Solution Section 2.6 – Properties of Division

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=2x^4-x^3+7x-12;\quad p(x)=x^2-3$

Solution

$$
\begin{array}{r|l}
 & 2x^2-x+6 \\
\hline
x^2-3 & 2x^4-x^3+0x^2+7x-12 \\
 & \underline{2x^4\qquad -6x^2} \\
 & \qquad -x^3+6x^2+7x \\
 & \qquad \underline{-x^3\qquad +3x} \\
 & \qquad\qquad 6x^2+4x-12 \\
 & \qquad\qquad \underline{6x^2\qquad -18} \\
 & \qquad\qquad\qquad 4x+6
\end{array}
$$

$$Q(x)=2x^2-x+6;\quad R(x)=4x+6$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=3x^3+2x-4;\quad p(x)=2x^2+1$

Solution

$$
\begin{array}{r|l}
 & \frac{3}{2}x \\
\hline
2x^2+1 & 3x^3+0x^2+2x-4 \\
 & \underline{3x^3\qquad +\frac{3}{2}x} \\
 & \qquad\qquad \frac{1}{2}x-4
\end{array}
$$

$$Q(x)=\frac{3}{2}x;\quad R(x)=\frac{1}{2}x-4$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=7x+2;\quad p(x)=2x^2-x-4$

Solution

$$
\begin{array}{r|l}
 & \frac{2}{7}x-\frac{11}{49} \\
\hline
7x+2 & 2x^2-x-4 \\
 & 2x^2+\frac{4}{7}x \\
\hline
 & -\frac{11}{7}x-4 \\
 & -\frac{11}{7}x-\frac{22}{49} \\
\hline
 & -\frac{174}{49}
\end{array}
$$

$$Q(x)=\frac{2}{7}x-\frac{11}{49} \qquad R(x)=-\frac{174}{49}$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=9x+4$; $p(x)=2x-5$

Solution

$$
\begin{array}{r|l}
 & \frac{9}{2} \\
\hline
2x-5 & 9x+4 \\
 & 9x-\frac{45}{2} \\
\hline
 & -\frac{37}{2}
\end{array}
$$

$$Q(x)=\frac{9}{2}; \quad R(x)=-\frac{37}{2}$$

Exercise

Use the remainder theorem to find $f(c)$: $f(x)=x^4-6x^2+4x-8$; $c=-3$

Solution

$$f(-3)=(-3)^4-6(-3)^2+4(-3)-8$$

$$=7$$

Exercise

Use the remainder theorem to find $f(c)$: $f(x)=x^4+3x^2-12$; $c=-2$

Solution

$$f(-2)=(-2)^4+3(-2)^2-12$$

$$=16$$

Exercise

Use the factor theorem to show that $x - c$ is a factor of $f(x)$: $f(x) = x^3 + x^2 - 2x + 12$; $c = -3$

Solution

$$f(-3) = (-3)^3 + (-3)^2 - 2(-3) + 12$$
$$= 0$$

From the factor theorem; $x + 3$ is a factor of $f(x)$.

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $2x^3 - 3x^2 + 4x - 5$; $x - 2$

Solution

$$\begin{array}{r|rrrr} 2 & 2 & -3 & 4 & -5 \\ & & 4 & 2 & 12 \\ \hline & 2 & 1 & 6 & \boxed{7} \end{array}$$

$Q(x) = 2x^2 + x + 6 \qquad R(x) = 7$

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $5x^3 - 6x^2 + 15$; $x - 4$

Solution

$$\begin{array}{r|rrrr} 4 & 5 & -6 & 0 & 15 \\ & & 20 & 56 & 224 \\ \hline & 5 & 14 & 56 & \boxed{239} \end{array}$$

$Q(x) = 5x^2 + 14x + 56 \qquad R(x) = 239$

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second: $9x^3 - 6x^2 + 3x - 4$; $x - \frac{1}{3}$

Solution

$$\begin{array}{r|rrrr}
\frac{1}{3} & 9 & -6 & 3 & -4 \\
 & & 3 & -1 & \frac{2}{3} \\
\hline
 & 9 & -3 & 2 & \boxed{-\frac{10}{3}}
\end{array}$$

$$Q(x) = 9x^2 - 3x + 2 \qquad R(x) = -\frac{10}{3}$$

Exercise

Use the synthetic division to find $f(c)$: $f(x) = 2x^3 + 3x^2 - 4x + 4; \quad c = 3$

Solution

$$\begin{array}{r|rrrr}
3 & 2 & 3 & -4 & 4 \\
 & & 9 & 36 & 93 \\
\hline
 & 3 & 12 & 32 & \boxed{97}
\end{array}$$

$$f(3) = 97$$

Exercise

Use the synthetic division to find $f(c)$: $f(x) = 8x^5 - 3x^2 + 7; \quad c = \frac{1}{2}$

Solution

$$\begin{array}{r|rrrrrr}
\frac{1}{2} & 8 & 0 & 0 & -3 & 0 & 7 \\
 & & 4 & 2 & 1 & -1 & -\frac{1}{2} \\
\hline
 & 8 & 4 & 2 & -2 & -1 & \boxed{\frac{13}{2}}
\end{array}$$

$$f\left(\frac{1}{2}\right) = \frac{13}{2}$$

Exercise

Use the synthetic division to find $f(c)$: $f(x) = x^3 - 3x^2 - 8; \quad c = 1 + \sqrt{2}$

Solution

$$\begin{array}{r|rrrr}
1+\sqrt{2} & 3 & -3 & 0 & -8 \\
 & & 3+3\sqrt{2} & 6+3\sqrt{2} & 12+9\sqrt{2} \\
\hline
 & 3 & 3\sqrt{2} & 6+3\sqrt{2} & \boxed{4+9\sqrt{2}}
\end{array}$$

$$f\left(1+\sqrt{2}\right) = 4 + 9\sqrt{2}$$

Exercise

Use the synthetic division to show that c is a zero of $f(x)$: $f(x)=3x^4+8x^3-2x^2-10x+4;\quad c=-2$

Solution

$$\begin{array}{r|rrrrr} -2 & 3 & 8 & -2 & -10 & 4 \\ & & -6 & -4 & 12 & -4 \\ \hline & 3 & 2 & -6 & 2 & \boxed{0} \end{array}$$

$$\boxed{f(-2)=0}$$

Exercise

Use the synthetic division to show that c is a zero of $f(x)$: $f(x)=27x^4-9x^3+3x^2+6x+1;\quad c=-\frac{1}{3}$

Solution

$$\begin{array}{r|rrrrr} -\frac{1}{3} & 27 & -9 & 3 & 6 & 1 \\ & & -9 & 6 & -3 & -1 \\ \hline & 27 & -18 & 9 & 3 & \boxed{0} \end{array}$$

$$\boxed{f\left(-\frac{1}{3}\right)=0}$$

Exercise

Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

$f(x)=kx^3+x^2+k^2x+3k^2+11;\quad x+2$

Solution

$$\begin{array}{r|cccc} -2 & k & 1 & k^2 & 3k^2+11 \\ & & -2k & 4k-2 & -2k^2-8k+4 \\ \hline & k & 1-2k & k^2+4k-2 & k^2-8k+15 \end{array}$$

$$k^2-8k+15=0 \Rightarrow \boxed{k=3,\ 5}$$

Exercise

Find all solutions of the equation: $\quad x^3-x^2-10x-8=0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{8}{1}\right\} = \pm\{1,\ 2,\ 4,\ 8\}$

$$\begin{array}{r|rrrr} -1 & 1 & -1 & -10 & -8 \\ & & -1 & 2 & 8 \\ \hline & 1 & -2 & -8 & 0 \end{array} \rightarrow \quad x^2 - 2x - 8 = 0$$

The solutions are: $\underline{x = -1,\ -2,\ 4}|$

Exercise

Find all solutions of the equation: $x^3 + x^2 - 14x - 24 = 0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{24}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24\}$

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -14 & -24 \\ & & -2 & 2 & 24 \\ \hline & 1 & -1 & -12 & 0 \end{array} \rightarrow \quad x^2 - x - 12 = 0$$

The solutions are: $\underline{x = -2,\ -3,\ 4}|$

Exercise

Find all solutions of the equation: $2x^3 - 3x^2 - 17x + 30 = 0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{30}{2}\right\} = \pm\left\{1,\ 2,\ 3,\ 5,\ 6,\ 10,\ 15,\ 30,\ \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \frac{15}{2}\right\}$

$$\begin{array}{r|rrrr} 2 & 2 & -3 & -17 & 30 \\ & & 4 & 2 & -30 \\ \hline & 2 & 1 & -15 & \boxed{0} \end{array} \rightarrow \quad 2x^2 + x - 15 = 0$$

