

Solution **Section 2.1 – Introducing the Derivative**

Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P ; then graph the curve and the tangent line.

$$y = 4 - x^2; \quad P(-1, 3)$$

Solution

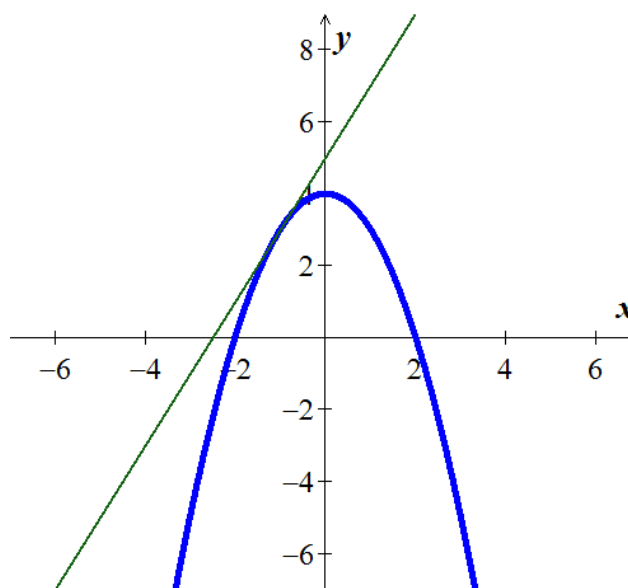
$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{4 - (x+h)^2 - (4 - x^2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{4 - (-1+h)^2 - (4 - (-1)^2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{4 - (1 - 2h + h^2) - (4 - 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{4 - 1 + 2h - h^2 - 3}{h} \\ &= \lim_{h \rightarrow 0} \frac{2h - h^2}{h} \\ &= \lim_{h \rightarrow 0} (2 - h) \\ &= \underline{2} \end{aligned}$$

$$y - y_1 = m(x - x_1)$$

$$\text{At } (-1, 3) \Rightarrow y - 3 = 2(x - (-1))$$

$$y - 3 = 2x + 2$$

$$\underline{y = 2x + 5}$$



Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P ; then graph the curve and the tangent line.

$$y = \frac{1}{x^2}; \quad P(-1, 1)$$

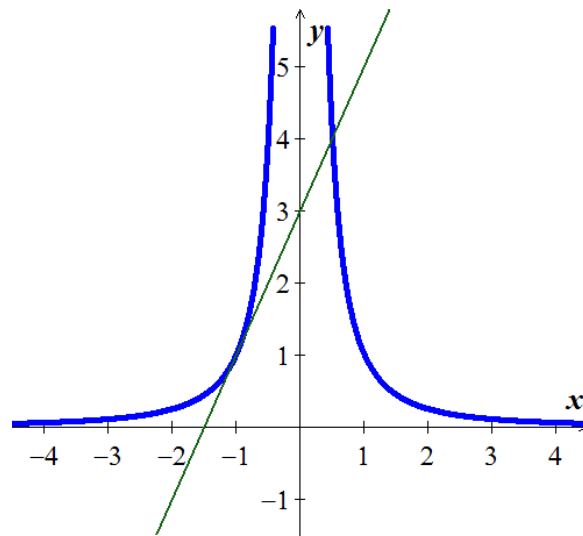
Solution

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{(-1+h)^2} - \frac{1}{1} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1 - (1 - 2h + h^2)}{(-1+h)^2} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1 - 1 + 2h - h^2}{(-1+h)^2} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{2h - h^2}{(-1+h)^2} \right) \\ &= \lim_{h \rightarrow 0} \frac{h}{h} \left(\frac{2 - h}{(-1+h)^2} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{2 - h}{(-1+h)^2} \right) \\ &= \frac{2 + 0}{(-1 + 0)^2} \\ &= 2 \end{aligned}$$

$$\text{At } (-1, 3) \Rightarrow y - 1 = 2(x - (-1))$$

$$y - 1 = 2x + 2$$

$$\underline{y = 2x + 3}$$



Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P ; then graph the curve and the tangent line.

