Solution   *Section* 1.4 – Limits at Infinity

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $h(x) = \dfrac{-5+\frac{7}{x}}{3-\frac{1}{x^2}}$

Solution

$$\lim_{x\to\infty} \frac{-5+\frac{7}{x}}{3-\frac{1}{x^2}} = -\frac{5}{3}$$

$$\lim_{x\to-\infty} \frac{-5+\frac{7}{x}}{3-\frac{1}{x^2}} = -\frac{5}{3}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{2x+3}{5x+7}$

Solution

$$\lim_{x\to\infty} \frac{2x+3}{5x+7} = \lim_{x\to\infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}}$$

$$= \frac{2}{5}$$

$$\lim_{x\to-\infty} \frac{2x+3}{5x+7} = \lim_{x\to-\infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}}$$

$$= \frac{2}{5}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{2x^3+7}{x^3-x^2+x+7}$

Solution

$$\lim_{x\to\infty} \frac{2x^3+7}{x^3-x^2+x+7} = \lim_{x\to\infty} \frac{2+\frac{7}{x^3}}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}}$$

$$\underline{=2}$$

$$\lim_{x\to-\infty}\frac{2x^3+7}{x^3-x^2+x+7}=\lim_{x\to-\infty}\frac{2+\frac{7}{x^3}}{1-\frac{1}{x}+\frac{1}{x^2}+\frac{7}{x^3}}$$

$$\underline{=2}$$

Exercise

Find the limit as $x\to\infty$ and as $x\to-\infty$ of $f(x)=\frac{x+1}{x^2+3}$

Solution

$$\lim_{x\to\infty}\frac{x+1}{x^2+3}=\lim_{x\to\infty}\frac{\frac{x}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{3}{x^2}}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}}$$

$$\underline{=0}$$

$$\lim_{x\to-\infty}\frac{x+1}{x^2+3}=\lim_{x\to-\infty}\frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}}$$

$$\underline{=0}$$

Exercise

Find the limit as $x\to\infty$ and as $x\to-\infty$ of $f(x)=\frac{7x^3}{x^3-3x^2+6x}$

Solution

$$\lim_{x\to\infty}\frac{7x^3}{x^3-3x^2+6x}=\lim_{x\to\infty}\frac{7}{1-\frac{3}{x}+\frac{6}{x^2}}$$

$$\underline{=7}$$

$$\lim_{x\to-\infty}\frac{7x^3}{x^3-3x^2+6x}=\lim_{x\to-\infty}\frac{7}{1-\frac{3}{x}+\frac{6}{x^2}}$$

$$\underline{=7}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

Solution

$$\lim_{x\to\infty} \frac{9x^4 + x}{2x^4 + 5x^2 - x + 6} = \lim_{x\to\infty} \frac{\frac{9x^4}{x^4} + \frac{x}{x^4}}{\frac{2x^4}{x^4} + \frac{5x^2}{x^4} - \frac{x}{x^4} + \frac{6}{x^4}}$$

$$= \lim_{x\to\infty} \frac{9 + \frac{1}{x^3}}{2 + \frac{5}{x^2} - \frac{1}{x^3} + \frac{6}{x^4}}$$

$$= \frac{9}{2}$$

$$\lim_{x\to-\infty} \frac{9x^4 + x}{2x^4 + 5x^2 - x + 6} = \lim_{x\to-\infty} \frac{9 + \frac{1}{x^3}}{2 + \frac{5}{x^2} - \frac{1}{x^3} + \frac{6}{x^4}}$$

$$= \frac{9}{2}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$

Solution

$$\lim_{x\to\infty} \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x} = \lim_{x\to\infty} \frac{-2 - \frac{2}{x^2} + \frac{3}{x^3}}{3 + \frac{3}{x} - \frac{5}{x^2}}$$

$$= -\frac{2}{3}$$

$$\lim_{x\to-\infty} \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x} = \lim_{x\to-\infty} \frac{-2 - \frac{2}{x^2} + \frac{3}{x^3}}{3 + \frac{3}{x} - \frac{5}{x^2}}$$

$$= -\frac{2}{3}$$

Exercise

Find $\lim\limits_{x \to \infty} x^{12}$

Solution

$$\lim_{x \to \infty} x^{12} \underline{= \infty}$$

Exercise

Find $\lim\limits_{x \to -\infty} 3x^{9}$

Solution

$$\lim_{x \to -\infty} 3x^{9} \underline{= -\infty}$$

Exercise

Find $\lim\limits_{x \to -\infty} x^{-8}$

Solution

$$\lim_{x \to -\infty} x^{-8} = \frac{1}{(-\infty)^{8}}$$

$$\underline{= 0}$$

Exercise

Find $\lim\limits_{x \to -\infty} x^{-9}$

Solution

$$\lim_{x \to -\infty} x^{-9} = \frac{1}{(-\infty)^{9}}$$

$$\underline{= 0}$$

Exercise

Find $\lim\limits_{x \to -\infty} 2x^{-6}$

Solution

$$\lim_{x \to -\infty} 2x^{-6} = \frac{2}{\infty}$$

$$\underline{= 0}$$

Exercise

Find $\lim\limits_{x\to\infty}\left(3x^{12}-9x^{7}\right)$

Solution

$$\lim_{x\to\infty}\left(3x^{12}-9x^{7}\right)\ \underline{=\infty\ }\rfloor$$

Exercise

Find $\lim\limits_{x\to-\infty}\left(3x^{7}+x^{2}\right)$

Solution

$$\lim_{x\to-\infty}\left(3x^{7}+x^{2}\right)=\lim_{x\to-\infty}x^{2}\left(3x^{5}+1\right)$$

$$\underline{=-\infty\ }\rfloor$$

Exercise

Find $\lim\limits_{x\to-\infty}\left(-2x^{16}+2\right)$

Solution

$$\lim_{x\to-\infty}\left(-2x^{16}+2\right)\ \underline{=-\infty\ }\rfloor$$

Exercise

Find $\lim\limits_{x\to-\infty}\left(2x^{-6}+4x^{5}\right)$

Solution

$$\lim_{x\to-\infty}\left(2x^{-6}+4x^{5}\right)=\lim_{x\to-\infty}x^{-6}\left(2+4x^{11}\right) \qquad +\infty(-\infty)$$

$$\underline{=-\infty\ }\rfloor$$

Exercise

Find $\lim\limits_{x\to-\infty}\dfrac{\cos x}{3x}$

Solution

$$-\frac{1}{3x} \le \frac{\cos x}{3x} \le \frac{1}{3x}, \quad -1 \le \cos x \le 1$$

$$\lim_{x \to -\infty} \frac{\cos x}{3x} = 0 \qquad \textit{By the Sandwich Theorem}$$

Exercise

Find $\lim\limits_{x \to \infty} \dfrac{x + \sin x}{2x + 7 - 5\sin x}$

Solution

$$\lim_{x \to \infty} \frac{x + \sin x}{2x + 7 - 5\sin x} = \lim_{x \to \infty} \frac{1 + \frac{\sin x}{x}}{2 + \frac{7}{x} - \frac{5\sin x}{x}}$$

$$= \frac{1 + 0}{2 + 0 - 0}$$

$$= \frac{1}{2}$$

Exercise

Find $\lim\limits_{x \to \infty} \sqrt{\dfrac{8x^2 - 3}{2x^2 + x}}$

Solution

$$\lim_{x \to \infty} \sqrt{\frac{8x^2 - 3}{2x^2 + x}} = \lim_{x \to \infty} \sqrt{\frac{8 - \frac{3}{x^2}}{2 + \frac{1}{x}}}$$

$$= \sqrt{\frac{8}{2}}$$

$$= 2$$

Exercise

Find $\lim\limits_{x \to -\infty} \left(\dfrac{x^2 + x - 1}{8x^2 - 3} \right)^{1/3}$

Solution

$$\lim_{x \to -\infty} \left(\frac{x^2 + x - 1}{8x^2 - 3} \right)^{1/3} = \lim_{x \to -\infty} \left(\frac{1 + \frac{1}{x} - \frac{1}{x^2}}{8 - \frac{3}{x^2}} \right)^{1/3}$$

$$= \left(\frac{1}{8}\right)^{1/3}$$

$$= \frac{1}{2}$$

Exercise

Find $\displaystyle\lim_{x\to\infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7}$

