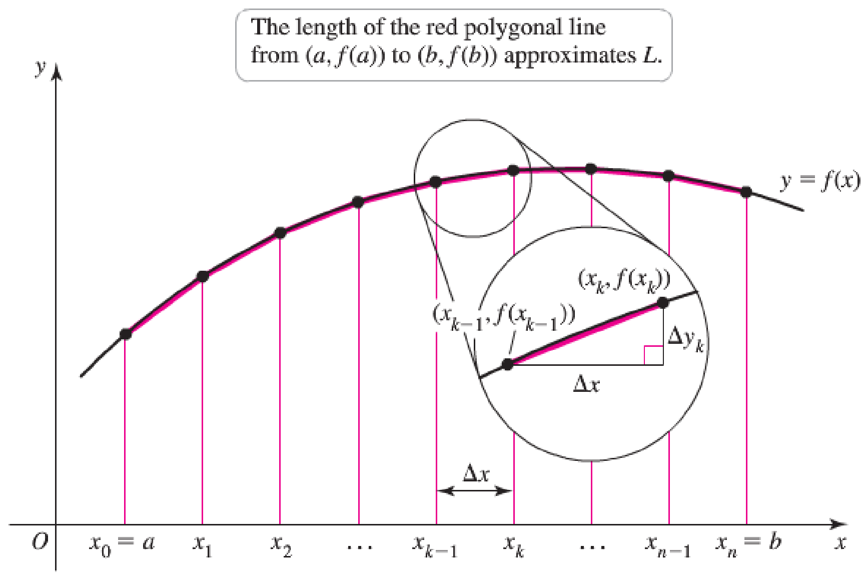
Section 1.5 – Length of Curves

Length of a curve $y = f(x)$

We assume that f has a continuous derivative at every point of $[a,\ b]$. Such function is called ***smooth***, and its graph is a ***smooth curve*** because it doesn't have any breaks, corners, or cusps.



Definition

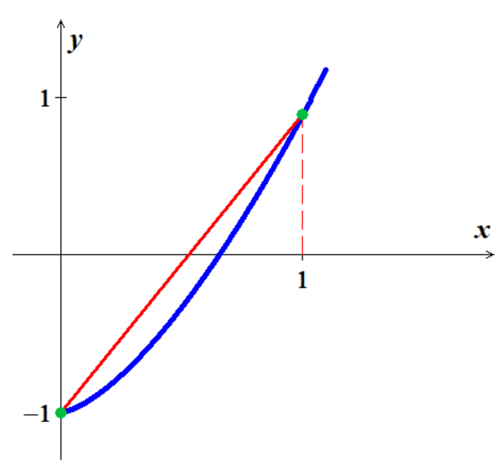
If f' is continuous on $[a,\ b]$, then the length (arc length) of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\ dx$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Example

Find the length of the curve $y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \quad 0 \le x \le 1$



Solution

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3}\frac{3}{2}x^{1/2}$$

$$= 2\sqrt{2}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}x^{1/2}\right)^2$$
$$= 8x$$

$$L = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$
$$= \int_0^1 (1+8x)^{1/2}\ dx \qquad \text{or}\ \ u = 1+8x \quad du = 8dx \to dx = \frac{du}{8}$$
$$= \frac{1}{8}\int_0^1 (1+8x)^{1/2}\ d(1+8x)$$
$$= \frac{1}{8}\ \left(\frac{2}{3}(1+8x)^{3/2}\ \right|_0^1$$
$$= \frac{1}{12}\left[\left(1+8(1)\right)^{3/2} - \left(1+8(0)\right)^{3/2}\right]$$
$$= \frac{1}{12}\ \left((9)^{3/2} - (1)^{3/2}\right)$$
$$= \frac{1}{12}[27-1]$$
$$= \frac{1}{12}(26)$$
$$= \frac{13}{6} \quad unit$$
$$\approx 2.17 \quad unit$$

Length of a curve $y = f(x)$:

If $f(x) = ax^m + bx^n$, then

$$L = \int_c^d \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx$$

$$= \left(ax^m - bx^n \right|_c^d$$

Iff $f(x)$ satisfies these 2 conditions:

