SOLUTION Section 4.1 – First-Order Systems

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $x'' + 3x' + 7x = t^2$

Solution

Let $x_1 = x, \quad x_2 = x' = x_1'$

Yield the system $\begin{cases} x_1' = x_2 \\ x_2' = -7x_1 - 3x_2 + t^2 \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $x^{(4)} + 6x'' - 3x' + x = \cos 3t$

Solution

Let $x_1 = x, \quad x_2 = x' = x_1', \quad x_3 = x'' = x_2', \quad x_4 = x''' = x_3'$

Yield the system $\begin{cases} x_1' = x_2, \quad x_2' = x_3, \quad x_3' = x_4 \\ x_4' = -x_1 + 3x_2 - 6x_3 + \cos 3t \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $t^2 x'' + tx' + \left(t^2 - 1\right)x = 0$

Solution

Let $x_1 = x, \quad x_2 = x' = x_1'$

Yield the system $\begin{cases} x_1' = x_2 \\ t^2 x_2' = \left(1 - t^2\right)x_1 - tx_2 \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1st-order differential equation $t^3 x^{(3)} - 2t^2 x'' + 3tx' + 5x = \ln t$

Solution

Let $\quad x_1 = x, \quad x_2 = x' = x'_1, \quad x_3 = x'' = x'_2$

Yield the system $\begin{cases} x'_1 = x_2, \quad x'_2 = x_3, \quad x'_3 = x_4 \\ t^3 x'_3 = -5x_1 - 3tx_2 + 2t^2 x_3 + \ln t \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1^{st}-order differential equation $\quad x'' - 5x + 4y = 0, \quad y'' + 4x - 5y = 0$

Solution

Let $\begin{array}{ll} x_1 = x & x_2 = x' = x'_1 \\ y_1 = y & y_2 = y' = y'_1 \end{array}$

$\Rightarrow \begin{cases} x'_1 = x_2 \\ x'_2 = 5x_1 - 4y_1 \end{cases} \qquad \begin{cases} y'_1 = y_2 \\ y'_2 = -4x_1 + 5y_1 \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1^{st}-order differential equation $\quad x'' - 3x' + 4x - 2y = 0, \quad y'' + 2y' - 3x + y = \cos t$

Solution

Let $\begin{array}{ll} x_1 = x & x_2 = x' = x'_1 \\ y_1 = y & y_2 = y' = y'_1 \end{array}$

$\Rightarrow \begin{cases} x'_1 = x_2 \\ x'_2 = -4x_1 + 2y_1 + 3x_2 \end{cases} \qquad \begin{cases} y'_1 = y_2 \\ y'_2 = 3x_1 - y_1 - 2y_2 + \cos t \end{cases}$

Exercise

Transform the given differential equation or system into an equivalent system of 1^{st}-order differential equation $\quad x'' = 3x - y + 2z, \quad y'' = x + y - 4z, \quad z'' = 5x - y - z$

Solution

Let $\begin{array}{ll} x_1 = x & x_2 = x' = x'_1 \\ y_1 = y & y_2 = y' = y'_1 \\ z_1 = z & z_2 = z' = z'_1 \end{array} \quad \Rightarrow \quad \begin{cases} x'_1 = x_2, \quad y'_1 = y_2, \quad z'_1 = z_2 \\ x'_2 = 3x_1 - y_1 + 2z_1 \\ y'_2 = x_1 + y_1 - 4z_1 \\ z'_2 = 5x_1 - y_1 - z_1 \end{cases}$

Exercise

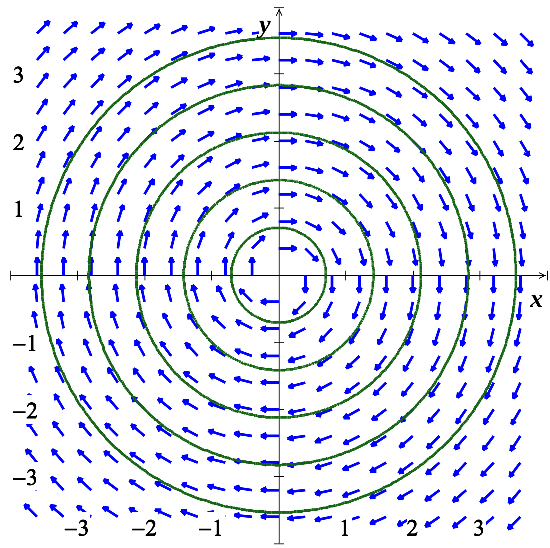
Transform the given differential equation or system into an equivalent system of 1st-order differential equation $\quad x'' = (1-y)x, \quad y'' = (1-x)y$

Solution

Let $\quad \begin{matrix} x_1 = x & x_2 = x' = x_1' \\ y_1 = y & y_2 = y' = y_1' \end{matrix} \quad \Rightarrow \quad \begin{cases} x_1' = x_2, \quad y_1' = y_2 \\ x_2' = (1-y_1)x_1 \\ y_2' = (1-x_1)y_1 \end{cases}$

Exercise

Find the general solution $\quad x' = y, \quad y' = -x$



Solution

$x'' = y' = -x$

$x'' + x = 0 \quad \Rightarrow \quad \lambda^2 + 1 = 0$

The eigenvalues are: $\lambda_{1,2} = \pm i$

$x(t) = C_1 \cos t + C_2 \sin t$. Given $y = x'$

$\therefore$ General solution: $\begin{cases} x(t) = C_1 \cos t + C_2 \sin t \\ y(t) = -C_1 \sin t + C_2 \cos t \end{cases}$

Exercise

Find the general solution $\quad x' = y, \quad y' = -9x + 6y$

Solution

$x'' = y' = -9x + 6y$

$x'' = -9x + 6x'$

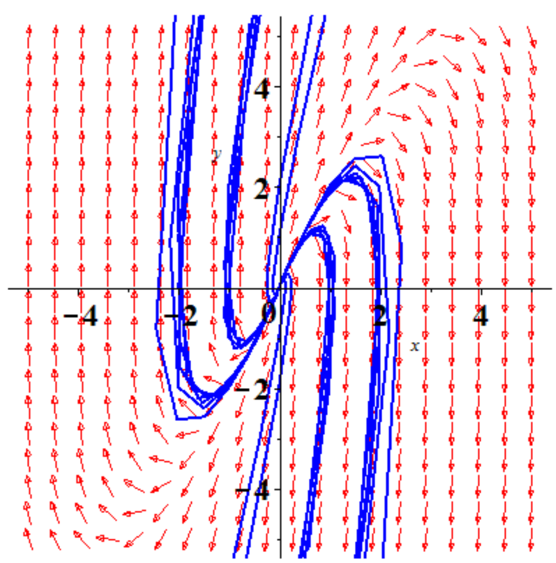
$x'' - 6x' + 9x = 0 \quad \Rightarrow \quad \lambda^2 - 6\lambda + 9 = 0$

The eigenvalues are: $\lambda_{1,2} = 3$

$x(t) = (C_1 + C_2 t)e^{3t}$

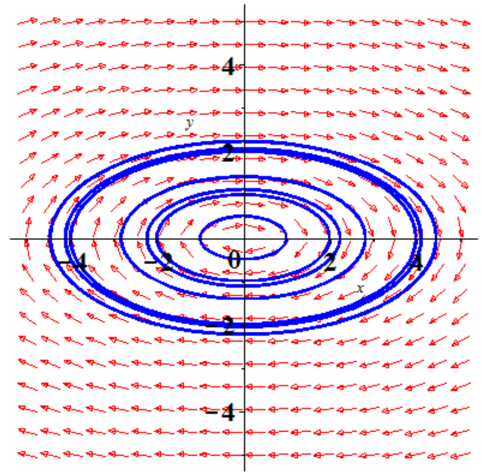
Given $y = x' = C_2 e^{3t} + 3(C_1 + C_2 t)e^{3t}$

$\therefore$ General solution: $\begin{cases} x(t) = (C_1 + C_2 t)e^{3t} \\ y(t) = (3C_1 + C_2 + 3C_2 t)e^{3t} \end{cases}$



Exercise

Find the general solution $\quad x' = 8y, \quad y' = -2x$



Solution

$$x'' = 8y' = -16x$$

$$x'' + 16x = 0 \quad \Rightarrow \quad \lambda^2 + 16 = 0$$

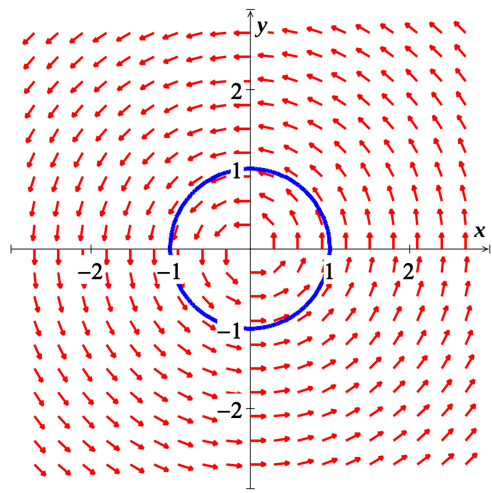
The eigenvalues are: $\lambda_{1,2} = \pm 4i$

$x(t) = C_1 \cos 4t + C_2 \sin 4t$. Given $y = \frac{1}{8}x'$

$$\therefore \text{ General solution: } \begin{cases} x(t) = C_1 \cos 4t + C_2 \sin 4t \\ y(t) = -\frac{1}{2}C_1 \sin 4t + \frac{1}{2}C_2 \cos 4t \end{cases}$$

Exercise

Find the general solution $\quad x' = -2y, \quad y' = 2x; \quad x(0) = 1, \quad y(0) = 0$



Solution

$$x'' = -2y' = -4x$$

$$x'' + 4x = 0 \quad \Rightarrow \quad \lambda^2 + 4 = 0$$

The eigenvalues are: $\lambda_{1,2} = \pm 2i$

$x(t) = C_1 \cos 2t + C_2 \sin 2t$.

Given $y = -\frac{1}{2}x' \Rightarrow y(t) = C_1 \sin 2t - C_2 \cos 2t$

$x(0) = C_1 = 1 \quad and \quad y(0) = -C_2 = 0$

$$\therefore \text{ General solution: } \begin{cases} x(t) = \cos 2t \\ y(t) = \sin 2t \end{cases}$$

Exercise

Find the general solution $\quad x' = y, \quad y' = 6x - y; \quad x(0) = 1, \quad y(0) = 2$

Solution

$$x'' = y' = 6x - y$$

$$x'' = 6x - x'$$

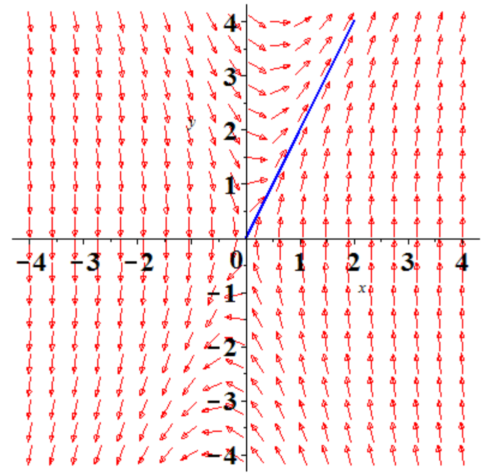
$$x'' + x' - 6x = 0 \quad \Rightarrow \quad \lambda^2 + \lambda - 6 = 0$$

The eigenvalues are: $\lambda_1 = -3 \quad \lambda_2 = 2$

$$x(t) = C_1 e^{-3t} + C_2 e^{2t} \quad \Rightarrow \quad x(0) = C_1 + C_2 = 1$$

$$y(t) = x'(t) = -3C_1e^{-3t} + 2C_2e^{2t} \quad \Rightarrow \quad y(0) = -3C_1 + 2C_2 = 2$$

$$\begin{cases} C_1 + C_2 = 1 \\ -3C_1 + 2C_2 = 2 \end{cases} \quad \rightarrow \quad C_1 = 0, \; C_2 = 1$$

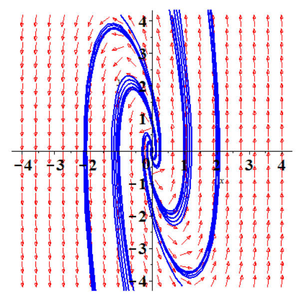


$\therefore$ General solution: $\begin{cases} x(t) = e^{2t} \\ y(t) = 2e^{2t} \end{cases}$

Exercise

Find the general solution $\quad x' = -y, \quad y' = 13x + 4y; \quad x(0) = 0, \quad y(0) = 3$

Solution



$$x'' = -y' = -13x - 4y$$

$$x'' + 4x' + 13x = 0 \quad \Rightarrow \quad \lambda^2 + 4\lambda + 13 = 0$$

The eigenvalues are: $\lambda = \dfrac{-4 \pm \sqrt{-36}}{2} \quad \lambda_{1,2} = -2 \pm 3i$

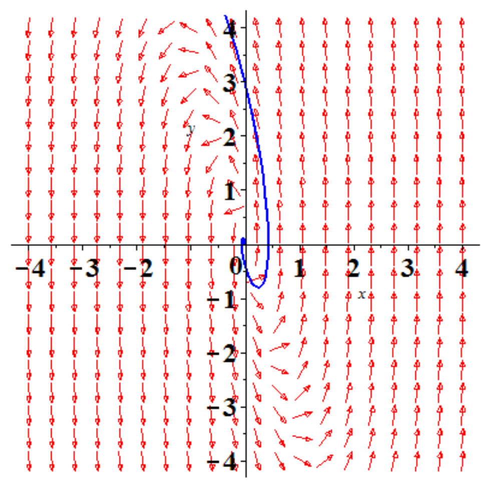
$$x(t) = e^{-2t}\left(C_1 \cos 3t + C_2 \sin 3t\right) \quad \Rightarrow \quad x(0) = \boxed{C_1 = 0}$$

Given $y = -x'$

$$\Rightarrow y(t) = \left(-3C_1 \sin 3t + C_2 \cos 3t\right)e^{-2t} - 2\left(C_1 \cos 3t + C_2 \sin 3t\right)e^{-2t}$$

$$y(t) = \left(-\left(3C_1 + 2C_2\right)\sin 3t + \left(C_2 - 2C_1\right)\cos 3t\right)e^{-2t}$$

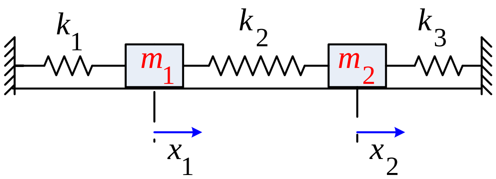
$$y(0) = C_2 - 2C_1 = 3 \quad \Rightarrow \quad \boxed{C_2 = 3}$$



$\therefore$ General solution: $\begin{cases} x(t) = (3\sin 3t)e^{-2t} \\ y(t) = (-6\sin 3t + 3\cos 3t)e^{-2t} \end{cases}$

Exercise

Derive the equations $\begin{cases} m_1 x_1'' = -\left(k_1 + k_2\right)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - \left(k_2 + k_3\right)x_2 \end{cases}$



For the displacements (from equilibrium) of the 2 masses.

Solution

First spring is stretched by x_1

Second spring is stretched by $x_2 - x_1$

Third spring is stretched by x_2

Newton's second law gives:

For m_1 $\Rightarrow$ $m_1 x_1'' = -k_1 x_1 + k_2\left(x_2 - x_1\right)$

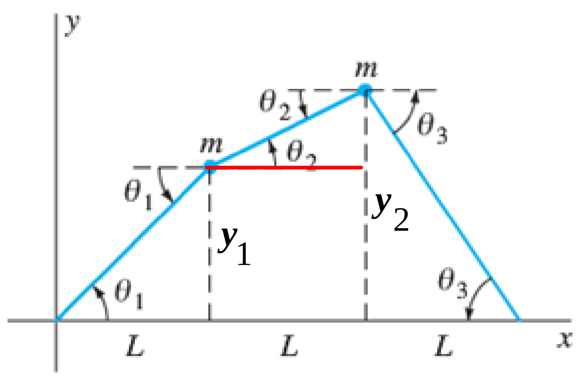
For m_2 $\Rightarrow$ $m_2 x_2'' = -k_2\left(x_2 - x_1\right) - k_3 x_2$

That implies to: $\begin{cases} m_1 x_1'' = -\left(k_1 + k_2\right)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - \left(k_2 + k_3\right)x_2 \end{cases}$

Exercise

Two particles each of mass *m* are attached to a string under (constant) tension *T*. Assume that the particles oscillate vertically (that is, parallel to the *y*-axis) with amplitudes so small that the sines of the angles shown are accurately approximated by their tangents. Show that the displacement y_1 and y_2 satisfy the equations

$$\begin{cases} k y_1'' = -2y_1 + y_2 \\ k y_2'' = y_1 - 2y_2 \end{cases} \qquad \text{where } k = \frac{mL}{T}$$



Solution

For the first mass:

$$m y_1'' = -T\sin\theta_1 + T\sin\theta_2$$

$$\approx -T\tan\theta_1 + T\tan\theta_2$$

$$m y_1'' = -T\frac{y_1}{L} + T\frac{y_2 - y_1}{L}$$

$$\frac{L}{T} m y_1'' = -\frac{L}{T} T\frac{y_1}{L} + \frac{L}{T} T\frac{y_2 - y_1}{L} \qquad \text{where } k = \frac{mL}{T}$$

$$\boxed{k y_1''} = -y_1 + y_2 - y_1 \; \boxed{= -2y_1 + y_2}$$

For the second mass:

$$my_2'' = -T\sin\theta_2 + T\sin\theta_3$$
$$\approx -T\tan\theta_2 + T\tan\theta_3$$

$$my_2'' = -T\frac{y_2 - y_1}{L} + T\frac{y_2}{L}$$

$$\frac{L}{T}my_2'' = -\frac{L}{T}T\frac{y_2 - y_1}{L} + \frac{L}{T}T\frac{y_2}{L} \qquad \text{where } k = \frac{mL}{T}$$

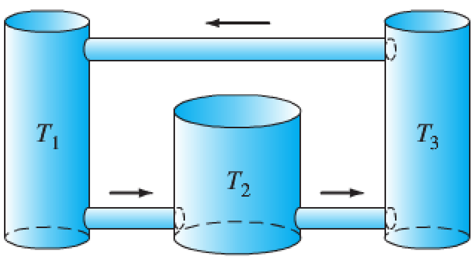
$$\boxed{ky_2'' = -y_2 + y_1 - y_2 = y_1 - 2y_2}$$

$$\Rightarrow \begin{cases} ky_1'' = -2y_1 + y_2 \\ ky_2'' = y_1 - 2y_2 \end{cases} \qquad \text{where } k = \frac{mL}{T}$$

Exercise

There 100-gal fermentation vats are connected, and the mixtures in each tank are kept uniform by stirring. Denote by $x_i(t)$ the amount (in pounds) of alcohol in tank T_i at time *t* (*i* = 1, 2, 3). Suppose that the mixture circulates between the tanks at the rate of 10 *gal/min*. Derive the equations

$$\begin{cases} 10x_1' = -x_1 \qquad + x_3 \\ 10x_2' = x_1 - x_2 \\ 10x_3' = \qquad x_2 - x_3 \end{cases}$$



Solution

Concentration of salt in each tank is: $c_i = \frac{X}{V} = \frac{x_i}{100}$

Rate in/out = Volume Rate x Concentration

Rate of change = Rate in – rate out

For T_1: $\quad x_1' = r\frac{x_3}{100} - r\frac{x_1}{100} = \frac{1}{10}(x_3 - x_1)$

For T_2: $\quad x_2' = r\frac{x_1}{100} - r\frac{x_2}{100} = \frac{1}{10}(x_1 - x_2)$

For T_3: $\quad x_3' = r\frac{x_2}{100} - r\frac{x_3}{100} = \frac{1}{10}(x_2 - x_3)$

That implies:

$$\begin{cases} 10x_1' = -x_1 \qquad + x_3 \\ 10x_2' = x_1 - x_2 \\ 10x_3' = \qquad x_2 - x_3 \end{cases}$$

Exercise

Suppose that a particle with mass m and electrical charge q moves in the xy-plane under the influence of the magnetic field $\vec{B}=B\hat{k}$ (thus a uniform field parallel to the z-axis), so the force on the particle is $\vec{F}=q\vec{v}\times\vec{B}$ if its velocity is $\vec{v}$. Show that the equations of motion of the particle are

$$mx''=+qBy', \quad my''=-qBx'$$

Solution

Let $\vec{r}=(x,\ y,\ z)$ be the position vector, then Newton's law

$$\vec{F}=mx''$$

$$\begin{aligned}\vec{F}=mx''&=q\left(\vec{v}\times\vec{B}\right)\\ &=q\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x' & y' & z'\\ 0 & 0 & B\end{vmatrix}\\ &=qBy'\hat{i}-qBx'\hat{j}\end{aligned}$$

$$\Rightarrow\ mx''=+qBy', \quad my''=-qBx'$$

SOLUTION *Section* 4.2 – Matrices and Linear Systems

Exercise

Write the given system in the form $\boldsymbol{x}' = \boldsymbol{P}(t)x + \boldsymbol{f}(t)$ $\qquad x' = -3y, \quad y' = 3x$

Solution

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \boldsymbol{P}(t) = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \quad \boldsymbol{f}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Exercise

Write the given system in the form $\boldsymbol{x}' = \boldsymbol{P}(t)x + \boldsymbol{f}(t)$ $\qquad x' = 3x - 2y, \quad y' = 2x + y$

Solution

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \boldsymbol{P}(t) = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} \quad \boldsymbol{f}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Exercise

Write the given system in the form $\boldsymbol{x}' = \boldsymbol{P}(t)x + \boldsymbol{f}(t)$ $\qquad x' = tx - e^t y + \cos t, \quad y' = e^{-t}x + t^2 y - \sin t$

Solution

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \boldsymbol{P}(t) = \begin{bmatrix} t & -e^t \\ e^{-t} & t^2 \end{bmatrix} \quad \boldsymbol{f}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$

Exercise

Write the given system in the form $\boldsymbol{x}' = \boldsymbol{P}(t)x + \boldsymbol{f}(t)$ $\qquad x' = y + z, \quad y' = z + x, \quad z' = x + y$

Solution

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \boldsymbol{P}(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \boldsymbol{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise

Write the given system in the form $\boldsymbol{x}' = \boldsymbol{P}(t)x + \boldsymbol{f}(t)$ $\qquad x' = 2x - 3y, \quad y' = x + y + 2z, \quad z' = 5y - 7z$

Solution

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \boldsymbol{P}(t) = \begin{bmatrix} 2 & -3 & 0 \\ 1 & 1 & 2 \\ 0 & 5 & -7 \end{bmatrix} \quad \boldsymbol{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise

Write the given system in the form $\boldsymbol{x}' = \boldsymbol{P}(t)x + \boldsymbol{f}(t)$

$$x' = 3x - 4y + z + t, \quad y' = x - 3z + t^2, \quad z' = 6y - 7z + t^3$$

Solution

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \boldsymbol{P}(t) = \begin{bmatrix} 3 & -4 & 1 \\ 1 & 0 & -3 \\ 0 & 6 & -7 \end{bmatrix} \quad \boldsymbol{f}(t) = \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$$

Exercise

Write the given system in the form $\boldsymbol{x}' = \boldsymbol{P}(t)x + \boldsymbol{f}(t)$ $\quad x_1' = x_2, \quad x_2' = 2x_3, \quad x_3' = 3x_4, \quad x_4' = 4x_1$

Solution

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \boldsymbol{P}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{bmatrix} \quad \boldsymbol{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise

Write the given system in the form $\boldsymbol{x}' = \boldsymbol{P}(t)x + \boldsymbol{f}(t)$

$$x_1' = x_2 + x_3 + 1, \quad x_2' = x_3 + x_4 + t, \quad x_3' = x_1 + x_4 + t^2, \quad x_4' = 4x_1 + x_2 + t^3$$

Solution

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \boldsymbol{P}(t) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad \boldsymbol{f}(t) = \begin{bmatrix} 0 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

Exercise

$$\boldsymbol{x}' = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \boldsymbol{x}; \quad \vec{x}_1 = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$$

a) Verify that the given vectors are solutions of the given system.

b) Use the Wronskian to show that they are linearly independent.

c) Write the general solution of the system.

