Solution *Section* 4.6 – Infinite Sequences and Summation Notation

Exercise

Find the first four terms and the eight term of the sequence: $\{12-3n\}$

Solution

$$a_n = 12 - 3n$$

$$a_1 = 12 - 3(1) \underline{= 9}$$

$$a_2 = 12 - 3(2) = 6$$

$$a_3 = 12 - 3(3) \underline{= 3}$$

$$a_4 = 12 - 3(4) \underline{= 0}$$

$$a_8 = 12 - 3(8) \underline{= -12}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\dfrac{3n-2}{n^2+1}\right\}$

Solution

$$a_n = \frac{3n-2}{n^2+1}$$

$$a_1 = \frac{3-2}{1^2+1} = \frac{1}{2}$$

$$a_2 = \frac{3(2)-2}{2^2+1} = \frac{4}{5}$$

$$a_3 = \frac{3(3)-2}{3^2+1} = \frac{7}{10}$$

$$a_4 = \frac{3(4)-2}{4^2+1} = \frac{10}{17}$$

$$a_8 = \frac{3(8)-2}{8^2+1} = \frac{22}{65}$$

Exercise

Find the first four terms and the eight term of the sequence: $\{9\}$

Solution

$\boxed{a_1 = 9}$

$\boxed{a_2 = 9}$

$\boxed{a_3 = 9}$

$\boxed{a_4 = 9}$

$\boxed{a_8 = 9}$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{(-1)^{n-1}\frac{n+7}{2n}\right\}$

Solution

$a_1 = (-1)^{1-1}\frac{1+7}{2(1)}\ \boxed{= 4}$

$a_2 = (-1)^{2-1}\frac{2+7}{2(2)}\ \boxed{= -\frac{9}{4}}$

$a_3 = (-1)^{3-1}\frac{3+7}{2(3)}\ \boxed{= \frac{5}{3}}$

$a_4 = (-1)^{4-1}\frac{4+7}{2(4)}\ \boxed{= -\frac{11}{8}}$

$a_8 = (-1)^{8-1}\frac{8+7}{2(8)}\ \boxed{= -\frac{15}{16}}$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\frac{2^n}{n^2+2}\right\}$

Solution

$a_1 = \frac{2^1}{1^2+2}\ \boxed{= \frac{2}{3}}$

$a_2 = \frac{2^2}{2^2+2}\ \boxed{= \frac{2}{3}}$

$$a_3 = \frac{2^3}{3^2 + 2} = \frac{8}{11}$$

$$a_4 = \frac{2^4}{4^2 + 2} = \frac{8}{9}$$

$$a_8 = \frac{2^8}{8^2 + 2} = \frac{128}{33}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{(-1)^{n-1}\frac{n}{2n-1}\right\}$

Solution

$$a_1 = (-1)^0 \frac{1}{2-1} = 1$$

$$a_2 = (-1)^1 \frac{2}{4-1} = -\frac{2}{3}$$

$$a_3 = (-1)^2 \frac{3}{6-1} = \frac{3}{5}$$

$$a_4 = (-1)^3 \frac{4}{8-1} = -\frac{4}{7}$$

$$a_8 = (-1)^7 \frac{8}{16-1} = -\frac{8}{15}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\frac{2^n}{3^n+1}\right\}$

Solution

$$a_1 = \frac{2^1}{3^1+1} = \frac{2}{4} = \frac{1}{2}$$

$$a_2 = \frac{2^2}{3^2+1} = \frac{4}{10} = \frac{2}{5}$$

$$a_3 = \frac{2^3}{3^3+1} = \frac{8}{28} = \frac{2}{7}$$

$$a_4 = \frac{2^4}{3^4+1} = \frac{16}{82} = \frac{8}{41}$$

$$a_8 = \frac{2^8}{3^8+1} = \frac{256}{6562} = \frac{128}{3281}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\frac{n^2}{2^n}\right\}$