Solution

$$\lim_{x\to\infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7} = \lim_{x\to\infty} \frac{\frac{2\sqrt{x}}{x} + \frac{x^{-1}}{x}}{3 - \frac{7}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{2}{x^{1/2}} + \frac{1}{x^2}}{3 - \frac{7}{x}}$$

$$= 0$$

Exercise

Find $\displaystyle\lim_{x\to\infty} \frac{x^{-1} + x^{-4}}{x^{-2} + x^{-3}}$

Solution

$$\lim_{x\to\infty} \frac{x^{-1} + x^{-4}}{x^{-2} + x^{-3}} = \lim_{x\to\infty} \frac{\frac{x^{-1}}{x^{-2}} + \frac{x^{-4}}{x^{-2}}}{\frac{x^{-2}}{x^{-2}} + \frac{x^{-3}}{x^{-2}}}$$

$$= \lim_{x\to\infty} \frac{x + \frac{1}{x^2}}{1 + \frac{1}{x}}$$

$$= \infty$$

Exercise

Find $\displaystyle\lim_{x\to-\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$

Solution

$$\lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}} = \lim_{x\to-\infty} \frac{\frac{4-3x^3}{\sqrt{x^6}}}{\frac{\sqrt{x^6+9}}{\sqrt{x^6}}}$$

$$= \lim_{x\to-\infty} \frac{\frac{4-3x^3}{x^3}}{\sqrt{\frac{x^6+9}{x^6}}}$$

$$= \lim_{x\to-\infty} \frac{\frac{4}{x3}-3}{\sqrt{1+\frac{9}{x^6}}}$$

$$= \frac{-3}{\sqrt{1}}$$

$$\underline{= -3}$$

Exercise

Find $\lim_{x\to-\infty} \left(\sqrt{x^2+3}+x\right)$

Solution

$$\lim_{x\to-\infty} \left(\sqrt{x^2+3}+x\right) = \lim_{x\to-\infty} \left(\sqrt{x^2+3}+x\right)\frac{\sqrt{x^2+3}-x}{\sqrt{x^2+3}-x}$$

$$= \lim_{x\to-\infty} \frac{x^2+3-x^2}{\sqrt{x^2+3}-x}$$

$$= \lim_{x\to-\infty} \frac{3}{\sqrt{x^2+3}-x}$$

$$= \lim_{x\to-\infty} \frac{\frac{3}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{3}{x^2}}-\frac{x}{x}}$$

$$= \lim_{x\to-\infty} \frac{\frac{3}{x}}{\sqrt{1+\frac{3}{x^2}}+1}$$

$$= \frac{0}{\sqrt{1}+1}$$

$$\underline{= 0}$$

Exercise

Find $\lim\limits_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right)$

Solution

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right) &= \lim_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right)\frac{\sqrt{x^2+3x}+\sqrt{x^2-2x}}{\sqrt{x^2+3x}+\sqrt{x^2-2x}} \\
&= \lim_{x\to\infty}\frac{\left(x^2+3x\right)-\left(x^2-2x\right)}{\sqrt{x^2+3x}+\sqrt{x^2-2x}} \\
&= \lim_{x\to\infty}\frac{x^2+3x-x^2+2x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}} \\
&= \lim_{x\to\infty}\frac{5x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}} \\
&= \lim_{x\to\infty}\frac{\frac{5x}{\sqrt{x^2}}}{\sqrt{\frac{x^2}{x^2}+\frac{3x}{x^2}}+\sqrt{\frac{x^2}{x^2}-\frac{2x}{x^2}}} \\
&= \lim_{x\to\infty}\frac{5}{\sqrt{1+\frac{3}{x}}+\sqrt{1-\frac{2}{x}}} \\
&= \frac{5}{\sqrt{1}+\sqrt{1}} \\
&= \frac{5}{2}
\end{aligned}$$

Exercise

Find $\lim\limits_{x\to\infty}\frac{2x-3}{4x+10}$

Solution

$$\lim_{x\to\infty}\frac{2x-3}{4x+10}=\frac{1}{2}$$

Exercise

Find $\lim\limits_{x\to\infty}\frac{x^4-1}{x^5+2}$

Solution

$$\lim_{x\to\infty} \frac{x^4-1}{x^5+2} = 0$$

Exercise

Find $\lim\limits_{x\to-\infty} \left(-3x^3+5\right)$

Solution

$$\lim_{x\to-\infty} \left(-3x^3+5\right) = \infty$$

Exercise

Find $\lim\limits_{x\to\infty} \left(e^{-2x}+\frac{2}{x}\right)$

Solution

$$\lim_{x\to\infty} \left(e^{-2x}+\frac{2}{x}\right) = e^{-\infty}+0$$

$$= 0$$

Exercise

Find $\lim\limits_{x\to\infty} \frac{1}{\ln x+1}$

Solution

$$\lim_{x\to\infty} \frac{1}{\ln x+1} = \frac{1}{\infty}$$

$$= 0$$

Exercise

Find $\lim\limits_{x\to\infty} \left(3+\frac{10}{x^2}\right)$

Solution

$$\lim_{x\to\infty} \left(3+\frac{10}{x^2}\right) = 3+0$$

$$= 3$$

Exercise

Find $\lim\limits_{x\to\infty}\left(5+\frac{1}{x}+\frac{10}{x^2}\right)$

Solution

$$\lim_{x\to\infty}\left(5+\frac{1}{x}+\frac{10}{x^2}\right)=5+0+0$$

$$\underline{=5\,|}$$

Exercise

Find $\lim\limits_{x\to\infty}\frac{4x^2+2x+3}{x^2}$

Solution

$$\lim_{x\to\infty}\frac{4x^2+2x+3}{x^2}=\lim_{x\to\infty}\frac{4x^2}{x^2}$$

$$\underline{=4\,|}$$

Exercise

Find $\lim\limits_{x\to\infty}\left(5+\frac{100}{x}+\frac{\sin^4 x^3}{x^2}\right)$

Solution

$$-1\le\sin\theta\le 1$$

$$0\le\sin^4\theta\le 1$$

$$0\le\frac{\sin^4\theta}{x^2}\le\frac{1}{x^2}\quad\to 0$$

$$\lim_{x\to\infty}\left(5+\frac{100}{x}+\frac{\sin^4 x^3}{x^2}\right)\ \underline{=5\,|}$$

Exercise

Find $\lim\limits_{\theta\to\infty}\frac{\cos\theta}{\theta^2}$

Solution

$$-1\le\cos\theta\le 1$$

$$-\frac{1}{\theta^2}\le\frac{\cos\theta}{\theta^2}\le\frac{1}{\theta^2}\quad\to 0$$

$$\lim_{\theta\to\infty} \frac{\cos\theta}{\theta^2} = 0$$

Exercise

Find $\lim_{\theta\to\infty} \frac{\cos\theta^5}{\sqrt{\theta}}$

Solution

$$-1 \le \cos\theta^5 \le 1$$

$$-\frac{1}{\sqrt{\theta}} \le \frac{\cos\theta^5}{\sqrt{\theta}} \le \frac{1}{\sqrt{\theta}} \quad \to 0$$

$$\lim_{\theta\to\infty} \frac{\cos\theta^5}{\sqrt{\theta}} = 0$$

Exercise

Find $\lim_{x\to\infty} \frac{4x}{20x+1}$

Solution

$$\lim_{x\to\infty} \frac{4x}{20x+1} = \frac{4}{20}$$

$$= \frac{1}{5}$$

Exercise

Find $\lim_{x\to-\infty} \frac{4x}{20x+1}$

Solution

$$\lim_{x\to-\infty} \frac{4x}{20x+1} = \lim_{x\to-\infty} \frac{4x}{20x}$$

$$= \frac{1}{5}$$

Exercise

Find $\lim_{x\to\infty} \frac{3x^2-7}{x^2+5x}$

Solution

$$\lim_{x\to\infty} \frac{3x^2-7}{x^2+5x} \underline{= 3}$$

Exercise

Find $\lim_{x\to-\infty} \frac{3x^2-7}{x^2+5x}$

Solution

$$\lim_{x\to-\infty} \frac{3x^2-7}{x^2+5x} = \lim_{x\to-\infty} \frac{3x^2}{x^2}$$

$$\underline{= 3}$$

Exercise

Find $\lim_{x\to\infty} \frac{6x^2-9x+8}{3x^2+2}$

Solution

$$\lim_{x\to\infty} \frac{6x^2-9x+8}{3x^2+2} = \lim_{x\to\infty} \frac{6x^2}{3x^2}$$

$$= \frac{6}{3}$$

$$\underline{= 2}$$

Exercise

Find $\lim_{x\to-\infty} \frac{6x^2-9x+8}{3x^2+2}$

Solution

$$\lim_{x\to-\infty} \frac{6x^2-9x+8}{3x^2+2} = \lim_{x\to-\infty} \frac{6x^2}{3x^2}$$

$$= \frac{6}{3}$$

$$\underline{= 2}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{4x^2-7}{8x^2+5x+2}$

