Solution   *Section* 1.3 – Matrices and Matrix operations

Exercise

For the matrices: $A = \begin{bmatrix} p & 0 \\ q & r \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, when does $AB = BA$

Solution

$$AB = \begin{pmatrix} p & 0 \\ q & r \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} p & p \\ q & q+r \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & 0 \\ q & r \end{pmatrix}$$

$$= \begin{pmatrix} p+q & r \\ q & r \end{pmatrix}$$

$$AB = BA$$

$$\begin{pmatrix} p & p \\ q & q+r \end{pmatrix} = \begin{pmatrix} p+q & r \\ q & r \end{pmatrix}$$

$$\begin{cases} p = p+q \\ \boxed{p = r} \\ q+r = r \end{cases} \Rightarrow \begin{cases} \boxed{q = 0} \\ \\ q = 0 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal. $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

Solution

$$\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$$

$$\Rightarrow \begin{cases} w = 9 & x = 17 \\ y = 8 & z = -12 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal. $\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$

Solution

$$\begin{cases} \underline{x = 12} \\ y+3=5 \to \underline{y=2} \\ 2z = 6 \to \underline{z=3} \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal. $\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$

Solution

$$\begin{bmatrix} 5 = y+3 & x-4=2 & 9=9 \\ 2 = z+4 & -3=-3 & 8=8 \\ 6=6 & 0=0 & 5=w \end{bmatrix}$$

$$\to \begin{cases} y=2 & z=-2 \\ x=6 & w=5 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

Solution

$$\begin{bmatrix} a-5 & 5b & 4c+6 \\ -2d & 7f-6 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

$$\begin{cases} a-5=15 & \to & a=20 \\ 5b=25 & \to & b=5 \\ 4c+6=6 & \to & 4c=0 \to c=0 \\ -2d=-8 & \to & d=4 \\ 7f-6=1 & \to & 7f=7 \to f=1 \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} a+11 & 12z+1 & 5m \\ 11k & 3 & 1 \end{bmatrix} + \begin{bmatrix} 9a & 9z & 4m \\ 12k & 5 & 3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

Solution

$$\begin{bmatrix} a+11+9a & 12z+1+9z & 5m+4m \\ 11k+12k & 3+5 & 1+3 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 10a+11 & 21z+1 & 9m \\ 23k & 8 & 4 \end{bmatrix} = \begin{bmatrix} 41 & -62 & 72 \\ 92 & 8 & 4 \end{bmatrix}$$

$$10a+11=41 \quad \to \quad 10a=30$$

$$\underline{a=3}$$

$$21z+1=-62 \quad \to \quad 21z=-63$$

$$\underline{z=-3}$$

$$9m=72 \quad \to \quad \underline{m=8}$$

$$23k=92 \quad \to \quad k=\frac{92}{23}=4$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} x+2 & 3y+1 & 5z \\ 8w & 2 & 3 \end{bmatrix} + \begin{bmatrix} 3x & 2y & 5z \\ 2w & 5 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 4x+2 & 5y+1 & 10z \\ 10w & 7 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 7 & -2 \end{bmatrix}$$

$$\begin{cases} 4x+2=10 & \to \quad \underline{x=2} \\ 5y+1=-14 & \to \quad \underline{y=-3} \\ 10z=80 & \to \quad \underline{z=8} \\ 10w=10 & \to \quad \underline{w=1} \end{cases}$$

Exercise

Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} 2x-3 & y-2 & 2z+1 \\ 5 & 2w & 7 \end{bmatrix} + \begin{bmatrix} 3x-3 & y+2 & z-1 \\ -5 & 5w+1 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 5x-6 & 2y & 3z \\ 0 & 7w+1 & 10 \end{bmatrix} = \begin{bmatrix} 20 & 8 & 9 \\ 0 & 8 & 10 \end{bmatrix}$$

$$\begin{cases} 5x-6=20 & \rightarrow \quad \underline{x=\frac{26}{5}|} \\ 2y=8 & \rightarrow \quad \underline{y=4|} \\ 3z=9 & \rightarrow \quad \underline{z=3|} \\ 7w+1=8 & \rightarrow \quad \underline{w=1|} \end{cases}$$

Exercise

Find a combination $x_1w_1 + x_2w_2 + x_3w_3$ that gives the zero vector:

$$w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \quad w_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}; \quad w_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Those vectors are independent or dependent?
The vectors lie in a ______.
The matrix W with those columns is not invertible.

