Lecture Two – Partial Derivatives

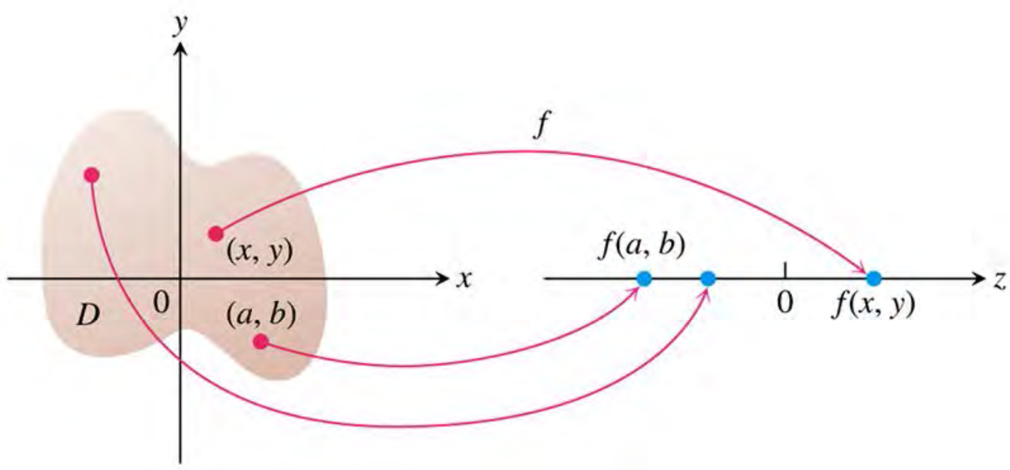
Section 2.1 – Graphs and Level Curves

Definitions

Suppose D is a set of n-tuples of real numbers $\left(x_1,\ x_2,\ \ldots,\ x_n\right)$. A ***real-valued function*** f on D is a rule that assigns a unique (single) real number

$$w = f\left(x_1,\ x_2,\ \ldots,\ x_n\right)$$

To each element in D. The set D is the function's ***domain***. The set of w-values taken on by f is the function's ***range***. The symbol w is the ***dependent variable*** of f, and f is said to be a function of the n ***independent variables*** x_1 to x_n. We also call the x_j's the function's ***input variables*** and call w the function's ***output variable***.



Domains and Ranges

Functions of two variables

Function	***Domain***	***Range***
$z = \sqrt{y - x^2}$	$y \ge x^2$	$\left[0,\ \infty\right)$
$z = \dfrac{1}{xy}$	$xy \ne 0$	$\left(-\infty,\ 0\right) \cup \left(0,\ \infty\right)$
$z = \sin xy$	*Entire plane*	$\left[-1,\ 1\right]$

Functions of three variables

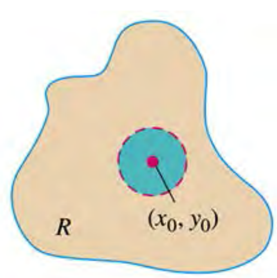
Function	***Domain***	***Range***
$w=\sqrt{x^2+y^2+z^2}$	*Entire plane*	$[0,\ \infty)$
$w=\dfrac{1}{x^2+y^2+z^2}$	$(x,\ y,\ z)\neq(0,\ 0,\ 0)$	$(0,\ \infty)$
$w=xy\ln z$	*Half – space* $z>0$	$(-\infty,\ \infty)$

Functions of Two Variables

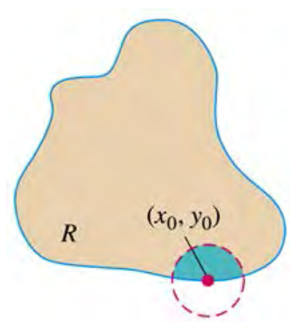
Definitions

A point $(x_0,\ y_0)$ in a region (set) R in the xy-plane is an ***interior point*** of R if it is the center of a disk of positive radius that lies entirely in R. A point $(x_0,\ y_0)$ is a ***boundary point*** of R if every disk centered at $(x_0,\ y_0)$ contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

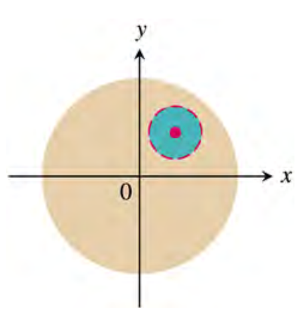
The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its ***boundary***. A region is ***open*** if it consists entirely of interior points. A region is ***closed*** if it contains all its boundary points.



Interior point

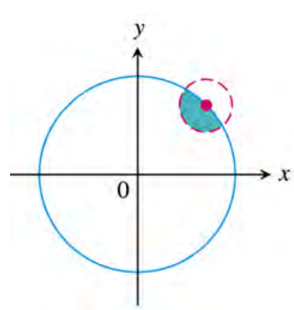


Boundary point



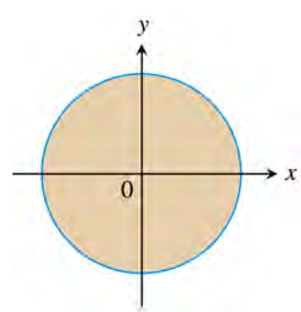
$x^2+y^2<1$

Open unit disk



$x^2+y^2=1$

Boundary of unit disk

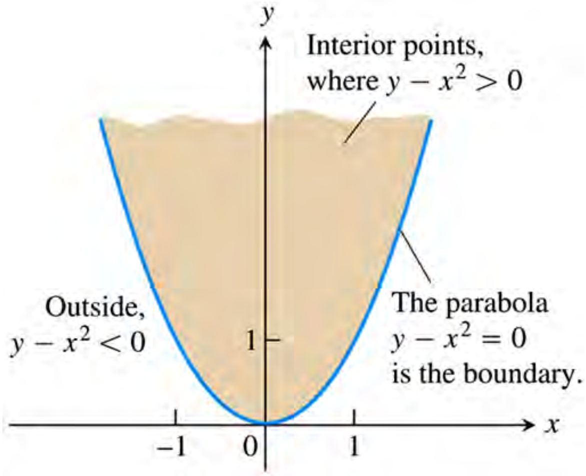


$x^2+y^2\le 1$

Closed unit disk

Definitions

A region in the plane is ***bounded*** if it lies inside a disk of fixed radius. A region is ***unbounded*** if it is not bounded.



Graphs, Level Curves, and contours of Functions of two Variables

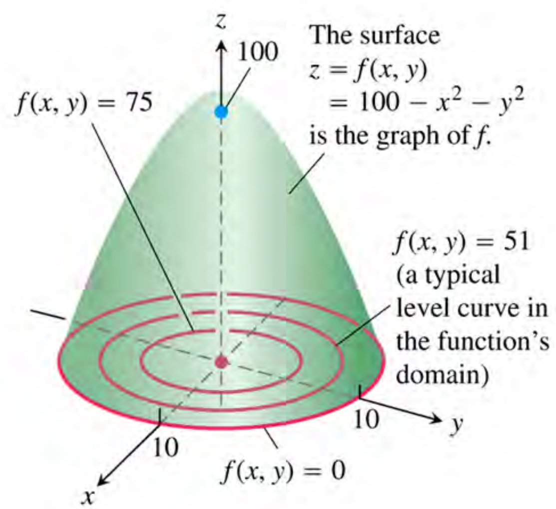
Definitions

The set of points in the plane where a function $f(x, y) = c$ is called a ***level curve*** of *f*. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the ***graph*** of *f*. The graph of *f* is also called the ***surface*** $z = f(x, y)$

Example

Graph $f(x,y)=100-x^2-y^2$ and plot the level curves $f(x,y)=0$, $f(x,y)=51$, *and* $f(x,y)=75$ in the domain of *f* in the plane.

Solution



The domain of *f* is the entire *xy*-plane, and the range of *f* is the set of real numbers less than or equal to 100.

The graph is the paraboloid $z=100-x^2-y^2$, the positive portion of which is shown in the picture.

At $f(x,y)=0 \Rightarrow x^2+y^2=100$

Which is the circle of radius 10 centered at the origin (level curve).

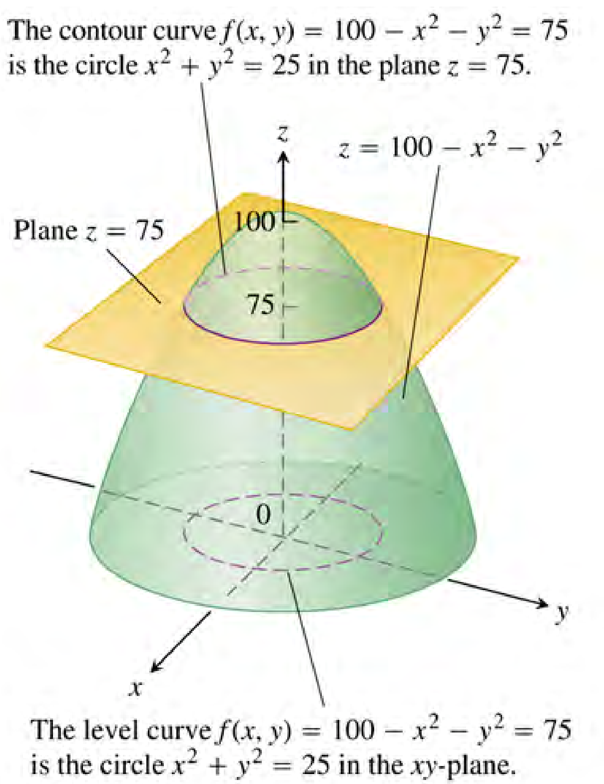
At $f(x,y)=51 \Rightarrow x^2+y^2=49$

Which is the circle of radius 7 centered at the origin.

At $f(x,y)=75 \Rightarrow x^2+y^2=25$

Which is the circle of radius 5 centered at the origin.

If $x^2+y^2>100$, then the values of $f(x,y)$ are negative.



Functions of Three Variables

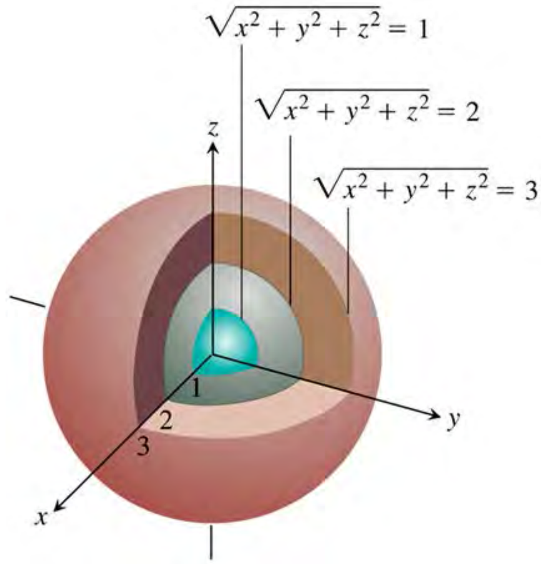
Definition

The set of points $(x,\ y,\ z)$ in space where a function of three independent variables has a constant value $f(x,\ y,\ z) = c$ is called a ***level surface*** of *f*.

Example

Describe the level surfaces of the function

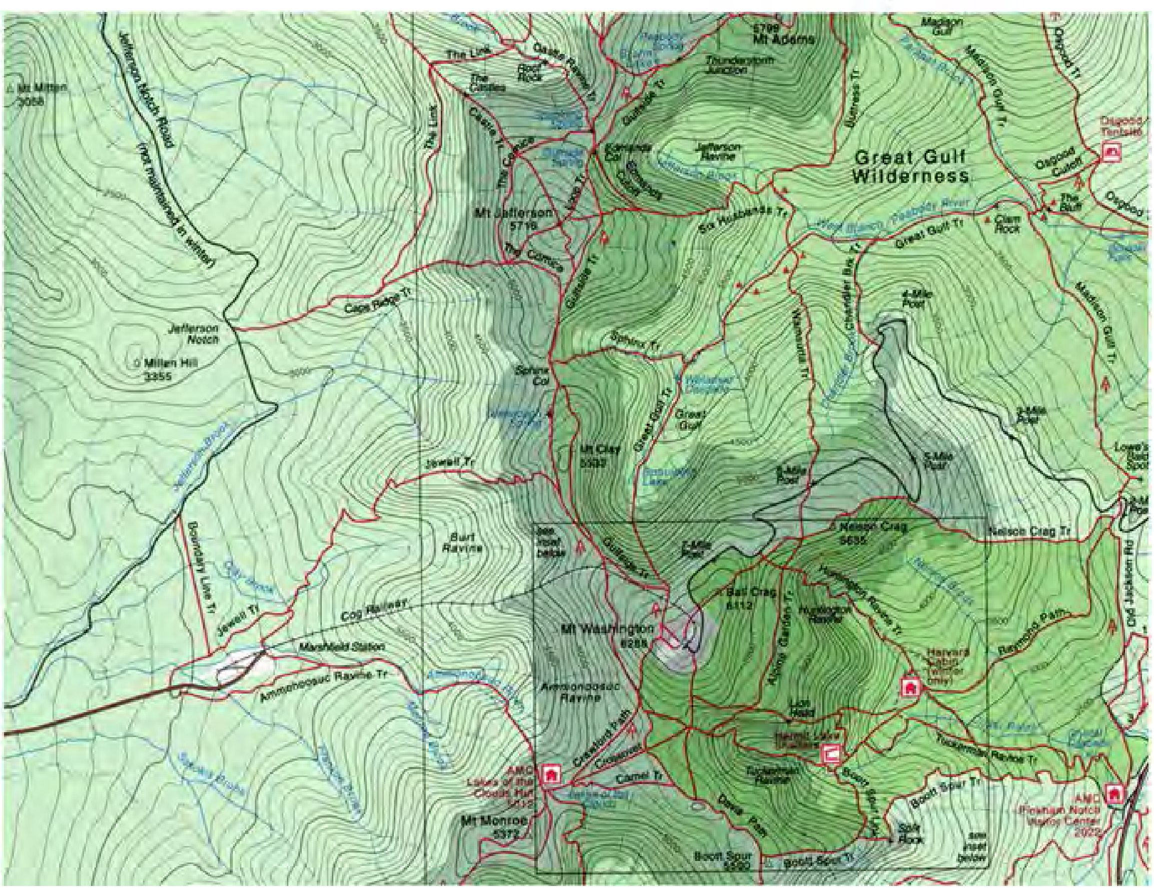
$$f(x,\ y,\ z) = \sqrt{x^2 + y^2 + z^2}$$



Solution

The value of *f* is the distance from the origin to the point $(x,\ y,\ z)$.

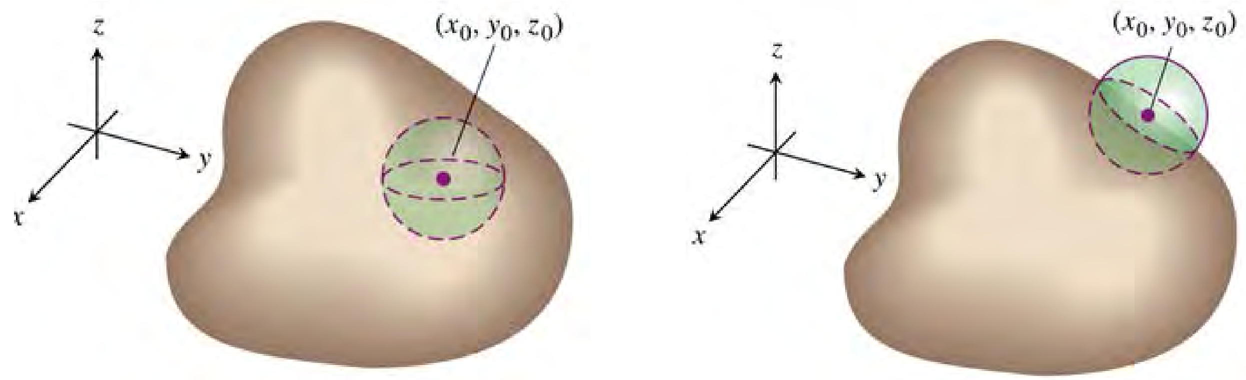
Each surface $\sqrt{x^2 + y^2 + z^2} = c\ (>0)$, is a sphere of radius c centered at the origin.



Definitions

A point $\left(x_0, y_0, z_0\right)$ in a region ***R*** in space is an ***interior point*** of ***R*** if it is the center of a solid ball that lies entirely in ***R***. A point $\left(x_0, y_0, z_0\right)$ is a ***boundary point*** of ***R*** if every solid ball centered at $\left(x_0, y_0, z_0\right)$ contains points that lie outside of ***R*** as well as that lie inside ***R***. The ***interior*** of R is the set of interior points of ***R***. The ***boundary*** of ***R*** is the set of boundary points of ***R***.

A region is ***open*** if it consists entirely of interior points. A region is ***closed*** if it contains its entire boundary.

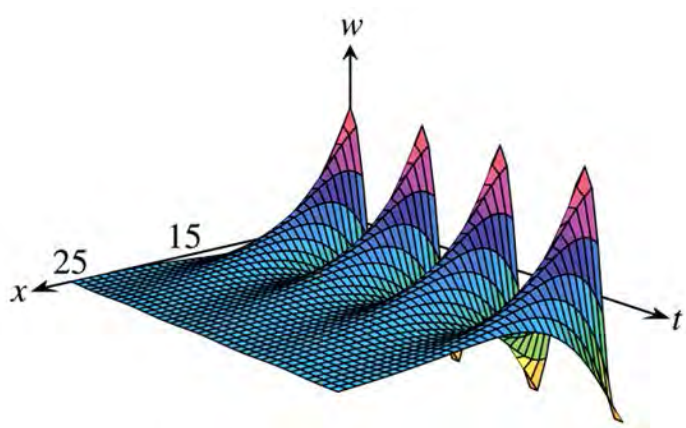


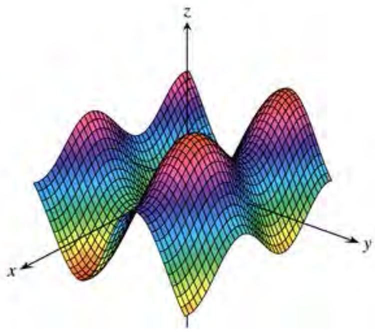
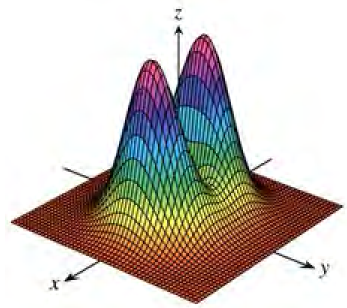
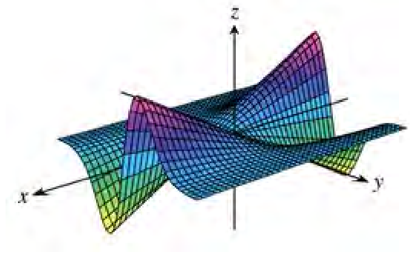
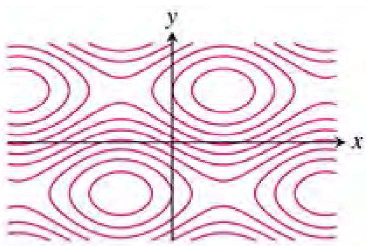
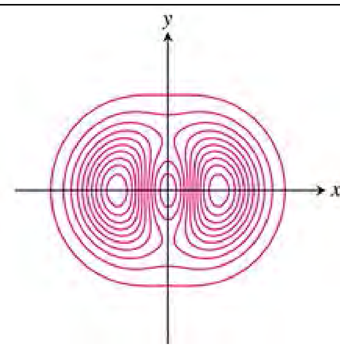
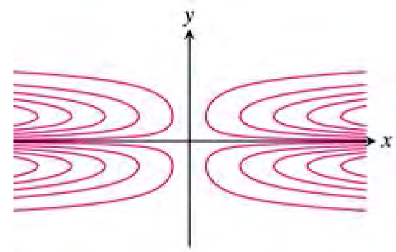
Example

The temperature w beneath the Earth's surface is a function of the depth x beneath the surface and the time t of the year. If we measure x in feet and t as the number of days elapsed from the expected date of the yearly highest surface temperature, we can model the variation in temperature with the function

$$w = \cos\left(1.7 \times 10^{-2} t - 0.2x\right) e^{-0.2x}$$

The temperature at 9 *ft* is scaled to vary from +1 to –1, so that the variation at x *ft*. can be interpreted as a fraction of the variation at the surface.



		
		
$z = \sin x + 2\sin y$	$z = \left(4x^2 + y^2\right)e^{-x^2-y^2}$	$z = xye^{-y^2}$

Exercises *Section* 2.1 – Graphs and Level Curves

1. Find the specific values for $f(x,y,z)=\dfrac{x-y}{y^2+z^2}$

a) $f(3,-1,2)$ *b)* $f\left(1,\ \frac{1}{2},\ -\frac{1}{4}\right)$ *c)* $f\left(0,\ -\frac{1}{3},\ 0\right)$ *d)* $f(2,\ 2,\ 100)$

2. Find the specific values for $f(x,y,z)=\sqrt{49-x^2-y^2-z^2}$

a) $f(0,\ 0,\ 0)$ *b)* $f(2,\ -3,\ 6)$ *c)* $f(-1,\ 2,\ 3)$ *d)* $f\left(\frac{4}{\sqrt{2}},\ \frac{5}{\sqrt{2}},\ \frac{6}{\sqrt{2}}\right)$

Find and sketch the domain for each function

3. $f(x,y)=\sqrt{y-x-2}$

4. $f(x,y)=\ln\left(x^2+y^2-4\right)$

5. $f(x,y)=\dfrac{\sin(xy)}{x^2+y^2-25}$

6. $f(x,y)=\ln(xy+x-y-1)$

7. $f(x,y)=\sqrt{\left(x^2-4\right)\left(y^2-9\right)}$

Find and sketch the level curves $f(x,y)=c$ on the same set of coordinate axes for the given values of c, we refer to these level curves as a contour map.

