Section 1.5 – Exponential and Logarithmic Equations

Properties of Logarithms For $M > 0$ and $N > 0$

***Product* Rule** $\log_b MN = \log_b M + \log_b N$

***Power* Rule** $\log_b M^p = p\log_b M$

***Quotient* Rule** $\log_b \frac{M}{N} = \log_b M - \log_b N$

Example

Express $\log_a \frac{x^3\sqrt{y}}{z^2}$ in terms of logarithms of x, y, and z.

Solution

$$\begin{aligned}\log_a \frac{x^3\sqrt{y}}{z^2} &= \log_a x^3y^{1/2} - \log_a z^2 && \textit{Quotient Rule}\\ &= \log_a x^3 + \log_a y^{1/2} - \log_a z^2 && \textit{Product Rule}\\ &= 3\log_a x + \tfrac{1}{2}\log_a y - 2\log_a z && \textit{Power Rule}\end{aligned}$$

Example

Express as one logarithm: $\frac{1}{3}\log_a\left(x^2-1\right) - \log_a y - 4\log_a z$

Solution

$$\begin{aligned}\tfrac{1}{3}\log_a\left(x^2-1\right) - \log_a y - 4\log_a z &= \log_a\left(x^2-1\right)^{1/3} - \log_a y - \log_a z^4 && \textit{Power Rule}\\ &= \log_a \sqrt[3]{x^2-1} - \left(\log_a y + \log_a z^4\right) && \textit{Factor } (-)\\ &= \log_a \sqrt[3]{x^2-1} - \left(\log_a yz^4\right) && \textit{Product Rule}\\ &= \log_a \frac{\sqrt[3]{x^2-1}}{yz^4} && \textit{Quotient Rule}\end{aligned}$$

Exponential Functions are One-to-One

$$b^{M} = b^{N} \quad \leftrightarrow \quad M = N \quad \text{for any b} > 0, \neq 1$$

Example

Solve $8^{x+2} = 4^{x-3}$

Solution

$$\left(2^3\right)^{x+2} = \left(2^2\right)^{x-3}$$

$$2^{3(x+2)} = 2^{2(x-3)}$$

$$3(x+2) = 2(x-3)$$

$$3x + 6 = 2x - 6$$

$$3x - 2x = -6 - 6$$

$$\boxed{x = -12}$$

Using *Natural Logarithms*

1. Isolate the exponential expression
2. Take the natural logarithm on both sides of the equation
3. Simplify using one of the following properties: $\ln b^{x} = x \ln b$ *or* $\ln e^{x} = x$
4. Solve for the variable

Example

Solve the equation $3^{x} = 21$

Solution

1st *method*	2nd *method*
$3^{x} = 21$ ***ln both sides*** $\ln 3^{x} = \ln 21$ $x \ln 3 = \ln 21$ $x = \dfrac{\ln 21}{\ln 3}$	$3^{x} = 21 \Rightarrow x = \log_3 21$ ***Convert to log form*** $x = \dfrac{\ln 21}{\ln 3}$ ***Change of base***

Example

Solve the equation $5^{2x+1} = 6^{x-2}$

Solution

$$\ln 5^{2x+1} = \ln 6^{x-2}$$

$$(2x+1)\ln 5 = (x-2)\ln 6$$

$$2x\ln 5 + \ln 5 = x\ln 6 - 2\ln 6$$

$$2x\ln 5 - x\ln 6 = -2\ln 6 - \ln 5$$

$$x(2\ln 5 - \ln 6) = -\ln 6^2 - \ln 5$$

$$x\left(\ln 5^2 - \ln 6\right) = -(\ln 36 + \ln 5)$$

$$x\left(\ln \frac{25}{6}\right) = -\ln(36 \times 5)$$

$$\underline{|x} = -\frac{\ln(180)}{\ln \frac{25}{6}} \ \underline{\approx -3.64|}$$

Example

Solve the equation $\dfrac{5^x - 5^{-x}}{2} = 3$

Solution

$5^x - 5^{-x} = 6$ *Multiply by 2 both sides*

$5^x 5^x - 5^{-x} 5^x = 6 5^x$ *Multiply by 5^x both sides*

$$\left(5^x\right)^2 - 1 = 6\left(5^x\right)$$

$$\left(5^x\right)^2 - 6\left(5^x\right) - 1 = 0$$

$$5^x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-1)}}{2(1)} = \frac{6 \pm \sqrt{40}}{2} = \frac{6 \pm 2\sqrt{10}}{2} = \begin{cases} 3 + \sqrt{10} \\ 3 - \sqrt{10} < 0 \end{cases}$$

$$5^x = 3 + \sqrt{10}$$

$$\ln 5^x = \ln\left(3 + \sqrt{10}\right)$$

$$x\ln 5 = \ln\left(3 + \sqrt{10}\right)$$

$$\underline{|x} = \frac{\ln\left(3+\sqrt{10}\right)}{\ln 5} \ \underline{\approx 1.13|}$$

