Solution Section 2.5 – Subspaces, Span and Null Space

Exercise

Suppose S and T are two subspaces of a vector space $\boldsymbol{V}$.

a) The sum $S+T$ contains all sums $s+t$ of a vector s in S and a vector t in T. Show that $S+T$ satisfies the requirements (addition and scalar multiplication) for a vector space.

b) If S and T are lines in $\mathbb{R}^m$, what is the difference between $S+T$ and $S \cup T$? That union contains all vectors from S and T or both. Explain this statement: The span of $S \cup T$ is $S+T$.

Solution

a) Let $s,\ s'$ be vectors in S, Let $t,\ t'$ be vectors in T, and let c be a scalar. Then

$$(s+t)+(s'+t')=(s+s')+(t+t') \text{ and}$$

$$c(s+t)=cs+ct$$

Thus $S+T$ is closed under addition and scalar multiplication, it satisfies the two requirements for a vector space.

b) If S and T are distinct lines, then S and T is a plane, whereas $S \cup T$ is not even closed under addition. The span of $S \cup T$ is the set of all combinations of vectors in this union. In particular, it contains all sums $s+t$ of a vector $\boldsymbol{s}$ in S and a vector $\boldsymbol{t}$ in T, and these sums form $S+T$. $S+T$ contains both S and T; so, it contains $S \cup T$. $S+T$ is a vector space.

c) So, it contains all combinations of vectors in itself; in particular, it contains the span of $S \cup T$. Thus, the span of $S \cup T$ is $S+T$.

Exercise

Determine which of the following are subspaces of $\mathbb{R}^3$?

a) All vectors of the form $(a, 0, 0)$

b) All vectors of the form $(a, 1, 1)$

c) All vectors of the form (a, b, c), where $b = a + c$

d) All vectors of the form (a, b, c), where $b = a + c + 1$

e) All vectors of the form $(a, b, 0)$

Solution

a) $(a_1,\ 0,\ 0)+(a_2,\ 0,\ 0)=(a_1+a_2,\ 0,\ 0)$

$k(a,\ 0,\ 0)=(ka,\ 0,\ 0)$

This is a subspace of $\mathbb{R}^3$

b) $(a_1, 1, 1)+(a_2, 1, 1)=(a_1+a_2, 2, 2)$ which is not in the set.

Therefore, this is not a subspace of $\mathbb{R}^3$

c) $(a_1, b_1, c_1)+(a_2, b_2, c_2)=(a_1+a_2, b_1+b_2, c_1+c_2)$

$$=(a_1+a_2, a_1+c_1+a_2+c_2, c_1+c_2)$$

$$=(a_1+a_2, (a_1+a_2)+(c_1+c_2), c_1+c_2)$$

$$=(a_1, a_1+c_1, c_1)+(a_2, a_2+c_2, c_2)$$

$$k(a, b, c)=(ka, kb, kc)$$

$$=(ka, k(a+c), kc)$$

$$=k(a, (a+c), c)$$

This is a subspace of $\mathbb{R}^3$

d) $k(a+c+1)\neq ka+kc+1$ so $k(a,b,c)$ is not in the set.

Therefore, this is not a subspace of $\mathbb{R}^3$

e) $(a_1, b_1, 0)+(a_2, b_2, 0)=(a_1+a_2, b_1+b_2, 0)$

$k(a, b, 0)=(ka, kb, 0)$

This is a subspace of $\mathbb{R}^3$

Exercise

Determine which of the following are subspaces of $\mathbb{R}^\infty$?

a) All sequences $\vec{v}$ in $\mathbb{R}^\infty$ of the form $\vec{v} = (v, 0, v, 0, \ldots)$

b) All sequences $\vec{v}$ in $\mathbb{R}^\infty$ of the form $\vec{v} = (v, 1, v, 1, \ldots)$

c) All sequences $\vec{v}$ in $\mathbb{R}^\infty$ of the form $\vec{v} = (v, 2v, 4v, 8v, 16v, \ldots)$

Solution

a) Let $\vec{v}_1=(v_1, 0, v_1, 0, \ldots)$ $\vec{v}_2=(v_2, 0, v_2, 0, \ldots)$

$$\vec{v}_1+\vec{v}_2=(v_1, 0, v_1, 0, \ldots)+(v_2, 0, v_2, 0, \ldots)$$

$$=(v_1+v_2, 0, v_1+v_2, 0, \ldots) \qquad w=v_1+v_2$$

$$=(w, 0, w, 0, \ldots)$$

$$k\vec{v} = k(v,\ 0,\ v,\ 0,\ \ldots)$$

$$= (kv,\ 0,\ kv,\ 0,\ \ldots) \qquad w = kv$$

$$= (w,\ 0,\ w,\ 0,\ \ldots)$$

This is a ***subspace*** of $\mathbb{R}^{\infty}$

b) Let $\vec{v} = (v,\ 1,\ v,\ 1,\ \ldots)$

$$k\vec{v} = k(v,\ 1,\ v,\ 1,\ \ldots)$$

$$= (kv,\ k,\ kv,\ k,\ \ldots)$$

$$\neq (kv,\ 1,\ kv,\ 1,\ \ldots)$$

$k\vec{v}$ is not in the set

Since $k \neq 1$, then is ***not*** a ***subspace*** of $\mathbb{R}^{\infty}$

c) Let $\vec{v}_1 = (v_1,\ 2v_1,\ 4v_1,\ 8v_1,\ \ldots)$ $\quad \vec{v}_2 = (v_2,\ 2v_2,\ 4v_2,\ 8v_2,\ \ldots)$

$$\vec{v}_1 + \vec{v}_2 = (v_1,\ 2v_1,\ 4v_1,\ 8v_1,\ \ldots) + (v_2,\ 2v_2,\ 4v_2,\ 8v_2,\ \ldots)$$

$$= (v_1 + v_2,\ 2v_1 + 2v_2,\ 4v_1 + 4v_2,\ 8v_1 + 8v_2,\ \ldots)$$

$$= (v_1 + v_2,\ 2(v_1 + v_2),\ 4(v_1 + v_2),\ 8(v_1 + v_2),\ \ldots) \qquad w = v_1 + v_2$$

$$= (w,\ 2w,\ 4w,\ 8w,\ \ldots)$$

$$k\vec{v} = k(v,\ 2v,\ 4v,\ 8v,\ 16v,\ \ldots)$$

$$= (kv,\ 2kv,\ 4kv,\ 8kv,\ 16kv,\ \ldots)$$

$$= (w,\ 2w,\ 4w,\ 8w,\ 16w,\ \ldots) \qquad w = kv$$

This is a ***subspace*** of $\mathbb{R}^{\infty}$

Exercise

Which of the following are linear combinations of $\vec{u}$ = (0, –2, 2) and $\vec{v}$ = (1, 3, –1)?

a) (2, 2, 2) *b)* (3, 1, 5) *c)* (0, 4, 5) *d)* (0, 0 ,0)

Solution

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \\ 2 & -1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

a) $b = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 0 & 1 & 2 \\ -2 & 3 & 2 \\ 2 & -1 & 2 \end{array}\right] \quad \textit{Switch } R_1 \ \& \ R_2$$

$$\left[\begin{array}{cc|c} -2 & 3 & 2 \\ 0 & 1 & 2 \\ 2 & -1 & 2 \end{array}\right] \quad R_3 + R_1$$

$$\left[\begin{array}{cc|c} -2 & 3 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{array}\right] \quad -\tfrac{1}{2}R_1$$

$$\left[\begin{array}{cc|c} 1 & -\frac{3}{2} & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{array}\right] \quad \begin{array}{l} R_1 + \frac{3}{2}R_2 \\ \\ R_3 - 2R_2 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$(2,\ 2,\ 2) = 2\vec{u} + 2\vec{v}$ is a linear combination of $\vec{u}$ and $\vec{v}$.

b) $b = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 0 & 1 & 3 \\ -2 & 3 & 1 \\ 2 & -1 & 5 \end{array}\right] \quad \textit{Switch } R_1 \ \& \ R_2$$

$$\left[\begin{array}{cc|c} -2 & 3 & 1 \\ 0 & 1 & 3 \\ 2 & -1 & 5 \end{array}\right] \quad R_3 + R_1$$

$$\left[\begin{array}{cc|c} -2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{array}\right] \quad \begin{array}{l} R_1 - 3R_2 \\ \\ R_3 - 2R_2 \end{array}$$

$$\left[\begin{array}{cc|c} -2 & 0 & -8 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \quad -\tfrac{1}{2}R_1$$

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$(3,\ 1,\ 5) = 4\vec{u} + 3\vec{v}$ is a linear combination of $\vec{u}$ and $\vec{v}$.

