Solution — Section 2.3 - Harmonic Motion

Exercise

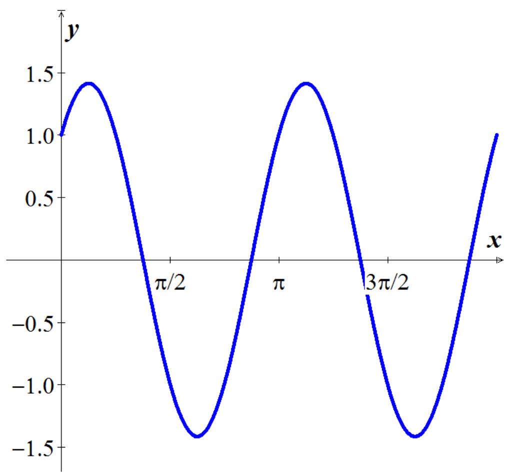
$y = \cos 2t + \sin 2t$

i. Plot the function

ii. Place the solution in the form $y = A\cos(\omega_0 t - \phi)$ and compare the graph with the plot in (***i***)

Solution

i. Plot the function



ii. Place the solution in the form $y = A\cos(\omega_0 t - \phi)$ and compare the graph with the plot in (***i***)

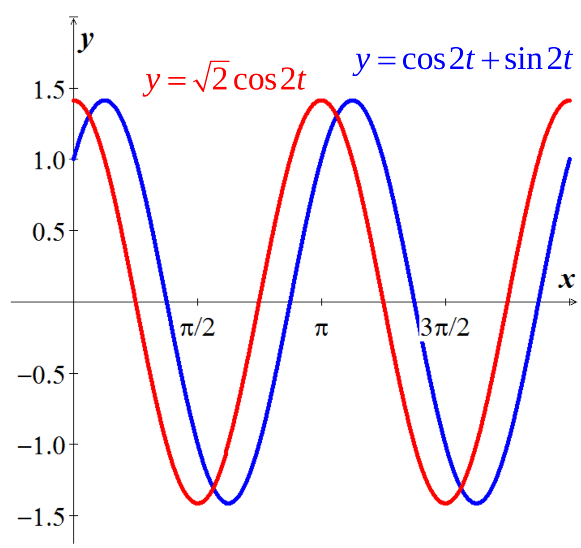
$y = 1.\cos 2t + 1.\sin 2t$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos 2t + \frac{1}{\sqrt{2}}\sin 2t\right)$ $\qquad \cos\phi = \frac{1}{\sqrt{2}} = \sin\phi$

$= \sqrt{2}\left(\cos\phi\cos 2t + \sin\phi\sin 2t\right)$

$= \sqrt{2}\cos(2t - \phi)$

$\boxed{= \sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)}$



Exercise

$y = \cos 4t + \sqrt{3}\sin 4t$

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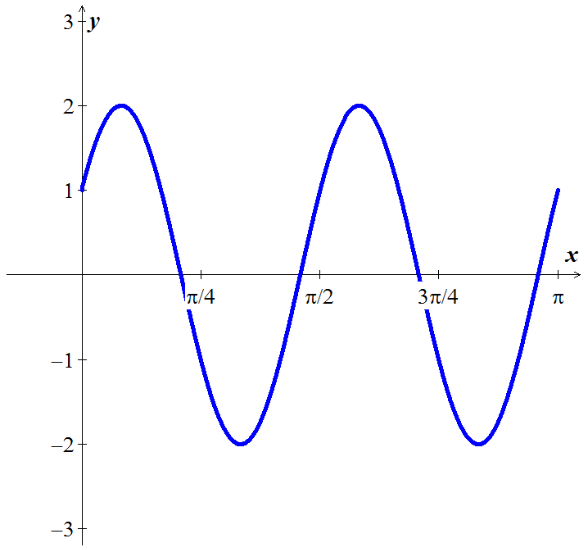
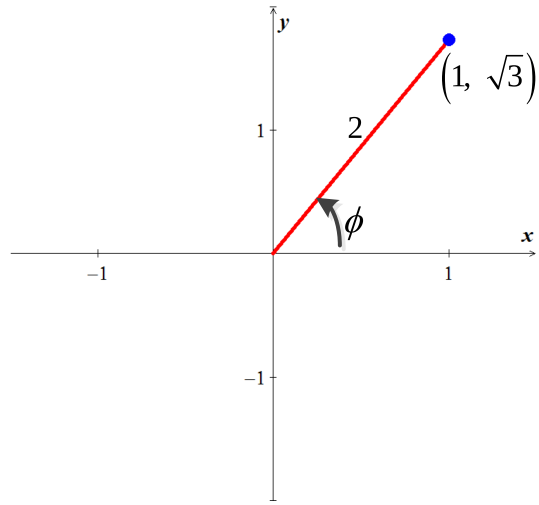
Solution

i. Plot the function

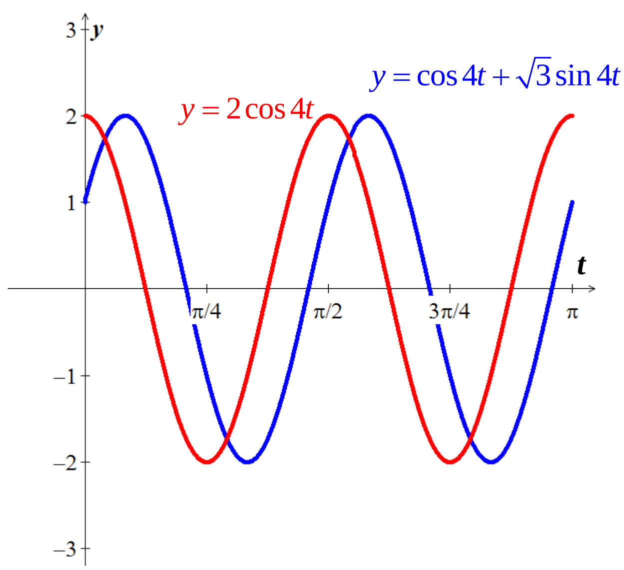




ii. Place the solution in the form $y = A\cos\left(\omega_0 t - \phi\right)$ and compare the graph with the plot in (***i***)

$$
\begin{aligned}
y &= 1 \cdot \cos 4t + \sqrt{3} \cdot \sin 4t \\
&= 2\left(\frac{1}{2}\cos 4t + \frac{\sqrt{3}}{2}\sin 4t\right) \\
&= 2\left(\cos\phi \cos 4t + \sin\phi \sin 4t\right) \\
&= 2\cos\left(4t - \phi\right) \\
&= 2\cos\left(4t - \frac{\pi}{3}\right) \\
&= 2\cos 4\left(t - \frac{\pi}{12}\right)
\end{aligned}
$$

$\cos\phi = \frac{1}{2} \quad \sin\phi = \frac{\sqrt{3}}{2}$



Exercise

A 1-*kg* mass, when attached to a large spring, stretches the spring a distance of 4.9 *m*.

a) Calculate the spring constant.

b) The system is placed in a viscous medium that supplies a damping constant $\mu = 3\ kg/s$. The system is allowed to come to rest. Then the mass is displaced 1 *m* in the downward direction and given a sharp tap, imparting an instantaneous velocity of 1 *m/s* in the downward direction. Find the position of the mass as a function of time and plot the solution.

Solution

a) By Hooke's law:

$$k = \frac{F}{y} = \frac{mg}{y} = \frac{(1kg)\left(9.8m/s^2\right)}{4.9m} = 2N/m$$

b) ***Given:*** $m = 1;\ \mu = 3;\ k = 2;\ y(0) = 1;\ y'(0) = 1$

$$y'' + 3y' + 2y = 0$$

The characteristic equation is: $\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda_{1,2} = -1, -2$

The general solution: $\boxed{y(t) = C_1 e^{-t} + C_2 e^{-2t}}$

Exercise

The undamped system $\frac{2}{5}x'' + kx = 0, \quad x(0) = 2 \quad x'(0) = v_0$ is observed to have period $\frac{\pi}{2}$ and amplitude 2. Find k and v_0

Solution

$$x'' + \frac{5}{2}kx = 0 \quad \Leftrightarrow \quad x'' + \omega_0^2 x = 0 \quad \left(\omega_0^2 = \frac{5k}{2}\right)$$

The characteristic equation is: $\lambda^2 + \omega_0^2 = 0 \quad \Rightarrow \lambda = \pm\omega_0 i$

It is a complex root, thus we have a complex solution:

$$x'(t) = -C_1\omega_0 \sin\omega_0 t + C_2\omega_0 \cos\omega_0 t \ z(t) = e^{i\omega_0 t} = \cos\omega_0 t + i\sin\omega_0 t = \text{cis}\,\omega_0 t$$

The general solution: $x(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$

This solution is periodic with period $T = \frac{2\pi}{\omega_0} = \frac{\pi}{2}$ (since the period is $\frac{\pi}{2}$ given)

$$\omega_0 = 4 \quad \Rightarrow \quad \boxed{k = \frac{2\omega_0^2}{5} = \frac{32}{5}}$$

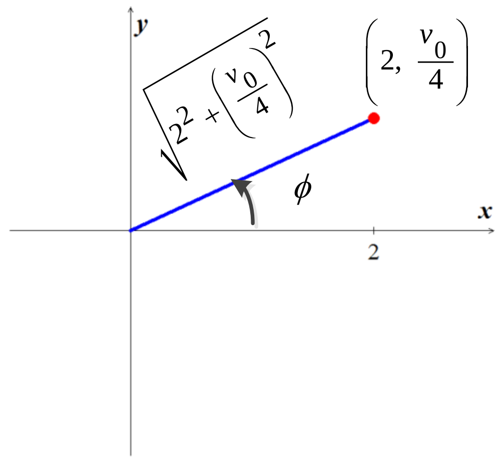
$$x(0) = C_1 \cos\omega_0 0 + C_2 \sin\omega_0 0 \quad \Rightarrow \quad \boxed{C_1 = 2}$$

$$x'(0) = -C_1\omega_0 \sin\omega_0 0 + C_2\omega_0 \cos\omega_0 0 \quad \Rightarrow \quad \boxed{C_2 = \frac{v_0}{\omega_0}}$$

$$x(t) = 2\cos 4t + \frac{v_0}{4}\sin 4t$$

$$x(t) = \sqrt{4 + \frac{v_0^2}{16}}\ \cos(4t - \phi)$$

$$\tan\phi = \frac{v_0}{8}$$



The amplitude is 2, therefore:

$$\sqrt{4 + \frac{v_0^2}{16}} = 2$$

$$4 + \frac{v_0^2}{16} = 4$$

$$\frac{v_0^2}{16} = 0$$

$$\boxed{v_0 = 0}$$

Exercise

A body with mass m = 0.5 *kg* is attached to the end of a spring that is stretched 2 *m* by a force of 100 *N*. It is set in motion with initial position $x_0 = 1\ m$ and initial velocity $v_0 = -5m/s$. (Note that these initial conditions indicate that he body is displaced to the right and is moving to the left at time t = 0.) Find the position function of the body as well as the amplitude, frequency, period of oscillation, and time lag of its motion.

Solution

Spring constant: $\boxed{k = \frac{100N}{2m} = 50\ N/m}$

$$my'' + \mu y' + ky = F(t)$$

The differential equation can be written as:

$$0.5x'' + 50x = 0 \quad \Rightarrow \quad x'' + 100x = 0$$

$\boxed{\omega_0 = \sqrt{100} = 10\ rad/s}$

Period: $\boxed{T = \frac{2\pi}{\omega_0} = \frac{2\pi}{10} = \frac{\pi}{5} \approx 0.6283\ \sec}$

Frequency: $\boxed{\upsilon = \frac{1}{T} = \frac{5}{\pi} \approx 1.5915 Hz}$

Given: $x(0) = 1, \quad x'(0) = -5$

$x(t) = A\cos 10t + B\sin 10t \quad \to \quad x(0) = \boxed{A = 1}$

$x'(t) = -10A\sin 10t + 10B\cos 10t \quad \to \quad x'(0) = 10B = -5 \Rightarrow \boxed{B = -\frac{1}{2}}$

$\boxed{x(t) = \cos 10t - \frac{1}{2}\sin 10t}$

Amplitude of motion is: $A = \sqrt{1^2 + \left(-\frac{1}{2}\right)^2} \boxed{= \frac{\sqrt{5}}{2} m}$

Time lag?

$x(t) = \frac{\sqrt{5}}{2}\left(\frac{2}{\sqrt{5}}\cos 10t - \frac{1}{\sqrt{5}}\sin 10t\right)$

$\boxed{= \frac{\sqrt{5}}{2}\cos(10t - \varphi)}$

Phase angle φ: $\hat{\varphi} = \tan^{-1}\left(\frac{\frac{1}{\sqrt{5}}}{\frac{2}{\sqrt{5}}}\right) \approx 0.46365$

Since $\cos\varphi = \frac{2}{\sqrt{5}} > 0 \quad and \quad \sin\varphi = -\frac{1}{\sqrt{5}} < 0$

$\Rightarrow \quad \varphi = 2\pi - 0.46365 \boxed{= 5.8195}$

Time lag of the motion is:

$\delta = \frac{\varphi}{\omega_0} \approx \frac{5.8195}{10} \boxed{\approx 0.58195\ \sec}$

$\boxed{x(t) = \frac{\sqrt{5}}{2}\cos(10t - 5.8195)}$

Exercise

Given the mass, damping, and spring constants of an undriven spring-mass system $my'' + \mu y' + ky = 0$

$$m = 1\ kg, \quad \mu = 0\ kg/s, \quad k = 4 kg/s^2, \quad y(0) = -2\ m, \quad y'(0) = -2\ m/s$$

a) Provide separate plots of the position versus time (y vs. t) and the velocity versus time (v vs. t)

b) Provide a combined plot of both position and velocity versus time

c) Provide a plot of the velocity versus position (v vs. y) in the yv phase plane.

Solution

$$my'' = -\mu y' - ky$$

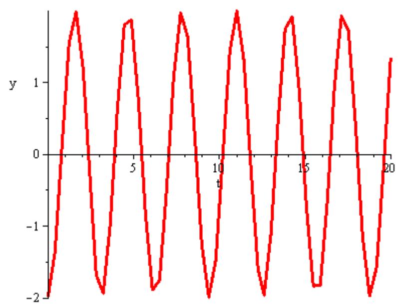
$$y'' = -\frac{\mu}{m} y' - \frac{k}{m} y$$

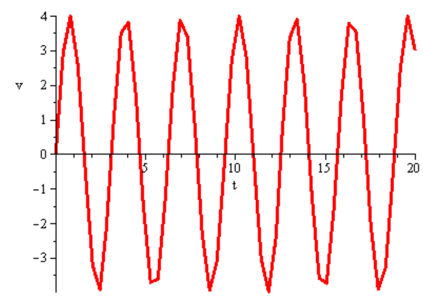
Let $\boxed{v = y'}$ $\Rightarrow v' = -\frac{\mu}{m} v - \frac{k}{m} y$

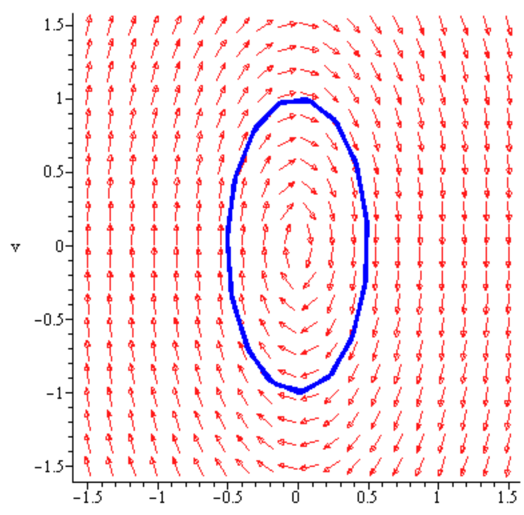
$$= -\frac{0}{1} v - \frac{4}{1} y$$

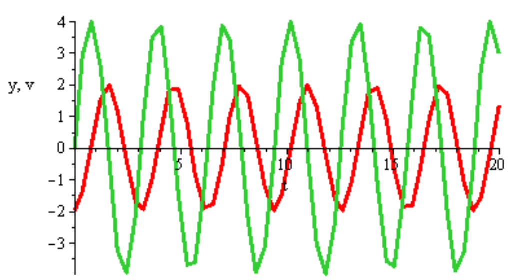
$$\boxed{v' = -4y}$$

$$y(0) = -2, \quad y'(0) = -2 = v(0)$$









Exercise

Given the mass, damping, and spring constants of an undriven spring-mass system $my'' + \mu y' + ky = 0$

$$m = 1\ kg, \quad \mu = 2\ kg/s, \quad k = 1 kg/s^2, \quad y(0) = -3\ m, \quad y'(0) = -2\ m/s$$

a) Provide separate plots of the position versus time (y vs. t) and the velocity versus time (v vs. t)

b) Provide a combined plot of both position and velocity versus time

c) Provide a plot of the velocity versus position (v vs. y) in the yv phase plane.

Solution

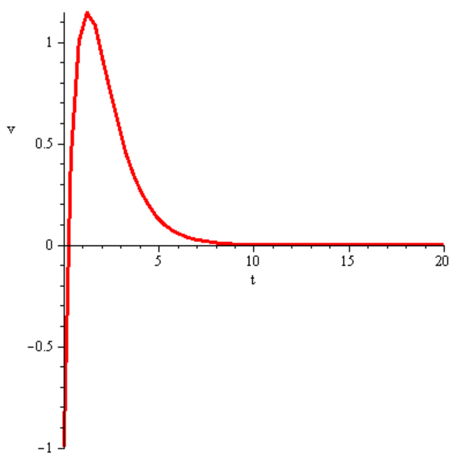
$$my'' = -\mu y' - ky \quad \Rightarrow \quad y'' = -\frac{\mu}{m} y' - \frac{k}{m} y$$

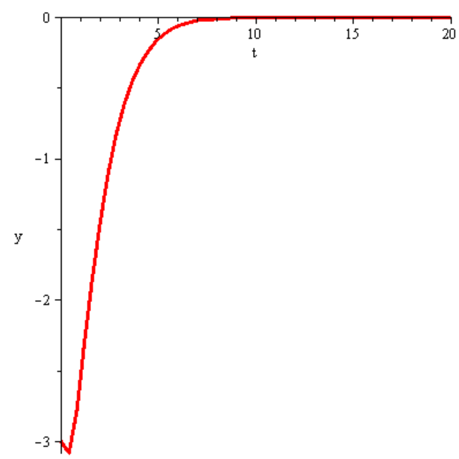
Let $v = y'$ $\Rightarrow v' = -\frac{\mu}{m} v - \frac{k}{m} y$

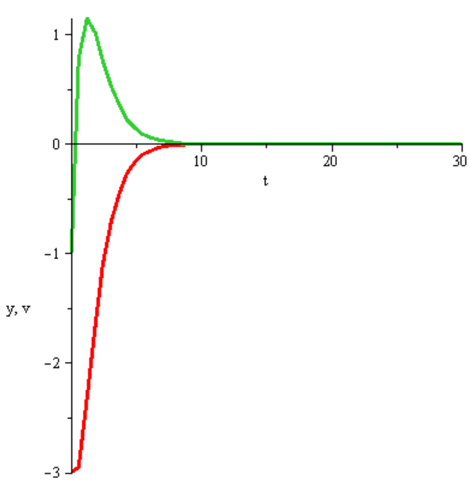
$$= -\frac{2}{1} v - \frac{1}{1} y$$

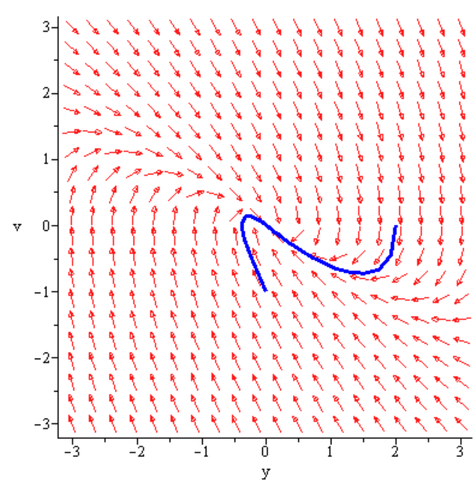
$$= -2v - y$$

The following system of the first-order equations: $\begin{cases} y' = v \\ v' = -2v - y \end{cases}$ *with* $\begin{matrix} y(0) = -3 \\ y'(0) = -2 = v(0) \end{matrix}$





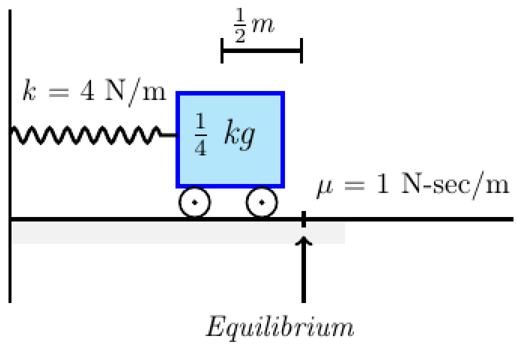




Exercise

A $\frac{1}{4}$-*kg* mass is attached to a spring with a stiffness $k = 4\ N/m$. The damping constant $\mu = 1\ N\text{-}sec\,/\,m$.

