Solution   *Section* 4.3 – Legendre's Equation

Exercise

Establish the recursion formula using the following two steps

a) Differentiate both sides of equation

$$g(t,\ x)=\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^{\infty}P_n(x)t^n \quad \text{with respect to } t \text{ to show that}$$

$$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\left(1-2xt+t^2\right)\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

b) Equate the coefficients of t^n in this equation to show that

$$P_1(x)=xP_0(x) \qquad \textit{and} \qquad (n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x) \quad \textit{for}\ \ n\ge 1$$

Solution

a) Let: $\left(1-2xt+t^2\right)^{-1/2}=\sum_{n=0}^{\infty}P_n(x)t^n$

Differentiate both sides with respect to *t*: $\left(\left(1-2xt+t^2\right)^{-1/2}\right)'=\left(\sum_{n=0}^{\infty}P_n(x)t^n\right)'$

$$-\frac{1}{2}(-2x+2t)\left(1-2xt+t^2\right)^{-3/2}=\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

$$(x-t)\left(1-2xt+t^2\right)^{-1/2}=\left(1-2xt+t^2\right)\sum_{n=1}^{\infty}nP_n(x)t^{n-1} \qquad \textit{Multiply both sides by: } 1-2xt+t^2$$

$$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\left(1-2xt+t^2\right)\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

b) $$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\sum_{n=0}^{\infty}xP_n(x)t^n-\underbrace{\sum_{n=0}^{\infty}P_n(x)t^{n+1}}_{n=n+1}$$

$$=\sum_{n=0}^{\infty}xP_n(x)t^n-\sum_{n=1}^{\infty}P_{n-1}(x)t^n$$

$$\left(1-2xt+t^2\right)\sum_{n=1}^{\infty} nP_n(x)t^{n-1} = \sum_{n=1}^{\infty} nP_n(x)t^{n-1} - \sum_{n=1}^{\infty} 2xnP_n(x)t^n + \sum_{n=1}^{\infty} nP_n(x)t^{n+1}$$

$$= \sum_{n=0}^{\infty} (n+1)P_{n+1}(x)t^n - \sum_{n=1}^{\infty} 2nxP_n(x)t^n + \sum_{n=2}^{\infty} (n-1)P_{n-1}(x)t^n$$

Thus,

$$\sum_{n=0}^{\infty} xP_n(x)t^n - \sum_{n=1}^{\infty} P_{n-1}(x)t^n = \sum_{n=0}^{\infty} (n+1)P_{n+1}(x)t^n - \sum_{n=1}^{\infty} 2nxP_n(x)t^n + \sum_{n=2}^{\infty} (n-1)P_{n-1}(x)t^n$$

Therefore;

$$0 = \left[xP_0(x) - P_1(x)\right]t^0 + \left[xP_1(x) - P_0(x) - 2P_2(x) + 2xP_1(x)\right]t^1$$
$$+\sum_{n=0}^{\infty}\left[xP_n(x) - P_{n-1}(x) - (n+1)P_{n+1}(x) + 2nxP_n(x) - (n-1)P_{n-1}(x)\right]t^n$$

$$0 = \left[xP_0(x) - P_1(x)\right]t^0 + \left[3xP_1(x) - P_0(x) - 2P_2(x)\right]t^1$$
$$+\sum_{n=0}^{\infty}\left[(2n+1)xP_n(x) - nP_{n-1}(x) - (n+1)P_{n+1}(x)\right]t^n$$

That implies:

$$xP_0(x) - P_1(x) = 0 \quad \Rightarrow \quad P_1(x) = xP_0(x)$$

$$3xP_1(x) - P_0(x) - 2P_2(x) = 0 \quad \Rightarrow \quad 2P_2(x) = P_0(x) - 3xP_1(x)$$

$$(2n+1)xP_n(x) - nP_{n-1}(x) - (n+1)P_{n+1}(x) = 0$$
$$\Rightarrow \quad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

