

Solution **Section 3.2 – Angle and Orthogonality in Inner Product Spaces**

Exercise

Which of the following form orthonormal sets?

- a) $(1, 0), (0, 2)$ in \mathbb{R}^2
- b) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in \mathbb{R}^2
- c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in \mathbb{R}^2
- d) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ in \mathbb{R}^3
- e) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ in \mathbb{R}^3
- f) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ in \mathbb{R}^3

Solution

a) $(1, 0) \cdot (0, 2) = 1(0) + 0(2)$
 $\quad \quad \quad = 0$

They are **orthonormal** sets

b) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}$
 $\quad \quad \quad = \frac{1}{2} - \frac{1}{2}$
 $\quad \quad \quad = 0$

They are **orthonormal** sets

c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}$
 $\quad \quad \quad = -\frac{1}{2} - \frac{1}{2}$
 $\quad \quad \quad = -1 \neq 0$

They are **not orthonormal** sets

d) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$$\begin{aligned}
&= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{3}} \left(-\frac{1}{\sqrt{2}} \right) + 0 + \frac{1}{\sqrt{2}} \left(-\frac{1}{\sqrt{3}} \right) \frac{1}{\sqrt{2}} \\
&= -\frac{1}{2} \frac{1}{\sqrt{3}} - \frac{1}{2} \frac{1}{\sqrt{3}} \\
&= -\frac{1}{\sqrt{3}} \neq 0
\end{aligned}$$

They are **not orthonormal** sets

$$\begin{aligned}
e) \quad &\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) \cdot \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right) \\
&= \frac{2}{3} \frac{2}{3} \frac{1}{3} + \left(-\frac{2}{3} \right) \frac{1}{3} \frac{2}{3} + \frac{1}{3} \left(-\frac{2}{3} \right) \frac{2}{3} \\
&= \frac{4}{27} - \frac{4}{27} - \frac{4}{27} \\
&= -\frac{4}{27} \neq 0
\end{aligned}$$

They are **not orthonormal** sets

$$\begin{aligned}
f) \quad &\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) = \frac{1}{\sqrt{6}} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}} \left(-\frac{1}{\sqrt{2}} \right) + 0 \\
&= 0
\end{aligned}$$

They are **orthonormal** sets

Exercise

Find the cosine of the angle between \vec{u} and \vec{v} .

$$a) \quad \vec{u} = (1, -3), \quad \vec{v} = (2, 4)$$

$$e) \quad \vec{u} = (1, 0, 1, 0), \quad \vec{v} = (-3, -3, -3, -3)$$

$$b) \quad \vec{u} = (-1, 0), \quad \vec{v} = (3, 8)$$

$$f) \quad \vec{u} = (2, 1, 7, -1), \quad \vec{v} = (4, 0, 0, 0)$$

$$c) \quad \vec{u} = (-1, 5, 2), \quad \vec{v} = (2, 4, -9)$$

$$g) \quad \vec{u} = (1, 3, -5, 4), \quad \vec{v} = (2, -4, 4, 1)$$

$$d) \quad \vec{u} = (4, 1, 8), \quad \vec{v} = (1, 0, -3)$$

$$h) \quad \vec{u} = (1, 2, 3, 4), \quad \vec{v} = (-1, -2, -3, -4)$$

Solution

$$a) \quad \vec{u} = (1, -3), \quad \vec{v} = (2, 4)$$

$$\begin{aligned}
\|\vec{u}\| &= \sqrt{1^2 + (-3)^2} \\
&= \sqrt{10}
\end{aligned}$$

$$\begin{aligned}
\|\vec{v}\| &= \sqrt{2^2 + 4^2} \\
&= \sqrt{20}
\end{aligned}$$

$$\begin{aligned}\langle \vec{u}, \vec{v} \rangle &= 1(2) + (-3)(4) \\ &= -10\end{aligned}$$

$$\begin{aligned}\cos \theta &= \frac{-10}{\sqrt{10} \sqrt{20}} \\ &= -\frac{10}{\sqrt{200}} \\ &= -\frac{1}{\sqrt{2}}\end{aligned}$$

$$\cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

$$b) \quad \vec{u} = (-1, 0); \quad \vec{v} = (3, 8)$$

$$\begin{aligned}\|\vec{u}\| &= \sqrt{(-1)^2 + 0^2} \\ &= 1\end{aligned}$$

$$\begin{aligned}\|\vec{v}\| &= \sqrt{3^2 + 8^2} \\ &= \sqrt{73}\end{aligned}$$

$$\begin{aligned}\langle \vec{u}, \vec{v} \rangle &= (-1)(3) + (0)(8) \\ &= -3\end{aligned}$$

$$\begin{aligned}\cos \theta &= \frac{-3}{1\sqrt{73}} \\ &= -\frac{3}{\sqrt{73}}\end{aligned}$$

$$\cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

$$c) \quad \vec{u} = (-1, 5, 2); \quad \vec{v} = (2, 4, -9)$$

$$\begin{aligned}\|\vec{u}\| &= \sqrt{(-1)^2 + 5^2 + 2^2} \\ &= \sqrt{30}\end{aligned}$$

$$\begin{aligned}\|\vec{v}\| &= \sqrt{2^2 + 4^2 + (-9)^2} \\ &= \sqrt{101}\end{aligned}$$

$$\begin{aligned}\langle \vec{u}, \vec{v} \rangle &= (-1)(2) + (5)(4) + (2)(-9) \\ &= 0\end{aligned}$$

$$\cos \theta = 0$$

$$\cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

$$d) \quad \vec{u} = (4, 1, 8); \quad \vec{v} = (1, 0, -3)$$

$$\|\vec{u}\| = \sqrt{4^2 + 1^2 + 8^2}$$

$$\begin{aligned}
 &= 9 \mid \\
 \|\vec{v}\| &= \sqrt{1+0+9} \\
 &= \sqrt{10} \mid
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{u}, \vec{v} \rangle &= (4)(1) + (1)(0) + (8)(-3) \\
 &= -20 \mid
 \end{aligned}$$

$$\cos \theta = \frac{-\frac{20}{9\sqrt{10}}}{\left| \right.} \qquad \cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

$$e) \quad \vec{u} = (1, 0, 1, 0); \quad \vec{v} = (-3, -3, -3, -3)$$

$$\|\vec{u}\| = \sqrt{2} \mid$$

$$\begin{aligned}
 \|\vec{v}\| &= \sqrt{9+9+9+9} \\
 &= 12 \mid
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{u}, \vec{v} \rangle &= -3+0-3+0 \\
 &= -6 \mid
 \end{aligned}$$

