3.4 Comparison Test

$$d_n \le a_n \le c_n$$

1- If $\sum c_n$ converges $\Rightarrow$ $\sum a_n$ converges

2. If $\sum d_n$ diverges $\Rightarrow$ $\sum a_n$ diverges

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{5}{5n-1}$ Conv? div?

$$\frac{5}{5n-1} > \frac{5}{5n} = \frac{1}{n}.$$

$\sum \frac{1}{n}$ diverges by p-series ($p = 1 \le 1$)

By the Comparison Test, the given series diverges

$$5n > n$$
$$5n-1 > n$$
$$\frac{1}{5n-1} < \frac{1}{n}$$
$$\frac{5}{5n-1} > \frac{1}{n}$$

Ex $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} < 1 + \sum \frac{1}{2^n}$

$$= 1 + \sum \left(\frac{1}{2}\right)^n$$

converges

$r = \frac{1}{2} < 1 \Rightarrow \checkmark$ By the Geometric series

$$S = 1 + \frac{1}{1-\frac{1}{2}}$$

$$= 3$$

By the Comparison, the given series converges

Limit Comparison Test

$a_n > 0, \quad b_n > 0$

1. $\lim \frac{a_n}{b_n} = c > 0 \Rightarrow \sum a_n, \sum b_n$ both $\begin{cases} \text{converge} \\ \text{diverge} \end{cases}$

2. $\lim \frac{a_n}{b_n} = 0 \qquad \sum b_n$ converges $\Rightarrow \sum a_n$ converges

3 – $\lim \frac{a}{b_n} = \infty \qquad \sum b_n$ diverges $\Rightarrow \sum a_n$ diverges

Ex $\frac{3}{4}+\frac{5}{9}+\frac{7}{16}+\frac{9}{25}+\cdots$ Conv? div?

$$\frac{3}{4}+\frac{5}{9}+\frac{7}{16}+\cdots=\sum_{n=1}^{\infty}\frac{2n+1}{(n+1)^2}$$

$2^2 \quad 3^2 \quad 4^2$

$3, 5, 7 \cdots \quad d=2 \quad a_n = 3+(n-1)2$
$= 2n+1$

$$\sum_{n=1}^{\infty}\frac{2n+1}{n^2+2n+1}$$

$$a_n = \frac{2n+1}{n^2+2n+1} \longrightarrow \frac{2n}{n^2} = \frac{2}{n} > \frac{1}{n}$$

$b_n = \frac{1}{n} \rightarrow$ diverges by p-series ($p=1\leq 1$)

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2n+1}{n^2+2n+1}\cdot\frac{n}{1}$$
$$= \lim_{n\to\infty}\frac{2n^2}{n^2}$$
$$= \boxed{2}$$

$\therefore$ By the Limit Comparison Test, the given series diverges.

Ex

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{7}+\frac{1}{15}+\cdots=\sum_{n=1}^{\infty}\frac{1}{2^n-1} \quad ?$$

$$a_n=\frac{1}{2^n-1}, \quad b_n=\frac{1}{2^n}$$

$$\sum b_n=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \qquad r=\frac{1}{2}<1.$$

$\sum b_n$ converges by Geometric series

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{1}{2^n-1}\cdot 2^n$$

$$=\lim_{n\to\infty}\frac{2^n}{2^n-1}$$

$$=\boxed{1}$$

By the Limit Comparison Test, the given series converges

Ex

$$\frac{1+2\ln 2}{9}+\frac{1+3\ln 3}{14}+\cdots=\sum_{n=2}^{\infty}\frac{1+n\ln n}{n^2+5}$$

$$a_n=\frac{1+n\ln n}{n^2+5}\to\frac{n\ln n}{n^2}=\frac{\ln n}{n}>\frac{1}{n}=b_n$$

$\sum b_n$ diverges by p-series $(p=1\le 1)$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{1+n\ln n}{n^2+5}\cdot\frac{n}{1}$$

$$=\lim_{n\to\infty}\frac{n^2\ln n}{n^2}$$

$$=\lim_{n\to\infty}\ln n$$

$$=\boxed{\infty}$$

By the Limit Comparison Test, the given series diverges

Ex $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^{3/2}}$ converge?

