# Solution  *Section* 3.1 – Integrals over Rectangular Regions

### *Exercise*

Evaluate the iterated integral $\int_1^2 \int_0^4 2xy\ dydx$

***Solution***

$$\begin{aligned}\int_1^2 \int_0^4 2xy\ dydx &= \int_1^2 x\left[y^2\right]_0^4 dx \\ &= \int_1^2 16x dx \\ &= 8\left[x^2\right]_1^2 \\ &= 8(4-1) \\ &= 24\end{aligned}$$

### *Exercise*

Evaluate the iterated integral $\int_0^2 \int_{-1}^1 (x-y)\ dydx$

***Solution***

$$\begin{aligned}\int_0^2 \int_{-1}^1 (x-y)\ dydx &= \int_0^2 \left[xy - \frac{1}{2}y^2\right]_{-1}^1 dx \\ &= \int_0^2 \left[x - \frac{1}{2} - \left(-x - \frac{1}{2}\right)\right] dx \\ &= \int_0^2 2x\ dx \\ &= x^2 \Big|_0^2 \\ &= 4\end{aligned}$$

## *Exercise*

Evaluate the iterated integral $\int_0^1 \int_0^1 \left(1-\frac{x^2+y^2}{2}\right) dxdy$

***Solution***

$$\begin{aligned}
\int_0^1 \int_0^1 \left(1-\frac{x^2+y^2}{2}\right) dxdy &= \int_0^1 \left[x-\frac{1}{6}x^3-\frac{1}{2}y^2x\right]_0^1 dy \\
&= \int_0^1 \left(1-\frac{1}{6}-\frac{1}{2}y^2\right) dy \\
&= \int_0^1 \left(\frac{5}{6}-\frac{1}{2}y^2\right) dy \\
&= \left[\frac{5}{6}y-\frac{1}{6}y^3\right]_0^1 \\
&= \frac{5}{6}-\frac{1}{6} \\
&= \frac{4}{6} \\
&= \frac{2}{3}
\end{aligned}$$

## *Exercise*

Evaluate the iterated integral $\int_0^3 \int_{-2}^0 \left(x^2y-2xy\right) dydx$

***Solution***

$$\begin{aligned}
\int_0^3 \int_{-2}^0 \left(x^2y-2xy\right) dydx &= \int_0^3 \left[\frac{1}{2}x^2y^2-xy^2\right]_{-2}^0 dx \\
&= \int_0^3 \left(-2x^2+4x\right) dx \\
&= \left[-\frac{2}{3}x^3+2x^2\right]_0^3 \\
&= -18+18 \\
&= 0
\end{aligned}$$

## *Exercise*

Evaluate the iterated integral $\int_0^1 \int_0^1 \frac{y}{1+xy}\,dxdy$

***Solution***

$$\int_0^1 \int_0^1 \frac{y}{1+xy}\,dxdy = \int_0^1 \int_0^1 \frac{d(1+xy)}{1+xy}\,dy \qquad d(1+xy) = ydx$$

$$= \int_0^1 \left[\ln|1+xy|\right]_0^1 dy$$

$$= \int_0^1 \ln|1+y|\,dy \qquad d(1+y) = dy$$

$$= \left[(y+1)\ln|1+y| - (y+1)\right]_0^1 \qquad \int \ln u\ du = u\ln u - u$$

$$= 2\ln 2 - 2 + 1$$

$$= 2\ln 2 - 1$$

## *Exercise*

Evaluate the iterated integral $\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y}\,dydx$

***Solution***

$$\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y}\,dydx = \int_0^{\ln 2} \left[e^{2x+y}\right]_1^{\ln 5} dx$$

$$= \int_0^{\ln 2} \left(e^{2x+\ln 5} - e^{2x+1}\right) dx$$

$$= \int_0^{\ln 2} \left(e^{2x}e^{\ln 5} - e^{2x+1}\right) dx$$

$$= \int_0^{\ln 2} \left(5e^{2x} - e^{2x+1}\right) dx$$

$$= \left[\frac{5}{2}e^{2x} - \frac{1}{2}e^{2x+1}\right]_0^{\ln 2}$$

$$= \frac{5}{2}e^{2\ln 2} - \frac{1}{2}e^{2\ln 2}e - \left(\frac{5}{2} - \frac{1}{2}e\right)$$

$$= \frac{5}{2}e^{\ln 2^2} - \frac{1}{2}ee^{\ln 2^2} - \frac{5}{2} + \frac{1}{2}e$$

$$= 10 - 2e - \frac{5}{2} + \frac{1}{2}e$$

### *Exercise*

Evaluate the iterated integral $\int_0^1 \int_1^2 xye^x \, dydx$

***Solution***

$$\int_0^1 \int_1^2 xye^x \, dydx = \int_0^1 xe^x \left[\frac{1}{2}y^2\right]_1^2 dx$$

$$= \frac{3}{2}\int_0^1 xe^x \, dx$$

$$= \frac{3}{2}\left[xe^x - e^x\right]_0^1$$

$$= \frac{3}{2}(e - e + 1)$$

$$\underline{\underline{= \frac{3}{2}}}$$

### *Exercise*

Evaluate the iterated integral $\int_\pi^{2\pi} \int_0^\pi (\sin x + \cos y) \, dxdy$

