Solution Section 2.4 – Integration of Rational Functions by Partial Fractions

Exercise

Evaluate $\displaystyle\int \frac{dx}{x^2+2x}$

Solution

$$\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2}$$

$$1 = Ax + 2A + Bx$$

$$x \qquad 2A = 1 \qquad \to A = \frac{1}{2}$$

$$x^0 \qquad A + B = 0 \quad \to B = -\frac{1}{2}$$

$$\int \frac{1}{x^2+2x}dx = \frac{1}{2}\int \frac{1}{x}dx \; - \; \frac{1}{2}\int \frac{1}{x+2}dx$$

$$= \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + C$$

Exercise

Evaluate $\displaystyle\int \frac{2x+1}{x^2-7x+12}dx$

Solution

$$\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3}$$

$$2x+1 = Ax - 3A + Bx - 4B$$

$$x \qquad A + B = 2$$

$$x^0 \qquad -3A - 4B = 1$$

$$A = \frac{\begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -3 & -4 \end{vmatrix}} = \frac{-9}{-1} = 9 \qquad\qquad B = \frac{\begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix}}{-1} = \frac{7}{-1} = -7$$

$$\int \frac{2x+1}{x^2-7x+12}dx = 9\int \frac{dx}{x-4} - 7\int \frac{dx}{x-3}$$

$$= 9\ln|x-4| - 7\ln|x-3| + C$$

$$\boxed{= \ln\left|\frac{(x-4)^9}{(x-3)^7}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x+3}{2x^3-8x}\,dx$

Solution

$$\frac{x+3}{2x^3-8x} = \frac{1}{2}\frac{x+3}{x\left(x^2-4\right)}$$

$$= \frac{1}{2}\left(\frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}\right)$$

$$= \frac{1}{2}\frac{A(x+2)(x-2) + Bx(x-2) + Cx(x+2)}{x(x+2)(x-2)}$$

$$Ax^2 - 4A + Bx^2 - 2Bx + Cx^2 + 2Cx = x+3$$

$$x^2 \quad A + B + C = 0$$

$$x \quad -2B + 2C = 1$$

$$x^0 \quad -4A = 3 \quad \to \boxed{A = -\frac{3}{4}}$$

$$\begin{cases} B + C = \frac{3}{4} \\ -2B + 2C = 1 \end{cases} \Rightarrow \quad \begin{aligned} 2B + 2C &= \frac{3}{2} \\ -2B + 2C &= 1 \end{aligned}$$

$$4C = \frac{5}{2} \quad \to \quad \boxed{C = \frac{5}{8}}$$

$$B = \frac{3}{4} - \frac{5}{8} \quad \to \quad \boxed{B = \frac{1}{8}}$$

$$\int \frac{x+3}{2x^3-8x}\,dx = \frac{1}{2}\int -\frac{3}{4}\frac{dx}{x} + \frac{1}{2}\int \frac{1}{8}\frac{dx}{x+2} + \frac{1}{2}\int \frac{5}{8}\frac{dx}{x-2}$$

$$= -\frac{3}{8}\ln|x| + \frac{1}{16}\ln|x+2| + \frac{5}{16}\ln|x-2| + K$$

$$= \frac{1}{16}\left(\ln|x+2| + 5\ln|x-2| - 6\ln|x|\right) + K$$

$$\boxed{= \frac{1}{16}\ln\left|\frac{(x+2)(x-2)^5}{x^6}\right| + K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2}{(x-1)\left(x^2+2x+1\right)}dx$

Solution

$$\frac{x^2}{(x-1)\left(x^2+2x+1\right)} = \frac{x^2}{(x-1)(x+1)^2}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$$= Ax^2 + 2Ax + A + Bx^2 - B + Cx - C$$

$$\begin{array}{lll} x^2 & A+B=1 & \to B = 1-A \\ x & 2A+C=0 & \to C=-2A \\ x^0 & A-B-C=0 & \to A-1+A+2A=0 \end{array}$$

$$\boxed{A=\frac{1}{4} \quad B=\frac{3}{4} \quad C=-\frac{1}{2}}$$

$$\int \frac{x^2}{(x-1)\left(x^2+2x+1\right)}dx = \frac{1}{4}\int\frac{dx}{x-1} + \frac{3}{4}\int\frac{dx}{x+1} - \frac{1}{2}\int\frac{dx}{(x+1)^2}$$

$$= \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2}\frac{1}{(x+1)} + K$$

$$= \frac{1}{4}\left(\ln|x-1| + \ln|x+1|^3\right) + \frac{1}{2(x+1)} + K$$

$$\boxed{= \frac{1}{4}\ln\left|(x-1)(x+1)^3\right| + \frac{1}{2(x+1)} + K}$$

Exercise

Evaluate $\displaystyle\int \frac{8x^2+8x+2}{\left(4x^2+1\right)^2}dx$

Solution

$$\frac{8x^2+8x+2}{\left(4x^2+1\right)^2} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{\left(4x^2+1\right)^2}$$

$$= \frac{(Ax+B)\left(4x^2+1\right)+Cx+D}{\left(4x^2+1\right)^2}$$

$$8x^2+8x+2=4Ax^3+4Bx^2+(A+C)x+B+D$$

$$\begin{cases} x^3 & A=0 \\ x^2 & 4B=8 \\ x & A+C=8 \\ x^0 & B+D=2 \end{cases} \rightarrow \boxed{A=0} \quad \boxed{B=2} \quad \boxed{C=8} \quad \boxed{D=0}$$

$$\int \frac{8x^2+8x+2}{\left(4x^2+1\right)^2}dx = \int \frac{2}{4x^2+1}dx + \int \frac{8x}{\left(4x^2+1\right)^2}dx \qquad d\left(4x^2+1\right)=8xdx$$

$$= \int \frac{2}{4x^2+1}\,dx + \int \frac{d\left(4x^2+1\right)}{\left(4x^2+1\right)^2} \qquad \int \frac{du}{u^2}=-\frac{1}{u} \quad \int \frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \tan^{-1}2x - \frac{1}{4x^2+1}+K$$

Exercise

Evaluate $\displaystyle\int \frac{x^2+x}{x^4-3x^2-4}dx$

Solution

$$\frac{x^2+x}{x^4-3x^2-4} = \frac{x^2+x}{\left(x^2-4\right)\left(x^2+1\right)}$$

$$= \frac{A}{x-2}+\frac{B}{x+2}+\frac{Cx+D}{x^2+1}$$

$$x^2+x = A(x+2)\left(x^2+1\right)+B(x-2)\left(x^2+1\right)+(Cx+D)\left(x^2-4\right)$$

$$= Ax^3+Ax+2Ax^2+2A+Bx^3+Bx-2Bx^2-2B+Cx^3-4Cx+Dx^2-4D$$

$$= (A+B+C)x^3+(2A-2B+D)x^2+(A+B-4C)x+2A-2B-4D$$

$$\begin{cases} x^3 & A+B+C=0 & (1) \\ x^2 & 2A-2B+D=1 & (2) \\ x & A+B-4C=1 & (3) \\ x^0 & 2A-2B-4D=0 & (4) \end{cases}$$

$$(1)-(3) \rightarrow \quad 5C=-1 \quad C=-\frac{1}{5}$$

$$(2)-(4) \rightarrow \quad 5D=1 \quad D=\frac{1}{5}$$

$$\begin{cases} A+B=\frac{1}{5} \\ 2A-2B=\frac{4}{5} \end{cases} \to \begin{cases} 2A+2B=\frac{2}{5} \\ 2A-2B=\frac{4}{5} \end{cases}$$

$$4A=\frac{6}{5} \quad \to \quad \underline{A=\frac{3}{10}}$$

$$B=\frac{1}{5}-\frac{3}{10} \quad \to \quad \underline{B=-\frac{1}{10}}$$

$$\int \frac{x^2+x}{x^4-3x^2-4}dx = \frac{3}{10}\int \frac{1}{x-2}dx - \frac{1}{10}\int \frac{1}{x+2}dx + \frac{1}{5}\int \frac{-x+1}{x^2+1}dx$$

$$= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{5}\int \frac{x}{x^2+1}dx + \frac{1}{5}\int \frac{1}{x^2+1}dx \qquad d\left(x^2+1\right)=2xdx$$

$$= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{10}\int \frac{d\left(x^2+1\right)}{x^2+1} + \frac{1}{5}\int \frac{1}{x^2+1}dx \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\underline{= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{10}\ln\left(x^2+1\right) + \frac{1}{5}\tan^{-1}x + K}$$

Exercise

Evaluate $\displaystyle\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{\left(\theta^2+1\right)^3}d\theta$

Solution

$$\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{\left(\theta^2+1\right)^3} = \frac{A\theta+B}{\theta^2+1} + \frac{C\theta+D}{\left(\theta^2+1\right)^2} + \frac{E\theta+F}{\left(\theta^2+1\right)^3}$$

$$\theta^4-4\theta^3+2\theta^2-3\theta+1 = (A\theta+B)\left(\theta^2+1\right)^2 + (C\theta+D)\left(\theta^2+1\right) + E\theta+F$$

$$= (A\theta+B)\left(\theta^4+2\theta^2+1\right) + C\theta^3 + C\theta + D\theta^2 + D + E\theta + F$$

$$= A\theta^5 + B\theta^4 + (2A+C)\theta^3 + (2B+D)\theta^2 + (A+C+E)\theta + B+D+F$$

$$\begin{cases} \boxed{A=0} \\ \boxed{B=1} \\ 2A+C=-4 \\ 2B+D=2 \\ A+C+E=-3 \\ B+D+F=1 \end{cases} \to \boxed{C=-4} \quad \boxed{D=0} \quad \boxed{E=1} \quad \boxed{F=0}$$

$$\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{\left(\theta^2+1\right)^3} d\theta = \int \frac{1}{\theta^2+1} d\theta - 4\int \frac{\theta}{\left(\theta^2+1\right)^2} d\theta + \int \frac{\theta}{\left(\theta^2+1\right)^3} d\theta$$

$$= \int \frac{1}{\theta^2+1} d\theta - 2\int \frac{d\left(\theta^2+1\right)}{\left(\theta^2+1\right)^2} + \frac{1}{2}\int \frac{d\left(\theta^2+1\right)}{\left(\theta^2+1\right)^3} \qquad d\left(\theta^2+1\right) = 2\theta d\theta$$

$$= \tan^{-1}\theta + 2\frac{1}{\theta^2+1} - \frac{1}{4}\frac{1}{\left(\theta^2+1\right)^2} + K$$

Exercise

Evaluate $\int \frac{x^4}{x^2-1} dx$

Solution

$$\frac{x^4}{x^2-1} = x^2 + 1 + \frac{1}{(x-1)(x+1)}$$

$$\begin{array}{r} x^2+1 \\ x^2-1 \overline{\smash{)}\, x^4 \qquad} \\ \underline{x^4 - x^2} \\ x^2 \\ \underline{x^2 - 1} \\ 1 \end{array}$$

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$1 = Ax + A + Bx - B$$

$$\begin{cases} A + B = 0 \\ A - B = 1 \end{cases} \rightarrow A = \frac{1}{2}, \quad B = -\frac{1}{2}$$

$$\int \frac{x^4}{x^2-1} dx = \int \left(x^2+1\right) dx + \frac{1}{2}\int \frac{1}{x-1} dx - \frac{1}{2}\int \frac{1}{x+1} dx$$

$$= \frac{1}{3}x^3 + x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

$$= \frac{1}{3}x^3 + x + \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) + C$$

$$= \frac{1}{3}x^3 + x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

Exercise

Evaluate $\int \frac{16x^3}{4x^2-4x+1} dx$

Solution

$$\frac{16x^3}{4x^2-4x+1} = 4x+4+\frac{12x-4}{(2x-1)^2}$$

$$= 4x+4+\frac{A}{2x-1}+\frac{B}{(2x-1)^2}$$

$$\begin{array}{r} 4x+4 \\ 4x^2-4x+1 \overline{)\,16x^3} \\ \underline{16x^3-16x^2+4x} \\ 16x^2-4x \\ \underline{16x^2-16x+4} \\ 12x-4 \end{array}$$

$$12x-4 = 2Ax - A + B$$

$$\begin{cases} 2A = 12 \\ -A+B=-4 \end{cases} \to \underline{A=6, \quad B=2}$$

$$\int \frac{16x^3}{4x^2-4x+1}dx = \int (4x+4)dx + 6\int \frac{dx}{2x-1} + 2\int \frac{dx}{(2x-1)^2}$$

$$= 2x^2+4x+6\left(\frac{1}{2}\right)\ln|2x-1| + 2\left(-\frac{1}{2}\right)\frac{1}{2x-1}+C$$

$$= 2x^2+4x+3\ln|2x-1| - \frac{1}{2x-1}+C$$

Exercise

Evaluate $\displaystyle\int \frac{e^{4x}+2e^{2x}-e^x}{e^{2x}+1}dx$

Solution

$$\int \frac{e^{4x}+2e^{2x}-e^x}{e^{2x}+1}dx = \int \frac{e^x\left(e^{3x}+2e^x-1\right)}{e^{2x}+1}dx \qquad y=e^x \Rightarrow dy = e^x dx$$

$$= \int \frac{y^3+2y-1}{y^2+1}dy$$

$$= \int \left(y+\frac{y-1}{y^2+1}\right)dy$$

$$= \int ydy + \int \frac{y}{y^2+1}dy - \int \frac{1}{y^2+1}dy$$

$$= \int ydy + \frac{1}{2}\int \frac{1}{y^2+1}d\left(y^2+1\right) - \int \frac{1}{y^2+1}dy \qquad d\left(y^2+1\right) = 2ydy$$

$$= \frac{1}{2}y^2 + \frac{1}{2}\ln\left(y^2+1\right) - \tan^{-1}y + C$$

$$= \frac{1}{2}e^{2x} + \frac{1}{2}\ln\left(e^{2x}+1\right) - \tan^{-1}e^x + C$$

Exercise

Evaluate $\displaystyle\int \frac{\sin\theta d\theta}{\cos^2\theta + \cos\theta - 2}$

Solution

Let $y = \cos\theta \quad \Rightarrow \quad dy = -\sin\theta\, d\theta$

$$\int \frac{\sin\theta d\theta}{\cos^2\theta + \cos\theta - 2} = -\int \frac{dy}{y^2 + y - 2}$$

$$\frac{1}{y^2 + y - 2} = \frac{1}{(y+2)(y-1)} = \frac{A}{y+2} + \frac{B}{y-1}$$

$$1 = (A+B)y - A + 2B$$

$$\begin{cases} A + B = 0 \\ -A + 2B = 1 \end{cases} \quad \rightarrow \quad \boxed{A = -\tfrac{1}{3}, \quad B = \tfrac{1}{3}}$$

$$\begin{aligned}
\int \frac{\sin\theta d\theta}{\cos^2\theta + \cos\theta - 2} &= -\left(-\frac{1}{3}\int \frac{dy}{y+2} + \frac{1}{3}\int \frac{dy}{y-1}\right) \\
&= \frac{1}{3}\ln|y+2| - \frac{1}{3}\ln|y-1| + C \\
&= \frac{1}{3}\left(\ln|y+2| - \ln|y-1|\right) + C \\
&= \frac{1}{3}\ln\left|\frac{y+2}{y-1}\right| + C \\
&= \boxed{\frac{1}{3}\ln\left|\frac{\cos\theta + 2}{\cos\theta - 1}\right| + C}
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int \frac{(x-2)^2 \tan^{-1}(2x) - 12x^3 - 3x}{\left(4x^2+1\right)(x-2)^2}\, dx$

Solution

$$\begin{aligned}
\int \frac{(x-2)^2 \tan^{-1}(2x) - 12x^3 - 3x}{\left(4x^2+1\right)(x-2)^2}\, dx &= \int \frac{(x-2)^2 \tan^{-1}(2x)}{\left(4x^2+1\right)(x-2)^2}\, dx - \int \frac{12x^3 + 3x}{\left(4x^2+1\right)(x-2)^2}\, dx \\
&= \int \frac{\tan^{-1}(2x)}{4x^2+1}\, dx - \int \frac{3x\left(4x^2+1\right)}{\left(4x^2+1\right)(x-2)^2}\, dx \\
&= \int \frac{\tan^{-1}(2x)}{4x^2+1}\, dx - \int \frac{3x}{(x-2)^2}\, dx
\end{aligned}$$

$$d\left(\tan^{-1}2x\right)=\frac{dx}{(2x)^2+1}=\frac{dx}{4x^2+1}$$

$$\frac{3x}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}$$

$$=\frac{Ax-2A+B}{(x-2)^2}$$

$$\begin{cases} \underline{A=3} \\ -2A+B=0 \end{cases} \rightarrow \underline{B=6}$$

$$\int\frac{(x-2)^2\tan^{-1}(2x)-12x^3-3x}{\left(4x^2+1\right)(x-2)^2}dx=\frac{1}{2}\int\tan^{-1}(2x)\,d\left(\tan^{-1}(2x)\right)-3\int\frac{dx}{x-2}-6\int\frac{dx}{(x-2)^2}$$

$$=\frac{1}{4}\left(\tan^{-1}(2x)\right)^2-3\int\frac{d(x-2)}{x-2}-6\int\frac{d(x-2)}{(x-2)^2}$$

$$=\frac{1}{4}\left(\tan^{-1}(2x)\right)^2-3\ln|x-2|-\frac{6}{x-2}+C$$

Exercise

Evaluate $\displaystyle\int\frac{\sqrt{x+1}}{x}dx$

Solution

Let $x+1=u^2 \quad\Rightarrow\quad dx=2udu$

$$\int\frac{\sqrt{x+1}}{x}dx=\int\frac{u}{u^2-1}2udu$$

$$=2\int\frac{u^2}{u^2-1}du$$

$$=2\int\left(1+\frac{1}{u^2-1}\right)du$$

$$=2\int du+2\int\frac{1}{u^2-1}du$$

$$\begin{array}{r|l} & 1 \\ \hline u^2-1 & u^2 \\ & \underline{u^2-1} \\ & 1 \end{array}$$

$$\frac{1}{u^2-1}=\frac{A}{u-1}+\frac{B}{u+1}$$

$$=\frac{(A+B)u+A-B}{(u-1)(u+1)}$$

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \Rightarrow \quad \underline{A=\frac{1}{2}, \quad B=-\frac{1}{2}}$$

$$= 2\int du + 2\int\left(\frac{1}{2}\frac{1}{u-1} - \frac{1}{2}\frac{1}{u+1}\right)du$$

$$= 2u + \int \frac{1}{u-1}du - \int \frac{1}{u+1}du$$

$$= 2u + \ln|u-1| - \ln|u+1| + C$$

$$= 2\sqrt{x+1} + \ln\left|\sqrt{x+1}-1\right| - \ln\left|\sqrt{x+1}+1\right| + C$$

$$= 2\sqrt{x+1} + \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{x^3-2x^2+3x-4}{x^2+1}dx$