The solutions are: $\underline{x = 2,\ -3,\ \frac{5}{2}}|$

Exercise

Find all solutions of the equation: $12x^3 + 8x^2 - 3x - 2 = 0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{2}{12}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{2},\ \frac{1}{6},\ \frac{1}{12}\right\}$

$$\begin{array}{r|rrrr} \frac{1}{2} & 12 & 8 & -3 & -2 \\ & & 6 & 7 & 2 \\ \hline & 12 & 14 & 4 & \boxed{0} \end{array} \rightarrow \ 12x^2+14x+4=0$$

The solutions are: $\boxed{x=\frac{1}{2},\ -\frac{1}{2},\ -\frac{2}{3}}$

Exercise

Find all solutions of the equation: $x^3+x^2-6x-8=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{8}{1}\right\}=\pm\{1,\ 2,\ 4,\ 8\}$

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -6 & -8 \\ & & -2 & 2 & 8 \\ \hline & 1 & -1 & -4 & \boxed{0} \end{array} \rightarrow \ x^2-x-4=0$$

$$x=\frac{1\pm\sqrt{1+16}}{2}$$

The solutions are: $\boxed{x=-2,\ \frac{1\pm\sqrt{17}}{2}}$

Exercise

Find all solutions of the equation: $x^3-19x-30=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{30}{1}\right\}=\pm\{1,\ 2,\ 3,\ 5,\ 6,\ 15,\ 30\}$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -19 & -30 \\ & & -2 & 4 & 30 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array} \rightarrow \ x^2-2x-15$$

$$\begin{aligned} x &= \frac{2\pm\sqrt{4+60}}{2} \\ &= \begin{cases} \frac{2-8}{2}=-3 \\ \frac{2+8}{2}=5 \end{cases} \end{aligned}$$

The solutions are: $\boxed{x=-2,\ -3,\ 5}$

Exercise

Find all solutions of the equation: $2x^3 + x^2 - 25x + 12 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{12}{2}\right\} = \pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 12,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$$\begin{array}{r|rrrr} 3 & 2 & 1 & -25 & 12 \\ & & 6 & 21 & -12 \\ \hline & 2 & 7 & -4 & \boxed{0} \end{array} \quad \rightarrow \quad 2x^2 + 7x - 4$$

$$x = \frac{-7 \pm \sqrt{49+32}}{4}$$

$$= \begin{cases} \frac{-7-9}{4} = -4 \\ \frac{-7+9}{4} = \frac{1}{2} \end{cases}$$

The solutions are: $x = -4,\ \frac{1}{2},\ 3$

Exercise

Find all solutions of the equation: $3x^3 + 11x^2 - 6x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{8}{3}\right\} = \pm\left\{1,\ 2,\ 4,\ 8,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{8}{3}\right\}$

$$\begin{array}{r|rrrr} 1 & 3 & 11 & -6 & -8 \\ & & 3 & 14 & 8 \\ \hline & 3 & 14 & 8 & \boxed{0} \end{array} \quad \rightarrow \quad 3x^2 + 14x + 8$$

$$x = \frac{-14 \pm \sqrt{196-96}}{6}$$

$$= \begin{cases} \frac{-14-10}{6} = -4 \\ \frac{-14+10}{6} = -\frac{2}{3} \end{cases}$$

The solutions are: $x = -4,\ -\frac{2}{3},\ 1$

Exercise

Find all solutions of the equation: $2x^3 + 9x^2 - 2x - 9 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{9}{2}\right\} = \pm\left\{1,\ 3,\ 9,\ \frac{1}{2},\ \frac{3}{2},\ \frac{9}{2}\right\}$

$$\begin{array}{r|rrrr} 1 & 2 & 9 & -2 & -9 \\ & & 2 & 11 & 9 \\ \hline & 2 & 11 & 9 & \boxed{0} \end{array} \quad \to \quad 2x^2+11x+9$$

$x=-1,\ -\frac{9}{2}$ $\qquad a-b+c=0 \ \to \ x=-1,\ -\frac{c}{a}$

The solutions are: $x=-\frac{9}{2},\ -1,\ 1$

Exercise

Find all solutions of the equation: $\quad x^3+3x^2-6x-8=0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{8}{1}\right\} = \pm\{1,\ 2,\ 4,\ 8\}$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & -6 & -8 \\ & & -1 & -2 & 8 \\ \hline & 1 & 2 & -8 & \boxed{0} \end{array} \to \quad x^2+2x-8=0$$

$$x=\frac{-2\pm\sqrt{4+32}}{2}$$

$$=\begin{cases} \frac{-2-6}{2}=-4 \\ \frac{-2+6}{2}=2 \end{cases}$$

The solutions are: $x=-4,\ -1,\ 2$

Exercise

Find all solutions of the equation: $\quad 3x^3-x^2-6x+2=0$

Solution

possibilities for $\frac{c}{d}: \pm\left\{\frac{2}{3}\right\} = \pm\left\{1,\ 2,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -1 & -6 & 2 \\ & & 1 & 0 & -2 \\ \hline & 3 & 0 & -6 & \boxed{0} \end{array} \quad \to \quad 3x^2-6=0$$

$x^2=2$

The solutions are: $x=\frac{1}{3},\ \pm\sqrt{2}$

Exercise

Find all solutions of the equation: $x^3 - 8x^2 + 8x + 24 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{24}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24\}$

$$\begin{array}{r|rrrr} 6 & 1 & -8 & 8 & 24 \\ & & 6 & -12 & -24 \\ \hline & 1 & -2 & -4 & \boxed{0} \end{array} \quad \rightarrow \quad x^2 - 2x - 4 = 0$$

$$x = \frac{2 \pm \sqrt{4+16}}{2}$$

$$= \frac{2 \pm 2\sqrt{5}}{2}$$

The solutions are: $\underline{x = 6,\ 1 \pm \sqrt{5}}$

Exercise

Find all solutions of the equation: $x^3 - 7x^2 - 7x + 69 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{69}{1}\right\} = \pm\{1,\ 3,\ 23,\ 69\}$

$$\begin{array}{r|rrrr} -3 & 1 & -7 & -7 & 69 \\ & & -3 & 30 & -69 \\ \hline & 1 & -10 & 23 & \boxed{0} \end{array} \quad \rightarrow \quad x^2 - 10x + 23 = 0$$

$$x = \frac{10 \pm \sqrt{100-92}}{2}$$

$$= \frac{10 \pm 2\sqrt{2}}{2}$$

The solutions are: $\underline{x = -3,\ 5 \pm \sqrt{2}}$

Exercise

Find all solutions of the equation: $x^3 - 3x - 2 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array} \quad \to \quad x^2 - x - 2 = 0$$

$\underline{x = -1,\ 2|}$ $\qquad a - b + c = 0 \quad \to \quad x = -1,\ -\frac{c}{a}$

The solutions are: $\underline{x = -1,\ \ -1,\ 2\ |}$

Exercise

Find all solutions of the equation: $\quad x^3 - 2x + 1 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\{1\}$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -2 & 1 \\ & & 1 & 1 & 1 \\ \hline & 1 & 1 & -1 & \boxed{0} \end{array} \quad \to \quad x^2 + x - 1 = 0$$

$$x = \frac{-1 \pm \sqrt{5}}{2}$$

The solutions are: $x = 1,\ \dfrac{-1 \pm \sqrt{5}}{2}$

Exercise

Find all solutions of the equation: $\quad x^3 - 2x^2 - 11x + 12 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{12}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -11 & 12 \\ & & 1 & -1 & 12 \\ \hline & 1 & -1 & -12 & \boxed{0} \end{array} \quad \to \quad x^2 - x - 12 = 0$$

$$\begin{aligned} x &= \frac{1 \pm \sqrt{1 + 48}}{2} \\ &= \begin{cases} \dfrac{1-7}{2} = -3 \\ \dfrac{1+7}{2} = 4 \end{cases} \end{aligned}$$

The solutions are: $\underline{x = -3,\ 1,\ 4\ |}$

Exercise

Find all solutions of the equation: $x^3 - 2x^2 - 7x - 4 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{4}{1}\right\} = \pm\{1,\ 2,\ 4\}$

$$\begin{array}{r|rrrr} -1 & 1 & -2 & -7 & -4 \\ & & -1 & 3 & 4 \\ \hline & 1 & -3 & -4 & \boxed{0} \end{array} \quad \to \quad x^2 - 3x - 4 = 0$$

$x = -1,\ 4$ $\qquad a - b + c = 0 \quad \to \quad x = -1,\ -\frac{c}{a}$

The solutions are: $x = -1,\ -1,\ 4$

Exercise

Find all solutions of the equation: $x^3 - 10x - 12 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{12}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -10 & -12 \\ & & -2 & 4 & 12 \\ \hline & 1 & -2 & -6 & \boxed{0} \end{array} \quad \to \quad x^2 - 2x - 6 = 0$$

$$x = \frac{2 \pm \sqrt{4 + 24}}{2}$$

$$= \frac{2 \pm 2\sqrt{7}}{2}$$

The solutions are: $x = -2,\ 1 \pm \sqrt{7}$

Exercise

Find all solutions of the equation: $x^3 - 5x^2 + 17x - 13 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{13}{1}\right\} = \pm\{1,\ 13\}$

