Solution *Section* 2.1 – Definition of the Derivative

Exercise

Find the derivative of *y* with the respect to *t* for the function $y = \frac{4}{t}$

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta t \to 0} \frac{\frac{4}{t+\Delta t} - \frac{4}{t}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\frac{4t - 4(t+\Delta t)}{t(t+\Delta t)}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \frac{4t - 4(t+\Delta t)}{t(t+\Delta t)}$$

$$= \lim_{\Delta t \to 0} \frac{-4\Delta t}{t(t+\Delta t)\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{-4}{t(t+\Delta t)}$$

$$= -\frac{4}{t^2}$$

Exercise

Find the equation of the tangent line to $f(x) = x^2 + 1$ that is parallel to $2x + y = 0$

Solution

$2x + y = 0 \Rightarrow y = -2x \Rightarrow$ slope $= -2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + 1 - (x^2 + 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + \Delta x^2 + 2x\Delta x + 1 - x^2 - 1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x^2 + 2x\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \Delta x + 2x = 2x$$

$$f' = 2x = -2 \quad \Rightarrow x = -1 \Rightarrow f(-1) = (-1)^2 + 1 = 2 \qquad \rightarrow \text{(-1, 2)}$$

The line equation is given by

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -2(x + 1)$$

$$y - 2 = -2x - 2$$

$$y = -2x$$

Exercise

Use the definition of limits to find the derivative: $f(x) = \dfrac{3}{\sqrt{x}}$

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(\frac{3}{\sqrt{x + \Delta x}}\right) - \left(\frac{3}{\sqrt{x}}\right)}{\Delta x} \cdot \frac{\sqrt{x} \cdot \sqrt{x + \Delta x}}{\sqrt{x} \cdot \sqrt{x + \Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{3\sqrt{x} - 3\sqrt{x + \Delta x}}{\Delta x\left(\sqrt{x} \cdot \sqrt{x + \Delta x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{3\left(\sqrt{x} - \sqrt{x + \Delta x}\right)}{\Delta x\left(\sqrt{x} \cdot \sqrt{x + \Delta x}\right)} \cdot \frac{\sqrt{x} + \sqrt{x + \Delta x}}{\sqrt{x} + \sqrt{x + \Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{3\left(x - (x + \Delta x)\right)}{\Delta x\left(\sqrt{x} \cdot \sqrt{x + \Delta x}\right)\left(\sqrt{x} + \sqrt{x + \Delta x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{-3\Delta x}{\Delta x\left(\sqrt{x} \cdot \sqrt{x + \Delta x}\right)\left(\sqrt{x} + \sqrt{x + \Delta x}\right)}$$

$$= \frac{-3}{x\left(2\sqrt{x}\right)} = \frac{-3}{2x^{3/2}}$$

Exercise

Use the definition of limits to find the derivative: $f(x) = \sqrt{x+2}$

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+2} - \sqrt{x+2}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x+2} + \sqrt{x+2}}{\sqrt{x+\Delta x+2} + \sqrt{x+2}}$$

$$= \lim_{\Delta x \to 0} \frac{x+\Delta x+2-(x+2)}{\Delta x\left(\sqrt{x+\Delta x+2} + \sqrt{x+2}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{x+\Delta x+2} + \sqrt{x+2}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x+2} + \sqrt{x+2}}$$

$$= \frac{1}{2\sqrt{x+2}}$$

Exercise

Find the first derivative of $f(x) = -2$

Solution

$$f'(x) = 0$$

Exercise

Find the first derivative of $y = \pi$

Solution

$$y' = 0$$

Exercise

Find the first derivative of $y = \sqrt{5}$

Solution

$$y' = 0$$

Exercise

Find the first derivative of $f(x) = x^4$

Solution

$$f'(x) = 4x^{4-1}$$

$$\underline{= 4x^3}$$

Exercise

Find the first derivative of $s(t) = \frac{1}{t}$

Solution

$$s(t) = t^{-1}$$

$$s'(t) = (-1)t^{-1-1}$$

$$\underline{= -t^{-2}}$$

Exercise

Find the first derivative of $y = 4x^2$

Solution

$$\begin{aligned} y' &= 4(2)x^{2-1} \\ &= \boxed{8x} \end{aligned}$$

Exercise

Find the first derivative of $y = \dfrac{9}{4x^2}$

Solution

$$y = \frac{9}{4x^2}$$

$$= \frac{9}{4}x^{-2}$$

$$\rightarrow y' = \frac{9}{4}(-2)x^{-3}$$

$$= -\frac{9}{2x^3}$$

Exercise

Find the first derivative of $y = \dfrac{9}{(4x)^2}$

Solution

$$y = \frac{9}{(4x)^2}$$

$$= \frac{9}{4^2x^2}$$

$$= \frac{9}{16}x^{-2}$$

$$\rightarrow y' = \frac{9}{16}(-2)x^{-3}$$

$$= -\frac{9}{8x^3}$$

Exercise

Find the first derivative of $\quad y=\sqrt{5x}$

Solution

$$y=\sqrt{5}x^{1/2}$$

$$\rightarrow y'=\sqrt{5}\left(\frac{1}{2}\right)x^{1/2-1}$$

$$=\frac{\sqrt{5}}{2x^{1/2}}$$

$$=\frac{\sqrt{5}}{2\sqrt{x}}$$

Exercise

Find the first derivative of $\quad y=\sqrt[3]{x}$

Solution

$$y=x^{1/3}$$

$$\rightarrow y'=\frac{1}{3}x^{(1/3)-1}$$

$$=\frac{1}{3}x^{-2/3}$$

$$\boxed{=\frac{1}{3\sqrt[3]{x^2}}}$$

Exercise

Find the derivative of $\quad y=\dfrac{0.4}{\sqrt{x^3}}$

Solution

$$y'=0.4\frac{d}{dx}\frac{1}{x^{3/2}}$$

$$=0.4\frac{d}{dx}x^{-3/2}$$

$$=0.4\left(-\frac{3}{2}\right)x^{-3/2-1}$$

$$\boxed{=-0.6x^{-5/2}} \quad or \quad =-\frac{0.6}{\sqrt{x^5}}$$

Exercise

Find the derivative of $y=-\dfrac{2}{\sqrt[3]{x}}$

Solution

$$y=-\frac{2}{x^{1/3}}=-2x^{-1/3}$$

$$y'=-2\left(-\tfrac{1}{3}\right)x^{-1/3-1}$$

$$=\tfrac{2}{3}x^{-4/3}$$

$$=\frac{2}{3x^{4/3}}$$

$$\boxed{=\frac{2}{3x\sqrt[3]{x}}}$$

Exercise

Find the derivative of $y=\dfrac{1}{\sqrt[3]{x}}$

Solution

$$y=\frac{1}{x^{1/3}}=x^{-1/3}$$

$$y'=-\left(-\tfrac{1}{3}\right)x^{-4/3}$$

$$=\frac{1}{3x^{4/3}}$$

$$x^{4/3}=\sqrt[3]{x^4}=\sqrt[3]{x^3\cdot x}=x\sqrt[3]{x}$$

$$\boxed{=\frac{1}{3x\sqrt[3]{x}}}$$

Exercise

Find the derivative of $y=\dfrac{x^3-4x}{\sqrt{x}}$

Solution

$$y=\frac{x^3}{x^{1/2}}-4\frac{x}{x^{1/2}}=x^{5/2}-4x^{1/2}$$

$$y'=\tfrac{5}{2}x^{3/2}-4\tfrac{1}{2}x^{-1/2}$$

$$\boxed{=\tfrac{5}{2}x\sqrt{x}-\frac{2}{\sqrt{x}}}$$

Exercise

Find the derivative of $\quad f(x)=3x^2+2x$

Solution

$$f'(x)=\boxed{6x+2}$$

Exercise

Find the derivative of $\quad f(x)=4+2x^3-3x^{-1}$

Solution

$$f(x)=0+6x^2+3x^{-2}$$

$$\boxed{=6x^2+3x^{-2}}$$

Exercise

Find the derivative of $\quad f(x)=\frac{5}{3x^2}-\frac{2}{x^4}+\frac{x^3}{9}$

Solution

$$f(x)=\frac{5}{3}x^{-2}-2x^{-4}+\frac{x^3}{9}$$

$$f'(x)=\frac{5}{3}\left(-2x^{-3}\right)-2\left(-4x^{-5}\right)+\frac{3x^2}{9}$$

$$\boxed{=-\frac{10}{3x^3}+\frac{8}{x^5}+\frac{x^2}{3}}$$

Exercise

Find the derivative of $\quad f(x)=\frac{3}{x^{3/5}}-\frac{6}{x^{1/2}}$

Solution

$$f(x)=3x^{-3/5}-6x^{-1/2}$$

$$f'(x)=3\left(-\frac{3}{5}\right)x^{-8/5}-6\left(-\frac{1}{2}\right)x^{-3/2}$$

$$\boxed{=-\frac{9}{5x^{8/5}}+\frac{3}{x^{3/2}}}$$

Exercise

Find the derivative of $f(x)=\frac{5}{x^{1/5}}-\frac{8}{x^{3/2}}$

Solution

$$f(x)=5x^{-1/5}-8x^{-3/2}$$

$$f'(x)=5\left(-\frac{1}{5}\right)x^{-6/5}-8\left(-\frac{3}{2}\right)x^{-5/2}$$

$$=-\frac{1}{x^{6/5}}+\frac{12}{x^{5/2}}$$

Exercise

Find the derivative of $y=\frac{1.2}{\sqrt{x}}-3.2x^{-2}+x$

Solution

$$y'=-\frac{1.2}{2x\sqrt{x}}+6.4x^{-3}+1$$

$$\left(\frac{1}{\sqrt{x}}\right)'=-\frac{1}{2x\sqrt{x}}$$

$$=-\frac{0.6}{x\sqrt{x}}+6.4x^{-3}+1$$

Exercise

Find the derivative of $f(x)=x^2-3x-4\sqrt{x}$

Solution

$$f'(x)=2x-3-\frac{2}{\sqrt{x}}$$

$$\left(\sqrt{x}\right)'=\frac{1}{2\sqrt{x}}$$

Exercise

Find the derivative of $f(x)=3\sqrt[3]{x^4}-2x^3+4x$

Solution

$$f(x)=3x^{4/3}-2x^3+4x$$

$$f'(x)=4x^{1/3}-6x^2+4$$

$$=4\sqrt[3]{x}-6x^2+4$$

Exercise

Find the derivative of $f(x) = 0.05x^4 + 0.1x^3 - 1.5x^2 - 1.6x + 3$

Solution

$$f'(x) = \underline{0.2x^3 + 0.3x^2 - 3x - 1.6}$$

Exercise

Find the derivative of $y = 3x^4 - 6x^3 + \frac{x^2}{8} + 5$

Solution

$$y' = 3(4)x^3 - 6(3)x^2 + \frac{2}{8}x + 0$$

$$= 12x^3 - 18x^2 + \frac{1}{4}x$$

Exercise

Find the derivative of $f(t) = -3t^2 + 2t - 4$

Solution

$$f'(t) = \underline{-6t + 2}$$

Exercise

Find the derivative of $g(x) = 4\sqrt[3]{x} + 2$

Solution

$$g(x) = 4x^{1/3} + 2$$

$$g'(x) = \frac{4}{3}x^{-2/3}$$

Exercise

Find the derivative of $f(x) = x\left(x^2 + 1\right)$

Solution

$$f(x) = x^3 + x$$

$$f(x) = \underline{3x^2 + 1}$$

Exercise

Find the derivative of $f(x) = \frac{2x^2 - 3x + 1}{x}$

Solution

$$f(x) = \frac{2x^2}{x} - \frac{3x}{x} + \frac{1}{x}$$

$$= 2x - 3 + \frac{1}{x}$$

$$f'(x) = \boxed{2 - \frac{1}{x^2}} \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

Exercise

Find the derivative of $f(x) = \frac{4x^3 - 3x^2 + 2x + 5}{x^2}$

Solution

$$f(x) = 4\frac{x^3}{x^2} - 3\frac{x^2}{x^2} + 2\frac{x}{x^2} + \frac{5}{x^2}$$

$$= 4x - 3 + 2\frac{1}{x} + 5x^{-2}$$

$$f'(x) = 4 - 2\frac{1}{x^2} - 10x^{-3} \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$\boxed{= 4 - \frac{2}{x^2} - \frac{10}{x^3}}$$

Exercise

Find the derivative of $f(x) = \frac{-6x^3 + 3x^2 - 2x + 1}{x}$

Solution

$$f(x) = -6\frac{x^3}{x} + 3\frac{x^2}{x} - 2\frac{x}{x} + \frac{1}{x}$$

$$= -6x^2 + 3x - 2 + \frac{1}{x}$$

$$f'(x) = \boxed{-12x + 3 - \frac{1}{x^2}} \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

Exercise

Find the slope of the graph of $f(x)=x^2-5x+1$ at the point (2, –5)

Solution

$$f'(x)=2x-5$$

Slope = $f'(2)=2(2)-5=\underline{-1}$

Exercise

Find an equation of the tangent line to the graph of $f(x)=-x^2+3x-2$ at the point (2, 0)

Solution

$$f'(x)=-2x+3$$

Slope = $f'(2)$

$$=-2(2)+3=-1$$

$$=-1$$

$y-0=-1(x-2)$

$\Rightarrow y=-x+2$

Exercise

Find the slope of the graph of $f(x)=x^3$ when $x=-1$, 0, and 1.

Solution

$$f'(x)=3x^2$$

$x=-1 \Rightarrow \quad m=f'(x)=3(-1)^2=3$

$x=0 \Rightarrow \quad m=f'(x)=3(0)^2=0$

$x=1 \Rightarrow \quad m=f'(x)=3$

Exercise

The height h (in feet) of a free-falling object at time (in seconds) is given by $h=-16t^2+180$. Find the average velocity of the object over each interval.

a. [0, 1]

b. [1, 2]

Solution

a) $h(0)=180$,

$h(1) = 164$

$$\rightarrow \frac{\Delta h}{\Delta t} = \frac{164-180}{1-0} = -16\ ft/\sec$$

b) $h(2) = 116$

$$\rightarrow \frac{\Delta h}{\Delta t} = \frac{164-116}{2-1} = -48\ ft/\sec$$

Exercise

Give the position function of a diver who jumps from a board 12 feet high with initial velocity 16 feet per second. Then find the diver's velocity function.

Solution

$h_0 = 12\ ft$ and $v_0 = 16\ ft/\sec$

$$\Rightarrow\ h = -16t^2 + 16t + 12$$

$$v(t) = h' = -32t + 16$$

Exercise

An analyst has found that a company's costs and revenues in dollars for one product are given by

$$C(x) = 2x \qquad\qquad R(x) = 6x - \frac{x^2}{1000}$$

Respectively, where x is the number of items produced.

a) Find the marginal cost function

b) Find the marginal revenue function

c) Using the fact that profit is the difference between revenue and costs, find the marginal profit function.

d) What value of x makes the marginal profit is 0.

e) Find the profit when the marginal profit is 0.

Solution

a) $C'(x) = 2$

b) $R'(x) = 6 - \frac{2x}{1000} = 6 - \frac{x}{500}$

c) $P = R - C$

$$= 6x - \frac{x^2}{1000} - 2x$$

$$= 4x - \frac{x^2}{1000}$$

$$P'(x) = 4 - \frac{x}{500}$$

d) $P'(x) = 4 - \frac{x}{500} = 0$

$$\frac{x}{500} = 4$$

$$x = 2{,}000$$

e) $P(x) = 4x - \frac{x^2}{1000}$

$$= 4(2000) - \frac{2000^2}{1000}$$

$$\underline{= \$4{,}000}|$$

Exercise

A business sells 2000 units per month at a price \$10 each. If monthly sales increases 200 units for each \$0.10 reduction in price.

Solution

$$x = 2000 + 200\left(\frac{10-p}{0.1}\right)$$

$$= 2000 + 2000(10 - p)$$

$$= 2000 + 20000 - 2000p$$

$$x = 22000 - 2000p$$

$$\Rightarrow x - 22000 = -2000p$$

$$\Rightarrow -x + 22000 = 2000p$$

$$p = \frac{22000}{2000} - \frac{x}{2000}$$

$$\underline{= 11 - \frac{x}{2000}}|$$

Exercise

From 1998 through 2005, the revenue per share R (in dollars) for McDonald's Corporation can be modeled by

$$R = 0.0598t^2 - 0.379t + 8.44 \qquad\qquad 8 \le t \le 15$$

Where t represents the year, with t = 8 corresponding to 1998. At what rate was McDonald's revenue per share changing in 2003?

