Section 4.3 – Eigenvalue Method for Linear System

A homogeneous first-order system with constant coefficients is given by

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \ \dots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \ \dots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ \dots + a_{nn}x_n \end{aligned}$$

We can find *n* linear independent solution vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ and the linear combination

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \ \dots \ + c_n\vec{x}_n$$

We apply the characteristics root method for solving a single homogeneous equation with constant coefficients.

$$\vec{x}(t) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \\ \vdots \\ v_n e^{\lambda t} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} e^{\lambda t} = \vec{v} e^{\lambda t}$$

Theorem

Let λ be an eigenvalue of the constant coefficient matrix *A* of the first-order linear system

$$\frac{dx}{dt} = Ax$$

If $\vec{v}$ is an eigenvector associated with λ, then

$$\vec{x}(t) = \vec{v} e^{\lambda t} \qquad \vec{v} \neq \vec{0}$$

is a nontrivial solution of the system

If $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct eigenvalues of *A* with corresponding $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, then

$$\vec{x}_1 = e^{\lambda_1 t}\vec{v}_1, \quad \vec{x}_2 = e^{\lambda_2 t}\vec{v}_2, \quad \dots, \quad \vec{x}_n = e^{\lambda_k t}\vec{v}_n$$

form a fundamental set of solutions of $\vec{x}' = A\vec{x}$

And $\vec{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 + \ \dots \ + C_n\vec{x}_n$ is the general solution.

Note

- Recall that an eigenvalue λ of the matrix *A* is a solution of the characteristic equation $|A - \lambda I| = 0$
- An eigenvector $\vec{v}$ associated with λ is then a solution of the eigenvector equation $(A - \lambda I)\vec{v} = 0$

Distinct Real *Eigenvalues*

Examples

Find a general solution of the system

$$\begin{cases} x_1' = 4x_1 + 2x_2 \\ x_2' = 3x_1 - x_2 \end{cases}$$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda) - 6$$

$$= \lambda^2 - 3\lambda - 10 = 0$$

The distinct real eigenvalues: $\lambda_1 = -2, \quad \lambda_2 = 5$

For $\lambda_1 = -2 \quad \Rightarrow \quad (A+2I)V_1 = 0$

$$\begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 3x_1 + y_1 = 0 \ \rightarrow \ y_1 = -3x_1$$

$$\rightarrow V_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-2t}$$

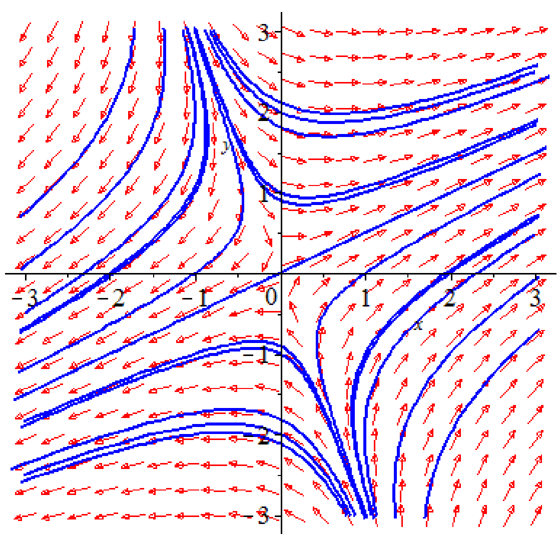
For $\lambda_2 = 5 \quad \Rightarrow \quad (A-5I)V_2 = 0$

$$\begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad -x_2 + 2y_2 = 0 \ \rightarrow \ x_2 = 2y_2$$

$$\rightarrow V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \Rightarrow \quad x_2(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}$$

$$x_1(t) = \begin{pmatrix} e^{-2t} \\ -3e^{-2t} \end{pmatrix} \qquad x_2(t) = \begin{pmatrix} 2e^{5t} \\ e^{5t} \end{pmatrix}$$

Using Wronskian: $\begin{vmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{vmatrix} = 7e^{3t} \neq 0$

