

Solution

Section 1.4 – Inverse Matrices - Finding A^{-1}

Exercise

Apply Gauss-Jordan method to find the inverse of this triangular “Pascal matrix”

$$\text{Triangular Pascal matrix} \quad A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

Solution

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 3 & 1 & -1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ \\ R_3 - 2R_2 \\ R_4 - 3R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & 1 & 2 & -3 & 0 & 1 \end{array} \right] \begin{array}{l} \\ \\ \\ R_4 - 3R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{array} \right]$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$

✚ The inverse matrix A^{-1} looks like A , except odd-numbered diagonals are multiplied by -1.

Exercise

If A is invertible and $AB = AC$, prove that $B = C$

Solution

$$AB = AC$$

Multiply by A^{-1} both sides.

$$A^{-1}(AB) = A^{-1}(AC)$$

Multiplication is associative

$$(A^{-1}A)B = (A^{-1}A)C$$

$$A^{-1}A = I$$

$$IB = IC$$

$$\boxed{B = C}$$

Exercise

If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find two matrices $B \neq C$ such that $AB = AC$

Solution

$$\text{Let } B = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\boxed{B \neq C \Rightarrow AB = AC}$$

Exercise

If A has $\text{row } 1 + \text{row } 2 = \text{row } 3$, show that A is not invertible

- a) Explain why $Ax = (1, 0, 0)$ can't have a solution.
- b) Which right sides (b_1, b_2, b_3) might allow a solution to $Ax = b$
- c) What happens to $\text{row } 3$ in elimination?

Solution

- a) Let A_1, A_2, A_3 be the row vectors of A and x is a solution to $Ax = (1, 0, 0)$.

Then $A_1 \cdot x = 1, A_2 \cdot x = 0, A_3 \cdot x = 0$.

Since $A_1 + A_2 = A_3$

Means $A_1 \cdot x + A_2 \cdot x = A_3 \cdot x$

Implies $1 + 0 = 0$ a contradiction

- b) If $Ax = (b_1, b_2, b_3) \Rightarrow A_1 \cdot x = b_1, A_2 \cdot x = b_2, A_3 \cdot x = b_3$

Since $A_1 + A_2 = A_3$

$A_1 \cdot x + A_2 \cdot x = A_3 \cdot x$

$\Rightarrow b_1 + b_2 = b_3$

- c) In the elimination matrix, the third row will be zero.

Exercise

True or false (with a counterexample if false and a reason if true):

- a) A 4 by 4 matrix with a row of zeros is not invertible.
- b) A matrix with 1's down the main diagonal is invertible.
- c) If A is invertible then A^{-1} is invertible.
- d) If A is invertible then A^2 is invertible.

Solution

- a) True, because it can have at most 3 pivots.

- b) False, if the matrix: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and only has 2 pivots, thus is not invertible.

- c) True, If A is invertible then necessarily A^{-1} is invertible.

- d) True, $A^2 x = 0$ where x is nonzero matrix.

$$A^{-1}A^2x = (A^{-1}A)Ax = IAx = Ax = 0$$

Since A is invertible, this can only be true if x was zero to begin with. Thus A^2 must also be invertible.

Exercise

Do there exist 2 by 2 matrices A and B with real entries such that $AB - BA = I$, where I is the identity matrix?

Solution

$$\text{Let } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$\begin{aligned} AB &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ &= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{pmatrix} \end{aligned}$$

$$\begin{aligned} AB - BA &= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} - \begin{pmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{pmatrix} \\ &= \begin{pmatrix} bg - cf & af + bh - be - df \\ ce + dg - ag - ch & cf - bg \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{cases} bg - cf = 1 \\ af + bh - be - df = 0 \\ ce + dg - ag - ch = 0 \\ cf - bg = 1 \end{cases}$$

$$\rightarrow \begin{cases} bg - cf = 1 \\ cf - bg = 1 \\ \hline 0 = 2 \end{cases}$$

Therefore, $AB - BA \neq I$ for any 2 by 2 matrices.

Exercise

If B is the inverse of A^2 , show that AB is the inverse of A .

Solution

Since B is the inverse of A^2 that implies: $\underline{B = (A^2)^{-1} = (AA)^{-1} = A^{-1}A^{-1}}$

Show that AB is the inverse of A

$$\begin{aligned}(AB)A &= \left(A(A^{-1}A^{-1})\right)A \\ &= \left((AA^{-1})A^{-1}\right)A \\ &= (IA^{-1})A \\ &= A^{-1}A \\ &= \underline{I}\end{aligned}$$

Therefore, AB is the inverse of A .

Exercise

Find and check the inverses (assuming they exist) of these block matrices.

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}$$

Solution

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ C+A & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \Rightarrow C+A=0 \rightarrow \mathbf{A = -C}$$

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{C} & \mathbf{I} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{-C} & \mathbf{I} \end{pmatrix}$$

$$\begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \begin{bmatrix} E & 0 \\ F & G \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} AE & 0 \\ CE+DF & DG \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\Rightarrow \begin{cases} AE = I \\ CE + DF = 0 \\ DG = I \end{cases} \rightarrow \begin{cases} E = A^{-1} \\ G = D^{-1} \end{cases}$$

$$\begin{aligned} CE + DF = 0 &\rightarrow CA^{-1} + DF = 0 \\ DF &= -CA^{-1} \\ D^{-1}DF &= -D^{-1}CA^{-1} \\ IF &= -D^{-1}CA^{-1} \\ F &= -D^{-1}CA^{-1} \end{aligned}$$

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{pmatrix}$$

$$\begin{aligned} \begin{bmatrix} 0 & I \\ I & D \end{bmatrix} \begin{bmatrix} A & I \\ I & B \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ \begin{bmatrix} I & B \\ A+D & I+DB \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ \rightarrow \begin{cases} B=0 \\ A+D=0 \\ I+DB=I \end{cases} &\Rightarrow \begin{cases} A=-D \\ DB=0 \end{cases} \end{aligned}$$

$$\begin{pmatrix} 0 & I \\ I & D \end{pmatrix}^{-1} = \begin{pmatrix} -D & I \\ I & 0 \end{pmatrix}$$

