Section 4.3 – Inferences About Two Means: Independent

Hypothesis Test for Two Means: Independent Samples

$$t=\frac{\left(\bar{x}_1-\bar{x}_2\right)-\left(\mu_1-\mu_2\right)}{\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}}$$ (where $\mu_1-\mu_2$ is ***often*** ***assumed*** to be 0)

approximately follows Student's *t*-distribution with the smaller of n_1-1 or n_2-1 degrees of freedom

Notation

μ_i = population mean

n_1 = size of the first sample

s_i = sample standard deviation

Testing Hypotheses Regarding the Difference of Two Means

To test hypotheses regarding the mean difference of matched-pairs data, the following must be satisfied:

- ✓ The sample is obtained using simple random sampling;
- ✓ The sample are independent;
- ✓ The populations from which the samples are drawn are normally distributed or the sample sizes are large $\left(n_1\geq 30,\ n_2\geq 30\right)$;
- ✓ For each sample, the sample size is no more than 5% of the population size.

Degrees of freedom

1. We use this simple and conservative estimate: d*f* = smaller of n_1-1 and n_2-1.
2. Statistically software typically use the more accurate but more difficult estimate formula

$$\boldsymbol{df}=\frac{(A+B)^2}{\frac{A^2}{n_1-1}+\frac{B^2}{n_2-1}}\qquad \textbf{where}\qquad A=\frac{s_1^2}{n_1}\quad B=\frac{s_2^2}{n_2}$$

Confidence Interval Estimate of $\mu_1-\mu_2$: Independent Samples

The confidence interval estimate of the difference $\mu_1-\mu_2$ is

$$\left(\bar{x}_1-\bar{x}_2\right)-E<\left(\mu_1-\mu_2\right)<\left(\bar{x}_1-\bar{x}_2\right)+E\qquad \textbf{Where}\qquad E=t_{\alpha/2}\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}$$

***Step* 1:** Determine the null and alternative hypotheses. The hypotheses can structured in one of three ways.

Two–Tailed	***Left–Tailed***	***Right–Tailed***
$H_0 : \mu_1 = \mu_2$	$H_0 : \mu_1 = \mu_2$	$H_0 : \mu_1 = \mu_2$
$H_1 : \mu_1 \neq \mu_2$	$H_1 : \mu_1 < \mu_2$	$H_1 : \mu_1 > \mu_2$

***Step* 2:** Select a level of significance, α, based on the seriousness of making a Type I error.

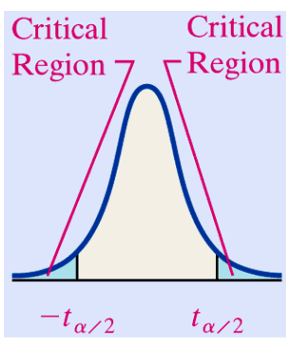
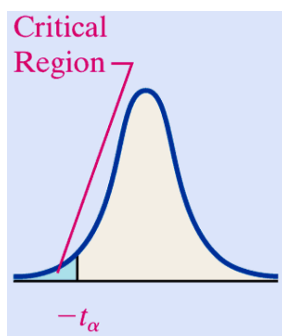
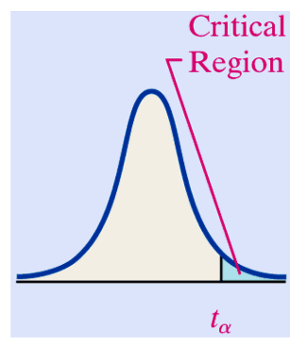
***Step* 3:** Compute the test statistic

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

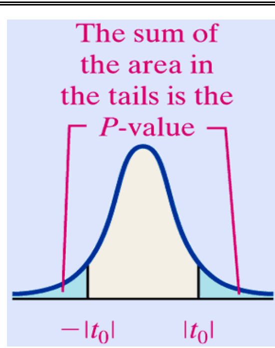
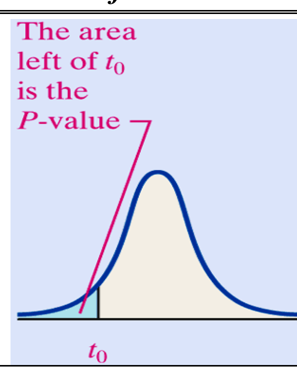
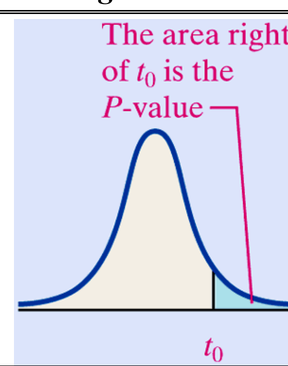
which approximately follows Student's t-distribution.

