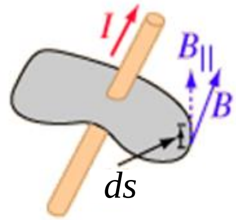
4.2 – Ampere's Law

Consider a small path element $d\vec{s}$ of a closed path. Let the magnetic field at the location of $d\vec{s}$ be $\vec{B}$ as shown.



Ampere law is based on the product of the path element ds and the component of the magnetic field in the direction element $d\vec{s}$. That is $ds \bullet B_{||}$.

If the angle between $d\vec{s}$ & $\vec{B}$ is θ then $B_{||} = B\cos\theta$.

$$ds\ B_{||} = ds\ B\cos\theta$$

But $ds\ B\cos\theta$ is also equal to the dot product of $\vec{B}$ & $d\vec{s}$

$$\Rightarrow \quad ds\ B_{||} = \vec{B} \bullet d\vec{s}$$

Ampere's Law

States that the integral (or the summation) of $\vec{B} \bullet d\vec{s}$ along a closed path is equal to μ_0 times the net current crossing the closed path

$$\oint_{\substack{closed\\ path}} \vec{B} \bullet d\vec{s} = \mu_0 I$$

$\vec{B}$ → Magnetic field at the location of $d\vec{s}$

I → Current crosses the closed path

André-Maris Ampère

Applications of Ampere's law

1. Magnetic field due to an infinitely loop current carrying straight wire

The following conclusions can be inferred from symmetry

a) The magnetic field lines must be circular lines concentric with the wire.
b) The magnitude of the magnetic field on a circle concentric with the wire must be constant.

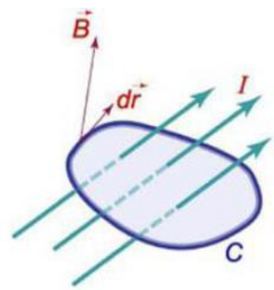
Consider a wire penetration the plane of the paper perpendicularly.

Even though the choice of the closed path is arbitrary, it should be chosen in such a way as it simplifies the problem, In this case it is easier to choose the closed path to be one of the concentric circular field lines. Since the field is tangent to the field lines, on this path $\vec{B}$ & $d\vec{s}$ are parallel.

$$\therefore \quad d\vec{s} \cdot B = Bds$$

Now applying Ampere's Law

$$\oint_{\substack{closed\\path}} \vec{B} \bullet d\vec{s} = \mu_0 I$$ Where I is the current in the wire



$$\oint_{\substack{closed\\path}} Bds = \mu_0 I$$

Since B is constant over the entire circular path (by symmetry), it can be factored out from the integral

$$\Rightarrow \oint_{\substack{closed\\path}} Bds = B \oint ds = \mu_0 I$$

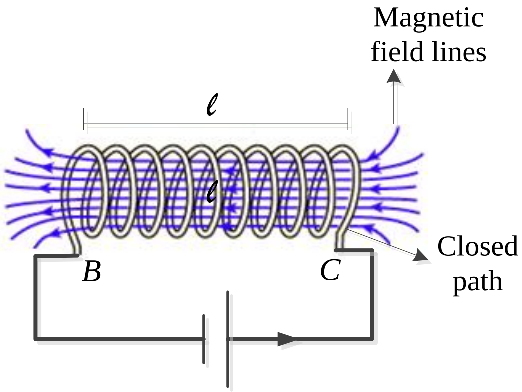
$\int ds$ is equal to the circumference of the circle (sum of all $ds's$)

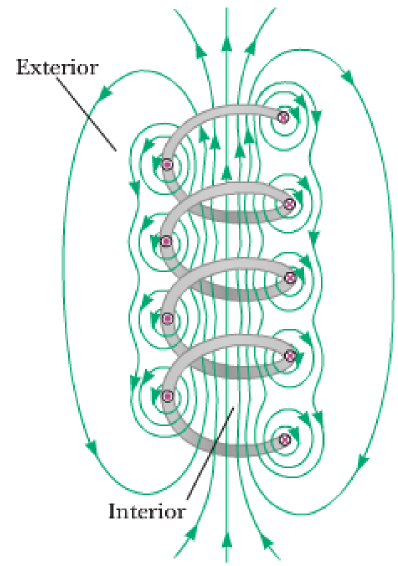
$$B 2\pi r_\perp = \mu_0 I$$

$$\Rightarrow \boxed{B = \frac{\mu_0 I}{2\pi r_\perp}}$$

Which is the same with the expression obtained in previous section using Biot-Savart Law.

2. Magnetic field inside a solenoid





The following conclusions can be made about a solenoid from symmetry

a) The magnetic field just outside a solenoid is approximately zero.

b) The field inside a solenoid is approximately constant & parallel to the axis of the solenoid.

Let the number of turns of the solenoid is N and let its length be ℓ. Let the current be I.

To take advantage of the symmetries the closed path is chosen to a rectangle with one of the side inside (|| to its axis) & the other side outside as shown.

This closed path is being crossed by the wires N times. Therefore the net current crossing the closed path is N time I.

$$\oint_{\substack{closed\\path}} \vec{B}\bullet d\vec{s} = \mu_0 NI$$

The integral is zero on the side $\left(\overline{BC}\right)$ that lies outside the solenoid because the field is approximately zero over there. The integral is also zero on the vertical sides $\left(\overline{AB}\ \&\ \overline{CD}\right)$ because outside $B = 0$ & inside $\vec{B}$ & $d\vec{s}$ are $\perp$ to each other $(Bds\cos 90° = 0)$. Therefore the non-zero contribution comes only from the side $\left(\overline{AD}\right)$ inside the solenoid

$$\int_{over\ \overline{AD}} \vec{B}\bullet d\vec{s} = \mu_0 NI$$

Since $\vec{B}$ is approximately parallel to the axis $\vec{B}$ & $d\vec{s}$ are parallel

$$\vec{B}\cdot d\vec{s} = Bds\cos(0) = Bds$$

$$\int_{\overline{AD}} Bds = \mu_0 NI$$

Since B is approximately constant inside, B can be factored out from the integral

$$B \int_{over\,\overline{AD}} ds = \mu_0 NI$$

$$\int_{over\,\overline{AD}} ds = length\ of\ \overline{AD} = \ell$$

$$B\ell = \mu_0 NI \quad \Rightarrow \quad \boxed{B = \frac{\mu_0 NI}{\ell}}$$

$B \rightarrow$ Field inside a solenoid

$N \rightarrow$ Number of turns

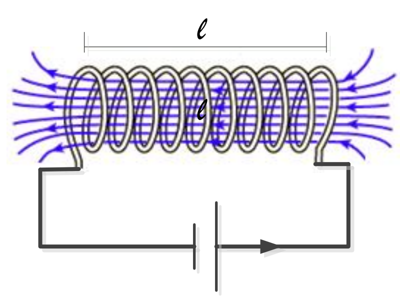
$\ell \rightarrow$ Length of solenoid

Customarily the ratio $\frac{N}{\ell}$ which is number of turns per a unit length & is denoted by n

$$B = \mu_0 nI \qquad where \quad n = \frac{N}{\ell}$$

The direction of the current and the direction of the field are related by the right hand rule. If fingers are wrapped around the solenoid in the direction of the current, then thumb will point in the direction of the field

Example



The solenoid shown has 50 turns.
It is 10 cm long & carries a current of 2A. Calculate the strength of the magnetic field inside the solenoid and determine whether the right end or lower end is the North Pole.