If the mass is displaced $x_0 = \frac{1}{2}\ m$ to the left and given an initial velocity of $v_0 = 1\ m/s$ to the left.



a) Find the equation of motion.

b) What is the maximum displacement that the mass will attain?

Solution

a) $\frac{1}{4}x'' + x' + 4x = 0\ ;\quad x(0) = -\frac{1}{2},\quad x'(0) = -1 \qquad mx'' + \mu x' + kx = F(t)$

b) $\lambda^2 + 4\lambda + 16 = 0 \quad \to \quad \lambda_{1,2} = \frac{-4 \pm 4\sqrt{3}}{2} = -2 \pm 2i\sqrt{3}$

$$x(t) = e^{-2t}\left(C_1 \cos 2\sqrt{3}t + C_2 \sin 2\sqrt{3}t\right)$$

$$x(0) = -\frac{1}{2} \quad \to \quad C_1 = -\frac{1}{2}$$

$$x'(t) = e^{-2t}\left(-2\sqrt{3}C_1 \sin 2\sqrt{3}t + 2\sqrt{3}C_2 \cos 2\sqrt{3}t - 2C_1 \cos 2\sqrt{3}t - 2C_2 \sin 2\sqrt{3}t\right)$$

$$x'(0) = -1 \quad \to \quad 2\sqrt{3}C_2 - 2C_1 = -1 \quad C_2 = -\frac{1}{\sqrt{3}}$$

$$x(t) = e^{-2t}\left(-\frac{1}{2}\cos 2\sqrt{3}t - \frac{1}{\sqrt{3}}\sin 2\sqrt{3}t\right)$$

$$x'(t) = e^{-2t}\left(\sqrt{3}\sin\left(2\sqrt{3}t\right) - 2\cos\left(2\sqrt{3}t\right) + \cos\left(2\sqrt{3}t\right) + \frac{2}{\sqrt{3}}\sin\left(2\sqrt{3}t\right)\right)$$

$$= e^{-2t}\left(\frac{5}{\sqrt{3}}\sin\left(2\sqrt{3}t\right) - \cos\left(2\sqrt{3}t\right)\right) = 0$$

$$\frac{5}{\sqrt{3}}\sin\left(2\sqrt{3}t\right) = \cos\left(2\sqrt{3}t\right)$$

$$\tan\left(2\sqrt{3}t\right) = \frac{\sqrt{3}}{5} \quad \to \quad 2\sqrt{3}t = \arctan\frac{\sqrt{3}}{5}$$

$$t = \frac{1}{2\sqrt{3}}\arctan\frac{\sqrt{3}}{5} \approx 0.096$$

$$x(0.096) = e^{-2(0.096)}\left(-\frac{1}{2}\cos 2\sqrt{3}(0.096) - \frac{1}{\sqrt{3}}\sin 2\sqrt{3}(0.096)\right)$$

$$\approx -0.55\ m$$

Exercise

A 2-*kg* mass is attached to a spring with a stiffness $k = 50\ N/m$. The mass is displaced $\frac{1}{4}\ m$ to the left of the equilibrium point and given a velocity of 1 *m/s* to the left. Neglecting the damping,

a) Find the equation of motion of the mass along with the amplitude, period, and frequency.

b) How long after release does the mass pass through the equilibrium position?

Solution

a) *Given*: $\mu = 0$

$$2x'' + 50x = 0\ ; \quad x(0) = -\frac{1}{4}, \quad x'(0) = -1 \qquad mx'' + \mu x' + kx = 0$$

$$2\lambda^2 + 50 = 0 \ \rightarrow\ \lambda_{1,2} = \pm 5i$$

$$x(t) = C_1 \cos 5t + C_2 \sin 5t$$

$$x(0) = -\frac{1}{4} \ \rightarrow\ C_1 = -\frac{1}{4}$$

$$x'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t$$

$$x'(0) = -1 \ \rightarrow\ 5C_2 = -1 \quad C_2 = -\frac{1}{5}$$

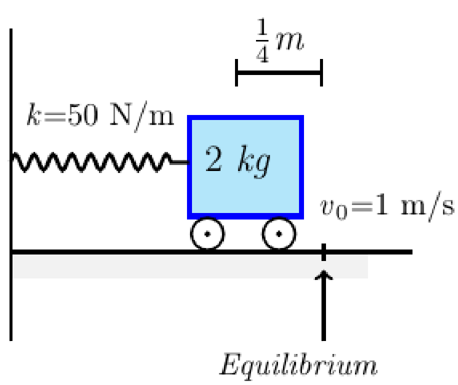
$$x(t) = -\frac{1}{4}\cos 5t - \frac{1}{5}\sin 5t$$

Amplitude: $A = \sqrt{\frac{1}{16} + \frac{1}{25}} = \frac{\sqrt{41}}{20}$

The angular velocity: $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{50}{2}} = 5$

Period: $T = \frac{2\pi}{\omega} = \frac{2\pi}{5}$

Natural *frequency* $= \frac{1}{T} = \frac{5}{2\pi}$



b) $x(t) = -\frac{1}{4}\cos 5t - \frac{1}{5}\sin 5t = 0$

$$\frac{1}{5}\sin 5t = -\frac{1}{4}\cos 5t$$

$$\tan 5t = -\frac{5}{4} \ \rightarrow\ 5t = \pi - \arctan\left(-\frac{5}{4}\right)$$

$$t = \frac{1}{5}\left(\pi - \arctan\left(\frac{5}{4}\right)\right) \approx 0.45\ sec$$

Exercise

A 3-*kg* mass is attached to a spring with a stiffness $k = 48\ N/m$. The mass is displaced $\frac{1}{2}\ m$ to the left of the equilibrium point and given a velocity of $2\ m/s$ to the left. Neglecting the damping,

a) Find the equation of motion of the mass

b) Find the amplitude, period, and frequency.

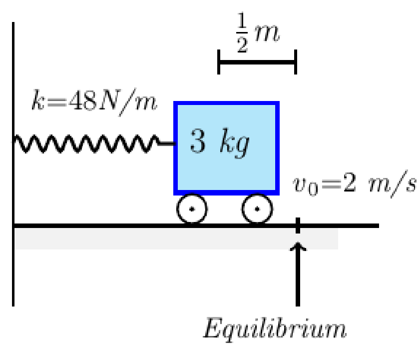
c) How long after release does the mass pass through the equilibrium position?

Solution

a) ***Given***: $\mu = 0$

$$3x'' + 48x = 0\ ; \quad x(0) = -\frac{1}{2}, \quad x'(0) = -2 \qquad\qquad mx'' + \mu x' + kx = 0$$

$$\lambda^2 + 16 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 4i}$$



$$x(t) = C_1 \cos 4t + C_2 \sin 4t$$

$$x(0) = -\frac{1}{2} \quad \rightarrow \quad \boxed{C_1 = -\frac{1}{2}}$$

$$x'(t) = -4C_1 \sin 4t + 4C_2 \cos 4t$$

$$x'(0) = -2 \quad \rightarrow 4C_2 = -2 \quad \boxed{C_2 = -\frac{1}{2}}$$

$$\boxed{x(t) = -\frac{1}{2}\cos 5t - \frac{1}{2}\sin 5t}$$

b) Amplitude: $A = \sqrt{\frac{1}{4} + \frac{1}{4}}\ \boxed{= \frac{\sqrt{2}}{2}\ m}$

The angular velocity: $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{48}{3}}\ \boxed{= 4}$

Period: $T = \frac{2\pi}{\omega}\ \boxed{= \frac{\pi}{2}}$

Natural *frequency* $= \frac{1}{T}\ \boxed{= \frac{2}{\pi}}$

c) $x(t) = -\frac{1}{2}\cos 4t - \frac{1}{2}\sin 4t = 0$

$$\frac{1}{2}\sin 4t = -\frac{1}{2}\cos 4t$$

$$\tan 4t = -1 \quad \rightarrow \quad 4t = \frac{3\pi}{4}$$

$$\boxed{t = \frac{3\pi}{16} \approx 0.59\ \ sec}$$

Exercise

A 20-*kg* mass is attached to a spring with a stiffness $k = 200$ *N/m*. The damping constant $\mu = 140$ *N - sec / m*. If the mass is pulled 25 *cm* to the right of the equilibrium point and given an initial velocity of 1 *m/s*. Neglecting the damping,

a) Find the equation of motion.

b) When will it first return to its equilibrium position?

Solution

a) $20x'' + 140x' + 200x = 0 \qquad mx'' + \mu x' + kx = 0$

$x'' + 7x' + 10x = 0\ ; \quad x(0) = \frac{1}{4}, \quad x'(0) = -1$

$$\lambda^2 + 7\lambda + 10 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = -2,\ -5}$$

$x(t) = C_1 e^{-2t} + C_2 e^{-5t}$

$$x(0) = \frac{1}{4} \quad \rightarrow \quad C_1 + C_2 = \frac{1}{4}$$

$$x(t) = -2C_1 e^{-2t} - 5C_2 e^{-5t}$$

$$x'(0) = -1 \quad \rightarrow \quad 2C_1 + 5C_2 = 1$$

$$\begin{cases} C_1 + C_2 = \frac{1}{4} \\ 2C_1 + 5C_2 = 1 \end{cases} \rightarrow 3C_2 = \frac{1}{2} \quad \underline{C_2 = \frac{1}{6},\ C_1 = \frac{1}{12}}$$

$\underline{x(t) = \frac{1}{12} e^{-2t} + \frac{1}{6} e^{-5t}}$

b) $x(t) = \frac{1}{12} e^{-2t} + \frac{1}{6} e^{-5t} \neq 0$

The mass will not return to the equilibrium position.

Exercise

A $\frac{1}{4}$-*kg* mass is attached to a spring with a stiffness $k = 8$ *N/m*. The damping constant $\mu = \frac{1}{4}$ *N - sec / m*. If the mass is displaced $x_0 = 1\ m$ to the left of equilibrium and released, what is the maximum displacement to the right that the mass will attain?

Solution

$\frac{1}{4} x'' + \frac{1}{4} x' + 8x = 0 \qquad mx'' + \mu x' + kx = 0$

$x'' + x' + 32x = 0\ ; \quad x(0) = -1, \quad x'(0) = 0$

$$\lambda^2 + \lambda + 32 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = \frac{1}{2} \pm i \frac{\sqrt{127}}{2}}$$

$$x(t)=e^{-t/2}\left(C_1\cos\left(\frac{\sqrt{127}}{2}t\right)+C_2\sin\left(\frac{\sqrt{127}}{2}t\right)\right)$$

$$x(0)=-1 \quad\to\quad \underline{C_1=-1}$$

$$x'(t)=e^{-t/2}\left(-\frac{\sqrt{127}}{2}C_1\sin\frac{\sqrt{127}}{2}t+\frac{\sqrt{127}}{2}C_2\cos\frac{\sqrt{127}}{2}t-\frac{1}{2}C_1\cos\frac{\sqrt{127}}{2}t-\frac{1}{2}C_2\sin\frac{\sqrt{127}}{2}t\right)$$

$$x'(0)=0 \quad\to\quad \frac{\sqrt{127}}{2}C_2-\frac{1}{2}C_1=0 \quad\Rightarrow\quad \underline{C_2=-\frac{1}{\sqrt{127}}}$$

$$\underline{x(t)=e^{-t/2}\left(-\cos\left(\frac{\sqrt{127}}{2}t\right)-\frac{1}{\sqrt{127}}\sin\left(\frac{\sqrt{127}}{2}t\right)\right)}$$

$$x'(t)=e^{-t/2}\left(\frac{\sqrt{127}}{2}\sin\frac{\sqrt{127}}{2}t-\frac{1}{2}\cos\frac{\sqrt{127}}{2}t+\frac{1}{2}\cos\frac{\sqrt{127}}{2}t+\frac{1}{2\sqrt{127}}\sin\frac{\sqrt{127}}{2}t\right)$$

$$=\frac{64}{\sqrt{127}}e^{-t/2}\sin\frac{\sqrt{127}}{2}t=0$$

$$\frac{\sqrt{127}}{2}t=n\pi \quad (t>0)\to n=1$$

$$\underline{t=\frac{2\pi}{\sqrt{127}}}$$

$$x(t)=e^{-\pi/\sqrt{127}}\left(-\cos(\pi)-\frac{1}{\sqrt{127}}\sin(\pi)\right)$$

$$\underline{=e^{-\pi/\sqrt{127}}\approx 0.7567\ m}$$

Exercise

A $\frac{1}{4}$-*kg* mass is attached to a spring with a stiffness $k=8$ *N/m*. The damping constant $\mu=2$ *N - sec / m*. If the mass is pushed 50 *cm* to the left of equilibrium and given a leftward velocity of 2 *m/sec*, when will the mass attain its maximum displacement to the left?

Solution

$$\frac{1}{4}x''+2x'+8x=0 \qquad\qquad mx''+\mu x'+kx=0$$

$$x''+8x'+32x=0\ ; \quad x(0)=-0.5\ m, \quad x'(0)=-2$$

$$\lambda^2+8\lambda+32=0 \quad\to\quad \lambda_{1,2}=\frac{-8\pm 8i}{2}\underline{=-4\pm 4i}$$

$$x(t)=e^{-4t}\left(C_1\cos 4t+C_2\sin 4t\right)$$

$$x(0)=-\frac{1}{2} \quad\to\quad \underline{C_1=-\frac{1}{2}}$$

$$x'(t) = e^{-4t}\left(-4C_1 \sin 4t + 4C_2 \cos 4t - 4C_1 \cos 4t - 4C_2 \sin 4t\right)$$

$$x'(0) = -2 \quad \rightarrow \quad 4C_2 - 4C_1 = -2 \quad \Rightarrow \quad \boxed{C_2 = -1}$$

$$x(t) = -e^{-4t}\left(\frac{1}{2}\cos 4t + \sin 4t\right)$$

$$x'(t) = e^{-4t}\left(2\sin 4t - 4\cos 4t + 2\cos 4t + 4\sin 4t\right)$$

$$= e^{-4t}\left(6\sin 4t - 2\cos 4t\right) = 0$$

$$3\sin 4t = \cos 4t$$

$$\tan 4t = \frac{1}{3} \quad \rightarrow \quad t = \frac{1}{4}\arctan\frac{1}{3} \ \boxed{\approx 0.08 \ sec}$$

Exercise

A 8-*lb* mass weight stretches a spring 2 *feet*. Assuming that a damping force numerically equal to 2 times the instantaneous velocity acts on the system, determine the equation if motion if the mass released from the equilibrium position with an upward velocity of 3 *ft/s*

Solution

$$m = \frac{8}{32} = \frac{1}{4} \ slug \qquad\qquad W = mg$$

$$k(2\ ft) = 8 \quad \rightarrow \quad k = 4 \qquad\qquad kx = mg$$

$$\frac{1}{4}x'' + 2x' + 4x = 0 \qquad\qquad mx'' + \mu x' + kx = 0$$

$$x'' + 8x' + 16x = 0\ ; \quad x(0) = 0, \quad x'(0) = -3$$

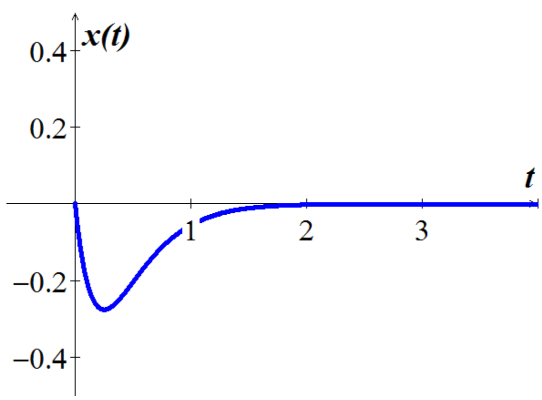
$$\lambda^2 + 8\lambda + 16 = (\lambda + 4)^2 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -4}$$

$$x(t) = \left(C_1 + C_2 t\right)e^{-4t}$$

$$x(0) = 0 \quad \rightarrow \quad \boxed{C_1 = 0}$$

$$x'(t) = \left(C_2 - 4C_1 - 4C_2 t\right)e^{-4t}$$

$$x'(0) = -3 \quad \rightarrow \quad \Rightarrow \quad \boxed{C_2 = -3}$$



$$\boxed{x(t) = -3te^{-4t}}$$

Exercise

A 8-*lb* mass weight is attached to the end of a spring, causing the spring to stretch a spring 6 *in*. beyond its natural length. The block is then pulled down 3 *in*. and released. Determine the motion of the block, assuming there are no damping or external applied force.

Solution

$$m = \frac{8}{32} = \frac{1}{4}\ slug \qquad W = mg$$

$$k\left(6\ in\frac{1\ ft}{12\ in}\right) = 8 \quad \rightarrow \quad k = 16\ lb/ft \qquad kx = mg$$

$$\frac{1}{4}y'' + 16y = 0\ ; \quad y(0) = \frac{3}{12} = \frac{1}{4}, \quad y'(0) = 0 \qquad mx'' + \mu x' + kx = 0$$

$$\frac{1}{4}\lambda^2 + 16 = 0 \quad \rightarrow \lambda^2 = -64 \quad \boxed{\lambda_{1,2} = -8i}$$

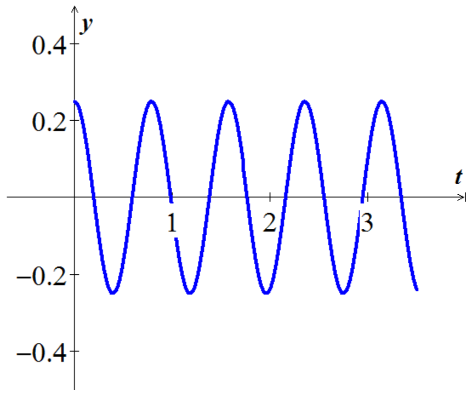
$$y(t) = C_1 \cos 8t + C_2 \sin 8t$$

$$y(0) = \frac{1}{4} \quad \rightarrow \quad \boxed{C_1 = \frac{1}{4}}$$

$$y'(t) = -8C_1 \sin 8t + 8C_2 \cos 8t$$

$$y'(0) = 0 \quad \rightarrow \quad \Rightarrow \boxed{C_2 = 0}$$

$$\boxed{y(t) = \frac{1}{4}\cos 8t}$$



Exercise

A 8-*lb* mass weight is attached to the end of a spring, causing the spring to stretch a spring 6 *in*. beyond its natural length. The block is then pulled down 3 *in*. and released. Determine the motion of the block, assuming there damping is present and that the damping coefficient is $\mu = 1$ *lb-sec/ft* and external applied force.

