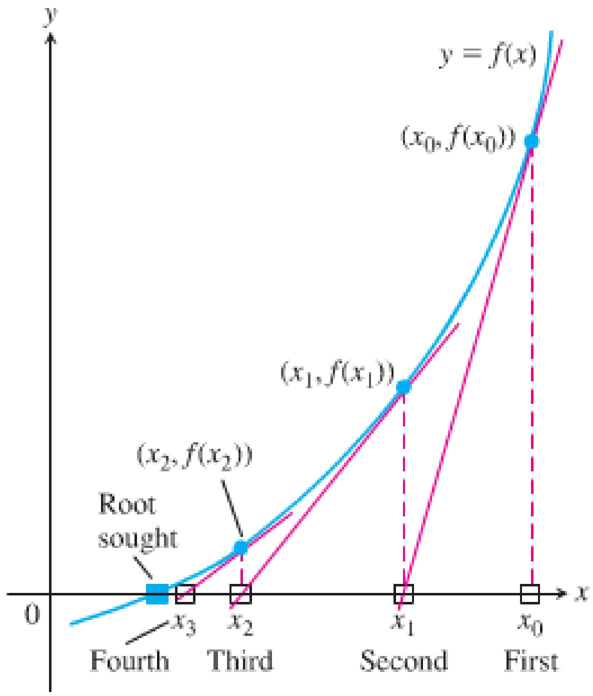
# *Section* 3.6 – Newton's Method

## Procedure for *Newton's Method*

The goal of Newton's method, also called the ***Newton-Raphson*** method, for estimating a solution of an equation $f(x)=0$ is to produce a sequence of approximations that approach the solution.




We begin with the first number $x_0$ of the sequence. Then the function is approximated by its tangent line, and one computes the *x*-intercept of this tangent line. At each step the method approximates a zero of *f* with a zero of one of its linearization.

> Initial estimates, $x_0$, the method than uses the tangent curve $y=f(x)$ @ $\left(x_0, f\left(x_0\right)\right)$ to approximate the curve, calling the point $x_1$ where the tangent meets the *x*-axis. The number $x_1$ usually a better approximation to the solution that is $x_0$. The point $x_2$ where the tangent to the curve at $\left(x_1, f\left(x_1\right)\right)$ crosses the *x*-axis is the next approximation in the sequence. We continue on using each approximation to generate the next, until we are close enough to the root to stop.

The point-slope equation for the tangent to the curve at $\left(x_n, f\left(x_n\right)\right)$ is

$$y=f\left(x_n\right)+f'\left(x_n\right)\cdot\left(x-x_n\right)$$

We can find where it crosses the *x*-axis by setting $y = 0$.

$$0 = f\left(x_n\right) + f'\left(x_n\right)\cdot\left(x - x_n\right) \quad \Rightarrow \quad x - x_n = -\frac{f\left(x_n\right)}{f'\left(x_n\right)}$$

$$\boxed{x = x_n - \frac{f\left(x_n\right)}{f'\left(x_n\right)}}$$

### *Newton's Method*

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f\left(x_n\right)}{f'\left(x_n\right)} \quad if\ f'\left(x_n\right) \neq 0$$

### *Example*

Find the positive root of the equation $f(x) = x^2 - 2 = 0$

***Solution***

$$f'(x) = 2x$$

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n^2 - 2}{2x_n} \\ &= x_n - \frac{x_n}{2} + \frac{1}{x_n} \\ &= \frac{x_n}{2} + \frac{1}{x_n} \end{aligned}$$

### *Example*

Find the $x$-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

**Solution**

-- $x^3 - x = 1$

$x^3 - x - 1 = 0$

$f(x) = x^3 - x - 1$

$$\begin{cases} f(1) = -1 \\ f(2) = 5 \end{cases}$$

$f'(x) = 3x^2 - 1$

| **n** | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|
| 0 | 1 | –1 | 2 | 1.5 |
| 1 | 1.5 | 0.875 | 5.75 | 1.347826087 |
| 2 | 1.347826087 | 0.100682173 | 4.449905482 | 1.325200399 |
| 3 | 1.325200399 | 0.002058362 | 4.268468292 | 1.324718174 |
| 4 | 1.324718174 | 0.000000924 | 4.264634722 | 1.324717957 |
| 5 | 1.324717957 | –1.8672 E–13 | 4.264632999 | 1.324717957 |

The result: $\boxed{x = 1.324717957}$

# *Exercises* Section 3.6 – Newton's Method

**1.** Use Newton's method to estimate the on real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find $x_2$

**2.** Use Newton's method to estimate the on real solution of $x^4 + x - 3 = 0$. Start with $x_0 = -1$ for the left-hand zero and with $x_0 = 1$ for the zero on the right. Then, in each case, find $x_2$

**3.** Use Newton's method to estimate the on real solution of $2x - x^2 + 1 = 0$. Start with $x_0 = 0$ for the left-hand zero and with $x_0 = 2$ for the zero on the right. Then, in each case, find $x_2$

**4.** Use Newton's method to estimate the on real solution of $x^4 - 2 = 0$. Start with $x_0 = 1$ and then find $x_2$

Use the Newton's method to approximate the roots to ten digits of

**5.** $f(x) = 3x^3 - 4x^2 + 1$ **6.** $f(x) = e^{-2x} + 2e^x - 6$ **7.** $f(x) = 2x^5 - 6x^3 - 4x + 2$