Section 4.6 – Surfaces Integrals

We have defined curves in the plane in three different ways:

Explicit form: $y = f(x)$

Implicit form: $F(x, y) = 0$

Parametric vector form: $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} \quad a \le t \le b$

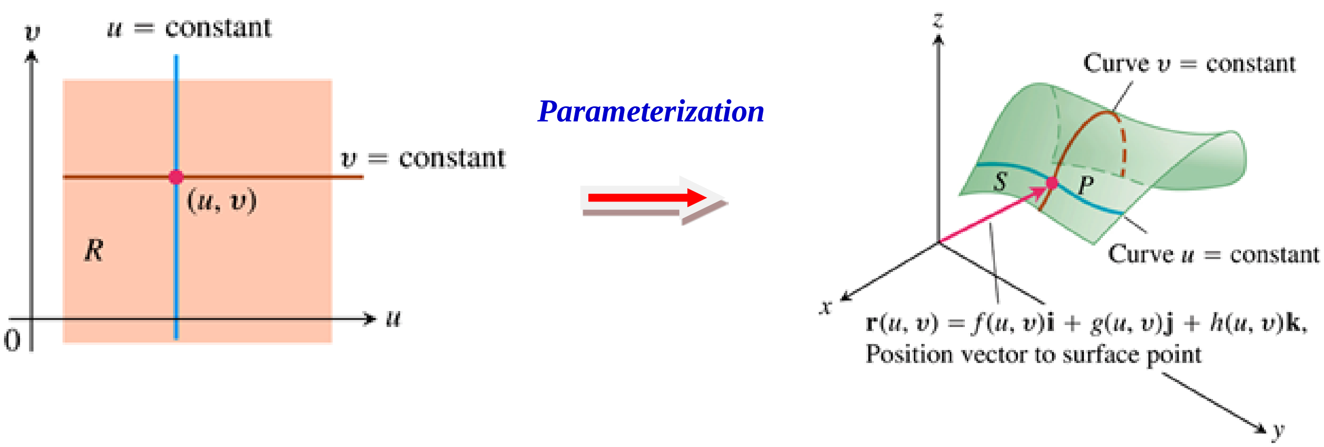
And

Explicit form: $z = f(x, y)$

Implicit form: $F(x, y, z) = 0$

Parameterizations of Surfaces

Suppose:



We call the range of ***r*** the ***surface*** *S* defined or traced by ***r***.

u and *v*: variable parameters

R: parameter domain

Example

Find a parameterization of the cone $z = \sqrt{x^2 + y^2}, \quad 0 \le z \le 1$

Solution

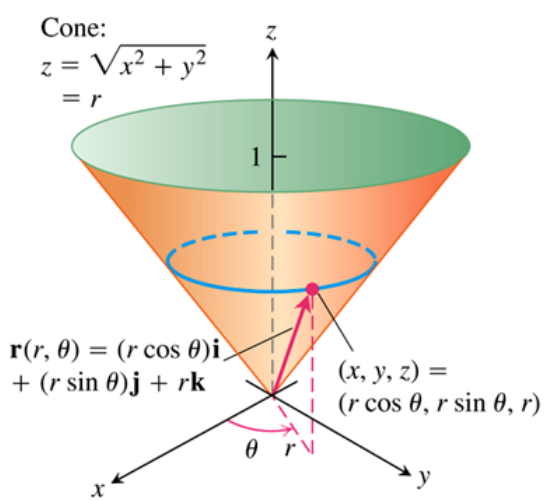
$x = r\cos\theta, \quad y = r\sin\theta$

$z = \sqrt{x^2 + y^2} = r$

Assume $u = r \quad and \quad v = \theta$

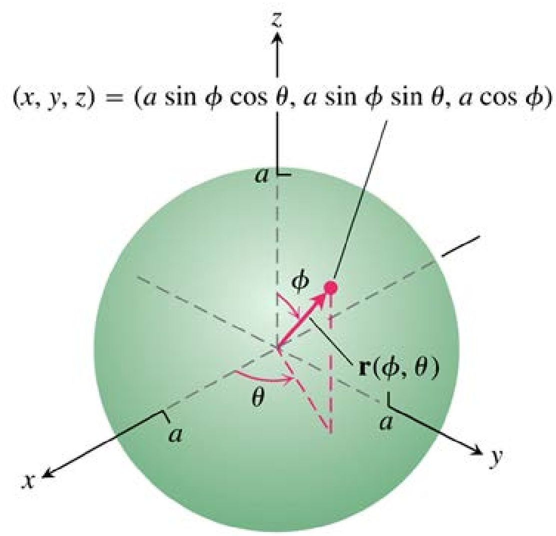
$\vec{r}(r,\ \theta) = (r\cos\theta)\hat{i} + (r\sin\theta)\hat{j} + r\hat{k}$

$0 \le r \le 1, \quad 0 \le \theta \le 2\pi$



Example

Find a parameterization of the cone $x^2+y^2+z^2=a^2$



Solution

A typical point $(x,\ y,\ z)$ on the sphere has

$$x=a\sin\phi\cos\theta, \quad y=a\sin\phi\sin\theta, \quad z=a\cos\phi$$

$$0\le\phi\le 2\pi, \quad 0\le\theta\le 2\pi$$

Taking $u=\phi$ *and* $v=\theta$

$$\vec{r}(\phi,\ \theta)=(a\sin\phi\cos\theta)\hat{\boldsymbol{i}}+(a\sin\phi\sin\theta)\hat{\boldsymbol{j}}+(a\cos\phi)\hat{\boldsymbol{k}}$$

