2.8 Applications

$$F = mg$$
$$= ma$$
$$= m\frac{dv}{dt} = -kv \quad (1) \qquad (k>0)$$

Newton's 2nd Law

$$(1) \quad m\frac{dv}{dt} = -kv$$

$$\int \frac{dv}{v} = -\frac{k}{m}\int dt$$

$$\ln v = -\frac{k}{m}t + C$$

$$v = e^{-kt/m + C}$$

$$= e^{-kt/m}e^{C} \qquad e^{C} = v_0$$

$$\boxed{v(t) = v_0 e^{-kt/m}} \quad (1)$$

Pounds = 32 × slugs

Ex

$$m = 192 \text{ lb} \qquad k = \frac{1}{3} \text{ slug/sec}$$
$$= \frac{192}{32} = 6 \text{ slugs}$$

$$\underset{v_0}{11 \text{ ft/sec}} \longrightarrow \underset{v(t)}{1 \text{ ft/sec}} \qquad \boxed{\frac{k}{m} = \frac{1}{3}\cdot\frac{1}{6} = \frac{1}{18}}$$

$$(1) \quad 1 = 11e^{-t/18}$$

$$-t\frac{k}{m} = \ln\left(\frac{v}{v_0}\right) \qquad \ln(0<\,<1)<0$$

$$t = -18\ln\left(\frac{1}{11}\right) \qquad \ln\frac{1}{x} = -\ln x$$

$$\boxed{= 18\ln 11}$$

$$\boxed{\approx 43 \text{ sec}}$$

$$d = \frac{v_0 m}{k} \qquad v = \frac{d}{t}$$

$$= \frac{11(6)}{1/3}$$

$$\boxed{= 198 \text{ ft}}$$

Mixtures $\qquad \dfrac{dy}{dt} = Rate_{in} - Rate_{out}$ $\qquad$ Balance law

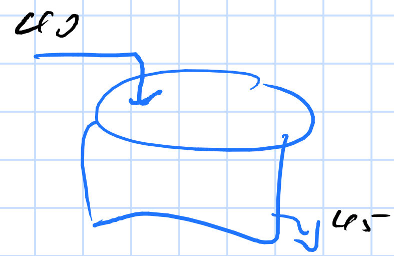
$Rate = Vol \dfrac{gal}{min} \times Concentration \dfrac{lb}{gal}$ $\qquad$ lb/min

* $Rate_{out} = \dfrac{y(t)}{V(t)} \cdot \text{outflow rate}$

Ex $\quad V_0 = 2k$ gal, $y_0 = 100$ lb

In: 40 gal/min $\quad$ 2 lb/gal ⟸

Out: 45 gal/min



Soln

$$V(t) = 2{,}000 + (40 - 45)t$$
$$= 2{,}000 - 5t = 0 \qquad t = \frac{2000}{5} = 400 \text{ min}$$

$$R_{out} = \frac{y(t)}{2000-5t}(45)$$
$$= \frac{45y}{2{,}000 - 5t}$$

$$Rate_{in} = 40(2)$$
$$= 80 \text{ lb/min}$$

$$\frac{dy}{dt} = 80 - \frac{45y}{2000-5t}$$

$$y' + \frac{45}{2000-5t}y = 80$$

$$e^{\int \frac{45}{2000-5t}dt} = e^{-9\int \frac{d(2000-5t)}{2000-5t}}$$
$$= e^{-9\ln(2000-5t)}$$
$$= (2000-5t)^{-9}$$

$$\int 80(2000-5t)^{-9}dt = -16\int (2000-5t)^{-9} d(2000-5t)$$
$$= 2(2\times 10^3 - 5t)^{-8}$$