Solution

$$m = \frac{8}{32} = \frac{1}{4}\ slug \qquad W = mg$$

$$k\left(6\ in\frac{1\ ft}{12\ in}\right) = 8 \quad \rightarrow \quad k = 16\ lb/ft \qquad kx = mg$$

$$\frac{1}{4}y'' + y' + 16y = 0\ ; \quad y(0) = \frac{3}{12} = \frac{1}{4}, \quad y'(0) = 0 \qquad mx'' + \mu x' + kx = 0$$

$$\frac{1}{4}\lambda^2 + \lambda + 16 = 0 \quad \rightarrow \lambda^2 + 4\lambda + 64 = 0$$

$$\lambda_{1,2} = \frac{-4 \pm i\sqrt{240}}{2} = \frac{-4 \pm 4i\sqrt{15}}{2} \boxed{= -2 \pm 2i\sqrt{15}}$$

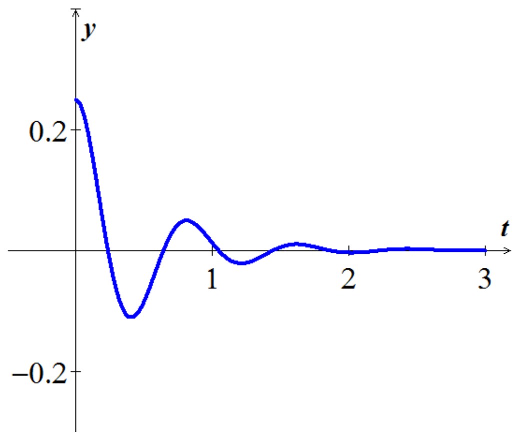
$$y(t)=e^{-2t}\left(C_1\cos\left(2\sqrt{15}\,t\right)+C_2\sin\left(2\sqrt{15}\,t\right)\right)$$

$$y(0)=\frac{1}{4} \quad\to\quad \boxed{C_1=\frac{1}{4}}$$

$$y'(t)=e^{-2t}\left(-2C_1\cos\left(2\sqrt{15}\,t\right)-2C_2\sin\left(2\sqrt{15}\,t\right)-2\sqrt{15}C_1\sin\left(2\sqrt{15}\,t\right)+2\sqrt{15}C_2\cos\left(2\sqrt{15}\,t\right)\right)$$

$$y'(0)=0 \quad\to -2\left(\frac{1}{4}\right)+2\sqrt{15}C_2=0 \quad\Rightarrow\quad \boxed{C_2=\frac{1}{4\sqrt{15}}}$$

$$\boxed{y(t)=e^{-2t}\left(\frac{1}{4}\cos\left(2\sqrt{15}\,t\right)+\frac{1}{4\sqrt{15}}\sin\left(2\sqrt{15}\,t\right)\right)}$$



Exercise

A 16-*lb* mass weight is attached to a 5-*foot* spring. At equilibrium the spring measures 8.2 feet. If the mass is initially released from rest at a point $x_0 = 2\ ft$ above the equilibrium position, find the displacements $x(t)$ if it is further known that the surrounding medium offers a resistance numerically equal to the instantaneous velocity.

Solution

$m=\frac{16}{32}=\frac{1}{2}$ *slug* $\qquad\qquad W=mg$

$k(8.2-5)=16 \quad\to\quad k=\frac{16}{3.2}=5\ \ lb/ft \qquad\qquad kx=mg$

$\frac{1}{2}x''+x'+5x=0 \qquad\qquad mx''+\mu x'+kx=0$

$x''+2x'+10x=0\ ;\quad x(0)=-2,\quad x'(0)=0$

$$\lambda^2+2\lambda+10=0 \quad\to\quad \lambda_{1,2}=\frac{-2\pm 6i}{2}\ \boxed{=-1\pm 3i}$$

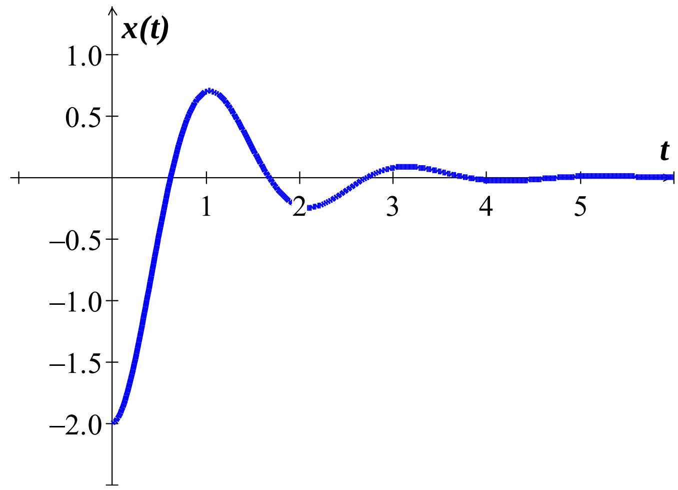
$$x(t)=e^{-t}\left(C_1\cos 3t+C_2\sin 3t\right)$$

$$x(0)=-2 \quad\to\quad \boxed{C_1=-2}$$

$$x'(t) = e^{-t}\left(-3C_1 \sin 3t + 3C_2 \cos 3t - C_1 \cos 3t - C_2 \sin 3t\right)$$

$$x'(0) = 0 \quad \to 3C_2 - C_1 = 0 \quad \Rightarrow \quad \underline{C_2 = -\frac{2}{3}}$$

$$\underline{x(t) = -e^{-t}\left(2\cos 3t + \frac{2}{3}\sin 3t\right)}$$



Exercise

A 16-*lb* mass weight is attached to a spring, stretches $\frac{8}{9}$ *ft* by itself. There is no damping and no external forces acting on the system. The spring is initially displaced 6 *inches* upwards from its equilibrium position and given an initial velocity of 1 *ft/sec* downward. Find the displacement $y(t)$ at any time *t*.

Solution

$$m = \frac{16}{32} = \frac{1}{2} \text{ slug} \qquad\qquad W = mg$$

$$k\left(\frac{8}{9}\right) = 16 \quad \to \quad k = 18 \ \ lb/ft \qquad\qquad kL = mg$$

$$\frac{1}{2}y'' + 18y = 0 \qquad\qquad my'' + \mu y' + ky = 0$$

$$y'' + 36y = 0\ ; \quad y(0) = -\frac{6 \ in}{12 \ in} = -\frac{1}{2} \ ft, \quad y'(0) = 1 \ \ ft/sec$$

$$\lambda^2 + 36 = 0 \quad \to \quad \underline{\lambda_{1,2} = \pm 6i}$$

$$y(t) = C_1 \cos 6t + C_2 \sin 6t$$

$$y(0) = -\frac{1}{2} \quad \to \quad \underline{C_1 = -\frac{1}{2}}$$

$$y' = -6C_1 \sin 6t + 6C_2 \cos 6t$$

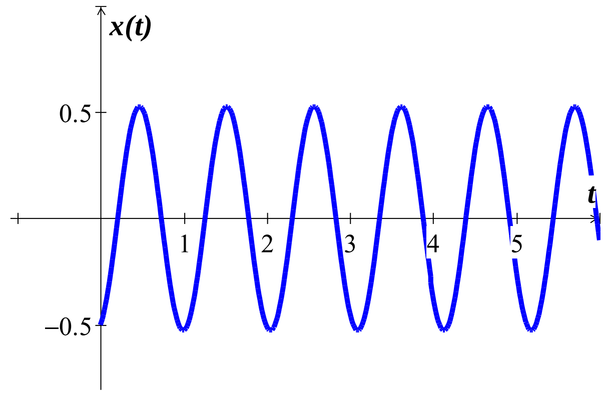
$$y'(0) = 1 \quad \to \quad 6C_2 = 1 \quad \Rightarrow \quad \underline{C_2 = \frac{1}{6}}$$

$$\boxed{y(t) = -\frac{1}{2}\cos 6t + \frac{1}{6}\sin 6t}$$

$$A = \sqrt{\frac{1}{4} + \frac{1}{36}} = \boxed{\frac{\sqrt{10}}{6}}$$

$$\varphi = \pi - \tan^{-1}\frac{1}{3} \approx 2.81984$$

$$\boxed{y(t) = \frac{\sqrt{10}}{6}\cos(6t - 2.81984)}$$



Exercise

A 16-*lb* mass weight is attached to a spring, stretches $\frac{8}{9}$ *ft* by itself. A damper to the mass that will exert of 12 *lbs*. when the velocity is 2 *ft/sec* . The spring is initially displaced 6 *inches* upwards from its equilibrium position and given an initial velocity of 1 *ft/sec* downward. Find the displacement $y(t)$ at any time *t*.

Solution

$$m = \frac{16}{32} = \frac{1}{2} \ slug \qquad W = mg$$

$$k\left(\frac{8}{9}\right) = 16 \quad \to \quad k = 18 \ \ lb/ft \qquad kL = mg$$

$$12 = \mu(2) \quad \to \quad \mu = 6 \qquad F = \mu v$$

$$\frac{1}{2}y'' + 6y' + 18y = 0 \qquad my'' + \mu y' + ky = 0$$

$$y'' + 12y' + 36y = 0\ ; \quad y(0) = -\frac{6\ in}{12\ in} = -\frac{1}{2}\ ft, \quad y'(0) = 1\ \ ft/sec$$

$$\lambda^2 + 12\lambda + 36 = 0 \quad \to \quad \boxed{\lambda_{1,2} = -6,\ -6}$$

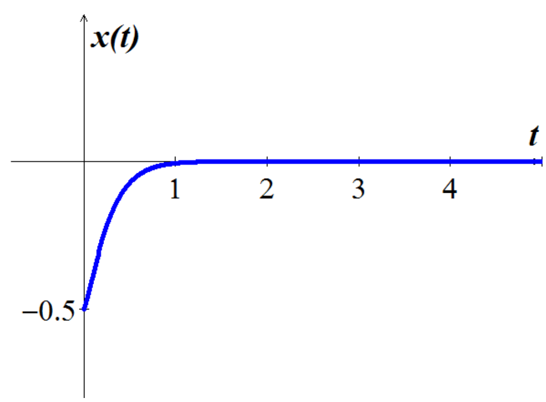
$$y(t) = (C_1 + C_2 t)e^{-6t}$$

$$y(0) = -\frac{1}{2} \quad \to \quad \boxed{C_1 = -\frac{1}{2}}$$

$$y' = (C_2 - 6C_1 - 6C_2 t)e^{-6t}$$

$$y'(0) = 1 \quad \to C_2 + 3 = 1 \quad \Rightarrow \ \boxed{C_2 = -2}$$

$$\boxed{y(t) = \left(-\frac{1}{2} - 2t\right)e^{-6t}}$$



Exercise

A 16-*lb* mass weight is attached to a spring, stretches $\frac{8}{9}$ *ft* by itself. A damper to the mass that will exert of 5 *lbs*. when the velocity is 2 *ft/sec* . The spring is initially displaced 6 *inches* upwards from its equilibrium position and given an initial velocity of 1 *ft/sec* downward. Find the displacement $y(t)$ at any time *t*.

Solution

$$m = \frac{16}{32} = \frac{1}{2} \text{ slug} \qquad\qquad W = mg$$

$$k\left(\frac{8}{9}\right) = 16 \quad \rightarrow \quad k = 18 \ \ lb/ft \qquad\qquad kL = mg$$

$$5 = \mu(2) \quad \rightarrow \quad \mu = \frac{5}{2} \qquad\qquad F = \mu v$$

$$\frac{1}{2}y'' + \frac{5}{2}y' + 18y = 0 \qquad\qquad my'' + \mu y' + ky = 0$$

$$y'' + 5y' + 36y = 0\ ; \quad y(0) = -\frac{6 \text{ in}}{12 \text{ in}} = -\frac{1}{2} \ ft, \quad y'(0) = 1 \ \ ft/sec$$

$$\lambda^2 + 5\lambda + 36 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \frac{-5 \pm i\sqrt{119}}{2}}$$

$$y(t) = \left(C_1 \cos\frac{\sqrt{119}}{2}t + C_2 \sin\frac{\sqrt{119}}{2}t\right)e^{-5t/2}$$

$$y(0) = -\frac{1}{2} \quad \rightarrow \quad \boxed{C_1 = -\frac{1}{2}}$$

$$y' = \left(-\frac{5}{2}C_1 \cos\frac{\sqrt{119}}{2}t - \frac{5}{2}C_2 \sin\frac{\sqrt{119}}{2}t - \frac{\sqrt{119}}{2}C_1 \sin\frac{\sqrt{119}}{2}t + \frac{\sqrt{119}}{2}C_2 \cos\frac{\sqrt{119}}{2}t\right)e^{-5t/2}$$

$$y'(0) = 1 \quad \rightarrow \quad -\frac{5}{2}\left(-\frac{1}{2}\right) + \frac{\sqrt{119}}{2}C_2 = 1 \quad \Rightarrow \quad \boxed{C_2 = -\frac{1}{2\sqrt{119}}}$$

$$\boxed{y(t) = \left(-\frac{1}{2}\cos\frac{\sqrt{119}}{2}t - \frac{\sqrt{119}}{238}\sin\frac{\sqrt{119}}{2}t\right)e^{-5t/2}}$$

$$A = \sqrt{\frac{1}{4} + \frac{119}{238^2}} \ \boxed{= 0.502096}$$

$$\varphi = \pi + \tan^{-1}\frac{2\sqrt{119}}{238} \ \boxed{\approx 3.2321}$$

$$\boxed{y(t) = 0.502096\, e^{-5t/2} \cos\left(\frac{\sqrt{119}}{2}t - 3.2321\right)}$$

Exercise

A mass weighing 4-*lb* is attached to a spring whose spring constant is 16 *lb/ft*.

a) Find the equation of motion.

b) What is the period of simple harmonic motion?

Solution

$m = \frac{4}{32} = \frac{1}{8}$ $\qquad W = mg$

a) $\frac{1}{8}x'' + 16x = 0$ $\qquad mx'' + \mu x' + kx = 0$

$\frac{1}{8}\lambda^2 + 16 = 0 \Rightarrow \lambda^2 = -128 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 8i\sqrt{2}}$

$\boxed{x(t) = C_1 \cos 5t + C_2 \sin 5t}$

b) The angular velocity: $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{16}{1/8}} \boxed{= 8\sqrt{2}}$

Period: $T = \frac{2\pi}{\omega} = \frac{2\pi}{8\sqrt{2}} \boxed{= \frac{\pi\sqrt{2}}{8}}$

Exercise

A 20-*kg* mass is attached to a spring. If the frequency of simple harmonic motion is $\frac{2}{\pi}$ *cycles/s*.

a) What is the spring constant *k*?

b) Find the equation of motion.

c) What is the frequency of simple harmonic motion if the original mass is replaced with an 80-*kg* mass.?

Solution

a) $20x'' + kx = 0$ $\qquad mx'' + \mu x' + kx = 0$

$20\lambda^2 + k = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm\frac{1}{2}i\sqrt{\frac{k}{5}}} \quad (k > 0)$

$x(t) = C_1 \cos\frac{1}{2}\sqrt{\frac{k}{5}}t + C_2 \sin\frac{1}{2}\sqrt{\frac{k}{5}}t$

$\omega = \frac{1}{2}\sqrt{\frac{k}{5}} = 2\pi f$

$\sqrt{\frac{k}{5}} = 4\pi\frac{2}{\pi} = 8$

$k = 5\left(8^2\right) \boxed{= 320\ N/m}$

b) $\boxed{x(t) = C_1 \cos 4t + C_2 \sin 4t}$

c) $80x'' + 320x = 0$

$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{320}{80}} \boxed{= 2}$

Natural *frequency* = $f = \frac{\omega}{2\pi} \boxed{= \frac{1}{\pi}\ cycle/s}$

$80\lambda^2 + 320 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 2i}$

$\boxed{x(t) = C_1 \cos 2t + C_2 \sin 2t}$

Exercise

A 24-*lb* mass weight is attached to the end of a spring, stretches it 4 *inches*. Initially, the mass is released from rest from a point 3 *inches* above the equilibrium position.

a) Find the equation of the motion.

b) If the mass is initially released from the equilibrium position with a downward velocity of 2 *ft/s*

Solution

$m = \frac{24}{32} = \frac{3}{4}$ $\qquad W = mg$

$k\left(4\ in \frac{1\ ft}{12\ in}\right) = 24$ $\qquad kx = mg$

$k = 72\ \ lb/ft$

a) $\frac{3}{4}y'' + 72y = 0$ $\qquad my'' + \mu y' + ky = 0$

$\frac{3}{4}\lambda^2 + 72 = 0 \Rightarrow \lambda^2 = -96 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 4i\sqrt{6}}$

$y(t) = C_1 \cos 4t\sqrt{6} + C_2 \sin 4t\sqrt{6}\ ; \quad y(0) = -\frac{3}{12} = -\frac{1}{4}, \quad y'(0) = 0$

$y(0) = -\frac{1}{4} \quad \rightarrow \boxed{C_1 = -\frac{1}{4}}$

$y'(t) = -4\sqrt{6}C_1 \sin 4t\sqrt{6} + 4\sqrt{6}C_2 \cos 4t\sqrt{6}$

$y'(0) = 0 \quad \rightarrow \boxed{C_2 = 0}$

$\boxed{y(t) = -\frac{1}{4}\cos 4\sqrt{6}t}$

b) $y(t) = C_1 \cos 4t\sqrt{6} + C_2 \sin 4t\sqrt{6}\ ; \quad y(0) = 0, \quad y'(0) = 2$

$y(0) = 0 \quad \rightarrow \boxed{C_1 = 0}$

$y'(t) = -4\sqrt{6}C_1 \sin 4t\sqrt{6} + 4\sqrt{6}C_2 \cos 4t\sqrt{6}$

$y'(0) = 2 \quad \rightarrow 4\sqrt{6}C_2 = 2 \quad \Rightarrow \boxed{C_2 = \frac{\sqrt{6}}{12}}$

$\boxed{y(t) = \frac{\sqrt{6}}{12}\sin 4\sqrt{6}t}$

Exercise

The motion of a mass-spring system with damping is given by:

$$y'' + 4y' + ky = 0\ ; \quad y(0) = 1, \quad y'(0) = 0$$

Find the equation of motion and sketch its graph for $k = 2,\ 4,\ 6,\ and\ 8$.

