

Exercise 1

Ox and Oy are bisector to 2 adjacent acute angles, \widehat{AOB} and \widehat{BOC} where the difference is 36° and $\widehat{AOC} = 90^\circ$. Oz is the bisector of the angle \widehat{xOy} . Determine the angle \widehat{BOz}

Solution

$$\widehat{BOC} - \widehat{AOB} = 36^\circ$$

$$\widehat{BOC} + \widehat{AOB} = 90^\circ$$

$$2 \widehat{BOC} = 126^\circ$$

$$\widehat{BOC} = 63^\circ$$

$$\widehat{AOB} = 27^\circ$$

$$\widehat{xOB} = \frac{1}{2} \widehat{AOB}$$

$$= \frac{27^\circ}{2}$$

$$\widehat{BOy} = \frac{63^\circ}{2}$$

$$\widehat{xOy} = \widehat{xOB} + \widehat{BOy}$$

$$= \frac{1}{2}(63^\circ + 27^\circ)$$

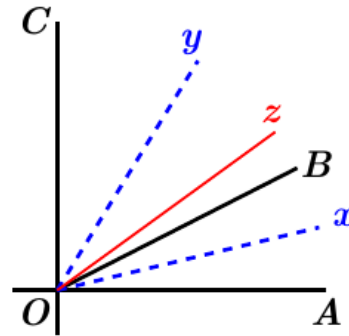
$$= 45^\circ$$

$$\widehat{xOz} = \frac{45^\circ}{2}$$

$$\widehat{BOz} = \widehat{xOz} - \widehat{xOB}$$

$$= \frac{1}{2}(45^\circ - 27^\circ)$$

$$= 9^\circ$$



Exercise 2

Ox and Oy are bisector to 2 adjacent acute angles, \widehat{AOB} and \widehat{BOC} where the difference is 36° . Oz is the bisector of the angle \widehat{xOy} . Determine the angle \widehat{BOz} .

Solution

$$Ox \text{ is the bisector } \widehat{AOB} \quad (1)$$

$$Oy \text{ is the bisector } \widehat{BOC} \quad (2)$$

$$Om \text{ is the bisector } \widehat{AOC} \quad (3)$$

$$Oz \text{ is the bisector } \widehat{xOy} \quad (4)$$

$$Oy \text{ is the bisector } \widehat{BOC} \quad (5)$$

$$\widehat{BOC} - \widehat{AOB} = 36^\circ$$

$$\widehat{BOC} - \widehat{BOD} = 36^\circ$$

$$\widehat{DOC} = 36^\circ$$

$$\begin{aligned} (3) \rightarrow \widehat{AOM} &= \frac{1}{2} \widehat{AOC} \\ &= \frac{1}{2} (2\widehat{AOB} + \widehat{DOC}) \\ &= \frac{1}{2} (2\widehat{AOB} + 36^\circ) \\ &= \widehat{AOB} + 18^\circ \end{aligned}$$

$$\begin{aligned} \widehat{BOM} &= \widehat{AOM} - \widehat{AOB} \\ &= \widehat{AOB} + 18^\circ - \widehat{AOB} \\ &= 18^\circ \end{aligned}$$

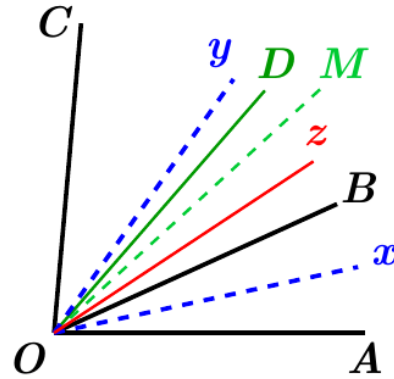
$$(1) \rightarrow \widehat{BOx} = \frac{1}{2} \widehat{AOB}$$

$$(4) \rightarrow \widehat{BOy} = \frac{1}{2} \widehat{BOC}$$

$$(1) + (4) \rightarrow \widehat{xOy} = \frac{1}{2} \widehat{AOC}$$

$$(3) \rightarrow \widehat{AOM} = \frac{1}{2} \widehat{AOC} = \widehat{xOy}$$

$$\begin{aligned} \widehat{BOz} &= \widehat{xOz} - \widehat{xOB} \\ &= \frac{1}{2} (\widehat{xOy} - \widehat{AOB}) \\ &= \frac{1}{2} (\widehat{AOM} - \widehat{AOB}) \\ &= \frac{1}{2} \widehat{BOM} \\ &= 9^\circ \end{aligned}$$



Exercise 3

Four consecutive half-lines (segments): OA , OB , OC , and OD formed angles such as

$$\widehat{DOA} = \widehat{COB} = 2\widehat{AOB} \quad \text{and} \quad \widehat{COD} = 3\widehat{AOB}$$

Calculate the angles to demonstrate that the bisectors of \widehat{AOB} and \widehat{COD} are in a straight line.

Solution

$$\widehat{AOB} + \widehat{BOC} + \widehat{COD} + \widehat{DOA} = 360^\circ$$

$$\widehat{AOB} + 2\widehat{AOB} + 3\widehat{AOB} + 2\widehat{AOB} = 360^\circ$$

$$8\widehat{AOB} = 360^\circ$$

$$\widehat{AOB} = 45^\circ$$

$$\widehat{DOA} = \widehat{COB} = 90^\circ$$

$$\widehat{COD} = 135^\circ$$

Let:

Ox is the bisector \widehat{AOB}

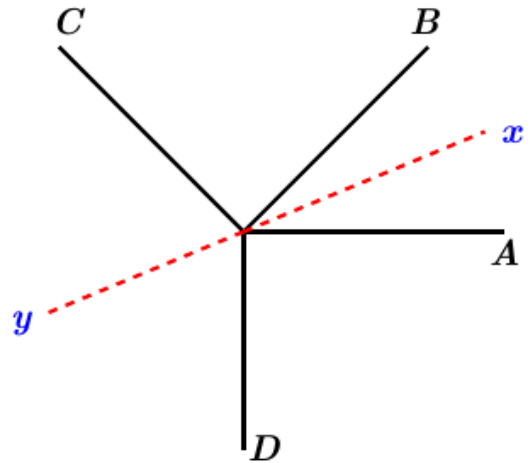
Oy is the bisector \widehat{COD}

$$\widehat{xOy} = \widehat{xOB} + \widehat{BOC} + \widehat{COy}$$

$$= \frac{1}{2}\widehat{AOB} + 90^\circ + \frac{1}{2}\widehat{COD}$$

$$= \frac{1}{2}(45^\circ + 135^\circ) + 90^\circ$$

$$= 180^\circ$$



Therefore; the bisectors of \widehat{AOB} and \widehat{COD} are in a straight line

Exercise 4

The segments OA and OB formed with OX the angles α and β .

