# *Solution* *Section* 3.1 – Proving Identities

### *Exercise*

Prove the identity $\cos\theta\cot\theta+\sin\theta=\csc\theta$

***Solution***

$$\begin{aligned}
\cos\theta\cot\theta+\sin\theta &= \cos\theta\frac{\cos\theta}{\sin\theta}+\sin\theta \\
&= \frac{\cos^2\theta}{\sin\theta}+\sin\theta \\
&= \frac{\cos^2\theta}{\sin\theta}+\sin\theta\frac{\sin\theta}{\sin\theta} \\
&= \frac{\cos^2\theta}{\sin\theta}+\frac{\sin^2\theta}{\sin\theta} \\
&= \frac{\cos^2\theta+\sin^2\theta}{\sin\theta} \\
&= \frac{1}{\sin\theta} \\
&= \csc\theta
\end{aligned}$$

### *Exercise*

Prove the identity $\sec\theta\cot\theta-\sin\theta=\frac{\cos^2\theta}{\sin\theta}$

***Solution***

$$\begin{aligned}
\sec\theta\cot\theta-\sin\theta &= \frac{1}{\cos\theta}\frac{\cos\theta}{\sin\theta}-\sin\theta \\
&= \frac{1}{\sin\theta}-\sin\theta \\
&= \frac{1-\sin^2\theta}{\sin\theta} \\
&= \frac{\cos^2\theta}{\sin\theta}
\end{aligned}$$

### *Exercise*

Prove the identity $\dfrac{\csc\theta\tan\theta}{\sec\theta}=1$

***Solution***

$$\begin{aligned}\frac{\csc\theta\tan\theta}{\sec\theta}&=\csc\theta\tan\theta\frac{1}{\sec\theta}\\&=\frac{1}{\sin\theta}\frac{\sin\theta}{\cos\theta}\cos\theta\\&=1\end{aligned}$$

### *Exercise*

Prove the identity $(\sin\theta+\cos\theta)^2=1+2\sin\theta\cos\theta$

***Solution***

$$\begin{aligned}(\sin\theta+\cos\theta)^2&=\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta\\&=1+2\sin\theta\cos\theta\end{aligned}$$

### *Exercise*

Prove the identity $\sin\theta(\sec\theta+\cot\theta)=\tan\theta+\cos\theta$

***Solution***

$$\begin{aligned}\sin\theta(\sec\theta+\cot\theta)&=\sin\theta\left(\frac{1}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)\\&=\sin\theta\frac{1}{\cos\theta}+\sin\theta\frac{\cos\theta}{\sin\theta}\\&=\frac{\sin\theta}{\cos\theta}+\cos\theta\\&=\tan\theta+\cos\theta\end{aligned}$$

### *Exercise*

Prove the identity $\cos\theta(\csc\theta+\tan\theta)=\cot\theta+\sin\theta$

***Solution***

$$\begin{aligned}\cos\theta(\csc\theta+\tan\theta)&=\cos\theta\frac{1}{\sin\theta}+\cos\theta\frac{\sin\theta}{\cos\theta}\\&=\cot\theta+\sin\theta\end{aligned}$$

***Exercise***

Prove the identity $\cos\theta(\sec\theta - \cos\theta) = \sin^2\theta$

***Solution***

$$\cos\theta(\sec\theta - \cos\theta) = \cos\theta\frac{1}{\cos\theta} - \cos^2\theta$$

$$= 1 - \cos^2\theta$$

$$= \sin^2\theta$$

***Exercise***

Prove the identity $\cot\theta + \tan\theta = \csc\theta\sec\theta$

***Solution***

$$\cot\theta + \tan\theta = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta}\frac{1}{\cos\theta}$$

$$= \csc\theta\sec\theta$$

***Exercise***

Prove $\tan x(\cos x + \cot x) = \sin x + 1$

***Solution***

$$\tan x(\cos x + \cot x) = \frac{\sin x}{\cos x}\left(\cos x + \frac{\cos x}{\sin x}\right)$$

$$= \cos x\frac{\sin x}{\cos x} + \frac{\sin x}{\cos x}\frac{\cos x}{\sin x}$$

$$= \sin x + 1$$

***Exercise***

Prove $\quad \dfrac{1-\cos^4\theta}{1+\cos^2\theta} = \sin^2\theta$

***Solution***

$$\frac{1-\cos^4\theta}{1+\cos^2\theta} = \frac{(1+\cos^2\theta)(1-\cos^2\theta)}{1+\cos^2\theta}$$

$$= 1-\cos^2\theta$$

$$= \sin^2\theta$$

***Exercise***

Prove $\quad \dfrac{1-\sec x}{1+\sec x} = \dfrac{\cos x - 1}{\cos x + 1}$

***Solution***

$$\frac{1-\sec x}{1+\sec x} = \frac{1-\frac{1}{\cos x}}{1+\frac{1}{\cos x}}$$

$$= \frac{\frac{\cos x - 1}{\cos x}}{\frac{\cos x + 1}{\cos x}}$$

$$= \frac{\cos x - 1}{\cos x + 1}$$

***Exercise***

Prove $\quad \dfrac{\cos x}{1-\sin x} - \dfrac{1-\sin x}{\cos x} = 0$

***Solution***

$$\frac{\cos x}{1+\sin x} - \frac{1-\sin x}{\cos x} = \frac{\cos x}{\cos x}\frac{\cos x}{1-\sin x} - \frac{1+\sin x}{1+\sin x}\frac{1-\sin x}{\cos x}$$

$$= \frac{\cos^2 x - (1-\sin^2 x)}{\cos x(1+\sin x)}$$

$$= \frac{\cos^2 x - 1 + \sin^2 x)}{\cos x(1+\sin x)}$$

$$= \frac{1-1}{\cos x(1+\sin x)}$$

$$= \frac{0}{\cos x(1+\sin x)}$$

$$= 0$$

### *Exercise*

Prove $\quad \dfrac{1+\cot^3 t}{1+\cot t} = \csc^2 t - \cot t$

***Solution***

$$\frac{1+\cot^3 t}{1+\cot t} = \frac{1+\frac{\cos^3 t}{\sin^3 t}}{1+\frac{\cos t}{\sin t}}$$

$$= \frac{\frac{\sin^3 t+\cos^3 t}{\sin^3 t}}{\frac{\sin t+\cos t}{\sin t}}$$

$$= \frac{\sin^3 t+\cos^3 t}{\sin^3 t} \cdot \frac{\sin t}{\sin t+\cos t}$$

$$= \frac{(\sin t+\cos t)(\sin^2 t-\sin t\cos t+\cos^2 t)}{\sin^2 t} \cdot \frac{1}{\sin t+\cos t}$$

$$= \frac{1-\sin t\cos t}{\sin^2 t}$$

$$= \frac{1}{\sin^2 t} - \frac{\sin t\cos t}{\sin^2 t}$$

$$= \csc^2 t - \frac{\cos t}{\sin t}$$

$$= \csc^2 t - \cot t$$

### *Exercise*

Prove: $\quad \tan x + \cot x = \sec x \csc x$

***Solution***

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x}$$

$$= \frac{1}{\cos x \sin x}$$

$$= \frac{1}{\cos x}\frac{1}{\sin x}$$

$$= \sec x \csc x$$

### *Exercise*

Prove: $\dfrac{\tan x-\cot x}{\sin x\cos x}=\sec^2 x-\csc^2 x$

***Solution***

$$\frac{\tan x-\cot x}{\sin x\cos x}=\frac{\tan x}{\sin x\cos x}-\frac{\cot x}{\sin x\cos x}$$

$$=\tan x\frac{1}{\sin x\cos x}-\cot x\frac{1}{\sin x\cos x}$$

$$=\frac{\sin x}{\cos x}\frac{1}{\sin x\cos x}-\frac{\cos x}{\sin x}\frac{1}{\sin x\cos x}$$

$$=\frac{1}{\cos^2 x}-\frac{1}{\sin^2 x}$$

$$=\sec^2 x-\csc^2 x$$

### *Exercise*

Prove: $\dfrac{\sec x+\tan x}{\sec x-\tan x}=\dfrac{1+2\sin x+\sin^2 x}{\cos^2 x}$

***Solution***

$$\frac{\sec x+\tan x}{\sec x-\tan x}=\frac{\frac{1}{\cos x}+\frac{\sin x}{\cos x}}{\frac{1}{\cos x}-\frac{\sin x}{\cos x}}\frac{\cos x}{\cos x}$$

$$=\frac{\frac{1}{\cos x}\cos x+\frac{\sin x}{\cos x}\cos x}{\frac{1}{\cos x}\cos x-\frac{\sin x}{\cos x}\cos x}$$

$$=\frac{1+\sin x}{1-\sin x}$$

$$=\frac{1+\sin x}{1-\sin x}\frac{1+\sin x}{1+\sin x}$$

$$=\frac{(1+\sin x)^2}{1-\sin^2 x}$$

$$=\frac{1+2\sin x+\sin^2 x}{\cos^2 x}$$

***Exercise***

Prove the identity: $\sin^2 x - \cos^2 x = 2\sin^2 x - 1$

***Solution***

$$\begin{aligned}\sin^2 x - \cos^2 x &= \sin^2 x - \left(1 - \sin^2 x\right) \\ &= \sin^2 x - 1 + \sin^2 x \\ &= 2\sin^2 x - 1\end{aligned}$$

***Exercise***

Prove the identity: $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$

***Solution***

$$\begin{aligned}\sin^4 x - \cos^4 x &= \left(\sin^2 x + \cos^2 x\right)\left(\sin^2 x - \cos^2 x\right) \\ &= (1)\left(\sin^2 x - \cos^2 x\right) \\ &= \sin^2 x - \cos^2 x\end{aligned}$$

***Exercise***

Prove the identity: $\dfrac{\cos\alpha}{1+\sin\alpha} = \sec\alpha - \tan\alpha$

***Solution***

$$\begin{aligned}\frac{\cos\alpha}{1+\sin\alpha} &= \frac{\cos\alpha}{1+\sin\alpha}\frac{1-\sin\alpha}{1-\sin\alpha} \\ &= \frac{\cos\alpha - \cos\alpha\sin\alpha}{1-\sin^2\alpha} \\ &= \frac{\cos\alpha - \cos\alpha\sin\alpha}{\cos^2\alpha} \\ &= \frac{\cos\alpha}{\cos^2\alpha} - \frac{\cos\alpha\sin\alpha}{\cos^2\alpha} \\ &= \frac{1}{\cos\alpha} - \frac{\sin\alpha}{\cos\alpha} \\ &= \sec\alpha - \tan\alpha\end{aligned}$$

### ***Exercise***

Prove the identity: $\dfrac{\sin\alpha}{1-\sin\alpha}-\dfrac{\cos\alpha}{1-\sin\alpha}=\dfrac{1-\cot\alpha}{\csc\alpha-1}$

***Solution***

$$\frac{\sin\alpha}{1-\sin\alpha}-\frac{\cos\alpha}{1-\sin\alpha}=\frac{\sin\alpha-\cos\alpha}{1-\sin\alpha}$$

$$=\frac{\frac{\sin\alpha-\cos\alpha}{\sin\alpha}}{\frac{1-\sin\alpha}{\sin\alpha}}$$

$$=\frac{\frac{\sin\alpha}{\sin\alpha}-\frac{\cos\alpha}{\sin\alpha}}{\frac{1}{\sin\alpha}-\frac{\sin\alpha}{\sin\alpha}}$$

$$=\frac{1-\cot\alpha}{\csc\alpha-1}$$

### ***Exercise***

Prove the identity: $\dfrac{\frac{1}{\tan x}+\cot x}{\frac{1}{\tan x}+\tan x}=\dfrac{2}{\sec^2 x}$

***Solution***

$$\frac{\frac{1}{\tan x}+\cot x}{\frac{1}{\tan x}+\tan x}=\frac{\frac{1}{\tan x}+\frac{1}{\tan x}}{\frac{1+\tan^2 x}{\tan x}}$$

$$=\frac{\frac{2}{\tan x}}{\frac{\sec^2 x}{\tan x}}$$

$$=\frac{2}{\sec^2 x}$$

### ***Exercise***

Prove the following equation is an identity: $\dfrac{\cot^2\theta+3\cot\theta-4}{\cot\theta+4}=\cot\theta-1$

***Solution***

$$\frac{\cot^2\theta+3\cot\theta-4}{\cot\theta+4}=\frac{(\cot\theta+4)(\cot\theta-1)}{\cot\theta+4}$$

$$=\cot\theta-1$$

### *Exercise*

Prove the following equation is an identity: $\frac{\sin\theta}{1+\cos\theta}=\frac{1-\cos\theta}{\sin\theta}$

***Solution***

$$\frac{\sin\theta}{1+\cos\theta}=\frac{\sin\theta}{1+\cos\theta}\frac{1-\cos\theta}{1-\cos\theta}$$

$$=\frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta}$$

$$=\frac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$$

$$=\frac{1-\cos\theta}{\sin\theta}$$

### *Exercise*

Prove the following equation is an identity: $\tan x(\csc x-\sin x)=\cos x$

***Solution***

$$\tan x(\csc x-\sin x)=\frac{\sin x}{\cos x}\left(\frac{1}{\sin x}-\sin x\right)$$

$$=\frac{\sin x}{\cos x}\left(\frac{1-\sin^2 x}{\sin x}\right)$$

$$=\frac{1}{\cos x}\left(\frac{\cos^2 x}{1}\right)$$

$$=\cos x$$

### *Exercise*

Prove the following equation is an identity: $\sin x(\tan x\cos x-\cot x\cos x)=1-2\cos^2 x$

***Solution***

$$\sin x(\tan x\cos x-\cot x\cos x)=\sin x\cos x\left(\frac{\sin x}{\cos x}-\frac{\cos x}{\sin x}\right)$$

$$=\sin x\cos x\left(\frac{\sin^2 x-\cos^2 x}{\cos x\sin x}\right)$$

$$=1-\cos^2 x-\cos^2 x$$

$$=1-2\cos^2 x$$

### *Exercise*

Prove the following equation is an identity: $(1+\tan x)^2+(\tan x-1)^2=2\sec^2 x$

***Solution***

$$\begin{aligned}(1+\tan x)^2+(\tan x-1)^2 &= 1+2\tan x+\tan^2 x+1-2\tan x+\tan^2 x\\ &= 2+2\tan^2 x\\ &= 2\left(1+\tan^2 x\right)\\ &= 2\sec^2 x\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\sec x+\tan x=\dfrac{\cos x}{1-\sin x}$

***Solution***

$$\begin{aligned}\sec x+\tan x &= \frac{1}{\cos x}+\frac{\sin x}{\cos x}\\ &= \frac{1+\sin x}{\cos x}\frac{1-\sin x}{1-\sin x}\\ &= \frac{1-\sin^2 x}{\cos x(1-\sin x)}\\ &= \frac{\cos^2 x}{\cos x(1-\sin x)}\\ &= \frac{\cos x}{1-\sin x}\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{\tan x-1}{\tan x+1}=\dfrac{1-\cot x}{1+\cot x}$

***Solution***

$$\begin{aligned}\frac{\tan x-1}{\tan x+1} &= \frac{\frac{1}{\cot x}-1}{\frac{1}{\cot x}+1}\\ &= \frac{\frac{1-\cot x}{\cot x}}{\frac{1+\cot x}{\cot x}}\\ &= \frac{1-\cot x}{1+\cot x}\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $7\csc^2 x - 5\cot^2 x = 2\csc^2 x + 5$

***Solution***

$$\begin{aligned} 7\csc^2 x - 5\cot^2 x &= 7\csc^2 x - 5\left(\csc^2 x - 1\right) \\ &= 7\csc^2 x - 5\csc^2 x + 5 \\ &= 2\csc^2 x + 5 \end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $1 - \dfrac{\cos^2 x}{1 - \sin x} = -\sin x$

