

Solution Section 1.5 – Calculus of Vector-Valued Functions

Exercise

$\vec{r}(t)$ is the position of a particle in the xy -plane at time t . Find an equation in x and y whose is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of t .

$$\vec{r}(t) = (t+1)\hat{i} + (t^2 - 1)\hat{j}, \quad t = 1$$

Solution

$$x = t + 1, \quad y = t^2 - 1$$

$$y = (x-1)^2 - 1$$
$$= x^2 - 2x$$

$$\vec{v}(t) = \vec{r}' = \hat{i} + 2t\hat{j}$$
$$\vec{v}(t=1) = \hat{i} + 2\hat{j}$$

$$\vec{a} = \vec{v}' = 2\hat{j}$$
$$\vec{a}(t=1) = 2\hat{j}$$

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$$\vec{r}(t) = \frac{t}{t+1}\hat{i} + \frac{1}{t}\hat{j}, \quad t = -\frac{1}{2}$$

Solution

$$x = \frac{t}{t+1}, \quad y = \frac{1}{t} \rightarrow t = \frac{1}{y}$$

$$x = \frac{\frac{1}{y}}{\frac{1}{y} + 1}$$
$$= \frac{1}{1+y}$$

$$1+y = \frac{1}{x}$$
$$y = \frac{1}{x} - 1$$

$$\left(\frac{t}{t+1}\right)' = \frac{1}{(t+1)^2} \quad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

$$\vec{v}(t) = \frac{1}{(t+1)^2} \hat{i} - \frac{1}{t^2} \hat{j}$$

$$\begin{aligned} \vec{v}\left(t = -\frac{1}{2}\right) &= \frac{1}{\left(-\frac{1}{2}+1\right)^2} \hat{i} - \frac{1}{\frac{1}{4}} \hat{j} \\ &= \underline{4\hat{i} - 4\hat{j}} \end{aligned}$$

$$\left(\frac{1}{(t+1)^2}\right)' = \frac{-2}{(t+1)^3} \quad \left(\frac{1}{U^n}\right)' = -\frac{nU'}{U^{n+1}}$$

$$\vec{a} = \vec{v}' = \frac{-2}{(t+1)^3} \hat{i} + \frac{2}{t^3} \hat{j}$$

$$\begin{aligned} \vec{a}\left(t = -\frac{1}{2}\right) &= \frac{-2}{\left(-\frac{1}{2}+1\right)^3} \hat{i} + \frac{2}{\left(-\frac{1}{2}\right)^3} \hat{j} \\ &= \frac{-2}{-\frac{1}{8}} \hat{i} + \frac{2}{-\frac{1}{8}} \hat{j} \\ &= \underline{16\hat{i} - 16\hat{j}} \end{aligned}$$

Exercise

$\vec{r}(t)$ is the position of a particle in the xy -plane at time t . Find an equation in x and y whose is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of t .

$$\vec{r}(t) = e^t \hat{i} + \frac{2}{9} e^{2t} \hat{j}, \quad t = \ln 3$$

Solution

$$x = e^t, \quad y = \frac{2}{9} e^{2t} = \frac{2}{9} (e^t)^2$$

$$y = \frac{2}{9} x^2$$

$$\vec{v}(t) = e^t \hat{i} + \frac{4}{9} e^{2t} \hat{j}$$

$$\begin{aligned} \vec{v}(t = \ln 3) &= e^{\ln 3} \hat{i} + \frac{4}{9} e^{2 \ln 3} \hat{j} \\ &= 3\hat{i} + \frac{4}{9} e^{\ln 3^2} \hat{j} \\ &= \underline{3\hat{i} + 4\hat{j}} \end{aligned}$$

$$e^{\ln 3^2} = e^{\ln 9} = 9$$

$$\vec{a}(t) = e^t \hat{i} + \frac{8}{9} e^{2t} \hat{j}$$

$$\begin{aligned}\vec{a}(t = \ln 3) &= e^{\ln 3} \hat{i} + \frac{8}{9} e^{2 \ln 3} \hat{j} \\ &= 3 \hat{i} + \frac{8}{9} e^{\ln 9} \hat{j} \\ &= \underline{3 \hat{i} + 8 \hat{j}}\end{aligned}$$

Exercise

$\vec{r}(t)$ is the position of a particle in the xy -plane at time t . Find an equation in x and y whose is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of t .

