Lecture R – Calculus I – Review

Section R.1 – Derivative

Constant Rule

$$\frac{d}{dx}[c]=0 \qquad c \text{ is constant}$$

Example

Find the derivative:

a) $f(x)=-2$ $\qquad f'(x)=0$

b) $y=\pi$ $\qquad y'=0$

c) $g(w)=\sqrt{5}$ $\qquad g'(w)=0$

d) $s(t)=320.5$ $\qquad s'(t)=0$

Power Rule

$$\frac{d}{dx}[x^n]=nx^{n-1} \qquad n \text{ is any real number}$$

Constant Times a Function

$$\frac{d}{dx}\left[cf(x)\right]=cf'(x)$$

Example

Find the derivative each function

a. $y=\dfrac{9}{4x^2}$

Solution

$$y=\frac{9}{4}x^{-2}$$

$$\to y'=\frac{9}{4}(-2)x^{-3} \boxed{=-\frac{9}{2x^3}}$$

b. $y=\sqrt[3]{x}$

Solution

$$y=x^{1/3}$$

$$\to y'=\frac{1}{3}x^{(1/3)-1}=\frac{1}{3}x^{-2/3} \boxed{=\frac{1}{3\sqrt[3]{x^2}}}$$

The Product Rule

The derivative of the product of two differentiable functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$(f.g)' = f.g' + f'.g$$

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'gh + fg'h + fgh'$$

Example

Find the derivative of $y = \left(4x + 3x^2\right)(6 - 3x)$

Solution

$$y = 24x + 6x^2 - 9x^3$$

$$\begin{aligned} y' &= \left(4x + 3x^2\right)\frac{d}{dx}(6 - 3x) + (6 - 3x)\frac{d}{dx}\left(4x + 3x^2\right) \\ &= \left(4x + 3x^2\right)(-3) + (6 - 3x)(4 + 6x) \\ &= -12x - 9x^2 + 24 + 36x - 12x - 18x^2 \\ &= -27x^2 + 12x + 24 \end{aligned}$$

Example

Find the derivative of $y = \left(\frac{1}{x} + 1\right)(2x + 1)$

Solution

$$\begin{aligned} y' &= \left(x^{-1} + 1\right)\frac{d}{dx}(2x + 1) + (2x + 1)\frac{d}{dx}\left(x^{-1} + 1\right) \\ &= \left(x^{-1} + 1\right)(2) + (2x + 1)\left(-x^{-2}\right) \\ &= \frac{2}{x} + 2 + (2x + 1)\left(-\frac{1}{x^2}\right) \\ &= \frac{2}{x} + 2 - \frac{2x}{x^2} - \frac{1}{x^2} \\ &= \frac{2}{x} + 2 - \frac{2}{x} - \frac{1}{x^2} \\ &= 2 - \frac{1}{x^2} \\ &= \frac{2x^2 - 1}{x^2} \end{aligned}$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}=\frac{gf'-fg'}{g^2}$$

Example

Find the derivative of $y=\frac{x+4}{5x-2}$

Solution

$$y'=\frac{(5x-2)\frac{d}{dx}[(x+4)]-(x+4)\frac{d}{dx}[(5x-2)]}{(5x-2)^2}$$

$$=\frac{(5x-2)(1)-(x+4)(5)}{(5x-2)^2}$$

$$=\frac{5x-2-5x-20}{(5x-2)^2}$$

$$=-\frac{22}{(5x-2)^2}$$

Example

Find the derivative of $y=\frac{3-\frac{2}{x}}{x+4}$

Solution

$$y=\frac{\frac{3x-2}{x}}{x+4}=\frac{3x-2}{x}\cdot\frac{1}{x+4}=\frac{3x-2}{x^2+4x}$$

$$y'=\frac{\left(x^2+4x\right)(3)-(3x-2)(2x+4)}{\left[x(x+4)\right]^2}$$

$$=\frac{3x^2+12x-6x^2-12x+4x+8}{x^2(x+4)^2}$$

$$=\boxed{\frac{-3x^2+4x+8}{x^2(x+4)^2}}$$

Chain Rule

The General Power Rule

$$\frac{dy}{dx} = \frac{d}{dx}\left[u(x)^n\right]$$

$$= n\, u^{n\text{-}1}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[u^n\right] = n\, u^{n\text{-}1}u'$$

Example

Find the derivative of $y = \left(x^2 + 3x\right)^4$

Solution

$$u = x^2 + 3x$$

$$y' = n \quad (u)^{n-1} \quad \frac{d}{dx}[u]$$

$$= 4\left(x^2 + 3x\right)^3 \frac{d}{dx}[x^2 + 3x]$$

$$\boxed{= 4\left(x^2 + 3x\right)^3 (2x + 3)}$$

Formula $\left(U^m V^n W^p\right)' = U^{m-1}V^{n-1}W^{p-1}\left(mU'VW + nUV'W + pUVW'\right)$