Solution

a) $\vec{x}_1{}' = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}' = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}$ $\qquad$ $\boldsymbol{x}'\,\vec{x}_1 = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} = \vec{x}_1{}'$ ✓

$\vec{x}_2{}' = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}' = \begin{bmatrix} 2e^{2t} \\ -2e^{2t} \end{bmatrix}$ $\qquad$ $\boldsymbol{x}'\,\vec{x}_2 = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ -2e^{2t} \end{bmatrix} = \vec{x}_2{}'$ ✓

b) $W = \begin{vmatrix} 2e^t & e^{2t} \\ -3e^t & -e^{2t} \end{vmatrix} = e^{3t} \neq 0$

The solutions x_1 *and* x_2 are linearly independent.

c) $\boldsymbol{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 = C_1\begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix} + C_2\begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} = \begin{pmatrix} 2C_1e^t + C_2e^{2t} \\ -3C_1e^t - C_2e^{2t} \end{pmatrix}$

Exercise

$\boldsymbol{x}' = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\boldsymbol{x};\quad \vec{x}_1 = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix},\quad \vec{x}_2 = \begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix},\quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 5 \end{cases}$

a) Verify that the given vectors are solutions of the given system.
b) Use the Wronskian to show that they are linearly independent.
c) Write the general solution of the system.
d) Find the particular solution that satisfies the given initial conditions

Solution

a) $\vec{x}_1{}' = \begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix}' = \begin{bmatrix} 3e^{3t} \\ 9e^{3t} \end{bmatrix}$ $\qquad$ $\boldsymbol{x}' \cdot \vec{x}_1 = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} e^{3t} \\ 3e^{3t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ 9e^{3t} \end{bmatrix} = \vec{x}_1{}'$ ✓

$\vec{x}_2{}' = \begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix}' = \begin{bmatrix} -4e^{-2t} \\ -2e^{-2t} \end{bmatrix}$ $\qquad$ $\boldsymbol{x}' \cdot \vec{x}_2 = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 2e^{-2t} \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} -4e^{2t} \\ -2e^{2t} \end{bmatrix} = \vec{x}_2{}'$ ✓

b) $W = \begin{vmatrix} e^{3t} & 2e^{-2t} \\ 3e^{3t} & e^{-2t} \end{vmatrix} = -5e^t \neq 0$

The solutions x_1 *and* x_2 are linearly independent.

c) $\boldsymbol{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 = C_1\begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} + C_2\begin{pmatrix} 2e^{-2t} \\ e^{-2t} \end{pmatrix} = \begin{pmatrix} C_1e^{3t} + 2C_2e^{-2t} \\ 3C_1e^{3t} + C_2e^{-2t} \end{pmatrix}$

d) $x_1 = C_1e^{3t} + 2C_2e^{-2t} \qquad x_2 = 3C_1e^{3t} + C_2e^{-2t}$

$$x_1(0) = C_1 + 2C_2 = 0 \qquad x_2(0) = 3C_1 + C_2 = 5$$

$$\Rightarrow \quad C_1 = 2 \quad C_2 = -1$$

$$\begin{cases} x_1 = 2e^{3t} - 2e^{-2t} \\ x_2 = 6e^{3t} - e^{-2t} \end{cases}$$

Exercise

$$\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\mathbf{x}; \quad \vec{x}_1 = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = e^{-2t}\begin{bmatrix} 1 \\ 5 \end{bmatrix}, \quad \begin{cases} x_1(0) = 5 \\ x_2(0) = -3 \end{cases}$$

a) Verify that the given vectors are solutions of the given system.
b) Use the Wronskian to show that they are linearly independent.
c) Write the general solution of the system.
d) Find the particular solution that satisfies the given initial conditions

Solution

a) $\vec{x}_1{}' = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}' = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} \end{bmatrix} \qquad \mathbf{x}' \cdot \vec{x}_1 = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} \end{bmatrix} = \vec{x}_1{}' \ \checkmark$

$\vec{x}_2{}' = \begin{bmatrix} e^{-2t} \\ 5e^{-2t} \end{bmatrix}' = \begin{bmatrix} -2e^{-2t} \\ -10e^{-2t} \end{bmatrix} \qquad \mathbf{x}' \cdot \vec{x}_2 = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} e^{-2t} \\ 5e^{-2t} \end{bmatrix} = \begin{bmatrix} -2e^{2t} \\ -10e^{2t} \end{bmatrix} = \vec{x}_2{}' \ \checkmark$

b) $W = \begin{vmatrix} e^{2t} & e^{-2t} \\ e^{2t} & 5e^{-2t} \end{vmatrix} = 4 \neq 0$

The solutions x_1 *and* x_2 are linearly independent.

c) $\mathbf{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 = C_1\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} + C_2\begin{pmatrix} e^{-2t} \\ 5e^{-2t} \end{pmatrix} = \begin{pmatrix} C_1e^{2t} + C_2e^{-2t} \\ C_1e^{2t} + 5C_2e^{-2t} \end{pmatrix}$

d) $x_1 = C_1e^{2t} + C_2e^{-2t} \qquad x_2 = C_1e^{2t} + 5C_2e^{-2t}$

$x_1(0) = C_1 + C_2 = 5 \qquad x_2(0) = C_1 + 5C_2 = -3$

$\Rightarrow \quad C_1 = 7 \quad C_2 = -2$

$$\begin{cases} x_1 = 7e^{2t} - 2e^{-2t} \\ x_2 = 7e^{2t} - 10e^{-2t} \end{cases}$$

Exercise

$$\mathbf{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\mathbf{x}; \quad \vec{x}_1 = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 8 \\ x_2(0) = 0 \end{cases}$$

a) Verify that the given vectors are solutions of the given system.

b) Use the Wronskian to show that they are linearly independent.

c) Write the general solution of the system.

d) Find the particular solution that satisfies the given initial conditions

Solution

a) $\vec{x}_1{}' = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}' = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} \qquad \mathbf{x}' \cdot \vec{x}_1 = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} = \vec{x}_1{}' \ \checkmark$

$\vec{x}_2{}' = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}' = \begin{bmatrix} -5e^{-2t} \\ -15e^{-2t} \end{bmatrix} \qquad \mathbf{x}' \cdot \vec{x}_2 = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} = \begin{bmatrix} -5e^{2t} \\ -15e^{2t} \end{bmatrix} = \vec{x}_2{}' \ \checkmark$

b) $W = \begin{vmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{vmatrix} = 7e^{-3t} \neq 0$

The solutions x_1 *and* x_2 are linearly independent.

c) $\mathbf{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 = C_1\begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix} + C_2\begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix} = \begin{pmatrix} 3C_1e^{2t} + C_2e^{-5t} \\ 2C_1e^{2t} + 3C_2e^{-5t} \end{pmatrix}$

d) $x_1 = 3C_1e^{2t} + C_2e^{-5t} \qquad x_2 = 2C_1e^{2t} + 3C_2e^{-5t}$

$x_1(0) = 3C_1 + C_2 = 8 \qquad x_2(0) = 2C_1 + 3C_2 = 0 \ \Rightarrow \ \underline{C_1 = \frac{24}{7} \quad C_2 = -\frac{16}{7}}$

$$\begin{cases} x_1 = \frac{72}{7}e^{2t} - \frac{16}{7}e^{-2t} \\ x_2 = \frac{48}{7}e^{2t} - \frac{48}{7}e^{-2t} \end{cases}$$

Exercise

$$\mathbf{x}' = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\mathbf{x}; \quad \vec{x}_1 = \begin{bmatrix} 2e^{t} \\ 2e^{t} \\ e^{t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 0 \\ x_2(0) = 0 \\ x_3(0) = 4 \end{cases}$$

a) Verify that the given vectors are solutions of the given system.

b) Use the Wronskian to show that they are linearly independent.

c) Write the general solution of the system.

d) Find the particular solution that satisfies the given initial conditions

Solution

a) $\vec{x}_1{}' = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}' = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}$ $\qquad$ $\mathbf{x}' \cdot \vec{x}_1 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix} = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix} = \vec{x}_1{}'$ $\checkmark$

$\vec{x}_2{}' = \begin{bmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}' = \begin{bmatrix} -6e^{3t} \\ 0 \\ 3e^{3t} \end{bmatrix}$ $\qquad$ $\mathbf{x}' \cdot \vec{x}_2 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix} = \begin{bmatrix} -6e^{3t} \\ 0 \\ 3e^{3t} \end{bmatrix} = \vec{x}_2{}'$ $\checkmark$

$\vec{x}_3{}' = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}' = \begin{bmatrix} 10e^{5t} \\ -10e^{5t} \\ 5e^{5t} \end{bmatrix}$ $\qquad$ $\mathbf{x}' \cdot \vec{x}_3 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix} = \begin{bmatrix} 10e^{5t} \\ -10e^{5t} \\ 5e^{5t} \end{bmatrix} = \vec{x}_3{}'$ $\checkmark$

b) $W = \begin{vmatrix} 2e^t & -2e^{3t} & 2e^{5t} \\ 2e^t & 0 & -2e^{5t} \\ e^t & e^{3t} & e^{5t} \end{vmatrix} = 4e^{9t} + 4e^{9t} + 4e^{9t} + 4e^{9t} = 16e^{9t} \neq 0$

The solutions $x_1,\ x_2$ *and* x_3 are linearly independent.

c) $\mathbf{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 + C_3\vec{x}_3 = C_1\begin{pmatrix} 2e^t \\ 2e^t \\ e^t \end{pmatrix} + C_2\begin{pmatrix} -2e^{3t} \\ 0 \\ e^{3t} \end{pmatrix} + C_3\begin{pmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{pmatrix}$

$$= \begin{pmatrix} 2C_1e^t - 2C_2e^{3t} + 2C_3e^{5t} \\ 2C_1e^t \qquad\qquad - 2C_3e^{5t} \\ C_1e^t + C_2e^{3t} + C_3e^{5t} \end{pmatrix}$$

d) $x_1 = 2C_1e^t - 2C_2e^{3t} + 2C_3e^{5t}$ $\qquad$ $x_2 = 2C_1e^t - 2C_3e^{5t}$ $\qquad$ $x_3 = C_1e^t + C_2e^{3t} + C_3e^{5t}$

$\begin{cases} x_1(0) = 2C_1 - 2C_2 + 2C_3 = 0 \\ x_2(0) = 2C_1 - 2C_3 = 0 \\ x_3(0) = C_1 + C_2 + C_3 = 4 \end{cases} \to \left[\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & 1 & 1 & 4 \end{array}\right] \qquad \Rightarrow\ C_1 = 1\quad C_2 = 2\quad C_3 = 1$

$$\begin{cases} x_1(t) = 2e^t - 4e^{3t} + 2e^{5t} \\ x_2(t) = 2e^t - 2e^{5t} \\ x_3(t) = e^t + 2e^{3t} + e^{5t} \end{cases}$$

Exercise

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\mathbf{x}; \quad \vec{x}_1 = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix}, \quad \begin{cases} x_1(0) = 10 \\ x_2(0) = 12 \\ x_3(0) = -1 \end{cases}$$

a) Verify that the given vectors are solutions of the given system.

b) Use the Wronskian to show that they are linearly independent.

c) Write the general solution of the system.

d) Find the particular solution that satisfies the given initial conditions

Solution

a)
$$\vec{x}_1{}' = \begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix}' = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{pmatrix} \qquad \mathbf{x}' \cdot \vec{x}_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{pmatrix} = \vec{x}_1{}' \ \checkmark$$

$$\vec{x}_2{}' = \begin{pmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix}' = \begin{pmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{pmatrix} \qquad \mathbf{x}' \cdot \vec{x}_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{pmatrix} = \vec{x}_2{}' \ \checkmark$$

$$\vec{x}_3{}' = \begin{pmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{pmatrix}' = \begin{pmatrix} 0 \\ -e^{-t} \\ e^{-t} \end{pmatrix} \qquad \mathbf{x}' \cdot \vec{x}_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} 0 \\ -e^{-t} \\ e^{-t} \end{pmatrix} = \vec{x}_3{}' \ \checkmark$$

b) $W = \begin{vmatrix} e^{2t} & e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & -e^{-t} & -e^{-t} \end{vmatrix} = 3 \neq 0$ The solutions $x_1,\ x_2$ *and* x_3 are linearly independent.

c)
$$\mathbf{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 + C_3\vec{x}_3 = C_1\begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix} + C_2\begin{pmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix} + C_3\begin{pmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} C_1e^{2t} + C_2e^{-t} \\ C_1e^{2t} \quad + C_3e^{-t} \\ C_1e^{2t} - C_2e^{-t} - C_3e^{-t} \end{pmatrix}$$

d) $x_1 = C_1e^{2t} + C_2e^{-t} \qquad x_2 = C_1e^{2t} + C_3e^{-t} \qquad x_3 = C_1e^{2t} - C_2e^{-t} - C_3e^{-t}$

$$\begin{cases} x_1(0) = C_1 + C_2 = 10 \\ x_2(0) = C_1 + C_3 = 12 \\ x_3(0) = C_1 - C_2 - C_3 = -1 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 1 & 0 & 1 & 12 \\ 1 & -1 & -1 & -1 \end{array}\right] \qquad \Rightarrow \underline{C_1 = 7 \quad C_2 = 3 \quad C_3 = 5}$$

$$\begin{cases} x_1(t) = 7e^{2t} + 3e^{-t} \\ x_2(t) = 7e^{2t} + 5e^{-t} \\ x_3(t) = 7e^{2t} - 8e^{-t} \end{cases}$$

Exercise

$$\mathbf{x}' = \begin{bmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{bmatrix}\mathbf{x}; \quad \vec{x}_1 = \begin{bmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{bmatrix}, \quad \vec{x}_4 = \begin{bmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{bmatrix}, \quad \begin{cases} x_1(0) = 1 \\ x_2(0) = 3 \\ x_3(0) = 4 \\ x_4(0) = 7 \end{cases}$$

a) Verify that the given vectors are solutions of the given system.

b) Use the Wronskian to show that they are linearly independent.

c) Write the general solution of the system.

d) Find the particular solution that satisfies the given initial conditions

Solution

a) $$\vec{x}_1{}' = \begin{pmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{pmatrix}' = \begin{pmatrix} -e^{-t} \\ 0 \\ 0 \\ -e^{-t} \end{pmatrix} \qquad \mathbf{x}' \cdot \vec{x}_1 = \begin{pmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ 0 \\ 0 \\ -e^{-t} \end{pmatrix} = \vec{x}_1{}' \quad \checkmark$$

$$\vec{x}_2{}' = \begin{pmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{pmatrix}' = \begin{pmatrix} 0 \\ 0 \\ -e^{-t} \\ 0 \end{pmatrix} \qquad \mathbf{x}' \cdot \vec{x}_2 = \begin{pmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -e^{-t} \\ 0 \end{pmatrix} = \vec{x}_2{}' \quad \checkmark$$

$$\vec{x}_3{}' = \begin{pmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{pmatrix}' = \begin{pmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{pmatrix} \qquad \mathbf{x}' \cdot \vec{x}_3 = \begin{pmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{pmatrix} = \begin{pmatrix} 0 \\ e^{t} \\ 0 \\ -2e^{t} \end{pmatrix} = \vec{x}_3{}' \quad \checkmark$$

$$\vec{x}_4{}' = \begin{pmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{pmatrix}' = \begin{pmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{pmatrix} \qquad \mathbf{x}' \cdot \vec{x}_3 = \begin{pmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{pmatrix}\begin{pmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{t} \\ 0 \\ 3e^{t} \\ 0 \end{pmatrix} = \vec{x}_3{}' \quad \checkmark$$

b) $W = \begin{vmatrix} e^{-t} & 0 & 0 & e^t \\ 0 & 0 & e^t & 0 \\ 0 & e^{-t} & 0 & 3e^t \\ e^{-t} & 0 & -2e^t & 0 \end{vmatrix} = e^{-t}\begin{vmatrix} 0 & e^t & 0 \\ e^{-t} & 0 & 3e^t \\ 0 & -2e^t & 0 \end{vmatrix} - e^t\begin{vmatrix} 0 & 0 & e^t \\ 0 & e^{-t} & 0 \\ e^{-t} & 0 & -2e^t \end{vmatrix} = 0-(-1) = 1 \neq 0$

The solutions x_1, x_2 *and* x_3 are linearly independent.

c) $\boldsymbol{x}(t) = C_1\vec{x}_1 + C_2\vec{x}_2 + C_3\vec{x}_3 + C_4\vec{x}_4 = C_1\begin{pmatrix} e^{-t} \\ 0 \\ 0 \\ e^{-t} \end{pmatrix} + C_2\begin{pmatrix} 0 \\ 0 \\ e^{-t} \\ 0 \end{pmatrix} + C_3\begin{pmatrix} 0 \\ e^t \\ 0 \\ -2e^t \end{pmatrix} + C_4\begin{pmatrix} e^t \\ 0 \\ 3e^t \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} C_1e^{-t} + C_4e^t \\ C_3e^t \\ C_2e^{-t} + 3C_4e^t \\ C_1e^{-t} - 2C_3e^t \end{pmatrix}$$

d) $x_1(t) = C_1e^{-t} + C_4e^t, \quad x_2(t) = C_3e^t, \quad x_3(t) = C_2e^{-t} + 3C_4e^t, \quad x_4(t) = C_1e^{-t} - 2C_3e^t$

$$\begin{cases} x_1(0) = C_1 + C_4 = 1 \\ x_2(0) = C_3 = 3 \\ x_3(0) = C_2 + 3C_4 = 4 \\ x_4(0) = C_1 - 2C_3 = 7 \end{cases} \qquad \Rightarrow \quad C_1 = 13 \quad C_2 = 40 \quad C_3 = 3 \quad C_4 = -12$$

$$\begin{cases} x_1(t) = 13e^{-t} - 12e^t \\ x_2(t) = 3e^t \\ x_3(t) = 40e^{-t} - 36e^t \\ x_3(t) = 13e^{-t} - 6e^t \end{cases}$$

SOLUTION *Section* 4.3 – Eigenvalue Method for Linear System

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = x_1 + 2x_2, \quad x_2' = 2x_1 + x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4$$

$$= \lambda^2 - 2\lambda - 3 = 0$$

The distinct real eigenvalues: $\lambda_1 = -1, \quad \lambda_2 = 3$

For $\lambda_1 = -1 \;\Rightarrow\; (A + I)V_1 = 0$

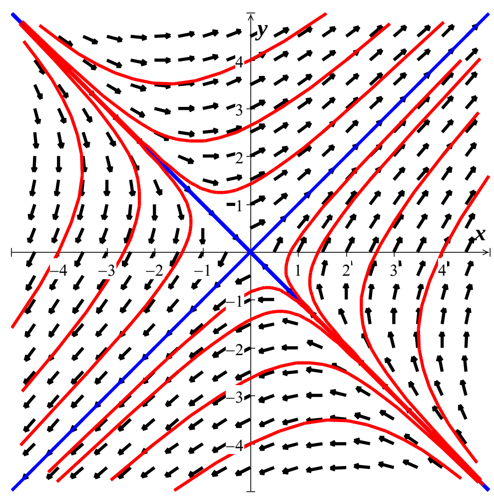
$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\Rightarrow\; 2x_1 + 2y_1 = 0 \;\rightarrow\; y_1 = -x_1$$

$$\rightarrow V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \;\Rightarrow\; x_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

For $\lambda_2 = 3 \;\Rightarrow\; (A - 3I)V_2 = 0$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\Rightarrow\; -2x_2 + 2y_2 = 0 \;\rightarrow\; x_2 = y_2$$

$$\rightarrow V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \;\Rightarrow\; x_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$



$$x_1(t) = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} \quad x_2(t) = \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$$

Using Wronskian: $\begin{vmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{vmatrix} = 2e^{2t} \neq 0$

The general solution: $x(t) = C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$ ***OR*** $\begin{cases} x_1(t) = C_1 e^{-t} + C_2 e^{3t} \\ x_2(t) = -C_1 e^{-t} + C_2 e^{3t} \end{cases}$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = 2x_1 + 3x_2, \quad x_2' = 2x_1 + x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 6$$

$$= \lambda^2 - 3\lambda - 4 = 0$$

The distinct real eigenvalues: $\underline{\lambda_1 = -1, \quad \lambda_2 = 4}$

For $\lambda_1 = -1 \;\Rightarrow\; (A+I)V_1 = 0$

$$\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\Rightarrow\; 3x_1 + 3y_1 = 0 \to y_1 = -x_1$$

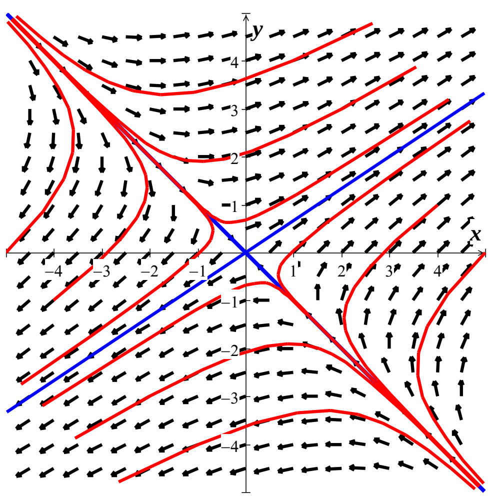
$$\to V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \;\Rightarrow\; x_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

For $\lambda_2 = 4 \;\Rightarrow\; (A-4I)V_2 = 0$

$$\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\Rightarrow\; 2x_2 = 3y_2$$

$$\to V_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \;\Rightarrow\; x_2(t) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$$

$$x_1(t) = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} \quad x_2(t) = \begin{pmatrix} 3e^{4t} \\ 2e^{4t} \end{pmatrix}$$



The general solution: $\underline{x(t) = C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}}$

OR $\begin{cases} x_1(t) = C_1 e^{-t} + 3C_2 e^{4t} \\ x_2(t) = -C_1 e^{-t} + 2C_2 e^{4t} \end{cases}$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = 6x_1 - 7x_2, \quad x_2' = x_1 - 2x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 6-\lambda & -7 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 5 = 0$$

The distinct real eigenvalues: $\underline{\lambda_1 = -1, \quad \lambda_2 = 5}$

For $\lambda_1 = -1 \Rightarrow (A+I)V_1 = 0$

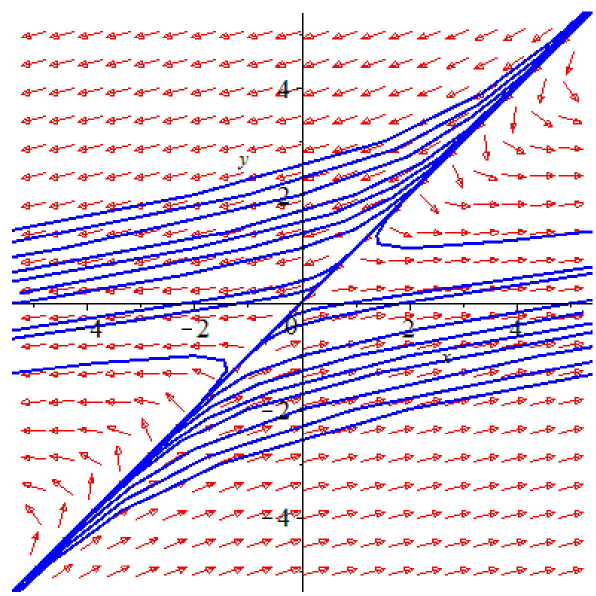
$$\begin{pmatrix} 7 & -7 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = y_1$$

$$\to V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow x_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

For $\lambda_2 = 5 \Rightarrow (A-5I)V_2 = 0$

$$\begin{pmatrix} 1 & -7 \\ 1 & -7 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_2 = 7y_2$$

$$\to V_2 = \begin{pmatrix} 7 \\ 1 \end{pmatrix} \Rightarrow x_2(t) = \begin{pmatrix} 7 \\ 1 \end{pmatrix} e^{5t}$$



The general solution: $\underline{x(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 7 \\ 1 \end{pmatrix} e^{5t}}$

OR $\begin{cases} x_1(t) = C_1 e^{-t} + 7C_2 e^{5t} \\ x_2(t) = C_1 e^{-t} + C_2 e^{5t} \end{cases}$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = -3x_1 + 4x_2, \quad x_2' = 6x_1 - 5x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -3 & 4 \\ 6 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} -3-\lambda & 4 \\ 6 & -5-\lambda \end{vmatrix} = \lambda^2 + 8\lambda - 9 = 0$$

The distinct real eigenvalues: $\underline{\lambda_1 = -9, \quad \lambda_2 = 1|}$

For $\lambda_1 = -9 \quad \Rightarrow \quad (A + 9I)V_1 = 0$

$$\begin{pmatrix} 6 & 4 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 6x_1 = -4y_1$$

$$\to V_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{-9t}$$

For $\lambda_2 = 1 \quad \Rightarrow \quad (A - I)V_2 = 0$

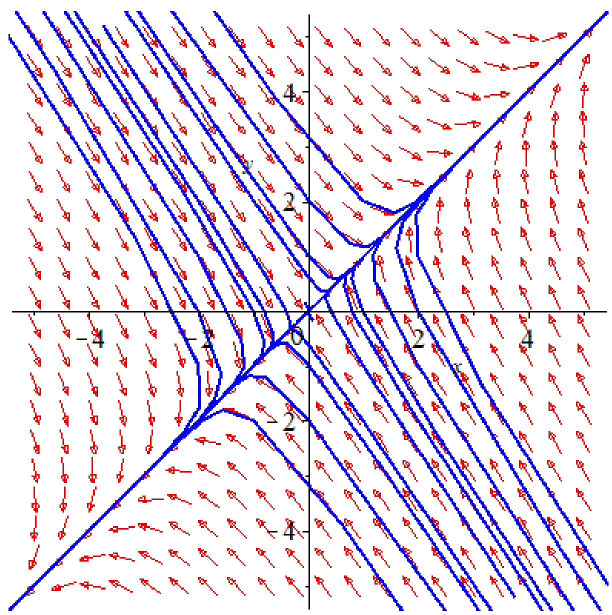
$$\begin{pmatrix} -4 & 4 \\ 6 & -6 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad x_2 = y_2$$

$$\to V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad x_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$$

The general solution:

$$x(t) = C_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{-9t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$$

$$\begin{cases} x_1(t) = 2C_1 e^{-9t} + C_2 e^{t} \\ x_2(t) = -3C_1 e^{-9t} + C_2 e^{t} \end{cases}$$



Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = x_1 - 5x_2, \quad x_2' = x_1 - x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + 4 = 0$$

The distinct real eigenvalues: $\lambda_{1,2} = \pm 2i$

For $\lambda = 2i \quad \Rightarrow \quad (A - \lambda I)V = 0$

$$\begin{pmatrix} 1-2i & -5 \\ 1 & -1-2i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad (1-2i)x - 5y = 0 \ \rightarrow \ (1-2i)x = 5y$$

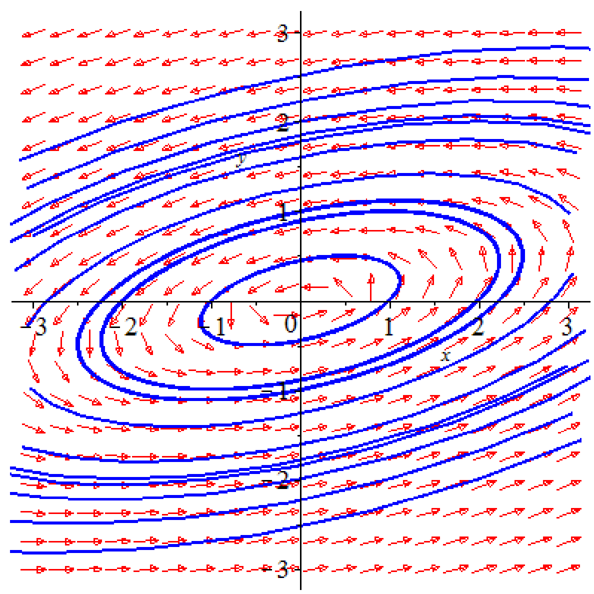
$$\rightarrow V = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix}$$

$$x(t) = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix} e^{2it} \qquad\qquad e^{ait} = \cos at + i \sin at$$

$$= \begin{pmatrix} 5 \\ 1-2i \end{pmatrix} (\cos 2t + i \sin 2t)$$

$$= \begin{pmatrix} 5\cos 2t + 5i \sin 2t \\ \cos 2t + 2\sin 2t + i(\sin 2t - 2\cos 2t) \end{pmatrix}$$

$$\begin{cases} x_1(t) = 5C_1 \cos 2t + 5C_2 \sin 2t \\ x_2(t) = C_1 (\cos 2t + 2\sin 2t) + C_2 (\sin 2t - 2\cos 2t) \\ \qquad = (C_1 - 2C_2)\cos 2t + (2C_1 + C_2)\sin 2t \end{cases}$$



Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = -3x_1 - 2x_2, \quad x_2' = 9x_1 + 3x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -3 & -2 \\ 9 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} -3-\lambda & -2 \\ 9 & 3-\lambda \end{vmatrix} = \lambda^2 + 9 = 0$$

The distinct real eigenvalues: $\underline{\lambda_{1,2} = \pm 3i|}$

For $\lambda = 3i \quad \Rightarrow \quad (A - \lambda I)V = 0$

$$\begin{pmatrix} -3-3i & -2 \\ 9 & 3-3i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad (-3-3i)x - 2y = 0 \ \rightarrow \ (3+3i)x = -2y$$