Solution

$$R' = 0.1196t - 0.379$$

$2003 \Rightarrow t = 13$

$$\to R' = 0.1196(13) - 0.379 \underline{= 1.1758|}$$

Exercise

The cost ***C*** (in dollars) of producing *x* units of a product is given by $C = 3.6\sqrt{x} + 500$

a) Find the additional cost when the production increases from 9 to 10 units.

b) Find the marginal cost when $x = 9$

c) Compare the results of parts (*a*) and (*b*)

Solution

a) $C(9) = 3.6\sqrt{9} + 500 \underline{= \$510.8|}$

$C(10) = 3.6\sqrt{10} + 500 \underline{= \$511.38|}$

Additional cost: $= 511.38 - 510.8 = \$0.58$

b) $C' = \frac{1}{2}3.6\left(x^{-1/2}\right)$

$C'(9) = 1.8\left((9)^{-1/2}\right) \quad \underline{= \$0.60|}$

c) Similar

Exercise

The revenue $\boldsymbol{R}$ (in dollars) of renting x apartments can be modeled by $R=2x\left(900+32x-x^2\right)$

a) Find the additional revenue when the number of rentals is increased from 14 to 15
b) Find the marginal revenue when $x=14$
c) Compare the results of parts (*a*) and (*b*)

Solution

a) $R(14)=2(14)\left(900+32(14)-(14)^2\right)\ \underline{=\$32,256.00}$

$R(15)=2(15)\left(900+32(15)-(15)^2\right)\ \underline{=\$34,650.00}$

Additional revenue: $34,650.00-32.256=\$2394.00$

b) $R=1800x+64x^2-2x^3$

$R'=1800+128x-6x^2$

$R'(14)=1800+128(14)-6(14)^2\ \underline{=\$2416.00}$

c) $2416-2394\ \underline{=\$22}$

Exercise

The profit $\boldsymbol{P}$ (in dollars) of selling x units of calculus textbooks is given by

$$P=-0.05x^2+20x-1000$$

a) Find the additional profit when the sales increase from 150 to 151 units.
b) Find the marginal profit when $x=150$
c) Compare the results of parts (*a*) and (*b*)

Solution

a) $P(150)=-0.05(150)^2+20(150)-1000\ \underline{=\$875.00}$

$P(151)=-0.05(151)^2+20(151)-1000\ \underline{=\$879.95}$

Additional profit: $879.95-875.00=\$4.95$

b) $P'=-0.1x+20$

$P'(150)=-0.1(150)+20\ \underline{=\$5.00}$

c) Nearly the same \$0.05

Exercise

The profit derived from selling x units, is given by $P = 0.0002x^3 + 10x$, find the marginal profit for a production level of 100 units. Compare this with the actual gain in profit by increasing production from 100 to 101 units.

Solution

$$\frac{dP}{dx} = P' = 0.0006x^2 + 10$$

$$\Rightarrow P'(100) = 0.0006(100)^2 + 10 \ \underline{= \$16|}$$

$$P(100) = 0.0002(100)^3 + 10(100) \ \underline{= \$1200.00|}$$

$$P(101) = 0.0002(101)^3 + 10(101) \ \underline{= \$1216.06|}$$

Actual Gain = 1216.06- 1200 = \$16.06

Exercise

The Cost of producing x hamburgers is $C = 5000 + 0.56x, \quad 0 \le x \le 50,000$ and the revenue function is given by

$$R = \frac{1}{20000}\left(60000x - x^2\right)$$

Compare the marginal profit when 10,000 units are produced with the actual increase in profit from 10,000 units to 10,001 units

Solution

$$\Rightarrow P = R - C = \frac{1}{20000}(60000x - x^2) - (5000 + 0.56x)$$

$$= 2.44x - \frac{x^2}{20000} - 5000$$

$$\frac{dP}{dx} = P' = 2.44 - \frac{x}{10000}$$

For x = 10000 $\Rightarrow P'(10000) = 2.44 - \frac{10000}{10000} \ \underline{= \$1.44 / unit|}$

$$P(10000) = 2.44(10000) - \frac{(10000)^2}{20000} - 5000 \ \underline{= \$14400|}$$

$$P(10001) = 2.44(10001) - \frac{(10001)^2}{20000} - 5000 \quad \underline{= \$14401.44|}$$

$$P(10001) - P(10000) = 14,401.44 - 14,400 = 1.44$$

$$\Rightarrow \boxed{\$1.44 / unit}$$

Exercise

An object moves along the *y*-axis (marked in feet) so that its position at time *x* (in seconds) is

$$f(x) = x^3 - 6x^2 + 9x$$

a) Find the instantaneous velocity function *v*.

b) Find the velocity at $x = 2$ and $x = 5$ seconds

c) Find the time(s) when the velocity is 0.

Solution

a) $v = f'(x) = \underline{3x^2 - 12x + 9}$

b) $v(2) = 3(2)^2 - 12(2) + 9 = \underline{-3\ ft/\sec}$

$v(5) = 3(5)^2 - 12(5) + 9 = \underline{24\ ft/\sec}$

c) $v = 3x^2 - 12x + 9 = 0$ ***Solve for x***

$\underline{x = 1,\ 3}$

So, $v = 0$ *at* $x = 1$ sec *and* $x = 5$ sec

Exercise

A company's total sales (in millions of dollars) *t* months from now are given by

$$S(t) = 0.03t^3 + 0.5t^2 + 2t + 3$$

a) Find $S'(t)$.

b) Find $S(5)$ and $S'(5)$ (to two decimal places). Write a brief verbal interpretation of these results.

c) Find $S(10)$ and $S'(10)$ (to two decimal places). Write a brief verbal interpretation of these results.

Solution

a) $S'(t) = 0.09t^2 + t + 2$

b) $S(5) = 0.03(5)^3 + 0.5(5)^2 + 2(5) + 3 \ \underline{=18}$

$S'(5) = 0.09(5)^2 + (5) + 2 \ \underline{=9.25}$

After 5 months, sales are \$18 million and are increasing at the rate of \$9.25 million per month.

c) $S(10) = 0.03(10)^3 + 0.5(10)^2 + 2(10) + 3 \ \underline{=103}$

$S'(10) = 0.09(10)^2 + 10 + 2 \ \underline{=21}$

After 5 months, sales are \$103 million and are increasing at the rate of \$21 million per month.

Exercise

A company's total sales (in millions of dollars) t months from now are given by

$$S(t) = 0.015t^4 + 0.4t^3 + 3.4t^2 + 10t - 3$$

a) Find $S'(t)$.

b) Find $S(4)$ and $S'(4)$ (to two decimal places). Write a brief verbal interpretation of these results.

c) Find $S(8)$ and $S'(8)$ (to two decimal places). Write a brief verbal interpretation of these results.

Solution

a) $S'(t) = 0.06t^3 + 1.2t^2 + 6.8t + 10$

b) $S(4) = 0.015(4)^4 + 0.4(4)^3 + 3.4(4)^2 + 10(4) - 3 \underline{= 120.84|}$

$S'(4) = 0.06(4)^3 + 1.2(4)^2 + 6.8(4) + 10 \underline{= 60.24|}$

After 5 months, sales are \$120.84 million and are increasing at the rate of \$60.24 million per month.

c) $S(8) = 0.015(8)^4 + 0.4(8)^3 + 3.4(8)^2 + 10(8) - 3 \underline{= 560.84|}$

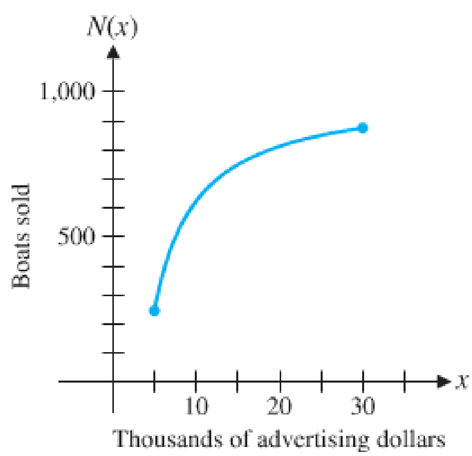
$S'(8) = 0.06(8)^3 + 1.2(8)^2 + 6.8(8) + 10 \underline{= 171.92|}$

After 5 months, sales are \$560.84 million and are increasing at the rate of \$171.92 million per month.

Exercise

A marine manufacturer will sell $N(x)$ power boats after spending \$$x$ thousand on advertising, as given by

$$N(x) = 1{,}000 - \frac{3{,}780}{x} \quad 5 \le x \le 30$$



a) Find $N'(x)$.

b) Find $N(20)$ and $N'(20)$ (to two decimal places). Write a brief verbal interpretation of these results.

Solution

a) $N'(x) = \dfrac{3{,}780}{x^2}$

b) $N(20) = 1{,}000 - \dfrac{3{,}780}{20} \underline{= 811|}$

$N'(20) = \dfrac{3{,}780}{20^2} \underline{= 9.45|}$

After \$811,000 on advertising, a marine manufacturer will sell 811 power boats.

Exercise

A company manufactures and sells x transistor radios per week. If the weekly cost and revenue equations are

$$C(x)=5,000+2x \quad R(x)=10x-\frac{x^2}{1,000} \quad 0\le x\le 8,000$$

Then find the approximate changes in revenue and profit if production is increased from 2,000 to 2,010 per week.

Solution

$$dx=2,010-2,000=10$$

$$dR=\left(10-\frac{x}{500}\right)dx$$

$$=\left(10-\frac{2,000}{500}\right)(10)$$

$$=\$60 \quad \textit{per week}$$

$$P(x)=R(x)-C(x)$$

$$=10x-\frac{x^2}{1,000}-(5,000+2x)$$

$$=10x-\frac{x^2}{1,000}-5,000-2x$$

$$=8x-\frac{x^2}{1,000}-5,000$$

$$dP=\left(8-\frac{x}{500}\right)dx$$

$$=\left(8-\frac{2,000}{500}\right)(10)$$

$$=\$40 \quad \textit{per week}$$

Exercise

A company manufactures fuel tanks for cars. The total weekly cost (in dollars) of producing x tanks given by

$$C(x)=10,000+90x-0.05x^2$$

a) Find the marginal cost function.
b) Find the marginal cost at a production level of 500 tanks per week.
c) Interpret the result of part b.
d) Find the exact cost of producing the 501st item.

Solution

a) $C'(x)=90-0.1x$

b) $C'(500) = 90 - 0.1(500)\ \boxed{= \$40}$

c) At a production level of 500 tanks per week, the total production costs are increasing at the rate of \$40 per tank.

d) $C(501) = 10,000 + 90(501) - 0.05(501)^2$

$$\boxed{= \$42,539.95}$$

$C(500) = 10,000 + 90(500) - 0.05(500)^2$

$$\boxed{= \$42,500.00}$$

$C(501) - C(500) = 42,539.95 - 42,500.00$

$\boxed{= \$39.95}$ Exact cost of producing the 501[st] tank.

Exercise

A company's market research department recommends the manufacture and marketing of a new headphone set for MP3 players. After suitable test marketing, the research department presents the following *price-demand* equation:

$$x = 10,000 - 1,000p \quad \rightarrow \quad p = 10 - 0.001x$$

Where x is the number of headphones that retailers are likely to buy at \$$p$ per set.
The financial department provides the cost function

$$C(x) = 7,000 + 2x$$

Where \$7,000 is the estimate of fixed costs (tooling and overhead) and \$2 is the estimate of variable costs per headphone set (materials, labor, marketing, transportation, storage, etc.).

a) Find the domain of the function defined by the price demand function.

b) Find and interpret the marginal cost function $C'(x)$.

c) Find the revenue function as a function of x and find its domain.

d) Find the marginal revenue at x = 2,000, 5,000, and 7,000. Interpret these results.

e) Graph the cost function and the revenue function in the same coordinate system, Find the intersection points of these two graphs and interpret the results.

f) Find the profit function and its domain and sketch the graph of the function.

g) Find the marginal profit at x = 1,000, 4,000, and 6,000. Interpret these results.

Solution

a) Since price p and demand x must be non-negative, we have $x \ge 0$

$$p = 10 - 0.001x \ge 0$$

$$10 \ge 0.001x$$

$$10,000 \ge x$$

The permissible values of x are $0 \le x \le 10,000$

b) The marginal cost is $C'(x) = 2$. Since this is a constant, it costs an additional \$2 to produce one more headphone set at any production level.

c) The revenue is the amount of money *R* received by the company for manufacturing and selling *x* headphone sets at \$*p* per set and is given by

$$\begin{aligned} R(x) &= (\textit{number of headphone sets sold})\ (\textit{price per headphone set}) \\ &= xp \\ &= x(10-0.001x) \\ &= 10x-0.001x^2 \qquad\qquad 0 \le x \le 10{,}000 \end{aligned}$$

d) The marginal revenue is:

$$R'(x)=10-0.002x$$

$$R'(2{,}000)=10-0.002(2{,}000)\ \underline{=6}$$

$$R'(5{,}000)=10-0.002(5{,}000)\ \underline{=0}$$

$$R'(7{,}000)=10-0.002(7{,}000)\ \underline{=-4}$$

At production levels of 2,000, 5,000, and 7,000, the respective approximate changes in revenue per unit change in production are \$6, \$0, and −\$4.
At the \$2,000 output level, revenue increases as production increases.
At the \$5,000 output level, revenue does not change with a *small* change in production.
At the \$7,000 output level, revenue decreases as production increases.

e) The intersection points are called the ***break-even points***, because revenue equals cost at these production levels.

$$C(x)=R(x)$$

$$7{,}000+2x=10x-0.001x^2$$

$$0.001x^2-8x+7{,}000=0$$

Solve for *x*: $x=1{,}000,\quad 7{,}000$

$$R(1{,}000)=10(1{,}000)-0.001(1{,}000)^2\ \underline{=9{,}000}$$

$$C(1{,}000)=7{,}000+2(1{,}000)\underline{=9{,}000}$$

$$R(7{,}000)=10(7{,}000)-0.001(7{,}000)^2\ \underline{=21{,}000}\ \ C(7{,}000)=7{,}000+2(7{,}000)\underline{=21{,}000}$$

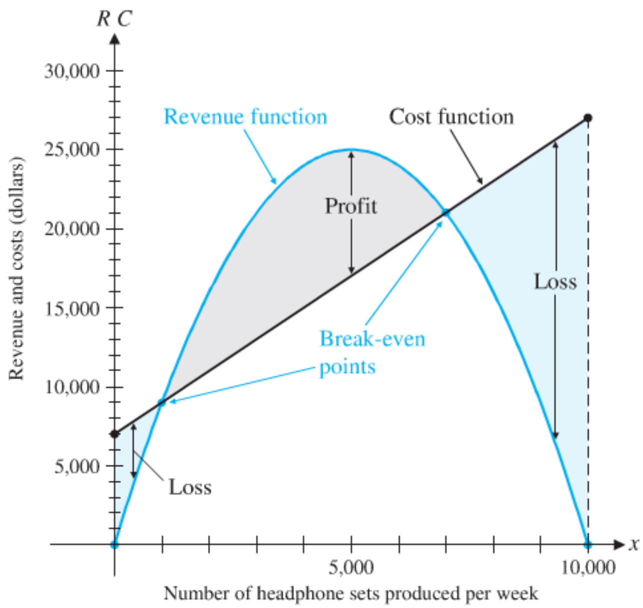
The *break–even* points are:

$(1{,}000,\ \ 9{,}000)$ and $(7{,}000,\ \ 21{,}000)$

f) The profit function is:

$$\begin{aligned} P(x) &= R(x)-C(x) \\ &= 10x-0.001x^2-(7{,}000+2x) \\ &= -0.001x^2+8x-7{,}000 \end{aligned}$$

The domain of the cost function is $x \ge 0$,
The domain of the revenue function is $0 \le x \le 10{,}000$
The domain of the profit function is $0 \le x \le 10{,}000$



g) The marginal profit is

$P'(x) = -0.002x + 8$

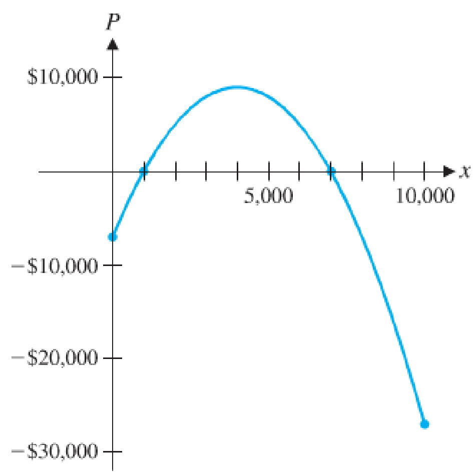
$$P'(1{,}000) = -0.002(1{,}000) + 8 = 6$$

$$P'(4{,}000) = -0.002(4{,}000) + 8 = 0$$

$$P'(6{,}000) = -0.002(6{,}000) + 8 = -4$$

This means that at productions of 1,000, 4,000, and 6,000, the respective approximate changes in revenue per unit change in production are \$6, \$0, and –\$4.
At the \$1,000 output level, profit will increase if production is increased.
At the \$4,000 output level, profit does not change for *small* changes in production.
At the \$6,000 output level, profit will decrease as production is increased.

