Lecture Two – Partial Derivatives

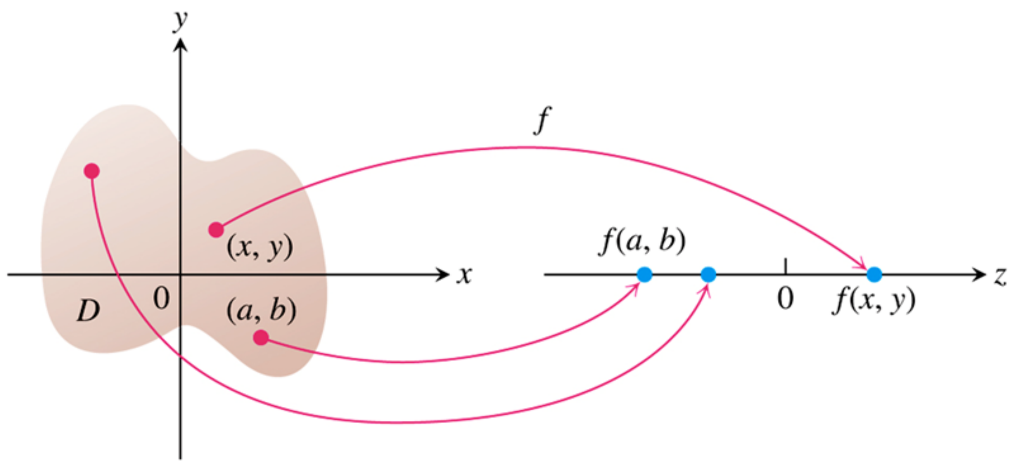
Section 2.1 – Graphs and Level Curves

Definitions

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A ***real-valued function*** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \ldots, x_n)$$

To each element in D. The set D is the function's ***domain***. The set of w-values taken on by f is the function's ***range***. The symbol w is the ***dependent variable*** of f, and f is said to be a function of the n ***independent variables*** x_1 to x_n. We also call the x_j's the function's ***input variables*** and call w the function's ***output variable***.



Domains and Ranges

Functions of two variables

Function	***Domain***	***Range***
$z=\sqrt{y-x^2}$	$y \ge x^2$	$[0, \infty)$
$z=\dfrac{1}{xy}$	$xy \ne 0$	$(-\infty, 0)\cup(0, \infty)$
$z=\sin xy$	*Entire plane*	$[-1, 1]$

Functions of three variables

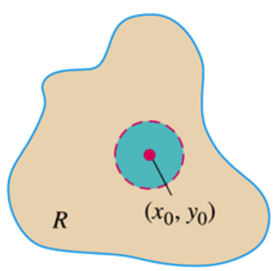
Function	***Domain***	***Range***
$w=\sqrt{x^2+y^2+z^2}$	*Entire plane*	$[0,\ \infty)$
$w=\dfrac{1}{x^2+y^2+z^2}$	$(x,\ y,\ z)\neq(0,\ 0,\ 0)$	$(0,\ \infty)$
$w=xy\ln z$	*Half – space* $z>0$	$(-\infty,\ \infty)$

Functions of Two Variables

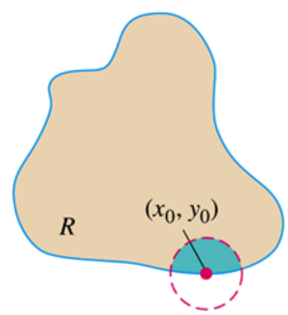
Definitions

A point (x_0, y_0) in a region (set) R in the xy-plane is an ***interior point*** of R if it is the center of a disk of positive radius that lies entirely in R. A point (x_0, y_0) is a ***boundary point*** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

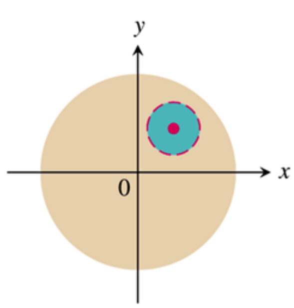
The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its ***boundary***. A region is ***open*** if it consists entirely of interior points. A region is ***closed*** if it contains all its boundary points.



