Solution *Section* 2.8 – Row and Column Spaces

Exercise

List the row vectors and column vectors of the matrix

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ 3 & 5 & 7 & -1 \\ 1 & 4 & 2 & 7 \end{bmatrix}$$

Solution

Row vectors:

$$r_1 = \begin{bmatrix} 2 & -1 & 0 & 1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 3 & 5 & 7 & -1 \end{bmatrix}, \quad r_3 = \begin{bmatrix} 1 & 4 & 2 & 7 \end{bmatrix}$$

Column vectors:

$$c_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}, \quad c_4 = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$$

Exercise

Express the product $A\vec{x}$ as a linear combination of the column vectors of A. $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Solution

$$\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Exercise

Express the product $A\vec{x}$ as a linear combination of the column vectors of A. $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$

Solution

$$\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = -2\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 6 \\ -1 \end{bmatrix} + 5\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$$

Exercise

Express the product $A\vec{x}$ as a linear combination of the column vectors of A. $\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$

Solution

$$\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = -1\begin{bmatrix} -3 \\ 5 \\ 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 6 \\ -4 \\ 3 \\ 8 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}$$

Exercise

Determine whether $\vec{b}$ is in the column space of A, and if so, express $\vec{b}$ as a linear combination of the column vectors of A.

$$A = \begin{bmatrix} 1 & 3 \\ 4 & -6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$$

Solution

$$\left[\begin{array}{cc|c} 1 & 3 & -2 \\ 4 & -6 & 10 \end{array}\right] \quad R_2 - 4R_1$$

$$\left[\begin{array}{cc|c} 1 & 3 & -2 \\ 0 & -18 & 18 \end{array}\right] \quad -\tfrac{1}{18}R_2$$

$$\left[\begin{array}{cc|c} 1 & 3 & -2 \\ 0 & 1 & -1 \end{array}\right] \quad R_1 - 3R_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

$$\begin{bmatrix} -2 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ -6 \end{bmatrix}$$

Exercise

Determine whether $\vec{b}$ is in the column space of *A*, and if so, express $\vec{b}$ as a linear combination of the column vectors of *A*.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 9 & 3 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right] \quad \begin{array}{l} R_2 - 9R_1 \\ R_3 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 12 & -8 & -44 \\ 0 & 2 & 0 & -6 \end{array}\right] \quad \tfrac{1}{4}R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 3 & -2 & -11 \\ 0 & 2 & 0 & -6 \end{array}\right] \quad \begin{array}{l} 3R_1 + R_2 \\ \\ 3R_3 - 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 0 & 1 & 4 \\ 0 & 3 & -2 & -11 \\ 0 & 0 & 4 & 4 \end{array}\right] \quad \tfrac{1}{4}R_3$$

$$\left[\begin{array}{ccc|c} 3 & 0 & 1 & 4 \\ 0 & 3 & -2 & -11 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 - R_3 \\ R_2 + 2R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & 3 \\ 0 & 3 & 0 & -9 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} \tfrac{1}{3}R_1 \\ \tfrac{1}{3}R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 9 \\ 1 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The system $A\vec{x} = \vec{b}$ is consistent and $\vec{b}$ is in the column space of *A*.

Exercise

Determine whether $\vec{b}$ is in the column space of *A*, and if so, express $\vec{b}$ as a linear combination of the column vectors of *A*.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 2 \end{array}\right] \quad \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & 4 \end{array}\right] \quad \begin{array}{l} R_1 + R_2 \\ \\ R_3 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right] \quad \begin{array}{l} \\ -R_2 \\ \frac{1}{3}R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \\ R_2 + R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \\ \\ \cancel{0 = 1} \end{array}$$

The system $A\vec{x} = \vec{b}$ is inconsistent and $\vec{b}$ is not in the column space of *A*.