If n = 1 then: $2P_2(x) = 3xP_1(x) - P_0(x)$ $\surd$

Exercise

Show that $P_{2n+1}(0)=0$ *and* $P_{2n}(0)=(-1)^n\dfrac{2n!}{2^{2n}(n!)^2}$

Solution

Given the formula:

$$nP_n(x)=(2n-1)xP_{n-1}(x)-(n-1)P_{n-2}(x) \qquad \textit{for } n\ge 2$$

By letting $x=0$, then the formula can be:

$$nP_n(0)=-(n-1)P_{n-2}(0)$$

Replacing *n* with 2*n*, then

$$2nP_{2n}(0)=-(2n-1)P_{2n-2}(0)$$

$$P_{2n}(0)=\frac{1-2n}{2n}P_{2n-2}(0)$$

$$P_2(0)=\frac{1-2}{2}P_0(0)=-\frac{1}{2}P_0(0)$$

$$P_4(0)=\frac{1-4}{4}P_2(0)=\frac{1-2}{2}\cdot\frac{1-4}{4}P_0(0)=\frac{1\cdot 3}{2^2\cdot 1\cdot 2}P_0(0)$$

$$P_6(0)=\frac{1-6}{6}P_4(0)=\frac{1-2}{2}\cdot\frac{1-4}{4}\cdot\frac{1-6}{6}P_0(0)=-\frac{1\cdot 3\cdot 5}{2^3\cdot 1\cdot 2\cdot 3}P_0(0)$$

$$\vdots \qquad \vdots \quad \vdots$$

$$P_{2n}(0)=\frac{1-2}{2}\cdot\frac{1-4}{4}\cdot\frac{1-6}{6}\cdots\frac{1-2n}{2n}P_0(0)$$

$$=(-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\cdot 1\cdot 2\cdot 3\cdots n}P_0(0)$$

$$1\cdot 3\cdot 5\cdots(2n-1)=\frac{1\cdot 2\cdot 3\cdot 4\cdots(2n-1)(2n)}{2\cdot 4\cdot 6\cdots(2n)}$$

$$=\frac{(2n)!}{2^n n!}$$

$$=(-1)^n\frac{(2n)!}{2^n\cdot(n!)^2}P_0(0)$$

With $P_0(0)=1$

$$\boxed{P_{2n}(0)=(-1)^n\frac{(2n)!}{2^n\cdot(n!)^2}}$$

Exercise

Show that $P'_n(0)=(-1)^{n+1}P'_n(-1)=\frac{n(n+1)}{2}$

Hint: Use Legendre's equation $\left(1-x^2\right)y''-2xy'+n(n+1)y=0$

Solution

Because $P_n(x)$ is a solution of Legendre's equation, then

$$\left(1-x^2\right)P''_n(x)-2xP'_n(x)+n(n+1)P_n(x)=0$$

Let $x=1$, then

$$-2P'_n(1)+n(n+1)P_n(1)=0$$

$$P'_n(1)=\frac{n(n+1)}{2}P_n(1)$$

Let $x=-1$, then

$$2P'_n(-1)+n(n+1)P_n(-1)=0$$

$$P'_n(-1)=-\frac{n(n+1)}{2}P_n(-1)$$

However, $P_n(1)=P_n(-1)=1$

$$\boxed{(-1)^{n+1}P'_n(-1)=\frac{n(n+1)}{2}}$$

Exercise

The differential equation $y''+xy=0$ is called ***Airy's equation***, and its solutions are called ***Airy functions***. Find the series for the solutions y_1 and y_2 where $y_1(0)=1$ and $y'_1(0)=0$, while $y_2(0)=0$ and $y'_2(0)=1$. What is the radius of convergence for these two series?