c) $b = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & 3 & 4 \\ 2 & -1 & 5 \end{array}\right] \quad \textit{Switch } R_1 \ \& \ R_2$$

$$\left[\begin{array}{cc|c} -2 & 3 & 4 \\ 0 & 1 & 0 \\ 2 & -1 & 5 \end{array}\right] \quad \begin{array}{l} \\ \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{cc|c} -2 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 2 & 9 \end{array}\right] \quad \begin{array}{l} R_1 - 3R_2 \\ \\ R_3 - 2R_2 \end{array}$$

$$\left[\begin{array}{cc|c} -2 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{array}\right] \quad \begin{array}{l} -\tfrac{1}{2}R_1 \\ \\ \tfrac{1}{9}R_3 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \quad R_1 + 2R_3$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

(0, 4, 5) is not a linear combination of $\vec{u}$ and $\vec{v}$.

d) $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & 3 & 0 \\ 2 & -1 & 0 \end{array}\right] \quad \textit{Switch } R_1 \ \& \ R_2$$

$$\left[\begin{array}{cc|c} -2 & 3 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right] \quad \begin{array}{l} \\ \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{cc|c} -2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right] \quad \begin{array}{l} R_1 - 3R_2 \\ \\ R_3 - 2R_2 \end{array}$$

$$\left[\begin{array}{cc|c} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} -\frac{1}{2}R_1 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$(0,\ 0,\ 0) = 0\vec{u} + 0\vec{v}$ is a linear combination of $\vec{u}$ and $\vec{v}$.

Exercise

Which of the following are linear combinations of $\vec{u}$ = (2, 1, 4), $\vec{v}$ = (1, –1, 3) and $\vec{w}$ = (3, 2, 5)?

a) (–9, –7, –15)

b) (6, 11, 6)

c) (0, 0 ,0)

Solution

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{pmatrix}$$

a)

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{array}\right] \quad \begin{array}{l} \\ 2R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & -9 \\ 0 & -3 & 1 & -5 \\ 0 & 1 & -1 & 3 \end{array}\right] \quad \begin{array}{l} 3R_1 + R_2 \\ \\ 3R_3 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 6 & 0 & 10 & -32 \\ 0 & -3 & 1 & -5 \\ 0 & 0 & -2 & 4 \end{array}\right] \begin{array}{l} R_1 + 5R_3 \\ 2R_2 + R_3 \\ \ \end{array}$$

$$\left[\begin{array}{ccc|c} 6 & 0 & 0 & -12 \\ 0 & -6 & 0 & -6 \\ 0 & 0 & -2 & 4 \end{array}\right] \begin{array}{l} \frac{1}{6}R_1 \\ -\frac{1}{6}R_2 \\ -\frac{1}{2}R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Therefore, $(-9,\ -7,\ -15) = -2\vec{u} + 1\vec{v} - 2\vec{w}$

b)

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array}\right] \begin{array}{l} \ \\ 2R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 0 & -3 & 1 & 16 \\ 0 & 1 & -1 & -6 \end{array}\right] \begin{array}{l} 3R_1 + R_2 \\ \ \\ 3R_3 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 6 & 0 & 10 & 34 \\ 0 & -3 & 1 & 16 \\ 0 & 0 & -2 & -2 \end{array}\right] \begin{array}{l} R_1 + 5R_3 \\ 2R_2 + R_3 \\ \ \end{array}$$

$$\left[\begin{array}{ccc|c} 6 & 0 & 0 & 24 \\ 0 & -6 & 0 & 30 \\ 0 & 0 & -2 & -2 \end{array}\right] \begin{array}{l} \frac{1}{6}R_1 \\ -\frac{1}{6}R_2 \\ -\frac{1}{2}R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore, $(6,\ 11,\ 6) = 4\vec{u} - 5\vec{v} + 1\vec{w}$

c)

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 0 \end{array}\right] \begin{array}{l} \ \\ 2R_2 - R_1 \\ R_2 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \quad \begin{array}{l} 2R_2 - R_1 \\ R_2 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 6 & 0 & 10 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \quad \begin{array}{l} R_1 + 5R_3 \\ 2R_2 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 6 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \quad \begin{array}{l} \frac{1}{6}R_1 \\ -\frac{1}{6}R_2 \\ -\frac{1}{2}R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, $(0,\ 0,\ 0) = 0\vec{u} + 0\vec{v} + 0\vec{w}$

Exercise

Determine whether the given vectors span $\mathbb{R}^3$

a) $\vec{v}_1 = (2,\ 2,\ 2),\quad \vec{v}_2 = (0,\ 0,\ 3),\quad \vec{v}_3 = (0,\ 1,\ 1)$

b) $\vec{v}_1 = (2,\ -1,\ 3),\quad \vec{v}_2 = (4,\ 1,\ 2),\quad \vec{v}_3 = (8,\ -1,\ 8)$

c) $\vec{v}_1 = (3,\ 1,\ 4),\quad \vec{v}_2 = (2,\ -3,\ 5),\quad \vec{v}_3 = (5,\ -2,\ 9),\quad \vec{v}_4 = (1,\ 4,\ -1)$

Solution

a) $det\begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{pmatrix} \underline{= -6 \neq 0}$

The system is consistent for all values so the given vectors span $\boldsymbol{R}^3$.

b) $det\begin{pmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{pmatrix} = 0$

The system is not consistent for all values so the given vectors do not span $\boldsymbol{R}^3$.

c) $\left[\begin{array}{cccc|c} 3 & 2 & 5 & 1 & b_1 \\ 1 & -3 & -2 & 4 & b_2 \\ 4 & 5 & 9 & -1 & b_3 \end{array}\right] \begin{array}{l} \\ 3R_2 - R_1 \\ 3R_3 - 4R_1 \end{array}$

$$\left[\begin{array}{cccc|c} 3 & 2 & 5 & 1 & b_1 \\ 0 & -11 & -11 & 11 & 3b_2 - b_1 \\ 0 & 7 & 7 & -7 & 3b_3 - 4b_1 \end{array}\right] \begin{array}{l} 11R_1 + 2R_2 \\ \\ 11R_3 + 7R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 33 & 0 & 33 & 33 & 9b_1 + 6b_2 \\ 0 & -11 & -11 & 11 & 3b_2 - b_1 \\ 0 & 0 & 0 & 0 & 33b_3 - 51b_1 + 21b_2 \end{array}\right] \begin{array}{l} \frac{1}{33}R_1 \\ -\frac{1}{11}R_2 \\ \frac{1}{33}R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & \frac{3}{11}b_1 + \frac{2}{11}b_2 \\ 0 & 1 & 1 & -1 & \frac{1}{11}b_1 - \frac{3}{11}b_2 \\ 0 & 0 & 0 & 0 & -\frac{17}{11}b_1 + \frac{7}{11}b_2 + b_3 \end{array}\right]$$

The system has a solution only if $-\frac{17}{11}b_1 + \frac{7}{11}b_2 + b_3 = 0$. But since this is a restriction that the given vectors don't span on all of $\mathbb{R}^3$. So the given vectors do not span $\mathbb{R}^3$.

Exercise

Which of the following are linear combinations of $A = \begin{pmatrix} 4 & 0 \\ -2 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}$

a) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ *b)* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *c)* $\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$

Solution

$$\begin{pmatrix} 4 & 1 & 0 \\ 0 & -1 & 2 \\ -2 & 2 & 1 \\ -2 & 3 & 4 \end{pmatrix}$$

a) $\left[\begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -8 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \end{array}\right] \begin{array}{l} \\ \\ 2R_3 + R_1 \\ 2R_4 + R_1 \end{array}$

$$\left[\begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -8 \\ 0 & 5 & 2 & 4 \\ 0 & 7 & 8 & -10 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ \\ R_3 + 5R_2 \\ R_4 + 7R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & -2 \\ 0 & -1 & 2 & -8 \\ 0 & 0 & 12 & -36 \\ 0 & 0 & 22 & -66 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & -2 \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 12 & -36 \\ 0 & 0 & 22 & -66 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{12}R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & -2 \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 22 & -66 \end{array}\right] \begin{array}{l} R_1 - 2R_3 \\ R_3 + 2R_3 \\ \\ R_4 - 22R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \frac{1}{4}R_1 \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = 1A + 2B - 3C$ is a linear combinations of A, B, and C.

b) $$\left[\begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ -2 & 2 & 1 & 0 \\ -2 & 3 & 4 & 0 \end{array}\right] \begin{array}{l} \\ \\ 2R_3 + R_1 \\ 2R_4 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 7 & 8 & 0 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ \\ R_3 + 5R_2 \\ R_4 + 7R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 22 & 0 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 22 & 0 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{12}R_3 \\ \frac{1}{22}R_4 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 - 2R_3 \\ R_3 + 2R_3 \\ \\ R_4 - R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \frac{1}{4}R_1 \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0A + 0B + 0C$ is a linear combinations of *A*, *B*, and *C*.

c) $$\left[\begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & 0 \\ -2 & 2 & 1 & 3 \\ -2 & 3 & 4 & 8 \end{array}\right] \begin{array}{l} \\ \\ 2R_3 + R_1 \\ 2R_4 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 5 & 2 & 12 \\ 0 & 7 & 8 & 22 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ \\ R_3 + 5R_2 \\ R_4 + 7R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 12 & 12 \\ 0 & 0 & 22 & 22 \end{array}\right] \begin{array}{l} \\ -R_2 \\ \frac{1}{12}R_3 \\ \frac{1}{22}R_4 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 2 & 6 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 - 2R_3 \\ R_2 + 2R_3 \\ \\ R_4 - R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \frac{1}{4}R_1 \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix} = 1A + 2B + 1C$ is a linear combination of A, B, and C.