Solution

$$\lim_{x\to\infty} \frac{4x^2-7}{8x^2+5x+2} = \lim_{x\to\infty} \frac{4x^2}{8x^2}$$

$$= \frac{4}{8}$$

$$= \frac{1}{2}$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{4x^2-7}{8x^2+5x+2}$

Solution

$$\lim_{x\to-\infty} \frac{4x^2-7}{8x^2+5x+2} = \lim_{x\to-\infty} \frac{4x^2}{8x^2}$$

$$= \frac{4}{8}$$

$$= \frac{1}{2}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

Solution

$$\lim_{x\to\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4} = \lim_{x\to\infty} \frac{\sqrt{16x^4}+x^2}{2x^2}$$

$$= \lim_{x\to\infty} \frac{4x^2+x^2}{2x^2}$$

$$= \lim_{x\to\infty} \frac{5x^2}{2x^2}$$

$$= \frac{5}{2}$$

Exercise

Find $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

Solution

$$\lim_{x\to-\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4} = \lim_{x\to-\infty} \frac{\sqrt{16x^4}+x^2}{2x^2}$$

$$= \lim_{x\to-\infty} \frac{4x^2+x^2}{2x^2}$$

$$= \lim_{x\to-\infty} \frac{5x^2}{2x^2}$$

$$= \frac{5}{2}$$

Exercise

Find $\displaystyle\lim_{x\to\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

Solution

$$\lim_{x\to\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144} = \lim_{x\to\infty} \frac{3x^4}{x^4}$$

$$= 3$$

Exercise

Find $\displaystyle\lim_{x\to-\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

Solution

$$\lim_{x\to-\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144} = \lim_{x\to-\infty} \frac{3x^4}{x^4}$$

$$= 3$$

Exercise

Find $\displaystyle\lim_{x\to\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right)$

Solution

$$\lim_{x\to\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right)=\infty-\infty$$

$$=\lim_{x\to\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right)\cdot\frac{4x^2+\sqrt{16x^4+1}}{4x^2+\sqrt{16x^4+1}}$$

$$=\lim_{x\to\infty} 16x^2\,\frac{16x^4-16x^4-1}{4x^2+\sqrt{16x^4+1}}$$

$$=\lim_{x\to\infty} 16x^2\,\frac{-1}{4x^2+\sqrt{16x^4+1}}$$

$$=\lim_{x\to\infty} \frac{-16x^2}{4x^2+4x^2}$$

$$=\lim_{x\to\infty} \frac{-16x^2}{8x^2}$$

$$\underline{=-2}$$

Exercise

Find $\displaystyle\lim_{x\to-\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right)$

Solution

$$\lim_{x\to-\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right)=\infty-\infty$$

$$=\lim_{x\to-\infty} 16x^2\left(4x^2-\sqrt{16x^4+1}\right)\cdot\frac{4x^2+\sqrt{16x^4+1}}{4x^2+\sqrt{16x^4+1}}$$

$$=\lim_{x\to-\infty} 16x^2\,\frac{16x^4-16x^4-1}{4x^2+\sqrt{16x^4+1}}$$

$$=\lim_{x\to-\infty} 16x^2\,\frac{-1}{4x^2+\sqrt{16x^4+1}}$$

$$=\lim_{x\to-\infty} \frac{-16x^2}{4x^2+4x^2}$$

$$=\lim_{x\to-\infty} \frac{-16x^2}{8x^2}$$

$$\underline{=-2}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{x-1}{x^{2/3}-1}$

Solution

$$\lim_{x\to\infty} \frac{x-1}{x^{2/3}-1} = \lim_{x\to\infty} \frac{x}{x^{2/3}}$$

$$= \lim_{x\to\infty} x^{1/3}$$

$$= \infty$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{x-1}{x^{2/3}-1}$

Solution

$$\lim_{x\to-\infty} \frac{x-1}{x^{2/3}-1} = \lim_{x\to-\infty} \frac{x}{x^{2/3}}$$

$$= \lim_{x\to-\infty} x^{1/3}$$

$$= -\infty$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+2x+6}-3}{x-1}$

Solution

$$\lim_{x\to\infty} \frac{\sqrt{x^2+2x+6}-3}{x-1} = \lim_{x\to\infty} \frac{\sqrt{x^2}}{x}$$

$$= \lim_{x\to\infty} \frac{x}{x}$$

$$= 1$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{\left|1-x^2\right|}{x(x+1)}$

Solution

$$\lim_{x\to\infty} \frac{\left|1-x^2\right|}{x(x+1)} = \lim_{x\to\infty} \frac{x^2-1}{x^2+1}$$

$$= \lim_{x\to\infty} \frac{x^2}{x^2}$$

$$\underline{=1}$$

Exercise

Find $\lim_{x\to\infty} \left(\sqrt{|x|}-\sqrt{|x-1|}\right)$

Solution

$$\lim_{x\to\infty} \left(\sqrt{|x|}-\sqrt{|x-1|}\right) = \infty-\infty \qquad x\to\infty \Rightarrow |x|=x \ \ \& \ \ |x-1|=x-1$$

$$= \lim_{x\to\infty} \left(\sqrt{x}-\sqrt{x-1}\right)\cdot\frac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}}$$

$$= \lim_{x\to\infty} \frac{x-x+1}{\sqrt{x}+\sqrt{x-1}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x}+\sqrt{x-1}}$$

$$= \frac{1}{\infty}$$

$$\underline{=0}$$

Exercise

Find $\lim_{x\to\infty} \frac{\tan^{-1} x}{x}$

Solution

$$-\frac{\pi}{2} \le \tan^{-1} x \le \frac{\pi}{2}$$

$$-\frac{\pi}{2x} \le \frac{\tan^{-1} x}{x} \le \frac{\pi}{2x} \quad \to \ 0$$

$$\lim_{x\to\infty} \frac{\tan^{-1} x}{x} \underline{=0}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{\cos x}{e^{3x}}$

Solution

$$-1 \le \cos x \le 1$$

$$-\frac{1}{e^{3x}} \le \frac{\cos x}{e^{3x}} \le \frac{1}{e^{3x}} \;\to\; 0$$

$$\lim_{x\to\infty} \frac{\cos x}{e^{3x}} = 0$$

Exercise

Find $\lim\limits_{x\to 0} \dfrac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}}$

Solution

$$\lim_{x\to 0} \frac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}} = \frac{2+10}{1+1}$$

$$= 6$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}}$

Solution

$$\lim_{x\to\infty} \frac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}} = \lim_{x\to\infty} \frac{2e^{x}}{e^{x}} \qquad \lim_{x\to\infty} e^{-x} = 0$$

$$= 2$$

Exercise

Find $\lim\limits_{x\to -\infty} \dfrac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}}$

Solution

$$\lim_{x\to\infty} \frac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}} = \lim_{x\to\infty} \frac{10e^{-x}}{e^{-x}} \qquad \lim_{x\to -\infty} e^{x} = 0$$

$$= 10$$

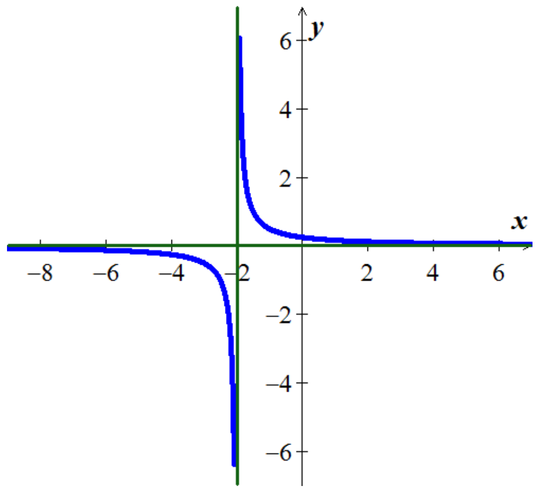
Exercise

Graph the rational function $y = \dfrac{1}{2x+4}$. Include the equations of the asymptotes.