The parameterization is one-to-one on the interior of the domain R, though not on its boundary "*poles*" where $\phi=0$ *or* $\theta=\pi$

Example

Find a parameterization of the cone $x^2+(y-3)^2=9, \quad 0\le z\le 5$

Solution

$$x=r\cos\theta, \quad y=r\sin\theta, \quad z=z$$

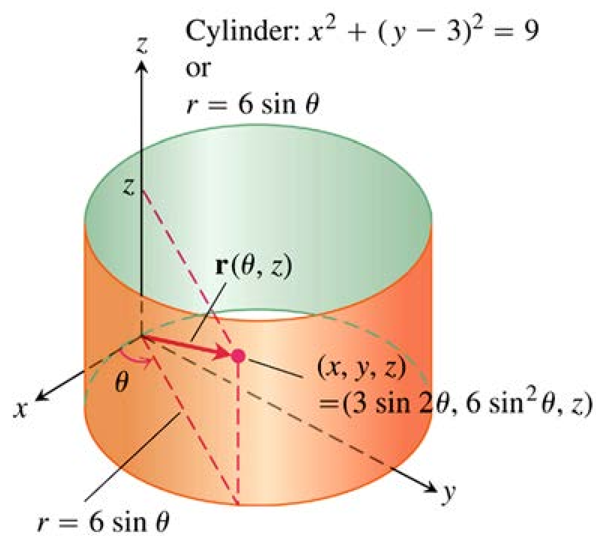
$$x^2+y^2-6y+9=9$$

$$x^2+y^2-6y=0$$

$$r^2-6r\sin\theta=0$$

$$r(r-6\sin\theta)=0$$

$$r=6\sin\theta, \quad 0\le\theta\le\pi$$



A typical point on the cylinder has

$$\begin{cases} x=r\cos\theta=6\sin\theta\cos\theta=3\sin 2\theta \\ y=r\sin\theta=6\sin^2\theta \\ z=z \end{cases}$$

Taking $u=\theta$ *and* $v=z$

$$\vec{r}(\theta,\ z)=(3\sin 2\theta)\hat{\boldsymbol{i}}+\left(6\sin^2\theta\right)\hat{\boldsymbol{j}}+z\hat{\boldsymbol{k}} \quad 0\le\theta\le\pi, \quad 0\le z\le 5$$

Surface Area

Calculating the area of a curved surface S based on the parameterization

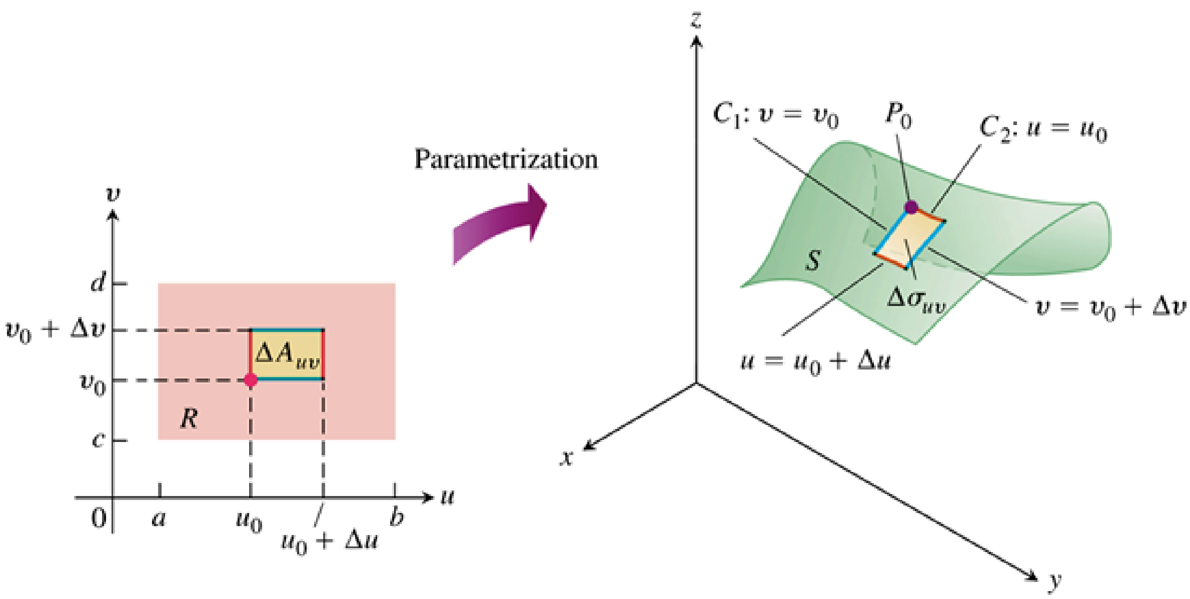
$$\vec{r}(u,\ v) = f(u,\ v)\hat{i} + g(u,\ v)\hat{j} + h(u,\ v)\hat{k}, \quad a \le u \le b, \quad c \le v \le d$$

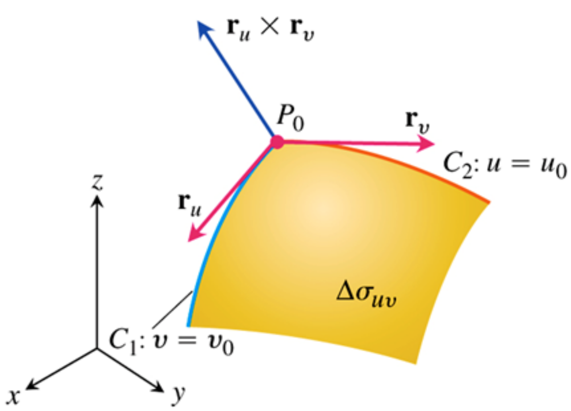
The definition of smoothness involves the partial derivatives of $\vec{r}$ with respect to u and v:

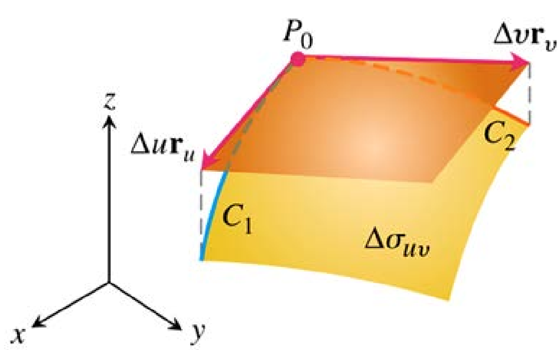
$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u} = \frac{\partial f}{\partial u}\hat{i} + \frac{\partial g}{\partial u}\hat{j} + \frac{\partial h}{\partial u}\hat{k} \qquad \vec{r}_v = \frac{\partial \vec{r}}{\partial v} = \frac{\partial f}{\partial v}\hat{i} + \frac{\partial g}{\partial v}\hat{j} + \frac{\partial h}{\partial v}\hat{k}$$

Definition

A ***parameterized*** surface $\vec{r}(u,v) = f(u,v)\hat{i} + g(u,v)\hat{j} + h(u,v)\hat{k}$ is smooth if $\vec{r}_u$ and $\vec{r}_v$ are continuous and $\vec{r}_u \times \vec{r}_v$ is never zero on the interior of the parameter domain.







Definition

The area of the smooth surface

$$\vec{r}(u, v) = f(u, v)\hat{i} + g(u, v)\hat{j} + h(u, v)\hat{k}, \quad a \le u \le b, \quad c \le v \le d$$

is $$Area = \iint_R \left|\vec{r}_u \times \vec{r}_v\right| dA = \int_c^d \int_a^b \left|\vec{r}_u \times \vec{r}_v\right| dudv$$

Surface area Differential for a Parameterized Surface

$$d\sigma = \left|\vec{r}_u \times \vec{r}_v\right| dudv \qquad \iint_S d\sigma$$

Surface area differential

Differntial formula for surface area

Example

Find the surface area of the cone $z = \sqrt{x^2 + y^2}, \quad 0 \le z \le 1$

Solution

$x = r\cos\theta, \quad y = r\sin\theta$, and $z = \sqrt{x^2 + y^2} = r$

$\vec{r}(r, \theta) = \langle r\cos\theta, \ r\sin\theta, \ r\rangle$

$\vec{r}_r = \langle \cos\theta, \ \sin\theta, \ 1\rangle$

$\vec{r}_\theta = \langle -r\sin\theta, \ r\cos\theta, \ 0\rangle$

$0 \le r \le 1, \quad 0 \le \theta \le 2\pi$

$$\begin{aligned}
\vec{r}_r \times \vec{r}_\theta &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} \\
&= -(r\cos\theta)\hat{i} - (r\sin\theta)\hat{j} + \left(r\cos^2\theta + r\sin^2\theta\right)\hat{k} \\
&= \langle -r\cos\theta, \ -r\sin\theta, \ r\rangle
\end{aligned}$$

$$\begin{aligned}
\left|\vec{r}_r \times \vec{r}_\theta\right| &= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} \\
&= \sqrt{r^2 + r^2} \\
&= r\sqrt{2}
\end{aligned}$$

$$A = \int_0^{2\pi} \int_0^1 \left|\vec{r}_r \times \vec{r}_\theta\right| \, dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 \sqrt{2}\, r \, dr$$

$$= \frac{\sqrt{2}}{2}(2\pi)\left(r^2\right)\Big|_0^1$$

$$\underline{= \pi\sqrt{2} \;\; units^2}$$

Example

Find the surface area of a sphere of radius a.

Solution

$$\vec{r}(\phi,\ \theta) = (a\sin\phi\cos\theta)\hat{i} + (a\sin\phi\sin\theta)\hat{j} + (a\cos\phi)\hat{k}$$

$$\vec{r}_\phi = \langle a\cos\phi\cos\theta,\ \ a\cos\phi\sin\theta,\ \ -a\sin\phi\rangle$$

$$\vec{r}_\theta = \langle -a\sin\phi\sin\theta,\ \ a\sin\phi\cos\theta,\ \ 0\rangle$$

$$0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \left(a^2\sin^2\phi\cos\theta\right)\hat{i} + \left(a^2\sin^2\phi\sin^2\theta\right)\hat{j} + \left(a^2\cos\phi\sin\phi\cos^2\theta + a^2\cos\phi\sin\phi\sin^2\theta\right)\hat{k}$$

$$= \left(a^2\sin^2\phi\cos\theta\right)\hat{i} + \left(a^2\sin^2\phi\sin\theta\right)\hat{j} + \left(a^2\cos\phi\sin\phi\right)\hat{k}$$

$$\left|\vec{r}_\phi \times \vec{r}_\theta\right| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\cos^2\phi\sin^2\phi}$$

$$= \sqrt{a^4\sin^4\phi\left(\cos^2\theta + \sin^2\theta\right) + a^4\cos^2\phi\sin^2\phi}$$

