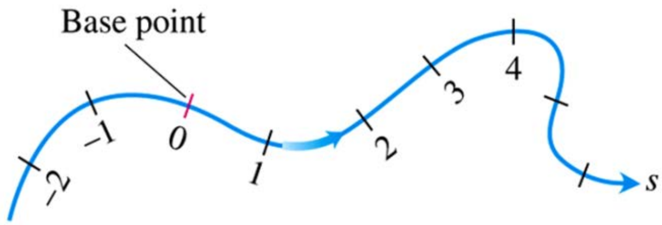
Section 1.7 – Length of Curves



Arc Length along a Space Curve

Definition

The ***length*** of a smooth curve $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}, \quad a \le t \le b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

Arc Length Formula $\qquad L = \int_a^b |\vec{v}|\ dt$

Example

A glider is soaring upward along the helix $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\,\hat{k}$. How long is the glider's path from $t = 0$ to $t = 2\pi$?

Solution

The path segment during this time corresponds to one full turn of the helix. The length of this portion of the curve is

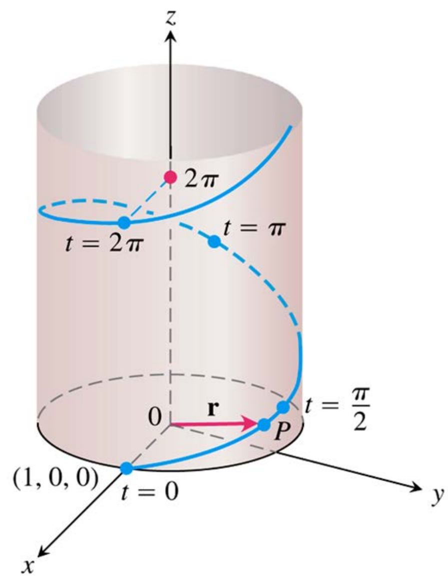
$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\ dt \\ &= \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1^2}\ dt \\ &= \int_0^{2\pi} \sqrt{2}\ dt \end{aligned}$$

$$= \sqrt{2}\left(t \right|_0^{2\pi}$$

$$\underline{= 2\pi\sqrt{2}|}$$

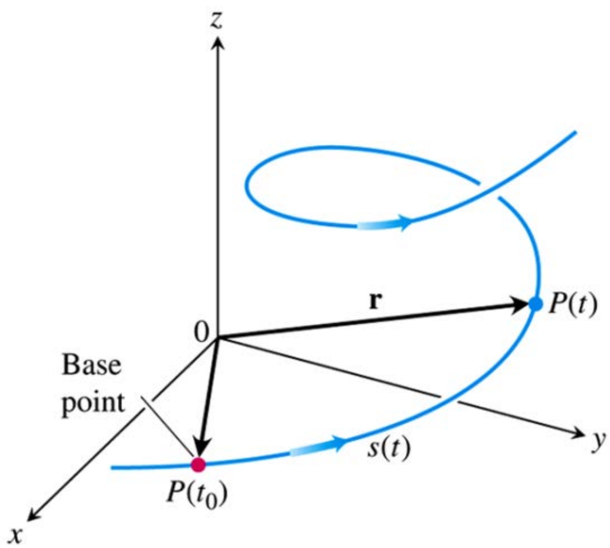
$\therefore$ This is $\sqrt{2}$ times the circumference of the circle in the *xy*-plane over which the helix stands.



Arc Length Parameter with Base Point $P(t_0)$

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau$$

$$= \int_{t_0}^{t} |\vec{v}(\tau)|\, d\tau$$

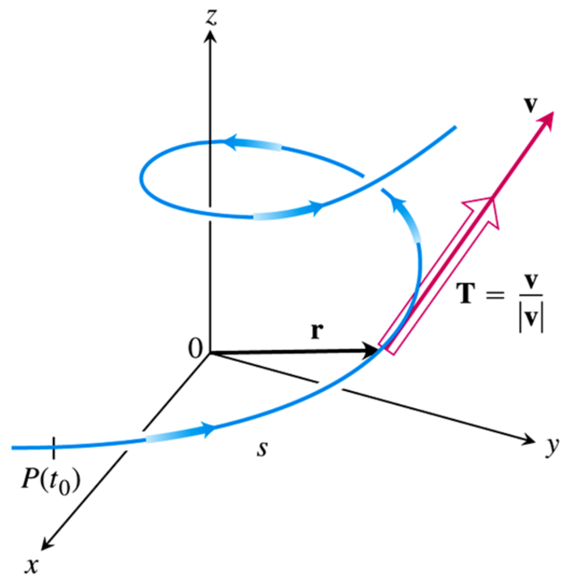


Unit Tangent Vector

The velocity vector $\vec{v} = \frac{d\vec{r}}{dt}$ is tangent to the curve $\vec{r}(t)$ and that the vector

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

A unit vector tangent to the (*smooth*) curve, called the ***unit tangent vector***.



Example

Find the unit tangent vector of the curve $\vec{r}(t) = (3\cos t)\hat{i} + (3\sin t)\hat{j} + t^2\hat{k}$ representing the path of the glider.