Interior point

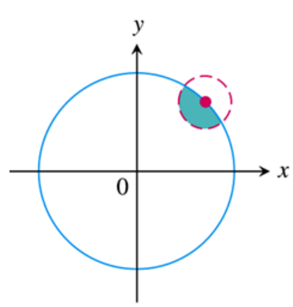


Boundary point



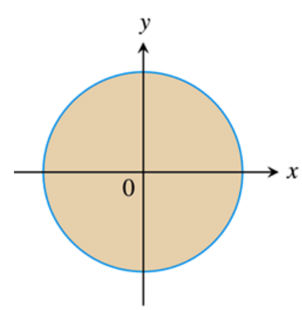
$x^2+y^2<1$

Open unit disk



$x^2+y^2=1$

Boundary of unit disk

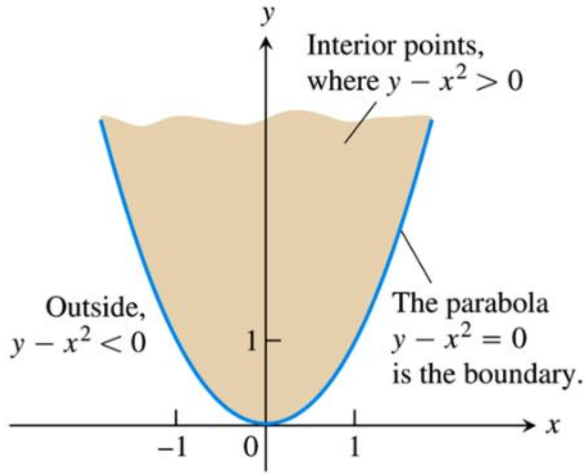


$x^2+y^2\le 1$

Closed unit disk

Definitions

A region in the plane is ***bounded*** if it lies inside a disk of fixed radius. A region is ***unbounded*** if it is not bounded.



Graphs, Level Curves, and contours of Functions of two Variables

Definitions

The set of points in the plane where a function $f(x, y) = c$ is called a ***level curve*** of *f*. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the ***graph*** of *f*. The graph of *f* is also called the ***surface*** $z = f(x, y)$

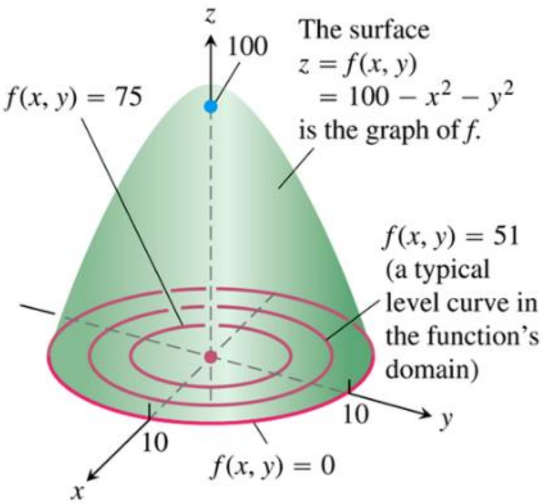
Example

Graph $f(x,y)=100-x^2-y^2$ and plot the level curves $f(x,y)=0,\quad f(x,y)=51,\quad \textit{and}\quad f(x,y)=75$ in the domain of *f* in the plane.

Solution

The domain of *f* is the entire *xy*-plane, and the range of *f* is the set of real numbers less than or equal to 100.

The graph is the paraboloid $z=100-x^2-y^2$, the positive portion of which is shown in the picture.



At $f(x,y)=0 \quad\Rightarrow\quad x^2+y^2=100$

Which is the circle of radius 10 centered at the origin (level curve).

