Solution Section 4.3 – Conservative Vector Fields

Exercise

Find the gradient field of the function $f(x,y,z)=\left(x^2+y^2+z^2\right)^{-1/2}$

Solution

$$\frac{\partial f}{\partial x}=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2x)$$
$$=-x\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial f}{\partial y}=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2y)$$
$$=-y\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial f}{\partial z}=-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2z)$$
$$=-z\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\nabla f=-x\left(x^2+y^2+z^2\right)^{-3/2}\boldsymbol{i}-y\left(x^2+y^2+z^2\right)^{-3/2}\boldsymbol{j}-z\left(x^2+y^2+z^2\right)^{-3/2}\boldsymbol{k}$$
$$=\frac{-x\boldsymbol{i}-y\boldsymbol{j}-z\boldsymbol{k}}{\left(x^2+y^2+z^2\right)^{3/2}}$$

Exercise

Find the gradient field of the function $f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}$

Solution

$$f(x,\ y,\ z)=\ln\sqrt{x^2+y^2+z^2}$$
$$=\ln\left(x^2+y^2+z^2\right)^{1/2}$$
$$=\frac{1}{2}\ln\left(x^2+y^2+z^2\right)$$

$$\frac{\partial f}{\partial x}=\frac{1}{2}\frac{2x}{x^2+y^2+z^2}=\frac{x}{x^2+y^2+z^2}$$

$$\frac{\partial f}{\partial y}=\frac{1}{2}\frac{2y}{x^2+y^2+z^2}=\frac{y}{x^2+y^2+z^2}$$

$$\frac{\partial f}{\partial z}=\frac{1}{2}\frac{2z}{x^2+y^2+z^2}=\frac{z}{x^2+y^2+z^2}$$

$$\nabla f=\boxed{\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{x^2+y^2+z^2}}$$

Exercise

Find the gradient field of the function $f(x,y,z)=e^z-\ln\left(x^2+y^2\right)$

Solution

$$\frac{\partial f}{\partial x}=-\frac{2x}{x^2+y^2}\qquad \frac{\partial f}{\partial y}=-\frac{2y}{x^2+y^2}\qquad \frac{\partial f}{\partial z}=e^z$$

$$\nabla f=\boxed{-\frac{2x}{x^2+y^2}\mathbf{i}-\frac{2y}{x^2+y^2}\mathbf{j}+e^z\mathbf{k}}$$

Exercise

Find the line integral of $\int_C (x-y)\,dx$ where $C:\ x=t,\quad y=2t+1,\quad \text{for } 0\le t\le 3$

Solution

$$x=t,\quad y=2t+1,\quad \text{for } 0\le t\le 3$$

$$dx=dt$$

$$\int_C (x-y)\,dx=\int_0^3 \left(t-(2t+1)\right)dt$$

$$=\int_0^3 (-t-1)\,dt$$

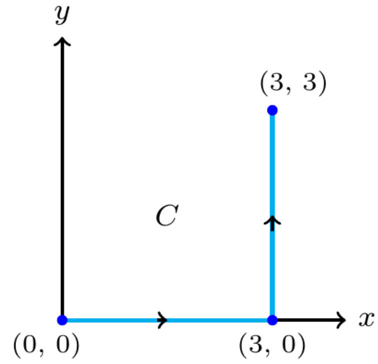
$$=-\left[\frac{1}{2}t^2+t\right]_0^3$$

$$=-\left(\frac{9}{2}+3\right)$$

$$=\boxed{-\frac{15}{2}}$$

Exercise

Find the line integral of $\int_C \left(x^2+y^2\right)dy$ where C is



Solution

$$C_1: \quad x=t, \quad y=0, \quad 0\le t\le 3 \qquad \Rightarrow dy=0$$

$$C_2: \quad x=3, \quad y=t, \quad 0\le t\le 3 \qquad \Rightarrow dy=dt$$

$$\int_C \left(x^2+y^2\right)dy = \int_{C_1} \left(x^2+y^2\right)dy + \int_{C_2} \left(x^2+y^2\right)dy$$

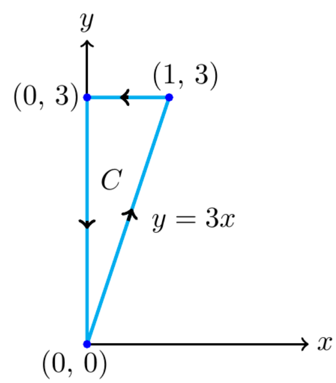
$$= \int_0^3 \left(t^2+0\right)(0) + \int_0^3 \left(9+t^2\right)dt$$

$$= \left[9t+\frac{1}{3}t^3\right]_0^3$$

$$\underline{= 36}$$

Exercise

Find the line integral of $\int_C \sqrt{x+y}\ dx$ where C is



Solution

$$C_1: \quad x=t, \quad y=3t, \quad 0\le t\le 1 \qquad \Rightarrow dx=dt$$

$$C_2: \quad x=1-t, \quad y=3, \quad 0\le t\le 1 \qquad \Rightarrow dx=-dt$$

$$C_3: \quad x=0, \quad y=3-t, \quad 0\le t\le 3 \qquad \Rightarrow dx=0$$

$$\int_C \sqrt{x+y}\ dx = \int_{C_1} \sqrt{x+y}\ dx + \int_{C_2} \sqrt{x+y}\ dx + \int_{C_3} \sqrt{x+y}\ dx$$

$$= \int_0^1 \sqrt{t+3t}dt + \int_0^1 \sqrt{1-t+3}(-dt) + \int_0^3 \sqrt{3-t}(0)$$

$$= \int_0^1 2\sqrt{t}\ dt + \int_0^1 \sqrt{4-t}\ d(4-t)$$

$$= 2\left[\frac{2}{3}t^{3/2}\right]_0^1 + \left[\frac{2}{3}(4-t)^{3/2}\right]_0^1$$

$$= \frac{4}{3} + \frac{2}{3}\left(3^{3/2}-4^{3/2}\right)$$

$$= \frac{4}{3} + \frac{2}{3}\left(3\sqrt{3} - 8\right)$$

$$= \frac{4 + 6\sqrt{3} - 16}{3}$$

$$= \frac{6\sqrt{3} - 12}{3}$$

$$\underline{= 2\sqrt{3} - 4}$$

Exercise

Find the work done by the force field $\vec{F} = xy\hat{i} + y\hat{j} - yz\hat{k}$ over the curve $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t\hat{k}, \quad 0 \le t \le 1$.

Solution

$$\frac{d\vec{r}}{dt} = \hat{i} + 2t\,\hat{j} + \hat{k}$$

$$\vec{F} = xy\hat{i} + y\hat{j} - yz\hat{k}$$

$$= t^3\hat{i} + t^2\hat{j} - t^3\hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left(t^3\hat{i} + t^2\hat{j} - t^3\hat{k}\right) \bullet \left(\hat{i} + 2t\hat{j} + \hat{k}\right)$$

$$= t^3 + 2t^3 - t^3$$

$$= 2t^3$$

$$Work = \int_0^1 \vec{F} \bullet \frac{d\vec{r}}{dt}\,dt$$

$$= \int_0^1 2t^3 dt$$

$$= \left[\frac{1}{2}t^4\right]_0^1$$

$$\underline{= \frac{1}{2}}$$

Exercise

Find the work done by the force field $\vec{F} = 2y\hat{i} + 3x\hat{j} + (x+y)\hat{k}$ over the curve $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + \frac{t}{6}\hat{k}, \quad 0 \le t \le 2\pi$.

Solution

$$\vec{F} = 2y\hat{i} + 3x\hat{j} + (x+y)\hat{k}$$

$$= (2\sin t)\hat{i} + (3\cos t)\hat{j} + (\cos t + \sin t)\hat{k}$$

$$\frac{d\boldsymbol{r}}{dt} = (-\sin t)\boldsymbol{i} + (\cos t)\boldsymbol{j} + \frac{1}{6}\boldsymbol{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left((2\sin t)\hat{i} + (3\cos t)\hat{j} + (\cos t + \sin t)\hat{k}\right) \bullet \left((-\sin t)\hat{i} + (\cos t)\hat{j} + \frac{1}{6}\hat{k}\right)$$

$$= -2\sin^2 t + 3\cos^2 t + \frac{1}{6}\cos t + \frac{1}{6}\sin t$$

$$= -2\left(\frac{1-\cos 2t}{2}\right) + 3\left(\frac{1+\cos 2t}{2}\right) + \frac{1}{6}\cos t + \frac{1}{6}\sin t$$

$$= \cos 2t - 1 + \frac{3}{2} + \frac{3}{2}\cos 2t + \frac{1}{6}\cos t + \frac{1}{6}\sin t$$

$$= \frac{1}{2} + \frac{5}{2}\cos 2t + \frac{1}{6}\cos t + \frac{1}{6}\sin t$$

$$Work = \int_0^{2\pi} \left(\frac{1}{2} + \frac{5}{2}\cos 2t + \frac{1}{6}\cos t + \frac{1}{6}\sin t\right) dt \qquad W = \int \vec{F} \bullet \frac{d\vec{r}}{dt}\, dt$$

$$= \left[\frac{1}{2}t + \frac{5}{4}\sin 2t + \frac{1}{6}\sin t - \frac{1}{6}\cos t\right]_0^{2\pi}$$

$$= \left(\pi - \frac{1}{6}\right) - \left(-\frac{1}{6}\right)$$

$$= \pi$$

Exercise

Find the work done by the force field $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ over the curve $\vec{r}(t) = (\sin t)\hat{i} + (\cos t)\hat{j} + t\hat{k}, \quad 0 \le t \le 2\pi$.

