Solution   *Section* 3.7 – Trigonometric Form

Exercise

Write $-\sqrt{3}+i$ in trigonometric form. (Use radian measure)

Solution

$$-\sqrt{3}+i \Rightarrow \begin{cases} x=-\sqrt{3} \\ y=1 \end{cases}$$

$$r=\sqrt{\left(-\sqrt{3}\right)^2+1^2}=2$$

$$\tan\theta=\frac{y}{x}=\frac{1}{-\sqrt{3}}=-\frac{\sqrt{3}}{3}$$

The reference angle for θ is $\frac{\pi}{6}$ and the angle is in quadrant II.

Therefore, $\boxed{\theta=\pi-\frac{\pi}{6}=\frac{5\pi}{6}}$

$$\boxed{-\sqrt{3}+i=2\ cis\frac{5\pi}{6}}$$

Exercise

Write $3-4i$ in trigonometric form.

Solution

$$3-4i \Rightarrow \begin{cases} r=\sqrt{3^2+(-4)^2}=5 \\ \hat{\theta}=\tan^{-1}\left(\frac{4}{3}\right)\approx 53° \end{cases}$$

The angle is in quadrant *II*; therefore, $\boxed{\theta=180°-53°=127°}$

$$\boxed{3-4i=5\ cis127°}$$

Exercise

Write $-21-20i$ in trigonometric form.

Solution

$$-21-20i \Rightarrow \begin{cases} r=\sqrt{(-21)^2+(-20)^2}=29 \\ \hat{\theta}=\tan^{-1}\left(\frac{20}{21}\right)\approx 43.6° \end{cases}$$

The angle is in quadrant *III*; therefore, $\boxed{\theta=180°+43.6°=223.6°}$

$$\boxed{-21-20i=29\ cis223.6°}$$

Exercise

Write $11+2i$ in trigonometric form.

Solution

$$11+2i \Rightarrow \begin{cases} r=\sqrt{11^2+2^2}=\sqrt{125}=5\sqrt{5} \\ \hat{\theta}=\tan^{-1}\left(\frac{2}{11}\right)\approx 10.3^\circ \end{cases}$$

The angle is in quadrant I; therefore, $\boxed{\theta=10.3^\circ}$

$$\boxed{11+2i=5\sqrt{5}\ cis10.3^\circ}$$

Exercise

Write $\sqrt{3}-i$ in trigonometric form.

Solution

$$r=\sqrt{3+1}=2$$

$$\hat{\theta}=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\approx 30^\circ \xrightarrow{QIV} \theta=360^\circ-30^\circ=330^\circ$$

$$\sqrt{3}-i\ \boxed{=2\ cis330^\circ}$$

Exercise

Write $1-\sqrt{3}i$ in trigonometric form.

Solution

$$r=\sqrt{1+3}=2$$

$$\hat{\theta}=\tan^{-1}\left(\sqrt{3}\right)\approx 60^\circ \xrightarrow{QIV} \theta=360^\circ-60^\circ=300^\circ$$

$$1-\sqrt{3}i\ \boxed{=2\ cis300^\circ}$$

Exercise

Write $9\sqrt{3}+9i$ in trigonometric form.

Solution

$$r=9\sqrt{3+1}=18$$

$$\theta=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\approx 30^\circ$$

$$9\sqrt{3}+9i\ \boxed{=18\ cis30^\circ}$$

Exercise

Write $-2+3i$ in trigonometric form.

Solution

$$r=\sqrt{4+9}=\sqrt{13}$$

$$\hat{\theta}=\tan^{-1}\left(\frac{3}{2}\right)\approx 56.31^\circ \xrightarrow{QII} \theta=180^\circ-56.31^\circ=123.69^\circ$$

$$-2+3i \ \underline{=\sqrt{13}\ cis123.69^\circ}$$

Exercise

Write $4\left(\cos 30^\circ+i\sin 30^\circ\right)$ in standard form.

