SOLUTION *Section* 3.4 – Comparison Tests

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+30}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$, which is a convergent p-series, since $p = 2 > 1$.

Both series have nonnegative terms for $n \ge 1$

$n^2 \le n^2 + 30$

$\dfrac{1}{n^2} \ge \dfrac{1}{n^2+30}$

Then, by Comparison Test, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+30}$ ***converges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty}\frac{n-1}{n^4+2}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$, which is a convergent p-series, since $p = 3 > 1$.

Both series have nonnegative terms for $n \ge 1$

$$n^4 \le n^4 + 2 \quad \Rightarrow \quad \frac{1}{n^4} \ge \frac{1}{n^4+2}$$

$$\frac{n}{n^4} \ge \frac{n}{n^4+2} \ge \frac{n-1}{n^4+2}$$

$$\frac{1}{n^3} \ge \frac{n}{n^4+2} \ge \frac{n-1}{n^4+2}$$

Then, by *Comparison Test*, the given series ***converges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, which is a divergent p-series, since $p = 1 \le 1$.

Both series have nonnegative terms for $n \ge 2$

$$n^2 - n \le n^2 \quad \Rightarrow \quad \frac{1}{n^2-n} \ge \frac{1}{n^2}$$

$$\frac{n}{n^2-n} \ge \frac{n}{n^2} = \frac{1}{n}$$

Then, by *Comparison Test*, the given series ***diverges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which is a convergent p-series, since $p = \frac{3}{2} > 1$.

Both series have nonnegative terms for $n \ge 1$

$$0 \le \cos^2 n \le 1$$

$$0 \le \frac{\cos^2 n}{n^{3/2}} \le \frac{1}{n^{3/2}}$$

Then, by *Comparison Test*, the given series ***converges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} < 1$.

Both series have nonnegative terms for $n \ge 1$

$\sqrt{n} \ge 1 \quad \Rightarrow \quad 2\sqrt{n} \ge 2$

$$2\sqrt{n} + 1 \ge 3$$

Then, by *Comparison Test*, the given series ***diverges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$, which is a convergent series.

Then $0 < \dfrac{1}{2^n + 1} < \dfrac{1}{2^n}$

Therefore, the given series ***converges*** by *comparison Test*.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{3n+1}{n^3 + 1}$

Solution

$$\frac{3n+1}{n^3+1} \xrightarrow[n\to\infty]{} \frac{3}{n^2}$$

$$\begin{aligned}\frac{3n+1}{n^3+1} &= \frac{3n}{n^3+1} + \frac{1}{n^3+1} \\ &< \frac{3n}{n^3} + \frac{1}{n^3} \\ &< \frac{3}{n^2} + \frac{1}{n^2} \\ &= \frac{4}{n^2}\end{aligned}$$

Therefore, by *Comparison Test,* the given series ***converges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

Solution

$< \ln n < n$

$\frac{1}{\ln n} > \frac{1}{n}$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ ***diverges*** to infinity (it is a harmonic series),

Therefore; by *Comparison Test* the series $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ ***diverges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

Solution

$2n-1 < 2n$

$\frac{1}{2n-1} > \frac{1}{2n} \qquad \text{for } n \ge 1$

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} > \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$$

By the ***p****-series* the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $p = 1$

Therefore; by *Comparison Test,* the given series ***diverges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$

Solution

$3n^2 + 2 > 3n^2 \quad \Rightarrow \quad \frac{1}{3n^2+2} < \frac{1}{3n^2}$

By the ***p**-series* the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $p > 1$

Therefore; by *Comparison Test,* the given series ***converges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

Solution

$$\sqrt{n} - 1 < \sqrt{n}$$

$$\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}} \qquad \textit{for } n \ge 2$$

By the ***p**-series* the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges $p = \frac{1}{2} < 1$

Therefore; by *Comparison Test,* the given series ***diverges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=0}^{\infty} \frac{4^n}{5^n + 3}$

Solution

$$\frac{4^n}{5^n + 3} < \frac{4^n}{5^n} = \left(\frac{4}{5}\right)^n$$

By the geometric series: $r = \frac{4}{5} < 1$ converges

Therefore; by *Comparison Test,* the given series ***converges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=2}^{\infty} \frac{\ln n}{n+1}$

