Section 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices

Transpose

Definition

The transpose of a matrix A is defined as the matrix that is obtained by interchanging the corresponding rows and columns in A. Then the transpose of A, denoted by A^T or A'.

The columns of A^T are the rows of A.

When A is an m by n matrix, the transpose is n by m:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \quad \textit{then} \quad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

The matrix ***flips over*** the main diagonal. The entry in row i, column j of A^T comes from row j, column i of the original A.

$$\left(A^T\right)_{ij} = A_{ji}$$

Properties of Transpose

a) $\left(A^T\right)^T = A$

b) $(A+B)^T = A^T + B^T$

c) $(A-B)^T = A^T - B^T$

d) $(kA)^T = kA^T$

e) $(AB)^T = B^T A^T$

The transpose of a product of any number of matrices is the product of the transposes in the reverse order.

Theorem

If A is an invertible matrix, then A^T is also invertible and

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

Proof

$$A^T\left(A^{-1}\right)^T=\left(A^{-1}A\right)^T$$
$$=I^T$$
$$=I$$

$$\left(A^{-1}\right)^T A^T=\left(AA^{-1}\right)^T$$
$$=I^T$$
$$=I$$

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$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad and \quad A^T=\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\left(A^T\right)^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$
$$=\begin{bmatrix} \frac{d}{ad-bc} & -\frac{c}{ad-bc} \\ -\frac{b}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

Trace

Definition

If A is a square matrix, then the trace of A, denoted by **tr(*A*)**, is defined to the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

Example

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$tr(A)=a_{11}+a_{22}+a_{33}$$

Diagonal

A square matrix in which all the entries off the main diagonal are zero is called a ***diagonal matrix***. A general $n \times n$ diagonal matrix can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

A diagonal matrix is invertible iff all of its diagonal entries are nonzero; the

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$$

Powers of diagonal matrices are:

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

Triangular Matrices

A square matrix in which all the entries above the main diagonal are zero is called ***lower diagonal triangular***.

A square matrix in which all the entries below the main diagonal are zero is called ***upper diagonal triangular***.

A matrix that is either upper triangular or lower triangular is called ***triangular***.

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ a_{51} & a_{51} & a_{53} & a_{54} & a_{55} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix}$$

lower diagonal triangular ***upper diagonal triangular***

Theorem

- ✓ The transpose of a lower triangular matrix is upper triangular, and the transpose of a upper triangular matrix is lower triangular.
- ✓ The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.
- ✓ A triangular matrix is invertible iff its diagonal entries are all nonzero.
- ✓ The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

Example

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \qquad AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix} \qquad BA = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

The goal is to describe Gaussian elimination in the most useful way by looking at them closely, which are factorizations of a matrix.

The factors are triangular matrices.

The factorization that comes from elimination is $A = LU$ **.**

Symmetric Matrices

Definition

A square matrix A is said to be ***symmetric*** if $A^T = A$. That means a square matrix must satisfies $a_{ij} = a_{ji}$

Example

$$A = \begin{pmatrix} 1 & -4 \\ -4 & 1 \end{pmatrix} = A^T$$

$$A = \begin{pmatrix} 6 & 5 & 1 \\ 5 & 0 & 7 \\ 1 & 7 & -1 \end{pmatrix} = A^T$$

- The ***inverse*** of a symmetric matrix is also ***symmetric***.

Example

Given $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, show that the inverse is symmetric too?

Solution

$$A^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

Theorem

If A and B are symmetric matrices with the same size, and if k is any scalar, then:

a) A^T is symmetric

b) $A + B$ and $A - B$ are symmetric.

c) kA is symmetric

- If A is an invertible symmetric matrix, then A^{-1} is *symmetric*.

Proof

Assume that A is symmetric and invertible then $A = A^T$

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1} = A^{-1}$$

Which proves that A^{-1} is *symmetric*

- Multiplying M by M^T gives a symmetric matrix.

Proof

The entry (i, j) of $M^T M$, it is the dot product of ***row*** i of M^T (column i of M) with column j of M.

The (i, j) entry is the same dot product, column j with column i. so $M^T M$ is symmetric.

The matrix $M.M^T$ is also symmetric and $M^T M$ is a different matrix from $M.M^T$.

- If A is an invertible symmetric matrix, then AA^T and $A^T A$ are also invertible.
- Matrix A is symmetric across its main diagonal. So is A^{-1}
- Matrix A is tridiagonal (only three nonzero diagonals). But A^{-1} is a full matrix with no zeros.

  (another reason we don't compute A^{-1})

Example

Given $M = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $M^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $M^T M$ and $M.M^T$

Solution

$$M^T M = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$MM^T = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \end{bmatrix}$$

Symmetric in **LDU**

When elimination is applied to a symmetric matrix, $A^T = A$ is an advantage.