Exercise

For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

Solution

$$c=0, A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & 7 & 0 \end{bmatrix} \text{ (zero column 2 / row 2)}$$

$$c=2, A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 8 & 7 & 2 \end{bmatrix} \text{ (equal rows)}$$

$$c=7, A = \begin{bmatrix} 2 & 7 & 7 \\ 7 & 7 & 7 \\ 8 & 7 & 7 \end{bmatrix} \text{ (equal columns)}$$

Exercise

Find A^{-1} and B^{-1} (if they exist) by elimination.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Solution

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array} \right) \frac{1}{2}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array} \right) \frac{2}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - \frac{1}{2}R_2 \\ R_3 - \frac{1}{2}R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{array} \right) \frac{3}{4}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array} \right) \begin{array}{l} R_1 - \frac{1}{3}R_3 \\ R_2 - \frac{1}{3}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array} \right) \frac{1}{2}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} R_2 + R_1 \\ R_3 + R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array} \right) R_3 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 1 & 1 \end{array} \right)$$

B^{-1} *doesn't* exist, and if we add the columns in B , the result is zero.

Exercise

Find A^{-1} using the Gauss-Jordan method, which has a remarkable inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) R_1 + R_2$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) R_2 + R_3$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) R_3 + R_4$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise

Find the inverse, if exists of $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$

Solution

$$\begin{aligned} \begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}^{-1} &= \frac{1}{12+12} \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{12} & \frac{1}{6} \\ \frac{1}{8} & \frac{1}{4} \end{bmatrix} \end{aligned}$$

Exercise

Find the inverse, if exists of $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$

Solution

$$\begin{aligned} \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}^{-1} &= \frac{1}{7-8} \begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix} \\ &= - \begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix} \end{aligned}$$

Exercise

Find the inverse, if exists of $\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$

Solution

$$\begin{aligned} \begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}^{-1} &= \frac{1}{-15-24} \begin{bmatrix} 5 & -6 \\ -4 & -3 \end{bmatrix} \\ &= -\frac{1}{39} \begin{bmatrix} 5 & -6 \\ -4 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{5}{39} & \frac{2}{13} \\ \frac{4}{39} & \frac{1}{13} \end{bmatrix} \end{aligned}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{-4+6} \begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix} \end{aligned}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{10-10} \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix} \\ &= \frac{1}{0} \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix} \end{aligned}$$

\therefore Inverse *doesn't exist*

Exercise

Find the inverse of $A = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$

Solution

$$\begin{aligned} \left[\begin{array}{cc|cc} -2 & 3 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{array} \right] & \quad -\frac{1}{2}R_1 & \quad \begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ -3 & 4 & 0 & 1 \end{array} \\ \\ \left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ -3 & 4 & 0 & 1 \end{array} \right] & \quad R_2 + 3R_1 & \quad \begin{array}{cccc} -3 & 4 & 0 & 1 \\ 3 & -\frac{9}{2} & -\frac{3}{2} & 0 \\ \hline 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array} \\ \\ \left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{array} \right] & \quad -2R_2 & \quad \begin{array}{cccc} 0 & 1 & 3 & -2 \end{array} \end{aligned}$$

$$\left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 3 & -2 \end{array} \right] R_1 + \frac{3}{2}R_2 \quad \begin{array}{cccc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & \frac{9}{2} & -3 \\ \hline 1 & 0 & 4 & -3 \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 4 & -3 \\ 0 & 1 & 3 & -2 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} a & b \\ 3 & 3 \end{bmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{3a-3b} \begin{bmatrix} 3 & -b \\ -3 & a \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{3(a-b)} & \frac{-b}{3(a-b)} \\ \frac{-3}{3(a-b)} & \frac{a}{3(a-b)} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{a-b} & \frac{-b}{3(a-b)} \\ \frac{-1}{a-b} & \frac{a}{3(a-b)} \end{bmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{bmatrix} -2 & a \\ 4 & a \end{bmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{-2a-4a} \begin{bmatrix} a & -a \\ -4 & -2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{a}{-6a} & \frac{-a}{-6a} \\ \frac{-4}{-6a} & \frac{-2}{-6a} \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} -\frac{1}{6} & \frac{1}{6} \\ \frac{2}{3a} & \frac{1}{3a} \end{bmatrix}$$

Exercise

Find the inverse of $A = \begin{bmatrix} 4 & 4 \\ b & a \end{bmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{4a-4b} \begin{bmatrix} a & -4 \\ -b & 4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{a}{4(a-b)} & \frac{-4}{4(a-b)} \\ \frac{-b}{4(a-b)} & \frac{4}{4(a-b)} \end{bmatrix} \\ &= \begin{bmatrix} \frac{a}{4(a-b)} & \frac{-1}{a-b} \\ \frac{-b}{4(a-b)} & \frac{1}{a-b} \end{bmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{-1+4} \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{3} \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= -\frac{1}{14} \begin{pmatrix} -1 & -4 \\ -3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{14} & \frac{2}{7} \\ \frac{3}{14} & -\frac{1}{7} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= -\frac{1}{5} \begin{pmatrix} -1 & -3 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{11} \begin{pmatrix} 5 & -3 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{11} & -\frac{3}{11} \\ \frac{2}{11} & \frac{1}{11} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{\textcolor{red}{0}} \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$$

\therefore Inverse *doesn't exist*

Exercise

Find the inverse of $A = \begin{pmatrix} -6 & 9 \\ 2 & -3 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{\textcolor{red}{18-18}} \begin{pmatrix} & \\ & \end{pmatrix}$$

\therefore Inverse *doesn't exist*

Exercise

Find the inverse of $A = \begin{pmatrix} -2 & 7 \\ 0 & 2 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{4} \begin{pmatrix} 2 & -7 \\ 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{7}{4} \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$