Exercise

Determine whether $\vec{b}$ is in the column space of *A*, and if so, express $\vec{b}$ as a linear combination of the column vectors of *A*.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 7 \end{bmatrix}$$

Solution

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 1 & 3 \\ 1 & 2 & 1 & 3 & 5 \\ 0 & 1 & 2 & 2 & 7 \end{array}\right] \quad R_3 - R_1$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 & 7 \end{array}\right] \quad \begin{array}{l} R_1 - 2R_2 \\ \\ \\ R_4 - R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -4 & -1 & -2 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad \begin{array}{l} R_1 + 4R_3 \\ R_2 - 2R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 7 & 2 \\ 0 & 1 & 0 & -3 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad \begin{array}{l} R_1 - 7R_4 \\ R_2 + 3R_4 \\ R_3 - 2R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -26 \\ 0 & 1 & 0 & 0 & 13 \\ 0 & 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$$\begin{bmatrix} 4 \\ 3 \\ 5 \\ 7 \end{bmatrix} = -26\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 13\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} - 7\begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

The system $A\vec{x} = \vec{b}$ is consistent and $\vec{b}$ is in the column space of A

Exercise

Suppose that $x_1 = -1, \quad x_2 = 2, \quad x_3 = 4, \quad x_4 = -3$ is a solution of a nonhomogeneous linear system $A\vec{x} = \vec{b}$ and that the solution set of the homogeneous system $A\vec{x} = \vec{0}$ is given by the formulas

$$x_1 = -3r + 4s, \quad x_2 = r - s, \quad x_3 = r, \quad x_4 = s$$

a) Find a vector form of the general solution of $A\vec{x} = \vec{0}$

b) Find a vector form of the general solution of $A\vec{x} = \vec{b}$

Solution

a) $x_1 = -3r + 4s, \quad x_2 = r - s, \quad x_3 = r, \quad x_4 = s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

b) Special Solution: $x_1 = -1, \quad x_2 = 2, \quad x_3 = 4, \quad x_4 = -3$

$$x_p = \begin{pmatrix} -1 \\ 2 \\ 4 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 4 \\ -3 \end{pmatrix} + r\begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 4 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Exercise

Find the vector form of the general solution of the given linear system $A\vec{x} = \vec{b}$; then use that result to find the vector form of the general solution of $A\vec{x} = \vec{0}$.

$$\begin{cases} x_1 - 3x_2 = 1 \\ 2x_1 - 6x_2 = 2 \end{cases}$$

Solution

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & -6 & 2 \end{array}\right] \quad R_2 - 2R_1$$

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \underline{x_1 = 1 + 3x_2}$$

The solution of $A\vec{x} = \vec{b}$ is

$$x_1 = 1 + 3t, \quad x_2 = t \ \textbf{\textit{or}}$$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

The general form of the solution $A\vec{x} = \vec{0}$ is $\vec{x} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Exercise

Find the vector form of the general solution of the given linear system $A\vec{x} = \vec{b}$; then use that result to find the vector form of the general solution of $A\vec{x} = \vec{0}$.

$$\begin{cases} x_1 + x_2 + 2x_3 = 5 \\ x_1 \qquad + x_3 = -2 \\ 2x_1 + x_2 + 3x_3 = 3 \end{cases}$$

Solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 1 & 0 & 1 & -2 \\ 2 & 1 & 3 & 3 \end{array}\right] \quad \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & -1 & -1 & -7 \\ 0 & -1 & -1 & -7 \end{array}\right] \quad \begin{array}{l} R_1 + R_2 \\ \\ R_3 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & -1 & -1 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \\ -R_2 \\ R \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \Rightarrow \underline{x_1 = -2 - x_3} \\ \Rightarrow \underline{x_2 = 7 - x_3} \end{array}$$

The solution of $A\vec{x} = \vec{b}$ is

$x_1 = -2 - t, \quad x_2 = 7 - t, \quad x_3 = t$ or

$$\vec{x} = \begin{bmatrix} -2 \\ 7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

The general form of the solution of $A\vec{x} = \vec{0}$ is

$$\vec{x} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Exercise

Find the vector form of the general solution of the given linear system $A\vec{x} = \vec{b}$; then use that result to find the vector form of the general solution of $A\vec{x} = \vec{0}$.

$$\begin{cases} x_1 + 2x_2 - 3x_3 + x_4 = 4 \\ -2x_1 + x_2 + 2x_3 + x_4 = -1 \\ -x_1 + 3x_2 - x_3 + 2x_4 = 3 \\ 4x_1 - 7x_2 \qquad - 5x_4 = -5 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 4 \\ -2 & 1 & 2 & 1 & -1 \\ -1 & 3 & -1 & 2 & 3 \\ 4 & -7 & 0 & -5 & -5 \end{array}\right] \quad \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 + R_1 \\ R_4 - 4R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 4 \\ 0 & 5 & -4 & 3 & 7 \\ 0 & 5 & -4 & 3 & 7 \\ 0 & -15 & 12 & -9 & 21 \end{array}\right] \quad \begin{array}{l} 5R_1 - 2R_2 \\ \\ R_3 - R_2 \\ R_4 + 3R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 5 & 0 & -7 & -1 & 6 \\ 0 & 5 & -4 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \frac{1}{5}R_1 \\ \frac{1}{5}R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -\frac{7}{5} & -\frac{1}{5} & \frac{6}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \Rightarrow \underline{x_1 = \frac{6}{5} + \frac{7}{5}x_3 + \frac{1}{5}x_4} \\ \Rightarrow \underline{x_2 = \frac{7}{5} + \frac{4}{5}x_3 - \frac{3}{5}x_4} \end{array}$$

