Solution *Section* 1.1 – The Binomial Theorem

Exercise

Find the fifth term in the expansion $\left(x^3+\sqrt{y}\right)^{13}$

Solution

$$\binom{13}{4}\left(x^3\right)^9\left(\sqrt{y}\right)^4=\frac{13!}{4!(13-4)!}x^{27}y^2$$

$$\boxed{=715x^{27}y^2}$$

Exercise

Find the term involving q^{10} in the binomial expansion $\left(\frac{1}{3}p+q^2\right)^{12}$

Solution

Given: $a=\frac{1}{3}p,\quad b=q^2,\quad n=12$

$$q^{10}=\left(q^2\right)^5=b^5$$

$$\binom{n}{k}a^{n-k}b^k=\binom{12}{5}\left(\frac{1}{3}p\right)^{12-5}\left(q^2\right)^5$$

$$=\frac{12!}{5!(12-5)!}\left(\frac{1}{3}p\right)^7q^{10}$$

$$\boxed{=\frac{88}{243}p^7q^{10}}$$

Exercise

Use the binomial theorem to expand and simplify: $(4x-y)^3$

Solution

$$(4x-y)^3=\binom{3}{0}(4x)^3(-y)^0+\binom{3}{1}(4x)^2(-y)^1+\binom{3}{2}(4x)^1(-y)^2+\binom{3}{3}(4x)^0(-y)^3$$

$$=64x^3+3\left(16x^2\right)(-y)+3(4x)y^2-y^3$$

$$\boxed{=64x^3-48x^2y+12xy^2-y^3}$$

Exercise

Use the binomial theorem to expand and simplify: $(x+y)^6$

Solution

$$(x+y)^6 = x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$$

Exercise

Use the binomial theorem to expand and simplify: $(a-b)^6$

Solution

$$(a-b)^6 = a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$$

Exercise

Use the binomial theorem to expand and simplify: $(x-y)^7$

Solution

$$(x-y)^7 = x^7-7x^6y+21x^5y^2-35x^4y^3+35x^3y^4-21x^2y^5+7xy^6-y^7$$

Exercise

Use the binomial theorem to expand and simplify: $(a+b)^8$

Solution

$$(a+b)^8 = a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8$$

Exercise

Use the binomial theorem to expand and simplify: $(3t-5x)^4$

Solution

$$\begin{aligned}(3t-5x)^4 &= (3t)^4+4(3t)^3(-5x)^1+6(3t)^2(-5x)^2+4(3t)^1(-5x)^3+(-5x)^4\\ &\underline{=81t^4-540t^3x+1350t^2x^2-1500tx^3+625x^4}\,\big|\end{aligned}$$

Exercise

Use the binomial theorem to expand and simplify: $\left(\frac{1}{3}x+y^2\right)^5$

Solution

$$\left(\frac{1}{3}x+y^2\right)^5=\left(\frac{1}{3}x\right)^5+5\left(\frac{1}{3}x\right)^4y^2+10\left(\frac{1}{3}x\right)^3\left(y^2\right)^2+10\left(\frac{1}{3}x\right)^2\left(y^2\right)^3+5\frac{1}{3}x\left(y^2\right)^4+\left(y^2\right)^5$$

$$=\frac{1}{243}x^5+\frac{5}{81}x^4y^2+\frac{10}{27}x^3y^4+\frac{10}{9}x^2y^6+\frac{5}{3}xy^8+y^{10}$$

Exercise

Use the binomial theorem to expand and simplify: $\left(\frac{1}{x^2}+3x\right)^6$

Solution

$$\left(\frac{1}{x^2}+3x\right)^6=\left(x^{-2}+3x\right)^6$$

$$=\left(x^{-2}\right)^6+6\left(x^{-2}\right)^5 3x+15\left(x^{-2}\right)^4(3x)^2+20\left(x^{-2}\right)^3(3x)^3$$

$$+15\left(x^{-2}\right)^2(3x)^4+15x^{-2}(3x)^5+(3x)^6$$

$$=x^{-12}+18x^{-9}+135x^{-6}+540x^{-3}+1215+1458x^3+729x^6$$

Exercise

Use the binomial theorem to expand and simplify: $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^5$

Solution

$$\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^5=\left(x^{1/2}+x^{-1/2}\right)^5$$

$$=\left(x^{1/2}\right)^5+5\left(x^{1/2}\right)^4x^{-1/2}+10\left(x^{1/2}\right)^3\left(x^{-1/2}\right)^2+10\left(x^{1/2}\right)^2\left(x^{-1/2}\right)^3$$

$$+5x^{1/2}\left(x^{-1/2}\right)^4+\left(x^{-1/2}\right)^5$$

$$=x^{5/2}+5x^2x^{-1/2}+10x^{3/2}x^{-1}+10xx^{-3/2}+5x^{1/2}x^{-2}+x^{-5/2}$$

$$=x^{5/2}+5x^{3/2}+10x^{1/2}+10x^{-1/2}+5x^{-3/2}+x^{-5/2}$$

Exercise

Expand and simplify: $(2y-3)^4$

Solution

$$(2y-3)^4 = (2y)^4 + 4(2y)^3(-3) + 6(2y)^2(-3)^2 + 4(2y)(-3)^3 + (-3)^4$$

$$\underline{= 16y^4 - 96y^3 + 216y^2 - 216y + 81}\,\big|$$

Exercise

Expand and simplify: $(x+2)^5$

Solution

$$(x+2)^5 = x^5 + 5x^4(2) + 10x^3(2)^2 + 10x^2(2)^3 + 5x(2)^4 + (2)^5$$

$$\underline{= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32}\,\big|$$

Exercise

Expand and simplify: $\left(x^2-y^2\right)^6$

Solution

$$\left(x^2-y^2\right)^6 = \left(x^2\right)^6 + 6\left(x^2\right)^5\left(-y^2\right) + 15\left(x^2\right)^4\left(-y^2\right)^2 + 20\left(x^2\right)^3\left(-y^2\right)^3$$

$$+15\left(x^2\right)^2\left(-y^2\right)^4 + 15\left(x^2\right)\left(-y^2\right)^5 + \left(-y^2\right)^6$$

$$\underline{= x^{12} - 6x^{10}y^2 + 15x^8y^4 - 20x^6y^6 + 15x^4y^8 - 15x^2y^{10} + y^{12}}\,\big|$$

Exercise

Expand and simplify: $(ax-by)^4$

Solution

$$(ax-by)^4 = (ax)^4 + 4(ax)^3(-by) + 6(ax)^2(-by)^2 + 4(ax)(-by)^3 + (-by)^4$$

$$\underline{= a^4x^4 - 4a^3x^3by + 6a^2x^2b^2y^2 - 4axb^3y^3 + b^4y^4}\,\big|$$

Exercise

Expand and simplify: $(ax+by)^5$

Solution

$$(ax+by)^5 = (ax)^5 + 5(ax)^4(by) + 10(ax)^3(by)^2 + 10(ax)^2(by)^3 + 5(ax)(by)^4 + (by)^5$$

$$= a^5x^5 + 5a^4x^4by + 10a^3x^3b^2y^2 + 10a^2x^2b^3y^3 + 5axb^4y^4 + b^5y^5$$

Exercise

Expand and simplify: $\left(\sqrt{x}-\sqrt{3}\right)^4$

Solution

$$\left(\sqrt{x}-\sqrt{3}\right)^4 = \left(\sqrt{x}\right)^4 + 4\left(\sqrt{x}\right)^3\left(-\sqrt{3}\right) + 6\left(\sqrt{x}\right)^2\left(-\sqrt{3}\right)^2 + 4\left(\sqrt{x}\right)\left(-\sqrt{3}\right)^3 + \left(-\sqrt{3}\right)^4$$

$$= x^2 - 4x\sqrt{3x} + 18x^2 - 13\sqrt{3x} + 9$$

Exercise

Expand and simplify: $\left(\sqrt{x}-\sqrt{2}\right)^6$

Solution

$$\left(\sqrt{x}-\sqrt{2}\right)^6 = \left(\sqrt{x}\right)^6 + 6\left(\sqrt{x}\right)^5\left(-\sqrt{2}\right) + 15\left(\sqrt{x}\right)^4\left(-\sqrt{2}\right)^2 + 20\left(\sqrt{x}\right)^3\left(-\sqrt{2}\right)^3$$

$$+15\left(\sqrt{x}\right)^2\left(-\sqrt{2}\right)^4 + 15\left(\sqrt{x}\right)\left(-\sqrt{2}\right)^5 + \left(-\sqrt{2}\right)^6$$

$$= x^3 - 6x^2\sqrt{2x} + 30x^2 - 40x\sqrt{2x} + 60x - 60\sqrt{2x} + 8$$

Exercise

Expand and simplify: $(2x-1)^{12}$

Solution

$$(2x-1)^{12} = (2x)^{12} + 12(2x)^{11}(-1) + 66(2x)^{10}(-1)^2 + 240(2x)^9(-1)^3 + 535(2x)^8(-1)^4$$

$$+812(2x)^7(-1)^5 + 924(2x)^6(-1)^6 + 812(2x)^5(-1)^7 + 535(2x)^4(-1)^8$$

$$+240(2x)^3(-1)^9 + 66(2x)^2(-1)^{10} + 12(2x)(-1)^{11} + (-1)^{12}$$

$$= 4096x^{12} - 24576x^{11} + 67584x^{10} - 122880x^9 + 136960x^8 - 103936x^7$$

$$+ 59136x^6 - 25984x^5 + 8560x^4 - 1920x^3 + 264x^2 - 24x + 1$$

Exercise

Expand and simplify: $\left(x-\frac{1}{x^2}\right)^9$

Solution

$$\left(x-\frac{1}{x^2}\right)^9 = x^9 + 9x^8\left(-\frac{1}{x^2}\right) + 36x^7\left(-\frac{1}{x^2}\right)^2 + 84x^6\left(-\frac{1}{x^2}\right)^3 + 126x^5\left(-\frac{1}{x^2}\right)^4 + 126x^4\left(-\frac{1}{x^2}\right)^5$$

$$+84x^3\left(-\frac{1}{x^2}\right)^6 + 36x^2\left(-\frac{1}{x^2}\right)^7 + 9x\left(-\frac{1}{x^2}\right)^8 + \left(-\frac{1}{x^2}\right)^9$$

$$= x^9 - 9x^6 + 36x^3 - 84 + 126x^{-3} - 126x^{-6} + 84x^{-9} - 36x^{-12} + 9x^{-15} - x^{-18}$$

Exercise

Expand and simplify: $\left(\frac{2}{x}-3y\right)^5$

Solution

$$\left(\frac{2}{x}-3y\right)^5 = \left(\frac{2}{x}\right)^5 + 5\left(\frac{2}{x}\right)^4(-3y) + 10\left(\frac{2}{x}\right)^3(-3y)^2 + 10\left(\frac{2}{x}\right)^2(-3y)^3 + 5\left(\frac{2}{x}\right)(-3y)^4 + (-3y)^5$$

$$= \frac{32}{x^5} - 240\frac{y}{x^4} + 720\frac{y^2}{x^3} - 1{,}080\frac{y^3}{x^2} + 810\frac{y^4}{x} - 243y^5$$

Exercise

Expand and simplify: $\left(3\sqrt{x}+\sqrt[4]{x}\right)^4$

Solution

$$\left(3\sqrt{x}+\sqrt[4]{x}\right)^4 = \left(3\sqrt{x}\right)^4 + 4\left(3\sqrt{x}\right)^3\left(\sqrt[4]{x}\right) + 6\left(3\sqrt{x}\right)^2\left(\sqrt[4]{x}\right)^2 + 4\left(3\sqrt{x}\right)\left(\sqrt[4]{x}\right)^3 + \left(\sqrt[4]{x}\right)^4$$

$$= 81x^2 + 108x^{3/2}x^{1/4} + 54x\sqrt{x} + 12x^{1/2}x^{3/4} + x$$

$$= 81x^2 + 108x^{7/4} + 54x\sqrt{x} + 12x^{5/4} + x$$

$$= 81x^2 + 108x\sqrt[4]{x^3} + 54x\sqrt{x} + 12x\sqrt[4]{x} + x$$

Exercise

Expand and simplify: $(x+1)^5$

Solution

$$(x+1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$$

Exercise

Expand and simplify: $(x-1)^5$

Solution

$$(x-1)^5=x^5-5x^4+10x^3-10x^2+5x-1$$

Exercise

Expand and simplify: $(x-2)^6$

Solution

$$(x-2)^6=x^6-12x^5+60x^4-160x^3+240x^2-192x+64$$

Exercise

Expand and simplify: $\left(\frac{1}{x^3}-2x\right)^5$

Solution

$$\begin{aligned}\left(\frac{1}{x^3}-2x\right)^5&=\frac{1}{x^{15}}-10\frac{x}{x^{12}}+10\frac{4x^2}{x^9}-10\frac{8x^3}{x^6}+5\frac{16x^4}{x^3}-32x^5\\&=\frac{1}{x^{15}}-\frac{10}{x^{11}}+\frac{40}{x^7}-\frac{80}{x^3}+80x-32x^5\end{aligned}$$

Exercise

Expand and simplify: $\left(\frac{1}{x}-2x\right)^6$

Solution

$$\begin{aligned}\left(\frac{1}{x}-2x\right)^6&=\frac{1}{x^6}-6\frac{1}{x^5}(2x)+15\frac{1}{x^4}(2x)^2-20\frac{1}{x^3}(2x)^3+15\frac{1}{x^2}(2x)^4-6\frac{1}{x}(2x)^5+(2x)^6\\&=\frac{1}{x^6}-\frac{12}{x^4}+\frac{60}{x^2}-160+240x^2-192x^4+64x^6\end{aligned}$$

Exercise

Expand and simplify: $\left(x^2-2y\right)^5$

Solution

$$\left(x^2-2y\right)^5=x^{10}-10x^8y+40x^6y^2-80x^4y^3+80x^2y^4-32y^5$$

Exercise

Expand and simplify: $\left(\frac{2}{x}+3\sqrt{x}\right)^4$

Solution

$$\begin{aligned}\left(\frac{2}{x}+3\sqrt{x}\right)^4&=\frac{16}{x^4}+\frac{32}{x^3}\left(3\sqrt{x}\right)+\frac{24}{x^2}(9x)+\frac{8}{x}\left(27x\sqrt{x}\right)+81x^2\\&=\frac{16}{x^4}+\frac{96\sqrt{x}}{x^3}+\frac{216}{x}+216\sqrt{x}+81x^2\end{aligned}$$

Exercise

Expand and simplify: $(2x+5y)^7$

Solution

$$\begin{aligned}(2x+5y)^7&=128x^7+7\left(64x^6\right)(5y)+21\left(32x^5\right)\left(25y^2\right)+35\left(16x^4\right)\left(125y^3\right)\\&\quad+35\left(8x^3\right)\left(625y^4\right)+21\left(4x^2\right)\left(3{,}125y^5\right)+7(2x)\left(5^6y^6\right)+(5y)^7\\&=128x^7+320x^6y+16{,}800x^5y^2+70{,}000x^4y^3+175{,}000x^3y^4+262{,}500x^2y^5\\&\quad+218{,}750xy^6+78{,}125y^7\end{aligned}$$

Exercise

Expand and simplify: $(2x-3)^{11}$

Solution

$$\begin{aligned}(2x-3)^{11}&=(2x)^{11}-33(2x)^{10}+495(2x)^9-4{,}995(2x)^8+350(3)^4(2x)^7-462(3)^5(2x)^6\\&\quad+462(3)^6(2x)^5-350(3)^7(2x)^4+185(3)^8(2x)^3-55(3)^9(2x)^2+22(3)^{10}x-3^{11}\\&=2{,}048x^{11}-33{,}792x^{10}+253{,}440x^9-1{,}278{,}720x^8+3{,}628{,}800x^7-7{,}185{,}024x^6\\&\quad+462(3)^6\,2^5x^5-5{,}600(3)^7x^4+1{,}480(3)^8x^3-220(3)^9x^2+22(3)^{10}x-3^{11}\end{aligned}$$

Exercise

Expand and simplify: $(2x-3y)^6$

Solution

$$(2x-3y)^6 = 64x^6 - 576x^5y + 2{,}160x^4y^2 - 4{,}320x^3y^3 + 4{,}860x^2y^4 - 2{,}196xy^5 + 729y^6$$

Exercise

Expand and simplify: $(2x+3y)^5$

Solution

$$(2x+3y)^5 = 32x^5 + 240x^4y + 720x^3y^2 + 1{,}080x^2y^3 + 810xy^4 + 243y^5$$

Exercise

Expand and simplify: $(3x-2y)^4$

Solution

$$(3x-2y)^4 = 81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4$$

Exercise

Expand and simplify: $\left(x^2+y^3\right)^3$

Solution

$$\left(x^2+y^3\right)^3 = x^6 + 3x^4y^3 + 3x^2y^6 + y^9$$

Exercise

Expand and simplify: $\left(x^2-y^2\right)^3$

Solution

$$\left(x^2-y^2\right)^3 = x^6 - 3x^4y^2 + 3x^2y^4 - y^6$$

Exercise

Expand and simplify: $(2+i)^6$

Solution

$$\begin{aligned}(2+i)^6 &= 64+6(32)i+15(16)i^2+20(8)i^3+15(4)i^4+12i^5+i^6 \\ &= 64+192i-240-160i+60+12i-1 \\ &\underline{= -117+44i}\end{aligned}$$

Exercise

Expand and simplify: $(2-i)^6$

Solution

$$\begin{aligned}(2-i)^6 &= 64-6(32)i+15(16)i^2-20(8)i^3+15(4)i^4-12i^5+i^6 \\ &= 64-192i-240+160i+60-12i-1 \\ &\underline{= -117-44i}\end{aligned}$$

Exercise

Expand and simplify: $\left(\sqrt{2}+i\right)^5$

Solution

$$\begin{aligned}\left(\sqrt{2}+i\right)^5 &= 2\sqrt{2}+20i+20\sqrt{2}\ i^2+20i^3+\sqrt{2}\ i^4+i^5 \\ &= 2\sqrt{2}+20i-20\sqrt{2}-20i+\sqrt{2}+i \\ &\underline{= -17\sqrt{2}+i}\end{aligned}$$

Exercise

Expand and simplify: $(3-i)^4$

Solution

$$\begin{aligned}(3-i)^4 &= 84-108i+54i^2-12i^3+i^4 \\ &= 84-108i-54+12i+1 \\ &\underline{= 31-96i}\end{aligned}$$

Solution *Section* 1.2 – Functions

Exercise

Find the domain: $f(x)=7x+4$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x)=|3x-2|$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x)=3x+\pi$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x)=\sqrt{7}x+\frac{1}{2}$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x)=-2x^2+3x-5$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x) = x^3 - 2x^2 + x - 3$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain: $f(x) = x^2 - 2x - 15$

Solution

Domain: $\mathbb{R}$

Exercise

Find the domain $f(x) = 4 - \frac{2}{x}$

Solution

Domain: $x \neq 0$

Exercise

Find the domain $f(x) = \frac{1}{x^4}$

Solution

Domain: $x \neq 0$

Exercise

Find the domain: $g(x) = \frac{3}{x - 4}$

Solution

Domain: $x \neq 4$

Exercise

Find the domain $y = \frac{2}{x - 3}$

Solution

Domain: $x \neq 3$

Exercise

Find the domain $y = \dfrac{-7}{x-5}$

Solution

Domain: $\underline{x \neq 5\,|}$

Exercise

Find the domain $f(x) = \dfrac{x+5}{2-x}$

Solution

$2 - x \neq 0$

Domain: $\underline{x \neq 2\,|}$

Exercise

Find the domain $f(x) = \dfrac{8}{x+4}$

Solution

$x + 4 \neq 0$

Domain: $\underline{x \neq -4\,|}$

Exercise

Find the domain $f(x) = \dfrac{1}{x+4}$

Solution

Domain: $\underline{x \neq -4\,|}$

Exercise

Find the domain $f(x) = \dfrac{1}{x-4}$

Solution

Domain: $\underline{x \neq 4\,|}$

Exercise

Find the domain $f(x)=\dfrac{3x}{x+2}$

Solution

Domain: $\underline{x\neq -2}$

Exercise

Find the domain $f(x)=x-\dfrac{2}{x-3}$

Solution

Domain: $\underline{x\neq 3}$

Exercise

Find the domain $f(x)=x+\dfrac{3}{x-5}$

Solution

Domain: $\underline{x\neq 5}$

Exercise

Find the domain $f(x)=\dfrac{1}{2}x-\dfrac{8}{x+7}$

Solution

Domain: $\underline{x\neq -7}$

Exercise

Find the domain $f(x)=\dfrac{1}{x-3}-\dfrac{8}{x+7}$

Solution

Domain: $\underline{x\neq -7,\ 3}$

Exercise

Find the domain $f(x)=\dfrac{1}{x+4}-\dfrac{2x}{x-4}$

Solution

Domain: $\underline{x\neq \pm 4}$

Exercise

Fib+cnd the domain $f(x)=\frac{3x^2}{x+3}-\frac{4x}{x-2}$

Solution

Domain: $\boxed{x \neq -3,\ 2}$

Exercise

Find the domain $f(x)=\frac{1}{x^2-2x+1}$

Solution

$x^2-2x+1 \neq 0 \qquad a+b+c=0 \ \rightarrow \ x=1,\ \frac{c}{a}$

Domain: $\boxed{x \neq 1}$

Exercise

Find the domain $f(x)=\frac{x}{x^2+3x+2}$

Solution

$x^2+3x+2 \neq 0 \qquad a-b+c=0 \ \rightarrow \ x=-1,\ -\frac{c}{a}$

Domain: $\boxed{x \neq -1,\ -2}$

Exercise

Find the domain $f(x)=\frac{x^2}{x^2-5x+4}$

Solution

$x^2-5x+4 \neq 0 \qquad a+b+c=0 \ \rightarrow \ x=1,\ \frac{c}{a}$

Domain: $\boxed{x \neq 1,\ 4}$

Exercise

Find the domain $f(x)=\frac{1}{x^2-4x-5}$

Solution

$x^2-4x-5 \neq 0 \qquad a-b+c=0 \ \rightarrow \ x=-1,\ -\frac{c}{a}$

Domain: $\boxed{x \neq -1,\ 5}$

Exercise

Find the domain $g(x)=\dfrac{2}{x^2+x-12}$

Solution

$$x^2+x-12\neq 0$$
$$(x+4)(x-3)\neq 0$$

Domain: $\boxed{x\neq -4,\ 3}$ $\boxed{(-\infty,-4)\cup(-4,3)\cup(3,\infty)}$

Exercise

Find the domain $h(x)=\dfrac{5}{\frac{4}{x}-1}$

Solution

$x\neq 0$ $\dfrac{4}{x}-1\neq 0$

$$\frac{4-x}{x}\neq 0$$
$$4-x\neq 0$$
$$x\neq 4$$

Domain: $\boxed{x\neq 0,\ 4}$ $\boxed{(-\infty,0)\cup(0,4)\cup(4,\infty)}$

Exercise

Find the domain $y=\sqrt{x}$

Solution

$$x\geq 0$$

Domain: $\boxed{x\geq 0}$ $\boxed{[0,\ \infty)}$

Exercise

Find the domain $f(x)=\sqrt{8-3x}$

Solution

$$8-3x\geq 0$$
$$8\geq 3x$$

Domain: $\boxed{x\leq \frac{8}{3}}$ $\boxed{\left(-\infty,\ \frac{8}{3}\right]}$

Exercise

Find the domain $y = \sqrt{4x+1}$

Solution

$$4x+1 \ge 0 \Rightarrow x \ge -\frac{1}{4}$$

Domain: $\boxed{x \ge -\frac{1}{4}}$ $\left[-\frac{1}{4}, \infty\right)$

Exercise

Find the domain $y = \sqrt{7-2x}$

Solution

$$7-2x \ge 0$$

$$-2x \ge -7$$

Domain: $\boxed{x \le \frac{7}{2}}$ $\left(-\infty, \frac{7}{2}\right]$

Exercise

Find the domain $f(x) = \sqrt{8-x}$

Solution

$$8-x \ge 0$$

Domain: $\boxed{x \le -8}$ $(-\infty,\ 8]$

Exercise

Find the domain $f(x) = \sqrt{3-2x}$

Solution

Domain: $\boxed{x \le \frac{3}{2}}$

Exercise

Find the domain $f(x) = \sqrt{3+2x}$

Solution

Domain: $\boxed{x \ge -\frac{3}{2}}$

Exercise

Find the domain $f\left(x\right)=\sqrt{5-x}$

Solution

Domain: $x \le 5$

Exercise

Find the domain $f\left(x\right)=\sqrt{x-5}$

Solution

Domain: $x \ge 5$

Exercise

Find the domain $f\left(x\right)=\sqrt{6-3x}$

Solution

Domain: $x \le 2$

Exercise

Find the domain $f\left(x\right)=\sqrt{3x-6}$

Solution

Domain: $x \ge 2$

Exercise

Find the domain $f(x)=\sqrt{2x+7}$

Solution

Domain: $x \ge -\frac{7}{2}$

Exercise

Find the domain $f(x)=\sqrt{x^2-16}$

Solution

$$x^2-16=0$$

$$x^2 = 16$$
$$x = \pm 4$$

Domain: $\underline{x \le -4 \quad x \ge 4}\,|$

Exercise

Find the domain $\qquad f(x) = \sqrt{16 - x^2}$

Solution

$$x = \pm 4$$

Domain: $\underline{-4 \le x \le 4}\,|$

Exercise

Find the domain $\qquad f(x) = \sqrt{9 - x^2}$

Solution

$$x = \pm 3$$

Domain: $\underline{-3 \le x \le 3}\,|$

Exercise

Find the domain $\qquad f(x) = \sqrt{x^2 - 25}$

Solution

$$x = \pm 5$$

Domain: $\underline{x \le -5 \quad x \ge 5}\,|$

Exercise

Find the domain $\qquad f(x) = \sqrt{x^2 - 5x + 4}$

Solution

$$x^2 - 5x + 4 \qquad\qquad a + b + c = 0 \quad \rightarrow \quad x = 1, \ \frac{c}{a}$$

$$x = 1, \ 4$$

Domain: $\underline{x \le 1 \quad x \ge 4}\,|$

Exercise

Find the domain $f(x)=\sqrt{x^2+5x+4}$

Solution

$x^2+5x+4 \qquad a-b+c=0 \;\rightarrow\; x=-1,\; -\frac{c}{a}$

$x=-1,\; -4$

Domain: $\underline{x \le -4 \quad x \ge -1}\Big|$

Exercise

Find the domain $f(x)=\sqrt{x^2+3x+2}$

Solution

$x^2+3x+2 \qquad a-b+c=0 \;\rightarrow\; x=-1,\; -\frac{c}{a}$

$x=-1,\; -2$

Domain: $\underline{x \le -2 \quad x \ge -1}\Big|$

Exercise

Find the domain $f(x)=\sqrt{x^2-3x+2}$

Solution

$x^2-3x+2 \qquad a+b+c=0 \;\rightarrow\; x=1,\; \frac{c}{a}$

$x=1,\; 2$

Domain: $\underline{x \le 1 \quad x \ge 2}\Big|$

Exercise

Find the domain $f(x)=\sqrt{x-4}+\sqrt{x+1}$

Solution

$x \ge 4 \quad x \ge -1$

Domain: $\underline{x \ge 4}\Big|$

Exercise

Find the domain $f(x)=\sqrt{3-x}+\sqrt{x-2}$

Solution

$x \le 3 \quad x \ge 2$

Domain: $\underline{2 \le x \le 3}\,\big|$

Exercise

Find the domain $f(x)=\sqrt{1-x}+\sqrt{4-x}$

Solution

$x \le 1 \quad x \le 4$

Domain: $\underline{x \le 1}\,\big|$

Exercise

Find the domain $f(x)=\sqrt{1-x}-\sqrt{x-3}$

Solution

$x \le 1 \quad x \ge 3$

Domain: $\underline{\varnothing}\,\big|$

Exercise

Find the domain $f(x)=\sqrt{x+4}-\sqrt{x-1}$

Solution

$x \ge -4 \quad x \ge 1$

Domain: $\underline{x \ge 1}\,\big|$

Exercise

Find the domain $f(x)=\dfrac{\sqrt{x+1}}{x}$

Solution

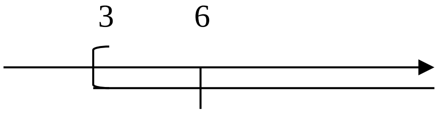
$x+1 \ge 0 \qquad\qquad x \ne 0$

$x \ge -1$

Domain: $\underline{x \ge -1 \quad x \ne 0}\,\big| \qquad \underline{[-1,\ 0) \cup (0,\ \infty)}\,\big|$

Exercise

Find the domain $g(x)=\dfrac{\sqrt{x-3}}{x-6}$



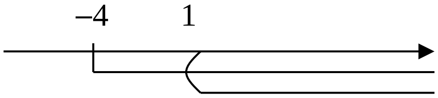
Solution

$$\rightarrow \begin{cases} x \ge 3 \\ x \ne 6 \end{cases}$$

Domain: $x \ge 3 \quad x \ne 6$ $\quad [3,\ 6)\cup(6,\ \infty)$

Exercise

Find the domain $f(x)=\dfrac{\sqrt{x+4}}{\sqrt{x-1}}$



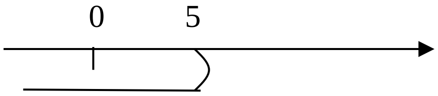
Solution

$$\rightarrow \begin{cases} x \ge -4 \\ x > 1 \end{cases}$$

Domain: $x > 1$ $\quad (1,\ \infty)$

Exercise

Find the domain $f(x)=\dfrac{\sqrt{5-x}}{x}$



Solution

$x \le 5 \quad x \ne 0$

Domain: $x \le 5 \quad x \ne 0$ $\quad (-\infty,\ 0)\cup(0,\ 5]$

Exercise

Find the domain $f(x)=\dfrac{x}{\sqrt{5-x}}$

Solution

Domain: $x < 5$ $\quad (-\infty,\ 5)$

Exercise

Find the domain $f(x)=\dfrac{1}{x\sqrt{5-x}}$

Solution

$x<5 \quad x\neq 0$

Domain: $\underline{x<5 \quad x\neq 0}$

Exercise

Find the domain $f(x)=\dfrac{x+1}{x^3-4x}$

Solution

$x^3-4x\neq 0$

$x\left(x^2-4\right)\neq 0$

Domain: $\underline{x\neq 0, \ \pm 2}$

Exercise

Find the domain $f(x)=\dfrac{\sqrt{x+5}}{x}$

Solution

$x\geq -5 \quad x\neq 0$

Domain: $\underline{x\geq -5 \quad x\neq 0}$

Exercise

Find the domain $f(x)=\dfrac{x}{\sqrt{x+5}}$

Solution

$x>-5$

Domain: $\underline{x>-5}$

Exercise

Find the domain $f(x)=\frac{1}{x\sqrt{x+5}}$

Solution

$x>-5 \quad x\neq 0$

Domain: $x>-5 \quad x\neq 0$

Exercise

Find the domain $f(x)=\frac{x+3}{\sqrt{x-3}}$

Solution

Domain: $x>3$

Exercise

Find the domain $f(x)=\frac{\sqrt{x+3}}{\sqrt{x-3}}$

Solution

$x\geq -3 \quad x>3$

Domain: $x>3$

Exercise

Find the domain $f(x)=\frac{\sqrt{x-2}}{\sqrt{x+2}}$

Solution

$x\geq 2 \quad x>-2$

Domain: $x\geq 2$

Exercise

Find the domain $f(x)=\frac{\sqrt{2-x}}{\sqrt{x+2}}$

Solution

$x\leq 2 \quad x>-2$

Domain: $-2<x\leq 2$

Exercise

Find the domain $f(x)=\dfrac{x-4}{\sqrt{x-2}}$

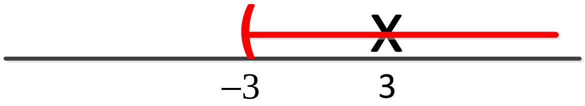
Solution

Domain: $x>2$

Exercise

Find the domain of $f(x)=\dfrac{1}{(x-3)\sqrt{x+3}}$

Solution



$x-3\neq 0 \qquad x+3>0$

$x\neq 3 \qquad x>-3$

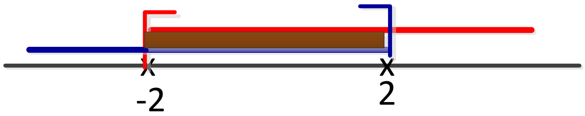
Domain: $\{x \mid x>-3 \ \ and \ \ x\neq 3\}$

$(-3,\ 3)\cup(3,\ \infty)$

Exercise

Find the domain of $f(x)=\sqrt{x+2}+\sqrt{2-x}$

Solution



$x+2\geq 0 \qquad 2-x\geq 0$

$x\geq -2 \qquad -x\geq -2\rightarrow x\leq 2$

Domain: $\{x \mid -2\leq x\leq 2\}$

Exercise

Find the domain of $f(x)=\sqrt{(x-2)(x-6)}$

Solution

$x-2\geq 0 \quad x-6\geq 0$

$x\geq 2 \quad x\geq 6$

Domain: $\{x \mid x\leq 2,\ x\geq 6\}$

2 6

–	+	+
–	–	+
+	–	+

Exercise

Find the domain of $f(x)=\sqrt{x+3}-\sqrt{4-x}$

Solution

$x\ge -3 \quad x\le 4$

Domain: $\underline{-3\le x\le 4}|$

Exercise

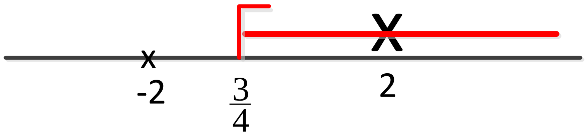
Find the domain of $f(x)=\dfrac{\sqrt{4x-3}}{x^2-4}$

Solution

$$4x-3\ge 0 \qquad x^2-4\ne 0$$

$$4x\ge 3 \qquad x\ne \pm 2$$

$$x\ge \frac{3}{4}$$



Domain: $\left[\frac{3}{4},\ 2\right)\cup(2,\ \infty)$

Exercise

Find the domain of $f(x)=\dfrac{4x}{6x^2+13x-5}$

Solution

$$6x^2+13x-5\ne 0$$

$$x=\frac{-13\pm\sqrt{169+120}}{12}$$

$$=\begin{cases}\dfrac{-13-17}{12}=-\dfrac{5}{2}\\ \dfrac{-13+17}{12}=\dfrac{1}{3}\end{cases}$$

Domain: $\underline{x\ne -\frac{5}{2},\ \frac{1}{3}}|$

Exercise

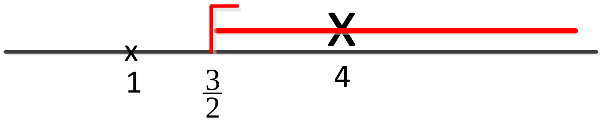
Find the domain of $f(x)=\dfrac{\sqrt{2x-3}}{x^2-5x+4}$

Solution

$$2x-3\ge 0 \qquad x^2-5x+4\ne 0$$

$2x \ge 3 \qquad\qquad x \ne 1,\ 4$

$x \ge \frac{3}{2}$



Domain: $x \ge \frac{3}{2},\ \ x \ne 4$ $\qquad \left[\frac{3}{2},\ 4\right) \cup (4,\ \infty)$

Exercise

Find the domain of $\quad f(x) = \dfrac{x^2}{\sqrt{x^2 - 5x + 4}}$

Solution

$x^2 - 5x + 4 \qquad\qquad a + b + c = 0 \ \rightarrow\ x = 1,\ \frac{c}{a}$

$x = 1,\ \ 4$

Domain: $x < 1 \quad x > 4$

Exercise

Find the domain of $\quad f(x) = \dfrac{x + 2}{\sqrt{x^2 + 5x + 4}}$

Solution

$x^2 + 5x + 4 \qquad\qquad a - b + c = 0 \ \rightarrow\ x = -1,\ \ -\frac{c}{a}$

$x = -1,\ \ -4$

Domain: $x < -4 \quad x > -1$

Exercise

Find the domain of $\quad f(x) = \dfrac{\sqrt{x + 2}}{\sqrt{x^2 + 3x + 2}}$

Solution

$x^2 + 3x + 2 \qquad\qquad a - b + c = 0 \ \rightarrow\ x = -1,\ \ -\frac{c}{a}$

$x < -2 \quad x > -1$

$\sqrt{x + 2} \ \rightarrow\ x \ge -2$

Domain: $x > -1$

Exercise

Find the domain of $f(x)=\dfrac{\sqrt{2x+3}}{x^2-6x+5}$

Solution

$x^2-6x+5 \qquad\qquad a+b+c=0 \;\to\; x=1,\ \dfrac{c}{a}$

$x\neq 1,\ 5$

$\sqrt{2x+3} \;\to\; x\geq -\dfrac{3}{2}$

Domain: $\boxed{x\geq -\dfrac{3}{2} \quad x\neq 1,\ 5}$

Exercise

For the function *f* given by $f(x)=9x+5$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$f(x+h)=9(x+h)+5=9x+9h+5$

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{\overset{f(x+h)}{9x+9h+5}\;-\;\overset{f(x)}{(9x+5)}}{h}\\&=\frac{9x+9h+5\;-\;9x-5}{h}\\&=\frac{9h}{h}\\&\boxed{=9}\end{aligned}$$

Exercise

For the function *f* given by $f(x)=6x+2$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{6(x+h)+2-(6x+2)}{h}\\&=\frac{6x+6h+2-6x-2}{h}\\&=\frac{6h}{h}\\&\boxed{=6}\end{aligned}$$

Exercise

For the function *f* given by $f(x) = 4x + 11$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{4(x+h) + 11 - (4x+11)}{h} \\ &= \frac{4x + 4h + 11 - 4x - 11}{h} \\ &= \frac{4h}{h} \\ &= 4\end{aligned}$$

Exercise

For the function *f* given by $f(x) = 3x - 5$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{3(x+h) - 5 - 3x + 5}{h} \\ &= \frac{3x + 3h - 5 - 3x + 5}{h} \\ &= \frac{3h}{h} \\ &= 3\end{aligned}$$

Exercise

For the function *f* given by $f(x) = -2x - 3$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{-2(x+h) - 3 + 2x + 3}{h} \\ &= \frac{-2x - 2h - 3 + 2x + 3}{h} \\ &= \frac{-2h}{h} \\ &= -2\end{aligned}$$

Exercise

For the function *f* given by $f(x) = -4x + 3$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{-4(x+h)+3+4x-3}{h}\\ &= \frac{-4x-4h+3+4x-3}{h}\\ &= \frac{-4h}{h}\\ &\underline{= -4}\end{aligned}$$

Exercise

For the function *f* given by $f(x) = 3x - 6$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{3(x+h)-6-3x+6}{h}\\ &= \frac{3x+3h-6-3x+6}{h}\\ &= \frac{3h}{h}\\ &\underline{= 3}\end{aligned}$$

Exercise

For the function *f* given by $f(x) = -5x - 7$, find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{-5(x+h)-7+5x+7}{h}\\ &= \frac{-5x-5h-7+5x+7}{h}\\ &= \frac{-5h}{h}\\ &\underline{= -5}\end{aligned}$$

Exercise

Given the function: $f(x)=2x^2$. Find and simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned} f(x+h) &= 2(x+h)^2 \\ &= 2\left(x^2+2hx+h^2\right) \\ &= 2x^2+4hx+2h^2 \end{aligned}$$

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{2x^2+4hx+2h^2-2x^2}{h} \\ &= \frac{4hx+2h^2}{h} \\ &= \frac{4hx}{h}+\frac{2h^2}{h} \\ &= 4x+2h \end{aligned}$$

Exercise

For the function *f* given by $f(x)=5x^2$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{5(x+h)^2-5x^2}{h} \\ &= \frac{5\left(x^2+2hx+h^2\right)-5x^2}{h} \\ &= \frac{5x^2+10hx+5h^2-5x^2}{h} \\ &= \frac{10hx+5h^2}{h} \\ &= 10x+5h \end{aligned}$$

Exercise

For the function *f* given by $f(x)=3x^2-4x$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\frac{f(x+h)-f(x)}{h}=\frac{3(x+h)^2-4(x+h)-3x^2+4x}{h}$$

$$= \frac{3\left(x^2 + 2hx + h^2\right) - 4x - 4h - 3x^2 + 4x}{h}$$

$$= \frac{3x^2 + 6hx + 3h^2 - 4x - 4h - 3x^2 + 4x}{h}$$

$$= \frac{6hx + 3h^2 - 4h}{h}$$

$$\underline{= 6x + 3h - 4\,}|$$

Exercise

For the function *f* given by $f(x) = 2x^2 - 3x$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$f(x+h) = 2(\_\_)^2 - 3(\_\_)$$

$$= 2(x+h)^2 - 3(x+h) \qquad\qquad (a+b)^2 = a^2 + 2ab + b^2$$

$$= 2\left(x^2 + 2xh + h^2\right) - 3x - 3h$$

$$= 2x^2 + 4xh + 2h^2 - 3x - 3h$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\overbrace{2x^2 + 4xh + 2h^2 - 3x - 3h}^{f(x+h)} \; - \; \overbrace{(2x^2 - 3x)}^{f(x)}}{h}$$

$$= \frac{2x^2 + 4xh + 2h^2 - 3x - 3h - 2x^2 + 3x}{h}$$

$$= \frac{4xh + 2h^2 - 3h}{h}$$

$$= \frac{4xh}{h} + \frac{2h^2}{h} - \frac{3h}{h}$$

$$\underline{= 4x + 2h - 3}|$$

Exercise

For the function f given by $f(x) = 2x^2 - x - 3$, find the difference quotient $\frac{f(x+h) - f(x)}{h}$

Solution

$$f(x+h) = 2(x+h)^2 - (x+h) - 3$$

$$= 2\left(x^2 + 2hx + h^2\right) - x - h - 3$$

$$= 2x^2 + 4hx + 2h^2 - x - h - 3$$

$$\frac{f(x+h)-f(x)}{h}=\frac{2x^2+2h^2+4hx-x-h-3-\left(2x^2-x-3\right)}{h}$$

$$=\frac{2x^2+2h^2+4hx-x-h-3-2x^2+x+3}{h}$$

$$=\frac{2h^2+4hx-h}{h}$$

$$=\frac{2h^2}{h}+\frac{4hx}{h}-\frac{h}{h}$$

$$\underline{=2h+4x-1}$$

Exercise

For the given function $f(x)=2x^2-x-3$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\frac{f(x+h)-f(x)}{h}=\frac{2(x+h)^2-(x+h)-3-2x^2+x+3}{h}$$

$$=\frac{2\left(x^2+2hx+h^2\right)-x-h-2x^2+x}{h}$$

$$=\frac{2x^2+4hx+2h^2-h-2x^2}{h}$$

$$=\frac{4hx+2h^2-h}{h}$$

$$\underline{=4x+2h-1}$$

Exercise

For the given function $f(x)=x^2-2x+5$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$\frac{f(x+h)-f(x)}{h}=\frac{(x+h)^2-2(x+h)+5-x^2+2x-5}{h}$$

$$=\frac{x^2+2hx+h^2-2x-2h-x^2+2x}{h}$$

$$=\frac{2hx+h^2-2h}{h}$$

$$\underline{=2x+h-2}$$

Exercise

For the given function $f(x)=3x^2-2x+5$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$
\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{3(x+h)^2-2(x+h)+5-3x^2+2x-5}{h} \\
&= \frac{3\left(x^2+2hx+h^2\right)-2x-2h-3x^2+2x}{h} \\
&= \frac{3x^2+6hx+3h^2-2h-3x^2}{h} \\
&= \frac{6hx+3h^2-2h}{h} \\
&\boxed{= 6x+3h-2}
\end{aligned}
$$

Exercise

For the given function $f(x)=-2x^2-3x+7$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$
\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{-2(x+h)^2-3(x+h)+7+2x^2+3x-7}{h} \\
&= \frac{-2\left(x^2+2hx+h^2\right)-3x-3h+2x^2+3x}{h} \\
&= \frac{-2x^2-4hx-2h^2-3h+2x^2}{h} \\
&= \frac{-4hx-2h^2-3h}{h} \\
&\boxed{= -4x-2h-3}
\end{aligned}
$$

Exercise

For the function *f* given by $f(x)=\sqrt{x-3}$, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

Solution

$$
\frac{f(x+h)-f(x)}{h} \boxed{=\frac{\sqrt{x+h-3}-\sqrt{x-3}}{h}}
$$

Exercise

Let $f(x)=4x-3$ and $g(x)=5x+7$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

Solution

a) $(f+g)(x)=4x-3+5x+7$

$$\boxed{=9x+4}$$

Domain: $\boxed{\mathbb{R}}$

b) $(f-g)(x)=4x-3-(5x+7)$

$$=4x-3-5x-7$$

$$\boxed{=-x-10}$$

Domain: $\boxed{\mathbb{R}}$

c) $(fg)(x)=(4x-3)(5x+7)$

$$\boxed{=20x^2+13x-21}$$

Domain: $\boxed{\mathbb{R}}$

d) $\boxed{\left(\frac{f}{g}\right)(x)=\frac{4x-3}{5x+7}}$

Domain: $\boxed{x\neq-\frac{7}{5}}$

Exercise

Let $f(x)=2x^2+3$ and $g(x)=3x-4$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

Solution

a) $(f+g)(x)=2x^2+3+3x-4$

$$\boxed{=2x^2+3x-1}$$

Domain: $\boxed{\mathbb{R}}$

b) $(f-g)(x)=2x^2+3-(3x-4)$

$$=2x^2+3-3x+4$$

$$= 2x^2 - x + 7$$

Domain: $\mathbb{R}$

c) $(fg)(x) = \left(2x^2 + 3\right)(3x - 4)$

$$= 6x^2 + x - 12$$

Domain: $\mathbb{R}$

d) $\left(\frac{f}{g}\right)(x) = \frac{2x^2 + 3}{3x - 4}$

Domain: $x \neq -\frac{4}{3}$

Exercise

Let $f(x) = x^2 - 2x - 3$ and $g(x) = x^2 + 3x - 2$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

Solution

a) $(f+g)(x) = x^2 - 2x - 3 + x^2 + 3x - 2$

$$= 2x^2 + x - 5$$

Domain: $\mathbb{R}$

b) $(f-g)(x) = x^2 - 2x - 3 - x^2 - 3x + 2$

$$= -5x - 1$$

Domain: $\mathbb{R}$

c) $(fg)(x) = \left(x^2 - 2x - 3\right)\left(x^2 + 3x - 2\right)$

$$= x^4 + 3x^3 - 2x^2 - 2x^3 - 6x^2 + 4x - 3x^2 - 9x + 6$$

$$= x^4 + x^3 - 11x^2 - 5x + 6$$

Domain: $\mathbb{R}$

d) $\left(\frac{f}{g}\right)(x) = \frac{x^2 - 2x - 3}{x^2 + 3x - 2}$

Domain: $x \neq \frac{-3 \pm \sqrt{17}}{2}$

Exercise

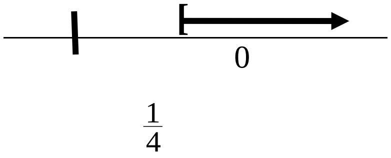
Let $f(x)=\sqrt{4x-1}$ and $g(x)=\frac{1}{x}$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

Solution

a) $(f+g)(x)$

$$(f+g)(x)=\sqrt{4x-1}+\frac{1}{x}$$



$4x-1\geq 0$ $\quad x\neq 0$

$x\geq\frac{1}{4}$

Domain: $\left[\frac{1}{4},\infty\right)$

b) $(f-g)(x)$

$$(f-g)(x)=\sqrt{4x-1}-\frac{1}{x}$$

$4x-1\geq 0$ $\quad x\neq 0$

$x\geq\frac{1}{4}$

Domain: $\left[\frac{1}{4},\infty\right)$

c) $(fg)(x)=\sqrt{4x-1}\left(\frac{1}{x}\right)$

$$=\frac{\sqrt{4x-1}}{x}$$

$4x-1\geq 0$ $\quad x\neq 0$

$x\geq\frac{1}{4}$

Domain: $\left[\frac{1}{4},\infty\right)$

d) $\left(\frac{f}{g}\right)(x)=\frac{\sqrt{4x-1}}{\frac{1}{x}}$ *Domain:* $x\neq 0$

$$=x\sqrt{4x-1}$$

$4x-1\geq 0$

$x\geq\frac{1}{4}$

Domain: $\left[\frac{1}{4},\ \infty\right)$

Exercise

Find $(f+g)(x)$, $(f-g)(x)$, $(f\cdot g)(x)$, *and* $(f/g)(x)$ and the domain of

$f(x)=\sqrt{3-2x}$, $g(x)=\sqrt{x+4}$

Solution

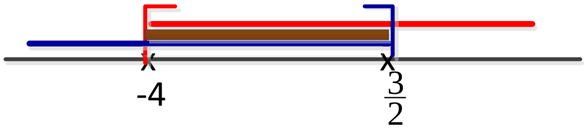
$$f(x)+g(x)=\sqrt{3-2x}+\sqrt{x+4}$$

$3-2x\ge 0 \qquad x+4\ge 0$

$-2x\ge -3 \qquad x\ge -4$

$x\le \frac{3}{2}$

Domain: $\left\{x\,|\,-4\le x\le \frac{3}{2}\right\}$



$$f(x)-g(x)=\sqrt{3-2x}-\sqrt{x+4}$$

$3-2x\ge 0 \qquad x+4\ge 0$

$-2x\ge -3 \qquad x\ge -4$

$x\le \frac{3}{2}$

Domain: $\left\{x\,|\,-4\le x\le \frac{3}{2}\right\}$

$$(f\cdot g)(x)=\left(\sqrt{3-2x}\right)\left(\sqrt{x+4}\right)=\sqrt{(3-2x)(x+4)}=\sqrt{-2x^2-5x+12}$$

$3-2x\ge 0 \qquad x+4\ge 0$

$-2x\ge -3 \qquad x\ge -4$

$x\le \frac{3}{2}$

Domain: $\left\{x\,|\,-4\le x\le \frac{3}{2}\right\}$

$$(f/g)(x)=\frac{\sqrt{3-2x}}{\sqrt{x+4}}\,\frac{\sqrt{x+4}}{\sqrt{x+4}}=\frac{\sqrt{-2x^2-5x+12}}{x+4}$$

$3-2x\ge 0 \qquad x+4>0$

$-2x\ge -3 \qquad x>-4$

$x\le \frac{3}{2}$

Domain: $\left\{x\,|\,-4< x\le \frac{3}{2}\right\}$

$$\boxed{\left(-4,\ \frac{3}{2}\right]}$$

Exercise

Find $\left(f+g\right)(x),\ \left(f-g\right)(x),\ \left(f\cdot g\right)(x),$ *and* $\left(f\,/\,g\right)(x)$ and the domain of

$$f(x)=\frac{2x}{x-4},\quad g(x)=\frac{x}{x+5}$$

Solution

$$\begin{aligned}\left(f+g\right)(x)&=\frac{2x}{x-4}+\frac{x}{x+5}\\&=\frac{2x\left(x+5\right)+x\left(x-4\right)}{\left(x-4\right)\left(x+5\right)}\\&=\frac{2x^2+10x+x^2-4x}{\left(x-4\right)\left(x+5\right)}\\&=\frac{3x^2+6x}{\left(x-4\right)\left(x+5\right)}\end{aligned}$$

$x-4\neq 0 \qquad x+5\neq 0$

$x\neq 4 \qquad x\neq -5$

Domain: $\{x\mid x\neq -5,4\}$ $\qquad \left(-\infty,\ -5\right)\cup\left(-5,\ 4\right)\cup\left(4,\ \infty\right)$

$$\begin{aligned}\left(f-g\right)(x)&=\frac{2x}{x-4}-\frac{x}{x+5}\\&=\frac{2x\left(x+5\right)-x\left(x-4\right)}{\left(x-4\right)\left(x+5\right)}\\&=\frac{2x^2+10x-x^2+4x}{\left(x-4\right)\left(x+5\right)}\\&=\frac{x^2+14x}{\left(x-4\right)\left(x+5\right)}\end{aligned}$$

$x\neq 4 \qquad x\neq -5$

Domain: $\{x\mid x\neq -5,4\}$

$$\left(f\bullet g\right)(x)=\frac{2x}{x-4}\frac{x}{x+5}=\frac{2x^2}{\left(x-4\right)\left(x+5\right)}$$

$x\neq 4 \qquad x\neq -5$

Domain: $\{x\mid x\neq -5,4\}$

$$\left(f\,/\,g\right)(x)=\frac{2x}{x-4}\div\frac{x}{x+5}==\frac{2x}{x-4}\frac{x+5}{x}=2\frac{x+5}{x-4}$$

$x\neq 4 \qquad x\neq -5$

Domain: $\{x\mid x\neq -5,4\}$

Exercise

Given that $f(x)=x+1$ and $g(x)=\sqrt{x+3}$

a) Find $(f+g)(x)$

b) Find the domain of $(f+g)(x)$

c) Find: $(f+g)(6)$

Solution

a) $(f+g)(x)=f(x)+g(x)$

$$=x+1+\sqrt{x+3}$$

b) $x+3\geq 0\rightarrow x\geq -3$

Domain $=[-3,\ \infty)$

c) $(f+g)(6)=6+1+\sqrt{6+3}=10$

Exercise

Given that $f(x)=x^2-4$ and $g(x)=x+2$

a) Find $(f+g)(x)$ and its domain

b) Find $(f/g)(x)$ and its domain

Solution

a) $(f+g)(x)=x^2-4+x+2$

$$=x^2+x-2$$

Domain $=\mathbb{R}$

b) $\frac{f}{g}(x)=\frac{f(x)}{g(x)}=\frac{x^2-4}{x+2}$

Domain: $\underline{x\neq 2}\,|$

Exercise

Let $f(x)=x^2+1$ and $g(x)=3x+5$. Find $(f+g)(1)$, $(f-g)(-3)$, $(fg)(5)$, and $\left(\frac{f}{g}\right)(0)$

Solution

a) $(f+g)(1)=f(1)+g(1)$

$$=1^2+1+3(1)+5$$
$$=1+1+3+5$$
$$\underline{=10}\,|$$

b) $(f-g)(-3) = f(-3) - g(-3)$

$$= (-3)^2 + 1 - \left(3(-3) + 5\right)$$

$$\boxed{= 10}$$

c) $(fg)(5) = f(5)\cdot g(5)$

$$= \left(5^2 + 1\right)\cdot\left(3(5) + 5\right)$$

$$= (26)\cdot(20)$$

$$\boxed{= 520}$$

d) $\left(\dfrac{f}{g}\right)(0) = \dfrac{f(0)}{g(0)}$

$$= \frac{0^2 + 1}{3(0) + 5}$$

$$\boxed{= \frac{1}{5}}$$

Exercise

Find $(f \circ g)(x)$, $(g \circ f)(x)$, $f\left(g(-2)\right)$ *and* $g\left(f(3)\right)$: $f(x) = 2x^2 + 3x - 4$, $g(x) = 2x - 1$

Solution

$$f\left(g(x)\right) = f\left(2x - 1\right)$$

$$= 2\left(2x - 1\right)^2 + 3\left(2x - 1\right) - 4$$

$$= 2\left(4x^2 - 4x + 1\right) + 6x - 3 - 4$$

$$= 8x^2 - 8x + 2 + 6x - 7$$

$$\boxed{= 8x^2 - 2x - 5}$$

$$g\left(f(x)\right) = g\left(2x^2 + 3x - 4\right)$$

$$= 2\left(2x^2 + 3x - 4\right) - 1$$

$$= 4x^2 + 6x - 8 - 1$$

$$\boxed{= 4x^2 + 6x - 9}$$

$$f\left(g(-2)\right) = 8(-2)^2 - 2(-2) - 5$$

$$\boxed{= 31}$$

$$g\left(f(3)\right) = 4(3)^2 + 6(3) - 9$$

$$\boxed{= 45}$$

Exercise

Find $(f\circ g)(x),\quad (g\circ f)(x),\quad f\big(g(-2)\big)\quad$ *and* $g\big(f(3)\big):\ f(x)=x^3+2x^2,\quad g(x)=3x$

Solution

$$\begin{aligned}f\big(g(x)\big) &= f(3x)\\ &=(3x)^3+2(3x)^2\\ &\underline{=27x^3+18x^2\ }\big|\end{aligned}$$

$$\begin{aligned}g\big(f(x)\big) &= g\left(x^3+2x^2\right)\\ &=3\left(x^3+2x^2\right)\\ &\underline{=3x^3+6x^2\ }\big|\end{aligned}$$

$$\begin{aligned}f\big(g(-2)\big) &= 27(-2)^3+18(-2)^2\\ &\underline{=-144\ }\big|\end{aligned}$$

$$\begin{aligned}g\big(f(3)\big) &= 3(3)^3+6(3)^2\\ &\underline{=135\ }\big|\end{aligned}$$

Exercise

Find $(f\circ g)(x),\quad (g\circ f)(x),\quad f\big(g(-2)\big)\quad$ *and* $g\big(f(3)\big):\ f(x)=|x|,\quad g(x)=-7$

Solution

$$\begin{aligned}f\big(g(x)\big) &= f(-7)\\ &=|-7|\\ &\underline{=7\ }\big|\end{aligned}$$

$$\begin{aligned}g\big(f(x)\big) &= g\left(|x|\right)\\ &\underline{=-7\ }\big|\end{aligned}$$

$$f\big(g(-2)\big)\ \underline{=7\ }\big|$$

$$g\big(f(3)\big)\ \underline{=-7\ }\big|$$

Exercise

Given $f(x) = x - 3$ *and* $g(x) = x + 3$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(x+3)$ ***Domain***: $\mathbb{R}$

$= (x-3) + 3$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g(f(x)) = g(x-3)$ ***Domain***: $\mathbb{R}$

$= (x+3) - 3$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = \frac{2}{3}x$ *and* $g(x) = \frac{3}{2}x$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\frac{3}{2}x\right)$ ***Domain***: $\mathbb{R}$

$= \frac{2}{3}\left(\frac{3}{2}x\right)$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g(f(x)) = g\left(\frac{2}{3}x\right)$ ***Domain***: $\mathbb{R}$

$= \frac{3}{2}\left(\frac{2}{3}x\right)$

$= x$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = x - 1$ *and* $g(x) = 3x^2 - 2x - 1$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(3x^2 - 2x - 1\right)$ ***Domain***: $\mathbb{R}$

$= 3(x-1)^2 - 2(x-1) - 1$

$= 3\left(x^2 - 2x + 1\right) - 2x + 2 - 1$

$= 3x^2 - 6x + 3 - 2x + 1$

$= 3x^2 - 8x + 4$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g(f(x)) = g(x-1)$ ***Domain***: $\mathbb{R}$

$= 3x^2 - 2x - 1 - 1$

$= 3x^2 - 2x - 2$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = 3x - 2$ *and* $g(x) = x^2 - 5$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(x^2 - 5\right)$ ***Domain***: $\mathbb{R}$

$= 3\left(x^2 - 5\right) - 2$

$= 3x^2 - 15 - 2$

$= 3x^2 - 17$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g(f(x)) = g(3x-2)$ ***Domain***: $\mathbb{R}$

$= (3x-2)^2 - 5$

$= 9x^2 - 12x + 4 - 5$

$\underline{= 9x^2 - 12x - 1}$ **Domain**: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}$

Exercise

Given $f(x) = x^2 - 2$ *and* $g(x) = 4x - 3$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(4x - 3)$ **Domain**: $\mathbb{R}$

$= (4x - 3)^2 - 2$

$= 16x^2 - 24x + 9 - 2$

$\underline{= 16x^2 - 24x + 7}$ **Domain**: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}$

b) $g(f(x)) = g(x^2 - 2)$ **Domain**: $\mathbb{R}$

$= 4(x^2 - 2) - 3$

$= 4x^2 - 8 - 3$

$\underline{= 4x^2 - 11}$ **Domain**: $\mathbb{R}$

Domain: $\underline{\mathbb{R}}$

Exercise

Given $f(x) = 4x^2 - x + 10$ *and* $g(x) = 2x - 7$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f(2x - 7)$ **Domain**: $\mathbb{R}$

$= 4(2x - 7)^2 - (2x - 7) + 10$

$= 4(4x^2 - 28x + 49) - 2x + 7 + 10$

$= 16x^2 - 112x + 196 - 2x + 17$

$\boxed{= 16x^2 - 114x + 213}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

b) $g\left(f(x)\right) = g\left(4x^2 - x + 10\right)$ ***Domain***: $\mathbb{R}$

$= 2\left(4x^2 - x + 10\right) - 7$

$= 8x^2 - 2x + 20 - 7$

$\boxed{= 8x^2 - 2x + 13}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x) = \sqrt{x}$ *and* $g(x) = x + 3$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(x + 3\right)$ ***Domain***: $\mathbb{R}$

