# Matrices

## Elementary Matrix Theory

It is often desirable to use matrix notation to simplify complex mathematical expressions. The simplifying matrix notation usually makes the equations much easier to handle and manipulate. Let consider the following set of n simultaneous algebraic equations:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = y_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = y_2$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = y_n$$

We may simplify to $AX = Y$; $A$, $X$, and $Y$ are defined as matrices. These three matrices are defined to be:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ \\ y_n \end{bmatrix} \qquad \text{(D-1)}$$

where the bracketed arrays are simplified of coefficients and variables.

## *Definition of a Matrix*

A matrix is a collection of elements arranged in a rectangular or square array. Several ways of representing a matrix are as follows:

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 4 \end{pmatrix} = \begin{Vmatrix} 0 & 1 \\ 2 & 4 \end{Vmatrix} = \begin{bmatrix} a_{ij} \end{bmatrix}_{2,2} \qquad \text{(D-2)}$$

## Matrix Elements/ Entries:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$a_{ij}$ is defined as the element or entry in the $i$th row and the $j$th column of the matrix.

**Order of Matrix:** refers to the total number of rows and columns of the matrix. In general, a matrix with $n$ rows and $m$ columns is termed "$n \times m$" or "$n$ by $m$".

**Square Matrix:** When number of rows equals to number of columns of the matrix $(m = n)$. In general, a matrix with $n$ rows and $n$ columns is termed "$n \times n$".

**Column Matrix:** one that has one column and more than one row: $m \times 1$ matrix $m > 1$.

**Row Matrix:** is one that has one row and more than one column: $1 \times n$ matrix $n > 1$.

**Diagonal Matrix:** is a square matrix with $a_{ij} = 0$ for all $i \neq j$

$$
\begin{bmatrix}
a_{11} & 0 & 0 \\
0 & a_{22} & 0 \\
0 & 0 & a_{33}
\end{bmatrix}
\tag{D-3}
$$

**Unit Matrix (Identity Matrix):** is a diagonal matrix with all the elements on the main diagonal $(i = j)$ equal to 1. A unity matrix is often designated by either I or U.

$$
\text{Unity example is } I =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\tag{D-4}
$$

**Null Matrix:** is one whose elements are equal to zero for example:

$$
O =
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\tag{D-5}
$$

**Symmetric Matrix:** is a square matrix that satisfies the condition $a_{ij} = a_{ji}$. For example:

$$
\begin{bmatrix}
6 & 5 & 1 \\
5 & 0 & 7 \\
1 & 7 & -1
\end{bmatrix}
;\quad
\begin{bmatrix}
1 & -4 \\
-4 & 1
\end{bmatrix}
$$

**Determinant of a Matrix:** with each square matrix a determinant having the same elements and order. The determinant of a square matrix $A$ is designated by:

$$Det\ A = \Delta_A = D_A = |A|$$

Determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is define as $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
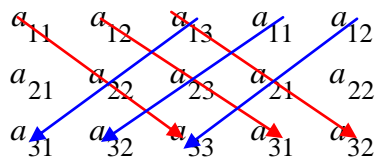
Consider the matrix: $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & 0 \end{bmatrix}$

Then: $|A| = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & 0 \end{vmatrix}$

$$= 1\begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix}$$

$$= 1(0-2) - 0(0-2) + (-1)(0-3)$$

$$= 1$$



$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$-a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

**Singular Matrix:** is said to be singular if the value of its determinant is zero. Where a matrix is singular, it usually means that not all the rows or not all the columns of the matrix are independent of each other. Let consider the following set of equations:

$$2x_1 - 3x_2 + x_3 = 0$$
$$-1x_1 + x_2 + x_3 = 0$$
$$-x_1 - 2x_2 + 2x_3 = 0$$

The third equation is equal to the sum of the first two equations. Therefore these three equations are not completely independent. In matrix form, these equations may be represented by $AX = 0$

Where $A = \begin{bmatrix} 2 & -3 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & 2 \end{bmatrix}$ & $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Determinant of A:

$$|A| = \begin{vmatrix} 2 & -3 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & 2 \end{vmatrix} = 0$$

$\Rightarrow$ Therefore the matrix is ***singular***.

**Transpose of a Matrix:** The transpose of a matrix $A$ is defined as the matrix that is obtained by interchanging the corresponding rows and columns in $A$. Let $A$ be an $n \times m$ matrix which is represented by $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{n,m}$.

Then the transpose of $A$, denoted by $A'$ is given by $A'$ = transpose of $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{m,n}$.

For example: $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & -1 & 5 \end{bmatrix}$ $\Rightarrow A' = \begin{bmatrix} 3 & 0 \\ 2 & -1 \\ 1 & 5 \end{bmatrix}$

**Some Operations of a Matrix Transpose:**

$$\left( A' \right)' = A$$

$$\left( kA \right)' = kA' \quad ; \text{ where } k \text{ is a scalar}$$

$$\left( A + B \right)' = A' + B'$$

$$\left( AB \right)' = B'A'$$

**Skew-Symmetric Matrix:** is a square matrix that equals its negative transpose, that is:

$$A = -A'$$

**Adjoint of a Matrix:** Let $A$ be a square matrix of order $n$. The Adjoint matrix of $A$ denoted by $Adj(A)$ is defined as $Adj(A) = \begin{bmatrix} ij \ cofactor \ of \ DetA \end{bmatrix}'_{n,n}$; where the $ij$ cofactor of the determinant of $A$ is the determinant obtained by

omitting the $i^{\text{th}}$ row and the $j^{\text{th}}$ column of $|A|$ and then multiplying by $(-1)^{i+j}$.