Solution

$$\frac{\ln n}{n+1} > \frac{1}{n+1} \qquad \text{(and by integral test)}$$

The given series ***converges*** by *Comparison Test* with the divergent series.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

Solution

$$\frac{1}{\sqrt{n^3+1}} < \frac{1}{\sqrt{n^3}}$$

By the ***p****-series* the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges $p = \frac{3}{2} > 1$

Therefore; by *Comparison Test,* the given series ***converges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=0}^{\infty} \frac{1}{n!}$

Solution

$$\frac{1}{n^2} > \frac{1}{n!} \quad \text{For } n > 3$$

By the ***p****-series* the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $p = 2 > 1$

Therefore; by *Comparison Test,* the given series ***converges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$

Solution

$$\frac{1}{4\sqrt[3]{n}-1} > \frac{1}{4\sqrt[3]{n}}$$

By the ***p****-series* the series $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ diverges $p = \frac{1}{3} < 1$

Therefore; by *Comparison Test,* the given series ***diverges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=0}^{\infty} e^{-n^2}$

Solution

$$\frac{1}{e^{n^2}} \le \frac{1}{e^n}$$

Geometric series: $r = \frac{1}{e} < 1$ converges

Therefore; by *Comparison Test,* the given series ***converges***.

Exercise

Use the *Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$

Solution

$$\frac{3^n}{2^n - 1} > \left(\frac{3}{2}\right)^n$$

Geometric series: $r = \frac{3}{2} > 1$ *diverges*

Therefore; by *Comparison Test,* the given series ***diverges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n-2}{n^3 - n^2 + 3}$

Solution

Comparing with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent *p*-series, since $p = 2 > 1$.

$$a_n = \frac{n-2}{n^3 - n^2 + 3} \quad \Rightarrow \quad b_n = \frac{1}{n^2}$$

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n-2}{n^3 - n^2 + 3} \cdot \frac{n^2}{1} \\ &= \lim_{n\to\infty} \frac{n^3 - 2n^2}{n^3 - n^2 + 3} \qquad \text{or } \textbf{\textit{L'Hopital Rule}} \\ &= 1 > 0 \end{aligned}$$

Therefore; by *Limit Comparison Test,* the given series ***converges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)}$

Solution

Comparing with $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent *p*-series, since $p = 1 \le 1$.

$$a_n = \frac{n(n+1)}{(n^2+1)(n-1)} \quad \Rightarrow \quad b_n = \frac{n^2}{n^3} = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n(n+1)}{(n^2+1)(n-1)} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{n^3+n^2}{n^3-n^2+n-1}$$

$$\underline{= 1 > 0\;|}$$

Therefore; by *Limit Comparison Test,* the given series ***diverges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$, which is a convergent geometric, since $|r| = \frac{1}{2} < 1$.

$$a_n = \frac{2^n}{3+4^n} \quad \Rightarrow \quad b_n = \frac{1}{2^n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{3+4^n} \cdot \frac{2^n}{1}$$

$$= \lim_{n\to\infty} \frac{2^{2n}}{3+4^n}$$

$$= \lim_{n\to\infty} \frac{4^n}{3+4^n}$$

$$= \lim_{n\to\infty} \frac{4^n \ln 4}{4^n \ln 4}$$

$= 1 > 0$

Therefore; by *Limit Comparison Test,* the given series ***converges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\,4^n}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent *p*-series, since $p = \frac{1}{2} < 1$.

$$a_n = \frac{5^n}{\sqrt{n}\,4^n} \quad \Rightarrow \quad b_n = \frac{1}{\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5^n}{\sqrt{n}\,4^n} \cdot \frac{\sqrt{n}}{1}$$

$$= \lim_{n\to\infty} \frac{5^n}{4^n}$$

$$= \lim_{n\to\infty} \left(\frac{5}{4}\right)^n$$

$$= \infty$$

Therefore; by *Limit Comparison Test,* the given series ***diverges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$, which is a convergent geometric, since $|r| = \frac{2}{5} < 1$.