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 17 & -13 \\ & & 1 & -4 & 13 \\ \hline & 1 & -4 & 13 & \boxed{0} \end{array} \quad \to \quad x^2 - 4x + 13 = 0$$

$$x = \frac{4 \pm \sqrt{16-52}}{2}$$
$$= \frac{4 \pm 6i}{2}$$

The solutions are: $\boxed{x = 1,\ 2 \pm 3i}$

Exercise

Find all solutions of the equation: $6x^3 + 25x^2 - 24x + 5 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{5}{6}\right\} = \pm\left\{1,\ 5,\ \frac{1}{2},\ \frac{5}{2},\ \frac{1}{3},\ \frac{5}{3},\ \frac{1}{6},\ \frac{5}{6}\right\}$

$$\begin{array}{r|rrrr} -5 & 6 & 25 & -24 & 5 \\ & & -30 & 25 & -5 \\ \hline & 6 & -5 & 1 & \boxed{0} \end{array} \quad \rightarrow \quad 6x^2 - 5x + 1 = 0$$

$$x = \frac{5 \pm \sqrt{25-24}}{12}$$
$$= \begin{cases} \frac{5-1}{12} = \frac{1}{3} \\ \frac{5+1}{12} = \frac{1}{2} \end{cases}$$

The solutions are: $\boxed{x = -5,\ \frac{1}{3},\ \frac{1}{2}}$

Exercise

Find all solutions of the equation: $8x^3 + 18x^2 + 45x + 27 = 0$

Solution

possibilities : $\pm\left\{\frac{27}{8}\right\} = \pm\left\{\frac{1,\ 3,\ 9,\ 27}{1,\ 2,\ 4,\ 8}\right\}$

$= \pm\left\{1,\ 3,\ 9,\ 27,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{3}{2},\ \frac{3}{4},\ \frac{3}{8},\ \frac{9}{2},\ \frac{9}{4},\ \frac{9}{8},\ \frac{27}{2},\ \frac{27}{4},\ \frac{27}{8}\right\}$

$$\begin{array}{r|rrrr} -\frac{3}{4} & 8 & 18 & 45 & 27 \\ & & -6 & -9 & -27 \\ \hline & 8 & 12 & 36 & \boxed{0} \end{array} \quad \rightarrow \quad 8x^2 + 12x + 36 = 0$$

The solutions are: $\boxed{x = -\frac{3}{4},\ -\frac{3}{4} \pm i\frac{3\sqrt{7}}{4}}$

Exercise

Find all solutions of the equation: $3x^3 - x^2 + 11x - 20 = 0$

Solution

$$possibilities: \ \pm\left\{\frac{20}{3}\right\} = \pm\left\{\frac{1,\ 2,\ 4,\ 5,\ 10,\ 20}{1,\ 3}\right\}$$

$$= \pm\left\{1,\ 2,\ 4,\ 5,\ 10,\ 20,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{5}{3},\ \frac{10}{3},\ \frac{20}{3}\right\}$$

$$\begin{array}{c|cccc} \frac{4}{3} & 3 & -1 & 11 & -20 \\ & & 4 & 4 & 20 \\ \hline & 3 & 3 & 15 & \boxed{0} \end{array} \quad \to 3x^2 + 3x + 15 = 0$$

$$x = \frac{-3 \pm \sqrt{9 - 180}}{6}$$

$$= \frac{-3 \pm 3\sqrt{19}}{6}$$

The solutions are: $x = \frac{4}{3},\ -\frac{1}{2} \pm i\frac{\sqrt{19}}{2}$

Exercise

Find all solutions of the equation: $x^4 - x^3 - 9x^2 + 3x + 18 = 0$

Solution

$$possibilities\ for\ \frac{c}{d}: \pm\left\{\frac{18}{1}\right\} = \pm\{1,\ 2,\ 3,\ 6,\ 9,\ 18\}$$

$$\begin{array}{c|ccccc} -2 & 1 & -1 & -9 & 3 & 18 \\ & & -2 & 6 & 6 & -18 \\ \hline 3 & 1 & -3 & -3 & 9 & 0 \\ & & 3 & 0 & -9 & \\ \hline & 1 & 0 & -3 & 0 & \end{array}$$

$\to x^3 - 3x^2 - 3x + 9 = 0 \ \to \ \pm\left\{\frac{9}{1}\right\} = \pm\{1,\ 3,\ 9\}$

$\to x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3}$

The solutions are: $x = -2,\ 3,\ \pm\sqrt{3}$

Exercise

Find all solutions of the equation: $2x^4 - 9x^3 + 9x^2 + x - 3 = 0$

Solution

$$possibilities\ for\ \frac{c}{d}: \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$$

$$\begin{array}{r|rrrrr}
1 & 2 & -9 & 9 & 1 & -3 \\
 & & 2 & -7 & 2 & 3 \\ \hline
1 & 2 & -7 & 2 & 3 & 0 \\
 & & 2 & -5 & -3 & \\ \hline
 & 2 & -5 & -3 & 0 &
\end{array}$$

$\rightarrow\ 2x^3 - 7x^2 + 2x + 3 = 0 \ \rightarrow\ \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$\rightarrow\ 2x^2 - 5x - 3 = 0$

$$x = \frac{5 \pm \sqrt{25+24}}{4}$$

$$= \begin{cases} \frac{5-7}{4} = -\frac{1}{2} \\ \frac{5+7}{4} = 3 \end{cases}$$

The solutions are: $\boxed{x = 1,\ 1,\ -\frac{1}{2},\ 3}$

Exercise

Find all solutions of the equation: $6x^4 + 5x^3 - 17x^2 - 6x = 0$

Solution

$x\left(6x^3 + 5x^2 - 17x - 6\right) = 0 \ \rightarrow\ \underline{x = 0}$

possibilities : $\pm\left\{\frac{6}{6}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrr}
-2 & 6 & 5 & -17 & -6 \\
 & & -12 & 14 & 6 \\ \hline
 & 6 & -7 & -3 & \boxed{0}
\end{array}$$

$\rightarrow\ 6x^2 - 7x - 3 = 0$

$$x = \frac{7 \pm \sqrt{49+72}}{12}$$

$$= \begin{cases} \frac{7-11}{12} = -\frac{1}{3} \\ \frac{7+11}{12} = \frac{3}{2} \end{cases}$$

The solutions are: $\boxed{x = 0,\ -2,\ -\frac{1}{3},\ \frac{3}{2}}$

Exercise

Find all solutions of the equation: $x^4-2x^2-16x-15=0$

Solution

$$possibilities: \ \pm\left\{\frac{15}{1}\right\}=\pm\{1,\ 3,\ 5,\ 15\}$$

$$\begin{array}{r|rrrrr}
-1 & 1 & 0 & -2 & -16 & -15 \\
 & & -1 & 1 & 1 & 15 \\
\hline
3 & 1 & -1 & -1 & -15 & 0 \\
 & & 3 & 6 & 15 & \\
\hline
 & 1 & 2 & 5 & 0 &
\end{array}$$

$\to x^3-x^2-x-15=0 \to \pm\left\{\frac{15}{1}\right\}=\pm\{1,\ 3,\ 5,\ 15\}$

$\to x^2+2x+5=0$

$$x=\frac{-2\pm\sqrt{-16}}{2}$$

$$=-1\pm 2i$$

The solutions are: $\boxed{x=-1,\ 3,\ \ -1\pm 2i}$

Exercise

Find all solutions of the equation: $x^4-2x^3-5x^2+8x+4=0$

Solution

$$possibilities: \ \pm\left\{\frac{4}{1}\right\}=\pm\{1,\ 2,\ 4\}$$

$$\begin{array}{r|rrrrr}
2 & 1 & -2 & -5 & 8 & 4 \\
 & & 2 & 0 & -10 & -4 \\
\hline
-2 & 1 & 0 & -5 & -2 & 0 \\
 & & -2 & 4 & 2 & \\
\hline
 & 1 & -2 & -1 & 0 &
\end{array}$$

$\to x^3-5x-2=0 \to \pm\left\{\frac{2}{1}\right\}=\pm\{1,\ 2\}$

$\to x^2-2x-1=0$

$$x=\frac{2\pm\sqrt{8}}{2}$$

$$=\frac{2\pm 2\sqrt{2}}{2}$$

The solutions are: $\boxed{x=-2,\ 2,\ 1\pm\sqrt{2}}$

Exercise

Find all solutions of the equation: $2x^4-17x^3+4x^2+35x-24=0$

Solution

$possibilities: \ \pm\left\{\frac{24}{2}\right\} = \pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$$\begin{array}{r|rrrrr}
1 & 2 & -17 & 4 & 35 & -24 \\
 & & 2 & -15 & -11 & 24 \\
\hline
1 & 2 & -15 & -11 & 24 & 0 \\
 & & 2 & -13 & 24 & \\
\hline
 & 2 & -13 & -24 & 0 &
\end{array}$$

