z\left(\frac{\pi}{4},\frac{\pi}{4},0\right) = \frac{\sqrt{2}}{2}$ $\Rightarrow L(x,y,z) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}\left(y-\frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}(z-0) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y + \frac{\sqrt{2}}{2}z$

59. $V = \pi r^2 h \Rightarrow dV = 2\pi rh\, dr + \pi r^2\, dh \Rightarrow dV|_{(1.5,5280)} = 2\pi(1.5)(5280)\, dr + \pi(1.5)^2\, dh = 15{,}840\pi\, dr + 2.25\pi\, dh$. You should be more careful with the diameter since it has a greater effect on dV.

60. $df = (2x - y)\, dx + (-x + 2y)\, dy \Rightarrow df|_{(1,2)} = 3\, dy \Rightarrow$ f is more sensitive to changes in y; in fact, near the point $(1, 2)$ a change in x does not change f.

61. $dI = \frac{1}{R}\, dV - \frac{V}{R^2}\, dR \Rightarrow dI\big|_{(24,100)} = \frac{1}{100}\, dV - \frac{24}{100^2}\, dR \Rightarrow dI\big|_{dV=-1,dR=-20} = -0.01 + (480)(.0001) = 0.038$, or increases by 0.038 amps; % change in $V = (100)\left(-\frac{1}{24}\right) \approx -4.17\%$; % change in $R = \left(-\frac{20}{100}\right)(100) = -20\%$; $I = \frac{24}{100} = 0.24 \Rightarrow$ estimated % change in $I = \frac{dI}{I} \times 100 = \frac{0.038}{0.24} \times 100 \approx 15.83\% \Rightarrow$ more sensitive to voltage change.

62. $A = \pi ab \Rightarrow dA = \pi b\, da + \pi a\, db \Rightarrow dA|_{(10,16)} = 16\pi\, da + 10\pi\, db$; $da = \pm 0.1$ and $db = \pm 0.1$ $\Rightarrow dA = \pm 26\pi(0.1) = \pm 2.6\pi$ and $A = \pi(10)(16) = 160\pi \Rightarrow \left|\frac{dA}{A} \times 100\right| = \left|\frac{2.6\pi}{160\pi} \times 100\right| \approx 1.625\%$

63. (a) $y = uv \Rightarrow dy = v\, du + u\, dv$; percentage change in $u \le 2\% \Rightarrow |du| \le 0.02$, and percentage change in $v \le 3\%$ $\Rightarrow |dv| \le 0.03$; $\frac{dy}{y} = \frac{v\,du + u\,dv}{uv} = \frac{du}{u} + \frac{dv}{v} \Rightarrow \left|\frac{dy}{y} \times 100\right| = \left|\frac{du}{u} \times 100 + \frac{dv}{v} \times 100\right| \le \left|\frac{du}{u} \times 100\right| + \left|\frac{dv}{v} \times 100\right|$ $\le 2\% + 3\% = 5\%$

(b) $z = u + v \Rightarrow \frac{dz}{z} = \frac{du + dv}{u + v} = \frac{du}{u + v} + \frac{dv}{u + v} \le \frac{du}{u} + \frac{dv}{v}$ (since $u > 0, v > 0$) $\Rightarrow \left|\frac{dz}{z} \times 100\right| \le \left|\frac{du}{u} \times 100 + \frac{dv}{v} \times 100\right| = \left|\frac{dy}{y} \times 100\right|$

64. $C = \frac{7}{71.84 w^{0.425} h^{0.725}} \Rightarrow C_w = \frac{(-0.425)(7)}{71.84 w^{1.425} h^{0.725}}$ and $C_h = \frac{(-0.725)(7)}{71.84 w^{0.425} h^{1.725}}$ $\Rightarrow dC = \frac{-2.975}{71.84 w^{1.425} h^{0.725}}\, dw + \frac{-5.075}{71.84 w^{0.425} h^{1.725}}\, dh$; thus when $w = 70$ and $h = 180$ we have $dC|_{(70,180)} \approx -(0.00000225)\, dw - (0.00000149)\, dh \Rightarrow$ 1 kg error in weight has more effect

65. $f_x(x, y) = 2x - y + 2 = 0$ and $f_y(x, y) = -x + 2y + 2 = 0 \Rightarrow x = -2$ and $y = -2 \Rightarrow (-2, -2)$ is the critical point; $f_{xx}(-2, -2) = 2$, $f_{yy}(-2, -2) = 2$, $f_{xy}(-2, -2) = -1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(-2, -2) = -8$

66. $f_x(x, y) = 10x + 4y + 4 = 0$ and $f_y(x, y) = 4x - 4y - 4 = 0 \Rightarrow x = 0$ and $y = -1 \Rightarrow (0, -1)$ is the critical point; $f_{xx}(0, -1) = 10$, $f_{yy}(0, -1) = -4$, $f_{xy}(0, -1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -56 < 0 \Rightarrow$ saddle point with $f(0, -1) = 2$

67. $f_x(x, y) = 6x^2 + 3y = 0$ and $f_y(x, y) = 3x + 6y^2 = 0 \Rightarrow y = -2x^2$ and $3x + 6\left(4x^4\right) = 0 \Rightarrow x\left(1 + 8x^3\right) = 0$ $\Rightarrow x = 0$ and $y = 0$, or $x = -\frac{1}{2}$ and $y = -\frac{1}{2} \Rightarrow$ the critical points are $(0, 0)$ and $\left(-\frac{1}{2}, -\frac{1}{2}\right)$. For $(0, 0)$: $f_{xx}(0, 0) = 12x|_{(0,0)} = 0$, $f_{yy}(0, 0) = 12y|_{(0,0)} = 0$, $f_{xy}(0, 0) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -9 < 0 \Rightarrow$ saddle point with $f(0, 0) = 0$. For $\left(-\frac{1}{2}, -\frac{1}{2}\right)$: $f_{xx} = -6$, $f_{yy} = -6$, $f_{xy} = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 27 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum value of $f\left(-\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{4}$

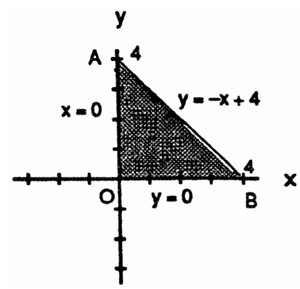
68. $f_x(x, y) = 3x^2 - 3y = 0$ and $f_y(x, y) = 3y^2 - 3x = 0 \Rightarrow y = x^2$ and $x^4 - x = 0 \Rightarrow x\left(x^3 - 1\right) = 0 \Rightarrow$ the critical points are $(0, 0)$ and $(1, 1)$. For $(0, 0)$: $f_{xx}(0, 0) = 6x|_{(0,0)} = 0$, $f_{yy}(0, 0) = 6y|_{(0,0)} = 0$, $f_{xy}(0, 0) = -3$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -9 < 0 \Rightarrow$ saddle point with $f(0, 0) = 15$. For $(1, 1)$: $f_{xx}(1, 1) = 6$, $f_{yy}(1, 1) = 6$, $f_{xy}(1, 1) = -3$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 27 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(1, 1) = 14$

69. $f_x(x, y) = 3x^2 + 6x = 0$ and $f_y(x, y) = 3y^2 - 6y = 0 \Rightarrow x(x + 2) = 0$ and $y(y - 2) = 0 \Rightarrow x = 0$ or $x = -2$ and $y = 0$ or $y = 2 \Rightarrow$ the critical points are $(0, 0)$, $(0, 2)$, $(-2, 0)$, and $(-2, 2)$. For $(0, 0)$: $f_{xx}(0, 0) = 6x + 6|_{(0,0)}$ $= 6$, $f_{yy}(0, 0) = 6y - 6|_{(0,0)} = -6$, $f_{xy}(0, 0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point with $f(0, 0) = 0$. For $(0, 2)$: $f_{xx}(0, 2) = 6$, $f_{yy}(0, 2) = 6$, $f_{xy}(0, 2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of

$f(0,2) = -4$. For $(-2,0)$: $f_{xx}(-2,0) = -6$, $f_{yy}(-2,0) = -6$, $f_{xy}(-2,0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0$ $\Rightarrow$ local maximum value of $f(-2,0) = 4$. For $(-2,2)$: $f_{xx}(-2,2) = -6$, $f_{yy}(-2,2) = 6$, $f_{xy}(-2,2) = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point with $f(-2,2) = 0$.

70. $f_x(x,y) = 4x^3 - 16x = 0 \Rightarrow 4x\,(x^2 - 4) = 0 \Rightarrow x = 0, 2, -2$; $f_y(x,y) = 6y - 6 = 0 \Rightarrow y = 1$. Therefore the critical points are $(0,1)$, $(2,1)$, and $(-2,1)$. For $(0,1)$: $f_{xx}(0,1) = 12x^2 - 16\big|_{(0,1)} = -16$, $f_{yy}(0,1) = 6$, $f_{xy}(0,1) = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -96 < 0 \Rightarrow$ saddle point with $f(0,1) = -3$. For $(2,1)$: $f_{xx}(2,1) = 32$, $f_{yy}(2,1) = 6$, $f_{xy}(2,1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 192 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(2,1) = -19$. For $(-2,1)$: $f_{xx}(-2,1) = 32$, $f_{yy}(-2,1) = 6$, $f_{xy}(-2,1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 192 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(-2,1) = -19$.

71. (i) On OA, $f(x,y) = f(0,y) = y^2 + 3y$ for $0 \le y \le 4$ $\Rightarrow f'(0,y) = 2y + 3 = 0 \Rightarrow y = -\frac{3}{2}$. But $\left(0, -\frac{3}{2}\right)$ is not in the region.
Endpoints: $f(0,0) = 0$ and $f(0,4) = 28$.

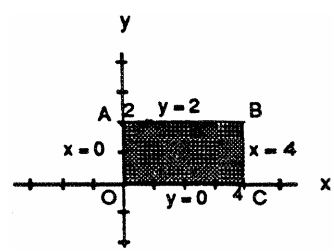


(ii) On AB, $f(x,y) = f(x, -x+4) = x^2 - 10x + 28$ for $0 \le x \le 4 \Rightarrow f'(x, -x+4) = 2x - 10 = 0$ $\Rightarrow x = 5, y = -1$. But $(5,-1)$ is not in the region.
Endpoints: $f(4,0) = 4$ and $f(0,4) = 28$.

(iii) On OB, $f(x,y) = f(x,0) = x^2 - 3x$ for $0 \le x \le 4 \Rightarrow f'(x,0) = 2x - 3 \Rightarrow x = \frac{3}{2}$ and $y = 0 \Rightarrow \left(\frac{3}{2}, 0\right)$ is a critical point with $f\left(\frac{3}{2}, 0\right) = -\frac{9}{4}$.
Endpoints: $f(0,0) = 0$ and $f(4,0) = 4$.

(iv) For the interior of the triangular region, $f_x(x,y) = 2x + y - 3 = 0$ and $f_y(x,y) = x + 2y + 3 = 0 \Rightarrow x = 3$ and $y = -3$. But $(3,-3)$ is not in the region. Therefore the absolute maximum is 28 at $(0,4)$ and the absolute minimum is $-\frac{9}{4}$ at $\left(\frac{3}{2}, 0\right)$.