Exercise

Suppose that $\vec{v}_1 = (2,\ 1,\ 0,\ 3)$, $\vec{v}_2 = (3,\ -1,\ 5,\ 2)$, $\vec{v}_3 = (-1,\ 0,\ 2,\ 1)$. Which of the following vectors are in span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

a) (2, 3, –7, 3) *b)* (0, 0, 0, 0) *c)* (1, 1, 1, 1) *d)* (–4, 6, –13, 4)

Solution

In order to be span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, there must exists scalars *a*, *b*, *c* that $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \boldsymbol{w}$

$$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

a) (2, 3, –7, 3)

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{array}\right] \begin{array}{l} \\ 2R_2 - R_1 \\ \\ 2R_4 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 0 & -5 & 1 & 4 \\ 0 & 5 & 2 & -7 \\ 0 & -5 & 5 & 0 \end{array}\right] \begin{array}{l} 5R_1 + 3R_2 \\ \\ R_3 + R_2 \\ R_4 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & -2 & 22 \\ 0 & -5 & 1 & 4 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 4 & -4 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{3}R_3 \\ \frac{1}{4}R_4 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & -2 & 22 \\ 0 & -5 & 1 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 + 2R_3 \\ R_2 - R_3 \\ \\ R_4 - R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & 0 & 20 \\ 0 & -5 & 0 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \frac{1}{10}R_1 \\ -\frac{1}{5}R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system is consistent, it has only solution is $a = 2$, $b = -1$, $c = -1$

$2\vec{v}_1 - 1\vec{v}_2 - 1\vec{v}_3 = (2,\ 3,\ -7,\ 3)$

Therefore, (2, 3, –7, 3) is in span $\{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\}$

b) The vector (0, 0 , 0, 0) is obviously in span $\{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\}$

Since $0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = (0,\ 0,\ 0,\ 0)$

c) For the vector $b = (1, 1, 1, 1)$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 5 & 2 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \begin{array}{l} \\ 2R_2 - R_1 \\ \\ 2R_4 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 0 & -5 & 1 & 1 \\ 0 & 5 & 2 & 1 \\ 0 & -5 & 5 & -1 \end{array}\right] \begin{array}{l} 5R_1 + 3R_2 \\ \\ R_3 + R_2 \\ R_4 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & -2 & 8 \\ 0 & -5 & 1 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 4 & -2 \end{array}\right] \begin{array}{l} 3R_1 + 2R_3 \\ 3R_2 - R_3 \\ \\ 3R_4 - 4R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & 0 & 28 \\ 0 & -5 & 0 & 1 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & -14 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{3}R_3 \\ -\frac{1}{14}R_4 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & 0 & 28 \\ 0 & -5 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - 28R_4 \\ R_2 - R_4 \\ R_3 + R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \frac{1}{10}R_1 \\ -\frac{1}{5}R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

This system is inconsistent, therefore (1, 1, 1, 1) is *not* in span $\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\}$

d) For the vector $b = (-4, 6, -13, 4)$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & -4 \\ 1 & -1 & 0 & 6 \\ 0 & 5 & 2 & -13 \\ 3 & 2 & 1 & 4 \end{array}\right] \quad \begin{array}{l} \\ 2R_2 - R_1 \\ \\ 2R_4 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & -4 \\ 0 & -5 & 1 & 16 \\ 0 & 5 & 2 & -13 \\ 0 & -5 & 5 & 20 \end{array}\right] \quad \begin{array}{l} 5R_1 + 3R_2 \\ \\ R_3 + R_2 \\ R_4 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & -2 & 28 \\ 0 & -5 & 1 & 16 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 4 & 4 \end{array}\right] \quad \begin{array}{l} \\ \\ \frac{1}{3}R_3 \\ \frac{1}{4}R_4 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & -2 & 28 \\ 0 & -5 & 1 & 16 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 + 2R_3 \\ R_2 - R_3 \\ \\ R_4 - R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 10 & 0 & 0 & 30 \\ 0 & -5 & 0 & 15 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \frac{1}{10}R_1 \\ -\frac{1}{5}R_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system is consistent, it has only solution is $a = 3$, $b = -3$, $c = 1$

$$3\vec{v}_1 - 3\vec{v}_2 + 1\vec{v}_3 = (-4,\ 6,\ -13,\ 4)$$

Therefore, $(-4, 6, -13, 4)$ is in span $\{\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\}$

Exercise

Let $\boldsymbol{f} = \cos^2 x$ and $\boldsymbol{g} = \sin^2 x$. Which of the following lie in the space spanned by $\boldsymbol{f}$ and $\boldsymbol{g}$

a) $\cos 2x$ *b)* $3+x^2$ *c)* $\sin x$ *d)* 0

Solution

a) $\cos 2x = \cos^2 x - \sin^2 x$, therefore $\cos 2x$ is in span $\{\boldsymbol{f}, \boldsymbol{g}\}$

b) In order for $3+x^2$ to be in span $\{\boldsymbol{f}, \boldsymbol{g}\}$, there must exist scalars *a* and *b* such that

$$a\cos^2 x + b\sin^2 x = 3 + x^2$$

When $\left.\begin{array}{l} x=0 \quad \Rightarrow \quad a=3 \\ x=\pi \quad \Rightarrow \quad a=3+\pi^2 \end{array}\right\} \rightarrow \textit{contradiction}$

Therefore $3+x^2$ is *not* in span $\{\boldsymbol{f}, \boldsymbol{g}\}$

c) In order for $\sin x$ to be in span $\{\boldsymbol{f}, \boldsymbol{g}\}$, there must exist scalars *a* and *b* such that

$$a\cos^2 x + b\sin^2 x = \sin x$$

When $\left.\begin{array}{l} x=\frac{\pi}{2} \quad \Rightarrow \quad b=1 \\ x=-\frac{\pi}{2} \quad \Rightarrow \quad b=-1 \end{array}\right\} \rightarrow \textit{contradiction}$

Therefore $\sin x$ is *not* in span $\{\boldsymbol{f}, \boldsymbol{g}\}$

d) In order for 0 to be in span $\{\boldsymbol{f}, \boldsymbol{g}\}$, there must exist scalars *a* and *b* such that

$$0\cos^2 x + 0\sin^2 x = 0$$

Therefore $\mathbf{0}$ is in span $\{\boldsymbol{f}, \boldsymbol{g}\}$

Exercise

Let $S = \left\{(x, y) \mid x^2 + y^2 = 0;\ x, y \in \mathbb{R}\right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^2$?

Solution

$$x^2 + y^2 = 0 \quad \rightarrow \quad x = y = 0 \quad (x, y \in \mathbb{R})$$

a) Let $\vec{u} = (x_1, y_1) \ni x_1^2 + y_1^2 = 0$ & $x_1 = y_1 = 0$, and

$$\vec{v} = (x_2, y_2) \ \ni \ x_2^2 + y_2^2 = 0 \ \ \& \ \ x_2 = y_2 = 0$$

$$\vec{u} + \vec{v} = (x_1 + x_2,\ y_1 + y_2)$$

$$\begin{aligned}(x_1 + x_2)^2 + (y_1 + y_2)^2 &= x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2 \\ &= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2(x_1x_2 + y_1y_2) \\ &= (0) + (0) + 2(0 + 0) \qquad x_i = y_i = 0 \\ &= 0\end{aligned}$$

S is closed under addition

b) $k\vec{u} = k(x_1, y_1)$

$$= (kx_1,\ ky_1)$$

$$\begin{aligned}(kx_1)^2 + (ky_1)^2 &= k^2x_1^2 + k^2y_1^2 \\ &= k^2(x_1^2 + y_1^2) \\ &= k^2(0) \\ &= 0\end{aligned}$$

S is closed under scalar multiplication

c) Since S is closed under addition and scalar multiplication, then S is a subspace of $\mathbb{R}^2$.

Exercise

Let $S = \left\{ (x, y) \middle| \ x^2 + y^2 = 0; \ \ x, y \in \mathbb{C} \right\}$, Determine:

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of $\mathbb{C}^2$?