$$a_n = \left(\frac{2n+3}{5n+4}\right)^n \quad \Rightarrow \quad b_n = \left(\frac{2}{5}\right)^n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{2n+3}{5n+4}\right)^n \cdot \left(\frac{5}{2}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{10n+15}{10n+8}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{10n}{10n}\right)^n$$

$$= \lim_{n\to\infty} 1^n$$

$$\boxed{= 1 > 0}$$

Therefore; by *Limit Comparison Test,* the given series ***converges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

Solution

Comparing with $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent *p*-series, since $p = 1 \le 1$.

$$a_n = \frac{1}{\ln n} \quad \Rightarrow \quad b_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\ln n} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{n}{\ln n}$$

$$= \lim_{n\to\infty} \frac{1}{1/n}$$

$$= \lim_{n\to\infty} n$$

$$\boxed{= \infty}$$

Therefore; by *Limit Comparison Test,* the given series ***diverges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$

Solution

Let $b_n = \dfrac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{1}{1+\sqrt{n}} \Big/ \frac{1}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}}$$

$$\boxed{= 1}$$

Since the ***p****-series* diverges to infinity $\left(p = \frac{1}{2}\right)$

Therefore; by *Limit Comparison Test,* the given series ***diverges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n+5}{n^3 - 2n + 3}$

Solution

Let $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{n+5}{n^3 - 2n + 3} \Big/ \frac{1}{n^2} = \lim_{n\to\infty} \frac{n^3 + 5n^2}{n^3 - 2n + 3}$$

$$= 1 < \infty$$

Since the ***p**-series* converges $(p = 2)$

Therefore; by *Limit Comparison Test,* the given series ***converges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$

Solution

$b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{n}{n^2 + 1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^2}{n^2 + 1}$$

$$= 1 < \infty$$

Since the ***p**-series* diverges to infinity $(p = 1)$

Therefore; by *Limit Comparison Test,* the given series ***diverges***.

Exercise

Use the Limit Comparison Test to determine if the series converges or diverges. $\sum_{n=1}^{\infty} \frac{5}{4^n + 1}$

Solution

$b_n = \frac{1}{4^n} = \left(\frac{1}{4}\right)^n$

$$\lim_{n\to\infty} \frac{5}{4^n + 1} \cdot \frac{4^n}{1} = 5$$

By geometric series $\left(r = \frac{1}{4} < 1\right)$ converges

Therefore; by *Limit Comparison Test,* the given series ***converges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$

Solution

$$b_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} \cdot \frac{n}{1} = 1 < \infty$$

Since the ***p**-series* diverges to infinity $(p=1)$

Therefore; by *Limit Comparison Test,* the given series ***diverges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{5^n+1}$

Solution

$$b_n = \left(\frac{2}{5}\right)^n$$

$$\lim_{n\to\infty} \frac{2^n+1}{5^n+1} \cdot \frac{5^n}{2^n} = 1$$

By geometric series $\left(r = \frac{2}{5} < 1\right)$ converges

Therefore; by *Limit Comparison Test,* the given series ***converges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$

Solution

Let $b_n = \frac{1}{n^3}$

$$\lim_{n\to\infty} \frac{2n^2-1}{3n^5+2n+1} \Big/ \frac{1}{n^3} = \lim_{n\to\infty} \frac{2n^5-n^3}{3n^5+2n+1}$$

$$= \frac{2}{3} < \infty$$

Since the ***p**-series* converges $(p=3)$

Therefore; by *Limit Comparison Test,* the given series ***converges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2(n+3)}$

Solution

Let $b_n = \dfrac{1}{n^3}$ By the ***p****-series* converges $(p=3)$

$$\lim_{n\to\infty} \frac{1}{n^2(n+3)} \Big/ \frac{1}{n^3} = \lim_{n\to\infty} \frac{n^3}{n^2(n+3)}$$

$$= 1 < \infty$$

Therefore; by *Limit Comparison Test,* the given series ***converges***.