Solution

$$\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$$

$$= t\hat{i} + (\sin t)\hat{j} + (\cos t)\hat{k}$$

$$\frac{d\vec{r}}{dt} = (\cos t)\hat{i} + (-\sin t)\hat{j} + \hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left(t\hat{i} + (\sin t)\hat{j} + (\cos t)\hat{k}\right) \bullet \left((\cos t)\hat{i} + (-\sin t)\hat{j} + \hat{k}\right)$$

$$= t\cos t - \sin^2 t + \cos t$$

$$= t\cos t - \frac{1}{2} + \frac{1}{2}\cos 2t + \cos t$$

		$\int \cos t$
+	t	$\sin t$
−	1	$-\cos t$

$$Work = \int_0^{2\pi} \vec{F} \bullet \frac{d\vec{r}}{dt}\, dt$$

$$= \int_0^{2\pi} \left(t\cos t - \frac{1}{2} + \frac{1}{2}\cos 2t + \cos t \right) dt$$

$$= \left[t\sin t + \cos t - \frac{1}{2}t + \frac{1}{4}\sin 2t + \sin t \right]_0^{2\pi}$$

$$= (1-\pi) - (1)$$

$$= -\pi$$

Exercise

Find the work required to move an object with given force field $\vec{F} = \langle -y,\ z,\ x \rangle$ on the path consisting of the line segments from $(0,\ 0,\ 0)$ to $(0,\ 1,\ 0)$ followed by the line segment from $(0,\ 1,\ 0)$ to $(0,\ 1,\ 4)$

Solution

$(0,\ 0,\ 0)$ to $(0,\ 1,\ 0)$ $\rightarrow \vec{r}_1(t) = \langle 0,\ t,\ 0 \rangle$

$(0,\ 1,\ 0)$ to $(0,\ 1,\ 4)$ $\rightarrow \vec{r}_2(t) = \langle 0,\ 1,\ 4t \rangle$

$\vec{r}_1'(t) = \langle 0,\ 1,\ 0 \rangle$

$\vec{r}_2'(t) = \langle 0,\ 0,\ 4 \rangle$

$\vec{F} \bullet \vec{r}_1'(t) = \langle -t,\ 0,\ 0 \rangle \bullet \langle 0,\ 1,\ 0 \rangle = 0$

$\vec{F} \bullet \vec{r}_2'(t) = \langle -1,\ 4t,\ 0 \rangle \bullet \langle 0,\ 0,\ 4 \rangle = 0$

$$W = \int_0^1 (0+0)\,dt \qquad\qquad W = \int_C \vec{F} \cdot d\vec{r}$$

$$= 0$$

Exercise

Find the work required to move an object with given force field $\vec{F} = \dfrac{\langle x,\ y,\ z \rangle}{\left(x^2+y^2+z^2\right)^{3/2}}$ on the path

$\boldsymbol{r}(t) = \left\langle t^2,\ 3t^2,\ -t^2 \right\rangle$ for $1 \le t \le 2$

Solution

$\vec{r}'(t) = \langle 2t,\ 6t,\ -2t \rangle$

$$W = \int_1^2 \frac{\left\langle t^2,\ 3t^2,\ -t^2 \right\rangle \bullet \langle 2t,\ 6t,\ -2t \rangle}{\left(t^4+9t^4+t^4\right)^{3/2}}\,dt \qquad\qquad W = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_1^2 \frac{2t^3 + 18t^3 + 2t^3}{\left(11t^4\right)^{3/2}} dt$$

$$= \frac{1}{11\sqrt{11}} \int_1^2 \frac{22t^3}{t^6} dt$$

$$= \frac{2}{\sqrt{11}} \int_1^2 t^{-3} dt$$

$$= -\frac{1}{\sqrt{11}} t^{-2} \Big|_1^2$$

$$= -\frac{1}{\sqrt{11}} \left(\frac{1}{4} - 1\right)$$

$$= \frac{3}{4\sqrt{11}}$$

Exercise

Evaluate $\int_C \vec{F} \cdot \vec{T}\, ds$ for the vector field $\vec{F} = x^2\hat{i} - y\hat{j}$ along the curve $x = y^2$ from $(4,\ 2)$ to $(1,\ -1)$

Solution

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$= y^2\hat{i} + y\hat{j} \qquad -1 \le y \le 2$$

$$\vec{F} = x^2\hat{i} - y\hat{j}$$

$$= y^4\hat{i} - y\hat{j}$$

$$\frac{d\vec{r}}{dy} = 2y\hat{i} + \hat{j}$$

$$\vec{F} \cdot \frac{d\vec{r}}{dy} = \left(y^4\hat{i} - y\hat{j}\right) \cdot \left(2y\hat{i} + \hat{j}\right)$$

$$= 2y^5 - y$$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_2^{-1} \vec{F} \cdot \frac{d\vec{r}}{dy}\, dy$$

$$= \int_2^{-1} \left(2y^5 - y\right) dy$$

$$= \left[\frac{1}{3}y^6 - \frac{1}{2}y^2\right]_2^{-1}$$

$$= \left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{64}{3} - 2\right)$$

$$= -\frac{39}{2}$$

Exercise

Find the circulation and flux of the fields $\vec{F}_1 = x\hat{i} + y\hat{j}$ *and* $\vec{F}_2 = -y\hat{i} + x\hat{j}$ around and across each of the following curves.

a) The circle $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}, \quad 0 \le t \le 2\pi$

b) The ellipse $\vec{r}(t) = (\cos t)\hat{i} + (4\sin t)\hat{j}, \quad 0 \le t \le 2\pi$

Solution

a) $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j}, \quad 0 \le t \le 2\pi$

$$\frac{d\vec{r}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\vec{F}_1 = x\hat{i} + y\hat{j}$$

$$= (\cos t)\hat{i} + (\sin t)\hat{j}$$

$$\vec{F}_1 \bullet \frac{d\vec{r}}{dt} = \left((\cos t)\hat{i} + (\sin t)\hat{j}\right) \bullet \left((-\sin t)\hat{i} + (\cos t)\hat{j}\right)$$

$$= -\cos t \sin t + \sin t \cos t$$

$$= 0$$

$$\vec{F}_2 = -y\hat{i} + x\hat{j}$$

$$= -(\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\vec{F}_2 \bullet \frac{d\vec{r}}{dt} = \left(-(\sin t)\hat{i} + (\cos t)\hat{j}\right) \bullet \left((-\sin t)\hat{i} + (\cos t)\hat{j}\right)$$

$$= \sin^2 t + \cos^2 t$$

$$= 1$$

$$Cir_1 = \int_0^{2\pi} \left(\vec{F}_1 \bullet \frac{d\vec{r}}{dt}\right) dt$$

$$= \int_0^{2\pi} 0\,dt$$

$$= 0$$

$$Cir_2 = \int_0^{2\pi} \left(\vec{F}_2 \bullet \frac{d\vec{r}}{dt} \right) dt$$

$$= \int_0^{2\pi} dt$$

$$\underline{= 2\pi}$$

$$dx = -\sin t\ dt, \quad dy = \cos t\ dt$$

$$M_1 = x = \cos t, \quad N_1 = y = \sin t$$

$$M_2 = -y = -\sin t, \quad N_2 = x = \cos t$$

$$Flux_1 = \int_C M_1 dy - N_1 dx$$

$$= \int_0^{2\pi} \left(\cos^2 t + \sin^2 t \right) dt$$

$$= \int_0^{2\pi} dt$$

$$= \underline{2\pi}$$

$$Flux_2 = \int_C M_2 dy - N_2 dx$$

$$= \int_0^{2\pi} \left(-\sin t \cos t + \sin t \cos t \right) dt$$

$$= \int_0^{2\pi} (0)\, dt$$

$$= \underline{0}$$

b) $\vec{r}(t) = (\cos t)\hat{i} + (4\sin t)\hat{j}, \quad 0 \le t \le 2\pi$

$$\frac{d\vec{r}}{dt} = (-\sin t)\hat{i} + (4\cos t)\hat{j}$$

$$\vec{F}_1 = x\hat{i} + y\hat{j}$$

$$= (\cos t)\hat{i} + (4\sin t)\hat{j}$$

$$\vec{F}_1 \bullet \frac{d\vec{r}}{dt} = \left((\cos t)\hat{i} + (4\sin t)\hat{j} \right) \bullet \left((-\sin t)\hat{i} + (4\cos t)\hat{j} \right)$$

