Section 2.2 – Trigonometric Integrals

Products of Powers of *Sines* and *Cosines*

We begin with integrals of the form

$$\int \sin^m x \ \cos^n x \ dx$$

Example

Evaluate $\int \sin^3 x \cos^2 x \ dx$

Solution

$$\int \sin^3 x \cos^2 x \ dx = \int \sin x \sin^2 x \cos^2 x \ dx$$

$$= \int \left(1-\cos^2 x\right)\cos^2 x \ \left(-d\left(\cos x\right)\right) \qquad d\left(\cos x\right) = -\sin x dx \quad \Rightarrow \sin x dx = -d\left(\cos x\right)$$

$$= -\int \left(\cos^2 x - \cos^4 x\right) \ d\left(\cos x\right) \qquad \text{or} \quad \textit{Assume} \quad u = \cos x$$

$$= -\left(\frac{1}{3}\cos^3 x - \frac{1}{5}\cos^5 x\right) + C$$

$$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$$

Example

Evaluate $\int \cos^5 x \ dx$

Solution

$$\int \cos^5 x \ dx = \int \cos^4 x \cos x \ dx \qquad \cos x dx = d\left(\sin x\right) \qquad \cos^2 x = 1 - \sin^2 x$$

$$= \int \left(1-\sin^2 x\right)^2 \ d\left(\sin x\right)$$

$$= \int \left(1 - 2\sin^2 x + \sin^4 x\right) \ d\sin x$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

Example

Evaluate $\int \sin^2 x \cos^4 x \, dx$

Solution

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx \qquad \sin^2 x = \frac{1-\cos 2x}{2} \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{1}{8}\int (1-\cos 2x)\left(1+2\cos 2x+\cos^2 2x\right) dx$$

$$= \frac{1}{8}\int \left(1+2\cos 2x+\cos^2 2x-\cos 2x-2\cos^2 2x-\cos^3 2x\right) dx$$

$$= \frac{1}{8}\int \left(1+\cos 2x-\cos^2 2x-\cos^3 2x\right) dx$$

$$= \frac{1}{8}\left[x+\frac{1}{2}\sin 2x-\int \left(\cos^3 2x+\cos^2 2x\right) dx\right]$$

$$\int \cos^3 2x dx = \int \left(1-\sin^2 2x\right)\cos 2x dx$$

$$= \frac{1}{2}\int \left(1-\sin^2 2x\right) d(\sin 2x)$$

$$= \frac{1}{2}\left(\sin 2x-\frac{1}{3}\sin^3 2x\right)$$

$$\int \cos^2 2x dx = \frac{1}{2}\int (1+\cos 4x)\, dx$$

$$= \frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)$$

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{8}\left[x+\frac{1}{2}\sin 2x-\frac{1}{2}\left(\sin 2x-\frac{1}{3}\sin^3 2x\right)-\frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)\right]+C$$

$$= \frac{1}{8}\left[x+\frac{1}{2}\sin 2x-\frac{1}{2}\sin 2x+\frac{1}{6}\sin^3 2x-\frac{1}{2}x-\frac{1}{8}\sin 4x\right]+C$$

$$= \frac{1}{8}\left(\frac{1}{2}x+\frac{1}{6}\sin^3 2x-\frac{1}{8}\sin 4x\right)+C$$

$$\boxed{= \frac{1}{16}\left(x+\frac{1}{3}\sin^3 2x-\frac{1}{4}\sin 4x\right)+C}$$

Example

Evaluate $\int_0^{\pi/4} \sqrt{1+\cos 4x}\ dx$

Solution

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \quad \Rightarrow \quad 1+\cos 2\theta = 2\cos^2\theta$$

$$\theta = 2x \quad \Rightarrow \quad 1+\cos 4x = 2\cos^2 2x$$

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\ dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\ dx \qquad \sqrt{2\cos^2 2x} = \sqrt{2}\sqrt{\cos^2 2x} = \sqrt{2}\,|\cos 2x|$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\ dx \qquad \cos 2x \ge 0 \quad on \quad \left[0, \tfrac{\pi}{4}\right]$$

$$= \sqrt{2}\int_0^{\pi/4} \cos 2x\ dx$$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4}$$