Solution

$$4\left(\cos 30^\circ+i\sin 30^\circ\right)=4\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)$$

$$\underline{=2\sqrt{3}+2i}$$

Exercise

Write $\sqrt{2}\ cis\frac{7\pi}{4}$ in standard form.

Solution

$$\sqrt{2}\ cis\frac{7\pi}{4}=\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$$

$$=\sqrt{2}\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)$$

$$\underline{=1-i}$$

Exercise

Write $3cis210^\circ$ in standard form.

Solution

$$3cis210^\circ=3\left(\cos 210^\circ+i\sin 210^\circ\right)$$

$$\underline{=-\frac{3\sqrt{3}}{2}-\frac{3}{2}i}$$

Exercise

Write $4\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$ in standard form.

Solution

$$4\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)=4\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)$$
$$=2\sqrt{2}-2i\sqrt{2}$$

Exercise

Write $4cis\frac{\pi}{2}$ in standard form.

Solution

$$4cis\frac{\pi}{2}=4\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$
$$=4i$$

Exercise

Find the quotient $\dfrac{20cis(75°)}{4cis(40°)}$. Write the result in rectangular form.

Solution

$$\frac{20cis(75°)}{4cis(40°)}=\frac{20}{4}cis(75°-40°)$$
$$=5cis(35°)$$
$$=5(\cos 35°+i\sin 35°)$$
$$=4.1+2.87i$$

Exercise

Divide $z_1=1+i\sqrt{3}$ *by* $z_2=\sqrt{3}+i$. Write the result in rectangular form.

Solution

$$\frac{z_1}{z_2}=\frac{1+i\sqrt{3}}{\sqrt{3}+i}$$
$$=\frac{1+i\sqrt{3}}{\sqrt{3}+i}\frac{\sqrt{3}-i}{\sqrt{3}-i}$$
$$=\frac{\sqrt{3}-i+3i-\sqrt{3}\,i^2}{3+1}$$

$\boxed{or}$ $1+i\sqrt{3}:\begin{cases} r=\sqrt{1^2+\left(\sqrt{3}\right)^2} \\ \theta=\tan^{-1}\frac{\sqrt{3}}{1}=\frac{\pi}{3}\end{cases}$

$\sqrt{3}+i:\begin{cases} r=\sqrt{\left(\sqrt{3}\right)^2+1^2} \\ \theta=\tan^{-1}\frac{1}{\sqrt{3}}=\frac{\pi}{6}\end{cases}$

$$\frac{z_1}{z_2}=\frac{2cis\frac{\pi}{3}}{2cis\frac{\pi}{6}}$$
$$=\frac{2}{2}cis\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$$

$$= \frac{2\sqrt{3}+2i}{4} \qquad\qquad = \frac{2}{2}cis\left(\frac{\pi}{6}\right)$$

$$= \frac{2\sqrt{3}}{4}+\frac{2i}{4} \qquad\qquad = cis\left(\frac{\pi}{6}\right)$$

$$\boxed{= \frac{\sqrt{3}}{2}+\frac{i}{2}} \qquad = \cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)$$

Exercise

Find $(1+i)^8$ and express the result in rectangular form.

Solution

$$1+i \;\Rightarrow\; \begin{cases} r=\sqrt{1^2+1^2}=\sqrt{2} \\ \theta=\tan^{-1}1=\frac{\pi}{4} \end{cases} \qquad \to 1+i=\sqrt{2}cis\frac{\pi}{4}$$

$$\begin{aligned}(1+i)^8 &= \left(\sqrt{2}cis\frac{\pi}{4}\right)^8 \\ &= \left(\sqrt{2}\right)^8 cis\left[8\left(\frac{\pi}{4}\right)\right] \\ &= 16cis2\pi \\ &= 16(\cos 2\pi + i\sin 2\pi) \\ &= 16(1+i0) \\ &\boxed{=16}\end{aligned}$$

Exercise

Find $(1+i)^{10}$ and express the result in rectangular form.