The solution of $A\vec{x} = \vec{b}$ is

$$\vec{x} = \begin{bmatrix} \frac{6}{5} \\ \frac{7}{5} \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} \frac{7}{5} \\ \frac{4}{5} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{5} \\ -\frac{3}{5} \\ 0 \\ 1 \end{bmatrix}$$

The general form of the solution of $A\vec{x} = \vec{0}$ is

$$\vec{x} = s\begin{bmatrix} \frac{7}{5} \\ \frac{4}{5} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{5} \\ -\frac{3}{5} \\ 0 \\ 1 \end{bmatrix}$$

Exercise

Find the vector form of the general solution of the given linear system $A\vec{x} = \vec{b}$; then use that result to find the vector form of the general solution of $A\vec{x} = \vec{0}$.

$$\begin{cases} x_1 - 2x_2 + x_3 + 2x_4 = -1 \\ 2x_1 - 4x_2 + 2x_3 + 4x_4 = -2 \\ -x_1 + 2x_2 - x_3 - 2x_4 = 1 \\ 3x_1 - 6x_2 + 3x_3 + 6x_4 = -3 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & -1 \\ 2 & -4 & 2 & 4 & -2 \\ -1 & 2 & -1 & -2 & 1 \\ 3 & -6 & 3 & 6 & -3 \end{array}\right] \quad \begin{array}{c} R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 - 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \underline{x_1 = -1 + 2x_2 - x_3 - 2x_4}$$

Let $x_2 = s \quad x_3 = t \quad x_4 = r$

The solution of $A\vec{x} = \vec{b}$ is

$$\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The general form of the solution of $A\vec{x} = \vec{0}$ is

$$\vec{x} = s\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Exercise

Given the vectors $\vec{v}_1 = (1,\ 2,\ 0)$ and $\vec{v}_2 = (2,\ 3,\ 0)$

a) Are they linearly independent?

b) Are they a basis for any space?

c) What space **V** do they span?

d) What is the dimension of that space?

e) What matrices ***A*** have **V** as their column space?

f) Which matrices have **V** as their nullspace?

g) Describe all vectors $\vec{v}_3$ that complete a basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$ for $\mathbb{R}^3$.

Solution

a) $\vec{v}_1, \vec{v}_2$ are independent – the only combination to give $\vec{0}$ is $0.\vec{v}_1 + 0.\vec{v}_2$.

b) Yes, they are a basis for whatever space **V** they span.

c) That space **V** contains all vectors $(x,\ y,\ 0)$. It is the *xy* plane in $\mathbb{R}^3$.

d) The dimension of **V** is 2 since the basis contains 2 vectors.

e) This **V** is the column space of any 3 by *n* matrix ***A*** of rank 2, if every column is a combination of $\vec{v}_1$ *and* $\vec{v}_2$. In particular ***A*** could just have columns $\vec{v}_1$ *and* $\vec{v}_2$.

f) This **V** is the nullspace of any m by 3 matrix $\boldsymbol{B}$ of rank 1, if every row is a multiple of $(0,\ 0,\ 1)$. In particular, take $B=\begin{bmatrix}0 & 0 & 1\end{bmatrix}$. Then $B\vec{v}_1=\vec{0}$ *and* $B\vec{v}_2=\vec{0}$.

g) Any third vector $\vec{v}_3=(a,\ b,\ c)$ will complete a basis for $\mathbb{R}^3$ provided $c\neq 0$.