$$y(t) = (2000-5t)^9\left(2(2000-5t)^{-8} + C\right)$$

$$y(t) = 4{,}000 - 10t + C(2000 - 5t)^9 \leftarrow \quad y(0) = 100$$

$$100 = 4{,}000 + C(2 \times 10^3)^9$$

$$C = -\frac{3{,}900}{(2\times10^3)^9}$$

$$y(t) = 4{,}000 - 10t - \frac{3{,}900}{(2\times10^3)^9}(2000 - 5t)^9$$

$$y(20) = 4000 - 200 - \frac{3900}{2000^9}(2000 - 100)^9$$

$$\boxed{\approx 1{,}342 \text{ lb}}$$

RL circuit

R : resistor Ω

L : inductor H

Ohm's law: $V = RI$

I or i current

j : complex

$$L\frac{dI}{dt}$$

$$L\frac{di}{dt} + Ri = V$$

$$\frac{di}{dt} + \frac{R}{L} i = \frac{V}{L}$$

$$e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t} \quad (1)$$

$$\int \frac{V}{L} e^{Rt/L} dt = \frac{V}{R} e^{\frac{Rt}{L}} \quad (2)$$

$$i(t) = e^{-\frac{R}{L}t}\left(\frac{V}{R} e^{\frac{R}{L}t} + C\right)$$

$$\boxed{= \frac{V}{R} + C e^{-\frac{R}{L}t}}$$

Review Exam 2

$x^n e^{ax} \leftrightarrow x^n$ Trig

Trig e^{ax}

#1 $\int x^5 e^{4x}\,dx$

		$\int e^{4x}\,dx$
$+$	x^5	$\frac{1}{4}e^{4x}$
$-$	$5x^4$	$\frac{1}{2^4}e^{4x}$
$+$	5×2^2 $20x^3$	$\frac{1}{2^6}e^{4x}$
$-$	$15\cdot 2^2$ $60x^2$	$\frac{1}{2^8}e^{4x}$
$+$	$15\cdot 2^3$ $120x$	$\frac{1}{2^{10}}e^{4x}$
$-$	15×2^3 120	$\frac{1}{2^{12}}e^{4x}$

$$\int x^5 e^{4x}\,dx = e^{4x}\left(\frac{1}{4}x^5 - \frac{5}{16}x^4 + \frac{5}{16}x^3 - \frac{15}{64}x^2 + \frac{15}{128}x - \frac{15}{512}\right) + C$$

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2) $\int \cos 2x\, e^{3x}\,dx$

		$\int \cos 2x$
$+$	e^{3x}	$\frac{1}{2}\sin 2x$
$-$	$3e^{3x}$	$-\frac{1}{4}\cos 2x$
$+$	$9e^{3x}$	

$$\int (\cos 2x)e^{3x}\,dx = e^{3x}\left(\frac{1}{2}\sin 2x + \frac{3}{4}\cos 2x\right) - \frac{9}{4}\int e^{3x}\cos 2x\,dx$$

$$\rightarrow \frac{13}{4}\int e^{3x}\cos 2x\,dx = \frac{1}{4}e^{3x}(2\sin 2x + 3\cos 2x)$$

$$\int e^{3x}\cos 2x\,dx = \frac{1}{13}e^{3x}(2\sin 2x + 3\cos 2x) + C$$

#3 $\int x^6 \cos 4x\,dx$

		$\int \cos 4x\,dx$
$+$	x^6	$\frac{1}{4}\sin 4x$
$-$	$6x^5$	$-\frac{1}{2^4}\cos 4x$
$+$	$2\times 15\,x^4$	$-\frac{1}{2^6}\sin 4x$
$-$	$2^3\times 15\,x^3$	$\frac{1}{2^8}\cos 4x$
$+$	$2^3(45)x^2$	$\frac{1}{2^{10}}\sin 4x$
$-$	$2^4(45)x$	$-\frac{1}{2^{12}}\cos 4x$
$+$	$2^4(45)$	$-\frac{1}{2^{14}}\sin 4x$

$$\int x^6\cos 4x\,dx = \left(\frac{x^6}{4} - \frac{15}{32}x^4 + \frac{45}{128}x^2 - \frac{45}{1024}\right)\sin 4x$$
$$+\left(\frac{3}{8}x^5 - \frac{15}{32}x^3 + \frac{45}{256}x\right)\cos 4x + C$$

#1 $\int \sin^4 x\,dx = \frac{1}{4}\int (1-\cos 2x)^2\,dx$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right)dx$$
$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right)dx$$
$$= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + C$$

2/ $\int \cos^7 x\,dx = \int \cos^6 x \cos x\,dx$ $\qquad (\cos^2 x)^3$

$$= \int (1-\sin^2 x)^3\,d(\sin x)$$
$$= \int (1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x)\,d(\sin x)$$
$$= \sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$$