$$= -\cos t \sin t + 16 \sin t \cos t$$

$$= 15 \sin t \cos t$$

$$\vec{F}_2 = -y\hat{i} + x\hat{j}$$
$$= -(4\sin t)\hat{i} + (\cos t)\,\hat{j}$$

$$\vec{F}_2 \bullet \frac{d\vec{r}}{dt} = \left((-4\sin t)\hat{i} + (\cos t)\,\hat{j}\right) \bullet \left((-\sin t)\hat{i} + (4\cos t)\,\hat{j}\right)$$
$$= 4\sin^2 t + 4\cos^2 t$$
$$\underline{= 4}\rfloor$$

$$Cir_1 = \int_0^{2\pi} \left(\vec{F}_1 \bullet \frac{d\vec{r}}{dt}\right) dt$$
$$= \int_0^{2\pi} 15\sin t\cos t dt \qquad\qquad d(\sin t) = \cos t\ dt$$
$$= 15\int_0^{2\pi} \sin t\ d(\sin t)$$
$$= \frac{15}{2}\left[\sin^2 t\right]_0^{2\pi}$$
$$= \frac{15}{2}(1-1)$$
$$\underline{= 0}\rfloor$$

$$Cir_2 = \int_0^{2\pi} \left(\vec{F}_2 \bullet \frac{d\vec{r}}{dt}\right) dt$$
$$= \int_0^{2\pi} 4dt$$
$$= 4t\Big|_0^{2\pi}$$
$$\underline{= 8\pi}\rfloor$$

$$dx = -\sin t\ dt, \quad dy = 4\cos t\ dt$$
$$M_1 = x = \cos t, \quad N_1 = y = 4\sin t$$
$$M_2 = -y = -4\sin t, \quad N_2 = x = \cos t$$

$$Flux_1 = \int_C M_1 dy - N_1 dx$$
$$= \int_0^{2\pi} \left(4\cos^2 t + 4\sin^2 t\right) dt$$

$$= 4\int_0^{2\pi} dt$$

$$= \underline{8\pi}$$

$$Flux_2 = \int_C M_2\,dy - N_2\,dx$$

$$= -15\int_0^{2\pi} (\sin t\cos t)\,dt$$

$$= -15\int_0^{2\pi} \sin t\ d(\sin t)$$

$$= -15\left[\frac{1}{2}\sin^2 t\right]_0^{2\pi}$$

$$\underline{= 0}$$

Exercise

Find the circulation and flux of the fields $\vec{F}_1 = 2x\hat{i} - 3y\hat{j}$ *and* $\vec{F}_2 = 2x\hat{i} + (x-y)\,\hat{j}$ across the circle $\vec{r}(t) = (a\cos t)\,\hat{i} + (a\sin t)\,\hat{j},\quad 0 \le t \le 2\pi$

Solution

$$\frac{d\vec{r}}{dt} = (-a\sin t)\,\hat{i} + (a\cos t)\,\hat{j}$$

$$\vec{F}_1 = 2x\hat{i} - 3y\hat{j}$$

$$= (2a\cos t)\,\hat{i} - (3a\sin t)\,\hat{j}$$

$$\vec{F}_1 \bullet \frac{d\vec{r}}{dt} = \left((2a\cos t)\,\hat{i} - (3a\sin t)\,\hat{j}\right) \bullet \left((-a\sin t)\,\hat{i} + (a\cos t)\,\hat{j}\right)$$

$$= -5a^2\cos t\sin t$$

$$\vec{F}_2 = 2x\hat{i} + (x-y)\,\hat{j}$$

$$= (2a\cos t)\,\hat{i} + a(\cos t - \sin t)\,\hat{j}$$

$$\vec{F}_2 \bullet \frac{d\vec{r}}{dt} = \left((2a\cos t)\,\hat{i} + a(\cos t - \sin t)\,\hat{j}\right) \bullet \left((-a\sin t)\,\hat{i} + (a\cos t)\,\hat{j}\right)$$

$$= -2a^2\cos t\sin t + a^2\cos^2 t - a^2\cos t\sin t$$

$$= a^2\left(\cos^2 t - 3\cos t\sin t\right)$$

$$Cir_1 = \int_0^{2\pi} \left(\vec{F}_1 \bullet \frac{d\vec{r}}{dt}\right) dt$$

$$= -5a^2 \int_0^{2\pi} \sin t \cos t dt$$

$$= -5a^2 \int_0^{2\pi} \sin t \; d(\sin t)$$

$$= -5a^2 \left[\sin^2 t\right]_0^{2\pi}$$

$$\underline{= 0}$$

$$Cir_2 = \int_0^{2\pi} \left(\vec{F}_2 \bullet \frac{d\vec{r}}{dt}\right) dt$$

$$= a^2 \int_0^{2\pi} \left(\cos^2 t - 3\cos t \sin t\right) dt$$

$$= a^2 \left[\int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right) dt - 3\int_0^{2\pi} (\sin t)\, d(\sin t)\right]$$

$$= a^2 \left[\frac{1}{2}t + \frac{1}{4}\sin 2t - 0\right]_0^{2\pi}$$

$$\underline{= \pi a^2}$$

$$dx = -a\sin t \; dt, \quad dy = a\cos t \; dt$$
$$M_1 = 2x = 2a\cos t, \quad N_1 = -3y = -3a\sin t$$
$$M_2 = 2a\cos t, \quad N_2 = a\cos t - a\sin t$$

$$Flux_1 = \int_C M_1 dy - N_1 dx$$

$$= \int_0^{2\pi} \left(2a^2\cos^2 t - 3a^2\sin^2 t\right) dt \qquad \cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t, \quad \sin^2 t = \frac{1}{2} - \frac{1}{2}\cos 2t$$

$$= a^2 \int_0^{2\pi} \left(1 + \cos 2t - \frac{3}{2} + \frac{3}{2}\cos 2t\right) dt$$

$$= a^2 \int_0^{2\pi} \left(\frac{5}{2}\cos 2t - \frac{1}{2}\right) dt$$

$$= a^2\left[\frac{5}{4}\sin 2t - \frac{1}{2}t\right]_0^{2\pi}$$

$$= a^2\left[0 - \frac{1}{2}(2\pi)\right]$$

$$= \underline{-\pi a^2}$$

$$Flux_2 = \int_C M_2 dy - N_2 dx$$

$$= \int_0^{2\pi}\left(2a^2\cos^2 t - a^2\sin^2 t + a^2\cos t\sin t\right)dt$$

$$= a^2\left[\int_0^{2\pi}\left(1 + \cos 2t - \frac{1}{2} + \frac{1}{2}\cos 2t\right)dt + \int_0^{2\pi}(\sin t)\,d(\sin t)\right]$$

$$= a^2\left[\frac{1}{2}t + \frac{3}{4}\sin 2t + \frac{1}{2}\sin^2 t\right]_0^{2\pi}$$

$$= a^2\frac{1}{2}(2\pi)$$

$$\underline{= \pi a^2}$$

Exercise

Find a field $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$ in the *xy*-plane with the property that at each point $(x,y) \neq (0,0)$, $\vec{F}$ points toward the origin and $|\vec{F}|$ is

a) The distance from (x, y) to the origin

b) Inversely proportional to the distance from (x, y) to the origin. (The field is undefined at (0, 0).)

Solution

a) The slope of the line through the origin and a point (x,y) is: $m = \frac{y}{x}$

The vector parallel to the line is given by: $\vec{v} = x\hat{i} + y\hat{j}$

Pointing away from the origin: $\vec{F} = -\frac{\vec{v}}{|\vec{v}|} = -\frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}}$ is the unit vector pointing toward the origin.

$$|\vec{F}| = \sqrt{x^2 + y^2}$$

$$\vec{F} = \sqrt{x^2 + y^2}\left(-\frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}}\right)$$

$$= -x\hat{i} - y\hat{j}$$

b) $\left|\vec{F}\right| = \dfrac{C}{\sqrt{x^2+y^2}}, \quad C \neq 0$

$$\vec{F} = \frac{C}{\sqrt{x^2+y^2}}\left(-\frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}}\right)$$

$$\boxed{= -C\left(\frac{x\hat{i}+y\hat{j}}{x^2+y^2}\right)}$$

Exercise

A fluid's velocity field is $\vec{F} = -4xy\hat{i} + 8y\hat{j} + 2\hat{k}$. Find the flow along the curve $\vec{r}(t) = t\hat{i} + t^2\hat{j} + \hat{k}, \quad 0 \le t \le 2$