1. $m + n = 2$
2. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = max^{m-1} + nbx^{n-1}$$

$$1 + (f')^2 = 1 + \left(max^{m-1} + nbx^{n-1}\right)^2$$

$$= 1 + m^2a^2x^{2m-2} + 2abmnx^{m+n-2} + n^2b^2x^{2n-2}$$

We need to combined to a perfect square $\qquad a^2 - 2ab + b^2 = (a-b)^2$

- If $x^{m+n-2} = 1 = x^0 \quad \rightarrow \quad \boxed{m + n = 2}$

$$= m^2a^2x^{2m-2} + (1 + 2abmn) + n^2b^2x^{2n-2} \qquad a^2 - 2ab + b^2 = (a-b)^2$$

- Let $1 + 2abmn = -2abmn \quad \rightarrow \quad \boxed{abmn = -\frac{1}{4}}$

$$= m^2a^2x^{2m-2} - 2abmn + n^2b^2x^{2n-2} \qquad x^{2(m+n-2)} = 1$$

$$= \left(max^{m-1} - nbx^{n-1}\right)^2$$

$$L = \int_c^d \sqrt{\left(max^{m-1} - nbx^{n-1}\right)^2}\, dx$$

$$= \int_c^d \left(max^{m-1} - nbx^{n-1}\right) dx$$

$$= \left(ax^m - bx^n \right|_c^d \quad \checkmark$$

Example

Find the length of the graph of $f(x)=\frac{x^3}{12}+\frac{1}{x}, \quad 1\le x\le 4$

Solution

$$a=\frac{1}{12}, \quad m=3, \quad b=1, \quad n=-1$$

1. $m+n=3-1=2$ ✓

2. $abmn=\frac{1}{12}(1)(3)(-1)=-\frac{1}{4}$ ✓

$$L=\left(\frac{x^3}{12}-\frac{1}{x}\right|_1^4$$

$$=\left(\frac{4^3}{12}-\frac{1}{4}\right)-\left(\frac{1}{12}-\frac{1}{1}\right)$$

$$=\frac{72}{12}$$

$$=6 \quad unit$$

Discontinuities in $\frac{dy}{dx}$

Formula for the length of $x=g(y), \quad c\le y\le d$

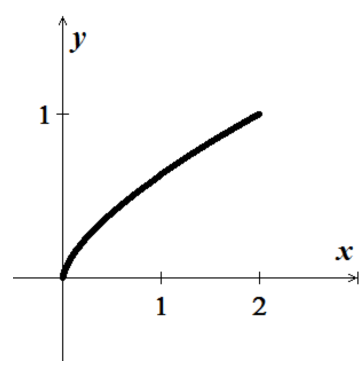
If g' is continuous on $[c, d]$, the length of the curve $x=g(y)$ from the point $A=(g(c),c)$ to the point $B=(g(d),d)$ is the value of the integral

$$L=\int_c^d\sqrt{1+\left[g'(y)\right]^2}\,dy \quad = \int_c^d\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Example

Find the length of the curve $y=\left(\frac{x}{2}\right)^{2/3}$ from $x=0$ to $x=2$.

Solution



$$\frac{dy}{dx}=\frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right)$$

$$=\frac{1}{3}\left(\frac{2}{x}\right)^{1/3} \qquad \boxed{x\ne 0} \ (CP)$$

$$y=\left(\frac{x}{2}\right)^{2/3} \qquad \textit{Raised both sides to the power 3/2}$$

$y^{3/2} = \frac{x}{2}$

$x = 2y^{3/2}$

$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2}$

$\quad = 3y^{1/2}$

$$\rightarrow \begin{cases} x = 0 & \Rightarrow y = 0 \\ x = 2 & \Rightarrow y = \left(\frac{2}{2}\right)^{2/3} = 1 \end{cases}$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

$$= \int_0^1 \sqrt{1 + \left(3y^{1/2}\right)^2}\ dy$$

$$= \int_0^1 \sqrt{1 + 9y}\ dy$$

$$= \int_0^1 (1 + 9y)^{1/2}\ dy$$

$$= \frac{1}{9}\frac{2}{3}\ (1 + 9y)^{3/2}\ \Big|_0^1$$

$$= \frac{2}{27}\left[(1 + 9)^{3/2} - (1 + 0)^{3/2}\right]$$

$$= \frac{2}{27}\left(10^{3/2} - 1\right)$$

$$= \frac{2}{27}\left(10\sqrt{10} - 1\right) \quad unit$$

$$\approx 2.27 \ \ unit$$

If $f(x) = ae^{mx} + be^{nx}$, then

$$\boldsymbol{L} = \int_c^d \sqrt{1+\left(\frac{df}{dx}\right)^2}\, dx$$

$$= \left(ae^{mx} - be^{nx} \right|_c^d$$

Iff $f(x)$ satisfies these 2 conditions:

1. $m = -n$

2. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = ame^{mx} + bne^{nx}$$

$$1+(f')^2 = 1+\left(ame^{mx} + bne^{nx}\right)^2$$

$$= 1 + m^2a^2e^{2mx} + 2abmne^{(m+n)x} + n^2b^2e^{2nx}$$

- If $e^{(m+n)x} = 1 = e^{(x=0)} \quad \to \quad \boxed{m-n}$

$$= m^2a^2x^{2m-2} + (1+2abmn) + n^2b^2x^{2n-2} \qquad a^2 - 2ab + b^2 = (a-b)^2$$

- Let $1+2abmn = -2abmn \quad \to \quad \boxed{abmn = -\frac{1}{4}}$

$$= m^2a^2e^{2mx} - 2abmne^{(m+n)x} + n^2b^2e^{2nx} \qquad x^{2(m+n-2)} = 1$$

$$= \left(ame^{mx} - bne^{nx}\right)^2$$

$$\left(ame^{mx} - bne^{nx}\right)^2 = m^2a^2e^{2mx} - 2abmne^{(m+n)x} + n^2b^2e^{2nx}$$

$$\boldsymbol{L} = \int_c^d \sqrt{\left(ame^{mx} - bne^{nx}\right)^2}\, dx$$

$$= \int_c^d \left(ame^{mx} - bne^{nx}\right) dx$$

$$= \left(ame^{mx} - bne^{nx} \right|_c^d \quad \checkmark$$

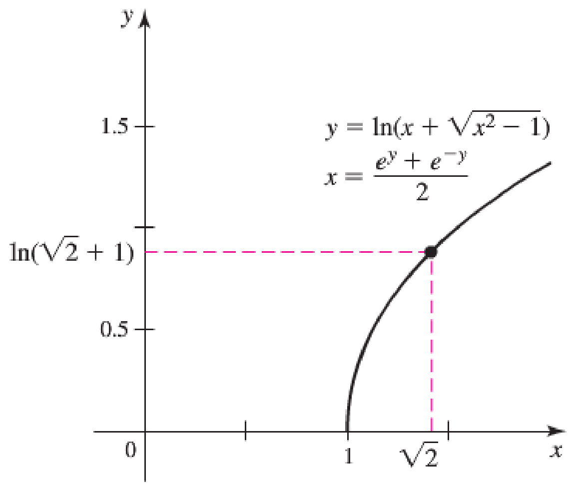
Example

Find the arc length function for the curve $f(x)=\ln\left(x+\sqrt{x^2-1}\right)$ on the interval $\left[1,\ \sqrt{2}\right]$

