Solution   *Section* 1.2 – Solutions to Separable Equations

Exercise

Find the general solution of the differential equation $y' = xy$

Solution

$$\frac{dy}{dx} = xy$$

$$\frac{dy}{y} = xdx$$

$$\int \frac{dy}{y} = \int xdx$$

$$\ln|y| = \frac{1}{2}x^2 + C$$

$$|y| = e^{x^2/2+C}$$

$$y(x) = \pm e^{x^2/2}e^C$$

$$= Ae^{x^2/2} \qquad \text{Where } A = \pm e^C$$

Exercise

Find the general solution of the differential equation $xy' = 2y$

Solution

$$x\frac{dy}{dx} = 2y$$

$$\frac{dy}{y} = 2\frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{2}{x}dx$$

$$\ln|y| = 2\ln|x| + C$$

$$= \ln x^2 + C$$

$$y(x) = \pm e^{\ln x^2 + C}$$

$$= \pm e^C x^2$$

$$= Ax^2$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = e^{x-y}$

Solution

$$\frac{dy}{dx} = e^x e^{-y}$$

$$\frac{dy}{e^{-y}} = e^x dx$$

$$\int e^y dy = \int e^x dx$$

$$e^y = e^x + C$$

$$\boxed{y(x) = \ln\left(e^x + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \left(1+y^2\right)e^x$

Solution

$$\frac{dy}{dx} = \left(1+y^2\right)e^x$$

$$\int \frac{dy}{1+y^2} = \int e^x dx$$

$$\tan^{-1} y = e^x + C$$

$$\boxed{y(x) = \tan\left(e^x + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = xy + y$

Solution

$$\frac{dy}{dx} = (x+1)y$$

$$\int \frac{dy}{y} = \int (x+1)dx$$

$$\ln y = \frac{1}{2}x^2 + x + C$$

$$\rightarrow \ \boxed{y(x) = e^{x^2/2+x+C}}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$y' = ye^x - 2e^x + y - 2$

Solution

$$\frac{dy}{dx} = (y-2)e^x + y - 2$$

$$\frac{dy}{dx} = (y-2)\left(e^x + 1\right)$$

$$\frac{dy}{y-2} = \left(e^x + 1\right)dx$$

$$\int \frac{dy}{y-2} = \int \left(e^x + 1\right)dx$$

$$\ln|y-2| = e^x + x + C$$

$$y - 2 = \pm e^{e^x + x + C}$$

$$y - 2 = \pm e^C e^{e^x + x}$$

$$\boxed{y(x) = De^{e^x + x} + 2} \quad D = \pm e^C$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \dfrac{x}{y+2}$

Solution

$$\frac{dy}{dx} = \frac{x}{y+2}$$

$$(y+2)dy = xdx$$

$$\int (y+2)dy = \int xdx$$

$$\tfrac{1}{2}y^2 + 2y = \tfrac{1}{2}x^2 + C$$

$$y^2 + 4y = x^2 + 2C$$

$$y^2 + 4y - x^2 - D = 0, \quad (D = 2C)$$

$$y = \frac{-4 \pm \sqrt{16 - 4\left(-x^2 - D\right)}}{2} = \frac{-4 \pm \sqrt{16 + 4x^2 + 4D}}{2} = \frac{-4 \pm 2\sqrt{x^2 + (4+D)}}{2} \underline{= -2 \pm \sqrt{x^2 + E}} \qquad E = 4 + D$$

$$\boxed{y(x) = -2 \pm \sqrt{x^2 + E}}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \dfrac{xy}{x-1}$

Solution

$$\frac{dy}{dx} = y\left(\frac{x}{x-1}\right)$$

$$\frac{dy}{y} = \left(\frac{x}{x-1}\right)dx$$

$$\int \frac{dy}{y} = \int \left(1 + \frac{1}{x-1}\right)dx$$

$$\ln|y| = x + \ln|x-1| + C$$

$$y(x) = \pm e^{x+\ln|x-1|+C}$$

$$= \pm e^{C} e^{x} e^{\ln|x-1|}$$

$$\boxed{= De^{x}|x-1|}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$y' = \frac{y^2 + ty + t^2}{t^2}$$

Solution

$$y' = \frac{y^2}{t^2} + \frac{y}{t} + 1 \qquad \text{Let } x = \frac{y}{t} \Rightarrow y = xt \to y' = x + tx'$$

$$y' = \frac{y^2}{t^2} + \frac{y}{t} + 1 = x^2 + x + 1$$

$$x + tx' = x^2 + x + 1$$

$$t\frac{dx}{dt} = x^2 + 1$$

$$\int \frac{dx}{x^2+1} = \int \frac{dt}{t}$$

$$\tan^{-1} x = \ln|t| + C$$

$$\tan^{-1}\frac{y}{t} = \ln|t| + C$$

$$\frac{y}{t} = \tan\left(\ln|t| + C\right)$$

$$\boxed{y(t) = t\tan\left(\ln|t| + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$$

Solution

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$$

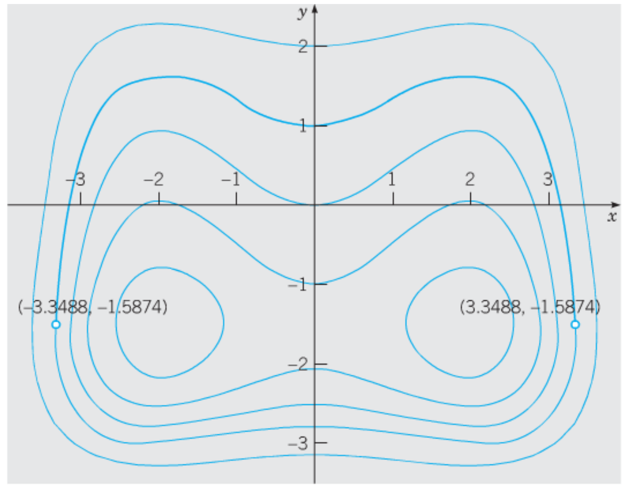
$$\left(4 + y^3\right)dy = \left(4x - x^3\right)dx$$

$$\int \left(4 + y^3\right)dy = \int \left(4x - x^3\right)dx$$

$$4y + \frac{1}{4}y^4 = 2x^2 - \frac{1}{4}x^4 + C_1$$

$$16y + y^4 = 8x^2 - x^4 + C$$

$$\boxed{y^4 + 16y + x^4 - 8x^2 = +C}$$



Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$y' = \frac{2xy + 2x}{x^2 - 1}$$

Solution

$$\frac{dy}{dx} = \frac{2x(y+1)}{x^2 - 1}$$

$$\frac{dy}{y+1} = \frac{2x}{x^2 - 1}dx \qquad d\left(x^2 - 1\right) = 2xdx$$

$$\int \frac{d(y+1)}{y+1} = \int \frac{d\left(x^2 - 1\right)}{x^2 - 1}$$

$$\ln|y+1| = \ln\left|x^2 - 1\right| + C$$

$$y + 1 = e^{\ln\left|x^2-1\right| + C}$$

$$y = e^C e^{\ln\left|x^2-1\right|} - 1$$

$$\boxed{y(x) = Ae^{\ln\left|x^2-1\right|} - 1}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx}=\sin 5x$

Solution

$$\int dy=\int \sin 5x dx$$

$$\boxed{y(x)=-\frac{1}{5}\cos 5x+C}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx}=(x+1)^2$

Solution

$$\int dy=\int \left(x^2+2x+1\right)dx$$

$$\boxed{y(x)=\frac{1}{3}x^3+x^2+x+C}$$

Exercise

Find the general solution of the differential equation $dx+e^{3x}dy=0$

Solution

$$\int dy=-\int e^{-3x}dx$$

$$\boxed{y(x)=\frac{1}{3}e^{-3x}+C}$$

Exercise

Find the general solution of the differential equation $dy-(y-1)^2\,dx=0$

Solution

$$\int \frac{dy}{(y-1)^2}=\int dx$$

$$\int \frac{d(y-1)}{(y-1)^2}=\int dx$$

$$-\frac{1}{y-1}=x+C$$

$$\boxed{y(x)=1-\frac{1}{x+C}}$$

Exercise

Find the general solution of the differential equation $x\dfrac{dy}{dx} = 4y$

Solution

$$\int \frac{dy}{y} = 4\int \frac{dx}{x}$$

$$\ln y = 4\ln x + \ln C$$

$$\ln y = \ln Cx^4$$

$$\underline{y(x) = Cx^4}$$

Exercise

Find the general solution of the differential equation $\dfrac{dx}{dy} = y^2 - 1$

Solution

$$\int dx = \int \left(y^2 - 1\right) dy$$

$$\underline{x = \frac{1}{3}y^3 - y + C}$$

Exercise

Find the general solution of the differential equation $\dfrac{dy}{dx} = e^{2y}$

Solution

$$\int e^{-2y}\,dy = \int dx$$

$$-\frac{1}{2}e^{-2y} = x + C$$

$$e^{-2y} = -2x + C_1$$

$$-2y = \ln\left(C_1 - 2x\right)$$

$$\underline{y(x) = -\frac{1}{2}\ln\left(C_1 - 2x\right)}$$

Exercise

Find the general solution of the differential equation $\dfrac{dy}{dx} + 2xy^2 = 0$

Solution

$$\frac{dy}{dx} = -2xy^2$$

$$-\int \frac{dy}{y^2} = \int 2x\,dx$$

$$\frac{1}{y} = x^2 + C$$

$$\boxed{y(x) = \frac{1}{x^2 + C}}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = e^{3x+2y}$

Solution

$$\frac{dy}{dx} = e^{3x}e^{2y}$$

$$\int e^{-2y}\,dy = \int e^{3x}\,dx$$

$$-\frac{1}{2}e^{-2y} = \frac{1}{3}e^{3x} + C$$

$$e^{-2y} = C_1 - \frac{2}{3}e^{3x}$$

$$-2y = \ln\left(C_1 - \frac{2}{3}e^{3x}\right)$$

$$\boxed{y(x) = -\frac{1}{2}\ln\left(C_1 - \frac{2}{3}e^{3x}\right)}$$

Exercise

Find the general solution of the differential equation $e^x y\frac{dy}{dx} = e^{-y} + e^{-2x-y}$

Solution

$$e^x y\frac{dy}{dx} = e^{-y}\left(1 + e^{-2x}\right)$$

$$ye^y\,dy = e^{-x}\left(1 + e^{-2x}\right)dx$$

$$\int ye^y\,dy = \int\left(e^{-x} + e^{-3x}\right)dx$$

		$\int e^y$
+	y	e^y
−	1	e^y

$$\boxed{(y-1)e^y = -e^{-x} - \frac{1}{3}e^{-3x} + C}$$

Exercise

Find the general solution of the differential equation $y\ln x\frac{dx}{dy} = \left(\frac{y+1}{x}\right)^2$

Solution

$$x^2 \ln x dx = \frac{1}{y}\left(y^2 + 2y + 1\right)dy$$

$$\int x^2 \ln x dx = \int \left(y + 2 + \frac{1}{y}\right)dy$$

$$u = \ln x \qquad dv = x^2 dx$$

$$du = \frac{dx}{x} \qquad v = \frac{1}{3}x^3$$

$$\frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx = \frac{1}{2}y^2 + 2y + \ln|y| + C$$

$$\boxed{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 = \frac{1}{2}y^2 + 2y + \ln|y| + C}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = \left(\frac{2y+3}{4x+5}\right)^2$

Solution

$$\int \frac{dy}{(2y+3)^2} = \int \frac{dx}{(4x+5)^2}$$

$$\frac{1}{2}\int \frac{d(2y+3)}{(2y+3)^2} = \frac{1}{4}\int \frac{d(4x+5)}{(4x+5)^2}$$

$$\frac{1}{2}\frac{-1}{2y+3} = \frac{1}{4}\frac{-1}{4x+5} + C$$

$$\boxed{\frac{2}{2y+3} = \frac{1}{4x+5} + C}$$

Exercise

Find the general solution of the differential equation $\csc y dx + \sec^2 x dy = 0$

Solution

$$\csc y dx = -\sec^2 x dy$$

$$\frac{dy}{\csc y} = -\frac{dx}{\sec^2 x}$$

$$\sin y dy = -\cos^2 x dx$$

$$\int \sin y \, dy = -\frac{1}{2}\int (1 + \cos 2x)dx$$

$$-\cos y = -\frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$

$$\boxed{\cos y = \frac{1}{2}x + \frac{1}{4}\sin 2x + C}$$

Exercise

Find the general solution of the differential equation $\sin 3x dx + 2y\cos^3 3x dy = 0$