Solution

$$\frac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j}$$

$$\vec{F} = -4xy\hat{i} + 8y\hat{j} + 2\hat{k}$$

$$= -4t^3\hat{i} + 8t^2\hat{j} + 2\hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left(-4t^3\hat{i} + 8t^2\hat{j} + 2\hat{k}\right) \bullet \left(\hat{i} + 2t\,\hat{j}\right)$$

$$= -4t^3 + 16t^3 = 12t^3$$

$$Flow = \int_R \vec{F} \bullet \frac{d\vec{r}}{dt}\,dt$$

$$= \int_0^2 12t^3 dt$$

$$= 3t^4 \Big|_0^2$$

$$\boxed{= 48}$$

Exercise

A fluid's velocity field is $\vec{F} = x^2\hat{i} + yz\hat{j} + y^2\hat{k}$. Find the flow along the curve $\vec{r}(t) = 3t\,\hat{j} + 4t\,\hat{k}, \quad 0 \le t \le 1$

Solution

$$\frac{d\vec{r}}{dt} = 3\hat{i} + 4\hat{j}$$

$$\vec{F} = x^2\hat{i} + yz\hat{j} + y^2\hat{k}$$

$$= 12t^2\hat{j} + 9t^2\hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left(12t^2\hat{j} + 9t^2\hat{k}\right) \bullet \left(3\hat{i} + 4\hat{j}\right)$$

$$= 36t^2 + 36t^2 = 72t^2$$

$$Flow = \int_0^1 72t^2 dt \qquad\qquad Flow = \int_C \vec{F} \bullet \frac{d\vec{r}}{dt}\,dt$$

$$= 24t^3 \Big|_0^1$$

$$\underline{= 24}$$

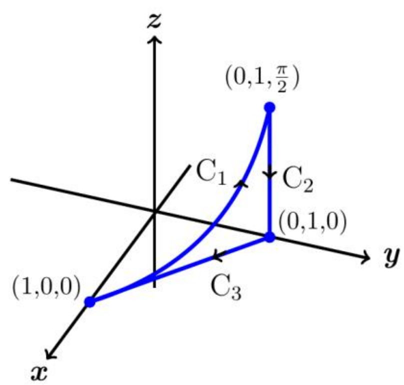
Exercise

Find the circulation of $\vec{F} = 2x\hat{i} + 2z\hat{j} + 2y\hat{k}$ around the closed path consisting of the following three curves traversed in the direction of increasing t.

$$C_1: \ \ \vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\,\hat{k}, \quad 0 \le t \le \frac{\pi}{2}$$

$$C_2: \ \ \vec{r}(t) = \hat{j} + \frac{\pi}{2}(1-t)\hat{k}, \quad 0 \le t \le 1$$

$$C_3: \ \ \vec{r}(t) = t\,\hat{i} + (1-t)\hat{j}, \quad 0 \le t \le 1$$



Solution

$$C_1: \ \ \vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\,\hat{k}, \quad 0 \le t \le \frac{\pi}{2}$$

$$\frac{d\vec{r}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j} + \hat{k}$$

$$\vec{F} = 2x\hat{i} + 2z\hat{j} + 2y\hat{k}$$

$$= (2\cos t)\hat{i} + 2t\,\hat{j} + (2\sin t)\hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left((2\cos t)\hat{i} + 2t\,\hat{j} + (2\sin t)\hat{k}\right) \bullet \left((-\sin t)\hat{i} + (\cos t)\hat{j} + \hat{k}\right)$$

$$= -2\sin t\cos t + 2t\cos t + 2\sin t$$

$$= -\sin 2t + 2t\cos t + 2\sin t$$

$$Flow_1 = \int_0^{\pi/2} \left(-\sin 2t + 2t\cos t + 2\sin t\right) dt$$

		$\int \cos t$
+	t	$\sin t$
−	1	$-\cos t$

$$= \left[\frac{1}{2}\cos 2t + 2t\sin t + 2\cos t - 2\cos t\right]_0^{\pi/2}$$

$$= \left[\frac{1}{2}\cos 2t + 2t\sin t\right]_0^{\pi/2}$$

$$= \left(-\frac{1}{2} + 2\frac{\pi}{2}\right) - \left(\frac{1}{2}\right)$$

$$\underline{= \pi - 1}$$

C_2 : $\vec{r}(t) = \hat{j} + \frac{\pi}{2}(1-t)\hat{k}, \quad 0 \le t \le 1$

$$\frac{d\vec{r}}{dt} = -\frac{\pi}{2}\hat{k}$$

$$\vec{F} = 2x\hat{i} + 2z\hat{j} + 2y\hat{k}$$

$$= \pi(1-t)\,\hat{j} + 2\hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left(\pi(1-t)\,\hat{j} + 2\hat{k}\right) \bullet \left(-\frac{\pi}{2}\hat{k}\right)$$

$$= -\pi$$

$$Flow_2 = \int_0^1 (-\pi)\,dt$$

$$= -\pi t\Big|_0^1$$

$$\underline{= -\pi}$$

C_3 : $\vec{r}(t) = t\,\hat{i} + (1-t)\,\hat{j}, \quad 0 \le t \le 1$

$$\frac{d\vec{r}}{dt} = \hat{i} - \hat{j}$$

$$\vec{F} = 2x\hat{i} + 2z\hat{j} + 2y\hat{k}$$

$$= 2t\hat{i} + 2(1-t)\hat{k}$$

$$\vec{F} \bullet \frac{d\vec{r}}{dt} = \left(2t\hat{i} + 2(1-t)\hat{k}\right) \bullet \left(\hat{i} - \hat{j}\right)$$

$$= 2t$$

$$Flow_3 = \int_0^1 (2t)\,dt$$

$$= t^2\Big|_0^1$$

$$\underline{= 1}$$

$$Circulation = Flow_1 + Flow_2 + Flow_3$$

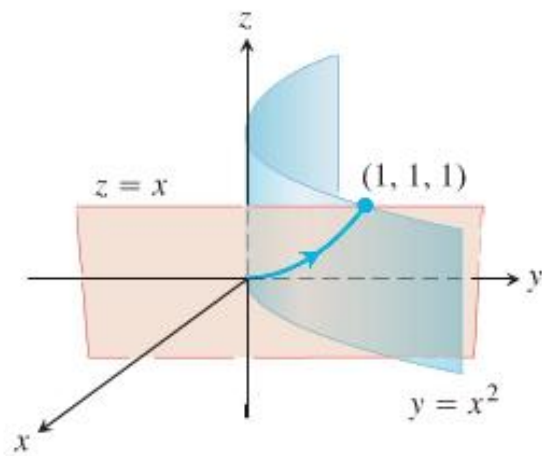
$$= \pi - 1 - \pi + 1$$

$$\underline{= 0}$$

Exercise

The field $\vec{F} = xy\hat{i} + y\hat{j} - yz\hat{k}$ is the velocity field of a flow in space. Find the flow from $(0,\ 0,\ 0)$ to $(1,\ 1,\ 1)$ along the curve of intersection of the cylinder $y = x^2$ and the plane $z = x$. (***Hint***: Use $t = x$ as the parameter.)

Solution



Let $x = t \quad \Rightarrow \quad y = x^2 = t^2$

$z = x = t$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= t\hat{i} + t^2\hat{j} + t\hat{k} \qquad 0 \le t \le 1$$

$$\frac{d\vec{r}}{dt} = \hat{i} + 2t\,\hat{j} + \hat{k}$$

$$\vec{F} = xy\hat{i} + y\hat{j} - yz\hat{k}$$

$$= t^3\hat{i} + t^2\hat{j} - t^3\hat{k}$$

$$\vec{F} \cdot \frac{d\vec{r}}{dt} = \left(t^3\hat{i} + t^2\hat{j} - t^3\hat{k}\right) \cdot \left(\hat{i} + 2t\,\hat{j} + \hat{k}\right)$$

$$= t^3 + 2t^3 - t^3 = 2t^3$$

$$Flow = \int_0^1 \left(2t^3\right) dt$$

$$= \frac{1}{2}t^4 \Big|_0^1$$

$$= \frac{1}{2}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ for the vector fields $\vec{F}$ and curves C.

$$\vec{F} = \nabla\left(x^2 y\right); \quad C: \vec{r}(t) = \left\langle 9 - t^2,\ t\right\rangle, \quad for \quad 0 \le t \le 3$$

Solution

$$\vec{F} = \nabla\left(x^2 y\right)$$

$$= \left\langle 2xy,\ x^2\right\rangle$$

$$= \left\langle 18t - 2t^3,\ 81 - 18t^2 + t^4\right\rangle$$

$$\vec{r}'(t) = \langle -2t,\ 1\rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^3 \left\langle 18t - 2t^3,\ \ 81 - 18t^2 + t^4 \right\rangle \cdot \langle -2t,\ 1\rangle\, dt$$

$$= \int_0^3 \left(-36t^2 + 4t^4 + \ 81 - 18t^2 + t^4 \right) dt$$

$$= \int_0^3 \left(5t^4 - 54t^2 + \ 81 \right) dt$$

$$= \left(t^5 - 18t^3 + \ 81t \right) \Big|_0^3$$

$$= 243 - 486 + 243$$

$$\underline{= 0}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \nabla(xyz); \quad C: \vec{r}(t) = \left\langle \cos t,\ \sin t,\ \frac{t}{\pi} \right\rangle, \quad \textit{for} \quad 0 \le t \le \pi$

