Solution Section 1.1 – Idea of Limits

Exercise

Find the average rate of change of the function $f(x)=x^3+1$ over the interval $[2,\ 3]$

Solution

$$\frac{\Delta f}{\Delta x}=\frac{f(3)-f(2)}{3-2}$$

$$=\frac{3^3+1-\left(2^3+1\right)}{1}$$

$$=27+1-(8+1)$$

$$=19$$

Exercise

Find the average rate of change of the function $f(x)=x^2$ over the interval $[-1,\ 1]$

Solution

$$\frac{\Delta f}{\Delta x}=\frac{f(1)-f(-1)}{1-(-1)}$$

$$=\frac{1^2-(-1)^2}{2}$$

$$=\frac{0}{2}$$

$$=0$$

Exercise

Find the average rate of change of the function $f(t)=2+\cos t$ over the interval $[-\pi,\ \pi]$

Solution

$$\frac{\Delta f}{\Delta x}=\frac{f(\pi)-f(-\pi)}{\pi-(-\pi)}$$

$$=\frac{2+\cos\pi-\left(2+\cos(-\pi)\right)}{2\pi}$$

$$=\frac{2-1-(2-1)}{2}$$

$$=0$$

Exercise

Find the slope of $y = x^2 - 3$ at the point $P(2,\ 1)$ and an equation of the tangent line at this P.

Solution

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1 + h) - f(x_1)}{h}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(2+h) - f(2)}{h} \\ &= \frac{(2+h)^2 - 3 - (2^2 - 3)}{h} \\ &= \frac{4 + 4h + h^2 - 3 - (4-3)}{h} \\ &= \frac{4h + h^2}{h} \\ &= 4 + h\end{aligned}$$

As h approaches 0. Then the secant slope $h + 4 \to 4 = slope$

$$y = m(x - x_1) + y_1$$

$$\begin{aligned}&y = 4(x-2) + 1 \\ &y - 1 + 1 = 4x - 8 + 1 \\ &\underline{y = 4x - 7}\end{aligned}$$

Exercise

Find the slope of $y = x^2 - 2x - 3$ at the point $P(2,\ -3)$ and an equation of the tangent line at this P.

Solution

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(2+h) - f(2)}{h} \\ &= \frac{(2+h)^2 - 2(2+h) - 3 - (2^2 - 2(2) - 3)}{h} \\ &= \frac{4 + 4h + h^2 - 4 - 2h - 3 - (-3)}{h} \\ &= \frac{2h + h^2}{h} \\ &= 2 + h\end{aligned}$$

As h approaches 0. Then the secant slope $2 + h \to 2 = slope$

$$y = m(x - x_1) + y_1$$

$$\begin{aligned}&y + 3 = 2(x-2) \\ &y = 2x - 4 - 3 \\ &\underline{y = 2x - 7}\end{aligned}$$

Exercise

Find the slope of $y = x^3$ at the point $P(2,\ 8)$ and an equation of the tangent line at this *P*.

Solution

$$\frac{\Delta y}{\Delta x} = \frac{f(2+h) - f(2)}{h}$$

$$= \frac{(2+h)^3 - 2^3}{h}$$

$$= \frac{8 + 12h + 6h^2 + h^3 - 8}{h}$$

$$= 12 + 6h + h^2 \qquad \text{As } h \text{ approaches 0. Then } slope = 12$$

$$y - 8 = 12(x-2) \qquad\qquad y = m(x - x_1) + y_1$$

$$y = 12x - 24 + 8$$

$$\boxed{y = 12x - 16}$$

Exercise

Make a table of values for the function $f(x) = \frac{x+2}{x-2}$ at the points

$$x = 1.2, \quad x = \frac{11}{10}, \quad x = \frac{101}{100}, \quad x = \frac{1001}{1000}, \quad x = \frac{10001}{10000}, \quad \textit{and } x = 1$$

a) Find the average rate of change of $f(x)$ over the intervals $[1,\ x]$ for each $x \ne 1$ in the table

b) Extending the table if necessary, try to determine the rate of change of $f(x)$ at $x = 1$.

Solution

a)

x	1.2	1.1	1.01	1.001	1.0001	1
$f(x)$	-4.0	$-3.\overline{4}$	$-3.\overline{04}$	$-3.\overline{004}$	$-3.\overline{004}$	-3

$$\frac{\Delta y}{\Delta x} = \frac{-4 - (-3)}{1.2 - 1} = -5.0$$

$$\frac{\Delta y}{\Delta x} = \frac{-3.\overline{4} - (-3)}{1.1 - 1} = -4.\overline{4}$$

$$\frac{\Delta y}{\Delta x} = \frac{-3.\overline{04} - (-3)}{1.01 - 1} = -4.\overline{04}$$

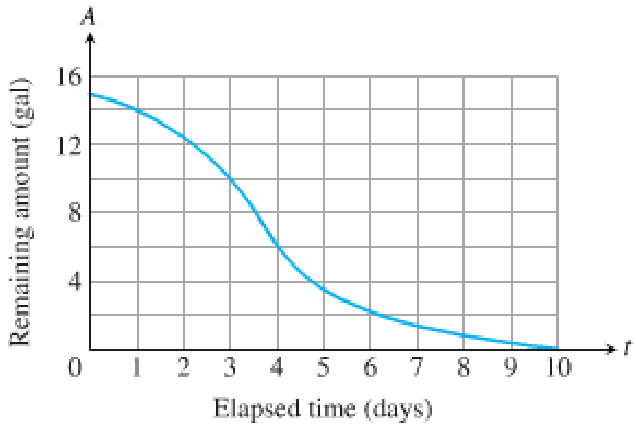
$$\frac{\Delta y}{\Delta x} = \frac{-3.\overline{004} - (-3)}{1.001 - 1} = -4.\overline{004}$$

$$\frac{\Delta y}{\Delta x} = \frac{-3.\overline{0004} - (-3)}{1.0001 - 1} = -4.\overline{0004}$$

b) The rate of change of $f(x)$ at $x = 1$ is $\boxed{-4}$

Exercise

The accompanying graph shows the total amount of gasoline A in the gas tank of an automobile after being driven for t days.



a) Estimate the average rate of gasoline consumption over the time intervals

$$[0,\ 3],\ [0,\ 5],\ \textit{and}\ [7,\ 10]$$

b) Estimate the instantaneous rate of gasoline consumption over the time $t=1,\ t=4,\ \textit{and}\ t=8$

Solution

a) Average rate of gasoline consumption over the time intervals:

$$[0,\ 3] \quad \Rightarrow \quad \textit{Average Rate} = \frac{10-15}{3-0} \approx = -1.67\ gal\ /\ day$$

$$[0,\ 5] \quad \Rightarrow \quad \textit{Average Rate} = \frac{3.9-15}{3-0} \approx -2.2\ gal\ /\ day$$

$$[7,\ 10] \quad \Rightarrow \quad \textit{Average Rate} = \frac{0-1.4}{10-7} \approx -0.5\ gal\ /\ day$$

b) At $t=1 \quad \rightarrow P(1,\ 14)$

At $t=4 \quad \rightarrow P(4,\ 6)$

At $t=8 \quad \rightarrow P(8,\ 1)$