Section 4.3 – Closures of Relations

Closures

The ***reflexive closure*** of R can be formed by adding to R all pairs of the form (a, a) with $a \in A$, not already in R.
The reflexive closure of R equals $R \cup \Delta$ where
$\Delta = \{(a,a) \mid a \in A\}$ is the ***diagonal relation*** on A.

Example

What is the reflexive closure of the relation $R = \{(a,b) \mid a < b\}$ on the set of integers?

Solution

The reflexive closure of R is the relation

$$
\begin{aligned}
R \cup \Delta &= \{(a,b) \mid a < b\} \cup \{(a,a) \mid a \in \mathbb{Z}\} \\
&= \{(a,b) \mid a \le b\}
\end{aligned}
$$

Example

What is the symmetric closure of the relation $R = \{(a,b) \mid a > b\}$ on the set of positive integers?

Solution

The symmetric closure of R is the relation

$$
\begin{aligned}
R \cup R^{-1} &= \{(a,b) \mid a > b\} \cup \{(b,a) \mid a > b\} \\
&= \{(a,b) \mid a \ne b\}
\end{aligned}
$$

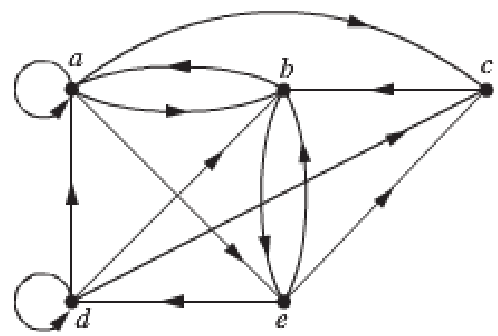
Path in Directed Graphs

Definition

A path from a to b in the directed graph G is a sequence of edges (x_0, x_1), (x_1, x_2), (x_2, x_3), …, (x_{n-1}, x_n) in G, where n is nonnegative integer, and $x_0 = a$ and $x_n = b$, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. The path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ and has length n. We view the empty set of edges as a path of length zero from a to a. A path of length $n \ge 1$ that begins and ends at the same vertex is called a ***circuit*** or ***cycle***.

Example

Which of the following are paths in the directed graph: *a, b, e, d; a, e, c, d, b; b ,a, c, b, a, a, b; d, c; c, b, a; e, b, a, b, a, b, e*?
What are the lengths of those that are paths?
Which of the paths in this list are circuits?



Solution

Each of (*a, b*), (*b, e*), and (*e, d*) is an edge *a, b, e, d* is a path of length 3
(*c, d*) is not an edge, therefore *a, e, c, d, b* is not a path
b ,a, c, b, a, a, b is a path of length 6
d, c is a path of length 1
c, b, a is a path of length 2
e, b, a, b, a, b, e is a path of length 6

The 2 paths *b, a, c, b, a, a, b* and *e, b, a, b, a, b, e* are circuits because they begin and end the same vertex.
The paths *a, b, e, d; c, b, a;* and *d, c* are not circuits

Theorem

Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a,b) \in R^n$

Proof

Using mathematical induction
There is a path from a to b of length one if and only if $(a,b) \in R$, which is true when $n = 1$.
Assume that the theorem is true for a positive integer n.
We need to prove that there is a path of length $n + 1$ from a to b if and only if $c \in A$ such there is a path of length 1 from a to c, so $(a,c) \in R$, and path of length n from c to b $(c,b) \in R^n$.
Consequently, by the inductive hypothesis, there is a path of length $n + 1$ from a to b if and only if there is an element c with $(a,c) \in R$ and $(c,b) \in R^n$. But there is such an element iff $(a,b) \in R^{n+1}$.
Therefore, there is a path of length $n + 1$ from a to b iff $(a,b) \in R^{n+1}$. This completes the proof.

Transitive Closures

Definition

Let R be a relation on a set A. The ***connectivity relation*** R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.

Example

Let R be the relation on the set of all people in the world that contains (a, b) if a has met b. What is R^n, where n is a positive integer greater than one? What is R^*?

Solution

The relation R^* contain (a, b) if there is a person c such that $(a, c) \in R$ and , that is, if there is a person c such that a has met c and c has met b.

Similarly, R^n consists of those pairs (a, b) such that there are people $x_1, x_2, \ldots, x_{n-1}$ such that a has met x_1. x_1 has met $x_2, \ldots, x_{n-1}$ has met b.

The relation R^* contains (a, b) if there is a sequence of people, starting with a and ending with b, such that each person in the sequence has met next person in the sequence.

Example

Let R be the relation on the set of all states in U.S. that contains (a, b) if state a and state b have a common border. What is R^n, where n is a positive integer? What is R^*?

Solution

The relation R^n contain (a, b), where it is possible to go from state a to state b by crossing exactly n state borders. The relation R^*consists of the ordered pairs (a, b), where it is possible to go from state a to state b crossing as many borders as necessary.

