Lecture Two – Second & Higher Order Equations

Section 2.1- Definitions of Second and Higher Order Equations

A second-order differential equation is an equation involving the independent variable t and unknown function y.

$$y'' = f(t, y, y')$$

Linear equation: $$y'' + p(t)y' + q(t)y = g(t)$$

The coefficient $p, q, \text{ and } g$ can be arbitrary functions.

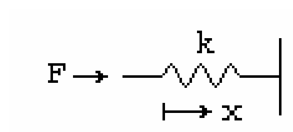
The equation is said to be ***homogeneous*** when:

$$y'' + p(t)y' + q(t)y = 0$$

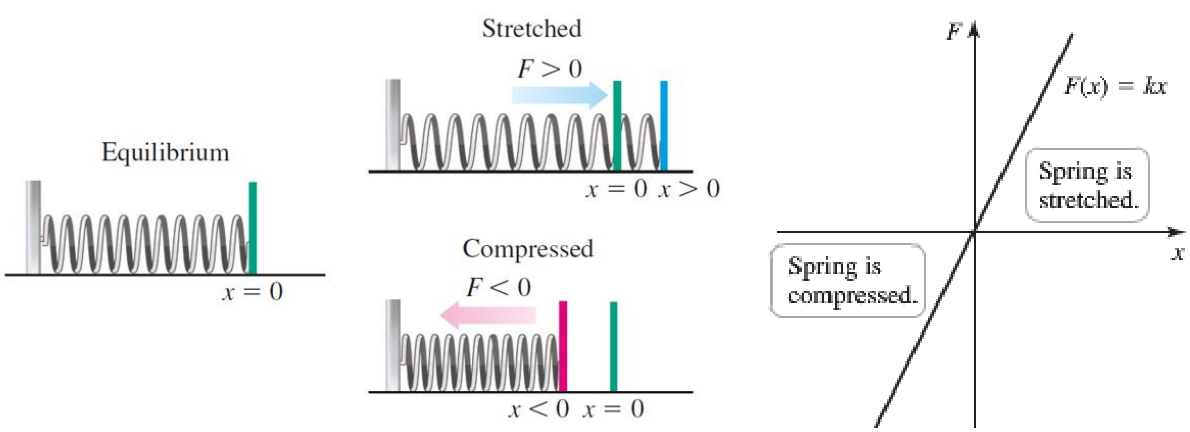
Newton's - Hooke's Law for Springs: *F = kx*

Hooke's Law says that the force required to hold a stretched or compressed spring x units from its natural (unstressed) length is proportional to x. In symbols

$$F = kx$$



The constant k, measured in force units per unit length, is a characteristic of the spring, called ***the force constant*** (or ***spring constant***) of the spring.



- To stretch the spring to a position $x > 0$, a force $F > 0$ (in the ***positive*** direction) is required.
- To compress the spring to a position $x < 0$, a force $F < 0$ (in the ***negative*** direction) is required.

$1\ kg.m/s^2 = 1\ N$ (*Newton*)

Example

A 4-*kg* weight is suspended from a spring. The displacement of the spring-mass equilibrium from the spring equilibrium is measured to be 49 *cm*. What is the spring constant?

Solution

$$mg = kx_0$$

$$k = \frac{mg}{x_0}$$

$$= \frac{4(9.8)}{0.49}$$

$$\underline{= 80\ N/m|}$$

Proposition

$$y'' + p(t)y' + q(t)y = 0$$

Solutions: $y = C_1 y_1 + C_2 y_2$

$C_1,\ C_2$ are any constant.

$y_1(t)\ \&\ y_2(t)$ are linearly independent solutions forming a ***fundamental set of solutions***.

Definition

A linear combination of the two functions $u\ \&\ v$ is any function of the form

$$w = Au + Bv$$

Definition

Two functions $u\ \&\ v$ are said to be linearly independent on the interval $(\alpha,\ \beta)$, if neither is a constant multiple of the order on that interval. If one is a constant multiple of the other on $(\alpha,\ \beta)$, they said to be linearly dependent there.

Wronskian

The Wronskian is a function named after the Polish mathematician Józef Hoene-Wroński and it is used to determine whether a set of differentiable functions (solutions) is ***linearly independent*** on a given interval.

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & & f_n(x) \\ f_1'(x) & f_2'(x) & & f_n'(x) \\ & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & & f_n^{(n-1)}(x) \end{vmatrix}$$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu'$$

If $W = 0 \Rightarrow u$ & v are linearly dependent.

If $W \neq 0 \Rightarrow u$ & v are linearly independent.