Solution

$$\sin 3x dx = -2y\cos^3 3x dy$$

$$\int \frac{\sin 3x}{\cos^3 3x} dx = -\int 2y dy$$

$$-\frac{1}{3}\int \cos^{-3} 3x \, d(\cos 3x) = -\int 2y dy$$

$$-\frac{1}{6}\cos^{-2} 3x + C = y^2$$

$$\boxed{y^2 = -\frac{1}{6}\sec^2 3x + C}$$

Exercise

Find the general solution of the differential equation $\left(e^y + 1\right)^2 e^{-y} dx + \left(e^x + 1\right)^3 e^{-x} dy = 0$

Solution

$$\left(e^y + 1\right)^2 e^{-y} dx = -\left(e^x + 1\right)^3 e^{-x} dy$$

$$\frac{e^x}{\left(e^x + 1\right)^3} dx = -\frac{e^y}{\left(e^y + 1\right)^2} dy$$

$$\int \left(e^x + 1\right)^{-3} d\left(e^x + 1\right) = -\int \frac{1}{\left(e^y + 1\right)^2} d\left(e^y + 1\right)$$

$$\boxed{-\frac{1}{2}\frac{1}{\left(e^x + 1\right)^2} + C = \frac{1}{e^y + 1}}$$

Exercise

Find the general solution of the differential equation $x\left(1 + y^2\right)^{1/2} dx = y\left(1 + x^2\right)^{1/2} dy$

Solution

$$\int x\left(1 + x^2\right)^{-1/2} dx = \int y\left(1 + y^2\right)^{-1/2} dy$$

$$\frac{1}{2}\int\left(1+x^2\right)^{-1/2}d\left(1+x^2\right)=\frac{1}{2}\int\left(1+y^2\right)^{-1/2}d\left(1+y^2\right)$$

$$2\left(1+y^2\right)^{1/2}=2\left(1+x^2\right)^{1/2}+C$$

$$\boxed{\left(1+y^2\right)^{1/2}=\left(1+x^2\right)^{1/2}+C}$$

Exercise

Find the general solution of the differential equation. $\dfrac{dy}{dx}=y\sin x$

Solution

$$\int\frac{dy}{y}=\int\sin x\,dx$$

$$\ln|y|=-\cos x+C$$

$$y=e^{-\cos x+C}$$

$$\boxed{=Ae^{-\cos x}}$$

Exercise

Find the general solution of the differential equation. $(1+x)\dfrac{dy}{dx}=4y$

Solution

$$\int\frac{dy}{y}=\int\frac{4}{1+x}dx$$

$$\ln|y|=4\ln|1+x|+\ln C$$

$$=\ln(1+x)^4+\ln C$$

$$=\ln C(1+x)^4$$

$$\boxed{y(x)=C(1+x)^4}$$

Exercise

Find the general solution of the differential equation. $2\sqrt{x}\dfrac{dy}{dx}=\sqrt{1-y^2}$

Solution

$$\int\frac{dy}{\sqrt{1-y^2}}=\frac{1}{2}\int x^{-1/2}\,dx$$

$$\boxed{\arcsin y=\sqrt{x}+C}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = 3\sqrt{xy}$

Solution

$$\int y^{1/2} dy = 3\int x^{1/2} dx$$

$$\frac{2}{3} y^{3/2} = 2x^{3/2} + C$$

$$\boxed{y^{3/2} = 3x^{3/2} + C_1}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = (64xy)^{1/3}$

Solution

$$\int y^{-1/3} dy = \int 4x^{1/3} dx$$

$$\frac{3}{2} y^{2/3} = 3x^{4/3} + C_1$$

$$\boxed{y^{2/3} = 2x^{4/3} + C}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = 2x \sec y$

Solution

$$\int \cos y \; dy = \int 2x \; dx$$

$$\boxed{\sin y = x^2 + C}$$

Exercise

Find the general solution of the differential equation. $\left(1 - x^2\right)\frac{dy}{dx} = 2y$

Solution

$$\int \frac{dy}{y} = 2\int \frac{1}{1 - x^2} dx$$

$$\int \frac{dy}{y} = \int \left(\frac{1}{1 + x} + \frac{1}{1 - x}\right) dx$$

$$\ln|y| = \ln|1 + x| - \ln|1 - x| + \ln C$$

$$\ln|y| = \ln C\left|\frac{1+x}{1-x}\right|$$

$$\boxed{y(x) = C\frac{1+x}{1-x}}$$

Exercise

Find the general solution of the differential equation. $(1+x)^2\frac{dy}{dx} = (1+y)^2$

Solution

$$\int\frac{dy}{(1+y)^2} = \int\frac{1}{(1+x)^2}dx$$

$$-\frac{1}{1+y} = -\frac{1}{1+x} + C$$

$$\frac{1}{1+y} = \frac{1+C+Cx}{1+x}$$

$$y+1 = \frac{1+x}{C_1+Cx}$$

$$y = \frac{1+x}{C_1+Cx} - 1$$

$$= \frac{1+x-C_1-Cx}{C_1+Cx} \qquad A = 1-C_1 \quad B = 1-C$$

$$\boxed{= \frac{A+Bx}{C_1+Cx}}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = xy^3$

Solution

$$\int y^{-3}dy = \int xdx$$

$$-\frac{1}{2y^2} = \frac{1}{2}x^2 + C_1$$

$$\frac{1}{y^2} = -x^2 + C$$

$$\boxed{y^2 = \frac{1}{-x^2+C}}$$

Exercise

Find the general solution of the differential equation. $y\frac{dy}{dx} = x\left(y^2+1\right)$

Solution

$$\int \frac{y}{y^2+1}dy = \int xdx$$

$$\frac{1}{2}\int \frac{1}{y^2+1}d\left(y^2+1\right) = \frac{1}{2}x^2 + C$$

$$\ln\left(y^2+1\right) = x^2 + C$$

$$y^2+1 = e^{x^2+C}$$

$$\boxed{y^2 = Ae^{x^2} - 1}$$

Exercise

Find the general solution of the differential equation. $y^3\frac{dy}{dx} = \left(y^4+1\right)\cos x$

Solution

$$\int \frac{y^3}{y^4+1}dy = \int \cos x dx$$

$$\frac{1}{4}\ln\left(y^4+1\right) = \sin x + C$$

$$\ln\left(y^4+1\right) = 4\sin x + C$$

$$y^4+1 = e^{4\sin x + C}$$

$$\boxed{y^4 = A\,e^{4\sin x} - 1}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = \frac{1+\sqrt{x}}{1+\sqrt{y}}$

Solution

$$\int \left(1+y^{1/2}\right)dy = \int \left(1+x^{1/2}\right)dx$$

$$\boxed{y + \frac{2}{3}y^{3/2} = x + \frac{2}{3}x^{3/2} + C}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx}=\frac{(x-1)y^5}{x^2\left(2y^3-y\right)}$

Solution

$$\left(\frac{2y^3-y}{y^5}\right)dy=\left(\frac{x-1}{x^2}\right)dx$$

$$\int\left(2\frac{1}{y^2}-\frac{1}{y^4}\right)dy=\int\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$$

$$-\frac{2}{y}+\frac{1}{3y^3}=\ln|x|+\frac{1}{x}+C$$

$$\boxed{\frac{1-6y^2}{3y^3}=\ln|x|+\frac{1}{x}+C}$$

Exercise

Find the general solution of the differential equation. $\left(x^2+1\right)(\tan y)y'=x$

Solution

$$\int\tan y\ dy=\int\frac{x}{x^2+1}dx$$

$$\ln|\sec y|=\frac{1}{2}\ln\left(x^2+1\right)+\ln C$$

$$=\ln C\sqrt{x^2+1}$$

$$\boxed{\sec y=C\sqrt{x^2+1}}$$

Exercise

Find the general solution of the differential equation. $x^2y'=1-x^2+y^2-x^2y^2$

Solution

$$x^2y'=1-x^2+\left(1-x^2\right)y^2$$

$$x^2y'=\left(1-x^2\right)\left(1+y^2\right)$$

$$\int\frac{1}{1+y^2}dy=\int\frac{1-x^2}{x^2}dx$$

$$\int \frac{1}{1+y^2}\,dy = \int\left(\frac{1}{x^2}-1\right)\,dx$$

$$\boxed{\arctan y = -\frac{1}{x} - x + C}$$

Exercise

Find the general solution of the differential equation. $\quad xy' + 4y = 0$

Solution

$$x\frac{dy}{dx} = -4y$$

$$\int \frac{dy}{y} = -4\int \frac{dx}{x}$$

$$\ln|y| = -4\ln|x| + C$$

$$\ln|y| = \ln x^{-4} + C$$

$$y(x) = e^{\ln x^{-4}+C}$$

$$= e^C e^{\ln x^{-4}}$$

$$\boxed{= Ax^{-4}}$$

Exercise

Find the general solution of the differential equation. $\quad \left(x^2+1\right)y' + 2xy = 0$

Solution

$$\left(x^2+1\right)\frac{dy}{dx} = -2xy$$

$$\int \frac{dy}{y} = -\int \frac{2x}{x^2+1}\,dx$$

$$\ln|y| = -\ln\left(x^2+1\right) + \ln C$$

$$\ln|y| = \ln\frac{C}{x^2+1}$$

$$\boxed{y(x) = \frac{C}{x^2+1}}$$

Exercise

Find the general solution of the differential equation. $\dfrac{y'}{\left(x^2+1\right)y}=3$

Solution

$$\int \frac{1}{y}dy=\int \left(3x^2+3\right)dx$$

$$\ln|y|=x^3+3x+C$$

$$\boxed{y(x)=e^{x^3+3x+C}}$$

Exercise

Find the general solution of the differential equation. $y+e^x y'=0$

Solution

$$e^x\frac{dy}{dx}=-y$$

$$\int \frac{dy}{y}=-\int e^{-x}dx$$

$$\ln|y|=e^{-x}+C$$

$$\boxed{y(x)=e^{e^{-x}+C}}$$

Exercise

Find the general solution of the differential equation. $\dfrac{dx}{dt}=3xt^2$

Solution

$$\int \frac{dx}{x}=\int 3t^2dt$$

$$\ln|x|=t^3+C$$

$$x(t)=e^{t^3+C}\ \boxed{=Ae^{t^3}}$$

Exercise

Find the general solution of the differential equation. $x\dfrac{dy}{dx}=\dfrac{1}{y^3}$

Solution

$$\int y^3dy=\int \frac{1}{x}dx$$

$$\frac{1}{4}y^4 = \ln|x| + C_1$$

$$y^4 = 4\ln|x| + C$$

$$\boxed{y^4 = \ln x^4 + C}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = \frac{x}{y^2\sqrt{x+1}}$

Solution

$$\int y^2 dy = \int \frac{x}{\sqrt{x+1}} dx$$

Let $u = x+1 \quad \rightarrow x = u-1 \quad \rightarrow du = dx$

$$\frac{1}{3}y^3 = \int \frac{u-1}{u^{1/2}} du$$

$$\frac{1}{3}y^3 = \int \left(u^{1/2} - u^{-1/2}\right) du$$

$$\frac{1}{3}y^3 = \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C_1$$

$$\boxed{y^3 = 2(x+1)^{3/2} - 6(x+1)^{1/2} + C}$$

Exercise

Find the general solution of the differential equation. $\frac{dx}{dt} - x^3 = x$

Solution

$$\frac{dx}{dt} = x^3 + x$$

$$\int \frac{dx}{x\left(x^2+1\right)} = \int dt$$

$$\frac{1}{x\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$Ax^2 + A + Bx^2 + Cx = 1$$

$$\begin{cases} x^2 & A + B = 0 \\ x & \boxed{C = 0} \quad \rightarrow \boxed{B = -1} \\ x^0 & \boxed{A = 1} \end{cases}$$

$$\int \frac{dx}{x} - \int \frac{dx}{x^2+1} = t + K$$

$$\boxed{\ln|x| - \arctan x = t + K}$$

Exercise

Find the general solution of the differential equation. $\dfrac{dy}{dx}=\dfrac{x}{ye^{x+2y}}$