72. (i) On OA, $f(x,y) = f(0,y) = -y^2 + 4y + 1$ for $0 \le y \le 2 \Rightarrow f'(0,y) = -2y + 4 = 0 \Rightarrow y = 2$ and $x = 0$. But $(0,2)$ is not in the interior of OA.
Endpoints: $f(0,0) = 1$ and $f(0,2) = 5$.



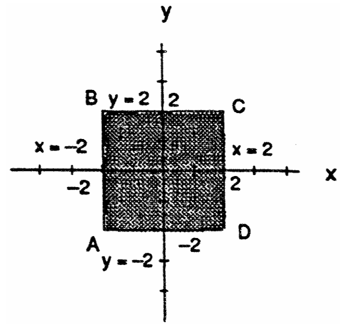
(ii) On AB, $f(x,y) = f(x,2) = x^2 - 2x + 5$ for $0 \le x \le 4$ $\Rightarrow f'(x,2) = 2x - 2 = 0 \Rightarrow x = 1$ and $y = 2$ $\Rightarrow (1,2)$ is an interior critical point of AB with $f(1,2) = 4$. Endpoints: $f(4,2) = 13$ and $f(0,2) = 5$.

(iii) On BC, $f(x,y) = f(4,y) = -y^2 + 4y + 9$ for $0 \le y \le 2 \Rightarrow f'(4,y) = -2y + 4 = 0 \Rightarrow y = 2$ and $x = 4$. But $(4,2)$ is not in the interior of BC. Endpoints: $f(4,0) = 9$ and $f(4,2) = 13$.

(iv) On OC, $f(x,y) = f(x,0) = x^2 - 2x + 1$ for $0 \le x \le 4 \Rightarrow f'(x,0) = 2x - 2 = 0 \Rightarrow x = 1$ and $y = 0 \Rightarrow (1,0)$ is an interior critical point of OC with $f(1,0) = 0$. Endpoints: $f(0,0) = 1$ and $f(4,0) = 9$.

(v) For the interior of the rectangular region, $f_x(x,y) = 2x - 2 = 0$ and $f_y(x,y) = -2y + 4 = 0 \Rightarrow x = 1$ and $y = 2$. But $(1,2)$ is not in the interior of the region. Therefore the absolute maximum is 13 at $(4,2)$ and the absolute minimum is 0 at $(1,0)$.

73. (i) On AB, $f(x, y) = f(-2, y) = y^2 - y - 4$ for $-2 \le y \le 2 \Rightarrow f'(-2, y) = 2y - 1 \Rightarrow y = \frac{1}{2}$ and $x = -2 \Rightarrow \left(-2, \frac{1}{2}\right)$ is an interior critical point in AB with $f\left(-2, \frac{1}{2}\right) = -\frac{17}{4}$. Endpoints: $f(-2, -2) = 2$ and $f(2, 2) = -2$.



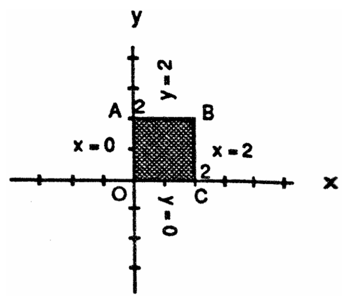
(ii) On BC, $f(x, y) = f(x, 2) = -2$ for $-2 \le x \le 2$ $\Rightarrow f'(x, 2) = 0 \Rightarrow$ no critical points in the interior of BC. Endpoints: $f(-2, 2) = -2$ and $f(2, 2) = -2$.

(iii) On CD, $f(x, y) = f(2, y) = y^2 - 5y + 4$ for $-2 \le y \le 2 \Rightarrow f'(2, y) = 2y - 5 = 0 \Rightarrow y = \frac{5}{2}$ and $x = 2$. But $\left(2, \frac{5}{2}\right)$ is not in the region. Endpoints: $f(2, -2) = 18$ and $f(2, 2) = -2$.

(iv) On AD, $f(x, y) = f(x, -2) = 4x + 10$ for $-2 \le x \le 2 \Rightarrow f'(x, -2) = 4 \Rightarrow$ no critical points in the interior of AD. Endpoints: $f(-2, -2) = 2$ and $f(2, -2) = 18$.

(v) For the interior of the square, $f_x(x, y) = -y + 2 = 0$ and $f_y(x, y) = 2y - x - 3 = 0 \Rightarrow y = 2$ and $x = 1$ $\Rightarrow (1, 2)$ is an interior critical point of the square with $f(1, 2) = -2$. Therefore the absolute maximum is 18 at $(2, -2)$ and the absolute minimum is $-\frac{17}{4}$ at $\left(-2, \frac{1}{2}\right)$.

74. (i) On OA, $f(x, y) = f(0, y) = 2y - y^2$ for $0 \le y \le 2$ $\Rightarrow f'(0, y) = 2 - 2y = 0 \Rightarrow y = 1$ and $x = 0 \Rightarrow$ $(0, 1)$ is an interior critical point of OA with $f(0, 1) = 1$. Endpoints: $f(0, 0) = 0$ and $f(0, 2) = 0$.



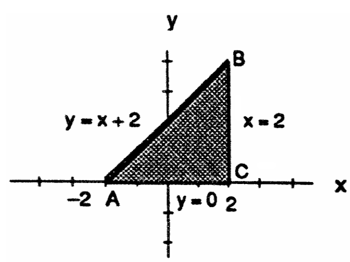
(ii) On AB, $f(x, y) = f(x, 2) = 2x - x^2$ for $0 \le x \le 2$ $\Rightarrow f'(x, 2) = 2 - 2x = 0 \Rightarrow x = 1$ and $y = 2$ $\Rightarrow (1, 2)$ is an interior critical point of AB with $f(1, 2) = 1$. Endpoints: $f(0, 2) = 0$ and $f(2, 2) = 0$.

(iii) On BC, $f(x, y) = f(2, y) = 2y - y^2$ for $0 \le y \le 2$ $\Rightarrow f'(2, y) = 2 - 2y = 0 \Rightarrow y = 1$ and $x = 2$ $\Rightarrow (2, 1)$ is an interior critical point of BC with $f(2, 1) = 1$. Endpoints: $f(2, 0) = 0$ and $f(2, 2) = 0$.

(iv) On OC, $f(x, y) = f(x, 0) = 2x - x^2$ for $0 \le x \le 2 \Rightarrow f'(x, 0) = 2 - 2x = 0 \Rightarrow x = 1$ and $y = 0 \Rightarrow (1, 0)$ is an interior critical point of OC with $f(1, 0) = 1$. Endpoints: $f(0, 0) = 0$ and $f(0, 2) = 0$.

(v) For the interior of the rectangular region, $f_x(x, y) = 2 - 2x = 0$ and $f_y(x, y) = 2 - 2y = 0 \Rightarrow x = 1$ and $y = 1 \Rightarrow (1, 1)$ is an interior critical point of the square with $f(1, 1) = 2$. Therefore the absolute maximum is 2 at $(1, 1)$ and the absolute minimum is 0 at the four corners $(0, 0)$, $(0, 2)$, $(2, 2)$, and $(2, 0)$.

75. (i) On AB, $f(x, y) = f(x, x + 2) = -2x + 4$ for $-2 \le x \le 2 \Rightarrow f'(x, x + 2) = -2 = 0 \Rightarrow$ no critical points in the interior of AB. Endpoints: $f(-2, 0) = 8$ and $f(2, 4) = 0$.

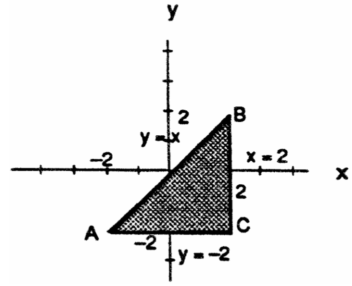


(ii) On BC, $f(x, y) = f(2, y) = -y^2 + 4y$ for $0 \le y \le 4$ $\Rightarrow f'(2, y) = -2y + 4 = 0 \Rightarrow y = 2$ and $x = 2$ $\Rightarrow (2, 2)$ is an interior critical point of BC with $f(2, 2) = 4$. Endpoints: $f(2, 0) = 0$ and $f(2, 4) = 0$.

(iii) On AC, $f(x, y) = f(x, 0) = x^2 - 2x$ for $-2 \le x \le 2$ $\Rightarrow f'(x, 0) = 2x - 2 \Rightarrow x = 1$ and $y = 0 \Rightarrow (1, 0)$ is an interior critical point of AC with $f(1, 0) = -1$. Endpoints: $f(-2, 0) = 8$ and $f(2, 0) = 0$.

(iv) For the interior of the triangular region, $f_x(x, y) = 2x - 2 = 0$ and $f_y(x, y) = -2y + 4 = 0 \Rightarrow x = 1$ and $y = 2 \Rightarrow (1, 2)$ is an interior critical point of the region with $f(1, 2) = 3$. Therefore the absolute maximum is 8 at $(-2, 0)$ and the absolute minimum is -1 at $(1, 0)$.

76. (i) On AB, $f(x, y) = f(x, x) = 4x^2 - 2x^4 + 16$ for $-2 \le x \le 2 \Rightarrow f'(x, x) = 8x - 8x^3 = 0 \Rightarrow x = 0$ and $y = 0$, or $x = 1$ and $y = 1$, or $x = -1$ and $y = -1$ $\Rightarrow (0, 0), (1, 1), (-1, -1)$ are all interior points of AB with $f(0, 0) = 16$, $f(1, 1) = 18$, and $f(-1, -1) = 18$. Endpoints: $f(-2, -2) = 0$ and $f(2, 2) = 0$.

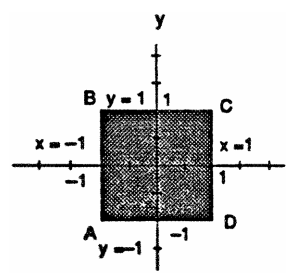


(ii) On BC, $f(x, y) = f(2, y) = 8y - y^4$ for $-2 \le y \le 2$ $\Rightarrow f'(2, y) = 8 - 4y^3 = 0 \Rightarrow y = \sqrt[3]{2}$ and $x = 2$ $\Rightarrow \left(2, \sqrt[3]{2}\right)$ is an interior critical point of BC with $f\left(2, \sqrt[3]{2}\right) = 6\sqrt[3]{2}$. Endpoints: $f(2, -2) = -32$ and $f(2, 2) = 0$.

(iii) On AC, $f(x, y) = f(x, -2) = -8x - x^4$ for $-2 \le x \le 2 \Rightarrow f'(x, -2) = -8 - 4x^3 = 0 \Rightarrow x = \sqrt[3]{-2}$ and $y = -2$ $\Rightarrow \left(\sqrt[3]{-2}, -2\right)$ is an interior critical point of AC with $f\left(\sqrt[3]{-2}, -2\right) = 6\sqrt[3]{2}$. Endpoints: $f(-2, -2) = 0$ and $f(2, -2) = -32$.