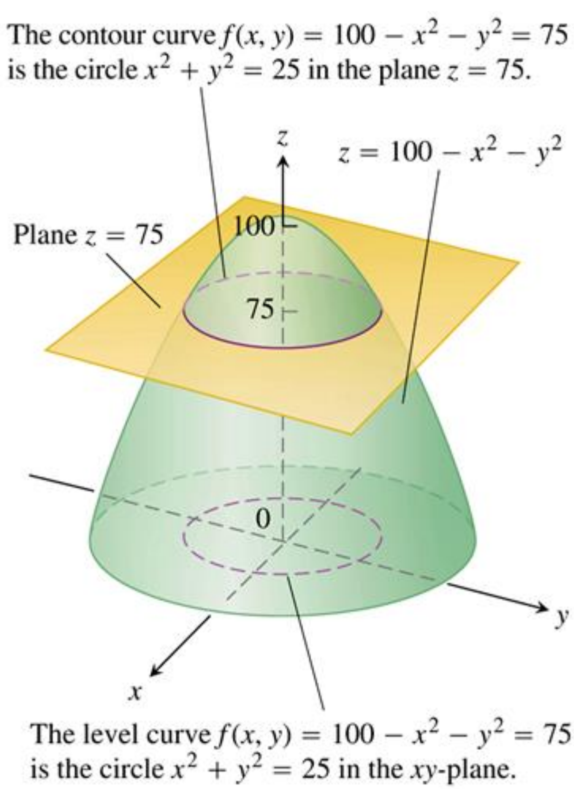
At $f(x,y)=51 \quad\Rightarrow\quad x^2+y^2=49$

Which is the circle of radius 7 centered at the origin.

At $f(x,y)=75 \quad\Rightarrow\quad x^2+y^2=25$

Which is the circle of radius 5 centered at the origin.

If $x^2+y^2>100$, then the values of $f(x,y)$ are negative.



Functions of Three Variables

Definition

The set of points $(x,\ y,\ z)$ in space where a function of three independent variables has a constant value $f(x,\ y,\ z) = c$ is called a ***level surface*** of *f*.

Example

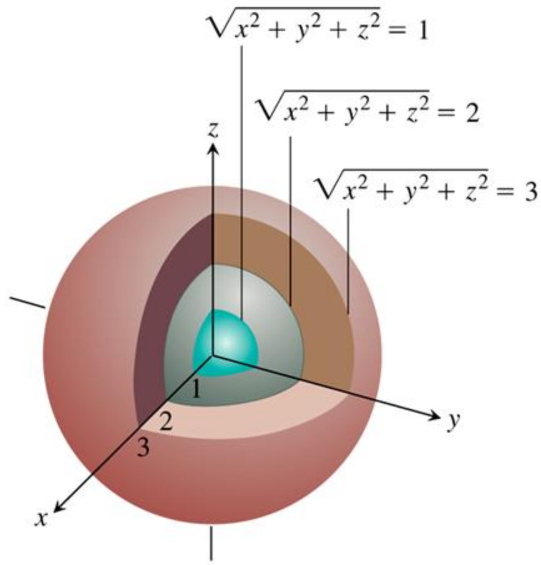
Describe the level surfaces of the function