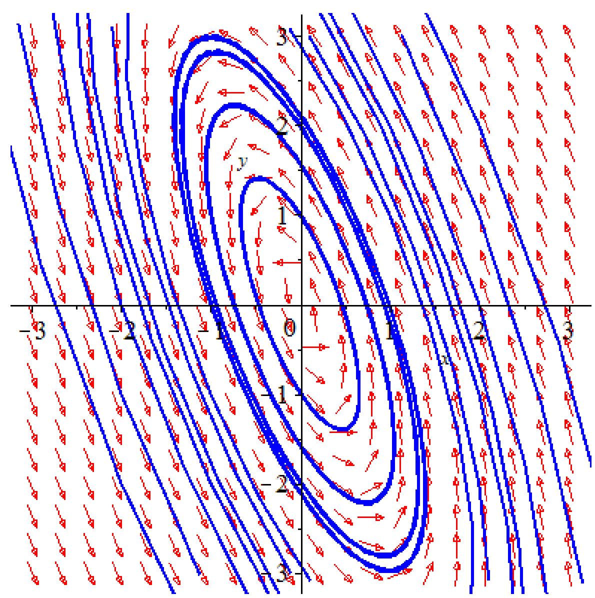
$$\rightarrow V = \begin{pmatrix} -2 \\ 3+3i \end{pmatrix}$$

$$x(t) = \begin{pmatrix} -2 \\ 3+3i \end{pmatrix} e^{3it} \qquad\qquad e^{ait} = \cos at + i \sin at$$

$$= \begin{pmatrix} -2 \\ 3+3i \end{pmatrix}(\cos 3t + i \sin 3t)$$

$$= \begin{pmatrix} -2\cos 3t - 2i \sin 3t \\ 3\cos 3t - 3\sin 3t + i(3\sin 3t + 3\cos 3t) \end{pmatrix}$$

$$\begin{cases} x_1(t) = -2C_1 \cos 3t - 2C_2 \sin 3t \\ x_2(t) = 3C_1(\cos 3t - \sin 3t) + 3C_2(\sin 3t + \cos 3t) \\ \qquad\quad = 3(C_1 + C_2)\cos 3t + 3(C_2 - C_1)\sin 3t \end{cases}$$



Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = x_1 - 5x_2, \quad x_2' = x_1 + 3x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & -5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

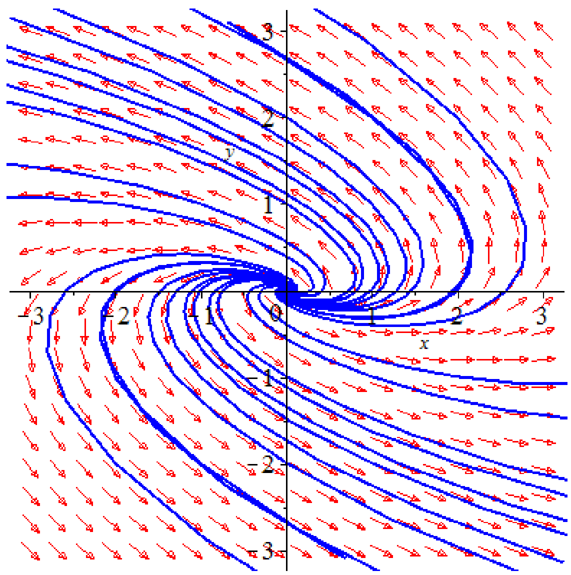
$$\begin{vmatrix} 1-\lambda & -5 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 8 = 0$$

The distinct real eigenvalues: $\underline{\lambda_{1,2} = 2 \pm 2i}$

For $\lambda = 2+2i \quad \Rightarrow \quad (A - \lambda I)V = 0$

$$\begin{pmatrix} -1-2i & -5 \\ 1 & 1-2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad (1+2i)x = -5y$$

$$\rightarrow V = \begin{pmatrix} -5 \\ 1+2i \end{pmatrix}$$



$$\begin{aligned} x(t) &= \begin{pmatrix} -5 \\ 1+2i \end{pmatrix} e^{(2+2i)t} \\ &= \begin{pmatrix} -5 \\ 1+2i \end{pmatrix} e^{2t} e^{2it} \\ &= \begin{pmatrix} -5 \\ 1+2i \end{pmatrix} e^{2t}(\cos 2t + i\sin 2t) \\ &= \underline{\begin{pmatrix} -5\cos 2t - 5i\sin 2t \\ \cos 2t - 2\sin 2t + i(2\cos 2t + \sin 2t) \end{pmatrix} e^{2t}} \end{aligned}$$

$$\begin{cases} x_1(t) = \left(-5C_1\cos 2t - 2C_2\sin 2t\right) e^{2t} \\ x_2(t) = \left[C_1(\cos 2t - 2\sin 2t) + C_2(2\cos 2t + \sin 2t)\right] e^{2t} \\ \qquad\quad = \left[(C_1 + 2C_2)\cos 2t + (C_2 - 2C_1)\sin 2t\right] e^{2t} \end{cases}$$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = 5x_1 - 9x_2, \quad x_2' = 2x_1 - x_2$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 5 & -9 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

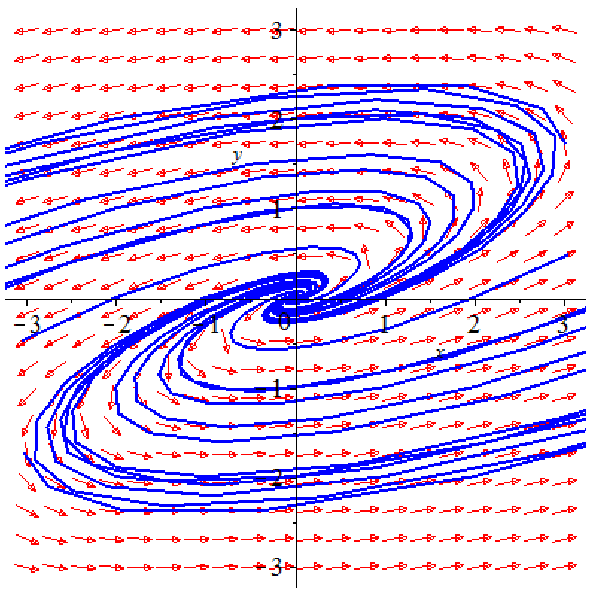
$$\begin{vmatrix} 5-\lambda & -9 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 13 = 0$$

The distinct real eigenvalues: $\underline{\lambda_{1,2} = 2 \pm 3i}$

For $\lambda = 2+3i \quad \Rightarrow \quad (A - \lambda I)V = 0$

$$\begin{pmatrix} 3-3i & -9 \\ 2 & -3-3i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 3(1-i)x = 9y$$

$$\to V = \begin{pmatrix} 3 \\ 1-i \end{pmatrix}$$



$$\begin{aligned} x(t) &= \begin{pmatrix} 3 \\ 1-i \end{pmatrix} e^{(2+3i)t} \\ &= \begin{pmatrix} 3 \\ 1-i \end{pmatrix} e^{2t} e^{3it} \\ &= \begin{pmatrix} 3 \\ 1-i \end{pmatrix} e^{2t}(\cos 3t + i\sin 3t) \\ &= \underline{\begin{pmatrix} 3\cos 3t + 3i\sin 3t \\ \cos 3t + \sin 3t + i(\sin 3t - \cos 3t) \end{pmatrix} e^{2t}} \end{aligned}$$

$$\begin{cases} x_1(t) = \left(3C_1 \cos 3t + 3C_2 \sin 3t\right) e^{2t} \\ x_2(t) = \left[C_1(\cos 3t + \sin 3t) + C_2(\sin 3t - \cos 3t)\right] e^{2t} \\ \qquad\quad = \left[(C_1 - C_2)\cos 3t + (C_1 + C_2)\sin 3t\right] e^{2t} \end{cases}$$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = 3x_1 + 4x_2, \quad x_2' = 3x_1 + 2x_2; \quad x_1(0) = x_2(0) = 1$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 5\lambda - 6 = 0$$

The distinct real eigenvalues: $\underline{\lambda_1 = -1, \quad \lambda_2 = 6|}$

For $\lambda_1 = -1 \quad \Rightarrow \quad (A+I)V_1 = 0$

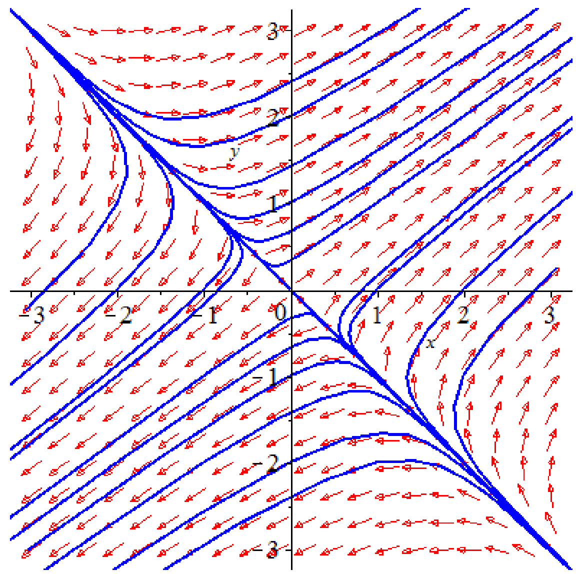
$$\begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad x_1 = -y_1$$

$$\rightarrow V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

For $\lambda_2 = 6 \quad \Rightarrow \quad (A-6I)V_2 = 0$

$$\begin{pmatrix} -3 & 4 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 3x_2 = 4y_2$$

$$\rightarrow V_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad \Rightarrow \quad x_2(t) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{6t}$$



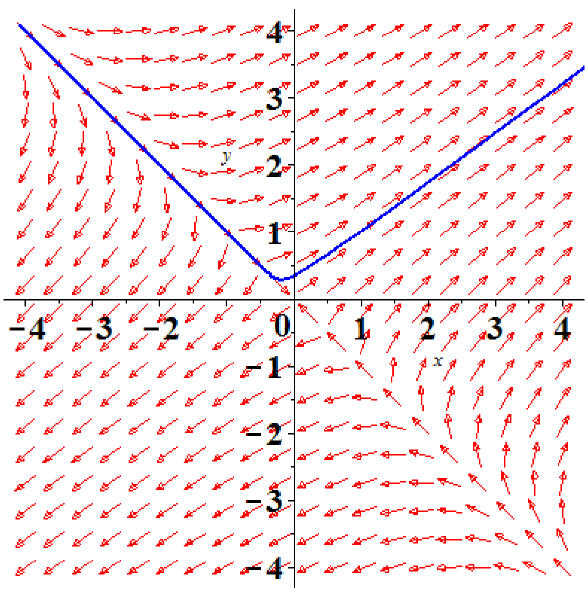
The general solution: $x(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{6t}$

$$\begin{cases} x_1(t) = C_1 e^{-t} + 4C_2 e^{6t} \\ x_2(t) = -C_1 e^{-t} + 3C_2 e^{6t} \end{cases}$$

Given: $\begin{cases} x_1(0) = C_1 + 4C_2 = 1 \\ x_2(0) = -C_1 + 3C_2 = 1 \end{cases}$

$$\rightarrow \underline{C_2 = \tfrac{2}{7}, \ C_1 = -\tfrac{1}{7}|}$$

$$\begin{cases} x_1(t) = -\frac{1}{7} e^{-t} + \frac{8}{7} e^{6t} \\ x_2(t) = \frac{1}{7} e^{-t} + \frac{6}{7} e^{6t} \end{cases}$$



Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = 9x_1 + 5x_2$, $x_2' = -6x_1 - 2x_2$; $x_1(0) = 1$, $x_2(0) = 0$

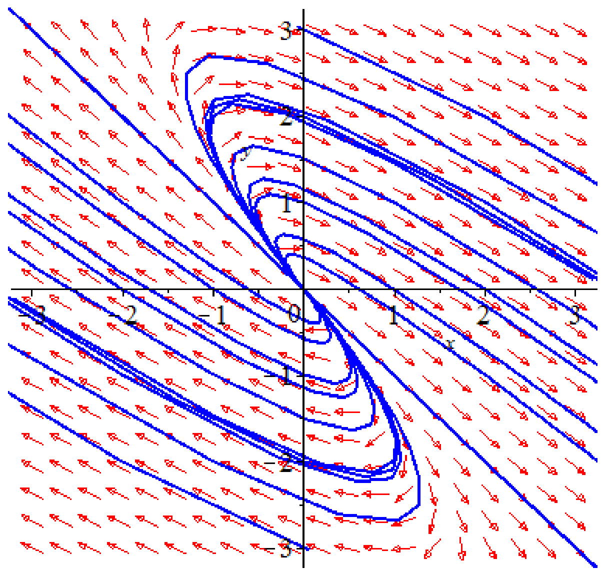
Solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 9 & 5 \\ -6 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 9-\lambda & 5 \\ -6 & -2-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 12 = 0$$

The distinct real eigenvalues: $\underline{\lambda_1 = 3, \quad \lambda_2 = 4|}$



For $\lambda_1 = 3 \implies (A - 3I)V_1 = 0$

$$\begin{pmatrix} 6 & 5 \\ -6 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies 6x_1 = -5y_1$$

$$\rightarrow V_1 = \begin{pmatrix} 5 \\ -6 \end{pmatrix} \implies x_1(t) = \begin{pmatrix} 5 \\ -6 \end{pmatrix} e^{3t}$$

For $\lambda_2 = 4 \implies (A - 4I)V_2 = 0$

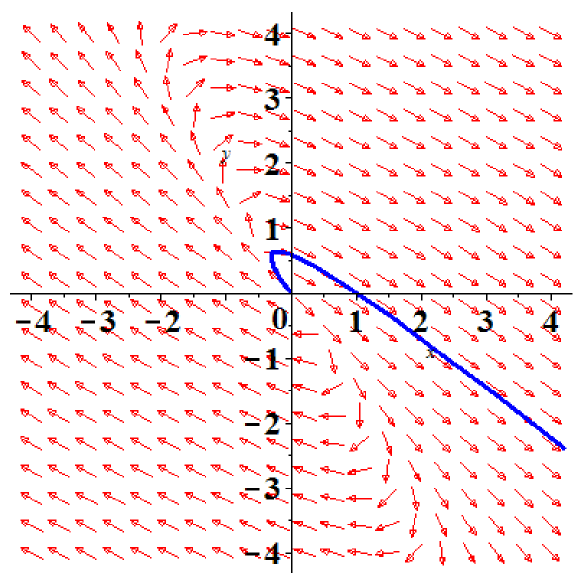
$$\begin{pmatrix} 5 & 5 \\ -6 & -6 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x_2 = -y_2$$

$$\rightarrow V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \implies x_2(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$$

The general solution:

$$x(t) = C_1\begin{pmatrix} 5 \\ -6 \end{pmatrix} e^{3t} + C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$$

$$\begin{cases} x_1(t) = 5C_1 e^{3t} + C_2 e^{4t} \\ x_2(t) = -6C_1 e^{3t} - C_2 e^{4t} \end{cases}$$



Given: $\begin{cases} x_1(0) = 5C_1 + C_2 = 1 \\ x_2(0) = -6C_1 - C_2 = 0 \end{cases}$

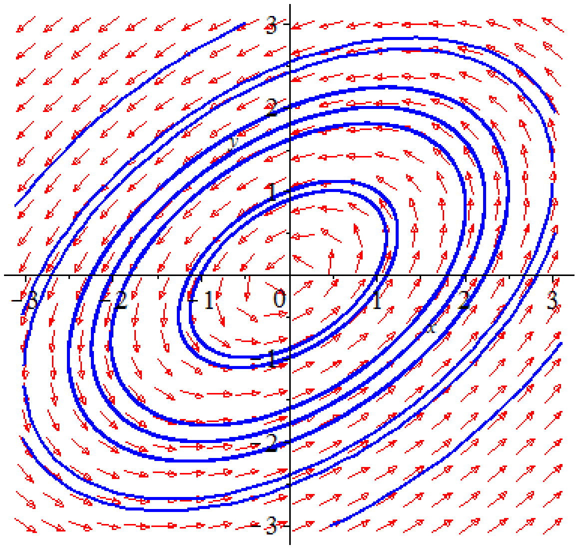
$$\rightarrow \underline{C_1 = -1, \quad C_2 = 6|}$$

$$\begin{cases} x_1(t) = -5e^{3t} + 6e^{4t} \\ x_2(t) = 6e^{3t} - 6e^{4t} \end{cases}$$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = 2x_1 - 5x_2, \quad x_2' = 4x_1 - 2x_2; \quad x_1(0) = 2,\ x_2(0) = 3$

Solution



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 2-\lambda & -5 \\ 4 & -2-\lambda \end{vmatrix} = \lambda^2 + 16 = 0$$

The distinct real eigenvalues: $\underline{\lambda = \pm 4i|}$

For $\lambda = 4i \quad \Rightarrow \quad (A - \lambda I)V = 0$

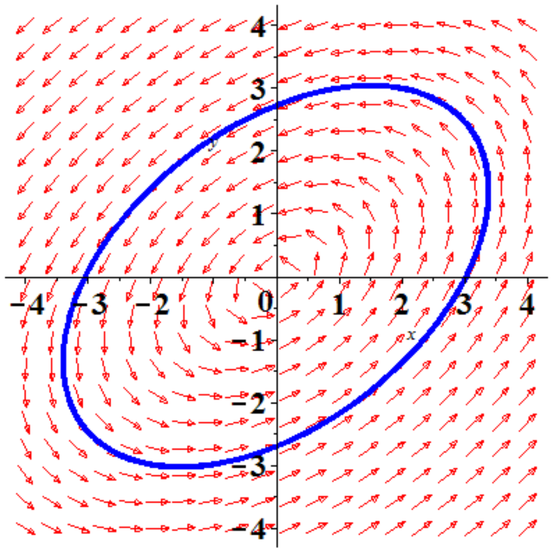
$$\begin{pmatrix} 2-4i & -5 \\ 4 & -2-4i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad (2-4i)x = 5y$$

$$\to V = \begin{pmatrix} 5 \\ 2-4i \end{pmatrix}$$

$$x(t) = \begin{pmatrix} 5 \\ 2-4i \end{pmatrix} e^{4it} \qquad\qquad e^{ait} = \cos at + i \sin at$$

$$= \begin{pmatrix} 5 \\ 2-4i \end{pmatrix} (\cos 4t + i \sin 4t)$$

$$= \begin{pmatrix} 5\cos 4t + 5i \sin 4t \\ 2\cos 4t + 4\sin 4t + i(2\sin 4t - 4\cos 4t) \end{pmatrix}$$



$$\begin{cases} x_1(t) = 5C_1 \cos 4t + 5C_2 \sin 4t \\ x_2(t) = C_1 (2\cos 4t + 4\sin 4t) + C_2 (2\sin 4t - 4\cos 4t) \end{cases}$$

Given: $x_1(0) = 2, \quad x_2(0) = 3$

$$\begin{cases} x_1(0) = 5C_1 = 2 \\ x_2(0) = 2C_1 - 4C_2 = 3 \end{cases} \to \underline{C_1 = \frac{2}{5}, \quad C_2 = -\frac{11}{20}|}$$

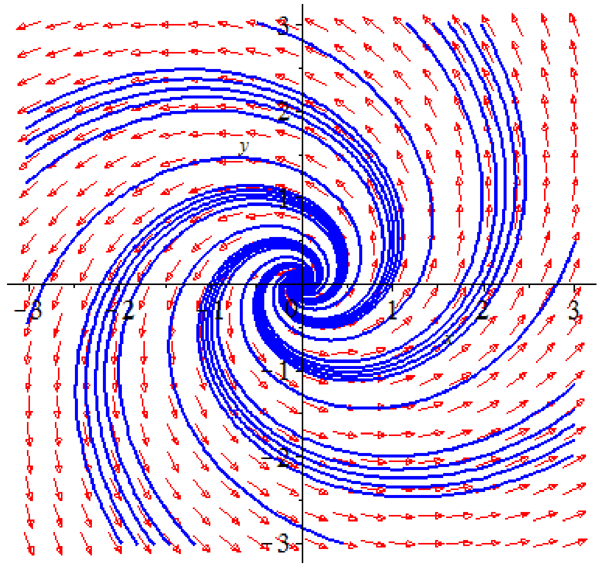
$$\begin{cases} x_1(t) = 2\cos 4t - \frac{11}{4} \sin 4t \\ x_2(t) = \frac{2}{5}(2\cos 4t + 4\sin 4t) - \frac{11}{20}(2\sin 4t - 4\cos 4t) \end{cases}$$

$$\begin{cases} x_1(t) = 2\cos 4t - \frac{11}{4} \sin 4t \\ x_2(t) = 3\cos 4t + \frac{1}{2} \sin 4t \end{cases}$$

Exercise

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system. $x_1' = x_1 - 2x_2, \quad x_2' = 2x_1 + x_2; \quad x_1(0) = 0, \; x_2(0) = 4$

Solution



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5 = 0$$

The distinct real eigenvalues: $\lambda = 1 \pm 2i$

For $\lambda = 1 - 2i \;\Rightarrow\; (A - \lambda I)V = 0$

$$\begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\Rightarrow\; (2i)x = 2y$$

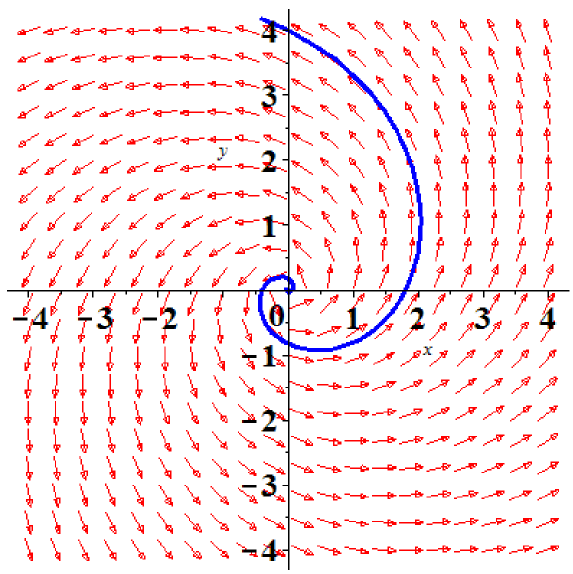
$$\to V = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$x(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(1-2i)t} \qquad\qquad e^{ait} = \cos at + i \sin at$$

$$= \begin{pmatrix} 1 \\ i \end{pmatrix} (\cos 2t - i \sin 2t)\, e^t$$

$$= \begin{pmatrix} \cos 2t - i \sin 2t \\ \sin 2t + i \cos 2t \end{pmatrix} e^t$$

$$\begin{cases} x_1(t) = (C_1 \cos 2t - C_2 \sin 2t)\, e^t \\ x_2(t) = (C_1 \sin 2t + C_2 \cos 2t)\, e^t \end{cases}$$



Given: $x_1(0) = 0, \quad x_2(0) = 4$

$$\begin{cases} x_1(0) = C_1 = 0 \\ x_2(0) = C_2 = 4 \end{cases}$$

$$\begin{cases} x_1(t) = -4e^t \sin 2t \\ x_2(t) = 4e^t \cos 2t \end{cases}$$

Exercise

Find the general solution of the given system.

$$x_1' = 4x_1 + x_2 + 4x_3, \quad x_2' = x_1 + 7x_2 + x_3, \quad x_3' = 4x_1 + x_2 + 4x_3$$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 4-\lambda & 1 & 4 \\ 1 & 7-\lambda & 1 \\ 4 & 1 & 4-\lambda \end{vmatrix} = (4-\lambda)^2(7-\lambda) + 8 - 112 + 16\lambda - 8 + 2\lambda$$

$$= \left(16 - 8\lambda + \lambda^2\right)(7-\lambda) + 18\lambda - 112$$

$$= -\lambda^3 + 15\lambda^2 - 54\lambda = 0$$

The distinct real eigenvalues: $\lambda_1 = 0; \quad \lambda_2 = 6; \quad \lambda_3 = 9$

For $\lambda_1 = 0 \quad \Rightarrow \quad (A - 0I)V_1 = 0$

$$\begin{pmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} 4a_1 + b_1 + 4c_1 = 0 \\ a_1 + 7b_1 + c_1 = 0 \end{cases}$$

$$\text{Let } b_1 = 0 \quad \Rightarrow \quad a_1 = -c_1 = 1 \quad \rightarrow V_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $\lambda_2 = 6 \quad \Rightarrow \quad (A - 6I)V_2 = 0$

$$\begin{pmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} a_2 = c_2 \\ b_2 = -2c_2 \end{cases} \rightarrow V_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 9 \quad \Rightarrow \quad (A - 9I)V_3 = 0$

$$\begin{pmatrix} -5 & 1 & 4 \\ 1 & -2 & 1 \\ 4 & 1 & -5 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} a_3 = c_3 \\ b_3 = c_3 \end{cases} \rightarrow V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x_1(t) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad x_2(t) = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} e^{6t} \quad x_3(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{9t}$$

$$\begin{cases} x_1(t) = C_1 + C_2 e^{6t} + C_3 e^{9t} \\ x_2(t) = -2C_2 e^{6t} + C_3 e^{9t} \\ x_3(t) = -C_1 - C_2 e^{6t} + C_3 e^{9t} \end{cases}$$

Exercise

Find the general solution of the given system.

$$x_1' = x_1 + 2x_2 + 2x_3, \quad x_2' = 2x_1 + 7x_2 + x_3, \quad x_3' = 2x_1 + x_2 + 7x_3$$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 7-\lambda & 1 \\ 2 & 1 & 7-\lambda \end{vmatrix} = (1-\lambda)(7-\lambda)^2 + 8 - 28 + 4\lambda - 1 + \lambda - 28 + 4\lambda$$

$$= (1-\lambda)\left(49 - 14\lambda + \lambda^2\right) + 9\lambda - 49$$

$$= -\lambda^3 + 15\lambda^2 - 54\lambda = 0$$

The distinct real eigenvalues: $\underline{\lambda_1 = 0; \quad \lambda_2 = 6; \quad \lambda_3 = 9}|$

For $\lambda_1 = 0 \quad \Rightarrow \quad (A - 0I)V_1 = 0$

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \Rightarrow \quad \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_1 = -4c_1 \\ b_1 = c_1 \end{cases} \quad \to V_1 = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 6 \quad \Rightarrow \quad (A - 6I)V_2 = 0$

$$\begin{pmatrix} -5 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_2 = 0 \\ b_2 = -c_2 \end{cases} \quad \to V_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

For $\lambda_3 = 9 \quad \Rightarrow \quad (A - 9I)V_3 = 0$

$$\begin{pmatrix} -8 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_3 = \frac{1}{2}c_3 \\ b_3 = c_3 \end{cases} \to V_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$x_1(t)=\begin{pmatrix}-4\\1\\1\end{pmatrix}\quad x_2(t)=\begin{pmatrix}0\\1\\-1\end{pmatrix}e^{6t}\quad x_3(t)=\begin{pmatrix}1\\2\\2\end{pmatrix}e^{9t}$$

$$\begin{cases}x_1(t)=-4C_1+C_3e^{9t}\\x_2(t)=C_1+C_2e^{6t}+2C_3e^{9t}\\x_3(t)=C_1-C_2e^{6t}+2C_3e^{9t}\end{cases}$$

Exercise

Find the general solution of the given system.

$$x_1'=4x_1+x_2+x_3,\quad x_2'=x_1+4x_2+x_3,\quad x_3'=x_1+x_2+4x_3$$

Solution

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}'=\begin{pmatrix}4&1&1\\1&4&1\\1&1&4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix}4-\lambda&1&1\\1&4-\lambda&1\\1&1&4-\lambda\end{vmatrix}=(4-\lambda)^3+1+1-3(4-\lambda)$$

$$=64-48\lambda+12\lambda^2-\lambda^3-10+3\lambda$$

$$=-\lambda^3+12\lambda^2-45\lambda+54=0$$

The distinct real eigenvalues: $\lambda_{1,2}=3;\quad \lambda_3=6$

For $\lambda_1=3 \quad\Rightarrow\quad (A-3I)V_1=0$

$$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad\Rightarrow\quad a_1+b_1+c_1=0\qquad \to V_1=\begin{pmatrix}1\\-1\\0\end{pmatrix}\quad\to V_2=\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