(iv) For the interior of the triangular region, $f_x(x, y) = 4y - 4x^3 = 0$ and $f_y(x, y) = 4x - 4y^3 = 0 \Rightarrow x = 0$ and $y = 0$, or $x = 1$ and $y = 1$ or $x = -1$ and $y = -1$. But neither of the points $(0, 0)$ and $(1, 1)$, or $(-1, -1)$ are interior to the region. Therefore the absolute maximum is 18 at $(1, 1)$ and $(-1, -1)$, and the absolute minimum is -32 at $(2, -2)$.

77. (i) On AB, $f(x, y) = f(-1, y) = y^3 - 3y^2 + 2$ for $-1 \le y \le 1 \Rightarrow f'(-1, y) = 3y^2 - 6y = 0 \Rightarrow y = 0$ and $x = -1$, or $y = 2$ and $x = -1 \Rightarrow (-1, 0)$ is an interior critical point of AB with $f(-1, 0) = 2$; $(-1, 2)$ is outside the boundary. Endpoints: $f(-1, -1) = -2$ and $f(-1, 1) = 0$.



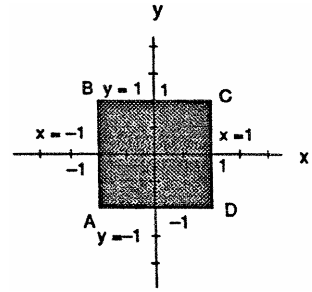
(ii) On BC, $f(x, y) = f(x, 1) = x^3 + 3x^2 - 2$ for $-1 \le x \le 1 \Rightarrow f'(x, 1) = 3x^2 + 6x = 0 \Rightarrow x = 0$ and $y = 1$, or $x = -2$ and $y = 1 \Rightarrow (0, 1)$ is an interior critical point of BC with $f(0, 1) = -2$; $(-2, 1)$ is outside the boundary. Endpoints: $f(-1, 1) = 0$ and $f(1, 1) = 2$.

(iii) On CD, $f(x, y) = f(1, y) = y^3 - 3y^2 + 4$ for $-1 \le y \le 1 \Rightarrow f'(1, y) = 3y^2 - 6y = 0 \Rightarrow y = 0$ and $x = 1$, or $y = 2$ and $x = 1 \Rightarrow (1, 0)$ is an interior critical point of CD with $f(1, 0) = 4$; $(1, 2)$ is outside the boundary. Endpoints: $f(1, 1) = 2$ and $f(1, -1) = 0$.

(iv) On AD, $f(x, y) = f(x, -1) = x^3 + 3x^2 - 4$ for $-1 \le x \le 1 \Rightarrow f'(x, -1) = 3x^2 + 6x = 0 \Rightarrow x = 0$ and $y = -1$, or $x = -2$ and $y = -1 \Rightarrow (0, -1)$ is an interior point of AD with $f(0, -1) = -4$; $(-2, -1)$ is outside the boundary. Endpoints: $f(-1, -1) = -2$ and $f(1, -1) = 0$.

(v) For the interior of the square, $f_x(x, y) = 3x^2 + 6x = 0$ and $f_y(x, y) = 3y^2 - 6y = 0 \Rightarrow x = 0$ or $x = -2$, and $y = 0$ or $y = 2 \Rightarrow (0, 0)$ is an interior critical point of the square region with $f(0, 0) = 0$; the points $(0, 2)$, $(-2, 0)$, and $(-2, 2)$ are outside the region. Therefore the absolute maximum is 4 at $(1, 0)$ and the absolute minimum is -4 at $(0, -1)$.

78. (i) On AB, $f(x, y) = f(-1, y) = y^3 - 3y$ for $-1 \le y \le 1$ $\Rightarrow f'(-1, y) = 3y^2 - 3 = 0 \Rightarrow y = \pm 1$ and $x = -1$ yielding the corner points $(-1, -1)$ and $(-1, 1)$ with $f(-1, -1) = 2$ and $f(-1, 1) = -2$.



(ii) On BC, $f(x, y) = f(x, 1) = x^3 + 3x + 2$ for $-1 \le x \le 1 \Rightarrow f'(x, 1) = 3x^2 + 3 = 0 \Rightarrow$ no solution. Endpoints: $f(-1, 1) = -2$ and $f(1, 1) = 6$.

(iii) On CD, $f(x, y) = f(1, y) = y^3 + 3y + 2$ for $-1 \le y \le 1 \Rightarrow f'(1, y) = 3y^2 + 3 = 0 \Rightarrow$ no solution. Endpoints: $f(1, 1) = 6$ and $f(1, -1) = -2$.

(iv) On AD, $f(x, y) = f(x, -1) = x^3 - 3x$ for $-1 \le x \le 1 \Rightarrow f'(x, -1) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$ and $y = -1$ yielding the corner points $(-1, -1)$ and $(1, -1)$ with $f(-1, -1) = 2$ and $f(1, -1) = -2$

(v) For the interior of the square, $f_x(x, y) = 3x^2 + 3y = 0$ and $f_y(x, y) = 3y^2 + 3x = 0 \Rightarrow y = -x^2$ and $x^4 + x = 0 \Rightarrow x = 0$ or $x = -1 \Rightarrow y = 0$ or $y = -1 \Rightarrow (0, 0)$ is an interior critical point of the square region with $f(0, 0) = 1$; $(-1, -1)$ is on the boundary. Therefore the absolute maximum is 6 at $(1, 1)$ and the absolute minimum is -2 at $(1, -1)$ and $(-1, 1)$.

79. $\nabla f = 3x^2\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 3x^2\mathbf{i} + 2y\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 3x^2 = 2x\lambda$ and $2y = 2y\lambda \Rightarrow \lambda = 1$ or $y = 0$.

CASE 1: $\lambda = 1 \Rightarrow 3x^2 = 2x \Rightarrow x = 0$ or $x = \frac{2}{3}$; $x = 0 \Rightarrow y = \pm 1$ yielding the points $(0, 1)$ and $(0, -1)$; $x = \frac{2}{3}$ $\Rightarrow y = \pm \frac{\sqrt{5}}{3}$ yielding the points $\left(\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$ and $\left(\frac{2}{3}, -\frac{\sqrt{5}}{3}\right)$.

CASE 2: $y = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$ yielding the points $(1, 0)$ and $(-1, 0)$.

Evaluations give $f(0, \pm 1) = 1$, $f\left(\frac{2}{3}, \pm \frac{\sqrt{5}}{3}\right) = \frac{23}{27}$, $f(1, 0) = 1$, and $f(-1, 0) = -1$. Therefore the absolute maximum is 1 at $(0, \pm 1)$ and $(1, 0)$, and the absolute minimum is -1 at $(-1, 0)$.

80. $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow y = 2\lambda x$ and $xy = 2\lambda y \Rightarrow x = 2\lambda(2\lambda x) = 4\lambda^2 x \Rightarrow x = 0$ or $4\lambda^2 = 1$.

CASE 1: $x = 0 \Rightarrow y = 0$ but $(0, 0)$ does not lie on the circle, so no solution.

CASE 2: $4\lambda^2 = 1 \Rightarrow \lambda = \frac{1}{2}$ or $\lambda = -\frac{1}{2}$. For $\lambda = \frac{1}{2}$, $y = x \Rightarrow 1 = x^2 + y^2 = 2x^2 \Rightarrow x = y = \pm \frac{1}{\sqrt{2}}$ yielding the points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. For $\lambda = -\frac{1}{2}$, $y = -x \Rightarrow 1 = x^2 + y^2 = 2x^2 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$ and $y = -x$ yielding the points $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Evaluations give the absolute maximum value $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$ and the absolute minimum value $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$.

81. (i) $f(x, y) = x^2 + 3y^2 + 2y$ on $x^2 + y^2 = 1 \Rightarrow \nabla f = 2x\mathbf{i} + (6y + 2)\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g$ $\Rightarrow 2x\mathbf{i} + (6y + 2)\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 2x = 2x\lambda$ and $6y + 2 = 2y\lambda \Rightarrow \lambda = 1$ or $x = 0$.

CASE 1: $\lambda = 1 \Rightarrow 6y + 2 = 2y \Rightarrow y = -\frac{1}{2}$ and $x = \pm \frac{\sqrt{3}}{2}$ yielding the points $\left(\pm \frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

CASE 2: $x = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$ yielding the points $(0, \pm 1)$.

Evaluations give $f\left(\pm \frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{1}{2}$, $f(0, 1) = 5$, and $f(0, -1) = 1$. Therefore $\frac{1}{2}$ and 5 are the extreme values on the boundary of the disk.

(ii) For the interior of the disk, $f_x(x, y) = 2x = 0$ and $f_y(x, y) = 6y + 2 = 0 \Rightarrow x = 0$ and $y = -\frac{1}{3}$ $\Rightarrow \left(0, -\frac{1}{3}\right)$ is an interior critical point with $f\left(0, -\frac{1}{3}\right) = -\frac{1}{3}$. Therefore the absolute maximum of f on the disk is 5 at $(0, 1)$ and the absolute minimum of f on the disk is $-\frac{1}{3}$ at $\left(0, -\frac{1}{3}\right)$.

82. (i) $f(x, y) = x^2 + y^2 - 3x - xy$ on $x^2 + y^2 = 9 \Rightarrow \nabla f = (2x - 3 - y)\mathbf{i} + (2y - x)\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow (2x - 3 - y)\mathbf{i} + (2y - x)\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 2x - 3 - y = 2x\lambda$ and $2y - x = 2y\lambda$ $\Rightarrow 2x(1 - \lambda) - y = 3$ and $-x + 2y(1 - \lambda) = 0 \Rightarrow 1 - \lambda = \frac{x}{2y}$ and $(2x)\left(\frac{x}{2y}\right) - y = 3 \Rightarrow x^2 - y^2 = 3y$ $\Rightarrow x^2 = y^2 + 3y$. Thus, $9 = x^2 + y^2 = y^2 + 3y + y^2 \Rightarrow 2y^2 + 3y - 9 = 0 \Rightarrow (2y - 3)(y + 3) = 0$ $\Rightarrow y = -3, \frac{3}{2}$. For $y = -3$, $x^2 + y^2 = 9 \Rightarrow x = 0$ yielding the point $(0, -3)$. For $y = \frac{3}{2}$, $x^2 + y^2 = 9$ $\Rightarrow x^2 + \frac{9}{4} = 9 \Rightarrow x^2 = \frac{27}{4} \Rightarrow x = \pm\frac{3\sqrt{3}}{2}$. Evaluations give $f(0, -3) = 9$, $f\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) = 9 + \frac{27\sqrt{3}}{4}$ ≈ 20.691, and $f\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) = 9 - \frac{27\sqrt{3}}{4} \approx -2.691$.

(ii) For the interior of the disk, $f_x(x, y) = 2x - 3 - y = 0$ and $f_y(x, y) = 2y - x = 0 \Rightarrow x = 2$ and $y = 1$ $\Rightarrow (2, 1)$ is an interior critical point of the disk with $f(2, 1) = -3$. Therefore, the absolute maximum of f on the disk is $9 + \frac{27\sqrt{3}}{4}$ at $\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ and the absolute minimum of f on the disk is -3 at $(2, 1)$.

83. $\nabla f = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} - \mathbf{j} + \mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 1 = 2x\lambda$, $-1 = 2y\lambda$, $1 = 2z\lambda \Rightarrow x = -y = z = \frac{1}{\lambda}$. Thus $x^2 + y^2 + z^2 = 1 \Rightarrow 3x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$ yielding the points $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$. Evaluations give the absolute maximum value of $f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{3}{\sqrt{3}} = \sqrt{3}$ and the absolute minimum value of $f\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = -\sqrt{3}$.

84. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance to the origin and $g(x, y, z) = x^2 - zy - 4$. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = 2x\mathbf{i} - z\mathbf{j} - y\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = 2\lambda x$, $2y = -\lambda z$, and $2z = -\lambda y \Rightarrow x = 0$ or $\lambda = 1$.

CASE 1: $x = 0 \Rightarrow zy = -4 \Rightarrow z = -\frac{4}{y}$ and $y = -\frac{4}{z} \Rightarrow 2\left(-\frac{4}{y}\right) = -\lambda y$ and $2\left(-\frac{4}{z}\right) = -\lambda z \Rightarrow \frac{8}{\lambda} = y^2$ and $\frac{8}{\lambda} = z^2 \Rightarrow y^2 = z^2 \Rightarrow y = \pm z$. But $y = x \Rightarrow z^2 = -4$ leads to no solution, so $y = -z \Rightarrow z^2 = 4$ $\Rightarrow z = \pm 2$ yielding the points $(0, -2, 2)$ and $(0, 2, -2)$.

CASE 2: $\lambda = 1 \Rightarrow 2z = -y$ and $2y = -z \Rightarrow 2y = -\left(-\frac{y}{2}\right) \Rightarrow 4y = y \Rightarrow y = 0 \Rightarrow z = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow$ $x = \pm 2$ yielding the points $(-2, 0, 0)$ and $(2, 0, 0)$.

Evaluations give $f(0, -2, 2) = f(0, 2, -2) = 8$ and $f(-2, 0, 0) = f(2, 0, 0) = 4$. Thus the points $(-2, 0, 0)$ and $(2, 0, 0)$ on the surface are closest to the origin.

85. The cost is $f(x, y, z) = 2axy + 2bxz + 2cyz$ subject to the constraint $xyz = V$. Then $\nabla f = \lambda \nabla g$ $\Rightarrow 2ay + 2bz = \lambda yz$, $2ax + 2cz = \lambda xz$, and $2bx + 2cy = \lambda xy \Rightarrow 2axy + 2bxz = \lambda xyz$, $2axy + 2cyz = \lambda xyz$, and $2bxz + 2cyz = \lambda xyz \Rightarrow 2axy + 2bxz = 2axy + 2cyz \Rightarrow y = \left(\frac{b}{c}\right)x$. Also $2axy + 2bxz = 2bxz + 2cyz \Rightarrow z = \left(\frac{a}{c}\right)x$. Then $x\left(\frac{b}{c}x\right)\left(\frac{a}{c}x\right) = V \Rightarrow x^3 = \frac{c^2V}{ab} \Rightarrow$ width $= x = \left(\frac{c^2V}{ab}\right)^{1/3}$, Depth $= y = \left(\frac{b}{c}\right)\left(\frac{c^2V}{ab}\right)^{1/3} = \left(\frac{b^2V}{ac}\right)^{1/3}$, and Height $= z = \left(\frac{a}{c}\right)\left(\frac{c^2V}{ab}\right)^{1/3} = \left(\frac{a^2V}{bc}\right)^{1/3}$.

86. The volume of the pyramid in the first octant formed by the plane is $V(a, b, c) = \frac{1}{3}\left(\frac{1}{2}ab\right)c = \frac{1}{6}abc$. The point $(2, 1, 2)$ on the plane $\Rightarrow \frac{2}{a} + \frac{1}{b} + \frac{2}{c} = 1$. We want to minimize V subject to the constraint $2bc + ac + 2ab = abc$. Thus, $\nabla V = \frac{bc}{6}\mathbf{i} + \frac{ac}{6}\mathbf{j} + \frac{ab}{6}\mathbf{k}$ and $\nabla g = (c + 2b - bc)\mathbf{i} + (2c + 2a - ac)\mathbf{j} + (2b + a - ab)\mathbf{k}$ so that $\nabla V = \lambda \nabla g$ $\Rightarrow \frac{bc}{6} = \lambda(c + 2b - bc)$, $\frac{ac}{6} = \lambda(2c + 2a - ac)$, and $\frac{ab}{6} = \lambda(2b + a - ab) \Rightarrow \frac{abc}{6} = \lambda(ac + 2ab - abc)$, $\frac{abc}{6} = \lambda(2bc + 2ab - abc)$, and $\frac{abc}{6} = \lambda(2bc + ac - abc) \Rightarrow \lambda ac = 2\lambda bc$ and $2\lambda ab = 2\lambda bc$. Now $\lambda \neq 0$ since $a \neq 0$, $b \neq 0$, and $c \neq 0 \Rightarrow ac = 2bc$ and $ab = bc \Rightarrow a = 2b = c$. Substituting into the constraint equation gives $\frac{2}{a} + \frac{2}{a} + \frac{2}{a} = 1 \Rightarrow a = 6 \Rightarrow b = 3$ and $c = 6$. Therefore the desired plane is $\frac{x}{6} + \frac{y}{3} + \frac{z}{6} = 1$ or $x + 2y + z = 6$.

87. $\nabla f = (y+z)\mathbf{i} + x\mathbf{j} + x\mathbf{k}$, $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$, and $\nabla h = z\mathbf{i} + x\mathbf{k}$ so that $\nabla f = \lambda \nabla g + \mu \nabla h$ $\Rightarrow (y+z)\mathbf{i} + x\mathbf{j} + x\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) + \mu(z\mathbf{i} + x\mathbf{k}) \Rightarrow y + z = 2\lambda x + \mu z$, $x = 2\lambda y$, $x = \mu x \Rightarrow x = 0$ or $\mu = 1$.

CASE 1: $x = 0$ which is impossible since $xz = 1$.

CASE 2: $\mu = 1 \Rightarrow y + z = 2\lambda x + z \Rightarrow y = 2\lambda x$ and $x = 2\lambda y \Rightarrow y = (2\lambda)(2\lambda y) \Rightarrow y = 0$ or $4\lambda^2 = 1$. If $y = 0$, then $x^2 = 1 \Rightarrow x = \pm 1$ so with $xz = 1$ we obtain the points $(1, 0, 1)$ and $(-1, 0, -1)$. If $4\lambda^2 = 1$, then $\lambda = \pm\frac{1}{2}$. For $\lambda = -\frac{1}{2}$, $y = -x$ so $x^2 + y^2 = 1 \Rightarrow x^2 = \frac{1}{2}$ $\Rightarrow x = \pm\frac{1}{\sqrt{2}}$ with $xz = 1 \Rightarrow z = \pm\sqrt{2}$, and we obtain the points $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}\right)$. For $\lambda = \frac{1}{2}$, $y = x \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$ with $xz = 1 \Rightarrow z = \pm\sqrt{2}$, and we obtain the points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$.

Evaluations give $f(1,0,1) = 1$, $f(-1,0,-1) = 1$, $f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2}\right) = \frac{1}{2}$, $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}\right) = \frac{1}{2}$, $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right) = \frac{3}{2}$, and $f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right) = \frac{3}{2}$. Therefore the absolute maximum is $\frac{3}{2}$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$, and the absolute minimum is $\frac{1}{2}$ at $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2}\right)$.

88. Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\nabla g = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\nabla h = 4x\mathbf{i} + 4y\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g + \mu \nabla h \Rightarrow 2x = \lambda + 4x\mu$, $2y = \lambda + 4y\mu$, and $2z = \lambda - 2z\mu \Rightarrow \lambda = 2x(1 - 2\mu) = 2y(1 - 2\mu) = 2z(1 + 2\mu) \Rightarrow x = y$ or $\mu = \frac{1}{2}$.

CASE 1: $x = y \Rightarrow z^2 = 4x^2 \Rightarrow z = \pm 2x$ so that $x + y + z = 1 \Rightarrow x + x + 2x = 1$ or $x + x - 2x = 1$ (impossible) $\Rightarrow x = \frac{1}{4} \Rightarrow y = \frac{1}{4}$ and $z = \frac{1}{2}$ yielding the point $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$.

CASE 2: $\mu = \frac{1}{2} \Rightarrow \lambda = 0 \Rightarrow 0 = 2z(1+1) \Rightarrow z = 0$ so that $2x^2 + 2y^2 = 0 \Rightarrow x = y = 0$. But the origin $(0,0,0)$ fails to satisfy the first constraint $x + y + z = 1$.

Therefore, the point $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ on the curve of intersection is closest to the origin.

89. (a) y, z are independent with $w = x^2 e^{yz}$ and $z = x^2 - y^2 \Rightarrow \frac{\partial w}{\partial y} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$ $= (2xe^{yz})\frac{\partial x}{\partial y} + (zx^2e^{yz})(1) + (yx^2e^{yz})(0)$; $z = x^2 - y^2 \Rightarrow 0 = 2x\frac{\partial x}{\partial y} - 2y \Rightarrow \frac{\partial x}{\partial y} = \frac{y}{x}$; therefore, $\left(\frac{\partial w}{\partial y}\right)_z = (2xe^{yz})\left(\frac{y}{x}\right) + zx^2e^{yz} = (2y + zx^2)e^{yz}$

(b) z, x are independent with $w = x^2e^{yz}$ and $z = x^2 - y^2 \Rightarrow \frac{\partial w}{\partial z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z}$ $= (2xe^{yz})(0) + (zx^2e^{yz})\frac{\partial y}{\partial z} + (yx^2e^{yz})(1)$; $z = x^2 - y^2 \Rightarrow 1 = 0 - 2y\frac{\partial y}{\partial z} \Rightarrow \frac{\partial y}{\partial z} = -\frac{1}{2y}$; therefore, $\left(\frac{\partial w}{\partial z}\right)_x = (zx^2e^{yz})\left(-\frac{1}{2y}\right) + yx^2e^{yz} = x^2e^{yz}\left(y - \frac{z}{2y}\right)$

(c) z, y are independent with $w = x^2e^{yz}$ and $z = x^2 - y^2 \Rightarrow \frac{\partial w}{\partial z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z}$ $= (2xe^{yz})\frac{\partial x}{\partial z} + (zx^2e^{yz})(0) + (yx^2e^{yz})(1)$; $z = x^2 - y^2 \Rightarrow 1 = 2x\frac{\partial x}{\partial z} - 0 \Rightarrow \frac{\partial x}{\partial z} = \frac{1}{2x}$; therefore, $\left(\frac{\partial w}{\partial z}\right)_y = (2xe^{yz})\left(\frac{1}{2x}\right) + yx^2e^{yz} = (1 + x^2y)e^{yz}$

90. (a) T, P are independent with $U = f(P, V, T)$ and $PV = nRT \Rightarrow \frac{\partial U}{\partial T} = \frac{\partial U}{\partial P}\frac{\partial P}{\partial T} + \frac{\partial U}{\partial V}\frac{\partial V}{\partial T} + \frac{\partial U}{\partial T}\frac{\partial T}{\partial T}$ $= \left(\frac{\partial U}{\partial P}\right)(0) + \left(\frac{\partial U}{\partial V}\right)\left(\frac{\partial V}{\partial T}\right) + \left(\frac{\partial U}{\partial T}\right)(1)$; $PV = nRT \Rightarrow P\frac{\partial V}{\partial T} = nR \Rightarrow \frac{\partial V}{\partial T} = \frac{nR}{P}$; therefore, $\left(\frac{\partial U}{\partial T}\right)_P = \left(\frac{\partial U}{\partial V}\right)\left(\frac{nR}{P}\right) + \frac{\partial U}{\partial T}$

