6. Curvilinear coordinates
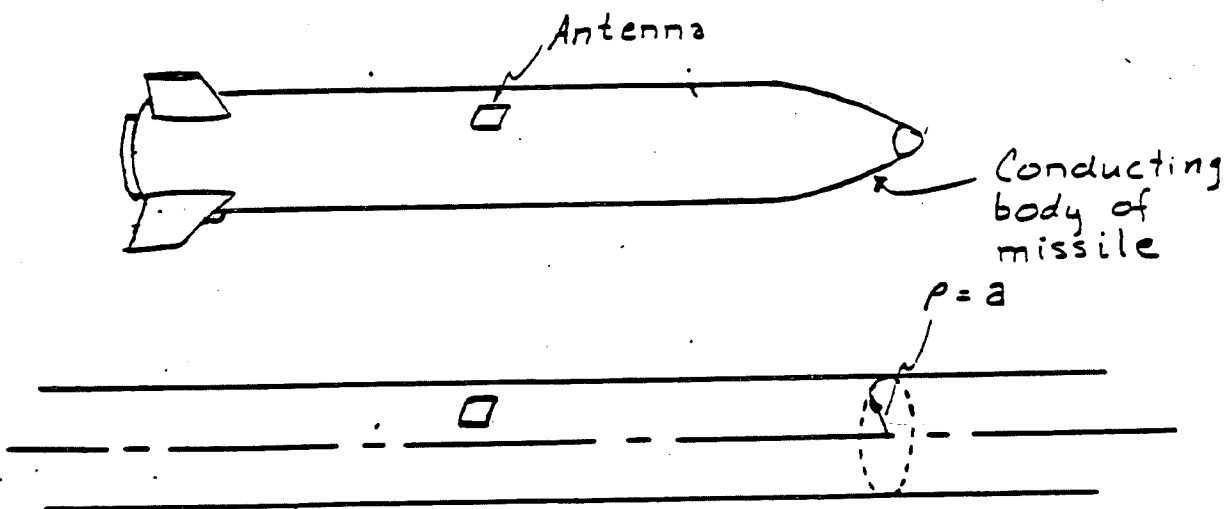+*+*+*+*+*+*+*+*+*+*+*+*+*+*+*+*+

a) Coordinate system
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Why not use rectangular coordinates for any physical problem?

Most often, physical problems of interest involve some associated physical structure.

For example, suppose we wish to find the antenna radiation pattern produced by an antenna physically mounted on the side of a missile.

Model of missile: Conducting cylinder described by the simple equation, $\rho = a$. This is a <u>constant</u> <u>"coordinate surface."</u>

The electromagnetic field MUST obey
certain contitions at the surface
of the missile (boundary condi-
tions).

It is MUCH easier to specify the
shape of the missile (at least ap-
proximately) in cylindrical
coordinates than in rectangular
coordinates.

In general, whenever a physical
boundary can be specified by a con-
stant coordinate surface in some
coordinate system, it is much
easier to attack the problem using
that coordinate system than the
rectangular system.

WHAT IS A COORDINATE SYSTEM?

If we have three functions u_1, u_2, and u_3, which map RXRXR into U_1, U_2, and U_3, respectively, in a 1:1 manner (except at a finite number of singular points), then we say that

$$(u_1, u_2, u_3) \quad \text{for} \quad u_k \in U_k, \quad k = 1, 2, 3$$

forms a coordinate system.

Eg:
$$u_1 = \sqrt{x^2 + y^2}, \qquad U_1 = \{\rho \mid \rho > 0\}$$
$$u_2 = \tan^{-1} \frac{y}{x}, \qquad U_2 = \{\theta \mid 0 \le \theta < 2\pi\}$$
$$u_3 = z \qquad\qquad U_3 = R = \text{set of real numbers}$$

forms a cylindrical coordinate system.

b) Orthogonal coordinate system
-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-

(1-) Definition
<<<<<<<<<<<<<<<<>>>>>>>>>>>>>>>>>>

When the COORDINATE SURFACES inter-
sect one another (for every non-
singular point) at right angles,
then we have an ORTHOGONAL SYSTEM.

$$u_1 = u_1^0$$

$$\hat{u}_1 = \frac{\nabla u_1}{|\nabla u_1|}$$

$$u_2 = u_2^0$$

$$\hat{u}_2 = \frac{\nabla u_2}{|\nabla u_2|}$$

$$\hat{u}_3 = \frac{\nabla u_3}{|\nabla u_3|}$$

$$u_3 = u_3^0$$

$$\left.\begin{array}{l} \hat{u}_1 = \nabla u_1 / |\nabla u_1| \\ \hat{u}_2 = \nabla u_2 / |\nabla u_2| \\ \hat{u}_3 = \nabla u_3 / |\nabla u_3| \end{array}\right\}$$ In general are $\underline{not}$ constant but $\underline{depend}$ on (u_1, u_2, u_3).