Solution

$$\frac{dy}{dx}=\frac{x}{ye^{2y}e^{x}}$$

$$\int ye^{2y}dy=\int xe^{-x}dx$$

$$\frac{1}{2}ye^{2y}-\frac{1}{4}e^{2y}=-xe^{-x}-e^{-x}+C_1$$

$$\boxed{(2y-1)e^{2y}=-4(x+1)e^{-x}+C}$$

		$\int e^{2y}$
+	y	$\frac{1}{2}e^{2y}$
−	1	$\frac{1}{4}e^{2y}$

		$\int e^{-x}$
+	x	$-e^{-x}$
−	1	e^{-x}

Exercise

Find the general solution of the differential equation. $\dfrac{dy}{dx}=\dfrac{\sec^2 y}{1+x^2}$

Solution

$$\int \cos^2 y\ dy=\int\frac{dx}{1+x^2}$$

$$\frac{1}{2}\int(1+\cos 2y)dy=\arctan x+C$$

$$\boxed{\frac{1}{2}\left(y+\frac{1}{2}\sin 2y\right)=\arctan x+C}$$

Exercise

Find the general solution of the differential equation. $x\dfrac{dv}{dx}=\dfrac{1-4v^2}{3v}$

Solution

$$\int\frac{3v}{1-4v^2}dv=\int\frac{dx}{x}$$

$$-\frac{3}{8}\int\frac{1}{1-4v^2}d\left(1-4v^2\right)=\int\frac{dx}{x}$$

$$-\frac{3}{8}\ln\left|1-4v^2\right|=\ln|x|+\ln C$$

$$\ln\left(\left|1-4v^2\right|\right)^{-3/8}=\ln|Cx|$$

$$\boxed{\left(1-4v^2\right)^{-3/8}=Cx}$$

Exercise

Find the general solution of the differential equation. $\dfrac{dy}{dx}=3x^2\left(1+y^2\right)^{3/2}$

Solution

$$\int\left(1+y^2\right)^{-3/2}dy=\int 3x^2dx$$

$y=\tan\theta \qquad \sqrt{1+y^2}=\sec\theta$

$dy=\sec^2\theta\ d\theta$

$$\int\sec^{-3}\theta\ \sec^2\theta\ d\theta=x^3+C$$

$$\int\sec\theta\ d\theta=x^3+C$$

$$\ln\left|\sec\theta+\tan\theta\right|=x^3+C$$

$$\boxed{\frac{1}{\sqrt{1+y^2}}+y=C_1e^{x^3}}$$

Exercise

Find the general solution of the differential equation. $\dfrac{1}{y}dy+ye^{\cos x}\sin xdx=0$

Solution

$$\int\frac{1}{y^2}dy=-\int e^{\cos x}\sin xdx$$

$$-\frac{1}{y}=e^{\cos x}+C$$

$$\boxed{y(x)=\frac{-1}{e^{\cos x}+C}}$$

Exercise

Find the general solution of the differential equation. $\left(x+xy^2\right)dx+e^{x^2}ydy=0$

Solution

$$x\left(1+y^2\right)dx=-e^{x^2}ydy$$

$$\int xe^{-x^2}dx=-\int\frac{y}{1+y^2}dy$$

$$-\frac{1}{2}\int e^{-x^2}d\left(e^{-x^2}\right)=-\frac{1}{2}\int\frac{1}{1+y^2}d\left(1+y^2\right)$$

$$\boxed{e^{-x^2}=\ln\left(1+y^2\right)+C}$$

Exercise

Find the exact solution of the initial value problem. $y' = \frac{y}{x}, \quad y(1) = -2$

Solution

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln|y| = \ln|x| + C$$

$$\begin{aligned} y &= \pm e^{\ln|x| + C} \\ &= \pm e^{C} e^{\ln|x|} \\ &= D|x| \\ &= Dx \end{aligned}$$

$$y = Dx \quad \Rightarrow \; D = \frac{y}{x} = \frac{-2}{1} = -2$$

$$\boxed{y(x) = -2x}$$

Exercise

Find the exact solution of the initial value problem. $y' = -\frac{2t\left(1+y^2\right)}{y}, \quad y(0) = 1$

Solution

$$\frac{dy}{dt} = -\frac{2t\left(1+y^2\right)}{y}$$

$$\int \frac{ydy}{1+y^2} = \int -2tdt$$

$$\frac{1}{2}\int \frac{1}{1+y^2} d\left(1+y^2\right) = -2\int tdt$$

$$\frac{1}{2}\ln\left(1+y^2\right) = -t^2 + C$$

$$\ln\left(1+y^2\right) = -2t^2 + 2C$$

$$1 + y^2 = e^{-2t^2 + 2C}$$

$$1 + y^2 = e^{2C} e^{-2t^2}$$

$$1+y^2=De^{-2t^2}$$

$$1+1^2=De^{-2(0)^2} \;\to\; \underline{2=D}$$

$$y^2=2e^{-2t^2}-1$$

$$y^2=2e^{-2t^2}-1$$

$$y=\pm\sqrt{2e^{-2t^2}-1}$$

$$\boxed{y(x)=\sqrt{2e^{-2t^2}-1}}$$

$$2e^{-2t^2}-1>0$$

$$2e^{-2t^2}>1$$

$$e^{-2t^2}>\frac{1}{2}$$

$$-2t^2>\ln\left(\frac{1}{2}\right)$$

$$t^2<-\frac{1}{2}\ln\left(\frac{1}{2}\right)=\frac{1}{2}\ln 2$$

$$t^2<\ln\sqrt{2}$$

$$t<\left|\ln\sqrt{2}\right|$$

The interval of existence: $\left(-\ln\sqrt{2},\ \ln\sqrt{2}\right)$

Exercise

Find the exact solution of the initial value problem. Indicate the interval of existence.

$$y'=\frac{\sin x}{y}, \qquad y\left(\frac{\pi}{2}\right)=1$$

Solution

$$\frac{dy}{dx}=\frac{\sin x}{y}$$

$$ydy=\sin xdx$$

$$\int ydy=\int \sin xdx$$

$$\frac{1}{2}y^2=-\cos x+C_1$$

$$y^2=-2\cos x+C \qquad \left(C=2C_1\right)$$

$$y(x)=\pm\sqrt{-2\cos x+C}$$

$$y\left(\frac{\pi}{2}\right)=\sqrt{-2\cos\frac{\pi}{2}+C}$$

$1 = \sqrt{C} \quad \Rightarrow \quad \boxed{C = 1}$

$\boxed{y(x) = \sqrt{1 - 2\cos x}}$

The interval of existence will be the interval containing $\frac{\pi}{2}$ and $1 - 2\cos x > 0$

$\cos x < \frac{1}{2} \quad \Rightarrow \quad \boxed{\frac{\pi}{3} < x < \frac{5\pi}{3}}$

Exercise

Find the exact solution of the initial value problem. $4tdy = \left(y^2 + ty^2\right)dt, \quad y(1) = 1$

Solution

$4tdy = y^2(1+t)dt$

$4\int \frac{dy}{y^2} = \int \left(\frac{1}{t} + 1\right)dt \qquad\qquad \int \frac{dx}{x^2} = -\frac{1}{x}$

$-\frac{4}{y} = \ln|t| + t + C$

$y = \frac{-4}{\ln|t| + t + C}$

$1 = \frac{-4}{\ln|1| + 1 + C} \quad \Rightarrow \quad 1 = \frac{-4}{1 + C} \quad \to \quad 1 + C = -4 \quad \Rightarrow \quad \boxed{C = -5}$

$\boxed{y = \frac{-4}{\ln|t| + t - 5}}$

Exercise

Find the exact solution of the initial value problem. $y' = \frac{1 - 2t}{y}, \quad y(1) = -2$

Solution

$y\frac{dy}{dt} = 1 - 2t$

$\int ydy = \int (1 - 2t)dt$

$\frac{1}{2}y^2 = t - t^2 + C_1$

$y^2 = 2t - 2t^2 + C$

$(-2)^2 = 2(1) - 2(1)^2 + C \quad \Rightarrow \quad \boxed{C = 4}$

$\boxed{y = -\sqrt{2t - 2t^2 + 4}}$

The negative value is taken to satisfy the initial condition.

Exercise

Find the exact solution of the initial value problem. $y' = y^2 - 4, \quad y(0) = 0$

Solution

$$\frac{dy}{dt} = y^2 - 4$$

$$\frac{dy}{y^2 - 4} = dt \qquad \frac{1}{y^2-4} = \frac{A}{y-2} + \frac{B}{y+2}$$

$$\frac{1}{y^2 - 4} = \frac{(A+B)y + 2A - 2B}{y - 2}$$

$$\rightarrow \begin{cases} A + B = 0 \\ 2A - 2B = 1 \end{cases} \Rightarrow A = \frac{1}{4} \quad B = -\frac{1}{4}$$

$$\left(\frac{1}{4(y-2)} - \frac{1}{4(y+2)}\right)dy = dt$$

$$\int\left(\frac{1}{4(y-2)} - \frac{1}{4(y+2)}\right)dy = \int dt$$

$$\frac{1}{4}\left(\ln|y-2| - \ln|y+2|\right) = t + C$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4t + C$$

$$\frac{y-2}{y+2} = \pm e^{4t+C}$$

$$\frac{y-2}{y+2} = \pm e^{C} e^{4t} = ke^{4t}$$

$$y - 2 = ke^{4t} y + 2ke^{4t}$$

$$y - ke^{4t} y = 2 + 2ke^{4t}$$

$$y\left(1 - ke^{4t}\right) = 2 + 2ke^{4t}$$

$$y = \frac{2 + 2ke^{4t}}{1 - ke^{4t}}$$

$$0 = \frac{2 + 2ke^{4(0)}}{1 - ke^{4(0)}}$$

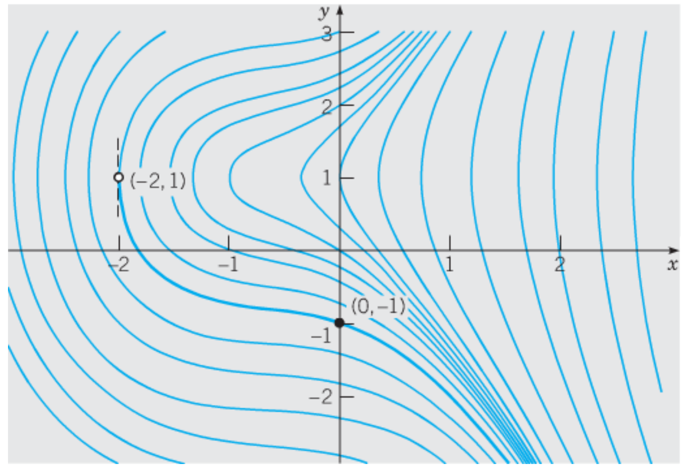
$$0 = 2 + 2k \quad \Rightarrow \quad \boxed{k = -1}$$

$$y = \frac{2 - 2e^{4t}}{1 + e^{4t}}$$

Exercise

Find the exact solution of the initial value problem. Indicate the interval of existence.

$$\frac{dy}{dx}=\frac{3x^2+4x+2}{2(y-1)}, \quad y(0)=-1$$



Solution

$$\frac{dy}{dx}=\frac{3x^2+4x+2}{2y-2}$$

$$(2y-2)dy=\left(3x^2+4x+2\right)dx$$

$$\int(2y-2)dy=\int\left(3x^2+4x+2\right)dx$$

$$y^2-2y=x^3+2x^2+2x+C$$

$$y(0)=-1$$

$$(-1)^2-2(-1)=(0)^3+2(0)^2+2(0)+C$$

$$\Rightarrow \boxed{C=3}$$

$$y^2-2y=x^3+2x^2+2x+3$$

Exercise

Find the exact solution of the initial value problem. $y'=\frac{x}{1+2y}, \quad y(-1)=0$

Solution

$$\frac{dy}{dx}=\frac{x}{1+2y}$$

$$\int(1+2y)dy=\int xdx$$

$$y+y^2=\frac{1}{2}x^2+C \qquad y(-1)=0$$

$$0=\frac{1}{2}(-1)^2+C \quad \Rightarrow \quad C=-\frac{1}{2}$$

$$y+y^2=\frac{1}{2}x^2-\frac{1}{2}$$

Exercise

Find the exact solution of the initial value problem $\left(e^{2y}-y\right)\cos x\frac{dy}{dx}=e^y\sin 2x, \quad y(0)=0$

