Section 1.10 - Autonomous Equations and Stability

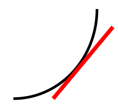
A first-order autonomous equation is an equation of the form

$$x' = f(x)$$

$$\frac{dy}{dx} = f(x, y)$$

Definition

The value $f(x, y)$ where the function f assigns to the point represent the slope of a line (line segment) call ***a lineal element***.

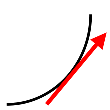


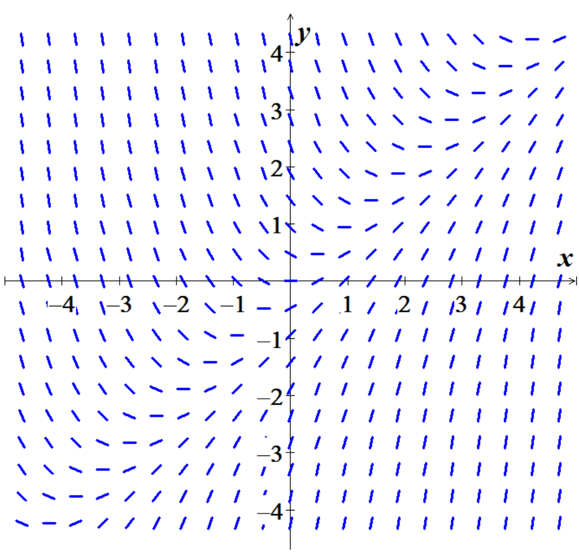
Example: Given $\frac{dy}{dx} = 0.2xy$ and consider the point $(2,\ 3)$

The slope of the lineal element is $\frac{dy}{dx} = 0.2xy = 0.2(2)(3) = 1.2$ (*positive sign*)

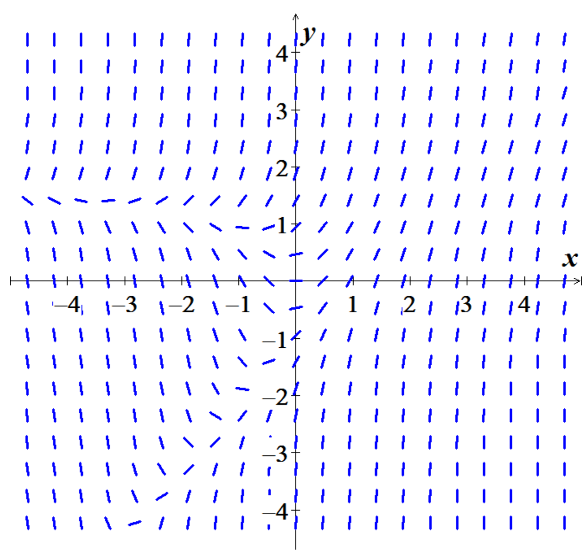
The Direction Fields

What we draw a lineal element at each point (x, y) with slope $f(x, y)$ then the collection of these lineal elements is called a ***direction field*** or a ***slope field*** of the differential equation $\frac{dy}{dx} = f(x, y)$.

