x-axis $S = 2\pi \int_a^b y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

y-axis $S = 2\pi \int_a^b x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Ex S?
$$\begin{cases} x = \cos t \\ y = 1 + \sin t \end{cases} \quad 0 \le t \le 2\pi \quad \text{(x-axis)}$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-\sin t)^2 + (\cos t)^2}$$
$$= \boxed{1}$$

$$S = 2\pi \int_0^{2\pi} (1 + \sin t)\, dt$$
$$= 2\pi \left(t - \cos t \right|_0^{2\pi}$$
$$= 2\pi (2\pi - 1 + 1)$$
$$= \boxed{4\pi \ \text{unit}^2}$$

17/ $x = \frac{1}{t+1}$ $y = \frac{t}{t-1}$ $t=2$

$$\frac{d^2y}{dx^2}$$

$$\frac{dx}{dt} = \frac{-1}{(t+1)^2}$$

$$\frac{dy}{dt} = \frac{-1}{(t-1)^2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{+1}{(t-1)^2} \cdot (t+1)^2$$

$$= \left(\frac{t+1}{t-1}\right)^2$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = 2\left(\frac{t+1}{t-1}\right)\frac{-2}{(t-1)^2}$$

$$= -4\,\frac{t+1}{(t-1)^3}$$

$$\frac{d^2y}{dx^2} = \left.\frac{+4(t+1)^3}{(t-1)^3}\right|_{t=2}$$

$$= \boxed{108}$$

5) A? $\begin{cases} x = t - t^2 \quad y = 1 + e^{-t} \\ y\text{-axis} \Rightarrow x = 0 \end{cases}$

$$x = t - t^2 = 0$$

$$\boxed{t = 0, 1}$$

$$\text{Area} = \int_0^1 x\, dy$$

$$= \int_0^1 (t - t^2)(-e^{-t})\, dt$$

$$= \int_0^1 (t^2 - t)\, e^{-t}\, dt$$

		e^{-t}
$+$	$t^2 - t$	$-e^{-t}$
$-$	$2t - 1$	e^{-t}
$+$	2	$-e^{-t}$

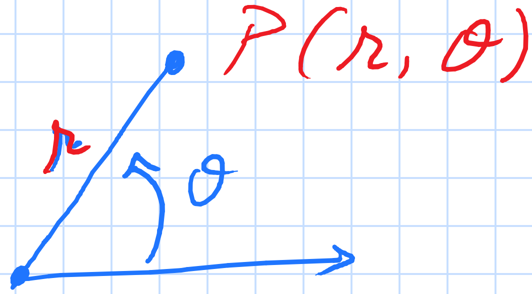
$$= (-t^2 + t - 2t + 1 - 2)\, e^{-t} \Big|_0^1$$