$$f(x) = 2\sqrt{x}; \quad P(1, 2)$$

Solution

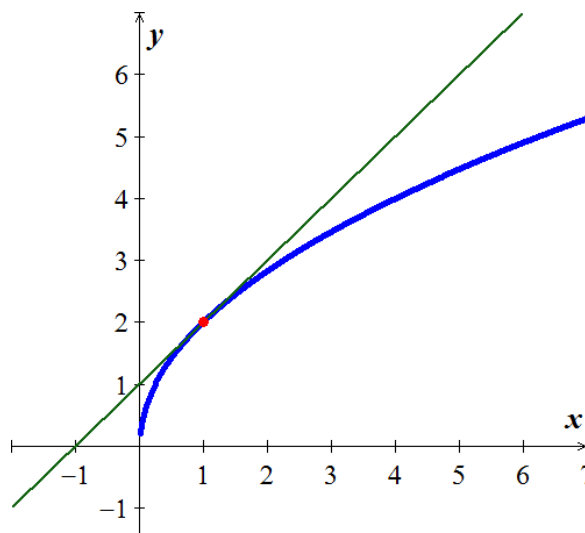
$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{2\sqrt{x+h} - 2\sqrt{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{2\sqrt{1+h} - 2\sqrt{x}}{h} \cdot \frac{2\sqrt{1+h} + 2}{2\sqrt{1+h} + 2} \\ &= \lim_{h \rightarrow 0} \frac{4(1+h) - 4}{h(2\sqrt{1+h} + 2)} \\ &= \lim_{h \rightarrow 0} \frac{4 + 4h - 4}{h(2\sqrt{1+h} + 2)} \\ &= \lim_{h \rightarrow 0} \frac{4h}{h(2\sqrt{1+h} + 2)} \\ &= \lim_{h \rightarrow 0} \frac{4}{2\sqrt{1+h} + 2} \\ &= \frac{4}{2+2} \\ &= 1 \end{aligned}$$

$$\text{At } (1, 2) \Rightarrow y - 2 = (x - 1)$$

$$y - y_1 = m(x - x_1)$$

$$y - 2 = x - 1$$

$$\underline{y = x + 1}$$



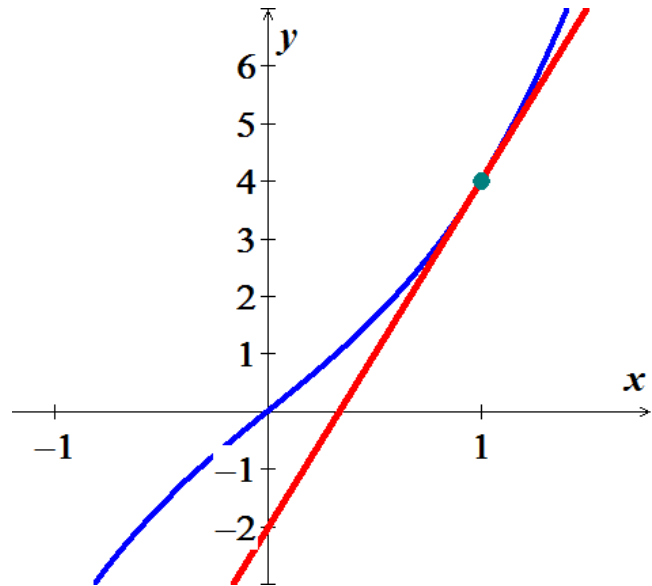
Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P ; then graph the curve and the tangent line.

$$f(x) = x^3 + 3x; \quad P(1, 4)$$

Solution

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{(x+h)^3 + 3(x+h) - (x^3 + 3x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 + 3h + 3h^2 + h^3 + 3 + 3h - (4)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3h + 3h^2 + h^3 + 3h}{h} \\ &= \lim_{h \rightarrow 0} \frac{3h^2 + h^3 + 6h}{h} \\ &= 6 \end{aligned}$$



At $(1, 4)$

$$y - 4 = 6(x - 1) \qquad y - y_1 = m(x - x_1)$$

$$y - 4 = 6x - 6$$

$$y = 6x - 2$$

Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P ; then graph the curve and the tangent line.