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''+xy=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}a_{n-1} x^{n} = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}a_{n-1} x^{n} = 0$$

$$2a_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_{n-1}\right]x^{n} = 0$$

$$2a_2 = 0 \quad \text{or} \quad (n+2)(n+1)a_{n+2} + a_{n-1} = 0$$

$$a_2 = 0 \quad \text{or} \quad a_{n+2} = \frac{-a_{n-1}}{(n+2)(n+1)} \qquad n \geq 1$$

$$a_3 = \frac{-a_0}{3\cdot 2} \qquad a_4 = -\frac{a_1}{4\cdot 3} \qquad a_5 = -\frac{a_2}{5\cdot 4} = 0$$

$$a_6 = -\frac{a_3}{6\cdot 5} = \frac{a_0}{6\cdot 5\cdot 3\cdot 2} \qquad a_7 = -\frac{a_4}{7\cdot 6} = \frac{a_1}{7\cdot 6\cdot 4\cdot 3} \qquad a_8 = -\frac{a_5}{8\cdot 7} = 0$$

$$a_9 = -\frac{a_6}{9\cdot 8} = -\frac{a_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2} \qquad a_{10} = -\frac{a_7}{10\cdot 9} = -\frac{a_1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3} \qquad a_{11} = 0$$

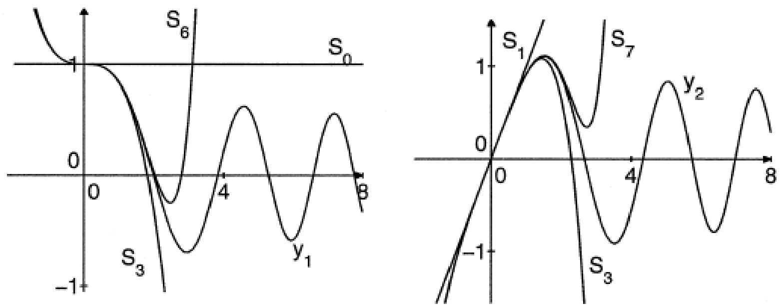
$$\vdots \quad \vdots \qquad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \qquad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots$$

$$a_{3n} = \frac{(-1)^n a_0}{(2\cdot 3)\cdot(5\cdot 6)\cdots(3n-1)(3n)} \qquad a_{3n+1} = \frac{(-1)^n a_1}{(3\cdot 4)\cdot(6\cdot 7)\cdots(3n)(3n+1)} \qquad a_{3n+2} = 0$$

$$y(x) = a_0\left[1 - \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 - \cdots\right] + a_1\left[x - \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}x^7 - \cdots\right]$$

$$y_1(x) = 1 - \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 - \cdots = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^{3n}}{(2\cdot 3)\cdot(5\cdot 6)\cdots(3n-1)(3n)}$$

$$y_2(x) = x - \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}x^7 - \cdots = x + \sum_{n=1}^{\infty}\frac{(-1)^n x^{3n+1}}{(3\cdot 4)\cdot(6\cdot 7)\cdots(3n)(3n+1)}$$



Exercise

The Hermite equation of order α is $\quad y'' - 2xy' + 2\alpha y = 0$

a) Find the general solution is $y(x) = a_0 y_1(x) + a_1 y_2(x)$

Show that $y_1(x)$ is a polynomial if α is an even integer, whereas $y_2(x)$ is a polynomial if α is an odd integer.

b) When $\alpha = n$, use $y_1(x)$ to find polynomial solutions for $n = 0$, $n = 2$, and $n = 4$, then use $y_2(x)$ to find polynomial solutions for $n = 1$, $n = 3$, and $n = 5$.

c) The Hermite polynomial of degree n is denoted by $H_n(x)$. It is the nth-degree polynomial solution of Hermite's equation, multiplied by a suitable constant so that the coefficient of x^n is 2^n. Use part (b) to show the first six Hermite polynomials are

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

d) A general formula for the Hermite polynomials is

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

Verify that this formula does in fact give an nth-degree polynomial.

Solution

a) $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''-2xy'+2\alpha y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+2\alpha\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}2(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}2\alpha a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n-\sum_{n=1}^{\infty}2n a_n x^n+\sum_{n=0}^{\infty}2\alpha a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}-2(n-\alpha)a_n\right]x^n=0$$

$$(n+1)(n+2)a_{n+2}-2(n-\alpha)a_n=0$$

$$a_{n+2}=\frac{2(n-\alpha)}{(n+1)(n+2)}a_n$$

$$a_0$$

$$n=0 \;\rightarrow\; a_2=-\frac{2\alpha}{2}a_0$$

$$n=2 \;\rightarrow\; a_4=\frac{2(2-\alpha)}{3\cdot 4}a_2=-\frac{2^2\alpha(2-\alpha)}{4!}a_0$$