Solution

$$A = \frac{1}{-16+16} \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$$

\therefore Inverse ***doesn't exist***

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

Solution

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 2 & 1 \\ a & a \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{a} \begin{pmatrix} a & -1 \\ -a & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\frac{1}{a} \\ -1 & \frac{2}{a} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} b & 3 \\ b & 2 \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= -\frac{1}{b} \begin{pmatrix} 2 & -3 \\ -b & b \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2}{b} & \frac{3}{b} \\ 1 & -1 \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 1 & a \\ 3 & a \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= -\frac{1}{2a} \begin{pmatrix} a & -a \\ -3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2a} & -\frac{1}{2a} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$

Solution

$$\begin{aligned} A^{-1} &= \frac{1}{a^2 - 4} \begin{pmatrix} a & -2 \\ -2 & a \end{pmatrix} \\ &= \begin{pmatrix} \frac{a}{a^2 - 4} & \frac{-2}{a^2 - 4} \\ \frac{-2}{a^2 - 4} & \frac{a}{a^2 - 4} \end{pmatrix} \end{aligned}$$

Exercise

Find the inverse of $A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{\textcolor{red}{0}} \begin{pmatrix} & \\ & \end{pmatrix}$$

\therefore Inverse *doesn't exist*

Exercise

Find the inverse of $A = \begin{pmatrix} -3 & \frac{1}{2} \\ 6 & -1 \end{pmatrix}$

Solution

$$A^{-1} = \frac{1}{\textcolor{red}{0}} \begin{pmatrix} & \\ & \end{pmatrix}$$

\therefore Inverse *doesn't exist*

Exercise

Find A^{-1} if $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \quad \begin{array}{cccccc} 2 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -2 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \end{array} \quad \begin{array}{cccccc} 3 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & -3 & 0 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array} \right] -\frac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array} \right] -\frac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array} \right] \begin{array}{l} R_1 - R_3 \\ R_2 - \frac{3}{2}R_3 \\ \end{array} \quad \begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array} \quad \begin{array}{cccccc} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

Exercise

Find A^{-1} , where $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 + 2R_1 \end{array}$$

$$\begin{array}{cccccc} 3 & 5 & 3 & 0 & 1 & 0 \\ -3 & -6 & 3 & -3 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \end{array} \quad \begin{array}{cccccc} 2 & 4 & 3 & 0 & 0 & 1 \\ -2 & -4 & 2 & -2 & 0 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right] -R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_1 - 2R_2 \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 12 & -6 & 2 & 0 \\ 1 & 0 & 11 & -5 & 2 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right] \frac{1}{5}R_3$$

$$\begin{array}{cccccc} 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array} \right] \begin{array}{l} R_1 - 11R_3 \\ R_2 + 6R_3 \\ \end{array}$$

$$\begin{array}{cccccc} 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 6 & -\frac{12}{5} & 0 & \frac{6}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \end{array} \quad \begin{array}{cccccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 0 & -11 & \frac{22}{5} & 0 & -\frac{11}{5} \\ 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{bmatrix}$$

Exercise

Find A^{-1} , where $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{array}{cccccc} -2 & 0 & 1 & 0 & 1 & 0 \\ 2 & 4 & -2 & 2 & 0 & 0 \\ 0 & 4 & -1 & 2 & 1 & 0 \end{array} \quad \begin{array}{cccccc} 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & -2 & 1 & -1 & 0 & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 4 & -1 & 2 & 1 & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array} \right] \frac{1}{4}R_2$$

$$0 \quad 1 \quad -\frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{4} \quad 0$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\begin{array}{cccccc} 0 & -3 & 1 & -1 & 0 & 1 \\ 0 & 3 & -\frac{3}{4} & \frac{3}{2} & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{array} \quad \begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & \frac{1}{2} & -1 & -\frac{1}{2} & 0 \\ 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{array} \right] 4R_3$$

$$0 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array} \right] \begin{array}{l} R_1 + \frac{1}{2}R_3 \\ R_2 + \frac{1}{4}R_3 \\ \end{array}$$

$$\begin{array}{cccccc} 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 \\ 1 & 0 & 0 & 1 & 1 & 2 \end{array} \quad \begin{array}{cccccc} 0 & 1 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

Exercise

Find A^{-1} , where $A = \begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 3 & 1 & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array} \right] \quad \frac{1}{-2}R_1 \quad \begin{array}{cccccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 4 & -1 & 3 & 0 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array} \right] \quad R_2 - 4R_1 \quad \begin{array}{cccccc} 4 & -1 & 3 & 0 & 1 & 0 \\ -4 & 10 & 6 & 2 & 0 & 0 \\ \hline 0 & 9 & 9 & 2 & 1 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 7 & -2 & 5 & 0 & 0 & 1 \end{array} \right] \quad R_3 - 7R_1 \quad \begin{array}{cccccc} 7 & -2 & 5 & 0 & 0 & 1 \\ -7 & \frac{35}{2} & \frac{21}{2} & \frac{7}{2} & 0 & 0 \\ \hline 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 2 & 1 & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array} \right] \quad \frac{1}{9}R_2 \quad \begin{array}{cccccc} 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \end{array} \right] \quad R_3 - \frac{31}{2}R_2 \quad \begin{array}{cccccc} 0 & \frac{31}{2} & \frac{31}{2} & \frac{7}{2} & 0 & 1 \\ 0 & -\frac{31}{2} & -\frac{31}{2} & -\frac{31}{9} & -\frac{31}{18} & 0 \\ \hline 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{9} & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{18} & -\frac{31}{18} & 1 \end{array} \right]$$