Solution

$w_1 - 2w_2 + w_3 = 0$; Therefore those vectors are dependent

The vectors lie in a plane.

Exercise

The very last words say that the 5 by 5 centered difference matrix is not invertible, Write down the 5 equations $Cx = b$. Find a combination of left sides that gives zero. What combination of b_1, b_2, b_3, b_4, b_5 must be zero?

Solution

The 5 by 5 centered difference matrix is

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

The five equations $Cx = b$ are:

$$x_2 = b_1, \quad -x_1 + x_3 = b_2, \quad -x_2 + x_4 = b_3, \quad -x_3 + x_5 = b_4, \quad -x_4 = b_5.$$

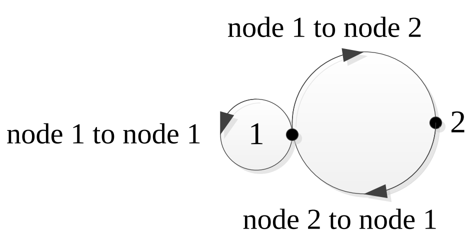
Observe that the sum of the first

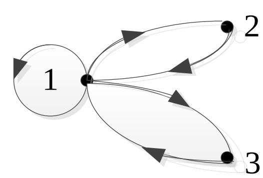
$$x_2 - x_2 + x_4 - x_4 = b_1 + b_2 + b_5$$

$$0 = b_1 + b_2 + b_5$$

Exercise

A direct graph starts with *n* nodes. There are n^2 possible edges, each edge leaves one of the *n* nodes and enters one of the *n* nodes (possibly itself). The *n* by *n* adjacency matrix has $a_{ij} = 1$ when edge leaves node *i* and enter node *j*; if no edge then $a_{ij} = 0$. Here are directed graphs and their adjacency matrices:





$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The *i*, *j* entry of A^2 is $a_{i1}a_{1j} + \ldots + a_{in}a_{nj}$.

Why does that sum count the two-step paths from *i* to any node to *j*?

The *i*, *j* entry of A^k counts *k*-steps paths:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{matrix} \textit{counts the paths} \\ \textit{with two edges} \end{matrix} \quad \begin{bmatrix} 1 \text{ to } 2 \text{ to } 1, 1 \text{ to } 1 \text{ to } 1 & 1 \text{ to } 1 \text{ to } 2 \\ 2 \text{ to } 1 \text{ to } 1 & 2 \text{ to } 1 \text{ to } 2 \end{bmatrix}$$

List all 3-step paths between each pair of nodes and compare with A^3. When A^k has ***no zeros***, that number *k* is the diameter of the graph – the number of edges needed to connect the most pair of nodes. What is the diameter of the second graph?

Solution

The number $a_{ik}a_{kj}$ will be "**1**" if there is an edge from node i to k and an edge from k to j.

This is a 2-step path. The number $a_{ik}a_{kj}$ will be "**0**" if either of those edge (from node i to k and from k to j) is missing.

The sum of $a_{ik}a_{kj}$ is the number of 2-step paths leaving i and entering j.

Matrix multiplication is right for this count.

The 3-step paths are counted by A^3; we look at paths to node 2:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{matrix} \textit{counts the paths} \\ \textit{with three steps} \end{matrix} \quad \begin{bmatrix} \dots & 1 \text{ to } 1 \text{ to } 1 \text{ to } 2,\ 1 \text{ to } 2 \text{ to } 1 \text{ to } 2 \\ \dots & 2 \text{ to } 1 \text{ to } 1 \text{ to } 2 \end{bmatrix}$$

The A^k contain Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, 13, ….

Fibonacci's rule $F_{k+2} = F_{k+1} + F_k$ show up in $(A)\left(A^k\right) = A^{k+1}$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} = \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix} = A^{k+1}$$

There are ***13 six-step*** paths from node one to node 1.

