Solution   *Section* 2.7 – Coordinates, Basis and Dimension

Exercise

Suppose $v_1, \ldots, v_n$ is a basis for R^n and the *n* by *n* matrix *A* is invertible. Show that $Av_1, \ldots, Av_n$ is also a basis for R^n.

Solution

Put the basis vectors $v_1, \ldots, v_n$ in the columns of an invertible matrix **V**. then $Av_1, \ldots, Av_n$ are the columns of ***A*V**. Since ***A*** is invertible, so is ***A*V** and its column give a basis.

Suppose $c_1 Av_1 + \ldots + c_n Av_n = 0$. This is $Av = 0$ with $v = c_1 v_1 + \ldots + c_n v_n$. Multiply by A^{-1} to get $v = 0$. By linear independence of $v's$, all $c_i = 0$. So, the $Av's$ are independent.

Exercise

Consider the matrix $A = \begin{pmatrix} 1 & 0 & a \\ 2 & -1 & b \\ 1 & 1 & c \\ -2 & 1 & d \end{pmatrix}$

a) Which vectors $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ will make the columns of ***A*** linearly dependent?

b) Which vectors $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ will make the columns of ***A*** a basis for $\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : y + w = 0 \right\}$?

c) For $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$, compute a basis for the four subspaces.

Solution

a) All linear combination of $\begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

b) To satisfy $b + d = 0$. For example, $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$A\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + B\begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix} + C\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} \;;\; A \neq 0$$

c) $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix} + 4\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & 5 \\ -2 & 1 & 2 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{matrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -4 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{pmatrix} \begin{matrix} \\ \\ R_3 + R_2 \\ R_4 + R_2 \end{matrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{cases} x_1 + x_3 = 0 \\ -x_2 - 4x_3 = 0 \\ \\ \end{cases}$$

The first 2 columns span the column space $C(\boldsymbol{A})$.

If $x_3 = 1$ that implies that the nullspace $N(\boldsymbol{A})$: $\left\{ \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} \right\}$

Rank($\boldsymbol{A}$) = 2 and $\begin{bmatrix} -1 & -4 & 1 \end{bmatrix}^T$ is a basis for the one-dimensional $N(\boldsymbol{A})$.

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Exercise

Find a basis for $x-2y+3z=0$ in $\boldsymbol{R}^3$.

Find a basis for the intersection of that plane with *xy* plane. Then find a basis for all vectors perpendicular to the plane.

Solution

This plane is the nullspace of the matrix

$$A=\begin{pmatrix}1 & -2 & 3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

The special solutions: $s_1=\begin{pmatrix}2\\1\\0\end{pmatrix}\quad s_2=\begin{pmatrix}-3\\0\\1\end{pmatrix}$ give a basis for the nullspace, and for the plane.

The intersection of this plane with the *xy*-plane is a line $(x,\ -2x,\ 3x)$ and the vector $(1,\ -2,\ 3)^T$ lies in the *xy*-plane.

The vector $v_3=\begin{pmatrix}1\\-2\\3\end{pmatrix}$ is perpendicular to both vectors s_1 *and* s_2: the space vectors perpendicular to a plane $\mathbb{R}^3$ is one-dimensional, it gives a basis.

Exercise

U comes from ***A*** by subtracting row 1 from row 3:

$$A=\begin{bmatrix}1 & 3 & 2\\ 0 & 1 & 1\\ 1 & 3 & 2\end{bmatrix}\quad and\quad U=\begin{bmatrix}1 & 3 & 2\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$$

Find the bases for the two column spaces. Find the bases for the two row spaces. Find bases for the two nullspaces.