$$n=4 \;\rightarrow\; a_6=\frac{2(4-\alpha)}{5\cdot 6}a_4=-\frac{2^3\alpha(2-\alpha)(4-\alpha)}{6!}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_1(x)=1-\frac{2\alpha}{2!}x^2-\frac{2^2(2-\alpha)}{4!}x^4-\frac{2^3\alpha(2-\alpha)(4-\alpha)}{6!}x^6-\cdots$$

$$=1-\frac{2\alpha}{2!}x^2+\frac{2^2(\alpha-2)}{4!}x^4-\frac{2^3\alpha(\alpha-2)(\alpha-4)}{6!}x^6+\cdots$$

$$a_1$$

$$n=1 \;\rightarrow\; a_3=\frac{2(1-\alpha)}{6}a_1=\frac{2(1-\alpha)}{3!}a_1$$

$$n=3 \;\to\; a_5=\frac{2(3-\alpha)}{4\cdot 5}a_3=\frac{2^2(1-\alpha)(3-\alpha)}{5!}a_1$$

$$n=5 \;\to\; a_7=\frac{2(3-\alpha)}{6\cdot 7}a_5=\frac{2^3(1-\alpha)(3-\alpha)(5-\alpha)}{7!}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y_2(x)=x+\frac{2(1-\alpha)}{3!}x^3+\frac{2^2(1-\alpha)(3-\alpha)}{5!}x^5+\frac{2^3(1-\alpha)(3-\alpha)(5-\alpha)}{7!}x^7+\cdots$$

$$=x-\frac{2(\alpha-1)}{3!}x^3+\frac{2^2(\alpha-1)(\alpha-3)}{5!}x^5-\frac{2^3(\alpha-1)(\alpha-3)(\alpha-5)}{7!}x^7+\cdots$$

$$y(x)=a_0y_1(x)+a_1y_2(x)$$

$$=a_0\left(1-\frac{2\alpha}{2!}x^2+\frac{2^2\alpha(\alpha-2)}{4!}x^4-\frac{2^3\alpha(\alpha-2)(\alpha-4)}{6!}x^6+\cdots\right)$$

$$+a_1\left(x-\frac{2(\alpha-1)}{3!}x^3+\frac{2^2(\alpha-1)(\alpha-3)}{5!}x^5-\frac{2^3(\alpha-1)(\alpha-3)(\alpha-5)}{7!}x^7+\cdots\right)$$

$$=a_0+a_0\sum_{m=1}^{\infty}(-1)^m\frac{2^m\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!}x^{2m}$$

$$+a_1x+a_1\sum_{m=1}^{\infty}(-1)^m\frac{2^m(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!}x^{2m+1}$$

$$\boxed{y_1(x)=1+\sum_{m=1}^{\infty}(-1)^m\frac{2^m\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!}x^{2m}}$$

$$\boxed{y_2(x)=x+\sum_{m=1}^{\infty}(-1)^m\frac{2^m(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!}x^{2m+1}}$$

b) $n=\alpha=0 \;\to\; y_1(x)\ \boxed{=1}$

$n=\alpha=2 \;\to\; y_1(x)=1-\alpha x^2\ \boxed{=1-2x^2}$

$n=\alpha=4 \;\to\; y_1(x)=1-\alpha x^2+\frac{\alpha(\alpha-2)}{6}x^4\ \boxed{=1-4x^2+\frac{4}{3}x^4}$