$\boxed{= \sqrt{x + 3}}$ ***Domain***: $x \ge -3$

Domain: $\boxed{x \ge -3}$

b) $g\left(f(x)\right) = g\left(\sqrt{x}\right)$ ***Domain***: $x \ge 0$

$\boxed{= \sqrt{x} + 3}$ ***Domain***: $x \ge 0$

Domain: $\boxed{x \ge 0}$

Exercise

Given $f(x) = \sqrt{x}$ *and* $g(x) = 2 - 3x$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right) = f\left(2 - 3x\right)$ ***Domain***: $\mathbb{R}$

$\boxed{= \sqrt{2 - 3x}}$ ***Domain***: $x \le \frac{2}{3}$

Domain: $\boxed{x \le \frac{2}{3}}$

b) $g\left(f(x)\right)=g\left(\sqrt{x}\right)$ **Domain**: $x\geq 0$

$=2-3\sqrt{x}$ **Domain**: $x\geq 0$

Domain: $x\geq 0$

Exercise

Given $f(x)=3x+2$ *and* $g(x)=\sqrt{x}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f\left(g(x)\right)=f\left(\sqrt{x}\right)$ **Domain**: $x\geq 0$

$=3\sqrt{x}+2$ **Domain**: $x\geq 0$

Domain: $x\geq 0$

b) $g\left(f(x)\right)=g(3x+2)$ **Domain**: $\mathbb{R}$

$=\sqrt{3x+2}$ **Domain**: $x\geq -\frac{2}{3}$

Domain: $x\geq -\frac{2}{3}$

Exercise

Given $f(x)=x^4$ *and* $g(x)=\sqrt[4]{x}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f\left(g(x)\right)=f\left(\sqrt[4]{x}\right)$ **Domain**: $x\geq 0$

$=\left(\sqrt[4]{x}\right)^4$

$=x$ **Domain**: $\mathbb{R}$

Domain: $x\geq 0$

b) $g\left(f(x)\right)=g\left(x^4\right)$ **Domain**: $\mathbb{R}$

$=\sqrt[4]{x^4}$

$=x$ **Domain**: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = x^n$ *and* $g(x) = \sqrt[n]{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\sqrt[n]{x}\right)$ **Domain**: $\begin{cases} \textit{If } n \text{ is } \textit{even} & x \geq 0 \\ \textit{If } n \text{ is } \textit{odd} & \mathbb{R} \end{cases}$

$$= \left(\sqrt[n]{x}\right)^n$$

$= x$ **Domain**: $\mathbb{R}$

Domain: $\begin{cases} \textit{If } n \text{ is } \textit{even} & x \geq 0 \\ \textit{If } n \text{ is } \textit{odd} & \mathbb{R} \end{cases}$

b) $g(f(x)) = g\left(x^n\right)$ **Domain**: $\mathbb{R}$

$$= \sqrt[n]{x^n}$$

$= x$ **Domain**: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = x^2 - 3x$ *and* $g(x) = \sqrt{x+2}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\sqrt{x+2}\right)$ $x + 2 \geq 0 \Rightarrow x \geq -2$

$$= \left(\sqrt{x+2}\right)^2 - 3\sqrt{x+2}$$

$= x + 2 - 3\sqrt{x+2}$ $x + 2 \geq 0 \Rightarrow x \geq -2$

Domain: $\{x \mid x \geq -2\}$

b) $g(f(x)) = g\left(x^2 - 3x\right)$ $\mathbb{R}$

$\underline{= \sqrt{x^2 - 3x + 2}}$ $\qquad$ $x^2 - 3x + 2 \ge 0 \Rightarrow (x = 1,\ 2) \leftrightarrow x \le 1,\ x \ge 2$

Domain: $\{x \mid x \le 1,\ x \ge 2\}$

Exercise

Given $f(x) = \sqrt{x - 2}$ *and* $g(x) = \sqrt{x + 5}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\sqrt{x+5}\right)$ $\qquad$ $x + 5 \ge 0 \Rightarrow x \ge -5$

$= \sqrt{\sqrt{x+5} - 2}$ $\qquad$ $\sqrt{x+5} - 2 \ge 0 \Rightarrow \sqrt{x+5} \ge 2$

$x + 5 \ge 4$

$x \ge -1$

Domain: $\{x \mid x \ge -1\}$

b) $g(f(x)) = g\left(\sqrt{x-2}\right)$ $\qquad$ $x - 2 \ge 0 \Rightarrow x \ge 2$

$= \sqrt{\sqrt{x-2} + 5}$ $\qquad$ $\sqrt{x-2} + 5 \ge 0 \Rightarrow \sqrt{x-2} \ge -5$ *Always true when* $x \ge 2$

Domain: $\{x \mid x \ge 2\}$

Exercise

Given $f(x) = x^2 + 2$ *and* $g(x) = \sqrt{3 - x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\sqrt{3-x}\right)$ $\qquad$ **Domain**: $x \le 3$

$= \left(\sqrt{3-x}\right)^2 + 2$

$= 3 - x + 2$

$\underline{= 5 - x}$ $\qquad$ **Domain**: $\mathbb{R}$

Domain: $\underline{x \le 3}$

b) $g(f(x)) = g\left(x^2 + 2\right)$ $\qquad$ **Domain**: $\mathbb{R}$

$$=\sqrt{3-x^2-2}$$

$$\boxed{=\sqrt{1-x^2}}$$ **Domain**: $-1\le x\le 1$

Domain: $\boxed{-1\le x\le 1}$

Exercise

Given $f(x)=x^5-2$ *and* $g(x)=\sqrt[5]{x+2}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\sqrt[5]{x+2}\right)$ **Domain**: $\mathbb{R}$

$$=\left(\sqrt[5]{x+2}\right)^5-2$$

$$=x+2-2$$

$$\boxed{=x}$$ **Domain**: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

b) $g(f(x))=g\left(x^5-2\right)$ **Domain**: $\mathbb{R}$

$$=\sqrt[5]{x^5-2+2}$$

$$=\sqrt[5]{x^5}$$

$$\boxed{=x}$$ **Domain**: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x)=1-x^2$ *and* $g(x)=\sqrt{x^2-25}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\sqrt{x^2-25}\right)$ **Domain**: $x\le -5 \quad x\ge 5$

$$=1-\left(\sqrt{x^2-25}\right)^2$$

$$=1-\left(x^2-25\right)$$

$$=1-x^2+25$$

$$\boxed{=26-x^2}$$ **Domain**: $\mathbb{R}$

Domain: $\boxed{x\le -5 \quad x\ge 5}$

b) $g\left(f(x)\right)=g\left(1-x^2\right)$ **Domain**: $\mathbb{R}$

$$=\sqrt{\left(1-x^2\right)^2-25}$$

$$=\sqrt{1-2x^2+x^4-25}$$

$$\boxed{=\sqrt{x^4-2x^2-24}}$$

$$x^2=\frac{2\pm\sqrt{4+96}}{2}$$

$$=\begin{cases}\frac{2-10}{2}=-4 \text{✗}\\ \frac{2+10}{2}=6\end{cases}$$

$x^2=6 \quad\to\quad \boxed{x=\pm\sqrt{6}}$ **Domain**: $x\le -\sqrt{6} \quad x\ge\sqrt{6}$

Domain: $\boxed{x\le -\sqrt{6} \quad x\ge\sqrt{6}}$

Exercise

Given $f(x)=2x+3$ *and* $g(x)=\frac{x-3}{2}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f\left(g(x)\right)=f\left(\frac{x-3}{2}\right)$ **Domain**: $\mathbb{R}$

$$=2\left(\frac{x-3}{2}\right)+3$$

$$=x-3+3$$

$$\boxed{=x}$$ **Domain**: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

b) $g\left(f(x)\right)=g(2x+3)$ **Domain**: $\mathbb{R}$

$$=\frac{1}{2}(2x+3-3)$$

$\boxed{= x}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x) = 4x - 5$ *and* $g(x) = \dfrac{x+5}{4}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\dfrac{x+5}{4}\right)$ ***Domain***: $\mathbb{R}$

$$= 4\left(\frac{x+5}{4}\right) - 5$$

$$= x + 5 - 5$$

$\boxed{= x}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

b) $g(f(x)) = g(4x - 5)$ ***Domain***: $\mathbb{R}$

$$= \frac{1}{4}(4x - 5 + 5)$$

$\boxed{= x}$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x) = \dfrac{4}{1-5x}$ *and* $g(x) = \dfrac{1}{x}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\dfrac{1}{x}\right)$ ***Domain***: $x \neq 0$

$$= \frac{4}{1 - 5\frac{1}{x}}$$

$\boxed{= \dfrac{4x}{x-5}}$ ***Domain***: $x \neq 5$

Domain: $\boxed{x \neq 0,\ 5}$

b) $g(f(x)) = g\left(\dfrac{4}{1-5x}\right)$ ***Domain***: $x \neq \dfrac{1}{5}$

$$\boxed{=\frac{1-5x}{4}}$$ **Domain**: $\mathbb{R}$

Domain: $\boxed{x \neq \frac{1}{5}}$

Exercise

Given $f(x)=\frac{1}{x-2}$ *and* $g(x)=\frac{x+2}{x}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f\left(g(x)\right)=f\left(\frac{x+2}{x}\right)$ **Domain**: $x\neq 0$

$$=\frac{1}{\frac{x+2}{x}-2}$$

$$=\frac{1}{\frac{x+2-2x}{x}}$$

$$\boxed{=\frac{x}{2-x}}$$ **Domain**: $x\neq 2$

Domain: $\boxed{x\neq 0,\ 2}$ $(-\infty,\ 0)\cup(0,\ 2)\cup(2,\ \infty)$

b) $g\left(f(x)\right)=g\left(\frac{1}{x-2}\right)$ **Domain**: $x\neq 2$

$$=\frac{\frac{1}{x-2}+2}{\frac{1}{x-2}}$$

$$=\frac{\frac{1+2x-4}{x-2}}{\frac{1}{x-2}}$$

$$\boxed{=2x-3}$$ **Domain**: $\mathbb{R}$

Domain: $\boxed{x\neq 2}$

$$\boxed{(-\infty,\ 2)\cup(2,\ \infty)}$$

Exercise

Given $f(x)=\dfrac{3x+5}{2}$ *and* $g(x)=\dfrac{2x-5}{3}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f\left(g(x)\right)=f\left(\dfrac{2x-5}{3}\right)$ **Domain**: $\mathbb{R}$

$$=\frac{3\frac{2x-5}{3}+5}{2}$$

$$=\frac{2x-5+5}{2}$$

$$=\frac{2x}{2}$$

$$\underline{=x\,|}$$

Domain: $\underline{\mathbb{R}\,|}$

b) $g\left(f(x)\right)=g\left(\dfrac{3x+5}{2}\right)$ **Domain**: $\mathbb{R}$

$$=\frac{2\frac{3x+5}{2}-5}{3}$$

$$=\frac{3x+5-5}{3}$$

$$=\frac{3x}{3}$$

$$\underline{=x\,|}$$

Domain: $\underline{\mathbb{R}\,|}$

Exercise

Given $f(x)=\dfrac{1}{1+x}$ *and* $g(x)=\dfrac{1-x}{x}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f\left(g(x)\right)=f\left(\dfrac{1-x}{x}\right)$ **Domain**: $x\neq 0$

$$=\frac{1}{1+\frac{1-x}{x}}$$

$$=\frac{x}{x+1-x}$$

$$\boxed{=x}$$ **Domain**: $\mathbb{R}$

Domain: $\boxed{x \neq 0}$

b) $g\left(f(x)\right)=g\left(\frac{1}{x+1}\right)$ **Domain**: $x \neq -1$

$$=\frac{1-\frac{1}{x+1}}{\frac{1}{x+1}}$$

$$=x+1-1$$

$$\boxed{=x}$$ **Domain**: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x)=\frac{x-1}{x-2}$ *and* $g(x)=\frac{x-3}{x-4}$

a) Find $\left(f\circ g\right)(x)$ and the domain of $f\circ g$

b) Find $\left(g\circ f\right)(x)$ and the domain of $g\circ f$

Solution

a) $f\left(g(x)\right)=f\left(\frac{x-3}{x-4}\right)$ **Domain**: $x \neq 4$

$$=\frac{\frac{x-3}{x-4}-1}{\frac{x-3}{x-4}-2}$$

$$=\frac{\frac{x-3-(x-4)}{x-4}}{\frac{x-3-2(x-4)}{x-4}}$$

$$=\frac{x-3+x+4}{x-3-2x+8}$$

$$\boxed{=\frac{2x+1}{-x+5}}$$ **Domain**: $x \neq 5$

Domain: $\{x \mid x \neq 4,\ 5\}$

b) $g\left(f(x)\right)=g\left(\frac{x-1}{x-2}\right)$ **Domain**: $x \neq 2$

$$=\frac{\frac{x-1}{x-2}-3}{\frac{x-1}{x-2}-4}$$

$$=\frac{x-1-3(x-2)}{x-1-4(x-2)}$$

$$=\frac{x-1-3x+6}{x-1-4x+8}$$

$$\boxed{=\frac{-2x+5}{-3x+7}} \qquad \textbf{Domain}:\ x\neq\frac{7}{3}$$

Domain: $\left\{x\mid x\neq 2,\ \frac{7}{3}\right\}$

Exercise

Given $f(x)=\frac{6}{x-3}$ *and* $g(x)=\frac{1}{x}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $(f\circ g)(x)$

$$f(g(x))=f\left(\frac{1}{x}\right) \qquad \textbf{Domain}:\ x\neq 0$$

$$=\frac{6}{\frac{1}{x}-3}$$

$$=\frac{6}{\frac{1-3x}{x}}$$

$$=\frac{6x}{1-3x} \qquad \textbf{Domain}:\ x\neq\frac{1}{3}$$

Domain: $\boxed{x\neq 0,\ \frac{1}{3}}$ $\qquad \boxed{(-\infty,0)\cup\left(0,\frac{1}{3}\right)\cup\left(\frac{1}{3},\infty\right)}$

b) $(g\circ f)(x)$

$$g(f(x))=g\left(\frac{6}{x-3}\right) \qquad \textbf{Domain}:\ x\neq 3$$

$$=\frac{1}{\frac{6}{x-3}}$$

$$=\frac{x-3}{6} \qquad \textbf{Domain}:\ (-\infty,\infty)$$

Domain: $\boxed{x\neq 3}$ $\qquad \boxed{(-\infty,3)\cup(3,\infty)}$

Exercise

Given $f(x)=\frac{6}{x}$ *and* $g(x)=\frac{1}{2x+1}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\frac{1}{2x+1}\right)$ ***Domain***: $x\neq -\frac{1}{2}$

$$=\frac{6}{\frac{1}{2x+1}}$$

$$\boxed{=12x+6}$$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{x\neq -\frac{1}{2}}$

b) $g(f(x))=g\left(\frac{6}{x}\right)$ ***Domain***: $x\neq 0$

$$=\frac{1}{2\frac{6}{x}+1}$$

$$\boxed{=\frac{x}{12+x}}$$ ***Domain***: $x\neq -12$

Domain: $\boxed{x\neq -12,\ 0}$

Exercise

Given $f(x)=3x-7$ *and* $g(x)=\frac{x+7}{3}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f(g(x))=f\left(\frac{x+7}{3}\right)$ ***Domain***: $\mathbb{R}$

$$=3\frac{x+7}{3}-7$$

$$=x+7-7$$

$$\boxed{=x}$$ ***Domain***: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

b) $g(f(x))=g(3x-7)$ ***Domain***: $\mathbb{R}$

$$=\frac{3x-7+7}{3}$$

$= x$ **Domain**: $\mathbb{R}$

Domain: $\mathbb{R}$

Exercise

Given $f(x) = \dfrac{2x+3}{x-4}$ *and* $g(x) = \dfrac{4x+3}{x-2}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(\dfrac{4x+3}{x-2}\right)$ **Domain**: $x \neq 2$

$$= \frac{2\dfrac{4x+3}{x-2}+3}{\dfrac{4x+3}{x-2}-4}$$

$$= \frac{8x+6+3x-6}{4x+3-4x+8}$$

$$= \frac{11x}{11}$$

$= x$ **Domain**: $\mathbb{R}$

Domain: $x \neq 2$

b) $g(f(x)) = g\left(\dfrac{2x+3}{x-4}\right)$ **Domain**: $x \neq 4$

$$= \frac{4\dfrac{2x+3}{x-4}+3}{\dfrac{2x+3}{x-4}-2}$$

$$= \frac{8x+12+3x-4}{2x+3-2x+8}$$

$$= \frac{11x}{11}$$

$= x$ **Domain**: $\mathbb{R}$

Domain: $x \neq 4$

Exercise

Given $f(x) = \dfrac{2x+3}{x+4}$ *and* $g(x) = \dfrac{-4x+3}{x-2}$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f\left(g(x)\right)=f\left(\frac{-4x+3}{x-2}\right)$ ***Domain***: $x\neq 2$

$$=\frac{2\frac{-4x+3}{x-2}+3}{\frac{4x+3}{x-2}+4}$$

$$=\frac{-8x+6+3x-6}{4x+3+4x-8}$$

$$=\frac{-5x}{-5}$$

$=x$ ***Domain***: $\mathbb{R}$

Domain: $x\neq 2$

b) $g\left(f(x)\right)=g\left(\frac{2x+3}{x+4}\right)$ ***Domain***: $x\neq -4$

$$=\frac{-4\frac{2x+3}{x+4}+3}{\frac{2x+3}{x+4}-2}$$

$$=\frac{-8x-12+3x+12}{2x+3-2x-8}$$

$$=\frac{-5x}{-5}$$

$=x$ ***Domain***: $\mathbb{R}$

Domain: $x\neq -4$

Exercise

Given $f(x)=x+1$ *and* $g(x)=x^3-5x^2+3x+7$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $f\left(g(x)\right)=f\left(x^3-5x^2+3x+7\right)$ ***Domain***: $\mathbb{R}$

$$=x^3-5x^2+3x+7+1$$

$=x^3-5x^2+3x+8$ ***Domain***: $\mathbb{R}$

Domain: $\mathbb{R}$

b) $g\left(f(x)\right)=g\left(x+1\right)$ ***Domain***: $\mathbb{R}$

$$=(x+1)^3-5(x+1)^2+3(x+1)+7$$

$$=x^3+3x^2+3x+1-5\left(x^2+2x+1\right)+3x+3+7$$

$= x^3 + 3x^2 + 6x + 11 - 5x^2 - 10x - 5$

$\boxed{= x^3 - 2x^2 - 4x + 6}$ **Domain**: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x) = x - 1$ *and* $g(x) = x^3 + 2x^2 - 3x - 9$

a) Find $(f \circ g)(x)$ and the domain of $f \circ g$

b) Find $(g \circ f)(x)$ and the domain of $g \circ f$

Solution

a) $f(g(x)) = f\left(x^3 + 2x^2 - 3x - 9\right)$ **Domain**: $\mathbb{R}$

$= x^3 + 2x^2 - 3x - 9 - 1$

$\boxed{= x^3 + 2x^2 - 3x - 10}$ **Domain**: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

b) $g(f(x)) = g(x - 1)$ **Domain**: $\mathbb{R}$

$= (x-1)^3 + 2(x-1)^2 - (x-1) - 9$

$= x^3 - 3x^2 + 3x - 1 + 2\left(x^2 - 2x + 1\right) - 3x + 3 - 9$

$= x^3 - 3x^2 - 7 + 2x^2 - 4x + 2$

$\boxed{= x^3 - x^2 - 4x - 5}$ **Domain**: $\mathbb{R}$

Domain: $\boxed{\mathbb{R}}$

Exercise

Given $f(x) = \sqrt{x}$ and $g(x) = x + 3$, find $(f \circ g)(x)$, $(g \circ f)(x)$ and their domain.

Solution

$(f \circ g)(x) = f(g(x))$

$= f(x + 3)$ **Domain**: $(-\infty, \infty)$

$= \sqrt{x + 3}$

$x + 3 \geq 0 \quad \Rightarrow \quad x \geq -3$

Domain: $\boxed{x \geq -3}$

$(g \circ f)(x) = g(f(x))$

$$= g\left(\sqrt{x}\right) \qquad \textbf{\textit{Domain}}: \ x \ge 0$$

$$= \sqrt{x} + 3$$

Domain: $\underline{x \ge 0}\,|$

Exercise

Given that $f(x) = \sqrt{x}$ and $g(x) = 2 - 3x$, find $(f \circ g)(x)$, $(g \circ f)(x)$ and their domain.

Solution

$$(f \circ g)(x) = f(g(x))$$

$$= f(2 - 3x) \qquad \textbf{\textit{Domain}}: \ (-\infty, \infty)$$

$$\underline{= \sqrt{2 - 3x}}\,|$$

$$2 - 3x \ge 0 \to -3x \ge -2 \Rightarrow \boxed{x \le \frac{2}{3}}$$

Domain: $\left(-\infty, \ \frac{2}{3}\right]$

$$g(f(x)) = g\left(\sqrt{x}\right) \qquad \textbf{\textit{Domain}}: \ x \ge 0$$

$$= 2 - 3\sqrt{x}$$

$$x \ge 0$$

Domain: $[0, \infty)$

Exercise

Given that $f(x) = \dfrac{1}{x-2}$ and $g(x) = \dfrac{x+2}{x}$, find $(f \circ g)(x)$, $(g \circ f)(x)$ and their domain.

Solution

$$f(g(x)) = f\left(\frac{x+2}{x}\right) \qquad \textbf{\textit{Domain}}: \ \boxed{x \ne 0}$$

$$= \frac{1}{\frac{x+2}{x} - 2}$$

$$= \frac{1}{\frac{x+2-2x}{x}}$$

$$= \frac{x}{2-x} \qquad \textbf{\textit{Domain}}: \ \boxed{x \ne 2}$$

Domain: $\underline{x \ne 0, \ 2}\,|$

$g\left(f(x)\right) = g\left(\frac{1}{x-2}\right)$ **Domain**: $\boxed{x \neq 2}$

$$= \frac{\frac{1}{x-2}+2}{\frac{1}{x-2}} \quad \underline{(-\infty,0) \cup (0,2) \cup (2,\infty)}$$

$$= \frac{\frac{1+2x-4}{x-2}}{\frac{1}{x-2}}$$

$= 2x - 3$ **Domain**: $\boxed{\mathbb{R}}$

Domain: $\underline{x \neq 2}$

Exercise

Given that $f(x) = 2x - 5$ and $g(x) = x^2 - 3x + 8$, find $(f \circ g)(x)$, $(g \circ f)(x)$ and their domain then find $(f \circ g)(7)$

Solution

$f\left(g(x)\right) = f\left(x^2 - 3x + 8\right)$ *Domain:* $(-\infty, \infty)$

$= 2(\text{------------}) - 5$

$= 2\left(2x^2 - 3x + 8\right) - 5$

$= 2x^2 - 6x + 16 - 5$

$= 2x^2 - 6x + 11$ *Domain:* $(-\infty, \infty)$

Domain: $\underline{\mathbb{R}}$

$g\left(f(x)\right) = g(2x - 5)$ *Domain:* $(-\infty, \infty)$

$= (---)^2 - 3(---) + 8$

$= (2x-5)^2 - 3(2x-5) + 8$

$= 4x^2 - 20x + 25 - 6x + 15 + 8$

$= 4x^2 - 26x + 48$ *Domain:* $(-\infty, \infty)$

Domain: $\underline{\mathbb{R}}$

$f\left(g(7)\right) = 2(7)^2 - 6(7) + 11$

$\underline{= 67}$

Exercise

Given that $f(x)=\sqrt{x}$ and $g(x)=x-1$, find

a) $(f\circ g)(x)=f(g(x))$ *b)* $(g\circ f)(x)=g(f(x))$ *c)* $(f\circ g)(2)=f(g(2))$

Solution

a) $$\begin{aligned}(f\circ g)(x)&=f(g(x))\\&=f(x-1)\\&\boxed{=\sqrt{x-1}}\end{aligned}$$

b) $$\begin{aligned}(g\circ f)(x)&=g(f(x))\\&=g(\sqrt{x})\\&\boxed{=\sqrt{x}-1}\end{aligned}$$

c) $$\begin{aligned}(f\circ g)(2)&=f(g(2))\\&=\sqrt{2-1}\\&\boxed{=1}\end{aligned}$$

$$=\sqrt{x-1}$$

Exercise

Given that $f(x)=\dfrac{x}{x+5}$ and $g(x)=\dfrac{6}{x}$, find

a) $(f\circ g)(x)=f(g(x))$ *b)* $(g\circ f)(x)=g(f(x))$ *c)* $(f\circ g)(2)=f(g(2))$

Solution

a) $$\begin{aligned}(f\circ g)(x)&=f(g(x))\\&=f\left(\frac{6}{x}\right)\\&=\frac{\frac{6}{x}}{\frac{6}{x}+5}\\&=\frac{\frac{6}{x}}{\frac{6+5x}{x}}\\&\boxed{=\frac{6}{6+5x}}\end{aligned}$$

b) $$\begin{aligned}(g\circ f)(x)&=g(f(x))\\&=g\left(\frac{x}{x+5}\right)\\&=\frac{6}{\frac{x}{x+5}}\end{aligned}$$

$$\boxed{=\frac{6(x+5)}{x}}$$

c) $(f\circ g)(2)=f(g(2))$

$$=\frac{6}{6+5(2)}\quad=\frac{6}{16}$$

$$\boxed{=\frac{3}{8}}$$

Exercise

Determine whether f is even, odd, or neither: $f(x)=3x^4+2x^2-5$

Solution

$$f(-x)=3(-x)^4+2(-x)^2-5$$
$$=3x^4+2x^2-5$$
$$=f(x)$$

$\therefore$ The function is ***even***.

Exercise

Determine whether f is even, odd, or neither: $f(x)=8x^3-3x^2$

Solution

$$f(-x)=8(-x)^3-3(-x)^2$$
$$=-8x^3-3x^2$$

$\therefore$ The function is ***neither***.

Exercise

Determine whether f is even, odd, or neither: $f(x)=\sqrt{x^2+4}$

Solution

$$f(-x)=\sqrt{(-x)^2+4}$$
$$=\sqrt{x^2+4}$$
$$=f(x)$$

$\therefore$ The function is ***even***.

Exercise

Determine whether f is even, odd, or neither: $f(x)=3x^2-5x+1$

Solution

$$f(-x)=3(-x)^2-5(-x)+1$$
$$=3x^2+5x+1$$

$\therefore$ The function is ***neither***.

Exercise

Determine whether f is even, odd, or neither: $f(x)=\sqrt[3]{x^3-x}$

Solution

$$f(-x)=\sqrt[3]{(-x)^3-(-x)}$$
$$=\sqrt[3]{-x^3+x}$$
$$=\sqrt[3]{-\left(x^3-x\right)}$$
$$=-\sqrt[3]{x^3-x}$$
$$=-f(x)$$

$\therefore$ The function is ***odd***.

Exercise

Determine whether f is even, odd, or neither: $f(x)=|x|-3$

Solution

$$f(-x)=|-x|-3$$
$$=|(-)x|-3$$
$$=|-1||x|-3$$
$$=|x|-3$$
$$=f(x)$$

$\therefore$ The function is ***even***.

Exercise

Determine whether f is even, odd, or neither: $f(x)=x^3-\frac{1}{x}$

Solution

$$f(-x) = (-x)^3 - \frac{1}{(-x)}$$
$$= -x^3 + \frac{1}{x}$$
$$= -\left(x^3 - \frac{1}{x}\right)$$
$$= -f(x)$$

$\therefore$ The function is ***odd***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = -x^3 + 2x$

Solution

$$f(-x) = -(-x)^3 + 2(-x)$$
$$= x^3 - 2x$$
$$= -f(x)$$

$\therefore$ The function is ***odd***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = x^5 - 2x^3$

Solution

$$f(-x) = (-x)^5 - 2(-x)^3$$
$$= -x^5 + 2x^3$$
$$= -f(x)$$

$\therefore$ The function is ***odd***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = .5x^4 - 2x^2 + 6$

Solution

$$f(-x) = .5(-x)^4 - 2(-x)^2 + 6$$
$$= .5x^4 - 2x^2 + 6$$
$$= f(x)$$

$\therefore$ The function is ***even***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = .75x^2 + |x| + 4$

Solution

$$f(-x) = .75(-x)^2 + |-x| + 4$$

$$= .75x^2 + |x| + 4$$

$$= f(x)$$ ∴ The function is ***even***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = x^3 - x + 9$

Solution

$$f(-x) = (-x)^3 - (-x) + 9$$

$$= -x^3 + x + 9$$

∴ The function is ***neither***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = x^4 - 5x + 8$

Solution

$$f(-x) = (-x)^4 - 5(-x) + 8$$

$$= x^4 + 5x + 8$$

∴ The function is ***neither***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = x^3 + x$

Solution

$$f(-x) = (-x)^3 + (-x)$$

$$= -x^3 - x$$

$$= -f(x)$$

∴ The function is ***odd***.

Exercise

Decide whether each function is even, odd, or neither $g(x) = x^2 - x$

Solution

$$\begin{aligned} g(-x) &= (-x)^2 + (-x) \\ &= x^2 - x \end{aligned}$$

$\therefore$ The function is ***neither***.

Exercise

Decide whether each function is even, odd, or neither $h(x) = 2x^2 + x^4$

Solution

$$\begin{aligned} h(-x) &= 2(-x)^2 + (-x)^4 \\ &= 2x^2 + x^4 \\ &= h(x) \end{aligned}$$

$\therefore$ The function is ***even***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = 2x^2 + x^4 + 1$

Solution

$$\begin{aligned} f(-x) &= 2(-x)^2 + (-x)^4 + 1 \\ &= 2x^2 + x^4 + 1 \\ &= f(x) \end{aligned}$$

$\therefore$ The function is ***even***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = \frac{1}{5}x^6 - 3x^2$

Solution

$$\begin{aligned} f(-x) &= \tfrac{1}{5}(-x)^6 - 3(-x)^2 \\ &= \tfrac{1}{5}x^6 - 3x^2 \\ &= f(x) \end{aligned}$$

$\therefore$ The function is ***even***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = x\sqrt{1 - x^2}$

Solution

$$\begin{aligned} f(-x) &= -x\sqrt{1-(-x)^2} \\ &= -x\sqrt{1-x^2} \\ &= -f(x) \end{aligned}$$

$\therefore$ The function is ***odd***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = x^2\sqrt{1-x^2}$

Solution

$$\begin{aligned} f(-x) &= (-x)^2\sqrt{1-(-x)^2} \\ &= x^2\sqrt{1-x^2} \\ &= f(x) \end{aligned}$$

$\therefore$ The function is ***even***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = 5x^7 - 6x^3 - 2x$

Solution

$$\begin{aligned} f(-x) &= 5(-x)^7 - 6(-x)^3 - 2(-x) \\ &= -5x^7 + 6x^3 + 2x \\ &= -(5x^7 - 6x^3 - 2x) \\ &= -f(x) \end{aligned}$$

$\therefore$ The function is ***odd***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = 5x^6 - 3x^2 - 7$

Solution

$$\begin{aligned} f(-x) &= 5(-x)^6 - 3(-x)^2 - 7 \\ &= 5x^6 - 3x^2 - 7 \\ &= f(x) \end{aligned}$$

$\therefore$ The function is ***even***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = x^2 + 6$

Solution

$$f(-x) = (-x)^2 + 6$$
$$= x^2 + 6$$
$$= f(x)$$

$\therefore$ The function is ***even***.

Exercise

Decide whether each function is even, odd, or neither $f(x) = 7x^3 - x$

Solution

$$f(-x) = 7(-x)^3 - (-x)$$
$$= -7x^3 + x$$
$$= -(7x^3 - x)$$
$$= -f(x)$$

$\therefore$ The function is ***odd***.

Exercise

Decide whether each function is even, odd, or neither $h(x) = x^5 + 1$

Solution

$$h(-x) = (-x)^5 + 1$$
$$= -x^5 + 1 \quad \begin{cases} \neq x^5 + 1 \\ \neq -(x^5 + 1) \end{cases}$$

$\therefore$ The function is ***neither***.

Exercise

$$f(x) = \begin{cases} 2 + x & if \quad x < -4 \\ -x & if \quad -4 \le x \le 2 \\ 3x & if \quad x > 2 \end{cases} \qquad \text{Find: } f(-5),\ f(-1),\ f(0),\ and\ f(3)$$

Solution

$$f(-5) = 2 - 5 = -3$$

$f(-1) = -(-1) = 1$

$f(0) = -0 = 0$

$f(3) = 3(3) = 9$

Exercise

$$f(x) = \begin{cases} -2x & if \quad x < -3 \\ 3x - 1 & if \quad -3 \le x \le 2 \\ -4x & if \quad x > 2 \end{cases}$$

Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

Solution

$f(-5) = -2(-5) = 10$

$f(-1) = 3(-1) - 1 = -4$

$f(0) = 3(0) - 1 = -1$

$f(3) = -4(3) = -12$

Exercise

$$f(x) = \begin{cases} x^3 + 3 & if \quad -2 \le x \le 0 \\ x + 3 & if \quad 0 < x < 1 \\ 4 + x - x^2 & if \quad 1 \le x \le 3 \end{cases}$$

Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

Solution

$f(-5) =$ *doesn't exist*

$f(-1) = (-1)^3 + 3 = 2$

$f(0) = (0)^3 + 3 = 3$

$f(3) = 4 + (3) - (3)^2 = -2$

Exercise

$$h(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & if \quad x \ne 3 \\ 6 & if \quad x = 3 \end{cases}$$

Find: $h(5),\ h(0),\ and\ h(3)$

Solution

$h(5) = \dfrac{5^2 - 9}{5 - 3} = 8$

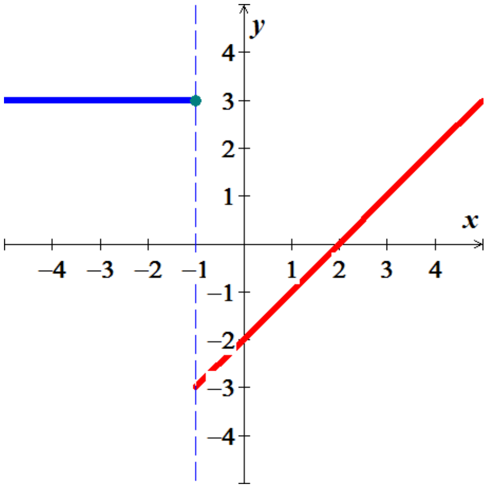
$$h(0) = \frac{0^2 - 9}{0 - 3} = 3$$

$$h(3) = 6$$

Exercise

Graph the piecewise function defined by $f(x) = \begin{cases} 3 & if \quad x \le -1 \\ x - 2 & if \quad x > -1 \end{cases}$

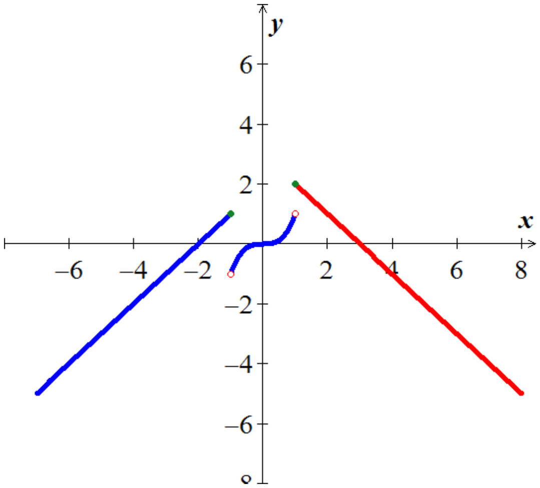
Solution



Exercise

Sketch the graph $f(x) = \begin{cases} x + 2 & if \quad x \le -1 \\ x^3 & if \quad -1 < x < 1 \\ -x + 3 & if \quad x \ge 1 \end{cases}$

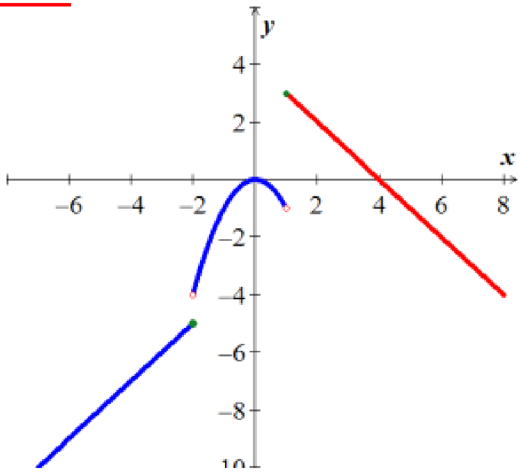
Solution



Exercise

Sketch the graph $f(x) = \begin{cases} x-3 & if \quad x \le -2 \\ -x^2 & if \quad -2 < x < 1 \\ -x+4 & if \quad x \ge 1 \end{cases}$

Solution



Solution *Section* 1.3 – Polynomial Functions & Graphs

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=2x^4-x^3+7x-12; \quad p(x)=x^2-3$

Solution

$$\begin{array}{r}
2x^2-x+6 \\
x^2-3\overline{\big)\,2x^4-x^3+0x^2+7x-12} \\
\underline{2x^4 \qquad -6x^2 \qquad\qquad} \\
-x^3+6x^2+7x \quad\; \\
\underline{-x^3 \qquad +3x \quad\;} \\
6x^2+4x-12 \\
\underline{6x^2 \qquad -18} \\
4x+6 \quad
\end{array}$$

$$\boxed{Q(x)=2x^2-x+6; \quad R(x)=4x+6}$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=3x^3+2x-4; \quad p(x)=2x^2+1$

Solution

$$\begin{array}{r}
\frac{3}{2}x \qquad \\
2x^2+1\overline{\big)\,3x^3+0x^2+2x-4} \\
\underline{3x^3 \qquad\quad +\frac{3}{2}x \qquad} \\
\frac{1}{2}x-4 \quad
\end{array}$$

$$\boxed{Q(x)=\frac{3}{2}x; \quad R(x)=\frac{1}{2}x-4}$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x)=7x+2; \quad p(x)=2x^2-x-4$

Solution

$$P(x)=\frac{7x+2}{2x^2-x-4}$$

$$\boxed{Q(x)=0; \quad R(x)=7x+2}$$

Exercise

Find the quotient and remainder if $f(x)$ is divided by $p(x)$: $f(x) = 9x + 4; \quad p(x) = 2x - 5$

Solution

$$
\begin{array}{r|l}
 & \frac{9}{2} \\ \hline
2x-5 \,) & 9x+4 \\
 & \underline{9x-\frac{45}{2}} \\
 & \quad -\frac{37}{2}
\end{array}
$$

$$\boxed{Q(x) = \frac{9}{2}; \quad R(x) = -\frac{37}{2}}$$

Exercise

Use the remainder theorem to find $f(c)$: $f(x) = x^4 - 6x^2 + 4x - 8; \quad c = -3$

Solution

$$f(-3) = (-3)^4 - 6(-3)^2 + 4(-3) - 8$$

$$\boxed{= 7}$$

Exercise

Use the remainder theorem to find $f(c)$: $f(x) = x^4 + 3x^2 - 12; \quad c = -2$

Solution

$$f(-2) = (-2)^4 + 3(-2)^2 - 12$$

$$\boxed{= 16}$$

Exercise

Use the factor theorem to show that $x - c$ is a factor of $f(x)$: $f(x) = x^3 + x^2 - 2x + 12; \quad c = -3$

Solution

$$f(-3) = (-3)^3 + (-3)^2 - 2(-3) + 12$$

$$\boxed{= 0}$$

From the factor theorem; $x + 3$ is a factor of $f(x)$.

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second:

$2x^3 - 3x^2 + 4x - 5; \quad x - 2$

Solution

$$\begin{array}{r|rrrr} 2 & 2 & -3 & 4 & -5 \\ & & 4 & 2 & 12 \\ \hline & 2 & 1 & 6 & \boxed{7} \end{array}$$

$Q(x) = 2x^2 + x + 6 \qquad R(x) = 7$

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second:

$5x^3 - 6x^2 + 15; \quad x - 4$

Solution

$$\begin{array}{r|rrrr} 4 & 5 & -6 & 0 & 15 \\ & & 20 & 56 & 224 \\ \hline & 5 & 14 & 56 & \boxed{239} \end{array}$$

$Q(x) = 5x^2 + 14x + 56 \qquad R(x) = 239$

Exercise

Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second:

$9x^3 - 6x^2 + 3x - 4; \quad x - \frac{1}{3}$

Solution

$$\begin{array}{r|rrrr} \frac{1}{3} & 9 & -6 & 3 & -4 \\ & & 3 & -1 & \frac{2}{3} \\ \hline & 9 & -3 & 2 & \boxed{-\frac{10}{3}} \end{array}$$

$Q(x) = 9x^2 - 3x + 2 \qquad R(x) = -\frac{10}{3}$

Exercise

Use the synthetic division to find $f(c)$: $f(x)=2x^3+3x^2-4x+4;\quad c=3$

Solution

$$\begin{array}{r|rrrr} 3 & 2 & 3 & -4 & 4 \\ & & 6 & 27 & 69 \\ \hline & 2 & 9 & 23 & \boxed{73} \end{array}$$

$$f(3)=73$$

Exercise

Use the synthetic division to find $f(c)$: $f(x)=8x^5-3x^2+7;\quad c=\frac{1}{2}$

Solution

$$\begin{array}{r|rrrrrr} \frac{1}{2} & 8 & 0 & 0 & -3 & 0 & 7 \\ & & 4 & 2 & 1 & -1 & -\frac{1}{2} \\ \hline & 8 & 4 & 2 & -2 & -1 & \boxed{\frac{13}{2}} \end{array}$$

$$f\left(\frac{1}{2}\right)=\frac{13}{2}$$

Exercise

Use the synthetic division to find $f(c)$: $f(x)=x^3-3x^2-8;\quad c=1+\sqrt{2}$

Solution

$$\begin{array}{r|cccc} 1+\sqrt{2} & 3 & -3 & 0 & -8 \\ & & 3+3\sqrt{2} & 6+3\sqrt{2} & 12+9\sqrt{2} \\ \hline & 3 & 3\sqrt{2} & 6+3\sqrt{2} & \boxed{4+9\sqrt{2}} \end{array}$$

$$f\left(1+\sqrt{2}\right)=4+9\sqrt{2}$$

Exercise

Use the synthetic division to show that c is a zero of $f(x)$: $f(x)=3x^4+8x^3-2x^2-10x+4;\quad c=-2$

Solution

$$\begin{array}{r|rrrrr} -2 & 3 & 8 & -2 & -10 & 4 \\ & & -6 & -4 & 12 & -4 \\ \hline & 3 & 2 & -6 & 2 & \boxed{0} \end{array}$$

$$\boxed{f(-2)=0}$$

Exercise

Use the synthetic division to show that c is a zero of $f(x)$: $f(x)=27x^4-9x^3+3x^2+6x+1;\quad c=-\frac{1}{3}$

Solution

$$\begin{array}{r|rrrrr} -\frac{1}{3} & 27 & -9 & 3 & 6 & 1 \\ & & -9 & 6 & -3 & -1 \\ \hline & 27 & -18 & 9 & 3 & \boxed{0} \end{array}$$

$$\boxed{f\left(-\frac{1}{3}\right)=0}$$

Exercise

Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

$f(x)=kx^3+x^2+k^2x+3k^2+11;\quad x+2$

Solution

$$\begin{array}{r|cccc} -2 & k & 1 & k^2 & 3k^2+11 \\ & & -2k & 4k-2 & -2k^2-8k+4 \\ \hline & k & 1-2k & k^2+4k-2 & k^2-8k+15 \end{array}$$

$$k^2-8k+15=0 \Rightarrow \boxed{k=3,\ 5}$$

Exercise

Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

$f(x)=x^3+k^3x^2++2kx-2k^4;\quad x-1.6$

Solution

$$\begin{array}{r|cccc} 1.6 & 1 & k^3 & 2k & -2k^4 \\ & & 1.6 & 1.6k^3+2.56 & 2.56k^3+3.2k+4.096 \\ \hline & 1 & k^3+1.6 & 1.6k^3+2k+2.56 & -2k^4+2.56k^3+3.2k+4.096 \end{array}$$

$$-2k^4 + 2.56k^3 + 3.2k + 4.096 = 0$$

Using the calculator, the result will show that the solutions are: $\underline{x = -0.75,\ \ 1.96}|$ $\quad 0.032 \pm 1.18i$

Exercise

Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

$f(x) = k^2x^3 - 4kx + 3; \quad x - 1$

Solution

$$\begin{array}{r|cccc} 1 & k^2 & 0 & -4k & 3 \\ & & k^2 & k^2-4k & k^2-4k \\ \hline & k^2 & k^2 & k^2-4k & k^2-4k+3 \end{array}$$

$k^2 - 4k + 3 = 0 \quad \Rightarrow \quad \underline{k = 1,\ 3}|$

Exercise

Find all solutions of the equation: $\quad x^3 - x^2 - 10x - 8 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{8}{1}\right\} = \pm\{1,\ 2,\ 4,\ 8\}$

$$\begin{array}{r|rrrr} -1 & 1 & -1 & -10 & -8 \\ & & -1 & 2 & 8 \\ \hline & 1 & -2 & -8 & 0 \end{array} \rightarrow \quad x^2 - 2x - 8 = 0$$

The solutions are: $\underline{x = -1,\ \ -2,\ 4}|$

Exercise

Find all solutions of the equation: $\quad x^3 + x^2 - 14x - 24 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{24}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24\}$

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -14 & -24 \\ & & -2 & 2 & 24 \\ \hline & 1 & -1 & -12 & 0 \end{array} \rightarrow \quad x^2 - x - 12 = 0$$

The solutions are: $\underline{x = -2,\ \ -3,\ 4}|$

Exercise

Find all solutions of the equation: $2x^3 - 3x^2 - 17x + 30 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{30}{2}\right\}$

$$= \pm\left\{1,\ 2,\ 3,\ 5,\ 6,\ 10,\ 15,\ 30,\ \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \frac{15}{2}\right\}$$

$$\begin{array}{r|rrrr} 2 & 2 & -3 & -17 & 30 \\ & & 4 & 2 & -30 \\ \hline & 2 & 1 & -15 & \boxed{0} \end{array} \rightarrow\ 2x^2 + x - 15 = 0$$

The solutions are: $x = 2,\ -3,\ \frac{5}{2}$

Exercise

Find all solutions of the equation: $12x^3 + 8x^2 - 3x - 2 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{2}{12}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{2},\ \frac{1}{6},\ \frac{1}{12}\right\}$

$$\begin{array}{r|rrrr} \frac{1}{2} & 12 & 8 & -3 & -2 \\ & & 6 & 7 & 2 \\ \hline & 12 & 14 & 4 & \boxed{0} \end{array} \rightarrow\ 12x^2 + 14x + 4 = 0$$

$$6x^2 + 7x + 2 = 0$$

$$x = \frac{-7 \pm \sqrt{49 - 48}}{12}$$

$$= \begin{cases} \frac{-7-1}{12} = -\frac{2}{3} \\ \frac{-7+1}{12} = -\frac{1}{2} \end{cases}$$

The solutions are: $x = \frac{1}{2},\ -\frac{1}{2},\ -\frac{2}{3}$

Exercise

Find all solutions of the equation: $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{56}{1}\right\} = \pm\{1,\ 2,\ 4,\ 7,\ 8,\ 14,\ 28,\ 56\}$

$$\begin{array}{r|rrrrr}
4 & 1 & 3 & -3 & -6 & 56 \\
 & & 4 & 28 & -8 & -56 \\ \hline
-7 & 1 & 7 & -2 & -14 & 0 \\
 & & -7 & 0 & 14 & \\ \hline
 & 1 & 0 & -2 & &
\end{array}$$

$$\rightarrow \quad x^3+7x^2-2x-14=0 \quad \Rightarrow \quad \frac{c}{d}=\pm\left\{\frac{14}{1}\right\}=\pm\{1,\ 2,\ 7,\ 14\}$$

$$\rightarrow \quad x^2-2=0 \ \Rightarrow \ \underline{x=\pm\sqrt{2}}|$$

The solutions are: $\underline{x=4,\ \ -7,\ \ \pm\sqrt{2}}|$

Exercise

Find all solutions of the equation: $3x^5-10x^4-6x^3+24x^2+11x-6=0$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{6}{3}\right\}=\pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrrrr}
-1 & 3 & -10 & -6 & 24 & 11 & -6 \\
 & & -3 & 13 & -7 & -17 & 6 \\ \hline
-1 & 3 & -13 & 7 & 17 & -6 & 0 \\
 & & -3 & 16 & -23 & 6 & \\ \hline
2 & 3 & -16 & 23 & -6 & 0 & \\
 & & 6 & 20 & 6 & & \\ \hline
 & 3 & -10 & 3 & 0 & &
\end{array}$$

$$x^4-13x^3+7x^2+17x-6=0 \rightarrow \pm\left\{\frac{6}{3}\right\}=\pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$$

$$3x^3-16x^2+26x-6=0 \rightarrow \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$$

$$3x^2-10x+3=0$$

$$3x^2-10x+3=0$$

$$x=\frac{10\pm\sqrt{100-36}}{6}$$

$$=\begin{cases}\frac{10-8}{6}=\frac{1}{3}\\[4pt] \frac{10+8}{6}=3\end{cases}$$

The solutions are: $\underline{x=-1,\ \ -1,\ \ \frac{1}{3},\ 2,\ 3}|$

Exercise

Find all solutions of the equation: $6x^5+19x^4+x^3-6x^2=0$

Solution

$$x^2\left(6x^3+19x^2+x-6\right)=0 \quad \rightarrow \quad \boxed{x=0,\ 0}$$

$$6x^3+19x^2+x-6=0$$

$$\textit{possibilities for } \frac{c}{d}: \pm\left\{\frac{6}{6}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3}\right\}$$

$$\begin{array}{r|rrrr}
-3 & 6 & 19 & 1 & -6 \\
 & & -18 & -3 & 6 \\
\hline
 & 6 & 1 & -2 & \boxed{0}
\end{array} \quad 6x^2 + x - 2 = 0$$

$$x = \frac{-1 \pm \sqrt{1+48}}{12}$$

$$= \begin{cases} \dfrac{-1-7}{12} = -\dfrac{2}{3} \\ \dfrac{-1+7}{12} = \dfrac{1}{2} \end{cases}$$

The solutions are: $x = 0,\ 0,\ -\frac{2}{3},\ -3,\ \frac{1}{2}$

Exercise

Find all solutions of the equation: $x^4 - x^3 - 9x^2 + 3x + 18 = 0$

Solution

$$\textit{possibilities for } \frac{c}{d}: \pm\left\{\frac{18}{1}\right\} = \pm\{1,\ 2,\ 3,\ 6,\ 9,\ 18\}$$

$$\begin{array}{r|rrrrr}
-2 & 1 & -1 & -9 & 3 & 18 \\
 & & -2 & 6 & 6 & -18 \\
\hline
3 & 1 & -3 & -3 & 9 & 0 \\
 & & 3 & 0 & -9 & \\
\hline
 & 1 & 0 & -3 & 0 &
\end{array}$$

$$\to x^3 - 3x^2 - 3x + 9 = 0 \to \pm\left\{\frac{9}{1}\right\} = \pm\{1,\ 3,\ 9\}$$

$$\to x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3}$$

The solutions are: $x = -2,\ 3,\ \pm\sqrt{3}$

Exercise

Find all solutions of the equation: $2x^4 - 9x^3 + 9x^2 + x - 3 = 0$

Solution

$$\textit{possibilities for } \frac{c}{d}: \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$$

$$\begin{array}{r|rrrrr} 1 & 2 & -9 & 9 & 1 & -3 \\ & & 2 & -7 & 2 & 3 \\ \hline 1 & 2 & -7 & 2 & 3 & 0 \\ & & 2 & -5 & -3 & \\ \hline & 2 & -5 & -3 & 0 & \end{array}$$

$\rightarrow 2x^3 - 7x^2 + 2x + 3 = 0 \rightarrow \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$\rightarrow 2x^2 - 5x - 3 = 0$

$$x = \frac{5 \pm \sqrt{25+24}}{4}$$

$$= \begin{cases} \frac{5-7}{4} = -\frac{1}{2} \\ \frac{5+7}{4} = 3 \end{cases}$$

The solutions are: $\boxed{x = 1,\ 1,\ -\frac{1}{2},\ 3}$

Exercise

Find all solutions of the equation: $8x^3 + 18x^2 + 45x + 27 = 0$

Solution

possibilities : $\pm\left\{\frac{27}{8}\right\} = \pm\left\{\frac{1,\ 3,\ 9,\ 27}{1,\ 2,\ 4,\ 8}\right\}$

$$= \pm\left\{1,\ 3,\ 9,\ 27,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{3}{2},\ \frac{3}{4},\ \frac{3}{8},\ \frac{9}{2},\ \frac{9}{4},\ \frac{9}{8},\ \frac{27}{2},\ \frac{27}{4},\ \frac{27}{8}\right\}$$

$$\begin{array}{r|rrrr} -\frac{3}{4} & 8 & 18 & 45 & 27 \\ & & -6 & -9 & -27 \\ \hline & 8 & 12 & 36 & \boxed{0} \end{array}$$

$\rightarrow 8x^2 + 12x + 36 = 0$

$2x^2 + 3x + 9 = 0$

$$x = \frac{-3 \pm \sqrt{9-72}}{4}$$

$$= \frac{-3 \pm \sqrt{-63}}{4}$$

$$= \frac{-3 \pm 3i\sqrt{7}}{4}$$

The solutions are: $\boxed{x = -\frac{3}{4},\ -\frac{3}{4} \pm i\frac{3\sqrt{7}}{4}}$

Exercise

Find all solutions of the equation: $3x^3 - x^2 + 11x - 20 = 0$

Solution

$$possibilities: \ \pm\left\{\frac{20}{3}\right\} = \pm\left\{\frac{1,\ 2,\ 4,\ 5,\ 10,\ 20}{1,\ 3}\right\}$$

$$= \pm\left\{1,\ 2,\ 4,\ 5,\ 10,\ 20,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{5}{3},\ \frac{10}{3},\ \frac{20}{3}\right\}$$

A result will show that one solution is: $x = \frac{4}{3}$

$$\begin{array}{c|cccc} \frac{4}{3} & 3 & -1 & 11 & -20 \\ & & 4 & 4 & 20 \\ \hline & 3 & 3 & 15 & \boxed{0} \end{array} \ \rightarrow\ 3x^2 + 3x + 15 = 0$$

$$x^2 + x + 5 = 0$$

$$x = \frac{-1 \pm \sqrt{1 - 20}}{2}$$

The solutions are: $x = \frac{4}{3},\ -\frac{1}{2} \pm i\frac{\sqrt{19}}{2}$

Exercise

Find all solutions of the equation: $6x^4 + 5x^3 - 17x^2 - 6x = 0$

Solution

$$x\left(6x^3 + 5x^2 - 17x - 6\right) = 0 \ \rightarrow\ x = 0$$

$$possibilities: \ \pm\left\{\frac{6}{6}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3}\right\}$$

$$\begin{array}{c|cccc} -2 & 6 & 5 & -17 & -6 \\ & & -12 & 14 & 6 \\ \hline & 6 & -7 & -3 & \boxed{0} \end{array} \ \rightarrow\ 6x^2 - 7x - 3 = 0$$

$$x = \frac{7 \pm \sqrt{49 + 72}}{12}$$

$$= \begin{cases} \frac{7-11}{12} = -\frac{1}{3} \\ \frac{7+11}{12} = \frac{3}{2} \end{cases}$$

The solutions are: $x = 0,\ -2,\ -\frac{1}{3},\ \frac{3}{2}$

Exercise

If $f(x) = 3x^3 - kx^2 + x - 5k$, find a number k such that the graph of f contains the point $(-1, 4)$.

Solution

$$f(-1) = 3(-1)^3 - k(-1)^2 + (-1) - 5k$$

$$4 = -3 - k - 1 - 5k$$

$$4 = -4 - 6k$$

Add 4 on both side

$$8 = -6k$$

$$k = -\frac{8}{6}$$

$$= -\frac{4}{3}$$

Exercise

If $f(x) = kx^3 + x^2 - kx + 2$, find a number k such that the graph of f contains the point $(2, 12)$.

Solution

$$f(2) = k(2)^3 + (2)^2 - k(2) + 2$$

$$12 = 8k + 4 - 2k + 2$$

$$12 = 6k + 6$$

$$6k = 6$$

$$k = 1$$

Exercise

If one zero of $f(x) = x^3 - 2x^2 - 16x + 16k$ is 2, find two other zeros.

Solution

$$f(x) = x^2(x-2) - 16(x-k) \qquad k = 2$$

$$= (x-2)(x^2-16)$$

$$= (x-2)(x-4)(x+4)$$

The other zeros are: 4, –4

Exercise

If one zero of $f(x) = x^3 - 3x^2 - kx + 12$ is –2, find two other zeros.

Solution

$$f(x)=x^2(x-3)-k\left(x-\frac{12}{k}\right) \qquad \frac{12}{k} \textit{ has to be equal to } 3. \Rightarrow k=4$$

$$\begin{aligned} f(x)&=x^2(x-3)-4(x-3)\\ &=(x-3)\left(x^2-4\right)\\ &=(x-3)(x-2)(x+2) \end{aligned}$$

The zeros of $f(x)$ are: 3, –2, 2

Exercise

Find a polynomial $f(x)$ of degree 3 that has the zeros –1, 2, 3; and satisfies the given condition: $f(-2)=80$

Solution

$$\begin{aligned} f(x)&=k(x+1)(x-2)(x-3)\\ &=k\left(x^2-x-2\right)(x-3)\\ &=k\left(x^3-3x^2-x^2+3x-2x+6\right)\\ &=k\left(x^3-4x^2+x+6\right) \end{aligned}$$

$$f(-2)=k\left((-2)^3-4(-2)^2+(-2)+6\right)$$

$$80=k(-20)$$

$$k=\frac{80}{-20}=-4$$

$$f(x)=-4\left(x^3-4x^2+x+6\right)$$

$$f(x)=-4x^3+16x^2-4x-24$$

Exercise

Find a polynomial $f(x)$ of degree 3 that has the zeros $-2i$, $2i$, 3; and satisfies the given condition: $f(1)=20$

Solution

$$\begin{aligned} f(x)&=k(x+2i)(x-2i)(x-3)\\ &=k\left(x^2+4\right)(x-3)\\ &=k\left(x^3-3x^2+4x-12\right) \end{aligned}$$

$$f(1)=k\left((1)^3-3(1)^2+4(1)-12\right)$$

$$20=k(-10)$$

$k = -2$

$f(x) = -2\left(x^3 - 3x^2 + 4x - 12\right)$

$\underline{f(x) = -2x^3 + 6x^2 - 8x + 24 \,|}$

Exercise

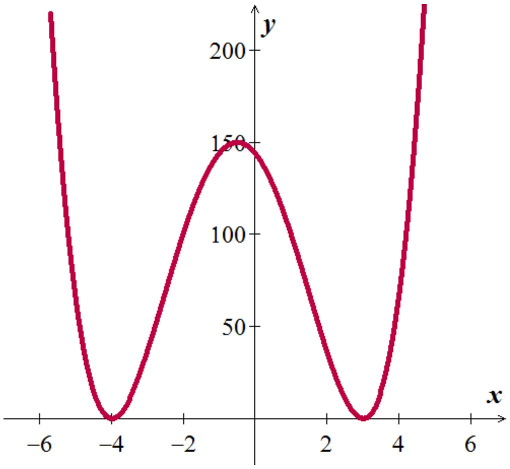
Find a polynomial $f(x)$ of degree 4 with leading coefficient 1 such that both -4 and 3 are zeros of multiplicity 2, and sketch the graph of f.

Solution

$f(x) = a\left(x - x_1\right)\left(x - x_2\right)\left(x - x_3\right)\left(x - x_4\right)$

$a = 1 \quad x_1 = x_2 = -4 \quad x_3 = x_4 = 3$

$$\begin{aligned} f(x) &= (x+4)(x+4)(x-3)(x-3) \\ &= \left(x^2 + 8x + 16\right)\left(x^2 - 6x + 9\right) \\ &= x^4 - 6x^3 + 9x^2 + 8x^3 - 48x^2 + 72x + 16x^2 - 96x + 144 \\ &= x^4 + 2x^3 - 23x^2 - 24x + 144 \end{aligned}$$



Exercise

Find the zeros of $f(x) = x^2(3x+2)(2x-5)^3$, and state the multiplicity of each zero.