Exercise

Use the *Limit Comparison Test* to determine if the series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

Solution

Let $b_n = \dfrac{1}{n^2}$ By the ***p****-series* converges $(p=2)$,

$$\lim_{n\to\infty} \frac{1}{n\sqrt{n^2+1}} \cdot \frac{n^2}{1} = 1 < \infty$$

Therefore; by *Limit Comparison Test,* the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+\sqrt[3]{n}}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} < 1$.

$$a_n = \frac{1}{2\sqrt{n}+\sqrt[3]{n}} \Rightarrow b_n = \frac{1}{\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{2\sqrt{n}+\sqrt[3]{n}} \cdot \frac{\sqrt{n}}{1}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{2\sqrt{n}+\sqrt[3]{n}}$$

$$= \lim_{n\to\infty} \frac{1}{2+n^{1/3-1/2}}$$

$$= \lim_{n\to\infty} \frac{1}{2+n^{-1/6}}$$

$$= \frac{1}{2} > 0$$

Then, by Comparison Test, the given series ***diverges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$, which is a convergent geometric, since $|r| = \frac{1}{2} < 1$.

By the Direct Comparison Test: $\dfrac{\sin^2 n}{2^n} \le \dfrac{1}{2^n}$

The given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which is a convergent p-series, since $p = \frac{3}{2} > 1$.

$$a_n = \frac{n+1}{n^2\sqrt{n}} \quad \Rightarrow \quad b_n = \frac{1}{n^{3/2}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{n^2\sqrt{n}} \cdot \frac{n^{3/2}}{1}$$

$$= \lim_{n\to\infty} \frac{n+1}{n}$$

$$= 1 > 0$$

Then, by Comparison Test, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$

Solution

Comparing with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series, since $p = 2 > 1$.

$$a_n = \frac{10n+1}{n(n+1)(n+2)} \quad \Rightarrow \quad b_n = \frac{1}{n^2}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{10n+1}{n(n+1)(n+2)} \cdot \frac{n^2}{1} \\
&= \lim_{n\to\infty} \frac{10n^3+n^2}{n\left(n^2+3n+2\right)} \\
&= \lim_{n\to\infty} \frac{10n^3+n^2}{n^3+3n^2+2} \\
&= 10 > 0
\end{aligned}$$

Then, by Comparison Test, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$

Solution

By the Direct Comparison Test: $\left(\frac{n}{3n+1}\right)^n < \left(\frac{n}{3n}\right)^n = \left(\frac{1}{3}\right)^n$

$\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$, which is a convergent geometric, since $|r| = \frac{1}{3} < 1$.

Therefore, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$

Solution

Comparing with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series, since $p = 2 > 1$.

$$a_n = \frac{(\ln n)^2}{n^3} \quad \Rightarrow \quad b_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(\ln n)^2}{n^3} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{(\ln n)^2}{n}$$

$$= \lim_{n\to\infty} \frac{2\ln n\left(\frac{1}{n}\right)}{1}$$

$$= 2 \lim_{n\to\infty} \frac{\ln n}{n}$$

$$\underline{= 0 \,|}$$

Then, by Comparison Test, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{1+\sin n}{n^2}$

Solution

Let $b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{1+\sin n}{n^2} \Big/ \frac{1}{n^2} = \lim_{n\to\infty} (1+\sin n)$ which does not exist.

$\frac{1+\sin n}{n^2} \le \frac{2}{n^2}$, then the given series converges by comparison test

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

Solution

$$a_n = \frac{1}{2+3^n} < \frac{1}{3^n} = b_n$$

So, by the Direct Comparison Test, the series converges.

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$

Solution

$$2+\sqrt{n} < n \quad \Rightarrow \quad \frac{1}{2+\sqrt{n}} \geq \frac{1}{n}$$

By the ***p**-series* the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $p=1$

The given series ***diverges*** by Comparison Test using ***p**-series.*

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{1}{an+b}$

Solution

$$an+b < n \quad \Rightarrow \quad \frac{1}{an+b} \geq \frac{1}{n} \qquad a,\ b > 0$$

By the ***p**-series* the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $p=1$

The given series ***diverges*** by Comparison Test using ***p**-series.*

$$\lim_{n\to\infty} \frac{1}{an+b} \cdot \frac{n}{1} = \frac{1}{a} > 0$$

The given series ***diverges*** by Limit Comparison Test

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

Solution

Let $b_n = \frac{1}{n^{3/2}}$ By the ***p**-series* converges $\left(p = \frac{3}{2} > 1\right)$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n^2+1} \cdot \frac{n^{3/2}}{1} = 1$$

Therefore, the given series ***converges*** by the limit comparison test.

