

## ***Solution***      **Section 1.1 – Idea of Limits**

### ***Exercise***

Find the average rate of change of the function  $f(x) = x^3 + 1$  over the interval  $[2, 3]$

### **Solution**

$$\begin{aligned}\frac{\Delta f}{\Delta x} &= \frac{f(3) - f(2)}{3 - 2} \\ &= \frac{3^3 + 1 - (2^3 + 1)}{1} \\ &= 27 + 1 - (8 + 1) \\ &= 19\end{aligned}$$

### ***Exercise***

Find the average rate of change of the function  $f(x) = x^2$  over the interval  $[-1, 1]$

### **Solution**

$$\begin{aligned}\frac{\Delta f}{\Delta x} &= \frac{f(1) - f(-1)}{1 - (-1)} \\ &= \frac{1^2 - (-1)^2}{2} \\ &= \frac{0}{2} \\ &= 0\end{aligned}$$

### ***Exercise***

Find the average rate of change of the function  $f(t) = 2 + \cos t$  over the interval  $[-\pi, \pi]$

### **Solution**

$$\begin{aligned}\frac{\Delta f}{\Delta x} &= \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)} \\ &= \frac{2 + \cos \pi - (2 + \cos(-\pi))}{2\pi} \\ &= \frac{2 - 1 - (2 - 1)}{2} \\ &= 0\end{aligned}$$

### Exercise

Find the slope of  $y = x^2 - 3$  at the point  $P(2, 1)$  and an equation of the tangent line at this  $P$ .

### Solution

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(2+h) - f(2)}{h} & \frac{\Delta y}{\Delta x} &= \frac{f(x_1 + h) - f(x_1)}{h} \\ &= \frac{(2+h)^2 - 3 - (2^2 - 3)}{h} \\ &= \frac{4 + 4h + h^2 - 3 - (4 - 3)}{h} \\ &= \frac{4h + h^2}{h} \\ &= 4 + h\end{aligned}$$

As  $h$  approaches 0. Then the secant slope  $h + 4 \rightarrow 4 = \text{slope}$

$$y = 4(x - 2) + 1$$

$$y - 1 + 1 = 4x - 8 + 1$$

$$\underline{y = 4x - 7}$$

$$y = m(x - x_1) + y_1$$

### Exercise

Find the slope of  $y = x^2 - 2x - 3$  at the point  $P(2, -3)$  and an equation of the tangent line at this  $P$ .

### Solution

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(2+h) - f(2)}{h} \\ &= \frac{(2+h)^2 - 2(2+h) - 3 - (2^2 - 2(2) - 3)}{h} \\ &= \frac{4 + 4h + h^2 - 4 - 2h - 3 - (-3)}{h} \\ &= \frac{2h + h^2}{h} \\ &= 2 + h\end{aligned}$$

As  $h$  approaches 0. Then the secant slope  $2 + h \rightarrow 2 = \text{slope}$

$$y + 3 = 2(x - 2)$$

$$y = 2x - 4 - 3$$

$$\underline{y = 2x - 7}$$

$$y = m(x - x_1) + y_1$$

### Exercise

Find the slope of  $y = x^3$  at the point  $P(2, 8)$  and an equation of the tangent line at this  $P$ .

### Solution

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(2+h) - f(2)}{h} \\ &= \frac{(2+h)^3 - 2^3}{h} \\ &= \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\ &= \underline{12 + 6h + h^2} \quad \text{As } h \text{ approaches } 0. \text{ Then } \text{slope} = 12\end{aligned}$$

$$y - 8 = 12(x - 2)$$

$$y = 12x - 24 + 8$$

$$\underline{y = 12x - 16}$$

$$y = m(x - x_1) + y_1$$

### Exercise

Make a table of values for the function  $f(x) = \frac{x+2}{x-2}$  at the points

$$x = 1.2, \quad x = \frac{11}{10}, \quad x = \frac{101}{100}, \quad x = \frac{1001}{1000}, \quad x = \frac{10001}{10000}, \quad \text{and } x = 1$$

- a) Find the average rate of change of  $f(x)$  over the intervals  $[1, x]$  for each  $x \neq 1$  in the table  
b) Extending the table if necessary, try to determine the rate of change of  $f(x)$  at  $x = 1$ .

### Solution

a)

$x$	1.2	1.1	1.01	1.001	1.0001	1
$f(x)$	-4.0	-3.4	-3.04	-3.004	-3.0004	-3

$$\frac{\Delta y}{\Delta x} = \frac{-4 - (-3)}{1.2 - 1} = -5.0$$

$$\frac{\Delta y}{\Delta x} = \frac{-3.4 - (-3)}{1.1 - 1} = -4.4$$

$$\frac{\Delta y}{\Delta x} = \frac{-3.04 - (-3)}{1.01 - 1} = -4.04$$

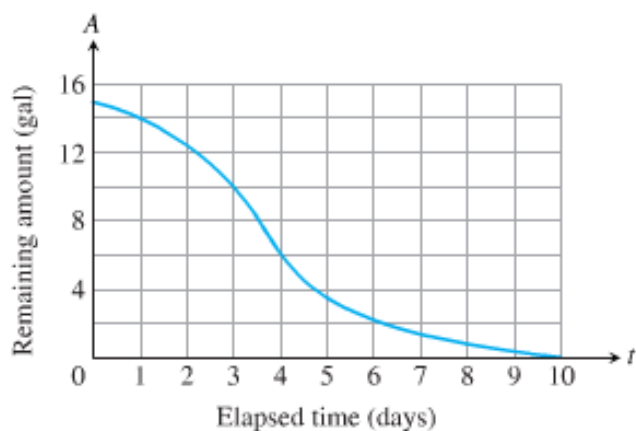
$$\frac{\Delta y}{\Delta x} = \frac{-3.004 - (-3)}{1.001 - 1} = -4.004$$

$$\frac{\Delta y}{\Delta x} = \frac{-3.0004 - (-3)}{1.0001 - 1} = -4.0004$$

b) The rate of change of  $f(x)$  at  $x = 1$  is -4

### Exercise

The accompanying graph shows the total amount of gasoline  $A$  in the gas tank of an automobile after being driven for  $t$  days.



- a) Estimate the average rate of gasoline consumption over the time intervals  $[0, 3]$ ,  $[0, 5]$ , and  $[7, 10]$
- b) Estimate the instantaneous rate of gasoline consumption over the time  $t = 1$ ,  $t = 4$ , and  $t = 8$

### Solution

- a) Average rate of gasoline consumption over the time intervals:

$[0, 3]$

$$\begin{aligned} \text{Average Rate} &= \frac{10 - 15}{3 - 0} \\ &= -\frac{5}{3} \\ &\approx -1.67 \text{ gal / day} \end{aligned}$$

$[0, 5]$

$$\begin{aligned} \text{Average Rate} &= \frac{3.9 - 15}{5 - 0} \\ &\approx -2.2 \text{ gal / day} \end{aligned}$$

$[7, 10]$

$$\begin{aligned} \text{Average Rate} &= \frac{0 - 1.4}{10 - 7} \\ &= -\frac{1.4}{3} \\ &= -\frac{7}{15} \\ &\approx -0.5 \text{ gal / day} \end{aligned}$$

- b) At  $t = 1 \rightarrow P(1, 14)$   
At  $t = 4 \rightarrow P(4, 6)$   
At  $t = 8 \rightarrow P(8, 1)$

## ***Solution***      **Section 1.2 – Definitions / Techniques of Limits**

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 3} (-1)$

### **Solution**

$$\lim_{x \rightarrow 3} (-1) = \underline{-1}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow -1} (3)$

### **Solution**

$$\lim_{x \rightarrow -1} (3) = \underline{3}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 1000} 18\pi^2$

### **Solution**

$$\lim_{x \rightarrow 1000} 18\pi^2 = \underline{18\pi^2}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 1} \sqrt{5x+6}$

### **Solution**

$$\lim_{x \rightarrow 1} \sqrt{5x+6} = \underline{\sqrt{11}}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 9} \sqrt{x}$

### **Solution**

$$\begin{aligned} \lim_{x \rightarrow 9} \sqrt{x} &= \sqrt{9} \\ &= \underline{3} \end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow -3} (x^2 + 3x)$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow -3} (x^2 + 3x) &= (-3)^2 + 3(-3) \\ &= 9 - 9 \\ &= \underline{0}\end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow -4} |x - 4|$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow -4} |x - 4| &= |-4 - 4| \\ &= |-8| \\ &= \underline{8}\end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 4} (x + 2)$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow 4} (x + 2) &= 4 + 2 \\ &= \underline{6}\end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 4} (x - 4)$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow 4} (x - 4) &= 4 - 4 \\ &= \underline{0}\end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 2} (5x - 6)^{3/2}$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow 2} (5x - 6)^{3/2} &= (10 - 6)^{3/2} \\ &= \sqrt{4^3} \\ &= 8\end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 9} \frac{x - 9}{\sqrt{x} - 3}$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow 9} \frac{x - 9}{\sqrt{x} - 3} &= \frac{9 - 9}{3 - 3} = \frac{0}{0} \\ &= \lim_{x \rightarrow 9} \frac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{\sqrt{x} - 3} \\ &= \lim_{x \rightarrow 9} (\sqrt{x} + 3) \\ &= 6\end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 1} (2x + 4)$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow 1} (2x + 4) &= 2(1) + 4 \\ &= 6\end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 1} \frac{x^2 - 4}{x - 2}$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^2 - 4}{x - 2} &= \frac{1^2 - 4}{1 - 2} \\ &= \frac{-3}{-1}\end{aligned}$$

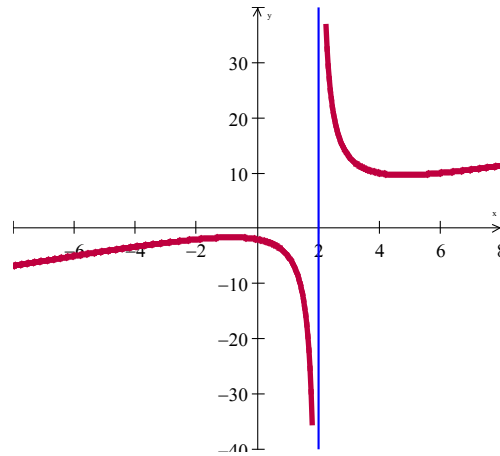
$$= 3$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 2} \frac{x^2 + 4}{x - 2}$

### Solution

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2 + 4}{x - 2} &= \frac{2^2 + 4}{2 - 2} \\ &= \frac{8}{0} \\ &= \infty \end{aligned} \quad (\text{Doesn't exist})$$



### Exercise

Find the limit:  $\lim_{x \rightarrow 0} \frac{|x|}{x}$

### Solution

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{|x|}{x} &= \frac{0}{0} \\ \lim_{x \rightarrow 0^-} \frac{|x|}{x} &= \frac{x}{-x} = -1 \\ \lim_{x \rightarrow 0^+} \frac{|x|}{x} &= \frac{x}{x} = 1 \end{aligned}$$

Doesn't exist

### Exercise

Find:  $\lim_{x \rightarrow 3} \frac{x^2 - x - 1}{\sqrt{x} + 1}$

### Solution

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{x^2 - x - 1}{\sqrt{x} + 1} &= \frac{3^2 - 3 - 1}{\sqrt{3} + 1} \\ &= \frac{5}{2} \end{aligned}$$

### Exercise

Find:  $\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2}$

### Solution



$$\begin{aligned}
 \lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2} &= \frac{2^2 + 2 - 6}{2 - 2} = \frac{0}{0} \\
 &= \lim_{x \rightarrow 2} \frac{(x + 3)(x - 2)}{x - 2} \\
 &= \lim_{x \rightarrow 2} (x + 3) \\
 &= 5
 \end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 0} (3x - 2)$

#### **Solution**

$$\begin{aligned}
 \lim_{x \rightarrow 0} (3x - 2) &= 3(0) - 2 \\
 &= -2
 \end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 1} (2x^2 - x + 4)$

#### **Solution**

$$\begin{aligned}
 \lim_{x \rightarrow 1} (2x^2 - x + 4) &= 2(1)^2 - (1) + 4 \\
 &= 5
 \end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow -2} (x^3 - 2x^2 + 4x + 8)$

#### **Solution**

$$\begin{aligned}
 \lim_{x \rightarrow -2} (x^3 - 2x^2 + 4x + 8) &= (-2)^3 - 2(-2)^2 + 4(-2) + 8 \\
 &= -16
 \end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$

#### **Solution**

$$\begin{aligned}
 \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} &= \frac{2^2 - 4}{2 - 2} = \frac{0}{0} \\
 &= \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} \\
 &= \lim_{x \rightarrow 2} (x + 2) \\
 &= 4
 \end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2}$

### **Solution**

$$\begin{aligned}
 \lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2} &= \frac{0}{0} \\
 &= \lim_{x \rightarrow 2} \frac{(x - 2)(x^2 + 2x + 4)}{x - 2} \\
 &= \lim_{x \rightarrow 2} x^2 + 2x + 4 \\
 &= 2^2 + 2(2) + 4 \\
 &= 12
 \end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 3} \frac{x^2 + x - 12}{x - 3}$

### **Solution**

$$\begin{aligned}
 \lim_{x \rightarrow 3} \frac{x^2 + x - 12}{x - 3} &= \frac{0}{0} \\
 &= \lim_{x \rightarrow 3} \frac{(x - 3)(x + 4)}{x - 3} \\
 &= \lim_{x \rightarrow 3} (x + 4) \\
 &= 7
 \end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 0} \frac{\sqrt{x+4}-2}{x}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{x+4}-2}{x} &= \frac{\sqrt{4}-2}{0} = \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{\sqrt{x+4}-2}{x} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} \\&= \lim_{x \rightarrow 0} \frac{x+4-4}{x(\sqrt{x+4}+2)} \\&= \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{x+4}+2)} \\&= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+4}+2} \\&= \frac{1}{\sqrt{4}+2} \\&= \frac{1}{4} \quad \left| \right.\end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 0} \frac{3}{\sqrt{3x+1}+1}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{3}{\sqrt{3x+1}+1} &= \frac{3}{\sqrt{3(0)+1}+1} \\&= \frac{3}{1+1} \\&= \frac{3}{2} \quad \left| \right.\end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 0} f(x)$   $f(x) = \begin{cases} x^2+1 & x < 0 \\ 2x+1 & x > 0 \end{cases}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow 0^-} x^2+1 &= 1 \\ \lim_{x \rightarrow 0^+} 2x+1 &= 1\end{aligned}$$

$$\lim_{x \rightarrow 0} f(x) = 1$$

### Exercise

Find the limit:  $\lim_{x \rightarrow -2} \frac{5}{x+2}$

#### Solution

$$\lim_{x \rightarrow -2} \frac{5}{x+2} = \frac{5}{0}$$

$$= \infty$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 3} \frac{\sqrt{x+1}-1}{x}$

#### Solution

$$\lim_{x \rightarrow 3} \frac{\sqrt{x+1}-1}{x} = \frac{\sqrt{3+1}-1}{3} = \frac{2-1}{3}$$

$$= \frac{1}{3}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 1} \frac{x^2-1}{x-1}$

#### Solution

$$\lim_{x \rightarrow 1} \frac{x^2-1}{x-1} = \frac{0}{0}$$

$$= \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x \rightarrow 1} (x+1)$$

$$= 2$$

### Exercise

Find the limit:  $\lim_{x \rightarrow -2} \frac{|x+2|}{x+2}$

### Solution

$$\lim_{x \rightarrow -2} \frac{|x+2|}{x+2} = \frac{|-2+2|}{-2+2} = \frac{0}{0}$$

$$\lim_{x \rightarrow -2^+} \frac{|x+2|}{x+2} = \frac{(x+2)}{(x+2)} \\ = 1$$

$$\lim_{x \rightarrow -2^-} \frac{|x+2|}{x+2} = \frac{(x+2)}{-(x+2)} \\ = -1$$

*Doesn't exist*

### Exercise

Find the limit:  $\lim_{x \rightarrow 0} (2x-8)^{1/3}$

### Solution

$$\lim_{x \rightarrow 0} (2x-8)^{1/3} = (2(0)-8)^{1/3} \\ = (-8)^{1/3} \\ = -2$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 2} \frac{x^2 - 7x + 10}{x - 2}$

### Solution

$$\lim_{x \rightarrow 2} \frac{x^2 - 7x + 10}{x - 2} = \frac{2^2 - 7(2) + 10}{2 - 2} = \frac{0}{0} \\ = \lim_{x \rightarrow 2} \frac{(x-2)(x-5)}{x-2} \\ = \lim_{x \rightarrow 2} (x-5) \\ = 2 - 5 \\ = -3$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 0} \frac{5x^3 + 8x^2}{3x^4 - 16x^2}$

#### Solution

$$\lim_{x \rightarrow 0} \frac{5x^3 + 8x^2}{3x^4 - 16x^2} = \frac{0}{0}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{5x^3 + 8x^2}{3x^4 - 16x^2} &= \lim_{x \rightarrow 0} \frac{x^2(5x + 8)}{x^2(3x^2 - 16)} \\ &= \lim_{x \rightarrow 0} \frac{5x + 8}{3x^2 - 16} \\ &= \frac{8}{-16} \\ &= -\frac{1}{2} \end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{x - 1}$

#### Solution

$$\lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{x - 1} = \frac{1 - 1}{1 - 1} = \frac{0}{0}$$

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{\frac{1 - x}{x}}{x - 1} \\ &= \lim_{x \rightarrow 1} \left( \frac{1 - x}{x} \right) \left( \frac{1}{x - 1} \right) \\ &= \lim_{x \rightarrow 1} \left( \frac{-(x - 1)}{x} \right) \left( \frac{1}{x - 1} \right) \\ &= \lim_{x \rightarrow 1} \frac{-1}{x} \\ &= -1 \end{aligned}$$

### Exercise

Find the limit:  $\lim_{u \rightarrow 1} \frac{u^4 - 1}{u^3 - 1}$

#### Solution

$$\begin{aligned}
\lim_{u \rightarrow 1} \frac{u^4 - 1}{u^3 - 1} &= \lim_{u \rightarrow 1} \frac{(u^2 - 1)(u^2 + 1)}{(u - 1)(u^2 + u + 1)} \\
&= \lim_{u \rightarrow 1} \frac{(u - 1)(u + 1)(u^2 + 1)}{(u - 1)(u^2 + u + 1)} \\
&= \lim_{u \rightarrow 1} \frac{(u + 1)(u^2 + 1)}{u^2 + u + 1} \\
&= \frac{(1 + 1)(1^2 + 1)}{1^2 + 1 + 1} \\
&= \frac{4}{3}
\end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 1} \frac{x - 1}{\sqrt{x + 3} - 2}$

### **Solution**

$$\begin{aligned}
\lim_{x \rightarrow 1} \frac{x - 1}{\sqrt{x + 3} - 2} &= \frac{1 - 1}{\sqrt{1 + 3} - 2} \\
&= \frac{0}{\sqrt{4} - 2} = \frac{0}{0} \\
&= \lim_{x \rightarrow 1} \frac{x - 1}{\sqrt{x + 3} - 2} \cdot \frac{\sqrt{x + 3} + 2}{\sqrt{x + 3} + 2} \\
&= \lim_{x \rightarrow 1} \frac{(x - 1)(\sqrt{x + 3} + 2)}{x + 3 - 4} \\
&= \lim_{x \rightarrow 1} \frac{(x - 1)(\sqrt{x + 3} + 2)}{x - 1} \\
&= \lim_{x \rightarrow 1} (\sqrt{x + 3} + 2) \\
&= \sqrt{1 + 3} + 2 \\
&= 4
\end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1} &= \frac{\sqrt{(-1)^2 + 8} - 3}{-1 + 1} = \frac{\sqrt{9} - 3}{0} = \frac{0}{0} \\&= \lim_{x \rightarrow -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1} \cdot \frac{\sqrt{x^2 + 8} + 3}{\sqrt{x^2 + 8} + 3} \\&= \lim_{x \rightarrow -1} \frac{x^2 + 8 - 9}{(x + 1)(\sqrt{x^2 + 8} + 3)} \\&= \lim_{x \rightarrow -1} \frac{x^2 - 1}{(x + 1)(\sqrt{x^2 + 8} + 3)} \\&= \lim_{x \rightarrow -1} \frac{(x - 1)(x + 1)}{(x + 1)(\sqrt{x^2 + 8} + 3)} \\&= \lim_{x \rightarrow -1} \frac{(x - 1)}{\sqrt{x^2 + 8} + 3} \\&= \frac{-2}{\sqrt{9} + 3} \\&= \frac{-2}{6} \\&= -\frac{1}{3}\end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow -3} \frac{2 - \sqrt{x^2 - 5}}{x + 3}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow -3} \frac{2 - \sqrt{x^2 - 5}}{x + 3} &= \frac{2 - \sqrt{(-3)^2 - 5}}{-3 + 3} \\&= \frac{2 - \sqrt{9 - 5}}{0} \\&= \frac{2 - \sqrt{4}}{0} = \frac{0}{0}\end{aligned}$$



$$\begin{aligned}
&= \lim_{x \rightarrow -3} \frac{2 - \sqrt{x^2 - 5}}{x + 3} \cdot \frac{2 + \sqrt{x^2 - 5}}{2 + \sqrt{x^2 - 5}} \\
&= \lim_{x \rightarrow -3} \frac{4 - (x^2 - 5)}{(x + 3)(2 + \sqrt{x^2 - 5})} \\
&= \lim_{x \rightarrow -3} \frac{4 - x^2 + 5}{(x + 3)(2 + \sqrt{x^2 - 5})} \\
&= \lim_{x \rightarrow -3} \frac{9 - x^2}{(x + 3)(2 + \sqrt{x^2 - 5})} \\
&= \lim_{x \rightarrow -3} \frac{(x - 3)(x + 3)}{(x + 3)(2 + \sqrt{x^2 - 5})} \\
&= \lim_{x \rightarrow -3} \frac{(x - 3)}{2 + \sqrt{x^2 - 5}} \\
&= \frac{-6}{2 + \sqrt{9 - 5}} \\
&= \frac{-6}{2 + \sqrt{4}} \\
&= -\frac{6}{4} \\
&= -\frac{3}{2}
\end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 0} (2 \sin x - 1)$

### **Solution**

$$\begin{aligned}
\lim_{x \rightarrow 0} (2 \sin x - 1) &= 2 \sin(0) - 1 \\
&= 0 - 1 \\
&= -1
\end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 0} \sin^2 x$

### **Solution**

$$\lim_{x \rightarrow 0} \sin^2 x = \sin^2(0)$$

$$\underline{= 0}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 0} \sec x$

#### **Solution**

$$\lim_{x \rightarrow 0} \sec x = \sec(0)$$

$$= \frac{1}{\cos(0)}$$

$$\underline{= 1}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 0} \frac{1+x+\sin x}{3\cos x}$

#### **Solution**

$$\lim_{x \rightarrow 0} \frac{1+x+\sin x}{3\cos x} = \frac{1+0+\sin(0)}{3\cos(0)}$$

$$\underline{= \frac{1}{3}}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow -\pi} \sqrt{x+4} \cos(x+\pi)$

#### **Solution**

$$\lim_{x \rightarrow -\pi} \sqrt{x+4} \cos(x+\pi) = \sqrt{-\pi+4} \cos(-\pi+\pi)$$

$$= \sqrt{-\pi+4} \cos(0)$$

$$\underline{= \sqrt{4-\pi}}$$

### ***Exercise***

Find  $\lim_{x \rightarrow -0.5^-} \sqrt{\frac{x+2}{x+1}}$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow -0.5^-} \sqrt{\frac{x+2}{x+1}} &= \sqrt{\frac{-0.5+2}{-0.5+1}} \\ &= \sqrt{\frac{1.5}{0.5}} \\ &= \sqrt{3} \quad | \end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 1^+} \sqrt{\frac{x-1}{x+2}}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 1^+} \sqrt{\frac{x-1}{x+2}} &= \sqrt{\frac{1-1}{1+2}} \\ &= 0 \quad | \end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow -2^+} \left( \frac{x}{x+1} \right) \left( \frac{2x+5}{x^2+x} \right)$

### Solution

$$\begin{aligned}\lim_{x \rightarrow -2^+} \left( \frac{x}{x+1} \right) \left( \frac{2x+5}{x^2+x} \right) &= \left( \frac{-2}{-2+1} \right) \left( \frac{2(-2)+5}{(-2)^2+(-2)} \right) \\ &= \left( \frac{-2}{-1} \right) \left( \frac{1}{2} \right) \\ &= 1 \quad | \end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x} &= \frac{\sqrt{5}-\sqrt{5}}{0} = \frac{0}{0} \\ &= \lim_{x \rightarrow 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x} \cdot \frac{\sqrt{x^2+4x+5}+\sqrt{5}}{\sqrt{x^2+4x+5}+\sqrt{5}} \end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 0^+} \frac{x^2 + 4x + 5 - 5}{x \left( \sqrt{x^2 + 4x + 5} + \sqrt{5} \right)} \\
&= \lim_{x \rightarrow 0^+} \frac{x^2 + 4x}{x \left( \sqrt{x^2 + 4x + 5} + \sqrt{5} \right)} \\
&= \lim_{x \rightarrow 0^+} \frac{x(x+4)}{x \left( \sqrt{x^2 + 4x + 5} + \sqrt{5} \right)} \\
&= \lim_{x \rightarrow 0^+} \frac{x+4}{\sqrt{x^2 + 4x + 5} + \sqrt{5}} \\
&= \frac{\textcolor{red}{0} + 4}{\sqrt{\textcolor{red}{0}^2 + 4(\textcolor{red}{0}) + 5} + \sqrt{5}} \\
&= \frac{4}{\sqrt{5} + \sqrt{5}} \\
&= \frac{4}{2\sqrt{5}} \\
&= \frac{\textcolor{blue}{2}}{\sqrt{5}} \Big|
\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow -2^+} (x+3) \frac{|x+2|}{x+2}$