Therefore, the best production level to produce a maximum profit is 4,000.



Exercise

A small machine shop manufactures drill bits used in the petroleum industry. The manager estimates that the total daily cost (in dollars) of producing x bits is

$$C(x) = 1{,}000 + 25x - 0.1x^2$$

a) Find $\overline{C}(x)$ and $\overline{C}'(x)$

b) Find $\overline{C}(10)$ and $\overline{C}'(10)$. Interpret these quantities.

c) Use the results in part (***b***) to estimate the average cost per bit at a production level of 11 bits per day.

Solution

a) $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{1{,}000 + 25x - 0.1x^2}{x}$

$$= \frac{1{,}000}{x} + 25 - 0.1x$$

$$\overline{C}'(x) = -\frac{1{,}000}{x^2} - 0.1 \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

b) $\overline{C}(10) = \dfrac{1{,}000}{10} + 25 - 0.1(10)\ \underline{= \$124}$

$$\overline{C}'(10) = -\frac{1{,}000}{10^2} - 0.1\ \underline{= -\$10.10}$$

At a production level of 10 bits per day, the average cost of producing a bit is \$124. This cost is decreasing at the rate of \$10.10 per bit.

c) If the production is increased by 1 bit, then the average cost per bit will decrease by approximately \$10.10. So, the average cost per bit at a production level of 11 bits per day is approximately

$$\$124 - \$10.10 = \$113.90$$

Exercise

The total profit (in dollars) from the sale of x calendars is

$$P(x) = 22x - 0.2x^2 - 400 \qquad 0 \le x \le 100$$

a) Find the exact profit from the sale of the 41st calendar.

b) Use the marginal profit to approximate the profit from the sale of the 41st calendar.

Solution

a) $P(41) - P(40) = 22(41) - 0.2(41)^2 - 400 - \left(22(40) - 0.2(40)^2 - 400\right)$

$$\underline{= \$5.80}$$

b) $P'(x) = 22 - 0.4x$

$$P'(40) = 22 - 0.4(40)\ \underline{= \$6}$$

Exercise

The total profit (in dollars) from the sale of x cameras is

$$P(x) = 12x - 0.02x^2 - 1{,}000 \qquad 0 \le x \le 600$$

Evaluate the marginal profit at the given values of x, and interpret the results.

a) $x = 200$.
b) $x = 350$.

Solution

$$P'(x) = 12 - 0.04x$$

a) $P'(200) = 12 - 0.04(200) \ \boxed{= \$4}$

At a production level of 200 cameras, the profit is increasing at the rate of \$4.00 per camera.

b) $P'(350) = 12 - 0.04(350) \ \boxed{= -\$2}$

At a production level of 350 cameras, the profit is decreasing at the rate of \$2.00 per camera.

Exercise

The total profit (in dollars) from the sale of x gas grills is

$$P(x) = 20x - 0.02x^2 - 320 \qquad 0 \le x \le 1{,}000$$

a) Find the average profit per grill if 40 grills are produced.
b) Find the marginal average profit at a production level of 40 grills and interpret the results.
c) Use the results from parts (*a*) and (*b*) to estimate the average profit per grill if 41 grills are produced.

Solution

Average profit: $\overline{P}(x) = \dfrac{P(x)}{x} = \dfrac{20x - 0.02x^2 - 320}{x}$

$$= 20 - 0.02x - \frac{320}{x}$$

a) $\overline{P}(40) = 20 - 0.02(40) - \dfrac{320}{40} \ \boxed{= \$11.20}$

b) $\overline{P}'(x) = -0.02 + \dfrac{320}{x^2}$ $\qquad \left(\dfrac{1}{x}\right)' = -\dfrac{1}{x^2}$

$\overline{P}'(40) = -0.02 + \dfrac{320}{40^2} \ \boxed{= \$0.18}$

At a production level of 40 grills, the average profit is increasing at the rate of \$0.18 per grill.

c) The average profit per grill if 41 grills are produced is $\$11.20 + \$0.18 \ \boxed{= \$11.38}$

Exercise

The price *p* (in dollars) and the demand *x* for a particular steam iron are related by the equation

$$x = 1{,}000 - 20p$$

a) Express the price *p* in terms of the demand *x*, and find the domain of this function.

b) Find the revenue $R(x)$ from the sale of x steam irons. What is the domain of R?

c) Find the marginal revenue at a production level of 400 steam irons and interpret the results.

d) Find the marginal revenue at a production level of 650 steam irons and interpret the results.

Solution

a) $20p = 1{,}000 - x$

$p = 50 - 0.05x \qquad 0 \le x \le 1{,}000$

b) $R(x) = xp = x(50 - 0.05x)$

$= 50x - 0.05x^2 \qquad 0 \le x \le 1{,}000$

c) $R'(x) = 50 - 0.1x$

$R'(400) = 50 - 0.1(400) = 10$

At a production level of 400 steam irons, the revenue is increasing at the rate of \$10 per steam iron.

d) $R'(650) = 50 - 0.1(650) = -15$

At a production level of 650 steam irons, the revenue is decreasing at the rate of \$15 per steam iron.

Exercise

The price-demand equation and the cost function for the production of TVs are given respectively, by

$$x = 9,000 - 30p \quad \text{and} \quad C(x) = 150,000 + 30x$$

Where *x* is the number of TVs that can be sold at a price of \$*p* per TV and $C(x)$ is the total cost (in dollars) of producing *x* TVs.

a) Express the price *p* as a function of the demand *x*, and find the domain of this function.
b) Find the marginal cost.
c) Find the revenue function and state its domain.
d) Find the marginal revenue.
e) Find $R'(3,000)$ and $R'(6,000)$ and interpret these quantities.
f) Graph the cost function and the revenue function on the same coordinate system for $0 \le x \le 9,000$. Find the break–even points and indicate regions of loss and profit.
g) Find the profit function in terms of *x*.
h) Find the marginal profit.
i) Find $P'(1,500)$ and $P'(4,500)$ and interpret these quantities

Solution

a) $30p = 9,000 - x \quad \rightarrow \quad p = 300 - \frac{1}{30}x \qquad 0 \le x \le 9,000$

b) $C'(x) = 30$

c) $R(x) = xp = x\left(300 - \frac{1}{30}x\right)$

$$= 300x - \frac{1}{30}x^2 \qquad 0 \le x \le 9,000$$

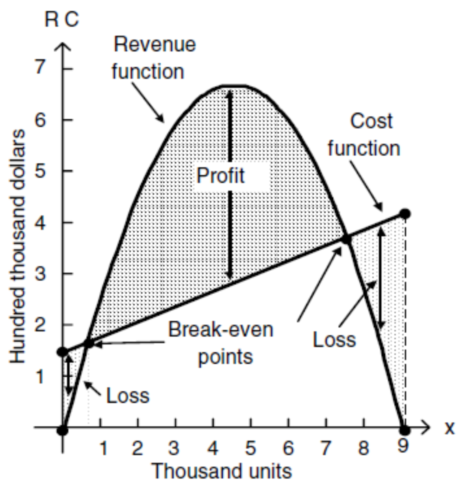
d) $R'(x) = 300 - \frac{1}{15}x$

e) $R'(3,000) = 300 - \frac{1}{15}(3,000) \ \underline{= 100}$

At a production level of 3,000 sets, the revenue is *increasing* at the rate of \$100 per set.

$R'(6,000) = 300 - \frac{1}{15}(6,000) \ \underline{= -100}$

At a production level of 6,000 sets, the revenue is *decreasing* at the rate of \$100 per set.



f) The break–even points are:

$C(x) = R(x)$

$150,000 + 30x = 300x - \frac{1}{30}x^2$

$\frac{1}{30}x^2 - 270x + 150,000 = 0$

$x = 600 \quad or \quad x = 7,500$

$C(600) = 150,000 + 30(600) = 168,000$

$C(7{,}500) = 150{,}000 + 30(7{,}500) = 375{,}000$

Thus, the break–even points are $(600,\ 168{,}000)$ and $(7{,}500,\ 375{,}000)$

g) $P(x) = R(x) - C(x)$

$$= 300x - \frac{1}{30}x^2 - (150{,}000 + 30x)$$

$$= -\frac{1}{30}x^2 + 270x - 150{,}000$$

h) $P'(x) = -\frac{1}{15}x + 270$

i) $P'(1{,}500) = -\frac{1}{15}(1{,}500) + 270 \underline{= 170}$

At a production level of 1,500 sets, the profit is *increasing* at the rate of \$170 per set.

$P'(4{,}500) = -\frac{1}{15}(4{,}500) + 270 \underline{= -30}$

At a production level of 4,500 sets, the revenue is *decreasing* at the rate of \$30 per set.

Exercise

The total cost and the total revenue (in dollars) for the production and sale of x hair dryers are given, respectively, by

$$C(x) = 5x + 2,340 \quad \text{and} \quad R(x) = 40x - 0.1x^2 \quad 0 \le x \le 400$$

a) Find the value of x where the graph of $R(x)$ has a horizontal tangent line.

b) Find the profit function $P(x)$.

c) Find the value of x where the graph of $P(x)$ has a horizontal tangent line.

d) Graph $C(x)$, $R(x)$, and $P(x)$ on the same coordinate system for $0 \le x \le 400$. Find the break–even points. Find the x intercept of the graph of $P(x)$.

Solution

a) $R'(x) = 40 - 0.2x$

The graph has a horizontal tangent line at the value(s) of x where $R'(x) = 0$

$40 - 0.2x = 0 \quad \rightarrow \quad \boxed{x = 200}$

b) $P(x) = R(x) - C(x)$

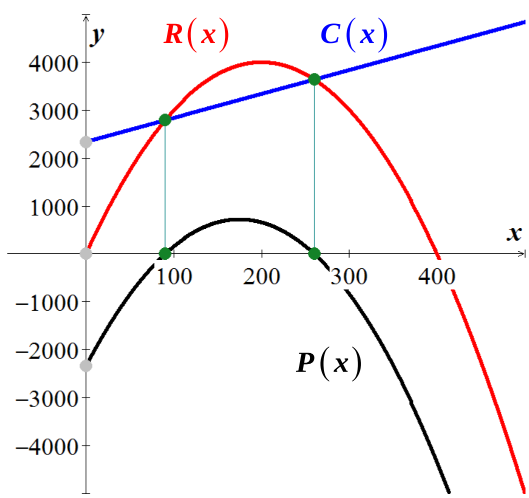
$$= 40x - 0.1x^2 - (5x + 2,340)$$

$$= -0.1x^2 + 35x - 2,340$$

c) $P'(x) = -0.2x + 35$

$P'(x) = -0.2x + 35 = 0$

$\boxed{x = 175}$



d) The break–even points are:

$R(x) = C(x)$

$40x - 0.1x^2 = 5x + 2,340$

$-0.1x^2 + 35x - 2,340 = 0$

$x = 90 \quad or \quad x = 260$

$C(90) = 5(90) + 2,340 = 2,790$

$C(260) = 5(260) + 2,340 = 3,640$

Thus, the break–even points are $(90,\ 2,790)$ and $(260,\ 3,640)$

the x intercept of the graph of $P(x)$ are $P(x) = -0.1x^2 + 35x - 2,340 = 0$

Thus $x = 90 \quad and \quad x = 260$ are x intercepts of $P(x)$

Solution Section 2.3 – Derivatives of Products and Quotients

Exercise

Find the first derivative $y=(x+1)\left(\sqrt{x}+2\right)$

Solution

$$y'=(1)\left(x^{1/2}+2\right)+(x+1)\left(\tfrac{1}{2}x^{-1/2}\right)$$
$$=x^{1/2}+2+\tfrac{1}{2}x^{1/2}+\tfrac{1}{2}x^{-1/2}$$
$$=\tfrac{3}{2}x^{1/2}+\tfrac{1}{2}x^{-1/2}+2$$

Exercise

Find the first derivative $y=\left(4x+3x^2\right)(6-3x)$

Solution

or $y=24x+6x^2-9x^3$

$$u=4x+3x^2 \qquad v=6-3x$$
$$u'=4+6x \qquad v'=-3$$

$$y'=(6-3x)(4+6x)+(-3)\left(4x+3x^2\right)$$
$$=24+36x-12x-18x^2-12x-9x^2$$
$$=-27x^2+12x+24$$

Exercise

Find the first derivative $y=\left(\frac{1}{x}+1\right)(2x+1)$

Solution

$$y=\left(\frac{1}{x}+1\right)(2x+1)$$
$$=2+\frac{1}{x}+2x+1$$
$$=\frac{1}{x}+2x+3$$
$$y'=-\frac{1}{x^2}+2$$
$$=\frac{2x^2-1}{x^2}$$

Exercise

Find the first derivative $y = 3x\left(2x^2 + 5x\right)$

Solution

$$y = 6x^3 + 15x^2$$

$$\Rightarrow \boxed{y' = 18x^2 + 30x}$$

Exercise

Find the first derivative $y = 3\left(2x^2 + 5x\right)$

Solution

$$y = 6x^2 + 15x$$

$$\boxed{y' = 12x + 15}$$

Exercise

Find the derivative of $y = \dfrac{x^2 + 4x}{5}$

Solution

$$y = \frac{1}{5}\left[x^2 + 4x\right]$$

$$\boxed{y' = \frac{1}{5}(2x + 4)}$$

Exercise

Find the first derivative $y = \dfrac{3x^4}{5}$

Solution

$$y = \frac{3}{5}x^4$$

$$y' \boxed{= \frac{12}{5}x^3}$$

Exercise

Find the first derivative $y=\dfrac{3-\frac{2}{x}}{x+4}$

Solution

$$y=\dfrac{\frac{3x-2}{x}}{x+4}$$

$$=\frac{3x-2}{x}\cdot\frac{1}{x+4}$$

$$=\frac{3x-2}{x^2+4x}$$

$$u=3x-2 \qquad v=x^2+4x$$
$$u'=3 \qquad v'=2x+4$$

$$y'=\frac{3\left(x^2+4x\right)-\left(2x+4\right)(3x-2)}{\left(x^2+4x\right)^2} \qquad \left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{3x^2+12x-6x^2-12x+4x+8}{\left(x^2+4x\right)^2}$$

$$=\frac{-3x^2+4x+8}{\left(x^2+4x\right)^2}$$

Exercise

Find the first derivative: $f(x)=\dfrac{(3-4x)(5x+1)}{7x-9}$

Solution

$$u=(3-4x)(5x+1)=15x+3-20x^2-4x$$

$$u=11x+3-20x^2 \qquad v=7x-9$$

$$u'=11-40x \qquad v'=7$$

$$f'(x)=\frac{(11-40x)(7x-9)-(7)\left(-20x^2+11x+3\right)}{(7x-9)^2} \qquad \left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{77x-99-280x^2+360x+140x^2-77x-21}{(7x-9)^2}$$

$$=\boxed{\frac{-140x^2+360x-120}{(7x-9)^2}}$$

Exercise

Find the derivative $g(x)=\dfrac{x^2-4x+2}{x^2+3}$

Solution

$$g'=\frac{(2x-4)\left(x^2+3\right)-\left(x^2-4x+2\right)(2x)}{\left(x^2+3\right)^2} \qquad \left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{2x^3+6x-4x^2-12-2x^3+8x^2-4x}{\left(x^2+3\right)^2}$$

$$=\boxed{\frac{4x^2+2x-12}{\left(x^2+3\right)^2}}$$

Exercise

Find the derivative of $y=\frac{x+4}{5x-2}$

Solution

$$u=x+4 \qquad v=5x-2$$
$$u'=1 \qquad v'=5$$

$$y'=\frac{(1)(5x-2)-(5)(x+4)}{(5x-2)^2} \qquad \left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{5x-2-5x-20}{(5x-2)^2}$$

$$\boxed{=-\frac{22}{(5x-2)^2}}$$

Exercise

Find the derivative of $f(x)=x\left(1-\frac{2}{x+1}\right)$

Solution

$$f(x)=x-\frac{2x}{x+1}$$

$$\left(\frac{2x}{x+1}\right)' \Rightarrow \qquad u=2x \quad v=x+1 \qquad u'=2 \quad v'=1 \qquad \left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$f'(x)=1-\frac{2(x+1)-2x}{(x+1)^2}$$

$$=1-\frac{2x+2-2x}{(x+1)^2}$$

$$\boxed{=1-\frac{2}{(x+1)^2}}$$

Exercise

Find the derivative of $g(s)=\dfrac{s^2-2s+5}{\sqrt{s}}$

Solution

$$g(s)=\frac{s^2}{s^{1/2}}-2\frac{s}{s^{1/2}}+\frac{5}{s^{1/2}}$$
$$=s^{3/2}-2s^{1/2}+5s^{-1/2}$$

$$g'(s)=\tfrac{3}{2}s^{1/2}-2\tfrac{1}{2}s^{-1/2}+5\left(-\tfrac{1}{2}\right)s^{-3/2}$$
$$=\tfrac{3}{2}s^{1/2}-s^{-1/2}-\tfrac{5}{2}s^{-3/2}$$
$$=\frac{3}{2}\sqrt{s}-\frac{1}{\sqrt{s}}-\frac{5}{2s^{3/2}}$$
$$=\frac{3}{2}\sqrt{s}-\frac{1}{\sqrt{s}}-\frac{5}{2s\sqrt{s}}$$