8. $f(x,y)=x+y-1,\quad c=-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3$

9. $f(x,y)=x^2+y^2,\quad c=0,\ 1,\ 4,\ 9,\ 16,\ 25$

10. For the function: $f(x,y)=4x^2+9y^2$:

a) Find the function's domain
b) Find the function's range
c) Find the function's level curves
d) Find the boundary of the function's domain
e) Determine if the domain is an open region, a closed region, or neither
f) Decide if the domain is bounded or unbounded

11. For the function: $f(x,y)=xy$:

a) Find the function's domain
b) Find the function's range
c) Find the function's level curves
d) Find the boundary of the function's domain
e) Determine if the domain is an open region, a closed region, or neither
f) Decide if the domain is bounded or unbounded

12. For the function: $f(x,y)=e^{-\left(x^2+y^2\right)}$:

a) Find the function's domain
b) Find the function's range
c) Find the function's level curves
d) Find the boundary of the function's domain
e) Determine if the domain is an open region, a closed region, or neither
f) Decide if the domain is bounded or unbounded

13. For the function: $f(x,y)=\ln\left(9-x^2-y^2\right)$:

a) Find the function's domain
b) Find the function's range
c) Find the function's level curves
d) Find the boundary of the function's domain
e) Determine if the domain is an open region, a closed region, or neither
f) Decide if the domain is bounded or unbounded

14. Find an equation for $f(x,y)=16-x^2-y^2$ and sketch the graph of the level curve of the function $f(x,y)$ that passes through the point $\left(2\sqrt{2},\ \sqrt{2}\right)$

15. Find an equation for $f(x,y)=\dfrac{2y-x}{x+y+1}$ and sketch the graph of the level curve of the function $f(x,y)$ that passes through the point $(-1,\ 1)$

Sketch a typical level surface for the function

16. $f(x,y,z)=x^2+y^2+z^2$

17. $f(x,y,z)=\ln\left(x^2+y^2+z^2\right)$

18. $f(x,y,z)=y^2+z^2$

19. $f(x,y,z)=z-x^2-y^2$

Section 2.2 – Limits and Continuity

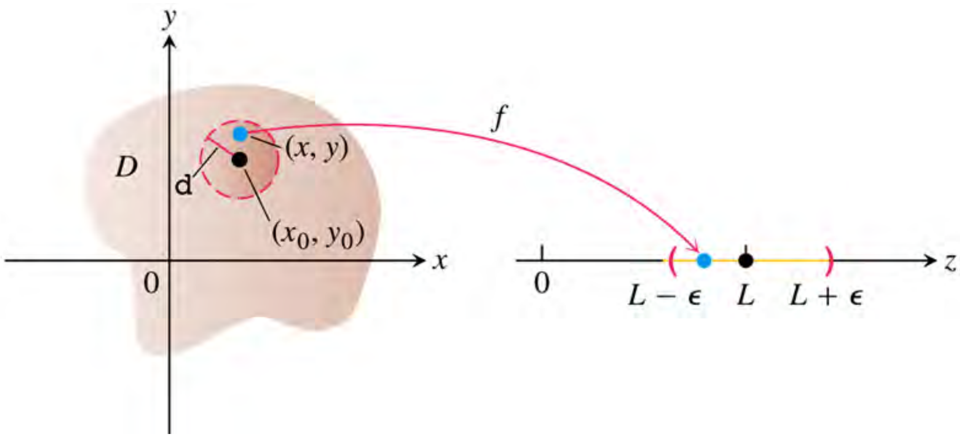
Definition

We say that a function $f(x, y)$ approaches the limit *L*, as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L$$

If, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of *f*.

$$|f(x, y) - L| < \varepsilon \quad \textit{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$



Theorem

The following rules hold if *L*, *M*, *K* are real numbers and

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L \quad \textit{and} \quad \lim_{(x, y)\to(x_0, y_0)} g(x, y) = M$$

Sum Rule: $\displaystyle\lim_{(x, y)\to(x_0, y_0)} \left(f(x, y) + g(x, y)\right) = L + M$

Difference Rule: $\displaystyle\lim_{(x, y)\to(x_0, y_0)} \left(f(x, y) - g(x, y)\right) = L - M$

Constant Multiple Rule: $\displaystyle\lim_{(x, y)\to(x_0, y_0)} kf(x, y) = kL$

Product Rule: $\displaystyle\lim_{(x, y)\to(x_0, y_0)} \left(f(x, y) \cdot g(x, y)\right) = L \cdot M$

Quotient Rule: $\displaystyle\lim_{(x, y)\to(x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M}$

Power Rule: $$\lim_{(x,y)\to(x_0,y_0)} \left[f(x,y)\right]^n = L^n \quad (\boldsymbol{n} \text{ a positive integer})$$

Root Rule: $$\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n} \quad (\boldsymbol{n} \text{ a positive integer})$$

Example

Find $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2y+5xy-y^3}$

Solution

$$\lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2y+5xy-y^3} = \frac{0-(0)(1)+3}{(0)^2(1)+5(0)(1)-(1)^3} = \underline{-3}$$

Example

Find $\displaystyle\lim_{(x,y)\to(3,-4)} \sqrt{x^2+y^2}$

Solution

$$\lim_{(x,y)\to(3,-4)} \sqrt{x^2+y^2} = \sqrt{(3)^2+(-4)^2} = \underline{5}$$

Example

Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

Solution

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \frac{0}{0}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} \cdot \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \frac{x(x-y)\left(\sqrt{x}+\sqrt{y}\right)}{x-y}$$

$$= x\left(\sqrt{x}+\sqrt{y}\right)$$

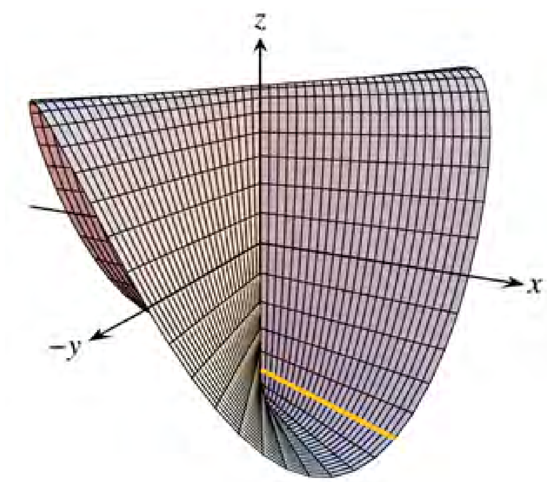
$$\underline{=0}$$

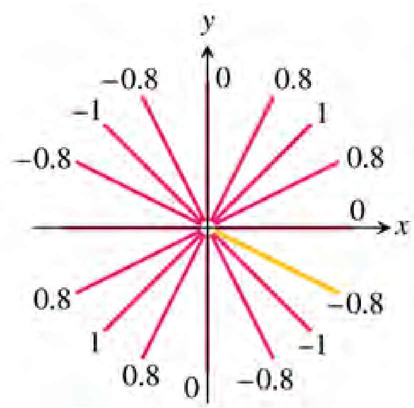
Definition

A function $f(x, y)$ is ***continuous at the point*** (x_0, y_0) if

1. f is defined at (x_0, y_0)
2. $\lim\limits_{(x, y)\to(x_0, y_0)} f(x, y)$ exists
3. $\lim\limits_{(x, y)\to(x_0, y_0)} f(x, y) = f(x_0, y_0)$

A function is ***continuous*** if it is continuous at every point of its domain.





Two-Path Test for Nonexistence of a Limit

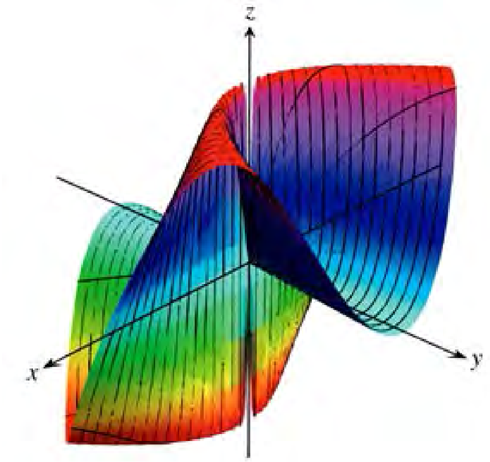
If a function $f(x, y)$ has different limits along two different paths in the domain of *f* as (x, y) approaches (x_0, y_0), then $\lim\limits_{(x, y)\to(x_0, y_0)} f(x, y)$ does not exist.

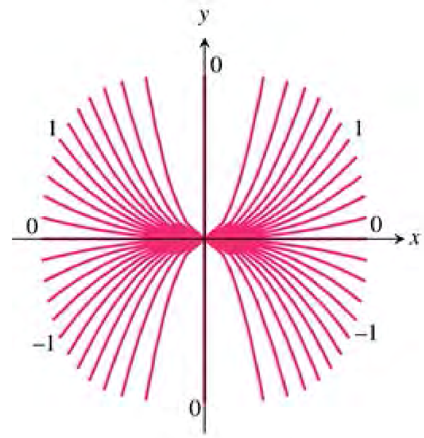
Example

Show that the function $f(x,y) = \dfrac{2x^2y}{x^4 + y^2}$ has no limit as (x, y) approaches $(0, 0)$

Solution

$$\lim_{(x, y)\to(0,0)} \frac{2x^2y}{x^4 + y^2} = \frac{0}{0}$$





We examine the curve $y = kx^2, \quad x \neq 0$

$$\left.\frac{2x^2 y}{x^4 + y^2}\right|_{y=kx^2} = \frac{2x^2\left(kx^2\right)}{x^4 + \left(kx^2\right)^2}$$

$$= \frac{2kx^4}{x^4 + k^2x^4}$$

$$= \frac{2kx^4}{x^4\left(1+k^2\right)}$$

$$= \frac{2k}{1+k^2}$$

$$\lim_{\substack{(x,\,y)\to(0,0)\\ along\ y=kx^2}} f(x,y) = \left.\frac{2k}{1+k^2}\right|$$

This limit varies with the path of approach. If $(x,\ y)$ approaches (0, 0) along the parabola $y = x^2$.

Continuity of Composites

If *f* is continuous at $\left(x_0,\, y_0\right)$ and *g* is a single-variable function continuous at $f\left(x_0,\, y_0\right)$, then the composite function $h = g \circ f$ defined by $h(x,y) = g\left(f\left(x,y\right)\right)$ is continuous at $\left(x_0,\, y_0\right)$.

Functions of More Than Two Variables

The definitions of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, products, quotients, powers, and composites all extend to functions of three or more variables. Functions like

$$\ln\left(x+y+z\right) \quad and \quad \frac{y\sin z}{x-1}$$

Exercises *Section* 2.2 – Limits and Continuity

Find the limits

1. $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$

2. $\lim\limits_{(x,y)\to(0,4)} \dfrac{x}{\sqrt{y}}$

3. $\lim\limits_{(x,y)\to(3,4)} \sqrt{x^2 + y^2 - 1}$

4. $\lim\limits_{(x,y)\to(0,0)} \cos\dfrac{x^2 + y^3}{x + y + 1}$

5. $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^y \sin x}{x}$

6. $\lim\limits_{(x,y)\to\left(\frac{\pi}{2},0\right)} \dfrac{\cos y + 1}{y - \sin x}$

7. $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq y}} \dfrac{x^2 - 2xy + y^2}{x - y}$

8. $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq y}} \dfrac{x^2 - y^2}{x - y}$

9. $\lim\limits_{\substack{(x,y)\to(2,-4)\\ x\neq x^2,\ y\neq -4}} \dfrac{y + 4}{x^2 y - xy + 4x^2 - 4x}$

10. $\lim\limits_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \dfrac{\sqrt{x} - \sqrt{y + 1}}{x - y - 1}$

11. $\lim\limits_{(x,y)\to(1,-1)} \dfrac{x^3 + y^3}{x + y}$

12. $\lim\limits_{(x,y)\to(2,2)} \dfrac{x - y}{x^4 - y^4}$

13. $\lim\limits_{P\to(1,3,4)} \left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right)$

14. $\lim\limits_{P\to(1,-1,-1)} \dfrac{2xy + yz}{x^2 + z^2}$

15. $\lim\limits_{P\to(\pi,0,2)} ze^{-2y} \cos 2x$

16. $\lim\limits_{P\to(2,-3,6)} \ln\sqrt{x^2 + y^2 + z^2}$

At what points (x, y, z) in space are the functions continuous

17. $f(x,y,z) = x^2 + y^2 - 2z^2$

18. $f(x,y,z) = \sqrt{x^2 + y^2 - 1}$

19. $f(x,y,z) = \ln(xyz)$

20. $f(x,y,z) = e^{x+y} \cos z$

21. $h(x,y,z) = \dfrac{1}{|y| + |z|}$

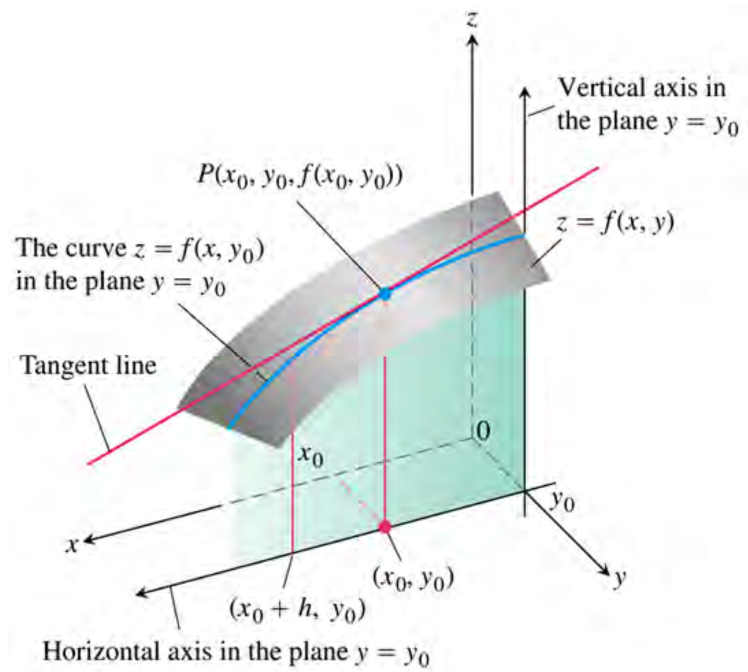
22. $h(x,y,z) = \dfrac{1}{z - \sqrt{x^2 + y^2}}$

Section 2.3 – Partial Derivatives

Partial Derivatives of a Function of Two Variables

We define the partial derivative of *f* with respect to *x* at the point $\left(x_0, y_0\right)$ as the ordinary derivative of $f\left(x, y_0\right)$ with respect to *x* at the point $x = x_0$.

To distinguish partial derivatives from ordinary derivatives we use the symbol ∂ rather than the **d** symbol.



Definition

The ***partial derivative*** of $f(x, y)$ with ***respect to x*** at the point $\left(x_0, y_0\right)$ is

$$\left.\frac{\partial f}{\partial x}\right|_{\left(x_0, y_0\right)} = \lim_{h\to 0} \frac{f\left(x_0, y_0 + h\right) - f\left(x_0, y_0\right)}{h}$$

provided the limit exists.

Definition

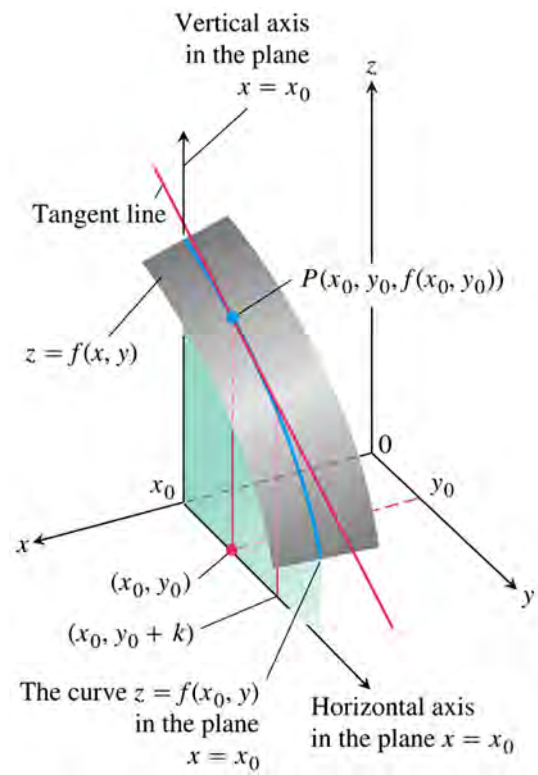
The ***partial derivative*** of $f(x, y)$ with ***respect to y*** at the point $\left(x_0, y_0\right)$ is

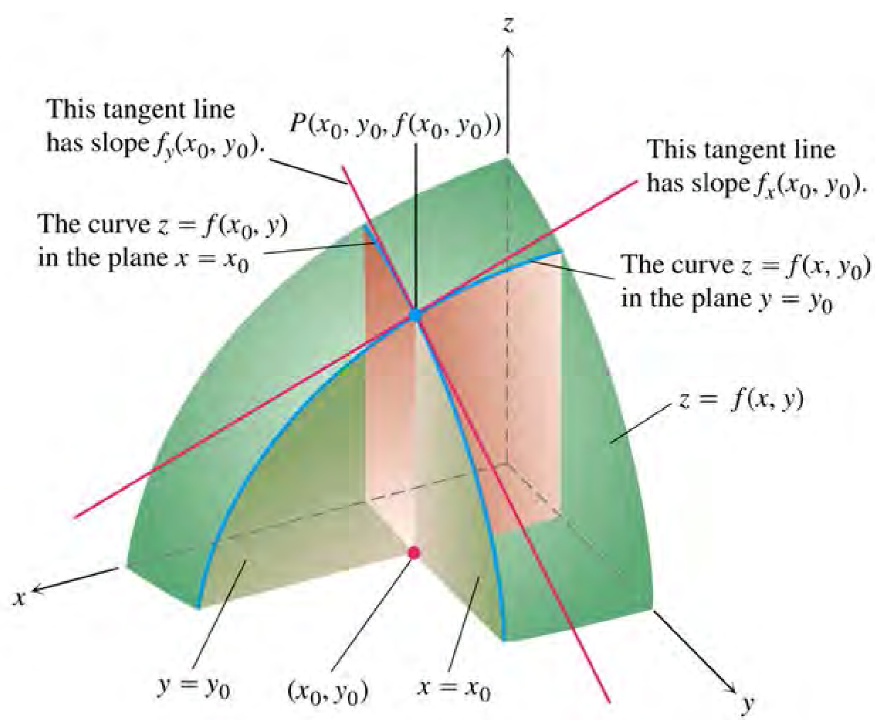
$$\left.\frac{\partial f}{\partial y}\right|_{\left(x_0, y_0\right)} = \left.\frac{d}{dy} f\left(x_0, y\right)\right|_{y=y_0} = \lim_{h\to 0} \frac{f\left(x_0, y_0 + h\right) - f\left(x_0, y_0\right)}{h}$$

provided the limit exists.

The partial derivative with respect to y is denoted:

$$\frac{\partial f}{\partial y}\left(x_0, y_0\right), \quad f_y\left(x_0, y_0\right), \quad \frac{\partial f}{\partial y}, \quad f_y$$





Calculations

Example

Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4, \ -5)$ if $\ f(x, y) = x^2 + 3xy + y - 1$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + 3xy + y - 1\right) = 2x + 3y$$

$$\left.\frac{\partial f}{\partial x}\right|_{(4,-5)} = 2(4) + 3(-5) = -7$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + 3xy + y - 1\right) = 3x + 1$$

$$\left.\frac{\partial f}{\partial y}\right|_{(4,-5)} = 3(4) + 1 = 13$$

Example

Find $\frac{\partial f}{\partial y}$ as a function if $\ f(x, y) = y \sin xy$

Solution

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y \sin xy) \\ &= \sin xy \frac{\partial}{\partial y}(y) + y \frac{\partial}{\partial y}(\sin xy) \\ &= \sin xy + (y \cos xy)\frac{\partial}{\partial y}(xy) \\ &= \sin xy + xy \cos xy\end{aligned}$$

Example

Find f_x and f_y as a function if $\ f(x, y) = \dfrac{2y}{y + \cos x}$

Solution

$$\begin{aligned}f_x &= \frac{\partial}{\partial x}\left(\frac{2y}{y + \cos x}\right) \\ &= \frac{(y + \cos x)\frac{\partial}{\partial x}(2y) - (2y)\frac{\partial}{\partial x}(y + \cos x)}{(y + \cos x)^2}\end{aligned}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$= \frac{(y+\cos x)(0) - 2y(-\sin x)}{(y+\cos x)^2}$$

$$\boxed{= \frac{2y\sin x}{(y+\cos x)^2}}$$

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y+\cos x}\right)$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$= \frac{(y+\cos x)(2) - (2y)(1)}{(y+\cos x)^2}$$

$$= \frac{2y + 2\cos x - 2y}{(y+\cos x)^2}$$

$$\boxed{= \frac{2\cos x}{(y+\cos x)^2}}$$

Example

Find $\frac{\partial z}{\partial x}$ if the equation $yz - \ln z = x + y$ defines z as a function of the two independent variables x and y and the partial derivative exist.