Logarithmic Equations

1. Express the equation in the form $\log_b M = c$
2. Use the definition of a logarithm to rewrite the equation in exponential form:

$$\log_b M = c \quad \Rightarrow \quad b^c = M$$

3. Solve for the variable
4. Check proposed solution in the original equation. Include only the set for $M > 0$

Example

Solve: $\log x + \log(x-3) = 1$

Solution

$\log[x(x-3)] = 1$ — ***Product Rule***

$x(x-3) = 10^1$ — ***Convert to exponential form***

$x^2 - 3x = 10$

$x^2 - 3x - 10 = 0$ — ***Solve for x***

$\Rightarrow x = -2, 5$

Check: $x = -2 \Rightarrow \cancel{\log(-2) + \log(x-3) = 1}$

$x = 5 \Rightarrow \log(5) + \log(5-3) = 1$

Example

Solve the equation $\log_2 x + \log_2 (x+2) = 3$

Solution

$\log_2 [x(x+2)] = 3$ — ***Product Rule***

$x(x+2) = 2^3$ — ***Change to exponential form***

$x^2 + 2x - 8 = 0$ — ***Solve for x***

$x = -4 \quad x = 2$

Check: $\log_2 (-4) + \log_2 (-4+2) = 3$ Not a solution (negative inside the log)

$\log_2 (2) + \log_2 (2+2) = 3$ Only solution

Property of Logarithmic Equality

The logarithmic function with base ***b*** is 1-1. Thus the following equivalent conditions are satisfied for positive real numbers M and N.

For any $M > 0$, $N > 0$, $b > 0, \neq 1$

$$\text{If } \log_b M = \log_b N \quad \Rightarrow \quad M = N$$

$$\text{If } M \neq N \quad \Rightarrow \quad \log_b M \neq \log_b N$$

Example

Solve the equation $\log_6(4x-5) = \log_6(2x+1)$

Solution

$$\log_6(4x-5) = \log_6(2x+1)$$

$$4x-5 = 2x+1$$

$$4x-2x = 5+1$$

$$2x = 6$$

$$x = 3$$

Check: $\log_6(4(3)-5) = \log_6(2(3)+1)$

$\log_6(7) = \log_6(7)$ ***True statement***

$\boxed{x=3}$ is a solution

Example

Solve the equation $\ln(x+6) - \ln 10 = \ln(x-1) - \ln 2$

Solution

$$\ln(x+6) - \ln 10 = \ln(x-1) - \ln 2$$

$$\ln(x+6) - \ln(x-1) = \ln 10 - \ln 2$$

$$\ln\left(\frac{x+6}{x-1}\right) = \ln\frac{10}{2}$$

$$\frac{x+6}{x-1} = 5$$

$$x+6 = 5(x-1)$$

$$x+6 = 5x-5$$

$$x-5x = -5-6$$

$$-4x = -11$$

$x = \frac{-11}{-4} = \boxed{\frac{11}{4}}$

Check: $\ln\left(\frac{11}{4} + 6\right) - \ln 10 = \ln\left(\frac{11}{4} - 1\right) - \ln 2$

$\ln\left(\frac{35}{4}\right) - \ln 10 = \ln\left(\frac{7}{4}\right) - \ln 2$

$\boxed{x = \frac{11}{4}}$ is the solution

Example

Solve the equation $\log \sqrt[3]{x} = \sqrt{\log x}$ for x.

Solution

$\log x^{1/3} = \sqrt{\log x}$

$\left(\frac{1}{3}\log x\right)^2 = \left(\sqrt{\log x}\right)^2$

$\frac{1}{9}(\log x)^2 = \log x$

$(\log x)^2 = 9\log x$

$(\log x)^2 - 9\log x = 0$

$\log x(\log x - 9) = 0$

$\log x = 0 \qquad \log x - 9 = 0$

$\boxed{x = 1} \qquad \log x = 9$

$\boxed{x = 10^9}$

Check: $x = 1 \quad \Rightarrow \quad \log \sqrt[3]{1} = \sqrt{\log 1} \rightarrow 0 = 0$

$x = 10^9 \quad \Rightarrow \quad \log \sqrt[3]{10^9} = \sqrt{\log 10^9} \rightarrow 3 = 3$

The equation has two solutions: $\boxed{x = 1,\ 10^9}$

Example (*hyperbolic secant function*)

Solve the equation $y=\dfrac{2}{e^{x}+e^{-x}}$ for x in terms of y.