(b) V, T are independent with $U = f(P, V, T)$ and $PV = nRT \Rightarrow \frac{\partial U}{\partial V} = \frac{\partial U}{\partial P}\frac{\partial P}{\partial V} + \frac{\partial U}{\partial V}\frac{\partial V}{\partial V} + \frac{\partial U}{\partial T}\frac{\partial T}{\partial V}$ $= \left(\frac{\partial U}{\partial P}\right)\left(\frac{\partial P}{\partial V}\right) + \left(\frac{\partial U}{\partial V}\right)(1) + \left(\frac{\partial U}{\partial T}\right)(0)$; $PV = nRT \Rightarrow V\frac{\partial P}{\partial V} + P = (nR)\left(\frac{\partial T}{\partial V}\right) = 0 \Rightarrow \frac{\partial P}{\partial V} = -\frac{P}{V}$; therefore, $\left(\frac{\partial U}{\partial V}\right)_T = \left(\frac{\partial U}{\partial P}\right)\left(-\frac{P}{V}\right) + \frac{\partial U}{\partial V}$

91. Note that $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. Thus,
$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x} = \left(\frac{\partial w}{\partial r}\right)\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \left(\frac{\partial w}{\partial \theta}\right)\left(\frac{-y}{x^2+y^2}\right) = (\cos\theta)\frac{\partial w}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial w}{\partial \theta}$;
$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\theta y} = \left(\frac{\partial w}{\partial r}\right)\left(\frac{y}{\sqrt{x^2+y^2}}\right) + \left(\frac{\partial w}{\partial \theta}\right)\left(\frac{x}{x^2+y^2}\right) = (\sin\theta)\frac{\partial w}{\partial r} + \left(\frac{\cos\theta}{r}\right)\frac{\partial w}{\partial \theta}$

92. $z_x = f_u\frac{\partial u}{\partial x} + f_v\frac{\partial v}{\partial x} = af_u + af_v$, and $z_y = f_u\frac{\partial u}{\partial y} + f_v\frac{\partial v}{\partial y} = bf_u - bf_v$

93. $\frac{\partial u}{\partial y} = b$ and $\frac{\partial u}{\partial x} = a \Rightarrow \frac{\partial w}{\partial x} = \frac{dw}{du}\frac{\partial u}{\partial x} = a\frac{dw}{du}$ and $\frac{\partial w}{\partial y} = \frac{dw}{du}\frac{\partial u}{\partial y} = b\frac{dw}{du} \Rightarrow \frac{1}{a}\frac{\partial w}{\partial x} = \frac{dw}{du}$ and $\frac{1}{b}\frac{\partial w}{\partial y} = \frac{dw}{du}$
$\Rightarrow \frac{1}{a}\frac{\partial w}{\partial x} = \frac{1}{b}\frac{\partial w}{\partial y} \Rightarrow b\frac{\partial w}{\partial x} = a\frac{\partial w}{\partial y}$

94. $\frac{\partial w}{\partial x} = \frac{2x}{x^2+y^2+2z} = \frac{2(r+s)}{(r+s)^2+(r-s)^2+4rs} = \frac{2(r+s)}{2(r^2+2rs+s^2)} = \frac{1}{r+s}$, $\frac{\partial w}{\partial y} = \frac{2y}{x^2+y^2+2z} = \frac{2(r-s)}{2(r+s)^2} = \frac{r-s}{(r+s)^2}$,
and $\frac{\partial w}{\partial z} = \frac{2}{x^2+y^2+2z} = \frac{1}{(r+s)^2} \Rightarrow \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = \frac{1}{r+s} + \frac{r-s}{(r+s)^2} + \left[\frac{1}{(r+s)^2}\right](2s) = \frac{2r+2s}{(r+s)^2}$
$= \frac{2}{r+s}$ and $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = \frac{1}{r+s} - \frac{r-s}{(r+s)^2} + \left[\frac{1}{(r+s)^2}\right](2r) = \frac{2}{r+s}$

95. $e^u\cos v - x = 0 \Rightarrow (e^u\cos v)\frac{\partial u}{\partial x} - (e^u\sin v)\frac{\partial v}{\partial x} = 1$; $e^u\sin v - y = 0 \Rightarrow (e^u\sin v)\frac{\partial u}{\partial x} + (e^u\cos v)\frac{\partial v}{\partial x} = 0$.
Solving this system yields $\frac{\partial u}{\partial x} = e^{-u}\cos v$ and $\frac{\partial v}{\partial x} = -e^{-u}\sin v$. Similarly, $e^u\cos v - x = 0$
$\Rightarrow (e^u\cos v)\frac{\partial u}{\partial y} - (e^u\sin v)\frac{\partial v}{\partial y} = 0$ and $e^u\sin v - y = 0 \Rightarrow (e^u\sin v)\frac{\partial u}{\partial y} + (e^u\cos v)\frac{\partial v}{\partial y} = 1$. Solving this
second system yields $\frac{\partial u}{\partial y} = e^{-u}\sin v$ and $\frac{\partial v}{\partial y} = e^{-u}\cos v$. Therefore $\left(\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j}\right)\cdot\left(\frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j}\right)$
$= [(e^{-u}\cos v)\mathbf{i} + (e^{-u}\sin v)\mathbf{j}]\cdot[(-e^{-u}\sin v)\mathbf{i} + (e^{-u}\cos v)\mathbf{j}] = 0 \Rightarrow$ the vectors are orthogonal $\Rightarrow$ the angle between the vectors is the constant $\frac{\pi}{2}$.

96. $\frac{\partial g}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = (-r\sin\theta)\frac{\partial f}{\partial x} + (r\cos\theta)\frac{\partial f}{\partial y}$
$\Rightarrow \frac{\partial^2 g}{\partial \theta^2} = (-r\sin\theta)\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial \theta}\right) - (r\cos\theta)\frac{\partial f}{\partial x} + (r\cos\theta)\left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial \theta}\right) - (r\sin\theta)\frac{\partial f}{\partial y}$
$= (-r\sin\theta)\left(\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\right) - (r\cos\theta) + (r\cos\theta)\left(\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\right) - (r\sin\theta)$
$= (-r\sin\theta + r\cos\theta)(-r\sin\theta + r\cos\theta) - (r\cos\theta + r\sin\theta) = (-2)(-2) - (0+2) = 4 - 2 = 2$ at
$(r,\theta) = \left(2, \frac{\pi}{2}\right)$.

97. $(y+z)^2 + (z-x)^2 = 16 \Rightarrow \nabla f = -2(z-x)\mathbf{i} + 2(y+z)\mathbf{j} + 2(y+2z-x)\mathbf{k}$; if the normal line is parallel to the yz-plane, then x is constant $\Rightarrow \frac{\partial f}{\partial x} = 0 \Rightarrow -2(z-x) = 0 \Rightarrow z = x \Rightarrow (y+z)^2 + (z-z)^2 = 16 \Rightarrow y+z = \pm 4$.
Let $x = t \Rightarrow z = t \Rightarrow y = -t \pm 4$. Therefore the points are $(t, -t \pm 4, t)$, t a real number.

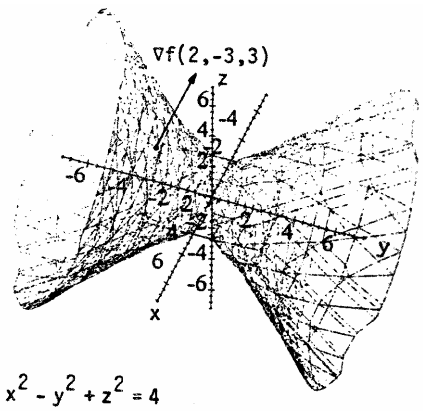
98. Let $f(x,y,z) = xy + yz + zx - x - z^2 = 0$. If the tangent plane is to be parallel to the xy-plane, then ∇f is perpendicular to the xy-plane $\Rightarrow \nabla f\cdot\mathbf{i} = 0$ and $\nabla f\cdot\mathbf{j} = 0$. Now $\nabla f = (y+z-1)\mathbf{i} + (x+z)\mathbf{j} + (y+x-2z)\mathbf{k}$ so that $\nabla f\cdot\mathbf{i} = y+z-1 = 0 \Rightarrow y+z = 1 \Rightarrow y = 1-z$, and $\nabla f\cdot\mathbf{j} = x+z = 0 \Rightarrow x = -z$. Then
$-z(1-z) + (1-z)z + z(-z) - (-z) - z^2 = 0 \Rightarrow z - 2z^2 = 0 \Rightarrow z = \frac{1}{2}$ or $z = 0$. Now $z = \frac{1}{2} \Rightarrow x = -\frac{1}{2}$ and $y = \frac{1}{2}$
$\Rightarrow \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is one desired point; $z = 0 \Rightarrow x = 0$ and $y = 1 \Rightarrow (0,1,0)$ is a second desired point.

99. $\nabla f = \lambda(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \Rightarrow \frac{\partial f}{\partial x} = \lambda x \Rightarrow f(x,y,z) = \frac{1}{2}\lambda x^2 + g(y,z)$ for some function g $\Rightarrow \lambda y = \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}$
$\Rightarrow g(y,z) = \frac{1}{2}\lambda y^2 + h(z)$ for some function h $\Rightarrow \lambda z = \frac{\partial f}{\partial z} = \frac{\partial g}{\partial z} = h'(z) \Rightarrow h(z) = \frac{1}{2}\lambda z^2 + C$ for some arbitrary constant C $\Rightarrow g(y,z) = \frac{1}{2}\lambda y^2 + \left(\frac{1}{2}\lambda z^2 + C\right) \Rightarrow f(x,y,z) = \frac{1}{2}\lambda x^2 + \frac{1}{2}\lambda y^2 + \frac{1}{2}\lambda z^2 + C \Rightarrow f(0,0,a) = \frac{1}{2}\lambda a^2 + C$
and $f(0,0,-a) = \frac{1}{2}\lambda(-a)^2 + C \Rightarrow f(0,0,a) = f(0,0,-a)$ for any constant a, as claimed.

100. $\left(\frac{df}{ds}\right)_{\mathbf{u},(0,0,0)} = \lim_{s \to 0} \frac{f(0+su_1, 0+su_2, 0+su_3) - f(0,0,0)}{s}, \; s > 0$
$= \lim_{s \to 0} \frac{\sqrt{s^2u_1^2 + s^2u_2^2 + s^2u_3^2} - 0}{s}, \; s > 0$
$= \lim_{s \to 0} \frac{s\sqrt{u_1^2 + u_2^2 + u_3^2}}{s} = \lim_{s \to 0} |\mathbf{u}| = 1;$
however, $\nabla f = \frac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{k}$ fails to exist at the origin $(0,0,0)$

101. Let $f(x,y,z) = xy + z - 2 \Rightarrow \nabla f = y\mathbf{i} + x\mathbf{j} + \mathbf{k}$. At $(1,1,1)$, we have $\nabla f = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow$ the normal line is $x = 1 + t, y = 1 + t, z = 1 + t$, so at $t = -1 \Rightarrow x = 0, y = 0, z = 0$ and the normal line passes through the origin.