Proof

$$\left(U^m V^n W^p\right)' = \left(U^m\right)' V^n W^p + U^m\left(V^n\right)' W^p + U^m V^n\left(W^p\right)'$$

$$= mU^{m-1}U'V^nW^p + nU^mV^{n-1}V'W^p + pU^mV^nW^{p-1}W' \qquad \textit{factor}\ \ U^{m-1}V^{n-1}W^{p-1}$$

$$= U^{m-1}V^{n-1}W^{p-1}\left(mU'VW + nUV'W + pUVW'\right)$$

Derivatives of Trigonometric Functions

$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$	$(\tan x)' = \sec^2 x$
$(\csc x)' = -\csc x \cot x$	$(\sec x)' = \sec x \tan x$	$(\cot x)' = -\csc^2 x$

Example

Find the derivatives

a) $y = \sin x \cos x$

$$y' = \sin x(\cos x)' + \cos x(\sin x)'$$

$$= \sin x(-\sin x) + \cos x(\cos x)$$

$$\underline{= \cos^2 x - \sin^2 x}$$

b) $y = \dfrac{\cos x}{1-\sin x}$

$$y' = \frac{(1-\sin x)(\cos x)' - \cos x(1-\sin x)'}{(1-\sin x)^2}$$

$$= \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$$

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$

$$= \frac{1-\sin x}{(1-\sin x)^2}$$

$$\underline{= \frac{1}{1-\sin x}}$$

Derivatives of Logarithmic

Derivative of $y = \ln x$ $\quad \boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}} \quad x \neq 0$

The chain rule extends: $\quad \boxed{\frac{d}{dx}\ln u = \frac{1}{u}\cdot\frac{du}{dx} = \frac{u'}{u}} \quad u > 0$

Example

Find $\quad \frac{d}{dx}\ln 2x$

Solution

$$
\begin{aligned}
\frac{d}{dx}\ln 2x &= \frac{(2x)'}{2x} \\
&= \frac{2}{2x} \\
&= \frac{1}{x}
\end{aligned}
$$

Example

Find the derivative of $\ln\left(x^2+3\right)$

Solution

$$\frac{d}{dx}\ln\left(x^2+3\right) = \frac{2x}{x^2+3}$$

Derivative $\quad \boxed{\frac{d}{dx}\left(\log_a u\right) = \frac{1}{u}\cdot\frac{1}{\ln a}\frac{du}{dx}}$

Example

- $\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{(3x+1)\cdot\ln 10}\frac{d}{dx}(3x+1) = \frac{3}{(3x+1)\cdot\ln 10}$

Derivatives of Exponential Functions

If u is any differentiable function of x, then $\boxed{\frac{d}{dx}e^u = e^u \frac{du}{dx}}$ $\boxed{\left(e^u\right)' = u'e^u}$

Example

Find the derivative of $\frac{d}{dx}\left(5e^x\right)$

Solution

$$\frac{d}{dx}\left(5e^x\right) = 5\frac{d}{dx}e^x = 5e^x$$

Example

Find the derivative of $\frac{d}{dx}\left(e^{\sin x}\right)$

Solution

$$\frac{d}{dx}\left(e^{\sin x}\right) = e^{\sin x}\frac{d}{dx}(\sin x) = e^{\sin x}\cdot\cos x$$

Example

Find the derivative of $\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right)$

Solution

$$\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right) = e^{\sqrt{3x+1}}\cdot\frac{1}{2}(3x+1)^{-1/2}\cdot 3$$

$$= \frac{3}{2\sqrt{3x+1}}e^{\sqrt{3x+1}}$$

Definition

If $a > 0$ and u is a differentiable of x, then a^u is a differentiable function of x and

$$\boxed{\frac{d}{dx}a^u = a^u \ln a\frac{du}{dx}}$$

Example

- $\frac{d}{dx}3^x = 3^x \ln 3$
- $\frac{d}{dx}3^{-x} = 3^{-x}\ln 3\frac{d}{dx}(-x) = -3^{-x}\ln 3$
- $\frac{d}{dx}3^{\sin x} = 3^{\sin x}\ln 3\frac{d}{dx}(\sin x) = 3^{\sin x}\ln 3(\cos x)$

Derivatives of Inverse Trigonometric Functions

$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad \lvert u\rvert < 1$	$\left(\sin^{-1}u\right)' = \frac{u'}{\sqrt{1-u^2}}$
$\frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad \lvert u\rvert < 1$	$\left(\cos^{-1}u\right)' = -\frac{u'}{\sqrt{1-u^2}}$
$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$	$\left(\tan^{-1}u\right)' = \frac{u'}{1+u^2}$
$\frac{d}{dx}\cot^{-1}u = -\frac{1}{1+u^2}\frac{du}{dx}$	$\left(\cot^{-1}u\right)' = -\frac{u'}{1+u^2}$
$\frac{d}{dx}\sec^{-1}u = \frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}, \quad \lvert u\rvert > 1$	$\left(\sec^{-1}u\right)' = \frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$
$\frac{d}{dx}\csc^{-1}u = -\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}, \quad \lvert u\rvert > 1$	$\left(\csc^{-1}u\right)' = -\frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$