Solution

$$y=\frac{2}{e^{x}+e^{-x}}$$

$$y\left(e^{x}+e^{-x}\right)=2$$

$$ye^{x}+ye^{-x}=2$$

$$ye^{x}e^{x}+ye^{-x}e^{x}=2e^{x}$$

$$y\left(e^{x}\right)^{2}-2e^{x}+y=0$$

$$\begin{aligned} e^{x}&=\frac{2\pm\sqrt{4-4y^{2}}}{2y}\\ &=\frac{2\pm\sqrt{4\left(1-y^{2}\right)}}{2y}\\ &=\frac{2\pm2\sqrt{1-y^{2}}}{2y}\\ &=\frac{1\pm\sqrt{1-y^{2}}}{y}\end{aligned}$$

$$\ln e^{x}=\ln\left(\frac{1\pm\sqrt{1-y^{2}}}{y}\right)$$

$$x=\ln\frac{1\pm\sqrt{1-y^{2}}}{y}$$

Exercises Section 1.5 – Exponential and Logarithmic Equations

(1 – 31) Express the following in terms of sums and differences of logarithms

1. $\log_3(ab)$

2. $\log_7(7x)$

3. $\log\dfrac{x}{1000}$

4. $\log_5\left(\dfrac{125}{y}\right)$

5. $\log_b x^7$

6. $\ln\sqrt[7]{x}$

7. $\log_a \dfrac{x^2\, y}{z^4}$

8. $\log_b \dfrac{x^2 y}{b^3}$

9. $\log_b\left(\dfrac{x^3 y}{z^2}\right)$

10. $\log_b\left(\dfrac{\sqrt[3]{x}y^4}{z^5}\right)$

11. $\log\left(\dfrac{100x^3\ \sqrt[3]{5-x}}{3(x+7)^2}\right)$

12. $\log_a \sqrt[4]{\dfrac{m^8\ n^{12}}{a^3\ b^5}}$

13. $\log_p \sqrt[3]{\dfrac{m^5 n^4}{t^2}}$

14. $\log_b \sqrt[n]{\dfrac{x^3 y^5}{z^m}}$

15. $\log_a \sqrt[3]{\dfrac{a^2\ b}{c^5}}$

16. $\log_b\left(x^4\ \sqrt[3]{y}\right)$

17. $\log_5\left(\dfrac{\sqrt{x}}{25y^3}\right)$

18. $\log_a \dfrac{x^3 w}{y^2 z^4}$

19. $\log_a \dfrac{\sqrt{y}}{x^4\sqrt[3]{z}}$

20. $\ln\sqrt[4]{\dfrac{x^7}{y^5 z}}$

21. $\ln x\sqrt[3]{\dfrac{y^4}{z^5}}$

22. $\log_b \sqrt[5]{\dfrac{m^4 n^5}{x^2 ab^{10}}}$

23. $\log_b \dfrac{a^5 b^{10}}{c^2\ \sqrt[4]{d^3}}$

24. $\ln\left(x^2\sqrt{x^2+1}\right)$

25. $\ln\dfrac{x^2}{x^2+1}$

26. $\ln\left(\dfrac{x^2(x+1)^3}{(x+3)^{1/2}}\right)$

27. $\ln\sqrt{\dfrac{(x+1)^5}{(x+2)^{20}}}$

28. $\ln\dfrac{\left(x^2+1\right)^5}{\sqrt{1-x}}$

29. $\ln\left(\sqrt[3]{\dfrac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}}\right)$

30. $\ln\left(\sqrt{\dfrac{1}{x(x+1)}}\right)$

31. $\ln\left(\sqrt{\left(x^2+1\right)(x-1)^2}\right)$

(32 – 55) Write the expression as a single logarithm and simplify if necessary

32. $\log(x+5)+2\log x$

33. $3\log_b x-\frac{1}{3}\log_b y+4\log_b z$

34. $\frac{1}{2}\log_b(x+5)-5\log_b y$

35. $\ln\left(x^2-y^2\right)-\ln(x-y)$

36. $\ln(xz)-\ln\left(x\sqrt{y}\right)+2\ln\dfrac{y}{z}$

37. $\log\left(x^2 y\right)-\log z$

38. $\log\left(z^2\sqrt{y}\right)-\log z^{1/2}$

39. $2\log_a x+\frac{1}{3}\log_a(x-2)-5\log_a(2x+3)$

40. $5\log_a x-\frac{1}{2}\log_a(3x-4)-3\log_a(5x+1)$

41. $\log\left(x^3y^2\right)-2\log\left(x\sqrt[3]{y}\right)-3\log\left(\frac{x}{y}\right)$

