Solution *Section* 2.8 – Derivatives of Logarithmic & Exponential Functions

Exercise

Find the derivative of $y = \ln\sqrt{x+5}$

Solution

$$y = \ln(x+5)^{1/2}$$

$$= \frac{1}{2}\ln(x+5)$$

$$y' = \frac{1}{2(x+5)}$$

Exercise

Find the Derivatives of $y = (3x+7)\ln(2x-1)$

Solution

$$f = 3x+7 \quad f' = 3$$

$$g = \ln(2x-1) \quad g' = \frac{2}{2x-1}$$

$$y' = 3\ln(2x-1) + \frac{2(3x+7)}{2x-1}$$

Exercise

Find the Derivatives of $f(x) = \ln\sqrt[3]{x+1}$

Solution

$$f(x) = \ln(x+1)^{1/3}$$

$$= \frac{1}{3}\ln(x+1)$$

$$u = x+1 \Rightarrow \frac{du}{dx} = 1$$

$$f'(x) = \frac{1}{3}\frac{1}{x+1}$$

$$= \frac{1}{3(x+1)}$$

Exercise

Find the Derivatives of $f(x) = \ln\left(x^2 \sqrt{x^2+1}\right)$

Solution

$$f(x) = \ln\left(x^2\right) + \ln\sqrt{x^2+1} \qquad\qquad \textit{Product Property}$$

$$f(x) = \ln\left(x^2\right) + \ln\left(x^2+1\right)^{1/2}$$

$$f(x) = 2\ln x + \frac{1}{2}\ln\left(x^2+1\right) \qquad\qquad \textit{Power Property}$$

$$f'(x) = 2\frac{1}{x} + \frac{1}{2}\frac{2x}{x^2+1} \qquad\qquad \textit{Differentiate}$$

$$= \frac{2}{x} + \frac{x}{x^2+1}$$

Exercise

Find the Derivatives of $y = \ln\dfrac{x^2}{x^2+1}$

Solution

$$y = \ln x^2 - \ln x^2 + 1$$

$$y' = \frac{2x}{x^2} - \frac{2x}{x^2+1}$$

$$= \frac{2}{x} - \frac{2x}{x^2+1}$$

Exercise

Find the Derivatives of $y = \ln\left[\dfrac{x^2(x+1)^3}{(x+3)^{1/2}}\right]$

Solution

$$y = \ln\left[x^2(x+1)^3\right] - \ln(x+3)^{1/2} \qquad\qquad \textit{Quotient Rule}$$

$$= \ln x^2 + \ln(x+1)^3 - \ln(x+3)^{1/2} \qquad\qquad \textit{Product Rule}$$

$$= 2\ln x + 3\ln(x+1) - \frac{1}{2}\ln(x+3) \qquad\qquad \textit{Power Rule}$$

261

$$y' = \frac{2}{x} + \frac{3}{x+1} - \frac{1}{2(x+3)}$$

Exercise

Find the Derivatives of $y = \ln\left(x^2 + 1\right)$

Solution

$$y' = \frac{2x}{x^2 + 1} \qquad\qquad (\ln U)' = \frac{U'}{U}$$

Exercise

Find the Derivatives of $f(x) = \ln(x^2 - 4)$

Solution

Let $u = x^2 - 4 \implies \frac{du}{dx} = 2x$

$$f'(x) = \frac{1}{x^2 - 4}(2x)$$

$$= \frac{2x}{x^2 - 4}$$

Exercise

Find the derivative $\quad f(x) = 2\ln\left(x^2 - 3x + 4\right)$

Solution

$$f'(x) = 2\frac{2x - 3}{x^2 - 3x + 4}$$

$$= \frac{4x - 6}{x^2 - 3x + 4}$$

Exercise

Find the derivative $\quad f(x) = 3\ln\left(1 + x^2\right)$

Solution

$$f'(x) = 3\frac{2x}{1+x^2}$$

$$= \frac{6x}{1+x^2}$$

Exercise

Find the derivative $f(x) = (1 + \ln x)^3$

Solution

$$f'(x) = 3(1 + \ln x)^2 (1 + \ln x)'$$

$$= 3(1 + \ln x)^2 \left(\frac{1}{x}\right)$$

$$= \frac{3}{x}(1 + \ln x)^2$$

Exercise

Find the derivative $f(x) = (x - 2\ln x)^4$

Solution

$$f'(x) = 4(x - 2\ln x)^3 (x - 2\ln x)'$$

$$= 4(x - 2\ln x)^3 \left(1 - \frac{2}{x}\right)$$

$$= 4(x - 2\ln x)^3 \left(\frac{x-2}{x}\right)$$

$$= \frac{4x-8}{x}(x - 2\ln x)^3$$

Exercise

Find the Derivatives of $f(x) = x^2 \ln x$

Solution

$$f' = x^2 \left(\frac{1}{x}\right) + 2x \ln x \qquad\qquad (fg)' = f'g + fg'$$

$$= x + 2x \ln x$$

$$= x(1 + 2\ln x)$$

Exercise

Find the Derivatives of $f(x) = -\dfrac{\ln x}{x^2}$

Solution

$$f' = -\frac{x^2 \frac{d}{dx}(\ln x) - \ln x \frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= -\frac{x^2 \frac{1}{x} - 2x \ln x}{x^4}$$