Solution

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\left(\frac{yz-1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{z}{yz-1}}$$

Example

The plane $x=1$ intersects the paraboloid $z=x^2+y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1,\ 2,\ 5)$.

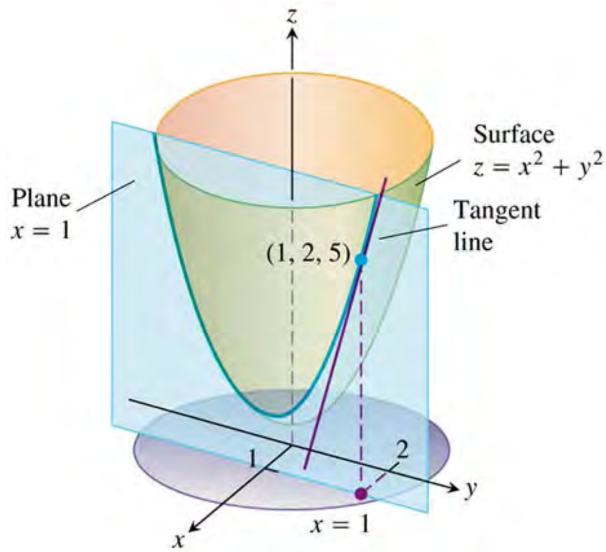
Solution

The slope is the value of the partial derivative $\frac{\partial z}{\partial y}$ at $(1,\ 2)$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)}=\left.\frac{\partial}{\partial y}\left(x^2+y^2\right)\right|_{(1,2)}=\left.2y\right|_{(1,2)}\underline{=4|}$$

The plane $x=1$ intersects the paraboloid $z=x^2+y^2$ in a parabola. $\Rightarrow\ z=1+y^2$

$$\left.\frac{\partial z}{\partial y}\right|_{y=2}=\left.\frac{\partial}{\partial y}\left(1+y^2\right)\right|_{y=2}=\left.2y\right|_{y=2}\underline{=4|}$$



Functions of More than Two Variables

The partial derivatives of more than two variables are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

Example

If *x*, *y*, and *z* are independent variables and $f(x,\ y,\ z)=x\sin(y+3z)$. Find $\frac{\partial f}{\partial z}$

Solution

$$\frac{\partial f}{\partial z}=\frac{\partial}{\partial z}\left(x\sin(y+3z)\right)$$

$$\underline{=3x\cos(y+3z)|}$$

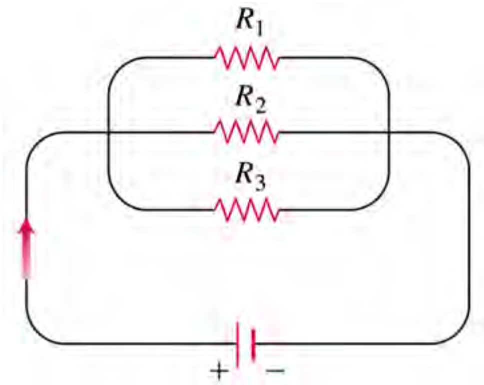
Example

If resistors of $R_1, R_2, \text{ and } R_3$ ohms are connected in parallel to make an R-ohm resistor, the value of R can be found from the equation.

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}$$

Find the value of $\dfrac{\partial R}{\partial R_2}$ when $R_1=30\ \Omega,\ R_2=45\ \Omega,\ \text{and } R_3=90\ \Omega$

Solution



$$\frac{\partial}{\partial R_2}\left(\frac{1}{R}\right)=\frac{\partial}{\partial R_2}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)$$

$$-\frac{1}{R^2}\frac{\partial R}{\partial R_2}=\frac{\partial}{\partial R_2}\left(\frac{1}{R_2}\right)$$

$$-\frac{1}{R^2}\frac{\partial R}{\partial R_2}=-\frac{1}{R_2^2}$$

$$\frac{\partial R}{\partial R_2}=\frac{R^2}{R_2^2}=\left(\frac{R}{R_2}\right)^2$$

$$\frac{1}{R}=\frac{1}{30}+\frac{1}{45}+\frac{1}{90}=\frac{6}{90}=\frac{1}{15}$$

$$\Rightarrow\ R=15$$

$$\frac{\partial R}{\partial R_2}=\left(\frac{15}{45}\right)^2=\frac{1}{9}$$

A small change in the resistance R_2 leads to a change in R about $\frac{1}{9}th$ as large.

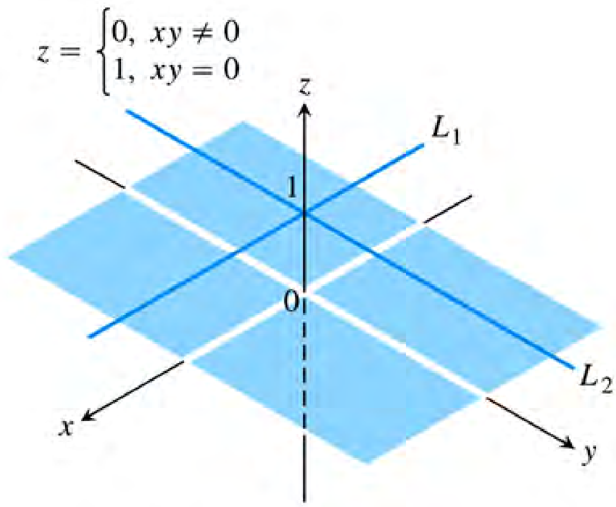
Partial Derivatives and Continuity

Example

Let $f(x,y)=\begin{cases}0, & xy\neq 0\\ 1, & xy=0\end{cases}$

a) Find the limit of f as (x, y) approaches $(0, 0)$ along the line $y = x$.
b) Prove that f is not continuous at the origin.
c) Show that both partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at the origin.

Solution



a) Since $f(x, y)$ is constantly zero along the line $y = x$ (except at the origin)

$$\lim_{(x,y)\to(0,0)} f(x, y)\Big|_{y=x} = \lim_{(x,y)\to(0,0)} 0 = 0$$

b) Since $f(0, 0)=1$, and the limit proves that f is not continuous at $(0, 0)$.

c) $\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \left.\frac{\partial}{\partial x}1\right|_{(0,0)} = 0$ is the slope of the line at any x.

The slope of the line at any y, $\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 0$

Second-Order Partial Derivatives

The second-order derivatives are denoted by

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \quad \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}, \quad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}, \quad \textit{and} \quad \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$\frac{\partial^2 f}{\partial y \partial x}$ or $f_{xy} = \left(f_x\right)_y$ Differentiate first with respect to *x*, then with respect to *y*.

Example

If $f(x, y) = x\cos y + ye^x$. Find the second derivatives $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$, *and* $\frac{\partial^2 f}{\partial x \partial y}$

Solution

$$\frac{\partial f}{\partial x} = \cos y + ye^x \qquad \frac{\partial f}{\partial y} = -x\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\cos y + ye^x\right) = ye^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\cos y + ye^x\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(-x\sin y + e^x\right) = -x\cos y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(-x\sin y + e^x\right) = -\sin y + e^x$$

Theorem – The Mixed Derivative Theorem

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, *and* f_{yx} are defined throughout an open region containing a point (*a*, *b*) and are all continuous at (*a*, *b*), then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Example

Find $\dfrac{\partial^2 w}{\partial x \partial y}$ if $w = xy + \dfrac{e^y}{y^2+1}$

Solution

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(xy + \frac{e^y}{y^2+1}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(x + \frac{e^y\left(y^2+1\right) - 2ye^y}{\left(y^2+1\right)^2}\right)$$

$$\underline{=1|}$$

Partial Derivatives of Still Higher Order

Example

Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$

Solution

$$f_y = -4xyz + x^2$$

$$f_{yx} = -4yz + 2x$$

$$f_{yxy} = -4z$$

$$\underline{f_{yxyz} = -4|}$$

Differentiability

Theorem – The Increment Theorem for Functions of Two Variables

Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region R containing a point (x_0, y_0) and that f_x *and* f_y are continuous at (x_0, y_0). Then the change

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

In the value of *f* that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

In which each of $\varepsilon_1, \varepsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$

Definition

A function $z = f(x, y)$ is ***differentiable at*** (x_0, y_0) If $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and Δz satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

In which each of $\varepsilon_1, \varepsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$. We call f ***differentiable*** if it is differentiable at every point in its domain, and say that its graph is a ***smooth surface***.

Exercises *Section* 2.3 – Partial Derivatives

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

1. $f(x,y)=2x^2-3y-4$

2. $f(x,y)=x^2-xy+y^2$

3. $f(x,y)=5xy-7x^2-y^2+3x-6y+2$

4. $f(x,y)=(xy-1)^2$

5. $f(x,y)=\left(x^3+\frac{y}{2}\right)^{2/3}$

6. $f(x,y)=\frac{1}{x+y}$

7. $f(x,y)=\frac{x}{x^2+y^2}$

8. $f(x,y)=\tan^{-1}\frac{y}{x}$

9. $f(x,y)=e^{-x}\sin(x+y)$

10. $f(x,y)=e^{xy}\ln y$

11. $f(x,y)=\sin^2(x-3y)$

12. $f(x,y)=\cos^2\left(3x-y^2\right)$

13. $f(x,y)=x^y$

Find f_x, f_y, *and* f_z

14. $f(x,y,z)=1+xy^2-2z^2$

15. $f(x,y,z)=xy+yz+xz$

16. $f(x,y,z)=x-\sqrt{y^2+z^2}$

17. $f(x,y,z)=\left(x^2+y^2+z^2\right)^{-1/2}$

18. $f(x,y,z)=\sec^{-1}(x+yz)$

19. $f(x,y,z)=\ln(x+2y+3z)$

20. $f(x,y,z)=e^{-\left(x^2+y^2+z^2\right)}$

21. $f(x,y,z)=\tanh(x+2y+3z)$

22. $f(x,y,z)=\sinh\left(xy-z^2\right)$

Find all the second-order partial derivatives of

23. $f(x,y)=x+y+xy$

24. $f(x,y)=\sin xy$

25. $g(x,y)=x^2y+\cos y+y\sin x$

26. $r(x,y)=\ln(x+y)$

27. $w=x^2\tan(xy)$

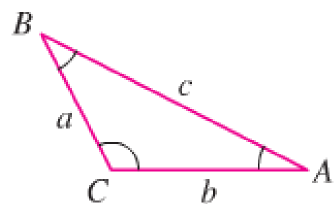
28. $w=ye^{x^2-y}$

29. Let $f(x,y)=2x+3y-4$. Find the slope of the line tangent to this surface at the point $(2,\ -1)$ and lying in the ***a***. plane $x=2$ ***b***. plane $y=-1$.

30. Let $w=f(x,y,z)$ be a function of three independent variables and writs the formal definition of the partial derivative $\frac{\partial f}{\partial y}$ at $\left(x_0,y_0,z_0\right)$. Use this definition to find $\frac{\partial f}{\partial y}$ at $(-1,\ 0,\ 3)$ for $f(x,y,z)=-2xy^2+yz^2$.

31. Find the value of $\frac{\partial x}{\partial z}$ at the point $(1,-1,-3)$ if the equation $xz + y\ln x - x^2 + 4 = 0$ defines x as a function of the two independent variables y and z and the partial derivative exists.

32. Express A implicitly as a function of a, b, and c and calculate $\frac{\partial A}{\partial a}$ and $\frac{\partial A}{\partial b}$.



33. An important partial differential equation that describes the distribution of heat in a region at time t can be represented by the *one-dimensional heat equation*

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

Show that $u(x,t) = \sin(\alpha x) \cdot e^{-\beta t}$ satisfies the heat equation for constants α and β. What is the relationship between α and β for this function to be a solution?

Section 2.4 – Chain Rule

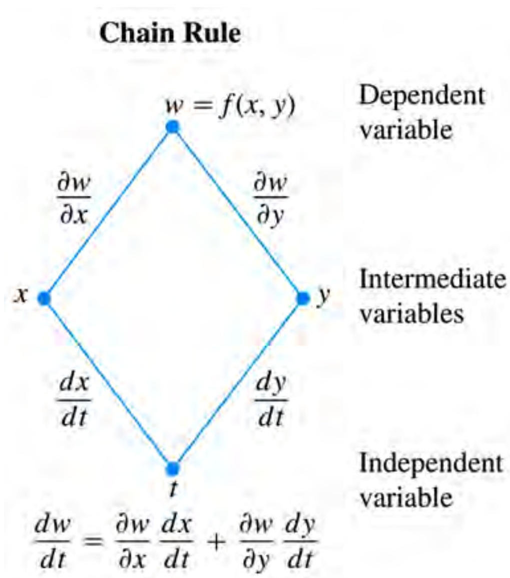
Functions of Two Variables

Theorem – Chain Rule for Functions of Two Independent Variables

If $w = f(x, y)$ is differentiable and if $x = x(t),\ y = y(t)$ are differentiable functions of *t*, then the composite $w = f(x(t),\ y(t))$ is a differentiable function of *t* and

$$\frac{dw}{dt} = f_x(x(t),\ y(t)) \bullet x'(t) + f_y(x(t),\ y(t)) \bullet y'(t)$$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$



Example

Use the Chain Rule to find the derivative of $w = xy$ with respect to *t* along the path $x = \cos t, \quad y = \sin t$. What is the derivative's value at $t = \frac{\pi}{2}$?

Solution

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial(xy)}{\partial x}\frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y}\frac{d}{dt}(\sin t) \\
&= y(-\sin t) + x(\cos t) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) \\
&= -\sin^2 t + \cos^2 t \\
&= \cos 2t
\end{aligned}$$

$w = xy$

$$= \cos t \sin t$$
$$= \frac{1}{2}\sin 2t$$

$$\frac{dw}{dt} = \frac{1}{2}(2\cos 2t)$$
$$\underline{= \cos 2t}$$

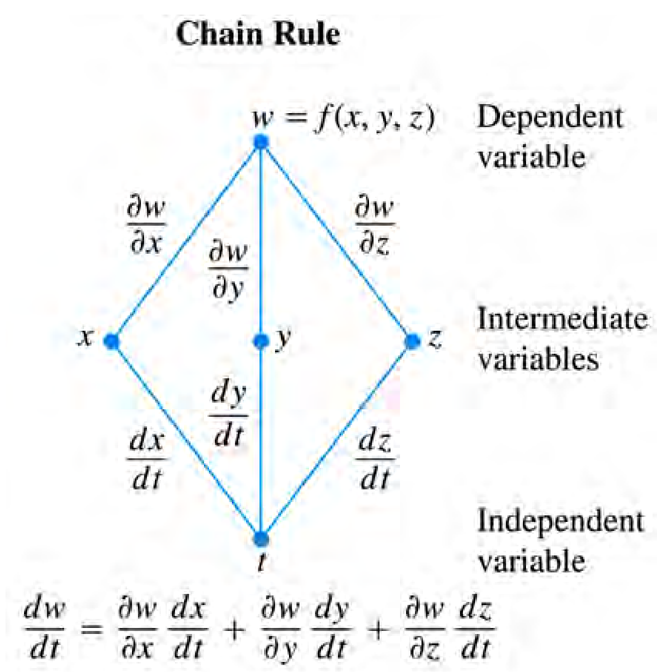
$$\left.\frac{dw}{dt}\right|_{t=\pi/2} = \cos 2\left(\frac{\pi}{2}\right)$$
$$= \cos(\pi)$$
$$\underline{= -1}$$

Functions of Three Variables

Theorem – Chain Rule for Functions of Three Independent Variables

If $w = f(x, y, z)$ is differentiable and if $x, y, and\ z$ are differentiable functions of *t*, then w is a differentiable function of *t* and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$



Example

Find $\frac{dw}{dt}$ if $w = xy + z, \quad x = \cos t, \quad y = \sin t, \quad z = t$

In this example the values of $w(t)$ are changing along the path of a helix as t changes. What is the derivative's value at $t = 0$?

Solution

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= (y)(-\sin t) + (x)(\cos t) + (1)(1)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1$$

$$= -\sin^2 t + \cos^2 t + 1$$

$$\underline{= \cos 2t + 1}$$

$$\left.\frac{dw}{dt}\right|_{t=0} = \cos(0) + 1 \ \underline{= 2}$$

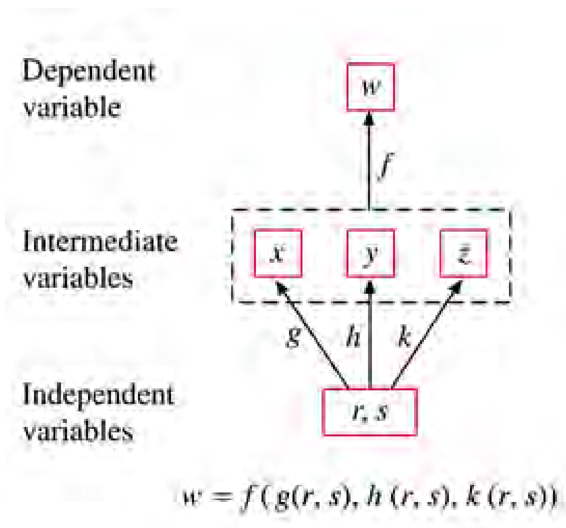
Functions Defined on Surfaces

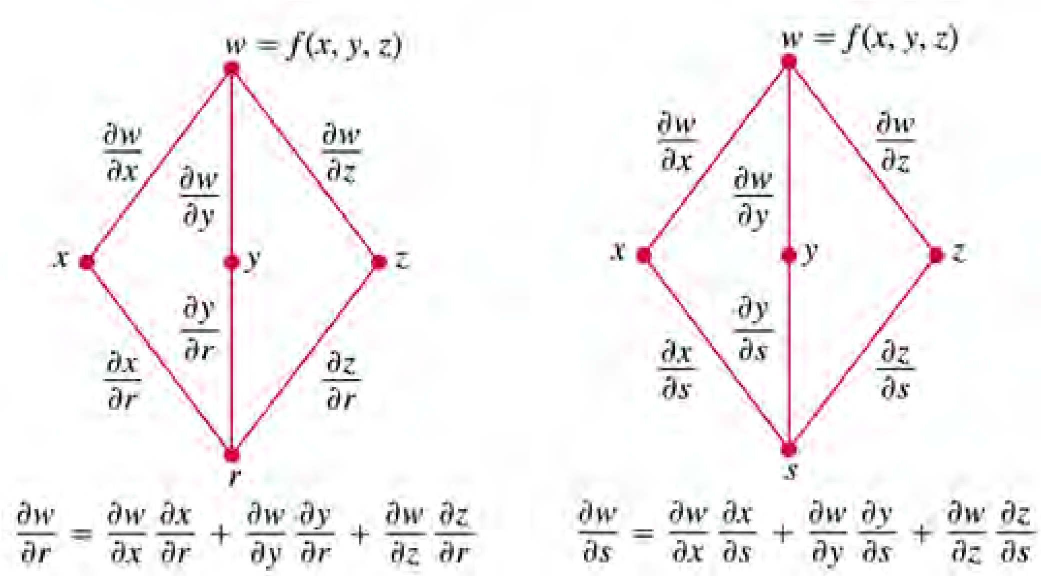
Theorem – Chain Rule for Two Independent Variables and Three Intermediate Variables

Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{dr} + \frac{\partial w}{\partial y}\frac{\partial y}{dr} + \frac{\partial w}{\partial z}\frac{\partial z}{dr}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$





Example

Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if $\quad w = x + 2y + z^2, \quad x = \frac{r}{s}, \quad y = r^2 + \ln s, \quad z = 2r$

Solution

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{dr} + \frac{\partial w}{\partial y}\frac{\partial y}{dr} + \frac{\partial w}{\partial z}\frac{\partial z}{dr}$$

$$= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2)$$

$$= \frac{1}{s} + 4r + (4r)(2)$$

$$= \frac{1}{s} + 12r$$

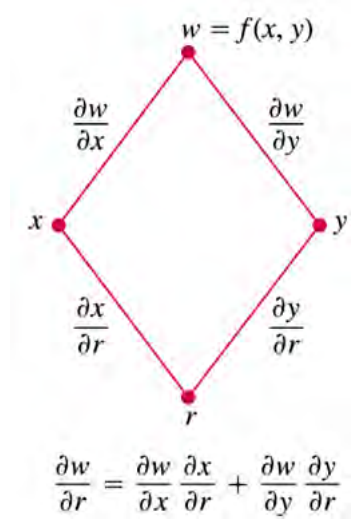
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0)$$

$$= -\frac{r}{s^2} + \frac{2}{s}$$

- If $w = f(x, y)$, $x = g(r, s)$ and $y = h(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{dr} + \frac{\partial w}{\partial y}\frac{\partial y}{dr} \quad \textit{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$