Theorem

Suppose that y_1 and y_2 are two solutions of the homogeneous second-order linear equation

$$y'' + p(x)y' + q(x)y = 0$$

On an open interval I on which p and q are continuous

1. If y_1 and y_2 are linearly dependent, then $W(y_1, y_2) \equiv 0$ on I.

2. If y_1 and y_2 are linearly independent, then $W(y_1, y_2) \neq 0$ at each point of I.

Example

Use the Wronskian to show that $\mathbf{f}_1 = x, \quad \mathbf{f}_2 = \sin x$ are linearly independence

Solution

The Wronskian is

$$W(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x \neq 0$$

This function is not identically zero. Thus the functions are linearly independent.

System Equations

A Planar System of 1^{st}- order equations is a set of two first-order differential equations involving two unknown

$$x' = f(t,x,y)$$
$$y' = g(t,x,y)$$

where f *and* g are functions of the independent variable t and the unknown x and y.

Second-Order Equations and Planar Systems

$$y'' = f(t,\ y,\ y')$$

Let's re-write in first-order system:

$$y' = v$$
$$v' = F(t,y,v)$$

$$y'' + p(t)y' + q(t)y = F(t)$$
$$y'' = F(t) - p(t)y' - q(t)y$$
$$v' = F(t) - p(t)v - q(t)y$$

$$\begin{aligned} y' &= v \\ v' &= F(t) - p(t)v - q(t)y \end{aligned}$$

Example

Consider a damped unforced spring: $y'' + 0.4y' + 3y = 0$; which satisfies the initial conditions $y(0) = 2$ and $v(0) = y'(0) = -1$

Solution

$$\begin{cases} y' = v \\ v' = -0.4v - 3y \end{cases}$$

$$v' + 0.4v = -3y$$

$$e^{\int 0.4dy} = e^{0.4y}$$

$$\int -3ye^{0.4y}dy = -\frac{3}{.16}e^{0.4y}(0.4y-1) \qquad \int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$v(y) = \underline{-18.75e^{0.4y}(0.4y-1) + C}$$

$$v = -7.5y + 18.75 + Ce^{-0.4y}$$

$$v(0) = -7.5(0) + 18.75 + Ce^{-0.4(0)}$$

$$-1 = 18.75 + C \quad \rightarrow \quad \boxed{C = -19.75}$$

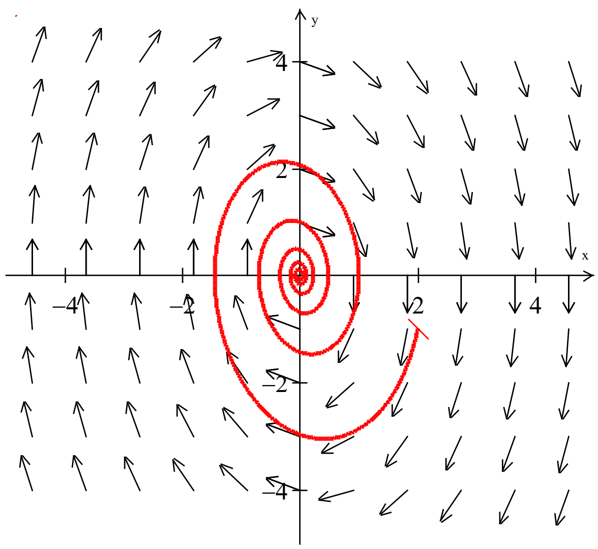
$$v(y) = \boxed{-7.5y + 18.75 - 19.75e^{-0.4y}}$$

$$y' = v = -7.5y + 18.75 - 19.75e^{-0.4y}$$

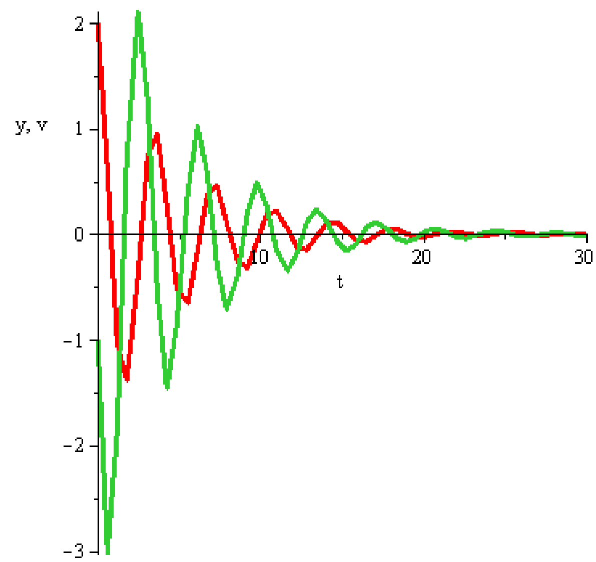
$$\frac{dy}{dt} = -7.5y + 18.75 - 19.75e^{-0.4y}$$

$$\boxed{y(t) = -\frac{3\sqrt{74}}{74}e^{-t/5}\sin\left(\frac{\sqrt{74}}{5}t\right) + 2e^{-t/5}\cos\left(\frac{\sqrt{74}}{5}t\right)}$$

The *yv*-plane is called the ***phase plane***.