If $\hat{u}_1 \times \hat{u}_2 = \hat{u}_3$, then
this is a right handed system.

(2) Test for orthogonality
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From the diagram above it is clear
that a system is orthogonal if and
only if the gradients of the three
functions, ∇u_1, ∇u_2, and ∇u_3 are
mutually perpendicular for every
point in the systém.

$$\nabla u_i \cdot \nabla u_j = 0 \quad \text{for } i \neq j.$$

c) Arc length and scale factors
- = - = - = - = - = - = - = - = - = - = - = - = - = - = - = - = - = - = -

The arc length between the
coordinate surface $u_1 = u_1^o$ and
$u_1 = u_1^o + du_1$ is **NOT** du_1.

Arc length = ds

The line, $u_2 = u_2^o$, $u_1 = u_1^o + du_1$

The line, $u_2 = u_2^o$, $u_3 = u_3^o$

This is so since u_1 might be an
angle or some other non-linear
measure.   (Its units may not even
be-units of distance!)

We can find what the distance
between these two surfaces is by
computing the directional deriva-
tive of u_1 in the ·grad(u_1) direc-
tion.

This is (by DEFINITION of the
gradient)

$$\frac{du_1}{ds} = (\nabla u_1) \cdot \frac{\nabla u_1}{|\nabla u_1|} = |\nabla u_1|.$$

Solving for ds, the arc length is

$$ds = \frac{1}{|\nabla u_1|} \cdot du_1.$$

The reciprocal magnitude of the
gradient of u_K is called the "scale
factor" of the u_K coordinate.

$$h_K = \frac{1}{|\nabla u_K|} = \frac{1}{\sqrt{\left(\frac{\partial u_K}{\partial x}\right)^2 + \left(\frac{\partial u_K}{\partial y}\right)^2 + \left(\frac{\partial u_K}{\partial z}\right)^2}}$$

$$ds_K = h_K \, du_K$$

Let's try an example. Suppose that we wish to compute the scale factors for cylindrical coordinates.

Usually, we write x, y, and z in terms of ρ, φ, and z, instead of the other way around.

$$x = \rho \cos \varphi$$
$$y = \rho \sin \varphi$$
$$z = z$$

$$u_1 = \rho$$
$$u_2 = \varphi$$
$$u_3 = z$$

But according to the formula that we derived, we must take a gradient and so we need to write ρ, φ, and z in terms of x, y, and z.

$$\rho = \sqrt{x^2 + y^2}$$
$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$

Then the gradients and scale factors are

$$\nabla \rho = \frac{x}{\sqrt{x^2+y^2}}\hat{x} + \frac{y}{\sqrt{x^2+y^2}}\hat{y} \Rightarrow h_\rho = \frac{1}{|\nabla \rho|} = \frac{1}{1} = 1$$

$$\nabla \varphi = \frac{-y\hat{x}}{x^2+y^2} + \frac{x\hat{y}}{x^2+y^2} \Rightarrow h_\varphi = \frac{1}{|\nabla \varphi|} = \sqrt{x^2+y^2}$$

$$\nabla z = \hat{z} \Rightarrow \qquad h_z = \frac{1}{|\nabla z|} = 1$$

But these scale factors are in
terms of x, y, and z which is not
what we wanted. We must now write
this in terms of ρ, φ, and z by
substituting the expressions for
x, y, and z in terms of these
variables

$$h_\rho = 1$$

$$h_\varphi = \sqrt{x^2+y^2} = \sqrt{\rho^2 \cos^2\varphi + \rho^2 \sin^2\varphi} = \rho$$

$$h_z = 1$$

This is a lot of work. For-
tunately, there is an easier way
to find the scale factors.

All we really need to do to find,
say h_1, is to find the distance,
ds, between the points (u_1, u_2, u_3)
and $(u_1 + du_1, u_2, u_3)$ and to divide
this distance by du_1.