Solution

$$\int \frac{x^3-2x^2+3x-4}{x^2+1}dx = \int\left(x-2+\frac{2x-2}{x^2+1}\right)dx$$

$$\begin{array}{r} x-2 \\ x^2+1\overline{\big)\,x^3-2x^2+3x-4} \\ \underline{x^3 \qquad\quad +x} \\ -2x^2+2x-4 \\ \underline{-2x^2 \qquad\quad -2} \\ 2x-2 \end{array}$$

$$= \int(x-2)dx + \int\frac{2x}{x^2+1}dx - 2\int\frac{1}{x^2+1}dx$$

$$= \int(x-2)dx + \int\frac{d\left(x^2+1\right)}{x^2+1} - 2\int\frac{1}{x^2+1}dx$$

$$= \frac{1}{2}x^2 - 2x + \ln\left(x^2+1\right) - 2\tan^{-1}(x) + C$$

Exercise

Evaluate $\displaystyle\int \frac{4x^2+2x+4}{x+1}dx$

Solution

$$\int \frac{4x^2+2x+4}{x+1}dx = \int\left(4x+2+\frac{6}{x+1}\right)dx$$

$$= \int (4x-2)dx + \int \frac{6}{x+1}dx$$

$$= \int (4x-2)\, dx \;+\; 6\int \frac{d(x+1)}{x+1} \qquad\qquad \int \frac{d(U)}{U} = \ln|U|$$

$$\boxed{= 2x^2 - 2x + 6\ln|x+1| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{3x^2+7x-2}{x^3-x^2-2x}dx$

Solution

$$\frac{3x^2+7x-2}{x^3-x^2-2x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

$$3x^2+7x-2 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1)$$

$$\begin{aligned} &= Ax^2 - Ax - 2A \\ &\quad Bx^2 - 2Bx \\ &\quad Cx^2 + Cx \end{aligned}$$

$$\begin{cases} A+B+C=3 \\ -A-2B+C=7 \\ -2A=-2 \end{cases} \quad \rightarrow \boxed{A=1}$$

$$\begin{cases} B+C=2 \\ -2B+C=8 \end{cases} \rightarrow \boxed{B=-2} \quad \boxed{C=4}$$

$$\int \frac{3x^2+7x-2}{x^3-x^2-2x}\,dx = \int \left(\frac{1}{x} - \frac{2}{x+1} + \frac{4}{x-2}\right)dx$$

$$= \ln|x| - 2\ln|x+1| + 4\ln|x-2| + K$$

$$\boxed{= \ln\frac{|x|(x-2)^4}{(x+1)^2} + K}$$

Exercise

Evaluate $\displaystyle\int \frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}dx$

Solution

$$\frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}=\frac{A}{x-1}+\frac{B}{x-5}+\frac{C}{x+4}$$

$$3x^2+2x+5=(A+B+C)x^2+(-A+3B-6C)x-20A-4B+5C$$

$$\begin{cases} x^2 & A+B+C=3 \\ x & -A+3B-6C=2 \\ x^0 & -20A-4B+5C=5 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 1 \\ -1 & 3 & -6 \\ -20 & -4 & 5 \end{vmatrix}=180 \qquad D_A=\begin{vmatrix} 3 & 1 & 1 \\ 2 & 3 & -6 \\ 5 & -4 & 5 \end{vmatrix}=-90$$

$$D_B=\begin{vmatrix} 1 & 3 & 1 \\ -1 & 2 & -6 \\ -20 & 5 & 5 \end{vmatrix}=450 \qquad D_C=\begin{vmatrix} 1 & 1 & 3 \\ -1 & 3 & 2 \\ -20 & -4 & 5 \end{vmatrix}=180$$

$$\underline{A=\frac{1}{2},\quad B=\frac{5}{2},\quad C=1}$$

$$\int \frac{3x^2+2x+5}{(x-1)\left(x^2-x-20\right)}\,dx=\int\left(\frac{1}{2}\frac{1}{x-1}+\frac{5}{2}\frac{1}{x-5}+\frac{1}{x+4}\right)dx$$

$$=\frac{1}{2}\ln|x-1|+\frac{5}{2}\ln|x-5|+\ln|x+4|+K$$

Exercise

Evaluate $\displaystyle\int \frac{5x^2-3x+2}{x^3-2x^2}\,dx$

Solution

$$\frac{5x^2-3x+2}{x^3-2x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}$$

$$5x^2-3x+2=Ax^2-2Ax+Bx-2B+Cx^2$$

$$\begin{cases} x^2 & A+C=5 & \underline{C=4} \\ x & -2A+B=-3 & \underline{A=1} \\ x^0 & -2B=2 & \rightarrow \underline{B=-1} \end{cases}$$

$$\int \frac{5x^2-3x+2}{x^3-2x^2}\,dx=\int\frac{dx}{x}-\int\frac{dx}{x^2}+4\int\frac{dx}{x-2}$$

$$=\ln|x|+\frac{1}{x}+4\ln|x-2|+K$$

Exercise

Evaluate $\displaystyle\int \frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}dx$

Solution

$$\frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}=\frac{A}{x-2}+\frac{Bx+C}{x^2-2x+3}$$

$$7x^2-13x+13=Ax^2-2Ax+3A+Bx^2-2Bx+Cx-2C$$

$$\begin{cases} x^2 & A+B=7 \\ x^1 & -2A-2B+C=-13 \\ x^0 & 3A-2C=13 \end{cases}$$

$$D=\begin{vmatrix} 1 & 1 & 0 \\ -2 & -2 & 1 \\ 3 & 0 & -2 \end{vmatrix}=3 \qquad D_A=\begin{vmatrix} 7 & 1 & 0 \\ -13 & -2 & 1 \\ 13 & 0 & -2 \end{vmatrix}=15$$

$$D_B=\begin{vmatrix} 1 & 7 & 0 \\ -2 & -13 & 1 \\ 3 & 13 & -2 \end{vmatrix}=6 \qquad D_C=\begin{vmatrix} 1 & 1 & 7 \\ -2 & -2 & -13 \\ 3 & 0 & 13 \end{vmatrix}=3$$

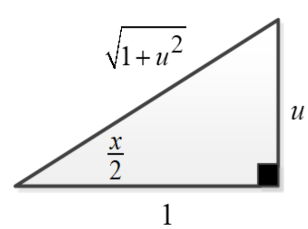
$$\underline{A=5;\ \ B=2;\ \ C=1}$$

$$\begin{aligned}\int \frac{7x^2-13x+13}{(x-2)\left(x^2-2x+3\right)}dx &= 5\int\frac{dx}{x-2}+\int\frac{2x+1}{x^2-2x+3}dx \\ &= 5\ln|x-2|+\int\frac{2x-2+3}{x^2-2x+3}dx \\ &= 5\ln|x-2|+\int\frac{2x-2}{x^2-2x+3}dx+\int\frac{3}{(x-1)^2+3}dx \\ &= 5\ln|x-2|+\ln\left(x^2-2x+3\right)+\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+K\end{aligned}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{1+\sin x}$

Solution



Let $u=\tan\left(\frac{x}{2}\right) \quad x=2\tan^{-1}u$

$$dx = \frac{2du}{1+u^2}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}}$$

$$= \frac{2u}{1+u^2}$$

$$\int \frac{dx}{1+\sin x} = \int \frac{1}{1+\frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2}\, du$$

$$= \int \frac{2}{u^2+2u+1}\, du$$

$$= \int \frac{2}{(u+1)^2}\, d(u+1)$$

$$= -\frac{2}{u+1} + C$$

$$\boxed{= -\frac{2}{\tan\left(\frac{x}{2}\right)+1} + C}$$

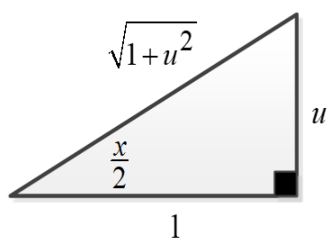
Exercise

Evaluate $\displaystyle\int \frac{dx}{2+\cos x}$

Solution

$$Let \quad u = \tan\left(\frac{x}{2}\right) \qquad x = 2\tan^{-1} u$$

$$dx = \frac{2du}{1+u^2}$$



$$\cos x = 2\cos^2\frac{x}{2} - 1$$

$$= 2\frac{1}{1+u^2} - 1$$

$$= \frac{1-u^2}{1+u^2}$$

$$\int \frac{dx}{2+\cos x} = \int \frac{1}{2+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}\, du$$

$$= 2\int \frac{1}{u^2+3}\,du \qquad\qquad \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C$$

$$\boxed{= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) + C}$$

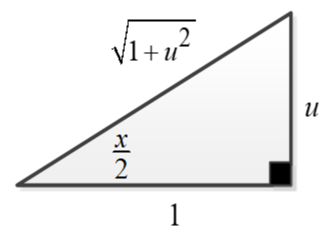
Exercise

Evaluate $\displaystyle\int \frac{dx}{1-\cos x}$

Solution

Let $u = \tan\left(\frac{x}{2}\right) \quad x = 2\tan^{-1} u$

$$dx = \frac{2du}{1+u^2}$$



$$\cos x = 2\cos^2\frac{x}{2} - 1$$

$$= 2\frac{1}{1+u^2} - 1$$

$$= \frac{1-u^2}{1+u^2}$$

$$\int \frac{dx}{1-\cos x} = \int \frac{1}{1-\frac{1-u^2}{1+u^2}}\cdot\frac{2}{1+u^2}\,du$$

$$= \int \frac{1}{u^2}\,du$$

$$= -\frac{1}{u} + C$$

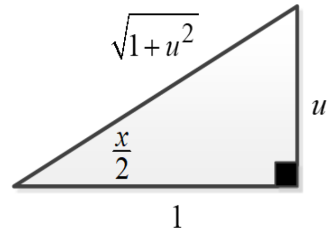
$$= -\frac{1}{\tan\frac{x}{2}} + C$$

$$\boxed{= -\cot\frac{x}{2} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

Solution



Let $u=\tan\left(\frac{x}{2}\right) \quad \to \quad x=2\tan^{-1}u$

$$dx=\frac{2du}{1+u^2}$$

$$\cos x=2\cos^2\frac{x}{2}-1$$

$$=2\frac{1}{1+u^2}-1$$

$$=\frac{1-u^2}{1+u^2}$$

$$\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$=2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}}$$

$$=\frac{2u}{1+u^2}$$

$$\int\frac{dx}{1+\sin x+\cos x}=\int\frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\cdot\frac{2}{1+u^2}\,du$$

$$=2\int\frac{1}{2+2u}\,du$$

$$=\int\frac{1}{1+u}\,d(1+u)$$

$$=\ln|1+u|+C$$

$$=\ln\left|1+\tan\frac{x}{2}\right|+C$$

Exercise

Evaluate $\displaystyle\int\frac{1}{x^2-5x+6}\,dx$

Solution

$$\frac{1}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3}$$

$$Ax - 3A + Bx - 2B = 1$$

$$\rightarrow \begin{cases} A + B = 0 \\ -3A - 2B = 1 \end{cases} \rightarrow \quad A = -1 \quad B = 1$$

$$\int \frac{1}{x^2 - 5x + 6}\, dx = \int \left(\frac{-1}{x-2} + \frac{1}{x-3} \right) dx$$

$$= \ln|x-3| - \ln|x-2| + C$$

$$\boxed{= \ln\left|\frac{x-3}{x-2}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^2 - 5x + 5}\, dx$

Solution

$$\frac{1}{x^2 - 5x + 5} = \frac{A}{x - \frac{5+\sqrt{5}}{2}} + \frac{B}{x - \frac{5-\sqrt{5}}{2}}$$

$$Ax - \left(\frac{5-\sqrt{5}}{2}\right)A + Bx - \left(\frac{5+\sqrt{5}}{2}\right)B = 1$$

$$\begin{cases} A + B = 0 \\ -\frac{5-\sqrt{5}}{2}A - \frac{5+\sqrt{5}}{2}B = 1 \end{cases}$$

$$\begin{cases} \frac{5-\sqrt{5}}{2}A + \frac{5-\sqrt{5}}{2}B = 0 \\ -\frac{5-\sqrt{5}}{2}A - \frac{5+\sqrt{5}}{2}B = 1 \end{cases}$$

$$-\sqrt{5}B = 1 \rightarrow \boxed{B = -\frac{1}{\sqrt{5}}} \quad \Rightarrow \quad \boxed{A = \frac{1}{\sqrt{5}}}$$

$$\int \frac{1}{x^2 - 5x + 5}\, dx = \int \left(\frac{\sqrt{5}}{5} \frac{2}{2x - 5 - \sqrt{5}} - \frac{\sqrt{5}}{5} \frac{2}{2x - 5 + \sqrt{5}} \right) dx$$

$$\boxed{= \frac{\sqrt{5}}{5} \ln\left|2x - 5 - \sqrt{5}\right| - \frac{\sqrt{5}}{5} \ln\left|2x - 5 + \sqrt{5}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$

Solution

$$\frac{5x^2+20x+6}{x^3+2x^2+x}=\frac{5x^2+20x+6}{x(x+1)^2}$$

$$=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$Ax^2+2Ax+A+Bx^2+Bx+Cx=5x^2+20x+6$$

$$\begin{cases} A+B=5 \\ 2A+B+C=20 \quad \rightarrow \quad \underline{B=-1} \quad \underline{C=9} \\ \underline{A=6} \end{cases}$$

$$\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx=\int\left(\frac{6}{x}-\frac{1}{x+1}+\frac{9}{(x+1)^2}\right)dx$$

$$=6\ln|x|-\ln|x+1|-\frac{9}{x+1}+C$$

$$=\ln\frac{x^6}{|x+1|}-\frac{9}{x+1}+C$$

Exercise

Evaluate $\displaystyle\int \frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}\,dx$

Solution

$$\frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}=\frac{2x^3-4x-8}{x(x-1)\left(x^2+4\right)}=\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+4}$$

$$Ax^3-Ax^2+4Ax-4A+Bx^3+4Bx+Cx^3-Cx^2+Dx^2-Dx=2x^3-4x-8$$

$$\begin{cases} x^3 & A+B+C=2 \\ x^2 & -A-C+D=0 \\ x^1 & 4A+4B-D=-4 \\ x^0 & -4A=-8 \end{cases} \rightarrow \begin{cases} B+C=0 \\ -C+D=2 \\ 4B-D=-12 \\ \underline{A=2} \end{cases}$$

$$\Rightarrow \begin{cases} B+D=2 \\ 4B-D=-12 \end{cases}$$

$$\underline{A=2 \quad B=-2 \quad C=2 \quad D=4}$$

$$\int \frac{2x^3-4x-8}{\left(x^2-x\right)\left(x^2+4\right)}dx = \int \left(\frac{2}{x}-\frac{2}{x-1}+\frac{2x}{x^2+4}+\frac{4}{x^2+4}\right)dx \qquad \int \frac{dx}{x^2+a^2}=\frac{1}{a}\arctan\frac{x}{a}$$

$$= 2\ln|x| - 2\ln|x-1| + \ln\left(x^2+4\right) + 2\tan^{-1}\frac{x}{2} + C$$

Exercise

Evaluate $\displaystyle\int \frac{8x^3+13x}{\left(x^2+2\right)^2}dx$

Solution

$$\frac{8x^3+13x}{\left(x^2+2\right)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{\left(x^2+2\right)^2}$$

$$Ax^3+2Ax+Bx^2+2B+Cx+D = 8x^3+13x$$

$$\begin{cases} x^3 & A=8 \\ x^2 & B=0 \\ x^1 & 2A+C=13 \\ x^0 & D=0 \end{cases} \rightarrow C=-3$$

$$\int \frac{8x^3+13x}{\left(x^2+2\right)^2}dx = \int \frac{8x}{x^2+2}dx - \int \frac{3x}{\left(x^2+2\right)^2}dx$$

$$= 2\int \frac{1}{x^2+2}d\left(x^2+2\right) - \frac{3}{2}\int \frac{1}{\left(x^2+2\right)^2}d\left(x^2+2\right)$$

$$= 2\ln\left(x^2+2\right) + \frac{3}{2}\frac{1}{x^2+2} + C$$

Exercise

Evaluate $\displaystyle\int \frac{\sin x}{\cos x+\cos^2 x}dx$

Solution

$$\frac{\sin x}{\cos x+\cos^2 x} = \frac{A}{\cos x} + \frac{B}{1+\cos x}$$

$$A + A\cos x + B\cos x = \sin x$$

$$\begin{cases} \boxed{A = \sin x} \\ A + B = 0 \end{cases} \rightarrow \boxed{B = -\sin x}$$

$$\int \frac{\sin x}{\cos x + \cos^2 x}\, dx = \int \frac{\sin x}{\cos x} dx - \int \frac{\sin x}{1 + \cos x} dx$$

$$= -\int \frac{1}{\cos x}\, d(\cos x) + \int \frac{1}{1 + \cos x}\, d(1 + \cos x)$$

$$= -\ln|\cos x| + \ln|1 + \cos x| + C$$

$$\boxed{= \ln\left|\frac{1 + \cos x}{\cos x}\right| + C}$$

$$\boxed{= \ln|\sec x + 1| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{5\cos x}{\sin^2 x + 3\sin x - 4}\, dx$

Solution

$$\frac{5\cos x}{\sin^2 x + 3\sin x - 4} = \frac{A}{\sin x - 1} + \frac{B}{\sin x + 4}$$

$$A\sin x + 4A + B\sin x - B = 5\cos x$$

$$\begin{cases} 4A - B = 5\cos x \\ A + B = 0 \end{cases}$$

$$\boxed{A = \cos x} \quad \boxed{B = -\cos x}$$

$$\int \frac{5\cos x}{\sin^2 x + 3\sin x - 4}\, dx = \int \frac{\cos x}{\sin x - 1} dx - \int \frac{\cos x}{\sin x + 4} dx$$

$$= \int \frac{1}{\sin x - 1} d(\sin x - 1) - \int \frac{1}{\sin x + 4} d(\sin x + 4)$$

$$= \ln|\sin x - 1| - \ln|\sin x + 4| + C$$

$$\boxed{= \ln\left|\frac{\sin x - 1}{\sin x + 4}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{e^x}{\left(e^x - 1\right)\left(e^x + 4\right)}\, dx$

Solution

Let $u = e^x \quad \to \quad du = e^x dx$

$$\int \frac{e^x}{\left(e^x - 1\right)\left(e^x + 4\right)}\, dx = \int \frac{du}{(u-1)(u+4)}$$

$$\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}$$

$$Au + 4A + Bu - B = 1$$

$$\Rightarrow \begin{cases} A + B = 0 \\ 4A - B = 1 \end{cases} \quad \to \quad \boxed{A = \frac{1}{5}, \quad B = -\frac{1}{5}}$$

$$\int \frac{du}{(u-1)(u+4)} = \frac{1}{5}\int \frac{1}{u-1}\, du \; + \; \frac{4}{5}\int \frac{1}{u+4}\, du$$

$$= \frac{1}{5}\int \frac{1}{u-1}\, d(u-1) \; + \; \frac{4}{5}\int \frac{1}{u+4}\, d(u+4)$$

$$= \frac{1}{5}\ln\left|e^x - 1\right| - \frac{1}{5}\ln\left(e^x + 4\right) + C$$

$$\boxed{= \frac{1}{5}\ln\left|\frac{e^x - 1}{e^x + 4}\right| + C}$$

Exercise

Evaluate $\displaystyle \int \frac{e^x}{\left(e^{2x} + 1\right)\left(e^x - 1\right)}\, dx$