$$a_n = \frac{\ln n}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}}$$

$n!$ 2^n

$\ln$ $n^{1/4}$

$$b_n = \frac{1}{n^{5/4}} \quad \overset{\text{converges}}{\checkmark} \text{ by } \quad p\text{-series} \left(p = \tfrac{5}{4} > 1\right)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^{3/2}} \cdot \frac{n^{5/4}}{1}$$

$$= \lim_{n\to\infty} \frac{\ln n}{n^{1/4}} \quad = \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{4} n^{-3/4}}$$

$$= 4 \lim_{n\to\infty} \frac{1}{n^{1/4}}$$

$$= 0$$

By the Limit Comparison Test, the given series converges

$$\underline{\frac{\ln n}{n^{3/2}}} > \underline{\frac{1}{n^{3/2}}} \text{ converges}$$

$$\int \frac{\ln x}{x^{3/2}}\,dx$$

$y = \ln x \to x = e^y$

$dy = \frac{dx}{x} \to dx = e^y\,dy$

$$= \int (e^y)^{-3/2}\, y\, e^y\,dy$$

$$= \int_2^{\infty} y\, e^{-y/2}\,dy = e^{-y/2}\left(-\frac{1}{2}y - \frac{1}{4}\right)\Big|_2^{\infty}$$

$= \frac{3}{4e}$

#1 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+30}$

$$\frac{1}{n^2+30} < \frac{1}{n^2} \longrightarrow \begin{array}{l} n^2 = n^2 \\ n^2+30 > n^2 \\ \dfrac{1}{n^2+30} < \dfrac{1}{n^2} \end{array}$$

$\sum \frac{1}{n^2}$ converges by p-series ($p=2>1$)

By the Comparison Test, the given series converges

2/ $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$ $\quad \dfrac{n-1}{n^4+2} \longrightarrow \dfrac{n}{n^4} = \dfrac{1}{n^3}$

$$n^4 < n^4+2$$

$$\frac{1}{n^4} > \frac{1}{n^4+2}$$

$$\frac{n}{n^4} > \frac{n}{n^4+2}$$

$$\frac{1}{n^3} > \frac{n}{n^4+2} \Rightarrow \frac{n}{n^4+2} < \frac{1}{n^3}$$

$\sum \frac{1}{n^3}$ converges by p-series ($p=3>1$)

By the Comparison Test, the given series converges.

4/ $$\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$$

$$-1 \le \cos n \le 1$$

$$0 \le \cos^2 n \le 1$$

$$\frac{\cos^2 n}{n^{3/2}} \le \frac{1}{n^{3/2}}$$

$\sum \frac{1}{n^{3/2}}$ converges by p-series ($p = \frac{3}{2} > 1$)

By the Comparison Test, the given series converges.

3.5 Ratio & Root Tests

Ratio Test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

1. $\rho < 1 \rightarrow$ converges
2. $\rho > 1 \rightarrow$ diverges (∞)
3. $\rho = 1 \rightarrow$ inconclusive

Ex $$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} = \sum \frac{2^n}{3^n} + \sum 5\left(\frac{1}{3}\right)^n$$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}+5}{3^{n+1}} \cdot \frac{3^n}{2^n+5}$$

$$= \frac{1}{3} \quad \frac{2^{n+1}+5}{2^n+5}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{3} \lim_{n\to\infty} \frac{2^{n+1}+5}{2^n+5}$$

$$= \frac{1}{3} \lim_{n\to\infty} \frac{2^{n+1}}{2^n}$$

$$= \frac{2}{3} < 1$$

$\therefore$ By the Ratio Test, the given series converges.

$$\sum \left(\frac{2}{3}\right)^n + \sum 5\left(\frac{1}{3}\right)^n$$ Geometric series