Solution

$$\begin{bmatrix}1 & 3 & 2\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\xrightarrow{R_1-3R_2}\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\to\begin{cases}x_1-x_3=0\\ x_2+x_3=0\end{cases}$$

a) The pivots are in the first two columns, so one possible basis for $C(\boldsymbol{A})$ is $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}3\\1\\3\end{bmatrix}\right\}$ and for $C(\boldsymbol{U})$ is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}3\\1\\0\end{bmatrix}\right\}$.

b) Both $\boldsymbol{A}$ and $\boldsymbol{U}$ have the same nullspace $N(\boldsymbol{A}) = N(\boldsymbol{U})$, with basis $\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$

c) Both $\boldsymbol{A}$ and $\boldsymbol{U}$ have the same row space $C\left(A^T\right)=C\left(U^T\right)$, *with basis* $\left\{\begin{bmatrix}1\\3\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$

Exercise

Write a 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives $c_1P_1+\ldots+c_5P_5=0$, and check entries to prove c_i is zero.) The five permutation matrices are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

Solution

$$P_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\quad P_2=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\quad P_3=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\quad P_4=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\quad P_5=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

$$P_1+P_2+P_3=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} \text{ and } P_4+P_5=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$$

$$P_1+P_2+P_3-P_4-P_5=I$$

$$c_1P_1+c_2P_2+c_3P_3+c_4P_4+c_5P_5=\begin{pmatrix}c_3&c_1+c_4&c_2+c_5\\c_1+c_5&c_2&c_3+c_4\\c_2+c_4&c_3+c_5&c_1\end{pmatrix}=0$$

$$c_1=c_2=c_3=0\ (diagonal)\Rightarrow\begin{pmatrix}0&0+c_4&0+c_5\\0+c_5&0&0+c_4\\0+c_4&0+c_5&0\end{pmatrix}=0\Rightarrow c_4=c_5=0$$

Exercise

Choose three independent columns of $A=\begin{pmatrix}2&3&4&1\\4&12&15&2\\0&0&0&9\\0&6&7&0\end{pmatrix}$. Then choose a different three independent columns. Explain whether either of these choices forms a basis for $C(A)$.

Solution

$$\begin{pmatrix}2&3&4&1\\4&12&15&2\\0&0&0&9\\0&6&7&0\end{pmatrix}\begin{matrix}\\R_2-2R_1\\ \\ \\ \end{matrix}\begin{pmatrix}2&3&4&1\\0&6&7&0\\0&0&0&9\\0&6&7&0\end{pmatrix}\begin{matrix}\\ \\ \\R_4-R_2\end{matrix}\begin{pmatrix}2&3&4&1\\0&6&7&0\\0&0&0&9\\0&0&0&0\end{pmatrix}$$

$$\begin{pmatrix}2&0&\frac{1}{2}&1\\0&6&7&0\\0&0&0&9\\0&0&0&0\end{pmatrix}\begin{matrix}\frac{1}{2}R_1\\\frac{1}{6}R_2\\\frac{1}{9}R_3\\ \\ \end{matrix}\begin{pmatrix}1&0&\frac{1}{4}&\frac{1}{2}\\0&1&\frac{7}{6}&0\\0&0&0&1\\0&0&0&0\end{pmatrix}\begin{matrix}R_1-\frac{1}{2}R_3\\ \\ \\ \\ \end{matrix}\begin{pmatrix}1&0&\frac{1}{4}&0\\0&1&\frac{7}{6}&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$$

Rank($\boldsymbol{A}$) = 3, the columns space is 3 which form a basis of $C(A)$. The variable is x_3

If $x_3=1 \Rightarrow \begin{pmatrix}1&0&\frac{1}{4}&0\\0&1&\frac{7}{6}&0\\0&0&0&1\\0&0&0&0\end{pmatrix} \to \begin{cases}x_1+\frac{1}{4}x_3=0\\x_2+\frac{7}{6}x_3=0\\x_4=0\end{cases} \to x_1=-\frac{1}{4}\quad x_2=-\frac{7}{6}$

$N(A)$ is spanned by $x_n=\begin{pmatrix}-\frac{1}{4}\\-\frac{7}{6}\\1\\0\end{pmatrix}$, which gives the relation of the columns. The special solution x_n gives a relation $-\frac{1}{4}v_1-\frac{7}{6}v_2+v_3=0$. If we take any two columns from the first three columns and the column 4, they will span a three-dimensional space since there will be no relation among them. Hence, they form a basis of $C(A)$.