$$f(x) = 4x^2 - 7x + 5; \quad P(2, 7)$$

Solution

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{4(x+h)^2 - 7(x+h) + 5 - 4x^2 + 7x - 5}{h} \\ &= \lim_{h \rightarrow 0} \frac{4(x^2 + 2xh + h^2) - 7x - 7h - 4x^2 + 7x}{h} \\ &= \lim_{h \rightarrow 0} \frac{4x^2 + 8xh + 4h^2 - 7h - 4x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{8xh + 4h^2 - 7h}{h} \end{aligned}$$

$$= \lim_{h \rightarrow 0} (8xh + 4h - 7)$$

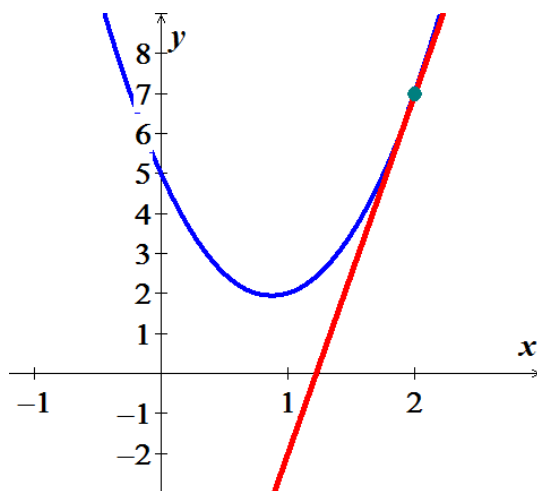
$$= \underline{8x - 7}$$

At $(2, 7) \rightarrow m = 9$

$$y = 9(x - 2) + 7$$

$$y = m(x - x_1) + y_1$$

$$= \underline{9x - 11}$$



Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P ; then graph the curve and the tangent line.

$$f(x) = 5x^3 + x; \quad P(1, 6)$$

Solution

$$m = \lim_{h \rightarrow 0} \frac{5(x+h)^3 + (x+h) - 5x^3 - x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{5(x^3 + 3x^2h + 3xh^2 + h^3) + h - 5x^3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{15x^2h + 15xh^2 + 5h^3 + h}{h}$$

$$= \lim_{h \rightarrow 0} (15x^2 + 15xh + 5h^2 + 1)$$

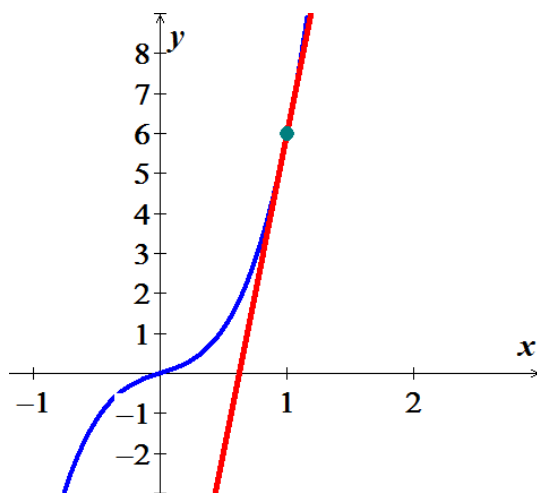
$$= 15x^2 + 1 \Big|_{(1, 6)}$$

$$= \underline{16}$$

$$y = 16(x - 1) + 6$$

$$y = m(x - x_1) + y_1$$

$$= \underline{16x - 10}$$



Exercise

Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P ; then graph the curve and the tangent line.

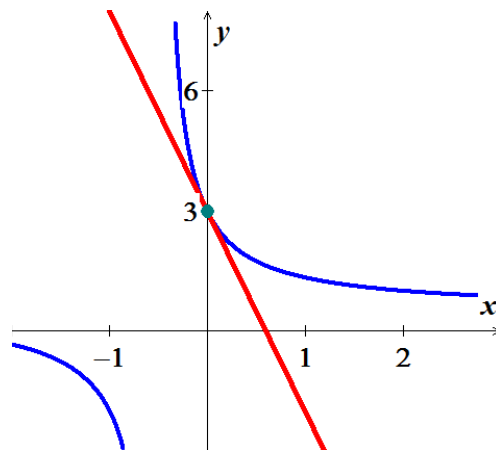
$$f(x) = \frac{x+3}{2x+1}; \quad P(0, 3)$$