$\rightarrow\ 2x^3 - 15x^2 - 11x + 24 = 0\ \rightarrow\ \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$\rightarrow\ 2x^2 - 13x - 24 = 0$

$$x = \frac{13 \pm \sqrt{169+192}}{4}$$

$$= \begin{cases} \frac{13-19}{4} = -\frac{3}{2} \\ \frac{13+19}{4} = 8 \end{cases}$$

The solutions are: $\underline{x = -\frac{3}{2},\ 1,\ 1,\ 8}$

Exercise

Find all solutions of the equation: $x^4 + x^3 - 3x^2 - 5x - 2 = 0$

Solution

$possibilities: \ \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$$\begin{array}{r|rrrrr}
-1 & 1 & 1 & -3 & -5 & -2 \\
 & & -1 & 0 & 3 & 2 \\
\hline
-1 & 1 & 0 & -3 & -2 & 0 \\
 & & -1 & 1 & 2 & \\
\hline
 & 1 & -1 & -2 & 0 &
\end{array}$$

$\rightarrow\ x^3 - 3x - 2 = 0\ \rightarrow\ \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$\rightarrow\ x^2 - x - 2 = 0$

$$x = \frac{1 \pm \sqrt{9}}{2}$$

$$= \begin{cases} \frac{1-3}{2} = -1 \\ \frac{1+3}{2} = 2 \end{cases}$$

The solutions are: $\underline{x = -1,\ \ -1,\ \ -1,\ 2}$

Exercise

Find all solutions of the equation: $6x^4 - 17x^3 - 11x^2 + 42x = 0$

Solution

$$x\left(6x^3 - 17x^2 - 11x + 42\right) = 0$$

$$x = 0 \quad 6x^3 - 17x^2 - 11x + 42 = 0$$

$$\textit{possibilities}: \ \pm\left\{\frac{42}{6}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ 7,\ 14,\ 21,\ 42,\ \frac{1}{2},\ \frac{3}{2},\ \frac{7}{2},\ \frac{21}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{7}{3},\ \frac{14}{3},\ \frac{1}{6},\ \frac{7}{6},\ \frac{21}{6}\right\}$$

$$\begin{array}{r|rrrr} 2 & 6 & -17 & -11 & 42 \\ & & 12 & -10 & -42 \\ \hline & 6 & -5 & -21 & \boxed{0} \end{array} \ \rightarrow\ 6x^2 - 5x - 21 = 0$$

$$x = \frac{5 \pm \sqrt{25 + 504}}{12}$$

$$= \begin{cases} \frac{5 - 23}{12} = -\frac{3}{2} \\ \frac{5 + 23}{12} = \frac{7}{3} \end{cases}$$

The solutions are: $x = -\frac{3}{2},\ 0,\ 2,\ \frac{7}{3}$

Exercise

Find all solutions of the equation: $x^4 - 5x^2 - 2x = 0$

Solution

$$x\left(x^3 - 5x - 2\right) = 0$$

$$x = 0 \quad x^3 - 5x - 2 = 0$$

$$\textit{possibilities}: \ \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -5 & -2 \\ & & -2 & 4 & 2 \\ \hline & 1 & -2 & -1 & \boxed{0} \end{array} \ \rightarrow\ x^2 - 2x - 1 = 0$$

$$x = \frac{2 \pm \sqrt{4 + 4}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}}{2}$$

The solutions are: $x = -2,\ 0,\ 1 \pm \sqrt{2}$

Exercise

Find all solutions of the equation: $3x^4 - 4x^3 - 11x^2 + 16x - 4 = 0$

Solution

possibilities : $\pm\left\{\frac{4}{3}\right\} = \pm\left\{1,\ 2,\ 4,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3}\right\}$

$$\begin{array}{r|rrrrr}
1 & 3 & -4 & -11 & 16 & -4 \\
 & & 3 & -1 & -12 & 4 \\
\hline
\underline{2} & 3 & -1 & -12 & 4 & 0 \\
 & & 6 & 10 & -4 & \\
\hline
 & 3 & 5 & -2 & 0 &
\end{array}$$

$\to 3x^3 - x^2 - 12x + 4 = 0 \to \pm\left\{\frac{4}{3}\right\} = \pm\left\{1,\ 2,\ 4,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3}\right\}$

$\to 3x^2 + 5x - 2 = 0$

$$x = \frac{-5 \pm \sqrt{25+24}}{6}$$

$$= \begin{cases} \frac{-5-7}{6} = -2 \\ \frac{-5+7}{6} = \frac{1}{3} \end{cases}$$

The solutions are: $x = -2,\ \frac{1}{3},\ 1,\ 2$

Exercise

Find all solutions of the equation: $6x^4 + 23x^3 + 19x^2 - 8x - 4 = 0$

Solution

possibilities : $\pm\left\{\frac{4}{6}\right\} = \pm\left\{1,\ 2,\ 4,\ \frac{1}{6},\ \frac{2}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrrr}
-2 & 6 & 23 & 19 & -8 & -4 \\
 & & -12 & -22 & 6 & 4 \\
\hline
\underline{-2} & 6 & 11 & -3 & -2 & 0 \\
 & & -12 & 2 & 2 & \\
\hline
 & 6 & -1 & -1 & 0 &
\end{array}$$

$\to 6x^3 + 11x^2 - 3x - 2 = 0 \to \pm\left\{\frac{2}{6}\right\} = \pm\left\{1,\ 2,\ \frac{1}{6},\ \frac{1}{3}\right\}$

$\to 6x^2 - x - 1 = 0$

$$x = \frac{1 \pm \sqrt{25}}{12}$$

$$= \begin{cases} \frac{1-5}{12} = -\frac{1}{3} \\ \frac{1+5}{12} = \frac{1}{2} \end{cases}$$

The solutions are: $x = -2,\ -2,\ -\frac{1}{3},\ \frac{1}{2}$

Exercise

Find all solutions of the equation: $4x^4 - 12x^3 + 3x^2 + 12x - 7 = 0$

Solution

possibilities : $\pm\left\{\frac{7}{4}\right\} = \pm\left\{1,\ 7,\ \frac{1}{2},\ \frac{7}{2},\ \frac{1}{4},\ \frac{7}{4}\right\}$

$$\begin{array}{r|rrrrr}
1 & 4 & -12 & 3 & 12 & -7 \\
 & & 4 & -8 & -5 & 7 \\
\hline
-1 & 4 & -8 & -5 & 7 & 0 \\
 & & -4 & 12 & -7 & \\
\hline
 & 4 & -12 & 7 & 0 &
\end{array}$$

$\rightarrow 4x^3 - 8x^2 - 5x + 7 = 0 \rightarrow \pm\left\{\frac{7}{4}\right\} = \pm\left\{1,\ 7,\ \frac{1}{2},\ \frac{7}{2},\ \frac{1}{4},\ \frac{7}{4}\right\}$

$\rightarrow 4x^2 - 12x + 7 = 0$

$$x = \frac{12 \pm \sqrt{144 - 112}}{8}$$

$$= \frac{12 \pm 4\sqrt{2}}{8}$$

The solutions are: $\boxed{x = -1,\ 1,\ \frac{3 \pm \sqrt{2}}{2}}$

Exercise

Find all solutions of the equation: $2x^4 - 9x^3 - 2x^2 + 27x - 12 = 0$

Solution

possibilities : $\pm\left\{\frac{12}{2}\right\} = \pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 12,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$$\begin{array}{r|rrrrr}
4 & 2 & -9 & -2 & 27 & -12 \\
 & & 8 & -4 & -24 & 12 \\
\hline
\frac{1}{2} & 2 & -1 & -6 & 3 & 0 \\
 & & 1 & 0 & -3 & \\
\hline
 & 2 & 0 & -6 & 0 &
\end{array}$$

$\rightarrow 2x^3 - x^2 - 6x + 3 = 0 \rightarrow \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$\rightarrow 2x^2 - 6 = 0$

$$x^2 = 3$$

The solutions are: $\boxed{x = \frac{1}{2},\ 4,\ \pm\sqrt{3}}$

Exercise

Find all solutions of the equation: $2x^4 - 19x^3 + 51x^2 - 31x + 5 = 0$

Solution

possibilities : $\pm\left\{\frac{5}{2}\right\} = \pm\left\{1,\ 5,\ \frac{1}{2},\ \frac{5}{2}\right\}$

$$\begin{array}{r|rrrrr}
5 & 2 & -19 & 51 & -31 & 5 \\
 & & 10 & -45 & 30 & -5 \\
\hline
\frac{1}{2} & 2 & -9 & 6 & -1 & 0 \\
 & & 1 & -4 & 1 & \\
\hline
 & 2 & -8 & 2 & 0 &
\end{array}$$