- a) Demonstrate that the bisector OC of the angle \widehat{AOB} made with OX an angle $\frac{\alpha + \beta}{2}$.
- b) Examine the cases where
 - i. $\alpha + \beta = 90^\circ$
 - ii. $\alpha + \beta = 180^\circ$

Solution

Given:

$$\widehat{XOA} = \alpha \quad \& \quad \widehat{XOB} = \beta$$

$$\begin{aligned} \widehat{AOC} &= \frac{1}{2} \widehat{AOB} \\ &= \frac{\beta - \alpha}{2} \end{aligned}$$

$$\begin{aligned} \text{a) } \widehat{XOC} &= \widehat{XOA} + \widehat{AOC} \\ &= \alpha + \frac{\beta - \alpha}{2} \\ &= \frac{\alpha + \beta}{2} \end{aligned}$$

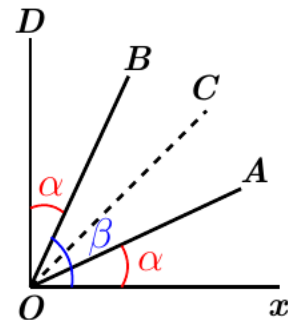
- b) **i.** If $\alpha + \beta = 90^\circ$, then

$$\widehat{XOC} = 45^\circ$$

Let: $\widehat{XOD} = 90^\circ$ that implies OC is the bisector of \widehat{XOD}

Since OC is the bisector of \widehat{AOB} , then

$$\begin{aligned} \widehat{BOD} &= 90^\circ - \beta \\ &= 90^\circ - 90^\circ + \alpha \\ &= \alpha \end{aligned} \quad \beta = 90^\circ - \alpha$$



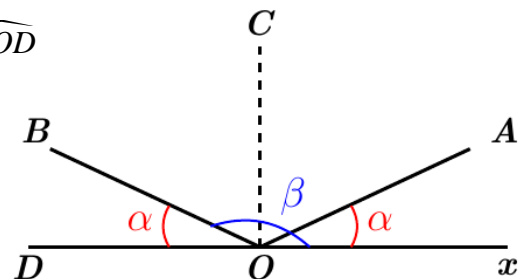
- ii.** If $\alpha + \beta = 180^\circ$, then

$$\widehat{XOC} = 90^\circ$$

Let: $\widehat{XOD} = 180^\circ$ that implies OC is the bisector of \widehat{XOD}

Since OC is the bisector of \widehat{AOB} , then

$$\begin{aligned} \widehat{BOD} &= 180^\circ - \beta \\ &= 180^\circ - 180^\circ + \alpha \\ &= \alpha \end{aligned} \quad \beta = 180^\circ - \alpha$$



Exercise 5

A point O takes on an infinite right $x'Ox$ be conducted the same side half-lines OA and OB , as well as the bisectors of angles \widehat{xOA} , \widehat{AOB} , and $\widehat{BOx'}$.

Calculate the angles of the figure such that the bisector of the angle \widehat{AOB} is perpendicular to $x'Ox$ and the bisectors of the extreme angles formed an angle of 100° .

Solution

Given: $\widehat{zOz'} = 100^\circ$

$$\widehat{xOC} = 90^\circ$$

OC is the bisector \widehat{AOB}

$$\widehat{AOC} = \widehat{COB}$$

Oz is the bisector \widehat{xOA}

$$\widehat{xOz} = \widehat{zOA}$$

Oz' is the bisector $\widehat{x'OB}$

$$\widehat{x'Oz'} = \widehat{z'OB}$$

$$\widehat{xOz} = \frac{180^\circ - 100^\circ}{2}$$

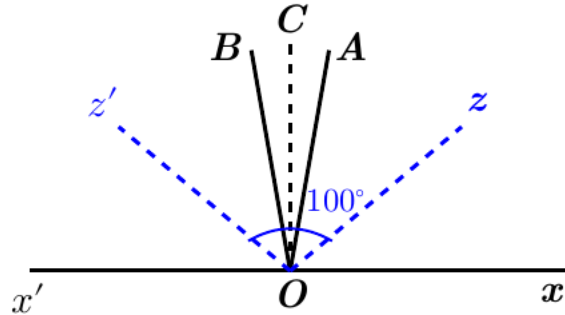
$$= 40^\circ$$

$$\widehat{AOB} = 2\widehat{AOC}$$

$$= 2(90^\circ - 2\widehat{xOz})$$

$$= 2(90^\circ - 80^\circ)$$

$$= 20^\circ$$



Exercise 6

Four consecutive half-lines OA , OB , OC , and OD formed four adjacent consecutive angles which are between them like 1, 2, 3, 4.

Calculate the angles and the adjacent consecutive angles formed by their bisectors.