Now, in rectangular coordinates,
the point corresponding to point
(u_1, u_2, u_3) is the point (x, y, z)

$$x = x(u_1, u_2, u_3)$$
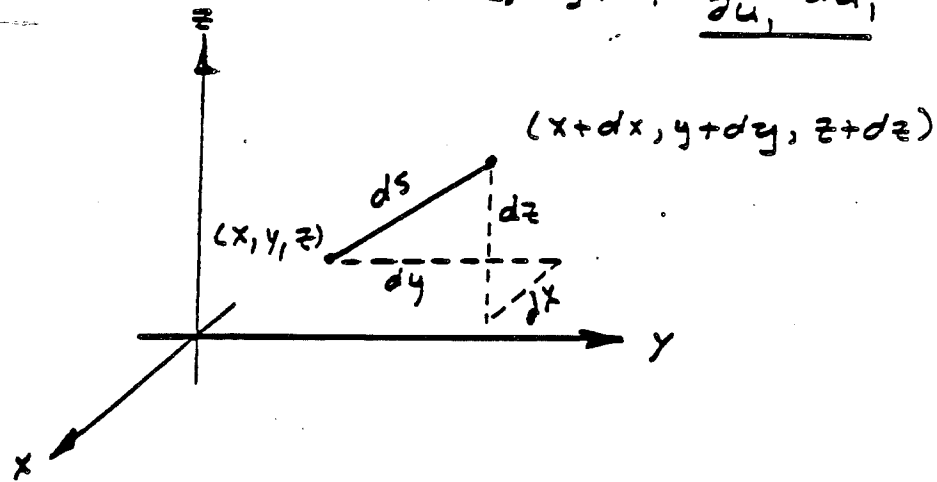$$y = y(u_1, u_2, u_3)$$
$$z = z(u_1, u_2, u_3)$$

The displaced point,
$(u_1 + du_1, u_2, u_3)$ is

$$x(u_1 + du_1, u_2, u_3) = x(u_1, u_2, u_3) + \frac{\partial x}{\partial u_1} \cdot du_1 = x + \underline{dx}$$

$$y(u_1 + du_1, u_2, u_3) = y(u_1, u_2, u_3) + \frac{\partial y}{\partial u_1} \cdot du_1 = y + \underline{dy}$$

$$z(u_1 + du_1, u_2, u_3) = z(u_1, u_2, u_3) + \frac{\partial z}{\partial u_1} \cdot du_1 = z + \underline{dz}$$

Thus, the distance, ds, is just

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2} \; du$$

and the scale factor, h_1, is

$$h_1 = \frac{ds}{du_1} = \sqrt{\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2}$$

Of course, in general,

$$h_K = \sqrt{\left(\frac{\partial x}{\partial u_K}\right)^2 + \left(\frac{\partial y}{\partial u_K}\right)^2 + \left(\frac{\partial z}{\partial u_K}\right)^2}$$

Let's use this formula to find the
scale factors for cylindrical
coordinates

$$x = \rho \cos \phi$$

$$y = \rho \sin \phi$$

$$z = z$$

$$\frac{\partial x}{\partial \rho} = \cos \phi, \quad \frac{\partial y}{\partial \rho} = \sin \phi, \quad \frac{\partial z}{\partial \rho} = 0$$

$$\frac{\partial x}{\partial \phi} = -\rho \sin \phi, \quad \frac{\partial y}{\partial \phi} = \rho \cos \phi, \quad \frac{\partial z}{\partial \phi} = 0$$

$$\frac{\partial x}{\partial z} = 0, \quad \frac{\partial y}{\partial z} = 0, \quad \frac{\partial z}{\partial z} = 1$$

$$\therefore \quad h_\rho = \sqrt{\cos^2 \phi + \sin^2 \phi} = 1, \quad h_\phi = \sqrt{\rho^2 \sin^2 \phi + \rho^2 \cos^2 \phi}$$

$$= \rho,$$

$$h_z = 1.$$

A corollary to these results is the
interesting result that for or-
thogonal coordinate systems,

$$\sqrt{\left(\frac{\partial x}{\partial u_k}\right)^2 + \left(\frac{\partial y}{\partial u_k}\right)^2 + \left(\frac{\partial z}{\partial u_k}\right)^2}$$

$$= 1 / \sqrt{\left(\frac{\partial u_k}{\partial x}\right)^2 + \left(\frac{\partial u_k}{\partial y}\right)^2 + \left(\frac{\partial u_k}{\partial z}\right)^2}$$

d) Volume

- = - = - = - = - = - = - = - = - = - = - = - = - = - = - = -

(1) The volume element

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Using the scale factors, it is easy
to construct the volume element in
the coordinate system.

