Section 3.8 – Hypothesis Tests for a Population Standard Deviation

Objective

Test a claim about a population standard deviation σ (or population variance σ^2) by using a formal method of hypothesis testing.

Notation

n = Sample size
s = sample standard deviation
s^2 = sample variance
σ = claimed value of the population standard deviation
σ^2 = claimed value of the population variance

Requirements for Testing Claims About a Population Mean (with σ Not Known)

1. The sample is a simple random sample.
2. The population has a normal distribution. (This is a much stricter requirement than the requirement of a normal distribution when testing claims about means.)

Chi-Square Distribution

If a simple random sample of size n is obtained from a normally distributed population with mean μ and standard deviation σ, then

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

where s^2 is a sample variance has a chi-square distribution with $n - 1$ degrees of freedom.

Properties of Chi-Square Distribution

1. It is ***not symmetric***.
2. The shape of the chi-square distribution depends on the degrees of freedom, just as with Student's t-distribution.
3. As the number of degrees of freedom increases, the chi-square distribution becomes more nearly symmetric.
4. The values of χ^2 are nonnegative (greater than or equal to 0).

Caution

The χ^2 test of this section is not *robust* against a departure from normality, meaning that the test does not work well if the population has a distribution that is far from normal. The condition of a normally distributed population is therefore a much stricter requirement in this section

Chi-Square Distribution Table

Chi-Square Distribution Table is based on cumulative areas from the right (unlike the entries in Standard Normal Distribution Table, which are cumulative areas from the left). Critical values are found in Chi-Square $\left(\chi^2\right)$ Distribution Table by first locating the row corresponding to the appropriate number of degrees of freedom (where df = n –1). Next, the significance level α is used to determine the correct column. The following examples are based on a significance level of α = 0.05, but any other significance level can be used in a similar manner.

Right-tailed test: Because the area to the right of the critical value is 0.05, locate 0.05 at the top of Chi-Square Distribution Table.

Left-tailed test: With a left-tailed area of 0.05, the area to the right of the critical value is 0.95, so locate 0.95 at the top of Chi-Square Distribution Table.

Two-tailed test: Unlike the normal and Student *t* distributions, the critical values in this χ^2 test will be two different positive values (instead of something like ± 1.96). Divide a significance level of 0.05 between the left and right tails, so the areas to the right of the two critical values are 0.975 and 0.025, respectively. Locate 0.975 and 0.025 at the top of Chi-Square Distribution Table

Testing Hypotheses about a Population Variance or Standard Deviation

To test hypotheses about the population variance or standard deviation, we can use the following steps, provided that

- The sample is obtained using simple random sampling.
- The population is normally distributed.

***Step* 1**: Determine the null and alternative hypotheses. The hypotheses can be structured in one of three ways:

Two–Tailed	***Left–Tailed***	***Right–Tailed***
$H_0: \sigma = \sigma_0$	$H_0: \sigma = \sigma_0$	$H_0: \sigma = \sigma_0$
$H_1: \sigma \neq \sigma_0$	$H_1: \sigma < \sigma_0$	$H_1: \sigma > \sigma_0$

σ_0 is assumed value of the population standard deviation.

***Step* 2:** Select a level of significance, α, based on the seriousness of making a Type I error.

Step* 3**: Compute the ***test statistic $\chi^2 = \dfrac{(n-1)s^2}{\sigma^2}$

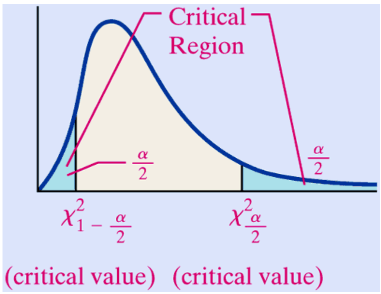
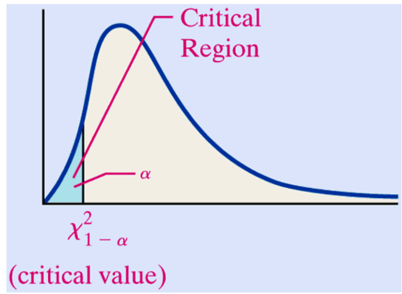
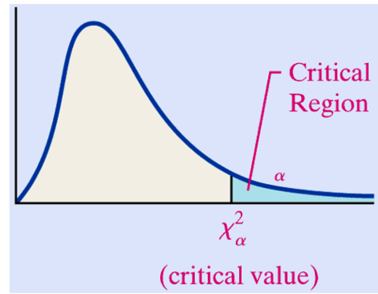
Classical Approach

Determine the critical value using n – 1 degrees of freedom.