Solution

$\boldsymbol{VA}$: $2x = 4 = 0 \quad \Rightarrow \quad \boxed{x = -2}$

$\boldsymbol{HA}$: $\boxed{y = 0}$



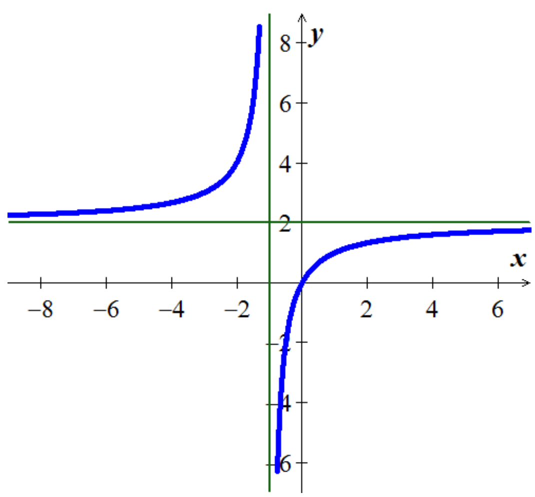
Exercise

Graph the rational function $y = \dfrac{2x}{x+1}$. Include the equations of the asymptotes.

Solution

$\boldsymbol{VA}$: $\boxed{x = -1}$

$\boldsymbol{HA}$: $\boxed{y = 2}$



Exercise

Graph the rational function $y = \frac{x^2}{x-1}$. Include the equations of the asymptotes.

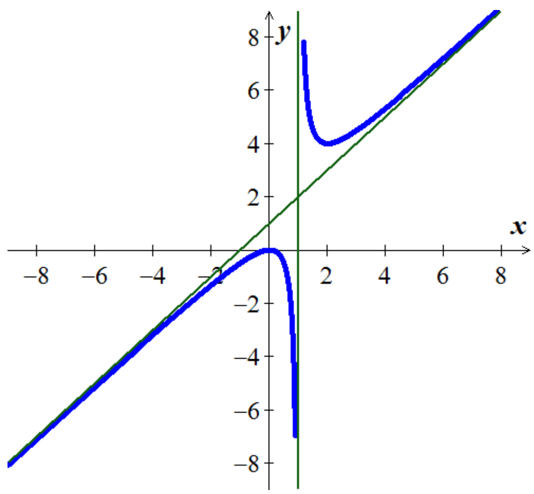
Solution

$$\begin{array}{r} x+1 \\ x-1 \overline{\big) \; x^2 \qquad} \\ \underline{x^2 - x} \\ x \quad \\ \underline{x-1} \\ \boxed{1} \end{array}$$

$$\begin{aligned} y &= \frac{x^2}{x-1} \\ &= x+1+\frac{1}{x-1} \end{aligned}$$

VA: $\underline{x=1}$

Oblique Asymptote: $\underline{y = x+1}$



Exercise

Graph the rational function $y = \frac{x^3+1}{x^2}$. Include the equations of the asymptotes.

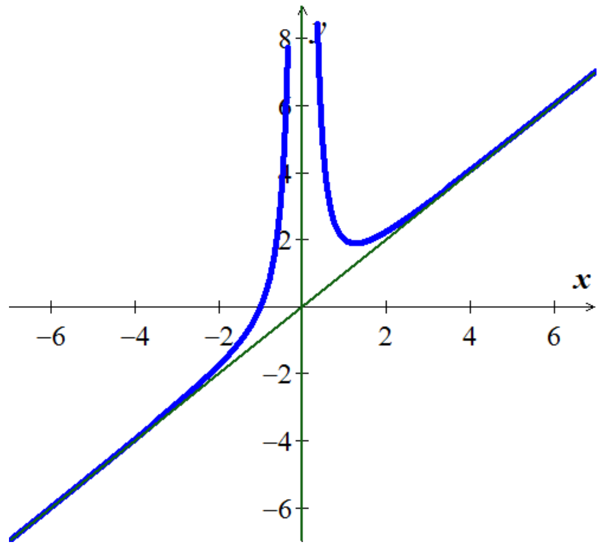
Solution

$$\begin{array}{r} x \quad \\ x^2 \overline{\big) \; x^3 + 1} \\ \underline{x^3 \qquad} \\ \boxed{1} \end{array}$$

$$y = \frac{x^3+1}{x^2} = x + \frac{1}{x^2}$$

VA: $\underline{x = 0}$

Oblique Asymptote: $\underline{y = x}$



Exercise

Let $\quad f(x) = \dfrac{x^2 - 5x + 6}{x^2 - 2x}$

a) Analyze $\lim\limits_{x \to 0^-} f(x)$, $\lim\limits_{x \to 0^+} f(x)$, $\lim\limits_{x \to 2^-} f(x)$, and $\lim\limits_{x \to 2^+} f(x)$

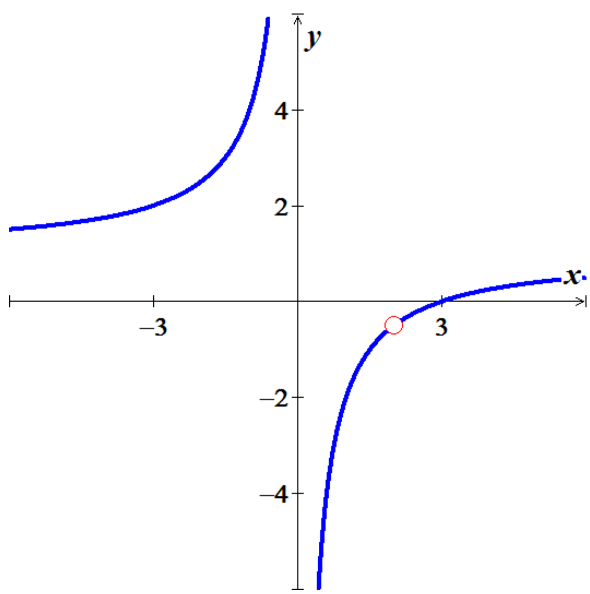
b) Does the graph of f have any vertical asymptotes? Explain?

Solution

$$f(x) = \frac{x^2 - 5x + 6}{x^2 - 2x}$$

$$= \frac{(x-2)(x-3)}{x(x-2)}$$

$$= \frac{x-3}{x}$$



a) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \dfrac{x-3}{x}$

$$= \frac{-3}{0^-}$$

$$\underline{= \infty}$$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \dfrac{x-3}{x}$

$$= \frac{-3}{0^+}$$

$$\boxed{= -\infty}$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x-3}{x}$$

$$= \frac{2-3}{2}$$

$$\boxed{= -\frac{1}{2}}$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x-3}{x}$$

$$= \frac{2-3}{2}$$

$$\boxed{= -\frac{1}{2}}$$

b) ***VA*** $: x = 0$ ***Hole*** $: x = 2 \to f(2) = -\frac{1}{2}$

HA $: y = 1$ ***OA*** $: n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \frac{3x}{1-x}$

Solution

VA $: x = 1,$ ***Hole*** $: n/a,$ ***HA*** $: y = -3,$ ***OA*** $: n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \frac{x^2}{x^2+9}$

Solution

VA $: n/a;$ ***Hole*** $: n/a;$ ***HA*** $: y = 1;$ ***OA*** $: n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y = \frac{x-2}{x^2-4x+3}$

Solution

VA $: x = 1,\ 3;$ ***Hole*** $: n/a;$ ***HA*** $: y = 0;$ ***OA*** $: n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{5x-1}{1-3x}$

Solution

$$\boldsymbol{VA}:\ x=\frac{1}{3};\quad \boldsymbol{Hole}:\ n/a;\quad \boldsymbol{HA}:\ y=-\frac{5}{3};\quad \boldsymbol{OA}:\ n/a$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{3}{x-5}$

Solution

$$\boldsymbol{VA}: x=5,\quad \boldsymbol{Hole}:\ n/a,\quad \boldsymbol{HA}: y=0,\quad \boldsymbol{OA}:\ n/a$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{x^3-1}{x^2+1}$

Solution

$$\begin{array}{r|l} & x \\ \hline x^2+1 & x^3-1 \\ & \underline{x^3+x} \\ & \boxed{-x-1} \end{array}$$

$$\begin{aligned} y &= \frac{x^3-1}{x^2+1} \\ &= x+\frac{-x-1}{x^2+1} \\ &= x-\frac{x+1}{x^2+1} \end{aligned}$$

$$\boldsymbol{VA}:\ n/a,\quad \boldsymbol{Hole}:\ n/a,\quad \boldsymbol{HA}:\ n/a,\quad \boldsymbol{OA}:\ y=x$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{3x^2-27}{(x+3)(2x+1)}$