Exercise

Which of the following sets of vectors are bases for $\boldsymbol{R}^2$?

a) $\{(2,1),\ (3,0)\}$

b) $\{(0,0),\ (1,3)\}$

Solution

a) $k_1(2,1)+k_2(3,0)=(0,0)$

$k_1(2,1)+k_2(3,0)=(b_1,b_2)$

$\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} = -3 \neq 0$

Therefore, the vectors $\{(2,1),\ (3,0)\}$ are linearly independent and span $\boldsymbol{R}^2$, so they form a basis for $\boldsymbol{R}^2$.

b) $k_1(0,0)+k_2(1,3)=(0,0)$

$k_1(0,0)+k_2(1,3)=(b_1,b_2)$

$\begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} = 0$

Therefore; the vectors $\{(0,0),\ (1,3)\}$ are linearly dependent, so they don't form a basis for $\boldsymbol{R}^2$.

Exercise

Which of the following sets of vectors are bases for $\boldsymbol{R}^3$?

a) $\{(1,0,0),\ (2,2,0),\ (3,3,3)\}$

b) $\{(3,1,-4),\ (2,5,6),\ (1,4,8)\}$

c) $\{(2,-3,1),\ (4,1,1),\ (0,-7,1)\}$

d) $\{(1,\ 6,\ 4),\ (2,\ 4,-1),\ (-1,\ 2,\ 5)\}$

Solution

a) $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix} = 6 \neq 0$ Therefore, the set of vectors are linearly independent.

The set form a basis for $\boldsymbol{R}^3$.

b) $\begin{vmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{vmatrix} = 26 \neq 0$ Therefore, the set of vectors are linearly independent.

The set form a basis for $\boldsymbol{R}^3$.

c) $\begin{vmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{vmatrix} = 0$ Therefore, the set of vectors are linearly dependent.

The set don't form a basis for $\boldsymbol{R}^3$.

d) $\begin{vmatrix} 1 & 2 & -1 \\ 6 & 4 & 2 \\ 4 & -1 & 5 \end{vmatrix} = 0$

Therefore; the set of vectors are linearly dependent.

The set don't form a basis for $\boldsymbol{R}^3$.

Exercise

Let V be the space spanned by $v_1 = \cos^2 x, \quad v_2 = \sin^2 x, \quad v_3 = \cos 2x$

a) Show that $S = \{v_1, v_2, v_3\}$ is not a basis for V.

b) Find a basis for V.

Solution

a) $\cos 2x = \cos^2 x - \sin^2 x$

$k_1 \cos^2 x + k_2 \sin^2 x + k_3 \cos 2x = 0$

$k_1 \cos^2 x + k_2 \sin^2 x + k_3 \left(\cos^2 x - \sin^2 x\right) = 0$

$\left(k_1 + k_3\right)\cos^2 x + \left(k_2 - k_3\right)\sin^2 x = 0 \quad \Rightarrow \begin{cases} k_1 + k_3 = 0 & \rightarrow k_1 = -k_3 \\ k_2 - k_3 = 0 & \rightarrow k_2 = k_3 \end{cases}$

If $k_3 = -1 \quad \Rightarrow k_1 = 1, \quad k_2 = -1$

$(1)\cos^2 x + (-1)\sin^2 x + (-1)\cos 2x = 0$

This shows that $\{v_1, v_2, v_3\}$ is linearly dependent, therefore it is not a basis for V.

b) For $c_1 \cos^2 x + c_2 \sin^2 x = 0$ to hold for all real x values, we must have $c_1 = 0 \quad (x = 0)$ and $c_2 = 0 \quad \left(x = \frac{\pi}{2}\right)$. Therefore, the vectors $v_1 = \cos^2 x \quad v_2 = \sin^2 x$ are linearly independent.