\therefore The inverse matrix ***doesn't exist***

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 + R_1 \\ \end{array} \quad \begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ \hline 0 & 4 & 4 & 1 & 1 & 0 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 4 & 1 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array} \right) \frac{1}{4} R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 4R_2 \end{array} \quad \begin{array}{cccccc} 0 & 4 & 3 & 0 & 0 & 1 \\ 0 & -4 & -4 & -1 & -1 & 0 \\ \hline 0 & 0 & -1 & -1 & -1 & 1 \end{array} \quad \begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ \hline 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array} \right) -R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array} \right) \begin{array}{l} R_1 + R_3 \\ R_2 - R_3 \\ \end{array} \quad \begin{array}{cccccc} 1 & 0 & -1 & \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ \hline 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1 \end{array} \quad \begin{array}{cccccc} 0 & 1 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \\ \hline 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & -1 \\ 0 & 1 & 0 & -\frac{3}{4} & -\frac{3}{4} & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{7}{4} & \frac{3}{4} & -1 \\ -\frac{3}{4} & -\frac{3}{4} & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ -2 & -3 & 0 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_3 + 2R_1 \end{array} \quad \begin{array}{cccccc} -2 & -3 & 0 & 0 & 0 & 1 \\ 2 & -2 & 2 & 2 & 0 & 0 \\ \hline 0 & -5 & 2 & 2 & 0 & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array} \right) -\frac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 + R_2 \\ R_3 + 5R_2 \end{array} \quad \begin{array}{cccccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ \hline 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \end{array} \quad \begin{array}{cccccc} 0 & -5 & 2 & 2 & 0 & 1 \\ 0 & 5 & -\frac{5}{2} & 0 & -\frac{5}{2} & 0 \\ \hline 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \end{array} \right) -2R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -4 & 5 & -2 \end{array} \right) \begin{array}{l} R_1 - \frac{1}{2}R_3 \\ R_2 + \frac{1}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & -4 & 5 & -2 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 + R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array} \right) \frac{1}{2}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array} \right) R_3 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array} \right) 2R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right) \begin{array}{l} R_1 - 2R_3 \\ R_2 - \frac{5}{2}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 1 & 2 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & -1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -4 & -3 & 1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array} \right) -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 2R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & 0 & 7 & 3 & -2 & 1 \end{array} \right) \frac{1}{7}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{array} \right) \begin{array}{l} R_1 + 3R_3 \\ R_2 - 4R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ 0 & 1 & 0 & \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 3 & 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array} \right) \frac{1}{3}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} R_2 - R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & -3 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - R_2 \\ R_3 + 3R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{7}{3} & -\frac{5}{3} & 3 & 1 \end{array} \right) \frac{3}{7}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array} \right) \begin{array}{l} R_1 + \frac{1}{3}R_3 \\ R_2 - \frac{2}{3}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ 0 & 1 & 0 & \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{pmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} -3 & 1 & -1 \\ 1 & -4 & -7 \\ 1 & 2 & 5 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} -3 & 1 & -1 & 1 & 0 & 0 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array} \right) \quad -\frac{1}{3}R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array} \right) \quad \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{11}{3} & -\frac{22}{3} & \frac{1}{3} & 1 & 0 \\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1 \end{array} \right) \quad -\frac{3}{11}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0 \\ 0 & \frac{7}{3} & \frac{14}{3} & \frac{1}{3} & 0 & 1 \end{array} \right) \quad R_3 - \frac{7}{3}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 2 & -\frac{1}{11} & -\frac{3}{11} & 0 \\ 0 & 0 & 0 & -\frac{1}{11} & -\frac{3}{11} & 0 \end{array} \right)$$

\therefore Inverse **does not exist**

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 1 & -3 \\ 2 & -4 & 1 \\ -5 & 7 & 1 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 2 & -4 & 1 & 0 & 1 & 0 \\ -5 & 7 & 1 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 5R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & -6 & 7 & -2 & 1 & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array} \right) -\frac{1}{6}R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 12R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & 0 \end{array} \right)$$

\therefore Inverse **does not exist**

Exercise

Find the inverse, if exists, of $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{pmatrix}$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} \\ R_3 - 3R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right) -R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right) R_1 - 2R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 5 & -2 & 0 & 1 \end{array} \right) \frac{1}{5}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -5 & 2 & 0 \\ 0 & 1 & -6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array} \right) \begin{array}{l} R_1 - 11R_3 \\ R_2 + 6R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & 2 & -\frac{11}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -1 & \frac{6}{5} \\ 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{pmatrix}$$

Exercise

Find the inverse, if exists of $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] R_3 - R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array} \right] R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array} \right] -\frac{1}{2}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array} \right] \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Exercise

Find the inverse, if exists of $A = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{pmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & 1 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} & 0 & 1 & 0 \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} & 0 & 0 & 1 \end{array} \right] \begin{array}{l} 5R_1 \\ 10R_2 \\ 10R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 2 & 2 & 1 & 0 & 10 & 0 \\ 2 & -8 & 1 & 0 & 0 & 10 \end{array} \right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 0 & 5 & -10 & 10 & 0 \\ 0 & -10 & 5 & -10 & 0 & 10 \end{array} \right] \begin{array}{l} \\ R_2 - \frac{1}{10}R_3 \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & \frac{9}{2} & -9 & 10 & -1 \\ 0 & -10 & 5 & -10 & 0 & 10 \end{array} \right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 10R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{13}{2} & 14 & -10 & 1 \\ 0 & 1 & \frac{9}{2} & -9 & 10 & -1 \\ 0 & 0 & 50 & -100 & 100 & 0 \end{array} \right] \frac{1}{50}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{13}{2} & 14 & -10 & 1 \\ 0 & 1 & \frac{9}{2} & -9 & 10 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array} \right] \begin{array}{l} R_1 + \frac{13}{2}R_3 \\ R_2 - \frac{9}{2}R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{bmatrix}$$

Exercise

Find the inverse, if exists of $A = \begin{pmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solution

$$\left[\begin{array}{ccc|ccc} \sqrt{2} & 3\sqrt{2} & 0 & 1 & 0 & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 + 4R_1 \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} \sqrt{2} & 3\sqrt{2} & 0 & 1 & 0 & 0 \\ 0 & 13\sqrt{2} & 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} 13R_1 - 3R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 13\sqrt{2} & 0 & 0 & 1 & -3 & 0 \\ 0 & 13\sqrt{2} & 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \frac{1}{13\sqrt{2}}R_1 \\ \frac{1}{13\sqrt{2}}R_2 \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{13\sqrt{2}} & -\frac{3}{13\sqrt{2}} & 0 \\ 0 & 1 & 0 & \frac{4}{13\sqrt{2}} & \frac{1}{13\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$A^{-1} = \begin{pmatrix} \frac{\sqrt{2}}{26} & -\frac{3\sqrt{2}}{26} & 0 \\ \frac{2\sqrt{2}}{13} & \frac{\sqrt{2}}{26} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise

Find the inverse, if exists of $A = \begin{pmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{pmatrix}$

Solution

$$\begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}^{-1} = \text{doesn't exist}$$

Since this matrix is *singular*, row 3 all zeros.

