SOLUTION *Section* 4.3 – Closures of Relations

Exercise

Let R be the relation on the set {0, 1, 2, 3} containing the ordered pairs (0, 1), (1, 1), (1, 2), (2, 0), (2, 2), and (3, 0). Find the

a) Reflexive closure of R.
b) Symmetric closure of R.

Solution

a) The reflexive closure of R is R with all (a, a). In this case the closure of R is
{(0, 0), (0, 1), (1, 1), (1, 2), (2,0), (2, 2), (3, 0), (3, 3)}

b) The symmetric closure of R is R with (b, a) for which (a, b) is in R.
In this case the symmetric of R is
{(0, 1), (0, 2), (0, 3), (1, 0) (1, 1), (1, 2), (2,0), (2, 1) (2, 2), (3, 0)}

Exercise

Let R be the relation $\{(a,\ b) \mid a \neq b\}$ on the set of integers. What is the reflexive closure of R?

Solution

When we add all the pairs (x, x) to the given relation we have all of $\mathbb{Z} \times \mathbb{Z}$, which the relation will always holds.

Exercise

Let R be the relation $\{(a,\ b) \mid a \text{ divides } b\}$ on the set of integers. What is the symmetric closure of R?

Solution

To form the symmetric closure, we need to add all the pairs (b, a) such that (a, b) is in R.
We need to include pairs (b, a) such that a divides b, which is equivalent to saying that we need to include all the pairs (a, b) such that b divides a.
Thus the closure is $\{(a, b) \mid a$ divides b **or** b divides $a\}$

Exercise

How can the directed graph representing the reflexive closure of a relation on a finite set be constructed from the directed graph of the relation?

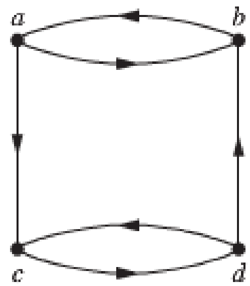
Solution

To form a reflexive closure, we simply need to add a loop at each vertex that does not already have one.

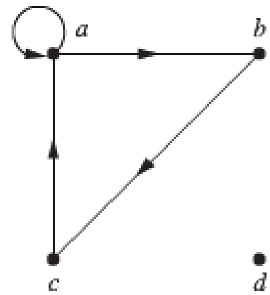
Exercise

Draw the directed graph of the *reflexive*, *symmetric*, and *both reflexive and symmetric* closure of the relations with the directed graph shown

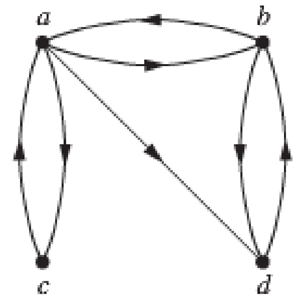
a)



b)