$$Volume = dV$$

$$h_1 du_1 \cdot h_2 du_2 \cdot h_3 du_3$$

$$= h_1 h_2 h_3 \, du_1 du_2 du_3$$

(2) The Jacobian

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In terms of dx, dy, dz, the total
differentials of u_1, u_2, and u_3,
are

$$du_1 = \frac{\partial u_1}{\partial x} dx + \frac{\partial u_1}{\partial y} dy + \frac{\partial u_1}{\partial z} dz$$

$$du_2 = \frac{\partial u_2}{\partial x} dx + \frac{\partial u_2}{\partial y} dy + \frac{\partial u_2}{\partial z} dz$$

$$du_3 = \frac{\partial u_3}{\partial x} dx + \frac{\partial u_3}{\partial y} dy + \frac{\partial u_3}{\partial z} dz$$

$$\longleftrightarrow \begin{bmatrix} du_1 \\ du_2 \\ du_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

The determinant of this matrix is called the Jacobian, J, and (recall) is simply, the triple scalar product of the vectors formed by the rows. Thus

$$\nabla u_1 \cdot \nabla u_2 \times \nabla u_3 = J.$$

Since, the coordinate system is orthogonal, $\nabla u_2 \times \nabla u_3$ is parallel to ∇u_1 and hence

$$J = \nabla u_1 \cdot \nabla u_2 \times \nabla u_3 = |\nabla u_1||\nabla u_2||\nabla u_3| = \frac{1}{h_1 h_2 h_3}.$$

$$\therefore \; dx\,dy\,dz = \frac{du_1\,du_2\,du_3}{J}$$
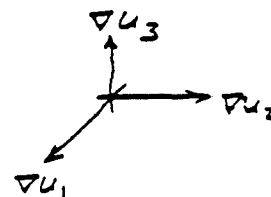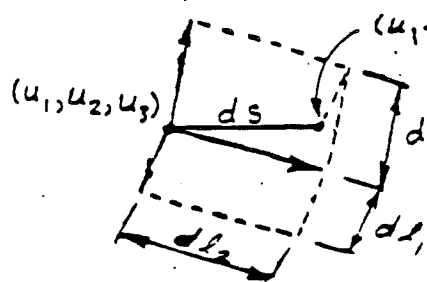
e) Gradient

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Using the DEFINITION of grad, we can now construct grad $w(u_1, u_2, u_3)$ in this system:

$$\frac{dw}{ds} = \frac{\partial w}{\partial u_1}\frac{du_1}{ds} + \frac{\partial w}{\partial u_2}\frac{du_2}{ds} + \frac{\partial w}{\partial u_3}\frac{du_3}{ds}.$$

$$dl_1 = h_1\,du_1$$
$$dl_2 = h_2\,du_2$$
$$dl_3 = h_3\,du_3$$

$$\therefore \frac{dw}{ds} = \frac{\partial w}{\partial u_1}\cdot\frac{1}{h_1}\frac{dl_1}{ds} + \frac{\partial w}{\partial u_2}\frac{1}{h_2}\frac{dl_2}{ds} + \frac{\partial w}{\partial u_3}\frac{1}{h_3}\frac{dl_3}{ds}$$

$$= \left[\frac{1}{h_1}\frac{\partial w}{\partial u_1}\hat{u}_1 + \frac{1}{h_2}\frac{\partial w}{\partial u_2}\hat{u}_2 + \frac{1}{h_3}\frac{\partial w}{\partial u_3}\hat{u}_3\right]\cdot \underset{\text{grad } w}{}$$