### Solution

$$\lim_{x \rightarrow -2^+} (x+3) \frac{|x+2|}{x+2} = (x+3) \frac{\textcolor{red}{-2} + 2}{\textcolor{red}{-2} + 2} = \frac{\textcolor{red}{0}}{\textcolor{red}{0}}$$

$$\text{Since } x \rightarrow -2^+ \Rightarrow x > -2$$

$$\Rightarrow |x+2| = (x+2)$$

$$\begin{aligned}
\lim_{x \rightarrow -2^+} (x+3) \frac{|x+2|}{x+2} &= \lim_{x \rightarrow -2^+} (x+3) \frac{x+2}{x+2} \\
&= \lim_{x \rightarrow -2^+} (x+3) \\
&= -2 + 3 \\
&= \textcolor{blue}{1} \Big|
\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|}$

### Solution

$$\lim_{x \rightarrow 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|} = \frac{\sqrt{2(1)}(1-1)}{|1-1|} = \frac{0}{0}$$

$$\begin{aligned} \text{Since } x \rightarrow 1^+ &\Rightarrow x > 1 \\ &\Rightarrow |x-1| = x-1 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|} &= \lim_{x \rightarrow 1^+} \frac{\sqrt{2x}(x-1)}{x-1} \\ &= \lim_{x \rightarrow 1^+} \sqrt{2x} \\ &= \sqrt{2} \end{aligned}$$

### Exercise

Find  $\lim_{\theta \rightarrow 0} \frac{\sin \sqrt{2}\theta}{\sqrt{2}\theta}$

### Solution

Let:  $\sqrt{2}\theta = x \rightarrow 0$

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{\sin \sqrt{2}\theta}{\sqrt{2}\theta} &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \\ &= 1 \end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 0} \frac{\sin 3x}{4x}$

### Solution

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 3x}{4x} &= \lim_{x \rightarrow 0} \frac{\sin 3x}{4x} \cdot \frac{3}{3} \\ &= \frac{3}{4} \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \\ &= \frac{3}{4} \lim_{u \rightarrow 0} \frac{\sin u}{u} \\ &= \frac{3}{4} \end{aligned}$$

Let:  $3x = u$

*By definition:*  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

### Exercise

Find  $\lim_{x \rightarrow 0^-} \frac{x}{\sin 3x}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow 0^-} \frac{x}{\sin 3x} &= \lim_{x \rightarrow 0^-} \frac{x}{\sin 3x} \left( \frac{3}{3} \right) \\&= \frac{1}{3} \lim_{x \rightarrow 0^-} \frac{3x}{\sin 3x} \\&= \frac{1}{3} \lim_{x \rightarrow 0^-} \frac{1}{\frac{\sin 3x}{3x}} \\&= \frac{1}{3} \quad \left| \right.\end{aligned}$$

*By definition:*  $\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$

### Exercise

Find  $\lim_{x \rightarrow 0} \frac{\tan 2x}{x}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\tan 2x}{x} &= \lim_{x \rightarrow 0} \frac{\frac{\sin 2x}{\cos 2x}}{x} \\&= \lim_{x \rightarrow 0} \left( \frac{\sin 2x}{x} \cdot \frac{1}{\cos 2x} \right) \\&= \lim_{x \rightarrow 0} \left( 2 \frac{\sin 2x}{2x} \right) \lim_{x \rightarrow 0} \left( \frac{1}{\cos 2x} \right) \\&= 2 \frac{1}{\cos 0} \\&= 2 \quad \left| \right.\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 0} 6x^2 (\cot x) (\csc 2x)$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow 0} 6x^2 (\cot x) (\csc 2x) &= \lim_{x \rightarrow 0} 6x^2 \left( \frac{\cos x}{\sin x} \right) \left( \frac{1}{\sin 2x} \right) \\&= \lim_{x \rightarrow 0} 3 \cos x \left( \frac{x}{\sin x} \right) \left( \frac{2x}{\sin 2x} \right) \\&= 3 \lim_{x \rightarrow 0} (\cos x) \cdot \lim_{x \rightarrow 0} \left( \frac{x}{\sin x} \right) \cdot \lim_{2x \rightarrow 0} \left( \frac{2x}{\sin 2x} \right) \\&= (3)(1)(1)(1) \\&= 3 \quad \left| \right.\end{aligned}$$

### Exercise

Find  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta}$

### Solution

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta} &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta} \frac{2\theta}{2\theta} \\ &= \frac{1}{2} \lim_{\theta \rightarrow 0} \left( \frac{2\theta}{\sin 2\theta} \cdot \frac{\sin \theta}{\theta} \right) \\ &= \frac{1}{2} (1)(1) \\ &= \frac{1}{2} \end{aligned}$$

### Exercise

Find  $\lim_{h \rightarrow 0} \frac{\sin(\sin h)}{\sin h}$

### Solution

Let:  $\sin h = \theta \quad \theta = \sin h \xrightarrow{h \rightarrow 0} 0$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\sin(\sin h)}{\sin h} &= \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} \\ &= 1 \end{aligned}$$

### Exercise

Find  $\lim_{\theta \rightarrow 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta}$

### Solution

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta} &= \lim_{\theta \rightarrow 0} \frac{\theta \frac{\cos 4\theta}{\sin 4\theta}}{\sin^2 \theta \frac{\cos^2 2\theta}{\sin^2 2\theta}} \\ &= \lim_{\theta \rightarrow 0} \theta \frac{\cos 4\theta}{2 \sin 2\theta \cos 2\theta} \frac{\sin^2 2\theta}{\sin^2 \theta \cos^2 2\theta} \\ &= \lim_{\theta \rightarrow 0} \left( \frac{1}{2} \cdot \theta \cdot \cos 4\theta \cdot \frac{2 \sin \theta \cos \theta}{\sin^2 \theta} \cdot \frac{1}{\cos^3 2\theta} \right) \\ &= \lim_{\theta \rightarrow 0} \left( \cos 4\theta \cdot \frac{\theta}{\sin \theta} \cdot \cos \theta \cdot \frac{1}{\cos^3 2\theta} \right) \end{aligned}$$

$$\begin{aligned}
&= \lim_{\theta \rightarrow 0} (\cos 4\theta) \quad \lim_{\theta \rightarrow 0} \left( \frac{\theta}{\sin \theta} \right) \quad \lim_{\theta \rightarrow 0} \left( \frac{\cos \theta}{\cos^3 2\theta} \right) \\
&= (1)(1)(1) \\
&= 1
\end{aligned}$$

### Exercise

Find  $\lim_{\theta \rightarrow \pi/4} \frac{\sin^2 \theta - \cos^2 \theta}{\sin \theta - \cos \theta}$

### Solution

$$\begin{aligned}
\lim_{\theta \rightarrow \pi/4} \frac{\sin^2 \theta - \cos^2 \theta}{\sin \theta - \cos \theta} &= \frac{\frac{1}{2} - \frac{1}{2}}{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}} = \frac{0}{0} \\
&= \lim_{\theta \rightarrow \pi/4} \frac{(\sin \theta - \cos \theta)(\sin \theta + \cos \theta)}{\sin \theta - \cos \theta} \\
&= \lim_{\theta \rightarrow \pi/4} (\sin \theta + \cos \theta) \\
&= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \\
&= \sqrt{2}
\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow \pi/2} \frac{\frac{1}{\sqrt{\sin x}} - 1}{x + \frac{\pi}{2}}$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow \pi/2} \frac{\frac{1}{\sqrt{\sin x}} - 1}{x + \frac{\pi}{2}} &= \frac{1 - 1}{\frac{\pi}{2} + \frac{\pi}{2}} \\
&= 0
\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 1} \frac{x^3 - 7x^2 + 12x}{4 - x}$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow 1} \frac{x^3 - 7x^2 + 12x}{4 - x} &= \frac{1 - 7 + 12}{4 - 1} \\
&= 2
\end{aligned}$$



### Exercise

Find  $\lim_{x \rightarrow 4} \frac{x^3 - 7x^2 + 12x}{4 - x}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 4} \frac{x^3 - 7x^2 + 12x}{4 - x} &= \frac{64 - 112 + 48}{4 - 4} = \frac{0}{0} \\ &= \lim_{x \rightarrow 4} \frac{x(x-3)(x-4)}{4-x} \\ &= \lim_{x \rightarrow 4} -x(x-3) \\ &= \underline{-4}\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 1} \frac{1 - x^2}{x^2 - 8x + 7}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{1 - x^2}{x^2 - 8x + 7} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 1} \frac{(1-x)(1+x)}{(x-1)(x-7)} \\ &= - \lim_{x \rightarrow 1} \frac{1+x}{x-7} \\ &= \underline{\frac{1}{3}}\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 3} \frac{\sqrt{3x+16} - 5}{x-3}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{\sqrt{3x+16} - 5}{x-3} &= \frac{\sqrt{9+16} - 5}{3-3} = \frac{5-5}{0} = \frac{0}{0} \\ &= \lim_{x \rightarrow 3} \frac{\sqrt{3x+16} - 5}{x-3} \cdot \frac{\sqrt{3x+16} + 5}{\sqrt{3x+16} + 5} \\ &= \lim_{x \rightarrow 3} \frac{3x+16-25}{(x-3)(\sqrt{3x+16} + 5)} \\ &= \lim_{x \rightarrow 3} \frac{3(x-3)}{(x-3)(\sqrt{3x+16} + 5)}\end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 3} \frac{3}{\sqrt{3x+16}+5} \\
&= \frac{3}{5+5} \\
&= \frac{3}{10}
\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 3} \frac{1}{x-3} \left( \frac{1}{\sqrt{x+1}} - \frac{1}{2} \right)$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow 3} \frac{1}{x-3} \left( \frac{1}{\sqrt{x+1}} - \frac{1}{2} \right) &= \frac{1}{0} \left( \frac{1}{2} - \frac{1}{2} \right) = \frac{0}{0} \\
&= \lim_{x \rightarrow 3} \frac{1}{x-3} \left( \frac{2-\sqrt{x+1}}{\sqrt{x+1}} \right) \left( \frac{2+\sqrt{x+1}}{2+\sqrt{x+1}} \right) \\
&= \lim_{x \rightarrow 3} \frac{1}{x-3} \left( \frac{4-x-1}{2\sqrt{x+1}+x+1} \right) \\
&= \lim_{x \rightarrow 3} \frac{x-3}{x-3} \left( \frac{-1}{2\sqrt{x+1}+x+1} \right) \\
&= \lim_{x \rightarrow 3} \frac{-1}{2\sqrt{x+1}+x+1} \\
&= -\frac{1}{8}
\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 1/3} \frac{x - \frac{1}{3}}{(3x-1)^2}$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow 1/3} \frac{x - \frac{1}{3}}{(3x-1)^2} &= \frac{\frac{1}{3} - \frac{1}{3}}{\left(3\frac{1}{3} - 1\right)^2} = \frac{0}{0} \\
&= \lim_{x \rightarrow 1/3} \frac{x - \frac{1}{3}}{9\left(x - \frac{1}{3}\right)^2} \\
&= \lim_{x \rightarrow 1/3} \frac{1}{9\left(x - \frac{1}{3}\right)} = \frac{1}{0} \\
&= \infty
\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 3} \frac{x^4 - 81}{x - 3}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{x^4 - 81}{x - 3} &= \frac{81 - 81}{3 - 3} = \frac{0}{0} \\ &= \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)(x^2 + 9)}{x - 3} \quad a^4 - b^4 = (a^2 - b^2)(a^2 + b^2) = (a - b)(a + b)(a^2 + b^2) \\ &= \lim_{x \rightarrow 3} (x + 3)(x^2 + 9) = 6(18) \\ &= 108\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 1} \frac{x^5 - 1}{x - 1}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^5 - 1}{x - 1} &= \frac{1 - 1}{1 - 1} = \frac{0}{0} \quad (a^5 - b^5) = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4) \\ &= \lim_{x \rightarrow 1} \frac{(x - 1)(x^4 + x^3 + x^2 + x + 1)}{x - 1} \\ &= \lim_{x \rightarrow 1} (x^4 + x^3 + x^2 + x + 1) \\ &= 5\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 81} \frac{\sqrt[4]{x} - 3}{x - 81}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 81} \frac{\sqrt[4]{x} - 3}{x - 81} &= \frac{3 - 3}{81 - 81} = \frac{0}{0} \\ &= \lim_{x \rightarrow 81} \frac{\sqrt[4]{x} - 3}{(\sqrt{x} + 9)(\sqrt{x} - 9)} \\ &= \lim_{x \rightarrow 81} \frac{\sqrt[4]{x} - 3}{(\sqrt{x} + 9)(\sqrt[4]{x} + 3)(\sqrt[4]{x} - 3)}\end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 81} \frac{1}{(\sqrt{x} + 9)(\sqrt[4]{x} + 3)} \\
&= \frac{1}{(18)(6)} \\
&= \frac{1}{108}
\end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{x - 1}$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{x - 1} &= \frac{0}{0} \\
&= \lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{(\sqrt[3]{x})^3 - 1^3} \\
&= \lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{(\sqrt[3]{x} - 1)(x^{2/3} + \sqrt[3]{x} + 1)} \\
&= \lim_{x \rightarrow 1} \frac{1}{x^{2/3} + \sqrt[3]{x} + 1} \\
&= \frac{1}{3}
\end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 2} \frac{x^5 - 32}{x - 2}$

### Solution

$$\begin{array}{c|cccccc}
2 & 1 & 0 & 0 & 0 & 0 & -32 \\
& & 2 & 4 & 8 & 16 & 32 \\
\hline
& 1 & 2 & 4 & 8 & 16 & 0
\end{array}$$

$$\begin{aligned}
\lim_{x \rightarrow 2} \frac{x^5 - 32}{x - 2} &= \frac{2^5 - 32}{2 - 2} = \frac{0}{0} \\
&= \lim_{x \rightarrow 2} \frac{(x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16)}{x - 2} \\
&= \lim_{x \rightarrow 2} (x^4 + 2x^3 + 4x^2 + 8x + 16) \\
&= 16 + 16 + 16 + 16 + 16 \\
&= 80
\end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 1} \frac{x^6 - 1}{x - 1}$

#### Solution

$$\begin{array}{c|ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ & & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}$$

$$\lim_{x \rightarrow 1} \frac{x^6 - 1}{x - 1} = \frac{0}{0}$$

$$= \lim_{x \rightarrow 1} \frac{(x-1)(x^5 + x^4 + x^3 + x^2 + x + 1)}{x - 1}$$

$$= \lim_{x \rightarrow 1} (x^5 + x^4 + x^3 + x^2 + x + 1)$$

$$= 6$$

### Exercise

Find the limit:  $\lim_{x \rightarrow -1} \frac{x^7 + 1}{x + 1}$

#### Solution

$$\begin{array}{c|ccccccc} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ & & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 \end{array}$$

$$\lim_{x \rightarrow -1} \frac{x^7 + 1}{x + 1} = \frac{-1 + 1}{-1 + 1} = \frac{0}{0}$$

$$= \lim_{x \rightarrow -1} \frac{(x+1)(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)}{x + 1}$$

$$= \lim_{x \rightarrow -1} (x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)$$

$$= 1$$

### Exercise

Find the limit:  $\lim_{x \rightarrow a} \frac{x^5 - a^5}{x - a}$

#### Solution

$$\begin{array}{c|cccccc}
 a & 1 & 0 & 0 & 0 & 0 & -a^5 \\
 & & a & a^2 & a^3 & a^4 & a^5 \\
 \hline
 & 1 & a & a^2 & a^3 & a^4 & 0
 \end{array}$$

$$\begin{aligned}
 \lim_{x \rightarrow a} \frac{x^5 - a^5}{x - a} &= \frac{a^5 - a^5}{a - a} = \frac{0}{0} \\
 &= \lim_{x \rightarrow a} \frac{(x - a)(x^4 + ax^3 + a^2x^2 + a^3x + a^4)}{x - a} \\
 &= \lim_{x \rightarrow a} (x^4 + ax^3 + a^2x^2 + a^3x + a^4) \\
 &= a^4 + a^4 + a^4 + a^4 + a^4 \\
 &= 5a^4
 \end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} \quad n \in \mathbb{Z}^+$

### Solution

$$\begin{array}{c|ccccccc}
 a & 1 & 0 & 0 & 0 & \dots & 0 & -a^n \\
 & & a & a^2 & a^3 & \dots & a^{n-1} & a^n \\
 \hline
 & 1 & a & a^2 & a^3 & \dots & a^{n-1} & 0
 \end{array}$$

$$\begin{aligned}
 \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} &= \frac{a^n - a^n}{a - a} = \frac{0}{0} \\
 &= \lim_{x \rightarrow a} \frac{(x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-2}x + a^{n-1})}{x - a} \\
 &= \lim_{x \rightarrow a} (x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-2}x + a^{n-1}) \\
 &= a^{n-1} + a^{n-1} + a^{n-1} + \dots + a^{n-1} + a^{n-1} \\
 &= na^{n-1}
 \end{aligned}$$

### Exercise

Find the limit:  $\lim_{h \rightarrow 0} \frac{100}{(10h - 1)^{11} + 2}$

### Solution

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{100}{(10h-1)^{11} + 2} &= \frac{100}{(-1)^{11} + 2} \\ &= \frac{100}{-1 + 2} \\ &= 100\end{aligned}$$

### Exercise

Find the limit:  $\lim_{h \rightarrow 0} \frac{(5+h)^2 - 25}{h}$

#### Solution

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{(5+h)^2 - 25}{h} &= \frac{5^2 - 25}{0} = \frac{0}{0} \\ &= \lim_{h \rightarrow 0} \frac{((5+h)-5)((5+h)+5)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(h+10)}{h} \\ &= \lim_{h \rightarrow 0} (h+10) \\ &= 10\end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 3} \frac{\frac{1}{x^2 + 2x} - \frac{1}{15}}{x - 3}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{\frac{1}{x^2 + 2x} - \frac{1}{15}}{x - 3} &= \frac{\frac{1}{15} - \frac{1}{15}}{0} = \frac{0}{0} \\ &= \lim_{x \rightarrow 3} \frac{1}{x-3} \left( \frac{1}{x(x+2)} - \frac{1}{15} \right) \\ &= \lim_{x \rightarrow 3} \frac{1}{x-3} \left( \frac{15 - x^2 - 2x}{15x(x+2)} \right) \\ &= \lim_{x \rightarrow 3} \frac{-(x-3)(x+5)}{15x(x+2)(x-3)} \\ &= \lim_{x \rightarrow 3} \frac{-(x+5)}{15x(x+2)}\end{aligned}$$

$$= -\frac{8}{15(3)(5)}$$

$$= -\frac{8}{225}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 1} \frac{\sqrt{10x-9}-1}{x-1}$

### Solution

$$\lim_{x \rightarrow 1} \frac{\sqrt{10x-9}-1}{x-1} = \frac{1-1}{0} = \frac{0}{0}$$

$$= \lim_{x \rightarrow 1} \frac{\sqrt{10x-9}-1}{x-1} \cdot \frac{\sqrt{10x-9}+1}{\sqrt{10x-9}+1}$$

$$= \lim_{x \rightarrow 1} \frac{10x-9-1}{(x-1)(\sqrt{10x-9}+1)}$$

$$= \lim_{x \rightarrow 1} \frac{10(x-1)}{(x-1)(\sqrt{10x-9}+1)}$$

$$= \lim_{x \rightarrow 1} \frac{10}{\sqrt{10x-9}+1}$$

$$= \frac{10}{2}$$

$$= 5$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 2} \left( \frac{1}{x-2} - \frac{2}{x^2-2x} \right)$

### Solution

$$\lim_{x \rightarrow 2} \left( \frac{1}{x-2} - \frac{2}{x^2-2x} \right) = \frac{1}{0} - \frac{2}{0} = \infty - \infty$$

$$= \lim_{x \rightarrow 2} \left( \frac{1}{x-2} - \frac{2}{x(x-2)} \right)$$

$$= \lim_{x \rightarrow 2} \frac{x-2}{x(x-2)}$$

$$= \lim_{x \rightarrow 2} \frac{1}{x}$$

$$= \frac{1}{2}$$



### Exercise

Find the limit:  $\lim_{x \rightarrow c} \frac{x^2 - 2cx + c^2}{x - c}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow c} \frac{x^2 - 2cx + c^2}{x - c} &= \frac{c^2 - 2c^2 + c^2}{0} = \frac{0}{0} \\ &= \lim_{x \rightarrow c} \frac{(x - c)^2}{x - c} \\ &= \lim_{x \rightarrow c} (x - c) \\ &= 0\end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow -c} \frac{x^2 + 5cx + 4c^2}{x^2 + cx}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow -c} \frac{x^2 + 5cx + 4c^2}{x^2 + cx} &= \frac{c^2 - 5c^2 + 4c^2}{c^2 - c^2} = \frac{0}{0} \\ &= \lim_{x \rightarrow -c} \frac{(x + c)(x + 4c)}{x(x + c)} \\ &= \lim_{x \rightarrow -c} \frac{x + 4c}{x} \\ &= \frac{-c + 4c}{-c} \\ &= \frac{3c}{-c} \\ &= -3\end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 16} \frac{\sqrt[4]{x} - 2}{x - 16}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow 16} \frac{\sqrt[4]{x} - 2}{x - 16} &= \frac{\sqrt[4]{16} - 2}{16 - 16} = \frac{2 - 2}{0} = \frac{0}{0} \\ \lim_{x \rightarrow 16} \frac{\sqrt[4]{x} - 2}{(\sqrt[4]{x})^4 - 2^4}\end{aligned}$$

$$a^4 - b^4 = (a^2 + b^2)(a - b)(a + b)$$

$$\begin{aligned}
&= \lim_{x \rightarrow 16} \frac{\sqrt[4]{x} - 2}{(\sqrt{x} + 2^2)(\sqrt[4]{x} + 2)(\sqrt[4]{x} - 2)} \\
&= \lim_{x \rightarrow 16} \frac{1}{(\sqrt{x} + 4)(\sqrt[4]{x} + 2)} \\
&= \frac{1}{(\sqrt{16} + 4)(\sqrt[4]{16} + 2)} \\
&= \frac{1}{(4 + 4)(2 + 2)} \\
&= \frac{1}{(8)(4)} \\
&= \frac{1}{32}
\end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x}-1}$

#### **Solution**

$$\begin{aligned}
\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x}-1} &= \frac{0}{0} \\
&= \lim_{x \rightarrow 1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}-1} \\
&= \lim_{x \rightarrow 1} (\sqrt{x}+1) \\
&= 2
\end{aligned}$$

### ***Exercise***

Find the limit:  $\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{4x+5}-3}$

#### **Solution**

$$\begin{aligned}
\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{4x+5}-3} &= \frac{0}{0} \\
&= \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{4x+5}-3} \cdot \frac{\sqrt{4x+5}+3}{\sqrt{4x+5}+3} \\
&= \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{4x+5}+3)}{4x+5-9}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{4x+5}+3)}{4(x-1)} \\
&= \frac{1}{5} \lim_{x \rightarrow 1} (\sqrt{4x+5}+3) \\
&= \frac{6}{5} \quad |
\end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}} &= \frac{0}{3-3} = \frac{0}{0} \\
&= \lim_{x \rightarrow 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}} \cdot \frac{3+\sqrt{x+5}}{3+\sqrt{x+5}} \\
&= 3 \lim_{x \rightarrow 4} \frac{(x-4)(3+\sqrt{x+5})\sqrt{x+5}}{9-(x+5)} \\
&= 3 \lim_{x \rightarrow 4} \frac{(x-4)(3+\sqrt{x+5})\sqrt{x+5}}{4-x} \\
&= -3 \lim_{x \rightarrow 4} (3+\sqrt{x+5})\sqrt{x+5} \\
&= -3 (6)(3) \\
&= -54 \quad |
\end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 0} \frac{x}{\sqrt{ax+1}-1} \quad (a \neq 0)$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{x}{\sqrt{ax+1}-1} &= \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{x}{\sqrt{ax+1}-1} \cdot \frac{\sqrt{ax+1}+1}{\sqrt{ax+1}+1} \\
&= \lim_{x \rightarrow 0} \frac{x(\sqrt{ax+1}+1)}{ax+1-1} \\
&= \lim_{x \rightarrow 0} \frac{x(\sqrt{ax+1}+1)}{ax}
\end{aligned}$$

$$= \frac{1}{a} \lim_{x \rightarrow 0} (\sqrt{ax+1} + 1)$$

$$= \frac{2}{a} \quad |$$

### Exercise

Find the limit:  $\lim_{x \rightarrow \pi} \frac{\cos^2 x + 3 \cos x + 2}{\cos x + 1}$