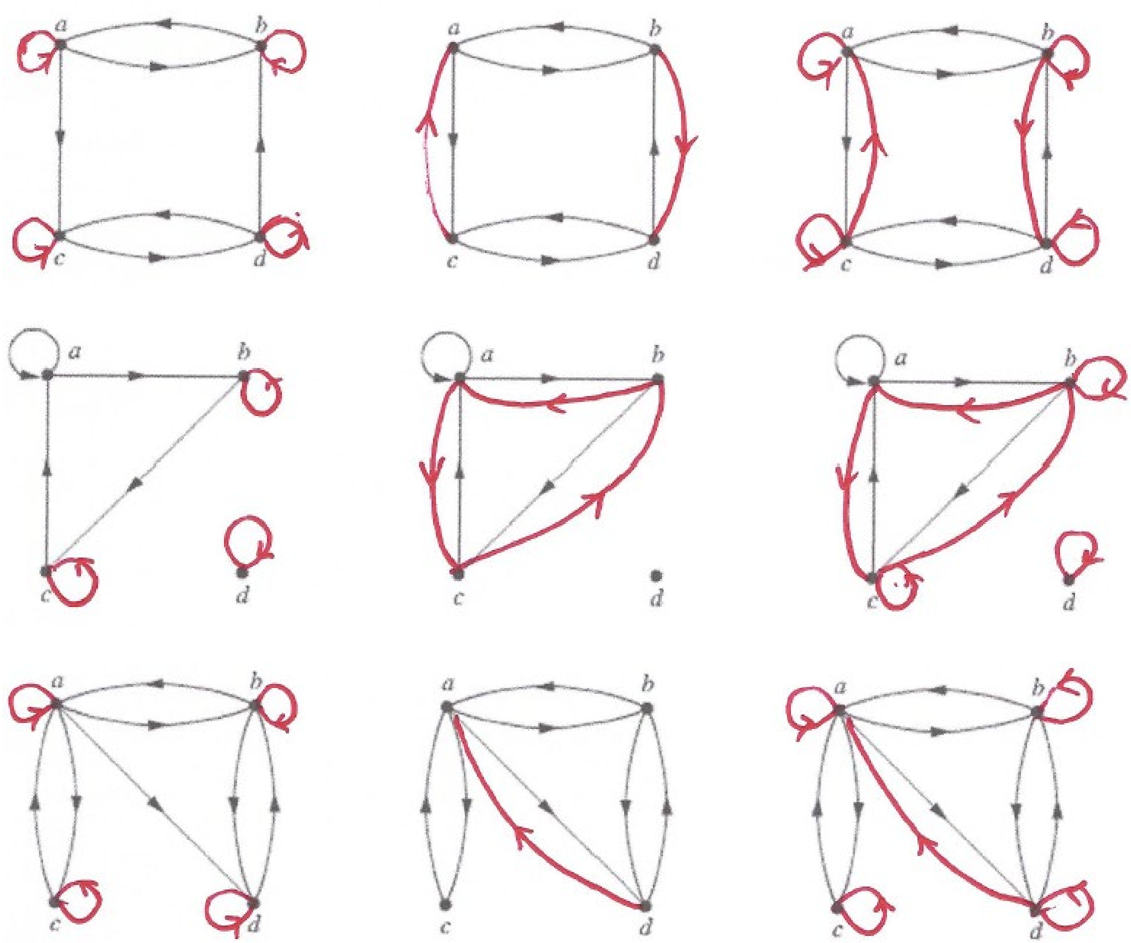


c)



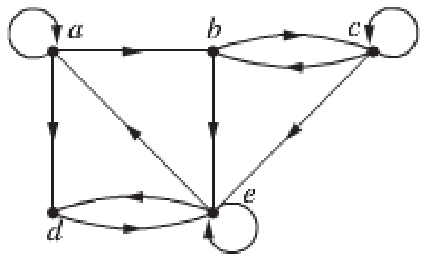
Solution

Reflexive ***Symmetric*** ***Reflexive** and **Symmetric***



Exercise

1. Determine whether these sequences of vertices are paths in this directed graph



a) *a, b, c, e*
b) *b, e, c, b, e*
c) *a, a, b, e, d, e*
d) *b, c, e, d, a, a, b*
e) *b, c, c, b, e, d, e, d*
f) *a, a, b, b, c, ,c, b, e, d*

2. Find all circuits of length three in the directed graph

Solution

a) This is a path
b) This is not a path (no edge from *e* to *c*)
c) This is a path
d) This is not a path (no edge from *d* to *a*)
e) This is a path
f) This is not a path (no loop at *b*)

2. A circuit of length 3 can be written as a sequence of 4 vertices.

Start @ ***b***: *bccb* and *beab*

Start @ ***c***: *ccbc* and *cbcc*

Start @ ***d***: *deed*, *eede* and *edee*

eabe, dead, eade, abea, adea, aaaa, cccc, and eeee

Exercise

Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs (1, 3), (2, 4), (3, 1), (3, 5), (4, 3), (5, 1), and (5, 2). Find

a) R^2 ***b***) R^3 ***c***) R^4 ***d***) R^5 ***e***) R^6 ***f***) R^*

Solution

$$M_R = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

a) $M_{R^2} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$

b) $M_{R^3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$

c) $M_{R^4} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

d) $M_{R^5} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

e) $M_{R^6} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

f) $M_{R^*} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

Exercise

Let R be the relation on the pair (a, b) if a and b are cities such that there is a direct non-stop airline flight from a to b. When is (a, b) in

a) R^2 ***b***) R^3 ***c***) R^*

Solution

a) The pair (a, b) is in R^2 precisely when there is a city c such that there is a direct flight from a to c and a direct flight from c to b – when it is possible to fly from a to b with a scheduled stop in some intermediate city.

b) The pair (a, b) is in R^3 precisely when there are cities c and d such that there is a direct flight from a to c, a direct flight from c to d, and a direct flight from d to b – when it is possible to fly from a to b with two scheduled stops in some intermediate cities.

c) The pair (a, b) is in $R*$ precisely when it is possible to fly from a to b.

Exercise

Let R be the relation on the set of all students containing the ordered pair (a, b) if a and b are in at least one common class and $a \neq b$. When is (a, b) in

a) R^2 ***b***) R^3 ***c***) R^*

Solution

a) The pair $(a,\ b) \in R^2$ if there is a person c other than a or b who is in a class with a and a class with b. $(a,\ a) \in R^2$ as long a is taking a class that has at least one other person in it, that person serves as the "c".

b) The pair $(a,\ b) \in R^3$ if there are persons c different from a and d different from b ***and*** c such that c is in a class with a, c is in class with d, and d is in class with b.

c) The pair $(a,\ b) \in R^*$ if there is a sequence of persons $c_0,\ c_1,\ c_2,\ \ldots,\ c_n$, with $n \geq 1$ such that $c_0 = a,\ c_n = b$, and for each i from 1 to n, $c_{i-1} \neq c_i$ and c_{i-1} is at least one class with c_i

Exercise

Suppose that the relation R is reflexive. Show that R^* is reflexive.

Solution

Since $R \subseteq R^*$, clearly if $\Delta \subseteq R$, then $\Delta \subseteq R^*$

Exercise

Suppose that the relation R is symmetric. Show that R^* is symmetric.

Solution

Suppose $(a,\ b) \in R^*$, then there is a path from a to b in R. Given such a path, if R is symmetric, then the reverse of every edge in the path is also in R; therefore there is a path from b to a in R.
This means that $(b,\ a) \in R^*$ whenever (a, b) is.

Exercise

Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive.

Solution

It is certainly possibly for R^2 to contain some pairs (a, a).
For example: $R = \{(1, 2), (2, 1)\}$