

## Section 1.8 – Applications

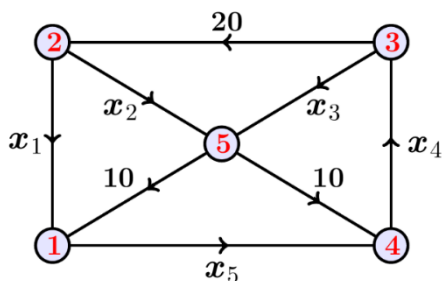
### Network Analysis

Networks composed of branches and junctions are used as models in such fields as economics, traffic analysis, and electrical engineering.

In a network model, you assume that the total flow into a junction is equal to the total flow out of the junction

#### Example

Set up a system of linear equations to represent the network shown below. Then solve the system for  $x_i$ ,  $i = 1, 2, 3, 4, 5$ .



#### Solution

$$1 \rightarrow x_1 + 10 = x_5 \Rightarrow x_1 - x_5 = -10$$

$$2 \rightarrow x_1 + x_2 = 20$$

$$3 \rightarrow x_4 = x_3 + 20 \Rightarrow -x_3 + x_4 = 20$$

$$4 \rightarrow x_4 = x_5 + 10 \Rightarrow x_4 - x_5 = 10$$

$$5 \rightarrow x_2 + x_3 = 10 + 10 = 20$$

$$\left( \begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 1 & 1 & 0 & 0 & 20 \end{array} \right) \quad R_2 - R_1$$

$$\left( \begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 1 & 1 & 0 & 0 & 20 \end{array} \right) \quad R_5 - R_2$$

$$\left( \begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 1 & 0 & -1 & -10 \end{array} \right) \quad R_5 + R_3$$

$$\left( \begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 10 \end{array} \right) \quad R_5 - R_4$$

$$\left( \begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad \begin{array}{l} \rightarrow x_1 - x_5 = -10 \quad \Rightarrow \underline{x_1 = x_5 - 10} \\ \rightarrow x_2 + x_5 = 30 \quad \Rightarrow \underline{x_2 = 30 - x_5} \\ \rightarrow -x_3 + x_4 = 20 \quad \Rightarrow \underline{x_3 = x_5 - 10} \\ \rightarrow x_4 - x_5 = 10 \quad \Rightarrow \underline{x_4 = x_5 + 10} \end{array}$$

**Solution:**  $\underline{(x_5 - 10, \ 30 - x_5, \ x_5 - 10, \ 10 + x_5, \ x_5)}$

## 2<sup>nd</sup> Method

$$\begin{aligned} \left( \begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{array} \right) &= 1 \left( \begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & -20 \\ 0 & 0 & 1 & -1 & 0 & 20 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{array} \right) - 1 \left( \begin{array}{ccccc|c} 0 & 0 & 0 & -1 & -10 & -10 \\ 0 & -1 & 1 & 0 & 0 & -20 \\ 0 & 0 & 1 & -1 & 0 & 20 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{array} \right) \\ &= 1 \left( \begin{array}{ccccc|c} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -20 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right) - 1 \left( \begin{array}{ccccc|c} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -20 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \\ &= -1 + 1 \\ &= 0 \end{aligned}$$

Infinite solution:

$$1 \rightarrow \underline{x_1 = x_5 - 10}$$

$$2 \rightarrow \underline{x_2 = 20 - x_1 = 30 - x_5}$$

$$4 \rightarrow \underline{x_4 = x_5 + 10}$$

$$3 \rightarrow \underline{x_3 = x_4 - 20 = x_5 - 10}$$

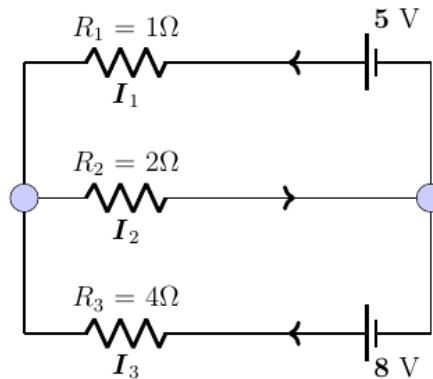
## Electrical network

An electrical network is another type of network where analysis is commonly applied. An analysis of such a system uses two properties of electrical networks known as **Kirchhoff's Laws**.