Solution

Given: $\mu_0 = 4\pi \times 10^{-7}, \quad N = 50, \quad \ell = 0.1m, \quad I = 2A$

$$B = \frac{\mu_0 NI}{\ell}$$

$$= \frac{4\pi \times 10^{-7}(50)(2)}{0.1}$$

$$= 4\pi \times 10^{-4}\ T$$

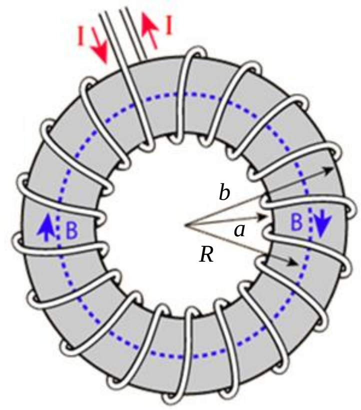
From the right hand rule the direction of the field is as indicated. Thus the left end is the North Pole

3. Magnetic field inside a toroid

From symmetry, the following conclusions can be inferred

a) The field inside the toroid is approximately constant

b) The field line inside the toroid are circles concentric with the center on the toroid



Let its radius be R, number of turns N & current I.

Let's choose the closed path to be a circle (concentric with its center) inside the toroid as shown.

Since the field line is also circular $\vec{B}$ (which is tangent to the circle) & $d\vec{s}$ are parallel:

$$\vec{B} \bullet d\vec{s} = Bds\cos\theta = Bds$$

This closed path is being crosses N times by the coils and thus the net crossing current is NI.

$$\oint_{\substack{close\\path}} \vec{B} \bullet d\vec{s} = \mu_0 NI \quad \Rightarrow \quad \oint_{\substack{close\\path}} Bds = \mu_0 NI$$

And since B is approximately constant over the entire path B can be factored out

$$\oint_{\substack{close\\path}} Bds = B \oint_{\substack{close\\path}} ds = \mu_0 NI$$

$$\oint_{\substack{close\\path}} ds = circumference\ of\ \ the\ closed\ \ path = 2\pi R$$

$$B \oint_{\substack{close\\path}} ds = B(2\pi R) = \mu_0 NI$$

$$B = \frac{\mu_0 NI}{2\pi R}$$ Magnetic field inside a toroid of radius R.

The direction of the field inside a toroid (that is whether clockwise or counterclockwise) is related with the direction of the current by the right hand rule. If fingers are wrapped around the toroid in the direction of the current then thumb will point in the direction of the field.

Example

A toroid with 100 turns has a radius of 5 *cm*. If it is carrying a current of 5*A*, determine the magnetic & direction of the field inside the toroid. The direction of the current is indicated in the diagram.

Solution

Given: $\mu_0 = 4\pi \times 10^{-7}, \quad N = 100, \quad R = 0.5m, \quad I = 5A$

$$B = \frac{\mu_0 NI}{2\pi R}$$

$$= \frac{4\pi \times 10^{-7}(100)(5)}{2\pi(0.05)}$$

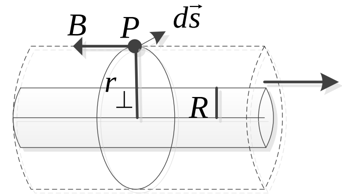
$$= 2 \times 10^{-3} \ T$$

From the right hand rule the direction of the magnetic field inside the toroid is counterclockwise.

4. Field due to an infinitely long thick cylindrical wire

Consider a wire of radius R carrying current I.

a) For points outside the wire $\left(r_{\perp} > R\right)$



From symmetry, the field lines are circular concentric with the cylinder and the magnitude is constant on this circle.