***Solution***

$$\int_\pi^{2\pi} \int_0^\pi (\sin x + \cos y) \, dxdy = \int_\pi^{2\pi} \left[-\cos x + x\cos y\right]_0^\pi dy$$

$$= \int_\pi^{2\pi} (1 + \pi\cos y + 1) \, dy$$

$$= \int_\pi^{2\pi} (2 + \pi\cos y) \, dy$$

$$= \left[2y + \pi\sin y\right]_\pi^{2\pi}$$

$$= 4\pi - 2\pi$$

$$\underline{\underline{= 2\pi}}$$

### Exercise

Evaluate the double integral over the given region $R$ $\iint_R \left(6y^2 - 2x\right) dA \quad R: \ 0 \le x \le 1, \quad 0 \le y \le 2$

**Solution**

$$\iint_R \left(6y^2 - 2x\right) dA = \int_0^1 \int_0^2 \left(6y^2 - 2x\right) dy dx$$

$$= \int_0^1 \left[2y^3 - 2xy\right]_0^2 dx$$

$$= \int_0^1 (16 - 4x) dx$$

$$= \left[16x - 2x^2\right]_0^1$$

$$= 14$$

### Exercise

Evaluate the double integral over the given region $R$ $\iint_R \left(\frac{\sqrt{x}}{y^2}\right) dA \quad R: \ 0 \le x \le 4, \quad 1 \le y \le 2$

**Solution**

$$\iint_R \left(\frac{\sqrt{x}}{y^2}\right) dA = \int_0^4 \int_1^2 \left(\frac{\sqrt{x}}{y^2}\right) dy dx$$

$$= \int_0^4 \left[-\frac{\sqrt{x}}{y}\right]_1^2 dx$$

$$= \int_0^4 -\sqrt{x}\left(\frac{1}{2} - 1\right) dx$$

$$= \frac{1}{2}\int_0^4 x^{1/2} dx$$

$$= \frac{1}{3}\left[x^{3/2}\right]_0^4$$

$$= \frac{8}{3}$$

### Exercise

Evaluate the double integral over the given region $R$ $\iint\limits_R y\sin(x+y)\,dA \quad R: \ -\pi \le x \le 0, \quad 0 \le y \le \pi$

***Solution***

$$\iint\limits_R y\sin(x+y)\,dA = \int_{-\pi}^{0}\int_{0}^{\pi} y\sin(x+y)\,dx\,dy$$

| | | $\int \sin(x+y)$ |
|---|---|---|
| + | $y$ | $-\cos(x+y)$ |
| − | 1 | $-\sin(x+y)$ |

$$= \int_{-\pi}^{0}\left[-y\cos(x+y)+\sin(x+y)\right]_0^{\pi} dx$$

$$= \int_{-\pi}^{0}\left[\sin(x+\pi)-\pi\cos(x+\pi)-\sin x\right] dx$$

$$= \left[-\cos(x+\pi)-\pi\sin(x+\pi)+\cos x\right]_{-\pi}^{0}$$

$$= -(-1)+1-(-1-1)$$

$$\underline{= 4}$$

### Exercise

Evaluate the double integral over the given region $R$. $\iint\limits_R e^{x-y}\,dA \quad R: \ 0 \le x \le \ln 2, \quad 0 \le y \le \ln 2$

***Solution***

$$\iint\limits_R e^{x-y}\,dA = \int_0^{\ln 2}\int_0^{\ln 2} e^{x-y}\,dy\,dx$$

$$= \int_0^{\ln 2}\left[-e^{x-y}\right]_0^{\ln 2} dx$$

$$= \int_0^{\ln 2}\left(-e^{x-\ln 2}+e^{x}\right) dx$$

$$= \left[-e^{x-\ln 2}+e^{x}\right]_0^{\ln 2}$$

$$= -1+e^{\ln 2}+e^{-\ln 2}-1 \qquad e^{-\ln 2} = e^{\ln 2^{-1}} = 2^{-1} = \frac{1}{2}$$

$$= -2+2+\frac{1}{2}$$

$$\underline{= \frac{1}{2}}$$

## Exercise

Evaluate the double integral over the given region $R$. $\iint\limits_R \frac{y}{x^2y^2+1}dA \quad R: \ 0 \le x \le 1, \quad 0 \le y \le 1$