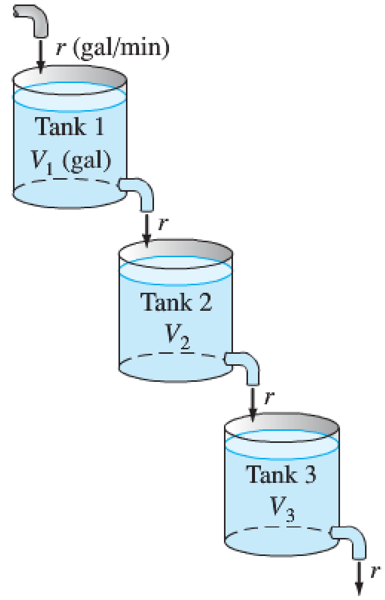


The general solution: $x(t) = C_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}$

OR $\begin{cases} x_1(t) = C_1 e^{-2t} + 2C_2 e^{5t} \\ x_2(t) = -3C_1 e^{-2t} + C_2 e^{5t} \end{cases}$

Examples

If $V_1 = 20$ *gal*, $V_2 = 40$ *gal*, $V_3 = 50$ *gal*, $r = 10$ *gal* / min and the initial amounts of salt in 3 brine tanks, in lbs, are $x_1(0) = 15 \quad x_2(0) = x_3(0) = 0$. Find the amount of salt in each tank at time $t \ge 0$.



Solution

$$\begin{cases} x_1' = -k_1 x_1 \\ x_2' = k_1 x_1 - k_2 x_2 \\ x_3' = \qquad k_2 x_2 - k_3 x_3 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i = 1, 2, 3$$

$$k_1 = \frac{10}{20} = .5 \quad k_2 = \frac{10}{40} = .25 \quad k_3 = \frac{10}{50} = .2$$

$$\begin{cases} x_1' = -.5x_1 \\ x_2' = .5x_1 - .25x_2 \\ x_3' = \qquad .25x_2 - .2x_3 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -.5 & 0 & 0 \\ .5 & -.25 & 0 \\ 0 & .25 & -.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \textit{with } x(0) = \begin{pmatrix} 15 \\ 0 \\ 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -.5-\lambda & 0 & 0 \\ .5 & -.25-\lambda & 0 \\ 0 & .25 & -.2-\lambda \end{vmatrix}$$

$$= (-.5-\lambda)(-.25-\lambda)(-.2-\lambda) = 0$$

The eigenvalues are: $\lambda_1 = -.5 \quad \lambda_2 = -.25 \quad \lambda_3 = -.2$

For $\lambda_1 = -.5 \quad \Rightarrow \quad (A + .5I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ .5 & .25 & 0 \\ 0 & .25 & .3 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} .5a_1 + .25b_1 = 0 \ \rightarrow \ 2a_1 = -b_1 \\ .25b_1 + .3c_1 = 0 \ \rightarrow \ 6c_1 = -5b_1 \end{cases}$$

$$\rightarrow V_1 = \begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix} e^{-.5t}$$

For $\lambda_2 = -.25 \quad \Rightarrow \quad (A + .25I)V_2 = 0$

$$\begin{pmatrix} -.25 & 0 & 0 \\ .5 & 0 & 0 \\ 0 & .25 & .05 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} a_2 = 0 \\ .25b_2 + .05c_2 = 0 \ \rightarrow \ c_2 = -5b_2 \end{cases}$$

$$\to V_2 = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} \quad \Rightarrow \quad x_2(t) = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} e^{-.25t}$$

For $\lambda_3 = -.2 \quad \Rightarrow \quad (A + .2I)V_3 = 0$

$$\begin{pmatrix} -.3 & 0 & 0 \\ .5 & -.05 & 0 \\ 0 & .25 & 0 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} a_3 = 0 \\ b_3 = 0 \\ 0c_3 = 0 \to c_3 = 1 \end{cases}$$

$$\to V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad x_3(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-.2t}$$