Exercise

a) Let $A=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

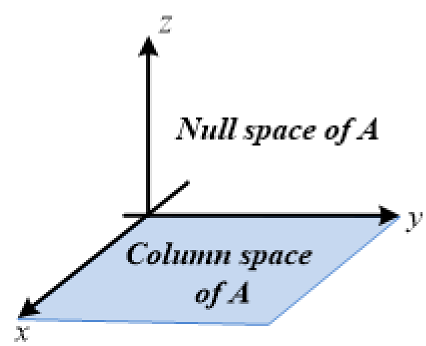
Show that relative to an *xyz*-coordinate system in 3-space the null space of *A* consists of all points on the *z*-axis and that the column space consists of all points in the *xy*-plane.

b) Find a 3 x 3 matrix whose null space is the *x*-axis and whose column space is the *yz*-plane.

Solution

a) $A=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ *Interchange* R_1 & R_2

$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$ $\begin{array}{l} x=0\\ y=0\\ z=t\end{array}$



The general form of the solution of $A\vec{x}=\vec{0}$ is,

$$t\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$$

Therefore, the null space of *A* is the *z*-axis, and the column space is the span of

$c_1=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix},\quad c_2=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$ which is all linear combinations of *y* and *x* (*xy*-plane)

b) $\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

Exercise

If we add an extra column $\vec{b}$ to a matrix A, then the column space gets larger unless _____. Give an example where the column space gets larger and an example where it doesn't. Why is $A\vec{x}=\vec{b}$ solvable exactly when the column space doesn't get larger – it is the same for A and $\begin{bmatrix} A & \vec{b} \end{bmatrix}$?

Solution

If we add an extra column $\vec{b}$ to a matrix $\boldsymbol{A}$, then the column space gets larger unless ***it contains*** $\vec{b}$ that is a linear combination of the columns of $\boldsymbol{A}$.

Let $A=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; then the column space gets larger if $\vec{b}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and it doesn't if $\vec{b}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The equation $A\vec{x}=\vec{b}$ is solvable exactly when $\vec{b}$ is a (nontrivial) linear combination of the column of $\boldsymbol{A}$.

The equation $A\vec{x}=\vec{b}$ is solvable exactly when $\vec{b}$ lies in the column space, when the column space doesn't get larger.

Exercise

For which right sides (find a condition on $b_1,\ b_2,\ b_3$) are these solvable. (Use the column space $C(A)$ and the equation $A\vec{x}=\vec{b}$)

a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Solution

a) The column space consists of the vectors for

$$\begin{pmatrix} x_1+4x_2+2x_3 \\ 2x_1+8x_2+4x_3 \\ -x_1-4x_2-2x_3 \end{pmatrix} \text{ is } \begin{pmatrix} z \\ 2z \\ -z \end{pmatrix}=z\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

They are scalar multiples of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

b) By substituting $x_1 + 4x_2$ with new variable z, then the column space consists of the vectors

$$\begin{pmatrix} x_1 + 4x_2 \\ 2x_1 + 9x_2 \\ -x_1 - 4x_2 \end{pmatrix} = z\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

They are linear combinations of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Exercise

Show that the matrices A and $\begin{bmatrix} A & AB \end{bmatrix}$ (with extra columns) have the same column space. But find a square matrix with $C\left(A^2\right)$ smaller than $C(A)$. Important point: An n by n matrix has $C(A) = \mathbb{R}^n$ exactly when A is an ______ matrix.

Solution

Each column of $\boldsymbol{AB}$ is a combination of the columns of $\boldsymbol{A}$ (the combining coefficients are the entries in the corresponding column of B). So, any combination of the columns of $\begin{bmatrix} A & AB \end{bmatrix}$ is a combination of the columns of $\boldsymbol{A}$ alone. Thus, $\boldsymbol{A}$ and $\begin{bmatrix} A & AB \end{bmatrix}$ have the same column space.

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; then $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so $C\left(A^2\right) = Z$.

$C(A)$ is the line through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Any n by n matrix has $C(A) = \boldsymbol{R}^n$ exactly when $\boldsymbol{A}$ is an ***invertible*** matrix, because $Ax = b$ is solvable for any given $\boldsymbol{b}$ when $\boldsymbol{A}$ is invertible.

Exercise

The column of AB are combinations of the columns of A. This means: The column space of AB is contained in (possibly equal to) to the column space of A. Give an example where the column spaces A and AB are not equal.

Solution

The column space of $\boldsymbol{AB}$ is contained in (possibly equal to) to the column space of $\boldsymbol{A}$.
$B = 0$ *and* $A \neq 0$ is a case when $AB = 0$ has a smaller column space than $\boldsymbol{A}$.