$$= \sqrt{a^4\sin^4\phi + a^4\cos^2\phi\sin^2\phi}$$

$$= \sqrt{a^4\sin^2\phi\left(\sin^2\phi + \cos^2\phi\right)}$$

$$= a^2\sin\phi$$

$$A = \int_0^{2\pi} \int_0^{\pi} a^2\sin\phi \, d\phi d\theta$$

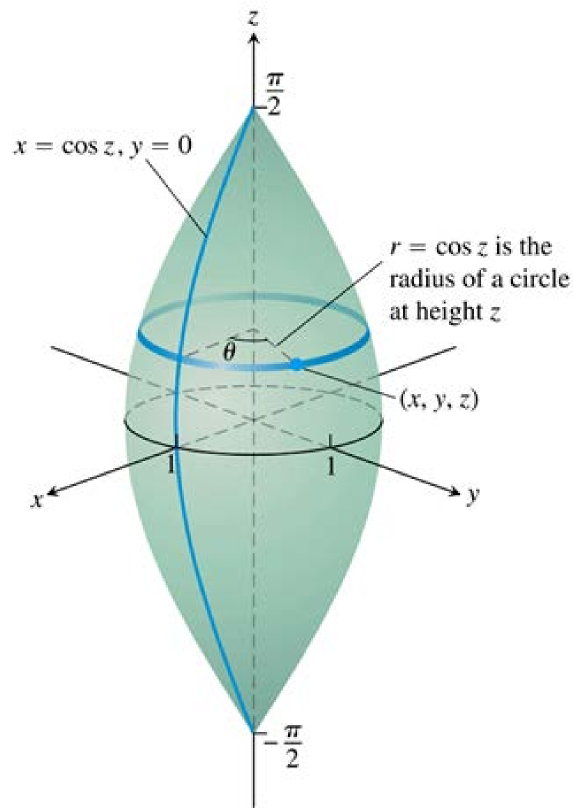
$$= a^2(-\cos\phi)\Big|_0^{\pi} \int_0^{2\pi} d\theta$$

$$= 4\pi a^2 \ \ unit^2$$

Example

Let S be the "football" surface formed by rotating the curve $x = \cos z, \quad y = 0, \quad -\frac{\pi}{2} \le z \le \frac{\pi}{2}$ around the z-axis. Find the parameterization for S and compute its surface area.

Solution



Let (x, y, z) be an arbitrary point on the circle.

The parameters: $u = z$ and $v = \theta$.

We have:

$$\begin{cases} x = r\cos\theta = \cos u \cos v \\ y = r\sin\theta = \cos u \sin v \\ z = u \end{cases}$$

$$\vec{r}(u, v) = (\cos u \cos v)\hat{i} + (\cos u \sin v)\hat{j} + u\hat{k}$$

$$\vec{r}_u = (-\sin u \cos v)\hat{i} - (\sin u \sin v)\hat{j} + \hat{k}$$

$$\vec{r}_v = (-\cos u \sin v)\hat{i} + (\cos u \cos v)\hat{j}$$

$$-\frac{\pi}{2} \le u \le \frac{\pi}{2}, \quad 0 \le v \le 2\pi$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin u \cos v & -\sin u \sin v & 1 \\ -\cos u \sin v & \cos u \cos v & 0 \end{vmatrix}$$

$$= (-\cos u \cos v)\hat{i} - (\cos u \sin v)\hat{j} - \left(\sin u \cos u \cos^2 v + \cos u \sin u \sin^2 v\right)\hat{k}$$

$$\left|\vec{r}_u \times \vec{r}_v\right| = \sqrt{\cos^2 u \cos^2 v + \cos^2 u \sin^2 v + \left(\sin u \cos u\left(\cos^2 v + \sin^2 v\right)\right)^2}$$

$$= \sqrt{\cos^2 u\left(\cos^2 v + \sin^2 v\right) + \sin^2 u \cos^2 u}$$

$$= \sqrt{\cos^2 u\left(1 + \sin^2 u\right)}$$

$$= \cos u\sqrt{1 + \sin^2 u}$$

$$A=\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\cos u\sqrt{1+\sin^2 u}\;dudv$$

$$w=\sin u \;\Rightarrow\; dw=\cos u du \to \begin{cases} u=-\frac{\pi}{2} & \to w=-1 \\ u=\frac{\pi}{2} & \to w=1 \end{cases}$$

$$=\int_0^{2\pi}\int_{-1}^{1}\sqrt{1+w^2}\;dwdv$$

$$=\int_0^{2\pi}\left[\frac{w}{2}\sqrt{1+w^2}+\frac{1}{2}\ln\left(w+\sqrt{1+w^2}\right)\right]_{-1}^{1}dv$$

$$=\int_0^{2\pi}\left[\frac{1}{2}\sqrt{2}+\frac{1}{2}\ln\left(1+\sqrt{2}\right)+\frac{1}{2}\sqrt{2}-\frac{1}{2}\ln\left(-1+\sqrt{2}\right)\right]dv$$

$$\ln\left(-1+\sqrt{2}\cdot\frac{1+\sqrt{2}}{1+\sqrt{2}}\right)=\ln\left(\frac{1}{1+\sqrt{2}}\right)=-\ln\left(1+\sqrt{2}\right)$$

$$=\int_0^{2\pi}\left[\sqrt{2}+\frac{1}{2}\ln\left(1+\sqrt{2}\right)+\frac{1}{2}\ln\left(1+\sqrt{2}\right)\right]dv$$