Solution

Let $u = e^x \quad \to \quad du = e^x dx$

$$\int \frac{e^x}{\left(e^{2x} + 1\right)\left(e^x - 1\right)}\, dx = \int \frac{du}{\left(u^2 + 1\right)(u-1)}$$

$$\frac{1}{\left(u^2 + 1\right)(u-1)} = \frac{Au + B}{u^2 + 1} + \frac{C}{u-1}$$

$$Au^2 - Au + Bu - B + Cu^2 + C = 1$$

$$\begin{cases} u^2 & A+C=0 \\ u^1 & -A+B=0 \\ u^0 & -B+C=1 \end{cases} \to \begin{cases} B + C = 0 \\ -B + C = 1 \end{cases}$$

$$\boxed{C = \frac{1}{2} \quad B = -\frac{1}{2} \quad A = -\frac{1}{2}}$$

$$\int \frac{du}{\left(u^2+1\right)(u-1)} = -\frac{1}{2}\int \frac{u}{u^2+1}du - \frac{1}{2}\int \frac{du}{u^2+1} + \frac{1}{2}\int \frac{du}{u-1}$$

$$= -\frac{1}{4}\int \frac{1}{u^2+1}d\left(u^2+1\right) - \frac{1}{2}\arctan u + \frac{1}{2}\ln|u-1|$$

$$\boxed{= -\frac{1}{4}\ln\left(e^{2x}+1\right) - \frac{1}{2}\arctan e^x + \frac{1}{2}\ln\left|e^x-1\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{\sqrt{x}}{x-4}\, dx$

Solution

Let $u = \sqrt{x}$

$$du = \frac{1}{2\sqrt{x}}dx \Rightarrow 2udu = dx$$

$$\int \frac{\sqrt{x}}{x-4}\, dx = \int \frac{u}{u^2-4}\, 2u\, du$$

$$= \int \frac{2u^2}{u^2-4}\, du$$

$$= \int \left(2 + \frac{8}{u^2-4}\right) du$$

$$\frac{8}{u^2-4} = \frac{A}{u-2} + \frac{B}{u+2}$$

$$Au + 2A + Bu - 2B = 8$$

$$\begin{cases} A+B=0 \\ 2A-2B=8 \end{cases} \Rightarrow \boxed{A=2 \quad B=-2}$$

$$= \int \left(2 + \frac{2}{u-2} - \frac{2}{u+2}\right) du$$

$$= 2\sqrt{x} + 2\ln\left|\sqrt{x}-2\right| - 2\ln\left|\sqrt{x}+2\right| + C$$

$$\boxed{= 2\sqrt{x} + 2\ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right| + C}$$

Exercise

Evaluate $\quad \displaystyle\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$

Solution

Let $\quad u = x^{1/6} \to u^6 = x \quad \to \quad 6u^5 du = dx$

$u^2 = x^{1/3} \quad u^3 = x^{1/2}$

$$\begin{array}{r|l} & 6u^2+6u+6 \\ \hline u-1 & 6u^3 \\ & \underline{-6u^3+6u^2} \\ & \quad 6u^2 \\ & \quad \underline{-6u^2+6u} \\ & \qquad 6u \\ & \qquad \underline{-6u+6} \\ & \qquad\quad 6 \end{array}$$

$$\begin{aligned}
\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx &= \int \frac{6u^5}{u^3-u^2}\,du \\
&= \int \frac{6u^3}{u-1}\,du \\
&= \int \left(6u^2+6u+6+\frac{6}{u-1}\right)du \\
&= 2u^3+3u^2+6u+6\ln|u-1|+C \\
&= \boxed{2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+6\ln\left|\sqrt[6]{x}-1\right|+C}
\end{aligned}$$

Exercise

Evaluate $\quad \displaystyle\int \frac{1}{x^2-9}\,dx$

Solution

$$\frac{1}{x^2-9} = \frac{A}{x-3}+\frac{B}{x+3}$$

$$Ax+3A+Bx-3B = 1$$

$$\begin{cases} A+B=0 \\ 3A-3B=1 \end{cases} \to \boxed{A=\frac{1}{6} \quad B=-\frac{1}{6}}$$

$$\begin{aligned}
\int \frac{1}{x^2-9}\,dx &= \frac{1}{6}\int \frac{1}{x-3}\,dx-\frac{1}{6}\int \frac{1}{x+3}\,dx \\
&= \frac{1}{6}\ln|x-3|-\frac{1}{6}\ln|x+3|+C \\
&= \boxed{\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|+C}
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int \frac{2}{9x^2-1}\,dx$

Solution

$$\frac{2}{9x^2-1}=\frac{A}{3x-1}+\frac{B}{3x+1}$$

$$3Ax+A+3Bx-B=2$$

$$\begin{cases}3A+3B=0\\ A-B=2\end{cases} \rightarrow \underline{A=1\quad B=-1}$$

$$\int \frac{2}{9x^2-1}\,dx=\int \frac{1}{3x-1}\,dx-\int \frac{1}{3x+1}\,dx$$

$$=\frac{1}{3}\ln|3x-1|-\frac{1}{3}\ln|3x+1|+C$$

$$=\frac{1}{3}\ln\left|\frac{3x-1}{3x+1}\right|+C$$

Exercise

Evaluate $\displaystyle\int \frac{5}{x^2+3x-4}\,dx$

Solution

$$\frac{5}{x^2+3x-4}=\frac{A}{x-1}+\frac{B}{x+4}$$

$$Ax+4A+Bx-B=5$$

$$\begin{cases}A+B=0\\ 4A-B=5\end{cases} \rightarrow \underline{A=1\quad B=-1}$$

$$\int \frac{5}{x^2+3x-4}\,dx=\int \frac{1}{x-1}\,dx-\int \frac{1}{x+4}\,dx$$

$$=\ln|x-1|-\ln|x+4|+C$$

$$=\ln\left|\frac{x-1}{x+4}\right|+C$$

Exercise

Evaluate $\displaystyle\int \frac{3-x}{3x^2-2x-1}\,dx$

Solution

$$\frac{3-x}{3x^2-2x-1} = \frac{A}{x-1} + \frac{B}{3x+1}$$

$$3Ax + A + Bx - B = 3 - x$$

$$\begin{cases} 3A + B = -1 \\ A - B = 3 \end{cases} \rightarrow \underline{A = \frac{1}{2} \quad B = -\frac{5}{2}}$$

$$\int \frac{3-x}{3x^2-2x-1}\,dx = \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{5}{2}\int \frac{1}{3x+1}\,dx$$

$$= \frac{1}{2}\ln|x-1| - \frac{5}{6}\ln|3x+1| + C$$

Exercise

Evaluate $\displaystyle\int \frac{x^2+12x+12}{x^3-4x}\,dx$

Solution

$$\frac{x^2+12x+12}{x^3-4x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

$$Ax^2 - 4A + Bx^2 + 2Bx + Cx^2 - 2Cx = x^2 + 12x + 12$$

$$\begin{cases} x^2 & A + B + C = 1 \\ x^1 & 2B - 2C = 12 \\ x^0 & -4A = 12 \end{cases} \quad \rightarrow \underline{A = -3}$$

$$\begin{cases} B + C = 4 \\ B - C = 6 \end{cases} \qquad \underline{B = 5 \quad C = -1}$$

$$\int \frac{x^2+12x+12}{x^3-4x}\,dx = -\frac{3}{x} + \frac{5}{x-2} - \frac{1}{x+2}$$

$$= -3\ln|x| + 5\ln|x-2| - \ln|x+2| + C$$

Exercise

Evaluate $\displaystyle\int \frac{x^3-x+3}{x^2+x-2}\,dx$

Solution

$$\frac{x^3-x+3}{x^2+x-2} = x - 1 + \frac{2x+1}{x^2+x-2}$$

$$\begin{array}{r} x-1 \\ x^2+x-2\,\overline{)\,x^3-x+3} \\ \underline{-x^3-x^2+2x} \\ -x^2+x+3 \\ \underline{x^2+x-2} \\ 2x-1 \end{array}$$

$$\frac{2x+1}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$Ax + 2A + Bx - B = 2x + 1$$

$$\begin{cases} A + B = 2 \\ 2A - B = 1 \end{cases} \rightarrow \underline{A = 1 \quad B = 1}$$

$$\int \frac{x^3 - x + 3}{x^2 + x - 2}\, dx = \int \left(x - 1 + \frac{1}{x-1} + \frac{1}{x+2} \right) dx$$

$$= \frac{1}{2}x^2 - x + \ln|x-1| + \ln|x+2| + C$$

Exercise

Evaluate $\displaystyle\int \frac{5x-2}{(x-2)^2}\, dx$

Solution

$$\frac{5x-2}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$$

$$Ax - 2A + B = 5x - 2$$

$$\Rightarrow \begin{cases} \underline{A = 5} \\ -2A + B = -2 \end{cases} \rightarrow \underline{B = 8}$$

$$\int \frac{5x-2}{(x-2)^2}\, dx = \frac{5}{x-2} + \frac{8}{(x-2)^2}$$

$$= 5\ln|x-2| - \frac{8}{x-2} + C$$

Exercise

Evaluate $\displaystyle\int \frac{2x^3 - 4x^2 - 15x + 4}{x^2 - 2x - 8}\, dx$

Solution

$$\begin{array}{r|l} & 2x \\ \hline x^2 - 2x - 8 & 2x^3 - 4x^2 - 15x + 4 \\ & \underline{2x^3 - 4x^2 - 16x} \\ & \qquad\qquad x + 4 \end{array}$$

$$\int \frac{2x^3 - 4x^2 - 15x + 4}{x^2 - 2x - 8}\, dx = \int 2x\, dx + \int \frac{x+4}{x^2 - 2x - 8}\, dx$$

$$\frac{x+4}{x^2-2x-8}=\frac{A}{x-4}+\frac{B}{x+2}$$

$$Ax+2A+Bx-4B=x+4$$

$$\begin{cases} A+B=1 \\ 2A-4B=4 \end{cases} \rightarrow \boxed{A=\frac{4}{3} \quad B=-\frac{1}{3}}$$

$$\int \frac{2x^3-4x^2-15x+4}{x^2-2x-8}\,dx=x^2+\frac{4}{3}\int \frac{1}{x-4}\,dx-\frac{1}{3}\int \frac{1}{x+2}\,dx$$

$$\boxed{=x^2+\frac{4}{3}\ln|x-4|-\frac{1}{3}\ln|x+2|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{x+2}{x^2+5x}\,dx$

Solution

$$\frac{x+2}{x^2+5x}=\frac{A}{x}+\frac{B}{x+5}$$

$$Ax+5A+Bx=x+2$$

$$\begin{cases} A+B=1 \\ 5A=2 \end{cases} \rightarrow \boxed{A=\frac{2}{5} \quad B=\frac{3}{5}}$$

$$\int \frac{x+2}{x^2+5x}\,dx=\frac{2}{5}\int \frac{1}{x}\,dx+\frac{3}{5}\int \frac{1}{x+5}\,dx$$

$$\boxed{=\frac{2}{5}\ln|x|+\frac{3}{5}\ln|x+5|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{\sec^2 x}{\tan^2 x+5\tan x+6}\,dx$

Solution

Let $u=\tan x \quad du=\sec^2 x\,dx$

$$\int \frac{\sec^2 x}{\tan^2 x+5\tan x+6}\,dx=\int \frac{1}{u^2+5u+6}\,du$$

$$\frac{1}{u^2+5u+6}=\frac{1}{(u+2)(u+3)}=\frac{A}{u+2}+\frac{B}{u+3}$$

$$1=Au+3A+Bu+2B$$

$$\begin{cases} A+B=0 \\ 3A+2B=1 \end{cases} \quad \to \quad \boxed{A=1,\quad B=-1}$$

$$\int \frac{\sec^2 x}{\tan^2 x+5\tan x+6}\,dx = \int \frac{1}{u+2}du - \int \frac{1}{u+3}du$$

$$= \ln|\tan x+2| - \ln|\tan x+3| + C$$

$$\boxed{= \ln\left|\frac{\tan x+2}{\tan x+3}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{\sec^2 x}{\tan x(\tan x+1)}\,dx$

Solution

Let $u=\tan x \quad du=\sec^2 x\,dx$

$$\int \frac{\sec^2 x}{\tan x(\tan x+1)}\,dx = \int \frac{du}{u(u+1)}$$

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

$$1 = Au + A + Bu$$

$$\begin{cases} A+B=0 \\ \boxed{A=1} \end{cases} \quad \to \quad \boxed{B=-1}$$

$$\int \frac{\sec^2 x}{\tan x(\tan x+1)}\,dx = \int \frac{du}{u} - \int \frac{du}{u+1}$$

$$= \ln|\tan x| - \ln|\tan x+1| + C$$

$$\boxed{= \ln\left|\frac{\tan x}{\tan x+1}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x\ dx}{x^2+4x+3}$

Solution

$$\frac{x}{x^2+4x+3} = \frac{A}{x+1} + \frac{B}{x+3}$$

$$x = Ax + 3A + Bx + B$$

$$\begin{cases} x & A+B=1 \\ x^0 & 3A+B=0 \end{cases} \rightarrow \boxed{A=-\frac{1}{2} \quad B=\frac{3}{2}}$$

$$\int \frac{x\ dx}{x^2+4x+3} = -\frac{1}{2}\int \frac{dx}{x+1} + \frac{3}{2}\int \frac{dx}{x+3}$$

$$= -\frac{1}{2}\ln|x+1| + \frac{3}{2}\ln|x+3| + C$$

$$= \frac{1}{2}\left(3\ln|x+3| - \ln|x+1|\right) + C$$

$$= \frac{1}{2}\ln\left|\frac{(x+3)^3}{x+1}\right| + C$$

$$\boxed{= \ln\sqrt{\left|\frac{(x+3)^3}{x+1}\right|} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x+1}{x^2(x-1)}\,dx$

Solution

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$x+1 = Ax^2 - Ax + Bx - B + Cx^2$$

$$\begin{array}{lll} x^2 & A+C=0 & \boxed{C=2} \\ x & -A+B=1 & \boxed{A=-2} \\ x^0 & -B=1 & \rightarrow \boxed{B=-1} \end{array}$$

$$\int \frac{x+1}{x^2(x-1)}\,dx = -2\int \frac{dx}{x} - \int \frac{dx}{x^2} + 2\int \frac{dx}{x-1}$$

$$\boxed{= -2\ln|x| + \frac{1}{x} + 2\ln|x-1| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\,dx$

Solution

$$\begin{array}{r|l} & 2x-3 \\ \hline x^2-2x-8 & 2x^3+x^2-21x+24 \\ & \underline{2x^3-4x^2-16x} \\ & \quad -3x^2-5x+24 \\ & \quad \underline{-3x^2-6x+24} \\ & \qquad\quad x \end{array}$$

$$\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\,dx = \int\left(2x-3+\frac{x}{x^2+2x-8}\right)dx$$

$$\frac{x}{x^2+2x-8} = \frac{A}{x+4} + \frac{B}{x-2}$$

$$x = Ax - 2A + Bx + 4B$$

$$\begin{cases} x & A+B=1 \\ x^0 & -2A+4B=0 \end{cases} \rightarrow \ B=\frac{1}{3} \quad A=\frac{2}{3}$$

$$\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\,dx = \int(2x-3)\,dx + \frac{2}{3}\int\frac{dx}{x+4} + \frac{1}{3}\int\frac{dx}{x-2}$$

$$= x^2 - 3x + \frac{2}{3}\ln|x+4| + \frac{1}{3}\ln|x-2| + C$$

Exercise

Evaluate $\displaystyle\int \frac{8x+5}{2x^2+3x+1}\,dx$

Solution

$$\frac{8x+5}{2x^2+3x+1} = \frac{A}{2x+1} + \frac{B}{x+1}$$

$$8x+5 = Ax + A + 2Bx + B$$

$$\begin{cases} x & A+2B=8 \\ x^0 & A+B=5 \end{cases} \rightarrow \ B=3 \quad A=2$$

$$\int \frac{8x+5}{2x^2+3x+1}\,dx = \int \frac{2}{2x+1}\,dx + \int\frac{3}{x+1}\,dx$$

$$= \int \frac{1}{2x+1}\,d(2x+1) + 3\int\frac{1}{x+1}\,d(x+1)$$

$$= \ln|2x+1| + 3\ln|x+1| + C$$

Exercise

Evaluate $\displaystyle\int \frac{3x^3+4x^2+6x}{(x+1)^2(x^2+4)}\,dx$

Solution

$$\frac{3x^3+4x^2+6x}{(x+1)^2(x^2+4)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+4}$$

$$3x^3+4x^2+6x = A(x+1)(x^2+4) + Bx^2 + 4B + (Cx+D)(x^2+2x+1)$$

$$= Ax^3 + 4Ax + Ax^2 + 4A + Bx^2 + 4B + Cx^3 + 2Cx^2 + Cx + Dx^2 + 2Dx + D$$

$$\begin{cases} x^3 & A+C=3 \quad \to A=3-C \\ x^2 & A+B+2C+D=4 \\ x & 4A+C+2D=6 \\ x^0 & 4A+4B+D=0 \end{cases}$$

$$\begin{cases} B+C+D=1 \\ -3C+2D=-6 \\ 4B-4C+D=-12 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & 2 \\ 4 & -4 & 1 \end{vmatrix} = 25 \qquad \Delta_B = \begin{vmatrix} 1 & 1 & 1 \\ -6 & -3 & 2 \\ -12 & -4 & 1 \end{vmatrix} = -25 \qquad \Delta_C = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -6 & 2 \\ 4 & -12 & 1 \end{vmatrix} = 50$$

$B = \frac{-25}{25} = -1$ $\qquad C = \frac{50}{25} = 2 \quad \to \quad A = 3-2 = 1$

$2D = -6+6 \quad \to \quad D = 0$

$$\int \frac{3x^3+4x^2+6x}{(x+1)^2(x^2+4)}\,dx = \int \frac{1}{x+1}\,dx - \int \frac{1}{(x+1)^2}\,dx + \int \frac{2x}{x^2+4}\,dx$$

$$= \int \frac{1}{x+1}\,d(x+1) - \int \frac{1}{(x+1)^2}\,d(x+1) + \int \frac{1}{x^2+4}\,d(x^2+4)$$

$$= \ln|x+1| + \frac{1}{x+1} + \ln(x^2+4) + K$$

Exercise

Evaluate $\displaystyle\int \frac{x^2-4}{x^2+4}\,dx$

Solution

$$x^2+4\,\overset{\displaystyle 1}{\overline{\big)\,x^2-4}}$$
$$\underline{x^2+4}$$
$$8$$

$$\int \frac{x^2-4}{x^2+4}\,dx = \int\left(1+\frac{8}{x^2+4}\right)dx$$
$$\boxed{= x + 8\arctan\frac{x}{2} + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{x^2-2x-15}$