Solution

$$\begin{aligned}(1+i)^{10} &= \left(\sqrt{2}cis\frac{\pi}{4}\right)^{10} \\ &= \left(\sqrt{2}\right)^{10} cis\left[10\left(\frac{\pi}{4}\right)\right] \\ &= 32cis\frac{5\pi}{2} \\ &= 32cis\frac{\pi}{2} \\ &= 32\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right) \\ &= 32(0+i) \\ &\boxed{=32i}\end{aligned}$$

Exercise

Find and express the result in rectangular form $(1-i)^5$

Solution

$$r=\sqrt{1+1}=\sqrt{2}$$

$$\hat{\theta}=\tan^{-1}1=45^\circ \xrightarrow{QIV} \theta=360^\circ-45^\circ=315^\circ$$

$$\begin{aligned}(1-i)^5&=\left(\sqrt{2}\ cis315^\circ\right)^5\\&=4\sqrt{2}\left(\ cis(5\times 315^\circ)\right)\\&=4\sqrt{2}\left(\ cis(1575)\right) && 1575^\circ-4\times 360^\circ=135^\circ\\&=4\sqrt{2}\left(\cos 135^\circ+i\sin 135^\circ\right)\\&=4\sqrt{2}\left(-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)\\&=-4+4i\end{aligned}$$

Exercise

Find and express the result in rectangular form $\left(1-\sqrt{5}i\right)^8$

Solution

$$r=\sqrt{1+5}=\sqrt{6}$$

$$\hat{\theta}=\tan^{-1}\sqrt{5}\approx 66^\circ \xrightarrow{QIV} \theta=360^\circ-66^\circ=294^\circ$$

$$\begin{aligned}\left(1-\sqrt{5}i\right)^8&=\left(\sqrt{6}cis294^\circ\right)^8\\&=\left(\sqrt{6}\right)^8\left(cis2352^\circ\right) && 2352^\circ-6\times 360^\circ=192^\circ\\&=1296\left(\cos 192^\circ+i\sin 192^\circ\right)\\&=1296(-.978-0.208i)\\&=-1267.488-269.568\ i\end{aligned}$$

Exercise

Find and express the result in rectangular form $(3cis80^\circ)^3$

Solution

$$\begin{aligned}(3cis80^\circ)^3&=3^3\left(cis240^\circ\right)\\&=27\left(\cos 240^\circ+i\sin 240^\circ\right)\\&=27\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\\&=-\frac{27}{2}-i\frac{27\sqrt{3}}{2}\end{aligned}$$

Exercise

Find and express the result in rectangular form $\left(\sqrt{3}cis10°\right)^6$

Solution

$$\left(\sqrt{3}cis10°\right)^6 = 27\left(cis60°\right)$$
$$= 27\left(\cos 60° + i\sin 60°\right)$$
$$\boxed{= \frac{27}{2} + i\frac{27\sqrt{3}}{2}}$$

Exercise

Find and express the result in rectangular form $\left(\sqrt{2} - i\right)^6$

Solution

$$r = \sqrt{2+1} = \sqrt{3}$$
$$\hat{\theta} = \tan^{-1}\frac{1}{\sqrt{2}} \approx 35.26° \rightarrow \theta = 360° - 35.26° = 324.74°$$
$$\left(\sqrt{2} - i\right)^6 = \left(\sqrt{3}\ cis324.74°\right)^6$$
$$= 27\left(cis1948.44°\right) \qquad 1948.44° - 5\times 360° = 148.44°$$
$$= 27\left(\cos 148.44° + i\sin 148.44°\right)$$
$$\boxed{= -23 + 14.142i}$$