### Solution

$$\lim_{x \rightarrow \pi} \frac{\cos^2 x + 3 \cos x + 2}{\cos x + 1} = \frac{1 - 3 + 2}{-1 + 1} = \frac{0}{0}$$

$$= \lim_{x \rightarrow \pi} \frac{(\cos x + 1)(\cos x + 2)}{\cos x + 1}$$

$$= \lim_{x \rightarrow \pi} (\cos x + 2)$$

$$= -1 + 2$$

$$= 1 \quad |$$

### Exercise

Find the limit:  $\lim_{x \rightarrow \frac{3\pi}{2}} \frac{\sin^2 x + 6 \sin x + 5}{\sin^2 x - 1}$

### Solution

$$\lim_{x \rightarrow \frac{3\pi}{2}} \frac{\sin^2 x + 6 \sin x + 5}{\sin^2 x - 1} = \frac{1 - 6 + 5}{1 - 1} = \frac{0}{0}$$

$$= \lim_{x \rightarrow \frac{3\pi}{2}} \frac{(\sin x + 1)(\sin x + 5)}{(\sin x - 1)(\sin x + 1)}$$

$$= \lim_{x \rightarrow \frac{3\pi}{2}} \frac{\sin x + 5}{\sin x - 1}$$

$$= \frac{-1 + 5}{-1 - 1}$$

$$= -2 \quad |$$

### Exercise

Find the limit:  $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x - 1}{\sqrt{\sin x} - 1}$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x - 1}{\sqrt{\sin x} - 1} &= \frac{1-1}{1-1} = \frac{0}{0} \\
&= \lim_{x \rightarrow \frac{\pi}{2}} \frac{(\sqrt{\sin x} - 1)(\sqrt{\sin x} + 1)}{\sqrt{\sin x} - 1} \\
&= \lim_{x \rightarrow \frac{\pi}{2}} (\sqrt{\sin x} + 1) \\
&= 2
\end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 0} \frac{\frac{1}{2 + \sin x} - \frac{1}{2}}{\sin x}$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{\frac{1}{2 + \sin x} - \frac{1}{2}}{\sin x} &= \frac{\frac{1}{2} - \frac{1}{2}}{0} = \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{1}{\sin x} \cdot \frac{2 - \sin x - 2}{2(2 + \sin x)} \\
&= \frac{1}{2} \lim_{x \rightarrow 0} \frac{1}{\sin x} \frac{-\sin x}{(2 + \sin x)} \\
&= -\frac{1}{2} \lim_{x \rightarrow 0} \frac{1}{2 + \sin x} \\
&= -\frac{1}{2} \left( \frac{1}{2} \right) \\
&= -\frac{1}{4}
\end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{e^x - 1}$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{e^x - 1} &= \frac{1-1}{1-1} = \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{(e^x - 1)(e^x + 1)}{e^x - 1} \\
&= \lim_{x \rightarrow 0} (e^x + 1) \\
&= 2
\end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow \frac{\pi}{4}} \csc x$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow \frac{\pi}{4}} \csc x &= \csc \frac{\pi}{4} \\ &= \frac{1}{\cos \frac{\pi}{4}} \\ &= \sqrt{2} \quad | \end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 4} \frac{x-5}{(x^2-10x+24)^2}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow 4} \frac{x-5}{(x^2-10x+24)^2} &= \frac{-1}{(16-41+24)^2} \\ &= -1 \quad | \end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin^2 x}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin^2 x} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{\cos x - 1}{(1 - \cos x)(1 + \cos x)} \\ &= - \lim_{x \rightarrow 0} \frac{1}{1 + \cos x} \\ &= -\frac{1}{2} \quad | \end{aligned}$$

### Exercise

Find the limit:  $\lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{\sin x}$

#### Solution

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{\sin x} &= \frac{0}{0} \\
 &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sin x} \\
 &= \lim_{x \rightarrow 0} \sin x \\
 &= 0
 \end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 0} \frac{x^3 - 5x^2}{x^2}$

### Solution

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{x^3 - 5x^2}{x^2} &= \frac{0}{0} \\
 &= \lim_{x \rightarrow 0} (x - 5) \\
 &= -5
 \end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 5} \frac{4x^2 - 100}{x - 5}$

### Solution

$$\begin{aligned}
 \lim_{x \rightarrow 5} \frac{4x^2 - 100}{x - 5} &= \frac{0}{0} \\
 &= \lim_{x \rightarrow 5} \frac{4(x - 5)(x + 5)}{x - 5} \\
 &= \lim_{x \rightarrow 5} 4(x + 5) \\
 &= 40
 \end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 3} \frac{\sqrt{9 - 6x + x^2}}{x - 3}$

### Solution

$$\lim_{x \rightarrow 3} \frac{\sqrt{9 - 6x + x^2}}{x - 3} = \frac{\sqrt{9 - 18 + 9}}{3 - 3} = \frac{0}{0}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 3} \frac{\sqrt{(x-3)^2}}{x-3} \\
&= \lim_{x \rightarrow 3} \frac{x-3}{x-3} \\
&= 1
\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 3} \frac{\sqrt{9+6x+x^2}}{x-3}$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow 3} \frac{\sqrt{9+6x+x^2}}{x-3} &= \frac{\sqrt{9+18+9}}{3-3} \\
&= \frac{\sqrt{36}}{0} \\
&= \infty
\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 3} \frac{\sqrt{x^2-9}}{x-3}$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow 3} \frac{\sqrt{x^2-9}}{x-3} &= \frac{\sqrt{9-9}}{3-3} = \frac{0}{0} \\
&= \lim_{x \rightarrow 3} \frac{\sqrt{(x-3)(x+3)}}{x-3} \\
&= \lim_{x \rightarrow 3} \sqrt{\frac{x+3}{x-3}} \\
&= \sqrt{\frac{6}{0}} \\
&= \infty
\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow \frac{4\pi}{3}} \sin x$

### Solution



$$\lim_{x \rightarrow \frac{4\pi}{3}} \sin x = \sin \frac{4\pi}{3}$$

$$= -\frac{\sqrt{3}}{2}$$

### Exercise

Find  $\lim_{x \rightarrow \frac{2\pi}{3}} \cos x$

### Solution

$$\lim_{x \rightarrow \frac{2\pi}{3}} \cos x = \cos \frac{2\pi}{3}$$

$$= -\frac{1}{2}$$

### Exercise

Find  $\lim_{x \rightarrow \frac{7\pi}{4}} \sin x$

### Solution

$$\lim_{x \rightarrow \frac{7\pi}{4}} \sin x = \sin \frac{7\pi}{4}$$

$$= -\frac{\sqrt{2}}{2}$$

### Exercise

Find  $\lim_{x \rightarrow 1} \frac{\sin \sqrt{1-x}}{\sqrt{1-x^2}}$

### Solution

$$\lim_{x \rightarrow 1} \frac{\sin \sqrt{1-x}}{\sqrt{1-x^2}} = \frac{\sin 0}{0} = \frac{0}{0}$$

$$= \lim_{x \rightarrow 1} \frac{\sin \sqrt{1-x}}{\sqrt{1-x}} \cdot \frac{1}{\sqrt{1+x}}$$

$$= \lim_{(1-x) \rightarrow 0} \frac{\sin \sqrt{1-x}}{\sqrt{1-x}} \lim_{x \rightarrow 1} \frac{1}{\sqrt{1+x}}$$

$$= 1 \left( \frac{1}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{2}}$$

### Exercise

Find  $\lim_{x \rightarrow 2} \frac{\sin \sqrt{2-x}}{\sqrt{4-x^2}}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{\sin \sqrt{2-x}}{\sqrt{4-x^2}} &= \frac{\sin 0}{0} = \frac{0}{0} \\&= \lim_{x \rightarrow 2} \frac{\sin \sqrt{2-x}}{\sqrt{2-x}} \cdot \frac{1}{\sqrt{2+x}} \\&= \lim_{\sqrt{2-x} \rightarrow 0} \frac{\sin \sqrt{2-x}}{\sqrt{2-x}} \quad \lim_{x \rightarrow 2} \frac{1}{\sqrt{2+x}} \\&= 1 \left( \frac{1}{2} \right) \\&= \frac{1}{2} \quad \left| \right.\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 0} \frac{\sin(\sqrt{5} x)}{\sin(\sqrt{3} x)}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin(\sqrt{5} x)}{\sin(\sqrt{3} x)} &= \frac{\sin 0}{\sin 0} = \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{\sqrt{5} x}{\sqrt{3} x} \cdot \frac{\sin(\sqrt{5} x)}{\sqrt{5} x} \cdot \frac{1}{\frac{\sin(\sqrt{3} x)}{\sqrt{3} x}} \\&= \frac{\sqrt{5}}{\sqrt{3}} \lim_{\sqrt{5} x \rightarrow 0} \frac{\sin(\sqrt{5} x)}{\sqrt{5} x} \cdot \frac{1}{\lim_{\sqrt{3} x \rightarrow 0} \frac{\sin(\sqrt{3} x)}{\sqrt{3} x}} \\&= \frac{\sqrt{5}}{\sqrt{3}} \quad \left| \right.\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 0} \frac{\sin(\sqrt{15} x)}{\sin(\sqrt{3} x)}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin(\sqrt{15} x)}{\sin(\sqrt{3} x)} &= \frac{\sin 0}{\sin 0} = \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{\sqrt{15} x}{\sqrt{3} x} \cdot \frac{\sin(\sqrt{3} x)}{\sqrt{15} x} \cdot \frac{1}{\frac{\sin(\sqrt{3} x)}{\sqrt{3} x}} \\&= \sqrt{\frac{15}{3}} \lim_{x \rightarrow 0} \frac{\sin(\sqrt{15} x)}{\sqrt{15} x} \cdot \frac{1}{\lim_{x \rightarrow 0} \frac{\sin(\sqrt{3} x)}{\sqrt{3} x}} \\&= \sqrt{3} \quad | \end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 0^+} \frac{x - \sqrt{x}}{\sqrt{\sin x}}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{x - \sqrt{x}}{\sqrt{\sin x}} &= \frac{0}{0} \\&= \lim_{x \rightarrow 0^+} \frac{x - \sqrt{x}}{\sqrt{\sin x}} \cdot \frac{1}{\sqrt{x}} \\&= \lim_{x \rightarrow 0^+} \frac{1}{\sqrt{\frac{\sin x}{x}}} \cdot \lim_{x \rightarrow 0^+} \frac{x - \sqrt{x}}{\sqrt{x}} \\&= (1) \lim_{x \rightarrow 0^+} \left( \frac{x}{\sqrt{x}} - \frac{\sqrt{x}}{\sqrt{x}} \right) \\&= \lim_{x \rightarrow 0^+} (\sqrt{x} - 1) \\&= -1 \quad | \end{aligned}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 1} \frac{x - \sqrt{x}}{\sqrt{\sin x}}$

### **Solution**

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x - \sqrt{x}}{\sqrt{\sin x}} &= \frac{0}{\sqrt{\sin 1}} \\ &= 0\end{aligned}$$

### ***Exercise***

Find  $\lim_{x \rightarrow \pi} \frac{x - \sqrt{x}}{\sqrt{\sin x}}$

### **Solution**

$$\begin{aligned}\lim_{x \rightarrow \pi} \frac{x - \sqrt{x}}{\sqrt{\sin x}} &= \frac{\pi - \sqrt{\pi}}{\sqrt{\sin \pi}} \\ &= \frac{\pi - \sqrt{\pi}}{0} \\ &= \infty\end{aligned}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 0} e^{x^3}$

### **Solution**

$$\begin{aligned}\lim_{x \rightarrow 0} e^{x^3} &= e^0 \\ &= 1\end{aligned}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 1} e^{x^2}$

### **Solution**

$$\begin{aligned}\lim_{x \rightarrow 1} e^{x^2} &= e^1 \\ &= e\end{aligned}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 1} e^{x^3-1}$

### **Solution**

$$\begin{aligned}\lim_{x \rightarrow 1} e^{x^3-1} &= e^{1-1} \\ &= e^0 \\ &= \underline{1} \end{aligned}$$

### ***Exercise***

Find  $\lim_{x \rightarrow -1} e^{x^3-1}$

### **Solution**

$$\begin{aligned}\lim_{x \rightarrow -1} e^{x^3-1} &= e^{-1-1} \\ &= e^{-2} \\ &= \underline{\frac{1}{e^2}} \end{aligned}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 2} \left( e^{x^2} - \ln x \right)$

### **Solution**

$$\lim_{x \rightarrow 2} \left( e^{x^2} - \ln x \right) = \underline{e^4 - \ln 2}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 1} \left( e^{x^2} - \ln x \right)$

### **Solution**

$$\begin{aligned}\lim_{x \rightarrow 1} \left( e^{x^2} - \ln x \right) &= e - \ln 1 \\ &= \underline{e} \end{aligned}$$

### ***Exercise***

Find  $\lim_{x \rightarrow e} \ln x$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow e} \ln x &= \ln e \\ &= 1\end{aligned}$$

### ***Exercise***

Find  $\lim_{x \rightarrow e} \ln x^2$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow e} \ln x^2 &= \ln e^2 \\ &= 2 \ln e \\ &= 2\end{aligned}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 0^+} \ln x$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow 0^+} \ln x &= \ln 0^+ \\ &= -\infty\end{aligned}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 1} \frac{1}{\ln x}$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{1}{\ln x} &= \frac{1}{\ln 1} \\ &= \frac{1}{0} \\ &= \infty\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow e} \ln e^{2x}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow e} \ln e^{2x} &= \ln e^{2e} \\ &= 2e \ln e \\ &= \underline{2e}\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow 1} \ln e^{x^2}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow 1} \ln e^{x^2} &= \ln e \\ &= \underline{1}\end{aligned}$$

### Exercise

For the function  $f(t)$  graphed, find the following limits or explain why they do not exist.

$$a) \lim_{t \rightarrow -2} f(t) \quad b) \lim_{t \rightarrow -1} f(t) \quad c) \lim_{t \rightarrow 0} f(t) \quad d) \lim_{t \rightarrow -0.5} f(t)$$

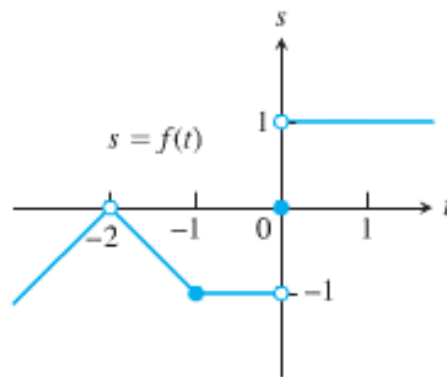
#### Solution

$$a) \lim_{t \rightarrow -2} f(t) = \underline{0}$$

$$b) \lim_{t \rightarrow -1} f(t) = \underline{-1}$$

$$c) \lim_{t \rightarrow 0} f(t) = \text{doesn't exist}$$

$$d) \lim_{t \rightarrow -0.5} f(t) = \underline{-1}$$



### Exercise

Suppose  $\lim_{x \rightarrow c} f(x) = 5$  and  $\lim_{x \rightarrow c} g(x) = -2$ . Find

a)  $\lim_{x \rightarrow c} f(x)g(x)$

b)  $\lim_{x \rightarrow c} 2f(x)g(x)$

c)  $\lim_{x \rightarrow c} (f(x) + 3g(x))$

d)  $\lim_{x \rightarrow c} \frac{f(x)}{f(x) - g(x)}$

### Solution

$$\begin{aligned} \text{a) } \lim_{x \rightarrow c} f(x)g(x) &= \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x) \\ &= (5)(-2) \\ &= -10 \end{aligned}$$

$$\begin{aligned} \text{b) } \lim_{x \rightarrow c} 2f(x)g(x) &= 2 \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x) \\ &= 2(-10) \\ &= -20 \end{aligned}$$

$$\begin{aligned} \text{c) } \lim_{x \rightarrow c} (f(x) + 3g(x)) &= \lim_{x \rightarrow c} f(x) + 3 \lim_{x \rightarrow c} g(x) \\ &= 5 + 3(-2) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \text{d) } \lim_{x \rightarrow c} \frac{f(x)}{f(x) - g(x)} &= \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} g(x)} \\ &= \frac{5}{5 - (-2)} \\ &= \frac{5}{7} \end{aligned}$$

### Exercise

Explain why the limits do not exist for  $\lim_{x \rightarrow 0} \frac{x}{|x|}$

### Solution

$$\lim_{x \rightarrow 0} \frac{x}{|x|} = \frac{0}{0}$$

$$\lim_{x \rightarrow 0^-} \frac{x}{|x|} = \frac{-x}{x} = -1$$

$$\lim_{x \rightarrow 0^+} \frac{x}{|x|} = \frac{x}{x} = 1$$

*Doesn't exist*



### Exercise

Evaluate the limit using the form  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$  for  $f(x) = x^2$ ,  $x = 1$

### Solution

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\&= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\&= \lim_{h \rightarrow 0} \left( \frac{2xh}{h} + \frac{h^2}{h} \right) \\&= \lim_{h \rightarrow 0} (2x + h) \\&= 2x \quad | \end{aligned}$$

### Exercise

Evaluate the limit using the form  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$  for  $f(x) = \sqrt{3x+1}$ ,  $x = 0$

### Solution

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h} \\&= \lim_{h \rightarrow 0} \frac{\sqrt{3x+3h+1} - \sqrt{3x+1}}{h} \\&= \lim_{h \rightarrow 0} \frac{\sqrt{3x+3h+1} - \sqrt{3x+1}}{h} \cdot \frac{\sqrt{3x+3h+1} + \sqrt{3x+1}}{\sqrt{3x+3h+1} + \sqrt{3x+1}} \\&= \lim_{h \rightarrow 0} \frac{3x+3h+1 - (3x+1)}{h(\sqrt{3x+3h+1} + \sqrt{3x+1})} \\&= \lim_{h \rightarrow 0} \frac{3x+3h+1-3x-1}{h(\sqrt{3x+3h+1} + \sqrt{3x+1})} \\&= \lim_{h \rightarrow 0} \frac{3h}{h(\sqrt{3x+3h+1} + \sqrt{3x+1})} \\&= \lim_{h \rightarrow 0} \frac{3}{\sqrt{3x+3h+1} + \sqrt{3x+1}} \\&= \frac{3}{\sqrt{3(0)+1} + \sqrt{3(0)+1}} \quad \text{Given : } x = 0 \\&= \frac{3}{2} \quad | \end{aligned}$$

### Exercise

If  $\lim_{x \rightarrow 4} \frac{f(x) - 5}{x - 2} = 1$ , find  $\lim_{x \rightarrow 4} f(x)$

### Solution

$$\lim_{x \rightarrow 4} \frac{f(x) - 5}{x - 2} = 1$$

$$\frac{\lim_{x \rightarrow 4} f(x) - 5}{4 - 2} = 1$$

$$\frac{\lim_{x \rightarrow 4} f(x) - 5}{2} = 1$$

*Multiply both sides by 2*

$$\lim_{x \rightarrow 4} f(x) - 5 = 2$$

*Add 5 on both sides*

$$\lim_{x \rightarrow 4} f(x) = 7$$

### Exercise

If  $\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 1$ , find  $\lim_{x \rightarrow 0} f(x)$  and  $\lim_{x \rightarrow 0} \frac{f(x)}{x}$

### Solution

$$\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 1$$

$$\frac{\lim_{x \rightarrow 0} f(x)}{\lim_{x \rightarrow 0} x^2} = 1$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x^2$$
$$\underline{= 0}$$

$$\lim_{x \rightarrow 0} \frac{f(x)}{x} = \lim_{x \rightarrow 0} \left( \frac{f(x)}{x^2} \cdot x \right)$$

$$= \lim_{x \rightarrow 0} \frac{f(x)}{x^2} \cdot \lim_{x \rightarrow 0} x$$

$$= 1 \cdot 0$$

$$\underline{= 0}$$

### Exercise

If  $x^4 \leq f(x) \leq x^2$ ;  $-1 \leq x \leq 1$  and  $x^2 \leq f(x) \leq x^4$ ;  $x < -1$  and  $x > 1$ . At what points  $c$  do you automatically know  $\lim_{x \rightarrow c} f(x)$ ? What can you say about the value of the limits at these points?

### Solution

$$\lim_{x \rightarrow c} x^4 = \lim_{x \rightarrow c} x^2 \Rightarrow c^4 = c^2$$

$$c^4 - c^2 = 0$$

$$c^2(c^2 - 1) = 0$$

$$c^2 = 0$$

$$c^2 - 1 = 0$$

$$\boxed{c = 0}$$

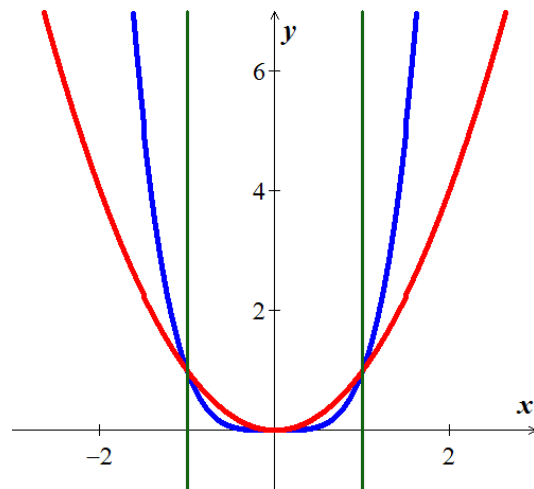
$$\boxed{c = \pm 1}$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x^2$$

$$\underline{= 0}$$

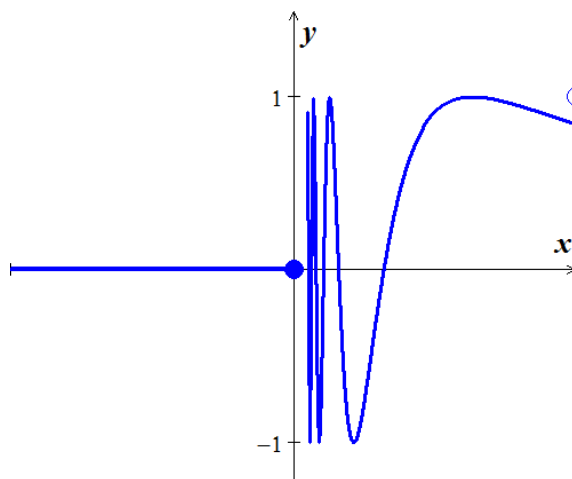
$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow 1} f(x)$$

$$\underline{= 1}$$



### Exercise

$$\text{Let } f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0 \end{cases}$$



a) Does  $\lim_{x \rightarrow 0^+} f(x)$  exist? If so, what is it? If not, why not?

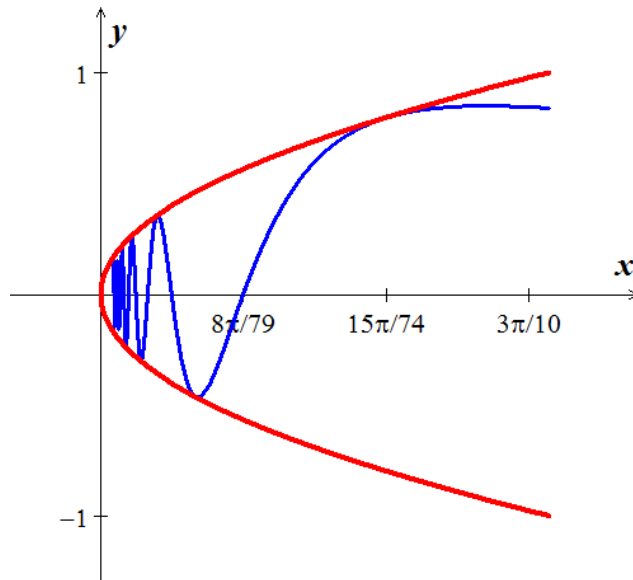
- b) Does  $\lim_{x \rightarrow 0^-} f(x)$  exist? If so, what is it? If not, why not?
- c) Does  $\lim_{x \rightarrow 0} f(x)$  exist? If so, what is it? If not, why not?

### Solution

- a)  $\lim_{x \rightarrow 0^+} f(x)$  doesn't exist, since  $\sin\left(\frac{1}{x}\right)$  doesn't approach any single value as  $x \rightarrow 0$
- b)  $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} 0 = 0$
- c)  $\lim_{x \rightarrow 0} f(x)$  doesn't exist, since  $\lim_{x \rightarrow 0^+} f(x)$  doesn't exist

### **Exercise**

Let  $g(x) = \sqrt{x} \sin \frac{1}{x}$



- a) Does  $\lim_{x \rightarrow 0^+} g(x)$  exist? If so, what is it? If not, why not?
- b) Does  $\lim_{x \rightarrow 0^-} g(x)$  exist? If so, what is it? If not, why not?
- c) Does  $\lim_{x \rightarrow 0} g(x)$  exist? If so, what is it? If not, why not?

