Ex $\vec{F} = \langle z-y, x, -x \rangle$

$S: x^2+y^2+z^2=4 \quad z \geq 0$

$C: x^2+y^2=4 \quad$ ccw

Sol.

$\vec{r}(t) = \langle x, y, z \rangle$

$= \langle 2\cos t, 2\sin t, 0 \rangle \quad 0 \leq t \leq 2\pi$

$\vec{F} = \langle -2\sin t, 2\cos t, -2\cos t \rangle$

$\frac{d\vec{r}}{dt} = \langle -2\sin t, 2\cos t, 0 \rangle$

$\oint \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (4\sin^2 t + 4\cos^2 t)\, dt$

$= 4 \int_0^{2\pi} dt$

$= 8\pi$

$\iint (\nabla \times \vec{F}) \cdot \vec{n}\, ds$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-y & x & -x \end{vmatrix}$$

$= 2\hat{j} + 2\hat{k}$

Ans: $x^2+y^2+z^2 = 4 \Rightarrow z = \sqrt{4-(x^2+y^2)}$

$2x + 2z\, z_x = 0 \Rightarrow z_x = -\frac{x}{z}$

$2y + 2z\, z_y = 0 \Rightarrow z_y = -\frac{y}{z}$

$\vec{n} = \langle -z_x, -z_y, 1 \rangle$

$= \langle \frac{x}{z}, \frac{y}{z}, 1 \rangle$

$\nabla \times \vec{F} = \langle 0, 2, 2 \rangle$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} = \iint_R \left(2\frac{y}{z} + 2 \right) dA$$

$$= \iint_R \left(\frac{2y}{\sqrt{4-(x^2+y^2)}} + 2 \right) dA$$

$$= \int_0^{2\pi}\int_0^2 \left(\frac{2r\sin\theta}{\sqrt{4-r^2}} + 2 \right) r\,dr\,d\theta$$

$$= \int_0^2\int_0^{2\pi} \left(\frac{2r^2\sin\theta}{\sqrt{4-r^2}} + 2r \right) d\theta\,dr$$

$$= \int_0^2 \left(\frac{-2r^2}{\sqrt{4-r^2}}\cos\theta + 2r\theta \Big|_0^{2\pi} \right) dr$$

$$= \int_0^2 \left(\frac{-2r^2}{\sqrt{4-r^2}}(1-1) + 4\pi r \right) dr$$

$$= 4\pi \int_0^2 r\,dr$$

$$= 2\pi r^2 \Big|_0^2$$

$$= 8\pi$$

Ex $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, ds \quad \vec{F} = \langle -xz, yz, xye^z \rangle$

$z = 5 - x^2 - y^2 \qquad z = 3 \uparrow$

Soln

$$3 = 5 - x^2 - y^2$$

$$x^2 + y^2 = 2 \rightarrow r = \sqrt{2}$$

$$\vec{r}(t) = \langle \overset{\sqrt{2}}{r}\cos t, \overset{\sqrt{2}}{r}\sin t, \overset{3}{z} \rangle \quad z = 3$$

$$\frac{d\vec{r}}{dt} = \langle -\sqrt{2}\sin t, \sqrt{2}\cos t, 0 \rangle$$

$$\vec{r}(t) = \langle \sqrt{2}\cos t, \sqrt{2}\sin t, 3 \rangle$$

$$\vec{F} = \langle -3\sqrt{2}\cos t, 3\sqrt{2}\sin t, 2\cos t \sin t\, e^3 \rangle$$

$$\vec{F} \cdot \frac{d\vec{r}}{dt} = 6\cos t \sin t + 6\cos t \sin t$$

$$= 12\cos t \sin t$$

$$= 6\sin 2t$$

$$\iint_S (\nabla \times \vec{F} \cdot \vec{n})\, ds = \int_0^{2\pi} 6\sin 2t\, dt$$

$$= -3\cos 2t \Big|_0^{2\pi}$$

$$= 0$$

-11 16 $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, ds$

$\vec{F} = \langle 2y, -z, x - y - z \rangle$

$S: x^2 + y^2 + z^2 = 25$

$3 \leq x \leq 5$

$x = 3 \Rightarrow y^2 + z^2 = 16 \rightarrow (r = 4)$

$\boxed{x = 5 \rightarrow y^2 + z^2 = 0}$

$\vec{r}(t) = \langle 3, 4\cos t, 4\sin t \rangle$

$\frac{d\vec{r}}{dt} = \langle 0, -4\sin t, 4\cos t \rangle$

$\vec{F} = \langle 8\cos t, -4\sin t, 3 - 4\cos t - 4\sin t \rangle$

$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (16\sin^2 t + 12\cos t - 16\cos^2 t - 16\cos t \sin t)\, dt$