Solution

$$\boldsymbol{VA}: x=-3,\ -\frac{1}{2};\quad \boldsymbol{Hole}:\ n/a;\quad \boldsymbol{HA}: y=\frac{3}{2};\quad \boldsymbol{OA}:\ n/a$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{x^3+3x^2-2}{x^2-4}$

Solution

$$\begin{array}{r|l} & x+3 \\ \hline x^2-4 & x^3+3x^2-2 \\ & \underline{x^3 \quad\; -4x} \\ & \quad 3x^2+4x-2 \end{array}$$

$$\begin{aligned} y &= \frac{x^3+3x^2-2}{x^2-4} \\ &= x+3+\frac{4x+10}{x^2-4} \end{aligned}$$

$\boldsymbol{VA}: \ x=\pm 2, \quad \boldsymbol{Hole}: \ n/a, \quad \boldsymbol{HA}: \ n/a, \quad \boldsymbol{OA}: \ y=x+3$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{x-3}{x^2-9}$

Solution

$\boldsymbol{VA}: \ x=-3; \quad \boldsymbol{Hole}: \ x=3; \quad \boldsymbol{HA}: \ y=0; \quad \boldsymbol{OA}: \ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{6}{\sqrt{x^2-4x}}$

Solution

$\boldsymbol{VA}: \ x=0, \ 4; \quad \boldsymbol{Hole}: \ n/a; \quad \boldsymbol{HA}: \ y=0; \quad \boldsymbol{OA}: \ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{4x^3+1}{1-x^3}$

Solution

$\boldsymbol{VA}: \ x=1; \quad \boldsymbol{Hole}: \ n/a; \quad \boldsymbol{HA}: \ y=-4; \quad \boldsymbol{OA}: \ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x) = \dfrac{x+1}{\sqrt{9x^2+x}}$

Solution

$\textbf{VA}:\ x = 0,\ -\frac{1}{9};\quad \textbf{Hole}:\ n/a;\quad \textbf{HA}:\ y = \frac{1}{3};\quad \textbf{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x) = 1 - e^{-2x}$

Solution

$\textbf{VA}:\ n/a;\quad \textbf{Hole}:\ n/a;\quad \textbf{HA}:\ y = 1;\quad \textbf{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x) = \dfrac{1}{\ln x^2}$

Solution

$\textbf{VA}:\ x = 0;\quad \textbf{Hole}:\ n/a;\quad \textbf{HA}:\ y = 0;\quad \textbf{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x) = \dfrac{1}{\tan^{-1} x}$

Solution

$\textbf{VA}:\ x = 0;\quad \textbf{Hole}:\ n/a;\quad \textbf{HA}:\ y = \dfrac{1}{\frac{\pi}{2}} = \dfrac{2}{\pi};\quad \textbf{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x) = \dfrac{2x^2+6}{2x^2+3x-2}$

Solution

$\textbf{VA}:\ x = -2,\ \frac{1}{2};\quad \textbf{Hole}:\ n/a;\quad \textbf{HA}:\ y = 1;\quad \textbf{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{3x^2+2x-1}{4x+1}$

Solution

$$\begin{array}{r|l} & \frac{3}{4}x+\frac{5}{16} \\ \hline 4x+1 & 3x^2+2x-1 \\ & \underline{3x^2+\frac{3}{4}x} \\ & \qquad \frac{5}{4}x-1 \end{array}$$

$\boldsymbol{VA}:\ x=-\frac{1}{4};\quad \boldsymbol{Hole}:\ n/a;\quad \boldsymbol{HA}:\ n/a;\quad \boldsymbol{OA}:\ y=\frac{3}{4}x+\frac{5}{16}$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{9x^2+4}{(2x-1)^2}$

Solution

$\boldsymbol{VA}:\ x=\frac{1}{2};\quad \boldsymbol{Hole}:\ n/a;\quad \boldsymbol{HA}:\ y=\frac{9}{4};\quad \boldsymbol{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{1+x-2x^2-x^3}{x^2+1}$

Solution

$$\begin{array}{r|l} & \qquad -x-2 \\ \hline x^2+1 & -x^3-2x^2+x+1 \\ & \underline{-x^3 \qquad\quad -x} \\ & \quad -2x^2+2x \end{array}$$

$\boldsymbol{VA}:\ n/a;\quad \boldsymbol{Hole}:\ n/a;\quad \boldsymbol{HA}:\ n/a;\quad \boldsymbol{OA}:\ y=-x-2$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{x(x+2)^3}{3x^2-4x}$

Solution

$$f(x)=\frac{x\left(x^3+6x^2+12x+8\right)}{x(3x-4)}$$

$$=\frac{x^3+6x^2+12x+8}{3x-4}$$

$$\begin{array}{r|l} & \frac{1}{3}x^2+\frac{22}{9}x+\frac{196}{27} \\ \hline 3x-4 & x^3+6x^2+12x+8 \\ & x^3-\frac{4}{3}x^2 \\ \hline & \frac{22}{3}x^2+12x \\ & \frac{22}{3}x^2-\frac{88}{9}x \\ \hline & \frac{196}{9}x \end{array}$$

VA: $x=\frac{4}{3}$; ***Hole***: $(0,\ -2)$; ***HA***: n/a; ***OA***: $y=\frac{1}{3}x^2+\frac{22}{9}x+\frac{196}{27}$

Exercise

Find the limit $\displaystyle\lim_{x\to 0}\frac{x^2-4x+4}{x^3+5x^2-14x}$

Solution

$$\lim_{x\to 0}\frac{x^2-4x+4}{x^3+5x^2-14x}=\frac{4}{0}$$

$$=\infty$$

Exercise

Find the limit $\displaystyle\lim_{x\to 2}\frac{x^2-4x+4}{x^3+5x^2-14x}$

Solution

$$\lim_{x\to 2}\frac{x^2-4x+4}{x^3+5x^2-14x}=\frac{4-8+4}{8+20-28}=\frac{0}{0}$$

$$=\lim_{x\to 2}\frac{(x-2)(x-2)}{x(x-2)(x+7)}$$

$$=\lim_{x\to 2}\frac{x-2}{x(x+7)}$$

$$=\frac{0}{18}$$

$$=0$$

Exercise

Find the limit $\displaystyle\lim_{x\to a} \frac{x^2-a^2}{x^4-a^4}$

Solution

$$\lim_{x\to a} \frac{x^2-a^2}{x^4-a^4} = \frac{a^2-a^2}{a^4-a^4} = \frac{0}{0}$$

$$= \lim_{x\to a} \frac{x^2-a^2}{\left(x^2-a^2\right)\left(x^2+a^2\right)}$$

$$= \lim_{x\to a} \frac{1}{x^2+a^2}$$

$$= \frac{1}{a^2+a^2}$$

$$= \frac{1}{2a^2}$$

Exercise

Find the limit $\displaystyle\lim_{x\to 0} \frac{(x+h)^2-x^2}{h}$

Solution

$$\lim_{x\to 0} \frac{(x+h)^2-x^2}{h} = \frac{h^2}{h}$$

$$= h$$

Exercise

Find the limit $\displaystyle\lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$