For $\lambda_3=6 \quad\Rightarrow\quad (A-6I)V_3=0$

$$\begin{pmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{pmatrix}\begin{pmatrix}a_3\\b_3\\c_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad rref\quad\Rightarrow\quad\begin{pmatrix}1&0&-1\\0&1&-1\\0&0&0\end{pmatrix}\to\begin{cases}a_3=c_3\\b_3=c_3\end{cases}\to V_3=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$x_1(t)=\begin{pmatrix}1\\-1\\0\end{pmatrix}e^{3t}\quad x_2(t)=\begin{pmatrix}1\\0\\-1\end{pmatrix}e^{3t}\quad x_3(t)=\begin{pmatrix}1\\1\\1\end{pmatrix}e^{6t}$$

$$\begin{cases}x_1(t)=C_1e^{3t}+C_2e^{3t}+C_3e^{6t}\\ x_2(t)=-C_1e^{3t}\qquad\quad +C_3e^{6t}\\ x_3(t)=\qquad\quad -C_2e^{3t}+C_3e^{6t}\end{cases}$$

Exercise

Find the general solution of the given system.

$$x_1'=5x_1+x_2+3x_3,\quad x_2'=x_1+7x_2+x_3,\quad x_3'=3x_1+x_2+5x_3$$

Solution

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}'=\begin{pmatrix}5&1&3\\1&7&1\\3&1&5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix}5-\lambda&1&3\\1&7-\lambda&1\\3&1&5-\lambda\end{vmatrix}=(7-\lambda)(5-\lambda)^2+6-9(7-\lambda)-5+\lambda-5+\lambda$$

$$=(7-\lambda)\left(25-10\lambda+\lambda^2\right)-67+11\lambda$$

$$=-\lambda^3+17\lambda^2-84\lambda+108=0$$

The distinct real eigenvalues: $\underline{\lambda_1=2;\quad \lambda_2=6;\quad \lambda_3=9}|$

For $\lambda_1=2\ \Rightarrow\ (A-2I)V_1=0$

$$\begin{pmatrix}3&1&3\\1&5&1\\3&1&3\end{pmatrix}\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad rref\Rightarrow\quad \begin{pmatrix}1&0&1\\0&1&0\\0&0&0\end{pmatrix}\rightarrow\begin{cases}a_1=-c_1\\b_1=0\end{cases}\quad\rightarrow V_1=\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

For $\lambda_2=6\ \Rightarrow\ (A-6I)V_2=0$

$$\begin{pmatrix}-1&1&3\\1&1&1\\3&1&-1\end{pmatrix}\begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad rref\quad\Rightarrow\quad \begin{pmatrix}1&0&-1\\0&1&2\\0&0&0\end{pmatrix}\rightarrow\begin{cases}a_2=c_2\\b_2=-2c_2\end{cases}\quad\rightarrow V_2=\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

For $\lambda_3=9\ \Rightarrow\ (A-9I)V_3=0$

$$\begin{pmatrix} -4 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} a_3 = c_3 \\ b_3 = c_3 \end{cases} \rightarrow V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x_1(t) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{2t} \quad x_2(t) = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{6t} \quad x_3(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{9t}$$

$$\begin{cases} x_1(t) = -C_1 e^{2t} + C_2 e^{6t} + C_3 e^{9t} \\ x_2(t) = \qquad -2C_2 e^{6t} + C_3 e^{9t} \\ x_3(t) = C_1 e^{2t} + C_2 e^{6t} + C_3 e^{9t} \end{cases}$$

Exercise

Find the general solution of the given system.

$$x_1' = 5x_1 - 6x_3, \quad x_2' = 2x_1 - x_2 - 2x_3, \quad x_3' = 4x_1 - 2x_2 - 4x_3$$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 5-\lambda & 0 & -6 \\ 2 & -1-\lambda & -2 \\ 4 & -2 & -4-\lambda \end{vmatrix} = (-1-\lambda)(5-\lambda)(-4-\lambda) + 24 + 24(-1-\lambda) - 4(5-\lambda)$$

$$= (-1-\lambda)\left(-20 - \lambda + \lambda^2\right) - 24\lambda - 20 + 4\lambda$$

$$= -\lambda^3 + \lambda = 0$$

The distinct real eigenvalues: $\lambda_1 = -1; \quad \lambda_2 = 0; \quad \lambda_3 = 1$

For $\lambda_1 = -1 \quad \Rightarrow \quad (A+I)V_1 = 0$

$$\begin{pmatrix} 6 & 0 & -6 \\ 2 & 0 & -2 \\ 4 & -2 & -3 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \Rightarrow \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} a_1 = c_1 \\ b_1 = \frac{1}{2} c_1 \end{cases} \rightarrow V_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 0 \quad \Rightarrow \quad (A - 0I)V_2 = 0$

$$\begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \color{red}{rref} \quad \Rightarrow \begin{pmatrix} 1 & 0 & -\frac{6}{5} \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_2=\frac{6}{5}c_2 \\ b_2=\frac{2}{5}c_2 \end{cases} \to V_2=\begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix}$$

For $\lambda_3=1 \;\Rightarrow\; (A-I)V_3=0$

$$\begin{pmatrix} 4 & 0 & -6 \\ 2 & -2 & -2 \\ 4 & -2 & -5 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_3=\frac{3}{2}c_3 \\ b_3=\frac{1}{2}c_3 \end{cases} \to V_3=\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$x_1(t)=\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}e^{-t} \quad x_2(t)=\begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} \quad x_3(t)=\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}e^{t}$$

$$\begin{cases} x_1(t)=2C_1e^{-t}+6C_2+3C_3e^{t} \\ x_2(t)=C_1e^{-t}+2C_2+C_3e^{t} \\ x_3(t)=2C_1e^{-t}+5C_2+2C_3e^{t} \end{cases}$$

Exercise

Find the general solution of the given system.

$$x_1'=3x_1+2x_2+2x_3, \quad x_2'=-5x_1-4x_2-2x_3, \quad x_3'=5x_1+5x_2+3x_3$$

Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}'=\begin{pmatrix} 3 & 2 & 2 \\ -5 & -4 & -2 \\ 5 & 5 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The characteristic equation:

$$\begin{vmatrix} 3-\lambda & 2 & 2 \\ -5 & -4-\lambda & -2 \\ 5 & 5 & 3-\lambda \end{vmatrix}=(3-\lambda)^2(-4-\lambda)-20-50-10(-4-\lambda)+20(3-\lambda)$$

$$=\left(9-6\lambda+\lambda^2\right)(-4-\lambda)+30-10\lambda$$

$$=-\lambda^3+2\lambda^2+5\lambda-6=0$$

The distinct real eigenvalues: $\lambda_1=-2; \quad \lambda_2=1; \quad \lambda_3=3$

For $\lambda_1=-2 \;\Rightarrow\; (A+2I)V_1=0$

$$\begin{pmatrix} 5 & 2 & 2 \\ -5 & -2 & -2 \\ 5 & 5 & 5 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_1 = 0 \\ b_1 = -c_1 \end{cases} \quad \to V_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1 \quad \Rightarrow \quad (A - I)V_2 = 0$

$$\begin{pmatrix} 2 & 2 & 2 \\ -5 & -5 & -2 \\ 5 & 5 & 2 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_2 = -b_2 \\ c_2 = 0 \end{cases} \quad \to V_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

For $\lambda_3 = 3 \quad \Rightarrow \quad (A - 3I)V_3 = 0$

$$\begin{pmatrix} 0 & 2 & 2 \\ -5 & -7 & -2 \\ 5 & 5 & 0 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad rref \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{cases} a_3 = c_3 \\ b_3 = -c_3 \end{cases} \to V_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

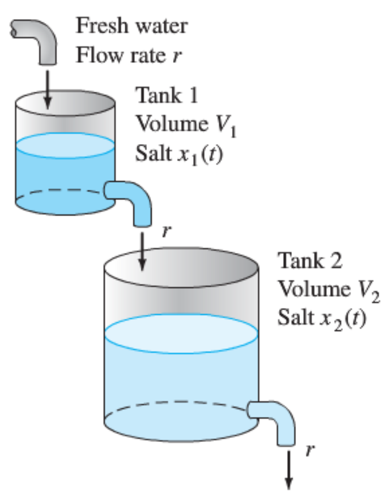
$$x_1(t) = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{-2t} \quad x_2(t) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{t} \quad x_3(t) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{3t}$$

$$\begin{cases} x_1(t) = \qquad\qquad C_2 e^{t} + C_3 e^{3t} \\ x_2(t) = -C_1 e^{-2t} - C_2 e^{t} - C_3 e^{3t} \\ x_3(t) = C_1 e^{-2t} \qquad\qquad + C_3 e^{3t} \end{cases}$$

Exercise

Find the amount $x_1(t)$, $x_2(t)$ of salt in each tank at time $t \ge 0$, with $x_1(0) = 15\ lb \quad x_2(0) = 0$. If $V_1 = 50\ gal, \quad V_2 = 25\ gal, \quad r = 10\ gal/\min$

Solution



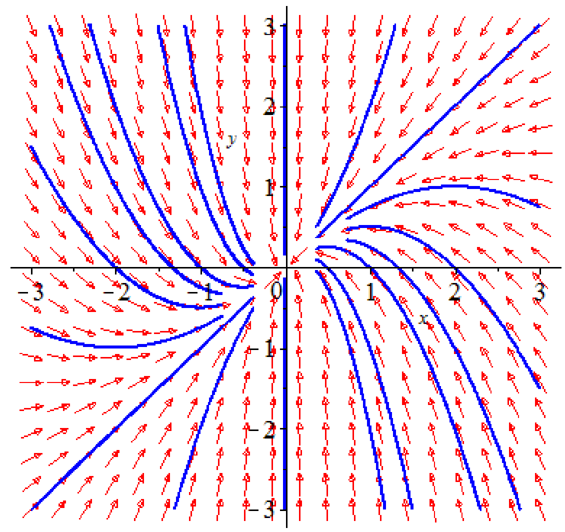
$$\begin{cases} x_1' = -k_1 x_1 \\ x_2' = k_1 x_1 - k_2 x_2 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i = 1, 2$$

$$k_1 = \frac{10}{50} = .2 \quad k_2 = \frac{10}{25} = .4 \to \begin{cases} x_1' = -.2x_1 \\ x_2' = .2x_1 - .4x_2 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -.2 & 0 \\ .2 & -.4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad with \quad x(0) = \begin{pmatrix} 15 \\ 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -.2 - \lambda & 0 \\ .2 & -.4 - \lambda \end{vmatrix} = (-.2 - \lambda)(-.4 - \lambda) = 0$$

The eigenvalues are: $\boxed{\lambda_1 = -.4 \quad \lambda_2 = -.2}$



For $\lambda_1 = -.4 \quad \Rightarrow \quad (A + .4I)V_1 = 0$

$$\begin{pmatrix} .2 & 0 \\ .2 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a_1 = 0 \quad \rightarrow \quad V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -.2 \quad \Rightarrow \quad (A + .2I)V_2 = 0$

$$\begin{pmatrix} 0 & 0 \\ .2 & -.2 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a_2 = b_2 \quad \rightarrow \quad V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow x(t) = C_1\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-.4t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-.2t}$$

The general solution:

$$\begin{cases} x_1(t) = C_2 e^{-.2t} \\ x_2(t) = C_1 e^{-.4t} + C_2 e^{-.2t} \end{cases}$$

$$\begin{cases} x_1(0) = C_2 = 15 \\ x_2(0) = C_1 + C_2 = 0 \end{cases} \Rightarrow \quad \boxed{C_2 = 15, \; C_1 = -15}$$

$$\begin{cases} x_1(t) = 15e^{-.2t} \\ x_2(t) = 15e^{-.2t} - 15e^{-.4t} \end{cases}$$

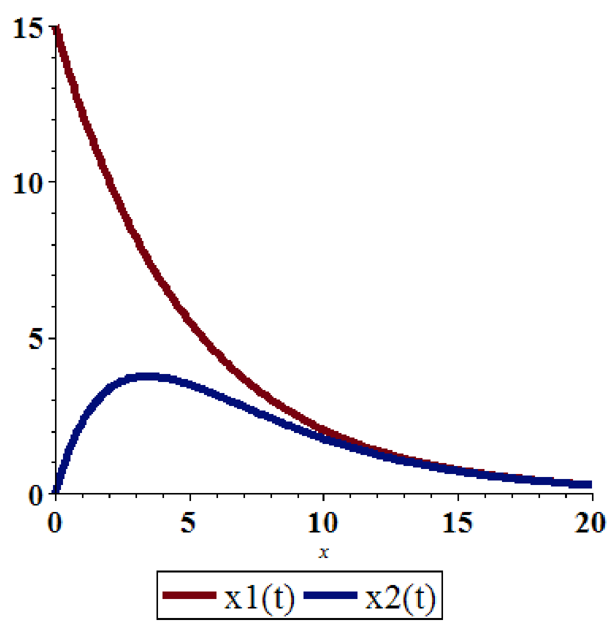
***Tank* 2**: $x_2'(t) = -3e^{-.2t} + 6e^{-.4t} = 0$

$$e^{-.2t} = 2e^{-.4t}$$

$$\ln e^{-.2t} = \ln\left(2e^{-.4t}\right)$$

$$-.2t = \ln(2) - .4t$$

$$\boxed{t = \frac{1}{.2}\ln 2 = 5\ln 2}$$



The maximum values of salt in tank 2 is:

$$x_2(t = 5\ln 2) = 15e^{-.2(5\ln 2)} - 15e^{-.4(5\ln 2)}$$

$$= 15\left(2^{-1} - 2^{-2}\right)$$

$$\boxed{= 3.75 \;\; lb.}$$

There is no maximum values of salt in tank 1.

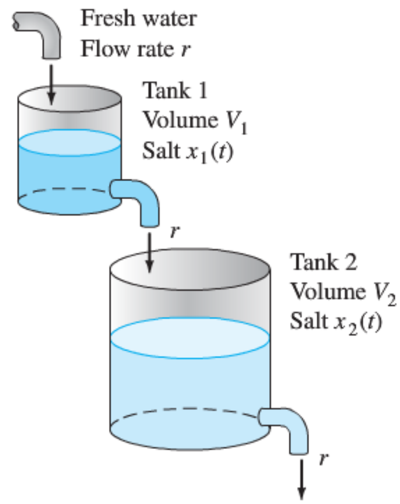
$$x_1'(t) = -3e^{-.2t} \neq 0$$

Exercise

Find the amount $x_1(t),\ x_2(t)$ of salt in each tank at time $t \ge 0$, with $x_1(0) = 15\ lb \quad x_2(0) = 0$. If $V_1 = 25\ gal,\quad V_2 = 40\ gal,\quad r = 10\ gal/\min$

Solution

$$\begin{cases} x_1' = -k_1 x_1 \\ x_2' = k_1 x_1 - k_2 x_2 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i = 1, 2$$



$$k_1 = \frac{10}{25} = .4 \quad k_2 = \frac{10}{40} = .25 \rightarrow \begin{cases} x_1' = -.4x_1 \\ x_2' = .4x_1 - .25x_2 \end{cases}$$

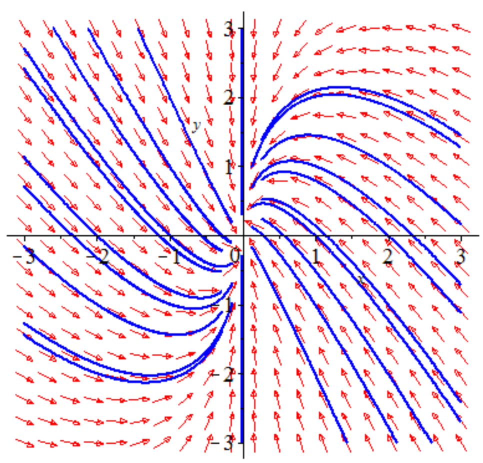
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -.4 & 0 \\ .4 & -.25 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad with \quad x(0) = \begin{pmatrix} 15 \\ 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -.4 - \lambda & 0 \\ .4 & -.25 - \lambda \end{vmatrix} = (-.25 - \lambda)(-.4 - \lambda) = 0$$

The eigenvalues are: $\lambda_1 = -.4 \quad \lambda_2 = -.25$

For $\lambda_1 = -.4 \quad \Rightarrow \quad (A + .4I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 \\ .4 & .15 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow .4a_1 = -.15b_1 \quad \rightarrow \quad V_1 = \begin{pmatrix} 3 \\ -8 \end{pmatrix}$$



For $\lambda_2 = -.25 \quad \Rightarrow \quad (A + .25I)V_2 = 0$

$$\begin{pmatrix} .15 & 0 \\ .4 & 0 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a_2 = 0 \quad \rightarrow \quad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow x(t) = C_1 \begin{pmatrix} 3 \\ -8 \end{pmatrix} e^{-.4t} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-.25t}$$

The general solution:

$$\begin{cases} x_1(t) = 3C_1 e^{-.4t} \\ x_2(t) = -8C_1 e^{-.4t} + C_2 e^{-.25t} \end{cases}$$

$$\begin{cases} x_1(0) = 3C_1 = 15 \\ x_2(0) = -8C_1 + C_2 = 0 \end{cases} \Rightarrow \quad C_1 = 5,\ C_2 = 40$$

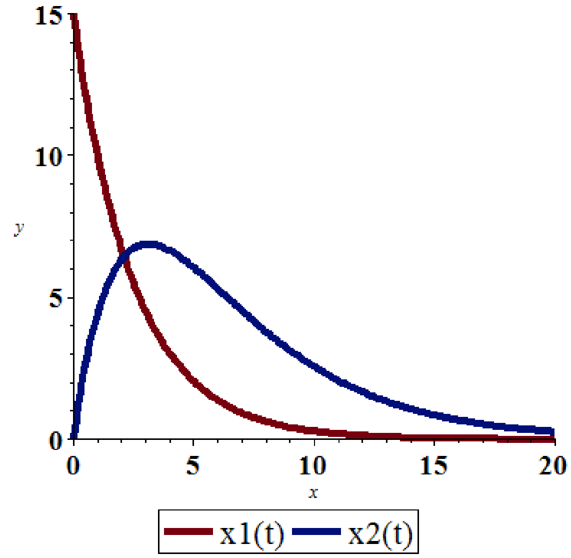
$$\begin{cases} x_1(t) = 15e^{-.4t} \\ x_2(t) = -40e^{-.4t} + 40e^{-.25t} \end{cases}$$

There is no maximum values of salt in tank 1.

$$x_1'(t) = -6e^{-.4t} \ne 0$$

Tank 2: $x_2'(t) = 16e^{-.4t} - 10e^{-.25t} = 0$

$$8e^{-.4t} = 5e^{-.25t}$$
$$\ln\left(e^{-.4t}\right) = \ln\left(\frac{5}{8}e^{-.25t}\right)$$
$$-.4t = \ln\left(\frac{5}{8}\right) - .25t$$
$$-.15t = \ln\left(\frac{5}{8}\right)$$
$$t = \frac{1}{.15}\ln\frac{8}{5} = \boxed{\frac{20}{3}\ln\frac{8}{5}}$$



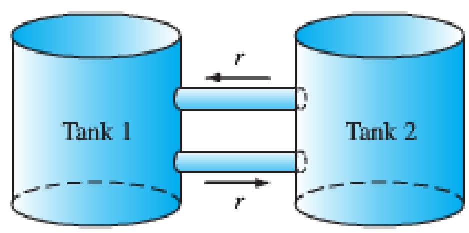
The maximum values of salt in tank 2 is:

$$x_2\left(t = \frac{20}{3}\ln\frac{8}{5}\right) = -40e^{-.4\left(\frac{20}{3}\ln\frac{8}{5}\right)} + 40e^{-.25\left(\frac{20}{3}\ln\frac{8}{5}\right)}$$
$$\boxed{= 6.85\ \ lb.}$$

Exercise

Find the amount $x_1(t),\ x_2(t)$ of salt in each tank at time $t \ge 0$, with $x_1(0) = 15\ lb \quad x_2(0) = 0$. If $V_1 = 50\ gal,\quad V_2 = 25\ gal,\quad r = 10\ gal/\min$

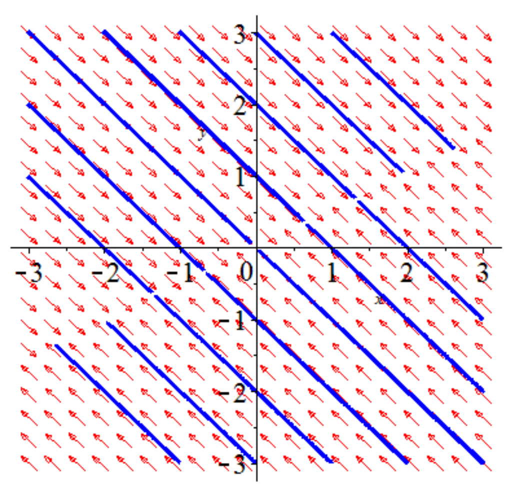
Solution



$$\begin{cases} x_1' = -k_1x_1 + k_2x_2 \\ x_2' = k_1x_1 - k_2x_2 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i = 1, 2$$

$$k_1 = \frac{10}{50} = .2 \quad k_2 = \frac{10}{25} = .4 \ \rightarrow \begin{cases} x_1' = -.2x_1 + .4x_2 \\ x_2' = .2x_1 - .4x_2 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -.2 & .4 \\ .2 & -.4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \textit{with} \quad x(0) = \begin{pmatrix} 15 \\ 0 \end{pmatrix}$$



$$|A - \lambda I| = \begin{vmatrix} -.2-\lambda & .4 \\ .2 & -.4-\lambda \end{vmatrix}$$
$$= (-.2-\lambda)(-.4-\lambda) - .08$$
$$= \lambda^2 + .6\lambda = 0$$

The eigenvalues are: $\boxed{\lambda_1 = -.6 \quad \lambda_2 = 0}$

For $\lambda_1 = -.6 \ \Rightarrow\ (A + .6I)V_1 = 0$

$$\begin{pmatrix} .4 & .4 \\ .2 & .2 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow .4a_1 = -.4b_1 \ \rightarrow\ V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = 0 \quad \Rightarrow \quad (A - 0I)V_2 = 0$

$$\begin{pmatrix} -.2 & .4 \\ .2 & -.4 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow .2a_2 = .4b_2 \quad \rightarrow \quad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

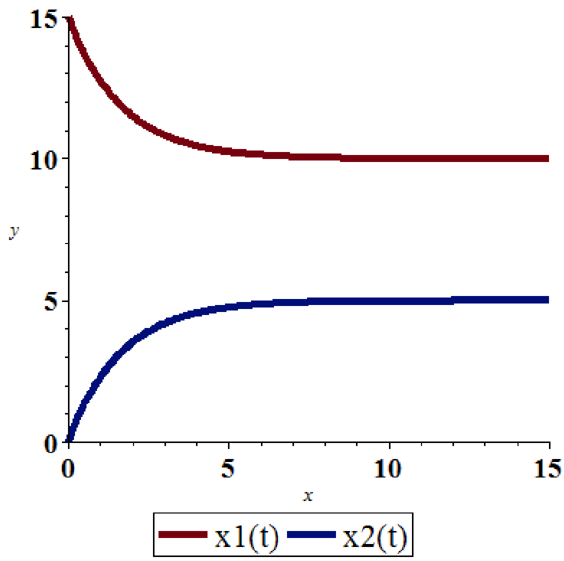
$$\Rightarrow x(t) = C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-.6t} + C_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The general solution:

$$\begin{cases} x_1(t) = C_1 e^{-0.6t} + 2C_2 \\ x_2(t) = -C_1 e^{-0.6t} + C_2 \end{cases}$$

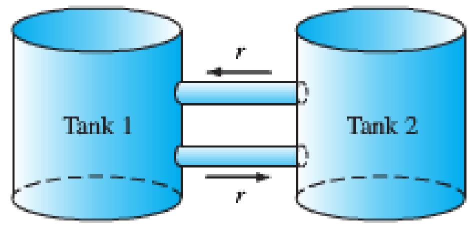
$$\begin{cases} x_1(0) = C_1 + 2C_2 = 15 \\ x_2(0) = -C_1 + C_2 = 0 \end{cases} \quad \Rightarrow \quad \underline{C_1 = 5,\ C_2 = 5|}$$

$$\begin{cases} x_1(t) = 10 + 5e^{-0.6t} \\ x_2(t) = 5 - 5e^{-0.6t} \end{cases}$$



Exercise

Find the amount $x_1(t),\ x_2(t)$ of salt in each tank at time $t \ge 0$, with $x_1(0) = 15\ lb \quad x_2(0) = 0$. If $V_1 = 25\ gal,\quad V_2 = 40\ gal,\quad r = 10\ gal/\min$



Solution

$$\begin{cases} x_1' = -k_1x_1 + k_2x_2 \\ x_2' = k_1x_1 - k_2x_2 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i = 1, 2$$

$$k_1 = \frac{10}{25} = .4 \quad k_2 = \frac{10}{40} = .25 \to \begin{cases} x_1' = -.4x_1 + .25x_2 \\ x_2' = .4x_1 - .25x_2 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -.4 & .25 \\ .4 & -.25 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \textit{with} \quad x(0) = \begin{pmatrix} 15 \\ 0 \end{pmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -.4 - \lambda & .25 \\ .4 & -.25 - \lambda \end{vmatrix} \\ &= (-.25 - \lambda)(-.4 - \lambda) - .1 \\ &= \lambda^2 + .65\lambda = 0 \end{aligned}$$

The eigenvalues are: $\underline{\lambda_1 = 0 \quad \lambda_2 = -.65}$

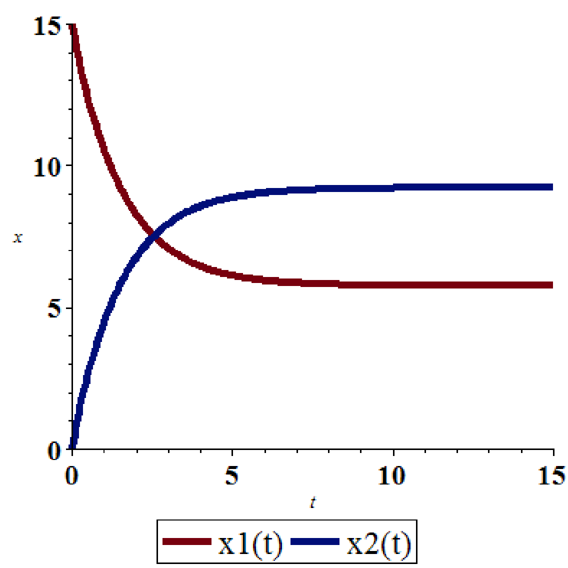
For $\lambda_1 = 0 \Rightarrow (A - 0I)V_1 = 0$

$$\begin{pmatrix} -.4 & .25 \\ .4 & -.25 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow .4a_1 = .25b_1 \quad \to \quad V_1 = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

For $\lambda_2 = -.65 \Rightarrow (A + .65I)V_2 = 0$

$$\begin{pmatrix} .25 & .25 \\ .4 & .4 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow .25a_2 = -.25b_2 \quad \to \quad V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\Rightarrow x(t) = C_1\begin{pmatrix} 5 \\ 8 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-.65t}$$