Exercise

Find and express the result in rectangular form $\left(4cis40°\right)^6$

Solution

$$\left(4cis40°\right)^6 = 4^6\left(cis\left(6\times 40°\right)\right)$$
$$= 4^6\left(\cos 240° + i\sin 240°\right)$$
$$= 4096\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$\boxed{= -2048 + 2048\ i\sqrt{3}}$$

Exercise

Find and express the result in rectangular form $\left(2cis30°\right)^5$

Solution

$$\left(2cis30°\right)^5 = 2^5 cis\left(5(30°)\right)$$

$$= 32\left(\cos 150° + i\sin 150°\right)$$

$$= 32\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$\underline{= -16\sqrt{3} + 16i}$$

Exercise

Find and express the result in rectangular form $\left(\frac{1}{2}cis72°\right)^5$

Solution

$$\left(\frac{1}{2}cis72°\right)^5 = \frac{1}{2^5}cis\left(5\times 72°\right)$$

$$= \frac{1}{32}cis\left(\cos 360° + i\sin 360°\right)$$

$$\underline{= \frac{1}{32}}$$

Exercise

Find fifth roots of $z = 1 + i\sqrt{3}$ and express the result in rectangular form.

Solution

$$1 + i\sqrt{3} \Rightarrow \begin{cases} r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2 \\ \theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60° \end{cases}$$

$$\left(1 + i\sqrt{3}\right)^{1/5} = \left(2cis60°\right)^{1/5}$$

$$= \sqrt[5]{2}\left(cis\frac{60°}{5} + \frac{360°k}{5}\right)$$

$$= \sqrt[5]{2}\ cis\left(12° + 72°k\right)$$

$$If\ \ k = 0 \Rightarrow \sqrt[5]{2}\ cis\left(12° + 72°0\right) = \underline{\sqrt[5]{2}\ cis12°}$$

$$If\ \ k = 1 \Rightarrow \sqrt[5]{2}\ cis\left(12° + 72°.(1)\right) = \underline{\sqrt[5]{2}\ cis84°}$$

$$If\ \ k = 2 \Rightarrow \sqrt[5]{2}\ cis\left(12° + 72°.(2)\right) = \underline{\sqrt[5]{2}\ cis156°}$$

$$If\ \ k = 3 \Rightarrow \sqrt[5]{2}\ cis\left(12° + 72°.(3)\right) = \underline{\sqrt[5]{2}\ cis228°}$$

$$If\ \ k = 4 \Rightarrow \sqrt[5]{2}\ cis\left(12° + 72°.(4)\right) = \underline{\sqrt[5]{2}\ cis300°}$$

Exercise

Find the fourth roots of $z = 16cis60°$

Solution

$$\sqrt[4]{z} = \sqrt[4]{16}\ cis\left(\frac{60°}{4} + \frac{360°}{4}k\right)$$
$$= 2cis\left(15° + 90°k\right)$$

$$If\ \ k = 0 \Rightarrow 2\ cis\left(15° + 90°(0)\right) = \boxed{2cis15°}$$

$$If\ \ k = 1 \Rightarrow 2\ cis\left(15° + 90°(1)\right) = \boxed{2cis105°}$$

$$If\ \ k = 2 \Rightarrow 2\ cis\left(15° + 90°(2)\right) = \boxed{2cis195°}$$

$$If\ \ k = 3 \Rightarrow 2\ cis\left(15° + 90°(3)\right) = \boxed{2cis285°}$$

Exercise

Find the fourth roots of $\sqrt{3} - i$

Solution

$$r = \sqrt{3+1} = 2$$

$$\hat{\theta} = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6} \xrightarrow{QIV} \theta = \frac{11\pi}{6}$$

$$\sqrt[4]{\sqrt{3} - i} = \sqrt[4]{2\ cis\frac{11\pi}{6}}$$
$$= \sqrt[4]{2}\ cis\left(\frac{1}{4}\frac{11\pi}{6} + \frac{2\pi k}{4}\right)$$
$$= \sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + \frac{\pi k}{2}\right)$$

