

## Solution

### Section 1.6 – Precise Definition of Limits

#### Exercise

Sketch the interval  $(a, b)$  on the  $x$ -axis with the point  $x_0$  inside. Then find a value of  $\delta > 0$  such that for

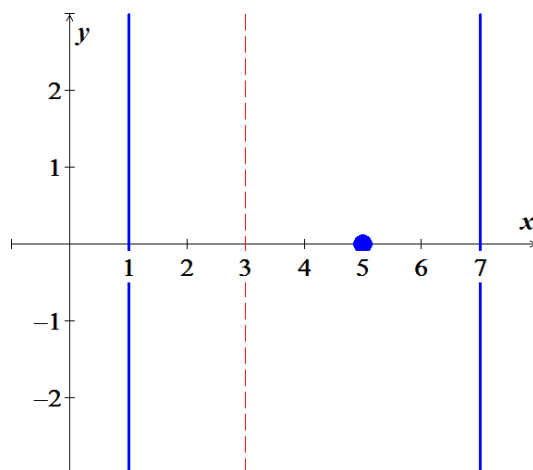
$$\text{all } x, 0 < |x - x_0| < \delta \Rightarrow a < x < b \text{ for } a = 1, \quad b = 7, \quad x_0 = 5$$

#### Solution

$$\begin{aligned} |x - 5| < \delta &\Rightarrow -\delta < x - 5 < \delta \\ &\Rightarrow -\delta + 5 < x < \delta + 5 \end{aligned}$$

$$-\delta + 5 = 1 \Rightarrow \underline{\delta = 4}$$

$$\delta + 5 = 7 \Rightarrow \underline{\delta = 2}$$



#### Exercise

Sketch the interval  $(a, b)$  on the  $x$ -axis with the point  $x_0$  inside. Then find a value of  $\delta > 0$  such that for

$$\text{all } x, 0 < |x - x_0| < \delta \Rightarrow a < x < b \text{ for } a = -\frac{7}{2}, \quad b = -\frac{1}{2}, \quad x_0 = -\frac{3}{2}$$

#### Solution

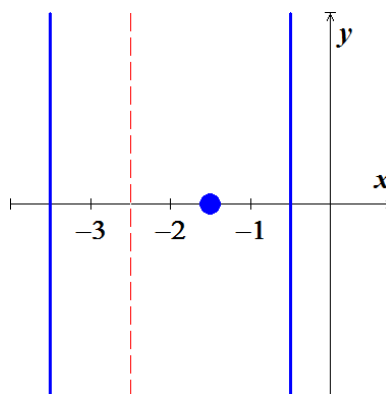
$$\left|x + \frac{3}{2}\right| < \delta$$

$$-\delta < x + \frac{3}{2} < \delta$$

$$-\delta - \frac{3}{2} < x < \delta - \frac{3}{2}$$

$$-\delta - \frac{3}{2} = -\frac{7}{2} \Rightarrow \underline{\delta = \frac{7}{2} - \frac{3}{2} = 2}$$

$$\delta - \frac{3}{2} = -\frac{1}{2} \Rightarrow \underline{\delta = \frac{1}{2} - \frac{3}{2} = -1}$$





## Exercise

Use the graph to find a  $\delta > 0$  such that for all  $x$   $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$

### Solution

**Given:**  $a = -3.1$ ,  $b = -2.9$ ,  $x_0 = -3$

$$|x + 3| < \delta$$

$$-\delta < x + 3 < \delta$$

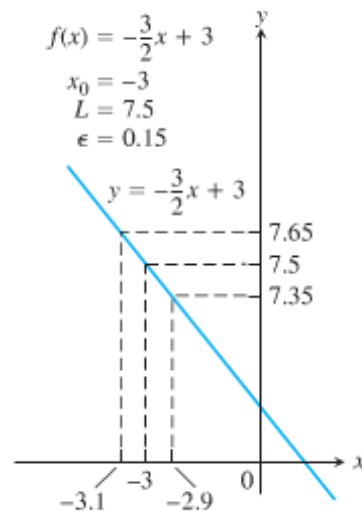
$$-\delta - 3 < x < \delta - 3$$

$$-\delta - 3 = -3.1$$

$$\Rightarrow |\underline{\delta} = 3.1 - 3 = \underline{0.1}|$$

$$\delta - 3 = -2.9$$

$$\Rightarrow |\underline{\delta} = 3 - 2.9 = \underline{0.1}|$$



## Exercise

Find an open interval about  $x_0$  on which the inequality  $|f(x) - L| < \epsilon$  holds. Then give a value for  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - x_0| < \delta$  the inequality  $|f(x) - L| < \epsilon$  holds.

