Section 2.2 – Trigonometric Integrals

Products of Powers of *Sines* and *Cosines*

We begin with integrals of the form

$$\int \sin^m x \cos^n x\, dx$$

Example

Evaluate $\int \sin^3 x \cos^2 x\, dx$

Solution

$$\begin{aligned}
\int \sin^3 x \cos^2 x\, dx &= \int \sin x \sin^2 x \cos^2 x\, dx \\
&= \int \left(1-\cos^2 x\right)\cos^2 x\, \left(-d\left(\cos x\right)\right) \qquad d\left(\cos x\right) = -\sin x dx \quad \Rightarrow \sin x dx = -d\left(\cos x\right) \\
&= -\int \left(\cos^2 x - \cos^4 x\right)\, d\left(\cos x\right) \qquad or \quad Assume \quad u = \cos x \\
&= -\left(\frac{1}{3}\cos^3 x - \frac{1}{5}\cos^5 x\right) + C \\
&= \boxed{\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C}
\end{aligned}$$

Example

Evaluate $\int \cos^5 x\, dx$

Solution

$$\begin{aligned}
\int \cos^5 x\, dx &= \int \cos^4 x \cos x\, dx \qquad \cos x dx = d\left(\sin x\right) \qquad \cos^2 x = 1-\sin^2 x \\
&= \int \left(1-\sin^2 x\right)^2\, d\left(\sin x\right) \\
&= \int \left(1-2\sin^2 x+\sin^4 x\right)\, d\sin x \\
&= \boxed{\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C}
\end{aligned}$$

Example

Evaluate $\displaystyle\int \sin^2 x \cos^4 x \, dx$

Solution

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx \qquad \sin^2 x = \frac{1-\cos 2x}{2} \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{1}{8}\int (1-\cos 2x)\left(1+2\cos 2x+\cos^2 2x\right) dx$$

$$= \frac{1}{8}\int \left(1+2\cos 2x+\cos^2 2x-\cos 2x-2\cos^2 2x-\cos^3 2x\right) dx$$

$$= \frac{1}{8}\int \left(1+\cos 2x-\cos^2 2x-\cos^3 2x\right) dx$$

$$= \frac{1}{8}\left[x+\frac{1}{2}\sin 2x-\int\left(\cos^3 2x+\cos^2 2x\right) dx\right]$$

$$\int \cos^3 2x dx = \int \left(1-\sin^2 2x\right)\cos 2x dx$$

$$= \frac{1}{2}\int \left(1-\sin^2 2x\right) d(\sin 2x)$$

$$= \frac{1}{2}\left(\sin 2x-\frac{1}{3}\sin^3 2x\right)$$

$$\int \cos^2 2x dx = \frac{1}{2}\int (1+\cos 4x) dx$$

$$= \frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)$$

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{8}\left[x+\frac{1}{2}\sin 2x-\frac{1}{2}\left(\sin 2x-\frac{1}{3}\sin^3 2x\right)-\frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)\right]+C$$

$$= \frac{1}{8}\left[x+\frac{1}{2}\sin 2x-\frac{1}{2}\sin 2x+\frac{1}{6}\sin^3 2x-\frac{1}{2}x-\frac{1}{8}\sin 4x\right]+C$$

$$= \frac{1}{8}\left(\frac{1}{2}x+\frac{1}{6}\sin^3 2x-\frac{1}{8}\sin 4x\right)+C$$

$$= \frac{1}{16}\left(x+\frac{1}{3}\sin^3 2x-\frac{1}{4}\sin 4x\right)+C$$

Example

Evaluate $\displaystyle\int_0^{\pi/4} \sqrt{1+\cos 4x}\ dx$

Solution

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \quad\Rightarrow\quad 1+\cos 2\theta = 2\cos^2\theta$$

$$\theta = 2x \quad\Rightarrow\quad 1+\cos 4x = 2\cos^2 2x$$

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\ dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\ dx \qquad\qquad \sqrt{2\cos^2 2x} = \sqrt{2}\sqrt{\cos^2 2x} = \sqrt{2}\,|\cos 2x|$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\ dx \qquad\qquad \cos 2x \ge 0 \quad on \quad \left[0, \tfrac{\pi}{4}\right]$$

$$= \sqrt{2}\int_0^{\pi/4} \cos 2x\ dx$$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4}$$

$$= \frac{\sqrt{2}}{2}\left[\sin\frac{\pi}{2} - 0\right]$$

$$\boxed{= \frac{\sqrt{2}}{2}}$$

Example

Evaluate $\displaystyle\int \sin^3 x\cos^{-2} x\ dx$

Solution

$$\int \sin^3 x\cos^{-2} x\ dx = \int \sin^2 x\cos^{-2} x\ \sin x\ dx$$

$$= -\int \left(1-\cos^2 x\right)\cos^{-2} x\ \ d(\cos x)$$

$$= -\int \left(\cos^{-2} x - 1\right)\ d(\cos x)$$

$$= -\left(-\cos^{-1} x - \cos x\right) + C$$

$$\boxed{= \cos x + \sec x + C}$$

Products of Powers of *tan x* and *sec x*

Example

Evaluate $\displaystyle\int \tan^4 x\, dx$

Solution

$$\int \tan^4 x dx = \int \tan^2 x \cdot \tan^2 x\, dx \qquad\qquad \tan^2 x = \sec^2 x - 1$$

$$= \int \tan^2 x \cdot \left(\sec^2 x - 1\right)\, dx$$

$$= \int \tan^2 x \sec^2 x dx - \int \tan^2 x\, dx$$

$$= \int \tan^2 x \sec^2 x dx - \int \left(\sec^2 x - 1\right)\, dx$$

$$= \int \tan^2 x \sec^2 x dx - \int \sec^2 x\, dx + \int dx$$

$$= \int \tan^2 x\, d(\tan x) - \int \sec^2 x\, dx + \int dx \qquad\qquad d(\tan x) = \sec^2 x\, dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x + C$$

Example

Evaluate $\displaystyle\int \sec^3 x\, dx$

Solution

Let: $u = \sec x \qquad dv = \sec^2 x dx$

$du = \sec x \tan x dx \qquad v = \tan x$

$$\int \sec^3 x dx = \sec x \tan x - \int \tan x (\sec x \tan x dx)$$

$$= \sec x \tan x - \int \tan^2 x \sec x dx$$

$$= \sec x \tan x - \int \left(\sec^2 x - 1\right) \sec x\, dx$$

$$= \sec x \tan x - \int \left(\sec^3 x - \sec x\right)\, dx$$

$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$\int \sec^3 x\, dx + \int \sec^3 x\, dx = \sec x \tan x - \int \sec^3 x\, dx + \int \sec^3 x\, dx + \int \sec x\, dx$$

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

$$= \sec x \tan x + \ln\left|\sec x + \tan x\right| + C_1$$

$$\int \sec^3 x\, dx = \boxed{\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C}$$

Products of Sines and Cosines

Recall the identities

$$\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$$

$$\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$$

$$\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$$

Example

Evaluate $\int \sin 3x \cos 5x dx$

Solution

$$\int \sin 3x \cos 5x dx = \frac{1}{2}\int \left[\sin(-2x) + \sin 8x\right] dx$$

$$= \frac{1}{2}\int \left[-\sin(2x) + \sin 8x\right] dx$$

$$= \frac{1}{2}\left(\frac{1}{2}\cos 2x - \frac{1}{8}\cos 8x\right) + C$$

$$\boxed{= \frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C}$$

Guidelines for Cosine & Sine

Case* 1** If ***m is ***odd***, we write m as $2k+1$ and use the identity $\sin^2 x = 1-\cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = \left(\sin^2 x\right)^k \sin x = \left(1-\cos^2 x\right)^k \sin x$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x dx = -d\left(\cos x\right)$

Case* 2** If ***m is ***even* and *n*** is ***odd***, in $\int \sin^m x \cos^n x dx$ we write n as $2k+1$ and use the identity $\cos^2 x = 1-\sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = \left(\cos^2 x\right)^k \cos x = \left(1-\sin^2 x\right)^k \cos x$$

Then we combine the single $\cos x$ with dx in the integral and set $\cos x dx = d\left(\sin x\right)$

Case* 3** If both ***m* and *n are ***even***, in $\int \sin^m x \cos^n x dx$, we substitute

To reduce the integrand to one in lower powers of $\cos 2x$ $\quad \int \cos ax dx = \frac{1}{a}\sin ax + C$

Guidelines for Tangent & Secant

Case* 1** When the power of the tangent is ***odd and positive.

$$\begin{aligned}\int \sec^m x \; \tan^{2k+1} x \, dx &= \int \sec^{m-1} x \left(\tan^2 x\right)^k \sec x \tan x \, dx \\ &= \int \sec^{m-1} x \left(\sec^2 x - 1\right)^k d\left(\sec x\right)\end{aligned}$$

Case* 2** When the power of the secant is ***even and positive.

$$\int \sec^{2k} x \; \tan^n x \, dx = \int \left(\sec^2 x\right)^{k-1} \tan^n x \; \sec^2 x \, dx = \int \left(1+\tan^2 x\right)^{k-1} \tan^n x \, d\left(\tan x\right)$$

***Case* 3** When there are no secant factors

$$\int \tan^n x \, dx = \int \tan^{n-2} x \left(\tan^2 x\right) dx = \int \tan^{n-2} x \left(\sec^2 x - 1\right) dx$$

***Case* 4** When there are only secant, use integration by parts.

***Case* 5** Otherwise, convert to cosines and sines.

Wallis's Formulas

1. If n is odd $(n \geq 3)$, then $\displaystyle\int_0^{\pi/2} \cos^n x\, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right) \cdots \left(\frac{n-1}{n}\right)$
2. If n is even $(n \geq 2)$, then $\displaystyle\int_0^{\pi/2} \cos^n x\, dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) \cdots \left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$

Formulas

$$\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx$$

$$\int \sec^n x\, dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx$$

$$\int \sin^n x\, dx = \frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx$$