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= \underset{L}{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}$$

- If $A = A^T$ can be factored into LDU with no row exchanges, then $U = L^T$. The ***symmetric factorization of a symmetric matrix is*** $A = LDL^T$

Exercises *Section* 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices

1. Solve $Lc = b$ to find $\boldsymbol{c}$. Then solve $Ux = c$ to find $\boldsymbol{x}$. What was A?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

2. Find L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots

3. Determine whether the given matrix is invertible

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

4. Find $A^2,\ A^{-2},\ and\ A^{-k}$ by inspection

$a)\ A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ $\qquad b)\ A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$ $\qquad c)\ A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

5. Decide whether the given matrix is symmetric

$a)\ \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ $\qquad b)\ \begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 1 \\ 3 & 1 & 7 \end{bmatrix}$ $\qquad c)\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

6. Find all values of the unknown constant(s) in order for A to be symmetric

$$A = \begin{bmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 7 \end{bmatrix}$$

7. Find a diagonal matrix A that satisfies the given condition $A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. Let A be an $n \times n$ symmetric matrix

a) Show that A^2 is symmetric

b) Show that $2A^2 - 3A + I$ is symmetric

9. Prove if $A^T A = A$, then A is symmetric and $A = A^2$

10. A square matrix A is called ***skew-symmetric*** if $A^T = -A$. Prove

a) If A is an invertible skew-symmetric matrix, then A^{-1} is skew-symmetric.

b) If A and B are skew-symmetric matrices, then so are A^T, $A + B$, $A - B$, *and* kA for any scalar k.

c) Every square matrix A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

$\left[\textit{Hint}: \text{ Note the identity } A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right) \right]$

11. Suppose R is rectangular (m by n) and A is symmetric (m by m)

a) Transpose $R^T AR$ to show its symmetric

b) Show why $R^T R$ has no negative numbers on its diagonal.

12. If L is a lower-triangular matrix, then $\left(L^{-1}\right)^T$ is _______Triangular

13. True or False

a) The block matrix $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is automatically symmetric

b) If A and B are symmetric then their product is symmetric

c) If A is not symmetric then A^{-1} is not symmetric

d) When A, B, C are symmetric, the transpose of ABC is CBA.

e) The transpose of a diagonal matrix is a diagonal.

f) The transpose of an upper triangular matrix is an upper triangular matrix.

g) The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.

h) All entries of a symmetric matrix are determined by the entries occurring on and above the main diagonal.

i) All entries of an upper triangular matrix are determined by the entries occurring on and above the main diagonal.

j) The inverse of an invertible lower triangular matrix is an upper triangular matrix.
k) A diagonal matrix is invertible if and only if all of its diagonal entries are positive.
l) The sum of a diagonal matrix and a lower triangular matrix is a lower triangular matrix.
m) A matrix that is both symmetric and upper triangular must be a diagonal matrix.
n) If A and B are $n \times n$ matrices such that $A + B$ is symmetric, then A and B are symmetric.
o) If A and B are $n \times n$ matrices such that $A + B$ is upper triangular, then A and B are upper triangular.
p) If A^2 is a symmetric matrix, then A is a symmetric matrix.
q) If kA is a symmetric matrix for some $k \neq 0$, then A is a symmetric matrix.

14. Find 2 by 2 symmetric matrices $A = A^T$ with these properties

a) A is not invertible

b) A is invertible but cannot be factored into LU (row exchanges needed)

c) A can be factored into LDL^T but not into LL^T (because of negative D)

15. A group of matrices includes AB and A^{-1} if it includes A and B. "Products and inverses stay in the group." Which of these sets are groups?

Lower triangular matrices L with 1's on the diagonal, symmetric matrices S, positive matrices M, diagonal invertible matrices D, permutation matrices P, matrices with $Q^T = Q^{-1}$. ***Invent two more matrix groups***.

16. Write $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$ as the product EH of an elementary row operation matrix E and a symmetric matrix H.

17. When is the product of two symmetric matrices symmetric? Explain your answer.

18. Express $\left((AB)^{-1}\right)^T$ in terms of $\left(A^{-1}\right)^T$ and $\left(B^{-1}\right)^T$

19. Find the transpose of the given matrix: $\begin{bmatrix} 8 & -1 \\ 3 & 5 \\ -2 & 5 \\ 1 & 2 \\ -3 & -5 \end{bmatrix}$

20. Show that if A is symmetric and invertible, then A^{-1} is also symmetric.

21. Prove that $(AB)^T = B^T A^T$

22. For the given matrix, compute A^T, $\left(A^T\right)^{-1}$, A^{-1}, and $\left(A^{-1}\right)^T$, then compare $\left(A^T\right)^{-1}$ and $\left(A^{-1}\right)^T$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

23. Show that a 2×2 lower triangular matrix is invertible if and only if $a_{11}a_{22} \neq 0$ and in this case the inverse is also lower triangular.

24. Let A be any 2×2 diagonal matrix. Give a necessary and sufficient condition on the diagonal entries so that A has an inverse. Compute the inverse of any such matrix.