$$f(x) = x + 1, \quad L = 5, \quad x_0 = 4, \quad \epsilon = 0.01$$

### Solution

$$|(x + 1) - 5| < .01$$

$$|x - 4| < .01$$

$$-.01 < x - 4 < .01$$

$$-.01 + 4 < x - 4 + 4 < .01 + 4$$

$$3.99 < x < 4.01$$

$$|x - 4| < \delta$$

$$-\delta < x - 4 < \delta$$

$$-\delta + 4 < x < \delta + 4$$

$$-\delta + 4 = 3.99$$

$$|\underline{\delta} = 4 - 3.99 = \underline{0.01}|$$

$$\delta + 4 = 4.01$$

$$|\underline{\delta} = 4.01 - 4 = \underline{0.01}|$$

$$\Rightarrow \underline{\delta = .01}$$

### Exercise

Find an open interval about  $x_0$  on which the inequality  $|f(x) - L| < \varepsilon$  holds. Then give a value for  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - x_0| < \delta$  the inequality  $|f(x) - L| < \varepsilon$  holds.

$$f(x) = 2x - 1, \quad L = 3, \quad x_0 = 2, \quad \varepsilon = 0.1$$

### Solution

$$|2x - 1 - 3| < .1$$

$$|2x - 4| < .1$$

$$-.1 < 2x - 4 < .1$$

$$-.1 + 4 < 2x - 4 + 4 < .1 + 4$$

$$3.9 < 2x < 4.1$$

$$\frac{3.9}{2} < x < \frac{4.1}{2}$$

$$1.95 < x < 2.05$$

$$|x - 2| < \delta$$

$$-\delta < x - 2 < \delta$$

$$-\delta + 2 < x < \delta + 2$$

$$-\delta + 2 = 1.95$$

$$\delta = 2 - 1.95 = \underline{0.05}$$

$$\delta + 2 = 2.05$$

$$\delta = 2.05 - 2 = \underline{0.05}$$

$$\Rightarrow \underline{\delta = .05}$$

### Exercise

Find an open interval about  $x_0$  on which the inequality  $|f(x) - L| < \varepsilon$  holds. Then give a value for  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - x_0| < \delta$  the inequality  $|f(x) - L| < \varepsilon$  holds.

$$f(x) = x + 2, \quad L = 3, \quad x_0 = 1, \quad \varepsilon = 0.001$$

### Solution

$$|x + 2 - 3| < .001$$

$$|x - 1| < .001$$

$$-.001 < x - 1 < .001$$

$$-.001 + 1 < x - 1 + 1 < .001 + 1$$

$$0.999 < x < 1.001$$

$$|x-1| < \delta$$

$$-\delta < x-1 < \delta$$

$$-\delta+1 < x < \delta+1$$

$$-\delta+1 = .999$$

$$\delta = 1 - .999 = \underline{0.001}$$

$$\delta+1 = 1.001$$

$$\delta = 1.001 - 1 = \underline{0.001}$$

$$\Rightarrow \delta = \underline{.001}$$

### ***Exercise***

Find an open interval about  $x_0$  on which the inequality  $|f(x) - L| < \varepsilon$  holds. Then give a value for  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - x_0| < \delta$  the inequality  $|f(x) - L| < \varepsilon$  holds.

$$f(x) = 3x + 2, \quad L = 2, \quad x_0 = 0, \quad \varepsilon = 0.1$$

### ***Solution***

$$|3x + 2 - 2| < .1$$

$$|3x| < .1$$

$$-.1 < 3x < .1$$

$$-\frac{.1}{3} < x < \frac{.1}{3}$$

$$-\frac{1}{30} < x < \frac{1}{30}$$

$$|x-0| < \delta$$

$$-\delta < x < \delta$$

$$-\delta = -\frac{1}{30}$$

$$\delta = \underline{\frac{1}{30}}$$

$$\Rightarrow \delta = \underline{\frac{1}{30}}$$

### Exercise

Find an open interval about  $x_0$  on which the inequality  $|f(x) - L| < \varepsilon$  holds. Then give a value for  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - x_0| < \delta$  the inequality  $|f(x) - L| < \varepsilon$  holds.