The general solution: $\begin{cases} x_1(t) = 5C_1 + C_2e^{-0.65t} \\ x_2(t) = 8C_1 - C_2e^{-0.65t} \end{cases}$

$$\begin{cases} x_1(0) = 5C_1 + C_2 = 15 \\ x_2(0) = 8C_1 - C_2 = 0 \end{cases} \Rightarrow \underline{C_1 = \frac{15}{13},\ C_2 = \frac{120}{13}}$$

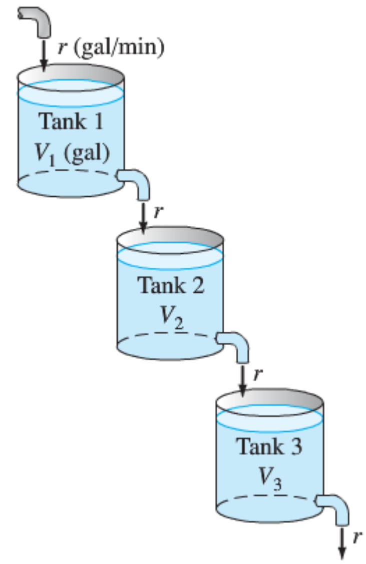
$$\begin{cases} x_1(t) = \frac{15}{13}\left(5 + 8e^{-0.6t}\right) \\ x_2(t) = \frac{120}{13}\left(1 - e^{-0.6t}\right) \end{cases}$$

Exercise

Find the amount $x_1(t),\ x_2(t),\ x_3(t)$ of salt in each tank at time $t \ge 0$, if

$V_1 = 30\ gal,\quad V_2 = 15\ gal,\quad V_3 = 10\ gal,\quad r = 30\ gal/\min \quad x_1(0) = 27\ lb \quad x_2(0) = x_3(0) = 0$

Solution



$$\begin{cases} x_1' = -k_1 x_1 \\ x_2' = k_1 x_1 - k_2 x_2 \\ x_3' = \qquad\quad k_2 x_2 - k_3 x_3 \end{cases} \qquad \text{where } k_i = \frac{r}{V_i} \quad i = 1,2,3$$

$$k_1 = \frac{30}{30} = 1 \quad k_2 = \frac{30}{15} = 2 \quad k_3 = \frac{30}{10} = 3$$

$$\rightarrow \begin{cases} x_1' = -x_1 \\ x_2' = x_1 - 2x_2 \\ x_3' = \qquad 2x_2 - 3x_3 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \textit{with} \quad x(0) = \begin{pmatrix} 27 \\ 0 \\ 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 0 & 0 \\ 1 & -2-\lambda & 0 \\ 0 & 2 & -3-\lambda \end{vmatrix} = (-1-\lambda)(-2-\lambda)(-3-\lambda) = 0$$

The eigenvalues are: $\lambda_1 = -3 \quad \lambda_2 = -2 \quad \lambda_3 = -1$

For $\lambda_1 = -3 \quad \Rightarrow \quad (A + 3I)V_1 = 0$

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} 2a_1 = 0 \ \rightarrow\ a_1 = 0 \\ a_1 = -b_1 \ \rightarrow\ b_1 = 0 \end{cases} \quad \rightarrow V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-3t}$$

For $\lambda_2 = -2 \quad \Rightarrow \quad (A + 2I)V_2 = 0$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & -1 \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} a_2 = 0 \\ 2b_2 = c_2 \end{cases} \quad \rightarrow V_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \Rightarrow \quad x_2(t) = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} e^{-2t}$$

For $\lambda_3 = -1 \quad \Rightarrow \quad (A + I)V_3 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} a_3 = b_3 \\ 2b_3 = 2c_3 \end{cases} \quad \rightarrow V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad x_3(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-t}$$

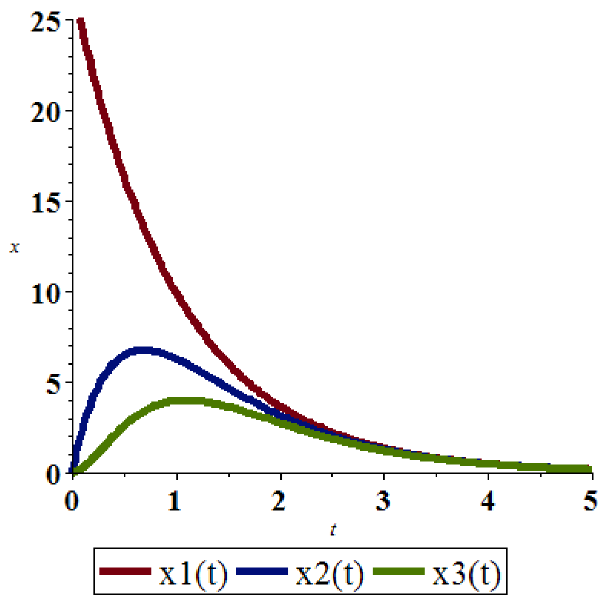
$$\Rightarrow \ x(t) = C_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-3t} + C_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} e^{-2t} + C_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-t}$$

$$\begin{cases} x_1(t) = C_3 e^{-t} \\ x_2(t) = C_2 e^{-2t} + C_3 e^{-t} \\ x_3(t) = C_1 e^{-3t} + 2C_2 e^{-2t} + C_3 e^{-t} \end{cases}$$

With *initial* values

$$\begin{cases} 27 = C_3 \\ 0 = C_2 + C_3 \\ 0 = C_1 + 2C_2 + C_3 \end{cases} \rightarrow \begin{cases} \boxed{C_3 = 27} \\ \boxed{C_2 = -27} \\ \boxed{C_1 = -27 - 2(-27) = 27} \end{cases}$$

$$\begin{cases} x_1(t) = 27e^{-t} \\ x_2(t) = 27e^{-t} - 27e^{-2t} \\ x_3(t) = 27e^{-t} - 54e^{-2t} + 27e^{-3t} \end{cases}$$



***Tank* 2**: $x_2'(t) = -27e^{-t} + 54e^{-2t} = 0$

$$e^{-t} = 2e^{-2t} \quad \Rightarrow \quad -t = \ln 2 - 2t$$

$$\boxed{t = \ln 2}$$

The maximum values of salt in tank 2 is:

$$x_2(\ln 2) = 27\left(e^{-\ln 2} - e^{-2\ln 2}\right) = 27\left(\tfrac{1}{2} - \tfrac{1}{4}\right)$$

$$= \boxed{\frac{27}{4}\ lbs}$$

***Tank* 3**: $x_3'(t) = 27\left(-e^{-t} + 4e^{-2t} - 3e^{-3t}\right) = 0$

$$e^{3t}\left(-e^{-t} + 4e^{-2t} - 3e^{-3t}\right) = 0$$

$$e^{2t} - 4e^{t} + 3 = 0 \quad \begin{cases} e^{t} = 1 \rightarrow t = 0 \\ e^{t} = 3 \rightarrow \boxed{t = \ln 3} \end{cases}$$

The maximum values of salt in tank 3 is:

$$x_3(\ln 3) = 27\left(e^{-\ln 3} - 2e^{-2\ln 3} + e^{-3\ln 3}\right) = 27\left(\tfrac{1}{3} - \tfrac{2}{9} + \tfrac{1}{27}\right)$$

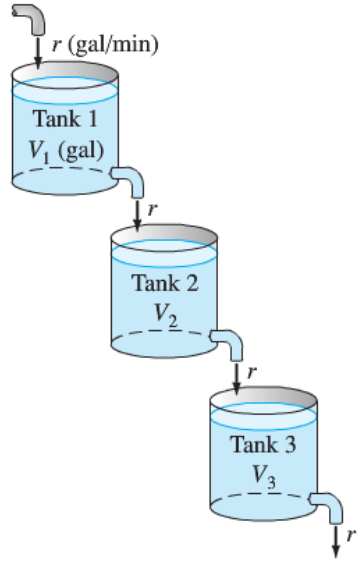
$$\boxed{= 4\ lbs}$$

Exercise

Find the amount $x_1(t),\ x_2(t),\ x_3(t)$ of salt in each tank at time $t \ge 0$, if

$V_1 = 20\ gal,\quad V_2 = 30\ gal,\quad V_3 = 60\ gal,\quad r = 60\ gal/\min \quad x_1(0) = 45\ lb \quad x_2(0) = x_3(0) = 0$

Solution



$$\begin{cases} x_1' = -k_1 x_1 \\ x_2' = k_1 x_1 - k_2 x_2 \\ x_3' = \qquad\quad k_2 x_2 - k_3 x_3 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i = 1,2,3$$

$$k_1 = \frac{60}{20} = 3 \quad k_2 = \frac{60}{30} = 2 \quad k_3 = \frac{60}{60} = 1$$

$$\to \begin{cases} x_1' = -3x_1 \\ x_2' = 3x_1 - 2x_2 \\ x_3' = \qquad 2x_2 - x_3 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -3 & 0 & 0 \\ 3 & -2 & 0 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \textit{with} \quad x(0) = \begin{pmatrix} 45 \\ 0 \\ 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -3-\lambda & 0 & 0 \\ 3 & -2-\lambda & 0 \\ 0 & 2 & -1-\lambda \end{vmatrix} = (-3-\lambda)(-2-\lambda)(-1-\lambda) = 0$$

The eigenvalues are: $\underline{\lambda_1 = -3 \quad \lambda_2 = -2 \quad \lambda_3 = -1|}$

For $\lambda_1 = -3 \quad \Rightarrow \quad (A + 3I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 3a_1 = -b_1 & \to a_1 = 1 \\ 2c_1 = -2b_1 & \to b_1 = -3 \\ & \quad\; c_1 = 3 \end{cases} \to V_1 = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} \Rightarrow x_1(t) = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} e^{-3t}$$

For $\lambda_2 = -2 \quad \Rightarrow \quad (A + 2I)V_2 = 0$

$$\begin{pmatrix} -1 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a_2 = 0 \\ 2b_2 = -c_2 \end{cases} \to V_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \Rightarrow x_2(t) = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} e^{-2t}$$

For $\lambda_3 = -1 \quad \Rightarrow \quad (A + I)V_3 = 0$

$$\begin{pmatrix} -2 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a_3 = b_3 = 0 \to V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow x_3(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t}$$

$$\Rightarrow\ x(t)=C_1\begin{pmatrix}1\\-3\\3\end{pmatrix}e^{-3t}+C_2\begin{pmatrix}0\\1\\-2\end{pmatrix}e^{-2t}+C_3\begin{pmatrix}0\\0\\1\end{pmatrix}e^{-t}$$

$$\begin{cases}x_1(t)=C_1e^{-3t}\\ x_2(t)=-3C_1e^{-3t}+\ C_2e^{-2t}\\ x_3(t)=3C_1e^{-3t}-2C_2e^{-2t}+C_3e^{-t}\end{cases}$$

With *initial* values

$$\begin{cases}45=C_1\\ 0=-3C_1+C_2\\ 0=3C_1-2C_2+C_3\end{cases}\quad\rightarrow\begin{cases}\underline{C_1=45}\\ \underline{C_2=135}\\ \underline{C_3}=-3(45)+2(-135)\underline{=135}\end{cases}$$

$$\begin{cases}x_1(t)=45e^{-3t}\\ x_2(t)=-135e^{-3t}+135e^{-2t}\\ x_3(t)=135e^{-3t}-270e^{-2t}+135e^{-t}\end{cases}$$

***Tank* 2**: $x_2'(t)=3e^{-3t}-2e^{-2t}=0$

$$1.5e^{-3t}=e^{-2t}\quad\Rightarrow\quad \ln 1.5-3t=-2t$$

$$\underline{t=\ln 1.5}$$

The maximum values of salt in tank 2 is:

$$x_2(\ln 1.5)=135\left(-e^{-3\ln 1.5}+e^{-2\ln 1.5}\right)=135\left(-\tfrac{8}{27}+\tfrac{4}{9}\right)$$

$$\underline{=20\ lbs}$$

***Tank* 3**: $x_3'(t)=135\left(-3e^{-3t}+4e^{-2t}-e^{-t}\right)=0$

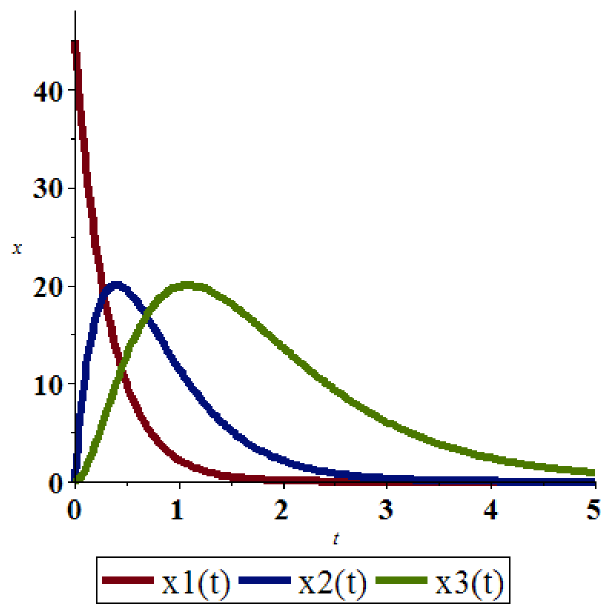
$$e^{3t}\left(-3e^{-3t}+4e^{-2t}-e^{-t}\right)=0$$

$$-3+4e^{t}-e^{2t}=0\qquad\begin{cases}e^{t}=1\rightarrow t=0\\ e^{t}=3\rightarrow \underline{t=\ln 3}\end{cases}$$

The maximum values of salt in tank 3 is:

$$x_2(\ln 3)=135\left(e^{-3\ln 3}-2e^{-2\ln 3}+e^{-\ln 3}\right)=135\left(\tfrac{1}{27}-\tfrac{2}{9}+\tfrac{1}{3}\right)$$

$$\underline{=20\ lbs}$$

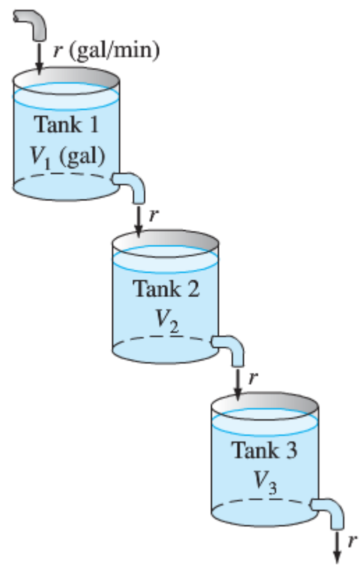


Exercise

Find the amount $x_1(t)$, $x_2(t)$, $x_3(t)$ of salt in each tank at time $t \ge 0$, if

$V_1 = 15\ gal$, $V_2 = 10\ gal$, $V_3 = 30\ gal$, $r = 60\ gal/\min$ $x_1(0) = 45\ lb$ $x_2(0) = x_3(0) = 0$

Solution



$$\begin{cases} x_1' = -k_1 x_1 \\ x_2' = k_1 x_1 - k_2 x_2 \\ x_3' = \qquad k_2 x_2 - k_3 x_3 \end{cases} \qquad \text{where } k_i = \frac{r}{v_i} \quad i = 1,2,3$$

$$k_1 = \frac{60}{15} = 4 \quad k_2 = \frac{60}{10} = 6 \quad k_3 = \frac{60}{30} = 2$$

$$\to \begin{cases} x_1' = -4x_1 \\ x_2' = 4x_1 - 6x_2 \\ x_3' = \qquad 6x_2 - 2x_3 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -4 & 0 & 0 \\ 4 & -6 & 0 \\ 0 & 6 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad with \quad x(0) = \begin{pmatrix} 45 \\ 0 \\ 0 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -4-\lambda & 0 & 0 \\ 4 & -6-\lambda & 0 \\ 0 & 6 & -2-\lambda \end{vmatrix} = (-4-\lambda)(-6-\lambda)(-2-\lambda) = 0$$

The eigenvalues are: $\underline{\lambda_1 = -4 \quad \lambda_2 = -6 \quad \lambda_3 = -2|}$

For $\lambda_1 = -4 \quad \Rightarrow \quad (A + 4I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 4 & -2 & 0 \\ 0 & 6 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} 4a_1 = 2b_1 & \to a_1 = 1 \\ 2c_1 = -6b_1 & \to b_1 = 2 \\ & \quad\ c_1 = -6 \end{cases} \quad \to V_1 = \begin{pmatrix} 1 \\ 2 \\ -6 \end{pmatrix} \quad \Rightarrow \quad x_1(t) = \begin{pmatrix} 1 \\ 2 \\ -6 \end{pmatrix} e^{-4t}$$

For $\lambda_2 = -6 \quad \Rightarrow \quad (A + 6I)V_2 = 0$

$$\begin{pmatrix} 2 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 6 & 4 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} a_2 = 0 \\ 6b_2 = -4c_2 \end{cases} \quad \to V_2 = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} \quad \Rightarrow \quad x_2(t) = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} e^{-6t}$$

For $\lambda_3 = -2 \quad \Rightarrow \quad (A + 2I)V_3 = 0$

$$\begin{pmatrix} -2 & 0 & 0 \\ 4 & -4 & 0 \\ 0 & 6 & 0 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a_3 = b_3 = 0 \quad \to V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad x_3(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-2t}$$

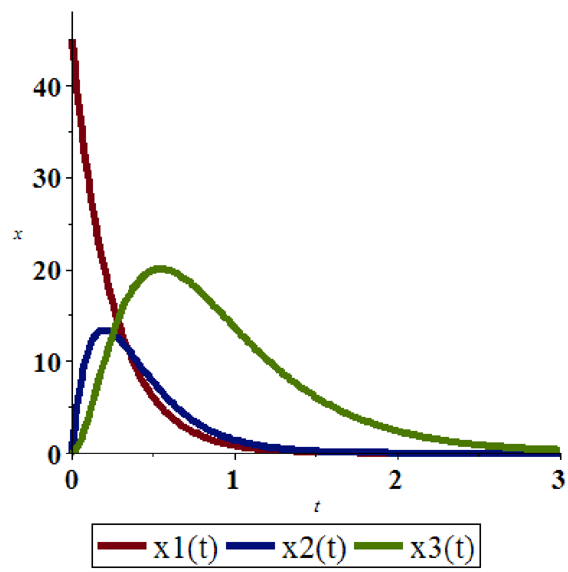
$$\Rightarrow\ x(t)=C_1\begin{pmatrix}1\\2\\-6\end{pmatrix}e^{-4t}+C_2\begin{pmatrix}0\\2\\-3\end{pmatrix}e^{-6t}+C_3\begin{pmatrix}0\\0\\1\end{pmatrix}e^{-2t}$$

$$\begin{cases}x_1(t)=C_1e^{-4t}\\ x_2(t)=2C_1e^{-4t}+2C_2e^{-6t}\\ x_3(t)=-6C_1e^{-4t}-3C_2e^{-6t}+C_3e^{-2t}\end{cases}$$

With *initial* values

$$\begin{cases}45=C_1\\ 0=2C_1+2C_2\\ 0=-6C_1-3C_2+C_3\end{cases}\ \rightarrow\begin{cases}\underline{C_1=45}|\\ \underline{C_2=-45}|\\ \underline{C_3}=6(45)+3(-45)\underline{=135}|\end{cases}$$

$$\begin{cases}x_1(t)=45e^{-4t}\\ x_2(t)=90e^{-4t}-90e^{-6t}\\ x_3(t)=-270e^{-4t}+135e^{-6t}+135e^{-2t}\end{cases}$$



***Tank* 2**: $x_2'(t)=-360e^{-4t}+540e^{-6t}=0$

$$2e^{-4t}=3e^{-6t}\quad\Rightarrow\quad \ln(2)-4t=\ln(3)-6t$$

$$\underline{t=\frac{1}{2}\ln 1.5}|$$

The maximum values of salt in tank 2 is:

$$x_2\left(\frac{1}{2}\ln 1.5\right)=90\left(e^{-2\ln 1.5}-e^{-3\ln 1.5}\right)=90\left(\frac{4}{9}-\frac{8}{27}\right)$$

$$\underline{=13.3\ lbs}|$$

***Tank* 3**: $x_3'(t)=135\left(8e^{-4t}-6e^{-6t}-2e^{-2t}\right)=0$

$$-2e^{-6t}\left(4e^{2t}-3-e^{4t}\right)=0$$

$$e^{4t}-4e^{2t}+3=0\qquad\begin{cases}e^{2t}=1\rightarrow t=0\\ e^{2t}=3\rightarrow \underline{t=\frac{1}{2}\ln 3}|\end{cases}$$

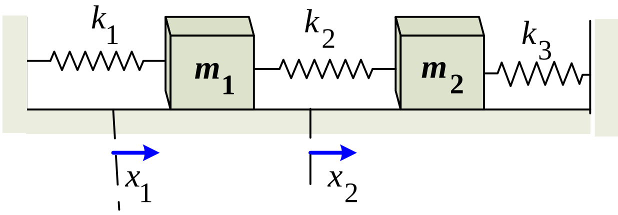
The maximum values of salt in tank 3 is:

$$x_2\left(\frac{1}{2}\ln 3\right)=135\left(-2e^{-2\ln 3}+e^{-3\ln 3}+e^{-\ln 3}\right)=135\left(-\frac{2}{9}+\frac{1}{27}+\frac{1}{3}\right)$$

$$\underline{=20\ lbs}|$$

Exercise

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

$$m_1 = m_2 = 1; \quad k_1 = 0,\ k_2 = 2,\ k_3 = 0 \quad (\textit{no walls})$$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 \end{cases} \Rightarrow \begin{cases} x_1'' = -2x_1 + 2x_2 \\ x_2'' = 2x_1 - 2x_2 \end{cases}$$

$$x'' = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\vec{x} \rightarrow A = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -2-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} \\ &= (-2-\lambda)^2 - 4 \\ &= \lambda^2 + 4\lambda = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = 0, \quad \lambda_2 = -4$

The natural frequencies: $\omega_1 = 0 \quad \textit{and} \quad \omega_2 = \sqrt{-(-4)} = 2$

For $\lambda_1 = 0 \quad \Rightarrow \quad (A - 0I)V_1 = 0$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = b \rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1(t) = (a_1 + b_1 t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -4 \quad \Rightarrow \quad (A + 4I)V_2 = 0$

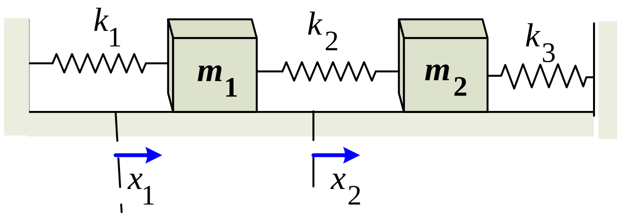
$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = -b \rightarrow V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \vec{x}_2(t) = (a_2 \cos 2t + b_2 \sin 2t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1 + b_1 t + a_2 \cos 2t + b_2 \sin 2t \\ \vec{x}_2(t) = a_1 + b_1 t - a_2 \cos 2t - b_2 \sin 2t \end{cases}$$

In the degenerate natural mode with frequency $\omega_1 = 0$ the 2 masses move by translation without oscillating. At frequency $\omega_2 = 2$ they oscillate in opposite directions with equal amplitudes.

Exercise

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

$$m_1 = m_2 = 1; \quad k_1 = 1,\ k_2 = 2,\ k_3 = 1$$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 \end{cases} \Rightarrow \begin{cases} x_1'' = -3x_1 + 2x_2 \\ x_2'' = 2x_1 - 3x_2 \end{cases}$$

$$x'' = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix} \vec{x} \quad \rightarrow \quad A = \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -3-\lambda & 2 \\ 2 & -3-\lambda \end{vmatrix} \\ &= (-3-\lambda)^2 - 4 \\ &= \lambda^2 + 4\lambda + 5 = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = -1, \quad \lambda_2 = -5$

The natural frequencies: $\omega_1 = 1 \quad and \quad \omega_2 = \sqrt{5}$

For $\lambda_1 = -1 \quad \Rightarrow \quad (A + I)V_1 = 0$

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = b \quad \rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1(t) = (a_1 \cos t + b_1 \sin t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -5 \quad \Rightarrow \quad (A + 5I)V_2 = 0$

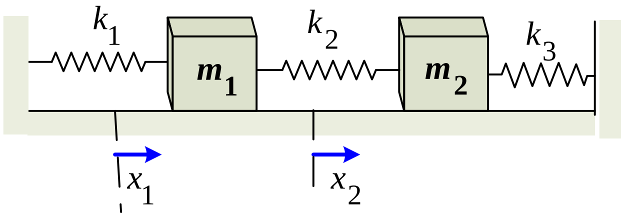
$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = -b \quad \rightarrow V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \vec{x}_2(t) = \left(a_2 \cos t\sqrt{5} + b_2 \sin t\sqrt{5}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1 \cos t + b_1 \sin t + a_2 \cos t\sqrt{5} + b_2 \sin t\sqrt{5} \\ \vec{x}_2(t) = a_1 \cos t + b_1 \sin t - a_2 \cos t\sqrt{5} - b_2 \sin t\sqrt{5} \end{cases}$$

In the degenerate natural mode with frequency $\omega_1 = 1$ the 2 masses move in the same direction with equal amplitudes of oscillation. At frequency $\omega_2 = \sqrt{5}$ they oscillate in opposite directions with equal amplitudes.

Exercise

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

$$m_1 = m_2 = 1; \quad k_1 = 2,\ k_2 = 1,\ k_3 = 2$$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 \end{cases} \Rightarrow \begin{cases} x_1'' = -3x_1 + x_2 \\ x_2'' = x_1 - 3x_2 \end{cases}$$

$$x'' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\vec{x} \quad \to \quad A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} \\ &= (-3-\lambda)^2 - 1 \\ &= \lambda^2 + 4\lambda + 8 = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = -2, \quad \lambda_2 = -4$

The natural frequencies: $\omega_1 = \sqrt{2} \quad and \quad \omega_2 = 2$

For $\lambda_1 = -2 \quad \Rightarrow \quad (A + 2I)V_1 = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = b \quad \to V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1(t) = \left(a_1 \cos t\sqrt{2} + b_1 \sin t\sqrt{2}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -4 \quad \Rightarrow \quad (A + 4I)V_2 = 0$

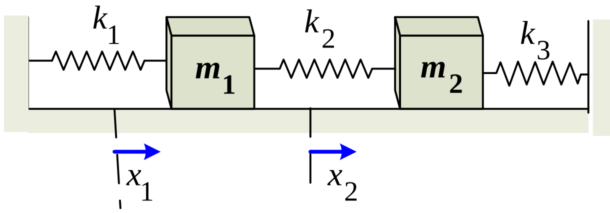
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = -b \quad \to V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \vec{x}_2(t) = \left(a_2 \cos 2t + b_2 \sin 2t\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1 \cos t\sqrt{2} + b_1 \sin t\sqrt{2} + a_2 \cos 2t + b_2 \sin 2t \\ \vec{x}_2(t) = a_1 \cos t\sqrt{2} + b_1 \sin t\sqrt{2} - a_2 \cos 2t - b_2 \sin 2t \end{cases}$$

In the degenerate natural mode with frequency $\omega_1 = \sqrt{2}$ the 2 masses move in the same direction with equal amplitudes of oscillation. At frequency $\omega_2 = 2$ they oscillate in opposite directions with equal amplitudes.