Solution

$f(x) = x^2(3x+2)(2x-5)^3 = 0$

The zeros are: $x = 0 \quad (\textit{multiplicity of } \ 2)$

$x = -\frac{2}{3}$

$x = \frac{5}{2} \quad (\textit{multiplicity of } \ 3)$

Exercise

Find the zeros of $f(x) = 4x^5 + 12x^4 + 9x^3$, and state the multiplicity of each zero.

Solution

$$\begin{aligned} f(x) &= x^3\left(4x^2 + 12x + 9\right) = 0 \\ &= x^3(2x+3)^2 = 0 \end{aligned}$$

The zeros are: $x = 0$ $(\textit{multiplicity of } 3)$

$$x = -\frac{3}{2} \quad (\textit{multiplicity of } 2)$$

Exercise

Find the zeros of $f(x) = \left(x^2 + x - 12\right)^3 \left(x^2 - 9\right)^2$, and state the multiplicity of each zero.

Solution

$$f(x) = \left(x^2 + x - 12\right)^3 \left(x^2 - 9\right)^2 = 0$$

$$x^2 + x - 12 = 0 \qquad x^2 - 9 = 0$$

$$x = -4,\ 3 \qquad x = \pm 3$$

The zeros are: $x = -4$ $(\textit{multiplicity of } 3)$

$$x = -3 \quad (\textit{multiplicity of } 2)$$

$$x = 3 \quad (\textit{multiplicity of } 5)$$

Exercise

Find the zeros of $f(x) = \left(6x^2 + 7x - 5\right)^4 \left(4x^2 - 1\right)^2$, and state the multiplicity of each zero.

Solution

$$f(x) = \left(6x^2 + 7x - 5\right)^4 \left(4x^2 - 1\right)^2 = 0$$

$$6x^2 + 7x - 5 = 0 \qquad 4x^2 - 1 = 0 \to x^2 = \frac{1}{4}$$

$$x = -\frac{5}{3},\ \frac{1}{2} \qquad x = \pm\frac{1}{2}$$

The zeros are: $x = -\frac{5}{3}$ $(\textit{multiplicity of } 4)$

$$x = -\frac{1}{2} \quad (\textit{multiplicity of } 2)$$

$$x = \frac{1}{2} \quad (\textit{multiplicity of } 6)$$

Exercise

Find the zeros of $f(x) = x^4 + 7x^2 - 144$, and state the multiplicity of each zero.

Solution

$$f(x) = x^4 + 7x^2 - 144$$

$$= \left(x^2 - 9\right)\left(x^2 + 16\right) = 0$$

$$x^2 - 9 = 0 \qquad\qquad x^2 + 16 = 0$$

$$x = \pm 3 \qquad\qquad x^2 = -16 \ \ (\mathbb{C})$$

The zeros are: $\underline{x = \pm 3}\,|$

Exercise

Find the zeros of $f(x) = x^4 + 21x^2 - 100$, and state the multiplicity of each zero.

Solution

$$f(x) = x^4 + 21x^2 - 100$$

$$= \left(x^2 - 4\right)\left(x^2 + 25\right) = 0$$

$$x^2 - 4 = 0 \qquad\qquad x^2 + 25 = 0$$

$$x = \pm 2 \qquad\qquad x^2 = -25 \ \ (\mathbb{C})$$

The zeros are: $\underline{x = \pm 2}\,|$

Exercise

Let $f(x) = x^4 - 4x^2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$f(x) = x^2\left(x^2 - 4\right)$$

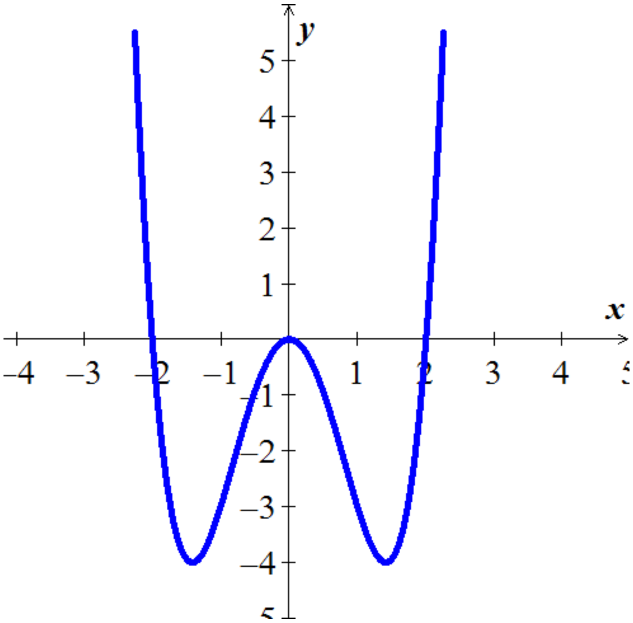
$$= x^2(x-2)(x+2)$$

The zeros are: 0, 0, 2, −2.

$-\infty$	-2	**0,0**	2	∞
+		−		+

$f(x) < 0$ $\quad \underline{(-2,\ 0) \cup (0,\ 2)}\,|$

$f(x) > 0$ $\quad \underline{(-\infty,\ -2) \cup (2,\ \infty)}\,|$



Exercise

Let $f(x)=x^4+3x^3-4x^2$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

Solution

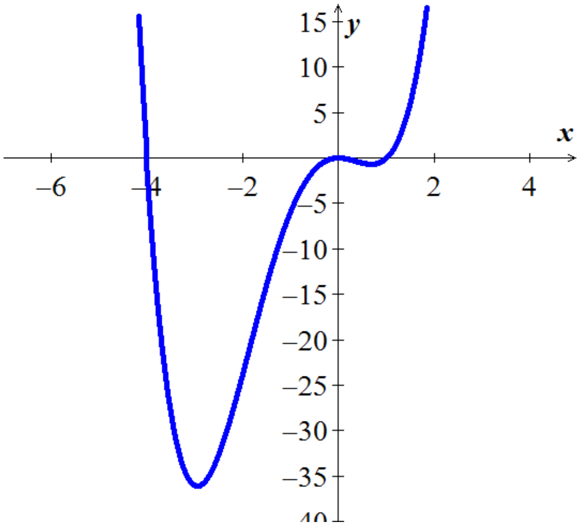
$$f(x)=x^2\left(x^2+3x-4\right)$$

The zeros are: 0, 0, 1, −4.

−∞	−4	**0,0**	1	∞
+		−		+

$f(x)>0$ $\quad \underline{(-\infty,\ -4)\cup(1,\ \infty)\,|}$

$f(x)<0$ $\quad \underline{(-4,\ 0)\cup(0,\ 1)\,|}$



Exercise

Let $f(x)=x^3+2x^2-4x-8$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

Solution

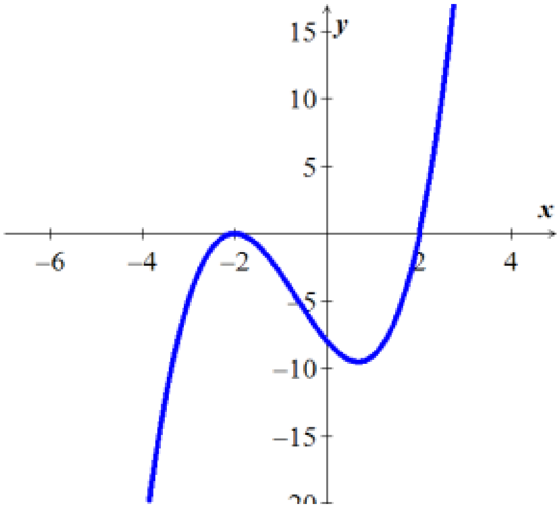
$$\begin{aligned} f(x)&=x^2(x+2)-4(x+2)\\ &=(x+2)\left(x^2-4\right)\\ &=(x+2)(x+2)(x-2)=0 \end{aligned}$$

The zeros are: 2, −2, −2

−∞	−2	0	2	∞
−		−		+

$f(x)>0$ $\quad \underline{(2,\ \infty)\,|}$

$f(x)<0$ $\quad \underline{(-\infty,\ -2)\cup(-2,\ 2)\,|}$



Exercise

Let $f(x) = x^3 - 3x^2 - 9x + 27$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$f(x) = x^2(x-3) - 9(x-3)$$
$$= (x-3)\left(x^2-9\right)$$
$$= (x-3)(x-3)(x+3)$$

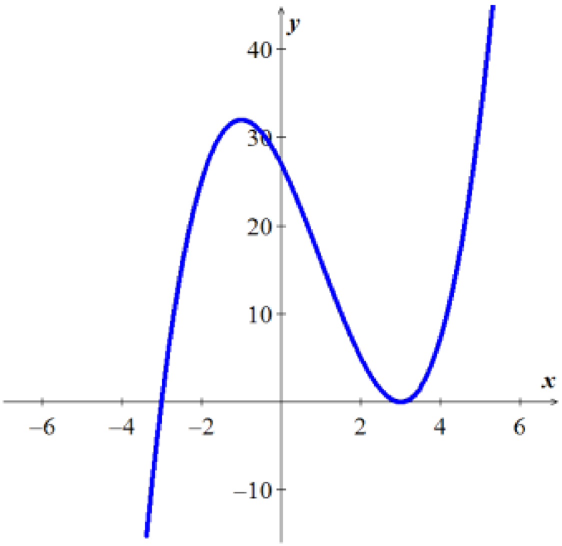
The zeros are: –3, 3 (multiplicity)

$-\infty \qquad -3 \qquad 0 \qquad 3 \qquad \infty$

–	+	+

$f(x) > 0 \quad (-3,\ 3) \cup (3,\ \infty)$

$f(x) < 0 \quad (-\infty,\ -3)$



Exercise

Let $f(x) = -x^4 + 12x^2 - 27$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

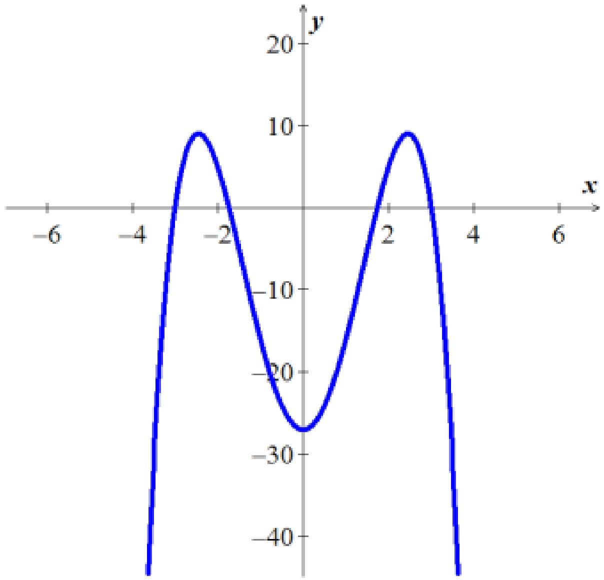
$$x^2 = \frac{-12 \pm \sqrt{36}}{-2}$$
$$= \begin{cases} \dfrac{-12-6}{-2} = 9 \\[2ex] \dfrac{-12+6}{-2} = 3 \end{cases}$$
$$\rightarrow \begin{cases} x^2 = 9 \\ x^2 = 3 \end{cases} \Rightarrow \quad \begin{matrix} x = \pm 3 \\ x = \pm\sqrt{3} \end{matrix}$$

$-3 \qquad -\sqrt{3} \qquad \sqrt{3} \qquad 3$

–	+	–	+	–

$f(x) > 0 \quad \left(-3,\ -\sqrt{3}\right) \cup \left(\sqrt{3},\ 3\right)$

$f(x) < 0 \quad (-\infty,\ -3) \cup \left(-\sqrt{3},\ \sqrt{3}\right) \cup (3,\ \infty)$



Exercise

Let $f(x)=x^2(x+2)(x-1)^2(x-2)$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

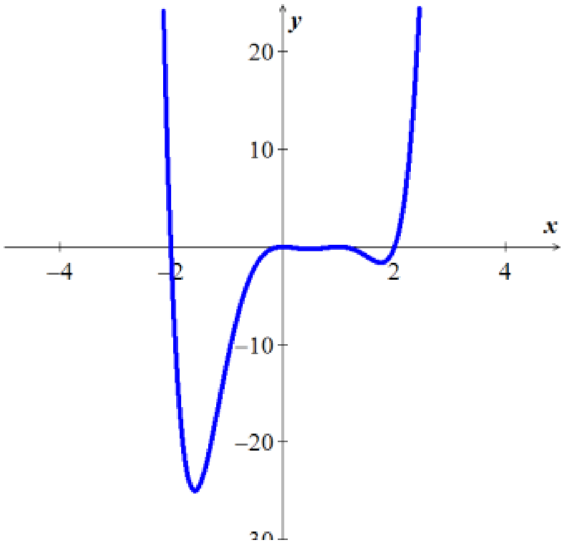
Solution

The zeros are: −2, 2, 0, 0, 1, 1

−2	**0,0** **1,1**	2
+	−	+

$f(x)>0 \quad (-\infty,\ -2)\cup(2,\ \infty)$

$f(x)<0 \quad (-2,\ 0)\cup(0,\ 1)\cup(1,\ 2)$



Exercise

Let $f(x)=2x^3+11x^2-7x-6$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

Solution

$$\text{possibilities}: \pm\left\{\frac{6}{2}\right\}=\pm\left\{\frac{1,\ 2,\ 3,\ 6}{1,\ 2}\right\}$$

$$=\pm\left\{1,\ 2,\ 3,\ 5,\ \frac{1}{2},\ \frac{3}{2}\right\}$$

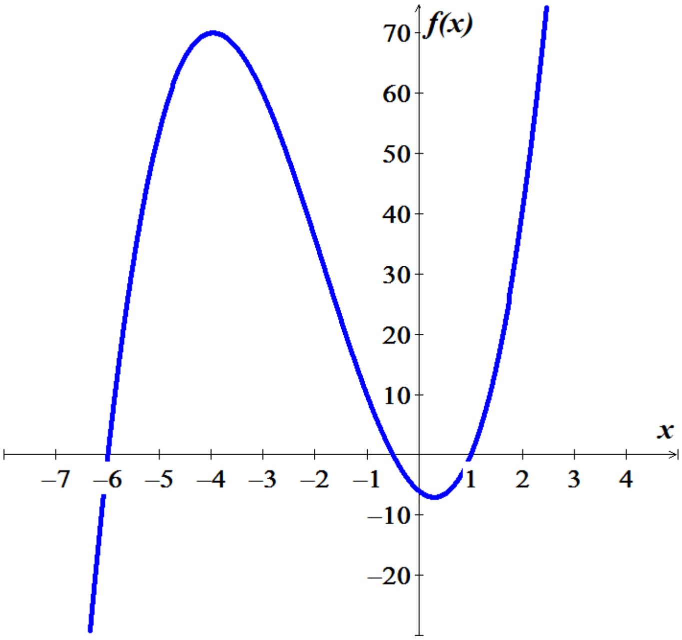
$$\begin{array}{c|cccc} 1 & 2 & 11 & -7 & -6 \\ & & 2 & 13 & 6 \\ \hline & 2 & 13 & 6 & \boxed{0} \end{array} \quad \to \ 2x^2+13x+6=0$$

The zeros are: $x=1,\ -\frac{1}{2},\ -6$

	−6	$-\frac{1}{2}$	1
−	+	−	+

$f(x)>0 \quad \left(-6,\ -\frac{1}{2}\right)\cup(1,\ \infty)$

$f(x)<0 \quad (-\infty,\ -6)\cup\left(-\frac{1}{2},\ 1\right)$



Exercise

Let $f(x) = x^3 + 2x^2 - 5x - 6$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$possibilities\ : \pm\left\{\frac{6}{1}\right\} = \pm\{1,\ 2,\ 3,\ 6\}$$

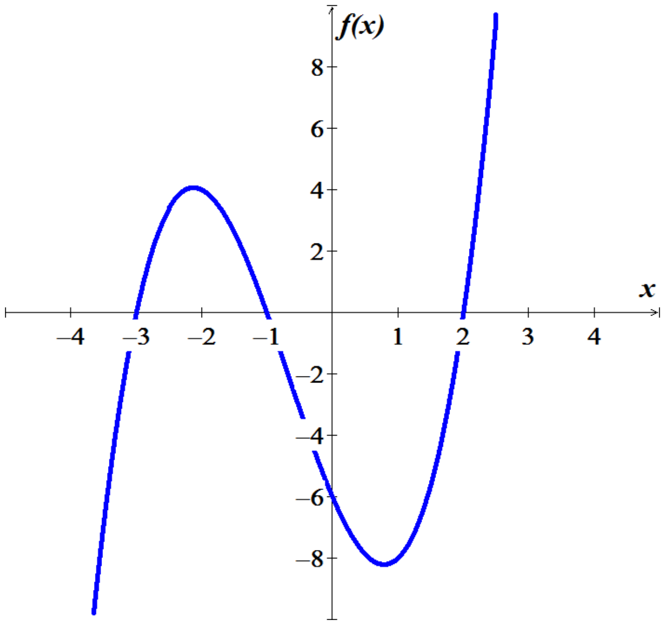
$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & & -1 & -1 & 6 \\ \hline & 1 & 1 & -6 & \boxed{0} \end{array} \quad \to\ x^2 + x - 6 = 0$$

The zeros are: $x = -1,\ -3,\ 2$

$$\begin{array}{c|c|c|c} & -3 & -1 & 2 \\ - & + & - & + \end{array}$$

$f(x) > 0$ $\quad (-3,\ -1) \cup (2,\ \infty)$

$f(x) < 0$ $\quad (-\infty,\ -3) \cup (-1,\ 2)$



Exercise

Let $f(x) = x^3 + 8x^2 + 11x - 20$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$possibilities\ : \pm\left\{\frac{20}{1}\right\} = \pm\{1,\ 2,\ 4,\ 5,\ 20,\ 20\}$$

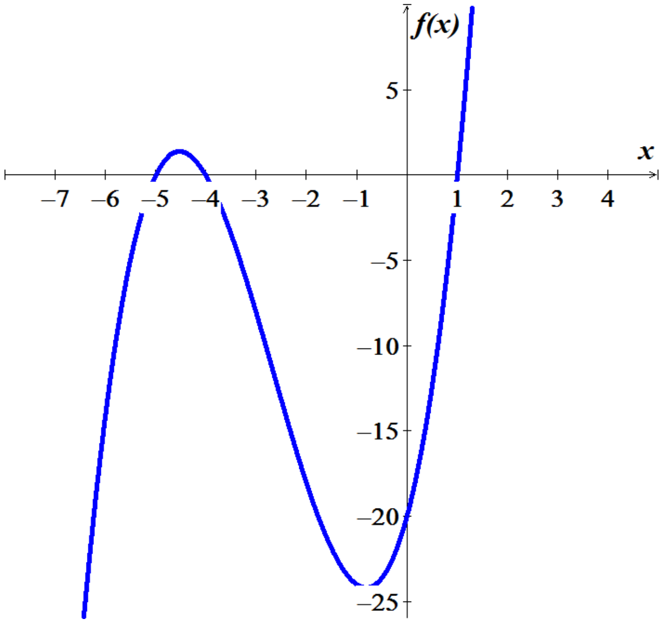
$$\begin{array}{r|rrrr} 1 & 1 & 8 & 11 & -20 \\ & & 1 & 9 & 20 \\ \hline & 1 & 9 & 20 & \boxed{0} \end{array} \quad \to\ x^2 + 9x + 20 = 0$$

The zeros are: $x = -5,\ -4,\ 1$

$$\begin{array}{c|c|c|c} & -5 & -4 & 1 \\ - & + & - & + \end{array}$$

$f(x) > 0$ $\quad (-5,\ -1) \cup (1,\ \infty)$

$f(x) < 0$ $\quad (-\infty,\ -5) \cup (-4,\ 1)$



Exercise

Let $f(x) = x^4 + x^2 - 2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$possibilities: \ \pm\{1,\ 2\}$

$$\begin{array}{r|rrrrr} 1 & 1 & 0 & 1 & 0 & -2 \\ & & 1 & 1 & 2 & 1 \\ \hline -1 & 1 & 1 & 2 & 2 & 0 \\ & & -1 & 0 & -2 & \\ \hline & 1 & 0 & 2 & 0 & \end{array}$$

$\rightarrow x^3 + x^2 + 2x + 1 = 0 \ \rightarrow \pm\{1,\ 2\}$

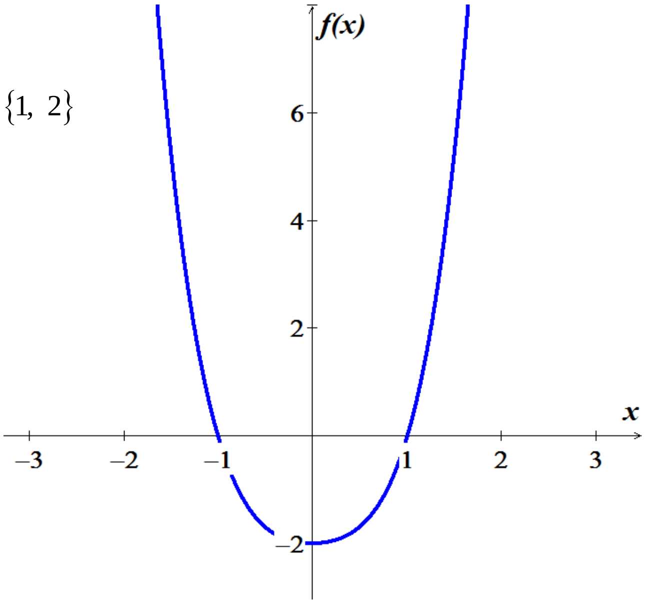
$\rightarrow x^2 + 2 = 0 \quad \Rightarrow \quad \boxed{x = \pm i\sqrt{2}}$

The zeros are: $\boxed{x = \pm 1}$

$$\begin{array}{ccccc} & -1 & & 1 & \\ \hline + & \vert & - & \vert & + \end{array}$$

$f(x) > 0 \quad \boxed{(-\infty,\ -1) \cup (1,\ \infty)}$

$f(x) < 0 \quad \boxed{(-1,\ 1)}$



Exercise

Let $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$possibilities: \ \pm\{1,\ 2,\ 4,\ 8\}$

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & -6 & 4 & 8 \\ & & -1 & 2 & 4 & -8 \\ \hline -2 & 1 & -2 & -4 & 8 & 0 \\ & & -2 & 8 & -8 & \\ \hline & 1 & -4 & 4 & 0 & \end{array}$$

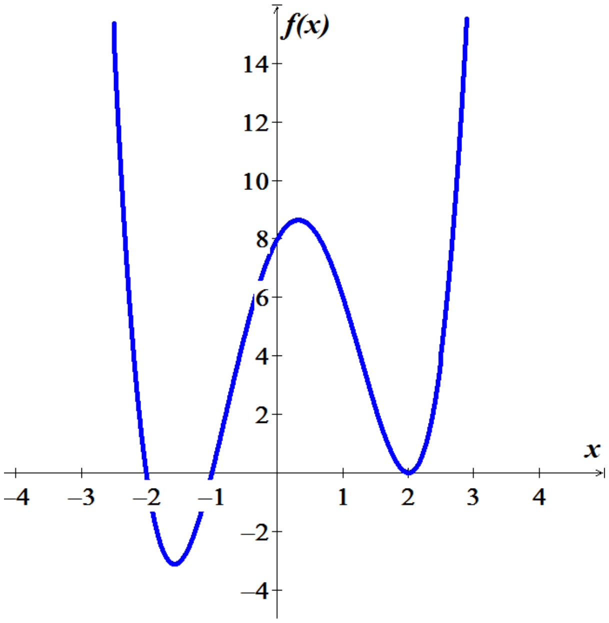
$\rightarrow x^3 - 2x^2 - 4x + 8 = 0$

$\rightarrow \pm\{1,\ 2,\ 4,\ 8\}$

$\rightarrow x^2 - 4x + 4 = 0 \ \Rightarrow \ \boxed{x = 2,\ 2}$

The zeros are: $\boxed{x = -2,\ \ -1,\ 2,\ 2}$

$$\begin{array}{ccccccc} & -2 & & -1 & & 2 & \\ \hline + & \vert & - & \vert & + & \Vert & + \end{array}$$



$f(x) > 0 \quad \boxed{(-\infty,\ -1) \cup (1,\ \infty)}$

$f(x) < 0 \quad \boxed{(-1,\ 1)}$

Exercise

Let $f(x) = 2x^4 - x^3 - 5x^2 + 2x + 2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

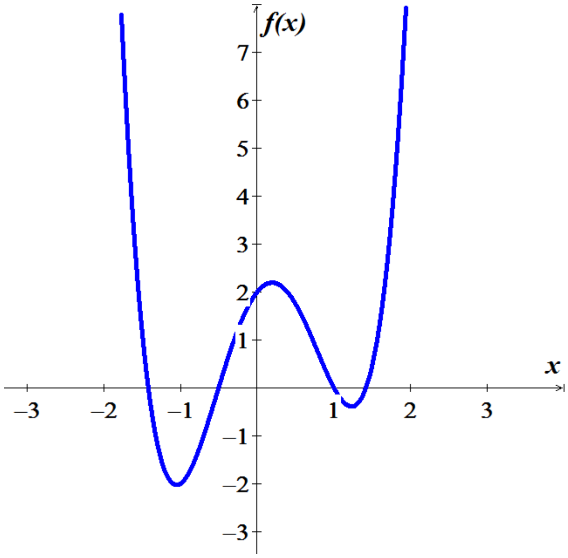
$$possibilities: \ \pm\left\{\frac{2}{2}\right\} = \pm\left\{1,\ 2,\ \frac{1}{2}\right\}$$

$$\begin{array}{r|rrrrr} 1 & 2 & -1 & -5 & 2 & 2 \\ & & 2 & 1 & -4 & -2 \\ \hline -\frac{1}{2} & 2 & 1 & -4 & -2 & 0 \\ & & -1 & 0 & 2 & \\ \hline & 2 & 0 & -4 & 0 & \end{array} \quad \begin{array}{l} \to 2x^3+x^2-4x-2=0 \\ \to \pm\left\{1,\ 2,\ \frac{1}{2}\right\} \end{array}$$

$$\to 2x^2 - 4 = 0 \Rightarrow \boxed{x = \pm\sqrt{2}}$$

The zeros are: $\boxed{x = -\frac{1}{2},\ 1,\ -\sqrt{2},\ \sqrt{2}}$

$$\begin{array}{ccccccccc} & -\sqrt{2} & & -\frac{1}{2} & & 1 & & \sqrt{2} & \\ + & | & - & | & + & | & - & | & + \end{array}$$



$f(x) > 0 \quad \boxed{(-\infty,\ -\sqrt{2}) \cup \left(-\frac{1}{2},\ 1\right) \cup (\sqrt{2},\ \infty)}$

$f(x) < 0 \quad \boxed{\left(-\sqrt{2},\ -\frac{1}{2}\right) \cup (1,\ \sqrt{2})}$

Exercise

Let $f(x) = 4x^5 - 8x^4 - x + 2$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$\begin{aligned} f(x) &= 4x^4(x-2) - (x-2) \\ &= (x-2)\left(4x^4 - 1\right) = 0 \end{aligned}$$

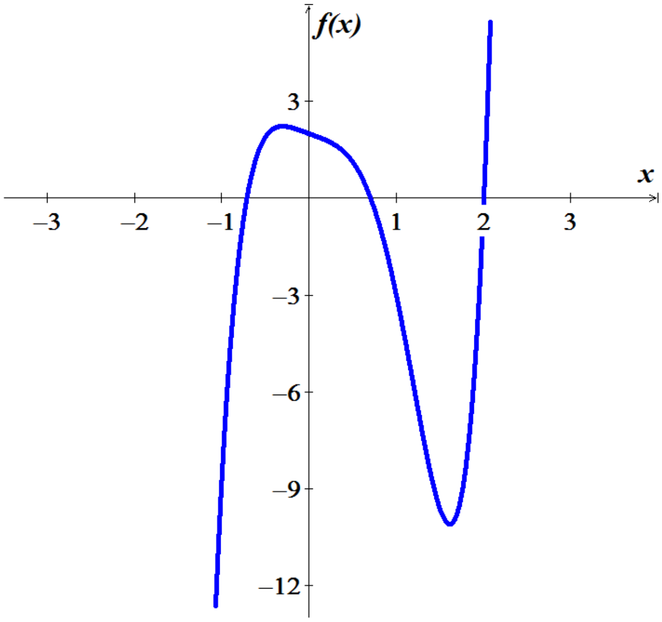
$$4x^4 - 1 = 0 \quad \Rightarrow \quad \begin{cases} x^2 = -\frac{1}{2} & \mathbb{C} \\ x^2 = \frac{1}{2} & x = \pm\frac{\sqrt{2}}{2} \end{cases}$$

The zeros are: $x = 2,\ -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}$

	$-\frac{\sqrt{2}}{2}$		$\frac{\sqrt{2}}{2}$		2	
$-$		$+$		$-$		$+$

$f(x) > 0 \quad \left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right) \cup (2,\ \infty)$

$f(x) < 0 \quad \left(-\infty,\ -\frac{\sqrt{2}}{2}\right) \cup \left(\frac{\sqrt{2}}{2},\ 2\right)$



Exercise

Let $f(x) = x^5 - 7x^4 + 19x^3 - 37x^2 + 60x - 36$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

possibilities: $\pm\left\{\frac{36}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12,\ 18,\ 36\}$

1	1	−7	19	−37	60	−36
		1	−6	13	−24	36
3	1	−6	13	−24	36	0
		3	−9	12	−36	
3	1	−3	4	−12	0	
		3	0	12		
	1	0	4	0		

$x^4 - 6x^3 + 13x^2 - 24x + 36 = 0 \ \rightarrow\ \pm\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

$x^3 - 3x^2 + 4x - 12 = 0 \ \rightarrow\ \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

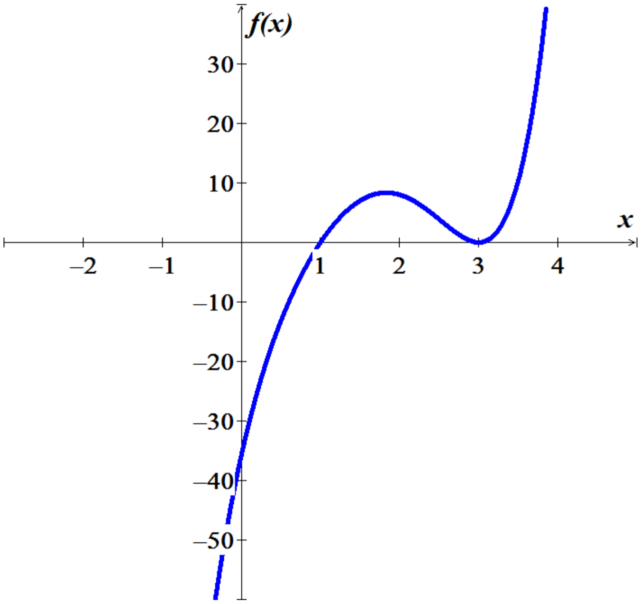
$x^2 + 4 = 0 \quad \Rightarrow \quad x = \pm 2i$

The zeros are: $x = 1,\ 3,\ 3$

	1		3	
$-$		$+$		$+$

$f(x) > 0 \quad (1,\ 3) \cup (3,\ \infty)$

$f(x) < 0 \quad (-\infty,\ 1)$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

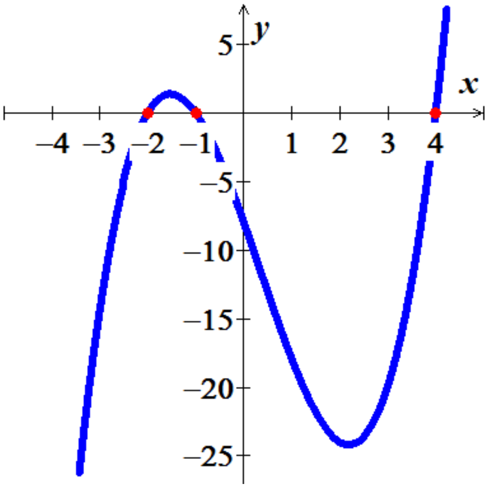
$$f(x) = x^3 - x^2 - 10x - 8$$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{8}{1}\right\} = \pm\{1,\ 2,\ 4,\ 8\}$

$$\begin{array}{r|rrrr} -1 & 1 & -1 & -10 & -8 \\ & & -1 & 2 & 8 \\ \hline & 1 & -2 & -8 & 0 \end{array} \rightarrow \quad x^2 - 2x - 8 = 0$$

$x = -1,\ -2,\ 4$



$-2 \qquad -1 \qquad 4$

–	+	–	+

$f(x) > 0 \quad (-2,\ -1) \cup (4,\ \infty)$

$f(x) < 0 \quad (-\infty,\ -2) \cup (-1,\ 4)$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

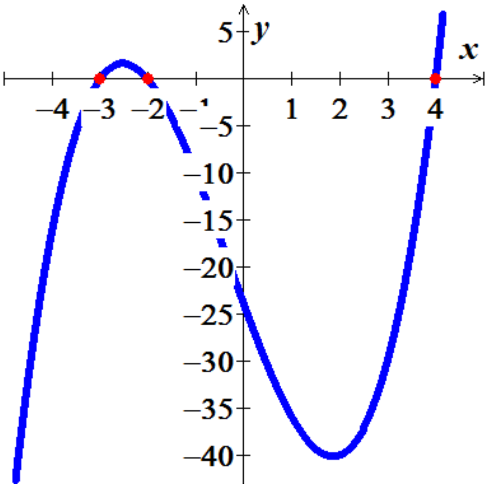
$$f(x) = x^3 + x^2 - 14x - 24$$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{24}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24\}$

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -14 & -24 \\ & & -2 & 2 & 24 \\ \hline & 1 & -1 & -12 & 0 \end{array} \rightarrow \quad x^2 - x - 12 = 0$$

$x = -2,\ -3,\ 4$



$-3 \qquad -2 \qquad 4$

–	+	–	+

$f(x) > 0 \quad (-3,\ -2) \cup (4,\ \infty)$

$f(x) < 0 \quad (-\infty,\ -3) \cup (-2,\ 4)$

Exercise

Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of $f(x)$

$$f(x)=2x^3-3x^2-17x+30$$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{30}{2}\right\}=\pm\left\{1,\ 2,\ 3,\ 5,\ 6,\ 10,\ 15,\ 30,\ \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \frac{15}{2}\right\}$

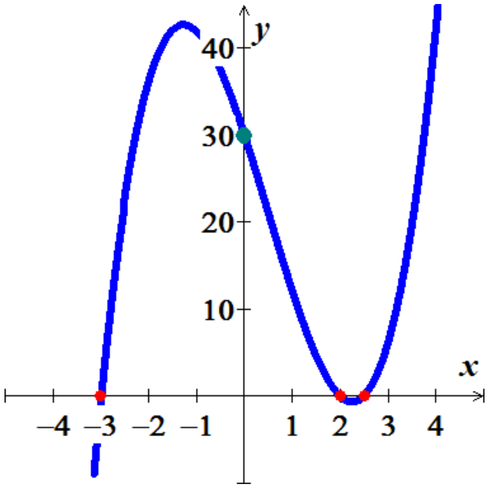
$$\begin{array}{r|rrrr} 2 & 2 & -3 & -17 & 30 \\ & & 4 & 2 & -30 \\ \hline & 2 & 1 & -15 & \boxed{0} \end{array} \rightarrow\ 2x^2+x-15=0$$

$x=2,\ -3,\ \frac{5}{2}$

$$\begin{array}{ccccccc} & -3 & & 2 & & \frac{5}{2} & \\ \hline - & \vert & + & \vert & - & \vert & + \end{array}$$

$f(x)>0\quad (-3,\ 2)\cup\left(\frac{5}{2},\ \infty\right)$

$f(x)<0\quad (-\infty,\ -3)\cup\left(2,\ \frac{5}{2}\right)$



Exercise

Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of $f(x)$

$$f(x)=12x^3+8x^2-3x-2$$

Solution

possibilities for $\frac{c}{d}:\pm\left\{\frac{2}{12}\right\}=\pm\left\{1,\ 2,\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{2},\ \frac{1}{6},\ \frac{1}{12}\right\}$

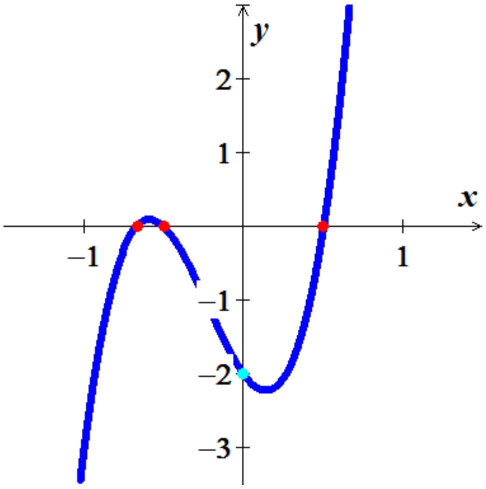
$$\begin{array}{r|rrrr} \frac{1}{2} & 12 & 8 & -3 & -2 \\ & & 6 & 7 & 2 \\ \hline & 12 & 14 & 4 & \boxed{0} \end{array} \rightarrow\ 12x^2+14x+4=0$$

$x=\frac{1}{2},\ -\frac{1}{2},\ -\frac{2}{3}$

$$\begin{array}{ccccccc} & -\frac{2}{3} & & -\frac{1}{2} & & \frac{1}{2} & \\ \hline - & \vert & + & \vert & - & \vert & + \end{array}$$

$f(x)>0\quad \left(-\frac{2}{3},\ -\frac{1}{2}\right)\cup\left(\frac{1}{2},\ \infty\right)$

$f(x)<0\quad \left(-\infty,\ -\frac{2}{3}\right)\cup\left(-\frac{1}{2},\ \frac{1}{2}\right)$



Exercise

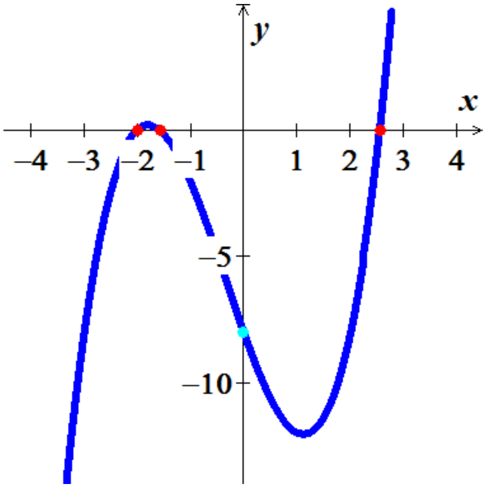
Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 + x^2 - 6x - 8$$

Solution

$$\textit{possibilities for } \frac{c}{d} : \pm\left\{\frac{8}{1}\right\} = \pm\{1,\ 2,\ 4,\ 8\}$$

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -6 & -8 \\ & & -2 & 2 & 8 \\ \hline & 1 & -1 & -4 & \boxed{0} \end{array} \rightarrow\ x^2 - x - 4 = 0$$



$$x = \frac{1 \pm \sqrt{1+16}}{2}$$

$$\boxed{x = -2,\ \frac{1 \pm \sqrt{17}}{2}}$$

$$\begin{array}{c|c|c|c} \multicolumn{1}{c}{} & -2 & \frac{1-\sqrt{17}}{2} & \frac{1+\sqrt{17}}{2} \\ - & + & - & + \end{array}$$

$$f(x) > 0 \quad \boxed{\left(-2,\ \frac{1-\sqrt{17}}{2}\right) \cup \left(\frac{1+\sqrt{17}}{2},\ \infty\right)}$$

$$f(x) < 0 \quad \boxed{(-\infty,\ -2) \cup \left(\frac{1-\sqrt{17}}{2},\ \frac{1+\sqrt{17}}{2}\right)}$$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 19x - 30$$

Solution

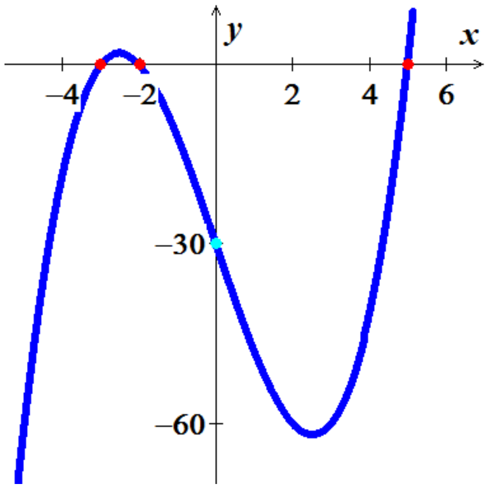
$$\textit{possibilities for } \frac{c}{d} : \pm\left\{\frac{30}{1}\right\} = \pm\{1,\ 2,\ 3,\ 5,\ 6,\ 15,\ 30\}$$

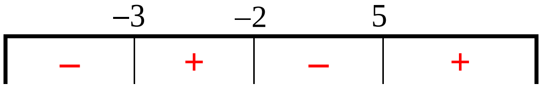
$$\begin{array}{r|rrrr} -2 & 1 & 0 & -19 & -30 \\ & & -2 & 4 & 30 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array} \rightarrow\ x^2 - 2x - 15$$

$$x = \frac{2 \pm \sqrt{4+60}}{2}$$

$$=\begin{cases}\frac{2-8}{2}=-3\\[4pt] \frac{2+8}{2}=5\end{cases}$$

$x=-2,\ \ -3,\ 5$





$f(x)>0\quad (-3,\ -2)\cup(5,\ \infty)$

$f(x)<0\quad (-\infty,\ -3)\cup(-2,\ 5)$

Exercise

Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of $f(x)$

$$f(x)=2x^3+x^2-25x+12$$

Solution

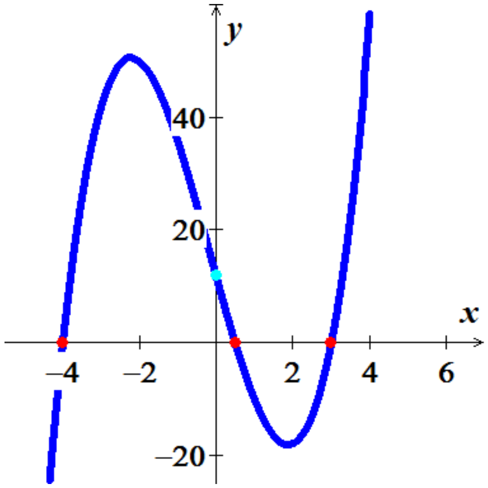
possibilities for $\frac{c}{d}:\pm\left\{\frac{12}{2}\right\}=\pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 12,\ \frac{1}{2},\ \frac{3}{2}\right\}$

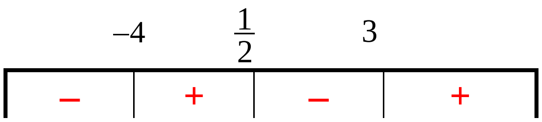
$$\begin{array}{r|rrrr} 3 & 2 & 1 & -25 & 12 \\ & & 6 & 21 & -12 \\ \hline & 2 & 7 & -4 & \boxed{0} \end{array} \quad \rightarrow \quad 2x^2+7x-4$$

$$x=\frac{-7\pm\sqrt{49+32}}{4}$$

$$=\begin{cases}\frac{-7-9}{4}=-4\\[4pt] \frac{-7+9}{4}=\frac{1}{2}\end{cases}$$

$x=-4,\ \frac{1}{2},\ 3$





$f(x)>0\quad \left(-4,\ \frac{1}{2}\right)\cup(3,\ \infty)$

$f(x)<0\quad (-\infty,\ -1)\cup\left(\frac{1}{2},\ 3\right)$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 3x^3 + 11x^2 - 6x - 8$$

Solution

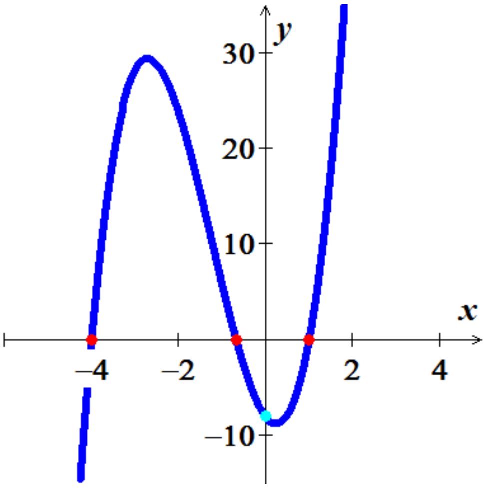
possibilities for $\frac{c}{d} : \pm \left\{ \frac{8}{3} \right\} = \pm \left\{ 1,\ 2,\ 4,\ 8,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{8}{3} \right\}$

$$\begin{array}{r|rrrr} 1 & 3 & 11 & -6 & -8 \\ & & 3 & 14 & 8 \\ \hline & 3 & 14 & 8 & \boxed{0} \end{array} \quad \rightarrow \quad 3x^2 + 14x + 8$$

$$x = \frac{-14 \pm \sqrt{196 - 96}}{6}$$

$$= \begin{cases} \frac{-14 - 10}{6} = -4 \\ \frac{-14 + 10}{6} = -\frac{2}{3} \end{cases}$$

$$x = -4,\ -\frac{2}{3},\ 1$$



	-4		$-\frac{2}{3}$		1	
$-$		$+$		$-$		$+$

$f(x) > 0 \quad \left(-4,\ -\frac{2}{3}\right) \cup (1,\ \infty)$

$f(x) < 0 \quad (-\infty,\ -4) \cup \left(-\frac{2}{3},\ 1\right)$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 2x^3 + 9x^2 - 2x - 9$$

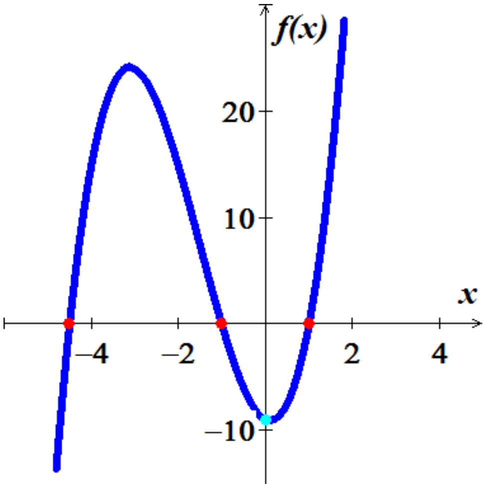
Solution

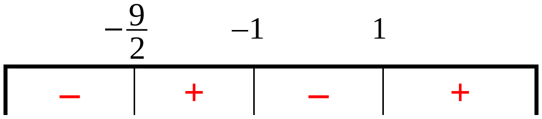
possibilities for $\frac{c}{d} : \pm \left\{ \frac{9}{2} \right\} = \pm \left\{ 1,\ 3,\ 9,\ \frac{1}{2},\ \frac{3}{2},\ \frac{9}{2} \right\}$

$$\begin{array}{r|rrrr} 1 & 2 & 9 & -2 & -9 \\ & & 2 & 11 & 9 \\ \hline & 2 & 11 & 9 & \boxed{0} \end{array} \quad \rightarrow \quad 2x^2 + 11x + 9$$

$x = -1,\ -\frac{9}{2}$ $\qquad a - b + c = 0 \ \rightarrow\ x = -1,\ -\frac{c}{a}$

$x = -\frac{9}{2},\ -1,\ 1$





$f(x) > 0 \quad \left(-\frac{9}{2},\ -1\right) \cup (1,\ \infty)$

$f(x) < 0 \quad \left(-\infty,\ -\frac{9}{2}\right) \cup (-1,\ 1)$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 + 3x^2 - 6x - 8$$

Solution

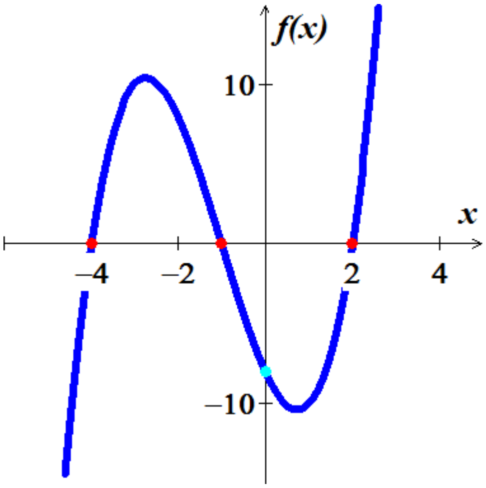
possibilities for $\frac{c}{d} : \pm\left\{\frac{8}{1}\right\} = \pm\{1,\ 2,\ 4,\ 8\}$

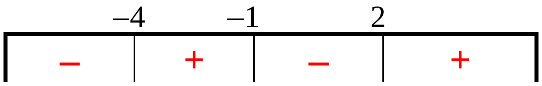
$$\begin{array}{r|rrrr} -1 & 1 & 3 & -6 & -8 \\ & & -1 & -2 & 8 \\ \hline & 1 & 2 & -8 & \boxed{0} \end{array} \rightarrow \quad x^2 + 2x - 8 = 0$$

$$x = \frac{-2 \pm \sqrt{4+32}}{2}$$

$$= \begin{cases} \frac{-2-6}{2} = -4 \\ \frac{-2+6}{2} = 2 \end{cases}$$

$x = -4,\ -1,\ 2$





$f(x) > 0 \quad (-4,\ -1) \cup (2,\ \infty)$

$f(x) < 0 \quad (-\infty,\ -4) \cup (-1,\ 2)$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

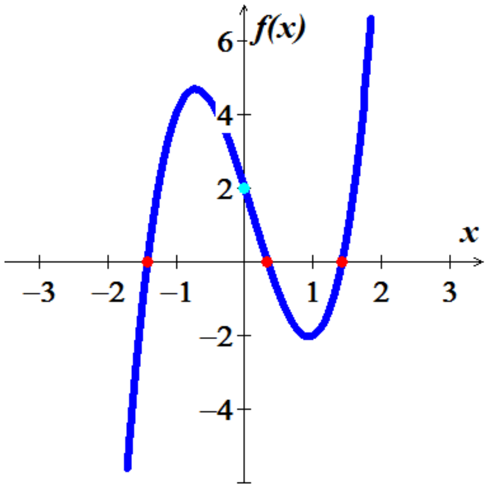
$$f(x) = 3x^3 - x^2 - 6x + 2$$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{2}{3}\right\} = \pm\left\{1,\ 2,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -1 & -6 & 2 \\ & & 1 & 0 & -2 \\ \hline & 3 & 0 & -6 & \boxed{0} \end{array} \quad \rightarrow \quad 3x^2 - 6 = 0$$

$$x = \frac{1}{3},\ \pm\sqrt{2}$$



$-\sqrt{2}$ $\frac{1}{3}$ $\sqrt{2}$

–	+	–	+

$f(x) > 0$ $\left(-\sqrt{2},\ \frac{1}{3}\right) \cup \left(\sqrt{2},\ \infty\right)$

$f(x) < 0$ $\left(-\infty,\ -\sqrt{2}\right) \cup \left(\frac{1}{3},\ \sqrt{2}\right)$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 8x^2 + 8x + 24$$

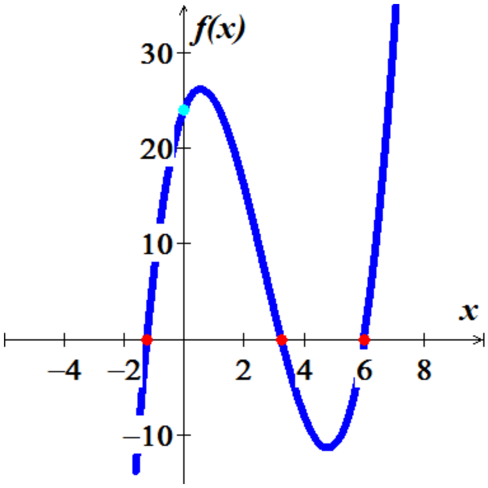
Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{24}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24\}$

$$\begin{array}{r|rrrr} 6 & 1 & -8 & 8 & 24 \\ & & 6 & -12 & -24 \\ \hline & 1 & -2 & -4 & \boxed{0} \end{array} \quad \rightarrow \quad x^2 - 2x - 4 = 0$$

$$x = \frac{2 \pm \sqrt{4+16}}{2}$$

$$x = 6,\ 1 \pm \sqrt{5}$$



$1-\sqrt{5}$ $1+\sqrt{5}$ 6

–	+	–	+

$f(x) > 0 \quad \left(1-\sqrt{5},\ 1+\sqrt{5}\right) \cup (6,\ \infty)$

$f(x) < 0 \quad \left(-\infty,\ 1-\sqrt{5}\right) \cup \left(1+\sqrt{5},\ 6\right)$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 7x^2 - 7x + 69$$

Solution

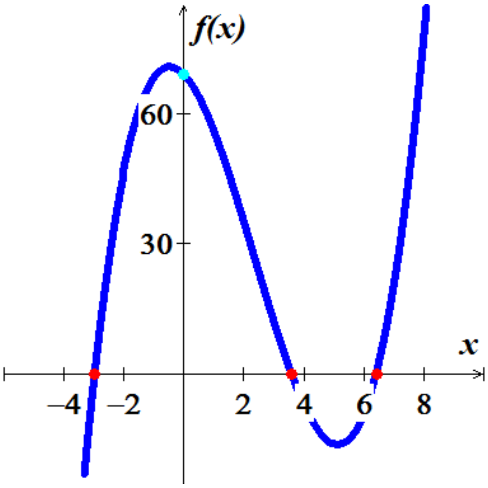
possibilities for $\frac{c}{d} : \pm\left\{\frac{69}{1}\right\} = \pm\{1,\ 3,\ 23,\ 69\}$

$$\begin{array}{r|rrrr} -3 & 1 & -7 & -7 & 69 \\ & & -3 & 30 & -69 \\ \hline & 1 & -10 & 23 & \boxed{0} \end{array} \quad \to \quad x^2 - 10x + 23 = 0$$

$$x = \frac{10 \pm \sqrt{100-92}}{2}$$

$$= \frac{10 \pm 2\sqrt{2}}{2}$$

$$x = -3,\ 5 \pm \sqrt{2}$$



$$\begin{array}{ccccccc} & -3 & & 5-\sqrt{2} & & 5+\sqrt{2} & \\ - & | & + & | & - & | & + \end{array}$$

$f(x) > 0 \quad \left(-3,\ 5-\sqrt{2}\right) \cup \left(5+\sqrt{2},\ \infty\right)$

$f(x) < 0 \quad (-\infty,\ -3) \cup \left(5-\sqrt{2},\ 5+\sqrt{2}\right)$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 3x - 2$$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array} \quad \to \quad x^2 - x - 2 = 0$$

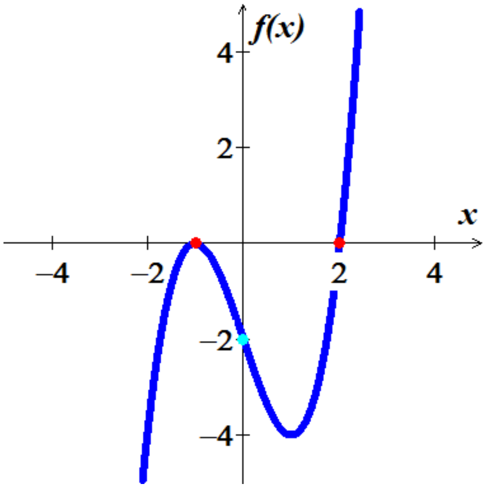
$x = -1,\ 2$ $\qquad\qquad$ $a - b + c = 0 \quad \to \quad x = -1,\ -\frac{c}{a}$

$x = -1,\ -1,\ 2$

	-1		2	
$-$		$-$		$+$

$f(x) > 0 \quad (2,\ \infty)$

$f(x) < 0 \quad (-\infty,\ -1) \cup (-1,\ 2)$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 2x + 1$$

Solution

possibilities for $\frac{c}{d} : \pm\{1\}$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -2 & 1 \\ & & 1 & 1 & 1 \\ \hline & 1 & 1 & -1 & \boxed{0} \end{array} \quad \to \quad x^2 + x - 1 = 0$$

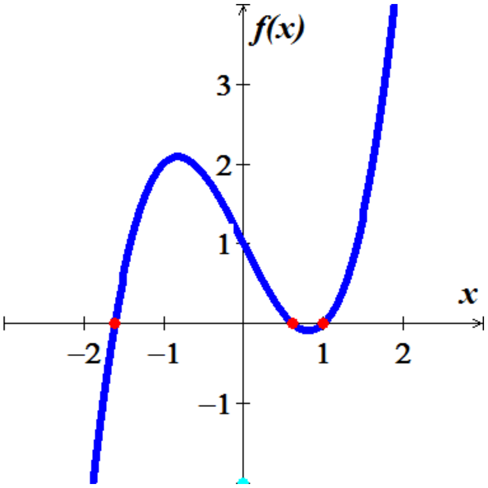
$x = \dfrac{-1 \pm \sqrt{5}}{2}$

$x = 1,\ \dfrac{-1 \pm \sqrt{5}}{2}$

	$\frac{-1-\sqrt{5}}{2}$		$\frac{-1+\sqrt{5}}{2}$		1	
$-$		$+$		$-$		$+$

$f(x) > 0 \quad \left(\dfrac{-1-\sqrt{5}}{2},\ \dfrac{-1+\sqrt{5}}{2}\right) \cup (1,\ \infty)$

$f(x) < 0 \quad \left(-\infty,\ \dfrac{-1-\sqrt{5}}{2}\right) \cup \left(\dfrac{-1+\sqrt{5}}{2},\ 1\right)$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 2x^2 - 11x + 12$$

Solution

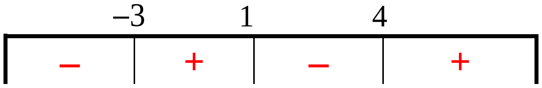
possibilities for $\frac{c}{d} : \pm\left\{\frac{12}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -11 & 12 \\ & & 1 & -1 & 12 \\ \hline & 1 & -1 & -12 & \boxed{0} \end{array} \quad \to \quad x^2 - x - 12 = 0$$

$$x = \frac{1 \pm \sqrt{1+48}}{2}$$

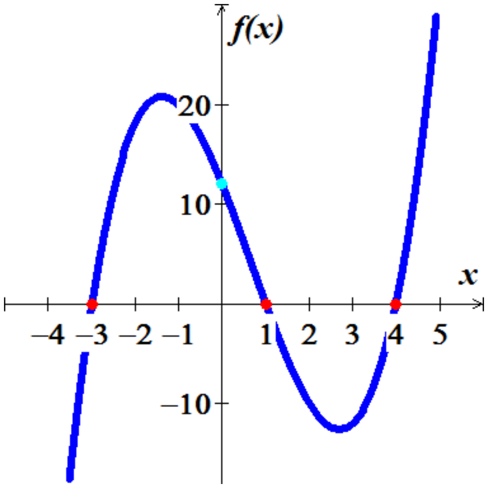
$$= \begin{cases} \frac{1-7}{2} = -3 \\ \frac{1+7}{2} = 4 \end{cases}$$

$\underline{x = -3,\ 1,\ 4}|$



$f(x) > 0 \quad \underline{(-3,\ 1) \cup (4,\ \infty)}|$

$f(x) < 0 \quad \underline{(-\infty,\ -3) \cup (1,\ 4)}|$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 2x^2 - 7x - 4$$

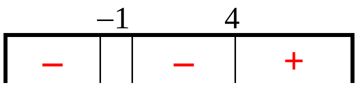
Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{4}{1}\right\} = \pm\{1,\ 2,\ 4\}$

$$\begin{array}{r|rrrr} -1 & 1 & -2 & -7 & -4 \\ & & -1 & 3 & 4 \\ \hline & 1 & -3 & -4 & \boxed{0} \end{array} \quad \to \quad x^2 - 3x - 4 = 0$$

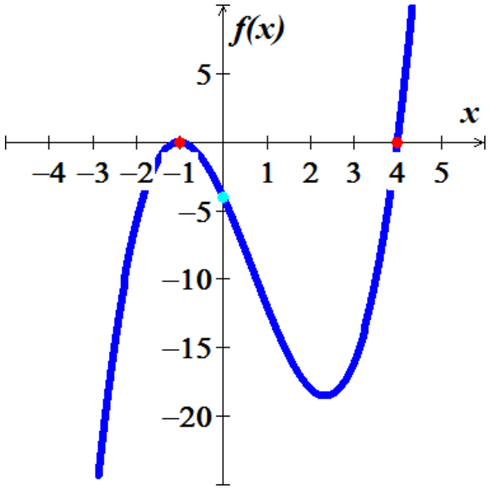
$\underline{x = -1,\ 4}|$ $\qquad a - b + c = 0 \quad \to \quad x = -1,\ -\frac{c}{a}$