$\boldsymbol{v} = k_1 \cos^2 x + k_2 \sin^2 x + k_3 \cos 2x$

$= \left(k_1 + k_3\right)\cos^2 x + \left(k_2 - k_3\right)\sin^2 x$

This proves that the vectors $v_1 = \cos^2 x \quad and \quad v_2 = \sin^2 x$ span V. We can conclude that $v_1 = \cos^2 x \quad and \quad v_2 = \sin^2 x$ can form a basis for V.

Exercise

Find the coordinate vector of $\boldsymbol{w}$ relative to the basis $S=\{u_1,u_2\}$ for $\boldsymbol{R}^2$

a) $u_1=(1,\ 0),\quad u_2=(0,\ 1),\quad w=(3,\ -7)$

b) $u_1=(2,\ -4),\quad u_2=(3,\ 8),\quad w=(1,\ 1)$

c) $u_1=(1,\ 1),\quad u_2=(0,\ 2),\quad w=(a,\ b)$

d) $u_1=(1,\ -1),\quad u_2=(1,\ 1),\quad w=(0,\ 1)$

e) $u_1=(1,\ -1),\quad u_2=(1,\ 1),\quad w=(1,\ 1)$

Solution

a) We must first express $\boldsymbol{w}$ as a linear combination of the vectors in *S;* $\boldsymbol{w}=c_1u_1+c_2u_2$

$$(3,-7)=3(1,0)-7(0,1)$$
$$=3u_1-7u_2$$

Therefore, $(w)_S=\boxed{(3,-7)}$

b) Solve $c_1u_1+c_2u_2=\boldsymbol{w}\ \Rightarrow\ c_1(2,-4)+c_2(3,8)=(1,1)$

$$\rightarrow\begin{cases}2c_1+3c_2=1\\-4c_1+8c_2=1\end{cases}$$

$$\left[\begin{array}{cc|c}2&3&1\\-4&8&1\end{array}\right]\xrightarrow{rref}\left[\begin{array}{cc|c}1&0&\frac{5}{28}\\0&1&\frac{3}{14}\end{array}\right]$$

Therefore, $(w)_S=\boxed{\left(\frac{5}{28},\ \frac{3}{14}\right)}$

c) Solve $c_1u_1+c_2u_2=\boldsymbol{w}\ \Rightarrow\ c_1(1,1)+c_2(0,2)=(a,b)$

$$\rightarrow\begin{cases}\boxed{c_1=a}\\c_1+2c_2=b\end{cases}\Rightarrow\boxed{c_2=\frac{b-a}{2}}$$

Therefore, $(w)_S=\boxed{\left(a,\ \frac{b-a}{2}\right)}$

d) Solve $c_1u_1+c_2u_2=\boldsymbol{w}\ \Rightarrow\ c_1(1,-1)+c_2(1,1)=(0,1)$

$$\rightarrow\begin{cases}c_1+c_2=0\\-c_1+c_2=1\end{cases}$$

$$\left[\begin{array}{cc|c}1&1&0\\-1&1&1\end{array}\right]\xrightarrow{rref}\left[\begin{array}{cc|c}1&0&-\frac{1}{2}\\0&1&\frac{1}{2}\end{array}\right]\rightarrow\boxed{c_1=-\frac{1}{2}}\quad\boxed{c_2=\frac{1}{2}}$$

Therefore, $(w)_S=\boxed{\left(-\frac{1}{2},\ \frac{1}{2}\right)}$

e) Solve $c_1u_1 + c_2u_2 = \boldsymbol{w} \quad \Rightarrow \quad c_1(1,-1) + c_2(1,1) = (1,1)$

$$\rightarrow \begin{cases} c_1 + c_2 = 1 \\ -c_1 + c_2 = 1 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right] \rightarrow \boxed{c_1 = 0} \quad \boxed{c_2 = 1}$$

Therefore, $(w)_S = \underline{(0,\ 1)|}$

Exercise

Find the coordinate vector of v relative to the basis $S = \{v_1, v_2, v_3\}$

a) $v = (2,-1,3), \quad v_1 = (1,0,0), \quad v_2 = (2,2,0), \quad v_3 = (3,3,3)$

b) $v = (5,-12,3), \quad v_1 = (1,2,3), \quad v_2 = (-4,5,6), \quad v_3 = (7,-8,9)$