Solution

$$\frac{1}{x^2-2x-15} = \frac{A}{x-5} + \frac{B}{x+3}$$

$$1 = Ax + 3A + Bx - 5B$$

$$\begin{cases} x & A + B = 0 & \rightarrow B = -A \\ x^0 & 3A - 5B = 1 & \rightarrow 8A = 1 \Rightarrow \boxed{A = \frac{1}{8}} \end{cases}$$

$$\boxed{B = -\frac{1}{8}}$$

$$\int \frac{dx}{x^2-2x-15} = \frac{1}{8}\int \frac{dx}{x-5} - \frac{1}{8}\int \frac{dx}{x+3}$$
$$= \frac{1}{8}\ln|x-5| - \frac{1}{8}\ln|x+3| + C$$
$$\boxed{= \frac{1}{8}\ln\left|\frac{x-5}{x+3}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{3x^2+x-3}{x^2-1}\,dx$

Solution

$$x^2-1\,\overset{\displaystyle 3}{\overline{\big)\,3x^2+x-3}}$$
$$\underline{3x^2-3}$$
$$x$$

$$\int \frac{3x^2+x-3}{x^2-1}\,dx = \int\left(3+\frac{x}{x^2-1}\right)dx$$

$$= 3x + \frac{1}{2}\int \frac{1}{x^2-1}\,d\left(x^2-1\right)$$

$$\boxed{= 3x + \frac{1}{2}\ln\left|x^2-1\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{2x^2-4x}{x^2-4}\,dx$

Solution

$$\begin{array}{r|l} & \quad 2 \\ x^2-4 & 2x^2-4x \\ & \underline{2x^2 \quad -8} \\ & \quad -4x+8 \end{array}$$

$$\int \frac{2x^2-4x}{x^2-4}\,dx = \int\left(2-4\frac{x-2}{x^2-4}\right)dx$$

$$= \int\left(2-4\frac{x-2}{(x-2)(x+2)}\right)dx$$

$$= \int\left(2-\frac{4}{x+2}\right)dx$$

$$\boxed{= 2x + 4\ln|x+2| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{x^3-2x^2}$

Solution

$$\frac{1}{x^3-2x^2} = \frac{1}{x^2(x-2)}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$1 = Ax(x-2) + Bx - 2B + Cx^2$$

$$x^2 \qquad A + C = 0 \qquad \boxed{C = \frac{1}{4}}$$

$$x \qquad -2A + B = 0 \qquad \boxed{A = -\frac{1}{4}}$$

$$x^0 \qquad -2B = 1 \qquad \to \boxed{B = -\frac{1}{2}}$$

$$\int \frac{dx}{x^3 - 2x^2} = -\frac{1}{4}\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{1}{x^2}\,dx + \frac{1}{4}\int \frac{1}{x-2}\,dx$$

$$= -\frac{1}{4}\ln|x| + \frac{1}{2}\frac{1}{x} + \frac{1}{4}\ln|x-2| + K$$

$$\boxed{= \frac{1}{2x} + \frac{1}{4}\ln\left|\frac{x-2}{x}\right| + K}$$

Exercise

Evaluate $\displaystyle\int \frac{dx}{x^2 - x - 2}$

Solution

$$\frac{1}{x^2 - x - 2} = \frac{A}{x+1} + \frac{B}{x-2}$$

$$1 = Ax - 2A + Bx + B$$

$$\begin{cases} x & A + B = 0 \\ x^0 & -2A + B = 1 \end{cases}$$

$$\boxed{A = -\frac{1}{3} \quad B = \frac{1}{3}}$$

$$\int \frac{dx}{x^2 - x - 2} = -\frac{1}{3}\int \frac{dx}{x+1} + \frac{1}{3}\int \frac{dx}{x-2}$$

$$= -\frac{1}{3}\ln|x+1| + \frac{1}{3}\ln|x-2| + K$$

$$\boxed{= \frac{1}{3}\ln\left|\frac{x-2}{x+1}\right| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{4x^2 + 13x - 9}{x^3 + 2x^2 - 3x}\,dx$

Solution

$$\frac{4x^2+13x-9}{x^3+2x^2-3x} = \frac{4x^2+13x-9}{x\left(x^2+2x-3\right)}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+3}$$

$$4x^2+13x-9 = A\left(x^2+2x-3\right) + Bx(x+3) + Cx(x-1)$$

x^2 $\quad A+B+C=4$

x $\quad 2A+3B-C=13$

x^0 $\quad -3A=-9 \quad \rightarrow \underline{A=3}$

$$\begin{cases} B+C=1 \\ 3B-C=7 \end{cases} \quad \rightarrow \underline{B=2 \quad C=-1}$$

$$\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\, dx = \int \left(\frac{3}{x} + \frac{2}{x-1} - \frac{1}{x+3}\right) dx$$

$$\underline{= 3\ln|x| + 2\ln|x-1| - \ln|x+3| + K}$$

Exercise

Evaluate $\quad \displaystyle\int \frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3}\, dx$

Solution

$$\frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

$$3x^3-18x^2+29x-4 = A(x-2)^3 + B(x+1)(x-2)^2 + C(x+1)(x-2) + Dx + D$$

$$= A\left(x^3-6x^2+12x-8\right) + B(x+1)\left(x^2-4x+4\right) + C\left(x^2-x-2\right) + Dx + D$$

x^3 $\quad A+B=3 \quad \rightarrow A=3-B$

x^2 $\quad -6A-3B+C=-18$

x $\quad 12A-C+D=29$

x^0 $\quad -8A+4B-2C+D=-4$

$$\begin{cases} -18+6B-3B+C=-18 \\ 36-12B-C+D=29 \\ -24+8B+4B-2C+D=-4 \end{cases}$$

$$\begin{cases} 3B + C = 0 \\ -12B - C + D = -7 \\ 12B - 2C + D = 20 \end{cases}$$

$$\Delta = \begin{vmatrix} 3 & 1 & 0 \\ -12 & -1 & 1 \\ 12 & -2 & 1 \end{vmatrix} = 27 \qquad \Delta_B = \begin{vmatrix} 0 & 1 & 0 \\ -7 & -1 & 1 \\ 20 & -2 & 1 \end{vmatrix} = 27$$

$B = \frac{27}{27} \underline{=1}|$

$C = -3B \underline{= -3}|$

$D = -7 + 12 - 3 \underline{= 2}|$

$A = 3 - 1 \underline{= 2}|$

$$\int \frac{3x^3 - 18x^2 + 29x - 4}{(x+1)(x-2)^3} dx = \int \frac{2}{x+1} dx + \int \frac{1}{x-2} dx - 3\int \frac{1}{(x-2)^2} dx + 2\int \frac{1}{(x-2)^3} dx$$

$$= 2\int \frac{d(x+1)}{x+1} + \int \frac{d(x-2)}{x-2} - 3\int \frac{d(x-2)}{(x-2)^2} + 2\int (x-2)^{-3} d(x-2)$$

$$= 2\ln|x+1| + \ln|x-2| + \frac{3}{x-2} - \frac{1}{(x-2)^2} + K$$

Exercise

Evaluate $\displaystyle\int \frac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4} dx$

Solution

$$2x^3 - x^2 + 8x - 4 = x^2(2x-1) + 4(2x-1)$$

$$= (2x-1)\left(x^2+4\right)$$

$$\frac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4} = \frac{A}{2x-1} + \frac{Bx + C}{x^2 + 4}$$

$$x^2 - x - 21 = Ax^2 + 4A + 2Bx^2 - Bx + 2Cx - C$$

$x^2 \quad A + 2B = 1 \qquad A = 1 - 2B$

$x \quad -B + 2C = -1 \quad C = \frac{1}{2}(B-1)$

$x^0 \quad 4A - C = -21 \qquad (1)$

$(1) \quad 4 - 8B - \frac{1}{2}B + \frac{1}{2} = -21$

$$-\frac{17}{2}B = -21 - \frac{9}{2}$$

$$\frac{17}{2}B = \frac{51}{2} \quad \to \quad \underline{B = 3}$$

$$C = \frac{1}{2}(3-1) \ \underline{= 1}$$

$$A = 1 - 6 \ \underline{= 5}$$

$$\int \frac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4}\,dx = \int \frac{5}{2x-1}\,dx + \int \frac{3x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$$

$$= \frac{5}{2}\int \frac{1}{2x-1}\,d(2x-1) + \frac{3}{2}\int \frac{1}{x^2+4}\,d\left(x^2+4\right) + \int \frac{1}{x^2+4}\,dx$$

$$= \frac{5}{2}\ln|2x-1| + \frac{3}{2}\ln\left(x^2+4\right) + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + K$$

Exercise

Evaluate $\displaystyle\int \frac{5x^3 - 3x^2 + 7x - 3}{\left(x^2+1\right)^2}\,dx$

Solution

$$\frac{5x^3 - 3x^2 + 7x - 3}{\left(x^2+1\right)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{\left(x^2+1\right)^2}$$

$$5x^3 - 3x^2 + 7x - 3 = Ax^3 + Ax + Bx^2 + B + Cx + D$$

x^3 $\quad \underline{A = 5}$

x^2 $\quad \underline{B = -3}$

x $\quad A + C = 7 \quad \to \quad \underline{C = 2}$

x^0 $\quad B + D = -3 \quad \to \quad \underline{D = 0}$

$$\int \frac{5x^3 - 3x^2 + 7x - 3}{\left(x^2+1\right)^2}\,dx = \int \frac{5x-3}{x^2+1}\,dx + \int \frac{2x}{\left(x^2+1\right)^2}\,dx$$

$$= \frac{5}{2}\int \frac{1}{x^2+1}\,d\left(x^2+1\right) - 3\int \frac{1}{x^2+1}\,dx + \int \frac{1}{\left(x^2+1\right)^2}\,d\left(x^2+1\right)$$

$$= \frac{5}{2}\ln\left(x^2+1\right) - 3\arctan(x) - \frac{1}{x^2+1} + K$$

Exercise

Evaluate $\displaystyle\int \frac{2x^4 - 2x^3 + 6x^2 - 5x + 1}{x^3 - x^2 + x - 1}\, dx$

Solution

$$\begin{array}{r|l} & 2x \\ \hline x^3 - x^2 + x - 1 & 2x^4 - 2x^3 + 6x^2 - 5x + 1 \\ & \underline{2x^4 - 2x^3 - 2x^2 + 2x} \\ & 8x^2 - 7x + 1 \end{array}$$

$$\frac{2x^4 - 2x^3 + 6x^2 - 5x + 1}{x^3 - x^2 + x - 1} = 2x + \frac{8x^2 - 7x + 1}{x^3 - x^2 + x - 1}$$

$$\begin{aligned} x^3 - x^2 + x - 1 &= x^2(x-1) + (x-1) \\ &= (x-1)\left(x^2+1\right) \end{aligned}$$

$$\frac{8x^2 - 7x + 1}{x^3 - x^2 + x - 1} = \frac{A}{x-1} + \frac{Bx + C}{x^2+1}$$

$$8x^2 - 7x + 1 = Ax^2 + A + Bx^2 - Bx + Cx - C$$

$$\begin{array}{lll} x^2 & A + B = 8 & A = 8 - B \\ x & -B + C = -7 & C = B - 7 \\ x^0 & A - C = 1 & (1) \end{array}$$

$(1) \rightarrow \quad 8 - B - B + 7 = 1 \quad \Rightarrow \underline{B = 7}|$

$A = 8 - 7 \ \underline{= 1}|$

$C = 7 - 7 \ \underline{= 0}|$

$$\begin{aligned} \int \frac{2x^4 - 2x^3 + 6x^2 - 5x + 1}{x^3 - x^2 + x - 1}\, dx &= \int 2x\, dx + \int \frac{1}{x-1}\, dx + \int \frac{7x}{x^2+1}\, dx \\ &= x^2 + \ln|x-1| + \frac{7}{2}\int \frac{1}{x^2+1}\, d\left(x^2+1\right) \\ &= x^2 + \ln|x-1| + \frac{7}{2}\ln\left(x^2+1\right) + K \end{aligned}$$

Exercise

Evaluate $\displaystyle\int \frac{81}{x^3 - 9x^2}\, dx$

Solution

$$\frac{81}{x^3-9x^2}=\frac{81}{x^2(x-9)}$$

$$=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-9}$$

$$81=Ax^2-9Ax+Bx-9B+Cx^2$$

$$x^2 \quad A+C=0 \quad \rightarrow \quad \underline{C=1}$$

$$x \quad -9A+B=0 \quad \rightarrow \quad \underline{A=-1}$$

$$x^0 \quad -9B=81 \quad \rightarrow \quad \underline{B=-9}$$

$$\int \frac{81}{x^3-9x^2}\,dx=\int\left(-\frac{1}{x}-\frac{9}{x^2}+\frac{1}{x-9}\right)dx$$

$$=-\ln|x|+\frac{9}{x}+\ln|x-9|+K$$

$$\underline{=\frac{9}{x}+\ln\left|\frac{x-9}{x}\right|+K}$$

Exercise

Evaluate $\displaystyle\int \frac{10x}{x^2-2x-24}\,dx$

Solution

$$\frac{10x}{x^2-2x-24}=\frac{A}{x-6}+\frac{B}{x+4}$$

$$10x=Ax+4A+Bx-6B$$

$$x \quad A+B=10$$

$$x^0 \quad 4A-6B=0$$

$$\Delta=\begin{vmatrix}1 & 1\\ 4 & -6\end{vmatrix}=-10 \qquad \Delta_A=\begin{vmatrix}10 & 1\\ 0 & -6\end{vmatrix}=-60$$

$$\underline{A=6 \quad B=4}$$

$$\int \frac{10x}{x^2-2x-24}\,dx=\int\left(\frac{6}{x-6}+\frac{4}{x+4}\right)dx$$

$$\underline{=6\ln|x-6|+4\ln|x+4|+C}$$

Exercise

Evaluate $\displaystyle\int \frac{x+1}{x^2\left(x^2+4\right)}\,dx$

Solution

$$\frac{x+1}{x^2\left(x^2+4\right)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+4}$$

$$x+1=Ax^3+4Ax+Bx^2+4B+Cx^3+Dx^2$$

$$\begin{array}{lll} x^3 & A+C=0 & \rightarrow \boxed{C=-\frac{1}{4}} \\ x^2 & B+D=0 & \rightarrow \boxed{D=-\frac{1}{4}} \\ x & 4A=1 & \rightarrow \boxed{A=\frac{1}{4}} \\ x^0 & 4B=1 & \rightarrow \boxed{B=\frac{1}{4}} \end{array}$$

$$\int \frac{x+1}{x^2\left(x^2+4\right)}\,dx=\frac{1}{4}\int\frac{1}{x}\,dx+\frac{1}{4}\int\frac{1}{x^2}\,dx-\frac{1}{4}\int\frac{x}{x^2+4}\,dx-\frac{1}{4}\int\frac{1}{x^2+4}\,dx$$

$$=\frac{1}{4}\ln|x|-\frac{1}{4}\frac{1}{x}-\frac{1}{8}\int\frac{x}{x^2+4}\,d\left(x^2+4\right)-\frac{1}{8}\arctan\left(\frac{x}{2}\right)+K$$

$$\boxed{=\frac{1}{4}\ln|x|-\frac{1}{4x}-\frac{1}{8}\ln\left(x^2+4\right)-\frac{1}{8}\arctan\left(\frac{x}{2}\right)+K}$$

Exercise

Evaluate $\displaystyle\int \frac{1+x^2}{(x+1)^3}\,dx$

Solution

$$\frac{1+x^2}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$

$$x^2+1=A(x+1)^2+Bx+B+C$$

$$\begin{array}{lll} x^2 & \boxed{A=1} & \\ x & 2A+B=0 & \rightarrow \boxed{B=-2} \\ x^0 & A+B+C=1 & \rightarrow \boxed{C=2} \end{array}$$

$$\int \frac{1+x^2}{(x+1)^3}\,dx=\int\frac{1}{x+1}\,dx-\int\frac{2}{(x+1)^2}\,dx+\int\frac{2}{(x+1)^3}\,dx$$

$$= \int \frac{1}{x+1}\, d(x+1) - 2\int \frac{1}{(x+1)^2}\, d(x+1) + 2\int (x+1)^{-3}\, d(x+1)$$

$$= \ln|x+1| + \frac{2}{x+1} - \frac{1}{(x+1)^2} + K$$

Exercise

Evaluate $\displaystyle\int \frac{6}{x^2-1}\, dx$

Solution

$$\frac{6}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$6 = Ax + A + Bx - B$$

$$x \qquad A + B = 0$$

$$x^0 \qquad A - B = 6$$

$$2A = 6 \;\rightarrow\; A = 3 \quad B = -3$$

$$\int \frac{6}{x^2-1}\, dx = \int \frac{3}{x-1}\, dx - \int \frac{3}{x+1}\, dx$$

$$= 3\ln|x-1| - 3\ln|x+1| + C$$

$$= 3\ln\left|\frac{x-1}{x+1}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{21x^2}{x^3-x^2-12x}\, dx$

Solution

$$\int \frac{21x^2}{x^3-x^2-12x}\, dx = \int \frac{21x}{x^2-x-12}\, dx$$

$$\frac{21x^2}{x^3-x^2-12x} = \frac{21x^2}{x\left(x^2-x-12\right)}$$

$$= \frac{21x^2}{x(x+3)(x-4)}$$

$$\frac{21x}{x^2-x-12} = \frac{A}{x+3} + \frac{B}{x-4}$$

$$21x = Ax - 4A + Bx + 3B$$

$x \qquad A + B = 21$

$x^0 \qquad -4A + 3B = 0$

$$\Delta = \begin{vmatrix} 1 & 1 \\ -4 & 3 \end{vmatrix} = 7 \qquad\qquad \Delta_A = \begin{vmatrix} 21 & 1 \\ 0 & 3 \end{vmatrix} = 63$$

$$\boxed{A = 9 \quad B = 12}$$

$$\int \frac{21x^2}{x^3 - x^2 - 12x}\,dx = \int \frac{9}{x+3}\,dx + \int \frac{12}{x-4}\,dx$$

$$\boxed{= 9\ln|x+3| + 12\ln|x-4| + C}$$

Exercise

Evaluate $\displaystyle\int \frac{x+1}{x^3 + 3x^2 - 18x}\,dx$

Solution

$$\frac{x+1}{x^3 + 3x^2 - 18x} = \frac{x+1}{x\left(x^2 + 3x - 18\right)}$$

$$= \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+6}$$

$$x + 1 = Ax^2 + 3Ax - 18A + Bx^2 + 6Bx + Cx^2 - 3Cx$$

$x^2 \qquad A + B + C = 0$

$x \qquad 3A + 6B - 3C = 1$

$x^0 \qquad -18A = 1 \qquad \rightarrow \boxed{A = -\frac{1}{18}}$

$$\begin{cases} B + C = \frac{1}{18} \\ 6B - 3C = \frac{7}{6} \end{cases} \qquad \rightarrow \begin{cases} 3B + 3C = \frac{1}{6} \\ 6B - 3C = \frac{7}{6} \end{cases}$$

$$9B = \frac{4}{3} \quad \rightarrow \quad \boxed{B = \frac{4}{27}}$$

$$C = \frac{1}{18} - \frac{4}{27} \ \boxed{= -\frac{5}{54}}$$

$$\int \frac{x+1}{x^3 + 3x^2 - 18x}\,dx = \int \left(-\frac{1}{18}\frac{1}{x} + \frac{4}{27}\frac{1}{x-3} - \frac{5}{54}\frac{1}{x+6}\right)dx$$

$$\boxed{= -\frac{1}{18}\ln|x| + \frac{4}{27}\ln|x-3| - \frac{5}{54}\ln|x+6| + K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2+12x-4}{x^3-4x}\,dx$