$\underline{x = -1,\ -1,\ 4}|$



$f(x) > 0 \quad \boxed{(4,\ \infty)}$

$f(x) < 0 \quad \boxed{(-\infty,\ -1) \cup (-1,\ 4)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 10x - 12$$

Solution

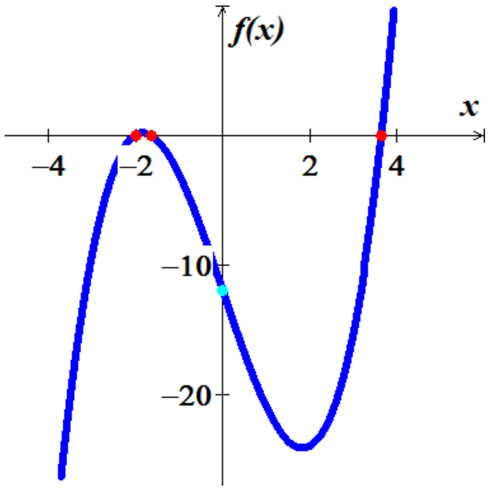
possibilities for $\frac{c}{d} : \pm\left\{\frac{12}{1}\right\} = \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

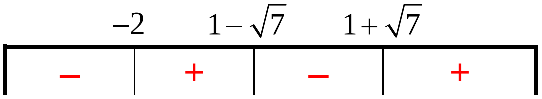
$$\begin{array}{r|rrrr} -2 & 1 & 0 & -10 & -12 \\ & & -2 & 4 & 12 \\ \hline & 1 & -2 & -6 & \boxed{0} \end{array} \quad \to \quad x^2 - 2x - 6 = 0$$

$$x = \frac{2 \pm \sqrt{4+24}}{2}$$

$$= \frac{2 \pm 2\sqrt{7}}{2}$$

$$\boxed{x = -2,\ 1 \pm \sqrt{7}}$$





$f(x) > 0 \quad \boxed{\left(-2,\ 1-\sqrt{7}\right) \cup \left(1+\sqrt{7},\ \infty\right)}$

$f(x) < 0 \quad \boxed{(-\infty,\ -2) \cup \left(1-\sqrt{7},\ 1+\sqrt{7}\right)}$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^3 - 5x^2 + 17x - 13$$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{13}{1}\right\} = \pm\{1,\ 13\}$

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 17 & -13 \\ & & 1 & -4 & 13 \\ \hline & 1 & -4 & 13 & \boxed{0} \end{array} \quad \rightarrow \quad x^2 - 4x + 13 = 0$$

$$x = \frac{4 \pm \sqrt{16 - 52}}{2}$$

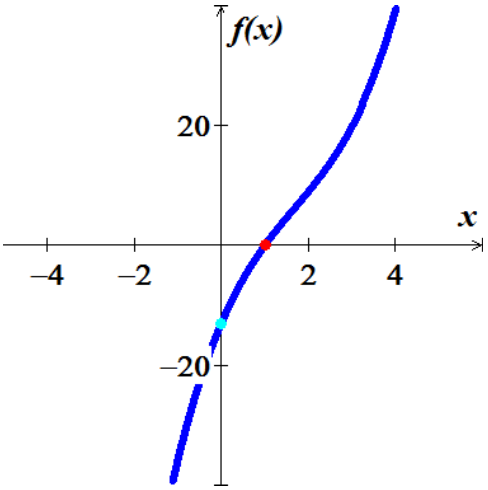
$$= \frac{4 \pm 6i}{2}$$

$$x = 1,\ 2 \pm 3i$$

$$\begin{array}{c|c} \multicolumn{2}{c}{1} \\ \hline - & + \end{array}$$

$f(x) > 0 \quad (1,\ \infty)$

$f(x) < 0 \quad (-\infty,\ 1)$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 6x^3 + 25x^2 - 24x + 5$$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{5}{6}\right\} = \pm\left\{1,\ 5,\ \frac{1}{2},\ \frac{5}{2},\ \frac{1}{3},\ \frac{5}{3},\ \frac{1}{6},\ \frac{5}{6}\right\}$

$$\begin{array}{r|rrrr} -5 & 6 & 25 & -24 & 5 \\ & & -30 & 25 & -5 \\ \hline & 6 & -5 & 1 & \boxed{0} \end{array} \quad \rightarrow \quad 6x^2 - 5x + 1 = 0$$

$$x = \frac{5 \pm \sqrt{25 - 24}}{12}$$

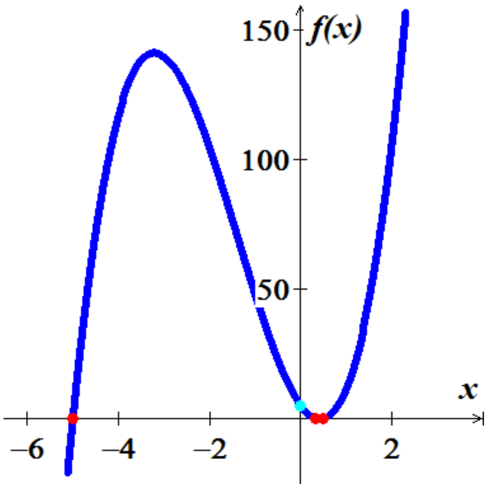
$$= \begin{cases} \frac{5-1}{12} = \frac{1}{3} \\ \frac{5+1}{12} = \frac{1}{2} \end{cases}$$

$x = -5,\ \frac{1}{3},\ \frac{1}{2}$

	-5		$\frac{1}{3}$		$\frac{1}{2}$	
$-$		$+$		$-$		$+$

$f(x) > 0 \quad \left(-5,\ \frac{1}{3}\right) \cup \left(\frac{1}{2},\ \infty\right)$

$f(x) < 0 \quad (-\infty,\ -5) \cup \left(\frac{1}{3},\ \frac{1}{2}\right)$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 8x^3 + 18x^2 + 45x + 27$$

Solution

$$\text{possibilities}: \quad \pm\left\{\frac{27}{8}\right\} = \pm\left\{\frac{1,\ 3,\ 9,\ 27}{1,\ 2,\ 4,\ 8}\right\}$$

$$= \pm\left\{1,\ 3,\ 9,\ 27,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{3}{2},\ \frac{3}{4},\ \frac{3}{8},\ \frac{9}{2},\ \frac{9}{4},\ \frac{9}{8},\ \frac{27}{2},\ \frac{27}{4},\ \frac{27}{8}\right\}$$

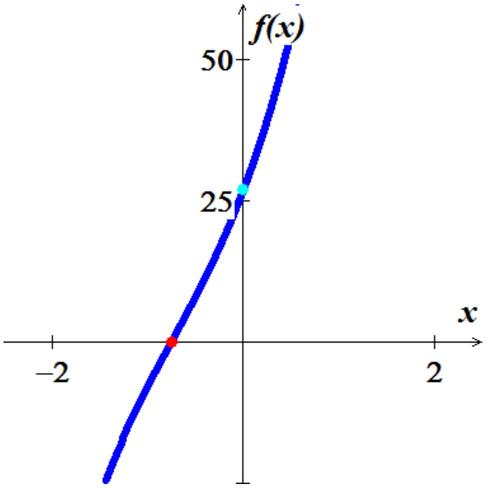
$$\begin{array}{r|rrrr} -\frac{3}{4} & 8 & 18 & 45 & 27 \\ & & -6 & -9 & -27 \\ \hline & 8 & 12 & 36 & \boxed{0} \end{array} \quad \to \ 8x^2 + 12x + 36 = 0$$

$x = -\frac{3}{4},\ -\frac{3}{4} \pm i\frac{3\sqrt{7}}{4}$

	$-\frac{3}{4}$	
$-$		$+$

$f(x) > 0 \quad \left(-\frac{3}{4},\ \infty\right)$

$f(x) < 0 \quad \left(-\infty,\ -\frac{3}{4}\right)$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 3x^3 - x^2 + 11x - 20$$

Solution

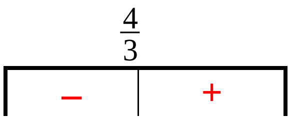
$$possibilities: \ \pm\left\{\frac{20}{3}\right\} = \pm\left\{\frac{1,\ 2,\ 4,\ 5,\ 10,\ 20}{1,\ 3}\right\}$$

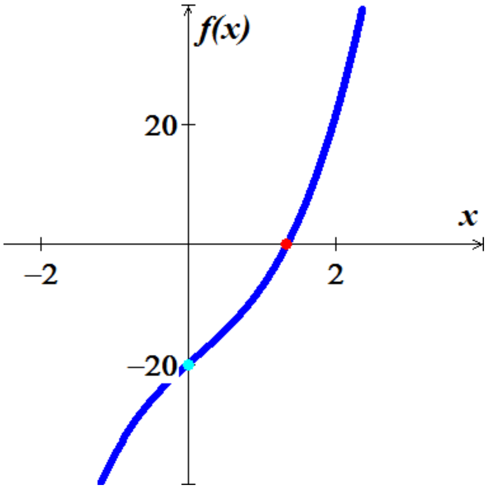
$$= \pm\left\{1,\ 2,\ 4,\ 5,\ 10,\ 20,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{5}{3},\ \frac{10}{3},\ \frac{20}{3}\right\}$$

$$\begin{array}{r|rrrr} \frac{4}{3} & 3 & -1 & 11 & -20 \\ & & 4 & 4 & 20 \\ \hline & 3 & 3 & 15 & \boxed{0} \end{array} \quad \to \ 3x^2 + 3x + 15 = 0$$

$$x = \frac{-3 \pm \sqrt{9 - 180}}{6}$$

$$\underline{x = \frac{4}{3},\ -\frac{1}{2} \pm i\frac{\sqrt{19}}{2}}$$





$f(x) > 0$ $\quad \left(\frac{4}{3},\ \infty\right)$

$f(x) < 0$ $\quad \left(-\infty,\ \frac{4}{3}\right)$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^4 - x^3 - 9x^2 + 3x + 18$$

Solution

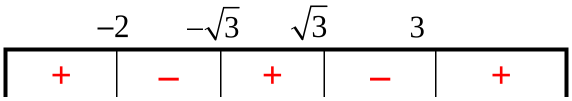
$$possibilities\ for\ \frac{c}{d}: \pm\left\{\frac{18}{1}\right\} = \pm\{1,\ 2,\ 3,\ 6,\ 9,\ 18\}$$

$$\begin{array}{r|rrrrr} -2 & 1 & -1 & -9 & 3 & 18 \\ & & -2 & 6 & 6 & -18 \\ \hline 3 & 1 & -3 & -3 & 9 & 0 \\ & & 3 & 0 & -9 & \\ \hline & 1 & 0 & -3 & 0 & \end{array}$$

$$\to x^3 - 3x^2 - 3x + 9 = 0 \ \to \ \pm\left\{\frac{9}{1}\right\} = \pm\{1,\ 3,\ 9\}$$

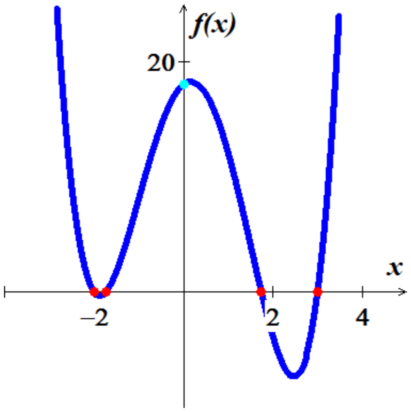
$$\to x^2 - 3 = 0 \Rightarrow \underline{x = \pm\sqrt{3}}$$

$$\underline{x = -2,\ 3,\ \pm\sqrt{3}}$$



$f(x) > 0$ $\quad \boxed{(-\infty,\ -2)\cup\left(-\sqrt{3},\ \sqrt{3}\right)\cup(3,\ \infty)}$

$f(x) < 0$ $\quad \boxed{\left(-2,\ -\sqrt{3}\right)\cup\left(\sqrt{3},\ 3\right)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 2x^4 - 9x^3 + 9x^2 + x - 3$$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$$\begin{array}{r|rrrrr}
1 & 2 & -9 & 9 & 1 & -3 \\
 & & 2 & -7 & 2 & 3 \\
\hline
1 & 2 & -7 & 2 & 3 & 0 \\
 & & 2 & -5 & -3 & \\
\hline
 & 2 & -5 & -3 & 0 &
\end{array}$$

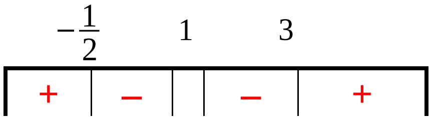
$\to\ 2x^3 - 7x^2 + 2x + 3 = 0\ \to\ \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$\to\ 2x^2 - 5x - 3 = 0$

$$x = \frac{5 \pm \sqrt{25 + 24}}{4}$$

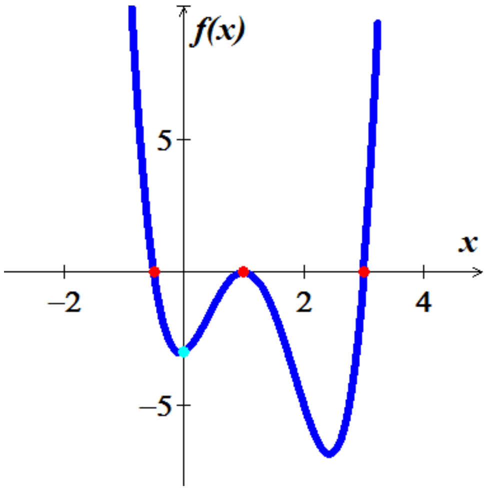
$$= \begin{cases} \frac{5-7}{4} = -\frac{1}{2} \\ \frac{5+7}{4} = 3 \end{cases}$$

$\boxed{x = 1,\ 1,\ -\frac{1}{2},\ 3}$



$f(x) > 0$ $\quad \boxed{\left(-\infty,\ -\frac{1}{2}\right)\cup(3,\ \infty)}$

$f(x) < 0$ $\quad \boxed{\left(-\frac{1}{2},\ 1\right)\cup(1,\ 3)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 6x^4 + 5x^3 - 17x^2 - 6x$$

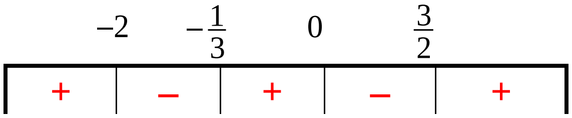
Solution

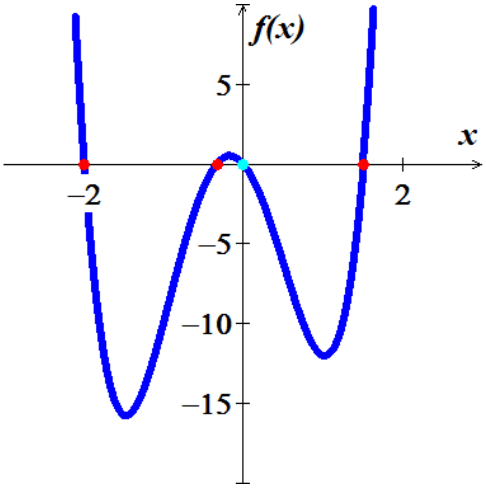
$$x\left(6x^3 + 5x^2 - 17x - 6\right) = 0 \quad \rightarrow \quad \underline{x = 0}$$

$$possibilities: \ \pm\left\{\frac{6}{6}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3}\right\}$$

$$\begin{array}{r|rrrr} -2 & 6 & 5 & -17 & -6 \\ & & -12 & 14 & 6 \\ \hline & 6 & -7 & -3 & \boxed{0} \end{array} \quad \rightarrow \ 6x^2 - 7x - 3 = 0$$

$$\underline{x = 0,\ -2,\ -\frac{1}{3},\ \frac{3}{2}}$$





$$f(x) > 0 \quad \underline{(-\infty,\ -2) \cup \left(-\frac{1}{3},\ 0\right) \cup \left(\frac{3}{2},\ \infty\right)}$$

$$f(x) < 0 \quad \underline{\left(-2,\ -\frac{1}{3}\right) \cup \left(0,\ \frac{3}{2}\right)}$$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^4 - 2x^2 - 16x - 15$$

Solution

$$possibilities: \ \pm\left\{\frac{15}{1}\right\} = \pm\{1,\ 3,\ 5,\ 15\}$$

$$\begin{array}{r|rrrrr} -1 & 1 & 0 & -2 & -16 & -15 \\ & & -1 & 1 & 1 & 15 \\ \hline 3 & 1 & -1 & -1 & -15 & 0 \\ & & 3 & 6 & 15 & \\ \hline & 1 & 2 & 5 & 0 & \end{array} \quad \rightarrow \ x^3 - x^2 - x - 15 = 0 \ \rightarrow \ \pm\left\{\frac{15}{1}\right\} = \pm\{1,\ 3,\ 5,\ 15\}$$

$$\rightarrow \ x^2 + 2x + 5 = 0$$

$$x = \frac{-2 \pm \sqrt{-16}}{2}$$

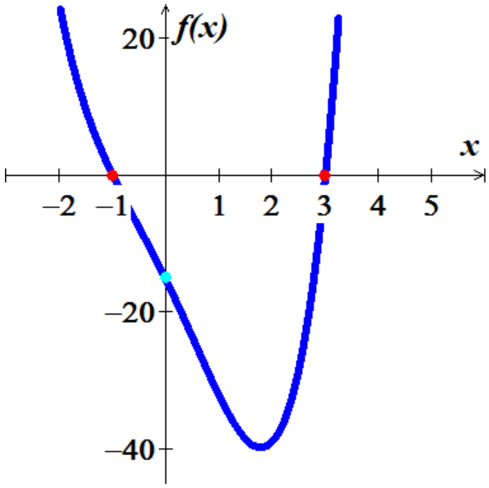
$$= -1 \pm 2i$$

$\underline{x=-1,\ 3,\ \ -1\pm 2i}\,|$

	-1		3	
$+$		$-$		$+$

$f(x)>0 \quad \underline{(-\infty,\ -1)\cup(3,\ \infty)}\,|$

$f(x)<0 \quad \underline{(-1,\ 3)}\,|$



Exercise

Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of $f(x)$

$$f(x)=x^4-2x^3-5x^2+8x+4$$

Solution

possibilities : $\pm\left\{\frac{4}{1}\right\}=\pm\{1,\ 2,\ 4\}$

$$\begin{array}{r|rrrrr} 2 & 1 & -2 & -5 & 8 & 4 \\ & & 2 & 0 & -10 & -4 \\ \hline -2 & 1 & 0 & -5 & -2 & 0 \\ & & -2 & 4 & 2 & \\ \hline & 1 & -2 & -1 & 0 & \end{array}$$

$\rightarrow x^3-5x-2=0 \rightarrow \pm\left\{\frac{2}{1}\right\}=\pm\{1,\ 2\}$

$\rightarrow x^2-2x-1=0$

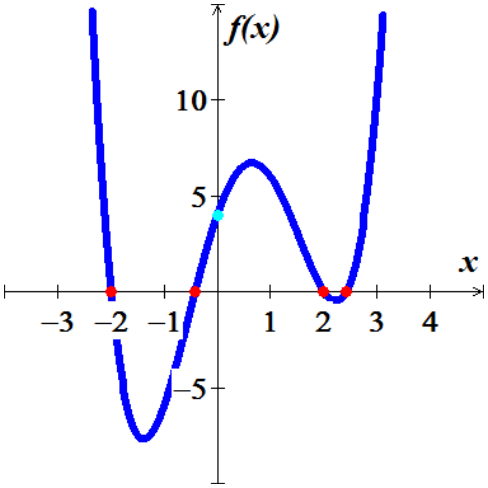
$$x=\frac{2\pm\sqrt{8}}{2}$$

$\underline{x=-2,\ 2,\ 1\pm\sqrt{2}}\,|$

	-2		$1-\sqrt{2}$		2		$1+\sqrt{2}$	
$+$		$-$		$+$		$-$		$+$

$f(x)>0 \quad \underline{(-\infty,\ -2)\cup\left(1-\sqrt{2},\ 2\right)\cup\left(1+\sqrt{2},\ \infty\right)}\,|$

$f(x) < 0 \quad \left(-2,\ 1-\sqrt{2}\right) \cup \left(2,\ 1+\sqrt{2}\right)$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 2x^4 - 17x^3 + 4x^2 + 35x - 24$$

Solution

possibilities : $\pm\left\{\frac{24}{2}\right\} = \pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 24,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$$\begin{array}{r|rrrrr}
1 & 2 & -17 & 4 & 35 & -24 \\
 & & 2 & -15 & -11 & 24 \\
\hline
1 & 2 & -15 & -11 & 24 & 0 \\
 & & 2 & -13 & 24 & \\
\hline
 & 2 & -13 & -24 & 0 &
\end{array}$$

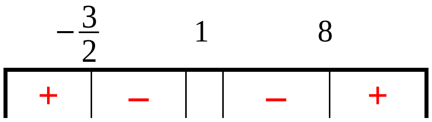
$\to 2x^3 - 15x^2 - 11x + 24 = 0 \to \pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$\to 2x^2 - 13x - 24 = 0$

$$x = \frac{13 \pm \sqrt{169 + 192}}{4}$$

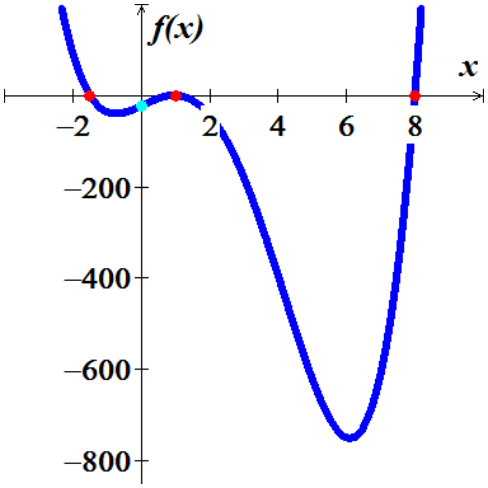
$$= \begin{cases} \frac{13-19}{4} = -\frac{3}{2} \\ \frac{13+19}{4} = 8 \end{cases}$$

$$x = -\frac{3}{2},\ 1,\ 1,\ 8$$



$f(x) > 0 \quad \left(-\infty,\ -\frac{3}{2}\right) \cup (8,\ \infty)$

$f(x) < 0 \quad \left(-\frac{3}{2},\ 1\right) \cup (1,\ 8)$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^4 + x^3 - 3x^2 - 5x - 2$$

Solution

$$possibilities: \ \pm\left\{\frac{2}{1}\right\} = \pm\{1, \ 2\}$$

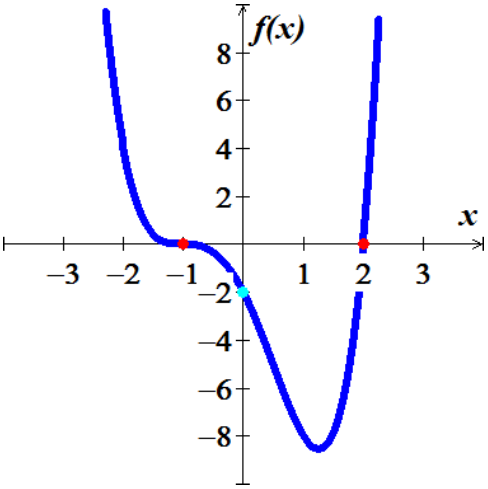
$$\begin{array}{r|rrrrr}
-1 & 1 & 1 & -3 & -5 & -2 \\
 & & -1 & 0 & 3 & 2 \\
\hline
-1 & 1 & 0 & -3 & -2 & 0 \\
 & & -1 & 1 & 2 & \\
\hline
 & 1 & -1 & -2 & 0 &
\end{array}
\begin{array}{l}
\\ \\
\to \ x^3 - 3x - 2 = 0 \ \to \ \pm\left\{\frac{2}{1}\right\} = \pm\{1, \ 2\} \\
\\
\\
\to \ x^2 - x - 2 = 0
\end{array}$$

$$x = \frac{1 \pm \sqrt{1+8}}{2}$$

$$= \begin{cases} \frac{1-3}{2} = -1 \\ \frac{1+3}{2} = 2 \end{cases}$$

$$\underline{x = -1, \ -1, \ -1, \ 2}$$

$$\begin{array}{|c|c|c|}
 & -1 & 2 \\
\hline
+ & - & +
\end{array}$$



$f(x) > 0 \quad \underline{(-\infty, \ -1) \cup (2, \ \infty)}$

$f(x) < 0 \quad \underline{(-2, \ 2)}$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 6x^4 - 17x^3 - 11x^2 + 42x$$

Solution

$$x\left(6x^3 - 17x^2 - 11x + 42\right) = 0$$

$$x = 0 \quad 6x^3 - 17x^2 - 11x + 42 = 0$$

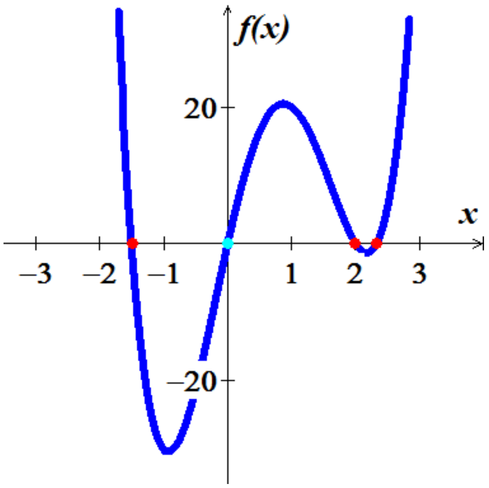
$$possibilities: \ \pm\left\{\frac{42}{6}\right\} = \pm\left\{1, \ 2, \ 3, \ 6, \ 7, \ 14, \ 21, \ 42, \ \frac{1}{2}, \ \frac{3}{2}, \ \frac{7}{2}, \ \frac{21}{2}, \ \frac{1}{3}, \ \frac{2}{3}, \ \frac{7}{3}, \ \frac{14}{3}, \ \frac{1}{6}, \ \frac{7}{6}, \ \frac{21}{6}\right\}$$

$$\begin{array}{r|rrrr}
2 & 6 & -17 & -11 & 42 \\
 & & 12 & -10 & -42 \\
\hline
 & 6 & -5 & -21 & \boxed{0}
\end{array}
\ \to \ 6x^2 - 5x - 21 = 0$$

$$x = \frac{5 \pm \sqrt{25+504}}{12}$$

$$= \begin{cases} \frac{5-23}{12} = -\frac{3}{2} \\ \frac{5+23}{12} = \frac{7}{3} \end{cases}$$

$$\underline{x = -\frac{3}{2},\ 0,\ 2,\ \frac{7}{3}}\Big|$$



	$-\frac{3}{2}$	0	2	$\frac{7}{3}$
+	–	+	–	+

$f(x) > 0$ $\quad \left(-\infty,\ -\frac{3}{2}\right) \cup (0,\ 2) \cup \left(\frac{7}{3},\ \infty\right)$

$f(x) < 0$ $\quad \left(-\frac{3}{2},\ 0\right) \cup \left(2,\ \frac{7}{3}\right)$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^4 - 5x^2 - 2x$$

Solution

$$x\left(x^3 - 5x - 2\right) = 0$$

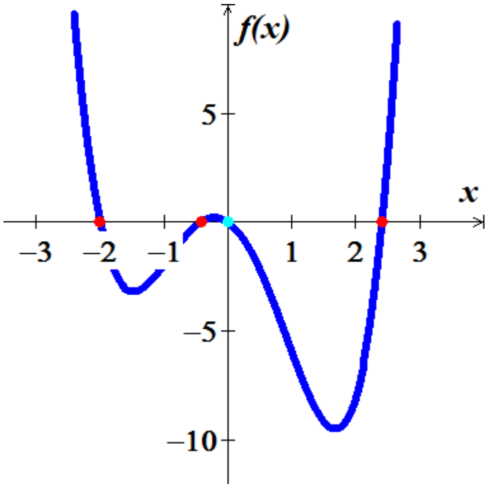
$$x = 0 \quad x^3 - 5x - 2 = 0$$

possibilities : $\pm\left\{\frac{2}{1}\right\} = \pm\{1,\ 2\}$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -5 & -2 \\ & & -2 & 4 & 2 \\ \hline & 1 & -2 & -1 & \boxed{0} \end{array} \rightarrow x^2 - 2x - 1 = 0$$

$$x = \frac{2 \pm \sqrt{8}}{2}$$

$$\underline{x = -2,\ 0,\ 1 \pm \sqrt{2}}\Big|$$



	-2	$1-\sqrt{2}$	2	$1+\sqrt{2}$
+	–	+	–	+

$f(x) > 0$ $\quad (-\infty,\ -2) \cup \left(1-\sqrt{2},\ 2\right) \cup \left(1+\sqrt{2},\ \infty\right)$

$f(x) < 0$ $\quad \left(-2,\ 1-\sqrt{2}\right) \cup \left(2,\ 1+\sqrt{2}\right)$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 3x^4 - 4x^3 - 11x^2 + 16x - 4$$

Solution

$$possibilities: \ \pm\left\{\frac{4}{3}\right\} = \pm\left\{1,\ 2,\ 4,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3}\right\}$$

$$\begin{array}{r|rrrrr} 1 & 3 & -4 & -11 & 16 & -4 \\ & & 3 & -1 & -12 & 4 \\ \hline -2 & 3 & -1 & -12 & 4 & 0 \\ & & 6 & 10 & -4 & \\ \hline & 3 & 5 & -2 & 0 & \end{array}$$

$$\to 3x^3 - x^2 - 12x + 4 = 0 \ \to \ \pm\left\{\frac{4}{3}\right\} = \pm\left\{1,\ 2,\ 4,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3}\right\}$$

$$\to 3x^2 + 5x - 2 = 0$$

$$x = \frac{-5 \pm \sqrt{25+24}}{6}$$

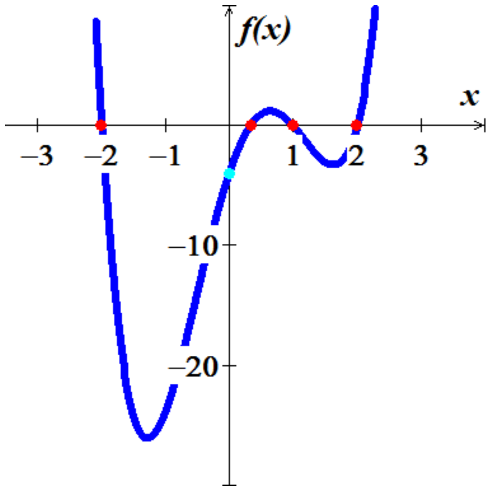
$$= \begin{cases} \frac{-5-7}{6} = -2 \\ \frac{-5+7}{6} = \frac{1}{3} \end{cases}$$

$$\boxed{x = -2,\ \frac{1}{3},\ 1,\ 2}$$

	-2		$\frac{1}{3}$		1		2	
+		−		+		−		+

$f(x) > 0$ $\quad \boxed{(-\infty,\ -2) \cup \left(\frac{1}{3},\ 1\right) \cup (2,\ \infty)}$

$f(x) < 0$ $\quad \boxed{\left(-2,\ \frac{1}{3}\right) \cup (1,\ 2)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 6x^4 + 23x^3 + 19x^2 - 8x - 4$$

Solution

$$possibilities: \ \pm\left\{\frac{4}{6}\right\} = \pm\left\{1,\ 2,\ 4,\ \frac{1}{6},\ \frac{2}{3},\ \frac{2}{3}\right\}$$

$$\begin{array}{r|rrrrr}
-2 & 6 & 23 & 19 & -8 & -4 \\
 & & -12 & -22 & 6 & 4 \\
\hline
-2 & 6 & 11 & -3 & -2 & \color{red}{0} \\
 & & -12 & 2 & 2 & \\
\hline
 & 6 & -1 & -1 & \color{red}{0} &
\end{array}$$

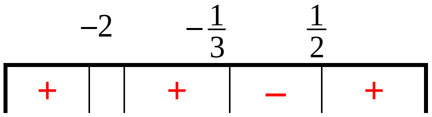
$\rightarrow 6x^3+11x^2-3x-2=0 \rightarrow \pm\left\{\frac{2}{6}\right\}=\pm\left\{1,\ 2,\ \frac{1}{6},\ \frac{1}{3}\right\}$

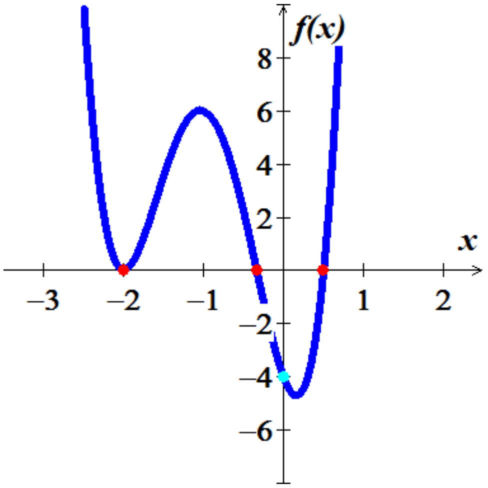
$\rightarrow 6x^2-x-1=0$

$$x=\frac{1\pm\sqrt{25}}{12}$$

$$=\begin{cases}\frac{1-5}{12}=-\frac{1}{3}\\ \frac{1+5}{12}=\frac{1}{2}\end{cases}$$

$$\boxed{x=-2,\ -2,\ -\frac{1}{3},\ \frac{1}{2}}$$





$f(x)>0 \quad \boxed{(-\infty,\ -2)\cup\left(-2,\ -\frac{1}{3}\right)\cup\left(\frac{1}{2},\ \infty\right)}$

$f(x)<0 \quad \boxed{\left(-\frac{1}{3},\ \frac{1}{2}\right)}$

Exercise

Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of $f(x)$

$$f(x)=4x^4-12x^3+3x^2+12x-7$$

Solution

possibilities : $\pm\left\{\frac{7}{4}\right\}=\pm\left\{1,\ 7,\ \frac{1}{2},\ \frac{7}{2},\ \frac{1}{4},\ \frac{7}{4}\right\}$

$$\begin{array}{r|rrrrr}
1 & 4 & -12 & 3 & 12 & -7 \\
 & & 4 & -8 & -5 & 7 \\
\hline
-1 & 4 & -8 & -5 & 7 & \color{red}{0} \\
 & & -4 & 12 & -7 & \\
\hline
 & 4 & -12 & 7 & \color{red}{0} &
\end{array}$$

$\rightarrow 4x^3-8x^2-5x+7=0 \rightarrow \pm\left\{\frac{7}{4}\right\}=\pm\left\{1,\ 7,\ \frac{1}{2},\ \frac{7}{2},\ \frac{1}{4},\ \frac{7}{4}\right\}$

$\rightarrow 4x^2-12x+7=0$

$$x=\frac{12\pm\sqrt{144-112}}{8}$$

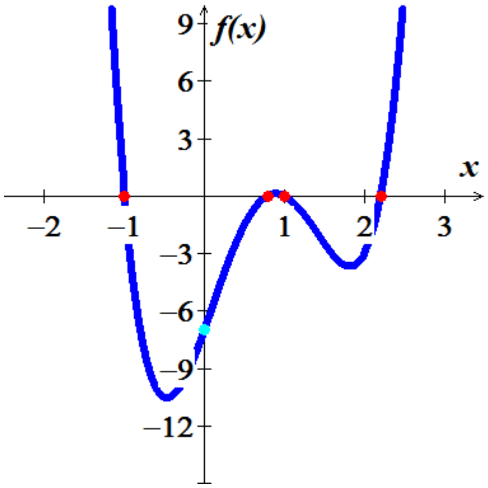
$$=\frac{12\pm4\sqrt{2}}{8}$$

$$\boxed{x = -1,\ 1,\ \frac{3\pm\sqrt{2}}{2}}$$

+	-1	−	$\frac{3-\sqrt{2}}{2}$	+	1	−	$\frac{3+\sqrt{2}}{2}$	+

$f(x) > 0$ $\quad \boxed{(-\infty,\ -1)\cup\left(\frac{3-\sqrt{2}}{2},\ 1\right)\cup\left(\frac{3+\sqrt{2}}{2},\ \infty\right)}$

$f(x) < 0$ $\quad \boxed{\left(-1,\ \frac{3-\sqrt{2}}{2}\right)\cup\left(1,\ \frac{3+\sqrt{2}}{2}\right)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 2x^4 - 9x^3 - 2x^2 + 27x - 12$$

Solution

possibilities : $\pm\left\{\frac{12}{2}\right\} = \pm\left\{1,\ 2,\ 3,\ 4,\ 6,\ 12,\ \frac{1}{2},\ \frac{3}{2}\right\}$

$$\begin{array}{r|rrrrr}
4 & 2 & -9 & -2 & 27 & -12 \\
 & & 8 & -4 & -24 & 12 \\
\hline
\frac{1}{2} & 2 & -1 & -6 & 3 & 0 \\
 & & 1 & 0 & -3 & \\
\hline
 & 2 & 0 & -6 & 0 &
\end{array}$$

$\to 2x^3 - x^2 - 6x + 3 = 0 \ \to\ \pm\left\{\frac{3}{2}\right\} = \pm\left\{1,\ 3,\ \frac{1}{2},\ \frac{3}{2}\right\}$

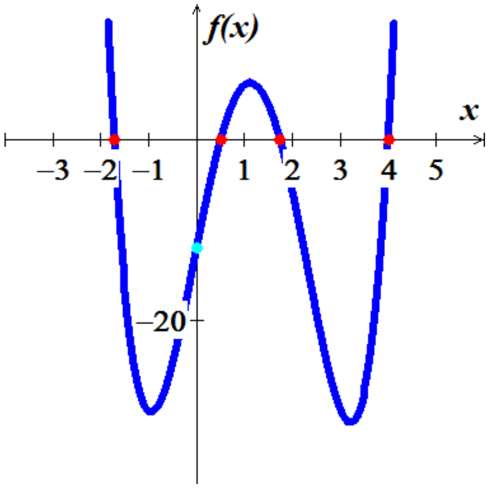
$\to 2x^2 - 6 = 0$

$$\boxed{x = \frac{1}{2},\ 4,\ \pm\sqrt{3}}$$

+	$-\sqrt{3}$	−	$\frac{1}{2}$	+	$\sqrt{3}$	−	4	+

$f(x) > 0$ $\quad \boxed{\left(-\infty,\ -\sqrt{3}\right)\cup\left(\frac{1}{2},\ \sqrt{3}\right)\cup(4,\ \infty)}$

$f(x) < 0$ $\quad \boxed{\left(-\sqrt{3},\ \frac{1}{2}\right)\cup\left(\sqrt{3},\ 4\right)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 2x^4 - 19x^3 + 51x^2 - 31x + 5$$

Solution

$$\text{possibilities}: \ \pm\left\{\frac{5}{2}\right\} = \pm\left\{1,\ 5,\ \frac{1}{2},\ \frac{5}{2}\right\}$$

$$\begin{array}{r|rrrrr}
5 & 2 & -19 & 51 & -31 & 5 \\
 & & 10 & -45 & 30 & -5 \\
\hline
\frac{1}{2} & 2 & -9 & 6 & -1 & 0 \\
 & & 1 & -4 & 1 & \\
\hline
 & 2 & -8 & 2 & 0 &
\end{array}$$

$$\to 2x^3 - 9x^2 + 6x - 1 = 0 \ \to \ \pm\left\{\frac{1}{2}\right\}$$

$$\to 2x^2 - 8x + 2 = 0$$

$$x = \frac{8 \pm \sqrt{64-16}}{4}$$

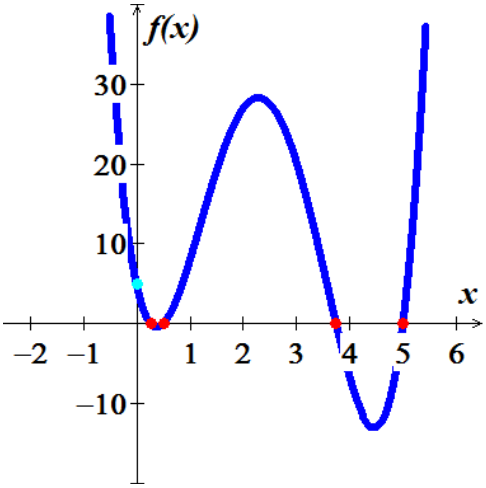
$$= \frac{8 \pm 4\sqrt{3}}{4}$$

$$\boxed{x = \frac{1}{2},\ 5,\ 2 \pm \sqrt{3}}$$

	$2-\sqrt{3}$		$\frac{1}{2}$		$2+\sqrt{3}$		5	
+		–		+		–		+

$$f(x) > 0 \quad \boxed{\left(-\infty,\ 2-\sqrt{3}\right) \cup \left(\frac{1}{2},\ 2+\sqrt{3}\right) \cup (5,\ \infty)}$$

$$f(x) < 0 \quad \boxed{\left(2-\sqrt{3},\ \frac{1}{2}\right) \cup \left(2+\sqrt{3},\ 5\right)}$$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 4x^4 - 35x^3 + 71x^2 - 4x - 6$$

Solution

$$\text{possibilities}: \ \pm\left\{\frac{6}{4}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{4},\ \frac{3}{4}\right\}$$

$$\begin{array}{r|rrrrr}
3 & 4 & -35 & 71 & -4 & -6 \\
 & & 12 & -69 & 6 & 6 \\
\hline
-\frac{1}{4} & 4 & -23 & 2 & 2 & 0 \\
 & & -1 & 6 & -2 & \\
\hline
 & 4 & -24 & 8 & 0 &
\end{array}$$

$$\rightarrow 4x^3-23x^2+2x+2=0 \rightarrow \pm\left\{\frac{2}{4}\right\}=\pm\left\{1,\ 2,\ \frac{1}{2},\ \frac{1}{4}\right\}$$

$$\rightarrow 4x^2-24x+8=0$$

$$x^2-6x+2=0$$

$$x=\frac{6\pm\sqrt{36-8}}{2}$$

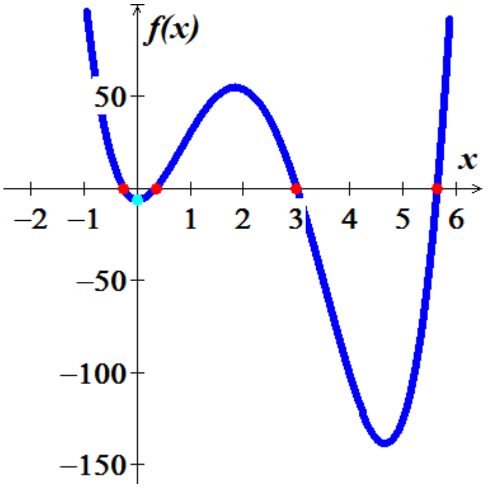
$$=\frac{6\pm 2\sqrt{7}}{4}$$

$$\boxed{x=-\frac{1}{4},\ 3,\ 3\pm\sqrt{7}}$$

$$\begin{array}{|c|c|c|c|c|}
\multicolumn{5}{c}{-\frac{1}{4} \qquad 3-\sqrt{7} \qquad 3 \qquad 3+\sqrt{7}} \\
\hline
+ & - & + & - & +
\end{array}$$

$f(x)>0$ $\boxed{\left(-\infty,\ -\frac{1}{4}\right)\cup\left(3-\sqrt{7},\ 3\right)\cup\left(3+\sqrt{7},\ \infty\right)}$

$f(x)<0$ $\boxed{\left(-\frac{1}{4},\ 3-\sqrt{7}\right)\cup\left(3,\ 3+\sqrt{7}\right)}$



Exercise

Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of $f(x)$

$$f(x)=2x^4+3x^3-4x^2-3x+2$$

Solution

possibilities : $\pm\left\{\frac{2}{2}\right\}=\pm\left\{1,\ 2,\ \frac{1}{2}\right\}$

$$\begin{array}{r|rrrrr}
1 & 2 & 3 & -4 & -3 & 2 \\
 & & 2 & 5 & 1 & -2 \\
\hline
-1 & 2 & 5 & 1 & -2 & 0 \\
 & & -2 & -3 & 2 & \\
\hline
 & 2 & 3 & -2 & 0 &
\end{array}$$

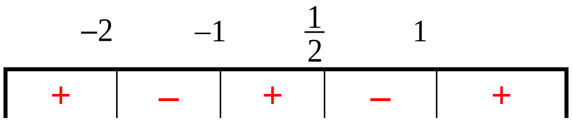
$$\rightarrow 2x^3-23x^2+2x-2=0 \rightarrow \pm\left\{\frac{2}{2}\right\}=\pm\left\{1,\ 2,\ \frac{1}{2}\right\}$$

$$\rightarrow 2x^2+3x-2=0$$

$$x=\frac{-3\pm\sqrt{9+16}}{4}$$

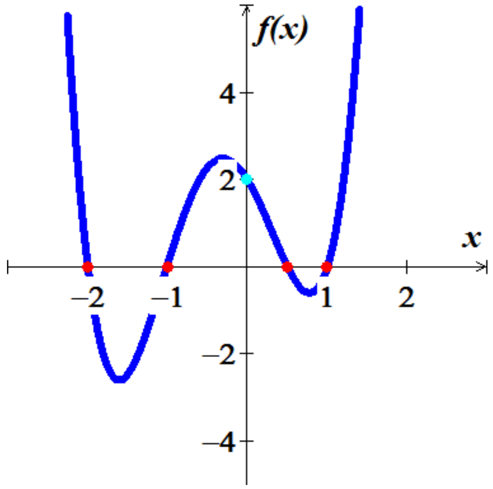
$$=\begin{cases}\frac{-3-5}{4}=-2\\ \frac{-3+5}{4}=\frac{1}{2}\end{cases}$$

$\underline{x=-2,\ -1,\ \frac{1}{2},\ 1}|$



$f(x)>0 \quad \underline{(-\infty,\ -2)\cup\left(-1,\ \tfrac{1}{2}\right)\cup(1,\ \infty)}|$

$f(x)<0 \quad \underline{(-2,\ -1)\cup\left(\tfrac{1}{2},\ 1\right)}|$



Exercise

Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of $f(x)$

$$f(x)=x^4+3x^3-30x^2-6x+56$$

Solution

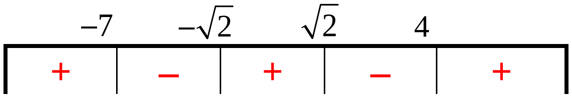
possibilities for $\frac{c}{d}:\pm\left\{\frac{56}{1}\right\}=\pm\{1,\ 2,\ 4,\ 7,\ 8,\ 14,\ 28,\ 56\}$

$$\begin{array}{r|rrrrr} 4 & 1 & 3 & -3 & -6 & 56 \\ & & 4 & 28 & -8 & -56 \\ \hline -7 & 1 & 7 & -2 & -14 & 0 \\ & & -7 & 0 & 14 & \\ \hline & 1 & 0 & -2 & & \end{array}$$

$\rightarrow\ x^3+7x^2-2x-14=0 \Rightarrow \frac{c}{d}=\pm\left\{\frac{14}{1}\right\}=\pm\{1,\ 2,\ 7,\ 14\}$

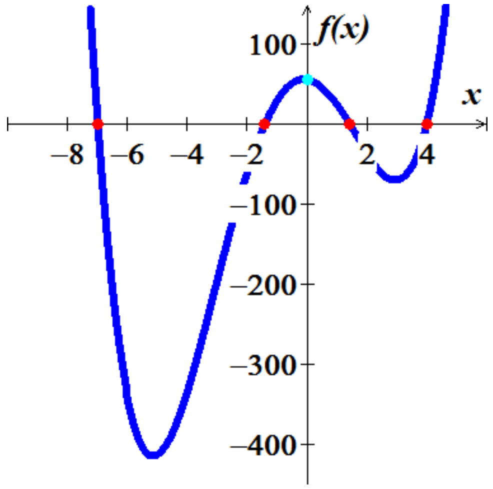
$\rightarrow\ x^2-2=0 \Rightarrow \underline{x=\pm\sqrt{2}}|$

$\underline{x=4,\ -7,\ \pm\sqrt{2}}|$



$f(x)>0 \quad \underline{(-\infty,\ -7)\cup\left(-\sqrt{2},\ \sqrt{2}\right)\cup(4,\ \infty)}|$

$f(x)<0 \quad \underline{\left(-7,\ -\sqrt{2}\right)\cup\left(\sqrt{2},\ 4\right)}|$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6$$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{6}{3}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrrrr}
-1 & 3 & -10 & -6 & 24 & 11 & -6 \\
 & & -3 & 13 & -7 & -17 & 6 \\
\hline
-1 & 3 & -13 & 7 & 17 & -6 & 0 \\
 & & -3 & 16 & -23 & 6 & \\
\hline
2 & 3 & -16 & 23 & -6 & 0 & \\
 & & 6 & 20 & 6 & & \\
\hline
 & 3 & -10 & 3 & 0 & &
\end{array}$$

$$x^4 - 13x^3 + 7x^2 + 17x - 6 = 0 \rightarrow \pm\left\{\frac{6}{3}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$$

$$3x^3 - 16x^2 + 26x - 6 = 0 \rightarrow \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{3},\ \frac{2}{3}\right\}$$

$$3x^2 - 10x + 3 = 0$$

$$x = \frac{10 \pm \sqrt{100 - 36}}{6}$$

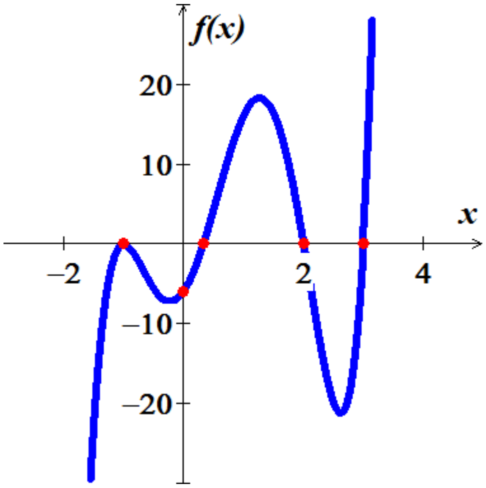
$$= \begin{cases} \frac{10 - 8}{6} = \frac{1}{3} \\ \frac{10 + 8}{4} = 3 \end{cases}$$

$$\boxed{x = -1,\ -1,\ \frac{1}{3},\ 2,\ 3}$$

$$\begin{array}{ccccccccc}
 & -1 & & \frac{1}{3} & & 2 & & 3 & \\
- & | & - & | & + & | & - & | & +
\end{array}$$

$f(x) > 0$ $\quad \boxed{\left(\frac{1}{3},\ 2\right) \cup (3,\ \infty)}$

$f(x) < 0$ $\quad \boxed{(-\infty,\ -1) \cup \left(-1,\ \frac{1}{3}\right) \cup (2,\ 3)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 6x^5 + 19x^4 + x^3 - 6x^2$$

Solution

$$x^2\left(6x^3 + 19x^2 + x - 6\right) = 0 \quad \rightarrow \quad \underline{x = 0,\ 0|}$$

$$6x^3 + 19x^2 + x - 6 = 0$$

possibilities for $\frac{c}{d} : \pm\left\{\frac{6}{6}\right\} = \pm\left\{1,\ 2,\ 3,\ 6,\ \frac{1}{2},\ \frac{3}{2},\ \frac{1}{3},\ \frac{2}{3}\right\}$

$$\begin{array}{r|rrrr} -3 & 6 & 19 & 1 & -6 \\ & & -18 & -3 & 6 \\ \hline & 6 & 1 & -2 & \boxed{0} \end{array} \qquad 6x^2 + x - 2 = 0$$

$$x = \frac{-1 \pm \sqrt{1+48}}{12}$$

$$= \begin{cases} \frac{-1-7}{12} = -\frac{2}{3} \\ \frac{-1+7}{12} = \frac{1}{2} \end{cases}$$

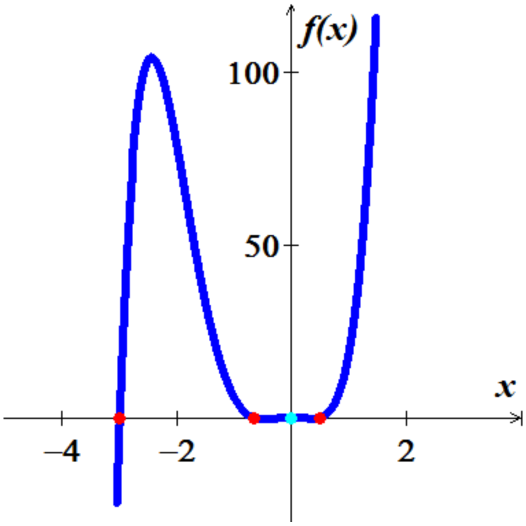
$$\underline{x = 0,\ 0,\ -\frac{2}{3},\ -3,\ \frac{1}{2}|}$$

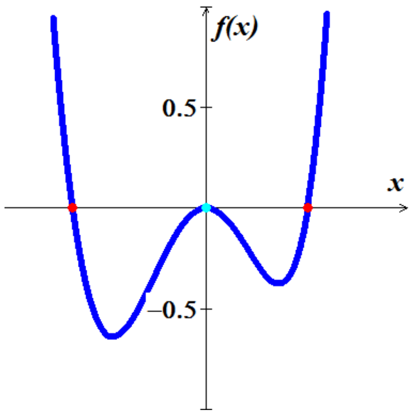
$-3 \qquad -\frac{2}{3} \qquad 0 \qquad \frac{1}{2}$

−	+	−	−	+

$f(x) > 0 \quad \underline{\left(-3,\ -\frac{2}{3}\right) \cup \left(\frac{1}{2},\ \infty\right)|}$

$f(x) < 0 \quad \underline{(-\infty,\ -3) \cup \left(-\frac{2}{3},\ 0\right) \cup \left(0,\ \frac{1}{2}\right)|}$





Exercise

Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of $f(x)$

$$f(x)=x^5+5x^4+10x^3+10x^2+5x+1$$

Solution

$$x^5+5x^4+10x^3+10x^2+5x+1=(x+1)^5=0$$

possibilities for $\frac{c}{d}:\pm\{1\}$

$$\begin{array}{r|rrrrrr} -1 & 1 & 5 & 10 & 10 & 5 & 1\\ & & -1 & -4 & -6 & -4 & -1\\ \hline -1 & 1 & 4 & 6 & 4 & 1 & 0 \\ & & -1 & -3 & -3 & -1 & \\ \hline -1 & 1 & 3 & 3 & 1 & 0 & \\ & & -1 & -2 & -1 & & \\ \hline & 1 & 2 & 1 & 0 & & \end{array}$$

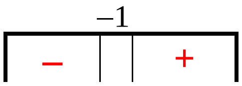
$\rightarrow\ x^4+4x^3+6x^2+4x+1=0\rightarrow\pm\{1\}$

$\rightarrow\ x^3+3x^2+3x+1=0\rightarrow\pm\{1\}$

$\rightarrow\ x^2+2x+1=0$

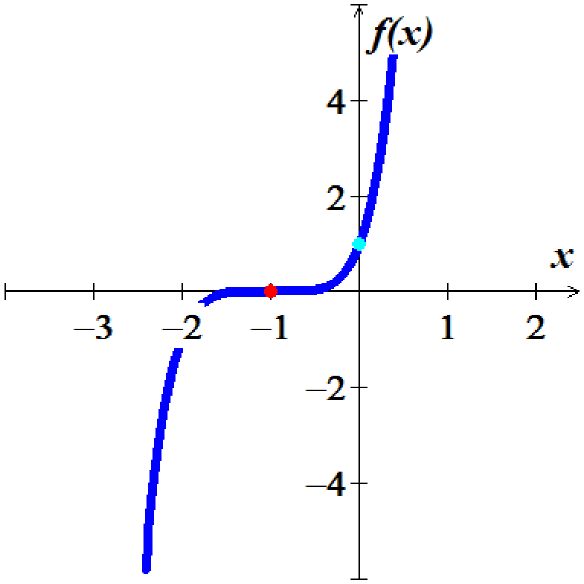
$$x^2+2x+1=(x+1)^2$$

$\underline{x=-1}\ \ (\textit{multiplicity}\text{ of }5)$



$f(x)>0\quad \underline{(-1,\ \infty)}$

$f(x)<0\quad \underline{(-\infty,\ -1)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^5 - x^4 - 7x^3 + 7x^2 + 12x - 12$$

Solution

possibilities for $\frac{c}{d} : \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$$\begin{array}{r|rrrrrr}
1 & 1 & -1 & -7 & 7 & 12 & -12 \\
 & & 1 & 0 & -7 & 0 & 12 \\
\hline
2 & 1 & 0 & -7 & 0 & 12 & 0 \\
 & & 2 & 4 & -6 & -12 & \\
\hline
-2 & 1 & 2 & -3 & -6 & 0 & \\
 & & -2 & 0 & 6 & & \\
\hline
 & 1 & 0 & -3 & 0 & &
\end{array}$$

$\rightarrow x^4 - 7x^2 - 12 = 0 \rightarrow \pm\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$

$\rightarrow x^3 + 2x^2 - 3x - 6 = 0 \rightarrow \pm\{1,\ 2,\ 3,\ 6\}$

$\rightarrow x^2 - 3 = 0$

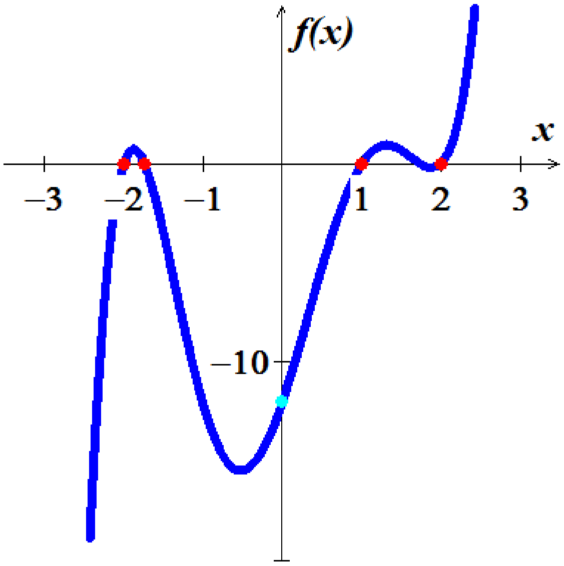
$x^2 = 3$

$\underline{x = -2,\ 1,\ 2,\ \pm\sqrt{3}}$

$$\begin{array}{cccccccccccc}
 & -2 & & -\sqrt{3} & & 1 & & \sqrt{3} & & 2 & \\
- & | & + & | & - & | & + & | & - & | & +
\end{array}$$

$f(x) > 0$ $\quad \underline{(-2,\ -\sqrt{3}) \cup (1,\ \sqrt{3}) \cup (2,\ \infty)}$

$f(x) < 0$ $\quad \underline{(-\infty,\ -2) \cup (-\sqrt{3},\ 1) \cup (\sqrt{3},\ 2)}$



Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = x^5 - 2x^3 - 8x$$

Solution

$$x\left(x^4 - 2x^2 - 8\right) = 0$$

$$\boxed{x = 0}$$

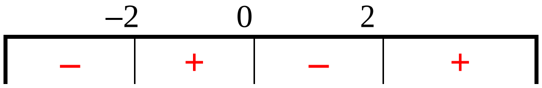
$$x^4 - 2x^2 - 8 = 0.$$

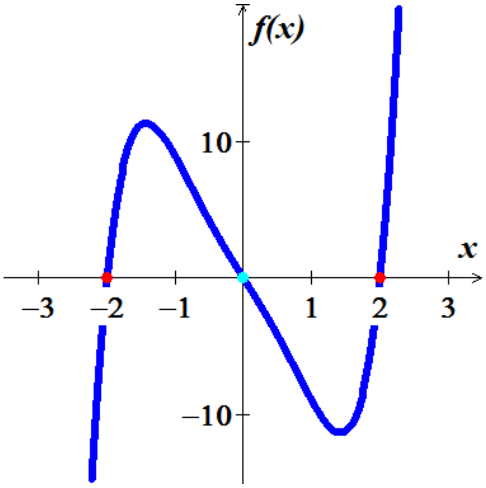
$$x^2 = \frac{2 \pm \sqrt{4+32}}{2}$$

$$= \begin{cases} \frac{2-6}{2} = -2 \\ \frac{2+6}{2} = 4 \end{cases}$$

$$\begin{cases} x^2 = -2 & \rightarrow \quad x = \pm i\sqrt{2} \\ x^2 = 4 & \rightarrow \quad x = \pm 2 \end{cases}$$

$$\boxed{x = 0, \ \pm 2, \ \pm i\sqrt{2}}$$





$f(x) > 0$ $\boxed{(-2, \ 0) \cup (2, \ \infty)}$

$f(x) < 0$ $\boxed{(-\infty, \ -2) \cup (0, \ 2)}$

Exercise

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of $f(x)$

$$f(x) = 3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24$$

Solution

possibilities for $\frac{c}{d} : \pm\left\{\frac{24}{3}\right\} = \pm\left\{1, \ 2, \ 3, \ 4, \ 6, \ 8, \ 12, \ 24, \ \frac{1}{3}, \ \frac{2}{3}, \ \frac{4}{3}, \ \frac{8}{3}\right\}$

1	3	−10	−29	34	50	−24	−24
		3	−7	−36	−2	48	24
−1	3	−7	−36	−2	48	24	0
		−3	10	26	−24	−24	
−2	3	−10	−26	24	24	0	
		−6	32	−12	−24		
$-\frac{2}{3}$	3	−16	6	12	0		
		−2	12	−12			
	3	−18	18	0			

$\rightarrow\ 3x^5 - 7x^4 - 36x^3 - 2x^2 + 48x + 24 = 0$

$\rightarrow\ 3x^4 - 10x^3 - 26x^2 + 24x + 24 = 0$

$\rightarrow\ 3x^3 - 16x^2 + 12x - 12 = 0$

$\rightarrow\ 3x^2 - 18x + 18 = 0$

$x^2 - 6x + 6 = 0$

$$x = \frac{6 \pm \sqrt{36-24}}{2}$$

$$= \frac{6 \pm 2\sqrt{3}}{2}$$

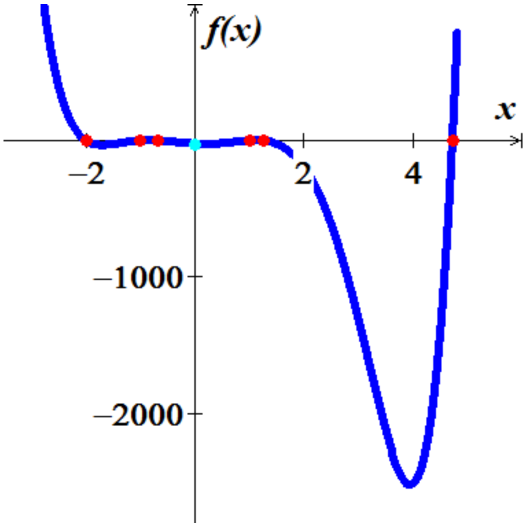
$$\boxed{x = -2,\ -1,\ 1,\ -\frac{2}{3},\ 3 \pm \sqrt{3}}$$

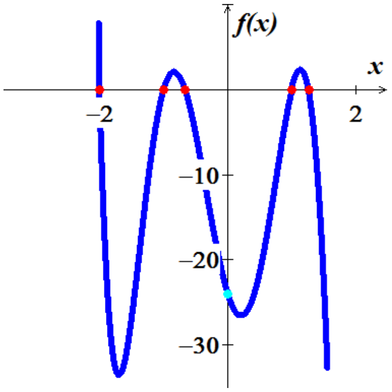
$-2 \qquad -1 \qquad -\frac{2}{3} \qquad 1 \qquad 3-\sqrt{3} \qquad 3+\sqrt{3}$

+	−	+	−	+	−	+

$f(x) > 0 \quad \boxed{(-\infty,\ -2) \cup \left(-1,\ -\frac{2}{3}\right) \cup \left(1,\ 3-\sqrt{3}\right) \cup \left(3+\sqrt{3},\ \infty\right)}$

$f(x) < 0 \quad \boxed{(-2,\ -1) \cup \left(-\frac{2}{3},\ 1\right) \cup \left(3-\sqrt{3},\ 3+\sqrt{3}\right)}$





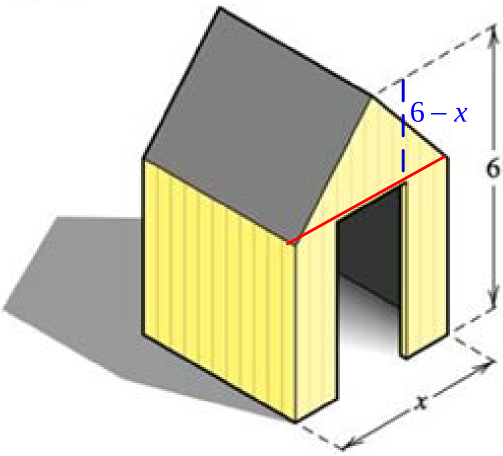
Exercise

A storage shelter is to be constructed in the shape of a cube with a triangular prism forming the roof. The length x of a side of the cube is yet to be determined.

a) If the total height of the structure is 6 *feet*, show that its volume V is given by $V=x^3+\frac{1}{2}x^2(6-x)$

b) Determine x so that the volume is $80\ ft^3$

Solution



a) $V=V_{cube}+V_{triangle}$

$$=x^3+\frac{1}{2}x(x)(6-x)$$

$$=\frac{1}{2}x^2(2x+6-x)$$

$$\underline{=\frac{1}{2}x^2(x+6)}$$

b) $V=\frac{1}{2}x^2(x+6)=80$

$$x^3+6x^2-160=0$$

possibilities : $\pm\left\{\frac{160}{1}\right\}=\pm\{1,\ 2,\ 4,\ 5,\ 8,\ 10,\ 16,\ 20,\ 32,\ 40,\ 80,\ 160\}$

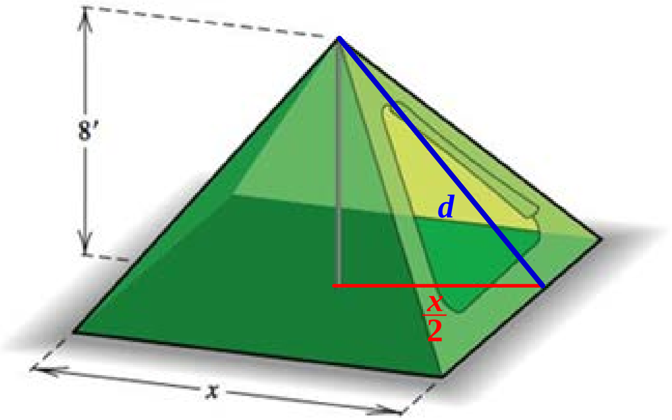
$$\begin{array}{r|rrrr} 4 & 1 & 6 & 0 & -160 \\ & & 4 & 40 & 160 \\ \hline & 1 & 10 & 40 & \boxed{0} \end{array} \rightarrow x^2+10x+40=0 \Rightarrow \underline{x=-5\pm i\sqrt{15}}$$

The solution is: $\underline{x=4}$

Exercise

A canvas camping tent is to be constructed in the shape of a pyramid with a square base. An 8–foot pole will form the center support. Find the length x of a side of the base so that the total amount of canvas needed for the sides and bottom is $384\ ft^2$.