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -6 & 1 \\ -2 & -2 & 5 & 1 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} -2 & -3 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad -\frac{1}{2}R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 5 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad R_4 + 2R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array} \right] R_4 - R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & \frac{3}{2} & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

\therefore Inverse *does not exist*

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 1 & -14 & 7 & 38 \\ -1 & 2 & 1 & -2 \\ 1 & 2 & -1 & -6 \\ 1 & -2 & 3 & 6 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & -6 & 0 & 0 & 1 & 0 \\ 1 & -2 & 3 & 6 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_2 + R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & -12 & 8 & 36 & 1 & 1 & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array} \right] -\frac{1}{12}R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & -14 & 7 & 38 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 16 & -8 & -44 & -1 & 0 & 1 & 0 \\ 0 & 12 & -4 & -32 & -1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 + 14R_2 \\ R_3 - 16R_2 \\ R_4 - 12R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & \frac{8}{3} & 4 & \frac{1}{3} & \frac{4}{3} & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array} \right] \frac{3}{8}R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -\frac{7}{3} & -4 & -\frac{1}{6} & -\frac{7}{6} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & -3 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 4 & 4 & 0 & 1 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 + \frac{7}{3}R_3 \\ R_2 + \frac{2}{3}R_3 \\ \\ R_4 - 4R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & -2 & -\frac{1}{2} & -1 & -\frac{3}{2} & 1 \end{array} \right] -\frac{1}{2}R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{8} & 0 & \frac{7}{8} & 0 \\ 0 & 1 & 0 & -2 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array} \right] \begin{array}{l} R_1 + \frac{1}{2}R_4 \\ R_2 + 2R_4 \\ R_3 - \frac{3}{2}R_4 \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ 0 & 0 & 1 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{5}{4} & \frac{7}{4} & -1 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \end{bmatrix}$$

Exercise

Find the inverse, if exists, of $A = \begin{bmatrix} 10 & 20 & -30 & 15 \\ 3 & -7 & 14 & -8 \\ -7 & -2 & -1 & 2 \\ 4 & 4 & -3 & 1 \end{bmatrix}$

Solution

$$\left[\begin{array}{cccc|cccc} 10 & 20 & -30 & 15 & 1 & 0 & 0 & 0 \\ 3 & -7 & 14 & -8 & 0 & 1 & 0 & 0 \\ -7 & -2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & -3 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad \frac{1}{10}R_1$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 3 & -7 & 14 & -8 & 0 & 1 & 0 & 0 \\ -7 & -2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & -3 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 + 7R_1 \\ R_4 - 4R_1 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 0 & -13 & 23 & -\frac{25}{2} & -\frac{3}{10} & 1 & 0 & 0 \\ 0 & 12 & -22 & \frac{25}{2} & \frac{7}{10} & 0 & 1 & 0 \\ 0 & -4 & 9 & -5 & -\frac{2}{5} & 0 & 0 & 1 \end{array} \right] \quad -\frac{1}{13}R_2$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & -3 & \frac{3}{2} & \frac{1}{10} & 0 & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 12 & -22 & \frac{25}{2} & \frac{7}{10} & 0 & 1 & 0 \\ 0 & -4 & 9 & -5 & -\frac{2}{5} & 0 & 0 & 1 \end{array} \right] \quad \begin{array}{l} R_1 - 2R_2 \\ R_3 - 12R_2 \\ R_4 + 4R_2 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 0 & -\frac{10}{13} & \frac{25}{26} & \frac{11}{26} & \frac{12}{13} & 1 & 0 \\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1 \end{array} \right] \quad -\frac{13}{10}R_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \frac{7}{13} & -\frac{11}{26} & \frac{7}{130} & \frac{2}{13} & 0 & 0 \\ 0 & 1 & -\frac{23}{13} & \frac{25}{26} & \frac{3}{130} & -\frac{1}{13} & 0 & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & \frac{25}{13} & -\frac{15}{13} & -\frac{4}{13} & -\frac{4}{13} & 0 & 1 \end{array} \right] \begin{array}{l} R_1 - \frac{7}{13}R_3 \\ R_2 + \frac{23}{13}R_3 \\ R_4 - \frac{25}{13}R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{19}{20} & -\frac{11}{5} & -\frac{23}{10} & 0 \\ 0 & 0 & 1 & -\frac{5}{4} & -\frac{11}{20} & -\frac{6}{5} & -\frac{13}{10} & 0 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array} \right] \begin{array}{l} R_2 + R_4 \\ R_3 + R_4 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 2 & \frac{5}{2} & 1 \end{array} \right] \frac{4}{5}R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{1}{4} & \frac{7}{20} & \frac{4}{5} & \frac{7}{10} & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array} \right] R_1 - \frac{1}{4}R_4$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & 0 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} & 1 \\ \frac{1}{5} & \frac{4}{5} & \frac{6}{5} & 1 \\ \frac{3}{5} & \frac{8}{5} & 2 & \frac{4}{5} \end{bmatrix}$$

Exercise

Show that A is not invertible for any values of the entries

$$A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

Solution

Since the matrix A had zero's on its diagonals, therefore A is not invertible.

Exercise

Prove that if A is an invertible matrix and B is row equivalent to A , then B is also invertible.

Solution

Since B is row equivalent to A , there exist some elementary matrices E_1, E_2, \dots, E_n such that $B = E_n \dots E_1 A$. Because E_1, E_2, \dots, E_n and A are invertible, then B is also invertible.

