Solution *Section* 4.2 – Series Solutions near Ordinary Points

Exercise

Find a power series solution. $y' = 3y$

Solution

The equation $y' = 3y$ is separable with solution

$$\frac{dy}{dx} = 3y \;\Rightarrow\; \frac{dy}{y} = 3dx \quad \to \quad \underline{y = Ce^{3x}|}$$

$$\underline{\ln(y) = 3x + C|}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y' - 3y = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - 3\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1) a_{n+1} - 3a_n\right] x^n = 0$$

$$(n+1) a_{n+1} - 3a_n = 0 \;\Rightarrow\; a_{n+1} = \frac{3a_n}{n+1}; \;\; n \ge 0$$

With $y(0) = 3a_0$

$$a_1 = 3a_0$$

$$a_2 = \frac{3}{2} a_1 = \frac{3 \cdot 3}{2} a_0$$

$$a_3 = \frac{3}{3} a_2 = \frac{3 \cdot 3 \cdot 3}{2 \cdot 3} a_0$$

$$a_4 = \frac{3}{4} a_3 = \frac{3 \cdot 3 \cdot 3 \cdot 3}{2 \cdot 3 \cdot 4} a_0$$

$$\vdots \qquad \vdots$$

$$\underline{a_n = \frac{3^n}{n!} a_0 |}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{3^n}{n!} a_0 x^n$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$

$$\boxed{= a_0 e^{3x}} \quad \checkmark$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Exercise

Find a power series solution. $y' = 4y$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y' = 4y$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = 4 \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} 4 a_n x^n$$

$$(n+1) a_{n+1} x^n = 4 a_n x^n$$

$$(n+1) a_{n+1} = 4 a_n$$

$$\boxed{a_{n+1} = \frac{4}{n+1} a_n}$$

$$n = 0 \quad \to \quad a_1 = 4 a_0$$

$$n = 1 \quad \to \quad a_2 = \frac{4}{2} a_1 = \frac{4^2}{2!} a_0$$

$$n = 2 \quad \to \quad a_3 = \frac{4}{3} a_2 = \frac{4^3}{3!} a_0$$

$$n = 3 \quad \to \quad a_4 = \frac{4}{4} a_3 = \frac{4^4}{4!} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = \frac{4^n}{n!}a_0}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{4^n}{n!}a_0 x$$

$$= a_0\left(1 + 4x + \frac{4^2}{2!}x^2 + \frac{4^3}{3!}x^3 + \ \dots\right)$$

$$\boxed{= a_0 e^{4x}}$$

Exercise

Find a power series solution. $y' = x^2 y$

Solution

$$\frac{dy}{dx} = x^2 y$$

$$\int \frac{dy}{y} = \int x^2 dx$$

$$\ln y = \frac{1}{3}x^3 + C_1$$

$$y = e^{\frac{1}{3}x^3 + C_1}$$

$$y = Ce^{\frac{1}{3}x^3}$$

$$y(0) = C(1) = a_0 \quad \rightarrow \quad C = a_0$$

$$\boxed{y = a_0 e^{x^3/3}}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$$

$$y' - x^2 y = 0$$

$$\sum_{n=1}^{\infty} na_n x^{n-1} - x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{k=-2}^{\infty} (k+3)a_{k+3}x^{k+2}$$

$$= \sum_{n=-2}^{\infty} (n+3)a_{n+3}x^{n+2}$$

$$\sum_{n=-2}^{\infty} (n+3)a_{n+3}x^{n+2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$a_1 + 2a_2 x + \sum_{n=-2}^{\infty} (n+3)a_{n+3}x^{n+2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$a_1 + 2a_2 x + \sum_{n=-2}^{\infty} \left[(n+3)a_{n+3} - a_n\right]x^{n+2} = 0$$

If we set $a_1 = a_2 = 0$, then

$$(n+3)a_{n+3} - a_n = 0 \;\Rightarrow\; a_{n+3} = \frac{a_n}{n+3}, \quad n \geq 0$$

$$a_3 = \frac{1}{3}a_0$$

$$a_4 = \frac{1}{4}a_1 = 0$$

$$a_5 = \frac{1}{5}a_2 = 0$$

$$a_6 = \frac{1}{6}a_3 = \frac{1}{3\cdot 6}a_0$$

$$a_7 = \frac{1}{7}a_4 = 0$$

$$a_9 = \frac{1}{9}a_6 = \frac{1}{3\cdot 6\cdot 9}a_0 = \frac{1}{3^3(1\cdot 2\cdot 3)}a_0$$

$$a_{12} = \frac{1}{12}a_9 = \frac{1}{3\cdot 6\cdot 9\cdot 12}a_0 = \frac{1}{3^4(1\cdot 2\cdot 3\cdot 4)}a_0$$

$$\boxed{a_{3n} = \frac{1}{3^n\cdot n!}a_0}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{3^n\cdot n!}a_0 x^{3n}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{x^3}{3}\right)^n$$

$$\boxed{= a_0 e^{x^3/3}} \quad \checkmark \qquad\qquad P = \lim_{n\to\infty}\left|\frac{3^k k!}{1}\right| \boxed{= \infty}$$

Exercise

Find a power series solution. $y' + 2xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y' + 2xy = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2 a_n x^{n+1} = 0$$

$$a_1 + \sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1} + \sum_{n=0}^{\infty} 2 a_n x^{n+1} = 0$$

$$a_1 + \sum_{n=0}^{\infty} \left[(n+2) a_{n+2} + 2a_n \right] x^{n+1} = 0$$

$$\begin{cases} \boxed{a_1 = 0} \\ (n+2) a_{n+2} + 2a_n = 0 \quad \rightarrow \quad \boxed{a_{n+2} = -\dfrac{2}{n+2} a_0} \end{cases}$$

$$n = 0 \quad \rightarrow \quad a_2 = -a_1 \qquad\qquad n = 1 \quad \rightarrow \quad a_3 = -\tfrac{2}{3} a_1 = 0$$

$$n = 2 \quad \rightarrow \quad a_4 = -\tfrac{1}{2} a_2 = \tfrac{1}{2} a_0 \qquad\qquad n = 3 \quad \rightarrow \quad a_5 = -\tfrac{2}{7} a_3 = 0$$

$$n = 4 \quad \rightarrow \quad a_6 = -\tfrac{1}{3} a_4 = -\tfrac{1}{2 \cdot 3} a_0 \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n = 6 \quad \rightarrow \quad a_8 = -\tfrac{1}{4} a_6 = \tfrac{1}{4!} a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_{2k} = \frac{(-1)^k}{k!} a_0}$$

$$y(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} a_0 x^{2k}$$

$$= a_0 \left(1 - x^2 + \tfrac{1}{2!} x^4 - \tfrac{1}{3!} x^6 + \cdots \right)$$

$$\boxed{= a_0 e^{-x^2}} \qquad\qquad P = \lim_{n \to \infty} \left| \frac{n+2}{-2} \right| \boxed{= \infty}$$

Exercise

Find a power series solution. $(x-2)y' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$(x-2)\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$x\sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n a_n x^n - 2\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[n a_n - 2(n+1)a_{n+1} + a_n \right] x^n = 0$$

$$n a_n - 2(n+1)a_{n+1} + a_n = 0$$

$$2(n+1)a_{n+1} = (n+1)a_n$$

$$\boxed{a_{n+1} = \frac{1}{2}a_n}$$

$$n = 0 \quad \to \quad a_1 = \frac{1}{2}a_0$$

$$n = 1 \quad \to \quad a_2 = \frac{1}{2}a_1 = \frac{1}{2^2}a_0$$

$$n = 2 \quad \to \quad a_3 = \frac{1}{2}a_2 = \frac{1}{2^3}a_0$$

$$n = 3 \quad \to \quad a_4 = \frac{1}{2}a_3 = \frac{1}{2^4}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = \frac{1}{2^n}a_0}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} a_0 x^n$$

$$= a_0 \left(1 + \frac{1}{2}x + \frac{1}{2^2}x^2 + \frac{1}{2^3}x^3 + \cdots \right)$$

$$= a_0 \left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \cdots \right)$$

$$= a_0 \frac{1}{1 - \frac{x}{2}}$$

$$\boxed{= \frac{2a_0}{2 - x}}$$

Exercise

Find a power series solution. $(2x-1)y' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$(2x-1)\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$0 + \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[2n a_n - (n+1) a_{n+1} + 2a_n \right] x^n = 0$$

$$2n a_n - (n+1) a_{n+1} + 2a_n = 0$$

$$(n+1) a_{n+1} = 2(n+1) a_n$$

$$\boxed{a_{n+1} = 2a_n}$$

$$n = 0 \quad \rightarrow \quad a_1 = 2a_0$$

$$n = 1 \quad \rightarrow \quad a_2 = 2a_1 = 2^2 a_0$$

$$n = 2 \quad \rightarrow \quad a_3 = 2a_2 = 2^3 a_0$$

$$n = 3 \quad \rightarrow \quad a_4 = 2a_3 = 2^4 a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n = 2^n a_0}$$

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} 2^n a_0 x^n \\
&= a_0\left(1+2x+2^2x^2+2^3x^3+\cdots\right) \\
&= a_0\left(1+2x+(2x)^2+(2x)^3+\cdots\right) \\
&= \boxed{\frac{a_0}{1-2x}}
\end{aligned}$$

Exercise

Find a power series solution. $2(x-1)y' = 3y$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$(2x-2)\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} 3a_n x^n$$

$$2x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 3a_n x^n$$

$$0 + \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 3a_n x^n$$

$$\sum_{n=0}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} 3a_n x^n$$

$$\sum_{n=0}^{\infty}\left[2n a_n - 2(n+1)a_{n+1}\right]x^n = \sum_{n=0}^{\infty} 3a_n x^n$$

$$2n a_n - 2(n+1)a_{n+1} = 3a_n$$

$$-2(n+1)a_{n+1} = (3-2n)a_n$$

$$\boxed{a_{n+1} = \frac{2n-3}{2n+2}a_n} \qquad\qquad \rho = \lim_{n\to\infty}\frac{2n-3}{2n+2} \boxed{=1}$$

$$n=0 \quad \rightarrow \quad a_1 = -\frac{3}{2}a_0$$

$$n=1 \quad \rightarrow \quad a_2 = -\frac{1}{4}a_1 = \frac{3}{8}a_0$$

$$n=2 \quad \rightarrow \quad a_3 = \frac{1}{6}a_2 = \frac{1}{16}a_0$$

$$n=3 \quad \rightarrow \quad a_4 = \frac{3}{8}a_3 = \frac{3}{128}a_0$$

$$\vdots \quad \vdots \qquad \vdots \qquad \vdots$$

$$\boxed{y(x) = a_0\left(1-\frac{3}{2}x+\frac{3}{8}x^2+\frac{1}{16}x^3+\frac{3}{128}x^4+\cdots\right)}$$

$$\frac{y'}{y} = \frac{3}{2}\frac{1}{x-1}$$

$$\int \frac{dy}{y} = \frac{3}{2}\int \frac{1}{x-1}dx$$

$$\ln y = \frac{3}{2}\ln(x-1)+\ln C$$

$$\ln y = \ln C(x-1)^{3/2}$$

$$\boxed{y(x) = C(x-1)^{3/2}}$$

Exercise

Find a power series solution. $(1+x)y' - y = 0$

Solution

$$(1+x)\frac{dy}{dx} = y$$

$$\frac{dy}{y} = \frac{dx}{1+x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\ln(y) = \ln(x+1)+C \quad \Rightarrow \quad \boxed{y = C(x+1)}$$

With $y(0) = C(0+1) = a_0 \quad \rightarrow \quad C = a_0$

$$\Rightarrow \quad \boxed{y = a_0(x+1)}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$(1+x)y' - y = 0$$

$$(1+x)\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=1}^{\infty}na_n x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}na_n x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left((n+1)a_{n+1}x^n + na_n x^n - a_n x^n\right) = 0$$

$$(n+1)a_{n+1} + (n-1)a_n = 0 \quad \rightarrow a_{n+1} = \frac{1-n}{n+1}a_n; \quad n \ge 0$$

$$a_1 = a_0 \qquad a_2 = 0a_1 = 0 \qquad a_3 = \frac{-1}{3}a_2 = 0$$

$$a_n = 0 \quad \text{for} \quad n \ge 2$$

$$\begin{aligned} y(x) &= a_0 + a_1 x \\ &= a_0 + a_0 x \\ &= \underline{a_0(1+x)} \end{aligned} \quad \checkmark$$

Exercise

Find a power series solution. $(2-x)y' + 2y = 0$

Solution

$$(2-x)\frac{dy}{dx} + 2y = 0$$

$$(2-x)\frac{dy}{dx} = -2y$$

$$\int \frac{dy}{y} = \int \frac{2d(2-x)}{2-x}$$

$$\ln y = 2\ln(2-x) + C_1$$

$$\ln y = \ln(2-x)^2 + C_1$$

$$\ln y = \ln C(2-x)^2$$

$$y = C(2-x)^2$$

$$y(0) = C(2-0)^2 = a_0 \quad \rightarrow \quad \boxed{C = \tfrac{1}{4}a_0}$$

$$\boxed{y = \tfrac{1}{4}a_0(2-x)^2}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$(2-x)y' + 2y = 0$$

$$(2-x)\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$2\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0 \qquad \sum_{n=1}^{\infty} n a_n x^n = 0 + \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n$$

$$2\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[2(n+1)a_{n+1} - n a_n + 2a_n \right] x^n = 0$$

$$2(n+1)a_{n+1} - (n-2)a_n = 0$$

$$2(n+1)a_{n+1} = (n-2)a_n$$

$$a_{n+1} = \frac{n-2}{2(n+1)} a_n, \qquad n \ge 0$$

$$a_1 = \frac{-2}{2}a_0 = -a_0$$

$$a_2 = \frac{-1}{4}a_1 = \frac{1}{4}a_0$$

$$a_3 = \frac{0}{6}a_0 = 0$$

$$\vdots$$

$$a_n = 0$$

$$\begin{aligned} y(x) &= a_0 - a_0 x + \tfrac{1}{4}a_0 x^2 \\ &= a_0\left(1 - x + \tfrac{1}{4}x^2\right) \\ &= \tfrac{1}{4}a_0\left(4 - 4x + x^2\right) \\ &= \boxed{\tfrac{1}{4}a_0(2-x)^2} \quad \checkmark \end{aligned}$$

Exercise

Find a power series solution. $(x-4)y' + y = 0$

Solution

$$y(x)=\sum_{n=0}^{\infty} a_n x^n \qquad y'(x)=\sum_{n=1}^{\infty} na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$(x-4)y' + y = 0$$

$$(x-4)\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}4(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=1}^{\infty}na_n x^n+\sum_{n=0}^{\infty}\left[a_n-4(n+1)a_{n+1}\right]x^n=0$$

$$\sum_{n=0}^{\infty}na_n x^n+a_0-4a_1+\sum_{n=1}^{\infty}\left[a_n-4(n+1)a_{n+1}\right]x^n=0$$

$$a_0-4a_1+\sum_{n=1}^{\infty}\left[(n+1)a_n-4(n+1)a_{n+1}\right]x^n=0$$

$$a_0-4a_1=0 \quad\rightarrow\quad \boxed{a_1=\frac{1}{4}a_0}$$

$$(n+1)a_n-4(n+1)a_{n+1}=0 \quad\rightarrow\quad a_{n+1}=\frac{1}{4}a_n$$

$$a_2=\frac{1}{4}a_1=\frac{1}{4^2}a_0$$

$$a_3=\frac{1}{4}a_2=\frac{1}{4^3}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_n=\frac{1}{4^n}a_0}$$

$$y(x)=\sum_{n=0}^{\infty}\frac{1}{4^n}a_0x^n$$

$$=a_0\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n$$

$$= a_0\left(\frac{1}{1-\frac{x}{4}}\right)$$

$$= a_0\left(\frac{4}{4-x}\right)$$

$$\boxed{= \frac{-4a_0}{x-4}} \quad \checkmark$$

Exercise

Find a power series solution. $x^2y' = y - x - 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$x^2y' = y - x - 1$$

$$x^2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = -x-1+\sum_{n=0}^{\infty}a_n x^n$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+2} = -x-1+\sum_{n=0}^{\infty}a_n x^n$$

$$\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n = -x-1+\sum_{n=0}^{\infty}a_n x^n$$

$$\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n = -x-1+a_0+a_1x+\sum_{n=2}^{\infty}a_n x^n$$

$$-x-1+a_0+a_1x = 0$$

$$a_0 + a_1x = 1 + x \quad \Rightarrow \quad a_0 = 1; \quad a_1 = 1$$

$$(n-1)a_{n-1} = a_n$$

$$a_2 = a_1 = 1$$

$$a_3 = 2a_2 = 2$$

$$a_4 = 3a_3 = 1\cdot 2\cdot 3$$

$$\vdots \quad \vdots$$

$$a_n = (n-1)!$$

$$\boxed{y(x) = 1 + x + x^2 + 2!x^3 + 3!x^4 + \cdots}$$

Exercise

Find a power series solution. $(x-3)y'+2y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n \qquad y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$(x-3)y'+2y=0$$

$$(x-3)\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+2\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}3(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=1}^{\infty}na_n x^n-\sum_{n=0}^{\infty}3(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}na_n x^n-\sum_{n=0}^{\infty}3(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[na_n-3(n+1)a_{n+1}+2a_n\right]x^n=0$$

$$-3(n+1)a_{n+1}+(n+2)a_n=0 \quad\to\quad \boxed{a_{n+1}=\frac{n+2}{3(n+1)}a_n} \qquad n=0,\ 1,\ 2,\ \cdots$$

$$a_1=\frac{2}{3}a_0 \qquad\qquad a_2=\frac{3}{3\cdot 2}a_1=\frac{3}{3^2}a_0$$

$$a_3=\frac{4}{3\cdot 3}a_2=\frac{4}{3^3}a_0 \qquad\qquad a_4=\frac{5}{3\cdot 4}a_3=\frac{5}{3^4}a_0$$

$$\vdots \quad \vdots$$

$$a_n=\frac{n+1}{3^n}a_0 \qquad n\ge 1$$

$$y(x)=\left(1+\frac{2}{3}x+\frac{3}{3^2}x^2+\frac{4}{3^3}x^3+\dots\right)=a_0\sum_{n=0}^{\infty}\frac{n+1}{3^n}x^n$$

$$\rho=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\frac{3n+3}{n+2}=3$$

$$\boxed{y(x)=\frac{1}{(3-x)^2}}$$

Exercise

Find a power series solution. $xy' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$xy' + y = 0$

$$x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_n x^n = 0$$

$$(n+1)a_n = 0 \quad \rightarrow \quad \underline{a_n = 0}$$

$$\underline{y(x) \equiv 0}$$

$\therefore$ The equation has no non-trivial power series.

Exercise

Find a power series solution. $x^3 y' - 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$x^3 y' - 2y = 0$

$$x^3 \sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} na_n x^{n+2} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=3}^{\infty} (n-2)a_{n-2} x^n - 2\left(a_0 + a_1 x + a_2 x^2\right) - \sum_{n=3}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=3}^{\infty} \left[(n-2)a_{n-2} - 2a_n\right] x^n - 2\left(a_0 + a_1 x + a_2 x^2\right) = 0$$

$$\begin{cases} a_0 = a_1 = a_2 = 0 \\ (n-2)a_{n-2} = 2a_n \end{cases} \rightarrow ka_k = 2a_{k+2} \quad (k = n-2)$$

$$\boxed{a_{k+2} = \frac{k}{2} a_k}$$

$$a_3 = \frac{1}{2} a_1 = 0$$

$$a_4 = a_2 = 0$$

$$a_n \equiv 0$$

$$\boxed{y(x) \equiv 0}$$

∴ The equation has no non-trivial power series.