Solution

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x+h+3}{2x+2h+1} - \frac{x+3}{2x+1} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{2x^2 + 2hx + 6x + x + h + 3 - 2x^2 - 2hx - x - 6x - 6h - 3}{(2x+2h+1)(2x+1)} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-5h}{(2x+2h+1)(2x+1)} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{-5}{(2x+2h+1)(2x+1)} \right) \\ &= \frac{-5}{(2x+1)^2} \Big|_{(0, 3)} \\ &= -5 \end{aligned}$$

$$\underline{y = -5x + 3}$$

$$y = m(x - x_1) + y_1$$



Exercise

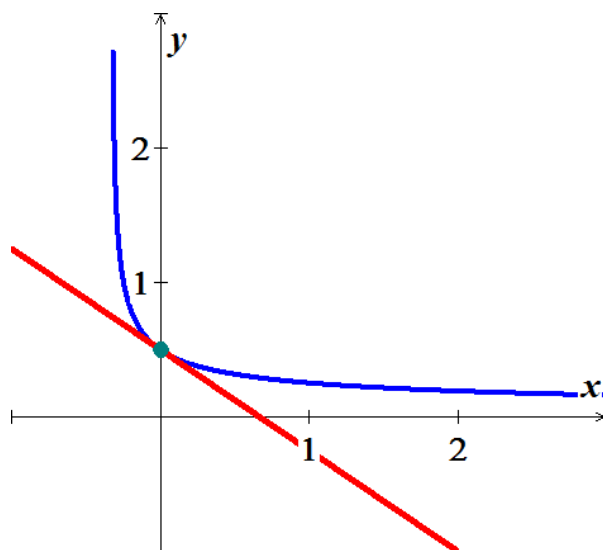
Use the definition of the derivative to determine the slope of the curve $y = f(x)$. Find an equation of the line tangent to the curve $y = f(x)$ at P ; then graph the curve and the tangent line.

$$f(x) = \frac{1}{2\sqrt{3x+1}}; \quad P\left(0, \frac{1}{2}\right)$$

Solution

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{2\sqrt{3x+3h+1}} - \frac{1}{2\sqrt{3x+1}} \right] \\ &= \frac{1}{2} \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\sqrt{3x+1} - \sqrt{3x+3h+1}}{\sqrt{3x+3h+1} \sqrt{3x+1}} \right) = \frac{0}{0} \\ &= \frac{1}{2} \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\sqrt{3x+1} - \sqrt{3x+3h+1}}{\sqrt{3x+3h+1} \sqrt{3x+1}} \right) \frac{\sqrt{3x+1} + \sqrt{3x+3h+1}}{\sqrt{3x+1} + \sqrt{3x+3h+1}} \\ &= \frac{1}{2} \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{3x+1-3x-3h-1}{\sqrt{3x+3h+1} \sqrt{3x+1} (\sqrt{3x+1} + \sqrt{3x+3h+1})} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-3h}{\sqrt{3x+3h+1} \sqrt{3x+1} (\sqrt{3x+1} + \sqrt{3x+3h+1})} \right) \\
&= -\frac{3}{2} \lim_{h \rightarrow 0} \left(\frac{1}{\sqrt{3x+3h+1} \sqrt{3x+1} (\sqrt{3x+1} + \sqrt{3x+3h+1})} \right) \\
&= -\frac{3}{2} \frac{1}{(3x+1)(2\sqrt{3x+1})} \\
&= -\frac{3}{4} \frac{1}{(3x+1)^{3/2}} \left| \left(0, \frac{1}{2}\right) \right. \\
&= -\frac{3}{4} \left| \right. \\
\underline{y = -\frac{3}{4}x + \frac{1}{2}} & \quad y = m(x - x_1) + y_1
\end{aligned}$$



Exercise

Find the slope of the curve $y = 1 - x^2$ at the point $x = 2$

Solution

$$\begin{aligned}
m &= \lim_{h \rightarrow 0} \frac{1 - (x+h)^2 - (1 - x^2)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1 - (2+h)^2 - (1 - 2^2)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1 - (4 + 4h + h^2) - (-3)}{h}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{1 - 4 - 4h - h^2 + 3}{h} \\
&= \lim_{h \rightarrow 0} \frac{-4h - h^2}{h} \\
&= \lim_{h \rightarrow 0} (-4 - h) \\
&= -4
\end{aligned}$$