Solution

a) Solve $c_1v_1 + c_2v_2 + c_3v_3 = \boldsymbol{v} \quad \Rightarrow \quad c_1(1,0,0) + c_2(2,2,0) + c_2(3,3,3) = (2,-1,3)$

$$\rightarrow \begin{cases} c_1 + 2c_2 + 3c_3 = 2 & \rightarrow c_1 = 2 - 2c_2 - 3c_3 = 3 \\ \qquad 2c_2 + 3c_3 = -1 & \rightarrow c_2 = \dfrac{-3c_3 - 1}{2} = -2 \\ \qquad\qquad 3c_3 = 3 & \rightarrow c_3 = 1 \end{cases}$$

Therefore, $(v)_S = \underline{(3,\ -2,\ 1)|}$

b) Solve $c_1v_1 + c_2v_2 + c_3v_3 = \boldsymbol{v} \quad \Rightarrow \quad c_1(1,2,3) + c_2(-4,5,6) + c_2(7,-8,9) = (5,-12,3)$

$$\rightarrow \begin{cases} c_1 - 4c_2 + 7c_3 = 5 \\ 2c_1 + 5c_2 - 8c_3 = -12 \\ 3c_1 + 6c_2 + 9c_3 = 3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -4 & 7 & 5 \\ 2 & 5 & -8 & -12 \\ 3 & 6 & 9 & 3 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{cases} c_1 = -2 \\ c_2 = 0 \\ c_3 = 1 \end{cases}$$

Therefore, $(v)_S = \underline{(-2,\ 0,\ 1)|}$

Exercise

Show that $\{A_1, A_2, A_3, A_4\}$ is a basis for M_{22}, and express A as a linear combination of the basis vectors

a) $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $A = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$

b) $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

c) $A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$, $A_1 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Solution

a) Matrices $\{A_1, A_2, A_3, A_4\}$ are linearly independent if the equation

$$k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4 = \mathbf{0}$$

$$k_1 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + k_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k_4 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \mathbf{0}$$

Has only the trivial solution.

For these matrices to span M_{22}, it must be expressed every matrix $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ as

$$k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4 = \mathbf{A}$$

$$k_1 \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + k_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k_4 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$$

The 2 equations can be written as linear systems

$$\begin{cases} k_1 + k_2 + k_3 = 0 \\ k_2 = 0 \\ k_1 + k_4 = 0 \\ k_3 = 0 \end{cases} \quad \text{and} \quad \begin{cases} k_1 + k_2 + k_3 = a_1 \\ k_2 = a_2 \\ k_1 + k_4 = a_3 \\ k_3 = a_4 \end{cases}$$

$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -1 \neq 0$, that the homogeneous system has only the trivial solution.

$\{A_1, A_2, A_3, A_4\}$ span M_{22}

$$\rightarrow \begin{cases} k_1 + k_2 + k_3 & = 6 \\ k_2 & = 2 \\ k_1 + k_4 & = 5 \\ k_3 & = 3 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad \begin{array}{l} k_1 = 1 \\ k_2 = 2 \\ k_3 = 3 \\ k_4 = 4 \end{array}$$

$\mathbf{A} = \underline{A_1 + 2A_2 + 3A_3 + 4A_4}$

b) Matrices $\{A_1, A_2, A_3, A_4\}$ are linearly independent if the equation

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{0}$$

$$k_1\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + k_2\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + k_3\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + k_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{0}$$

Has only the trivial solution.

For these matrices to span M_{22}, it must be expressed every matrix $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ as

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = \mathbf{A}$$

$$k_1\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + k_2\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + k_3\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + k_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

The 2 equations can be written as linear systems

$$\begin{array}{ll} k_1 & = 0 \\ k_1 + k_2 & = 0 \\ k_1 + k_2 + k_3 & = 0 \\ k_1 + k_2 + k_3 + k_4 & = 0 \end{array} \quad \textit{and} \quad \begin{array}{ll} k_1 & = a_1 \\ k_1 + k_2 & = a_2 \\ k_1 + k_2 + k_3 & = a_3 \\ k_1 + k_2 + k_3 + k_4 & = a_4 \end{array}$$

$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$, that the homogeneous system has only the trivial solution.