Solution



$$d=\sqrt{64+\frac{x^2}{4}}=\frac{1}{2}\sqrt{x^2+256}$$

$$A_{bottom}=x^2$$

$$A_{1-side}=\frac{1}{2}xd$$

$$=\frac{1}{4}x\sqrt{x^2+256}$$

$$A_{total}=A_{bottom}+4A_{1-side}$$

$$= x^2 + x\sqrt{x^2+256} = 384$$

$$x\sqrt{x^2+256} = 384 - x^2$$

$$\left(x\sqrt{x^2+256}\right)^2 = \left(384 - x^2\right)^2$$

$$x^2\left(x^2+256\right) = 147,456 - 768x^2 + x^4$$

$$-1,024x^2 + 147,456 = 0$$

$$x = \pm\sqrt{\frac{147,456}{1,024}}$$

$$\underline{= 12\ \ ft|}$$

Exercise

Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k) = \frac{1}{6}\left(k^3 + 3k^2 + 2k\right)$$

Where *k* is the number of levels in the pyramid. If 220 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

Solution

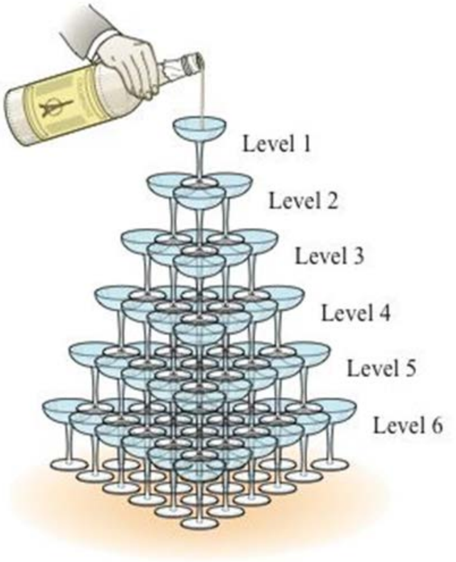
$$\frac{1}{6}\left(k^3 + 3k^2 + 2k\right) = 220$$

$$k^3 + 3k^2 + 2k - 1,320 = 0$$

$$\begin{array}{r|rrrr} 10 & 1 & 3 & 2 & -1320 \\ & & 10 & 130 & 1320 \\ \hline & 1 & 13 & 132 & 0 \end{array} \quad \to \quad k^2 + 13k + 132 = 0$$

$$k = \frac{-13 \pm \sqrt{-359}}{2} \quad \mathbb{C}$$

The are **10** levels in the pyramid.

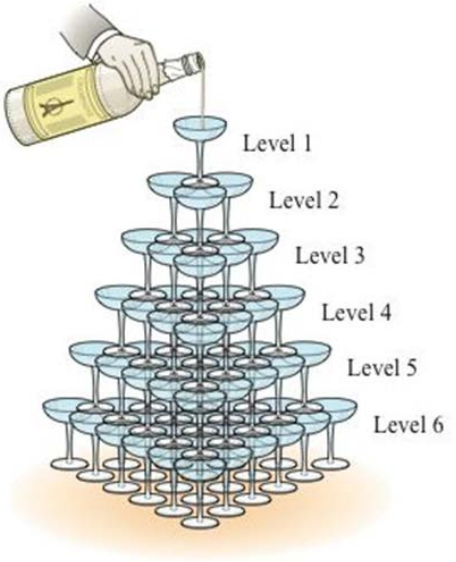


Exercise

Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k) = \frac{1}{6}\left(2k^3 + 3k^2 + k\right)$$

Where *k* is the number of levels in the pyramid. If 140 glasses are used to form a triangle pyramid, how many levels are in the pyramid?



Solution

$$\frac{1}{6}\left(2k^3+3k^2+k\right)=150$$

$$2k^3+3k^2+k-840=0$$

$$\begin{array}{r|rrrr} 7 & 2 & 3 & 1 & -840 \\ & & 14 & 119 & 840 \\ \hline & 2 & 17 & 120 & 0 \end{array} \quad \rightarrow \quad 2k^2+17k+120=0$$

$$k=\frac{-17\pm\sqrt{-671}}{4} \qquad \mathbb{C}$$

The are 7 levels in the pyramid.

Exercise

A carbon dioxide cartridge for a paintball rifle has the shape of a right circular cylinder with a hemisphere at each end. The cylinder is 4 *inches* long, and the volume of the cartridge is $2\pi \ \ in^3$.
The common interior radius of the cylinder and the hemispheres is denoted by *x*. Estimate the length of the radius *x*.

Solution

Volume of the Cartridge = 2 × (Volume of Hemisphere) + Volume of Cylinder

$$Volume\ of\ \ Sphere=\frac{4}{3}\pi x^3$$

$$Volume\ of\ \ Cylinder=4\pi x^2$$

$$Volume\ of\ \ Cartridge=\frac{4}{3}\pi x^3+4\pi x^2$$

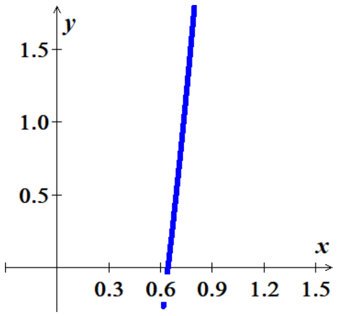
$$2\pi=\frac{4}{3}\pi x^3+4\pi x^2$$

$$2x^3+6x^2=3$$

$$2x^3+6x^2-3=0$$

Using Graph:

$\underline{x\approx 0.64\ \ in.}$

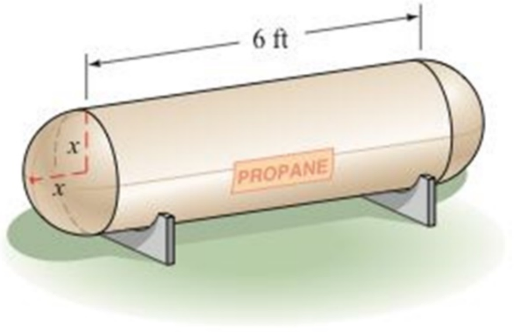


Exercise

A propane tank has the shape of a circular cylinder with a hemisphere at each end. The cylinder is 6 *feet* long and the volume of the tank is $9\pi \ ft^3$. Find the length of the radius x.

Solution

Volume of the Cartridge = 2 × (Volume of Hemisphere) + Volume of Cylinder



$$Volume\ of\ \ Sphere = \frac{4}{3}\pi x^3$$

$$Volume\ of\ \ Cylinder = 6\pi x^2$$

$$Volume\ of\ \ Cartridge = \frac{4}{3}\pi x^3 + 6\pi x^2$$

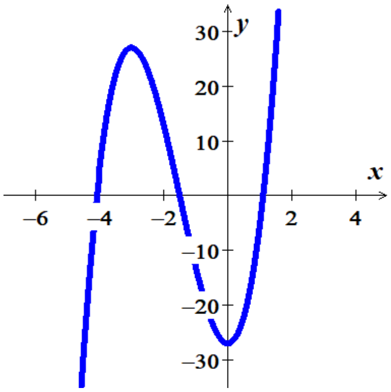
$$9\pi = \frac{4}{3}\pi x^3 + 6\pi x^2$$

$$27 = 4x^3 + 18x^2$$

$$4x^3 + 18x^2 - 27 = 0$$

$$possibilities\ for\ \ \frac{c}{d} : \pm\left\{\frac{27}{4}\right\} = \pm\left\{1,\ 3,\ 9,\ 27,\ \frac{1}{2},\ \frac{3}{2},\ \frac{9}{2},\ \frac{27}{2},\ \frac{1}{4},\ \frac{3}{4},\ \frac{9}{4},\ \frac{27}{4}\right\}$$

$$\begin{array}{r|rrrr} -\frac{3}{2} & 4 & 18 & 0 & -27 \\ & & -6 & -18 & 27 \\ \hline & 4 & 12 & -18 & 0 \end{array} \quad \rightarrow \quad 4x^2 + 12x - 18 = 0$$



$$2x^2 + 6x - 9 = 0$$

$$x = \frac{-6 \pm \sqrt{36+72}}{4}$$

$$= \frac{-6 \pm 6\sqrt{3}}{4}$$

$$= \frac{-3 \pm 3\sqrt{3}}{2}$$

$$x = -\frac{3}{2},\ \frac{-3-3\sqrt{3}}{2},\ \frac{-3+3\sqrt{3}}{2}$$

∴ the length of the radius x is $\frac{-3+3\sqrt{3}}{2} \approx 1.1$ *foot*

Exercise

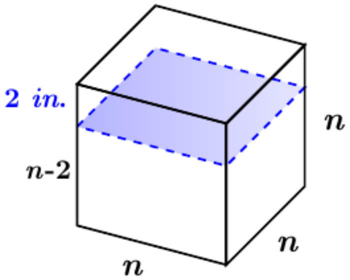
A cube measures n inches on each edge. If a slice 2 *inches* thick is cut from one face of the cube, the resulting solid has a volume of $567 \ in^3$. Find n.

Solution

$$Volume = n^2(n-2)$$

$$n^3 - 2n^2 = 567$$

$$n^3 - 2n^2 - 567 = 0$$



$$possibilities\ for\ \frac{c}{d} := \pm\{1,\ 3,\ 7,\ 9,\ 21,\ 27,\ 63,\ 81,\ 189,\ 567\}$$

$$\begin{array}{r|rrrr} 9 & 1 & -2 & 0 & -567 \\ & & 9 & 63 & 567 \\ \hline & 1 & 7 & 63 & 0 \end{array} \rightarrow \quad n^2 + 7n + 63 = 0$$

$$n = \frac{-7 \pm \sqrt{49 - 252}}{2}$$

$$= \frac{-7 \pm i\sqrt{203}}{2} \quad \times$$

$$\therefore \quad \underline{n = 9}\,|$$

Exercise

A cube measures n inches on each edge. If a slice 1 *inch* thick is cut from one face of the cube and then a slice 3 inches thick is cut from another face of the cube, the resulting solid has a volume of 1560 in^3. Find the dimensions of the original cube.

Solution

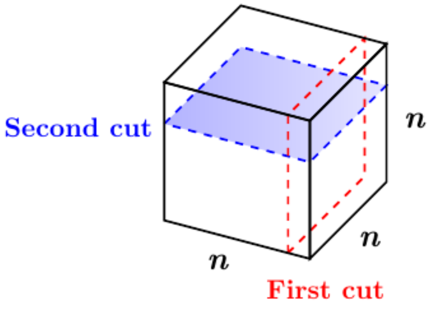
$$Volume = n(n-1)(n-3)$$

$$n^3 - 4n^2 + 3n = 1560$$

$$n^3 - 4n^2 + 3n - 1560 = 0$$

$$possibilities\ for\ \frac{c}{d} := \pm\left\{\begin{array}{l} 1,\ 2,\ 4,\ 5,\ 6,\ 8,\ 10,\ 12,\ 13,\ 15,\ 20,\ 24,\ 26,\ 30,\ 39, \\ 40,\ 52,\ 60,\ 65,\ 78,\ 104,\ 120,\ 130,\ 156,\ 195,\ 260,\ 312,\ 390,\ 780,\ 1560 \end{array}\right\}$$

$$\begin{array}{r|rrrr} 13 & 1 & -4 & 3 & -1560 \\ & & 13 & 117 & 1560 \\ \hline & 1 & 9 & 120 & 0 \end{array} \rightarrow \quad n^2 + 9n + 120 = 0$$



$$n = \frac{-9 \pm \sqrt{81 - 480}}{2}$$

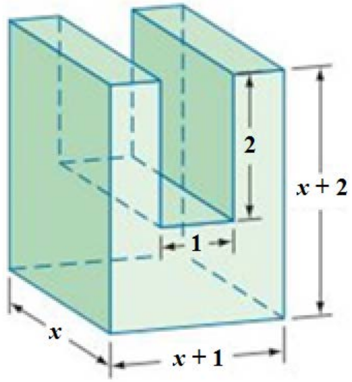
$$= \frac{-9 \pm i\sqrt{399}}{2} \quad \times$$

$$\therefore \quad \underline{n = 13}\,|$$

Exercise

For what value of x will the volume of the following solid be $112 \ \ in^3$

Solution



Volume of the bottom portion $= x^2(x+1)$

$$\text{Volume of one side portion } = 2x\left(\frac{1}{2}x\right)$$

$$= x^2$$

Total Volume $= x^2(x+1) + 2x^2$

$112 = x^3 + 3x^2$

$x^3 + 3x^2 - 112 = 0$

possibilities for $\frac{c}{d} := \pm\{1,\ 2,\ 4,\ 8,\ 14,\ 28,\ 56,\ 112\}$

$$\begin{array}{r|rrrr} 4 & 1 & 3 & 0 & -112 \\ & & 4 & 28 & 112 \\ \hline & 1 & 7 & 28 & 0 \end{array} \quad \rightarrow \quad x^2 + 7x + 28 = 0$$

$$x = \frac{-7 \pm \sqrt{49 - 112}}{2}$$

$$= \frac{-7 \pm 3i\sqrt{7}}{2} \quad \times$$

$\therefore \ \ \boxed{x = 4}$

Exercise

For what value of x will the volume of the following solid be $208 \ \ in^3$

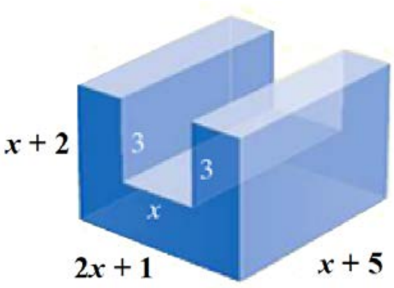
Solution

Volume of the bottom portion $= (2x+1)(x+5)(x+2-3)$

$$= \left(2x^2 + 11x + 5\right)(x-1)$$

$$= 2x^3 + 11x^2 + 5x - 2x^2 - 11x - 5$$

$$= 2x^3 + 9x^2 - 6x - 5$$



$$\text{Volume of one side portion } = (3)\frac{1}{2}(2x+1-x)(x+5)$$

$$= \frac{3}{2}(x+1)(x+5)$$

$$= \frac{3}{2}\left(x^2 + 6x + 5\right)$$

$$\text{Total Volume } = 2x^3 + 9x^2 - 6x - 5 + 2\left(\frac{3}{2}\right)\left(x^2 + 6x + 5\right)$$

$$208 = 2x^3 + 9x^2 - 6x - 5 + 3x^2 + 18x + 15$$

$$2x^3 + 12x^2 + 12x - 198 = 0$$

$$x^3 + 6x^2 + 6x - 99 = 0$$

possibilities for $\frac{c}{d} := \pm\{1,\ 3,\ 9,\ 11,\ 33,\ 99\}$

$$\begin{array}{r|rrrr} 3 & 1 & 6 & 6 & -99 \\ & & 3 & 27 & 99 \\ \hline & 1 & 9 & 33 & 0 \end{array} \quad \rightarrow \quad x^2 + 9x + 33 = 0$$

$$x = \frac{-9 \pm \sqrt{81 - 132}}{2}$$

$$= \frac{-9 \pm i\sqrt{51}}{2} \quad ✗$$

$$\therefore \quad \underline{x = 3}\,|$$

Exercise

The length of rectangular box is 1 *inch* more than twice the height of the box, and the width is 3 *inches* more than the height. If the volume of the box is 126 in^3, find the dimensions of the box.

Solution

$$Volume = x(2x + 1)(x + 3)$$

$$2x^3 + 7x^2 + 3x = 126$$

$$2x^3 + 7x^2 + 3x - 126 = 0$$

possibilities for $\frac{c}{d} := \pm\left\{\frac{126}{2}\right\}$

$$= \pm\left\{1,\ 2,\ 3,\ 6,\ 9,\ 14,\ 21,\ 42,\ 63,\ 126,\ \frac{1}{2},\ \frac{3}{2},\ \frac{9}{2},\ \frac{21}{2},\ \frac{63}{2}\right\}$$

$$\begin{array}{r|rrrr} 3 & 2 & 7 & 3 & -126 \\ & & 6 & 39 & 126 \\ \hline & 2 & 13 & 42 & 0 \end{array} \quad \rightarrow \quad 2x^2 + 13x + 42 = 0$$

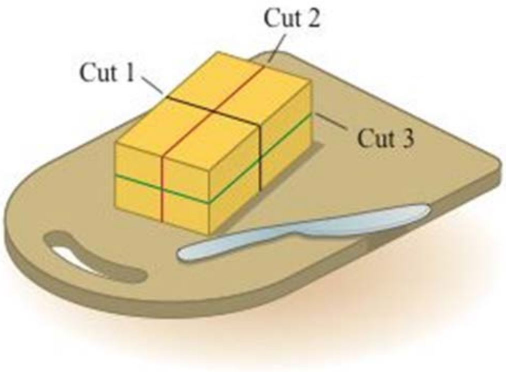
$$x = \frac{-13 \pm \sqrt{169 - 336}}{4}$$

$$= \frac{-13 \pm i\sqrt{167}}{4} \quad ✗$$

$$\therefore \quad \underline{x = 3}\,|$$

Exercise

One straight cut through a thick piece of cheese produces two pieces. Two straight cuts can produce a maximum of four pieces. Two straight cuts can produce a maximum of four pieces. Three straight cuts can produce a maximum of eight pieces.



You might be inclined to think that every additional cut doubles number of pieces. However, for four straight cuts, you get a maximum of 15 pieces. The maximum number of pieces P that can be produced by n straight cuts is given by

$$P(n)=\frac{n^3+5n+6}{6}$$

a) Determine number of pieces that can be produces by five straight cuts.

b) What is the fewest number of straight cuts that are needed to produce 64 pieces?

Solution

a) $P(5)=\frac{5^3+25+6}{6}$

$= 26$

b) $\frac{n^3+5n+6}{6}=64$

$n^3+5n+6=384$

$n^3+5n-378=0$

possibilities for $\frac{c}{d}:=\pm\{378\}$

$=\pm\{1,\ 2,\ 3,\ 6,\ 7,\ 9,\ 14,\ 18,\ 21,\ 27,\ 42,\ 54,\ 63,\ 126,\ 189,\ 378\}$

$$\begin{array}{r|rrrr} 7 & 1 & 0 & 5 & -378 \\ & & 7 & 49 & 378 \\ \hline & 1 & 7 & 54 & 0 \end{array} \quad \to \quad n^2+7n+54=0$$

$n=\frac{-7\pm\sqrt{49-216}}{2}$

$=\frac{-7\pm i\sqrt{167}}{2}$ X

$\therefore\ n=7$

Exercise

The number of ways one can select three cards from a group of *n* cards (the order of the selection matters), where $n \geq 3$, is given by $P(n) = n^3 - 3n^2 + 2n$. For a certain card trick, a magician has determined that there are exactly 504 *ways* to choose three cards from a given group. How many cards are in the group?

Solution

$$P(n) = n^3 - 3n^2 + 2n = 504$$

$$n^3 - 3n^2 + 2n - 504 = 0$$

possibilities for $\frac{c}{d} := \pm\{504\}$

$$= \pm\left\{\begin{matrix} 1,\ 2,\ 3,\ 4,\ 6,\ 7,\ 8,\ 9,\ 12,\ 14,\ 18,\ 21, \\ 24,\ 28,\ 36,\ 42,\ 56,\ 63,\ 72,\ 84,\ 126,\ 168,\ 252,\ 504 \end{matrix}\right\}$$

$$\begin{array}{c|cccc} 9 & 1 & -3 & 2 & -504 \\ & & 9 & 54 & 504 \\ \hline & 1 & 6 & 56 & 0 \end{array} \rightarrow \quad n^2 + 6n + 56 = 0$$

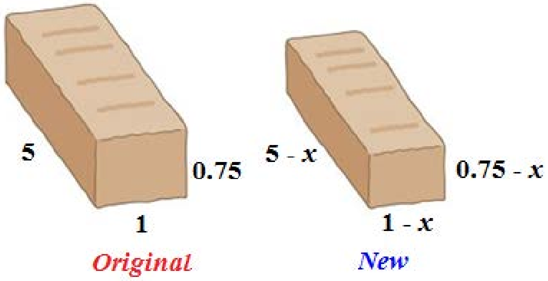
$$n = \frac{-6 \pm \sqrt{36 - 224}}{2}$$

$$= -3 \pm i\sqrt{47} \quad \times$$

$$\therefore \quad \underline{n = 9}|$$

Exercise

A nutrition bar in the shape of a rectangular solid measure 0.75 *in*. by 1 *in*. by 5 *inches*.



To reduce costs, the manufacturer has decided to decrease each of the dimensions of the nutrition bar by *x inches*, what value of *x* will produce a new bar with a volume that is $0.75\ in^3$ less than the present bar's volume.

Solution

$$V_{original} = (5)(1)\left(\frac{3}{4}\right)$$

$$= \frac{15}{4}$$

$$V_{new} = (5-x)(1-x)\left(\frac{3}{4}-x\right) \qquad \left(x<\frac{3}{4}\right)$$

$$\left(5-6x+x^2\right)\left(\frac{3-4x}{4}\right) = \frac{15}{4}-\frac{3}{4}$$

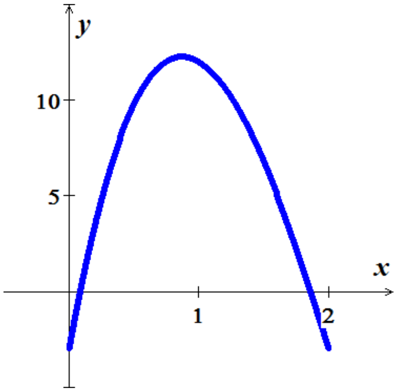
$$15-20x-18x+24x^2+3x^2-4x^3=4(3)$$

$$4x^3-27x^2+38x-3=0$$

From graph table:

```
0.08200  -0.06334
0.08400   0.00386
```

$\underline{x \approx 0.083 \ \ in.}|$

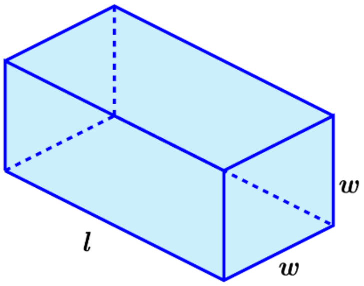


Exercise

A rectangular box is square on two ends and has length plus girth of 81 *inches*. (Girth: distance *around* the box). Determine the possible lengths l $(l>w)$ of the box if its volume is $4900 \ \ in^3$.

Solution

$$81 = l+4w$$
$$l = 81-4w$$
$$V = lw^2$$
$$= (81-4w)w^2$$



$$-4w^3+81w^2=4900$$

$$4w^3-81w^2+4900=0$$

$$\text{possibilities for } \frac{c}{d} := \pm\left\{\frac{4900}{4}\right\} = \pm\left\{\begin{matrix}1,\ 2,\ 4,\ 7,\ 10,\ 14,\ 20,\ 28,\ 49, \\ 100,\ 175,\ 245,\ 350,\ 490,\ 700,\ 1225,\ 2450,\ 4900,\cdots\end{matrix}\right\}$$

$$\begin{array}{r|rrrr} 14 & 4 & -81 & 0 & 4900 \\ & & 56 & -350 & -4900 \\ \hline & 4 & -25 & -350 & 0 \end{array} \quad \to \quad 4w^2-25w-350=0$$

$$w = \frac{25\pm\sqrt{625+5600}}{8}$$
$$= \frac{25\pm5\sqrt{249}}{8}$$
$$= \begin{cases} \dfrac{25-5\sqrt{249}}{8} < 0 \\ \dfrac{25+5\sqrt{249}}{8} \approx 13 \end{cases}$$

$$l = 81-4(14) = 25$$

$l = 81 - 4(13) = 29$

$\therefore$ the possible lengths l are around **25** *in*. ***or*** **29** *in*.

Solution Section 1.4 – Rational Functions

Exercise

Determine all asymptotes of the function: $y=\dfrac{3x}{1-x}$

Solution

VA: $x=1$ **HA**: $y=-3$

Hole: n/a **Oblique asymptote**: n/a

Exercise

Determine all asymptotes of the function: $y=\dfrac{x^2}{x^2+9}$

Solution

VA: n/a $x^2+9\neq 0$ **HA**: $y=1$

Hole: n/a **Oblique asymptote**: n/a

Exercise

Determine all asymptotes of the function: $y=\dfrac{x-2}{x^2-4x+3}$

Solution

$$x^2-4x+3=0 \Rightarrow x=1,\ 3$$

$$y=\frac{x}{x^2}\to 0$$

VA: $x=1,\ \ x=3$ **HA**: $y=0$

Hole: n/a **Oblique asymptote**: n/a

Exercise

Determine all asymptotes of the function: $y=\dfrac{3}{x-5}$

Solution

VA: $x = 5$ **HA**: $y = 0$

Hole: n/a **Oblique asymptote**: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{x^3 - 1}{x^2 + 1}$

Solution

VA: *none*

HA: *none*

Hole: n / a

Oblique asymptote: $y = x$

$$\begin{array}{r|l} x^2+1 & \overline{x^3 \quad -1}^{\;x} \\ & \underline{-x^3 - x} \\ & \quad -x - 1 \end{array}$$

$$y = x - \frac{x+1}{x^2+1}$$

Exercise

Determine all asymptotes of the function: $y = \dfrac{x^3 + 3x^2 - 2}{x^2 - 4}$

Solution

VA: $x = \pm 2$

HA: n / a

Hole: n / a

Oblique asymptote: $y = x + 3$

$$\begin{array}{r|l} x^2-4 & \overline{x^3 + 3x^2 - 2}^{\;x+3} \\ & \underline{-x^3 + 4x} \\ & \quad 3x^2 + 4x - 2 \\ & \quad \underline{-3x^2 \quad + 12} \\ & \qquad 4x + 10 \end{array}$$

$$y = x + 3 + \frac{4x+10}{x^2-4}$$

Exercise

Determine all asymptotes of the function: $y = \dfrac{3x^2 - 27}{(x+3)(2x+1)}$

Solution

$$y = \frac{3x^2 - 27}{(x+3)(2x+1)} = \frac{3\left(x^2 - 9\right)}{(x+3)(2x+1)} = \frac{3(x+3)(x-3)}{(x+3)(2x+1)} = \frac{3(x-3)}{(2x+1)}$$

VA: $x = -3, \; -\frac{1}{2}$ ***HA***: $y = \frac{3}{2}$

Hole: n / a ***Oblique asymptote***: n / a

Exercise

Determine all asymptotes of the function: $y = \dfrac{x-3}{x^2-9}$

Solution

$$x^2 - 9 = 0 \to \boxed{x = \pm 3}$$

$$y = \frac{x-3}{(x-3)(x+3)}$$

$$= \frac{1}{x+3}$$

VA: $x = 3$ ***HA***: $y = 0$

Hole: $x = 3 \to y = \frac{1}{6}$ ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{6}{\sqrt{x^2 - 4x}}$

Solution

$$x^2 - 4x = 0$$

$$\Rightarrow x(x-4) = 0 \to \boxed{x = 0, 4}$$

VA: $x = 0,\ x = 4$ ***HA***: $y = 0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $y = \dfrac{5x-1}{1-3x}$

Solution

VA: $x = \frac{1}{3}$ ***HA***: $y = -\frac{5}{3}$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $f(x) = \dfrac{2x-11}{x^2+2x-8}$

Solution

VA: $x = 2,\ x = -4$ ***HA***: $y = 0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $f(x)=\dfrac{x^2-4x}{x^3-x}$

Solution

$$f(x)=\frac{x(x-4)}{x\left(x^2-1\right)}$$

$$=\frac{x-4}{x^2-1}$$

VA: $x=-1,\ x=1$ ***HA***: $y=0$

Hole: $x=0\rightarrow y=4$ ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function: $f(x)=\dfrac{x-2}{x^3-5x}$

Solution

VA: $x=0,\ x=\pm\sqrt{5}$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{4x}{x^2+10x}$

Solution

$$x^2+10x=0 \quad\rightarrow\quad x=0,\ -10$$

Domain: $(-\infty,\ -10)\cup(-10,\ 0)\cup(0,\ \infty)$

$$f(x)=\frac{4x}{x(x+10)}$$

$$=\frac{4}{x+10}$$

VA: $x=-10$ ***HA***: $y=0$

Hole: $x=0 \quad\rightarrow\quad y=\frac{4}{10}\Rightarrow$ *hole* $\left(0,\ \frac{2}{5}\right)$ ***Oblique asymptote***: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{3-x}{(x-4)(x+6)}$

Solution

VA: $x=-6 \quad and \quad x=4$ **HA**: $y=0$

Hole: n/a **Oblique asymptote**: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3}{2x^3-x^2-3x}$

Solution

$$2x^3-x^2-3x=x\left(2x^2-x-3\right)=0 \quad \to \quad x=0,\ -1,\ \frac{3}{2}$$

$$\begin{aligned} f(x)&=\frac{x^3}{2x^3-x^2-3x} \\ &=\frac{x^3}{x\left(2x^2-x-3\right)} \\ &=\frac{x^2}{2x^2-x-3} \end{aligned}$$

VA: $x=-1 \quad and \quad x=\frac{3}{2}$ **HA**: $y=\frac{1}{2}$

Hole: $x=0 \quad \to \quad y=0 \Rightarrow$ *hole* $(0,\ 0)$ **Oblique asymptote**: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{3x^2+5}{4x^2-3}$

Solution

$$4x^2-3=0 \quad \to \quad x=\pm\frac{\sqrt{3}}{2}$$

Domain: $\left(-\infty,\ -\frac{\sqrt{3}}{2}\right)\cup\left(-\frac{\sqrt{3}}{2},\ \frac{\sqrt{3}}{2}\right)\cup\left(\frac{\sqrt{3}}{2},\ \infty\right)$

VA: $x=-\frac{\sqrt{3}}{2} \quad and \quad x=\frac{\sqrt{3}}{2}$ **HA**: $y=\frac{3}{4}$

Hole: n/a **Oblique asymptote**: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x+6}{x^3+2x^2}$

Solution

$$x^3+2x^2=x^2(x+2)=0 \quad \to \quad x=0,-2$$

Domain: $(-\infty,\ -2)\cup(-2,\ 0)\cup(0,\ \infty)$

VA: $x=0$ *and* $x=2$ **HA**: $y=0$

Hole: n/a **Oblique asymptote**: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2+4x-1}{x+3}$

Solution

VA: $x=-3$

HA: n/a

Hole: n/a

Oblique asymptote: $y=x+1$

$$\begin{array}{r} x+1 \\ x+3\overline{\big)\,x^2+4x-1} \\ \underline{-x^2-3x} \\ x-1 \\ \underline{-x-3} \\ -4 \end{array}$$

$$f(x)=\frac{x^2+4x-1}{x+3}=x+1-\frac{4}{x+3}$$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2-6x}{x-5}$

Solution

$x-5=0 \quad\rightarrow\quad x=5$

Domain: $(-\infty,\ 5)\cup(5,\ \infty)$

$$\begin{aligned} f(x)&=\frac{x^2-6x}{x-5} \\ &=x-1-\frac{5}{x-5} \end{aligned}$$

$$\begin{array}{r} x-1 \\ x-5\overline{\big)\,x^2-6x} \\ \underline{-x^2+5x} \\ -x \\ \underline{x-5} \\ -5 \end{array}$$

VA: $x=5$

HA: N/A

Hole: N/A

Oblique asymptote: $y=x-1$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3-x^2+x-4}{x^2+2x-1}$

Solution

$$x^2+2x-1=0 \quad \to \quad x=-1\pm\sqrt{2}$$

Domain: $\left(-\infty,\ -1-\sqrt{2}\right)\cup\left(-1-\sqrt{2},\ -1+\sqrt{2}\right)\cup\left(-1+\sqrt{2},\ \infty\right)$

$$\begin{aligned} f(x)&=\frac{x^3-x^2+x-4}{x^2+2x-1} \\ &=x-3+\frac{8x-7}{x^2+2x-1} \end{aligned}$$

$$\begin{array}{r} x-3 \\ x^2+2x-1\overline{)\,x^3-x^2+x-4} \\ \underline{-x^3-2x^2+x} \\ -3x^2+2x-4 \\ \underline{3x^2+6x-3} \\ 8x-7 \end{array}$$

VA: $x=-1\pm\sqrt{2}$

HA: n/a

Hole: n/a

Oblique asymptote: $y=x-3$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{4x}{x^2+10x}$

Solution

$x^2+10x=0 \quad \to \quad x=0,\ -10$ **Domain**: $(-\infty,\ -10)\cup(-10,\ 0)\cup(0,\ \infty)$

$$f(x)=\frac{4x}{x(x+10)}=\frac{4}{x+10}$$

VA: $x=-10$

HA: $y=0$

Hole: $x=0 \quad \to \quad y=\frac{4}{10} \Rightarrow \text{hole } \left(0,\ \frac{2}{5}\right)$

Oblique asymptote: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{3-x}{(x-4)(x+6)}$

Solution

Domain: $(-\infty,\ -6)\cup(-6,\ 4)\cup(4,\ \infty)$

VA: $x=-6$ *and* $x=4$

HA: $y=0$

Hole: n/a **Oblique asymptote**: n/a

Exercise

Determine all asymptotes of the function $f(x)=\frac{x^3}{2x^3-x^2-3x}$

Solution

$$2x^3-x^2-3x=x\left(2x^2-x-3\right)=0 \quad \rightarrow \quad x=0,\ -1,\ \frac{3}{2}$$

Domain: $(-\infty,\ -1)\cup(-1,\ 0)\cup\left(0,\ \frac{3}{2}\right)\cup\left(\frac{3}{2},\ \infty\right)$

$$f(x)=\frac{x^3}{2x^3-x^2-3x}=\frac{x^3}{x\left(2x^2-x-3\right)}=\frac{x^2}{2x^2-x-3}$$

VA: $x=-1$ *and* $x=\frac{3}{2}$ **HA**: $y=\frac{1}{2}$

Hole: $x=0 \quad \rightarrow \quad y=0 \Rightarrow$ *hole* $(0,\ 0)$

Oblique asymptote: n/a

Exercise

Determine all asymptotes of the function $f(x)=\frac{3x^2+5}{4x^2-3}$

Solution

$4x^2-3=0 \quad \rightarrow \quad x=\pm\frac{\sqrt{3}}{2}$ **Domain**: $\left(-\infty,\ -\frac{\sqrt{3}}{2}\right)\cup\left(-\frac{\sqrt{3}}{2},\ \frac{\sqrt{3}}{2}\right)\cup\left(\frac{\sqrt{3}}{2},\ \infty\right)$

VA: $x=-\frac{\sqrt{3}}{2}$ *and* $x=\frac{\sqrt{3}}{2}$ **HA**: $y=\frac{3}{4}$

Hole: n/a **Oblique asymptote**: n/a

Exercise

Determine all asymptotes of the function $f(x)=\frac{x+6}{x^3+2x^2}$

Solution

$x^3+2x^2=x^2(x+2)=0 \quad \rightarrow \quad x=0,-2$ **Domain**: $(-\infty,\ -2)\cup(-2,\ 0)\cup(0,\ \infty)$

VA: $x=0$ *and* $x=2$ **HA**: $y=0$

Hole: n/a **Oblique asymptote**: n/a

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2+4x-1}{x+3}$

Solution

$x+3=0 \quad \rightarrow \quad x=-3$ **Domain**: $(-\infty,\ -3)\cup(-3,\ \infty)$

$$\begin{array}{r|l} & x+1 \\ \hline x+3 & x^2+4x-1 \\ & \underline{-x^2-3x} \\ & \ x-1 \\ & \ \underline{-x-3} \\ & \ -4 \end{array}$$

$$f(x)=\frac{x^2+4x-1}{x+3}=x+1-\frac{4}{x+3}$$

VA: $x=-3$ ***HA***: n/a

Hole: n/a ***Oblique asymptote***: $y=x+1$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^2-6x}{x-5}$

Solution

$x-5=0 \quad \rightarrow \quad x=5$ **Domain**: $(-\infty,\ 5)\cup(5,\ \infty)$

$$\begin{array}{r|l} & x-1 \\ \hline x-5 & x^2-6x \\ & \underline{-x^2+5x} \\ & \ -x \\ & \ \underline{x-5} \\ & \ -5 \end{array}$$

VA: $x=5$

HA: N/A

Hole: N/A

Oblique asymptote: $y=x-1$

$$f(x)=\frac{x^2-6x}{x-5}=x-1-\frac{5}{x-5}$$

Exercise

Determine all asymptotes of the function $f(x)=\dfrac{x^3-x^2+x-4}{x^2+2x-1}$

Solution

$x^2+2x-1=0 \quad \rightarrow \quad x=-1\pm\sqrt{2}$

Domain: $\left(-\infty,\ -1-\sqrt{2}\right)\cup\left(-1-\sqrt{2},\ -1+\sqrt{2}\right)\cup\left(-1+\sqrt{2},\ \infty\right)$

$$\begin{array}{r}
x-3 \\
x^2+2x-1\overline{\smash{)}\,x^3-x^2+x-4} \\
\underline{-x^3-2x^2+x} \\
-3x^2+2x-4 \\
\underline{3x^2+6x-3} \\
8x-7
\end{array}$$

VA: $x=-1\pm\sqrt{2}$

HA: N/A

Hole: N/A

Oblique asymptote: $y=x-3$

$$f(x)=\frac{x^3-x^2+x-4}{x^2+2x-1}$$

$$=x-3+\frac{8x-7}{x^2+2x-1}$$

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

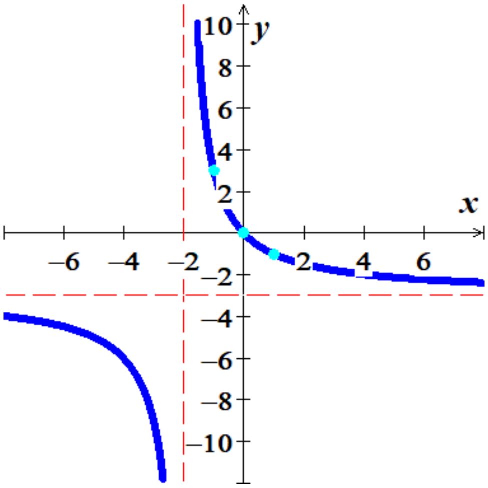
$$f(x)=\frac{-3x}{x+2}$$

Solution

VA: $x=-2$ **HA**: $y=-3$

Hole: n/a **OA**: n/a

x	**y**
0	0
1	–1
–1	3



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

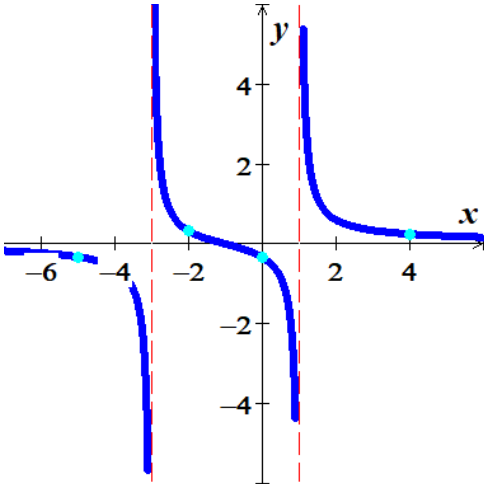
$$f(x)=\frac{x+1}{x^2+2x-3}$$

Solution

VA: $x=1,\ x=-3$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

x	y
–5	–0.33
–2	0.33
0	–1/3
4	0.24



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

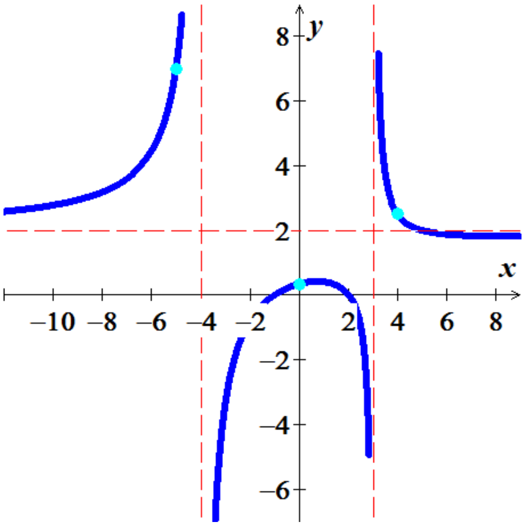
$$f(x)=\frac{2x^2-2x-4}{x^2+x-12}$$

Solution

VA: $x=-4,\ 3$ ***HA***: $y=2$

Hole: n/a ***OA***: n/a

x	y
–5	7
–2	–0.8
0	1/3
4	2.5
5	2



Exercise

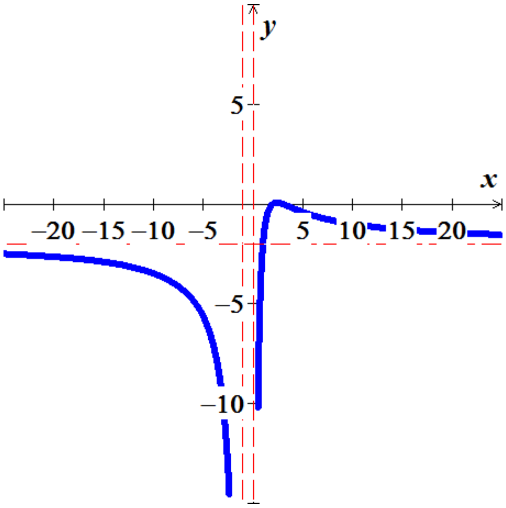
Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

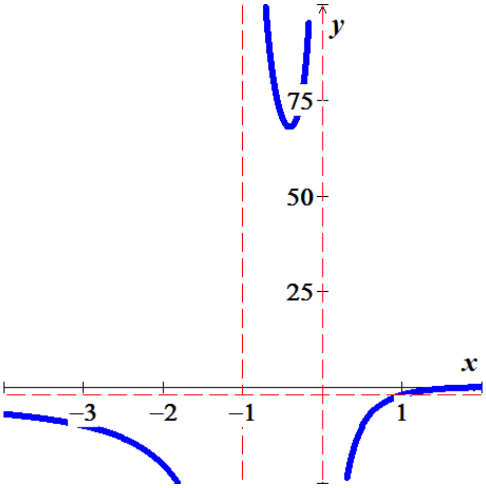
$$f(x) = \frac{-2x^2 + 10x - 12}{x^2 + x}$$

Solution

VA: $x = -1,\ 0$ ***HA***: $y = -2$

Hole: n/a ***OA***: n/a





Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x) = \frac{x^2 - x - 6}{x + 1}$$

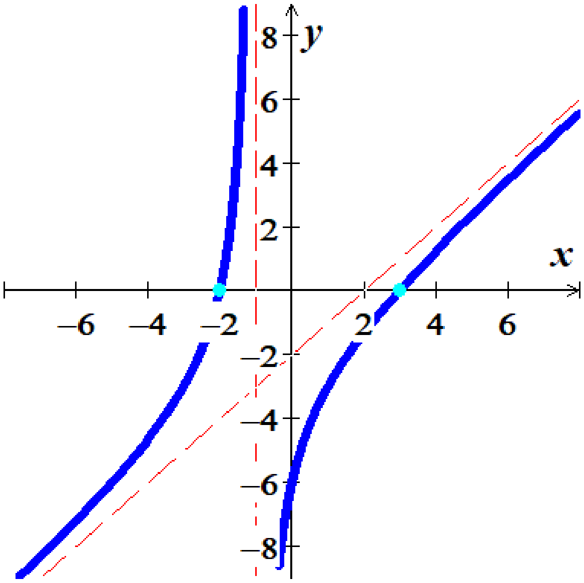
Solution

$$
\begin{array}{rl}
 & x-2 \\
x+1 \,\big) & x^2 - x - 6 \\
 & \underline{x^2 + x} \\
 & -2x-6 \\
 & \underline{-2x-2} \\
 & \boxed{-4}
\end{array}
$$

VA: $x = -1$ ***HA***: n/a

Hole: n/a ***OA***: $y = x - 2$

x	**y**
2	0
–2	0
0	–6



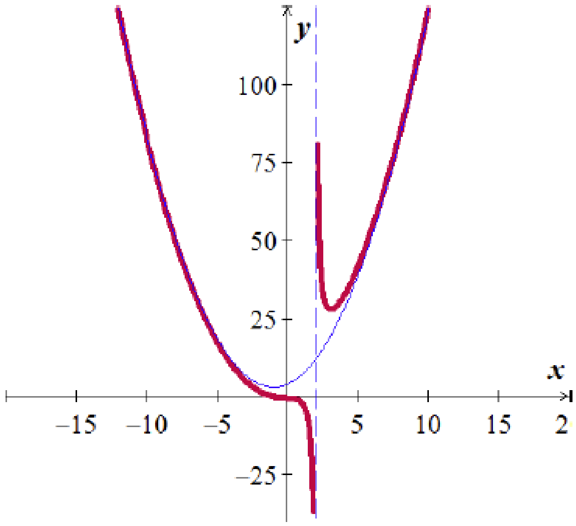
Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x^3+1}{x-2}$$

Solution

$$\begin{array}{r|l} & x^2+2x+4 \\ \hline x-2 & x^3-1 \\ & \underline{x^3-2x^2} \\ & 2x^2 \\ & \underline{2x^2-4x} \\ & 4x-1 \\ & \underline{4x-8} \\ & 7 \end{array}$$



VA: $x=2$ ***HA***: n/a

Hole: n/a ***OA***: $y=x^2+2x+4$

Exercise

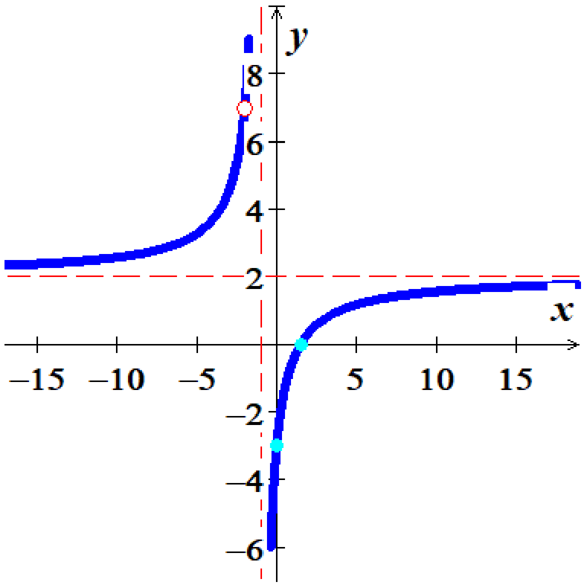
Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{2x^2+x-6}{x^2+3x+2}$$

Solution

$$f(x)=\frac{(2x-3)(x+2)}{(x+1)(x+2)}$$

$$=\frac{2x-3}{x+1}$$



VA: $x=-1$ ***HA***: $y=2$

Hole: $(-2,\ 7)$ ***OA***: n/a

x	**y**
0	−3
$-\frac{3}{2}$	0

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x-1}{1-x^2}$$

Solution

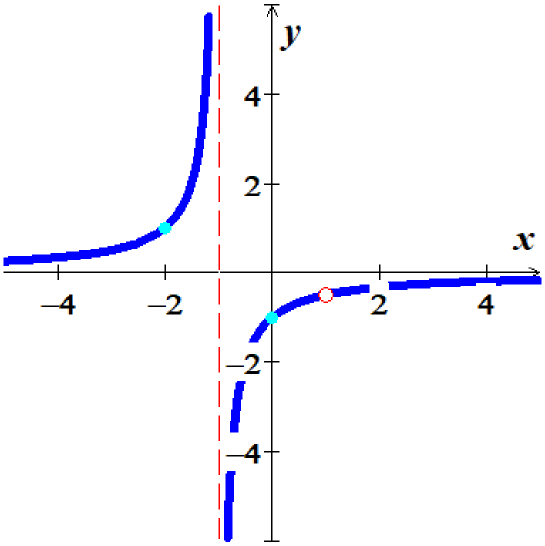
$$f(x)=\frac{x-1}{(x+1)(1-x)}$$

$$=-\frac{1}{x+1}$$

VA: $x=-1$ ***HA***: $y=0$

Hole: $\left(1,\ -\frac{1}{2}\right)$ ***OA***: n/a

x	***y***
0	–1
–2	1



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x^2+x-2}{x+2}$$

Solution

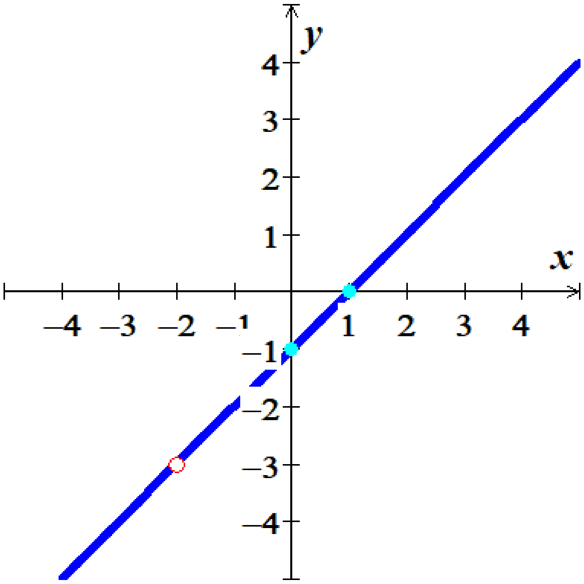
$$f(x)=\frac{(x+2)(x-1)}{x+2}$$

$$=x-1$$

VA: n/a ***HA***: n/a

Hole: $(-2,\ -3)$ ***OA***: n/a

x	***y***
0	–1
1	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x^3-2x^2-4x+8}{x-2}$$

Solution

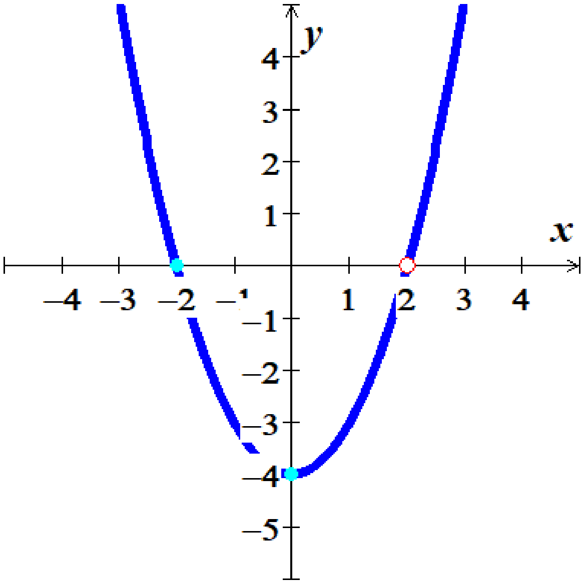
$$f(x)=\frac{\left(x^2-4\right)(x-2)}{x-2}$$

$$=x^2-4$$

VA: n/a ***HA***: n/a

Hole: $(2,\ 0)$ ***OA***: n/a

x	***y***
0	–4
–2	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{2x^2-3x-1}{x-2}$$

Solution

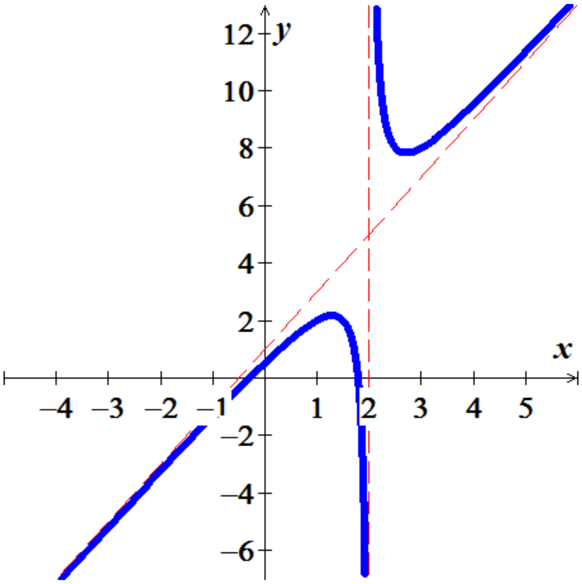
$$\begin{array}{r} 2x+1 \\ x-2\overline{)2x^2-3x-1} \\ \underline{-2x^2+4x} \\ x-1 \\ \underline{-x+2} \\ 1 \end{array}$$

$$f(x)=\frac{2x^2-3x-1}{x-2}$$

$$=(2x+1)+\frac{1}{x-2}$$

VA: $x=2$ ***HA***: $y=1$

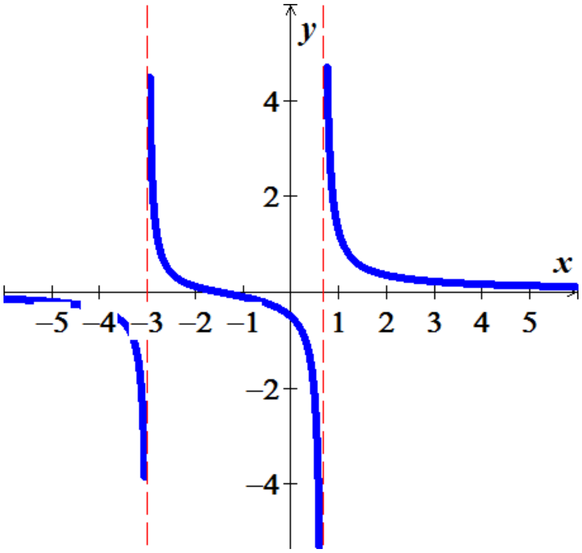
Hole: n/a ***OA***: $y=2x+1$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{2x+3}{3x^2+7x-6}$$