Taking the closed path to be one of these circles, the current crossing the loop is I & the path element $d\vec{s}$ and magnitude field $\vec{B}$ are parallel

$$\oint_{\substack{close\\path}} \vec{B}\cdot d\vec{s} = \mu_0 I$$

$$\vec{B}\cdot d\vec{s} = Bds \quad \text{(because the field is tangent to field lines)}$$

$$\oint_{\substack{close\\path}} Bds = B\oint_{\substack{close\\path}} ds = \mu_0 I$$

$$\text{But} \quad \oint_{\substack{close\\path}} ds = 2\pi r_{\perp}$$

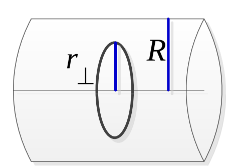
$$B 2\pi r_{\perp} = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r_{\perp}} \quad if \quad r_{\perp} > R$$

b) For points inside the wire $\left(r_{\perp} < R\right)$

Even though the same symmetry conditions apply, now with the circular closed path of radius $r_{\perp} < R$, only part of the current crosses this closed path. If the current density is J then

$$I = J\left(\pi R^2\right) \qquad \text{(remember } I = JA\text{)}$$



$A \rightarrow$ Area
$I \rightarrow$ Total current in the wire

If $I_{\perp}$ is the current crossing the circular path of radius $r_{\perp} < R$, then

$$I_{\perp} = J\pi r_{\perp}^2$$

$$\frac{I_{\perp}}{I} = \frac{J\pi r_{\perp}^{2}}{J\pi R^{2}} = \frac{r_{\perp}^{2}}{R^{2}}$$

$$I_{\perp} = \frac{r_{\perp}^{2}}{R^{2}} I$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\perp} = \mu_0 \frac{r_{\perp}^{2}}{R^{2}} I$$

$$\oint B ds = B \oint ds = \mu_0 \frac{r_{\perp}^{2}}{R^{2}} I$$

But $\oint ds = circumference = 2\pi r_{\perp}$

$$B 2\pi r_{\perp} = \frac{\mu_0 r_{\perp}^{2} I}{R^{2}}$$

$$\boxed{B = \frac{\mu_0 r_{\perp} I}{2\pi R^{2}}} \quad if \quad r_{\perp} < R$$

Modification of Ampere's Law

Ampere's law which states that the integral of $\vec{E} \cdot d\vec{s}$ along a closed path is equal to μ_0 times the current crossing the closed path, was deduced by Ampere experimentally. But when Maxwell was developing theory if electromagnetism, he found it necessary to include another kind if current in Ampere's law to make it consistent with the theory, This new kind of current is called ***displacement current*** $\left(I_0\right)$. Later it was found that his theoretical prediction was correct, with this new type of current, the modified Ampere's law is written as

$$\oint_{\substack{closed\\path}} \vec{B} \bullet d\vec{s} = \mu_0 \left(I + I_0\right)$$

Maxwell found that the displacement current is proportional to the rate of change of electric flux with time.

$$I_0 = \varepsilon_0 \frac{d\phi_E}{dt}$$

Where $\phi_E = \int \vec{E} \bullet d\vec{A}$

If ϕ_E is constant, $\phi_E = EA$

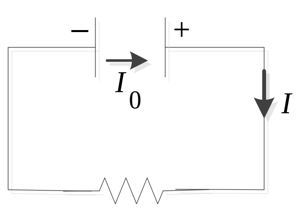
$E \rightarrow$ Electric field

$A \rightarrow$ Area

Therefore the modified Ampere's law is written as

$$\oint_{\substack{closed\\path}} \vec{B} \bullet d\vec{s} = \mu_0 \left(I + \varepsilon_0 \frac{d\phi_E}{dt}\right)$$

A very good example of a displacement current is the kind of current that exists between the plates of a capacitor. Consider a charges capacitor connected to an external resistor as shown.