Solution

$$a_1 = \frac{1^2}{2^1} \underline{= \frac{1}{2}}|$$

$$a_2 = \frac{2^2}{2^2} \underline{= 1}|$$

$$a_3 = \frac{3^2}{2^3} \underline{= \frac{9}{8}}|$$

$$a_4 = \frac{4^2}{2^4} = \frac{16}{16} \underline{= 1}|$$

$$a_8 = \frac{8^2}{2^8} = \frac{64}{256} \underline{= \frac{1}{4}}|$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{\frac{n}{e^n}\right\}$

Solution

$$a_1 = \frac{1}{e^1} \underline{= \frac{1}{e}}|$$

$$a_2 \underline{= \frac{2}{e^2}}|$$

$$a_3 \underline{= \frac{3}{e^3}}|$$

$$a_4 \underline{= \frac{4}{e^4}}|$$

$$a_8 \underline{= \frac{8}{e^8}}|$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{c_n\right\} = \left\{(-1)^{n+1} n^2\right\}$

Solution

$$c_1 = (-1)^2 1^2 \ \underline{=1}$$

$$c_2 = (-1)^3 2^2 \ \underline{=-4}$$

$$c_3 = (-1)^4 3^2 \ \underline{=9}$$

$$c_4 = (-1)^5 4^2 \ \underline{=-16}$$

$$c_8 = (-1)^9 8^2 \ \underline{=-64}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{c_n\right\} = \left\{\dfrac{(-1)^n}{(n+1)(n+2)}\right\}$

Solution

$$c_1 = \frac{(-1)^1}{2\cdot 3} \ \underline{=-\frac{1}{6}}$$

$$c_2 = \frac{(-1)^2}{3\cdot 4} \ \underline{=\frac{1}{12}}$$

$$c_3 = \frac{(-1)^3}{4\cdot 5} \ \underline{=-\frac{1}{20}}$$

$$c_4 = \frac{(-1)^4}{5\cdot 6} \ \underline{=\frac{1}{30}}$$

$$c_8 = \frac{(-1)^8}{9\cdot 10} \ \underline{=\frac{1}{90}}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{c_n\right\}=\left\{\left(\frac{4}{3}\right)^n\right\}$

Solution

$$c_1=\left(\frac{4}{3}\right)^1 \boxed{=\frac{4}{3}}$$

$$c_2=\left(\frac{4}{3}\right)^2 \boxed{=\frac{16}{9}}$$

$$c_3=\left(\frac{4}{3}\right)^3 \boxed{=\frac{64}{27}}$$

$$c_4=\left(\frac{4}{3}\right)^4 \boxed{=\frac{256}{81}}$$

$$c_8=\left(\frac{4}{3}\right)^8 \boxed{=\frac{65,536}{6,561}}$$

Exercise

Find the first four terms and the eight term of the sequence: $\left\{b_n\right\}=\left\{\frac{3^n}{n}\right\}$

Solution

$$b_1=\frac{3^1}{1} \boxed{=3}$$

$$b_2=\frac{3^2}{2} \boxed{=\frac{9}{2}}$$

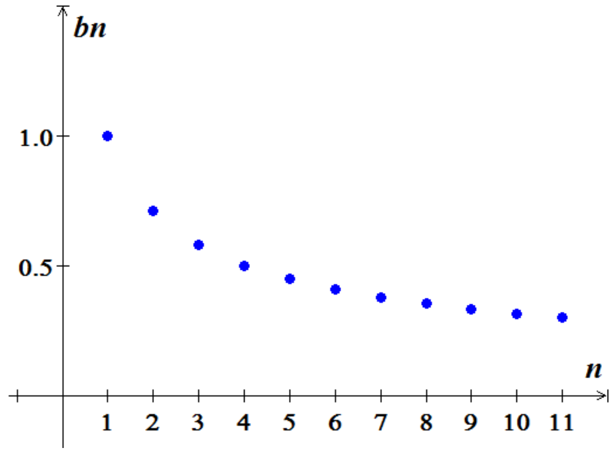
$$b_3=\frac{3^3}{3} \boxed{=9}$$

$$b_4=\frac{3^4}{4} \boxed{=\frac{81}{4}}$$

$$c_8=\frac{3^8}{8} \boxed{=\frac{6,561}{8}}$$

Exercise

Graph the sequence $\left\{\frac{1}{\sqrt{n}}\right\}$



Solution

$$\left\{\frac{1}{\sqrt{n}}\right\} = \left\{\frac{1}{\sqrt{1}},\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{4}},\ \cdots\right\}$$

$$\approx \{1,\ 0.71,\ 0.58,\ 0.5,\ 0.45\}$$

Exercise

Find the first four terms of the sequence of partial sums for the given sequence. $\left\{3+\frac{1}{2}n\right\}$