Solution

$$\frac{e^{2y}-y}{e^y}dy=\frac{2\sin x\cos x}{\cos x}dx$$

$$\int\left(e^{y}-ye^{-y}\right)dy=\int 2\sin x\,dx$$

$$e^{y}+ye^{-y}+e^{-y}=-2\cos x+C$$

$$y(0)=0 \quad 1+1=-2+C \quad \to \underline{C=4}$$

$$\boxed{e^{y}+ye^{-y}+e^{-y}=4-2\cos x}$$

		$\int e^{-y}dy$
+	y	$-e^{-y}$
−	1	e^{-y}

Exercise

Find the exact solution of the initial value problem $\dfrac{dy}{dx}=e^{-x^2}, \quad y(3)=5$

Solution

$$\int_3^x \frac{dy}{dt}dt=\int_3^x e^{-t^2}dt$$

$$y(x)-y(3)=\int_3^x e^{-t^2}dt$$

$$\boxed{y(x)=5+\int_3^x e^{-t^2}dt}$$

Exercise

Find the exact solution of the initial value problem. $\dfrac{dy}{dx}+2y=1, \quad y(0)=\dfrac{5}{2}$

Solution

$$\frac{dy}{dx}=1-2y$$

$$\frac{dy}{1-2y}=dx$$

$$-\frac{1}{2}\int\frac{d(1-2y)}{1-2y}=\int dx$$

$$-\frac{1}{2}\ln|1-2y|=x+C$$

$$\ln|1-2y|=-2x+C \qquad y(0)=\frac{5}{2}$$

$$\ln|1-5|=C \quad \to \quad \underline{C=\ln 4}$$

$$1-2y=e^{-2x+\ln 4}$$

$$1-2y=e^{-2x}e^{\ln 4}$$

$$\boxed{y=\frac{1}{2}-2e^{-2x}}$$

Exercise

Find the exact solution of the initial value problem. $\sqrt{1-y^2}\,dx-\sqrt{1-x^2}\,dy=0, \quad y(0)=\frac{\sqrt{3}}{2}$

Solution

$$\int\frac{dx}{\sqrt{1-x^2}}=\int\frac{dy}{\sqrt{1-y^2}} \qquad\qquad \int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}$$

$$\sin^{-1}x+C=\sin^{-1}y \qquad y(0)=\frac{\sqrt{3}}{2}$$

$$\sin^{-1}0+C=\sin^{-1}\frac{\sqrt{3}}{2} \quad\Rightarrow\quad \boxed{C=\frac{\pi}{3}}$$

$$\sin^{-1}y=\sin^{-1}x+\frac{\pi}{3}$$

$$y=\sin\left(\sin^{-1}x+\frac{\pi}{3}\right)$$

$$=\sin\left(\sin^{-1}x\right)\cos\frac{\pi}{3}+\cos\left(\sin^{-1}x\right)\sin\frac{\pi}{3} \qquad \alpha=\sin^{-1}x\to\sin\alpha=x \quad \cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-x^2}$$

$$\boxed{y(x)=\frac{x}{2}+\frac{\sqrt{3}}{2}\sqrt{1-x^2}}$$

Exercise

Find the exact solution of the initial value problem. $\left(1+x^4\right)dy+x\left(1+4y^2\right)dx=0, \quad y(1)=0$

Solution

$$\int\frac{1}{1+(2y)^2}dy=-\int\frac{x}{1+\left(x^2\right)^2}dx$$

$$\int\frac{1}{1+(2y)^2}dy=-\frac{1}{2}\int\frac{1}{1+\left(x^2\right)^2}d\left(x^2\right) \qquad\qquad \int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\frac{1}{2}\tan^{-1}2y=-\frac{1}{2}\tan^{-1}x^2+C$$

$$\tan^{-1}2y+\tan^{-1}x^2=C_1 \qquad y(1)=0 \quad \tan^{-1}0+\tan^{-1}1=C_1 \quad\Rightarrow \boxed{C_1=\frac{\pi}{4}}$$

$$\boxed{\tan^{-1}2y+\tan^{-1}x^2=\frac{\pi}{4}}$$

$$2y=\tan\left(\frac{\pi}{4}-\tan^{-1}x^2\right)$$

$$=\frac{\tan\frac{\pi}{4}-\tan\left(\tan^{-1}x^2\right)}{1+\tan\left(\frac{\pi}{4}\right)\tan\left(\tan^{-1}x^2\right)}$$

$$\boxed{y(x)=\frac{1}{2}\frac{1-x^2}{1+x^2}}$$

Exercise

Find the exact solution of the initial value problem. $e^{-2t}\frac{dy}{dt} = \frac{1+e^{-2t}}{y}, \quad y(0)=0$

Solution

$$ydy = \left(1+e^{-2t}\right)e^{2t}dt$$

$$\int ydy = \int \left(e^{2t}+1\right)dt$$

$$\frac{1}{2}y^2 = \frac{1}{2}e^{2t} + t + C_1$$

$$y^2 = e^{2t} + 2t + C \qquad y(0)=0$$

$$0 = 1 + C \quad \rightarrow \quad C = -1$$

$$\boxed{y^2 = e^{2t} + 2t - 1}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt} = \frac{t+2}{y}, \quad y(0)=2$

Solution

$$\int ydy = \int (t+2)dt$$

$$\frac{1}{2}y^2 = \frac{1}{2}t^2 + 2t + C_1$$

$$y^2 = t^2 + 4t + C \qquad y(0)=2 \qquad 4 = C$$

$$\boxed{y = \sqrt{t^2 + 4t + 4}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{1}{t^2}\frac{dy}{dt} = y, \quad y(0)=1$

Solution

$$\int \frac{1}{y}dy = \int t^2 dt$$

$$\ln|y| = \frac{1}{3}t^3 + C \qquad y(0)=1$$

$$\ln|1| = C \rightarrow C = 0$$

$$\ln|y| = \frac{1}{3}t^3$$

$$\boxed{y(t) = e^{t^3/3}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt} = -y^2 e^{2t}; \quad y(0) = 1$

Solution

$$-\int \frac{1}{y^2} dy = \int e^{2t} dt$$

$$\frac{1}{y} = \frac{1}{2}e^{2t} + C \qquad y(0) = 1 \qquad 1 = \frac{1}{2} + C \rightarrow \boxed{C = \frac{1}{2}}$$

$$\frac{1}{y} = \frac{1}{2}\left(e^{2t} + 1\right)$$

$$\boxed{y(t) = \frac{2}{e^{2t} + 1}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt} - (2t+1)y = 0; \quad y(0) = 2$

Solution

$$\frac{dy}{dt} = (2t+1)y$$

$$\int \frac{dy}{y} = \int (2t+1)dt$$

$$\ln|y| = t^2 + t + C \qquad y(0) = 2$$

$$\boxed{\ln 2 = C}$$

$$\ln|y| = t^2 + t + \ln 2$$

$$y(t) = e^{\ln 2} e^{t^2 + t}$$

$$\boxed{= 2e^{t^2 + t}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt} + 4ty^2 = 0; \quad y(0) = 1$

Solution

$$-\int \frac{dy}{y^2} = \int 4t dt$$

$$\frac{1}{y} = 2t^2 + C \qquad y(0) = 1 \qquad \boxed{1 = C}$$

$$\frac{1}{y} = 2t^2 + 1$$

$$\boxed{y(t) = \frac{1}{2t^2 + 1}}$$

Exercise

Find the exact solution of the initial value problem $\quad \frac{dy}{dx} = ye^x; \quad y(0) = 2e$

Solution

$$\int \frac{dy}{y} = \int e^x dx$$

$$\ln|y| = e^x + \ln C$$

$$y(0) = 2e \quad \rightarrow \quad \ln|2e| = 1 + \ln C$$

$$\ln 2 + 1 = 1 + \ln C \quad \Rightarrow \quad \boxed{C = 2}$$

$$\ln|y| = e^x + \ln 2$$

$$y(x) = e^{e^x + \ln 2}$$

$$= e^{e^x} e^{\ln 2}$$

$$\boxed{= 2e^{e^x}}$$

Exercise

Find the exact solution of the initial value problem $\quad \frac{dy}{dx} = 3x^2\left(y^2 + 1\right); \quad y(0) = 1$

Solution

$$\int \frac{1}{y^2 + 1} dy = \int 3x^2 dx$$

$$\arctan y = x^3 + C$$

$$y(0) = 1 \quad \rightarrow \quad \arctan 1 = C \quad \Rightarrow \boxed{C = \frac{\pi}{4}}$$

$$\boxed{y(x) = \tan\left(x^3 + \frac{\pi}{4}\right)}$$

Exercise

Find the exact solution of the initial value problem $\quad 2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2 - 16}}; \quad y(5) = 2$

Solution

$$\int 2y dy = \int \frac{x}{\sqrt{x^2 - 16}} dx$$

$$y^2 = \frac{1}{2}\int \left(x^2 - 16\right)^{-1/2} d\left(x^2 - 16\right)$$

$$y^2 = \left(x^2 - 16\right)^{1/2} + C$$

$$y(5) = 2 \quad \rightarrow 4 = (9)^{1/2} + C \quad \Rightarrow C = 4 - 3 \underline{= 1}$$

$$\underline{y^2 = 1 + \sqrt{x^2 - 16}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 4x^3 y - y; \quad y(1) = -3$

Solution

$$\frac{dy}{dx} = \left(4x^3 - 1\right)y$$

$$\int \frac{dy}{y} = \int \left(4x^3 - 1\right)dx$$

$$\ln|y| = x^4 - x + C$$

$$y = Ce^{x^4 - x}$$

$$y(1) = -3 \quad \underline{-3 = C}$$

$$\underline{y(x) = -3e^{x^4 - x}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} + 1 = 2y; \quad y(1) = 1$

Solution

$$\int \frac{dy}{2y - 1} = \int dx$$

$$\frac{1}{2}\ln(2y - 1) = x + C$$

$$\ln(2y - 1) = 2x + C$$

$$2y - 1 = e^{2x + C}$$

$$y(x) = Ae^{2x} + \frac{1}{2}$$

$$y(1) = 1 \quad 1 = Ae^2 + 1 \quad \rightarrow \quad \underline{A = e^{-2}}$$

$$\underline{y(x) = e^{2x - 2} + \frac{1}{2}}$$

Exercise

Find the exact solution of the initial value problem $(\tan x)\frac{dy}{dx} = y;\quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

Solution

$$\int \frac{dy}{y} = \int \frac{dx}{\tan x} = \int \frac{\cos x dx}{\sin x}$$

$$\ln y = \ln(\sin x) + \ln C$$

$$y(x) = C\sin x$$

$$y\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \quad \rightarrow \underline{\frac{\pi}{2} = C}$$

$$\underline{y(x) = \frac{\pi}{2}\sin x}$$

Exercise

Find the exact solution of the initial value problem $x\frac{dy}{dx} - y = 2x^2 y;\quad y(1) = 1$

Solution

$$x\frac{dy}{dx} = 2x^2 y + y$$

$$x\frac{dy}{dx} = \left(2x^2 + 1\right)y$$

$$\int \frac{dy}{y} = \int \left(2x + \frac{1}{x}\right)dx$$

$$\ln y = x^2 + \ln x + \ln C$$

$$y(x) = e^{x^2 + \ln x + \ln C}$$

$$= Cxe^{x^2}$$

$$y(1) = 1 \quad \rightarrow 1 = Ce \quad \Rightarrow \underline{C = e^{-1}}$$

$$\underline{y(x) = xe^{x^2 - 1}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 2xy^2 + 3x^2 y^2;\quad y(1) = -1$

Solution

$$\frac{dy}{dx} = \left(2x + 3x^2\right)y^2$$

$$\int \frac{dy}{y^2} = \int \left(2x + 3x^2\right)dx$$

$$-\frac{1}{y}=x^2+x^3+C$$

$$y(x)=\frac{-1}{x^2+x^3+C}$$

$$y(1)=-1 \quad \rightarrow \quad -1=\frac{-1}{C} \quad \Rightarrow \quad \underline{C=1}$$

$$\boxed{y(x)=\frac{-1}{x^2+x^3+1}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx}=6e^{2x-y}; \quad y(0)=0$

Solution

$$\int e^y dy=\int 6e^{2x}dx$$

$$e^y=3e^{2x}+C$$

$$y(x)=\ln\left(3e^{2x}+C\right)$$

$$y(0)=0 \quad \rightarrow \quad 0=\ln(3+C) \quad \Rightarrow 3+C=1 \rightarrow \underline{C=-2}$$

$$\boxed{y(x)=\ln\left(3e^{2x}-2\right)}$$

Exercise

Find the exact solution of the initial value problem $2\sqrt{x}\frac{dy}{dx}=\cos^2 y; \quad y(4)=\frac{\pi}{4}$