Solution

$$\lambda^2 + 4\lambda + k = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-4 \pm \sqrt{16-4k}}{2} = -2 \pm \sqrt{4-k}$$

For $\boldsymbol{k = 2}$

$$\lambda_{1,2} = -2 \pm \sqrt{2}$$

$$y(t) = C_1 e^{\left(-2-\sqrt{2}\right)t} + C_2 e^{\left(-2+\sqrt{2}\right)t}$$

$$y(0) = 1 \quad \rightarrow \quad C_1 + C_2 = 1$$

$$y'(t) = \left(-2-\sqrt{2}\right)C_1 e^{\left(-2-\sqrt{2}\right)t} + \left(-2+\sqrt{2}\right)C_2 e^{\left(-2+\sqrt{2}\right)t}$$

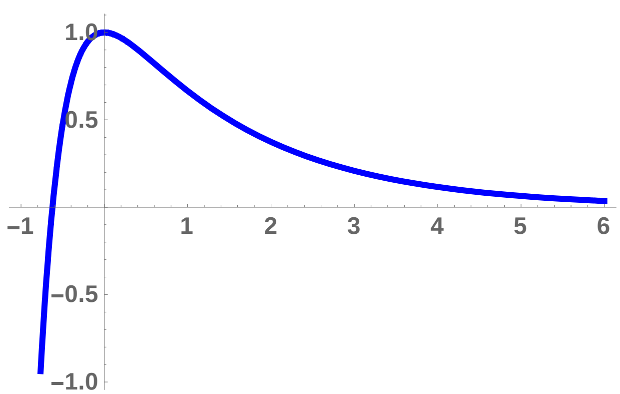
$$y'(0) = 0 \quad \rightarrow \quad \left(-2-\sqrt{2}\right)C_1 + \left(-2+\sqrt{2}\right)C_2 = 0$$

$$\begin{cases} C_1 + C_2 = 1 \\ \left(-2-\sqrt{2}\right)C_1 + \left(-2+\sqrt{2}\right)C_2 = 0 \end{cases}$$

$$\rightarrow \Delta = \begin{vmatrix} 1 & 1 \\ -2-\sqrt{2} & -2+\sqrt{2} \end{vmatrix} = 2\sqrt{2} \qquad \Delta_{C_1} = \begin{vmatrix} 1 & 1 \\ 0 & -2+\sqrt{2} \end{vmatrix} = -2+\sqrt{2}$$

$$C_1 = \frac{-2+\sqrt{2}}{2\sqrt{2}} = \frac{1-\sqrt{2}}{2} \qquad C_2 = \frac{1+\sqrt{2}}{2}$$

$$y(t) = \frac{1-\sqrt{2}}{2} e^{\left(-2-\sqrt{2}\right)t} + \frac{1+\sqrt{2}}{2} e^{\left(-2+\sqrt{2}\right)t}$$



For $\boldsymbol{k = 4}$

$$\lambda_{1,2} = -2$$

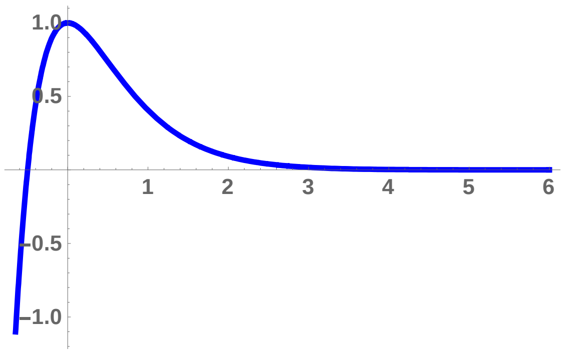
$$y(t)=\left(C_1+C_2t\right)e^{-2t}$$

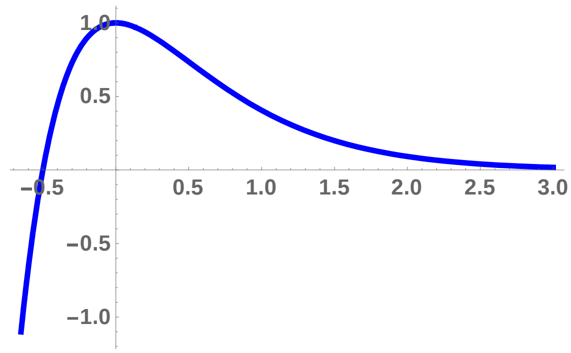
$$y(0)=1 \quad \rightarrow \quad \boxed{C_1=1}$$

$$y'(t)=\left(C_2-2C_1-2C_2t\right)e^{-2t}$$

$$y'(0)=0 \quad \rightarrow C_2-2C_1=0 \quad C_2=2$$

$$\boxed{y(t)=(1+2t)e^{-2t}}$$





For $\boldsymbol{k}$ **= 6**

$$\boxed{\lambda_{1,2}=-2\pm i\sqrt{2}}$$

$$y(t)=e^{-2t}\left(C_1\cos\sqrt{2}t+C_2\sin\sqrt{2}t\right)$$

$$y(0)=1 \quad \rightarrow \quad \boxed{C_1=1}$$

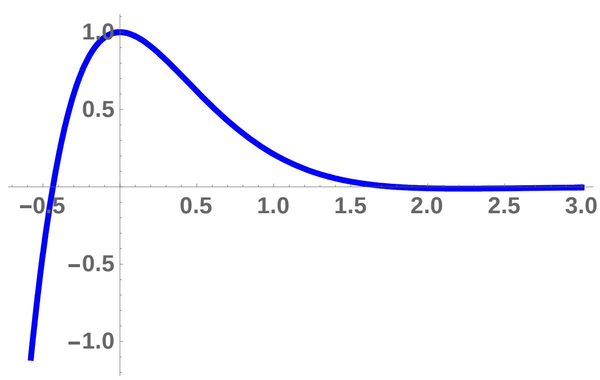
$$y'(t)=e^{-2t}\left(-2C_1\cos\sqrt{2}t-2C_2\sin\sqrt{2}t-\sqrt{2}C_1\sin\sqrt{2}t+\sqrt{2}C_2\cos\sqrt{2}t\right)$$

$$y'(0)=0 \quad \rightarrow \; -2C_1+\sqrt{2}C_2=0 \quad C_2=\sqrt{2}$$

$$\boxed{y(t)=e^{-2t}\left(\cos\sqrt{2}t+\sqrt{2}\sin\sqrt{2}t\right)}$$

$$A=\sqrt{1+2}\ \boxed{=\sqrt{3}} \quad \& \quad \phi=\tan^{-1}\frac{1}{\sqrt{2}}\ \boxed{\approx 0.615}$$

$$\boxed{y(t)=\sqrt{3}\,e^{-2t}\sin\left(\sqrt{2}t-0.615\right)}$$



For $\boldsymbol{k} = \mathbf{8}$

$$\boxed{\lambda_{1,2} = -2 \pm 2i}$$

$$y(t) = e^{-2t}\left(C_1 \cos 2t + C_2 \sin 2t\right)$$

$$y(0) = 1 \quad \to \quad \boxed{C_1 = 1}$$

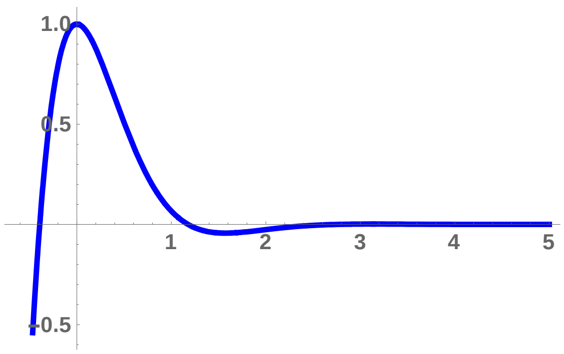
$$y'(t) = e^{-2t}\left(-2C_1 \cos 2t - 2C_2 \sin 2t - 2C_1 \sin 2t + 2C_2 \cos 2t\right)$$

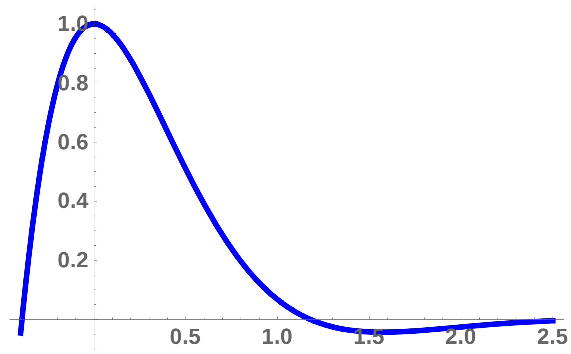
$$y'(0) = 0 \quad \to \quad -2C_1 + 2C_2 = 0 \quad C_2 = 1$$

$$\boxed{y(t) = e^{-2t}\left(\cos 2t + \sin 2t\right)}$$

$$A = \sqrt{1+1} = \boxed{\sqrt{2}} \quad \& \quad \phi = \tan^{-1}\frac{1}{1} = \boxed{\frac{\pi}{4}}$$

$$\boxed{y(t) = \sqrt{2}\, e^{-2t} \sin\left(2t - \frac{\pi}{4}\right)}$$





Exercise

A 10-*lb* mass weight is attached to the end of a spring, stretches it 3 *inches*. This mass is removed and replaced with a mass of 1.6 *slugs*, which initially released from a point 4 inches above the equilibrium position with a downward velocity of. $\frac{5}{4}$ *ft/s*

a) Find the equation of the motion.
b) Find the amplitude, phase angle, period and the frequency.
c) Express the motion equation in amplitude and phase angle form.
d) Determine the times the mass attains a displacement below the equilibrium position numerically equal to $\frac{1}{2}$ the amplitude of motion.

Solution

a) $1.6y'' + 40y = 0\ ;\quad y(0) = -\frac{1}{3},\quad y'(0) = \frac{5}{4}$

$$1.6\lambda^2 + 40 = 0 \quad \to \quad \lambda_{1,2} = \pm 5i$$

$$y(t) = C_1 \cos 5t + C_2 \sin 5t$$

$$y(0) = -\frac{1}{3} \quad \rightarrow \quad \boxed{C_1 = -\frac{1}{3}}$$

$$y'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t$$

$$y'(0) = \frac{5}{4} \quad \rightarrow \quad \boxed{C_2 = \frac{1}{4}}$$

$$\boxed{y(t) = -\frac{1}{3}\cos 5t + \frac{1}{4}\sin 5t}$$

b) Amplitude: $A = \sqrt{\frac{1}{9} + \frac{1}{16}} = \boxed{\frac{5}{12}}$

Phase angle: $\phi = \tan^{-1}\left|-\frac{3}{4}\right| \boxed{\approx 0.9273}$

Period: $\boxed{T = \frac{2\pi}{5}}$ $\qquad T = \frac{2\pi}{\omega_0}$

Frequency: $\boxed{f = \frac{5}{2\pi}}$ $\qquad f = \frac{1}{T}$

c) $\boxed{y(t) = \frac{5}{12}\sin(5t - 0.9273)}$

d) $y(t) = \frac{1}{2}A = \frac{5}{24}$

$$\frac{5}{24} = \frac{5}{12}\sin(5t - 0.9273) \quad \rightarrow \quad \sin(5t - 0.9273) = \frac{1}{2}$$

$$5t - 0.9273 = \frac{\pi}{6} + 2n\pi \quad \& \quad 5t - 0.9273 = \frac{5\pi}{6} + 2n\pi$$

$$\boxed{t = \frac{1}{5}\left(\frac{\pi}{6} + 0.9273 + 2n\pi\right)} \quad \& \quad \boxed{t = \frac{1}{5}\left(\frac{5\pi}{6} + 0.9273 + 2n\pi\right)}$$

Exercise

A 64-*lb* mass weight is attached to the end of a spring, stretches it 0.32 *foot*. This mass is initially released from a point 8 *inches* above the equilibrium position with a downward velocity of 5 *ft/s*.

a) Find the equation of the motion.
b) Find the amplitude, phase angle, period and the frequency.
c) Write the motion equation with phase angle form.
d) How many complete cycles will the mass have completed at the end of 3π *sec*.
e) At what time does the mass pass through the equilibrium position heading downward for the second time?
f) At what times does the mass attain its extreme displacements on either side of the equilibrium position?
g) What is the position of the mass at $t = 3$ *sec*?
h) What is the instantaneous velocity at $t = 3$ *sec*?
i) What is the acceleration at $t = 3$ *sec*?

j) What is the instantaneous velocity at the times when the mass passes through the equilibrium position?

k) At what times is the mass 5 *inches* below the equilibrium position?

l) At what times is the mass 5 *inches* below the equilibrium position heading in the upward direction?

Solution

$m = \frac{64}{32} = 2 \;\; slugs$ $\qquad W = mg$

$k(0.32\, f)t = 64 \quad \to \quad k = 200 \;\; lb/ft$ $\qquad kx = mg$

a) $2y'' + 200y = 0\,; \quad y(0) = -\frac{2}{3}, \quad y'(0) = 5$ $\qquad my'' + \mu y' + ky = 0$

$$\lambda^2 + 100 = 0 \quad \to \quad \lambda_{1,2} = \pm 10i$$

$$y(t) = C_1 \cos 10t + C_2 \sin 10t$$

$$y(0) = -\frac{2}{3} \quad \to C_1 = -\frac{2}{3}$$

$$y'(t) = -10C_1 \sin 10t + 10C_2 \cos 10t$$

$$y'(0) = 5 \quad \to C_2 = \frac{1}{2}$$

$$y(t) = -\frac{2}{3}\cos 10t + \frac{1}{2}\sin 10t$$

b) ***Amplitude***: $\quad A = \sqrt{\frac{4}{9} + \frac{1}{4}} = \frac{5}{6}$

Phase angle: $\quad \phi = \tan^{-1}\left|-\frac{4}{3}\right| \approx 0.9273$

Period: $\quad T = \frac{2\pi}{10} = \frac{\pi}{5}$ $\qquad T = \frac{2\pi}{\omega_0}$

Frequency: $\quad f = \frac{5}{\pi}$ $\qquad f = \frac{1}{T}$

c) $y(t) = \frac{5}{6}\sin(10t - 0.9273)$

d) $T = \frac{\pi}{5}n = 3\pi \quad \to \quad n = 15 \;\; cycles$

e) Mass passes through the equilibrium position: $y(t) = 0$

$$y(t) = \frac{5}{6}\sin(10t - 0.9273) = 0$$

$10t - 0.9273 = n\pi \quad \to$ *second time* $\quad 10t - 0.9273 = 2\pi$

$$t = \frac{2\pi + .9273}{10} \approx 0.721 \;\; sec$$

f) $y'(t) = \frac{25}{3}\cos(10t - 0.9273) = 0$

$$10t - 0.9273 = \frac{\pi}{2} + n\pi$$

$$\boxed{t = \frac{(2n+1)\pi}{20} + 0.09273}$$

g) $y(t=3) = \frac{5}{6}\sin(30 - 0.9273)\ \boxed{\approx -0.597\ \ ft}$

h) $y'(t=3) = \frac{25}{3}\cos(30 - 0.9273)\ \boxed{\approx -5.814\ \ ft/s}$

i) $y''(t) = -\frac{250}{3}\sin(10t - 0.9273)$

$$y''(t=3) = -\frac{250}{3}\sin(30 - 0.9273)\ \boxed{\approx -59.685\ \ ft/s^2}$$

j) $y(t) = 0 \quad \to 10t - 0.9273 = n\pi \qquad t = \frac{1}{10}(n\pi + 0.9273)$

$$y'\left(t = \frac{1}{10}(n\pi + 0.9273)\right) = \frac{25}{3}\cos(n\pi + .9273 - 0.9273)$$

$$= \frac{25}{3}\cos(n\pi) \qquad\qquad \cos(n\pi) = \pm 1$$

$$\boxed{\approx \pm 8.33\ \ ft/s}$$

k) $y = 5\ in\frac{1\ ft}{12\ in} = \frac{5}{12}\ ft$

$$y(t) = \frac{5}{6}\sin(10t - 0.9273) = \frac{5}{12}$$

$$\sin(10t - 0.9273) = \frac{1}{2}$$

$$10t - 0.9273 = \frac{\pi}{6} + 2n\pi \quad \& \quad 10t - 0.9273 = \frac{5\pi}{6} + 2n\pi$$

$$\boxed{t = \frac{1}{10}\left(\frac{\pi}{6} + 2n\pi + 0.9273\right)} \quad \& \quad \boxed{t = \frac{1}{10}\left(\frac{5\pi}{6} + 2n\pi + 0.9273\right)}$$

l) $y = \frac{5}{12}\ ft\ \&\ y'(t) < 0$

$$\boxed{t = \frac{1}{10}\left(\frac{5\pi}{6} + 2n\pi + 0.9273\right)}$$

Exercise

If it is underdamped, write the position function in the form $x(t) = C_1 e^{-pt}\cos(\omega_1 t - \alpha_1)$.

Also, find the undamped position function $u(t) = C_0\cos(\omega_0 t - \alpha_0)$ that would result if the mass on the spring were set in motion with the same initial position and velocity, but with the dashpot disconnected (so $c = 0$). Then, construct a figure that illustrates the effect of damping by comparing the graphs of $x(t)$ and $u(t)$ $\qquad m = \frac{1}{2}, \quad c = 3, \quad k = 4; \quad x_0 = 2, \quad v_0 = 0$

Solution

With damping motion

$$\frac{1}{2}x'' + 3x' + 4x = 0 \qquad\qquad mx'' + cx' + kx = 0$$

$$\lambda^2 + 6\lambda + 8 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -4,\ -2}$$

$$x(t) = C_1 e^{-4t} + C_2 e^{-2t}$$

$$x(0) = \boxed{C_1 + C_2 = 2}$$

$$x'(t) = -4C_1 e^{-4t} - 2C_2 e^{-2t} \ \rightarrow\ x'(0) = \boxed{-4C_1 - 2C_2 = 0}$$

$$\begin{cases} C_1 + C_2 = 2 \\ -4C_1 - 2C_2 = 0 \end{cases} \rightarrow \boxed{C_1 = -2} \quad \boxed{C_2 = 4}$$

$$\boxed{x(t) = -2e^{-4t} + 4e^{-2t}} \qquad (\textit{Overdamped motion})$$

Without damping $(c = 0)$

$$\frac{1}{2}x'' + 4x = 0 \qquad\qquad mx'' + kx = 0$$

$$\lambda^2 + 8 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 2i\sqrt{2}}$$

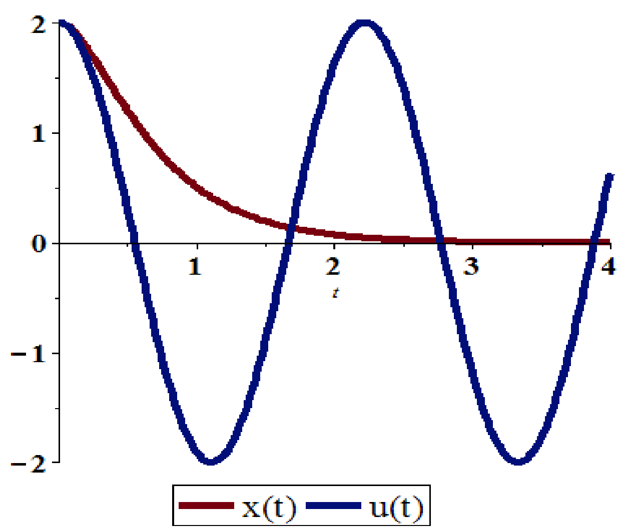
$$u(t) = A\cos\left(2\sqrt{2}t\right) + B\sin\left(2\sqrt{2}t\right)$$

$$u(0) = \boxed{A = 2}$$

$$u'(t) = -2A\sqrt{2}\sin\left(2\sqrt{2}t\right) + 2B\sqrt{2}\cos\left(2\sqrt{2}t\right)$$

$$\rightarrow \quad u'(0) = 2B\sqrt{2} = 0 \Rightarrow \boxed{B = 0}$$

$$\boxed{u(t) = 2\cos\left(2\sqrt{2}t\right)}$$



Exercise

If it is underdamped, write the position function in the form $x(t) = C_1 e^{-pt}\cos(\omega_1 t - \alpha_1)$.

Also find the undamped position function $u(t) = C_0 \cos(\omega_0 t - \alpha_0)$ that would result if the mass on the spring were set in motion with the same initial position and velocity, but with the dashpot disconnected (so $c = 0$). Then, construct a figure that illustrates the effect of damping by comparing the graphs of $x(t)$ and $u(t)$

$$m = 1, \quad c = 8, \quad k = 16; \quad x_0 = 5, \quad v_0 = -10$$

Solution

With damping motion

$x'' + 8x' + 16x = 0 \qquad\qquad mx'' + cx' + kx = 0$

$\lambda^2 + 8\lambda + 16 = 0 \ \rightarrow\ \boxed{\lambda_{1,2} = -4}$

$x(t) = (C_1 + C_2 t)e^{-4t}$

$x(0) = \boxed{C_1 = 5}$

$x'(t) = (C_2 - 4C_1 - 4C_2 t)e^{-4t}$

$x'(0) = C_2 - 4C_1 = -10 \quad \Rightarrow \boxed{C_2 = 10}$

$\boxed{x(t) = (5 + 10t)e^{-2t}} \qquad$ (*Overdamped motion*)

Without damping $(c = 0)$

$x'' + 16x = 0 \qquad\qquad mx'' + kx = 0$

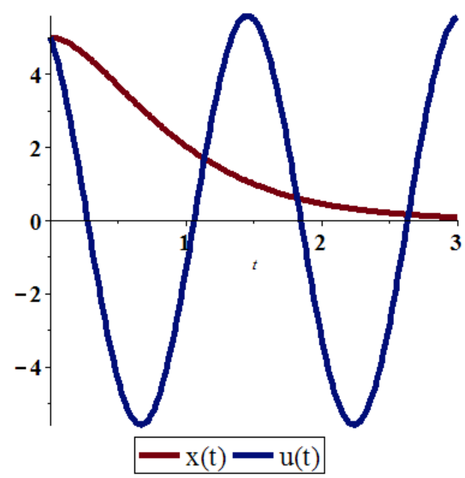
$\lambda^2 + 16 = 0 \ \rightarrow\ \boxed{\lambda_{1,2} = \pm 4i}$

$u(t) = A\cos(4t) + B\sin(4t)$

$u(0) = \boxed{A = 5}$

$u'(t) = -4A\sin(4t) + 4B\cos(4t)$

$\rightarrow\ u'(0) = 4B = -10 \Rightarrow \boxed{B = -\frac{5}{2}}$



$A = \sqrt{25 + \frac{25}{4}} = \boxed{\frac{5\sqrt{5}}{2}} \qquad \phi = \tan^{-1}\left(-\frac{1}{2}\right) = \boxed{-0.4636}$

$u(t) = 5\cos(4t) - \frac{5}{2}\sin(4t)$

$\boxed{= \frac{5\sqrt{5}}{2}\cos(4t + 0.4636)}$

Exercise

If it is underdamped, write the position function in the form $x(t)=C_1 e^{-pt}\cos\left(\omega_1 t-\alpha_1\right)$.