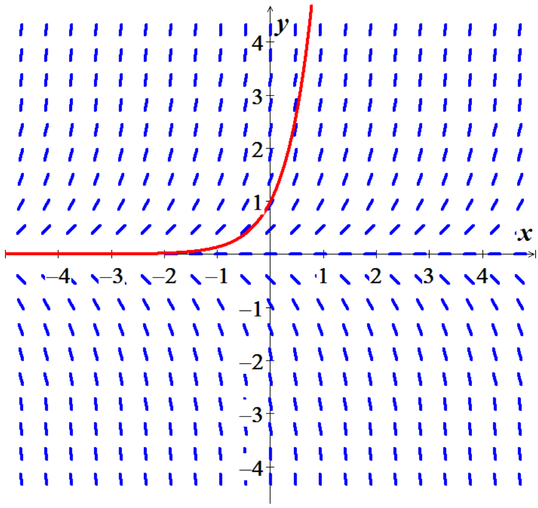


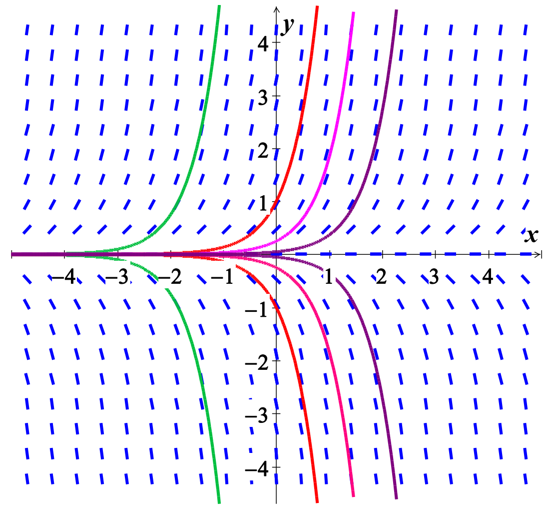


$y' = x - y$



$y' = y^2 - xy + 2x$





$y' = 2y, \quad \text{with } y(0) = 1 \quad \Rightarrow \quad y = e^{2x}$

Example

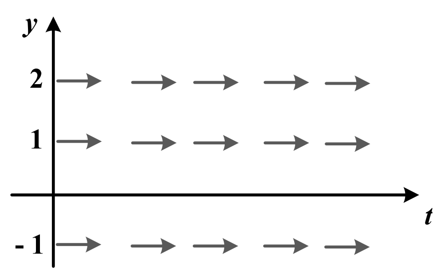
Sketch the direction field for the following differential equation. Sketch the set of integral curves for this differential equation, how the solutions behave as $t \to \infty$ and if this behavior depends on the value of $y(0)$ describe this dependency

$$y' = \left(y^2 - y - 2\right)(1 - y)^2$$

Solution

$y' = 0 \quad \Rightarrow \quad \left(y^2 - y - 2\right)(1 - y)^2 = 0$

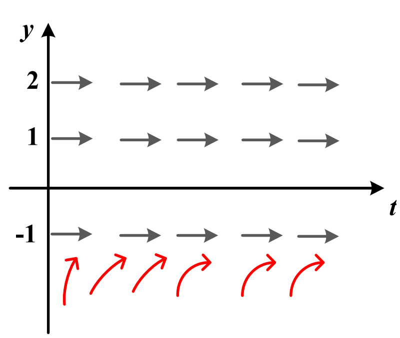
$y = \pm 1, \ 2$ *Slope of the tangent lines*



This divided into 4 regions.

For $y < -1$, assume $y = -2 \Rightarrow y' = \left(4^2 + 2 - 2\right)(1 + 2)^2 = 36 > \mathbf{0}$ $(\nearrow)$

$y = -1$, the slopes will flatten out while staying positive

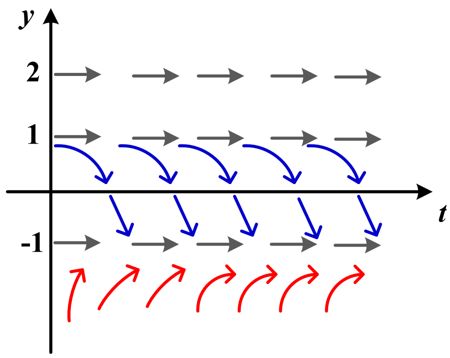


For $-1 < y < 1$, assume $y = 0 \Rightarrow y' = (-2)(1)^2 = -2 < \mathbf{0}$ $(\searrow)$

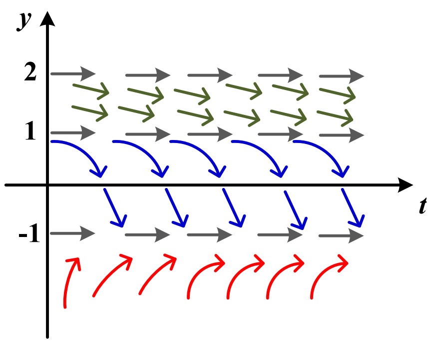
Therefore, tangent lines in this region will have negative slopes and apparently not very steep.