Exercise

Find a power series solution. $y'' = 4y$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$y'' = 4y$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n = 4\sum_{n=0}^{\infty} a_n x^n$$

$$(n+2)(n+1)a_{n+2} = 4a_n$$

$$a_{n+2} = \frac{4}{(n+2)(n+1)}a_n$$

$n=0 \quad \rightarrow \quad a_2 = 2a_0 = \frac{4}{2}a_0 \qquad\qquad n=1 \quad \rightarrow \quad a_3 = \frac{4}{2\cdot 3}a_1$

$n=2 \quad \rightarrow \quad a_4 = \frac{4}{3\cdot 4}a_2 = \frac{4^2}{4!}a_0 \qquad\qquad n=3 \quad \rightarrow \quad a_5 = \frac{4}{4\cdot 5}a_3 = \frac{4^2}{5!}a_1$

$n=4 \quad \rightarrow \quad a_6 = \frac{4}{5\cdot 6}a_4 = \frac{4^3}{6!}a_0 \qquad\qquad n=5 \quad \rightarrow \quad a_7 = \frac{4}{6\cdot 7}a_5 = \frac{4^3}{7!}a_1$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{a_{2k} = \frac{2^{2k}}{(2k)!}a_0} \qquad\qquad \boxed{a_{2k+1} = \frac{2^{2k}}{(2k+1)!}a_1}$$

$$y(x) = a_0\left(1 + \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 + \frac{2^6}{6!}x^6 + \cdots\right) + a_1\left(x + \frac{2^3}{3!}x^3 + \frac{2^5}{5!}x^5 + \cdots\right)$$

$$= a_0\left(1 + \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 + \frac{1}{6!}(2x)^6 + \cdots\right) + a_1\left(x + \frac{1}{3!}(2x)^3 + \frac{1}{5!}(2x)^5 + \cdots\right)$$

$$\boxed{= a_0 \cosh 2x + a_1 \sinh 2x}$$

Exercise

Find a power series solution. $y'' = 9y$

Solution

The equation $y'' = 9y$ has a characteristic equation $\lambda^2 - 9 = 0 \quad \Rightarrow \quad \boxed{\lambda = \pm 3}$

$\therefore$ The general solution: $y(x) = C_1 e^{3x} + C_2 e^{-3x}$

With $y(0) = a_0 \quad$ *and* $\quad y'(0) = a_1$

$$y(0) = C_1 e^{3(0)} + C_2 e^{-3(0)} \quad \rightarrow \quad C_1 + C_2 = a_0$$

$$y'(x) = 3C_1 e^{3x} - 3C_2 e^{-3x}$$

$$y(0) = 3C_1 e^{3(0)} - 3C_2 e^{-3(0)} \quad \rightarrow \quad 3C_1 - 3C_2 = a_1$$

$$\begin{cases} C_1 + C_2 = a_0 \\ 3C_1 - 3C_2 = a_1 \end{cases} \quad \rightarrow \quad \begin{cases} 3C_1 + 3C_2 = 3a_0 \\ 3C_1 - 3C_2 = a_1 \end{cases}$$

$$6C_1 = 3a_0 + a_1 \quad \rightarrow \quad \boxed{C_1 = \frac{3a_0 + a_1}{6}}$$

$$C_2 = a_0 - C_1 \quad \rightarrow \quad \boxed{C_2 = a_0 - \frac{3a_0 + a_1}{6} = \frac{3a_0 - a_1}{6}}$$

$$\boxed{y(x)=\frac{3a_0+a_1}{6}e^{3x}+\frac{3a_0-a_1}{6}e^{-3x}}$$

$$y(x)=\sum_{n=0}^{\infty}a_n x^n \qquad y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y''-9y=0$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}-9\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-9\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-9a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}-9a_n=0$$

$$a_{n+2}=\frac{9}{(n+2)(n+1)}a_n, \quad n\ge 0$$

$$a_2=\frac{9}{(2)(1)}a_0=\frac{9}{2}a_0 \qquad a_3=\frac{9}{(3)(2)}a_1=\frac{9}{2\cdot 3}a_1$$

$$a_4=\frac{3^2}{(4)(3)}a_2=\frac{9\cdot 9}{2\cdot 3\cdot 4}a_0=\frac{3^4}{2\cdot 3\cdot 4}a_0 \qquad a_5=\frac{9}{(5)(4)}a_3=\frac{3^4}{2\cdot 3\cdot 4\cdot 5}a_1$$

$$a_6=\frac{3^2}{(6)(5)}a_4=\frac{3^6}{6!}a_0 \qquad a_7=\frac{9}{(7)(6)}a_5=\frac{3^6}{7!}a_1$$

$$a_{2n}=\frac{3^{2n}}{(2n)!}a_0 \qquad a_{2n+1}=\frac{3^{2n}}{(2n+1)!}a_1$$

$$y(x)=a_0\left(1+\frac{3^2}{2!}x^2+\frac{3^4}{4!}x^4+\frac{3^6}{6!}x^6+\cdots\right)+a_1\left(x+\frac{3^2}{3!}x^3+\frac{3^4}{5!}x^5+\frac{3^6}{7!}x^7+\cdots\right)$$

$$y(x)=\frac{3a_0+a_1}{6}e^{3x}+\frac{3a_0-a_1}{6}e^{-3x}$$

$$=\frac{3a_0+a_1}{6}\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\cdots\right]+\frac{3a_0-a_1}{6}\left[1-3x+\frac{(-3x)^2}{2!}+\frac{(-3x)^3}{3!}+\cdots\right]$$

$$=\frac{3a_0}{6}\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots\right]+\frac{a_1}{6}\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots\right]$$

$$+\frac{3a_0}{6}\left[1-3x+\frac{(3x)^2}{2!}-\frac{(3x)^3}{3!}+\cdots\right]-\frac{a_1}{6}\left[1-3x+\frac{(3x)^2}{2!}-\frac{(3x)^3}{3!}+\cdots\right]$$

$$=\frac{1}{2}a_0\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots+1-3x+\frac{(3x)^2}{2!}-\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots\right]$$

$$+\frac{a_1}{6}\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\cdots-1+3x-\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}-\frac{(3x)^4}{4!}+\cdots\right]$$

$$=\frac{1}{2}a_0\left[2+2\frac{(3x)^2}{2!}+2\frac{(3x)^4}{4!}+\cdots\right]+\frac{a_1}{6}\left[6x+2\frac{(3x)^3}{3!}+2\frac{(3x)^5}{5!}+\cdots\right]$$

$$\boxed{=a_0\left[1+\frac{(3x)^2}{2!}+\frac{(3x)^4}{4!}+\cdots\right]+a_1\left[x+\frac{3^2x^3}{3!}+\frac{3^4x^5}{5!}+\cdots\right]}$$

Which are identical.

Exercise

Find a power series solution. $y''+y=0$

Solution

The equation $y''+y=0$ has a characteristic equation $\lambda^2+1=0 \quad\Rightarrow\quad \boxed{\lambda=\pm i}$

$\therefore$ The general solution: $y(x)=C_1\sin x+C_2\cos x$

With $y(0)=a_0$ *and* $y'(0)=a_1$

$$y(0)=C_1\sin(0)+C_2\cos(0)\quad\to\quad C_2=a_0$$

$$y'(x)=C_1\cos x-C_2\sin x$$

$$y(0)=C_1\cos(0)-C_2\sin(0)\quad\to\quad C_1=a_1$$

$$y(x)=a_1\sin x+a_0\cos x$$

$$\boxed{=a_0\cos x+a_1\sin x}$$

$$y(x)=\sum_{n=0}^{\infty}a_nx^n \qquad y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y'' + y = 0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_n\right]x^n = 0$$

$$(n+2)(n+1)a_{n+2} + a_n = 0 \quad \rightarrow \quad a_{n+2} = \frac{-1}{(n+2)(n+1)}a_n, \quad n \ge 0$$

$$a_2 = \frac{1}{(2)(1)}a_0 = -\frac{1}{2}a_0 \qquad a_3 = \frac{1}{(3)(2)}a_1 = -\frac{1}{2\cdot 3}a_1$$

$$a_4 = -\frac{1}{(4)(3)}a_2 = \frac{1}{2\cdot 3\cdot 4}a_0 \qquad a_5 = -\frac{1}{(5)(4)}a_3 = \frac{1}{2\cdot 3\cdot 4\cdot 5}a_1$$

$$a_6 = -\frac{1}{(6)(5)}a_4 = -\frac{1}{6!}a_0 \qquad a_7 = -\frac{1}{(7)(6)}a_5 = -\frac{1}{7!}a_1$$

$$a_{2n} = \frac{(-1)^n}{(2n)!}a_0 \qquad a_{2n+1} = \frac{(-1)^n}{(2n+1)!}a_1$$

$$y(x) = a_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots\right) + a_1\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots\right)$$

$$\boxed{= a_0 \cos x + a_1 \sin x} \quad \checkmark$$

Exercise

Find a power series solution. $y'' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y'' - y = 0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-a_n\right]x^n=0$$

$$\boxed{a_{n+2}=\frac{1}{(n+2)(n+1)}a_n}$$

$n=0 \quad \rightarrow \quad a_2=\frac{1}{1\cdot 2}a_0$

$n=2 \quad \rightarrow \quad a_4=\frac{1}{3\cdot 4}a_2=\frac{1}{4!}a_0$

$n=4 \quad \rightarrow \quad a_6=\frac{1}{5\cdot 6}a_4=\frac{1}{6!}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{a_{2k}=\frac{1}{(2k)!}a_0}$$

$n=1 \quad \rightarrow \quad a_3=\frac{1}{2\cdot 3}a_1$

$n=3 \quad \rightarrow \quad a_5=\frac{1}{4\cdot 5}a_3=\frac{1}{5!}a_1$

$n=5 \quad \rightarrow \quad a_7=\frac{1}{6\cdot 7}a_5=\frac{1}{7!}a_1$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{a_{2k+1}=\frac{1}{(2k+1)!}a_1}$$

$$y(x)=a_0\left(1+\frac{1}{2!}x^2+\frac{1}{4!}x^4+\frac{1}{6!}x^6+\ \cdots\right)+a_1\left(x+\frac{1}{3!}x^3+\frac{1}{5!}x^5+\ \cdots\right)$$

$$\boxed{=a_0\cosh x+a_1\sinh x}$$

Exercise

Find a power series solution. $y''+y=x$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y''+y=x$

$$\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+\sum_{n=0}^{\infty}a_nx^n=x$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n+a_0+a_1x+\sum_{n=2}^{\infty}a_nx^n-x=0$$

$$\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_n\right]x^n+a_0+2a_2+\left(6a_3+a_1-1\right)x=0$$

$$\begin{cases} a_0+2a_2=0 \\ 6a_3+a_1-1=0 \\ (n+2)(n+1)a_{n+2}+a_n=0 \end{cases}$$

$$\begin{cases} a_2=-\frac{1}{2}a_0 \\ a_3=-\frac{1}{6}\left(a_1-1\right) \\ a_{n+2}=-\frac{1}{(n+2)(n+1)}a_n \end{cases}$$

$$n=2 \rightarrow a_4=-\frac{1}{3\cdot 4}a_2=\frac{1}{4!}a_0 \qquad n=3 \rightarrow a_5=-\frac{1}{4\cdot 5}a_3=\frac{1}{5!}\left(a_1-1\right)$$

$$n=4 \rightarrow a_6=-\frac{1}{5\cdot 6}a_4=-\frac{1}{6!}a_0 \qquad n=5 \rightarrow a_7=-\frac{1}{6\cdot 7}a_5=-\frac{1}{7!}\left(a_1-1\right)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{a_{2k}=\frac{(-1)^k}{(2k)!}a_0} \qquad \boxed{a_{2k+1}=\frac{(-1)^k}{(2k+1)!}\left(a_1-1\right)}$$

$$\begin{aligned} y(x) &= a_0+a_1x+a_0\sum_{k=2}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}+\left(a_1-1\right)\sum_{k=3}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1} \\ &= a_0+a_0\left(-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots\right)+\left(a_1-1\right)\left(-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots\right)+a_1x-x+x \\ &= a_0\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots\right)+\left(a_1-1\right)\left(-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots\right)+\left(a_1-1\right)x+x \\ &= x+a_0\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots\right)+\left(a_1-1\right)\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots\right) \\ &= \boxed{x+a_0\cos x+\left(a_1-1\right)\sin x} \end{aligned}$$

Exercise

Find a power series solution. $y''-xy=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty} na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y''-xy=0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}a_n x^{n+1}=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}a_{n-1} x^{n}=0$$

$$2a_2+\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}a_{n-1} x^{n}=0$$

$$2a_2+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}-a_{n-1}\right]x^n=0$$

$$2a_2=0\rightarrow \boxed{a_2=0}$$

$$(n+2)(n+1)a_{n+2}-a_{n-1}=0 \quad\rightarrow\quad a_{n+2}=\frac{a_{n-1}}{(n+1)(n+2)}$$

$$a_3=\frac{a_0}{2\cdot 3}=\frac{1}{6}a_0 \qquad a_4=\frac{a_1}{3\cdot 4}=\frac{1}{12}a_1 \qquad a_5=\frac{a_2}{4\cdot 5}=0$$

$$a_6=\frac{a_3}{5\cdot 6}=\frac{1}{180}a_0 \qquad a_7=\frac{a_3}{6\cdot 7}=\frac{1}{504}a_1 \qquad a_8=\frac{a_5}{7\cdot 8}=0$$

$$\vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x)=\left(1+\frac{1}{6}x^3+\frac{1}{180}x^6+\cdots\right)a_0 \\ y_2(x)=\left(x+\frac{1}{12}x^4+\frac{1}{504}x^7+\cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $y''+xy=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''+xy=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}a_n x^{n+1}=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}a_{n-1}x^n=0$$

$$2a_2+\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}a_{n-1}x^n=0$$

$$2a_2+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_{n-1}\right]x^n=0$$

$$\begin{cases}2a_2=0 \;\rightarrow\; \boxed{a_2=0}\\ (n+2)(n+1)a_{n+2}+a_{n-1}=0\end{cases}$$

$$\boxed{a_{n+2}=-\frac{a_{n-1}}{(n+1)(n+2)}}$$

a_0	a_1	$a_2=0$
$n=1 \rightarrow a_3=-\frac{a_0}{2\cdot 3}=-\frac{1}{6}a_0$	$n=2 \rightarrow a_4=-\frac{1}{3\cdot 4}a_1$	$n=3 \rightarrow a_5=-\frac{a_2}{20}=0$
$n=4 \rightarrow a_6=-\frac{a_3}{5\cdot 6}=\frac{1}{180}a_0$	$n=5 \rightarrow a_7=-\frac{a_3}{6\cdot 7}=\frac{1}{504}a_1$	$n=6 \rightarrow a_8=-\frac{a_5}{56}=0$
$n=7 \rightarrow a_9=-\frac{a_6}{8\cdot 9}=-\frac{1}{12{,}960}a_0$		
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\begin{cases} y_1(x) = \left(1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \frac{1}{12,960}x^9 + \cdots\right)a_0 \\ y_2(x) = \left(x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \cdots\right)a_1 \end{cases}$$

$$\boxed{y(x) = \left(1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \frac{1}{12,960}x^9 + \cdots\right)a_0 + \left(x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \cdots\right)a_1}$$

Exercise

Find a power series solution. $y'' + xy' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + xy' + y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n+1) a_n\right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + (n+1) a_n = 0$$

$$\boxed{a_{n+2} = -\frac{a_n}{n+2}}$$

a_0 | a_1

$n = 0 \rightarrow a_2 = -\frac{1}{2} a_0$ | $n = 1 \rightarrow a_3 = -\frac{1}{3} a_1$

$$n=2 \ \rightarrow \ a_4 = -\frac{1}{4}a_2 = \frac{1}{4\cdot 2}a_0 \qquad\qquad n=3 \ \rightarrow \ a_5 = -\frac{a_3}{5} = \frac{1}{3\cdot 5}a_1$$

$$n=4 \ \rightarrow \ a_6 = -\frac{a_4}{6} = -\frac{1}{6\cdot 4\cdot 2}a_0 \qquad\qquad n=5 \ \rightarrow \ a_7 = -\frac{a_5}{7} = -\frac{1}{7\cdot 5\cdot 3}a_1$$

$$n=6 \ \rightarrow \ a_8 = -\frac{a_6}{8} = \frac{1}{8\cdot 6\cdot 4\cdot 2}a_0 \qquad\qquad n=7 \ \rightarrow \ a_9 = -\frac{a_7}{9} = \frac{1}{9\cdot 7\cdot 5\cdot 3}a_1$$

$$\vdots \quad \vdots \qquad \vdots \qquad \vdots \qquad\qquad\qquad \vdots \quad \vdots \qquad \vdots \qquad \vdots$$

$$a_{2n} = \frac{(-1)^n a_0}{(2n)(2n-2)\cdots 4\cdot 2} = \frac{(-1)^n}{n!\ 2^n}a_0 \qquad\qquad a_{2n} = \frac{(-1)^n a_1}{(2n+1)(2n-1)\cdots 5\cdot 3} = \frac{(-1)^n}{(2n+1)!!}a_1$$

$$y(x) = a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\ 2^n}x^{2n} + a_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!!}x^{2n+1}$$

$$y(x) = a_0\left(1-\frac{1}{2}x^2+\frac{1}{8}x^4-\frac{1}{48}x^6+\cdots\right)+a_1\left(x-\frac{1}{3}x^3+\frac{1}{15}x^5-\frac{1}{105}x^7+\cdots\right)$$

Exercise

Find a power series solution. $y''-xy'-y=0$

Solution

$$y(x) = \sum_{n=0}^{\infty}a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty}na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty}n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''-xy'-y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=1}^{\infty}na_n x^{n-1} - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}na_n x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}na_n x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-(n+1)a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}-(n+1)a_n=0$$

$$\boxed{a_{n+2}=\frac{a_n}{n+2}}$$

a_0

$$n=0 \;\rightarrow\; a_2=\frac{1}{2}a_0$$

$$n=2 \;\rightarrow\; a_4=\frac{1}{4}a_2=\frac{1}{4\cdot 2}a_0$$

$$n=4 \;\rightarrow\; a_6=\frac{a_4}{6}=\frac{1}{6\cdot 4\cdot 2}a_0$$

$$n=6 \;\rightarrow\; a_8=\frac{a_6}{8}=\frac{1}{8\cdot 6\cdot 4\cdot 2}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n}=\frac{a_0}{(2n)(2n-2)\cdots 4\cdot 2}=\frac{1}{n!\;2^n}a_0$$

a_1

$$n=1 \;\rightarrow\; a_3=\frac{1}{3}a_1$$

$$n=3 \;\rightarrow\; a_5=\frac{a_3}{5}=\frac{1}{3\cdot 5}a_1$$

$$n=5 \;\rightarrow\; a_7=\frac{a_5}{7}=\frac{1}{7\cdot 5\cdot 3}a_1$$

$$n=7 \;\rightarrow\; a_9=\frac{a_7}{9}=\frac{1}{9\cdot 7\cdot 5\cdot 3}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n}=\frac{a_1}{(2n+1)(2n-1)\cdots 5\cdot 3}=\frac{1}{(2n+1)!!}a_1$$

$$\boxed{y(x)=a_0\sum_{n=0}^{\infty}\frac{1}{n!\;2^n}x^{2n}+a_1\sum_{n=0}^{\infty}\frac{1}{(2n+1)!!}x^{2n+1}}$$

$$\boxed{y(x)=a_0\left(1+\frac{x^2}{2}+\frac{x^4}{2^2\cdot 2}+\cdots+\frac{x^{2n}}{2^n\cdot n!}\right)+a_1\left(x+\frac{x^3}{3}+\frac{x^5}{5\cdot 3}+\cdots+\frac{2^n\cdot n!}{(2n+1)!!}x^{2n+1}\right)}$$

Exercise

Find a power series solution. $y''+x^2y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y'' + x^2 y = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x^2\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}a_{n-2} x^{n} = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}a_{n-2} x^{n} = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_{n-2}\right]x^n = 0$$

$$2a_2 = 0 \to \underline{a_2 = 0|} \qquad 6a_3 = 0 \to \underline{a_3 = 0|}$$

$$(n+2)(n+1)a_{n+2} + a_{n-2} = 0 \quad \to \quad a_{n+2} = \frac{-a_{n-2}}{(n+1)(n+2)}$$

$$a_4 = \frac{-a_0}{3\cdot 4} = -\frac{1}{12}a_0 \qquad a_5 = \frac{-a_1}{4\cdot 5} = -\frac{1}{20}a_1 \qquad a_6 = \frac{-a_2}{5\cdot 6} = 0 \qquad a_7 = \frac{-a_3}{6\cdot 7} = 0$$

$$a_8 = \frac{-a_4}{7\cdot 8} = \frac{1}{672}a_0 \qquad a_9 = \frac{-a_5}{8\cdot 9} = \frac{1}{1,440}a_1 \qquad a_{10} = \frac{-a_6}{9\cdot 10} = 0 \qquad a_{11} = \frac{a_7}{10\cdot 11} = 0$$

$$\vdots \quad \vdots \qquad \vdots \quad \vdots \qquad \vdots \quad \vdots \qquad \vdots \quad \vdots$$

$$\underline{\begin{cases} y_1(x) = \left(1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - \cdots\right)a_0 \\ y_2(x) = \left(x - \frac{1}{20}x^5 + \frac{1}{1,440}x^9 + \cdots\right)a_1 \end{cases}}\Bigg|$$