### Solution

- a)  $\lim_{x \rightarrow 0^+} g(x)$  exists, by the sandwich theorem  $-\sqrt{x} \leq g(x) \leq \sqrt{x}$ . for  $x > 0$
- b)  $\lim_{x \rightarrow 0^-} g(x)$  doesn't exist, since  $\sqrt{x}$  is not defined for  $x < 0$

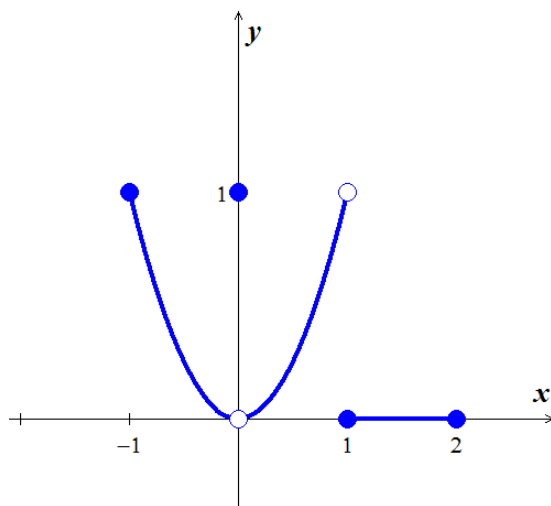
- c)  $\lim_{x \rightarrow 0} g(x)$  doesn't exist, since  $\lim_{x \rightarrow 0^-} g(x)$  doesn't exist.

### Exercise

Which of the following statements about the function  $y = f(x)$  graphed here are true, and which are false?

#### Solution

- a)  $\lim_{x \rightarrow -1^+} f(x) = 1$  **True**
- b)  $\lim_{x \rightarrow 0^-} f(x) = 0$  **True**
- c)  $\lim_{x \rightarrow 0^-} f(x) = 1$  **False**
- d)  $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$  **True**
- e)  $\lim_{x \rightarrow 0} f(x)$  exists **True**
- f)  $\lim_{x \rightarrow 0} f(x) = 0$  **True**
- g)  $\lim_{x \rightarrow 0} f(x) = 1$  **False**
- h)  $\lim_{x \rightarrow 1} f(x) = 1$  **False**
- i)  $\lim_{x \rightarrow 1} f(x) = 0$  **False**
- j)  $\lim_{x \rightarrow 2^-} f(x) = 2$  **False**
- k)  $\lim_{x \rightarrow -1^-} f(x) = 0$  does not exist **True**
- l)  $\lim_{x \rightarrow 2^+} f(x) = 0$  **False**



## ***Solution***      ***Section 1.3 – Infinite Limits***

### ***Exercise***

Find  $\lim_{x \rightarrow 5} \frac{x-7}{x(x-5)^2}$

### **Solution**

$$\lim_{x \rightarrow 5} \frac{x-7}{x(x-5)^2} = \frac{-2}{0} \\ = \infty$$

### ***Exercise***

Find  $\lim_{x \rightarrow -5^+} \frac{x-5}{x+5}$

### **Solution**

$$\lim_{x \rightarrow -5^+} \frac{x-5}{x+5} = \frac{-10}{0^+} \\ = -\infty$$

### ***Exercise***

Find  $\lim_{x \rightarrow 3^-} \frac{x-4}{x^2-3x}$

### **Solution**

$$\lim_{x \rightarrow 3^-} \frac{x-4}{x^2-3x} = \frac{-1}{0^-} \\ = \infty$$

### ***Exercise***

Find  $\lim_{x \rightarrow 0^+} \frac{1}{3x}$

### **Solution**

$$\lim_{x \rightarrow 0^+} \frac{1}{3x} = \frac{1}{0^+} \\ = \infty$$

### ***Exercise***

Find  $\lim_{x \rightarrow -5^-} \frac{3x}{2x+10}$

#### **Solution**

$$\lim_{x \rightarrow -5^-} \frac{3x}{2x+10} = \lim_{x \rightarrow -5^-} \frac{3}{2 + \frac{10}{x}} \\ = \underline{\underline{\infty}}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 0} \frac{1}{x^{2/3}}$

#### **Solution**

$$\lim_{x \rightarrow 0} \frac{1}{x^{2/3}} = \lim_{x \rightarrow 0} \frac{1}{\left(x^{1/3}\right)^2} \\ = \underline{\underline{\infty}}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 0^-} \frac{1}{3x^{1/3}}$

#### **Solution**

$$\lim_{x \rightarrow 0^-} \frac{1}{3x^{1/3}} = \frac{1}{0^-} \\ = \underline{\underline{-\infty}}$$

### ***Exercise***

Find  $\lim_{x \rightarrow \left(-\frac{\pi}{2}\right)^+} \sec x$

#### **Solution**

$$\lim_{x \rightarrow \left(-\frac{\pi}{2}\right)^+} \sec x = \underline{\underline{\infty}}$$

### ***Exercise***

Find  $\lim_{\theta \rightarrow 0^-} (1 + \csc \theta)$

#### **Solution**

$$\lim_{\theta \rightarrow 0^-} (1 + \csc \theta) = \lim_{\theta \rightarrow 0^-} \left(1 + \frac{1}{\sin \theta}\right)$$

$$\underline{= -\infty}$$

### Exercise

Find  $\lim_{\theta \rightarrow 0^+} \csc \theta$

#### Solution

$$\lim_{\theta \rightarrow 0^+} \csc \theta = \lim_{\theta \rightarrow 0^+} \frac{1}{\sin \theta}$$

$$\underline{= +\infty}$$

As  $\theta \rightarrow 0^+$   $\sin \theta > 0$

### Exercise

Find  $\lim_{x \rightarrow 0^+} (-10 \cot x)$

#### Solution

$$\lim_{x \rightarrow 0^+} (-10 \cot x) = -10 \lim_{x \rightarrow 0^+} \frac{\cos \theta}{\sin \theta} = -10 \left(\frac{1}{0}\right)$$

$$\underline{= -\infty}$$

As  $x \rightarrow 0^+$   $\cos \theta > 0$ ;  $\sin \theta > 0$

### Exercise

Find  $\lim_{\theta \rightarrow \frac{\pi}{2}^+} \frac{1}{3} \tan \theta$

#### Solution

$$\lim_{\theta \rightarrow \frac{\pi}{2}^+} \frac{1}{3} \tan \theta = \frac{1}{3} \lim_{\theta \rightarrow \frac{\pi}{2}^+} \frac{\sin \theta}{\cos \theta} = \frac{1}{3} \left(-\frac{1}{0}\right)$$

$$\underline{= -\infty}$$

As  $\theta \rightarrow \frac{\pi}{2}^+$   $\cos \theta < 0$ ;  $\sin \theta > 0$

### Exercise

Find  $\lim_{x \rightarrow 2^+} \frac{1}{x-2}$

#### Solution

$$\lim_{x \rightarrow 2^+} \frac{1}{x-2} = \frac{1}{2^+ - 2} = \frac{1}{0^+}$$

$$\underline{= \infty}$$



### ***Exercise***

Find  $\lim_{x \rightarrow 2^-} \frac{1}{x-2}$

#### **Solution**

$$\lim_{x \rightarrow 2^-} \frac{1}{x-2} = \frac{1}{2^- - 2} = \frac{1}{0^-} \\ \underline{\underline{= -\infty}}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 2} \frac{1}{x-2}$

#### **Solution**

$$\lim_{x \rightarrow 2} \frac{1}{x-2} = \frac{1}{0} \\ \underline{\underline{= \infty}}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 3^+} \frac{2}{(x-3)^3}$

#### **Solution**

$$\lim_{x \rightarrow 3^+} \frac{2}{(x-3)^3} = \frac{2}{0^+} \\ \underline{\underline{= \infty}}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 3^-} \frac{2}{(x-3)^3}$

#### **Solution**

$$\lim_{x \rightarrow 3^-} \frac{2}{(x-3)^3} = \frac{2}{0^-} \\ \underline{\underline{= -\infty}}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 3} \frac{2}{(x-3)^3}$

#### **Solution**

$$\lim_{x \rightarrow 3} \frac{2}{(x-3)^3} = \frac{2}{0}$$

$$\underline{\underline{= \infty}}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 4^+} \frac{x-5}{(x-4)^2}$

#### **Solution**

$$\lim_{x \rightarrow 4^+} \frac{x-5}{(x-4)^2} = \frac{-1}{0}$$

$$\underline{\underline{= -\infty}}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 4^-} \frac{x-5}{(x-4)^2}$

#### **Solution**

$$\lim_{x \rightarrow 4^-} \frac{x-5}{(x-4)^2} = \frac{-1}{0}$$

$$\underline{\underline{= -\infty}}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 4} \frac{x-5}{(x-4)^2}$

#### **Solution**

$$\lim_{x \rightarrow 4^-} \frac{x-5}{(x-4)^2} = \frac{-1}{0}$$

$$\underline{\underline{= -\infty}}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 1^+} \frac{x-2}{(x-1)^3}$

#### **Solution**

$$\lim_{x \rightarrow 1^+} \frac{x-2}{(x-1)^3} = \frac{-1}{0^+}$$

$$\underline{\underline{= -\infty}}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 1^-} \frac{x-2}{(x-1)^3}$

### **Solution**

$$\lim_{x \rightarrow 1^-} \frac{x-2}{(x-1)^3} = \frac{-1}{0^-}$$
$$\underline{= \infty}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 1} \frac{x-2}{(x-1)^3}$

### **Solution**

$$\lim_{x \rightarrow 1} \frac{x-2}{(x-1)^3} = \frac{-1}{0^+}$$
$$\underline{= \cancel{\infty}}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 3^+} \frac{(x-1)(x-2)}{x-3}$

### **Solution**

$$\lim_{x \rightarrow 3^+} \frac{(x-1)(x-2)}{x-3} = \frac{2}{0}$$
$$\underline{= \infty}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 3^-} \frac{(x-1)(x-2)}{x-3}$

### **Solution**

$$\lim_{x \rightarrow 3^-} \frac{(x-1)(x-2)}{x-3} = \frac{2}{0^-}$$
$$\underline{= -\infty}$$

### Exercise

Find  $\lim_{x \rightarrow 3} \frac{(x-1)(x-2)}{x-3}$

#### Solution

$$\lim_{x \rightarrow 3^-} \frac{(x-1)(x-2)}{x-3} = \frac{2}{0^-} \\ = -\infty$$

$$\lim_{x \rightarrow 3^+} \frac{(x-1)(x-2)}{x-3} = \frac{2}{0^+} \\ = \infty$$

$$\lim_{x \rightarrow 3} \frac{(x-1)(x-2)}{x-3} = \text{DNE}$$

### Exercise

Find  $\lim_{x \rightarrow -2^+} \frac{x-4}{x(x+2)}$

#### Solution

$$\lim_{x \rightarrow -2^+} \frac{x-4}{x(x+2)} = \frac{-6}{0^+} \\ = \infty$$

### Exercise

Find  $\lim_{x \rightarrow -2^-} \frac{x-4}{x(x+2)}$

#### Solution

$$\lim_{x \rightarrow -2^-} \frac{x-4}{x(x+2)} = \frac{-6}{0^-} \\ = -\infty$$

### Exercise

Find  $\lim_{x \rightarrow -2} \frac{x-4}{x(x+2)}$

#### Solution

$$\lim_{x \rightarrow -2} \frac{x-4}{x(x+2)} = \infty$$

$$\lim_{x \rightarrow -2^-} \frac{x-4}{x(x+2)} = -\infty$$

$$\lim_{x \rightarrow -2} \frac{x-4}{x(x+2)} = \underline{\underline{\text{DNE}}}$$

### Exercise

Find  $\lim_{x \rightarrow 2^+} \frac{x^2 - 4x + 3}{(x-2)^2}$

### Solution

$$\lim_{x \rightarrow 2^+} \frac{x^2 - 4x + 3}{(x-2)^2} = \frac{-1}{0^+} = \underline{\underline{-\infty}}$$

### Exercise

Find  $\lim_{x \rightarrow 2^-} \frac{x^2 - 4x + 3}{(x-2)^2}$

### Solution

$$\lim_{x \rightarrow 2^-} \frac{x^2 - 4x + 3}{(x-2)^2} = \frac{-1}{0^+} = \underline{\underline{-\infty}}$$

### Exercise

Find  $\lim_{x \rightarrow 2} \frac{x^2 - 4x + 3}{(x-2)^2}$

### Solution

$$\lim_{x \rightarrow 2} \frac{x^2 - 4x + 3}{(x-2)^2} = \frac{-1}{0} = \underline{\underline{-\infty}}$$

### Exercise

Find  $\lim_{x \rightarrow -2^+} \frac{x^3 - 5x^2 + 6x}{x^4 - 4x^2}$

### Solution

$$\lim_{x \rightarrow -2^+} \frac{x^3 - 5x^2 + 6x}{x^4 - 4x^2} = \lim_{x \rightarrow -2^+} \frac{x(x-2)(x-3)}{x^2(x-2)(x+2)}$$

$$= \lim_{x \rightarrow -2^+} \frac{x-3}{x(x+2)} \quad \frac{-}{-(+)} \\ = \infty$$

### Exercise

Find  $\lim_{x \rightarrow -2^-} \frac{x^3 - 5x^2 + 6x}{x^4 - 4x^2}$

#### Solution

$$\lim_{x \rightarrow -2^-} \frac{x^3 - 5x^2 + 6x}{x^4 - 4x^2} = \lim_{x \rightarrow -2^-} \frac{x(x-2)(x-3)}{x^2(x-2)(x+2)} \\ = \lim_{x \rightarrow -2^-} \frac{x-3}{x(x+2)} \quad \frac{-}{-(-)} \\ = -\infty$$

### Exercise

Find  $\lim_{x \rightarrow -2} \frac{x^3 - 5x^2 + 6x}{x^4 - 4x^2}$

#### Solution

$$\lim_{x \rightarrow -2} \frac{x^3 - 5x^2 + 6x}{x^4 - 4x^2} = \frac{-8 - 20 - 12}{16 - 16} \\ = \frac{-40}{0} \\ = -\infty$$

### Exercise

Find  $\lim_{u \rightarrow 0^+} \frac{u-1}{\sin u}$

#### Solution

$$\lim_{u \rightarrow 0^+} \frac{u-1}{\sin u} = \frac{-1}{0^+} \\ = -\infty$$

### Exercise

Find  $\lim_{x \rightarrow 0^-} \frac{2}{\tan x}$

#### Solution

$$\lim_{x \rightarrow 0^-} \frac{2}{\tan x} = \frac{2}{0^-}$$

$$\underline{= -\infty}$$

### Exercise

Find  $\lim_{x \rightarrow 1^+} \frac{x^2 - 5x + 6}{x - 1}$

### Solution

$$\lim_{x \rightarrow 1^+} \frac{x^2 - 5x + 6}{x - 1} = \frac{2}{0^+}$$

$$\underline{= \infty}$$

### Exercise

Find  $\lim_{x \rightarrow 2\pi^-} \csc x$

### Solution

$$\lim_{x \rightarrow 2\pi^-} \csc x = \frac{1}{\sin(2\pi^-)} = \frac{1}{0^-}$$

$$\underline{= -\infty}$$

### Exercise

Find  $\lim_{x \rightarrow 0^+} e^{\sqrt{x}}$

### Solution

$$\lim_{x \rightarrow 0^+} e^{\sqrt{x}} \underline{= 1}$$

### Exercise

Find  $\lim_{x \rightarrow \frac{\pi}{2}^-} \frac{1 + \sin x}{\cos x}$

### Solution

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \frac{1 + \sin x}{\cos x} = \frac{2}{0^+}$$

$$\underline{= \infty}$$

### ***Exercise***

Find  $\lim_{x \rightarrow \frac{\pi}{2}^+} \frac{1 + \sin x}{\cos x}$

### **Solution**

$$\lim_{x \rightarrow \frac{\pi}{2}^+} \frac{1 + \sin x}{\cos x} = \frac{2}{0^-}$$
$$\underline{\underline{= -\infty}}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 0^-} \frac{e^x}{1 + e^x}$

### **Solution**

$$\lim_{x \rightarrow 0^-} \frac{e^x}{1 - e^x} = \frac{1}{0^+}$$
$$\underline{\underline{= \infty}}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 0^+} \frac{e^x}{1 - e^x}$

### **Solution**

$$\lim_{x \rightarrow 0^+} \frac{e^x}{1 - e^x} = \frac{1}{0^-}$$
$$\underline{\underline{= -\infty}}$$

### ***Exercise***

Find  $\lim_{x \rightarrow 1^-} \frac{x}{\ln x}$

### **Solution**

$$\lim_{x \rightarrow 1^-} \frac{x}{\ln x} = \frac{1}{0^-}$$
$$\underline{\underline{= -\infty}}$$



### Exercise

Find  $\lim_{x \rightarrow 0^+} \frac{x}{\ln x}$

### Solution

$$\lim_{x \rightarrow 0^+} \frac{x}{\ln x} = \frac{0}{-\infty} \\ = 0 \quad |$$

### Exercise

Find  $\lim_{x \rightarrow 0^-} \frac{2e^x + 5e^{3x}}{e^{2x} - e^{3x}}$

### Solution

$$\lim_{x \rightarrow 0^-} \frac{2e^x + 5e^{3x}}{e^{2x} - e^{3x}} = \lim_{x \rightarrow 0^-} \frac{2e^x + 5e^{3x}}{e^{2x}(1 - e^x)} \\ = \frac{7}{0} \\ = \infty \quad |$$

### Exercise

Find  $\lim_{x \rightarrow 0^+} \frac{2e^x + 5e^{3x}}{e^{2x} - e^{3x}}$

### Solution

$$\lim_{x \rightarrow 0^+} \frac{2e^x + 5e^{3x}}{e^{2x} - e^{3x}} = \lim_{x \rightarrow 0^+} \frac{2e^x + 5e^{3x}}{e^{2x}(1 - e^x)} \\ = \frac{7}{0^-} \\ = -\infty \quad |$$

### Exercise

Find  $\lim_{x \rightarrow 1^-} \frac{\ln x}{\sin^{-1} x}$

### Solution

$$\lim_{x \rightarrow 1^-} \frac{\ln x}{\sin^{-1} x} = \frac{\ln 1}{\sin^{-1} 1}$$

$$= \frac{0}{\frac{\pi}{2}}$$

$$\underline{= 0}$$

### Exercise

Let  $f(x) = \frac{x^2 - 7x + 12}{x - a}$

- a) For what values of  $a$ , if any, does  $\lim_{x \rightarrow a^+} f(x)$  equal a finite number?
- b) For what values of  $a$ , if any, does  $\lim_{x \rightarrow a^+} f(x) = \infty$ ?
- c) For what values of  $a$ , if any, does  $\lim_{x \rightarrow a^+} f(x) = -\infty$ ?

### Solution

$$f(x) = \frac{x^2 - 7x + 12}{x - a} = \frac{(x-3)(x-4)}{x - a}$$

a) If  $a = 3$ , then

$$\lim_{x \rightarrow 3} \frac{(x-3)(x-4)}{x-3} = \lim_{x \rightarrow 3} (x-4)$$

$$\underline{= -1}$$

If  $a = 4$ , then

$$\lim_{x \rightarrow 4} \frac{(x-3)(x-4)}{x-4} = \lim_{x \rightarrow 4} (x-3)$$

$$\underline{= 1}$$

b)  $\lim_{x \rightarrow a^+} f(x) = \infty$  for any number other than 3 or 4.

As  $x \rightarrow a^+$ , then  $(x - a)$  is always positive.

$$(x-3)(x-4) > 0 \Rightarrow (-\infty, 3) \cup (4, \infty)$$

c)  $\lim_{x \rightarrow a^+} f(x) = -\infty$  for any number other than 3 or 4.

As  $x \rightarrow a^+$ , then  $(x - a)$  is always positive, and  $(3, 4)$

### ***Exercise***

Analyze  $\lim_{x \rightarrow 1^+} \sqrt{\frac{x-1}{x-3}}$  and  $\lim_{x \rightarrow 1^-} \sqrt{\frac{x-1}{x-3}}$

### **Solution**

$$\lim_{x \rightarrow 1^+} \sqrt{\frac{x-1}{x-3}} = \sqrt{\frac{0^+}{-2}} \quad \nexists$$

$$\lim_{x \rightarrow 1^-} \sqrt{\frac{x-1}{x-3}} = \sqrt{\frac{0^-}{-2}} \\ = 0$$

## ***Solution***      **Section 1.4 – Limits at Infinity**

### ***Exercise***

Find the limit as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$  of  $h(x) = \frac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}}$

### **Solution**

$$\lim_{x \rightarrow \infty} \frac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}} = \underline{-\frac{5}{3}}$$

$$\lim_{x \rightarrow -\infty} \frac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}} = \underline{-\frac{5}{3}}$$

### ***Exercise***

Find the limit as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$  of  $f(x) = \frac{2x+3}{5x+7}$

### **Solution**

$$\lim_{x \rightarrow \infty} \frac{2x+3}{5x+7} = \lim_{x \rightarrow \infty} \frac{2 + \frac{3}{x}}{5 + \frac{7}{x}} = \underline{\frac{2}{5}}$$

$$\lim_{x \rightarrow -\infty} \frac{2x+3}{5x+7} = \lim_{x \rightarrow -\infty} \frac{2 + \frac{3}{x}}{5 + \frac{7}{x}} = \underline{\frac{2}{5}}$$

### ***Exercise***

Find the limit as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$  of  $f(x) = \frac{2x^3+7}{x^3-x^2+x+7}$

### **Solution**

$$\lim_{x \rightarrow \infty} \frac{2x^3+7}{x^3-x^2+x+7} = \lim_{x \rightarrow \infty} \frac{2 + \frac{7}{x^3}}{1 - \frac{1}{x} + \frac{1}{x^2} + \frac{7}{x^3}}$$

$$= 2 \mid$$

$$\lim_{x \rightarrow -\infty} \frac{2x^3 + 7}{x^3 - x^2 + x + 7} = \lim_{x \rightarrow -\infty} \frac{2 + \frac{7}{x^3}}{1 - \frac{1}{x} + \frac{1}{x^2} + \frac{7}{x^3}}$$

$$= 2 \mid$$

### Exercise

Find the limit as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$  of  $f(x) = \frac{x+1}{x^2+3}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{x+1}{x^2+3} = \lim_{x \rightarrow \infty} \frac{\frac{x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{3}{x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 + \frac{3}{x^2}}$$

$$= 0 \mid$$

$$\lim_{x \rightarrow -\infty} \frac{x+1}{x^2+3} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 + \frac{3}{x^2}}$$

$$= 0 \mid$$

### Exercise

Find the limit as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$  of  $f(x) = \frac{7x^3}{x^3 - 3x^2 + 6x}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{7x^3}{x^3 - 3x^2 + 6x} = \lim_{x \rightarrow \infty} \frac{7}{1 - \frac{3}{x} + \frac{6}{x^2}}$$

$$= 7 \mid$$

$$\lim_{x \rightarrow -\infty} \frac{7x^3}{x^3 - 3x^2 + 6x} = \lim_{x \rightarrow -\infty} \frac{7}{1 - \frac{3}{x} + \frac{6}{x^2}}$$

$$= 7 \mid$$

### Exercise

Find the limit as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$  of  $f(x) = \frac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{9x^4 + x}{2x^4 + 5x^2 - x + 6} &= \lim_{x \rightarrow \infty} \frac{\frac{9x^4}{x^4} + \frac{x}{x^4}}{\frac{2x^4}{x^4} + \frac{5x^2}{x^4} - \frac{x}{x^4} + \frac{6}{x^4}} \\ &= \lim_{x \rightarrow \infty} \frac{9 + \frac{1}{x^3}}{2 + \frac{5}{x^2} - \frac{1}{x^3} + \frac{6}{x^4}} \\ &= \frac{9}{2}\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{9x^4 + x}{2x^4 + 5x^2 - x + 6} &= \lim_{x \rightarrow -\infty} \frac{9 + \frac{1}{x^3}}{2 + \frac{5}{x^2} - \frac{1}{x^3} + \frac{6}{x^4}} \\ &= \frac{9}{2}\end{aligned}$$

### Exercise

Find the limit as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$  of  $f(x) = \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x} &= \lim_{x \rightarrow \infty} \frac{-2 - \frac{2}{x^2} + \frac{3}{x^3}}{3 + \frac{3}{x} - \frac{5}{x^2}} \\ &= -\frac{2}{3}\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x} &= \lim_{x \rightarrow -\infty} \frac{-2 - \frac{2}{x^2} + \frac{3}{x^3}}{3 + \frac{3}{x} - \frac{5}{x^2}} \\ &= -\frac{2}{3}\end{aligned}$$

### ***Exercise***

Find  $\lim_{x \rightarrow \infty} x^{12}$

#### **Solution**

$$\lim_{x \rightarrow \infty} x^{12} = \underline{\infty}$$

### ***Exercise***

Find  $\lim_{x \rightarrow -\infty} 3x^9$

#### **Solution**

$$\lim_{x \rightarrow -\infty} 3x^9 = \underline{-\infty}$$

### ***Exercise***

Find  $\lim_{x \rightarrow -\infty} x^{-8}$

#### **Solution**

$$\lim_{x \rightarrow -\infty} x^{-8} = \frac{1}{(-\infty)^8} = \underline{0}$$

### ***Exercise***

Find  $\lim_{x \rightarrow -\infty} x^{-9}$

#### **Solution**

$$\lim_{x \rightarrow -\infty} x^{-9} = \frac{1}{(-\infty)^9} = \underline{0}$$

### ***Exercise***

Find  $\lim_{x \rightarrow -\infty} 2x^{-6}$

#### **Solution**

$$\lim_{x \rightarrow -\infty} 2x^{-6} = \frac{2}{\infty} = \underline{0}$$

### ***Exercise***

Find  $\lim_{x \rightarrow \infty} (3x^{12} - 9x^7)$

### **Solution**

$$\lim_{x \rightarrow \infty} (3x^{12} - 9x^7) = \underline{\infty}$$

### ***Exercise***

Find  $\lim_{x \rightarrow -\infty} (3x^7 + x^2)$

### **Solution**

$$\lim_{x \rightarrow -\infty} (3x^7 + x^2) = \lim_{x \rightarrow -\infty} x^2(3x^5 + 1) = \underline{-\infty}$$