Example

Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if $w = x^2 + y^2$, $x = r - s$, $y = r + s$

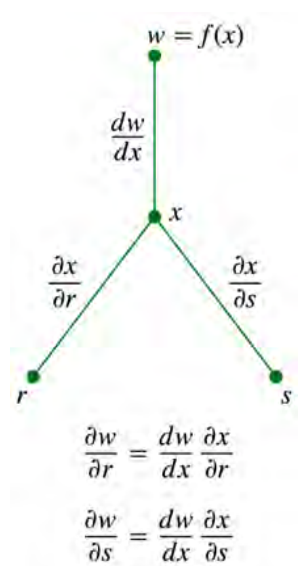
Solution

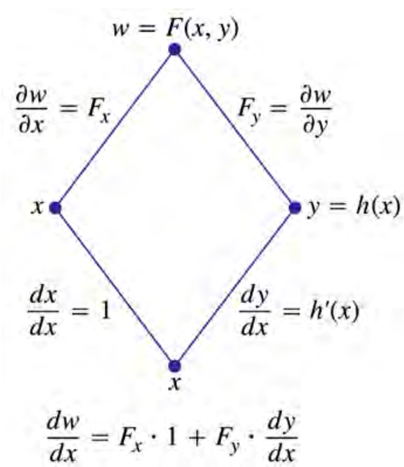
$$\begin{aligned} \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{dr} + \frac{\partial w}{\partial y}\frac{\partial y}{dr} \\ &= (2x)(1) + (2y)(1) \\ &= 2(r - s) + 2(r + s) \\ &= 2r - 2s + 2r + 2s \\ &= 4r \end{aligned}$$

$$\begin{aligned} \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \\ &= (2x)(-1) + (2y)(1) \\ &= -2(r - s) + 2(r + s) \\ &= -2r + 2s + 2r + 2s \\ &= 4s \end{aligned}$$

- If $w = f(x)$, $x = g(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{dr} \quad \textit{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s}$$





Implicit Differentiation Revisited

Theorem – A Formula for Implicit Differentiation

Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y)=0$ defines *y* as a differentiable function of *x*. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$\frac{dz}{dx} = -\frac{F_x}{F_z} \qquad \frac{dz}{dy} = -\frac{F_y}{F_z}$$

Example

Find $\frac{dy}{dx}$ if $y^2 - x^2 - \sin xy = 0$

Solution

$$F(x, y) = y^2 - x^2 - \sin xy$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{-2x - y\cos xy}{2y - x\cos xy}$$

$$\underline{= \frac{2x + y\cos xy}{2y - x\cos xy}}$$

Example

Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at (0, 0, 0) if $x^3 + z^2 + ye^{xz} + z\cos y = 0$

Solution

$$F(x, y, z) = x^3 + z^2 + ye^{xz} + z\cos y$$

$$F_x = 3x^2 + yze^{xz}, \quad F_y = e^{xz} - z\sin y, \quad \textit{and} \quad F_z = 2z + xye^{xz} + \cos y$$

$$F(0, 0, 0) = 0 \quad F_z = 1 \neq 0$$

$$\frac{dz}{dx} = -\frac{F_x}{F_z} = -\frac{3x^2 + yze^{xz}}{2z + xye^{xz} + \cos y} \qquad \frac{dz}{dy} = -\frac{F_y}{F_z} = -\frac{e^{xz} - z\sin y}{2z + xye^{xz} + \cos y}$$

$$\left.\frac{dz}{dx}\right|_{(0,0,0)} = -\frac{0}{1} \ \underline{= 0} \qquad \frac{dz}{dy} = -\frac{F_y}{F_z} = -\frac{1}{1} \ \underline{= -1}$$

Exercises *Section* 2.4 – Chain Rule

Express $\frac{dw}{dt}$ as a function of t, then evaluate $\frac{dw}{dt}$ at the given value of *t*.

1. $w = x^2 + y^2, \quad x = \cos t, \quad y = \sin t, \quad t = \pi$

2. $w = x^2 + y^2, \quad x = \cos t + \sin t, \quad y = \cos t - \sin t, \quad t = 0$

3. $w = \ln\left(x^2 + y^2 + z^2\right), \quad x = \cos t, \quad y = \sin t, \quad z = 4\sqrt{t}, \quad t = 3$

4. $w = z - \sin xy, \quad x = t, \quad y = \ln t, \quad z = e^{t-1}, \quad t = 1$

5. Express $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as functions of *u* and *v* if $z = 4e^x \ln y, \quad x = \ln(u\cos v), \quad y = u\sin v$, then evaluate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at the point $(u,v) = \left(2, \frac{\pi}{4}\right)$.

6. Express $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ as functions of *u* and *v* if $w = xy + yz + xz, \quad x = u + v, \quad y = u - v, \quad z = uv$, then evaluate $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ at the point $(u,v) = \left(\frac{1}{2}, 1\right)$.

7. Express $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ as functions of *x*, *y* and *z* if $u = e^{qr}\sin^{-1} p, \quad p = \sin x, \quad q = z^2 \ln y, \quad r = \frac{1}{z}$, then evaluate $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ at the point $(x,y,z) = \left(\frac{\pi}{4}, \frac{1}{2}, -\frac{1}{2}\right)$.

8. Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z^3 - xy + yz + y^3 - 2 = 0$ at the point $(1, 1, 1)$

9. Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $\sin(x+y) + \sin(y+z) + \sin(x+z) = 0$ at the point (π, π, π)

10. Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0$ at the point $(1, \ln 2, \ln 3)$

11. Find $\frac{\partial w}{\partial r}$ when $r = 1, s = -1$ if $w = (x+y+z)^2, \quad x = r - s, \quad y = \cos(r+s), \quad z = \sin(r+s)$

12. Find $\frac{\partial z}{\partial u}$ when $u = 0, v = 1$ if $z = \sin xy + x\sin y, \quad x = u^2 + v^2, \quad y = uv$

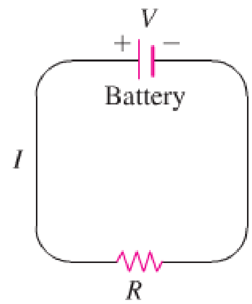
13. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ when $u = \ln 2, v = 1$ if $z = 5\tan^{-1} x, \quad x = e^u + \ln v$

14. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ when $u = 1, v = -2$ if $z = \ln q, \quad q = \sqrt{v+3}\ \tan^{-1} u$

15. Assume that $w = f\left(s^3 + t^2\right)$ and $f'(x) = e^x$. Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$

16. The voltage V in a circuit that satisfies the law $V = IR$ is slowly dropping as the battery wears out. At the same time, the resistance R is increasing as the resistor heats up. Use the equation

$$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt}$$



To find how the current is changing at the instant when $R = 600\ \Omega$, $I = 0.04A$, $\frac{dR}{dt} = 0.5\ ohm/\sec$, and $\frac{dV}{dt} = -0.01\ volt/\sec$

17. The lengths a, b, and c of the edges of a rectangular box are changing with time. At the instant in question, $a = 1\ m$, $b = 2\ m$, $c = 3\ m$, $\frac{da}{dt} = \frac{db}{dt} = 1\ m/\sec$, and $\frac{dc}{dt} = -3\ m/\sec$. At what rates the box's volume V and surface area S changing at that instant? Are the box's interior diagonals increasing in length or decreasing?

18. Let $T = f(x, y)$ be the temperature at the point (x, y) on the circle $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$ and suppose that

$$\frac{\partial T}{\partial x} = 8x - 4y, \quad \frac{\partial T}{\partial y} = 8y - 4x$$

a) Find where the maximum and minimum temperatures on the circle occur by examining the derivatives $\frac{dT}{dt}$ and $\frac{d^2T}{dt^2}$.

b) Suppose that $T = 4x^2 - 4xy + 4y^2$. Find the maximum and minimum values of T on the circle.

Evaluate $\frac{dy}{dx}$

19. $x^2 - 2y^2 - 1 = 0$

20. $x^3 + 3xy^2 - y^5 = 0$

21. $2\sin xy = 1$

22. $ye^{xy} - 2 = 0$

23. $\sqrt{x^2 + 2xy + y^4} = 3$

24. $y\ln\left(x^2 + y^2 + 4\right) = 3$

Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ at the given point.

25. $z^3 - xy + yz + y^3 - 2 = 0; \quad (1,\ 1,\ 1)$

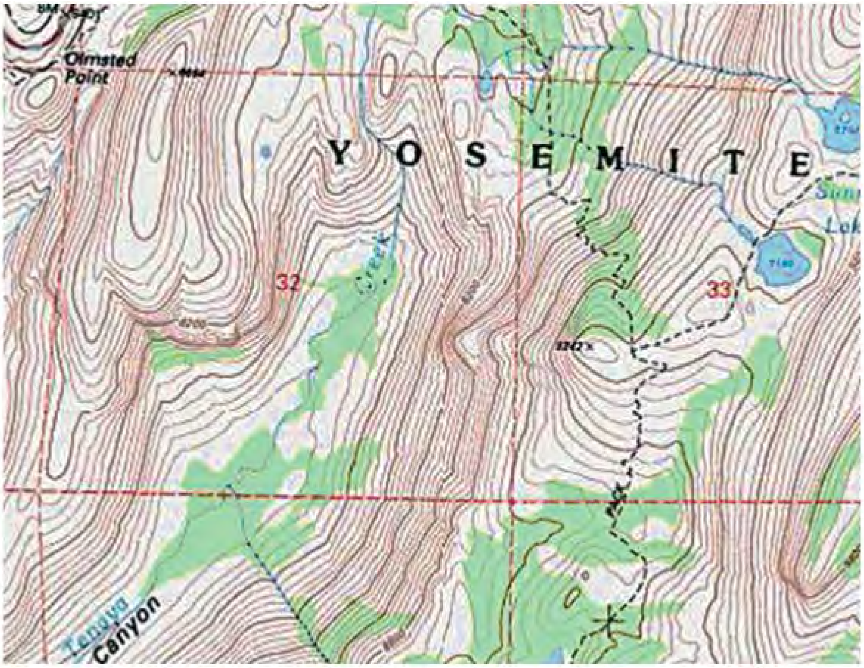
26. $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0; \quad (2,\ 3,\ 6)$

27. $\sin(x+y) + \sin(y+z) + \sin(x+z) = 0; \quad (\pi,\ \pi,\ \pi)$

28. $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0; \quad (1,\ \ln 2,\ \ln 3)$

Section 2.5 – Directional Derivatives and the Gradient

You notice that the streams flow perpendicular to the contours. The streams are following paths of steepest descent so the waters reach lower elevation as quickly as possible. Therefore, the fastest instantaneous rate of change in a stream's elevation above the sea level has a particular direction.



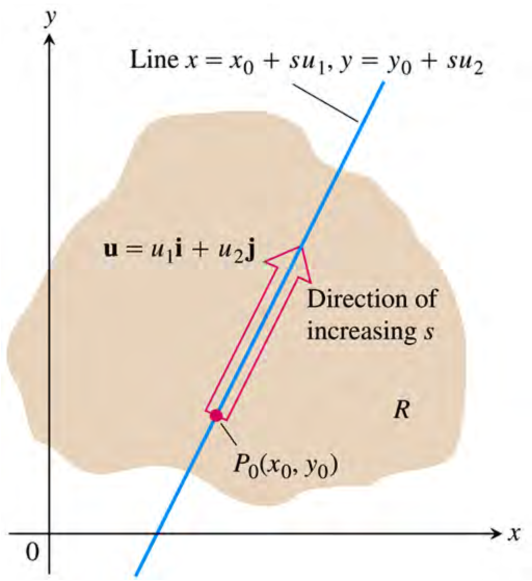
Directional Derivatives in the Plane

The rate at which f changes with respect to t along a differentiable curve $x = g(t),\ y = h(t)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Suppose that the function $f(x, y)$ is defined throughout a region R in the xy-plane, that $P_0(x_0, y_0)$ is a point in R, and that $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j}$ is a unit vector. Then the equations

$x = x_0 + su_1, \quad y = y_0 + su_2$



Definition

The derivative of *f* at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \lim_{s\to 0} \frac{f\left(x_0 + su_1, y_0 + su_2\right) - f\left(x_0, y_0\right)}{s}$$

provided the limit exists.

The directional derivative is also noted by: $\left(D_{\mathbf{u}} f\right)_{P_0}$

Example

Find the derivative of $f(x, y) = x^2 + xy$ at $P_0(1, 2)$ in the direction of the unit vector $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

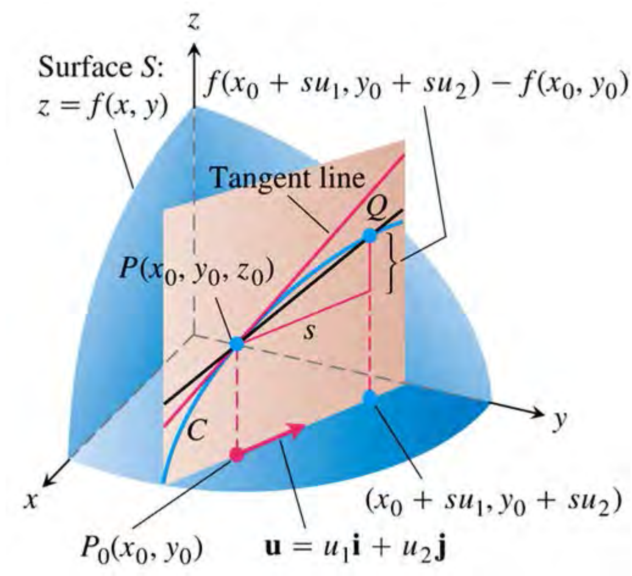
Solution

$$\begin{aligned}
\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} &= \lim_{s\to 0} \frac{f\left(x_0 + su_1, y_0 + su_2\right) - f\left(x_0, y_0\right)}{s} \\
&= \lim_{s\to 0} \frac{f\left(1 + s\frac{1}{\sqrt{2}}, 2 + s\frac{1}{\sqrt{2}}\right) - f(1, 2)}{s} \\
&= \lim_{s\to 0} \frac{\left(1 + \frac{s}{\sqrt{2}}\right)^2 + \left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right) - \left(1^2 + 1\cdot 2\right)}{s} \\
&= \lim_{s\to 0} \frac{1 + \frac{2}{\sqrt{2}}s + \frac{1}{2}s^2 + 2 + \frac{3}{\sqrt{2}}s + \frac{1}{2}s^2 - 3}{s} \\
&= \lim_{s\to 0} \frac{s^2 + \frac{5}{\sqrt{2}}s}{s} \\
&= \lim_{s\to 0} \left(s + \frac{5}{\sqrt{2}}\right) \\
&= \frac{5}{\sqrt{2}}
\end{aligned}$$

The rate of change of $f(x, y) = x^2 + xy$ at $P_0(1, 2)$ in the direction $\mathbf{u}$ is $\frac{5}{\sqrt{2}}$

Interpretation of the Directional Derivative

The equation $z = f(x, y)$ represents a surface S in space. If $z_0 = f(x_0, y_0)$, then the point $P_0(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P and $P_0(x_0, y_0, z_0)$ parallel to $\boldsymbol{u}$ intersects S in a curve C.



When $\boldsymbol{u} = \boldsymbol{i}$, the directional derivative at P_0 is $\dfrac{\partial f}{\partial x}$ evaluated at (x_0, y_0).

When $\boldsymbol{u} = \boldsymbol{j}$, the directional derivative at P_0 is $\dfrac{\partial f}{\partial y}$ evaluated at (x_0, y_0).

The directional derivative generalizes the two partial derivatives.

Calculation and Gradients

$$\left.\left(\frac{df}{ds}\right)\right|_{\boldsymbol{u},P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0}\frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\frac{dy}{ds}$$

$$= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2$$

$$= \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)_{P_0}\boldsymbol{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\boldsymbol{j}\right]}_{\text{Gradient of at } P_0} \bullet \underbrace{\left(u_1\boldsymbol{i} + u_2\boldsymbol{j}\right)}_{\text{Directional } u}$$

Definition

The ***gradient vector*** (*gradient*) of $f(x,y)$ at a point $P_0(x_0,y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}$$

obtained by evaluating the partial derivatives of *f* at P_0

Theorem – The directional Derivative is a Dot Product

If $f(x,y)$ is differentiable in an open region containing $P_0(x_0,y_0)$, then

$$\left(\frac{df}{ds}\right)_{\boldsymbol{u},P_0} = (\nabla f)_{P_0} \cdot \boldsymbol{u}$$

The dot product of the gradient ∇f at P_0 and ***u***.

Example

Find the derivative of $f(x,y) = xe^y + \cos(xy)$ at the point (2, 0) in the direction of $\boldsymbol{v} = 3\boldsymbol{i} - 4\boldsymbol{j}$

Solution

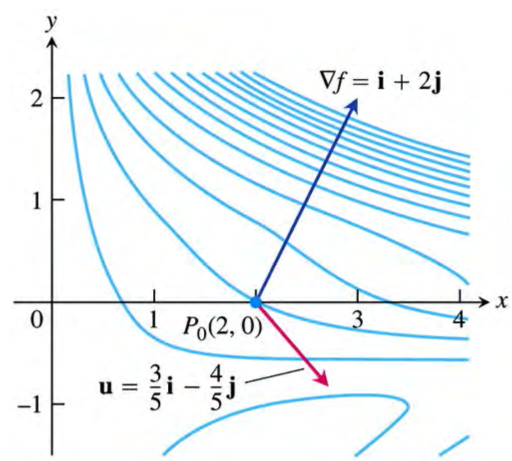
$$\boldsymbol{u} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{3\boldsymbol{i} - 4\boldsymbol{j}}{\sqrt{3^2+4^2}} = \frac{3}{5}\boldsymbol{i} - \frac{4}{5}\boldsymbol{j}$$

The partial derivatives of *f* are continuous and at (2, 0)

$$f_x(2,0) = \left(e^y - y\sin(xy)\right)_{(2,0)} = e^0 - 0 = 1$$

$$f_y(2,0) = \left(xe^y - x\sin(xy)\right)_{(2,0)} = 2e^0 - 0 = 2$$

$$\begin{aligned}\nabla f &= f_x(2,0)\boldsymbol{i} + f_y(2,0)\boldsymbol{j} \\ &= \boldsymbol{i} + 2\boldsymbol{j}\end{aligned}$$



Therefore, the derivative of *f* at (2, 0) in the direction of ***v*** is

$$\begin{aligned}\left(D_{\boldsymbol{u}} f\right)\Big|_{(2,0)} &= (\nabla f)_{(2,0)} \cdot \boldsymbol{u} \\ &= (\boldsymbol{i} + 2\boldsymbol{j}) \cdot \left(\frac{3}{5}\boldsymbol{i} - \frac{4}{5}\boldsymbol{j}\right) \\ &= \frac{3}{5} - \frac{8}{5} \\ &= -1\end{aligned}$$

Properties of the Directional Derivative $D_{\boldsymbol{u}} f = \nabla f \cdot \boldsymbol{u} = |\nabla f|\cos\theta$

1. The function *f* increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\boldsymbol{u}$ is the direction of ∇f. That is, at each point *P* in its domain, *f* increases most rapidly in the direction of the gradient vector ∇f at *P*. The derivative in this direction is $D_{\boldsymbol{u}} f = |\nabla f|\cos 0 = |\nabla f|$
2. Similarly, *f* decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\boldsymbol{u}} f = |\nabla f|\cos(\pi) = -|\nabla f|$
3. Any direction $\boldsymbol{u}$ orthogonal to a gradient ∇f is a direction of zero change in *f* because θ then equals $\frac{\pi}{2}$ and $D_{\boldsymbol{u}} f = |\nabla f|\cos\left(\frac{\pi}{2}\right) = |\nabla f| \cdot (0) = 0$

Example

Find the directions in which $f(x,y) = \dfrac{x^2}{2} + \dfrac{y^2}{2}$

a) Increases most rapidly at the point (1, 1)
b) decreases most rapidly at the point (1, 1)
c) What are the directions of zero change in *f* at (1, 1)

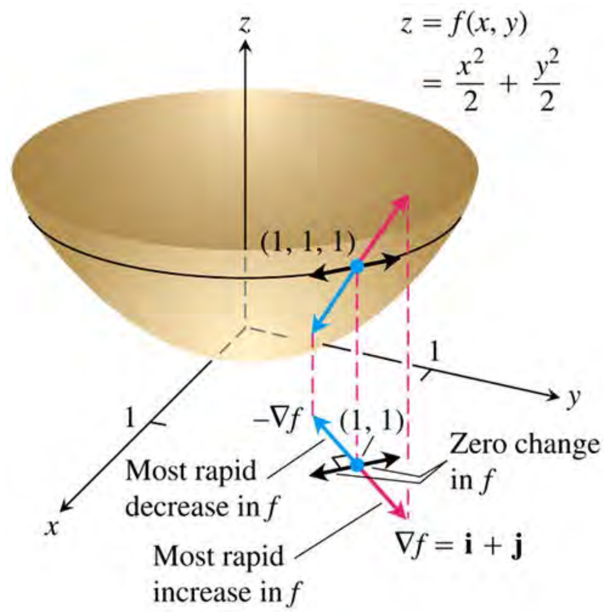
Solution

a) The function increases most rapidly at the point (1, 1).
The gradient is

$$\nabla f = f_x \boldsymbol{i} + f_y \boldsymbol{j} = x\boldsymbol{i} + y\boldsymbol{j}$$

$$(\nabla f)_{(1,1)} = (x\boldsymbol{i} + y\boldsymbol{j})_{(1,1)} = \boldsymbol{i} + \boldsymbol{j}$$

Its direction is: $\boldsymbol{u} = \dfrac{\boldsymbol{i}+\boldsymbol{j}}{|\boldsymbol{i}+\boldsymbol{j}|} = \dfrac{1}{\sqrt{2}}\boldsymbol{i} + \dfrac{1}{\sqrt{2}}\boldsymbol{j}$



b) The function decreases most rapidly at the point (1, 1).