Exercise

A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

a) AB *c)* ABD *e)* ABC *g)* $A(B+C)$

b) BA *d)* DBA *f)* $ABCD$

Solution

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

a) $AB: (3\times 5)(5\times 3) = (3\times 3)$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{pmatrix}$$

b) $BA: (5\times 3)(3\times 5) = (5\times 5)$

$$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{pmatrix}=\begin{pmatrix}3&3&3&3&3\\3&3&3&3&3\\3&3&3&3&3\\3&3&3&3&3\\3&3&3&3&3\end{pmatrix}$$

c) $ABD:(3\times 5)(5\times 3)(3\times 1)=(3\times 1)$

$$\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}5&5&5\\5&5&5\\5&5&5\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$=\begin{pmatrix}15\\15\\15\end{pmatrix}$$

d) $DBA:(3\times 1)(5\times 3)(3\times 5)=NA$

e) $ABC:(3\times 5)(5\times 3)(5\times 1)=NA$

f) $ABCD:(3\times 5)(5\times 3)(5\times 1)(3\times 1)=NA$

g) $A(B+C):(3\times 5)((5\times 3)+(5\times 1))=NA$

Matrices B and C are not the same size.

Exercise

What rows or columns or matrices do you multiply to find.

a) The third column of AB?

b) The second column of AB?

c) The first row of AB?

d) The second row of AB?

e) The entry in row 3, column 4 of AB?

f) The entry in row 2, column 3 of AB?

Solution

a) A (column 3 of B)

b) A (column 2 of B)

c) (Row 1 of A) B

d) (Row 2 of A) B

e) (Row 3 of A) (Column 4 of B)

f) (Row 2 of A) (Column 3 of B)

Exercise

Add AB to AC and compare with $A(B+C)$:

$$A=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix} \quad \textit{and} \quad B=\begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix} \quad \textit{and} \quad C=\begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix}$$

Solution

$$AB=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}\begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 7\\ 0 & 7\end{bmatrix}$$

$$AC=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}\begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix}=\begin{bmatrix}3 & 1\\ 6 & 2\end{bmatrix}$$

$$\begin{aligned}A(B+C)&=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}\left(\begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix}+\begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix}\right)\\ &=\begin{bmatrix}1 & 5\\ 2 & 3\end{bmatrix}\begin{bmatrix}3 & 3\\ 0 & 1\end{bmatrix}\\ &=\begin{bmatrix}3 & 8\\ 6 & 9\end{bmatrix}\end{aligned}$$

$$\begin{aligned}AB+AC&=\begin{bmatrix}0 & 7\\ 0 & 7\end{bmatrix}+\begin{bmatrix}3 & 1\\ 6 & 2\end{bmatrix}\\ &=\begin{bmatrix}3 & 8\\ 6 & 9\end{bmatrix}\end{aligned}$$

$$\underline{A(B+C)=AB+AC}$$

Exercise

True or False

a) If A^2 is defined then A is necessarily square.

b) If AB and BA are defined then A and B are square.

c) If AB and BA are defined then AB and BA are square.

d) If $AB=B$, then $A=I$

Solution

a) True

b) False, if A has an order m by n and B n by m: $AB: m\times m \qquad BA: n\times n$

c) True; $AB: m\times m \qquad BA: n\times n$

d) False, if B is the matrix of all zeros.

Exercise

a) Find a nonzero matrix A such that $A^2 = 0$

b) Find a matrix that has $A^2 \neq 0$ but $A^3 = 0$

Solution

a) A nonzero matrix A such that $A^2 = 0$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

b) A matrix that has $A^2 \neq 0$ but $A^3 = 0$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = A^2 A$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise

Suppose you solve $Ax = b$ for three special right sides b:

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If the three solutions x_1, x_2, x_3 are the columns of a matrix X, what is A times X?

Solution

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Therefore, $Ax = I$

Exercise

Show that $(A+B)^2$ is different from $A^2 + 2AB + B^2$, when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

Write down the correct rule for $(A+B)(A+B) = A^2 +$ __________ $+ B^2$

Solution

$$A + B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}$$

$$(A+B)^2 = \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}$$

$$2AB = 2\begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 14 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix} \neq \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$$

$$\boxed{(A+B)^2 \neq A^2 + 2AB + B^2}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

$$A^2 + AB + BA + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 4 \\ 5 & 6 \end{bmatrix}$$

$$(A+B)(A+B) = A^2 + \underline{\ } AB + BA \underline{\ } + B^2$$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

Solution

By rows: $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 5 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \end{pmatrix}$

$$= \begin{pmatrix} 14 \\ 22 \end{pmatrix}$$

By columns: $\begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 4\begin{pmatrix} 2 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 14 \\ 22 \end{pmatrix}$$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Solution