### ***Exercise***

Find  $\lim_{x \rightarrow -\infty} (-2x^{16} + 2)$

### **Solution**

$$\lim_{x \rightarrow -\infty} (-2x^{16} + 2) = \underline{-\infty}$$

### ***Exercise***

Find  $\lim_{x \rightarrow -\infty} (2x^{-6} + 4x^5)$

### **Solution**

$$\lim_{x \rightarrow -\infty} (2x^{-6} + 4x^5) = \lim_{x \rightarrow -\infty} x^{-6}(2 + 4x^{11}) = \underline{-\infty} \quad +\infty(-\infty)$$

### ***Exercise***

Find  $\lim_{x \rightarrow -\infty} \frac{\cos x}{3x}$

### **Solution**



$$-\frac{1}{3x} \leq \frac{\cos x}{3x} \leq \frac{1}{3x}, \quad -1 \leq \cos x \leq 1$$

$$\lim_{x \rightarrow -\infty} \frac{\cos x}{3x} = \underline{0} \quad \text{By the Sandwich Theorem}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{x + \sin x}{2x + 7 - 5 \sin x}$

#### Solution

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x + \sin x}{2x + 7 - 5 \sin x} &= \lim_{x \rightarrow \infty} \frac{1 + \frac{\sin x}{x}}{2 + \frac{7}{x} - \frac{5 \sin x}{x}} \\ &= \frac{1 + 0}{2 + 0 - 0} \\ &= \underline{\frac{1}{2}} \end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \sqrt{\frac{8x^2 - 3}{2x^2 + x}}$

#### Solution

$$\begin{aligned} \lim_{x \rightarrow \infty} \sqrt{\frac{8x^2 - 3}{2x^2 + x}} &= \lim_{x \rightarrow \infty} \sqrt{\frac{8 - \frac{3}{x^2}}{2 + \frac{1}{x}}} \\ &= \sqrt{\frac{8}{2}} \\ &= \underline{2} \end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \left( \frac{x^2 + x - 1}{8x^2 - 3} \right)^{1/3}$

#### Solution

$$\lim_{x \rightarrow -\infty} \left( \frac{x^2 + x - 1}{8x^2 - 3} \right)^{1/3} = \lim_{x \rightarrow -\infty} \left( \frac{1 + \frac{1}{x} - \frac{1}{x^2}}{8 - \frac{3}{x^2}} \right)^{1/3}$$

$$= \left(\frac{1}{8}\right)^{1/3}$$

$$= \frac{1}{2}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7} = \lim_{x \rightarrow \infty} \frac{\frac{2\sqrt{x}}{x} + \frac{x^{-1}}{x}}{3 - \frac{7}{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{2}{x^{1/2}} + \frac{1}{x^2}}{3 - \frac{7}{x}}$$

$$= 0$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{x^{-1} + x^{-4}}{x^{-2} + x^{-3}}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{x^{-1} + x^{-4}}{x^{-2} + x^{-3}} = \lim_{x \rightarrow \infty} \frac{\frac{x^{-1}}{x^{-2}} + \frac{x^{-4}}{x^{-2}}}{\frac{x^{-2}}{x^{-2}} + \frac{x^{-3}}{x^{-2}}}$$

$$= \lim_{x \rightarrow \infty} \frac{x + \frac{1}{x^2}}{1 + \frac{1}{x}}$$

$$= \infty$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6} + 9}$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow -\infty} \frac{4-3x^3}{\sqrt{x^6+9}} &= \lim_{x \rightarrow -\infty} \frac{\frac{4-3x^3}{\sqrt{x^6}}}{\frac{\sqrt{x^6+9}}{\sqrt{x^6}}} \\
&= \lim_{x \rightarrow -\infty} \frac{\frac{4-3x^3}{x^3}}{\sqrt{\frac{x^6+9}{x^6}}} \\
&= \lim_{x \rightarrow -\infty} \frac{\frac{4}{x^3}-3}{\sqrt{1+\frac{9}{x^6}}} \\
&= \frac{-3}{\sqrt{1}} \\
&= \underline{-3}
\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \left( \sqrt{x^2+3} + x \right)$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow -\infty} \left( \sqrt{x^2+3} + x \right) &= \lim_{x \rightarrow -\infty} \left( \sqrt{x^2+3} + x \right) \frac{\sqrt{x^2+3}-x}{\sqrt{x^2+3}-x} \\
&= \lim_{x \rightarrow -\infty} \frac{x^2+3-x^2}{\sqrt{x^2+3}-x} \\
&= \lim_{x \rightarrow -\infty} \frac{3}{\sqrt{x^2+3}-x} \\
&= \lim_{x \rightarrow -\infty} \frac{\frac{3}{x}}{\sqrt{\frac{x^2}{x^2} + \frac{3}{x^2} - \frac{x}{x}}} \\
&= \lim_{x \rightarrow -\infty} \frac{\frac{3}{x}}{\sqrt{1 + \frac{3}{x^2} + 1}} \\
&= \frac{0}{\sqrt{1+1}} \\
&= \underline{0}
\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \left( \sqrt{x^2 + 3x} - \sqrt{x^2 - 2x} \right)$

### Solution

$$\begin{aligned} \lim_{x \rightarrow \infty} \left( \sqrt{x^2 + 3x} - \sqrt{x^2 - 2x} \right) &= \lim_{x \rightarrow \infty} \left( \sqrt{x^2 + 3x} - \sqrt{x^2 - 2x} \right) \frac{\sqrt{x^2 + 3x} + \sqrt{x^2 - 2x}}{\sqrt{x^2 + 3x} + \sqrt{x^2 - 2x}} \\ &= \lim_{x \rightarrow \infty} \frac{(x^2 + 3x) - (x^2 - 2x)}{\sqrt{x^2 + 3x} + \sqrt{x^2 - 2x}} \\ &= \lim_{x \rightarrow \infty} \frac{x^2 + 3x - x^2 + 2x}{\sqrt{x^2 + 3x} + \sqrt{x^2 - 2x}} \\ &= \lim_{x \rightarrow \infty} \frac{5x}{\sqrt{x^2 + 3x} + \sqrt{x^2 - 2x}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{5x}{\sqrt{x^2}}}{\sqrt{\frac{x^2}{x^2} + \frac{3x}{x^2}} + \sqrt{\frac{x^2}{x^2} - \frac{2x}{x^2}}} \\ &= \lim_{x \rightarrow \infty} \frac{5}{\sqrt{1 + \frac{3}{x}} + \sqrt{1 - \frac{2}{x}}} \\ &= \frac{5}{\sqrt{1} + \sqrt{1}} \\ &= \frac{5}{2} \end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{2x - 3}{4x + 10}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{2x - 3}{4x + 10} = \frac{1}{2}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{x^4 - 1}{x^5 + 2}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{x^4 - 1}{x^5 + 2} = 0$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} (-3x^3 + 5)$

### Solution

$$\lim_{x \rightarrow -\infty} (-3x^3 + 5) = \infty$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \left( e^{-2x} + \frac{2}{x} \right)$

### Solution

$$\lim_{x \rightarrow \infty} \left( e^{-2x} + \frac{2}{x} \right) = e^{-\infty} + 0 = 0$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{1}{\ln x + 1}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{1}{\ln x + 1} = \frac{1}{\infty} = 0$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \left( 3 + \frac{10}{x^2} \right)$

### Solution

$$\lim_{x \rightarrow \infty} \left( 3 + \frac{10}{x^2} \right) = 3 + 0 = 3$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \left( 5 + \frac{1}{x} + \frac{10}{x^2} \right)$

### Solution

$$\lim_{x \rightarrow \infty} \left( 5 + \frac{1}{x} + \frac{10}{x^2} \right) = 5 + 0 + 0$$
$$\underline{= 5}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{4x^2 + 2x + 3}{x^2}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{4x^2 + 2x + 3}{x^2} = \lim_{x \rightarrow \infty} \frac{4x^2}{x^2}$$
$$\underline{= 4}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \left( 5 + \frac{100}{x} + \frac{\sin^4 x^3}{x^2} \right)$

### Solution

$$-1 \leq \sin \theta \leq 1$$

$$0 \leq \sin^4 \theta \leq 1$$

$$0 \leq \frac{\sin^4 \theta}{x^2} \leq \frac{1}{x^2} \rightarrow 0$$

$$\lim_{x \rightarrow \infty} \left( 5 + \frac{100}{x} + \frac{\sin^4 x^3}{x^2} \right) \underline{= 5}$$

### Exercise

Find  $\lim_{\theta \rightarrow \infty} \frac{\cos \theta}{\theta^2}$

### Solution

$$-1 \leq \cos \theta \leq 1$$

$$-\frac{1}{\theta^2} \leq \frac{\cos \theta}{\theta^2} \leq \frac{1}{\theta^2} \rightarrow 0$$

$$\lim_{\theta \rightarrow \infty} \frac{\cos \theta}{\theta^2} = \underline{0}$$

### Exercise

Find  $\lim_{\theta \rightarrow \infty} \frac{\cos \theta^5}{\sqrt{\theta}}$

### Solution

$$-1 \leq \cos \theta^5 \leq 1$$

$$-\frac{1}{\sqrt{\theta}} \leq \frac{\cos \theta^5}{\sqrt{\theta}} \leq \frac{1}{\sqrt{\theta}} \rightarrow 0$$

$$\lim_{\theta \rightarrow \infty} \frac{\cos \theta^5}{\sqrt{\theta}} = \underline{0}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{4x}{20x+1}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{4x}{20x+1} = \frac{4}{20} = \underline{\frac{1}{5}}$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \frac{4x}{20x+1}$

### Solution

$$\lim_{x \rightarrow -\infty} \frac{4x}{20x+1} = \lim_{x \rightarrow -\infty} \frac{4x}{20x} = \underline{\frac{1}{5}}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{3x^2-7}{x^2+5x}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{3x^2 - 7}{x^2 + 5x} = 3$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \frac{3x^2 - 7}{x^2 + 5x}$

### Solution

$$\lim_{x \rightarrow -\infty} \frac{3x^2 - 7}{x^2 + 5x} = \lim_{x \rightarrow -\infty} \frac{3x^2}{x^2} = 3$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{6x^2 - 9x + 8}{3x^2 + 2}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{6x^2 - 9x + 8}{3x^2 + 2} = \lim_{x \rightarrow \infty} \frac{6x^2}{3x^2} = \frac{6}{3} = 2$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \frac{6x^2 - 9x + 8}{3x^2 + 2}$

### Solution

$$\lim_{x \rightarrow -\infty} \frac{6x^2 - 9x + 8}{3x^2 + 2} = \lim_{x \rightarrow -\infty} \frac{6x^2}{3x^2} = \frac{6}{3} = 2$$



### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{4x^2 - 7}{8x^2 + 5x + 2}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{4x^2 - 7}{8x^2 + 5x + 2} &= \lim_{x \rightarrow \infty} \frac{4x^2}{8x^2} \\ &= \frac{4}{8} \\ &= \frac{1}{2}\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \frac{4x^2 - 7}{8x^2 + 5x + 2}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{4x^2 - 7}{8x^2 + 5x + 2} &= \lim_{x \rightarrow -\infty} \frac{4x^2}{8x^2} \\ &= \frac{4}{8} \\ &= \frac{1}{2}\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{\sqrt{16x^4 + 64x^2} + x^2}{2x^2 - 4}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\sqrt{16x^4 + 64x^2} + x^2}{2x^2 - 4} &= \lim_{x \rightarrow \infty} \frac{\sqrt{16x^4} + x^2}{2x^2} \\ &= \lim_{x \rightarrow \infty} \frac{4x^2 + x^2}{2x^2} \\ &= \lim_{x \rightarrow \infty} \frac{5x^2}{2x^2} \\ &= \frac{5}{2}\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \frac{\sqrt{16x^4 + 64x^2 + x^2}}{2x^2 - 4}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{\sqrt{16x^4 + 64x^2 + x^2}}{2x^2 - 4} &= \lim_{x \rightarrow -\infty} \frac{\sqrt{16x^4 + x^2}}{2x^2} \\ &= \lim_{x \rightarrow -\infty} \frac{4x^2 + x^2}{2x^2} \\ &= \lim_{x \rightarrow -\infty} \frac{5x^2}{2x^2} \\ &= \underline{\underline{\frac{5}{2}}}\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{3x^4 + 3x^3 - 36x^2}{x^4 - 25x^2 + 144}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{3x^4 + 3x^3 - 36x^2}{x^4 - 25x^2 + 144} &= \lim_{x \rightarrow \infty} \frac{3x^4}{x^4} \\ &= \underline{\underline{3}}\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \frac{3x^4 + 3x^3 - 36x^2}{x^4 - 25x^2 + 144}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{3x^4 + 3x^3 - 36x^2}{x^4 - 25x^2 + 144} &= \lim_{x \rightarrow -\infty} \frac{3x^4}{x^4} \\ &= \underline{\underline{3}}\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} 16x^2 \left( 4x^2 - \sqrt{16x^4 + 1} \right)$

#### Solution

$$\begin{aligned}
\lim_{x \rightarrow \infty} 16x^2 \left( 4x^2 - \sqrt{16x^4 + 1} \right) &= \infty - \infty \\
&= \lim_{x \rightarrow \infty} 16x^2 \left( 4x^2 - \sqrt{16x^4 + 1} \right) \cdot \frac{4x^2 + \sqrt{16x^4 + 1}}{4x^2 + \sqrt{16x^4 + 1}} \\
&= \lim_{x \rightarrow \infty} 16x^2 \frac{16x^4 - 16x^4 - 1}{4x^2 + \sqrt{16x^4 + 1}} \\
&= \lim_{x \rightarrow \infty} 16x^2 \frac{-1}{4x^2 + \sqrt{16x^4 + 1}} \\
&= \lim_{x \rightarrow \infty} \frac{-16x^2}{4x^2 + 4x^2} \\
&= \lim_{x \rightarrow \infty} \frac{-16x^2}{8x^2} \\
&= \underline{-2}
\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} 16x^2 \left( 4x^2 - \sqrt{16x^4 + 1} \right)$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow -\infty} 16x^2 \left( 4x^2 - \sqrt{16x^4 + 1} \right) &= \infty - \infty \\
&= \lim_{x \rightarrow -\infty} 16x^2 \left( 4x^2 - \sqrt{16x^4 + 1} \right) \cdot \frac{4x^2 + \sqrt{16x^4 + 1}}{4x^2 + \sqrt{16x^4 + 1}} \\
&= \lim_{x \rightarrow -\infty} 16x^2 \frac{16x^4 - 16x^4 - 1}{4x^2 + \sqrt{16x^4 + 1}} \\
&= \lim_{x \rightarrow -\infty} 16x^2 \frac{-1}{4x^2 + \sqrt{16x^4 + 1}} \\
&= \lim_{x \rightarrow -\infty} \frac{-16x^2}{4x^2 + 4x^2} \\
&= \lim_{x \rightarrow -\infty} \frac{-16x^2}{8x^2} \\
&= \underline{-2}
\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{x-1}{x^{2/3}-1}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{x-1}{x^{2/3}-1} &= \lim_{x \rightarrow \infty} \frac{x}{x^{2/3}} \\ &= \lim_{x \rightarrow \infty} x^{1/3} \\ &= \infty \quad | \end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \frac{x-1}{x^{2/3}-1}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{x-1}{x^{2/3}-1} &= \lim_{x \rightarrow -\infty} \frac{x}{x^{2/3}} \\ &= \lim_{x \rightarrow -\infty} x^{1/3} \\ &= -\infty \quad | \end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2+2x+6}-3}{x-1}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\sqrt{x^2+2x+6}-3}{x-1} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2}}{x} \\ &= \lim_{x \rightarrow \infty} \frac{x}{x} \\ &= 1 \quad | \end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{|1-x^2|}{x(x+1)}$

**Solution**

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{|1-x^2|}{x(x+1)} &= \lim_{x \rightarrow \infty} \frac{x^2-1}{x^2+1} \\ &= \lim_{x \rightarrow \infty} \frac{x^2}{x^2} \\ &= 1\end{aligned}$$

***Exercise***

Find  $\lim_{x \rightarrow \infty} (\sqrt{|x|} - \sqrt{|x-1|})$

**Solution**

$$\begin{aligned}\lim_{x \rightarrow \infty} (\sqrt{|x|} - \sqrt{|x-1|}) &= \infty - \infty & x \rightarrow \infty \Rightarrow |x| = x \quad \& \quad |x-1| = x-1 \\ &= \lim_{x \rightarrow \infty} (\sqrt{x} - \sqrt{x-1}) \cdot \frac{\sqrt{x} + \sqrt{x-1}}{\sqrt{x} + \sqrt{x-1}} \\ &= \lim_{x \rightarrow \infty} \frac{x - x + 1}{\sqrt{x} + \sqrt{x-1}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x} + \sqrt{x-1}} \\ &= \frac{1}{\infty} \\ &= 0\end{aligned}$$

***Exercise***

Find  $\lim_{x \rightarrow \infty} \frac{\tan^{-1} x}{x}$

**Solution**

$$\begin{aligned}-\frac{\pi}{2} &\leq \tan^{-1} x \leq \frac{\pi}{2} \\ -\frac{\pi}{2x} &\leq \frac{\tan^{-1} x}{x} \leq \frac{\pi}{2x} \rightarrow 0 \\ \lim_{x \rightarrow \infty} \frac{\tan^{-1} x}{x} &= 0\end{aligned}$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{\cos x}{e^{3x}}$

### Solution

$$-1 \leq \cos x \leq 1$$

$$-\frac{1}{e^{3x}} \leq \frac{\cos x}{e^{3x}} \leq \frac{1}{e^{3x}} \rightarrow 0$$

$$\lim_{x \rightarrow \infty} \frac{\cos x}{e^{3x}} = 0$$

### Exercise

Find  $\lim_{x \rightarrow 0} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}}$

### Solution

$$\lim_{x \rightarrow 0} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}} = \frac{2+10}{1+1} = 6$$

### Exercise

Find  $\lim_{x \rightarrow \infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}}$

### Solution

$$\lim_{x \rightarrow \infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}} = \lim_{x \rightarrow \infty} \frac{2e^x}{e^x} = 2$$

$$\lim_{x \rightarrow \infty} e^{-x} = 0$$

### Exercise

Find  $\lim_{x \rightarrow -\infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}}$

### Solution

$$\lim_{x \rightarrow -\infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}} = \lim_{x \rightarrow -\infty} \frac{10e^{-x}}{e^{-x}} = 10$$

$$\lim_{x \rightarrow -\infty} e^x = 0$$

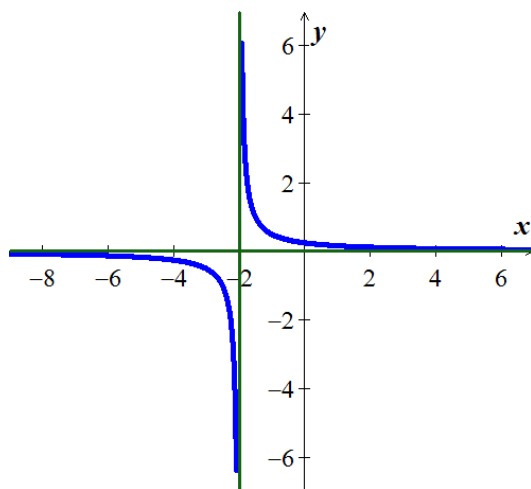
### Exercise

Graph the rational function  $y = \frac{1}{2x+4}$ . Include the equations of the asymptotes.

### Solution

$$VA: 2x + 4 = 0 \Rightarrow \underline{x = -2}$$

$$HA: \underline{y = 0}$$



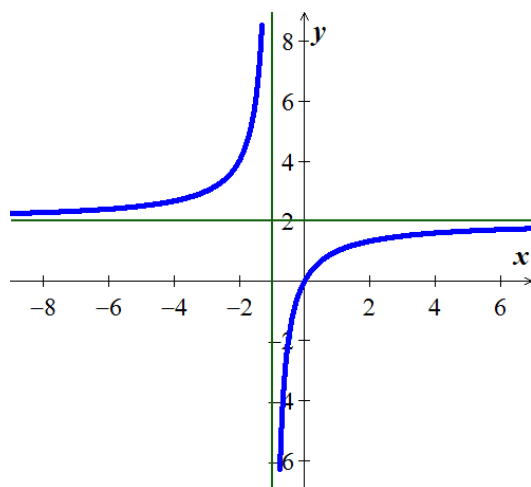
### Exercise

Graph the rational function  $y = \frac{2x}{x+1}$ . Include the equations of the asymptotes.

### Solution

$$VA: \underline{x = -1}$$

$$HA: \underline{y = 2}$$



### Exercise

Graph the rational function  $y = \frac{x^2}{x-1}$ . Include the equations of the asymptotes.

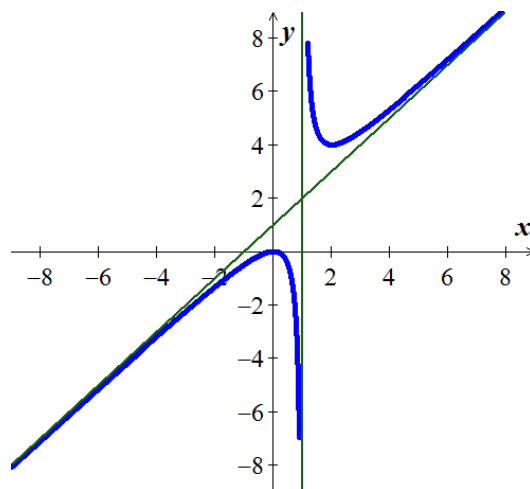
### Solution

$$\begin{array}{r} x+1 \\ x-1 \overline{) x^2} \\ \underline{x^2 - x} \phantom{+1} \\ x \phantom{+1} \\ \underline{x-1} \\ 1 \end{array}$$

$$\begin{aligned} y &= \frac{x^2}{x-1} \\ &= x+1 + \frac{1}{x-1} \end{aligned}$$

**VA:**  $x=1$

**Oblique Asymptote:**  $y = x+1$



### Exercise

Graph the rational function  $y = \frac{x^3+1}{x^2}$ . Include the equations of the asymptotes.

### Solution

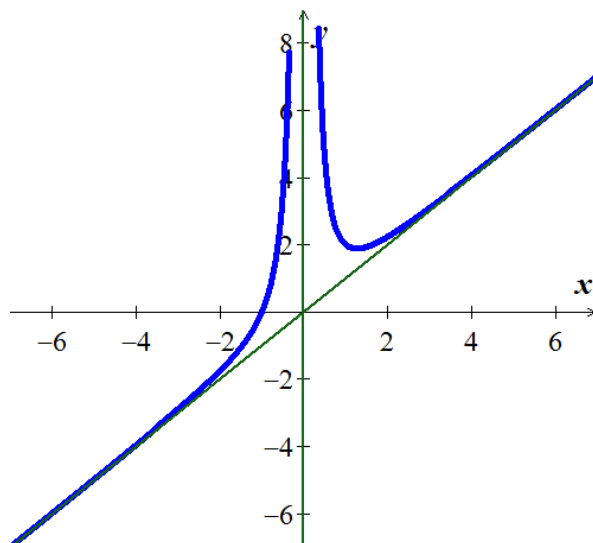
$$\begin{array}{r} x \\ x^2 \overline{) x^3 + 1} \\ \underline{x^3} \phantom{+1} \\ 1 \end{array}$$

$$y = \frac{x^3+1}{x^2} = x + \frac{1}{x^2}$$



**VA:**  $x = 0$

**Oblique Asymptote:**  $y = x$



### Exercise

Let  $f(x) = \frac{x^2 - 5x + 6}{x^2 - 2x}$

a) Analyze  $\lim_{x \rightarrow 0^-} f(x)$ ,  $\lim_{x \rightarrow 0^+} f(x)$ ,  $\lim_{x \rightarrow 2^-} f(x)$ , and  $\lim_{x \rightarrow 2^+} f(x)$

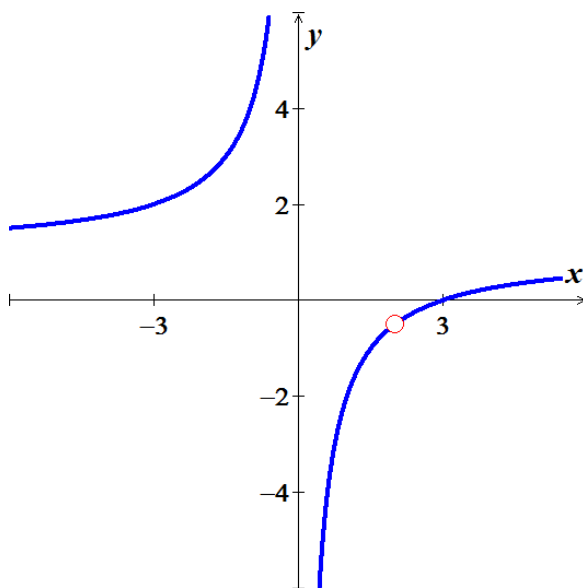
b) Does the graph of  $f$  have any vertical asymptotes? Explain?