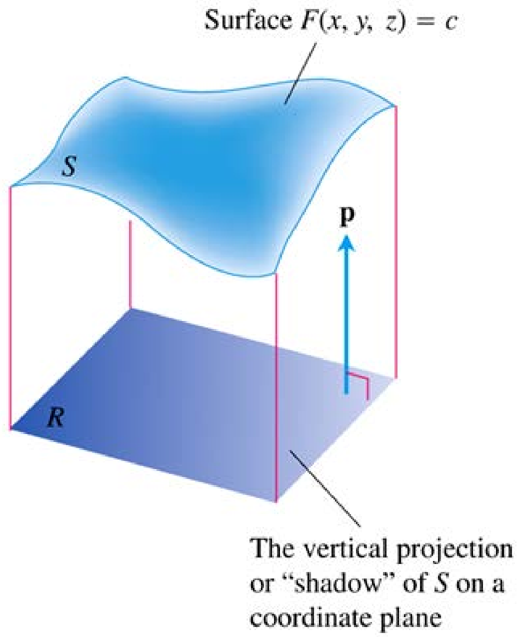
$$=\int_0^{2\pi}\left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]dv$$

$$=\left(\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right)\;v\Big|_0^{2\pi}$$

$$\boxed{=2\pi\left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]\;\; unit^2}$$

Implicit Surfaces

Surfaces are often presented as level sets of a function $F(x, y, z) = c$ for some constant c. Such a level surface does not come with an explicit parameterization, and is called am *implicit defined surface*.



The surface is defined by the equation $F(x, y, z) = c$ and $\vec{p}$ is a unit vector normal to the plane region *R*.

$$\begin{aligned}\nabla F \bullet \vec{p} &= \nabla F \bullet \hat{k} \\ &= F_z \neq 0\end{aligned}$$

Define the parameters *u* and *v* by *u* = *x* and *v* = *y*. Then *z* = *h*(*u*, *v*) and

$$\vec{r}(u, v) = u\hat{i} + v\hat{j} + h(u, v)\hat{k}$$

Calculating the partial derivatives of $\vec{r}$,

$$\vec{r}_u = \hat{i} + \frac{\partial h}{\partial u}\hat{k} \quad and \quad \vec{r}_v = \hat{j} + \frac{\partial h}{\partial v}\hat{k}$$

$$\frac{\partial h}{\partial u} = -\frac{F_x}{F_z} \quad and \quad \frac{\partial h}{\partial v} = -\frac{F_y}{F_z}$$

$$\vec{r}_u = \hat{i} - \frac{F_x}{F_z}\hat{k} \quad and \quad \vec{r}_v = \hat{j} - \frac{F_y}{F_z}\hat{k}$$

$$\begin{aligned}\vec{r}_u \times \vec{r}_v &= \frac{F_x}{F_z}\hat{i} + \frac{F_y}{F_z}\hat{j} + \hat{k} \\ &= \frac{1}{F_z}\left(F_x\hat{i} + F_y\hat{j} + F_z\hat{k}\right) \\ &= \frac{\nabla F}{F_z}\end{aligned}$$

$$= \frac{\nabla F}{\nabla F \bullet \hat{k}}$$

$$= \frac{\nabla F}{\nabla F \bullet \vec{p}}$$

Therefore, the surface area differential is given by

$$d\sigma = \left|\vec{r}_u \times \vec{r}_v\right| du dv = \frac{|\nabla F|}{|\nabla F \bullet \vec{p}|} dx dy \qquad u = x \ \ and \ \ v = y$$

Formula for the Surface Area of an Implicit Surface

The area of the surface $F(x, y, z) = c$ over a closed and bounded plane region R is

$$Surface\ area = \iint_R \frac{|\nabla F|}{|\nabla F \bullet \vec{p}|}\, dA$$

Where $\vec{p} = \hat{i},\ \hat{j},\ or\ \hat{k}$ is normal to R and $\nabla F \bullet \vec{p} \neq 0$

Example

Find the area of the surface cut from the bottom of the paraboloid $x^2+y^2-z=0$ by the plane $z=4$.

Solution

Let $F(x,y,z)=x^2+y^2-z=0$ and R the disk $x^2+y^2\le 4$

$\nabla F=2x\hat{i}+2y\hat{j}-\hat{k}$

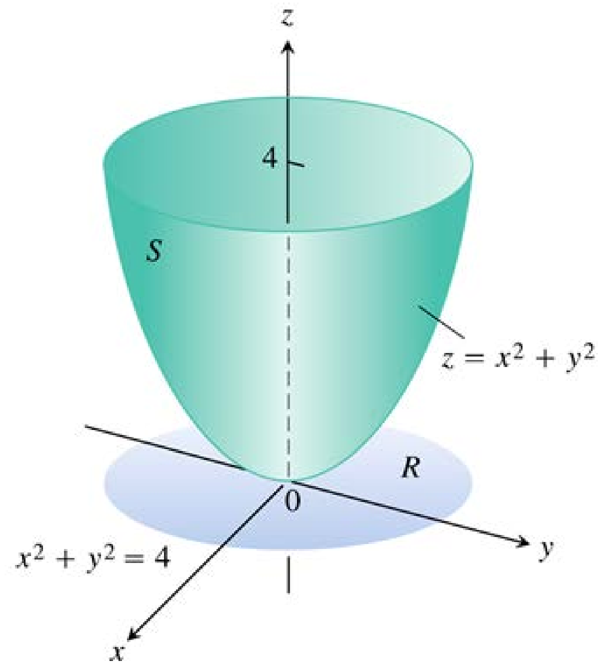
$$\left|\nabla F\right|=\sqrt{(2x)^2+(2y)^2+(-1)^2}$$