*Example D-1:*

Determine the Adjoint matrix of: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

The determinant of $A$ is $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \Rightarrow adjA = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

*Example D-2:*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$Adj(A) = \begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & -\left(a_{12}a_{33} - a_{13}a_{32}\right) & a_{12}a_{23} - a_{13}a_{22} \\ -\left(a_{21}a_{33} - a_{23}a_{31}\right) & a_{11}a_{33} - a_{13}a_{31} & -\left(a_{11}a_{23} - a_{13}a_{21}\right) \\ a_{21}a_{32} - a_{22}a_{31} & -\left(a_{11}a_{32} - a_{12}a_{31}\right) & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}$$

## Summation or Subtraction of Matrix:

Given two *m x n* matrices $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ their sum is

$$A + B = \begin{bmatrix} a_{ij} + b_{ij} \end{bmatrix}$$
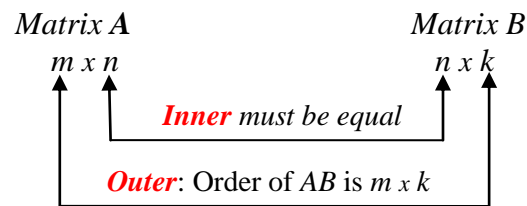
And their difference is

$$A - B = \begin{bmatrix} a_{ij} - b_{ij} \end{bmatrix}$$

## Scalar Multiplication: The scalar product of a number $k$ and a matrix $A$ is denoted by $kA$.

$$kA = k\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$$

## Multiplication of Matrices:

Let $A$ be an $m \times n$ matrix and let $B$ be an $n \times k$ matrix. To find the element in the $i^{th}$ row and $j^{th}$ column of the product matrix $AB$, multiply each element in the $i^{th}$ row of $A$ by the corresponding element in the $j^{th}$ column of $B$, and then add these products. The product matrix $AB$ is an $m \times k$ matrix.

$$\begin{matrix} \text{Matrix } A & & \text{Matrix } B \\ m \times n & & n \times k \end{matrix}$$

**Inner** *must be equal*

**Outer**: Order of $AB$ is $m \times k$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$a_{11} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & - \\ - & - \end{bmatrix}$$

$$a_{12} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} - & af+bh \\ - & - \end{bmatrix}$$

$$a_{21} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} - & - \\ ce+dg & - \end{bmatrix}$$

$$a_{22} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} - & - \\ - & cf+dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$$C = AB = \left| a_{ij} \right|_{m,n} \left| b_{ij} \right|_{n,k}$$

$$= \left| c_{ij} \right|_{m,k}$$

$$c_{ij} = \sum_{n=1}^{k} a_{in} b_{nj} \qquad \text{for } i = 1, 2, \ldots, k \quad \text{and } j = 1, 2, \ldots, k$$

***Example D-3:*** Given the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$$

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \end{bmatrix}$$

***Example D-4:***

$$AB = \begin{bmatrix} 1 & -3 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+(-3)3 & 1(0)+(-3)1 & 1(-1)+(-3)4 & 1(2)+(-3)(-1) \\ 7(1)+2(3) & 7(0)+2(1) & 7(-1)+2(4) & 7(2)+2(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -8 & -3 & -13 & 5 \\ 13 & 2 & 1 & 12 \end{bmatrix}$$

**Inverse of a matrix:**  $A$ is $A^{-1} = \dfrac{adjA}{|A|}$

**Properties of the Inverse Matrix:**

✓ $AA^{-1} = A^{-1}A = I$

✓ $\left(A^{-1}\right)^{-1} = A$

✓ $\left(AB\right)^{-1} = B^{-1}A^{-1}$

***Example D-5:***

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}; \qquad |A| = 1 \implies A^{-1} = \begin{bmatrix} -2 & -1 & 2 \\ 3 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

# *Properties of Matrix*

## Addition and Scalar Multiplication

$A+B=B+A$                 *Commutative Property of Addition*

$A+(B+C)=(A+B)+C$       *Associative Property of Addition*

$(kl)A=k(lA)$                *Associative Property of Scalar Multiplication*

$k(A+B)=kA+kB$         *Distributive Property*

$(k+l)A=kA+lA$          *Distributive Property*

$A+0=0+A=A$           *Additive Identity Property*

$A+(-A)=(-A)+A=0$      *Additive Inverse Property*


## *Multiplication*

$A(BC)=(AB)C$              *Associative Property of Multiplication*

$A(B+C)=AB+AC$        *Distributive Property*

$(B+C)A=BA+CA$        *Distributive Property*