The gradient is $-(\nabla f)_{(1,1)} = -\boldsymbol{i} - \boldsymbol{j}$

Its direction is: $-\boldsymbol{u} = -\dfrac{1}{\sqrt{2}}\boldsymbol{i} - \dfrac{1}{\sqrt{2}}\boldsymbol{j}$

c) The directions of zero change at (1, 1) are the direction orthogonal to ∇f:

$$\boldsymbol{n} = -\frac{1}{\sqrt{2}}\boldsymbol{i} + \frac{1}{\sqrt{2}}\boldsymbol{j} \quad \textit{and} \quad -\boldsymbol{n} = \frac{1}{\sqrt{2}}\boldsymbol{i} - \frac{1}{\sqrt{2}}\boldsymbol{j}$$

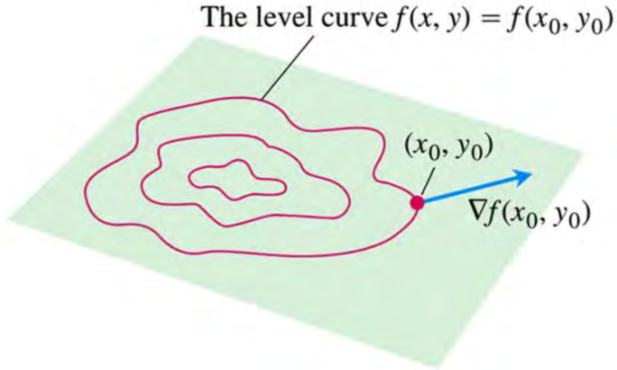
Gradients and Tangents to Level Curves

If a differentiable function $f(x,y)$ has a constant value c along a smooth curve $r = g(t)\boldsymbol{i} + h(t)\boldsymbol{j}$, then $f(g(t),h(t)) = c$. Differentiating both sides of this equation with respect to t leads to the equations

$$\frac{d}{dt} f(g(t),h(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\boldsymbol{i} + \frac{dh}{dt}\boldsymbol{j}\right)}_{\frac{d\boldsymbol{r}}{dt}} = 0$$



- At every point (x_0, y_0) in the domain of a differentiable function $f(x,y)$, the gradient of f is normal to the level curve through (x_0, y_0).

Example

Find an equation for the tangent to the ellipse $\frac{x^2}{4} + y^2 = 2$ at the point $(-2,\ 1)$

Solution

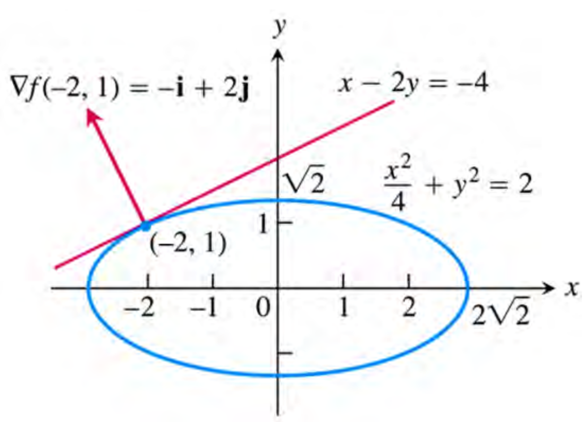
$$f(x,y) = \frac{x^2}{4} + y^2$$

The gradient of f at $(-2,\ 1)$ is

$$\nabla f = f_x\boldsymbol{i} + f_y\boldsymbol{j} = \frac{x}{2}\boldsymbol{i} + 2y\boldsymbol{j}$$

$$\nabla f\Big|_{(-2,1)} = \left(\frac{x}{2}\boldsymbol{i} + 2y\boldsymbol{j}\right)_{(-2,1)}$$

$$\underline{= -\boldsymbol{i} + 2\boldsymbol{j}}$$



The equation of the line is given by: $\quad f_x(x - x_0) + f_y(y - y_0) = 0$

$$(-1)(x - (-2)) + (2)(y - 1) = 0$$

$$-x - 2 + 2y - 2 = 0$$

$$-x + 2y = 4 \quad \rightarrow \quad \boxed{x - 2y = -4}$$

Algebra Rules for Gradients

Sum Rule:	$\nabla(f+g)=\nabla f+\nabla g$
Difference Rule:	$\nabla(f-g)=\nabla f-\nabla g$
Constant Multiple Rule:	$\nabla(kf)=k\nabla f \qquad \forall k$
Product Rule:	$\nabla(fg)=f\nabla g+g\nabla f$
Quotient Rule:	$\nabla\left(\frac{f}{g}\right)=\frac{g\nabla f-f\nabla g}{g^2}$

Functions of Three Variables

For a differentiable function $f(x,y,z)$ and a unit $\boldsymbol{u}=u_1\boldsymbol{i}+u_2\boldsymbol{j}+u_3\boldsymbol{k}$ in space, we have

$$\nabla f=\frac{\partial f}{\partial x}\boldsymbol{i}+\frac{\partial f}{\partial y}\boldsymbol{j}+\frac{\partial f}{\partial z}\boldsymbol{k}$$

$$D_{\boldsymbol{u}}f=\nabla f\bullet\boldsymbol{u}=\frac{\partial f}{\partial x}u_1+\frac{\partial f}{\partial y}u_2+\frac{\partial f}{\partial z}u_3$$

The directional derivative can be written in the form

$$D_{\boldsymbol{u}}f=\nabla f\bullet\boldsymbol{u}=|\nabla f||\boldsymbol{u}|\cos\theta=|\nabla f|\cos\theta$$

Example

a) Find the derivative of $f(x,y,z)=x^3-xy^2-z$ at $P_0(1,\ 1,\ 0)$ in the direction of $\boldsymbol{v}=2\boldsymbol{i}-3\boldsymbol{j}+6\boldsymbol{k}$

b) In what directions does *f* change most rapidly at P_0, and what are the rates of change in these directions?

Solution

a) The direction of ***v*** is obtained by dividing ***v*** by its length:

$$|\boldsymbol{v}|=\sqrt{2^2+(-3)^2+6^2}=7$$

$$\boldsymbol{u}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=\frac{2}{7}\boldsymbol{i}-\frac{3}{7}\boldsymbol{j}+\frac{6}{7}\boldsymbol{k}$$

The partial derivatives of *f* at P_0 are

$$f_x\big|_{(1,1,0)}=\left(3x^2-y^2\right)\Big|_{(1,1,0)}=2,\quad f_y\big|_{(1,1,0)}=-2y\big|_{(1,1,0)}=-2,\quad f_z\big|_{(1,1,0)}=-1$$

The gradient of f at P_0 is

$$\nabla f\Big|_{(1,1,0)} = \left(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}\right)\Big|_{(1,1,0)}$$

$$= 2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$$

Therefore, the derivative of f at P_0 in the direction of $\boldsymbol{v}$ is

$$D_{\boldsymbol{u}} f\Big|_{(1,1,0)} = \nabla f\Big|_{(1,1,0)} \cdot \boldsymbol{u}$$

$$= (2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) \cdot \left(\frac{2}{7}\boldsymbol{i} - \frac{3}{7}\boldsymbol{j} + \frac{6}{7}\boldsymbol{k}\right)$$

$$= (2)\left(\frac{2}{7}\right) + (-2)\left(-\frac{3}{7}\right) + (-1)\left(\frac{6}{7}\right)$$

$$= \frac{4}{7}$$

b) The function increases most rapidly in the direction of $\nabla f = 2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$ and decreases most rapidly in the direction of $-\nabla f$.

The rates of change in the directions are

$$|\nabla f| = \sqrt{4+4+1} = 3 \quad \textit{and} \quad -|\nabla f| = -3$$

Exercises   *Section* 2.5 – Directional Derivatives and the Gradient

Find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point

1. $f(x,y) = y - x, \quad (2,\ 1)$

2. $f(x,y) = \ln\left(x^2 + y^2\right), \quad (1,\ 1)$

3. $f(x,y) = \sqrt{2x+3y}, \quad (-1,\ 2)$

Find ∇f at the given point

4. $f(x,y,z) = x^2 + y^2 - 2z^2 + z\ln x, \quad (1,\ 1,\ 1)$

5. $f(x,y,z) = 2x^3 - 3\left(x^2 + y^2\right)z + \tan^{-1} xz, \quad (1,\ 1,\ 1)$

6. $f(x,y,z) = e^{x+y}\cos z + (y+1)\sin^{-1} x, \quad \left(0,\ 0,\ \frac{\pi}{6}\right)$

7. Find the derivative of the function $f(x,y) = 2xy - 3y^2$ at $P_0(5,\ 5)$ in the direction of $\boldsymbol{v} = 4\boldsymbol{i} + 3\boldsymbol{j}$

8. Find the derivative of the function $f(x,y) = \dfrac{x-y}{xy+2}$ at $P_0(1,\ -1)$ in the direction of $\boldsymbol{v} = 12\boldsymbol{i} + 5\boldsymbol{j}$

9. Find the derivative of the function $h(x,y) = \tan^{-1}\left(\dfrac{y}{x}\right) + \sqrt{3}\sin^{-1}\left(\dfrac{xy}{2}\right)$ at $P_0(1,\ 1)$ in the direction of $\boldsymbol{v} = 3\boldsymbol{i} - 2\boldsymbol{j}$

10. Find the derivative of the function $f(x,y,z) = xy + yz + zx$ at $P_0(1,\ -1,\ 2)$ in the direction of $\boldsymbol{v} = 3\boldsymbol{i} + 6\boldsymbol{j} - 2\boldsymbol{k}$

11. Find the derivative of the function $g(x,y,z) = 3e^x\cos yz$ at $P_0(0,\ 0,\ 0)$ in the direction of $\boldsymbol{v} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

12. Find the derivative of the function $h(x,y,z) = \cos xy + e^{yz} + \ln zx$ at $P_0\left(1,\ 0,\ \frac{1}{2}\right)$ in the direction of $\boldsymbol{v} = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$

13. Find the directions in which the function $f(x,y) = x^2 + xy + y^2$ increase and decrease most rapidly at $P_0(-1,\ 1)$. Then find the derivatives of the function in these directions.

14. Find the directions in which the function $f(x,y) = x^2y + e^{xy}\sin y$ increase and decrease most rapidly at $P_0(1,\ 0)$. Then find the derivatives of the function in these directions.

15. Find the directions in which the function $g(x, y, z) = xe^y + z^2$ increase and decrease most rapidly at $P_0\left(1,\ \ln 2,\ \frac{1}{2}\right)$. Then find the derivatives of the function in these directions.

16. Find the directions in which the function $h(x, y, z) = \ln\left(x^2 + y^2 - 1\right) + y + 6z$ increase and decrease most rapidly at $P_0\,(1,\ 1,\ 0)$. Then find the derivatives of the function in these directions.

17. Sketch the curve $x^2 + y^2 = 4;$ $\left(f(x, y) = c\right)$ together with ∇f and the tangent line at the point $\left(\sqrt{2},\ \sqrt{2}\right)$. Then write an equation for the tangent line.

18. Sketch the curve $x^2 - y = 1;$ $\left(f(x, y) = c\right)$ together with ∇f and the tangent line at the point $\left(\sqrt{2},\ 1\right)$. Then write an equation for the tangent line.

19. Sketch the curve $x^2 - xy + y^2 = 7;$ $\left(f(x, y) = c\right)$ together with ∇f and the tangent line at the point $(-1,\ 2)$. Then write an equation for the tangent line.

20. In what direction is the derivative of $f(x, y) = xy + y^2$ at $P(3,\ 2)$ equal to zero?

Section 2.6 – Tangent Planes and Linear Approximation

Tangent Planes and Normal Lines

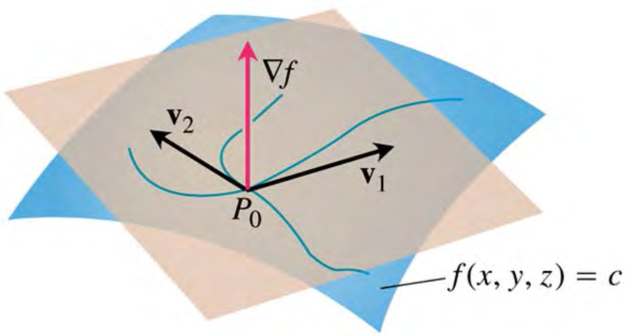
If $\boldsymbol{r} = g(t)\boldsymbol{i} + h(t)\boldsymbol{j} + k(t)\boldsymbol{k}$ is a smooth curve on the level surface $f(x,y,z) = c$ of a differentiable function *f*, then $f(g(t),h(t),k(t)) = c$.

Differentiating both sides of this equation with respect to *t* leads to

$$\frac{d}{dt} f(g(t),h(t),k(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} + \frac{\partial f}{\partial z}\frac{dk}{dt} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\boldsymbol{i} + \frac{dh}{dt}\boldsymbol{j} + \frac{dk}{dt}\boldsymbol{k}\right)}_{\frac{d\boldsymbol{r}}{dt}} = 0$$



Definition

The ***tangent plane*** at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x,y,z) = c$ of a differentiable function *f* is the plane through P_0 normal to $\nabla f\big|_{P_0}$.

The ***normal line*** of the surface at P_0 is the line through P_0 parallel to $\nabla f\big|_{P_0}$.

Normal Line to $f(x,y,z) = c$ at $P_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$$

Tangent Plane to $f(x,y,z) = c$ at $P_0(x_0, y_0, z_0)$

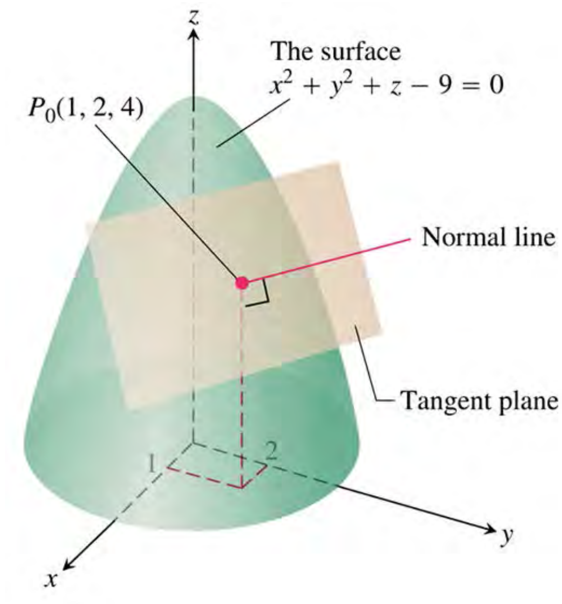
$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

Example

Find the tangent plane and normal line of the surface $f(x,y,z) = x^2 + y^2 + z - 9 = 0$ at the point $P_0(1,\ 2,\ 4)$

Solution

The tangent plane is the plane through P_0 perpendicular to the gradient of f at P_0.



The gradient is:

$$\nabla f\Big|_{P_0} = (2x\boldsymbol{i} + 2y\boldsymbol{j} + \boldsymbol{k})\Big|_{(1,2,4)} = 2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$$

The tangent plane is the plane

$$2(x-1) + 4(y-2) + (z-4) = 0$$
$$2x + 4y + z = 14$$

The line normal to the surface at P_0 is

$$x = 1 + 2t, \quad y = 2 + 4t, \quad z = 4 + t$$

Plane Tangent to a Surface $z = f(x,y)$ at $(x_0, y_0, f(x_0, y_0))$

The plane tangent to the surface $z = f(x,y)$ of a differentiable function f at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

Example

Find the plane tangent to the surface $z = x\cos y - ye^x$ at (0, 0, 0)

Solution

$$f(x,y) = x\cos y - ye^x$$

$$f_x(0,0) = \left(\cos y - ye^x\right)_{(0,0)} = 1 - 0 = 1$$

$$f_y(0,0) = \left(-x\sin y - e^x\right)_{(0,0)} = 0 - 1 = -1$$

Therefore, the tangent plane is

$$1(x-0) - (y-0) - (z-0) = 0$$

$$\boxed{x - y - z = 0}$$

Example

The surfaces $f(x,y,z) = x^2 + y^2 - 2 = 0$ and $g(x,y,z) = x + z - 4 = 0$ meet in an ellipse E. Find the parametric equations for the line tangent to E at the point $P_0(1,\ 1,\ 3)$

Solution

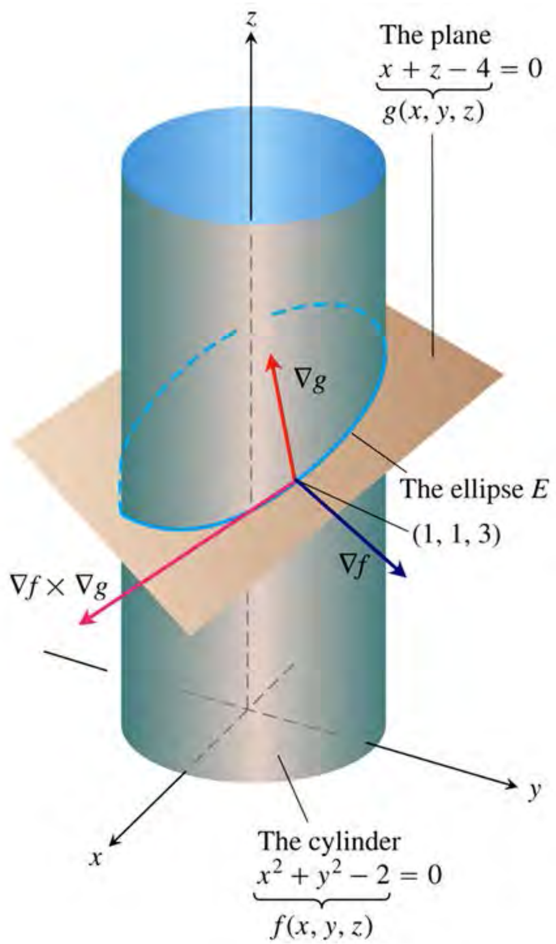
The tangent line is orthogonal to both ∇f and ∇g at P_0 and therefore parallel to $\boldsymbol{v} = \nabla f \times \nabla g$.

The components of $\boldsymbol{v}$ and the coordinates of P_0 give us equations for the line.

$$\nabla f\Big|_{(1,1,3)} = (2x\boldsymbol{i} + 2y\boldsymbol{j})\Big|_{(1,1,3)}$$
$$= 2\boldsymbol{i} + 2\boldsymbol{j}$$

$$\nabla g\Big|_{(1,1,3)} = (\boldsymbol{i} + \boldsymbol{k})\Big|_{(1,1,3)}$$
$$= \boldsymbol{i} + \boldsymbol{k}$$

$$\boldsymbol{v} = (2\boldsymbol{i} + 2\boldsymbol{j}) \times (\boldsymbol{i} + \boldsymbol{k})$$
$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$
$$= 2\boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$$



The tangent line is

$$x = 1 + 2t, \quad y = 1 - 2t, \quad z = 3 - 2t$$

Estimating Change in a Specific Direction

How much the value of a function *f* changes if we move a small distance *ds* from a point P_0 to another point neary.

$$df = f'(P_0)\,ds \qquad (\textit{single variable})$$

$$df = \left(\nabla f\Big|_{P_0} \cdot \boldsymbol{u}\right) ds \qquad (\textit{two or more variables})$$

u is the direction of the motion away from P_0.