Solution

$$\frac{x^2+12x-4}{x^3-4x}=\frac{x^2+12x-4}{x\left(x^2-4\right)}$$

$$=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2}$$

$$x^2+12x-4=Ax^2-4A+Bx^2+2Bx+Cx^2-2Cx$$

$$x^2 \quad A+B+C=1$$

$$x \quad 2B-2C=12$$

$$x^0 \quad -4A=-4 \quad \rightarrow \underline{A=1}$$

$$\rightarrow \begin{cases} B+C=0 \\ 2B-2C=12 \end{cases}$$

$$3B=12 \quad \rightarrow \quad \underline{B=4} \quad \underline{C=-4}$$

$$\int \frac{x^2+12x-4}{x^3-4x}\,dx=\int\left(\frac{1}{x}+\frac{4}{x-2}-\frac{4}{x+2}\right)dx$$

$$=\ln|x|+4\ln|x-2|-4\ln|x+2|+K$$

Exercise

Evaluate $\displaystyle\int \frac{6x^2}{x^4-5x^2+4}\,dx$

Solution

$$\frac{6x^2}{x^4-5x^2+4}=\frac{6x^2}{\left(x^2-1\right)\left(x^2-4\right)}$$

$$=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x-2}+\frac{D}{x+2}$$

$$6x^2=A(x+1)\left(x^2-4\right)+B(x-1)\left(x^2-4\right)+C(x+2)\left(x^2-1\right)+D(x-2)\left(x^2-1\right)$$

$$x^3 \quad A+B+C+D=0$$

$$x^2 \quad A-B+2C-2D=6$$

$$x^1 \quad -4A-4B-C-D=0$$

$$x^0 \quad -4A+4B-2C+2D=0$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -2 \\ -4 & -4 & -1 & -1 \\ -4 & 4 & -2 & 2 \end{vmatrix} = 72 \qquad \Delta_A = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 6 & -1 & 2 & -2 \\ 0 & -4 & -1 & -1 \\ 0 & 4 & -2 & 2 \end{vmatrix} = -72$$

$$\Delta_B = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 6 & 2 & -2 \\ -4 & 0 & -1 & -1 \\ -4 & 0 & -2 & 2 \end{vmatrix} = 72 \qquad \Delta_C = \begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 6 & -2 \\ -4 & -4 & 0 & -1 \\ -4 & 4 & 0 & 2 \end{vmatrix} = 144$$

$\underline{A=-1 \quad B=1 \quad C=2 \quad D=-2}$

$$\int \frac{6x^2}{x^4-5x^2+4}\,dx = \int\left(\frac{-1}{x-1}+\frac{1}{x+1}+\frac{2}{x-2}-\frac{2}{x+2}\right)dx$$

$$\underline{= -\ln|x-1| + \ln|x+1| + 2\ln|x-2| - 2\ln|x+2| + K}$$

Exercise

Evaluate $\displaystyle\int \frac{4x-2}{x^3-x}\,dx$

Solution

$$\frac{4x-2}{x^3-x} = \frac{4x-2}{x\left(x^2-1\right)}$$

$$= \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$$

$$4x-2 = Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx$$

$x^2 \quad A+B+C=0$

$x^1 \quad B-C=4$

$x^0 \quad -A=-2 \quad \rightarrow \underline{A=2}$

$$\begin{cases} B+C=-2 \\ B-C=4 \end{cases} \rightarrow \underline{B=1} \quad \underline{C=-3}$$

$$\int \frac{4x-2}{x^3-x}\,dx = \int\left(\frac{2}{x}+\frac{1}{x-1}-\frac{3}{x+1}\right)dx$$

$$\underline{= 2\ln|x| + \ln|x-1| - 3\ln|x+1| + K}$$

Exercise

Evaluate $\displaystyle\int \frac{16x^2}{(x-6)(x+2)^2}\,dx$

Solution

$$\frac{16x^2}{(x-6)(x+2)^2} = \frac{A}{x-6} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$16x^2 = A(x+2)^2 + Bx^2 - 4Bx - 12B + Cx - 6C$$

$x^2 \quad A + B = 16$

$x^1 \quad 4A - 4B + C = 0$

$x^0 \quad 4A - 12B - 6C = 0$

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ 4 & -4 & 1 \\ 4 & -12 & -6 \end{vmatrix} = 64 \qquad \Delta_A = \begin{vmatrix} 16 & 1 & 0 \\ 0 & -4 & 1 \\ 0 & -12 & -6 \end{vmatrix} = 576$$

$A = \frac{576}{64} \underline{=9|}$

$B = 16 - 9 \underline{=7|}$

$C = -36 + 28 \underline{=-8|}$

$$\int \frac{16x^2}{(x-6)(x+2)^2}\,dx = \int \frac{9}{x-6}\,dx + \int \frac{7}{x+2}\,dx - \int \frac{8}{(x+2)^2}\,dx$$

$$= \int \frac{9}{x-6}\,d(x-6) + \int \frac{7}{x+2}\,d(x+2) - \int \frac{8}{(x+2)^2}\,d(x+2)$$

$$\underline{= 9\ln|x-6| + 7\ln|x+2| + \frac{8}{x+2} + K}$$

Exercise

Evaluate $\displaystyle\int \frac{8\left(x^2+4\right)}{x\left(x^2+8\right)}\,dx$

Solution

$$\int \frac{8\left(x^2+4\right)}{x\left(x^2+8\right)}\,dx = 8\int \frac{x^2+4}{x\left(x^2+8\right)}\,dx$$

$$\frac{x^2+4}{x\left(x^2+8\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+8}$$

$$x^2+4=Ax^2+8A+Bx^2+Cx$$

$x^2 \quad A+B=1 \qquad \boxed{B=\frac{1}{2}}$

$x^1 \quad \boxed{C=0}$

$x^0 \quad 8A=4 \qquad \boxed{A=\frac{1}{2}}$

$$8\int \frac{x^2+4}{x\left(x^2+8\right)}dx = 8\int\left(\frac{1}{2}\frac{1}{x}+\frac{1}{2}\frac{x}{x^2+8}\right)dx$$

$$=4\int\frac{1}{x}dx+2\int\frac{x}{x^2+8}d\left(x^2+8\right)$$

$$\boxed{=4\ln|x|+2\ln\left(x^2+8\right)+K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^2+x+2}{(x+1)\left(x^2+1\right)}dx$

Solution

$$\frac{x^2+x+2}{(x+1)\left(x^2+1\right)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$$

$$x^2+x+2=Ax^2+A+Bx^2+Bx+Cx+C$$

$x^2 \quad A+B=1 \quad \rightarrow A=1-B$

$x^1 \quad B+C=1 \quad \rightarrow C=1-B$

$x^0 \quad A+C=2 \qquad \rightarrow \qquad 1-B+1-B=2$

$\boxed{B=0 \quad A=1 \quad C=1}$

$$\int\frac{x^2+x+2}{(x+1)\left(x^2+1\right)}dx=\int\frac{1}{x+1}dx+\int\frac{1}{x^2+1}dx$$

$$\boxed{=\ln|x+1|+\arctan(x)+K}$$

Exercise

Evaluate $\displaystyle\int \frac{2}{x\left(x^2+1\right)^2}\,dx$

Solution

$$\frac{2}{x\left(x^2+1\right)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{\left(x^2+1\right)^2}$$

$$2=A\left(x^2+1\right)^2+(Bx+C)\left(x^3+x\right)+Dx^2+Ex$$

x^4 $\quad A+B=0 \quad \rightarrow \underline{B=-2}$

x^3 $\quad \underline{C=0}$

x^2 $\quad 2A+B+D=0 \quad \rightarrow \underline{D=-2}$

x^1 $\quad C+E=0 \quad \rightarrow \underline{E=0}$

x^0 $\quad \underline{A=2}$

$$\int \frac{2}{x\left(x^2+1\right)^2}\,dx=\int \frac{2}{x}\,dx-\int \frac{2x}{x^2+1}\,dx-\int \frac{2x}{\left(x^2+1\right)^2}\,dx$$

$$=2\ln|x|-\int \frac{1}{x^2+1}\,d\left(x^2+1\right)-\int \frac{1}{\left(x^2+1\right)^2}\,d\left(x^2+1\right)$$

$$=2\ln|x|-\ln\left(x^2+1\right)+\frac{1}{x^2+1}+K$$

Exercise

Evaluate $\displaystyle\int \frac{1}{(x+1)\left(x^2+2x+2\right)^2}\,dx$

Solution

$$\frac{1}{(x+1)\left(x^2+2x+2\right)^2}=\frac{A}{x+1}+\frac{Bx+C}{x^2+2x+2}+\frac{Dx+E}{\left(x^2+2x+2\right)^2}$$

$$1=A\left(x^2+2x+2\right)^2+(Bx+C)(x+1)\left(x^2+2x+2\right)+(Dx+E)(x+1)$$

$$=Ax^4+4Ax^3+8Ax^2+8Ax+4A+\left(Bx^2+Bx+Cx+C\right)\left(x^2+2x+2\right)+Dx^2+Dx+Ex+E$$

$x^4 \quad A+B=0$

$x^3 \quad 4A+3B+C=0$

$x^2 \quad 8A+4B+3C+D=0$

$x^1 \quad 8A+2B+4C+D+E=0$

$x^0 \quad 4A+2C+E=1$

$\underline{A=1, \quad B=-1, \quad C=-1, \quad D=-1, \quad E=-1}$

$$\int \frac{1}{(x+1)\left(x^2+2x+2\right)^2}\,dx = \int \frac{dx}{x+1} - \int \frac{x+1}{x^2+2x+2}\,dx - \int \frac{x+1}{\left(x^2+2x+2\right)^2}\,dx$$

$$= \ln|x+1| - \frac{1}{2}\int \frac{d\left(x^2+2x+2\right)}{x^2+2x+2} - \frac{1}{2}\int \frac{d\left(x^2+2x+2\right)}{\left(x^2+2x+2\right)^2}$$

$$= \ln|x+1| - \frac{1}{2}\ln\left|x^2+2x+2\right| + \frac{1}{2}\frac{1}{x^2+2x+2} + K$$

Exercise

Evaluate $\displaystyle\int \frac{2-x}{x^2+x}\,dx$

Solution

$$\frac{2-x}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}$$

$2-x = Ax + A + Bx$

$x^1 \quad A+B=-1 \quad \to \underline{B=-3}$

$x^0 \quad \underline{A=2}$

$$\int \frac{2-x}{x^2+x}\,dx = \int \left(\frac{2}{x} - \frac{3}{x+1}\right)dx$$

$$= 2\ln|x| - 3\ln|x+1| + C$$

Exercise

Evaluate $\displaystyle\int \frac{3x+11}{(x+2)(x+3)}\,dx$

Solution

$$\frac{3x+11}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$3x+11 = Ax+3A+Bx+2B$

$x^1 \quad A+B=3$

$x^0 \quad 3A+2B=11$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -1 \qquad \Delta_A = \begin{vmatrix} 3 & 1 \\ 11 & 2 \end{vmatrix} = -5 \qquad \Delta_B = \begin{vmatrix} 1 & 3 \\ 3 & 11 \end{vmatrix} = 2$$

$\underline{A=5 \quad B=-2}$

$$\int \frac{3x+11}{(x+2)(x+3)}\,dx = \int\left(\frac{5}{x+2} - \frac{1}{x+3}\right)dx$$

$$= 5\ln|x+2| - 2\ln|x+3| + C$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^2-a^2}\,dx$

Solution

$$\frac{1}{x^2-a^2} = \frac{A}{x-a} + \frac{B}{x+a}$$

$1 = Ax+aA+Bx-aB$

$x^1 \quad A+B=0$

$x^0 \quad aA-aB=1$

$$\Delta = \begin{vmatrix} 1 & 1 \\ a & -a \end{vmatrix} = -2a \qquad \Delta_A = \begin{vmatrix} 0 & 1 \\ 1 & -a \end{vmatrix} = -1 \qquad \Delta_B = \begin{vmatrix} 1 & 0 \\ a & 1 \end{vmatrix} = 1$$

$\underline{A=\frac{1}{2a} \quad B=-\frac{1}{2a}}$

$$\int \frac{1}{x^2-a^2}\,dx = \int\left(\frac{1}{2a}\frac{1}{x-a} - \frac{1}{2a}\frac{1}{x+a}\right)dx$$

$$= \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C$$

Exercise

Evaluate $\displaystyle\int \frac{1}{x^2+5x+6}\,dx$

Solution

$$\frac{1}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3}$$

$1 = Ax + 3A + Bx + 2B$

$x^1 \quad A + B = 0$

$x^0 \quad 3A + 2B = 1$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -1 \qquad \Delta_A = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1 \qquad \Delta_B = \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1$$

$\underline{A = 1 \quad B = -1}$

$$\int \frac{1}{x^2 + 5x + 6}\, dx = \int \left(\frac{1}{x+2} - \frac{1}{x+3} \right) dx$$

$$= \ln|x + 2| - \ln|x - 3| + C$$

$$= \ln\left|\frac{x+2}{x-3}\right| + C$$

Exercise

Evaluate $\displaystyle\int \frac{x^3 + 6x^2 + 3x + 6}{x^3 + 2x^2}\, dx$

Solution

$$\begin{array}{r|l} & \quad 1 \\ x^3 + 2x^2 & x^3 + 6x^2 + 3x + 6 \\ & \underline{x^3 + 2x^2} \\ & \qquad 4x^2 + 3x + 6 \end{array}$$

$$\frac{x^3 + 6x^2 + 3x + 6}{x^3 + 2x^2} = 1 + \frac{4x^2 + 3x + 6}{x^3 + 2x^2}$$

$$\frac{4x^2 + 3x + 6}{x^3 + 2x^2} = \frac{4x^2 + 3x + 6}{x^2(x+2)}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

$4x^2 + 3x + 6 = Ax(x+2) + Bx + 2B + Cx^2$

$x^2 \quad A + C = 4 \quad \rightarrow \underline{C = 4}$

$x^1 \quad 2A + B = 3 \quad \rightarrow \underline{A = 0}$

$x^0 \quad 2B = 6 \quad \rightarrow \underline{B = 3}$

$$\int \frac{x^3 + 6x^2 + 3x + 6}{x^3 + 2x^2}\, dx = \int \left(1 + \frac{0}{x} + \frac{3}{x^2} + \frac{4}{x+2} \right) dx$$

$$= x - \frac{3}{x} + 4\ln|x + 2| + K$$

Exercise

Evaluate $\displaystyle\int \frac{2x^2+5x-1}{x^3+x^2-2x}\,dx$

Solution

$$\frac{2x^2+5x-1}{x^3+x^2-2x}=\frac{2x^2+5x-1}{x\left(x^2+x-2\right)}$$

$$=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+2}$$

$$2x^2+5x-1=Ax^2+Ax-2A+Bx^2+2Bx+Cx^2-Cx$$

$$\begin{array}{lll} x^2 & A+B+C=2 & B+C=\frac{3}{2} \\ x^1 & A+2B-C=5 & 2B-C=\frac{9}{2} \\ x^0 & -2A=-1 & \rightarrow \ \underline{A=\frac{1}{2}} \end{array}$$

$3B=6 \quad \rightarrow \ \underline{B=2}$

$C=\frac{3}{2}-2 \quad \rightarrow \ \underline{C=-\frac{1}{2}}$

$$\int \frac{2x^2+5x-1}{x^3+x^2-2x}\,dx=\int\left(\frac{1}{2}\frac{1}{x}+\frac{2}{x-1}-\frac{1}{2}\frac{1}{x+2}\right)dx$$

$$=\frac{1}{2}\ln|x|+2\ln|x-1|-\frac{1}{2}\ln|x+2|+K$$

Exercise

Evaluate $\displaystyle\int \frac{3x+6}{x^3+2x^2-3x}\,dx$

Solution

$$\frac{3x+6}{x^3+2x^2-3x}=\frac{3x+6}{x\left(x^2+2x-3\right)}$$

$$=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+3}$$

$$3x+6=Ax^2+2Ax-3A+Bx^2+3Bx+Cx^2-Cx$$

$$\begin{array}{lll} x^2 & A+B+C=0 & B+C=2 \\ x^1 & 2A+3B-C=3 & 3B-C=7 \\ x^0 & -3A=6 & \rightarrow \ \underline{A=-2} \end{array}$$

$4B=9 \quad \rightarrow \ \underline{B=\frac{9}{4}}$

$$C = 2 - \frac{9}{4} \quad \to \quad \boxed{C = -\frac{1}{4}}$$

$$\int \frac{3x+6}{x^3 + 2x^2 - 3x}\, dx = \int \left(-\frac{2}{x} + \frac{9}{4}\frac{1}{x-1} - \frac{1}{4}\frac{1}{x+3} \right) dx$$

$$\boxed{= -2\ln|x| + \frac{9}{4}\ln|x-1| - \frac{1}{4}\ln|x+3| + K}$$

Exercise

Evaluate $\displaystyle\int \frac{3x^2 + 2x - 2}{x^3 - 1}\, dx$

Solution

$$\frac{3x^2+2x-2}{x^3-1} = \frac{3x^2+2x-2}{(x-1)\left(x^2+x+1\right)}$$

$$= \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$3x^2 + 2x - 2 = Ax^2 + Ax + A + Bx^2 - Bx + Cx - C$$

$x^2 \quad A + B = 3 \quad \to B = 3 - A$

$x^1 \quad A - B + C = 2 \quad \to A - 3 + A + A + 2 = 2$

$x^0 \quad A - C = -2 \quad \to C = A + 2$

$3A = 3 \quad \to \quad \boxed{A = 1}$

$B = 3 - 1 \ \boxed{= 2}$

$C = 1 + 2 \ \boxed{= 3}$

$$\int \frac{3x^2+2x-2}{x^3-1}\, dx = \int \left(\frac{1}{x-1} + \frac{2x+3}{x^2+x+1} \right) dx$$

$$= \ln|x-1| + \int \frac{2x+1+2}{x^2+x+1}\, dx$$

$$= \ln|x-1| + \int \frac{2x+1}{x^2+x+1}\, dx + \int \frac{2}{x^2+x+1}\, dx$$

$$= \ln|x-1| + \int \frac{d\left(x^2+x+1\right)}{x^2+x+1} + \int \frac{2}{\left(x+\frac{1}{2}\right)^2 + 1 - \frac{1}{4}}\, dx$$

$$= \ln|x-1| + \ln\left(x^2+x+1\right) + \int \frac{2}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\, d\left(x+\frac{1}{2}\right)$$

$$= \ln|x-1| + \ln\left(x^2+x+1\right) + \int \frac{2}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} d\left(x+\frac{1}{2}\right)$$

$$= \ln|x-1| + \ln\left(x^2+x+1\right) + 2\frac{2}{\sqrt{3}}\arctan\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + K$$

$$\boxed{= \ln|x-1| + \ln\left(x^2+x+1\right) + \frac{4\sqrt{3}}{3}\arctan\frac{2x+1}{\sqrt{3}} + K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^3+5x^2+2x-4}{x^4-1}\,dx$

Solution

$$\frac{x^3+5x^2+2x-4}{x^4-1} = \frac{x^3+5x^2+2x-4}{(x-1)(x+1)\left(x^2+1\right)}$$

$$= \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$x^3+5x^2+2x-4 = A(x+1)\left(x^2+1\right) + B(x-1)\left(x^2+1\right) + (Cx+D)\left(x^2-1\right)$$

$x^3 \quad A+B+C=1 \quad (1)$

$x^2 \quad A-B+D=5 \quad (2)$

$x^1 \quad A+B-C=2 \quad (3)$

$x^0 \quad A-B-D=-4 \quad (4)$

$(1)+(3) \rightarrow \quad 2A+2B=3$

$(2)+(4) \rightarrow \quad 2A-2B=1$

$$\Delta = \begin{vmatrix} 2 & 2 \\ 2 & -2 \end{vmatrix} = -8 \qquad \Delta_A = \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} = -8 \qquad \Delta_B = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = -4$$

$\boxed{A=1 \quad B=\frac{1}{2}}$

$(1) \rightarrow \quad C = 1-1-\frac{1}{2} \boxed{= -\frac{1}{2}}$

$(2) \rightarrow \quad D = 5-1+\frac{1}{2} \boxed{= \frac{9}{2}}$

$$\int \frac{x^3+5x^2+2x-4}{x^4-1}\,dx = \int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{x}{x^2+1}\,dx + \frac{9}{2}\int \frac{1}{x^2+1}\,dx$$

$$= \ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\int \frac{1}{x^2+1}\, d\left(x^2+1\right) + \frac{9}{2}\arctan x + K$$

$$= \ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln\left(x^2+1\right) + \frac{9}{2}\arctan x + K$$

Exercise

Evaluate $\displaystyle\int \frac{x^2+4x}{\left(x^2+4\right)(x-2)^2}\, dx$

Solution

$$\frac{x^2+4x}{\left(x^2+4\right)(x-2)^2} = \frac{Ax+B}{x^2+4} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

$$x^2+4x = (Ax+B)\left(x^2-4x+4\right) + C\left(x^2+4\right)(x-2) + Dx^2 + 4D$$

x^3 $\quad A + C = 0 \quad \to A = -C$

x^2 $\quad -4A + B - 2C + D = 1$

x^1 $\quad 4A - 4B + 4C = 4$

x^0 $\quad 4B - 8C + 4D = 0$

$$\begin{cases} B + 2C + D = 1 & \qquad 2C + D = 2 \\ -4B = 4 \qquad \underline{B = -1} & \\ 4B - 8C + 4D = 0 & \qquad -8C + 4D = 4 \end{cases}$$

$$\begin{cases} 2C + D = 2 \\ -2C + D = 1 \end{cases}$$

$$2D = 3 \quad \to \quad \underline{D = \frac{3}{2}}$$

$$2C = 2 - \frac{3}{2} = \frac{1}{2} \quad \to \quad \underline{C = \frac{1}{4}}$$

$$\underline{A = -\frac{1}{4}}$$

$$\int \frac{x^2+4x}{\left(x^2+4\right)(x-2)^2}\, dx = -\frac{1}{4}\int \frac{x}{x^2+4}\, dx - \int \frac{1}{x^2+4}\, dx + \frac{1}{4}\int \frac{1}{x-2}\, dx + \frac{3}{2}\int \frac{1}{(x-2)^2}\, dx$$

$$= -\frac{1}{8}\int \frac{1}{x^2+4}\, d\left(x^2+4\right) - \frac{1}{2}\arctan\frac{x}{2} + \frac{1}{4}\ln|x-2| + \frac{3}{2}\int \frac{1}{(x-2)^2}\, d(x-2)$$

$$= -\frac{1}{8}\ln\left(x^2+4\right) - \frac{1}{2}\arctan\frac{x}{2} + \frac{1}{4}\ln|x-2| - \frac{3}{2}\frac{1}{x-2} + K$$

Exercise

Evaluate $\displaystyle\int \frac{x^2+2x+3}{(x-1)(x+1)^2}\,dx$

Solution

$$\frac{x^2+2x+3}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

$$x^2+2x+3=Ax^2+2Ax+A+B(x+1)(x-1)+Cx-C$$

x^2 $\quad A+B=1 \qquad B=1-A$

x^1 $\quad 2A+C=2 \qquad C=2-2A$

x^0 $\quad A-B-C=3 \qquad (1)$

$(1)\rightarrow \quad A-1+A-2+2A=3 \quad \Rightarrow \boxed{A=\frac{3}{2}}$

$B=1-\frac{3}{2}\ \boxed{=-\frac{1}{2}}$

$C=2-3\ \boxed{=-1}$

$$\int \frac{x^2+2x+3}{(x-1)(x+1)^2}\,dx=\frac{3}{2}\int\frac{1}{x-1}\,dx-\frac{1}{2}\int\frac{1}{x+1}\,dx-\int\frac{1}{(x+1)^2}\,d(x+1)$$

$$\boxed{=\frac{3}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+\frac{1}{x+1}+K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^4-x^3+3x^2-x+2}{(x-1)\left(x^2+2\right)^2}\,dx$

Solution

$$\frac{x^4-x^3+3x^2-x+2}{(x-1)\left(x^2+2\right)^2}=\frac{A}{x-1}+\frac{Bx+C}{x^2+2}+\frac{Dx+E}{\left(x^2+2\right)^2}$$

$$x^4-x^3+3x^2-x+2=A\left(x^4+4x^2+4\right)+(Bx+C)(x-1)\left(x^2+2\right)+(Dx+E)(x-1)$$

$$=Ax^4+4Ax^2+4A+(Bx+C)\left(x^3+2x-x^2-2\right)+Dx^2-Dx+Ex-E$$

$$
\begin{array}{lll}
x^4 & A+B=1 & \to A=1-B \\
x^3 & -B+C=-1 & \to C=B-1 \\
x^2 & 4A+2B-C+D=3 & \\
x^1 & -2B+2C-D+E=1 & \\
x^0 & 4A-2C-E=2 &
\end{array}
$$

$$
\begin{cases}
4-4B+2B-B+1+D=3 \\
-2B+2B-2-D+E=1 \\
4-4B-2B+2-E=2
\end{cases}
$$

$$
\begin{cases}
\left.\begin{array}{l} -3B+D=-2 \\ -D+E=3 \end{array}\right\} \\
-6B-E=-4
\end{cases}
\to \quad \begin{array}{l} -3B+E=1 \\ -6B-E=-4 \end{array}
$$

$$-9B=-3 \quad \to \quad \boxed{B=\frac{1}{3}}$$

$$E=1+1 \ \boxed{=2}$$

$$A=1-\frac{1}{3} \ \boxed{=\frac{2}{3}}$$

$$C=\frac{1}{3}-1 \ \boxed{=-\frac{2}{3}}$$

$$D=-2+1 \ \boxed{=-1}$$

$$
\begin{aligned}
\int \frac{x^4-x^3+3x^2-x+2}{(x-1)\left(x^2+2\right)^2}\,dx &= \frac{2}{3}\int \frac{1}{x-1}dx+\frac{1}{3}\int \frac{x}{x^2+2}dx-\frac{2}{3}\int \frac{1}{x^2+2}dx-\int \frac{x}{\left(x^2+2\right)^2}dx \\
&\quad +\int \frac{2}{\left(x^2+2\right)^2}dx \\
&= \frac{2}{3}\ln|x-1|+\frac{1}{6}\int \frac{d\left(x^2+2\right)}{x^2+2}-\frac{2}{3\sqrt{2}}\arctan\frac{x}{\sqrt{2}}-\frac{1}{2}\int \frac{d\left(x^2+2\right)}{\left(x^2+2\right)^2} \\
&\quad +\int \frac{2}{\left(x^2+2\right)^2}dx \\
&= \frac{2}{3}\ln|x-1|+\frac{1}{6}\ln\left(x^2+2\right)-\frac{2}{3\sqrt{2}}\arctan\frac{x}{\sqrt{2}}+\frac{1}{2\left(x^2+2\right)}+\int \frac{2}{\left(x^2+2\right)^2}dx
\end{aligned}
$$

$$x=\sqrt{2}\tan\theta \quad \to dx=\sqrt{2}\sec^2\theta\ d\theta$$

$$x^2+2=2\sec^2\theta$$

$$\int \frac{2}{\left(x^2+2\right)^2}\,dx = \int \frac{2}{4\sec^4\theta}\sqrt{2}\sec^2\theta\,d\theta$$

$$= \frac{\sqrt{2}}{2}\int \frac{d\theta}{\sec^2\theta}$$

$$= \frac{\sqrt{2}}{2}\int \cos^2\theta\,d\theta$$

$$= \frac{\sqrt{2}}{4}\int (1+\cos 2\theta)\,d\theta$$

$$= \frac{\sqrt{2}}{4}\left(\theta+\frac{1}{2}\sin 2\theta\right)$$

$$= \frac{\sqrt{2}}{4}(\theta+\sin\theta\cos\theta)$$

$$= \frac{\sqrt{2}}{4}\left(\theta+\frac{\tan\theta}{\sec^2\theta}\right)$$

$$= \frac{\sqrt{2}}{4}\left(\arctan\frac{x}{\sqrt{2}}+\frac{x}{\sqrt{2}}\frac{2}{x^2+2}\right)$$

$$\int \frac{x^4-x^3+3x^2-x+2}{(x-1)\left(x^2+2\right)^2}\,dx = \frac{2}{3}\ln|x-1|+\frac{1}{6}\ln\left(x^2+2\right)-\frac{2}{3\sqrt{2}}\arctan\frac{x}{\sqrt{2}}+\frac{1}{2\left(x^2+2\right)}$$

$$+\frac{\sqrt{2}}{4}\arctan\frac{x}{\sqrt{2}}+\frac{1}{2}\frac{x}{x^2+2}+K$$

Exercise

Evaluate $\displaystyle\int \frac{-x^2+11x+18}{(x-1)(x+1)\left(x^2+3x+3\right)}\,dx$

Solution

$$\frac{-x^2+11x+18}{(x-1)(x+1)\left(x^2+3x+3\right)} = \frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+3x+3}$$

$$-x^2+11x+18 = A(x+1)\left(x^2+3x+3\right)+B(x-1)\left(x^2+3x+3\right)+(Cx+D)\left(x^2-1\right)$$

$x^3 \quad A+B+C=0 \qquad (1)$

$x^2 \quad 4A+2B+D=-1 \qquad (2)$

$x^1 \quad 6A-C=11 \quad \rightarrow \; C=6A-11$

$x^0 \quad 3A-3B-D=18 \qquad (3)$

$$\begin{cases} (1) \rightarrow & 7A+B=11 \\ (2)+(3) \rightarrow & 7A-B=17 \end{cases}$$

$14A=28 \quad \rightarrow \quad \underline{A=2}$

$B=11-14 \underline{=-3}$

$C=12-11 \underline{=1}$

$(2) \rightarrow \quad D=-1-8+6 \underline{=-3}$

$$\int \frac{-x^2+11x+18}{(x-1)(x+1)\left(x^2+3x+3\right)} dx = \int \frac{2}{x-1} dx - \int \frac{3}{x+1} dx + \int \frac{x-3}{x^2+3x+3} dx$$

$$= 2\ln|x-1| - 3\ln|x+1| + \int \frac{x-3}{x^2+3x+3} dx$$

$$\int \frac{x-3}{x^2+3x+3} dx = \frac{1}{2}\int \frac{2x-6+3-3}{x^2+3x+3} dx$$

$$= \frac{1}{2}\int \frac{2x+3}{x^2+3x+3} dx - \frac{9}{2}\int \frac{1}{\left(x+\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} dx$$

$$= \frac{1}{2}\int \frac{1}{x^2+3x+3} d\left(x^2+3x+3\right) - \frac{9}{2}\frac{2}{\sqrt{3}}\arctan\frac{x+\frac{3}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{2}\ln\left|x^2+3x+3\right| - 3\sqrt{3}\arctan\frac{2x+3}{\sqrt{3}}$$

$$\int \frac{-x^2+11x+18}{(x-1)(x+1)\left(x^2+3x+3\right)} dx = \underline{2\ln|x-1| - 3\ln|x+1| + \frac{1}{2}\ln\left|x^2+3x+3\right| - 3\sqrt{3}\arctan\frac{2x+3}{\sqrt{3}} + K}$$

Exercise

Evaluate $\displaystyle\int \frac{x^3+5x^2+2x-4}{x\left(x^2+4\right)^2} dx$

Solution

$$\frac{x^3+5x^2+2x-4}{x\left(x^2+4\right)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+4}+\frac{Dx+E}{\left(x^2+4\right)^2}$$

$$x^3+5x^2+2x-4=A\left(x^4+8x^2+16\right)+\left(Bx^2+Cx\right)\left(x^2+4\right)+Dx^2+Ex$$

$x^4 \quad A+B=0 \quad \to \underline{B=\frac{1}{4}}$

$x^3 \quad \underline{C=1}$

$x^2 \quad 8A+4B+D=5 \quad \to \underline{D=6}$

$x^1 \quad 4C+E=2 \quad \to \underline{E=-2}$

$x^0 \quad 16A=-4 \quad \to \underline{A=-\frac{1}{4}}$

$$\int \frac{x^3+5x^2+2x-4}{x\left(x^2+4\right)^2}\,dx=-\frac{1}{4}\int\frac{dx}{x}+\frac{1}{4}\int\frac{x}{x^2+4}dx+\int\frac{dx}{x^2+4}+\int\frac{6x}{\left(x^2+4\right)^2}dx-\int\frac{2}{\left(x^2+4\right)^2}dx$$

$$=-\frac{1}{4}\ln|x|+\frac{1}{8}\int\frac{d\left(x^2+4\right)}{x^2+4}+\frac{1}{2}\arctan\frac{x}{2}+3\int\frac{d\left(x^2+4\right)}{\left(x^2+4\right)^2}-\int\frac{2}{\left(x^2+4\right)^2}dx$$

$$=-\frac{1}{4}\ln|x|+\frac{1}{8}\ln\left(x^2+4\right)+\frac{1}{2}\arctan\frac{x}{2}-\frac{3}{x^2+4}-\int\frac{2}{\left(x^2+4\right)^2}dx$$

$x=2\tan\theta \quad \to dx=2\sec^2\theta\, d\theta$

$x^2+4=4\sec^2\theta$

$$\int\frac{2}{\left(x^2+4\right)^2}dx=\int\frac{2}{16\sec^4\theta}2\sec^2\theta\,d\theta$$

$$=\frac{1}{4}\int\frac{d\theta}{\sec^2\theta}$$

$$=\frac{1}{4}\int\cos^2\theta\,d\theta$$

$$=\frac{1}{8}\int(1+\cos 2\theta)\,d\theta$$

$$=\frac{1}{8}\left(\theta+\frac{1}{2}\sin 2\theta\right)$$

$$=\frac{1}{8}(\theta+\sin\theta\cos\theta)$$

$$=\frac{1}{8}\left(\theta+\frac{\tan\theta}{\sec^2\theta}\right)$$

$$=\frac{1}{8}\left(\arctan\frac{x}{2}+\frac{x}{2}\frac{4}{x^2+4}\right)$$

$$\underline{=\frac{1}{8}\arctan\frac{x}{2}+\frac{1}{4}\frac{x}{x^2+4}}$$

$$\int\frac{x^3+5x^2+2x-4}{x\left(x^2+4\right)^2}dx=-\frac{1}{4}\ln|x|+\frac{1}{8}\ln\left(x^2+4\right)+\frac{1}{2}\arctan\frac{x}{2}-\frac{3}{x^2+4}-\frac{1}{8}\arctan\frac{x}{2}-\frac{1}{4}\frac{x}{x^2+4}+K$$

$$\underline{=-\frac{1}{4}\ln|x|+\frac{1}{8}\ln\left(x^2+4\right)+\frac{3}{8}\arctan\frac{x}{2}-\frac{1}{4}\frac{x+12}{x^2+4}+K}$$

Exercise

Evaluate $\displaystyle\int_{-1}^{2}\frac{5x}{x^2-x-6}dx$

Solution

$$\frac{5x}{x^2-x-6}=\frac{A}{x-3}+\frac{B}{x+2}$$

$$5x=Ax+2A+Bx-3B$$

$x^1 \quad A+B=5$

$x^0 \quad 2A-3B=0$

$$\Delta=\begin{vmatrix}1&1\\2&-3\end{vmatrix}=-5 \qquad \Delta_A=\begin{vmatrix}5&1\\0&-3\end{vmatrix}=-15 \qquad \Delta_B=\begin{vmatrix}1&5\\2&0\end{vmatrix}=-10$$

$$A=\frac{-15}{-5}\ \underline{=3} \qquad B=\frac{-10}{-5}\ \underline{=2}$$

$$\int_{-1}^{2}\frac{5x}{x^2-x-6}dx=\int_{-1}^{2}\left(\frac{3}{x-3}+\frac{2}{x+2}\right)dx$$

$$=3\ln|x-3|+2\ln|x+2|\ \Big|_{-1}^{2}$$

$$=3\ln|-1|+2\ln 4-3\ln|-4|-2\ln 1$$

$$=2\ln 4-3\ln 4$$

$$\underline{=-\ln 4}$$

Exercise

Evaluate $\displaystyle\int_0^5 \frac{2}{x^2-4x-32}\,dx$

Solution

$$\frac{2}{x^2-4x-32}=\frac{A}{x-8}+\frac{B}{x+4}$$

$$2=Ax+4A+Bx-8B$$

$$x^1 \quad A+B=0$$

$$x^0 \quad 4A-8B=2$$

$$\Delta=\begin{vmatrix}1 & 1\\ 4 & -8\end{vmatrix}=-12 \qquad \Delta_A=\begin{vmatrix}0 & 1\\ 2 & -8\end{vmatrix}=-2 \qquad \Delta_B=\begin{vmatrix}1 & 0\\ 4 & 2\end{vmatrix}=2$$

$$A=\frac{-2}{-12}=\frac{1}{6} \qquad B=\frac{2}{-12}=-\frac{1}{6}$$

$$\begin{aligned}
\int_0^5 \frac{2}{x^2-4x-32}\,dx &= \frac{1}{6}\int_0^5\left(\frac{1}{x-8}-\frac{1}{x+4}\right)dx\\
&=\frac{1}{6}\left(\ln|x-8|-\ln|x+4|\right.\Big|_0^5\\
&=\frac{1}{6}\left(\ln|-3|-\ln 9-\ln|-8|+\ln 4\right)\\
&=\frac{1}{6}\left(\ln 3-\ln 3^2-\ln 8+\ln 4\right)\\
&=\frac{1}{6}\left(\ln 3-2\ln 3-3\ln 2+2\ln 2\right)\\
&=\frac{1}{6}\left(-\ln 3-\ln 2\right)\\
&=-\frac{1}{6}\left(\ln 3+\ln 2\right)\\
&=-\frac{\ln 6}{6}
\end{aligned}$$

