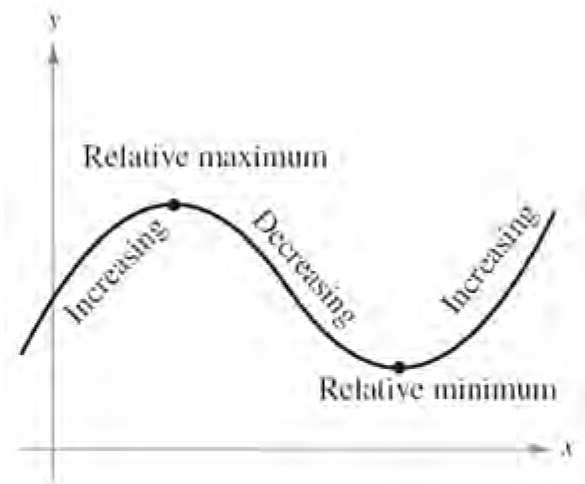
Section 3.2 –Extrema and the First-Derivative Test



First-Derivative Test for the Relative Extrema

Let f be continuous on the interval (a, b) in which x is the only critical number. If f is differentiable on the interval (except possibly at c), then $f(c)$ can be classified as a relative minimum, a relative maximum, or neither, as shown

1. On the interval (a, b), If $f'(x)$ is negative to the left of $x = c$ and positive to the right of $x = c$, then $f(c)$ is a relative minimum (**RMIN**).

2. On the interval (a, b), If $f'(x)$ is positive to the left of $x = c$ and negative to the right of $x = c$, then $f(c)$ is a relative maximum (**RMAX**).

3. On the interval (a, b), If $f'(x)$ is positive on both sides of $x = c$ or negative on both sides of $x = c$, then $f(c)$ is not a relative extremum of f.

Example

Find all relative Extrema as well as where the function is increasing and decreasing $f(x) = 2x^3 - 6x + 1$

Solution

$$f'(x) = 6x^2 - 6 = 0$$

$$\Rightarrow 6x^2 = 6$$

$$\Rightarrow x^2 = 1 \quad \rightarrow x = \pm 1 \quad (CN)$$

$$\begin{cases} x = 1 \rightarrow y = f(1) = -3 \\ x = -1 \rightarrow y = f(-1) = 5 \end{cases} \qquad (-1,5),\ (1,-3)$$

-∞ **-1**	**1**	**∞**
$f'(-2) > 0$ *Increasing*	$f'(0) < 0$ *Decreasing*	$f'(2) > 0$ *Increasing*

RMAX: (-1, 5);

RMIN: (1, -3)

Increasing*:*(-∞, -1) and (1, ∞);

Decreasing*:* (-1, 1)

Example

Find all relative Extrema of $f(x) = 6x^{2/3} - 4x$ and Find the open intervals on which is increasing or decreasing

Solution

$$f'(x) = 4x^{-1/3} - 4$$

$$= 4\left(\frac{1}{x^{1/3}} - 1\right)$$

$$f'(x) = 4\left(\frac{1}{x^{1/3}} - 1\right) = 0 \qquad \boxed{x \neq 0}$$

$$\frac{1}{x^{1/3}} - 1 = 0$$

$$\frac{1}{x^{1/3}} = 1 \qquad \textit{Multiply both sides by } x^{1/3}$$

$$1 = x^{1/3}$$

$$\lfloor x = 1^3 = 1 \rfloor$$

CN: $x = 0,\ 1$

$$\begin{cases} x = 0 \to y = 0 \\ x = 1 \to y = 2 \end{cases} \qquad (0, 0) \text{ and } (1, 2)$$

$-\infty$ **0**	**1**	∞
$f'(-1) < 0$ *Decreasing*	$f'\left(\frac{1}{2}\right) > 0$ *Increasing*	$f'(2) < 0$ *Decreasing*

RMIN: (0, 0)

RMAX: (1, 2)

Decreasing: $(-\infty, 0)$ and $(1, \infty)$

Increasing: (0, 1)

Example

Find all relative Extrema as well as where the function is increasing and decreasing $f(x) = xe^{2-x^2}$

<u>*Solution*</u>

$$\begin{aligned} f'(x) &= e^{2-x^2} - 2x^2e^{2-x^2} \\ &= e^{2-x^2}\left(1 - 2x^2\right) \\ &= 0 \end{aligned}$$

$1 - 2x^2 = 0$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}}$ ***CN***: $\boxed{x = \pm\frac{1}{\sqrt{2}}} \approx \pm 0.707$

$-\infty$ $\quad -\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	∞
$f'(-1) < 0$ *Decreasing*	$f'(0) > 0$ *Increasing*	$f'(1) < 0$ *Decreasing*

RMIN: $\left(-\frac{1}{\sqrt{2}}, -3.17\right)$

RMAX: $\left(\frac{1}{\sqrt{2}}, 3.17\right)$

Decreasing: $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$

Increasing: $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Example

A small company manufactures and sells bicycles. The production manager has determined that the cost and demand functions for q $(q \ge 0)$ bicycles per week are

$$C(q) = 10 + 5q + \frac{1}{60}q^3 \qquad \textit{and} \qquad p = D(q) = 90 - q$$

Where p is the price per bicycle

a) Find the maximum weekly revenue

b) Find the maximum weekly profit

c) Find the price the company should charge to realize maximum profit.