Use Table to determine the critical value using $n_1 - 1$ and $n_2 - 1$ degrees of freedom.

Classical Approach

Two–Tailed	***Left–Tailed***	***Right–Tailed***
$t_0 < -t_{\alpha/2}$ *or* $t_0 > t_{\alpha/2}$	$t_0 < -t_\alpha$	$t_0 > t_\alpha$
Reject the null hypothesis	*Reject the null hypothesis*	*Reject the null hypothesis*
		

***Step* 4**: Estimate the P–value

Two–Tailed	***Left–Tailed***	***Right–Tailed***
		

*If **P-value < α**, reject the null hypothesis*

Example

A headline in *USA Today* proclaimed that "Men, women are equal talkers." That headline referred to a study of the numbers of words that samples of men and women spoke in a day. Given below are the results from the study. .

Number of Words Spoken in a Day

Men	***Women***
$n_1 = 186$	$n_2 = 210$
$\bar{x}_1 = 15{,}668.5$	$\bar{x}_2 = 16{,}215.0$
$s_1 = 8632.5$	$s_2 = 7301.2$

a) Use a 0.05 significance level to test the claim that men and women speak the same mean number of words in a day. Does there appear to be a difference?

b) Construct a 95% confidence interval estimate of the difference between the mean number of words spoken by men and the mean number of words spoken by women.

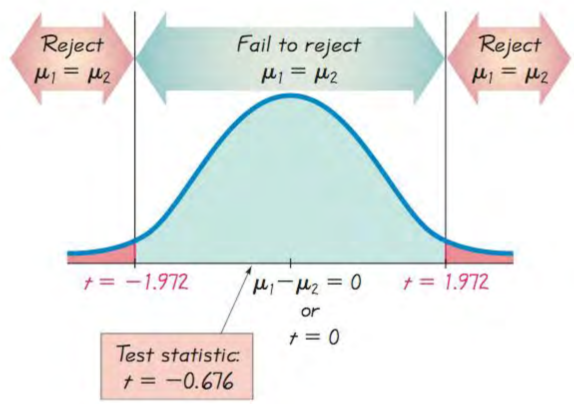
Solution

a) Two population standard deviations are not known and not assumed to be equal, independent samples, simple random samples, both samples are large.

$H_0: \ \mu_1 = \mu_2$

$H_1: \ \mu_1 \neq \mu_2$

Assume: $\mu_1 = \mu_2$ or $\mu_1 - \mu_2 = 0$.



$$t = \frac{\left(\bar{X}_1 - \bar{X}_2\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{(15{,}668.5 - 16{,}215.0) - 0}{\sqrt{\frac{8632.5^2}{186} + \frac{7301.2^2}{210}}}$$

$= -0.676$

Use t Distribution Table: area in two tails is 0.05, $df = 185$, which is not in the table, the closest value is $t = \pm 1.972$

Degrees of Freedom	Area in Two Tails 0.01	0.02	0.05	0.10	0.20
200	**2.601**	2.345	**1.972**	1.653	1.286

Because the test statistic does not fall within the critical region, fail to reject the null hypothesis: $\mu_1 = \mu_2$ (or $\mu_1 - \mu_2 = 0$).

Conclusion

There is not sufficient evidence to warrant rejection of the claim that men and women speak the same mean number of words in a day. There does not appear to be a significant difference between the two means.

b) $E = t_{\alpha/2}\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}} = 1.972\sqrt{\frac{8632.5^2}{186}+\frac{7301.2^2}{210}}$