42. $\ln y^3+\frac{1}{3}\ln\left(x^3y^6\right)-5\ln y$

43. $2\ln x-4\ln\left(\frac{1}{y}\right)-3\ln(xy)$

44. $4\ln x+7\ln y-3\ln z$

45. $\frac{1}{3}\left[5\ln(x+6)-\ln x-\ln(x^2-25)\right]$

46. $\frac{2}{3}\left[\ln\left(x^2-4\right)-\ln(x+2)\right]+\ln(x+y)$

47. $\frac{1}{2}\log_b m+\frac{3}{2}\log_b 2n-\log_b m^2n$

48. $\frac{1}{2}\log_y p^3q^4-\frac{2}{3}\log_y p^4q^3$

49. $\frac{1}{2}\log_a x+4\log_a y-3\log_a x$

50. $\frac{2}{3}\left[\ln\left(x^2-9\right)-\ln(x+3)\right]+\ln(x+y)$

51. $\frac{1}{4}\log_b x-2\log_b 5-10\log_b y$

52. $2\ln(x+4)-\ln x-\ln\left(x^2-3\right)$

53. $\ln x+\ln(y+3)+\ln(y+2)-\ln\left(y^2+5y+6\right)$

54. $\ln x+\ln(x+4)+\ln(x+1)-\ln\left(x^2+5x+4\right)$

55. $\ln\left(x^2-25\right)-2\ln(x+5)+\ln(x-5)$

(**56 – 170**) Solve the equations

56. $2^x=128$

57. $3^x=243$

58. $5^x=70$

59. $6^x=50$

60. $5^x=134$

61. $7^x=12$

62. $9^x=\frac{1}{\sqrt[3]{3}}$

63. $49^x=\frac{1}{343}$

64. $2^{5x+3}=\frac{1}{16}$

65. $\left(\frac{2}{5}\right)^x=\frac{8}{125}$

66. $2^{3x-7}=32$

67. $4^{2x-1}=64$

68. $3^{1-x}=\frac{1}{27}$

69. $2^{-x^2}=5$

70. $2^{-x}=8$

71. $\left(\frac{1}{3}\right)^x=81$

72. $3^{-x}=120$

73. $27=3^{5x}\ 9^{x^2}$

74. $4^{x+3}=3^{-x}$

75. $2^{x+4}=8^{x-6}$

76. $8^{x+2}=4^{x-3}$

77. $7^x=12$

78. $5^{x+4}=4^{x+5}$

79. $5^{x+2}=4^{1-x}$

80. $3^{2x-1}=0.4^{x+2}$

81. $4^{3x-5}=16$

82. $4^{x+3}=3^{-x}$

83. $7^{2x+1}=3^{x+2}$

84. $3^{x-1}=7^{2x+5}$

85. $4^{x-2} = 2^{3x+3}$

86. $3^{5x-8} = 9^{x+2}$

87. $3^{x+4} = 2^{1-3x}$

88. $3^{2-3x} = 4^{2x+1}$

89. $4^{x+3} = 3^{-x}$

90. $7^{x+6} = 7^{3x-4}$

91. $2^{-100x} = (0.5)^{x-4}$

92. $4^{x}\left(\frac{1}{2}\right)^{3-2x} = 8.\left(2^{x}\right)^{2}$

93. $5^{x} + 125\left(5^{-x}\right) = 30$

94. $4^{x} - 3\left(4^{-x}\right) = 8$

95. $5^{3x-6} = 125$

96. $e^{x} = 15$

97. $e^{x+1} = 20$

98. $9e^{x} = 107$

99. $e^{x\ln 3} = 27$

100. $e^{x^2} = e^{7x-12}$

101. $f(x) = xe^{x} + e^{x}$

102. $f(x) = x^{3}\left(4e^{4x}\right) + 3x^{2}e^{4x}$

103. $e^{2x} - 2e^{x} - 3 = 0$

104. $e^{0.08t} = 2500$

105. $e^{x^2} = 200$

106. $e^{2x+1} \cdot e^{-4x} = 3e$

107. $e^{2x} - 8e^{x} + 7 = 0$

108. $e^{2x} + 2e^{x} - 15 = 0$

109. $e^{x} + e^{-x} - 6 = 0$

110.