102. (b) $f(x,y,z) = x^2 - y^2 + z^2 = 4$
$\Rightarrow \nabla f = 2x\mathbf{i} - 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow$ at $(2,-3,3)$
the gradient is $\nabla f = 4\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$ which is normal to the surface
(c) Tangent plane: $4x + 6y + 6z = 8$ or
$2x + 3y + 3z = 4$
Normal line: $x = 2 + 4t, y = -3 + 6t, z = 3 + 6t$



$x^2 - y^2 + z^2 = 4$

CHAPTER 14 ADDITIONAL AND ADVANCED EXERCISES

1. By definition, $f_{xy}(0,0) = \lim_{h \to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$ so we need to calculate the first partial derivatives in the numerator. For $(x,y) \neq (0,0)$ we calculate $f_x(x,y)$ by applying the differentiation rules to the formula for $f(x,y)$: $f_x(x,y) = \frac{x^2y - y^3}{x^2+y^2} + (xy)\frac{(x^2+y^2)(2x) - (x^2-y^2)(2x)}{(x^2+y^2)^2} = \frac{x^2y - y^3}{x^2+y^2} + \frac{4x^2y^3}{(x^2+y^2)^2} \Rightarrow f_x(0,h) = -\frac{h^3}{h^2} = -h$.
For $(x,y) = (0,0)$ we apply the definition: $f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$. Then by definition $f_{xy}(0,0) = \lim_{h \to 0} \frac{-h-0}{h} = -1$. Similarly, $f_{yx}(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$, so for $(x,y) \neq (0,0)$ we have $f_y(x,y) = \frac{x^3 - xy^2}{x^2+y^2} - \frac{4x^3y^2}{(x^2+y^2)^2} \Rightarrow f_y(h,0) = \frac{h^3}{h^2} = h$; for $(x,y) = (0,0)$ we obtain $f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h}$ $= \lim_{h \to 0} \frac{0-0}{h} = 0$. Then by definition $f_{yx}(0,0) = \lim_{h \to 0} \frac{h-0}{h} = 1$. Note that $f_{xy}(0,0) \neq f_{yx}(0,0)$ in this case.

2. $\frac{\partial w}{\partial x} = 1 + e^x \cos y \Rightarrow w = x + e^x \cos y + g(y)$; $\frac{\partial w}{\partial y} = -e^x \sin y + g'(y) = 2y - e^x \sin y \Rightarrow g'(y) = 2y$ $\Rightarrow g(y) = y^2 + C$; $w = \ln 2$ when $x = \ln 2$ and $y = 0 \Rightarrow \ln 2 = \ln 2 + e^{\ln 2}\cos 0 + 0^2 + C \Rightarrow 0 = 2 + C$ $\Rightarrow C = -2$. Thus, $w = x + e^x \cos y + g(y) = x + e^x \cos y + y^2 - 2$.

3. Substitution of $u + u(x)$ and $v = v(x)$ in $g(u,v)$ gives $g(u(x), v(x))$ which is a function of the independent variable x. Then, $g(u,v) = \int_u^v f(t)\,dt \Rightarrow \frac{dg}{dx} = \frac{\partial g}{\partial u}\frac{du}{dx} + \frac{\partial g}{\partial v}\frac{dv}{dx} = \left(\frac{\partial}{\partial u}\int_u^v f(t)\,dt\right)\frac{du}{dx} + \left(\frac{\partial}{\partial v}\int_u^v f(t)\,dt\right)\frac{dv}{dx}$ $= \left(-\frac{\partial}{\partial u}\int_v^u f(t)\,dt\right)\frac{du}{dx} + \left(\frac{\partial}{\partial v}\int_u^v f(t)\,dt\right)\frac{dv}{dx} = -f(u(x))\frac{du}{dx} + f(v(x))\frac{dv}{dx} = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}$

4. Applying the chain rules, $f_x = \frac{df}{dr}\frac{\partial r}{\partial x} \Rightarrow f_{xx} = \left(\frac{d^2f}{dr^2}\right)\left(\frac{\partial r}{\partial x}\right)^2 + \frac{df}{dr}\frac{\partial^2 r}{\partial x^2}$. Similarly, $f_{yy} = \left(\frac{d^2f}{dr^2}\right)\left(\frac{\partial r}{\partial y}\right)^2 + \frac{df}{dr}\frac{\partial^2 r}{\partial y^2}$ and $f_{zz} = \left(\frac{d^2f}{dr^2}\right)\left(\frac{\partial r}{\partial z}\right)^2 + \frac{df}{dr}\frac{\partial^2 r}{\partial z^2}$. Moreover, $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} \Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{y^2+z^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$; $\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}$ $\Rightarrow \frac{\partial^2 r}{\partial y^2} = \frac{x^2+z^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$; and $\frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}} \Rightarrow \frac{\partial^2 r}{\partial z^2} = \frac{x^2+y^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$. Next, $f_{xx} + f_{yy} + f_{zz} = 0$

$\Rightarrow \left(\frac{d^2f}{dr^2}\right)\left(\frac{x^2}{x^2+y^2+z^2}\right) + \left(\frac{df}{dr}\right)\left(\frac{y^2+z^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right) + \left(\frac{d^2f}{dr^2}\right)\left(\frac{y^2}{x^2+y^2+z^2}\right) + \left(\frac{df}{dr}\right)\left(\frac{x^2+z^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right)$

$+ \left(\frac{d^2f}{dr^2}\right)\left(\frac{z^2}{x^2+y^2+z^2}\right) + \left(\frac{df}{dr}\right)\left(\frac{x^2+y^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right) = 0 \Rightarrow \frac{d^2f}{dr^2} + \left(\frac{2}{\sqrt{x^2+y^2+z^2}}\right)\frac{df}{dr} = 0 \Rightarrow \frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr} = 0$

$\Rightarrow \frac{d}{dr}(f') = \left(-\frac{2}{r}\right)f'$, where $f' = \frac{df}{dr} \Rightarrow \frac{df'}{f'} = -\frac{2\,dr}{r} \Rightarrow \ln f' = -2\ln r + \ln C \Rightarrow f' = Cr^{-2}$, or $\frac{df}{dr} = Cr^{-2} \Rightarrow f(r) = -\frac{C}{r} + b = \frac{a}{r} + b$ for some constants a and b (setting $a = -C$)

5. (a) Let $u = tx$, $v = ty$, and $w = f(u, v) = f(u(t, x), v(t, y)) = f(tx, ty) = t^n f(x, y)$, where t, x, and y are independent variables. Then $nt^{n-1}f(x, y) = \frac{\partial w}{\partial t} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial t} = x\frac{\partial w}{\partial u} + y\frac{\partial w}{\partial v}$. Now,
$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = \left(\frac{\partial w}{\partial u}\right)(t) + \left(\frac{\partial w}{\partial v}\right)(0) = t\frac{\partial w}{\partial u} \Rightarrow \frac{\partial w}{\partial u} = \left(\frac{1}{t}\right)\left(\frac{\partial w}{\partial x}\right)$. Likewise,
$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = \left(\frac{\partial w}{\partial u}\right)(0) + \left(\frac{\partial w}{\partial v}\right)(t) \Rightarrow \frac{\partial w}{\partial v} = \left(\frac{1}{t}\right)\left(\frac{\partial w}{\partial y}\right)$. Therefore,
$nt^{n-1}f(x, y) = x\frac{\partial w}{\partial u} + y\frac{\partial w}{\partial v} = \left(\frac{x}{t}\right)\left(\frac{\partial w}{\partial x}\right) + \left(\frac{y}{t}\right)\left(\frac{\partial w}{\partial y}\right)$. When $t = 1$, $u = x$, $v = y$, and $w = f(x, y)$
$\Rightarrow \frac{\partial w}{\partial x} = \frac{\partial f}{\partial x}$ and $\frac{\partial w}{\partial y} = \frac{\partial f}{\partial x} \Rightarrow nf(x, y) = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$, as claimed.

(b) From part (a), $nt^{n-1}f(x, y) = x\frac{\partial w}{\partial u} + y\frac{\partial w}{\partial v}$. Differentiating with respect to t again we obtain
$n(n-1)t^{n-2}f(x, y) = x\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial t} + x\frac{\partial^2 w}{\partial v\partial u}\frac{\partial v}{\partial t} + y\frac{\partial^2 w}{\partial u\partial v}\frac{\partial u}{\partial t} + y\frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial t} = x^2\frac{\partial^2 w}{\partial u^2} + 2xy\frac{\partial^2 w}{\partial u\partial v} + y^2\frac{\partial^2 w}{\partial v^2}$.
Also from part (a), $\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}\left(t\frac{\partial w}{\partial u}\right) = t\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial x} + t\frac{\partial^2 w}{\partial v\partial u}\frac{\partial v}{\partial x} = t^2\frac{\partial^2 w}{\partial u^2}$, $\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)$
$= \frac{\partial}{\partial y}\left(t\frac{\partial w}{\partial v}\right) = t\frac{\partial^2 w}{\partial u\partial v}\frac{\partial u}{\partial y} + t\frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial y} = t^2\frac{\partial^2 w}{\partial v^2}$, and $\frac{\partial^2 w}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial y}\left(t\frac{\partial w}{\partial u}\right) = t\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial y} + t\frac{\partial^2 w}{\partial v\partial u}\frac{\partial v}{\partial y}$
$= t^2\frac{\partial^2 w}{\partial v\partial u} \Rightarrow \left(\frac{1}{t^2}\right)\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial u^2}$, $\left(\frac{1}{t^2}\right)\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial v^2}$, and $\left(\frac{1}{t^2}\right)\frac{\partial^2 w}{\partial y\partial x} = \frac{\partial^2 w}{\partial v\partial u}$
$\Rightarrow n(n-1)t^{n-2}f(x, y) = \left(\frac{x^2}{t^2}\right)\left(\frac{\partial^2 w}{\partial x^2}\right) + \left(\frac{2xy}{t^2}\right)\left(\frac{\partial^2 w}{\partial y\partial x}\right) + \left(\frac{y^2}{t^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right)$ for $t \neq 0$. When $t = 1$, $w = f(x, y)$ and we have $n(n-1)f(x, y) = x^2\left(\frac{\partial^2 f}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 f}{\partial x\partial y}\right) + y^2\left(\frac{\partial^2 f}{\partial y^2}\right)$ as claimed.