- All the current flowing into a junction must flow out of it.
- The sum of the products  $IR$  ( $I$  is current and  $R$  is resistance) around a closed path is equal to the total voltage in the path.

### Example

Determine the currents  $I_1$ ,  $I_2$ , and  $I_3$  for the electrical network



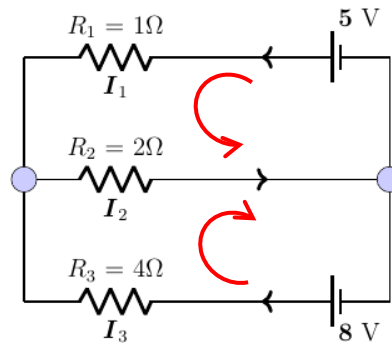
### Solution

$$I_2 = I_1 + I_3$$

$$I_1 + 2I_2 = 5$$

$$2I_2 + 4I_3 = 8$$

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ I_1 + 2I_2 = 5 \\ I_2 + 2I_3 = 4 \end{cases}$$



$$D = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 7$$

$$D_1 = \begin{vmatrix} 0 & -1 & 1 \\ 5 & 2 & 0 \\ 4 & 1 & 2 \end{vmatrix} = 7$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 5 & 0 \\ 0 & 4 & 2 \end{vmatrix} = 14$$

$$D_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 5 \\ 0 & 1 & 4 \end{vmatrix} = 7$$

$$\underline{I_1 = 1 \text{ A}} \quad \underline{I_2 = 2 \text{ A}} \quad \underline{I_3 = 1 \text{ A}}$$

## Cryptography

A **cryptogram** is a message written according to a secret code (the Greek word *kryptos* means “hidden”). One method of using matrix multiplication to **encode** and **decode** messages.

Let assign a number to each letter in the alphabet (with 0 assigned to a blank space), as shown

0 = _	4 = D	8 = H	12 = L	16 = P	20 = T	24 = X
1 = A	5 = E	9 = I	13 = M	17 = Q	21 = U	25 = Y
2 = B	6 = F	10 = J	14 = N	18 = R	22 = V	26 = Z
3 = C	7 = G	11 = K	15 = O	19 = S	23 = W	

### Example

Consider the invertible matrix:  $A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix}$

The message: **MEET ME MONDAY**

- Write the uncoded row matrices  $1 \times 3$  for the message.
- Use the matrix  $A$  to encode the message.
- Decode a message from part  $b$ ) given the matrix  $A$ .

### Solution

$a)$

$$\begin{array}{cccccccccccccc} M & E & E & T & _ & M & E & _ & M & O & N & D & A & Y & _ \\ [13 & 5 & 5] & [20 & 0 & 13] & [5 & 0 & 13] & [15 & 14 & 4] & [1 & 25 & 0] \end{array}$$

$b)$  Let encode the message **MEET ME MONDAY**

$$[13 \ 5 \ 5] \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [13 \ -26 \ 21]$$

$$[20 \ 0 \ 13] \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [33 \ -53 \ -12]$$

$$[5 \ 0 \ 13] \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [18 \ -23 \ -42]$$

$$\begin{bmatrix} 15 & 14 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 5 & -20 & 56 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 25 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} -24 & 23 & 77 \end{bmatrix}$$

The sequence of coded row matrices is

$$\begin{bmatrix} 13 & -26 & -21 \end{bmatrix} \quad \begin{bmatrix} 33 & -53 & -12 \end{bmatrix} \quad \begin{bmatrix} 18 & -23 & -42 \end{bmatrix} \quad \begin{bmatrix} 5 & -20 & 56 \end{bmatrix} \quad \begin{bmatrix} -24 & 23 & 77 \end{bmatrix}$$

The cryptogram:

$$13 \quad -26 \quad -21 \quad 33 \quad -53 \quad -12 \quad 18 \quad -23 \quad -42 \quad 5 \quad -20 \quad 56 \quad -24 \quad 23 \quad 77$$

c) To decode a message given the matrix A.

$$|A| = \begin{vmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{vmatrix} = 1$$

$$A^{-1} = \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$$

With the cryptogram:

$$\begin{bmatrix} 13 & -26 & -21 \end{bmatrix} \quad \begin{bmatrix} 33 & -53 & -12 \end{bmatrix} \quad \begin{bmatrix} 18 & -23 & -42 \end{bmatrix} \quad \begin{bmatrix} 5 & -20 & 56 \end{bmatrix} \quad \begin{bmatrix} -24 & 23 & 77 \end{bmatrix}$$

$$\begin{bmatrix} 13 & -26 & -21 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 13 & 5 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 33 & -53 & -12 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 18 & -23 & -42 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -20 & 56 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 15 & 14 & 4 \end{bmatrix}$$

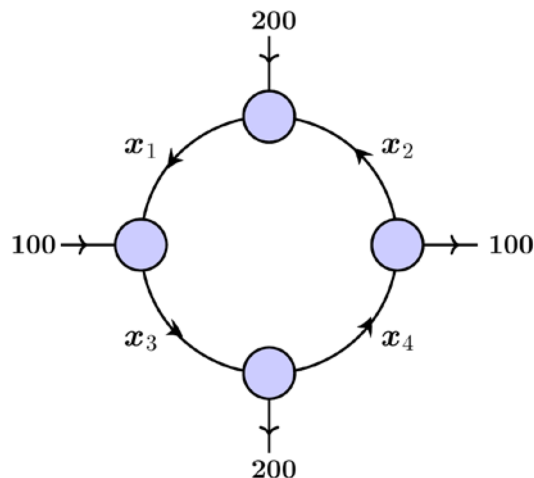
$$\begin{bmatrix} -24 & 23 & 77 \end{bmatrix} \begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 25 & 0 \end{bmatrix}$$

The message is:

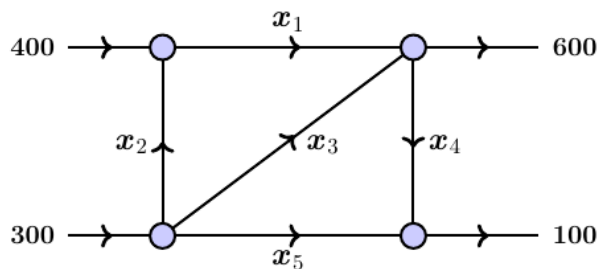
13 5 5 20 0 13 5 0 13 15 14 4 1 25 0  
*M E E T \_ M E \_ M O N D A Y \_*

## Exercises      Section 1.8 – Applications

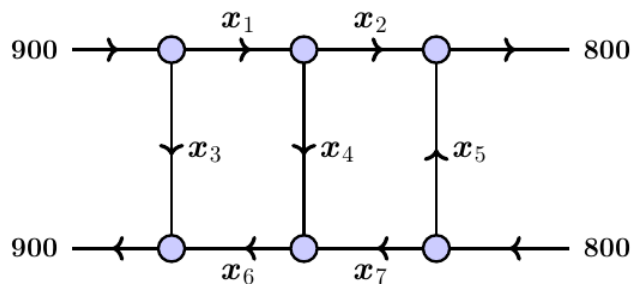
1. The flow of traffic, in vehicles per hour, through a network of streets as is shown below



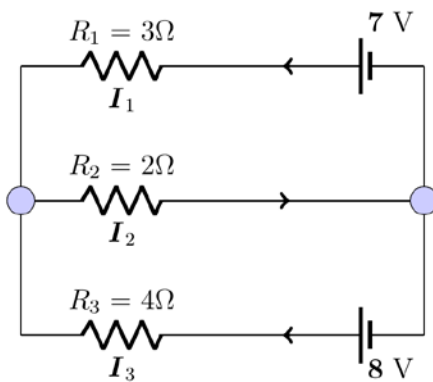
- Solve this system for  $x_i$ ,  $i = 1, 2, 3, 4$ .
  - Find the traffic flow when  $x_4 = 0$ .
  - Find the traffic flow when  $x_4 = 100$ .
  - Find the traffic flow when  $x_1 = 2x_2$ .
2. Through a network, Express  $x_n$ 's in terms of the parameters  $s$  and  $t$ .