Exercise

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

$$m_1 = 1,\ m_2 = 2; \quad k_1 = 2,\ k_2 = k_3 = 4$$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 \end{cases} \Rightarrow \begin{cases} x_1'' = -6x_1 + 4x_2 \\ 2x_2'' = 4x_1 - 8x_2 \end{cases} \rightarrow \begin{cases} x_1'' = -6x_1 + 4x_2 \\ x_2'' = 2x_1 - 4x_2 \end{cases}$$

$$x'' = \begin{pmatrix} -6 & 4 \\ 2 & -4 \end{pmatrix}\vec{x} \quad \rightarrow \quad A = \begin{pmatrix} -6 & 4 \\ 2 & -4 \end{pmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -6-\lambda & 4 \\ 2 & -4-\lambda \end{vmatrix} \\ &= (-6-\lambda)(-4-\lambda) - 8 \\ &= \lambda^2 + 10\lambda + 16 = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = -2, \quad \lambda_2 = -8$

The natural frequencies: $\omega_1 = \sqrt{2}$ *and* $\omega_2 = 2\sqrt{2}$

For $\lambda_1 = -2 \Rightarrow (A + 2I)V_1 = 0$

$$\begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = b \quad \rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1(t) = \left(a_1 \cos t\sqrt{2} + b_1 \sin t\sqrt{2}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -8 \Rightarrow (A + 8I)V_2 = 0$

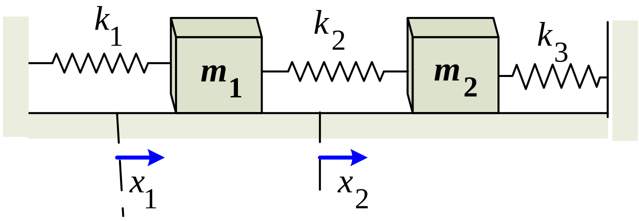
$$\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = -2b \rightarrow V_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow \vec{x}_2(t) = \left(a_2 \cos t\sqrt{8} + b_2 \sin t\sqrt{8}\right)\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1 \cos t\sqrt{2} + b_1 \sin t\sqrt{2} + 2a_2 \cos t\sqrt{8} + 2b_2 \sin t\sqrt{8} \\ \vec{x}_2(t) = a_1 \cos t\sqrt{2} + b_1 \sin t\sqrt{2} - a_2 \cos t\sqrt{8} - b_2 \sin t\sqrt{8} \end{cases}$$

In the degenerate natural mode with frequency $\omega_1 = \sqrt{2}$ the 2 masses move in the same direction with equal amplitudes of oscillation. At frequency $\omega_2 = \sqrt{8}$ they oscillate in opposite directions with amplitude of oscillation of m_1 twice that of m_2.

Exercise

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



The mass-and-spring system is set in motion from rest $x_1'(0) = x_2'(0) = 0$ in its equilibrium position $x_1(0) = x_2(0) = 0$.

Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

For the given external forces $F_1(t)$ and $F_2(t)$ acting on the masses m_1 and m_2, respectively.

Find the resulting motion of the system and describe it as a superposition of oscillations at three different frequencies.

$$m_1 = m_2 = 1; \quad k_1 = 1,\ k_2 = 4,\ k_3 = 1 \quad F_1(t) = 96\cos 5t,\ \ F_2(t) = 0$$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 + 96\cos 5t \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 \end{cases} \Rightarrow \begin{cases} x_1'' = -5x_1 + 4x_2 + 96\cos 5t \\ x_2'' = 4x_1 - 5x_2 \end{cases}$$

$$A = \begin{pmatrix} -5 & 4 \\ 4 & -5 \end{pmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -5-\lambda & 4 \\ 4 & -5-\lambda \end{vmatrix} \\ &= (-5-\lambda)^2 - 16 \\ &= \lambda^2 + 10\lambda + 9 = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = -1, \quad \lambda_2 = -9$

The natural frequencies: $\omega_1 = 1 \quad \omega_2 = 3 \quad \omega_3 = 5$

For $\lambda_1 = -1 \ \Rightarrow \ (A + I)V_1 = 0$

$$\begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \Rightarrow \ a = b \quad \to V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ \Rightarrow \vec{x}_1(t) = (a_1\cos t + b_1 \sin t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -9 \ \Rightarrow \ (A + 9I)V_2 = 0$

$$\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \Rightarrow \ a = -b \quad \to V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \ \Rightarrow \vec{x}_2(t) = (a_2\cos 3t + b_2 \sin 3t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1\cos t + b_1\sin t + a_2\cos 3t + b_2\sin 3t + c_1\cos 5t \\ \vec{x}_2(t) = a_1\cos t + b_1\sin t - a_2\cos 3t - b_2\sin 3t + c_2\cos 5t \end{cases}$$

$$\begin{cases} \vec{x}_1''(t) = -a_1\cos t - b_1\sin t - 9a_2\cos 3t - 9b_2\sin 3t - 25c_1\cos 5t \\ \vec{x}_2''(t) = -a_1\cos t - b_1\sin t + 9a_2\cos 3t + 9b_2\sin 3t - 25c_2\cos 5t \end{cases}$$

$$x_1'' = -5x_1 + 4x_2 + 96\cos 5t$$

$$-a_1\cos t - b_1\sin t - 9a_2\cos 3t - 9b_2\sin 3t - 25c_1\cos 5t =$$

$$-5a_1\cos t - 5b_1\sin t - 5a_2\cos 3t - 5b_2\sin 3t - 5c_1\cos 5t$$

$$+4a_1\cos t + 4b_1\sin t - 4a_2\cos 3t - 4b_2\sin 3t + 4c_2\cos 5t + 96\cos 5t$$

$$-25c_1\cos 5t = -5c_1\cos 5t + 4c_2\cos 5t + 96\cos 5t$$

$$-20c_1 - 4c_2 = 96 \;\rightarrow\; \boxed{5c_1 + c_2 = -24}$$

$$x_2'' = 4x_1 - 5x_2$$

$$-25c_2\cos 5t = 4c_1\cos 5t - 5c_2\cos 5t \qquad \rightarrow\; \boxed{c_1 = -5c_2}$$

$$5(-5c_2) + c_2 = -24 \;\Rightarrow\; \boxed{c_2 = 1,\; c_1 = -5}$$

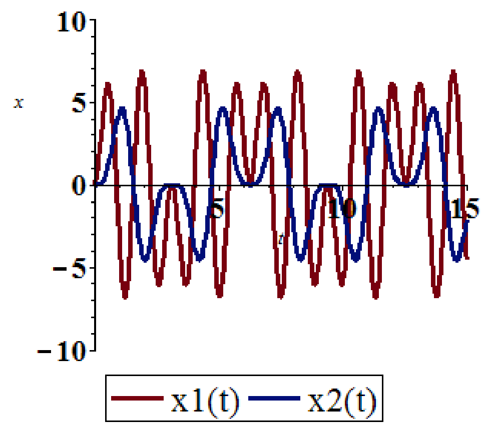
$$\begin{cases} \vec{x}_1(t) = a_1\cos t + b_1\sin t + a_2\cos 3t + b_2\sin 3t - 5\cos 5t \\ \vec{x}_2(t) = a_1\cos t + b_1\sin t - a_2\cos 3t - b_2\sin 3t + \cos 5t \end{cases}$$

Given initial values: $x_1'(0) = x_2'(0) = 0$ and $x_1(0) = x_2(0) = 0$.

$$\begin{cases} \vec{x}_1(0) = a_1 + a_2 - 5 = 0 \\ \vec{x}_2(0) = a_1 - a_2 + 1 = 0 \end{cases} \;\rightarrow\; \boxed{a_1 = 2,\; a_2 = 3}$$

$$\begin{cases} \vec{x}_1'(0) = b_1 + 3b_2 = 0 \\ \vec{x}_2'(0) = b_1 - 3b_2 = 0 \end{cases} \;\rightarrow\; \boxed{b_1 = b_2 = 0}$$

$$\begin{cases} \vec{x}_1(t) = 2\cos t + 3\cos 3t - 5\cos 5t \\ \vec{x}_2(t) = 2\cos t - 3\cos 3t + \cos 5t \end{cases}$$



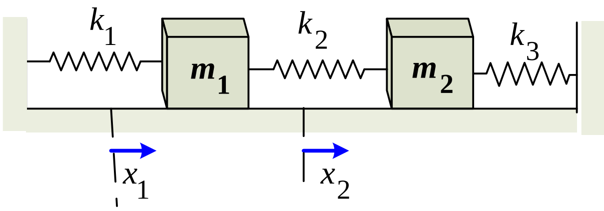
At frequency $\omega_1 = 1$ the 2 masses move in the same direction with equal amplitudes of oscillation.

At frequency $\omega_2 = 3$ the 2 masses move in the opposite direction with equal amplitudes of oscillation.

At frequency $\omega_3 = 5$ they oscillate in opposite directions with amplitude of oscillation of m_1 **5** times that of m_2.

Exercise

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



The mass-and-spring system is set in motion from rest $x_1'(0)=x_2'(0)=0$ in its equilibrium position $x_1(0)=x_2(0)=0$.

Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

For the given external forces $F_1(t)$ and $F_2(t)$ acting on the masses m_1 and m_2, respectively.

Find the resulting motion of the system and describe it as a superposition of oscillations at three different frequencies.

$$m_1=1,\ m_2=2;\quad k_1=1,\ k_2=k_3=2;\quad F_1(t)=0,\ F_2(t)=120\cos 3t$$

Solution

$$\begin{cases} m_1x_1''=-\left(k_1+k_2\right)x_1+k_2x_2 \\ m_2x_2''=k_2x_1-\left(k_2+k_3\right)x_2+120\cos 3t \end{cases} \Rightarrow \begin{cases} x_1''=-3x_1+2x_2 \\ 2x_2''=2x_1-4x_2+120\cos 3t \end{cases}$$

$$\to \begin{cases} x_1''=-3x_1+2x_2 \\ x_2''=x_1-2x_2+60\cos 3t \end{cases} \qquad A=\begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}$$

$$\begin{aligned} \left|A-\lambda I\right| &= \begin{vmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} \\ &= (-3-\lambda)(-2-\lambda)-2 \\ &= \lambda^2+5\lambda+4=0 \end{aligned}$$

The eigenvalues are: $\lambda_1=-1,\quad \lambda_2=-4$

The natural frequencies: $\omega_1=1\quad \omega_2=2\quad \omega_3=3$

For $\lambda_1=-1\ \Rightarrow\ (A+I)V_1=0$

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a=b \quad \to V_1=\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1(t)=\left(a_1\cos t+b_1\sin t\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2=-4\ \Rightarrow\ (A+4I)V_2=0$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a=-2b \quad \to V_2=\begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow \vec{x}_2(t)=\left(a_2\cos 2t+b_2\sin 2t\right)\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1 \cos t + b_1 \sin t + 2a_2 \cos 2t + 2b_2 \sin 2t + c_1 \cos 3t \\ \vec{x}_2(t) = a_1 \cos t + b_1 \sin t - a_2 \cos 2t - b_2 \sin 2t + c_2 \cos 3t \end{cases}$$

$$\begin{cases} \vec{x}''_{1p} = -9c_1 \cos 3t \\ \vec{x}''_{2p} = -9c_2 \cos 3t \end{cases}$$

$$x''_1 = -3x_1 + 2x_2$$

$$-9c_1 \cos 3t = -3c_1 \cos 3t + 2c_2 \cos 3t \quad \Rightarrow \quad -6c_1 = 2c_2 \quad \rightarrow \boxed{-3c_1 = c_2}$$

$$x''_2 = x_1 - 2x_2 + 60 \cos 3t$$

$$-9c_2 \cos 3t = c_1 \cos 3t - 2c_2 \cos 3t + 60 \cos 3t \quad \Rightarrow \quad \boxed{c_1 + 7c_2 = -60}$$

$$c_1 + 7(-3c_1) = -60 \quad \Rightarrow \quad \boxed{c_1 = 3, \ c_2 = -9}$$

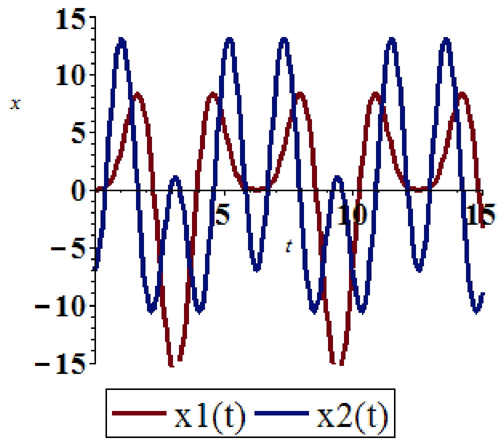
$$\begin{cases} \vec{x}_1(t) = a_1 \cos t + b_1 \sin t + 2a_2 \cos 2t + 2b_2 \sin 2t + 3 \cos 3t \\ \vec{x}_2(t) = a_1 \cos t + b_1 \sin t - a_2 \cos 2t - b_2 \sin 2t - 9 \cos 3t \end{cases}$$

Given initial values: $x'_1(0) = x'_2(0) = 0$ and $x_1(0) = x_2(0) = 0$.

$$\begin{cases} \vec{x}_1(0) = a_1 + 2a_2 + 3 = 0 \\ \vec{x}_2(0) = a_1 - a_2 - 9 = 0 \end{cases} \rightarrow \boxed{a_1 = 5, \ a_2 = -4}$$

$$\begin{cases} \vec{x}'_1(0) = b_1 + 4b_2 = 0 \\ \vec{x}'_2(0) = b_1 - 2b_2 = 0 \end{cases} \rightarrow \boxed{b_1 = b_2 = 0}$$

$$\begin{cases} \vec{x}_1(t) = 5 \cos t - 8 \cos 2t + 3 \cos 3t \\ \vec{x}_2(t) = 5 \cos t + 4 \cos 2t - 9 \cos 3t \end{cases}$$



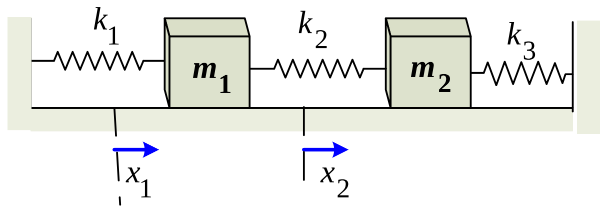
At frequency $\omega_1 = 1$ the 2 masses oscillate in the same direction with equal amplitudes.

At frequency $\omega_2 = 2$ the 2 masses oscillate in opposite directions with equal amplitudes of m_1 ***twice*** that of m_2.

At frequency $\omega_3 = 3$ they oscillate in opposite directions with amplitude of oscillation of m_1 **3** times that of m_2.

Exercise

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



The mass-and-spring system is set in motion from rest $x_1'(0) = x_2'(0) = 0$ in its equilibrium position $x_1(0) = x_2(0) = 0$.

Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

For the given external forces $F_1(t)$ and $F_2(t)$ acting on the masses m_1 and m_2, respectively.

Find the resulting motion of the system and describe it as a superposition of oscillations at three different frequencies.

$$m_1 = m_2 = 1; \quad k_1 = 4, \ k_2 = 6, \ k_3 = 4; \quad F_1(t) = 30\cos t, \ \ F_2(t) = 60\cos t$$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 + 30\cos t \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 + 60\cos t \end{cases} \Rightarrow \begin{cases} x_1'' = -10x_1 + 6x_2 + 30\cos t \\ x_2'' = 6x_1 - 10x_2 + 60\cos t \end{cases}$$

$$A = \begin{pmatrix} -10 & 6 \\ 6 & -10 \end{pmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -10-\lambda & 6 \\ 6 & -10-\lambda \end{vmatrix} \\ &= (-10-\lambda)^2 - 36 \\ &= \lambda^2 + 20\lambda + 64 = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = -4, \quad \lambda_2 = -16$

The natural frequencies: $\omega_1 = 2 \quad \omega_2 = 4 \quad \omega_3 = 1$

For $\lambda_1 = -4 \quad \Rightarrow \quad (A + 4I)V_1 = 0$

$$\begin{pmatrix} -6 & 6 \\ 6 & -6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \ a = b \ \rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ \Rightarrow \vec{x}_1(t) = (a_1 \cos 2t + b_1 \sin 2t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -16 \quad \Rightarrow \quad (A + 16I)V_2 = 0$

$$\begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \ 2a = -3b$$

$$\rightarrow V_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \ \Rightarrow \vec{x}_2(t) = (a_2 \cos 4t + b_2 \sin 4t)\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = a_1\cos 2t + b_1\sin 2t + 3a_2\cos 3t + 3b_2\sin 3t + c_1\cos t \\ \vec{x}_2(t) = a_1\cos 2t + b_1\sin 2t - 2a_2\cos 3t - 2b_2\sin 3t + c_2\cos t \end{cases}$$

$$\begin{cases} \vec{x}''_{1p} = -c_1\cos t \\ \vec{x}''_{2p} = -c_2\cos t \end{cases}$$

$$x''_1 = -10x_1 + 6x_2 + 30\cos t$$

$$-c_1\cos t = -10c_1\cos t + 6c_2\cos t + 30\cos t \quad \Rightarrow 9c_1 - 6c_2 = 30 \quad \Rightarrow \boxed{3c_1 - 2c_2 = 10}$$

$$x''_2 = 6x_1 - 10x_2 + 60\cos t$$

$$-c_2\cos t = 6c_1\cos t - 10c_2\cos t + 60\cos t \quad \Rightarrow -6c_1 + 9c_2 = 60 \quad \Rightarrow \boxed{-2c_1 + 3c_2 = 20}$$

$$5c_1 = 70 \quad \Rightarrow \quad \boxed{c_1 = 14, \ c_2 = 16}$$

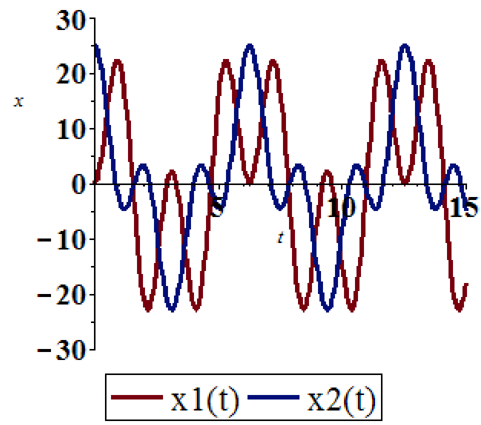
$$\begin{cases} \vec{x}_1(t) = a_1\cos 2t + b_1\sin 2t + 3a_2\cos 3t + 3b_2\sin 3t + 14\cos t \\ \vec{x}_2(t) = a_1\cos 2t + b_1\sin 2t - 2a_2\cos 3t - 2b_2\sin 3t + 16\cos t \end{cases}$$

Given initial values: $x'_1(0) = x'_2(0) = 0$ and $x_1(0) = x_2(0) = 0$.

$$\begin{cases} \vec{x}_1(0) = a_1 + 3a_2 + 14 = 0 \\ \vec{x}_2(0) = a_1 - 2a_2 + 16 = 0 \end{cases} \quad \to \quad \boxed{a_1 = 1, \ a_2 = -5}$$

$$\begin{cases} \vec{x}'_1(0) = 2b_1 + 9b_2 = 0 \\ \vec{x}'_2(0) = 2b_1 - 6b_2 = 0 \end{cases} \quad \to \quad \boxed{b_1 = b_2 = 0}$$

$$\begin{cases} \vec{x}_1(t) = \cos 2t - 15\cos 3t + 14\cos t \\ \vec{x}_2(t) = \cos 2t + 10\cos 3t + 16\cos t \end{cases}$$



At frequency $\omega_1 = 2$ the 2 masses oscillate in the same direction of m_1 ***twice*** that of m_2.

At frequency $\omega_2 = 3$ the 2 masses oscillate in opposite directions with equal amplitudes of m_1 **3** *times* that of m_2.

At frequency $\omega_3 = 1$ they oscillate in the same direction with equal amplitudes of oscillation.

Exercise

Consider a mass-and-spring system containing two masses $m_1 = m_2 = 1$ whose displacement functions $x(t)$ and $y(t)$ satisfy the differential equations

$$x'' = -40x + 8y$$
$$y'' = 12x - 60y$$

a) Describe the two fundamental modes of free oscillation of the system.

b) Assume that the two masses start in motion with the initial conditions

$x(0) = 19,\ x'(0) = 12 \quad and \quad y(0) = 3,\ y'(0) = 6$

And are acted on by the same force, $F_1(t) = F_2(t) = -195\cos 7t$. Describe the resulting motion as a superposition of oscillations at three different frequencies.

Solution

a) $A = \begin{pmatrix} -40 & 8 \\ 12 & -60 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -40-\lambda & 8 \\ 12 & -60-\lambda \end{vmatrix}$$
$$= (-40-\lambda)(-60-\lambda) - 96$$
$$= \lambda^2 + 100\lambda + 144 = 0$$

The eigenvalues are: $\lambda_1 = -36, \quad \lambda_2 = -64$

The natural frequencies: $\omega_1 = 6 \quad \omega_2 = 8$

For $\lambda_1 = -36 \quad \Rightarrow \quad (A + 36I)V_1 = 0$

$$\begin{pmatrix} -4 & 8 \\ 12 & -24 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = 2b \quad \rightarrow V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \Rightarrow \vec{x}_1(t) = (a_1 \cos 6t + b_1 \sin 6t)\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -64 \quad \Rightarrow \quad (A + 64I)V_2 = 0$

$$\begin{pmatrix} 24 & 8 \\ 12 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad 3a = -b \quad \rightarrow V_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad \Rightarrow \vec{x}_2(t) = (a_2 \cos 8t + b_2 \sin 8t)\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\begin{cases} \vec{x}(t) = 2a_1 \cos 6t + 2b_1 \sin 6t + a_2 \cos 8t + b_2 \sin 8t \\ \vec{y}(t) = a_1 \cos 6t + b_1 \sin 6t - 3a_2 \cos 8t - 3b_2 \sin 8t \end{cases}$$

In mode 1: At frequency $\omega_1 = 6$, the 2 masses oscillate in the same direction of m_1 ***twice*** of m_2.

In mode 2: At frequency $\omega_2 = 8$, the 2 masses oscillate in opposite directions of oscillation of m_1 **3 *times*** that of m_2.

b) ***Given*** $x(0) = 19,\ x'(0) = 12 \quad y(0) = 3,\ y'(0) = 6$ and $F_1(t) = F_2(t) = -195\cos 7t$

$$x'' = -40x + 8y - 195\cos 7t$$
$$y'' = 12x - 60y - 195\cos 7t$$

$$\begin{cases} \vec{x}(t) = 2a_1 \cos 6t + 2b_1 \sin 6t + a_2 \cos 8t + b_2 \sin 8t + c_1 \cos 7t \\ \vec{y}(t) = a_1 \cos 6t + b_1 \sin 6t - 3a_2 \cos 8t - 3b_2 \sin 8t + c_2 \cos 7t \end{cases}$$

$$\begin{cases} x''_p = -49c_1 \cos 7t \\ y''_p = -49c_2 \cos 7t \end{cases}$$

$$x'' = -40x + 8y - 195\cos 7t$$

$$-49c_1 \cos 7t = -40c_1 \cos 7t + 8c_2 \cos 7t - 195\cos 7t \quad \Rightarrow \boxed{9c_1 + 8c_2 = 195}$$

$$y'' = 12x - 60y - 195\cos 7t$$

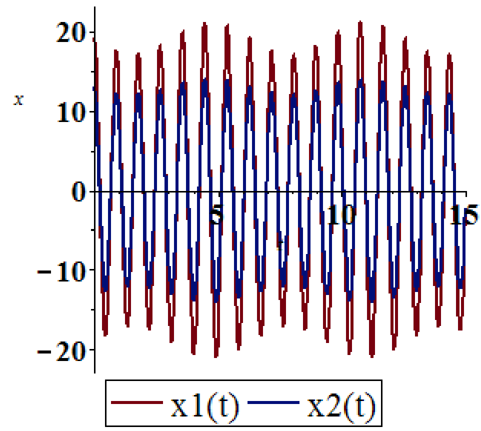
$$-49c_2 \cos 7t = 12c_1 \cos 7t - 60c_2 \cos 7t - 195\cos 7t \quad \Rightarrow \boxed{12c_1 - 11c_2 = 195}$$

$$\Rightarrow \quad \boxed{c_1 = 19, \ \ c_2 = 3}$$

$$\begin{cases} \vec{x}(t) = 2a_1 \cos 6t + 2b_1 \sin 6t + a_2 \cos 8t + b_2 \sin 8t + 19\cos 7t \\ \vec{y}(t) = a_1 \cos 6t + b_1 \sin 6t - 3a_2 \cos 8t - 3b_2 \sin 8t + 3\cos 7t \end{cases}$$

$$\begin{cases} x(0) = 2a_1 + a_2 + 19 = 19 \\ y(0) = a_1 - 3a_2 + 3 = 3 \end{cases} \rightarrow \begin{cases} 2a_1 + a_2 = 0 \\ a_1 - 3a_2 = 0 \end{cases} \Rightarrow \boxed{a_1 = 0\ , a_2 = 0}$$

$$\Rightarrow \begin{cases} x(t) = 2b_1 \sin 6t + b_2 \sin 8t + 19\cos 7t \\ y(t) = b_1 \sin 6t - 3b_2 \sin 8t + 3\cos 7t \end{cases}$$



$$\begin{cases} x'(0) = 12b_1 + 8b_2 = 12 \\ y'(0) = 6b_1 - 24b_2 = 6 \end{cases} \Rightarrow \boxed{b_1 = 1\ , b_2 = 0}$$

$$\Rightarrow \begin{cases} x(t) = 2\sin 6t + 19\cos 7t \\ y(t) = \ \sin 6t + 3\cos 7t \end{cases}$$

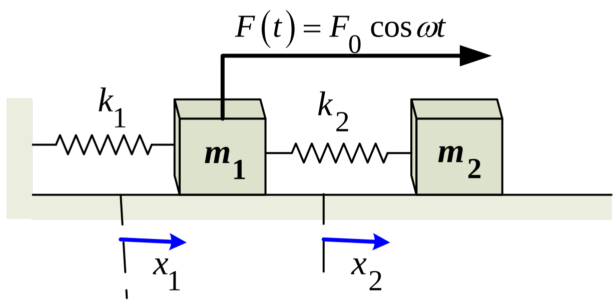
At frequency $\omega_1 = 6$, the 2 masses oscillate in the same direction with amplitude of motion of m_1 ***twice*** that of m_2.

At frequency $\omega_3 = 7$, the 2 masses oscillate in the same direction with amplitude of motion of m_1 being $\frac{19}{3}$ *times* that of m_2.

At frequency $\omega_2 = 8$, the expected oscillation is missing.

Exercise

Consider a mass-and-spring system shown below. Assume that $m_1 = 1;\ \ k_1 = 50,\ k_2 = 10;\ \ F_0 = 5$ in mks units, and that $\omega = 10$. Then find m_2 so that in the resulting steady periodic oscillations, the mass m_1 will remain at rest (!).



Thus the effect of the second mass-and-spring pair will be to neutralize the effect of the force on the first mass. This is an example of a dynamic damper. It has an electrical analogy that some cable companies use to prevent your reception of certain cable channels.