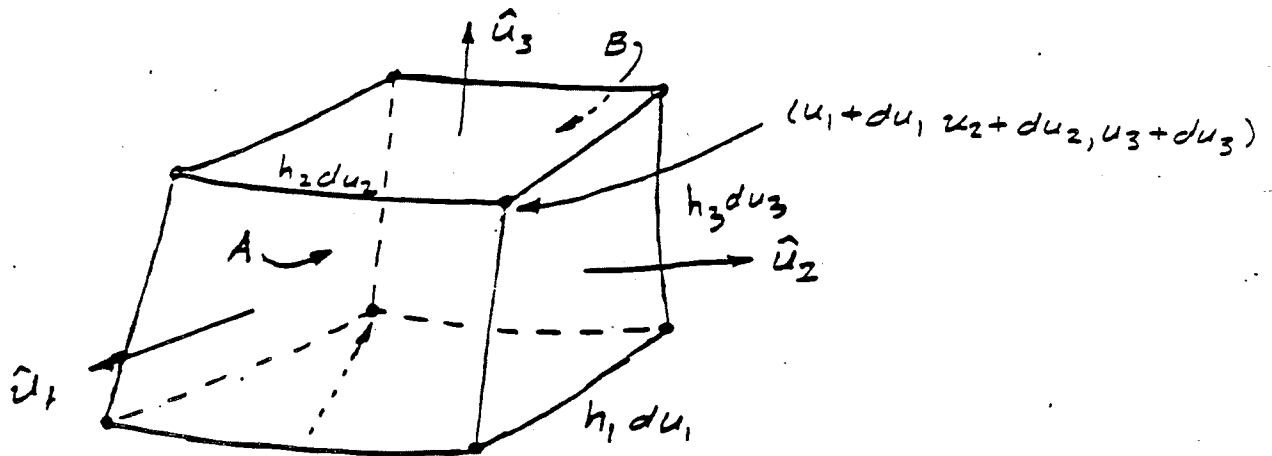$$\left[\frac{dl_1}{ds}\hat{u}_1 + \frac{dl_2}{ds}\hat{u}_2 + \frac{dl_3}{ds}\hat{u}_3\right]$$

← unit vector

i) Divergence
- -

Using the DEFINITION of div, we can now construct
div $\vec{F}(u_1, u_2, u_3)$ in this system:

$$\vec{F} = \hat{u}_1 F_1 + \hat{u}_2 F_2 + \hat{u}_3 F_3 .$$

$$\text{div } \vec{F} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_{\Delta S} \vec{F} \cdot \hat{n} \, ds .$$

$\hat{u}_3$ B)

$(u_1 + du_1, u_2 + du_2, u_3 + du_3)$

$h_2 du_2$ $h_3 du_3$

A $\hat{u}_2$

$\hat{u}_1$ $h_1 du_1$

(u_1, u_2, u_3)

Flux through face A:

$F_1(u_1 + du_1, u_2, u_3) h_2(u_1 + du_1, u_2, u_3) h_3(u_1 + du_1, u_2, u_3) \, du_2 \, du_3$

Flux through face B:

$- F_1(u_1, u_2, u_3) h_2(u_1, u_2, u_3) h_3(u_1, u_2, u_3) \, du_2 \, du_3$

In a similar way the flux through the remaining faces can be ~~found~~.

Therefore, the total flux per unit volume is

$$\frac{1}{\Delta V}\oiint_{\Delta S} \vec{F}\cdot d\vec{S} =$$

$$\frac{1}{h_1 h_2 h_3 \, du_1\, du_2\, du_3}\cdot \Bigg\{$$

$$\Big[F_1(u_1+du_1, u_2, u_3)\, h_2(u_1+du_1, u_2, u_3)\, h_3(u_1+du_1, u_2, u_3)$$
$$- F_1(u_1, u_2, u_3)\, h_2(u_1, u_2, u_3)\, h_3(u_1, u_2, u_3)\Big] du_2\, du_3$$

$$+ \Big[F_2(u_1, u_2+du_2, u_3)\, h_1(u_1, u_2+du_2, u_3)\, h_3(u_1, u_2+du_2, u_3)$$
$$- F_2(u_1, u_2, u_3)\, h_1(u_1, u_2, u_3)\, h_3(u_1, u_2, u_3)\Big] du_1\, du_3$$

$$+ \Big[F_3(u_1, u_2, u_3+du_3)\, h_1(u_1, u_2, u_3+du_3)\, h_2(u_1, u_2, u_3+du_3)$$
$$- F_3(u_1, u_2, u_3)\, h_1(u_1, u_2, u_3)\, h_2(u_1, u_2, u_3)\Big] du_1\, du_2\Bigg\}$$