***Solution***

$$\begin{aligned} 1 - \frac{\cos^2 x}{1 - \sin x} &= 1 - \frac{1 - \sin^2 x}{1 - \sin x} \\ &= 1 - \frac{(1 - \sin x)(1 + \sin x)}{1 - \sin x} \\ &= 1 - (1 + \sin x) \\ &= 1 - 1 - \sin x \\ &= -\sin x \end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{1 - \cos x}{1 + \cos x} = \dfrac{\sec x - 1}{\sec x + 1}$

***Solution***

$$\begin{aligned} \frac{1 - \cos x}{1 + \cos x} &= \frac{\frac{1}{\cos x} - \frac{\cos x}{\cos x}}{\frac{1}{\cos x} + \frac{\cos x}{\cos x}} \\ &= \frac{\sec x - 1}{\sec x + 1} \end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{\sec x - 1}{\tan x} = \dfrac{\tan x}{\sec x + 1}$

***Solution***

$$\frac{\sec x-1}{\tan x}=\frac{\sec x-1}{\tan x}\frac{\sec x+1}{\sec x+1}$$

$$=\frac{\sec^2 x-1}{\tan x(\sec x+1)}$$

$$=\frac{\tan^2 x}{\tan x(\sec x+1)}$$

$$=\frac{\tan x}{\sec x+1}$$

### *Exercise*

Prove the following equation is an identity: $\frac{\cos x}{\cos x-\sin x}=\frac{1}{1-\tan x}$

***Solution***

$$\frac{\cos x}{\cos x-\sin x}=\frac{\frac{\cos x}{\cos x}}{\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}}$$

$$=\frac{1}{1-\tan x}$$

### *Exercise*

Prove the following equation is an identity: $(\sec x+\tan x)^2=\frac{1+\sin x}{1-\sin x}$

***Solution***

$$(\sec x+\tan x)^2=\left(\frac{1}{\cos x}+\frac{\sin x}{\cos x}\right)^2$$

$$=\left(\frac{1+\sin x}{\cos x}\right)^2$$

$$=\frac{(1+\sin x)^2}{\cos^2 x}$$

$$=\frac{(1+\sin x)^2}{1-\sin^2 x}$$

$$=\frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)}$$

$$=\frac{1+\sin x}{1-\sin x}$$

***Exercise***

Prove the following equation is an identity: $\frac{\cos x}{1+\tan x}-\frac{\sin x}{1+\cot x}=\cos x-\sin x$

***Solution***

$$\frac{\cos x}{1+\tan x}-\frac{\sin x}{1+\cot x}=\frac{\cos x}{1+\frac{\sin x}{\cos x}}-\frac{\sin x}{1+\frac{\cos x}{\sin x}}$$

$$=\frac{\cos x}{\frac{\cos x+\sin x}{\cos x}}-\frac{\sin x}{\frac{\sin x+\cos x}{\sin x}}$$

$$=\frac{\cos^2 x}{\cos x+\sin x}-\frac{\sin^2 x}{\sin x+\cos x}$$

$$=\frac{\cos^2 x-\sin^2 x}{\cos x+\sin x}$$

$$=\frac{(\cos x+\sin x)(\cos x-\sin x)}{\cos x+\sin x}$$

$$=\cos x-\sin x$$

***Exercise***

Prove the following equation is an identity: $\frac{\cot x+\csc x-1}{\cot x-\csc x+1}=\csc x+\cot x$

***Solution***

$$\frac{\cot x+\csc x-1}{\cot x-\csc x+1}=\frac{\cot x+\csc x-\left(\csc^2 x-\cot^2 x\right)}{\cot x-\csc x+1}$$

$$=\frac{\cot x+\csc x-(\csc x-\cot x)(\csc x+\cot x)}{\cot x-\csc x+1}$$

$$=\frac{(\csc x+\cot x)\big(1-(\csc x-\cot x)\big)}{\cot x-\csc x+1}$$

$$=\frac{(\csc x+\cot x)(1-\csc x+\cot x)}{\cot x-\csc x+1}$$

$$=\csc x+\cot x$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{\tan x+\cot x}{\tan x-\cot x}=\dfrac{1}{\sin^2 x-\cos^2 x}$

***Solution***

$$\frac{\tan x+\cot x}{\tan x-\cot x}=\frac{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}}{\frac{\sin x}{\cos x}-\frac{\cos x}{\sin x}}$$

$$=\frac{\frac{\sin^2 x+\cos^2 x}{\cos x\sin x}}{\frac{\sin^2 x-\cos^2 x}{\cos x\sin x}}$$

$$=\frac{1}{\sin^2 x-\cos^2 x}$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{1-\cot^2 x}{1+\cot^2 x}+1=2\sin^2 x$

***Solution***

$$\frac{1-\cot^2 x}{1+\cot^2 x}+1=\frac{1-\cot^2 x+1+\cot^2 x}{1+\cot^2 x}$$

$$=\frac{2}{\csc^2 x}$$

$$=2\sin^2 x$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{1+\cos x}{1-\cos x}-\dfrac{1-\cos x}{1+\cos x}=4\cot x\csc x$

***Solution***

$$\frac{1+\cos x}{1-\cos x}-\frac{1-\cos x}{1+\cos x}=\frac{(1+\cos x)^2-(1-\cos x)^2}{1-\cos^2 x}$$

$$=\frac{(1+\cos x+1-\cos x)(1+\cos x-1+\cos x)}{\sin^2 x} \qquad a^2-b^2=(a-b)(a+b)$$

$$=\frac{(2)(2\cos x)}{\sin^2 x}$$

$$=4\frac{\cos x}{\sin x}\frac{1}{\sin x}$$

$$=4\cot x\csc x$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{\sin^3 x-\cos^3 x}{\sin x-\cos x}=1+\sin x\cos x$

***Solution***

$$\frac{\sin^3 x-\cos^3 x}{\sin x-\cos x}=\frac{(\sin x-\cos x)\left(\sin^2 x+\sin x\cos x+\cos^2 x\right)}{\sin x-\cos x}$$

$$a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$$

$$=1+\sin x\cos x$$

### *Exercise*

Prove the following equation is an identity: $1+\sec^2 x\sin^2 x=\sec^2 x$

***Solution***

$$\begin{aligned}1+\sec^2 x\sin^2 x&=1+\frac{1}{\cos^2 x}\sin^2 x\\&=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}\\&=\frac{1}{\cos^2 x}\\&=\sec^2 x\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{1+\csc x}{\sec x}=\cos x+\cot x$

***Solution***

$$\begin{aligned}\frac{1+\csc x}{\sec x}&=\frac{1}{\sec x}+\frac{\csc x}{\sec x}\\&=\cos x+\frac{\frac{1}{\sin x}}{\frac{1}{\cos x}}\\&=\cos x+\frac{\cos x}{\sin x}\\&=\cos x+\cot x\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\tan^2 x = \sec^2 x - \sin^2 x - \cos^2 x$

***Solution***

$$\begin{aligned}
\sec^2 x - \sin^2 x - \cos^2 x &= \frac{1}{\cos^2 x} - \left(\sin^2 x + \cos^2 x\right) \\
&= \frac{1}{\cos^2 x} - 1 \\
&= \frac{1-\cos^2 x}{\cos^2 x} \\
&= \frac{\sin^2 x}{\cos^2 x} \\
&= \tan^2 x
\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\frac{\sin x}{1-\cos x} + \frac{\sin x}{1+\cos x} = 2\csc x$

***Solution***

$$\begin{aligned}
\frac{\sin x}{1-\cos x} + \frac{\sin x}{1+\cos x} &= \sin x\left(\frac{1}{1-\cos x} + \frac{1}{1+\cos x}\right) \\
&= \sin x\left(\frac{1+\cos x + 1 - \cos x}{1-\cos^2 x}\right) \\
&= \sin x\left(\frac{2}{\sin^2 x}\right) \\
&= \frac{2}{\sin x} \\
&= 2\csc x
\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\cos^2(\alpha-\beta) - \cos^2(\alpha+\beta) = \sin^2(\alpha+\beta) - \sin^2(\alpha-\beta)$

***Solution***

$$\begin{aligned}
\cos^2(\alpha-\beta) - \cos^2(\alpha+\beta) &= 1 - \sin^2(\alpha-\beta) - \left[1 - \sin^2(\alpha+\beta)\right] \\
&= 1 - \sin^2(\alpha-\beta) - 1 + \sin^2(\alpha+\beta) \\
&= \sin^2(\alpha+\beta) - \sin^2(\alpha-\beta)
\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\tan x \csc x - \sec^2 x \cos x = 0$

***Solution***

$$\tan x \csc x - \sec^2 x \cos x = \frac{\sin x}{\cos x}\frac{1}{\sin x} - \frac{1}{\cos^2 x}\cos x$$

$$= \frac{1}{\cos x} - \frac{1}{\cos x}$$

$$= 0$$

### *Exercise*

Prove the following equation is an identity: $(1+\tan x)^2 - 2\tan x = \frac{1}{(1-\sin x)(1+\sin x)}$

***Solution***

$$(1+\tan x)^2 - 2\tan x = 1 + 2\tan x + \tan^2 x - 2\tan x$$

$$= 1 + \tan^2 x$$

$$= \sec^2 x$$

$$= \frac{1}{\cos^2 x}$$

$$= \frac{1}{1-\sin^2 x}$$

$$= \frac{1}{(1-\sin x)(1+\sin x)}$$

### *Exercise*

Prove the following equation is an identity: $\frac{3\csc^2 x - 5\csc x - 28}{\csc x - 4} = \frac{3}{\sin x} + 7$

***Solution***

$$\frac{3\csc^2 x - 5\csc x - 28}{\csc x - 4} = \frac{(3\csc x + 7)(\csc x - 4)}{\csc x - 4}$$

$$= 3\csc x + 7$$

$$= \frac{3}{\sin x} + 7$$

***Exercise***

Prove the following equation is an identity: $\left(\sec^2 x - 1\right)\left(\sec^2 x + 1\right) = \tan^4 x + 2\tan^2 x$

***Solution***

$$\begin{aligned}\left(\sec^2 x - 1\right)\left(\sec^2 x + 1\right) &= \sec^4 x - 1 \qquad\qquad (a-b)(a+b) = a^2 - b^2 \quad a = \sec^2 x\\ &= \left(\sec^2 x\right)^2 - 1\\ &= \left(1 + \tan^2 x\right)^2 - 1\\ &= 1 + 2\tan^2 x + \tan^4 x - 1\\ &= \tan^4 x + 2\tan^2 x\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\dfrac{\csc x}{\cot x} - \dfrac{\cot x}{\csc x} = \dfrac{\sin x}{\cot x}$

***Solution***

$$\begin{aligned}\frac{\csc x}{\cot x} - \frac{\cot x}{\csc x} &= \frac{\csc^2 x - \cot^2 x}{\cot x \csc x}\\ &= \frac{\csc^2 x - \left(\csc^2 x - 1\right)}{\cot x \csc x}\\ &= \frac{\csc^2 x - \csc^2 x + 1}{\cot x \csc x}\\ &= \frac{1}{\cot x \frac{1}{\sin x}}\\ &= \frac{\sin x}{\cot x}\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\dfrac{1 - \cos^2 x}{1 + \cos x} = \dfrac{\sec x - 1}{\sec x}$

***Solution***

$$\begin{aligned}\frac{1 - \cos^2 x}{1 + \cos x} &= \frac{(1 - \cos x)(1 + \cos x)}{1 + \cos x}\\ &= 1 - \cos x\end{aligned}$$

$$=1-\frac{1}{\sec x}$$

$$=\frac{\sec x-1}{\sec x}$$

### Exercise

Prove the following equation is an identity: $\frac{\cos x}{1+\cos x}=\frac{\sec x-1}{\tan^2 x}$

**Solution**

$$\frac{\cos x}{1+\cos x}=\frac{\cos x}{1+\cos x}\frac{1-\cos x}{1-\cos x}$$

$$=\frac{\cos x-\cos^2 x}{\cos^2 x-1}$$

$$=\frac{\cos x-\cos^2 x}{\sin^2 x}\frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}}$$

$$=\frac{\frac{1}{\cos x}-1}{\frac{\sin^2 x}{\cos^2 x}}$$

$$=\frac{\sec x-1}{\tan^2 x}$$

### Exercise

Prove the following equation is an identity: $\frac{1-2\sin^2 x}{1+2\sin x\cos x}=\frac{\cos x-\sin x}{\cos x+\sin x}$

**Solution**

$$\frac{1-2\sin^2 x}{1+2\sin x\cos x}=\frac{\cos^2 x+\sin^2 x-2\sin^2 x}{\cos^2 x+\sin^2 x+2\sin x\cos x}$$

$$=\frac{\cos^2 x-\sin^2 x}{(\cos x+\sin x)^2}$$

$$=\frac{(\cos x-\sin x)(\cos x+\sin x)}{(\cos x+\sin x)^2}$$

$$=\frac{\cos x-\sin x}{\cos x+\sin x}$$

***Exercise***

Prove the following equation is an identity: $(\cos x - \sin x)^2 + (\cos x + \sin x)^2 = 2$

***Solution***

$$\begin{aligned}(\cos x - \sin x)^2 + (\cos x + \sin x)^2 &= \cos^2 x - 2\sin x\cos x + \sin^2 x + \cos^2 x + 2\sin x\cos x + \sin^2 x \\ &= \cos^2 x + \sin^2 x + \cos^2 x + \sin^2 x \\ &= 1 + 1 \\ &= 2\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\dfrac{\sin x}{1+\cos x} + \dfrac{1+\cos x}{\sin x} = 2\csc x$

***Solution***

$$\begin{aligned}\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} &= \frac{\sin x\sin x + (1+\cos x)(1+\cos x)}{(1+\cos x)\sin x} \\ &= \frac{\sin^2 x + 1 + 2\cos x + \cos^2 x}{(1+\cos x)\sin x} \\ &= \frac{1+1+2\cos x}{(1+\cos x)\sin x} \\ &= \frac{2+2\cos x}{(1+\cos x)\sin x} \\ &= \frac{2(1+\cos x)}{(1+\cos x)\sin x} \\ &= \frac{2}{\sin x} \\ &= 2\csc x\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\dfrac{\sin x + \tan x}{\cot x + \csc x} = \sin x \tan x$

***Solution***

$$\begin{aligned}\frac{\sin x + \tan x}{\cot x + \csc x} &= \frac{\sin x + \tan x}{\frac{1}{\tan x} + \frac{1}{\sin x}} \\ &= \frac{\sin x + \tan x}{\frac{\sin x + \tan x}{\tan x \sin x}} \\ &= (\sin x + \tan x)\frac{\tan x \sin x}{\sin x + \tan x} \\ &= \tan x \sin x\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\csc^2 x \sec^2 x = \sec^2 x + \csc^2 x$

***Solution***

$$\begin{aligned}
\csc^2 x \sec^2 x &= \frac{1}{\sin^2 x}\frac{1}{\cos^2 x} \\
&= \frac{1}{\sin^2 x \cos^2 x} \\
&= \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} \\
&= \frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x} \\
&= \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \\
&= \sec^2 x + \csc^2 x
\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\cos^2 x + 1 = 2\cos^2 x + \sin^2 x$

***Solution***

$$\begin{aligned}
\cos^2 x + 1 &= \cos^2 x + \cos^2 x + \sin^2 x \\
&= 2\cos^2 x + \sin^2 x
\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $1 - \dfrac{\cos^2 x}{1 + \sin x} = \sin x$

***Solution***

$$\begin{aligned}
1 - \frac{\cos^2 x}{1 + \sin x} &= 1 - \frac{1 - \sin^2 x}{1 + \sin x} \\
&= 1 - \frac{(1 - \sin x)(1 + \sin x)}{1 + \sin x} \\
&= 1 - (1 - \sin x) \\
&= 1 - 1 + \sin x \\
&= \sin x
\end{aligned}$$