Solution

$$\frac{dy}{\cos^2 y}=\frac{1}{2}x^{-1/2}dx$$

$$\int \sec^2 y\ dy=\int \frac{1}{2}x^{-1/2}\ dx$$

$$\tan y=\sqrt{x}+C$$

$$y(x)=\tan^{-1}\left(\sqrt{x}+C\right)$$

$$y(4)=\frac{\pi}{4} \quad \rightarrow \quad \frac{\pi}{4}=\arctan(2+C) \quad \rightarrow 2+C=1 \Rightarrow \underline{C=-1}$$

$$\boxed{y(x)=\tan^{-1}\left(\sqrt{x}-1\right)}$$

Exercise

Find the exact solution of the initial value problem $\quad y' + 3y = 0; \quad y(0) = -3$

Solution

$$\frac{dy}{dx} = -3y$$

$$\int \frac{dy}{y} = -3\int dx$$

$$\ln|y| = -3x + C$$

$$y(x) = e^{-3x+C}$$

$$= Ae^{-3x}$$

$$y(0) = -3 \quad \to \quad \underline{-3 = A}$$

$$\boxed{y(x) = -3e^{-3x}}$$

Exercise

Find the exact solution of the initial value problem $\quad 2y' - y = 0; \quad y(-1) = 2$

Solution

$$2\frac{dy}{dx} = y$$

$$\int \frac{dy}{y} = \frac{1}{2}\int dx$$

$$\ln|y| = \frac{1}{2}x + C$$

$$y(x) = e^{x/2+C}$$

$$= Ae^{x/2}$$

$$y(-1) = 2 \quad \to \quad 2 = Ae^{-1/2} \quad \Rightarrow \quad \boxed{A = 2e^{1/2}}$$

$$y(x) = 2e^{1/2}e^{x/2} \quad \boxed{= 2e^{(x+1)/2}}$$

Exercise

Find the exact solution of the initial value problem $\quad 2xy - y' = 0; \quad y(1) = 3$

Solution

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{dy}{y} = \int 2x\, dx$$

$$\ln|y| = x^2 + C$$

$$y(x) = e^{x^2 + C}$$

$$= Ae^{x^2}$$

$$y(1) = 3 \quad \to \quad 3 = Ae \quad \Rightarrow \quad \boxed{A = 3e^{-1}}$$

$$\boxed{y(x) = \frac{3}{e}e^{x^2}}$$

Exercise

Find the exact solution of the initial value problem $\quad y\frac{dy}{dx} - \sin x = 0; \quad y\left(\frac{\pi}{2}\right) = -2$

Solution

$$\int y\ dy = \int \sin x\ dx$$

$$\frac{1}{2}y^2 = -\cos x + C$$

$$y\left(\frac{\pi}{2}\right) = -2 \quad \to \quad \boxed{2 = C}$$

$$\frac{1}{2}y^2 = -\cos x + 2$$

$$y^2 = 4 - 2\cos x$$

$$\boxed{y(x) = -\sqrt{4 - 2\cos x}} \quad \textit{Initial value is negative}$$

Exercise

Find the exact solution of the initial value problem $\quad \frac{dy}{dt} = \frac{1}{y^2}; \quad y(1) = 2$

Solution

$$\int y^2 dy = \int dt$$

$$\frac{1}{3}y^3 = t + C$$

$$y(1) = 2 \quad \to \quad \frac{8}{3} = 1 + C \quad \Rightarrow \quad \boxed{C = \frac{5}{3}}$$

$$\frac{1}{3}y^3 = t + \frac{5}{3}$$

$$y^3 = 3t + 5$$

$$\boxed{y(t) = (3t + 5)^{1/3}}$$

Exercise

Find the exact solution of the initial value problem $y' + \dfrac{1}{y+1} = 0; \quad y(1) = 0$

Solution

$$\frac{dy}{dx} = -\frac{1}{y+1}$$

$$\int (y+1)\,dy = -\int dx$$

$$\frac{1}{2}y^2 + y = -x + C$$

$$y(1) = 0 \quad \rightarrow \quad \underline{C = 1}$$

$$\frac{1}{2}y^2 + y = -x + 1$$

$$y^2 + 2y + 2(x-1) = 0 \quad \rightarrow \quad y = \frac{-2 \pm 2\sqrt{1 - 2x + 2}}{2}$$

$$\boxed{y(x) = -1 + \sqrt{3 - 2x}} \quad (\textit{initial condition} + \textit{sign})$$

Exercise

Find the exact solution of the initial value problem $y' + e^y t = e^y \sin t; \quad y(0) = 0$

Solution

$$\frac{dy}{dt} = (-t + \sin t)e^y$$

$$\int e^{-y}\,dy = \int (-t + \sin t)\,dt$$

$$-e^{-y} = -\frac{1}{2}t^2 - \cos t + C$$

$$e^{-y} = \frac{1}{2}t^2 + \cos t + C$$

$$y(0) = 0 \quad \rightarrow 1 = 1 + C \;\Rightarrow\; \underline{C = 0}$$

$$e^{-y} = \frac{1}{2}t^2 + \cos t$$

$$-y = \ln\left(\frac{1}{2}t^2 + \cos t\right)$$

$$\boxed{y(x) = -\ln\left(\frac{1}{2}t^2 + \cos t\right)}$$

Exercise

Find the exact solution of the initial value problem $y' - 2ty^2 = 0; \quad y(0) = -1$

Solution

$$\frac{dy}{dt} = 2ty^2$$

$$\int \frac{dy}{y^2} = \int 2t\ dt$$

$$-\frac{1}{y} = t^2 + C$$

$$y(0) = -1 \quad \Rightarrow \quad \boxed{C = 1}$$

$$-\frac{1}{y} = t^2 + 1$$

$$\boxed{y(x) = \frac{-1}{t+1}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 1 + y^2; \quad y\left(\frac{\pi}{4}\right) = -1$

Solution

$$\int \frac{dy}{1+y^2} = \int dx$$

$$\tan^{-1} y = x + C$$

$$y\left(\frac{\pi}{4}\right) = -1 \quad \rightarrow -\frac{\pi}{4} = \frac{\pi}{4} + C \ \Rightarrow \ \boxed{C = -\frac{\pi}{2}}$$

$$\tan^{-1} y = x - \frac{\pi}{2}$$

$$\boxed{y(x) = \tan\left(x - \frac{\pi}{2}\right)}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dt} = t - ty^2; \quad y(0) = \frac{1}{2}$

Solution

$$\frac{dy}{dt} = t\left(1 - y^2\right)$$

$$\int \frac{dy}{1-y^2} = \int t\ dt$$

$$\frac{1}{1-y^2} = \frac{A}{1-y} + \frac{B}{1+y}$$

$$1 = A + Ay + B - By$$

$$\begin{cases} A - B = 0 \\ A + B = 1 \end{cases} \quad \rightarrow \quad \boxed{A = \frac{1}{2} = B}$$

$$\int \left(\frac{1}{2}\frac{1}{1-y} + \frac{1}{2}\frac{1}{1+y}\right) dy = \int t\ dt$$

$$-\frac{1}{2}\ln|1-y|+\frac{1}{2}\ln|1+y|=\frac{1}{2}t+C$$

$$\ln|1+y|-\ln|1-y|=t+C$$

$$\ln\left|\frac{1+y}{1-y}\right|=t+C$$

$$\frac{1+y}{1-y}=Ae^t$$

$$y(0)=\frac{1}{2} \quad \to \frac{\frac{3}{2}}{\frac{1}{2}}=A \;\Rightarrow\; \underline{A=3}$$

$$\frac{1+y}{1-y}=3e^t$$

$$1+y=3e^t-3ye^t$$

$$y\left(1+3e^t\right)=3e^t-1$$

$$\boxed{y(x)=\frac{3e^t-1}{1+3e^t}}$$

Exercise

Find the exact solution of the initial value problem $3y^2\frac{dy}{dt}+2t=1;\quad y(-1)=-1$

Solution

$$\int 3y^2dy=\int(1-2t)\,dt$$

$$y^3=t-t^2+C$$

$$y(-1)=-1 \quad \to -1=-1-1+C \;\Rightarrow\; \underline{C=1}$$

$$y^3=t-t^2+1$$

$$\boxed{y(t)=\left(t-t^2+1\right)^{1/3}}$$

Exercise

Find the exact solution of the initial value problem $e^x y'+(\cos y)^2=0;\quad y(0)=\frac{\pi}{4}$

Solution

$$e^x\frac{dy}{dx}=-(\cos y)^2$$

$$\int \sec^2 y\;dy=-\int e^{-x}dx$$

$$\tan y=e^{-x}+C$$

$$y(0)=\frac{\pi}{4} \quad \to 1=1+C \Rightarrow \underline{C=0}$$

$$\tan y = e^{-x}$$

$$\boxed{y(x)=\arctan\left(e^{-x}\right)}$$

Exercise

Find the exact solution of the initial value problem $\quad (2y-\sin y)y'+x=\sin x; \quad y(0)=0$

Solution

$$(2y-\sin y)\frac{dy}{dx}=-x+\sin x$$

$$\int(2y-\sin y)dy=\int(-x+\sin x)dx$$

$$y^2+\cos y=-\frac{1}{2}x^2-\cos x+C$$

$$y(0)=0 \quad \to 1=-1+C \Rightarrow \underline{C=2}$$

$$\boxed{y^2+\cos y=-\frac{1}{2}x^2-\cos x+2}$$

Exercise

Find the exact solution of the initial value problem $\quad e^y y'+\frac{x}{y+1}=\frac{2}{y+1}; \quad y(1)=2$

Solution

$$e^y\frac{dy}{dx}=\frac{2-2x}{y+1}$$

$$\int(y+1)e^y dy=\int(2-2x)dx$$

$$ye^y=2x-x^2+C$$

$$y(1)=2 \quad \to 2e^2=2-1+C \Rightarrow \underline{C=2e^2-1}$$

$$\boxed{ye^y=2x-x^2+2e^2-1}$$

Exercise

Find the exact solution of the initial value problem $\quad (\ln y)y'+x=1; \quad y(3)=e$

Solution

$$(\ln y)\frac{dy}{dx}=1-x$$

$$\int (\ln y)\,dy = \int (1-x)\,dx \qquad \begin{matrix} u = \ln y & dv = dy \\ du = \frac{1}{y}dy & v = y \end{matrix}$$

$$y\ln y - \int dy = x - \frac{1}{2}x^2 + C$$

$$y\ln y - y = x - \frac{1}{2}x^2 + C$$

$$y(3) = e \quad \rightarrow \quad e\ln e = 3 - \frac{9}{2} + C \;\Rightarrow\; \boxed{C = e + \frac{3}{2}}$$

$$\boxed{y\ln y - y = x - \frac{1}{2}x^2 + e + \frac{3}{2}}$$

Exercise

Find the exact solution of the initial value problem $y' = x^3(1-y);\quad y(0) = 3$

Solution

$$\int \frac{dy}{1-y} = \int x^3 dx$$

$$-\ln|1-y| = \frac{1}{4}x^4 + C_1$$

$$\ln|1-y| = -\frac{1}{4}x^4 + C$$

$$y(0) = 3 \quad \rightarrow \quad \boxed{\ln 2 = C}$$

$$1 - y = e^{-\frac{1}{4}x^4 + C}$$

$$y = 1 - e^{\ln 2}e^{-\frac{1}{4}x^4}$$

$$\boxed{y(x) = 1 - 2e^{-x^4/4}}$$

Exercise

Find the exact solution of the initial value problem $y' = \left(1 + y^2\right)\tan x;\quad y(0) = \sqrt{3}$

Solution

$$\int \frac{1}{1+y^2}dy = \int \tan x\; dx$$

$$\tan^{-1} y = \ln|\sec x| + C$$

$$y(0) = \sqrt{3} \quad \rightarrow \quad \frac{\pi}{3} = \ln 1 + C \;\Rightarrow\; \boxed{C = \frac{\pi}{3}}$$

$$\boxed{y(x) = \tan\left(\ln|\sec x| + \frac{\pi}{3}\right)}$$

Exercise

Find the exact solution of the initial value problem $\frac{1}{2}\frac{dy}{dx}=\sqrt{1+y}\ \cos x;\quad y(\pi)=0$