Solution

$$x^2 + y^2 = 0 \quad \rightarrow \quad x = \pm iy \quad (x,\ y \in \mathbb{C})$$

a) Let $\vec{u} = (x_1, y_1) \ \ni \ x_1^2 + y_1^2 = 0 \ \rightarrow \ x_1 = i\,y_1$, and

$$\vec{v} = (x_2, y_2) \ \ni \ x_2^2 + y_2^2 = 0 \ \rightarrow \ x_2 = -i\,y_2$$

$$\vec{u} + \vec{v} = (x_1 + x_2,\ y_1 + y_2)$$

$$\begin{aligned}\left(x_1+x_2\right)^2+\left(y_1+y_2\right)^2 &= x_1^2+x_2^2+2x_1x_2+y_1^2+y_2^2+2y_1y_2\\ &=\left(x_1^2+y_1^2\right)+\left(x_2^2+y_2^2\right)+2\left(x_1x_2+y_1y_2\right)\\ &=\left(x_1^2+y_1^2\right)+\left(x_2^2+y_2^2\right)+2\left(iy_1\left(-iy_2\right)+y_1y_2\right)\\ &=0+0+2\left(-i^2y_1y_2+y_1y_2\right)\\ &=2\left(y_1y_2+y_1y_2\right)\\ &=4y_1y_2\\ &\neq 0\end{aligned}$$

S is not closed under addition

b) $k\vec{u}=k\left(x_1,\ y_1\right)$

$=\left(kx_1,\ ky_1\right)$

$$\begin{aligned}\left(kx_1\right)^2+\left(ky_1\right)^2 &= k^2x_1^2+k^2y_1^2\\ &=k^2\left(x_1^2+y_1^2\right)\\ &=k^2(0)\\ &=0\end{aligned}$$

S is closed under scalar multiplication

c) Since S is *not* closed under addition, then S is ***not*** a subspace of $\mathbb{C}^2$.

Exercise

Let $S=\left\{(x,\ y)\,\middle|\ x^2-y^2=0;\ \ x,\ y\in\mathbb{R}\right\}$, Determine:

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of $\mathbb{R}^2$?

Solution

$x^2-y^2=0 \quad\to\quad x=\pm y \quad (x,\ y\in\mathbb{R})$

a) Let $\vec{u}=\left(x_1,\ y_1\right)\ \ni\ x_1^2-y_1^2=0 \ \to\ x_1=y_1$, and

$\vec{v}=\left(x_2,\ y_2\right)\ \ni\ x_2^2-y_2^2=0\ \to\ x_2=-y_2$

$\vec{u}+\vec{v}=\left(x_1+x_2,\ y_1+y_2\right)$

$$\begin{aligned}\left(x_1+x_2\right)^2-\left(y_1+y_2\right)^2 &= x_1^2+x_2^2+2x_1x_2-y_1^2-y_2^2-2y_1y_2\\ &=\left(x_1^2-y_1^2\right)+\left(x_2^2-y_2^2\right)+2\left(x_1x_2-y_1y_2\right)\\ &=(0)+(0)+2\left(y_1\left(-y_2\right)-y_1y_2\right)\\ &=2\left(-y_1y_2-y_1y_2\right)\\ &=-4y_1y_2\\ &\underline{\neq 0}\end{aligned}$$

S is not closed under addition

b) $k\vec{u}=k\left(x_1,y_1\right)$

$=\left(kx_1,ky_1\right)$

$$\begin{aligned}\left(kx_1\right)^2-\left(ky_1\right)^2 &= k^2x_1^2-k^2y_1^2\\ &=k^2\left(x_1^2-y_1^2\right)\\ &=k^2(0)\\ &\underline{=0}\end{aligned}$$

S is closed under scalar multiplication

c) Since *S* is *not* closed under addition, then *S* is ***not*** a subspace of $\mathbb{R}^2$.

Exercise

Let $S=\left\{(x,\ y)\,\middle|\ x-y=0;\ \ x,\ y\in\mathbb{R}\right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^2$?

Solution

$x-y=0 \quad\rightarrow\quad x=y \quad (x,\ y\in\mathbb{R})$

a) Let $\vec{u}=\left(x_1,y_1\right)\ \ni\ x_1-y_1=0$, and

$\vec{v}=\left(x_2,y_2\right)\ \ni\ x_2-y_2=0$

$$\vec{u}+\vec{v}=\left(x_1+x_2,\ y_1+y_2\right)$$

$$\begin{aligned}\left(x_1+x_2\right)-\left(y_1+y_2\right)&=x_1+x_2-y_1-y_2\\&=\left(x_1-y_1\right)+\left(x_2-y_2\right)\\&=0\end{aligned}$$

S is closed under addition

b) $k\vec{u}=k\left(x_1,y_1\right)$

$$=\left(kx_1,\ ky_1\right)$$

$$\begin{aligned}kx_1-ky_1&=k\left(x_1-y_1\right)\\&=k(0)\\&=0\end{aligned}$$

S is closed under scalar multiplication

c) Since S is closed under addition and scalar multiplication, then S is a subspace of $\mathbb{R}^2$.

Exercise

Let $S=\{(x,\ y)|\ x-y=1;\ \ x,\ y\in\mathbb{R}\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^2$?

Solution

$x-y=0 \quad\rightarrow\quad x=y \quad (x,\ y\in\mathbb{R})$

a) Let $\vec{u}=\left(x_1,y_1\right)\ \ni\ x_1-y_1=1$, and

$\vec{v}=\left(x_2,y_2\right)\ \ni\ x_2-y_2=1$

$$\vec{u}+\vec{v}=\left(x_1+x_2,\ y_1+y_2\right)$$

$$\begin{aligned}\left(x_1+x_2\right)-\left(y_1+y_2\right)&=x_1+x_2-y_1-y_2\\&=\left(x_1-y_1\right)+\left(x_2-y_2\right)\\&=1+1\\&=2\ \neq 1\end{aligned}$$

S is not closed under addition

b) $k\vec{u} = k\left(x_1, y_1\right)$

$$= \left(kx_1, ky_1\right)$$

$$kx_1 - ky_1 = k\left(x_1 - y_1\right)$$

$$= k(1)$$

$$\underline{= k \ \neq 1}$$

S is not closed under scalar multiplication

c) Since S is not closed under addition and not closed scalar multiplication, then S is ***not*** a subspace of $\mathbb{R}^2$.

Exercise

$V = \mathbb{R}^3, \quad S = \left\{(0, s, t) \mid s, t \ \text{are real numbers}\right\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of V?

Solution

d) Let $\vec{u} = \left(0, s_1, t_1\right)$ *and* $\vec{v} = \left(0, s_2, t_2\right)$

$$\vec{u} + \vec{v} = \left(0, s_1 + s_2, t_1 + t_2\right)$$

$$= (0, s, t)$$

Yes, S is closed under addition

e) $k\vec{u} = \left(0, ks_1, kt_1\right)$

$$= (0, s, t)$$

Yes, S is closed under scalar multiplication

f) Since S is closed under addition and scalar multiplication, then S is a subspace of V.

Exercise

$V = \mathbb{R}^3, \quad S = \left\{(x, y, z) \mid x, y, z \geq 0\right\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of V?

Solution

a) Let $\vec{u} = (x_1, y_1, z_1)$ *and* $\vec{v} = (x_2, y_2, z_2)$

$\vec{u} + \vec{v} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$ *where* $x = x_1 + x_2, y = y_1 + y_2, z = z_1 + z_2$

$= (x, y, z)$

Yes, S is closed under addition

b) $(-1)\vec{u} = (-x_1, -y_1, -z_1)$

S is ***not*** closed under scalar multiplication since $x_1 \ge 0 \Rightarrow -x_1 \le 0$

c) Since S is closed under addition but it is not closed scalar multiplication, then S is ***not*** a subspace of V.

Exercise

$V = \mathbb{R}^3, \quad S = \{(x, y, z) \mid z = x + y + 1\}$ where V is a vector space and S is subset of V

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of V?

Solution

a) Let $\vec{u} = (0, 1, 2)$ *and* $\vec{v} = (1, 2, 4)$

$\vec{u} + \vec{v} = (1, 3, 6)$

$\neq (1, 3, 1 + 3 + 1)$

S is *not* closed under addition

b) $k\vec{u} = (kx_1, ky_1, kz_1)$

$= (kx_1, ky_1, k(x_1 + y_1 + 1))$

$= (kx_1, ky_1, kx_1 + ky_1 + k)$ *Where* $x = kx_1, \quad y = ky_1, \quad z = k(x_1 + y_1 + 1)$

$= (x, y, z)$

S is closed under scalar multiplication

c) Since S is not closed under addition and closed scalar multiplication, then S is ***not*** a subspace of V.