Solution

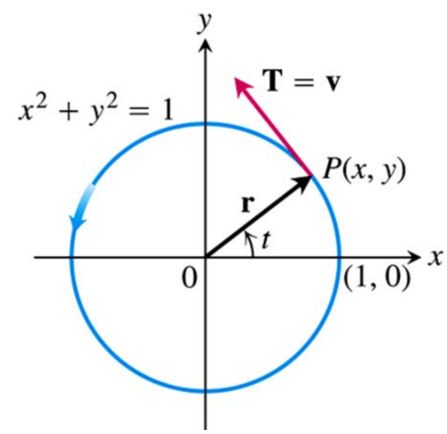
$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -(3\sin t)\hat{i} + (3\cos t)\hat{j} + 2t\hat{k}$$

$$|\vec{v}| = \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2}$$

$$= \sqrt{9 + 4t^2}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$= -\frac{3\sin t}{\sqrt{9+4t^2}}\hat{i} + \frac{3\cos t}{\sqrt{9+4t^2}}\hat{j} + \frac{2t}{\sqrt{9+4t^2}}\hat{k}$$



Exercises Section 1.7 – Length of Curves

(1 – 6) Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

1. $\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j} + \sqrt{5}t\,\hat{k}; \quad 0 \le t \le \pi$

2. $\vec{r}(t) = t\hat{i} + \frac{2}{3}t^{3/2}\hat{k}; \quad 0 \le t \le 8$

3. $\vec{r}(t) = (2+t)\hat{i} - (t+1)\hat{j} + t\,\hat{k}; \quad 0 \le t \le 3$

4. $\vec{r}(t) = \left(\cos^3 t\right)\hat{i} + \left(\sin^3 t\right)\hat{k}; \quad 0 \le t \le \frac{\pi}{2}$

5. $\vec{r}(t) = (t\sin t + \cos t)\hat{i} + (t\cos t - \sin t)\hat{j}; \quad \sqrt{2} \le t \le 2$

6. $\vec{r}(t) = (t\cos t)\hat{i} + (t\sin t)\hat{j} + \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\hat{k}; \quad 0 \le t \le \pi$

7. Find the point on the curve $\vec{r}(t) = (5\sin t)\hat{i} + (5\cos t)\hat{j} + 12t\,\hat{k}$ at a distance 26π units along the curve from the point (0, 5, 0) in the direction of increasing arc length.

(8 – 13) Find the arc length parameter along the curve from the point. Also, find the length of the indicated portion of the curve.

8. $\vec{r}(t) = (4\cos t)\hat{i} + (4\sin t)\hat{j} + 3t\,\hat{k}; \quad 0 \le t \le \frac{\pi}{2}$

9. $\vec{r}(t) = \left(e^t \cos t\right)\hat{i} + \left(e^t \sin t\right)\hat{j} + e^t\,\hat{k}; \quad -\ln 4 \le t \le 0$

10. $\vec{r}(t) = (1+2t)\hat{i} + (1+3t)\hat{j} + (6-6t)\hat{k}; \quad -1 \le t \le 0$

11. $\vec{r}(t) = \left\langle 2t^{9/2},\ t^3 \right\rangle \quad \text{for} \quad 0 \le t \le 2$

12. $\vec{r}(t) = \left\langle t^2,\ \frac{4\sqrt{2}}{3}t^{3/2},\ 2t \right\rangle \quad \text{for} \quad 1 \le t \le 3$

13. $\vec{r}(t) = \left\langle t,\ \ln\sec t,\ \ln(\sec t + \tan t) \right\rangle \quad \text{for} \quad 0 \le t \le \frac{\pi}{4}$

(14 – 15) Find the lengths of the curves

14. $\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j} + t^2\,\hat{k}; \quad 0 \le t \le \frac{\pi}{4}$

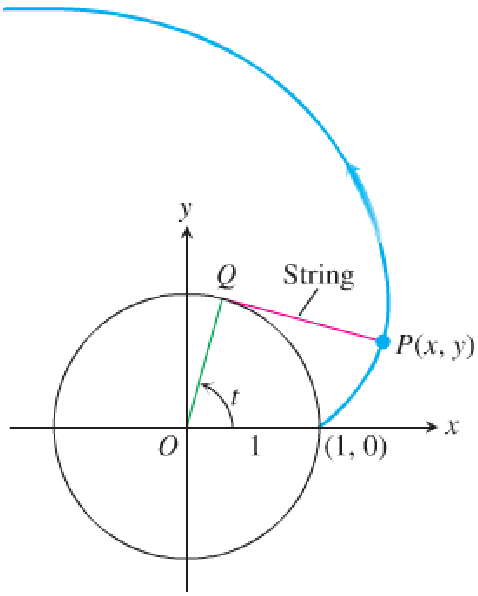
15. $\vec{r}(t) = (3\cos t)\hat{i} + (3\sin t)\hat{j} + 2t^{3/2}\,\hat{k}; \quad 0 \le t \le 3$

16. The acceleration of a wayward firework is given by $\vec{a}(t) = \sqrt{2}\hat{j} + 2t\,\hat{k} \quad \text{for} \quad 0 \le t \le 3$. Suppose the initial velocity of the firework is $\vec{v}(0) = 1$.

a) Find the velocity of the firework, for $0 \le t \le 3$.

b) Find the length of the trajectory of the firework over the interval $0 \le t \le 3$

17. If a string wound around a fixed circle in unwound while held taut in the plane of the circle, its end P traces an involute of the circle. The circle in question is the circle $x^2 + y^2 = 1$ and the tracing point starts at (1, 0). The unwound portion of the string is tangent to the circle at Q, and t is the radian measure of the angle from the position x-axis to segment OQ.



Derive the parametric equations $x = \cos t + t\sin t, \quad y = \sin t - t\cos t, \quad t > 0$ of the point $P(x, y)$ for the involute.