Exercise

Find the derivative of $f(x)=\dfrac{x+1}{\sqrt{x}}$

Solution

$$f(x)=\frac{x}{x^{1/2}}+\frac{1}{x^{1/2}}$$
$$=x^{1/2}+x^{-1/2}$$
$$f'(x)=\tfrac{1}{2}x^{-1/2}-\tfrac{1}{2}x^{-3/2}$$
$$=\frac{1}{2x^{1/2}}-\frac{1}{2x^{3/2}}$$

Exercise

Find the derivative $f(x)=\dfrac{x^2}{2x+1}$

Solution

$$u=x^2 \qquad v=2x+1$$
$$u'=2x \qquad v'=2$$

$$f'(x)=\frac{2x(2x+1)-x^2(2)}{(2x+1)^2} \qquad \left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{4x^2+2x-2x^2}{(2x+1)^2}$$

$$=\frac{2x^2+2x}{(2x+1)^2}$$

Exercise

Find the derivative $f(x)=\dfrac{x^2-x}{x^3+1}$

Solution

$$u=x^2-x \qquad v=x^3+1$$
$$u'=2x-1 \qquad v'=3x^2$$

$$f'(x)=\frac{(2x-1)\left(x^3+1\right)-\left(x^2-x\right)\left(3x^2\right)}{\left(x^3+1\right)^2} \qquad \left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{2x^4+2x-x^3-1-3x^4+3x^3}{\left(x^3+1\right)^2}$$

$$=\frac{-x^4+2x^3+2x-1}{\left(x^3+1\right)^2}$$

Exercise

Find the derivative $f(x)=\dfrac{2x}{x^2+3}$

Solution

$$u=2x \qquad v=x^2+3$$
$$u'=2 \qquad v'=2x$$

$$f'(x)=\frac{2\left(x^2+3\right)-2x(2x)}{\left(x^2+3\right)^2}$$

$$\left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{2x^2+6-4x^2}{\left(x^2+3\right)^2}$$

$$=\boxed{\frac{-2x^2+6}{\left(x^2+3\right)^2}}$$

Exercise

Find the derivative $y=\dfrac{t^3-3t}{t^2-4}$

Solution

$$u=t^3-3t \qquad v=t^2-4$$
$$u'=3t^2-3 \qquad v'=2t$$

$$y'=\frac{\left(3t^2-3\right)\left(t^2-4\right)-\left(t^3-3t\right)(2t)}{\left(t^2-4\right)^2}$$

$$\left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{3t^4-15t^2+12-2t^4+2t^2}{\left(t^2-4\right)^2}$$

$$=\boxed{\frac{t^4-13t^2+12}{\left(t^2-4\right)^2}}$$

Exercise

Find the derivative $f(x)=5x^2\left(x^3+2\right)$

Solution

$$f(x)=5x^5+10x^2$$

$$f'(x)\ \boxed{=25x^4+20x}$$

Exercise

Find the derivative $f(x)=\dfrac{3x-4}{2x+3}$

Solution

$$u=3x-4 \quad v=2x+3$$

$$u'=3 \qquad v'=2$$

$$f'(x)=\frac{3(2x+3)-2(3x-4)}{(2x+3)^2} \qquad \left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$=\frac{6x+9-6x+8}{(2x+3)^2}$$

$$\boxed{=\frac{17}{(2x+3)^2}}$$

Exercise

Find the derivative $f(x)=\dfrac{3x+5}{x^2-3}$

Solution

$$u=3x+5 \quad v=x^2-3$$

$$u'=3 \qquad v'=2x$$

$$f'(x)=\frac{3x^2-9-6x^2-10x}{\left(x^2-3\right)^2} \qquad \left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$$

$$\boxed{=\frac{-3x^2-10x-9}{\left(x^2-3\right)^2}}$$

Exercise

Find the derivative $f(x)=\left(x^2-4\right)\left(x^2+5\right)$

Solution

$u=x^2-4 \quad v=x^2+5$

$u'=2x \qquad v'=2x$

$$f'(x)=2x\left(x^2+5\right)+2x\left(x^2-4\right)$$

$$=2x^3+10x+2x^3-8x$$

$$\underline{=4x^3+2x}$$

$$f(x)=x^4+5x^2-4x^2-20$$

$$=x^4+x^2-20$$

$$\underline{f'(x)=4x^3+2x}$$

Exercise

A company that manufactures bicycles has determined that a new employee can assemble $M(d)$ bicycles per day after d days of on-the-job training, where

$$M(d)=\frac{100d^2}{3d^2+10}$$

a) Find the rate of change function for the number of bicycles assembled with respect to time.

b) Find and interpret $M'(2)$ and $M'(5)$

Solution

a) Find the rate of change function for the number of bicycles assembled with respect to time.

$$M'=\frac{(200d)\left(3d^2+10\right)-100d^2(6d)}{\left(3d^2+10\right)^2}$$

$$=\frac{600d^3+2000d-600d^3}{\left(3d^2+10\right)^2}$$

$$=\frac{2000d}{\left(3d^2+10\right)^2}$$

b) Find and interpret $M'(2)$ and $M'(5)$

$$M'(2)=\frac{2000(2)}{\left(3(2)^2+10\right)^2}$$

$$\underline{=8.3}$$

After 2 days of training, employee can assemble about 8.3 bicycles per day.

$$M'(5) = \frac{2000(5)}{\left(3(5)^2 + 10\right)^2}$$

$$= 1.4$$

After 4 days of training, employee can assemble about 1.4 bicycles per day.

Exercise

Find an equation of the tangent line to the graph of $y = \frac{x^2 - 4}{2x + 5}$ when $x = 0$

Solution

$$y' = \frac{(2x)(2x+5) - (2)(x^2 - 4)}{(2x+5)^2} \qquad \left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$= \frac{4x^2 + 10x - 2x^2 + 8}{(2x+5)^2}$$

$$= \frac{2x^2 + 10x + 8}{(2x+5)^2}$$

$$\Rightarrow x = 0 \rightarrow y' = \frac{8}{25} = m$$

$$x = 0 \rightarrow y = \frac{x^2 - 4}{2x + 5} = -\frac{4}{5}$$

$$y - y_1 = m\left(x - x_1\right)$$

$$\rightarrow y + \frac{4}{5} = \frac{8}{25}(x - 0) \qquad \rightarrow y = \frac{8}{25}x - \frac{4}{5}$$

Exercise

A small business invests \$25,000.00 in a new product. In addition, the product will cost \$0.75 per unit to produce. Find the cost function and the average cost function. What is the limit of the average cost function as production increase?

Solution

C = 0.75x + 25000

$$\overline{C} = \frac{C}{x} = \frac{0.75x + 25000}{x}$$

$$= \frac{0.75x}{x} + \frac{25000}{x}$$

$$= 0.75 + \frac{25000}{x}$$

$$\lim_{x\to\infty} \overline{C} = \lim_{x\to\infty}\left(0.75+\frac{25000}{x}\right)$$

$$= \lim_{x\to\infty} 0.75 + \lim_{x\to\infty}\frac{25000}{x}$$

$$= 0.75 + 0$$

$$\boxed{= \$0.75 / unit}$$

Exercise

A communications company has installed a new cable TV system in a city. The total number N (in thousands) of subscribers t months after the installation of the system is given by

$$N(t) = \frac{180t}{t+4}$$

a) Find $N'(t)$

b) Find $N(16)$ and $N'(16)$. Write a brief interpretation of these results.

c) Use the results from part (***b***) to estimate the total number of subscribers after 17 months.

Solution

a) $N'(t) = \dfrac{180(t+4)-180t}{(t+4)^2}$ $\quad u = 180t \quad v = t+4 \quad \left(\dfrac{u}{v}\right)' = \dfrac{u'v - v'u}{v^2}$

$\quad u' = 180 \quad v' = 1$

$$= \frac{180t+720-180t}{(t+4)^2}$$

$$\boxed{= \frac{720}{(t+4)^2}}$$

b) $N(16) = \dfrac{180(16)}{16+4}$ $\boxed{=144}$

$N'(16) = \dfrac{720}{(16+4)^2}$ $\boxed{=1.8}$

After 16 months, the total number of subscribers is 144,000 and is increasing at a rate of 1,800 subscribers per month.

c) The total subscribers after 17 months will be approximately 145,800

Exercise

One hour after a dose of x milligrams of a particular drug is administered to a person, the change in body temperature $T(x)$, in degrees Fahrenheit, is given approximately by

$$T(x) = x^2\left(1 - \frac{x}{9}\right) \quad 0 \le x \le 7$$

The rate $T'(x)$ at which T changes with respect to the size of the dosage x is called the sensitivity of the body to the dosage.

a) Find $T'(x)$

b) Find $T'(1)$, $T'(3)$, and $T'(6)$

Solution

a)

$u = x^2 \quad v = 1 - \frac{x}{9}$

$u = 2x \quad v' = -\frac{1}{9}$

$$T'(x) = 2x\left(1 - \frac{x}{9}\right) + x^2\left(-\frac{1}{9}\right)$$

$$= 2x - \frac{2}{9}x^2 - \frac{1}{9}x^2$$

$$\boxed{= 2x - \frac{1}{3}x^2}$$

b) $T'(1) = 2(1) - \frac{1}{3}(1)^2 \ \boxed{= \frac{5}{3} \ \textit{per mg of drug}}$

$T'(3) = 2(3) - \frac{1}{3}(3)^2 \ \boxed{= 3 \ \textit{per mg of drug}}$

$T'(6) = 2(6) - \frac{1}{3}(6)^2 \ \boxed{= 0 \ \textit{per mg of drug}}$

Exercise

According to economic theory, the supply *x* of a quantity in a free market increases as the price *p* increases. Suppose that the number *x* of DVD players a retail chain is willing to sell per week at a price of $*p* is given by

$$x = \frac{100p}{0.1p + 1} \quad 10 \le p \le 70$$

a) Find $\frac{dx}{dp}$

b) Find the supply and the instantaneous rate of change of supply with respect to price is $40. Write a brief interpretation of these results.

c) Use the results from part (***b***) to estimate the supply if the price is increased to $41.

Solution

a) $u = 100p \quad v = 0.1p + 1$

$u' = 100 \qquad v' = 0.1$

$$\frac{dx}{dp} = \frac{100(0.1p+1) - 100p(0.1)}{(0.1p+1)^2}$$

$$= \frac{10p + 100 - 10p}{(0.1p+1)^2}$$

$$\boxed{= \frac{100}{(0.1p+1)^2}}$$

b) $x(40) = \dfrac{100(40)}{0.1(40)+1} \ \boxed{= 800}$

$$\left.\frac{dx}{dp}\right|_{40} = \frac{100}{(0.1(40)+1)^2} \ \boxed{= 4}$$

At price level of \$40, the supply is 800 DVD players and is increasing at the rate of 4 players per dollars.

c) At a price of \$41, the demand will be approximately 800 + 4 = 804 DVD players

Exercise

Find the derivative of $y=\left(3x^4+1\right)^4\left(x^3+4\right)$

Solution

$$u=\left(3x^4+1\right)^4 \qquad v=x^3+4$$

$$u'=4\left(3x^4+1\right)^3\left(12x^3\right) \qquad v'=3x^2$$

$$y'=4\left(12x^3\right)\left(3x^4+1\right)^3\left(x^3+4\right)+3x^2\left(3x^4+1\right)^4$$

$$=48x^3\left(3x^4+1\right)^3\left(x^3+4\right)+3x^2\left(3x^4+1\right)^4$$

$$=3x^2\left(3x^4+1\right)^3\left[16x\left(x^3+4\right)+3x^4+1\right]$$

$$=3x^2\left(3x^4+1\right)^3\left[16x^4+64x+3x^4+1\right]$$

$$=3x^2\left(3x^4+1\right)^3\left[19x^4+64x+1\right]$$

Exercise

Find the derivative of $y=\left(x^3+1\right)^2$

Solution

$$u=x^3+1 \quad \rightarrow y=u^2$$

$$\frac{d}{dx}y=\frac{dy}{du}\frac{du}{dx}=2u\left(3x^2\right)$$

$$y'=2\left(x^3+1\right)\left(3x^2\right)$$

$$=6x^2\left(x^3+1\right)$$

Exercise

Find the derivative of $y = \left(x^2 + 3x\right)^4$

Solution

$$u = x^2 + 3x$$

$$y' = n \quad (u)^{n-1} \quad \frac{d}{dx}[u]$$

$$= 4\left(x^2 + 3x\right)^3 \frac{d}{dx}\left[x^2 + 3x\right]$$

$$= 4\left(x^2 + 3x\right)^3 (2x + 3)$$

Exercise

Find the derivative of $y = \frac{4}{2x+1}$

Solution

$$y = 4(2x+1)^{-1}$$

$$y' = -4(2x+1)^{-2}(2)$$

$$= -8(2x+1)^{-2}$$

$$= -\frac{8}{(2x+1)^2}$$

Exercise

Find the derivative of $y = \frac{2}{(x-1)^3}$

Solution

$$y = 2(x-1)^{-3}$$

$$y' = 2(-3)(x-1)^{-4}(1)$$

$$= -\frac{6}{(x-1)^4}$$

Exercise

Find and simplify the derivative of $y = x^2\sqrt{x^2+1}$

Solution

$$y = x^2\left(x^2+1\right)^{1/2}$$

$$y' = x^2 \frac{d}{dx}\left[(x^2+1)^{1/2}\right] + (x^2+1)^{1/2}\frac{d}{dx}\left[x^2\right]$$

$$= x^2\left[\frac{1}{2}(x^2+1)^{-1/2}(2x)\right] + (x^2+1)^{1/2}[2x]$$

$$= x^3(x^2+1)^{-1/2} + 2x(x^2+1)^{1/2}$$

$$= \frac{(x^2+1)^{1/2}}{(x^2+1)^{1/2}}\left[x^3(x^2+1)^{-1/2} + 2x(x^2+1)^{1/2}\right]$$

$$= \frac{x^3(x^2+1)^{-1/2}(x^2+1)^{1/2} + 2x(x^2+1)^{1/2}(x^2+1)^{1/2}}{(x^2+1)^{1/2}}$$

$$= \frac{x^3+2x(x^2+1)}{(x^2+1)^{1/2}}$$

$$= \frac{x^3+2x^3+2x}{\sqrt{x^2+1}}$$

$$= \frac{3x^3+2x}{\sqrt{x^2+1}}$$

$$= \frac{x(3x^2+2)}{\sqrt{x^2+1}}$$

Exercise

Find and simplify the derivative of $y = \left(\frac{x+1}{x-5}\right)^2$

Solution

$$y' = 2\left(\frac{x+1}{x-5}\right)\frac{d}{dx}\left[\frac{x+1}{x-5}\right]$$

$$= 2\left(\frac{x+1}{x-5}\right)\left[\frac{(1)(x-5)-(1)(x+1)}{(x-5)^2}\right]$$