3/ $\displaystyle\int_0^{\pi/2} \cos^{12} x\,dx = \frac{11}{\cancel{12}_4}\,\frac{\cancel{9}}{\cancel{10}_2}\,\frac{7}{8}\,\frac{\cancel{5}}{\cancel{6}_2}\,\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}$

$$= \boxed{\frac{231}{2^{11}}\,\pi}$$

4/ $\displaystyle\int_0^{\pi/2} \cos^{15} x\,dx = \frac{\overset{2}{\cancel{14}}}{15}\,\frac{\overset{2^2}{\cancel{12}}}{13}\,\frac{\overset{2}{\cancel{10}}}{11}\,\frac{8}{\cancel{9}}\,\frac{\overset{2}{\cancel{6}}}{\cancel{7}}\,\frac{4}{\cancel{5}}\,\frac{2}{3}$

$$= \boxed{\frac{2^{11}}{6{,}435}}$$

5/ $\displaystyle\int \sqrt{9-4x^2}\,dx$ $\qquad \boxed{2x = 3\sin\theta}$ $\qquad \sqrt{9-4x^2} = 3\cos\theta$

$dx = \frac{3}{2}\cos\theta\,d\theta$

$$\int \sqrt{9-4x^2}\,dx = \int 3\cos\theta \,\frac{3}{2}\cos\theta\,d\theta$$

$$= \frac{9}{2}\int \cos^2\theta\,d\theta$$

$$= \frac{9}{4}\int (1+\cos 2\theta)\,d\theta$$

$$= \frac{9}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right)$$

$$= \frac{9}{4}(\theta + \sin\theta\cos\theta)$$

$$= \frac{9}{4}\left(\sin^{-1}\frac{2x}{3} + \frac{2x}{3}\,\frac{\sqrt{9-4x^2}}{3}\right)$$

$$= \boxed{\frac{9}{4}\sin^{-1}\left(\frac{2x}{3}\right) + \frac{1}{2}x\sqrt{9-4x^2} + C}$$

6. $\displaystyle\int \frac{dx}{\sqrt{x^2-25}}$

$x = 5\sec\theta$ $\quad \sqrt{x^2-25} = 5\tan\theta$

$dx = 5\sec\theta\tan\theta\, d\theta$

$$\int \frac{dx}{\sqrt{x^2-25}} = \int \frac{5\sec\theta\tan\theta}{5\tan\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta|$$

$$= \boxed{\ln\left|\frac{x}{5} + \frac{\sqrt{x^2-25}}{5}\right| + C}$$

7. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+36}}$

$x = 6\tan\theta$ $\quad \sqrt{x^2+36} = 6\sec\theta$

$dx = 6\sec^2\theta\, d\theta$

$$\int \frac{dx}{x^2\sqrt{x^2+36}} = \int \frac{6\sec^2\theta\, d\theta}{36\tan^2\theta\,(6\sec\theta)}$$

$$= \frac{1}{36}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$$= \frac{1}{36}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{36}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{36}\int \frac{d(\sin\theta)}{\sin^2\theta}$$

$$= -\frac{1}{36}\,\frac{1}{\sin\theta}$$

$\sin\theta = \dfrac{\tan\theta}{\sec\theta}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$= -\frac{1}{36}\,\frac{\sec\theta}{\tan\theta}$$

$$= -\frac{1}{36}\,\frac{\sqrt{x^2+36}}{6}\cdot\frac{6}{x}$$

$$= \boxed{-\frac{1}{36}\,\frac{\sqrt{x^2+36}}{x} + C}$$

$$\int \frac{dx}{x^2+2x} \qquad \frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2}$$

$$1 = A(x+2) + Bx$$

x^1 $\quad A + B = 0 \implies \boxed{B = -\frac{1}{2}}$

x^0 $\quad 2A = 1 \to \boxed{A = \frac{1}{2}}$

$$\int \frac{dx}{x^2+2x} = \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x+2}$$

$$= \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + C$$

$$\int \frac{2x+1}{x^2-7x+12}\,dx$$

$$\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3}$$

$$2x+1 = A(x-3) + B(x-4)$$

x^1 $\quad A + B = 2 \qquad (-1)$

x^0 $\quad -3A - 4B = 1$

$\boxed{A = +9} \qquad \boxed{B = -7}$

$$\int \frac{2x+1}{x^2-7x+12} = 9\int \frac{dx}{x-4} - 7\int \frac{dx}{x-3}$$

$$\boxed{= 9\ln|x-4| - 7\ln|x-3| + C}$$

$$\int \frac{dx}{x+a} = \ln|x+a| \qquad \int \frac{du}{u^2} = -\frac{1}{u}$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x^2+x}{x^4-3x^2-4}\,dx \qquad x^4-3x^2-4=(x^2+1)(x^2-4)$$