Current will flow in the circuit until the capacitor is fully discharged. Even though the plates of a capacitor are separated by an insulator, current flows continuously in the circuit. This is because there is a different kind of current between the plates of the capacitor that we called displacement current. As the charge in the plates decreases, giving rise to a non-zero rate of change of electric flux

$$\frac{d\phi_E}{dt} = \frac{d}{dt}(EA) = A\frac{dE}{dt}$$

Actually it can be shown that the displacement current between the plates is equal to the traditional current in the wire.

$$I_0 = \varepsilon_0 \frac{d\phi_E}{dt}$$

$$\phi_E = \int \vec{E} \bullet d\vec{A} \quad but \quad \vec{E} \parallel d\vec{A} \quad \& \; E = constant$$

$\phi_E = EA$ Where A is the area of the plate.

$$I_0 = \varepsilon_0 \frac{d\phi_E}{dt} = \varepsilon_0 \frac{d}{dt}(EA) = \varepsilon_0 A \frac{dE}{dt}$$

The electric field between the parallel plates is given by

$E = \frac{\sigma}{\varepsilon_0}$ Where σ → Charge density in the plates

$\sigma = \frac{Q}{Q}$ Where Q → Charge in the plates

$$I_0 = \varepsilon_0 A \frac{dE}{dt} = \varepsilon_0 A \frac{d}{dt}\left(\frac{Q}{A\varepsilon_0}\right)$$

$$= \frac{\varepsilon_0 A}{\varepsilon_0 A} \frac{d}{dt}(Q)$$

$$= \frac{dQ}{dt}$$

But $\frac{dQ}{dt} = I$ → Current in the wires

$$\Rightarrow \boxed{I_0 = I}$$

Magnetic Flux

Magnetic Flux is a measure of the amount of magnitude field that crosses a certain area perpendicularly.

Consider a small area element $d\vec{A}$ (remember area is a vector quantity whose direction is $\perp$ to the plane of the area, thumb gives the direction of the area).

If the magnetic field at the location of $d\vec{A}$, is $\vec{B}$ then the magnetic flux is defined to be $B_{\perp}dA$ where $B_{\perp}$ is the component of the magnetic field perpendicularly.

If the angle between $d\vec{A}$ & $\vec{B}$ is θ, then $B_{\perp} = B\cos\phi$ and the flux (ϕ_B) is given by

$$d\phi_B = B\cos\theta \, dA$$

But this also equal to the dot product between $d\vec{A}$ & $\vec{B}$

$$d\phi_B = \vec{B}\bullet d\vec{A}$$

The total flux crossing a certain area is obtained by integrating this on the whole area

$$\boxed{\phi_B = \int d\phi_B = \int \vec{B}\bullet d\vec{A}}$$

If B & θ are constant over the entire area

$$\vec{B}\bullet d\vec{A} = BdA\ \cos\theta$$

$$\& \quad \phi_B = \int BdA\ \cos\theta = B\cos\theta\int dA \qquad \text{But } \int dA = A$$

$$\phi_B = BA\cos\theta$$

$\phi_B \rightarrow$ Magnetic flux

$B \rightarrow$ Magnitude of field

The unit of measurement for magnetic flux is Tm^2 which is sometimes called the ***Weber***.

Gauss' Law for Magnetic Field

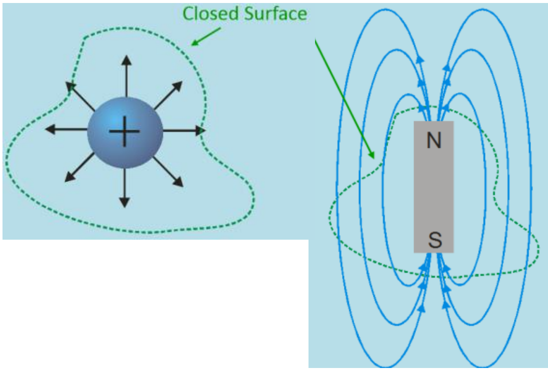
Gauss' law for magnetic field states that the magnetic flux crossing any closed surface is zero.

$$\oint_{\substack{closed \\ surface}} \vec{B} \bullet d\vec{A} = 0$$

This is basically a statement of the fact that magnetic field lines do not originate or sink anywhere but form complete loops.

Example

Consider the closed surface placed near a permanent magnet



<u>Solution</u>

From Gauss' law $\phi_B = 0$