$$\vec{r}(t) = (\cos 2t) \hat{i} + (3 \sin 2t) \hat{j}, \quad t = 0$$

Solution

$$x = \cos 2t, \quad y = 3 \sin 2t \rightarrow \sin 2t = \frac{y}{3}$$

$$\cos^2 2t + \sin^2 2t = 1$$

$$x^2 + \frac{y^2}{9} = 1$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -(2 \sin 2t) \hat{i} + (6 \cos 2t) \hat{j}$$

$$\begin{aligned}\vec{v}(t = 0) &= -(2 \sin 0) \hat{i} + (6 \cos 0) \hat{j} \\ &= \underline{6 \hat{j}}\end{aligned}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = -(4 \cos 2t) \hat{i} - (12 \sin 2t) \hat{j}$$

$$\begin{aligned}\vec{a}(t = 0) &= -(4 \cos 2t) \hat{i} - (12 \sin 2t) \hat{j} \\ &= \underline{-4 \hat{i}}\end{aligned}$$

Exercise

Give the position vectors of particles moving along various curves in the xy -plane. Find the particle's velocity and acceleration vectors at the stated times and sketch them as vectors on the curve

$$\text{Motion on the circle } x^2 + y^2 = 1 \quad \vec{r}(t) = (\sin t) \hat{i} + (\cos t) \hat{j}, \quad t = \frac{\pi}{4} \text{ and } \frac{\pi}{2}$$

Solution

$$\vec{v} = \frac{d\vec{r}}{dt} = (\cos t) \hat{i} - (\sin t) \hat{j}$$

$$\vec{v}\left(t = \frac{\pi}{4}\right) = \left(\cos \frac{\pi}{4}\right) \hat{i} - \left(\sin \frac{\pi}{4}\right) \hat{j}$$

$$= \frac{\sqrt{2}}{2} \hat{i} - \frac{\sqrt{2}}{2} \hat{j} \Big|$$

$$\vec{v}\left(t = \frac{\pi}{2}\right) = \left(\cos \frac{\pi}{2}\right) \hat{i} - \left(\sin \frac{\pi}{2}\right) \hat{j}$$

$$= -\hat{j} \Big|$$

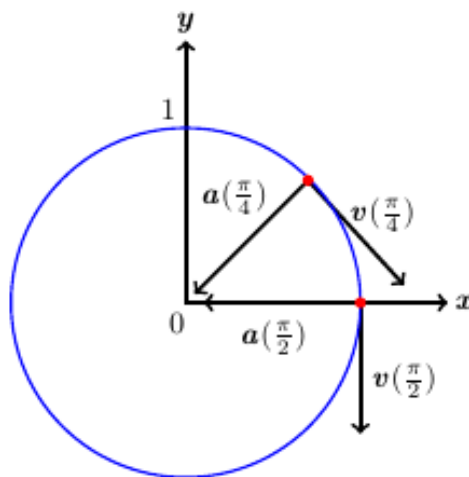
$$\vec{a} = \frac{d\vec{v}}{dt} = -(\sin t) \hat{i} - (\cos t) \hat{j}$$

$$\vec{a}\left(t = \frac{\pi}{4}\right) = -\left(\sin \frac{\pi}{4}\right) \hat{i} - \left(\cos \frac{\pi}{4}\right) \hat{j}$$

$$= -\frac{\sqrt{2}}{2} \hat{i} - \frac{\sqrt{2}}{2} \hat{j} \Big|$$

$$\vec{a}\left(t = \frac{\pi}{2}\right) = -\left(\sin \frac{\pi}{2}\right) \hat{i} - \left(\cos \frac{\pi}{2}\right) \hat{j}$$

$$= -\hat{i} \Big|$$



Exercise

Give the position vectors of particles moving along various curves in the xy -plane. Find the particle's velocity and acceleration vectors at the stated times and sketch them as vectors on the curve

Motion on the cycloid $x = t - \sin t$, $y = 1 - \cos t$

$$\vec{r}(t) = (t - \sin t) \hat{i} + (1 - \cos t) \hat{j}; \quad t = \pi \text{ and } \frac{3\pi}{2}$$

Solution

$$\vec{v} = \frac{d\vec{r}}{dt} = (1 - \cos t) \hat{i} + (\sin t) \hat{j}$$

$$\vec{v}(t = \pi) = (1 - \cos \pi) \hat{i} + (\sin \pi) \hat{j}$$

$$= 2\hat{i} \Big|$$

$$\vec{v}\left(t = \frac{3\pi}{2}\right) = \left(1 - \cos \frac{3\pi}{2}\right) \hat{i} + \left(\sin \frac{3\pi}{2}\right) \hat{j}$$