Solution

$$F(t) = F_0 \cos \omega t = 5\cos 10t$$

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 + 5\cos 10t \\ m_2 x_2'' = -k_2(x_2 - x_1) \end{cases} \Rightarrow \begin{cases} x_1'' = -60x_1 + 10x_2 + 5\cos 10t \\ m_2 x_2'' = 10x_1 - 10x_2 \end{cases}$$

$$\begin{cases} x_{1p} = c_1 \cos 10t \\ x_{2p} = c_2 \cos 10t \end{cases} \rightarrow \begin{cases} x_{1p}'' = -100c_1 \cos 10t \\ x_{2p}'' = -100c_2 \cos 10t \end{cases}$$

$$x_1'' = -60x_1 + 10x_2 + 5\cos 10t$$

$$-100c_1 \cos 10t = -60c_1 \cos 10t + 10c_2 \cos 10t + 5\cos 10t \quad \rightarrow \quad \boxed{-40c_1 - 10c_2 = 5}$$

$$m_2 x_2'' = 10x_1 - 10x_2$$

$$-100m_2 c_2 \cos 10t = 10c_1 \cos 10t - 10c_2 \cos 10t \quad \rightarrow \quad \boxed{c_1 - (1 - 10m_2)c_2 = 0}$$

$$-40(1 - 10m_2)c_2 - 10c_2 = 5 \qquad\qquad c_1 = (1 - 10m_2)c_2$$

$$390m_2 c_2 = 45 \quad \Rightarrow \quad \boxed{c_2 = \frac{3}{26m_2}} \quad \rightarrow c_1 = (1 - 10m_2)\frac{3}{26m_2} = \boxed{\frac{3}{26m_2} - \frac{15}{13}}$$

$$-40\left(\frac{3}{26m_2} - \frac{15}{13}\right) - 10\frac{3}{26m_2} = 5$$

$$-4.615 + 46.154m_2 - 1.154 = 5m_2$$

$$41.154m_2 = 5.769$$

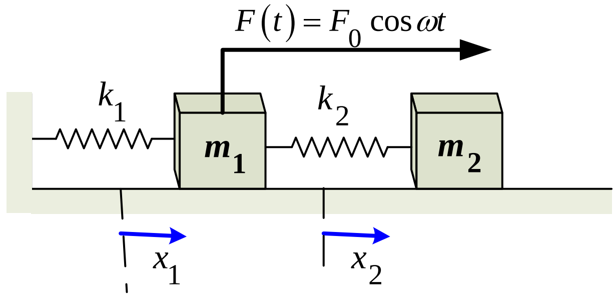
$$\boxed{m_2 \approx 0.1 \ \ slug} \qquad \Rightarrow \quad c_1 = \frac{3}{26m_2} - \frac{15}{13} \boxed{\approx 0} \quad c_2 = \frac{3}{26m_2} \boxed{\approx 1.15}$$

Since $c_1 = 0$, so the mass m_1 remains at rest.

Exercise

Consider a mass-and-spring system shown below. Assume that

$m_1 = 2,\ m_2 = \frac{1}{2};\quad k_1 = 75,\ k_2 = 25;\quad F_0 = 100\quad and\ \omega = 10$ (in mks units).



Find the solution of the system $M\vec{x}'' = K\vec{x} + F$ that satisfies the initial conditions $\vec{x}(0) = \vec{x}'(0) = \mathbf{0}$

Solution

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 + 100\cos 10t \\ m_2 x_2'' = -k_2(x_2 - x_1) \end{cases} \Rightarrow \begin{cases} 2x_1'' = -100x_1 + 25x_2 + 100\cos 10t \\ \frac{1}{2}x_2'' = 25x_1 - 25x_2 \end{cases}$$

$$\begin{cases} x_1'' = -50x_1 + \frac{25}{2}x_2 + 50\cos 10t \\ x_2'' = 50x_1 - 50x_2 \end{cases} \rightarrow A = \begin{bmatrix} -50 & \frac{25}{2} \\ 50 & -50 \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -50-\lambda & \frac{25}{2} \\ 50 & -50-\lambda \end{vmatrix} \\ &= (-50-\lambda)^2 - 625 \\ &= \lambda^2 + 100\lambda - 1875 = 0 \end{aligned}$$

The eigenvalues are: $\lambda_1 = -25,\quad \lambda_2 = -75$

The natural frequencies: $\omega_1 = 5\quad \omega_2 = 5\sqrt{3}$

For $\lambda_1 = -25 \ \Rightarrow\ (A + 25I)V_1 = 0$

$$\begin{pmatrix} -25 & \frac{25}{2} \\ 50 & -25 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2a = b \quad \rightarrow V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \vec{x}_1(t) = (a_1 \cos 5t + b_1 \sin 5t)\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = -75 \ \Rightarrow\ (A + 75I)V_2 = 0$

$$\begin{pmatrix} 25 & \frac{25}{2} \\ 50 & 25 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2a = -b \ \rightarrow V_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow \vec{x}_2(t) = \left(a_2 \cos 5t\sqrt{3} + b_2 \sin 5t\sqrt{3}\right)\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\begin{cases} x_1(t) = a_1 \cos 5t + b_1 \sin 5t + a_2 \cos 5t\sqrt{3} + b_2 \sin 5t\sqrt{3} \\ x_2(t) = 2a_1 \cos 5t + 2b_1 \sin 5t - 2a_2 \cos 5t\sqrt{3} - 2b_2 \sin 5t\sqrt{3} \end{cases}$$

$$\begin{cases} x_{1p} = c_1 \cos 10t \\ x_{2p} = c_2 \cos 10t \end{cases} \rightarrow \begin{cases} x''_{1p} = -100c_1 \cos 10t \\ x''_{2p} = -100c_2 \cos 10t \end{cases}$$

$$x''_1 = -50x_1 + \frac{25}{2}x_2 + 50\cos 10t$$

$$-100c_1 \cos 10t = -50c_1 \cos 10t + \frac{25}{2}c_2 \cos 10t + 50\cos 10t$$

$$\Rightarrow\ 50c_1 + \frac{25}{2}c_2 = -50 \quad \Rightarrow\ \underline{4c_1 + c_2 = -4|}$$

$$x''_2 = 50x_1 - 50x_2$$

$$-100c_2 = 50c_1 - 50c_2 \quad \Rightarrow\ \underline{c_1 + c_2 = 0|}$$

$$\underline{c_1 = -\frac{4}{3},\ c_2 = \frac{4}{3}|}$$

$$\begin{cases} x_1(t) = a_1 \cos 5t + b_1 \sin 5t + a_2 \cos 5t\sqrt{3} + b_2 \sin 5t\sqrt{3} - \frac{4}{3}\cos 10t \\ x_2(t) = 2a_1 \cos 5t + 2b_1 \sin 5t - 2a_2 \cos 5t\sqrt{3} - 2b_2 \sin 5t\sqrt{3} + \frac{4}{3}\cos 10t \end{cases}$$

$$\begin{cases} x_1(0) = a_1 + a_2 - \frac{4}{3} = 0 \\ x_2(0) = 2a_1 - 2a_2 + \frac{4}{3} = 0 \end{cases} \quad \begin{cases} a_1 + a_2 = \frac{4}{3} \\ 2a_1 - 2a_2 = -\frac{4}{3} \end{cases} \quad \underline{a_1 = \frac{1}{3},\ a_2 = 1|}$$

$$\begin{cases} x'_1(t) = -5a_1 \sin 5t + 5b_1 \cos 5t - 5a_2\sqrt{3}\sin 5t\sqrt{3} + 5b_2\sqrt{3}\cos 5t\sqrt{3} + \frac{40}{3}\sin 10t \\ x'_2(t) = -10a_1 \sin 5t + 10b_1 \cos 5t + 10a_2\sqrt{3}\sin 5t\sqrt{3} - 10b_2\sqrt{3}\cos 5t\sqrt{3} - \frac{40}{3}\sin 10t \end{cases}$$

$$\begin{cases} x'_1(0) = 5b_1 + 5\sqrt{3}b_2 = 0 \\ x'_2(0) = 10b_1 - 10\sqrt{3}b_2 = 0 \end{cases} \quad \Rightarrow\ \underline{b_1 = b_2 = 0|}$$

$$\begin{cases} x_1(t) = \frac{1}{3}\cos 5t + \cos 5t\sqrt{3} - \frac{4}{3}\cos 10t \\ x_2(t) = \frac{2}{3}\cos 5t - 2\cos 5t\sqrt{3} + \frac{4}{3}\cos 10t \end{cases}$$

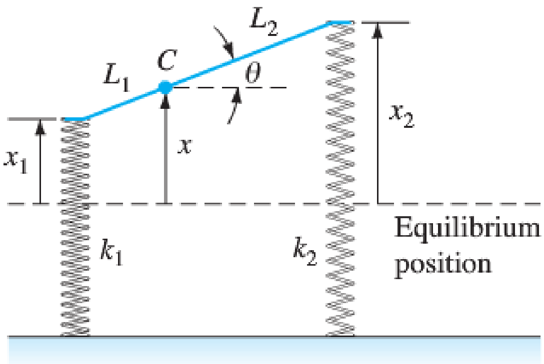
At frequency $\omega_1 = 5,$ the 2 masses oscillate in the same direction with amplitude of motion of m_1 ***half*** that of m_2.

At frequency $\omega_2 = 5\sqrt{3},$ the 2 masses oscillate in opposite directions with amplitude of motion of m_1 being ***half*** that of m_2.

At frequency $\omega_3 = 10$ the 2 masses oscillate in opposite directions with equal amplitudes.

Exercise

A car with two axles and with separate front and rear suspension systems.



We assume that the car body acts as would a solid bar of mass m and length $L = L_1 + L_2$. It has moment of inertia I about its center of mass *C*, which is at distance L_1 from the front of the car. The car has front and back suspension springs with Hooke's constants k_1 and k_2, respectively. When the car is in motion, let $x(t)$ denote the vertical displacement of the center of mass of the car from equilibrium; let $\theta(t)$ denote its angular displacement (in radians) from the horizontal. Then Newton's laws of motion for linear and angular acceleration can be used to derive the equations.

$$mx'' = -(k_1 + k_2)x + (k_1L_1 - k_2L_2)\theta$$

$$I\theta'' = (k_1L_1 - k_2L_2)x - \left(k_1L_1^2 + k_2L_2^2\right)\theta$$

Suppose that $m = 75$ *slugs* (the car weighs 2400 *lb*), $L_1 = 7\,ft$, $L_2 = 3\,ft$ (it's a rear engine car), $k_1 = k_2 = 2000$ lb/ft, and $I = 1000$ $ft.lb.s^2$.

a) Find the two natural frequencies ω_1 and ω_2 of the car.

b) Now suppose that the car is driven at a speed of v $ft/\sec$ along a washboard surface shaped like a sine curve with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1$ *or* $\omega = \omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

Solution

a) $\begin{cases} 75x'' = -4000x + 8000\theta \\ 1000\theta'' = 8000x - (98000 + 18000)\theta \end{cases}$

$$\begin{cases} x'' = -\frac{160}{3}x + \frac{320}{3}\theta \\ \theta'' = 8x - 116\theta \end{cases} \quad \rightarrow A = \begin{bmatrix} -\frac{160}{3} & \frac{320}{3} \\ 8 & -116 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\frac{160}{3} - \lambda & \frac{320}{3} \\ 8 & -116 - \lambda \end{vmatrix}$$

$$= \left(-\frac{160}{3} - \lambda\right)(-116 - \lambda) - \frac{2560}{3}$$

$$= \lambda^2 + \frac{508}{3}\lambda - \frac{48640}{3} = 0$$

The eigenvalues are: $\lambda_1 \approx -41.8285, \quad \lambda_2 \approx -127.5049$

The natural frequencies: $\omega_1 \approx \underline{6.4675 \ \ rad/\sec}| \quad \omega_2 \approx \underline{11.2918 \ \ rad/\sec}|$

$$\omega_1 = \frac{6.4675}{2\pi} \approx \underline{1.0293 \ \ Hz}| \quad \omega_2 = \frac{11.2918}{2\pi} \approx \underline{1.7971 \ \ Hz}|$$

b) $\omega = \frac{\pi}{20}v \quad \Rightarrow \quad v = \frac{20}{\pi}\omega$

$$v_1 = \frac{20}{\pi}\omega_1 = \frac{(20)(6.4675)}{\pi} \ \underline{\approx 41 \ \ ft/\sec}| \qquad (41)(0.681818) \ \underline{\approx 28 \ \ mph}|$$

$$v_2 = \frac{20}{\pi}\omega_2 = \frac{(20)(11.2918)}{\pi} \ \underline{\approx 72 \ \ ft/\sec}| \qquad (72)(0.681818) \ \underline{\approx 49 \ \ mph}|$$

Exercise

The system is taken as a model for an undamped car with the given parameters in *fps* units.

$m = 100; \quad I = 800; \quad L_1 = L_2 = 5; \quad k_1 = k_2 = 2000$

a) Find the two natural frequencies ω_1 and ω_2 of the car (in hertz).

b) Assume that his car is driven along a sinusoidal washboard surface with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1 \ or \ \omega = \omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

Solution

a) $$\begin{cases} mx'' = -(k_1 + k_2)x + (k_1L_1 - k_2L_2)\theta \\ I\theta'' = (k_1L_1 - k_2L_2)x - \left(k_1L_1^2 + k_2L_2^2\right)\theta \end{cases} \rightarrow \begin{cases} 100x'' = -4000x \\ 800\theta'' = -100{,}000\theta \end{cases}$$

$$\begin{cases} x'' = -40x \\ \theta'' = -125\theta \end{cases} \qquad \rightarrow A = \begin{bmatrix} -40 & 0 \\ 0 & -125 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -40-\lambda & 0 \\ 0 & -125-\lambda \end{vmatrix} = (-40-\lambda)(-125-\lambda) = 0$$

The eigenvalues are: $\lambda_1 = -40, \quad \lambda_2 = -125$

The natural frequencies: $\omega_1 = \sqrt{40} \approx \underline{6.325 \ \ rad/\sec}| \quad \omega_2 = \sqrt{125} \ \underline{\approx 11.180 \ \ rad/\sec}|$

$$\omega_1 = \frac{6.325}{2\pi} \approx \underline{1.0067 \ \ Hz}| \quad \omega_2 = \frac{11.180}{2\pi} \approx \underline{1.779 \ \ Hz}|$$

b) $$v_1 = \frac{20}{\pi}\omega_1 = \frac{(20)(6.325)}{\pi} \ \underline{\approx 40.26 \ \ ft/\sec}| \qquad (40.26)(0.681818) \ \underline{\approx 27 \ \ mph}|$$

$$v_2 = \frac{20}{\pi}\omega_2 = \frac{(20)(11.180)}{\pi} \ \underline{\approx 71.18 \ \ ft/\sec}| \qquad (71.18)(0.681818) \ \underline{\approx 49 \ \ mph}|$$

Exercise

The system is taken as a model for an undamped car with the given parameters in *fps* units.

$m = 100; \quad I = 1000; \quad L_1 = 6, \ L_2 = 4; \quad k_1 = k_2 = 2000$

a) Find the two natural frequencies ω_1 and ω_2 of the car (in hertz).

b) Assume that his car is driven along a sinusoidal washboard surface with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1$ *or* $\omega = \omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

Solution

a) $\begin{cases} mx'' = -\left(k_1 + k_2\right)x + \left(k_1L_1 - k_2L_2\right)\theta \\ I\theta'' = \left(k_1L_1 - k_2L_2\right)x - \left(k_1L_1^2 + k_2L_2^2\right)\theta \end{cases} \rightarrow \begin{cases} 100x'' = -4000x + 4000\theta \\ 1000\theta'' = 4000x - 104{,}000\theta \end{cases}$

$$\begin{cases} x'' = -40x + 40\theta \\ \theta'' = 4x - 104\theta \end{cases} \rightarrow A = \begin{bmatrix} -40 & 40 \\ 4 & -104 \end{bmatrix}$$

$$\begin{aligned} \left|A - \lambda I\right| &= \begin{vmatrix} -40-\lambda & 40 \\ 4 & -104-\lambda \end{vmatrix} \\ &= \left(-40-\lambda\right)\left(-104-\lambda\right) - 160 \\ &= \lambda^2 + 144\lambda + 4000 = 0 \end{aligned} \qquad \lambda_{1,2} = -72 \pm 4\sqrt{74}$$

The eigenvalues are: $\lambda_1 \approx -37.591, \quad \lambda_2 \approx -106.409$

The natural frequencies: $\omega_1 = \sqrt{37.591} \approx \underline{6.131 \ rad/\sec|} \quad \omega_2 = \sqrt{106.409} \approx \underline{10.315 \ rad/\sec|}$

$$\omega_1 = \frac{6.131}{2\pi} \approx \underline{.9758 \ Hz|} \quad \omega_2 = \frac{10.315}{2\pi} \approx \underline{1.6417 \ Hz|}$$

b) $v_1 = \frac{20}{\pi}\omega_1 = \frac{(20)(6.131)}{\pi} \approx \underline{39.03 \ ft/\sec|} \quad (39.03)(0.681818) \approx \underline{27 \ mph|}$

$v_2 = \frac{20}{\pi}\omega_2 = \frac{(20)(10.315)}{\pi} \approx \underline{65.67 \ ft/\sec|} \quad (65.67)(0.681818) \approx \underline{45 \ mph|}$

Exercise

The system is taken as a model for an undamped car with the given parameters in *fps* units.

$m = 100; \quad I = 800; \quad L_1 = L_2 = 5; \quad k_1 = 1000, \ k_2 = 2000$

a) Find the two natural frequencies ω_1 and ω_2 of the car (in hertz).

b) Assume that his car is driven along a sinusoidal washboard surface with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1$ *or* $\omega = \omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

Solution

a) $\begin{cases} mx'' = -\left(k_1 + k_2\right)x + \left(k_1L_1 - k_2L_2\right)\theta \\ I\theta'' = \left(k_1L_1 - k_2L_2\right)x - \left(k_1L_1^2 + k_2L_2^2\right)\theta \end{cases} \rightarrow \begin{cases} 100x'' = -3000x - 5000\theta \\ 800\theta'' = -5000x - 75{,}000\theta \end{cases}$

$$\begin{cases} x'' = -30x - 50\theta \\ \theta'' = -\frac{25}{4}x - \frac{375}{4}\theta \end{cases} \rightarrow A = \begin{bmatrix} -30 & -50 \\ -\frac{25}{4} & -\frac{375}{4} \end{bmatrix}$$

$$\left|A - \lambda I\right| = \begin{vmatrix} -30-\lambda & -50 \\ -\frac{25}{4} & -\frac{375}{4} - \lambda \end{vmatrix}$$

$$= \left(-30-\lambda\right)\left(-\frac{375}{4} - \lambda\right) - \frac{625}{2}$$

$$= \lambda^2 + \frac{495}{4}\lambda + 2500 = 0 \qquad \lambda_{1,2} = \frac{-495 \pm 5\sqrt{3401}}{8}$$

The eigenvalues are: $\lambda_1 \approx -25.426, \quad \lambda_2 \approx -98.234$

The natural frequencies: $\omega_1 = \sqrt{25.426} \approx \underline{5.0424 \ \ rad/\sec}$ $\quad \omega_2 = \sqrt{98.234} \ \underline{\approx 9.9158 \ \ rad/\sec}$

$$\omega_1 = \frac{5.0424}{2\pi} \underline{\approx .8025 \ \ Hz} \quad \omega_2 = \frac{9.9158}{2\pi} \underline{\approx 1.5781 \ \ Hz}$$

b) $v_1 = \frac{20}{\pi}\omega_1 = \frac{(20)(5.0424)}{\pi} \ \underline{\approx 32.10 \ \ ft/\sec} \quad (32.1)(0.681818) \ \underline{\approx 22 \ \ mph}$

$v_2 = \frac{20}{\pi}\omega_2 = \frac{(20)(9.9158)}{\pi} \ \underline{\approx 63.13 \ \ ft/\sec} \quad (63.13)(0.681818) \ \underline{\approx 43 \ \ mph}$

SOLUTION Section 4.5 – Multiple Eigenvalues Solutions

Exercise

Find the general solution $\quad \boldsymbol{x}' = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix}\boldsymbol{x}$

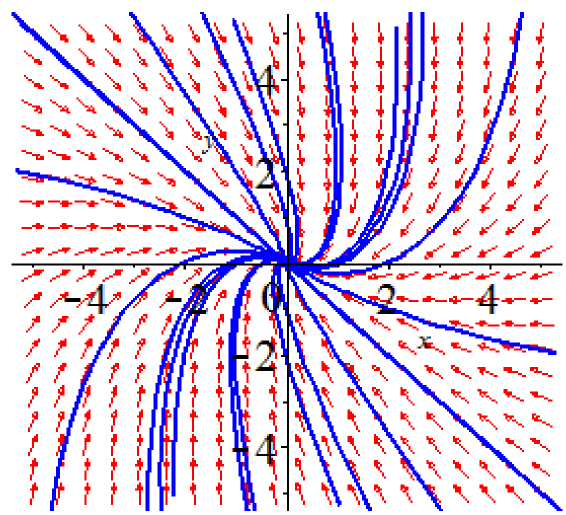
Solution

$$|A-\lambda I| = \begin{vmatrix} -2-\lambda & 1 \\ -1 & -4-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 9 = 0 \qquad \text{The eigenvalues are: } \lambda_{1,2} = -3 \quad (\textit{multiplicity } 2)$$

$$(A+3I)^2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$(A-\lambda I)^2 \vec{v}_2 = \vec{0} \;\Rightarrow\; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\vec{v}_2 = \vec{0} \text{ and } \vec{v}_2 \text{ is a nonzero vector, we can let } \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(A+3I)\vec{v}_2 = \vec{v}_1 \;\Rightarrow\; \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{v}_1}$$



$$\vec{x}_1(t) = \vec{v}_1 e^{\lambda t} \quad \textit{and} \quad \vec{x}_2(t) = \left(\vec{v}_1 t + \vec{v}_2\right)e^{\lambda t}$$

$$\rightarrow \begin{cases} \vec{x}_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} \\ \vec{x}_2(t) = \begin{pmatrix} t+1 \\ -t \end{pmatrix} e^{-3t} \end{cases}$$

The general solution: $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$

$$\begin{cases} x_1(t) = \left(c_1 + c_2 + c_2 t\right)e^{-3t} \\ x_2(t) = \left(-c_1 - c_2 t\right)e^{-3t} \end{cases}$$

Exercise

Find the general solution $\quad \boldsymbol{x}' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\boldsymbol{x}$

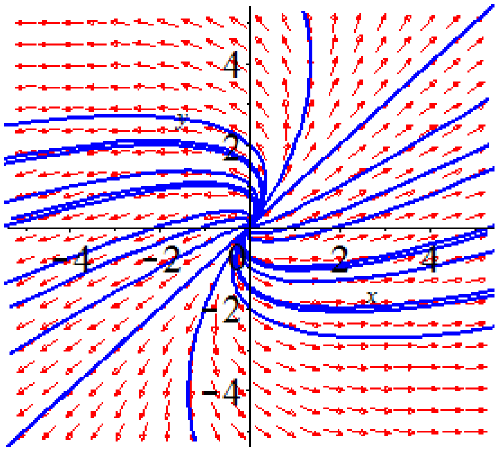
Solution

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 2 = 0 \qquad \text{The eigenvalues are: } \lambda_{1,2} = 2 \quad (\textit{multiplicity } 2)$$

$$(A-2I)^2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$(A-\lambda I)^2 \vec{v}_2 = \vec{0} \;\Rightarrow\; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\vec{v}_2 = \vec{0} \text{ and } \vec{v}_2 \text{ is a nonzero vector, we can let } \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(A-2I)\vec{v}_2=\vec{v}_1 \quad \Rightarrow \quad \begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\underline{\left|\begin{pmatrix}1\\ 1\end{pmatrix}=\vec{v}_1\right|}$$



$$\vec{x}_1(t)=\vec{v}_1e^{\lambda t} \quad \textit{and} \quad \vec{x}_2(t)=\left(\vec{v}_1t+\vec{v}_2\right)e^{\lambda t}$$

$$\begin{cases}\vec{x}_1(t)=\begin{pmatrix}1\\ 1\end{pmatrix}e^{2t}\\ \vec{x}_2(t)=\begin{pmatrix}t+1\\ t\end{pmatrix}e^{2t}\end{cases}$$

The general solution: $\vec{x}(t)=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)$

$$\begin{cases}x_1(t)=\left(c_1+c_2+c_2t\right)e^{2t}\\ x_2(t)=\left(c_1+c_2t\right)e^{2t}\end{cases}$$

Exercise

Find the general solution $\quad \boldsymbol{x}'=\begin{bmatrix}1 & -2\\ 2 & 5\end{bmatrix}\boldsymbol{x}$

Solution

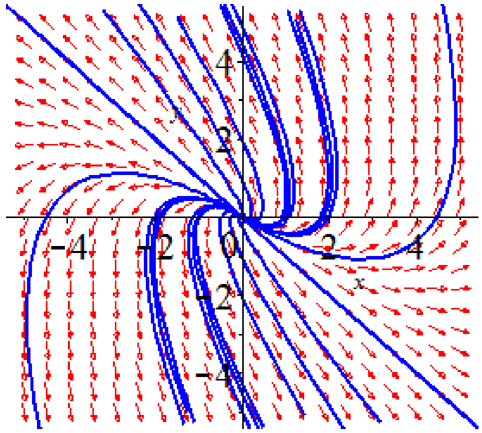
$$\left|A-\lambda I\right|=\begin{vmatrix}1-\lambda & -2\\ 2 & 5-\lambda\end{vmatrix}=\lambda^2-6\lambda+9=0$$

The eigenvalues are: $\lambda_{1,2}=3 \quad (\textit{multiplicity } 2)$

$$(A-3I)^2=\begin{pmatrix}-2 & -2\\ 2 & 2\end{pmatrix}\begin{pmatrix}-2 & -2\\ 2 & 2\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$(A-\lambda I)^2\vec{v}_2=\vec{0} \quad \Rightarrow \quad \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\vec{v}_2=\vec{0}$ and $\vec{v}_2$ is a nonzero vector, we can let $\vec{v}_2=\begin{pmatrix}1\\ 0\end{pmatrix}$

$$(A-3I)\vec{v}_2=\vec{v}_1 \quad \Rightarrow \quad \begin{pmatrix}-2 & -2\\ 2 & 2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\underline{\left|\begin{pmatrix}-2\\ 2\end{pmatrix}=\vec{v}_1\right|}$$



$$\vec{x}_1(t)=\vec{v}_1e^{\lambda t} \quad \textit{and} \quad \vec{x}_2(t)=\left(\vec{v}_1t+\vec{v}_2\right)e^{\lambda t}$$

$$\begin{cases}\vec{x}_1(t)=\begin{pmatrix}-2\\ 2\end{pmatrix}e^{3t}\\ \vec{x}_2(t)=\begin{pmatrix}-2t+1\\ 2t\end{pmatrix}e^{3t}\end{cases}$$

The general solution: $\vec{x}(t)=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)$

$$\begin{cases}x_1(t)=\left(-2c_1+c_2-2c_2t\right)e^{3t}\\ x_2(t)=\left(2c_1+2c_2t\right)e^{3t}\end{cases}$$