$$= -\frac{x - 2x \ln x}{x^4}$$

$$= -\frac{x(1 - 2\ln x)}{x^4}$$

$$= -\frac{1 - 2\ln x}{x^3}$$

Exercise

Find the derivative of $y = \ln\left(t^2\right)$

Solution

$$y' = \frac{\left(t^2\right)'}{t^2}$$

$$= \frac{2t}{t^2}$$

$$= \frac{2}{t}$$

Exercise

Find the derivative of $y = \ln(2\theta + 2)$

Solution

$$y' = \frac{2}{2\theta + 2}$$

$$= \frac{1}{\theta + 1}$$

264

Exercise

Find the derivative of $y = (\ln x)^3$

$$y' = 3(\ln x)^2 \frac{1}{x}$$

$$= \frac{3(\ln x)^2}{x}$$

Exercise

Find the derivative of $y = x(\ln x)^2$

$$y' = (\ln x)^2 + x\left(2(\ln x)\frac{1}{x}\right)$$

$$= (\ln x)^2 + 2\ln x$$

Exercise

Find the derivative of $y = \frac{x^4}{4}\ln x - \frac{x^4}{16}$

$$y' = \frac{4x^3}{4}\ln x + \frac{x^4}{4}\frac{1}{x} - \frac{4x^3}{16}$$

$$= x^3 \ln x + \frac{1}{4}x^3 - \frac{1}{4}x^3$$

$$= x^3 \ln x$$

Exercise

Find the derivative of $y = \frac{1 + \ln t}{t}$

$$y' = \frac{\frac{1}{t}t - (1 + \ln t)}{t^2}$$

$$= \frac{1 - 1 - \ln t}{t^2}$$

$$= -\frac{\ln t}{t^2}$$

265

Exercise

Find the derivative $\quad f(x) = \dfrac{\ln x}{1 + x}$

Solution

$$f'(x) = \frac{\left(\frac{1}{x}\right)(1+x) - \ln x}{(1+x)^2}$$

$$= \frac{1}{x} \frac{1 + x - x\ln x}{(1+x)^2}$$

$$= \frac{1 + x - x\ln x}{x(1+x)^2}$$

$$u = \ln x \quad v = 1 + x$$

$$u' = \frac{1}{x} \qquad v' = 1$$

Exercise

Find the derivative $\quad f(x) = \dfrac{2x}{1 + \ln x}$

Solution

$$f'(x) = \frac{2(1 + \ln x) - (2x)\frac{1}{x}}{(1 + \ln x)^2}$$

$$= \frac{2 + 2\ln x - 2}{(1 + \ln x)^2}$$

$$= \frac{2\ln x}{(1 + \ln x)^2}$$

$$u = 2x \quad v = 1 + \ln x$$

$$u' = 2 \qquad v' = \frac{1}{x}$$

Exercise

Find the derivative $\quad f(x) = x^3 \ln x$

Solution

$$u = x^3 \qquad v = \ln x$$

$$u' = 3x^2 \qquad v' = \frac{1}{x}$$

$$f'(x) = 3x^2 \ln x + x^3 \frac{1}{x}$$

$$= 3x^2 \ln x + x^2$$

$$= (3\ln x + 1)x^2$$

266

Exercise

Find the derivative $\quad f(x) = 6x^4 \ln x$

Solution

$$u = 6x^4 \qquad v = \ln x$$

$$u' = 24x^3 \qquad v' = \frac{1}{x}$$

$$f'(x) = 24x^3 \ln x + 6x^4 \frac{1}{x}$$

$$= 24x^3 \ln x + 6x^3$$

$$= 6x^3 (4\ln x + 1)$$

Exercise

Find the derivative $\quad f(x) = \ln x^8$

Solution

$$f(x) = \ln x^8 = 8\ln x$$

$$f'(x) = \frac{8}{x}$$

Power Rule

$$(\ln x)' = \frac{1}{x}$$

Exercise

Find the derivative $\quad f(x) = 5x - \ln x^5$

Solution

$$f(x) = 5x - \ln x^5$$

$$= 5x - 5\ln x$$

Power Rule

$$f'(x) = 5 - \frac{5}{x}$$

$$(\ln x)' = \frac{1}{x}$$

Exercise

Find the derivative $\quad f(x) = \ln x^{10} + 2\ln x$

Solution

$$f(x) = 10\ln x + 2\ln x$$

Power Rule

$$= 12\ln x$$

$$f'(x) = \frac{12}{x}$$

$$(\ln x)' = \frac{1}{x}$$

Exercise

Find the derivative $\quad f(x) = \dfrac{\ln x}{2x+5}$

Solution

$$u = \ln x \quad v = 2x+5$$

$$u' = \frac{1}{x} \qquad v' = 2$$

$$f'(x) = \frac{\frac{1}{x}(2x+5) - (2)\ln x}{(2x+5)^2} \cdot \frac{x}{x}$$

$$= \frac{2x+5 - 2x\ln x}{x(2x+5)^2}$$

Exercise

Find the derivative $\quad f(x) = -2\ln x + x^2 - 4$

Solution

$$f'(x) = -\frac{2}{x} + 2x$$

Exercise

Find the derivative of $\quad y = \ln\left(\dfrac{1}{x\sqrt{x+1}}\right)$

Solution

$$y = \ln(1) - \ln\left(x\sqrt{x+1}\right)$$

$$= -\ln x - \ln(x+1)^{1/2}$$

$$= -\ln x - \frac{1}{2}\ln(x+1)$$

$$y' = -\frac{1}{x} - \frac{1}{2}\frac{1}{x+1}$$

$$= -\frac{3x+2}{2x(x+1)}$$

268

Exercise

Find the derivative of $y = \ln\big(\ln(\ln x)\big)$

Solution

$$y' = \frac{1}{\ln(\ln x)} \cdot \big(\ln(\ln x)\big)'$$

$$= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot (\ln x)'$$

$$= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$$

$$= \frac{1}{x(\ln x)\big(\ln(\ln x)\big)}$$

Exercise

Find the derivative of $y = \ln\big(\sec(\ln x)\big)$

Solution

$$y' = \frac{1}{\sec(\ln x)} \cdot \big(\sec(\ln x)\big)'$$

$$= \frac{1}{\sec(\ln x)} \cdot \big(\sec(\ln x)\tan(\ln x)\big) \cdot (\ln x)'$$

$$= \frac{\sec(\ln x)}{\sec(\ln x)}\tan(\ln x) \cdot \frac{1}{x}$$

$$= \frac{\tan(\ln x)}{x}$$

Exercise

Find the derivative of $\qquad y = \ln\left(\dfrac{\left(x^2 + 1\right)^5}{\sqrt{1-x}}\right)$