Solution

$$S_1 = a_1$$
$$= 3+\frac{1}{2}(1)$$
$$= \frac{7}{2}$$

$$S_2 = S_1 + a_2$$
$$= \frac{7}{2}+3+\frac{1}{2}(2)$$
$$= \frac{15}{2}$$

$$S_3 = S_2 + a_3$$
$$= \frac{15}{2}+3+\frac{1}{2}(3)$$
$$= 12$$

$$S_4 = S_3 + a_4$$
$$= 12+3+\frac{1}{2}(4)$$
$$= 17$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_{k+1} = 3a_k - 5$

Solution

$$k = 1 \rightarrow \quad a_2 = 3a_1 - 5$$
$$= 3(2) - 5$$
$$\boxed{= 1}$$

$$k = 2 \rightarrow \quad a_3 = 3a_2 - 5$$
$$= 3(1) - 5$$
$$\boxed{= -2}$$

$$k = 3 \rightarrow \quad a_4 = 3a_3 - 5$$
$$= 3(-2) - 5$$
$$\boxed{= -11}$$

$$k = 4 \rightarrow \quad a_5 = 3a_4 - 5$$
$$= 3(-11) - 5$$
$$\boxed{= -38}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = -3, \quad a_{k+1} = a_k^2$

Solution

$$k = 1 \rightarrow \quad a_2 = a_1^2$$
$$= (-3)^2$$
$$\boxed{= 9}$$

$$k = 2 \rightarrow \quad a_3 = a_2^2$$
$$= (9)^2$$
$$\boxed{= 81}$$

$$k = 3 \rightarrow \quad a_4 = a_3^2$$
$$= \left(3^4\right)^2$$
$$\boxed{= 3^8}$$

$$k = 4 \rightarrow \quad a_5 = a_4^2$$
$$= \left(3^8\right)^2$$
$$\boxed{= 3^{16}}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 5, \quad a_{k+1} = ka_k$

Solution

$$k = 1 \rightarrow \quad a_2 = 1a_1$$
$$\boxed{= 5}$$

$$k = 2 \rightarrow \quad a_3 = 2a_2$$
$$= 2(5)$$
$$\boxed{= 10}$$

$$k = 3 \rightarrow \quad a_4 = 3a_3$$
$$= 3(10)$$
$$\boxed{= 30}$$

$$k = 4 \rightarrow \quad a_5 = 4a_4$$
$$= 4(30)$$
$$\boxed{= 120}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_n = 3 + a_{n-1}$

Solution

$$a_2 = 3 + a_1 = 3 + 2 \ \boxed{= 5}$$

$$a_3 = 3 + a_2 = 3 + 5 \ \boxed{= 8}$$

$$a_4 = 3 + a_3 = 3 + 8 \ \boxed{= 11}$$

$$a_5 = 3 + a_4 = 3 + 11 \ \boxed{= 14}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 5, \quad a_n = 2a_{n-1}$

Solution

$$\begin{aligned} a_2 &= 2a_1 \\ &= 2(5) \\ &= 10 \end{aligned}$$

$$\begin{aligned} a_3 &= 2a_2 \\ &= 2(10) \\ &= 20 \end{aligned}$$

$$\begin{aligned} a_4 &= 2a_3 \\ &= 2(20) \\ &= 40 \end{aligned}$$

$$\begin{aligned} a_5 &= 2a_4 \\ &= 2(40) \\ &= 80 \end{aligned}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = \sqrt{2}, \quad a_n = \sqrt{2 + a_{n-1}}$