Estimating the Change in *f* in a Direction *u*

To estimate the change in the value of a differentiable function *f* when we move a small distance *ds* from a point P_0 in a paricular direction $\boldsymbol{u}$ is given by

$$df = \underbrace{\left(\nabla f \Big|_{P_0} \cdot \boldsymbol{u} \right)}_{\substack{\text{Directional}\\ \text{derivative}}} \cdot \underbrace{ds}_{\text{Distance}}$$

Example

Estimate how much the value of $f(x,y,z) = y\sin x + 2yz$ will change if the point $P(x,y,z)$ moves 0.1 unit from $P_0(0,\ 1,\ 0)$ straight toward $P_1(2,\ 2,\ -2)$

Solution

$$\overrightarrow{P_0P_1} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$$

The direction of the vector is:

$$\begin{aligned} \boldsymbol{u} &= \frac{\overrightarrow{P_0P_1}}{\left|\overrightarrow{P_0P_1}\right|} \\ &= \frac{2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}}{\sqrt{4+1+4}} \\ &= \frac{2}{3}\boldsymbol{i} + \frac{1}{3}\boldsymbol{j} - \frac{2}{3}\boldsymbol{k} \end{aligned}$$

$$\begin{aligned} \nabla f \Big|_{(0,1,0)} &= \left((y\cos x)\boldsymbol{i} + (\sin x + 2z)\boldsymbol{j} + 2y\boldsymbol{k} \right)\Big|_{(0,1,0)} \\ &= \boldsymbol{i} + 2\boldsymbol{k} \end{aligned}$$

$$\begin{aligned} \nabla f \Big|_{P_0} \bullet \boldsymbol{u} &= (= \boldsymbol{i} + 2\boldsymbol{k}) \bullet \left(\frac{2}{3}\boldsymbol{i} + \frac{1}{3}\boldsymbol{j} - \frac{2}{3}\boldsymbol{k} \right) \\ &= \frac{2}{3} - \frac{4}{3} \\ &= -\frac{2}{3} \end{aligned}$$

The change *df* in *f* that results from moving $ds = 0.1$ unit away from P_0 in the direction of $\boldsymbol{u}$ is

$$\begin{aligned} df &= \left(\nabla f \Big|_{P_0} \cdot u \right)(ds) \\ &= \left(-\frac{2}{3}\right)(0.1) \\ &\approx -0.067 \ \ unit \end{aligned}$$

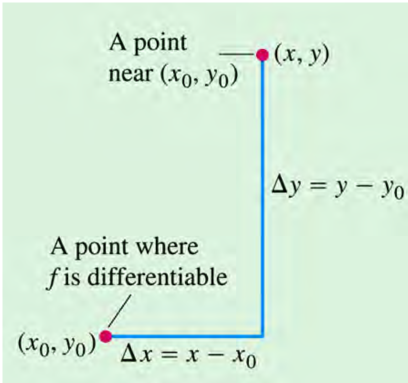
Definition

The ***linearization*** of a function $f(x,y)$ at a point (x_0, y_0) where *f* is differentiable is the function

$$L(x,y) = f(x_0,y_0) + f_x\Big|_{(x_0,y_0)}(x-x_0) + f_y\Big|_{(x_0,y_0)}(y-y_0)$$

The approximation $\qquad f(x,y) \approx L(x,y)$

is the ***standard linear*** approximation of *f* at (x_0, y_0)



Example

Find the linearization of $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$ at the point (3, 2)

Solution

$$f(3,2) = 3^2 - (3)(2) + \tfrac{1}{2}2^2 + 3 = 8$$

$$f_x(3,2) = \frac{\partial}{\partial x}\left(x^2 - xy + \tfrac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = 2x - y\Big|_{(3,2)} = 2(3) - 2 = 4$$

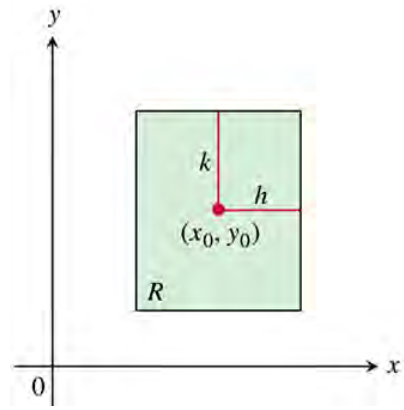
$$f_y(3,2) = -x + y\Big|_{(3,2)} = -3 + 2 = -1$$

$$L(x,y) = 8 + 4(x-3) - 1(y-2)$$

$$= 4x - y - 2$$

The Error in the Standard Linear Approximation

If *f* has continuous first and second partial derivatives throughout an open set containing a rectangle *R* centered at (x_0, y_0) and if *M* is any upper bound for the values of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on *R*, then the error $E(x,y)$ incurred in replacing $F(x,y)$ on *R* by its linearization



$$L(x,y) = f\left(x_0,y_0\right) + f_x\left(x_0,y_0\right)\left(x-x_0\right) + f_y\left(x_0,y_0\right)\left(y-y_0\right)$$

Satisfies the inequality: $\quad |E(x,y)| \le \frac{1}{2}M\left(\left|x-x_0\right| + \left|y-y_0\right|\right)^2$

$R: \quad \left|x-x_0\right| \le h, \quad \left|y-y_0\right| \le k$

Example

Find an upper bound for the error in the approximation $f(x,y) \approx L(x,y)$ of

$f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$ over the rectangle

$$R: \quad |x-3| \le 0.1, \quad |y-2| \le 0.1$$

Express the upper bound as a percentage of $f(3,2)$, the value of *f* at the center of the rectangle.

Solution

$$f_{xx} = \frac{\partial}{dx}(2x-y) = 2 \quad \rightarrow \quad \left|f_{xx}\right| = 2$$

$$f_{yy} = \frac{\partial}{dy}(-x+y) = 1 \quad \rightarrow \quad \left|f_{yy}\right| = 1$$

$$f_{xy} = \frac{\partial}{dy}(2x-y) = -1 \quad \rightarrow \quad \left|f_{xy}\right| = |-1| = 1$$

The largest of these is 2, so let $M = 2$.

$$\begin{aligned} |E(x,y)| &\le \frac{1}{2}M\left(\left|x-x_0\right| + \left|y-y_0\right|\right)^2 \\ &= \frac{1}{2}(2)\left(|x-3| + |y-2|\right)^2 \\ &= \left(|x-3| + |y-2|\right)^2 \end{aligned}$$

Since $|x-3| \le 0.1, \quad |y-2| \le 0.1$

$|E(x,y)| \le (0.1+0.1)^2 = \underline{0.04}$

As a percentage of $f(3,2)=8$, the error is no greater than

$$\frac{0.04}{8} \times 100 \underline{= 0.5\%}$$

Differentials

Definition

If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

In the linearization of *f* is called the ***total differential of f***.

Example

Suppose that a cylindrical can is designed to have a radius of 1 *in*. and a height of 5 *in*., but that the radius and height are off the amounts $dr = +0.03$ and $dh = -0.1$. Estimate the resulting absolute change in the volume of the can.

Solution

To estimate the absolute change in $V = \pi r^2 h$,

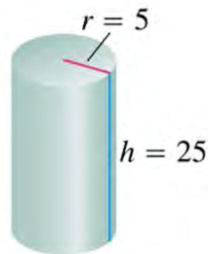
$$\Delta V \approx dV = V_r(r_0,\ h_0)dr + V_h(r_0,\ h_0)dh$$

$$\begin{aligned} dV &= (2\pi r_0 h_0)(0.03) + (\pi r_0^2)(-0.1) \\ &= 2\pi(1)(5)(0.03) + \pi(1)^2(-0.1) \\ &= 0.2\pi \\ &\approx 0.63\ in^3 \end{aligned}$$

Example

Your company manufactures right circular cylindrical molasses storage tanks that are 25 ft with a radius of 5 ft. How sensitive are the tanks' volumes to small variations in height and radius?

Solution



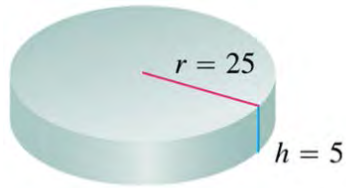
$$V = \pi r^2 h$$

$$\begin{aligned} dV &= V_r(r_0,\ h_0)dr + V_h(r_0,\ h_0)dh \\ &= V_r(5,\ 25)dr + V_h(5,\ 25)dh \\ &= (2\pi rh)_{(5,\ 25)}dr + (\pi r^2)_{(5,\ 25)}dh \\ &= 250\pi dr + 25\pi dh \end{aligned}$$

A 1-unit change in *r* will change *V* about 250π units.

A 1-unit change in *h* will change *V* about 25π units.

The tanks' volume is 10 times more sensitive to a small change in r than it is to a small change of equal size in h.



$$dV = (2\pi rh)_{(25,\ 5)}\, dr + \left(\pi r^2\right)_{(25,\ 5)} dh$$
$$= 250\pi dr + 625\pi dh$$

Now the volume is more sensitive to changes in h than to changes in r.

The general rule is that functions are most sensitive to small changes in the variables that generated the largest partial derivatives.

Example

The volume $V = \pi r^2 h$ of a right circular cylinder is to be calculated from measured values of r and h. Suppose that r is measured with an error of no more than 2% and h with an error of no more than 0.5%. Estimate the resulting possible percentage error in the calculation of V.

Solution

$$\left|\frac{dr}{r}\times 100\right| \le 2 \qquad \left|\frac{dh}{h}\times 100\right| \le 0.5$$

$$\frac{dV}{V} = \frac{2\pi rh dr + \pi r^2 dh}{\pi r^2 h}$$
$$= \frac{2dr}{r} + \frac{dh}{h}$$

$$\left|\frac{dV}{V}\right| = \left|\frac{2dr}{r} + \frac{dh}{h}\right|$$
$$\le \left|2\frac{dr}{r}\right| + \left|\frac{dh}{h}\right|$$
$$\le 2(0.02) + 0.005$$
$$= 0.045$$

The error in the volume is at the most 4.5%

Functions of More Than Two Variables

- The linearization of $f(x,y,z)$ at a point $P_0(x_0,y_0,z_0)$ is

$$L(x,y,z)=f(P_0)+f_x(P_0)(x-x_0)+f_y(P_0)(y-y_0)+f_z(P_0)(z-z_0)$$

- Suppose that R is a closed rectangular solid centered at P_0 and lying in an open region on which the second partial derivatives of *f* are continuous. Suppose also that $|f_{xx}|$, $|f_{yy}|$, $|f_{zz}|$, $|f_{xy}|$, $|f_{xz}|$, and $|f_{yz}|$ are all less than or equal to *M* throughout *R*. Then the error $E(x,y,z)=f(x,y,z)-L(x,y,z)$ in the approximation of *f* by *L* is bounded throughout *R* by the inequality

$$|E(x,y)|\le \tfrac{1}{2}M\left(|x-x_0|+|y-y_0|+|z-z_0|\right)^2$$

- If the second partial derivatives of *f* are continuous and if *x*, *y*, and *z* change from $x_0, y_0, and\ z_0$ by small amounts *dx*, *dy*, and *dz*, the total differential

$$df=f_x(P_0)dx+f_y(P_0)dy+f_z(P_0)dz$$

Example

Find the linearization $L(x,y,z)$ of $f(x,y,z)=x^2-xy+3\sin z$ at the point (2, 1, 0).

Find the upper bound for the error incurred in replacing *f* by *L* on the rectangle

$$R:\quad |x-2|\le 0.01,\quad |y-1|\le 0.02,\quad |z|\le 0.01$$

Solution

$f(2,1,0)=2^2-(2)(1)+3\sin 0=2$

$f_x(2,1,0)=2x-y=3$	$f_{xx}=2$	$f_{xy}=-1$
$f_y(2,1,0)=-x=-2$	$f_{yy}=0$	$f_{xz}=0$
$f_z(2,1,0)=3\cos z=3$	$f_{zz}=-3\sin z$	$f_{yz}=0$

$|-3\sin z|\le 3\sin 0.01\approx 0.03$

Let $M = 2$.

$|E|\le \tfrac{1}{2}2(0.01+0.02+0.01)^2 = 0.0016$

Exercises *Section* 2.6 – Tangent Planes and Linear Approximation

1. Find the tangent plane and normal line of the surface $x^2+y^2+z^2=3$ at the point $P_0(1,\ 1,\ 1)$

2. Find the tangent plane and normal line of the surface $x^2+2xy-y^2+z^2=7$ at the point $P_0(1,\ -1,\ 3)$

3. Find the tangent plane and normal line of the surface $\cos \pi x - x^2 y + e^{xz} + yz = 4$ at the point $P_0(0,\ 1,\ 2)$

4. Find the tangent plane and normal line of the surface $x^2-xy-y^2-z=0$ at the point $P_0(1,\ 1,\ -1)$

5. Find the tangent plane and normal line of the surface $x^2+y^2-2xy-x+3y-z=-4$ at the point $P_0(2,\ -3,\ 18)$

6. Find an equation for the plane that is tangent to the surface $z=\ln\left(x^2+y^2\right)$ at the point $(1,\ 0,\ 0)$

7. Find an equation for the plane that is tangent to the surface $z=e^{-x^2-y^2}$ at the point $(0,\ 0,\ 1)$

8. Find an equation for the plane that is tangent to the surface $z=\sqrt{y-x}$ at the point $(1,\ 2,\ 1)$

9. Find parametric equation for the line tangent to the curve of intersection of the surfaces $x+y^2+2z=4, \quad x=1$ at the point $(1,\ 1,\ 1)$

10. Find parametric equation for the line tangent to the curve of intersection of the surfaces $xyz=1, \quad x^2+2y^2+3z^2=6$ at the point $(1,\ 1,\ 1)$

11. Find parametric equation for the line tangent to the curve of intersection of the surfaces $x^3+3x^2y^2+y^3+4xy-z^2=0, \quad x^2+y^2+z^2=11$ at the point $(1,\ 1,\ 3)$

12. Find parametric equation for the line tangent to the curve of intersection of the surfaces $x^2+y^2=4, \quad x^2+y^2-z=0$ at the point $\left(\sqrt{2},\ \sqrt{2},\ 4\right)$

13. By about how much will $f(x,y,z)=\ln\sqrt{x^2+y^2+z^2}$ change if the point $P(x,y,z)$ moves from $P_0(3,\ 4,\ 12)$ a distance of $ds=0.1$ unit in the direction of $3\boldsymbol{i}+6\boldsymbol{j}-2\boldsymbol{k}$?

14. By about how much will $f(x,y,z)=e^x\cos yz$ change if the point $P(x,y,z)$ moves from the origin at distance of $ds=0.1$ unit in the direction of $2\boldsymbol{i}+2\boldsymbol{j}-2\boldsymbol{k}$?

15. Find the linearization $L(x,y)$ of $f(x,y)=x^2+y^2+1$ at the point (0, 0) and (1, 1)

16. Find the linearization $L(x,y)$ of $f(x,y)=(x+y+2)^2$ at the point (0, 0) and (1, 2)

17. Find the linearization $L(x,y)$ of $f(x,y)=x^3y^4$ at the point (1, 1) and (0, 0)

18. Find the linearization $L(x,y)$ of $f(x,y)=e^{2y-x}$ at the point (0, 0) and (1, 2)

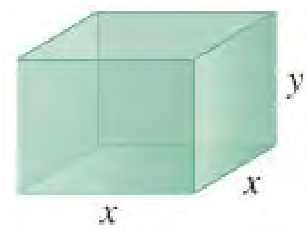
19. Find the linearization $L(x,y,z)$ of $f(x,y,z)=x^2+y^2+z^2$ at the point (1, 1, 1)

20. Find the linearization $L(x,y,z)$ of $f(x,y,z)=\sqrt{x^2+y^2+z^2}$ at the point (1, 2, 2)

21. Find the linearization $L(x,y,z)$ of $f(x,y,z)=\dfrac{\sin xy}{z}$ at the point $\left(\frac{\pi}{2},\ 1,\ 1\right)$

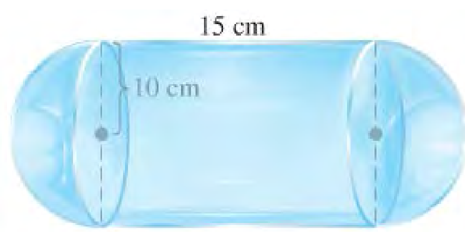
22. Find the linearization $L(x,y,z)$ of $f(x,y,z)=e^x+\cos(y+z)$ at the point $\left(0,\ \frac{\pi}{4},\ \frac{\pi}{4}\right)$

23. Consider a closed rectangular box with a square base. If *x* is measured with error at most 2% and y is measured with error at most 3% use a differential to estimate the corresponding percentage error in computing the box's



a) Surface area
b) Volume

24. Consider a closed container in the shape of a cylinder of radius 10 *cm* and height 15 *cm* with a hemisphere on each end.



The container is coated with a layer of ice $\frac{1}{2}$ cm thick. Use a differential to estimate the total volume of ice. (*Hint*: assume *r* is radius with $dr=\frac{1}{2}$ and *h* is height with $dh=0$)

25. A standard 12-fl-oz can of soda is essentially a cylinder of radius $r=1$ *in* and height $h=5$ *in.*
a) At these dimensions, how sensitive is the can's volume to a small change in radius versus a small change in height?
b) Could you design a soda can that appears to hold more soda but in fact holds the same 12-fl-oz? What might its dimensions be? (There is more than one correct answer.)

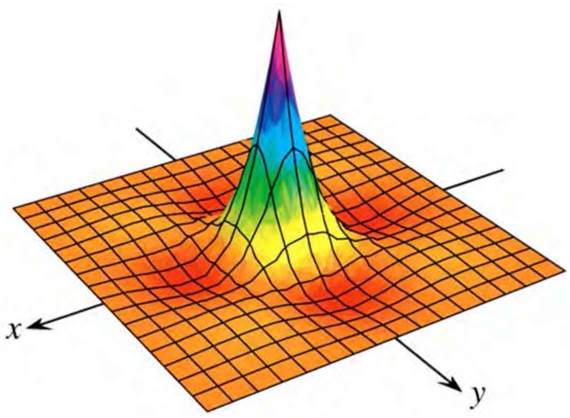
Section 2.7 – Maximum/Minimum Problems

Derivative Tests for Local Extreme Values

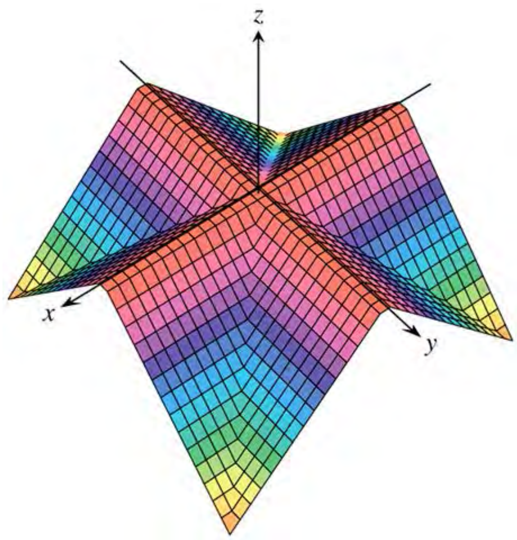
Definition

Let $f(x,y)$ be defined on a region R containing the point $(a,\ b)$. Then

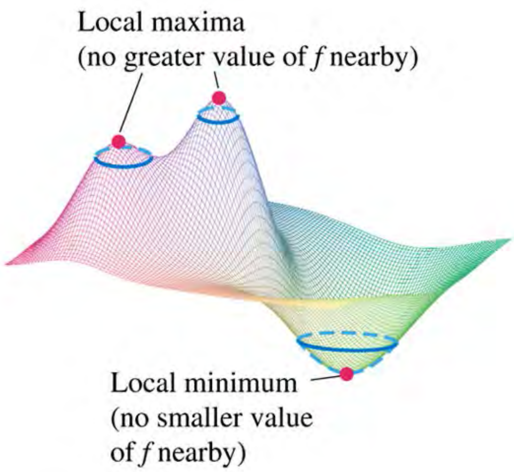
- $f(a,\ b)$ is a local maximum value of *f* if $f(a,\ b) \ge f(x,\ y)$ for all domain points $(x,\ y)$ in an open disk centered at $(a,\ b)$.
- $f(a,\ b)$ is a local minimum value of *f* if $f(a,\ b) \le f(x,\ y)$ for all domain points $(x,\ y)$ in an open disk centered at $(a,\ b)$.



$z = (\cos x)(\cos y)e^{-\sqrt{x^2+y^2}}$

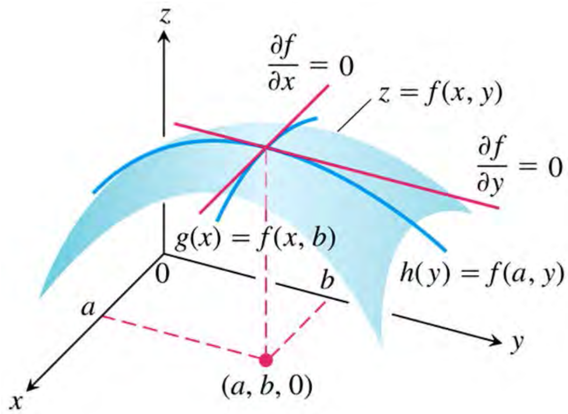


$z = \frac{1}{2}\left(\left||x|-|y|\right|-|x|-|y|\right)$



Theorem – **First derivative Test for Local Extreme Values**

If $f(x, y)$ has a local maximum or minimum value at an interior point $(a,\ b)$ of its domain and if the first partial derivatives exist there, then $f_x(a,\ b) = 0$ *and* $f_y(a,\ b) = 0$

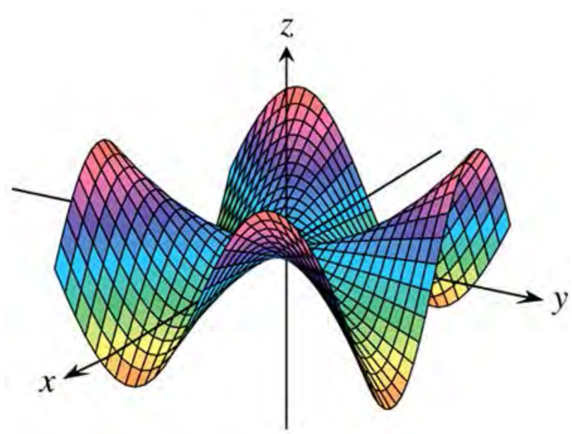


Definition

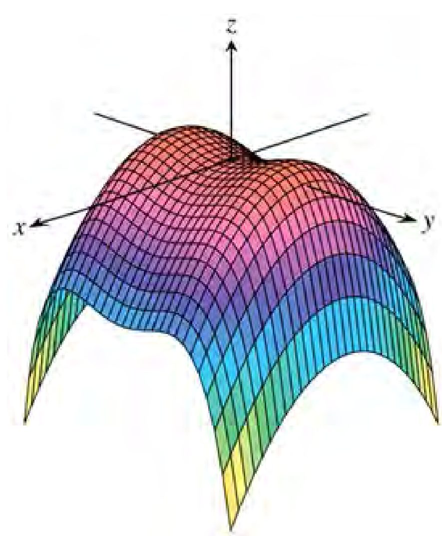
An interior point of the domain $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a ***critical point*** of *f*.