$\{A_1, A_2, A_3, A_4\}$ span M_{22}

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \quad \begin{array}{l} k_1 = 1 \\ k_2 = -1 \\ k_3 = 1 \\ k_4 = -1 \end{array}$$

$\mathbf{A} = \underline{A_1 - A_2 + A_3 - A_4}$

c) $k_1\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + k_2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + k_3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + k_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \quad \begin{array}{l} k_1 = 1 \\ k_2 = 1 \\ k_3 = -1 \\ k_4 = 3 \end{array}$$

$\mathbf{A} = \underline{A_1 + A_2 - A_3 + 3A_4}$

Exercise

Consider the eight vectors

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

a) List all of the one-element. Linearly dependent sets formed from these.
b) What are the two-element, linearly dependent sets?
c) Find a three-element set spanning a subspace of dimension three, and dimension of two? One? Zero?
d) Which four-element sets are linearly dependent? Explain why.

Solution

a) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ zero vector is the only linearly dependent.

b) The set that contains zero vector and any other vector.

c) 2-dimension:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad or \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad or \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

1-dimensional subspace if we allow duplicates (zero vector) $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$

d) All four-element sets are linearly dependent in three-dimensional space.

Exercise

Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space

a) $\begin{cases} x_1 + x_2 - x_3 = 0 \\ -2x_1 - x_2 + 2x_3 = 0 \\ -x_1 \qquad + x_3 = 0 \end{cases}$

b) $\begin{cases} 3x_1 + x_2 + x_3 + x_4 = 0 \\ 5x_1 - x_2 + x_3 - x_4 = 0 \end{cases}$

c) $\begin{cases} x_1 - 3x_2 + x_3 = 0 \\ 2x_1 - 6x_2 + 2x_3 = 0 \\ 3x_1 - 9x_2 + 3x_3 = 0 \end{cases}$

d) $\begin{cases} x + y + z = 0 \\ 3x + 2y - 2z = 0 \\ 4x + 3y - z = 0 \\ 6x + 5y + z = 0 \end{cases}$

e) $\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 \qquad + 5x_3 = 0 \\ \qquad x_2 + x_3 = 0 \end{cases}$

Solution

a) $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad x_1 - x_3 = 0 \to x_1 = x_3$, $\quad x_2 = 0$

The solution: $(x_1,\ 0,\ x_1) = x_1(1,\ 0,\ 1)$

The solution space has dimension 1 and a basis $\underline{(1,\ 0,\ 1)}$

b) $\left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{array}\right]$ $\quad x_3 = s \quad x_1 = -\frac{1}{4}x_3 = s$

$\quad x_4 = t \quad x_2 = -\frac{1}{4}x_3 - x_4 = -\frac{1}{4}s - t$

The solution: $(x_1,\ x_2,\ x_3,\ x_4) = \left(-\frac{1}{4}s,\ -\frac{1}{4}s - t,\ s,\ t\right)$

$$= s\left(-\frac{1}{4},\ -\frac{1}{4},\ 1,\ 0\right) + t(0,\ -1,\ 0,\ 1)$$

The solution space has dimension 2 and a basis $\underline{\left(-\frac{1}{4},\ -\frac{1}{4},\ 1,\ 0\right),\ (0,\ -1,\ 0,\ 1)}$

$$\textbf{c)}\ \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad x_1 - 3x_2 + x_3 = 0 \to x_1 = 3x_2 - x_3$$

The solution: $\left(x_1,\ x_2,\ x_3\right) = \left(3x_2 - x_3,\ x_2,\ x_3\right)$

$$= x_2\left(3,\ 1,\ 0\right) + x_3\left(-1,\ 0,\ 1\right)$$

The solution space has dimension 2 and a basis $\underline{\left(3,\ 1,\ 0\right)\ \textit{and}\ \left(-1,\ 0,\ 1\right)}$

$$\textbf{d)}\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 2 & -2 & 0 \\ 4 & 3 & -1 & 0 \\ 6 & 5 & 1 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x = 4z \\ y = -5z \end{array}$$

