Solution *Section* 1.5 – Introduction to Proofs

Exercise

Show that the square of an even number is an even number

Solution

We can rewrite the statement as: if n is even, then n^2 is even

Assume n is even, thus $n = 2k$ for some k.

$$n^2 = (2k)^2 = 4k^2 = 2\left(2k^2\right)$$

As n^2 is 2 times an integer, n^2 is thus even

Exercise

Prove that if n is an integer and $n^3 + 5$ is odd, then n is even

Solution

By indirect proof:

Using the contrapositive: If n is odd, then $n^3 + 5$ is even

Assume n is odd, let show that $n^3 + 5$ is even

$n = 2k + 1$ for some integer k (definition of odd numbers)

$$n^3 + 5 = (2k+1)^3 + 5 = 8k^3 + 12k^2 + 6k + 6 = 2\left(4k^3 + 6k^2 + 3k + 3\right)$$

As $n^3 + 5$ is 2 times an integer, it is even

Assume that $n^3 + 5$ is odd, let show that n is odd, and Assume p is true and q is false

$n = 2k + 1$ for some integer k (definition of odd numbers)

$$n^3 + 5 = (2k+1)^3 + 5 = 8k^3 + 12k^2 + 6k + 6 = 2\left(4k^3 + 6k^2 + 3k + 3\right)$$

As $n^3 + 5$ is 2 times an integer, it must be even. ***Contradiction***!

The indirect proof proved that the contrapositive: $\neg q \rightarrow \neg p$

If n is odd, then $n^3 + 5$ is even

The proof by contradiction assumed that the implication was false, and showed a contradiction

- If we assume p and $\neg q$, we can show that implies q
- The contradiction is q and $\neg q$
- Note that both used similar steps, but are different means of proving the implication

Exercise

Show that $m^2 = n^2$ if and only if $m = n \quad or \quad m = -n$

Solution

Rephrased: $m^2 = n^2 \leftrightarrow \left[(m = n) \vee (m = -n) \right]$. Proof by cases!

$Case$ 1: $(m = n) \rightarrow \left(m^2 = n^2 \right)$

$$(m)^2 = m^2 \quad and \quad (n)^2 = n^2$$, this case is proven.

$Case$ 1: $(m = -n) \rightarrow \left(m^2 = n^2 \right)$

$$(m)^2 = m^2 \quad and \quad (-n)^2 = n^2$$, this case is proven.

$$m^2 = n^2 \leftrightarrow \left[(m = n) \vee (m = -n) \right]$$

$$m^2 - n^2 = n^2 - n^2$$

$$m^2 - n^2 = 0 \quad \Rightarrow \quad (m - n)(m + n) = 0$$

$$m - n = 0 \quad or \quad m + n = 0$$

$$m = n \quad or \quad m = -n$$

Exercise

Use a direct proof to show that the sum of two odd integers is even.

Solution

Let m and n be two odd integers. Then there exists a and b such that $n = 2a + 1$ and $m = 2b + 1$.

$$\begin{aligned} n + m &= 2a + 1 + 2b + 1 \\ &= 2a + 2b + 2 \\ &= 2(a + b + 1) \end{aligned}$$

Since this represents $n + m$ as 2 times $a + b + 1$, we conclude that $n + m$ is even, as desired.

Exercise

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$$\begin{aligned} n + m &= 2a + 2b \\ &= 2(a + b) \end{aligned}$$

Since this represents $n + m$ as 2 times $a + b$, we conclude that $n + m$ is even, as desired.

Exercise

Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

Solution

Let r is a rational number an s is irrational number then $t = r + s$ is an irrational.

Suppose that t is rational, then if $t = \frac{a}{b}$ and $r = \frac{c}{d}$ where a, b, c, and d are integers with $b \neq 0$ and $d \neq 0$. Then, $t + (-r) = \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$ which is rational.

$t + (-r) = r + s - r = s$, forcing that s is rational. This contradicts the hypothesis that s is irrational.

Therefore the assumption that t was rational was incorrect, and we conclude that t is irrational.

Exercise

Prove or disprove that the product of two irrational numbers is irrational.

Solution

Let $\sqrt{2}$ be the irrational number,. If we take the product of the irrational number $\sqrt{2}$ and the irrational number $\sqrt{2}$, then we obtain the rational number 2. This counterexample refutes the proposition.

Exercise

Prove that if x is irrational, then $\frac{1}{x}$ is irrational.

Solution

The contrapositive is: if $\frac{1}{x}$ is rational, then x is rational.

Since $\frac{1}{x}$ exists, then $x \neq 0$. If $\frac{1}{x}$ is rational then by definition $\frac{1}{x} = \frac{q}{p}$ and $p \neq 0$. Since $\frac{1}{x}$ can't be zero, then we would have the contradiction $1 = x \cdot 0$.