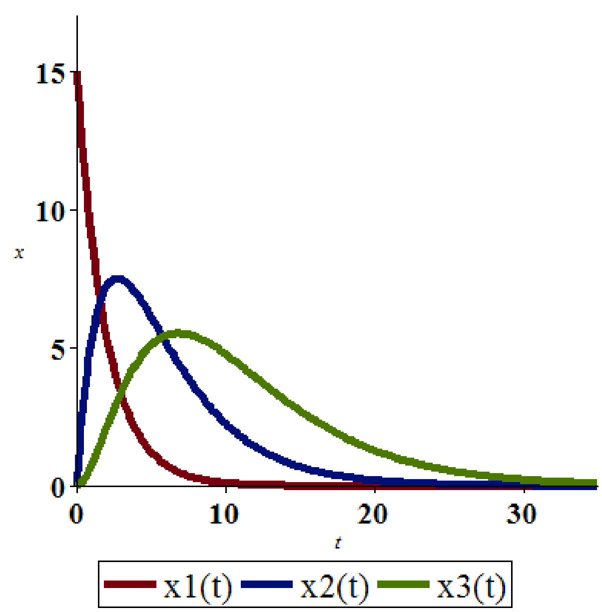
$$\Rightarrow \; x(t) = C_1 \begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix} e^{-.5t} + C_2 \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} e^{-.25t} + C_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-.2t}$$

$$\begin{cases} x_1(t) = 3C_1 e^{-.5t} \\ x_2(t) = -6C_1 e^{-.5t} + C_2 e^{-.25t} \\ x_3(t) = 5C_1 e^{-.5t} - 5C_2 e^{-.25t} + C_3 e^{-.2t} \end{cases}$$

With *initial* values

$$\begin{cases} 15 = 3C_1 \\ 0 = -6C_1 + C_2 \\ 0 = 5C_1 - 5C_2 + C_3 \end{cases} \to \begin{cases} \boxed{5 = C_1} \\ \boxed{C_2 = 30} \\ \boxed{C_3 = -5(5) + 5(30) = 125} \end{cases}$$

$$\begin{cases} x_1(t) = 15e^{-.5t} \\ x_2(t) = -30e^{-.5t} + 30e^{-.25t} \\ x_3(t) = 25e^{-.5t} - 150e^{-.25t} + 125e^{-.2t} \end{cases}$$



Complex *Eigenvalues*

Examples

Find a general solution of the system

$$\begin{cases} x_1' = 4x_1 - 3x_2 \\ x_2' = 3x_1 + 4x_2 \end{cases}$$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 + 9 = 0$$

$$(4-\lambda)^2 = -9 \quad \Rightarrow \quad 4-\lambda = \pm 3i$$

The distinct real eigenvalues: $\lambda_{1,2} = 4 \pm 3i$

For $\lambda_1 = 4-3i \quad \Rightarrow \quad \big(A-(4-3i)I\big)V = 0$

$$\begin{pmatrix} 3i & -3 \\ 3 & 3i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 3ia - 3b = 0 \ \rightarrow \ b = ia \quad \rightarrow V = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

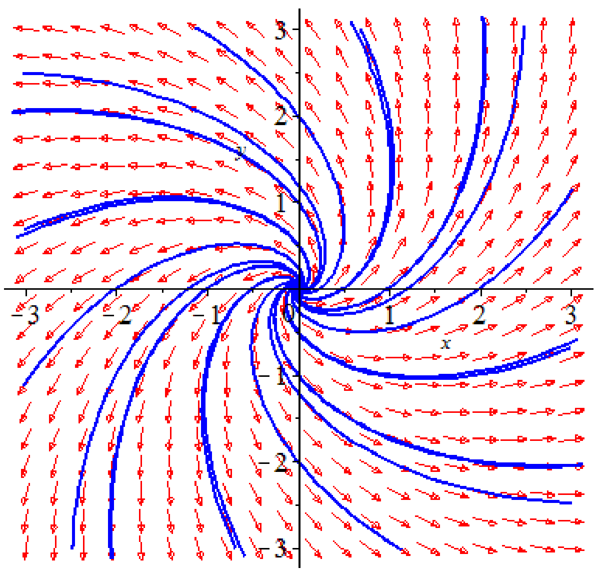
$$x(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(4-3i)t}$$