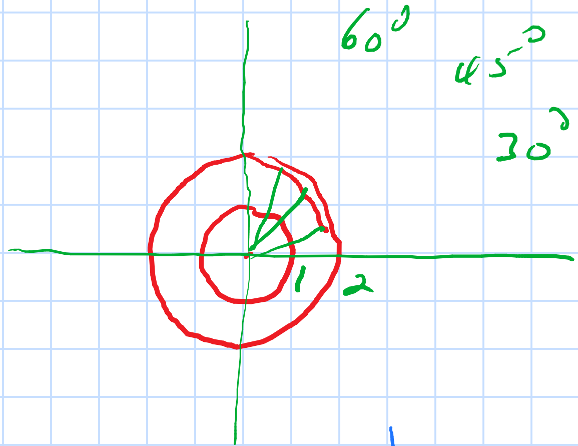
$$= (-t^2 - t - 1)\, e^{-t} \Big|_0^1$$

$$= -3e^{-1} + 1$$

$$= \boxed{1 - \frac{3}{e} \ \text{unit}^2}$$

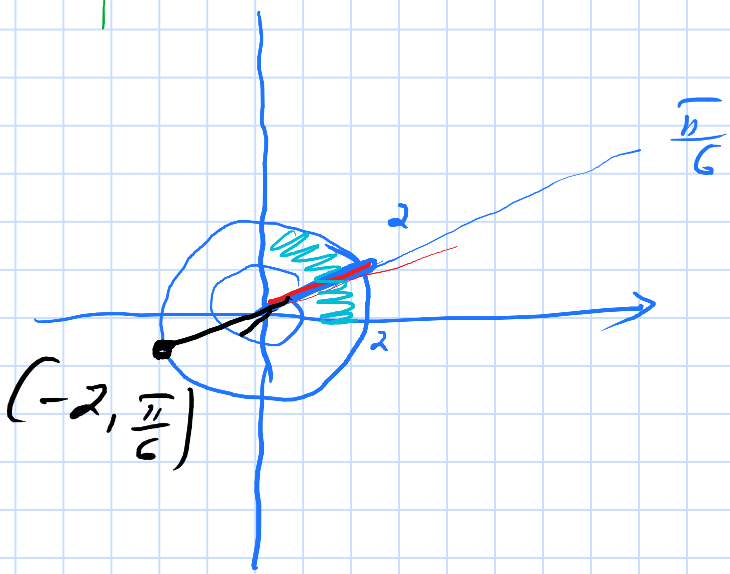
4.3 Polar Coordinates





$P\left(2, \frac{\pi}{6}\right)$

$P\left(-2, \frac{\pi}{6}\right)$



$1 \le r \le 2$

$0 \le \theta \le \frac{\pi}{2}$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$

$r\cos\theta = 2 \longrightarrow y = 2$

$r^2\cos\theta\sin\theta = 4$

$(r\cos\theta)(r\sin\theta) = 4$

$xy = 4$

$r^2\cos^2\theta - r^2\sin^2\theta = 1$

$x^2 - y^2 = 1$

$r = 1 + 2r\cos\theta$

$\left(\sqrt{x^2+y^2}\right)^2 = (1+2x)^2$

$x^2 + y^2 = 1 + 4x + 4x^2$

$y^2 - 3x^2 - 4x - 1 = 0$

$r = 1 - \cos\theta$?

$rr = r - r\cos\theta$

$x^2 + y^2 = \sqrt{x^2+y^2} - x$

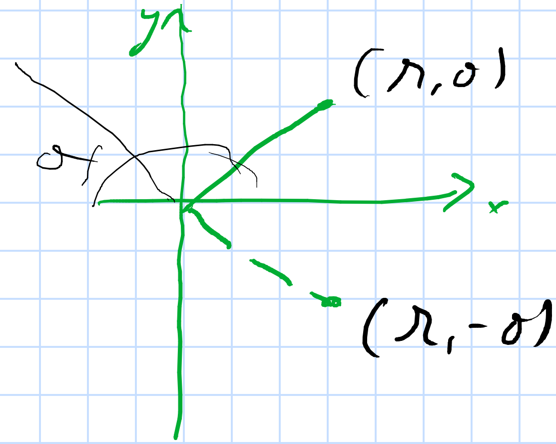
$(x^2+y^2-x)^2 = \left(\sqrt{x^2+y^2}\right)^2$