Exercise

Prove that if x is rational and $x \neq 0$, then $\frac{1}{x}$ is rational.

Solution

if x is rational and $x \neq 0$, then by definition we can write $x = \frac{p}{q}$, where p and q are nonzero integers.

Since $\frac{1}{x} = \frac{q}{p}$ and $p \neq 0$, we can conclude that $\frac{1}{x}$ is rational.

Exercise

Prove the proposition $P(0)$, where $P(n)$ is the proposition "If *n* is a positive integer greater than 1, then $n^2 > n$." What kind of proof did you use?

Solution

The proposition that we are trying to prove is If 0 is a positive integer gr2ater than 1, then $0^2 = 0$. Our proof is a vacuous one.
Since the hypothesis is false, the implication is automatically true.

Exercise

Let $P(n)$ be the proposition "If *a* and *b* are positive real numbers, then $(a+b)^n \ge a^n + b^n$." Prove that $P(1)$ is true. What kind of proof did you use?

Solution

Our proof is a direct one. By the definition of exponential, any real number to the power 1 is itself. Hence $(a+b)^1 = a+b = a^1 + b^1$. Finally, by the addition rule, we can conclude from $(a+b)^1 = a^1 + b^1$ that $(a+b)^1 \ge a^1 + b^1$.

Exercise

Show that these statements about the integer *x* are equivalent:

$\boldsymbol{i})\ 3x+2\ \ is\ even \quad \boldsymbol{ii})\ x+5\ \ is\ odd \quad \boldsymbol{iii})\ x^2\ \ is\ even$

Solution

If *x* is even, then $x = 2k$ for some integer *k*.

$3x+2 = 3\cdot 2k + 2 = 6k + 2 = 2(3k+1)$ which is even.

$x+5 = 2k+4+1 = 2(k+2)+1$, so $x+5$ is odd

$x^2 = (2k)^2 = 4k^2 = 2\left(2k^2\right)$, so x^2 is odd

If *x* is odd, then $x = 2k+1$ for some integer *k*.

$3x+2 = 3\cdot(2k+1)+2 = 6k+3+2 = 6k+4+1 = 2(3k+2)+1$ which is odd *not* even.

$x+5 = 2k+1+5 = 2k+6 = 2(k+3)$, so $x+5$ is even not odd.

$x^2 = (2k+1)^2 = 4k^2+4k+1 = 2\left(2k^2+2k\right)+1$, so x^2 is odd

Exercise

Show that these statements about the real number x are equivalent:

$\boldsymbol{i})\ x$ *is irrational* $\quad \boldsymbol{ii})\ 3x+2$ *is irrational* $\quad \boldsymbol{iii})\ \frac{x}{2}$ *is irrational*

Solution

The simplest way is to approach in indirect proof.

$\boldsymbol{i}) \to \boldsymbol{ii})$

Suppose that $3x+2$ is rational, that $3x+2=\frac{p}{q}$ for some integers p and q with $q \neq 0$. Then

$3x=\frac{p}{q}-2=\frac{p-2q}{q} \quad \Rightarrow \quad x=\frac{p-2q}{3q}$ where $3q \neq 0$. This shows that x is rational.

Suppose that x is rational, that $x=\frac{p}{q}$ for some integers p and q with $q \neq 0$. Then

$3x+2=3\frac{p}{q}-2=\frac{3p-2q}{q}$ where $q \neq 0$. This shows that $3x+2$ is rational.

$\boldsymbol{i}) \to \boldsymbol{iii})$

Suppose that $\frac{x}{2}$ is rational, that $\frac{x}{2}=\frac{p}{q}$ for some integers p and q with $q \neq 0$. Then $x=\frac{2p}{q}$ where $q \neq 0$. This shows that x is rational.

Suppose that x is rational, that $x=\frac{p}{q}$ for some integers p and q with $q \neq 0$. Then $\frac{x}{2}=\frac{p}{2q}$ where $2q \neq 0$. This shows that $\frac{x}{2}$ is rational.

Exercise

Prove that at least one of the real numbers $a_1,\ a_2,\ \ldots,\ a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?

Solution

Using proof of contradiction, then suppose all the number $a_1,\ a_2,\ \ldots,\ a_n$ are less than their average.

$a_1 + a_2 + \ldots + a_n < nA$

By definition: $A=\frac{a_1 + a_2 + \ldots + a_n}{n}$

The two displayed formulas clearly contradict each other, however: they imply that $nA < nA$. Thus our assumption must have been incorrect, and we conclude that at least one of the numbers a_1 is greater than or equal to their average.