By rows: $\begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} (3 \quad 6)(2 \quad -1) \\ (6 \quad 12)(2 \quad -1) \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

By columns: $\begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 6 \end{pmatrix} - 1\begin{pmatrix} 6 \\ 12 \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

Solution

By rows: $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} (1 \quad 2 \quad 4)(3 \quad 1 \quad 1) \\ (2 \quad 0 \quad 1)(3 \quad 1 \quad 1) \end{pmatrix}$

$$= \begin{pmatrix} 1(3) + 2(1) + 4(1) \\ 2(3) + 0(1) + 1(1) \end{pmatrix}$$

$$= \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

By columns: $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

Exercise

Find the product of the 2 matrices by rows or by columns: $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

Solution

By rows: $\begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} (1 & 2 & 4)(2 & 2 & 3) \\ (-2 & 3 & 1)(2 & 2 & 3) \\ (-4 & 1 & 2)(2 & 2 & 3) \end{pmatrix}$

$$= \begin{pmatrix} 18 \\ 5 \\ 0 \end{pmatrix}$$

By columns: $\begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} 18 \\ 5 \\ 0 \end{pmatrix}$$

Exercise

Given $A = \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix}$ $\quad$ Find $A+B, \quad 2A, \quad \mathit{and} \quad -B$

Solution

$$A+B = \begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix} + \begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -2 \\ 8 & -2 & 0 \end{bmatrix}$$

$$2A = 2\begin{bmatrix} 4 & 1 & 3 \\ 3 & -1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 2 & 6 \\ 6 & -2 & -4 \\ 0 & 0 & 8 \end{bmatrix}$$

$$-B = -\begin{bmatrix} -3 & -2 & -3 \\ -1 & 0 & 0 \\ 8 & -2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 3 \\ 1 & 0 & 0 \\ -8 & 2 & 4 \end{bmatrix}$$

Exercise

Given $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}$. Find AB and BA.

Solution

$$AB = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 13 \\ 4 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2 & 7 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} -16 & 29 \\ -4 & 10 \end{bmatrix}$$

$$Note: \quad AB \neq BA$$

Exercise

Given $A = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$ $B = \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & 17 \\ 6 & -8 \end{pmatrix}$$

$$BA = \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 14 \\ 1 & -20 \end{pmatrix}$$

Exercise

Given $A=\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$ $B=\begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} -3 & -11 \\ 4 & 0 \end{pmatrix}$$

$$BA=\begin{pmatrix} -1 & -1 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} -7 & 1 \\ 16 & 4 \end{pmatrix}$$

Exercise

Given $A=\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}$ $B=\begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}$$

$$=\begin{pmatrix} 10 & 6 \\ 14 & -7 \end{pmatrix}$$

$$BA=\begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}$$

$$=\begin{pmatrix} 14 & -1 \\ 0 & -11 \end{pmatrix}$$

Exercise

Given $A=\begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}$ $B=\begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$. Find AB and BA.

Solution

$$AB=\begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} -4 & 2 \\ 4 & -4 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}$$
$$= \begin{pmatrix} 4 & -4 \\ 14 & -12 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -4 & 5 \\ 6 & 0 & 3 \\ -3 & -2 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & -2 \end{pmatrix}$$
$$= \begin{pmatrix} 5 & -13 \\ 3 & -4 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 5 & 4 \\ 2 & -3 & 8 \\ -3 & 8 & -5 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -14 & 7 \\ -4 & 5 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 24 & 4 \\ 2 & -6 & -2 \\ -13 & 12 & 10 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 6 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 0 & -1 \\ -3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 8 \\ -10 & 10 \end{pmatrix}$$

Exercise

Given $A = \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ Find AB *and* BA *if possible*

Solution

$$AB = \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3(3)+2(0)-3(1) & 3(-4)+2(1)-3(0) \\ 0(3)+1(0)+0(1) & 0(-4)+1(1)+0(0) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -10 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & -4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3(3)-4(0) & 3(2)-4(1) & 3(-3)-4(0) \\ 0(3)+1(0) & 0(2)+1(1) & 0(-3)+1(0) \\ 1(3)+0(0) & 1(2)+0(1) & 1(-3)+0(0) \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 2 & -9 \\ 0 & 1 & 0 \\ 3 & 2 & -3 \end{bmatrix}$$