### Exercise

Prove the following equation is an identity: $\cot^2 x = (\csc x - 1)(\csc x + 1)$

***Solution***

$$\cot^2 x = \csc^2 x - 1$$
$$= (\csc x - 1)(\csc x + 1)$$

### Exercise

Prove the following equation is an identity: $\dfrac{\sec x - 1}{\tan x} = \dfrac{\tan x}{\sec x + 1}$

***Solution***

$$\frac{\sec x - 1}{\tan x} = \frac{\sec x - 1}{\tan x}\frac{\sec x + 1}{\sec x + 1}$$
$$= \frac{\sec^2 x - 1}{\tan x(\sec x + 1)}$$
$$= \frac{\tan^2 x}{\tan x(\sec x + 1)}$$
$$= \frac{\tan x}{\sec x + 1}$$

### Exercise

Prove the following equation is an identity: $10\csc^2 x - 6\cot^2 x = 4\csc^2 x + 6$

***Solution***

$$10\csc^2 x - 6\cot^2 x = 10\csc^2 x - 6\left(\csc^2 x - 1\right)$$
$$= 10\csc^2 x - 6\csc^2 x + 6$$
$$= 4\csc^2 x + 6$$

### Exercise

Prove the following equation is an identity: $\dfrac{\csc x + \cot x}{\tan x + \sin x} = \csc x \cot x$

***Solution***

$$\frac{\csc x + \cot x}{\tan x + \sin x} = \frac{\csc x + \cot x}{\frac{1}{\cot x} + \frac{1}{\csc x}}$$

$$= \frac{\csc x + \cot x}{\frac{\csc x + \cot x}{\cot x \csc x}}$$

$$= \csc x + \cot x \frac{\cot x \csc x}{\csc x + \cot x}$$

$$= \cot x \csc x$$

### *Exercise*

Prove the following equation is an identity: $\frac{1-\sec x}{\tan x} + \frac{\tan x}{1-\sec x} = -2\csc x$

***Solution***

$$\frac{1-\sec x}{\tan x} + \frac{\tan x}{1-\sec x} = \frac{(1-\sec x)(1-\sec x) + \tan^2 x}{\tan x(1-\sec x)}$$

$$= \frac{(1-\sec x)^2 + \sec^2 x - 1}{\tan x(1-\sec x)}$$

$$= \frac{(1-\sec x)^2 + (\sec x + 1)(\sec x - 1)}{\tan x(1-\sec x)}$$

$$= \frac{(1-\sec x)^2 - (\sec x + 1)(1 - \sec x)}{\tan x(1-\sec x)}$$

$$= \frac{(1-\sec x)\left[(1-\sec x) - (\sec x + 1)\right]}{\tan x(1-\sec x)}$$

$$= \frac{1 - \sec x - \sec x - 1}{\tan x}$$

$$= \frac{-2\sec x}{\tan x}$$

$$= -2\frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}}$$

$$= -2\frac{1}{\sin x}$$

$$= -2\csc x$$

***Exercise***

Prove the following equation is an identity: $\csc x - \sin x = \cos x \cot x$

***Solution***

$$\begin{aligned}\csc x - \sin x &= \frac{1}{\sin x} - \sin x \\ &= \frac{1 - \sin^2 x}{\sin x} \\ &= \frac{\cos^2 x}{\sin x} \\ &= \cos x \frac{\cos x}{\sin x} \\ &= \cos x \cot x\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\frac{\tan x + \sec x}{\sec x} - \frac{\tan x + \sec x}{\tan x} = -\cos x \cot x$

***Solution***

$$\begin{aligned}\frac{\tan x + \sec x}{\sec x} - \frac{\tan x + \sec x}{\tan x} &= \frac{(\tan x + \sec x)\tan x - \sec x(\tan x + \sec x)}{\sec x \tan x} \\ &= \frac{\tan^2 x + \sec x \tan x - \sec x \tan x - \sec^2 x}{\sec x \tan x} \\ &= \frac{\tan^2 x - \sec^2 x}{\sec x \tan x} \qquad\qquad 1 + \tan^2 \alpha = \sec^2 \alpha \\ &= \frac{-1}{\sec x \tan x} \\ &= -\frac{1}{\sec x}\frac{1}{\tan x} \\ &= -\cos x \cot x\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\cot^3 x = \cot x\left(\csc^2 x - 1\right)$

***Solution***

$$\begin{aligned}\cot^3 x &= \cot x \cot^2 x \\ &= \cot x\left(\csc^2 x - 1\right)\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\dfrac{\cot^2 x}{\csc x - 1} = \dfrac{1+\sin x}{\sin x}$

***Solution***

$$\begin{aligned}\frac{\cot^2 x}{\csc x - 1} &= \frac{\csc^2 x - 1}{\csc x - 1} \\ &= \frac{(\csc x - 1)(\csc x + 1)}{\csc x - 1} \\ &= \csc x + 1 \\ &= \frac{1}{\sin x} + 1 \\ &= \frac{1+\sin x}{\sin x}\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\cot^2 x + \csc^2 x = 2\csc^2 x - 1$

***Solution***

$$\begin{aligned}\cot^2 x + \csc^2 x &= \csc^2 x - 1 + \csc^2 x \\ &= 2\csc^2 x - 1\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\dfrac{\cot^2 x}{1+\csc x} = \csc x - 1$

***Solution***

$$\begin{aligned}\frac{\cot^2 x}{1+\csc x} &= \frac{\csc^2 x - 1}{1+\csc x} \\ &= \frac{(\csc x - 1)(\csc x + 1)}{1+\csc x} \\ &= \csc x - 1\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\sec^4 x - \tan^4 x = \sec^2 x + \tan^2 x$

***Solution***

$$\begin{aligned}\sec^4 x - \tan^4 x &= \left(\sec^2 x + \tan^2 x\right)\left(\sec^2 x - \tan^2 x\right) \qquad a^2 - b^2 = (a-b)(a+b) \\ &= \left(\sec^2 x + \tan^2 x\right)(1) \\ &= \sec^2 x + \tan^2 x\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\frac{\cos x}{1+\sin x}+\frac{1+\sin x}{\cos x}=2\sec x$

***Solution***

$$\begin{aligned}
\frac{\cos x}{1+\sin x}+\frac{1+\sin x}{\cos x} &= \frac{\cos^2 x+(1+\sin x)^2}{(1+\sin x)\cos x} \\
&= \frac{\cos^2 x+1+2\sin x+\sin^2 x}{(1+\sin x)\cos x} \\
&= \frac{2+2\sin x}{(1+\sin x)\cos x} \\
&= \frac{2(1+\sin x)}{(1+\sin x)\cos x} \\
&= \frac{2}{\cos x} \\
&= 2\sec x
\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\frac{\sin x+\cos x}{\sin x-\cos x}=\frac{1+2\sin x\cos x}{2\sin^2 x-1}$

***Solution***

$$\begin{aligned}
\frac{\sin x+\cos x}{\sin x-\cos x} &= \frac{\sin x+\cos x}{\sin x-\cos x}\,\frac{\sin x+\cos x}{\sin x+\cos x} \\
&= \frac{\sin^2 x+2\sin x\cos x+\cos^2 x}{\sin^2 x-\cos^2 x} \\
&= \frac{1+2\sin x\cos x}{\sin^2 x-\left(1-\sin^2 x\right)} \\
&= \frac{1+2\sin x\cos x}{\sin^2 x-1+\sin^2 x} \\
&= \frac{1+2\sin x\cos x}{2\sin^2 x-1}
\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\frac{\csc x - 1}{\csc x + 1} = \frac{\cot^2 x}{\csc^2 x + 2\csc x + 1}$

***Solution***

$$\begin{aligned}
\frac{\csc x - 1}{\csc x + 1} &= \frac{\csc x - 1}{\csc x + 1}\frac{\csc x + 1}{\csc x + 1} \\
&= \frac{\csc^2 x - 1}{\csc^2 x + 2\csc x + 1} \\
&= \frac{\cot^2 x}{\csc^2 x + 2\csc x + 1}
\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\csc^4 x - \cot^4 x = \csc^2 x + \cot^2 x$

***Solution***

$$\begin{aligned}
\csc^4 x - \cot^4 x &= \left(\csc^2 x + \cot^2 x\right)\left(\csc^2 x - \cot^2 x\right) \\
&= \left(\csc^2 x + \cot^2 x\right)(1) \\
&= \csc^2 x + \cot^2 x
\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\tan\left(\frac{\pi}{4} + x\right) = \cot\left(\frac{\pi}{4} - x\right)$

***Solution***

$$\begin{aligned}
\tan\left(\frac{\pi}{4} + x\right) &= \cot\left[\frac{\pi}{2} - \left(\frac{\pi}{4} + x\right)\right] \\
&= \cot\left[\frac{\pi}{2} - \frac{\pi}{4} - x\right] \\
&= \cot\left(\frac{\pi}{4} - x\right)
\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\frac{\sin\theta}{1+\sin\theta}-\frac{\sin\theta}{1-\sin\theta}=-2\tan^2\theta$

***Solution***

$$\begin{aligned}\frac{\sin\theta}{1+\sin\theta}-\frac{\sin\theta}{1-\sin\theta}&=\sin\theta\left[\frac{1-\sin\theta-(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}\right]\\&=\sin\theta\left[\frac{1-\sin\theta-1-\sin\theta}{1-\sin^2\theta}\right]\\&=\sin\theta\left(\frac{-2\sin\theta}{\cos^2\theta}\right)\\&=-2\frac{\sin^2\theta}{\cos^2\theta}\\&=-2\tan^2\theta\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\csc^2 x-\cos^2 x\csc^2 x=1$

***Solution***

$$\begin{aligned}\csc^2 x-\cos^2 x\csc^2 x&=\csc^2 x\left(1-\cos^2 x\right)\\&=\frac{1}{\sin^2 x}\left(\sin^2 x\right)\\&=1\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $1-2\sin^2 x=2\cos^2 x-1$

***Solution***

$$\begin{aligned}1-2\sin^2 x&=1-2\left(1-\cos^2 x\right)\\&=1-2+2\cos^2 x\\&=2\cos^2 x-1\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\csc^2 x - \cos x \sec x = \cot^2 x$

***Solution***

$$\begin{aligned}\csc^2 x - \cos x \sec x &= \frac{1}{\sin^2 x} - \cos x \frac{1}{\cos x} \\ &= \frac{1}{\sin^2 x} - 1 \\ &= \frac{1-\sin^2 x}{\sin^2 x} \\ &= \frac{\cos^2 x}{\sin^2 x} \\ &= \cot^2 x\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $(\sec x - \tan x)(\sec x + \tan x) = 1$

***Solution***

$$\begin{aligned}(\sec x - \tan x)(\sec x + \tan x) &= \sec^2 x - \tan^2 x \\ &= 1 + \tan^2 x - \tan^2 x \\ &= 1\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\left(1+\tan^2 x\right)\left(1-\sin^2 x\right) = 1$

***Solution***

$$\begin{aligned}\left(1+\tan^2 x\right)\left(1-\sin^2 x\right) &= \sec^2 x \cos^2 x \\ &= \frac{1}{\cos^2 x}\cos^2 x \\ &= 1\end{aligned}$$

### Exercise

Prove the identity $\cos(A+B)+\cos(A-B)=2\cos A\cos B$

**Solution**

$$\begin{aligned}\cos(A+B)+\cos(A-B) &= \cos A\cos B-\sin A\sin B+\cos A\cos B+\sin A\sin B\\ &= \cos A\cos B+\cos A\cos B\\ &= 2\cos A\cos B\end{aligned}$$

### Exercise

Prove the identity $\sec(A+B)=\dfrac{\cos(A-B)}{\cos^2 A-\sin^2 B}$

**Solution**

$$\begin{aligned}\sec(A+B) &= \frac{1}{\cos(A+B)}\\ &= \frac{1}{\cos A\cos B-\sin A\sin B}\\ &= \frac{1}{\cos A\cos B-\sin A\sin B}\frac{\cos(A-B)}{\cos(A-B)}\\ &= \frac{1}{\cos A\cos B-\sin A\sin B}\frac{\cos(A-B)}{\cos A\cos B+\sin A\sin B}\\ &= \frac{\cos(A-B)}{\cos^2 A\cos^2 B-\sin^2 A\sin^2 B}\\ &= \frac{\cos(A-B)}{\cos^2 A(1-\sin^2 B)-(1-\cos^2 A)\sin^2 B}\\ &= \frac{\cos(A-B)}{\cos^2 A-\cos^2 A\sin^2 B-\sin^2 B+\cos^2 A\sin^2 B}\\ &= \frac{\cos(A-B)}{\cos^2 A-\sin^2 B}\end{aligned}$$

### *Exercise*

Prove the identity $\dfrac{\cos 4\alpha}{\sin\alpha}-\dfrac{\sin 4\alpha}{\cos\alpha}=\dfrac{\cos 5\alpha}{\sin\alpha\cos\alpha}$

***Solution***

$$\begin{aligned}\frac{\cos 4\alpha}{\sin\alpha}-\frac{\sin 4\alpha}{\cos\alpha}&=\frac{\cos 4\alpha\cos\alpha-\sin 4\alpha\sin\alpha}{\sin\alpha\cos\alpha}\\&=\frac{\cos(4\alpha+\alpha)}{\sin\alpha\cos\alpha}\\&=\frac{\cos 5\alpha}{\sin\alpha\cos\alpha}\end{aligned}$$

### *Exercise*

Show that $\sin\left(x-\frac{\pi}{2}\right)=-\cos x$

***Solution***

$$\begin{aligned}\sin\left(x-\frac{\pi}{2}\right)&=\sin x\cos\frac{\pi}{2}-\cos x\sin\frac{\pi}{2}\\&=\sin x\cdot(0)-\cos x\cdot(1)\\&=-\cos x\end{aligned}$$

### *Exercise*

Prove the identity $\sin\left(\frac{\pi}{4}+x\right)+\sin\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$

***Solution***

$$\begin{aligned}\sin\left(\frac{\pi}{4}+x\right)+\sin\left(\frac{\pi}{4}-x\right)&=\sin\frac{\pi}{4}\cos x+\sin x\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\cos x-\sin x\cos\frac{\pi}{4}\\&=\sin\frac{\pi}{4}\cos x+\sin\frac{\pi}{4}\cos x\\&=2\sin\frac{\pi}{4}\cos x\\&=2\frac{\sqrt{2}}{2}\cos x\\&=\sqrt{2}\cos x\end{aligned}$$

***Exercise***

Prove the identity $\dfrac{\sin(A-B)}{\cos A\cos B} = \tan A - \tan B$

***Solution***

$$\begin{aligned}\frac{\sin(A-B)}{\cos A\cos B} &= \frac{\sin A\cos B - \sin B\cos A}{\cos A\cos B} \\ &= \frac{\sin A\cos B}{\cos A\cos B} - \frac{\sin B\cos A}{\cos A\cos B} \\ &= \frac{\sin A}{\cos A} - \frac{\sin B}{\cos B} \\ &= \tan A - \tan B\end{aligned}$$

***Exercise***

Write the expression as a single trigonometric function $\sin 8x\cos x - \cos 8x\sin x$

***Solution***

$$\begin{aligned}\sin 8x\cos x - \cos 8x\sin x &= \sin(8x - x) \\ &= \sin 7x\end{aligned}$$

### *Exercise*

If $\sin A = \frac{4}{5}$ with $A$ in QII, and $\cos B = -\frac{5}{13}$ with $B$ in QIII, find $\sin(A+B)$, $\cos(A+B)$, and $\tan(A+B)$