6. (a) $\lim_{r\to 0}\frac{\sin 6r}{6r} = \lim_{t\to 0}\frac{\sin t}{t} = 1$, where $t = 6r$

(b) $f_r(0, 0) = \lim_{h\to 0}\frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h\to 0}\frac{\left(\frac{\sin 6h}{6h}\right) - 1}{h} = \lim_{h\to 0}\frac{\sin 6h - 6h}{6h^2} = \lim_{h\to 0}\frac{6\cos 6h - 6}{12h}$
$= \lim_{h\to 0}\frac{-36\sin 6h}{12} = 0$ (applying l'Hôpital's rule twice)

(c) $f_\theta(r, \theta) = \lim_{h\to 0}\frac{f(r, \theta+h) - f(r, \theta)}{h} = \lim_{h\to 0}\frac{\left(\frac{\sin 6r}{6r}\right) - \left(\frac{\sin 6r}{6r}\right)}{h} = \lim_{h\to 0}\frac{0}{h} = 0$

7. (a) $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \Rightarrow r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$ and $\nabla r = \frac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{k} = \frac{\mathbf{r}}{r}$

(b) $r^n = \left(\sqrt{x^2+y^2+z^2}\right)^n$
$\Rightarrow \nabla(r^n) = nx\left(x^2+y^2+z^2\right)^{(n/2)-1}\mathbf{i} + ny\left(x^2+y^2+z^2\right)^{(n/2)-1}\mathbf{j} + nz\left(x^2+y^2+z^2\right)^{(n/2)-1}\mathbf{k} = nr^{n-2}\mathbf{r}$

(c) Let $n = 2$ in part (b). Then $\frac{1}{2}\nabla(r^2) = \mathbf{r} \Rightarrow \nabla\left(\frac{1}{2}r^2\right) = \mathbf{r} \Rightarrow \frac{r^2}{2} = \frac{1}{2}\left(x^2+y^2+z^2\right)$ is the function.

(d) $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \Rightarrow \mathbf{r}\cdot d\mathbf{r} = x\,dx + y\,dy + z\,dz$, and $dr = r_x\,dx + r_y\,dy + r_z\,dz = \frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz$
$\Rightarrow r\,dr = x\,dx + y\,dy + z\,dz = \mathbf{r}\cdot d\mathbf{r}$

(e) $\mathbf{A} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \Rightarrow \mathbf{A}\cdot\mathbf{r} = ax + by + cz \Rightarrow \nabla(\mathbf{A}\cdot\mathbf{r}) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \mathbf{A}$

8. $f(g(t), h(t)) = c \Rightarrow 0 = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)\cdot\left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right)$, where $\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$ is the tangent vector
$\Rightarrow \nabla f$ is orthogonal to the tangent vector

9. $f(x, y, z) = xz^2 - yz + \cos xy - 1 \Rightarrow \nabla f = (z^2 - y\sin xy)\mathbf{i} + (-z - x\sin xy)\mathbf{j} + (2xz - y)\mathbf{k} \Rightarrow \nabla f(0, 0, 1) = \mathbf{i} - \mathbf{j}$
$\Rightarrow$ the tangent plane is $x - y = 0$; $\mathbf{r} = (\ln t)\mathbf{i} + (t\ln t)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{r}' = \left(\frac{1}{t}\right)\mathbf{i} + (\ln t + 1)\mathbf{j} + \mathbf{k}$; $x = y = 0$, $z = 1$
$\Rightarrow t = 1 \Rightarrow \mathbf{r}'(1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Since $(\mathbf{i} + \mathbf{j} + \mathbf{k})\cdot(\mathbf{i} - \mathbf{j}) = \mathbf{r}'(1)\cdot\nabla f = 0$, $\mathbf{r}$ is parallel to the plane, and $\mathbf{r}(1) = 0\mathbf{i} + 0\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}$ is contained in the plane.

10. Let $f(x, y, z) = x^3 + y^3 + z^3 - xyz \Rightarrow \nabla f = (3x^2 - yz)\,\mathbf{i} + (3y^2 - xz)\,\mathbf{j} + (3z^2 - xy)\,\mathbf{k} \Rightarrow \nabla f(0, -1, 1) = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ $\Rightarrow$ the tangent plane is $x + 3y + 3z = 0$; $\mathbf{r} = \left(\frac{t^3}{4} - 2\right)\mathbf{i} + \left(\frac{4}{t} - 3\right)\mathbf{j} + (\cos(t-2))\,\mathbf{k}$ $\Rightarrow \mathbf{r}' = \left(\frac{3t^2}{4}\right)\mathbf{i} - \left(\frac{4}{t^2}\right)\mathbf{j} - (\sin(t-2))\,\mathbf{k}$; $x = 0, y = -1, z = 1 \Rightarrow t = 2 \Rightarrow \mathbf{r}'(2) = 3\mathbf{i} - \mathbf{j}$. Since $\mathbf{r}'(2) \cdot \nabla f = 0 \Rightarrow \mathbf{r}$ is parallel to the plane, and $\mathbf{r}(2) = -\mathbf{i} + \mathbf{k} \Rightarrow \mathbf{r}$ is contained in the plane.

11. $\frac{\partial z}{\partial x} = 3x^2 - 9y = 0$ and $\frac{\partial z}{\partial y} = 3y^2 - 9x = 0 \Rightarrow y = \frac{1}{3}x^2$ and $3\left(\frac{1}{3}x^2\right)^2 - 9x = 0 \Rightarrow \frac{1}{3}x^4 - 9x = 0$ $\Rightarrow x(x^3 - 27) = 0 \Rightarrow x = 0$ or $x = 3$. Now $x = 0 \Rightarrow y = 0$ or $(0, 0)$ and $x = 3 \Rightarrow y = 3$ or $(3, 3)$. Next $\frac{\partial^2 z}{\partial x^2} = 6x$, $\frac{\partial^2 z}{\partial y^2} = 6y$, and $\frac{\partial^2 z}{\partial x \partial y} = -9$. For $(0, 0)$, $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = -81 \Rightarrow$ no extremum (a saddle point), and for $(3, 3)$, $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 243 > 0$ and $\frac{\partial^2 z}{\partial x^2} = 18 > 0 \Rightarrow$ a local minimum.

12. $f(x, y) = 6xye^{-(2x+3y)} \Rightarrow f_x(x, y) = 6y(1 - 2x)e^{-(2x+3y)} = 0$ and $f_y(x, y) = 6x(1 - 3y)e^{-(2x+3y)} = 0 \Rightarrow x = 0$ and $y = 0$, or $x = \frac{1}{2}$ and $y = \frac{1}{3}$. The value $f(0, 0) = 0$ is on the boundary, and $f\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{e^2}$. On the positive y-axis, $f(0, y) = 0$, and on the positive x-axis, $f(x, 0) = 0$. As $x \to \infty$ or $y \to \infty$ we see that $f(x, y) \to 0$. Thus the absolute maximum of f in the closed first quadrant is $\frac{1}{e^2}$ at the point $\left(\frac{1}{2}, \frac{1}{3}\right)$.

13. Let $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \Rightarrow \nabla f = \frac{2x}{a^2}\,\mathbf{i} + \frac{2y}{b^2}\,\mathbf{j} + \frac{2z}{c^2}\,\mathbf{k} \Rightarrow$ an equation of the plane tangent at the point $P_0(x_0, y_0, y_0)$ is $\left(\frac{2x_0}{a^2}\right)x + \left(\frac{2y_0}{b^2}\right)y + \left(\frac{2z_0}{c^2}\right)z = \frac{2x_0^2}{a^2} + \frac{2y_0^2}{b^2} + \frac{2z_0^2}{c^2} = 2$ or $\left(\frac{x_0}{a^2}\right)x + \left(\frac{y_0}{b^2}\right)y + \left(\frac{z_0}{c^2}\right)z = 1$. The intercepts of the plane are $\left(\frac{a^2}{x_0}, 0, 0\right)$, $\left(0, \frac{b^2}{y_0}, 0\right)$ and $\left(0, 0, \frac{c^2}{z_0}\right)$. The volume of the tetrahedron formed by the plane and the coordinate planes is $V = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{a^2}{x_0}\right)\left(\frac{b^2}{y_0}\right)\left(\frac{c^2}{z_0}\right) \Rightarrow$ we need to maximize $V(x, y, z) = \frac{(abc)^2}{6}(xyz)^{-1}$ subject to the constraint $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Thus, $\left[-\frac{(abc)^2}{6}\right]\left(\frac{1}{x^2yz}\right) = \frac{2x}{a^2}\lambda$, $\left[-\frac{(abc)^2}{6}\right]\left(\frac{1}{xy^2z}\right) = \frac{2y}{b^2}\lambda$, and $\left[-\frac{(abc)^2}{6}\right]\left(\frac{1}{xyz^2}\right) = \frac{2z}{c^2}\lambda$. Multiply the first equation by a^2yz, the second by b^2xz, and the third by c^2xy. Then equate the first and second $\Rightarrow a^2y^2 = b^2x^2 \Rightarrow y = \frac{b}{a}x$, $x > 0$; equate the first and third $\Rightarrow a^2z^2 = c^2x^2 \Rightarrow z = \frac{c}{a}x$, $x > 0$; substitute into $f(x, y, z) = 0 \Rightarrow x = \frac{a}{\sqrt{3}} \Rightarrow y = \frac{b}{\sqrt{3}} \Rightarrow z = \frac{c}{\sqrt{3}} \Rightarrow V = \frac{\sqrt{3}}{2}abc$.

14. $2(x - u) = -\lambda$, $2(y - v) = \lambda$, $-2(x - u) = \mu$, and $-2(y - v) = -2\mu v \Rightarrow x - u = v - y$, $x - u = -\frac{\mu}{2}$, and $y - v = \mu v \Rightarrow x - u = -\mu v = -\frac{\mu}{2} \Rightarrow v = \frac{1}{2}$ or $\mu = 0$.

CASE 1: $\mu = 0 \Rightarrow x = u$, $y = v$, and $\lambda = 0$; then $y = x + 1 \Rightarrow v = u + 1$ and $v^2 = u \Rightarrow v = v^2 + 1$
$\Rightarrow v^2 - v + 1 = 0 \Rightarrow v = \frac{1 \pm \sqrt{1-4}}{2} \Rightarrow$ no real solution.

CASE 2: $v = \frac{1}{2}$ and $u = v^2 \Rightarrow u = \frac{1}{4}$; $x - \frac{1}{4} = \frac{1}{2} - y$ and $y = x + 1 \Rightarrow x - \frac{1}{4} = -x - \frac{1}{2} \Rightarrow 2x = -\frac{1}{4} \Rightarrow x = -\frac{1}{8}$
$\Rightarrow y = \frac{7}{8}$. Then $f\left(-\frac{1}{8}, \frac{7}{8}, \frac{1}{4}, \frac{1}{2}\right) = \left(-\frac{1}{8} - \frac{1}{4}\right)^2 + \left(\frac{7}{8} - \frac{1}{2}\right)^2 = 2\left(\frac{3}{8}\right)^2 \Rightarrow$ the minimum distance is $\frac{3}{8}\sqrt{2}$.
(Notice that f has no maximum value.)