$$\cos \theta = \frac{-6}{12\sqrt{2}} \qquad \cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

$$\frac{-\frac{1}{2\sqrt{2}}}{\left| \right.}$$

$$f) \quad \vec{u} = (2, 1, 7, -1); \quad \vec{v} = (4, 0, 0, 0)$$

$$\begin{aligned}
 \|\vec{u}\| &= \sqrt{2^2+1^2+7^2+(-1)^2} \\
 &= \sqrt{55} \mid
 \end{aligned}$$

$$\begin{aligned}
 \|\vec{v}\| &= \sqrt{4^2+0} \\
 &= 4 \mid
 \end{aligned}$$

$$\begin{aligned}
 \langle \vec{u}, \vec{v} \rangle &= (2)(4) + (1)(0) + (7)(0) + (-1)(0) \\
 &= 8 \mid
 \end{aligned}$$

$$\cos \theta = \frac{8}{4\sqrt{55}} \qquad \cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}$$

$$\frac{\frac{2}{\sqrt{55}}}{\left| \right.}$$

$$g) \quad \vec{u} = (1, 3, -5, 4), \quad \vec{v} = (2, -4, 4, 1)$$

$$\|\vec{u}\| = \sqrt{1+9+25+16}$$

$$\begin{aligned}
&= \sqrt{51} \mid \\
\|\vec{v}\| &= \sqrt{4+16+16+1} \\
&= \sqrt{37} \mid \\
\langle \vec{u}, \vec{v} \rangle &= 2-12-20+4 \\
&= -26 \mid \\
\cos \theta &= \frac{-26}{\sqrt{51}\sqrt{37}} \mid \qquad \cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}
\end{aligned}$$

$$h) \quad \vec{u} = (1, 2, 3, 4), \quad \vec{v} = (-1, -2, -3, -4)$$

$$\begin{aligned}
\|\vec{u}\| &= \sqrt{1+4+9+16} \\
&= \sqrt{30} \mid \\
\|\vec{v}\| &= \sqrt{1+4+9+16} \\
&= \sqrt{30} \mid \\
\langle \vec{u}, \vec{v} \rangle &= -1-4-9-16 \\
&= -30 \mid \\
\cos \theta &= \frac{-30}{\sqrt{30}\sqrt{30}} \\
&= -1 \mid \qquad \cos \theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}
\end{aligned}$$

Exercise

Find the cosine of the angle between A and B .

$$a) \quad A = \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$$

$$c) \quad A = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$b) \quad A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$$

$$d) \quad A = \begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$$

Solution

$$\begin{aligned}
a) \quad A &= \begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} \\
\|A\| &= \sqrt{\langle A, A \rangle} \\
&= \sqrt{2^2 + 6^2 + 1^2 + (-3)^2} \\
&= \sqrt{50}
\end{aligned}$$

$$= 5\sqrt{2} \mid$$

$$\begin{aligned}\|B\| &= \sqrt{\langle B, B \rangle} \\ &= \sqrt{9+4+1+0} \\ &= \sqrt{14} \mid\end{aligned}$$

$$\begin{aligned}\langle A, B \rangle &= 2(3) + 6(2) + 1(1) + (-3)(0) \\ &= 19 \mid\end{aligned}$$

$$\begin{aligned}\cos \theta &= \frac{19}{5\sqrt{2}\sqrt{14}} \\ &= \frac{19}{10\sqrt{7}} \mid\end{aligned} \qquad \cos \theta = \frac{\langle A, B \rangle}{\|A\| \|B\|}$$

$$b) \quad A = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$$

$$\begin{aligned}\|A\| &= \sqrt{\langle A, A \rangle} \\ &= \sqrt{2^2 + 4^2 + (-1)^2 + 3^2} \\ &= \sqrt{30} \mid\end{aligned}$$

$$\begin{aligned}\|B\| &= \sqrt{\langle B, B \rangle} \\ &= \sqrt{(-3)^2 + 1^2 + 4^2 + 2^2} \\ &= \sqrt{30} \mid\end{aligned}$$

$$\begin{aligned}\langle A, B \rangle &= 2(-3) + 4(1) + (-1)(4) + 3(2) \\ &= 0 \mid\end{aligned}$$

$$\begin{aligned}\cos \theta &= \frac{0}{30} \\ &= 0 \mid\end{aligned} \qquad \cos \theta = \frac{\langle A, B \rangle}{\|A\| \|B\|}$$

$$c) \quad A = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$\begin{aligned}\|A\| &= \sqrt{81+64+49+36+25+16} \\ &= \sqrt{271} \mid\end{aligned}$$

$$\|A\| = \sqrt{\langle A, A \rangle}$$

$$\begin{aligned}\|B\| &= \sqrt{1+4+9+16+25+36} \\ &= \sqrt{91} \mid\end{aligned}$$

$$\|B\| = \sqrt{\langle B, B \rangle}$$

$$\langle A, B \rangle = 9 + 16 + 21 + 24 + 25 + 24$$

$$= 119$$

$$\cos \theta = \frac{119}{\sqrt{271}\sqrt{91}}$$

$$\cos \theta = \frac{\langle A, B \rangle}{\|A\| \|B\|}$$

$$d) \quad A = \begin{pmatrix} 1 & -2 & 7 \\ 6 & -3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 6 & 5 & -4 \end{pmatrix}$$

$$\|A\| = \sqrt{1^2 + (-2)^2 + 7^2 + 6^2 + (-3)^2 + 4^2}$$

$$= \sqrt{115}$$

$$\|A\| = \sqrt{\langle A, A \rangle}$$

$$\|B\| = \sqrt{1^2 + (-2)^2 + 3^2 + 6^2 + 5^2 + (-4)^2}$$

$$= \sqrt{91}$$

$$\|B\| = \sqrt{\langle B, B \rangle}$$

$$\langle A, B \rangle = 1 + 4 + 21 + 36 - 15 - 16$$

$$= 31$$

$$\cos \theta = \frac{31}{\sqrt{115}\sqrt{91}}$$

$$\cos \theta = \frac{\langle A, B \rangle}{\|A\| \|B\|}$$

Exercise

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

$$a) \quad \vec{u} = (-1, 3, 2), \quad \vec{v} = (4, 2, -1)$$

$$d) \quad \vec{u} = (-4, 6, -10, 1), \quad \vec{v} = (2, 1, -2, 9)$$

$$b) \quad \vec{u} = (a, b), \quad \vec{v} = (-b, a)$$

$$e) \quad \vec{u} = (-4, 6, -10, 1), \quad \vec{v} = (2, 1, -2, 9)$$

$$c) \quad \vec{u} = (-2, -2, -2), \quad \vec{v} = (1, 1, 1)$$

Solution

$$a) \quad \langle \vec{u}, \vec{v} \rangle = (-1)(4) + 3(2) + 2(-1)$$

$$= 0$$

Therefore, the given vectors are orthogonal.

$$b) \quad \langle \vec{u}, \vec{v} \rangle = a(-b) + b(a)$$

$$= 0$$

Therefore, the given vectors are orthogonal.

$$c) \quad \langle \vec{u}, \vec{v} \rangle = (-2)(1) + (-2)(1) + (-2)(1)$$

$$= -6$$

Therefore, the given vectors are **not** orthogonal.

$$\begin{aligned} d) \quad \langle \vec{u}, \vec{v} \rangle &= (-4)(2) + (6)(1) + (-10)(-2) + (1)(9) \\ &= 27 \quad | \quad \neq 0 \end{aligned}$$

Therefore, the given vectors are **not** orthogonal.

$$\begin{aligned} e) \quad \|\vec{u}\| &= \sqrt{(-4)^2 + 6^2 + (-10)^2 + 1^2} \\ &= \sqrt{153} \\ &= 3\sqrt{17} \quad | \end{aligned}$$

$$\begin{aligned} \|\vec{v}\| &= \sqrt{2^2 + 1^2 + (-2)^2 + 9^2} \\ &= \sqrt{90} \\ &= 3\sqrt{10} \quad | \end{aligned}$$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= (-4)(2) + 6(1) - 10(-2) + 1(9) \\ &= 27 \quad | \end{aligned}$$

$$\begin{aligned} \cos \theta &= \frac{27}{3\sqrt{17}(3\sqrt{10})} & \cos \theta &= \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|} \\ &= \frac{3}{\sqrt{170}} \quad | \end{aligned}$$

The vectors \vec{u} and \vec{v} are **not** orthogonal with respect to the Euclidean

Exercise

Do there exist scalars k and l such that the vectors $\vec{u} = (2, k, 6)$, $\vec{v} = (l, 5, 3)$, and $\vec{w} = (1, 2, 3)$ are mutually orthogonal with respect to the Euclidean inner product?

Solution

$$\begin{aligned} \langle \vec{u}, \vec{w} \rangle &= (2)(1) + (k)(2) + (6)(3) \\ &= 20 + 2k = 0 \\ \Rightarrow k &= -10 \quad | \end{aligned}$$

$$\begin{aligned} \langle \vec{v}, \vec{w} \rangle &= (l)(1) + (5)(2) + (3)(3) \\ &= l + 19 = 0 \\ \Rightarrow l &= -19 \quad | \end{aligned}$$

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= (2)(l) + (k)(5) + (6)(3) \\ &= 2l + 5k + 18 = 0 \end{aligned}$$

$$2(-19) + 5(-10) + 18 = -70 \neq 0$$

Thus, there are no scalars such that the vectors are mutually orthogonal.

Exercise

Let \mathbb{R}^3 have the Euclidean inner product. For which values of k are \vec{u} and \vec{v} orthogonal?

$$a) \quad \vec{u} = (2, 1, 3), \quad \vec{v} = (1, 7, k)$$

$$b) \quad \vec{u} = (k, k, 1), \quad \vec{v} = (k, 5, 6)$$

Solution

$$\begin{aligned} a) \quad \langle \vec{u}, \vec{v} \rangle &= (2)(1) + (1)(7) + (3)(k) \\ &= 9 + 3k = 0 \end{aligned}$$

\vec{u} and \vec{v} are orthogonal for $\underline{k = -3}$

$$\begin{aligned} b) \quad \langle \vec{u}, \vec{v} \rangle &= (k)(k) + (k)(5) + (1)(6) \\ &= k^2 + 5k + 6 = 0 \end{aligned}$$

\vec{u} and \vec{v} are orthogonal for $\underline{k = -2, -3}$

Exercise

Let V be an inner product space. Show that if \vec{u} and \vec{v} are orthogonal unit vectors in V , then $\|\vec{u} - \vec{v}\| = \sqrt{2}$

Solution

$$\begin{aligned} \|\vec{u} - \vec{v}\|^2 &= \langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle \\ &= \langle \vec{u}, \vec{u} - \vec{v} \rangle - \langle \vec{v}, \vec{u} - \vec{v} \rangle \\ &= \langle \vec{u}, \vec{u} \rangle - \langle \vec{u}, \vec{v} \rangle - \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{v} \rangle \\ &= \|\vec{u}\|^2 - 0 - 0 + \|\vec{v}\|^2 \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

since \vec{u} and \vec{v} are orthogonal unit vectors

Thus $\underline{\|\vec{u} - \vec{v}\| = \sqrt{2}}$

Exercise

Let S be a subspace of \mathbb{R}^n . Explain what $(S^\perp)^\perp = S$ means and why it is true.

Solution

$(S^\perp)^\perp$ is the orthogonal complement of S^\perp , which is itself the orthogonal complement of S , so $(S^\perp)^\perp = S$ means that S is the orthogonal of its orthogonal complement.

We need to show that S is contained in $(S^\perp)^\perp$ and, conversely, that $(S^\perp)^\perp$ is contained in S to be true.

i. Suppose $\vec{v} \in S^\perp$ and $\vec{w} \in S^\perp$. Then $\langle \vec{v}, \vec{w} \rangle = 0$ by definition of S^\perp .

Thus, S is certainly contained in $(S^\perp)^\perp$ (which consists of all vectors in \mathbb{R}^n which are orthogonal to S^\perp).

ii. Suppose $\vec{v} \in (S^\perp)^\perp$ (means \vec{v} is orthogonal to all vectors in S^\perp); then we need to show that $\vec{v} \in S$.

Let assume $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p\}$ be a basis for S and let $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_q\}$ be a basis for S^\perp . If

$\vec{v} \notin S$, then $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p, \vec{v}\}$ is linearly independent set. Since each vector in that set is orthogonal to all of S^\perp , the set $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p, \vec{v}, \vec{w}_1, \vec{w}_2, \dots, \vec{w}_q\}$ is linearly independent.

Since there are $p + q + 1$ vectors in this set, this means that $p + q + 1 \leq n \Leftrightarrow p + q \leq n - 1$.

On the other hand, If A is the matrix whose i^{th} row is \vec{u}_i^T , then the row space of A is S and the nullspace of A is S^\perp .

Since S is p -dimensional, the rank of A is p , meaning that the dimension of $\text{nul}(A) = S^\perp$ is $q = n - p$. Therefore,

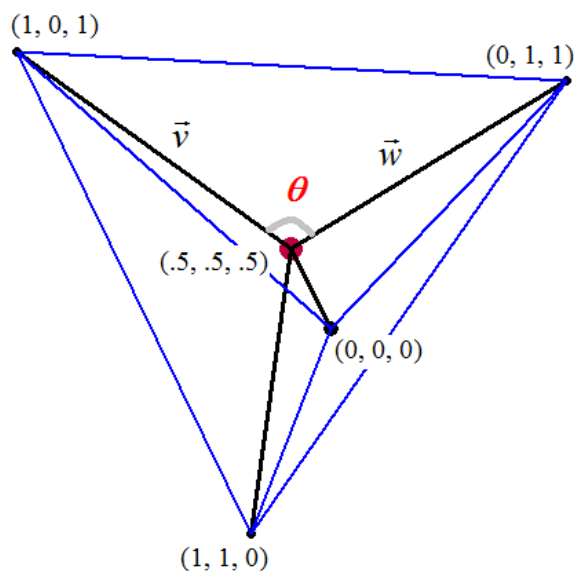
$$p + q = p + (n - p) = n$$

Which contradict the fact that $p + q \leq n - 1$. From this, we see that, if $\vec{v} \in (S^\perp)^\perp$, it must be the case that $\vec{v} \in S$.

Exercise

The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ – (**note** that all six edges have length $\sqrt{2}$, so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ to the vertices?

Solution



Let \vec{v} be the vector of the segment $(1, 0, 1)$ and $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$$\begin{aligned}\vec{v} &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\end{aligned}$$

Let be the vector of the segment $(0, 1, 1)$ and $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$$\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

We have:

$$\begin{aligned} \cos \theta &= \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \|\vec{w}\|} \\ &= \frac{\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \cdot \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}{\sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}}} \\ &= \frac{-\frac{1}{4}}{\frac{3}{4}} \\ &= -\frac{1}{3} \end{aligned}$$

$$\theta \approx 109.47^\circ$$

Exercise

Determine if the given vectors are orthogonal.

$$\vec{x}_1 = (1, 0, 1, 0), \quad \vec{x}_2 = (0, 1, 0, 1), \quad \vec{x}_3 = (1, 0, -1, 0), \quad \vec{x}_4 = (1, 1, -1, -1)$$

Solution

$$\begin{aligned} \vec{x}_1 \cdot \vec{x}_2 &= (1, 0, 1, 0) \cdot (0, 1, 0, 1) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{x}_1 \cdot \vec{x}_3 &= (1, 0, 1, 0) \cdot (1, 0, -1, 0) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{x}_1 \cdot \vec{x}_4 &= (1, 0, 1, 0) \cdot (1, 1, -1, -1) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{x}_2 \cdot \vec{x}_3 &= (0, 1, 0, 1) \cdot (1, 0, -1, 0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{x}_2 \cdot \vec{x}_4 &= (0, 1, 0, 1) \cdot (1, 1, -1, -1) \\ &= 1 - 1 \end{aligned}$$

$$\underline{=0}$$

$$\vec{x}_3 \cdot \vec{x}_4 = (1, 0, -1, 0) \cdot (1, 1, -1, -1)$$

$$= 1 - 1$$

$$\underline{=0}$$

The given vectors are *orthogonal*.

Exercise

Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

$$a) \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right) \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)$$

$$b) \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right) \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right)$$

Solution

$$a) \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right) = \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}}$$

$$\underline{=0}$$

$$\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) = -\frac{1}{2} + 0 + \frac{1}{2}$$

$$\underline{=0}$$

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right) \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) = -\frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}}$$

$$\underline{= -\frac{2}{\sqrt{6}} \neq 0}$$

Therefore, the given vectors are *not* orthogonal.

$$b) \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) \cdot \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right) = \frac{4}{9} - \frac{2}{9} - \frac{2}{9}$$

$$\underline{=0}$$

$$\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right) = \frac{2}{9} - \frac{4}{9} + \frac{2}{9}$$

$$\underline{=0}$$

$$\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right) = \frac{2}{9} + \frac{2}{9} - \frac{4}{9}$$

$$\underline{=0}$$

Therefore, the given vectors are orthogonal.

Exercise

Consider vectors $\vec{u} = (2, 3, 5)$ $\vec{v} = (1, -4, 3)$ in \mathbb{R}^3

- a) $\langle \vec{u}, \vec{v} \rangle$ b) $\|\vec{u}\|$ c) $\|\vec{v}\|$ d) Cosine between \vec{u} and \vec{v}

Solution

$$\begin{aligned} a) \quad \langle \vec{u}, \vec{v} \rangle &= (2, 3, 5) \cdot (1, -4, 3) \\ &= 2 - 12 + 15 \\ &= 5 \end{aligned}$$

$$\begin{aligned} b) \quad \|\vec{u}\| &= \sqrt{4 + 9 + 25} \\ &= \sqrt{38} \end{aligned}$$

$$\begin{aligned} c) \quad \|\vec{v}\| &= \sqrt{1 + 16 + 9} \\ &= \sqrt{26} \end{aligned}$$

$$\begin{aligned} d) \quad \cos \theta &= \frac{5}{\sqrt{38}\sqrt{26}} \quad \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \end{aligned}$$

Exercise

Consider vectors $\vec{u} = (1, 1, 1)$ $\vec{v} = (1, 2, -3)$ in \mathbb{R}^3

- a) $\langle \vec{u}, \vec{v} \rangle$ b) $\|\vec{u}\|$ c) $\|\vec{v}\|$ d) Cosine θ between \vec{u} and \vec{v}

Solution

$$\begin{aligned} a) \quad \langle \vec{u}, \vec{v} \rangle &= (1, 1, 1) \cdot (1, 2, -3) \\ &= 1 + 2 - 3 \\ &= 0 \end{aligned}$$

$$\begin{aligned} b) \quad \|\vec{u}\| &= \sqrt{1 + 1 + 1} \\ &= \sqrt{3} \end{aligned}$$

$$\begin{aligned} c) \quad \|\vec{v}\| &= \sqrt{1 + 4 + 9} \\ &= \sqrt{14} \end{aligned}$$

$$\begin{aligned} d) \quad \cos \theta &= 0 \quad \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \end{aligned}$$

\vec{u} and \vec{v} are orthogonal vectors.

Exercise

Consider vectors $\vec{u} = (1, 2, 5)$ $\vec{v} = (2, -3, 5)$ $\vec{w} = (4, 2, -3)$ in \mathbb{R}^3

- | | | |
|---------------------------------------|------------------|--|
| a) $\langle \vec{u}, \vec{v} \rangle$ | d) $\ \vec{u}\ $ | g) Cosine α between \vec{u} and \vec{v} |
| b) $\langle \vec{u}, \vec{w} \rangle$ | e) $\ \vec{v}\ $ | h) Cosine β between \vec{u} and \vec{w} |
| c) $\langle \vec{v}, \vec{w} \rangle$ | f) $\ \vec{w}\ $ | i) Cosine θ between \vec{v} and \vec{w} |
| | | j) $(\vec{u} + \vec{v}) \cdot \vec{w}$ |

Solution

$$\begin{aligned} \text{a) } \langle \vec{u}, \vec{v} \rangle &= (1, 2, 5) \cdot (2, -3, 5) \\ &= 2 - 6 + 25 \\ &= \underline{21} \end{aligned}$$

$$\begin{aligned} \text{b) } \langle \vec{u}, \vec{w} \rangle &= (1, 2, 5) \cdot (4, 2, -3) \\ &= 4 + 4 - 15 \\ &= \underline{-7} \end{aligned}$$

$$\begin{aligned} \text{c) } \langle \vec{v}, \vec{w} \rangle &= (2, -3, 5) \cdot (4, 2, -3) \\ &= 8 - 6 - 15 \\ &= \underline{-13} \end{aligned}$$

$$\begin{aligned} \text{d) } \|\vec{u}\| &= \sqrt{1 + 4 + 25} \\ &= \underline{\sqrt{30}} \end{aligned}$$

$$\begin{aligned} \text{e) } \|\vec{v}\| &= \sqrt{4 + 9 + 25} \\ &= \underline{\sqrt{38}} \end{aligned}$$

$$\begin{aligned} \text{f) } \|\vec{w}\| &= \sqrt{16 + 4 + 9} \\ &= \underline{\sqrt{29}} \end{aligned}$$

$$\text{g) } \cos \alpha = \frac{21}{\sqrt{30}\sqrt{38}} \quad \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$\text{h) } \cos \beta = \frac{-7}{\sqrt{30}\sqrt{29}} \quad \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$\text{i) } \cos \theta = \frac{-13}{\sqrt{38}\sqrt{29}} \quad \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$\begin{aligned} \text{j) } (\vec{u} + \vec{v}) \cdot \vec{w} &= [(1, 2, 5) + (2, -3, 5)] \cdot (4, 2, -3) \\ &= (3, -1, 10) \cdot (4, 2, -3) \\ &= 12 - 2 - 30 \\ &= \underline{-20} \end{aligned}$$

Exercise

Consider polynomial $f(t) = 3t - 5$; $g(t) = t^2$ in $\mathbb{P}(t)$

a) $\langle f, g \rangle$

b) $\|f\|$

c) $\|g\|$

d) Cosine between f and g

Solution

$$\begin{aligned} \text{a) } \langle f, g \rangle &= \int_0^1 f(t)g(t)dt \\ &= \int_0^1 (3t-5)t^2 dt \\ &= \int_0^1 (3t^3 - 5t^2) dt \\ &= \left. \frac{3}{4}t^4 - \frac{5}{3}t^3 \right|_0^1 \\ &= \frac{3}{4} - \frac{5}{3} \\ &= \underline{\underline{-\frac{11}{12}}} \end{aligned}$$

$$\begin{aligned} \text{b) } \langle f, f \rangle &= \int_0^1 f(t)f(t)dt \\ &= \int_0^1 (3t-5)^2 dt \\ &= \frac{1}{3} \int_0^1 (3t-5)^2 d(3t-5) \\ &= \left. \frac{1}{9}(3t-5)^3 \right|_0^1 \\ &= \frac{1}{9}(8-125) \\ &= \underline{\underline{13}} \end{aligned}$$

$$\begin{aligned} \|f\| &= \sqrt{\langle f, f \rangle} \\ &= \underline{\underline{\sqrt{13}}} \end{aligned}$$

$$\text{c) } \langle g, g \rangle = \int_0^1 g(t)g(t)dt$$

$$\begin{aligned}
&= \int_0^1 t^4 dt \\
&= \frac{1}{5} t^5 \Big|_0^1 \\
&= \frac{1}{5} \Big| \\
\|g\| &= \sqrt{|\langle g, g \rangle|} \\
&= \frac{1}{\sqrt{5}} \Big|
\end{aligned}$$

$$\begin{aligned}
d) \quad \cos \theta &= \frac{-\frac{11}{12}}{\sqrt{13} \frac{\sqrt{5}}{5}} \\
&= \frac{-55}{12\sqrt{65}} \Big|
\end{aligned}
\qquad
\cos \theta = \frac{f \cdot g}{\|f\| \|g\|}$$

Exercise

Consider polynomial $f(t) = t + 2$; $g(t) = 3t - 2$; $h(t) = t^2 - 2t - 3$ in $\mathbb{P}(t)$

- | | | |
|---------------------------|------------|--|
| a) $\langle f, g \rangle$ | d) $\ f\ $ | g) Cosine α between f and g |
| b) $\langle f, h \rangle$ | e) $\ g\ $ | h) Cosine β between f and h |
| c) $\langle g, h \rangle$ | f) $\ h\ $ | i) Cosine θ between g and h |

Solution

$$\begin{aligned}
a) \quad \langle f, g \rangle &= \int_0^1 (t+2)(3t-2) dt \\
&= \int_0^1 (3t^2 + 4t - 4) dt \\
&= t^3 + 2t^2 - 4t \Big|_0^1 \\
&= 1 + 2 - 4 \\
&= -1 \Big|
\end{aligned}
\qquad
\langle f, g \rangle = \int_0^1 f(t)g(t) dt$$

$$b) \quad \langle f, h \rangle = \int_0^1 (t+2)(t^2 - 2t - 3) dt \qquad \langle f, h \rangle = \int_0^1 f(t)h(t) dt$$

$$\begin{aligned}
&= \int_0^1 (t^3 - 7t - 6) dt \\
&= \left. \frac{1}{4}t^4 - \frac{7}{2}t^2 - 6t \right|_0^1 \\
&= \frac{1}{4} - \frac{7}{2} - 6 \\
&= \underline{-\frac{37}{4}}
\end{aligned}$$

$$\begin{aligned}
c) \quad \langle g, h \rangle &= \int_0^1 (3t-2)(t^2-2t-3) dt \\
&= \int_0^1 (3t^3 - 8t^2 - 5t + 6) dt \\
&= \left. \frac{3}{4}t^4 - \frac{8}{3}t^3 - \frac{5}{2}t^2 + 6t \right|_0^1 \\
&= \frac{3}{4} - \frac{8}{3} - \frac{5}{2} + 6 \\
&= \underline{\frac{9}{4}}
\end{aligned}$$

$$\begin{aligned}
d) \quad \langle f, f \rangle &= \int_0^1 (t+2)^2 dt \\
&= \left. \frac{1}{3}(t+2)^3 \right|_0^1 \\
&= \frac{1}{3}(27-8) \\
&= \underline{\frac{19}{3}}
\end{aligned}$$

$$\begin{aligned}
\|f\| &= \sqrt{\langle f, f \rangle} \\
&= \underline{\sqrt{\frac{19}{3}}}
\end{aligned}$$

$$\begin{aligned}
e) \quad \langle g, g \rangle &= \int_0^1 (3t-2)^2 dt \\
&= \frac{1}{3} \int_0^1 (3t-2)^2 d(3t-2) \\
&= \left. \frac{1}{9}(3t-2)^3 \right|_0^1
\end{aligned}$$

$$\langle f, h \rangle = \int_0^1 g(t)h(t)dt$$

$$\langle f, f \rangle = \int_0^1 f(t)f(t)dt$$

$$\langle g, g \rangle = \int_0^1 g(t)g(t)dt$$

$$= \frac{1}{9}(1+8)$$

$$= 1$$

$$\|g\| = \sqrt{\langle g, g \rangle}$$

$$= 1$$

$$f) \quad \langle g, g \rangle = \int_0^1 (t^2 - 2t - 3)^2 dt$$

$$\langle h, h \rangle = \int_0^1 h(t)h(t)dt$$

$$= \int_0^1 (t^4 - 4t^3 - 2t^2 + 12t + 9) dt$$

$$= \left(\frac{1}{5}t^5 - t^4 - \frac{2}{3}t^3 + 6t^2 + 9t \right) \Big|_0^1$$

$$= \frac{1}{5} - 1 - \frac{2}{3} + 6 + 9$$

$$= \frac{203}{15}$$

$$\|h\| = \sqrt{\langle h, h \rangle}$$

$$= \sqrt{\frac{203}{15}}$$

$$g) \quad \cos \alpha = \frac{-1}{\sqrt{\frac{19}{3}}}$$

$$\cos \alpha = \frac{f \cdot g}{\|f\| \|g\|}$$

$$= -\sqrt{\frac{3}{19}}$$

$$h) \quad \cos \beta = -\frac{37}{4} \frac{1}{\sqrt{\frac{19}{3}} \sqrt{\frac{203}{15}}}$$

$$\cos \beta = \frac{f \cdot g}{\|f\| \|g\|}$$

$$= -\frac{111}{4} \sqrt{\frac{5}{3,857}}$$

$$i) \quad \cos \theta = \frac{9}{4} \frac{1}{\sqrt{\frac{203}{15}}}$$

$$\cos \theta = \frac{f \cdot g}{\|f\| \|g\|}$$

$$= \frac{9}{4} \sqrt{\frac{15}{203}}$$

Exercise

Suppose $\langle \vec{u}, \vec{v} \rangle = 3 + 2i$ in a complex inner product space V . Find:

$$a) \langle (2-4i)\vec{u}, \vec{v} \rangle \quad b) \langle \vec{u}, (4+3i)\vec{v} \rangle \quad c) \langle (3-6i)\vec{u}, (5-2i)\vec{v} \rangle \quad d) \|\vec{u}, \vec{v}\|$$

Solution

$$\begin{aligned} a) \quad \langle (2-4i)\vec{u}, \vec{v} \rangle &= (2-4i)\langle \vec{u}, \vec{v} \rangle \\ &= (2-4i)(3+2i) \\ &= 6+4i-12i+8 \quad i^2 = -1 \\ &= \underline{14-8i} \end{aligned}$$

$$\begin{aligned} b) \quad \langle \vec{u}, (4+3i)\vec{v} \rangle &= (4+3i)\langle \vec{u}, \vec{v} \rangle \\ &= (4+3i)(3+2i) \\ &= 12+8i+9i-6 \quad i^2 = -1 \\ &= \underline{14-8i} \end{aligned}$$

$$\begin{aligned} c) \quad \langle (3-6i)\vec{u}, (5-2i)\vec{v} \rangle &= (3-6i)(5-2i)\langle \vec{u}, \vec{v} \rangle \\ &= (15-36i-12)(3+2i) \\ &= (3-36i)(3+2i) \\ &= 9-102i+72 \\ &= \underline{81-102i} \end{aligned}$$

$$\begin{aligned} d) \quad \|\vec{u}, \vec{v}\| &= \sqrt{\langle \vec{u}, \vec{v} \rangle} \\ &= \sqrt{9+4} \\ &= \underline{\sqrt{13}} \end{aligned}$$

Exercise

Find the Fourier coefficient c and the projection $c\vec{v}$ of $\vec{u} = (3+4i, 2-3i)$ along $\vec{v} = (5+i, 2i)$ in \mathbb{C}^2

Solution

$$\begin{aligned} c &= \frac{(3+4i)(\overline{5+i}) + (2-3i)(\overline{2i})}{5^2 + 1^2 + 2^2} & c &= \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \\ &= \frac{(3+4i)(5-i) + (2-3i)(-2i)}{25+1+4} \\ &= \frac{15+17i+4-4i-6}{30} \\ &= \underline{\frac{13+13i}{30}} \end{aligned}$$

$$= \frac{13}{30} + \frac{13}{30}i \mid$$

$$\text{proj}(\vec{u}, \vec{v}) = c\vec{v}$$

$$\begin{aligned} &= \left(\frac{13}{30} + \frac{13}{30}i \right) (5 + i, 2i) \\ &= \left(\frac{13}{6} + \frac{13}{6}i + \frac{13}{30}i - \frac{13}{30}, \frac{13}{15}i - \frac{13}{15} \right) \\ &= \left(\frac{52}{30} + \frac{78}{30}i, -\frac{13}{15} + \frac{13}{15}i \right) \\ &= \left(\frac{26}{15} + \frac{39}{30}i, -\frac{13}{15} + \frac{13}{15}i \right) \mid \end{aligned}$$

Exercise

Suppose $\vec{v} = (1, 3, 5, 7)$. Find the projection of \vec{v} onto W or find $\vec{w} \in W$ that minimizes $\|\vec{v} - \vec{w}\|$, where

W is the subspace of \mathbb{R}^4 spanned by:

- a) $\vec{u}_1 = (1, 1, 1, 1)$ and $\vec{u}_2 = (1, -3, 4, -2)$
b) $\vec{v}_1 = (1, 1, 1, 1)$ and $\vec{v}_2 = (1, 2, 3, 2)$

Solution

$$\begin{aligned} \text{a) } \vec{u}_1 \cdot \vec{u}_2 &= (1, 1, 1, 1) \cdot (1, -3, 4, -2) \\ &= 1 - 3 + 4 - 2 \\ &= 0 \mid \end{aligned}$$

Therefore, \vec{u}_1 and \vec{u}_2 are orthogonal.

$$\begin{aligned} c_1 &= \frac{\langle \vec{v}, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \\ &= \frac{(1, 3, 5, 7) \cdot (1, 1, 1, 1)}{\|(1, 1, 1, 1)\|^2} \\ &= \frac{1+3+5+7}{1+1+1+1} \\ &= \frac{16}{4} \\ &= 4 \mid \end{aligned}$$

$$c_2 = \frac{\langle \vec{v}, \vec{u}_2 \rangle}{\langle \vec{u}_2, \vec{u}_2 \rangle}$$

$$\begin{aligned}
&= \frac{(1, 3, 5, 7) \cdot (1, -3, 4, -2)}{\|(1, -3, 4, -2)\|^2} \\
&= \frac{1-9+20-14}{1+9+16+4} \\
&= \frac{-2}{30} \\
&= \frac{1}{15} \Big|
\end{aligned}$$

$$\begin{aligned}
w &= \text{proj}(\vec{v}, W) \\
&= c_1 \vec{u}_1 + c_2 \vec{u}_2 \\
&= 4(1, 1, 1, 1) + \frac{1}{15}(1, -3, 4, -2) \\
&= \left(\frac{59}{15}, \frac{63}{5}, \frac{56}{15}, \frac{62}{15} \right) \Big|
\end{aligned}$$

$$\begin{aligned}
b) \quad \vec{v}_1 \cdot \vec{v}_2 &= (1, 1, 1, 1) \cdot (1, 2, 3, 2) \\
&= 1 + 2 + 3 + 2 \\
&= 8 \neq 0 \Big|
\end{aligned}$$

Therefore, \vec{v}_1 and \vec{v}_2 are *not* orthogonal.

Applying Gram-Schmidt algorithm

$$\vec{w}_1 = \vec{v}_1 = (1, 1, 1, 1) \Big|$$

$$\begin{aligned}
\vec{w}_2 &= (1, 2, 3, 2) - \frac{(1, 2, 3, 2) \cdot (1, 1, 1, 1)}{4}(1, 1, 1, 1) & \vec{w}_2 &= \vec{v}_2 - \frac{\langle \vec{v}, \vec{w}_1 \rangle}{\|\vec{w}_1\|^2} \vec{w}_1 \\
&= (1, 2, 3, 2) - 2(1, 1, 1, 1) \\
&= (-1, 0, 1, 0) \Big|
\end{aligned}$$

$$\begin{aligned}
c_1 &= \frac{(1, 3, 5, 7) \cdot (1, 1, 1, 1)}{\|(1, 1, 1, 1)\|^2} & c_1 &= \frac{\langle \vec{v}, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \\
&= \frac{1+3+5+7}{1+1+1+1} \\
&= \frac{16}{4} \\
&= 4 \Big|
\end{aligned}$$

$$\begin{aligned}
c_2 &= \frac{(1, 3, 5, 7) \cdot (-1, 0, 1, 0)}{\|(-1, 0, 1, 0)\|^2} & c_2 &= \frac{\langle \vec{v}, \vec{w}_2 \rangle}{\langle \vec{w}_2, \vec{w}_2 \rangle}
\end{aligned}$$

$$= \frac{-1+0+5+0}{2}$$

$$= -3 \mid$$

$$w = \text{proj}(\vec{v}, W)$$

$$= c_1 \vec{w}_1 + c_2 \vec{w}_2$$

$$= 4(1, 1, 1, 1) - 3(-1, 0, 1, 0)$$

$$= (7, 4, 1, 4) \mid$$

Exercise

Suppose $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ is an orthogonal set of vectors. Prove that (Pythagoras)

$$\|\vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n\|^2 = \|\vec{u}_1\|^2 + \|\vec{u}_2\|^2 + \dots + \|\vec{u}_n\|^2$$

Solution

$$\begin{aligned} \|\vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n\|^2 &= \langle \vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n, \vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n \rangle \\ &= \langle \vec{u}_1, \vec{u}_1 \rangle + \langle \vec{u}_2, \vec{u}_2 \rangle + \dots + \langle \vec{u}_n, \vec{u}_n \rangle \\ &= \|\vec{u}_1\|^2 + \|\vec{u}_2\|^2 + \dots + \|\vec{u}_n\|^2 \end{aligned}$$

Exercise

Suppose A is an orthogonal matrix. Show that $\langle \vec{u}A, \vec{v}A \rangle = \langle \vec{u}, \vec{v} \rangle$ for any $\vec{u}, \vec{v} \in V$

Solution

$$A \text{ is an orthogonal matrix } \Rightarrow AA^T = I$$

$$\text{And } \langle \vec{u}, \vec{v} \rangle = \vec{u}^T \vec{v}$$

$$\langle \vec{u}A, \vec{v}A \rangle = (A\vec{u})^T (A\vec{v})$$

$$= \vec{u}^T (A^T A) \vec{v}$$

$$= \vec{u}^T I \vec{v}$$

$$= \vec{u}^T \vec{v}$$

$$= \langle \vec{u}, \vec{v} \rangle \quad \checkmark$$

Exercise

Suppose A is an orthogonal matrix. Show that $\|\vec{u}A\| = \|\vec{u}\|$ for every $\vec{u} \in V$