$$= \begin{pmatrix} 1 \\ i \end{pmatrix} e^{4t} e^{-3it}$$

$$e^{ait} = \cos at + i \sin at$$

$$= \begin{pmatrix} 1 \\ i \end{pmatrix} e^{4t} (\cos 3t - i \sin 3t)$$

$$= \begin{pmatrix} \cos 3t - i \sin 3\text{t} \\ i \cos 3t + \sin 3t \end{pmatrix} e^{4t}$$



$$\vec{x}_1(t) = \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} e^{4t} \quad \vec{x}_2(t) = \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix} e^{4t}$$

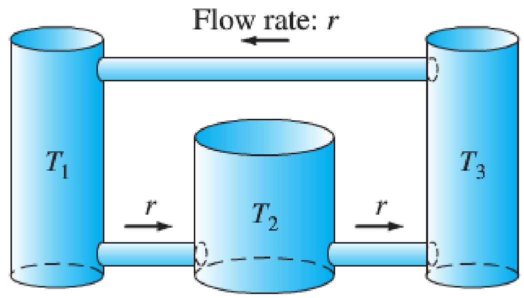
$$x(t) = C_1 \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix} e^{4t}$$

$$\begin{cases} x_1(t) = \left(C_1 \cos 3t - C_2 \sin 3t\right) e^{4t} \\ x_2(t) = \left(-C_1 \sin 3t + C_2 \cos 3t\right) e^{4t} \end{cases}$$

Examples

If $V_1 = 50\ gal,\quad V_2 = 25\ gal,\quad V_3 = 50\ gal,\quad r = 10\ gal/\min$, find the amount $x_1(t),\ x_2(t),\ x_3(t)$ of salt in each tank at time $t \ge 0$

Solution



$$\begin{cases} x_1' = -k_1x_1 \qquad + k_3x_3 \\ x_2' = k_1x_1 - k_2x_2 \\ x_3' = \qquad k_2x_2 - k_3x_3 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i = 1,2,3$$

$$k_1 = \frac{10}{50} = .2 \quad k_1 = \frac{10}{25} = .4 \quad k_1 = \frac{10}{50} = .2$$

$$\begin{cases} x_1' = -.2x_1 \qquad + .2x_3 \\ x_2' = .2x_1 - .4x_2 \\ x_3' = \qquad .4x_2 - .2x_3 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -.2 & 0 & .2 \\ .2 & -.4 & 0 \\ 0 & .4 & -.2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -.2-\lambda & 0 & .2 \\ .2 & -.4-\lambda & 0 \\ 0 & .4 & -.2-\lambda \end{vmatrix}$$

$$= (-.2-\lambda)(-.4-\lambda)(-.2-\lambda) + (.2)(.2)(.4)$$

$$= -\lambda^3 - .8\lambda^2 - .2\lambda$$

$$= -\lambda\left(\lambda^2 + .8\lambda + .2\right) = 0 \qquad\qquad \lambda^2 + .8\lambda + .2 = 0 \quad \lambda = \frac{-.8 \pm \sqrt{.64 - .8}}{2} = -.4 \pm .2i$$

The eigenvalues are: $\underline{\lambda_1 = 0 \quad \lambda_{2,3} = -.4 \pm .2i}|$