Solution

$$\begin{aligned} a_2 &= \sqrt{2 + a_1} \\ &= \sqrt{2 + \sqrt{2}} \end{aligned}$$

$$\begin{aligned} a_3 &= \sqrt{2 + a_2} \\ &= \sqrt{2 + \sqrt{2 + \sqrt{2}}} \end{aligned}$$

$$\begin{aligned} a_4 &= \sqrt{2 + a_3} \\ &= \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \end{aligned}$$

$$\begin{aligned} a_5 &= \sqrt{2 + a_4} \\ &= \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}} \end{aligned}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_{n+1} = 7 - 2a_n$

Solution

$$a_2 = 7 - 2a_1$$
$$= 7 - 4$$
$$\underline{= 3}$$

$$a_3 = 7 - 2a_2$$
$$= 7 - 6$$
$$\underline{= 1}$$

$$a_4 = 7 - 2a_3$$
$$= 7 - 2$$
$$\underline{= 5}$$

$$a_5 = 7 - 2a_4$$
$$= 7 - 10$$
$$\underline{= -5}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 128, \quad a_{n+1} = \frac{1}{4}a_n$

Solution

$$a_2 = \frac{1}{4}a_1$$
$$= \frac{1}{4}128$$
$$\underline{= 32}$$

$$a_3 = \frac{1}{4}a_2$$
$$= \frac{32}{4}$$
$$\underline{= 8}$$

$$a_4 = \frac{1}{4}a_3$$
$$\underline{= 2}$$

$$a_5 = \frac{1}{4}a_4$$
$$\underline{= \frac{1}{2}}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_{n+1} = (a_n)^n$

Solution

$$a_2 = (a_1)^1$$
$$\underline{= 2}$$

$$a_3 = (a_2)^2$$
$$= 2^2$$
$$\underline{= 4}$$

$$a_4 = (a_3)^3$$
$$= 4^3$$
$$\underline{= 64}$$

$$a_5 = (a_4)^4$$
$$\underline{= 64^4}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = A, \quad a_n = a_{n-1} + d$

Solution

$$a_2 = a_1 + d$$
$$\underline{= A + d}$$

$$a_3 = a_2 + d$$
$$= A + d + d$$
$$\underline{= A + 2d}$$

$$a_4 = a_3 + d$$
$$\underline{= A + 3d}$$

$$a_5 = a_4 + d$$
$$\underline{= A + 4d}$$

Exercise

Find the first five terms of the recursively defined infinite sequence: $a_1 = A, \quad a_n = ra_{n-1}, \quad r \neq 0$

Solution

$$a_2 = ra_1$$
$$\boxed{= rA}$$

$$a_3 = ra_2$$
$$\boxed{= Ar^2}$$

$$a_4 = ra_3$$
$$\boxed{= Ar^3}$$

$$a_5 = ra_4$$
$$\boxed{= Ar^4}$$

Exercise

Find the first 5 terms of the recursively defined infinite sequence: $a_1 = 2, \quad a_2 = 2; \quad a_n = a_{n-1} \bullet a_{n-2}$

Solution

$$a_3 = a_2 \cdot a_1$$
$$= 2 \cdot 2$$
$$\boxed{= 4}$$

$$a_4 = a_3 \cdot a_2$$
$$= 4 \cdot 2$$
$$\boxed{= 8}$$

$$a_5 = a_4 \cdot a_3$$
$$= 8 \cdot 4$$
$$\boxed{= 32}$$

$$a_6 = a_5 \cdot a_4$$
$$= 32 \cdot 8$$
$$\boxed{= 256}$$

Exercise

Express each sum using summation notation $1+2+3+\ldots+20$

Solution

$$1+2+3+4+\cdots+20=\sum_{k=1}^{20} k$$

Exercise

Express each sum using summation notation $1+2+3+\ldots+40$

Solution

$$1+2+3+\ldots+40=\sum_{k=1}^{40} k$$

Exercise

Express each sum using summation notation $1^3+2^3+3^3+\ldots+8^3$

Solution

$$1^3+2^3+3^3+\ldots+8^3=\sum_{k=1}^{8} k^3$$

Exercise

Express each sum using summation notation $1^2+2^2+3^2+\ldots+15^2$

Solution

$$1^2+2^2+3^2+\ldots+15^2=\sum_{k=1}^{15} k^2$$

Exercise

Express each sum using summation notation $2^2+2^3+2^4+\ldots+2^{11}$

Solution

$$2^2+2^3+2^4+\ldots+2^{11}=\sum_{k=2}^{11} 2^k$$

Exercise

Express each sum using summation notation $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{13}{14}$