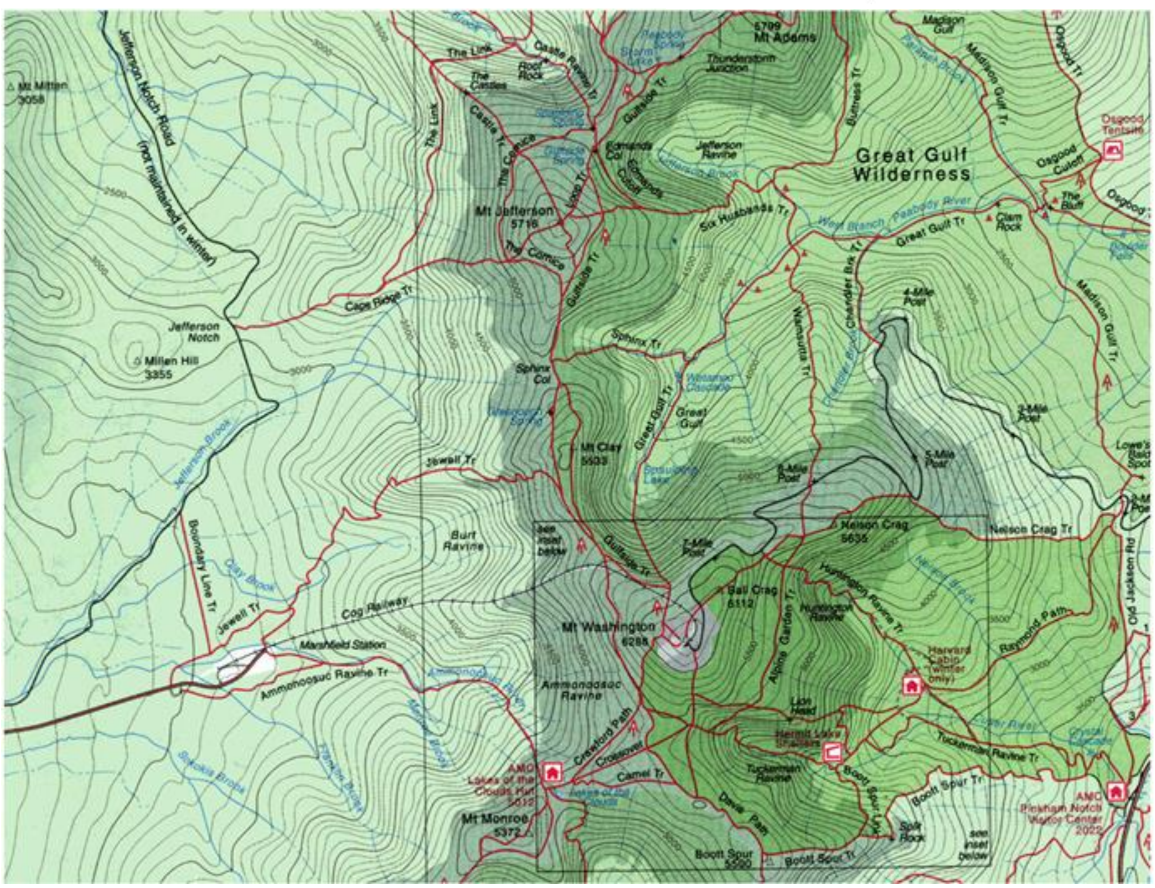
$$f(x,\ y,\ z) = \sqrt{x^2 + y^2 + z^2}$$

Solution

The value of *f* is the distance from the origin to the point $(x,\ y,\ z)$.

Each surface $\sqrt{x^2 + y^2 + z^2} = c\ \ (>0)$, is a sphere of radius c centered at the origin.

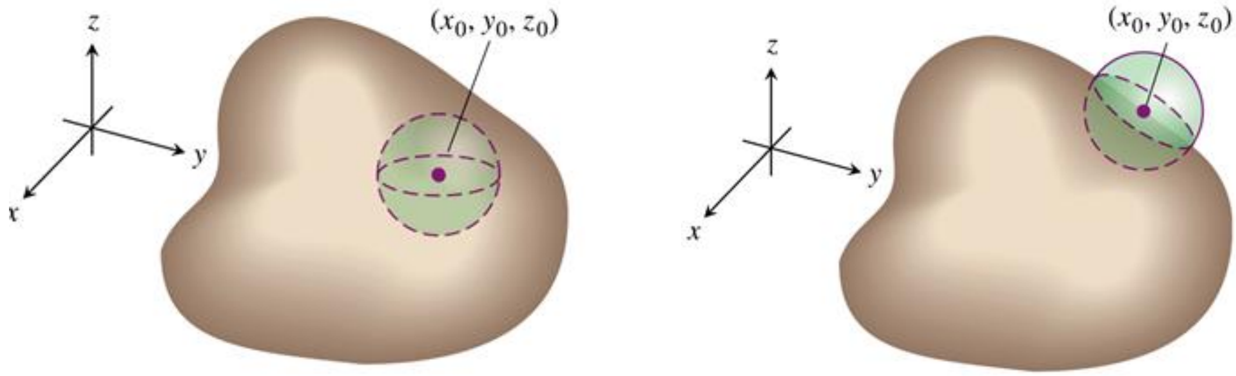




Definitions

A point $\left(x_0, y_0, z_0\right)$ in a region ***R*** in space is an ***interior point*** of ***R*** if it is the center of a solid ball that lies entirely in ***R***. A point $\left(x_0, y_0, z_0\right)$ is a ***boundary point*** of ***R*** if every solid ball centered at $\left(x_0, y_0, z_0\right)$ contains points that lie outside of ***R*** as well as that lie inside ***R***. The ***interior*** of R is the set of interior points of ***R***. The ***boundary*** of ***R*** is the set of boundary points of ***R***.