$\boxed{= 1595.4}$

Construct the confidence interval use E = 1595.4 and

$(\bar{x}_1 - \bar{x}_2) - E < (\mu_1 - \mu_2) < (\bar{x}_1 - \bar{x}_2) + E$

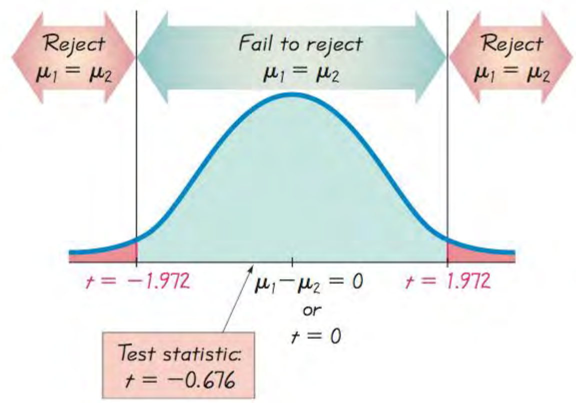
$(15{,}668.5 - 16{,}215.0) - 1595.4 < (\mu_1 - \mu_2) < (15{,}668.5 - 16{,}215.0) + 1595.4$

$-2141.9 < (\mu_1 - \mu_2) < 1048.9$

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}}$$

$$= \frac{(15{,}668.5 - 16{,}215.0) - 0}{\sqrt{\frac{8632.5^2}{186}+\frac{7301.2^2}{210}}}$$

$\boxed{= -0.676}$



Because the test statistic does not fall within the critical region, fail to reject the null hypothesis: $\mu_1 = \mu_2$ (or $\mu_1 - \mu_2 = 0$).

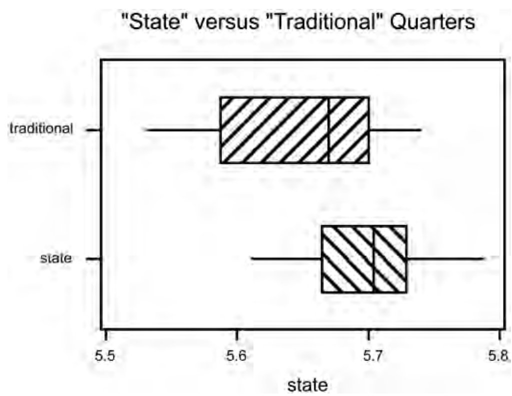
We are 95% confident that the limits of −2141.9 words and 1048.9 words actually do contain the difference between the two population means. Because those limits do contain 0, there is not sufficient evidence to warrant rejection of the claim that men and women speak the same mean number of words in a day. There does not appear to be a significant difference between the two means.

Example

A researcher wanted to know whether “state” quarters had a weight that is more than “traditional” quarters. He randomly selected 18 “state” quarters and 16 “traditional” quarters, weighed each of them and obtained the following data.

STATE (grams)		***TRADITIONAL (grams)***	
5.70	5.67	5.67	5.55
5.73	5.61	5.70	5.61
5.70	5.67	5.72	5.58
5.65	5.62	5.66	5.74
5.73	5.65	5.70	5.68
5.79	5.73	5.68	5.53
5.77	5.71	5.67	5.55
5.70	5.76	5.61	5.74
5.73	5.72		

Test the claim that “state” quarters have a mean weight that is more than “traditional” quarters at the α = 0.05 level of significance.



Solution

Step 1: We want to determine whether state quarters weigh more than traditional quarters:

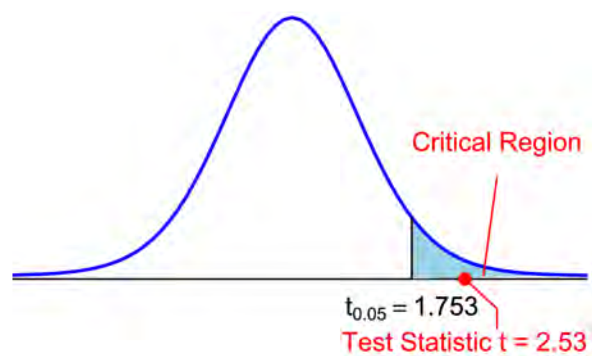
$$\begin{cases} H_0 : \mu_1 = \mu_2 \\ H_1 : \mu_1 \neq \mu_2 \end{cases}$$

Step 2: The level of significance is $\alpha = 0.05$.