$$= \frac{\sqrt{2}}{2}\left[\sin\frac{\pi}{2} - 0\right]$$

$$= \frac{\sqrt{2}}{2}$$

Example

Evaluate $\int \sin^3 x \cos^{-2} x\ dx$

Solution

$$\int \sin^3 x \cos^{-2} x\ dx = \int \sin^2 x \cos^{-2} x\ \sin x\ dx$$

$$= -\int \left(1-\cos^2 x\right)\cos^{-2} x\ \ d(\cos x)$$

$$= -\int \left(\cos^{-2} x - 1\right)\ d(\cos x)$$

$$= -\left(-\cos^{-1} x - \cos x\right) + C$$

$$= \cos x + \sec x + C$$

Products of Powers of *tan x* and *sec x*

Example

Evaluate $\int \tan^4 x\, dx$

Solution

$$\int \tan^4 x dx = \int \tan^2 x \cdot \tan^2 x\, dx \qquad \tan^2 x = \sec^2 x - 1$$

$$= \int \tan^2 x \cdot \left(\sec^2 x - 1\right) dx$$

$$= \int \tan^2 x \sec^2 x dx - \int \tan^2 x\, dx$$

$$= \int \tan^2 x \sec^2 x dx - \int \left(\sec^2 x - 1\right) dx$$

$$= \int \tan^2 x \sec^2 x dx - \int \sec^2 x\, dx + \int dx$$

$$= \int \tan^2 x\, d(\tan x) - \int \sec^2 x\, dx + \int dx \qquad d(\tan x) = \sec^2 x\, dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x + C$$

Example

Evaluate $\int \sec^3 x\, dx$

Solution

Let: $u = \sec x \qquad dv = \sec^2 xdx$

$du = \sec x \tan xdx \qquad v = \tan x$

$$\int \sec^3 xdx = \sec x \tan x - \int \tan x (\sec x \tan xdx)$$

$$= \sec x \tan x - \int \tan^2 x \sec xdx$$

$$= \sec x \tan x - \int \left(\sec^2 x - 1\right)\sec x\, dx$$

$$= \sec x \tan x - \int \left(\sec^3 x - \sec x\right) dx$$

$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$\int \sec^3 x\, dx + \int \sec^3 x\, dx = \sec x \tan x - \int \sec^3 x\, dx + \int \sec^3 x\, dx + \int \sec x\, dx$$

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

$$= \sec x \tan x + \ln\left|\sec x + \tan x\right| + C_1$$

$$\int \sec^3 x\, dx = \underline{\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C}$$

Products of Sines and Cosines

Recall the identities

$$\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$$

$$\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$$

$$\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$$

Example

Evaluate $\int \sin 3x \cos 5x dx$

Solution

$$\int \sin 3x \cos 5x dx = \frac{1}{2}\int \left[\sin(-2x) + \sin 8x\right] dx$$

$$= \frac{1}{2}\int \left[-\sin(2x) + \sin 8x\right] dx$$

$$= \frac{1}{2}\left(\frac{1}{2}\cos 2x - \frac{1}{8}\cos 8x\right) + C$$

$$\underline{= \frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C}$$

Guidelines for Cosine & Sine

Case* 1** If ***m is ***odd***, we write m as $2k+1$ and use the identity $\sin^2 x = 1-\cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = \left(\sin^2 x\right)^k \sin x = \left(1-\cos^2 x\right)^k \sin x$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x dx = -d\left(\cos x\right)$

Case* 2** If ***m is ***even* and *n*** is ***odd***, in $\int \sin^m x \cos^n x dx$ we write n as $2k+1$ and use the identity $\cos^2 x = 1-\sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = \left(\cos^2 x\right)^k \cos x = \left(1-\sin^2 x\right)^k \cos x$$

Then we combine the single $\cos x$ with dx in the integral and set $\cos x dx = d\left(\sin x\right)$

Case* 3** If both ***m* and *n are ***even***, in $\int \sin^m x \cos^n x dx$, we substitute

To reduce the integrand to one in lower powers of $\cos 2x$ $\quad \int \cos ax dx = \frac{1}{a}\sin ax + C$

Guidelines for Tangent & Secant

Case* 1** When the power of the tangent is ***odd and positive.

$$\begin{aligned}\int \sec^m x \ \tan^{2k+1} x \, dx &= \int \sec^{m-1} x \left(\tan^2 x\right)^k \sec x \tan x \, dx \\ &= \int \sec^{m-1} x \left(\sec^2 x - 1\right)^k d\left(\sec x\right)\end{aligned}$$

Case* 2** When the power of the secant is ***even and positive.