<table>
<tr><th>Two–Tailed</th><th>Left–Tailed</th><th>Right–Tailed</th></tr>
<tr><td>$\chi_0^2 < \chi_{1-\alpha/2}^2$ or $\chi_0^2 > \chi_{\alpha/2}^2$

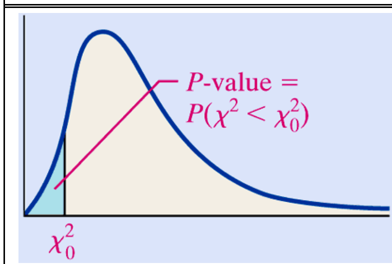
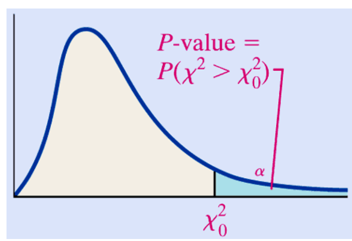
Reject the null hypothesis</td><td>$\chi_0^2 < \chi_{1-\alpha/2}^2$

Reject the null hypothesis</td><td>$\chi_0^2 > \chi_{\alpha/2}^2$

Reject the null hypothesis</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
</table>

P-Value Approach

To approximate the *P*-value for a left- or right-tailed test by determining the area under the chi-square distribution with $n - 1$ degrees of freedom to the left (for a left-tailed test) or right (for a right-tailed test) of the test statistic. For two-tailed tests, it is recommended that technology be used to find the *P*-value or obtain a confidence interval.

<table>
<tr><th>Left–Tailed</th><th>Right–Tailed</th></tr>
<tr><td>
</td><td>
</td></tr>
</table>

***Step* 4:** If the *P*-value $< \alpha$, reject the null hypothesis.

***Step* 5:** State the conclusion.

CAUTION!

The procedures in this section are not robust.

Therefore, if analysis of the data indicates that the variable does not come from a population that is normally distributed, the procedures presented in this section are not valid.

Example

A can of 7-Up states that the contents of the can are 355 *ml*. A quality control engineer is worried that the filling machine is miscalibrated. In other words, she wants to make sure the machine is not under- or over-filling the cans. She randomly selects 9 cans of 7-Up and measures the contents. She obtains the following data.

351	360	358	356	359	358	355	361	352

We assumed the population standard deviation is 3.2. Test the claim that the population standard deviation, σ, is greater than 3.2 ml at the α = 0.05 level of significance.

Solution

Step 1: $H_0 : \sigma = 3.2$ *vs.* $H_1 : \sigma \neq 3.2$ This is a right-tailed test.

Step 2: The level of significance is α = 0.05.

Step 3: The sample standard deviation is computed $s = 3.464$. The test statistic is

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \frac{(9-1)(3.464)^2}{3.2^2} = 9.374$$

Classical Approach

Step 4: Since this is a right-tailed test, we determine the critical value at the α = 0.05 level of significance with 9 – 1 = 8 degrees of freedom to be $\chi^2_{0.05} = 15.507$.

Step 5: Since the test statistic, $\chi^2_0 = 9.374$, is less than the critical value, 15.507, we fail to reject the null hypothesis.

P-Value Approach

Step 4: Since this is a right-tailed test, the *P*-value is the area under the χ^2 distribution with $9-1=8$ to the right of the test statistic $\chi^2_0 = 9.374$. That is

$$P-value = P\left(\chi^2_0 > 9.37\right) > 0.1$$

Step 5: Since the *P*-value is greater than the level of significance, we fail to reject the null hypothesis.

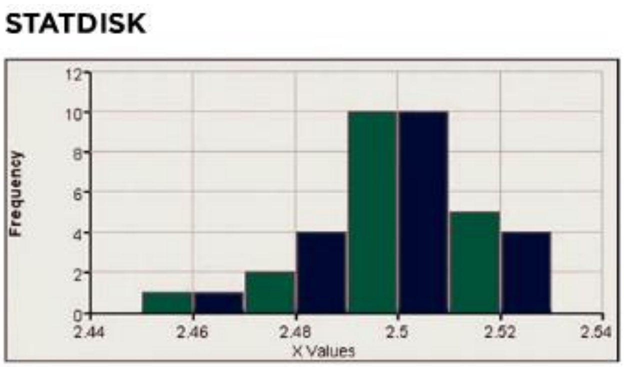
***Step* 6**: There is insufficient evidence at the α = 0.05 level of significance to conclude that the standard deviation of the can content of 7-Up is greater than 3.2 *ml*.