Exercise

Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ a_1=3a_2\ \ and\ \ a_3=-a_2\right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

Solution

a) Let $\vec{u}=\left(a_1,a_2,a_3\right)\quad and\quad \vec{v}=\left(b_1,b_2,b_3\right)$

$$\begin{aligned}\vec{u}+\vec{v}&=\left(a_1,a_2,a_3\right)+\left(b_1,b_2,b_3\right)\\&=\left(3a_2,a_2,\ -a_2\right)+\left(3b_2,b_2,\ -b_2\right)\\&=\left(3a_2+3b_2,a_2+b_2,\ -a_2-b_2\right)\\&=\left(3\left(a_2+b_2\right),a_2+b_2,\ -\left(a_2+b_2\right)\right)\qquad\qquad c_2=a_2+b_2\\&=\left(3c_2,c_2,\ -c_2\right)\\&=\left(c_1,c_2,c_3\right):\quad c_1=3c_2\quad c_3=-c_2\end{aligned}$$

S is closed under addition

b) $$\begin{aligned}k\vec{u}&=k\left(a_1,a_2,a_3\right)\\&=k\left(3a_2,a_2,\ -a_2\right)\\&=\left(3ka_2,ka_2,\ -ka_2\right)\qquad\qquad c_2=ka_2\\&=\left(3c_2,c_2,\ -c_2\right)\\&=\left(c_1,c_2,c_3\right):\quad c_1=3c_2\quad c_3=-c_2\end{aligned}$$

S is closed under scalar multiplication.

c) Since *S* is closed under addition and scalar multiplication, then *S* is a subspace of *V*.

Exercise

Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ a_1=a_3+2\right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

Solution

a) Let $\vec{u}=(2,\ 1,\ 0)$ *and* $\vec{v}=(3,\ 0,\ 1)$ $\quad a_1=a_3+2$

$\vec{u}+\vec{v}=(2,\ 1,\ 0)+(3,\ 0,\ 1)$

$=(5,\ 1,\ 1) \qquad 5\overset{?}{=}1+2$

$\neq(3,\ 1,\ 1)$

S is *not* closed under addition

b) $k\vec{u}=k\left(a_1,\ a_2,\ a_3\right)$

$=k\left(a_3+2,\ a_2,\ a_3\right)$

$=\left(ka_3+2k,\ ka_2,\ ka_3\right)$

$a_1=a_3+2 \ \rightarrow\ ka_3+2k=a_3+2$

$2k\neq2 \quad (\forall k)$

S is *not* closed under scalar multiplication.

c) Since *S* is not closed under addition and not closed scalar multiplication, then *S* is ***not*** a subspace of *V*.

Exercise

Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ 2a_1-7a_2+a_3=0\right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

Solution

a) Let $\vec{u}=\left(a_1,\ a_2,\ a_3\right)$ *and* $\vec{v}=\left(b_1,\ b_2,\ b_3\right)$

$2a_1-7a_2+a_3=0\ \rightarrow\ a_3=7a_2-2a_1$

$$\vec{u}+\vec{v}=\left(a_1, a_2, a_3\right)+\left(b_1, b_2, b_3\right)$$

$$=\left(a_1, a_2, 7a_2-2a_1\right)+\left(b_1, b_2, 7b_2-2b_1\right)$$

$$=\left(a_1+b_1, a_2+b_2, 7a_2-2a_1+7b_2-2b_1\right)$$

$$=\left(a_1+b_1, a_2+b_2, 7\left(a_2+b_2\right)-2\left(a_1+b_1\right)\right) \qquad \text{Let } c_1=a_1+b_1 \quad c_2=a_2+b_2$$

$$=\left(c_1, c_2, 7c_2-2c_1\right) \qquad c_3=7c_2-2c_1 \;\rightarrow\; 2c_1-7c_2+c_3=0$$

$$=\left(c_1, c_2, c_3\right)$$

S is closed under addition

b) $k\vec{u}=k\left(a_1, a_2, a_3\right)$

$$=k\left(a_1, a_2, 7a_2-2a_1\right)$$

$$=\left(ka_1, ka_2, 7ka_2-2ka_1\right) \qquad \text{Let } c_1=ka_1 \quad c_2=ka_2$$

$$=\left(c_1, c_2, 7c_2-2c_1\right) \qquad c_3=7c_2-2c_1 \;\rightarrow\; 2c_1-7c_2+c_3=0$$

$$=\left(c_1, c_2, c_3\right)$$

S is closed under scalar multiplication.

c) Since *S* is closed under addition and scalar multiplication, then *S* is a subspace of *V*.

Exercise

Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ a_1-4a_2-a_3=0\right\}$, Determine:

a) Is *S* closed under addition?

b) Is *S* closed under scalar multiplication?

c) Is *S* a subspace of $\mathbb{R}^3$?

Solution

$$a_1-4a_2-a_3=0 \;\rightarrow\; a_1=4a_2+a_3$$

a) Let $\vec{u}=\left(a_1, a_2, a_3\right)$ *and* $\vec{v}=\left(b_1, b_2, b_3\right)$

$$\vec{u}+\vec{v}=\left(a_1, a_2, a_3\right)+\left(b_1, b_2, b_3\right)$$

$$=\left(4a_2+a_3, a_2, a_3\right)+\left(4b_2+b_3, b_2, b_3\right)$$

$$=\left(4a_2+a_3+4b_2+b_3, a_2+b_2, a_3+b_3\right)$$

$$=\left(4\left(a_2+b_2\right)+\left(a_3+b_3\right),\ a_2+b_2,\ a_3+b_3\right) \qquad \text{Let } c_2=a_2+b_2 \quad c_3=a_3+b_3$$

$$=\left(4c_2+c_3,\ c_2,\ c_3\right) \qquad c_1-4c_2-c_3=0 \ \rightarrow\ c_1=4c_2+c_3$$

$$=\left(c_1,\ c_2,\ c_3\right)$$

S is closed under addition

b) $k\vec{u}=k\left(a_1,\ a_2,\ a_3\right)$

$$=k\left(4a_2+a_3,\ a_2,\ a_3\right)$$

$$=\left(4ka_2+ka_3,\ ka_2,\ ka_3\right) \qquad \text{Let } c_2=ka_2 \quad c_3=ka_3$$

$$=\left(4c_2+c_3,\ c_2,\ c_3\right) \qquad c_1=4c_2+c_3 \ \rightarrow\ c_1-4c_2-c_3=0$$

$$=\left(c_1,\ c_2,\ c_3\right)$$

S is closed under scalar multiplication.

c) Since S is closed under addition and scalar multiplication, then S is a subspace of V.

Exercise

Let $S=\left\{\left(a_1,\ a_2,\ a_3\right)\in\mathbb{R}^3:\ a_1+2a_2-3a_3=0\right\}$, Determine:

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of $\mathbb{R}^3$?

Solution

$$a_1+2a_2-3a_3=0 \quad\rightarrow\quad a_1=-2a_2+3a_3$$

a) Let $\vec{u}=\left(a_1,\ a_2,\ a_3\right)$ *and* $\vec{v}=\left(b_1,\ b_2,\ b_3\right)$

$$\vec{u}+\vec{v}=\left(a_1,\ a_2,\ a_3\right)+\left(b_1,\ b_2,\ b_3\right)$$

$$=\left(-2a_2+3a_3,\ a_2,\ a_3\right)+\left(-2b_2+3b_3,\ b_2,\ b_3\right)$$

$$=\left(-2a_2+3a_3-2b_2+3b_3,\ a_2+b_2,\ a_3+b_3\right)$$

$$=\left(-2\left(a_2+b_2\right)+3\left(a_3+b_3\right),\ a_2+b_2,\ a_3+b_3\right) \qquad \text{Let } c_2=a_2+b_2 \quad c_3=a_3+b_3$$

$$=\left(-2c_2+3c_3,\ c_2,\ c_3\right) \qquad c_1+2c_2-3c_3=0 \ \rightarrow\ c_1=-2c_2+3c_3$$

$$=\left(c_1,\ c_2,\ c_3\right)$$

S is closed under addition

b) $k\vec{u} = k\left(a_1, a_2, a_3\right)$

$= k\left(4a_2 + a_3, a_2, a_3\right)$

$= \left(-2ka_2 + 3ka_3, ka_2, ka_3\right)$ Let $c_2 = ka_2 \quad c_3 = ka_3$

$= \left(-2c_2 + 3c_3, c_2, c_3\right)$ $c_1 = -2c_2 + 3c_3 \rightarrow c_1 - 2c_2 + 3c_3 = 0$

$= \left(c_1, c_2, c_3\right)$

S is closed under scalar multiplication.

c) Since S is closed under addition and scalar multiplication, then S is a subspace of V.

Exercise

Let $S = \left\{\left(a_1, a_2, a_3\right) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1\right\}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of $\mathbb{R}^3$?

