Section 1.5 – Limits and Asymptotes

Definition of the Limit of a Function

If $f(x)$ becomes arbitrary close to a single number L as x approaches c from either side, then

$$\lim_{x \to c} f(x) = L$$

Which is read as "the limit of $f(x)$ as x approaches c is L."

Limit of a Polynomial Function

If p is a polynomial function and c is any real number, then

$$\lim_{x \to c} p(x) = p(c)$$

Example

Find the limit: $\lim_{x \to 1} (2x+4)$

Solution

$$\lim_{x \to 1} (2x+4) = 2*(1) + 4 = 6$$

Example

Find the limit: $\lim_{x \to 1} \frac{x^2-4}{x-2}$

Solution

$$\lim_{x \to 1} \frac{x^2-4}{x-2} = \frac{1^2-4}{1-2}$$

$$= \frac{-3}{-1}$$

$$= 3$$

Unbounded Behavior

Example

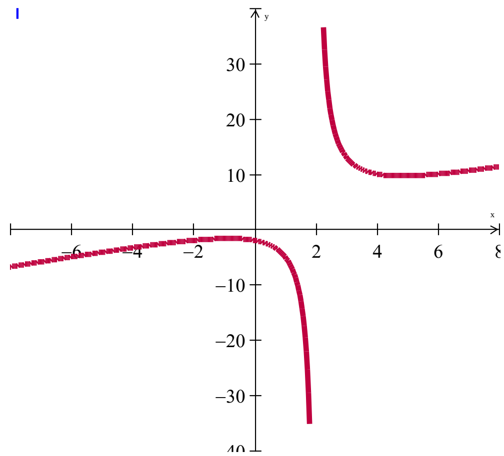
Find the limit: $\lim_{x \to 2} \frac{x^2+4}{x-2}$

<u>*Solution*</u>

$$\lim_{x \to 2} \frac{x^2+4}{x-2} = \frac{2^2+4}{2-2}$$

$$= \frac{8}{0}$$

$$= \infty \quad (\textbf{\textit{Doesn't exist}})$$



$$\lim_{x \to 2^-} \frac{x^2+4}{x-2} = \frac{(-2)^2+4}{2^- - 2}$$

$$= -\infty$$

$$\lim_{x \to 2^+} \frac{x^2+4}{x-2} = \frac{(-2)^2+4}{2^+ - 2}$$

$$= +\infty$$

Example

Find the limit: $\lim\limits_{x \to 0} \frac{|x|}{x}$

<u>*Solution*</u>

$$\lim_{x \to 0} \frac{|x|}{x} = \frac{0}{0}$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \frac{x}{-x} = -1$$

$$\lim_{x \to 0^+} \frac{|x|}{x} = \frac{x}{x} = 1$$

Doesn't exist

On-Sided limits

Example

Find the limit: $\lim\limits_{x \to 2^-} \frac{|x-2|}{x-2}$

<u>*Solution*</u>

$$\lim_{x \to 2^-} \frac{|x-2|}{x-2} = \frac{(x-2)}{-(x-2)} = -1$$

Find the limit: $\lim\limits_{x \to 2+} \frac{|x-2|}{x-2}$

$$\lim_{x \to 2^+} \frac{|x-2|}{x-2} = \frac{(x-2)}{(x-2)} = 1$$

Example

Find: $\lim\limits_{x \to 3} \frac{x^2 - x - 1}{\sqrt{x+1}}$

<u>*Solution*</u>

$$\lim_{x \to 3} \frac{x^2 - x - 1}{\sqrt{x+1}} = \frac{3^2 - 3 - 1}{\sqrt{3+1}}$$

$$= \frac{5}{2}$$

Example

Suppose $\lim\limits_{x\to 2} f(x)=3$ and $\lim\limits_{x\to 2} g(x)=4$

Find $\lim\limits_{x\to 2} \dfrac{[f(x)]^2}{\ln g(x)}$

Solution

$$\lim_{x\to 2} \frac{[f(x)]^2}{\ln g(x)} = \frac{\lim\limits_{x\to 2} [f(x)]^2}{\lim\limits_{x\to 2} \ln g(x)}$$

$$= \frac{\left[\lim\limits_{x\to 2} f(x)\right]^2}{\ln\left(\lim\limits_{x\to 2} g(x)\right)}$$

$$= \frac{[3]^2}{\ln(4)}$$

$$\approx \frac{9}{1.38629}$$

$$\approx 6.492$$

Example

Find: $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x-2}$

Solution

$$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \frac{2^2+2-6}{2-2}$$

$$= \frac{0}{0}$$

$$\lim_{x\to 2} \frac{(x+3)(x-2)}{x-2} = \lim_{x\to 2} (x+3)$$

$$= 5$$

Vertical Asymptotes and Infinite Limits

Definition

If $f(x)$ approaches infinity ($\pm\infty$) as x approaches c $(a \to c)$ from the right or from the left, then the line $x = c$ is a vertical asymptote of the graph *f*.