$$\frac{x^2+x}{x^4-3x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1}$$

$$x^2+x = A(x+2)(x^2+1) + B(x-2)(x^2+1) + (Cx+D)(x^2-4)$$

$$\begin{array}{ll}
x^3 & A + B + C = 0 \\
x^2 & 2A - 2B + D = 1 \\
x^1 & A + B - 4C = 1 \\
x^0 & 2A - 2B - 4D = 0
\end{array} \qquad ① - ②$$

$$5C = -1 \Rightarrow \boxed{C = -\frac{1}{5}}$$

$$5D = 1 \Rightarrow \boxed{D = \frac{1}{5}}$$

$$① \quad A + B = \frac{1}{5}$$

$$② \quad 2A - 2B = 1 - \frac{1}{5} = \frac{4}{5}$$

$$4A = \frac{6}{5} \Rightarrow \boxed{A = \frac{3}{10}}$$

$$\boxed{B = \frac{1}{5} - \frac{3}{10} = -\frac{1}{10}}$$

$$\int \frac{x^2+4x}{x^4-3x^2-4}\,dx = \frac{3}{10}\int\frac{dx}{x-2} - \frac{1}{10}\int\frac{dx}{x+2} - \frac{1}{5}\int\frac{x\,dv}{x^2+1}$$

$$+ \frac{1}{5}\int\frac{dx}{x^2+1}$$

$$= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{10}\int\frac{d(x^2+1)}{x^2+1}$$

$$+ \frac{1}{5}\arctan x$$

$$= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{10}\ln|x^2+1|$$

$$\boxed{+ \frac{1}{5}\tan^{-1}x + K}$$

$A + B = \frac{1}{5}$

$2A - 2B = \frac{4}{5}$

$5A + 5B = 1$

$10A - 10B = 4$

$5A + 5B = 1$

$5A - 5B = 2$

10/12 $\rightarrow$ Exam Review

10/14 Exam 2

$y' + y = e^t \qquad y(0) = 1$

$e^{\int dt} = e^t$

$\int e^{2t} dt = \frac{1}{2} e^{2t}$

$y(t) = \frac{1}{e^t}\left(\frac{1}{2} e^{2t} + C\right.$

$= \frac{1}{2} e^t + \frac{C}{e^t}$

$1 = \frac{1}{2} + C$

$C = \frac{1}{2}$

$$\boxed{y(t) = \frac{1}{2} e^t + \frac{1}{2e^t}}$$

$$(1+e^x)\,dy + (ye^x - e^{-x})\,dx = 0$$

$$(1+e^x)\frac{dy}{dx} + ye^x - e^{-x} = 0$$

$$(1+e^x)\,y' + e^x y = e^{-x} = \frac{1}{e^x}$$

$$y' + \frac{e^x}{1+e^x}\,y = \frac{1}{e^x(1+e^x)}$$

$$e^{\int \frac{e^x}{1+e^x}dx} = e^{\int \frac{d(1+e^x)}{1+e^x}} = e^{\ln(1+e^x)}$$

$$= 1+e^x$$

$$\int (1+e^x)\frac{1}{e^x(1+e^x)}\,dx = \int e^{-x}dx = -e^{-x}$$

$$y(x) = \frac{1}{1+e^x}\left(-e^{-x} + C\right)$$

$$\frac{C - e^{-x}}{1+e^x}$$

$$y' + py = f$$

$$e^{\int p\,dx}$$

$$\int \cdot f\,dx$$

$$\int_0^\infty \frac{dx}{x^2+1} = \tan^{-1}x \Big|_0^\infty$$
$$= \tan^{-1}\infty - \underbrace{\tan^{-1}0}_{=0}$$
$$= \boxed{\frac{\pi}{2}}$$

$$\frac{1}{0} = \infty$$

$$\int_{-\infty}^\infty 2xe^{-x^2}dx = -\int_{-\infty}^\infty e^{-x^2}d(-x^2)$$
$$= -e^{-x^2}\Big|_{-\infty}^\infty$$
$$= -(0-0)$$
$$= \boxed{0}$$

$$e^{-\infty} \to 0$$

$$\int_0^1 (-\ln x)dx = -(x\ln x - x \Big|_0^1$$
$$= -(-1)$$
$$= \boxed{1}$$

$$\int_0^\infty \frac{dx}{(x+1)^3} = \int_0^\infty (x+1)^{-3}d(x+1)$$
$$= -\frac{1}{2}\frac{1}{(x+1)^2}\Big|_0^\infty$$
$$= -\frac{1}{2}(0-1)$$
$$= \boxed{\frac{1}{2}}$$