***Solution***

| | |
|---|---|
| $\cos A = -\frac{3}{5}$ $\quad$ $\tan A = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$ | $\sin B = -\frac{12}{13}$ $\quad$ $\tan B = \frac{-\frac{12}{13}}{-\frac{5}{13}} = \frac{12}{5}$ |
| $\sin(A+B) = \sin A\cos B + \sin B\cos A$ <br> $= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) + \left(-\frac{12}{13}\right)\left(-\frac{3}{5}\right)$ <br> $= -\frac{20}{65} + \frac{36}{65}$ <br> $= \frac{16}{65}$ | $\cos(A+B) = \cos A\cos B - \sin A\sin B$ <br> $= \left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)$ <br> $= \frac{15}{65} + \frac{48}{65}$ <br> $= \frac{63}{65}$ |
| $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$ <br> $= \frac{\frac{16}{65}}{\frac{63}{65}}$ <br> $= \frac{16}{63}$ | $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$ <br> $= \frac{-\frac{4}{3} + \frac{12}{5}}{1 - \left(-\frac{4}{3}\right)\left(\frac{12}{5}\right)}$ <br> $= \frac{\frac{-20+36}{15}}{1 + \frac{48}{15}}$ <br> $= \frac{\frac{16}{15}}{\frac{63}{15}}$ <br> $= \frac{16}{63}$ |

### *Exercise*

If $\sin A = \frac{1}{\sqrt{5}}$ with $A$ in QI, and $\tan B = \frac{3}{4}$ with $B$ in QI, find $\sin(A+B)$, $\cos(A+B)$, and $\tan(A+B)$

***Solution***

| | |
|---|---|
| $\cos A = \sqrt{1-\sin^2 A} \quad A \in \text{ QI}$<br>$\cos A = \sqrt{1-\frac{1}{5}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$ | $\sin B = \frac{3}{5}$<br>$\cos B = \frac{4}{5}$ |
| $\sin(A+B) = \sin A\cos B + \sin B\cos A$<br>$= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{4}{5}\right) + \left(\frac{3}{5}\right)\left(\frac{2}{\sqrt{5}}\right)$<br>$= \frac{4}{5\sqrt{5}} + \frac{6}{5\sqrt{5}}$<br>$= \frac{10}{5\sqrt{5}}$<br>$= \frac{2}{\sqrt{5}}$ | $\cos(A+B) = \cos A\cos B - \sin A\sin B$<br>$= \left(\frac{2}{\sqrt{5}}\right)\left(\frac{4}{5}\right) - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{3}{5}\right)$<br>$= \frac{8}{5\sqrt{5}} - \frac{3}{5\sqrt{5}}$<br>$= \frac{5}{5\sqrt{5}}$<br>$= \frac{1}{\sqrt{5}}$ |
| $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$<br>$= \frac{\frac{2}{\sqrt{5}}}{\frac{1}{\sqrt{5}}}$<br>$= 2$ | |

### *Exercise*

If $\sec A = \sqrt{5}$ with $A$ in QI, and $\sec B = \sqrt{10}$ with $B$ in QI, find $\sec(A+B)$

***Solution***

$$\sec(A+B) = \frac{1}{\cos(A+B)}$$

$$\sec A = \sqrt{5} \Rightarrow \cos A = \frac{1}{\sqrt{5}} \qquad \sin A = \frac{2}{\sqrt{5}}$$

$$\sec B = \sqrt{10} \Rightarrow \cos B = \frac{1}{\sqrt{10}} \qquad \sin B = \sqrt{1-\frac{1}{10}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{5}}$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$= \frac{1}{\sqrt{5}}\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{5}}\frac{3}{\sqrt{10}}$$

$$= \frac{1}{\sqrt{50}} - \frac{6}{\sqrt{50}}$$

$$= \frac{5}{\sqrt{50}}$$

$$= \frac{5}{5\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}$$

$$\boxed{\sec(A+B) = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}}$$

***Exercise***

Prove the following equation is an identity: $\sin(x-y) - \sin(y-x) = 2\sin x \cos y - 2\cos x \sin y$

***Solution***

$$\begin{aligned}\sin(x-y) - \sin(y-x) &= \sin x \cos y - \sin y \cos x - (\sin y \cos x - \sin x \cos y) \\ &= \sin x \cos y - \sin y \cos x - \sin y \cos x + \sin x \cos y \\ &= 2\sin x \cos y - 2\sin y \cos x\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\cos(x-y) + \cos(y-x) = 2\cos x \cos y + 2\sin x \sin y$

***Solution***

$$\begin{aligned}\cos(x-y) + \cos(y-x) &= \cos x \cos y + \sin x \sin y + \cos y \cos x + \sin y \sin x \\ &= 2\cos x \cos y + 2\sin x \sin y\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\tan(x+y)\tan(x-y) = \dfrac{\tan^2 x - \tan^2 y}{1 - \tan^2 x \tan^2 y}$

***Solution***

$$\begin{aligned}\tan(x+y)\tan(x-y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} \frac{\tan x - \tan y}{1 + \tan x \tan y} \qquad (a+b)(a-b) = a^2 - b^2 \\ &= \frac{\tan^2 x + \tan^2 y}{1 - \tan x^2 \tan^2 y}\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\dfrac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)}=\dfrac{1-\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$

***Solution***

$$\frac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)}=\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\cos\beta+\sin\beta\cos\alpha}$$

$$=\frac{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\beta\cos\alpha}{\cos\alpha\cos\beta}}$$

$$=\frac{1+\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$$

***Exercise***

Prove the following equation is an identity: $\sec(x+y)=\dfrac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\sin^2 y}$

***Solution***

$$\sec(x+y)=\frac{1}{\cos(x+y)}\frac{\cos(x-y)}{\cos(x-y)}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{(\cos x\cos y-\sin x\sin y)(\cos x\cos y+\sin x\sin y)}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x\left(1-\sin^2 y\right)-\left(1-\cos^2 x\right)\sin^2 y}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\cos^2 x\sin^2 y-\sin^2 y+\cos^2 x\sin^2 y}$$

$$=\frac{\cos x\cos y+\sin x\sin y}{\cos^2 x-\sin^2 y}$$

### *Exercise*

Prove the following equation is an identity: $\csc(x-y) = \dfrac{\sin x\cos y + \cos x\sin y}{\sin^2 x - \sin^2 y}$

### ***Solution***

$$\csc(x-y) = \frac{1}{\sin(x-y)}\frac{\sin x\cos y+\cos x\sin y}{\sin x\cos y+\cos x\sin y}$$

$$= \frac{\sin x\cos y + \cos x\sin y}{(\sin x\cos y - \cos x\sin y)(\sin x\cos y + \cos x\sin y)}$$

$$= \frac{\sin x\cos y + \cos x\sin y}{\sin^2 x\cos^2 y - \cos^2 x\sin^2 y}$$

$$= \frac{\sin x\cos y + \cos x\sin y}{\sin^2 x\left(1-\sin^2 y\right) - \left(1-\sin^2 x\right)\sin^2 y}$$

$$= \frac{\sin x\cos y + \cos x\sin y}{\sin^2 x - \sin^2 x\sin^2 y - \sin^2 y + \sin^2 x\sin^2 y}$$

$$= \frac{\sin x\cos y + \cos x\sin y}{\sin^2 x - \sin^2 y}$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{\cos(x+y)}{\cos(x-y)} = \dfrac{\cot y - \tan x}{\cot y + \tan x}$

### ***Solution***

$$\frac{\cos(x+y)}{\cos(x-y)} = \frac{\cos x\cos y - \sin x\sin y}{\cos x\cos y + \sin x\sin y}$$

$$= \frac{\frac{\cos x\cos y}{\cos x\sin y} - \frac{\sin x\sin y}{\cos x\sin y}}{\frac{\cos x\cos y}{\cos x\sin y} + \frac{\sin x\sin y}{\cos x\sin y}}$$

$$= \frac{\cot y - \tan x}{\cot y + \tan x}$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{\cot y + \cot x}{\cot y - \cot x}$

### ***Solution***

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\sin x\cos y + \sin y\cos x}{\sin x\cos y - \sin y\cos x}$$

$$= \frac{\frac{\sin x\cos y}{\sin x\sin y} + \frac{\sin y\cos x}{\sin x\sin y}}{\frac{\sin x\cos y}{\sin x\sin y} - \frac{\sin y\cos x}{\sin x\sin y}}$$

$$= \frac{\cot y + \cot x}{\cot y - \cot x}$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{\cos(x+y)}{\cos(x-y)} = \dfrac{\cot y - \tan x}{\cot y + \tan x}$

***Solution***

$$\frac{\cos(x+y)}{\cos(x-y)} = \frac{\cos x\cos y - \sin x\sin y}{\cos x\cos y + \sin x\sin y}$$

$$= \frac{\frac{\cos x\cos y}{\cos x\sin y} - \frac{\sin x\sin y}{\cos x\sin y}}{\frac{\cos x\cos y}{\cos x\sin y} + \frac{\sin x\sin y}{\cos x\sin y}}$$

$$= \frac{\cot y - \tan x}{\cot y + \tan x}$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{\sin(x-y)}{\sin x\cos y} = 1 - \cot x\tan y$

***Solution***

$$\frac{\sin(x-y)}{\sin x\cos y} = \frac{\sin x\cos y - \cos x\sin y}{\sin x\cos y}$$

$$= \frac{\sin x\cos y}{\sin x\cos y} - \frac{\cos x\sin y}{\sin x\cos y}$$

$$= 1 - \cot x\tan y$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{\sin(x-y)}{\sin x \sin y} = \cot y - \cot x$

***Solution***

$$\begin{aligned}
\frac{\sin(x-y)}{\sin x \sin y} &= \frac{\sin x \cos y - \cos x \sin y}{\sin x \sin y} \\
&= \frac{\sin x \cos y}{\sin x \sin y} - \frac{\cos x \sin y}{\sin x \sin y} \\
&= \frac{\cos y}{\sin y} - \frac{\cos x}{\sin x} \\
&= \cot y - \cot x
\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{\cos(x+y)}{\cos x \sin y} = \cot y - \tan x$

***Solution***

$$\begin{aligned}
\frac{\cos(x+y)}{\cos x \sin y} &= \frac{\cos x \cos y - \sin x \sin y}{\cos x \sin y} \\
&= \frac{\cos x \cos y}{\cos x \sin y} - \frac{\sin x \sin y}{\cos x \sin y} \\
&= \frac{\cos y}{\sin y} - \frac{\sin x}{\cos x} \\
&= \cot y - \tan x
\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\tan(x+y) + \tan(x-y) = \dfrac{2\tan x}{\cos^2 y\left(1 - \tan^2 x \tan^2 y\right)}$

***Solution***

$$\begin{aligned}
\tan(x+y) + \tan(x-y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} + \frac{\tan x - \tan y}{1 + \tan x \tan y} \\
&= \frac{(\tan x + \tan y)(1 + \tan x \tan y) + (\tan x - \tan y)(1 - \tan x \tan y)}{(1 - \tan x \tan y)(1 + \tan x \tan y)} \\
&= \frac{\tan x + \tan^2 x \tan y + \tan y + \tan x \tan^2 y + \tan x - \tan^2 x \tan y - \tan y + \tan x \tan^2 y}{\left(1 - \tan^2 x \tan^2 y\right)}
\end{aligned}$$

$$= \frac{2\tan x + 2\tan x \tan^2 y}{\left(1 - \tan^2 x \tan^2 y\right)}$$

$$= \frac{2\tan x\left(1 + \tan^2 y\right)}{\left(1 - \tan^2 x \tan^2 y\right)}$$

$$= \frac{2\tan x \sec^2 y}{\left(1 - \tan^2 x \tan^2 y\right)}$$

$$= \frac{2\tan x}{\cos^2 y\left(1 - \tan^2 x \tan^2 y\right)}$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{\sin(x+y)}{\cos(x-y)} = \dfrac{1+\cot x \tan y}{\cot x + \tan y}$

***Solution***

$$\frac{\sin(x+y)}{\cos(x-y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y + \sin x \sin y}$$

$$= \frac{\frac{\sin x \cos y}{\sin x \cos y} + \frac{\cos x \sin y}{\sin x \cos y}}{\frac{\cos x \cos y}{\sin x \cos y} + \frac{\sin x \sin y}{\sin x \cos y}}$$

$$= \frac{1+\cot x \tan y}{\cot x + \tan y}$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{\cos(x-y)}{\cos(x+y)} = \dfrac{1+\tan x \tan y}{1-\tan x \tan y}$

***Solution***

$$\frac{\cos(x-y)}{\cos(x+y)} = \frac{\cos x \cos y + \sin x \sin y}{\cos x \cos y - \sin x \sin y}$$

$$= \frac{\frac{\cos x \cos y}{\cos x \cos y} + \frac{\sin x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} - \frac{\sin x \sin y}{\cos x \cos y}}$$

$$= \frac{1+\tan x \tan y}{1-\tan x \tan y}$$

# Solution Section 3.3 – Double-angle Formulas

### Exercise

Let $\sin A = -\frac{3}{5}$ with A in QIII and find $\cos 2A$

**Solution**

$$\cos 2A = 1 - 2\sin^2 A$$

$$= 1 - 2\left(-\frac{3}{5}\right)^2$$

$$= 1 - 2\left(\frac{9}{25}\right)$$

$$= \frac{25-18}{25}$$

$$= \frac{7}{25}$$

### Exercise

Let $\cos x = \frac{1}{\sqrt{10}}$ with $x$ in QIV and find $\cot 2x$

**Solution**

$x$ in QIV $\Rightarrow \sin x < 0$

$$\sin x = -\sqrt{1-\cos^2 x}$$

$$= -\sqrt{1-\frac{1}{10}}$$

$$= -\sqrt{\frac{9}{10}}$$

$$= -\frac{3}{\sqrt{10}}$$

$$\cot 2x = \frac{\cos 2x}{\sin 2x}$$

$$= \frac{2\cos^2 x - 1}{2\sin x \cos x}$$

$$= \frac{2\left(\frac{1}{\sqrt{10}}\right)^2 - 1}{2\frac{1}{\sqrt{10}}\left(-\frac{3}{\sqrt{10}}\right)}$$

$$= \frac{2\frac{1}{10} - 1}{-\frac{6}{10}}$$

$$= \frac{\frac{2-10}{10}}{-\frac{6}{10}}$$

$$= \frac{-8}{-6}$$

$$= \frac{4}{3}$$

***Exercise***

Verify: $(\cos x - \sin x)(\cos x + \sin x) = \cos 2x$

***Solution***

$$(\cos x - \sin x)(\cos x + \sin x) = \cos^2 x - \sin^2 x$$
$$= \cos 2x$$

$$(a+b)(a-b) = a^2 - b^2$$

***Exercise***

Prove: $\cot x \sin 2x = 1 + \cos 2x$

***Solution***

$$\cot x \sin 2x = \frac{\cos x}{\sin x}(2\sin x \cos x)$$
$$= 2\cos^2 x$$
$$= \cos 2x + 1$$

$$\cos 2x = 2\cos^2 x - 1 \Rightarrow 2\cos^2 x = \cos 2x + 1$$

***Exercise***

Prove: $\cot\theta = \dfrac{\sin 2\theta}{1-\cos 2\theta}$

***Solution***

$$\begin{aligned}\frac{\sin 2\theta}{1-\cos 2\theta} &= \frac{2\sin\theta\cos\theta}{1-\left(1-2\sin^2\theta\right)}\\ &= \frac{2\sin\theta\cos\theta}{1-1+2\sin^2\theta}\\ &= \frac{2\sin\theta\cos\theta}{2\sin^2\theta}\\ &= \frac{\cos\theta}{\sin\theta}\\ &= \cot\theta\end{aligned}$$