A region is ***open*** if it consists entirely of interior points. A region is ***closed*** if it contains its entire boundary.

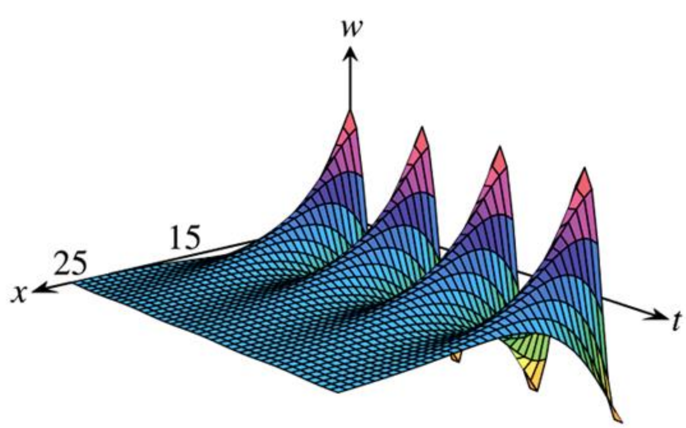


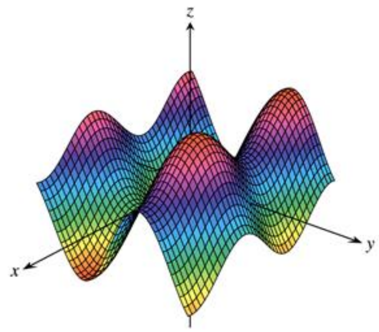
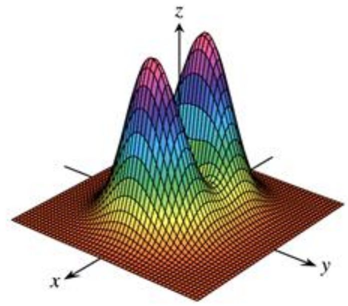
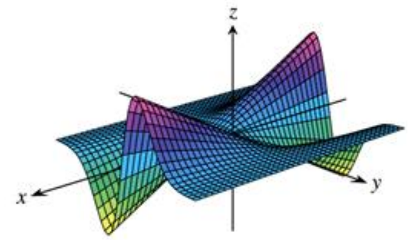
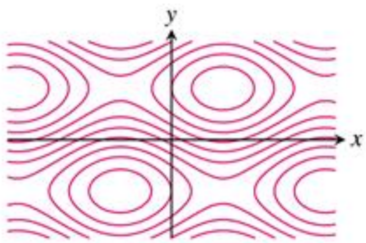
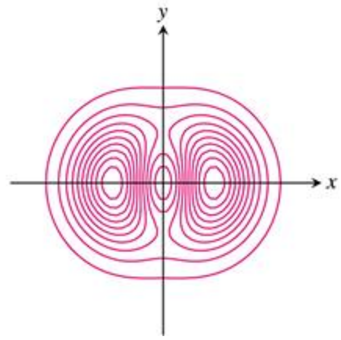
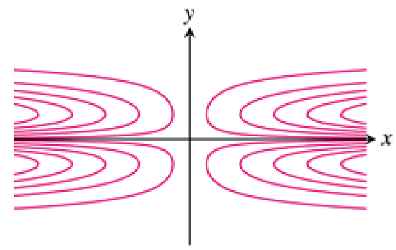
Example

The temperature *w* beneath the Earth's surface is a function of the depth *x* beneath the surface and the time *t* of the year. If we measure *x* in feet and *t* as the number of days elapsed from the expected date of the yearly highest surface temperature, we can model the variation in temperature with the function

$$w = \cos\left(1.7 \times 10^{-2} t - 0.2x\right) e^{-0.2x}$$

The temperature at 9 *ft* is scaled to vary from +1 to −1, so that the variation at *x ft*. can be interpreted as a fraction of the variation at the surface.