$$=\sqrt{4x^2+4y^2+1}$$

$$\left|\nabla F\cdot\vec{p}\right|=\left|\nabla F\cdot\hat{k}\right|$$

$$=\left|-1\right|$$

$$=1$$



In the region R, $dA=dxdy$. Therefore,

$$Surface\ area=\iint_R \frac{\left|\nabla F\right|}{\left|\nabla F\cdot\vec{p}\right|}dA$$

$$=\iint_{x^2+y^2\le 4}\sqrt{4\left(x^2+y^2\right)+1}\ dxdy$$

$$=\int_0^{2\pi}d\theta\ \int_0^2\sqrt{4r^2+1}\ r\ dr$$

$$=2\pi\left(\frac{1}{8}\right)\int_0^2\sqrt{4r^2+1}\ d\left(4r^2+1\right)$$

$$=(2\pi)\frac{1}{12}\left(4r^2+1\right)^{3/2}\Big|_0^2$$

$$=\frac{\pi}{6}\left(17^{3/2}-1\right)$$

$$=\frac{\pi}{6}\left(17\sqrt{17}-1\right)\ unit^2$$

Formula for the Surface Area of a Graph $z = f(x,y)$

For a graph $z = f(x,y)$ over the region R in the *xy*-plane, the surface area formula is

$$A = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\; dxdy$$

Surface	***Equation***	***Explicit Description***	
		Normal Vector $\pm\langle -z_x,\ -z_y,\ 1\rangle$	***Magnitude*** $\left|\langle -z_x,\ -z_y,\ 1\rangle\right|$
Cylinder	$x^2 + y^2 = a^2$ $0 \le z \le h$	$\langle x,\ y,\ 0\rangle$	a
Cone	$z^2 = x^2 + y^2$ $0 \le z \le h$	$\left\langle \frac{x}{z},\ \frac{y}{z},\ -1\right\rangle$	$\sqrt{2}$
Sphere	$x^2 + y^2 + z^2 = a^2$	$\left\langle \frac{x}{z},\ \frac{y}{z},\ 1\right\rangle$	$\frac{a}{z}$
Paraboloid	$z = x^2 + y^2$ $0 \le z \le h$	$\langle 2x,\ 2y,\ -1\rangle$	$\sqrt{1 + 4\left(x^2 + y^2\right)}$

Surface	***Equation***	***Parametric Description***	
		Normal Vector $\vec{r}_u \times \vec{r}_v$	***Magnitude*** $\left|\vec{r}_u \times \vec{r}_v\right|$
Cylinder	$\vec{r} = \langle a\cos u,\ a\sin u,\ v\rangle$ $0 \le u \le 2\pi,\quad 0 \le v \le h$	$\langle a\cos u,\ a\sin u,\ 0\rangle$	a
Cone	$\vec{r} = \langle v\cos u,\ v\sin u,\ v\rangle$ $0 \le u \le 2\pi,\quad 0 \le v \le h$	$\langle v\cos u,\ v\sin u,\ -v\rangle$	$\sqrt{2}\, v$
Sphere	$\vec{r} = \langle a\sin u\cos v,\ a\sin u\sin v,\ a\cos u\rangle$ $0 \le u \le \pi,\quad 0 \le v \le 2\pi$	$\langle a^2\sin^2 u\cos v,\ a^2\sin^2 u\sin v,$ $a^2\sin^2 u\cos u\rangle$	$a^2\sin u$
Paraboloid	$\vec{r} = \left\langle v\cos u,\ v\sin u,\ v^2\right\rangle$ $0 \le u \le 2\pi,\quad 0 \le v \le \sqrt{h}$	$\left\langle 2v^2\cos u,\ 2v^2\sin u,\ -v\right\rangle$	$v\sqrt{1 + 4v^2}$

Exercises *Section* 4.6 – Surfaces Integrals

(**1–9**) Find a parametrization of the surface:

1. The paraboloid $z = x^2 + y^2, \ z \le 4$

2. The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 4$

3. The sphere $x^2 + y^2 + z^2 = 8$ cuts by the plane $z = -2$

4. The plane $2x - 4y + 3z = 16$

5. The cap of the sphere $x^2 + y^2 + z^2 = 16$ for $2\sqrt{2} \le z \le 4$

6. The frustum of the cone $z^2 = x^2 + y^2$ for $2 \le z \le 8$

7. The cone $z^2 = 4\left(x^2 + y^2\right)$ for $0 \le z \le 4$

8. The portion of the cylinder $x^2 + y^2 = 9$ in the first octant, for $0 \le z \le 3$

9. The cylinder $y^2 + z^2 = 36$ for $0 \le x \le 9$

(**10–19**) Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of

10. A plane $y + 2z = 2$ inside the cylinder $x^2 + y^2 = 1$

11. A cone $z = \dfrac{\sqrt{x^2 + y^2}}{3}$ between the planes $z = 1$ and $z = \dfrac{4}{3}$

12. A cylinder $x^2 + z^2 = 10$ between the planes $y = -1$ and $y = 1$

13. Cap cut from the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 4$

14. The half cylinder $\left\{(r, \ \theta, \ z): \quad r = 4, \quad 0 \le \theta \le \pi. \quad 0 \le z \le 7\right\}$

15. The plane $z = 3 - x - 3y$ in the first octant

16. The plane $z = 10 - x - y$ above the square $|x| \le 2, \quad |y| \le 2$

17. The hemisphere $x^2 + y^2 + z^2 = 100, \quad z \ge 0$

18. A cone with base radius *r* and height *h*, where *r* and *h* are positive constants.

19. The cap of the sphere $x^2 + y^2 + z^2 = 4, \quad 1 \le z \le 2$

(**20–39**) Use a surface integral to find the area of

20. Cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 2$.

21. Portion $x^2 - 2z = 0$ that lies above the triangle bounded by the lines $x = \sqrt{3},\ y = 0$, and $y = x$ in the *xy*-plane.