Exercise

Find a power series solution. $y'' + k^2 x^2 y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty}a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty}n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''+k^2x^2y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+k^2x^2\sum_{n=0}^{\infty}a_n x^n=0$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}k^2a_{n-2}x^n=0$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}+k^2a_{n-2}\right]x^n=0$$

$$\begin{cases} a_2=0 \\ a_3=0 \\ (n+2)(n+1)a_{n+2}+k^2a_{n-2}=0 \end{cases}$$

$$\boxed{a_{n+2}=-\frac{k^2}{(n+1)(n+2)}a_{n-2}} \quad (n\geq 2)$$

$n=2 \to a_4=-\frac{k^2}{3\cdot 4}a_0$

$n=6 \to a_8=-\frac{k^2}{7\cdot 8}a_4=\frac{k^4}{3\cdot 4\cdot 7\cdot 8}a_0$

$n=10 \to a_{12}=-\frac{k^2}{11\cdot 12}a_8=\frac{k^6}{3\cdot 4\cdot 7\cdot 8\cdot 11\cdot 12}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{4m}=-\frac{k^2}{(4m)(4m-1)}a_{4m-4}$$

$n=3 \to a_5=-\frac{k^2}{4\cdot 5}a_1$

$n=7 \to a_9=-\frac{k^2}{8\cdot 9}a_5=\frac{k^4}{4\cdot 5\cdot 8\cdot 9}a_1$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{4m+1}=-\frac{k^2}{(4m)(4m+1)}a_{4m-3}$$

$n=4 \to a_6=-\frac{k^2}{5\cdot 6}a_2=0$

$n=8 \to a_{10}=-\frac{k^2}{7\cdot 8}a_6=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$n=5 \to a_7=-\frac{k^2}{6\cdot 7}a_3=0$

$n=9 \to a_{11}=-\frac{k^2}{10\cdot 11}a_7=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\begin{aligned} y(x)&=a_0\left(1-\frac{k^2}{3\cdot 4}x^4+-\frac{k^4}{3\cdot 4\cdot 7\cdot 8}x^8-\frac{k^6}{3\cdot 4\cdot 7\cdot 8\cdot 11\cdot 12}x^{12}+\cdots\right) \\ &\quad +a_1\left(x-\frac{k^2}{4\cdot 5}x^5+-\frac{k^4}{4\cdot 5\cdot 8\cdot 9}x^9-\frac{k^6}{4\cdot 5\cdot 8\cdot 9\cdot 12\cdot 13}x^{13}+\cdots\right) \end{aligned}$$

Exercise

Find a power series solution. $y'' + 3xy' + 3y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$y'' + 3xy' + 3y = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 3x \sum_{n=1}^{\infty} n a_n x^{n-1} + 3 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (3n+3) a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + 3(n+1) a_n = 0$$

$$\boxed{a_{n+2} = -\frac{3}{n+2} a_n}$$

a_0	a_1
$n=2 \to a_2 = -\frac{3}{2} a_0$	$n=3 \to a_3 = -\frac{3}{3} a_1$
$n=4 \to a_4 = -\frac{3}{4} a_2 = \frac{3^2}{2^2 \cdot 2} a_0$	$n=5 \to a_5 = -\frac{3}{5} a_3 = \frac{3^2}{3 \cdot 5} a_1$
$n=6 \to a_6 = -\frac{3}{6} a_3 = \frac{3^3}{2^3 \cdot 2 \cdot 3} a_0$	$n=7 \to a_7 = -\frac{3}{7} a_5 = \frac{3^3}{3 \cdot 5 \cdot 7} a_1$
$n=8 \to a_8 = -\frac{3}{8} a_6 = \frac{3^4}{2^4 \cdot 2 \cdot 3 \cdot 4} a_0$	$n=9 \to a_9 = -\frac{3}{9} a_7 = \frac{3^3}{3 \cdot 5 \cdot 7 \cdot 9} a_1$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$a_{2k} = \frac{(-3)^k}{2^k k!} a_0$	$a_{2k+1} = \frac{(-3)^k}{3 \cdot 5 \cdot 7 \cdots (2k+1)} a_1$

$$y(x) = a_0\left(1 + \sum_{k=1}^{\infty}\frac{(-3)^k}{2^k k!}x^{2k}\right) + a_1\left(x + \sum_{k=1}^{\infty}\frac{(-3)^k}{3\cdot 5\cdot 7\cdots(2k+1)}x^{2k+1}\right)$$

$$\boxed{= a_0\left(1 - \frac{3}{2}x^2 + \frac{9}{8}x^4 - \frac{27}{56}x^6 + \cdots\right) + a_1\left(x - x^3 + \frac{3^2}{3\cdot 5}x^5 - \frac{27}{3\cdot 5\cdot 7}x^7 + \cdots\right)}$$

Exercise

Find a power series solution. $y'' - 2xy' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y'' - 2xy' + y = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - 2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}2na_n x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}2na_n x^n + a_0 + \sum_{n=1}^{\infty}a_n x^n = 0$$

$$2a_2 + a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - 2na_n + a_n\right]x^n = 0$$

$$2a_2 + a_0 = 0 \ \rightarrow\ \boxed{a_2 = -\frac{1}{2}a_0}$$

$$(n+2)(n+1)a_{n+2} - (2n-1)a_n = 0 \ \rightarrow\ a_{n+2} = \frac{(2n-1)a_n}{(n+1)(n+2)}$$

$$a_3 = \frac{a_1}{2\cdot 3} = \frac{1}{6}a_1 \qquad\qquad a_4 = \frac{3a_2}{3\cdot 4} = -\frac{1}{4}\frac{1}{4}a_0 = -\frac{1}{8}a_0$$

$$a_5 = \frac{5a_3}{4\cdot 5} = \frac{1}{2\cdot 3\cdot 4}a_1 = \frac{1}{4!}a_1 \qquad\qquad a_6 = \frac{7a_4}{5\cdot 6} = -\frac{7}{240}a_0$$

$$\vdots \quad \vdots$$

$$a_7 = \frac{9a_5}{6\cdot 7} = \frac{1}{112}a_1$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6 - \cdots\right)a_0 \\ y_2(x) = \left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{112}x^7 + \cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $y'' - xy' + 2y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' + 2y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + 2a_0 + \sum_{n=1}^{\infty} 2a_n x^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n + 2a_n\right] x^n = 0$$

$$2a_2 + 2a_0 = 0 \ \rightarrow\ \boxed{a_2 = -a_0}$$

$$(n+2)(n+1) a_{n+2} - (n-2) a_n = 0 \ \rightarrow\ a_{n+2} = \frac{(n-2)a_n}{(n+1)(n+2)}$$

$$a_3 = \frac{-a_1}{2\cdot 3} = -\frac{1}{6}a_1$$

$$a_5 = \frac{a_3}{4\cdot 5} = -\frac{1}{5!}a_1$$

$$a_7 = \frac{3a_5}{6\cdot 7} = \frac{3}{7!}a_1$$

$$a_9 = \frac{5a_7}{8\cdot 9} = \frac{3\cdot 5}{9!}a_1$$

$$\vdots \quad \vdots$$

$$a_4 = 0$$

$$a_6 = \frac{2a_4}{5\cdot 6} = 0$$

$$a_8 = \frac{4a_6}{5\cdot 6} = 0$$

$$\vdots \quad \vdots$$

$$\begin{cases} y_1(x) = 1 - x^2 \\ y_2(x) = \left(x - \frac{1}{6}x^3 + \frac{1}{5!}x^5 + \frac{3}{7!}x^7 + \cdots\right)a_1 \end{cases}$$

Exercise

Find a power series solution. $y'' - xy' - x^2 y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' - x^2 y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x\sum_{n=1}^{\infty} n a_n x^{n-1} - x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n - a_1 x - \sum_{n=2}^{\infty} n a_n x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$2a_2 + 6a_3 x - a_1 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} - na_n - a_{n-2} \right] x^n = 0$$

$$\begin{cases} 2a_2 = 0 \ \rightarrow \ \boxed{a_2 = 0} \\ (6a_3 - a_1)x = 0 \ \rightarrow \ \boxed{a_3 = \frac{1}{6}a_1} \\ (n+1)(n+2)a_{n+2} - na_n - a_{n-2} = 0 \end{cases}$$

$$\boxed{a_{n+2} = \frac{na_n + a_{n-2}}{(n+1)(n+2)}}$$

a_0

$a_2 = 0$

$n = 2 \ \rightarrow \ a_4 = \frac{2a_2 + a_0}{3 \cdot 4} = \frac{1}{12}a_0$

$n = 4 \ \rightarrow \ a_6 = \frac{4a_4 + a_2}{5 \cdot 6} = \frac{1}{90}a_0$

$n = 6 \ \rightarrow \ a_8 = \frac{6a_6 + a_4}{7 \cdot 8} = \frac{1}{56}\left(\frac{1}{15} + \frac{1}{12}\right)a_0 = \frac{3}{1120}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

a_1

$a_3 = \frac{1}{6}a_1$

$n = 3 \ \rightarrow \ a_5 = \frac{3a_3 + a_1}{20} = \frac{1}{20}\left(\frac{3}{6} + 1\right)a_1 = \frac{1}{12}a_1$

$n = 5 \ \rightarrow \ a_7 = \frac{5a_5 + a_3}{6 \cdot 7} = \frac{1}{42}\left(\frac{5}{12} + \frac{1}{6}\right)a_1 = \frac{1}{72}a_1$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = a_0\left(1 + \frac{1}{12}x^2 + \frac{1}{90}x^4 + \frac{3}{1120}x^6 + \cdots\right) + a_1\left(x + \frac{1}{12}x^3 + \frac{1}{72}x^5 + \cdots\right)}$$

Exercise

Find a power series solution. $y'' + x^2 y' + xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$y'' + x^2 y' + xy = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + x^2 \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n + x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+2}+\sum_{n=0}^{\infty}a_n x^{n+1}=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n+\sum_{n=1}^{\infty}a_{n-1}x^n=0$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n+a_0x+\sum_{n=2}^{\infty}a_{n-1}x^n=0$$

$$2a_2+\left(6a_3+a_0\right)x+\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}+(n-1)a_{n-1}+a_{n-1}\right]x^n=0$$

$$2a_2+\left(6a_3+a_0\right)x=0 \Rightarrow \begin{cases} a_2=0 \\ a_3=-\frac{1}{6}a_0 \end{cases}$$

$$(n+2)(n+1)a_{n+2}+na_{n-1}=0 \Rightarrow a_{n+2}=-\frac{n}{(n+1)(n+2)}a_{n-1}$$

$$a_4=-\frac{2}{3\cdot 4}a_1=-\frac{1}{6}a_1 \qquad a_5=-\frac{3}{4\cdot 5}a_2=0 \qquad a_6=-\frac{4}{5\cdot 6}a_3=\frac{1}{45}a_0$$

$$a_7=-\frac{5}{6\cdot 7}a_4=\frac{5}{252}a_1 \qquad a_8=-\frac{6}{7\cdot 8}a_5=0 \qquad a_9=-\frac{7}{8\cdot 9}a_3=-\frac{7}{3,240}a_0$$

$$\vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x)=\left(1-\frac{1}{6}x^3+\frac{1}{45}x^6-\frac{7}{3,240}x^9+\cdots\right)a_0 \\ y_2(x)=\left(x-\frac{1}{6}x^4+\frac{5}{252}x^7-\cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $y''+x^2y'+2xy=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y'' + x^2 y' + 2xy = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x^2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 2x\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+2} + \sum_{n=0}^{\infty}2a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}(n-1)a_{n-1}x^n + \sum_{n=1}^{\infty}2a_{n-1}x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty}(n-1)a_{n-1}x^n + 2a_0 x + \sum_{n=2}^{\infty}2a_{n-1}x^n = 0$$

$$2a_2 + \left(6a_3 + 2a_0\right)x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n-1)a_{n-1} + 2a_{n-1}\right]x^n = 0$$

$$2a_2 + \left(6a_3 + 2a_0\right)x = 0 \;\Rightarrow\; \begin{cases} a_2 = 0 \\ a_3 = -\frac{1}{3}a_0 \end{cases}$$

$$(n+2)(n+1)a_{n+2} + (n+1)a_{n-1} = 0 \;\Rightarrow a_{n+2} = -\frac{a_{n-1}}{n+2}$$

$$\boxed{a_{n+3} = -\frac{a_n}{n+3}}$$

a_0

$$a_3 = -\frac{1}{3}a_0$$

$$n = 3 \;\rightarrow\; a_6 = -\frac{a_3}{6} = \frac{1}{2\cdot 3^2}a_0$$

$$n = 6 \;\rightarrow\; a_9 = -\frac{a_6}{9} = -\frac{1}{3!\,3^3}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{3n} = \frac{(-1)^n}{n!\,3^n}$$

a_1

$$n = 1 \;\rightarrow\; a_4 = -\frac{1}{4}a_1$$

$$n = 4 \;\rightarrow\; a_7 = -\frac{a_4}{7} = \frac{1}{7\cdot 4}a_1$$

$$n = 7 \;\rightarrow\; a_{10} = -\frac{a_7}{10} = -\frac{1}{10\cdot 7\cdot 4\cdot 1}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{3n+1} = \frac{(-1)^n}{1\cdot 4\cdot 7\cdots(3n+1)}$$

$a_2 = 0$

$$n = 2 \rightarrow\; a_5 = -\frac{a_2}{5} = 0$$

$$n = 5 \rightarrow\; a_8 = -\frac{a_5}{8} = 0$$

$$n = 8 \rightarrow\; a_{11} = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{3n+2} = 0$$

$$\boxed{y(x) = a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,3^n}x^{3n} + a_1\sum_{n=0}^{\infty}\frac{(-1)^n}{1\cdot 4\cdot 7\cdots(3n+1)}x^{3n+1}}$$

$$\boxed{y(x) = a_0\left(1 - \frac{1}{3}x^3 + \frac{1}{18}x^6 - \frac{1}{162}x^9 + \cdots\right) + a_1\left(x - \frac{1}{4}x^4 + \frac{1}{28}x^7 - \frac{1}{280}x^{10} + \cdots\right)}$$

Exercise

Find a power series solution. $y'' - x^2y' - 3xy = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$y'' - x^2y' - 3xy = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x^2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - 3x\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+2} - \sum_{n=0}^{\infty}3a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty}(n-1)a_{n-1}x^n - \sum_{n=1}^{\infty}3a_{n-1}x^n = 0$$

$$2a_2 + 6a_3x + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty}(n-1)a_{n-1}x^n - 3a_0x - \sum_{n=2}^{\infty}3a_{n-1}x^n = 0$$

$$2a_2 + 3(2a_3 - a_0)x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n+2)a_{n-1}\right]x^n = 0$$

$$\begin{cases} 2a_2 = 0 & \rightarrow \underline{a_2 = 0} \\ 2a_3 - a_0 = 0 & \rightarrow \underline{a_3 = \frac{1}{2}a_0} \end{cases}$$

$$(n+2)(n+1)a_{n+2} = (n+2)a_{n-1}$$

$$a_{n+2} = \frac{a_{n-1}}{n+1} \quad \rightarrow \quad \underline{a_{n+3} = \frac{a_n}{n+2}}$$

a_0	a_1	$a_2 = 0$
$a_3 = \frac{1}{2}a_0$	$n=1 \rightarrow a_4 = \frac{1}{3}a_1$	$n=2 \rightarrow a_5 = \frac{a_2}{5} = 0$
$n=3 \rightarrow a_6 = \frac{a_3}{5} = \frac{1}{2\cdot 5}a_0$	$n=4 \rightarrow a_7 = \frac{a_4}{6} = \frac{1}{2\cdot 3^2}a_1$	$n=5 \rightarrow a_8 = 0$

$$n=6 \rightarrow a_9 = \frac{a_6}{8} = -\frac{1}{2\cdot5\cdot8}a_0 \qquad n=7 \rightarrow a_{10} = \frac{a_7}{9} = \frac{1}{2\cdot3\cdot3^3}a_1 \qquad n=8 \rightarrow a_{11} = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{3n} = \frac{1}{2\cdot5\cdot8\cdots(3n-1)} \qquad a_{3n+1} = \frac{1}{n!\,3^n} \qquad a_{3n+2} = 0$$

$$\boxed{y(x) = a_0\sum_{n=0}^{\infty}\frac{x^{3n}}{2\cdot5\cdot8\cdots\cdot(3n-1)} + a_1\sum_{n=0}^{\infty}\frac{x^{3n+1}}{n!\,3^n}}$$

$$\boxed{y(x) = a_0\left(1+\frac{1}{2}x^3+\frac{1}{10}x^6+\frac{1}{80}x^9+\cdots\right)+a_1\left(x+\frac{1}{3}x^4+\frac{1}{18}x^7+\frac{1}{162}x^{10}+\cdots\right)}$$

Exercise

Find a power series solution. $y''+2xy'+2y=0$

Solution

$$y(x) = \sum_{n=0}^{\infty}a_nx^n$$

$$y'(x) = \sum_{n=1}^{\infty}na_nx^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty}n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''+2xy'+2y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 2\sum_{n=0}^{\infty}a_nx^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}2(n+1)a_{n+1}x^{n+1} + \sum_{n=0}^{\infty}2a_nx^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}2na_nx^n + \sum_{n=0}^{\infty}2a_nx^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}2na_nx^n + 2a_0 + \sum_{n=1}^{\infty}2a_nx^n = 0$$

$$2a_2+2a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}+2na_n+2a_n\right]x^n=0$$

$$2a_2+2a_0=0 \;\rightarrow\; \boxed{a_2=-a_0}$$

$$(n+2)(n+1)a_{n+2}+2(n+1)a_n=0 \;\rightarrow\; a_{n+2}=-\frac{2}{n+2}a_n \qquad n=1,\ 2,\ \cdots$$

$$a_3=-\frac{2}{3}a_1 \qquad\qquad a_4=-\frac{2}{4}a_2=\frac{1}{2}a_0$$

$$a_5=-\frac{2}{5}a_3=\frac{4}{15}a_1 \qquad\qquad a_6=-\frac{2}{6}a_4=-\frac{1}{6}a_0$$

$$a_7=-\frac{2}{7}a_3=-\frac{8}{105}a_1 \qquad\qquad a_8=-\frac{2}{8}a_6=\frac{1}{24}a_0$$

$$\vdots\ \vdots \qquad\qquad \vdots\ \vdots$$

$$\boxed{\begin{cases} y_1(x)=\left(1-x^2+\frac{1}{2!}x^4-\frac{1}{3!}x^6+\frac{1}{4!}x^8-\cdots\right)a_0 \\ y_2(x)=\left(x-\frac{2}{3}x^3+\frac{4}{15}x^5-\frac{8}{105}x^7+\cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $2y''+xy'+y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$2y''+xy'+y=0$$

$$2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}na_nx^n+\sum_{n=0}^{\infty}a_nx^n=0$$

$$4a_2 + \sum_{n=1}^{\infty} 2(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} na_n x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n = 0$$

$$4a_2 + a_0 + \sum_{n=1}^{\infty}\left[2(n+2)(n+1)a_{n+2} + (n+1)a_n\right]x^n = 0$$

$$4a_2 + a_0 = 0 \quad \to \quad \boxed{a_2 = -\tfrac{1}{4}a_0}$$

$$2(n+2)(n+1)a_{n+2} + (n+1)a_n = 0$$

$$\boxed{a_{n+2} = -\frac{1}{2(n+2)}a_n}$$

a_0	a_1
$n=0 \to a_2 = -\frac{1}{4}a_0$	$n=1 \to a_3 = -\frac{1}{6}a_1$
$n=2 \to a_4 = -\frac{1}{8}a_2 = \frac{1}{2^4 \cdot 2}a_0$	$n=3 \to a_5 = -\frac{1}{2\cdot 5}a_3 = \frac{1}{2^2 \cdot 3 \cdot 5}a_1$
$n=4 \to a_6 = -\frac{1}{2\cdot 6}a_4 = -\frac{1}{2^6 \cdot 2 \cdot 3}a_0$	$n=5 \to a_7 = -\frac{1}{2\cdot 7}a_5 = -\frac{1}{2^3 \cdot 3\cdot 5\cdot 7}a_1$
$n=6 \to a_8 = -\frac{1}{2\cdot 8}a_6 = \frac{1}{2^8 \cdot 4!}a_0$	
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$n \ge 1 \quad a_{2n+1} = \frac{(-1)^n}{2^n\ 1\cdot 3\cdot 5\cdots(2n+1)}a_1 = \frac{(-1)^n}{2^n\ (2n+1)!!}a_1$$

$$n \ge 1 \quad a_{2n} = \frac{(-1)^n}{2^{2n}\ n!}a_0$$

$$\boxed{y(x) = a_0\left(1 - \tfrac{1}{4}x^2 + \tfrac{1}{32}x^4 - \ldots\right) + a_1\left(x - \tfrac{1}{6}x^3 + \tfrac{1}{60}x^5 - \cdots\right)}$$

$$\boxed{y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}\ n!}x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n\ (2n+1)!!}x^{2n+1}}$$