Solution

$a_1 + 2a_2 - 3a_3 = 1 \rightarrow a_1 = 1 - 2a_2 + 3a_3$

a) Let $\vec{u} = \left(a_1, a_2, a_3\right)$ *and* $\vec{v} = \left(b_1, b_2, b_3\right)$

$\vec{u} + \vec{v} = \left(a_1, a_2, a_3\right) + \left(b_1, b_2, b_3\right)$

$= \left(1 - 2a_2 + 3a_3, a_2, a_3\right) + \left(1 - 2b_2 + 3b_3, b_2, b_3\right)$

$= \left(1 - 2a_2 + 3a_3 + 1 - 2b_2 + 3b_3, a_2 + b_2, a_3 + b_3\right)$

$= \left(2 - 2\left(a_2 + b_2\right) + 3\left(a_3 + b_3\right), a_2 + b_2, a_3 + b_3\right)$ Let $c_2 = a_2 + b_2 \quad c_3 = a_3 + b_3$

$= \left(2 - 2c_2 + 3c_3, c_2, c_3\right)$ $c_1 + 2c_2 - 3c_3 = 1 \rightarrow c_1 = 1 - 2c_2 + 3c_3$

$\neq \left(1 - 2c_2 + 3c_3, c_2, c_3\right)$

S is *not* closed under addition

b) $\vec{u} = (2, 1, 1)$

$k\vec{u} = k(2, 1, 1)$

$$= (2k,\ k,\ k) \qquad\qquad a_1 + 2a_2 - 3a_3 = 1 \ \rightarrow\ 2k + 2k - 3k \stackrel{?}{=} 1$$

$$k \neq 1 \ \ (\forall k)$$

S is *not* closed under scalar multiplication.

c) Since S is not closed under addition and not closed scalar multiplication, then S is ***not*** a subspace of V.

Exercise

Let $S = \left\{ \left(a_1,\ a_2,\ a_3\right) \in \mathbb{R}^3 :\ 5a_1^2 - 3a_2^2 + 6a_3^2 = 0 \right\}$, Determine:

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of $\mathbb{R}^3$?

Solution

$$5a_1^2 - 3a_2^2 + 6a_3^2 = 0 \ \rightarrow\ a_2^2 = \frac{5}{3}a_1^2 + 2a_3^2$$

a) Let $\vec{u} = \left(0,\ \sqrt{2},\ 1\right)$ *and* $\vec{v} = \left(3,\ \sqrt{17},\ 1\right)$

$$\vec{u} + \vec{v} = \left(0,\ \sqrt{2},\ 1\right) + \left(3,\ \sqrt{17},\ 1\right)$$

$$= \left(3,\ \sqrt{2} + \sqrt{17},\ 2\right)$$

$$a_2^2 = \frac{5}{3}a_1^2 + 2a_3^2 \ \rightarrow\ \left(\sqrt{2} + \sqrt{17}\right)^2 \neq 15 + 8$$

S is *not* closed under addition

b) $k\vec{u} = k\left(a_1,\ a_2,\ a_3\right)$

$$= \left(ka_1,\ ka_2,\ ka_3\right)$$

$$5\left(ka_1\right)^2 - 3\left(ka_2\right)^2 + 6\left(ka_3\right)^2 = 0$$

$$5k^2a_1^2 - 3k^2a_2^2 + 6k^2a_3^2 = 0$$

$$5a_1^2 - 3a_2^2 + 6a_3^2 = 0$$

S is closed under scalar multiplication.

c) Since S is *not* closed under addition and is closed scalar multiplication, then S is ***not*** a subspace of V.

Exercise

Let $S = \left\{ \left(a_1,\ a_2,\ a_3 \right) \in \mathbb{R}^3 :\ a_3 = a_1 + a_2 \right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

Solution

a) Let $\vec{u} = \left(a_1, a_2, a_3\right)$ *and* $\vec{v} = \left(b_1, b_2, b_3\right)$

$$\begin{aligned}\vec{u} + \vec{v} &= \left(a_1, a_2, a_3\right) + \left(b_1, b_2, b_3\right)\\ &= \left(a_1, a_2, a_1 + a_2\right) + \left(b_1, b_2, b_1 + b_2\right)\\ &= \left(a_1 + b_1, a_2 + b_2, a_1 + a_2 + b_1 + b_2\right) && \text{Let } c_1 = a_1 + b_1 \quad c_2 = a_2 + b_2\\ &= \left(c_1, c_2, c_1 + c_2\right) && \text{Then, } c_3 = c_1 + c_2\\ &= \left(c_1, c_2, c_3\right)\end{aligned}$$

S is closed under addition

b)
$$\begin{aligned}k\vec{u} &= k\left(a_1, a_2, a_3\right)\\ &= k\left(a_1, a_2, a_1 + a_2\right)\\ &= \left(ka_1, ka_2, k\left(a_1 + a_2\right)\right)\\ &= \left(ka_1, ka_2, ka_1 + ka_2\right) && \textit{Where } c_1 = ka_1,\ c_2 = ka_2,\ c_3 = ka_1 + ka_2\\ &= \left(c_1, c_2, c_3\right) && c_3 = ka_1 + ka_2 = c_1 + c_2\end{aligned}$$

S is closed under scalar multiplication.

c) Since *S* is closed under addition and scalar multiplication, then *S* is a subspace of *V*.

Exercise

Let $S = \left\{ \left(a_1,\ a_2,\ a_3 \right) \in \mathbb{R}^3 :\ a_1 + a_2 + a_3 = 0 \right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

Solution

a) Let $\vec{u}=(a_1, a_2, a_3) \ni a_1+a_2+a_3=0$

$\vec{v}=(b_1, b_2, b_3) \ni b_1+b_2+b_3=0$

$\vec{u}+\vec{v}=(a_1, a_2, a_3)+(b_1, b_2, b_3)$

$=(a_1+b_1, a_2+b_2, a_3+b_3)$

Since $a_1+a_2+a_3=0$ & $b_1+b_2+b_3=0$

Then, $\rightarrow (a_1+b_1)+(a_2+b_2)+(a_3+b_3)=0$

S is closed under addition

b) $k\vec{u}=k(a_1, a_2, a_3)$

$=(ka_1, ka_2, ka_3)$

$ka_1+ka_2+ka_3=k(a_1+a_2+a_3)=k(0)=0$

S is closed under scalar multiplication.

c) Since *S* is closed under addition and scalar multiplication, then *S* is a subspace of *V*.

Exercise

Let $S=\{(x_1, x_2, 1): x_1 \text{ and } x_2 \text{ are real numbers}\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\mathbb{R}^3$?

Solution

a) Let $\vec{u}=(x_1, x_2, 1)$ & $\vec{v}=(y_1, y_2, 1)$

$\vec{u}+\vec{v}=(x_1, x_2, 1)+(y_1, y_2, 1)$

$=(x_1+y_1, x_2+y_2, 2)$ If we let $z_1=x_1+y_1 \quad z_2=x_2+y_2$

$=(z_1, z_2, 2)$

$\neq(z_1, z_2, 1)$

S is ***not*** closed under addition

b) $k\vec{u}=k(x_1, x_2, 1)$

$$= \left(kx_1, kx_2, k\right) \qquad \text{If we let } z_1 = kx_1 \quad z_2 = kx_2$$

$$\neq \left(z_1, z_2, 1\right) \qquad k \neq 1 \quad (\forall k)$$

S is ***not*** closed under scalar multiplication.

c) Since S is *not* closed under addition and is *not* closed scalar multiplication, then S is ***not*** a subspace of V.

Exercise

Let $S = \left\{\left(x_1, x_2, x_3\right) \in \mathbb{R}^3 : \ x_2 = x_1 + 2x_3\right\}$, Determine:

a) Is S closed under addition?

b) Is S closed under scalar multiplication?

c) Is S a subspace of $\mathbb{R}^3$?

Solution

a) Let $\vec{u} = \left(x_1, x_2, x_3\right) \quad \& \quad \vec{v} = \left(y_1, y_2, y_3\right)$

$$\vec{u} + \vec{v} = \left(x_1, x_2, x_3\right) + \left(y_1, y_2, y_3\right)$$

$$= \left(x_1 + y_1, x_2 + y_2, x_3 + y_3\right) \qquad S = \left\{\left(x_1, x_2, x_3\right) \in \mathbb{R}^3 : \ x_2 = x_1 + 2x_3\right\}$$

$$x_2 + y_2 = x_1 + 2x_3 + y_1 + 2y_3$$

$$= x_1 + y_1 + 2\left(x_3 + y_3\right)$$

S is closed under addition

b) $k\vec{u} = k\left(x_1, x_2, x_3\right)$

$$= \left(kx_1, kx_2, kx_3\right) \qquad \text{If we let } z_1 = kx_1 \quad z_2 = kx_2$$

$$kx_2 = kx_1 + 2kx_3$$

$$kx_2 = k\left(x_1 + 2x_3\right)$$

$$x_2 = x_1 + 2x_3$$

S is closed under scalar multiplication.

c) Since S is closed under addition and scalar multiplication, then S is a subspace of V.