Also find the undamped position function $u(t)=C_0\cos\left(\omega_0 t-\alpha_0\right)$ that would result if the mass on the spring were set in motion with the same initial position and velocity, but with the dashpot disconnected (so $c=0$). Then, construct a figure that illustrates the effect of damping by comparing the graphs of $x(t)$ and $u(t)$

$$m=1,\quad c=10,\quad k=125;\quad x_0=6,\quad v_0=50$$

Solution

With damping motion

$x''+10x'+125x=0 \qquad\qquad mx''+cx'+kx=0$

$\lambda^2+10\lambda+125=0 \quad\rightarrow\quad \underline{\lambda_{1,2}=-5\pm 10i}$

$x(t)=e^{-5t}\left(A\cos 10t+B\sin 10t\right) \qquad x(0)=\underline{A=6}$

$x'(t)=\left(-5A\cos 10t-5B\sin 10t-10A\sin 10t+10B\cos 10t\right)e^{-5t}$

$\rightarrow\ x'(0)=-5A+10B=50 \Rightarrow \underline{B=8}$

$A=\sqrt{36+64}\ \underline{=10} \qquad \phi=\tan^{-1}\left(\frac{4}{3}\right)\ \underline{=0.9273}$

$x(t)=e^{-5t}\left(6\cos 10t+8\sin 10t\right)$

$\underline{=10e^{-5t}\cos\left(10t-0.9273\right)}$ (*Overdamped motion*)

Without damping $(c=0)$

$x''+125x=0 \qquad\qquad mx''+kx=0$

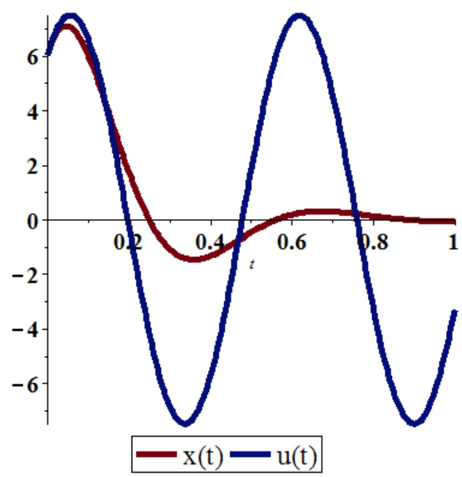
$\lambda^2+125=0 \quad\rightarrow\quad \underline{\lambda_{1,2}=\pm 5i\sqrt{5}}$

$u(t)=A\cos\left(5\sqrt{5}\ t\right)+B\sin\left(5\sqrt{5}\ t\right)$

$u(0)=\underline{A=6}$

$u'(t)=-5\sqrt{5}A\sin\left(5\sqrt{5}\ t\right)+5\sqrt{5}B\cos\left(5\sqrt{5}\ t\right)$

$\rightarrow\ x'(0)=5\sqrt{5}B=50 \ \Rightarrow \underline{B=2\sqrt{5}}$



$A=\sqrt{36+20}\ \underline{=2\sqrt{14}} \qquad \phi=\tan^{-1}\left(\frac{\sqrt{5}}{3}\right)\ \underline{=0.6405}$

$u(t)=6\cos\left(5\sqrt{5}t\right)+2\sqrt{5}\sin\left(5\sqrt{5}t\right)$

$\underline{=2\sqrt{14}\cos\left(5\sqrt{5}\ t-.6405\right)}$

Exercise

If it is underdamped, write the position function in the form $x(t)=C_1e^{-pt}\cos\left(\omega_1t-\alpha_1\right)$.

Also find the undamped position function $u(t)=C_0\cos\left(\omega_0t-\alpha_0\right)$ that would result if the mass on the spring were set in motion with the same initial position and velocity, but with the dashpot disconnected (so $c=0$). Then, construct a figure that illustrates the effect of damping by comparing the graphs of $x(t)$ and $u(t)$ $\qquad m=2,\quad c=12,\quad k=50;\quad x_0=0,\quad v_0=-8$

Solution

With damping motion

$2x''+12x'+50x=0 \qquad\qquad mx''+cx'+kx=0$

$\lambda^2+6\lambda+25=0 \quad\rightarrow\quad \boxed{\lambda_{1,2}=3\pm 4i}$

$x(t)=e^{3t}\left(A\cos 4t+B\sin 4t\right)$

$x(0)=\boxed{A=0}$

$x'(t)=\left(3A\cos 4t+3B\sin 4t-4A\sin 4t+4B\cos 4t\right)e^{3t}$

$\rightarrow\ x'(0)=3A+4B=-8 \quad\Rightarrow \boxed{B=-2}$

$x(t)=-2e^{3t}\sin 4t$

$\boxed{=2e^{3t}\cos\left(4t-\frac{3\pi}{2}\right)} \qquad (\textit{Overdamped motion})$

Without damping $\quad (c=0)$

$2x''+50x=0 \qquad\qquad mx''+kx=0$

$2\lambda^2+50=0 \quad\rightarrow\quad \boxed{\lambda_{1,2}=\pm 5i}$

$u(t)=A\cos(5t)+B\sin(5t)$

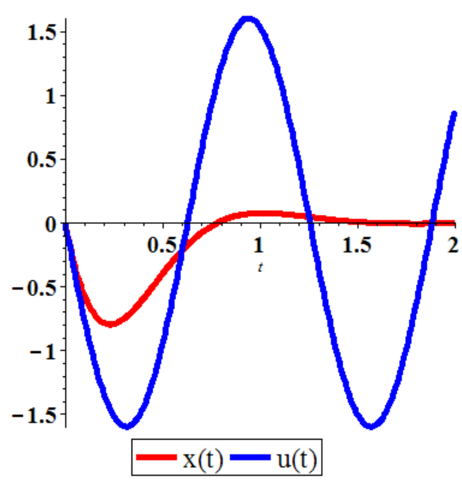
$u(0)=\boxed{A=0}$

$u'(t)=-5A\sin(\ t)+5B\cos(5t)$

$\rightarrow\ u'(0)=5B=-8 \ \Rightarrow \boxed{B=-\frac{8}{5}}$

$u(t)=-\frac{8}{5}\sin(5t)$

$\boxed{=\frac{8}{5}\cos\left(5t-\frac{3\pi}{2}\right)}$



Exercise

If it is underdamped, write the position function in the form $x(t)=C_1e^{-pt}\cos(\omega_1 t-\alpha_1)$.

Also find the undamped position function $u(t)=C_0\cos(\omega_0 t-\alpha_0)$ that would result if the mass on the spring were set in motion with the same initial position and velocity, but with the dashpot disconnected (so $c=0$). Then, construct a figure that illustrates the effect of damping by comparing the graphs of $x(t)$ and $u(t)$ $\qquad m=2,\quad c=16,\quad k=40;\quad x_0=5,\quad v_0=4$

Solution

With damping motion

$$2x''+16x'+40x=0 \qquad mx''+cx'+kx=0$$

$$\lambda^2+8\lambda+20=0 \rightarrow \boxed{\lambda_{1,2}=-4\pm 2i}$$

$$x(t)=e^{-4t}(A\cos 2t+B\sin 2t)$$

$$x(0)=\boxed{A=5}$$

$$x'(t)=(-4A\cos 2t+-4B\sin 2t-2A\sin 2t+2B\cos 2t)e^{-4t}$$

$$\rightarrow x'(0)=-4A+2B=4 \Rightarrow \boxed{B=12}$$

$$A=\sqrt{25+144}\ \boxed{=13} \quad \phi=\tan^{-1}\left(\frac{12}{5}\right)\boxed{=1.176}$$

$$x(t)=e^{-4t}(5\cos 2t+12\sin 2t)$$

$$\boxed{=12e^{-4t}\cos(2t-1.176)} \qquad (\textit{Overdamped motion})$$

Without damping $(c=0)$

$$2x''+40x=0 \qquad mx''+kx=0$$

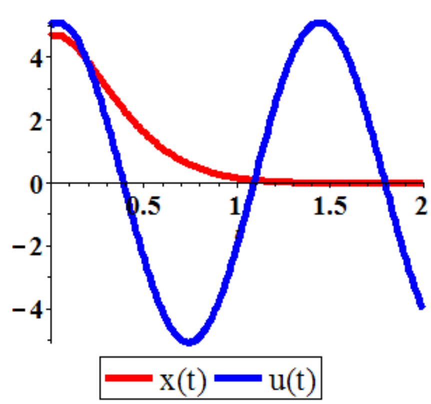
$$\lambda^2+20=0 \rightarrow \boxed{\lambda_{1,2}=\pm 2\sqrt{5}i}$$

$$u(t)=A\cos(2\sqrt{5}\,t)+B\sin(2\sqrt{5}\,t)$$

$$u(0)=\boxed{A=5}$$

$$u'(t)=-2\sqrt{5}A\sin(2\sqrt{5}\,t)+2\sqrt{5}B\cos(2\sqrt{5}\,t)$$

$$\rightarrow x'(0)=2\sqrt{5}B=4 \Rightarrow \boxed{B=\frac{2\sqrt{5}}{5}}$$



$$A=\sqrt{25+\frac{4}{5}}\ \boxed{=\sqrt{\frac{129}{5}}} \quad \phi=\tan^{-1}\left(\frac{2\sqrt{5}}{25}\right)\boxed{=0.177}$$

$$u(t)=5\cos(2\sqrt{5}\,t)-\frac{2\sqrt{5}}{5}\sin(2\sqrt{5}\,t)$$

$$\boxed{=\sqrt{\frac{129}{5}}\cos(2\sqrt{5}t-0.177)}$$

Exercise

If it is underdamped, write the position function in the form $x(t) = C_1 e^{-pt}\cos(\omega_1 t - \alpha_1)$.

Also find the undamped position function $u(t) = C_0 \cos(\omega_0 t - \alpha_0)$ that would result if the mass on the spring were set in motion with the same initial position and velocity, but with the dashpot disconnected (so $c = 0$). Then, construct a figure that illustrates the effect of damping by comparing the graphs of $x(t)$ and $u(t)$ $\quad m = 3, \quad c = 30, \quad k = 63; \quad x_0 = 2, \quad v_0 = 2$

Solution

With damping motion

$3x'' + 30x' + 63x = 0 \qquad mx'' + cx' + kx = 0$

$\lambda^2 + 10\lambda + 21 = 0 \;\rightarrow\; \lambda_{1,2} = -7, \; -3$

$x(t) = C_1 e^{-7t} + C_2 e^{-3t}$

$x(0) = C_1 + C_2 = 2$

$x'(t) = -7C_1 e^{-7t} - 3C_2 e^{-3t}$

$\rightarrow x'(0) = -7C_1 - 3C_2 = 2$

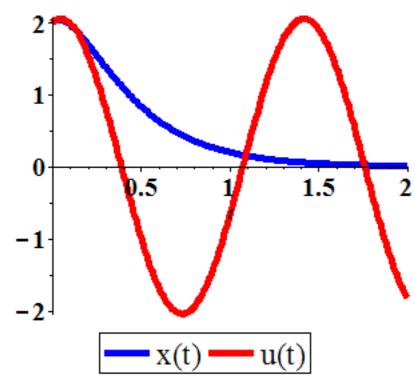
$$\begin{cases} C_1 + C_2 = 2 \\ -7C_1 - 3C_2 = 2 \end{cases} \rightarrow C_1 = -2 \quad C_2 = 4$$

$x(t) = 4e^{-3t} - 2e^{-7t}$ $\qquad$ (*Overdamped motion*)

Without damping $\quad (c = 0)$

$3x'' + 63x = 0 \qquad mx'' + kx = 0$

$\lambda^2 + 21 = 0 \;\rightarrow\; \lambda_{1,2} = \pm i\sqrt{21}$



$u(t) = A\cos(\sqrt{21}\,t) + B\sin(\sqrt{21}\,t)$

$u(0) = A = 2$

$u'(t) = -A\sqrt{21}\sin(\sqrt{21}\,t) + B\sqrt{21}\cos(\sqrt{21}\,t)$

$\rightarrow \; u'(0) = \sqrt{21}B = 2 \;\Rightarrow B = \dfrac{2}{\sqrt{21}}$

$A = \sqrt{4 + \dfrac{4}{21}} = 2\sqrt{\dfrac{22}{21}} \qquad \phi = \tan^{-1}\left(\dfrac{1}{\sqrt{21}}\right) = 0.2149$

$u(t) = 2\cos(\sqrt{21}\,t) + \dfrac{2}{\sqrt{21}}\sin(\sqrt{21}\,t)$

$= 2\sqrt{\dfrac{22}{21}}\cos(\sqrt{21}\,t - 0.2149)$

Exercise

If it is underdamped, write the position function in the form $x(t)=C_1e^{-pt}\cos\left(\omega_1 t-\alpha_1\right)$.

Also find the undamped position function $u(t)=C_0\cos\left(\omega_0 t-\alpha_0\right)$ that would result if the mass on the spring were set in motion with the same initial position and velocity, but with the dashpot disconnected (so $c=0$). Finally, construct a figure that illustrates the effect of damping by comparing the graphs of $x(t)$ and $u(t)$ $\quad m=4,\quad c=20,\quad k=169;\quad x_0=4,\quad v_0=16$

Solution

With damping motion

$4x''+20x'+169x=0 \qquad\qquad mx''+cx'+kx=0$

$4\lambda^2+20\lambda+169=0 \quad\rightarrow\quad \lambda_{1,2}=-\frac{5}{2}\pm 6i$

$x(t)=e^{-\frac{5}{2}t}\left(A\cos 6t+B\sin 6t\right)$

$x(0)=A=5$

$x'(t)=\left(-\frac{5}{2}A\cos 6t+-\frac{5}{2}B\sin 6t-6A\sin 6t+6B\cos 6t\right)e^{-\frac{5}{2}t}$

$\rightarrow\ x'(0)=-\frac{5}{2}A+6B=16 \quad\Rightarrow 6B=16+10\rightarrow B=\frac{13}{3}$

$A=\sqrt{16+\frac{169}{9}}=\frac{\sqrt{313}}{3} \qquad \phi=\tan^{-1}\left(\frac{13}{12}\right)=0.8254$

$x(t)=e^{-\frac{5}{2}t}\left(4\cos 6t+\frac{13}{3}\sin 6t\right)$

$=\frac{\sqrt{313}}{3}e^{-\frac{5}{2}t}\cos\left(6t-0.8254\right) \qquad$ (*Overdamped motion*)

Without damping $\quad (c=0)$

$4x''+169x=0 \qquad\qquad mx''+kx=0$

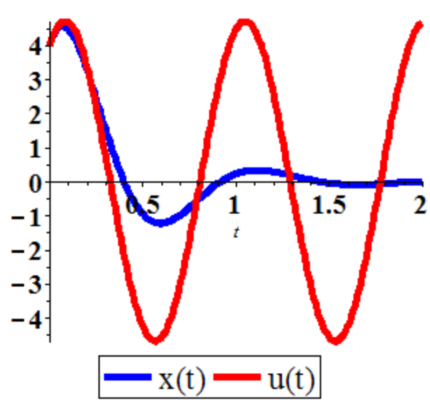
$4\lambda^2+169=0 \quad\rightarrow\quad \lambda_{1,2}=\pm\frac{13}{2}i$

$u(t)=A\cos\left(\frac{13}{2}t\right)+B\sin\left(\frac{13}{2}t\right)$

$u(0)=A=4$

$u'(t)=-\frac{13}{2}A\sin\left(\frac{13}{2}t\right)+\frac{13}{2}B\cos\left(\frac{13}{2}t\right)$

$\rightarrow\ x'(0)=\frac{13}{2}B=16 \quad\Rightarrow B=\frac{32}{13}$



$$A=\sqrt{16+\frac{1,024}{169}}=\underline{\frac{4\sqrt{233}}{13}}\qquad \phi=\tan^{-1}\left(\frac{8}{13}\right)\underline{=0.5517}$$

$$u(t)=4\cos\left(\frac{13}{2}t\right)+\frac{32}{13}\sin\left(\frac{13}{2}t\right)$$

$$=\frac{4\sqrt{233}}{13}\cos\left(\frac{13}{2}t-0.5517\right)$$

Exercise

Suppose that the mass in a mass–spring–dashpot system with $m=10,\ c=9,\ and\ k=2$ is set in motion with $x(0)=0$ and $x'(0)=5$

a) Find the position function $x(t)$ and graph the function

b) Find how far the mass moves to the right before starting back toward the origin.

Solution

a) $10x''+9x'+2x=0$ $\qquad mx''+cx'+kx=0$

The *characteristic equation*: $10\lambda^2+9\lambda+2=0 \quad\rightarrow\quad \lambda_{1,2}=\frac{-9\pm 1}{20}=\underline{-\frac{1}{2},\ -\frac{2}{5}}$

$$x(t)=C_1e^{-t/2}+C_2e^{-2t/5}$$

$$x(0)=0 \quad\rightarrow\quad C_1+C_2=0$$

$$x'(t)=-\frac{1}{2}C_1e^{-t/2}-\frac{2}{5}C_2e^{-2t/5}$$

$$x'(0)=5 \quad\rightarrow -\frac{1}{2}C_1-\frac{2}{5}C_2=5 \quad\Rightarrow 5C_1+4C_2=-50$$

$$-4\begin{cases} C_1+C_2=0 \\ 5C_1+4C_2=-50\end{cases} \quad\rightarrow\quad \underline{C_1=-50,\ \ C_2=50}$$

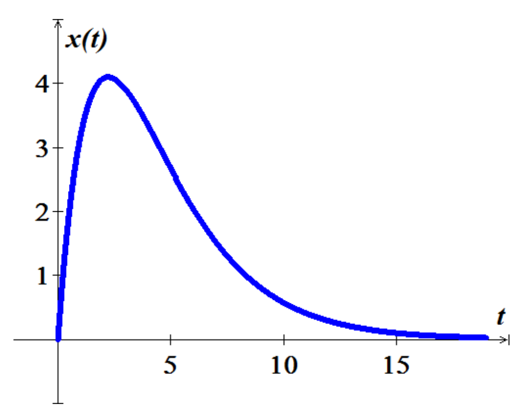
$$\underline{x(t)=-50e^{-t/2}+50e^{-2t/5}}$$

b) $x'(t)=25e^{-t/2}-20e^{-2t/5}=0$

$5e^{-t/2}=4e^{-2t/5}$

$e^{-t/10}=\frac{4}{5}$

$\underline{t=-10\ln\frac{4}{5}}\ \underline{\approx 2.2314}$



The farthest distance to the right of the mass is:

$$x\left(t=-10\ln\frac{4}{5}\right)=-50e^{5\ln 4/5}+50e^{4\ln 4/5}$$

$$=50\left(-\left(\frac{4}{5}\right)^5+\left(\frac{4}{5}\right)^4\right)$$

$$= 50\left(\frac{4}{5}\right)^4\left(1-\frac{4}{5}\right)$$

$$= 10\left(\frac{4}{5}\right)^4 \quad = 4.096$$

Exercise

Suppose that the mass in a mass–spring–dashpot system with $m = 25,\ c = 10,\ and\ k = 226$ is set in motion with $x(0) = 20$ and $x'(0) = 41$

a) Find the position function $x(t)$ and graph the function

b) Find the pseudoperiod of the oscillations and the equations of the "envelope curves" that are dashed.