$$k = 0 \Rightarrow \sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + 0\right) = \boxed{\sqrt[4]{2}\ cis\frac{11\pi}{24}}$$

$$k = 1 \Rightarrow \sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + \frac{\pi}{2}\right) = \boxed{\sqrt[4]{2}\ cis\frac{23\pi}{24}}$$

$$k = 2 \Rightarrow \sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + \pi\right) = \boxed{\sqrt[4]{2}\ cis\frac{35\pi}{24}}$$

$$k = 3 \Rightarrow \sqrt[4]{2}\ cis\left(\frac{11\pi}{24} + \frac{3\pi}{2}\right) = \boxed{\sqrt[4]{2}\ cis\frac{47\pi}{24}}$$

Exercise

Find the fourth roots of $4 - 4\sqrt{3}i$

Solution

$$r = 4\sqrt{1+3} = 8$$

$$\hat{\theta} = \tan^{-1}\sqrt{3} = \frac{\pi}{3} \xrightarrow{QIV} \theta = \frac{5\pi}{3}$$

$$\sqrt[4]{4-4\sqrt{3}i} = \sqrt[4]{8\ cis\frac{5\pi}{3}}$$

$$= \sqrt[4]{8}\ cis\left(\frac{5\pi}{12} + \frac{\pi k}{2}\right)$$

$$k = 0 \Rightarrow \sqrt[4]{8}\ cis\left(\frac{5\pi}{12} + 0\right) = \sqrt[4]{8}\ cis\frac{5\pi}{12}$$

$$k = 1 \Rightarrow \sqrt[4]{8}\ cis\left(\frac{5\pi}{12} + \frac{\pi}{2}\right) = \sqrt[4]{8}\ cis\frac{11\pi}{12}$$

$$k = 2 \Rightarrow \sqrt[4]{8}\ cis\left(\frac{5\pi}{12} + \pi\right) = \sqrt[4]{8}\ cis\frac{17\pi}{12}$$

$$k = 3 \Rightarrow \sqrt[4]{8}\ cis\left(\frac{5\pi}{12} + \frac{3\pi}{2}\right) = \sqrt[4]{8}\ cis\frac{23\pi}{12}$$

Exercise

Find the fourth roots of $-16i$

Solution

$$r = 16; \quad \theta = \frac{3\pi}{2}$$

$$\sqrt[4]{-16i} = \sqrt[4]{16\ cis\frac{3\pi}{2}}$$

$$= 2\ cis\left(\frac{3\pi}{8} + \frac{\pi k}{2}\right)$$

$$k = 0 \Rightarrow 2\ cis\left(\frac{3\pi}{8} + 0\right) = 2\ cis\frac{3\pi}{8}$$

$$k = 1 \Rightarrow 2\ cis\left(\frac{3\pi}{8} + \frac{\pi}{2}\right) = 2\ cis\frac{7\pi}{8}$$

$$k = 2 \Rightarrow 2\ cis\left(\frac{3\pi}{8} + \pi\right) = 2\ cis\frac{11\pi}{8}$$

$$k = 1 \Rightarrow 2\ cis\left(\frac{3\pi}{8} + \frac{3\pi}{2}\right) = 2\ cis\frac{15\pi}{8}$$

Exercise

Find the cube roots of 27.