Solution

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{13}{14}=\sum_{k=1}^{13}\frac{k}{k+1}$$

Exercise

Express each sum using summation notation $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\cdots+(-1)^6\frac{1}{3^6}$

Solution

$$1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\cdots+(-1)^6\frac{1}{3^6}=\sum_{k=0}^{6}(-1)^k\frac{1}{3^k}$$

Exercise

Express each sum using summation notation $\frac{2}{3}-\frac{4}{9}+\frac{8}{27}-\cdots+(-1)^{12}\left(\frac{2}{3}\right)^{11}$

Solution

$$\frac{2}{3}-\frac{4}{9}+\frac{8}{27}-\cdots+(-1)^{12}\left(\frac{2}{3}\right)^{11}=\sum_{k=1}^{11}(-1)^{k+1}\left(\frac{2}{3}\right)^k$$

Exercise

Express each sum using summation notation $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{14}{14+1}$

Solution

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{14}{14+1}=\sum_{k=1}^{14}\frac{k}{k+1}$$

Exercise

Express each sum using summation notation $\frac{1}{e}+\frac{2}{e^2}+\frac{3}{e^3}+\cdots+\frac{n}{e^n}$

Solution

$$\frac{1}{e}+\frac{2}{e^2}+\frac{3}{e^3}+\cdots+\frac{n}{e^n}=\sum_{k=1}^{n}\frac{k}{e^k}$$

Exercise

Find the sum: $\sum_{k=1}^{5}(2k-7)$

Solution

$$\sum_{k=1}^{5}(2k-7)=(-5)+(-3)+(-1)+1+3$$

$$\underline{=-5}$$

Exercise

Find the sum: $\sum_{k=0}^{5}k(k-2)$

Solution

$$\sum_{k=0}^{5}k(k-2)=0+(-1)+0+3+8+15$$

$$\underline{=25}$$

Exercise

Find the sum: $\sum_{k=1}^{5}(-3)^{k-1}$

Solution

$$\sum_{k=1}^{5}(-3)^{k-1}=1+(-3)+9+(-27)+81$$

$$\underline{=61}$$

Exercise

Find the sum: $\sum_{k=253}^{571}\left(\frac{1}{3}\right)$

Solution

$$\sum_{k=253}^{571}\left(\frac{1}{3}\right)=(571-253+1)\left(\frac{1}{3}\right)$$

$$\sum_{k=m}^{n} c=(n-m+1)c$$

$$=\frac{319}{3}$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{50} 8$

Solution

$$\sum_{k=1}^{50} 8=(50-1+1)8$$

$$\sum_{k=m}^{n} c=(n-m+1)c$$

$$=400$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{40} k$

Solution

$$\sum_{k=1}^{40} k=\frac{40(41)}{2}$$

$$\sum_{k=1}^{n} k=\frac{n(n+1)}{2}$$

$$=820$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{5}(3k)$

Solution

$$\sum_{k=1}^{5} 3k \ =3(1)+3(2)+3(3)+3(4)+3(5)$$

$$=45$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{10}\left(k^3+1\right)$

Solution

$$\sum_{k=1}^{10}\left(k^3+1\right)=\sum_{k=1}^{10}k^3+\sum_{k=1}^{10}1$$

$$\sum_{k=1}^{n}k^3=\frac{n^2(n+1)^2}{4}$$

$$=\frac{10^2(10+1)^2}{4}+10(1)$$

$$=\frac{12100}{4}+10$$

$$=3025+10$$

$$=3035$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{24}\left(k^2-7k+2\right)$