Solution

$$\frac{1}{2}\int(1+y)^{-1/2}\,dy=\int\cos x\,dx$$

$$\sqrt{1+y}=\sin x+C$$

$$y(\pi)=0\quad \underline{1=C}$$

$$\sqrt{1+y}=\sin x+1$$

$$\underline{y(x)=(\sin x+1)^2-1}$$

Exercise

Find the exact solution of the initial value problem $x^2\frac{dy}{dx}=\frac{4x^2-x-2}{(x+1)(y+1)};\quad y(1)=1$

Solution

$$(y+1)\,dy=\frac{4x^2-x-2}{x^2(x+1)}dx$$

$$\frac{4x^2-x-2}{x^2(x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$$

$$4x^2-x-2=Ax^2+Ax+Bx+B+Cx^2$$

$$\begin{cases} x^2 & A+C=4 & \underline{C=3} \\ x & A+B=-1 & \underline{A=1} \\ x^0 & & \underline{B=-2} \end{cases}$$

$$\int(y+1)\,dy=\int\frac{dx}{x}-\int\frac{2}{x^2}dx+3\int\frac{dx}{x+1}$$

$$\frac{1}{2}y^2+y=\ln|x|+\frac{2}{x}+3\ln|x+1|+C$$

$$y(1)=1\quad \rightarrow \frac{1}{2}+1=2+3\ln 2+C\quad \Rightarrow\ \underline{C=-\frac{1}{2}-\ln 8}$$

$$\underline{\frac{1}{2}y^2+y=\ln|x|+\frac{2}{x}+3\ln|x+1|-\frac{1}{2}-\ln 8}$$

Exercise

Find the exact solution of the initial value problem $\frac{1}{\theta}\frac{dy}{d\theta}=\frac{y\sin\theta}{y^2+1}\quad y(\pi)=1$

Solution

$$\int \frac{y^2+1}{y}dy = \int \theta \sin\theta \, d\theta$$

$$\int \left(y+\frac{1}{y}\right)dy = -\theta\cos\theta + \sin\theta + C$$

$$\frac{1}{2}y^2 + \ln|y| = -\theta\cos\theta + \sin\theta + C$$

$$y(\pi) = 1 \quad \frac{1}{2} = \pi + C \;\Rightarrow\; \underline{C = \frac{1}{2} - \pi}$$

$$\underline{\frac{1}{2}y^2 + \ln|y| = -\theta\cos\theta + \sin\theta + \frac{1}{2} - \pi}$$

Exercise

Find the exact solution of the initial value problem $x^2dx + 2ydy = 0; \quad y(0) = 2$

Solution

$$\int 2ydy = -\int x^2dx$$

$$y^2 = -\frac{1}{3}x^3 + C$$

$$y(0) = 2 \quad \rightarrow \quad \underline{4 = C}$$

$$\underline{y^2 = -\frac{1}{3}x^3 + 4}$$

Exercise

Find the exact solution of the initial value problem $\frac{1}{t}\frac{dy}{dt} = 2\cos^2 y; \quad y(0) = \frac{\pi}{4}$

Solution

$$\int \sec^2 y \, dy = \int 2t \, dt$$

$$\tan y = t^2 + C$$

$$y(0) = \frac{\pi}{4} \quad \rightarrow \quad \underline{1 = C}$$

$$\underline{y(t) = \tan^{-1}\left(t^2 + 1\right)}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 8x^3e^{-2y}; \quad y(1) = 0$

Solution

$$\int e^{2y}dy = \int 8x^3 dx$$

$$\frac{1}{2}e^{2y} = 2x^4 + C$$

$$y(1) = 0 \quad \rightarrow \quad \underline{\frac{1}{2} = C}$$

$$\frac{1}{2}e^{2y} = 2x^4 + \frac{1}{2}$$

$$e^{2y} = 4x^4 + 1$$

$$\underline{y(x) = \frac{1}{2}\ln\left(4x^4 + 1\right)}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = x^2(1+y); \quad y(0) = 3$

Solution

$$\int \frac{1}{1+y}dy = \int x^2 dx$$

$$\ln|1+y| = \frac{1}{3}x^3 + C$$

$$y(0) = 3 \quad \rightarrow \quad \underline{\ln 4 = C}$$

$$1 + y = e^{\frac{1}{3}x^3 + \ln 4}$$

$$\underline{y(x) = 4e^{x^3/3} - 1}$$

Exercise

Find the exact solution of the initial value problem $\sqrt{y}dx + (1+x)dy = 0; \quad y(0) = 1$

Solution

$$\int y^{-1/2}dy = -\int \frac{1}{x+1}dx$$

$$2\sqrt{y} = -\ln|x+1| + C$$

$$y(0) = 1 \quad \rightarrow \quad \underline{2 = C}$$

$$\underline{2\sqrt{y} = -\ln|x+1| + 2}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 6y^2x, \quad y(1) = \frac{1}{25}$

Solution

$$\int \frac{dy}{y^2} = \int 6x\,dx$$

$$-\frac{1}{y} = 3x^2 + C \qquad y(1) = \frac{1}{25}$$

$$-25 = 3 + C \quad \to \quad \underline{C = -28}$$

$$-\frac{1}{y} = 3x^2 - 28$$

$$\boxed{y(x) = \frac{1}{28 - 3x^2}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = \frac{3x^2 + 4x - 4}{2y - 4}, \quad y(1) = 3$

Solution

$$\int (2y - 4)\,dy = \int \left(3x^2 + 4x - 4\right) dx$$

$$y^2 - 4y = x^3 + 2x^2 - 4x + C \qquad y(1) = 3$$

$$9 - 12 = 1 + 2 - 4 + C \quad \to \quad \underline{C = -2}$$

$$\boxed{y^2 - 4y = x^3 + 2x^2 - 4x - 2}$$

Exercise

Find the exact solution of the initial value problem $y' = e^{-y}(2x - 4) \quad y(5) = 0$

Solution

$$\int e^y\,dy = \int (2x - 4)\,dx$$

$$e^y = x^2 - 4x + C \qquad y(5) = 0$$

$$e^0 = 25 - 20 + C \quad \to \quad \underline{C = -4}$$

$$e^y = x^2 - 4x - 4$$

$$\boxed{y(x) = \ln\left|x^2 - 4x - 4\right|}$$

Exercise

Find the exact solution of the initial value problem $\frac{dr}{d\theta} = \frac{r^2}{\theta}, \quad r(1) = 2$

Solution

$$\int \frac{dr}{r^2} = \int \frac{d\theta}{\theta}$$

$$-\frac{1}{r} = \ln|\theta| + C \qquad r(1) = 2$$

$$\boxed{-\frac{1}{2} = C}$$

$$-\frac{1}{r} = \ln|\theta| - \frac{1}{2}$$

$$\frac{1}{r} = \frac{1 - 2\ln|\theta|}{2}$$

$$\boxed{r(\theta) = \frac{2}{1 - 2\ln|\theta|}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dt} = e^{y-t}\left(1+t^2\right)\sec y, \quad y(0) = 0$

Solution

$$\frac{dy}{dt} = e^{-t}\left(1+t^2\right)e^{y}\sec y$$

$$\int \left(e^{-y}\cos y\right)dy = \int \left(1+t^2\right)e^{-t}dt$$

$$\int \left(e^{-y}\cos y\right)dy = e^{-y}(\sin y - \cos y) - \int \left(e^{-y}\cos y\right)dy$$

$$2\int \left(e^{-y}\cos y\right)dy = e^{-y}(\sin y - \cos y)$$

$$\int \left(e^{-y}\cos y\right)dy = \frac{1}{2}e^{-y}(\sin y - \cos y)$$

$$\int \left(1+t^2\right)e^{-t}dt = e^{-t}\left(-1-t^2-2t-2\right)$$

		$\int \cos y$		$\int e^{-t}$
+	e^{-y}	$\sin y$	$1+t^2$	$-e^{-t}$
−	$-e^{-y}$	$-\cos y$	$2t$	e^{-t}
+	e^{-y}		2	$-e^{-t}$

$$\frac{1}{2}e^{-y}(\sin y - \cos y) = -e^{-t}\left(t^2+2t+3\right) + C \qquad y(0) = 0$$

$$-\frac{1}{2} = -3 + C \quad \rightarrow \quad \boxed{C = \frac{5}{2}}$$

$$\boxed{\frac{1}{2}e^{-y}(\sin y - \cos y) = -e^{-t}\left(t^2+2t+3\right) + \frac{5}{2}}$$

Exercise

A thermometer reading 100°F is placed in a medium having a constant temperature of 70°F. After 6 *min*, the thermometer reads 80°F. What is the reading after 20 *min*?

Solution

Given: $T_0 = 100°F \quad \& \quad A = 70°F$ $\qquad \frac{dT}{dt} = -k(T - A) \quad \rightarrow \quad T = A + \left(T_0 - A\right)e^{-kt}$

$$T(t) = 70 + (100 - 70)e^{-kt}$$

$$= \boxed{70 + 30e^{-kt}}$$

$$T(6) = 70 + 30e^{-6k} = 80$$

$$e^{-6k} = \frac{1}{3} \quad \rightarrow \quad k = -\frac{1}{6}\ln\frac{1}{3} = \boxed{\frac{\ln 3}{6}}$$

$$T(t) = 70 + 30e^{-\frac{\ln 3}{6}t}$$

$$T(20) = 70 + 30e^{-\frac{\ln 3}{6}20} \boxed{\approx 70.77\ °F}$$

Exercise

Blood plasma is stored at 40°F. Before the plasma can be used, it must be at 90°F. When the plasma is placed in an oven at 120°F, it takes 45 *min* for the plasma to warm to 90°F. How long will it take for the plasma to warm to 90°F if the oven temperature is set at:

a) 100°F.
b) 140°F.
c) 80°F.

Solution

Given: $T_0 = 40°F \quad \& \quad A = 120°F$ $\qquad \frac{dT}{dt} = -k(T - A) \quad \rightarrow \quad T = A + \left(T_0 - A\right)e^{-kt}$

$$T(t) = 120 - 80e^{-kt}$$

$$T(45) = 120 - 80e^{-45k} = 90$$

$$e^{-45k} = \frac{3}{8} \quad \rightarrow \quad \boxed{k = -\frac{1}{45}\ln\frac{3}{8} \approx .021796}$$

$$\boxed{T(t) = 120 - 80e^{-.021796t}}$$

a) $T_0 = 40°F \quad \& \quad A = 100°F$

$$T(t) = 100 - 60e^{-.021796t} = 90$$

$$e^{-.021796t} = \frac{1}{6}$$

$$t = -\frac{1}{.021796}\ln\frac{1}{6} \boxed{\approx 82\ \ min}$$

b) $T_0 = 40°F \quad \& \;\; A = 140°F$

$$T(t) = 140 - 100e^{-.021796t} = 90$$

$$e^{-.021796t} = \frac{1}{2}$$

$$t = -\frac{1}{.021796}\ln\frac{1}{2} \;\; \underline{\approx 31.8 \;\; min|}$$

c) $T_0 = 40°F \quad \& \;\; A = 80°F$

$$T(t) = 80 - 40e^{-.021796t} = 90$$

$$e^{-.021796t} = \frac{1}{4}$$

$$t = -\frac{1}{.021796}\ln\frac{1}{4} \;\; \underline{\approx 63.6 \;\; min|}$$

Exercise

A pot of boiling water at $100°C$ is removed from a stove at time $t = 0$ and left to cool in the kitchen. After 5 *min*, the water temperature has decreased to $80°C$, and another 5 *min* later it has dropped to $65°C$. Assuming Newton's law for cooling, determine the (constant) temperature of the kitchen.