Solution

$3x^2+7x-6=0 \quad \Rightarrow \quad x=-3,\ \frac{2}{3}$

VA: $x=-3$ *and* $x=\frac{2}{3}$

HA: $y=0$

Hole: n/a

OA: n/a

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph

$$f(x)=\frac{x^2-1}{x^2+x-6}$$

Solution

$x^2+x-6=0 \quad \Rightarrow \quad \underline{x=-3,\ 2|}$

VA: $x=-3$ *and* $x=2$ ***HA***: $y=1$

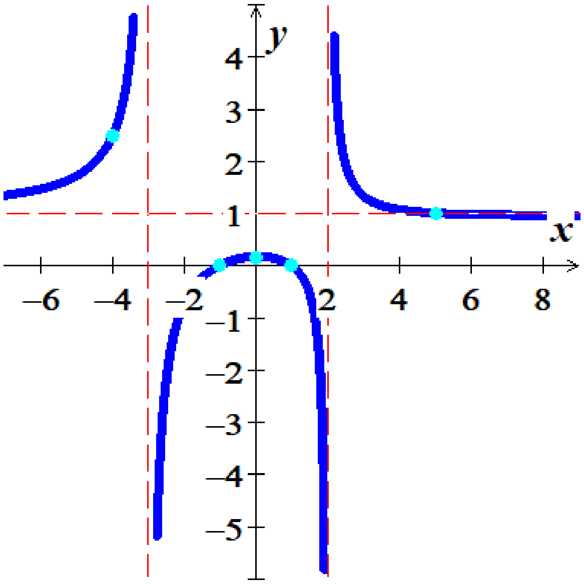
Hole: n/a ***OA:*** n/a

$1=\frac{x^2-1}{x^2+x-6}$

$x^2+x-6=x^2-1$

$\underline{x=5|}$

x	***y***
0	$\frac{1}{6}$
5	1
±1	0
−4	$\frac{5}{2}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{-2x^2-x+15}{x^2-x-12}$$

Solution

$x^2-x-12=0 \quad \Rightarrow \quad \underline{x=-3,\ 4|}$

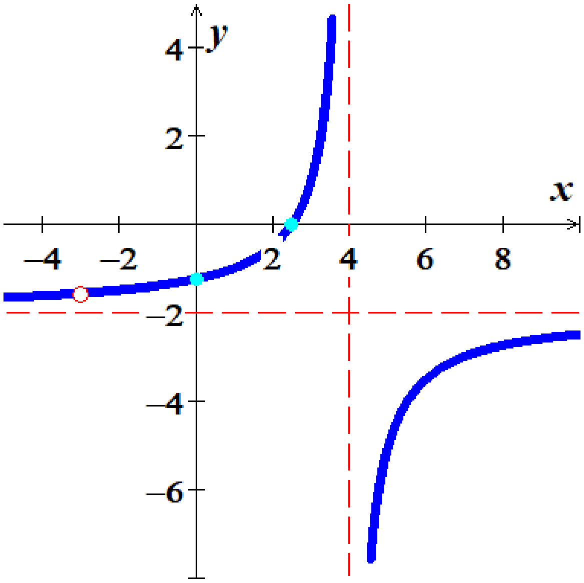
Domain: $(-\infty,-3)\cup(-3,4)\cup(4,\infty)$

$$f(x)=\frac{(-2x+5)(x+3)}{(x-4)(x+3)}$$
$$=\frac{-2x+5}{x-4}$$

VA: $x=4$ ***HA***: $y=-2$

Hole: $\left(-3,\ -\frac{11}{7}\right)$ ***OA:*** n/a

x	**y**
0	$-\frac{5}{4}$
$\frac{5}{2}$	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

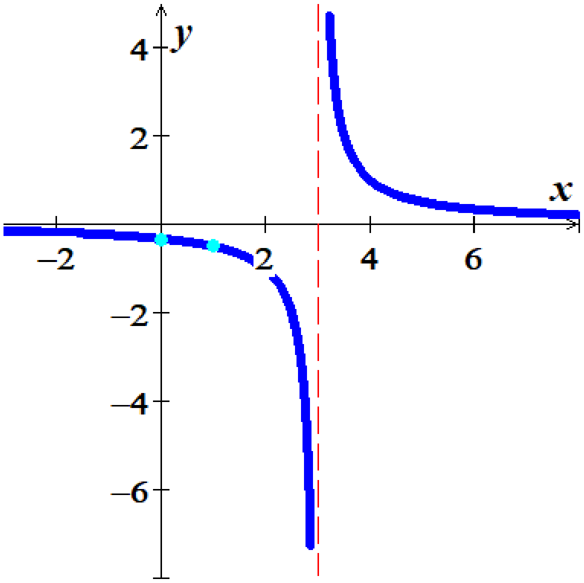
$$f(x)=\frac{1}{x-3}$$

Solution

VA: $x=3$ ***HA***: $y=0$

Hole: n/a ***OA:*** n/a

x	**y**
0	$-\frac{1}{3}$
1	$-\frac{1}{2}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

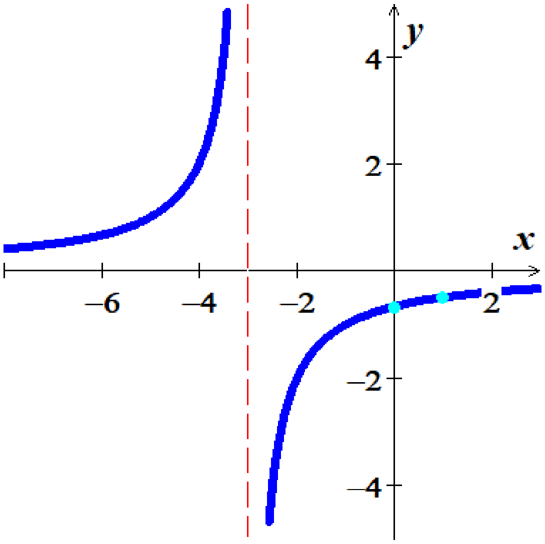
$$f(x)=\frac{-2}{x+3}$$

Solution

VA: $x=-3$ **HA**: $y=0$

Hole: n/a **OA:** n/a

x	*y*
0	$-\frac{2}{3}$
1	$-\frac{1}{2}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

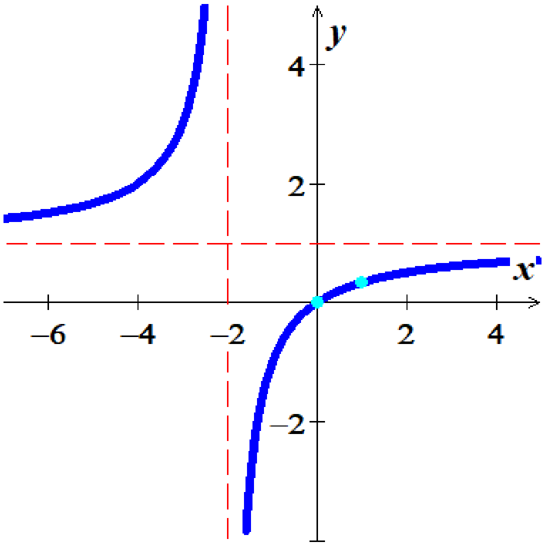
$$f(x)=\frac{x}{x+2}$$

Solution

VA: $x=-2$ **HA**: $y=1$

Hole: n/a **OA:** n/a

x	*y*
0	0
1	$\frac{1}{3}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

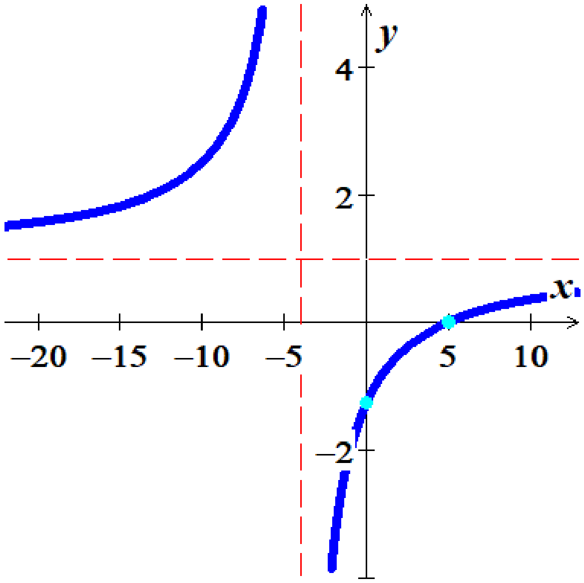
$$f(x)=\frac{x-5}{x+4}$$

Solution

VA: $x=-4$ **HA**: $y=1$

Hole: n/a **OA:** n/a

x	y
0	$-\frac{5}{4}$
5	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{2x^2-2}{x^2-9}$$

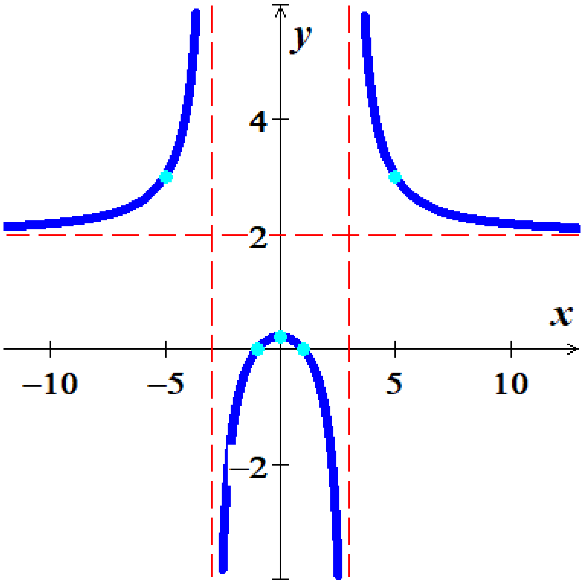
Solution

$x^2=9 \quad \rightarrow \quad \underline{x=\pm 3|}$

VA: $x=\pm 3$ **HA**: $y=2$

Hole: n/a **OA:** n/a

x	y
0	$\frac{2}{9}$
± 1	0
± 5	3



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

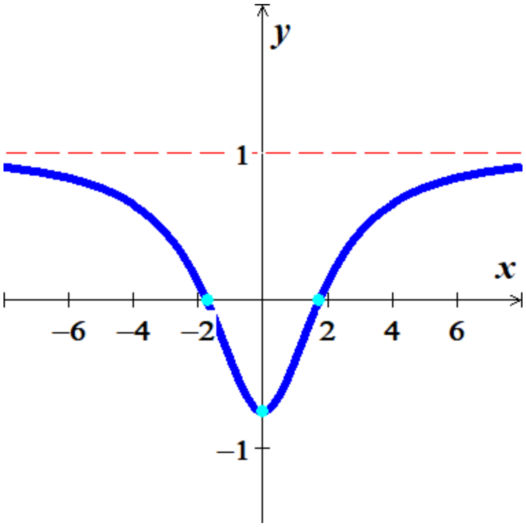
$$f(x)=\frac{x^2-3}{x^2+4}$$

Solution

VA: n/a **HA**: $y=1$

Hole: n/a **OA:** n/a

x	y
0	$-\frac{3}{4}$
$\pm\sqrt{3}$	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2+4}{x^2-3}$$

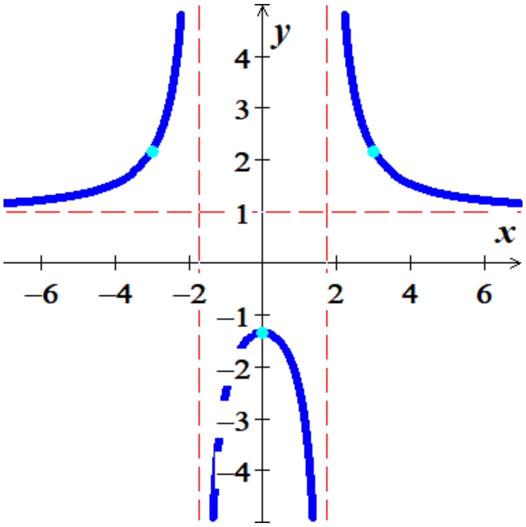
Solution

$$x^2-3=0 \quad \rightarrow \quad x=\pm\sqrt{3}$$

VA: $x=\pm\sqrt{3}$ **HA**: $y=1$

Hole: n/a **OA:** n/a

x	y
0	$-\frac{4}{3}$
± 3	$\frac{13}{6}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2}{x^2-6x+9}$$

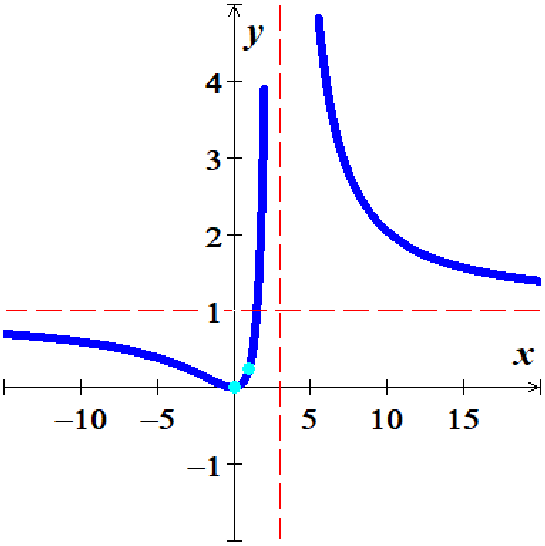
Solution

$x^2-6x+9=0 \quad \rightarrow \quad x=3$

VA: $x=3$ **HA**: $y=1$

Hole: n/a **OA:** n/a

x	y
0	0
1	$\frac{1}{4}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2+x+4}{x^2+2x-1}$$

Solution

$x^2+2x-1=0$

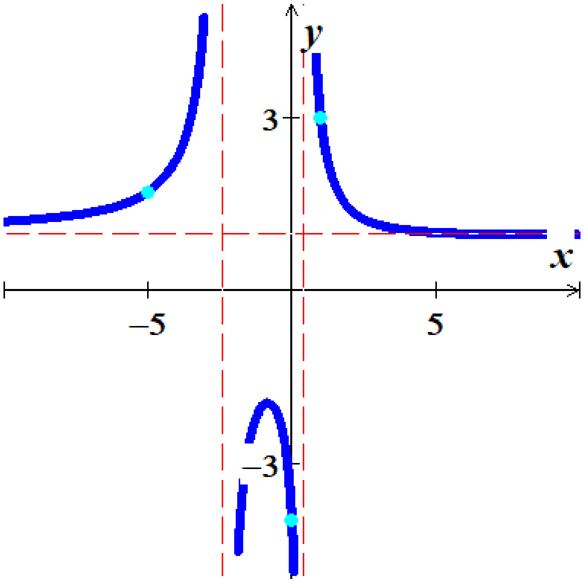
$x=\frac{-2\pm\sqrt{8}}{2}$

$=-1\pm\sqrt{2}$

VA: $x=-1\pm\sqrt{2}$ **HA**: $y=1$

Hole: n/a **OA:** n/a

x	y
0	−4
1	3
−5	$\frac{12}{7}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

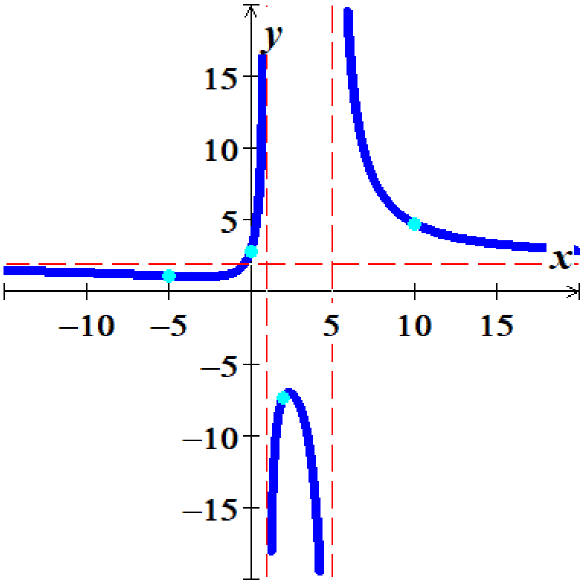
$$f(x)=\frac{2x^2+14}{x^2-6x+5}$$

Solution

VA: $x=1,\ 5$ **HA**: $y=2$

Hole: n/a **OA:** n/a

x	**y**
0	$\frac{14}{5}$
2	$-\frac{22}{3}$
–5	$\frac{16}{15}$
10	$\frac{214}{45}$



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2-4x-5}{2x+5}$$

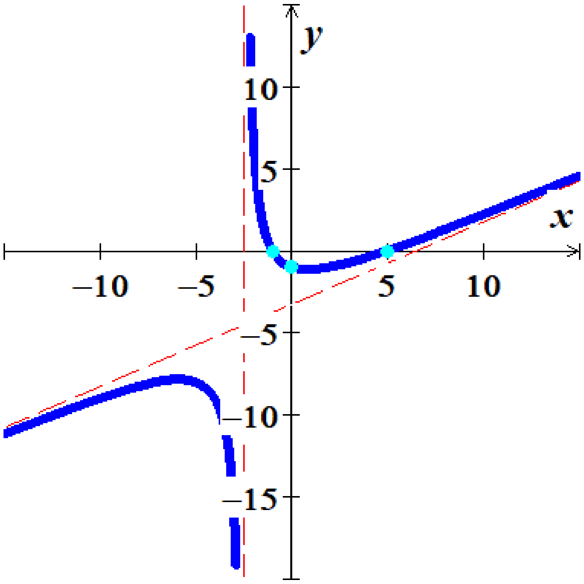
Solution

$$\begin{array}{r} \frac{1}{2}x-\frac{13}{4} \\ 2x+5\overline{\big)x^2-4x-5} \\ \underline{x^2+\frac{5}{2}x} \\ -\frac{13}{2}x-5 \end{array}$$

VA: $x=-\frac{5}{2}$ **HA**: n/a

Hole: n/a **OA:** $y=\frac{1}{2}x-\frac{13}{2}$

x	**y**
0	–1
–1, 5	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x-3}{x^2-3x+2}$$

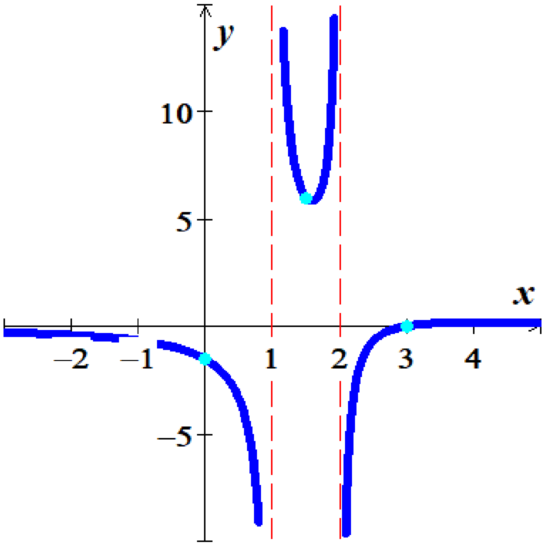
Solution

$x^2 - 3x + 2 \quad \to \quad \underline{x = 1,\ 2}\big|$

VA: $x = 1,\ 2$ ***HA***: $y = 0$

Hole: n/a ***OA:*** n/a

x	**y**
0	$-\frac{3}{2}$
3	0
$\frac{3}{2}$	6



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x) = \frac{x^2+2}{x^2+3x+2}$$

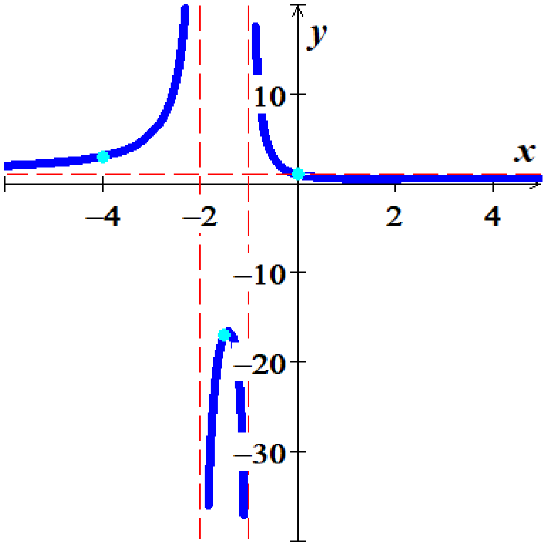
Solution

$x^2 + 3x + 2 \quad \to \quad \underline{x = -1,\ -2}\big|$

VA: $x = -1,\ -2$ ***HA***: $y = 1$

Hole: n/a ***OA:*** n/a

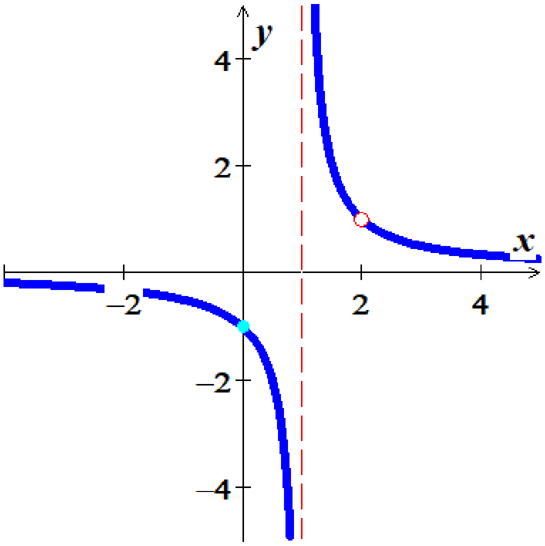
x	**y**
0	1
$-\frac{3}{2}$	-17
-4	3



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x-2}{x^2-3x+2}$$



Solution

$$x^2-3x+2 \quad \to \quad \underline{x=1,\ 2}|$$

$$f(x)=\frac{x-2}{(x-2)(x-1)}$$

$$=\frac{1}{x-1}$$

VA: $x=1$ ***HA***: $y=0$

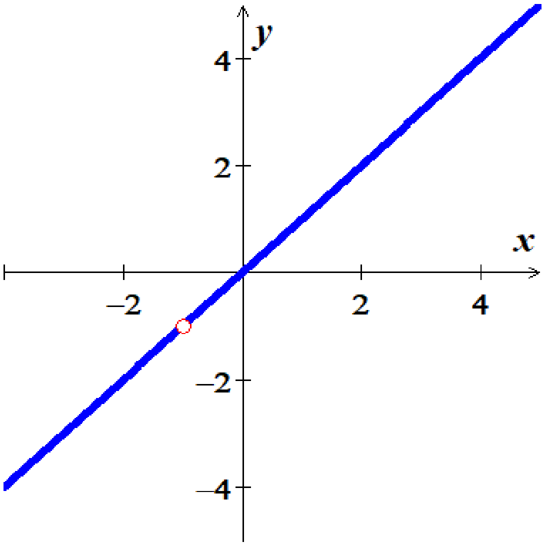
Hole: $(2,\ 1)$ ***OA:*** n/a

x	y
0	−1

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2+x}{x+1}$$



Solution

$$f(x)=\frac{x(x+1)}{x+1}$$

$$\underline{=x}|$$

VA: n/a ***HA***: n/a

Hole: $(-1,\ -1)$ ***OA:*** n/a

Hole: $\left(-3,\ -\frac{11}{7}\right)$ ***OA:*** n/a

x	y
0	0

Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2-2x}{x-2}$$

Solution

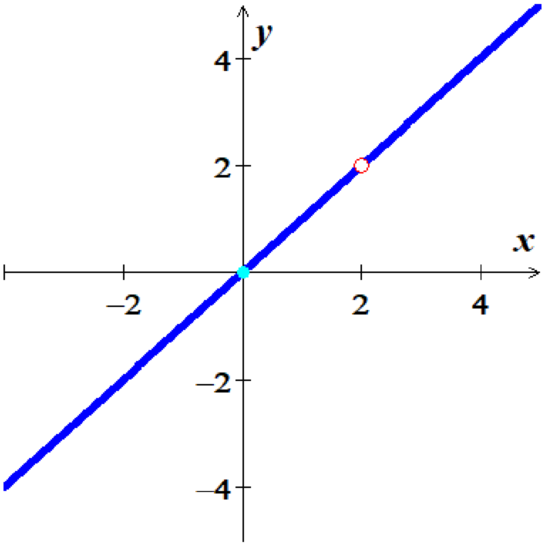
$$f(x)=\frac{x(x-2)}{x-2}$$
$$\underline{=x\ \big|}$$

VA: n/a **HA**: n/a

Hole: $(2,\ 2)$ **OA:** n/a

Hole: $\left(-3,\ -\frac{11}{7}\right)$ **OA:** n/a

x	**y**
0	0



Exercise

Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^2-3x}{x+3}$$

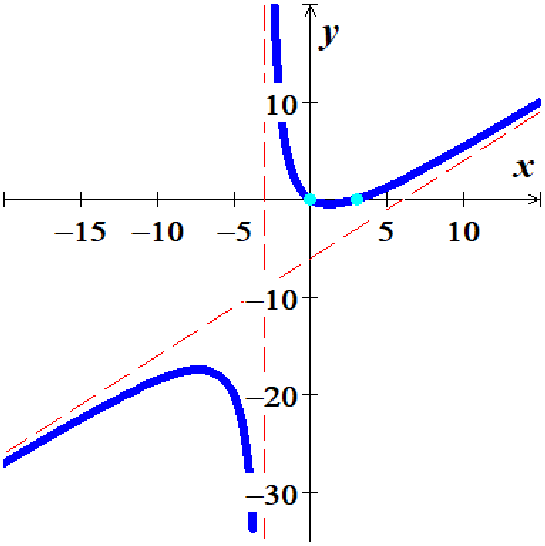
Solution

$$\begin{array}{rl} & \underline{\ x-6\ } \\ x+3\ \big) & x^2-3x \\ & \underline{x^2+3x} \\ & -6x-5 \end{array}$$

VA: $x=-3$ **HA**: n/a

Hole: n/a **OA:** $y=x-6$

x	**y**
0	0
3	0



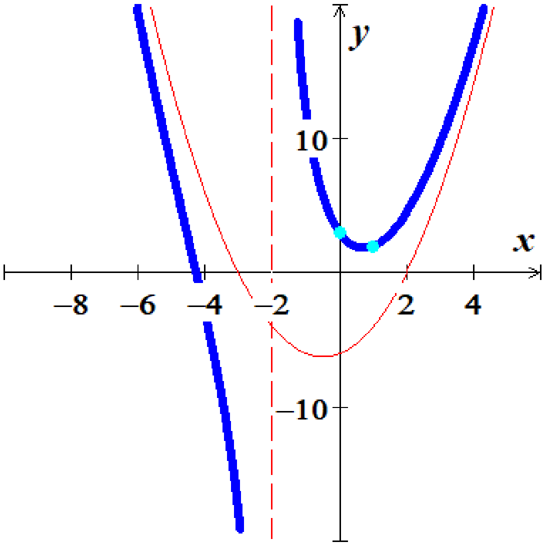
Exercise

Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

$$f(x)=\frac{x^3+3x^2-4x+6}{x+2}$$

Solution

$$\begin{array}{r|l} & x^2+x-6 \\ \hline x+2 & x^3+3x^2-4x+6 \\ & \underline{x^3+2x^2} \\ & x^2-4x \\ & \underline{x^2+2x} \\ & -6x+6 \end{array}$$



VA: $x=-2$ ***HA***: n/a

Hole: n/a ***OA:*** $y=x^2+x-6$

x	**y**
0	3
1	2

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} \text{vertical asymptote: } x=4 \\ \text{horizontal asymptote: } y=-1 \\ x-\text{intercept: } 3 \end{cases}$$

Solution

Vertical Asymptote: $f(x)=\frac{\qquad}{x-4}$

Horizontal Asymptote: $f(x)=\frac{-x+a}{x-4}$

x-intercept: $f(x=3)=\frac{-3+a}{3-4}=0 \quad\Rightarrow\quad \underline{a=3}$

$$f(x)=\frac{-x+3}{x-4}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote:\ x=-4, x=5 \\ horizontal\ asymptote:\ y=\frac{3}{2} \\ x-intercept:\ -2 \end{cases}$$

Solution

Vertical Asymptote: $f(x)=\frac{}{(x+4)(x-5)}$

Horizontal Asymptote: $f(x)=\frac{3}{2}\frac{(x+a)(x+b)}{(x+4)(x-5)}$

x-intercept: $f(x=-2)=\frac{3}{2}\frac{(-2+a)(-2+b)}{}$

$$0=(-2+a)(-2+b)$$

$$\boxed{a=b=2}$$

$$f(x)=\frac{3}{2}\frac{(x-2)^2}{x^2-x-20}$$

$$\boxed{=\frac{3x^2-12x+12}{2x^2-2x-40}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote:\ x=5 \\ horizontal\ asymptote:\ y=-1 \\ x-intercept:\ 2 \end{cases}$$

Solution

Vertical Asymptote: $f(x)=\frac{}{x-5}$

x-intercept : $f(x)=\frac{x-2}{x-5}$

Horizontal Asymptote: $f(x)=-\frac{x-2}{x-5}$

$$\boxed{f(x)=-\frac{x-2}{x-5}}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote:\ x=-2,\ x=0 \\ horizontal\ asymptote:\ y=0 \\ x-intercept:\ 2, \quad f(3)=1 \end{cases}$$

Solution

Vertical Asymptote: $f(x)=\dfrac{}{x(x+2)}$

x-intercept : $f(x)=\dfrac{x-2}{x(x+2)}$

Horizontal Asymptote: $f(x)=\dfrac{a(x-2)}{x(x+2)}$

$$f(3)=1 \quad \rightarrow \frac{a(1)}{(3)(5)}=1 \quad \Rightarrow \underline{a=15|}$$

$$f(x)=\frac{15x-30}{x^2+2x}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote:\ x=-3,\ x=1 \\ horizontal\ asymptote:\ y=0 \\ x-intercept:\ -1, \quad f(0)=-2 \\ \quad hole:\ x=2 \end{cases}$$

Solution

Vertical Asymptote: $f(x)=\dfrac{}{(x+3)(x-1)}$

x-intercept : $f(x)=\dfrac{(x+1)}{(x+3)(x-1)}$

Horizontal Asymptote: $f(x)=\dfrac{a(x+1)}{(x+3)(x-1)}$

$$f(0)=-2 \quad \rightarrow \frac{a}{-3}=-2 \quad \Rightarrow \underline{a=6|}$$

Hole at $x=2$**:** $f(x)=\dfrac{6(x+1)(x-2)}{(x^2+2x-3)(x-2)}$

$$f(x)=\frac{6x^2-6x-12}{x^3-7x+6}$$

Exercise

Find an equation of a rational function f that satisfies the given conditions

$$\begin{cases} vertical\ asymptote: \ x=-1, \ \ x=3 \\ horizontal\ asymptote: \ y=2 \\ x-intercept: \ \ -2, \ \ 1 \\ \ \ hole: \ \ x=0 \end{cases}$$

Solution

Vertical Asymptote: $f(x)=\frac{}{(x+1)(x-3)}$

Horizontal Asymptote: $f(x)=\frac{2}{(x+1)(x-3)}$

x-intercept : $f(x)=\frac{2(x+2)(x-1)}{(x+1)(x-3)}$

Hole at $x=0$**:** $f(x)=\frac{2x(x+2)(x-1)}{x(x+1)(x-3)}$

$$f(x)=\frac{2x^3+2x^2-4x}{x^3-2x^2-3x}$$

Solution *Section* 1.5 – Inverse, Exponential & Logarithmic Functions

Exercise

Determine whether the function is one-to-one: $f(x) = 3x - 7$

Solution

$$f(a) = f(b)$$
$$3a - 7 = 3b - 7$$
$$3a = 3b - 7 + 7$$
$$3a = 3b$$

Divide both sides by 3

$$a = b$$

$\therefore$ The function is one-to-one

Exercise

Determine whether the function is one-to-one: $f(x) = x^2 - 9$

Solution

$1 \neq -1$ $1^2 - 9 \neq (-1)^2 - 9$ $-8 = -8 \rightarrow$ Contradict the definition	$f(a) = f(b)$ $a^2 - 9 = b^2 - 9$ $a^2 = b^2$ $a = \pm b$

$\therefore$ The function is ***not*** one-to-one

Exercise

Determine whether the function is one-to-one: $f(x) = \sqrt{x}$

Solution

$$f(a) = f(b)$$
$$\sqrt{a} = \sqrt{b}$$
$$\left(\sqrt{a}\right)^2 = \left(\sqrt{b}\right)^2$$

Square both sides

$$a = b$$

$\therefore$ The function is one-to-one

Exercise

Determine whether the function is one-to-one: $f(x)=\sqrt[3]{x}$

Solution

$$f(a)=f(b)$$

$$\sqrt[3]{a}=\sqrt[3]{b}$$

$$\left(\sqrt[3]{a}\right)^3=\left(\sqrt[3]{b}\right)^3$$ ***cube both sides***

$$a=b$$

$\therefore$ The function is one-to-one

Exercise

Determine whether the function is one-to-one: $f(x)=|x|$

Solution

$$1\neq -1$$

$$|1|\neq|-1|$$

$$1\neq 1\ \left(\mathit{false}\right)$$

$\therefore$ The function is ***not*** one-to-one

Exercise

Determine whether the function is one-to-one $f(x)=\dfrac{2}{x+3}$

Solution

$$f(a)=f(b)$$

$$\frac{2}{a+3}=\frac{2}{b+3}$$

$$2(b+3)=2(a+3)$$

$$b+3=a+3$$

$$a=b$$

$\therefore f$ is one-to-one

Exercise

Determine whether the function is one-to-one $f(x)=\left(x-2\right)^3$

Solution

$$f(a)=f(b)$$

$$(a-2)^3=(b-2)^3$$

$$\left[(a-2)^3\right]^{1/3}=\left[(b-2)^3\right]^{1/3}$$

$a-2=b-2$ *Add 2 on both sides*

$a=b$

∴ Function is one-to-one

Exercise

Determine whether the function is one-to-one $y=x^2+2$

Solution

$f(a)=f(b)$

$a^2+2=b^2+2$ *Subtract 2*

$a^2=b^2$

$a=\pm\sqrt{b^2}$

∴ Function is ***not*** a one-to-one

The inverse function doesn't exist.

Exercise

Determine whether the function is one-to-one $f(x)=\dfrac{x+1}{x-3}$

Solution

$f(a)=f(b)$

$\dfrac{a+1}{a-3}=\dfrac{b+1}{b-3}$ ***Cross multiplication***

$(a+1)(b-3)=(b+1)(a-3)$

$ab-3a+b-3=ab-3b+a-3$

$-4a=-4b$ ***Divide by -4***

$a=b$

∴ Function is *one*-to-*one*

Exercise

Given the function $f(x) = (x+8)^3$

a) Find $f^{-1}(x)$

b) Graph f and f^{-1} in the same rectangular coordinate system

c) Find the domain and the range of f and f^{-1}

Solution

a) $y = (x+8)^3$ **Replace f(*x*) with *y***

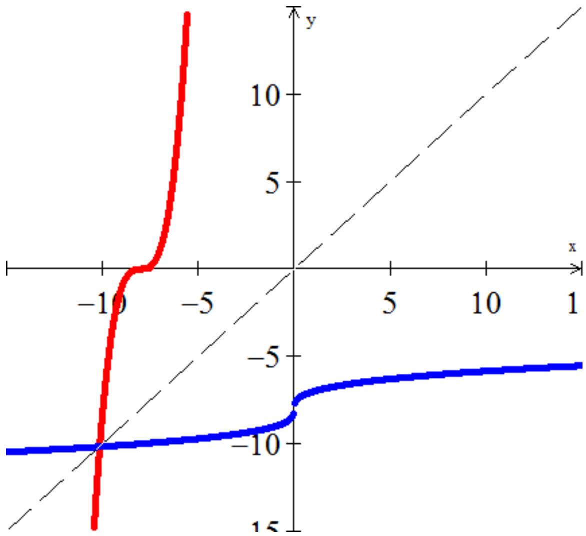
$x = (y+8)^3$ **Interchange *x* and *y***

$(x)^{1/3} = \left((y+8)^3\right)^{1/3}$

$x^{1/3} = y+8$ ***Subtract 8 from both sides.***

$\boxed{f^{-1}(x) = x^{1/3} - 8}$

b)



c) Domain of f = Range of f^{-1}: $(-\infty, \infty)$

Range of f = Domain of f^{-1}: $(-\infty, \infty)$

Exercise

For the given function $f(x)=\dfrac{2x}{x-1}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$$\frac{2a}{a-1}=\frac{2b}{b-1}$$

$$2ab-2a=2ab-2b$$

$$-2a=-2b$$

$$\boxed{a=b} \quad ✓$$

$\therefore\ f(x)$ is one-to-one function.

b) $y=\dfrac{2x}{x-1}$

$$x=\frac{2y}{y-1}$$

$$xy-x=2y$$

$$(x-2)y=x$$

$$\boxed{y=\frac{x}{x-2}=f^{-1}(x)}$$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\boxed{\mathbb{R}-\{1\}}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\boxed{\mathbb{R}-\{2\}}$

Exercise

For the given function $f(x)=\dfrac{x}{x-2}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$$\frac{a}{a-2}=\frac{b}{b-2}$$

$$ab-2a=ab-2b$$
$$-2a=-2b$$
$$\boxed{a=b} \qquad \checkmark$$

$\therefore\ f(x)$ is one-to-one function.

b) $y=\dfrac{x}{x-2}$

$$x=\frac{y}{y-2}$$
$$xy-2x=y$$
$$(x-1)y=2x$$
$$\boxed{f^{-1}(x)=\frac{2x}{x-1}}$$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\boxed{\mathbb{R}-\{2\}}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\boxed{\mathbb{R}-\{1\}}$

Exercise

For the given function $f(x)=\dfrac{x+1}{x-1}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$$\frac{a+1}{a-1}=\frac{b+1}{b-1}$$
$$ab-a+b-1=ab-b+a-1$$
$$-2a=-2b$$
$$\boxed{a=b} \qquad \checkmark$$

$\therefore\ f(x)$ is one-to-one function.

b) $y=\dfrac{x+1}{x-1}$

$$x=\frac{y+1}{y-1}$$
$$xy-x=y+1$$
$$(x-1)y=x+1$$

$$\boxed{f^{-1}(x)=\frac{x+1}{x-1}}$$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\boxed{\mathbb{R}-\{1\}}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\boxed{\mathbb{R}-\{1\}}$

Exercise $f(x)=\dfrac{2x+1}{x+3}$

For the given function

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$$\frac{2a+1}{a+3}=\frac{2b+1}{b+3}$$
$$2ab+6a+b+3=2ab+6b+a+3$$
$$5a=5b$$
$$\boxed{a=b} \quad \checkmark$$

$\therefore$ $f(x)$ is one-to-one function.

b) $y=\dfrac{2x+1}{x+3}$

$$x=\frac{2y+1}{y+3}$$
$$xy+3x=2y+1$$
$$(x-2)y=-3x+1$$
$$\boxed{f^{-1}(x)=\frac{-3x+1}{x-2}}$$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\boxed{\mathbb{R}-\{-3\}}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\boxed{\mathbb{R}-\{2\}}$

Exercise

For the given function $f(x)=\dfrac{3x-1}{x-2}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$$\frac{3a-1}{a-2}=\frac{3b-1}{b-2}$$
$$3ab-6a-b+2=3ab-6b-a+2$$
$$-5a=-5b$$
$$\boxed{a=b} \qquad \checkmark$$

$\therefore\ f(x)$ is one-to-one function.

b) $y=\dfrac{3x-1}{x-2}$

$$x=\frac{3y-1}{y-2}$$
$$xy-2x=3y-1$$
$$(x-3)y=2x-1$$
$$\boxed{f^{-1}(x)=\frac{2x-1}{x-3}}$$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\boxed{\mathbb{R}-\{2\}}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\boxed{\mathbb{R}-\{3\}}$

Exercise

For the given function $f(x)=\dfrac{3x-2}{x+4}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$$\frac{3a-2}{a+4}=\frac{3b-2}{b+4}$$
$$3ab+12a-2b-8=3ab+12b-2a-8$$

$14a = 14b$

$\boxed{a = b}$ ✓

$\therefore\ f(x)$ is one-to-one function.

b) $y = \dfrac{3x-2}{x+4}$

$x = \dfrac{3y-2}{y+4}$

$xy + 4x = 3y - 2$

$(x-3)y = -4x - 2$

$\boxed{f^{-1}(x) = \dfrac{-4x-2}{x-3}}$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\boxed{\mathbb{R} - \{-4\}}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\boxed{\mathbb{R} - \{3\}}$

Exercise

For the given function $f(x) = \dfrac{-3x-2}{x+4}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$\dfrac{-3a-2}{a+4} = \dfrac{-3b-2}{b+4}$

$-3ab - 12a - 2b - 8 = -3ab - 12b - 2a - 8$

$-10a = -10b$

$\boxed{a = b}$ ✓

$\therefore\ f(x)$ is one-to-one function.

b) $y = \dfrac{-3x-2}{x+4}$

$x = \dfrac{-3y-2}{y+4}$

$xy + 4x = -3y - 2$

$(x+3)y = -4x - 2$

$\boxed{f^{-1}(x) = \dfrac{-4x-2}{x+3}}$

c) Domain of $f^{-1}(x)$ = Range of $f(x)$: $\boxed{\mathbb{R}-\{-4\}}$

Range of $f^{-1}(x)$ = Domain of $f(x)$: $\boxed{\mathbb{R}-\{-3\}}$

Exercise

For the given function $f(x)=\sqrt{x-1} \quad x\ge 1$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$\sqrt{a-1}=\sqrt{b-1}$

$\left(\sqrt{a-1}\right)^2=\left(\sqrt{b-1}\right)^2$

$a-1=b-1$

$\boxed{a=b}$ ✓

$\therefore$ $f(x)$ is one-to-one function.

b) $y=\sqrt{x-1}$

$x=\sqrt{y-1}$

$x^2=y-1$

$y=x^2+1$

$\boxed{f^{-1}(x)=x^2+1 \quad x\ge 0}$

c) Domain of $f(x)$ = Range of $f^{-1}(x)$: $\boxed{[1,\ \infty)}$

Range of $f(x)$ = Domain of $f^{-1}(x)$: $\boxed{[0,\ \infty)}$

Exercise

For the given function $f(x)=\sqrt{2-x} \quad x\le 2$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$\sqrt{2-a} = \sqrt{2-b}$

$\left(\sqrt{2-a}\right)^2 = \left(\sqrt{2-b}\right)^2$

$2-a = 2-b$

$\boxed{a = b}$ ✓

$\therefore\ f(x)$ is one-to-one function.

b) $y = \sqrt{2-x}$

$x = \sqrt{2-y}$

$x^2 = 2-y$

$y = 2-x^2$

$\boxed{f^{-1}(x) = 2-x^2 \quad x \ge 0}$

c) Domain of $f(x)$ = Range of $f^{-1}(x)$: $\boxed{(-\infty,\ 2]}$

Range of $f(x)$ = Domain of $f^{-1}(x)$: $\boxed{[0,\ \infty)}$

Exercise

For the given function $f(x) = x^2 + 4x \quad x \ge -2$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

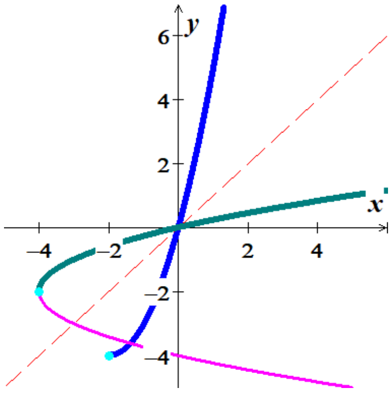
$x_{vertex} = -\frac{4}{2}$

$\boxed{= -2}$

$f(-2) = 4-8$

$\boxed{= -4}$

$Vertex = (-2,\ -4)$



a) Since, $f(x)$ is a restricted function with $x \ge -2$.

$x = -2$ is the line symmetry, therefore; $f(x)$ is one-to-one function.

b) $y = x^2 + 4x$

$x = y^2 + 4y$

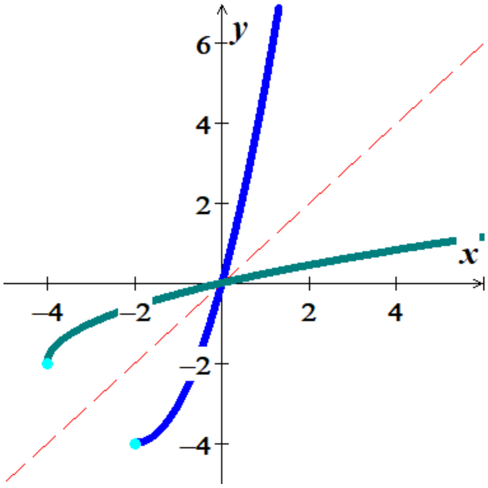
$y^2 + 4y - x = 0$

$y = \dfrac{-4 \pm \sqrt{16+4x}}{2}$

$= \dfrac{-4 \pm 2\sqrt{4+x}}{2}$

$= -2 + \sqrt{x+4}$

$\boxed{f^{-1}(x) = -2 + \sqrt{x+4} \quad x \ge 0}$



c) Domain of $f(x)$ = Range of $f^{-1}(x)$: $\boxed{[-2, \infty)}$

Range of $f(x)$ = Domain of $f^{-1}(x)$: $\boxed{[-4, \infty)}$

Exercise

For the given function $f(x) = 3x + 5$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$3a + 5 = 3b + 5$

$3a = 3b$

$a = b$

$\therefore$ $f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = 3x + 5$

$x = 3y + 5$ ***Interchange x and y***

$x - 5 = 3y$ ***Solve for y***

$\frac{x-5}{3} = y$

$\boxed{f^{-1}(x) = \frac{x-5}{3}}$

c) Domain of f^{-1} = Range of f: $\boxed{\mathbb{R}}$

Range of f^{-1} = Domain of f: $\boxed{\mathbb{R}}$

Exercise

For the given function $f(x)=\dfrac{1}{3x-2}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$$\frac{1}{3a-2}=\frac{1}{3b-2}$$

$$3b-2=3a-2$$

$$3b=3a$$

$$a=b$$

$\therefore$ $f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y=\dfrac{1}{3x-2}$

$x=\dfrac{1}{3y-2}$ *Interchange x and y*

$x(3y-2)=1$ *Solve for y*

$$3xy-2x=1$$

$$3xy=1+2x$$

$$f^{-1}(x)=\frac{1+2x}{3x}$$

c) Domain of f^{-1} = Range of f: $\mathbb{R}-\left\{\frac{2}{3}\right\}$

Range of f^{-1} = Domain of f: $\mathbb{R}-\{0\}$

Exercise

For the given function $f(x)=\dfrac{3x+2}{2x-5}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$$\frac{3a+2}{2a-5}=\frac{3b+2}{2b-5}$$

$$6ab-15a+4b-10=6ab-15b+4a-10$$

$19a = 19b$

$a = b$

$\therefore\ f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = \dfrac{3x+2}{2x-5}$

$x = \dfrac{3y+2}{2y-5}$ *Interchange x and y*

$2xy - 5x = 2y + 2$ *Solve for y*

$(2x-3)\,y = 5x + 2$

$\boxed{f^{-1}(x) = \dfrac{5x+2}{2x-3}}$

c) Domain of f^{-1} = Range of f: $\boxed{\mathbb{R} - \left\{\dfrac{5}{2}\right\}}$

Range of f^{-1} = Domain of f: $\boxed{\mathbb{R} - \left\{\dfrac{3}{2}\right\}}$

Exercise

For the given function $f(x) = \dfrac{4x}{x-2}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$\dfrac{4a}{a-2} = \dfrac{4b}{b-2}$

$4ab - 8a = 4ab - 8b$

$-8a = -8b$

$a = b$

$\therefore\ f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = \dfrac{4x}{x-2}$

$x = \dfrac{4y}{y-2}$

$xy - 2x = 4y$

$(x-4)\,y = 2x$

$\boxed{f^{-1}(x) = \dfrac{2x}{x-4}}$

c) Domain of f^{-1} = Range of f: $\boxed{\mathbb{R}-\{2\}}$

Range of f^{-1}= Domain of f: $\boxed{\mathbb{R}-\{4\}}$

Exercise

For the given function $f(x)=2-3x^2; \quad x \le 0$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$2-3a^2=2-3b^2$

$-3a^2=-3b^2$

$a^2=b^2$

$a=b$ since $x \le 0$

$\therefore$ $f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y=2-3x^2$

$x=2-3y^2$

$3y^2=2-x$

$y^2=\dfrac{2-x}{3}$

$\boxed{f^{-1}(x)=-\sqrt{\dfrac{2-x}{3}}}$ ***Since x < 0***

c) Domain of f^{-1} = Range of f: $\boxed{\mathbb{R}}$

Range of f^{-1}= Domain of f: $\boxed{\mathbb{R}}$

Exercise

For the given function $f(x)=2x^3-5$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$$2a^3 - 5 = 2b^3 - 5$$
$$a^3 = b^3$$
$$a = b$$

$\therefore$ $f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = 2x^3 - 5$

$$y + 5 = 2x^3$$
$$\frac{y+5}{2} = x^3$$
$$\boxed{f^{-1}(x) = \sqrt[3]{\frac{x+5}{2}}}$$

c) Domain of f^{-1} = Range of f: $\boxed{\mathbb{R}}$

Range of f^{-1} = Domain of f: $\boxed{\mathbb{R}}$

Exercise

For the given function $f(x) = \sqrt{3-x}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\left(\sqrt{3-a}\right)^2 = \left(\sqrt{3-b}\right)^2$$
$$3 - a = 3 - b$$
$$a = b$$

$\therefore$ $f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y = \sqrt{3-x}$ $\qquad$ $y \geq 0$

$$y = \sqrt{3-x}$$
$$y^2 = 3 - x$$
$$x = 3 - y^2 \qquad x \geq 0$$
$$\boxed{f^{-1}(x) = 3 - x^2}$$

c) Domain of f^{-1} = Range of f: $\boxed{(-\infty,\ 3]}$

Range of f^{-1} = Domain of f: $\boxed{[0,\ \infty)}$

Exercise

For the given function $f(x)=\sqrt[3]{x}+1$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$\sqrt[3]{a}+1=\sqrt[3]{b}+1$

$\left(\sqrt[3]{a}\right)^3=\left(\sqrt[3]{b}\right)^3$

$a=b$

$\therefore\ f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y=\sqrt[3]{x}+1$

$y=\sqrt[3]{x}+1$

$y-1=\sqrt[3]{x}$

$(y-1)^3=x$

$\boxed{f^{-1}(x)=(x-1)^3}$

c) Domain of f^{-1} = Range of f: $\mathbb{R}$

Range of f^{-1}= Domain of f: $\mathbb{R}$

Exercise

For the given function $f(x)=\left(x^3+1\right)^5$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$\left(a^3+1\right)^5=\left(b^3+1\right)^5$

$a^3+1=b^3+1$

$a^3=b^3$

$a=b$

$\therefore\ f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y=\left(x^3+1\right)^5$

$y=\left(x^3+1\right)^5$

$\sqrt[5]{y}=x^3+1$

$\sqrt[5]{y}-1=x^3$

$x=\sqrt[3]{\sqrt[5]{y}-1}$

$\boxed{f^{-1}(x)=\sqrt[3]{\sqrt[5]{x}-1}}$

c) Domain of f^{-1} = Range of f: $\mathbb{R}$

Range of f^{-1} = Domain of f: $\mathbb{R}$

Exercise

For the given function $f(x)=x^2-6x; \quad x\geq 3$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$a^2-6a=b^2-6b$

$a^2-b^2=6a-6b$

$(a-b)(a+b)=6(a-b)$

$a=b$

$\therefore$ $f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y=x^2-6x$

$x^2-6x-y=0$

$$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-y)}}{2(1)}$$

$$=\frac{6\pm 4\sqrt{9+y}}{2}$$

$$=3\pm\sqrt{9+y}$$

Since $x\geq 3 \Rightarrow$ we can select $x=3+\sqrt{y+9}$

$\therefore$ $\boxed{f^{-1}(x)=3+\sqrt{x+9}}$

c) Domain of f^{-1} = Range of f: $\underline{\mathbb{R}: \ \geq 3}$

Range of f^{-1} = Domain of f: $\underline{\geq -9}$

Exercise

For the given function $f(x)=(x-2)^3$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$(a-2)^3=(b-2)^3$

$a-2=b-2$

$a=b$

$\therefore$ $f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y=(x-2)^3$

$x=(y-2)^3$

$x^{1/3}=\left[(y-2)^3\right]^{1/3}$

$x^{1/3}=y-2$

$\sqrt[3]{x}+2=y$

$\therefore$ $\underline{f^{-1}(x)=\sqrt[3]{x}+2}$

c) Domain of f^{-1} = Range of f: $\underline{\mathbb{R}}$

Range of f^{-1} = Domain of f: $\underline{\mathbb{R}}$

Exercise

For the given function $f(x)=\dfrac{x+1}{x-3}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$$\frac{a+1}{a-3}=\frac{b+1}{b-3}$$
$$ab-3a+b-3=ab-3b+a-3$$
$$-4a=-4b$$
$$a=b$$

$\therefore\ f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y=\dfrac{x+1}{x-3}$

$$x=\frac{y+1}{y-3}$$
$$x(y-3)=y+1$$
$$xy-3x=y+1$$
$$xy-y=3x+1$$
$$y(x-1)=3x+1$$
$$y=\boxed{\frac{3x+1}{x-1}=f^{-1}(x)}$$

c) Domain of f^{-1} = Range of f: $\boxed{\mathbb{R}-\{3\}}$

Range of f^{-1} = Domain of f: $\boxed{\mathbb{R}-\{1\}}$

Exercise

For the given function $f(x)=\dfrac{2x+1}{x-3}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a)=f(b)$

$$\frac{2a+1}{a-3}=\frac{2b+1}{b-3}$$
$$2ab-6a+b-3=2ab-6b+a-3$$
$$-7a=-7b$$
$$a=b$$

$\therefore\ f(x)$ is **1–1 &** $f^{-1}(x)$ exists

b) $y=\dfrac{2x+1}{x-3}$

$$x=\frac{2y+1}{y-3}$$
$$xy-3x=2y+1$$

$$y(x-2)=3x+1$$

$$f^{-1}(x)=\frac{3x+1}{x-2}$$

c) Domain of f^{-1} = Range of f: $\mathbb{R}-\{3\}$

Range of f^{-1}= Domain of f: $\mathbb{R}-\{2\}$

Exercise

Simplify the expression $\dfrac{\left(e^{x}+e^{-x}\right)\left(e^{x}+e^{-x}\right)-\left(e^{x}-e^{-x}\right)\left(e^{x}-e^{-x}\right)}{\left(e^{x}+e^{-x}\right)^{2}}$

Solution

$$\frac{\left(e^{x}+e^{-x}\right)\left(e^{x}+e^{-x}\right)-\left(e^{x}-e^{-x}\right)\left(e^{x}-e^{-x}\right)}{\left(e^{x}+e^{-x}\right)^{2}}=\frac{\left(e^{x}+e^{-x}\right)^{2}-\left(e^{x}-e^{-x}\right)^{2}}{\left(e^{x}+e^{-x}\right)^{2}}$$

$$=\frac{\left[\left(e^{x}+e^{-x}\right)-\left(e^{x}-e^{-x}\right)\right]\left[\left(e^{x}+e^{-x}\right)+\left(e^{x}-e^{-x}\right)\right]}{\left(e^{x}+e^{-x}\right)^{2}}$$

$$=\frac{\left(e^{x}+e^{-x}-e^{x}+e^{-x}\right)\left(e^{x}+e^{-x}+e^{x}-e^{-x}\right)}{\left(e^{x}+e^{-x}\right)^{2}}$$

$$=\frac{\left(2e^{-x}\right)\left(2e^{x}\right)}{\left(e^{x}+e^{-x}\right)^{2}}$$

$e^{-x}e^{x}=e^{0}=1$

$$=\frac{4}{\left(e^{x}+e^{-x}\right)^{2}}$$

Exercise

Simplify the expression $\dfrac{\left(e^{x}-e^{-x}\right)^{2}-\left(e^{x}+e^{-x}\right)^{2}}{\left(e^{x}+e^{-x}\right)^{2}}$

Solution

$$\frac{\left(e^{x}-e^{-x}\right)^{2}-\left(e^{x}+e^{-x}\right)^{2}}{\left(e^{x}+e^{-x}\right)^{2}}=\frac{\left[\left(e^{x}-e^{-x}\right)-\left(e^{x}+e^{-x}\right)\right]\left[\left(e^{x}-e^{-x}\right)+\left(e^{x}+e^{-x}\right)\right]}{\left(e^{x}+e^{-x}\right)^{2}}$$

$$= \frac{\left(e^{x} - e^{-x} - e^{x} - e^{-x}\right)\left(e^{x} - e^{-x} + e^{x} + e^{-x}\right)}{\left(e^{x} + e^{-x}\right)^{2}}$$

$$= \frac{\left(-2e^{-x}\right)\left(2e^{x}\right)}{\left(e^{x} + e^{-x}\right)^{2}}$$

$$\boxed{= \frac{-4}{\left(e^{x} + e^{-x}\right)^{2}}}$$

Exercise

Write the equation in its equivalent logarithmic form $2^{6} = 64$

Solution

$$\boxed{6 = \log_{2} 64}$$

Exercise

Write the equation in its equivalent logarithmic form $5^{4} = 625$

Solution

$$\boxed{4 = \log_{5} 625}$$

Exercise

Write the equation in its equivalent logarithmic form $5^{-3} = \frac{1}{125}$

Solution

$$\boxed{-3 = \log_{5} \frac{1}{125}}$$

Exercise

Write the equation in its equivalent logarithmic form $\sqrt[3]{64} = 4$

Solution

$$64^{1/3} = 4$$

$$\boxed{\log_{64} = \frac{1}{3}}$$

Exercise

Write the equation in its equivalent logarithmic form $b^3 = 343$

Solution

$$\boxed{\log_b 343 = 3}$$

Exercise

Write the equation in its equivalent logarithmic form $8^y = 300$

Solution

$$\boxed{\log_8 300 = y}$$

Exercise

Write the equation in its equivalent logarithmic form: $\sqrt[n]{x} = y$

Solution

$$(x)^{1/n} = y$$

$$\boxed{\log_x (y) = \frac{1}{n}}$$

Exercise

Write the equation in its equivalent logarithmic form: $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$

Solution

$$\boxed{\log_{\frac{2}{3}} \left(\frac{27}{8}\right) = -3}$$

Exercise

Write the equation in its equivalent logarithmic form: $\left(\frac{1}{2}\right)^{-5} = 32$

Solution

$$\boxed{\log_{\frac{1}{2}} (32) = -5}$$

Exercise

Write the equation in its equivalent logarithmic form: $e^{x-2} = 2y$

Solution

$$\boxed{x-2=\ln|2y|}$$

Exercise

Write the equation in its equivalent logarithmic form: $e=3x$

Solution

$$\boxed{1=\ln|3x|}$$

Exercise

Write the equation in its equivalent logarithmic form: $\sqrt[3]{e^{2x}}=y$

Solution

$$e^{2x/3}=y$$

$$\boxed{\frac{2x}{3}=\ln|y|}$$

Exercise

Write the equation in its equivalent exponential form $\log_5 125=y$

Solution

$$\boxed{5^y=125}$$

Exercise

Write the equation in its equivalent exponential form $\log_4 16=x$

Solution

$$\boxed{16=4^x}$$

Exercise

Write the equation in its equivalent exponential form $\log_5 \frac{1}{5}=x$

Solution

$$\boxed{\frac{1}{5}=5^x}$$

Exercise

Write the equation in its equivalent exponential form $\log_2 \frac{1}{8}=x$

Solution

$$\boxed{\frac{1}{8} = 2^{x}}$$

Exercise

Write the equation in its equivalent exponential form $\log_6 \sqrt{6} = x$

Solution

$$\boxed{\sqrt{6} = 6^{x}}$$

Exercise

Write the equation in its equivalent exponential form $\log_3 \frac{1}{\sqrt{3}} = x$

Solution

$$\boxed{3^{-1/2} = 3^{x}}$$

Exercise

Write the equation in its equivalent exponential form: $6 = \log_2 64$

Solution

$$6 = \log_2 64 \iff \boxed{2^{6} = 64}$$

Exercise

Write the equation in its equivalent exponential form: $2 = \log_9 x$

Solution

$$2 = \log_9 x \iff \boxed{x = 2^{9}}$$

Exercise

Write the equation in its equivalent exponential form: $\log_{\sqrt{3}} 81 = 8$

Solution

$$\log_{\sqrt{3}} 81 = 8 \iff \boxed{81 = \left(\sqrt{3}\right)^{8}}$$

Exercise

Write the equation in its equivalent exponential form: $\log_4 \frac{1}{64} = -3$

Solution

$$\log_4 \frac{1}{64} = -3 \quad \Leftrightarrow \quad \boxed{\frac{1}{64} = x^{-3}}$$

Exercise

Write the equation in its equivalent exponential form: $\log_4 26 = y$

Solution

$$\log_4 26 = y \quad \Leftrightarrow \quad \boxed{26 = 4^y}$$

Exercise

Write the equation in its equivalent exponential form: $\ln M = c$

Solution

$$\ln M = c \quad \Leftrightarrow \quad \boxed{M = e^c}$$

Exercise

Evaluate the expression without using a calculator: $\log_4 16$

Solution

$$\log_4 16 = \log_4 4^2 \qquad\qquad \log_b b^x = x$$

$$\boxed{= 2}$$

Exercise

Evaluate the expression without using a calculator: $\log_2 \frac{1}{8}$

Solution

$$\log_2 \frac{1}{8} = \log_2 \frac{1}{2^3}$$

$$= \log_2 2^{-3} \qquad\qquad \log_b b^x = x$$

$$\boxed{= -3}$$

Exercise

Evaluate the expression without using a calculator: $\log_6 \sqrt{6}$

Solution

$$\log_6 \sqrt{6} = \log_6 6^{1/2}$$

$$\boxed{= \frac{1}{2}}$$

Exercise

Evaluate the expression without using a calculator: $\log_3 \frac{1}{\sqrt{3}}$

Solution

$$\log_3 \frac{1}{\sqrt{3}} = \log_3 \frac{1}{3^{1/2}}$$

$$= \log_3 3^{-1/2} \qquad \log_b b^x = x$$

$$\boxed{= -\frac{1}{2}}$$

Exercise

Evaluate the expression without using a calculator: $\log_3 \sqrt[7]{3}$

Solution

$$\log_3 3^{1/7} = x \qquad \textit{Converts to exponential}$$

$$3^{1/7} = 3^x$$

$$x = \frac{1}{7}$$

$$\boxed{\log_3 \sqrt[7]{3} = \frac{1}{7}}$$

Exercise

Evaluate the expression without using a calculator: $\log_3 \sqrt{9}$

Solution

$$\log_3 \sqrt{9} = \log_3 3 \qquad \log_b b^x = x$$

$$\boxed{= 1}$$

Exercise

Evaluate the expression without using a calculator: $\log_{\frac{1}{2}} \sqrt{\frac{1}{2}}$

Solution

$$\log_{\frac{1}{2}} \sqrt{\frac{1}{2}} = \log_{\frac{1}{2}} \left(\frac{1}{2}\right)^{\frac{1}{2}} \qquad \log_b b^x = x$$

$$= \frac{1}{2}$$

Exercise

Simplify $\log_5 1$

Solution

$$\log_5 1 = 0$$

Exercise

Simplify $\log_7 7^2$

Solution

$$\log_7 7^2 = 2$$

Exercise

Simplify $3^{\log_3 8}$

Solution

$$3^{\log_3 8} = 8$$

Exercise

Simplify $10^{\log 3}$

Solution

$$\boxed{10^{\log 3} = 3}$$

Exercise

Simplify $e^{2+\ln 3}$

Solution

$$e^{2+\ln 3} = e^{2}e^{\ln 3}$$

$$\boxed{= 3e^{2}}$$

Exercise

Simplify $\ln e^{-3}$

Solution

$$\boxed{\ln e^{-3} = -3}$$

Exercise

Simplify $\ln e^{x-5}$

Solution

$$\boxed{\ln e^{x-5} = x-5}$$

Exercise

Simplify $\log_{b} b^{n}$

Solution

$$\boxed{\log_{b} b^{n} = n}$$

Exercise

Simplify $\ln e^{x^2+3x}$

Solution

$$\ln e^{x^2+3x} = x^2 + 3x$$

Exercise

Find the domain of $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Solution

$e^x + e^{-x} > 0$

Domain: $\mathbb{R}$

Exercise

Find the domain of $f(x) = \dfrac{e^{|x|}}{1+e^x}$

Solution

$1 + e^x > 0$

Domain: $\mathbb{R}$

Exercise

Find the domain of $f(x) = \sqrt{1-e^x}$

Solution

$1 - e^x \ge 0$

$e^x \le 1$

$x \le \ln 1$

Domain: $x \le 0$

Exercise

Find the domain of $f(x) = \sqrt{e^x - e^{-x}}$

Solution

$e^x - e^{-x} \ge 0$

$e^x \ge e^{-x}$

$e^{2x} \ge 1$

$2x \ge \ln 1$

Domain: $\boxed{x \ge 0}$

Exercise

Find the domain of $f(x) = \log_5(x+4)$

Solution

Domain: $\boxed{x > -4}$

Exercise

Find the domain of $f(x) = \log_5(x+6)$

Solution

Domain: $\boxed{x > -6}$

Exercise

Find the domain of $f(x) = \log(2-x)$

Solution

Domain: $\boxed{x < 2}$

Exercise

Find the domain of $f(x) = \log(7-x)$

Solution

Domain: $\boxed{x < 7}$

Exercise

Find the domain of $f(x) = \ln(x-2)^2$

Solution

Domain: $\boxed{\mathbb{R} - \{2\}}$ $\boxed{(-\infty,\ 2) \cup (2,\ \infty)}$

Exercise

Find the domain of $f(x) = \ln(x-7)^2$

Solution

Domain: $\boxed{\mathbb{R} - \{7\}}$

$\boxed{(-\infty,\ 7) \cup (7,\ \infty)}$

Exercise

Find the domain of $f(x) = \log\left(x^2 - 4x - 12\right)$

Solution

$$x^2 - 4x - 12 > 0$$

$$x = \frac{4 \pm \sqrt{16+48}}{2}$$

$$= \begin{cases} \frac{4-8}{2} = -2 \\ \frac{4+8}{2} = 6 \end{cases}$$

Domain: $\boxed{x < -2 \quad x > 6}$

$\boxed{(-\infty,\ -2) \cup (6,\ \infty)}$

Exercise

Find the domain of $f(x) = \log\left(\frac{x-2}{x+5}\right)$

Solution

$$\begin{cases} x \ne 2 \\ x \ne -5 \end{cases}$$

<table>
<tr><td colspan="3">−5 0 2</td></tr>
<tr><td>+</td><td>–</td><td>+</td></tr>
</table>

Domain: $\boxed{x < -5 \quad x > 2}$

$\boxed{(-\infty,\ -5) \cup (2,\ \infty)}$

Exercise

Find the domain of $f(x) = \log\left(\frac{3-x}{x-2}\right)$

Solution

$$\begin{cases} x \neq 3 \\ x \neq 2 \end{cases}$$

0	2	3
$-$	$+$	$-$

Domain: $\underline{2 < x < 3}\,\big|$

$\underline{(2,\ 3)}\,\big|$

Exercise

Find the domain of $f(x) = \ln\left(x^2 - 9\right)$

Solution

$$x^2 - 9 > 0$$

Domain: $\underline{x < -3 \quad x > 3}\,\big|$

Exercise

Find the domain of $f(x) = \ln\left(\frac{x^2}{x-4}\right)$

Solution

$$\frac{x^2}{x-4} > 0$$

$$x^2 \ \rightarrow \ \mathbb{R}$$

$$x > 4$$

Domain: $\underline{x > 4}\,\big|$

Exercise

Find the domain of $f(x) = \log_3\left(x^3 - x\right)$

Solution

0,0	1	2
$-$	$-$	$+$

$$x^3 - x > 0$$

$$\underline{x = 0,\ 0,\ 1}\,\big|$$

Domain: $\underline{x > 1}\,\big|$

Exercise

Find the domain of $f(x) = \log\sqrt{2x-5}$

Solution

$2x - 5 > 0$

Domain: $\boxed{x > \frac{5}{2}}$

Exercise

Find the domain of $f(x) = 3\ln(5x - 6)$

Solution

$5x - 6 > 0$

Domain: $\boxed{x > \frac{6}{5}}$

Exercise

Find the domain of $f(x) = \log\left(\frac{x}{x-2}\right)$

Solution

$\frac{x}{x-2} > 0$

$\boxed{x = 0,\ 2}$

Domain: $\boxed{x < 0 \quad x > 2}$

Exercise

Find the domain of $f(x) = \log\left(4 - x^2\right)$

Solution

$4 - x^2 > 0$

$4 - x^2 = 0 \quad \rightarrow \quad x = \pm 2$

Domain: $\boxed{-2 < x < 2}$

Exercise

Find the domain of $f(x) = \ln\left(x^2 + 4\right)$

Solution

$x^2 + 4$ always positive.