Exercise

Find the slope of the curve $y = \frac{1}{x-1}$ at the point $x = 3$

Solution

$$\begin{aligned}
m &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\frac{1}{3+h-1} - \frac{1}{3-1}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{2-2-h}{2+h} \right) \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-h}{2+h} \right) \\
&= \lim_{h \rightarrow 0} \left(\frac{-1}{2+h} \right) \\
&= -\frac{1}{2}
\end{aligned}$$

Exercise

Find the slope of the curve $y = \frac{x-1}{x+1}$ at the point $x = 0$

Solution

$$\begin{aligned}
m &= \lim_{h \rightarrow 0} \frac{\frac{x+h-1}{x+h+1} - \frac{x-1}{x+1}}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{0+h-1}{0+h+1} - \frac{0-1}{0+1} \right) \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{h-1}{h+1} + 1 \right)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{h-1+h+1}{h+1} \right) \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{2h}{h+1} \right) \\
&= \lim_{h \rightarrow 0} \left(\frac{2}{h+1} \right) \\
&= \underline{2}
\end{aligned}$$

Exercise

Find equations of all lines having slope -1 that are tangent to the curve $y = \frac{1}{x-1}$

Solution

$$\begin{aligned}
m &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h} \\
-1 &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h} \\
-1 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{x-1-(x+h-1)}{x+h-1} \right) \\
-1 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{x-1-x-h+1}{x+h-1} \right) \\
-1 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-h}{x+h-1} \right) \\
-1 &= \lim_{h \rightarrow 0} \left(\frac{-1}{x+h-1} \right) \\
-1 &= \frac{-1}{x-1} \\
-x+1 &= -1 \\
x &= \underline{2}
\end{aligned}$$

Cross multiplication

$$\begin{aligned}
y &= \frac{1}{x-1} \\
&= \frac{1}{2-1} \\
&= \underline{1}
\end{aligned}$$

$$\begin{aligned}
\text{At } (2, 1) &\Rightarrow y-1 = -1(x-2) \\
&y-1 = -x+2 \\
&y = \underline{-x+3}
\end{aligned}$$

Exercise

What is the rate of change of the area of a circle $(A = \pi r^2)$ with respect to the radius when the radius is $r = 3$?

Solution

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{\pi(3+h)^2 - \pi(3)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\pi(9 + 6h + h^2) - 9\pi}{h} \\ &= \lim_{h \rightarrow 0} \frac{9\pi + 6\pi h + \pi h^2 - 9\pi}{h} \\ &= \lim_{h \rightarrow 0} \frac{6\pi h + \pi h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\pi h(6 + h)}{h} \\ &= \lim_{h \rightarrow 0} \pi(6 + h) \\ &= 6\pi \end{aligned}$$

Exercise

Find the slope of the tangent to the curve $y = \frac{1}{\sqrt{x}}$ at the point where $x = 4$

Solution

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\sqrt{4} - \sqrt{4+h}}{2\sqrt{4+h}} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{2 - \sqrt{4+h}}{2\sqrt{4+h}} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{2 - \sqrt{4+h}}{2\sqrt{4+h}} \cdot \frac{2 + \sqrt{4+h}}{2 + \sqrt{4+h}} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{4 - (4+h)}{2\sqrt{4+h}(2 + \sqrt{4+h})} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{-h}{2\sqrt{4+h}(2 + \sqrt{4+h})} \right) \end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \left(\frac{-1}{2\sqrt{4+h}(2+\sqrt{4+h})} \right) \\
&= \frac{-1}{2\sqrt{4}(2+\sqrt{4})} \\
&= \frac{-1}{2(2)(2+2)} \\
&= \underline{\underline{\frac{-1}{16}}}
\end{aligned}$$

Exercise

Find the values of the derivatives of the function $f(x) = 4 - x^2$. Then find the values of $f'(-3)$, $f'(0)$, $f'(1)$

Solution

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{4 - (x+h)^2 - (4 - x^2)}{h} \\
&= \frac{4 - (x^2 + 2xh + h^2) - (4 - x^2)}{h} \\
&= \frac{4 - x^2 - 2xh - h^2 - 4 + x^2}{h} \\
&= \frac{-2xh - h^2}{h} \\
&= \underline{\underline{-2x - h}}
\end{aligned}$$