Solution

a) Find the maximum weekly revenue

$$\begin{aligned} R(q) &= qp \\ &= q(90-q) \\ &= 90q - q^2 \end{aligned}$$

Maximum revenue = $R'(q)$

$$R' = 90 - 2q = 0$$

$$\Rightarrow \boxed{q = 45}$$

$$\begin{aligned} R(45) &= 90(45) - 45^2 \\ &= \$2025. \end{aligned}$$

b) Find the maximum weekly profit

$$\begin{aligned} P(q) &= R(q) - C(q) \\ &= 90q - q^2 - \left(10 + 5q + \frac{1}{60}q^3\right) \\ &= -\frac{1}{60}q^3 - q^2 + 85q - 10 \end{aligned}$$

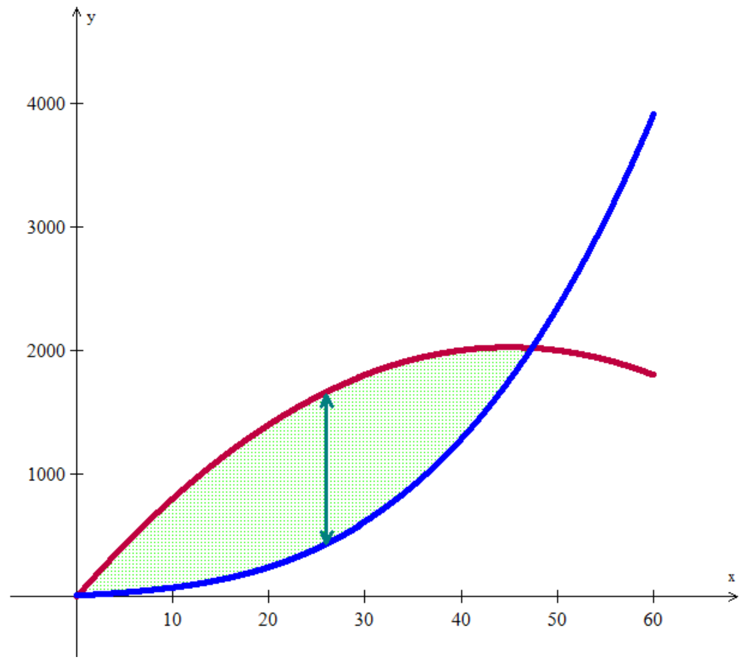
$$P'(q) = -\frac{1}{20}q^2 - 2q + 85 = 0$$

$$\boxed{q \approx 25.8} \qquad q \approx -65.8$$

$$\begin{aligned} P(26) &= -\frac{1}{60}(26)^3 - (26)^2 + 85(26) - 10 \\ &= \$1231.07 \end{aligned}$$

c) Find the price the company should charge to realize maximum profit.

If $q = 26 \Rightarrow \boxed{p = 90 - 26 = \$64}$



y
4000
3000
2000
1000
x
10
20
30
40
50
60

Exercise *Section* 3.2 – Extrema and the First-Derivative Test

1. Find all relative extrema of the function $f(x) = 6x^3 - 15x^2 + 12x$

Find all relative Extrema as well as where the function is increasing and decreasing

2. $f(x) = x^4 - 4x^3$

3. $f(x) = 3x^{2/3} - 2x$

4. $y = \sqrt{4 - x^2}$

5. $f(x) = x\sqrt{x+1}$

6. $f(x) = \dfrac{x}{x^2 + 1}$

7. $f(x) = x^4 - 8x^2 + 9$

8. $f(x) = 3xe^x + 2$

9. Coughing forces the trachea to contract, this in turn affects the velocity of the air through the trachea. The velocity of the air during coughing can be modeled by: $v = k(R - r)r^2, \quad 0 \le r < R$ where k is a constant, R is the normal radius of the trachea (also a constant) and r is the radius of the trachea during coughing. What radius r will produce the maximum air velocity?

10. When a telephone wire is hung between two poles, the wire forms a U-shape curve called a Catenary. For instance, the function $y = 30\left(e^{x/60} + e^{-x/60}\right) - 30 \le x \le 30$ models the shape of the telephone wire strung between two poles that are 60 *ft* apart (x & y are measured in *ft*). Show that the lowest point on the wire is midway between two poles. How much does the wire sag between the two poles?

11. The demand function for the product is modeled by $p = 50e^{-0.0000125x}$ where p is the price per unit in dollars and x is the number of units. What price will yield maximum revenue?

12. The annual revenue and cost functions for a manufacturer of grandfather clocks are approximately $R(x) = 520x - 0.03x^2$ and $C(x) = 200x + 100{,}000$, where x denotes the number of clocks made. What is the maximum annual profit?

13. Find the number of units, x, that produces the maximum profit P, if $C(x) = 30 + 20x$ and $p = 32 - 2x$

14. $P(x) = -x^3 + 15x^2 - 48x + 450, \quad x \geq 3$ is an approximation to the total profit (in thousands of dollars) from the sale of x hundred thousand tires. Find the number of hundred thousands of tires that must be sold to maximize profit.

15. $P(x) = -x^3 + 3x^2 + 360x + 5000; \quad 6 \leq x \leq 20$ is an approximation to the number of salmon swimming upstream to spawn, where x represents the water temperature in degrees Celsius. Find the temperature that produces the maximum number of salmon.