22. Cap cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$.

23. Ellipse cut from the plane $z = cx$ (c a constant) by the cylinder $x^2 + y^2 = 1$.

24. From the nose of the paraboloid $x = 1 - y^2 - z^2$ by *yz*-plane.

25. First octant cut from the cylinder $y = \frac{2}{3}z^{3/2}$ by the planes $x = 1$ and $y = \frac{16}{3}$

26. Helicoid $\vec{r}(r,\ \theta) = (r\cos\theta)\hat{\boldsymbol{i}} + (r\sin\theta)\hat{\boldsymbol{j}} + \theta\hat{\boldsymbol{k}},\quad 0 \le \theta \le 2\pi,\quad 0 \le r \le 1$

27. Surface $f(x,\ y) = \sqrt{2}\,xy$ above the origin $\{(r,\ \theta):\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

28. Hemisphere $x^2 + y^2 + z^2 = 9$, for $z \ge 0$ (excluding the base)

29. Frustum of the cone $z^2 = x^2 + y^2$, for $2 \le z \le 4$ (excluding the bases)

30. Area of the plane $z = 6 - x - y$ above the square $|x| \le 1,\ |y| \le 1$

31. The cone $z^2 = 4\left(x^2 + y^2\right),\quad 0 \le z \le 4$

32. The paraboloid $z = 2\left(x^2 + y^2\right),\quad 0 \le z \le 8$

33. The trough $z = x^2,\quad -2 \le x \le 2,\quad 0 \le y \le 4$

34. The part of the hyperbolic paraboloid $z = x^2 - y^2$ above the sector $R = \left\{(r,\ \theta):\quad 0 \le r \le 4,\quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}\right\}$

35. $f(x,\ y,\ z) = xy$, where S is the plane $z = 2 - x - y$ in the first octant

36. $f(x,\ y,\ z) = x^2 + y^2$, where S is the paraboloid $z = x^2 + y^2,\quad 0 \le z \le 4$

37. $f(x,\ y,\ z) = 25 - x^2 - y^2$, where S is the hemisphere centered at the origin with radius 5, for $z \ge 0$

38. $f(x,\ y,\ z) = e^x$, where S is the plane $z = 8 - x - 2y$ in the first octant

39. $f(x,\ y,\ z) = e^z$, where S is the plane $z = 8 - x - 2y$ in the first octant

(40–46) Evaluate the surface integrals

40. $\iint\limits_S (1 + yz)\,dS$; S is the plane $x + y + z = 2$ in the first octant.

41. $\iint\limits_S \langle 0, y, z\rangle \cdot \vec{n}\ dS$; S is the curve surface of the cylinder $y^2 + z^2 = a^2$, $|x| \le 8$ with outward normal vectors.

42. $\iint\limits_S (x - y + z)\, dS$; S is the entire surface including the base of the hemisphere $x^2 + y^2 + z^2 = 4$, for $z \ge 0$.

43. $\iint_S \nabla \ln|\vec{r}| \bullet \vec{n}\ dS$, where S is the hemisphere $x^2 + y^2 + z^2 = a^2$, for $z \ge 0$, and where $\vec{r} = \langle x,\ y,\ z\rangle$. Assume normal vectors point either outward or in the positive z-direction.

44. $\iint_S |\vec{r}|\ dS$, where S is the cylinder $x^2 + y^2 = 4$, for $0 \le z \le 8$, and where $\vec{r} = \langle x,\ y,\ z\rangle$. Assume normal vectors point either outward or in the positive z-direction.

45. $\iint\limits_S xyz\ dS$, where S is the part of the plane $z = 6 - y$ that lies on the cylinder $x^2 + y^2 = 4$. Assume normal vectors point either outward or in the positive z-direction.

46. $\iint\limits_S \dfrac{\langle x,\ 0,\ z\rangle}{\sqrt{x^2 + z^2}} \bullet \vec{n}\ dS$, where S is the cylinder $x^2 + y^2 = a^2$, $|y| \le 2$. Assume normal vectors point either outward or in the positive z-direction.

(**47–50**) Evaluate the surface integral $\iint_S f(x,\ y,\ z)\, dS$

47. $f(x,\ y,\ z) = x^2 + y^2$, where S is the hemisphere $x^2 + y^2 + z^2 = 36, \quad z \ge 0$

48. $f(x,\ y,\ z) = y$, where S is the cylinder $x^2 + y^2 = 9, \quad 0 \le z \le 3$

49. $f(x,\ y,\ z) = x$, where S is the cylinder $x^2 + z^2 = 1, \quad 0 \le y \le 3$

50. $f(\rho,\ \varphi,\ \theta) = \cos\varphi$, where S is the part of the unit shpere in the first octant

(**51–58**) Find the flux of the vector fields across the given surface with the specified orientation

51. $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|}$ across the sphere of radius a centered at the origin, where $\vec{r} = \langle x,\ y,\ z\rangle$. Assume the normal vectors to the surface point outward.