Exercise

Find a square matrix A where $C\left(A^2\right)$ (the column space of A^2 is smaller than $C(A)$.

Solution

For example, $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$; then $A^2=\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$.

Thus $C(A)$ is generated by vector $\begin{pmatrix}1\\0\end{pmatrix}$, which is of one dimensional, but $C\left(A^2\right)$ is a zero space.

Hence, $C\left(A^2\right)$ is strictly smaller than $C(A)$.

Exercise

Suppose $A\vec{x}=\vec{b}$ and $C\vec{x}=\vec{b}$ have the same (complete) solutions for every $\vec{b}$. Is true that $A=C$?

Solution

Yes, if $A=C$, let $\vec{y}$ be any vector of the correct size, and set $\vec{b}=A\vec{y}$. Then $\boldsymbol{y}$ is a solution to

$A\vec{x}=\vec{b}$ and it is also a solution to $C\vec{x}=\vec{b}$;

$\vec{b}=A\vec{y}=C\vec{y}$

Exercise

Apply Gauss-Jordan elimination to $U\vec{x}=0$ and $U\vec{x}=c$. Reach $R\vec{x}=0$ and $R\vec{x}=d$:

$$\begin{bmatrix}U & 0\end{bmatrix}=\begin{bmatrix}1&2&3&0\\0&0&4&0\end{bmatrix} \qquad \begin{bmatrix}U & c\end{bmatrix}=\begin{bmatrix}1&2&3&5\\0&0&4&8\end{bmatrix}$$

Solve $Rx=0$ to find x_n (its free variable is $x_2=1$).

Solve $Rx=d$ to find x_p (its free variable is $x_2=0$).

Solution

$$\begin{bmatrix}1&2&3&0\\0&0&4&0\end{bmatrix} \quad \tfrac{1}{4}R_2$$

$$\begin{bmatrix}1&2&3&0\\0&0&1&0\end{bmatrix} \quad R_1-3R_2$$

$$\begin{bmatrix}1&2&0&0\\0&0&1&0\end{bmatrix}$$

The free variable is x_2, since it is the only one. We have to let $x_2 = 1$

$$\begin{cases} x_1 + 2x_2 = 0 \\ x_3 = 0 \end{cases} \rightarrow x_1 = -2x_2$$

The special solution is $s_1\left(-2,\ 1,\ 0\right)$

$$\overline{x}_n = x_2\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix} \quad \tfrac{1}{4}R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad R_1 - 3R_2$$

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The free variable is x_2 that implies to $x_2 = 0$

$$\begin{cases} x_1 + 2x_2 = -1 \\ x_3 = 2 \end{cases} \rightarrow x_1 = -1$$

The particular solution is $\vec{x}_p = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

Exercise

Which of the following subsets of $\mathbb{R}^3$ are actually subspaces?

a) The plane of vectors $\left(b_1,\ b_2,\ b_3\right)$ with $b_1 = b_2$

b) The plane of vectors with $b_1 = 1$.

c) The vectors with $b_1 b_2 b_3 = 0$.

d) All linear combinations of $v = \left(1,\ 4,\ 0\right)$ and $w = \left(2,\ 2,\ 2\right)$.

e) All vectors that satisfies $b_1 + b_2 + b_3 = 0$

f) All vectors with $b_1 \le b_2 \le b_3$.

Solution

a) This is subspace

- For $\vec{v}=(b_1,\ b_2,\ b_3)$ with $b_1=b_2$ and $\vec{w}=(c_1,\ c_2,\ c_3)$ with $c_1=c_2$ the sum $\vec{v}+\vec{w}=(b_1+c_1,\ b_2+c_2,\ b_3+c_3)$ is in the same set as $b_1+c_1=b_2+c_2$
- For an element $\vec{v}=(b_1,\ b_2,\ b_3)$ with $b_1=b_2$, $c\vec{v}=(cb_1,\ cb_2,\ cb_3)$ and $cb_1=cb_2$, thus it is in the same set.

b) This is not a subspace. For example, for $\vec{v}=(1,\ 0,\ 0)$ and $c\vec{v}=-\vec{v}=(-1,\ 0,\ 0)$ is not in the set.

c) This is not a subspace. For example, for $\vec{v}=(1,\ 1,\ 0)$ and $\vec{w}=(1,\ 0,\ 1)$ are in the set, but their sum $\vec{v}+\vec{w}=(2,\ 1,\ 1)$ is not in the set.

d) This is subspace, by definition of linear combination.