$\to 2x^3 - 9x^2 + 6x - 1 = 0 \to \pm\left\{\frac{1}{2}\right\}$

$\to 2x^2 - 8x + 2 = 0$

$$x = \frac{8 \pm \sqrt{64-16}}{4}$$

$$= \frac{8 \pm 4\sqrt{3}}{4}$$

The solutions are: $\boxed{x = \frac{1}{2},\ 5,\ 2 \pm \sqrt{3}}$

Exercise

Find all solutions of the equation: $4x^4 - 35x^3 + 71x^2 - 4x - 6 = 0$

Solution

possibilities : $\pm\left\{\frac{6}{4}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{4},\ \frac{3}{4}\right\}$

$$\begin{array}{r|rrrrr}
3 & 4 & -35 & 71 & -4 & -6 \\
 & & 12 & -69 & 6 & 6 \\
\hline
-\frac{1}{4} & 4 & -23 & 2 & 2 & 0 \\
 & & -1 & 6 & -2 & \\
\hline
 & 4 & -24 & 8 & 0 &
\end{array}$$

$\to 4x^3 - 23x^2 + 2x + 2 = 0 \to \pm\left\{\frac{2}{4}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2},\ \frac{1}{4}\right\}$

$\to 4x^2 - 24x + 8 = 0$

$$x^2 - 6x + 2 = 0$$

$$x = \frac{6 \pm \sqrt{36-8}}{2}$$

$$= \frac{6 \pm 2\sqrt{7}}{4}$$

The solutions are: $\boxed{x = -\frac{1}{4},\ 3,\ 3 \pm \sqrt{7}}$

Exercise

Find all solutions of the equation: $2x^4 + 3x^3 - 4x^2 - 3x + 2 = 0$

Solution

$$\textit{possibilities}: \ \pm\left\{\frac{2}{2}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2}\right\}$$

$$\begin{array}{r|rrrrr}
1 & 2 & 3 & -4 & -3 & 2 \\
 & & 2 & 5 & 1 & -2 \\
\hline
-1 & 2 & 5 & 1 & -2 & 0 \\
 & & -2 & -3 & 2 & \\
\hline
 & 2 & 3 & -2 & 0 &
\end{array}$$

$\rightarrow 2x^3 - 23x^2 + 2x - 2 = 0 \ \rightarrow \ \pm\left\{\frac{2}{2}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2}\right\}$

$\rightarrow 2x^2 + 3x - 2 = 0$

$$x = \frac{-3 \pm \sqrt{9+16}}{4}$$

$$= \begin{cases} \frac{-3-5}{4} = -2 \\ \frac{-3+5}{4} = \frac{1}{2} \end{cases}$$

The solutions are: $\boxed{x = -2,\ \ -1,\ \ \frac{1}{2},\ 1}$

Exercise

Find all solutions of the equation: $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$

Solution

$$\textit{possibilities for } \frac{c}{d}: \pm\left\{\frac{56}{1}\right\} = \pm\{1,\ 2,\ 4,\ 7,\ 8,\ 14,\ 28,\ 56\}$$

$$\begin{array}{r|rrrrr}
4 & 1 & 3 & -3 & -6 & 56 \\
 & & 4 & 28 & -8 & -56 \\
\hline
-7 & 1 & 7 & -2 & -14 & 0 \\
 & & -7 & 0 & 14 & \\
\hline
 & 1 & 0 & -2 & &
\end{array}$$

$\rightarrow \ x^3 + 7x^2 - 2x - 14 = 0 \ \Rightarrow \ \frac{c}{d} = \pm\left\{\frac{14}{1}\right\} = \pm\{1,\ 2,\ 7,\ 14\}$

$\rightarrow \ x^2 - 2 = 0$

$\Rightarrow \boxed{x = \pm\sqrt{2}}$

The solutions are: $\boxed{x = 4,\ \ -7,\ \ \pm\sqrt{2}}$

Exercise

Find all solutions of the equation: $3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{6}{3}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrrrr}
-1 & 3 & -10 & -6 & 24 & 11 & -6 \\
 & & -3 & 13 & -7 & -17 & 6 \\
\hline
-1 & 3 & -13 & 7 & 17 & -6 & 0 \\
 & & -3 & 16 & -23 & 6 & \\
\hline
2 & 3 & -16 & 23 & -6 & 0 & \\
 & & 6 & 20 & 6 & & \\
\hline
 & 3 & -10 & 3 & 0 & &
\end{array}$$

$x^4 - 13x^3 + 7x^2 + 17x - 6 = 0 \to \pm\left\{\frac{6}{3}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$3x^3 - 16x^2 + 26x - 6 = 0 \to \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$3x^2 - 10x + 3 = 0$

$$x = \frac{10 \pm \sqrt{100 - 36}}{6}$$

$$= \begin{cases} \frac{10-8}{6} = \frac{1}{3} \\ \frac{10+8}{6} = 3 \end{cases}$$

The solutions are: $x = -1,\ -1,\ \frac{1}{3},\ 2,\ 3$

Exercise

Find all solutions of the equation: $6x^5 + 19x^4 + x^3 - 6x^2 = 0$

Solution

$x^2\left(6x^3 + 19x^2 + x - 6\right) = 0 \quad \to \quad x = 0,\ 0$

$6x^3 + 19x^2 + x - 6 = 0$

possibilities for $\frac{c}{d} : \pm\left\{\frac{6}{6}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrr}
-3 & 6 & 19 & 1 & -6 \\
 & & -18 & -3 & 6 \\
\hline
 & 6 & 1 & -2 & \boxed{0}
\end{array}$$

$6x^2 + x - 2 = 0$

$$x = \frac{-1 \pm \sqrt{1 + 48}}{12}$$

$$= \begin{cases} \frac{-1-7}{12} = -\frac{2}{3} \\ \frac{-1+7}{12} = \frac{1}{2} \end{cases}$$

The solutions are: $\boxed{x = 0,\ 0,\ -\frac{2}{3},\ -3,\ \frac{1}{2}}$

Exercise

Find all solutions of the equation: $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = 0$

Solution

$$x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = (x+1)^5 = 0$$

possibilities for $\frac{c}{d} : \pm\{1\}$

$$\begin{array}{r|rrrrrr} -1 & 1 & 5 & 10 & 10 & 5 & 1 \\ & & -1 & -4 & -6 & -4 & -1 \\ \hline -1 & 1 & 4 & 6 & 4 & 1 & 0 \\ & & -1 & -3 & -3 & -1 & \\ \hline -1 & 1 & 3 & 3 & 1 & 0 & \\ & & -1 & -2 & -1 & & \\ \hline & 1 & 2 & 1 & 0 & & \end{array}$$

$\rightarrow\ x^4 + 4x^3 + 6x^2 + 4x + 1 = 0 \rightarrow \pm\{1\}$

$\rightarrow\ x^3 + 3x^2 + 3x + 1 = 0 \rightarrow \pm\{1\}$

$\rightarrow\ x^2 + 2x + 1 = 0$

$$x^2 + 2x + 1 = (x+1)^2$$

The solutions are: $\boxed{x = -1,\ -1,\ -1,\ -1,\ -1}$

Exercise

Find all solutions of the equation: $x^5 - x^4 - 7x^3 + 7x^2 + 12x - 12 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$$\begin{array}{c|cccccc}
1 & 1 & -1 & -7 & 7 & 12 & -12 \\
 & & 1 & 0 & -7 & 0 & 12 \\
\hline
2 & 1 & 0 & -7 & 0 & 12 & 0 \\
 & & 2 & 4 & -6 & -12 & \\
\hline
-2 & 1 & 2 & -3 & -6 & 0 & \\
 & & -2 & 0 & 6 & & \\
\hline
 & 1 & 0 & -3 & 0 & &
\end{array}$$

$\rightarrow\ x^4 - 7x^2 - 12 = 0 \rightarrow \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$\rightarrow\ x^3 + 2x^2 - 3x - 6 = 0 \rightarrow \pm\{1,\ 2,\ 3,\ 6\}$

$\rightarrow\ x^2 - 3 = 0$

$x^2 = 3$

The solutions are: $\underline{x = -2,\ 1,\ 2,\ \pm\sqrt{3}}|$

Exercise

Find all solutions of the equation: $x^5 - 2x^3 - 8x = 0$

Solution

$x\left(x^4 - 2x^2 - 8\right) = 0$

$\underline{x = 0}|$

$x^4 - 2x^2 - 8 = 0$

$$x^2 = \frac{2 \pm \sqrt{4+32}}{2}$$

$$= \begin{cases} \frac{2-6}{2} = -2 \\ \frac{2+6}{2} = 4 \end{cases}$$

$$\begin{cases} x^2 = -2 & \rightarrow & x = \pm i\sqrt{2} \\ x^2 = 4 & \rightarrow & x = \pm 2 \end{cases}$$

The solutions are: $\underline{x = 0,\ \pm 2,\ \pm i\sqrt{2}}|$

Exercise

Find all solutions of the equation: $x^5 - 32 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\{1,\ 2,\ 4,\ 8,\ 16,\ 32\}$

$$\begin{array}{c|cccccc}
2 & 1 & 0 & 0 & 0 & 0 & -32 \\
 & & 2 & 4 & 8 & 16 & 32 \\
\hline
 & 1 & 2 & 4 & 8 & 16 & 0
\end{array}
\quad \to \quad x^4 + 2x^3 + 4x^2 + 8x + 16 = 0 \to \pm\{1,\ 2,\ 4,\ 8,\ 16\}$$

$x^4 + 2x^3 + 4x^2 + 8x + 16 = 0$

Cannot be solved using rational zero theorem.