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$

Solution

$$\frac{\sqrt[3]{n}}{n} = \frac{1}{n^{2/3}}$$

By the ***p**-series* the series $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges $p = \frac{2}{3} < 1$

The given series ***diverges*** by comparison test using ***p**-series.*

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=0}^{\infty} 5\left(-\frac{4}{3}\right)^{n}$

Solution

$$|r| = \frac{4}{3} > 1$$

The given series ***diverges*** by Geometric series

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=1}^{\infty} \frac{1}{5^n + 1}$

Solution

$$\frac{1}{5^n + 1} < \left(\frac{1}{5}\right)^n$$

The given series converges by a Direct Comparison with the convergent geometric series $\left(r = \frac{1}{5} < 1\right)$

Exercise

Use any method to determine if the series converges or diverges $\sum_{n=2}^{\infty} \frac{1}{n^3 - 8}$

Solution

Let $b_n = \frac{1}{n^3}$ By the ***p**-series* converges $(p = 3)$

$$\lim_{n\to\infty} \frac{1}{n^3-8} \cdot \frac{n^3}{1} = 1$$

Therefore, the given series ***converges*** by the limit comparison test with ***p**-series*.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n-2}$

Solution

$$\lim_{n\to\infty} \frac{2n}{3n-2} = \frac{2}{3} \neq 0$$

The given series ***diverges*** by the Limit.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

Solution

$$\begin{aligned}\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) &= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots \\ &= \frac{1}{2}\end{aligned}$$

The given series ***converges*** by telescoping series.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\left(n^2+1\right)^2}$

Solution

$$\begin{aligned}\int_1^{\infty} \frac{x}{\left(x^2+1\right)^2} dx &= \frac{1}{2}\int_1^{\infty} \frac{1}{\left(x^2+1\right)^2} d\left(x^2+1\right) \\ &= -\frac{1}{2}\frac{1}{x^2+1}\Big|_1^{\infty} \\ &= -\frac{1}{2}\left(0-\frac{1}{2}\right)\end{aligned}$$

$$= \boxed{\frac{1}{4}}$$

The given series ***converges*** by the *Integral Test*.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1}$

Solution

$$b_n = \frac{n^2}{2^n}$$

$$a_n = \frac{n2^n}{4n^3+1}$$

$$\lim_{k\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n2^n}{4n^3+1} \cdot \frac{n^2}{2^n}$$

$$= \boxed{\frac{1}{4}}$$

Therefore; by the *Limit Comparison Test*, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{k=1}^{\infty} \frac{|\sin k|}{k^2}$

Solution

$$|\sin k| \le 1$$

$$\frac{|\sin k|}{k^2} \le \frac{1}{k^2}$$

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by ***p**-series* $(p = 2 \ > 1)$

Therefore; by the *Comparison Test*, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$

Solution

$$0 \le \sin^2 k \le 1$$

$$0 \le \frac{\sin^2 k}{k^2} \le \frac{1}{k^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges by } \boldsymbol{p}\textit{-series } (p = 2 \ > 1)$$

Therefore; by the *Comparison Test*, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{k=1}^{\infty} \sin^2 \frac{1}{k}$

Solution

Let $b_k = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by ***p**-series* $(p = 2 \ > 1)$

$$a_k = \sin^2 \frac{1}{k}$$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sin^2 \frac{1}{k}}{\frac{1}{k^2}} = \frac{0}{0}$$

$$= \lim_{k\to\infty} \left(\frac{\sin \frac{1}{k}}{\frac{1}{k}} \right)^2 \qquad \text{Let } x = \frac{1}{k} \ \to 0$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x} \right)^2$$

$$\underline{= 1}$$

Therefore; by the *Limit Comparison Test*, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{k=1}^{\infty} \sin \frac{1}{k}$

Solution

Let $b_k = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges by ***p**-series* $(p = 1 \ \le 1)$

$$a_k = \sin\frac{1}{k}$$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sin\frac{1}{k}}{\frac{1}{k}} = \frac{0}{0} \qquad \text{Let } x = \frac{1}{k} \to 0$$

$$= \lim_{x\to 0} \frac{\sin x}{x}$$

$$= 1$$

Therefore; by the *Limit Comparison Test*, the given series ***diverges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}\sin\frac{1}{k}$