$n=\alpha=1 \;\to\; y_2(x)\ \boxed{=x}$

$n=\alpha=3 \;\to\; y_2(x)=x-\frac{2(\alpha-1)}{3!}x^3\ \boxed{=x-\frac{2}{3}x^3}$

$$n=\alpha=5 \quad \rightarrow \quad y_2(x)=x-\frac{2(\alpha-1)}{3!}x^3+\frac{2^2(\alpha-1)(\alpha-3)}{5!}x^5 \; \underline{=x-\frac{4}{3}x^3+\frac{4}{15}x^5}$$

c) $H_0(x)=2^0\cdot 1=1$

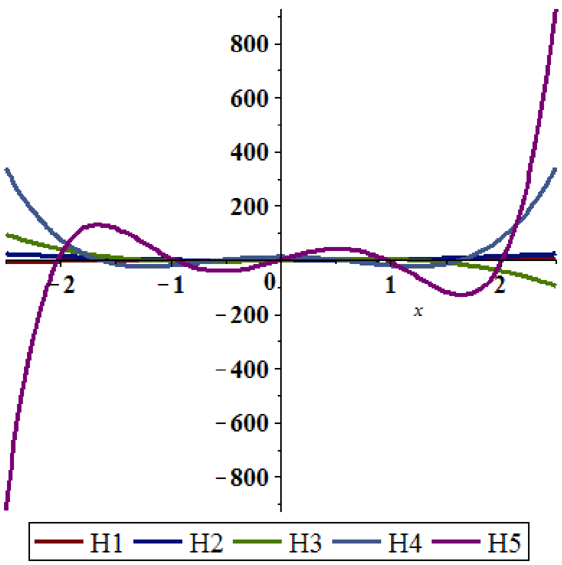
$H_1(x)=2^1\cdot x=2x$

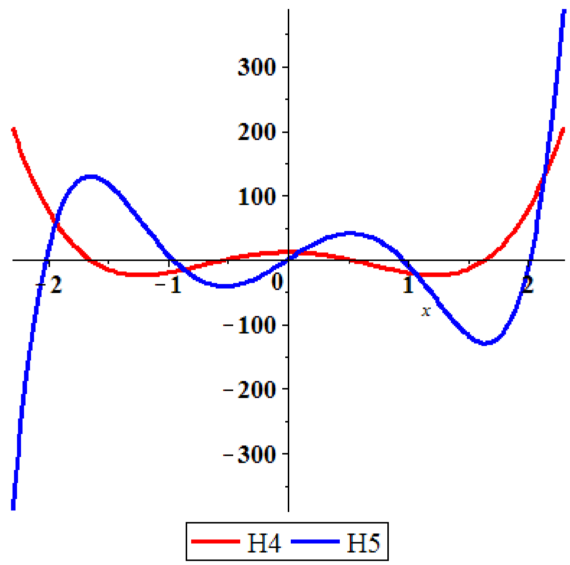
$H_2(x)=-2\left(1-2x^2\right)=4x^2-2$

$H_3(x)=-2^2\cdot 3\left(x-\frac{2}{3}x^3\right)=8x^3-12x$

$H_4(x)=2^2\cdot 3\left(1-4x^2+\frac{4}{3}x^4\right)=16x^4-48x^2+12$

$H_5(x)=2^3\cdot 3\cdot 5\left(\frac{4}{15}x^5-\frac{4}{3}x^3+x\right)=32x^5-160x^3+120x$





d) $\frac{d}{dx}\left(e^{-x^2}\right)=-2xe^{-x^2}$

$\frac{d^2}{dx^2}\left(e^{-x^2}\right)=-2\frac{d}{dx}\left(xe^{-x^2}\right)=-2\left(1-2x^2\right)e^{-x^2}$

$\frac{d^3}{dx^3}\left(e^{-x^2}\right)=2\frac{d}{dx}\left(\left(2x^2-1\right)e^{-x^2}\right)=2\left(4x-4x^3+2x\right)e^{-x^2}=\left(12x-8x^3\right)e^{-x^2}$

$\frac{d^4}{dx^4}\left(e^{-x^2}\right)=4\frac{d}{dx}\left(\left(3x-2x^3\right)e^{-x^2}\right)=4\left(3-6x^2-6x^2+4x^4\right)e^{-x^2}=\left(16x^4-48x^2+12\right)e^{-x^2}$

$H_1(x)=-e^{x^2}\frac{d}{dx}\left(e^{-x^2}\right)=2xe^{x^2}e^{-x^2}=2x \quad \surd$