Exercise

Given $A = \begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 4 & -2 \\ -2 & 0 \\ 9 & 1 \end{bmatrix}$ Find AB *and* BA *if possible*

Solution

$AB =$ ***Undefined***

$$BA = \begin{bmatrix} 4 & -2 \\ -2 & 0 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 12 \\ -10 & -6 \\ 44 & 27 \end{bmatrix}$$

Exercise

Given $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$ Find AB *and* BA *if possible*

Solution

$$AB = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}$$

$$BA = Undefined$$

Exercise

Given $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} \quad A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -2 & -6 \\ 0 & -1 & 2 \\ 5 & -3 & -4 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -3 & 9 \\ 2 & -3 & 4 \\ 4 & -6 & 3 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 8 & -2 \\ 3 & -8 & 4 \\ -2 & 13 & -5 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 5 & -1 \\ 11 & -5 & 6 \\ -8 & 7 & -4 \end{pmatrix}$$

Exercise

Given $A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}$ $B = \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}$. Find AB and BA.

Solution

$$AB = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}\begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -7 & 2 \\ -6 & 2 & 2 \\ -8 & -6 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} -3 & 1 & 0 \\ 1 & 4 & -1 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 6 & 1 \\ 7 & 0 & 5 \\ 4 & -4 & -2 \end{pmatrix}$$

Exercise

Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Compute the following (where possible):

a) $D+E$ ***b***) $D-E$ ***c***) $5A$ ***d***) $-7C$ ***e***) $2B-C$ ***f***) $-3(D+2E)$

Solution

a) $D+E=\begin{bmatrix}1&5&2\\-1&0&1\\3&2&4\end{bmatrix}+\begin{bmatrix}6&1&3\\-1&1&2\\4&1&3\end{bmatrix}$

$=\begin{bmatrix}7&6&5\\-2&1&3\\7&3&7\end{bmatrix}$

b) $D-E=\begin{bmatrix}1&5&2\\-1&0&1\\3&2&4\end{bmatrix}-\begin{bmatrix}6&1&3\\-1&1&2\\4&1&3\end{bmatrix}$

$=\begin{bmatrix}-5&4&-1\\0&-1&-1\\-1&1&1\end{bmatrix}$

c) $5A=5\begin{bmatrix}3&0\\-1&2\\1&1\end{bmatrix}$

$=\begin{bmatrix}15&0\\-5&10\\5&5\end{bmatrix}$

d) $-7C=-7\begin{bmatrix}1&4&2\\3&1&5\end{bmatrix}$

$=\begin{bmatrix}-7&-28&-14\\-21&-7&-35\end{bmatrix}$

e) Since B and C are not the same size

$2B-C$: *can't be calculated*

f) $-3(D+2E)=-3\left(\begin{bmatrix}1&5&2\\-1&0&1\\3&2&4\end{bmatrix}+\begin{bmatrix}12&2&6\\-2&2&4\\8&2&6\end{bmatrix}\right)$

$=-3\begin{bmatrix}13&7&8\\-3&2&5\\11&4&10\end{bmatrix}$

$=\begin{bmatrix}-39&-21&-24\\9&-6&-15\\-33&-12&-30\end{bmatrix}$

Exercise

Consider the matrices

$$A=\begin{bmatrix}3&0\\-1&2\\1&1\end{bmatrix}\qquad B=\begin{bmatrix}1&5&2\\-1&1&0\\-4&1&3\end{bmatrix}\qquad C=\begin{bmatrix}-3&-1\\2&1\\4&3\end{bmatrix}\qquad D=\begin{bmatrix}4&-1\\2&0\end{bmatrix}$$

Compute the following (where possible):

a) $A+B$ ***b***) $A+C$ ***c***) AB ***d***) BA ***e***) CD ***f***) DC

g) BD ***h***) DB ***i***) A^2 ***j***) B^2 ***k***) D^2

Solution

a) Since A and B are not the same size, then

$A+B=$ *can't be calculated*

b) $A+C=\begin{bmatrix}3&0\\-1&2\\1&1\end{bmatrix}+\begin{bmatrix}-3&-1\\2&1\\4&3\end{bmatrix}$

$=\begin{bmatrix}0&-1\\1&3\\5&4\end{bmatrix}$

c) $A:\ 3\times 2\quad B:\ 3\times 3$

AB *can't be calculated* , since the inner are not equal.