***Exercise***

Simplify $\cos^2 7x - \sin^2 7x$

***Solution***

$$\begin{aligned}\cos^2 7x - \sin^2 7x &= \cos\left(2(7x)\right)\\ &= \cos 14x\end{aligned}$$

$\cos 2x = \cos^2 x - \sin^2 x$

***Exercise***

Write $\sin 3x$ in terms of $\sin x$

***Solution***

$$\begin{aligned}\sin 3x &= \sin(2x+x)\\ &= \sin 2x\cos x + \cos 2x\sin x\\ &= (2\sin x\cos x)\cos x + \left(1-2\sin^2 x\right)\sin x\\ &= 2\sin x\cos^2 x + \sin x - 2\sin^3 x\\ &= 2\sin x\left(1-\sin^2 x\right) + \sin x - 2\sin^3 x\\ &= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x\\ &= 3\sin x - 4\sin^3 x\end{aligned}$$

$\cos^2 x = 1-\sin^2 x$

### *Exercise*

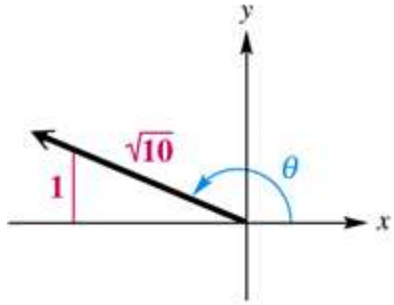
Find the values of the six trigonometric functions of $\theta$ if $\cos 2\theta = \frac{4}{5}$ *and* $90° < \theta < 180°$

***Solution***

| | |
|---|---|
| $\cos^2\theta = \frac{1+\cos 2\theta}{2} = \frac{1+\frac{4}{5}}{2} = \frac{\frac{9}{5}}{2} = \frac{9}{10}$ <br> $\cos\theta = -\sqrt{\frac{9}{10}} = -\frac{3}{\sqrt{10}}\frac{\sqrt{10}}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}$ | $\sin^2\theta = \frac{1-\cos 2\theta}{2} = \frac{1-\frac{4}{5}}{2} = \frac{\frac{1}{5}}{2} = \frac{1}{10}$ <br> $\sin\theta = \sqrt{\frac{1}{10}} = \frac{1}{\sqrt{10}}\frac{\sqrt{10}}{\sqrt{10}} = \frac{\sqrt{10}}{10}$ |
| $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{10}}{10}}{-\frac{3\sqrt{10}}{10}} = -\frac{\sqrt{10}}{10}\frac{10}{3\sqrt{10}} = -\frac{1}{3}$ | $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{-\frac{1}{3}} = -3$ |
| $\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{1}{\sqrt{10}}} = \sqrt{10}$ | $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{-\frac{3}{\sqrt{10}}} = -\frac{\sqrt{10}}{3}$ |

### *Exercise*

Use a right triangle in QII to find the value of $\cos\theta$ *and* $\tan\theta$




### ***Solution***

$$r = \sqrt{10},\ y = 1$$

$$\begin{aligned} x &= -\sqrt{r^2 - y^2} \\ &= -\sqrt{\left(\sqrt{10}\right)^2 - 1^2} \\ &= -\sqrt{10 - 1} \\ &= -\sqrt{9} \\ &= -3 \end{aligned}$$

$$\cos\theta = \frac{-3}{\sqrt{10}}\frac{\sqrt{10}}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}$$

$$\tan\theta = -\frac{1}{3}$$

### *Exercise*

Prove the following equation is an identity: $\sin 3x = \sin x\left(3\cos^2 x - \sin^2 x\right)$

### ***Solution***

$$\begin{aligned} \sin 3x &= \sin(x + 2x) \\ &= \sin x \cos 2x + \sin 2x \cos x \\ &= \sin x\left(\cos^2 x - \sin^2 x\right) + (2\sin x \cos x)\cos x \\ &= \sin x \cos^2 x - \sin^3 x + 2\sin x \cos^2 x \\ &= 3\sin x \cos^2 x - \sin^3 x \\ &= \sin x\left(3\cos^2 x - \sin^2 x\right) \end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\cos 3x = \cos^3 x - 3\cos x \sin^2 x$

***Solution***

$$\begin{aligned}\cos 3x &= \cos(x+2x)\\ &= \cos x\cos 2x - \sin x \sin 2x\\ &= \cos x\left(\cos^2 x - \sin^2 x\right) - \sin x\left(2\sin x\cos x\right)\\ &= \cos^3 x - \sin^2 x\cos x - 2\sin^2 x\cos x\\ &= \cos^3 x - 3\sin^2 x\cos x\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\cos^4 x - \sin^4 x = \cos 2x$

***Solution***

$$\begin{aligned}\cos^4 x - \sin^4 x &= \left(\cos^2 x - \sin^2 x\right)\left(\cos^2 x + \sin^2 x\right) \qquad (a-b)(a+b) = a^2 + b^2\\ &= (\cos 2x)(1)\\ &= \cos 2x\end{aligned}$$

***Exercise***

Prove: $\cot\theta = \dfrac{\sin 2\theta}{1-\cos 2\theta}$

***Solution***

$$\begin{aligned}\frac{\sin 2\theta}{1-\cos 2\theta} &= \frac{2\sin\theta\cos\theta}{1-\left(1-2\sin^2\theta\right)}\\ &= \frac{2\sin\theta\cos\theta}{1-1+2\sin^2\theta}\\ &= \frac{2\sin\theta\cos\theta}{2\sin^2\theta}\\ &= \frac{\cos\theta}{\sin\theta}\\ &= \cot\theta\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\sin 2x = -2\sin x \sin\left(x - \frac{\pi}{2}\right)$

***Solution***

$$\sin 2x = 2\sin x \cos x \qquad \cos x = \sin\left(\frac{\pi}{2} - x\right)$$

$$= 2\sin x \sin\left(\frac{\pi}{2} - x\right) \qquad \sin(-x) = -\sin x$$

$$= -2\sin x \sin\left(x - \frac{\pi}{2}\right)$$

### *Exercise*

Prove the following equation is an identity: $\frac{\sin 4t}{4} = \cos^3 t \sin t - \sin^3 t \cos t$

***Solution***

$$\frac{\sin 4t}{4} = \frac{1}{4}(2\sin 2t \cos 2t)$$

$$= \frac{1}{2}(2\sin t \cos t)\left(\cos^2 t - \sin^2 t\right)$$

$$= \sin t \cos t\left(\cos^2 t - \sin^2 t\right)$$

$$= \sin t \cos^3 t - \cos t \sin^3 t$$

### *Exercise*

Prove the following equation is an identity: $\frac{\cos 2x}{\sin^2 x} = \csc^2 x - 2$

***Solution***

$$\frac{\cos 2x}{\sin^2 x} = \frac{1 - 2\sin^2 x}{\sin^2 x}$$

$$= \frac{1}{\sin^2 x} - \frac{2\sin^2 x}{\sin^2 x}$$

$$= \csc^2 x - 2$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{\cos 2x+\cos 2y}{\sin x+\cos y}=2\cos y-2\sin x$

**Solution**

$$\begin{aligned}
\frac{\cos 2x+\cos 2y}{\sin x+\cos y} &= \frac{2\cos\left(\frac{2x+2y}{2}\right)\cos\left(\frac{2x-2y}{2}\right)}{\sin x+\cos y} \\
&= \frac{2\cos(x+y)\cos(x-y)}{\sin x+\cos y} \\
&= \frac{2(\cos x\cos y-\sin x\sin y)(\cos x\cos y+\sin x\sin y)}{\sin x+\cos y} \\
&= 2\frac{\cos^2 x\cos^2 y-\sin^2 x\sin^2 y}{\sin x+\cos y} \\
&= 2\frac{\left(1-\sin^2 x\right)\cos^2 y-\sin^2 x\left(1-\cos^2 y\right)}{\sin x+\cos y} \\
&= 2\frac{\cos^2 y-\sin^2 x\cos^2 y-\sin^2 x+\sin^2 x\cos^2 y}{\sin x+\cos y} \\
&= 2\frac{\cos^2 y-\sin^2 x}{\sin x+\cos y} \\
&= 2\frac{(\cos y-\sin x)(\cos y+\sin x)}{\sin x+\cos y} \\
&= 2(\cos y-\sin x) \\
&= 2\cos y-2\sin x
\end{aligned}$$

$$\begin{aligned}
\frac{\cos 2x+\cos 2y}{\sin x+\cos y} &= \frac{\cos^2 x-\sin^2 x+\cos^2 y-\sin^2 y}{\sin x+\cos y} \\
&= \frac{1-\sin^2 x-\sin^2 x+\cos^2 y-\left(1-\cos^2 y\right)}{\sin x+\cos y} \\
&= \frac{1-2\sin^2 x+\cos^2 y-1+\cos^2 y}{\sin x+\cos y} \\
&= \frac{2\cos^2 y-2\sin^2 x}{\sin x+\cos y} \\
&= 2\frac{\cos^2 y-\sin^2 x}{\sin x+\cos y} \\
&= 2\frac{(\cos y-\sin x)(\cos y+\sin x)}{\sin x+\cos y} \\
&= 2(\cos y-\sin x) \\
&= 2\cos y-2\sin x
\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\dfrac{\cos 2x}{\cos^2 x} = \sec^2 x - 2\tan^2 x$

***Solution***

$$\frac{\cos 2x}{\cos^2 x} = \frac{1-2\sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} - \frac{2\sin^2 x}{\cos^2 x}$$

$$= \sec^2 x - 2$$

***Exercise***

Prove the following equation is an identity: $\sin 4x = (4\sin x\cos x)\left(2\cos^2 x - 1\right)$

***Solution***

$$\sin 4x = \sin(2(2x))$$

$$= 2\sin 2x\cos 2x$$

$$= 2(2\sin x\cos x)\left(2\cos^2 x - 1\right)$$

$$= (4\sin x\cos x)\left(2\cos^2 x - 1\right)$$

***Exercise***

Prove the following equation is an identity: $\cos 4x = \cos^4 x - 6\sin^2 x\cos^2 x + \sin^4 x$

***Solution***

$$\cos 4x = \cos(2(2x))$$

$$= \cos^2 2x - \sin^2 2x$$

$$= (\cos 2x)^2 - (\sin 2x)^2$$

$$= \left(\cos^2 x - \sin^2 x\right)^2 - (2\sin x\cos x)^2$$

$$= \cos^4 x - 2\sin^2 x\cos^2 x - \sin^4 x - 4\sin^2 x\cos^2 x$$

$$= \cos^4 x - 6\sin^2 x\cos^2 x - \sin^4 x$$

***Exercise***

Prove the following equation is an identity: $\cos 2y = \dfrac{1-\tan^2 y}{1+\tan^2 y}$

***Solution***

| | |
|---|---|
| $\cos 2y = \cos^2 y - \sin^2 y$<br>$= \dfrac{\cos^2 y - \sin^2 y}{1}$<br>$= \dfrac{\cos^2 y - \sin^2 y}{\cos^2 y + \sin^2 y}$<br>$= \dfrac{\frac{\cos^2 y}{\cos^2 y} - \frac{\sin^2 y}{\cos^2 y}}{\frac{\cos^2 y}{\cos^2 y} + \frac{\sin^2 y}{\cos^2 y}}$<br>$= \dfrac{1-\tan^2 y}{1+\tan^2 y}$ | $\dfrac{1-\tan^2 y}{1+\tan^2 y} = \dfrac{1-\frac{\sin^2 y}{\cos^2 y}}{1+\frac{\sin^2 y}{\cos^2 y}}$<br>$= \dfrac{\frac{\cos^2 y - \sin^2 y}{\cos^2 y}}{\frac{\cos^2 y + \sin^2 y}{\cos^2 y}}$<br>$= \dfrac{\cos^2 y - \sin^2 y}{\cos^2 y + \sin^2 y}$<br>$= \dfrac{\cos^2 y - \sin^2 y}{1}$<br>$= \cos^2 y - \sin^2 y$ |

***Exercise***

Prove the following equation is an identity: $\tan^2 x(1+\cos 2x) = 1-\cos 2x$

***Solution***

$$\begin{aligned}
\tan^2 x(1+\cos 2x) &= \frac{\sin^2 x}{\cos^2 x}\left(1+2\cos^2 x-1\right)\\
&= \frac{\sin^2 x}{\cos^2 x}\left(2\cos^2 x\right)\\
&= 2\sin^2 x\\
&= 1-1+2\sin^2 x\\
&= 1-\left(1-2\sin^2 x\right)\\
&= 1-\cos 2x
\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{\cos 2x}{\sin^2 x} = 2\cot^2 x - \csc^2 x$

***Solution***

$$\frac{\cos 2x}{\sin^2 x} = \frac{\cos^2 x - \sin^2 x}{\sin^2 x}$$

$$= \frac{\cos^2 x}{\sin^2 x} - \frac{\sin^2 x}{\sin^2 x}$$

$$= \cot^2 x - 1 \qquad\qquad \cot^2 x + 1 = \csc^2 x$$

$$= \cot^2 x + \cot^2 x - \csc^2 x$$

$$= 2\cot^2 x - \csc^2 x$$

### *Exercise*

Prove the following equation is an identity: $\tan x + \cot x = 2\csc 2x$

***Solution***

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x}$$

$$= \frac{1}{\cos x \sin x}$$

$$= \frac{1}{\frac{1}{2}\sin 2x}$$

$$= 2\frac{1}{\sin 2x}$$

$$= 2\csc 2x$$

### *Exercise*

Prove the following equation is an identity: $\tan 2x = \dfrac{2}{\cot x - \tan x}$

***Solution***

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$= \frac{2\frac{\tan x}{\tan x}}{\frac{1}{\tan x} - \frac{\tan^2 x}{\tan x}}$$

$$= \frac{2}{\cot x - \tan x}$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{1-\tan x}{1+\tan x} = \dfrac{1-\sin 2x}{\cos 2x}$

***Solution***

$$\frac{1-\tan x}{1+\tan x} = \frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}$$

$$= \frac{\frac{\cos x - \sin x}{\cos x}}{\frac{\cos x + \sin x}{\cos x}}$$

$$= \frac{\cos x - \sin x}{\cos x + \sin x}\frac{\cos x - \sin x}{\cos x - \sin x}$$

$$= \frac{\cos^2 x - 2\cos x \sin x + \sin^2 x}{\cos^2 x + \sin^2 x}$$

$$= \frac{1-\sin 2x}{\cos 2x}$$

### *Exercise*

Prove the following equation is an identity: $\sin 2\alpha \sin 2\beta = \sin^2(\alpha+\beta) - \sin^2(\alpha-\beta)$

***Solution***

$$\begin{aligned}
\sin 2\alpha \sin 2\beta &= (2\sin\alpha\cos\alpha)(2\sin\beta\cos\beta) \\
&= (2\sin\alpha\cos\beta)(2\sin\beta\cos\alpha) \\
&= \left(2\tfrac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]\right)\left(2\tfrac{1}{2}\left[\sin(\beta+\alpha)+\sin(\beta-\alpha)\right]\right) \\
&= \left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)\left(\sin(\alpha+\beta)-\sin(\alpha-\beta)\right) \\
&= \sin^2(\alpha+\beta) - \sin^2(\alpha-\beta)
\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\cos^2(A-B) - \cos^2(A+B) = \sin 2A \ \sin 2B$

***Solution***

$$\begin{aligned}
\cos^2(A-B) - \cos^2(A+B) &= \left(\cos(A-B)-\cos(A+B)\right)\left(\cos(A-B)+\cos(A+B)\right) \\
&= (2\sin A\sin B)(2\cos A\cos B) \\
&= (2\sin A\cos A)(2\sin B\cos B) \\
&= \sin 2A \ \sin 2B
\end{aligned}$$