Solution

$$\vec{F} = \nabla(xyz)$$

$$= \langle yz,\ \ xz,\ \ xy\rangle$$

$$= \left\langle \frac{t}{\pi}\sin t,\ \ \frac{t}{\pi}\cos t,\ \ \cos t \sin t \right\rangle$$

$$\vec{r}'(t) = \left\langle -\sin t,\ \cos t,\ \frac{1}{\pi} \right\rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^\pi \left\langle \frac{t}{\pi}\sin t,\ \ \frac{t}{\pi}\cos t,\ \ \cos t \sin t \right\rangle \cdot \left\langle -\sin t,\ \cos t,\ \frac{1}{\pi} \right\rangle dt$$

$$= \int_0^\pi \left(-\frac{t}{\pi}\sin^2 t + \frac{t}{\pi}\cos^2 t + \frac{1}{\pi}\cos t \sin t \right) dt$$

$$= \frac{1}{\pi}\int_0^\pi \left(t\cos 2t + \frac{1}{2}\sin 2t \right) dt$$

$$= \frac{1}{\pi}\left(\frac{1}{2}t\sin 2t + \frac{1}{4}\cos 2t - \frac{1}{4}\cos 2t \right) \Big|_0^\pi$$

		$\int \cos 2t$
$+$	t	$\frac{1}{2}\sin 2t$
$-$	1	$-\frac{1}{4}\cos 2t$

$$= \frac{1}{2\pi}\left(t \sin 2t\right)\Big|_0^{\pi}$$

$$\underline{= 0}$$

Or

$$\vec{F} = \nabla(xyz) = \nabla\varphi$$

$$\int_C \vec{F}\cdot d\vec{r} = \varphi(\pi) - \varphi(0)$$

$$= \varphi\left(\cos\pi \sin\pi\left(\frac{\pi}{\pi}\right)\right) - \varphi\left(\cos 0 \sin 0\left(\frac{0}{\pi}\right)\right)$$

$$= 0 - 0$$

$$\underline{= 0}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \bullet d\vec{r}$ .for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \langle x, \ -y\rangle$; C is the square with vertices $(\pm 1, \ \pm 1)$ with counterclockwise orientation.

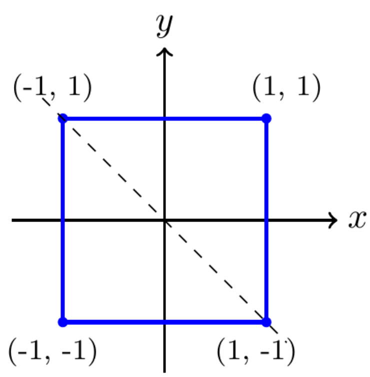
Solution

$(-1, \ -1) \ \rightarrow \ (1, \ -1)$

$$\begin{cases} x = -1 + (1+1)t \\ y = -1 + (-1+1)t \end{cases}$$

$$\vec{r}_1(t) = \langle -1 + 2t, \ -1\rangle$$

$$\vec{r}_1{}'(t) = \langle 2, \ 0\rangle$$



$(1, \ -1) \ \rightarrow \ (1, \ 1)$

$$\begin{cases} x = 1 + (1-1)t \\ y = -1 + (1+1)t \end{cases}$$

$$\vec{r}_2(t) = \langle 1, \ -1 + 2t\rangle$$

$$\vec{r}_2{}'(t) = \langle 0, \ 2\rangle$$

$(1, \ 1) \ \rightarrow \ (-1, \ 1)$

$$\vec{r}_3(t) = \langle 1 - 2t, \ 1\rangle$$

$$\vec{r}_3{}'(t) = \langle -2, \ 0\rangle$$

$(-1, \ 1) \ \rightarrow \ (-1, \ -1)$

$$\vec{r}_4(t) = \langle -1, \ 1 - 2t\rangle$$

$$\vec{r}_4{}'(t) = \langle 0, \ -2 \rangle$$

$$\vec{F}_1 = \langle -1+2t, \ 1 \rangle$$

$$\vec{F}_1 \cdot \vec{r}_1{}'(t) = \langle -1+2t, \ 1 \rangle \cdot \langle 2, \ 0 \rangle$$

$$= 4t - 2$$

$$\vec{F}_2 = \langle 1, \ 1-2t \rangle$$

$$\vec{F}_2 \cdot \vec{r}_2{}'(t) = \langle 1, \ 1-2t \rangle \cdot \langle 0, \ 2 \rangle$$

$$= 2 - 4t$$

$$\vec{F}_3 = \langle 1-2t, \ -1 \rangle$$

$$\vec{F}_3 \cdot \vec{r}_3{}'(t) = \langle 1-2t, \ -1 \rangle \cdot \langle -2, \ 0 \rangle$$

$$= 4t - 2$$

$$\vec{F}_4 = \langle -1, \ -1+2t \rangle$$

$$\vec{F}_4 \cdot \vec{r}_4{}'(t) = \langle -1, \ -1+2t \rangle \cdot \langle 0, \ -2 \rangle$$

$$= 2 - 4t$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (4t - 2 + 2 - 4t + 4t - 2 + 2 - 4t)\, dt$$

$$= 0$$

Or

$$\vec{F} = \nabla(xyz) = \nabla\varphi$$

$$= \nabla\left(\frac{1}{2}\left(x^2 + y^2\right)\right)$$

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

Since the integral around any closed curve is 0.

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \langle y, \ z, \ -x \rangle; \quad C: \ \vec{r}(t) = \langle \cos t, \ \sin t, \ 4 \rangle, \quad \textit{for} \quad 0 \le t \le 2\pi$

Solution

$$\vec{F} = \langle \sin t, \ 4, \ -\cos t \rangle$$

$$\vec{r}'(t) = \langle -\sin t,\ \cos t,\ 0\rangle$$

$$\vec{F}\cdot\vec{r}'(t) = \langle \sin t,\ 4,\ -\cos t\rangle \cdot \langle -\sin t,\ \cos t,\ 0\rangle$$

$$= -\sin^2 t + 4\cos t$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left(4\cos t - \sin^2 t\right)dt$$

$$= \int_0^{2\pi} \left(4\cos t - \frac{1}{2} - \frac{1}{2}\cos 2t\right)dt$$

$$= 4\sin t - \frac{1}{2}t - \frac{1}{2}\cos 2t \Big|_0^{2\pi}$$

$$\boxed{= -\pi}$$

Exercise

Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$ for the vector fields $\vec{F}$ and curves C

$\vec{F} = \left\langle y^2,\ x\right\rangle$; where C is the arc of the parabola $x = 4 - y^2$ from $(-5,\ -3)$ to $(0,\ 2)$

Solution

Let $y = t \quad \rightarrow \quad -3 \le t \le 2$

$$\vec{r}(t) = \left\langle 4 - t^2,\ t\right\rangle$$

$$\vec{r}'(t) = \langle -2t,\ 1\rangle$$

$$\vec{F} = \left\langle t^2,\ 4 - t^2\right\rangle$$

$$\vec{F}\cdot\vec{r}' = \left\langle t^2,\ 4 - t^2\right\rangle \cdot \langle -2t,\ 1\rangle$$

$$= -2t^3 + 4 - t^2$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{-3}^{2} \left(-2t^3 + 4 - t^2\right)dt$$

$$= \left(-\frac{1}{2}t^4 + 4t - \frac{1}{3}t^3\right)\Big|_{-3}^{2}$$

$$= -8 + 8 - \frac{8}{3} + \frac{81}{2} + 12 - 9$$

$$= \frac{-16 + 243 + 18}{6}$$

$$\boxed{= \frac{245}{6}}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ for the vector fields $\vec{F}$ and curves C

$\vec{F} = \left\langle x^2 + y^2,\ 4x + y^2 \right\rangle$; where C is the straight line segment from $(6,\ 3)$ to $(6,\ 0)$

Solution

$$\vec{r}(t) = \langle 6,\ 3 - 3t \rangle$$

$$\vec{r}'(t) = \langle 0,\ -3 \rangle$$

$$\vec{F} = \left\langle 36 + 9 - 18t + 9t^2,\ 24 + 9 - 18t + 9t^2 \right\rangle$$

$$= \left\langle 45 - 18t + 9t^2,\ 33 - 18t + 9t^2 \right\rangle$$

$$\vec{F} \cdot \vec{r}' = \left\langle 45 - 18t + 9t^2,\ 33 - 18t + 9t^2 \right\rangle \cdot \langle 0,\ -3 \rangle$$

$$= -99 + 54t - 27t^2$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left(-99 + 54t - 27t^2 \right) dt$$

$$= \left(-99t + 27t^2 - 9t^3 \right) \Big|_0^1$$

$$= -99 + 27 - 9$$

$$\underline{= -81}$$

OR

$(6,\ 3)$ to $(6,\ 0)$ is just a straight parallel to the x-axis,.