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} (-2x - h) \\
&= \underline{\underline{-2x}}
\end{aligned}$$

$$f'(-3) = \underline{\underline{6}}$$

$$f'(0) = \underline{\underline{0}}$$

$$f'(1) = \underline{\underline{-2}}$$

Exercise

Find the values of the derivatives of the function $r(s) = \sqrt{2s+1}$. Then find the values of $r'(0)$, $r'\left(\frac{1}{2}\right)$, $r'(1)$

Solution

$$\begin{aligned}
r'(s) &= \lim_{h \rightarrow 0} \frac{r(s+h) - r(s)}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sqrt{2(s+h)+1} - \sqrt{2s+1}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sqrt{2s+2h+1} - \sqrt{2s+1}}{h} \cdot \frac{\sqrt{2s+2h+1} + \sqrt{2s+1}}{\sqrt{2s+2h+1} + \sqrt{2s+1}} \\
&= \lim_{h \rightarrow 0} \frac{2s+2h+1 - (2s+1)}{h(\sqrt{2s+2h+1} + \sqrt{2s+1})} \\
&= \lim_{h \rightarrow 0} \frac{2s+2h+1 - 2s-1}{h(\sqrt{2s+2h+1} + \sqrt{2s+1})} \\
&= \lim_{h \rightarrow 0} \frac{2h}{h(\sqrt{2s+2h+1} + \sqrt{2s+1})} \\
&= \lim_{h \rightarrow 0} \frac{2}{\sqrt{2s+2h+1} + \sqrt{2s+1}} \\
&= \frac{2}{\sqrt{2s+1} + \sqrt{2s+1}} \\
&= \frac{2}{2\sqrt{2s+1}} \\
&= \frac{1}{\sqrt{2s+1}}
\end{aligned}$$

$$r'(0) = \frac{1}{\sqrt{2(0)+1}} = \underline{1}$$

$$r'\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2\frac{1}{2}+1}} = \underline{\frac{1}{\sqrt{2}}}$$

$$r'(1) = \frac{1}{\sqrt{2(1)+1}} = \underline{\frac{1}{\sqrt{3}}}$$

Exercise

Find the derivative of $f(x) = 3x^2 - 2x$

Solution

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{3(x+\Delta x)^2 - 2(x+\Delta x) - (3x^2 - 2x)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{3(x^2 + \Delta x^2 + 2x\Delta x) - 2x - 2\Delta x - 3x^2 + 2x}{\Delta x}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\Delta x \rightarrow 0} \frac{3x^2 + 3\Delta x^2 + 6x\Delta x - 2x - 2\Delta x - 3x^2 + 2x}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{3\Delta x^2 + 6x\Delta x - 2\Delta x}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} 3\Delta x + 6x - 2 \\
&= \underline{6x - 2}
\end{aligned}$$

Exercise

Find the derivative of y with the respect to t for the function $y = \frac{4}{t}$

Solution

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta t \rightarrow 0} \frac{\frac{4}{t + \Delta t} - \frac{4}{t}}{\Delta t} \\
&= \lim_{\Delta t \rightarrow 0} \frac{\frac{4t - 4(t + \Delta t)}{t(t + \Delta t)}}{\Delta t} \\
&= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \frac{4t - 4(t + \Delta t)}{t(t + \Delta t)} \\
&= \lim_{\Delta t \rightarrow 0} \frac{-4\Delta t}{t(t + \Delta t)\Delta t} \\
&= \lim_{\Delta t \rightarrow 0} \frac{-4}{t(t + \Delta t)} \\
&= \underline{-\frac{4}{t^2}}
\end{aligned}$$

Exercise

Find the derivative of $\frac{dy}{dx}$ if $y = 2x^3$

Solution

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{2(x + \Delta x)^3 - 2x^3}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{2\left(x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3\right) - 2x^3}{\Delta x}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\Delta x \rightarrow 0} \frac{2x^3 + 6x^2\Delta x + 6x(\Delta x)^2 + 3(\Delta x)^3 - 2x^3}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(6x^2 + 6x(\Delta x) + 3(\Delta x)^2)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} (6x^2 + 6x(\Delta x) + 3(\Delta x)^2) \\
&= \underline{6x^2}
\end{aligned}$$

Exercise

Differentiate the function $y = \frac{x+3}{1-x}$ and find the slope of the tangent line at the given value of the independent variable.