Solution

$$y = \ln\left(x^2 + 1\right)^5 - \ln(1-x)^{1/2}$$

$$= 5\ln\left(x^2 + 1\right) - \frac{1}{2}\ln(1-x)$$

$$y' = 5\frac{2x}{x^2 + 1} - \frac{1}{2}\frac{-1}{1-x}$$

269

$$= \frac{10x}{x^2+1} + \frac{1}{2(1-x)}$$

Exercise

Find the derivative of $y = \ln \sqrt{\dfrac{(x+1)^5}{(x+2)^{20}}}$

Solution

$$y = \frac{1}{2}\ln\left(\frac{(x+1)^5}{(x+2)^{20}}\right)$$

$$= \frac{1}{2}\left[\ln(x+1)^5 - \ln(x+2)^{20}\right]$$

$$= \frac{1}{2}\left(5\ln(x+1) - 20\ln(x+2)\right)$$

$$y' = \frac{1}{2}\left(5\frac{1}{x+1} - 20\frac{1}{x+2}\right)$$

$$= \frac{5}{2}\left(\frac{1}{x+1} - \frac{4}{x+2}\right)$$

$$= \frac{5}{2}\left(\frac{x+2-4x-4}{(x+1)(x+2)}\right)$$

$$= \frac{5}{2}\frac{-3x-2}{(x+1)(x+2)}$$

$$= -\frac{5}{2}\frac{3x+2}{(x+1)(x+2)}$$

Exercise

Find the derivative of $f(x) = e^{3x}$

Solution

$$f'(x) = 3e^{3x}$$

Exercise

Find the derivative of $f(x) = e^{-2x^3}$

Solution

$$f'(x) = e^{-2x^3}\left(-6x^2\right)$$

$$= -\frac{6x^2}{e^{2x^3}}$$

Exercise

Find the derivative of $f(x) = 4e^{x^2}$

Solution

$$f'(x) = 4e^{x^2}(2x)$$

$$= 8xe^{x^2}$$

Exercise

Find the derivative of $f(x) = x^2 e^x$

Solution

$$f'(x) = e^x \frac{d}{dx}[x^2] + x^2 \frac{d}{dx}[e^x]$$

$$= e^x(2x) + x^2 e^x$$

$$= xe^x(2 + x)$$

Exercise

Find the derivative $\quad f(x) = 2x^3 e^x$

Solution

$$f'(x) = 6x^2 e^x + 2x^3 e^x$$

$$= 2x^2 e^x(3 + x)$$

$$u = 2x^3 \qquad v = e^x$$

$$u' = 6x^2 \qquad v' = e^x$$

Exercise

Find the derivative $\quad f(x) = \dfrac{3e^x}{1 + e^x}$

Solution

$$f'(x) = \frac{3e^x(1 + e^x) - 3e^x e^x}{(1 + e^x)^2}$$

$$u = 3e^x \qquad v = 1 + e^x$$

$$u' = 3e^x \qquad v' = e^x$$

271

$$= \frac{3e^x + 3e^{2x} - 3e^{2x}}{\left(1 + e^x\right)^2}$$

$$= \frac{3e^x}{\left(1 + e^x\right)^2} \Bigg|$$

Exercise

Find the derivative $\quad f(x) = 5e^x + 3x + 1$

Solution

$$f'(x) = 5e^x + 3 \Bigg|$$

Exercise

Find the derivative of $f(x) = \frac{e^x + e^{-x}}{2}$

Solution

$$f(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f'(x) = \frac{1}{2}\left(\frac{d}{dx}\left[e^x\right] + \frac{d}{dx}\left[e^{-x}\right]\right)$$

$$= \frac{1}{2}\left(e^x - e^{-x}\right) \Bigg|$$

Exercise

Find the derivative of $f(x) = \frac{e^x}{x^2}$

Solution

$$f'(x) = \frac{x^2 e^x - e^x (2x)}{x^4}$$

$$= \frac{x^2 e^x - 2xe^x}{x^4}$$

$$= \frac{xe^x(x-2)}{x^4}$$

$$= \frac{e^x(x-2)}{x^3} \Bigg|$$

Exercise

Find the derivative of $f(x) = x^2 e^x - e^x$

Solution

$$f'(x) = e^x \frac{d}{dx}\left(x^2\right) + x^2 \frac{d}{dx}\left(e^x\right) - \frac{d}{dx}\left(e^x\right)$$

$$= e^x(2x) + x^2 e^x - e^x$$

$$\underline{= e^x\left(x^2 + 2x - 1\right)}$$

Exercise

Find the derivative of $f(x) = (1 + 2x)e^{4x}$

Solution

$$f'(x) = (2)e^{4x} + (1 + 2x)\left(4e^{4x}\right)$$

$$= 2e^{4x} + (1 + 2x)\left(4e^{4x}\right)$$

$$= 2e^{4x}\left(1 + 2(1 + 2x)\right)$$

$$= 2e^{4x}\left(1 + 2 + 4x\right)$$

$$\underline{= 2e^{4x}\left(3 + 4x\right)}$$

Exercise

Find the derivative of $y = x^2 e^{5x}$

Solution

$$y' = x^2\left(5e^{5x}\right) + 2x\left(e^{5x}\right)$$

$$\underline{= xe^{5x}(5x + 2)}$$

Exercise

Find the derivative of $y = x^2 e^{-2x}$

Solution

$$y' = 2xe^{-2x} - 2x^2 e^{-2x}$$

$$\underline{= 2xe^{-2x}(1 - x)}$$

Exercise

Find the derivative $f(x) = \dfrac{e^x}{x^2+1}$

Solution

$$f'(x) = \frac{e^x\left(x^2+1\right) - 2xe^x}{\left(x^2+1\right)^2}$$

$u = e^x \qquad v = x^2+1$

$u' = e^x \qquad v' = 2x$

$$= \frac{\left(x^2+1-2x\right)e^x}{\left(x^2+1\right)^2}$$

Exercise

Find the derivative $f(x) = \dfrac{1-e^x}{1+e^x}$

Solution

$$f'(x) = \frac{-e^x\left(1+e^x\right) - e^x\left(1-e^x\right)}{\left(1+e^x\right)^2}$$

$u = 1-e^x \qquad v = 1+e^x$

$u' = -e^x \qquad v' = e^x$

$$= \frac{-e^x - e^{2x} - e^x + e^{2x}}{\left(1+e^x\right)^2}$$

$$= -\frac{2e^x}{\left(1+e^x\right)^2}$$

Exercise

Find the derivative of $y = \dfrac{e^x + e^{-x}}{x}$

Solution

$$y = \frac{\left(e^x - e^{-x}\right)x - \left(e^x + e^{-x}\right)}{x^2}$$

$f = e^x + e^{-x} \qquad g = x$

$f' = e^x - e^{-x} \qquad g' = 1$

$$= \frac{xe^x - xe^{-x} - e^x - e^{-x}}{x^2}$$

$$= \frac{(x-1)e^x - (x+1)e^{-x}}{x^2}$$

Exercise

Find the derivative of $y = \sqrt{e^{2x^2} + e^{-2x^2}}$

Solution

$$y = \sqrt{e^{2x^2} + e^{-2x^2}} = \left(e^{2x^2} + e^{-2x^2}\right)^{1/2} = U^{1/2}$$

$$U = e^{2x^2} + e^{-2x^2} \qquad \left(e^{2x^2}\right)' = \left(2x^2\right)' e^{2x^2} = 4xe^{2x^2}$$

$$U' = 4xe^{2x^2} - 4xe^{-2x^2}$$

$$y' = \frac{1}{2}\left(4xe^{2x^2} - 4xe^{-2x^2}\right)\left(e^{2x^2} + e^{-2x^2}\right)^{-1/2}$$

$$= \frac{1}{2}\frac{4x\left(e^{2x^2} - e^{-2x^2}\right)}{\left(e^{2x^2} + e^{-2x^2}\right)^{1/2}}$$

$$= \frac{2x\left(e^{2x^2} - e^{-2x^2}\right)}{\sqrt{e^{2x^2} + e^{-2x^2}}}$$

Exercise

Find the derivative of $y = \dfrac{x}{e^{2x}}$

Solution

$$y' = \frac{1\left(e^{2x}\right) - x\left(2e^{2x}\right)}{\left(e^{2x}\right)^2}$$

$$= \frac{e^{2x}(1-2x)}{\left(e^{2x}\right)^2}$$

$$= \frac{1-2x}{e^{2x}}$$

$f = x \qquad g = e^{2x}$

$f' = 1 \quad g' = 2e^{2x}$

Exercise

Find the derivative of $y = 3\,e^{5x^3+1}$

$$y' = 3(15x^2)e^{5x^3+1}$$

$$y' = 45x^2 e^{5x^3+1}$$

$$f = x^2 \qquad g = e^{5x^3+1}$$

$$f' = 2x \quad g' = 15x^2 e^{5x^3+1}$$

$$y'' = 45\left(2xe^{5x^3+1} + \left(x^2\right)15x^2 e^{5x^3+1}\right)$$

$$= 45e^{5x^3+1}\left(2x + 15x^4\right)$$

$$= 45xe^{5x^3+1}\left(2 + 15x^3\right)$$

Exercise

Find the derivative of $\left(x^2 - 2x + 2\right)e^x$

Solution

$$y = \left(x^2 - 2x + 2\right)e^x$$

$$y' = (2x - 2)e^x + \left(x^2 - 2x + 2\right)e^x$$

$$= \left(2x - 2 + x^2 - 2x + 2\right)e^x$$

$$= x^2 e^x$$

Exercise

Find the derivative of $e^\theta\left(\sin\theta + \cos\theta\right)$

Solution

$$\frac{d}{d\theta}e^\theta\left(\sin\theta + \cos\theta\right) = e^\theta\left(\sin\theta + \cos\theta\right) + e^\theta\left(\cos\theta - \sin\theta\right)$$

$$= e^\theta\left(\sin\theta + \cos\theta + \cos\theta - \sin\theta\right)$$

$$= 2e^\theta\cos\theta$$

Exercise

Find the derivative of $\ln\left(3\theta e^{-\theta}\right)$

Solution

$$\frac{d}{d\theta}\ln\left(3\theta e^{-\theta}\right) = \frac{\left(3\theta e^{-\theta}\right)'}{3\theta e^{-\theta}}$$

$$= 3\frac{e^{-\theta} - \theta e^{-\theta}}{3\theta e^{-\theta}}$$

$$= \frac{e^{-\theta}(1-\theta)}{\theta e^{-\theta}}$$

$$= \underline{\frac{1-\theta}{\theta}}$$

$$\ln\left(3\theta e^{-\theta}\right) = \ln(3) + \ln(\theta) + \ln\left(e^{-\theta}\right)$$

$$= \ln 3 + \ln\theta - \theta$$

$$\frac{d}{d\theta}\ln\left(3\theta e^{-\theta}\right) = \underline{\frac{1}{\theta} - 1}$$

Exercise

Find the derivative of $\theta^3 e^{-2\theta}\cos 5\theta$

Solution

$$\frac{dy}{d\theta} = \left(\theta^3\right)' e^{-2\theta}\cos 5\theta + \theta^3\left(e^{-2\theta}\right)'\cos 5\theta + \theta^3 e^{-2\theta}(\cos 5\theta)'$$

$$= 3\theta^2 e^{-2\theta}\cos 5\theta - 2\theta^3 e^{-2\theta}\cos 5\theta - 5\theta^3 e^{-2\theta}\sin 5\theta$$

$$= \underline{\theta^3 e^{-2\theta}\left(3\cos 5\theta - 2\theta\cos 5\theta - 5\theta\sin\theta\right)}$$

Exercise

Find the derivative of $\ln\left(\dfrac{\sqrt{\theta}}{1+\sqrt{\theta}}\right)$

Solution

$$\frac{d}{d\theta}\ln\left(\frac{\sqrt{\theta}}{1+\sqrt{\theta}}\right) = \frac{d}{d\theta}\left[\ln\theta^{1/2} - \ln\left(1+\sqrt{\theta}\right)\right]$$

$$= \frac{d}{d\theta}\left[\frac{1}{2}\ln\theta - \ln\left(1+\sqrt{\theta}\right)\right]$$

$$= \frac{1}{2}\frac{1}{\theta} - \frac{\frac{1}{2}\theta^{-1/2}}{1+\sqrt{\theta}}$$

$$= \frac{1}{2\theta} - \frac{1}{2}\frac{1}{\sqrt{\theta}\left(1+\sqrt{\theta}\right)}$$

$$= \frac{1}{2}\left(\frac{1}{\theta} - \frac{1}{\sqrt{\theta}\left(1+\sqrt{\theta}\right)} \right)$$

$$= \frac{1}{2}\frac{\sqrt{\theta}\left(1+\sqrt{\theta}\right)-\theta}{\theta\sqrt{\theta}\left(1+\sqrt{\theta}\right)}$$

$$= \frac{1}{2}\frac{\sqrt{\theta}+\theta-\theta}{\theta\sqrt{\theta}\left(1+\sqrt{\theta}\right)}$$

$$= \frac{1}{2}\frac{\sqrt{\theta}}{\theta\sqrt{\theta}\left(1+\sqrt{\theta}\right)}$$

$$= \underline{\frac{1}{2\theta\left(1+\sqrt{\theta}\right)}}$$

Exercise

Find the derivative of $e^{(\cos t + \ln t)}$

Solution

$$e^{(\cos t + \ln t)} = e^{\cos t}e^{\ln t}$$

$$= t\ e^{\cos t}$$

$$\frac{d}{dt}e^{(\cos t + \ln t)} = \frac{d}{dt}\left(t\ e^{\cos t}\right)$$

$$= e^{\cos t} + te^{\cos t}\left(-\sin t\right)$$

$$= \underline{\left(1 - t\sin t\right)e^{\cos t}}$$

Exercise

Find the derivative of $e^{\sin t}\left(\ln t^2 + 1\right)$

Solution

$$\frac{d}{dt}\overset{u}{e^{\sin t}}\overset{v}{\left(\ln t^2 + 1\right)} = \overset{u'}{e^{\sin t}}\cos t\overset{v}{\left(\ln t^2 + 1\right)} + \overset{v'}{\frac{2}{t}}\overset{u}{e^{\sin t}}$$

$$= \underline{e^{\sin t}\left[\left(\ln t^2 + 1\right)\cos t + \frac{2}{t}\right]}$$

Exercise

Find the Derivatives of $y = e^{x^2} \ln x$

Solution

$$y' = 2xe^{x^2} \ln x + \frac{e^{x^2}}{x}$$

$$f = e^{x^2} \qquad g = \ln x$$

$$f' = 2xe^{x^2} \qquad g' = \frac{1}{x}$$

Exercise

Find the derivative $\quad f(x) = e^x + x - \ln x$

Solution

$$f'(x) = e^x + 1 - \frac{1}{x}$$

Exercise

Find the derivative $\quad f(x) = \ln x + 2e^x - 3x^2$

Solution

$$f'(x) = \frac{1}{x} + 2e^x - 6x$$

Exercise

Find the derivative $\quad f(x) = \ln x^2 + 4e^x$

Solution

$$f(x) = 2\ln x + 4e^x \qquad\qquad \textit{Power Rule}$$

$$f'(x) = \frac{2}{x} + 4e^x \qquad\qquad (\ln x)' = \frac{1}{x}$$

Exercise

Find the Derivatives of $y = \ln \dfrac{1 + e^x}{1 - e^x}$

Solution

$$y = \ln\left(1 + e^x\right) - \ln\left(1 - e^x\right)$$

$$y' = \frac{e^x}{1 + e^x} - \frac{-e^x}{1 - e^x}$$

279

$$= \frac{e^x}{1+e^x} + \frac{e^x}{1-e^x}$$

$$= \frac{e^x - e^{2x} + e^x + e^{2x}}{\left(1+e^x\right)\left(1-e^x\right)}$$

$$= \frac{2e^x}{\left(1+e^x\right)\left(1-e^x\right)}$$

Exercise

Find the Derivatives of $y = \dfrac{\ln x}{e^{2x}}$

Solution

$$y' = \frac{e^{2x}(1/x) - \ln x(2e^{2x})}{e^{4x}}$$

$$= \frac{e^{2x} - 2x\ln x(e^{2x})}{e^{4x}}$$

$$= \frac{e^{2x}\left(1 - 2x\ln x\right)}{e^{4x}}$$

Exercise

Find the Derivatives of $f(x) = e^{2x}\ln\left(xe^x + 1\right)$

Solution

$$f = e^{2x} \qquad\qquad U = 2x \rightarrow U' = 2 \qquad\qquad f' = 2e^{2x}$$

$$g = \ln\left(xe^x + 1\right) \quad U = xe^x + 1 \rightarrow U' = e^x + xe^x \quad g' = \frac{e^x + xe^x}{xe^x + 1}$$

$$f'(x) = 2e^{2x}\ln\left(xe^x + 1\right) + e^{2x}\frac{e^x + xe^x}{xe^x + 1}$$

$$= e^{2x}\left[2\ln\left(xe^x + 1\right) + \frac{e^x(1+x)}{xe^x + 1}\right]$$

Exercise

Find the Derivatives of $f(x) = \dfrac{xe^x}{\ln\left(x^2+1\right)}$

Solution

$$u = xe^x \qquad\qquad u' = e^x + xe^x$$

$$v = \ln\left(x^2+1\right) \qquad v' = \dfrac{2x}{x^2+1}$$

$$f'(x) = \dfrac{e^x(1+x)\ln\left(x^2+1\right) - \dfrac{2x}{x^2+1}xe^x}{\left[\ln\left(x^2+1\right)\right]^2}$$

$$= \dfrac{e^x\left[(1+x)\ln\left(x^2+1\right) - \dfrac{2x^2}{x^2+1}\right]}{\left[\ln\left(x^2+1\right)\right]^2}$$

$$= \dfrac{e^x\left[\dfrac{\left(x^2+1\right)(1+x)\ln\left(x^2+1\right) - 2x^2}{x^2+1}\right]}{\left[\ln\left(x^2+1\right)\right]^2}$$

$$= \dfrac{e^x\left[\left(x^2+1\right)(1+x)\ln\left(x^2+1\right) - 2x^2\right]}{\left(x^2+1\right)\left[\ln\left(x^2+1\right)\right]^2}$$

Exercise

Find the Derivatives of $\quad f(x) = xe^{-10x}$

Solution

$$f'(x) = e^{-10x} - 10xe^{-10x}$$

Exercise

Find the Derivatives of $\quad f(x) = x\ln^2 x$

Solution

$$f'(x) = \ln^2 x + x\left(2\frac{1}{x}\ln x\right)$$

$$= \ln^2 x + 2\ln x$$

Exercise

Find the Derivatives of $f(x) = e^{-x} \ln x$

Solution

$$f'(x) = -e^{-x} \ln x + \frac{e^{-x}}{x}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt{x(x+1)}$

Solution

$$\ln y = \ln\left(x(x+1)\right)^{1/2} = -\ln x - \frac{1}{2}\ln(x+1)$$

$$\frac{y'}{y} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x+1}\right)$$

$$\frac{y'}{y} = \frac{1}{2}\left(\frac{2x+1}{x(x+1)}\right)$$

$$y' = \frac{1}{2}\left(\frac{2x+1}{x(x+1)}\right) \cdot y$$

$$= \frac{1}{2}\left(\frac{2x+1}{x(x+1)}\right)\sqrt{x(x+1)}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt{\left(x^2+1\right)(x-1)^2}$

Solution

$$\ln y = \ln\left(\left(x^2+1\right)(x-1)^2\right)^{1/2}$$

$$= \frac{1}{2}\ln\left(\left(x^2+1\right)(x-1)^2\right)$$

$$= \frac{1}{2}\left[\ln\left(x^2+1\right) + \ln(x-1)^2\right]$$

$$= \frac{1}{2}\left[\ln\left(x^2+1\right) + 2\ln(x-1)\right]$$

$$= \frac{1}{2}\ln\left(x^2+1\right) + \ln(x-1)$$

$$\frac{y'}{y} = \frac{1}{2}\frac{2x}{x^2+1} + \frac{1}{x-1}$$

282

$$= \frac{x}{x^2+1} + \frac{1}{x-1}$$

$$= \frac{x(x-1) + \left(x^2+1\right)}{\left(x^2+1\right)(x-1)}$$

$$= \frac{x^2 - x + x^2 + 1}{\left(x^2+1\right)(x-1)}$$

$$= \frac{2x^2 - x + 1}{\left(x^2+1\right)(x-1)}$$

$$y' = \frac{2x^2 - x + 1}{\left(x^2+1\right)(x-1)} \cdot y$$

$$= \frac{2x^2 - x + 1}{\left(x^2+1\right)(x-1)} \sqrt{\left(x^2+1\right)(x-1)^2}$$

$$= \frac{2x^2 - x + 1}{\left(x^2+1\right)(x-1)} |x-1| \sqrt{\left(x^2+1\right)}$$

$$= \frac{\left(2x^2 - x + 1\right)|x-1|}{\left(x^2+1\right)(x-1)} \left(x^2+1\right)^{1/2}$$

$$= \frac{\left(2x^2 - x + 1\right)|x-1|}{\sqrt{x^2+1}(x-1)}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt{\dfrac{1}{t(t+1)}}$

Solution

$$y = \left(\frac{1}{t(t+1)}\right)^{1/2}$$

$$\ln y = \ln\left(\frac{1}{t(t+1)}\right)^{1/2}$$

$$\ln y = \frac{1}{2}\ln\left(\frac{1}{t(t+1)}\right)$$

$$= -\frac{1}{2}\ln\left(t(t+1)\right)$$

$$= -\frac{1}{2}\left[\ln t + \ln(t+1)\right]$$

$$\frac{y'}{y} = -\frac{1}{2}\left(\frac{1}{t} + \frac{1}{t+1}\right)$$

$$y' = -\frac{1}{2}\left(\frac{1}{t} + \frac{1}{t+1}\right)y$$

$$= -\frac{1}{2}\left(\frac{t+1+t}{t(t+1)}\right)\frac{1}{(t(t+1))^{1/2}}$$

$$= -\frac{1}{2}\frac{2t+1}{(t(t+1))^{3/2}}$$

$$= -\frac{2t+1}{2(t^2+t)^{3/2}}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \dfrac{\theta+5}{\theta\cos\theta}$

Solution

$$\ln y = \ln\left(\frac{\theta+5}{\theta\cos\theta}\right)$$

$$\ln y = \ln(\theta+5) - \ln(\theta\cos\theta)$$

$$\ln y = \ln(\theta+5) - \ln\theta - \ln(\cos\theta)$$

$$\frac{y'}{y} = \frac{1}{\theta+5} - \frac{1}{\theta} + \frac{\sin\theta}{\cos\theta}$$

$$y' = \left(\frac{1}{\theta+5} - \frac{1}{\theta} + \frac{\sin\theta}{\cos\theta}\right)y$$

$$y' = \left(\frac{\theta+5}{\theta\cos\theta}\right)\left(\frac{1}{\theta+5} - \frac{1}{\theta} + \tan\theta\right)$$

Exercise

Use logarithmic differentiation to find the derivative of $y = \sqrt[3]{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

Solution

$$\ln y = \ln\left(\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}\right)^{1/3}$$

$$= \frac{1}{3}\left[\ln x + \ln(x+1) + \ln(x-2) - \ln(x^2+1) - \ln(2x+3)\right]$$

$$\frac{y'}{y} = \frac{1}{3}\left(\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right)$$

$$y' = \frac{1}{3} \cdot \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}} \cdot \left(\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3} \right)$$

Exercise

Find the derivative of $y = t^{1-e}$

Solution

$$y' = (1-e)t^{1-e-1}$$

$$= (1-e)t^{-e}$$

Exercise

Find the derivative of $y = 2^{\sin 3t}$

Solution

$$y = a^u \quad \Rightarrow y' = a^u \ln a \cdot (u')$$

$$y' = \left(2^{\sin 3t} \ln 2\right)(\cos 3t)(3)$$

$$= 3(\ln 2)\cos 3t \left(2^{\sin 3t}\right)$$

Exercise

Find the derivative of $y = \log_3 (1 + \theta \ln 3)$

Solution

$$y = \frac{\ln(1 + \theta \ln 3)}{\ln 3}$$

$$y' = \frac{1}{\ln 3} \cdot \frac{\ln 3}{1 + \theta \ln 3} \qquad\qquad y = \ln u \quad \Rightarrow y' = \frac{u'}{u}$$

$$= \frac{1}{1 + \theta \ln 3}$$

Exercise

Find the derivative of $y = \log_{25} e^x - \log_5 \sqrt{x}$

Solution

$$y = \frac{\ln e^x}{\ln 25} - \frac{\ln x^{1/2}}{\ln 5}$$

$$= \frac{x}{2\ln 5} - \frac{1}{2}\frac{\ln x}{\ln 5}$$

$$= \frac{1}{2\ln 5}(x - \ln x)$$

$$y' = \frac{1}{2\ln 5}\left(1 - \frac{1}{x}\right)$$

$$\underline{= \frac{x-1}{2x\ln 5}}$$

Exercise

Find the derivative of $y = \log_3 r \cdot \log_9 r$

Solution

$$y = \frac{\ln r}{\ln 3} \cdot \frac{\ln r}{\ln 9}$$

$$= \frac{1}{\ln 3 \cdot \ln 9} \cdot \ln^2 r$$

$$y' = \frac{1}{\ln 3 \cdot \ln 9} \cdot (2\ln r)\left(\frac{1}{r}\right)$$

$$\underline{= \frac{2\ln r}{r \cdot \ln 3 \cdot \ln 9}}$$

Exercise

Find the derivative of $y = \log_7\left(\frac{\sin\theta\cos\theta}{e^\theta 2^\theta}\right)$

Solution

$$y = \frac{\ln(\sin\theta) + \ln(\cos\theta) - \ln(e^\theta) - \ln(2^\theta)}{\ln 7}$$

$$= \frac{1}{\ln 7}\left[\ln(\sin\theta) + \ln(\cos\theta) - \theta - \theta\ln(2)\right]$$

$$y' = \frac{1}{\ln 7}\left[\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} - 1 - \ln(2)\right]$$

$$\underline{= \frac{1}{\ln 7}(\cot\theta - \tan\theta - 1 - \ln 2)}$$

Exercise

Find the derivative of $y = 3\log_8\left(\log_2 t\right)$

Solution

$$y = 3\frac{\ln\left(\log_2 t\right)}{\ln 8} = \frac{3}{\ln 8}\ln\left(\frac{\ln t}{\ln 2}\right)$$

$$y' = \frac{3}{\ln 2^3}\left(\frac{1}{\frac{\ln t}{\ln 2}}\right)\left(\frac{1}{\ln 2}\cdot\frac{1}{t}\right)$$

$$= \frac{3}{2\ln 2}\left(\frac{\ln 2}{\ln t}\right)\left(\frac{1}{t\ln 2}\right)$$

$$= \underline{\frac{1}{t(\ln t)(\ln 2)}}$$

Exercise

Find the derivative of $y = t\,\log_3\left(e^{(\sin t)(\ln 3)}\right)$

Solution

$$y = t\frac{\ln e^{(\sin t)(\ln 3)}}{\ln 3}$$

$$= \frac{1}{\ln 3}t(\sin t)(\ln 3)$$

$$= \underline{t\sin t}$$

$$\underline{y' = \sin t + t\cos t}$$

Exercise

Find the derivative of $\quad f(x) = \log_3(x+8)$

Solution

$$\underline{f'(x) = \frac{1}{\ln 3}\left(\frac{1}{x+8}\right)}$$

$$\frac{d}{dx}\left[\log_a u\right] = \left(\frac{1}{\ln a}\right)\left(\frac{1}{u}\right)\frac{du}{dx}$$