$H_2(x)=e^{x^2}\frac{d^2}{dx^2}\left(e^{-x^2}\right)=-2e^{x^2}\left(1-2x^2\right)e^{-x^2}=4x^2-2 \quad \surd$

$H_3(x)=e^{x^2}\frac{d^3}{dx^3}\left(e^{-x^2}\right)=e^{x^2}\left(12x-8x^3\right)e^{-x^2}=12x-8x^3 \quad \surd$

$$H_4(x) = e^{x^2} \frac{d^4}{dx^4}\left(e^{-x^2}\right) = e^{x^2}\left(16x^4 - 48x^2 + 12x\right)e^{-x^2} = 16x^4 - 48x^2 + 12 \quad \surd$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

Exercise

Rodrigues's Formula is given by: $\quad P_n(x) = \dfrac{1}{n!\ 2^n} \dfrac{d^n}{dx^n}\left(x^2 - 1\right)^n$

For the nth-degree Legendre polynomial.

a) Show that $v = \left(x^2 - 1\right)^n$ satisfies the differential equation $\left(1 - x^2\right)v' + 2nxv = 0$

Differentiate each side of this equation to obtain

$$\left(1 - x^2\right)v'' + 2(n-1)xv' + 2nv = 0$$

b) Differentiate each side of the last equation n times in succession to obtain

$$\left(1 - x^2\right)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0$$

which satisfies Legendre's equation of order n.

c) Show that the coefficient of x^n in u is $\dfrac{(2n)!}{n!}$; then state why this proves Rodrigues. Formula.

Note: That the coefficient of x^n in $P_n(x)$ is $\dfrac{(2n)!}{2^n (n!)^2}$

$$u = v^{(n)} = D^n\left(x^2 - 1\right)^n$$

Solution

a) $v = \left(x^2 - 1\right)^n \quad \rightarrow \quad v' = 2nx\left(x^2 - 1\right)^{n-1}$

$$v' = 2nx\left(x^2 - 1\right)^{n-1}$$

$$\left(1 - x^2\right)v' + 2nxv = -2nx\left(x^2 - 1\right)\left(x^2 - 1\right)^{n-1} + 2nx\left(x^2 - 1\right)^n$$

$$= -2nx\left(x^2 - 1\right)^n + 2nx\left(x^2 - 1\right)^n$$

$$\underline{= 0}$$

$$\frac{d}{dx}\left(\left(1 - x^2\right)v' + 2nxv\right) = 0$$

$$\left(1 - x^2\right)v'' - 2xv' + 2nxv' + 2nv = 0$$

$$\boxed{\left(1-x^2\right)v''+2(n-1)xv'+2nv=0}$$

b) $\frac{d}{dx}\left(\left(1-x^2\right)v''-2xv'+2nxv'+2nv\right)=0$

$$\left(1-x^2\right)v^{(3)}-2xv''-2v'-2xv''+2nxv''+2nv'+2nv'$$

$$\boxed{\left(1-x^2\right)v^{(3)}+2(n-2)xv''+2(2n-1)v'=0}$$

$n=1 \quad \rightarrow \quad \left(1-x^2\right)v^{(3)}-2xv''+2v'=0$

$$\boxed{\left(1-x^2\right)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}=0} \quad \checkmark$$

$$\frac{d}{dx}\left(\left(1-x^2\right)v^{(3)}+2x(n-2)v''+2(2n-1)v'\right)=0$$

$$\left(1-x^2\right)v^{(4)}-2xv^{(3)}+2x(n-2)v^{(3)}+2(n-2)v''+2(2n-1)v''=0$$

$$\left(1-x^2\right)v^{(4)}+2x(n-3)v^{(3)}+6(n-1)v''=0$$

$$\boxed{\left(1-x^2\right)v^{(4)}+2(n-3)xv^{(3)}+3(2n-2)v''=0}$$

$n=2 \quad \rightarrow \quad \left(1-x^2\right)v^{(4)}-2xv^{(3)}+3\cdot 2v''=0$

$$\boxed{\left(1-x^2\right)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}=0} \quad \checkmark$$