$y = .9 \quad \Rightarrow \quad y' = -.0209$

$y = -.9 \quad \Rightarrow \quad y' = -1.0469$ (Steeper)

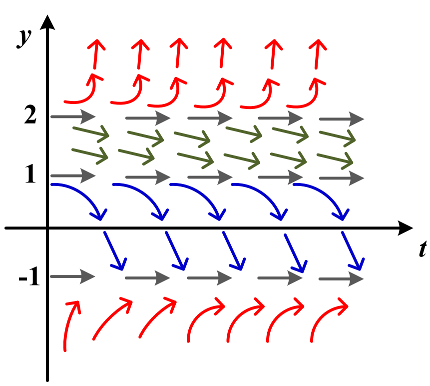


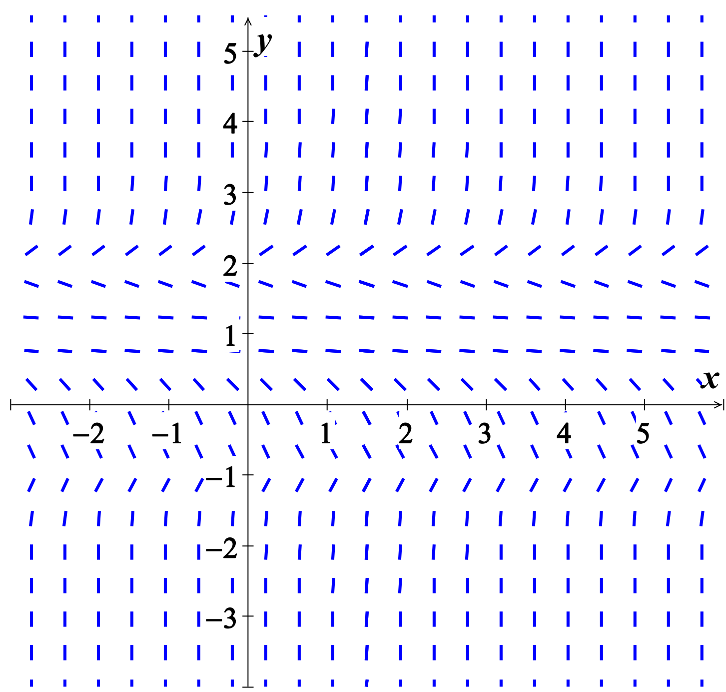
For $1 < y < 2$, assume $y = 1.5 \Rightarrow y' = \left(1.5^2 - 1.5 - 2\right)(-.5)^2 = -0.3125 < \mathbf{0}$ $(\searrow)$ Not to steep



For $y > 2$, assume $y = 3 \Rightarrow y' = (4)(-2)^2 = 16 > \mathbf{0}$ $(\nearrow)$

Start out fairly flat neary = 2, then will get fairly steep.





Value of $y(0)$	$t \to \infty$
$y(0) < -1$	$y \to -1$
$-1 \le y(0) < 2$	$y \to 1$
$y(0) = 2$	$y \to 2$
$y(0) > 2$	$y \to \infty$

Autonomous 1st order DE

A system $\dot{y} = rx - y - xz = 0$, which does not explicitly contain the independent variable t is called an ***autonomous system***. Otherwise, the system is called *non-autonomous system*.

Autonomous	*Not- Autonomous*
$x' = \sin x$	$x' = \sin(tx)$
$y' = y^2 + 1$	$y' = y^2 + t$
$z' = e^z$	$z' = t^2$

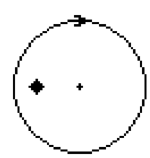
Equilibrium Points & Solutions

$x'(t) = 0 = f(x_0) \Rightarrow x_0$ *is an equilibrium point* and also called a ***critical point***.

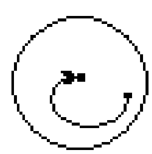
$x'(t) = x_0$ *called equilibrium solution*

From these equilibrium points, we can determine the stability of the system.

- An equilibrium point is ***stable*** if all nearby solutions stay nearby.



- An equilibrium point is ***asymptotically stable*** if all nearby solutions not only stay nearby, but also tend to the equilibrium point. An equilibrium point is stable if all nearby solutions.



1. If $f'(x_0) < 0$, then f is ***decreasing*** at x_0 and x_0 is asymptotically stable.
2. If $f'(x_0) > 0$, then f is ***increasing*** at x_0 and x_0 is unstable.
3. If $f'(x_0) = 0$, no conclusion can be drawn.

The family of all solution curves without the presence of the independent variable is called the ***phase portrait***.

When an independent variable ***t*** is interpreted as time and the solution curve $-P_+ < x < P_+$ could be thought of as the path of a particle moving in the solution space, then the system $f_\mu(x)$ is considered as a ***dynamical system***, where the solution curves are called ***trajectories*** or ***orbits***.

Example

Discover the behavior as $t \to \infty$ of all solutions to the differential equation

$$x' = f(x) = \left(x^2 - 1\right)(x - 2)$$

Solution

The equilibrium points: $f(x) = 0$

$\left(x^2 - 1\right)(x - 2) = 0$

$\Rightarrow x_1 = -1,\ x_2 = 1,\ x_3 = 2$ are equilibrium.

$f' = 3x^2 - 4x - 1$

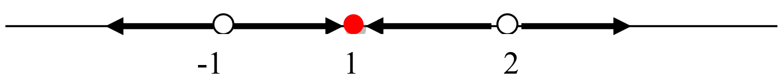
$f'(-1) = 3(-1)^2 - 4(-1) - 1 = 6 > 0$ ***unstable***

$f'(1) = 3(1)^2 - 4(1) - 1 = -2 < 0$ is ***asymptotically stable***

$f'(2) = 3(2)^2 - 4(2) - 1 = 3 > 0$ ***unstable***

$x(t) = -1,\ x(t) = 1,\ x(t) = 2$

These are constant functions, the position of the point the phase line modeled by them is also constant

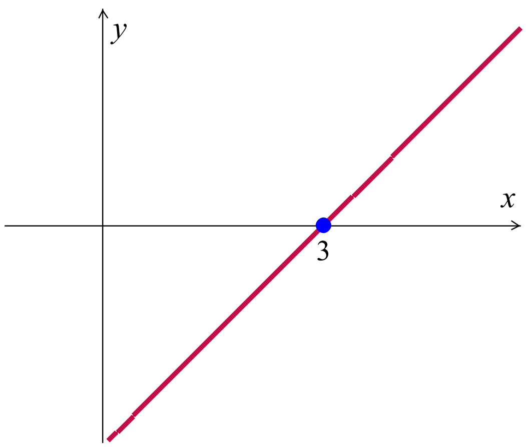


Phase Portrait

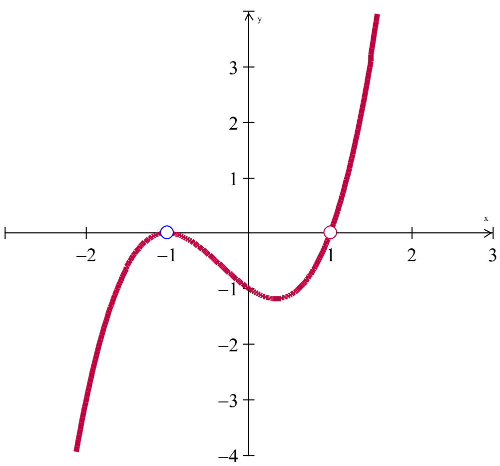
Exercises Section 1.10 - Autonomous Equations and Stability

The graph of the right-hand side $y' = f(y)$ is shown. Identify the equilibrium points and sketch the equilibrium solutions in the *ty*-plane. Classify each equilibrium point as either unstable or asymptotically stable.

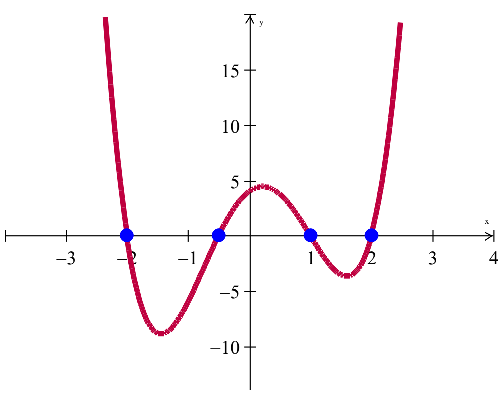
1.