$$= \hat{i} - \hat{j} \Big|$$

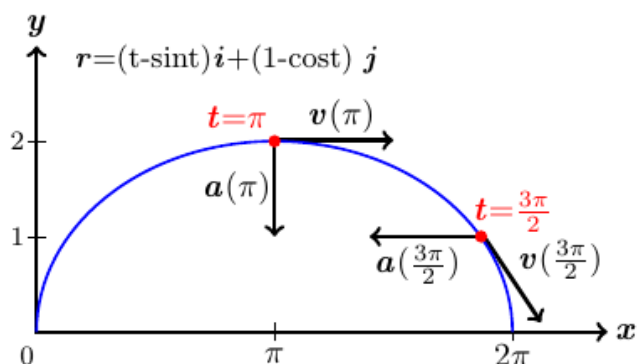
$$\vec{a} = \frac{d\vec{v}}{dt} = (\sin t) \hat{i} + (\cos t) \hat{j}$$

$$\vec{a}(t = \pi) = (\sin \pi) \hat{i} + (\cos \pi) \hat{j}$$

$$= -\hat{j} \Big|$$

$$\vec{a}\left(t = \frac{3\pi}{2}\right) = \left(\sin \frac{3\pi}{2}\right) \hat{i} + \left(\cos \frac{3\pi}{2}\right) \hat{j}$$

$$= -\hat{i} \Big|$$



Exercise

$\vec{r}(t)$ is the position of a particle in the xy -plane at time t . Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of t . Write the particle's velocity at that time as the product of its speed and direction.

$$\vec{r}(t) = (t+1)\hat{i} + (t^2 - 1)\hat{j} + 2t\hat{k}, \quad t = 1$$

Solution

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j} + 2\hat{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = 2\hat{j}$$

$$\vec{v}(t=1) = \hat{i} + 2\hat{j} + 2\hat{k}$$

$$\text{Speed: } |\vec{v}(1)| = \sqrt{1+4+4} = \underline{3}$$

$$\begin{aligned} \text{Direction: } \frac{\vec{v}(1)}{|\vec{v}(1)|} &= \frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k}) \\ &= \frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k} \end{aligned}$$

$$\underline{\vec{v}(1) = 3\left(\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right)}$$

Exercise

$\vec{r}(t)$ is the position of a particle in the xy -plane at time t . Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of t . Write the particle's velocity at that time as the product of its speed and direction.

$$\vec{r}(t) = (t+1)\hat{i} + \frac{t^2}{\sqrt{2}}\hat{j} + \frac{t^3}{3}\hat{k}, \quad t = 1$$

Solution

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i} + \frac{2}{\sqrt{2}}t\hat{j} + t^2\hat{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{2}{\sqrt{2}}\hat{j} + 2t\hat{k}$$

$$\vec{v}(t=1) = \hat{i} + \frac{2}{\sqrt{2}}\hat{j} + \hat{k}$$

$$\text{Speed: } |\vec{v}(1)| = \sqrt{1+2+1} = \underline{2}$$

$$\text{Direction: } \frac{\vec{v}(1)}{|\vec{v}(1)|} = \frac{1}{2}\left(\hat{i} + \frac{2}{\sqrt{2}}\hat{j} + \hat{k}\right)$$

$$= \frac{1}{2} \hat{i} + \frac{1}{\sqrt{2}} \hat{j} + \frac{1}{2} \hat{k}$$

$$\underline{\vec{v}(1) = 2 \left(\frac{1}{2} \hat{i} + \frac{1}{\sqrt{2}} \hat{j} + \frac{1}{2} \hat{k} \right)}$$

Exercise

$\vec{r}(t)$ is the position of a particle in the xy -plane at time t . Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of t . Write the particle's velocity at that time as the product of its speed and direction.

$$\vec{r}(t) = (2 \cos t) \hat{i} + (3 \sin t) \hat{j} + 4t \hat{k}, \quad t = \frac{\pi}{2}$$

Solution

$$\vec{v} = \frac{d\vec{r}}{dt} = -(2 \sin t) \hat{i} + (3 \cos t) \hat{j} + 4 \hat{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = -(2 \cos t) \hat{i} + (3 \sin t) \hat{j}$$

$$\begin{aligned} \vec{v}\left(t = \frac{\pi}{2}\right) &= -\left(2 \sin \frac{\pi}{2}\right) \hat{i} + \left(3 \cos \frac{\pi}{2}\right) \hat{j} + 4 \hat{k} \\ &= -2 \hat{i} + 4 \hat{k} \end{aligned}$$

$$\text{Speed: } \left| \vec{v}\left(\frac{\pi}{2}\right) \right| = \sqrt{4 + 16} = \underline{2\sqrt{5}}$$

$$\begin{aligned} \text{Direction: } \frac{\vec{v}\left(\frac{\pi}{2}\right)}{\left| \vec{v}\left(\frac{\pi}{2}\right) \right|} &= \frac{1}{2\sqrt{5}} (-2 \hat{i} + 4 \hat{k}) \\ &= -\frac{1}{\sqrt{5}} \hat{i} + \frac{2}{\sqrt{5}} \hat{k} \end{aligned}$$

$$\underline{\vec{v}\left(\frac{\pi}{2}\right) = 2\sqrt{5} \left(-\frac{1}{\sqrt{5}} \hat{i} + \frac{2}{\sqrt{5}} \hat{k} \right)}$$