# Solution    *Section* 3.4 – Half-Angle Formulas

### *Exercise*

Use half-angle formulas to find the exact value of $\sin 105^\circ$

***Solution***

$$\sin 105^\circ = \sin\frac{210^\circ}{2}$$

$$= \sqrt{\frac{1-\cos 210^\circ}{2}} \qquad\qquad \textit{reference}: 210^\circ - 180^\circ = 30^\circ$$

$$= \sqrt{\frac{1+\cos 30^\circ}{2}}$$

$$= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{2+\sqrt{3}}{4}}$$

$$= \frac{\sqrt{2+\sqrt{3}}}{2}$$

### *Exercise*

Find the exact of $\tan 22.5^\circ$

***Solution***

$$\tan 22.5^\circ = \tan\frac{45^\circ}{2}$$

$$= \frac{1-\cos 45^\circ}{\sin 45^\circ}$$

$$= \frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$

$$= \frac{\frac{2-\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$

$$=\frac{2-\sqrt{2}}{\sqrt{2}}$$

$$=\frac{2}{\sqrt{2}}-\frac{\sqrt{2}}{\sqrt{2}}$$

$$=\frac{2\sqrt{2}}{2}-1$$

$$=\sqrt{2}-1$$

***Exercise***

Given: $\cos x=\frac{2}{3},\ \ \frac{3\pi}{2}<x<2\pi$, find $\cos\frac{x}{2},\ \ \sin\frac{x}{2},$ and $\tan\frac{x}{2}$

**Solution**

$$\frac{3\pi}{2}<x<2\pi \Rightarrow \frac{3\pi}{4}<\frac{x}{2}<\pi \Rightarrow x\in QII$$

| | | |
|---|---|---|
| $\cos\frac{x}{2}=-\sqrt{\frac{1+\cos x}{2}}$ | $\sin\frac{x}{2}=\sqrt{\frac{1-\cos x}{2}}$ | $\tan\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$ |
| $=-\sqrt{\frac{1+\frac{2}{3}}{2}}$ | $=\sqrt{\frac{1-\frac{2}{3}}{2}}$ | $=\frac{\frac{\sqrt{6}}{6}}{-\frac{\sqrt{30}}{6}}$ |
| $=-\sqrt{\frac{1}{2}\frac{3+2}{3}}$ | $=\sqrt{\frac{1}{2}\frac{3-2}{3}}$ | $=-\frac{\sqrt{6}}{\sqrt{30}}\frac{\sqrt{30}}{\sqrt{30}}$ |
| $=-\sqrt{\frac{5}{6}}$ | $=\sqrt{\frac{1}{6}}$ | $=-\frac{6\sqrt{5}}{30}$ |
| $=-\frac{\sqrt{5}}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}}$ | $=\frac{1}{\sqrt{6}}\frac{\sqrt{6}}{\sqrt{6}}$ | $=-\frac{\sqrt{5}}{5}$ |
| $=-\frac{\sqrt{30}}{6}$ | $=\frac{\sqrt{6}}{6}$ | |

***Exercise***

Prove the identity $2\csc x \cos^2 \frac{x}{2} = \frac{\sin x}{1-\cos x}$

***Solution***

$$2\csc x \cos^2 \frac{x}{2} = 2\frac{1}{\sin x}\frac{1+\cos x}{2}$$

$$= \frac{1+\cos x}{\sin x}\frac{1-\cos x}{1-\cos x} \qquad (a-b)(a+b) = a^2 - b^2$$

$$= \frac{1-\cos^2 x}{\sin x(1-\cos x)}$$

$$= \frac{\sin^2 x}{\sin x(1-\cos x)}$$

$$= \frac{\sin x}{1-\cos x}$$

***Exercise***

Prove the identity $\tan\frac{\alpha}{2} = \sin\alpha + \cos\alpha \cot\alpha - \cot\alpha$

***Solution***

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} \qquad 1 = \sin^2\alpha + \cos^2\alpha$$

$$= \frac{\sin^2\alpha + \cos^2\alpha - \cos\alpha}{\sin\alpha}$$

$$= \frac{\sin^2\alpha}{\sin\alpha} + \frac{\cos^2\alpha}{\sin\alpha} - \frac{\cos\alpha}{\sin\alpha}$$

$$= \sin\alpha + \cos\alpha\frac{\cos\alpha}{\sin\alpha} - \cot\alpha$$

$$= \sin\alpha + \cos\alpha\cot\alpha - \cot\alpha$$

***Exercise***

Prove the following equation is an identity: $\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)=\frac{\sin^2 x}{4}$

***Solution***

$$\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)=\frac{1-\cos x}{2}\cdot\frac{1+\cos x}{2}$$

$$(a-b)(a+b)=a^2+b^2$$

$$=\frac{1-\cos^2 x}{4}$$

$$\cos^2\alpha+\sin^2\alpha=1$$

$$=\frac{\sin^2 x}{4}$$

***Exercise***

Prove the following equation is an identity: $\tan\frac{x}{2}+\cot\frac{x}{2}=2\csc x$

***Solution***

$$\tan\frac{x}{2}+\cot\frac{x}{2}=\tan\frac{x}{2}+\frac{1}{\tan\frac{x}{2}}$$

$$=\frac{\sin x}{1+\cos x}+\frac{\sin x}{1-\cos x}$$

$$=\sin x\frac{(1-\cos x)+(1+\cos x)}{1-\cos^2 x}$$

$$=\sin x\frac{2}{\sin^2 x}$$

$$=\frac{2}{\sin x}$$

$$=2\csc x$$

***Exercise***

Prove the following equation is an identity: $2\sin^2\left(\frac{x}{2}\right)=\frac{\sin^2 x}{1+\cos x}$

***Solution***

$$2\sin^2\left(\frac{x}{2}\right)=2\frac{1-\cos x}{2}$$

$$=1-\cos x\cdot\frac{1+\cos x}{1+\cos x}$$

$$=\frac{1-\cos^2 x}{1+\cos x}$$

$$=\frac{\sin^2 x}{1+\cos x}$$

### *Exercise*

Prove the following equation is an identity: $\tan^2\left(\frac{x}{2}\right)=\frac{\sec x+\cos x-2}{\sec x-\cos x}$

**Solution**

$$\tan^2\left(\frac{x}{2}\right)=\frac{1-\cos x}{1+\cos x}\qquad\qquad \tan^2\alpha=\frac{1-\cos 2\alpha}{1+\cos 2\alpha};\quad x=2\alpha;\quad \tan^2\frac{x}{2}=\frac{1-\cos x}{1+\cos x}$$

$$=\frac{1-\cos x}{1+\cos x}\frac{1-\cos x}{1-\cos x}$$

$$=\frac{1-2\cos x+\cos^2 x}{1-\cos^2 x}\ \frac{\frac{1}{\cos x}}{\frac{1}{\cos x}}$$

$$=\frac{\frac{1-2\cos x+\cos^2 x}{\cos x}}{\frac{1-\cos^2 x}{\cos x}}$$

$$=\frac{\frac{1}{\cos x}-\frac{2\cos x}{\cos x}+\frac{\cos^2 x}{\cos x}}{\frac{1}{\cos x}-\frac{\cos^2 x}{\cos x}}$$

$$=\frac{\sec x-2+\cos x}{\sec x-\cos x}$$

$$\frac{\sec x+\cos x-2}{\sec x-\cos x}=\frac{\frac{1}{\cos x}+\cos x-2}{\frac{1}{\cos x}-\cos x}$$

$$=\frac{\frac{1+\cos^2 x-2\cos x}{\cos x}}{\frac{1-\cos^2 x}{\cos x}}$$

$$=\frac{(1-\cos x)^2}{(1-\cos x)(1+\cos x)}$$

$$=\frac{1-\cos x}{1+\cos x}\qquad\qquad \tan^2\alpha=\frac{1-\cos 2\alpha}{1+\cos 2\alpha};\quad x=2\alpha;\quad \tan^2\frac{x}{2}=\frac{1-\cos x}{1+\cos x}$$

$$=\tan^2\left(\frac{x}{2}\right)$$

### *Exercise*

Prove the following equation is an identity: $\sec^2\left(\frac{x}{2}\right) = \frac{2\sec x + 2}{\sec x + 2 + \cos x}$

### ***Solution***

$$\sec^2\left(\frac{x}{2}\right) = \frac{1}{\cos^2\left(\frac{x}{2}\right)}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}} \Rightarrow \cos^2\left(\frac{\alpha}{2}\right) = \frac{1+\cos\alpha}{2}$$

$$= \frac{1}{\frac{1+\cos x}{2}}$$

$$= \frac{2}{1+\cos x}\,\frac{1+\cos x}{1+\cos x}$$

$$= \frac{2+2\cos x}{1+2\cos x+\cos^2 x}$$

$$= \frac{2+2\cos x}{1+2\cos x+\cos^2 x}\,\frac{\frac{1}{\cos x}}{\frac{1}{\cos x}}$$

$$= \frac{\frac{2}{\cos x}+2\frac{\cos x}{\cos x}}{\frac{1}{\cos x}+\frac{2\cos x}{\cos x}+\frac{\cos^2 x}{\cos x}}$$

$$= \frac{2\sec x+2}{\sec x+2+\cos x}$$

### *Exercise*

Prove the following equation is an identity: $\frac{1-\sin^2\left(\frac{x}{2}\right)}{1+\sin^2\left(\frac{x}{2}\right)} = \frac{1+\cos x}{3-\cos x}$

### ***Solution***

$$\frac{1-\sin^2\left(\frac{x}{2}\right)}{1+\sin^2\left(\frac{x}{2}\right)} = \frac{1-\frac{1-\cos x}{2}}{1+\frac{1-\cos x}{2}}$$

$$= \frac{\frac{2-1-\cos x}{2}}{\frac{2+1-\cos x}{2}}$$

$$= \frac{1-\cos x}{3-\cos x}$$

### ***Exercise***

Prove the following equation is an identity: $\dfrac{1-\cos^2\left(\frac{x}{2}\right)}{1-\sin^2\left(\frac{x}{2}\right)}=\dfrac{1-\cos x}{1+\cos x}$

***Solution***

$$\frac{1-\cos^2\left(\frac{x}{2}\right)}{1-\sin^2\left(\frac{x}{2}\right)}=\frac{1-\frac{1+\cos x}{2}}{1-\frac{1-\cos x}{2}}$$

$$=\frac{\frac{2-(1+\cos x)}{2}}{\frac{2-(1-\cos x)}{2}}$$

$$=\frac{\frac{2-1-\cos x}{2}}{\frac{2-1+\cos x}{2}}$$

$$=\frac{1-\cos x}{1+\cos x}$$

# *Solution* *Section* 3.5 – Additional Identities

### *Exercise*

Write $10\cos 5x\sin 3x$ as a sum or difference

***Solution***

$$10\cos 5x\sin 3x = 10\cdot\frac{1}{2}\left[\sin(5x+3x)-\sin(5x-3x)\right]$$

$$= 5(\sin 8x - \sin 2x)$$

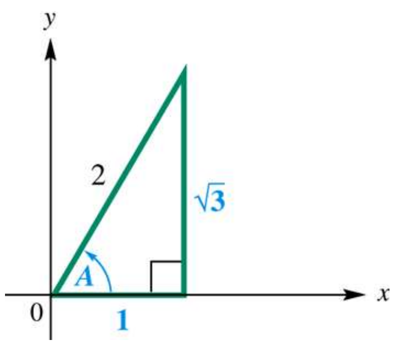
### *Exercise*

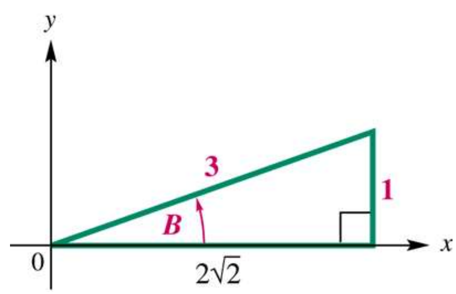
Evaluate without using the calculator $\cos\left(\arctan\sqrt{3}+\arcsin\frac{1}{3}\right)$

***Solution***

$$\alpha = \arctan\sqrt{3} \Rightarrow \tan\alpha = \sqrt{3}$$

$$\beta = \arcsin\frac{1}{3} \Rightarrow \sin\beta = \frac{1}{3}$$




$$\sin\alpha = \frac{\sqrt{3}}{2},\quad \cos\alpha = \frac{1}{2}$$

$$\sin\beta = \frac{1}{3},\quad \cos B = \frac{2\sqrt{2}}{3}$$

$$\cos\left(\arctan\sqrt{3}+\arcsin\frac{1}{3}\right) = \cos(\alpha+\beta)$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \frac{1}{2}\frac{2\sqrt{2}}{3} - \frac{\sqrt{3}}{2}\frac{1}{3}$$

$$= \frac{2\sqrt{2}-\sqrt{3}}{6}$$

***Exercise***

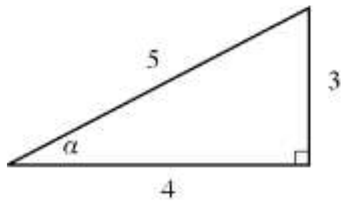
Evaluate without using the calculator $\cos(\arcsin\frac{3}{5} - \arctan 2)$

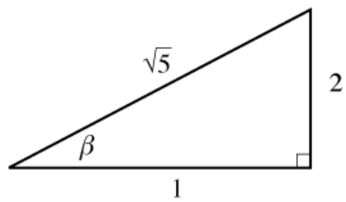
***Solution***

$$\cos(\arcsin\frac{3}{5} - \arctan 2) = \cos(\alpha - \beta)$$

$$\alpha = \arcsin\frac{3}{5}$$

$$\beta = \arctan 2$$



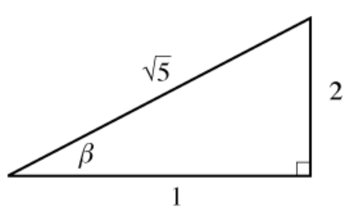




$$\begin{aligned}\cos(\arcsin\frac{3}{5} - \arctan 2) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ &= \frac{4}{5}\frac{1}{\sqrt{5}} + \frac{3}{5}\frac{2}{\sqrt{5}} \\ &= \frac{4}{5\sqrt{5}} + \frac{6}{5\sqrt{5}} \\ &= \frac{10}{5\sqrt{5}} \\ &= \frac{2}{\sqrt{5}}\end{aligned}$$

***Exercise***

Evaluate without using the calculator $\sin\left(2\cos^{-1}\frac{1}{\sqrt{5}}\right)$

***Solution***




$$\beta = \cos^{-1}\frac{1}{\sqrt{5}}$$

$$\cos\beta = \frac{1}{\sqrt{5}}$$

$$\sin\beta = \frac{2}{\sqrt{5}}$$

$$\begin{aligned}\sin(2\beta) &= 2\sin\beta\cos\beta \\ &= 2\frac{2}{\sqrt{5}}\frac{1}{\sqrt{5}} \\ &= \frac{4}{5}\end{aligned}$$

### *Exercise*

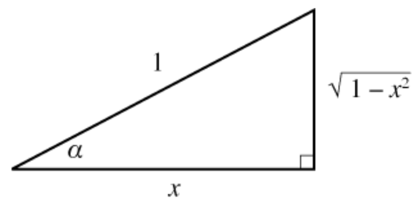
Write $\sin\left(2\cos^{-1}x\right)$ as an equivalent expression involving only $x$.

**Solution**

$$\alpha = \cos^{-1}x$$

$$\cos\alpha = x$$

$$\sin\alpha = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$




$$\begin{aligned}\sin\left(2\cos^{-1}x\right) &= \sin(2\alpha)\\ &= 2\sin\alpha\cos\alpha\\ &= 2\sqrt{1-x^2}\cdot x\\ &= 2x\sqrt{1-x^2}\end{aligned}$$

### *Exercise*

Write $\sec\left(\tan^{-1}\frac{x-2}{2}\right)$ as an equivalent expression involving only $x$.

**Solution**

$$\alpha = \tan^{-1}\frac{x-2}{2}$$

$$\tan\alpha = \frac{x-2}{2}$$

$$c = \sqrt{(x-2)^2+2^2}$$

$$c = \sqrt{x^2-4x+4+4}$$

$$c = \sqrt{x^2-4x+8}$$

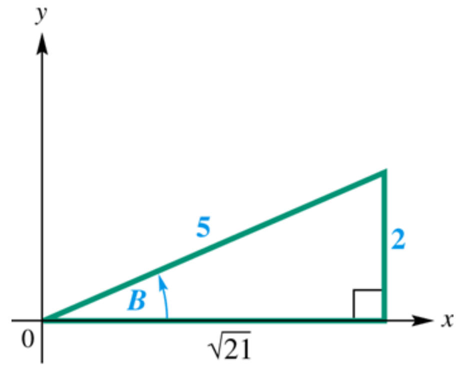
$$\cos\alpha = \frac{2}{\sqrt{x^2-4x+8}}$$

$$\begin{aligned}\sec\alpha &= \frac{1}{\cos\alpha}\\ &= \frac{1}{\frac{\sqrt{x^2-4x+8}}{2}}\\ &= \frac{\sqrt{x^2-4x+8}}{2}\end{aligned}$$

### *Exercise*

Evaluate without using the calculator $\tan\left(2\arcsin\frac{2}{5}\right)$

**Solution**




$$\alpha = \arcsin\frac{2}{5} \Rightarrow \sin\alpha = \frac{2}{5}$$

$$x = \sqrt{5^2 - 2^2} = \sqrt{21}$$

$$\tan(\alpha) = \frac{2}{\sqrt{21}}$$

$$\tan(2\alpha) = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

$$= \frac{2\frac{2}{\sqrt{21}}}{1 - \left(\frac{2}{\sqrt{21}}\right)^2}$$

$$= \frac{\frac{4}{\sqrt{21}}}{1 - \frac{4}{21}}$$

$$= \frac{\frac{4}{\sqrt{21}}}{\frac{21-4}{21}}$$

$$= \frac{\frac{4}{\sqrt{21}}}{\frac{17}{21}}$$

$$= \frac{4}{\sqrt{21}}\frac{21}{17}\frac{\sqrt{21}}{\sqrt{21}}$$

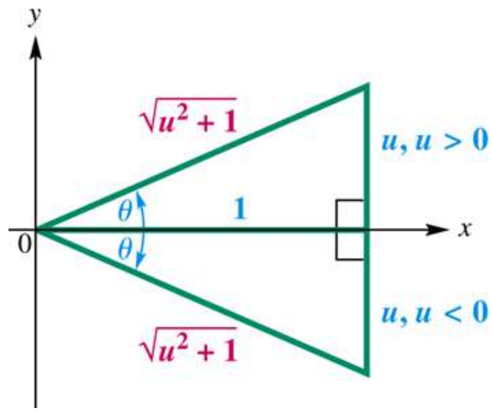
$$= \frac{4(21)\sqrt{21}}{21(17)}$$

$$\boxed{= \frac{4\sqrt{21}}{17}}$$

### *Exercise*

Evaluate without using the calculator $\sin\left(\tan^{-1}u\right)$

***Solution***




$$\theta = \tan^{-1}u \quad \Rightarrow \quad \tan\theta = u = \frac{u}{1}$$

$$r = \sqrt{x^2+y^2} = \sqrt{1+u^2}$$

$$\sin\theta = \frac{u}{\sqrt{u^2+1}}\frac{\sqrt{u^2+1}}{\sqrt{u^2+1}}$$

$$= \frac{u\sqrt{u^2+1}}{u^2+1}$$

### *Exercise*

Write $\cos\left(2\sin^{-1}u\right)$ as an equivalent expression involving only *x*.

***Solution***

$$\theta = \sin^{-1}u \quad \Rightarrow \quad \sin\theta = u$$

$$\cos\left(2\sin^{-1}u\right) = \cos 2\theta$$

$$= 1 - 2\sin^2\theta$$

$$\underline{= 1 - 2u^2}$$

### *Exercise*

Prove the identity: $\quad \dfrac{\sin 3x - \sin x}{\cos 3x - \cos x} = -\cot 2x$

***Solution***

$$\frac{\sin 3x - \sin x}{\cos 3x - \cos x} = \frac{2\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)}{-2\sin\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)}$$

$$= -\frac{\cos 2x \sin x}{\sin 2x \sin x}$$

$$= -\frac{\cos 2x}{\sin 2x}$$

$$= -\cot 2x$$

***Exercise***

Prove the following equation is an identity: $\sin(x+y)\cos(x-y)=\sin x\cos x+\cos y\sin y$

***Solution***

$$\begin{aligned}\sin(x+y)\cos(x-y)&=\tfrac{1}{2}\left[\sin(x+y+x-y)+\sin(x+y-x+y)\right]\\&=\tfrac{1}{2}\left[\sin(2x)+\sin(2y)\right]\\&=\tfrac{1}{2}\left[2\sin x\cos x+2\sin y\cos y\right]\\&=\sin x\cos x+\sin y\cos y\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $2\sin(x+y)\cos(x-y)=\sin 2x+\sin 2y$

***Solution***

$$\begin{aligned}2\sin(x+y)\cos(x-y)&=\sin(x+y+x-y)+\sin(x+y-(x-y))\\&=\sin(2x)+\sin(x+y-x+y)\\&=\sin(2x)+\sin(2y)\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\dfrac{\sin(26k)+\sin(8k)}{\cos(26k)-\cos(8k)}=-\cot(9k)$

***Solution***

$$\begin{aligned}\frac{\sin(26k)+\sin(8k)}{\cos(26k)-\cos(8k)}&=\frac{2\sin\left(\frac{26k+8k}{2}\right)\cos\left(\frac{26k-8k}{2}\right)}{-2\sin\left(\frac{26k+8k}{2}\right)\sin\left(\frac{26k-8k}{2}\right)}\\&=-\frac{\sin(17k)\cos(9k)}{\sin(17k)\sin(9k)}\\&=-\cot 9k\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\dfrac{\sin(26k)-\sin(12k)}{\sin(26k)+\sin(12k)}=\cot(19k)\tan(7k)$

***Solution***

$$\begin{aligned}
\frac{\sin(26k)-\sin(12k)}{\sin(26k)+\sin(12k)} &= \frac{2\cos\left(\frac{26k+12k}{2}\right)\sin\left(\frac{26k-12k}{2}\right)}{2\sin\left(\frac{26k+12k}{2}\right)\cos\left(\frac{26k-12k}{2}\right)} \\
&= \frac{\cos(19k)\sin(7k)}{\sin(19k)\cos(7k)} \\
&= \cot(19k)\tan(7k)
\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\sin(x+y)\cos(x-y)=\sin x\cos x+\cos y\sin y$

***Solution***

$$\begin{aligned}
\sin(x+y)\cos(x-y) &= \tfrac{1}{2}\left[\sin(x+y+x-y)+\sin(x+y-(x-y))\right] \\
&= \tfrac{1}{2}\left[\sin(2x)+\sin(x+y-x+y)\right] \\
&= \tfrac{1}{2}\left[\sin(2x)+\sin(2y)\right] \\
&= \tfrac{1}{2}\left[2\sin x\cos x+2\sin y\cos y\right] \\
&= \tfrac{1}{2}2(\sin x\cos x+\sin y\cos y) \\
&= \sin x\cos x+\sin y\cos y
\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $(\sin\alpha+\cos\alpha)(\sin\beta+\cos\beta)=\sin(\alpha+\beta)+\cos(\alpha-\beta)$

***Solution***

$$\begin{aligned}
(\sin\alpha+\cos\alpha)(\sin\beta+\cos\beta) &= \sin\alpha\sin\beta+\sin\alpha\cos\beta+\cos\alpha\sin\beta+\cos\alpha\cos\beta \\
&= \tfrac{1}{2}\cos(\alpha-\beta)-\tfrac{1}{2}\cos(\alpha+\beta)+\tfrac{1}{2}\sin(\alpha+\beta)+\tfrac{1}{2}\sin(\alpha-\beta) \\
&\quad +\tfrac{1}{2}\sin(\alpha+\beta)-\tfrac{1}{2}\sin(\alpha-\beta)+\tfrac{1}{2}\cos(\alpha+\beta)+\tfrac{1}{2}\cos(\alpha-\beta) \\
&= \cos(\alpha-\beta)+\sin(\alpha+\beta)
\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\dfrac{\cos x - \cos 3x}{\cos x + \cos 3x} = \tan 2x \tan x$

***Solution***

$$\frac{\cos x - \cos 3x}{\cos x + \cos 3x} = \frac{-2\sin\left(\frac{x+3x}{2}\right)\sin\left(\frac{x-3x}{2}\right)}{2\cos\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right)}$$

$$= -\frac{\sin(2x)\sin(-x)}{\cos(2x)\cos(-x)}$$

$$= -\tan(2x)\frac{-\sin(x)}{\cos(x)}$$

$$= \tan(2x)\tan(x)$$

***Exercise***

Prove the following equation is an identity: $\dfrac{\cos 5x + \cos 3x}{\cos 5x - \cos 3x} = -\cot 4x \cot x$

***Solution***

$$\frac{\cos 5x + \cos 3x}{\cos 5x - \cos 3x} = \frac{2\cos\left(\frac{5x+3x}{2}\right)\cos\left(\frac{5x-3x}{2}\right)}{-2\sin\left(\frac{5x+3x}{2}\right)\sin\left(\frac{5x-3x}{2}\right)}$$

$$= -\frac{\cos(4x)\cos(x)}{\sin(4x)\sin(x)}$$

$$= -\cot(4x)\cot(x)$$

***Exercise***

Prove the following equation is an identity: $\dfrac{\sin 3t - \sin t}{\cos 3t + \cos t} = \tan t$

***Solution***

$$\frac{\sin 3t - \sin t}{\cos 3t + \cos t} = \frac{2\cos\left(\frac{3t+t}{2}\right)\sin\left(\frac{3t-t}{2}\right)}{2\cos\left(\frac{3t+t}{2}\right)\cos\left(\frac{3t-t}{2}\right)}$$

$$= \frac{\cos(2t)\sin(t)}{\cos(2t)\cos(t)}$$

$$= \frac{\sin t}{\cos t}$$

$$= \tan t$$

***Exercise***

Prove the following equation is an identity: $\dfrac{\sin 3x+\sin 5x}{\sin 3x-\sin 5x}=-\dfrac{\tan 4x}{\tan x}$

***Solution***

$$\begin{aligned}
\frac{\sin 3x+\sin 5x}{\sin 3x-\sin 5x} &= \frac{2\sin\left(\frac{3x+5x}{2}\right)\cos\left(\frac{3x-5x}{2}\right)}{2\cos\left(\frac{3x+5x}{2}\right)\sin\left(\frac{3x-5x}{2}\right)} \\
&= \frac{\sin(4x)\cos(-x)}{\cos(4x)\sin(-x)} \\
&= \tan(4x)\frac{\cos(x)}{-\sin(x)} \\
&= -\tan(4x)\cot x \\
&= -\tan(4x)\frac{1}{\tan x} \\
&= -\frac{\tan 4x}{\tan x}
\end{aligned}$$

***Exercise***

Prove the following equation is an identity: $\cos^2 x-\cos^2 y=-\sin(x+y)\sin(x-y)$

***Solution***

$$\begin{aligned}
\cos^2 x-\cos^2 y &= (\cos x-\cos y)(\cos x+\cos y) \\
&= \left(-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right)\left(2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right) \\
&= -2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x+y}{2}\right)2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x-y}{2}\right) \\
&= -\sin 2\left(\frac{x+y}{2}\right)\sin 2\left(\frac{x-y}{2}\right) \\
&= -\sin(x+y)\sin(x-y)
\end{aligned}$$

### ***Exercise***

Prove the following equation is an identity: $\dfrac{\cos 8x - \cos 2x}{2\sin 5x} = -\sin 3x$

***Solution***

$$\frac{\cos 8x - \cos 2x}{2\sin 5x} = \frac{-2\sin\left(\frac{8x+2x}{2}\right)\sin\left(\frac{8x-2x}{2}\right)}{2\sin 5x}$$

$$= \frac{-\sin(5x)\sin(3x)}{\sin 5x}$$

$$= -\sin 3x$$

### ***Exercise***

Prove the following equation is an identity: $\dfrac{\sin 9x + \sin 3x}{\cos 9x + \cos 3x} = \tan 6x$

***Solution***

$$\frac{\sin 9x + \sin 3x}{\cos 9x + \cos 3x} = \frac{2\sin\left(\frac{9x+3x}{2}\right)\cos\left(\frac{9x-3x}{2}\right)}{2\cos\left(\frac{9x+3x}{2}\right)\cos\left(\frac{9x-3x}{2}\right)}$$

$$= \frac{2\sin(6x)\cos(3x)}{2\cos(6x)\cos(3x)}$$

$$= \frac{\sin(6x)}{\cos(6x)}$$

$$= \tan 6x$$

### ***Exercise***

Prove the following equation is an identity: $\dfrac{\cos 2x - \cos 6x}{\sin 2x + \sin 6x} = \tan 2x$

***Solution***

$$\frac{\cos 2x - \cos 6x}{\sin 2x + \sin 6x} = \frac{-2\sin\left(\frac{2x+6x}{2}\right)\sin\left(\frac{2x-6x}{2}\right)}{2\sin\left(\frac{2x+6x}{2}\right)\cos\left(\frac{2x-6x}{2}\right)}$$

$$= -\frac{\sin(4x)\sin(-2x)}{\sin(4x)\cos(-2x)}$$

$$= -\frac{-\sin 2x}{\cos 2x}$$

$$= \tan 2x$$

### ***Exercise***

Prove the following equation is an identity: $\dfrac{\sin 8x+\sin 2x}{\sin 8x-\sin 2x}=\dfrac{\tan 5x}{\tan 3x}$

***Solution***

$$\frac{\sin 8x+\sin 2x}{\sin 8x-\sin 2x}=\frac{2\sin\left(\frac{8x+2x}{2}\right)\cos\left(\frac{8x-2x}{2}\right)}{2\cos\left(\frac{8x+2x}{2}\right)\sin\left(\frac{8x-2x}{2}\right)}$$

$$=\frac{\sin(5x)\cos(3x)}{\cos(5x)\sin(3x)}$$

$$=\tan 5x\cot 3x$$

$$=\tan 5x\frac{1}{\tan 3x}$$

$$=\frac{\tan 5x}{\tan 3x}$$

### ***Exercise***

Prove the following equation is an identity: $\dfrac{\cos 6x-\cos 2x}{\cos 6x+\cos 2x}=-\tan 4x\tan 2x$

***Solution***

$$\frac{\cos 6x-\cos 2x}{\cos 6x+\cos 2x}=\frac{-2\sin\left(\frac{6x+2x}{2}\right)\sin\left(\frac{6x-2x}{2}\right)}{2\cos\left(\frac{6x+2x}{2}\right)\cos\left(\frac{6x-2x}{2}\right)}$$

$$=-\frac{\sin(4x)\sin(2x)}{\cos(4x)\cos(2x)}$$

$$=-\frac{\sin 4x}{\cos 4x}\frac{\sin 2x}{\cos 2x}$$

$$=-\tan 4x\tan 2x$$

### *Exercise*

Prove the following equation is an identity: $\sin x(\sin x+\sin 5x)=\cos 2x(\cos 2x-\cos 4x)$