Solution

$$\lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \frac{x^2-x^2}{0} = \frac{0}{0}$$

$$= \lim_{h\to 0} \frac{x^2+2hx+h^2-x^2}{h}$$

$$= \lim_{h\to 0} \frac{2hx+h^2}{h}$$

$$= \lim_{h\to 0} (2x+h)$$

$$\underline{= 2x}$$

Exercise

Find the limit $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}$

Solution

$$\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{1-\sqrt{x}}{\left(1-\sqrt{x}\right)\left(1+\sqrt{x}\right)}$$

$$= \lim_{x\to 1} \frac{1}{1+\sqrt{x}}$$

$$\underline{= \frac{1}{2}}$$

$$\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}}$$

$$= \lim_{x\to 1} \frac{1-x}{(1-x)\left(1+\sqrt{x}\right)}$$

$$= \lim_{x\to 1} \frac{1}{1+\sqrt{x}}$$

$$\underline{= \frac{1}{2}}$$

Exercise

Find the limit $\lim_{x\to 0} \dfrac{\frac{1}{2+x} - \frac{1}{2}}{x}$

Solution

$$\lim_{x\to 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = \frac{\frac{1}{2} - \frac{1}{2}}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1}{x}\left(\frac{2-2-x}{2(2+x)}\right)$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{1}{x}\left(\frac{-x}{2+x}\right)$$

$$= -\frac{1}{2}\lim_{x\to 0}\frac{1}{2+x}$$

$$= -\frac{1}{2}\left(\frac{1}{2}\right)$$

$$= -\frac{1}{4}$$

Exercise

Find the limit $\lim_{x\to 1}\frac{x^{1/3}-1}{\sqrt{x}-1}$

Solution

$$\lim_{x\to 1}\frac{x^{1/3}-1}{\sqrt{x}-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to 1}\frac{x^{1/3}-1}{\sqrt{x}-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}$$

$$= \lim_{x\to 1}\frac{\left(x^{1/3}-1\right)\left(\sqrt{x}+1\right)}{x-1}$$

$$= \lim_{x\to 1}\frac{\left(x^{1/3}-1\right)\left(\sqrt{x}+1\right)}{\left(x^{1/3}\right)^3-1^3} \qquad a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$$

$$= \lim_{x\to 1}\frac{\left(x^{1/3}-1\right)\left(\sqrt{x}+1\right)}{\left(x^{1/3}-1\right)\left(x^{2/3}+x^{1/3}+1\right)}$$

$$= \lim_{x\to 1}\frac{\sqrt{x}+1}{x^{2/3}+x^{1/3}+1}$$

$$= \frac{2}{3}$$

Exercise

Find the limit $\lim_{x\to 64}\frac{x^{2/3}-16}{\sqrt{x}-8}$

Solution

$$\lim_{x\to 64}\frac{x^{2/3}-16}{\sqrt{x}-8} = \frac{\left(4^3\right)^{2/3}-16}{8-8}$$

$$=\frac{16-16}{0}=\frac{0}{0}$$

$$=\lim_{x\to 64}\frac{\left(x^{1/3}\right)^2-16}{\sqrt{x}-8}\cdot\frac{\sqrt{x}+8}{\sqrt{x}+8}$$

$$=\lim_{x\to 64}\frac{\left(x^{1/3}-4\right)\left(x^{1/3}+4\right)\left(\sqrt{x}+8\right)}{x-64}$$

$$=\lim_{x\to 64}\frac{\left(x^{1/3}-4\right)\left(x^{1/3}+4\right)\left(\sqrt{x}+8\right)}{\left(x^{1/3}\right)^3-4^3}$$

$$a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$$

$$=\lim_{x\to 64}\frac{\left(x^{1/3}-4\right)\left(x^{1/3}+4\right)\left(\sqrt{x}+8\right)}{\left(x^{1/3}-4\right)\left(x^{2/3}+4x^{1/3}+16\right)}$$

$$=\lim_{x\to 64}\frac{\left(x^{1/3}+4\right)\left(\sqrt{x}+8\right)}{x^{2/3}+4x^{1/3}+16}$$

$$=\frac{(4+4)(8+8)}{16+16+16}$$

$$=\frac{8}{3}$$

Exercise

Find the limit $\displaystyle\lim_{x\to 0}\frac{\tan(2x)}{\tan(\pi x)}$

Solution

$$\lim_{x\to 0}\frac{\tan(2x)}{\tan(\pi x)}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{\sin 2x}{\cos 2x}\cdot\frac{\cos(\pi x)}{\sin(\pi x)}$$

$$=\lim_{x\to 0}\frac{\cos(\pi x)}{\cos 2x}\cdot\frac{\sin 2x}{2x}\cdot\frac{2x}{\pi x}\cdot\frac{\pi x}{\sin(\pi x)}$$

$$=\frac{2}{\pi}\,\frac{\cos 0}{\cos 0}\cdot\lim_{2x\to 0}\frac{\sin 2x}{2x}\quad\lim_{\pi x\to 0}\frac{1}{\frac{\sin\pi x}{\pi x}}$$

$$=\frac{2}{\pi}$$

Exercise

Find the limit $\lim\limits_{x\to\pi^-} \csc x$

Solution

$$\begin{aligned}\lim_{x\to\pi^-} \csc x &= \frac{1}{\sin \pi^-} \\ &= \frac{1}{0^-} \\ &= -\infty\end{aligned}$$

Exercise

Find the limit $\lim\limits_{x\to\pi} \sin\left(\frac{x}{2}+\sin x\right)$

Solution

$$\begin{aligned}\lim_{x\to\pi} \sin\left(\frac{x}{2}+\sin x\right) &= \sin\left(\frac{\pi}{2}+\sin \pi\right) \\ &= \sin\frac{\pi}{2} \\ &= 1\end{aligned}$$

Exercise

Find the limit $\lim\limits_{x\to\pi} \cos^2(x-\tan x)$

Solution

$$\begin{aligned}\lim_{x\to\pi} \cos^2(x-\tan x) &= \cos^2(\pi-\tan \pi) \\ &= \cos^2(\pi) \\ &= (-1)^2 \\ &= 1\end{aligned}$$

Exercise

Find the limit $\lim\limits_{x\to 0} \frac{8x}{3\sin x - x}$

Solution

$$\lim_{x\to 0} \frac{8x}{3\sin x - x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{8\frac{x}{x}}{3\frac{\sin x}{x} - \frac{x}{x}}$$

$$= \frac{8}{3\lim\limits_{x\to 0} \frac{\sin x}{x} - 1} \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$= \frac{8}{3-1}$$

$$\underline{= 4}$$

Exercise

Find the limit $\lim\limits_{x\to 0} \frac{\cos 2x - 1}{\sin x}$

Solution

$$\lim_{x\to 0} \frac{\cos 2x - 1}{\sin x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1 - 2\sin^2 x - 1}{\sin x}$$

$$= \lim_{x\to 0} \frac{-2\sin^2 x}{\sin x}$$

$$= -2 \lim_{x\to 0} \sin x$$

$$\underline{= 0}$$

Exercise

Find the limit $\lim\limits_{x\to -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$

Solution

$$\lim_{x\to -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}} = \lim_{x\to -\infty} \frac{3x^3}{\sqrt{x^6}}$$

$$= \lim_{x\to -\infty} \frac{3x^3}{x^3}$$

$$\underline{= 3}$$

Exercise

Find the limit $\lim\limits_{x\to-\infty} \dfrac{x^2-4x+8}{3x^3}$

Solution

$$\lim_{x\to-\infty} \frac{x^2-4x+8}{3x^3} = \lim_{x\to-\infty} \frac{x^2}{3x^3}$$

$$= \lim_{x\to-\infty} \frac{1}{3x}$$

$$\underline{= 0}$$

Exercise

Find the limit $\lim\limits_{x\to-\infty} \dfrac{2x^2+3}{5x^2+7}$

Solution

$$\lim_{x\to-\infty} \frac{2x^2+3}{5x^2+7} = \frac{2}{5}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{x^4+x^3}{12x^3+128}$

Solution

$$\lim_{x\to\infty} \frac{x^4+x^3}{12x^3+128} = \infty$$

Exercise

Find the limit $\lim\limits_{x\to-\infty} \dfrac{2+\sqrt{x}}{2-\sqrt{x}}$

Solution

Since $x\to-\infty$ and inside the square root can't be negative

$$\lim_{x\to-\infty} \frac{2+\sqrt{x}}{2-\sqrt{x}} = \nexists$$

Exercise

Find the limit $\displaystyle\lim_{x\to\infty} \frac{2+\sqrt{x}}{2-\sqrt{x}}$

Solution

$$\lim_{x\to\infty} \frac{2+\sqrt{x}}{2-\sqrt{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{-\sqrt{x}}$$

$$= -1$$

Exercise

Find the limit $\displaystyle\lim_{x\to-\infty} \frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}$

Solution

$$\lim_{x\to-\infty} \frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}} = \lim_{x\to-\infty} \frac{\sqrt[3]{x}}{\sqrt[3]{x}}$$

$$= 1$$

Exercise

Find the limit $\displaystyle\lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^4}}{\frac{1}{x^2}-\frac{1}{x^3}}$

Solution

$$\lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^4}}{\frac{1}{x^2}-\frac{1}{x^3}} = \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{\frac{x^3+1}{x^4}}{\frac{x-1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{x^3+1}{x-1}\cdot\frac{x^3}{x^4}$$

$$= \lim_{x\to\infty} \frac{x^3+1}{x(x-1)}$$

$$= \lim_{x\to\infty} \frac{x^3}{x^2}$$

$$= \infty$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{2x^{5/3}-x^{1/3}+7}{x^{8/5}+3x+\sqrt{x}}$