$$\int \sec^{2k} x \ \tan^n x \, dx = \int \left(\sec^2 x\right)^{k-1} \tan^n x \ \sec^2 x \, dx = \int \left(1+\tan^2 x\right)^{k-1} \tan^n x \, d\left(\tan x\right)$$

***Case* 3** When there are no secant factors

$$\int \tan^n x \, dx = \int \tan^{n-2} x \left(\tan^2 x\right) dx = \int \tan^{n-2} x \left(\sec^2 x - 1\right) dx$$

***Case* 4** When there are only secant, use integration by parts.

***Case* 5** Otherwise, convert to cosines and sines.

Wallis's Formulas

1. If n is odd $(n \geq 3)$, then $\displaystyle\int_0^{\pi/2} \cos^n x\, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right) \cdots \left(\frac{n-1}{n}\right)$

2. If n is even $(n \geq 2)$, then $\displaystyle\int_0^{\pi/2} \cos^n x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) \cdots \left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$

Formulas

$$\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx$$

$$\int \sec^n x\, dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx$$

$$\int \sin^n x\, dx = \frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx$$

$$\int \tan^n x\, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\, dx$$

$$\int \tan x\, dx = -\ln|\cos x| + C = \ln|\sec x| + C$$

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C$$

Exercises *Section* 2.2 – Trigonometric Integrals

(**1 – 149**) Evaluate the integrals

1. $\int \sin^5 \frac{x}{2}\, dx$
2. $\int \sin^4 6\theta\, d\theta$
3. $\int x^2 \sin^2 x\, dx$
4. $\int \sin^3 3x\, dx$
5. $\int \sin^5 x\, dx$
6. $\int 8\cos^4 2\pi x\, dx$
7. $\int x\cos^3 x\, dx$
8. $\int \cos^4 x\, dx$
9. $\int \cos^4 5x\, dx$
10. $\int \cos^2 3x\, dx$
11. $\int \cos^3 \frac{x}{3}\, dx$
12. $\int \cos^2 4x\, dx$
13. $\int \sqrt{1+\cos\frac{x}{2}}\, dx$
14. $\int \sec^4 2x\, dx$
15. $\int 6\sec^4 x\, dx$
16. $\int \sec^3 \pi x\, dx$
17. $\int \sec 4x\, dx$
18. $\int \csc^6 x\, dx$
19. $\int \tan^5 \frac{x}{2}\, dx$
20. $\int \tan^5 x\, dx$
21. $\int \tan^5 3x\, dx$
22. $\int \tan^6 3x\, dx$
23. $\int 20\tan^6 x\, dx$
24. $\int \tan^4 x\, dx$
25. $\int \tan^3 \theta\, d\theta$
26. $\int \tan^3 4x\, dx$
27. $\int \cot^3 2x\, dx$
28. $\int \cot^4 x\, dx$
29. $\int \cot^4 3x\, dx$
30. $\int \cot^5 3x\, dx$
31. $\int \sin^2 x\, \cos^2 x\, dx$
32. $\int \sin^2 x\, \cos^3 x\, dx$
33. $\int \sin^2 x\, \cos^4 x\, dx$
34. $\int \sin^2 x\, \cos^5 x\, dx$
35. $\int \sin^3 x\, \cos^5 x\, dx$
36. $\int \sin^3 x\, \cos^4 x\, dx$
37. $\int \sin^3 2x\, \cos^4 x\, dx$
38. $\int \sin^3 2x\, \cos^3 2x\, dx$
39. $\int \sin^4 x\cos^2 x\, dx$
40. $\int \sin^4 x\cos^3 x\, dx$
41. $\int \sin^4 x\, \cos^4 x\, dx$
42. $\int \sin^4 x\, \cos^5 x\, dx$
43. $\int \sin^5 x\, \cos^5 x\, dx$
44. $\int \sin^5 x\, \cos^{-2} x\, dx$
45. $\int \sin 3x\cos^6 3x\, dx$