Phase Plane Plot



Displacement *y* and the velocity *v*

Exercises *Section* 2.1 – Definitions of 2nd and Higher Order Equations

(*Exercises* 1- 4) Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous.

1. $t^2y'' = 4y' - \sin t$

2. $ty'' + (\sin t)y' = 4y - \cos 5t$

3. $t^2y'' + 4yy' = 0$

4. $y'' + 4y' + 7y = 3e^{-t}\sin t$

Show by direct substitution that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation. Then verify by direct substitution, that any linear combination $C_1y_1(t) + C_2y_2(t)$ of the 2 given solutions is also a solution.

5. $y'' + 4y = 0; \quad y_1(t) = \cos 2t \quad y_2(t) = \sin 2t$

6. $y'' - 2y' + 2y = 0; \quad y_1(t) = e^t\cos t \quad y_2(t) = e^t\sin t$

7. Explain why $y_1(t)$ and $y_2(t)$ are linearly independent solutions. Calculate Wronskian and use it to explain the independence of the given solutions.

$y'' + 9y = 0; \quad y_1(t) = \cos 3t \quad y_2(t) = \sin 3t$

8. Show that $y_1(t) = e^t$ and $y_2(t) = e^{-3t}$ form a fundamental set of solutions for $y'' + 2y' - 3y = 0,$ then find a solution satisfying $y(0) = 1$ and $y'(0) = -2$.

Use the Wronskian to show that are linearly independence

9. $y_1(x) = e^{-3x}, \quad y_2(x) = e^{3x}$

10. $\mathbf{f}_1 = 1, \quad \mathbf{f}_2 = e^x, \quad \mathbf{f}_3 = e^{2x}$

11. $\left\{e^x, \ xe^x, \ (x+1)e^x\right\}$

12. $y_1(x) = e^{-3x}, \quad y_2(x) = \cos 2x, \quad y_3(x) = \sin 2x$

13. $y_1(x) = e^x, \quad y_2(x) = e^{2x}, \quad y_3(x) = e^{3x}$

14. $y_1(x) = \cos^2 x, \quad y_2(x) = \sin^2 x, \quad y_3(x) = \sec^2 x, \quad y_4(x) = \tan^2 x$

Determine whether the functions $y_1(t)$ and $y_2(t)$ are linearly dependent on the interval $(0, \ 1)$

15. $y_1(t) = \cos t\sin t, \quad y_2(t) = \sin 2t$

16. $y_1(t) = e^{3t}, \quad y_2(t) = e^{-4t}$

17. $y_1(t) = te^{2t}, \quad y_2(t) = e^{2t}$

18. $y_1(t) = t^2\cos(\ln t), \quad y_2(t) = t^2\sin(\ln t)$

19. $y_1(t) = \tan^2 t - \sec^2 t, \quad y_2(t) = 3$

20. $y_1(t) \equiv 0, \quad y_2(t) = e^t$

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation.

21. $y'' + 2y' - 3y = 0$

22. $y'' + 3y' + 4y = 2\cos 2t$

23. $y'' + 2y' + 2y = 2\sin 2\pi t$

24. $y'' + \mu\left(t^2 - 1\right)y' + y = 0$

25. $4y'' + 4y' + y = 0$

Find a particular solution satisfying the given initial conditions

26. $y'' - 4y = 0$; $y_1(t) = e^{2t},\ y_2(t) = 2e^{-2t}$; $y(0) = 1,\ y'(0) = -2$

27. $y'' - y = 0$; $y_1(t) = 2e^{t},\ y_2(t) = e^{-t+3}$; $y(-1) = 1,\ y'(-1) = 0$

28. $y'' + y = 0$; $y_1(t) = 0,\ y_2(t) = \sin t$; $y\left(\frac{\pi}{2}\right) = 1,\ y'\left(\frac{\pi}{2}\right) = 1$

29. $y'' + y = 0$; $y_1(t) = \cos t,\ y_2(t) = \sin t$; $y\left(\frac{\pi}{2}\right) = 1,\ y'\left(\frac{\pi}{2}\right) = 1$

30. $y'' - 4y' + 4y = 0$; $y_1(t) = e^{2t},\ y_2(t) = te^{2t}$; $y(0) = 2,\ y'(0) = 0$

31. $2y'' - y' = 0$; $y_1(t) = 1,\ y_2(t) = e^{t/2}$; $y(2) = 0,\ y'(2) = 2$

32. $y'' - 3y' + 2y = 0$; $y_1(t) = 2e^{t},\ y_2(t) = e^{2t}$; $y(-1) = 1,\ y'(-1) = 0$