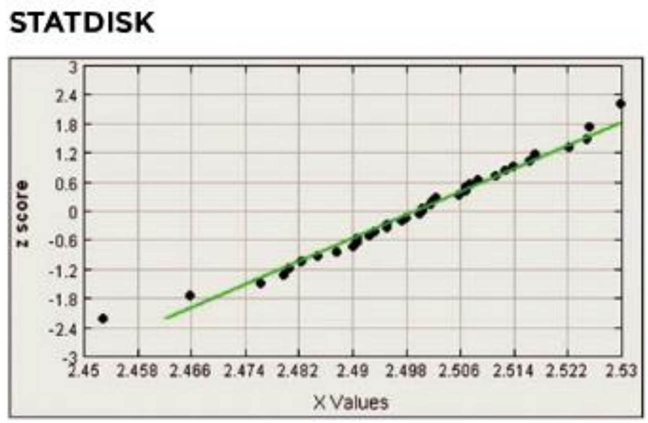
Example

A common goal in business and industry is to improve the quality of goods or services by reducing variation. Quality control engineers want to ensure that a product has an acceptable mean, but they also want to produce items of consistent quality so that there will be few defects. If weights of coins have a specified mean but too much variation, some will have weights that are too low or too high, so that vending machines will not work correctly (unlike the stellar performance that they now provide). Consider the simple random sample of the 37 weights of post-1983 pennies listed in Data Set 20 in Appendix B. Those 37 weights have a mean of 2.49910 g and a standard deviation of 0.01648 g. U.S. Mint specifications require that pennies be manufactured so that the mean weight is 2.500 g. A hypothesis test will verify that the sample appears to come from a population with a mean of 2.500 g as required, but use a 0.05 significance level to test the claim that the population of weights has a standard deviation less than the specification of 0.0230 g.

Solution

Requirements are satisfied: simple random sample; and STATDISK generated the histogram and quantile plot - sample appears to come from a population having a normal distribution.





Step 1: Express claim as $\sigma < 0.0230$ g

Step 2: If $\sigma < 0.0230$ g is false, then $\sigma \geq 0.0230$ g

Step 3: $\sigma < 0.0230$ g does not contain equality so it is the alternative hypothesis; null hypothesis is $\sigma = 0.0230$ g

H_0: $\sigma = 0.0230$ g

H_1: $\sigma < 0.0230$ g

Step 4: significance level is $\alpha = 0.05$

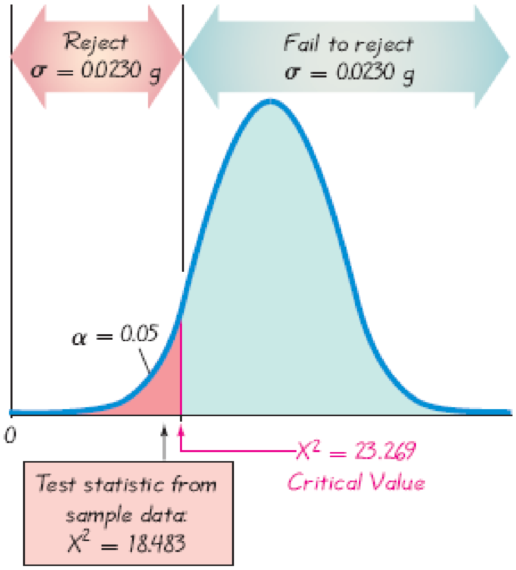
Step 5: Claim is about σ so use chi-square

Step 6: The test statistic is

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \frac{(37-1)(0.01648)^2}{0.0230^2} = 18.483$$

The critical value from Chi-Square $\left(\chi^2\right)$ Distribution Table corresponds to 36 degrees of freedom and an "area to the right" of 0.95 (based on the significance level of 0.05 for a left-tailed test). Chi-Square $\left(\chi^2\right)$ Distribution Table does not include 36 degrees of freedom, but Chi-Square $\left(\chi^2\right)$ Distribution Table shows that the critical value is between 18.493 and 26.509. (Using technology, the critical value is 23.269.)

Step 7: Because the test statistic is in the critical region, reject the null hypothesis.



✓ There is sufficient evidence to support the claim that the standard deviation of weights is less than 0.0230 g. It appears that the variation is less than 0.0230 g as specified, so the manufacturing process is acceptable.