		
		
$z = \sin x + 2\sin y$	$z = \left(4x^2 + y^2\right)e^{-x^2-y^2}$	$z = xye^{-y^2}$

Exercises *Section* 2.1 – Graphs and Level Curves

1. Find the specific values for $f(x,y,z)=\dfrac{x-y}{y^2+z^2}$

a) $f(3,-1,2)$ *b)* $f\left(1,\ \frac{1}{2},\ -\frac{1}{4}\right)$ *c)* $f\left(0,\ -\frac{1}{3},\ 0\right)$ *d)* $f(2,\ 2,\ 100)$

2. Find the specific values for $f(x,y,z)=\sqrt{49-x^2-y^2-z^2}$

a) $f(0,\ 0,\ 0)$ *b)* $f(2,\ -3,\ 6)$ *c)* $f(-1,\ 2,\ 3)$ *d)* $f\left(\frac{4}{\sqrt{2}},\ \frac{5}{\sqrt{2}},\ \frac{6}{\sqrt{2}}\right)$

Find and sketch the domain for each function

3. $f(x,y)=\sqrt{y-x-2}$

4. $f(x,y)=\ln\left(x^2+y^2-4\right)$

5. $f(x,y)=\dfrac{\sin(xy)}{x^2+y^2-25}$

6. $f(x,y)=\ln(xy+x-y-1)$

7. $f(x,y)=\sqrt{\left(x^2-4\right)\left(y^2-9\right)}$

8. $f(x,\ y)=\dfrac{1}{x^2+y^2}$

9. $f(x,\ y)=\ln xy$

10. $f(x,\ y)=\sqrt{x-y^2}$

11. $f(x,\ y)=\tan(x+y)$

Find and sketch the level curves $f(x,y)=c$ on the same set of coordinate axes for the given values of *c*, we refer to these level curves as a contour map.

12. $f(x,y)=x+y-1,\quad c=-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3$

13. $f(x,y)=x^2+y^2,\quad c=0,\ 1,\ 4,\ 9,\ 16,\ 25$

14. For the function: $f(x,y)=4x^2+9y^2$:

a) Find the function's domain
b) Find the function's range
c) Find the function's level curves
d) Find the boundary of the function's domain
e) Determine if the domain is an open region, a closed region, or neither
f) Decide if the domain is bounded or unbounded

15. For the function: $f(x,y)=xy$:

a) Find the function's domain
b) Find the function's range
c) Find the function's level curves

d) Find the boundary of the function's domain
e) Determine if the domain is an open region, a closed region, or neither
f) Decide if the domain is bounded or unbounded

16. For the function: $f(x,y) = e^{-\left(x^2+y^2\right)}$:

a) Find the function's domain
b) Find the function's range
c) Find the function's level curves
d) Find the boundary of the function's domain
e) Determine if the domain is an open region, a closed region, or neither
f) Decide if the domain is bounded or unbounded

17. For the function: $f(x,y) = \ln\left(9 - x^2 - y^2\right)$:

a) Find the function's domain
b) Find the function's range
c) Find the function's level curves
d) Find the boundary of the function's domain
e) Determine if the domain is an open region, a closed region, or neither
f) Decide if the domain is bounded or unbounded

18. Find an equation for $f(x,y) = 16 - x^2 - y^2$ and sketch the graph of the level curve of the function $f(x,y)$ that passes through the point $\left(2\sqrt{2},\ \sqrt{2}\right)$

19. Find an equation for $f(x,y) = \dfrac{2y - x}{x + y + 1}$ and sketch the graph of the level curve of the function $f(x,y)$ that passes through the point $(-1,\ 1)$

Sketch a typical level surface for the function

20. $f(x,y,z) = x^2 + y^2 + z^2$

21. $f(x,y,z) = \ln\left(x^2 + y^2 + z^2\right)$

22. $f(x,y,z) = y^2 + z^2$

23. $f(x,y,z) = z - x^2 - y^2$