d) $B:\ 3\times 3\quad A:\ 3\times 2$

$BA=\begin{bmatrix}1&5&2\\-1&1&0\\-4&1&3\end{bmatrix}\begin{bmatrix}3&0\\-1&2\\1&1\end{bmatrix}$

$=\begin{bmatrix}0&12\\-1&2\\-10&5\end{bmatrix}$

e) $C:\ 3\times 2\quad D:\ 2\times 2$

$CD=\begin{bmatrix}-3&-1\\2&1\\4&3\end{bmatrix}\begin{bmatrix}4&-1\\2&0\end{bmatrix}$

$=\begin{bmatrix}-14&3\\10&-2\\22&-4\end{bmatrix}$

f) $D:\ 2\times 2 \quad C:\ 3\times 2$

DC *can't be calculated*, since the inner are not equal.

g) $B:\ 3\times 3 \quad D:\ 2\times 2$

BD *can't be calculated*, since the inner are not equal.

h) $D:\ 2\times 2 \quad B:\ 3\times 3$

DB *can't be calculated*, since the inner are not equal.

i) A^2 *can't be calculated*, since A is not square matrix.

j) $B^2 = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} -12 & 12 & 8 \\ -2 & -4 & -2 \\ -17 & -16 & 1 \end{bmatrix}$

k) $D^2 = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 14 & -4 \\ 8 & -2 \end{bmatrix}$

Exercise

Let $B = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$, show that $B^4 = \begin{pmatrix} a^4 & 0 \\ a^3 + a^2b + ab^2 + b^3 & b^4 \end{pmatrix}$

Solution

$$B^2 = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$

$$= \begin{pmatrix} a^2 & 0 \\ a+b & b^2 \end{pmatrix}$$

$$B^4 = B^2B^2 = \begin{pmatrix} a^2 & 0 \\ a+b & b^2 \end{pmatrix}\begin{pmatrix} a^2 & 0 \\ a+b & b^2 \end{pmatrix}$$

$$= \begin{pmatrix} a^4 & 0 \\ a^3 + a^2b + ab^2 + b^3 & b^4 \end{pmatrix} \quad \checkmark$$

Exercise

Let $B=\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$, show that $B^n=\begin{pmatrix} a^n & 0 \\ \sum\limits_{k=0}^{n-1} a^{n-1-k}b^k & b^n \end{pmatrix}$

Solution

$$n=2 \quad \to \quad B^2=\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$

$$=\begin{pmatrix} a^2 & 0 \\ a+b & b^2 \end{pmatrix} \qquad \checkmark$$

Let assume $B^n=\begin{pmatrix} a^n & 0 \\ \sum\limits_{k=0}^{n-1} a^{n-1-k}b^k & b^n \end{pmatrix}$ is true

We need to also prove that it is true for $B^{n+1}=\begin{pmatrix} a^{n+1} & 0 \\ \sum\limits_{k=0}^{n} a^{n-k}b^k & b^{n+1} \end{pmatrix}$

$$B^{n+1}=B^nB=\begin{pmatrix} a^n & 0 \\ \sum\limits_{k=0}^{n-1} a^{n-1-k}b^k & b^n \end{pmatrix}\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$

$$=\begin{pmatrix} a^{n+1} & 0 \\ b^n+a\sum\limits_{k=0}^{n-1} a^{n-1-k}b^k & b^{n+1} \end{pmatrix}$$

$$=\begin{pmatrix} a^{n+1} & 0 \\ b^n+\sum\limits_{k=0}^{n} a^{n-k}b^k & b^{n+1} \end{pmatrix}$$

$$=\begin{pmatrix} a^{n+1} & 0 \\ \sum\limits_{k=0}^{n} a^{n-k}b^k & b^{n+1} \end{pmatrix} \qquad \checkmark$$

Exercise

Let $A=\begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix}$. Prove that $A^n=\begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$ if $n\ge 1$

Solution

Using the principle of mathematical induction.