Solution

$$f'(x)=\frac{1+x\left(x^2-1\right)^{-1/2}}{x+\sqrt{x^2-1}}$$

$$=\frac{\sqrt{x^2-1}+x}{x\sqrt{x^2-1}+x^2-1}$$



$$y=\ln\left(x+\sqrt{x^2-1}\right)$$

$$x+\sqrt{x^2-1}=e^y$$

$$\left(\sqrt{x^2-1}\right)^2=\left(e^y-x\right)^2$$

$$x^2-1=e^{2y}-2xe^y+x^2$$

$$2xe^y=e^{2y}+1$$

$$x=\frac{e^{2y}+1}{2e^y}\left(\frac{e^y}{e^y}\right)$$

$$=\frac{e^y+e^{-y}}{2}$$

$$y=\ln\left(x+\sqrt{x^2-1}\right)$$

$$x=\frac{e^y+e^{-y}}{2}=g(y)$$

$$x=1 \quad\to\quad y=0$$

$$x=\sqrt{2} \quad\to\quad y=\ln\left(\sqrt{2}+1\right)$$

$$a=\frac{1}{2},\quad m=1,\quad b=\frac{1}{2},\quad n=-1$$

1. $m=-n$ ✓

2. $abmn=-\frac{1}{4}$ ✓

$$L=\frac{1}{2}\left(e^y-e^{-y}\right)\Big|_0^{\ln\left(\sqrt{2}+1\right)}$$

$$=\frac{1}{2}\left(\sqrt{2}+1-\frac{1}{\sqrt{2}+1}-1+1\right)$$

$$=\frac{1}{2}\left(\frac{3+2\sqrt{2}-1}{\sqrt{2}+1}\right)$$

$$=\frac{1}{2}\left(\frac{2+2\sqrt{2}}{\sqrt{2}+1}\right)$$

$$\underline{=1 \quad \textit{unit}}$$

OR

$$L=\int_0^{\ln\left(\sqrt{2}+1\right)} \sqrt{1+g'(y)^2}\ dy$$

$$=\int_0^{\ln\left(\sqrt{2}+1\right)} \sqrt{1+\left(\frac{e^y-e^{-y}}{2}\right)^2}\ dy$$

$$=\frac{1}{2}\int_0^{\ln\left(\sqrt{2}+1\right)} \sqrt{4+e^{2y}-2+e^{-2y}}\ dy$$

$$=\frac{1}{2}\int_0^{\ln\left(\sqrt{2}+1\right)} \sqrt{e^{2y}+2+e^{-2y}}\ dy$$

$$=\frac{1}{2}\int_0^{\ln\left(\sqrt{2}+1\right)} \sqrt{\left(e^y+e^{-y}\right)^2}\ dy$$

$$=\frac{1}{2}\int_0^{\ln\left(\sqrt{2}+1\right)} \left(e^y+e^{-y}\right)\ dy$$

$$=\frac{1}{2}\left(e^y-e^{-y}\right)\Big|_0^{\ln\left(\sqrt{2}+1\right)}$$

$$=\frac{1}{2}\left(\sqrt{2}+1-\frac{1}{\sqrt{2}+1}-1+1\right)$$

$$=\frac{1}{2}\left(\frac{3+2\sqrt{2}-1}{\sqrt{2}+1}\right)$$

$$=\frac{1}{2}\left(\frac{2+2\sqrt{2}}{\sqrt{2}+1}\right)$$

$$\underline{=1 \quad \textit{unit}}$$

The differential Formula for Arc length

If $y = f(x)$ and if f' is continuous on [*a, b*], then by the Fundamental Theorem of Calculus, we can define a new function

$$s(x) = \int_a^x \sqrt{1 + \left[f'(t)\right]^2}\, dt$$

$$\frac{ds}{dx} = \sqrt{1 + \left[f'(t)\right]^2}$$

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \sqrt{(dx)^2 + (dx)^2 \frac{(dy)^2}{(dx)^2}}$$

$$\boxed{ds = \sqrt{dx^2 + dy^2}}$$

Example

Find the arc length function for the curve $f(x) = \frac{x^3}{12} + \frac{1}{x}$ taking $A = \left(1,\ \frac{13}{12}\right)$ as the starting point

Solution

$$1 + \left[f'(x)\right]^2 = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