Exercise

Find the derivative of $\quad f(x) = 2^{x^2-x}$

Solution

$$\underline{f'(x) = (2x-1)(\ln 2)2^{x^2-x}}$$

$$\frac{d}{dx}[a^u] = a^u \ln(a)\frac{du}{dx}$$

287

Exercise

Use logarithmic differentiation to find the derivative of $y = (x+1)^x$

Solution

$$\ln y = \ln(x+1)^x = x \cdot \ln(x+1)$$

$$\frac{y'}{y} = \ln(x+1) + x \cdot \frac{1}{x+1}$$

$$y' = (x+1)^x \left(\ln(x+1) + \frac{x}{x+1} \right)$$

Exercise

Use logarithmic differentiation to find the derivative of $y = x^2 + x^{2x}$

Solution

$$y - x^2 = x^{2x}$$

$$\ln\left(y - x^2\right) = \ln x^{2x} = 2x \ln x|$$

$$\frac{1}{y-x^2}(y' - 2x) = 2\ln x + 2x\frac{1}{x}$$

$$y' - 2x = \left(y - x^2\right)(2\ln x + 2)$$

$$y' - 2x = \left(x^2 + x^{2x} - x^2\right)(2\ln x + 2)$$

$$y' = 2x^{2x}(\ln x + 1) + 2x$$

$$= 2\left(x^{2x}\ln x + x^{2x} + x\right)$$

Exercise

Use logarithmic differentiation to find the derivative of $y = (\sin x)^x$

Solution

$$\ln y = \ln(\sin x)^x$$

$$\ln y = x\ln(\sin x)$$

$$u = x \quad v = \ln(\sin x)$$

$$u' = 1 \quad v' = \frac{\cos x}{\sin x}$$

$$\frac{y'}{y} = \ln(\sin x) + x\frac{\cos x}{\sin x}$$

$$y' = y\big(\ln(\sin x) + x\cot x\big)$$

$$= (\sin x)^x\big[\ln(\sin x) + x\cot x\big]$$

Exercise

Use logarithmic differentiation to find the derivative of $y = x^{\sin x}$

Solution

$$\ln y = \ln x^{\sin x}$$

$$\ln y = \sin x \ln x$$

$$\frac{y'}{y} = \cos x \ln x + \frac{\sin x}{x}$$

$$\frac{y'}{y} = \frac{x \cos x \ln x + \sin x}{x}$$

$$y' = y \frac{x \cos x \ln x + \sin x}{x}$$

$$= x^{\sin x} \frac{\sin x + x(\ln x)(\cos x)}{x}$$

Exercise

Use logarithmic differentiation to find the derivative of $y = (\ln x)^{\ln x}$

Solution

$$\ln y = \ln(\ln x)^{\ln x}$$

$$\ln y = (\ln x)\ln(\ln x)$$

$$\frac{y'}{y} = \frac{1}{x}\ln(\ln x) + \ln x \frac{\frac{1}{x}}{\ln x}$$

$$y' = \left(\frac{1}{x}\ln(\ln x) + \frac{1}{x}\right)y$$

$$= \left(\frac{\ln(\ln x) + 1}{x}\right)(\ln x)^{\ln x}$$

Exercise

Find the second derivative of $y = 3\,e^{5x^3 + 1}$

Solution

$$y' = 45x^2\, e^{5x^3 + 1}$$

$$y'' = \left(90x + 675x^5\right)e^{5x^3 + 1}$$

289

Exercise

Find the equations of the tangent lines to $f(x) = e^x$ at the points (0, 1)

Solution

$$f'(x) = e^x$$

$$(0, 1) \Rightarrow m = f'(x = 0)$$
$$= e^0$$
$$= 1$$

$$y - 1 = 1(x - 0) + 1 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$\underline{y = x + 1}$$

Exercise

Find the equations of the tangent lines to $f(x) = e^x$ at the points (1, e)

Solution

$$f'(x) = e^x$$

$$(1, e) \Rightarrow m = f'(x = 1) = e^1 = e$$

$$y = e(x - 1) + e \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$\underline{y = ex}$$

Exercise

Find the equations of the tangent lines to $y = 4xe^{-x} + 5$ at $x = 1$

Solution

$$y' = 4e^{-x} - 4xe^{-x} = 4e^{-x}(1 - x)$$

$$= 4e^{-x}(1 - x)$$

$$m = y'(x = 1)$$

$$= 4e^{-1}(1 - 1) = 0$$

$$\Rightarrow x = 1 \rightarrow y = 4e^{-1} + 5 \qquad \left(1,\ 4e^{-1} + 5\right)$$

$$y = 0(x - 1) + 4e^{-1} + 5 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$\underline{y = 4e^{-1} + 5}$$

290

Exercise

Find the equation of the tangent lines to $f(x) = 4e^{-8x}$ at the points (0, 4)

$f'(x) = -32e^{-8x}$

$m = f'(0) = -32e^{-8(0)} = -32$

$y = -32(x - 0) + 4$ $\qquad\qquad\qquad\qquad\qquad\qquad y = m\left(x - x_1\right) + y_1$

$\underline{y = -32x + 4}$

Exercise

The following formula accurately models the relationship between the size of a certain type of tumor and the amount of time that it has been growing:

$$V(t) = 450\left(1 - e - 0.0022t\right)^3$$

where t is in months and $V(t)$ is measured in cubic centimeters. Calculate the rate of change of tumor volume at 80 months.

$U = 1 - e - 0.0022t \qquad V = 450U^3$

$U' = -.0022 \qquad V' = 450(3)U^2U'$

$V'(t) = 450(3)\left(1 - e - 0.0022t\right)^2(-.0022)$

$\underline{= 2.97\left(1 - e - 0.0022t\right)^2}$

$V'(t = 80) = 2.97\left(1 - e - 0.0022(80)\right)^2$

$\underline{\approx 10.66}$

Exercise

A yeast culture at room temperature (68° *F*) is placed in a refrigerator set at a constant temperature of 38° *F*. After t hours, the temperature T of the culture is given approximately by

$$T = 30e^{-0.58t} + 38 \quad t \geq 0$$

What is the rate of change of temperature of the culture at the end of 1 *hour*? At the end of 4 *hours*?

$T' = 30(-0.58)e^{-0.58t} \underline{= -17.4e^{-0.58t}}$

$$T'(1) = -17.4e^{-0.58(1)} \approx -9.74° \ F \ / \ hr$$

$$T'(4) = -17.4e^{-0.58(4)} \approx -1.71° \ F \ / \ hr$$

Exercise

A mathematical model for the average age of a group of people learning to type is given by

$$N(t) = 10 + 6\ln t \quad t \geq 1$$

Where $N(t)$ is the number of words per minute typed after t *hours* of instruction and practice (2 hours per day, 5 days per week). What is the rate of learning after 10 *hours* of instruction and practice? After 100 *hours*?

Solution

$$N'(t) = \frac{6}{t}$$

$$N'(10) = \frac{6}{10} = 0.6$$

After 10 *hours* of instruction and practice, the rate of learning is 0.6 words/minute per hour of instruction and practice.

$$N'(100) = \frac{6}{100} = 0.06$$

After 100 *hours* of instruction and practice, the rate of learning is 0.06 words/minute per hour of instruction and practice.

Exercise

The population of coyotes in the northwestern portion of Alabama is given by the formula $P(t) = (t^2 + 100)\ln(t + 2)$, where t represents the time in years since 2000 (the year 2000 corresponds to $(t = 0)$ Find the rate of change of the coyote population in 2013 $(t = 13)$.

Solution

$$P'(t) = 2t\ln(t+2) + \frac{1}{t+2}(t^2 + 100) \qquad P' = f'g + g'f \qquad \begin{array}{ll} f = t^2 + 100 & g = \ln(t+2) \\ f' = 2t & g' = \frac{1}{t+2} \end{array}$$

$$= 2t\ln(t+2) + \frac{t^2 + 100}{t+2}$$

$$P'(t = 13) = 2(13)\ln(13+2) + \frac{13^2 + 100}{13+2} \qquad 2*13\ln(13+2) + (13 \wedge 2 + 100)/(13+2)$$

$$\approx 88.34$$