Solution

$$\widehat{AOB} + \widehat{BOC} + \widehat{COD} + \widehat{DOA} = 360^\circ$$

$$\widehat{AOB} + 2\widehat{AOB} + 3\widehat{AOB} + 4\widehat{AOB} = 360^\circ$$

$$10\widehat{AOB} = 360^\circ$$

$$\widehat{AOB} = 36^\circ$$

$$\widehat{BOC} = 72^\circ$$

$$\widehat{COD} = 108^\circ$$

$$\widehat{DOA} = 144^\circ$$

$$\widehat{xOy} = \frac{1}{2}\widehat{AOB} + \frac{1}{2}\widehat{BOC}$$

$$= \frac{1}{2}36^\circ + \frac{1}{2}72^\circ$$

$$= 18^\circ + 36^\circ$$

$$= 54^\circ$$

$$\widehat{yOz} = \frac{1}{2}\widehat{BOC} + \frac{1}{2}\widehat{COD}$$

$$= \frac{1}{2}72^\circ + \frac{1}{2}108^\circ$$

$$= 36^\circ + 54^\circ$$

$$= 90^\circ$$

$$\widehat{zOw} = \frac{1}{2}\widehat{COD} + \frac{1}{2}\widehat{DOA}$$

$$= \frac{1}{2}108^\circ + \frac{1}{2}144^\circ$$

$$= 54^\circ + 72^\circ$$

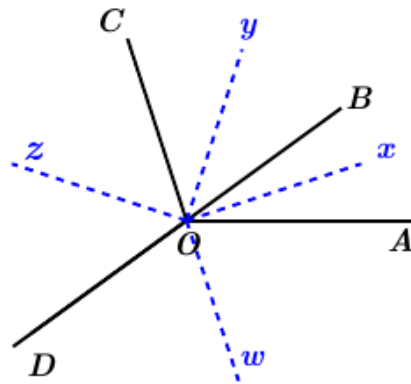
$$= 126^\circ$$

$$\widehat{wOx} = \frac{1}{2}\widehat{DOA} + \frac{1}{2}\widehat{AOB}$$

$$= \frac{1}{2}144^\circ + \frac{1}{2}36^\circ$$

$$= 72^\circ + 18^\circ$$

$$= 90^\circ$$



Exercise 7

A point P is on the base BC of an isosceles triangle ABC . The two points M and N are the middle points of the segments PB and PC , respectively, which lead the perpendicular to the base BC ; these perpendiculars meet AB in E , AC in F .

Demonstrate that the angle EPF is equal to A .

Solution

$$\widehat{BAC} = 180^\circ - \widehat{ABC} - \widehat{ACB}$$

M is the middle of the segment BP and $EM \perp$ to BP , therefore

$$EB = EP \quad \& \quad \widehat{EBP} = \widehat{EPB}$$

N is the middle of the segment CP and $FN \perp$ to CP , therefore

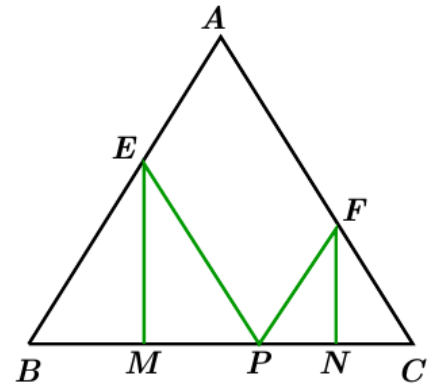
$$FP = FN \quad \& \quad \widehat{FPC} = \widehat{FCP}$$

$$\widehat{EPF} = 180^\circ - \widehat{CPF} - \widehat{BPE}$$

$$= 180^\circ - \widehat{PFC} - \widehat{PBE}$$

$$= 180^\circ - \widehat{ABC} - \widehat{ACB}$$

$$= \widehat{A} \quad \checkmark$$



Exercise 8

Given the triangle ABC and the bisectors BO and CO of the angles of the base, where the point O is the intersection of the 2 bisectors. A line DOE passes through the point O parallel to base BC .

Prove that $DE = DB + CE$

Solution

CO is the bisector of $\widehat{BCE} \Rightarrow \widehat{BCO} = \widehat{OCE}$

$OE \parallel BC \Rightarrow \widehat{COE} = \widehat{BOC}$

$\therefore \widehat{EOC} = \widehat{OCE} \rightarrow \underline{OE = EC}$

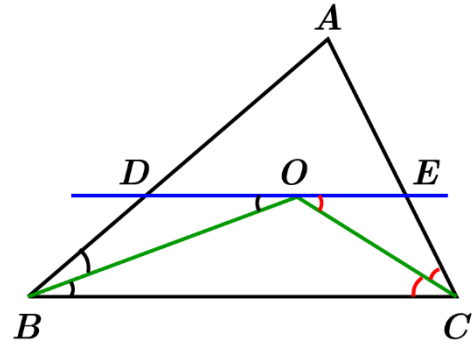
Similar; BO is the bisector of $\widehat{DBC} \Rightarrow \widehat{DBO} = \widehat{OBC}$

$DO \parallel BC \Rightarrow \widehat{DOB} = \widehat{OBC}$

$\therefore \widehat{DOB} = \widehat{OBC} \rightarrow \underline{DO = DB}$

$DE = DO + OE$

$\underline{= DB + CE}$



Exercise 9

A right triangle ABC at A with a height AH . We drop perpendiculars HE and HD from H to sides AB and AC respectively.

- Prove that $DE = AH$
- Prove that AM is perpendicular to DE , where M is the middle point of BC .
- Prove that MN (N is the middle point of AB) and the segment Bx (parallel to DE) are intersect on AH .
- Prove that AM and HD are intersect on Bx .

Solution

- The triangles AEH and ADH are right triangles and angle A is right angle.

Then $AEHD$ is a rectangle.

Therefore, $DE = AH$

- A middle point of a hypotenuse of a right triangle is the center of the circle of that triangle.

Therefore, $MC = MA = MB$

That implies to: $\widehat{MAC} = \widehat{MCA}$

From the rectangle $ADHE$: $\widehat{EAH} = \widehat{EDH}$

$$\widehat{EAH} + \widehat{HAM} + \widehat{MAC} = 90^\circ \quad \widehat{HAM} + \widehat{MAC} = \widehat{HAC}$$

$$\widehat{EAH} + \widehat{HAC} = 90^\circ$$

$$\widehat{EAH} + 90^\circ - \widehat{MCA} = 90^\circ$$

$$\widehat{EAH} = \widehat{MCA} = \widehat{EDH} = \widehat{MAC}$$

$$\widehat{ADE} + \widehat{EDH} = 90^\circ$$

$$\widehat{ADE} + \widehat{MAD} = 90^\circ$$

Therefore, AM is perpendicular to DE .

- N is the middle point of $AB \Rightarrow NA = NB$

$$Bx \text{ parallel to } DE \Rightarrow \widehat{ABx} = \widehat{AED} = \widehat{EDH} = \widehat{EAH}$$

Let point P the intersection of Bx and AH . Since $\widehat{ABP} = \widehat{BAP}$, then the triangle BPA is an

isosceles. PN is the perpendicular to AB as well MN . Which gives us that points M, P, N are on the same line.

Therefore, segment MN and AH intersect at point P .

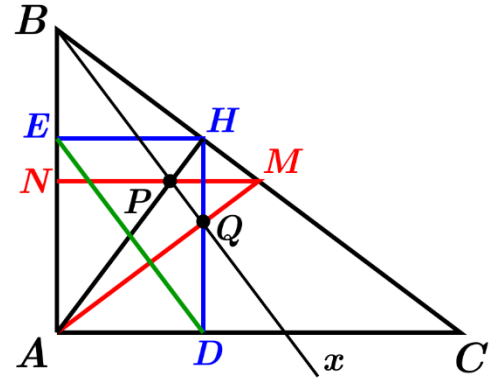
- Let Point Q be the intersection of AM and Bx .

$$\widehat{ABQ} = \widehat{BAH} \quad \& \quad \widehat{BAQ} = \widehat{ABH}$$

Then, the triangles BHA and BQA are equivalent, therefore $AQ \perp BQ$ with hypotenuse AB .

$HQ \parallel AB$, line HQ has to be perpendicular to AC .

AM and HD are intersect on Bx at Q .



Exercise 10

Given an isosceles triangle ABC with a peak at A . Extend base BC the length $CD = AB$, then extend AB of a length $BE = \frac{1}{2}BC$, at the end draw a line EHF , H is the middle point of BC and F is located on AD .

- Prove that $\widehat{ADB} = \frac{1}{2}\widehat{ABC}$
- Prove that $EA = HD$
- Prove that $FA = FD = FH$
- Calculate the value of the angles \widehat{AFH} and \widehat{ADB} where $\widehat{BAC} = 58^\circ$.

Solution

- Triangle ABC is isosceles, then $\widehat{ABC} = \widehat{ACB}$

Since, $CD = AB = AC$, then $\widehat{CAD} = \widehat{ADC}$

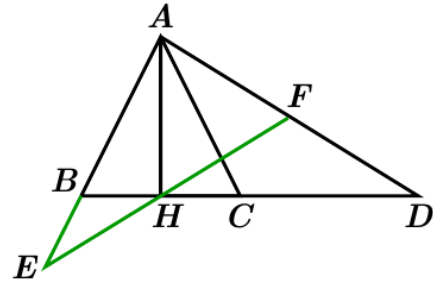
$$2\widehat{ADC} = 180^\circ - \widehat{ACD}$$

$$2\widehat{ADC} = 180^\circ - (180^\circ - \widehat{ACB})$$

$$2\widehat{ADC} = \widehat{ACB}$$

$$\widehat{ADB} = \frac{1}{2}\widehat{ACB}$$

$$= \frac{1}{2}\widehat{ABC}$$



- $BE = \frac{1}{2}BC$ H the middle point of BC
 $= HC$

$$CD = AB$$

$$HC + CD = BE + AB$$

$$EA = HD \quad \checkmark$$

- $\widehat{ADH} = \frac{1}{2}\widehat{ABD}$
 $= \frac{1}{2}(180^\circ - \widehat{HBE})$
 $= \frac{1}{2}(180^\circ - 180^\circ + 2\widehat{BHE})$
 $= \widehat{BHE}$

$$\Rightarrow \underline{FD = FH}$$

$$\widehat{AHF} = 90^\circ - \widehat{FHD}$$

$$= 90^\circ - \widehat{ADH} \quad (\triangle HDA)$$

$$= 90^\circ - (90^\circ - \widehat{HAF})$$

$$= \widehat{HAF}$$

$$\Rightarrow \underline{FA = FH}$$

$$FA = FD = FH \quad \checkmark$$

$$d) \widehat{BAC} = 58^\circ$$

$$\begin{aligned} \widehat{ADB} &= \frac{1}{2} \widehat{ACB} \\ &= \frac{1}{2} \left(\frac{1}{2} (180^\circ - \widehat{BAC}) \right) \\ &= \frac{1}{4} (180^\circ - 58^\circ) \\ &= \frac{122^\circ}{4} \\ &= \frac{61^\circ}{2} \quad \Bigg| \quad = 30.5^\circ \end{aligned}$$

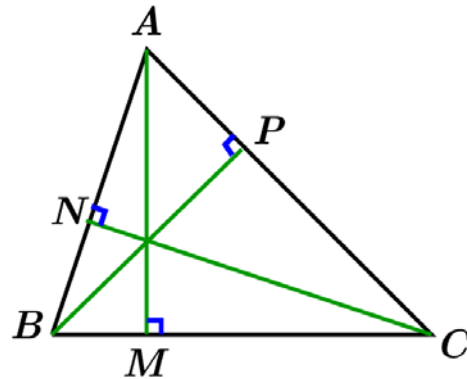
Triangle AFH is isosceles then,

$$\begin{aligned} \widehat{AFH} &= 180^\circ - \widehat{HFD} \\ &= 180^\circ - (180^\circ - 2\widehat{FDH}) \\ &= 2\widehat{FDH} \\ &= 2 \frac{61^\circ}{2} \\ &= 61^\circ \quad \Bigg| \end{aligned}$$

Exercise 11

Demonstrate that the heights of a triangle share the angles of triangle that equal to each other.

Solution



Consider the 2 right triangles APB and ANC , which they have the same angle A .

Therefore, $\widehat{ABP} = \widehat{ACN}$.

Similar, consider the 2 right triangles BPC and AMC , which they have the same angle C .

Therefore, $\widehat{MAC} = \widehat{CBP}$.

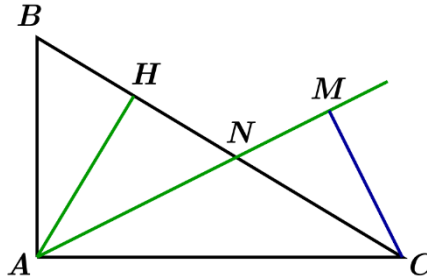
Similar, consider the 2 right triangles BNC and AMB , which they have the same angle B .

Therefore, $\widehat{BCN} = \widehat{BAM}$.

Exercise 12

A right triangle ABC at A where $AB < AC$, drop a perpendicular AH from A to the hypotenuse BC where $HN = HB$. From C drops a perpendicular CM at AN . Demonstrate that BC is the bisector of the angle \widehat{ACM} .

Solution



Consider the 2 right triangles ABC and ABH with a common angle B , then

$$\widehat{BAH} = \widehat{ACB}$$

Given: $HN = HB$, then $\widehat{HAN} = \widehat{BAH} = \widehat{ACB}$

$$\begin{aligned}\widehat{NAC} &= 90^\circ - \widehat{HAB} - \widehat{HAN} \\ &= 90^\circ - 2\widehat{HCA}\end{aligned}$$

Consider the 2 right triangles AHN and CMC , where $\widehat{HNA} = \widehat{MNC}$

Therefore, $\widehat{HAN} = \widehat{NCM}$

Since $\widehat{HAN} = \widehat{ACB}$

Then $\widehat{ACB} = \widehat{MCB}$

Therefore, BC is the bisector of the angle \widehat{ACM}

Exercise 13

On the sides of an angle that it takes the length OA and OB , so that $OA + OB = \ell$ (is given) and construct a parallelogram $OABC$. What is the place of the summit C of parallelogram?

Solution

Let segment OE extension of segment OA such that $OE = \ell$

Let segment OF extension of segment OB such that $OF = \ell$

Then, the triangle OEF is an isosceles.

$$\widehat{OEF} = \widehat{OFE} = 90^\circ - \frac{1}{2}\widehat{EOF}$$

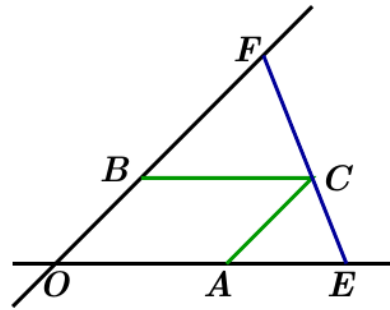
$$OA + OB = \ell$$

$$\begin{cases} OA + AC = \ell \\ OA + AE = \ell \end{cases} \Rightarrow AC = AE$$

$$\begin{cases} OB + BC = \ell \\ OB + BF = \ell \end{cases} \Rightarrow BC = BF$$

$$\widehat{OEF} = \widehat{OFE} = \widehat{FCB} = \widehat{ACE}$$

Therefore, the point C , E , and F are aligned.



Exercise 14

Demonstrate that the sum of distances from a point M on the base BC of an isosceles triangle ABC to the sides equal a constant.

Solution

The shortest distance from a point to a line is the perpendicular from that point to the line.

Therefore, let:

$$MP \perp AB$$

$$MQ \perp AC$$

Let $BH \perp AC$ (Shortest distance from B to side AC .)

Let D be the point of intersection ME and BH .

Let $ME \parallel AC$

Where the point E is the intersection of the lines MD and AB .

Since $MD \parallel AC$ then $\widehat{DMB} = \widehat{ACB}$

Since triangle ABC is an isosceles

$$\widehat{DMB} = \widehat{ACB} = \widehat{PBM}$$

The right triangles BPM and BDM at P & D and have the same hypotenuse, then

$$\Rightarrow |MP| = |BD|$$

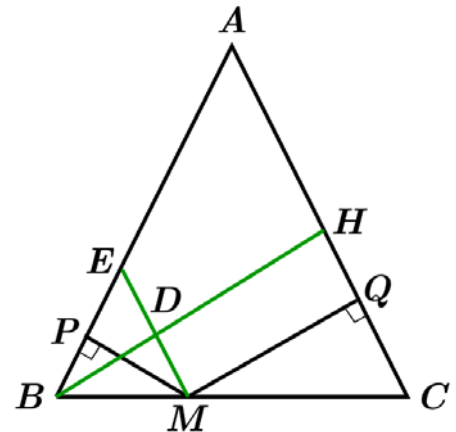
$MD \parallel HQ$ and $DH \perp MQ$

$$\Rightarrow |MQ| = |DH|$$

$$|MP| + |MQ| = |BD| + |DH|$$

$$= |BH|$$

$$= \text{constant}$$



Therefore; the sum of distances from a point M on the base BC of an isosceles triangle ABC to the sides equal a constant.

Exercise 15

Demonstrate that the difference of distances from a point M taken on the extension of the base BC of an isosceles triangle ABC to the sides equal a constant.

Solution

The shortest distance from a point to a line is the perpendicular from that point to the line.

Therefore, let:

$$MP \perp AB$$

$$MQ \perp AC$$

Let $BH \perp AC$ (Shortest distance from B to side AC .)

Let D be the point of intersection ME and BH .

Let $ME \parallel AC$

Where the point E is the intersection of the extensions of the lines MD and AB .

Since $MD \parallel AC$ then $\widehat{DMB} = \widehat{ACB}$

Since triangle ABC is an isosceles

$$\widehat{DMB} = \widehat{ACB} = \widehat{PBM}$$

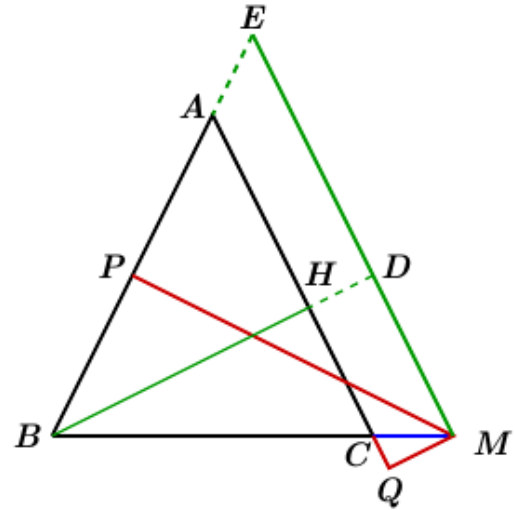
The right triangles BPM and BDM at P & D and have the same hypotenuse, then

$$\Rightarrow |MP| = |BD|$$

$MD \parallel HQ$ and $DH \perp MQ$

$$\Rightarrow |MQ| = |DH|$$

$$\begin{aligned} |MP| - |MQ| &= |BD| - |DH| \\ &= |BH| \\ &= \text{constant} \end{aligned}$$

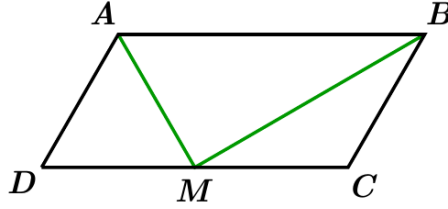


Therefore; the difference of distances from a point M taken on the extension of the base BC of an isosceles triangle ABC to the sides equal a constant.

Exercise 16

Consider a parallelogram $ABCD$ in which $CD = 2AD$. In the joint A and B the middle M of BC . Prove that the angle \widehat{AMB} is a right angle.

Solution



Since the point M is the middle of side BC , then

$$MD = MC = \frac{1}{2}CD$$

$$\Rightarrow MD = AD = BC$$

Therefore; the triangles ADM and BCM are isosceles at D and C respectively.

Which implies that $MA = MB$

Let O be the middle point of the side AB , and $OA = OB = AD$

O and M are middle of the parallelogram $ABCD$, that implies

$$OM = BC = AD$$

$$\Rightarrow OA = OB = OM$$

The triangle MAB inscribed in a circle with center at O and diameter AB , that will imply that is a right triangle at the point M .

Or

$$\widehat{AMD} = \frac{1}{2}(180^\circ - \widehat{MDA})$$

$$\widehat{BMC} = \frac{1}{2}(180^\circ - \widehat{MCB})$$

$$\widehat{ADM} + \widehat{MCB} = 180^\circ$$

$$\widehat{DMA} + \widehat{AMB} + \widehat{BMC} = 180^\circ$$

$$\widehat{AMB} = 180^\circ - (\widehat{BMC} + \widehat{DMA})$$

$$= 180^\circ - \left(90^\circ - \frac{1}{2}\widehat{MDA} + 90^\circ - \frac{1}{2}\widehat{MCB}\right)$$

$$= \frac{1}{2}(\widehat{MDA} + \widehat{MCB})$$

$$= \frac{1}{2}(180^\circ)$$

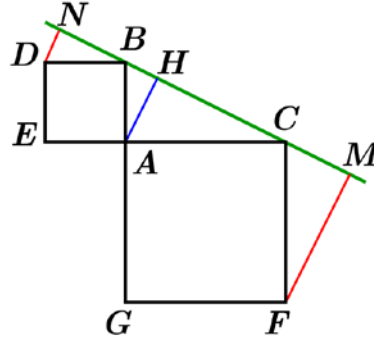
$$= \underline{90^\circ}$$

Exercise 17

From the sides AB and AC of a right triangle ABC at A , draw two squares $ABDE$ and $ACFG$. Then lead DN and FM perpendicular to the line BC .

- Prove that $DN + FM = BC$
- Prove that the points D, A, F on a straight line.
- Prove that the lines DE and FG contribute on the extension of the height AH .

Solution



- Let consider the 2 right triangles DNB & BHA at points N & H respectively, with $DB = AB$. Then

$$\begin{aligned}\widehat{HAB} &= 90^\circ - \widehat{ABH} \\ &= 90^\circ - (90^\circ - \widehat{NBD}) \\ &= \widehat{NBD} \\ \Rightarrow \widehat{BDN} &= \widehat{ABH}\end{aligned}$$

\therefore The 2 triangles are equals, which implies that $\underline{DN = BH}$

Similar, for the 2 right triangles CMF & AHC at points M & H respectively, with $AC = CF$. Then

$$\begin{aligned}\widehat{HAC} &= 90^\circ - \widehat{ACH} \\ &= 90^\circ - (90^\circ - \widehat{MCF}) \\ &= \widehat{MCF} \\ \Rightarrow \widehat{ACH} &= \widehat{CFM}\end{aligned}$$

\therefore The 2 triangles are equals, which implies that $\underline{FM = HC}$

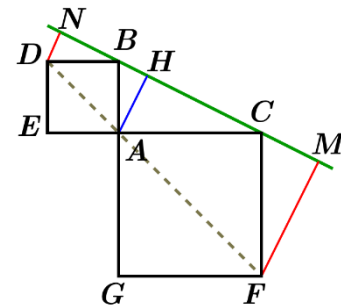
$$\begin{aligned}DN + FM &= BH + HC \\ &= \underline{BC} \quad \checkmark\end{aligned}$$

- Since $ABDE$ is a square, then $\widehat{BAD} = 45^\circ$

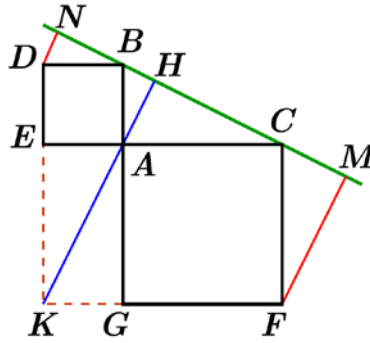
And $ACFG$ is a square, then $\widehat{CAF} = 45^\circ$

$$\begin{aligned}\widehat{DAF} &= \widehat{DAB} + \widehat{BAC} + \widehat{CAF} \\ &= 45^\circ + 90^\circ + 45^\circ \\ &= \underline{180^\circ}\end{aligned}$$

\therefore The points D, A , & F are on a straight line.



- c) Let the point K be the intersection of the extension of the sides DE and FG .
Which will result of $GKEA$ is a rectangle with $AE = GK$ & $EK = AG$



Consider the 2 right triangles BAC & KGA at points A & G respectively with $AE = AB = GK$
 $\widehat{ACB} = \widehat{GAK} = \widehat{ACH}$

From the right triangle AHC :

$$\widehat{HAC} + \widehat{ACH} = 90^\circ$$

$$\rightarrow \widehat{HAC} + \widehat{KAG} = 90^\circ$$

$$\begin{aligned} \widehat{HAC} + \widehat{CAG} + \widehat{KAG} &= (\widehat{HAC} + \widehat{KAG}) + \widehat{CAG} \\ &= 90^\circ + 90^\circ \\ &= \underline{180^\circ} \end{aligned}$$

∴ The points K , A , & H are on a straight line.

Exercise 18

Given a diamond $ABCD$; the peak B and D , the same the perpendiculars BM, BN, DP, DQ on opposite sides. These perpendiculars are intersected at E and F .

Demonstrate that the angles of the quadrilateral $BFDE$ are equals to the diamond and which is a diamond itself.

Solution

From the right triangles BPD & BMD , that implies $\widehat{MBD} = \widehat{PDB}$

$$\Rightarrow \widehat{EBD} = \widehat{EDB}$$

Similar, from the right triangles BND & BQD , that implies $\widehat{NBD} = \widehat{QDB}$

$$\Rightarrow \widehat{FBD} = \widehat{FDB}$$

$$\widehat{EBD} + \widehat{DBF} = \widehat{EDB} + \widehat{BDF}$$

$$\widehat{EBF} = \widehat{EDF}$$

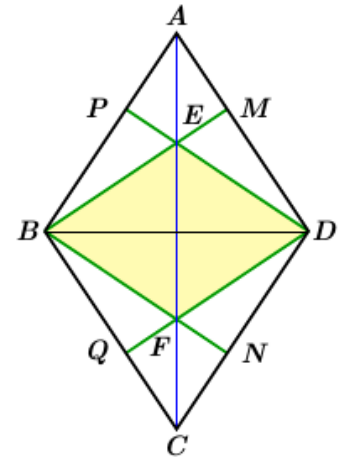
Since, $AC \perp BD$, then $EF \perp BD$

The 2 triangles EBF & EDF have EF as a common side and $\widehat{EBF} = \widehat{EDF}$, then

$$\widehat{BEF} = \widehat{DEF} = \widehat{BFE} = \widehat{DFE}$$

$$\widehat{BED} = \widehat{BFD}$$

Therefore; the angles of the quadrilateral $BFDE$ are equals to the diamond and which is a diamond itself.