Solution

Given: $T(0) = 100°C, \quad T(5) = 80°C, \quad T(10) = 65°C$

Let: $B = T_0 - A$ $\qquad\qquad \frac{dT}{dt} = -k(T - A) \;\; \to \;\; T = A + \left(T_0 - A\right)e^{-kt}$

$$T(t) = A + Be^{-kt}$$

$$T(0) = 100°C \quad \to A + B = 100 \quad \Rightarrow A = 100 - B$$

$$T(5) = 80°C \quad \to A + Be^{-5k} = 80$$

$$T(10) = 65°C \quad \to A + Be^{-10k} = 65$$

$$\to \begin{cases} 100 - B + Be^{-5k} = 80 \\ 100 - B + Be^{-10k} = 65 \end{cases} \qquad \begin{cases} B\left(1 - e^{-5k}\right) = 20 & (1) \\ B\left(1 - e^{-10k}\right) = 35 & (2) \end{cases}$$

$$(1) \to \quad B = \frac{20}{1 - e^{-5k}}$$

$$(2) \to \quad \frac{20}{1 - e^{-5k}}\left(1 - e^{-10k}\right) = 35$$

$$4 - 4e^{-10k} = 7 - 7e^{-5k}$$

$$4\left(e^{-5k}\right)^2 - 7e^{-5k} + 3 = 0 \quad \Rightarrow \begin{cases} e^{-5k} = 1 \\ e^{-5k} = \frac{3}{4} \end{cases}$$

$$e^{-5k} = 1 \rightarrow \cancel{k = 0} \qquad \boxed{e^{-5k} = \frac{3}{4}}$$

$$B = \frac{20}{1 - e^{-5k}} = \frac{20}{1 - \frac{3}{4}} = 80$$

$$A = 100 - B = 100 - 80 = 20\ °C$$

Exercise

A murder victim is discovered at midnight and the temperature of the body is recorded at 31°*C*. One hour later, the temperature of the body is 29°*C*. Assume that the surrounding air temperature remains constant at 21°*C*. Use Newton's law of cooling to calculate the victim's time of death. *Note*: The normal temperature of a living human being is approximately 37°*C*

Solution

Given: The initial temperature: $T(0) = 31\ °C$

At $t = 1\ hr \quad \Rightarrow \quad T(1) = 29\ °C$

The surrounding temperature: $A = 21\ °C$

The temperature is given by the formula: $T = A + \left(T_0 - A\right)e^{-kt}$

$$T = 21 + (31 - 21)e^{-kt} = 21 + 10e^{-kt}$$

$$29 = 21 + 10e^{-k(1)}$$

$$8 = 10e^{-k}$$

$$e^{-k} = \frac{8}{10}$$

$$-k = \ln(0.8)$$

$$\boxed{k \approx 0.2231}$$

$$T = 21 + 10e^{-0.2231t}$$

$$37 = 21 + 10e^{-0.2231t}$$

$$10e^{-0.2231t} = 16$$

$$e^{-0.2231t} = 1.6$$

$$-0.2231t = \ln 1.6$$

$$t = \frac{\ln 1.6}{-0.2231}$$

$$\approx -2.1\ hrs \qquad .1 * 60 = 6\ \text{min}$$

The murder occurred 2 *hours* and 6 *minutes* earlier.

Exercise

Suppose a cold beer at 40°F is placed into a warn room at 70°F. suppose 10 minutes later, the temperature of the beer is 48°F. Use Newton's law of cooling to find the temperature 25 *minutes* after the beer was placed into the room.

Solution

Given: The initial temperature: $T(0) = 40\ °F$.

At $t = 10$ min $\Rightarrow$ $T(10) = 48\ °F$

The surrounding temperature: $A = 70\ °F$

Let $T(t)$ be the temperature of the beer at time t minutes after being placed into the room.

From Newton's law of cooling: $T'(t) = k(A - T)$

$$\frac{dT}{dt} = k(70 - T)$$

$$\frac{dT}{70 - T} = kdt$$

$$\int \frac{dT}{70 - T} = \int kdt \qquad\qquad d(70 - T) = -dT$$

$$-\ln(70 - T) = kt + C \qquad 70 > T(t)$$

$$\ln(70 - T) = -kt - C$$

$$70 - T = e^{-kt - C} = e^{-C}e^{-kt} = ce^{-kt}$$

$$T(t) = 70 - ce^{-kt}$$

From the initial condition:

$$T(0) = 70 - ce^{-k(0)}$$

$$40 = 70 - c \quad \Rightarrow \quad c = 30$$

$$\Rightarrow T(t) = 70 - 30e^{-kt}$$

$$T(t = 10) = 70 - 30e^{-k(10)}$$

$$48 = 70 - 30e^{-10k}$$

$$30e^{-10k} = 70 - 48 = 22$$

$$e^{-10k} = \frac{22}{30}$$

$$-10k = \ln\left(\frac{22}{30}\right) \quad \Rightarrow \quad k = -\frac{\ln(11/15)}{10} \approx 0.031$$

$$T(t) = 70 - 30e^{-.031t}$$

$$T(t = 25) = 70 - 30e^{-.031(25)}$$

$$\approx 56.18°\ F$$

Exercise

A thermometer is removed from a room where the temperature is $70° F$ and is taken outside, where the air temperature is $10° F$. After one-half minute the thermometer reads $50° F$.

a) What is the reading of the thermometer at $t = 1$ *min*?

b) How long will it take for the thermometer to reach $15° F$?

Solution

$$\frac{dT}{dt} = -k(T - A) \quad \rightarrow \quad T = A + \left(T_0 - A\right)e^{-kt}$$

a) ***Given***: $T_0 = 70°F \quad \& \quad A = 10°F$

$$T(t) = 10 + (70 - 10)e^{-kt} = \boxed{10 + 60e^{-kt}}$$

$$T\left(t = \frac{1}{2}\right) = 10 + 60e^{-k/2} = 50$$

$$60e^{-k/2} = 40$$

$$e^{-k/2} = \frac{2}{3}$$

$$k = -2\ln\frac{2}{3} \approx \boxed{0.811}$$

$$T(t) = \boxed{10 + 60e^{-.811t}}$$

$$T(1) = 10 + 60e^{-.811} \approx \boxed{36.67°\ F}$$

b) $T(t) = 10 + 60e^{-.811t} = 15°\ F$

$$60e^{-.811t} = 5$$

$$e^{-.811t} = \frac{1}{12}$$

$$t = -\frac{1}{.811}\ln\frac{1}{12}$$

$$\approx \boxed{3.06\ min}$$

Exercise

A thermometer is taken from an inside room to the outside, where the air temperature is $5° F$. After 1 *minute* the thermometer reads $55° F$, and after 5 *minutes* the thermometer reads $30° F$. What is the initial temperature of the inside room?

Solution

Given: $A = 5°F$

$$\frac{dT}{dt} = -k(T - A) \quad \rightarrow \quad T = A + \left(T_0 - A\right)e^{-kt}$$

$$T(t) = 5 + \left(T_0 - 5\right)e^{-kt}$$

$$T(t = 1) = 5 + \left(T_0 - 5\right)e^{-k} = 55 \quad \rightarrow \quad \left(T_0 - 5\right)e^{-k} = 50$$

$$T(t=5)=5+\left(T_0-5\right)e^{-5k}=30 \quad \rightarrow \quad \left(T_0-5\right)e^{-5k}=25$$

$$\left(T_0-5\right)=\frac{50}{e^{-k}}=\frac{25}{e^{-5k}}$$

$$2e^{-5k}=e^{-k}$$

$$e^{-k}\left(2e^{-4k}-1\right)=0 \;\Rightarrow\; e^{-4k}=\frac{1}{2}$$

$$k=-\frac{1}{4}\ln\frac{1}{2} \approx 0.1733$$

$$T_0=\frac{50}{e^{-0.1733}}+5$$

$$\approx 64.461°\ F$$

Exercise

The temperature inside a house is $70°\ F$. A thermometer is taken outside after being inside the house for enough time for it to read $70°\ F$. The outside air temperature is $10°\ F$. After three *minutes* the thermometer reading is found to be $25°\ F$. Find the reading on the thermometer as a function of time.

Solution

Given: $T_0=70°F, \quad A=10°F$

$$T(t=3)=25°$$

$$T(t)=10+(70-10)e^{-kt} \qquad\qquad \frac{dT}{dt}=-k(T-A) \;\rightarrow\; T=A+\left(T_0-A\right)e^{-kt}$$

$$T(t)=10+60e^{-kt}$$

$$T(3)=25 \;\rightarrow\; 25=10+60e^{-3k}$$

$$60e^{-3k}=15$$

$$e^{-3k}=\frac{1}{4} \;\rightarrow\; k=-\frac{1}{3}\ln\frac{1}{4}\approx 0.462$$

$$T(t)=10+60e^{-0.462t}$$

Exercise

A metal bar at a temperature of $100°\ F$ is placed in a room at a constant temperature of $0°\ F$. If after 20 *minutes* the temperature of the bar is $50°\ F$.

a) Find the time it will take the bar to reach a temperature of $25°\ F$

b) Find the temperature of the bar after 10 *minutes*.

Solution

Given: $T_0=100°F, \quad A=0°F$

$$T(t=20)=50°$$

$$\frac{dT}{dt} = -k(T - A) \quad \rightarrow \quad T = A + \left(T_0 - A\right)e^{-kt}$$

a) $T(t) = 100e^{-kt}$

$$T(20) = 50 \quad \rightarrow \quad 50 = 100e^{-20k}$$

$$-20k = \ln\frac{1}{2} \quad \rightarrow \quad \underline{k \approx 0.035}$$

$$\underline{T(t) = 100e^{-0.035t}}$$

b) $T(t = 10) = 100e^{-0.035(10)}$

$$\underline{\approx 70.5^\circ}$$

Exercise

A small metal bar, whose initial temperature was $20^\circ\ C$, is dropped into a large container of boiling water.

a) How long will it take the bar to reach $90^\circ\ C$ if it is known that its temperature increases 2° in 1 *second*?

b) How long will it take the bar to reach $98^\circ\ C$

Solution

a) ***Given***: $T_0 = 20^\circ\ C \quad \& \quad A = 100^\circ\ C$

$$T(t) = 100 + (20 - 100)e^{-kt} \ \underline{= 100 - 80e^{-kt}} \qquad\qquad T = A + \left(T_0 - A\right)e^{-kt}$$

$$T(1) = 100 - 80e^{-k} = 22$$

$$e^{-k} = \frac{78}{80}$$

$$k = -\ln\frac{39}{40} \ \underline{\approx 0.0253}$$

$$T(t) = \underline{100 - 80e^{-0.0253t}}$$

$$100 - 80e^{-0.0253t} = 90$$

$$e^{-0.0253t} = \frac{1}{8}$$

$$\underline{t =} -\frac{1}{0.0253}\ln\frac{1}{8}$$

$$\underline{\approx 82.1\ sec}$$

b) $T = 100 - 80e^{-0.0253t} = 98 \quad \Rightarrow \quad e^{-0.0253t} = \frac{1}{40}$

$$\underline{t =} -\frac{1}{0.0253}\ln\frac{1}{40}$$

$$\underline{\approx 145.8\ sec}$$

Exercise

Two large containers ***A*** and ***B*** of the same size are filled with different fluids. The fluids in containers ***A*** and ***B*** are maintained at $0° \ C$ and $100° \ C$, respectively. A small metal bar, whose initial temperature is $100° \ C$, is lowered into container ***A***. After 1 *minute* the temperature of the bar is $90° \ C$. After 2 *minutes* the bar is removed and instantly transferred to the other container. After 1 *minute* in container ***B*** the temperature of the bar rises $10° \ C$. How long, measured from the start of the entire process, will it take the bar to reach $99.9° \ C$?

Solution

Given: *Tank* ***A***: $A = 0° \ C, \ \ T_1(0) = 100° \ C \ \ \& \ \ T_1(1) = 90° \ C$

Tank ***B***: $A = 100° \ C \ \ \& \ \ T_2(1) = 110° \ C$

$$T_1(t) = (100-0)e^{-kt} \ \underline{=100e^{-kt}} \qquad\qquad T = A + (T_0 - A)e^{-kt}$$

$$T_1(1) = 100e^{-k} = 90 \ \ \rightarrow \ k = -\ln\left(\frac{9}{10}\right) \approx 0.10536$$

$$T_1(t) \ \underline{=100e^{-0.10536t}}$$

$$T_1(2) = 100e^{-0.10536(2)} \ \underline{\approx 81° \ C}$$

Tank B: $\qquad T_1(2) \approx 81° \ C = T_2(0)$

$$T_2(t) = 100 + (81-100)e^{-kt} \ \underline{=100-19e^{-kt}} \qquad T = A + (T_0 - A)e^{-kt}$$

$$T_2(1) = 100 - 19e^{-k} = 81 + 10 = 91 \ \ \rightarrow \ k = -\ln\left(\frac{9}{19}\right) \approx 0.7472$$

$$T_2(t) \ \underline{=100-19e^{-0.7472t}}$$

$$T_2(t) \ = 100 - 19e^{-0.7472t} = 99.9$$

$$-0.7472t = \ln\frac{0.1}{19}$$

$$t = -\frac{1}{0.7472}\ln\frac{0.1}{19} \ \underline{\approx 7.02 \ \ min}$$

$\therefore$ The entire process will take the bar to reach $99.9° \ C$ is approximately 9.02 *minutes*.