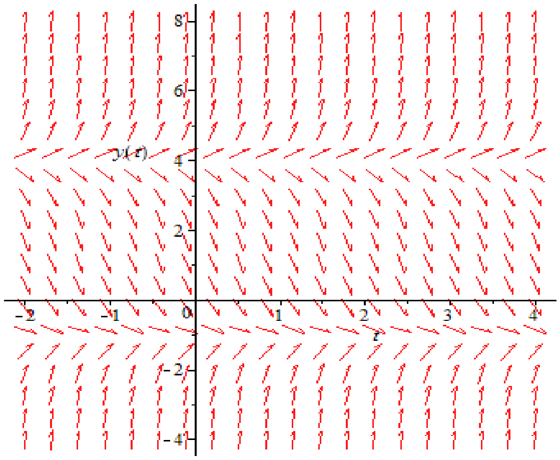
2.



3.



4. Impose the equilibrium solution(s), classifying each as either unstable or asymptotically stable



An autonomous differential equation is given. Perform each of the following exercises

a) Sketch a graph of $f(y)$

b) Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

c) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

5. $y' = 2 - y$

6. $y' = (y+1)(y-4)$

7. $y' = 9y - y^3$

8. $y' = \sin y$

9. $y' = y^2 - 3y$

10. $y' = y^2 - y^3$

11. $y' = (y-2)^4$

12. $y' = 10 + 3y - y^2$

13. $\dfrac{dy}{dt} = y^2\left(4 - y^2\right)$

14. $\dfrac{dy}{dt} = y(2-y)(4-y)$

15. $\dfrac{dy}{dt} = y\ln(y+2)$

16. $\dfrac{dy}{dt} = \dfrac{ye^y - 9y}{e^y}$

17. $y' = \dfrac{2}{\pi}y - \sin y$

18. $y' = 3y - ye^{y^2}$

19. $y' = (1-y)(y+1)^2$

20. $y' = \sin\dfrac{y}{2}$

Determine the stability of the equilibrium solutions

21. $x' = 4 - x^2$

22. $x' = x(x-1)(x+2)$

23. A tank contains 100 *gal* of pure water. A salt solution with concentration 3 *lb/gal* enters the tank at a rate of 2 *gal/min*. Solution drains from the tank at a rate of 2 *gal/min*. Use the qualitative analysis to find the eventual concentration of the salt in the tank.

24. A mathematical model for rate at which a drug disseminates into the bloodstream at time *t*.

$$\frac{dx}{dt} = r - kx$$

Where *r* and *k* are positive constants. The function $x(t)$ describes the concentration of the drug in the bloodstream at time *t*.

a) Since the *DE* is autonomous, use the phase portrait concept to find the limiting value of $x(t)$ as $t \to \infty$

b) Solve $x(t)$ subject to $x(0)=0$. Sketch the graph of $x(t)$ and verify your prediction in part (*a*). At what time is the concentration one-half this limiting value?

25. When forgetfulness is taken into account, the rate of memorization of a subject is given by

$$\frac{dA}{dt} = k_1 (M - A) - k_2 A$$

Where $k_1 > 0$, $k_2 > 0$, $A(t)$ is the amount memorized in time *t*, *M* is the total amount to be memorized, and $M - A$ is the amount remaining to be memorized.

a) Since the *DE* is autonomous, use the phase portrait concept to find the limiting value of $A(t)$ as $t \to \infty$. Interpret the result

b) Solve $A(t)$ subject to $A(0)=0$. Sketch the graph of $A(t)$ and verify your prediction in part (*a*).

26. The number $N(t)$ of supermarkets throughout the country that are using a computerized checkout system is described by the initial-value problem

$$\frac{dN}{dt} = N(1-0.0005N), \quad N(0)=1$$

a) Use the phase portrait concept to predict how many supermarkets are expected to adopt the new procedure over a long period of time. Sketch a solution curve of the given initial-value problem.

b) Solve the initial-value problem and then graph it to verify the solution in part (*a*)

c) How many companies are expected to adopt the new technology when $t = 10$?

27. For the linear ODE $\quad ty' + y = 2t$

a) Find all solution of the given *DE* equation.

b) Show that the initial value $y(0)=0$, has exactly one solution.

c) But if $y(0) = y_0 \neq 0$ there is no solution at all. Why doesn't this contradict the Existence and Uniqueness Theorem?

d) Plot several solutions of the *ODE* over the interval $-5 \le t \le 5$