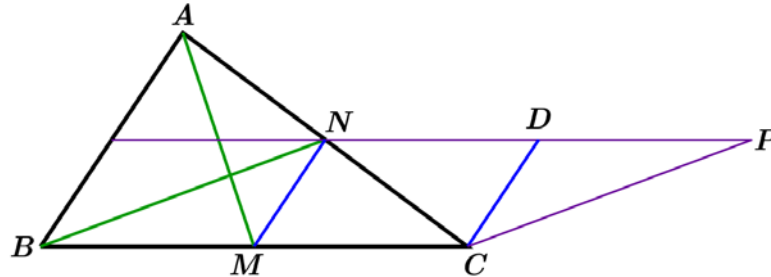


Exercise 19

In a triangle ABC , we trace the median AM and BN and from N a parallel to BC , from C a parallel to BN ; that the two sides intersect at a point P . Let D be the middle point of the segment PN .

Prove that CD is parallel to MN .

Solution



Since the points M & N are middle of the sides BC & AC of the triangle ABC , then
 $MN \parallel AB$

Given: $NP \parallel MC$

$BN \parallel CP$

Since M & D are the middle points of the segments BC and NP respectively, then
 $BN \parallel CP \parallel MD$

Therefore, $BNPC$ is a parallelogram, and $MC = ND$.

Since $MC = ND$ & $MN = CD$

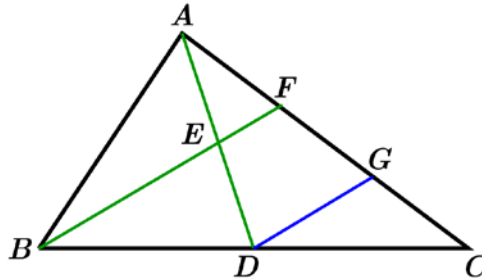
Therefore; $MCDN$ is a parallelogram which implies to CD parallel to MN

Exercise 20

The median AD of a given triangle ABC to the side BC . The same the median BE to the side AC which intersect AC at a point F .

Prove that where $AF = \frac{1}{3} AC$

Solution



Let DG be parallel to segment BEF .

Given: E is the middle point of the segment $AD \Rightarrow AE = ED$

D is the middle point of the segment $BC \Rightarrow BD = DC$

Since $EF \parallel DG$, and $AE = ED$, that implies $AF = FG$

Consider the triangles CDG and CBF :

$EF \parallel DG$, and $CD = DB$, that implies $GC = FG$

That will imply to: $AF = FG = GC$

$$\begin{aligned} AC &= AF + FG + GC \\ &= 3AF \end{aligned}$$

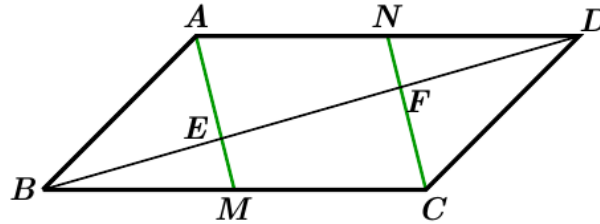
Therefore; $AF = \frac{1}{3} AC$

Exercise 21

In a parallelogram $ABCD$, from the points peak A and C joint the middle of opposite sides at M and N respectively.

Prove that the diagonal BD is divided in three equal parts.

Solution



M is the middle point of the segment BC , then $BM = CM$

N is the middle point of the segment AD , then $NA = ND$

From these, implies that $AM \parallel CN$.

From the triangles BEM & BCF , and since $ME \parallel CF$

It will give us that $BE = EF$

From the triangles DFN & DEA , and since $AE \parallel FN$

It will give us that $\Rightarrow DF = EF$

Therefore, $BE = EF = DF$

$$BD = BE + EF + FD$$

$$= \underline{3BE}$$

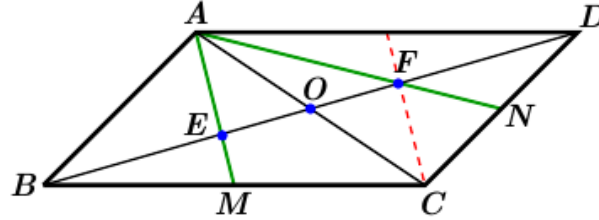
Therefore; the diagonal BD is divided in three equal parts

Exercise 22

In a parallelogram $ABCD$, from the point peak A , extend to the middle of sides BC and DC at M and N respectively.

Prove that the diagonal BD is divided in three equal parts.

Solution



Let a point E be the intersection of the segments AM & BD .

Let a point F be the intersection of the segments AN & BD .

Let O be the intersection of the both diagonal AC & BD .

From the triangles BEM & BCF , and since $ME \parallel CF$

$$\Rightarrow BE = EF$$

Similar, $\Rightarrow DF = EF$

$$BO = OF \rightarrow OE = OF$$

$$BO = BE + EO$$

$$= BE + \frac{1}{2} BE$$

$$= \frac{3}{2} BE$$

$$BE = \frac{2}{3} BO$$

$$= \frac{2}{3} \left(\frac{1}{2} BD \right)$$

$$= \frac{1}{3} BD$$

$$DF = \frac{2}{3} DO$$

$$= \frac{2}{3} \left(\frac{1}{2} BD \right)$$

$$= \frac{1}{3} BD$$

Therefore; the diagonal BD is divided in three equal parts

Exercise 23

Consider a triangle ABC with a bisector AF of the angle A . by F , we lead FE parallel to AB , and by E we lead ED parallel to BC .