Exercise

Let $S = \left\{ \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \in M_{2\times 2} \mid a,\ c,\ d \in \mathbb{R} \right\}$ and $V = M_{2,2}$, Determine:

a) Is *S* closed under addition?

b) Is *S* closed under scalar multiplication?

c) Is *S* a subspace of V?

Solution

a) Let $A = \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$ & $B = \begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix}$

$$A + B = \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + a_2 & 2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$ If we let $a = a_1 + a_2 \quad c = c_1 + c_2 \quad d = d_1 + d_2$

$$= \begin{pmatrix} a & 2 \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$$

S is ***not*** closed under addition

b) $kA = k\begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$

$$= \begin{pmatrix} ka_1 & k \\ kc_1 & kd_1 \end{pmatrix}$$ If we let $a = ka_1 \quad c = kc_1 \quad d = kd_1$

$$= \begin{pmatrix} a & k \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$$ $k \neq 1 \quad (\forall k)$

S is ***not*** closed under scalar multiplication.

c) Since *S* is *not* closed under addition and is *not* closed scalar multiplication, then *S* is ***not*** a subspace of *V*.

Exercise

Let $S = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in M_{2\times 2} \mid a,\ c,\ d \in \mathbb{R} \right\}$ and $V = M_{2,2}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V ?

Solution

a) Let $A = \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$ & $B = \begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix}$

$$A + B = \begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & 1 \\ c_2 & d_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + a_2 & 2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$ If we let $a = a_1 + a_2 \quad c = c_1 + c_2 \quad d = d_1 + d_2$

$$= \begin{pmatrix} a & 2 \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$$

S is ***not*** closed under addition

b) $kA = k\begin{pmatrix} a_1 & 1 \\ c_1 & d_1 \end{pmatrix}$

$$= \begin{pmatrix} ka_1 & k \\ kc_1 & kd_1 \end{pmatrix}$$ If we let $a = ka_1 \quad c = kc_1 \quad d = kd_1$

$$= \begin{pmatrix} a & k \\ c & d \end{pmatrix} \neq \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}$$ $k \neq 1 \quad (\forall k)$

S is ***not*** closed under scalar multiplication.

c) Since S is *not* closed under addition and is *not* closed scalar multiplication, then S is ***not*** a subspace of V.

Exercise

Let $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in M_{2\times 2} \mid a,\ d \in \mathbb{R}\ \ \&\ \ ad \geq 0 \right\}$ and $V = M_{2,2}$, Determine:

a) Is S closed under addition?
b) Is S closed under scalar multiplication?
c) Is S a subspace of V ?

Solution

a) Let $A=\begin{pmatrix}1&0\\0&2\end{pmatrix} \to 1(2)>0$ & $B=\begin{pmatrix}-2&0\\0&-1\end{pmatrix}\to(-2)(-1)>0$

$$A+B=\begin{pmatrix}1&0\\0&2\end{pmatrix}+\begin{pmatrix}-2&0\\0&-1\end{pmatrix}$$

$$=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$$

$ad\ge 0 \quad \to \quad (-1)(1)=-1<0$

S is ***not*** closed under addition

b) $kA=k\begin{pmatrix}a&0\\0&d\end{pmatrix}$

$$=\begin{pmatrix}ka&0\\0&kd\end{pmatrix}$$

$(ka)(kd)=k^2(ad)$

Since, $ad\ge 0 \quad \& \quad k^2\ge 0$

$k^2(ad)\ge 0$

S is closed under scalar multiplication.

c) Since *S* is *not* closed under addition and is closed scalar multiplication, then *S* is ***not*** a subspace of *V*.

Exercise

$V=M_{33}, \quad S=\{A|\ A \text{ is invertible}\}$ where *V* is a vector space and *S* is subset of *V*

a) Is *S* closed under addition?

b) Is *S* closed under scalar multiplication?

c) Is *S* a subspace of *V*?

Solution

a) Let assume: $A=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ *and* $B=\begin{pmatrix}1&1\\0&-1\end{pmatrix}$ are invertible

But $A+B=\begin{pmatrix}2&1\\0&0\end{pmatrix}$ is not invertible.

S is *not* closed under addition

b) *S* is *not* closed under scalar multiplication if $k=0$

c) Since *S* is *not* closed under addition and is *not* closed scalar multiplication, then *S* is ***not*** a subspace of *V*.

Exercise

Let $S = \left\{ p(t) = a + 2at + 3at^3 \ \mid a \in \mathbb{R} \ \ \& \ \ p(t) \in P_2 \right\}$ and $V = P_2$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of V?

Solution

a) Let $p_1(t) = a + 2at + 3at^3 \quad \& \quad p_2(t) = b + 2bt + 3bt^3$

$$p_1(t) + p_2(t) = a + 2at + 3at^3 + b + 2bt + 3bt^3$$

$$= (a+b) + 2(a+b)t + 3(a+b)t^3 \qquad \text{Let } c = a + b \ \in \mathbb{R}$$

$$= c + 2ct + 3ct^3$$

S is closed under addition

b) $kp_1(t) = k\left(a + 2at + 3at^3\right)$

$$= ka + 2kat + 3kat^3 \qquad \text{Let } c = ka \ \in \mathbb{R}$$

$$= c + 2ct + 3ct^3$$

S is closed under scalar multiplication.

c) Since *S* is closed under addition and is closed scalar multiplication, then *S* is a subspace of *V*.

Exercise

Let $S = \left\{ p(t) \ \mid \ p(t) \in \boldsymbol{P}[t] \ \textit{has degree } 3 \right\}$, Determine:

a) Is *S* closed under addition?
b) Is *S* closed under scalar multiplication?
c) Is *S* a subspace of $\boldsymbol{P}[t]$?

Solution

a) Let $p_1(t) = at^3 + b_1t^2 + c_1t + d_1 \quad \& \quad p_2(t) = -at^3 + b_2t^2 + c_2t + d_2$

$$p_1(t) + p_2(t) = at^3 + b_1t^2 + c_1t + d_1 - at^3 + b_2t^2 + c_2t + d_2$$

$$= \left(b_1 + b_2\right)t^2 + \left(c_1 + c_2\right)t + \left(d_1 + d_2\right)$$

$$= bt^2 + ct + d$$

Has no 3rd degree polynomial.

S is *not* closed under addition

b) $kp_1(t) = k\left(at^3 + b_1t^2 + c_1t + d_1\right)$

$$= kat^3 + kb_1t^2 + kc_1t + kd_1$$

$$= k_1t^3 + k_2t^2 + k_3t + k_4$$

It is 3rd degree polynomial.

S is closed under scalar multiplication.

c) Since S is *not* closed under addition and is closed scalar multiplication, then S is ***not*** a subspace of V.

Exercise

Let $S = \{p(t) \mid p(0) = 0,\ p(t) \in \boldsymbol{P}[t]\}$, Determine:

d) Is S closed under addition?
e) Is S closed under scalar multiplication?
f) Is S a subspace of $\boldsymbol{P}[t]$?

Solution

a) Let $p_1(t) = a_nt^n + a_{n-1}t^{n-1} + \ldots + a_1t \quad \Rightarrow \quad p_1(0) = 0$

$$p_2(t) = b_nt^n + b_{n-1}t^{n-1} + \ldots + b_1t \quad \Rightarrow \quad p_2(0) = 0$$

$$p_1(t) + p_2(t) = a_nt^n + a_{n-1}t^{n-1} + \ldots + a_1t + b_nt^n + b_{n-1}t^{n-1} + \ldots + b_1t$$

$$= \left(a_n + b_n\right)t^n + \left(a_{n-1} + b_{n-1}\right)t^{n-1} + \ldots + \left(a_1 + b_1\right)t$$

$$p_1(0) + p_2(0) = 0$$

S is closed under addition

b) $kp_1(t) = k\left(a_nt^n + a_{n-1}t^{n-1} + \ldots + a_1t\right)$

$$= ka_nt^n + ka_{n-1}t^{n-1} + \ldots + ka_1t$$

$$= k_1t^3 + k_2t^2 + k_3t + k_4$$

$$kp_1(0) = 0$$

S is closed under scalar multiplication.

c) Since S is closed under addition and is closed scalar multiplication, then S is a subspace of V.