Domain: $\boxed{\mathbb{R}}$

Exercise

Find the domain of $f(x) = \ln|4x-8|$

Solution

$$4x - 8 = 0 \quad \to \quad x = 2$$

Domain: $\boxed{\mathbb{R} - \{2\}}$

Exercise

Find the domain of $f(x) = \ln|5-x|$

Solution

$$5 - x = 0 \quad \to \quad x = 5$$

Domain: $\boxed{\mathbb{R} - \{5\}}$

Exercise

Find the domain of $f(x) = \ln(x-4)^2$

Solution

$$x - 4 = 0 \quad \to \quad x = 4$$

Domain: $\boxed{\mathbb{R} - \{4\}}$

Exercise

Find the domain of $f(x) = \ln\left(x^2 - 4\right)$

Solution

$$x^2 - 4 > 0$$

$$x^2 - 4 = 0 \quad \to \quad x = \pm 2$$

Domain: $\boxed{x < -2 \quad x > 2}$

Exercise

Find the domain of $f(x) = \ln\left(x^2 - 4x + 3\right)$

Solution

$$x^2 - 4x + 3 = 0 \quad \to \quad \boxed{x = 1,\ 3}$$

$$x^2-4x+3>0$$

Domain: $\boxed{x<1 \quad x>3}$

Exercise

Find the domain of $\quad f(x)=\ln\left(2x^2-5x+3\right)$

Solution

$$2x^2-5x+3=0 \quad \to \quad \boxed{x=1,\ \frac{3}{2}}$$

$$2x^2-5x+3>0$$

Domain: $\boxed{x<1 \quad x>\frac{3}{2}}$

Exercise

Find the domain of $\quad f(x)=\log\left(x^2+4x+3\right)$

Solution

$$x^2+4x+3=0 \quad \to \quad \boxed{x=-1,\ -3}$$

$$x^2+4x+3>0$$

Domain: $\boxed{x<-3 \quad x>-1}$

Exercise

Find the domain of $\quad f(x)=\ln\left(x^4-x^2\right)$

Solution

$$x^4-x^2=0$$

$$x^2\left(x^2-1\right)=0$$

$$\boxed{x=0,\ 0,\ \pm 1}$$

$$x^4-x^2>0$$

Domain: $\boxed{x<-1 \quad x>1}$

−1	0,0	1	2
+	–	–	+

Exercise

Sketch the graph: $f(x) = 2^x + 3$

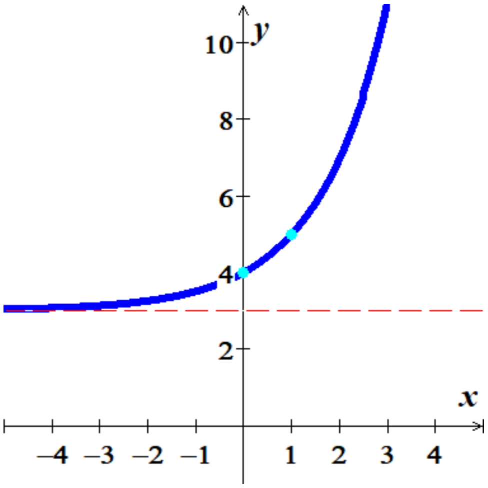
Solution

Asymptote: $y = 3$

Domain: $(-\infty,\ \infty)$

Range: $(3,\ \infty)$

x	$f(x)$
–1	3.5
0	4
1	5
2	7



Exercise

Sketch the graph: $f(x) = 2^{3-x}$

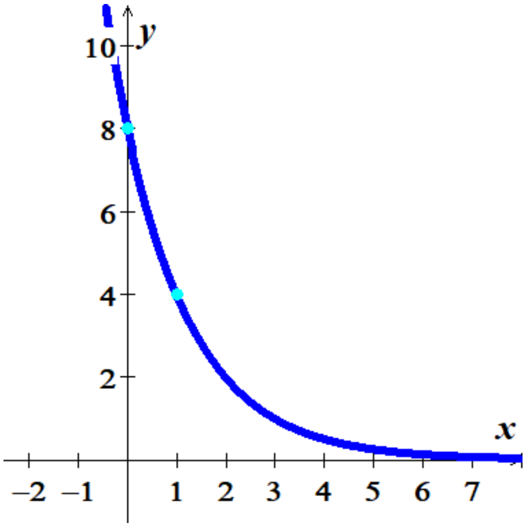
Solution

Asymptote: $y = 0$

Domain: $(-\infty,\ \infty)$

Range: $(0,\ \infty)$

x	$f(x)$
1	4
2	2
0	8



Exercise

Sketch the graph: $f(x) = \left(\frac{2}{5}\right)^{-x}$

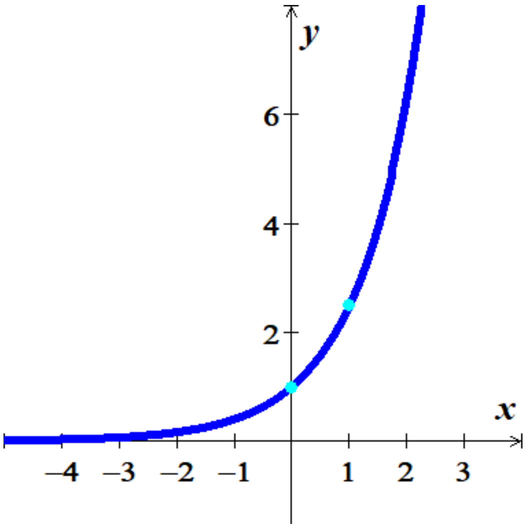
Solution

Asymptote: $y = 0$

Domain: $(-\infty,\ \infty)$

Range: $(0,\ \infty)$

x	$f(x)$
–1	0.4
0	1
1	2.5



Exercise

Sketch the graph: $f(x) = -\left(\frac{1}{2}\right)^x + 4$

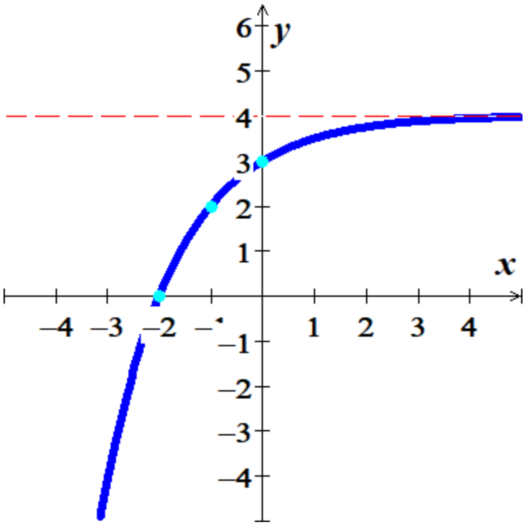
Solution

Asymptote: $y = 4$

Domain: $(-\infty,\ \infty)$

Range: $(-\infty,\ 4)$

x	$f(x)$
–2	0
–1	2
0	3



Exercise

Sketch the graph of $f(x) = 4^x$

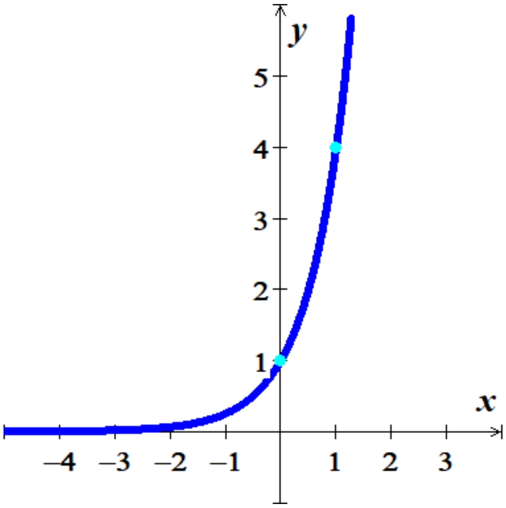
Solution

Asymptote: $y = 0$

Domain: $(-\infty,\ \infty)$

Range: $(0,\ \infty)$

x	$f(x)$
0	1
1	4



Exercise

Sketch the graph of $f(x) = 2 - 4^x$

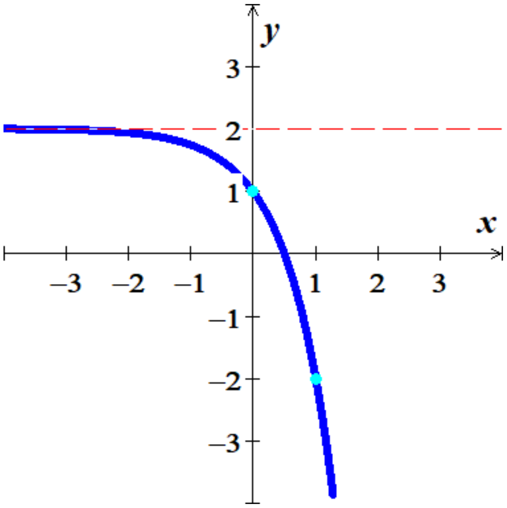
Solution

Asymptote: $y = 2$

Domain: $(-\infty,\ \infty)$

Range: $(-\infty,\ 2)$

x	$f(x)$
0	1
1	–2



Exercise

Sketch the graph of $f(x) = -3 + 4^{x-1}$

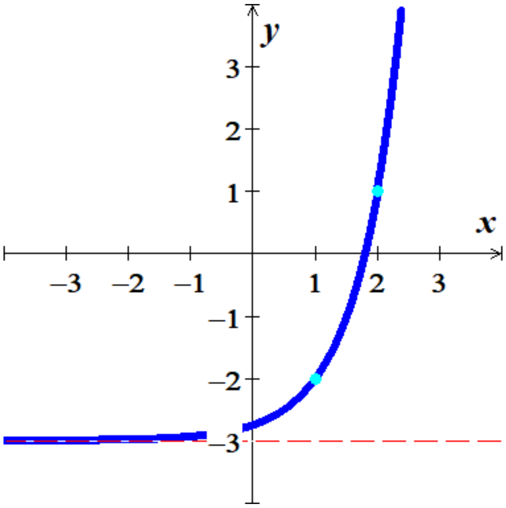
Solution

Asymptote: $y = -3$

Domain: $(-\infty,\ \infty)$

Range: $(-3,\ \infty)$

x	$f(x)$
1	-2
2	1



Exercise

Sketch the graph of $f(x) = 1 + \left(\frac{1}{4}\right)^{x+1}$

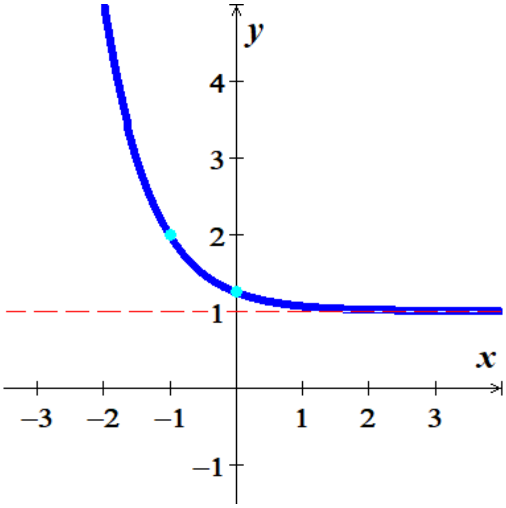
Solution

Asymptote: $y = 1$

Domain: $(-\infty,\ \infty)$

Range: $(1,\ \infty)$

x	$f(x)$
-1	2
0	$\frac{5}{4}$



Exercise

Sketch the graph of $f(x) = e^{x-2}$

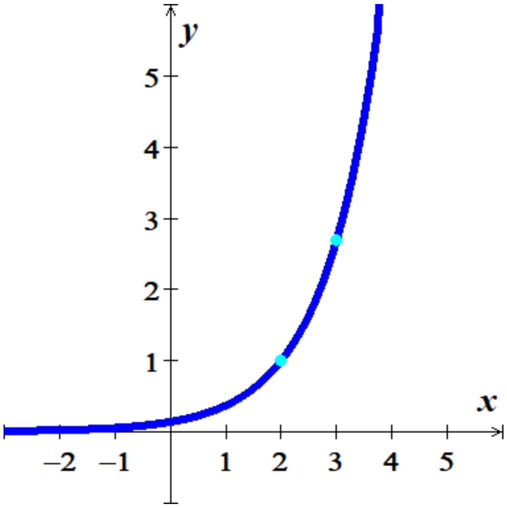
Solution

Asymptote: $y = 0$

Domain: $(-\infty,\ \infty)$

Range: $(0,\ \infty)$

x	$f(x)$
2	1
3	2.7



Exercise

Sketch the graph of $f(x) = 3 - e^{x-2}$

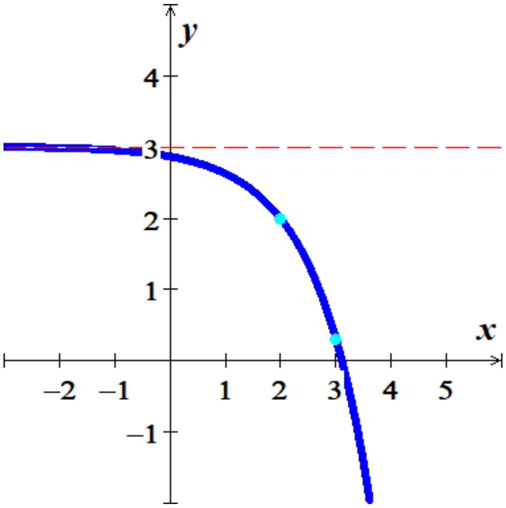
Solution

Asymptote: $y = 3$

Domain: $(-\infty,\ \infty)$

Range: $(-\infty,\ 3)$

x	$f(x)$
2	2
3	.3



Exercise

Sketch the graph of $f(x) = e^{x+4}$

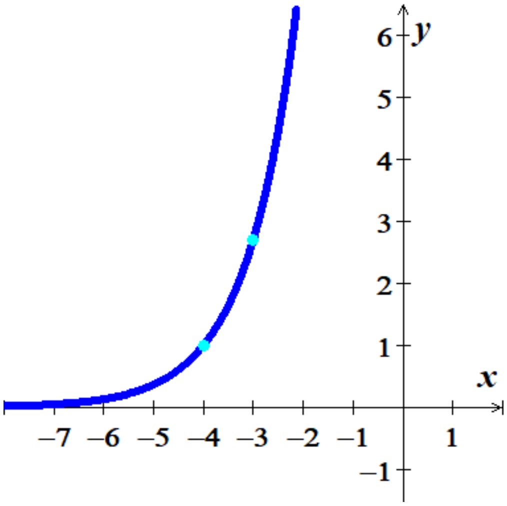
Solution

Asymptote: $y = 0$

Domain: $(-\infty,\ \infty)$

Range: $(0,\ \infty)$

x	$f(x)$
–4	1
–3	2.7



Exercise

Sketch the graph of $f\left(x\right) = 2 + e^{x-1}$

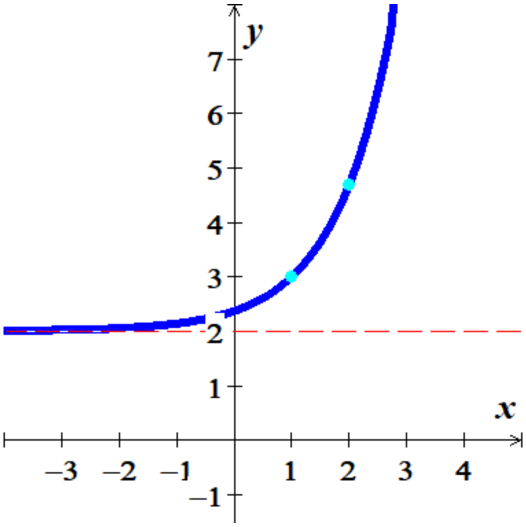
Solution

Asymptote: $y = 2$

Domain: $(-\infty,\ \infty)$

Range: $(2,\ \infty)$

x	$f(x)$
1	3
2	4.7



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given function. Then, sketch the graph $f(x) = \log_4 (x-2)$

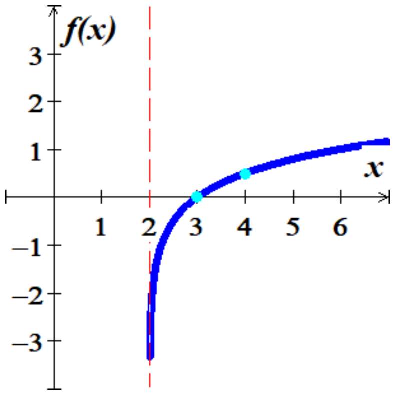
Solution

Asymptote: $x = 2$

Domain: $(2, \infty)$

Range: $(-\infty, \infty)$

x	$f(x)$
2	
3	0
4	.5



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given function. Then, sketch the graph $f(x) = \log_4 |x|$

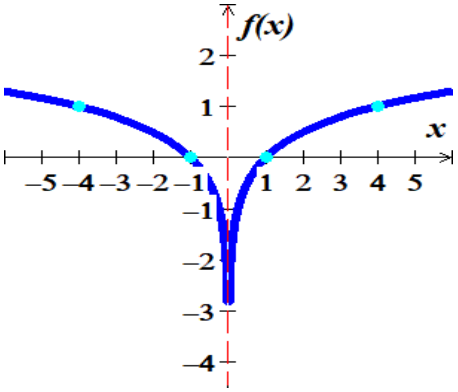
Solution

Asymptote: $x = 0$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, \infty)$

x	$f(x)$
0	
±1	0
±4	1



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given function. Then, sketch the graph $f(x) = \left(\log_4 x\right) - 2$

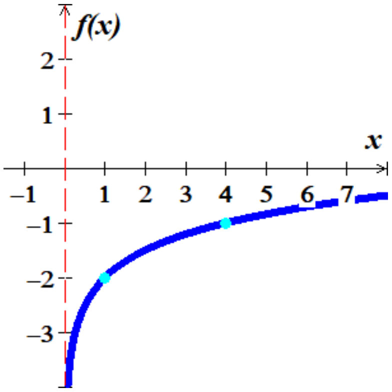
Solution

Asymptote: $x = 0$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

x	$f(x)$
0	
1	0
4	−1



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given function. Then, sketch the graph $f(x) = \log(3 - x)$

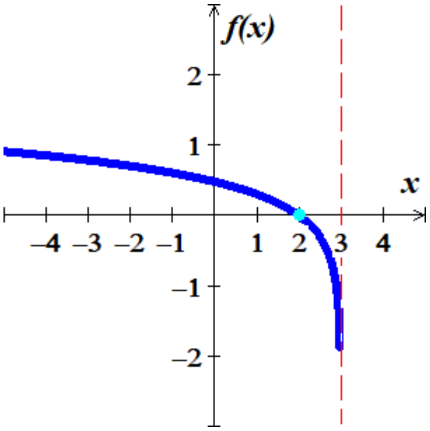
Solution

Asymptote: $x = 3$

Domain: $(-\infty,\ 3)$

Range: $(-\infty,\ \infty)$

x	$f(x)$
3	
2	0



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given function. Then, sketch the graph $f(x) = 2 - \log(x + 2)$

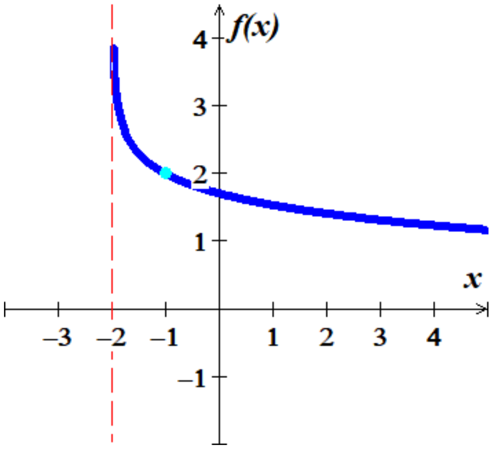
Solution

Asymptote: $x = -2$

Domain: $(-2,\ \infty)$

Range: $(-\infty,\ \infty)$

x	$f(x)$
–2	
–1	2



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given function. Then, sketch the graph $f(x) = \ln(x - 2)$

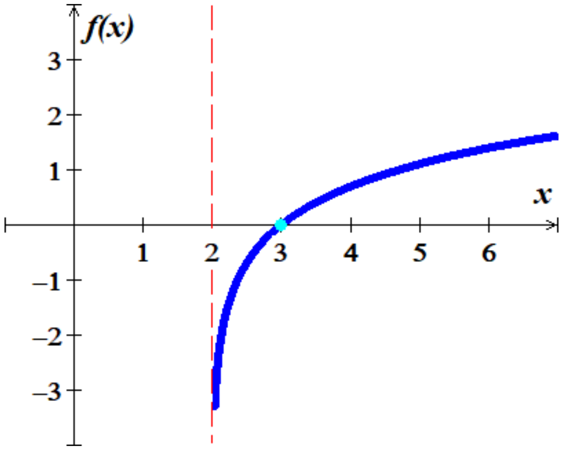
Solution

Asymptote: $x = 2$

Domain: $(2,\ \infty)$

Range: $(-\infty,\ \infty)$

x	$f(x)$
2	
3	0



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given function. Then, sketch the graph $f(x) = \ln(3-x)$

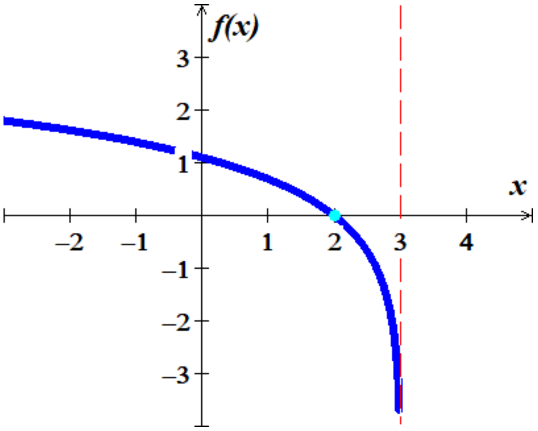
Solution

Asymptote: $x = 3$

Domain: $(-\infty,\ 3)$

Range: $(-\infty,\ \infty)$

x	$f(x)$
3	
2	0



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given function. Then, sketch the graph $f(x) = 2 + \ln(x+1)$

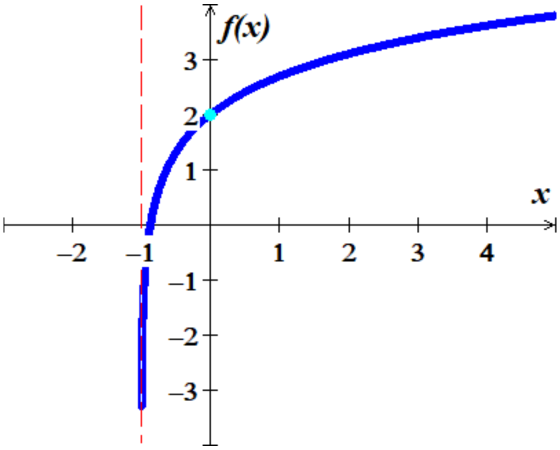
Solution

Asymptote: $x = -1$

Domain: $(-1,\ \infty)$

Range: $(-\infty,\ \infty)$

x	$f(x)$
–1	
0	2



Exercise

Find the ***asymptote***, ***domain***, and ***range*** of the given function. Then, sketch the graph $f(x) = 1 - \ln(x-2)$

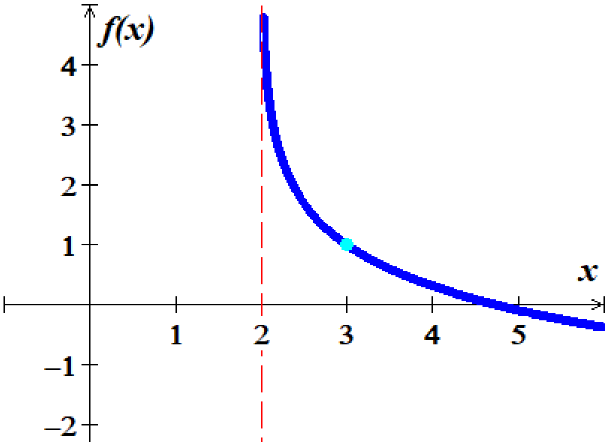
Solution

Asymptote: $x = 2$

Domain: $(2,\ \infty)$

Range: $(-\infty,\ \infty)$

x	$f(x)$
2	
3	1



Exercise

On a study by psychologists Bornstein and Bornstein, it was found that the average walking speed w, in feet per second, of a person living in a city of population P, in ***thousands***, is given by the function

$$w(P) = 0.37\ln P + 0.05$$

a) The population is 124,848. Find the average walking speed of people living in Hartford.
b) The population is 1,236,249. Find the average walking speed of people living in San Antonio.

Solution

124,848 = 124.848 ***thousand***

a) $w(124.848) = 0.37\ln(124.848) + 0.05$

$$\underline{\approx 1.8 \ \ ft/sec}$$

b) $w(1,236.249) = 0.37\ln(1,236.249) + 0.05$

$$\underline{\approx 2.7 \ \ ft/sec}$$

Exercise

The loudness of sounds is measured in a unit called a decibel. To measure with this unit, we first assign an intensity of I_0 to a very faint sound, called the threshold sound. If a particular sound has intensity I, then the decibel rating of this louder sound is

$$d = 10\log\frac{I}{I_0}$$

Find the exact decibel rating of a sound with intensity $10,000I_0$

Solution

$$d = 10\log\frac{10000I_0}{I_0}$$

$$= 10\log 10000$$

$$\underline{= 40 \ \ db}$$

Exercise

Students in an accounting class took a final exam and then took equivalent forms of the exam at monthly intervals thereafter. The average score *S(t)*, as a percent, after *t* months was found to be given by the function

$$S(t) = 78 - 15\ \log(t+1); \quad t \ge 0$$

a) What was the average score when the students initially took the test, $t = 0$?

b) What was the average score after 4 *months*? 24 *months*?

Solution

a) $S(0) = 78 - 15\ \log(1)$

$\approx 78\%$

b) After 4 *months*

$S(4) = 78 - 15\ \log(5)$

$\approx 67.5\%$

After 24 *months*

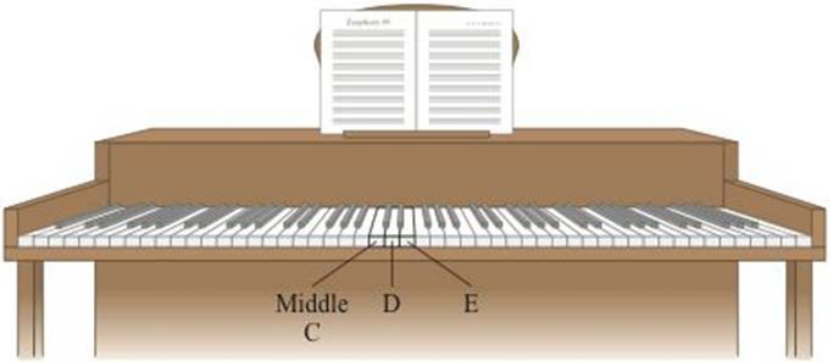
$S(24) = 78 - 15\ \log(25)$

$\approx 57\%$

Exercise

Starting on the left side of a standard 88–*key* piano, the frequency, in *vibrations* per *second*, of the *n*th note is given by

$$f(n) = (27.5)\ 2^{\frac{n-1}{12}}$$



a) Determine the frequency of middle *C*, key number 40 on an 88–*key* piano.

b) Is the difference in frequency between middle *C* (key number 40) and *D* (key number 42) the same as the difference in frequency between *D* (key number 42) and *E* (key number 44)?

Solution

a) $f(40) = (27.5)\ 2^{\frac{40-1}{12}}$

≈ 261.63

the frequency of middle *C* is ≈ 262 *vibrations* per *second*.

b) $f(42) = (27.5)\ 2^{(41/12)}$

≈ 293.66

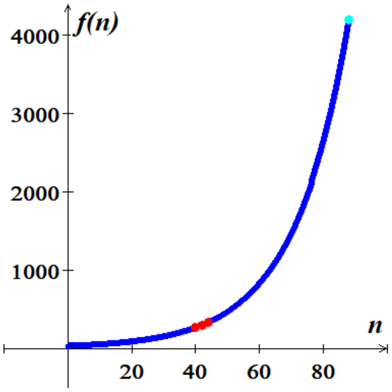
The difference between the frequency of middle *C* and *D* is: $293.66 - 261.66 \approx 32$

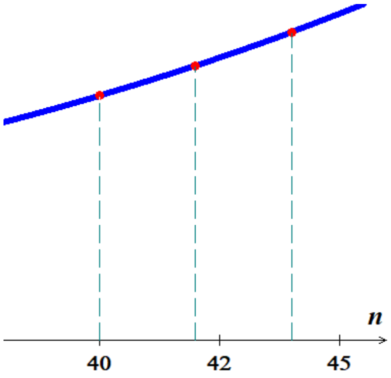
$f(44) = (27.5)\ 2^{(43/12)}$

≈ 329.63

The difference between the frequency of middle *D* and *E* is: $329.63 - 293.66 \approx 36$

∴ The differences are ***not*** the same since the function is *not* linear function.





Solution Section 1.6 – Exponential and Logarithmic Equations

Exercise

Express the following in terms of sums and differences of logarithms: $\log_3(ab)$

Solution

$$\boxed{\log_3(ab) = \log_3 a + \log_3 b}$$

Exercise

Express the following in terms of sums and differences of logarithms: $\log_7(7x)$

Solution

$$\log_7(7x) = \log_7 7 + \log_7 x$$
$$\boxed{= 1 + \log_7 x}$$

Exercise

Express the following in terms of sums and differences of logarithms: $\log \frac{x}{1000}$

Solution

$$\log \frac{x}{1000} = \log x - \log 10^3$$
$$\boxed{= \log x - 3}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_5\left(\frac{125}{y}\right)$

Solution

$$\log_5\left(\frac{125}{y}\right) = \log_5 5^3 - \log_5 y$$
$$\boxed{= 3 - \log_5 y}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b x^7$

Solution

$$\boxed{\log_b x^7 = 7\log_b x}$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln \sqrt[7]{x}$

Solution

$$\ln \sqrt[7]{x} = \ln x^{1/7}$$

$$\boxed{= \frac{1}{7}\ln x}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \dfrac{x^2 y}{z^4}$

Solution

$$\log_a \frac{x^2 y}{z^4} = \log_a x^2 y - \log_a z^4 \qquad \textit{Quotient Rule}$$

$$= \log_a x^2 + \log_a y - \log_a z^4 \qquad \textit{Product Rule}$$

$$\boxed{= 2\log_a x + \log_a y - 4\log_a z} \qquad \textit{Power Rule}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \dfrac{x^2 y}{b^3}$

Solution

$$\log_b\left(\frac{x^2 y}{b^3}\right) = \log_b x^2 y - \log_b b^3$$

$$= \log_b x^2 + \log_b y - \log_b b^3$$

$$= 2\log_b x + \log_b y - 3\log_b b$$

$$\boxed{= 2\log_b x + \log_b y - 3}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b\left(\frac{x^3 y}{z^2}\right)$

Solution

$$\begin{aligned}\log_b\left(\frac{x^3 y}{z^2}\right) &= \log_b\left(x^3 y\right) - \log_b z^2 \\ &= \log_b x^3 + \log_b y - \log_b z^2 \\ &\boxed{= 3\log_b x + \log_b y - 2\log_b z}\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b\left(\frac{\sqrt[3]{x}\, y^4}{z^5}\right)$

Solution

$$\begin{aligned}\log_b\left(\frac{\sqrt[3]{x}\, y^4}{z^5}\right) &= \log_b\left(\sqrt[3]{x}\, y^4\right) - \log_b\left(z^5\right) \\ &\boxed{= \log_b\left(x^{1/3}\right) + \log_b\left(y^4\right) - \log_b\left(z^5\right)}\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log\left(\frac{100x^3\ \sqrt[3]{5-x}}{3(x+7)^2}\right)$

Solution

$$\begin{aligned}\log\left(\frac{100x^3\ \sqrt[3]{5-x}}{3(x+7)^2}\right) &= \log\left(100x^3\ \sqrt[3]{5-x}\right) - \log\left(3(x+7)^2\right) \\ &= \log 10^2 + \log x^3 + \log\left(5-x\right)^{1/3} - \left[\log 3 + \log\left((x+7)^2\right)\right] \\ &= 2\log 10 + 3\log x + \tfrac{1}{3}\log\left(5-x\right) - \log 3 - 2\log(x+7) \\ &\boxed{= 2 + 3\log x + \tfrac{1}{3}\log\left(5-x\right) - \log 3 - 2\log(x+7)}\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \sqrt[4]{\frac{m^8\ n^{12}}{a^3\ b^5}}$

Solution

$$\begin{aligned}
\log_a \sqrt[4]{\frac{m^8\,n^{12}}{a^3\,b^5}} &= \log_a\left(\frac{m^8n^{12}}{a^3b^5}\right)^{1/4} && \textit{Power Rule}\\
&= \frac{1}{4}\log_a\left(\frac{m^8n^{12}}{a^3b^5}\right) && \textit{Quotient Rule}\\
&= \frac{1}{4}\left[\log_a m^8n^{12} - \log_a a^3b^5\right] && \textit{Product Rule}\\
&= \frac{1}{4}\left[\log_a m^8 + \log_a n^{12} - \left(\log_a a^3 + \log_a b^5\right)\right] && \textit{Power Rule}\\
&= \frac{1}{4}\left[8\log_a m + 12\log_a n - 3 - 5\log_a b\right]\\
&\underline{= 2\log_a m + 3\log_a n - \frac{3}{4} - \frac{5}{4}\log_a b\,\Big|}
\end{aligned}$$

Exercise

Use the properties of logarithms to rewrite: $\log_p \sqrt[3]{\frac{m^5n^4}{t^2}}$

Solution

$$\begin{aligned}
\log_p \sqrt[3]{\frac{m^5n^4}{t^2}} &= \log_p\left(\frac{m^5n^4}{t^2}\right)^{1/3} && \textit{Power Rule}\\
&= \frac{1}{3}\log_p\left(\frac{m^5n^4}{t^2}\right) && \textit{Quotient Rule}\\
&= \frac{1}{3}\left(\log_p m^5n^4 - \log_p t^2\right) && \textit{Product Rule}\\
&= \frac{1}{3}\left(\log_p m^5 + \log_p n^4 - \log_p t^2\right) && \textit{Power Rule}\\
&= \frac{1}{3}\left(5\log_p m + 4\log_p n - 2\log_p t\right)\\
&\underline{= \frac{5}{3}\log_p m + \frac{4}{3}\log_p n - \frac{2}{3}\log_p t\,\Big|}
\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \sqrt[n]{\frac{x^3y^5}{z^m}}$

Solution

$$\log_b \sqrt[n]{\frac{x^3y^5}{z^m}} = \log_b\left(\frac{x^3y^5}{z^m}\right)^{1/n}$$

$$=\frac{1}{n}\log_b\left(\frac{x^3y^5}{z^m}\right) \qquad \textit{Power Rule}$$

$$=\frac{1}{n}\left(\log_b x^3y^5-\log_b z^m\right) \qquad \textit{Quotient Rule}$$

$$=\frac{1}{n}\left(\log_b x^3+\log_b y^5-\log_b z^m\right) \qquad \textit{Product Rule}$$

$$=\frac{1}{n}\left(3\log_b x+5\log_b y-m\log_b z\right) \qquad \textit{Power Rule}$$

$$\boxed{=\frac{3}{n}\log_b x+\frac{5}{n}\log_b y-\frac{m}{n}\log_b z}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a \sqrt[3]{\frac{a^2\,b}{c^5}}$

Solution

$$\log_a \sqrt[3]{\frac{a^2\,b}{c^5}}=\log_a\left(\frac{a^2\,b}{c^5}\right)^{1/3} \qquad \textit{Convert the radical to power}$$

$$=\frac{1}{3}\log_a\left(\frac{a^2\,b}{c^5}\right) \qquad \textit{Power Rule}$$

$$=\frac{1}{3}\left[\log_a a^2\,b-\log_a c^5\right] \qquad \textit{Quotient Rule}$$

$$=\frac{1}{3}\left[\log_a a^2\;+\log_a b-\log_a c^5\right] \qquad \textit{Product Rule}$$

$$=\frac{1}{3}\left[2\log_a a\;+\log_a b-5\log_a c\right] \qquad \textit{Power Rule}$$

$$=\frac{2}{3}\log_a a\;+\frac{1}{3}\log_a b-\frac{5}{3}\log_a c$$

$$\boxed{=\frac{2}{3}+\frac{1}{3}\log_a b-\frac{5}{3}\log_a c}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b\left(x^4\,\sqrt[3]{y}\right)$

Solution

$$\log_b\left(x^4\,\sqrt[3]{y}\right)=\log_b\left(x^4\right)+\log_b\left(\sqrt[3]{y}\right)$$

$$=\log_b\left(x^4\right)+\log_b\left(y^{1/3}\right)$$

$$\boxed{=4\log_b x+\frac{1}{3}\log_b y}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_5\left(\frac{\sqrt{x}}{25y^3}\right)$

Solution

$$\log_5\left(\frac{\sqrt{x}}{25y^3}\right)=\log_5\left(x^{1/2}\right)-\log_5\left(25y^3\right)$$

$$=\log_5\left(x^{1/2}\right)-\left[\log_5\left(5^2\right)+\log_5\left(y^3\right)\right]$$

$$=\log_5\left(x^{1/2}\right)-\log_5\left(5^2\right)-\log_5\left(y^3\right)$$

$$=\frac{1}{2}\log_5(x)-2\log_5(5)-3\log_5(y)$$

$$\boxed{=\frac{1}{2}\log_5(x)-2-3\log_5(y)}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a\frac{x^3w}{y^2z^4}$

Solution

$$\log_a\frac{x^3w}{y^2z^4}=\log_a x^3w-\log_a y^2z^4 \qquad \textit{Quotient rule}$$

$$=\log_a x^3+\log_a w-\left(\log_a y^2+\log_a z^4\right) \qquad \textit{Product rule}$$

$$=\log_a x^3+\log_a w-\log_a y^2-\log_a z^4 \qquad \textit{Distribute minus}$$

$$\boxed{=3\log_a x+\log_a w-2\log_a y-4\log_a z} \qquad \textit{Power rule}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_a\frac{\sqrt{y}}{x^4\sqrt[3]{z}}$

Solution

$$\log_a\frac{\sqrt{y}}{x^4\sqrt[3]{z}}=\log_a y^{1/2}-\log_a x^4z^{1/3} \qquad \textit{Quotient rule}$$

$$=\log_a y^{1/2}-\left(\log_a x^4+\log_a z^{1/3}\right) \qquad \textit{Product rule}$$

$$= \log_a y^{1/2} - \log_a x^4 - \log_a z^{1/3}$$ ***Distribute minus***

$$\underline{= \frac{1}{2}\log_a y - 4\log_a x - \frac{1}{3}\log_a z}$$ ***Power rule***

Exercise

Express the following in terms of sums and differences of logarithms $\ln \sqrt[4]{\dfrac{x^7}{y^5 z}}$

Solution

$$\ln \sqrt[4]{\frac{x^7}{y^5 z}} = \ln\left(\frac{x^7}{y^5 z}\right)^{1/4}$$

$$= \frac{1}{4}\ln\left(\frac{x^7}{y^5 z}\right)$$ ***Power rule***

$$= \frac{1}{4}\left(\ln x^7 - \ln y^5 z\right)$$ ***Quotient rule***

$$= \frac{1}{4}\left(\ln x^7 - \left(\ln y^5 + \ln z\right)\right)$$ ***Product rule***

$$= \frac{1}{4}\left(\ln x^7 - \ln y^5 - \ln z\right)$$

$$= \frac{1}{4}\left(7\ln x - 5\ln y - \ln z\right)$$ ***Power rule***

$$\underline{= \frac{7}{4}\ln x - \frac{5}{4}\ln y - \ln z}$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln x \sqrt[3]{\dfrac{y^4}{z^5}}$

Solution

$$\ln x \sqrt[3]{\frac{y^4}{z^5}} = \ln x + \ln\left(\frac{y^4}{z^5}\right)^{1/3}$$ ***Product rule***

$$= \ln x + \ln\left(\frac{y^{4/3}}{z^{5/3}}\right)$$

$$= \ln x + \ln y^{4/3} - \ln z^{5/3}$$ ***Quotient rule***

$$\underline{= \ln x + \frac{4}{3}\ln y - \frac{5}{3}\ln z}$$ ***Power rule***

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \sqrt[5]{\frac{m^4n^5}{x^2ab^{10}}}$

Solution

$$\begin{aligned}
\log_b \sqrt[5]{\frac{m^4n^5}{x^2ab^{10}}} &= \log_b \left(\frac{m^4n^5}{x^2ab^{10}}\right)^{1/5} \\
&= \frac{1}{5}\log_b \left(\frac{m^4n^5}{x^2ab^{10}}\right) \\
&= \frac{1}{5}\left(\log_b \left(m^4n^5\right) - \log_b \left(x^2ab^{10}\right)\right) \\
&= \frac{1}{5}\left(\left(\log_b \left(m^4\right) + \log_b \left(n^5\right)\right) - \left(\log_b \left(x^2\right) + \log_b (a) + \log_b \left(b^{10}\right)\right)\right) \\
&= \frac{1}{5}\left(4\log_b m + 5\log_b n - 2\log_b x - \log_b a - 10\right) \\
&= \boxed{\frac{4}{5}\log_b m + \log_b n - \frac{2}{5}\log_b x - \frac{1}{5}\log_b (a) - 2}
\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\log_b \frac{a^5b^{10}}{c^2\sqrt[4]{d^3}}$

Solution

$$\begin{aligned}
\log_b \frac{a^5b^{10}}{c^2\sqrt[4]{d^3}} &= \log_b \left(a^5b^{10}\right) - \log_b \left(c^2\, d^{3/4}\right) \\
&= \log_b \left(a^5\right) + \log_b \left(b^{10}\right) - \left(\log_b \left(c^2\right) + \log_b \left(d^{3/4}\right)\right) \\
&= \boxed{5\log_b a + 10 - 2\log_b c - \frac{3}{4}\log_b d}
\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln\left(x^2\sqrt{x^2+1}\right)$

Solution

$$\begin{aligned}
\ln\left(x^2\sqrt{x^2+1}\right) &= \ln x^2 + \ln\left(x^2+1\right)^{1/2} \\
&= \boxed{2\ln x + \frac{1}{2}\ln\left(x^2+1\right)}
\end{aligned}$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln \frac{x^2}{x^2+1}$

Solution

$$\ln \frac{x^2}{x^2+1} = \ln x^2 - \ln\left(x^2+1\right)$$

$$\boxed{= 2\ln x - \ln\left(x^2+1\right)}$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln\left(\frac{x^2(x+1)^3}{(x+3)^{1/2}}\right)$

Solution

$$\ln\left(\frac{x^2(x+1)^3}{(x+3)^{1/2}}\right) = \ln\left(x^2(x+1)^3\right) - \ln(x+3)^{1/2}$$

$$= \ln x^2 + \ln(x+1)^3 - \frac{1}{2}\ln(x+3)$$

$$\boxed{= 2\ln x + 3\ln(x+1) - \frac{1}{2}\ln(x+3)}$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$

Solution

$$\ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}} = \ln\left(\frac{(x+1)^5}{(x+2)^{20}}\right)^{1/2}$$

$$= \frac{1}{2}\ln\left(\frac{(x+1)^5}{(x+2)^{20}}\right)$$

$$= \frac{1}{2}\left(\ln(x+1)^5 - \ln(x+2)^{20}\right)$$

$$= \frac{1}{2}\left(5\ln(x+1) - 20\ln(x+2)\right)$$

$$\boxed{= \frac{5}{2}\ln(x+1) - 10\ln(x+2)}$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln\frac{\left(x^2+1\right)^5}{\sqrt{1-x}}$

Solution

$$\ln\frac{\left(x^2+1\right)^5}{\sqrt{1-x}}=\ln\left(x^2+1\right)^5-\ln(1-x)^{1/2}$$

$$=5\ln\left(x^2+1\right)-\frac{1}{2}\ln(1-x)$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln\left(\sqrt[3]{\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}}\right)$

Solution

$$\ln\left(\sqrt[3]{\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}}\right)=\ln\left(\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}\right)^{1/3}$$

$$=\frac{1}{3}\ln\left(\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}\right)$$

$$=\frac{1}{3}\left(\ln\left(x(x+1)(x-2)\right)-\ln\left(\left(x^2+1\right)(2x+3)\right)\right)$$

$$=\frac{1}{3}\left(\ln x+\ln(x+1)+\ln(x-2)-\left(\ln\left(x^2+1\right)+\ln(2x+3)\right)\right)$$

$$=\frac{1}{3}\left(\ln x+\ln(x+1)+\ln(x-2)-\ln\left(x^2+1\right)-\ln(2x+3)\right)$$

$$=\frac{1}{3}\ln x+\frac{1}{3}\ln(x+1)+\frac{1}{3}\ln(x-2)-\frac{1}{3}\ln\left(x^2+1\right)-\frac{1}{3}\ln(2x+3)$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln\left(\sqrt{\frac{1}{x(x+1)}}\right)$

Solution

$$\ln\left(\sqrt{\frac{1}{x(x+1)}}\right)=\ln\left(\frac{1}{x(x+1)}\right)^{1/2}$$

$$= \frac{1}{2}\left(\ln 1 - \ln\left(x(x+1)\right)\right)$$

$$= -\frac{1}{2}\left(\ln x + \ln(x+1)\right)$$

$$\boxed{= -\frac{1}{2}\ln x - \frac{1}{2}\ln(x+1)}$$

Exercise

Express the following in terms of sums and differences of logarithms $\ln\left(\sqrt{\left(x^2+1\right)(x-1)^2}\right)$

Solution

$$\ln\left(\sqrt{\left(x^2+1\right)(x-1)^2}\right) = \ln\left(\left(x^2+1\right)(x-1)^2\right)^{1/2}$$

$$= \frac{1}{2}\ln\left(\left(x^2+1\right)(x-1)^2\right)$$

$$= \frac{1}{2}\left(\ln\left(x^2+1\right) + \ln(x-1)^2\right)$$

$$= \frac{1}{2}\left(\ln\left(x^2+1\right) + 2\ln(x-1)\right)$$

$$\boxed{= \frac{1}{2}\ln\left(x^2+1\right) + \ln(x-1)}$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $\log(x+5) + 2\log x$

Solution

$$\log(x+5) + 2\log x = \log(x+5) + \log x^2$$

$$\boxed{= \log\left(x^2(x+5)\right)}$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $3\log_b x - \frac{1}{3}\log_b y + 4\log_b z$

Solution

$$3\log_b x - \frac{1}{3}\log_b y + 4\log_b z = \log_b x^3 + \log_b z^4 - \log_b y^{1/3}$$

$$= \log_b\left(x^3 z^4\right) - \log_b \sqrt[3]{y}$$

$$\boxed{= \log_b\left(\frac{x^3 z^4}{\sqrt[3]{y}}\right)}$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $\frac{1}{2}\log_b (x+5) - 5\log_b y$

Solution

$$\frac{1}{2}\log_b (x+5) - 5\log_b y = \log_b (x+5)^{1/2} - \log_b y^5$$

$$= \log_b \left(\frac{\sqrt{x+5}}{y^5} \right)$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $\ln\left(x^2 - y^2\right) - \ln(x-y)$

Solution

$$\ln\left(x^2 - y^2\right) - \ln(x-y) = \ln \frac{x^2 - y^2}{x-y}$$

$$= \ln \frac{(x-y)(x+y)}{x-y}$$

$$= \ln(x+y)$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $\ln(xz) - \ln\left(x\sqrt{y}\right) + 2\ln\frac{y}{z}$

Solution

$$\ln(xz) - \ln\left(x\sqrt{y}\right) + 2\ln\frac{y}{z} = \ln(xz) + \ln\left(\frac{y}{z}\right)^2 - \ln\left(x\sqrt{y}\right)$$

$$= \ln\left(\frac{xzy^2}{z^2}\right) - \ln\left(x\sqrt{y}\right)$$

$$= \ln\left(\frac{xy^2}{z}\frac{1}{x\sqrt{y}}\right)$$

$$= \ln\left(\frac{y^{3/2}}{z}\right)$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $\log\left(x^2 y\right) - \log z$

Solution

$$\boxed{\log\left(x^2 y\right) - \log z = \log\left(\frac{x^2 y}{z}\right)}$$

Exercise

Write the expression as a single logarithm and simplify if necessary: $\log\left(z^2\sqrt{y}\right) - \log z^{1/2}$

Solution

$$\begin{aligned}\log\left(z^2\sqrt{y}\right) - \log z^{1/2} &= \log\left(\frac{z^2\sqrt{y}}{z^{1/2}}\right)\\ &= \log\left(z^{3/2}\ \sqrt{y}\right)\\ &\boxed{= \log\left(\sqrt{z^3 y}\right)}\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$2\log_a x + \frac{1}{3}\log_a (x-2) - 5\log_a (2x+3)$

Solution

$$\begin{aligned}2\log_a x + \tfrac{1}{3}\log_a (x-2) - 5\log_a (2x+3) &= \log_a x^2 + \log_a (x-2)^{1/3} - \log_a (2x+3)^5\\ &= \log_a x^2 (x-2)^{1/3} - \log_a (2x+3)^5\\ &\boxed{= \log_a \frac{x^2 (x-2)^{1/3}}{(2x+3)^5}}\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$5\log_a x - \frac{1}{2}\log_a (3x-4) - 3\log_a (5x+1)$

Solution

$$\begin{aligned}5\log_a x-\tfrac{1}{2}\log_a(3x-4)-3\log_a(5x+1)&=\log_a x^5-\log_a(3x-4)^{1/2}-\log_a(5x+1)^3\\&=\log_a x^5-\left[\log_a(3x-4)^{1/2}+\log_a(5x+1)^3\right]\\&=\log_a x^5-\left[\log_a(3x-4)^{1/2}(5x+1)^3\right]\\&\boxed{=\log_a\frac{x^5}{(3x-4)^{1/2}(5x+1)^3}}\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\log\left(x^3y^2\right)-2\log\left(x\sqrt[3]{y}\right)-3\log\left(\frac{x}{y}\right)$$

Solution

$$\begin{aligned}\log\left(x^3y^2\right)-2\log\left(x\sqrt[3]{y}\right)-3\log\left(\frac{x}{y}\right)&=\log\left(x^3y^2\right)-\log\left(xy^{1/3}\right)^2-\log\left(xy^{-1}\right)^3\\&=\log\left(x^3y^2\right)-\left[\log\left(x^2y^{2/3}\right)+\log\left(x^3y^{-3}\right)\right]\\&=\log\left(x^3y^2\right)-\log\left(x^2y^{2/3}x^3y^{-3}\right)\\&=\log\left(x^3y^2\right)-\log\left(x^5y^{-7/3}\right)\\&=\log\left(\frac{x^3y^2}{x^5y^{-7/3}}\right)\\&=\log\left(\frac{y^2y^{7/3}}{x^2}\right)\\&=\log\left(\frac{y^{13/3}}{x^2}\right)\\&=\log\left(\frac{\sqrt[3]{y^{13}}}{x^2}\right)\\&\boxed{=\log\left(\frac{y^4\sqrt[3]{y}}{x^2}\right)}\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\ln y^3 + \tfrac{1}{3}\ln\left(x^3y^6\right) - 5\ln y$$

Solution

$$\begin{aligned}
\ln y^3 + \tfrac{1}{3}\ln\left(x^3y^6\right) - 5\ln y &= \ln y^3 + \ln\left(x^3y^6\right)^{1/3} - \ln y^5 \\
&= \ln y^3 + \ln\left(x^{3/3}y^{6/3}\right) - \ln y^5 \\
&= \ln y^3 + \ln\left(xy^2\right) - \ln y^5 \\
&= \ln\left(y^3xy^2\right) - \ln y^5 \\
&= \ln\left(\frac{y^5x}{y^5}\right) \\
&= \ln x
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$2\ln x - 4\ln\left(\frac{1}{y}\right) - 3\ln(xy)$$

Solution

$$\begin{aligned}
2\ln x - 4\ln\left(\frac{1}{y}\right) - 3\ln(xy) &= \ln x^2 - \ln\left(\frac{1}{y}\right)^4 - \ln(xy)^3 \\
&= \ln x^2 - \left[\ln\left(y^{-4}\right) + \ln\left(x^3y^3\right)\right] \\
&= \ln x^2 - \ln\left(y^{-4}x^3y^3\right) \\
&= \ln x^2 - \ln\left(y^{-1}x^3\right) \\
&= \ln\frac{x^2}{y^{-1}x^3} \\
&= \ln\frac{y}{x}
\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$4\ln x + 7\ln y - 3\ln z$