Exercise

Find a power series solution. $3y'' + xy' - 4y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$3y''+xy'-4y=0$$

$$3\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-4\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}3(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}4a_n x^n=0$$

$$\sum_{n=0}^{\infty}3(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}na_n x^n-\sum_{n=0}^{\infty}4a_n x^n=0$$

$$6a_2+\sum_{n=1}^{\infty}3(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}na_n x^n-4a_0-\sum_{n=1}^{\infty}4a_n x^n=0$$

$$6a_2-4a_0+\sum_{n=1}^{\infty}\left[3(n+2)(n+1)a_{n+2}+(n-4)a_n\right]x^n=0$$

$$6a_2-4a_0 \quad\rightarrow\quad a_2=\frac{2}{3}a_0$$

$$3(n+2)(n+1)a_{n+2}+(n-4)a_n=0$$

$$a_{n+2}=-\frac{(n-4)}{3(n+1)(n+2)}a_n$$

a_0

$$a_2=\frac{2}{3}a_0$$

$$n=2\ \rightarrow\ a_4=\frac{2}{36}a_2=\frac{1}{27}a_0$$

$$n=4\ \rightarrow\ a_6=0$$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

a_1

$$n=1\ \rightarrow\ a_3=\frac{3}{2\cdot 3\cdot 3}a_1=\frac{1}{2\cdot 3}a_1$$

$$n=3\ \rightarrow\ a_5=\frac{1}{4\cdot 5\cdot 3}a_3=\frac{1}{5!\cdot 3}a_1$$

$$n=5\ \rightarrow\ a_7=-\frac{1}{3\cdot 6\cdot 7}a_5=-\frac{1}{7!\cdot 3^2}a_1$$

$$n=7\ \rightarrow\ a_9=-\frac{3}{3\cdot 9\cdot 8}a_7=\frac{3}{9!\cdot 3^3}a_1$$

$$n=9\ \rightarrow\ a_{11}=-\frac{5}{3\cdot 11\cdot 10}a_9=-\frac{3\cdot 5}{11!\cdot 3^4}a_1$$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$n\ge 3 \quad a_{2n+1}=\frac{1\cdot 3\cdot 5\cdots(2n-5)(-1)^n}{(2n+1)!\ 3^n}a_1$$

$$=\frac{(2n-5)!!(-1)^n}{(2n+1)!\ 3^n}a_1$$

$$\underline{y(x)=a_0\left(1+\frac{2}{3}x^2+\frac{1}{27}x^4\right)+a_1\left(x+\frac{1}{6}x^3+\frac{1}{360}x^5+\sum_{n=3}^{\infty}\frac{(2n-5)!!(-1)^n}{(2n+1)!\ 3^n}\right)}$$

$$\underline{=a_0\left(1+\frac{2}{3}x^2+\frac{1}{27}x^4\right)+a_1\left(x+\frac{1}{6}x^3+\frac{1}{360}x^5-\frac{1}{45,360}x^7+\cdots\right)}$$

Exercise

Find a power series solution. $5y''-2xy'+10y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$5y''-2xy'+10y=0$$

$$5\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+10\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}5(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}2(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}10a_n x^n=0$$

$$\sum_{n=0}^{\infty}5(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty}2na_n x^n+\sum_{n=0}^{\infty}10a_n x^n=0$$

$$10a_2+\sum_{n=1}^{\infty}5(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty}2na_n x^n+10a_0+\sum_{n=1}^{\infty}10a_n x^n=0$$

$$10a_2+10a_0+\sum_{n=1}^{\infty}\left[5(n+2)(n+1)a_{n+2}-2(n-5)a_n\right]x^n=0$$

$$10a_2+10a_0 \quad\rightarrow\quad \underline{a_2=-a_0}$$

$$5(n+2)(n+1)a_{n+2} - 2(n-5)a_n = 0$$

$$\boxed{a_{n+2} = \frac{2(n-5)}{5(n+1)(n+2)}a_n}$$

a_0

$$a_2 = -a_0$$

$$n=2 \ \rightarrow \ a_4 = -\frac{6}{60}a_2 = \frac{1}{10}a_0$$

$$n=4 \ \rightarrow \ a_6 = -\frac{2}{5\cdot 5\cdot 6}a_4 = -\frac{1}{750}a_0$$

$$n=6 \ \rightarrow \ a_8 = \frac{2}{5\cdot 7\cdot 8}a_6 = -\frac{2}{8!\cdot 5^2}a_0$$

$$n=8 \ \rightarrow \ a_{10} = \frac{2\cdot 3}{5\cdot 9\cdot 10}a_8 = -\frac{2^2\cdot 3}{10!\cdot 5^3}a_0$$

$$n=10 \ \rightarrow \ a_{12} = \frac{2\cdot 5}{5\cdot 11\cdot 12}a_8 = -\frac{2^3\cdot 3\cdot 5}{12!\cdot 5^4}a_0$$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$n \ge 4 \quad a_{2n} = -15\cdot\frac{2^n(2n-7)!!}{5^n\ (2n)!}a_0$$

a_1

$$n=1 \ \rightarrow \ a_3 = -\frac{8}{30}a_1 = -\frac{4}{15}a_1$$

$$n=3 \ \rightarrow \ a_5 = -\frac{4}{100}a_3 = \frac{4}{375}a_1$$

$$n=5 \ \rightarrow \ a_7 = 0$$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = a_0\left(1 - x^2 + \frac{1}{10}x^4 - \frac{1}{750}x^6 - \frac{1}{105,000}x^8 - \cdots\right) + a_1\left(x - \frac{4}{15}x^3 + \frac{4}{375}x^5\right)}$$

Exercise

Find a power series solution. $(x-1)y'' + y' = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$(x-1)y'' + y' = 0$$

$$(x-1)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=0$$

$$x\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^n-2a_2-\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n+a_1+\sum_{n=1}^{\infty}(n+1)a_{n+1}x^n=0$$

$$a_1-2a_2+\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}-(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}\right]x^n=0$$

$$a_1-2a_2+\sum_{n=1}^{\infty}\left[-(n+2)(n+1)a_{n+2}+(n+1)^2a_{n+1}\right]x^n=0$$

$$a_1-2a_2=0\ \rightarrow\ a_2=\frac{1}{2}a_1$$

$$-(n+2)(n+1)a_{n+2}+(n+1)^2a_{n+1}$$

$$\rightarrow\ a_{n+2}=\frac{n+1}{n+2}a_{n+1}\qquad n=1,\ 2,\ \ldots$$

$$a_3=\frac{2}{3}a_2=\frac{1}{3}a_1$$

$$a_4=\frac{3}{4}a_3=\frac{1}{4}a_1$$

$$a_5=\frac{4}{5}a_4=\frac{1}{5}a_1$$

$$\vdots\ \vdots$$

$$\begin{cases}y_1(x)=a_0\\ y_2(x)=\left(x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\cdots\right)a_1\end{cases}$$

Exercise

Find a power series solution. $(x+2)y'' + xy' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$(x+2)y'' + xy' - y = 0$$

$$(x+2)\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+1} + \sum_{n=0}^{\infty} 2(n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2(n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n(n+1) a_{n+1} x^n + 4a_2 + \sum_{n=1}^{\infty} 2(n+2)(n+1) a_{n+2} x^n + 4a_2 + \sum_{n=1}^{\infty} n a_n x^n - a_0 - \sum_{n=1}^{\infty} a_n x^n = 0$$

$$4a_2 - a_0 + \sum_{n=1}^{\infty} \left(n(n+1) a_{n+1} + 2(n+2)(n+1) a_{n+2} + (n-1) a_n \right) x^n = 0$$

$$4a_2 - a_0 = 0 \quad \to \quad \boxed{a_2 = \frac{1}{4} a_0} \quad \boxed{a_1}$$

$$n(n+1) a_{n+1} + 2(n+2)(n+1) a_{n+2} + (n-1) a_n = 0$$

$$\boxed{a_{n+2} = -\frac{n-1}{2(n+2)(n+1)} a_n - \frac{n}{2(n+2)} a_{n+1}} \quad n = 1,\ 2,\ \ldots$$

$$a_3 = -\frac{1}{6} a_2 = -\frac{1}{24} a_0$$

$$a_4 = -\frac{1}{24} a_2 - \frac{1}{4} a_3 = -\frac{1}{96} a_0 + \frac{1}{96} a_0 = 0$$

$$a_5 = -\frac{1}{20} a_3 - \frac{3}{10} a_4 = \frac{1}{480} a_0$$

$$a_6 = -\frac{3}{60}a_4 - \frac{1}{3}a_5 = -\frac{1}{1,440}a_0$$

$$\vdots \quad \vdots \qquad\qquad \vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = a_1 \\ y_2(x) = \left(1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 - \cdots\right)a_0 \end{cases}}$$

Exercise

Find a power series solution. $y'' - (x+1)y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$y'' - (x+1)y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - x\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_n\right]x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$$2a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_n\right]x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$$2a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_n - a_{n-1}\right]x^n = 0$$

$$2a_2 - a_0 = 0 \quad \rightarrow \quad \boxed{a_2 = \frac{1}{2}a_0}$$

$$(n+2)(n+1)a_{n+2} - a_n - a_{n-1} = 0$$

$$\boxed{a_{n+2}=\frac{a_n+a_{n-1}}{(n+1)(n+2)}}$$

$a_0 \qquad a_1$

$$a_2=\frac{1}{2}a_0$$

$$n=1 \;\rightarrow\; a_3=\frac{1}{6}\left(a_1+a_0\right)$$

$$n=2 \;\rightarrow\; a_4=\frac{1}{12}\left(a_2+a_1\right)=\frac{1}{12}\left(\frac{1}{2}a_0+a_1\right)$$

$$n=3 \;\rightarrow\; a_5=\frac{1}{20}\left(a_3+a_2\right)=\frac{1}{20}\left(\frac{2}{3}a_0+\frac{1}{6}a_1\right)$$

$$n=4 \;\rightarrow\; a_6=\frac{1}{30}\left(a_4+a_3\right)=\frac{1}{30}\left(\frac{1}{2}a_0+a_1+\frac{1}{6}a_0+\frac{1}{6}a_1\right)=\frac{1}{30}\left(\frac{2}{3}a_0+\frac{7}{6}a_1\right)$$

$a_0\neq 0 \quad a_1=0$	$a_0=0 \quad a_1\neq 0$
$a_2=\frac{1}{2}a_0$	$a_2=0$
$a_3=\frac{1}{6}a_0$	$a_3=\frac{1}{6}a_1$
$a_4=\frac{1}{24}a_0$	$a_4=\frac{1}{12}a_1$
$a_5=\frac{1}{30}a_0$	$a_5=\frac{1}{120}a_1$
$a_6=\frac{1}{45}a_0$	$a_6=\frac{7}{180}a_1$

$$\boxed{\begin{cases} y_1(x)=\left(1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{30}x^5+\frac{1}{45}x^6+\cdots\right)a_0 \\ y_2(x)=\left(x+\frac{1}{6}x^3+\frac{1}{12}x^4+\frac{1}{120}x^5+\frac{7}{180}x^6+\cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $y''-(x+1)y'-y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''-(x+1)y'-y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-(x+1)\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty}na_nx^n-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}a_nx^n=0$$

$$2a_2+\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty}na_nx^n-a_1-\sum_{n=1}^{\infty}(n+1)a_{n+1}x^n-a_0-\sum_{n=1}^{\infty}a_nx^n=0$$

$$2a_2-a_1-a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}-na_n-(n+1)a_{n+1}-a_n\right]x^n=0$$

$$2a_2-a_1-a_0=0 \quad\to\quad a_2=\frac{1}{2}a_0+\frac{1}{2}a_1$$

$$(n+2)(n+1)a_{n+2}-(n+1)a_{n+1}-(n+1)a_n=0$$

$$\boxed{a_{n+2}=\frac{1}{n+2}a_{n+1}+\frac{1}{n+2}a_n}$$

$$a_3=\frac{1}{3}a_2+\frac{1}{3}a_1=\frac{1}{6}a_0+\frac{1}{2}a_1$$

$$a_4=\frac{1}{4}a_3+\frac{1}{4}a_2=\frac{1}{24}a_0+\frac{1}{8}a_1+\frac{1}{8}a_0+\frac{1}{8}a_1=\frac{1}{6}a_0+\frac{1}{4}a_1$$

$$a_5=\frac{1}{5}a_4+\frac{1}{5}a_3=\frac{1}{30}a_0+\frac{1}{20}a_1+\frac{1}{30}a_0+\frac{1}{10}a_1=\frac{1}{15}a_0+\frac{3}{20}a_1$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x)=\left(1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{6}x^4+\cdots\right)a_0 \\ y_2(x)=\left(x+\frac{1}{2}x^2+\frac{1}{2}x^3+\frac{1}{4}x^4+\cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y''-6y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty} na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y''-6y=0$$

$$\left(x^2+1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-6\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-6\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-6\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n-6a_0-6a_1x-6\sum_{n=2}^{\infty}a_n x^n=0$$

$$2a_2-6a_0+\left(6a_3-6a_1\right)x+\sum_{n=2}^{\infty}\left[\left(n^2-n-6\right)a_n+(n+2)(n+1)a_{n+2}\right]x^n=0$$

$$2a_2-6a_0+\left(6a_3-6a_1\right)x=0 \quad\rightarrow\quad \begin{cases} a_2=3a_0 \\ a_3=a_1 \end{cases}$$

$$(n+2)(n-3)a_n+(n+2)(n+1)a_{n+2}=0$$

$$\Rightarrow \quad a_{n+2}=-\frac{n-3}{n+1}a_n \qquad n=2,3,\ldots$$

$$a_4=\frac{1}{3}a_2=a_0 \qquad\qquad a_5=0$$

$$a_6=-\frac{1}{5}a_4=-\frac{1}{5}a_0 \qquad\qquad a_7=-\frac{1}{3}a_5=0$$

$$a_8=-\frac{3}{7}a_6=\frac{3}{35}a_0$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x)=\left(1+3x^2+x^4-\frac{1}{5}x^6+\cdots\right)a_0 \\ y_2(x)=\left(x+x^3\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $\left(x^2+2\right)y''+3xy'-y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+2\right)y''+3xy'-y=0$$

$$\left(x^2+2\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+3x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}3(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}\left[2(n+2)(n+1)a_{n+2}-a_n\right]x^n+\sum_{n=1}^{\infty}3na_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+4a_2-a_0+\left(12a_3-a_1\right)x+\sum_{n=2}^{\infty}\left[2(n+2)(n+1)a_{n+2}-a_n\right]x^n$$

$$+3a_1x+\sum_{n=2}^{\infty}3na_n x^n=0$$

$$4a_2-a_0+\left(12a_3+2a_1\right)x+\sum_{n=2}^{\infty}\left[n(n-1)a_n+2(n+2)(n+1)a_{n+2}+(3n-1)a_n\right]x^n=0$$

$$4a_2-a_0+\left(12a_3+2a_1\right)x \quad\Rightarrow\quad \underline{a_2=\tfrac{1}{4}a_0}\,\big| \quad \underline{a_3=-\tfrac{1}{6}a_1}\,\big|$$

$$2(n+2)(n+1)a_{n+2}+\left(n^2+2n-1\right)a_n=0$$

$$\to\quad a_{n+2}=-\frac{n^2+2n-1}{2(n+2)(n+1)}a_n \qquad n=2,3,\ldots$$

$$a_4=-\frac{7}{24}a_2=-\frac{7}{96}a_0 \qquad\qquad a_5=-\frac{7}{20}a_3=\frac{7}{120}a_1$$

$$a_6 = -\frac{23}{60}a_4 = \frac{161}{5760}a_0 \qquad\qquad a_7 = -\frac{17}{42}a_5 = -\frac{17}{720}a_1$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = \left(1 + \frac{1}{4}x^2 - \frac{7}{96}x^4 + \frac{161}{5760}x^6 - \cdots\right)a_0 \\ y_2(x) = \left(1 - \frac{1}{6}x^3 + \frac{7}{120}x^5 - \frac{17}{720}x^7 + \cdots\right)a_1 \end{cases}}$$