For $\lambda_1 = 0 \quad \Rightarrow \quad (A - 0I)V_1 = 0$

$$\begin{pmatrix} -.2 & 0 & .2 \\ .2 & -.4 & 0 \\ 0 & .4 & -.2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} -.2a + .2c = 0 \ \rightarrow \ a = c \\ .2a - .4b = 0 \ \rightarrow \ a = 2b \end{cases}$$

$$\rightarrow V_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

For $\lambda = -.4 - .2i \quad \Rightarrow \quad \big(A + (.4 + .2i)\big)V_2 = 0$

$$\begin{pmatrix} .2+.2i & 0 & .2 \\ .2 & .2i & 0 \\ 0 & .4 & .2+.2i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} (.2+.2i)a=-.2c \\ .2a=-.2ib \end{cases}$$

Let $b=i \quad \Rightarrow a=1 \quad c=-1-i$

$$\rightarrow V_2=\begin{pmatrix} 1 \\ i \\ -1-i \end{pmatrix} \quad \Rightarrow \quad x_{2,3}(t)=\begin{pmatrix} 1 \\ i \\ -1-i \end{pmatrix} e^{-.4t}e^{-.2ti}$$

$$x_{2,3}(t)=\begin{pmatrix} 1 \\ i \\ -1-i \end{pmatrix} e^{-.4t}\left(\cos(.2t)-i\sin(.2t)\right)$$

$$=\begin{pmatrix} \cos .2t - i\sin .2t \\ \sin .2t + i\cos .2t \\ -\cos .2t - \sin .2t - i(\cos .2t - \sin .2t) \end{pmatrix} e^{-.4t}$$

$$x_1(t)=\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad x_2(t)=\begin{pmatrix} \cos .2t \\ \sin .2t \\ -\cos .2t - \sin .2t \end{pmatrix} e^{-.4t} \quad x_3(t)=\begin{pmatrix} -\sin .2t \\ \cos .2t \\ \sin .2t - \cos .2t \end{pmatrix} e^{-.4t}$$

$$\begin{cases} x_1(t)=2C_1+\left(C_2\cos 0.2t - C_3\sin 0.2t\right)e^{-.4t} \\ x_2(t)=C_1+\left(C_2\sin 0.2t + C_3\cos 0.2t\right)e^{-.4t} \\ x_3(t)=2C_1+\left(\left(-C_2-C_3\right)\cos 0.2t+\left(C_3-C_2\right)\sin 0.2t\right)e^{-.4t} \end{cases}$$

Exercises *Section* 4.3 – Eigenvalue Method for Linear System

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system.

1. $x_1' = x_1 + 2x_2, \quad x_2' = 2x_1 + x_2$

2. $x_1' = 2x_1 + 3x_2, \quad x_2' = 2x_1 + x_2$

3. $x_1' = 6x_1 - 7x_2, \quad x_2' = x_1 - 2x_2$

4. $x_1' = -3x_1 + 4x_2, \quad x_2' = 6x_1 - 5x_2$

5. $x_1' = x_1 - 5x_2, \quad x_2' = x_1 - x_2$

6. $x_1' = -3x_1 - 2x_2, \quad x_2' = 9x_1 + 3x_2$

7. $x_1' = x_1 - 5x_2, \quad x_2' = x_1 + 3x_2$

8. $x_1' = 5x_1 - 9x_2, \quad x_2' = 2x_1 - x_2$

9. $x_1' = 3x_1 + 4x_2, \quad x_2' = 3x_1 + 2x_2; \quad x_1(0) = x_2(0) = 1$

10. $x_1' = 9x_1 + 5x_2, \quad x_2' = -6x_1 - 2x_2; \quad x_1(0) = 1, \ x_2(0) = 0$

11. $x_1' = 2x_1 - 5x_2, \quad x_2' = 4x_1 - 2x_2; \quad x_1(0) = 2, \ x_2(0) = 3$

12. $x_1' = x_1 - 2x_2, \quad x_2' = 2x_1 + x_2; \quad x_1(0) = 0, \ x_2(0) = 4$

Find the general solution of the given system.