Step 3: The test statistic is

$$t_0 = \frac{5.7022 - 5.6494}{\sqrt{\frac{0.0497^2}{18} + \frac{0.0689^2}{16}}} = 2.53$$

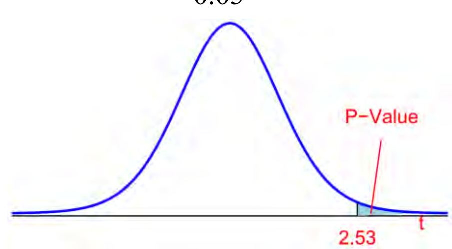
$$t_{0.05} = 1.753$$



Step 4: Since the test statistic, $t_0 = 2.53$ is greater than the critical value $t_{0.05} = 1.753$

We reject the null hypothesis.

Because this is a right-tailed test, the *P*-value is the area under the *t*-distribution to the right of the



test statistic $t_0 = 2.53$. That is, P-value = $P(t > 2.53) \approx 0.01$

Step 4: Since the P-value is less than the level of significance $\alpha = 0.05$, we reject the null hypothesis.

Step 5: There is sufficient evidence at the $\alpha = 0.05$ level to conclude that the state quarters weigh more than the traditional quarters.

Alternative Methods When σ_1 and σ_2 are Known

Requirements

1. The two population standard deviations are both known.
2. The two samples are independent.
3. Both samples are simple random samples.
4. Either or both of these conditions are satisfied: The two sample sizes are both large (with $n_1 > 30$ and $n_2 > 30$) or both samples come from populations having normal distributions.

Hypothesis Test for Two Means: Independent Samples with σ_1 and σ_2 Both Known

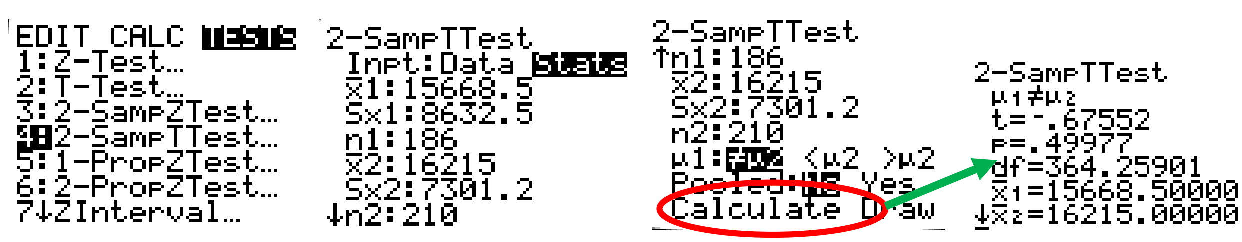
$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

P-values and critical values: Refer to Normal Distribution Table.

Confidence Interval: Independent Samples with σ_1 and σ_2 Both Known

$$(\bar{x}_1 - \bar{x}_2) - E < (\mu_1 - \mu_2) < (\bar{x}_1 - \bar{x}_2) + E$$

Where $E = z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$



Exercises *Section* 4.3 – Inferences about Two Means: Independent

1. If the pulse rates of men and women shown in the data below

Women:

76	72	88	60	72	68	80	64	68	68	80	76	68	72	96	72	68	72	64	80
64	80	76	76	76	80	104	88	60	76	72	72	88	80	60	72	88	88	124	64

Men:

68	64	88	72	64	72	60	88	76	60	96	72	56	64	60	64	84	76	84	88
72	56	68	64	60	68	60	60	56	84	72	84	88	56	64	56	56	60	64	72

These data are used to construct 95% confidence interval for the difference between the two population means, the result is $-12.2 < \mu_1 - \mu_2 < -1.6$, where pulse rates of men correspond to population 1 and pulse rates of women correspond to population 2. Express the confidence interval with pulse rates of women being population 1 and pulse rates of men being population 2.

2. Assume that you want to use a 0.01 significance level to test the claim that the mean pulse rate of men is less than the mean pulse rate of women. What confidence level should be used if you want to test that claim using a confidence interval?

3. To test the effectiveness of Lipitor, cholesterol levels are measured in 250 subjects before and after Lipitor treatments. Determine whether this sample is independent or dependent.

4. On each of 40 different days, you measured the voltage supplied to your home and you also measured the voltage produced by the gasoline-powered generator. One sample consists of the voltages in the house and the second sample consists of the voltages produced by the generator. Determine whether this sample is independent or dependent.