Solution

$$\lim_{x\to\infty} \frac{2x^{5/3}-x^{1/3}+7}{x^{8/5}+3x+\sqrt{x}} = \lim_{x\to\infty} \frac{2x^{5/3}}{x^{8/5}}$$

$$= \lim_{x\to\infty} 2x^{\left(\frac{5}{3}-\frac{8}{5}\right)}$$

$$= \lim_{x\to\infty} 2x^{\frac{1}{15}}$$

$$\underline{=\infty}$$

Exercise

Find the limit $\lim\limits_{x\to 2^+} \ln(x-2)$

Solution

$$\lim_{x\to 2^+} \ln(x-2) = \ln\left(0^+\right)$$

$$\underline{=-\infty}$$

Exercise

Find the limit $\lim\limits_{x\to 1} x^2\ln\left(2-\sqrt{x}\right)$

Solution

$$\lim_{x\to 1} x^2\ln\left(2-\sqrt{x}\right) = \ln(2-1)$$

$$= \ln 1$$

$$\underline{=0}$$

Exercise

Find the limit $\lim\limits_{\theta\to 0^+} \sqrt{\theta}\, e^{\cos\frac{\pi}{\theta}}$

Solution

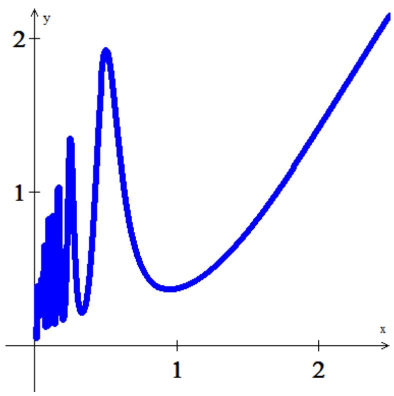
$$\lim_{\theta \to 0^+} \sqrt{\theta}\, e^{\cos\frac{\pi}{\theta}} = 0 \cdot e^{\cos\infty}$$

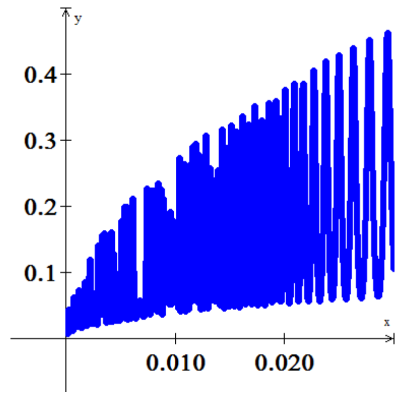
$$-1 \le \cos\frac{\pi}{\theta} \le 1$$

$$e^{-1} \le e^{\cos\frac{\pi}{\theta}} \le e$$

$$0 \cdot \frac{1}{e} \le 0 \cdot e^{\cos\frac{\pi}{\theta}} \le 0 \cdot e$$

$$\boxed{\lim_{\theta \to 0^+} \sqrt{\theta}\, e^{\cos\frac{\pi}{\theta}} = 0}$$





Exercise

Find the limit $\displaystyle\lim_{x \to \infty} \frac{2x-3}{5x+6}$

Solution

$$\lim_{x \to \infty} \frac{2x-3}{5x+6} = \lim_{x \to \infty} \frac{2x}{5x}$$

$$\boxed{= \frac{2}{5}}$$

Exercise

Find the limit $\displaystyle\lim_{x \to \infty} \frac{2x^2-3}{5x^2+6}$

Solution

$$\lim_{x \to \infty} \frac{2x^2-3}{5x^2+6} = \lim_{x \to \infty} \frac{2x^2}{5x^2}$$

$$\boxed{= \frac{2}{5}}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{2x-3}{5x^3+6}$

Solution

$$\lim_{x\to\infty} \frac{2x-3}{5x^3+6} = \lim_{x\to\infty} \frac{2x}{5x^3}$$

$$\underline{= 0}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{1}{5x^2-3x+6}$

Solution

$$\lim_{x\to\infty} \frac{1}{5x^2-3x+6} = \lim_{x\to\infty} \frac{1}{5x^2}$$

$$\underline{= 0}$$

Exercise

Find the limit $\lim\limits_{\theta\to 0} \dfrac{\theta \cot 4\theta}{\sin^2\theta \cot^2 2\theta}$

Solution

$$\lim_{\theta\to 0} \frac{\theta \cot 4\theta}{\sin^2\theta \cot^2 2\theta} = \frac{0}{0}$$

$$= \lim_{\theta\to 0} \frac{1}{\frac{\sin\theta}{\theta}} \cdot \frac{1}{\sin\theta} \cdot \frac{\cos 4\theta}{\sin 4\theta} \cdot \frac{\sin^2 2\theta}{\cos^2 2\theta}$$

$$= \lim_{\theta\to 0} \frac{1}{\frac{\sin\theta}{\theta}} \lim_{\theta\to 0} \frac{\cos 4\theta}{\cos^2 2\theta} \lim_{\theta\to 0} \frac{1}{\sin\theta} \cdot \frac{\sin 2\theta \sin 2\theta}{2\sin 2\theta \cos 2\theta}$$

$$= (1)(1) \lim_{\theta\to 0} \frac{1}{\sin\theta} \cdot \frac{2\sin\theta\cos\theta}{2\cos 2\theta}$$

$$= \lim_{\theta\to 0} \frac{\cos\theta}{\cos 2\theta}$$

$$\underline{= 1}$$

Exercise

Find the limit $\lim\limits_{x\to 0^+} \dfrac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}$

Solution

$$\lim_{x\to 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x} = \frac{\sqrt{5}-\sqrt{5}}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x} \cdot \frac{\sqrt{x^2+4x+5}+\sqrt{5}}{\sqrt{x^2+4x+5}+\sqrt{5}}$$

$$= \lim_{x\to 0^+} \frac{x^2+4x+5-5}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$= \lim_{x\to 0^+} \frac{x(x+4)}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$= \lim_{x\to 0^+} \frac{x+4}{\sqrt{x^2+4x+5}+\sqrt{5}}$$

$$= \frac{4}{\sqrt{5}+\sqrt{5}}$$

$$= \frac{4}{2\sqrt{5}}$$

$$= \frac{2}{\sqrt{5}}$$

Exercise

Find the limit $\lim\limits_{x\to 2} \dfrac{x^4-16}{x-2}$

Solution

$$\lim_{x\to 2} \frac{x^4-16}{x-2} = \frac{16-16}{2-2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(x-2)(x+2)\left(x^2+4\right)}{x-2} \qquad a^4-b^4=(a-b)(a+b)\left(a^2+b^2\right)$$

$$= \lim_{x\to 2} (x+2)\left(x^2+4\right)$$

$$=(4)(8)$$

$$= 32$$

Exercise

Find the limit $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x - 2}$

Solution

$$\lim\limits_{x \to 2} \frac{x^3 - 8}{x - 2} = \frac{0}{0}$$

$$= \lim\limits_{x \to 2} \frac{(x-2)\left(x^2 + 2x + 4\right)}{x - 2}$$

$a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$

$$= \lim\limits_{x \to 2} \left(x^2 + 2x + 4\right)$$

$$= 4 + 4 + 4$$

$$= 12$$

Exercise

Find the limit $\lim\limits_{x \to -\infty} \dfrac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$

Solution

$$\lim\limits_{x \to -\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4} = \lim\limits_{x \to -\infty} \frac{-5x}{2x}$$

$$= -\frac{5}{2}$$

Exercise

Find the limit $\lim\limits_{x \to -\infty} \dfrac{\sqrt{x^2 + 1}}{x + 1}$

Solution

$$\lim\limits_{x \to -\infty} \frac{\sqrt{x^2 + 1}}{x + 1} = \lim\limits_{x \to -\infty} \frac{\sqrt{x^2}}{x}$$

$$= \lim\limits_{x \to -\infty} \frac{|x|}{x}$$

$$= -1$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+1}}{x+1}$

Solution

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1} &= \lim_{x\to\infty} \frac{\sqrt{x^2}}{x} \\ &= \lim_{x\to\infty} \frac{|x|}{x} \\ &= 1\end{aligned}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{x-3}{\sqrt{4x^2+25}}$

Solution

$$\begin{aligned}\lim_{x\to\infty} \frac{x-3}{\sqrt{4x^2+25}} &= \lim_{x\to\infty} \frac{x}{2|x|} \\ &= \frac{1}{2}\end{aligned}$$