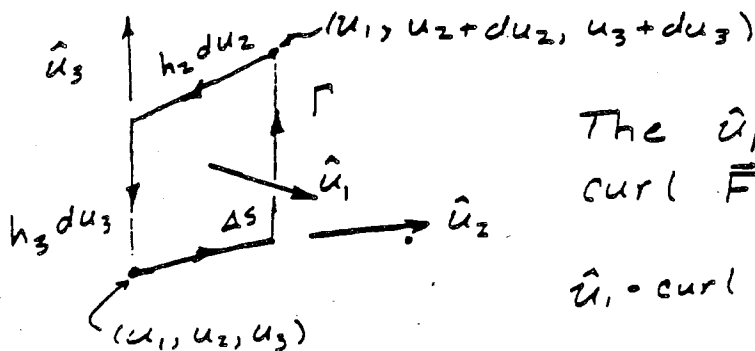$$\therefore \operatorname{div}\vec{F} = \frac{1}{h_1 h_2 h_3}\left\{ \frac{\partial[h_2 h_3 F_1]}{\partial u_1} + \frac{\partial[h_1 h_3 F_2]}{\partial u_2} + \frac{\partial[h_1 h_2 F_3]}{\partial u_3}\right\}$$

Compare this with the formula in <u>rectangular</u> coordinates

$$\operatorname{div}\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

g) Curl
_=_=_=_=_=_=_=_=_=_=_=_=_=_=_=_

Using the DEFINITION of curl, we
can now construct
curl $\vec{F}(u_1, u_2, u_3)$ in this system:

The $\hat{u}_1$ component of
curl $\vec{F}$ is defined as

$$\hat{u}_1 \cdot \text{curl}\, \vec{F} = \lim_{\Delta s \to 0} \frac{\oint_\Gamma \vec{F} \cdot d\vec{r}}{\Delta s}$$

The circulation of $\vec{F}$ about Γ is

$$F_2(u_1, u_2, u_3)\, h_2(u_1, u_2, u_3)\, du_2$$

$$- F_2(u_1, u_2, u_3 + du_3)\, h_2(u_1, u_2, u_3 + du_3)\, du_2$$

$$+ F_3(u_1, u_2 + du_2, u_3)\, h_3(u_1, u_2 + du_2, u_3)\, du_3$$

$$- F_3(u_1, u_2, u_3)\, h_3(u_1, u_2, u_3)\, du_3.$$

∴ The net circulation per unit area is

$$\frac{1}{h_2 h_3 \, du_2 \, du_3} \cdot$$

$$\left\{ du_3 \left[F_3(u_1, u_2+du_2, u_3) \, h_3(u_1, u_2+du_2, u_3) - F_3(u_1, u_2, u_3) \, h_3(u_1, u_2, u_3) \right. \right.$$

$$- du_2 \left[F_2(u_1, u_2, u_3+du_3) h_2(u_1, u_2, u_3+du_3) - F_2(u_1, u_2, u_3) h_2(u_1, u_2, u_3) \right.$$

$$= \frac{1}{h_2 h_3} \left\{ \frac{\partial}{\partial u_2}(h_3 F_3) - \frac{\partial}{\partial u_3}(h_2 F_2) \right\}$$

$$= \hat{u}_1 \cdot \text{curl} \, \vec{F}.$$

The other components are similarly found and

$$\text{curl} \, \vec{F} = \frac{1}{h_1 h_2 h_3} \cdot \left\{ \right.$$

$$\hat{u}_1 h_1 \left[\frac{\partial(h_3 F_3)}{\partial u_2} - \frac{\partial(h_2 F_2)}{\partial u_3} \right]$$

$$+ \hat{u}_2 h_2 \left[\frac{\partial(h_1 F_1)}{\partial u_3} - \frac{\partial(h_3 F_3)}{\partial u_1} \right]$$

$$+ \hat{u}_3 h_3 \left[\frac{\partial(h_2 F_2)}{\partial u_1} - \frac{\partial(h_1 F_1)}{\partial u_2} \right] \right\}$$

h) Laplacian
-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-

Using the results obtained above.,
we can now construct
$\nabla^2 w(u_1, u_2, u_3)$ in this system:

$$\nabla^2 w = \nabla \cdot (\nabla w) = \nabla \left[\frac{1}{h_1} \frac{\partial w}{\partial u_1} \hat{u}_1 + \frac{1}{h_2} \frac{\partial w}{\partial u_2} \hat{u}_2 + \frac{1}{h_3} \frac{\partial w}{\partial u_3} \hat{u}_3 \right]$$

$$= \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial w}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial w}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial w}{\partial u} \right. \right.$$

i) Delta functions
-=-=-=-=-=-=-=-=-=-=-=-=-=-=-

We will now consider how to
represent "point" sources in other
coordinate systems.