$$\frac{d}{dx}\left(\left(1-x^2\right)v^{(4)}+2(n-3)xv^{(3)}+3(2n-2)v''\right)=0$$

$$\left(1-x^2\right)v^{(5)}-2xv^{(4)}+2(n-3)xv^{(4)}+2(n-3)v^{(3)}+3(2n-2)v^{(3)}=0$$

$$\left(1-x^2\right)v^{(5)}+(2n-6-2)xv^{(4)}+(2n-6+6n-6)v^{(3)}=0$$

$$\left(1-x^2\right)v^{(5)}+(2n-8)xv^{(4)}+(8n-12)v^{(3)}=0$$

$$\boxed{\left(1-x^2\right)v^{(5)}+2(n-4)xv^{(4)}+4(2n-3)v^{(3)}=0}$$

$n=3 \quad \rightarrow \quad \left(1-x^2\right)v^{(5)}-2xv^{(4)}+4\cdot 3v^{(3)}=0$

$$\boxed{\left(1-x^2\right)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}=0} \quad \checkmark$$

After m differentiations:

$$\left(1-x^2\right)v^{(m+2)}+2\left(n-(m+1)\right)xv^{(m+1)}+(m+1)(2n-m)v^{(m)}=0$$

If we let $m=n$, then

$$\boxed{\left(1-x^2\right)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}=0}$$

Let assume that $\left(1-x^2\right)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}=0$ is true.

We need to prove that next derivative is also true.

$$\frac{d}{dx}\left(\left(1-x^2\right)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}\right)=0$$

$$\left(1-x^2\right)v^{(n+3)}-2xv^{(n+2)}-2v^{(n+1)}-2xv^{(n+2)}+(2n-n)(n+1)v^{(n+1)}=0$$

$$\left(1-x^2\right)v^{(n+3)}-4xv^{(n+2)}+(2n-n-2)(n+1)v^{(n+1)}=0$$

$$\left(1-x^2\right)v^{(n+3)}-2(2)xv^{(n+2)}+(n-1)(n+2)v^{(n+1)}=0$$

$$\left(1-x^2\right)v^{(n+3)}+2(n-n-2)xv^{(n+2)}+(n-1)(n+2)v^{(n+1)}=0$$

$$\left(1-x^2\right)v^{(n+3)}+2(n-(n+2))xv^{(n+2)}+(2n-(n+1))((n+1)+1)v^{(n+1)}=0$$

If we let $m=n+1$, then

$$\left(1-x^2\right)v^{(m+2)}+2(n-(m+1))xv^{(m+1)}+(2n-m)(m+1)v^{(m)}=0$$

If we let $m=n$, then

$$\left(1-x^2\right)v^{(n+2)}+2(n-n-1)xv^{(m+1)}+(2n-n)(n+1)v^{(n)}=0$$

$$\boxed{\left(1-x^2\right)v^{(n+2)}-2xv^{(n+1)}+n(n+1)v^{(n)}=0} \quad \checkmark$$

c) $u=v^{(n)}=D^n\left(x^2-1\right)^n$

$$=\frac{d^n}{dx^n}\left(x^{2n}-nx^{2n-1}+\cdots-1\right)$$

$$=2n(2n-1)\cdots(2n-(n-1))x^n-\frac{d^n}{dx^n}\left(nx^{2n-1}+\cdots-1\right)$$

$$=\frac{(2n)!}{n!}x^n-\frac{d^n}{dx^n}\left(nx^{2n-1}+\cdots-1\right)$$

Since $u=v^{(n)}$ satisfies Legendre's equation of order n, $\frac{u}{2^n n!}$

From the notes:

The solution of Legendre polynomial of degree ***n*** is

$$P_n(x)=\sum_{k=0}^{K}(-1)^k\frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!}x^{n-2k}$$

For the highest order, the result when:

$$k = 0 \quad \rightarrow \quad P_n(x) = \frac{(2n)!}{2^n (n!)^2} x^n$$

$$\frac{u}{2^n n!} = \frac{(2n)!}{2^n (n!)^2} x^n + \cdots$$

$$P_n(x) = \frac{1}{n!\, 2^n} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n$$