Solution

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\frac{x+\Delta x+3}{1-x-\Delta x} - \frac{x+3}{1-x}}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \left(\frac{1}{\Delta x} \right) \left(\frac{(x+\Delta x+3)(1-x) - (x+3)(1-x-\Delta x)}{(1-x-\Delta x)(1-x)} \right) \\
&= \lim_{\Delta x \rightarrow 0} \left(\frac{1}{\Delta x} \right) \left(\frac{x+\Delta x+3-x^2-x\Delta x-3x - (x-x^2-x\Delta x+3-3x-3\Delta x)}{(1-x-\Delta x)(1-x)} \right) \\
&= \lim_{\Delta x \rightarrow 0} \left(\frac{1}{\Delta x} \right) \left(\frac{x+\Delta x+3-x^2-x\Delta x-3x-x+x^2+x\Delta x-3+3x+3\Delta x}{(1-x-\Delta x)(1-x)} \right) \\
&= \lim_{\Delta x \rightarrow 0} \left(\frac{1}{\Delta x} \right) \left(\frac{4\Delta x}{(1-x-\Delta x)(1-x)} \right) \\
&= \lim_{\Delta x \rightarrow 0} \frac{4}{(1-x-\Delta x)(1-x)} \\
&= \frac{4}{(1-x)(1-x)} \\
&= \underline{\frac{4}{(1-x)^2}}
\end{aligned}$$

Exercise

Find the equation of the tangent line to $f(x) = x^2 + 1$ that is parallel to $2x + y = 0$

Solution

$$2x + y = 0 \Rightarrow y = -2x \Rightarrow \text{slope} = -2$$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 + 1 - (x^2 + 1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + \Delta x^2 + 2x\Delta x + 1 - x^2 - 1}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x^2 + 2x\Delta x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \Delta x + 2x = 2x \end{aligned}$$

$$f' = 2x = -2$$

$$\Rightarrow x = -1$$

$$f(-1) = (-1)^2 + 1 = 2$$

$$\rightarrow (-1, 2)$$

The line equation is given by $y = m(x - x_1) + y_1$

$$y = -2(x + 1) + 2$$

$$\underline{y = -2x}$$

Exercise

Use the definition of limits to find the derivative: $f(x) = \frac{3}{\sqrt{x}} y - 2 = -2x - 2$

Solution

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\left(\frac{3}{\sqrt{x + \Delta x}}\right) - \left(\frac{3}{\sqrt{x}}\right)}{\Delta x} \cdot \frac{\sqrt{x} \cdot \sqrt{x + \Delta x}}{\sqrt{x} \cdot \sqrt{x + \Delta x}} \end{aligned}$$

$$\begin{aligned}
&= \lim_{\Delta x \rightarrow 0} \frac{3\sqrt{x} - 3\sqrt{x + \Delta x}}{\Delta x (\sqrt{x} \cdot \sqrt{x + \Delta x})} \\
&= \lim_{\Delta x \rightarrow 0} \frac{3(\sqrt{x} - \sqrt{x + \Delta x})}{\Delta x (\sqrt{x} \cdot \sqrt{x + \Delta x})} \cdot \frac{\sqrt{x} + \sqrt{x + \Delta x}}{\sqrt{x} + \sqrt{x + \Delta x}} \\
&= \lim_{\Delta x \rightarrow 0} \frac{3(x - (x + \Delta x))}{\Delta x (\sqrt{x} \cdot \sqrt{x + \Delta x}) (\sqrt{x} + \sqrt{x + \Delta x})} \\
&= \lim_{\Delta x \rightarrow 0} \frac{-3\Delta x}{\Delta x (\sqrt{x} \cdot \sqrt{x + \Delta x}) (\sqrt{x} + \sqrt{x + \Delta x})} \\
&= \frac{-3}{x(2\sqrt{x})} \\
&= \frac{-3}{2x^{3/2}} \Big|
\end{aligned}$$