$$s(x) = \int_1^x \sqrt{1 + \left[f'(t)\right]^2}\, dt$$

$$= \int_1^x \left(\frac{t^2}{4} + \frac{1}{t^2}\right) dt$$

$$= \left(\frac{t^3}{12} - \frac{1}{t}\right|_1^x$$

$$= \left(\frac{x^3}{12} - \frac{1}{x}\right) - \left(\frac{1}{12} - 1\right)$$

$$= \frac{x^3}{12} - \frac{1}{x} + \frac{11}{12}$$

$$s(4) = \frac{4^3}{12} - \frac{1}{4} + \frac{11}{12}$$

$$= 6 \quad unit$$

Exercises *Section* 1.5 – Length of Curves

(1 – 30) Find the length of the curve of

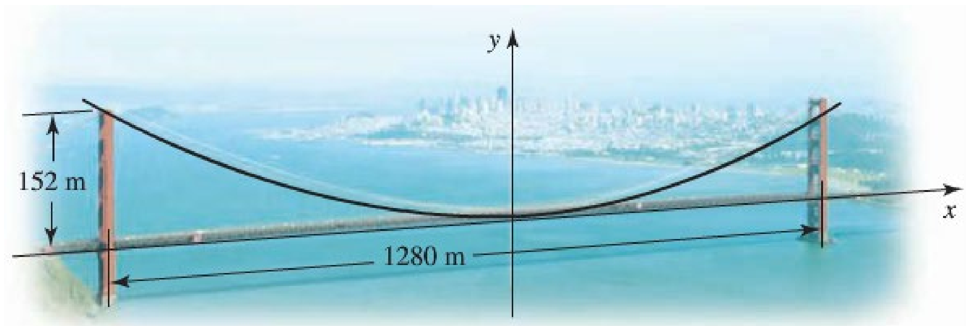
1. $y=\frac{1}{3}\left(x^2+2\right)^{3/2}$ from $x=0$ to $x=3$
2. $y=(x)^{3/2}$ from $x=0$ to $x=4$
3. $x=\frac{y^{3/2}}{3}-y^{1/2}$ from $y=1$ to $y=9$
4. $x=\frac{y^3}{6}+\frac{1}{2y}$ from $y=2$ to $y=3$
5. $f(x)=x^3+\frac{1}{12x}$ *for* $\frac{1}{2}\le x\le 2$
6. $f(x)=\frac{1}{5}x^5+\frac{1}{12x^3}$ $1\le x\le 2$
7. $y=\frac{1}{3}x^{1/2}-x^{3/2},\quad 0\le x\le \frac{1}{3}$
8. $y=\frac{1}{3}x^3+\frac{1}{4x},\quad 1\le x\le 2$
9. $y=2e^x+\frac{1}{8}e^{-x}$ $0\le x\le \ln 2$
10. $y=e^{2x}+\frac{1}{16}e^{-2x}$ $0\le x\le \ln 3$
11. $y=\ln(\cos x)$ $0\le x\le \frac{\pi}{4}$
12. $f(y)=2e^{\sqrt{2}y}+\frac{1}{16}e^{-\sqrt{2}y}$ $0\le y\le \frac{\ln 2}{\sqrt{2}}$
13. $y=\frac{x^3}{3}+x^2+x+1+\frac{1}{4x+4}$ $0\le x\le 2$
14. $y=\frac{x^3}{3}+x^2+x+1+\frac{1}{4x+4}$ $0\le x\le 4$
15. $y=\ln\left(e^x-1\right)-\ln\left(e^x+1\right)$ $\ln 2\le x\le \ln 3$
16. $f(x)=\frac{2}{3}x^{3/2}-\frac{1}{2}x^{1/2}$ $1\le x\le 4$
17. $f(x)=x^3+\frac{1}{12x}$ $1\le x\le 4$
18. $f(x)=\frac{1}{8}x^4+\frac{1}{4x^2}$ $1\le x\le 10$
19. $f(x)=\frac{1}{4}x^4+\frac{1}{8x^2}$ $3\le x\le 8$
20. $f(x)=\frac{1}{10}x^5+\frac{1}{6x^3}$ $1\le x\le 7$
21. $f(x)=\frac{3}{10}x^{1/3}-\frac{3}{2}x^{5/3}$ $0\le x\le 12$
22. $f(x)=x^{1/2}-\frac{1}{3}x^{3/2}$ $2\le x\le 9$
23. $y=x^{1/2}-\frac{1}{3}x^{3/2}$ $1\le x\le 4$
24. $x=y^{2/3},\quad 1\le y\le 8$
25. $y=2x+4$ $-2\le x\le 2$
26. $y=\frac{x^3}{6}+\frac{1}{2x}$ $x\in[1,\ 2]$
27. $f(x)=x^{1/2}-\frac{1}{3}x^{3/2}$ $1\le x\le 3$
28. $y=\frac{3}{4}x^{4/3}-\frac{3}{8}x^{2/3}+5,\quad 1\le x\le 8$
29. $y=\ln x-\frac{1}{8}x^2;\quad 1\le x\le 2$
30. $y=\frac{1}{2}x^2-\frac{1}{4}\ln x;\quad 1\le x\le 3$