46. $\int \sin^4 2x \cos 2x \, dx$

47. $\int \cos^3 2x \sin^5 2x \, dx$

48. $\int 16 \sin^2 x \cos^2 x \, dx$

49. $\int \sin 2x \cos 3x \, dx$

50. $\int \sin^2 \theta \cos 3\theta \, d\theta$

51. $\int \cos^3 \theta \sin 2\theta \, d\theta$

52. $\int \sin^{-3/2} x \cos^3 x \, dx$

53. $\int \sin^3 x \cos^{3/2} x \, dx$

54. $\int \sin \theta \sin 2\theta \sin 3\theta \, d\theta$

55. $\int \sin 3x \cos 6x \, dx$

56. $\int \sin 3x \cos 7x \, dx$

57. $\int \sin 5x \cos 4x \, dx$

58. $\int \cos 2\theta \cos 6\theta \, d\theta$

59. $\int \cos 5\theta \cos 3\theta \, d\theta$

60. $\int \sin 2\theta \cos 4\theta \, d\theta$

61. $\int \sin(-7\theta) \cos 6\theta \, d\theta$

62. $\int \sin \theta \sin 3\theta \, d\theta$

63. $\int \sin 5\theta \sin 4\theta \, d\theta$

64. $\int \sin x \cos^5 x \, dx$

65. $\int \sin^7 2x \cos 2x \, dx$

66. $\int \sin^3 2x \sqrt{\cos 2x} \, dx$

67. $\int \sin^3 x \cos^2 x \, dx$

68. $\int \frac{\cos^3 \theta}{\sqrt{\sin \theta}} \, d\theta$

69. $\int \frac{\cos^5 \theta}{\sqrt{\sin \theta}} \, d\theta$

70. $\int \frac{\cos^2 x}{\sin^5 x} \, dx$

71. $\int \frac{\sin^3 x}{\cos^4 x} \, dx$

72. $\int \frac{\sin^4 x}{\cos^6 x} \, dx$

73. $\int \frac{2\cos x + 3\sin x}{\sin^3 x} \, dx$

74. $\int \frac{\sin 2x}{1+\sin^2 x} \, dx$

75. $\int \frac{2+\sin x + 2\cos x}{1+\cos x} \, dx$

76. $\int \frac{dx}{1-\cos x}$

77. $\int \frac{dx}{1-\sin x}$

78. $\int \frac{\sin \theta \cos \theta}{2-\cos \theta} \, d\theta$

79. $\int \tan^3 x \sec^3 x \, dx$

80. $\int \sec x \tan^2 x \, dx$

81. $\int \sec^2 x \tan^2 x \, dx$

82. $\int \sec^4 x \tan^2 x \, dx$

83. $\int \sec^6 4x \tan 4x \, dx$

84. $\int \sec^2 \frac{x}{2} \tan \frac{x}{2} \, dx$

85. $\int \tan^3 2x \sec^3 2x \, dx$

86. $\int \tan^5 2x \sec^4 2x \, dx$

87. $\int \tan^3 x \sec^5 x \, dx$

88. $\int \tan^3 x \sec^4 x \, dx$

89. $\int \tan^5 \theta \sec^4 \theta \, d\theta$

90. $\int \tan^5 \theta \sec^7 \theta \, d\theta$

91. $\int \tan^7 \theta \sec^5 \theta \, d\theta$

92. $\int \sec^4 3x \tan^3 3x \, dx$

93. $\int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2} \, dx$

94. $\int \sec^{-2} x \tan^3 x \, dx$

95. $\int \sqrt{\tan x} \sec^4 x \, dx$

96. $\int \tan^5\theta \csc^2\theta\, d\theta$

97. $\int \csc^2 x \cot x\, dx$

98. $\int \csc^{10} x \cot x\, dx$

99. $\int (\cot 2x - \csc 2x)^2\, dx$

100. $\int \operatorname{sech}^4 x\, dx$

101. $\int \sinh^3 x \cosh^2 x\, dx$

102. $\int \operatorname{sech}^2 x \sinh x\, dx$

103. $\int \frac{\tan^3 x}{\sqrt{\sec x}}\, dx$

104. $\int \frac{\tan^2 x}{\sec x}\, dx$

105. $\int \frac{\sec x}{\tan^2 x}\, dx$

106. $\int \frac{\sec^2 x}{\tan^5 x}\, dx$

107. $\int \frac{\csc^4 x}{\cot^2 x}\, dx$

108. $\int \frac{\sec^4(\ln x)}{x}\, dx$

109. $\int e^x \sec\left(e^x+1\right) dx$

110. $\int e^x \sec^3 e^x\, dx$

111. $\int e^x \sqrt{\tan^2 e^x + 1}\, dx$

112. $\int_0^{\sqrt{\frac{\pi}{2}}} x \sin^3\left(x^2\right) dx$