$= \int_0^{2\pi} (-16\cos 2t + 12\cos t - 8\sin 2t)\, dt$

$= -8\sin 2t + 12\sin t + 4\cos 2t \Big|_0^{2\pi}$

$= 4 - 4$

$= \underline{0}$

$\vec{n}? \rightarrow x = 3$

$\vec{n} = \langle 3, 0, 0 \rangle$

$$\underbrace{\oiint \vec{F}\cdot\vec{n}\,dS}_{\text{flux}} = \iint (f_x + g_y)\,dA$$

$$= \iiint_D \nabla\cdot\vec{F}\,dV$$

ex $\vec{F} = \langle x, y, z \rangle$

$S: x^2+y^2+z^2 = a^2$

$$\nabla\cdot\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x, y, z \rangle$$

$$= 1 + 1 + 1$$

$$= 3$$

$$\iiint_D \nabla\cdot\vec{F}\,dV = \iiint_D 3\,dV$$

$= 3$ (volume of a sphere)

$$= 3\ \frac{4\pi}{3} a^3$$

$$= \underline{4\pi a^3}$$

$\vec{r} = \langle x, y, z \rangle \boxed{= \vec{F}}$

$= \langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi \rangle$

$t_\phi = \langle a\cos\phi\cos\theta,\ a\cos\phi\sin\theta,\ -a\sin\phi \rangle$

$$t_\phi \times t_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$= (a^2\sin^2\phi\sin\theta)\,\hat{i} + (a^2\sin^2\phi\sin\theta)\,\hat{j}$

$+ (a^2\cos\phi\sin\phi\cos^2\theta + a^2\sin\phi\cos\phi\sin^2\theta)\,\hat{k}$

$= (a^2\sin^2\phi\cos\theta)\,\hat{i} + (a^2\sin^2\phi\sin\theta)\,\hat{j}$

$+ (a^2\sin\phi\cos\phi)\,\hat{k}$

$\vec{F}\cdot(t_\phi \times t_\theta) = a^3\sin^3\phi\cos^2\theta + a^3\sin^3\phi\sin^2\theta$

$+ a^3\sin\phi\cos^2\phi$

$= a^3\sin\phi\left(\sin^2\phi(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) + \cos^2\phi\right)$

$= a^3\sin\phi.$

$$\iint_S \vec{F}\cdot\vec{n}\,dS = a^3\int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi$$

$= 2\pi a^3 \left(-\cos\phi \Big|_0^{\pi}\right.$

$\boxed{= 4\pi a^3}$

$\vec{F} = \langle -y,\ x-z,\ y \rangle$

$x^2+y^2+z^2=a^2 \qquad z \geq 0$

Net outward flux across S.

$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x-z) + \frac{\partial}{\partial z}(y)$

$= 0$

Ex $\quad \vec{F} = xyz\langle 1, 1, 1 \rangle$

$D = \{(x,y,z) : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1\}$

Soln

$\iiint_D \nabla \cdot \vec{F}\, dV =$

$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(xyz)$

$= yz + xz + xy.$

$\int_0^1 \int_0^1 \int_0^1 (yz + xz + xy)\, dx\, dy\, dz$

$= \int_0^1 \int_0^1 \left(yzx + \frac{1}{2}zx^2 + \frac{1}{2}yx^2\right)\Big|_0^1 dy\, dz$

$= \int_0^1 \int_0^1 \left(yz + \frac{1}{2}z + \frac{1}{2}y\right) dy\, dz$

$= \int_0^1 \left(\frac{1}{2}zy^2 + \frac{1}{2}zy + \frac{1}{4}y^2\right)\Big|_0^1 dz$