15. Let (x_0, y_0) be any point in R. We must show $\lim_{(x,y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$ or, equivalently that $\lim_{(h,k) \to (0,0)} |f(x_0 + h, y_0 + k) - f(x_0, y_0)| = 0$. Consider $f(x_0 + h, y_0 + k) - f(x_0, y_0)$ $= [f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)] + [f(x_0, y_0 + k) - f(x_0, y_0)]$. Let $F(x) = f(x, y_0 + k)$ and apply the Mean Value Theorem: there exists ξ with $x_0 < \xi < x_0 + h$ such that $F'(\xi)h = F(x_0 + h) - F(x_0) \Rightarrow hf_x(\xi, y_0 + k)$ $= f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)$. Similarly, $k\,f_y(x_0, \eta) = f(x_0, y_0 + k) - f(x_0, y_0)$ for some η with $y_0 < \eta < y_0 + k$. Then $|f(x_0 + h, y_0 + k) - f(x_0, y_0)| \le |hf_x(\xi, y_0 + k)| + |kf_y(x_0, \eta)|$. If M, N are positive real numbers such that $|f_x| \le M$ and $|f_y| \le N$ for all (x, y) in the xy-plane, then $|f(x_0 + h, y_0 + k) - f(x_0, y_0)|$ $\le M|h| + N|k|$. As $(h, k) \to 0$, $|f(x_0 + h, y_0 + k) - f(x_0, y_0)| \to 0 \Rightarrow \lim_{(h,k) \to (0,0)} |f(x_0 + h, y_0 + k) - f(x_0, y_0)|$ $= 0 \Rightarrow$ f is continuous at (x_0, y_0).

16. At extreme values, ∇f and $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ are orthogonal because $\frac{df}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt} = 0$ by the First Derivative Theorem for Local Extreme Values.

17. $\frac{\partial f}{\partial x} = 0 \Rightarrow f(x, y) = h(y)$ is a function of y only. Also, $\frac{\partial g}{\partial y} = \frac{\partial f}{\partial x} = 0 \Rightarrow g(x, y) = k(x)$ is a function of x only. Moreover, $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \Rightarrow h'(y) = k'(x)$ for all x and y. This can happen only if $h'(y) = k'(x) = c$ is a constant. Integration gives $h(y) = cy + c_1$ and $k(x) = cx + c_2$, where c_1 and c_2 are constants. Therefore $f(x, y) = cy + c_1$ and $g(x, y) = cx + c_2$. Then $f(1, 2) = g(1, 2) = 5 \Rightarrow 5 = 2c + c_1 = c + c_2$, and $f(0, 0) = 4 \Rightarrow c_1 = 4 \Rightarrow c = \frac{1}{2}$ $\Rightarrow c_2 = \frac{9}{2}$. Thus, $f(x, y) = \frac{1}{2}y + 4$ and $g(x, y) = \frac{1}{2}x + \frac{9}{2}$.

18. Let $g(x, y) = D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$. Then $D_{\mathbf{u}}g(x, y) = g_x(x, y)a + g_y(x, y)b$ $= f_{xx}(x, y)a^2 + f_{yx}(x, y)ab + f_{xy}(x, y)ba + f_{yy}(x, y)b^2 = f_{xx}(x, y)a^2 + 2f_{xy}(x, y)ab + f_{yy}(x, y)b^2$.

19. Since the particle is heat-seeking, at each point (x, y) it moves in the direction of maximal temperature increase, that is in the direction of $\nabla T(x, y) = (e^{-2y}\sin x)\,\mathbf{i} + (2e^{-2y}\cos x)\,\mathbf{j}$. Since $\nabla T(x, y)$ is parallel to the particle's velocity vector, it is tangent to the path $y = f(x)$ of the particle $\Rightarrow f'(x) = \frac{2e^{-2y}\cos x}{e^{-2y}\sin x} = 2\cot x$. Integration gives $f(x) = 2\ln|\sin x| + C$ and $f\left(\frac{\pi}{4}\right) = 0 \Rightarrow 0 = 2\ln\left|\sin\frac{\pi}{4}\right| + C \Rightarrow C = -2\ln\frac{\sqrt{2}}{2} = \ln\left(\frac{2}{\sqrt{2}}\right)^2$ $= \ln 2$. Therefore, the path of the particle is the graph of $y = 2\ln|\sin x| + \ln 2$.

20. The line of travel is $x = t$, $y = t$, $z = 30 - 5t$, and the bullet hits the surface $z = 2x^2 + 3y^2$ when $30 - 5t = 2t^2 + 3t^2 \Rightarrow t^2 + t - 6 = 0 \Rightarrow (t + 3)(t - 2) = 0 \Rightarrow t = 2$ (since $t > 0$). Thus the bullet hits the surface at the point (2, 2, 20). Now, the vector $4x\mathbf{i} + 6y\mathbf{j} - \mathbf{k}$ is normal to the surface at any (x, y, z), so that $\mathbf{n} = 8\mathbf{i} + 12\mathbf{j} - \mathbf{k}$ is normal to the surface at (2, 2, 20). If $\mathbf{v} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$, then the velocity of the particle after the ricochet is $\mathbf{w} = \mathbf{v} - 2\,\text{proj}_{\mathbf{n}}\,\mathbf{v} = \mathbf{v} - \left(\frac{2\mathbf{v}\cdot\mathbf{n}}{|\mathbf{n}|^2}\right)\mathbf{n} = \mathbf{v} - \left(\frac{2\cdot 25}{209}\right)\mathbf{n} = (\mathbf{i} + \mathbf{j} - 5\mathbf{k}) - \left(\frac{400}{209}\mathbf{i} + \frac{600}{209}\mathbf{j} - \frac{50}{209}\mathbf{k}\right)$ $= -\frac{191}{209}\mathbf{i} - \frac{391}{209}\mathbf{j} - \frac{995}{209}\mathbf{k}$.

21. (a) $\mathbf{k}$ is a vector normal to $z = 10 - x^2 - y^2$ at the point (0, 0, 10). So directions tangential to S at (0, 0, 10) will be unit vectors $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$. Also, $\nabla T(x, y, z) = (2xy + 4)\,\mathbf{i} + (x^2 + 2yz + 14)\,\mathbf{j} + (y^2 + 1)\,\mathbf{k}$ $\Rightarrow \nabla T(0, 0, 10) = 4\mathbf{i} + 14\mathbf{j} + \mathbf{k}$. We seek the unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ such that $D_{\mathbf{u}}T(0, 0, 10)$ $= (4\mathbf{i} + 14\mathbf{j} + \mathbf{k}) \cdot (a\mathbf{i} + b\mathbf{j}) = (4\mathbf{i} + 14\mathbf{j}) \cdot (a\mathbf{i} + b\mathbf{j})$ is a maximum. The maximum will occur when $a\mathbf{i} + b\mathbf{j}$ has the same direction as $4\mathbf{i} + 14\mathbf{j}$, or $\mathbf{u} = \frac{1}{\sqrt{53}}(2\mathbf{i} + 7\mathbf{j})$.

(b) A vector normal to S at (1, 1, 8) is $\mathbf{n} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Now, $\nabla T(1, 1, 8) = 6\mathbf{i} + 31\mathbf{j} + 2\mathbf{k}$ and we seek the unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}T(1, 1, 8) = \nabla T \cdot \mathbf{u}$ has its largest value. Now write $\nabla T = \mathbf{v} + \mathbf{w}$, where $\mathbf{v}$ is parallel to ∇T and $\mathbf{w}$ is orthogonal to ∇T. Then $D_{\mathbf{u}}T = \nabla T \cdot \mathbf{u} = (\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{u}$. Thus $D_{\mathbf{u}}T(1, 1, 8)$ is a maximum when $\mathbf{u}$ has the same direction as $\mathbf{w}$. Now, $\mathbf{w} = \nabla T - \left(\frac{\nabla T\cdot\mathbf{n}}{|\mathbf{n}|^2}\right)\mathbf{n}$ $= (6\mathbf{i} + 31\mathbf{j} + 2\mathbf{k}) - \left(\frac{12 + 62 + 2}{4 + 4 + 1}\right)(2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = \left(6 - \frac{152}{9}\right)\mathbf{i} + \left(31 - \frac{152}{9}\right)\mathbf{j} + \left(2 - \frac{76}{9}\right)\mathbf{k}$ $= -\frac{98}{9}\mathbf{i} + \frac{127}{9}\mathbf{j} - \frac{58}{9}\mathbf{k} \Rightarrow \mathbf{u} = \frac{\mathbf{w}}{|\mathbf{w}|} = -\frac{1}{\sqrt{29{,}097}}(98\mathbf{i} - 127\mathbf{j} + 58\mathbf{k})$.

22. Suppose the surface (boundary) of the mineral deposit is the graph of $z = f(x, y)$ (where the z-axis points up into the air). Then $-\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$ is an outer normal to the mineral deposit at (x, y) and $\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ points in the direction of steepest ascent of the mineral deposit. This is in the direction of the vector $\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ at (0, 0) (the location of the 1st borehole) that the geologists should drill their fourth borehole. To approximate this vector we use the fact that (0, 0, −1000), (0, 100, −950), and (100, 0, −1025) lie on the graph of $z = f(x, y)$. The plane containing these three points is a good approximation to the tangent plane to $z = f(x, y)$ at the point (0, 0, 0). A normal to this plane is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 100 & 50 \\ 100 & 0 & -25 \end{vmatrix}$

$= -2500\mathbf{i} + 5000\mathbf{j} - 10{,}000\mathbf{k}$, or $-\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$. So at $(0, 0)$ the vector $\frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}$ is approximately $-\mathbf{i} + 2\mathbf{j}$. Thus the geologists should drill their fourth borehole in the direction of $\frac{1}{\sqrt{5}}\,(-\mathbf{i} + 2\mathbf{j})$ from the first borehole.

23. $w = e^{rt} \sin \pi x \;\Rightarrow\; w_t = re^{rt} \sin \pi x$ and $w_x = \pi e^{rt} \cos \pi x \;\Rightarrow\; w_{xx} = -\pi^2 e^{rt} \sin \pi x$; $w_{xx} = \frac{1}{c^2}\, w_t$, where c^2 is the positive constant determined by the material of the rod $\;\Rightarrow\; -\pi^2 e^{rt} \sin \pi x = \frac{1}{c^2}\,(re^{rt} \sin \pi x)$
$\Rightarrow\; (r + c^2\pi^2)\, e^{rt} \sin \pi x = 0 \;\Rightarrow\; r = -c^2\pi^2 \;\Rightarrow\; w = e^{-c^2\pi^2 t} \sin \pi x$

24. $w = e^{rt} \sin kx \;\Rightarrow\; w_t = re^{rt} \sin kx$ and $w_x = ke^{rt} \cos kx \;\Rightarrow\; w_{xx} = -k^2 e^{rt} \sin kx$; $w_{xx} = \frac{1}{c^2}\, w_t$
$\Rightarrow\; -k^2 e^{rt} \sin kx = \frac{1}{c^2}\,(re^{rt} \sin kx) \;\Rightarrow\; (r + c^2k^2)\, e^{rt} \sin kx = 0 \;\Rightarrow\; r = -c^2k^2 \;\Rightarrow\; w = e^{-c^2k^2 t} \sin kx.$
Now, $w(L, t) = 0 \;\Rightarrow\; e^{-c^2k^2 t} \sin kL = 0 \;\Rightarrow\; kL = n\pi$ for n an integer $\;\Rightarrow\; k = \frac{n\pi}{L} \;\Rightarrow\; w = e^{-c^2n^2\pi^2 t/L^2} \sin\left(\frac{n\pi}{L}\, x\right)$.
As $t \to \infty$, $w \to 0$ since $\left|\sin\left(\frac{n\pi}{L}\, x\right)\right| \le 1$ and $e^{-c^2n^2\pi^2 t/L^2} \to 0$.