Exercise

Find a power series solution. $\left(x^2-1\right)y'' + xy' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2-1\right)y'' + xy' - y = 0$$

$$\left(x^2-1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2} - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n - \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_n\right]x^n + \sum_{n=1}^{\infty}na_n x^n = 0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n - \left(2a_2 + a_0\right) - \left(6a_3 + a_1\right)x - \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_n\right]x^n$$

$$+ a_1 x + \sum_{n=2}^{\infty}na_n x^n = 0$$

$$-2a_2 - a_0 - 6a_3 x + \sum_{n=2}^{\infty}\left[n(n-1)a_n - (n+2)(n+1)a_{n+2} + (n-1)a_n\right]x^n = 0$$

$$-2a_2 - a_0 - 6a_3 x = 0 \quad \boxed{\to a_2 = -\frac{1}{2}a_0} \quad \boxed{a_3 = 0}$$

$$-(n+2)(n+1)a_{n+2} + (n+1)(n-1)a_n = 0$$

$$\to \quad a_{n+2} = \frac{n-1}{n+2}a_n \qquad n = 2,3,\ldots$$

$$a_4 = \frac{1}{4}a_2 = -\frac{1}{8}a_0 \qquad\qquad a_5 = -\frac{2}{5}a_3 = 0$$

$$a_6 = \frac{1}{2}a_4 = -\frac{1}{16}a_0 \qquad\qquad a_7 = \frac{4}{7}a_5 = 0$$

$$\vdots \quad \vdots$$

$$\boxed{\begin{cases} y_1(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \cdots\right)a_0 \\ y_2(x) = a_1 \end{cases}}$$

$$\begin{cases} y_1(x) = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \cdots \\ y_2(x) = x \end{cases}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' + xy' - y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y'' + xy' - y = 0$$

$$\left(x^2+1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$x^2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n}+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}a_{n}x^{n}=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_{n}x^{n}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n}+\sum_{n=1}^{\infty}na_{n}x^{n}-\sum_{n=0}^{\infty}a_{n}x^{n}=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_{n}x^{n}+\sum_{n=1}^{\infty}na_{n}x^{n}+\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-a_{n}\right]x^{n}=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_{n}x^{n}+a_{1}x+\sum_{n=2}^{\infty}na_{n}x^{n}+\left(2a_{2}-a_{0}\right)+\left(6a_{3}-a_{1}\right)x+\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2}-a_{n}\right]x^{n}=0$$

$$2a_{2}-a_{0}+6a_{3}x+\sum_{n=2}^{\infty}\left[\left(n^{2}-n+n-1\right)a_{n}+(n+2)(n+1)a_{n+2}\right]x^{n}=0$$

$$\begin{cases}2a_{2}-a_{0}=0 & \rightarrow \underline{a_{2}=\frac{1}{2}a_{0}} \\ 6a_{3}x=0 & \rightarrow \underline{a_{3}=0}\end{cases}$$

$$(n-1)(n+1)a_{n}+(n+2)(n+1)a_{n+2}=0$$

$$\underline{a_{n+2}=-\frac{n-1}{n+2}a_{n}}$$

a_0

$n=0\rightarrow a_{2}=\frac{1}{2}a_{0}$

$n=2\rightarrow a_{4}=-\frac{1}{4}a_{2}=-\frac{1}{2^{3}}a_{0}$

$n=4\rightarrow a_{6}=-\frac{3}{6}a_{4}=\frac{1\cdot 3}{2^{3}\cdot 3!}a_{0}$

$n=6\rightarrow a_{8}=-\frac{5}{8}a_{6}=-\frac{1\cdot 3\cdot 5}{2^{4}\cdot 4!}a_{0}$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

a_1

$n=1\rightarrow a_{3}=0$

$n=3\rightarrow a_{5}=-\frac{2}{5}a_{3}=0$

$n=5\rightarrow a_{7}=0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{2n}=(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^{n}\cdot n!}a_{0}\qquad (n\geq 2)$$

$$\underline{y(x)=a_{0}\left(1+\frac{1}{2}x^{2}-\frac{1}{2^{2}\cdot 2!}x^{4}+\frac{1\cdot 3}{2^{3}\cdot 3!}x^{6}-\frac{1\cdot 3\cdot 5}{2^{4}\cdot 4!}x^{8}+\cdots\right)+a_{1}x}$$

$$\underline{=a_{0}\left(1+\frac{1}{2}x^{2}+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(2n-3)!!}{2^{n}\cdot n!}x^{2n}\right)+a_{1}x}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' - xy' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2+1\right)y'' - xy' + y = 0$$

$$\left(x^2+1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

$$x^2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2} + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} + \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}na_n x^n + \sum_{n=0}^{\infty}a_n x^n = 0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n - \sum_{n=1}^{\infty}na_n x^n + \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_n\right]x^n = 0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n - a_1 x - \sum_{n=2}^{\infty}na_n x^n + \left(2a_2 + a_0\right) + \left(6a_3 + a_1\right)x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_n\right]x^n = 0$$

$$2a_2 + a_0 + 6a_3 x + \sum_{n=2}^{\infty}\left[\left(n^2 - 2n + 1\right)a_n + (n+2)(n+1)a_{n+2}\right]x^n = 0$$

$$\begin{cases} 2a_2 + a_0 = 0 & \rightarrow \boxed{a_2 = -\frac{1}{2}a_0} \\ 6a_3 x = 0 & \rightarrow \boxed{a_3 = 0} \end{cases}$$

$$(n-1)^2 a_n + (n+1)(n+2)a_{n+2} = 0$$

$$\boxed{a_{n+2} = -\frac{(n-1)^2}{(n+1)(n+2)} a_n}$$

a_0

$n = 0 \to a_2 = -\frac{1}{2} a_0$

$n = 2 \to a_4 = -\frac{1}{12} a_2 = \frac{1}{4!} a_0$

$n = 4 \to a_6 = -\frac{3^2}{5 \cdot 6} a_4 = -\frac{3^2}{6!} a_0$

$n = 6 \to a_8 = -\frac{5^2}{7 \cdot 8} a_6 = \frac{1 \cdot 3^2 \cdot 5^2}{8!} a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

a_1

$n = 1 \to a_3 = 0$

$n = 3 \to a_5 = -\frac{2}{5} a_3 = 0$

$n = 5 \to a_7 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{2n} = (-1)^{n-1} \frac{1 \cdot 3^2 \cdot 5^2 \cdots (2n-3)^2}{(2n)!} a_0 \quad (n \ge 3)$$

$$\boxed{y(x) = a_0 \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{9}{6!}x^6 - \frac{1 \cdot 3^2 \cdot 5^2}{8!}x^8 - \cdots \right) + a_1 x}$$

$$\boxed{= a_0 \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \sum_{n=3}^{\infty} (-1)^n \frac{(2n-3)^2!!}{(2n)!} x^{2n} \right) + a_1 x}$$

Exercise

Find a power series solution. $\left(1 - x^2\right) y'' - 6xy' - 4y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(1 - x^2\right) y'' - 6xy' - 4y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 6x \sum_{n=1}^{\infty} n a_n x^{n-1} - 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty}n(n-1)a_n x^n - \sum_{n=1}^{\infty}6na_n x^n - \sum_{n=0}^{\infty}4a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}n(n-1)a_n x^n - \sum_{n=0}^{\infty}6na_n x^n - \sum_{n=0}^{\infty}4a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - \left(n(n-1)+6n+4\right)a_n\right]x^n = 0$$

$$(n+2)(n+1)a_{n+2} - \left(n^2+5n+4\right)a_n = 0$$

$$(n+2)(n+1)a_{n+2} = (n+4)(n+1)a_n$$

$$\boxed{a_{n+2} = \frac{n+4}{n+2}a_n}$$

a_0	a_1
$n=2 \rightarrow a_2 = 2a_0$	$n=3 \rightarrow a_3 = \frac{5}{3}a_1$
$n=4 \rightarrow a_4 = \frac{6}{4}a_2 = 3a_0$	$n=5 \rightarrow a_5 = \frac{7}{5}a_3 = \frac{7}{3}a_1$
$n=6 \rightarrow a_6 = \frac{8}{6}a_3 = 4a_0$	$n=7 \rightarrow a_7 = \frac{11}{7}a_5 = \frac{11}{3}a_1$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$a_{2k} = (k+1)a_0$	$a_{2k+1} = \frac{2k+3}{3}a_1$

$$y(x) = a_0\left(1+2x^2+3x^4+4x^6+\cdots\right) + a_1\left(x+\frac{5}{3}x^3+\frac{7}{3}x^5+\frac{11}{3}x^7+\cdots\right)$$

$$\boxed{= \frac{a_0}{\left(1-x^2\right)^2} + \frac{3x-x^3}{3\left(1-x^2\right)^2}a_1}$$

Exercise

Find a power series solution. $y'' + (x-1)^2 y' - 4(x-1)y = 0$

Solution

Let $z = x-1 \rightarrow dz = dx$

$$y(x) = \sum_{n=0}^{\infty}a_n z^n$$

$$y'(x) = \sum_{n=1}^{\infty}na_n z^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n$$

$$y''+z^2y'-4zy=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n+z^2\sum_{n=1}^{\infty}na_n z^{n-1}-4z\sum_{n=0}^{\infty}a_n z^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n+\sum_{n=1}^{\infty}na_n z^{n+1}-\sum_{n=0}^{\infty}4a_n z^{n+1}=0$$

$$2a_2+\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}z^n+\sum_{n=1}^{\infty}(n-4)a_n z^{n+1}=0$$

$$2a_2+\sum_{n=0}^{\infty}(n+2)(n+3)a_{n+3}z^{n+1}+\sum_{n=1}^{\infty}(n-4)a_n z^{n+1}=0$$

$$2a_2+\sum_{n=0}^{\infty}\left[(n+2)(n+3)a_{n+3}+(n-4)a_n\right]z^{n+1}=0$$

$$\boxed{a_2=0}$$

$$(n+2)(n+3)a_{n+3}+(n-4)a_n=0$$

$$\boxed{a_{n+3}=-\frac{n-4}{(n+2)(n+3)}a_n}$$

a_0

$$n=0\to\ a_3=\frac{4}{2\cdot 3}a_0$$

$$n=3\to\ a_6=\frac{1}{5\cdot 6}a_3=\frac{4}{2\cdot 3\cdot 5\cdot 6}a_0$$

$$n=6\to\ a_9=-\frac{2}{8\cdot 9}a_6=-\frac{8}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}a_0$$

$$\vdots\quad\vdots\quad\vdots\quad\vdots$$

a_1

$$n=1\to\ a_4=\frac{3}{3\cdot 4}a_1$$

$$n=4\to\ a_7=0$$

$$n=7\to\ a_{10}=0$$

$a_2=0$

$$n=2\ \to\ a_5=\frac{2}{20}a_2=0$$

$$n=5\ \to\ a_8=0$$

$$n=8\ \to\ a_5=0$$

$$\vdots\quad\vdots\quad\vdots\quad\vdots$$

$$y(x)=a_0\left(1+\frac{4}{2\cdot 3}z^3+\frac{4}{2\cdot 3\cdot 5\cdot 6}z^6-\frac{8}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 8}z^9+\cdots\right)+a_1\left(z+\frac{1}{4}z^4\right)$$

$$\boxed{=a_0\left(1+\tfrac{2}{3}(x-1)^3+\tfrac{1}{45}(x-1)^6-\tfrac{1}{1,620}(x-1)^9+\cdots\right)+a_1\left(x-1+\tfrac{1}{4}(x-1)^4\right)}$$

Exercise

Find a power series solution. $\left(2-x^2\right)y''-xy'+16y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(2-x^2\right)y''-xy'+16y=0$$

$$2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}-x\sum_{n=1}^{\infty}na_n x^{n-1}+16\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n-\sum_{n=2}^{\infty}n(n-1)a_n x^n-\sum_{n=1}^{\infty}na_n x^n+\sum_{n=0}^{\infty}16a_n x^n=0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}na_n x^n+\sum_{n=0}^{\infty}16a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[2(n+2)(n+1)a_{n+2}-\left(n^2-n+n-16\right)a_n\right]x^n=0$$

$$2(n+1)(n+2)a_{n+2}-\left(n^2-16\right)a_n=0$$

$$2(n+1)(n+2)a_{n+2}=(n+4)(n-4)a_n$$

$$\boxed{a_{n+2}=\frac{(n+4)(n-4)}{2(n+1)(n+2)}a_n}$$

a_0

$n=0\to\ a_2=-4a_0$

$n=2\ \to\ a_4=-\frac{1}{2}a_2=2a_0$

$n=4\ \to\ a_6=0$

$\vdots\quad\vdots\quad\vdots\quad\vdots$

a_1

$n=1\ \to\ a_3=-\frac{5}{4}a_1$

$n=3\ \to\ a_5=-\frac{7}{40}a_3=\frac{7}{32}a_1$

$n=5\ \to\ a_7=\frac{9}{70}a_5=\frac{9}{320}a_1$

$n=7\ \to\ a_9=\frac{33}{144}a_7=\frac{33}{5{,}120}a_1$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n+1} = \frac{(2n-5)!!\ (2n+3)!!}{2^n\ (2n+1)!} a_1$$

$$\underline{y(x) = a_0\left(1-4x^2+2x^4\right) + a_1\left(x - \frac{5}{4}x^3 + \frac{7}{32}x^5 + \frac{9}{320}x^7 + \cdots\right)}\Big|$$

$$\underline{= a_0\left(1-4x^2+2x^4\right) + a_1\sum_{n=0}^{\infty}\frac{(2n-5)!!\ (2n+3)!!}{2^n\ (2n+1)!}x^{2n+1}}\Big|$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' - y' + y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y'' - y' + y = 0$$

$$x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} + \left(n^2-n+1\right)a_n\right]x^n = 0$$

$$(n+1)(n+2)a_{n+2} - (n+1)a_{n+1} + \left(n^2-n+1\right)a_n = 0$$

$$\boxed{a_{n+2} = \frac{(n+1)a_{n+1} - \left(n^2 - n + 1\right)a_n}{(n+1)(n+2)}}$$

$$n = 0 \rightarrow \ a_2 = \frac{1}{2}\left(a_1 - a_0\right)$$

$$n = 1 \rightarrow \ a_3 = \frac{1}{6}\left(2a_2 - a_1\right) = \frac{1}{6}\left(a_1 - a_0 - a_1\right) = -\frac{1}{6}a_0$$

$$n = 2 \rightarrow \ a_4 = \frac{1}{12}\left(3a_3 - 3a_2\right) = \frac{1}{4}\left(-\frac{1}{6}a_0 - \frac{1}{2}a_1 + \frac{1}{2}a_0\right) = \frac{1}{12}a_0 - \frac{1}{8}a_1$$

$$n = 3 \rightarrow \ a_5 = \frac{1}{20}\left(4a_4 - 7a_3\right) = \frac{1}{20}\left(\frac{1}{3}a_0 - \frac{1}{2}a_1 + \frac{7}{6}a_0\right) = \frac{3}{40}a_0 - \frac{1}{40}a_1$$

$$n = 4 \rightarrow \ a_6 = \frac{1}{30}\left(5a_5 - 13a_4\right) = \frac{1}{30}\left(\frac{3}{8}a_0 - \frac{1}{8}a_1 - \frac{13}{12}a_0 + \frac{13}{8}a_1\right) = -\frac{17}{720}a_0 + \frac{1}{20}a_1$$

$$y(x) = a_0 + a_1 x + \left(\frac{1}{2}a_0 - \frac{1}{2}a_1\right)x^2 - \frac{1}{6}a_0 x^3 + \left(\frac{1}{12}a_0 - \frac{1}{8}a_1\right)x^4 + \left(\frac{3}{40}a_0 - \frac{1}{40}a_1\right)x^5$$
$$+\left(-\frac{17}{720}a_0 + \frac{1}{20}a_1\right)x^6 + \cdots$$

$$\boxed{\begin{aligned} y(x) &= a_0\left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 - \frac{17}{720}x^6 + \cdots\right) \\ &\quad + a_1\left(x - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{40}x^5 + \frac{1}{20}x^6 + \cdots\right) \end{aligned}}$$

Exercise

Find a power series solution. $\left(x^2 + 1\right)y'' + 6xy' + 4y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2 + 1\right)y'' + 6xy' + 4y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + 6x\sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} 6na_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 6na_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + \left(n^2+5n+4\right)a_n\right]x^n = 0$$

$$(n+1)(n+2)a_{n+2} + (n+1)(n+4)a_n = 0$$

$$\boxed{a_{n+2} = -\frac{n+4}{n+2}a_n}$$

a_0

$$n=0 \to a_2 = -2a_0$$

$$n=2 \to a_4 = -\frac{3}{2}a_2 = 3a_0$$

$$n=4 \to a_6 = -\frac{8}{6}a_4 = -4a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n} = (-1)^n(n+1)a_0$$

a_1

$$n=1 \to a_3 = -\frac{5}{3}a_1$$

$$n=3 \to a_5 = -\frac{7}{5}a_3 = \frac{7}{3}a_1$$

$$n=5 \to a_7 = -\frac{9}{7}a_5 = -\frac{9}{3}a_1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n+1} = (-1)^n(2n+3)a_1$$

$$\boxed{y(x) = a_0\sum_{n=0}^{\infty}(-1)^n(n+1)x^{2n} + \frac{1}{3}a_1\sum_{n=0}^{\infty}(-1)^n(2n+3)x^{2n+1}}$$

$$\boxed{y(x) = a_0\left(1-2x^2+3x^4-4x^6+\cdots\right) + \frac{1}{3}a_1\left(x-\frac{5}{3}x^3+\frac{7}{3}x^5-\frac{9}{3}x^7+\cdots\right)}$$

Exercise

Find a power series solution. $\left(x^2-1\right)y'' - 6xy' + 12y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2-1\right)y''-6xy'+12y=0$$

$$x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-6x\sum_{n=1}^{\infty}na_n x^{n-1}+12\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty}6na_n x^n+\sum_{n=0}^{\infty}12a_n x^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}6na_n x^n+\sum_{n=0}^{\infty}12a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[-(n+2)(n+1)a_{n+2}+\left(n^2-7n+12\right)a_n\right]x^n=0$$

$$-(n+1)(n+2)a_{n+2}+(n-3)(n-4)a_n=0$$

$$\boxed{a_{n+2}=\frac{(n-3)(n-4)}{(n+1)(n+2)}a_n}$$

a_0	a_1
$n=0\to a_2=6a_0$	$n=1\to a_3=a_1$
$n=2\to a_4=\frac{1}{6}a_2=a_0$	$n=3\to a_5=0$
$n=4\to a_6=0$	$\vdots\quad\vdots\quad\vdots\quad\vdots$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	

$$\boxed{y(x)=a_0\left(1+6x^2+x^4\right)+a_1\left(x+x^3\right)}$$

Exercise

Find a power series solution. $\left(x^2-1\right)y''+8xy'+12y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2-1\right)y''+8xy'+12y=0$$

$$x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+8x\sum_{n=1}^{\infty} na_n x^{n-1}+12\sum_{n=0}^{\infty} a_n x^n=0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}8na_n x^n+\sum_{n=0}^{\infty}12a_n x^n=0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}8na_n x^n+\sum_{n=0}^{\infty}12a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[-(n+2)(n+1)a_{n+2}+\left(n^2+7n+12\right)a_n\right]x^n=0$$

$$-(n+1)(n+2)a_{n+2}+(n+3)(n+4)a_n=0$$

$$a_{n+2}=\frac{(n+3)(n+4)}{(n+1)(n+2)}a_n$$

a_0	a_1
$n=0\to a_2=6a_0$	$n=1\to a_3=\frac{10}{3}a_1$
$n=2\to a_4=\frac{5}{2}a_2=15a_0$	$n=3\to a_5=\frac{42}{20}a_3=\frac{21}{3}a_1$
$n=4\to a_6=\frac{5}{2}a_4=28a_0$	$n=5\to a_7=\frac{12}{7}a_5=\frac{36}{3}a_1$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	$\vdots\quad\vdots\quad\vdots\quad\vdots$
$a_{2n}=(n+1)(2n+1)a_0$	$a_{2n+1}=\frac{1}{3}(n+1)(2n+3)a_0$

$$y(x)=a_0\left(1+6x^2+15x^4+28x^6+\cdots\right)+a_1\left(x+\frac{10}{3}x^3+7x^5+12x^7+\cdots\right)$$

$$=a_0\sum_{n=0}^{\infty}(n+1)(2n+1)x^{2n}+\frac{1}{3}a_1\sum_{n=0}^{\infty}(n+1)(2n+3)x^{2n+1}$$

Exercise

Find a power series solution. $\left(x^2-1\right)y''+4xy'+2y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2-1\right)y''+4xy'+2y=0$$

$$x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+4x\sum_{n=1}^{\infty}na_n x^{n-1}+2\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}4na_n x^n+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}4na_n x^n+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[-(n+2)(n+1)a_{n+2}+\left(n^2+3n+2\right)a_n\right]x^n=0$$

$$-(n+1)(n+2)a_{n+2}+(n+1)(n+2)a_n=0$$

$$\boxed{a_{n+2}=a_n}$$

a_0	a_1
$n=0\rightarrow\ a_2=a_0$	$n=1\ \rightarrow\ a_3=a_1$
$n=2\ \rightarrow\ a_4=a_2=a_0$	$n=3\ \rightarrow\ a_5=a_1$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	$\vdots\quad\vdots\quad\vdots\quad\vdots$

$$\boxed{y(x)=a_0\left(1+x^2+x^4+x^6+\cdots\right)+a_1\left(x+x^3+x^5+x^7+\cdots\right)}$$

$$=a_0\sum_{n=0}^{\infty}x^{2n}+a_1\sum_{n=0}^{\infty}x^{2n+1}$$

$$= \boxed{\frac{a_0 + a_1 x}{1 - x^2}}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' - 4xy' + 6y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2+1\right)y'' - 4xy' + 6y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 4x \sum_{n=1}^{\infty} n a_n x^{n-1} + 6 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} + \left(n^2 - 5n + 6\right) a_n \right] x^n = 0$$

$$(n+1)(n+2) a_{n+2} + (n-2)(n-3) a_n = 0$$

$$\boxed{a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)} a_n}$$

a_0	a_1
$n = 0 \rightarrow a_2 = -3a_0$	$n = 1 \rightarrow a_3 = -\frac{1}{3} a_1$
$n = 2 \rightarrow a_4 = 0$	$n = 3 \rightarrow a_5 = 0$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = a_0 \left(1 - 3x^2\right) + a_1 \left(x - \frac{1}{3} x^3\right)}$$