13. $x_1' = 4x_1 + x_2 + 4x_3, \quad x_2' = x_1 + 7x_2 + x_3, \quad x_3' = 4x_1 + x_2 + 4x_3$

14. $x_1' = x_1 + 2x_2 + 2x_3, \quad x_2' = 2x_1 + 7x_2 + x_3, \quad x_3' = 2x_1 + x_2 + 7x_3$

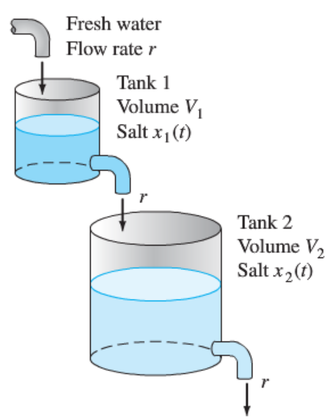
15. $x_1' = 4x_1 + x_2 + x_3, \quad x_2' = x_1 + 4x_2 + x_3, \quad x_3' = x_1 + x_2 + 4x_3$

16. $x_1' = 5x_1 + x_2 + 3x_3, \quad x_2' = x_1 + 7x_2 + x_3, \quad x_3' = 3x_1 + x_2 + 5x_3$

17. $x_1' = 5x_1 - 6x_3, \quad x_2' = 2x_1 - x_2 - 2x_3, \quad x_3' = 4x_1 - 2x_2 - 4x_3$

18. $x_1' = 3x_1 + 2x_2 + 2x_3, \quad x_2' = -5x_1 - 4x_2 - 2x_3, \quad x_3' = 5x_1 + 5x_2 + 3x_3$

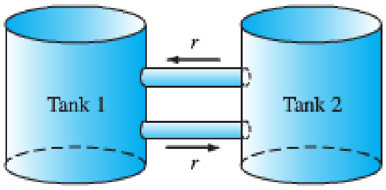
Find the amount $x_1(t), x_2(t)$ of salt in each tank at time $t \ge 0$, with $x_1(0) = 15\ lb \quad x_2(0) = 0$. If



19. $V_1 = 50\ gal, \quad V_2 = 25\ gal, \quad r = 10\ gal/\min$

20. $V_1 = 25\ gal, \quad V_2 = 40\ gal, \quad r = 10\ gal/\min$

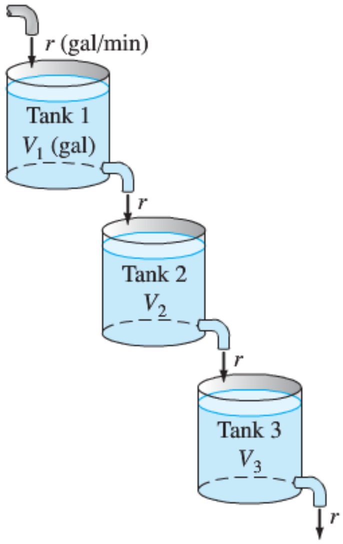
Find the amount $x_1(t),\ x_2(t)$ of salt in each tank at time $t \ge 0$, with $x_1(0) = 15\ lb \quad x_2(0) = 0$. If



21. $V_1 = 50\ gal, \quad V_2 = 25\ gal, \quad r = 10\ gal\ /\ \min$

22. $V_1 = 25\ gal, \quad V_2 = 40\ gal, \quad r = 10\ gal\ /\ \min$

Find the amount $x_1(t),\ x_2(t),\ x_3(t)$ of salt in each tank at time $t \ge 0$, if



23. $V_1 = 30\ gal, \quad V_2 = 15\ gal, \quad V_3 = 10\ gal, \quad r = 30\ gal\ /\ \min$

$x_1(0) = 27\ lb \quad x_2(0) = x_3(0) = 0$

24. $V_1 = 20\ gal, \quad V_2 = 30\ gal, \quad V_3 = 60\ gal, \quad r = 60\ gal\ /\ \min$

$x_1(0) = 45\ lb \quad x_2(0) = x_3(0) = 0$

25. $V_1 = 15\ gal, \quad V_2 = 10\ gal, \quad V_3 = 30\ gal, \quad r = 60\ gal\ /\ \min$

$x_1(0) = 45\ lb \quad x_2(0) = x_3(0) = 0$