5. In a randomized controlled trial conducted with children suffering from viral croup, 46 children were treated with low humidity while 46 other children were treated with high humidity. Researchers used the Westley Croup Score to assess the results after one hour. The low humidity group had a mean score of 0.98 with standard deviation of 1.22 while the high humidity group had a mean score of 1.09 with standard deviation of 1.11.

a) Use a 0.05 significance level to test the claim that the two groups are from populations with the same mean. What does the result suggest about the common treatment of humidity?
Assume that the two samples are independent simple random samples selected from normally distributed populations.

b) Assume that $\sigma_1 = \sigma_2$, how are the results affected by this additional assumption?

6. The mean tar content of a simple random sample of 25 unfiltered king size cigarettes is 21.1 mg, with a standard deviation of 3.2 mg. The mean tar content of a simple random sample of 25 filtered 100 mm cigarettes is 13.2 mg, with a standard deviation of 3.7 mg.
Assume that the two samples are independent simple random samples selected from normally distributed populations in part a and b.

a) Construct a 90% confidence interval estimate of the difference between the mean tar content of unfiltered king size cigarettes and the mean tar content of filtered 100 mm cigarettes. Does the result suggest that 100 mm filtered cigarettes have less tar than unfiltered king size cigarettes?

b) Use a 0.05 significance level to test the claim that unfiltered king size cigarettes have a mean tar content greater than that of filtered 100 mm cigarettes. What does the result suggest about the effectiveness of cigarette filters?

c) Assume that $\sigma_1 = \sigma_2$, how are the results affected by this additional assumption?

7. The heights are measured for the simple random sample of supermodels Crawford, Bundchen, Pestova, Christenson, Hume, Moss, Campbell, Schiffer, and Taylor. They have a mean of 70.0 in. and a standard deviation of 1.5 in. 40 women who are not supermodels, listed below and they have heights with means of 63.2 in. and a standard deviation of 2.7 in.

64.3	66.4	62.3	62.3	59.6	63.6	59.8	63.3	67.9	61.4	66.7	64.8	63.1	66.7	66.8
64.7	65.1	61.9	64.3	63.4	60.7	63.4	62.6	60.6	63.5	58.6	60.2	67.6	63.4	64.1
62.7	61.3	58.2	63.2	60.5	65.0	61.8	68.0	67.0	57.0					

a) Use a 0.01 significance level to test the claim that the mean height of supermodels is greater than the mean height of women who are not supermodels

b) Construct a 98% confidence interval level for the difference between the mean height of supermodels and the mean height of women who are not supermodels. What does the result suggest about those two means?

8. Many studies have been conducted to test the effects of marijuana use on mental abilities. In one such study, groups of light and heavy users of marijuana in college were tested for memory recall, with the results given below. Use a 0.01 significance level to test the claim that the population of heavy marijuana users has a lower mean than the light users. Should marijuana use be of concern to college students?

Items sorted correctly by light marijuana users: $n = 64, \quad \bar{x} = 53.3, \quad s = 3.6$

Items sorted correctly by heavy marijuana users: $n = 65, \quad \bar{x} = 51.3, \quad s = 4.5$

9. The trend of thinner Miss America winners has generated charges that the contest encourages unhealthy diet habits among young women. Listed below are body mass indexes (BMI) for Miss America winners from two different time periods. Consider the listed values to be simple random samples selected from larger populations.

a) Use a 0.05 significance level to test the claim that recent winners have a lower mean BMI than winners from the 1920s and 1930s.

b) Construct a 90% Confidence interval for the difference between the mean BMI of recent winners and the mean BMI of winners from the 1920s and 1930s.

BMI (from recent winners):	19.5	20.3	19.6	20.2	17.8	17.9	19.1	18.8	17.6	16.8
BMI (from 1920s and 1930s):	20.4	21.9	22.1	22.3	20.3	18.8	18.9	19.4	18.4	19.1

10. Listed below are amounts of strontium-90 (in millibecquerels or mBq per gram of calcium) in a simple random sample of baby teeth obtained from Pennsylvania residents and New York residents born after 1979.

a) Use a 0.05 significance level to test the claim that the mean amount of strontium-90 from Pennsylvania residents is greater than the mean amount from New York residents.

b) Construct a 90% Confidence interval for the difference between the mean amount of strontium-90 from Pennsylvania residents and the mean amount from New York residents.