Exercise

Evaluate $\displaystyle\int_0^1 \frac{dx}{(x+1)\left(x^2+1\right)}$

Solution

$$\frac{1}{(x+1)\left(x^2+1\right)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$$

$$1 = Ax^2 + A + Bx^2 + Bx + Cx + C$$

$$\begin{array}{lll} x^2 & A+B=0 & A=-B \\ x & B+C=0 & C=-B \\ x^0 & A+C=1 & (1) \end{array}$$

$$(1) \rightarrow \quad -B - B = 1 \quad \boxed{B = -\frac{1}{2}}$$

$$\boxed{A = C = \frac{1}{2}}$$

$$\int_0^1 \frac{dx}{(x+1)(x^2+1)} = \frac{1}{2}\int_0^1 \frac{1}{x+1}\,dx - \frac{1}{2}\int_0^1 \frac{x}{x^2+1}\,dx + \frac{1}{2}\int_0^1 \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln(x+1) - \frac{1}{4}\int_0^1 \frac{1}{x^2+1}\,d\left(x^2+1\right) + \frac{1}{2}\arctan x$$

$$= \left(\frac{1}{2}\ln(x+1) - \frac{1}{4}\ln\left(x^2+1\right) + \frac{1}{2}\arctan x\right)\Big|_0^1$$

$$= \frac{1}{2}\ln 2 - \frac{1}{4}\ln 2 + \frac{1}{2}\arctan 1 - \frac{1}{2}\ln 1 - \frac{1}{4}\ln 1 + \frac{1}{2}\arctan 0$$

$$\boxed{= \frac{1}{4}\ln 2 + \frac{\pi}{8}}$$

Exercise

Evaluate $\displaystyle\int_{-1/2}^{1/2} \frac{x^2+1}{x^2-1}\,dx$

Solution

$$\int_{-1/2}^{1/2} \frac{x^2+1}{x^2-1}\,dx = \int_{-1/2}^{1/2} \left(1 + \frac{2}{x^2-1}\right) dx$$

$$\frac{2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$2 = Ax + A + Bx - B$$

$$\begin{array}{ll} x & A+B=0 \\ x^0 & A-B=2 \end{array} \quad \rightarrow \quad A=1 \quad B=-1$$

$$\int_{-1/2}^{1/2} \frac{x^2+1}{x^2-1}\,dx = \int_{-1/2}^{1/2} \left(1 + \frac{1}{x-1} - \frac{1}{x+1}\right) dx$$

$$= x + \ln|x-1| - \ln|x+1| \Big|_{-1/2}^{1/2}$$

$$= x + \ln\left|\frac{x-1}{x+1}\right| \Big|_{-1/2}^{1/2}$$

$$= \frac{1}{2} + \ln\left|\frac{-\frac{1}{2}}{\frac{3}{2}}\right| + \frac{1}{2} - \ln\left|\frac{-\frac{3}{2}}{\frac{1}{2}}\right|$$

$$= 1 + \ln\left|\frac{1}{3}\right| - \ln|-3|$$

$$= 1 - \ln 3 - \ln 3$$

$$= 1 - 2\ln 3$$

Exercise

Evaluate $\displaystyle\int_0^2 \frac{3}{4x^2+5x+1}\, dx$

Solution

$$\frac{3}{4x^2+5x+1} = \frac{A}{x+1} + \frac{B}{4x+1}$$

$$4Ax + A + Bx + B = 3$$

$$\Rightarrow \begin{cases} 4A + B = 0 \\ A + B = 3 \end{cases} \rightarrow A = -1 \quad B = 4$$

$$\int_0^2 \frac{3}{4x^2+5x+1}\, dx = -\int_0^2 \frac{1}{x+1}\, dx + \int_0^2 \frac{4}{4x+1}\, dx$$

$$= -\ln(x+1) + \ln(4x+1) \Big|_0^2$$

$$= \ln\frac{4x+1}{x+1} \Big|_0^2$$

$$= \ln 3$$

Exercise

Evaluate $\displaystyle\int_1^5 \frac{x-1}{x^2(x+1)}\, dx$

Solution

$$\frac{x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$Ax^2 + Ax + Bx + B + Cx^2 = x - 1$$

$$\begin{cases} x^2 & A + C = 0 \\ x^1 & A + B = 1 \quad \rightarrow \underline{A = 2 \quad C = -2} \\ x^0 & \underline{B = -1} \end{cases}$$

$$\int_1^5 \frac{x-1}{x^2(x+1)}\,dx = \int_1^5 \left(\frac{2}{x} - \frac{1}{x^2} - \frac{2}{x+1}\right) dx$$

$$= 2\ln x + \frac{1}{x} - 2\ln(x+1) \Big|_1^5$$

$$= 2\ln 5 + \frac{1}{5} - 2\ln 6 - 1 + 2\ln 2$$

$$\underline{= 2\ln\frac{5}{3} - \frac{4}{5}}$$

Exercise

Evaluate $\displaystyle\int_1^2 \frac{x+1}{x(x^2+1)}\,dx$

Solution

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$Ax^2 + A + Bx^2 + Cx = x + 1$$

$$\begin{cases} x^2 & A+B=0 \\ x^1 & \underline{C=1} \quad \rightarrow \underline{B=-1} \\ x^0 & \underline{A=1} \end{cases}$$

$$\int_1^2 \frac{x+1}{x(x^2+1)}\,dx = \int_1^2 \frac{1}{x}\,dx - \int_1^2 \frac{x}{x^2+1}\,dx + \int_1^2 \frac{1}{x^2+1}\,dx$$

$$= \int_1^2 \frac{1}{x}\,dx - \frac{1}{2}\int_1^2 \frac{1}{x^2+1}\,d(x^2+1) + \int_1^2 \frac{1}{x^2+1}\,dx$$

$$= \ln x - \frac{1}{2}\ln(x^2+1) + \arctan x \Big|_1^2$$

$$= \ln 2 - \frac{1}{2}\ln 5 + \arctan 2 + \frac{1}{2}\ln 2 - \frac{\pi}{4}$$

$$= \frac{1}{2}(3\ln 2 - \ln 5) - \frac{\pi}{4} + \arctan 2$$

$$=\boxed{\frac{1}{2}\ln\frac{8}{5}-\frac{\pi}{4}+\arctan 2}$$

Exercise

Evaluate $\displaystyle\int_0^1 \frac{x^2-x}{x^2+x+1}\,dx$

Solution

$$\int_0^1 \frac{x^2-x}{x^2+x+1}\,dx=\int_0^1\left(1-\frac{2x+1}{x^2+x+1}\right)dx$$

$$=\int_0^1 dx-\int_0^1 \frac{1}{x^2+x+1}\,d\left(x^2+x+1\right)$$

$$=x-\ln\left(x^2+x+1\right)\Big|_0^1$$

$$=\boxed{1-\ln 3}$$

Exercise

Evaluate $\displaystyle\int_4^8 \frac{y\,dy}{y^2-2y-3}$

Solution

$$\frac{y}{y^2-2y-3}=\frac{A}{y-3}+\frac{B}{y+1}$$

$$y=Ay+A+By-3B$$

$$\to\begin{cases}A+B=1\\ A-3B=0\end{cases}$$

$$\Delta=\begin{vmatrix}1&1\\1&-3\end{vmatrix}=-4\qquad \Delta_A=\begin{vmatrix}1&1\\0&-3\end{vmatrix}=-3\qquad \Delta_B=\begin{vmatrix}1&1\\1&0\end{vmatrix}=-1$$

$$\Rightarrow \boxed{A=\frac{3}{4}}\quad \boxed{B=\frac{1}{4}}$$

$$\int_4^8 \frac{y\,dy}{y^2-2y-3}=\frac{3}{4}\int_4^8\frac{dy}{y-3}+\frac{1}{4}\int_4^8\frac{dy}{y+1}$$

$$=\frac{3}{4}\ln|y-3|+\frac{1}{4}\ln|y+1|\ \Big|_4^8$$

$= \frac{3}{4}\ln|5| + \frac{1}{4}\ln|9| - \left(\frac{3}{4}\ln|1| + \frac{1}{4}\ln|5|\right)$

$= \frac{3}{4}\ln 5 + \frac{1}{4}\ln 9 - \frac{1}{4}\ln 5$

$= \frac{1}{2}\ln 5 + \frac{1}{4}\ln 3^2$ ***Power Rule***

$= \frac{1}{2}\ln 5 + \frac{1}{2}\ln 3$

$= \frac{1}{2}(\ln 5 + \ln 3)$ ***Product Rule***

$\boxed{= \frac{1}{2}\ln 15}$

Exercise

Evaluate $\displaystyle\int_1^{\sqrt{3}} \frac{3x^2 + x + 4}{x^3 + x}\,dx$

Solution

$$\frac{3x^2 + x + 4}{x^3 + x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

$$3x^2 + x + 4 = Ax^2 + A + Bx^2 + Cx$$

$$\begin{cases} A + B = 3 & \to B = 3 - 4 \boxed{= -1} \\ \boxed{C = 1} \\ \boxed{A = 4} \end{cases}$$

$$\int_1^{\sqrt{3}} \frac{3x^2 + x + 4}{x^3 + x}\,dx = \int_1^{\sqrt{3}} \frac{4}{x}\,dx + \int_1^{\sqrt{3}} \frac{-x + 1}{x^2 + 1}\,dx$$

$$= 4\int_1^{\sqrt{3}} \frac{1}{x}\,dx - \int_1^{\sqrt{3}} \frac{x}{x^2 + 1}\,dx + \int_1^{\sqrt{3}} \frac{1}{x^2 + 1}\,dx \qquad d\left(x^2 + 1\right) = 2x\,dx$$

$$= 4\int_1^{\sqrt{3}} \frac{1}{x}\,dx - \frac{1}{2}\int_1^{\sqrt{3}} \frac{d\left(x^2 + 1\right)}{x^2 + 1} + \int_1^{\sqrt{3}} \frac{1}{x^2 + 1}\,dx \qquad \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \left[4\ln|x| - \frac{1}{2}\ln\left(x^2 + 1\right) + \tan^{-1} x\right]_1^{\sqrt{3}}$$

$$= 4\ln\sqrt{3} - \frac{1}{2}\ln 4 + \tan^{-1}\sqrt{3} - \left(4\ln 1 - \frac{1}{2}\ln 2 + \tan^{-1} 1\right)$$

$$= 4\ln 3^{1/2} - \frac{1}{2}\ln 2^2 + \frac{\pi}{3} + \frac{1}{2}\ln 2 - \frac{\pi}{4}$$

$$= 2\ln 3 - \ln 2 + \frac{\pi}{12} + \frac{1}{2}\ln 2$$

$$= 2\ln 3 - \frac{1}{2}\ln 2 + \frac{\pi}{12}$$

$$\boxed{= \ln\left(\frac{9}{\sqrt{2}}\right) + \frac{\pi}{12}}$$

Exercise

Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$

Solution

$u = \tan\left(\frac{x}{2}\right) \quad \rightarrow \; x = 2\tan^{-1} u$

$dx = \dfrac{2du}{1+u^2}$

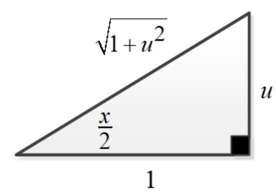
$$\cos x = 2\cos^2 \frac{x}{2} - 1$$

$$= 2\frac{1}{1+u^2} - 1$$

$$= \frac{1-u^2}{1+u^2}$$

$$\sin x = 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}}$$

$$= \frac{2u}{1+u^2}$$



$$\int_0^{\pi/2} \frac{dx}{\sin x + \cos x} = \int_0^{\pi/2} \frac{1}{\frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} du$$

$$= 2\int_0^{\pi/2} \frac{du}{2u + 1 - u^2}$$

$$= -2\int_0^{\pi/2} \frac{du}{u^2 - 2u - 1}$$

$$= -\frac{1}{\sqrt{2}}\int_0^{\pi/2} \left(\frac{1}{u-1-\sqrt{2}} - \frac{1}{u-1+\sqrt{2}}\right) du$$

$$\frac{2}{u^2 - 2u - 1} = \frac{A}{u-1-\sqrt{2}} + \frac{B}{u-1+\sqrt{2}}$$

$$2 = Au + \left(-1+\sqrt{2}\right)A + Bu + \left(-1-\sqrt{2}\right)B$$

$$\begin{cases} x & A+B=0 \\ x^0 & \left(-1+\sqrt{2}\right)A-\left(1+\sqrt{2}\right)B=2 \end{cases}$$

$$\rightarrow \begin{cases} B=-A=-\dfrac{1}{\sqrt{2}} \\ 2\sqrt{2}A=2 \end{cases}$$

$$=-\frac{1}{\sqrt{2}}\left(\ln\left|\frac{1}{u-1-\sqrt{2}}\right|-\ln\left|\frac{1}{u-1+\sqrt{2}}\right|\ \right|_0^{\pi/2}$$

$$=\frac{1}{\sqrt{2}}\left(\ln\left|\frac{u-1+\sqrt{2}}{u-1-\sqrt{2}}\right|\ \right|_0^{\pi/2}$$

$$=\frac{1}{\sqrt{2}}\left(\ln\left|\frac{\tan\frac{x}{2}-1+\sqrt{2}}{\tan\frac{x}{2}-1-\sqrt{2}}\right|\ \right|_0^{\pi/2}$$

$$=\frac{1}{\sqrt{2}}\left(\ln|-1|-\ln\left|\frac{-1+\sqrt{2}}{-1-\sqrt{2}}\right|\right)$$

$$\boxed{=\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)}$$

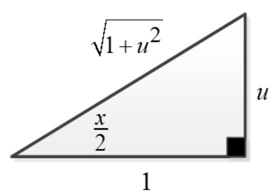
Exercise

Evaluate $\displaystyle\int_0^{\pi/3}\frac{\sin\theta}{1-\sin\theta}d\theta$

Solution

$u=\tan\left(\frac{x}{2}\right)\ \rightarrow\ x=2\tan^{-1}u$

$dx=\dfrac{2du}{1+u^2}$



$$\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$=2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}}$$

$$=\frac{2u}{1+u^2}$$

$$\int_0^{\pi/3}\frac{\sin\theta}{1-\sin\theta}d\theta=\int_0^{\pi/3}\frac{1}{\csc\theta-1}d\theta$$

$$= \int_0^{\pi/3} \frac{1}{\frac{1+u^2}{2u}-1} \cdot \frac{2}{1+u^2}\, du$$

$$= \int_0^{\pi/3} \frac{4u}{\left(1+u^2-2u\right)\left(1+u^2\right)}\, du$$

$$= \int_0^{\pi/3} \frac{4u}{(u-1)^2\left(1+u^2\right)}\, du$$

$$\frac{4u}{(u-1)^2\left(1+u^2\right)} = \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{Cu+D}{1+u^2}$$

$$4u = Au + Au^3 - A - Au^2 + B + Bu^2 + Cu^3 - 2Cu^2 + Cu + Du^2 - 2Du + D$$

$$\begin{cases} A + C = 0 & \rightarrow A = -C \\ -A + B - 2C + D = 0 & \\ C - 2D = 4 & \rightarrow D = \frac{1}{2}C - 2 \\ -A + B + D = 0 & \end{cases}$$

$$\begin{cases} C + B - 2C + \frac{1}{2}C - 2 = 0 \\ C + B + \frac{1}{2}C - 2 = 0 \end{cases}$$

$$\begin{cases} B - \frac{1}{2}C = 2 \\ B + \frac{3}{2}C = 2 \end{cases} \quad \rightarrow \quad \underline{C = 0} \quad \underline{B = 2}$$

$$\underline{A = 0; \quad D = -2}$$

$$= \int_0^{\pi/3} \left(\frac{2}{(u-1)^2} - \frac{2}{1+u^2} \right) du$$

$$= \frac{-2}{u-1} - 2\tan^{-1} u \,\Big|_0^{\pi/3}$$

$$= \frac{-2}{\tan\frac{x}{2} - 1} - 2\tan^{-1}\left(\tan\frac{x}{2}\right) \,\Big|_0^{\pi/3}$$

$$= \frac{-2}{\tan\frac{x}{2} - 1} - x \,\Big|_0^{\pi/3}$$

$$= \frac{-2}{\frac{1}{\sqrt{3}} - 1} - \frac{\pi}{3} - 2$$

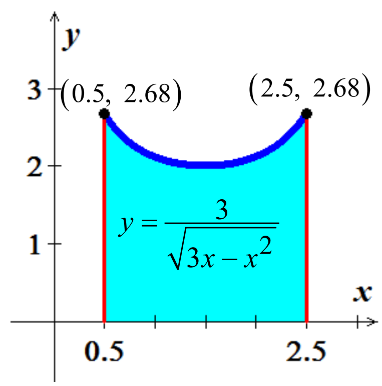
$$= \frac{-2\sqrt{3}}{1-\sqrt{3}} - \frac{\pi}{3} - 2$$

$$= \frac{-2}{1-\sqrt{3}} - \frac{\pi}{3} \qquad\qquad = \frac{-2}{1-\sqrt{3}}\frac{1+\sqrt{3}}{1+\sqrt{3}} - \frac{\pi}{3}$$

$$\boxed{= 1 + \sqrt{3} - \frac{\pi}{3}}$$

Exercise

Find the volume of the solid generated by the revolving the shaded region about x-axis



Solution

$$V = \pi \int_{0.5}^{2.5} y^2\, dx$$

$$= \pi \int_{0.5}^{2.5} \frac{9}{3x-x^2}\, dx$$

$$= 9\pi \int_{0.5}^{2.5} \frac{1}{3x-x^2}\, dx$$

$$\frac{1}{3x-x^2} = \frac{1}{x(3-x)}$$

$$= \frac{A}{x} + \frac{B}{3-x}$$

$$1 = 3A - Ax + Bx$$

$$\begin{cases} B - A = 0 \\ 3A = 1 \end{cases} \Rightarrow \boxed{A = \frac{1}{3}} \quad \boxed{B = \frac{1}{3}}$$

$$= 9\pi \int_{0.5}^{2.5} \frac{1}{3}\left(\frac{1}{x} + \frac{1}{3-x}\right) dx$$

$$= 3\pi \int_{0.5}^{2.5} \left(\frac{1}{x} - \frac{1}{x-3}\right) dx$$

$$= 3\pi \left[\int_{0.5}^{2.5} \frac{1}{x}\, dx - \int_{0.5}^{2.5} \frac{1}{x-3}\, dx\right]$$

$$= 3\pi \left(\ln|x| - \ln|x-3|\right\Big|_{0.5}^{2.5}$$

$$= 3\pi \left(\ln\left|\frac{x}{x-3}\right| \right|_{0.5}^{2.5}$$

$$= 3\pi\left[\ln\left|\frac{2.5}{-.5}\right| - \ln\left|\frac{0.5}{-2.5}\right| \right]$$

$$= 3\pi\left(\ln 5 - \ln\frac{1}{5} \right)$$

$$= 3\pi\ (\ln 5 + \ln 5)$$

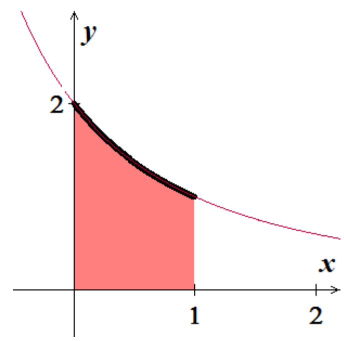
$$\boxed{= 6\pi \ln 5}$$

Exercise

Find the area of the region bounded by the graphs of $y = \dfrac{12}{x^2+5x+6}$, $y=0$, $x=0$, *and* $x=1$