$$f(x) = \sqrt{x+1}, \quad L = 1, \quad x_0 = 0, \quad \varepsilon = 0.1$$

### Solution

$$|\sqrt{x+1} - 1| < 0.1$$

$$-0.1 < \sqrt{x+1} - 1 < 0.1$$

$$-0.1 + 1 < \sqrt{x+1} - 1 + 1 < 0.1 + 1$$

$$.9 < \sqrt{x+1} < 1.1$$

$$(.9)^2 < (\sqrt{x+1})^2 < (1.1)^2$$

$$.81 < x+1 < 1.21$$

$$.81 - 1 < x+1 - 1 < 1.21 - 1$$

$$-0.19 < x < 0.21$$

$$|x - 0| < \delta \Rightarrow -\delta < x < \delta$$

$$-\delta = -0.19 \Rightarrow \begin{array}{l} \underline{\delta = 0.19} \\ \underline{\delta = 0.21} \end{array} \rightarrow \underline{\delta = 0.19}$$

### Exercise

Find an open interval about  $x_0$  on which the inequality  $|f(x) - L| < \varepsilon$  holds. Then give a value for  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - x_0| < \delta$  the inequality  $|f(x) - L| < \varepsilon$  holds.

$$f(x) = \sqrt{x-7}, \quad L = 4, \quad x_0 = 23, \quad \varepsilon = 1$$

### Solution

$$|\sqrt{x-7} - 4| < 1$$

$$-1 < \sqrt{x-7} - 4 < 1$$

$$3 < \sqrt{x-7} < 5$$

$$(3)^2 < (\sqrt{x-7})^2 < (5)^2$$

$$9 < x-7 < 25$$

$$9 + 7 < x-7 + 7 < 25 + 7$$

$$16 < x < 32$$

$$|x - 23| < \delta$$

$$-\delta < x - 23 < \delta$$

$$-\delta + 23 < x < \delta + 23$$

$$-\delta + 23 = 16 \Rightarrow \delta = 23 - 16 = \underline{7}$$

$$\delta + 23 = 32 \Rightarrow \delta = 32 - 23 = \underline{9}$$

$$\Rightarrow \underline{\delta = 7}$$

### Exercise

Find an open interval about  $x_0$  on which the inequality  $|f(x) - L| < \varepsilon$  holds. Then give a value for  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - x_0| < \delta$  the inequality  $|f(x) - L| < \varepsilon$  holds.

$$f(x) = x^2, \quad L = 3, \quad x_0 = \sqrt{3}, \quad \varepsilon = 0.1$$

### Solution

$$|x^2 - 3| < 0.1$$

$$-0.1 < x^2 - 3 < 0.1$$

$$2.9 < x^2 < 3.1$$

$$\sqrt{2.9} < x < \sqrt{3.1}$$

$$|x - \sqrt{3}| < \delta$$

$$-\delta < x - \sqrt{3} < \delta$$

$$-\delta + \sqrt{3} < x < \delta + \sqrt{3}$$

$$-\delta + \sqrt{3} = \sqrt{2.9} \Rightarrow \delta = \sqrt{3} - \sqrt{2.9} = \underline{.029}$$

$$\delta + \sqrt{3} = \sqrt{3.1} \Rightarrow \delta = \sqrt{3.1} - \sqrt{3} = \underline{.029}$$

$$\Rightarrow \underline{\delta = .029}$$

### Exercise

Find an open interval about  $x_0$  on which the inequality  $|f(x) - L| < \varepsilon$  holds. Then give a value for  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - x_0| < \delta$  the inequality  $|f(x) - L| < \varepsilon$  holds.