### Solution

$$\begin{aligned} f(x) &= \frac{x^2 - 5x + 6}{x^2 - 2x} \\ &= \frac{(x-2)(x-3)}{x(x-2)} \\ &= \frac{x-3}{x} \end{aligned}$$

$$\begin{aligned} a) \quad \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} \frac{x-3}{x} \\ &= \frac{-3}{0^-} \\ &= \infty \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \frac{x-3}{x} \\ &= \frac{-3}{0^+} \\ &= -\infty \end{aligned}$$



$$\underline{\underline{= -\infty}}$$

$$\begin{aligned}\lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} \frac{x-3}{x} \\ &= \frac{2-3}{2} \\ &= -\frac{1}{2}\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} \frac{x-3}{x} \\ &= \frac{2-3}{2} \\ &= -\frac{1}{2}\end{aligned}$$

$$b) \quad VA: x = 0 \quad \text{Hole: } x = 2 \rightarrow f(2) = -\frac{1}{2}$$

$$HA: y = 1 \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{3x}{1-x}$

#### Solution

$$VA: x = 1, \quad \text{Hole: } n/a, \quad HA: y = -3, \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{x^2}{x^2 + 9}$

#### Solution

$$VA: n/a; \quad \text{Hole: } n/a; \quad HA: y = 1; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{x-2}{x^2 - 4x + 3}$

#### Solution

$$VA: x = 1, 3; \quad \text{Hole: } n/a; \quad HA: y = 0; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{5x-1}{1-3x}$

#### Solution

$$VA: x = \frac{1}{3}; \quad \text{Hole: } n/a; \quad HA: y = -\frac{5}{3}; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{3}{x-5}$

#### Solution

$$VA: x = 5, \quad \text{Hole: } n/a, \quad HA: y = 0, \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{x^3-1}{x^2+1}$

#### Solution

$$\begin{array}{r} x \\ x^2+1 \overline{) x^3-1} \\ \underline{x^3+x} \phantom{-1} \\ -x-1 \end{array}$$

$$\begin{aligned} y &= \frac{x^3-1}{x^2+1} \\ &= x + \frac{-x-1}{x^2+1} \\ &= x - \frac{x+1}{x^2+1} \end{aligned}$$

$$VA: n/a, \quad \text{Hole: } n/a, \quad HA: n/a, \quad OA: y = x$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{3x^2-27}{(x+3)(2x+1)}$

#### Solution

$$VA: x = -3, \quad -\frac{1}{2}; \quad \text{Hole: } n/a; \quad HA: y = \frac{3}{2}; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{x^3 + 3x^2 - 2}{x^2 - 4}$

#### Solution

$$\begin{array}{r} x^2 - 4 \overline{) x^3 + 3x^2 - 2} \\ \underline{x^3 \phantom{+ 3x^2} - 4x} \phantom{- 2} \\ 3x^2 + 4x - 2 \end{array}$$

$$\begin{aligned} y &= \frac{x^3 + 3x^2 - 2}{x^2 - 4} \\ &= x + 3 + \frac{4x + 10}{x^2 - 4} \end{aligned}$$

$$VA: x = \pm 2, \quad \text{Hole: } n/a, \quad HA: n/a, \quad OA: y = x + 3$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{x - 3}{x^2 - 9}$

#### Solution

$$VA: x = -3; \quad \text{Hole: } x = 3; \quad HA: y = 0; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $y = \frac{6}{\sqrt{x^2 - 4x}}$

#### Solution

$$VA: x = 0, 4; \quad \text{Hole: } n/a; \quad HA: y = 0; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of  $f(x) = \frac{4x^3 + 1}{1 - x^3}$

#### Solution

$$VA: x = 1; \quad \text{Hole: } n/a; \quad HA: y = -4; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

$$f(x) = \frac{x+1}{\sqrt{9x^2+x}}$$

### Solution

$$VA: x = 0, -\frac{1}{9}; \quad \text{Hole: } n/a; \quad HA: y = \frac{1}{3}; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

$$f(x) = 1 - e^{-2x}$$

### Solution

$$VA: n/a; \quad \text{Hole: } n/a; \quad HA: y = 1; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

$$f(x) = \frac{1}{\ln x^2}$$

### Solution

$$VA: x = 0; \quad \text{Hole: } n/a; \quad HA: y = 0; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

$$f(x) = \frac{1}{\tan^{-1} x}$$

### Solution

$$VA: x = 0; \quad \text{Hole: } n/a; \quad HA: y = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

$$f(x) = \frac{2x^2+6}{2x^2+3x-2}$$

### Solution

$$VA: x = -2, \frac{1}{2}; \quad \text{Hole: } n/a; \quad HA: y = 1; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

$$f(x) = \frac{3x^2 + 2x - 1}{4x + 1}$$

### Solution

$$\begin{array}{r} \frac{3}{4}x + \frac{5}{16} \\ 4x + 1 \overline{) 3x^2 + 2x - 1} \\ \underline{3x^2 + \frac{3}{4}x} \phantom{-1} \\ \frac{5}{4}x - 1 \end{array}$$

$$VA: x = -\frac{1}{4}; \quad \text{Hole: } n/a; \quad HA: n/a; \quad OA: y = \frac{3}{4}x + \frac{5}{16}$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

$$f(x) = \frac{9x^2 + 4}{(2x - 1)^2}$$

### Solution

$$VA: x = \frac{1}{2}; \quad \text{Hole: } n/a; \quad HA: y = \frac{9}{4}; \quad OA: n/a$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

$$f(x) = \frac{1 + x - 2x^2 - x^3}{x^2 + 1}$$

### Solution

$$\begin{array}{r} -x - 2 \\ x^2 + 1 \overline{) -x^3 - 2x^2 + x + 1} \\ \underline{-x^3 \phantom{- 2x^2} - x} \phantom{+ 1} \\ -2x^2 + 2x \phantom{+ 1} \end{array}$$

$$VA: n/a; \quad \text{Hole: } n/a; \quad HA: n/a; \quad OA: y = -x - 2$$

### Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

$$f(x) = \frac{x(x + 2)^3}{3x^2 - 4x}$$

### Solution

$$\begin{aligned}
 f(x) &= \frac{x(x^3 + 6x^2 + 12x + 8)}{x(3x - 4)} \\
 &= \frac{x^3 + 6x^2 + 12x + 8}{3x - 4} \\
 &= \frac{\frac{1}{3}x^2 + \frac{22}{9}x + \frac{196}{27}}{3x - 4} \left( \begin{array}{l} x^3 + 6x^2 + 12x + 8 \\ x^3 - \frac{4}{3}x^2 \\ \hline \frac{22}{3}x^2 + 12x \\ \frac{22}{3}x^2 - \frac{88}{9}x \\ \hline \frac{196}{9}x \end{array} \right)
 \end{aligned}$$

$$VA: x = \frac{4}{3}; \quad Hole: (0, -2); \quad HA: n/a; \quad OA: y = \frac{1}{3}x^2 + \frac{22}{9}x + \frac{196}{27}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 0} \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x}$

#### Solution

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x} &= \frac{4}{0} \\
 &= \infty
 \end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 2} \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x}$

#### Solution

$$\begin{aligned}
 \lim_{x \rightarrow 2} \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x} &= \frac{4 - 8 + 4}{8 + 20 - 28} = \frac{0}{0} \\
 &= \lim_{x \rightarrow 2} \frac{(x-2)(x-2)}{x(x-2)(x+7)} \\
 &= \lim_{x \rightarrow 2} \frac{x-2}{x(x+7)} \\
 &= \frac{0}{18} \\
 &= 0
 \end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow a} \frac{x^2 - a^2}{x^4 - a^4}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow a} \frac{x^2 - a^2}{x^4 - a^4} &= \frac{a^2 - a^2}{a^4 - a^4} = \frac{0}{0} \\&= \lim_{x \rightarrow a} \frac{x^2 - a^2}{(x^2 - a^2)(x^2 + a^2)} \\&= \lim_{x \rightarrow a} \frac{1}{x^2 + a^2} \\&= \frac{1}{a^2 + a^2} \\&= \frac{1}{2a^2} \quad \Big| \end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{(x+h)^2 - x^2}{h} &= \frac{h^2}{h} \\&= h \quad \Big| \end{aligned}$$

### Exercise

Find the limit  $\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$

### Solution

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} &= \frac{x^2 - x^2}{0} = \frac{0}{0} \\&= \lim_{h \rightarrow 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \\&= \lim_{h \rightarrow 0} \frac{2hx + h^2}{h} \end{aligned}$$



$$\begin{aligned}
 &= \lim_{h \rightarrow 0} (2x + h) \\
 &= \underline{2x}
 \end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x}$

### Solution

$$\begin{aligned}
 \lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x} &= \frac{1 - 1}{1 - 1} = \frac{0}{0} \\
 &= \lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})} \\
 &= \lim_{x \rightarrow 1} \frac{1}{1 + \sqrt{x}} \\
 &= \underline{\frac{1}{2}}
 \end{aligned}$$

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$$\begin{aligned}
 \lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x} &= \lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x} \cdot \frac{1 + \sqrt{x}}{1 + \sqrt{x}} \\
 &= \lim_{x \rightarrow 1} \frac{1 - x}{(1 - x)(1 + \sqrt{x})} \\
 &= \lim_{x \rightarrow 1} \frac{1}{1 + \sqrt{x}} \\
 &= \underline{\frac{1}{2}}
 \end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x}$

### Solution

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} &= \frac{\frac{1}{2} - \frac{1}{2}}{0} = \frac{0}{0} \\
 &= \lim_{x \rightarrow 0} \frac{1}{x} \left( \frac{2 - 2 - x}{2(2+x)} \right) \\
 &= \frac{1}{2} \lim_{x \rightarrow 0} \frac{1}{x} \left( \frac{-x}{2+x} \right)
 \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2} \lim_{x \rightarrow 0} \frac{1}{2+x} \\
&= -\frac{1}{2} \left( \frac{1}{2} \right) \\
&= -\frac{1}{4}
\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{\sqrt{x} - 1}$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{\sqrt{x} - 1} &= \frac{1-1}{1-1} = \frac{0}{0} \\
&= \lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{\sqrt{x} - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} \\
&= \lim_{x \rightarrow 1} \frac{(x^{1/3} - 1)(\sqrt{x} + 1)}{x - 1} \\
&= \lim_{x \rightarrow 1} \frac{(x^{1/3} - 1)(\sqrt{x} + 1)}{(x^{1/3})^3 - 1^3} \\
&= \lim_{x \rightarrow 1} \frac{(x^{1/3} - 1)(\sqrt{x} + 1)}{(x^{1/3} - 1)(x^{2/3} + x^{1/3} + 1)} \\
&= \lim_{x \rightarrow 1} \frac{\sqrt{x} + 1}{x^{2/3} + x^{1/3} + 1} \\
&= \frac{2}{3}
\end{aligned}$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

### Exercise

Find the limit  $\lim_{x \rightarrow 64} \frac{x^{2/3} - 16}{\sqrt{x} - 8}$

### Solution

$$\lim_{x \rightarrow 64} \frac{x^{2/3} - 16}{\sqrt{x} - 8} = \frac{(4^3)^{2/3} - 16}{8 - 8}$$

$$\begin{aligned}
&= \frac{16-16}{0} = \frac{0}{0} \\
&= \lim_{x \rightarrow 64} \frac{(x^{1/3})^2 - 16}{\sqrt{x} - 8} \cdot \frac{\sqrt{x} + 8}{\sqrt{x} + 8} \\
&= \lim_{x \rightarrow 64} \frac{(x^{1/3} - 4)(x^{1/3} + 4)(\sqrt{x} + 8)}{x - 64} \\
&= \lim_{x \rightarrow 64} \frac{(x^{1/3} - 4)(x^{1/3} + 4)(\sqrt{x} + 8)}{(x^{1/3})^3 - 4^3} \\
&= \lim_{x \rightarrow 64} \frac{(x^{1/3} - 4)(x^{1/3} + 4)(\sqrt{x} + 8)}{(x^{1/3} - 4)(x^{2/3} + 4x^{1/3} + 16)} \\
&= \lim_{x \rightarrow 64} \frac{(x^{1/3} + 4)(\sqrt{x} + 8)}{x^{2/3} + 4x^{1/3} + 16} \\
&= \frac{(4+4)(8+8)}{16+16+16} \\
&= \frac{8}{3}
\end{aligned}$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

### Exercise

Find the limit  $\lim_{x \rightarrow 0} \frac{\tan(2x)}{\tan(\pi x)}$

### Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{\tan(2x)}{\tan(\pi x)} &= \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{\sin 2x}{\cos 2x} \cdot \frac{\cos(\pi x)}{\sin(\pi x)} \\
&= \lim_{x \rightarrow 0} \frac{\cos(\pi x)}{\cos 2x} \cdot \frac{\sin 2x}{2x} \cdot \frac{2x}{\pi x} \cdot \frac{\pi x}{\sin(\pi x)} \\
&= \frac{2}{\pi} \cdot \frac{\cos 0}{\cos 0} \cdot \lim_{2x \rightarrow 0} \frac{\sin 2x}{2x} \cdot \lim_{\pi x \rightarrow 0} \frac{1}{\frac{\sin \pi x}{\pi x}} \\
&= \frac{2}{\pi}
\end{aligned}$$

### ***Exercise***

Find the limit  $\lim_{x \rightarrow \pi^-} \csc x$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow \pi^-} \csc x &= \frac{1}{\sin \pi^-} \\ &= \frac{1}{0^-} \\ &= -\infty\end{aligned}$$

### ***Exercise***

Find the limit  $\lim_{x \rightarrow \pi} \sin\left(\frac{x}{2} + \sin x\right)$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow \pi} \sin\left(\frac{x}{2} + \sin x\right) &= \sin\left(\frac{\pi}{2} + \sin \pi\right) \\ &= \sin \frac{\pi}{2} \\ &= 1\end{aligned}$$

### ***Exercise***

Find the limit  $\lim_{x \rightarrow \pi} \cos^2(x - \tan x)$

#### **Solution**

$$\begin{aligned}\lim_{x \rightarrow \pi} \cos^2(x - \tan x) &= \cos^2(\pi - \tan \pi) \\ &= \cos^2(\pi) \\ &= (-1)^2 \\ &= 1\end{aligned}$$

### ***Exercise***

Find the limit  $\lim_{x \rightarrow 0} \frac{8x}{3 \sin x - x}$

#### **Solution**

$$\lim_{x \rightarrow 0} \frac{8x}{3\sin x - x} = \frac{0}{0}$$

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{8 \cancel{x}}{3 \frac{\sin \cancel{x}}{\cancel{x}} - \cancel{x}} \\ &= \frac{8}{3 \lim_{x \rightarrow 0} \frac{\sin x}{x} - 1} \\ &= \frac{8}{3 - 1} \\ &= 4 \end{aligned}$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

### Exercise

Find the limit  $\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\sin x}$

### Solution

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\sin x} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{1 - 2\sin^2 x - 1}{\sin x} \\ &= \lim_{x \rightarrow 0} \frac{-2\sin^2 x}{\sin x} \\ &= -2 \lim_{x \rightarrow 0} \sin x \\ &= 0 \end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$

### Solution

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}} &= \lim_{x \rightarrow -\infty} \frac{3x^3}{\sqrt{x^6}} \\ &= \lim_{x \rightarrow -\infty} \frac{3x^3}{x^3} \\ &= 3 \end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \frac{x^2 - 4x + 8}{3x^3}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{x^2 - 4x + 8}{3x^3} &= \lim_{x \rightarrow -\infty} \frac{x^2}{3x^3} \\ &= \lim_{x \rightarrow -\infty} \frac{1}{3x} \\ &= 0\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \frac{2x^2 + 3}{5x^2 + 7}$

#### Solution

$$\lim_{x \rightarrow -\infty} \frac{2x^2 + 3}{5x^2 + 7} = \frac{2}{5}$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{x^4 + x^3}{12x^3 + 128}$

#### Solution

$$\lim_{x \rightarrow \infty} \frac{x^4 + x^3}{12x^3 + 128} = \infty$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}}$

#### Solution

Since  $x \rightarrow -\infty$  and inside the square root can't be negative

$$\lim_{x \rightarrow -\infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}} = \text{not defined}$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{-\sqrt{x}} \\ &= -1\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}} &= \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x}}{\sqrt[3]{x}} \\ &= 1\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{1}{x^4}}{\frac{1}{x^2} - \frac{1}{x^3}}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{1}{x^4}}{\frac{1}{x^2} - \frac{1}{x^3}} &= \frac{0}{0} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{x^3 + 1}{x^4}}{\frac{x - 1}{x^3}} \\ &= \lim_{x \rightarrow \infty} \frac{x^3 + 1}{x - 1} \cdot \frac{x^3}{x^4} \\ &= \lim_{x \rightarrow \infty} \frac{x^3 + 1}{x(x - 1)} \\ &= \lim_{x \rightarrow \infty} \frac{x^3}{x^2} \\ &= \infty\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}} &= \lim_{x \rightarrow \infty} \frac{2x^{5/3}}{x^{8/5}} \\ &= \lim_{x \rightarrow \infty} 2x^{\left(\frac{5}{3} - \frac{8}{5}\right)} \\ &= \lim_{x \rightarrow \infty} 2x^{\frac{1}{15}} \\ &= \infty\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 2^+} \ln(x - 2)$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 2^+} \ln(x - 2) &= \ln(0^+) \\ &= -\infty\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 1} x^2 \ln(2 - \sqrt{x})$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 1} x^2 \ln(2 - \sqrt{x}) &= \ln(2 - 1) \\ &= \ln 1 \\ &= 0\end{aligned}$$

### Exercise

Find the limit  $\lim_{\theta \rightarrow 0^+} \sqrt{\theta} e^{\cos \frac{\pi}{\theta}}$

### Solution



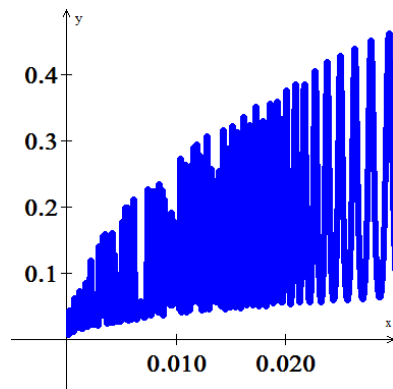
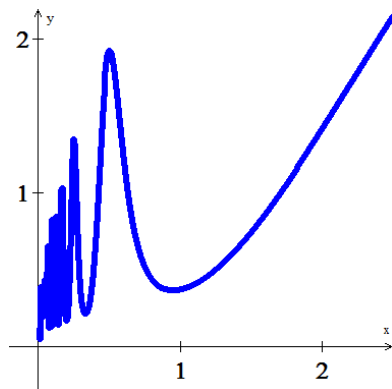
$$\lim_{\theta \rightarrow 0^+} \sqrt{\theta} e^{\cos \frac{\pi}{\theta}} = 0 \cdot e^{\cos \infty}$$

$$-1 \leq \cos \frac{\pi}{\theta} \leq 1$$

$$e^{-1} \leq e^{\cos \frac{\pi}{\theta}} \leq e$$

$$0 \cdot \frac{1}{e} \leq 0 \cdot e^{\cos \frac{\pi}{\theta}} \leq 0 \cdot e$$

$$\lim_{\theta \rightarrow 0^+} \sqrt{\theta} e^{\cos \frac{\pi}{\theta}} = 0$$



### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{2x-3}{5x+6}$

#### Solution

$$\lim_{x \rightarrow \infty} \frac{2x-3}{5x+6} = \lim_{x \rightarrow \infty} \frac{2x}{5x} = \frac{2}{5}$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{2x^2-3}{5x^2+6}$

#### Solution

$$\lim_{x \rightarrow \infty} \frac{2x^2-3}{5x^2+6} = \lim_{x \rightarrow \infty} \frac{2x^2}{5x^2} = \frac{2}{5}$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{2x-3}{5x^3+6}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2x-3}{5x^3+6} &= \lim_{x \rightarrow \infty} \frac{2x}{5x^3} \\ &= 0\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{1}{5x^2-3x+6}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{1}{5x^2-3x+6} &= \lim_{x \rightarrow \infty} \frac{1}{5x^2} \\ &= 0\end{aligned}$$

### Exercise

Find the limit  $\lim_{\theta \rightarrow 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta}$

### Solution

$$\begin{aligned}\lim_{\theta \rightarrow 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta} &= \frac{0}{0} \\ &= \lim_{\theta \rightarrow 0} \frac{1}{\frac{\sin \theta}{\theta}} \cdot \frac{1}{\sin \theta} \cdot \frac{\cos 4\theta}{\sin 4\theta} \cdot \frac{\sin^2 2\theta}{\cos^2 2\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{1}{\frac{\sin \theta}{\theta}} \lim_{\theta \rightarrow 0} \frac{\cos 4\theta}{\cos^2 2\theta} \lim_{\theta \rightarrow 0} \frac{1}{\sin \theta} \cdot \frac{\sin 2\theta \sin 2\theta}{2 \sin 2\theta \cos 2\theta} \\ &= (1)(1) \lim_{\theta \rightarrow 0} \frac{1}{\sin \theta} \cdot \frac{2 \sin \theta \cos \theta}{2 \cos 2\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{\cos \theta}{\cos 2\theta} \\ &= 1\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 0^+} \frac{\sqrt{x^2 + 4x + 5} - \sqrt{5}}{x}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{\sqrt{x^2 + 4x + 5} - \sqrt{5}}{x} &= \frac{\sqrt{5} - \sqrt{5}}{0} = \frac{0}{0} \\&= \lim_{x \rightarrow 0^+} \frac{\sqrt{x^2 + 4x + 5} - \sqrt{5}}{x} \cdot \frac{\sqrt{x^2 + 4x + 5} + \sqrt{5}}{\sqrt{x^2 + 4x + 5} + \sqrt{5}} \\&= \lim_{x \rightarrow 0^+} \frac{x^2 + 4x + 5 - 5}{x(\sqrt{x^2 + 4x + 5} + \sqrt{5})} \\&= \lim_{x \rightarrow 0^+} \frac{x(x + 4)}{x(\sqrt{x^2 + 4x + 5} + \sqrt{5})} \\&= \lim_{x \rightarrow 0^+} \frac{x + 4}{\sqrt{x^2 + 4x + 5} + \sqrt{5}} \\&= \frac{4}{\sqrt{5} + \sqrt{5}} \\&= \frac{4}{2\sqrt{5}} \\&= \frac{2}{\sqrt{5}}\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow 2} \frac{x^4 - 16}{x - 2}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^4 - 16}{x - 2} &= \frac{16 - 16}{2 - 2} = \frac{0}{0} \\&= \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)(x^2 + 4)}{x - 2} \\&= \lim_{x \rightarrow 2} (x + 2)(x^2 + 4) \\&= (4)(8) \\&= 32\end{aligned}$$
$$a^4 - b^4 = (a - b)(a + b)(a^2 + b^2)$$

### Exercise

Find the limit  $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2} &= \frac{0}{0} \\&= \lim_{x \rightarrow 2} \frac{(x - 2)(x^2 + 2x + 4)}{x - 2} \\&= \lim_{x \rightarrow 2} (x^2 + 2x + 4) \\&= 4 + 4 + 4 \\&= 12\end{aligned}$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4} &= \lim_{x \rightarrow -\infty} \frac{-5x}{2x} \\&= -\frac{5}{2}\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1}}{x + 1}$

#### Solution

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1}}{x + 1} &= \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2}}{x} \\&= \lim_{x \rightarrow -\infty} \frac{|x|}{x} \\&= -1\end{aligned}$$

### ***Exercise***

Find the limit  $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x + 1}$

### **Solution**

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x + 1} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2}}{x} \\ &= \lim_{x \rightarrow \infty} \frac{|x|}{x} \\ &= 1\end{aligned}$$

### ***Exercise***

Find the limit  $\lim_{x \rightarrow \infty} \frac{x - 3}{\sqrt{4x^2 + 25}}$

### **Solution**

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{x - 3}{\sqrt{4x^2 + 25}} &= \lim_{x \rightarrow \infty} \frac{x}{2|x|} \\ &= \frac{1}{2}\end{aligned}$$

### ***Exercise***

Find the limit  $\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$

### **Solution**

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}} &= \lim_{x \rightarrow -\infty} \frac{3x^3}{x^3} \\ &= 3\end{aligned}$$

### ***Exercise***

Find the limit  $\lim_{x \rightarrow \infty} \frac{x^4 - x}{15x^3 + 4}$

### **Solution**

$$\lim_{x \rightarrow \infty} \frac{x^4 - x}{15x^3 + 4} = \infty$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x}$

### Solution

$$-1 \leq \sin x \leq 1$$

$$\lim_{x \rightarrow \infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x} = \lim_{x \rightarrow \infty} \frac{x}{x} = 1$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{x^{2/3} - x^{-1}}{x^{2/3} + \cos^2 x}$

### Solution

$$-1 \leq \cos x \leq 1$$

$$0 \leq \cos^2 x \leq 1$$

$$\lim_{x \rightarrow \infty} \frac{x^{2/3} - \frac{1}{x}}{x^{2/3} + \cos^2 x} = \lim_{x \rightarrow \infty} \frac{x^{2/3}}{x^{2/3}} = 1$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{\sin 2x}{x}$