Therefore; using program

The solutions are: $\boxed{x = 2,\ \dfrac{-1-\sqrt{5} \pm i\sqrt{2}\sqrt{5-\sqrt{5}}}{2},\ \dfrac{-1+\sqrt{5} \pm i\sqrt{2}\sqrt{5-\sqrt{5}}}{2}}$

Exercise

Find all solutions of the equation: $\quad 3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{24}{3}\right\} = \pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{8}{3}\right\}$

$$\begin{array}{c|ccccccc}
1 & 3 & -10 & -29 & 34 & 50 & -24 & -24 \\
 & & 3 & -7 & -36 & -2 & 48 & 24 \\
\hline
-1 & 3 & -7 & -36 & -2 & 48 & 24 & 0 \\
 & & -3 & 10 & 26 & -24 & -24 & \\
\hline
-2 & 3 & -10 & -26 & 24 & 24 & 0 & \\
 & & -6 & 32 & -12 & -24 & & \\
\hline
-\frac{2}{3} & 3 & -16 & 6 & 12 & 0 & & \\
 & & -2 & 12 & -12 & & & \\
\hline
 & 3 & -18 & 18 & 0 & & &
\end{array}$$

$\to 3x^5 - 7x^4 - 36x^3 - 2x^2 + 48x + 24 = 0$

$\to 3x^4 - 10x^3 - 26x^2 + 24x + 24 = 0$

$\to 3x^3 - 16x^2 + 12x - 12 = 0$

$\to 3x^2 - 18x + 18 = 0$

$x^2 - 6x + 6 = 0$

$$x = \frac{6 \pm \sqrt{36-24}}{2}$$

$$= \frac{6 \pm 2\sqrt{3}}{2}$$

The solutions are: $\boxed{x = -2,\ \ -1,\ 1,\ \ -\frac{2}{3},\ 3 \pm \sqrt{3}}$

Exercise

Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k)=\frac{1}{6}\left(k^3+3k^2+2k\right)$$

Where k is the number of levels in the pyramid. If 220 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

Solution

$$\frac{1}{6}\left(k^3+3k^2+2k\right)=220$$

$$k^3+3k^2+2k-1,320=0$$

$$\begin{array}{r|rrrr} 10 & 1 & 3 & 2 & -1320 \\ & & 10 & 130 & 1320 \\ \hline & 1 & 13 & 132 & 0 \end{array} \quad \to \quad k^2+13k+132=0$$

$$k=\frac{-13\pm\sqrt{-359}}{2} \quad \mathbb{C}$$

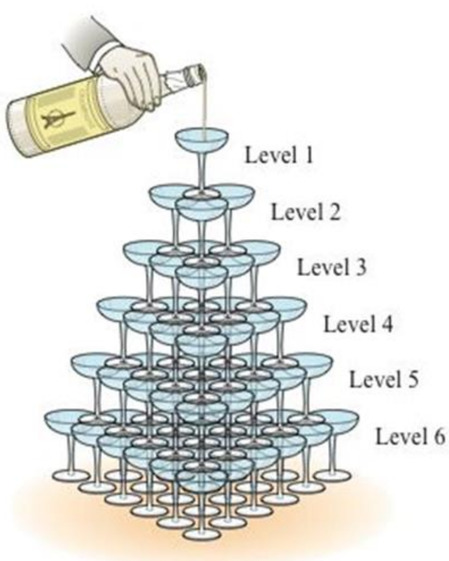
The are **10** levels in the pyramid.

Exercise

Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k)=\frac{1}{6}\left(2k^3+3k^2+k\right)$$

Where k is the number of levels in the pyramid. If 140 glasses are used to form a triangle pyramid, how many levels are in the pyramid?



Solution

$$\frac{1}{6}\left(2k^3+3k^2+k\right)=150$$

$$2k^3+3k^2+k-840=0$$

$$\begin{array}{r|rrrr} 7 & 2 & 3 & 1 & -840 \\ & & 14 & 119 & 840 \\ \hline & 2 & 17 & 120 & 0 \end{array} \quad \to \quad 2k^2+17k+120=0$$

$$k=\frac{-17\pm\sqrt{-671}}{4} \quad \mathbb{C}$$

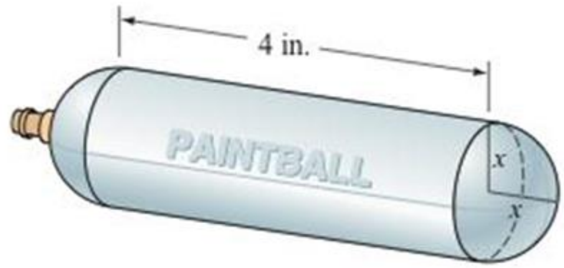
The are **7** levels in the pyramid.

Exercise

A carbon dioxide cartridge for a paintball rifle has the shape of a right circular cylinder with a hemisphere at each end. The cylinder is 4 *inches* long, and the volume of the cartridge is $2\pi \ \ in^3$.
The common interior radius of the cylinder and the hemispheres is denoted by *x*. Estimate the length of the radius *x*.

Solution

Volume of the Cartridge = 2 × (Volume of Hemisphere) + Volume of Cylinder



$$Volume\ of\ \ Sphere = \frac{4}{3}\pi x^3$$

$$Volume\ of\ \ Cylinder = 4\pi x^2$$

$$Volume\ of\ \ Cartridge = \frac{4}{3}\pi x^3 + 4\pi x^2$$

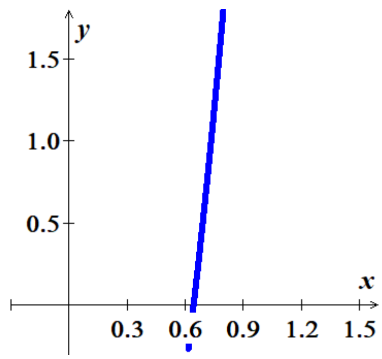
$$2\pi = \frac{4}{3}\pi x^3 + 4\pi x^2$$

$$2x^3 + 6x^2 = 3$$

$$2x^3 + 6x^2 - 3 = 0$$

Using Graph:

$\underline{x \approx 0.64 \ \ in.}|$

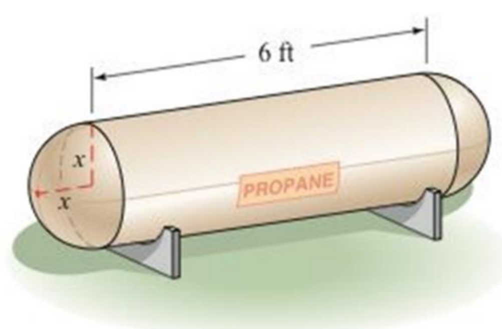


Exercise

A propane tank has the shape of a circular cylinder with a hemisphere at each end. The cylinder is 6 *feet* long and the volume of the tank is $9\pi \ \ ft^3$. Find the length of the radius *x*.