Exercise

Determine if the given matrix has an inverse, and find the inverse if it exists. Check your answer by multiplying $A \cdot A^{-1} = I$

$$a) \begin{bmatrix} 2 & 3 \\ -3 & -5 \end{bmatrix} \quad b) \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}$$

Solution

$$a) \quad 2(-5) - 3(-3) = -10 + 9 = -1$$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} -5 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ -3 & -2 \end{bmatrix}$$

$$AA^{-1} = \begin{pmatrix} 2 & 3 \\ -3 & -5 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$b) \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array} \right] \quad R_3 - 2R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 3 & 3 & -2 & 0 & 1 \end{array} \right] \quad R_3 - 3R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & * & * & * \end{array} \right]$$

The inverse matrix doesn't exist

Exercise

Show that the inverse of $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ is $\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$

Solution

$$\begin{aligned} & \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} \\ &= \begin{bmatrix} (\cos \theta)\cos(-\theta) - (\sin \theta)\sin(-\theta) & (\cos \theta)\sin(-\theta) - (\sin \theta)\cos(-\theta) \\ (-\sin \theta)\cos(-\theta) - (\cos \theta)\sin(-\theta) & (-\sin \theta)\sin(-\theta) + (\cos \theta)\cos(-\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta \cos \theta + \sin \theta \sin \theta & -\cos \theta \sin \theta - \sin \theta \cos \theta \\ -\sin \theta \cos \theta + \cos \theta \sin \theta & \sin \theta \sin \theta + \cos \theta \cos \theta \end{bmatrix} \quad \begin{cases} \cos(-\theta) = \cos \theta & (\text{even}) \\ \sin(-\theta) = -\sin \theta & (\text{odd}) \end{cases} \\ &= \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & 0 \\ 0 & \sin^2 \theta + \cos^2 \theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \underline{I} \end{aligned}$$

Exercise

If the product $C = AB$ is invertible (and A & B are square matrices), find a formula for A^{-1} that involves C^{-1} and B .

Hence, it is not possible to multiply a non-invertible matrix by another matrix and obtain an invertible matrix as a result.

Solution

Since $C = AB$ is invertible, the $CC^{-1} = C^{-1}C = I$

$$CC^{-1} = I$$

$$(AB)C^{-1} = I$$

$$A(BC^{-1}) = I$$

$$A^{-1}A(BC^{-1}) = A^{-1}I$$

$$I(BC^{-1}) = A^{-1}$$

$$\underline{BC^{-1} = A^{-1}} \quad \checkmark$$

Exercise

Prove that if A is an $m \times n$ matrix, there is an invertible matrix C such that CA is in reduced row-echelon form.

Solution

The reduced row-echelon form of A can be written in the form $E_n \dots E_2 E_1 A$ where

E_1, E_2, \dots, E_n are elementary matrices.

Let $C = E_n \dots E_2 E_1$, then C is invertible since E_1, E_2, \dots, E_n are invertible.

Hence, there exists such a matrix C .

Exercise

Prove that 2 $m \times n$ matrices A and B are row equivalent if and only if there exists a nonsingular matrix P such that $B = PA$

Solution

Suppose that $A \sim B$, then there exist elementary matrices E_1, E_2, \dots, E_n such that

$$B = E_n \dots E_2 E_1 A.$$

Let $P = E_n \dots E_2 E_1 \Rightarrow$ by the theorem, P is nonsingular.

Suppose that $B = PA$, for some nonsingular matrix P . By the theorem, P is row equivalent to I_k .

That is, $I_k = E_n \dots E_2 E_1 P$.

Thus, $B = E_n^{-1} E_{n-1}^{-1} \dots E_1^{-1} A$ and this implies that A is row equivalent to B .

Exercise

Let A and B be 2 $m \times n$ matrices. Suppose A is row equivalent to B . Prove that A is nonsingular if and only if B is nonsingular.

Solution

Suppose that A is row equivalent to B . Then, there exists a nonsingular matrix P such that $B = PA$.

If A is nonsingular then B is nonsingular.

Conversely, if B is nonsingular then $A = P^{-1}B$ is nonsingular.

Exercise

Show that if A and B are two $n \times n$ invertible matrices then A is row equivalent to B .

Solution

Since A is invertible, then A is row equivalent to I_n . That is, there exist elementary matrices

E_1, E_2, \dots, E_k such that $I_n = E_k E_{k-1} \cdots E_1 A$.

Similarly, there exist elementary matrices F_1, F_2, \dots, F_k such that $I_n = F_i F_{i-1} \cdots F_1 B$.

$$\begin{aligned} \text{Hence, } A &= E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n \\ &= E_1^{-1} E_2^{-1} \cdots E_k^{-1} (F_i F_{i-1} \cdots F_1 B) \\ &= (E_1^{-1} E_2^{-1} \cdots E_k^{-1} F_i F_{i-1} \cdots F_1 B) \end{aligned}$$

That is, A row equivalent to B .

Exercise

Prove that a square matrix A is nonsingular if and only if A is a product of elementary matrices.

Solution

Suppose that A is nonsingular. Then A is row equivalent to I_n . That is, there exist elementary

matrices E_1, E_2, \dots, E_k such that $I_n = E_k E_{k-1} \cdots E_1 A \rightarrow A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n$.

But each E_i^{-1} is an elementary matrix.

Conversely, suppose that $A = E_1 E_2 \cdots E_k$, then $(E_1 E_2 \cdots E_k)^{-1} A = I_n$

That is, A is nonsingular.

Exercise

Show that if $A \sim B$ (that is, if they are row equivalent), then $EA = B$ for some matrix E which is a product of elementary matrices.

Solution

If $A \sim B$, there is some sequence of elementary row operations which, when performed on A , produce B .

Further, multiplying on the left by the corresponding elementary matrix is the same as performing that row operation. So we have

$$\begin{aligned} A &\sim E_1 A \\ &\sim E_2 E_1 A \end{aligned}$$

$$\sim E_k E_{k-1} \dots E_2 E_1 A$$

$$= B$$

Thus, if $E = E_k \dots E_1$, then we have $EA = B$

Exercise

Show that if $EA = B$ for some matrix E which is a product of elementary matrices, then $AC \sim BC$ for every $n \times n$ matrix C .

Solution

Let $E = E_k E_{k-1} \dots E_1$, where each E_i is an elementary matrix.

$$\begin{aligned} AC &\sim E_1 AC \\ &\sim E_2 E_1 AC \\ &\sim E_k E_{k-1} \dots E_2 E_1 AC \\ &= EAC && \text{since } EA = B \\ &= BC \end{aligned}$$

Therefore; $AC \sim BC$

Exercise

Let $A\vec{x} = 0$ be a homogeneous system of n linear equations in n unknowns that has only the trivial solution. Show that if k is any positive integer, then the system $A^k \vec{x} = 0$ also has only trivial solution.