$$= 2\left(\frac{x+1}{x-5}\right)\left(\frac{x-5-x-1}{(x-5)^2}\right)$$

$$= 2\left(\frac{x+1}{x-5}\right)\left(\frac{-6}{(x-5)^2}\right)$$

$$\boxed{= -\frac{12(x+1)}{(x-5)^3}}$$

Exercise

Find the derivative of $p(t) = \frac{(2t+3)^3}{4t^2-1}$

Solution

$$P'(x) = \frac{2(3)(2t+3)^2\left(4t^2-1\right) - 8t(2t+3)^3}{\left(4t^2-1\right)^2}$$

$$= \frac{(2t+3)^2\left[6\left(4t^2-1\right) - 8t(2t+3)\right]}{\left(4t^2-1\right)^2}$$

$$= \frac{(2t+3)^2\left[24t^2 - 6 - 16t^2 - 24t\right]}{\left(4t^2-1\right)^2}$$

$$= \frac{(2t+3)^2\left(8t^2 - 24t - 6\right)}{\left(4t^2-1\right)^2}$$

$$\boxed{= \frac{2(2t+3)^2\left(4t^2-12t-3\right)}{\left(4t^2-1\right)^2}}$$

Exercise

Find the derivative of $s(t)=\sqrt{2t^2+5t+2}$

Solution

$$s(t)=\left(2t^2+5t+2\right)^{1/2} \qquad U=2t^2+5t+2 \quad \rightarrow \quad U'=4t+5$$

$$s'(t)=\frac{1}{2}(4t+5)\left(2t^2+5t+2\right)^{-1/2} \qquad \left(U^n\right)'=nU'U^{n-1}$$

$$\boxed{=\frac{1}{2}\frac{4t+5}{\sqrt{2t^2+5t+2}}}$$

Exercise

Find the derivative of $f(x)=\dfrac{1}{\left(x^2-3x\right)^2}$

Solution

$$f(x)=\left(x^2-3x\right)^{-2}$$

$$f'(x)=-2(2x-3)\left(x^2-3x\right)^{-3}$$

$$\boxed{=-\frac{2(2x-3)}{\left(x^2-3x\right)^3}}$$

Exercise

Find the derivative of $y=t^2\sqrt{t-2}$

Solution

$$f=t^2 \qquad g=(t-2)^{1/2}$$

$$f'=2t \quad g'=\tfrac{1}{2}(t-2)^{-1/2}$$

$$y'=2t\sqrt{t-2}+t^2\tfrac{1}{2}(t-2)^{-1/2}$$

$$=\left[2t(t-2)^{1/2}+t^2\tfrac{1}{2}(t-2)^{-1/2}\right]\frac{2(t-2)^{1/2}}{2(t-2)^{1/2}}$$

$$=\frac{4t(t-2)+t^2}{2(t-2)^{1/2}}$$

$$=\frac{4t^2-8t+t^2}{2\sqrt{t-2}}$$

$$=\boxed{\frac{5t^2-4t}{2\sqrt{t-2}}}$$

Exercise

Differentiate the function: $y=\frac{1}{(9x-4)^8}$

Solution

$$y=(9x-4)^{-8}$$

$U=9x-4 \qquad (U^n)'=nU^{n-1}U'$

$U'=9$

$$\frac{dy}{dx}=-8(9x-4)^{-9}(9)$$

$$=\boxed{-\frac{72}{(9x-4)^9}}$$

$y=\frac{u}{v} \rightarrow y'=\frac{u'v-v'u}{v^2}$

$u=1 \qquad v=(9x-4)^8$

$u'=0 \quad v'=8(9x-4)^7(9)=72(9x-4)^7$

$$\frac{dy}{dx}=\frac{0-72(9x-4)^7}{(9x-4)^{16}}$$

$$=\boxed{-\frac{72}{(9x-4)^9}}$$

Exercise

Find the derivative of $y=\left(\frac{6-5x}{x^2-1}\right)^2$

Solution

$f=6-5x \quad g=x^2-1$

$f'=-5 \quad g'=2x$

$$y=2\frac{-5\left(x^2-1\right)-2x(6-5x)}{\left(x^2-1\right)^2}\left(\frac{6-5x}{x^2-1}\right) \qquad (U^n)'=nU'U^{n-1}$$

$$=2\frac{-5x^2+5-12x+10x^2}{\left(x^2-1\right)^3}(6-5x)$$

$$=\boxed{\frac{2\left(5x^2-12x+5\right)(6-5x)}{\left(x^2-1\right)^3}}$$

Exercise

Find the derivative $f(x)=(3x+1)^4$

Solution

$U=3x+1 \quad \to \quad U'=3$

$f'(x)=4(3x+1)^3(3)$ $\qquad\qquad$ $\left(U^n\right)'=nU^{n-1}U'$

$\boxed{=12(3x+1)^3}$

Exercise

Find the derivative $f(x)=\dfrac{1}{\left(x^2+x+4\right)^3}$

Solution

$U=x^2+x+4 \quad \to \quad U'=2x+1$

$f'(x)=-\dfrac{3(2x+1)}{\left(x^2+x+4\right)^4}$ $\qquad\qquad$ $\left(\dfrac{1}{U^n}\right)'=-\dfrac{nU'}{U^{n+1}}$

$\boxed{=-\dfrac{6x+3}{\left(x^2+x+4\right)^4}}$

Exercise

Find the derivative $f(x)=\sqrt{3-x}$

Solution

$U=3-x \quad \to \quad U'=-1$

$f'(x)\ \boxed{=\dfrac{1}{2\sqrt{3-x}}}$ $\qquad\qquad$ $\left(\sqrt{U}\right)'=\dfrac{U'}{2\sqrt{U}}$

Exercise

Find the derivative $f(x)=\left(3x^2+5\right)^5$

Solution

$U=3x^2+5 \quad \to \quad U'=6x$

$$f'(x) = 5\left(3x^2+5\right)^4 (6x) \qquad \left(U^n\right)' = nU^{n-1}U'$$

$$= 30x\left(3x^2+5\right)^4$$

Exercise

Find the derivative $\quad f(x) = \left(5x^2-3\right)^6$

Solution

$$U = 5x^2 - 3 \rightarrow U' = 10x$$

$$f'(x) = 6\left(5x^2-3\right)^5 (10x) \qquad \left(U^n\right)' = nU^{n-1}U'$$

$$= 60x\left(5x^2-3\right)^5$$

Exercise

Find the derivative $\quad f(x) = \left(x^4+1\right)^{-2}$

Solution

$$U = x^4 + 1 \rightarrow U' = 4x^3$$

$$f'(x) = -2\left(x^4+1\right)^{-3} \left(4x^3\right) \qquad \left(U^n\right)' = nU^{n-1}U'$$

$$= -8x^3\left(x^4+1\right)^{-3}$$

Exercise

Find the derivative $\quad f(x) = (4x+3)^{1/2}$

Solution

$$U = 4x + 3 \rightarrow U' = 4$$

$$f'(x) = \frac{1}{2}(4x+3)^{-1/2}(4)$$

$$= \frac{2}{\sqrt{4x+3}}$$

Exercise

Suppose a demand function is given by $q = D(p) = 30\left(5 - \dfrac{p}{\sqrt{p^2+1}}\right)$

Where q is the demand for a product and p is the price per item in dollars. Find the rate of change in the demand for the product per unit change in price (i.e. find *dq/dp*)

Solution

$$\frac{dq}{dp} = 30\frac{1\sqrt{p^2+1} - \frac{1}{2}(2p)\left(p^2+1\right)^{-1/2}p}{\left(\sqrt{p^2+1}\right)^2}$$

$$= 30\frac{\left(p^2+1\right)^{1/2} - p^2\left(p^2+1\right)^{-1/2}}{p^2+1}\frac{\left(p^2+1\right)^{1/2}}{\left(p^2+1\right)^{1/2}}$$

$$= 30\frac{p^2+1-p^2}{\left(p^2+1\right)^{3/2}}$$

$$\boxed{= \frac{30}{\left(p^2+1\right)^{3/2}}}$$

Exercise

Find the tangent line to the graph of $y = \sqrt[3]{(x+4)^2}$ when x = 4.

Solution

$$y = (x+4)^{2/3}$$

$$y' = \frac{2}{3}(x+4)^{-1/3} = \frac{2}{3}\frac{1}{(x+4)^{1/3}}$$

$$= \frac{2}{3\sqrt[3]{x+4}}$$

$$x = 4 \to m = y' = \frac{2}{3\sqrt[3]{4+4}} = \frac{2}{3\sqrt[3]{2^3}} = \frac{2}{3(2)} \boxed{= \frac{1}{3}}$$

$$x = 4 \to y = \sqrt[3]{(4+4)^2} = 4$$

$$y = \frac{1}{3}(x-4)+4 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$y = \frac{1}{3}x - \frac{4}{3} + 4$$

$$\boxed{y = \frac{1}{3}x + \frac{8}{3}}$$

Exercise

The revenue realized by a small city from the collection of fines from parking tickets is given by

$$R(n)=\frac{8000n}{n+2}$$

where n is the number of work-hours each day that can be devoted to parking patrol. At the outbreak of a flu epidemic, 30 work-hours are used daily in parking patrol, but during the epidemic that number is decreasing at the rate of 6 work-hours per day. How fast is revenue from parking fines decreasing at the outbreak of the epidemic?

Solution

$$\frac{dn}{dt}=-6$$

$$\begin{aligned}\frac{dR}{dt}&=\frac{dR}{dn}\cdot\frac{dn}{dt}\\&=\frac{8000(n+2)-(1)8000n}{(n+2)^2}\bullet(-6)\\&=(-6)\frac{8000n+16000-8000n}{(n+2)^2}\\&=-\frac{96000}{(n+2)^2}\end{aligned}$$

$$\begin{aligned}\frac{dR}{dt}(30)&=-\frac{96000}{(30+2)^2}\\&=-93.75\approx-94\end{aligned}$$

The revenue is being lost at the rate of \$94 per day.

Exercise

To test an individual's use of a certain mineral, a researcher injects small amount form of that material into the person's bloodstream. The mineral remaining in the bloodstream is measured each day for several days. Suppose the amount of the mineral remaining in the bloodstream (in milligrams per cubic centimeter) t days after the injection is approximated by $C(t)=\frac{1}{2}(2t+1)^{-1/2}$. Find the rate of change of the mineral level with respect to time for 4 days.

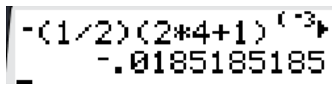
Solution

$u=2t+1 \quad\rightarrow\quad u'=2$

$$C'(t)=\frac{1}{2}\overset{n}{\left(-\frac{1}{2}\right)}\overbrace{(2t+1)^{-1/2-1}}^{u^{n-1}}\overset{u'}{(2)} \qquad\qquad (u^n)'=uu^{n-1}u'$$

$$\boxed{= -\frac{1}{2}(2t+1)^{-3/2}}$$

$$C'(t=4) = -\frac{1}{2}(2(4)+1)^{-3/2} = -0.0185 \ \boxed{\approx -0.02}$$



Exercise

The total cost (in hundreds of dollars) of producing x cameras per week is

$$C(x) = 6 + \sqrt{4x+4} \qquad 0 \le x \le 30$$

a) Find $C'(x)$

b) Find $C'(15)$ and $C'(24)$. Interpret the results

Solution

a) $C(x) = 6 + (4x+4)^{1/2}$

$$C'(x) = \frac{1}{2}(4x+4)^{-1/2}(4)$$

$$\boxed{= \frac{2}{(4x+4)^{1/2}}}$$

b) $C'(15) = \dfrac{2}{(4(15)+4)^{1/2}} = 0.25 \quad or \quad \boxed{\$25}$

$C'(24) = \dfrac{2}{(4(24)+4)^{1/2}} = 0.2 \quad or \quad \boxed{\$20}$

At a production level of 24 cameras, total costs are increasing at the rate of \$20 per camera. Therefore, the cost of purchasing the 25^{th} camera is approximately \$20.

Solution   *Section* 2.5 – Higher-Order Derivatives

Exercise

Find the second derivative: $f(x)=3\left(2-x^2\right)^3$

Solution

$$f'(x)=9(-2x)\left(2-x^2\right)^2 \qquad U=2-x^2 \Rightarrow U'=-2x$$

$$=-18x\left(2-x^2\right)^2$$

$$f=x \qquad f'=1$$

$$g=\left(2-x^2\right)^2 \qquad g'=-2x\left(2-x^2\right)$$

$$f''(x)=-18\left[\left(2-x^2\right)^2+x(-2x)\left(2-x^2\right)\right]$$

$$=-18\left(2-x^2\right)\left[2-x^2-2x^2\right]$$

$$=-18\left(2-x^2\right)\left(2-3x^2\right)$$

Exercise

Find the third derivative: $f(x)=5x(x+4)^3$

Solution

$$f'(x)=5\left[(x+4)^3+3x(x+4)^2\right]$$

$$=5(x+4)^2\left[(x+4)+3x\right]$$

$$=5(x+4)^2(4x+4)$$

$$=20(x+4)^2(x+1)$$

$$f''(x)=20\left[2(x+4)(x+1)+(x+4)^2\right]$$

$$=20(x+4)(2x+2+x+4)$$

$$=20(x+4)(3x+6)$$

$$=60(x+4)(x+2)$$

OR

$$f(x)=5x\left(x^3+12x^2+48x+64\right)$$

$$f(x)=5x^4+60x^3+240x^2+320x$$

$$f'(x)=20x^3+180x^2+480x$$

$$f''(x)=60x^2+360x$$

$$f'''(x)=120x+360$$

$$f'''(x) = 60\left[(x+2)+(x+4)\right]$$

$$= 60(2x+6)$$

$$= 120(x+3)$$

Exercise

Find the given value: $f(x) = \sqrt{4-x}$; $f'''(-5)$

Solution

$$f(x) = (4-x)^{1/2}$$

$$f'(x) = -\tfrac{1}{2}(4-x)^{-1/2}$$

$$f''(x) = -\tfrac{1}{2}\left(-\tfrac{1}{2}\right)(-1)(4-x)^{-1/2}$$

$$= -\tfrac{1}{4}(4-x)^{-3/2}$$

$$f''(x) = -\tfrac{1}{4}\left(-\tfrac{3}{2}\right)(-1)(4-x)^{-5/2}$$

$$= -\tfrac{3}{8}(4-x)^{-5/2}$$

$$f'''(-5) = -\tfrac{3}{8}(4-(-5))^{-5/2}$$

$$= -\tfrac{3}{8}(9)^{-5/2}$$

$$\underline{= -0.013889}$$

Exercise

Find the 4th derivative of $f(x) = x^4 + 2x^3 + 3x^2 - 5x + 7$

Solution

$$f'(x) = 4x^3 + 6x^2 + 6x - 5$$

$$f''(x) = 12x^2 + 12x + 6$$

$$f'''(x) = 24x + 12$$

$$f^{(4)}(x) = 24$$

Exercise

Find the second derivative of $f(x)=\left(x^2-1\right)^2$

Solution

$$f'(x)=2(2x)\left(x^2-1\right)$$

$$=4x\left(x^2-1\right)$$

$$=4x^3-4x$$

$$f''(x)=12x^2-4$$

Exercise

Find $f''(x)$ for $f(x)=\sqrt{x^2+36}$, then find $f''(0)$ and $f''(9)$

Solution

$$f(x)=\sqrt{x^2+36}$$

$$=\left(x^2+36\right)^{1/2} \qquad \left(U^{1/2}\right)'=\tfrac{1}{2}U'U^{1-1/2}$$

$$f'(x)=\tfrac{1}{2}(2x)\left(x^2+36\right)^{-1/2}$$

$$=x\left(x^2+36\right)^{-1/2}$$

$$f=x \qquad g=\left(x^2+36\right)^{-1/2}$$

$$f'=1 \qquad g'=-\tfrac{1}{2}(2x)\left(x^2+36\right)^{-3/2}=-x\left(x^2+36\right)^{-3/2}$$

$$f''(x)=(1)\left(x^2+36\right)^{-1/2}+(-x)\left(x^2+36\right)^{-3/2}(x)$$

$$=\left(x^2+36\right)^{-1/2}-x^2\left(x^2+36\right)^{-3/2}$$

$$=\left[\left(x^2+36\right)^{-1/2}-x^2\left(x^2+36\right)^{-3/2}\right]\frac{\left(x^2+36\right)^{3/2}}{\left(x^2+36\right)^{3/2}}$$

$$=\left[\left(x^2+36\right)^{-1/2}\frac{\left(x^2+36\right)^{3/2}}{\left(x^2+36\right)^{3/2}}-x^2\left(x^2+36\right)^{-3/2}\frac{\left(x^2+36\right)^{3/2}}{\left(x^2+36\right)^{3/2}}\right]$$

$$=\left[\frac{\left(x^2+36\right)-x^2}{\left(x^2+36\right)^{3/2}}\right]$$

$$=\frac{x^2+36-x^2}{\left(x^2+36\right)^{3/2}}$$

$$=\frac{36}{\left(x^2+36\right)^{3/2}}$$

$$f''(0)=\frac{36}{\left(0^2+36\right)^{3/2}}=\frac{36}{36^{3/2}}=\frac{1}{36^{1/2}}=\frac{1}{\sqrt{36}}=\underline{\frac{1}{6}}$$

$$f''(9)=\frac{36}{\left(9^2+36\right)^{3/2}}=\frac{36}{117^{3/2}}=0.028446\approx \underline{0.03}$$ ***Use the Calculator***