Solution

Volume of the Cartridge = 2 × (Volume of Hemisphere) + Volume of Cylinder



$$Volume\ of\ \ Sphere = \frac{4}{3}\pi x^3$$

$$Volume\ of\ \ Cylinder = 6\pi x^2$$

$$Volume\ of\ \ Cartridge = \frac{4}{3}\pi x^3 + 6\pi x^2$$

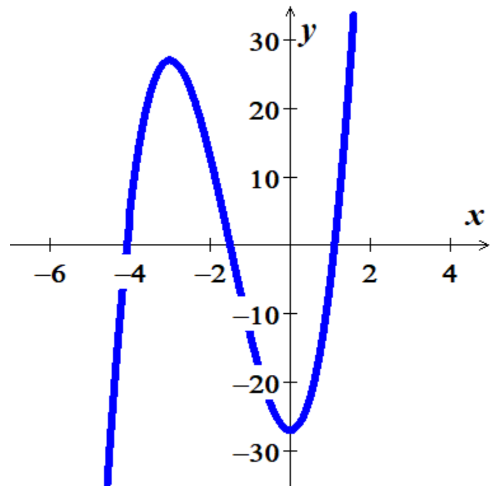
$$9\pi = \frac{4}{3}\pi x^3 + 6\pi x^2$$

$$27 = 4x^3 + 18x^2$$

$$4x^3 + 18x^2 - 27 = 0$$

$$possibilities\ for\ \ \frac{c}{d} : \pm\left\{\frac{27}{4}\right\} = \pm\left\{1,\ 3,\ 9,\ 27,\ \frac{1}{2},\ \frac{3}{2},\ \frac{9}{2},\ \frac{27}{2},\ \frac{1}{4},\ \frac{3}{4},\ \frac{9}{4},\ \frac{27}{4}\right\}$$

$$\begin{array}{r|rrrr} -\frac{3}{2} & 4 & 18 & 0 & -27 \\ & & -6 & -18 & 27 \\ \hline & 4 & 12 & -18 & 0 \end{array} \quad \to \quad 4x^2+12x-18=0$$



$$2x^2+6x-9=0$$

$$x=\frac{-6\pm\sqrt{36+72}}{4}$$

$$=\frac{-6\pm 6\sqrt{3}}{4}$$

$$=\frac{-3\pm 3\sqrt{3}}{2}$$

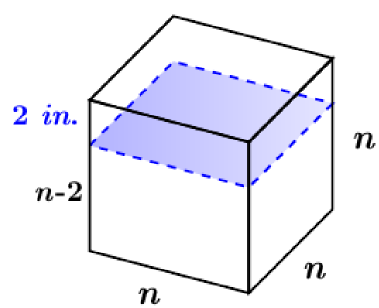
$$\boxed{x=-\frac{3}{2},\ \frac{-3-3\sqrt{3}}{2},\ \frac{-3+3\sqrt{3}}{2}}$$

$\therefore$ the length of the radius x is $\frac{-3+3\sqrt{3}}{2}\approx 1.1\ \ foot$

Exercise

A cube measures n inches on each edge. If a slice 2 *inches* thick is cut from one face of the cube, the resulting solid has a volume of $567\ \ in^3$. Find n.

Solution



$$Volume = n^2(n-2)$$

$$n^3-2n^2=567$$

$$n^3-2n^2-567=0$$

$$possibilities\ for\ \frac{c}{d} := \pm\{1,\ 3,\ 7,\ 9,\ 21,\ 27,\ 63,\ 81,\ 189,\ 567\}$$

$$\begin{array}{r|rrrr} 9 & 1 & -2 & 0 & -567 \\ & & 9 & 63 & 567 \\ \hline & 1 & 7 & 63 & 0 \end{array} \quad \to \quad n^2+7n+63=0$$

$$n=\frac{-7\pm\sqrt{49-252}}{2}$$

$$=\frac{-7\pm i\sqrt{203}}{2} \quad \times$$

$$\therefore\ \ \boxed{n=9}$$

Exercise

A cube measures n inches on each edge. If a slice 1 *inch* thick is cut from one face of the cube and then a slice 3 inches thick is cut from another face of the cube, the resulting solid has a volume of $1560\ in^3$. Find the dimensions of the original cube.

Solution

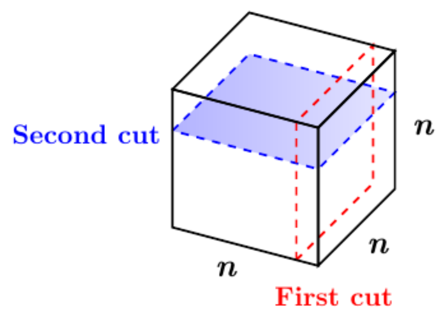
$$Volume = n(n-1)(n-3)$$

$$n^3 - 4n^2 + 3n = 1560$$

$$n^3 - 4n^2 + 3n - 1560 = 0$$

$$possibilities\ for\ \frac{c}{d} := \pm\left\{\begin{array}{l} 1,\ 2,\ 4,\ 5,\ 6,\ 8,\ 10,\ 12,\ 13,\ 15,\ 20,\ 24,\ 26,\ 30,\ 39, \\ 40,\ 52,\ 60,\ 65,\ 78,\ 104,\ 120,\ 130,\ 156,\ 195,\ 260,\ 312,\ 390,\ 780,\ 1560 \end{array}\right\}$$

$$\begin{array}{r|rrrr} 13 & 1 & -4 & 3 & -1560 \\ & & 13 & 117 & 1560 \\ \hline & 1 & 9 & 120 & 0 \end{array} \quad \to \quad n^2 + 9n + 120 = 0$$



$$n = \frac{-9 \pm \sqrt{81 - 480}}{2}$$

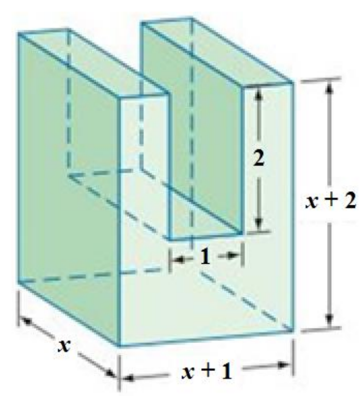
$$= \frac{-9 \pm i\sqrt{399}}{2} \quad \times$$

$$\therefore \quad \underline{n = 13}$$

Exercise

For what value of x will the volume of the following solid be $112\ in^3$

Solution



Volume of the bottom portion $= x^2(x+1)$

Volume of one side portion $= 2x\left(\frac{1}{2}x\right)$

$$= x^2$$

Total Volume $= x^2(x+1) + 2x^2$

$$112 = x^3 + 3x^2$$

$$x^3 + 3x^2 - 112 = 0$$

$$possibilities\ for\ \frac{c}{d} := \pm\{1,\ 2,\ 4,\ 8,\ 14,\ 28,\ 56,\ 112\}$$

$$\begin{array}{c|cccc} 4 & 1 & 3 & 0 & -112 \\ & & 4 & 28 & 112 \\ \hline & 1 & 7 & 28 & 0 \end{array} \rightarrow \quad x^2 + 7x + 28 = 0$$

$$x = \frac{-7 \pm \sqrt{49 - 112}}{2}$$

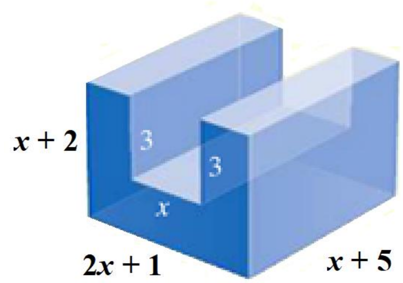
$$= \frac{-7 \pm 3i\sqrt{7}}{2}$$

$$\therefore \quad x = 4$$

Exercise

For what value of x will the volume of the following solid be $208 \ in^3$

Solution



Volume of the bottom portion $= (2x+1)(x+5)(x+2-3)$

$$= \left(2x^2 + 11x + 5\right)(x-1)$$

$$= 2x^3 + 11x^2 + 5x - 2x^2 - 11x - 5$$

$$= 2x^3 + 9x^2 - 6x - 5$$

Volume of one side portion $= (3)\frac{1}{2}(2x + 1 - x)(x+5)$

$$= \frac{3}{2}(x+1)(x+5)$$

$$= \frac{3}{2}\left(x^2 + 6x + 5\right)$$

Total Volume $= 2x^3 + 9x^2 - 6x - 5 + 2\left(\frac{3}{2}\right)\left(x^2 + 6x + 5\right)$

$$208 = 2x^3 + 9x^2 - 6x - 5 + 3x^2 + 18x + 15$$

$$2x^3 + 12x^2 + 12x - 198 = 0$$

$$x^3 + 6x^2 + 6x - 99 = 0$$

possibilities for $\frac{c}{d} := \pm\{1, \ 3, \ 9, \ 11, \ 33, \ 99\}$

$$\begin{array}{c|cccc} 3 & 1 & 6 & 6 & -99 \\ & & 3 & 27 & 99 \\ \hline & 1 & 9 & 33 & 0 \end{array} \rightarrow \quad x^2 + 9x + 33 = 0$$

$$x = \frac{-9 \pm \sqrt{81 - 132}}{2}$$

$$= \frac{-9 \pm i\sqrt{51}}{2}$$

$$\therefore \quad x = 3$$

Exercise

The length of rectangular box is 1 *inch* more than twice the height of the box, and the width is 3 *inches* more than the height. If the volume of the box is 126 in^3, find the dimensions of the box.