(1) Surface sources
<<<<<<<<<<<<<<<<<>>>>>>>>>>>>>>>>>>

Physical parameters such as charge
and mass which are distributed in
space are given in terms of a DEN-
SITY.

A density is a quantity <u>per unit</u>
<u>VOLUME</u>. If all the charge or mass
is concentrated on a single
coordinate surface, then the DEN-
SITY is represented by

Surface density
$\downarrow$ (eg. c/m^2)
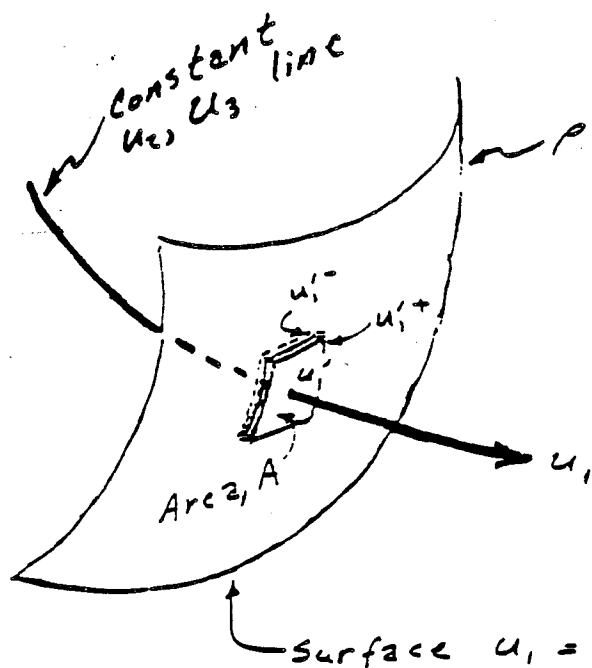
$\rho = \frac{1}{h_1} \delta(u_1 - u_1') \cdot 1$

constant u_2, u_3 line

$\overbrace{}^{dV}$

$$\iint_A \int_{u_1'^-}^{u_1'^+} \rho(u_1') \, d\ell_1 \, d\ell_2 \, d\ell_3$$

$$= A \int_{u_1'^-}^{u_1'^+} \rho(u_1') \, h_1 \, du_1$$

$$= A \int_{u_1'-}^{u_1'+} \frac{1}{h_1} \delta(u_1 - u_1') \, h_1 \, du_1$$

$$= 1 \cdot A$$

u_1

Area A

Surface $u_1 = u_1'$

$u_1'^- = u_1' - \epsilon$

$u_1'^+ = u_1' + \epsilon$

$\epsilon > c$

Without this
factor of $\frac{1}{h_1}$,
the integral of
ρ over this
volume would be
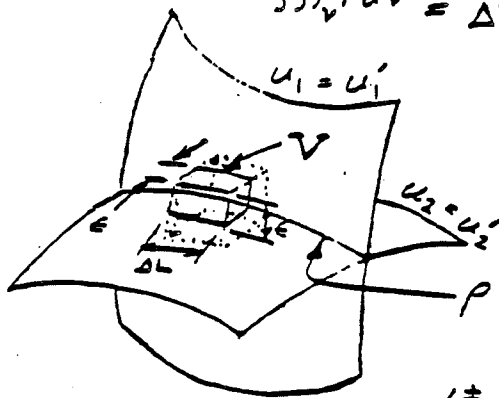$h_1 \cdot A$ not A
as it should be.

(2) Line source

<<<<<<<<<<<<<<<<<<<>>>>>>>>>>>>>>>>>>>

Suppose that all the charge is concentrated at the intersection of two coordinate surfaces.