***Solution***

$$\begin{aligned}
\sin x(\sin x+\sin 5x) &= \sin x\left(2\sin\left(\tfrac{x+5x}{2}\right)\cos\left(\tfrac{x-5x}{2}\right)\right)\\
&= \sin x\left(2\sin 3x\cos(-2x)\right)\\
&= 2\sin x\sin 3x\cos 2x\\
&= 2\cos 2x\left(\sin x\sin 3x\right)\\
&= 2\cos 2x\left(\tfrac{1}{2}\left[\cos(x-3x)-\cos(x+3x)\right]\right)\\
&= \cos 2x\left(\cos(-2x)-\cos 4x\right)\\
&= \cos 2x\left(\cos 2x-\cos 4x\right)
\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{\cos x+\cos y}{\sin x-\sin y}=\cot\dfrac{x-y}{2}$

***Solution***

$$\begin{aligned}
\frac{\cos x+\cos y}{\sin x-\sin y} &= \frac{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)}\\
&= \frac{\cos\left(\frac{x-y}{2}\right)}{\sin\left(\frac{x-y}{2}\right)}\\
&= \cot\left(\frac{x-y}{2}\right)
\end{aligned}$$

### *Exercise*

Prove the following equation is an identity: $\dfrac{\sin 6x+\sin 2x}{2\sin 4x}=\cos 2x$

***Solution***

$$\begin{aligned}
\frac{\sin 6x+\sin 2x}{2\sin 4x} &= \frac{2\sin\left(\frac{6x+2x}{2}\right)\cos\left(\frac{6x-2x}{2}\right)}{2\sin 4x}\\
&= \frac{\sin(4x)\cos(2x)}{\sin 4x}
\end{aligned}$$

# Solution — *Section* 3.6 – Solving Trigonometry Equations

### *Exercise*

Solve $2\cos\theta+\sqrt{3}=0$ *if* $0^\circ\le\theta<360^\circ$

***Solution***

$$2\cos\theta=-\sqrt{3}$$

$$\cos\theta=-\frac{\sqrt{3}}{2} \quad\Rightarrow\quad \hat{\theta}=\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\boxed{\theta=150^\circ,\ 210^\circ}$$

### *Exercise*

Solve $5\cos t+\sqrt{12}=\cos t$ *if* $0\le t<2\pi$

***Solution***

$$5\cos t-\cos t=-\sqrt{12}$$

$$\sqrt{12}=\sqrt{4.3}=2\sqrt{3}$$

$$4\cos t=-2\sqrt{3}$$

$$4\cos t=-2\sqrt{3}$$

$$\cos t=-\frac{2\sqrt{3}}{4}$$

$$\cos t=-\frac{\sqrt{3}}{2} \quad\Rightarrow\quad t=\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\boxed{\theta=\frac{5\pi}{6},\ \frac{7\pi}{6}}$$

### *Exercise*

Solve $\tan\theta-2\cos\theta\tan\theta=0$ *if* $0^\circ\le\theta<360^\circ$

***Solution***

$$\tan\theta(1-2\cos\theta)=0$$

$\tan\theta=0$

$\theta=0^\circ,\ 180^\circ$

$1-2\cos\theta=0$

$1=2\cos\theta$

$$\cos\theta=\frac{1}{2} \quad\Rightarrow\quad \theta=\cos^{-1}\left(\frac{1}{2}\right)$$

$\theta=60^\circ,\ 300^\circ$

$$\boxed{\theta=0^\circ,\ 60^\circ,\ 180^\circ,\ 300^\circ}$$

***Exercise***

Solve $2\sin^2\theta - 2\sin\theta - 1 = 0 \qquad if \;\; 0^\circ \le \theta < 360^\circ$

***Solution***

$$\sin\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{2 \pm \sqrt{12}}{4}$$

$$= \frac{2 \pm 2\sqrt{3}}{4}$$

$$= \frac{1 \pm \sqrt{3}}{2}$$

$\hat{\theta} = \sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right) = -21.47^\circ \qquad \sin\theta = \frac{1+\sqrt{3}}{2} = 1.366 > 1$

$\theta = 360^\circ - 21.47^\circ \; \underline{= 338.53^\circ}$

$\theta = 180^\circ + 21.47^\circ \; \underline{= 201.47^\circ}$

***Exercise***

Solve $\cos\left(A - \frac{\pi}{9}\right) = -\frac{1}{2}$

***Solution***

$-\frac{1}{2}$ *is negative* $\rightarrow$ *cosine is in QII or QIII.*

$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \qquad \cos^{-1}\left(-\frac{1}{2}\right) = \frac{4\pi}{3}$

$\cos\left(A - \frac{\pi}{9}\right) = -\frac{1}{2} = \cos\frac{2\pi}{3} \qquad \cos\left(A - \frac{\pi}{9}\right) = -\frac{1}{2} = \cos\frac{4\pi}{3}$

$A - \frac{\pi}{9} = \frac{2\pi}{3} + 2\pi k \qquad A - \frac{\pi}{9} = \frac{4\pi}{3} + 2\pi k$

$A = \frac{2\pi}{3} + \frac{\pi}{9} + 2\pi k \qquad A = \frac{4\pi}{3} + \frac{\pi}{9} + 2\pi k$

$\boxed{A = \frac{7\pi}{9} + 2\pi k} \qquad \boxed{A = \frac{13\pi}{9} + 2\pi k}$

### *Exercise*

Solve: $4\cos\theta - 3\sec\theta = 0 \;\; if \;\; 0° \le \theta < 360°$

***Solution***

$$4\cos\theta - 3\frac{1}{\cos\theta} = 0 \qquad \boxed{\cos\theta \ne 0}$$

$$4\cos\theta\cos\theta - 3\frac{1}{\cos\theta}\cos\theta = 0$$

$$4\cos^2\theta - 3 = 0$$

$$4\cos^2\theta = 3$$

$$\cos^2\theta = \frac{3}{4}$$

$$\cos\theta = \pm\frac{\sqrt{3}}{2} \Rightarrow \theta = \cos^{-1}\left(\pm\frac{\sqrt{3}}{2}\right)$$

***The solutions are:*** $\boxed{\theta = 30°,\ 150°,\ 210°,\ 330°}$

### *Exercise*

Solve: $2\sin^2 x - \cos x - 1 = 0 \;\; if \;\; 0 \le x < 2\pi$

***Solution***

$$2\left(1 - \cos^2 x\right) - \cos x - 1 = 0$$

$$2 - 2\cos^2 x - \cos x - 1 = 0$$

$$-2\cos^2 x - \cos x + 1 = 0$$

$$\cos x = -1 \qquad \cos x = \frac{1}{2}$$

$$x = \pi \qquad x = \frac{\pi}{3},\ \frac{5\pi}{3}$$

***The solutions are:*** $\boxed{x = \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}}$

***Exercise***

Solve: $\sin\theta - \sqrt{3}\cos\theta = 1 \qquad if \quad 0° \le \theta < 360°$

***Solution***

$$\sin\theta - 1 = -\sqrt{3}\cos\theta$$

$$(\sin\theta - 1)^2 = \left(-\sqrt{3}\cos\theta\right)^2$$

$$\sin^2\theta - 2\sin\theta + 1 = 3\cos^2\theta \qquad \cos^2\theta = 1 - \sin^2\theta$$

$$\sin^2\theta - 2\sin\theta + 1 = 3\left(1 - \sin^2\theta\right)$$

$$\sin^2\theta - 2\sin\theta + 1 = 3 - 3\sin^2\theta$$

$$\sin^2\theta - 2\sin\theta + 1 - 3 + 3\sin^2\theta = 0$$

$$4\sin^2\theta - 2\sin\theta - 2 = 0$$

$$\sin\theta = 1 \qquad \sin\theta = -\frac{1}{2}$$

$$\theta = 90° \qquad \theta = 210°,\ 330°$$

*Check*

| $\theta = 90°$ | $\theta = 210°$ | $\theta = 330°$ |
|---|---|---|
| $\sin 90° - \sqrt{3}\cos 90° \overset{?}{=} 1$ | $\sin 210° - \sqrt{3}\cos 210° \overset{?}{=} 1$ | $\sin 330° - \sqrt{3}\cos 330° \overset{?}{=} 1$ |
| $1 - \sqrt{3}(0) \overset{?}{=} 1$ | $-\frac{1}{2} - \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right) \overset{?}{=} 1$ | $-\frac{1}{2} - \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) \overset{?}{=} 1$ |
| $1 = 1$ | $-\frac{1}{2} + \frac{3}{2} \overset{?}{=} 1$ | $-\frac{1}{2} - \frac{3}{2} \overset{?}{=} 1$ |
| | $1 = 1$ | $-2 \ne 1$ |
| | | **(*False statement*)** |

***The solutions are:*** $\boxed{90°,\ 210°}$

### *Exercise*

Solve: $7\sin^2\theta - 9\cos 2\theta = 0 \qquad if \;\; 0° \le \theta < 360°$

***Solution***

$$7\sin^2\theta - 9\left(1-2\sin^2\theta\right) = 0 \qquad\qquad \cos^2\theta = 1-2\sin^2\theta$$

$$7\sin^2\theta - 9 + 18\sin^2\theta = 0$$

$$25\sin^2\theta - 9 = 0$$

$$25\sin^2\theta = 9$$

$$\sin^2\theta = \frac{9}{25} \quad \Rightarrow \quad \sin\theta = \pm\frac{3}{5}$$

$$\hat{\theta} = \sin^{-1}\left(\frac{3}{5}\right) \approx 36.87°$$

$$\theta \approx 36.87° \qquad\qquad \theta \approx 180° - 36.87° \approx 143.13°$$

$$\theta \approx 180° + 36.87° \approx 216.87° \qquad \theta \approx 360° - 36.87° \approx 323.13°$$

***The solutions are:*** $\boxed{36.87°,\ 143.13°,\ 216.87°,\ 323.13°}$

### *Exercise*

Solve: $2\cos^2 t - 9\cos t = 5 \qquad if \;\; 0 \le t < 2\pi$

***Solution***

$$2\cos^2 t - 9\cos t - 5 = 0$$

$$(2\cos t + 1)(\cos t - 5) = 0$$

$$2\cos t + 1 = 0 \qquad\qquad \cos t - 5 = 0$$

$$\cos t = -\frac{1}{2} \qquad\qquad \cos t = 5$$

$$\cos t = -\frac{1}{2} \qquad\qquad \cos t = 5$$

$$\hat{t} = \cos^{-1}\left(-\frac{1}{2}\right) \qquad\qquad \textit{No solution}$$

$$\hat{t} = \frac{\pi}{3}$$

*Negative sign → cosine is in QII or QIII*

$$t = \pi - \frac{\pi}{3} \qquad t = \pi + \frac{\pi}{3}$$

$$t = \frac{2\pi}{3} \qquad t = \frac{4\pi}{3}$$

***The solutions are:*** $\boxed{\frac{2\pi}{3},\ \frac{4\pi}{3}}$

## *Exercise*

Solve $\sin\theta\tan\theta = \sin\theta \quad if \;\; 0° \le \theta < 360°$

***Solution***

$$\sin\theta\tan\theta - \sin\theta = 0$$

$$\sin\theta(\tan\theta - 1) = 0$$

$\sin\theta = 0 \qquad\qquad \tan\theta - 1 = 0$

$\theta = 0°,\ 180° \qquad\qquad \tan\theta = 1$

$\theta = 45°,\ 225°$

***The solutions are:*** $\boxed{0°,\ 45°,\ 180°,\ 225°}$

## *Exercise*

Solve $\tan^2 x + \tan x - 2 = 0 \quad if \;\; 0 \le x < 2\pi$

***Solution***

$$\tan^2 x + \tan x - 2 = 0$$

$\tan x = 1 \qquad\qquad \tan x = -2$

$x = \frac{\pi}{4}, \frac{5\pi}{4} \qquad\qquad \hat{x} = \tan^{-1}(2) \approx 1.107 \qquad\qquad x \in QII,\ QIV$

$x = 2.034,\ 5.176$

***The solutions are:*** $\boxed{\frac{\pi}{4},\ \frac{5\pi}{4},\ 2.034,\ 5.176}$

***Exercise***

Solve $\tan x+\sqrt{3}=\sec x$ *if* $0\le x<2\pi$

***Solution***

$$\left(\tan x+\sqrt{3}\right)^2=(\sec x)^2$$

$$\tan^2 x+2\sqrt{3}\tan x+3=\sec^2 x$$

$$\tan^2 x+2\sqrt{3}\tan x+3=1+\tan^2 x$$

$$\tan^2 x+2\sqrt{3}\tan x+3-1-\tan^2 x=0$$

$$2\sqrt{3}\tan x+2=0$$

$$2\sqrt{3}\tan x=-2$$

$$\tan x=-\frac{2}{2\sqrt{3}}=-\frac{1}{\sqrt{3}}$$

$$x=\frac{5\pi}{6} \quad or \quad x=\frac{11\pi}{6}$$

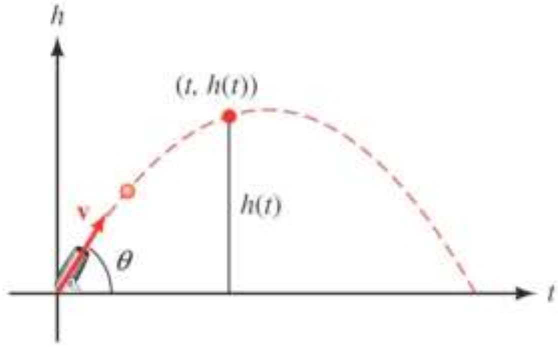
| | |
|---|---|
| $\tan\frac{5\pi}{6}+\sqrt{3}\overset{?}{=}\sec\frac{5\pi}{6}$<br>$-\frac{\sqrt{3}}{3}+\sqrt{3}\overset{?}{=}-\frac{2\sqrt{3}}{3}$<br>$\frac{2\sqrt{3}}{3}\neq-\frac{2\sqrt{3}}{3}$<br>***False*** | $\tan\frac{11\pi}{6}+\sqrt{3}\overset{?}{=}\sec\frac{11\pi}{6}$<br>$-\frac{\sqrt{3}}{3}+\sqrt{3}\overset{?}{=}\frac{2\sqrt{3}}{3}$<br>$\frac{2\sqrt{3}}{3}=\frac{2\sqrt{3}}{3}$ |

***The solutions are:*** $\boxed{\frac{11\pi}{6}}$

### *Exercise*

If a projectile (such as a bullet) is fired into the air with an initial velocity $\boldsymbol{v}$ at an angle of elevation $\theta$, then the height $h$ of the projectile at time $t$ is given by:

$$h(t) = -16t^2 + vt\sin\theta$$




*a)* Give the equation for the height, if $\boldsymbol{v}$ is 600 *ft./sec* and $\theta = 45°$.

*b)* Use the equation in part (*a*) to find the height of the object after $\sqrt{3}$ seconds.

*c)* Find the angle of elevation of $\theta$ of a rifle barrel, if a bullet fired at 1,500 ft./sec takes 3 seconds to reach a height of 750 feet. Give your answer in the nearest of a degree.

***Solution***

***a)*** $h(t) = -16t^2 + 600t\sin 45°$

$$= -16t^2 + 600t\,\frac{\sqrt{2}}{2}$$

$$= -16t^2 + 300\sqrt{2}\,t$$

***b)*** $h\left(t = \sqrt{3}\right) = -16\left(\sqrt{3}\right)^2 + 300\sqrt{2}\,\sqrt{3}$

$$\approx 686.8\ ft$$

***c)*** $h(t) = -16t^2 + vt\sin\theta$

$$750 = -16(3)^2 + 1500(3)\sin\theta$$

$$750 = -144 + 4500\sin\theta$$

$$750 + 144 = 4500\sin\theta$$

$$\frac{894}{4500} = \sin\theta$$

$$\theta = \sin^{-1}\left(\frac{894}{4500}\right) \approx 11.5°$$