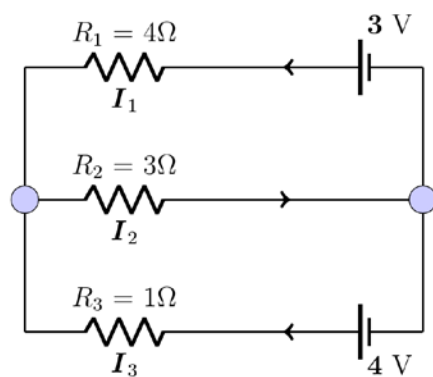
3. Water is flowing through a network of pipes. Express  $x_n$ 's in terms of the parameters  $s$  and  $t$ .



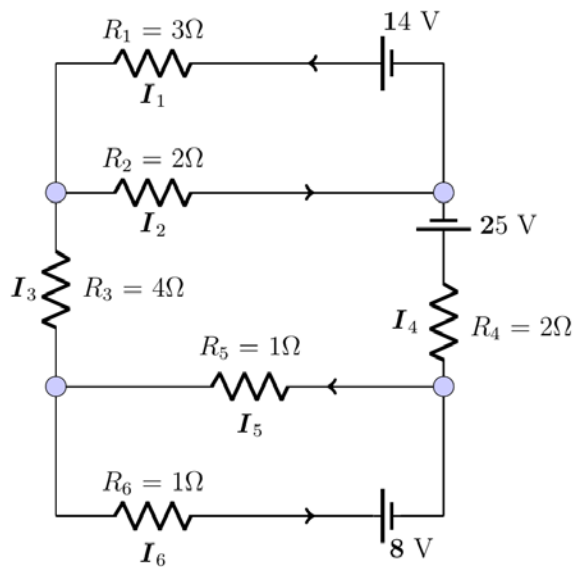
4. Determine the currents  $I_1$ ,  $I_2$ , and  $I_3$  for the electrical network shown below



5. Determine the currents  $I_1$ ,  $I_2$ , and  $I_3$  for the electrical network shown below

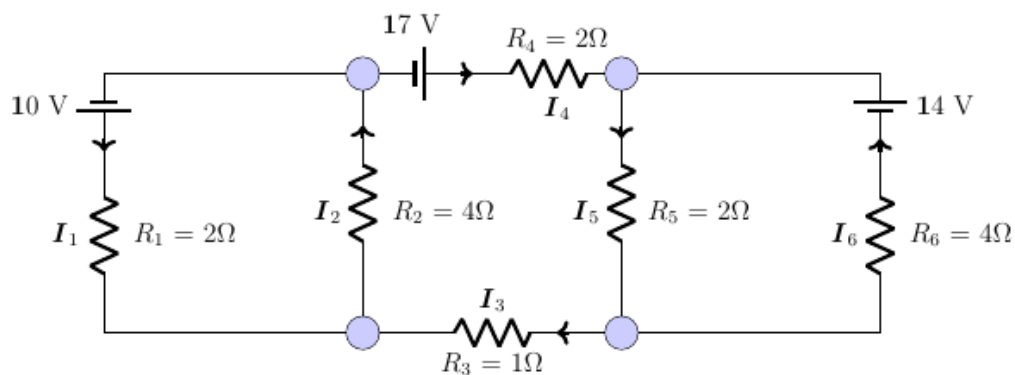


6. Determine the currents  $I_1$ ,  $I_2$ ,  $I_3$ ,  $I_4$ ,  $I_5$ , and  $I_6$  for the electrical network shown below





7. Determine the currents  $I_1$ ,  $I_2$ ,  $I_3$ ,  $I_4$ ,  $I_5$ , and  $I_6$  for the electrical network shown below



8. Consider the invertible matrix:  $A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -2 & -2 & 7 \end{pmatrix}$

The message: **ICEBERG DEAD AHEAD**

- Write the uncoded row matrices  $1 \times 3$  for the message.
  - Use the matrix  $A$  to encode the message.
  - Decode a message from part  $b$ ) given the matrix  $A$ .
9. You want to send the message: **LINEAR ALGEBRA** with a key word **MATH**
- Write the matrix  $A$ .
  - Write the uncoded row matrices  $1 \times 2$  for the message.
  - Use the matrix  $A$  to encode the message.
  - Decode a message from part  $b$ ) given the matrix  $A$ .