The solution: $\left(x,\ y,\ z\right) = \left(4z,\ -5z,\ z\right) = z\left(4,\ -5,\ 1\right)$

The solution space has dimension 1 and a basis $\left(4,\ -5,\ 1\right)$

$$\textbf{e)}\ \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{No basis and dimension} = 0$$

Exercise

If $AS = SA$ for the shift matrix S. Show that $\boldsymbol{A}$ must have this special form:

$$\textit{If}\ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\textit{then}\ \ A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

"The subspace of matrices that commute with the shift S has dimension ______."

Solution

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow d = g = h = 0 \quad b = f \quad a = e = i$$

$$A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

The subspace of matrices that commute with the shift *S* has dimension 3, because the matrix has only three variables.

Exercise

Find bases for the following subspaces of $\boldsymbol{R}^3$

a) All vectors of the form (*a*, *b*, *c*, 0)
b) All vectors of the form (*a*, *b*, *c*, *d*), where *d* = *a* + *b* and *c* = *a* – *b*.
c) All vectors of the form (*a*, *b*, *c*, *d*), where *a* = *b* = *c* = *d*.

Solution

a) The subspace can be expressed as span $S = \{(1,0,0,0),(0,1,0,0),(0,0,1,0)\}$ is a set of linearly independent vectors. Therefore; *S* forms a basis for the subspace, so its dimension is 3.

b) The subspace contains all vectors $(a,\ b,\ a+b,\ a-b) = a(1,0,1,1) + b(0,1,1,-1)$, the set $S = \{(1,0,1,1),(0,1,1,-1)\}$ is linearly independent vectors. Therefore; *S* forms a basis for the subspace, so its dimension is 2.

c) The subspace contains all vectors $(a,\ a,\ a,\ a) = a(1,1,1,1)$, we can express the set $S = \{(1,1,1,1)\}$ as span *S* and it is linearly independent. Therefore, *S* forms a basis for the subspace, so its dimension is 1.

Exercise

Find a basis for the null space of A.

a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

c) $A=\begin{bmatrix}1&4&5&6&9\\3&-2&1&4&-1\\-1&0&-1&-2&-1\\2&3&5&7&8\end{bmatrix}$

Solution

a) $A=\begin{bmatrix}1&-1&3\\5&-4&-4\\7&-6&2\end{bmatrix}\xrightarrow{rref}\begin{bmatrix}1&0&-16\\0&1&-19\\0&0&0\end{bmatrix}\begin{array}{l}\to x_1=16x_3=16t\\\to x_2=19x_3=19t\\ \end{array}$

The general form of the solution of A$\boldsymbol{x}$ = **0** is $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=t\begin{bmatrix}16\\19\\1\end{bmatrix}$, therefore the vector $\begin{bmatrix}16\\19\\1\end{bmatrix}$ forms a basis for the null space of *A*.

b) $A=\begin{bmatrix}1&4&5&2\\2&1&3&0\\-1&3&2&2\end{bmatrix}\xrightarrow{rref}\begin{bmatrix}1&0&1&-\frac{2}{7}\\0&1&1&\frac{4}{7}\\0&0&0&0\end{bmatrix}\begin{array}{l}\to x_1=-x_3+\frac{2}{7}x_4=-t+\frac{2}{7}s\\\to x_2=-x_3-\frac{4}{7}x_4=-t-\frac{4}{7}s\\ \end{array}$

The general form of the solution of A$\boldsymbol{x}$ = **0** is $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=t\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}+s\begin{bmatrix}\frac{2}{7}\\-\frac{4}{7}\\0\\1\end{bmatrix}$, therefore the vectors

$\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}$ *and* $\begin{bmatrix}\frac{2}{7}\\-\frac{4}{7}\\0\\1\end{bmatrix}$ form a basis for the null space of *A*.