33. $ty'' + y' = 0$; $y_1(t) = \ln t,\ y_2(t) = \ln 3t$; $y(3) = 0,\ y'(3) = 3$

34. $t^2y'' - ty' - 3y = 0$; $y_1(t) = t^3,\ y_2(t) = -\frac{1}{t}$; $y(-1) = 0,\ y'(-1) = -2$ $(t < 0)$

35. $y'' + \pi^2 y = 0$; $y_1(t) = \sin \pi t + \cos \pi t,\ y_2(t) = \sin \pi t - \cos \pi t$; $y\left(\frac{1}{2}\right) = 1,\ y'\left(\frac{1}{2}\right) = 0$

36. $x^3y^{(3)} - x^2y'' + 2xy' - 2y = 0$

$y(1) = 3,\quad y'(1) = 2,\quad y''(1) = 1 \qquad y_1(x) = x,\quad y_2(x) = x\ln x,\quad y_3(x) = x^2$

37. $y^{(3)} + 2y'' - y' - 2y = 0$

$y(0) = 1,\quad y'(0) = 2,\quad y''(0) = 0 \qquad y_1(x) = e^{x},\quad y_2(x) = e^{-x},\quad y_3(x) = e^{-2x}$

38. $y^{(3)} - 6y'' + 11y' - 6y = 0$

$y(0) = 0,\quad y'(0) = 0,\quad y''(0) = 3 \qquad y_1(x) = e^{x},\quad y_2(x) = e^{2x},\quad y_3(x) = e^{3x}$

39. $y^{(3)} - 3y'' + 3y' - y = 0$

$y(0) = 2,\quad y'(0) = 0,\quad y''(0) = 0 \qquad y_1(x) = e^{x},\quad y_2(x) = xe^{x},\quad y_3(x) = x^2e^{x}$

40. $y^{(3)} - 5y'' + 8y' - 4y = 0$

$y(0) = 1,\quad y'(0) = 4,\quad y''(0) = 0 \qquad y_1(x) = e^{x},\quad y_2(x) = e^{2x},\quad y_3(x) = xe^{2x}$

41. $y^{(3)} + 9y'' = 0$

$y(0) = 3,\quad y'(0) = -1,\quad y''(0) = 2 \qquad y_1(x) = 1,\quad y_2(x) = \cos 3x,\quad y_3(x) = \sin 3x$

42. $y^{(3)} - 3y'' + 4y' - 2y = 0$

$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0 \qquad y_1(x) = e^x, \quad y_2(x) = e^x \cos x, \quad y_3(x) = e^x \sin x$

43. $x^3 y^{(3)} - 3x^2 y'' + 6xy' - 6y = 0$

$y(1) = 6, \quad y'(1) = 14, \quad y''(1) = 1 \qquad y_1(x) = x, \quad y_2(x) = x^2, \quad y_3(x) = x^3$

44. $x^3 y^{(3)} + 6x^2 y'' + 4xy' - 4y = 0$

$y(1) = 1, \quad y'(1) = 5, \quad y''(1) = -11 \qquad y_1(x) = x, \quad y_2(x) = x^{-2}, \quad y_3(x) = x^{-2} \ln x$

(*Exercises* 17-18) Given the mass, damping, and spring constants of an undriven spring-mass system

$$my'' + \mu y' + ky = 0$$

a) Provide separate plots of the position versus time (y vs. t) and the velocity versus time (v vs. t)

b) Provide a combined plot of both position and velocity versus time

c) Provide a plot of the velocity versus position (v vs. y) in the yv phase plane.

45. $m = 1\ kg, \quad \mu = 0\ kg/s, \quad k = 4kg/s^2, \quad y(0) = -2\ m, \quad y'(0) = -2\ m/s$

46. $m = 1\ kg, \quad \mu = 2\ kg/s, \quad k = 1kg/s^2, \quad y(0) = -3\ m, \quad y'(0) = -2\ m/s$

47. When the values of a solution to a differential equation are specified at two different points, these conditions. (In contrast, initial conditions specify the values of a function and its derivative at the same point). The purpose of this is to show that for boundary value problems there is no existence-uniqueness theorem. Given that every solution to

$y'' + y = 0 \qquad$ is of the form $\qquad y(t) = c_1 \cos t + c_2 \sin t$

Where c_1 and c_2 are arbitrary constants, show that

a) There is a unique solution to the given differential equation that satisfies the boundary conditions $y(0) = 2$ and $y\left(\frac{\pi}{2}\right) = 0$

b) There is no solution to given equation that satisfies $y(2) = 0$ and $y(\pi) = 0$

c) There are infinitely many solution to the given DE equation that satisfy $y(0) = 2$ and $y(\pi) = -2$