- For 2 vectors $\vec{v}_1=\alpha_1\vec{v}+\beta_1\vec{w}$ and $\vec{v}_2=\alpha_2\vec{v}+\beta_2\vec{w}$ the sum

$$\begin{aligned}\vec{v}_1+\vec{v}_2&=\alpha_1\vec{v}+\beta_1\vec{w}+\alpha_2\vec{v}+\beta_2\vec{w}\\&=(\alpha_1+\alpha_2)\vec{v}+(\beta_1+\beta_2)\vec{w}\end{aligned}$$

  is still the linear combination of *v* and *w*.
- For an element $\vec{v}_1=\alpha_1\vec{v}+\beta_1\vec{w}$, $c\vec{v}_1=c\alpha_1\vec{v}+c\beta_1\vec{w}$ is still the linear combination of $\vec{v}$ and $\vec{w}$, thus it is the same set

e) This is subspace, these are the vectors orthogonal to $(1,\ 1,\ 1)$

- For $\vec{v}=(b_1,\ b_2,\ b_3)$ with $b_1+b_2+b_3=0$
  and $\vec{w}=(c_1,\ c_2,\ c_3)$ with $c_1+c_2+c_3=0$
  The sum $\vec{v}+\vec{w}=(b_1+c_1,\ b_2+c_2,\ b_3+c_3)$ is in the same set as
  $b_1+c_1+b_2+c_2+b_3+c_3=0$
- For an element $\vec{v}=(b_1,\ b_2,\ b_3)$ with $b_1+b_2+b_3=0$, $c\vec{v}=(cb_1,\ cb_2,\ cb_3)$ and $cb_1+cb_2+cb_3=0$, thus it is in the same set.

f) This is not a subspace. For example, for $\vec{v}=(1,\ 2,\ 3)$ and $-\vec{v}=(-1,\ -2,\ -3)$ is not in the set.

Exercise

We are given three different vectors $\vec{b}_1,\ \vec{b}_2,\ \vec{b}_3$. Construct a matrix so that the equations $A\vec{x}=\vec{b}_1$ and $A\vec{x}=\vec{b}_2$ are solvable, but $A\vec{x}=\vec{b}_3$ is not solvable.

a) How can you decide if this possible?

b) How could you construct A?

Solution

The equations $A\vec{x}=\vec{b}_1$ and $A\vec{x}=\vec{b}_2$ will be solvable.

$$A\vec{x}=\begin{bmatrix}\vec{b}_1 & \vec{b}_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\vec{b}_3 \ \text{(solvable?)}$$

If $A\vec{x}=\vec{b}_3$ is not solvable, we have the desired matrix A.

If $A\vec{x}=\vec{b}_3$ is solvable, then it is not possible to construct A.

When the column space contains $\vec{b}_1$ and $\vec{b}_2$, it will have to contain their linear combinations. So $\vec{b}_3$ would necessarily be in that column space and $A\vec{x}=\vec{b}_3$ would necessarily be solvable.

Exercise

For which vectors $(b_1,\ b_2,\ b_3)$ do these systems have a solution?

a) $\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}$

b) $\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}$

c) $\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}$

Solution

a) $\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}$ $\quad\begin{aligned}&\to x_1+x_2+x_3=b_1\\ &\to x_2+x_3=b_2\\ &\to x_3=b_3\end{aligned}$

$$\begin{cases} x_1 = b_1 - b_2 - b_3 \\ x_2 = b_2 - b_3 \\ x_3 = b_3 \end{cases} \Rightarrow \left(b_1 - b_2 - b_3,\ b_2 - b_3,\ b_3\right)$$

Solution for every b.

b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $\begin{array}{l} \to x_1 + x_2 + x_3 = b_1 \\ \to x_2 + x_3 = b_2 \\ \to 0x_3 = b_3 \end{array}$

$$\begin{cases} x_1 = b_1 - b_2 \\ x_2 = b_2 \\ 0 = b_3 \end{cases} \Rightarrow \left(b_1 - b_2,\ b_2,\ 0\right)$$

Solvable only if $b_3 = 0$

c) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $\quad R_2 - R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 0 & b_3 - b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right]$$