Exercise

Find $f''(x)$ for $f(x)=\sqrt{x^2+81}$, then find $f''(0)$ and $f''(2)$

Solution

$$f(x)=\sqrt{x^2+81}=\left(x^2+81\right)^{1/2}$$

$$\left(U^{1/2}\right)'=\tfrac{1}{2}U'U^{1-1/2}$$

$$f'(x)=\tfrac{1}{2}(2x)\left(x^2+81\right)^{-1/2}$$

$$=x\left(x^2+81\right)^{-1/2}$$

$u=x \qquad v=\left(x^2+81\right)^{-1/2}$

$u'=1 \qquad v'=-\tfrac{1}{2}(2x)\left(x^2+81\right)^{-3/2}$

$$=-x\left(x^2+81\right)^{-3/2}$$

$$f''(x)=(1)\left(x^2+81\right)^{-1/2}+(-x)\left(x^2+81\right)^{-3/2}(x)$$

$$=\left(x^2+81\right)^{-1/2}-x^2\left(x^2+81\right)^{-3/2}$$

$$=\left[\left(x^2+81\right)^{-1/2}\frac{\left(x^2+81\right)^{3/2}}{\left(x^2+81\right)^{3/2}}-x^2\left(x^2+81\right)^{-3/2}\frac{\left(x^2+81\right)^{3/2}}{\left(x^2+81\right)^{3/2}}\right]$$

$$= \frac{x^2+81-x^2}{\left(x^2+81\right)^{3/2}}$$

$$= \boxed{\frac{81}{\left(x^2+81\right)^{3/2}}}$$

$$f''(0) = \frac{81}{\left(0^2+81\right)^{3/2}} = \frac{81}{81^{3/2}} \boxed{= \frac{1}{9}}$$

$$f''(2) = \frac{81}{\left(2^2+81\right)^{3/2}} = 0.10336 \approx \boxed{0.10}$$

Use the Calculator

Exercise

The position function on Earth, where s is measured in meters, t is measured in seconds, v_0 is the initial velocity in meters per second, and h_0 is the initial height in meters, is

$$s = -4.9t^2 + v_0 t + h_0$$

If the initial velocity is 2.2 and the initial height is 3.6, what is the acceleration due to gravity on Earth in meters per second per second?

Solution

$$s = -4.9t^2 + 2.2t + 3.6$$

$$s' = -9.8t + 2.2$$

$$\boxed{a(t) = s'' = -9.8}$$

Solution   *Section* 2.6 – Exponential & Logarithmic Functions

Exercise

Solve $4^{2x-1} = 64$

Solution

$4^{2x-1} = 4^3$

$2x - 1 = 3$

$2x = 4$

$x = 2$

Exercise

Solve $3^{1-x} = \frac{1}{27}$

Solution

$3^{1-x} = \frac{1}{3^3}$

$3^{1-x} = 3^{-3}$

$1 - x = -3$

$-x = -4$

$x = 4$

Exercise

Solve $9^x = \frac{1}{\sqrt[3]{3}}$

Solution

$\left(3^3\right)^x = \frac{1}{3^{1/3}}$

$3^{3x} = 3^{-1/3}$

$3x = -\frac{1}{3}$

$x = -\frac{1}{9}$

Exercise

Solve $5^{3x-6} = 125$

Solution

$$5^{3x-6} = 5^3$$

$$\Rightarrow 3x - 6 = 3$$

$$\Rightarrow 3x = 9$$

$$\Rightarrow x = 3$$

Exercise

Solve $8^{x+2} = 4^{x-3}$

Solution

$$\left(2^3\right)^{x+2} = \left(2^2\right)^{x-3}$$

$$2^{3(x+2)} = 2^{2(x-3)}$$

$$3(x+2) = 2(x-3)$$

$$3x + 6 = 2x - 6$$

$$3x - 2x = -6 - 6$$

$$\underline{x = -12}$$

Exercise

Solve: $7e^{2x} - 5 = 58$

Solution

$$7e^{2x} = 63$$

$$e^{2x} = \frac{63}{7} = 9$$

$$\ln e^{2x} = \ln 9$$

$$2x \ln e = \ln 9$$

$$2x = \ln 9$$

$$x = \frac{\ln 9}{2} \; \underline{\approx 1.1}$$

Exercise

Solve: $4\ln(3x) = 8$

Solution

$$\ln(3x) = 2$$

$$3x = e^2$$

$$\boxed{x = \frac{e^2}{3}}$$

Exercise

Solve: $\ln(x-3) = \ln(7x-23) - \ln(x+1)$

Solution

$$\ln(x-3)=\ln\left(\frac{7x-23}{x+1}\right)$$ ***Quotient Rule***

$$x-3 = \frac{7x-23}{x+1}$$ ***Cross multiply***

$$(x-3)(x+1) = 7x-23$$

$$x^2 - 2x - 3 = 7x - 23$$

$$x^2 - 9x + 20 = 0$$ ***Solve for x***

$\Rightarrow x = 4, 5$

Check: $x = 4 \quad \Rightarrow \ln(4-3) = \ln(7(4)-23) - \ln(4+1)$

$x = 5 \quad \Rightarrow \ln(5-3) = \ln(7(5)-23) - \ln(5+1)$

Exercise

Use the properties of logarithms to rewrite $\log_b\left(\frac{x^3 y}{z^2}\right)$

Solution

$$\begin{aligned}\log_b\left(\frac{x^3 y}{z^2}\right) &= \log_b\left(x^3 y\right) - \log_b z^2 \\ &= \log_b x^3 + \log_b y - \log_b z^2 \\ &= 3\log_b x + \log_b y - 2\log_b z\end{aligned}$$

Exercise

Use the properties of logarithms to rewrite $\log_b \left(\frac{\sqrt[3]{x}\,y^4}{z^5}\right)$

Solution

$$\begin{aligned}\log_b \left(\frac{\sqrt[3]{x}\,y^4}{z^5}\right) &= \log_b \left(\sqrt[3]{x}\,y^4\right) - \log_b \left(z^5\right)\\ &= \log_b \left(x^{1/3}\right) + \log_b \left(y^4\right) - \log_b \left(z^5\right)\\ &= \frac{1}{3}\log_b (x) + 4\log_b (y) - 5\log_b (z)\end{aligned}$$

Exercise

Use the properties of logarithms to rewrite $\log_b \sqrt[n]{\frac{x^3y^5}{z^m}}$

Solution

$$\log_b \sqrt[n]{\frac{x^3y^5}{z^m}} = \log_b \left(\frac{x^3y^5}{z^m}\right)^{1/n}$$

$$= \frac{1}{n}\log_b \left(\frac{x^3y^5}{z^m}\right) \qquad \textit{Power Rule}$$

$$= \frac{1}{n}\left(\log_b x^3y^5 - \log_b z^m\right) \qquad \textit{Quotient Rule}$$

$$= \frac{1}{n}\left(\log_b x^3 + \log_b y^5 - \log_b z^m\right) \qquad \textit{Product Rule}$$

$$= \frac{1}{n}\left(3\log_b x + 5\log_b y - m\log_b z\right) \qquad \textit{Power Rule}$$

$$= \frac{3}{n}\log_b x + \frac{5}{n}\log_b y - \frac{m}{n}\log_b z$$

Exercise

Use the properties of logarithms to rewrite $\log_p \sqrt[3]{\frac{m^5n^4}{t^2}}$

Solution

$$\log_p \sqrt[3]{\frac{m^5n^4}{t^2}} = \log_p \left(\frac{m^5n^4}{t^2}\right)^{1/3}$$

$$= \frac{1}{3}\log_p\left(\frac{m^5n^4}{t^2}\right)$$

$$= \frac{1}{3}\left(\log_p m^5n^4 - \log_p t^2\right)$$

$$= \frac{1}{3}\left(\log_p m^5 + \log_p n^4 - \log_p t^2\right)$$

$$= \frac{1}{3}\left(5\log_p m + 4\log_p n - 2\log_p t\right)$$

$$= \frac{5}{3}\log_p m + \frac{4}{3}\log_p n - \frac{2}{3}\log_p t$$

Exercise

Use the properties of logarithms to rewrite $\log_a \sqrt[4]{\frac{m^8\,n^{12}}{a^3\,b^5}}$

Solution

$$\log_a \sqrt[4]{\frac{m^8\,n^{12}}{a^3\,b^5}} = \log_a\left(\frac{m^8n^{12}}{a^3b^5}\right)^{1/4}$$

$$= \frac{1}{4}\log_a\left(\frac{m^8n^{12}}{a^3b^5}\right)$$

$$= \frac{1}{4}\left[\log_a m^8n^{12} - \log_a a^3b^5\right]$$

$$= \frac{1}{4}[\log_a m^8 + \log_a n^{12} – (\log_a a^3 + \log_a b^5)]$$

$$= \frac{1}{4}[\log_a m^8 + \log_a n^{12} – \log_a a^3 - \log_a b^5]$$

$$= \frac{1}{4}[8\log_a m + 12\log_a n – 3\log_a a - 5\log_a b]$$

$$= 2\log_a m + 3\log_a n - \frac{3}{4} - \frac{5}{4}\log_a b$$

Exercise

Solve $\log_x \frac{8}{27} = 3$

Solution

$$\log_x \frac{8}{27} = 3 \qquad \textit{Write in exponential form}$$

$$\frac{8}{27} = x^3$$

$$x = \sqrt[3]{\frac{8}{27}}$$

$$= \frac{\sqrt[3]{8}}{\sqrt[3]{27}}$$

$$= \frac{2}{3}$$

Exercise

Solve $\log_3 \frac{1}{9} = x$

Solution

$$\frac{1}{3^2} = 3^x$$

$$3^{-2} = 3^x$$

$$x = -2$$

Exercise

Solve $3^x = 5$

Solution

$$\ln 3^x = \ln 5$$

$$x \ln 3 = \ln 5$$

$$x = \frac{\ln 5}{\ln 3}$$

$$\approx 1.465$$

Solution *Section* 2.7 – Derivatives of Exponential and Logarithmic Functions

Exercise

Find the derivative of $f(x) = e^{3x}$

Solution

$$f'(x) = 3e^{3x}$$

Exercise

Find the derivative of $f(x) = e^{-2x^3}$

Solution

$$f'(x) = e^{-2x^3} \frac{d}{dx}[-2x^3]$$

$$= e^{-2x^3}\left[-6x^2\right]$$

$$\boxed{= -\frac{6x^2}{e^{2x^3}}}$$

Exercise

Find the derivative of $f(x) = 4e^{x^2}$

Solution

$$f'(x) = 4e^{x^2} \frac{d}{dx}[x^2]$$

$$= 4e^{x^2}(2x)$$

$$\boxed{= 8xe^{x^2}}$$

Exercise

Find the derivative of $f(x) = e^{-2x}$

Solution

$$f'(x) = -2e^{-2x}$$

$$\boxed{= -\frac{2}{e^{2x}}}$$

Exercise

Find the derivative of $f(x) = x^2 e^x$

Solution

$$f'(x) = e^x \frac{d}{dx}[x^2] + x^2 \frac{d}{dx}[e^x]$$

$$= e^x(2x) + x^2 e^x$$

$$\boxed{= xe^x(2+x)}$$

Exercise

Find the derivative of $f(x) = \frac{e^x + e^{-x}}{2}$

Solution

$$f(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$$

$$f'(x) = \frac{1}{2}\left(\frac{d}{dx}\left[e^x\right] + \frac{d}{dx}\left[e^{-x}\right]\right)$$

$$\boxed{= \frac{1}{2}\left(e^x - e^{-x}\right)}$$

Exercise

Find the derivative of $f(x) = \frac{e^x}{x^2}$

Solution

$$f'(x) = \frac{x^2 e^x - e^x(2x)}{x^4}$$

$$= \frac{x^2 e^x - 2xe^x}{x^4}$$

$$= \frac{xe^x(x-2)}{x^4}$$

$$= \frac{e^x(x-2)}{x^3}$$

Exercise

Find the derivative of $f(x) = x^2 e^x - e^x$

Solution

$$f'(x) = e^x \frac{d}{dx}[x^2] + x^2 \frac{d}{dx}[e^x] - \frac{d}{dx}[e^x]$$

$$= e^x(2x) + x^2 e^x - e^x$$

$$= e^x(x^2 + 2x - 1)$$

Exercise

Find the derivative of $f(x) = (1+2x)e^{4x}$

Solution

$$f'(x) = (2)e^{4x} + (1+2x)(4e^{4x})$$

$$= 2e^{4x} + (1+2x)(4e^{4x})$$

$$= 2e^{4x}(1 + 2(1+2x))$$

$$= 2e^{4x}(1+2+4x)$$

$$= 2e^{4x}(3+4x)$$

Exercise

Find the derivative of $y = x^2 e^{5x}$

Solution

$$y' = x^2\left(5e^{5x}\right) + 2x\left(e^{5x}\right)$$

$$= xe^{5x}(5x+2)$$

Exercise

Find the derivative of $f(x) = \dfrac{100,000}{1+100e^{-0.3x}}$

Solution

$$f'(x) = \frac{(0)\left(1+100e^{-0.3x}\right) - 100,000\left(0+(-0.3)100e^{-0.3x}\right)}{\left(1+100e^{-0.3x}\right)^2}$$

$$= \frac{-100,000\left(-30e^{-0.3x}\right)}{\left(1+100e^{-0.3x}\right)^2}$$

$$= \frac{3,000,000e^{-0.3x}}{\left(1+100e^{-0.3x}\right)^2}$$

Exercise

Find the derivative of $y = x^2 e^{-2x}$

Solution

$$y' = 2xe^{-2x} - 2x^3 e^{-2x}$$

$$= 2xe^{-2x}\left(1-x^2\right)$$

Exercise

Find the derivative of $y = \dfrac{e^x + e^{-x}}{x}$

Solution

$$f = e^x + e^{-x} \qquad g = x$$

$$f' = e^x - e^{-x} \qquad g' = 1$$

$$y = \frac{\left(e^x - e^{-x}\right)x - \left(e^x + e^{-x}\right)}{x^2}$$

$$= \frac{xe^x - xe^{-x} - e^x - e^{-x}}{x^2}$$

$$= \boxed{\frac{(x-1)e^x - (x+1)e^{-x}}{x^2}}$$

Exercise

Find the derivative of $y = \sqrt{e^{2x^2} + e^{-2x^2}}$

Solution

$$y = \sqrt{e^{2x^2} + e^{-2x^2}} = \left(e^{2x^2} + e^{-2x^2}\right)^{1/2} = U^{1/2}$$

$$U = e^{2x^2} + e^{-2x^2} \qquad \left(e^{2x^2}\right)' = \left(2x^2\right)' e^{2x^2} = 4xe^{2x^2}$$

$$U' = 4xe^{2x^2} - 4xe^{-2x^2}$$

$$y' = \frac{1}{2}\left(4xe^{2x^2} - 4xe^{-2x^2}\right)\left(e^{2x^2} + e^{-2x^2}\right)^{-1/2}$$

$$= \frac{1}{2}\frac{4x\left(e^{2x^2} - e^{-2x^2}\right)}{\left(e^{2x^2} + e^{-2x^2}\right)^{1/2}}$$

$$= \boxed{\frac{2x\left(e^{2x^2} - e^{-2x^2}\right)}{\sqrt{e^{2x^2} + e^{-2x^2}}}}$$