$$\int \tan^n x\, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\, dx$$

$$\int \tan x\, dx = -\ln\left|\cos x\right| + C = \ln\left|\sec x\right| + C$$

$$\int \sec x\, dx = \ln\left|\sec x + \tan x\right| + C$$

$$\int \csc x\, dx = -\ln\left|\csc x + \cot x\right| + C$$

Exercises *Section* 2.2 – Trigonometric Integrals

Evaluate the integrals

1. $\int \sin^4 2x \cos 2x\, dx$

2. $\int \sin^5 \frac{x}{2}\, dx$

3. $\int \cos^3 2x \sin^5 2x\, dx$

4. $\int 8\cos^4 2\pi x\, dx$

5. $\int 16\sin^2 x \cos^2 x\, dx$

6. $\int \sec x \tan^2 x\, dx$

7. $\int \sec^2 x \tan^2 x\, dx$

8. $\int e^x \sec^3 e^x\, dx$

9. $\int \sec^4 x \tan^2 x\, dx$

10. $\int \sin 2x \cos 3x\, dx$

11. $\int \sin^2 \theta \cos 3\theta\, d\theta$

12. $\int \cos^3 \theta \sin 2\theta\, d\theta$

13. $\int \sin\theta \sin 2\theta \sin 3\theta\, d\theta$

14. $\int \frac{\sin^3 x}{\cos^4 x}\, dx$

15. $\int x\cos^3 x\, dx$

16. $\int \sin^4 x \cos^2 x\, dx$

17. $\int \tan^3 x \sec^4 x\, dx$

18. $\int \sin 3x \cos 7x\, dx$

19. $\int \sin^3 x \cos^4 x\, dx$

20. $\int \cos^4 x\, dx$

21. $\int \frac{\tan^3 x}{\sqrt{\sec x}}\, dx$

22. $\int \sec^4 3x \tan^3 3x\, dx$

23. $\int \frac{\sec x}{\tan^2 x}\, dx$

24. $\int \sin 5x \cos 4x\, dx$

25. $\int \sin x \cos^5 x\, dx$

26. $\int \sin^4 x \cos^3 x\, dx$

27. $\int \sin^7 2x\, \cos 2x\, dx$

28. $\int \sin^3 2x \sqrt{\cos 2x}\, dx$

29. $\int \frac{\cos^5 \theta}{\sqrt{\sin\theta}}\, d\theta$

30. $\int \sin^4 6\theta\, d\theta$

31. $\int \cos^2 3x\, dx$

32. $\int x^2 \sin^2 x\, dx$

33. $\int \sin^3 3x\, dx$

34. $\int \sin^3 x \cos^2 x\, dx$

35. $\int \cos^3 \frac{x}{3}\, dx$

36. $\int \sec^4 2x\, dx$

37. $\int \sec^4 2x\, dx$

38. $\int \sec^3 \pi x\, dx$

39. $\int \tan^6 3x\, dx$

40. $\int \tan^3 \frac{\pi x}{2}\, \sec^2 \frac{\pi x}{2}\, dx$

41. $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\, dx$

42. $\int_0^{\pi/4} \tan^4 x dx$

43. $\int_0^{\pi} 8\sin^4 y \cos^2 y\, dy$

44. $\int_0^{\pi/6} 3\cos^5 3x\, dx$

45. $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta\, d\theta$

46. $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\, dx$

47. $\displaystyle\int_0^{\pi} \sqrt{1-\cos^2\theta}\; d\theta$

48. $\displaystyle\int_0^{\pi/6} \sqrt{1+\sin x}\; dx$

49. $\displaystyle\int_{-\pi}^{\pi} \left(1-\cos^2 x\right)^{3/2} dx$

50. $\displaystyle\int_{\pi/4}^{\pi/2} \csc^4\theta\; d\theta$

51. $\displaystyle\int_{-\pi}^{\pi} \sin 3x \sin 3x\; dx$

52. $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x \cos 7x\; dx$

53. $\displaystyle\int_0^{\pi/2} \cos^{10}\theta\; d\theta$

54. $\displaystyle\int_0^{\pi/2} \cos^7 x\; dx$

55. $\displaystyle\int_0^{\pi/2} \cos^9 x\; dx$

56. $\displaystyle\int_0^{\pi/2} \sin^5 x\; dx$

57. $\displaystyle\int_0^{\pi/2} \sin^6 x\; dx$

58. $\displaystyle\int_0^{\pi/2} \sin^8 x\; dx$

59. Find the area of the region bounded by the graphs of $y = \tan x$ and $y = \sec x$ on the interval $\left[0, \frac{\pi}{4}\right]$

Find the area of the region bounded by the graphs of the equations

60. $y = \sin x, \quad y = \sin^3 x, \quad x = 0, \quad x = \frac{\pi}{2}$

61. $y = \sin^2 \pi x, \quad y = 0, \quad x = 0, \quad x = 1$

62. $y = \cos^2 x, \quad y = \sin^2 x, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$

63. $y = \cos^2 x, \quad y = \sin x \cos x, \quad x = -\frac{\pi}{2}, \quad x = \frac{\pi}{4}$

Find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis*

64. $y = \tan x, \quad y = 0, \quad x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}$

65. $y = \cos\frac{x}{2}, \quad y = \sin\frac{x}{2}, \quad x = 0, \quad x = \frac{\pi}{2}$

Find the ***volume*** of the solid generated by revolving the region bounded by the graphs of the equations about the *x-axis*, then find the ***centroid*** of the region

66. $y = \sin x, \quad y = 0, \quad x = 0, \quad x = \pi$

67. $y = \cos x, \quad y = \sin 0, \quad x = 0, \quad x = \frac{\pi}{2}$