Solution

$$
\begin{aligned}
4\ln x + 7\ln y - 3\ln z &= \ln x^4 + \ln y^7 - \ln z^3 \\
&= \ln\left(x^4 y^7\right) - \ln z^3 \\
&= \ln\left(\frac{x^4 y^7}{z^3}\right)
\end{aligned}
$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$\frac{1}{3}\left[5\ln(x+6) - \ln x - \ln(x^2 - 25)\right]$

Solution

$$
\begin{aligned}
\frac{1}{3}\left[5\ln(x+6) - \ln x - \ln(x^2-25)\right] &= \frac{1}{3}\left[5\ln(x+6) - \left(\ln x + \ln(x^2-25)\right)\right] \\
&= \frac{1}{3}\left[\ln(x+6)^5 - \ln x(x^2-25)\right] \\
&= \frac{1}{3}\left[\ln\frac{(x+6)^5}{x(x^2-25)}\right] \\
&= \ln\left(\frac{(x+6)^5}{x(x^2-25)}\right)^{1/3}
\end{aligned}
$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$\frac{2}{3}\left[\ln\left(x^2-4\right) - \ln\left(x+2\right)\right] + \ln(x+y)$

Solution

$$
\begin{aligned}
\frac{2}{3}\left[\ln\left(x^2-4\right) - \ln\left(x+2\right)\right] + \ln(x+y) &= \frac{2}{3}\left[\ln\frac{x^2-4}{x+2}\right] + \ln(x+y) \\
&= \frac{2}{3}\left[\ln\frac{(x+2)(x-2)}{x+2}\right] + \ln(x+y) \\
&= \frac{2}{3}\ln(x-2) + \ln(x+y)
\end{aligned}
$$

$$= \ln(x-2)^{2/3} + \ln(x+y)$$

$$= \ln(x-2)^{2/3}(x+y)$$

$$\boxed{= \ln(x+y)\sqrt[3]{(x-2)^2}}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\tfrac{1}{2}\log_b m + \tfrac{3}{2}\log_b 2n - \log_b m^2 n$$

Solution

$$\begin{aligned}\tfrac{1}{2}\log_b m + \tfrac{3}{2}\log_b 2n - \log_b m^2 n &= \log_b m^{1/2} + \log_b (2n)^{3/2} - \log_b m^2 n \\ &= \log_b \left(m^{1/2}(2n)^{3/2}\right) - \log_b m^2 n \\ &= \log_b \frac{m^{1/2}2^{3/2}n^{3/2}}{m^2 n} \\ &= \log_b \frac{2^{3/2}n^{1/2}}{m^{3/2}} \\ &= \log_b \left(\frac{2^3 n}{m^3}\right)^{1/2} \\ &\boxed{= \log_b \sqrt{\frac{8n}{m^3}}}\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\tfrac{1}{2}\log_y p^3q^4 - \tfrac{2}{3}\log_y p^4q^3$$

Solution

$$\begin{aligned}\tfrac{1}{2}\log_y p^3q^4 - \tfrac{2}{3}\log_y p^4q^3 &= \log_y \left(p^3q^4\right)^{1/2} - \log_y \left(p^4q^3\right)^{2/3} \\ &= \log_y \frac{\left(p^3q^4\right)^{1/2}}{\left(p^4q^3\right)^{2/3}} \\ &= \log_y \frac{\left(p^3\right)^{1/2}\left(q^4\right)^{1/2}}{\left(p^4\right)^{2/3}\left(q^3\right)^{2/3}}\end{aligned}$$

$$= \log_y \frac{p^{3/2}q^2}{p^{8/3}q^2}$$

$$= \log_y \frac{p^{3/2}}{p^{8/3}}$$

$$= \log_y \frac{1}{p^{8/3-3/2}}$$

$$\boxed{= \log_y \frac{1}{p^{7/6}}}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{1}{2}\log_a x + 4\log_a y - 3\log_a x$$

Solution

$$\frac{1}{2}\log_a x + 4\log_a y - 3\log_a x = 4\log_a y - \frac{5}{2}\log_a x$$

$$= \log_a y^4 - \log_a x^{5/2}$$

$$\boxed{= \log_a \frac{y^4}{\sqrt{x^5}}}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{2}{3}\left[\ln\left(x^2-9\right) - \ln(x+3)\right] + \ln(x+y)$$

Solution

$$\frac{2}{3}\left[\ln\left(x^2-9\right) - \ln(x+3)\right] + \ln(x+y) = \frac{2}{3}\ln\frac{x^2-9}{x+3} + \ln(x+y)$$

$$= \frac{2}{3}\ln\frac{(x+3)(x-3)}{x+3} + \ln(x+y)$$

$$= \frac{2}{3}\ln(x-3) + \ln(x+y)$$

$$= \ln(x-3)^{2/3} + \ln(x+y)$$

$$= \ln\left((x-3)^{2/3}(x+y)\right)$$

$$\boxed{= \ln\left((x+y)\sqrt[3]{(x-3)^2}\right)}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\frac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y$$

Solution

$$\begin{aligned}\frac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y &= \log_b x^{1/4} - \log_b 5^2 - \log_b y^{10} \\ &= \log_b x^{1/4} - \left[\log_b 5^2 + \log_b y^{10}\right] \\ &= \log_b x^{1/4} - \log_b \left(5^2 y^{10}\right) \\ &\boxed{= \log_b \frac{\sqrt[4]{x}}{25y^{10}}}\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$2\ln(x+4) - \ln x - \ln\left(x^2-3\right)$$

Solution

$$\begin{aligned}2\ln(x+4) - \ln x - \ln\left(x^2-3\right) &= \ln(x+4)^2 - \left(\ln x + \ln\left(x^2-3\right)\right) \\ &= \ln(x+4)^2 - \ln\left(x\left(x^2-3\right)\right) \\ &\boxed{= \ln\frac{(x+4)^2}{x\left(x^2-3\right)}}\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\ln x + \ln(y+3) + \ln(y+2) - \ln\left(y^2+5y+6\right)$$

Solution

$$\begin{aligned}\ln x + \ln(y+3) + \ln(y+2) - \ln\left(y^2+5y+6\right) &= \ln\left(x(y+3)(y+2)\right) - \ln\left((y+3)(y+2)\right) \\ &= \ln\left(\frac{x(y+3)(y+2)}{(y+3)(y+2)}\right) \\ &\boxed{= \ln x}\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\ln x+\ln(x+4)+\ln(x+1)-\ln\left(x^2+5x+4\right)$$

Solution

$$\begin{aligned}\ln x+\ln(x+4)+\ln(x+1)-\ln\left(x^2+5x+4\right)&=\ln\left(x(x+4)(x+1)\right)-\ln\left((x+4)(x+1)\right)\\&=\ln\left(\frac{x(x+4)(x+1)}{(x+4)(x+1)}\right)\\&=\ln x\end{aligned}$$

Exercise

Write the expression as a single logarithm and simplify if necessary:

$$\ln\left(x^2-25\right)-2\ln(x+5)+\ln(x-5)$$

Solution

$$\begin{aligned}\ln\left(x^2-25\right)-2\ln(x+5)+\ln(x-5)&=\ln\left(x^2-25\right)+\ln(x-5)-\ln(x+5)^2\\&=\ln\frac{(x-5)(x+5)(x-5)}{(x+5)^2}\\&=\ln\left(\frac{(x-5)^2}{x+5}\right)\end{aligned}$$

Exercise

Solve the equation: $2^x=128$

Solution

$2^x=2^7$

$x=7$

Exercise

Solve the equation: $3^x=243$

Solution

$3^x=3^5$

$x = 5$

Exercise

Solve the equation: $5^x = 70$

Solution

$x = \log_5 70$

Exercise

Solve the equation: $6^x = 50$

Solution

$x = \log_6 50$

Exercise

Solve the equation: $5^x = 134$

Solution

$x = \log_5 134$

Exercise

Solve the equation: $7^x = 12$

Solution

$$\boxed{x = \log_7 12}$$

Exercise

Solve the equation: $9^x = \frac{1}{\sqrt[3]{3}}$

Solution

$$\left(3^2\right)^x = \frac{1}{3^{1/3}}$$

$$3^{2x} = 3^{-1/3}$$

$$2x = -\frac{1}{3}$$

$$\boxed{x = -\frac{1}{6}}$$

Exercise

Solve the equation: $49^x = \frac{1}{343}$

Solution

$$\left(7^2\right)^x = \frac{1}{7^3}$$

$$7^{2x} = 7^{-3}$$

$$2x = -3$$

$$\boxed{x = -\frac{3}{2}}$$

Exercise

Solve the equation: $2^{5x+3} = \frac{1}{16}$

Solution

$$2^{5x+3} = 2^{-4}$$

$$5x + 3 = -4$$

$$5x = -7$$

$$\boxed{x = -\frac{7}{5}}$$

Exercise

Solve the equation: $\left(\frac{2}{5}\right)^{x}=\frac{8}{125}$

Solution

$$\left(\frac{2}{5}\right)^{x}=\left(\frac{2}{5}\right)^{3}$$

$$\underline{x=3}$$

Exercise

Solve the equation: $2^{3x-7}=32$

Solution

$$2^{3x-7}=32$$

$$=2^{5}$$

$3x-7=5$ *add 7 on both sides*

$3x=12$ *Divide by 3*

$$\underline{x=4}$$

Exercise

Solve the equation: $4^{2x-1}=64$

Solution

$$4^{2x-1}=4^{3}$$

$$2x-1=3$$

$$2x=4$$

$$\underline{x=2}$$

Exercise

Solve the equation: $3^{1-x}=\frac{1}{27}$

Solution

$$3^{1-x}=\frac{1}{3^{3}}$$

$$3^{1-x}=3^{-3}$$

$$1-x=-3$$

$\underline{x=4}|$

Exercise

Solve the equation: $2^{-x^2}=5$

Solution

$$\ln 2^{-x^2}=\ln 5$$
$$-x^2\ln 2=\ln 5$$
$$x^2=-\frac{\ln 5}{\ln 2} \quad \Rightarrow \quad \textit{No Solution}$$

Exercise

Solve the equation: $2^{-x}=8$

Solution

$$2^{-x}=2^3$$
$$-x=3$$
$$\underline{x=-3}|$$

Exercise

Solve the equation: $\left(\frac{1}{3}\right)^x=81$

Solution

$$\left(\frac{1}{3}\right)^x=81$$
$$\left(3^{-1}\right)^x=3^4$$
$$3^{-x}=3^4$$
$$-x=4$$
$$\underline{x=-4}|$$

Exercise

Solve the equation: $3^{-x}=120$

Solution

$$-x=\log_3 120 \qquad \textit{Convert to Log}$$
$$x=-\log_3 120$$

$$\boxed{=\log_3 \frac{1}{120}}$$

Exercise

Solve the equation: $27 = 3^{5x}\ 9^{x^2}$

Solution

$$3^3 = 3^{5x}\left(3^2\right)^{x^2}$$

$$= 3^{5x}\ 3^{2x^2}$$

$$= 3^{5x+2x^2}$$

$$2x^2 + 5x = 3$$

$$2x^2 + 5x - 3 = 0$$

$$x = \frac{-5 \pm \sqrt{25+24}}{6}$$

$$x = \begin{cases} \dfrac{-5-7}{6} & \boxed{=-2} \\ \dfrac{-5+7}{6} & \boxed{=\dfrac{1}{3}} \end{cases}$$

Exercise

Solve the equation: $4^{x+3} = 3^{-x}$

Solution

$$\ln 4^{x+3} = \ln 3^{-x}$$

$$(x+3)\ln 4 = -x\ln 3$$

$$x\ln 4 + 3\ln 4 = -x\ln 3$$

$$x\ln 4 + x\ln 3 = -3\ln 4$$

$$x(\ln 4 + \ln 3) = -3\ln 4$$

$$\boxed{x = \frac{-3\ln 4}{(\ln 4 + \ln 3)}}$$

Exercise

Solve the equation: $2^{x+4} = 8^{x-6}$

Solution

$$2^{x+4} = \left(2^3\right)^{x-6}$$

$$2^{x+4} = 2^{3x-18}$$

$$x+4=3x-18$$
$$2x=22$$
$$\boxed{x=11}$$

Exercise

Solve the equation: $8^{x+2}=4^{x-3}$

Solution

$$\left(2^3\right)^{x+2}=\left(2^2\right)^{x-3}$$
$$2^{3(x+2)}=2^{2(x-3)}$$
$$3(x+2)=2(x-3)$$
$$3x+6=2x-6$$
$$3x-2x=-6-6$$
$$\boxed{x=-12}$$

Exercise

Solve the equation: $7^x=12$

Solution

$\boxed{x=\log_7 12}$ ***Convert to Log***

Exercise

Solve the equation: $5^{x+4}=4^{x+5}$

Solution

$$\ln 5^{x+4}=\ln 4^{x+5}$$
$$(x+4)\ln 5=(x+5)\ln 4$$
$$x\ln 5+4\ln 5=x\ln 4+5\ln 4$$
$$(\ln 5-\ln 4)x=5\ln 4-4\ln 5$$
$$\boxed{x=\frac{5\ln 4-4\ln 5}{\ln 5-\ln 4}}$$

Exercise

Solve the equation: $5^{x+2}=4^{1-x}$

Solution

$$\ln 5^{x+2} = \ln 4^{1-x}$$
$$(x+2)\ln 5 = (1-x)\ln 4$$
$$x\ln 5 + 2\ln 5 = \ln 4 - x\ln 4$$
$$(\ln 5 + \ln 4)x = \ln 4 - 2\ln 5$$
$$\boxed{x = \frac{\ln 4 - 2\ln 5}{\ln 5 + \ln 4}}$$

Exercise

Solve the equation: $3^{2x-1} = 0.4^{x+2}$

Solution

$$\ln 3^{2x-1} = \ln\left(0.4^{x+2}\right)$$
$$(2x-1)\ln 3 = (x+2)\ln\frac{4}{10}$$
$$2x\ln 3 - \ln 3 = x\ln\frac{2}{5} + 2\ln\frac{2}{5}$$
$$\left(2\ln 3 - \ln\frac{2}{5}\right)x = \ln 3 + 2\ln\frac{2}{5}$$
$$\boxed{x = \frac{\ln 3 + 2\ln 0.4}{2\ln 3 - \ln 0.4}}$$

Exercise

Solve the equation: $4^{3x-5} = 16$

Solution

$$4^{3x-5} = 4^2$$
$$3x - 5 = 2$$
$$3x = 7$$
$$\boxed{x = \frac{7}{3}}$$

Exercise

Solve the equation: $4^{x+3} = 3^{-x}$

Solution

$$\ln 4^{x+3} = \ln 3^{-x}$$

$$(x+3)\ln 4 = -x\ln 3$$

$$x\ln 4 + 3\ln 4 = -x\ln 3$$

$$(\ln 4 + \ln 3)\,x = -3\ln 4$$

$$\boxed{x = -\frac{3\ln 4}{\ln 4 + \ln 3}}$$

Exercise

Solve the equation: $7^{2x+1} = 3^{x+2}$

Solution

$$\ln 7^{2x+1} = \ln 3^{x+2}$$

$$(2x+1)\ln 7 = (x+2)\ln 3$$

$$2x\ln 7 + \ln 7 = x\ln 3 + 2\ln 3$$

$$2x\ln 7 - x\ln 3 = 2\ln 3 - \ln 7$$

$$x(2\ln 7 - \ln 3) = 2\ln 3 - \ln 7$$

$$\boxed{x = \frac{2\ln 3 - \ln 7}{2\ln 7 - \ln 3}}$$

Exercise

Solve the equation: $3^{x-1} = 7^{2x+5}$

Solution

$$\ln 3^{x-1} = \ln 7^{2x+5}$$

$$(x-1)\ln 3 = (2x+5)\ln 7$$

$$x\ln 3 - \ln 3 = 2x\ln 7 + 5\ln 7$$

$$x\ln 3 - 2x\ln 7 = \ln 3 + 5\ln 7$$

$$x(\ln 3 - 2\ln 7) = \ln 3 + 5\ln 7$$

$$\boxed{x = \frac{\ln 3 + 5\ln 7}{\ln 3 - 2\ln 7}}$$

Exercise

Solve the equation: $4^{x-2} = 2^{3x+3}$

Solution

$$\left(2^2\right)^{x-2} = 2^{3x+3}$$

$$2^{2x-4} = 2^{3x+3}$$

$$2x-4 = 3x+3$$

$$2x-3x = 4+3$$

$$-x = 7$$

$$\boxed{x = -7}$$

Exercise

Solve the equation: $3^{5x-8} = 9^{x+2}$

Solution

$$3^{5x-8} = \left(3^2\right)^{x+2}$$

$$3^{5x-8} = 3^{2x+4}$$

$$5x-8 = 2x+4$$

$$5x-2x = 8+4$$

$$3x = 12$$

$$\boxed{x = 4}$$

Exercise

Solve the equation: $3^{x+4} = 2^{1-3x}$

Solution

$$\ln 3^{x+4} = \ln 2^{1-3x} \qquad \textit{'ln' both sides}$$

$$(x+4)\ln 3 = (1-3x)\ln 2 \qquad \textit{Power Rule}$$

$$x\ln 3 + 4\ln 3 = \ln 2 - 3x\ln 2 \qquad \textit{Distribute}$$

$$x\ln 3 + 3x\ln 2 = \ln 2 - 4\ln 3$$

$$x(\ln 3 + 3\ln 2) = \ln 2 - 4\ln 3$$

$$\boxed{x = \frac{\ln 2 - 4\ln 3}{\ln 3 + 3\ln 2}}$$

Exercise

Solve the equation: $3^{2-3x} = 4^{2x+1}$

Solution

$\ln 3^{2-3x} = \ln 4^{2x+1}$ ***'ln' both sides***

$(2-3x)\ln 3 = (2x+1)\ln 4$ ***Power Rule***

$2\ln 3 - 3x\ln 3 = 2x\ln 4 + \ln 4$

$-3x\ln 3 - 2x\ln 4 = \ln 4 - 2\ln 3$

$-x(3\ln 3 + 2\ln 4) = \ln 4 - 2\ln 3$

$$x = -\frac{\ln 4 - 2\ln 3}{3\ln 3 + 2\ln 4}$$

$$= -\frac{\ln 4 - \ln 3^2}{\ln 3^3 + \ln 4^2}$$

$$= \frac{\ln 9 - \ln 4}{\ln 27 + \ln 16}$$

$$= \frac{\ln \frac{9}{4}}{\ln 432}$$

$$= \log_{432} \frac{9}{4}$$

Exercise

Solve the equation: $4^{x+3} = 3^{-x}$

Solution

$\ln 4^{x+3} = \ln 3^{-x}$

$(x+3)\ln 4 = -x\ln 3$

$x\ln 4 + 3\ln 4 = -x\ln 3$

$x\ln 4 + x\ln 3 = -3\ln 4$

$x(\ln 4 + \ln 3) = -3\ln 4$

$$x = \frac{-3\ln 4}{(\ln 4 + \ln 3)}$$

Exercise

Solve the equation: $7^{x+6} = 7^{3x-4}$

Solution

$x + 6 = 3x - 4$

$4 + 6 = 3x - x$

$10 = 2x$

$x = 5$

Exercise

Solve the equation: $2^{-100x}=(0.5)^{x-4}$

Solution

$$2^{-100x}=\left(\frac{1}{2}\right)^{x-4}$$

$$2^{-100x}=\left(2^{-1}\right)^{x-4}$$

$$2^{-100x}=2^{-x+4}$$

$$-100x=-x+4$$

$$-100x+x=4$$

$$-99x=4$$

$$\underline{x=-\frac{4}{99}\Big|}$$

Exercise

Solve the equation: $4^{x}\left(\frac{1}{2}\right)^{3-2x}=8.\left(2^{x}\right)^{2}$

Solution

$$\left(2^{2}\right)^{x}\left(2^{-1}\right)^{3-2x}=2^{3}.2^{2x}$$

$$2^{2x}2^{2x-3}=2^{3+2x}$$

$$2^{2x+2x-3}=2^{3+2x}$$

$$2^{4x-3}=2^{3+2x}$$

$$4x-3=3+2x$$

$$4x-2x=3+3$$

$$2x=6$$

$$\underline{x=3\,|}$$

Exercise

$5^{x}+125\left(5^{-x}\right)=30$

Solution

$$5^{x}5^{x}+125\left(5^{-x}\right)5^{x}=30\left(5^{x}\right)$$

$$5^{2x}+125=30\left(5^{x}\right)$$

$$5^{2x} - 30\left(5^x\right) + 125 = 0 \qquad \textit{Solve for } 5^x$$

$$5^x = 5 \qquad 5^x = 25 = 5^2$$

$$x = 1 \qquad x = 2$$

$$\boxed{x = 1,\ 2}$$

Exercise

$$4^x - 3\left(4^{-x}\right) = 8$$

Solution

$$4^x 4^x - 3\left(4^{-x}\right)4^x = 8\left(4^x\right)$$

$$4^{2x} - 3 = 8\left(4^x\right)$$

$$4^{2x} - 8\left(4^x\right) - 3 = 0 \qquad \textit{Solve for } 4^x$$

$$4^x = 4 + \sqrt{19} \qquad 4^x = 4 - \sqrt{19} < 0$$

$$x \ln 4 = \ln\left(4 + \sqrt{19}\right)$$

$$\boxed{x = \frac{\ln\left(4+\sqrt{19}\right)}{\ln 4}}$$

Exercise

Solve the equation: $5^{3x-6} = 125$

Solution

$$5^{3x-6} = 5^3$$

$$3x - 6 = 3$$

$$3x = 9$$

$$\boxed{x = 3}$$

Exercise

Solve the equation: $e^x = 15$

Solution

$$\boxed{x = \ln 5} \qquad \textit{Convert to Log}$$

Exercise

Solve the equation: $e^{x+1} = 20$

Solution

$x + 1 = \ln 20$ ***Convert to Log***

$\underline{x = -1 + \ln 20}$

Exercise

Solve the equation: $9e^{x} = 107$

Solution

$$e^{x} = \frac{107}{9}$$

$$\ln e^{x} = \ln\left(\frac{107}{9}\right)$$

$$x \ln e = \ln\left(\frac{107}{9}\right)$$

$$\underline{x = \ln\left(\frac{107}{9}\right)}$$

Exercise

Solve the equation: $e^{x\ln 3} = 27$

Solution

$x \ln 3 = \ln 27$ ***Convert to Log***

$$x \ln 3 = \ln 3^{3}$$

$$x = \frac{3\ln 3}{\ln 3}$$

$$\underline{= 3}$$

Exercise

Solve the equation: $e^{x^2} = e^{7x-12}$

Solution

$$e^{x^2} = e^{7x-12}$$

$$x^2 = 7x - 12$$

$$x^2 - 7x + 12 = 0$$

$$\underline{x = 3,\ 4}$$

Exercise

Solve the equation: $f(x) = xe^{x} + e^{x}$

Solution

$$xe^{x} + e^{x} = 0$$

$$e^{x}(x+1) = 0$$

$$e^{x} \neq 0 \qquad x+1=0$$

$$\boxed{x=-1} \ (\textit{Only solution})$$

Exercise

Solve the equation $f(x) = x^{3}\left(4e^{4x}\right) + 3x^{2}e^{4x}$

Solution

$$x^{3}\left(4e^{4x}\right) + 3x^{2}e^{4x} = 0$$

$$x^{2}e^{4x}(4x+3) = 0$$

$$x^{2} = 0 \qquad 4x+3=0$$

$$x = 0,\ 0 \qquad x = -\frac{3}{4}$$

The solutions are: $\boxed{x = 0,\ 0,\ -\frac{3}{4}}$

Exercise

Solve the equation: $e^{2x} - 2e^{x} - 3 = 0$

Solution

$$\left(e^{x}\right)^{2} - 2e^{x} - 3 = 0$$

$$\begin{cases} e^{x} = -1 \ \times & \rightarrow \ \textit{Impossible} \\ e^{x} = 3 & \rightarrow \ \boxed{x = \ln 3} \end{cases}$$

Exercise

Solve the equation: $e^{0.08t} = 2500$

Solution

$$\ln\left(e^{0.08t}\right) = \ln 2500$$

$$0.08t = \ln(50)^2$$

$$t = \frac{200 \ln 50}{8}$$

$$\underline{= 25 \ln 50 \,|}$$

Exercise

Solve the equation: $e^{x^2} = 200$

Solution

$\ln e^{x^2} = \ln 200$ *Natural Log both sides*

$x^2 = \ln 200$ $\ln e = 1$

$\underline{x = \pm\sqrt{\ln 200} \,|}$

Exercise

Solve the equation: $e^{2x+1} \cdot e^{-4x} = 3e$

Solution

$$e^{2x+1-4x} = 3e$$

$$e^{-2x+1} = 3e$$

$e^{-2x}e = 3e$ *Divide by e*

$$e^{-2x} = 3$$

$$\ln e^{-2x} = \ln 3$$

$$-2x = \ln 3$$

$$\underline{x = -\frac{1}{2}\ln 3 \,|}$$

Exercise

Solve the equation: $e^{2x} - 8e^x + 7 = 0$

Solution

$\left(e^x\right)^2 - 8e^x + 7 = 0$ $a + b + c = 0 \rightarrow x = 1, \ \frac{c}{a}$

$$\begin{cases} e^x = 1 & \rightarrow \quad \underline{x = 0 \,|} \\ e^x = 7 & \rightarrow \quad \underline{x = \ln 7 \,|} \end{cases}$$

Exercise

Solve the equation without using the calculator: $e^{2x} + 2e^{x} - 15 = 0$

Solution

$(e^{x})^{2} + 2e^{x} - 15 = 0$ ***Solve for*** e^{x}

$e^{x} = 3$ $e^{x} \cancel{=} -5 < 0$

$\boxed{x = \ln 3}$

Exercise

Solve the equation: $e^{x} + e^{-x} - 6 = 0$

Solution

$$e^{x}e^{x} + e^{x}e^{-x} - e^{x}6 = e^{x}0$$

$$e^{2x} + 1 - 6e^{x} = 0$$

$$(e^{x})^{2} - 6e^{x} + 1 = 0$$

$$e^{x} = \frac{6 \pm \sqrt{36-4}}{2}$$

$$= \frac{6 \pm 4\sqrt{2}}{2}$$

$$e^{x} = 3 \pm 2\sqrt{2}$$

$$\boxed{x = \ln\left(3 \pm 2\sqrt{2}\right)}$$

Exercise

Solve the equation: $e^{1-3x} \cdot e^{5x} = 2e$

Solution

$$e^{1-3x+5x} = 2e$$

$$e^{1+2x} = 2e$$

$e^{1}e^{2x} = 2e$ *Divide by e*

$e^{2x} = 2$ *Natural Log both sides*

$$\ln e^{2x} = \ln 2$$

$$2x = \ln 2$$

$$\boxed{x = \frac{1}{2}\ln 2}$$

Exercise

Solve the equation: $6\ln(2x) = 30$

Solution

$$\ln(2x) = \frac{30}{6}$$

$$\ln(2x) = 5$$

$$2x = e^5$$

$$\boxed{x = \frac{1}{2}e^5}$$

Exercise

Solve the equation: $\log_5(x-7) = 2$

Solution

$$x - 7 = 5^2$$

$$x = 25 + 7$$

$$\boxed{x = 32}$$

Exercise

Solve the equation: $\log_4(5+x) = 3$

Solution

$$5 + x = 4^3$$

$$x = 64 - 5$$

$$\boxed{= 59}$$

Check: $\log_4(5+59)$

Exercise

Solve the equation: $\log(4x-18) = 1$

Solution

$$4x - 18 = 10$$

$$4x = 28$$

$$\boxed{x = 7}$$

Exercise

Solve the equation: $\log\left(x^2+19\right)=2$

Solution

$$x^2+19=10^2$$

$$x^2=81$$

$$\underline{x=\pm 9}\,\Big| \qquad (\pm 9)^2+19>0$$

Exercise

Solve the equation: $\ln\left(x^2-12\right)=\ln x$

Solution

$$\ln\left(x^2-12\right)=\ln x$$

$$x^2-12=x$$

$$x^2-x-12=0$$

$$\underline{x=-3,\ 4}\,\Big|$$

Check: $x=-3 \quad \ln(9-12)=\ln(-3)$ ✗

$x=4 \quad \ln(16-12)=\ln(4)$

$\therefore$ ***Solution***: $\underline{x=4}\,\Big|$

Exercise

Solve the equation: $\log\left(2x^2+3x\right)=\log(10x+30)$

Solution

$$\log\left(2x^2+3x\right)=\log(10x+30)$$

$$2x^2+3x=10x+30$$

$$2x^2-7x-30=0$$

$$x=\frac{7\pm\sqrt{49+240}}{4}$$

$$=\begin{cases}\frac{7-17}{4}=-\frac{5}{2}\\ \frac{7+17}{4}=6\end{cases}$$

Check: $x=-\frac{5}{2}\quad \log\left(\frac{25}{2}-\frac{15}{2}\right)=\log(-25+30)$

$x=4\quad \log(32+12)=\log(40+30)$

$\therefore$ **Solution**: $\boxed{x=-\frac{5}{2},\ 4}$

Exercise

Solve the equation: $\log_5(2x+3)=\log_5 11+\log_5 3$

Solution

$$\log_5(2x+3)=\log_5(11\times 3)$$

$$2x+3=33$$

$$2x=30$$

$\boxed{x=15}$ **Check:** $\log_5(30+3)$

Exercise

Solve the equation: $\log_3 x-\log_9(x+42)=0$

Solution

$$\frac{\log x}{\log 3}-\frac{\log(x+42)}{\log 9}=0$$

$$\frac{\log x}{\log 3}-\frac{\log(x+42)}{\log 3^2}=0$$

$$\frac{\log x}{\log 3}-\frac{1}{2}\frac{\log(x+42)}{\log 3}=0$$

$$\log x-\frac{1}{2}\log(x+42)=0$$

$$2\log x=\log(x+42)$$

$$\log x^2=\log(x+42)$$

$$x^2=x+42$$

$$x^2-x-42=0$$

$$\boxed{x=-6,\ 7}$$

Check: $x=-6\quad \log_3(-6)-\log_9(-6+42)$ ✗

$x=7\quad \log_3 7-\log_9(7+42)=0$

$\therefore$ **Solution**: $\boxed{x=7}$

Exercise

Solve the equation: $\log_5 x + \log_5 (4x-1) = 1$

Solution

$$\log_5 x(4x-1) = 1$$

$$4x^2 - x = 5$$

$$4x^2 - x - 5 = 0 \qquad a - b + c = 0 \;\rightarrow\; x = -1, \; -\frac{c}{a}$$

$$\boxed{x = -\frac{5}{2}, \; 4}$$

Check: $x = -\frac{5}{2} \quad \log_5\left(-\frac{5}{2}\right) + \log_5 (10-1)$ ✗

$x = 4 \quad \log_5 (4) + \log_5 (15)$

∴ ***Solution***: $\boxed{x = 4}$

Exercise

Solve the equation: $\log x - \log(x+3) = 1$

Solution

$$\log \frac{x}{x+3} = 1$$

$$\frac{x}{x+3} = 10$$

$$x = 10x + 30$$

$$9x = -30$$

$$\boxed{x = -\frac{10}{3}}$$

Check: $x = -\frac{10}{3} \quad \log\left(-\frac{10}{3}\right) - \log(x+3)$ ✗

∴ *No Solution*

Exercise

Solve the equation: $\log x + \log(x-9) = 1$

Solution

$$\log x(x-9) = 1$$

$$x^2 - 9x = 10$$

$$x^2 - 9x - 10 = 0 \qquad a - b + c = 0 \;\rightarrow\; x = -1, \; -\frac{c}{a}$$

$$\boxed{x = -1, \; 10}$$

Check: $x = -1 \quad \log(-1) + \log(x-9)$ ✗

$x = 10 \quad \log(10) + \log(10-9)$

∴ **Solution**: $\underline{x = 10}\,\big|$

Exercise

Solve the equation: $\log_2(x+1) + \log_2(x-1) = 3$

Solution

$$\log_2(x+1)(x-1) = 3$$

$$x^2 - 1 = 2^3$$

$$x^2 = 9$$

$$\underline{x = \pm 3}\,\big|$$

Check: $x = -3 \quad \log_2(-2) + \log_2(x-1)$ ✗

$x = 3 \quad \log_2(4) + \log_2(2)$

∴ **Solution**: $\underline{x = 3}\,\big|$

Exercise

Solve the equation: $\log_8(x+1) - \log_8 x = 2$

Solution

$$\log_8 \frac{x+1}{x} = 2$$

$$\frac{x+1}{x} = 8^2$$

$$x + 1 = 64x$$

$$63x = 1$$

$$\underline{x = \frac{1}{63}}\,\Big|$$

Check: $x = \frac{1}{63} \quad \log_8\left(\frac{1}{63} + 1\right) - \log_8 \frac{1}{63}$

∴ **Solution**: $\underline{x = \frac{1}{63}}\,\Big|$

Exercise

Solve the equation: $\ln(x+8)+\ln(x-1)=2\ln x$

Solution

$$\ln(x+8)(x-1)=\ln x^2$$

$$x^2+7x-8=x^2$$

$$7x-8=0$$

$$\boxed{x=\frac{8}{7}}$$

Check: $x=\frac{8}{7}$ $\ln\left(\frac{8}{7}+8\right)+\ln\left(\frac{8}{7}-1\right)=2\ln\frac{8}{7}$

$\therefore$ ***Solution***: $\boxed{x=\frac{8}{7}}$

Exercise

Solve the equation: $\ln(4x+6)-\ln(x+5)=\ln x$

Solution

$$\ln\frac{4x+6}{x+5}=\ln x$$

$$\frac{4x+6}{x+5}=x$$

$$4x+6=x^2+5x$$

$$x^2+x-6=0$$

$$\boxed{x=-3,\ 2}$$

Check: $x=-3$ $\ln(-6)-\ln(x+5)=\ln x$ ✗

$x=2$ $\ln(14)-\ln(7)=\ln 2$

$\therefore$ ***Solution***: $\boxed{x=2}$

Exercise

Solve the equation: $\ln(5+4x)-\ln(x+3)=\ln 3$

Solution

$$\ln\frac{5+4x}{x+3}=\ln 3$$

$$\frac{5+4x}{x+3}=3$$

$$5+4x=3x+9$$

$$\boxed{x=4}$$

Check: $x=4 \quad \ln(21)-\ln(7)=\ln 3$

$\therefore$ **Solution**: $\boxed{x=4}$

Exercise

Solve the equation: $\ln \sqrt[4]{x}=\sqrt{\ln x}$

Solution

Domain: $\boxed{x \ge 1}$

$$\ln x^{1/4}=\sqrt{\ln x}$$

$$\frac{1}{4}\ln x=\sqrt{\ln x}$$

$$\left(\frac{1}{4}\ln x\right)^2=\left(\sqrt{\ln x}\right)^2$$

$$\frac{1}{6}\ln^2 x=\ln x$$

$$\ln^2 x=6\ln x$$

$$\ln^2 x-6\ln x=0$$

$$(\ln x)(\ln x-6)=0$$

$$\begin{cases} \ln x=0 \ \rightarrow \ \boxed{x=1} \\ \ln x=6 \ \rightarrow \ \boxed{x=e^6} \end{cases}$$

$\therefore$ **Solution**: $\boxed{x=1,\ e^6}$

Exercise

Solve the equation: $\sqrt{\ln x}=\ln\sqrt{x}$

Solution

Domain: $\boxed{x \ge 1}$

$$\sqrt{\ln x}=\ln x^{1/2}$$

$$\sqrt{\ln x}=\frac{1}{2}\ln x$$

$$\left(\sqrt{\ln x}\right)^2=\left(\frac{1}{2}\ln x\right)^2$$

$$\ln x=\frac{1}{4}\ln^2 x$$

$$4\ln x=\ln^2 x$$

$$\ln^2 x-4\ln x=0$$

$$\ln x(\ln x-4)=0$$

$$\begin{cases} \ln x = 0 \ \rightarrow \ \underline{x = 1|} \\ \ln x = 4 \ \rightarrow \ \underline{x = e^4|} \end{cases}$$

$\therefore$ ***Solution***: $\underline{x = 1, \ e^4 \ |}$

Exercise

Solve the equation: $\log x^2 = (\log x)^2$

Solution

Domain: $\underline{x \ge 1 \ |}$

$$2\log x = (\log x)^2$$

$$(\log x)^2 - 2\log x = 0$$

$$\log x(\log x - 2) = 0$$

$$\begin{cases} \log x = 0 \ \rightarrow \ \underline{x = 1|} \\ \log x = 2 \ \rightarrow \ \underline{x = 100|} \end{cases}$$

$\therefore$ ***Solution***: $\underline{x = 1, \ 100 \ |}$

Exercise

Solve the equation: $\log x^3 = (\log x)^2$

Solution

Domain: $\underline{x \ge 1 \ |}$

$$3\log x = (\log x)^2$$

$$(\log x)^2 - 3\log x = 0$$

$$\log x(\log x - 3) = 0$$

$$\begin{cases} \log x = 0 \ \rightarrow \ \underline{x = 1|} \\ \log x = 3 \ \rightarrow \ \underline{x = 10^3|} \end{cases}$$

Convert to exponential

$\therefore$ ***Solution***: $\underline{x = 1, \ 10^3 \ |}$

Exercise

Solve the equation: $\log(\log x) = 1$

Solution

$\log x = 10$ ***Convert to exponential***

$\therefore$ ***Solution***: $\boxed{x = 10^{10}}$

Exercise

Solve the equation: $\log(\log x) = 2$

Solution

$\log x = 10^2$ ***Convert to exponential***

$\therefore$ ***Solution***: $\boxed{x = 10^{100}}$

Exercise

Solve the equation: $\ln(\ln x) = 2$

Solution

$\ln x = e^2$ ***Convert to exponential***

$\therefore$ ***Solution***: $\boxed{x = e^{e^2}}$

Exercise

Solve the equation: $\ln\left(e^{x^2}\right) = 64$

Solution

$e^{x^2} = e^{64}$ ***Convert to exponential***

$x^2 = 64$

$\therefore$ ***Solution***: $\boxed{x = \pm 8}$

Exercise

Solve the equation: $e^{\ln(x-1)} = 4$

Solution

$x - 1 = 4$

$\therefore$ ***Solution***: $\boxed{x = 5}$

Exercise

Solve the equation: $10^{\log(2x+5)} = 9$

Solution

$$2x + 5 = 9$$

$$2x = 4$$

$\therefore$ ***Solution***: $\boxed{x = 2}$

Exercise

Solve the equation: $\log\sqrt{x^3 - 9} = 2$

Solution

$$\sqrt{x^3 - 9} = 10^2$$

$$x^3 - 9 = 10^4$$

$$x^3 = 10,009$$

$\therefore$ ***Solution***: $\boxed{x = \sqrt[3]{10,009}}$

Exercise

Solve the equation: $\log\sqrt{x^3 - 17} = \frac{1}{2}$

Solution

$$\log\left(x^3 - 17\right)^{1/2} = \frac{1}{2}$$

$$\frac{1}{2}\log\left(x^3 - 17\right) = \frac{1}{2}$$

$$\log\left(x^3 - 17\right) = 1$$

$$x^3 - 17 = 10$$

$$x^3 = 27$$

$$\boxed{x = 3}$$

Check: $x = 3 \quad \log\sqrt{27 - 17}$

$\therefore$ ***Solution***: $\boxed{x = 3}$

Exercise

Solve the equation: $\log_4 x = \log_4 (8 - x)$

Solution

$$x = 8 - x$$
$$x + x = 8$$
$$2x = 8$$
$$x = 4$$

Check: $x = 4 \quad \log_4 4 = \log_4 (8 - 4)$

$\therefore$ ***Solution***: $x = 4$

Exercise

Solve the equation: $\log_7 (x - 5) = \log_7 (6x)$

Solution

$$x - 5 = 6x$$
$$x - 6x = 5$$
$$-5x = 5$$
$$x = -1$$

Check: $x = -1 \quad \log_7 (-6) = \log_7 (6x)$ ✗

$\therefore$ ***No Solution***

Exercise

Solve the equation: $\ln x^2 = \ln (12 - x)$

Solution

$$\ln x^2 = \ln (12 - x)$$
$$x^2 = 12 - x$$
$$x^2 + x - 12 = 0$$
$$x = -4,\ 3$$

Check: $x = -4 \quad \ln (16) = \ln (16)$

$x = 3 \quad \ln (9) = \ln (12 - 3)$

$\therefore$ ***Solution***: $x = -4,\ 3$

Exercise

Solve the equation $\log_2 (x+7) + \log_2 x = 3$

Solution

$$\log_2 x(x+7) = 3$$

$$x(x+7) = 2^3$$

Convert to Exponential Form

$$x^2 + 7x = 8$$

$$x^2 + 7x - 8 = 0$$

$$\underline{x = 1, \ -8|}$$

Check: $x = -8 \quad \log_2 (x+7) + \log_2 (-8) \ \times$

$x = 1 \quad \log_2 (1+7) + \log_2 1$

$\therefore$ ***Solution***: $\underline{x = 1 \,|}$

Exercise

Solve the equation $\ln x = 1 - \ln(x+2)$

Solution

$$\ln x + \ln(x+2) = 1$$

$$\ln x(x+2) = 1$$

$$x^2 + 2x = e$$

$$x^2 + 2x - e = 0$$

$$x = \frac{-2 \pm \sqrt{4+4e}}{2}$$

$$= \frac{-2 \pm 2\sqrt{1+e}}{2}$$

$$= \begin{cases} -1 - \sqrt{1+e} < 0 \\ -1 + \sqrt{1+e} > 0 \end{cases}$$

$\therefore$ ***Solution***: $\underline{x = -1 + \sqrt{1+e} \,|}$

Exercise

Solve the equation $\ln x = 1 + \ln(x+1)$

Solution

$$\ln x - \ln(x+1) = 1$$

$$\ln\frac{x}{x+1}=1$$

$$\frac{x}{x+1}=e^1$$

$$x=(x+1)e$$

$$x=ex+e$$

$$x-ex=e$$

$$x(1-e)=e$$

$$x=\frac{e}{1-e}<0$$

$\therefore$ ***No solution***

Exercise

Solve the equation $\log_6(2x-3)=\log_6 12-\log_6 3$

Solution

$$\log_6(2x-3)=\log_6\frac{12}{3}$$

$$\log_6(2x-3)=\log_6 4$$

$$2x-3=4$$

$$2x=7$$

$$\underline{x=\frac{7}{2}|}$$

Check: $x=\frac{7}{2}$ $\log_6(7-3)=\log_6 12-\log_6 3$

$\therefore$ ***Solution***: $\underline{x=\frac{7}{2}|}$

Exercise

Solve the equation: $\log(3x+2)+\log(x-1)=1$

Solution

Domain: $x>1$

$\log(3x+2)(x-1)=1$ *Convert to exponential form*

$3x^2-x-2=10$

$3x^2-x-12=0$ *Solve for x*

$$x=\frac{1\pm\sqrt{1+144}}{6}$$

$$= \begin{cases} \frac{1-\sqrt{145}}{6} < 0 \\ \frac{1+\sqrt{145}}{6} > 1 \end{cases}$$

$\therefore$ ***Solution***: $\boxed{x = \frac{1+\sqrt{145}}{6}}$

Exercise

Solve the equation: $\log_5(x+2) + \log_5(x-2) = 1$

Solution

$$\log_5(x+2)(x-2) = 1$$

$$(x+2)(x-2) = 5^1$$

$$x^2 - 4 = 5$$

$$x^2 = 5 + 4$$

$$x^2 = 9$$

$$\boxed{x = \pm 3}$$

Check: $x = -3 \quad \log_5(-1) + \log_5(x-2) \times$

$x = 3 \quad \log_5(3+2) + \log_5(3-2)$

$\therefore$ ***Solution***: $\boxed{x = 3}$

Exercise

Solve the equation: $\log_2 x + \log_2(x-4) = 2$

Solution

Domain: $x > 4$

$$\log_2 x(x-4) = 2$$

$$x^2 - 4x = 2^2$$

$$x^2 - 4x - 4 = 0$$

$$x = \frac{4 \pm \sqrt{32}}{2}$$

$$= \begin{cases} 2 - 2\sqrt{2} < 4 \times \\ 2 + 2\sqrt{2} > 4 \end{cases}$$

$\therefore$ ***Solution***: $\boxed{x = 2 + 2\sqrt{2}}$

Exercise

Solve the equation: $\log_3 x + \log_3 (x+6) = 3$

Solution

Domain: $x > 0$

$$\log_3 x(x+6) = 3$$

$$x^2 + 6x = 3^3$$

$$x^2 + 6x - 27 = 0$$

$$x = \frac{-6 \pm \sqrt{36+108}}{2}$$

$$= \begin{cases} \frac{-6-12}{2} = -9 < 0 \times \\ \frac{-6+12}{2} = 3 > 0 \end{cases}$$

∴ **Solution**: $\underline{x = 3}\,|$

Exercise

Solve the equation: $\log_3 (x+3) + \log_3 (x+5) = 1$

Solution

Domain: $x > -3$

$$\log_3 (x+3)(x+5) = 1$$

$$x^2 + 8x + 15 = 3$$

$$x^2 + 8x + 12 = 0$$

$$x = \frac{-8 \pm \sqrt{64-48}}{2}$$

$$= \begin{cases} \frac{-8-4}{2} = -6 < -3 \times \\ \frac{-8+4}{2} = -2 > -3 \end{cases}$$

∴ **Solution**: $\underline{x = -2}\,|$

Exercise

Solve the equation: $\ln x = \frac{1}{2}\ln\left(2x + \frac{5}{2}\right) + \frac{1}{2}\ln 2$

Solution

Domain: $x > 0$

$$2\ln x = \ln\left(2x + \frac{5}{2}\right) + \ln 2$$

$$\ln x^2 = \ln 2\left(2x + \frac{5}{2}\right)$$

$$x^2 = 4x + 5$$

$$x^2 - 4x - 5 = 0 \qquad\qquad a - b + c = 0 \quad \rightarrow \quad x = -1,\ -\frac{c}{a}$$

$$x = -1,\ 5$$

$\therefore$ ***Solution***: $x = 5$

Exercise

Solve the equation $\ln(-4 - x) + \ln 3 = \ln(2 - x)$

Solution

Domain: $x < 5$

$$\ln 3(-4 - x) = \ln(2 - x)$$
$$-12 - 3x = 2 - x$$
$$-12 - 2 = 3x - x$$
$$-14 = 2x$$

$\therefore$ ***Solution***: $x = -7$

Exercise

Solve the equation: $\log_4 x + \log_4 (x - 2) = \log_4 (15)$

Solution

Domain: $x > 2$

$$\log_4 x(x - 2) = \log_4 (15)$$

$$x^2 - 2x = 15$$

$$x^2 - 2x - 15 = 0$$

$$x = \frac{2 \pm \sqrt{4 + 60}}{2}$$

$$= \begin{cases} \frac{2 - 8}{2} = -4 < 2 \times \\ \frac{2 + 8}{2} = 5 > 2 \end{cases}$$

$\therefore$ ***Solution***: $x = 5$

Exercise

Solve the equation: $\ln(x-5)-\ln(x+4)=\ln(x-1)-\ln(x+2)$

Solution

Domain: $x>5$

$$\ln\frac{x-5}{x+4}=\ln\frac{x-1}{x+2}$$

$$\frac{x-5}{x+4}=\frac{x-1}{x+2}$$

$$(x-5)(x+2)=(x-1)(x+4)$$

$$x^2+2x-5x-10=x^2+4x-x-4$$

$$x^2-3x-10=x^2+3x-4$$

$$x^2-3x-10-x^2-3x+4=0$$

$$-6x-6=0$$

$$x=-1$$

$\therefore$ ***No solution***

Exercise

Solve the equation: $\log\left(x^2+4\right)-\log(x+2)=2+\log(x-2)$

Solution

Domain: $x>-2$

$$\log\left(x^2+4\right)-\log(x+2)-\log(x-2)=2$$

$$\log\left(x^2+4\right)-\left[\log(x+2)+\log(x-2)\right]=2$$

$$\log\left(x^2+4\right)-\log(x+2)(x-2)=2$$

$$\log\left(\frac{x^2+4}{x^2-4}\right)=2$$

$$\frac{x^2+4}{x^2-4}=10^2$$

$$x^2+4=100x^2-400$$

$$400+4=100x^2-x^2$$

$$99x^2=404$$

$$x^2=\frac{404}{99}$$

$\therefore$ **Solution**: $x=\frac{2\sqrt{101}}{3\sqrt{11}}$ is the only solution

Exercise

Solve the equation $\log_3(x-2) = \log_3 27 - \log_3(x-4) - 5^{\log_5 1}$

Solution

Domain: $x > 4$

$$\log_3(x-2) + \log_3(x-4) = \log_3 3^3 - 1$$

$$\log_3(x-2)(x-4) = 3 - 1$$

$$\log_3\left(x^2 - 6x + 8\right) = 2$$

$$x^2 - 6x + 8 = 3^2$$

$$x^2 - 6x + 8 = 9$$

$$x^2 - 6x - 1 = 0$$

$$\rightarrow \ \boxed{x = 3 \pm \sqrt{10}}$$

Check: $x = 3 + \sqrt{10} > 4$

$x = 3 - \sqrt{10} < 4$ ✗

$\therefore$ **Solution**: $\boxed{x = 3 + \sqrt{10}}$

Exercise

Solve the equation $\log_2(x+3) = \log_2(x-3) + \log_3 9 + 4^{\log_4 3}$

Solution

Domain: $x > 3$

$$\log_2(x+3) - \log_2(x-3) = 2 + 3$$

$$\log_2 \frac{x+3}{x-3} = 5$$

$$\frac{x+3}{x-3} = 2^5$$

$$x + 3 = 32(x-3)$$

$$x + 3 = 32x - 96$$

$$96 + 3 = 32x - x$$

$$31x = 99$$

$$x = \frac{99}{31} > 3$$

$\therefore$ **Solution**: $\boxed{x = \frac{99}{31}}$

Exercise

Solve the equation $\dfrac{10^x - 10^{-x}}{2} = 20$

Solution

$$\frac{10^x - 10^{-x}}{2} = 20$$

$$10^x - 10^{-x} = 40$$

$$10^x \times \quad 10^x - 40 - 10^{-x} = 0$$

$$\left(10^x\right)^2 - 40\left(10^x\right) - 1 = 0$$

$$\begin{aligned} 10^x &= \frac{40 \pm \sqrt{1604}}{2} \\ &= \frac{40 \pm 2\sqrt{401}}{2} \\ &= \begin{cases} 20 - \sqrt{401} < 0 \text{✗} \\ 20 + \sqrt{401} > 0 \end{cases} \end{aligned}$$

$$10^x = 20 + \sqrt{401}$$

$$\boxed{x = \log\left(20 + \sqrt{401}\right)}$$

Exercise

Solve the equation $\dfrac{10^x + 10^{-x}}{2} = 8$

Solution

$$10^x - 10^{-x} = 16$$

$$10^x \times \quad 10^x - 40 - 10^{-x} = 0$$

$$\left(10^x\right)^2 - 16\left(10^x\right) - 1 = 0$$

$$\begin{aligned} 10^x &= \frac{16 \pm \sqrt{260}}{2} \\ &= \frac{16 \pm 2\sqrt{65}}{2} \\ &= \begin{cases} 16 - \sqrt{65} < 0 \text{✗} \\ 16 + \sqrt{65} > 0 \end{cases} \end{aligned}$$

$$10^x = 16 + \sqrt{65}$$

$$\boxed{x = \log\left(16 + \sqrt{65}\right)}$$

Exercise

Solve the equation $\dfrac{10^x+10^{-x}}{10^x-10^{-x}}=5$

Solution

$$10^x+10^{-x}=5\left(10^x\right)-5\left(10^{-x}\right)$$

$$10^x\times\quad 4\left(10^x\right)=6\left(10^{-x}\right)$$

$$\left(10^x\right)^2=\frac{3}{2}$$

$$10^x=\pm\sqrt{\frac{3}{2}}$$

$$10^x=\sqrt{\frac{3}{2}} \qquad\qquad 10^x=-\sqrt{\frac{3}{2}}\ ✗$$

$$x=\log\left(\frac{3}{2}\right)^{1/2}$$

$$\boxed{=\frac{1}{2}\log\frac{3}{2}}$$

Exercise

Solve the equation $\dfrac{10^x+10^{-x}}{10^x-10^{-x}}=2$

Solution

$$10^x+10^{-x}=2\left(10^x\right)-2\left(10^{-x}\right)$$

$$10^x\times\quad\left(10^x\right)=3\left(10^{-x}\right)$$

$$\left(10^x\right)^2=3$$

$$10^x=\pm\sqrt{3}$$

$$10^x=\sqrt{3} \qquad\qquad 10^x=-\sqrt{3}\ ✗$$

∴ ***Solution***: $\boxed{x=\log\sqrt{3}}$

Exercise

Solve the equation $\dfrac{e^x+e^{-x}}{2}=15$

Solution

$$e^x+e^{-x}=30$$

$$e^x \times \quad e^x - 30 + e^{-x} = 0$$

$$\left(e^x\right)^2 - 30e^x + 1 = 0$$

$$e^x = \frac{30 \pm \sqrt{896}}{2}$$

$$= \frac{30 \pm 8\sqrt{14}}{2}$$

$$= 15 \pm 4\sqrt{14}$$

∴ **Solution**: $\underline{x = \ln\left(15 \pm 4\sqrt{14}\right)}$

Exercise

Solve the equation $\dfrac{e^x - e^{-x}}{2} = 15$

Solution

$$e^x - e^{-x} = 30$$

$$e^x \times \quad e^x - 30 - e^{-x} = 0$$

$$\left(e^x\right)^2 - 30e^x - 1 = 0$$

$$e^x = \frac{30 \pm \sqrt{904}}{2}$$

$$= \frac{30 \pm 2\sqrt{226}}{2}$$

$$15 - \sqrt{226} < 0$$

$$e^x = 15 + \sqrt{226}$$

∴ **Solution**: $\underline{x = \ln\left(15 + \sqrt{226}\right)}$

Exercise

Solve the equation $\dfrac{1}{e^x - e^{-x}} = 4$

Solution

$$4e^x - 4e^{-x} = 1$$

$$e^x \times \quad 4e^x - 1 - 4e^{-x} = 0$$

$$4\left(e^x\right)^2 - e^x - 4 = 0$$

$$e^x = \frac{1 \pm \sqrt{65}}{2}$$

∴ ***Solution***: $\boxed{x = \ln\left(\frac{1 \pm \sqrt{65}}{2}\right)}$

Exercise

Solve the equation $\dfrac{e^x + e^{-x}}{e^x - e^{-x}} = 3$

Solution

$$e^x + e^{-x} = 3e^x - 3e^{-x}$$

$$-2e^x = -4e^{-x}$$

$$e^x \times \quad e^x = 2e^{-x}$$

$$\left(e^x\right)^2 = 2$$

Since, e^x can't be negative, then

$$\boxed{e^x = \sqrt{2}}$$

∴ ***Solution***: $\boxed{x = \ln\sqrt{2}}$

Exercise

Solve the equation $\dfrac{e^x - e^{-x}}{e^x + e^{-x}} = 6$

Solution

$$e^x - e^{-x} = 6e^x + 6e^{-x}$$

$$-5e^x = 7e^{-x}$$

$$e^x \times \quad -5e^x = 7e^{-x}$$

$$\left(e^x\right)^2 = -\tfrac{7}{5} \times$$

∴ ***No Solution***

Exercise

Use common logarithms to solve for ***x*** in terms of ***y***: $y = \dfrac{10^x + 10^{-x}}{2}$

Solution

$$2y = 10^x + 10^{-x}$$

$$10^x\left(10^x\right) + 10^{-x}\left(10^x\right) - 2y\left(10^x\right) = 0$$

$$\left(10^x\right)^2 - 2y\left(10^x\right) + 1 = 0$$

Using the quadratic formula:

$$10^x = \frac{2y \pm \sqrt{(2y)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2y \pm \sqrt{4y^2 - 4}}{2}$$

$$= \frac{2y \pm 2\sqrt{y^2 - 1}}{2}$$

$$= y \pm \sqrt{y^2 - 1}$$

$y - \sqrt{y^2 - 1} > 0 \Rightarrow y > \sqrt{y^2 - 1} \Rightarrow y^2 > y^2 - 1$ (***True for any y* > 1**)

$y^2 - 1 \ge 0 \;\Rightarrow\; \cancel{y \le -1} \;\; or \;\; y \ge 1$

$10^x = y - \sqrt{y^2 - 1}$ $\qquad$ $10^x = y + \sqrt{y^2 - 1}$

$\boxed{x = \log\left(y - \sqrt{y^2 - 1}\right)}$ $\qquad$ $\boxed{x = \log\left(y + \sqrt{y^2 - 1}\right)}$

Exercise

Use common logarithms to solve for ***x*** in terms of ***y***: $y = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$

Solution

$$y\left(10^x + 10^{-x}\right) = 10^x - 10^{-x}$$

$$y10^x + y10^{-x} = 10^x - 10^{-x}$$

$$y10^x - 10^x = -10^{-x} - y10^{-x}$$

$$10^x(y-1) = -10^{-x}(1+y)$$

$$10^x 10^x(y-1) = -10^x 10^{-x}(1+y)$$

$$\left(10^x\right)^2(y-1) = -(1+y)$$

$$\left(10^x\right)^2 = -\frac{y+1}{y-1}$$

$$\left(10^x\right)^2 = \frac{y+1}{1-y}$$

$$10^x = \left(\frac{y+1}{1-y}\right)^{1/2}$$

$$\boxed{x = \log\left(\frac{y+1}{1-y}\right)^{1/2}}$$

Exercise

Use natural logarithms to solve for ***x*** in terms of ***y***: $y = \frac{e^x - e^{-x}}{2}$

Solution

$$2y = e^x - e^{-x}$$

$$2ye^x = e^x e^x - e^{-x}e^x$$

$$2ye^x = \left(e^x\right)^2 - 1$$

$$\left(e^x\right)^2 - 2ye^x - 1 = 0$$

$$e^x = \frac{2y \pm \sqrt{4y^2+4}}{2}$$

$$= \frac{2y \pm 2\sqrt{y^2+1}}{2}$$

$$\boxed{= y \pm \sqrt{y^2+1}}$$

$e^x = y - \sqrt{y^2+1} < 0$ (***not a solution***)

$$e^x = y + \sqrt{y^2+1}$$

$$\boxed{x = \ln\left(y + \sqrt{y^2+1}\right)}$$

Exercise

Use natural logarithms to solve for ***x*** in terms of ***y***: $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Solution

$$ye^x + ye^{-x} = e^x - e^{-x}$$

$$ye^{-x} + e^{-x} = e^{x} - ye^{x}$$

$$(y+1)e^{-x} = (1-y)e^{x}$$

$$(y+1)e^{-x}e^{x} = (1-y)e^{x}e^{x}$$

$$y+1 = (1-y)\left(e^{x}\right)^{2}$$

$$\left(e^{x}\right)^{2} = \frac{y+1}{1-y}$$

$$e^{x} = \pm\sqrt{\frac{y+1}{1-y}}$$

$$e^{x} = -\sqrt{\frac{y+1}{1-y}} < 0 \quad (\textbf{\textit{not a solution}})$$

$$e^{x} = \sqrt{\frac{y+1}{1-y}}$$

$$\boxed{x = \ln\sqrt{\frac{y+1}{1-y}}}$$

Exercise

Solve for t using logarithms with base a: $2a^{t/3} = 5$

Solution

$$a^{t/3} = \frac{5}{2}$$

$$\log a^{t/3} = \log\frac{5}{2}$$

$$\frac{t}{3}\log a = \log\frac{5}{2}$$

$$\frac{t}{3} = \frac{\log\frac{5}{2}}{\log a}$$

$$\frac{t}{3} = \log_{a}\frac{5}{2}$$

$$\boxed{t = 3\log_{a}\frac{5}{2}}$$

Exercise

Solve for t using logarithms with base a: $K = H - Ca^{t}$

Solution

$$Ca^t = H - K$$

$$a^t = \frac{H - K}{C}$$

$$\log a^t = \log \frac{H - K}{C}$$

$$t \log a = \log \frac{H - K}{C}$$

$$t = \frac{\log \frac{H-K}{C}}{\log a}$$

$$\boxed{= \log_a \frac{H-K}{C}}$$