The the density of charge is

$$\iiint_V \rho \, dv = \Delta L \int_{u_2'^-}^{u_2'^+}\int_{u_1'^-}^{u_1'^+} \rho \, d\ell_1 \, d\ell_2 = \Delta L \int_{u_2'^-}^{u_2'^+}\int_{u_1'^-}^{u_1'^+} \rho \, h_1 h_2 \, du_1$$

$$= \Delta L \int_{u_2'^-}^{u_2'^+}\int_{u_1'^-}^{u_1'^+} \frac{1}{h_1 h_2} \delta(u_1 - u_1') \delta(u_2 - u_2')$$

$$\cdot h_1 h_2 \, du_1 \, du_2 = 1 .$$

$$\rho = \frac{1}{h_1 h_2} \delta(u_1 - u_1') \delta(u_2 - u_2')$$

$$u_1'^{\pm} \equiv u_1' \pm \epsilon$$
$$u_2'^{\pm} \equiv u_2' \pm \epsilon$$

(3) Point source

<<<<<<<<<<<<<<<<.<<>>>>>>>>>>>>>>>>>>>

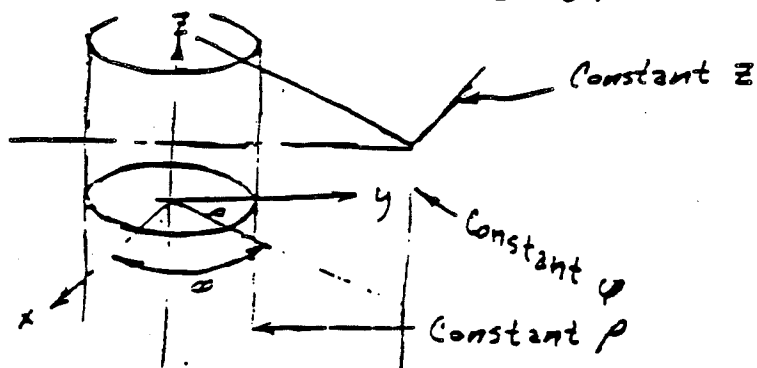Finally, the density of a point source which is located at the intersection of three coordinate surfaces is

$$\rho = \frac{1}{h_1 h_2 h_3} \delta(u_1 - u_1') \delta(u_2 - u_2') \delta(u_3 - u_3')$$

$$\iiint \rho \, dv = \iiint \rho \cdot h_1 h_2 h_3 \, du_1 \, du_2 \, du_3 = 1$$

j) Applications

- = - = - = - = - = - = - =- - = - = - = - = - = - = - = - = - = - = - = -

(1) Cylindrical coordinates

<<<<<<<<<<<<<<<<<>>>>>>>>>>>>>>>>>>>

The scale factors for cylindrical coordinates are:

$$h_\rho = 1$$

$$h_\varphi = \rho$$

$$h_z = 1$$

Therefore, the grad, div, curl, and Laplacian in cylindrical coordinates are:

$$\text{grad } w = \frac{\partial w}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial w}{\partial \varphi}\hat{\varphi} + \frac{\partial w}{\partial z}\hat{z}$$

$$\text{div } \bar{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \varphi}F_\varphi + \frac{\partial}{\partial z}F_z$$

$$\text{curl } \bar{F} = \hat{\rho}\left[\frac{1}{\rho}\frac{\partial F_z}{\partial \varphi} - \frac{\partial F_\varphi}{\partial z}\right] + \hat{\varphi}\left[\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right] + \hat{z}\frac{1}{\rho}\left[\frac{\partial(\rho F_\varphi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \varphi}\right.$$

$$\nabla^2 w = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial w}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{\partial^2 w}{\partial z^2}.$$

(2) Spherical coordinates

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The scale factors for spherical coordinates are

$$h_1 = 1$$
$$h_2 = r$$
$$h_3 = r\sin\theta$$

Therefore, the grad, div, curl, and laplacian in cylindrical coordinates are:

$$\text{grad } \omega = \frac{\partial \omega}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \omega}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial \omega}{\partial \varphi}\hat{\varphi}$$

$$\text{div } \vec{F} = \frac{1}{r^2}\frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial \varphi}$$

$$\text{curl } \vec{F} = \hat{r}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\varphi) - \frac{1}{r\sin\theta}\frac{\partial F_\theta}{\partial \varphi}\right]$$

$$+ \hat{\theta}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}F_r - \frac{1}{r}\frac{\partial}{\partial r}(r F_\varphi)\right]$$

$$+ \hat{\varphi}\left[\frac{1}{r}\frac{\partial(r F_\theta)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \theta}\right]$$

$$\nabla^2 \omega = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \omega}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$$