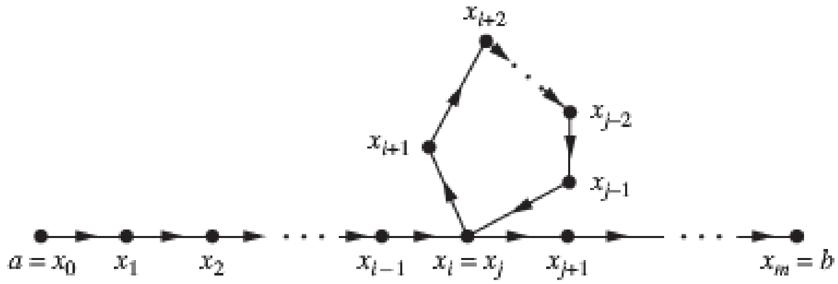
The only ordered pairs not in R^* are those containing sates that are not connected to the continental U.S.

Theorem

The transitive closure of a relation R equals the connectivity relation R^*.

Lemma

Let A be a set with n elements, and let R be the relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n. Moreover, when $a \neq b$, if there is a path of length at least one in R from a to b, then there is such a path with length not exceeding $n - 1$.



Proof

Suppose there is a path from a to b in R. Let m be the length of the shortest such path.

Suppose that $x_0, x_1, x_2, \ldots, x_{m-1}, x_m$, where $x_0 = a$ and $x_m = b$, is such a path.

Suppose that $a = b$ and that $m > n$, so that $m \geq n+1$.

By the pigeonhole principle, because there are n vertices in A, among m vertices $x_0, x_1, \ldots, x_m$, at least two are equal.

Suppose that $x_i = x_j$ with $0 \leq i < j \leq m-1$. Then the path contains a circuit from x_i to itself. This circuit can be deleted from the path from a to b, leaving a path, namely,

$x_0, x_1, \ldots, x_i, x_{j+1}, \ldots, x_m$, from a to b of shorter length. Hence, the path of shortest length must have less than or equal to n.

Exercises *Section* 4.3 – Closures of Relations

1. Let R be the relation on the set {0, 1, 2, 3} containing the ordered pairs (0, 1), (1, 1), (1, 2), (2, 0), (2, 2), and (3, 0). Find the

a) Reflexive closure of R.

b) Symmetric closure of R.

2. Let R be the relation $\{(a,\ b) \mid a \neq b\}$ on the set of integers. What is the reflexive closure of R?

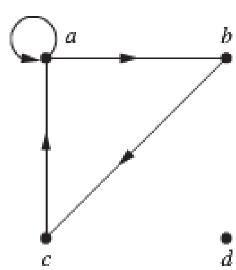
3. Let R be the relation $\{(a,\ b) \mid a \text{ divides } b\}$ on the set of integers. What is the symmetric closure of R?

4. How can the directed graph representing the reflexive closure of a relation on a finite set be constructed from the directed graph of the relation?

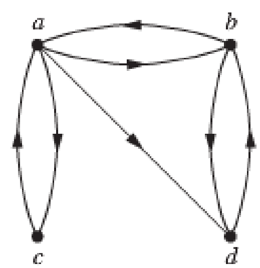
5. Draw the directed graph of the *reflexive*, *symmetric*, and *both reflexive and symmetric* closure of the relations with the directed graph shown

a)

b)



c)



6. 1. Determine whether these sequences of vertices are paths in this directed graph

a) *a, b, c, e*

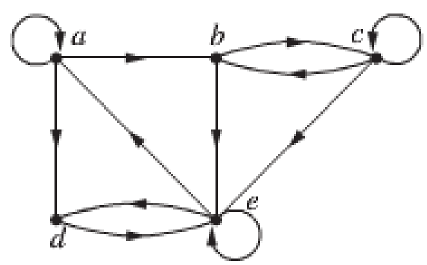
b) *b, e, c, b, e*

c) *a, a, b, e, d, e*

d) *b, c, e, d, a, a, b*

e) *b, c, c, b, e, d, e, d*

f) *a, a, b, b, c, ,c, b, e, d*



2. Find all circuits of length three in the directed graph

7. Let R be the relation on the set {1, 2, 3, 4, 5} containing the ordered pairs (1, 3), (2, 4), (3, 1), (3, 5), (4, 3), (5, 1), and (5, 2). Find

a) R^2 ***b***) R^3 ***c***) R^4 ***d***) R^5 ***e***) R^6 ***f***) R^*

8. Let R be the relation on the pair (a, b) if a and b are cities such that there is a direct non-stop airline flight from a to b. When is (a, b) in

a) R^2 ***b***) R^3 ***c***) R^*

9. Let R be the relation on the set of all students containing the ordered pair (a, b) if a and b are in at least one common class and $a \neq b$. When is (a, b) in

a) R^2 ***b***) R^3 ***c***) R^*

10. Suppose that the relation R is reflexive. Show that R^* is reflexive.

11. Suppose that the relation R is symmetric. Show that R^* is symmetric.

12. Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive.