Section 1.4 – Nested Quantifiers

Introduction

Nested quantifiers commonly occur in mathematics and computer science. Nested quantifiers can sometimes be difficult to understand.
We will see how to use nested quantifiers to express mathematical statements such as "The sum of two positive integers is always positive." We will show how nested quantifiers can be used to translate sentences such as "Everyone has exactly one best friend" into logical statements.

Example

Translate the statement $\forall x \forall y\big((x>0)\wedge(y<0)\rightarrow(xy<0)\big)$
Where the domain for both variables consists of all real numbers.

Solution

This statement says that for every real number *x* and every real number *y*, if $x>0$ and $y<0$, then $xy<0$. That is, this statement says that for all real numbers *x* and *y*, if *x* is positive and *y* is negative, then *xy* is negative.
This can be stated more succinctly as
"The product of a postitive real number and a negative real numer is always a negative real number."

The Order of Quantifiers

Example

Let $P(x,\ y)$ be the statement $"x+y=y+x"$. What are the truth values of the quantifications $\forall x \forall y\ P(x,\ y)$ and $\forall y \forall x\ P(x,\ y)$ where the domain for all variables consists of all real numbers?

Solution

The quantification $\forall x \forall y\ P(x,\ y)$ denotes the proposition

For all real numbers *x*, for all real numbers *y*, $x+y=y+x$

The quantification $\forall y \forall x\ P(x,\ y)$ denotes the proposition

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Which they have the same meaning.
Therefore; $\forall x \forall y\ P(x,\ y)$ and $\forall y \forall x\ P(x,\ y)$ have the same meaning, and both are true. This illustrates the principle that the order of nested universal quantifiers in a statement without other quantifiers can be changed without changing the meaning of the quantified statement.

Example

Let $P(x, y)$ be the statement "$x + y = 0$". What are the truth values of the quantifications $\exists y \forall x\, P(x, y)$ and $\forall x \exists y\, P(x, y)$ where the domain for all variables consists of all real numbers?

Solution

The quantification $\exists y \forall x\, P(x, y)$ denotes the proposition

There is a real number *y*, such that for every real number *x*, $P(x, y)$

No matter what value of *y* is chosen, there is only one value of *x* for which $x + y = 0$. Because there is no real number *y* such that $x + y = 0$ for all real numbers *x*, the statement $\exists y \forall x\, P(x, y)$ is false.

$x + 1 = 0$

The quantification $\forall x \exists y\, P(x, y)$ denotes the proposition

For every real number *x*, there is a real number *y* such that $P(x, y)$

- ✓ *"For all x, there exists a y such that $P(x, y)$"*

$x + y = 0 \quad \Rightarrow \quad y = -x$

Hence, the statement $\forall x \exists y\, P(x, y)$ is true.

- ✓ ***There exists an x such that for all y $P(x, y)$ is true"***

Quantifications of Two variables

Statement	***When True?***	***When False?***
$\forall x \forall y\, P(x, y)$ $\forall x \forall y\, P(x, y)$	$P(x, y)$ ***is true for every pair*** x, y	***There is a pair*** x, y ***for which*** $P(x, y)$ ***is false.***
$\forall x \exists y\, P(x, y)$	***For every x there is a y for which*** $P(x, y)$ ***is true.***	***There is an x such that*** $P(x, y)$ ***is false for every y.***
$\exists x \forall y\, P(x, y)$	***There is an x for which*** $P(x, y)$ ***is true for every y.***	***For every x there is a y for which*** $P(x, y)$ ***is false.***
$\exists x \exists y\, P(x, y)$ $\exists y \exists x\, P(x, y)$	***There is a pair*** x, y ***for which*** $P(x, y)$ ***is true.***	$P(x, y)$ ***is false for every pair*** x, y

Example

Let $P(x,\ y,\ z)$ be the statement "$x+y=z$". What are the truth values of the statements $\forall x \forall y \exists z\ P(x,\ y,\ z)$ and $\exists z \forall x \forall y\ P(x,\ y,\ z)$ where the domain for all variables consists of all real numbers?

Solution

The statement $\forall x \forall y \exists z\ P(x,\ y,\ z)$ denotes the proposition

For all real numbers *x* and for all real numbers *y* there is a real number *z* such that $x+y=z$

This statement is true.

The statement $\exists z \forall x \forall y\ P(x,\ y,\ z)$ denotes the proposition

There is a real number *z* such that for all real numbers *x* and for all real numbers *y* it is true that $x+y=z$

This statement is false, because there is no value of z that satisfies the equation $x+y=z$ for all values of *x* and *y*.

Translating Mathematical Statements into Statements Involving Nested Quantifiers

Example

Translate the statement "The sum of two positive integers is always positive" into a logical expression.

Solution

Let *x* and *y* be the positive integers variables which: "For all positive integers *x* and *y*, $x+y$ is positive."
We can express as:

$$\forall x \forall y \big((x>0) \wedge (y>0) \rightarrow (x+y>0)\big)$$

We also can translate this using the positive integers as the domain.

$$\forall x \forall y\ \ (x+y>0)$$

Where the domain for both variables consists of all positive integers

Example

Translate the statement $\forall x\left(C(x) \vee \exists y\left(C(y) \wedge F(x, y)\right)\right)$ into English, Where $C(x)$ is "*x* has a computer," $F(x,\ y)$ is "*x* and *y* are friends," and the domain for both *x* and *y* consists of all students in your school.

Solution

The statement says

For every student *x* in your school, *x* has a computer or there is a student *y* such that *y* has a computer and *x* and *y* are friends.

In other words

Every student in your school has a computer or has a friend who has a computer.

Example

Translate the statement $\exists x \forall y \forall z\left(F(x,y) \wedge F(x,z) \wedge (y \neq z)\right) \rightarrow \neg F(y,z)$ into English, where $F(a,b)$ means *a* and *b* are friends and the domain for both *x*, *y* and *z* consists of all students in your school.

Solution

The original statement says:

There is a student *x* such that for all students *y* and all students *z* other than *y*, if *x* and *y* are friends and *x* and *z* are friends, then *y* and *z* are not friends.

In other words

There is a student none of whose friends are also friends with each other.

Example

Express the statement "Everyone has exactly one best friend" as a logical expression involving predicates, quantifiers with domain consisting of all people, and logical connectives.

Solution

For every person *x*, *x* has exactly one best friend *y*. "$\forall x$ (person *x* has exactly one best friend)" with domain consisting of all people.

For every person *z*, if person *z* is not person *y*, then *z* is not the best friend of *x*.

Let $B(x,y)$ be the statement "*y* is the best friend of *x*".

Therefore; the statement can be expressed as:

$$\exists y\left(B(x,y) \wedge \forall z\left((z \neq y) \rightarrow \neg B(x,z)\right)\right)$$

The original statement can be expressed as:

$$\forall x \exists y\left(B(x,y) \wedge \forall z\left((z \neq y) \rightarrow \neg B(x,z)\right)\right)$$

Negating Multiple Quantifiers

The negation rules for single quantifiers:

- $\neg\forall x P(x) = \exists x \neg P(x)$
- $\neg\exists x P(x) = \forall x \neg P(x)$
- Essentially, you change the quantifier(s), and negate what it's quantifying

Example

Express the negation of the statement $\forall x \exists y (xy = 1)$ so that no negation precedes a quantifier

Solution

The negation is: $\neg\forall x \exists y (xy = 1)$

Which is equivalent: $\equiv \; \exists x \neg\exists y (xy = 1)$

$$\equiv \; \exists x \forall y \neg (xy = 1)$$

$$\equiv \; \exists x \forall y (xy \neq 1)$$

$$\neg\left(\forall x \exists y \forall z P(x, y, z)\right) = \exists x \neg\left(\exists y \forall z P(x, y, z)\right)$$

$$= \exists x \forall y \neg\left(\forall z P(x, y, z)\right)$$

$$= \exists x \forall y \exists z \neg P(x, y, z)$$

Consider $\neg(\forall x \exists y\ P(x,y)) = \exists x \forall y\ \neg P(x,y)$

- The left side is saying "for all *x*, there exists a *y* such that P is true"
- To disprove it (negate it), you need to show that "there exists an *x* such that for all *y*, P is false"

Consider $\neg(\exists x \forall y\ P(x,y)) = \forall x \exists y\ \neg P(x,y)$

- The left side is saying "there exists an *x* such that for all *y*, P is true"
- To disprove it (negate it), you need to show that "for all *x*, there exists a *y* such that P is false"

Exercises ***Section* 1.4 – Nested Quantifiers**

1. Translate these statements into English, where the domain for each variable consists of all real numbers

a) $\forall x \exists y (x < y)$

b) $\exists x \forall y (xy = y)$

c) $\forall x \forall y (((x \ge 0) \wedge (y < 0)) \rightarrow (x - y > 0))$

d) $\forall x \forall y (((x \ge 0) \wedge (y \ge 0)) \rightarrow (xy \ge 0))$

e) $\forall x \forall y \exists z (xy = z)$

f) $\forall x \forall y \exists z (x = y + z)$

2. Let $Q(x,\ y)$ be the statement "*x* has sent an e-mail message to *y*," where the domain for both *x* and *y* consists of all students in your class. Express each of these quantifications in English

a) $\exists x \exists y Q(x,\ y)$

b) $\exists x \forall y Q(x,\ y)$

c) $\forall x \exists y Q(x,\ y)$

d) $\exists y \forall x Q(x,\ y)$

e) $\forall y \exists x Q(x,\ y)$

f) $\forall x \forall y Q(x,\ y)$

3. Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators where the domain consists of all integers.

a) The product of two negative integers is positive.

b) The average of two positive integers is positive.

c) The difference of two negative integers is not necessarily negative.

d) The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers.

4. Rewrite these statements so that the negations only appear within the predicates

a) $\neg \exists y \forall x P(x, y)$

b) $\neg \forall x \exists y P(x, y)$

c) $\neg \exists y (Q(y) \wedge \forall x \neg R(x, y))$

5. Express the negations of each of these statements so that all negation symbols immediately precede predicates.

a) $\forall x \exists y \forall z T(x, y, z)$

b) $\forall x \exists y P(x, y) \vee \forall x \exists y Q(x, y)$

6. Let $T(x, y)$ mean that student *x* likes cuisine *y*, where the domain for *x* consists of all students at your school and the domain for *y* consists of all cuisines. Express each of these statements by a simple English sentence.

a) $\neg T(A, J)$

b) $\exists x T(x, Korean) \wedge \forall x T x(x, Mexican)$

c) $\exists y(T(Monique, y) \vee T(Jay, y))$

d) $\forall x \forall z \exists y((x \neq z) \rightarrow \neg(T(x, y) \wedge T(z, y)))$

e) $\exists x \exists z \forall y(T(x, y) \leftrightarrow T(z, y))$

f) $\forall x \forall z \exists y(T(x, y) \leftrightarrow T(z, y))$

7. Let $L(x, y)$ be the statement "*x* loves *y*", where the domain for both *x* and *y* consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody loves Jerry.
b) Everybody loves somebody.
c) There is somebody whom everybody loves.
d) Nobody loves everybody.
e) There is somebody whom Lois does not love.
f) There is somebody whom no one loves.
g) There is exactly one person whom everybody loves.
h) There are exactly two people whom L loves.
i) Everyone loves himself or herself.
j) There is someone who loves no one besides himself or herself.

8. Let $S(x)$ be the predicate "*x* is a student," $F(x)$ the predicate "*x* is a faculty member," $A(x, y)$ the predicate "*x* has asked *y* a question," where the domain consists of all people associated with your school. Use quantifiers to express each of these statements.

a) Lois asked Professor Fred a question.
b) Every student has asked Professor Fred a question.
c) Every faculty member has either asked Professor Fred a question or been asked a question by Professor Miller.
d) Some student has not asked any faculty member a question.
e) There is a faculty member who has never been asked a question by a student.
f) Some student has asked every faculty member a question.
g) There is a faculty member who has asked every other faculty member a question.
h) Some student has never been asked a question by a faculty member.

9. Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.

a) Every user has access to exactly one mailbox.
b) There is a process that continues to run during all error conditions only if the kernel is working correctly.
c) All users on the campus network can access all websites whose url has a .edu extension.

10. Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers

a) $\exists x \forall y (x + y = y)$

b) $\forall x \forall y \left(\left((x \geq 0) \wedge (y < 0) \right) \rightarrow (x - y > 0) \right)$

c) $\exists x \exists y \left(\left((x \leq 0) \wedge (y \leq 0) \right) \wedge (x - y > 0) \right)$

d) $\forall x \forall y \left(\left((x \neq 0) \wedge (y \neq 0) \right) \leftrightarrow (xy \neq 0) \right)$

11. Determine the truth value of each of these statements if the domain for all variables consists of all integers

a) $\forall n \exists m \left(n^2 < m \right)$

b) $\exists n \forall m \left(n < m^2 \right)$

c) $\forall n \exists m \ (n + m = 0)$

d) $\exists n \forall m \ (nm = m)$

e) $\exists n \exists m \left(n^2 + m^2 = 5 \right)$

f) $\exists n \exists m \left(n^2 + m^2 = 6 \right)$

g) $\exists n \exists m \ (n + m = 4 \ \wedge \ n - m = 1)$

h) $\exists n \exists m \ (n + m = 4 \ \wedge \ n - m = 2)$

i) $\forall n \forall m \exists p \left(p = \frac{m+n}{2} \right)$