Exercise

Find a power series solution. $\left(x^2+2\right)y''+4xy'+2y=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+2\right)y''+4xy'+2y=0$$

$$x^2\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}+2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+4x\sum_{n=1}^{\infty}na_n x^{n-1}+2\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}4na_n x^n+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}4na_n x^n+\sum_{n=0}^{\infty}2a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[2(n+2)(n+1)a_{n+2}+\left(n^2+3n+2\right)a_n\right]x^n=0$$

$$2(n+1)(n+2)a_{n+2}+(n+1)(n+2)a_n=0$$

$$\underline{a_{n+2}=-\frac{1}{2}a_n}$$

a_0	a_1
$n=0\to a_2=-\frac{1}{2}a_0$	$n=1\to a_3=-\frac{1}{2}a_1$
$n=2\to a_4=-\frac{1}{2}a_2=\frac{1}{2^2}a_0$	$n=3\to a_5=-\frac{1}{2}a_3=\frac{1}{2^2}a_1$
$n=4\to a_6=-\frac{1}{2}a_4=-\frac{1}{2^3}a_0$	$n=5\to a_7=-\frac{1}{2}a_5=-\frac{1}{2^3}a_1$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	$\vdots\quad\vdots\quad\vdots\quad\vdots$
$a_{2n}=(-1)^n\frac{1}{2^n}a_0$	$a_{2n+1}=(-1)^n\frac{1}{2^n}a_1$

$$\boxed{y(x) = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{4}x^4 - \frac{1}{8}x^6 + \cdots\right) + a_1\left(x - \frac{1}{2}x^3 + \frac{1}{4}x^5 - \frac{1}{8}x^7 + \cdots\right)}$$

$$\boxed{= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^{2n+1}}$$

Exercise

Find a power series solution. $\left(x^2 - 3\right)y'' + 2xy' = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2 - 3\right)y'' + 2xy' = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 3\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 2x \sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 3(n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} 2n a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 3(n+1)(n+2) a_{n+2} x^n + \sum_{n=0}^{\infty} 2n a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[-3(n+1)(n+2) a_{n+2} + \left(n^2 + n\right) a_n\right] x^n = 0$$

$$-3(n+1)(n+2) a_{n+2} + n(n+1) a_n = 0$$

$$\boxed{a_{n+2} = \frac{1}{3}\frac{n}{n+2} a_n}$$

a_0	a_1
$n = 0 \rightarrow\ a_2 = 0$	$n = 1\ \rightarrow\ a_3 = \frac{1}{3^2} a_1$

$n=2 \rightarrow a_4 = \frac{2}{12} a_2 = 0$

$n=4 \rightarrow a_6 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$n=3 \rightarrow a_5 = \frac{1}{3}\frac{3}{5} a_3 = \frac{1}{3^2 \cdot 5} a_1$

$n=5 \rightarrow a_7 = \frac{1}{3}\frac{5}{7} a_5 = \frac{1}{3^3 \cdot 7} a_1$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$a_{2n+1} = (-1)^n \frac{1}{(2n+1)3^n} a_1$$

$$y(x) = a_0 + a_1 \left(x + \frac{1}{9} x^3 + \frac{1}{45} x^5 + \frac{1}{189} x^7 + \cdots \right)$$

$$= a_0 + a_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)3^n} x^{2n+1}$$

Exercise

Find a power series solution. $\left(x^2+3\right)y'' - 7xy' + 16y = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2+3\right)y'' - 7xy' + 16y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 3\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - 7x\sum_{n=1}^{\infty} n a_n x^{n-1} + 16\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 3(n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 7n a_n x^n + \sum_{n=0}^{\infty} 16 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} 3(n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{\infty} 7n a_n x^n + \sum_{n=0}^{\infty} 16 a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[3(n+1)(n+2)a_{n+2}+\left(n^2-8n+16\right)a_n\right]x^n=0$$

$$3(n+1)(n+2)a_{n+2}+(n-4)^2a_n=0$$

$$\boxed{a_{n+2}=-\frac{(n-4)^2}{3(n+1)(n+2)}a_n}$$

a_0	a_1
$n=0\to\ a_2=-\frac{16}{6}a_0=-\frac{8}{3}a_0$	$n=1\ \to\ a_3=-\frac{9}{18}a_1=-\frac{1}{2}a_1$
$n=2\ \to\ a_4=-\frac{1}{9}a_2=\frac{8}{27}a_0$	$n=3\ \to\ a_5=-\frac{1}{60}a_3=\frac{1}{120}a_1$
$n=4\ \to\ a_6=0$	$n=5\ \to\ a_7=-\frac{1}{126}a_5=-\frac{1}{560\cdot 3^3}a_1$
$\vdots\quad\vdots\quad\vdots\quad\vdots$	$\vdots\quad\vdots\quad\vdots\quad\vdots$

$$\boxed{y(x)=a_0\left(1-\frac{8}{3}x^2+\frac{8}{27}x^4\right)+a_1\left(x-\frac{1}{2}x^3+\frac{1}{120}x^5+\frac{1}{15,120}x^7+\cdots\right)}$$

Exercise

Find the series solution to the initial value problem $y''+4y=0$; $y(0)=0,\quad y'(0)=3$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y''+4y=0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+4\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+4a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}+4a_n=0$$

$$\boxed{a_{n+2}=-\frac{4}{(n+1)(n+2)}a_n}$$

Given: $y(0)=0=a_0,\quad y'(0)=3=a_1$

$$a_0=0$$

$$n=0 \quad\rightarrow\quad a_2=-2a_0=0$$

$$n=2 \quad\rightarrow\quad a_4=-\frac{4}{12}a_2=0$$

$$n=4 \quad\rightarrow\quad a_6=0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_1=3$$

$$n=1 \quad\rightarrow\quad a_3=-\frac{4}{6}a_1=-\frac{2^2}{3!}a_1=-2$$

$$n=3 \quad\rightarrow\quad a_5=-\frac{4}{20}a_3=-\frac{2^4}{5!}a_1=\frac{2}{5}$$

$$n=5 \quad\rightarrow\quad a_7=-\frac{4}{42}a_5=-\frac{2^6}{7!}a_1=-\frac{4}{105}$$

$$a_{2k+1}=\frac{(-1)^k 2^{2k}}{(2k+1)!}a_1$$

$$\begin{aligned}
y(x)&=3x-2x^3+\frac{2}{5}x^5-\frac{4}{105}x^7+\cdots\\
&=3\left(x-\frac{2^2}{3!}x^3+\frac{2^4}{5!}x^5-\frac{2^6}{7!}x^7+\cdots\right)\\
&=\frac{3}{2}\left((2x)-\frac{1}{3!}(2x)^3+\frac{1}{5!}(2x)^5-\frac{1}{7!}(2x)^7+\cdots\right)\\
&\boxed{=\frac{3}{2}\sin 2x}
\end{aligned}$$

Exercise

Find the series solution to the initial value problem $y''+x^2y=0$; $\quad y(0)=1, \quad y'(0)=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''+x^2y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x^2\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}a_n x^{n+2}=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}a_{n-2}x^n=0$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}a_{n-2}x^n=0$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2}+a_{n-2}\right]x^n=0$$

$$(n+1)(n+2)a_{n+2}+a_{n-2}=0$$

$$\boxed{a_{n+2}=-\frac{1}{(n+1)(n+2)}a_{n-2}}$$

Given: $y(0)=1=a_0$, $y'(0)=0=a_1$

$2a_2+6a_3x=0 \quad \rightarrow \quad \boxed{a_2=a_3=0}$

$a_0=1$

$n=2 \quad \rightarrow \quad a_4=-\frac{1}{12}a_0=-\frac{1}{12}$

$a_1=a_2=a_3=0$

$n=3 \rightarrow a_5=-\frac{1}{20}a_1=0$

$n=4 \rightarrow a_6={}^*a_2=0$

$n=5 \rightarrow a_7={}^*a_3=0$

$n=6 \rightarrow a_8=-\frac{1}{56}a_4=\frac{1}{672}$

$n=10 \rightarrow a_{12}=-\frac{1}{132}a_8=\frac{1}{88,704}$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x)=1-\frac{1}{12}x^4+\frac{1}{672}x^8-\frac{1}{88,704}x^{12}+\cdots}$$

Exercise

Find the series solution to the initial value problem $y''-2xy'+8y=0;\quad y(0)=3,\quad y'(0)=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y''-2xy'+8y=0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+8\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}2(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}8a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+8a_n\right]x^n-\sum_{n=1}^{\infty}2na_n x^n=0$$

$$2a_2+8a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}+8a_n\right]x^n-\sum_{n=1}^{\infty}2na_n x^n=0$$

$$2a_2+8a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}+(8-2n)a_n\right]x^n=0$$

Given: $y(0)=3=a_0,\quad y'(0)=0=a_1$

$2a_2+8a_0=0 \quad\to\quad \boxed{a_2=-4a_0=-12}$

$(n+2)(n+1)a_{n+2}+(8-2n)a_n=0$

$\to a_{n+2}=\dfrac{2n-8}{(n+1)(n+2)}a_n \qquad n=1,2,\ldots$

$a_3=-a_1=0$ $\qquad a_4=-\frac{1}{3}a_2=4$

$a_5=-\frac{1}{10}a_3=0$ $\qquad a_6=0a_4=0$

$a_7=\frac{1}{21}a_5=0$ $\qquad a_6=0a_4=0$

$\boxed{y(x)=3-12x^2+4x^4}$

Exercise

Find the series solution to the initial value problem $y''+y'-2y=0$; $\quad y(0)=1, \quad y'(0)=-2$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$y''+y'-2y=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-2\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}-2a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}-2a_n=0$$

$$\boxed{a_{n+2}=\frac{2a_n-(n+1)a_{n+1}}{(n+1)(n+2)}}$$

Given: $y(0)=1=a_0$, $y'(0)=-2=a_1$

$a_0=1 \qquad\qquad a_1=-2$

$n=0 \ \rightarrow\ a_2=\frac{2a_0-a_1}{2}=2 \qquad\qquad n=1 \ \rightarrow\ a_3=\frac{2a_1-2a_2}{6}=-\frac{8}{6}$

$n=2 \ \rightarrow\ a_4=\frac{2a_2-3a_3}{12}=\frac{2}{3} \qquad\qquad n=3 \ \rightarrow\ a_5=\frac{2a_3-4a_4}{20}=-\frac{4}{15}$

$n=4 \ \rightarrow\ a_6=\frac{2a_4-5a_5}{30}=\frac{4}{45} \qquad\qquad n=5 \ \rightarrow\ a_7=\frac{2a_5-6a_6}{42}=\frac{8}{315}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x)=1-2x+2x^2-\frac{4}{3}x^3+\frac{2}{3}x^4-\frac{4}{15}x^5+\frac{4}{45}x^6-\frac{8}{315}x^7+\cdots}$$

$$=1+(-2x)+\frac{1}{2!}(-2x)^2+\frac{1}{3!}(-2x)^3+\frac{1}{4!}(-2x)^4+\frac{1}{5!}(-2x)^5+\frac{1}{6!}(-2x)^6+\frac{1}{7!}(-2x)^7+\cdots$$

$$\boxed{=e^{-2x}}$$

Exercise

Find the series solution to the initial value problem $y''-2y'+y=0$; $y(0)=0,$ $y'(0)=1$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y''-2y'+y=0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-2(n+1)a_{n+1}+a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}-2(n+1)a_{n+1}+a_n=0$$

$$\boxed{a_{n+2}=\frac{2(n+1)a_{n+1}-a_n}{(n+1)(n+2)}}$$

Given: $y(0)=0=a_0,\quad y'(0)=1=a_1$

$a_0=0 \qquad\qquad a_1=1$

$n=0 \;\rightarrow\; a_2=\dfrac{2a_1-a_0}{2}=1$

$n=1 \;\rightarrow\; a_3=\dfrac{4a_2-a_1}{6}=\dfrac{1}{2}$

$n=2 \;\rightarrow\; a_4=\dfrac{6a_3-a_2}{12}=\dfrac{1}{6}$

$n=3 \;\rightarrow\; a_5=\dfrac{8a_4-a_3}{20}=\dfrac{1}{20}\left(\dfrac{4}{3}-\dfrac{1}{2}\right)=\dfrac{1}{24}$

$n=4 \;\rightarrow\; a_6=\dfrac{10a_5-a_4}{30}=\dfrac{1}{30}\left(\dfrac{5}{12}-\dfrac{1}{6}\right)=\dfrac{1}{120}$

$n=5 \;\rightarrow\; a_7=\dfrac{12a_6-a_5}{42}=\dfrac{1}{42}\left(\dfrac{1}{10}-\dfrac{1}{24}\right)=\dfrac{1}{720}$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\begin{aligned}y(x)&=x+x^2+\frac{1}{2}x^3+\frac{1}{6}x^4+\frac{1}{24}x^5+\frac{1}{120}x^6+\frac{1}{720}x^7+\cdots\\&=x\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\cdots\right)\\&\boxed{=xe^x}\end{aligned}$$

Exercise

Find the series solution to the initial value problem $y''+xy'+y=0 \quad y(0)=1 \quad y'(0)=0$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$y''+xy'+y=0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x\sum_{n=1}^{\infty}na_n x^{n-1}+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}na_n x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+na_n+a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}+(n+1)a_n=0$$

$$(n+2)(n+1)a_{n+2}=-(n+1)a_n$$

$$a_{n+2}=-\frac{1}{n+2}a_n$$

$a_0=y(0)=1$ $\qquad$ $a_1=y'(0)=0$

$a_2=-\frac{1}{2}a_0=-\frac{1}{2}$ $\qquad$ $a_3=-\frac{1}{3}a_1=0$

$a_4=-\frac{1}{4}a_2=\frac{1}{2\cdot4}=\frac{1}{2^2\cdot1\cdot2}$ $\qquad$ $a_5=-\frac{1}{5}a_3=0$

$a_6=-\frac{1}{6}a_4=-\frac{1}{2^3\cdot1\cdot2\cdot3}$ $\qquad$ $a_7=-\frac{1}{7}a_7=0$

$$y(x)=\sum_{n=0}^{\infty}a_n x^n=a_0+a_1x+a_2x^2++a_3x^3+a_4x^4+a_5x^5+\cdots$$

$$=1-\frac{1}{2}x^2+\frac{1}{2^2 2!}x^4-\frac{1}{2^3 3!}x^5+\cdots$$

Exercise

Find the series solution to the initial value problem $y'' - xy' - y = 0 \quad y(0) = 2 \quad y'(0) = 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' - y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} - (n+1) a_n = 0$$

$$(n+2)(n+1) a_{n+2} = -(n+1) a_n$$

$$\boxed{a_{n+2} = \frac{1}{n+2} a_n}$$

$$a_0 = y(0) = 2 \qquad\qquad a_1 = y'(0) = 1$$

$$a_2 = \frac{1}{2} a_0 = 1 \qquad\qquad a_3 = \frac{1}{3} a_1 = \frac{1}{3}$$

$$a_4 = \frac{1}{4} a_2 = \frac{1}{4} \qquad\qquad a_5 = \frac{1}{5} a_3 = \frac{1}{3 \cdot 5}$$

$$a_6 = \frac{1}{6} a_4 = \frac{1}{4} \frac{1}{6} = \frac{1}{24} \qquad\qquad a_7 = \frac{1}{7} a_7 = \frac{1}{3 \cdot 5 \cdot 7}$$

$$\vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = 2 + x + x^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4 + \frac{1}{15} x^5 + \frac{1}{24} x^6 + \cdots}$$

Exercise

Find the series solution to the initial value problem $y'' - xy' - y = 0$; $y(0) = 1$ $y'(0) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' - xy' - y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - n a_n - a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} - (n+1) a_n = 0$$

$$(n+2)(n+1) a_{n+2} = -(n+1) a_n$$

$$a_{n+2} = \frac{1}{n+2} a_n$$

Given: $a_0 = y(0) = 1$ $\qquad$ $a_1 = y'(0) = 0$

$$a_2 = \frac{1}{2} a_0 = \frac{1}{2} \qquad a_3 = \frac{1}{3} a_1 = 0$$

$$a_4 = \frac{1}{4} a_2 = \frac{1}{2 \cdot 2^2} \qquad a_5 = \frac{1}{5} a_3 = 0$$

$$a_6 = \frac{1}{6} a_4 = \frac{1}{2^3 \cdot 3!} \qquad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots$$

$$y(x) = 1 + \frac{1}{2} x^2 + \frac{1}{8} x^4 + \frac{1}{48} x^6 + \cdots$$

Exercise

Find a power series solution. $y'' + xy' - 2y = 0$; $y(0) = 1$ $y'(0) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$y'' + xy' - 2y = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} + (n-2) a_n \right] x^n = 0$$

$$(n+2)(n+1) a_{n+2} + (n-2) a_n = 0$$

$$\boxed{a_{n+2} = -\frac{n-2}{(n+1)(n+2)} a_n}$$

$a_0 = y(0) = 1$ $a_1 = y'(0) = 0$

$n = 0 \to a_2 = \frac{2}{2} a_0 = 1$ $n = 1 \to a_3 = \frac{1}{6} a_1 = 0$

$n = 2 \to a_4 = 0$ $n = 3 \to a_5 = 0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$ $\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = 1 + x^2}$$

Exercise

Find the series solution to the initial value problem $y'' + (x-1)y' + y = 0 \qquad y(1) = 2 \qquad y'(1) = 0$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$y'' + (x-1)y' + y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + (x-1) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + \sum_{n=1}^{\infty} n a_n (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + n a_n + a_n \right] x^{n-1} = 0$$

$$(n+2)(n+1) a_{n+2} + (n+1) a_n = 0$$

$$\boxed{a_{n+2} = -\frac{1}{n+2} a_n}$$

$$a_0 = y(1) = 2 \qquad a_1 = y'(1) = 0$$

$$a_2 = -\frac{1}{2} a_0 = -1 \qquad a_3 = -\frac{1}{3} a_1 = 0$$

$$a_4 = -\frac{1}{4} a_2 = \frac{1}{2 \cdot 4} a_0 = \frac{1}{4} \qquad a_5 = -\frac{1}{5} a_3 = 0$$

$$a_6 = -\frac{1}{6} a_4 = -\frac{1}{24} \qquad a_7 = -\frac{1}{7} a_5 = 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n = a_0 + a_1 (x-1) + a_2 (x-1) + a_3 (x-1)^3 + a_4 (x-1)^4 + \cdots$$

$$\boxed{= 2 - (x-1)^2 + \frac{1}{4}(x-1)^4 - \frac{1}{24}(x-1)^6 + \cdots}$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{2n}}{n!\ 2^n}}$$

Exercise

Find the series solution to the initial value problem $(x-1)y''-xy'+y=0;\quad y(0)=-2,\quad y'(0)=6$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$(x-1)y''-xy'+y=0$$

$$(x-1)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^n-\sum_{n=1}^{\infty}na_n x^n+\sum_{n=0}^{\infty}\left[a_n-(n+2)(n+1)a_{n+2}\right]x^n=0$$

$$\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}-na_n\right]x^n+a_0-2a_2+\sum_{n=1}^{\infty}\left[a_n-(n+2)(n+1)a_{n+2}\right]x^n=0$$

$$a_0-2a_2+\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}-(n-1)a_n-(n+2)(n+1)a_{n+2}\right]x^n=0$$

Given: $y(0)=-2=a_0,\quad y'(0)=6=a_1$

$a_0-2a_2=0 \quad\rightarrow\quad a_2=\frac{1}{2}a_0=-1$

$n(n+1)a_{n+1}-(n-1)a_n-(n+2)(n+1)a_{n+2}=0$

$\rightarrow\; a_{n+2}=\frac{n}{n+2}a_{n+1}-\frac{n-1}{(n+2)(n+1)}a_n$

$a_3=\frac{1}{3}a_2-0a_1=-\frac{1}{3}$

$a_4=\frac{1}{2}a_3-\frac{1}{12}a_2=-\frac{1}{6}+\frac{1}{12}=-\frac{1}{12}$

$a_5=\frac{3}{5}a_4-\frac{1}{10}a_3=-\frac{3}{60}+\frac{1}{30}=-\frac{1}{60}$

$a_6=\frac{2}{3}a_5-\frac{1}{10}a_4=-\frac{1}{90}+\frac{1}{120}=-\frac{1}{360}$

$$y(x)=\left(-2-x^2-\frac{1}{3}x^3-\frac{1}{12}x^4-\ldots\right)+a_1x$$

$$= -2\left(1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\ldots\right)+6x$$

$$= -2\left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\ldots\right)+6x+2x$$

$$\underline{= 8x-2e^x}$$

Exercise

Find the series solution to the initial value problem

$$(x+1)y''-(2-x)y'+y=0; \quad y(0)=2, \quad y'(0)=-1$$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$(x+1)y''-(2-x)y'+y=0$$

$$(x+1)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-(2-x)\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}2(n+1)a_{n+1}x^n$$

$$+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+(n+1)a_{n+1}\right]x^{n+1}+\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-2(n+1)a_{n+1}+a_n\right]x^n=0$$

$$\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}+na_n\right]x^n+\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-2(n+1)a_{n+1}+a_n\right]x^n=0$$

$$\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}+na_n\right]x^n+2a_2-2a_1+a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}-2(n+1)a_{n+1}+a_n\right]x^n=0$$

$$2a_2-2a_1+a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}+(n-2)(n+1)a_{n+1}+(n+1)a_n\right]x^n=0$$