### Solution

$$-1 \leq \sin 2x \leq 1$$

$$-\lim_{x \rightarrow \infty} \frac{1}{x} \leq \lim_{x \rightarrow \infty} \frac{\sin 2x}{x} \leq \lim_{x \rightarrow \infty} \frac{1}{x}$$

$$0 \leq \lim_{x \rightarrow \infty} \frac{\sin 2x}{x} \leq 0$$

$$\lim_{x \rightarrow \infty} \frac{\sin 2x}{x} = 0$$

### Exercise

Find the limit  $\lim_{x \rightarrow 0} \frac{\sin 5x}{3x}$

### Solution

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 5x}{3x} &= \lim_{5x \rightarrow 0} \frac{\overset{5}{\cancel{5}} \cdot \sin \underset{5x}{\cancel{5}x}}{3 \cdot \underset{5x}{\cancel{5}}} \\ &= \frac{5}{3} \end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \frac{\cos x}{2x}$

### Solution

$$-1 \leq \cos x \leq 1$$

$$- \lim_{x \rightarrow \infty} \frac{1}{2x} \leq \lim_{x \rightarrow \infty} \frac{\cos x}{2x} \leq \lim_{x \rightarrow \infty} \frac{1}{2x}$$

$$0 \leq \lim_{x \rightarrow \infty} \frac{\cos x}{2x} \leq 0$$

$$\lim_{x \rightarrow \infty} \frac{\cos x}{2x} = 0$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \left( \frac{x^2 + x - 1}{8x^2 - 3} \right)^{1/3}$

### Solution

$$\begin{aligned} \lim_{x \rightarrow -\infty} \left( \frac{x^2 + x - 1}{8x^2 - 3} \right)^{1/3} &= \lim_{x \rightarrow -\infty} \left( \frac{x^2}{8x^2} \right)^{1/3} \\ &= \left( \frac{1}{8} \right)^{1/3} \\ &= \frac{1}{2} \end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1} &= \frac{3 - 3}{-1 + 1} = \frac{0}{0} \\&= \lim_{x \rightarrow -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1} \cdot \frac{\sqrt{x^2 + 8} + 3}{\sqrt{x^2 + 8} + 3} \\&= \lim_{x \rightarrow -1} \frac{x^2 + 8 - 9}{(x + 1)(\sqrt{x^2 + 8} + 3)} \\&= \lim_{x \rightarrow -1} \frac{x^2 - 1}{(x + 1)(\sqrt{x^2 + 8} + 3)} \\&= \lim_{x \rightarrow -1} \frac{(x - 1)(x + 1)}{(x + 1)(\sqrt{x^2 + 8} + 3)} \\&= \lim_{x \rightarrow -1} \frac{x - 1}{\sqrt{x^2 + 8} + 3} \\&= \frac{0}{6} \\&= 0\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \left( \frac{1 - x^3}{x^2 + 7x} \right)^5$

### Solution

$$\begin{aligned}\lim_{x \rightarrow -\infty} \left( \frac{1 - x^3}{x^2 + 7x} \right)^5 &= \lim_{x \rightarrow -\infty} \left( \frac{-x^3}{x^2} \right)^5 \\&= \lim_{x \rightarrow -\infty} (-x^5) \\&= \infty\end{aligned}$$



### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \sqrt{\frac{x^2 - 5x}{x^3 + x - 2}}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \sqrt{\frac{x^2 - 5x}{x^3 + x - 2}} &= \lim_{x \rightarrow \infty} \sqrt{\frac{x^2}{x^3}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} \\ &= 0\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow \infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7}$

### Solution

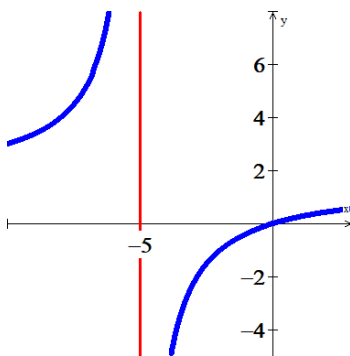
$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7} &= \lim_{x \rightarrow \infty} \frac{2\sqrt{x}}{3x} \\ &= \lim_{x \rightarrow \infty} \frac{2}{3\sqrt{x}} \\ &= 0\end{aligned}$$

### Exercise

Find the limit  $\lim_{x \rightarrow -5^-} \frac{3x}{2x + 10}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow -5^-} \frac{3x}{2x + 10} &= \frac{-15}{0^-} \\ &= \infty\end{aligned}$$

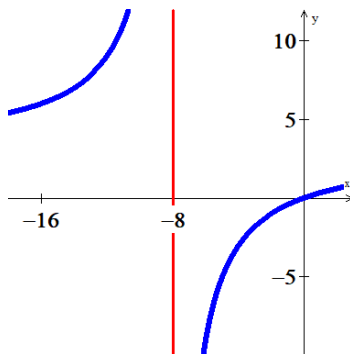


### Exercise

Find the limit  $\lim_{x \rightarrow -8^+} \frac{3x}{x + 8}$

### Solution

$$\begin{aligned}\lim_{x \rightarrow -8^+} \frac{3x}{x + 8} &= \frac{-24}{0^+} \\ &= -\infty\end{aligned}$$



### Exercise

Find the limit  $\lim_{x \rightarrow 0} \frac{-1}{x^2(x+1)}$

### Solution

$$\lim_{x \rightarrow 0} \frac{-1}{x^2(x+1)} = -\frac{1}{0} \\ = -\infty$$

### Exercise

Find the limit  $\lim_{x \rightarrow 7} \frac{4}{(x-7)^2}$

### Solution

$$\lim_{x \rightarrow 7} \frac{4}{(x-7)^2} = \frac{4}{0} \\ = \infty$$

### Exercise

Find the limit  $\lim_{x \rightarrow 0} \frac{1}{x^{2/3}}$

### Solution

$$\lim_{x \rightarrow 0} \frac{1}{x^{2/3}} = \infty$$

### Exercise

Find the limit  $\lim_{x \rightarrow -\infty} \left( x + \sqrt{x^2 - 4x + 2} \right)$

### Solution

$$\begin{aligned} \lim_{x \rightarrow -\infty} \left( x + \sqrt{x^2 - 4x + 2} \right) &= -\infty + \infty \\ &= \lim_{x \rightarrow -\infty} \left( x + \sqrt{x^2 - 4x + 2} \right) \cdot \frac{x - \sqrt{x^2 - 4x + 2}}{x - \sqrt{x^2 - 4x + 2}} \\ &= \lim_{x \rightarrow -\infty} \frac{x^2 - x^2 + 4x - 2}{x - \sqrt{x^2 - 4x + 2}}. \end{aligned}$$

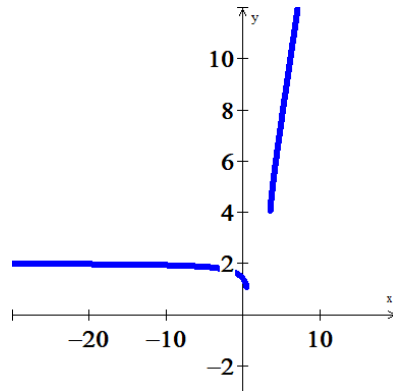
$$= \lim_{x \rightarrow -\infty} \frac{4x-2}{x-\sqrt{x^2-4x+2}}$$

$$= \lim_{x \rightarrow -\infty} \frac{4x-2}{x-|x|} \qquad x \rightarrow -\infty \quad (x < 0) \rightarrow |x| = -x$$

$$= \lim_{x \rightarrow -\infty} \frac{4x}{x+x}$$

$$= \lim_{x \rightarrow -\infty} \frac{4x}{2x}$$

$$= 2$$

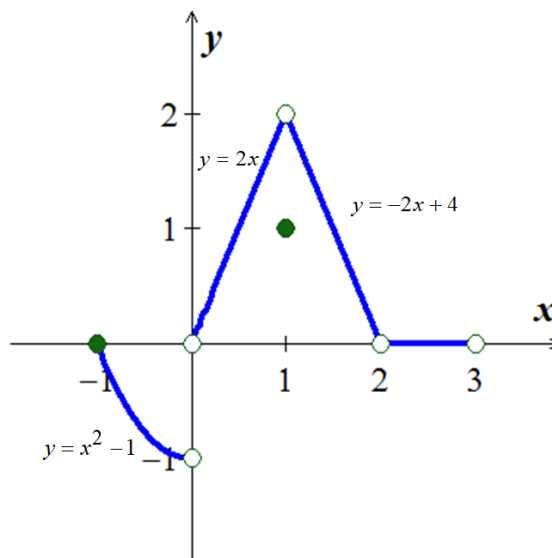


## ***Solution***      ***Section 1.5 – Continuity***

### ***Exercise***

Given the graphed function  $f(x)$

- a) Does  $f(-1)$  exist?
- b) Does  $\lim_{x \rightarrow -1^+} f(x)$  exist?
- c) Does  $\lim_{x \rightarrow -1^+} f(x) = f(-1)$ ?
- d) Is  $f$  continuous at  $x = -1$ ?
- e) Does  $f(1)$  exist?
- f) Does  $\lim_{x \rightarrow 1} f(x)$  exist?
- g) Does  $\lim_{x \rightarrow 1} f(x) = f(1)$ ?
- h) Is  $f$  continuous at  $x = 1$ ?



### ***Solution***

- a) Yes  $\underline{f(-1) = 0}$
- b) Yes,  $\lim_{x \rightarrow -1^+} f(x) = 0$
- c) Yes
- d) Yes
- e) Yes,  $\underline{f(1) = 1}$
- f) Yes,  $\lim_{x \rightarrow 1} f(x) = 2$
- g) No
- h) No

### ***Exercise***

At what points is the function  $y = \frac{1}{x-2} - 3x$  continuous?

### ***Solution***

The function is continuous everywhere except when  $x - 2 = 0 \Rightarrow x = 2$

### ***Exercise***

At what points is the function  $y = \frac{x+3}{x^2-3x-10}$  continuous?

#### **Solution**

The function is continuous everywhere except when  $x^2 - 3x - 10 = 0 \Rightarrow x = -2, 5$

### ***Exercise***

At what points is the function  $y = |x-1| + \sin x$  continuous?

#### **Solution**

The function is continuous everywhere

### ***Exercise***

At what points is the function  $y = \frac{x+2}{\cos x}$  continuous?

#### **Solution**

The function is continuous everywhere except when  $\cos x = 0 \Rightarrow x = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}$

### ***Exercise***

At what points is the function  $y = \tan \frac{\pi x}{2}$  continuous?

#### **Solution**

The function is continuous everywhere except when  $x = 2n-1, \quad n \in \mathbb{Z}$

### ***Exercise***

At what points is the function  $y = \frac{x \tan x}{x^2 + 1}$  continuous?

#### **Solution**

The function is continuous everywhere except when  $x = (2n-1)\frac{\pi}{2}, \quad n \in \mathbb{Z}$

### ***Exercise***

At what points is the function  $y = \frac{\sqrt{x^4+1}}{1+\sin^2 x}$  continuous?

#### **Solution**

The function is continuous everywhere

### Exercise

At what points is the function  $y = \sqrt{2x+3}$  continuous?

#### Solution

The function is continuous on the interval  $2x+3 \geq 0 \rightarrow x \geq -\frac{3}{2} \Rightarrow \left[-\frac{3}{2}, \infty\right)$ , and discontinuous when  $x < -\frac{3}{2}$

### Exercise

At what points is the function  $y = \sqrt[4]{3x-1}$  continuous?

#### Solution

The function is continuous on the interval  $3x-1 \geq 0 \rightarrow \left[\frac{1}{3}, \infty\right)$ , and discontinuous when  $x < \frac{1}{3}$

### Exercise

At what points is the function  $y = (2-x)^{1/5}$  continuous?

#### Solution

The function is continuous everywhere  $\forall x$

### Exercise

Find  $\lim_{x \rightarrow \pi} \sin(x - \sin x)$ , then is the function continuous at the point being approached?

#### Solution

$$\begin{aligned}\lim_{x \rightarrow \pi} \sin(x - \sin x) &= \sin(\pi - \sin \pi) \\ &= \sin(\pi - 0) \\ &= \sin(\pi) \\ &= 0\end{aligned}$$

The function is continuous at  $x = \pi$

### Exercise

Find  $\lim_{x \rightarrow 0} \tan\left(\frac{\pi}{4} \cos(\sin x^{1/3})\right)$ , then is the function continuous at the point being approached?

#### Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \tan\left(\frac{\pi}{4} \cos(\sin x^{1/3})\right) &= \tan\left(\frac{\pi}{4} \cos(\sin(0)^{1/3})\right) \\ &= \tan\left(\frac{\pi}{4} \cos(0)\right)\end{aligned}$$

$$= \tan\left(\frac{\pi}{4}\right)$$

$$= 1$$

The function is continuous at  $x = 0$

### Exercise

Find  $\lim_{t \rightarrow 0} \cos\left(\frac{\pi}{\sqrt{19 - 3 \sec 2t}}\right)$ , then is the function continuous at the point being approached?

### Solution

$$\begin{aligned} \lim_{t \rightarrow 0} \cos\left(\frac{\pi}{\sqrt{19 - 3 \sec 2t}}\right) &= \cos\left(\frac{\pi}{\sqrt{19 - 3 \sec 2(0)}}\right) \\ &= \cos\left(\frac{\pi}{\sqrt{19 - 3}}\right) \\ &= \cos\left(\frac{\pi}{\sqrt{16}}\right) \\ &= \cos\left(\frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2} \end{aligned}$$

$\therefore$  The function is continuous at  $t = 0$

### Exercise

Explain why the equation  $\cos x = x$  has at least one solution.

### Solution

$$\cos x - x = 0$$

$$\begin{cases} \text{if } x = -\frac{\pi}{2} & \rightarrow \cos\left(-\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) > 0 \\ \text{if } x = \frac{\pi}{2} & \rightarrow \cos\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right) < 0 \end{cases}$$

$$\Rightarrow \cos x - x = 0$$

for some  $x$  between  $-\frac{\pi}{2}$  and  $\frac{\pi}{2}$

According to the Intermediate Value Theorem, and the function  $\cos x = x$  is continuous and has at least one solution.

### Exercise

Show that the equation  $x^3 - 15x + 1 = 0$  has three solutions in the interval  $[-4, 4]$

#### Solution

$$f(-4) = (-4)^3 - 15(-4) + 1 = -3$$

$$f(-2) = (-2)^3 - 15(-2) + 1 = 23$$

$$f(-1) = (-1)^3 - 15(-1) + 1 = 15$$

$$f(1) = (1)^3 - 15(1) + 1 = -13$$

$$f(4) = (4)^3 - 15(4) + 1 = 5$$

By the Intermediate Value Theorem,  $f(x) = 0$  for some  $x$  in each of the intervals  $-4 < x < -1$ ,

$-1 < x < 1$ , and  $1 < x < 4$ . Thus,  $x^3 - 15x + 1 = 0$  has three solutions in  $[-4, 4]$ . Since the polynomial of degree 3 can have at most 3 solutions, these are the solutions.

### Exercise

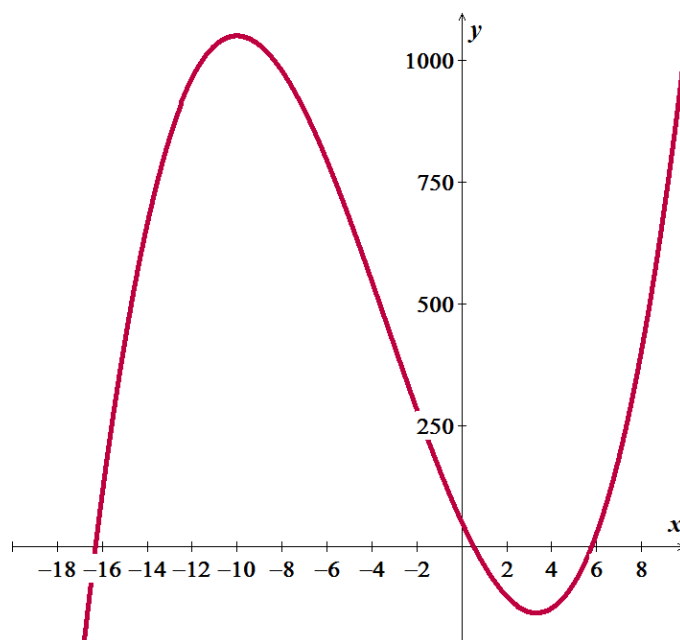
Show that the equation has three solutions in the given interval  $x^3 + 10x^2 - 100x + 50 = 0$ ;  $(-20, 10)$

#### Solution

$x$	$y$
-19	-1299
-18	-742
-17	-273
-16	114
-15	425
-14	666
-13	962
-12	1029
-10	1050
-9	1031
-8	978
-7	897
-6	794
-5	675
-4	546



-3	413
-2	282
-1	159
0	50
1	-39
2	-102
3	-133
4	-126
5	-75
6	26



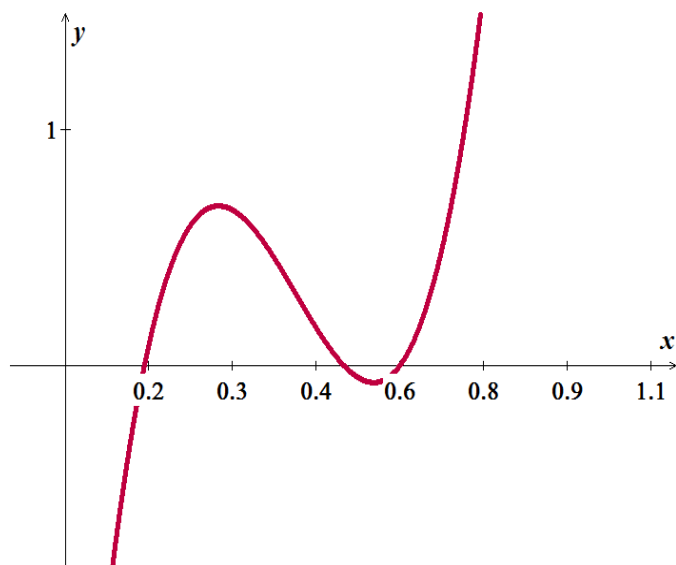
By the Intermediate Value Theorem,  $f(x) = 0$  for some  $x$  in each of the intervals  $-17 < x < -16$ ,  $0 < x < 1$ , and  $5 < x < 6$ .

### Exercise

Show that the equation has three solutions in the given interval  $70x^3 - 87x^2 + 32x - 3 = 0$ ;  $(0, 1)$

### Solution

$x$	$y$
.05	-1.6
.1	-0.6
.15	0.08
.2	.48
.25	.656
.3	.66
.35	.543
.4	.36
.45	.161
.5	0
.55	-.07
.6	0
.65	.266
.7	.78
.75	1.6
.8	2.76
.85	4.33
.9	6.36
.95	8.9



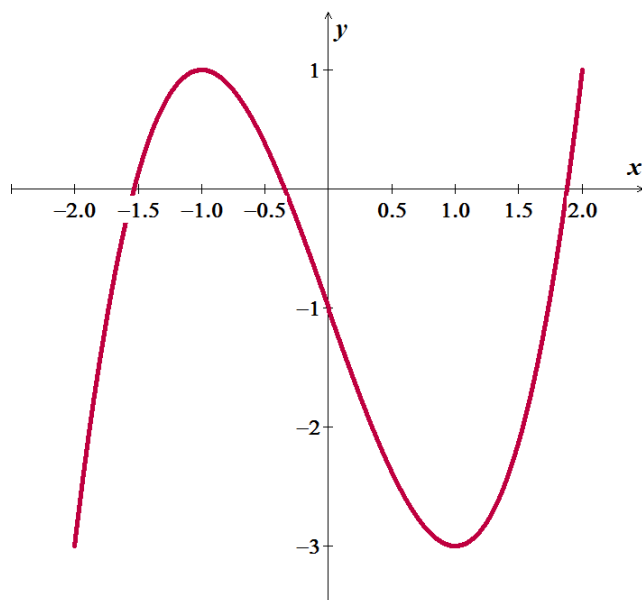
By the Intermediate Value Theorem,  $f(x) = 0$  for some  $x$  in each of the intervals  $0.1 < x < 0.15$ ,  $0.5 < x < 0.55$ , and  $0.55 < x < 0.6$ .

### Exercise

Show that the equation has three solutions in the given interval  $x^3 - 3x - 1 = 0$ ;  $[-2, 2]$

### Solution

$x$	$y$
-2	-3.0
-1.75	-1.109
-1.5	0.125
-1.25	0.797
-1.0	1
-0.75	0.828
-0.5	0.375
-0.25	-0.266
0	-1.0
0.25	-1.73
0.5	-2.375
0.75	-2.828
1.0	-3.0
1.25	-2.797
1.5	-2.12
1.75	-0.89
2.	1.0



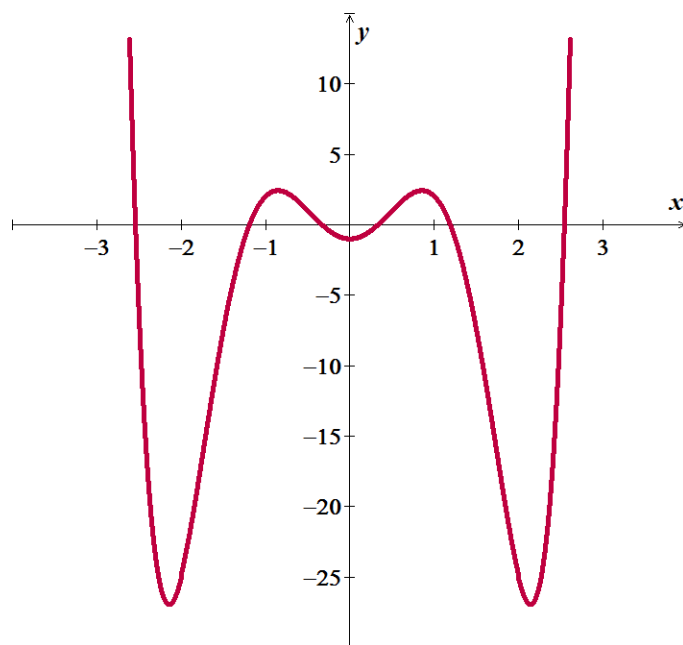
By the Intermediate Value Theorem,  $f(x) = 0$  for some  $x$  in each of the intervals  $-1.75 < x < -1.5$ ,  $-0.5 < x < -0.25$ , and  $1.75 < x < 2$ .

### Exercise

Show that the equation has six solutions in the given interval  $x^6 - 8x^4 + 10x^2 - 1 = 0$ ;  $[-3, 3]$

### Solution

$x$	$y$
-3.0	170.0
-2.5	-6.86
-2.0	-25.0
-1.5	-7.61
-1.0	2.0
-0.5	1.02
0.0	-1.0
0.5	1.01
1.0	2.0
1.5	-7.6
2.0	-25.0
2.5	-6.86
3.0	170.0



By the Intermediate Value Theorem,  $f(x) = 0$  for some  $x$  in each of the intervals  $-3.0 < x < -2.5$ ,  $-1.5 < x < -1.0$ ,  $-0.5 \leq x \leq 0$ ,  $0.0 \leq x \leq 0.5$ ,  $1.0 \leq x \leq 1.5$  and  $2.5 < x < 3.0$ .

### Exercise

If functions  $f(x)$  and  $g(x)$  are continuous for  $0 \leq x \leq 1$ , could  $\frac{f(x)}{g(x)}$  possibly be discontinuous at a point of  $[0, 1]$ ? Give reason for your answer.

### Solution

Yes, if we can get a value of  $g(x)$  is between  $[0, 1]$ ,  $x = \frac{1}{2} \Rightarrow g(x) = 2x - 1$  and  $f(x) = x$ .

Then  $\frac{f(x)}{g(x)} = \frac{x}{2x-1} \Rightarrow \frac{f(x)}{g(x)}$  is discontinuous at  $x = \frac{1}{2}$

### Exercise

Suppose that a function  $f$  is continuous on the closed interval  $[0, 1]$  and that  $0 \leq f(x) \leq 1$  for every  $x$  in  $[0, 1]$ . Show that there must exist a number  $c$  in  $[0, 1]$  such that  $f(c) = c$  ( $c$  is called a **fixed point** of  $f$ ).

### Solution

Let  $f(x) = x \Rightarrow f(0) = 0$  or  $f(1) = 1$ . In these cases,  $c = 0$  or  $c = 1$ .

Let  $f(0) = a > 0$  and  $f(1) = b < 1$  because  $0 \leq f(x) \leq 1$ .

Define  $g(x) = f(x) - x \Rightarrow g$  is continuous on  $[0, 1]$ .

$$\Rightarrow \begin{cases} g(0) = f(0) - 0 = a > 0 \\ g(1) = f(1) - 1 = b - 1 < 0 \end{cases}$$

By the Intermediate Value Theorem there is a number  $c$  in  $[0, 1]$  such that

$$g(c) = 0 \Rightarrow f(c) - c = 0 \Rightarrow f(c) = c$$

### Exercise

Use the Intermediate Value Theorem to show that the equation  $x^5 + 7x + 5 = 0$  has a solution in the interval  $(-1, 0)$ .

### Solution

$$f(-1) = -1 - 7 + 5 = -3 < 0$$

$$f(0) = 5 > 0$$

By Intermediate value theorem, the function has a solution in  $(-1, 0)$

### Exercise

The amount of an antibiotic (in *mg*) in the blood  $t$  hours after an intravenous line is opened is given by

$$m(t) = 100(e^{-0.1t} - e^{-0.3t})$$

- a) Use the Intermediate Value Theorem to show that the amount of drug is 30 *mg* at some time in the interval  $[0, 5]$  and again at some time in the interval  $[5, 15]$
- b) Estimate the times at which  $m = 30$  *mg*
- c) Is the amount of drug in the blood ever 50 *mg*?