Example

Find each limit.

a. $\lim\limits_{x\to 2^-} \frac{1}{x-2} = -\infty$

$\lim\limits_{x\to 2^+} \frac{1}{x-2} = \infty$

b. $\lim\limits_{x\to -3^-} \frac{-1}{x+3} = \infty$

$\lim\limits_{x\to -3^+} \frac{-1}{x+3} = -\infty$

Finding Vertical Asymptotes (***Think Domain***)

Example

$f(x) = \dfrac{x+2}{x^2 - 2x}$

Solution

$x^2 - 2x = 0$

$x(x-2) = 0$

$\to x = 0,\ 2$

Example

Find the vertical asymptote(s) of the graph of $f(x)=\frac{x+4}{x^2-4x}$

Solution

$$x^2-4x=0$$
$$x(x-4)=0$$
$$\to x=0,\ 4$$

Example

Find the vertical asymptote(s) of the graph of $f(x)=\frac{x^2+4x+3}{x^2-9}$

Solution

$$f(x)=\frac{(x+3)(x-1)}{(x+3)(x-3)}$$
$$=\frac{(x-1)}{(x-3)}$$

Vertical Asymptote (VA): $x = 3$

Hole: $x = -3$ (***undefined***)

Horizontal Asymptote

Definition

If *f* is a function and L_1 and L_2 are real numbers, the statements

$$\lim_{x\to\infty} f(x)=L_1 \qquad \textit{and} \qquad \lim_{x\to-\infty} f(x)=L_2$$

Denote limits at infinity. The lines $y=L_1$ and $y=L_2$ are ***horizontal asymptotes*** (***HA***) of the graph of *f*.

Example

Find the limit: $\lim\limits_{x\to\infty}\left(2+\frac{5}{x^2}\right)$

Solution

$$\lim_{x\to\infty}\left(2+\frac{5}{x^2}\right)=\lim_{x\to\infty}(2)+\lim_{x\to\infty}\left(\frac{5}{x^2}\right)$$
$$=2-5(0)$$
$$=2$$

HA: $y=2$

Horizontal Asymptotes of Rational Functions

Let $f(x)=\frac{p(x)}{q(x)}=\frac{a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0}{b_m x^m+b_{m-1}x^{m-1}+\ldots+b_1x+b_0}=\frac{a_n x^n}{b_m x^m}$ be a rational function.

1. If the degree of numerator is less than of denominator $(n < m) \Rightarrow y = 0$

$$y=\frac{2x+1}{4x^2+5} \quad \Rightarrow \boxed{y=0}$$

2. If the degree of numerator is equal of denominator $(n = m) \Rightarrow y=\frac{a_n}{b_m}$

$$y=\frac{2x^2+1}{4x^2+5} \quad \Rightarrow \boxed{y=\frac{2}{4}=\frac{1}{2}}$$

3. If the degree of numerator is greater than of denominator $(n > m) \Rightarrow$ No horizontal asymptote

$$y=\frac{2x^3+1}{4x^2+5} \quad \Rightarrow No\ \ HA$$

Example

Find the vertical and horizontal asymptotes (if any) of

1. $f(x)=\dfrac{x^2+2x-15}{(x+3)(x-4)}$

   VA: $x=-3$ & $x=4$

   HA: $y=1$

2. $g(x)=\dfrac{3x^2-2x+7}{2x^2+5}$

   No **VA**

   HA: $y=\frac{3}{2}$

Slant or Oblique Asymptotes

When the degree of the numerator is one greater than the degree of the numerator, the graph has a slant or oblique asymptote. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y=\frac{3x^2-1}{x+2}$$

$$\begin{array}{r} 3x-6 \\ x+2\overline{\smash{\big)}\,3x^2+0x-1} \\ 3x^2+6x \quad\quad \\ -6x-1 \\ -6x-12 \\ R=11 \end{array}$$

$$y=\frac{3x^2-1}{x+2}=(3x-6)+\frac{11}{x+2}$$

The slant asymptote is the line ***y = 3x - 6***

Exercises *Section* 1.5 – Limits and Asymptotes

Find the limit:

1. $\lim\limits_{x\to 1}(2x^2 - x + 4)$

2. $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}$

3. $\lim\limits_{x\to 2}\dfrac{x^3-8}{x-2}$

4. $\lim\limits_{x\to 3}\dfrac{x^2+x-12}{x-3}$

5. $\lim\limits_{x\to 0}\dfrac{\sqrt{x+4}-2}{x}$

6. $\lim\limits_{x\to 0} f(x) \qquad f(x) = \begin{cases} x^2+1 & x<0 \\ 2x+1 & x>0 \end{cases}$

7. $\lim\limits_{x\to -2}\dfrac{5}{x+2}$

8. $\lim\limits_{x\to 0}(3x-2)$

9. $\lim\limits_{x\to 3}\dfrac{\sqrt{x+1}-1}{x}$

10. $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}$

11. $\lim\limits_{x\to -2}\dfrac{|x+2|}{x+2}$

12. $\lim\limits_{x\to 2+}\dfrac{|x-2|}{x-2}$

Find the vertical and horizontal asymptotes (if any) of

13. $y = \dfrac{3x}{1-x}$

14. $y = \dfrac{x^2}{x^2+9}$

15. $y = \dfrac{x-2}{x^2-4x+3}$

16. $y = \dfrac{3}{x-5}$

17. $y=\dfrac{x^3-1}{x^2+1}$

18. $y=\dfrac{3x^2-27}{(x+3)(2x+1)}$

19. $y=\dfrac{x^3+3x^2-2}{x^2-4}$

20. $y=\dfrac{x-3}{x^2-9}$

21. $y=\dfrac{6}{\sqrt{x^2-4x}}$

22. $y=\dfrac{5x-1}{1-3x}$