Pennsylvania:	155	142	149	130	151	163	151	142	156	133	138	161
New York:	133	140	142	131	134	129	128	140	140	140	137	143

11. Listed below are the word counts for male and female psychology students.

a) Use a 0.05 significance level to test the claim that male and female psychology students speak the same mean number of words in a day.

b) Construct a 95% Confidence interval estimate of the difference between the mean number of words spoken in a day by male and female psychology students. Do the confidence interval limits include 0, and what does that suggest about the two means?

Male	21143	17791	36571	6724	15430	11552	11748	12169	15581	23858	5269
	12384	11576	17707	15229	18160	22482	18626	1118	5319		

Female	6705	21613	11935	15790	17865	13035	24834	7747	3852	11648	25862
	17183	11010	11156	11351	25693	13383	19992	14926	14128	10345	13516
	12831	9671	17011	28575	23557	13656	8231	10601	8124		

Assume that the two samples are independent simple random samples selected from normally distributed populations. Do not assume that the population standard deviations are equal.

12. Refer to the tables below and test the claim that they contain the same amount of cola, the mean weight of cola cans of regular Coke is the same as the mean weight of cola in cans of Diet Coke. If there is a difference in the mean weights, identify the most likely explanation for that difference.

Coke	0.8192	0.815	0.8163	0.8211	0.8181	0.8247	0.8062	0.8128	0.8172	0.811
	0.8251	0.8264	0.7901	0.8244	0.8073	0.8079	0.8044	0.817	0.8161	0.8194
	0.8189	0.8194	0.8176	0.8284	0.8165	0.8143	0.8229	0.815	0.8152	0.8244
	0.8207	0.8152	0.8126	0.8295	0.8161	0.8192				

Diet	0.7773	0.7758	0.7896	0.7868	0.7844	0.7861	0.7806	0.783	0.7852	0.7879
	0.7881	0.7826	0.7923	0.7852	0.7872	0.7813	0.7885	0.776	0.7822	0.7874
	0.7822	0.7839	0.7802	0.7892	0.7874	0.7907	0.7771	0.787	0.7833	0.7822
	0.7837	0.791	0.7879	0.7923	0.7859	0.7811				

Assume that the two samples are independent simple random samples selected from normally distributed populations. Do not assume that the population standard deviations are equal.

13. An Experiment was conducted to test the effects of alcohol. Researchers measured the breath alcohol levels for a treatment group of people who drank ethanol and another group given a placebo. The results are given in the accompanying table. Use a 0.05 significance level to test the claim that the two sample groups come from populations with the same mean.

Treatment Group:		$\bar{x}_1 = 0.049$	$s_1 = 0.015$
Placebo Group:	$n_2 = 22$	$\bar{x}_2 = 0.000$	$s_2 = 0.000$

14. A researcher was interested in comparing the GPAs $n_1 = 22$ of students at two different colleges. Independent simple populations. Do samples of 8 students from college *A* and 13 students from college *B* yielding the following GPAs.

College *A*	3.7	3.2	3.0	2.5	2.7	3.6	2.8	3.4					
College *B*	3.8	3.2	3.0	3.9	3.8	2.5	3.9	2.8	4.0	3.6	2.6	4.0	3.6

Construct a 95% confidence interval for $\mu_1 - \mu_2$. The difference between the mean GPA of college *A* students and the mean GPA of college *B* students.

$\left(Note: \bar{x}_1 = 3.1125,\ \bar{x}_2 = 3.4385,\ s_1 = 0.4357,\ s_2 = 0.5485\right)$

15. Assume that the two samples are independent simple random samples selected from normal distributed populations. Do not assume that the population standard deviations are equal.
A researcher was interested in comparing the heights of women in two different countries. Independent simple random samples of 9 women from country A and 9 women from B yielded to the following heights (in inches).

Country A	64.1	66.4	61.7	62.0	67.3	64.9	64.7	68.0	63.6
Country B	65.3	60.2	61.7	65.8	61.0	64.6	60.0	65.4	59.0

Construct a 90% confidence interval for $\mu_1 - \mu_2$ the difference between the mean height of women in country A and the mean height of women in country B. Round to two decimal places.

$\left(Note: \bar{x}_1 = 64.744\ in,\ \bar{x}_2 = 62.556\ in,\ s_1 = 2.192\ in,\ s_2 = 2.697\ in\right)$