Exercises Section 3.8 – Hypothesis Tests for a Population Standard Deviation

1. There is a claim that the lengths of men's hands have a standard deviation less than 200 mm. You plan to test that claim with a 0.01 significance level by constructing a confidence interval. What level of confidence should be used for the confidence interval? Will the conclusion based on the confidence interval be the same as the conclusion based on a hypothesis test that uses the traditional method or the *P*-value method?

2. There is a claim that daily rainfall amounts in Boston have a standard deviation equal to 0.25 in. Sample data show that daily rainfall amounts are from a population with a distribution that is very far from normal. Can the use of a very large sample compensate for the lack of normality, so that the methods of this section can be used for the hypothesis test?

3. There is a claim that men have foot breaths with a variance equal to 36 mm^2. Is a hypothesis test of the claim that the variance is equal to 36 mm^2 equivalent to a test of the claim that the standard deviation is equal to 6 *mm*.

4. Given: $H_1: \sigma \ne 696\ g, \quad \alpha = 0.05, \quad n = 25, \quad s = 645\ g$, Find

a) Find the test statistic
b) Find critical value(s)
c) Find *P*-value limits
d) Determine whether there is sufficient evidence to support the given alternative hypothesis.

5. Given: $H_1: \sigma < 29\ lb, \quad \alpha = 0.05, \quad n = 8, \quad s = 7.5\ lb$, Find

a) Find the test statistic
b) Find critical value(s)
c) Find *P*-value limits
d) Determine whether there is sufficient evidence to support the given alternative hypothesis.

6. Given: $H_1: \sigma > 3.5 \text{ min}, \quad \alpha = 0.01, \quad n = 15, \quad s = 4.8 \text{ min}$, Find

a) Find the test statistic
b) Find critical value(s)
c) Find *P*-value limits
d) Determine whether there is sufficient evidence to support the given alternative hypothesis.

7. Given: $H_1: \sigma \ne 0.25, \quad \alpha = 0.01, \quad n = 26, \quad s = 0.18$, Find

a) Find the test statistic
b) Find critical value(s)
c) Find *P*-value limits
d) Determine whether there is sufficient evidence to support the given alternative hypothesis.

8. A simple random sample of 40 men results in a standard deviation of 11.3 beats per minute. The normal range of pulse rates of adults is typically given as 60 to 100 beats per minute. If the range rule of thumb is applied to that normal range, the result is a standard deviation of 10 beats per minute. Use the sample results with a 0.05 significance level to test the claim that rates of men have a standard deviation greater than 10 beats per minute.

9. A simple random sample of 25 filtered 100 mm cigarettes is obtained, and the tar content of each cigarette is measured. The sample has a standard deviation of 3.7 mg. Use a 0.05 significance level to test the claim that the tar content of filtered 100 mm cigarettes has a standard deviation different from 3.2 mg, which is the standard deviation for unfiltered king size cigarettes.

10. When 40 people used the Weight Watchers diet for one year, their weight losses had a standard deviation of 4.9 lb. Use 0.01 significance level to test the claim that the amounts of weight loss have a standard deviation equal to 6.0 lb., which appears to be the standard deviation for the amounts of weight loss with the Zone diet.

11. Tests in the statistic classes have scores with a standard deviation equal to 14.1. One of the last classes has 27 test scores with a standard deviation of 9.3. Use a 0.01 significance level to test the claim that this class has less variation than other past classes. Does a lower standard deviation suggest that this last class is doing better?

12. A simple random sample of pulse rates of 40 women results in a standard deviation of 12.5 *beats/min*. The normal range of pulse rates of adults is typically given as 60 to 100 *beats/min*. If the range rule of thumb is applied to that normal range, the result is a standard deviation of 10 *beats/min*. Use the sample results with a 0.05 significance level to test the claim that pulse rates of women have a standard deviation equal to 10 *beats/min*.

13. Listed below are the playing times (in seconds) of songs that were popular at the time of this writing. Use a 0.05 significance level to test the claim that the songs are from a population with a standard deviation less than on minute.

448 242 231 246 246 293 280 227 244 213 262 239 213 258 255 257

14. Find the critical value or values of χ^2 based on the given information

a) $H_0: \sigma = 8.0, \quad \alpha = 0.01, \quad n = 10$

b) $H_1: \sigma > 3.5, \quad \alpha = 0.05, \quad n = 14$

c) $H_1: \sigma < 0.14, \quad \alpha = 0.10, \quad n = 23$

d) $H_1: \sigma \neq 9.3, \quad \alpha = 0.05, \quad n = 28$