Solution

Since A is a square matrix, thus A has only the trivial solution. That implies that A is invertible.

But A^k is also invertible so $A^k \vec{x} = 0$ has only trivial solution.

Exercise

Let $A\vec{x} = 0$ be a homogeneous system of n linear equations in n unknowns, and let Q be an invertible $n \times n$ matrix. Show that $A\vec{x} = 0$ has just trivial solution if and only if $(QA)\vec{x} = 0$ has just trivial solution.

Solution

A is a square $(n \times n)$ matrix. If $A\vec{x} = 0$ has just a trivial solution, then A is invertible. Since Q is an invertible $n \times n$ matrix, then QA is also invertible.

Thus, $(QA)\vec{x} = 0$ has trivial solution.

On the other hand, if $(QA)\vec{x} = 0$ has trivial solution, then QA is also invertible.

Since Q is invertible, then Q^{-1} is also invertible.

Thus, $A = Q^{-1}QA$ is invertible, i.e $A\vec{x} = 0$ has just trivial solution, equivalent $A\vec{x} = 0$ has just trivial solution if and only if $(QA)\vec{x} = 0$ has just trivial solution.

Exercise

Let $A\vec{x} = b$ be any consistent system of linear equations, and let \vec{x}_1 be a fixed solution. Show that every solution to the system can be written in the form $\vec{x} = \vec{x}_1 + \vec{x}_0$ where \vec{x}_0 is a solution to $A\vec{x} = 0$. Show also that every matrix of this form is a solution.

Solution

Since \vec{x}_0 is a solution to $A\vec{x} = 0$, we have $A\vec{x}_0 = 0$.

Adding $A\vec{x}_0 = 0$ to $A\vec{x} = b$, then

$$A\vec{x} + A\vec{x}_0 = b + 0$$

$$A(\vec{x} + \vec{x}_0) = b$$

As adding an equation to the original equation does not affect the solution.

If we let \vec{x}_1 be a fixed solution, then every solution to $A\vec{x} = b$ is $\vec{x} = \vec{x}_1 + \vec{x}_0$.

Besides,

$$\begin{aligned} A(\vec{x} + \vec{x}_0) &= A\vec{x} + A\vec{x}_0 \\ &= b + 0 \\ &= b \end{aligned}$$

So, every matrix (vector) in the form $\vec{x}_1 + \vec{x}_0$ is a solution to $A\vec{x} = b$

Exercise

If A and B are $n \times n$ matrices satisfying $A^2 = B^2 = (AB)^2 = I_n$. Prove that $AB = BA$.

Solution

Since $A^2 = B^2 = (AB)^2 = I_n$, then A, B, AB are nonsingular.

$$A^2 = I \rightarrow A = A^{-1}$$

$$B^2 = I \rightarrow B = B^{-1}$$

$$(AB)^2 = I \rightarrow AB = (AB)^{-1}$$

$$\begin{aligned}
 AB &= (AB)^{-1} \\
 &= B^{-1}A^{-1} \\
 &= \underline{BA} \quad \checkmark
 \end{aligned}$$

Exercise

Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix}$. Verify that $A^3 = 5I$, then find A^{-1} in term of A .

Solution

$$\begin{aligned}
 A^2 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & 0 & 1 \\ 5 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 A^3 &= AA^2 \\
 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 5 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} \\
 &= 5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 &= \underline{5I}
 \end{aligned}$$

Since $A^3 = AA^2 = 5I$

$$\frac{1}{5}(AA^2) = I$$

$$A\left(\frac{1}{5}A^2\right) = I$$

$$\underline{A^{-1} = \frac{1}{5}A^2}$$

Exercise

Consider $B(A, I) = (BA, B)$, thus if B is the inverse of A , then (BA, B) becomes (I, A^{-1}) . On the other hand B is a product of elementary matrices since it is invertible. This indicates that the inverse of A can be obtained by applying elementary row operations to (A, I) to get (I, A^{-1}) .

Find the inverses of

$$a) \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix}$$

$$b) \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{pmatrix}$$

Solution

$$a) \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix} \xrightarrow{R_3 - aR_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{pmatrix} \quad E_{31} = -a$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{pmatrix} \xrightarrow{R_3 - bR_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_{32} = -b$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & -b & 1 \end{pmatrix}$$

b) First, we have move row 4 to row 1, for the calculation

$$\begin{pmatrix} a & b & c & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} & \frac{c}{a} & \frac{d}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad E_{11} = \frac{1}{a}$$

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{c}{a} & \frac{d}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - \frac{b}{a}R_2} \begin{pmatrix} 1 & 0 & \frac{c}{a} & \frac{d}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad E_{12} = -\frac{b}{a}$$

$$\begin{pmatrix} 1 & 0 & \frac{c}{a} & \frac{d}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_1 - \frac{c}{a}R_3 \quad E_{13} = -\frac{c}{a}$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{d}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R_1 - \frac{d}{a}R_4 \quad E_{14} = -\frac{d}{a}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$E = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} & -\frac{c}{a} & -\frac{d}{a} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since we move Row 4 to Row 1, we must move Column 1 to Column 4 to get the inverse matrix.

$$B^{-1} = \begin{pmatrix} -\frac{b}{a} & -\frac{c}{a} & -\frac{d}{a} & \frac{1}{a} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Exercise