Definition

A differentiable function $f(x, y)$ has a ***saddle point*** at a critical point $(a,\ b)$ if in every open disk centered at $(a,\ b)$ there are domain points (x, y) where $f(x,\ y) > f(a,\ b)$ and domain points (x, y) where $f(x,\ y) < f(a,\ b)$. The corresponding point $(a,\ b,\ f(a,\ b))$ on the surface $z = f(x,\ y)$ is called a saddle point of the surface.



$$z = \frac{xy\left(x^2 - y^2\right)}{x^2 + y^2}$$



$$z = y^2 - y^4 - x^2$$

Example

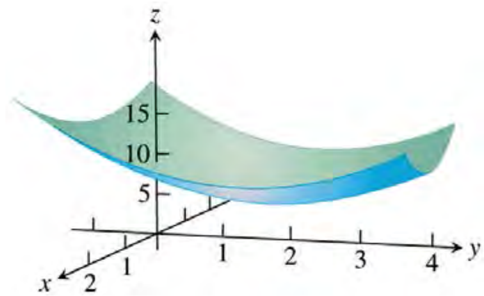
Find the local extreme values of $f(x,y) = x^2 + y^2 - 4y + 9$

Solution

The domain of *f* is the entire plane. The local extreme values occur:

$f_x = 2x = 0 \qquad f_y = 2y - 4 = 0$

Therefore, the critical point is $(0,\ 2)$ and the value $f(0,2) = 0 + 2^2 - 8 + 9 = 5$.



The critical point is a local minimum.

Example

Find the local extreme values of $f(x,y) = y^2 - x^2$

Solution

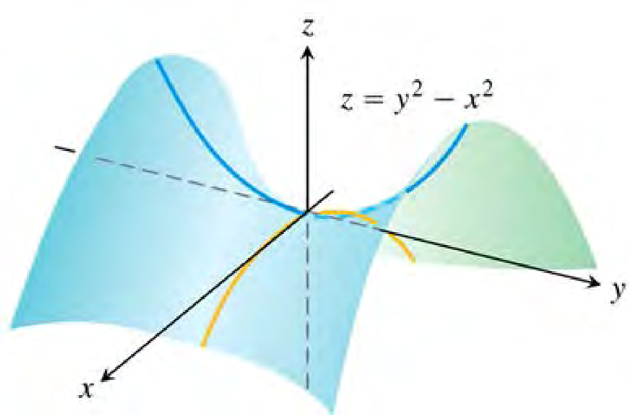
The domain of *f* is the entire plane.

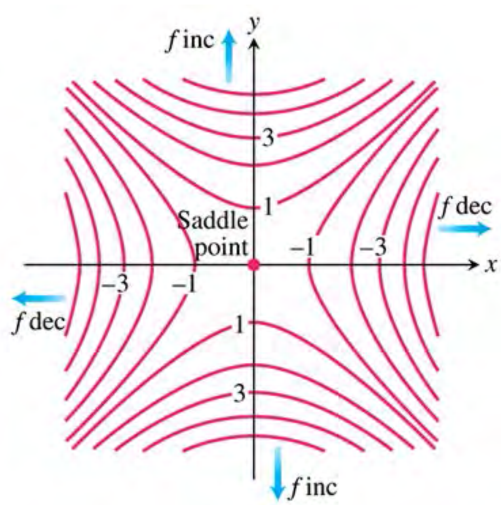
$f_x = -2x = 0 \qquad f_y = 2y = 0$

Therefore, the local extreme is the origin $(0,\ 0)$ and the value $f(0,\ 0) = 0$.

$f(0,y) = y^2 \ge 0 \qquad f(x,0) = -x^2 \le 0$

The function has a saddle point at the origin and no local extreme values.





Theorem – Second Derivative Test for Local Extreme Values

Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

- f has a ***local maximum*** at (a, b) if $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a, b).
- f has a ***local minimum*** at (a, b) if $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a, b).
- f has a ***saddle point*** at (a, b) if $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a, b).
- ***The test is inconclusive*** at (a, b) if $f_{xx} f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of *f* at (a, b).

Example

Find the local extreme values of $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

Solution

$$f_x = y - 2x - 2 = 0 \qquad f_y = x - 2y - 2 = 0$$

$$\begin{cases} -2x + y = 2 \\ x - 2y = 2 \end{cases} \rightarrow \boxed{x = y = -2}$$

Therefore, the critical point is $(-2, -2)$

$$f_{xx} = -2 \quad f_{yy} = -2 \quad f_{xy} = 1$$

$$f_{xx} f_{yy} - f_{xy}^2 = (-2)(-2) - 1^2 = 3 > 0$$

$$f_{xx} = -2 < 0$$

The function *f* has a local maximum at $(-2, -2)$ and the value is

$$f(-2, -2) = (-2)(-2) - (-2)^2 - (-2)^2 - 2(-2) - 2(-2) + 4 \underline{= 8}$$

Example

Find the local extreme values of $f(x,y)=3y^2-2y^3-3x^2+6xy$

Solution

$$f_x=-6x+6y=0 \quad \textit{and} \quad f_y=6y-6y^2+6x=0$$

$$\begin{cases} -6x+6y=0 \\ 6y-6y^2+6x=0 \end{cases} \to \quad x=y \qquad 6y-6y^2+6y=-6y(y-2)=0$$

$$\begin{cases} y=0=x & \boxed{(0,\ 0)} \\ y=2=x & \boxed{(2,\ 2)} \end{cases} \text{ are the critical points}$$

$$f_{xx}=-6 \quad f_y=6-12y \quad f_{xy}=6$$

$$\begin{aligned} f_{xx}f_{yy}-f_{xy}^2 &= (-6)(6-12y)-6^2 \\ &= -36+72y-36 \\ &= 72(y-1) \end{aligned}$$

At $(0,\ 0)$

$$f_{xx}f_{yy}-f_{xy}^2=-72<0$$

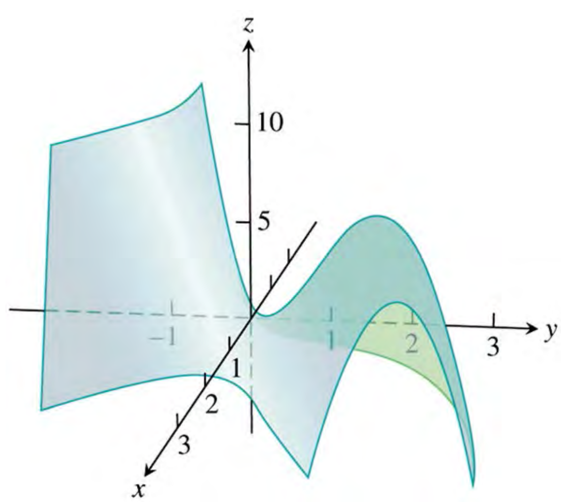
So the function has a saddle point at the origin.

At $(2,\ 2)$

$$f_{xx}f_{yy}-f_{xy}^2=72>0 \text{ and } f_{xx}=-6<0$$

So the function has a local maximum at $(2,\ 2)$ with a value of

$$f(2,\ 2)=12-26-12+24=8$$



Absolute Maxima and Minima on Closed Bounded Regions

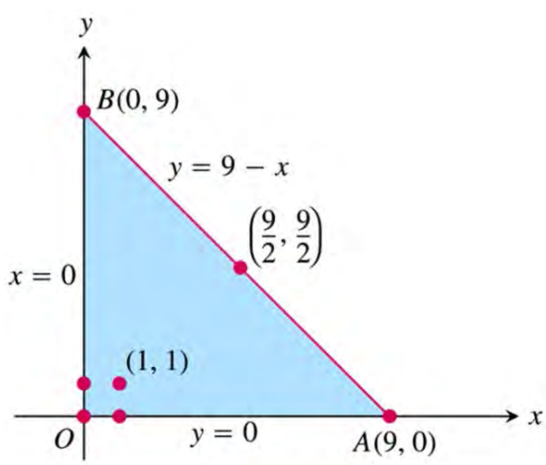
The absolute extrema of a continuous function $f(x, y)$ on a closed and bounded region *R* into three steps

1. *List the interior points of R* where *f* may have local maxima and minima and evaluate *f* at these points. These are the critical points of *f*.
2. *List the boundary points* of *R* where *f* may have local maxima and minima and evaluate *f* at these points.
3. *Look through the lists* for the maximum and minimum values of *f*. These will be the absolute maximum and minimum values of *f* on *R*. Since absolute maxima and minima are also local maxima and minima, the absolute maximum and minimum values of appear somewhere in the lists made in Steps 1 and 2

Example

Find the absolute maximum and minimum values of $f(x, y) = 2 + 2x + 2y - x^2 - y^2$ on the triangular region in the first quadrant bounded by the lines $x = 0, \quad y = 0, \quad y = 9 - x$

Solution



$f_x = 2 - 2x = 0 \quad f_y = 2 - 2y = 0$

$x = 1 \quad y = 1$

The critical point is $(1, 1)$. The value of *f* is

$f(1, 1) = 2 + 2 + 2 - 1 - 1 = \underline{4|}$

Boundary points:

i. On the segment *OA*, *y* = 0. The function

$$f(x, 0) = 2 + 2x - x^2$$

This function is defined on the closed interval $0 \le x \le 9$.

$$\begin{cases} x = 0 & \to f(0, 0) = \underline{2|} \\ x = 9 & \to f(9, 0) = 2 + 18 - 81 = \underline{-61|} \end{cases}$$

At the interior points where $f_x = 0$. The only point is $x = 1$ where $f(1, 0) = 3$

ii. On the segment *OB*, *x* = 0. The function

$$f(0, y) = 2 + 2y - y^2$$

$$f(0, 0) = 2, \quad f(0, 9) = -61, \quad f(1, 0) = 3$$

iii. Left the interior points of the segment *AB*. With $y = 9 - x$, then

$$f(x,\ y) = 2 + 2x + 2(9-x) - x^2 - (9-x)^2$$
$$= 2 + 2x + 18 - 2x - x^2 - 81 + 18x - x^2$$
$$= -2x^2 + 18x - 61$$

$$f'(x, 9-x) = -4x + 18 = 0 \quad \Rightarrow \quad \boxed{x = \frac{9}{2}}$$

At $x = \frac{9}{2} \quad \Rightarrow \quad y = 9 - x = \frac{9}{2}$

$$f\left(\frac{9}{2},\ \frac{9}{2}\right) = 2 + 2\left(\frac{9}{2}\right) + 2\left(9 - \frac{9}{2}\right) - \left(\frac{9}{2}\right)^2 - \left(9 - \frac{9}{2}\right)^2$$
$$= -\frac{41}{2}$$

$\therefore\ 4,\ 2,\ -61,\ 3,\ -\frac{41}{2}$. The maximum is 4, which *f* assumes at (1, 1). The minimum is –61, which *f* assumes at (0, 9) and (9, 0).

Example

A delivery company accepts only rectangular boxes the sum of whose length and girth (perimeter of a cross-section) does not exceed 108 *in*. Find the dimensions of an acceptable box of largest volume.

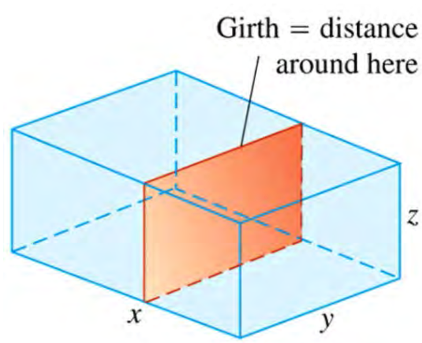
Solution

Let *x*, *y*, and *z* represent the length, width, and height.

The girth is: $= 2y + 2z\,(= P)$

Volume: $V = xyz$

We want to maximize the volume of the box satisfying:



$$x + 2y + 2z = 108$$
$$x = 108 - 2y - 2z$$

$$V(y,z) = (108 - 2y - 2z)yz$$
$$= 108yz - 2y^2z - 2yz^2$$

$$V_y(y,z) = 108z - 4yz - 2z^2$$
$$= 2z(54 - 2y - z) = 0$$

$$V_z(y,z) = 108y - 2y^2 - 4yz$$
$$= 2y(54 - y - 2z) = 0$$

$$\begin{cases} 2z(54-2y-z)=0 & \to \boxed{z=0} \quad 54-2y-z=0 \\ 2y(54-y-2z)=0 & \to \boxed{y=0} \quad 54-y-2z=0 \end{cases}$$

$$\begin{cases} 2y+z=54 \\ y+2z=54 \end{cases} \to \boxed{y=z=18}$$

$$\begin{cases} if\ \ y=0 \quad 54-2y-z=0 & \Rightarrow z=54 \to \boxed{(0,\ 54)} \\ if\ \ z=0 \quad 54-y-2z=0 & \Rightarrow y=54 \to \boxed{(54,\ 0)} \end{cases}$$

$\therefore$ The critical points are: $(0,\ 0),\ (0,\ 54),\ (54,\ 0),\ (18,\ 18)$

At (0, 0): $V(0,0)=108yz-2y^2z-2yz^2\Big|_{(0,0)}=0$

At (0, 54): $V(0,54)=108yz-2y^2z-2yz^2\Big|_{(0,54)}=0$

At (54, 0): $V(54,0)=108yz-2y^2z-2yz^2\Big|_{(54,0)}=0$

At (18, 18): $V(18,18)=108yz-2y^2z-2yz^2\Big|_{(18,18)}=11664$

$V_{yy}=-4z,\quad V_{zz}=-4y,\quad V_{yz}=108-4y-4z$

$$\begin{aligned} V_{xx}V_{yy}-V_{xy}^2 &= (-4z)(-4y)-(108-4y-4z)^2 \\ &= \left[16yz-16(27-y-z)^2\right]_{(18,18)} \\ &= 16(18)(18)-16(27-18-18)^2 \\ &= 3888 > 0 \end{aligned}$$

$V_{yy}(18,18)=-4(18)<0$

That implies (18, 18) give a maximum volume.

$\underline{x}=108-2(18)-2(18)\ \underline{=36}$

$\underline{V}=xyz=36(18)(18)\ \underline{=11,664}$

The dimensions of the package are: $\boldsymbol{x}=36$ *in*., $\boldsymbol{y}=18$ *in*, $\boldsymbol{z}=18$ *in*.

The maximum volume is $11,664\ \ in^3$

Summary of Max–Min Tests

The extreme values of $f(x, y)$ can occur only at

i. ***Boundary points*** of the domain of *f*.

ii. ***Critical points*** (interior points where $f_x = f_y = 0$ or points where f_x *or* f_y fail to exist)

If the first- and second-order partial derivatives of *f* are continuous throughout a disk centered at a point $(a,\ b)$ and $f_x(a,\ b) = f_y(a,\ b) = 0$, the nature of $f(a,\ b)$ can be tested with the ***Second Derivative Test***:

i. $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at $(a,\ b)$ $\Rightarrow$ ***local maximum***

ii. $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at $(a,\ b)$ $\Rightarrow$ ***local minimum***

iii. $f_{xx} f_{yy} - f_{xy}^2 < 0$ at $(a,\ b)$ $\Rightarrow$ ***saddle point***

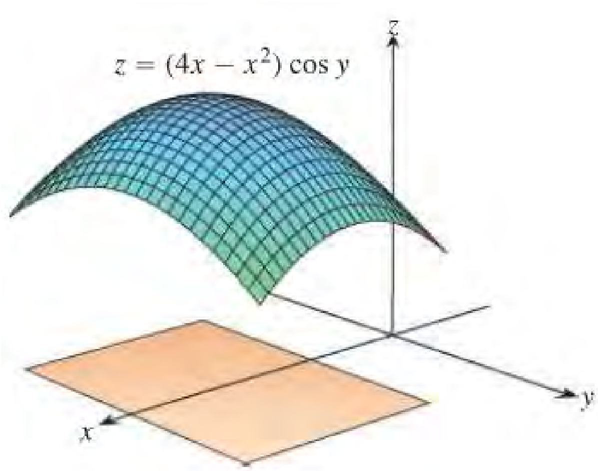
iv. $f_{xx} f_{yy} - f_{xy}^2 = 0$ at $(a,\ b)$ $\Rightarrow$ ***test is inconclusive.***

Exercises 2.7 – Maximum/Minimum Problems

Find all the local maxima, local minima, and saddle points of the function

1. $f(x,y)=x^2+xy+y^2+3x-3y+4$
2. $f(x,y)=2xy-5x^2-2y^2+4x+4y-4$
3. $f(x,y)=x^2-4xy+y^2+6y+2$
4. $f(x,y)=2x^2+3xy+4y^2-5x+2y$
5. $f(x,y)=x^2-y^2-2x+4y+6$
6. $f(x,y)=\sqrt{56x^2-8y^2-16x-31}+1-8x$
7. $f(x,y)=1-\sqrt[3]{x^2+y^2}$
8. $f(x,y)=x^3+y^3+3x^2-3y^2-8$
9. $f(x,y)=4xy-x^4-y^4$
10. $f(x,y)=\dfrac{1}{x^2+y^2-1}$
11. $f(x,y)=\dfrac{1}{x}+xy+\dfrac{1}{y}$
12. $f(x,y)=y\sin x$
13. $f(x,y)=e^{2x}\cos y$
14. $f(x,y)=e^y-ye^x$
15. $f(x,y)=e^{-y}\left(x^2+y^2\right)$
16. $f(x,y)=2\ln x+\ln y-4x-y$
17. $f(x,y)=\ln(x+y)+x^2-y$

18. Find the absolute maxima and minima of the function $f(x,y)=2x^2-4x+y^2-4y+1$ on the closed triangular plate bounded by the lines $x=0,\quad y=2,\quad y=2x$ in the first quadrant.
19. Find the absolute maxima and minima of the function $D(x,y)=x^2-xy+y^2+1$ on the closed triangular plate bounded by the lines $x=0,\quad y=4,\quad y=x$ in the first quadrant.
20. Find the absolute maxima and minima of the function $T(x,y)=x^2+xy+y^2-6x+2$ on the triangular plate $0\le x\le 5,\quad -3\le y\le 0$.
21. Find the absolute maxima and minima of the function $f(x,y)=\left(4x-x^2\right)\cos y$ on the triangular plate $1\le x\le 3,\quad -\frac{\pi}{4}\le y\le \frac{\pi}{4}$.



22. Find the point on the graph of $z=x^2+y^2+10$ nearest the plane $x+2y-z=0$

23. Find the minimum distance from the point $(2,\ -1,\ 1)$ to the plane $x + y - z = 2$

24. Find the maximum value of $s = xy + yz + xz$ where $x + y + z = 6$

Section 2.8 – Lagrange Multipliers

Constrained Maxima and Minima

We consider a problem where a constrained minimum can be found by eliminating a variable.

Example

Find the point $P(x,y,z)$ on the plane $2x+y-z-5=0$ that is closest to the origin.

Solution

$$\left|\overrightarrow{OP}\right| = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$$
$$= \sqrt{x^2+y^2+z^2}$$

Subject to the constraint that $2x+y-z-5=0$

Since $\left|\overrightarrow{OP}\right|$ has a minimum value wherever the function $f(x,y,z)=x^2+y^2+z^2$ has a minimum value.

$2x+y-z-5=0 \ \Rightarrow \ z=2x+y-5$

$$h(x,y)=f(x,\ y,\ 2x+y-5)$$
$$=x^2+y^2+(2x+y-5)^2$$

$h_x = 2x+2(2x+y-5)(2)$ $=10x+4y-20=0$	$h_y = 2y+2(2x+y-5)(1)$ $=4x+4y-10=0$

$$\to \begin{cases} 10x+4y=20 \\ 4x+4y=10 \end{cases} \ \Rightarrow \ \boxed{x=\frac{5}{3},\ y=\frac{5}{6}}$$

$$z=2\left(\frac{5}{3}\right)+\frac{5}{6}-5=-\frac{5}{6}$$

Therefore, the closest point to the origin is: $P\left(\frac{5}{3},\ \frac{5}{6},\ -\frac{5}{6}\right)$

The distance from P to the origin is: $\sqrt{\left(\frac{5}{3}\right)^2+\left(\frac{5}{6}\right)^2+\left(-\frac{5}{6}\right)^2} \ \approx 2.04$

Example

Find the points on the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ that are closest to the origin.

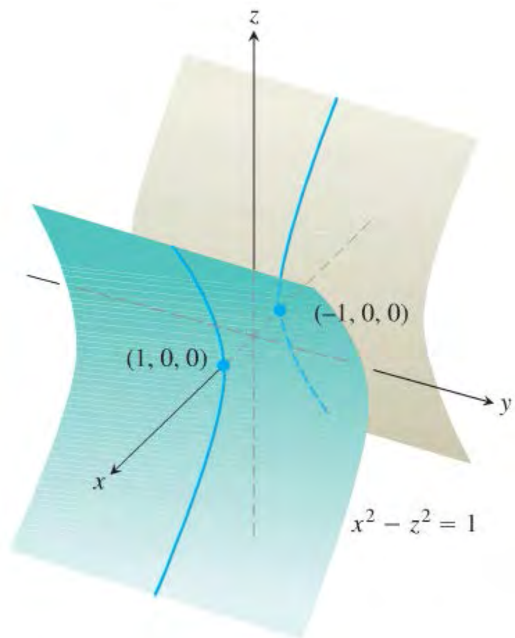
Solution

The points closest to the origin are the points whose coordinates minimize the value of the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint that $x^2 - z^2 - 1 = 0$

$x^2 - z^2 - 1 = 0 \quad \rightarrow \quad z^2 = x^2 - 1$

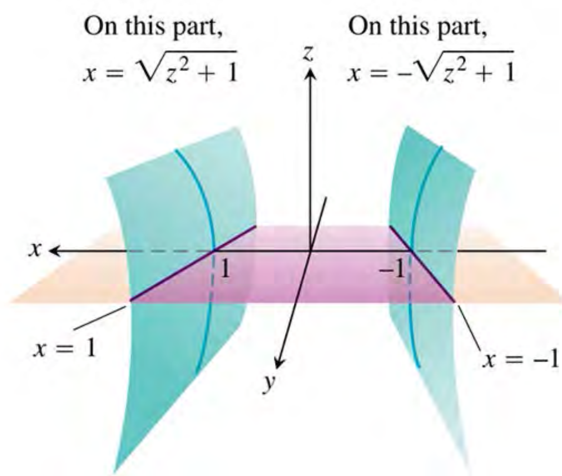
$$\begin{aligned} h(x, y) &= f\left(x,\ y,\ \sqrt{x^2 - 1}\right) \\ &= x^2 + y^2 + \left(x^2 - 1\right) \\ &= 2x^2 + y^2 - 1 \end{aligned}$$

$h_x = 4x = 0$	$h_y = 2y = 0$



That is, at the point (0, 0) ????

The domain of h is the entire *xy*-plane, the domain from which we can select the first two coordinates of the points (x, y, z) on the cylinder is restricted to the shadow of the cylinder on the xy-plane; it does not include the band between the lines $x = -1$ and $x = 1$.



$x^2 - z^2 - 1 = 0 \quad \rightarrow \quad x^2 = z^2 + 1$

$$\begin{aligned} k(y, z) &= f\left(z^2 + 1,\ y,\ z\right) \\ &= z^2 + 1 + y^2 + z^2 \\ &= y^2 + 2z^2 + 1 \end{aligned}$$

$k_y = 2y = 0$	$k_z = 4z = 0$

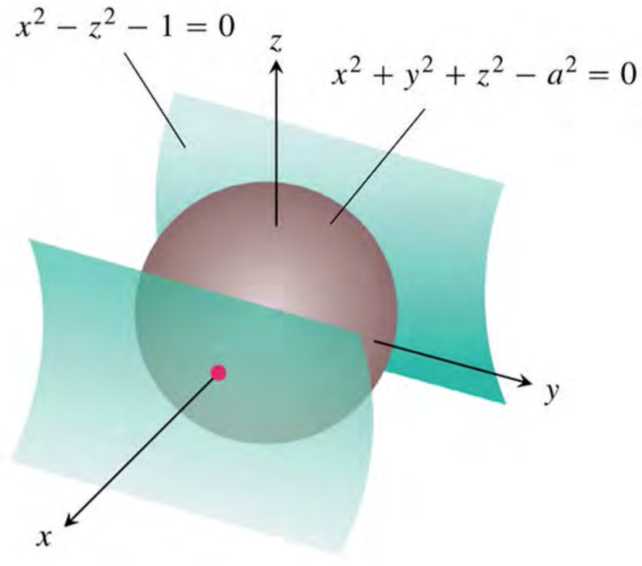
That implies to $y = z = 0$ and which leads to $x^2 = z^2 + 1 = 1 \quad \rightarrow \quad x = \pm 1$

The corresponding points on the cylinder are $(\pm 1,\ 0,\ 0)$.

$k(y, z) = y^2 + 2z^2 + 1 \geq 1$ gives a minimum value for *k*. We can also see that the minimum distance from the origin to a point on the cylinder is 1 unit.

Solution 2

Another way to find the points on the cylinder closet to the origin is to imagine a small sphere centered at the origin expanding until it touches the cylinder



$f(x,y,z) = x^2 + y^2 + z^2 - a^2$ *and* $g(x,y,z) = x^2 - z^2 - 1$

$\nabla f = \lambda \nabla g$

$2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k} = \lambda(2x\boldsymbol{i} - 2z\boldsymbol{k})$ $\qquad \nabla f = f_x\boldsymbol{i} + f_y\boldsymbol{j} + f_z\boldsymbol{k}$

$2x = 2\lambda x, \quad 2y = 0, \quad 2z = -2\lambda z$

Since that no point on the surface has a zero x-coordinate to conclude that $x \neq 0$.
Hence, $2x = 2\lambda x$ only if

$$2 = 2\lambda \quad \Rightarrow \quad \boxed{\lambda = 1}$$

For $\lambda = 1 \quad \rightarrow \quad 2z = -2\lambda z = -2z$, for this to satisfies, z must be zero.

Also $2y = 0 \Rightarrow y = 0$

We conclude that the points have coordinates of the form $(x,\ 0,\ 0)$

$x^2 = z^2 + 1 = 1 \quad \rightarrow \quad x = \pm 1$

The points on the cylinder closest to the origin are the points $(\pm 1,\ 0,\ 0)$.

The Method of *Lagrange* Multipliers

The method of Lagrange multipliers:

$$\nabla f = \lambda \nabla g$$

For some scalar λ (called a ***Lagrange multiplier***)

Theorem – The orthogonal Gradient Theorem

Suppose that $f(x,y,z)$ is differentiable in a region whose interior contains a smooth curve

$$C: \quad \boldsymbol{r}(t) = g(t)\boldsymbol{i} + h(t)\boldsymbol{j} + k(t)\boldsymbol{k}$$

If P_0 is a point on C where *f* has a local maximum or minimum relative to the values on C, then ∇f is orthogonal to *C* at P_0.

Corollary

At the points on a smooth curve $\boldsymbol{r}(t) = g(t)\boldsymbol{i} + h(t)\boldsymbol{j}$ where differentiable function $f(x,y)$ takes on its local maxima and minima relative to its values on the curve, $\nabla f \bullet \boldsymbol{v} = 0$ where $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$.

The Method of Lagrange Multipliers

Suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiable and $\nabla g \neq 0$ when $g(x,y,z) = 0$. To find the local maximum and minimum values of *f* subject to the constraint $g(x,y,z) = 0$ (if these exist), find the values of *x*, *y*, *z*, and λ that simultaneously satisfy the equations

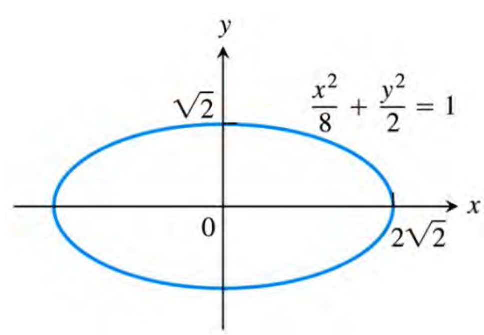
$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x,y,z) = 0$$

For functions of two independent variables, the condition is similar, but without the variables *z*.

Example

Find the greatest and smallest values that the function $f(x,y) = xy$ takes on the ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} = 1$$



Solution

$f(x,y) = xy$ subject to the constraint

$$g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$$

We need to find: $\nabla f = \lambda \nabla g$ *and* $g(x, y, z) = 0$

$y\boldsymbol{i} + x\boldsymbol{j} = \frac{1}{4}\lambda x\boldsymbol{i} + \lambda y\boldsymbol{j}$ $\qquad\qquad$ $\nabla f = f_x \boldsymbol{i} + f_y \boldsymbol{j}$

$y = \frac{1}{4}\lambda x, \quad x = \lambda y$

$y = \frac{1}{4}\lambda(\lambda y) = \frac{1}{4}\lambda^2 y$

$y = 0$ *or* $1 = \frac{1}{4}\lambda^2$

$\lambda^2 = 4 \Rightarrow \lambda = \pm 2$

Consider these two cases:

***Case* 1**: If $y = 0$, then $x = y = 0$. But (0, 0) is not on the ellipse. Hence, $y \neq 0$.

***Case* 2**: If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$.

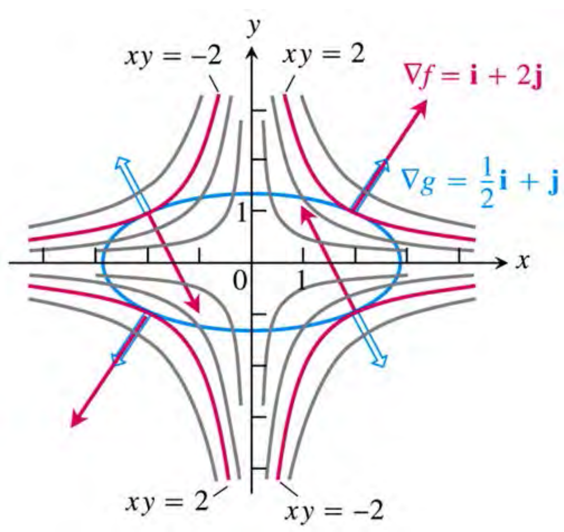
$$g(x, y) = \frac{(\pm 2y)^2}{8} + \frac{y^2}{2} - 1 = 0$$

$$\frac{y^2}{2} + \frac{y^2}{2} = 1$$

$$y^2 = 1 \Rightarrow \boxed{y = \pm 1}$$

Therefore, $f(x, y) = xy$ takes on its extreme values on the ellipse at the points $(\pm 2, \pm 1)$.

The extreme values are $xy = 2$ *and* $xy = -2$



The Geometry of the solution: The level curves of the function $f(x, y) = xy$ are the hyperbolas $xy = c$

At the point (2, 1): $\quad \nabla f = y\boldsymbol{i} + x\boldsymbol{j} = \boldsymbol{i} + 2\boldsymbol{j}, \quad \nabla g = \frac{1}{4}\lambda x\boldsymbol{i} + \lambda y\boldsymbol{j} = \frac{1}{2}\boldsymbol{i} + \boldsymbol{j}, \quad \nabla f = 2\nabla g$

At the point (–2, 1): $\quad \nabla f = \boldsymbol{i} - 2\boldsymbol{j}, \quad \nabla g = -\frac{1}{2}\boldsymbol{i} + \boldsymbol{j}, \quad \nabla f = -2\nabla g$

Example

Find the maximum and minimum values that the function $f(x,y)=3x+4y$ on the circle $x^2+y^2=1$

Solution

$f(x,y)=3x+4y \qquad g(x,y)=x^2+y^2-1=0$

$\nabla f = f_x\boldsymbol{i}+f_y\boldsymbol{j}=3\boldsymbol{i}+4\boldsymbol{j} \qquad \nabla g=2x\boldsymbol{i}+2y\boldsymbol{j}$

$\nabla f=\lambda\nabla g$

$3\boldsymbol{i}+4\boldsymbol{j}=2\lambda x\boldsymbol{i}+2\lambda y\boldsymbol{j}$

$2\lambda x=3, \quad 2\lambda y=4$

$x=\frac{3}{2\lambda}, \quad y=\frac{2}{\lambda}$

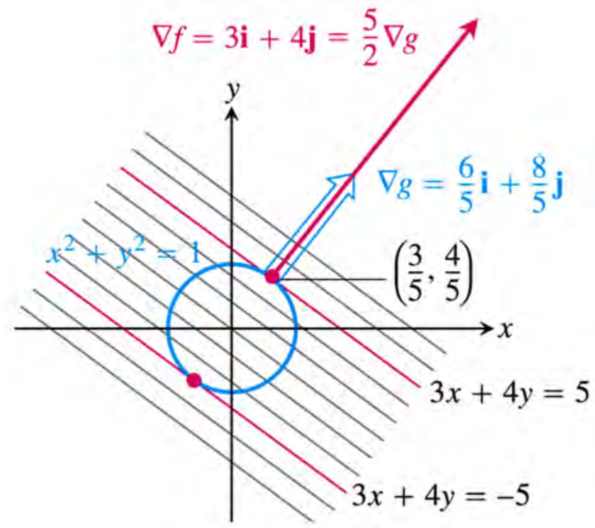
$\left(\frac{3}{2\lambda}\right)^2+\left(\frac{2}{\lambda}\right)^2-1=0$

$\frac{9}{4\lambda^2}+\frac{4}{\lambda^2}=1$

$9+16=4\lambda^2$

$25=4\lambda^2$

$\lambda^2=\frac{25}{4} \quad \rightarrow \quad \boxed{\lambda=\pm\frac{5}{2}}$



$x=\frac{3}{2\lambda}=\pm\frac{3}{5}, \quad y=\frac{2}{\lambda}=\pm\frac{4}{5}$

Therefore, $f(x,y)=3x+4y$ has extreme values $\left(\pm\frac{3}{5},\ \pm\frac{4}{5}\right)$

$f\left(\frac{3}{5},\ \frac{4}{5}\right)=3\left(\frac{3}{5}\right)+4\left(\frac{4}{5}\right)=5$

$f\left(-\frac{3}{5},\ -\frac{4}{5}\right)=3\left(-\frac{3}{5}\right)+4\left(-\frac{4}{5}\right)=-5$

The Geometry of the solution: The level curves of the function $f(x,y)=3x+4y$ are the lines $3x+4y=c$

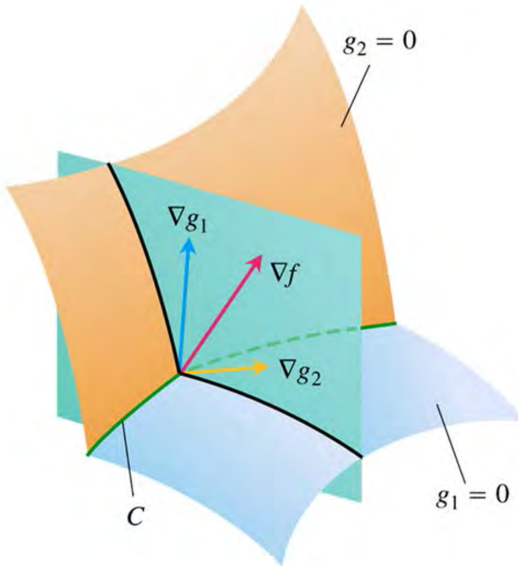
At the point $\left(\frac{3}{5},\ \frac{4}{5}\right)$: $\quad \nabla f=3\boldsymbol{i}+4\boldsymbol{j}, \quad \nabla g=\frac{6}{5}\boldsymbol{i}+\frac{8}{5}\boldsymbol{j}, \quad \nabla f=\frac{5}{2}\nabla g$

Lagrange Multipliers with Two Constraints

To find the extreme values of a differentiable function $f(x, y, z)$ whose variables are subject to two constraints. If the constraints are

$$g_1(x, y, z) = 0 \quad and \quad g_2(x, y, z) = 0$$

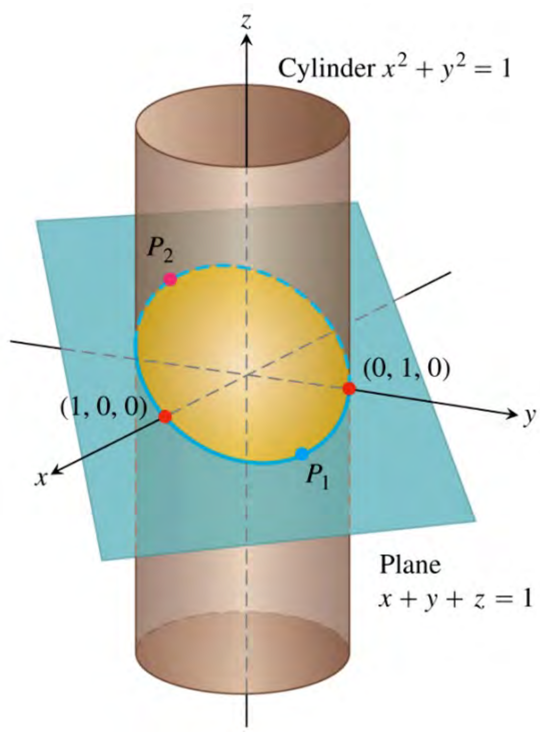
g_1 *and* g_2 are differentiable, with ∇g_1 not parallel to ∇g_2



$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \quad g_1(x, y, z) = 0, \quad g_2(x, y, z) = 0$$

Example

The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse. Find the points on the ellipse that lie closet to and farthest from the origin.



Solution

$$f(x, y, z) = x^2 + y^2 + z^2$$

$$g_1(x, y, z) = x^2 + y^2 - 1 = 0$$

$$g_2(x, y, z) = x + y + z - 1 = 0$$

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

$$2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k} = \lambda(2x\boldsymbol{i} + 2y\boldsymbol{j}) + \mu(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$$

$$2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k} = (2\lambda x + \mu)\boldsymbol{i} + (2\lambda y + \mu)\boldsymbol{j} + \mu\boldsymbol{k}$$

$2z = \mu$	$2x = 2\lambda x + \mu$ $2(1-\lambda)x = \mu = 2z$ $(1-\lambda)x = z$	$2y = 2\lambda y + \mu$ $2(1-\lambda)y = \mu = 2z$ $(1-\lambda)y = z$

$(1-\lambda)x = z = (1-\lambda)y$

These satisfy if either λ = 1 and z = 0 or λ ≠ 1 and $x = y = \frac{z}{1-\lambda}$

If z = 0,

$$\begin{cases} x^2 + y^2 - 1 = 0 \\ x + y - 1 = 0 \end{cases} \rightarrow \quad x = 1 - y$$

$$(1-y)^2 + y^2 - 1 = 0$$

$$1 - 2y + y^2 + y^2 - 1 = 0$$

$$2y(y-1) = 0 \quad \Rightarrow \quad \begin{cases} y = 0 & \rightarrow x = 1 \\ y = 1 & \rightarrow x = 0 \end{cases}$$

The points are: $(1, 0, 0)$ and $(0, 1, 0)$

If $x = y$,

$x^2 + y^2 - 1 = 0$ $x^2 + x^2 - 1 = 0$ $2x^2 = 1$ $x^2 = \frac{1}{2}$ $x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$	$x + y + z - 1 = 0$ $2x + z = 1$ $z = 1 - 2x$ $z = 1 \pm \sqrt{2}$

The points are: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1-\sqrt{2}\right)$ and $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1+\sqrt{2}\right)$

The points on the ellipse closest to the origin are $(1, 0, 0)$ and $(0, 1, 0)$. The point on the farthest from the origin is $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1+\sqrt{2}\right)$.

Exercises *Section* 2.8 – Lagrange Multipliers

1. Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x, y) = xy$ has its extreme values.

2. Find the extreme values of $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 10 = 0$.

3. Find the maximum value of $f(x, y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$.

4. Find the points on the curve $x^2 y = 2$ nearest the origin.

5. Use the method of Lagrange multipliers to find

a) The minimum value of $x + y$, subject to the constraints $xy = 16, \quad x > 0, \quad y > 0$

b) The maximum value of xy, subject to the constraints $x + y = 16$

6. Find the radius and height of the open right circular cylinder of largest surface area that can be inscribed in a sphere of radius *a*. What is the largest surface area?

7. Use the method of Lagrange multipliers to find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ with sides parallel to the coordinate axes.

8. Find the maximum and minimum values of $x^2 + y^2$ subject to the constraint $x^2 - 2x + y^2 - 4y = 0$

9. The temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant on the plate walks around the circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant?

10. Your firm has been asked to design a storage tank for liquid petroleum gas. The customer's specifications call for a cylindrical tank with hemispherical ends, and the tank is to hold $8000\ m^3$ of gas. He customer also wants to use the smallest amount of material possible in building the tank. What radius and height do you recommend for the cylindrical portion of the tank?

11. Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1,\ 1,\ 1)$

12. Find the point on the sphere $x^2 + y^2 + z^2 = 4$ farthest from the point $(1,\ -1,\ 1)$

13. Find the minimum distance from the surface $x^2 - y^2 - z^2 = 1$ to the origin

14. Find the maximum and minimum values of $f(x, y, z) = x - 2y + 5z$ on the sphere $x^2 + y^2 + z^2 = 30$

15. Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

16. A space probe in the shape of the ellipsoid $4x^2 + y^2 + 4z^2 = 16$ enters Earth's atmosphere and its surface begins to heat. After 1 hour, the temperature at the point (x, y, z) on the probe's surface is $T(x, y, z) = 8x^2 + 4yz - 16z + 600$.

Find the hottest point on the probe's surface.