Exercise

Find the derivative of $y = \dfrac{x}{e^{2x}}$

Solution

$$f = x \qquad g = e^{2x}$$

$$f' = 1 \qquad g' = 2e^{2x}$$

$$y' = \frac{1\left(e^{2x}\right) - x\left(2e^{2x}\right)}{\left(e^{2x}\right)^2}$$

$$= \frac{e^{2x}(1-2x)}{\left(e^{2x}\right)^2}$$

$$\boxed{= \frac{1-2x}{e^{2x}}}$$

Exercise

Find the second derivative of $y = 3e^{5x^3+1}$

Solution

$$y' = 3(15x^2)e^{5x^3+1}$$

$$y' = 45x^2e^{5x^3+1}$$

$$f = x^2 \qquad g = e^{5x^3+1}$$

$$f' = 2x \qquad g' = 15x^2e^{5x^3+1}$$

$$y'' = 45\left(2xe^{5x^3+1} + \left(x^2\right)15x^2e^{5x^3+1}\right)$$

$$= 45e^{5x^3+1}\left(2x + 15x^4\right)$$

$$\boxed{= 45xe^{5x^3+1}\left(2 + 15x^3\right)}$$

Exercise

Find the derivative of $y = \ln\sqrt{x+5}$

Solution

$$y = \ln(x+5)^{1/2}$$

$$= \frac{1}{2}\ln(x+5)$$

$$y' = \frac{1}{2(x+5)}$$

Exercise

Find the Derivatives of $y = (3x+7)\ln(2x-1)$

Solution

$$f = 3x+7 \quad f' = 3$$

$$g = \ln(2x-1) \quad g' = \frac{2}{2x-1}$$

$$y' = 3x\ln(2x-1) + \frac{2(3x+7)}{2x-1}$$

Exercise

Find the Derivatives of $y = e^{x^2}\ln x$

Solution

$$f = e^{x^2} \quad f' = 2xe^{x^2}$$

$$g = \ln x \quad g' = \frac{1}{x}$$

$$y' = 2xe^{x^2}\ln x + \frac{e^{x^2}}{x}$$

Exercise

Find the Derivatives of $y=\log_7 \sqrt{4x-3}$

Solution

$$y'=\frac{1}{\ln 7}\frac{\left(\sqrt{4x-3}\right)'}{\sqrt{4x-3}} \qquad \frac{d}{dx}\left[\log_a |g(x)|\right]=\frac{1}{(\ln a)}\cdot\frac{g'(x)}{g(x)}$$

$$\left(\sqrt{4x-3}\right)'=\left((4x-3)^{1/2}\right)'$$

$$=\frac{1}{2}(4)(4x-3)^{-1/2} \qquad \left(U^n\right)'=nU'U^{n-1}$$

$$=2(4x-3)^{-1/2}$$

$$y'=\frac{1}{\ln 7}\frac{2(4x-3)^{-1/2}}{\sqrt{4x-3}}$$

$$=\frac{1}{\ln 7}\frac{2}{(4x-3)^{1/2}(4x-3)^{1/2}}$$

$$=\frac{1}{\ln 7}\frac{2}{(4x-3)}$$

Exercise

Find the Derivatives of $f(x)=\ln\sqrt[3]{x+1}$

Solution

$$f(x)=\ln(x+1)^{1/3}$$

$$=\frac{1}{3}\ln(x+1)$$

$$u=x+1\Rightarrow\frac{du}{dx}=1$$

$$f'(x)=\frac{1}{3}\frac{1}{x+1}$$

$$=\frac{1}{3(x+1)}$$

Exercise

Find the Derivatives of $f(x) = \ln\left[x^2\sqrt{x^2+1}\right]$

Solution

$f(x) = \ln(x^2) + \ln\sqrt{x^2+1}$ *Product Property*

$f(x) = \ln(x^2) + \ln(x^2+1)^{1/2}$

$f(x) = 2\ln x + \frac{1}{2}\ln\left(x^2+1\right)$ *Power Property*

$f'(x) = 2\frac{1}{x} + \frac{1}{2}\frac{2x}{x^2+1}$ *Differentiate*

$= \frac{2}{x} + \frac{x}{x^2+1}$

Exercise

Find the Derivatives of $y = \ln\dfrac{1+e^x}{1-e^x}$

Solution

$y = \ln\left(1+e^x\right) - \ln\left(1-e^x\right)$

$$y' = \frac{e^x}{1+e^x} - \frac{-e^x}{1-e^x}$$

$$= \frac{e^x}{1+e^x} + \frac{e^x}{1-e^x}$$

$$= \frac{e^x - e^{2x} + e^x + e^{2x}}{\left(1+e^x\right)\left(1-e^x\right)}$$

$$= \frac{2e^x}{\left(1+e^x\right)\left(1-e^x\right)}$$

Exercise

Find the Derivatives of $y = \ln \dfrac{x^2}{x^2+1}$

Solution

$$y = \ln x^2 - \ln x^2 + 1$$

$$y' = \frac{2x}{x^2} - \frac{2x}{x^2+1}$$

$$= \frac{2}{x} - \frac{2x}{x^2+1}$$

Exercise

Find the Derivatives of $y = \ln\left[\dfrac{x^2(x+1)^3}{(x+3)^{1/2}}\right]$

Solution

$$y = \ln\left[x^2(x+1)^3\right] - \ln(x+3)^{1/2}$$ ***Quotient Rule***

$$= \ln x^2 + \ln(x+1)^3 - \ln(x+3)^{1/2}$$ ***Product Rule***

$$= 2\ln x + 3\ln(x+1) - \tfrac{1}{2}\ln(x+3)$$ ***Power Rule***

$$y' = \frac{2}{x} + \frac{3}{x+1} - \frac{1}{2(x+3)}$$

Exercise

Find the Derivatives of $y = \ln(x^2+1)$

Solution

$$y' = \frac{2x}{x^2+1}$$ $(\ln U)' = \frac{U'}{U}$

Exercise

Find the Derivatives of $y = \dfrac{\ln x}{e^{2x}}$

Solution

$$y' = \frac{e^{2x}(1/x) - \ln x(2e^{2x})}{e^{4x}}$$

$$= \frac{e^{2x} - 2x\ln x(e^{2x})}{e^{4x}}$$

$$= \frac{e^{2x}\left(1 - 2x\ln x\right)}{e^{4x}}$$

Exercise

Find the Derivatives of $f(x) = \ln(x^2 - 4)$

Solution

Let u = $x^2 - 4 \Rightarrow \frac{du}{dx} = 2x$

$$f'(x) = \frac{1}{u}\frac{du}{dx}$$

$$= \frac{1}{x^2-4}(2x)$$

$$= \frac{2x}{x^2-4}$$

Exercise

Find the Derivatives of $f(x) = x^2 \ln x$

Solution

$$f' = x^2 \frac{d}{dx}[\ln x] + \ln x \frac{d}{dx}\left[x^2\right] \qquad (fg)' = f'g + fg'$$

$$= x^2\left(\frac{1}{x}\right) + 2x\ln x$$

$$= x + 2x\ln x$$

$$= x\left(1 + 2\ln x\right)$$

Exercise

Find the Derivatives of $f(x) = -\frac{\ln x}{x^2}$

Solution

$$f' = -\frac{x^2 \frac{d}{dx}[\ln x] - \ln x \frac{d}{dx}\left[x^2\right]}{\left(x^2\right)^2}$$

$$= -\frac{x^2 \frac{1}{x} - 2x\ln x}{x^4}$$

$$= -\frac{x - 2x\ln x}{x^4}$$

$$= -\frac{x(1-2\ln x)}{x^4}$$

$$= -\frac{1-2\ln x}{x^3}$$

Exercise

Find the Derivative of $f(x) = \frac{e^{\sqrt{x}}}{\ln\left(\sqrt{x}+1\right)}$

Solution

$f = e^{\sqrt{x}}$ $\quad U = \sqrt{x} = x^{1/2} \Rightarrow U' = \frac{1}{2}x^{-1/2}$ $\quad f' = \frac{1}{2}x^{-1/2}e^{\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

$g = \ln\left(\sqrt{x}+1\right)$ $\quad U = x^{1/2}+1 \Rightarrow U' = \frac{1}{2}x^{-1/2}$ $\quad g' = \frac{\frac{1}{2}x^{-1/2}}{\sqrt{x}+1} = \frac{1}{2x^{1/2}\left(\sqrt{x}+1\right)}$

$$f'(x) = \frac{\frac{e^{\sqrt{x}}}{2\sqrt{x}}\ln\left(\sqrt{x}+1\right) - \frac{1}{2\sqrt{x}\left(\sqrt{x}+1\right)}e^{\sqrt{x}}}{\left(\sqrt{x}+1\right)^2}$$

$$= \frac{\frac{\left(\sqrt{x}+1\right)e^{\sqrt{x}}\ln\left(\sqrt{x}+1\right) - e^{\sqrt{x}}}{2\sqrt{x}\left(\sqrt{x}+1\right)}}{\left(\ln\left(\sqrt{x}+1\right)\right)^2}$$

$$= \frac{e^{\sqrt{x}}\left[\left(\sqrt{x}+1\right)\ln\left(\sqrt{x}+1\right) - 1\right]}{2\sqrt{x}\left(\sqrt{x}+1\right)\left(\ln\left(\sqrt{x}+1\right)\right)^2}$$

Exercise

Find the Derivative of $f(x)=e^{2x}\ln\left(xe^{x}+1\right)$

Solution

$$f=e^{2x} \qquad U=2x\to U'=2 \qquad f'=2e^{2x}$$

$$g=\ln\left(xe^{x}+1\right) \quad U=xe^{x}+1\to U'=e^{x}+xe^{x} \quad g'=\frac{e^{x}+xe^{x}}{xe^{x}+1}$$

$$f'(x)=2e^{2x}\ln\left(xe^{x}+1\right)+e^{2x}\frac{e^{x}+xe^{x}}{xe^{x}+1}$$

$$=e^{2x}\left[2\ln\left(xe^{x}+1\right)+\frac{e^{x}(1+x)}{xe^{x}+1}\right]$$

Exercise

Find the Derivative of $f(x)=\dfrac{xe^{x}}{\ln\left(x^2+1\right)}$

Solution

$$f=xe^{x} \qquad f'=e^{x}+xe^{x}$$

$$g=\ln\left(x^2+1\right) \qquad g'=\frac{2x}{x^2+1}$$

$$f'(x)=\frac{e^{x}(1+x)\ln\left(x^2+1\right)-\frac{2x}{x^2+1}xe^{x}}{\left[\ln\left(x^2+1\right)\right]^2}$$

$$=\frac{e^{x}\left[(1+x)\ln\left(x^2+1\right)-\frac{2x^2}{x^2+1}\right]}{\left[\ln\left(x^2+1\right)\right]^2}$$

$$=\frac{e^{x}\left[\frac{\left(x^2+1\right)(1+x)\ln\left(x^2+1\right)-2x^2}{x^2+1}\right]}{\left[\ln\left(x^2+1\right)\right]^2}$$

$$=\frac{e^{x}\left[\left(x^2+1\right)(1+x)\ln\left(x^2+1\right)-2x^2\right]}{\left(x^2+1\right)\left[\ln\left(x^2+1\right)\right]^2}$$

Exercise

Find the derivative $f(x)=2\ln\left(x^2-3x+4\right)$

Solution

$$f'(x)=2\frac{2x-3}{x^2-3x+4}$$

$$=\boxed{\frac{4x-6}{x^2-3x+4}}$$

Exercise

Find the derivative $f(x)=e^{x^2+3x+1}$

Solution

$$f'(x)=\boxed{(2x+3)e^{x^2+3x+1}}$$

Exercise

Find the derivative $f(x)=3\ln\left(1+x^2\right)$

Solution

$$f'(x)=3\frac{2x}{1+x^2}$$

$$=\boxed{\frac{6x}{1+x^2}}$$

Exercise

Find the derivative $f(x)=(1+\ln x)^3$

Solution

$$f'(x)=3(1+\ln x)^2\,(1+\ln x)'$$

$$=3(1+\ln x)^2\left(\frac{1}{x}\right)$$

$$=\frac{3}{x}(1+\ln x)^2$$

Exercise

Find the derivative $f(x) = (x - 2\ln x)^4$

Solution

$$f'(x) = 4(x - 2\ln x)^3 (x - 2\ln x)'$$

$$= 4(x - 2\ln x)^3 \left(1 - \frac{2}{x}\right)$$

$$= 4(x - 2\ln x)^3 \left(\frac{x-2}{x}\right)$$

$$\boxed{= \frac{4x-8}{x}(x - 2\ln x)^3}$$

Exercise

Find the derivative $f(x) = \dfrac{e^x}{x^2+1}$

Solution

$u = e^x \quad v = x^2 + 1$

$u' = e^x \quad v' = 2x$

$$f'(x) = \frac{e^x\left(x^2+1\right) - 2xe^x}{\left(x^2+1\right)^2}$$

$$\boxed{= \frac{\left(x^2+1-2x\right)e^x}{\left(x^2+1\right)^2}}$$

Exercise

Find the derivative $f(x) = \dfrac{1-e^x}{1+e^x}$

Solution

$u = 1 - e^x \quad v = 1 + e^x$

$u' = -e^x \quad v' = e^x$

$$f'(x) = \frac{-e^x\left(1+e^x\right) - e^x\left(1-e^x\right)}{\left(1+e^x\right)^2}$$

$$= \frac{-e^x - e^{2x} - e^x + e^{2x}}{\left(1+e^x\right)^2}$$

$$\boxed{= -\frac{2e^x}{\left(1+e^x\right)^2}}$$

Exercise

Find the derivative $f(x) = \dfrac{\ln x}{1+x}$

Solution

$u = \ln x \quad v = 1+x$

$u' = \dfrac{1}{x} \qquad v' = 1$

$$f'(x) = \frac{\left(\frac{1}{x}\right)(1+x) - \ln x}{(1+x)^2}$$

$$= \frac{1}{x}\frac{1+x-x\ln x}{(1+x)^2}$$

$$\boxed{= \frac{1+x-x\ln x}{x(1+x)^2}}$$

Exercise

Find the derivative $f(x) = \dfrac{2x}{1+\ln x}$

Solution

$u = 2x \quad v = 1+\ln x$

$u' = 2 \qquad v' = \dfrac{1}{x}$

$$f'(x) = \frac{2(1+\ln x) - (2x)\frac{1}{x}}{(1+\ln x)^2}$$

$$= \frac{2 + 2\ln x - 2}{(1+\ln x)^2}$$

$$= \boxed{\frac{2\ln x}{(1+\ln x)^2}}$$

Exercise

Find the derivative $\quad f(x) = x^2 e^x$

Solution

$$u = x^2 \qquad v = e^x$$

$$u' = 2x \qquad v' = e^x$$

$$f'(x) = 2xe^x + x^2 e^x$$

$$= \boxed{\left(2x + x^2\right)e^x}$$

Exercise

Find the derivative $\quad f(x) = x^3 \ln x$

Solution

$$u = x^3 \qquad v = \ln x$$

$$u' = 3x^2 \qquad v' = \frac{1}{x}$$

$$f'(x) = 3x^2 \ln x + x^3 \frac{1}{x}$$

$$= 3x^2 \ln x + x^2$$

$$= \boxed{(3\ln x + 1)x^2}$$

Exercise

Find the derivative $\quad f(x) = 6x^4 \ln x$

Solution

$$u = 6x^4 \qquad v = \ln x$$

$$u' = 24x^3 \qquad v' = \frac{1}{x}$$

$$f'(x) = 24x^3 \ln x + 6x^4 \frac{1}{x}$$

$$= 24x^3 \ln x + 6x^3$$

$$\boxed{= 6x^3 (4\ln x + 1)}$$

Exercise

Find the derivative $f(x)=2x^3e^x$

Solution

$$u=2x^3 \qquad v=e^x$$
$$u'=6x^2 \qquad v'=e^x$$

$$f'(x)=6x^2e^x+2x^3e^x$$
$$\underline{=2x^2e^x(3+x)}$$

Exercise

Find the derivative $f(x)=\dfrac{3e^x}{1+e^x}$

Solution

$$u=3e^x \qquad v=1+e^x$$
$$u'=3e^x \qquad v'=e^x$$

$$f'(x)=\frac{3e^x\left(1+e^x\right)-3e^xe^x}{\left(1+e^x\right)^2}$$
$$=\frac{3e^x+3e^{2x}-3e^{2x}}{\left(1+e^x\right)^2}$$
$$\underline{=\frac{3e^x}{\left(1+e^x\right)^2}}$$