$$f(x) = \frac{120}{x}, \quad L = 5, \quad x_0 = 24, \quad \varepsilon = 1$$

### Solution

$$\left| \frac{120}{x} - 5 \right| < 0.1$$

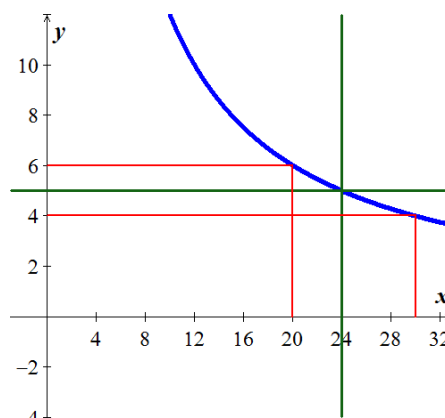
$$-1 < \frac{120}{x} - 5 < 1$$

$$4 < \frac{120}{x} < 6$$

$$\frac{1}{6} < \frac{x}{120} < \frac{1}{4}$$

$$\frac{1}{6}(120) < x < \frac{1}{4}(120)$$

$$20 < x < 30$$



$$|x - 24| < \delta$$

$$-\delta < x - 24 < \delta$$

$$-\delta + 24 < x < \delta + 24$$

$$-\delta + 24 = 20 \Rightarrow \delta = 24 - 20 = \underline{4}$$

$$\delta + 24 = 30 \Rightarrow \delta = 30 - 24 = \underline{6}$$

$$\Rightarrow \underline{\delta = 4}$$

## Exercise

Prove that  $\lim_{x \rightarrow 4} (9 - x) = 5$

## Solution

$$|(9 - x) - 5| < \varepsilon$$

$$-\varepsilon < 4 - x < \varepsilon$$

$$-\varepsilon - 4 < -x < \varepsilon - 4$$

$$\varepsilon + 4 > x > 4 - \varepsilon$$

$$4 - \varepsilon < x < \varepsilon + 4$$

*divide by (-).*

$$|x - 4| < \delta$$

$$-\delta < x - 4 < \delta$$

$$-\delta + 4 < x < \delta + 4$$

$$-\delta + 4 = 4 - \varepsilon \Rightarrow -\delta = -\varepsilon \Rightarrow \underline{\delta = \varepsilon}$$

$$\delta + 4 = \varepsilon + 4 \Rightarrow \underline{\delta = \varepsilon}$$

$$\Rightarrow \underline{\delta = \varepsilon}$$



### Exercise

Prove that  $\lim_{x \rightarrow 1} \frac{1}{x} = 1$

### Solution

$$\left| \frac{1}{x} - 1 \right| < \varepsilon$$

$$-\varepsilon < \frac{1}{x} - 1 < \varepsilon$$

$$-\varepsilon + 1 < \frac{1}{x} < \varepsilon + 1$$

$$\frac{1}{\varepsilon + 1} > x > \frac{1}{-\varepsilon + 1}$$

$$\frac{1}{1 + \varepsilon} < x < \frac{1}{1 - \varepsilon}$$

$$|x - 1| < \delta$$

$$-\delta < x - 1 < \delta$$

$$1 - \delta < x < 1 + \delta$$

$$1 - \delta = \frac{1}{1 + \varepsilon} \Rightarrow \delta = 1 + \frac{1}{1 + \varepsilon} = \frac{2 + \varepsilon}{1 + \varepsilon}$$

$$1 + \delta = \frac{1}{1 - \varepsilon} \Rightarrow \delta = \frac{1}{1 - \varepsilon} - 1 = \frac{\varepsilon}{1 - \varepsilon}$$

$$\text{The smallest: } \delta = \frac{\varepsilon}{1 - \varepsilon}$$

### Exercise

Prove that  $\lim_{x \rightarrow 5} \frac{x^2 - 25}{x - 5} = 10$

### Solution

$$\left| \frac{x^2 - 25}{x - 5} - 10 \right| < \varepsilon$$

$$-\varepsilon < \frac{(x - 5)(x + 5)}{x - 5} - 10 < \varepsilon$$

$$-\varepsilon + 10 < x + 5 < \varepsilon + 10$$

$$-\varepsilon + 5 < x < \varepsilon + 15$$

$$|x - 10| < \delta$$

$$-\delta < x - 10 < \delta$$

$$10 - \delta < x < 10 + \delta$$

$$10 - \delta = 5 - \varepsilon \Rightarrow \delta = 5 + \varepsilon$$

$$10 + \delta = \varepsilon + 15 \Rightarrow \underline{\delta = \varepsilon + 5}$$

The smallest :  $\underline{\delta = \varepsilon + 5}$

### Exercise

Prove that  $\lim_{x \rightarrow 0} f(x) = 0$  if  $f(x) = \begin{cases} 2x, & x < 0 \\ \frac{x}{2}, & x \geq 0 \end{cases}$