### Solution

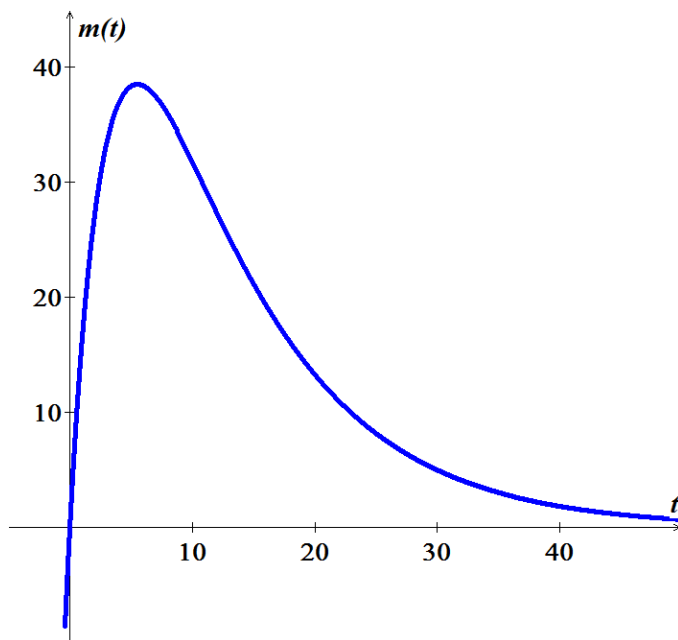
a)  $m(0) = 100(1 - 1) = 0$

$$m(5) \approx 38.34 > 30$$

$$m(15) \approx 21.2 < 30$$

30 is an intermediate value between for both  $[0, 5]$  and  $[5, 15]$ .

b)  $m(t) = 100(e^{-0.1t} - e^{-0.3t}) = 30$



$$e^{-0.1t} - e^{-0.3t} = 0.3 \xrightarrow{\text{software}} \begin{cases} t_1 \approx 2.4 \\ t_2 \approx 10.8 \end{cases}$$

- c) No, peak is 38.5 (using the graph)

### Exercise

Determine whether the following functions are continuous at  $a$ .  $f(x) = \frac{1}{x-5}$ ;  $a = 5$

#### Solution

$$f(5) \text{ is not defined}$$

The function is continuous everywhere except @  $x = 5$

### Exercise

Determine whether the following functions are continuous at  $a$ .  $h(x) = \sqrt{x^2 - 9}$ ;  $a = 3$

#### Solution

$$\lim_{x \rightarrow 3^-} h(x) \text{ is not defined} \quad \therefore h \text{ is discontinuous @ } 3$$

### Exercise

Determine whether the following functions are continuous at  $a$ .  $g(x) = \begin{cases} \frac{x^2 - 16}{x - 4} & \text{if } x \neq 4; \\ 9 & \text{if } x = 4 \end{cases}$ ;  $a = 4$

#### Solution

$$\lim_{x \rightarrow 4} g(x) = \lim_{x \rightarrow 4} \frac{(x-4)(x+4)}{x-4} = \lim_{x \rightarrow 4} (x+4) = 8 \neq 9 = g(4)$$

$\therefore g$  is discontinuous @ 4

### Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints  $f(x) = \sqrt{x^2 - 5}$

#### Solution

$$\sqrt{x^2 - 5} \geq 0 \Rightarrow x \leq -5 \text{ \& } x \geq 5$$

The function is continuous at  $-5$  to the left and right of  $x = 5$

### Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints  $f(x) = e^{\sqrt{x-2}}$

#### Solution

The function is continuous at and to the right of  $x = 2$

### Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the

endpoints  $f(x) = \frac{2x}{x^3 - 25x}$

### Solution

The function is continuous everywhere except at  $x = 0, \pm 5$

The function is continuous to the left of  $-5$ , then to the right of  $-5$  to the left of  $0$ , then to the right of  $0$  thru the left of  $5$  then to the right of  $5$ .

### Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the

endpoints  $f(x) = \cos e^x$

### Solution

The function is continuous everywhere.

### Exercise

$$\text{Let } g(x) = \begin{cases} 5x - 2 & \text{if } x < 1 \\ a & \text{if } x = 1 \\ ax^2 + bx & \text{if } x > 1 \end{cases}$$

Determine values of the constants  $a$  and  $b$  for which  $g(x)$  is continuous at  $x = 1$

### Solution

$$\begin{aligned} \lim_{x \rightarrow 1^-} g(x) &= g(1) \\ &= 5 - 2 \\ &= \underline{3 = a} \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 1^+} g(x) &= g(1) \\ &= a + b \\ &= 3 + b = 3 \end{aligned}$$

$$\rightarrow \underline{b = 0}$$



## ***Solution***      **Section 1.6 – Precise Definition of Limits**

### ***Exercise***

Sketch the interval  $(a, b)$  on the  $x$ -axis with the point  $x_0$  inside. Then find a value of  $\delta > 0$  such that for

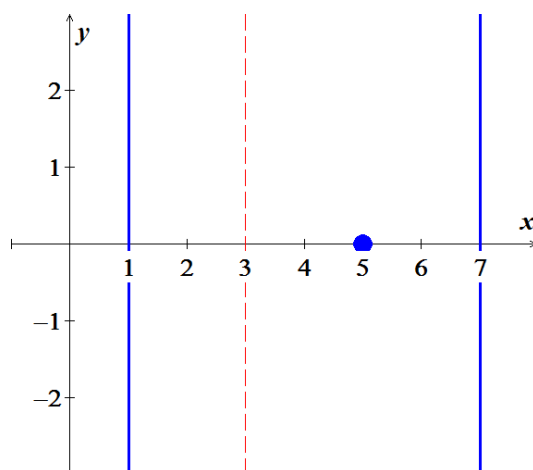
$$\text{all } x, 0 < |x - x_0| < \delta \Rightarrow a < x < b \text{ for } a = 1, \quad b = 7, \quad x_0 = 5$$

### **Solution**

$$\begin{aligned} |x - 5| < \delta &\Rightarrow -\delta < x - 5 < \delta \\ &\Rightarrow -\delta + 5 < x < \delta + 5 \end{aligned}$$

$$-\delta + 5 = 1 \Rightarrow \underline{\delta = 4}$$

$$\delta + 5 = 7 \Rightarrow \underline{\delta = 2}$$



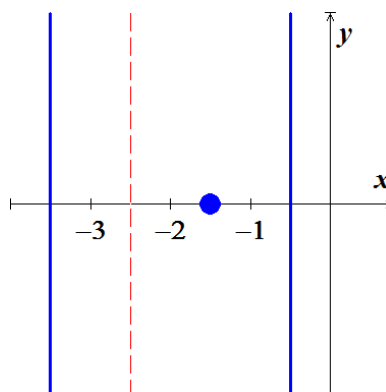
### ***Exercise***

Sketch the interval  $(a, b)$  on the  $x$ -axis with the point  $x_0$  inside. Then find a value of  $\delta > 0$  such that for

$$\text{all } x, 0 < |x - x_0| < \delta \Rightarrow a < x < b \text{ for } a = -\frac{7}{2}, \quad b = -\frac{1}{2}, \quad x_0 = -\frac{3}{2}$$

### **Solution**

$$\begin{aligned} \left|x + \frac{3}{2}\right| < \delta \\ -\delta < x + \frac{3}{2} < \delta \\ -\delta - \frac{3}{2} < x < \delta - \frac{3}{2} \\ -\delta - \frac{3}{2} = -\frac{7}{2} &\Rightarrow \underline{\delta = \frac{7}{2} - \frac{3}{2} = 2} \\ \delta - \frac{3}{2} = -\frac{1}{2} &\Rightarrow \underline{\delta = \frac{1}{2} - \frac{3}{2} = -1} \end{aligned}$$



### Exercise

Use the graph to find a  $\delta > 0$  such that for all  $x$   $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$

### Solution

**Given:**  $a = -3.1$ ,  $b = -2.9$ ,  $x_0 = -3$

$$|x + 3| < \delta$$

$$-\delta < x + 3 < \delta$$

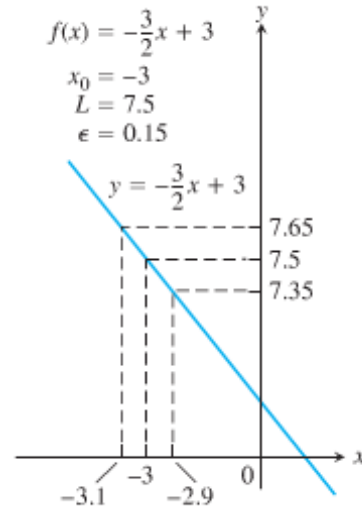
$$-\delta - 3 < x < \delta - 3$$

$$-\delta - 3 = -3.1$$

$$\Rightarrow |\underline{\delta} = 3.1 - 3 = \underline{0.1}|$$

$$\delta - 3 = -2.9$$

$$\Rightarrow |\underline{\delta} = 3 - 2.9 = \underline{0.1}|$$



### Exercise

Find an open interval about  $x_0$  on which the inequality  $|f(x) - L| < \epsilon$  holds. Then give a value for  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - x_0| < \delta$  the inequality  $|f(x) - L| < \epsilon$  holds.

$$f(x) = x + 1, \quad L = 5, \quad x_0 = 4, \quad \epsilon = 0.01$$

### Solution

$$|(x + 1) - 5| < .01$$

$$|x - 4| < .01$$

$$-.01 < x - 4 < .01$$

$$-.01 + 4 < x - 4 + 4 < .01 + 4$$

$$3.99 < x < 4.01$$

$$|x - 4| < \delta$$

$$-\delta < x - 4 < \delta$$

$$-\delta + 4 < x < \delta + 4$$

$$-\delta + 4 = 3.99$$

$$|\underline{\delta} = 4 - 3.99 = \underline{0.01}|$$

$$\delta + 4 = 4.01$$

$$|\underline{\delta} = 4.01 - 4 = \underline{0.01}|$$

$$\Rightarrow \underline{\delta = .01}$$

### Exercise

Find an open interval about  $x_0$  on which the inequality  $|f(x) - L| < \varepsilon$  holds. Then give a value for  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - x_0| < \delta$  the inequality  $|f(x) - L| < \varepsilon$  holds.

$$f(x) = 2x - 1, \quad L = 3, \quad x_0 = 2, \quad \varepsilon = 0.1$$

### Solution

$$|2x - 1 - 3| < .1$$

$$|2x - 4| < .1$$

$$-.1 < 2x - 4 < .1$$

$$-.1 + 4 < 2x - 4 + 4 < .1 + 4$$

$$3.9 < 2x < 4.1$$

$$\frac{3.9}{2} < x < \frac{4.1}{2}$$

$$1.95 < x < 2.05$$

$$|x - 2| < \delta$$

$$-\delta < x - 2 < \delta$$

$$-\delta + 2 < x < \delta + 2$$

$$-\delta + 2 = 1.95$$

$$\delta = 2 - 1.95 = \underline{0.05}$$

$$\delta + 2 = 2.05$$

$$\delta = 2.05 - 2 = \underline{0.05}$$

$$\Rightarrow \delta = \underline{.05}$$

### Exercise

Find an open interval about  $x_0$  on which the inequality  $|f(x) - L| < \varepsilon$  holds. Then give a value for  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - x_0| < \delta$  the inequality  $|f(x) - L| < \varepsilon$  holds.

$$f(x) = x + 2, \quad L = 3, \quad x_0 = 1, \quad \varepsilon = 0.001$$

### Solution

$$|x + 2 - 3| < .001$$

$$|x - 1| < .001$$

$$-.001 < x - 1 < .001$$

$$-.001 + 1 < x - 1 + 1 < .001 + 1$$

$$0.999 < x < 1.001$$

$$|x-1| < \delta$$

$$-\delta < x-1 < \delta$$

$$-\delta+1 < x < \delta+1$$

$$-\delta+1 = .999$$

$$\delta = 1 - .999 = \underline{0.001}$$

$$\delta+1 = 1.001$$

$$\delta = 1.001 - 1 = \underline{0.001}$$

$$\Rightarrow \underline{\delta = .001}$$

### ***Exercise***

Find an open interval about  $x_0$  on which the inequality  $|f(x) - L| < \varepsilon$  holds. Then give a value for  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - x_0| < \delta$  the inequality  $|f(x) - L| < \varepsilon$  holds.

$$f(x) = 3x + 2, \quad L = 2, \quad x_0 = 0, \quad \varepsilon = 0.1$$

### ***Solution***

$$|3x + 2 - 2| < .1$$

$$|3x| < .1$$

$$-.1 < 3x < .1$$

$$-\frac{.1}{3} < x < \frac{.1}{3}$$

$$-\frac{1}{30} < x < \frac{1}{30}$$

$$|x-0| < \delta$$

$$-\delta < x < \delta$$

$$-\delta = -\frac{1}{30}$$

$$\delta = \underline{\frac{1}{30}}$$

$$\Rightarrow \underline{\delta = \frac{1}{30}}$$

### Exercise

Find an open interval about  $x_0$  on which the inequality  $|f(x) - L| < \varepsilon$  holds. Then give a value for  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - x_0| < \delta$  the inequality  $|f(x) - L| < \varepsilon$  holds.

$$f(x) = \sqrt{x+1}, \quad L = 1, \quad x_0 = 0, \quad \varepsilon = 0.1$$

### Solution

$$|\sqrt{x+1} - 1| < 0.1$$

$$-0.1 < \sqrt{x+1} - 1 < 0.1$$

$$-0.1 + 1 < \sqrt{x+1} - 1 + 1 < 0.1 + 1$$

$$.9 < \sqrt{x+1} < 1.1$$

$$(.9)^2 < (\sqrt{x+1})^2 < (1.1)^2$$

$$.81 < x+1 < 1.21$$

$$.81 - 1 < x+1 - 1 < 1.21 - 1$$

$$-0.19 < x < 0.21$$

$$|x - 0| < \delta \Rightarrow -\delta < x < \delta$$

$$-\delta = -0.19 \Rightarrow \underline{\delta = 0.19} \rightarrow \underline{\delta = 0.19}$$

### Exercise

Find an open interval about  $x_0$  on which the inequality  $|f(x) - L| < \varepsilon$  holds. Then give a value for  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - x_0| < \delta$  the inequality  $|f(x) - L| < \varepsilon$  holds.

$$f(x) = \sqrt{x-7}, \quad L = 4, \quad x_0 = 23, \quad \varepsilon = 1$$

### Solution

$$|\sqrt{x-7} - 4| < 1$$

$$-1 < \sqrt{x-7} - 4 < 1$$

$$3 < \sqrt{x-7} < 5$$

$$(3)^2 < (\sqrt{x-7})^2 < (5)^2$$

$$9 < x-7 < 25$$

$$9 + 7 < x-7 + 7 < 25 + 7$$

$$16 < x < 32$$

$$|x - 23| < \delta$$

$$-\delta < x - 23 < \delta$$

$$-\delta + 23 < x < \delta + 23$$

$$-\delta + 23 = 16 \Rightarrow \delta = 23 - 16 = \underline{7}$$

$$\delta + 23 = 32 \Rightarrow \delta = 32 - 23 = \underline{9}$$

$$\Rightarrow \underline{\delta = 7}$$

### ***Exercise***

Find an open interval about  $x_0$  on which the inequality  $|f(x) - L| < \varepsilon$  holds. Then give a value for  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - x_0| < \delta$  the inequality  $|f(x) - L| < \varepsilon$  holds.

$$f(x) = x^2, \quad L = 3, \quad x_0 = \sqrt{3}, \quad \varepsilon = 0.1$$

### ***Solution***

$$|x^2 - 3| < 0.1$$

$$-0.1 < x^2 - 3 < 0.1$$

$$2.9 < x^2 < 3.1$$

$$\sqrt{2.9} < x < \sqrt{3.1}$$

$$|x - \sqrt{3}| < \delta$$

$$-\delta < x - \sqrt{3} < \delta$$

$$-\delta + \sqrt{3} < x < \delta + \sqrt{3}$$

$$-\delta + \sqrt{3} = \sqrt{2.9} \Rightarrow \delta = \sqrt{3} - \sqrt{2.9} = \underline{.029}$$

$$\delta + \sqrt{3} = \sqrt{3.1} \Rightarrow \delta = \sqrt{3.1} - \sqrt{3} = \underline{.029}$$

$$\Rightarrow \underline{\delta = .029}$$

### ***Exercise***

Find an open interval about  $x_0$  on which the inequality  $|f(x) - L| < \varepsilon$  holds. Then give a value for  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - x_0| < \delta$  the inequality  $|f(x) - L| < \varepsilon$  holds.

$$f(x) = \frac{120}{x}, \quad L = 5, \quad x_0 = 24, \quad \varepsilon = 1$$

### ***Solution***

$$\left| \frac{120}{x} - 5 \right| < 0.1$$

$$-1 < \frac{120}{x} - 5 < 1$$

$$4 < \frac{120}{x} < 6$$

$$\frac{1}{6} < \frac{x}{120} < \frac{1}{4}$$

$$\frac{1}{6}(120) < x < \frac{1}{4}(120)$$

$$20 < x < 30$$

$$|x - 24| < \delta$$

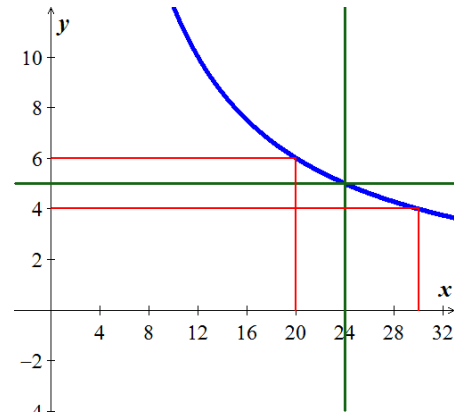
$$-\delta < x - 24 < \delta$$

$$-\delta + 24 < x < \delta + 24$$

$$-\delta + 24 = 20 \Rightarrow \delta = 24 - 20 = \underline{4}$$

$$\delta + 24 = 30 \Rightarrow \delta = 30 - 24 = \underline{6}$$

$$\Rightarrow \underline{\delta = 4}$$



## Exercise

Prove that  $\lim_{x \rightarrow 4} (9 - x) = 5$

## Solution

$$|(9 - x) - 5| < \varepsilon$$

$$-\varepsilon < 4 - x < \varepsilon$$

$$-\varepsilon - 4 < -x < \varepsilon - 4$$

$$\varepsilon + 4 > x > 4 - \varepsilon$$

$$4 - \varepsilon < x < \varepsilon + 4$$

*divide by (-).*

$$|x - 4| < \delta$$

$$-\delta < x - 4 < \delta$$

$$-\delta + 4 < x < \delta + 4$$

$$-\delta + 4 = 4 - \varepsilon \Rightarrow -\delta = -\varepsilon \Rightarrow \underline{\delta = \varepsilon}$$

$$\delta + 4 = \varepsilon + 4 \Rightarrow \underline{\delta = \varepsilon}$$

$$\Rightarrow \underline{\delta = \varepsilon}$$

### Exercise

Prove that  $\lim_{x \rightarrow 1} \frac{1}{x} = 1$

### Solution

$$\left| \frac{1}{x} - 1 \right| < \varepsilon$$

$$-\varepsilon < \frac{1}{x} - 1 < \varepsilon$$

$$-\varepsilon + 1 < \frac{1}{x} < \varepsilon + 1$$

$$\frac{1}{\varepsilon + 1} > x > \frac{1}{-\varepsilon + 1}$$

$$\frac{1}{1 + \varepsilon} < x < \frac{1}{1 - \varepsilon}$$

$$|x - 1| < \delta$$

$$-\delta < x - 1 < \delta$$

$$1 - \delta < x < 1 + \delta$$

$$1 - \delta = \frac{1}{1 + \varepsilon} \Rightarrow \delta = 1 + \frac{1}{1 + \varepsilon} = \frac{2 + \varepsilon}{1 + \varepsilon}$$

$$1 + \delta = \frac{1}{1 - \varepsilon} \Rightarrow \delta = \frac{1}{1 - \varepsilon} - 1 = \frac{\varepsilon}{1 - \varepsilon}$$

$$\text{The smallest : } \delta = \frac{\varepsilon}{1 - \varepsilon}$$

### Exercise

Prove that  $\lim_{x \rightarrow 5} \frac{x^2 - 25}{x - 5} = 10$

### Solution

$$\left| \frac{x^2 - 25}{x - 5} - 10 \right| < \varepsilon$$

$$-\varepsilon < \frac{(x - 5)(x + 5)}{x - 5} - 10 < \varepsilon$$

$$-\varepsilon + 10 < x + 5 < \varepsilon + 10$$

$$-\varepsilon + 5 < x < \varepsilon + 15$$

$$|x - 10| < \delta$$

$$-\delta < x - 10 < \delta$$

$$10 - \delta < x < 10 + \delta$$

$$10 - \delta = 5 - \varepsilon \Rightarrow \delta = 5 + \varepsilon$$



$$10 + \delta = \varepsilon + 15 \Rightarrow \underline{\delta = \varepsilon + 5}$$

The smallest:  $\underline{\delta = \varepsilon + 5}$

### Exercise

Prove that  $\lim_{x \rightarrow 0} f(x) = 0$  if  $f(x) = \begin{cases} 2x, & x < 0 \\ \frac{x}{2}, & x \geq 0 \end{cases}$

### Solution

$$\text{For } x < 0: |2x - 0| < \varepsilon$$

$$-\varepsilon < 2x < 0$$

$$-\frac{\varepsilon}{2} < x < 0$$

$$\text{For } x \geq 0: \left| \frac{x}{2} - 0 \right| < \varepsilon$$

$$0 \leq \frac{x}{2} < \varepsilon$$

$$0 \leq x < 2\varepsilon$$

$$|x - 0| < \delta \Rightarrow -\delta < x < \delta$$

$$-\delta = -\frac{\varepsilon}{2} \Rightarrow \delta = \frac{\varepsilon}{2} \rightarrow \text{the smallest: } \underline{\delta = \frac{\varepsilon}{2}}$$

### Exercise

Prove that  $\lim_{x \rightarrow 1} (5x - 2) = 3$

### Solution

$$|(5x - 2) - 3| < \varepsilon$$

$$-\varepsilon < 5x - 5 < \varepsilon$$

$$5 - \varepsilon < 5x < \varepsilon + 5$$

$$1 - \frac{1}{5}\varepsilon < x < 1 + \frac{1}{5}\varepsilon$$

$$|x - 3| < \delta$$

$$-\delta < x - 3 < \delta$$

$$3 - \delta < x < 3 + \delta$$

$$3 - \delta = 1 - \frac{1}{5}\varepsilon \Rightarrow \delta = \frac{1}{5}\varepsilon + 2$$

$$3 + \delta = 1 + \frac{1}{5}\varepsilon \Rightarrow \delta = \frac{1}{5}\varepsilon - 2 \rightarrow \text{the smallest: } \underline{\delta = \frac{1}{5}\varepsilon - 2}$$

### Exercise

Prove that  $\lim_{x \rightarrow 2} \frac{1}{(x-2)^4} = \infty$

### Solution

Let  $N > 0$  and let  $\delta = \frac{1}{\sqrt[4]{N}}$

Suppose that  $0 < |x - 2| < \delta$

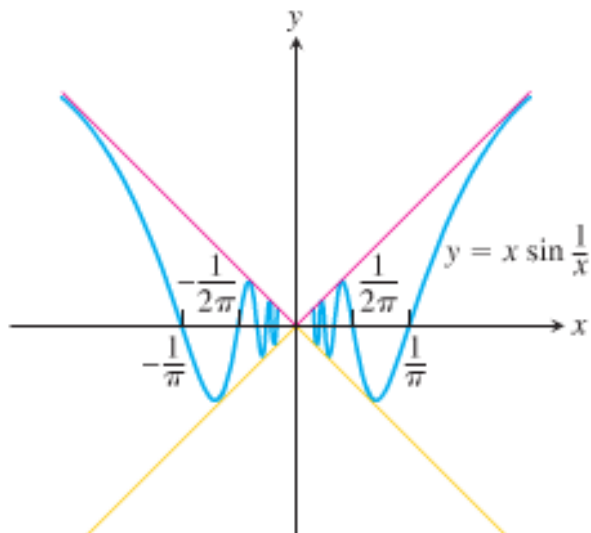
$$|x - 2| < \delta = \frac{1}{\sqrt[4]{N}}$$

$$\frac{1}{|x - 2|} > \sqrt[4]{N}$$

$$\frac{1}{(x - 2)^4} > N \quad \checkmark$$

### Exercise

Prove that  $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$



### Solution

$$\left. \begin{array}{l} -x \leq x \sin \frac{1}{x} \leq x, \quad \forall x > 0 \\ -x \geq x \sin \frac{1}{x} \geq x, \quad \forall x < 0 \end{array} \right\} \rightarrow \lim_{x \rightarrow 0} (-x) = \lim_{x \rightarrow 0} (x) = 0$$

Then by the sandwich theorem,  $\lim_{x \rightarrow 0} x \sin \left( \frac{1}{x} \right) = 0$