Let $A, B, C, X, Y, Z \in M_n(\mathbb{C})$, A and C are invertible. Find

$$a) \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}^{-1}$$

$$b) \begin{pmatrix} I & X & Y \\ 0 & I & Z \\ 0 & 0 & I \end{pmatrix}^{-1}$$

Solution

$$a) \left(A \quad B \mid I \quad 0 \right) \quad A^{-1}R_1$$

$$\left(\begin{array}{cc|cc} A^{-1}A & A^{-1}B & A^{-1}I & 0 \\ 0 & C & 0 & I \end{array} \right)$$

$$\left(\begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ 0 & C & 0 & I \end{array} \right) \quad \textcolor{red}{C}^{-1}R_2$$

$$\left(\begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ 0 & C^{-1}C & 0 & C^{-1}I \end{array} \right)$$

$$\left(\begin{array}{cc|cc} I & A^{-1}B & A^{-1} & 0 \\ 0 & I & 0 & C^{-1} \end{array} \right) \quad R_1 - \textcolor{red}{A}^{-1}\textcolor{red}{B}R_2$$

$$\left(\begin{array}{cc|cc} I & A^{-1}B - A^{-1}BI & A^{-1} & -A^{-1}BC^{-1} \\ 0 & I & 0 & C^{-1} \end{array} \right)$$

$$\left(\begin{array}{cc|cc} I & A^{-1}B - A^{-1}B & A^{-1} & -A^{-1}BC^{-1} \\ 0 & I & 0 & C^{-1} \end{array} \right)$$

$$\left(\begin{array}{cc|cc} I & 0 & A^{-1} & -A^{-1}BC^{-1} \\ 0 & I & 0 & C^{-1} \end{array} \right)$$

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}^{-1} = \begin{pmatrix} \textcolor{blue}{A}^{-1} & -\textcolor{blue}{A}^{-1}\textcolor{blue}{B}C^{-1} \\ \textcolor{blue}{0} & \textcolor{blue}{C}^{-1} \end{pmatrix}$$

$$b) \left(\begin{array}{ccc|ccc} I & X & Y & I & 0 & 0 \\ 0 & I & Z & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 & I \end{array} \right) \quad R_1 - \textcolor{red}{X}R_2$$

$$\left(\begin{array}{ccc|ccc} I & 0 & Y - XZ & I & -X & 0 \\ 0 & I & Z & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 & I \end{array} \right) \quad \begin{array}{l} R_1 - (\textcolor{red}{Y} - \textcolor{red}{X}\textcolor{red}{Z})R_3 \\ R_2 - \textcolor{red}{Z}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} I & 0 & 0 & I & -X & XZ - Y \\ 0 & I & 0 & 0 & I & -Z \\ 0 & 0 & I & 0 & 0 & I \end{array} \right)$$

$$\begin{pmatrix} I & X & Y \\ 0 & I & Z \\ 0 & 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} \textcolor{blue}{I} & -\textcolor{blue}{X} & \textcolor{blue}{X}Z - \textcolor{blue}{Y} \\ \textcolor{blue}{0} & \textcolor{blue}{I} & -\textcolor{blue}{Z} \\ \textcolor{blue}{0} & \textcolor{blue}{0} & \textcolor{blue}{I} \end{pmatrix}$$

Exercise

Suppose that A , B , and $A - B$ are invertible $n \times n$ matrices. Show that

$$(A - B)^{-1} = A^{-1} + A^{-1}(B^{-1} - A^{-1})^{-1}A^{-1}$$

Solution

A , B , and $A - B$ are invertible Then

$$AA^{-1} = A^{-1}A = I \quad BB^{-1} = B^{-1}B = I$$

$$(A - B)(A - B)^{-1} = (A - B)^{-1}(A - B) = I$$

Let:

$$(A - B)^{-1}(A - B) = I$$

Then, we need to prove that

$$\left(A^{-1} + A^{-1}(B^{-1} - A^{-1})^{-1}A^{-1} \right)(A - B) \stackrel{?}{=} I$$

$$\begin{aligned} \left(A^{-1} + A^{-1}(B^{-1} - A^{-1})^{-1}A^{-1} \right)(A - B) &= \left(A^{-1} + A^{-1} \left(A(B^{-1} - A^{-1}) \right)^{-1} \right)(A - B) \\ & \qquad \qquad \qquad \left(A(B^{-1} - A^{-1}) \right)^{-1} = (B^{-1} - A^{-1})^{-1}A^{-1} \end{aligned}$$

$$= \left(A^{-1} + A^{-1} \left(AB^{-1} - AA^{-1} \right)^{-1} \right)(A - B)$$

$$= \left(A^{-1} + A^{-1} \left(AB^{-1} - I \right)^{-1} \right)(A - B)$$

$$= \left(A^{-1} + A^{-1} \left(AB^{-1} - \textcolor{red}{BB}^{-1} \right)^{-1} \right)(A - B)$$

$$= \left(A^{-1} + A^{-1} \left((A - B)B^{-1} \right)^{-1} \right)(A - B)$$

$$\left((A - B)B^{-1} \right)^{-1} = B(A - B)^{-1}$$

$$= \left(A^{-1} + A^{-1} \left(B(A - B)^{-1} \right) \right)(A - B)$$

$$= \left(A^{-1} + A^{-1}B(A - B)^{-1} \right)(A - B)$$

$$= A^{-1}(A - B) + A^{-1}B(A - B)^{-1}(A - B)$$

$$= A^{-1}A - A^{-1}B + A^{-1}B \textcolor{red}{I}$$

$$= I - A^{-1}B + A^{-1}B$$

$$= I \quad \checkmark$$

$$\text{Therefore; } (A - B)^{-1} = A^{-1} + A^{-1} (B^{-1} - A^{-1})^{-1} A^{-1}$$

Exercise

Suppose P is invertible and $A = PBP^{-1}$. Solve for B in terms of A .

Solution

Since P is invertible, then $PP^{-1} = P^{-1}P = I$

$$A = PBP^{-1}$$

$$P^{-1}AP = P^{-1}PBP^{-1}P \quad PP^{-1} = P^{-1}P = I$$

$$P^{-1}AP = IBI \quad BI = B$$

$$\underline{P^{-1}AP = B} \quad \checkmark$$

Exercise

Suppose $(A - B)C = 0$, where A and B are $m \times n$ matrices and C is invertible. Show that $A = B$.

Solution

Since C is invertible, then $CC^{-1} = C^{-1}C = I$

$$(A - B)C = 0$$

$$(A - B)CC^{-1} = 0C^{-1}$$

$$(A - B)I = 0$$

$$A - B = 0$$

$$A - B + B = 0 + B$$

$$\underline{A = B} \quad \checkmark$$