111. $e^{1-3x} \cdot e^{5x} = 2e$

112. $6\ln(2x) = 30$

113. $\log_5(x-7) = 2$

114. $\log_4(5+x) = 3$

115. $\log(4x-18) = 1$

116. $\log_3 x = -2$

117. $\log\left(x^2+19\right) = 2$

118. $\ln\left(x^2-12\right) = \ln x$

119. $\log\left(2x^2+3x\right) = \log(10x+30)$

120. $\log_5(2x+3) = \log_5 11 + \log_5 3$

121. $\log_3 x - \log_9(x+42) = 0$

122. $\log_5 x + \log_5(4x-1) = 1$

123. $\log x - \log(x+3) = 1$

124. $\log x + \log(x-9) = 1$

125. $\log_2(x+1) + \log_2(x-1) = 3$

126. $\log_8(x+1) - \log_8 x = 2$

127. $\ln(x+8) + \ln(x-1) = 2\ln x$

128. $\ln(4x+6) - \ln(x+5) = \ln x$

129. $\ln(5+4x) - \ln(x+3) = \ln 3$

130. $\ln\sqrt[4]{x} = \sqrt{\ln x}$

131. $\sqrt{\ln x} = \ln\sqrt{x}$

132. $\log x^2 = (\log x)^2$

133. $\log x^3 = (\log x)^2$

134. $\log(\log x) = 1$

135. $\log(\log x) = 2$

136. $\ln(\ln x) = 2$

137. $\ln\left(e^{x^2}\right) = 64$

138. $e^{\ln(x-1)} = 4$

139. $10^{\log(2x+5)} = 9$

140. $\log\sqrt{x^3 - 9} = 2$

141. $\log\sqrt{x^3 - 17} = \frac{1}{2}$

142. $\log_4 x = \log_4 (8 - x)$

143. $\log_7 (x-5) = \log_7 (6x)$

144. $\ln x^2 = \ln(12 - x)$

145. $\log_2 (x+7) + \log_2 x = 3$

146. $\ln x = 1 - \ln(x+2)$

147. $\ln x = 1 + \ln(x+1)$

148. $\log_6 (2x-3) = \log_6 12 - \log_6 3$

149. $\log(3x+2) + \log(x-1) = 1$

150. $\log_5 (x+2) + \log_5 (x-2) = 1$

151. $\log_2 x + \log_2 (x-4) = 2$

152. $\log_3 x + \log_3 (x+6) = 3$

153. $\log_3 (x+3) + \log_3 (x+5) = 1$

154. $\ln x = \frac{1}{2}\ln\left(2x + \frac{5}{2}\right) + \frac{1}{2}\ln 2$

155. $\ln(-4 - x) + \ln 3 = \ln(2 - x)$

156. $\log_4 x + \log_4 (x-2) = \log_4 (15)$

157. $\ln(x-5) - \ln(x+4) = \ln(x-1) - \ln(x+2)$

158. $\ln(4-x) = \ln(x+8) + \ln(2x+13)$

159. $\log\left(x^2 + 4\right) - \log(x+2) = 2 + \log(x-2)$

160. $\log_3 (x-2) = \log_3 27 - \log_3 (x-4) - 5^{\log_5 1}$

161. $\log_2 (x+3) = \log_2 (x-3) + \log_3 9 + 4^{\log_4 3}$

162. $\dfrac{10^x - 10^{-x}}{2} = 20$

163. $\dfrac{10^x + 10^{-x}}{2} = 8$

164. $\dfrac{10^x + 10^{-x}}{10^x - 10^{-x}} = 5$

165. $\dfrac{10^x + 10^{-x}}{10^x - 10^{-x}} = 2$

166. $\dfrac{e^x + e^{-x}}{2} = 15$

167. $\dfrac{e^x - e^{-x}}{2} = 15$

168. $\dfrac{1}{e^x - e^{-x}} = 4$

169. $\dfrac{e^x + e^{-x}}{e^x - e^{-x}} = 3$

170. $\dfrac{e^x - e^{-x}}{e^x + e^{-x}} = 6$

(**171 – 174**) Use common logarithms to solve for ***x*** in terms of ***y***

171. $y = \dfrac{10^x + 10^{-x}}{2}$

172. $y = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$

173. $y = \dfrac{e^x - e^{-x}}{2}$

174. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

175. Solve for t using logarithms with base a: $2a^{t/3} = 5$

176. Solve for t using logarithms with base a: $K = H - Ca^t$