Solution

A is an orthogonal matrix

$$\Rightarrow AA^T = I \text{ and } \langle \vec{u}, \vec{u} \rangle = \vec{u}^T \vec{u}$$

$$\begin{aligned}\|\vec{u}A\|^2 &= \langle \vec{u}A, \vec{u}A \rangle \\ &= (A\vec{u})^T (A\vec{u}) \\ &= \vec{u}^T (A^T A) \vec{u} \\ &= \vec{u}^T I \vec{u} \\ &= \vec{u}^T \vec{u} \\ &= \langle \vec{u}, \vec{u} \rangle \quad \checkmark\end{aligned}$$

Exercise

Let V be an inner product space over \mathbb{R} or \mathbb{C} . Show that

$$\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$

If and only if

$$\|s\vec{u} + t\vec{v}\| = s\|\vec{u}\| + t\|\vec{v}\| \quad \text{for all } s, t \geq 0$$

Solution

Suppose that $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$. For $s, t \geq 0$

$$\begin{aligned}\|s\vec{u} + t\vec{v}\|^2 &= s^2 \|\vec{u}\|^2 + t^2 \|\vec{v}\|^2 + 2st \vec{u}\vec{v} \\ &\leq s^2 \|\vec{u}\|^2 + t^2 \|\vec{v}\|^2 \\ &\leq s\|\vec{u}\| + t\|\vec{v}\|\end{aligned}$$

$$\|s\vec{u} + t\vec{v}\| \leq s\|\vec{u}\| + t\|\vec{v}\|$$

$$\|s\vec{u} + t\vec{v}\| = \|s\vec{u} + t\vec{u} - t\vec{u} + t\vec{v}\|$$

$$= \|t\vec{u} + t\vec{v} - t\vec{u} + s\vec{u}\|$$

$$= \|t(\vec{u} + \vec{v}) - (t-s)\vec{u}\|$$

$$\geq |t\|\vec{u} + \vec{v}\| - (t-s)\|\vec{u}\||$$

$$= t\|\vec{u}\| + \|\vec{v}\| - t\|\vec{u}\| + s\|\vec{u}\|$$

$$\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$$

$$= t\|\vec{v}\| + s\|\vec{u}\|$$

$$\begin{cases} \|s\vec{u} + t\vec{v}\| \leq s\|\vec{u}\| + t\|\vec{v}\| \\ \text{and} \\ \|s\vec{u} + t\vec{v}\| \geq s\|\vec{u}\| + t\|\vec{v}\| \end{cases} \Rightarrow \|s\vec{u} + t\vec{v}\| = s\|\vec{u}\| + t\|\vec{v}\|$$

Exercise

Let V be an inner product vector space over \mathbb{R} .

- a) If e_1, e_2, e_3 are three vectors in V with pairwise product negative, that is,

$$\langle e_i, e_j \rangle < 0, \quad i, j = 1, 2, 3, \quad i \neq j$$

Show that e_1, e_2, e_3 are linearly independent.

- b) Is it possible for three vectors on the xy -plane to have pairwise negative products?
c) Does part (a) remain valid when the word “negative: is replaced with positive?”
d) Suppose \vec{u}, \vec{v} , and \vec{w} are three-unit vectors in the xy -plane. What are the maximum and minimum values that

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle$$

Can attain? And when?

Solution

- a) Suppose that e_1, e_2, e_3 are linearly dependent.

Then, assume that e_1, e_2, e_3 are unit vectors and that

$$e_3 = c_1 e_1 + c_2 e_2$$

Then

$$\langle e_1, e_3 \rangle = c_1 \langle e_1, e_1 \rangle + c_2 \langle e_1, e_2 \rangle \quad \langle e_1, e_1 \rangle = 1$$

$$= c_1 + c_2 \langle e_1, e_2 \rangle < 0$$

$$c_1 < -c_2 \langle e_1, e_2 \rangle$$

$$\langle e_2, e_3 \rangle = c_1 \langle e_2, e_1 \rangle + c_2 \langle e_2, e_2 \rangle \quad \langle e_2, e_2 \rangle = 1$$

$$= c_1 \langle e_2, e_1 \rangle + c_2 < 0$$

$$c_2 < -c_1 \langle e_2, e_1 \rangle$$

$$c_2 < -c_1 \langle e_2, e_1 \rangle$$

$$< -(-c_2 \langle e_1, e_2 \rangle) \langle e_2, e_1 \rangle$$

$$= c_2 \langle e_1, e_2 \rangle^2 \quad \langle e_1, e_2 \rangle^2 > 1$$

$$c_2 < c_2 \quad \text{Contradiction}$$

Therefore, e_1, e_2, e_3 are linearly independent.

b) To have all three vectors on the xy -plane which is in 2 dimensional.

Therefore, it is **impossible** for three to have pairwise negative products.

c) No

d) Given: \vec{u}, \vec{v} , and \vec{w} are three-unit vectors in the xy -plane and

$$|\langle \vec{u}, \vec{u} \rangle| = |\langle \vec{v}, \vec{v} \rangle| = |\langle \vec{w}, \vec{w} \rangle| = 1$$

$$\cos \alpha_1 = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|} \rightarrow \cos \alpha_1 = \langle \vec{u}, \vec{v} \rangle$$

$$\cos \alpha_2 = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \|\vec{w}\|} \rightarrow \cos \alpha_2 = \langle \vec{v}, \vec{w} \rangle$$

$$\cos \alpha_3 = \frac{\langle \vec{u}, \vec{w} \rangle}{\|\vec{u}\| \|\vec{w}\|} \rightarrow \cos \alpha_3 = \langle \vec{u}, \vec{w} \rangle$$

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = \cos \alpha_1 + \cos \alpha_2 + \cos \alpha_3$$

Since $-1 \leq \cos \theta \leq 1$

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = 1 + 1 + 1$$

$$\underline{= 3}$$

Since the 3 vectors are unit vectors in the xy -plane and which it will divide the plane into a three equal angles $\alpha = \frac{2\pi}{3}$

$$\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = \cos \frac{2\pi}{3} + \cos \frac{2\pi}{3} + \cos \frac{2\pi}{3}$$

$$= 3 \cos \frac{2\pi}{3}$$

$$= 3 \left(-\frac{1}{2} \right)$$

$$\underline{= -\frac{3}{2}}$$

Therefore, the minimum $\underline{\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = -\frac{3}{2}}$

The maximum: $\underline{\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{u} \rangle = 3}$