Solution

$$\sum_{k=1}^{24}\left(k^2-7k+2\right)=\frac{24(24+1)(2\cdot 24+1)}{6}-7\frac{24(24+1)}{2}+2(24)$$

$$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$=2848$$

Exercise

Find the sum: $\displaystyle\sum_{k=6}^{20}\left(4k^2\right)$

Solution

$$\sum_{k=6}^{20}\left(4k^2\right)=4\left(\sum_{k=1}^{20}k^2-\sum_{k=1}^{5}k^2\right)$$

$$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$=4\left(\frac{20(20+1)(2\cdot 20+1)}{6}-\frac{5(5+1)(2\cdot 5+1)}{6}\right)$$

$$= 4\left(\frac{20(21)(41)}{6} - \frac{5(6)(11)}{6}\right)$$

$$= 4(2870 - 55)$$

$$\underline{= 11,260}$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{16}\left(k^2 - 4\right)$

Solution

$$\sum_{k=1}^{16}\left(k^2 - 4\right) = \sum_{k=1}^{16} k^2 - \sum_{k=1}^{16} 4$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{16(16+1)(2\cdot 16+1)}{6} - 4(16)$$

$$= 1496 - 64$$

$$\underline{= 1432}$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{6}(10 - 3k)$

Solution

$$\sum_{k=1}^{6}(10 - 3k) = 7 + 4 + 1 - 2 - 5 - 8$$

$$\underline{= -3}$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{10}\left[1 + (-1)^k\right]$

Solution

$$\sum_{k=1}^{10}\left[1+(-1)^k\right]=0+2+0+2+0+2+0+2+0+2$$

$$\underline{=10}$$

Exercise

Find the sum: $\displaystyle\sum_{k=1}^{6}\frac{3}{k+1}$

Solution

$$\sum_{k=1}^{6}\frac{3}{k+1}=\frac{3}{2}+1+\frac{3}{4}+\frac{3}{5}+2+\frac{3}{7}$$

$$\underline{=\frac{879}{140}}$$

Exercise

Find the sum: $\displaystyle\sum_{k=137}^{428}2.1$

Solution

$$\sum_{k=137}^{428}2.1=(428-137+1)2.1=(292)2.1$$

$$\sum_{k=m}^{n}c=(n-m+1)c$$

$$\underline{=613.2}$$

Exercise

Write out each sum $\displaystyle\sum_{k=1}^{n}(k+2)$

Solution

$$\sum_{k=1}^{n}(k+2)\ \underline{=3+5+7+9+\cdots+(n+2)}$$

Exercise

Write out each sum $\sum\limits_{k=1}^{n} k^2$

Solution

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2$$

$$= 1 + 4 + 9 + 16 + \cdots + n^2$$

Exercise

Write out each sum $\sum\limits_{k=2}^{n} (-1)^k \ln k$

Solution

$$\sum_{k=2}^{n} (-1)^k \ln k = (-1)^2 \ln 2 + (-1)^3 \ln 3 + (-1)^4 \ln 4 + (-1)^5 \ln 5 + \cdots + (-1)^n \ln n$$

$$= \ln 2 - \ln 3 + \ln 4 - \ln 5 + \cdots + (-1)^n \ln n$$

Exercise

Write out each sum $\sum\limits_{k=3}^{n} (-1)^{k+1} 2^k$

Solution

$$\sum_{k=3}^{n} (-1)^{k+1}\ 2^k = (-1)^4 2^3 + (-1)^5 2^4 + (-1)^6 2^5 + (-1)^7 2^6 + \cdots + (-1)^{n+1} 2^n$$

$$= 8 - 16 + 32 - 64 + \cdots + (-1)^{n+1} 2^n$$

Exercise

Write out each sum $\sum\limits_{k=0}^{n} \frac{1}{3^k}$

Solution

$$\sum_{k=0}^{n} \frac{1}{3^k} \; \underline{= 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^n}}\Big|$$

Exercise

Fred has a balance of \$3,000 on his card which charges 1% interest per month on any unpaid balance. Fred can afford to pay \$100 toward the balance each *month*. His balance each month after making a \$100 payment is given by the recursively defined sequence

$$B_0 = \$3,000 \qquad B_n = 1.01B_{n-1} - 100$$

Determine Fred's balance after making the first payment. That is, determine B_1

Solution

$$\begin{aligned} B_1 &= 1.01B_0 - 100 \\ &= 1.01(3,000) - 100 \\ &\underline{= \$2,930}\,\Big| \end{aligned}$$

Fred's balance is \$2,930 after making the first payment.