Prove that $AE = BD$

Solution

Given: $\widehat{EAF} = \widehat{FAB}$

Since $FE \parallel AB$, then

$$\widehat{FEC} = \widehat{BAE} = 2\widehat{EAF}$$

$$\begin{aligned}\widehat{AEF} &= 180^\circ - \widehat{FEC} \\ &= 180^\circ - 2\widehat{EAF}\end{aligned}$$

Consider the triangle AEF :

$$\widehat{EAF} + \widehat{EFA} + \widehat{AEF} = 180^\circ$$

$$\widehat{EAF} + \widehat{EFA} + 180^\circ - 2\widehat{EAF} = 180^\circ$$

$$\widehat{EFA} - \widehat{EAF} = 0^\circ$$

$$\widehat{EFA} = \widehat{EAF}$$

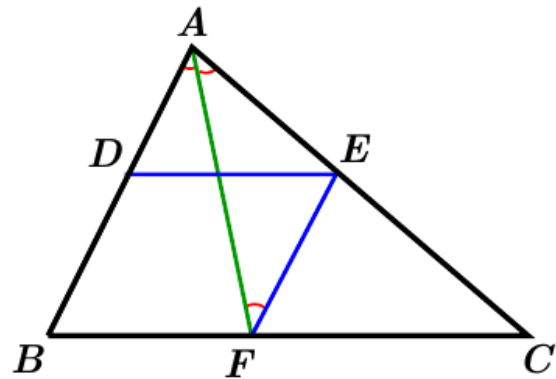
\therefore Triangle AEF is isosceles

$$\Rightarrow \underline{AE = EF}$$

Given $DE \parallel BF$ & $FE \parallel DB$

$FEDB$ is a parallelogram;

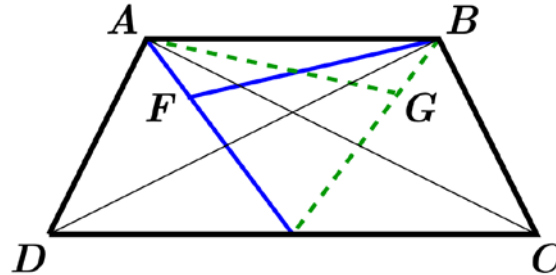
Then, $\underline{EF = DB = AE}$



Exercise 24

Given an isosceles trapezoid $ABCD$ ($AD = BC$) with diagonals AC and BD . The bisector of angles \widehat{DAB} and \widehat{DBA} intersect in F , and the bisector of angles \widehat{CBA} and \widehat{CAB} intersect in G . Demonstrate that FG is parallel to AB

Solution



Consider the 2 triangles ABD & ABC :

- Both has the AB as common
- $AD = BC$

That implies to: $\widehat{ABD} = \widehat{CAB}$

Since BF is the bisector of the angle \widehat{ABD}

$$\widehat{ABF} = \widehat{FBD}$$

$$\begin{aligned} \Rightarrow \widehat{ABF} &= \frac{1}{2} \widehat{ABD} \\ &= \frac{1}{2} \widehat{CAB} \\ &= \frac{1}{2} (2\widehat{BAG}) \\ &= \widehat{BAG} \end{aligned}$$

$$\widehat{ABF} = \widehat{BAG} \quad |$$

From the 2 triangles AFB & AGB

- Both has the AB as common
- $\widehat{ABF} = \widehat{BAG}$

$$\underline{FG \parallel AB} \quad |$$

Exercise 25

Let M and N be the middle points of the bases AB and CD of a trapezoid $ABCD$.

Let P and Q be the middle points of the diagonals AC and BD respectively.

Demonstrate that the angles \widehat{M} and \widehat{N} of quadrilateral $MNPQ$ are equals to the angle formed by extending the sides not parallel to BC and AD , where intersect at point E .

Solution

Since N is the mid-point of the side DC , and
 P is the mid-point of the side AC , then

$$\Rightarrow NP \parallel AD$$

Since M is the mid-point of the side AB , and
 Q is the mid-point of the side DB , then

$$\Rightarrow QM \parallel AD$$

$$\therefore NP \parallel QM \parallel AE$$

Since N is the mid-point of the side DC , and
 Q is the mid-point of the side DB , then

$$\Rightarrow NQ \parallel CB$$

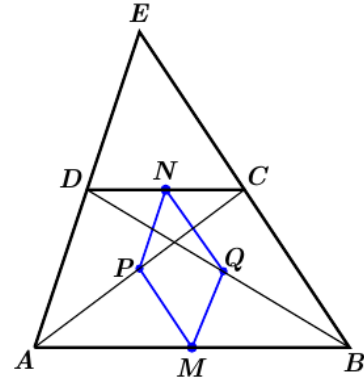
Since M is the mid-point of the side AB , and
 P is the mid-point of the side AC , then

$$\Rightarrow MP \parallel CB$$

$$\therefore NQ \parallel MP \parallel BE$$

$$\rightarrow \begin{cases} NP \parallel QM \\ NQ \parallel PM \end{cases}$$

$$\therefore \underline{\widehat{PMQ} = \widehat{PNQ}}$$



Exercise 26

In a triangle ABC , the medians segment BM and CN intersect in right angles and the measurement are 3 and 6 units respectively.

1. Construct a geometrical to the triangle ABC .
2. In the trace of third median AP which leads MN extension such the distance $MD = MN$, which lead to the segments AD and PD . Calculate AD and DP .
3. What is the nature of the triangle APD ?

Solution

1. Since M and N are the middle point of the sides AC & AB , then

$$\begin{aligned} BG &= \frac{2}{3} BM \\ &= \frac{2}{3}(3) \\ &= 2 \text{ units} \end{aligned}$$

Similar,

$$\begin{aligned} CG &= \frac{2}{3} CN \\ &= \frac{2}{3}(6) \\ &= 4 \text{ units} \end{aligned}$$

Wish, we lead to: $GM = 1$ & $GN = 2$

We can construct 2 perpendicular lines intersect at a point G , then we use to measure the distance from the point G to get the points B, C, M , & N .

By extending the segment BN and CM with equal distance and which it will intersect at point A .

2. Since $ND \parallel BC$ & $MD = MN$

The parallelogram $BPDM$, $BP = MD = MN$

Then $PD = MB = 3 \text{ units}$

$AD \parallel CN$ and M is the intersection of the diagonals of the parallelogram $ADCN$, then

$$AD = CN = 6 \text{ units}$$

3. $PD \parallel BN$ & $MB \perp CN$, then $PD \perp CN$
 $AD \parallel CN$ & $PD \perp CN$, then $AD \perp PD$.

Therefore; the triangle ADP is right triangle at point D .