Exercise

Find the general solution $\quad \boldsymbol{x}' = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} \boldsymbol{x}$

Solution

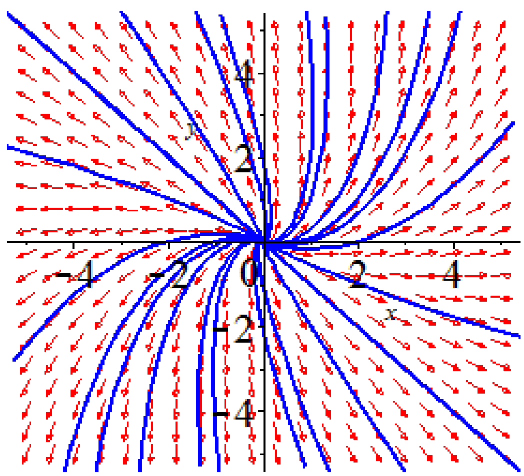
$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 16 = 0 \qquad \text{The eigenvalues are: } \lambda_{1,2} = 4 \quad (\textit{multiplicity } 2)$$

$$(A-4I)^2 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$(A-\lambda I)^2 \vec{v}_2 = \vec{0} \quad \Rightarrow \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \vec{v}_2 = \vec{0} \text{ and } \vec{v}_2 \text{ is a nonzero vector, we can let } \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(A-2I)\vec{v}_2 = \vec{v}_1 \quad \Rightarrow \quad \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \vec{v}_1$$

$$\vec{x}_1(t) = \vec{v}_1 e^{\lambda t} \quad \textit{and} \quad \vec{x}_2(t) = \left(\vec{v}_1 t + \vec{v}_2\right) e^{\lambda t}$$



$$\begin{cases} \vec{x}_1(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} \\ \vec{x}_2(t) = \begin{pmatrix} -t+1 \\ t \end{pmatrix} e^{4t} \end{cases}$$

The general solution: $\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$

$$\begin{cases} x_1(t) = \left(-c_1 + c_2 - c_2 t\right) e^{4t} \\ x_2(t) = \left(c_1 + c_2 t\right) e^{4t} \end{cases}$$

Exercise

Find the general solution $\quad \boldsymbol{x}' = \begin{bmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{bmatrix} \boldsymbol{x}$

Solution

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ -7 & 9-\lambda & 7 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (9-\lambda)(2-\lambda)^2 = 0$$

The eigenvalues are: $\lambda_1 = 9 \quad \lambda_{2,3} = 2$

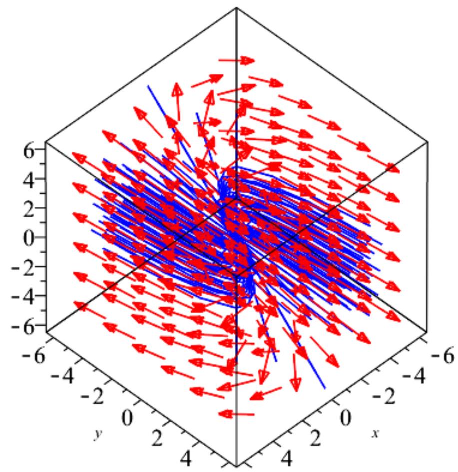
For $\lambda_1 = 9 \quad \Rightarrow \quad (A-9I)\vec{v}_1 = 0$

$$\begin{pmatrix} -7 & 0 & 0 \\ -7 & 0 & 7 \\ 0 & 0 & -7 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{l} a = 0 \\ \\ c = 0 \end{array} \quad \to \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \vec{x}_1(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{9t}$$

For $\lambda_{2,3}=2 \quad \Rightarrow \quad (A-2I)\vec{v}_2=0$

$$\begin{pmatrix} 0 & 0 & 0 \\ -7 & 7 & 7 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad -a+b+c=0$$

Let $b=0 \quad \Rightarrow \quad a=c=1 \rightarrow \vec{v}_2=\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \vec{x}_2(t)=\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}e^{2t}$



Let $c=0 \quad \Rightarrow \quad a=b=1 \rightarrow \vec{v}_3=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \vec{x}_3(t)=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{2t}$

The general solution: $\vec{x}(t)=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)+c_3\vec{x}_3(t)$

$$\begin{cases} x_1(t)=\left(c_2+c_3\right)e^{2t} \\ x_2(t)=c_1e^{9t}+c_3e^{2t} \\ x_3(t)=c_2e^{2t} \end{cases}$$

Exercise

Find the general solution $\quad \boldsymbol{x}'=\begin{bmatrix} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{bmatrix}\boldsymbol{x}$

Solution

$$|A-\lambda I|=\begin{vmatrix} 25-\lambda & 12 & 0 \\ -18 & -5-\lambda & 0 \\ 6 & 6 & 13-\lambda \end{vmatrix}=(25-\lambda)(-5-\lambda)(13-\lambda)+216(13-\lambda)$$

$$=(13-\lambda)\left(-125-20\lambda+\lambda^2+216\right)$$

$$=(13-\lambda)\left(\lambda^2-20\lambda+91\right)$$

$$=(13-\lambda)(\lambda-13)(\lambda-7)=0$$

The eigenvalues are: $\lambda_1=7 \quad \lambda_{2,3}=13$

For $\lambda_1=7 \quad \Rightarrow \quad (A-7I)\vec{v}_1=0$

$$\begin{pmatrix} 18 & 12 & 0 \\ -18 & -12 & 0 \\ 6 & 6 & 6 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} 3a=-2b \\ \\ a+b+c=0 \end{matrix} \quad \rightarrow \begin{cases} a=2 \\ b=-3 \\ c=-2+3=1 \end{cases}$$

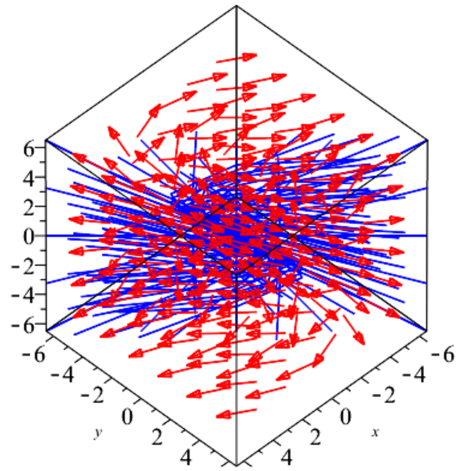
$$\rightarrow \vec{v}_1 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_1(t) = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} e^{7t}$$

For $\lambda_{2,3} = 13 \quad \Rightarrow \quad (A - 13I)V = 0$

$$\begin{pmatrix} 12 & 12 & 0 \\ -18 & -18 & 0 \\ 6 & 6 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = -b$$

Let $c = 0 \quad \& \quad a = 1,\ b = -1 \rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \Rightarrow \vec{x}_2(t) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{13t}$

Let $c = 1 \quad \Rightarrow \quad a = b = 0 \rightarrow \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_3(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{13t}$



The general solution: $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$

$$\begin{cases} x_1(t) = 2c_1e^{7t} + c_2e^{13t} \\ x_2(t) = -3c_1e^{7t} - c_2e^{13t} \\ x_3(t) = c_1e^{7t} \qquad\quad + c_3e^{13t} \end{cases}$$

Exercise

Find the general solution $\quad x' = \begin{bmatrix} -3 & 0 & -4 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix} x$

Solution

$$|A - \lambda I| = \begin{vmatrix} -3-\lambda & 0 & -4 \\ -1 & -1-\lambda & -1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (-3-\lambda)(-1-\lambda)(1-\lambda) + 4(-1-\lambda)$$

$$= (-1-\lambda)\left(\lambda^2 + 2\lambda + 1\right)$$

$$= -(\lambda + 1)^3 = 0$$

The eigenvalues are: $\lambda_{1,2,3} = -1 \quad (\textit{multiplicity } 3)$

For $\lambda = -1 \quad \Rightarrow \quad (A + I)V = 0$

$$\begin{pmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \quad a = -2c \quad \to V = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

The defect of $\lambda = -1$ is 2.

$$(A+I)^2 = \begin{pmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$(A+I)^2 \vec{v}_3 = 0$, therefore any nonzero vector $\vec{v}_3 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ will be a solution

$$\underline{\vec{v}_2} = (A+I)\vec{v}_3 = \begin{pmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

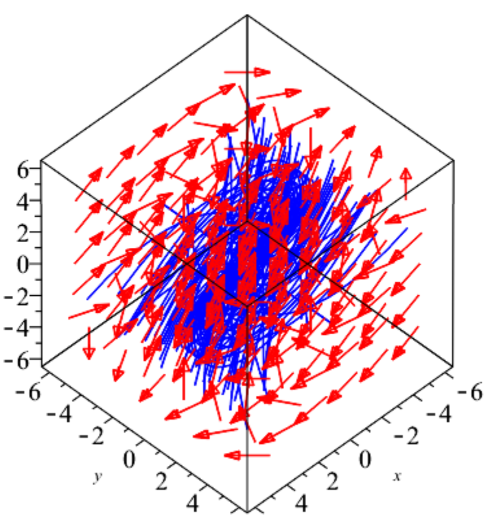
$$\underline{\vec{v}_1} = (A+I)\vec{v}_2 = \begin{pmatrix} -2 & 0 & -4 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = \vec{v}_1 e^{-t} \\ \vec{x}_2(t) = \left(\vec{v}_1 t + \vec{v}_2\right)e^{-t} \\ \vec{x}_3(t) = \left(\frac{1}{2}\vec{v}_1 t^2 + \vec{v}_2 t + \vec{v}_3\right)e^{-t} \end{cases} \to \begin{cases} \vec{x}_1(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-t} \\ \vec{x}_2(t) = \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}\right) e^{-t} \\ \vec{x}_3(t) = \left(\frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) e^{-t} \end{cases}$$

The general solution:

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

$$\begin{cases} x_1(t) = \left(-2c_2 + c_3 - 2c_3 t\right)e^{-t} \\ x_2(t) = \left(\frac{1}{2}c_3 t^2 - \left(c_3 + c_2\right)t - c_2 - c_1\right)e^{-t} \\ x_3(t) = \left(c_2 + c_3 t\right)e^{-t} \end{cases}$$



Exercise

Find the general solution $\quad \boldsymbol{x}' = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix} \boldsymbol{x}$

Solution

$$|A-\lambda I| = \begin{vmatrix} -1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -4 \\ 0 & 1 & -3-\lambda \end{vmatrix} = (-3-\lambda)(-1-\lambda)(1-\lambda) + 4(-1-\lambda)$$

$$= (-1-\lambda)\left(\lambda^2 + 2\lambda + 1\right)$$

$$= -(\lambda+1)^3 = 0$$

The eigenvalues are: $\lambda_{1,2,3} = -1 \ \ (multiplicity\ 3)$. The defect of $\lambda = -1$ is 2.

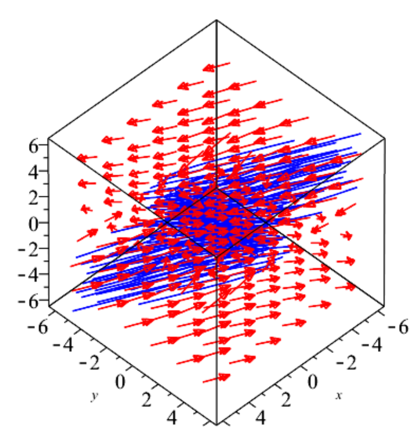
$$(A+I)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(A+I)^3 = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\vec{v}_2 = (A+I)\vec{v}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ Contradict the rule $\vec{v}_2 \neq 0$. Then, let assume $\rightarrow \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\underline{\vec{v}_2} = (A+I)\vec{v}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \qquad \underline{\vec{v}_1} = (A+I)\vec{v}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = \vec{v}_1 e^{-t} \\ \vec{x}_2(t) = \left(\vec{v}_1 t + \vec{v}_2\right) e^{-t} \\ \vec{x}_3(t) = \left(\frac{1}{2}\vec{v}_1 t^2 + \vec{v}_2 t + \vec{v}_3\right) e^{-t} \end{cases} \rightarrow \begin{cases} \vec{x}_1(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-t} \\ \vec{x}_2(t) = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}\right) e^{-t} \\ \vec{x}_3(t) = \left(\frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) e^{-t} \end{cases}$$



The general solution: $\begin{cases} x_1(t) = \left(c_1 + c_2 t + \frac{1}{2} c_3 t^2\right) e^{-t} \\ x_2(t) = \left(2c_2 + c_3 + 2c_3 t\right) e^{-t} \\ x_3(t) = \left(c_2 + c_3 t\right) e^{-t} \end{cases}$

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Exercise

Find the general solution $\quad \boldsymbol{x}' = \begin{bmatrix} 0 & 0 & 1 \\ -5 & -1 & -5 \\ 4 & 1 & -2 \end{bmatrix} \boldsymbol{x}$

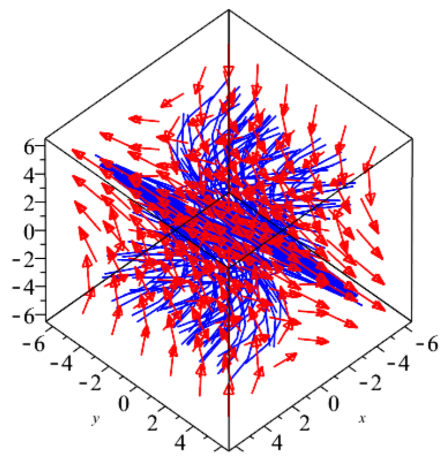
Solution

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 0 & 1 \\ -5 & -1-\lambda & -5 \\ 4 & 1 & -2-\lambda \end{vmatrix} = -\lambda(-1-\lambda)(-2-\lambda)-5-4(-1-\lambda)-5\lambda$$

$$= -\lambda(-1-\lambda)(-2-\lambda)-1-\lambda$$

$$= (-1-\lambda)\left(\lambda^2+2\lambda+1\right)$$

$$= -(\lambda+1)^3 = 0$$



The eigenvalues are: $\lambda_{1,2,3} = -1 \quad (\textit{multiplicity } 3)$

The defect of $\lambda = -1$ is 2.

$$(A+I)^2 = \begin{pmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 0 \\ -25 & -5 & 0 \\ -5 & -1 & 0 \end{pmatrix}$$

$$(A+I)^3 = \begin{pmatrix} 5 & 1 & 0 \\ -25 & -5 & 0 \\ -5 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \to \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{\vec{v}_2} = (A+I)\vec{v}_3 = \begin{pmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} \qquad \underline{\vec{v}_1} = (A+I)\vec{v}_2 = \begin{pmatrix} 1 & 0 & 1 \\ -5 & 0 & -5 \\ 4 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -25 \\ -5 \end{pmatrix}$$

$$\begin{cases} \vec{x}_1(t) = \vec{v}_1 e^{-t} \\ \vec{x}_2(t) = \left(\vec{v}_1 t + \vec{v}_2\right)e^{-t} \\ \vec{x}_3(t) = \left(\frac{1}{2}\vec{v}_1 t^2 + \vec{v}_2 t + \vec{v}_3\right)e^{-t} \end{cases} \to \begin{cases} \vec{x}_1(t) = \begin{pmatrix} 5 \\ -25 \\ -5 \end{pmatrix} e^{-t} \\ \vec{x}_2(t) = \left(\begin{pmatrix} 5 \\ -25 \\ -5 \end{pmatrix} t + \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}\right) e^{-t} \\ \vec{x}_3(t) = \left(\frac{1}{2}\begin{pmatrix} 5 \\ -25 \\ -5 \end{pmatrix} t^2 + \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) e^{-t} \end{cases}$$

The general solution:

$$\begin{cases} x_1(t) = \left(5c_1 + c_2 + c_3 + 5c_2 t + c_3 t + \frac{5}{2}c_3 t^2\right)e^{-t} \\ x_2(t) = \left(-25c_1 - 5c_2 - 25c_2 t - 5c_3 t - \frac{25}{2}c_3 t^2\right)e^{-t} \\ x_3(t) = \left(-5c_1 + 4c_2 - 5c_2 t + 4c_3 t - \frac{5}{2}c_3 t^2\right)e^{-t} \end{cases} \qquad \vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Exercise

Find the general solution $\quad \boldsymbol{x}' = \begin{bmatrix} 39 & 8 & -16 \\ -36 & -5 & 16 \\ 72 & 16 & -29 \end{bmatrix} \boldsymbol{x}$

Solution

$$|A-\lambda I| = \begin{vmatrix} 39-\lambda & 8 & -16 \\ -36 & -5-\lambda & 16 \\ 72 & 16 & -29-\lambda \end{vmatrix} = (39-\lambda)(-5-\lambda)(-29-\lambda) + 13032 - 1080\lambda - 9984 + 256\lambda - 8352 - 288\lambda$$

The eigenvalues are: $\lambda_1 = -1, \ \ \lambda_{2,3} = 3 \ (multiplicity\ 2)$

For $\lambda_1 = -1 \quad \Rightarrow \quad (A+I)\vec{v}_1 = 0$

$$\begin{pmatrix} 40 & 8 & -16 \\ -36 & -4 & 16 \\ 72 & 16 & -28 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} 5a+b-2c=0 \\ -9a-b+4c=0 \\ 18a+4b-7c=0 \end{matrix} \quad \to \begin{cases} 2a=c \\ 2b=-c \\ \end{cases}$$

$$\to \vec{v}_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \quad \Rightarrow \vec{x}_1(t) = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} e^{-t}$$

For $\lambda_{2,3} = 3 \quad \Rightarrow \quad (A-3I)V = 0$

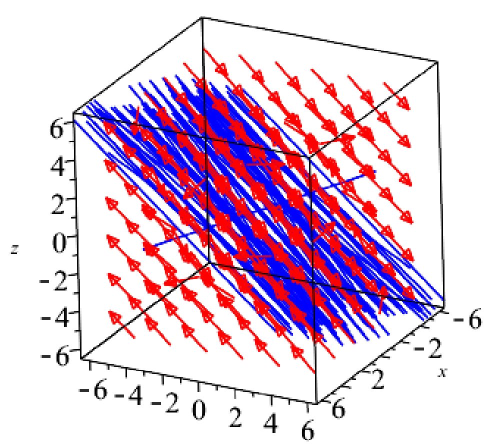
$$\begin{pmatrix} 36 & 8 & -16 \\ -36 & -8 & 16 \\ 72 & 16 & -32 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} 9a+2b-4c=0 \\ -9a-2b+4c=0 \\ 9a+2b-4c=0 \end{matrix} \quad \to \begin{cases} 9a+2b-4c=0 \\ \\ \end{cases}$$

Let $b=0 \quad \to 9a = 4c \quad a=4, \ c=9 \ \to \vec{v}_2 = \begin{pmatrix} 4 \\ 0 \\ 9 \end{pmatrix} \ \Rightarrow \vec{x}_2(t) = \begin{pmatrix} 4 \\ 0 \\ 9 \end{pmatrix} e^{3t}$

Let $c=0 \quad \to 9a = -2b \quad a=-2, \ b=9 \ \to \vec{v}_3 = \begin{pmatrix} -2 \\ 9 \\ 0 \end{pmatrix} \ \Rightarrow \vec{x}_3(t) = \begin{pmatrix} -2 \\ 9 \\ 0 \end{pmatrix} e^{3t}$

The general solution: $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$

$$\begin{cases} x_1(t) = 2c_1e^{-t} + 4c_2e^{3t} - 2c_3e^{3t} \\ x_2(t) = -2c_1e^{-t} \qquad\qquad + 9c_3e^{3t} \\ x_3(t) = c_1e^{-t} + 9c_2e^{3t} \end{cases}$$



Exercise

Find the general solution $\quad x' = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} x$

Solution

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 & 1 \\ 0 & 2-\lambda & 1 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^4 = 0$$

The eigenvalues are: $\lambda_{1,2,3,4} = 2 \ (\textit{multiplicity } 4)$ and defect 3.

$$(A-2I)^2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$$

$$(A-2I)^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$$

$$(A-2I)^4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \ let \ \vec{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{|\vec{v}_3} = (A-2I)\vec{v}_4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\underline{|\vec{v}_2} = (A-2I)\vec{v}_3 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{|\vec{v}_1} = (A-2I)\vec{v}_2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The general solution:

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t)$$

$$\vec{x}(t) = \left[c_1\vec{v}_1 + c_2\left(\vec{v}_1 t + \vec{v}_2\right) + c_3\left(\tfrac{1}{2}\vec{v}_1 t^2 + \vec{v}_2 t + \vec{v}_3\right) + c_4\left(\tfrac{1}{3!}\vec{v}_1 t^3 + \tfrac{1}{2}\vec{v}_2 t^2 + \vec{v}_3 t + \vec{v}_4\right)\right]e^{2t}$$

$$\begin{cases} x_1(t) = \left(c_1 + c_3 + c_2 t + c_4 t + \frac{1}{2}c_3 t^2 + \frac{1}{6}c_4 t^3\right)e^{2t} \\ x_2(t) = \left(c_2 + c_3 t + \frac{1}{2}c_4 t^2\right)e^{2t} \\ x_3(t) = \left(c_3 + c_4 t\right)e^{2t} \\ x_4(t) = c_4 e^{2t} \end{cases}$$

Exercise

Find the general solution $\qquad \boldsymbol{x}' = \begin{bmatrix} -1 & -4 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}\boldsymbol{x}$

Solution

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & -4 & 0 & 0 \\ 1 & 3-\lambda & 0 & 0 \\ 1 & 2 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^4 = 0$$

The eigenvalues are: $\lambda_{1,2,3,4} = 1$ $(multiplicity\ 4)$ and defect 2.

$$(A-I)^2 = \begin{pmatrix} -2 & -4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} -2 & -4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix} \neq 0$$

$$(A-I)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}\begin{pmatrix} -2 & -4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \to \quad let\ \ \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{|\vec{v}_2} = (A-I)\vec{v}_3 = \begin{pmatrix} -2 & -4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\underline{|\vec{v}_1} = (A-I)\vec{v}_2 = \begin{pmatrix} -2 & -4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad let\ \ \vec{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

The general solution:

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t)$$

$$\vec{x}(t) = \left[c_1\vec{v}_1 + c_2\left(\vec{v}_1 t + \vec{v}_2\right) + c_3\left(\tfrac{1}{2}\vec{v}_1 t^2 + \vec{v}_2 t + \vec{v}_3\right) + c_4\vec{v}_4\right]e^t$$

$$\begin{cases} x_1(t) = \left(-2c_2 + c_3 - 2c_3 t\right)e^t \\ x_2(t) = \left(c_2 + c_3 t\right)e^t \\ x_3(t) = \left(c_2 + c_4 + c_3 t\right)e^t \\ x_4(t) = \left(c_1 + c_2 t + \frac{1}{2}c_3 t^2\right)e^t \end{cases}$$

Exercise

The characteristic equation of the coefficient matrix $\boldsymbol{A}$ of the system

$$\boldsymbol{x}' = \begin{bmatrix} 3 & -4 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix}\boldsymbol{x} \qquad \text{is } \ p(\lambda) = \left(\lambda^2 - 6\lambda + 25\right)^2 = 0$$

Therefore, $\boldsymbol{A}$ has the repeated complex pair $3 \pm 4i$ of eigenvalues. First show that the complex vectors $\vec{v}_1 = \begin{bmatrix}1 & i & 0 & 0\end{bmatrix}^T$ *and* $\vec{v}_2 = \begin{bmatrix}0 & 0 & 1 & i\end{bmatrix}^T$ form a length 2 chain $\left\{\vec{v}_1,\ \vec{v}_2\right\}$ associated with the eigenvalue $\lambda = 3 - 4i$. Then calculate the real and imaginary parts of the complex-valued solutions

$$\vec{v}_1 e^{\lambda t} \quad \text{and} \quad \left(\vec{v}_1 t + \vec{v}_2\right)e^{\lambda t}$$

To find four independent real-valued solutions of $\boldsymbol{x}' = A\boldsymbol{x}$

Solution

For $\lambda = 3 - 4i \ \Rightarrow \ \left(A - (3-4i)I\right)\vec{v}_1 = 0$

$$A - \lambda I = \begin{pmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{pmatrix}$$

$$(A - \lambda I)^2 = \begin{pmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{pmatrix}\begin{pmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{pmatrix} = \begin{pmatrix} -32 & -32i & 8i & -8 \\ 32i & -32 & 8 & 8i \\ 0 & 0 & -32 & -32i \\ 0 & 0 & 32i & -32 \end{pmatrix} \to \begin{pmatrix} -4 & -4i & i & -1 \\ 4i & -4 & 1 & i \\ 0 & 0 & -4 & -4i \\ 0 & 0 & 4i & -4 \end{pmatrix}$$

$$\begin{pmatrix} -4 & -4i & i & -1 \\ 4i & -4 & 1 & i \\ 0 & 0 & -4 & -4i \\ 0 & 0 & 4i & -4 \end{pmatrix} \xrightarrow{R_2 + iR_1} \begin{pmatrix} -4 & -4i & i & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4i \\ 0 & 0 & 4i & -4 \end{pmatrix} \xrightarrow[\frac{1}{4}R_4]{-\frac{1}{4}R_3} \begin{pmatrix} -4 & -4i & i & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & -1 \end{pmatrix}$$

$$\begin{pmatrix} -4 & -4i & i & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & -1 \end{pmatrix} \xrightarrow[R_4 - iR_3]{R_1 - iR_3} \begin{pmatrix} -4 & -4i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{4}R_1} \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix}1 & i & 0 & 0\end{bmatrix}^T \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix}0 & 0 & 1 & i\end{bmatrix}^T$$

$$\vec{x}_1 = \vec{v}_1 e^{(3-4i)t} \quad and \quad \vec{x}_2 = \left(\vec{v}_1 t + \vec{v}_2\right) e^{(3-4i)t} \qquad\qquad e^{\alpha t i} = cis\alpha t$$

$$\vec{x}_1 = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} e^{-4t} e^{3t} = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} (\cos 4t - i\sin 4t) e^{3t} = \begin{pmatrix} \cos 4t - i\sin 4t \\ \sin 4t + i\cos 4t \\ 0 \\ 0 \end{pmatrix} e^{3t}$$

$$\vec{x}_2 = \left(\begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix} \right) e^{-4t} e^{3t} = \begin{pmatrix} t \\ ti \\ 1 \\ i \end{pmatrix} (\cos 4t - i\sin 4t) e^{3t} = \begin{pmatrix} t\cos 4t - it\sin 4t \\ t\sin 4t + it\cos 4t \\ \cos 4t - i\sin 4t \\ \sin 4t + i\cos 4t \end{pmatrix} e^{3t}$$

The general solution:

$$x_1(t) = \begin{pmatrix} \cos 4t \\ \sin 4t \\ 0 \\ 0 \end{pmatrix} e^{3t} \qquad x_2(t) = \begin{pmatrix} -\sin 4t \\ \cos 4t \\ 0 \\ 0 \end{pmatrix} e^{3t}$$

$$x_3(t) = \begin{pmatrix} t\cos 4t \\ t\sin 4t \\ \cos 4t \\ \sin 4t \end{pmatrix} e^{3t} \qquad x_4(t) = \begin{pmatrix} -t\sin 4t \\ t\cos 4t \\ -\sin 4t \\ \cos 4t \end{pmatrix} e^{3t}$$