Exercise

Use the definition of limits to find the derivative: $f(x) = \sqrt{x + 2}$

Solution

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x + 2} - \sqrt{x + 2}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x + 2} + \sqrt{x + 2}}{\sqrt{x + \Delta x + 2} + \sqrt{x + 2}} \\
&= \lim_{\Delta x \rightarrow 0} \frac{x + \Delta x + 2 - (x + 2)}{\Delta x (\sqrt{x + \Delta x + 2} + \sqrt{x + 2})} \\
&= \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x (\sqrt{x + \Delta x + 2} + \sqrt{x + 2})} \\
&= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x + 2} + \sqrt{x + 2}} \\
&= \frac{1}{2\sqrt{x + 2}} \Big|
\end{aligned}$$

Exercise

Suppose the height s of an object (in m) above the ground after t seconds is approximated by the function

$$s(t) = -4.9t^2 + 25t + 1$$

- a) Make a table showing the average velocities of the object from time $t = 1$ to $t = 1 + h$, for $h = 0.01, 0.001, 0.0001$, and 0.00001 .
- b) Use the table in part (a) to estimate the instantaneous velocity of the object at $t = 1$.
- c) Use limits to verify your estimate in part (b).

Solution

$$\begin{aligned} a) \quad \frac{f(1+h) - f(1)}{h} &= \frac{1}{h} \left(-4.9(1+h)^2 + 25(1+h) + 1 + 4.9 - 25 - 1 \right) \\ &= \frac{1}{h} \left(-4.9 - 9.8h - 4.9h^2 + 25h + 4.9 \right) \\ &= \frac{1}{h} \left(-4.9h^2 + 15.2h \right) \\ &= \underline{15.2 - 4.9h} \end{aligned}$$

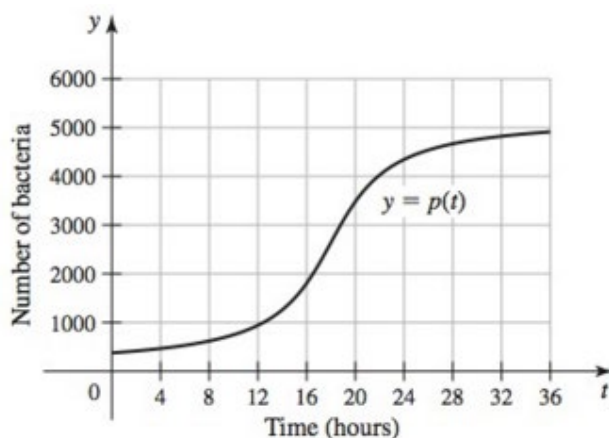
h	$\frac{f(1+h) - f(1)}{h}$
0.01	15.151
0.001	15.1951
0.0001	15.1995
0.00001	15.2
0.000001	15.2

$$\begin{aligned} b) \quad f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &\approx \underline{15.2 \text{ m/sec}} \end{aligned}$$

$$\begin{aligned} c) \quad f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &= \underline{15.2 \text{ m/sec}} \end{aligned}$$

Exercise

Suppose the following graph represents the number of bacteria in a culture t hours after the start of an experiment.



- a) At approximately what time is the instantaneous growth rate the greatest, for $0 \leq t \leq 36$? Estimate the growth rate at this time.
- b) At approximately what time is the instantaneous growth rate the least, for $0 \leq t \leq 36$? Estimate the growth rate at this time.
- c) What is the average growth rate over the interval $0 \leq t \leq 36$?

Solution

a) $t = \frac{36}{2} = 18$

$$\begin{aligned}\text{Point the rate} &= \frac{N(20) - N(16)}{20 - 16} \\ &= \frac{2500 - 1900}{4} \\ &= 400 \text{ bacteria/hr}\end{aligned}$$

b) It is smallest at $t = 0$ or $t = 36$

$$\begin{aligned}\frac{N(36) - N(32)}{4} &= \frac{4900 - 4800}{4} \\ &= 25 \text{ bacteria/hr}\end{aligned}$$

c) Growth rate $= \frac{N(36) - N(0)}{36}$

$$\begin{aligned}&\approx \frac{4900 - 400}{36} \\ &= 125 \text{ bacteria/hr}\end{aligned}$$