For $n=1 \;\rightarrow\; A=\begin{bmatrix} 1+6 & 4 \\ -9 & 1-6 \end{bmatrix}=\begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix}$ ✓ P_1 is true

Assume that P_n is true, $A^n=\begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$

We need to prove that P_{n+1}:

$$A^{n+1}=\begin{bmatrix} 1+6(n+1) & 4(n+1) \\ -9(n+1) & 1-6(n+1) \end{bmatrix}$$

$$=\begin{bmatrix} 7+6n & 4n+4 \\ -9n-9 & -6n-5 \end{bmatrix}$$ is also *true*.

$$A^{n+1}=AA^n$$

$$=\begin{bmatrix} 7 & 4 \\ -9 & -5 \end{bmatrix}\begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$$

$$=\begin{bmatrix} 7+42n-36n & 28n+4-24n \\ -9-54n+45n & -36n-5+30n \end{bmatrix}$$

$$=\begin{bmatrix} 7+6n & 4n+4 \\ -9n-9 & -6n-5 \end{bmatrix}$$ ✓ P_{n+1} is also true

$\therefore$ By mathematical induction, the proof of $A^n=\begin{bmatrix} 1+6n & 4n \\ -9n & 1-6n \end{bmatrix}$ is completed.

Exercise

Let $A=\begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix}$. Prove that $A^n=\begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix}$ if $n\ge 1$

Solution

Using the principle of mathematical induction.

For $n=1 \;\rightarrow\; A^1=\begin{bmatrix} (1+1)a & -a^2 \\ 1a^0 & (1-1)a \end{bmatrix}=\begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix}$ ✓ P_1 is true

Assume that P_n is true, $A^n = \begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix}$

We need to prove that P_{n+1}:

$A^{n+1} = \begin{bmatrix} (n+2)a^{n+1} & -(n+1)a^{n+2} \\ (n+1)a^n & -na^{n+1} \end{bmatrix}$ is also *true*.

$$
\begin{aligned}
A^{n+1} &= AA^n \\
&= \begin{bmatrix} 2a & -a^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix} \\
&= \begin{bmatrix} 2(n+1)a^{n+1} - na^{n+1} & -2na^{n+2} - (1-n)a^{n+2} \\ (n+1)a^n & -na^{n+1} \end{bmatrix} \\
&= \begin{bmatrix} (2n+2-n)a^{n+1} & -(2n+1-n)a^{n+2} \\ (n+1)a^n & -na^{n+1} \end{bmatrix} \\
&= \begin{bmatrix} (n+2)a^{n+1} & -(n+1)a^{n+2} \\ (n+1)a^n & -na^{n+1} \end{bmatrix}
\end{aligned}
$$
$\checkmark$ P_{n+1} is also true

$\therefore$ By mathematical induction, the proof of $A^n = \begin{bmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{bmatrix}$ is completed.

Exercise

The following system of recurrence relations holds for all $n \ge 0$

$$
\begin{cases} x_{n+1} = 7x_n + 4y_n \\ y_{n+1} = -9x_n - 5y_n \end{cases}
$$

Solve the system for x_n and y_n in terms of x_0 and y_0

Solution

$$
\begin{cases} x_{n+1} = 7x_n + 4y_n \\ y_{n+1} = -9x_n - 5y_n \end{cases}
$$

$$
\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}
$$

$$X_{n+1} = AX_n$$

$$A = \begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix} \quad X_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$X_1 = AX_0$$

$$X_2 = AX_1 = A\left(AX_0\right) = A^2 X_0$$

$$X_3 = AX_2 = A\left(A^2 X_0\right) = A^3 X_0$$

$$\vdots \qquad \vdots$$

$$X_n = A^n X_0$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix}^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Since, when $\begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix}$ that implies $A^n = \begin{pmatrix} 1+6n & 4n \\ -9n & 1-6n \end{pmatrix}$ (from previous prove).

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 1+6n & 4n \\ -9n & 1-6n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= \begin{pmatrix} (1+6n)x_0 + 4ny_0 \\ -9nx_0 + (1-6n)y_0 \end{pmatrix}$$

$$\therefore \begin{cases} x_n = (1+6n)x_0 + 4ny_0 \\ y_n = -9nx_0 + (1-6n)y_0 \end{cases}$$

Exercise

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, prove that $A^2 - (a+d)A + (ad-bc)I_{2\times 2} = 0$

Solution

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$$

$$A^2 - (a+d)A + (ad-bc)I = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} - \begin{pmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= \begin{pmatrix} a^2+bc-a^2-ad+ad-bc & ab+bd-ab-bd \\ ac+cd-ac-cd & bc+d^2-ad-d^2+ad-bc \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\underline{=0} \quad \checkmark$$