Exercise

$\vec{r}(t)$ is the position of a particle in the xy -plane at time t . Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of t . Write the particle's velocity at that time as the product of its speed and direction.

$$\vec{r}(t) = (2 \ln(t+1)) \hat{i} + t^2 \hat{j} + \frac{t^2}{2} \hat{k}, \quad t = 1$$

Solution

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{2}{t+1} \hat{i} + 2t \hat{j} + t \hat{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{-2}{(t+1)^2} \hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{v}(t=1) = \hat{i} + 2\hat{j} + \hat{k}$$

$$\text{Speed: } |\vec{v}(1)| = \sqrt{1+4+1} = \underline{\underline{\sqrt{6}}}$$

$$\begin{aligned} \text{Direction: } \frac{\vec{v}(1)}{|\vec{v}(1)|} &= \frac{1}{\sqrt{6}}(\hat{i} + 2\hat{j} + \hat{k}) \\ &= \frac{1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} + \frac{1}{\sqrt{6}}\hat{k} \end{aligned}$$

$$\underline{\underline{\vec{v}(1) = \sqrt{6} \left(\frac{1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} + \frac{1}{\sqrt{6}}\hat{k} \right)}}$$

Exercise

$\vec{r}(t)$ is the position of a particle in the xy -plane at time t . Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of t . Write the particle's velocity at that time as the product of its speed and direction.

$$\vec{r}(t) = (e^{-t})\hat{i} + (2 \cos 3t)\hat{j} + (2 \sin 3t)\hat{k}, \quad t = 0$$

Solution

$$\vec{v} = \frac{d\vec{r}}{dt} = -e^{-t}\hat{i} - 6 \sin 3t \hat{j} + 6 \cos 3t \hat{k} \quad \quad \quad \mathbf{v}(0) = -\mathbf{i} + 6\mathbf{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = e^{-t}\hat{i} - 18 \cos 3t \hat{j} - 18 \sin 3t \hat{k}$$

$$\text{Speed: } |\vec{v}(0)| = 1 + 36 = \underline{\underline{\sqrt{37}}}$$

$$\begin{aligned} \text{Direction: } \frac{\vec{v}(0)}{|\vec{v}(0)|} &= \frac{1}{\sqrt{37}}(-\hat{i} + 6\hat{k}) \\ &= -\frac{1}{\sqrt{37}}\hat{i} + \frac{6}{\sqrt{37}}\hat{k} \end{aligned}$$

$$\underline{\underline{\vec{v}(0) = \sqrt{37} \left(-\frac{1}{\sqrt{37}}\hat{i} + \frac{6}{\sqrt{37}}\hat{k} \right)}}$$

Exercise

Find all points on the ellipse $\vec{r}(t) = \langle 1, 8 \sin t, \cos t \rangle$, for $0 \leq t \leq 2\pi$, at which $\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal.

Solution

$$\vec{r}'(t) = \langle 0, 8 \cos t, -\sin t \rangle$$

$\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal that implies to $\vec{r}(t) \cdot \vec{r}'(t) = 0$

$$\begin{aligned}\vec{r}(t) \cdot \vec{r}'(t) &= \langle 1, 8 \sin t, \cos t \rangle \cdot \langle 0, 8 \cos t, -\sin t \rangle \\ &= 64 \sin t \cos t - \cos t \sin t \\ &= 63 \sin t \cos t = 0\end{aligned}$$

$$\rightarrow \begin{cases} \sin t = 0 & \Rightarrow t = 0, \pi, 2\pi \\ \cos t = 0 & \Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2} \end{cases}$$

$$t = 0, 2\pi \rightarrow \underline{(1, 0, 1)} \mid$$

$$t = \frac{\pi}{2} \rightarrow \underline{(1, 8, 0)} \mid$$

$$t = \pi \rightarrow \underline{(1, 0, -1)} \mid$$

$$t = \frac{3\pi}{2} \rightarrow \underline{(1, -8, 0)} \mid$$