$= \int_0^1 \left(z + \frac{1}{4}\right) dz$

$= \frac{1}{2}z^2 + \frac{1}{4}z \Big|_0^1$

$= \frac{1}{2} + \frac{1}{4}$

$= 3/4$

Ex $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^3}$

a) $D = \{(x, y, z) : a^2 \le x^2 + y^2 + z^2 \le b^2\}$

$$\nabla \cdot \vec{F} = \nabla \cdot \left(\frac{\vec{r}}{|\vec{r}|^3}\right)$$

$$= -\nabla \cdot \vec{r} \, \frac{1}{|\vec{r}|^3} - \vec{r} \cdot \vec{\nabla} \frac{1}{|\vec{r}|^3}$$

$$= \frac{3-p}{|\vec{r}|^p}$$

$$= \underline{0}$$

Flux is zero

b) Outward flux $\vec{F}$ any sphere $\to 0$

$$\iint_{S_1} \vec{F} \cdot \vec{n}_1 \, ds = \iint_{S_2} \vec{F} \cdot \vec{n}_2 \, ds$$

$$= \iint_{S_1} \frac{\vec{r}}{|\vec{r}|^3} \cdot \frac{\vec{r}}{|\vec{r}|} \, ds$$

$$= \iint_{S_1} \frac{\vec{r}^{\,2}}{|\vec{r}|^4} \, ds$$

$$= \iint_{S_1} \frac{1}{a^2} \, ds$$

$$= \frac{1}{a^2} \, 4\pi a^2$$

$$= \underline{4\pi}$$

1- work

2- Green's flux & Cir

3- divergence $\nabla \cdot \vec{F}$

4- $\vec{F} \times \vec{r}$

5- Surface

6- Stokes } Theorem.
7- diverg. }

4.8 #9 $\vec{F} = \langle x^2, 2xz, y^2 \rangle$ $0 \le x \le 1$

$x = 1, \quad y = 1 \quad z = 1$ $0 \le y \le 1$

$0 \le z \le 1$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(2xz) + \frac{\partial}{\partial z}(y^2)$$

$$= \underline{2x}$$

$$\iiint_D \nabla \cdot \vec{F}\, dV = \int_0^1 dz \int_0^1 dy \int_0^1 2x\, dx$$

$$= x^2 \Big|_0^1$$

$$= \underline{1}$$

11.[illegible] $\vec{F} = \langle y+z, x+z, x+y \rangle$

$|x| \le 1 \quad |y| \le 1 \quad |z| \le 1$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y}(x+z) + \frac{\partial}{\partial z}(x+y)$$

$$= 0$$

11.[illegible] $\vec{F} = \langle y-2x, x^3-y, y^2-z \rangle$

$x^2+y^2+z^2 = 4$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(y-2x) + \frac{\partial}{\partial y}(x^3-y) + \frac{\partial}{\partial z}(y^2-z)$$

$$= -2 - 1 - 1$$

$$= -4$$

$$\iiint_D \nabla \cdot \vec{F}\, dV = -4 \overbrace{\iiint_D dV}^{V_0}$$

$$= -4 \text{ (volume of a sphere of 2)}$$

$$= -4 \cdot \frac{4}{3}\pi (2)^2$$

$$= -\frac{64}{3}\pi$$

Σ $\oint_C \vec{F} \cdot d\vec{r}$ $\quad \vec{F} = \langle z, -z, x^2 - y^2 \rangle$

$z = 8 - 4x - 2y$ (1)

$R = \{(x,y) : 0 \le x \le 2,\ 0 \le y \le 4 - 2x\}$

soln. $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS$

(1) $4x + 2y + z = 8$

$\vec{n} = \langle 4, 2, 1 \rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -z & x^2 - y^2 \end{vmatrix}$$

dot •

$= (-2y + 1)\hat{i} + (1 - 2x)\hat{j}$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS = \int_0^2 \int_0^{4-2x} (4(1-2y) + 2 - 4x)\, dy\, dx$$

$$= \int_0^2 \int_0^{4-2x} (-8y - 4x + 6)\, dy\, dx$$

$$= \int_0^2 \left(-4y^2 - 4xy + 6y \Big|_0^{4-2x}\right) dx$$

$$= \int_0^2 (-4(16 - 16x + 4x^2) - 16x + 8x^2 + 24 - 12x)\, dx$$

$$= \int_0^2 (-8x^2 + 36x - 40)\, dx$$

$$= -\frac{8}{3}x^3 + 18x^2 - 40x \Big|_0^2$$

$$= -\frac{64}{3} + 72 - 80$$

$$= -\frac{88}{}$$