Exercise

If $A = \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}$, use the fact $A^2 = 4A - 3I$ and mathematical induction, to prove that

$$A^n = \frac{3^n - 1}{2}A + \frac{3 - 3^n}{2}I \quad if \quad n \ge 1$$

Solution

$$A^2 = \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & -12 \\ 4 & -3 \end{pmatrix}$$

$$4A - 3I = 4\begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & -12 \\ 4 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & -12 \\ 4 & -3 \end{pmatrix}$$

$$\underline{= A^2}$$

Using mathematical induction model

For $n = 1 \rightarrow A^1 = \frac{3^1 - 1}{2}A + \frac{3 - 3^1}{2}I$

$A = A + 0 = A$ $\checkmark$ is true for P_1

Assume is true for $P_k \rightarrow A^k = \frac{3^k - 1}{2}A + \frac{3 - 3^k}{2}I$

We need to prove that is also true for $P_{k+1} \rightarrow A^{k+1} = \frac{3^{k+1} - 1}{2}A + \frac{3 - 3^{k+1}}{2}I$

$A^{k+1} = AA^k$

$$= A\left(\frac{3^k-1}{2}A + \frac{3-3^k}{2}I\right)$$

$$= \frac{3^k-1}{2}A^2 + \frac{3-3^k}{2}(AI) \qquad A^2 = 4A-3I$$

$$= \frac{3^k-1}{2}(4A-3I) + \frac{3-3^k}{2}A$$

$$= 2\left(3^k-1\right)A - \frac{3\left(3^k-1\right)}{2}I + \frac{3-3^k}{2}A$$

$$= \left(2\cdot 3^k - 2 + \frac{3-3^k}{2}\right)A - \frac{3^{k+1}-3}{2}I$$

$$= \left(\frac{4\cdot 3^k - 4 + 3 - 3^k}{2}\right)A - \frac{3^{k+1}-3}{2}I$$

$$= \left(\frac{3\cdot 3^k - 1}{2}\right)A - \frac{3^{k+1}-3}{2}I$$

$$= \frac{3^{k+1}-1}{2}A + \frac{3-3^{k+1}}{2}I \qquad \checkmark \quad \text{is also true for } P_{k+1}$$

By mathematical induction, the proof that $A^n = \frac{3^n-1}{2}A + \frac{3-3^n}{2}I \quad if \quad n \ge 1$ is completed.

Exercise

A sequence of numbers $x_1, x_2, \ldots, x_n, \ldots$ satisfies the recurrence relation $x_{n+1} = ax_n + bx_{n-1}$ for $n \ge 1$, where a and b are constants. Prove that

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = A\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

Where $A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$ and hence express $\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix}$ in terms of $\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$.

If $a = 4$ and $b = -3$, use the previous question to find a formula for x_n in terms x_1 and x_0

Solution

$$x_{n+1} = ax_n + bx_{n-1}$$

$$= \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

$$x_n = x_n$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

$$= A \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$$

$$x_{n+1} = ax_n + bx_{n-1}$$

$$\underline{= 4x_n - 3x_{n-1}\,|}$$

$n = 1 \quad \to \quad x_2 = 4x_1 - 3x_0$

$n = 2 \quad \to \quad x_3 = 4x_2 - 3x_1$

$$= 4\left(4x_1 - 3x_0\right) - 3x_1 \qquad\qquad = \left(4^2 - 3\right)x_1 - 3x_0$$

$$= 13x_1 - 12x_0$$

$n = 3 \quad \to \quad x_4 = 4x_3 - 3x_2$

$$= 4\left(13x_1 - 12x_0\right) - 3\left(4x_1 - 3x_0\right)$$

$$= 40x_1 - 39x_0$$

$n = 4 \quad \to \quad x_5 = 4x_4 - 3x_3$

$$= 4\left(40x_1 - 39x_0\right) - 3\left(13x_1 - 12x_0\right)$$

$$= 121x_1 - 120x_0$$

	x_1	x_0
$n = 2 \to$	4	-3
$n = 3 \to$	13	-12
$n = 4 \to$	40	-39
$n = 5 \to$	121	-120

$$\underline{x_n = \frac{3^n - 1}{2} x_1 + \frac{3 - 3^n}{2} x_0\,|}$$