c) $\begin{bmatrix}1&4&5&6&9\\3&-2&1&4&-1\\-1&0&-1&-2&-1\\2&3&5&7&8\end{bmatrix}\xrightarrow{rref}\begin{bmatrix}1&0&1&2&1\\0&1&1&1&2\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$

$$\to\begin{cases}x_1=-x_3-2x_4-x_5=-r-2s-t\\x_2=-x_3-x_4-2x_5=-r-s-2t\end{cases}$$

The general form of the solution of $A\boldsymbol{x} = \mathbf{0}$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, therefore the vectors $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, *and* $\begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

Exercise

Find a basis for the subspace of $\boldsymbol{R}^4$ spanned by the given vectors

a) $(1,1,-4,-3),\quad (2,0,2,-2),\quad (2,-1,3,2)$

b) $(-1,1,-2,0),\quad (3,3,6,0),\quad (9,0,0,3)$

Solution

a) $\begin{pmatrix} 1 & 1 & -4 & -3 \\ 2 & 0 & 2 & -2 \\ 2 & -1 & 3 & 2 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 1 & -4 & -3 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 1 & -\frac{1}{2} \end{pmatrix}$

A basis for the subspace is $(1,1,-4,-3),\quad (0,1,-5,-2),\quad \left(0,0,1,-\frac{1}{2}\right)$

b) $\begin{pmatrix} -1 & 1 & -2 & 0 \\ 3 & 3 & 6 & 0 \\ 9 & 0 & 0 & 3 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{6} \end{pmatrix}$

A basis for the subspace is $(1,-1,2,0),\quad (0,1,0,0),\quad \left(0,0,1,-\frac{1}{6}\right)$

Exercise

Determine whether the given vectors form a basis for the given vector space

a) $\boldsymbol{v}_1(3,\ -2,\ 1),\quad \boldsymbol{v}_2(2,\ 3,\ 1),\quad \boldsymbol{v}_3(2,\ 1,\ -3),\quad in\ \mathbb{R}^3$

b) $\boldsymbol{v}_1 = (1,\ 1,\ 0,\ 0),\quad \boldsymbol{v}_2 = (0,\ 1,\ 1,\ 0),\quad \boldsymbol{v}_3 = (0,\ 0,\ 1,\ 1),\quad \boldsymbol{v}_4 = (1,\ 0,\ 0,\ 1),\quad for\ \mathbb{R}^4$

c) $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad M_{22}$

Solution

a) $\begin{vmatrix} 3 & 2 & 2 \\ -2 & 3 & 1 \\ 1 & 1 & -3 \end{vmatrix} = 14 \neq 0$

The given vectors are linearly independent and span $\boldsymbol{R}^3$, so they form a basis for $\boldsymbol{R}^3$.

b) $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 2 \neq 0$

The given vectors are linearly independent and span $\boldsymbol{R}^4$, so they form a basis for $\boldsymbol{R}^4$.

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad M_{22}$$

c) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$

They form a basis for M_{22}.

Exercise

Find a basis for, and the dimension of, the null space of the given matrix $\begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & -2 & -2 & 1 \end{bmatrix}$

Solution

$$\begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & -2 & -2 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -\frac{1}{2} & \frac{3}{8} \\ 0 & 1 & 0 & \frac{1}{4} \end{bmatrix} \quad \begin{aligned} x_1 - \tfrac{1}{2}x_3 + \tfrac{3}{8}x_4 &= 0 \\ x_2 + \tfrac{1}{4}x_4 &= 0 \end{aligned}$$

$$x_1 = \tfrac{1}{2}x_3 - \tfrac{3}{8}x_4$$

$$x_2 = -\tfrac{1}{4}x_4$$

$$\boldsymbol{x} = x_3 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{3}{8} \\ \frac{1}{4} \\ 0 \\ 1 \end{bmatrix}$$

The *bases* are: $\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} -\frac{3}{8} \\ \frac{1}{4} \\ 0 \\ 1 \end{bmatrix}$

Dimension: 2