Solution

$$Volume = x(2x+1)(x+3)$$

$$2x^3 + 7x^2 + 3x = 126$$

$$2x^3 + 7x^2 + 3x - 126 = 0$$

$$possibilities\ for\ \frac{c}{d} := \pm\left\{\frac{126}{2}\right\}$$

$$= \pm\left\{1,\ 2,\ 3,\ 6,\ 9,\ 14,\ 21,\ 42,\ 63,\ 126,\ \frac{1}{2},\ \frac{3}{2},\ \frac{9}{2},\ \frac{21}{2},\ \frac{63}{2}\right\}$$

$$\begin{array}{r|rrrr} 3 & 2 & 7 & 3 & -126 \\ & & 6 & 39 & 126 \\ \hline & 2 & 13 & 42 & 0 \end{array} \to\ 2x^2 + 13x + 42 = 0$$

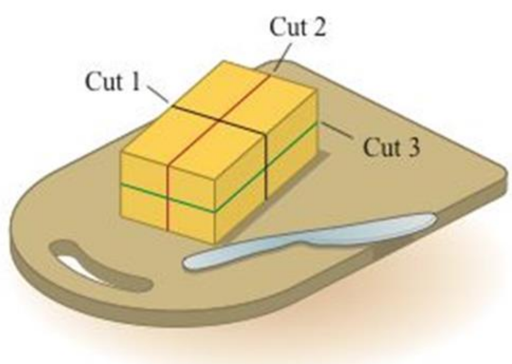
$$x = \frac{-13 \pm \sqrt{169 - 336}}{4}$$

$$= \frac{-13 \pm i\sqrt{167}}{4}\ \times$$

$$\therefore\ \ \underline{x = 3}$$

Exercise

One straight cut through a thick piece of cheese produces two pieces. Two straight cuts can produce a maximum of four pieces. Two straight cuts can produce a maximum of four pieces. Three straight cuts can produce a maximum of eight pieces.



You might be inclined to think that every additional cut doubles number of pieces. However, for four straight cuts, you get a maximum of 15 pieces. The maximum number of pieces P that can be produced by n straight cuts is given by

$$P(n) = \frac{n^3 + 5n + 6}{6}$$

a) Determine number of pieces that can be produces by five straight cuts.

b) What is the fewest number of straight cuts that are needed to produce 64 pieces?

Solution

a) $P(5) = \dfrac{5^3 + 25 + 6}{6}$

$= 26$

b) $\dfrac{n^3 + 5n + 6}{6} = 64$

$n^3 + 5n + 6 = 384$

$n^3 + 5n - 378 = 0$

$\textit{possibilities for } \frac{c}{d} := \pm\{378\}$

$= \pm\{1,\ 2,\ 3,\ 6,\ 7,\ 9,\ 14,\ 18,\ 21,\ 27,\ 42,\ 54,\ 63,\ 126,\ 189,\ 378\}$

$$\begin{array}{r|rrrr} 7 & 1 & 0 & 5 & -378 \\ & & 7 & 49 & 378 \\ \hline & 1 & 7 & 54 & 0 \end{array} \rightarrow \quad n^2 + 7n + 54 = 0$$

$n = \dfrac{-7 \pm \sqrt{49 - 216}}{2}$

$= \dfrac{-7 \pm i\sqrt{167}}{2}$ X

$\therefore \quad n = 7$

Exercise

The number of ways one can select three cards from a group of n cards (the order of the selection matters), where $n \geq 3$, is given by $P(n) = n^3 - 3n^2 + 2n$. For a certain card trick, a magician has determined that there are exactly 504 *ways* to choose three cards from a given group. How many cards are in the group?

Solution

$P(n) = n^3 - 3n^2 + 2n = 504$

$n^3 - 3n^2 + 2n - 504 = 0$

$\textit{possibilities for } \frac{c}{d} := \pm\{504\}$

$$= \pm\left\{\begin{array}{l} 1,\ 2,\ 3,\ 4,\ 6,\ 7,\ 8,\ 9,\ 12,\ 14,\ 18,\ 21, \\ \quad 24,\ 28,\ 36,\ 42,\ 56,\ 63,\ 72,\ 84,\ 126,\ 168,\ 252,\ 504 \end{array}\right\}$$

$$\begin{array}{r|rrrr} 9 & 1 & -3 & 2 & -504 \\ & & 9 & 54 & 504 \\ \hline & 1 & 6 & 56 & 0 \end{array} \rightarrow \quad n^2+6n+56=0$$

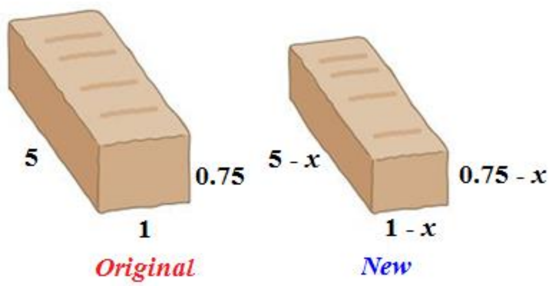
$$n = \frac{-6 \pm \sqrt{36-224}}{2}$$

$$= -3 \pm i\sqrt{47} \quad \times$$

$\therefore \quad n = 9$

Exercise

A nutrition bar in the shape of a rectangular solid measure 0.75 *in*. by 1 *in*. by 5 *inches*.



To reduce costs, the manufacturer has decided to decrease each of the dimensions of the nutrition bar by x *inches*, what value of x will produce a new bar with a volume that is $0.75 \ in^3$ less than the present bar's volume.

Solution

$$V_{original} = (5)(1)\left(\frac{3}{4}\right)$$

$$= \frac{15}{4}$$

$$V_{new} = (5-x)(1-x)\left(\frac{3}{4}-x\right) \qquad \left(x < \frac{3}{4}\right)$$

$$\left(5-6x+x^2\right)\left(\frac{3-4x}{4}\right) = \frac{15}{4} - \frac{3}{4}$$

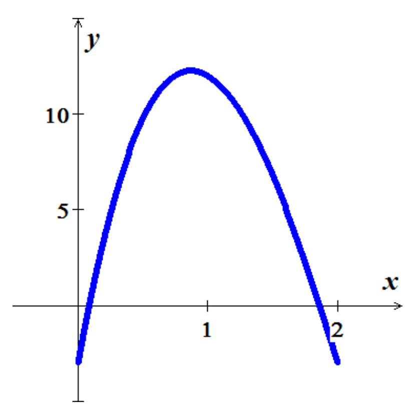
$$15-20x-18x+24x^2+3x^2-4x^3 = 4(3)$$

$$4x^3-27x^2+38x-3=0$$

From graph table:

0.08200	-0.06334
0.08400	0.00386

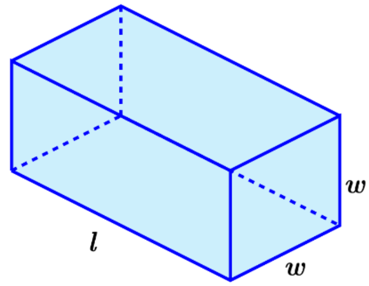
$x \approx 0.083 \ in.$



Exercise

A rectangular box is square on two ends and has length plus girth of 81 *inches*. (Girth: distance *around* the box). Determine the possible lengths $l\ (l > w)$ of the box if its volume is $4900\ \ in^3$.

Solution



$$81 = l + 4w$$

$$l = 81 - 4w$$

$$V = lw^2$$

$$= (81 - 4w)w^2$$

$$-4w^3 + 81w^2 = 4900$$

$$4w^3 - 81w^2 + 4900 = 0$$

$$\textit{possibilities for } \frac{c}{d} := \pm\left\{\frac{4900}{4}\right\} = \pm\left\{\begin{matrix} 1,\ 2,\ 4,\ 7,\ 10,\ 14,\ 20,\ 28,\ 49, \\ 100,\ 175,\ 245,\ 350,\ 490,\ 700,\ 1225,\ 2450,\ 4900, \cdots \end{matrix}\right\}$$

$$\begin{array}{r|rrrr} 14 & 4 & -81 & 0 & 4900 \\ & & 56 & -350 & -4900 \\ \hline & 4 & -25 & -350 & 0 \end{array} \quad \to \quad 4w^2 - 25w - 350 = 0$$

$$w = \frac{25 \pm \sqrt{625 + 5600}}{8}$$

$$= \frac{25 \pm 5\sqrt{249}}{8}$$

$$= \begin{cases} \dfrac{25 - 5\sqrt{249}}{8} < 0 \\ \dfrac{25 + 5\sqrt{249}}{8} \approx 13 \end{cases}$$

$$l = 81 - 4(14) = 25$$

$$l = 81 - 4(13) = 29$$

$\therefore$ the possible lengths l are around **25** *in*. ***or*** **29** *in*.