Exercise

Given: $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix}$

a) Find $NS(A)$

b) For which *n* is $NS(A)$ a subspace of $\mathbb{R}^n$

c) Sketch $NS(A)$ in $\mathbb{R}^2$ or $\mathbb{R}^3$

Solution

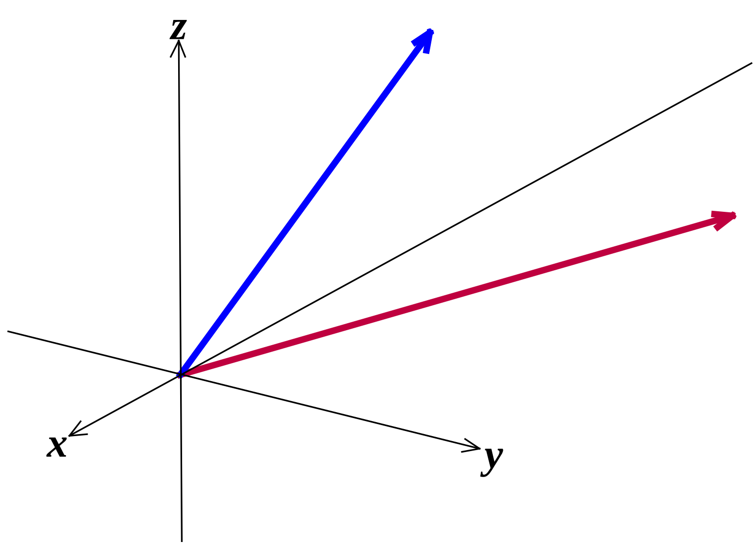
a) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix} \quad R_2 - 2R_1$

$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad x = -3y - 2z$

$$\left\{ y\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \middle| y, z \in \mathbb{R} \right\}$$

b) $n = 3$

c)



Exercise

Determine which of the following are subspaces of M_{22}

a) All 2×2 matrices with integer entries

b) All matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a+b+c+d=0$

Solution

a) Let $A=\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B=\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are integers.

$$A+B=\begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix}$$

where $a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4$ are integers too.

Then, it is closed under addition.

$$\frac{1}{2}A=\frac{1}{2}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix}$$

It is not closed under multiplication if the scalar is a real number.

Therefore; it is ***not*** a subspace of M_{22}

b) Let $A=\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ $a_1+a_2+a_3+a_4=0$

and $B=\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ $b_1+b_2+b_3+b_4=0$

$$A+B=\begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix}$$

$$a_1+a_2+a_3+a_4+b_1+b_2+b_3+b_4=0$$

$$\left(a_1+b_1\right)+\left(a_2+b_2\right)+\left(a_3+b_3\right)+\left(a_4+b_4\right)=0$$

Then, it is closed under addition.

$$kA = \begin{bmatrix} ka_1 & ka_2 \\ ka_3 & ka_4 \end{bmatrix}$$

$$ka_1 + ka_2 + ka_3 + ka_4 = k\left(a_1 + a_2 + a_3 + a_4\right) = k(0) = 0$$

It is closed under multiplication

Therefore; it is a subspace of M_{22}

Exercise

Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \ ad - bc = 1 \right\}$. Is V a vector space?

Solution

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

$$\begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k^2ad - k^2bc$$

$$= k^2(ad - bc)$$

$$= k^2 \neq k$$

$\therefore V$ is *not* a vector space

Exercise

Let $V = \left\{ (x, 0, y) : \ x \,\&\, y \text{ are arbitrary } \mathbb{R} \right\}$. Define addition and scalar multiplication as follows:

$$\begin{cases} \left(x_1, \ 0, \ y_1\right) + \left(x_2, \ 0, \ y_2\right) = \left(x_1 + x_2, \ y_1 + y_2\right) \\ \quad c(x, \ 0, \ y) = (cx, \ cy) \end{cases}$$

Is V a vector space?

Solution

Let $\vec{V}_1\left(x_1, \ 0, \ y_1\right)$ & $\vec{V}_2\left(x_2, \ 0, \ y_2\right)$

$$\vec{V}_1 + \vec{V}_2 = \left(x_1, \ 0, \ y_1\right) + \left(x_2, \ 0, \ y_2\right)$$

$$= \left(x_1 + x_2, \ y_1 + y_2\right)$$

$$\neq \left(x_1 + x_2, \ 0, \ y_1 + y_2\right)$$

$$=\vec{V}_1+\vec{V}_2$$

$\therefore$ *V* is *not* a vector space

Exercise

Construct a matrix whose column space contains $(1,\ 1,\ 0)$, $(0,\ 1,\ 1)$, and whose nullspace contains $(1,\ 0,\ 1)$ and $(0,\ 0,\ 1)$

Solution

It is *not* possible.
Since a matrix (*A*) must be 3×3.
Since the nullspace contains 2 independent vectors, then *A* can have at most $3-2=1$ pivot.
But the column space contains 2 independent vectors, *A* must have at least 2 pivots.

These 2 conditions can't both be met.

Exercise

How is the nullspace $N(C)$ related to the spaces $N(A)$ and $N(B)$, is $C=\begin{bmatrix} A \\ B \end{bmatrix}$?

Solution

$$N(C)=N(A)\cap N(B)$$

$$Cx=\begin{bmatrix} Ax \\ Bx \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Iff $Ax=0 \quad \& \quad Bx=0$

Exercise

True or False (check addition or give a counterexample)

a) If *V* is a vector space and *W* is a subset of *V* that is a vector space, then *W* is a subspace of *V*.

b) The empty set is a subspace of every vector space.

c) If *V* is a vector space other than the zero vector space, then *V* contains a subspace *W* such that $W\neq V$.

d) The intersection of any two subsets of *V* is a subspace of *V*.

e) Let *W* be the *xy*-plane in $\mathbb{R}^3$; that is, $W=\left\{\left(a_1,\ a_2,\ 0\right):\ a_1,\ a_2\in\mathbb{R}\right\}$. Then $W=\mathbb{R}^2$

Solution

a) False
W is a subset of *V*, but not necessary that the scalar of a vector in *W* is in *V*.

Therefore, W is *not* a subspace of V

b) False

Since not every subspace has an empty space, example $\mathbb{R}$

c) True

If V is a vector space in $\mathbb{R}^n$ and W is a vector space in $\mathbb{Z}^n$. Then V contains a subspace W and $W \neq V$

d) False

e) False

Exercise

Let $A\vec{x} = \vec{0}$ be a homogeneous system of n linear equations in n unknowns that has only the trivial solution. Show that of k is any positive integer, then the system $A^k\vec{x} = \vec{0}$ also has only trivial solution.

Solution

Since A is a square matrix, thus A has only the trivial solution that implies to A is invertible.

But A^k is also invertible so $A^k\vec{x} = \vec{0}$ has only trivial solution.

Exercise

Let $A\vec{x} = \vec{0}$ be a homogeneous system of n linear equations in n unknowns and let Q be an invertible $n \times n$ matrix. Show that of $A\vec{x} = \vec{0}$ has just trivial solution if and only if $(QA)\vec{x} = \vec{0}$ has just trivial solution.

Solution

Since A is a square matrix $n \times n$. If $A\vec{x} = \vec{0}$ has just trivial solution, then A is invertible. Since Q is an invertible $n \times n$ matrix that implies QA is also invertible. Thus, $(QA)\vec{x} = \vec{0}$ has trivial solution.

On the other hand, if $(QA)\vec{x} = 0$ has trivial solution then QA is invertible.

Since Q is invertible that implies Q^{-1} is also invertible.

Thus, $A = Q^{-1}QA$ is invertible i.e. $A\vec{x} = \vec{0}$ has just trivial solution.

$A\vec{x} = \vec{0}$ has just trivial solution *iff* $(QA)\vec{x} = \vec{0}$ has just trivial solution.

Exercise

Let $A\vec{x}=\vec{b}$ be a consistent system of linear equations and let $\vec{x}_1$ be a fixed solution. Show that every solution to the system can be written in the form $\vec{x}=\vec{x}_1+\vec{x}_0$ where $\vec{x}_0$ is a solution to $A\vec{x}=\vec{0}$. Show also that every matrix of this form is a solution.

Solution

Since $\vec{x}_0$ is a solution to $A\vec{x}=\vec{0}$, we have $A\vec{x}_0=\vec{0}$.

The sum of $A\vec{x}_0=\vec{0}$ and $A\vec{x}=\vec{b}$

$$\begin{array}{rr} & A\vec{x}_0=\vec{0} \\ + & A\vec{x}=\vec{b} \\ \hline & A\vec{x}_0+A\vec{x}=\vec{0}+\vec{b} \end{array}$$

$$A\left(\vec{x}+\vec{x}_0\right)=\vec{b}$$

As adding an equation to the original equation does not affect the solution.

If we let $\vec{x}_1$ be a fixed solution, then every solution to $A\vec{x}=\vec{b}$ is $\vec{x}=\vec{x}_1+\vec{x}_0$

Besides that

$$\begin{aligned} A\left(\vec{x}_1+\vec{x}_0\right) &= A\vec{x}_1+A\vec{x}_0 \\ &= \vec{b}+\vec{0} \\ &= \vec{b} \end{aligned}$$

So, every matrix (vector) in the form $\vec{x}_1+\vec{x}_0$ is a solution to $A\vec{x}=\vec{b}$.