$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

Symmetry

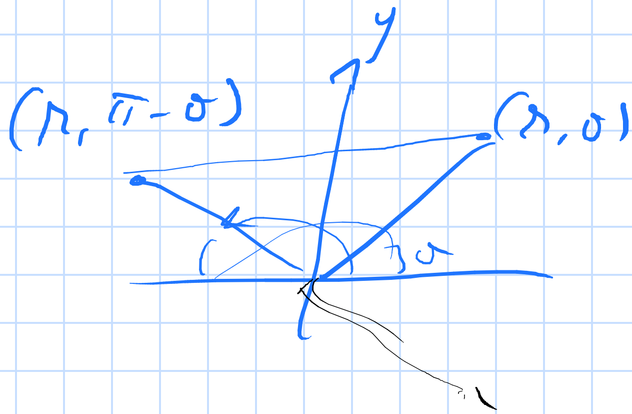
about x-axis

$$(r,\theta) \to \begin{cases} (r,-\theta) \\ (-r, \pi-\theta) \end{cases}$$



about y-axis

$$(r,\theta) \to \begin{cases} (r, \pi-\theta) \\ (-r, -\theta) \end{cases}$$

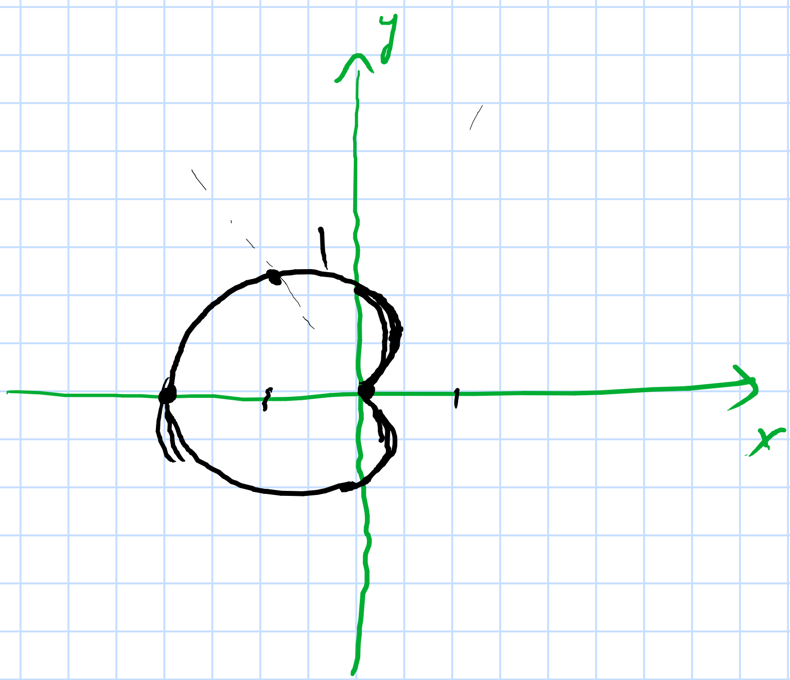


about Origin $\quad (-r,\theta)\,,\ (r, \pi+\theta)$

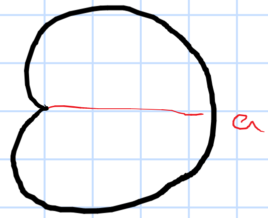
$$r = 1 - \cos\theta$$

$1-\cos(-\theta) = 1-\cos\theta = r$ symmetric x-axis

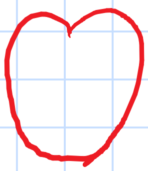
θ	r
0	0
$\pi/3$	$1/2$
$\pi/2$	1
$2\pi/3$	$3/2$
π	2



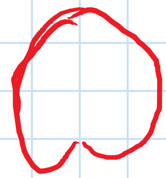
$a - \cos\theta$



$a + \cos\theta$



$a - \sin\theta$



$a + \sin\theta$

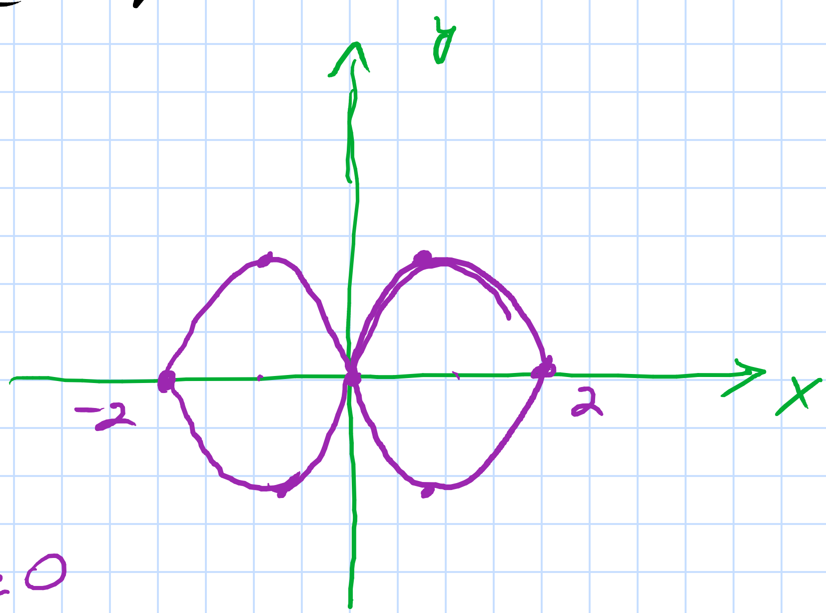
Ex $r^2 = 4\cos\theta$ x-axis

$(-r)^2 = 4\cos\theta$ origin

r^2

$r = \pm 2\sqrt{\cos\theta}$

$\frac{2}{\sqrt{2}} = \sqrt{2}$

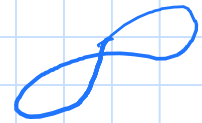


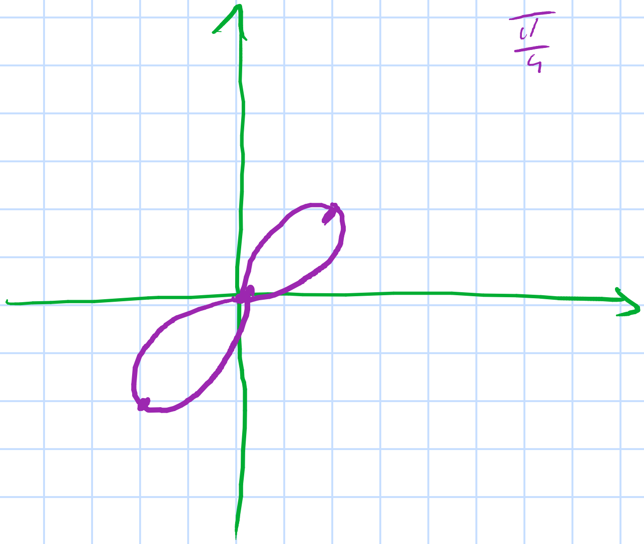
$\theta \geq 0$

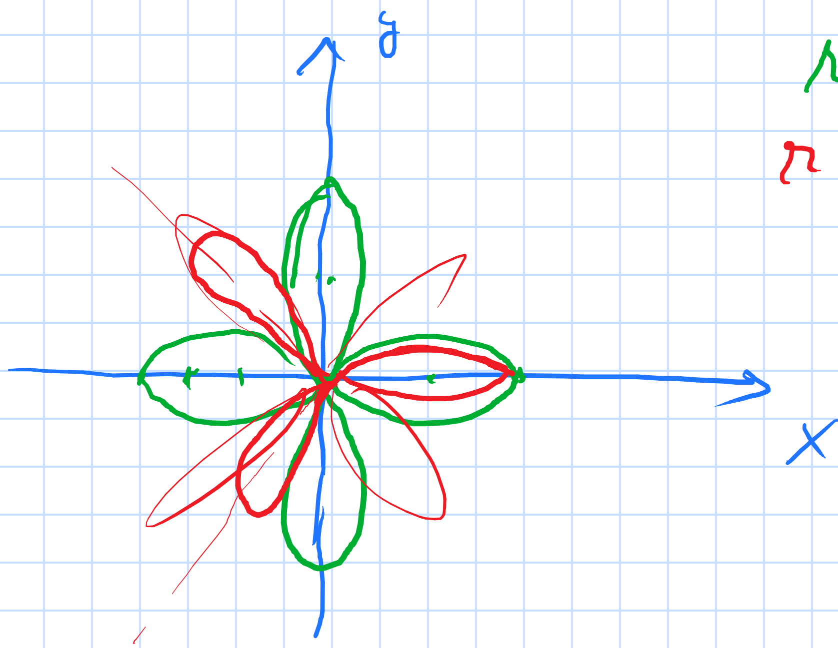
$-\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$

Ex $r^2 = \sin 2\theta$

$\frac{\pi}{4}$







$r = 2\cos 2\theta$

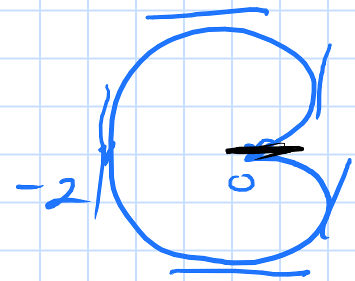
$r = 2\cos 3\theta$

4.4

$$\text{slope} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Ex $\boxed{\text{Cardioid}}$

$f(\theta) = r = 1-\cos\theta$



$r' = \sin\theta$

$$\text{slope} = \frac{\sin^2\theta + (1-\cos\theta)\cos\theta}{\sin\theta\cos\theta - (1-\cos\theta)\sin\theta}$$

$$= \frac{\sin^2\theta - \cos^2\theta + \cos\theta}{2\sin\theta\cos\theta - \sin\theta} = 0$$

$1-\cos^2\theta - \cos^2\theta + \cos\theta = 0$ ①

$\sin\theta(2\cos\theta - 1) = 0$ ②

① $-2\cos^2\theta + \cos\theta + 1 = 0$

$\cos\theta = 1$ $\qquad$ $\cos\theta = -\frac{1}{2}$

$\theta = 0$ $\qquad$ $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

② $\sin\theta = 0$ $\qquad$ $\cos\theta = \frac{1}{2}$

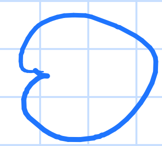
$\theta = 0, \pi$ $\qquad$ $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

Area

$A = \frac{1}{2} r^2 \theta$

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

Ex A? $r = 2(1+\cos\theta)$ enclosed



$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi} 4(1+\cos\theta)^2 \, d\theta$$

$$= 4\int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta) \, d\theta$$

$$= 4\int_0^{\pi} \left(1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= 4\int_0^{\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= 6\theta + 8\sin\theta + \sin 2\theta \Big|_0^{\pi}$$

$$= 6\pi \text{ unit}^2$$

$$A = \frac{1}{2}\int_{\alpha}^{\beta} (r_2^2 - r_1^2)\, d\theta$$

Ex A? in $r = 1$ out $r = 1 - \cos\theta$

soln

$r = 1 - \cos\theta = 1$

$\cos\theta = 0 \Rightarrow \theta = \pm\frac{\pi}{2}$

$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(1 - (1-\cos\theta)^2\right) d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta$$