Solution

a) $25x'' + 10x' + 226x = 0$ $\qquad\qquad mx'' + cx' + kx = 0$

The *characteristic equation*: $25\lambda^2 + 10\lambda + 226 = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-10 \pm 150}{50} = -\frac{1}{5} \pm 3i$

$$x(t) = e^{-t/5}\left(C_1 \cos 3t + C_2 \sin 3t\right)$$

$$x(0) = 20 \quad \rightarrow \quad C_1 = 20$$

$$x(t) = e^{-t/5}\left(-\frac{1}{5}C_1 \cos 3t - \frac{1}{5}C_2 \sin 3t - 3C_1 \sin 3t + 3C_2 \cos 3t\right)$$

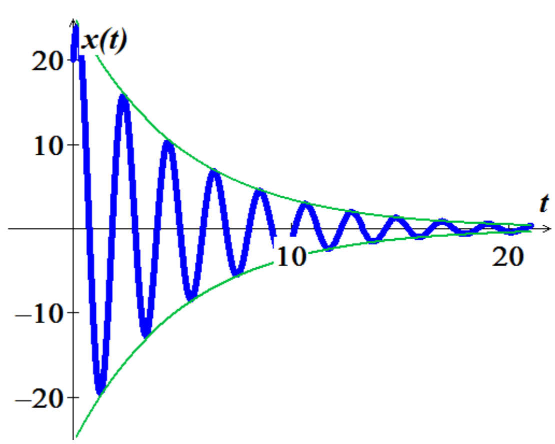
$$x'(0) = 41 \quad \rightarrow \quad -\frac{1}{5}C_1 + 3C_2 = 41 \quad \Rightarrow C_2 = 15$$

$$x(t) = e^{-t/5}\left(20\cos 3t + 15\sin 3t\right)$$

$$A = \sqrt{20^2 + 15^2} = 24$$

$$\phi = \tan^{-1}\frac{3}{4} \approx 0.6435$$

$$x(t) = 25e^{-t/5}\cos\left(3t - 0.6435\right)$$



b) Since $-1 \le \cos\theta \le 1$, then the oscillation are bounded by the curves $x(t) = \pm 25e^{-t/5}$ and pseudoperiod: $T = \frac{2\pi}{\omega} = \frac{2\pi}{3}$

Exercise

A mass of 1 *slug* is suspended from a spring, the spring constant is 9 *lb/ft*. The mass is initially released from a point 1 *foot* above the equilibrium position with an upward velocity of $\sqrt{3}$ *ft/s*. Find the times at which the mass is heading downward at a velocity of 3 *ft/s*

Solution

$$y'' + 9y = 0\ ; \quad y(0) = -1, \quad y'(0) = -\sqrt{3} \qquad\qquad mx'' + \mu x' + kx = 0$$

$$\lambda^2 + 9 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = \pm 3i|}$$

$$y_h = C_1 \cos 3t + C_2 \sin 3t$$

$$y(0) = -1 \quad \rightarrow \quad \underline{C_1 = -1|}$$

$$y_h' = -3C_1 \sin 3t + 3C_2 \cos 3t$$

$$y'(0) = -\sqrt{3} \quad \rightarrow \quad \underline{C_2 = -\frac{\sqrt{3}}{3}|}$$

$$\underline{y(t) = -\cos 3t + \frac{\sqrt{3}}{3}\sin 3t|}$$

$$A = \sqrt{(-1)^2 + \left(\frac{\sqrt{3}}{3}\right)^2} = \frac{2}{\sqrt{3}}$$

$$\phi = \tan^{-1}\frac{-\sqrt{3}}{3} = -\frac{\pi}{6}$$

$$\underline{y(t) = \frac{2}{\sqrt{3}}\cos\left(3t + \frac{\pi}{6}\right)|}$$

$$y'(t) = -2\sqrt{3}\sin\left(3t + \frac{\pi}{6}\right) = 3$$

$$\sin\left(3t + \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$3t + \frac{\pi}{6} = \frac{4\pi}{3} + 2n\pi \qquad\qquad 3t + \frac{\pi}{6} = \frac{5\pi}{3} + 2n\pi$$

$$\underline{t = \frac{7\pi}{18} + \frac{2n\pi}{3}|} \qquad\qquad \underline{t = \frac{\pi}{2} + \frac{2n\pi}{3}|}$$

Exercise

Two parallel springs, with constants k_1 and k_2, support a single mass, the effective spring constant of the system is given by $k=\dfrac{4k_1k_2}{k_1+k_2}$.

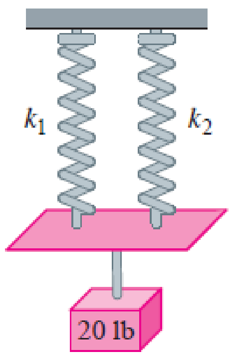
A mass weight 20 *pounds* stretches one spring 6 *inches* and another spring 2 *inches*. The springs are attached to a common rigid support and then to a metal plate. The mass is attached to the center of the plate in the double-spring constant arrangement.

a) Determine the effective spring constant of this system.

b) Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of $2\ ft/s$.

Solution

$$m=\frac{20}{32}=\frac{5}{8}\ slug \qquad\qquad W=mg$$



a) $k_1(0.5\ ft)=20 \quad\rightarrow\quad k_1=40\ \ lb/ft \qquad\qquad kx=mg$

$$k_2\left(\frac{2}{12}\ ft\right)=20 \quad\rightarrow\quad k_2=120\ \ lb/ft$$

$$k=\frac{4(40)(120)}{40+120} \qquad\qquad k=\frac{4k_1k_2}{k_1+k_2}$$

$$\boxed{=120\ \ lb/ft}$$

b) $\frac{5}{8}x''+120x=0$

$$x''+192x=0\ ; \quad x(0)=0, \quad x'(0)=2$$

$$\lambda^2+192=0 \quad\rightarrow\quad \boxed{\lambda_{1,2}=\pm 8i\sqrt{3}}$$

$$x(t)=C_1\cos 8\sqrt{3}t+C_2\sin 8\sqrt{3}t$$

$$x(0)=0 \quad\rightarrow\quad \boxed{C_1=0}$$

$$x'(t)=-8\sqrt{3}C_1\sin 8\sqrt{3}t+8\sqrt{3}C_2\cos 8\sqrt{3}t$$

$$x'(0)=2 \quad\rightarrow 8\sqrt{3}C_2=2 \quad \boxed{C_2=\frac{\sqrt{3}}{12}}$$

$$\boxed{x(t)=\frac{\sqrt{3}}{12}\sin 8\sqrt{3}t}$$

Exercise

A 12–*lb* weight is attached both to a vertically suspended spring that it stretches 6 *in*. and to a dashpot that provides 3 *lb*. of resistance for every foot per second of velocity.

a) If the weight is pulled down 1 *foot*. below its static equilibrium position and then released from rest at time $t=0$, find its position function $x(t)$.

b) Find the frequency, time-varying amplitude, and phase angle of the motion.

Solution

Given: $c = 3$ *lb-sec/ft*

$$m = \frac{12}{32} = \frac{3}{8} \ \textit{slug} \qquad\qquad W = mg$$

$$k\left(\frac{6}{12}\ \textit{ft}\right) = 12 \quad \to \quad k = 24 \ \textit{lb/ft} \qquad\qquad kx = mg$$

a) $\frac{3}{8}x'' + 3x + 24 = 0$

$$x'' + 8x + 64 = 0\ ; \quad x(0) = 1, \quad x'(0) = 0$$

$$\lambda^2 + 8\lambda + 64 = 0 \quad \to \quad \lambda_{1,2} = \frac{-8 \pm 8i\sqrt{3}}{2} = -4 \pm 4i\sqrt{3}$$

$$x(t) = \left(C_1 \cos 4\sqrt{3}t + C_2 \sin 4\sqrt{3}t\right)e^{-4t}$$

$$x(0) = 1 \quad \to \quad C_1 = 1$$

$$x'(t) = \left(-4\sqrt{3}C_1 \sin 4\sqrt{3}t + 4\sqrt{3}C_2 \cos 4\sqrt{3}t - 4C_1 \cos 4\sqrt{3}t - 4C_2 \sin 4\sqrt{3}t\right)e^{-4t}$$

$$x'(0) = 0 \quad \to 4\sqrt{3}C_2 - 4C_1 = 0 \quad \Rightarrow \quad C_2 = \frac{\sqrt{3}}{3}$$

$$x(t) = e^{-4t}\left(\cos 4\sqrt{3}t + \frac{\sqrt{3}}{3}\sin 4\sqrt{3}t\right)$$

b) *Frequency*: $4\sqrt{3} \approx 6.93$ *rad/sec*

Time-varying amplitude $\quad A = \sqrt{1 + \frac{1}{3}} = \frac{2}{\sqrt{3}}$

Phase angle $\quad \phi = \tan^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6}$

$$x(t) = \frac{2}{\sqrt{3}}e^{-4t}\cos\left(4\sqrt{3}t - \frac{\pi}{6}\right)$$

Exercise

A $\frac{1}{8}$-kg mass is attached to a spring with a spring constant $k = 16\ N/m$. The mass is displaced $\frac{1}{2}\ m$ to the right of the equilibrium point and given an outward velocity (to the right) of $\sqrt{2}\ m/sec$. Neglecting any damping or external forces that may be present,

a) Determine the equation of motion of the mass

b) Determine the equation of motion amplitude, period, and natural frequency.

c) How long after release does the mass pass through the equilibrium position?

Solution

a) $\frac{1}{8}x'' + 16x = 0 \qquad\qquad mx'' + \mu x' + kx = 0$

$x'' + 128x = 0\ ; \quad x(0) = \frac{1}{2}, \quad x'(0) = \sqrt{2}$

$\lambda^2 + 128 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 8i\sqrt{2}}$

$x_h = C_1 \cos 8\sqrt{2}t + C_2 \sin 8\sqrt{2}t$

$x(0) = \frac{1}{2} \quad \rightarrow \quad \boxed{C_1 = \frac{1}{2}}$

$x'_h = -8\sqrt{2}C_1 \sin 8\sqrt{2}t + 8\sqrt{2}C_2 \cos 8\sqrt{2}t$

$x'(0) = \sqrt{2} \quad \rightarrow 8\sqrt{2}C_2 = \sqrt{2} \quad \Rightarrow \quad \boxed{C_2 = \frac{1}{8}}$

$\boxed{x(t) = \frac{1}{2}\cos 8\sqrt{2}t + \frac{1}{8}\sin 8\sqrt{2}t}$

b) *Amplitude*: $\quad A = \sqrt{\frac{1}{4} + \frac{1}{64}} \boxed{= \frac{\sqrt{17}}{8}}$

Phase angle: $\quad \phi = \tan^{-1} 4 \boxed{\approx 1.326} \qquad\qquad C_1 \text{ and } C_2 \in QI$

$\boxed{x(t) = \frac{\sqrt{17}}{8}\sin\left(8\sqrt{2}t + 1.326\right)}$

Period: $\quad P = \frac{2\pi}{\omega} \boxed{= \frac{\pi\sqrt{2}}{8}}$

Natural Frequency: $\quad \frac{1}{P} \boxed{= \frac{8}{\pi\sqrt{2}}}$

c) $x(t) = \frac{\sqrt{17}}{8}\sin\left(8\sqrt{2}t + 1.326\right) = 0$

$8\sqrt{2}t + 1.326 = n\pi$

$t = \frac{n\pi - 1.326}{8\sqrt{2}}$

$n = 1 \quad \rightarrow \quad t = \frac{\pi - 1.326}{8\sqrt{2}} \boxed{\approx 0.16\ sec}$

Exercise

A 3-*kg* mass is attached to a spring with a spring constant 75 *N/m*. The mass is displaced $\frac{1}{4}$ *m* to the left and given a velocity of 1 *m/sec* to the right. The damping force is negligible.

a) Determine the equation of motion of the mass

b) Determine the equation of motion amplitude, period, and natural frequency.

c) How long after release does the mass pass through the equilibrium position?

Solution

Given: $m = 3 \quad k = 75 \quad c = 0$

a) $3x'' + 75x = 0\ ; \quad x(0) = -\frac{1}{4}, \quad x'(0) = 1 \qquad\qquad mx'' + \mu x' + kx = 0$

$$3\lambda^2 + 75 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 5i}$$

$$x_h = C_1 \cos 5t + C_2 \sin 5t$$

$$x(0) = -\frac{1}{4} \quad \rightarrow \quad \boxed{C_1 = -\frac{1}{4}}$$

$$x_h' = -5C_1 \sin 5t + 5C_2 \cos 5t$$

$$x'(0) = 1 \quad \rightarrow 5C_2 = 1 \quad \Rightarrow \quad \boxed{C_2 = \frac{1}{5}}$$

$$\boxed{x(t) = -\frac{1}{4}\cos 5t + \frac{1}{5}\sin 5t}$$

d) *Amplitude*: $\quad A = \sqrt{\frac{1}{16} + \frac{1}{25}} = \boxed{\frac{\sqrt{41}}{20}}$

Phase angle: $\quad \phi = \pi - \tan^{-1}\frac{5}{4} \approx \boxed{2.246} \qquad C_1 \text{ and } C_2 \in QII$

$$\boxed{x(t) = \frac{\sqrt{41}}{20}\sin(5t + 2.246)}$$

Period: $\quad P = \frac{2\pi}{\omega} = \boxed{\frac{2\pi}{5}}$

Natural Frequency: $\quad \frac{1}{P} = \boxed{\frac{5}{2\pi}}$

e) $x(t) = \frac{\sqrt{41}}{20}\sin(5t + 2.246) = 0$

$$5t + 2.246 = n\pi$$

$$t = \frac{n\pi - 2.246}{5}$$

$$n = 1 \quad \rightarrow \quad t = \frac{\pi - 2.246}{5} \approx \boxed{0.179\ sec}$$

Exercise

A 3-*kg* mass is attached to a spring with a spring constant 300 *N/m*. The mass is is pulled down 10 *cm* and released with downward velocity of 1 *m/sec*. The damping force is negligible.

a) Determine the equation of motion of the mass

b) Solve the equation to find the time when the maximum downward displacement of the mass from its equilibrium position is first achieved.

c) What is the maximum downward displacement?

Solution

Given: $m=3 \quad k=300 \quad \mu=0$

a) $3y''+300y=0\ ; \quad y(0)=0.1, \quad y'(0)=1 \qquad\qquad my''+\mu y'+ky=0$

$$3\lambda^2+300=0 \quad \rightarrow \lambda^2=-100 \quad \boxed{\lambda_{1,2}=\pm 10i}$$

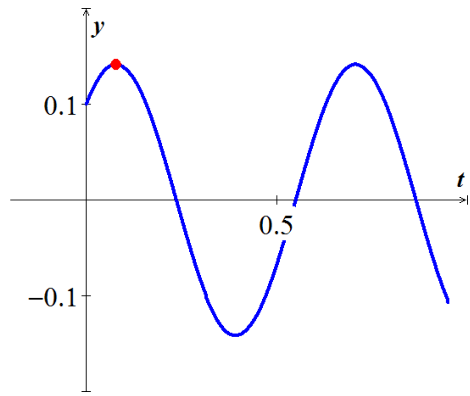
$$y(t)=C_1\cos 10t+C_2\sin 10t$$

$$x(0)=\frac{1}{10} \quad \rightarrow \quad \boxed{C_1=\frac{1}{10}}$$

$$y'(t)=-10C_1\sin 10t+10C_2\cos 10t$$

$$y'(0)=1 \quad \rightarrow 10C_2=1 \quad \Rightarrow \quad \boxed{C_2=\frac{1}{10}}$$

$$\boxed{y(t)=\frac{1}{10}(\cos 10t+\sin 10t)}$$



b) $y'(t)=\cos 10t-\sin 10t=0$

$$\cos 10t=\sin 10t \quad \rightarrow \quad 10t=\frac{\pi}{4}$$

The time when the maximum downward displacement of the mass from its equilibrium position is first achieved $\boxed{t=\frac{\pi}{40}}$

c) $y\left(t=\frac{\pi}{40}\right)=\frac{1}{10}\left(\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\right)$

$$=\frac{1}{10}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)$$

$$\boxed{=\frac{\sqrt{2}}{10}}$$

Exercise

A 10-*kg* mass is attached to the end of a spring hanging vertically, stretches the spring 0.03 *m*. The mass is pulled down another 7 *cm* and released (with no initial velocity).

a) Determine the spring constant *k*.
b) Determine the equation of motion of the mass

Solution

a) $k=\dfrac{10(9.8)}{0.03}\ \underline{=3,266.67\ N/m}|$ $\qquad ky=mg$

b) $10y''+3266.67y=0\ ;\quad y(0)=.07,\quad y'(0)=0 \qquad my''+cy'+ky=F(t)$

$$10\lambda^2+3266.67=0 \quad\rightarrow\quad \underline{\lambda=\pm 18.07i}|$$

$$y_h=C_1\cos 18.07t+C_2\sin 18.07t$$

$$y(0)=0.07 \quad\rightarrow\quad \underline{C_1=0.07}|$$

$$y_h'=-18.07C_1\sin 18.07t+18.07C_2\cos 18.07t$$

$$y'(0)=0 \quad\rightarrow\quad \underline{C_2=0}|$$

$$\underline{y(t)=0.07\cos 18.07t}|$$

Exercise

A 10-*kg* mass is attached to a spring with spring constant $k=300\ N/m$. At time $t=0$, the mass is pulled down another 10 *cm* and released with a downward velocity of 100 *cm/sec*.

a) Determine the equation of motion.
b) What is the maximum downward displacement?

Solution

a) $3y''+300y=0\ ;\quad y(0)=0.1,\quad y'(0)=1$

$$\lambda^2+100=0 \quad\rightarrow\quad \underline{\lambda_{1,2}=\pm 10i}|$$

$$y_h=C_1\cos 10t+C_2\sin 10t$$

$$y(0)=0.1 \quad\rightarrow\quad \underline{C_1=0.1}|$$

$$y_h'=-10C_1\sin 10t+10C_2\cos 10t$$

$$y'(0)=1 \quad\rightarrow\quad \underline{C_2=0.1}|$$

$$\underline{y(t)=0.1\cos 10t+0.1\sin 10t}|$$

b) $y'(t)=-\sin 10t+\cos 10t=0$

$$\sin 10t = \cos 10t \quad \rightarrow \quad \tan 10t = 1 \;\Rightarrow\; 10t = \frac{\pi}{4}$$

$$\boxed{t = \frac{\pi}{40}}$$

$$y\left(t = \frac{\pi}{40}\right) = \frac{1}{10}\cos\frac{\pi}{4} + \frac{1}{10}\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{20} + \frac{\sqrt{2}}{20}$$

$$\boxed{= \frac{\sqrt{2}}{10}}$$

Exercise

A 10-*kg* mass is attached to the end of a spring hanging vertically at rest. The mass is pulled down another 7 *cm* and released (with no initial velocity).

a) Determine the spring constant *k*.

b) Determine the equation of motion of the mass

Solution

a) $k = \dfrac{10(9.8)}{0.03}\ \boxed{= 3,266.67\ N/m}$ $\qquad ky = mg$

b) $10y'' + 3266.67y = 0\ ; \quad y(0) = .07, \quad y'(0) = 0 \qquad my'' + cy' + ky = F(t)$

$$10\lambda^2 + 3266.67 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 18.07i}$$

$$y_h = C_1 \cos 18.07t + C_2 \sin 18.07t$$

$$y(0) = 0.07 \quad \rightarrow \quad \boxed{C_1 = 0.07}$$

$$y_h' = -18.07C_1 \sin 18.07t + 18.07C_2 \cos 18.07t$$

$$y'(0) = 0 \quad \rightarrow \quad \boxed{C_2 = 0}$$

$$\boxed{y(t) = 0.07\cos 18.07t}$$

Exercise

A 10-*kg* mass is attached to the end of a spring hanging vertically, stretches the spring 0.7 *m*. The mass is started in motion from the equilibrium position with an initial velocity 1 *m/sec* in the upward direction. If the force due to air resistance is $-90y'\ N$

a) Determine the spring constant *k*.

b) Determine the equation of motion of the mass

Solution

a) $k = \dfrac{10(9.8)}{0.7}\ \boxed{= 140\ N/m}$ $\qquad ky = mg$

b) $10y'' + 90y' + 140y = 0$; $y(0) = 0, \quad y'(0) = 1$ $\qquad my'' + cy' + ky = F(t)$

$\lambda^2 + 9\lambda + 14 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -2, \ -7}$