Exercise

A thermometer reading $70° \ F$ is placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer records that the thermometer reads $110° \ F$ after $\frac{1}{2}$ *minute* and $145° \ F$ after 1 *minute*. How hot is the oven?

Solution

Given: $T_0 = 70° \ F$

$T(t) = A + (70 - A)e^{-kt}$ $\qquad\qquad$ $T = A + \left(T_0 - A\right)e^{-kt}$

$T\left(t = \frac{1}{2}\right) = A + (70 - A)e^{-k/2} = 110 \quad \rightarrow \quad e^{-k/2} = \frac{110 - A}{70 - A}$

$T(t = 1) = A + (70 - A)e^{-k} = 145 \quad \rightarrow \quad e^{-k} = \frac{145 - A}{70 - A}$

$e^{-k} = \left(e^{-k/2}\right)^2 = \left(\frac{110 - A}{70 - A}\right)^2 = \frac{145 - A}{70 - A}$

$\frac{(110 - A)^2}{70 - A} = 145 - A$

$12,100 - 220A + A^2 = 10,150 - 215A + A^2$

$5A = 1950 \quad \rightarrow \quad \underline{A = 390}$

$\therefore$ The temperature in the oven is 390° F.

Exercise

At $t = 0$ a sealed test tube containing a chemical is immersed in a liquid bath. The initial temperature of the chemical in the test tube is 80° F. the liquid bath has a controlled temperature given by $T_m(t) = 100 - 40e^{-0.1t}$, $t \ge 0$, where t is measured in *minutes*.

a) Assume that $k = -0.1$, describe in words what you expect the temperature $T(t)$ of the chemical to be like in the short term. In the long term.

b) Solve the initial-value problem.

c) Graph $T(t)$.

Solution

a) ***Given***: $T_0 = 80°\ F$

$T_m(0) = 100 - 40 = 60°\ F$

The temperature decreases (or cool off), in the short time.

Over time, the temperature will increase towards 100° since $e^{-0.1t}$ decrease from 1 to 0 as t approaches infinity. Thus, in the long term, the temperature of the chemical should increase or warm toward 100°.

b) $\frac{dT}{dt} = -0.1\left(T - 100 + 40e^{-0.1t}\right)$

$\frac{dT}{dt} + 0.1T = 10 - 4e^{-0.1t}$

$e^{\int 0.1dt} = \underline{e^{0.1t}}$

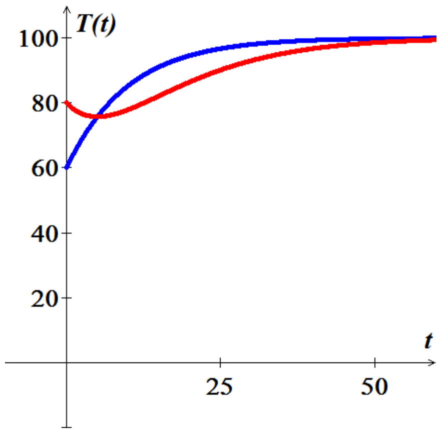
$\int \left(10 - 4e^{-0.1t}\right)e^{0.1t}dt = \int \left(10e^{0.1t} - 4\right)dt = \underline{100e^{0.1t} - 4t}$

$$T(t) = e^{-0.1t}\left(100e^{0.1t} - 4t + C\right) \qquad T_0 = 80$$

$$100 + C = 80 \ \rightarrow \ \boxed{C = -20}$$

$$\boxed{T(t) = 100 - 4(t+5)e^{-0.1t}}$$

c)



Exercise

The mathematical model for the shape of a flexible cable strung between two vertical supports is given by

$$\frac{dy}{dx} = \frac{W}{T_1}$$

Where W denotes the portion of the total vertical load between the points P_1 and P_2

The model is separable under the following conditions that describe a suspension bridge.

Let assume that the x-axis runs along the horizontal roadbed, and the y-axis passes through $(0,\ a)$, which is the lowest point on one cable over the span of the bridge, coinciding with the interval $\left[-\frac{L}{2},\ \frac{L}{2}\right]$.

In the case of a suspension bridge, the usual assumption is that the vertical load in the given equation is only a uniform roadbed, distributed along the horizontal axis. In other words, it is assumed that the weight of all cables is negligible in comparison to the weight of the roadbed and that the weight per unit length of the roadbed (lb/ft) is a constant ρ. Use this information to set up and solve an appropriate initial-value problem from which the shape (a curve with equation $y = \varphi(x)$) of each of the two cables in a suspension bridge is determined. Express the solution of the IVP in terms of the sag h and span L.

Solution

Since the tension T_1 (or magnitude T_1) acts at the lowest point of the cable, using the symmetry to solve the problem on the interval $\left[0,\ \frac{L}{2}\right]$.

The assumption that the roadbed is uniform (that is, weighs a constant ρ (lb/ft) implies

$$W = \rho x, \text{ where } 0 \le x \le \frac{L}{2}$$

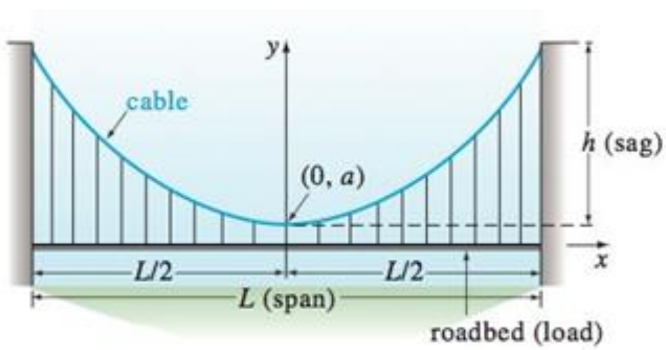
$$\frac{dy}{dx} = \frac{\rho}{T_1}x$$

$$\int dy = \frac{\rho}{T_1}\int x\,dx$$

$$y(x) = \frac{1}{2}\frac{\rho}{T_1}x^2 + C \qquad y(0) = a$$

$$y(0) = \boxed{C = a}$$

$$\boxed{y(x) = \frac{\rho}{2T_1}x^2 + a}$$



$$y\left(\frac{L}{2}\right) = h + a$$

$$\frac{\rho}{2T_1}\frac{L^2}{4} = h \;\Rightarrow\; \boxed{\frac{\rho}{2T_1} = \frac{4h}{L^2}}$$

$$\boxed{y(x) = \frac{4h}{L^2}x^2 + a} \quad -\frac{L}{2} \le x \le \frac{L}{2}$$

Exercise

The Brentano-Stevens Law in psychology models the way that a subject reacts to a stimulus. It states that if *R* represents the reaction to an amount *S* of stimulus, then the relative rates of increase are proportional:

$$\frac{1}{R}\frac{dR}{dt} = \frac{k}{S}\frac{dS}{dt}$$

Where k is a positive constant. Find *R* as a function of *S*.

Solution

$$\frac{1}{R}\frac{dR}{dt} = \frac{k}{S}\frac{dS}{dt}$$

$$\frac{d}{dt}(\ln R) = \frac{d}{dt}(k\ln S)$$

$$\ln R = k\ln S + C$$

$$R(S) = e^{\ln S^k + C}$$

$$= e^C e^{\ln S^k} \qquad A = e^C$$

$$\boxed{= AS^k}$$

Exercise

Barbara weighs 60 *kg* and is on a diet of 1600 *calories* per day, of which 850 are used automatically by basal metabolism. She spends about 15 *cal/kg/day* times her weight doing exercises. If 1 *kg* of fat contains 10,000 *cal*. and we assume that the storage of calories in the form of fat is 100% efficient, formulate a differential equation and solve it to find her weight as a function of time. Does her weight ultimately approach an equilibrium weight?

Solution

$m(t) = m$: mass at time t.

The net intake of calories per day at time t is

$c(t) = 1600 - 850 - 15m \;\; = 750 - 15m$

$m = \frac{1}{15}(750 - c(t))$

$\frac{dm}{dt} = \frac{c(t)}{10,000}$

$\quad = \frac{750 - 15m}{10,000}$

$\quad = -3\frac{m-50}{2000}$

$\int \frac{dm}{m-50} = -\frac{3}{2000}\int dt$

$\ln|m - 50| = -\frac{3}{2000}t + C$

$m - 50 = Ae^{-\frac{3}{2000}t}$

$\qquad m(0) = 60 \quad \rightarrow \underline{10 = A}$

$\boxed{m(t) = 10e^{-\frac{3}{2000}t} + 50}$

$\lim\limits_{t\to\infty} m(t) = \lim\limits_{t\to\infty}\left(10e^{-\frac{3}{2000}t} + 50\right) \underline{= 50}$

Thus, Barbara's mass gradually settles down to 50 kg.

Exercise

When a chicken is removed from an oven, its temperature is measured at 300° F. Three minutes later its temperature is 200° F. How long will it take for the chicken to cool off to a room temperature of 70° F.

Solution

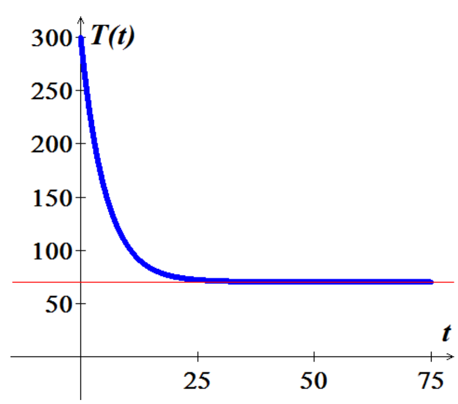
Given: $\quad T_0 = 300°F, \quad A = 70°F, \quad T(3) = 200°F$

$T(t) = 70 + (300 - 70)e^{-kt} \qquad T = A + (T_0 - A)e^{-kt}$

$T(3) = 70 + 230e^{-3k} = 200$

$e^{-3k} = \frac{130}{230}$

$k = -\frac{1}{3}\ln\left(\frac{13}{23}\right) \approx \underline{0.19018}$



$$\boxed{T(t) = 70 + 230e^{-0.19018t}}$$

$$70 + 230e^{-0.19018t} = 70$$

$$e^{-0.19018t} = 0 \quad \Rightarrow \quad \boxed{t \to \infty}$$

Exercise

Suppose that a corpse was discovered in a motel room at midnight and its temperature was $80° F$. The temperature of the room is kept constant at $60° F$. Two hours later the temperature of the corpse dropped to $75° F$. Find the time of death.

Solution

First we use the observed temperatures of the corpse to find the constant *k*. We have

$$k = -\frac{1}{2}\ln\left(\frac{75-60}{80-60}\right) = 0.1438$$

In order to find the time of death we need to remember that the temperature of a corpse at time of death is $98.6° F$ (assuming the dead person was not sick!). Then we have

$$\begin{aligned} t_d &= -\frac{1}{k}\ln\left(\frac{98.6-60}{80-60}\right) \\ &= -\frac{1}{0.1438}\ln\left(\frac{38.6}{20}\right) \\ &\boxed{\approx -4.57 \ \ hrs} \end{aligned}$$

$$4.57 \ hrs = \ \ 4 \quad .57 \times 60 = 34'$$

$$12 - (4 \ hrs \quad 34 \ \min)$$

Which means that the death happened around 7:26 P.M.

Exercise

Suppose that a corpse was discovered at 10 PM and its temperature was $85° F$. Two hours later, its temperature is $74° F$. If the ambient temperature is $68° F$. Estimate the time of death.

Solution

Given: $A = 68, \quad T_0 = 85, \quad T_2 = 74$

$$k = -\frac{1}{2}\ln\left(\frac{74-68}{85-68}\right) = 0.521$$

$$T(t) = 68 + (85-68)e^{-.521t} \qquad\qquad T(t) = A + \left(T_0 - A\right)e^{-kt}$$

$$= 68 + 17e^{-.521t}$$

$$t_d = -\frac{1}{.521}\ln\left(\frac{98.6-68}{85-68}\right) \boxed{\approx -1.13}$$

$1.13\ hrs = \ 1 \quad .13 \times 60 \approx 8'$

$10 - \left(1\ hr \quad 8\ \min\right)$

The death happened around 8:52 P.M