113. $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\, dx$

114. $\int_{\pi/6}^{\pi/2} \frac{dx}{\sin x}$

115. $\int_{\pi/6}^{\pi/3} \cot^3\theta\, d\theta$

116. $\int_0^{\pi/3} \tan^2 x\, dx$

117. $\int_0^{\pi/4} 6\tan^3 x\, dx$

118. $\int_0^{\pi/4} \tan^4 x\, dx$

119. $\int_0^{\pi} 8\sin^4 y \cos^2 y\, dy$

120. $\int_0^{\pi/6} 3\cos^5 3x\, dx$

121. $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta\, d\theta$

122. $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\, dx$

123. $\int_0^{\pi} \sqrt{1-\cos^2\theta}\, d\theta$

124. $\int_0^{\pi/6} \sqrt{1+\sin x}\, dx$

125. $\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2\theta - 1}\, d\theta$

126. $\int_0^{\pi} (1-\cos 2x)^{3/2}\, dx$

127. $\int_0^{\pi} \left(1-\cos^2 x\right)^{3/2} dx$

128. $\int_{-\pi}^{\pi} \left(1-\cos^2 x\right)^{3/2} dx$

129. $\int_{\pi/4}^{\pi/2} \csc^4\theta\, d\theta$

130. $\int_{-\pi}^{\pi} \sin 3x \sin 3x\, dx$

131. $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x\, dx$

132. $\int_0^{\pi/4} \cos^5 2x \sin^2 2x\, dx$

133. $\int_0^{\pi/6} \sin^5 x\, dx$

134. $\int_{-\pi}^{\pi} \sin^2 x\, dx$

135. $\int_{-\pi/2}^{\pi/2} \left(\sin^2 x + 1\right) dx$

136. $\int_0^{\pi/3} \sec^{3/2} x \tan x\, dx$

137. $\int_0^{\pi/2} \frac{\cos x}{1+\sin x}\, dx$

138. $\int_{\pi/6}^{\pi/3} \sin 6x \cos 4x\, dx$

139. $\int_{-\pi/2}^{\pi/2} 3\cos^3 x\, dx$

140. $\displaystyle\int_0^{\pi} \sec^2 x\,dx$

141. $\displaystyle\int_0^{\ln\left(\sqrt{3}+2\right)} \frac{\cosh x}{\sqrt{4-\sinh^2 x}}\,dx$

142. $\displaystyle\int_0^{\pi/2} \cos^4 x\,dx$

143. $\displaystyle\int_0^{\pi/2} \cos^{10}\theta\,d\theta$

144. $\displaystyle\int_0^{\pi/2} \cos^7 x\,dx$

145. $\displaystyle\int_0^{\pi/2} \cos^9 x\,dx$

146. $\displaystyle\int_0^{\pi/2} \sin^5 x\,dx$

147. $\displaystyle\int_0^{\pi/2} \sin^6 x\,dx$

148. $\displaystyle\int_0^{\pi/2} \sin^8 x\,dx$

149. $\displaystyle\int_0^{\pi/2} \tan^2 \frac{x}{2}\,dx$

150. Find the area of the region bounded by the graphs of $y=\tan x$ and $y=\sec x$ on the interval $\left[0,\ \frac{\pi}{4}\right]$

Find the area of the region bounded by the graphs of the equations

151. $y=\sin x,\quad y=\sin^3 x,\quad x=0,\quad x=\frac{\pi}{2}$

152. $y=\sin^2 \pi x,\quad y=0,\quad x=0,\quad x=1$

153. $y=\cos^2 x,\quad y=\sin^2 x,\quad x=-\frac{\pi}{4},\quad x=\frac{\pi}{4}$

154. $y=\cos^2 x,\quad y=\sin x\cos x,\quad x=-\frac{\pi}{2},\quad x=\frac{\pi}{4}$

Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis*

155. $y=\tan x,\quad y=0,\quad x=-\frac{\pi}{4},\quad x=\frac{\pi}{4}$

156. $y=\cos\frac{x}{2},\quad y=\sin\frac{x}{2},\quad x=0,\quad x=\frac{\pi}{2}$

Find the ***volume*** of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis*, then find the ***centroid*** of the region

157. $y=\sin x,\quad y=0,\quad x=0,\quad x=\pi$

158. $y=\cos x,\quad y=\sin 0,\quad x=0,\quad x=\frac{\pi}{2}$