$x = 6 \quad \& \ dx = 0$

$$\int_C \vec{F} \cdot d\vec{r} = \oint_C \left(x^2 + y^2 \right) dx + \left(4x + y^2 \right) dy$$

$$= \oint_C 0 + \left(24 + y^2 \right) dy$$

$$= \int_3^0 \left(24 + y^2 \right) dy$$

$$= \left(24y + \frac{1}{3} y^3 \right) \Big|_3^0$$

$$= -72 - 9$$

$$\underline{= -81}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \bullet \vec{T}\ ds$ for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \langle x,\ y \rangle$ on the parabola $\vec{r}(t) = \left\langle 4t,\ t^2 \right\rangle \quad 0 \le t \le 1$

Solution

$$\vec{F} = \left\langle 4t,\ t^2 \right\rangle$$

$$\vec{r}\,' = \langle 4,\ 2t \rangle$$

$$\begin{aligned} \vec{F} \bullet \vec{r}\,' &= \left\langle 4t,\ t^2 \right\rangle \bullet \langle 4,\ 2t \rangle \\ &= 16t + 2t^3 \end{aligned}$$

$$\begin{aligned} \int_C \vec{F} \bullet \vec{T}\ ds &= \int_0^1 \left(16t + 2t^3\right) dt \\ &= 8t^2 + \frac{1}{2}t^4 \Big|_0^1 \\ &= 8 + \frac{1}{2} \\ &= \frac{17}{2} \end{aligned}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \bullet \vec{T}\ ds$ for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \langle -y,\ x \rangle$ on the semicircle $\vec{r}(t) = \langle 4\cos t,\ 4\sin t \rangle \quad 0 \le t \le \pi$

Solution

$$\vec{F} = \langle -4\sin t,\ 4\cos t \rangle$$

$$\vec{r}\,' = \langle -4\sin t,\ 4\cos t \rangle$$

$$\begin{aligned} \vec{F} \bullet \vec{r}\,' &= \langle -4\sin t,\ 4\cos t \rangle \bullet \langle -4\sin t,\ 4\cos t \rangle \\ &= 16\sin^2 t + 16\cos^2 t \\ &= 16 \end{aligned}$$

$$\int_C \vec{F} \bullet \vec{T}\ ds = \int_0^{\pi} 16\ dt$$

$$= 16t \Big|_0^{\pi}$$

$$\underline{= 16\pi}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot \vec{T}\, ds$ for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \langle y,\ x \rangle$ on the line segment from $(1,\ 1)$ to $(5,\ 10)$

Solution

$$\vec{r}(t) = \langle (5-1)t + 1,\ (10-1)t + 1 \rangle$$
$$= \langle 4t + 1,\ 9t + 1 \rangle$$

$$\vec{F} = \langle 9t + 1,\ 4t + 1 \rangle$$

$$\vec{r}\,' = \langle 4,\ 9 \rangle$$

$$\vec{F} \cdot \vec{r}\,' = \langle 9t + 1,\ 4t + 1 \rangle \cdot \langle 4,\ 9 \rangle$$
$$= 36t + 4 + 36t + 9$$
$$= 72t + 13$$

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_0^1 (72t + 13)\, dt$$

$$= 36t^2 + 13t \Big|_0^1$$

$$= 36 + 13$$
$$\underline{= 49}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \cdot \vec{T}\, ds$ for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \langle -y,\ x \rangle$ on the parabola $y = x^2$ from $(0,\ 0)$ to $(1,\ 1)$

Solution

$$\vec{r}(t) = \left\langle t,\ t^2 \right\rangle \qquad \langle x = t,\ y \rangle$$

$$\vec{F} = \left\langle -t^2,\ t \right\rangle$$

$$\vec{r}\,' = \langle 1,\ 2t \rangle$$

$$\vec{F} \bullet \vec{r}' = \left\langle -t^2,\ t \right\rangle \bullet \langle 1,\ 2t \rangle$$

$$= t^2$$

$$\int_C \vec{F} \bullet \vec{T}\ ds = \int_0^1 t^2 dt$$

$$= \frac{1}{3} t^3 \Big|_0^1$$

$$\boxed{= \frac{1}{3}}$$

Exercise

Evaluate the line integral $\int_C \vec{F} \bullet \vec{T}\ ds$ for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \dfrac{\langle x,\ y \rangle}{\left(x^2 + y^2\right)^{3/2}}$ on the curve $\vec{r}(t) = \left\langle t^2,\ 3t^2 \right\rangle \quad 1 \le t \le 2$

Solution

$$\vec{F} = \frac{\left\langle t^2,\ 3t^2 \right\rangle}{\left(t^4 + 9t^4\right)^{3/2}}$$

$$= \frac{\left\langle t^2,\ 3t^2 \right\rangle}{\left(10t^4\right)^{3/2}}$$

$$= \frac{1}{10\sqrt{10}} \frac{\left\langle t^2,\ 3t^2 \right\rangle}{t^6}$$

$$= \frac{1}{10\sqrt{10}} \left\langle \frac{1}{t^4},\ \frac{3}{t^4} \right\rangle$$

$$\vec{r}' = \langle 2t,\ 6t \rangle$$

$$\vec{F} \bullet \vec{r}' = \frac{1}{10\sqrt{10}} \left\langle \frac{1}{t^4},\ \frac{3}{t^4} \right\rangle \bullet \langle 2t,\ 6t \rangle$$

$$= \frac{1}{10\sqrt{10}} \left(\frac{2}{t^3} + \frac{18}{t^3} \right)$$

$$= \frac{2}{\sqrt{10}} \frac{1}{t^3}$$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \frac{2}{\sqrt{10}}\int_1^2 t^{-3}dt$$

$$= -\frac{1}{\sqrt{10}}t^{-2}\Big|_1^2$$

$$= -\frac{\sqrt{10}}{10}\left(\frac{1}{4}-1\right)$$

$$\boxed{= \frac{3\sqrt{10}}{40}}$$

Exercise

Evaluate the line integral $\int_C \vec{F}\cdot\vec{T}\,ds$ for the vector fields $\vec{F}$ and curves C.

$\vec{F} = \dfrac{\langle x,\ y\rangle}{x^2+y^2}$ on the line $\vec{r}(t) = \langle t,\ 4t\rangle \quad 1 \le t \le 10$

Solution

$$\vec{F} = \frac{\langle t,\ 4t\rangle}{t^2+16t^2}$$

$$= \frac{1}{17}\left\langle \frac{1}{t},\ \frac{4}{t}\right\rangle$$

$$\vec{r}\,' = \langle 1,\ 4\rangle$$

$$\vec{F}\cdot\vec{r}\,' = \frac{1}{17}\left\langle \frac{1}{t},\ \frac{4}{t}\right\rangle \cdot \langle 1,\ 4\rangle$$

$$= \frac{1}{17}\left(\frac{1}{t}+\frac{16}{t}\right)$$

$$= \frac{1}{t}$$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_1^{10}\frac{1}{t}dt$$

$$= \ln t\Big|_1^{10}$$

$$\boxed{= \ln 10}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \langle y,\ -x \rangle$ on the path consisting of the line segment from $(1,\ 2)$ to $(0,\ 0)$ followed by the line segment from $(0,\ 0)$ to $(0,\ 4)$

Solution

$(1,\ 2)$ to $(0,\ 0)$

$$\vec{r}_1(t) = \langle 1-t,\ 2-2t \rangle$$

$$\vec{r}_1{}'(t) = \langle -1,\ -2 \rangle$$

$$\vec{F} = \langle 2-2t,\ t-1 \rangle$$

$$\begin{aligned}\vec{F} \cdot \vec{r}_1{}'(t) &= \langle 2-2t,\ t-1 \rangle \cdot \langle -1,\ -2 \rangle \\ &= -2+2t-2t+2 \\ &= 0\end{aligned}$$

$(0,\ 0)$ to $(0,\ 4)$

$$\vec{r}_2(t) = \langle 0,\ 4t \rangle$$

$$\vec{r}_2{}'(t) = \langle 0,\ 4 \rangle$$

$$\vec{F} = \langle 4t,\ 0 \rangle$$

$$\begin{aligned}\vec{F} \cdot \vec{r}_2{}'(t) &= \langle 4t,\ 0 \rangle \cdot \langle 0,\ 4 \rangle \\ &= 0\end{aligned}$$

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \vec{F} \cdot \vec{r}_1{}'\ dt + \int_0^1 \vec{F} \cdot \vec{r}_2{}'\ dt \\ &= 0\end{aligned}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \langle x,\ y \rangle$ on the path consisting of the line segment from $(-1,\ 0)$ to $(0,\ 8)$ followed by the line segment from $(0,\ 8)$ to $(2,\ 8)$

Solution

$(-1,\ 0)$ to $(0,\ 8)$

$$\vec{r}_1(t) = \langle t-1,\ 8t \rangle$$

$$\vec{r}_1{}'(t) = \langle 1,\ 8 \rangle$$

$$\vec{F} = \langle t-1,\ 8t\rangle$$

$$\begin{aligned}\vec{F}\cdot\vec{r}_1{}'(t) &= \langle t-1,\ 8t\rangle \cdot \langle 1,\ 8\rangle \\ &= t-1+64t \\ &= \underline{65t-1} \end{aligned}$$

$(0,\ 8)$ to $(2,\ 8)$

$$\vec{r}_2(t) = \langle 2t,\ 8\rangle$$

$$\vec{r}_2{}'(t) = \langle 2,\ 0\rangle$$

$$\vec{F} = \langle 2t,\ 8\rangle \text{o}$$

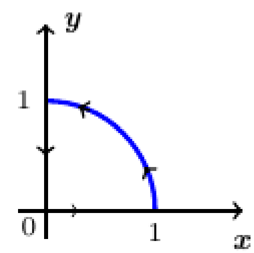
$$\begin{aligned}\vec{F}\cdot\vec{r}_2{}'(t) &= \langle 2t,\ 8\rangle \cdot \langle 2,\ 0\rangle \\ &= \underline{4t}\end{aligned}$$

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^1 \vec{F}\cdot\vec{r}_1{}'\ dt + \int_0^1 \vec{F}\cdot\vec{r}_2{}'\ dt \\ &= \int_0^1 (65t-1+4t)\,dt \\ &= \frac{69}{2}t^2 - t\Big|_0^1 \\ &= \frac{69}{2}-1 \\ &= \underline{\frac{67}{2}}\end{aligned}$$

Exercise

Find the work required to move an object on the given oriented curve $\vec{F} = \left\langle x^2,\ -xy\right\rangle$ on runs from $(1,\ 0)$ to $(0,\ 1)$ along the unit circle and then from $(0,\ 1)$ to $(0,\ 0)$ along the *y*-axis.

Solution



Along the unit circle: $\left(0 \le t \le \frac{\pi}{2}\right)$

$$\vec{r}_1(t) = \langle \cos t,\ \sin t\rangle$$

$$\vec{r}_1{}'(t) = \langle -\sin t,\ \cos t\rangle$$

$$\vec{F}_1 = \left\langle \cos^2 t,\ -\cos t \sin t\right\rangle$$

$$\vec{F_1} \cdot \vec{r}_1{}'(t) = \left\langle \cos^2 t,\ -\cos t \sin t \right\rangle \cdot \langle -\sin t,\ \cos t \rangle$$

$$= -\sin t \cos^2 t - \sin t \cos^2 t$$

$$\underline{= -2\sin t \cos^2 t}$$

$(0,\ 1)$ to $(0,\ 0)$: $\quad (0 \le t \le 1)$

$$\vec{r}_2(t) = \langle 0,\ t \rangle$$

$$\vec{r}_2{}'(t) = \langle 0,\ 1 \rangle$$

$$\vec{F}_1 = \langle 0,\ 0 \rangle$$

$$\vec{F_1} \cdot \vec{r}_1{}'(t) \ \underline{= 0}$$

$$W = \int_0^{\frac{\pi}{2}} \left(-2\sin t \cos^2 t \right) dt + 0$$

$$W = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= 2\int_0^{\frac{\pi}{2}} \cos^2 t \ d(\cos t)$$

$$= \frac{2}{3}\cos^3 t \Big|_0^{\frac{\pi}{2}}$$

$$\underline{= -\frac{2}{3}}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \langle y,\ x \rangle$ on the parabola $y = 2x^2$ from $(0,\ 0)$ to $(2,\ 8)$

Solution

$$\vec{r}(t) = \left\langle x,\ 2x^2 \right\rangle$$

$$= \left\langle 2t,\ 8t^2 \right\rangle \qquad 0 \le t \le 1$$

$$\vec{F} = \left\langle 8t^2,\ 2t \right\rangle$$

$$\vec{r}' = \langle 2,\ 16t \rangle$$

$$\vec{F} \cdot \vec{r}'(t) = \left\langle 8t^2,\ 2t \right\rangle \cdot \langle 2,\ 16t \rangle$$

$$= 16t^2 + 32t^2$$

$$= 48t^2$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 48t^2 \, dt$$

$$= 16t^3 \Big|_0^1$$

$$\underline{= 16}$$

Exercise

Find the work required to move an object on the given oriented curve
$\vec{F} = \langle y, \ -x \rangle$ on the line $y = 10 - 2x$ from $(1, \ 8)$ to $(3, \ 4)$

Solution

$$\vec{r}(t) = \langle 2t + 1, \ -4t + 8 \rangle$$

$$\vec{F} = \langle 8 - 4t, \ -2t - 1 \rangle$$

$$\vec{r}' = \langle 2, \ -4 \rangle$$

$$\begin{aligned} \vec{F} \cdot \vec{r}'(t) &= \langle 8 - 4t, \ -2t - 1 \rangle \cdot \langle 2, \ -4 \rangle \\ &= 16 - 8t + 8t + 4 \\ &= 20 \end{aligned}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 20 \, dt$$

$$\underline{= 20}$$

Exercise

Find the work required to move an object on the given oriented curve
$\vec{F} = \langle x, \ y, \ z \rangle$ on the tilted ellipse $\vec{r}(t) = \langle 4\cos t, \ 4\sin t, \ 4\cos t \rangle \quad 0 \le t \le 2\pi$

Solution

$$\vec{F} = \langle 4\cos t, \ 4\sin t, \ 4\cos t \rangle$$

$$\vec{r}' = \langle -4\sin t, \ 4\cos t, \ -4\sin t \rangle$$

$$\begin{aligned} \vec{F} \cdot \vec{r}' &= \langle 4\cos t, \ 4\sin t, \ 4\cos t \rangle \cdot \langle -4\sin t, \ 4\cos t, \ -4\sin t \rangle \\ &= -16\cos t \sin t + 16\sin t \cos t - 16\cos t \sin t \\ &= -16\cos t \sin t \end{aligned}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-16\cos t \sin t) \, dt$$

$$= \int_0^{2\pi} 16\sin t \ d(\cos t)$$

$$= 8\sin^2 t \Big|_0^{2\pi}$$

$$\underline{= 0}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \langle -y,\ x,\ z\rangle$ on the helix $\vec{r}(t) = \left\langle 2\cos t,\ 2\sin t,\ \frac{t}{2\pi}\right\rangle \quad 0 \le t \le 2\pi$

Solution

$$\vec{F} = \left\langle -2\sin t,\ 2\cos t,\ \frac{t}{2\pi}\right\rangle$$

$$\vec{r}\,' = \left\langle -2\sin t,\ 2\cos t,\ \frac{1}{2\pi}\right\rangle$$

$$\vec{F} \bullet \vec{r}\,' = \left\langle -2\sin t,\ 2\cos t,\ \frac{t}{2\pi}\right\rangle \bullet \left\langle -2\sin t,\ 2\cos t,\ \frac{1}{2\pi}\right\rangle$$

$$= 4\sin^2 t + 4\cos^2 t + \frac{t}{4\pi^2}$$

$$= 4 + \frac{1}{4\pi^2}t$$

$$\int_C \vec{F} \bullet d\vec{r} = \int_0^{2\pi} \left(4 + \frac{1}{4\pi^2}t\right) dt$$

$$= 4t + \frac{1}{8\pi^2}t^2 \Big|_0^{2\pi}$$

$$\underline{= 8\pi + \frac{1}{2}}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ on the line segment from $(1,\ 1,\ 1)$ to $(10,\ 10,\ 10)$

Solution

$$\vec{r}(t) = \langle t+1,\ t+1,\ t+1\rangle \quad 0 \le t \le 9$$

$$\vec{r}\,' = \langle 1,\ 1,\ 1\rangle$$

$$\vec{F} = \frac{\langle t+1,\ t+1,\ t+1\rangle}{\left(3(t+1)^2\right)^{3/2}}$$

$$= \frac{1}{3\sqrt{3}} \frac{\langle t+1,\ t+1,\ t+1\rangle}{(t+1)^3}$$

$$= \frac{1}{3\sqrt{3}} \left\langle \frac{1}{(t+1)^2},\ \frac{1}{(t+1)^2},\ \frac{1}{(t+1)^2} \right\rangle$$

$$\vec{F} \bullet \vec{r}\,' = \frac{1}{3\sqrt{3}} \left\langle \frac{1}{(t+1)^2},\ \frac{1}{(t+1)^2},\ \frac{1}{(t+1)^2} \right\rangle \bullet \langle 1,\ 1,\ 1\rangle$$

$$= \frac{1}{\sqrt{3}} \frac{1}{(t+1)^2}$$

$$\int_C \vec{F} \bullet d\vec{r} = \frac{1}{\sqrt{3}} \int_0^9 \frac{1}{(t+1)^2}\ dt$$

$$= \frac{1}{\sqrt{3}} \int_0^9 \frac{1}{(t+1)^2}\ d(t+1)$$

$$= -\frac{1}{\sqrt{3}} \frac{1}{t+1} \Big|_0^9$$

$$= -\frac{\sqrt{3}}{3}\left(\frac{1}{10} - 1\right)$$

$$= \frac{3\sqrt{3}}{10}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \dfrac{\langle x,\ y,\ z\rangle}{\left(x^2 + y^2 + z^2\right)^{3/2}}$ on the path $\vec{r}(t) = \left\langle t^2,\ 3t^2,\ -t^2\right\rangle,\quad 1 \le t \le 2$

Solution

$$\vec{r}\,' = \langle 2t,\ 6t,\ -2t\rangle$$

$$\vec{F} = \frac{\left\langle t^2,\ 3t^2,\ -t^2\right\rangle}{\left(t^4 + 9t^4 + t^4\right)^{3/2}}$$

$$= \frac{1}{11\sqrt{11}} \frac{\left\langle t^2,\ 3t^2,\ -t^2 \right\rangle}{t^6}$$

$$= \frac{1}{11\sqrt{11}} \left\langle \frac{1}{t^4},\ \frac{3}{t^4},\ -\frac{1}{t^4} \right\rangle$$

$$\vec{F} \bullet \vec{r}\,' = \frac{1}{11\sqrt{11}} \left\langle \frac{1}{t^4},\ \frac{3}{t^4},\ -\frac{1}{t^4} \right\rangle \bullet \langle 2t,\ 6t,\ -2t \rangle$$

$$= \frac{1}{11\sqrt{11}} \left(\frac{2}{t^3} + \frac{18}{t^3} + \frac{2}{t^3} \right)$$

$$= \frac{2\sqrt{11}}{11} \frac{1}{t^3}$$

$$W = \frac{2\sqrt{11}}{11} \int_1^2 t^{-3}\, dt \qquad\qquad W = \int_C \vec{F} \bullet d\vec{r}$$

$$= -\frac{\sqrt{11}}{11} t^{-2} \Big|_1^2$$

$$= -\frac{\sqrt{11}}{11} \left(\frac{1}{4} - 1 \right)$$

$$\boxed{= \frac{3\sqrt{11}}{44}}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \dfrac{\langle x,\ y \rangle}{\left(x^2 + y^2\right)^{3/2}}$ over the plane curve $\vec{r}\,(t) = \left\langle e^t \cos t,\ \ e^t \sin t \right\rangle$ from the point $(1,\ 0)$ to the point $\left(e^{2\pi},\ 0\right)$ by using the parametrization of the curve to evaluate the work integral

Solution

$$x = e^t \cos t \quad y = e^t \sin t$$

$$(1,\ 0) \ \Rightarrow \ \begin{cases} 1 = e^t \cos t \\ 0 = e^t \sin t & \rightarrow t = 0 \end{cases}$$

$$\left(e^{2\pi},\ 0\right) \ \Rightarrow \ \begin{cases} e^{2\pi} = e^t \cos t & \rightarrow \ t = 2\pi \\ \ 0 = e^t \sin t \end{cases}$$

$$\boxed{0 \le t \le 2\pi}$$

$$\vec{r}\,' = \left\langle e^t\left(\cos t - \sin t\right),\ \ e^t\left(\cos t + \sin t\right)\right\rangle$$

$$\vec{F} = \frac{\left\langle e^t \cos t,\ \ e^t \sin t\right\rangle}{\left(e^{2t}\cos^2 t + e^{2t}\sin^2 t\right)^{3/2}}$$

$$= \frac{\left\langle e^t \cos t,\ \ e^t \sin t\right\rangle}{e^{3t}}$$

$$= \left\langle \frac{\cos t}{e^{2t}},\ \ \frac{\sin t}{e^{2t}}\right\rangle$$

$$\vec{F} \bullet \vec{r}\,' = \left\langle \frac{\cos t}{e^{2t}},\ \ \frac{\sin t}{e^{2t}}\right\rangle \bullet \left\langle e^t\left(\cos t - \sin t\right),\ \ e^t\left(\cos t + \sin t\right)\right\rangle$$

$$= e^{-t}\left(\cos^2 t - \cos t \sin t + \sin^2 t + \cos t \sin t\right)$$

$$\underline{= e^{-t}}$$

$$W = \int_0^{2\pi} e^{-t}\ dt \qquad\qquad W = \int_C \vec{F} \bullet d\vec{r}$$

$$= -e^{-t}\Big|_0^{2\pi}$$

$$\underline{= 1 - e^{-2\pi}}$$

Exercise

Find the work required to move an object on the given oriented curve

$\vec{F} = \dfrac{\langle x,\ y,\ z\rangle}{x^2 + y^2 + z^2}$ on the line segment from $(1,\ 1,\ 1)$ to $(8,\ 4,\ 2)$

Solution

$$\vec{r}(t) = \langle 7t+1,\ 3t+1,\ t+1\rangle \quad 0 \le t \le 1$$

$$\vec{r}\,' = \langle 7,\ 3,\ 1\rangle$$

$$\vec{F} = \frac{\langle 7t+1,\ 3t+1,\ t+1\rangle}{(7t+1)^2 + (3t+1)^2 + (t+1)^2}$$

$$= \frac{\langle 7t+1,\ 3t+1,\ t+1\rangle}{49t^2 + 14t + 1 + 9t^2 + 6t + 1 + t^2 + 2t + 1}$$

$$= \frac{\langle 7t+1,\ 3t+1,\ t+1\rangle}{59t^2 + 22t + 3}$$

$$\vec{F} \bullet \vec{r}' = \frac{\langle 7t+1,\ 3t+1,\ t+1 \rangle}{59t^2+22t+3} \bullet \langle 7,\ 3,\ 1 \rangle$$

$$= \frac{49t+7+9t+3+t+1}{59t^2+22t+3}$$

$$= \frac{59t+11}{59t^2+22t+3}$$

$$W = \int_0^1 \frac{59t+11}{59t^2+22t+3}\, dt \qquad W = \int_C \vec{F} \bullet d\vec{r}$$

$$= \frac{1}{2}\int_0^1 \frac{1}{59t^2+22t+3}\, d\left(59t^2+22t+3\right)$$

$$= \frac{1}{2}\ln\left(59t^2+22t+3\right)\Big|_0^1$$

$$= \frac{1}{2}(\ln 84 - \ln 3)$$

$$= \frac{1}{2}\ln\frac{84}{3}$$

$$= \frac{1}{2}\ln 28$$

$$= \ln\left(2\sqrt{7}\right)$$

Exercise

Let C be the circle of radius 2 centered at the origin with counterclockwise orientation

a) Give the unit outward vector at any point $(x,\ y)$ on C.

b) Find the normal component of the vector field $\vec{F} = 2\langle y,\ -x \rangle$ at any point on C.

c) Find the normal component of the vector field $\vec{F} = \dfrac{\langle x,\ y \rangle}{x^2+y^2}$ at any point on C.

Solution

$r = 2$ @ origin, *ccw*.

a) $\langle x,\ y \rangle$ outward normal

$$|\langle x,\ y \rangle| = \sqrt{x^2+y^2}$$

$$= r$$

$$= 2$$

$\therefore$ unit outward normal: $\frac{1}{2}\langle x,\ y \rangle$

b) Normal component is:

$$\vec{F}\bullet\vec{n} = 2\langle y,\ -x\rangle \bullet \frac{1}{2}\langle x,\ y\rangle$$
$$= xy - xy$$
$$\boxed{= 0}$$

c) Normal component is:

$$\vec{F}\bullet\vec{n} = \frac{\langle x,\ y\rangle}{x^2+y^2} \bullet \frac{1}{2}\langle x,\ y\rangle$$
$$= \frac{1}{2}\frac{x^2+y^2}{x^2+y^2}$$
$$\boxed{= \frac{1}{2}}$$

Exercise

Find the flow of the field $\vec{F} = \nabla\left(x^2 z e^y\right)$

a) Once around the ellipse C in which the plane $x+y+z=1$ intersects the cylinder $x^2+z^2=25$, clockwise as viewed from the positive y-axis.

b) Along the curved boundary of the helicoid $\vec{r}(r,\ \theta) = (r\cos\theta)\hat{i} + (r\sin\theta)\hat{j} + \theta\,\hat{k}$ from $(1,\ 0,\ 0)$ to $(1,\ 0,\ 2\pi)$

Solution

a) For any closed path C.

$$\int_C \vec{F}\bullet d\vec{r}\ \boxed{= 0}$$

$\vec{F}$ is conservative.

b)
$$\int_C \vec{F}\bullet d\vec{r} = \int_{(1,\ 0,\ 0)}^{(1,\ 0,\ 2\pi)} \nabla\left(x^2 z e^y\right) dr$$
$$= \varphi(1,\ 0,\ 2\pi) - \varphi(1,\ 0,\ 0)$$
$$= x^2 z e^y\Big|_{(1,\ 0,\ 2\pi)} - x^2 z e^y\Big|_{(1,\ 0,\ 0)}$$
$$= 2\pi - 0$$
$$\boxed{= 2\pi}$$