Example

Find the derivative of $\frac{d}{dx}\left(\sin^{-1}x^2\right)$

Solution

$$\frac{d}{dx}\left(\sin^{-1}x^2\right) = \boxed{\frac{2x}{\sqrt{1-x^4}}}$$

Example

Find the derivative of $\frac{d}{dx}\left(\sec^{-1}5x^4\right)$

Solution

$$\frac{d}{dx}\left(\sec^{-1}5x^4\right) = \frac{\left(5x^4\right)'}{5x^4\sqrt{\left(5x^4\right)^2-1}}$$

$$5x^4-1>0$$

$$= \frac{20x^3}{5x^4\sqrt{25x^8-1}}$$

$$= \boxed{\frac{4}{x\sqrt{25x^8-1}}}$$

Exercises Section *R*.1 – Derivative

Find the derivative to the following functions

1. $f(t)=-3t^2+2t-4$
2. $g(x)=4\sqrt[3]{x}+2$
3. $f(x)=x\left(x^2+1\right)$
4. $f(x)=\frac{2x^2-3x+1}{x}$
5. $f(x)=\frac{4x^3-3x^2+2x+5}{x^2}$
6. $f(x)=\frac{-6x^3+3x^2-2x+1}{x}$
7. $f(x)=x\left(1-\frac{2}{x+1}\right)$
8. $g(s)=\frac{s^2-2s+5}{\sqrt{s}}$
9. $f(x)=\frac{x+1}{\sqrt{x}}$
10. $f(x)=3x\left(2x^2+5x\right)$
11. $y=3\left(2x^2+5x\right)$
12. $y=\frac{x^2+4x}{5}$
13. $y=\frac{3x^4}{5}$
14. $y=\frac{x^2-4}{2x+5}$
15. $y=\frac{(1+x)(2x-1)}{x-1}$
16. $y=\frac{4}{2x+1}$
17. $y=\frac{2}{(x-1)^3}$
18. $f(x)=\sqrt{2t^2+5t+2}$
19. $f(x)=\frac{1}{\left(x^2-3x\right)^2}$
20. $y=t^2\sqrt{t-2}$
21. $y=\left(\frac{6-5x}{x^2-1}\right)^2$
22. $y=x^2\sqrt{x^2+1}$
23. $y=\left(\frac{x+1}{x-5}\right)^2$
24. $y=\sqrt[3]{(x+4)^2}$
25. $y=x^2\sin x$
26. $y=\frac{\sin x}{x}$
27. $y=\frac{\cot x}{1+\cot x}$
28. $y=x^2\sin x+2x\cos x-2\sin x$
29. $y=x^3\sin x\cos x$
30. $y=\frac{4}{\cos x}+\frac{1}{\tan x}$
31. $f(x)=\frac{\left(x^2-6x\right)^5}{\left(3x^2+5x-2\right)^4}$
32. $y=\ln\sqrt{x+5}$
33. $y=(3x+7)\ln(2x-1)$
34. $f(x)=\ln\sqrt[3]{x+1}$
35. $f(x)=\ln\left[x^2\sqrt{x^2+1}\right]$
36. $y=\ln\frac{x^2}{x^2+1}$

37. $f(x) = e^{-2x^3}$

38. $f(x) = 4e^{x^2}$

39. $f(x) = 2x^3e^x$

40. $f(x) = \dfrac{3e^x}{1+e^x}$

41. $f(x) = 5e^x + 3x + 1$

42. $f(x) = x^2e^x$

43. $f(x) = \frac{e^x+e^{-x}}{2}$

44. $f(x) = \frac{e^x}{x^2}$

45. $f(x) = x^2e^x - e^x$

46. $f(x) = (1+2x)e^{4x}$

47. $y = x^2e^{5x}$

48. $y = x^2e^{-2x}$

49. $f(x) = \dfrac{e^x}{x^2+1}$

50. $f(x) = \dfrac{1-e^x}{1+e^x}$

51. $y = \dfrac{\ln x}{e^{2x}}$

52. $f(x) = e^{2x}\ln\left(xe^x + 1\right)$

53. $f(x) = \dfrac{xe^x}{\ln\left(x^2+1\right)}$

54. $y = \cos^{-1}\left(\frac{1}{x}\right)$

55. $y = \sin^{-1}\sqrt{2}t$

56. $y = \sec^{-1}(5s)$

57. $y = \cot^{-1}\sqrt{t-1}$

58. $y = \ln\left(\tan^{-1}x\right)$

59. $y = \tan^{-1}(\ln x)$