Exercise

Find the limit $\lim\limits_{x\to-\infty} \dfrac{4-3x^3}{\sqrt{x^6+9}}$

Solution

$$\begin{aligned}\lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}} &= \lim_{x\to-\infty} \frac{3x^3}{x^3} \\ &= 3\end{aligned}$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{x^4-x}{15x^3+4}$

Solution

$$\boxed{\lim_{x\to\infty} \frac{x^4 - x}{15x^3 + 4} = \infty}$$

Exercise

Find the limit $\displaystyle\lim_{x\to\infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x}$

Solution

$-1 \le \sin x \le 1$

$$\lim_{x\to\infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x} = \lim_{x\to\infty} \frac{x}{x}$$

$$\boxed{=1}$$

Exercise

Find the limit $\displaystyle\lim_{x\to\infty} \frac{x^{2/3} - x^{-1}}{x^{2/3} + \cos^2 x}$

Solution

$-1 \le \cos x \le 1$

$0 \le \cos^2 x \le 1$

$$\lim_{x\to\infty} \frac{x^{2/3} - \frac{1}{x}}{x^{2/3} + \cos^2 x} = \lim_{x\to\infty} \frac{x^{2/3}}{x^{2/3}}$$

$$\boxed{=1}$$

Exercise

Find the limit $\displaystyle\lim_{x\to\infty} \frac{\sin 2x}{x}$

Solution

$-1 \le \sin 2x \le 1$

$$-\lim_{x\to\infty} \frac{1}{x} \le \lim_{x\to\infty} \frac{\sin 2x}{x} \le \lim_{x\to\infty} \frac{1}{x}$$

$$0 \le \lim_{x\to\infty} \frac{\sin 2x}{x} \le 0$$

$$\boxed{\lim_{x\to\infty} \frac{\sin 2x}{x} = 0}$$

Exercise

Find the limit $\lim_{x\to 0} \frac{\sin 5x}{3x}$

Solution

$$\lim_{x\to 0} \frac{\sin 5x}{3x} = \lim_{5x\to 0} \frac{5}{3} \cdot \frac{\sin 5x}{5x}$$

$$\boxed{= \frac{5}{3}}$$

Exercise

Find the limit $\lim_{x\to -\infty} \frac{\cos x}{2x}$

Solution

$$-1 \le \cos x \le 1$$

$$-\lim_{x\to\infty} \frac{1}{2x} \le \lim_{x\to\infty} \frac{\cos x}{2x} \le \lim_{x\to\infty} \frac{1}{2x}$$

$$0 \le \lim_{x\to\infty} \frac{\cos x}{2x} \le 0$$

$$\boxed{\lim_{x\to\infty} \frac{\cos x}{2x} = 0}$$

Exercise

Find the limit $\lim_{x\to -\infty} \left(\frac{x^2 + x - 1}{8x^2 - 3}\right)^{1/3}$

Solution

$$\lim_{x\to -\infty} \left(\frac{x^2 + x - 1}{8x^2 - 3}\right)^{1/3} = \lim_{x\to -\infty} \left(\frac{x^2}{8x^2}\right)^{1/3}$$

$$= \left(\frac{1}{2^3}\right)^{1/3}$$

$$\boxed{= \frac{1}{2}}$$

Exercise

Find the limit $\lim\limits_{x \to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$

Solution

$$\lim_{x \to -1} \frac{\sqrt{x^2+8}-3}{x+1} = \frac{3-3}{-1+1} = \frac{0}{0}$$

$$= \lim_{x \to -1} \frac{\sqrt{x^2+8}-3}{x+1} \cdot \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$$

$$= \lim_{x \to -1} \frac{x^2+8-9}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x \to -1} \frac{x^2-1}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x \to -1} \frac{(x-1)(x+1)}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x \to -1} \frac{x+1}{\sqrt{x^2+8}+3}$$

$$= \frac{0}{6}$$

$$= 0$$

Exercise

Find the limit $\lim\limits_{x \to -\infty} \left(\dfrac{1-x^3}{x^2+7x}\right)^5$

Solution

$$\lim_{x \to -\infty} \left(\frac{1-x^3}{x^2+7x}\right)^5 = \lim_{x \to -\infty} \left(\frac{-x^3}{x^2}\right)^5$$

$$= \lim_{x \to -\infty} \left(-x^5\right)$$

$$= \infty$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \sqrt{\dfrac{x^2-5x}{x^3+x-2}}$

Solution

$$\lim_{x\to\infty} \sqrt{\frac{x^2-5x}{x^3+x-2}} = \lim_{x\to\infty} \sqrt{\frac{x^2}{x^3}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x}}$$

$$= 0$$

Exercise

Find the limit $\lim\limits_{x\to\infty} \dfrac{2\sqrt{x}+x^{-1}}{3x-7}$

Solution

$$\lim_{x\to\infty} \frac{2\sqrt{x}+x^{-1}}{3x-7} = \lim_{x\to\infty} \frac{2\sqrt{x}}{3x}$$

$$= \lim_{x\to\infty} \frac{2}{3\sqrt{x}}$$

$$= 0$$

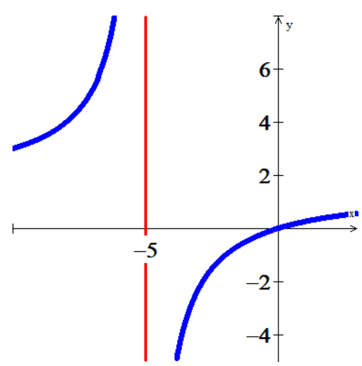
Exercise

Find the limit $\lim\limits_{x\to -5^-} \dfrac{3x}{2x+10}$

Solution

$$\lim_{x\to -5^-} \frac{3x}{2x+10} = \frac{-15}{0^-}$$

$$= \infty$$



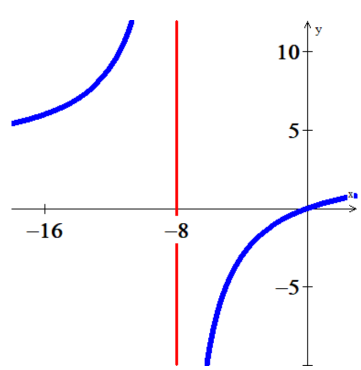
Exercise

Find the limit $\lim\limits_{x\to -8^+} \dfrac{3x}{x+8}$

Solution

$$\lim_{x\to -8^+} \frac{3x}{x+8} = \frac{-24}{0^+}$$

$$= -\infty$$



Exercise

Find the limit $\displaystyle\lim_{x\to 0} \frac{-1}{x^2(x+1)}$

Solution

$$\lim_{x\to 0} \frac{-1}{x^2(x+1)} = -\frac{1}{0}$$

$$\boxed{= -\infty}$$

Exercise

Find the limit $\displaystyle\lim_{x\to 7} \frac{4}{(x-7)^2}$

Solution

$$\lim_{x\to 7} \frac{4}{(x-7)^2} = \frac{4}{0}$$

$$\boxed{= \infty}$$

Exercise

Find the limit $\displaystyle\lim_{x\to 0} \frac{1}{x^{2/3}}$

Solution

$$\boxed{\lim_{x\to 0} \frac{1}{x^{2/3}} = \infty}$$

Exercise

Find the limit $\displaystyle\lim_{x\to -\infty} \left(x + \sqrt{x^2 - 4x + 2}\right)$

Solution

$$\lim_{x\to -\infty} \left(x + \sqrt{x^2 - 4x + 2}\right) = -\infty + \infty$$

$$= \lim_{x\to -\infty} \left(x + \sqrt{x^2 - 4x + 2}\right) \cdot \frac{x - \sqrt{x^2 - 4x + 2}}{x - \sqrt{x^2 - 4x + 2}}$$

$$= \lim_{x\to -\infty} \frac{x^2 - x^2 + 4x - 2}{x - \sqrt{x^2 - 4x + 2}}.$$

$$= \lim_{x \to -\infty} \frac{4x-2}{x-\sqrt{x^2-4x+2}}$$

$$= \lim_{x \to -\infty} \frac{4x-2}{x-|x|} \qquad x \to -\infty \quad (x<0) \to \quad |x| = -x$$

$$= \lim_{x \to -\infty} \frac{4x}{x+x}$$

$$= \lim_{x \to -\infty} \frac{4x}{2x}$$

$= 2$