Solution

$$\sqrt[3]{27} = \left(27 cis 0°\right)^{1/3}$$

$$= \sqrt[3]{27}\ cis\left(\frac{0°}{3} + \frac{360°}{3}k\right)$$

$$= 3\ cis\left(0° + 120°k\right)$$

$$k=0 \Rightarrow z = 3\ cis\left(0° + 120°(0)\right) = \underline{2cis0°}$$

$$k=1 \Rightarrow z = 3\ cis\left(0° + 120°(1)\right) = \underline{2cis120°}$$

$$k=2 \Rightarrow z = 3\ cis\left(0° + 120°(2)\right) = \underline{2cis240°}$$

Exercise

Find the cube roots of $8-8i$

Solution

$$r = 8\sqrt{1+1} = 8\sqrt{2}$$

$$\hat{\theta} = \tan^{-1}1 = \frac{\pi}{4} \xrightarrow{QIV} \theta = \frac{7\pi}{4}$$

$$\sqrt[3]{8-8i} = \sqrt[3]{8\sqrt{2}\ cis\frac{7\pi}{4}}$$

$$= 2\sqrt[3]{2}\ cis\left(\frac{7\pi}{12} + \frac{2\pi k}{3}\right)$$

$$k=0 \Rightarrow z = 2\sqrt[3]{2}\ cis\left(\frac{7\pi}{12} + 0\right) = \underline{2\sqrt[3]{2}\ cis\frac{7\pi}{12}}$$

$$k=1 \Rightarrow z = 2\sqrt[3]{2}\ cis\left(\frac{7\pi}{12} + \frac{2\pi}{3}\right) = \underline{2\sqrt[3]{2}\ cis\frac{15\pi}{12}}$$

$$k=2 \Rightarrow z = 2\sqrt[3]{2}\ cis\left(\frac{7\pi}{12} + \frac{4\pi}{3}\right) = \underline{2\sqrt[3]{2}\ cis\frac{23\pi}{12}}$$

Exercise

Find the cube roots of -8

Solution

$$r = 8; \quad \theta = \frac{3\pi}{2}$$

$$\sqrt[3]{-8} = \sqrt[3]{8\ cis\frac{3\pi}{2}}$$

$$= 2\ cis\left(\frac{\pi}{2} + \frac{2\pi k}{3}\right)$$

$$k=0 \Rightarrow z = 2\ cis\left(\frac{\pi}{2} + 0\right) = \underline{2\ cis\frac{\pi}{2}}$$

$$k=1 \Rightarrow z = 2\ cis\left(\frac{\pi}{2} + \frac{2\pi}{3}\right) = \underline{2\ cis\frac{7\pi}{6}}$$

$$k=2 \Rightarrow z = 2\ cis\left(\frac{\pi}{2} + \frac{4\pi}{3}\right) = \underline{2\ cis\frac{11\pi}{6}}$$

Exercise

Find all complex number solutions of $x^3+1=0$.

Solution

$$x^3+1=0 \Rightarrow x^3=-1$$

$$-1 \Rightarrow \begin{cases} r=\sqrt{(-1)^2+0^2}=1 \\ \theta=\tan^{-1}\left(\frac{0}{-1}\right)=\pi \end{cases}$$

$$x^3=-1=1\ cis\pi$$

$$x=(1\ cis\pi)^{1/3}$$

$$=(1)^{1/3}\ cis\left(\frac{\pi}{3}+\frac{2\pi}{3}k\right)$$

$$=cis\left(\frac{\pi}{3}+\frac{2\pi}{3}k\right)$$

$If\ \ k=0 \Rightarrow x=cis\left(\frac{\pi}{3}+\frac{2\pi}{3}(0)\right)=\underline{cis\frac{\pi}{3}}$ $\qquad x=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=\frac{1}{2}+i\frac{\sqrt{3}}{2}$

$If\ \ k=1 \Rightarrow x=cis\left(\frac{\pi}{3}+\frac{2\pi}{3}(1)\right)=cis\left(\frac{3\pi}{3}\right)=\underline{cis\pi}$ $\qquad \underline{x}=\cos\pi+i\sin\pi\ \underline{=-1}$

$If\ \ k=2 \Rightarrow x=cis\left(\frac{\pi}{3}+\frac{2\pi}{3}(2)\right)=\underline{cis\frac{5\pi}{3}}$ $\qquad x=\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}=\frac{1}{2}-i\frac{\sqrt{3}}{2}$