### Solution

$$\begin{aligned} \text{For } x < 0: |2x - 0| &< \varepsilon \\ -\varepsilon &< 2x < 0 \\ -\frac{\varepsilon}{2} &< x < 0 \end{aligned}$$

$$\begin{aligned} \text{For } x \geq 0: \left| \frac{x}{2} - 0 \right| &< \varepsilon \\ 0 \leq \frac{x}{2} &< \varepsilon \\ 0 \leq x &< 2\varepsilon \end{aligned}$$

$$|x - 0| < \delta \Rightarrow -\delta < x < \delta$$

$$\begin{aligned} -\delta = -\frac{\varepsilon}{2} \Rightarrow \delta = \frac{\varepsilon}{2} &\rightarrow \text{the smallest : } \underline{\delta = \frac{\varepsilon}{2}} \\ \delta = 2\varepsilon & \end{aligned}$$

### Exercise

Prove that  $\lim_{x \rightarrow 1} (5x - 2) = 3$

### Solution

$$\begin{aligned} |(5x - 2) - 3| &< \varepsilon \\ -\varepsilon &< 5x - 5 < \varepsilon \\ 5 - \varepsilon &< 5x < 5 + \varepsilon \\ 1 - \frac{1}{5}\varepsilon &< x < 1 + \frac{1}{5}\varepsilon \end{aligned}$$

$$\begin{aligned} |x - 3| &< \delta \\ -\delta &< x - 3 < \delta \\ 3 - \delta &< x < 3 + \delta \end{aligned}$$

$$\begin{aligned} 3 - \delta = 1 - \frac{1}{5}\varepsilon \Rightarrow \delta = \frac{1}{5}\varepsilon + 2 \\ 3 + \delta = 1 + \frac{1}{5}\varepsilon \Rightarrow \delta = \frac{1}{5}\varepsilon - 2 \end{aligned} \rightarrow \text{the smallest : } \underline{\delta = \frac{1}{5}\varepsilon - 2}$$

### Exercise

Prove that  $\lim_{x \rightarrow 2} \frac{1}{(x-2)^4} = \infty$

### Solution

Let  $N > 0$  and let  $\delta = \frac{1}{\sqrt[4]{N}}$

Suppose that  $0 < |x - 2| < \delta$

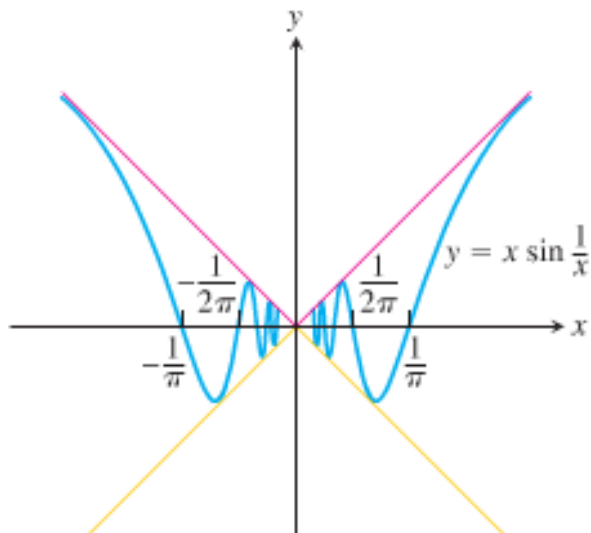
$$|x - 2| < \delta = \frac{1}{\sqrt[4]{N}}$$

$$\frac{1}{|x - 2|} > \sqrt[4]{N}$$

$$\frac{1}{(x - 2)^4} > N \quad \checkmark$$

### Exercise

Prove that  $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$



### Solution

$$\left. \begin{aligned} -x &\leq x \sin \frac{1}{x} \leq x, & \forall x > 0 \\ -x &\geq x \sin \frac{1}{x} \geq x, & \forall x < 0 \end{aligned} \right\} \rightarrow \lim_{x \rightarrow 0} (-x) = \lim_{x \rightarrow 0} (x) = 0$$

Then by the sandwich theorem,  $\lim_{x \rightarrow 0} x \sin \left( \frac{1}{x} \right) = 0$