Solution

Let $b_k = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by ***p**-series* $(p = 2 > 1)$

$$a_k = \frac{1}{k}\sin\frac{1}{k}$$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{1}{k}\sin\frac{1}{k}}{\frac{1}{k^2}}$$

$$= \lim_{k\to\infty} \frac{\sin\frac{1}{k}}{\frac{1}{k}} \qquad \text{Let } x = \frac{1}{k} \to 0$$

$$= \lim_{x\to 0} \frac{\sin x}{x}$$

$$= 1$$

Therefore; by the *Limit Comparison Test*, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}\sin\frac{1}{k}$

Solution

$$-1 \le \sin\frac{1}{k} \le 1$$

$$-\frac{1}{k^2} \le \frac{\sin\frac{1}{k}}{k^2} \le \frac{1}{k^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$ converges by ***p**-series* $(p = 2 \ > 1)$

Therefore; by the *Comparison Test*, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{k=1}^{\infty} (-1)^k \, k \sin\frac{1}{k}$

Solution

$$-1 \le \sin\frac{1}{k} \le 1$$

$$-k \le k\sin\frac{1}{k} \le k$$

$$\lim_{k\to\infty} k = \infty$$

Therefore; by the *Comparison Test*, the given series ***diverges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}\sin\frac{\pi}{2k}$

Solution

$$-1 \le \sin\frac{\pi}{2k} \le 1$$

$$-\frac{1}{k^2} \le \frac{1}{k^2}\sin\frac{\pi}{2k} \le \frac{1}{k^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$ converges by ***p**-series* $(p = 2 \ > 1)$

Therefore; by the *Comparison Test*, the given series ***converges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{k=1}^{\infty} \tan\frac{1}{k}$

Solution

Let $b_k = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges by ***p**-series* $(p = 1 \le 1)$

$a_k = \tan\frac{1}{k}$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\tan\frac{1}{k}}{\frac{1}{k}}$$

$$= \lim_{x\to 0} \frac{\tan x}{x} \qquad \text{Let } x = \frac{1}{k} \to 0$$

$$= 1$$

Therefore; by the *Limit Comparison Test*, the given series ***diverges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{k=1}^{\infty} (-1)^k \tan^{-1} k$

Solution

$$\lim_{k\to\infty} \tan^{-1} k = \tan^{-1} \infty$$

$$= \frac{\pi}{2} \ne 0$$

Therefore; by the *Divergence Test*, the given series ***diverges***.

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$

Solution

$|\cos n| \le 1$

$\frac{|\cos n|}{n^3} \le \frac{1}{n^3}$

$\sum \frac{1}{n^3}$ converges by ***p**-series* $(p = 3 > 1)$

Therefore; by the *Comparison Test*, the given series ***converges*** (absolutely)

Exercise

Use any method to determine if the series converges or diverges $\displaystyle\sum_{k=2}^{\infty} \frac{k}{\ln k}$

Solution

$$\lim_{k\to\infty} \frac{k}{\ln k} = \frac{\infty}{\infty}$$

$$= \lim_{k\to\infty} \frac{1}{\frac{1}{k}}$$

$$= \lim_{k\to\infty} k$$

$$\underline{= \infty}$$

Therefore; by the *Divergence Test*, the given series ***diverges***.

Exercise

Use any method to determine if the series converges or diverges $\dfrac{1}{1+\sqrt{1}} + \dfrac{1}{1+\sqrt{2}} + \dfrac{1}{1+\sqrt{3}} + \cdots$

Solution

$$\frac{1}{1+\sqrt{1}} + \frac{1}{1+\sqrt{2}} + \frac{1}{1+\sqrt{3}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$$

$$\sqrt{n} < n$$

$$1+\sqrt{n} < n$$

$$\frac{1}{1+\sqrt{n}} > \frac{1}{n}$$

$\displaystyle\sum \frac{1}{n}$ diverges by ***p**-series* $(p = 1 \ \le 1)$

Therefore; by the *Comparison Test*, the given series ***diverges***.