Exercise

A pond currently has 2,000 trout in it. A fish hatchery decides to add an additional 20 trout each month. Is it also known that the trout population is grwoing at a rate of 3% per *month*. The size of the population after *n* months is given bu the recursively defined sequence

$$P_0 = 2,000 \qquad P_n = 1.03P_{n-1} + 20$$

How many trout are in the pond after 2 months? That is, what is P_2 ?

Solution

$$\begin{aligned} P_1 &= 1.03P_0 + 20 \\ &= 1.03(2,000) + 20 \\ &\underline{= 2,080}\,\Big| \end{aligned}$$

$$\begin{aligned} P_2 &= 1.03P_1 + 20 \\ &= 1.03(2,080) + 20 \\ &\underline{= 2,162.4}\,\Big| \end{aligned}$$

There are approximately 2162 ***trout*** in the pond after 2 *months*.

Exercise

Fred bought a car by taking out a loan for $18,500 at 0.5% interest per month. Fred's normal monthly payment is $434.47 per month, but he decides that he can afford to pay $100 extra toward the balance each month. His balance each month is given by the recursively defined sequence

$$B_0 = \$18,500 \quad B_n = 1.005B_{n-1} - 534.47$$

Determine Fred's balance after making the first payment. That is, determine B_1

Solution

$$\begin{aligned} B_1 &= 1.005B_0 - 534.47 \\ &= 1.005(18,500) - 534.47 \\ &= \$18,058.03 \end{aligned}$$

Fred's balance is $18.058.03 after making the first payment.

Exercise

The Environmental Protection Agency (EPA) determines that Maple Lake has 250 *tons* of pollutant as a result of industrial waste and that 10% of the pollutant present is neuttralized by solar oxidation every year. The EPA imposes new pollution control laws that result in 15 *tons* of new pollutant entering the lake each year. The amount of pollutant in the lake after *n* years is given by the recursively defined sequence

$$P_0 = 250 \quad P_n = 0.9P_{n-1} + 15$$

Determine the amount of pollutant in the lake after 2 years? That is, what is P_2?

Solution

$$\begin{aligned} P_1 &= 0.9P_0 + 15 \\ &= 0.9(250) + 15 \\ &= 240 \end{aligned}$$

$$\begin{aligned} P_2 &= 0.9P_1 + 15 \\ &= 0.9(240) + 15 \\ &= 231 \end{aligned}$$

There are 231 ***tons*** of pollutants after 2 *years*.

Exercise

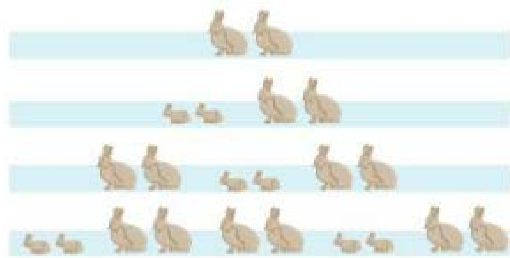
A colony of rabbits begins with one pair of mature rabbits, which will produce a pair of offspring (one male, one female) each month. Assume that all rabbits mature in 1 month and produce a pair of offspring (one male, one female) after 2 months. If no rabbits ever die, how many pairs of mature rabbits are there after 7 months?

1 mature pair

1 mature pair

2 mature pairs

3 mature pairs



Solution

$a_1 = 1$

$a_2 = 1$

$a_3 = 2$

$a_4 = 3$

$a_5 = 5$

$a_6 = 8$

$a_7 = 13$

$a_8 = 21$

$\vdots \quad \vdots$

$a_n = a_{n-1} + a_{n-2}$

After 7 *months* there are 21 *mature* pairs of rabbits.