Given: $y(0)=2=a_0,\quad y'(0)=-1=a_1$

$$2a_2-2a_1+a_0=0 \quad\to\quad \boxed{a_2=\frac{1}{2}(2a_1-a_0)=-2}$$

$$(n+2)(n+1)a_{n+2}+(n-2)(n+1)a_{n+1}+(n+1)a_n=0$$

$$\to\ a_{n+2}=-\frac{n-2}{n+2}a_{n+1}-\frac{1}{n+2}a_n$$

$$a_3=\frac{1}{3}a_2-\frac{1}{3}a_1=\frac{2}{3}+\frac{1}{3}=1 \qquad a_4=0a_3-\frac{1}{4}a_2=\frac{1}{2}$$

$$a_5=-\frac{1}{5}a_4-\frac{1}{5}a_3=-\frac{1}{10}-\frac{1}{5}=-\frac{3}{10} \qquad a_6=-\frac{1}{3}a_5-\frac{1}{6}a_4=\frac{1}{10}-\frac{1}{12}=\frac{1}{60}$$

$$\boxed{y(x)=2-x-2x^2+x^3+\frac{1}{2}x^4-\frac{3}{10}x^5+\dots}$$

Exercise

Find the series solution to the initial value problem

$$(1-x)y''+xy'-2y=0\ ;\quad y(0)=0,\quad y'(0)=1$$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$(1-x)y''+xy'-2y=0$$

$$(1-x)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-2\sum_{n=0}^{\infty}a_nx^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1}+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}-\sum_{n=0}^{\infty}2a_nx^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-2a_n\right]x^n-\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-(n+1)a_{n+1}\right]x^{n+1}=0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-2a_n\right]x^n-\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}-na_n\right]x^n=0$$

$$2a_2-2a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}-2a_n\right]x^n-\sum_{n=1}^{\infty}\left[n(n+1)a_{n+1}-na_n\right]x^n=0$$

$$2a_2-2a_0+\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}-n(n+1)a_{n+1}+(n-2)a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}-n(n+1)a_{n+1}+(n-2)a_n=0$$

$$\boxed{a_{n+2}=\frac{n(n+1)a_{n+1}-(n-2)a_n}{(n+1)(n+2)}}$$

Given: $y(0)=0=a_0,\quad y'(0)=1=a_1$

$$2a_2-2a_0=0\quad\rightarrow\quad \boxed{a_2=a_0=0}$$

$$n=1\rightarrow\ a_3=\frac{2a_2+a_1}{6}=\frac{1}{6}$$

$$n=2\ \rightarrow\ a_4=\frac{6a_3}{12}=\frac{1}{12}$$

$$n=3\rightarrow\ a_5=\frac{1}{20}\left(12a_4-a_3\right)=\frac{1}{20}\left(1-\frac{1}{6}\right)=\frac{1}{24}$$

$$n=4\rightarrow\ a_6=\frac{1}{30}\left(20a_5-2a_4\right)=\frac{1}{30}\left(\frac{5}{6}-\frac{1}{6}\right)=\frac{1}{45}$$

$$n=5\rightarrow\ a_7=\frac{1}{42}\left(30a_6-3a_5\right)=\frac{1}{30}\left(\frac{2}{3}-\frac{1}{8}\right)=\frac{13}{1008}$$

$$\vdots\quad\vdots\quad\vdots\quad\vdots$$

$$\boxed{y(x)=x+\frac{1}{6}x^3+\frac{1}{12}x^4+\frac{1}{24}x^5+\frac{1}{45}x^6+\frac{13}{1008}x^7+\cdots}$$

Exercise

Find the series solution to the initial value problem

$$\left(x^2+1\right)y''+2xy'=0;\quad y(0)=0,\quad y'(0)=1$$

Solution

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$y'(x)=\sum_{n=1}^{\infty}na_nx^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y''+2xy'=0$$

$$\left(x^2+1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}2(n+1)a_{n+1}x^{n+1}=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=1}^{\infty}2na_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+2a_2+6a_3x+\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n+2a_1x+\sum_{n=2}^{\infty}2na_n x^n=0$$

$$2a_2+\left(6a_3+2a_1\right)x+\sum_{n=2}^{\infty}\left[\left(n(n-1)+2n\right)a_n+(n+2)(n+1)a_{n+2}\right]x^n=0$$

Given: $y(0)=0=a_0,\quad y'(0)=1=a_1$

$$2a_2+\left(6a_3+2a_1\right)x=0\quad\rightarrow\quad\begin{cases}a_2=0\\ a_3=-\frac{1}{3}a_1=-\frac{1}{3}\end{cases}$$

$$n(n+1)a_n+(n+2)(n+1)a_{n+2}=0$$

$$\rightarrow\ a_{n+2}=-\frac{n}{n+2}a_n\qquad n=2,3,\ldots$$

$$a_4=-\frac{1}{2}a_2=0\qquad\qquad a_5=-\frac{3}{5}a_3=\frac{1}{5}$$

$$a_6=-\frac{2}{3}a_4=0\qquad\qquad a_7=-\frac{5}{7}a_5=-\frac{1}{7}$$

$$\boxed{y(x)=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots}$$

Exercise

Find the series solution to the initial value problem

$$\left(x^2-1\right)y''+3xy'+xy=0\ ;\qquad y(0)=4,\qquad y'(0)=6$$

Solution

$$y(x)=\sum_{n=0}^{\infty} a_n x^n$$

$$y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

$$y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2-1\right)y''+3xy'+xy=0$$

$$\left(x^2-1\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+3x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+x\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}3(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}a_n x^{n+1}=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^{n+2}+\sum_{n=0}^{\infty}\left[(3n+3)a_{n+1}+a_n\right]x^{n+1}-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\sum_{n=1}^{\infty}\left(3na_n+a_{n-1}\right)x^n-\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^n+\left(3a_1+a_0\right)x+\sum_{n=2}^{\infty}\left(3na_n+a_{n-1}\right)x^n-2a_2-6a_3x-\sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}x^n=0$$

$$-2a_2+\left(3a_1+a_0-6a_3\right)x+\sum_{n=2}^{\infty}\left[\left(n^2+2n\right)a_n+a_{n-1}-(n+2)(n+1)a_{n+2}\right]x^n=0$$

Given: $y(0)=4=a_0,\quad y'(0)=6=a_1$

$$\begin{cases} -2a_2=0 & \rightarrow \boxed{a_2=0} \\ 3a_1+a_0-6a_3=0 & \rightarrow \boxed{a_3=\frac{22}{6}=\frac{11}{3}} \end{cases}$$

$$\left(n^2+2n\right)a_n+a_{n-1}-(n+2)(n+1)a_{n+2}=0$$

$$\boxed{a_{n+2}=\frac{\left(n^2+2n\right)a_n+a_{n-1}}{(n+1)(n+2)}}$$

$$n=2\ \rightarrow\ a_4=\frac{8a_2+a_1}{12}=\frac{6}{12}=\frac{1}{2}$$

$$n=3 \to a_5 = \frac{1}{20}\left(15a_3 + a_2\right) = \frac{1}{20}(55) = \frac{11}{4}$$

$$n=4 \to a_6 = \frac{1}{30}\left(24a_4 + a_3\right) = \frac{1}{30}\left(12 + \frac{11}{3}\right) = \frac{47}{90}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = 4 + 6x + \frac{11}{3}x^3 + \frac{1}{2}x^4 + \frac{11}{4}x^5 + \frac{47}{90}x^6 + \cdots}$$

Exercise

Find the series solution to the initial value problem

$$\left(2+x^2\right)y'' - xy' + 4y = 0 \qquad y(0) = -1 \qquad y'(0) = 3$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(2+x^2\right)y'' - xy' + 4y = 0$$

$$\left(2+x^2\right)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} na_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} na_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$2(n+2)(n+1)a_{n+2} + n(n-1)a_n - na_n + 4a_n = 0$$

$$2(n+2)(n+1)a_{n+2} + \left(n^2 - 2n + 4\right)a_n = 0$$

$$\boxed{a_{n+2} = -\frac{n^2 - 2n + 4}{2(n+2)(n+1)} a_n}$$

$$a_0 = y(0) = -1 \qquad\qquad a_1 = y'(0) = 3$$

$n=0 \rightarrow \quad a_2 = -\frac{4}{4}a_0 = 1$ $\qquad$ $n=1 \rightarrow \quad a_3 = -\frac{3}{12}a_1 = -\frac{1}{4}(3) = -\frac{3}{4}$

$n=2 \rightarrow \quad a_4 = -\frac{4}{24}a_2 = -\frac{1}{6}$ $\qquad$ $n=3 \rightarrow \quad a_5 = -\frac{7}{40}a_3 = -\frac{7}{40}\left(-\frac{3}{4}\right) = \frac{21}{160}$

$$y(x) = -1 + 3x + x^2 - \frac{3}{4}x^3 - \frac{1}{6}x^4 + \frac{21}{160}x^5 + \cdots$$

Exercise

Find the series solution to the initial value problem

$$\left(2-x^2\right)y'' - xy' + 4y = 0 \qquad y(0)=1 \qquad y'(0)=0$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(2-x^2\right)y'' - xy' + 4y = 0$$

$$\left(2-x^2\right)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[2(n+1)(n+2)a_{n+2} - \left(n^2 - n + n - 4\right)a_n\right]x^n = 0$$

$$2(n+1)(n+2)a_{n+2} - (n-2)(n+2)a_n = 0$$

$$a_{n+2} = \frac{n-2}{2(n+1)}a_n$$

$a_0 = y(0) = 1 \qquad\qquad a_1 = y'(0) = 0$

$n = 0 \rightarrow \quad a_2 = \frac{-2}{2} a_0 = -1 \qquad\qquad n = 1 \rightarrow \quad a_3 = -\frac{1}{4} a_1 = 0$

$n = 2 \rightarrow \quad a_4 = 0 \qquad\qquad n = 3 \rightarrow \quad a_5 = {}^* a_3 = 0$

$\vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = 1 - x^2}$$

Exercise

Find the series solution to the initial value problem

$$\left(4 - x^2\right) y'' + 2y = 0 \qquad y(0) = 0 \qquad y'(0) = 1$$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(4 - x^2\right) y'' + 2y = 0$$

$$\left(4 - x^2\right) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 4(n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[4(n+1)(n+2) a_{n+2} - \left(n^2 - n - 2\right) a_n \right] x^n = 0$$

$$4(n+1)(n+2) a_{n+2} - (n+1)(n-2) a_n = 0$$

$$\boxed{a_{n+2} = \frac{n-2}{4(n+2)} a_n}$$

$a_0 = y(0) = 0$ $\qquad a_1 = y'(0) = 1$

$n = 0 \rightarrow \quad a_2 = \frac{-2}{8} a_0 = 0$ $\qquad n = 1 \rightarrow \quad a_3 = -\frac{1}{12} a_1 = -\frac{1}{12}$

$n = 2 \rightarrow \quad a_4 = 0$ $\qquad n = 3 \rightarrow \quad a_5 = \frac{1}{20} a_3 = -\frac{1}{240}$

$\vdots \quad \vdots \quad \vdots$ $\qquad n = 5 \rightarrow \quad a_7 = \frac{3}{28} a_5 = -\frac{1}{2,240}$

$\vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = x - \frac{1}{12}x^3 - \frac{1}{240}x^5 - \frac{1}{2240}x^7 - \frac{1}{16,128}x^9 - \cdots}$$

Exercise

Find a power series solution. $\left(x^2 - 4\right)y'' + 3xy' + y = 0$; $\quad y(0) = 4, \quad y'(0) = 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\left(x^2 - 4\right)y'' + 3xy' + y = 0$$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 3x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=2}^{\infty} 4n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 4(n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 3n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[\left(n^2 - n + 3n + 1\right) a_n - 4(n+2)(n+1) a_{n+2}\right] x^n = 0$$

$$\left(n^2 + 2n + 1\right) a_n - 4(n+2)(n+1) a_{n+2} = 0$$

$$4(n+2)(n+1)a_{n+2} = (n+1)^2 a_n$$

$$\boxed{a_{n+2} = \frac{n+1}{4(n+2)}a_n}$$

$$a_0 = y(0) = 4 \qquad\qquad a_1 = y'(0) = 1$$

$$n = 0 \quad \to \quad a_2 = \frac{1}{8}a_0 = \frac{1}{2} \qquad\qquad n = 1 \quad \to \quad a_3 = \frac{2}{4\cdot 3}a_1 = \frac{1}{6}$$

$$n = 2 \quad \to \quad a_4 = \frac{3}{16}a_2 = \frac{3}{32} \qquad\qquad n = 3 \quad \to \quad a_5 = \frac{1}{5}a_3 = \frac{1}{30}$$

$$n = 4 \quad \to \quad a_6 = \frac{5}{24}a_4 = \frac{5}{256} \qquad\qquad n = 5 \quad \to \quad a_7 = \frac{6}{4\cdot 7}a_5 = \frac{1}{140}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\boxed{y(x) = 4 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{3}{32}x^4 + \frac{1}{30}x^5 + \frac{5}{256}x^6 + \frac{1}{140}x^7 + \cdots}$$

Exercise

Find a power series solution. $\left(x^2+1\right)y'' + 2xy' - 2y = 0$; $y(0) = 0,\quad y'(0) = 1$

Solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$\left(x^2+1\right)y'' + 2xy' - 2y = 0$$

$$x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2x\sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 2n a_n x^n - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+\left(n^2+n-2\right)a_n\right]x^n=0$$

$$(n+1)(n+2)a_{n+2}+(n-1)(n+2)a_n=0$$

$$\boxed{a_{n+2}=-\frac{n-1}{n+1}a_n}$$

$$a_0=y(0)=0 \qquad\qquad a_1=y'(0)=1$$

$$n=0 \;\rightarrow\; a_2=a_0 \qquad\qquad n=1 \;\rightarrow\; a_3=0$$

$$n=2 \;\rightarrow\; a_4=-\frac{1}{3}a_2=-\frac{1}{3}a_0 \qquad\qquad n=3 \;\rightarrow\; a_5=0$$

$$n=4 \;\rightarrow\; a_6=-\frac{3}{5}a_4=\frac{1}{5}a_0 \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n=6 \;\rightarrow\; a_8=-\frac{5}{7}a_6=-\frac{1}{7}a_0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{2n}=\frac{(-1)^n}{2n-1}a_0$$

$$\boxed{y(x)=a_1x+a_0\left(1+x^2-\frac{1}{3}x^4+\frac{1}{5}x^6-\frac{1}{7}x^8+\cdots\right)}$$

$$y(x)=a_1x+a_0\left(1+x\left(x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots\right)\right)$$

$$\boxed{=a_1x+a_0\left(1+x\tan^{-1}x\right)}$$

Exercise

Find a power series solution. $\left(2x-x^2\right)y''-6(x-1)y'-4y=0$; $y(1)=0,\quad y'(1)=1$

Solution

Let $z=x-1 \;\Rightarrow\; \begin{cases} x=z+1 \\ dz=dx \end{cases}$

$$\left(2x-x^2\right)y''-6(x-1)y'-4y=0$$

$$\left(2z+2-z^2-2z-1\right)y''-6zy'-4y=0$$

$$\left(1-z^2\right)y''-6zy'-4y=0$$

$$y(z)=\sum_{n=0}^{\infty}a_nz^n$$

$$y'(z)=\sum_{n=1}^{\infty} n a_n z^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n$$

$$y''(z)=\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n$$

$$\left(1-z^2\right)y''-6zy'-4y=0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}-z^2\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}-6z\sum_{n=1}^{\infty} n a_n z^{n-1}-4\sum_{n=0}^{\infty} a_n z^n=0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}-\sum_{n=2}^{\infty} n(n-1)a_n z^{n}-\sum_{n=1}^{\infty} 6n a_n z^{n}-4\sum_{n=0}^{\infty} a_n z^n=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}z^n-\sum_{n=0}^{\infty} n(n-1)a_n z^{n}-\sum_{n=0}^{\infty} 6n a_n z^{n}-4\sum_{n=0}^{\infty} a_n z^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}-\left(n^2+5n+4\right)a_n\right]z^n=0$$

$$(n+1)(n+2)a_{n+2}-(n+1)(n+4)a_n=0$$

$$\boxed{a_{n+2}=\frac{n+4}{n+2}a_n}$$

Given: $y(1)=0=a_0,\quad y'(1)=1=a_1$

$$a_0=0$$

$n=0 \;\to\; a_2=2a_0=0$

$n=2 \;\to\; a_4=\frac{3}{2}a_2=0$

$n=4 \;\to\; a_6=0$

$\vdots \;\; \vdots \;\; \vdots \;\; \vdots \;\; \vdots$

$$a_1=1$$

$n=1 \;\to\; a_3=\frac{5}{3}a_1=\frac{5}{3}$

$n=3 \;\to\; a_5=\frac{7}{5}a_3=\frac{7}{3}$

$n=5 \;\to\; a_7=\frac{9}{7}a_5=\frac{9}{3}$

$n=7 \;\to\; a_9=\frac{11}{9}a_7=\frac{11}{3}$

$\vdots \;\; \vdots \;\; \vdots \;\; \vdots \;\; \vdots$

$$a_{2n+1}=\frac{2n+3}{3}$$

$$y(z)=z+\frac{5}{3}z^3+\frac{7}{3}z^5+3z^7+\frac{11}{3}z^9+\cdots+\frac{2n+3}{3}z^{2n+1}+\cdots$$

$$\boxed{y(x)=(x-1)+\frac{5}{3}(x-1)^3+\frac{7}{3}(x-1)^5+3(x-1)^7+\frac{11}{3}(x-1)^9+\cdots}$$

Exercise

Find a power series solution. $\left(x^2-6x+10\right)y''-4(x-3)y'+6y=0$; $y(3)=2,\quad y'(3)=0$

Solution

Let $z=x-3 \quad\Rightarrow\quad \begin{cases} x=z+3 \\ dz=dx \end{cases}$

$$\left(x^2-6x+10\right)y''-4(x-3)y'+6y=0$$

$$\left(z^2+6z+9-6z-18+10\right)y''-4zy'+6y=0$$

$$\boxed{\left(z^2+1\right)y''-4zy'+6y=0}$$

$$y(z)=\sum_{n=0}^{\infty}a_n z^n$$

$$y'(z)=\sum_{n=1}^{\infty}na_n z^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n$$

$$y''(z)=\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n$$

$$\left(z^2+1\right)y''-4zy'+6y=0$$

$$z^2\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}+\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}-4z\sum_{n=1}^{\infty}na_n z^{n-1}+6\sum_{n=0}^{\infty}a_n z^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n z^n+\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}-\sum_{n=1}^{\infty}4na_n z^n+\sum_{n=0}^{\infty}6a_n z^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n z^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n-\sum_{n=0}^{\infty}4na_n z^n+\sum_{n=0}^{\infty}6a_n z^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+\left(n^2-5n+6\right)a_n\right]z^n=0$$

$$(n+1)(n+2)a_{n+2}+(n-2)(n-3)a_n=0$$

$$\boxed{a_{n+2}=-\frac{(n-2)(n-3)}{(n+1)(n+2)}a_n}$$

Given: $y(3)=2=a_0,\quad y'(3)=0=a_1$

$$a_0 = 2 \qquad\qquad a_1 = 0$$

$$n=0 \quad \rightarrow \quad a_2 = -3a_0 = -6 \qquad\qquad n=1 \quad \rightarrow \quad a_3 = -\frac{2}{4}a_1 = 0$$

$$n=2 \quad \rightarrow \quad a_4 = 0 \qquad\qquad n=3 \quad \rightarrow \quad a_5 = 0$$

$$n=4 \quad \rightarrow \quad a_6 = 0 \qquad\qquad n=5 \quad \rightarrow \quad a_7 = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(z) = 2 - 6z^2$$

$$\boxed{y(x) = 2 - 6(x-3)^2}$$

Exercise

Find a power series solution. $\left(4x^2 + 16x + 17\right)y'' - 8y = 0$; $y(-2) = 1,\quad y'(-2) = 0$