***Solution***

$$\iint\limits_R \frac{y}{x^2y^2+1}dA = \int_0^1 \int_0^1 \frac{y}{(xy)^2+1}dxdy \qquad \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} \quad u = xy \to du = ydx$$

$$= \int_0^1 \left[\tan^{-1}(xy)\right]_0^1 dy$$

$$= \int_0^1 \tan^{-1} y \, dy \qquad \int \tan^{-1} ax dx = x\tan^{-1} ax - \frac{1}{2a}\ln\left(1+a^2x^2\right)$$

$$= \left[y\tan^{-1} y - \frac{1}{2}\ln\left|1+y^2\right|\right]_0^1$$

$$= \tan^{-1} 1 - \frac{1}{2}\ln 2$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

## Exercise

Integrate $f(x,y) = \frac{1}{xy}$ over the ***square*** $1 \le x \le 2, \quad 1 \le y \le 2$

***Solution***

$$\int_1^2 \int_1^2 \frac{1}{xy}dydx = \int_1^2 \frac{1}{x}[\ln y]_1^2 \, dx$$

$$= \int_1^2 \frac{1}{x}[\ln 2 - \ln 1]dx$$

$$= \ln 2 \int_1^2 \frac{1}{x}dx$$

$$= \ln 2 \, [\ln x]_1^2$$

$$= \ln 2 \cdot \ln 2$$

$$= (\ln 2)^2$$

## ***Exercise***

Integrate $f(x,y) = y\cos xy$ over the ***rectangle*** $0 \le x \le \pi, \quad 0 \le y \le 1$

***Solution***

$$\int_0^1 \int_0^\pi y\cos(xy)\,dxdy = \int_0^1 \left[\sin xy\right]_0^\pi dy$$

$$= \int_0^1 \sin(\pi y)\,dy$$

$$= -\frac{1}{\pi}\cos \pi y \Big|_0^1$$

$$= -\frac{1}{\pi}[-1-1]$$

$$= \frac{2}{\pi}$$

## ***Exercise***

Find the volume of the region bounded above the paraboloid $z = x^2 + y^2$ and below by the square $R: \quad -1 \le x \le 1, \quad -1 \le y \le 1$

***Solution***

$$V = \int_{-1}^1 \int_{-1}^1 \left(x^2 + y^2\right) dydx$$

$$= \int_{-1}^1 \left[x^2 y + \frac{1}{3}y^3\right]_{-1}^1 dx$$

$$= \int_{-1}^1 \left[x^2 + \frac{1}{3} - \left(-x^2 - \frac{1}{3}\right)\right] dx$$

$$= \int_{-1}^1 \left(2x^2 + \frac{2}{3}\right) dx$$

$$= \left[\frac{2}{3}x^3 + \frac{2}{3}x\right]_{-1}^1$$

$$= \frac{2}{3} + \frac{2}{3} - \left(-\frac{2}{3} - \frac{2}{3}\right)$$

$$= \frac{8}{3}$$

## *Exercise*

Find the volume of the region bounded above the plane $z = \frac{y}{2}$ and below by the rectangle

$R: \ 0 \le x \le 4, \quad 0 \le y \le 2$

***Solution***

$$V = \int_0^4 \int_0^2 \frac{y}{2} dydx$$

$$= \int_0^4 \left[ \frac{1}{4} y^2 \right]_0^2 dx$$

$$= \int_0^4 (1) dx$$

$$= x \Big|_0^4$$

$$\underline{\boxed{= 4}}$$

## *Exercise*

Find the volume of the region bounded above the surface $z = 4 - y^2$ and below by the rectangle

$R: \ 0 \le x \le 1, \quad 0 \le y \le 2$

***Solution***

$$V = \int_0^1 \int_0^2 \left(4 - y^2\right) dydx$$

$$= \int_0^1 \left[ 4y - \frac{1}{3} y^3 \right]_0^2 dx$$

$$= \int_0^1 \left(8 - \frac{8}{3}\right) dx$$

$$= \int_0^1 \frac{16}{3} dx$$

$$= \left[ \frac{16}{3} x \right]_0^1$$

$$\underline{\boxed{= \frac{16}{3}}}$$

## Exercise

Find the volume of the region bounded above the ellipitical paraboloid $z = 16 - x^2 - y^2$ and below by the square $R:\ \ 0 \le x \le 2,\quad 0 \le y \le 2$

Solution

$$V = \int_0^2 \int_0^2 \left(16 - x^2 - y^2\right) dydx$$

$$= \int_0^2 \left[16y - x^2 y - \frac{1}{3}y^3\right]_0^2 dx$$

$$= \int_0^2 \left(32 - 2x^2 - \frac{8}{3}\right) dx$$

$$= \int_0^2 \left(\frac{88}{3} - 2x^2\right) dx$$

$$= \left[\frac{88}{3}x - \frac{2}{3}x^3\right]_0^2$$

$$= \frac{176}{3} - \frac{16}{3}$$

$$= \boxed{\frac{160}{3}}$$