Exercise

Find the derivative $f(x)=5e^x+3x+1$

Solution

$$f'(x)\underline{=5e^x+3}$$

Exercise

Find the derivative $f(x)=\frac{\ln x}{2x+5}$

Solution

$$u=\ln x \qquad v=2x+5$$

$$u'=\frac{1}{x} \qquad v'=2$$

$$f'(x)=\frac{\frac{1}{x}(2x+5)-(2)\ln x}{(2x+5)^2}\cdot\frac{x}{x}$$

$$=\frac{2x+5-2x\ln x}{x(2x+5)^2}$$

Exercise

Find the derivative $f(x)=-2\ln x+x^2-4$

Solution

$$f'(x)=-\frac{2}{x}+2x$$

Exercise

Find the derivative $f(x)=e^x+x-\ln x$

Solution

$$f'(x)=e^x+1-\frac{1}{x}$$

Exercise

Find the derivative $f(x)=\ln x+2e^x-3x^2$

Solution

$$f'(x)=\frac{1}{x}+2e^x-6x$$

Exercise

Find the derivative $f(x)=\ln x^8$

Solution

$$f(x)=\ln x^8=8\ln x$$

Power Rule

$$f'(x)=\boxed{\frac{8}{x}}$$

$$(\ln x)'=\frac{1}{x}$$

Exercise

Find the derivative $f(x)=5x-\ln x^5$

Solution

$$f(x)=5x-\ln x^5$$
$$=5x-5\ln x$$

Power Rule

$$f'(x)=\boxed{5-\frac{5}{x}}$$

$$(\ln x)'=\frac{1}{x}$$

Exercise

Find the derivative $f(x)=\ln x^2+4e^x$

Solution

$$f(x)=2\ln x+4e^x$$

Power Rule

$$f'(x)=\boxed{\frac{2}{x}+4e^x}$$

$$(\ln x)'=\frac{1}{x}$$

Exercise

Find the derivative $f(x)=\ln x^{10}+2\ln x$

Solution

$$f(x)=10\ln x+2\ln x$$

Power Rule

$$=12\ln x$$

$$f'(x)=\boxed{\frac{12}{x}}$$

$$(\ln x)'=\frac{1}{x}$$

Exercise

The percentage of people of any particular age group that will die in a given year may be approximated by the formula

$$P(t) = 0.00239e^{0.0957t}$$

Where t is the age of the person in years

Solution

a) Find $P(25)$

$$P(25) = 0.00239e^{0.0957(25)} = \underline{0.02615}$$

b) Find $P'(25)$

$$P'(t) = 0.000228723e^{0.0957t}$$

$$P'(25) = 0.000228723e^{0.0957(25)} = \underline{0.0025}$$

Exercise

Find the equations of the tangent lines to $f(x) = e^x$ at the points (0, 1)

Solution

$$f'(x) = e^x$$

$$(0,\ 1) \Rightarrow m = f'(x = 0)$$

$$= e^0$$

$$= 1$$

$$y - y_1 = m\left(x - x_1\right)$$

$$y - 1 = 1(x - 0)$$

$$y - 1 = x$$

$$\underline{y = x + 1}$$

Exercise

Find the equations of the tangent lines to $f(x) = e^x$ at the points (1, e)

Solution

$$f'(x) = e^x$$

$(1,\,e)\ \Rightarrow m = f'(x=1) = e^1 = e$

$$y - e = e(x-1)$$

$$y - e = ex - e$$

$$\boxed{y = ex}$$

Exercise

Find the equations of the tangent lines to $y = 4xe^{-x} + 5$ at $x = 1$

Solution

$$y' = 4e^{-x} - 4xe^{-x} = 4e^{-x}(1-x)$$

$$= 4e^{-x}(1-x)$$

$$m = y'(x=1)$$

$$= 4e^{-1}(1-1) = 0$$

$$\Rightarrow x = 1 \rightarrow y = 4e^{-1} + 5$$

$$\left(1,\, 4e^{-1} + 5\right)$$

$$y - (4e^{-1} + 5) = 0(x-1)$$

$$\boxed{y = 4e^{-1} + 5}$$

Exercise

Find the equation of the tangent lines to $f(x) = 4e^{-8x}$ at the points (0, 4)

Solution

$$f'(x) = -32e^{-8x}$$

$$m = f'(0) = -32e^{-8(0)} = -32$$

$$y - 4 = -32\left(x - 0\right) \qquad y - y_1 = m\left(x - x_1\right)$$

$$y - 4 = -32x$$

$$\boxed{y = -32x + 4}$$

Exercise

Assume the cost of a gallon of milk is $2.90. With continuous compounding, find the time it would take the cost to be 5 times as much (to the nearest tenth of a year), at an annual inflation rate of 6 %.

Solution

$A = Pe^{rt}$

$5(2.90) = 2.9e^{.06t}$ ***Divide both sides by 2.9***

$5 = e^{.06t}$ ***"ln" both sides***

$\ln 5 = \ln e^{.06t}$ ***use inverse property***

$.06t = \ln 5$

$\boxed{t = \frac{\ln 5}{.06} = 26.8 \text{ years}}$

Exercise

The sales in thousands of a new type of product are given by $S(t) = 30 - 90e^{-0.5t}$, where t represents time in years. Find the rate of change of sales at the time when $t = 3$

Solution

$S' = -90(-.5)e^{-0.5t}$

$= 45e^{-0.5t}$

$S'(t=3) = 45e^{-0.5(3)}$

$\boxed{= 10.04}$

The rate of change of sales at the time when $t = 3$ is 10,040.

Exercise

A company's total cost, in millions of dollars, is given by $C(t) = 300 - 70e^{-t}$ where t = time in years. Find the marginal cost when $t = 3$.

Solution

$C'(t) = -70(-1)e^{-t}$

$= 70e^{-t}$

$C'(t=3) = 70e^{-3}$ $\boxed{= 3.485}$

The marginal cost is $3,485,000.

Exercise

A company's total cost, in millions of dollars, is given by $C(t) = 280 - 30e^{-t}$ where t = time in years. Find the marginal cost when $t = 2$.

Solution

$$C'(t) = 30e^{-t}$$

$$C'(t = 2) = 30e^{-2} \ \underline{= 4.06|}$$

The marginal cost is \$4,060,000.

Exercise

The demand function for a certain book is given by the function $x = D(p) = 70e^{-0.006p}$. Find the marginal demand $D'(p)$

Solution

$$D'(p) = 70(-.006)e^{-0.006p}$$

$$\underline{= -.42e^{-0.006p}|}$$

Exercise

Suppose that the amount in grams of a radioactive substance present at time t (in years) is given by $A(t) = 840e^{-0.63t}$. Find the rate of change of the quantity present at the time when $t = 2$.

Solution

$$A'(t) = 840(-0.63)e^{-0.63t}$$

$$= -529.2e^{-0.63t}$$

$$A'(t = 2) = -529.2e^{-0.63(2)}$$

$$\underline{= -150.11|}$$

Exercise

Researchers have found that the maximum number of successful trials that a laboratory rat can complete in a week is given by

$$P(t) = 53\left(1 - e^{-0.4t}\right)$$

where t is the number of weeks the rat has been trained. Find the rate of change $P'(t)$.

Solution

$$P'(t) = 53\left(-(-.4)e^{-0.4t}\right)$$
$$\boxed{= 21.2e^{-0.4t}}$$

Exercise

When a telephone wire is hung between two poles, the wire forms a U-shape curve called a Catenary. For instance, the function $y = 30\left(e^{x/60} + e^{-x/60}\right) \quad -30 \le x \le 30$ models the shape of the telephone wire strung between two poles that are 60 ft. apart (x & y are measured in ft.). Show that the lowest point on the wire is midway between two poles. How much does the wire sag between the two poles?

Solution

$$y' = 30\left(\frac{1}{60}e^{x/60} - \frac{1}{60}e^{-x/60}\right)$$
$$= \frac{1}{2}\left(e^{x/60} - e^{-x/60}\right)$$

Find the critical number(s)

$$y' = 0$$
$$\frac{1}{2}\left(e^{x/60} - e^{-x/60}\right) = 0$$
$$e^{x/60} - e^{-x/60} = 0$$
$$e^{x/60} = e^{-x/60}$$
$$\frac{x}{60} = -\frac{x}{60}$$
$$\Rightarrow x = 0$$

$$y(x = -30) = 30\left(e^{-30/60} + e^{-(-30)/60}\right) \approx 67.7\ ft$$

$$y(x = 0) = 30\left(e^{0} + e^{0}\right) = 30(2) = 60\ ft$$

$$y(x = 30) = 30\left(e^{30/60} + e^{-(30)/60}\right) \approx 67.7\ ft$$

Sag 7.7 *ft*

Exercise

Find $f''(x)$ for $f(x)=\dfrac{\ln x}{7x}$, then find $f''(0)$ and $f''(2)$

Solution

$$f(x)=\frac{\ln x}{7x} \qquad f=\ln x \quad f'=\frac{1}{x} \qquad g=7x \quad g'=7$$

$$f'(x)=\frac{\frac{1}{x}(7x)-7\ln x}{(7x)^2}$$

$$=\frac{7-7\ln x}{49x^2}$$

$$=\frac{7(1-\ln x)}{49x^2}$$

$$=\frac{1-\ln x}{7x^2}$$

$$f=1-\ln x \quad f'=-\frac{1}{x} \qquad g=7x^2 \quad g'=14x$$

$$f''(x)=\frac{-\frac{1}{x}\left(7x^2\right)-14x(1-\ln x)}{\left(7x^2\right)^2}$$

$$=\frac{-7x-14x+14x\ln x}{49x^4}$$

$$=\frac{-21x+14x\ln x}{49x^4}$$

$$=\frac{7x(-3+2\ln x)}{49x^4}$$

$$=\frac{-3+2\ln x}{7x^3}$$

$$\boxed{f''(x)=\frac{2\ln x-3}{7x^3}}$$

$$f''(x=0)=\frac{2\boxed{\ln(0)}-3}{7(0)^3}: \textit{undefined}$$ Inside log has to be > 0 Therefore is undefined for ln(0)

$$f''(x=7)=\frac{2\ln(7)-3}{7(7)^3}\approx 0.0004$$

Exercise

Suppose the average test score p and was modeled by $p = 92.3 - 16.9\ln(t+1)$, where t is the time in months. How would the rate at which the average test score changed after 1 year?

Solution

$$\frac{dp}{dt} = -\frac{16.9}{t+1}$$

$t = 1\ yr = 12\ mths$

$$\Rightarrow \frac{dp}{dt} = -\frac{16.9}{12+1}$$

$$= -\frac{16.9}{13}$$

$$\underline{= -1.3}$$

Exercise

Suppose that the population of a certain collection of rare ants is given by

$$P(t) = (t+100)\ln(t+2)$$

Where t represents the time in days. Find the rates of change of the population on the second day and on the eighth day.

Solution

$$P'(t) = \ln(t+2) + (t+100)\frac{1}{t+2}$$

$$= \ln(t+2) + \frac{t+100}{t+2}$$

$$P'(2) = \ln(2+2) + \frac{2+100}{2+2} = \underline{27.89}$$

$$P'(8) = \ln(8+2) + \frac{8+100}{8+2} = \underline{13.10}$$

Exercise

Suppose that the demand function for x units of a certain item is $P(x)=100+\dfrac{180\ln(x+5)}{x}$ where P is the price per unit, in dollars. Find the marginal revenue.

Solution

$$\begin{aligned}R&=x.P(x)\\&=x\left(100+\frac{180\ln(x+5)}{x}\right)\\&=100x+180\ln(x+5)\end{aligned}$$

$$\begin{aligned}R'(x)&=100+180\frac{1}{x+5}\\&=\frac{100(x+5)+180}{x+5}\\&=\frac{100x+500+180}{x+5}\\&=\boxed{\frac{100x+680}{x+5}}\end{aligned}$$

Exercise

The population of coyotes in the northwestern portion of Alabama is given by the formula $P(t)=\left(t^2+100\right)\ln(t+2)$, where t represents the time in years since 2000 (the year 2000 corresponds to $(t=0)$ Find the rate of change of the coyote population in $2013\ (t=13)$.

Solution

$$f=t^2+100\quad\to f'=2t$$

$$g=\ln(t+2)\quad\to g'=\frac{1}{t+2}$$

$$P'=f'g+g'f$$

$$\begin{aligned}P'(t)&=2t\ln(t+2)+\frac{1}{t+2}\left(t^2+100\right)\\&=2t\ln(t+2)+\frac{t^2+100}{t+2}\end{aligned}$$

$$P'(t=13)=2(13)\ln(13+2)+\frac{13^2+100}{13+2}\qquad\qquad 2*13\ln(13+2)+(13\wedge2+100)/(13+2)$$

$$\boxed{=88.34}$$

Exercise

Students in a math class took a final exam. They took equivalent forms of the exam in monthly intervals thereafter. The average score $S(t)$, in percent, after t months was found to be given by

$$S(t)=73-17\ln(t+1), \qquad t\ge 0$$

Find $S'(t)$.

Solution

$$S'(t)=-17\frac{1}{t+1}$$

$$=-\frac{17}{t+1}$$

Exercise

Suppose that the population of a town is given by $P(t)=8\ln\sqrt{8t+7}$ where t is the time in years after 1980 and P is the population of the town in thousands. Find $P'(t)$.

Solution

$$U=8t+7 \quad \rightarrow U'=8$$

$$V=\sqrt{8t+7}=(8t+7)^{1/2}=U^{1/2}$$

$$\rightarrow V'=nU'U^{n-1}=\tfrac{1}{2}8(8t+7)^{-1/2}$$

$$=\frac{4}{(8t+7)^{1/2}}$$

$$P'(t)=8\frac{V'}{V}$$

$$=8\underbrace{\frac{4}{(8t+7)^{1/2}}}_{V'}\underbrace{\frac{1}{(8t+7)^{1/2}}}_{V}$$

$$=\frac{32}{8t+7}$$

Exercise

The following formula accurately models the relationship between the size of a certain type of tumor and the amount of time that it has been growing:

$$V(t) = 450(1 - e - 0.0022t)^3$$

where t is in months and $V(t)$ is measured in cubic centimeters. Calculate the rate of change of tumor volume at 80 months.

Solution

$$U = 1 - e - 0.0022t \qquad V = 450U^3$$

$$U' = -.0022 \qquad V' = 450(3)U^2U'$$

$$V'(t) = 450(3)(1 - e - 0.0022t)^2(-.0022) \underline{= 2.97(1 - e - 0.0022t)^2}$$

$$V'(t = 80) = 2.97(1 - e - 0.0022(80))^2 \underline{= 10.66}$$

Exercise

A yeast culture at room temperature (68° F) is placed in a refrigerator set at a constant temperature of 38° F. After t hours, the temperature T of the culture is given approximately by

$$T = 30e^{-0.58t} + 38 \quad t \ge 0$$

What is the rate of change of temperature of the culture at the end of 1 hour? At the end of 4 hours?

Solution

$$T' = 30(-0.58)e^{-0.58t} \underline{= -17.4e^{-0.58t}}$$

$$T'(1) = -17.4e^{-0.58(1)} \underline{= -9.74° \ F/hr}$$

$$T'(4) = -17.4e^{-0.58(4)} \underline{= -1.71° \ F/hr}$$

Exercise

A mathematical model for the average age of a group of people learning to type is given by

$$N(t) = 10 + 6\ln t \quad t \ge 1$$

Where $N(t)$ is the number of words per minute typed after t hours of instruction and practice (2 hours per day, 5 days per week). What is the rate of learning after 10 hours of instruction and practice? After 100 hours?

Solution

$$N'(t) = \frac{6}{t}$$

$$N'(10) = \frac{6}{10} \underline{= 0.6}$$

After 10 hours of instruction and practice, the rate of learning is 0.6 words/minute per hour of instruction and practice.

$$N'(100) = \frac{6}{100} \underline{= 0.06}$$

After 100 hours of instruction and practice, the rate of learning is 0.06 words/minute per hour of instruction and practice.