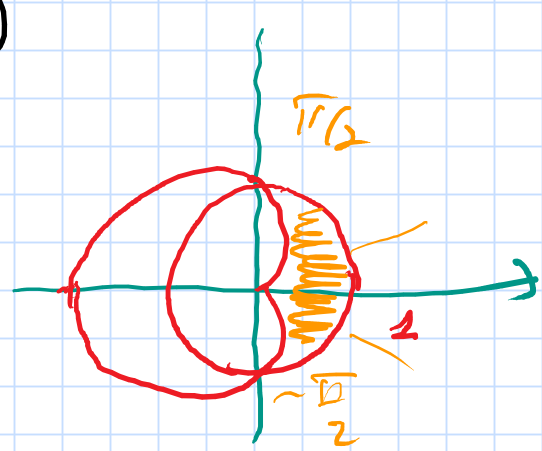
$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(2\cos\theta - \frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\left(2\sin\theta - \frac{1}{2}\theta - \frac{1}{4}\sin 4\theta \right|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2}\left(2 - \frac{\pi}{4} + 2 - \frac{\pi}{4}\right)$$

$$= \frac{1}{2}\left(4 - \frac{\pi}{2}\right)$$

$$= 2 - \frac{\pi}{4} \quad \text{unit}^2$$



$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

<u>Ex</u> $r = 1 - \cos\theta$ $\quad L?$

$$\frac{dr}{d\theta} = \sin\theta$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{(1-\cos\theta)^2 + \sin^2\theta}$$

$$= \sqrt{1 - 2\cos\theta + \underline{\cos^2\theta + \sin^2\theta}}$$

$$= \sqrt{2 - 2\cos\theta}$$

$$L = \int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta$$

$$2\sin^2\frac{\theta}{2} = \frac{1-\cos 2\theta/2}{\cancel{2}}$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta$$

$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta$$

$$= -4\cos\frac{\theta}{2}\Big|_0^{2\pi}$$

$$= -4(-1-1)$$

$$= 8 \text{ unit}$$

Surface

about

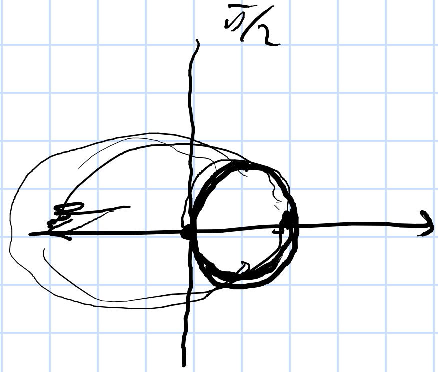
Polar axis: $S = 2\pi \int_{\alpha}^{\beta} f(\theta) \underline{\sin\theta} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta$

line $\theta = \pi/2$

$\sqrt{r^2 + r'^2}$

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta)\, \underline{\cos\theta} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta$$

Ex $f(\theta) = \cos\theta \qquad \theta = \pi/2$



$$\sqrt{r^2 + r'^2} = \sqrt{\cos^2\theta + \sin^2\theta}$$
$$= 1$$

$$S = 2\pi \int_0^{\pi} \cos^2\theta \, d\theta$$
$$= \pi \int_0^{\pi} (1 + \cos 2\theta)\, d\theta$$
$$= \pi \left(\theta + \tfrac{1}{2}\sin 2\theta \right|_0^{\pi}$$
$$= \pi(\pi)$$
$$= \pi^2 \text{ unit}^2$$

90/ $r = 6\cos\theta \qquad 0 \le \theta \le \frac{\pi}{2}$ Polar axis

$$\sqrt{r^2 + r'^2} = \sqrt{36\cos^2\theta + 36\sin^2\theta}$$

$$= 6$$

$$S = 2\pi \int_0^{\pi/2} (6\cos\theta)\sin\theta\,(6)\,d\theta$$

$$= 36\pi \int_0^{\pi/2} \sin 2\theta\,d\theta$$

$$= -18\pi \cos 2\theta \Big|_0^{\pi/2}$$

$$= -18\pi(-1-1)$$

$$= 36\pi \text{ unit}^2$$