Solution

$$A = \int_0^1 \frac{12}{x^2+5x+6}\,dx$$



$$\frac{12}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$12 = Ax + 3A + Bx + 2B$$

$$\begin{cases} A + B = 0 \\ 3A + 2B = 12 \end{cases} \rightarrow A = 12 \quad B = -12$$

$$A = \int_0^1 \frac{12}{x+2}\,dx - \int_0^1 \frac{12}{x+3}\,dx$$

$$= 12\ \left(\ln|x+2| - \ln|x+3| \right|_0^1$$

$$= 12\left(\ln 3 - \ln 4 - \ln 2 + \ln 3 \right)$$

$$= 12\left(2\ln 3 - 3\ln 2 \right)$$

$$= 12\left(\ln 9 - \ln 8 \right)$$

$$\boxed{= 12 \ln \frac{9}{8}}$$

Exercise

Find the area of the region bounded by the graphs of $y = \dfrac{7}{16 - x^2}$ *and* $y = 1$

Solution

$$A = 2\int_0^3 \left(1 - \frac{7}{16 - x^2}\right) dx$$

$$= 2\int_0^3 dx - 2\int_0^3 \frac{7}{16 - x^2} dx \qquad x = 4\sin\theta \quad 16 - x^2 = 16\cos^2\theta \quad dx = 4\cos\theta d\theta$$

$$= 2x \Big|_0^3 - 14\int_0^3 \frac{1}{4\cos\theta} d\theta$$

$$= 6 - \frac{7}{2}\int_0^3 \sec\theta \, d\theta$$

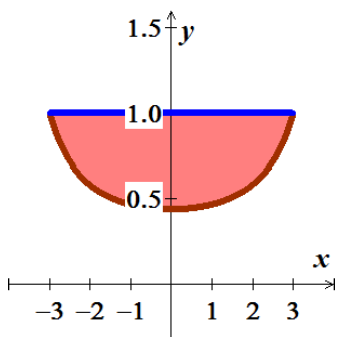
$$= 6 - \frac{7}{2}\ln\left|\sec\theta + \tan\theta\right| \Big|_0^3$$

$$= 6 - \frac{7}{2}\ln\left|\frac{4 + x}{\sqrt{16 - x^2}}\right| \Big|_0^3$$

$$= 6 - \frac{7}{2}\ln\left|\frac{7}{\sqrt{7}}\right|$$

$$= 6 - \frac{7}{2}\ln\sqrt{7}$$

$$\boxed{= 6 - \frac{7}{4}\ln 7} \quad \approx 2.595$$



Exercise

The region in the first quadrant that is enclosed by the x-axis, the curve $y = \dfrac{5}{x\sqrt{5 - x}}$, and the lines $x = 1$ and $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution

$$V = \pi\int_1^4 y^2 \, dx$$

$$= \pi\int_1^4 \left(\frac{5}{x\sqrt{5 - x}}\right)^2 dx$$

$$= \pi \int_1^4 \frac{25}{x^2(5-x)}\, dx$$

$$\frac{25}{x^2(5-x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{5-x}$$

$$25 = 5Ax - Ax^2 + 5B - Bx + Cx^2$$

x^2 $\quad -A + C = 0 \qquad \rightarrow \underline{C = 1}$

x^1 $\quad 5A - B = 0 \qquad \rightarrow \underline{A = 1}$

x^0 $\quad 5B = 25 \qquad \rightarrow \underline{B = 5}$

$$= \pi \int_1^4 \left(\frac{1}{x} + \frac{5}{x^2} + \frac{1}{5-x}\right) dx$$

$$= \pi \left(\ln x - \frac{5}{x} - \ln|5-x| \;\right|_1^4$$

$$= \pi \left(\ln\left|\frac{x}{5-x}\right| - \frac{5}{x} \;\right|_1^4$$

$$= \pi \left(\ln 4 - \frac{5}{4} - \ln\frac{1}{4} + 5\right)$$

$$= \pi \left(\ln 4 + \ln 4 + \frac{15}{4}\right)$$

$$\underline{= \pi \left(2\ln 4 + \frac{15}{4}\right)}$$

Exercise

Find the length of the graph of the function $y = \ln\left(1 - x^2\right), \quad 0 \le x \le \frac{1}{2}$

Solution

$$\frac{dy}{dx} = \frac{-2x}{1-x^2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{-2x}{1-x^2}\right)^2$$

$$= 1 + \frac{4x^2}{\left(1-x^2\right)^2}$$

$$= \frac{1 - 2x^2 + x^4 + 4x^2}{\left(1-x^2\right)^2}$$

$$= \frac{1+2x^2+x^4}{\left(1-x^2\right)^2}$$

$$= \left(\frac{1+x^2}{1-x^2}\right)^2$$

$$L = \int_0^{1/2} \frac{1+x^2}{1-x^2}\, dx$$

$$\begin{array}{r} -1 \\ -x^2+1 \overline{\smash{\big)} x^2+1} \\ \underline{x^2-1} \\ 2 \end{array}$$

$$= \int_0^{1/2} \left(-1+\frac{2}{1-x^2}\right) dx$$

$$\frac{2}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$$

$$2 = A + Ax + B - Bx$$

$$\begin{cases} x & A-B=0 \\ x^0 & A+B=2 \end{cases}$$

$$\boxed{A=1 \quad B=1}$$

$$= \int_0^{1/2} \left(-1+\frac{1}{1-x}+\frac{1}{1+x}\right) dx$$

$$= -x - \ln|1-x| + \ln|1+x| \Big|_0^{1/2}$$

$$= -x + \ln\left|\frac{1+x}{1-x}\right| \Big|_0^{1/2}$$

$$= -\frac{1}{2} + \ln\left|\frac{\frac{3}{2}}{\frac{1}{2}}\right| + 0 - \ln 1$$

$$\boxed{= -\frac{1}{2} + \ln 3}$$

Exercise

Consider the region bounded by the graphs $y = \dfrac{2x}{x^2+1}$, $y=0$, $x=0$, *and* $x=3$.

a) Find the volume of the solid generated by revolving the region about the x-*axis*

b) Find the centroid of the region.

Solution

a) $V = \pi \displaystyle\int_0^3 \left(\frac{2x}{x^2+1}\right)^2 dx$

$$= 4\pi \int_0^3 \frac{x^2}{\left(x^2+1\right)^2}\,dx$$

$$\frac{x^2}{\left(x^2+1\right)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{\left(x^2+1\right)^2}$$

$$x^2 = Ax^3 + Ax + Bx^2 + B + Cx + D$$

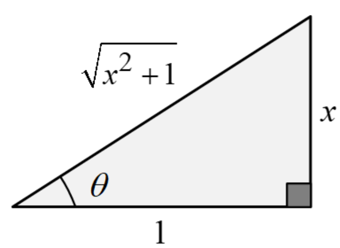
$$\begin{cases} x^3 & A = 0 \\ x^2 & B = 1 \\ x & A + C = 0 \rightarrow C = 0 \\ x^0 & B + D = 0 \rightarrow D = -1 \end{cases}$$

$$= 4\pi \int_0^3 \frac{1}{x^2+1]}\,dx - 4\pi \int_0^3 \frac{1}{\left(x^2+1\right)^2}\,dx$$

$x = \tan\theta \qquad x^2+1 = \sec^2\theta$

$dx = \sec^2\theta\, d\theta$

$$= 4\pi \arctan x \Big|_0^3 - 4\pi \int_0^3 \frac{1}{\sec^2\theta}\,d\theta$$



$$= 4\pi \arctan 3 - 4\pi \int_0^3 \cos^2\theta\, d\theta$$

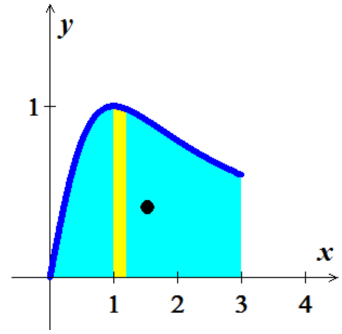
$$= 4\pi \arctan 3 - 2\pi \int_0^3 (1+\cos 2\theta)\, d\theta$$

$$= 4\pi \arctan 3 - 2\pi(\theta + \sin\theta\cos\theta \Big|_0^3$$

$$= 4\pi \arctan 3 - 2\pi \left(\arctan x + \frac{x}{x^2+1} \right|_0^3$$

$$= 4\pi \arctan 3 - 2\pi\left(\arctan 3 + \frac{3}{10}\right)$$

$$= 2\pi \arctan 3 - \frac{3\pi}{5} \quad unit^3 \qquad \approx 5.963 \quad unit^3$$



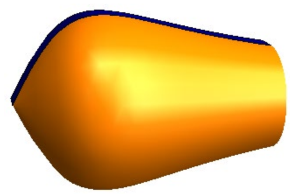
b) $$A = \int_0^3 \frac{2x}{x^2+1}\, dx$$

$$= \int_0^3 \frac{1}{x^2+1}\, d\left(x^2+1\right)$$

$$= \ln\left(x^2+1\right) \Big|_0^3$$

$$= \ln 10 \quad unit^2$$

$$\bar{x} = \frac{1}{\ln 10} \int_0^3 \frac{2x^2}{x^2+1} dx \qquad \bar{x} = \frac{1}{A}\int_a^b x \cdot f(x)\, dx$$



$$= \frac{1}{\ln 10} \int_0^3 \left(2 - \frac{2}{x^2+1}\right) dx$$

$$= \frac{1}{\ln 10} \left(2x - 2\arctan x \right. \Big|_0^3$$

$$= \frac{2}{\ln 10}(3 - \arctan 3) \qquad \approx 1.521$$

$$\bar{y} = \frac{1}{2}\frac{1}{\ln 10} \int_0^3 \left(\frac{2x}{x^2+1}\right)^2 dx \qquad \bar{x} = \frac{1}{A}\int_a^b x \cdot f(x)\, dx$$

$$= \frac{2}{\ln 10} \int_0^3 \frac{x^2}{\left(x^2+1\right)^2} dx$$

$$= \frac{2}{\ln 10} \int_0^3 \frac{1}{x^2+1} dx \; - \; \frac{2}{\ln 10} \int_0^3 \frac{1}{\left(x^2+1\right)^2} dx$$

$$= \frac{2}{\ln 10} \left(\arctan x - \frac{1}{2}\arctan x - \frac{1}{2}\frac{x}{x^2+1} \right. \Big|_0^3$$

$$= \frac{2}{\ln 10}\left(\frac{1}{2}\arctan 3 - \frac{3}{20}\right)$$

$$= \frac{1}{\ln 10}\left(\arctan 3 - \frac{3}{10}\right) \qquad \approx 0.412$$

$$(\bar{x}, \bar{y}) = \left(\frac{2}{\ln 10}(3 - \arctan 3), \quad \frac{1}{\ln 10}\left(\arctan 3 - \frac{3}{10}\right)\right) \qquad \approx (1.521, \quad 0.412)$$

Exercise

Consider the region bounded by the graph $y^2 = \dfrac{(2-x)^2}{(1+x)^2} \quad 0 \le x \le 1$.

Find the volume of the solid generated by revolving this region about the *x-axis*.

Solution

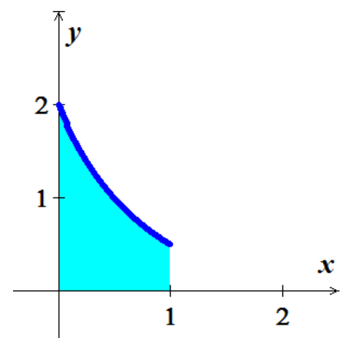
$$V = \pi \int_0^1 \frac{(2-x)^2}{(1+x)^2}\, dx$$

$$= 4\pi \int_0^1 \frac{1}{(1+x)^2}\, dx \;-\; 4\pi \int_0^1 \frac{x}{(1+x)^2}\, dx \;+\; \pi \int_0^1 \frac{x^2}{(1+x)^2}\, dx$$

$$\frac{x}{(1+x)^2} = \frac{A}{x+1} + \frac{B}{(1+x)^2}$$

$$Ax + A + B = x$$

$$A = 1, \quad B = -1$$



$$\frac{x^2}{x^2+2x+1} = 1 - \frac{2x+1}{(1+x)^2}$$

$$= 1 - \left(\frac{C}{x+1} + \frac{D}{(1+x)^2}\right)$$

$$Cx + C + D = 2x + 1$$

$$C = 2, \quad D = -1$$

$$= -4\pi \frac{1}{1+x}\Big|_0^1 - 4\pi \int_0^1 \frac{1}{x+1}\, dx + 4\pi \int_0^1 \frac{1}{(1+x)^2}\, dx + \pi \int_0^1 dx - 2\pi \int_0^1 \frac{1}{x+1}\, dx + \pi \int_0^1 \frac{1}{(1+x)^2}\, dx$$

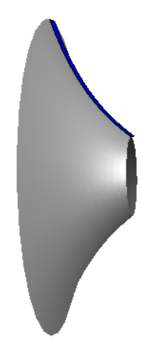
$$= 2\pi + \left(-4\pi \ln(x+1) - 4\pi \frac{1}{x+1} + \pi x - 2\pi \ln(x+1) - \pi \frac{1}{x+1}\right|_0^1$$

$$= 2\pi - \left(6\pi \ln(x+1) + 5\pi \frac{1}{x+1} - \pi x\right|_0^1$$

$$= 2\pi - \left(6\pi \ln(2) + \frac{5}{2}\pi - \pi - 5\pi\right)$$

$$= 2\pi - 6\pi \ln 2 + \frac{7}{2}\pi$$

$$= \frac{\pi}{2}(11 - 12\ln 2) \quad unit^3$$



Exercise

A single infected individual enters a community of n susceptible individuals. Let x be the number of newly infected individuals at time t. The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. So,

$\frac{dx}{dt} = k(x+1)(n-x)$ and you obtain

$\int \frac{1}{(x+1)(n-x)}\, dx = \int k\ dt$ Solve for x as a function of t.

Solution

$$\frac{1}{(x+1)(n-x)} = \frac{A}{x+1} + \frac{B}{n-x}$$

$$1 = An - Ax + Bx + B$$

$$\begin{cases} -A + B = 0 \\ nA + B = 1 \end{cases}$$

$$(n+1)A = 1 \Rightarrow \boxed{A = \frac{1}{n+1} = B}$$

$$\begin{aligned} \int \frac{1}{(x+1)(n-x)}\, dx &= \frac{1}{n+1}\int \frac{1}{x+1}\, dx + \frac{1}{n+1}\int \frac{1}{n-x}\, dx \\ &= \frac{1}{n+1}\left(\ln|x+1| - \ln|n-x|\right) \\ &= \frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right| \end{aligned}$$

$$\int k\ dt = kt + C$$

$$\frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right| = kt + C$$

$$x(t=0) = 0$$

$$\boxed{C = \frac{1}{n+1}\ln\left|\frac{1}{n}\right|}$$

$$\frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right| = kt + \frac{1}{n+1}\ln\left|\frac{1}{n}\right|$$

$$\ln\left|\frac{x+1}{n-x}\right| - \ln\left|\frac{1}{n}\right| = (n+1)kt$$

$$\ln\left|\frac{nx+n}{n-x}\right| = (n+1)kt$$

$$\frac{nx+n}{n-x} = e^{(n+1)kt}$$

$$nx + n = ne^{(n+1)kt} - xe^{(n+1)kt}$$

$$\left(n+e^{(n+1)kt}\right)x = ne^{(n+1)kt} - n$$

$$\boxed{x = \frac{ne^{(n+1)kt} - n}{n+e^{(n+1)kt}}} \qquad \lim_{t\to\infty} x = n$$

Exercise

Evaluate $\int_0^1 \frac{x}{1+x^4}\,dx$ in ***two*** different ways.

Solution

1- Partial method

$$\frac{x}{1+x^4} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}$$

$$x = Ax^3 - \sqrt{2}Ax^2 + Ax + Bx^2 - \sqrt{2}Bx + B + Cx^3 + \sqrt{2}Cx^2 + Cx + Dx^2 + \sqrt{2}Dx + D$$

$x^3 \quad A + C = 0 \to C = -A$

$x^2 \quad -\sqrt{2}A + B + \sqrt{2}C + D = 0$

$x \quad A - \sqrt{2}B + C + \sqrt{2}D = 1$

$x^0 \quad B + D = 0 \to D = -B$

$$\begin{cases} -2\sqrt{2}A = 0 \to \boxed{A = 0 = C} \\ -2\sqrt{2}B = 1 \Rightarrow \boxed{B = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}} \end{cases}$$

$$\to \boxed{D = \frac{\sqrt{2}}{4}}$$

$$\int_0^1 \frac{x}{1+x^4}\,dx = -\frac{\sqrt{2}}{4}\int_0^1 \frac{1}{x^2+\sqrt{2}x+1}\,dx + \frac{\sqrt{2}}{4}\int_0^1 \frac{1}{x^2-\sqrt{2}x+1}\,dx$$

$$= -\frac{\sqrt{2}}{4}\int_0^1 \frac{1}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\,dx + \frac{\sqrt{2}}{4}\int_0^1 \frac{1}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\,dx$$

$$= \frac{\sqrt{2}}{4}\left(-\sqrt{2}\arctan\sqrt{2}\left(x+\frac{\sqrt{2}}{2}\right) + \sqrt{2}\arctan\sqrt{2}\left(x-\frac{\sqrt{2}}{2}\right)\right)\Big|_0^1$$

$$= \frac{1}{2}\left(-\arctan\left(\sqrt{2}x+1\right) + \arctan\left(\sqrt{2}x-1\right)\right)\Big|_0^1$$

$$= \frac{1}{2}\left(-\arctan\left(\sqrt{2}+1\right) + \arctan\left(\sqrt{2}-1\right) + \arctan 1 - \arctan(-1)\right)$$

$$= \frac{1}{2}\left(-\arctan\left(\sqrt{2}+1\right)+\arctan\left(\sqrt{2}-1\right)+\frac{\pi}{2}\right)$$

2- Let $u = x^2 \rightarrow du = 2x dx$

$$\int_0^1 \frac{x}{1+x^4}\,dx = \frac{1}{2}\int_0^1 \frac{1}{1+u^2}\,du$$

$$= \frac{1}{2}\arctan x^2 \Big|_0^1$$

$$= \frac{\pi}{8}$$

$$\frac{1}{2}\left(\arctan\left(\sqrt{2}-1\right)-\arctan\left(\sqrt{2}+1\right)+\frac{\pi}{2}\right) = \frac{1}{2}\left(\arctan\left(\frac{\sqrt{2}-1-\sqrt{2}-1}{1+\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}\right)+\frac{\pi}{2}\right)$$

$$= \frac{1}{2}\left(\arctan(-1)+\frac{\pi}{2}\right)$$

$$= \frac{1}{2}\left(-\frac{\pi}{4}+\frac{\pi}{2}\right)$$

$$= \frac{\pi}{8}$$