$\Rightarrow b_3 - b_2 = 0$

$\underline{b_3 = b_2}|$

$$\begin{cases} x_1 = b_1 - 2b_3 \\ 0 = 0 \\ x_3 = b_3 \end{cases} \Rightarrow \left(b_1 - 2b_3,\ 0,\ b_3\right)$$

Solvable only if $b_3 = b_2$

Exercise

Find a basis for the null space of A. $\quad A=\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix} \quad \begin{array}{l} \\ \\ R_3-2R_1 \\ R_4-3R_1 \\ R_5+2R_1 \end{array}$$

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & 6 & 0 & -3 \end{bmatrix} \quad \begin{array}{l} R_1+R_2 \\ \\ R_3-R_2 \\ R_4-R_2 \\ R_5-R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 8 & 2 & -2 \\ 0 & 3 & 6 & 0 & -3 \\ 0 & 0 & -12 & 0 & 5 \\ 0 & 0 & -12 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \\ \frac{1}{3}R_2 \\ \\ R_4-R_3 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 8 & 2 & -2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & -12 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} 3R_1+2R_3 \\ 6R_3+R_3 \\ \\ \\ \\ \end{array}$$

$$\begin{bmatrix} 3 & 0 & 0 & 6 & 4 \\ 0 & 6 & 0 & 0 & -1 \\ 0 & 0 & -12 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \frac{1}{3}R_1 \\ \frac{1}{6}R_2 \\ -\frac{1}{12}R_3 \\ \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & \frac{4}{3} \\ 0 & 1 & 0 & 0 & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & -\frac{5}{12} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $x_4 = s \quad x_5 = t$

$$\begin{cases} x_1 = -2x_4 - \frac{4}{3}x_5 = 2s - \frac{4}{3}t \\ x_2 = \frac{1}{6}x_5 = \frac{1}{6}t \\ x_3 = \frac{5}{12}x_5 = \frac{5}{12}t \end{cases}$$

The general form of the solution of $A\vec{x} = \vec{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{4}{3} \\ \frac{1}{6} \\ \frac{5}{12} \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the vectors $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -\frac{4}{3} \\ \frac{1}{6} \\ \frac{5}{12} \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of *A*.

Exercise

Is it true that is $m = n$ then the row space of *A* equals the column space.

Solution

False

Counterexample, let $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$

We have $m = n = 2$, but the row space of *A* contains multiple of $(1,\ 2)$ while the column space of A contains multiples of $(1,\ 3)$.

Exercise

If the row space equals the column space the $A^T = A$

Solution

False,

Counterexample, let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Here, the row space and column space are both equal to all of $\mathbb{R}^2$ (since A is invertible).

But $A \neq A^T$

Exercise

If $A^T = -A$, then the row space of A equals the column space.

Solution

True,

The row space of A equals to the column space of A^T, which for this particular A equals the column space of $-A$.

Since A and $-A$ have the same fundamental subsequences. We conclude that the row space of A equals the column space of A.

Exercise

Does the matrices A and $-A$ share the same 4 subspaces?

Solution

True.

The nullspaces are identical because $A\vec{x} = 0 \quad \Leftrightarrow \quad -A\vec{x} = 0$

The column spaces are identical because any vector $\vec{v}$ that can be expressed as $\vec{v} = A\vec{x}$ for some $\vec{x}$ can also be expressed as $\vec{v} = (-A)(-\vec{x})$

Exercise

Is A and B share the same 4 subspaces then A is multiple of B.

Solution

False

Any invertible 2×2 matrix will have $\mathbb{R}^2$ as its column space and row space and zero vector as its (left and right) nullspace.

However, it is easy to produce 2 invertible 2×2 matrices that are not multiples of each other, as example

$$A=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\quad B=\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$$

Exercise

Suppose $A\vec{x}=\vec{b}$ & $C\vec{x}=\vec{b}$ have the same (complete) solutions for every $\vec{b}$. Is it true that $A=C$

Solution

If $A\vec{x}=C\vec{x}=\vec{b}$ for all vectors $\vec{x}$ of the correct size.

Then, it is true that $A=C$

Exercise

A and A^T have the same left nullspace?

Solution

False,

Counterexample, take any a 1×2 matrix, such as $A=\begin{bmatrix}1 & 1\end{bmatrix}$.

The left nullspace of *A* contains vectors in $\mathbb{R}$ while the left nullspace of A^T , which is the right nullspace of *A*, contains vectors in $\mathbb{R}^2$.

So, they can't be the same.