Solution

Let $z = x + 2 \quad \Rightarrow \quad \begin{cases} x = z - 2 \\ dz = dx \end{cases}$

$$\left(4z^2 - 16z + 16 + 16z - 32 + 17\right)y'' - 8y = 0$$

$$\boxed{\left(4z^2 + 1\right)y'' - 8y = 0}$$

$$y(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$y'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

$$y''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n$$

$$\left(4z^2 + 1\right)y'' - 8y = 0$$

$$4z^2 \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} - 8\sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=2}^{\infty} 4n(n-1) a_n z^n + \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} - 8\sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} 4n(n-1) a_n z^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n - 8\sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+\left(4n^2-4n-8\right)a_n\right]z^n=0$$

$$(n+1)(n+2)a_{n+2}+4(n+1)(n-2)a_n=0$$

$$\boxed{a_{n+2}=-\frac{4(n-2)}{n+2}a_n}$$

Given: $y(-2)=1=a_0, \quad y'(-2)=0=a_1$

$$a_0=1 \qquad\qquad a_1=0$$

$n=0 \quad\rightarrow\quad a_2=\frac{8}{2}a_0=4$ $\qquad$ $n=1 \quad\rightarrow\quad a_3=\frac{4}{3}a_1=0$

$n=2 \quad\rightarrow\quad a_4=-0a_2=0$ $\qquad$ $n=3 \quad\rightarrow\quad a_5=0$

$n=4 \quad\rightarrow\quad a_6=0$ $\qquad$ $n=5 \quad\rightarrow\quad a_7=0$

$\vdots \; \vdots \; \vdots \; \vdots \; \vdots$ $\qquad$ $\vdots \; \vdots \; \vdots \; \vdots \; \vdots$

$$y(z)=1+4z^2$$

$$\boxed{y(x)=1+4(x+2)^2}$$

Exercise

Find a power series solution. $\quad \left(x^2+6x\right)y''+(3x+9)y'-3y=0\ ; \quad y(-3)=0, \quad y'(-3)=2$

Solution

Let $\quad z=x+3 \quad\Rightarrow\quad \begin{cases} x=z-3 \\ dz=dx \end{cases}$

$$\left(x^2+6x\right)y''+(3x+9)y'-3y=0$$

$$\left(z^2-6z+9+6z-18\right)y''+3zy'-3y=0$$

$$\boxed{\left(z^2-9\right)y''+3zy'-3y=0}$$

$$y(z)=\sum_{n=0}^{\infty}a_nz^n$$

$$y'(z)=\sum_{n=1}^{\infty}na_nz^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n$$

$$y''(z)=\sum_{n=2}^{\infty}n(n-1)a_nz^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n$$

$$\left(z^2-9\right)y''+3zy'-3y=0$$

$$z^2\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}-9\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}+3z\sum_{n=1}^{\infty}na_n z^{n-1}-3\sum_{n=0}^{\infty}a_n z^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n z^n-\sum_{n=2}^{\infty}9n(n-1)a_n z^{n-2}+\sum_{n=1}^{\infty}3na_n z^n-\sum_{n=0}^{\infty}3a_n z^n=0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n z^n-\sum_{n=0}^{\infty}9(n+2)(n+1)a_{n+2}z^n+\sum_{n=0}^{\infty}3na_n z^n-\sum_{n=0}^{\infty}3a_n z^n=0$$

$$\sum_{n=0}^{\infty}\left[-9(n+2)(n+1)a_{n+2}+\left(n^2-n+3n-3\right)a_n\right]z^n=0$$

$$-9(n+2)(n+1)a_{n+2}+(n+3)(n-1)a_n=0$$

$$\boxed{a_{n+2}=\frac{(n+3)(n-1)}{9(n+1)(n+2)}a_n}$$

Given: $y(-3)=0=a_0, \quad y'(-3)=2=a_1$

$a_0=0$	$a_1=2$
$n=0 \ \rightarrow \ a_2=\frac{-3}{18}a_0=0$	$n=1 \ \rightarrow \ a_3=0a_1=0$
$n=2 \ \rightarrow \ a_4=\frac{5}{108}a_2=0$	$n=3 \ \rightarrow \ a_5=0$
$n=4 \ \rightarrow \ a_6=0$	$n=5 \ \rightarrow \ a_7=0$
$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$y(z)=2z$$

$$\boxed{y(x)=2x+6}$$

Exercise

Find the series solution near the given value $\qquad y''-(x-2)y'+2y=0\ ; \quad near\ x=2$

Solution

$$y=\sum_{n=0}^{\infty}a_n(x-2)^n$$

$$y'=\sum_{n=1}^{\infty}na_n(x-2)^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-2)^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n (x-2)^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} (x-2)^n$$

$$y'' - (x-2)y' + 2y = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} (x-2)^n - (x-2)\sum_{n=0}^{\infty} (n+1)a_{n+1} (x-2)^n + 2\sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} (x-2)^n - \sum_{n=0}^{\infty} (n+1)a_{n+1} (x-2)^{n+1} + \sum_{n=0}^{\infty} 2a_n (x-2)^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} + 2a_n\right](x-2)^n - \sum_{n=0}^{\infty} (n+1)a_{n+1} (x-2)^{n+1} = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1)(n+2)a_{n+2} + 2a_n\right](x-2)^n - \sum_{n=1}^{\infty} na_n (x-2)^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[(n+1)(n+2)a_{n+2} + 2a_n\right](x-2)^n - \sum_{n=1}^{\infty} na_n (x-2)^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[(n+1)(n+2)a_{n+2} - (n-2)a_n\right](x-2)^n = 0$$

For $n = 0 \quad \to \quad 2a_2 + 2a_0 = 0 \quad \Rightarrow \quad \boxed{a_2 = -a_0}$

$$(n+1)(n+2)a_{n+2} - (n-2)a_n = 0$$

$$\boxed{a_{n+2} = \frac{n-2}{(n+1)(n+2)} a_n}$$

a_0	a_1
$n = 0 \to a_2 = -a_0$	$n = 1 \to a_3 = -\frac{1}{6}a_1$
$n = 2 \to a_4 = 0$	$n = 3 \to a_5 = \frac{1}{20}a_3 = -\frac{1}{120}a_1$
$n = 4 \to a_6 = 0$	$n = 5 \to a_7 = \frac{3}{42}a_5 = -\frac{1}{1680}a_1$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\boxed{y(x) = a_0\left(1-(x-2)^2\right) + a_1\left((x-2) - \tfrac{1}{6}(x-2)^3 - \tfrac{1}{120}(x-2)^5 - \tfrac{1}{1680}(x-2)^7 - \cdots\right)}$$

Exercise

Find the series solution near the given value $y''+(x-1)^2 y'-4(x-1)y=0$; *near* $x=1$

Solution

$$y=\sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y'=\sum_{n=1}^{\infty} na_n (x-1)^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n$$

$$y''=\sum_{n=2}^{\infty} n(n-1)a_n (x-1)^{n-2}=\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n$$

$$y''+(x-1)^2 y'-4(x-1)y=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n+(x-1)^2\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n-4(x-1)\sum_{n=0}^{\infty}a_n(x-1)^n=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^{n+2}-\sum_{n=0}^{\infty}4a_n(x-1)^{n+1}=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}(x-1)^n-\sum_{n=1}^{\infty}4a_{n-1}(x-1)^n=0$$

$$2a_2+6a_3(x-1)+\sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}(x-1)^n$$

$$-4a_0(x-1)-\sum_{n=2}^{\infty}4a_{n-1}(x-1)^n=0$$

$$2a_2+\left(6a_3-4a_0\right)(x-1)+\sum_{n=2}^{\infty}\left[(n+1)(n+2)a_{n+2}+(n-5)a_{n-1}\right](x-1)^n=0$$

$$\begin{cases} 2a_2=0 & \rightarrow \underline{a_2=0|} \\ 6a_3-4a_0=0 & \rightarrow \underline{a_3=\frac{2}{3}a_0|} \end{cases}$$

$$(n+1)(n+2)a_{n+2}+(n-5)a_{n-1}=0$$

$$\underline{a_{n+2}=-\frac{(n-5)}{(n+1)(n+2)}a_{n-1}|}$$

$$a_0 \qquad\qquad a_1 \qquad\qquad a_2 = 0$$

$$n=1 \to a_3 = \tfrac{2}{3}a_0 \qquad n=2 \to a_4 = \tfrac{1}{4}a_1 \qquad n=3 \to a_5 = \frac{2}{20}a_2 = 0$$

$$n=4 \to a_6 = \frac{1}{30}a_3 = \frac{1}{45}a_0 \qquad n=5 \to a_7 = 0 \qquad n=6 \to a_8 = 0$$

$$n=7 \to a_9 = -\frac{2}{8\cdot 9}a_6 = -\frac{1}{1{,}620}a_0 \qquad n=8 \to a_{10} = 0 \qquad n=9 \to a_{11} = 0$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots \qquad\qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(x) = a_0\left(1 + \tfrac{2}{3}(x-1)^3 + \tfrac{1}{45}(x-1)^6 - \tfrac{1}{1{,}620}(x-1)^9 + \cdots\right) + a_1\left((x-1) + \tfrac{1}{4}(x-1)^4\right)$$

$$\boxed{y(x) = a_0\sum_{n=0}^{\infty}\frac{(-1)^n 4(x-1)^{3n}}{3^n(3n-1)(3n-4)n!} + a_1\left((x-1) + \tfrac{1}{4}(x-1)^4\right)}$$

Exercise

Find the series solution near the given value $y'' + (x-1)y = e^x$; *near* $x=1$

Solution

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y' = \sum_{n=1}^{\infty} na_n (x-1)^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n (x-1)^{n-2} = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n$$

$$y'' + (x-1)y = e^{x-1+1}$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n + (x-1)\sum_{n=0}^{\infty}a_n(x-1)^n = e\cdot e^{x-1} \qquad\qquad e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n + \sum_{n=0}^{\infty}a_n(x-1)^{n+1} = e\cdot\sum_{n=0}^{\infty}\frac{(x-1)^n}{n!}$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n + \sum_{n=1}^{\infty}a_{n-1}(x-1)^n = e\cdot\sum_{n=0}^{\infty}\frac{(x-1)^n}{n!}$$

$$2a_2 + \sum_{n=1}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n + \sum_{n=1}^{\infty}a_{n-1}(x-1)^n = e + e\cdot\sum_{n=1}^{\infty}\frac{(x-1)^n}{n!}$$

$$2a_2 + \sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2} + a_{n-1}\right](x-1)^n = e + e\cdot\sum_{n=1}^{\infty}\frac{(x-1)^n}{n!}$$

$$2a_2 = e \quad \to \quad \underline{a_2 = \frac{e}{2}}$$

$$(n+1)(n+2)a_{n+2} + a_{n-1} = \frac{e}{n!}$$

$$\underline{a_{n+2} = \frac{e}{(n+1)(n+2)\,n!} - \frac{1}{(n+1)(n+2)}a_{n-1}}$$

a_0 $\qquad\qquad\qquad\qquad\qquad$ a_1

$n=1 \ \to \ a_3 = \frac{e}{6} - \frac{1}{6}a_0$ $\qquad$ $n=2 \to \ a_4 = \frac{e}{24} - \frac{1}{12}a_1$

$n=4 \ \to \ a_6 = \frac{e}{720} - \frac{1}{30}a_3 = -\frac{11e}{720} + \frac{1}{180}a_0$ $\qquad$ $n=5 \to \ a_7 = \frac{e}{5040} - \frac{1}{42}a_4 = \frac{e}{1260} + \frac{1}{504}a_0$

$\vdots \quad \vdots \quad \vdots \quad \vdots$ $\qquad\qquad$ $\vdots \quad \vdots \quad \vdots \quad \vdots$

$a_2 = \frac{e}{2}$

$n=3 \ \to \ a_5 = \frac{e}{120} - \frac{1}{20}a_2 = \frac{e}{120} - \frac{e}{40} = -\frac{e}{60}$

$n=6 \ \to \ a_8 = \frac{e}{40,320} - \frac{1}{56}a_5 = \frac{e}{40,320} + \frac{e}{3360} = \frac{13e}{40,320}$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$\begin{aligned} y(x) &= a_0 + (x-1)a_1 + \frac{e}{2}(x-1)^2 + \left(\frac{e}{6} - \frac{1}{6}a_0\right)(x-1)^3 + \left(\frac{e}{24} - \frac{1}{12}a_1\right)(x-1)^4 - \frac{e}{60}(x-1)^5 + \cdots \\ &= a_0 + (x-1)a_1 + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + -\frac{1}{6}a_0(x-1)^3 + \frac{e}{24}(x-1)^4 \\ &\qquad -\frac{1}{12}a_1(x-1)^4 - \frac{e}{60}(x-1)^5 + \cdots \end{aligned}$$

$$\boxed{\begin{aligned} y(x) &= e\left(\frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4 - \frac{1}{60}(x-1)^5 + \cdots\right) \\ &\quad + a_0\left(1 - \frac{1}{6}(x-1)^3 + \cdots\right) + a_1\left((x-1) - \frac{1}{12}(x-1)^4 + \cdots\right) \end{aligned}}$$

Exercise

Find the series solution near the given value

$$y'' + xy' + (2x-1)y = 0\ ; \qquad near\ x=-1 \qquad y(-1)=2, \qquad y'(-1)=-2$$

Solution

$t = x+1 \quad \to \quad x = t-1$

$$y=\sum_{n=0}^{\infty}a_n t^n$$

$$y'=\sum_{n=1}^{\infty}na_n t^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n$$

$$y''=\sum_{n=2}^{\infty}n(n-1)a_n t^{n-2}=\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}t^n$$

$$y''+xy'+(2x-1)y=0$$

$$y''+(t-1)y'+(2t-3)y=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}t^n+(t-1)\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n+(2t-3)\sum_{n=0}^{\infty}a_n t^n=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}t^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}t^{n+1}-\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n+\sum_{n=0}^{\infty}2a_n t^{n+1}-\sum_{n=0}^{\infty}3a_n t^n=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}-(n+1)a_{n+1}-3a_n\right]t^n+\sum_{n=0}^{\infty}\left[(n+1)a_{n+1}+2a_n\right]t^{n+1}=0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}-(n+1)a_{n+1}-3a_n\right]t^n+\sum_{n=1}^{\infty}\left[na_n+2a_{n-1}\right]t^n=0$$

$$2a_2-a_1-3a_0+\sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2}-(n+1)a_{n+1}-3a_n\right]t^n+\sum_{n=1}^{\infty}\left[na_n+2a_{n-1}\right]t^n=0$$

$$2a_2-a_1-3a_0+\sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2}-(n+1)a_{n+1}+(n-3)a_n+2a_{n-1}\right]t^n=0$$

$$2a_2-a_1-3a_0=0\quad\rightarrow\quad \underline{a_2=\frac{1}{2}\left(a_1+3a_0\right)}$$

$$(n+1)(n+2)a_{n+2}-(n+1)a_{n+1}+(n-3)a_n+2a_{n-1}=0$$

$$\underline{a_{n+2}=\frac{1}{n+2}a_{n+1}-\frac{n-3}{(n+1)(n+2)}a_n-\frac{2}{(n+1)(n+2)}a_{n-1}}$$

Given: $t=x+1$

$y(x=-1)=y(t=0)=2=a_0,\qquad y'(x=-1)=y(t=0)=-2=a_1$

$$\underline{a_2}=\frac{1}{2}\left(a_1+3a_0\right)=\frac{1}{2}(-2+6)\ \underline{=2}$$

$$n=1 \;\rightarrow\; a_3 = \frac{1}{3}a_2 + \frac{1}{3}a_1 - \frac{1}{3}a_0 = \frac{2}{3} - \frac{2}{3} - \frac{2}{3} = \boxed{-\frac{2}{3}}$$

$$n=2 \;\rightarrow\; a_4 = \frac{1}{4}a_3 + \frac{1}{12}a_2 - \frac{1}{6}a_1 = -\frac{1}{6} + \frac{1}{6} + \frac{1}{3} = \boxed{\frac{1}{3}}$$

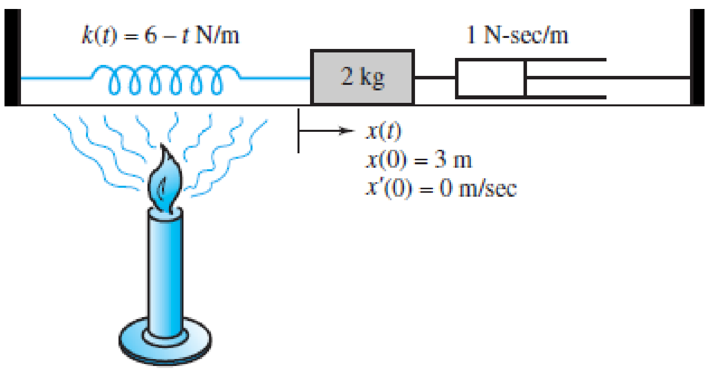
$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$y(t) = 2 - 2t + 3t^2 - \frac{1}{3}t^3 + \frac{1}{3}t^4 + \cdots$$

$$\boxed{y(x) = 2 - 2(x+1) + 3(x+1)^2 - \frac{1}{3}(x+1)^3 + \frac{1}{3}(x+1)^4 + \cdots}$$

Exercise

As a spring is heated, its spring "constant" decreases. Suppose the spring is heated so that the spring "constant" at time t is $k(t) = 6 - t$ *N/m*.



If the unforced mass-spring system has mass $m = 2$ *kg* and a damping constant $b = 1$ *N-sec/m* with initial conditions $x(0) = 3$ *m* and $x'(0) = 0$ *m/sec*, then the displacement $x(t)$ is governed by the initial value problem

$$2x''(t) + x'(t) + (6-t)x(t) = 0\;; \quad x(0) = 3, \quad x'(0) = 0$$

Find at least the first four nonzero terms in a power series expansion about $t = 0$ for the displacement.

Solution

$$x(t) = \sum_{n=0}^{\infty} a_n t^n$$

$$x'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} t^n$$

$$x''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n$$

$$2x'' + x' + (6-t)x = 0$$

$$2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}t^n + (6-t)\sum_{n=0}^{\infty}a_n t^n = 0$$

$$\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}t^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}t^n + \sum_{n=0}^{\infty}6a_n t^n - \sum_{n=0}^{\infty}a_n t^{n+1} = 0$$

$$\sum_{n=0}^{\infty}\left[2(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + 6a_n\right]t^n - \sum_{n=1}^{\infty}a_{n-1}t^n = 0$$

$$4a_2 + a_1 + 6a_0 + \sum_{n=1}^{\infty}\left[2(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + 6a_n\right]t^n - \sum_{n=1}^{\infty}a_{n-1}t^n = 0$$

$$4a_2 + a_1 + 6a_0 + \sum_{n=1}^{\infty}\left[2(n+1)(n+2)a_{n+2} + (n+1)a_{n+1} + 6a_n - a_{n-1}\right]t^n = 0$$

Given: $x(0) = 3 = a_0, \quad x'(0) = 0 = a_1$

$$4a_2 + a_1 + 6a_0 = 0 \quad \rightarrow \quad \boxed{a_2 = -\frac{9}{2}}$$

$$2(n+1)(n+2)a_{n+2} + (n+1)a_{n+1} + 6a_n - a_{n-1} = 0$$

$$\boxed{a_{n+2} = \frac{a_{n-1} - 6a_n - (n+1)a_{n+1}}{2(n+1)(n+2)}}$$

$$n = 1 \ \rightarrow \ a_3 = \frac{1}{12}\left(a_0 - 6a_1 - 2a_2\right) = \frac{1}{12}(3+9) = 1$$

$$n = 2 \ \rightarrow \ a_4 = \frac{1}{24}\left(a_1 - 6a_2 - 3a_3\right) = \frac{1}{24}(27-3) = 1$$

$$\boxed{x(t) = 3 - \frac{9}{2}t^2 + t^3 + t^4 + \cdots}$$