Exercise

Let $\quad u_n = \dfrac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}$

Define the *n*th term of a sequence

a) Show that $u_1 = 1$ and $u_2 = 1$

b) Show that $u_{n+2} = u_{n+1} + u_n$

c) Draw the conclusion that $\{u_n\}$ is a Fibonacci sequence

d) Find the first ten terms of the sequence from part (*c*)

Solution

a) $u_1 = \dfrac{\left(1+\sqrt{5}\right)^1 - \left(1-\sqrt{5}\right)^1}{2^1\ \sqrt{5}}$

$= \dfrac{2\sqrt{5}}{2\sqrt{5}}$

$\underline{=1}$

$u_2 = \dfrac{\left(1+\sqrt{5}\right)^2 - \left(1-\sqrt{5}\right)^2}{2^2\ \sqrt{5}}$ $\qquad$ $a^2 - b^2 = (a-b)(a+b)$

$= \dfrac{\left(1+\sqrt{5}-1+\sqrt{5}\right) - \left(1-\sqrt{5}+1-\sqrt{5}\right)}{2^2\ \sqrt{5}}$

$= \dfrac{4\sqrt{5}}{4\sqrt{5}}$

$\underline{=1}$

b) $u_{n+1} + u_n = \dfrac{\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}}{2^{n+1}\ \sqrt{5}} + \dfrac{\left(1+\sqrt{5}\right)^{n} - \left(1-\sqrt{5}\right)^{n}}{2^{n}\ \sqrt{5}}$

$$= \frac{\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1} + 2\left(1+\sqrt{5}\right)^{n} - 2\left(1-\sqrt{5}\right)^{n}}{2^{n+1}\ \sqrt{5}}$$

$$= \frac{\left(1+\sqrt{5}\right)^{n}\left(1+\sqrt{5}+2\right) - \left(1-\sqrt{5}\right)^{n}\left(1-\sqrt{5}+2\right)}{2^{n+1}\ \sqrt{5}}$$

$$= \frac{\left(1+\sqrt{5}\right)^{n}\left(3+\sqrt{5}\right) - \left(1-\sqrt{5}\right)^{n}\left(3-\sqrt{5}\right)}{2^{n+1}\ \sqrt{5}}$$

$$= \frac{\left(1+\sqrt{5}\right)^{n+2}\dfrac{3+\sqrt{5}}{\left(1+\sqrt{5}\right)^2} - \left(1-\sqrt{5}\right)^{n+2}\dfrac{3-\sqrt{5}}{\left(1-\sqrt{5}\right)^2}}{2^{n+1}\ \sqrt{5}}$$

$$= \frac{\left(1+\sqrt{5}\right)^{n+2}\dfrac{3+\sqrt{5}}{1+2\sqrt{5}+5} - \left(1-\sqrt{5}\right)^{n+2}\dfrac{3-\sqrt{5}}{1-2\sqrt{5}+5}}{2^{n+1}\ \sqrt{5}}$$

$$= \frac{\frac{1}{2}\left(1+\sqrt{5}\right)^{n+2}\dfrac{3+\sqrt{5}}{3+\sqrt{5}} - \frac{1}{2}\left(1-\sqrt{5}\right)^{n+2}\dfrac{3-\sqrt{5}}{3-\sqrt{5}}}{2^{n+1}\ \sqrt{5}}$$

$$= \frac{1}{2} \frac{\left(1+\sqrt{5}\right)^{n+2} - \left(1-\sqrt{5}\right)^{n+2}}{2^{n+1}\sqrt{5}}$$

$$= \frac{\left(1+\sqrt{5}\right)^{n+2} - \left(1-\sqrt{5}\right)^{n+2}}{2^{n+2}\sqrt{5}}$$

$$= u_{n+2} \quad \checkmark$$

c) Since $u_1 = 1$ and $u_2 = 1$ and $u_{n+2} = u_{n+1} + u_n$

$\therefore \{u_n\}$ is a Fibonacci sequence

e) $u_1 = 1$

$u_2 = 1$

$u_3 = u_2 + u_1 = 1 + 1 = 2$

$u_4 = u_3 + u_2 = 2 + 1 = 3$

$u_5 = u_4 + u_3 = 3 + 2 = 5$

$u_6 = u_5 + u_4 = 5 + 3 = 8$

$u_7 = u_6 + u_5 = 8 + 5 = 13$

$u_8 = u_7 + u_5 = 13 + 8 = 21$

$u_9 = u_8 + u_7 = 21 + 13 = 34$

$u_{10} = u_9 + u_8 = 34 + 21 = 55$