52. $\vec{F} = \langle x,\ y,\ z \rangle$ across the curved surface of the cylinder $x^2 + y^2 = 1$ for $|z| \le 8$

53. $\vec{F} = \langle 0,\ 0,\ -1 \rangle$ across the slanted face of the tetrahedron $z = 4 - x - y$ in the first octant; normal vectors point upward

54. $\vec{F} = \langle x,\ y,\ z \rangle$ across the slanted face of the tetrahedron $z = 10 - 2x - 5y$ in the first octant; normal vectors point upward

55. $\vec{F} = \langle x,\ y,\ z \rangle$ across the slanted face of the cone $z^2 = x^2 + y^2$ for $0 \le z \le 1$; normal vectors point upward

56. $\vec{F} = \left\langle e^{-y},\ 2z,\ xy \right\rangle$ across the curved sides of the surface

$S = \left\{ (x,\ y,\ z):\quad z = \cos y,\quad -\pi \le y \le \pi,\quad 0 \le x \le 4 \right\}$; normal vectors point upward.

57. $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^3}$ across the sphere of radius a centered at the origin, where $\vec{r} = \langle x,\ y,\ z \rangle$; normal vectors point outward

58. $\vec{F} = \langle -y,\ x,\ 1 \rangle$ across the cylinder $y = x^2$ for $0 \le x \le 1,\quad 0 \le z \le 4$; normal vectors point in the general direction of the positive y-axis

59. Consider the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, where a ,b, and c are positive real numbers.

a) Show that the surface is described by the parametric equations
$\vec{r}(u,v) = \langle a\cos u \sin v,\ \ b\sin u \sin v,\ \ c\cos v \rangle$ for $0 \le u \le 2\pi,\quad 0 \le v \le \pi$

b) Write an integral for the surface area of the ellipsoid.

60. The cone $z^2 = x^2 + y^2,\quad z \ge 0$, cuts the sphere $x^2 + y^2 + z^2 = 16$ along a curve C.

a) Find the surface area of the sphere below C, for $z \ge 0$
b) Find the surface area of the sphere above C.
c) Find the surface area of the cone below C, for $z \ge 0$

61. Consider the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $(x-1)^2 + y^2 = 1$ for $z \ge 0$.

a) Find the surface area of the cylinder inside the sphere
b) Find the surface area of the sphere inside the cylinder.

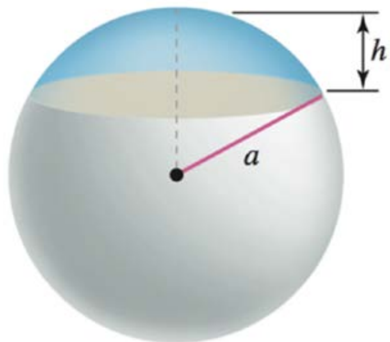
62. Find the upward flux of the field $\vec{F} = \langle x,\ y,\ z \rangle$ across the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ in the first octant.

Show that the flux equals c times the area if the base of the origin.

63. Consider the field $\vec{F} = \langle x,\ y,\ z\rangle$ and the cone $z^2 = \dfrac{x^2+y^2}{a^2}$, for $0 \le z \le 1$

a) Show that when $a = 1$, the outward flux across the cone is zero.
b) Find the outward flux (away from the *z*-axis); for any $a > 0$.

64. A sphere of radius *a* is sliced parallel to the equatorial plane at a distance $a - h$ from the equatorial plane. Find the general formula for the surface area of the resulting spherical cap (excluding the base) with thickness *h*.



65. Consider the radial field $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^p}$, where $\vec{r} = \langle x,\ y,\ z\rangle$ and *p* is a real number. Let *S* be he sphere of radius *a* centered at the origin. Show that the outward flux of $\vec{F}$ across the sphere is $\dfrac{4\pi}{a^{p-3}}$. It is instructive to do the calculation using both an explicit and parametric description of the sphere.

(**66–68**) The heat flow vector field for conducting objects is $\vec{F} = -k\nabla T$, where $T(x,\ y,\ z)$ is the temperature in the object and $k > 0$ is a constant that depends on the material. Compute the outward flux of $\vec{F}$ across the following surfaces *S* for the given temperature distributions. Assume $k = 1$.

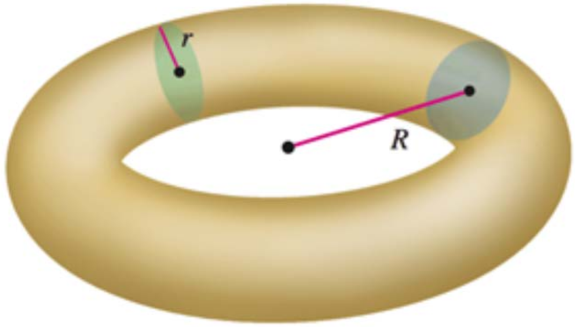
66. $T(x,\ y,\ z) = 100e^{-x-y}$; *S* consists of the faces of the cube $|x| \le 1,\quad |y| \le 1,\quad |z| \le 1$

67. $T(x,\ y,\ z) = 100e^{-x^2-y^2-z^2}$; *S* cis the sphere $x^2 + y^2 + z^2 = a^2$

68. $T(x,\ y,\ z) = -\ln\left(x^2 + y^2 + z^2\right)$; *S* cis the sphere $x^2 + y^2 + z^2 = a^2$

69. Given: $\vec{r}(u,\ v)=\langle (R+r\cos u)\cos v,\ \ (R+r\cos u)\sin v,\ \ r\sin u\rangle$

a) Show that a torus with radii $R>r$ may be described parametrically by $\vec{r}(u,\ v)$ for $0\le u\le 2\pi,\quad 0\le v\le 2\pi$



b) Show that the surface area of the torus is $4\pi^2 Rr$