$y(t) = C_1 e^{-2t} + C_2 e^{-7t}$

$y(0) = 0 \quad \rightarrow \quad C_1 + C_2 = 0$

$y' = -2C_1 e^{-2t} - 7C_2 e^{-7t}$

$y'(0) = 1 \quad \rightarrow \quad -2C_1 - 7C_2 = 1$

$\Delta = \begin{vmatrix} 1 & 1 \\ -2 & -7 \end{vmatrix} = -5 \quad \Delta_{C_1} = \begin{vmatrix} 0 & 1 \\ 1 & -7 \end{vmatrix} = -1 \quad \Delta_{C_2} = \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1$

$\boxed{C_1 = \frac{1}{5}, \quad C_2 = -\frac{1}{5}}$

$\boxed{y(t) = \frac{1}{5} e^{-2t} - \frac{1}{5} e^{-7t}}$

Exercise

A $\frac{1}{4}$-*slug* mass is attached to the end of a spring hanging vertically, stretches the spring 1.28 *ft*. The mass is started in motion from the equilibrium position with an initial velocity 4 *ft/sec* in the downward direction. If the force due to air resistance is $-2y'$ *lb*

a) Determine the spring constant *k*.

b) Determine the equation of motion of the mass

Solution

a) $k = \frac{1}{4}\frac{32}{1.28} \ \boxed{= 6.25 \ lb/ft}$ $\qquad ky = mg$

b) $\frac{1}{4}y'' + 2y' + 6.25y = 0$ $\qquad my'' + cy' + ky = F(t)$

$y'' + 8y' + 25y = 0$; $y(0) = 0, \quad y'(0) = 4$

$\lambda^2 + 8\lambda + 25 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -4 \pm 3i}$

$y(t) = e^{-4t}\left(C_1 \cos 3t + C_2 \sin 3t\right)$

$y(0) = 0 \quad \rightarrow \quad C_1 = 0$

$y' = e^{-4t}\left(-4C_1 \cos 3t - 4C_2 \sin 3t - 3C_1 \sin 3t + 3C_2 \cos 3t\right)$

$y'(0) = 4 \quad \rightarrow \quad \boxed{C_2 = \frac{4}{3}}$

$\boxed{y(t) = \frac{4}{3} e^{-4t} \sin 3t}$

Exercise

A 20-*kg* mass is attached to the end of a spring hanging vertically at rest. When given an initial downward velocity of 2 *m/s* from its equilibrium position the mass was observed to attain a maximum displacement of 0.2 *m* from its equilibrium position.

a) Determine the spring constant *k*.

b) Determine the equation of motion of the mass

Solution

a) $20y'' + ky = 0\ ; \quad y(0) = 0, \quad y'(0) = 2 \qquad\qquad my'' + cy' + ky = F(t)$

$$20\lambda^2 + k = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm\frac{1}{2}\sqrt{\frac{k}{5}}\,i}$$

$$y_h = C_1\cos\frac{1}{2}\sqrt{\frac{k}{5}}t + C_2\sin\frac{1}{2}\sqrt{\frac{k}{5}}t$$

$$y(0) = 0 \quad \rightarrow \quad \boxed{C_1 = 0}$$

$$y_h' = -\frac{1}{2}\sqrt{\frac{k}{5}}C_1\sin\frac{1}{2}\sqrt{\frac{k}{5}}t + \frac{1}{2}\sqrt{\frac{k}{5}}C_2\cos\frac{1}{2}\sqrt{\frac{k}{5}}t$$

$$y'(0) = 2 \quad \rightarrow \frac{1}{2}\sqrt{\frac{k}{5}}C_2 = 2 \quad \boxed{C_2 = 4\sqrt{\frac{5}{k}}}$$

$$\boxed{y(t) = 4\sqrt{\frac{5}{k}}\sin\frac{1}{2}\sqrt{\frac{k}{5}}t}$$

$$y'(t) = 2\cos\frac{1}{2}\sqrt{\frac{k}{5}}t = 0 \quad \rightarrow \quad \boxed{t = \frac{\pi}{2}}$$

$$y_{max} = 4\sqrt{\frac{5}{k}} = 0.2$$

$$\sqrt{\frac{5}{k}} = \frac{5}{100}$$

$$\frac{k}{5} = \left(\frac{100}{5}\right)^2 = 400$$

$$\boxed{k = 2{,}000\ N/m}$$

b) $\boxed{y(t) = \frac{1}{5}\sin 10t}$

Exercise

A steel ball weighing 128-*lb* is attached to the end of a spring, stretches 2 *ft* from its natural length. The ball is started in motion with no initial velocity by displacing it 6 *in* above the equilibrium position. Assuming no air resistance.

a) Determine the spring constant *k*.

b) Find the equation of the ball position at time *t*.

c) Find the position of the ball at $t = \frac{\pi}{12}$ *sec*

Solution

a) $m = \frac{128}{32} \boxed{= 4\ slugs}$ $\qquad w = mg$

$k = \frac{128}{2} \boxed{= 64\ N/m}$ $\qquad ky = mg$

b) $4y'' + 64y = 0$; $\quad y(0) = -\frac{6}{12} = -\frac{1}{2}, \quad y'(0) = 0$ $\qquad my'' + cy' + ky = F(t)$

$4\lambda^2 + 64 = 0 \quad \to \quad \boxed{\lambda_{1,2} = \pm 4i}$

$y(t) = C_1 \cos 4t + C_2 \sin 4t$

$y(0) = -\frac{1}{2} \quad \to \quad \boxed{C_1 = -\frac{1}{2}}$

$y' = -4C_1 \sin 4t + 4C_2 \cos 4t$

$y'(0) = 0 \quad \to \quad \boxed{C_2 = 0}$

$\boxed{y(t) = -\frac{1}{2}\cos 4t}$

c) $y\left(t = \frac{\pi}{12}\right) = -\frac{1}{2}\cos\frac{\pi}{3}$

$\boxed{= -\frac{1}{4}\ ft}$

$\boxed{= 3\ \ in}$

Exercise

A 9-*lb* mass is attached to the end of a spring hanging vertically with spring constant $k = 32\ lb/ft$, is perturbed from its equilibrium position with a certain upward initial velocity. The amplitude of the resulting vibrations is observed to be $4\ in$.

a) Determine the equation of motion.

b) What is the initial velocity?

c) Determine the period and frequency of the vibrations?

Solution

a) $\frac{9}{32}y'' + 32y = 0$; $\quad y(0) = 0, \quad y'(0) = y_0' < 0 \ (upward)$ $\qquad my'' + cy' + ky = F(t)$

$9\lambda^2 + 32^2 = 0 \quad \to \quad \boxed{\lambda_{1,2} = \pm\frac{32}{3}i}$

$y_h = C_1 \cos\frac{32}{3}t + C_2 \sin\frac{32}{3}t$

$y(0) = 0 \quad \to \quad \boxed{C_1 = 0}$

$$y(t) = C_2 \sin\frac{32}{3}t$$

Amplitude: $|A| = \frac{4}{12} = \frac{1}{3} \quad \to \quad \boxed{C_2 = -\frac{1}{3}}$

$$\boxed{y(t) = -\frac{1}{3}\sin\frac{32}{3}t}$$

b) $y'(t) = -\frac{32}{9}\cos\frac{32}{3}t$

$$\boxed{y'(0) = -\frac{32}{9}\ ft/sec}$$

c) *Period*: $\quad P = \frac{2\pi}{\omega} = \boxed{\frac{3\pi}{16}}$

Frequency: $\quad f = \frac{1}{P} = \boxed{\frac{16}{3\pi}\ Hz}$

Exercise

A 2-*kg* mass is suspended from a spring with a spring constant of 10 *N/m*. The mass is started in motion from the equilibrium position with an initial velocity 1.5 *m/sec*. Assuming no air resistance

a) Determine the equation of motion of the mass.

b) Determine the circular frequency, natural frequency, and period.

Solution

a) $2x'' + 10x = 0$

$x'' + 5x = 0\ ; \quad x(0) = 0; \quad x'(0) = 1.5$

$\lambda^2 + 5 = 0 \quad \to \quad \boxed{\lambda_{1,2} = \pm i\sqrt{5}}$

$x(t) = C_1 \cos\sqrt{5}t + C_2 \sin\sqrt{5}t$

$x(0) = 0 \quad \to \quad \boxed{C_1 = 0}$

$x' = -\sqrt{5}C_1 \sin\sqrt{5}t + \sqrt{5}C_2 \cos\sqrt{5}t$

$x'(0) = 1.5 \quad \to \quad \boxed{C_2 = \frac{3}{2\sqrt{5}}}$

$$\boxed{x(t) = \frac{3\sqrt{5}}{10}\sin\sqrt{5}t}$$

b) *Circular frequency*: $\quad \boxed{\omega = \sqrt{5} \approx 2.236\ \ Hz}$

Natural frequency: $\quad f = \frac{\omega}{2\pi} = \boxed{\frac{\sqrt{5}}{2\pi} \approx 0.3559\ Hz}$

Period: $\quad T = \frac{1}{f} = \boxed{\frac{2\pi}{\sqrt{5}} \approx 2.81\ \ sec}$

Exercise

A $\frac{1}{4}$-*slug* mass is attached to a spring having a spring constant of 1 *lb/ft*. The mass is started in motion initially displacing it 2 *ft* in the downward direction with an initial velocity 2 *ft/sec* in the upward direction. If the force due to air resistance is $-1x'$ *lb*. Find the subsequent motion of the mass

Solution

$$\frac{1}{4}x'' + x' + 1 = 0\ ; \quad x(0) = 2, \quad x'(0) = -2$$

$$\lambda^2 + 4\lambda + 4 = 0 \quad \to \quad \boxed{\lambda_{1,2} = -2}$$

$$x(t) = \left(C_1 + C_2 t\right)e^{-2t}$$

$$x(0) = 2 \quad \to \quad \boxed{C_1 = 2}$$

$$x' = \left(C_2 - 2C_1 - 2C_2 t\right)e^{-2t}$$

$$x'(0) = -2 \quad \to C_2 - 2C_1 = -2 \quad \Rightarrow \quad \boxed{C_2 = 2}$$

$$\boxed{x(t) = (2 - 2t)e^{-2t}}$$

Exercise

A spring with a mass of 2-*kg* has natural length 0.5 *m*. A force of 25.6 *N* is required to maintain it stretched to a length of 0.7 *m*. If the spring is stretched to a length of 0.7 *m* and then released with initial velocity zero. Find the position of the mass at any time *t*.

Solution

$$k = \frac{25.6}{0.7 - 0.5} \boxed{= 128} \qquad\qquad k\left(x_2 - x_1\right) = F$$

$$2x'' + 128 = 0\ ; \quad x(0) = 0.2, \quad x'(0) = 0$$

$$\lambda^2 + 64 = 0 \quad \to \quad \boxed{\lambda_{1,2} = \pm 8i}$$

$$x(t) = C_1 \cos 8t + C_2 \sin 8t$$

$$x(0) = 0.2 \quad \to \quad \boxed{C_1 = \frac{1}{5}}$$

$$x' = -8C_1 \sin 8t + 8C_2 \cos 8t$$

$$x'(0) = 0 \quad \to \quad \boxed{C_2 = 0}$$

$$\boxed{x(t) = \frac{1}{5}\cos 8t}$$

Exercise

A spring with a mass of 2-*kg* has natural length $0.5\ m$. A force of $25.6\ N$ is required to maintain it stretched to a length of $0.7\ m$. The spring is immersed in a fluid with damping constant $c = 40$. If the spring is started from the equilibrium position and is given a push to start it with initial velocity $0.6\ m/s$. Find the position of the mass at any time *t*.

Solution

$$k = \frac{25.6}{0.7 - 0.5} = 128 \qquad\qquad k\left(x_2 - x_1\right) = F$$

$$2x'' + 40x + 128 = 0\ ; \quad x(0) = 0, \quad x'(0) = 0.6$$

$$\lambda^2 + 20\lambda + 64 = 0 \quad \to \quad \lambda_{1,2} = -10 \pm 6 = -16,\ -4$$

$$x(t) = C_1 e^{-16t} + C_2 e^{-4t}$$

$$x(0) = 0 \quad \to \quad C_1 + C_2 = 0$$

$$x' = -16C_1 e^{-16t} - 4C_2 e^{-4t}$$

$$x'(0) = 0.6 \quad \to \quad -16C_1 - 4C_2 = 0.6$$

$$\Rightarrow -16C_1 + 4C_1 = 0.6 \quad \to \quad C_1 = -0.05,\ C_2 = 0.05$$

$$x(t) = -0.05e^{-16t} + 0.05e^{-4t}$$

Exercise

A spring with a mass of 3-*kg* is held stretched $0.6\ m$ beyond its natural length by a force of $20\ N$. If the spring begins at its equilibrium and with initial velocity $1.2\ m/s$. Find the position of the mass.

Solution

$$k = \frac{20}{0.6} = \frac{100}{3} \qquad\qquad kx = F$$

$$3x'' + \frac{100}{3}x = 0\ ; \quad x(0) = 0, \quad x'(0) = 1.2 \qquad\qquad mx'' + cx' + kx = F(t)$$

$$3\lambda^2 + \frac{100}{3} = 0 \quad \to \quad \lambda = \pm\frac{10}{3}i$$

$$x(t) = C_1 \cos\frac{10}{3}t + C_2 \sin\frac{10}{3}t$$

$$x(0) = 0 \quad \to \quad C_1 = 0$$

$$x' = -\frac{10}{3}C_1 \sin\frac{10}{3}t + \frac{10}{3}C_2 \cos\frac{10}{3}t$$

$$x'(0) = \frac{6}{5} \quad \to \frac{10}{3}C_2 = \frac{6}{5} \quad \Rightarrow \quad C_2 = \frac{9}{25}$$

$$\boxed{x(t) = \frac{9}{25}\sin\frac{10}{3}t}$$

Exercise

A spring with a mass of 2-*kg* is held stretched 0.5 *m*, has damping constant 14, and a force of 6 *N*. If the spring is stretched 1 *m* beyond at its equilibrium and with no initial velocity.

a) Find the position of the mass at any time *t*.

b) Find the mass that would produce critical damping.

Solution

a) $k = \frac{6}{.5} = 12 \qquad kx = F$

$2x'' + 14x' + 12x = 0\ ;\quad x(0) = 1,\quad x'(0) = 0 \qquad mx'' + cx' + kx = F(t)$

$\lambda^2 + 7\lambda + 6 = 0 \quad \to \quad \boxed{\lambda_1 = -6,\ \lambda_2 = -1}$

$x(t) = C_1 e^{-6t} + C_2 e^{-t}$

$x(0) = 1 \quad \to \quad C_1 + C_2 = 1$

$x(t) = -6C_1 e^{-6t} - C_2 e^{-t}$

$x'(0) = 0 \quad \to \quad -6C_1 - C_2 = 0$

$\Delta = \begin{vmatrix} 1 & 1 \\ -6 & -1 \end{vmatrix} = 5 \quad \Delta_{C_1} = \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1 \quad \Delta_{C_2} = \begin{vmatrix} 1 & 1 \\ -6 & 0 \end{vmatrix} = 6$

$\boxed{C_1 = -\frac{1}{5},\ C_2 = \frac{6}{5}}$

$\boxed{x(t) = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}}$

b) $m\lambda^2 + 14\lambda + 12 = 0 \quad \to \quad \lambda_{1,2} = \frac{-14 \pm \sqrt{196 - 48m}}{2m}$

For critical damping: $196 - 48m = 0$

$m = \frac{196}{48} \ \boxed{= \frac{49}{12}\ kg}$

Exercise

A spring has a mass of 1-*kg* and its spring constant $k = 100$. The spring is released at a point 0.1 *m* above its equilibrium position. Graph the position function for the following values of damping constant *c*: 10, 15, 20, 25, 30. What type of damping occurs each case?

Solution

Given: $m=1, \quad k=100, \quad x(0)=-0.1, \quad x'(0)=0$

$x''+cx'+100x=0 \ ; \quad x(0)=-0.1, \quad x'(0)=0$ $\qquad mx''+cx'+kx=F(t)$

For $c=10$

$$\lambda^2+10\lambda+100=0 \quad \rightarrow \quad \boxed{\lambda_{1,2}=-5\pm 5i\sqrt{3}}$$

$\therefore$ The motion is ***underdamped***

$$x(t)=e^{-5t}\left(C_1\cos 5\sqrt{3}t+C_2\sin 5\sqrt{3}t\right)$$

$$x(0)=-0.1 \quad \rightarrow \quad \boxed{C_1=-0.1=-\frac{1}{10}}$$

$$x'=e^{-5t}\left(-5C_1\cos 5\sqrt{3}t-5C_2\sin 5\sqrt{3}t-5\sqrt{3}C_1\sin 5\sqrt{3}t+5\sqrt{3}C_2\cos 5\sqrt{3}t\right)$$

$$x'(0)=0 \quad \rightarrow -5C_1+5\sqrt{3}C_2=0 \quad \boxed{C_2=-\frac{1}{10\sqrt{3}}}$$

$$\boxed{x(t)=-\frac{1}{10}e^{-5t}\left(\cos 5\sqrt{3}t+\frac{\sqrt{3}}{3}\sin 5\sqrt{3}t\right)}$$

For $c=15$

$$\lambda^2+15\lambda+100=0 \quad \rightarrow \quad \boxed{\lambda_{1,2}=-\frac{15}{2}\pm i\frac{5\sqrt{7}}{2}}$$

$\therefore$ The motion is ***underdamped***

$$x(t)=e^{-15t/2}\left(C_1\cos\frac{5\sqrt{7}}{2}t+C_2\sin\frac{5\sqrt{7}}{2}t\right)$$

$$x(0)=-0.1 \quad \rightarrow \quad \boxed{C_1=-0.1=-\frac{1}{10}}$$

$$x'=e^{-15t/2}\left(-\frac{15}{2}C_1\cos\frac{5\sqrt{7}}{2}t-\frac{15}{2}C_2\sin\frac{5\sqrt{7}}{2}t-\frac{5\sqrt{7}}{2}C_1\sin\frac{5\sqrt{7}}{2}t+\frac{5\sqrt{7}}{2}C_2\cos\frac{5\sqrt{7}}{2}t\right)$$

$$x'(0)=0 \quad \rightarrow -\frac{15}{2}C_1+\frac{5\sqrt{7}}{2}C_2=0 \quad \boxed{C_2=-\frac{3}{10\sqrt{7}}}$$

$$\boxed{x(t)=e^{-15t/2}\left(-\frac{1}{10}\cos\frac{5\sqrt{7}}{2}t-\frac{3}{10\sqrt{7}}\sin\frac{5\sqrt{7}}{2}t\right)}$$

For $c=20$

$$\lambda^2+20\lambda+100=0 \quad \rightarrow \quad \boxed{\lambda_{1,2}=-10}$$

$\therefore$ The motion is ***critically damped***

$$x(t)=\left(C_1+C_2t\right)e^{-10t}$$

$$x(0)=-0.1 \quad \rightarrow \quad \boxed{C_1=-0.1=-\frac{1}{10}}$$

$$x' = \left(C_2 - 10C_1 - 10C_2 t\right)e^{-10t}$$

$$x'(0) = 0 \quad \to C_2 + 1 = 0 \quad \boxed{C_2 = -1}$$

$$\boxed{x(t) = (-0.1 - t)e^{-10t}}$$

For $c = 25$

$$\lambda^2 + 25\lambda + 100 = 0 \quad \to \quad \lambda_{1,2} = \frac{-25 \pm 15}{2} \boxed{= -20, \ -5}$$

$\therefore$ The motion is ***overdamped***

$$x(t) = C_1 e^{-20t} + C_2 e^{-5t}$$

$$x(0) = -0.1 \quad \to \quad C_1 + C_2 = -0.1$$

$$x' = -20C_1 e^{-20t} - 5C_2 e^{-5t}$$

$$x'(0) = 0 \quad \to \quad -20C_1 - 5C_2 = 0 \ \Rightarrow \ 4C_1 + C_2 = 0$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} = -3 \quad \Delta_{C_1} = \begin{vmatrix} -0.1 & 1 \\ 0 & 1 \end{vmatrix} = -0.1 \quad \Delta_{C_2} = \begin{vmatrix} 1 & -0.1 \\ 4 & 0 \end{vmatrix} = 0.4$$

$$\boxed{C_1 = \frac{1}{30}, \ C_2 = -\frac{4}{30} = -\frac{2}{15}}$$

$$\boxed{x(t) = \frac{1}{10}e^{-10t} - \frac{1}{5}e^{-5t}}$$

For $c = 30$

$$\lambda^2 + 30\lambda + 100 = 0 \quad \to \quad \boxed{\lambda_{1,2} = -15 \pm 5\sqrt{5}}$$