31. Find the length of the curve $y=\int_{-2}^{x}\sqrt{2t^4-1}\,dt$ $-2\le x\le -1$

32. Find the length of the curve $x=\int_{0}^{y}\sqrt{\sec^4 t-1}\,dt$ $-\frac{\pi}{4}\le y\le \frac{\pi}{4}$

33. Find the length of the curve $y = 3 - 2x \quad 0 \le x \le 2$. Check your answer by finding the length of the segment as the hypotenuse of a right triangle.

34. The profile of the cables on a suspension bridge may be modeled by a parabola. The central span of the Golden Gate Bridge is 1280 *m* long and 152 *m* high. The parabola $y = 0.00037x^2$ gives a good fit to the shape of the cables, where $|x| \le 640$, and x and y are measured in meters. Approximate the length of the cables that stretch between the tops of the two towers.

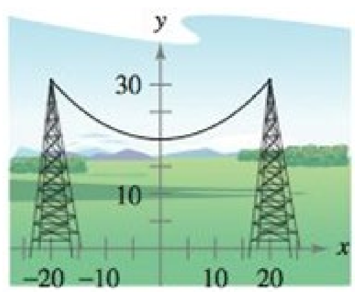


35. Find a curve through the origin in the *xy*-plane whose length from $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \tfrac{1}{4}e^x}\, dx$$

36. Confirm that the circumference of a circle of radius a is $2\pi a$

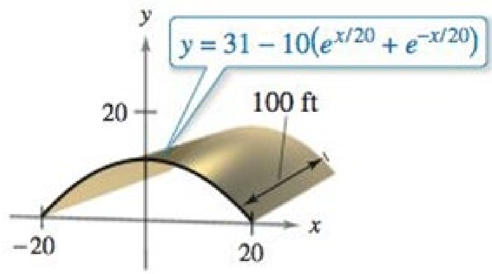
37. Electrical wires suspended between two towers form a caternary modeled by the equation



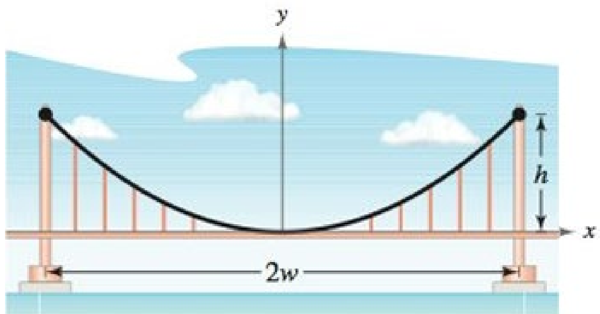
$$y = 20\cosh\frac{x}{20}, \quad -20 \le x \le 20$$

Where x and y are measured in meters. The towers are 40 meters apart. Find the length of the suspended cable.

38. A barn is 100 feet long and 40 feet wide. A cross section of the roof is the inverted caternary $y = 31 - 10\left(e^{x/20} + e^{-x/20}\right)$. Find the number of square feet of roofing on the barn.



39. A cable for a suspension bridge has the shape of a parabola with equation $y = kx^2$. Let h represent the height of the cable from it lowest point to its highest point and let $2w$ represent the total span of the bridge.



Show that the length C of the cable is given by $C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\,dx$

40. Find the total length of the graph of the astroid $x^{2/3} + y^{2/3} = 4$

41. Find the arc length from $(0,\ 3)$ clockwise to $(2,\ \sqrt{5})$ along the circle $x^2 + y^2 = 9$

42. Find the arc length from $(-3,\ 4)$ clockwise to $(4,\ 3)$ along the circle $x^2 + y^2 = 25$. Show that the result is one-fourth the circumference of the circle.

43. $y = \ln x$ between $x = 1$ and $x = b > 1$ that

$$\int \frac{\sqrt{x^2 + a^2}}{x}dx = \sqrt{x^2 + a^2} - a\ln\left(\frac{a + \sqrt{x^2 + a^2}}{x}\right) + C$$

Use any means to approximate the value of b for which the curve has length 2.