$\therefore$ The motion is ***overdamped***

$$x(t) = C_1 e^{\left(-15-5\sqrt{5}\right)t} + C_2 e^{\left(-15+5\sqrt{5}\right)t}$$

$$x(0) = -0.1 \quad \to \quad C_1 + C_2 = -0.1$$

$$x' = \left(-15 - 5\sqrt{5}\right)C_1 e^{\left(-15-5\sqrt{5}\right)t} + \left(-15 + 5\sqrt{5}\right)C_2 e^{\left(-15+5\sqrt{5}\right)t}$$

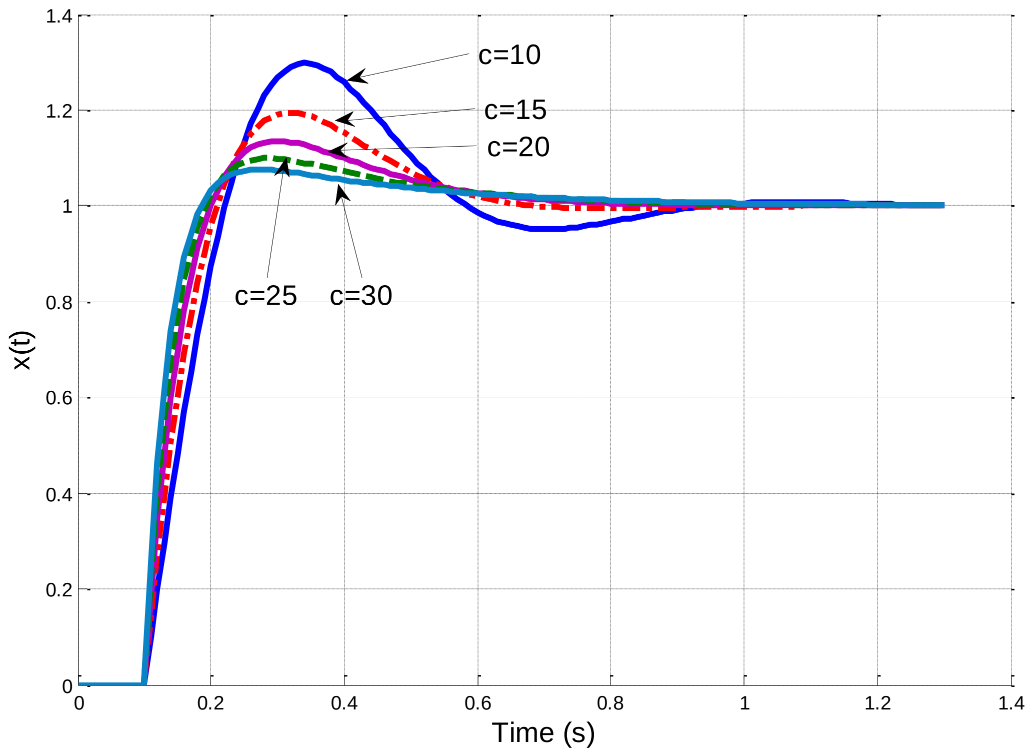
$$x'(0) = 0 \quad \to \quad \left(-15 - 5\sqrt{5}\right)C_1 + \left(-15 + 5\sqrt{5}\right)C_2 = 0$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ -15 - 5\sqrt{5} & -15 + 5\sqrt{5} \end{vmatrix} = 10\sqrt{5}$$

$$\Delta_{C_1} = \begin{vmatrix} -0.1 & 1 \\ 0 & -15 + 5\sqrt{5} \end{vmatrix} = 1.5 - .5\sqrt{5} \quad \Delta_{C_2} = \begin{vmatrix} 1 & -0.1 \\ -15 - 5\sqrt{5} & 0 \end{vmatrix} = -1.5 - .5\sqrt{5}$$

$$C_1 = \frac{15 - 5\sqrt{5}}{100\sqrt{5}} \boxed{= \frac{5 - 3\sqrt{5}}{100}} \qquad C_2 = \frac{-15 - 5\sqrt{5}}{100\sqrt{5}} \boxed{= \frac{-5 - 3\sqrt{5}}{100}}$$

$$\boxed{x(t)=\frac{5-3\sqrt{5}}{100}e^{\left(-15-5\sqrt{5}\right)t}+\frac{-5-3\sqrt{5}}{100}e^{\left(-15+5\sqrt{5}\right)t}}$$



Exercise

A 4-*kg* mass is attached to a spring and set in motion. A record of the displacements was made and found to be described by $y(t)=25\cos\left(2t-\frac{\pi}{6}\right)$, with displacement measured in centimeters and time in seconds.

a) Determine the displacement y_0.

b) Determine the initial velocity y'_0 ?

c) Determine the spring constant *k*.

d) Determine the period and frequency of the vibrations?

Solution

a) $4y''+ky=0$; $y(0)=y_0$, $y'(0)=y'_0$ $\quad\quad my''+cy'+ky=F(t)$

Given: $y(t)=25\cos\left(2t-\frac{\pi}{6}\right)\ cm$

$$=0.25\cos\left(2t-\frac{\pi}{6}\right)\ m$$

$$y(0)=\frac{1}{4}\cos\left(-\frac{\pi}{6}\right)$$

$$\boxed{y_0=\frac{\sqrt{3}}{8}\ m}$$

b) $y' = -\frac{1}{2}\sin\left(2t - \frac{\pi}{6}\right)$

$y'(0) = -\frac{1}{2}\sin\left(-\frac{\pi}{6}\right) = \frac{1}{4}\ \underline{= .25\ \ m/s}$

c) $2 = \sqrt{\frac{k}{4}} \quad \rightarrow \quad \underline{k = 16\ N/m}$ $\qquad\qquad \omega = \sqrt{\frac{k}{m}}$

d) $P = \frac{2\pi}{2}\ \underline{= \pi\ \ sec}$

$f = \frac{1}{P}\ \underline{= \frac{1}{\pi}\ Hz}$

Exercise

A 10-*kg* mass is attached to a spring with a spring constant $k = 100$ *N/m*; the dashpot has damping constant 7 *kg/sec*. At time $t = 0$, the system is set into motion by pulling the mass down 0.5 *m* from its equilibrium rest position while simultaneously giving it an initial downward velocity of 1 *m/s*

a) Solve the equation of motion.

b) What is the $\lim\limits_{t\to\infty} y(t)$

c) Plot the solution.

d) How long it takes for the magnitude of the vibrations to be reduced to 0.1 *m*.

(Estimate the smallest time, τ, for which $|y(t)| \le 0.1\ m, \quad \tau \le t < \infty$)

Solution

a) $10y'' + 7y' + 100y = 0$; $\quad y(0) = 0.5, \quad y'(0) = 1 \qquad my'' + cy' + ky = 0$

$$10\lambda^2 + 7\lambda + 100 = 0 \quad \rightarrow \quad \lambda_{1,2} = -\frac{7}{20} \pm i\frac{3\sqrt{439}}{20}$$

$$y(t) = e^{-7t/20}\left(C_1 \cos\frac{3\sqrt{439}}{20}t + C_2 \sin\frac{3\sqrt{439}}{20}t\right)$$

$$y(0) = \frac{1}{2} \quad \rightarrow \quad C_1 = \frac{1}{2}$$

$$y' = e^{-7t/20}\left(-\frac{7}{20}C_1 \cos\frac{3\sqrt{439}}{20}t - \frac{7}{20}C_2 \sin\frac{3\sqrt{439}}{20}t - \frac{3\sqrt{439}}{20}C_1 \sin\frac{3\sqrt{439}}{20}t + \frac{3\sqrt{439}}{20}C_2 \cos\frac{3\sqrt{439}}{20}t\right)$$

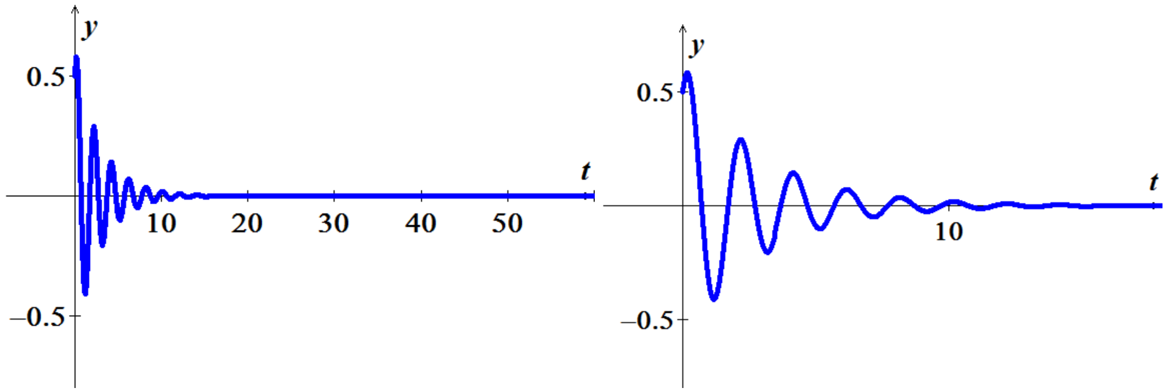
$$y'(0) = 1 \quad \rightarrow -\frac{7}{20}\frac{1}{2} + \frac{3\sqrt{439}}{20}C_2 = 1 \quad C_2 = \frac{47}{6\sqrt{439}}$$

$$y(t) = e^{-7t/20}\left(\frac{1}{2}\cos\frac{3\sqrt{439}}{20}t + \frac{47}{6\sqrt{439}}\sin\frac{3\sqrt{439}}{20}t\right)$$

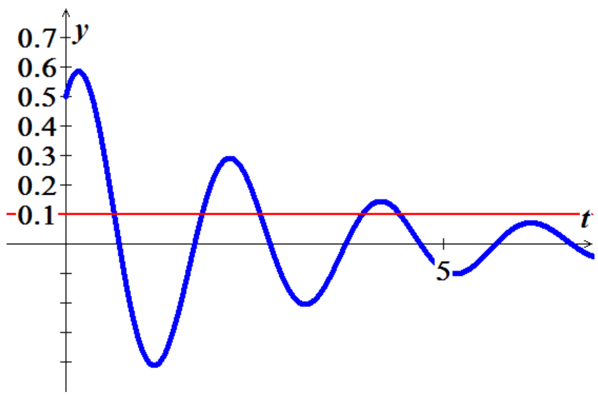
$$y(t) = e^{-0.35t}\left(0.5\cos 3.14285t + 0.37386\sin 3.14285t\right)$$

b) $\lim\limits_{t\to\infty} y(t) = 0$

c)



d)



From the graph:

τ	y
0.63960	0.10046
0.64000	0.09983

Exercise

A spring and dashpot system is to be designed for a 32-*lb* weight so that the overall system is critically damped

a) How must the damping constant c and the spring constant k be related?

b) Assume the system is to be designed so that the mass, when given initial velocity of 4 *ft/sec* from its rest position, will have a maximum displacement of 6 *in*. What values of damping constant c and spring constant k are required?

c) It is observed that the time interval between successive zero crossing is 20% larger for the damped vibration displacement than for the undamped vibration displacement. What is the damping constant c? (Spring constant k remains same from part (b)).

Solution

a) $\frac{32}{32}y'' + cy' + ky = 0$ $\qquad$ $my'' + cy' + ky = F(t)$

$y'' + cy' + ky = 0$

$$\lambda^2 + c\lambda + k = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4k}}{2}$$

Since the system is critically damped, then:

$$c^2 - 4k = 0 \quad \rightarrow \quad \boxed{c^2 = 4k} \quad c = 2\sqrt{k}$$

b) $y'' + 2\sqrt{k}\,y' + ky = 0$; $y(0) = 0,\quad y'(0) = 4$

$$\rightarrow \quad \boxed{\lambda_{1,2} = -\frac{c}{2}}$$

$$y(t) = \left(C_1 + C_2 t\right)e^{-ct/2}$$

$$y(0) = 0 \quad \rightarrow \quad \boxed{C_1 = 0}$$

$$y'(t) = \left(C_2 - \tfrac{c}{2}C_1 - \tfrac{c}{2}C_2 t\right)e^{-ct/2}$$

$$y(0) = 4 \quad \rightarrow C_2 - \tfrac{c}{2}C_1 = 4 \quad \boxed{C_2 = 4}$$

$$\boxed{y(t) = 4te^{-ct/2}}$$

$$y'(t) = (4 - 2ct)e^{-ct/2} = 0 \quad \rightarrow \quad \boxed{t = \frac{2}{c}}$$

$$y\left(t = \frac{2}{c}\right) = \frac{8}{c}e^{-1} = \frac{6}{12} = \frac{1}{2}$$

$$\boxed{c = \frac{16}{e} \approx 5.886 \;\; lb.sec/ft}$$

$$\boxed{k = \frac{64}{e^2} \approx 8.66 \;\; lb/ft}$$

c) Since, the time interval (τ) between successive zero crossing is 20% larger of undamped .

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4k}}{2} = -\frac{c}{2} \pm i\frac{\sqrt{4k - c^2}}{2}$$

$$1.2 \; damped = undamped\,(c = 0)$$

$$\frac{\sqrt{4k - c^2}}{2}(1.2\tau) = \frac{\sqrt{4k}}{2}\tau$$

$$0.6\sqrt{4k - c^2} = \sqrt{k}$$

$$\frac{1}{e}\sqrt{256 - e^2c^2} = \frac{8}{0.6e} = \frac{40}{3e}$$

$$\sqrt{256 - c^2} = \frac{40}{3}$$

$$256 - e^2c^2 = \frac{1600}{9}$$

$$e^2c^2 = \frac{704}{9} \quad \rightarrow \quad \boxed{c = \frac{8\sqrt{11}}{3e} \approx 3.254 \;\; lb.sec/ft}$$

Exercise

Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L = 0.25\ H$, $R = 10\ \Omega$, $C = 0.001\ F$, $E(t) = 0$, $q(0) = q_0\ C$, and $i(0) = 0$.

Solution

$$0.25q'' + 10q' + \frac{1}{0.001}q = 0 \qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 40q' + 4{,}000q = 0$$

$$\lambda^2 + 40\lambda + 4000 = 0 \quad \to \quad \lambda_{1,2} = \frac{-40 \pm 120i}{2} \ \underline{= -20 \pm i60}$$

$$q(t) = e^{-20t}\left(C_1 \cos 60t + C_2 \sin 60t\right)$$

$$q(0) = q_0 \quad \to \quad \underline{C_1 = q_0}$$

$$q'(t) = e^{-20t}\left(-20C_1 \cos 60t - 20C_2 \sin 60t - 60C_1 \sin 60t + 60C_2 \cos 60t\right)$$

$$q'(0) = i(0) = 0 \quad \to -20C_1 + 60C_2 = 0 \quad \underline{C_2 = \tfrac{1}{3}q_0}$$

$$\underline{q(t) = q_0 e^{-20t}\left(\cos 60t + \tfrac{1}{3}\sin 60t\right)}$$

$$A = \sqrt{1 + \frac{1}{9}} \ \underline{= \frac{\sqrt{10}}{3}}$$

$$\phi = \tan^{-1} 3 \ \underline{\approx 1.249}$$

$$\underline{q(t) = \frac{q_0\sqrt{10}}{3} e^{-20t} \sin(60t + 1.249)}$$

Exercise

Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit at $t = 0.01\ sec$ when $L = 0.05\ h$, $R = 2\ \Omega$, $C = 0.01\ f$, $E(t) = 0$, $q(0) = 5\ C$, and $i(0) = 0\ A$. Determine the first time at which the charge on the capacitor is equal to zero.

Solution

$$0.05q'' + 2q' + \frac{1}{0.01}q = 0 \qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 40q' + 2{,}000q = 0$$

$$\lambda^2 + 40\lambda + 2000 = 0 \quad \to \quad \lambda_{1,2} \ \underline{= -20 \pm i40}$$

$$q(t) = e^{-20t}\left(C_1 \cos 40t + C_2 \sin 40t\right)$$

$$q(0) = 5 \quad \to \quad \underline{C_1 = 5}$$

$$q'(t) = e^{-20t}\left(-20C_1\cos 40t - 20C_2\sin 40t - 40C_1\sin 40t + 40C_2\cos 40t\right)$$

$$q'(0) = i(0) = 0 \quad \to -20C_1 + 40C_2 = 0 \quad \boxed{C_2 = \frac{5}{2}}$$

$$\boxed{q(t) = e^{-20t}\left(5\cos 40t + \frac{5}{2}\sin 40t\right)}$$

$$A = \sqrt{25 + \frac{25}{4}} = \frac{5\sqrt{5}}{2}$$

$$\phi = \tan^{-1} 2 \approx \boxed{1.1071} \qquad \phi = \tan^{-1}\frac{a}{b}$$

$$\boxed{q(t) = \frac{5\sqrt{5}}{2}e^{-20t}\sin(40t + 1.1071)}$$

$$q(0.01) = \frac{5\sqrt{5}}{2}e^{-20t}\sin(40(.01) + 1.1071) \ \boxed{\approx 4.5676\ C}$$

$$q(t) = \frac{5\sqrt{5}}{2}e^{-20t}\sin(40t + 1.1071) = 0$$

$$40t + 1.1071 = \pi$$

$$\boxed{t \approx 0.0509\ \ sec}$$

Exercise

Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L = 0.25\ h$, $R = 20\ \Omega$, $C = \frac{1}{300}\ f$, $E(t) = 0$, $q(0) = 4\ C$, and $i(0) = 0\ A$. Is the charge on the capacitor ever equal to zero.

Solution

$$\frac{1}{4}q'' + 20q' + 300q = 0 \qquad\qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 80q' + 1{,}200q = 0$$

$$\lambda^2 + 80\lambda + 1200 = 0 \quad \to \quad \lambda_{1,2} = -40 \pm 20 = \boxed{-60,\ -20}$$

$$q(t) = C_1e^{-60t} + C_2e^{-20t}$$

$$q(0) = 4 \quad \to \quad C_1 + C_2 = 4$$

$$q'(t) = -60C_1e^{-60t} - 20C_2e^{-20t}$$

$$q'(0) = i(0) = 0 \quad \to \quad -60C_1 - 20C_2 = 0$$

$$\begin{cases} C_1 + C_2 = 4 \\ 3C_1 + C_2 = 0 \end{cases} \quad \to \quad \boxed{C_1 = -2,\ \ C_2 = 6}$$

$$\boxed{q(t) = 6e^{-20t} - 2e^{-60t}}$$

$$q(t) = 6e^{-20t} - 2e^{-60t} = 0$$

$$3e^{-20t} = e^{-60t}$$

$$e^{40t} = \frac{1}{3}$$

$$t = \frac{1}{40}\ln\frac{1}{3} \underline{\approx -0.0275 < 0}|$$

Therefore; the charge will never equal to zero.

Exercise

Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L = \frac{5}{3}\ h$, $R = 10\ \Omega$, $C = \frac{1}{30}\ f$, $E(t) = 0$, $q(0) = 4\ C$, and $i(0) = 0\ A$.

Solution

$$\frac{5}{3}q'' + 10q' + 30q = 0 \qquad\qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 6q' + 18q = 0$$

$$\lambda^2 + 6\lambda + 18 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = -3 \pm 3i}|$$

$$q(t) = e^{-3t}\left(C_1 \cos 3t + C_2 \sin 3t\right)$$

$$q(0) = 4 \quad \rightarrow \quad \underline{C_1 = 4}|$$

$$q'(t) = e^{-3t}\left(-3C_1 \cos 3t - 3C_2 \sin 3t - 3C_1 \sin 3t + 3C_2 \cos 3t\right)$$

$$q'(0) = i(0) = 0 \quad \rightarrow -3C_1 + 3C_2 = 0 \quad \underline{C_2 = 4}|$$

$$\underline{q(t) = e^{-3t}\left(4\cos 3t + 4\sin 3t\right)}|$$