$r_1 = \frac{2}{3} < 1$ $\qquad$ $r_2 = \frac{1}{3} < 1$

$$\sum = \frac{1}{1-\frac{2}{3}} + \frac{5}{1-\frac{1}{3}}$$
$$= 3 + \frac{15}{2}$$
$$= \frac{21}{2}$$

Ex $$\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(n+1)!\,(n+1)!} \cdot \frac{n!\,n!}{(2n)!}$$
$$= \frac{(2n+1)(2n+2)}{(n+1)(n+1)}$$

$2n! \neq 2(n!)$

$(2n)!$

$2(n+1)$

$\frac{(2n+2)!}{(2n)!}$

$(2n)!\ 2n+1$

$\frac{n!}{(n+1)!} = \frac{n!}{n!\,(n+1)}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+1)}$$
$$= \lim_{n\to\infty} \frac{4n^2}{n^2}$$
$$= 4 > 1$$

$\therefore$ By the Ratio Test, the given series diverges

Ex

$$\sum_{n=1}^{\infty} \frac{4^n\, n!\, n!}{(2n)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(n+1)!\,(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{4^n\, n!\, n!}$$

$$= \frac{4(n+1)(n+1)}{(2n+1)(2n+2)} \longrightarrow \frac{4n^2}{4n^2} = 1$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{4(n+1)(n+1)}{(2n+1)(2n+2)}$$

$$= \lim_{n\to\infty} \frac{4n^2}{4n^2}$$

$$= 1$$

$n=1 \rightarrow 2$

$n=2 \rightarrow \frac{16\times \cancel{2\times 2}}{2\times 3\times \cancel{4}} = \frac{8}{3}$

$a_{n+1} \;?\; a_n$

$$\frac{4^{n+1}(n+1)!\,(n+1)!}{(2n+2)!} \;?\; \frac{4^n\, n!\, n!}{(2n)!}$$

$$4\,\frac{(n+1)!\,(n+1)!}{n!\; n!} \;?\; \frac{(2n+2)!}{(2n)!}$$

$$4(n+1)(n+1) \;?\; (2n+1)(2n+2)$$

$$2(2n+1)(n+1)$$

$$2(n+1) \;?\; 2n+1$$

$$2n+2 > 2n+1$$

$$a_{n+1} > a_n$$

$\therefore$ the given series diverges

Root Test

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$$

$\rho > 1 \Rightarrow \sum a_n$ diverges

$\rho < 1 \Rightarrow$ " converges

$\rho = 1 \rightarrow$ Inconclusive ☹

Ex

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

$$\sqrt[n]{\frac{n^2}{2^n}} = \frac{n^{2/n}}{2} \qquad \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}}$$

$$\rho = \lim_{n\to\infty} \frac{n^{2/n}}{2} = \infty^0$$

$$= \frac{1}{2} < 1$$

By the Root Test, the given series converges

Ex

$$\sum_{n=1}^{\infty} \frac{2^n}{n^3}$$

$$\sqrt[n]{\frac{2^n}{n^3}} = \frac{2}{n^{3/n}}$$

$$\rho = \lim_{n\to\infty} \frac{2}{n^{3/n}}$$

$$= \frac{2}{\infty^0}$$

$$= \frac{2}{1}$$

$$= 2 > 1$$

By the Root Test, the given series diverges

Ex

$$\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$$

$$\sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \frac{1}{n+1}$$

$$\rho = \lim_{n\to\infty} \frac{1}{1+n} \qquad \frac{1}{\infty}$$

$$= 0 < 1$$

$\therefore$ By the Root Test, the given series converges

1) Ratio

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$$

$$= \frac{2}{n+1}$$

$$\rho = \lim_{n \to \infty} \frac{2}{n+1}$$

$$= 0 < 1$$

$\therefore$ By the Ratio Test, the given series converges

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3)

$$\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$$

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+3}}{\ln(n+1)} \cdot \frac{\ln n}{3^{n+2}}$$

$$= 3 \, \frac{\ln n}{\ln(n+1)}$$

$$\rho = 3 \lim_{n \to \infty} \frac{\ln n}{\ln(n+1)} = \frac{\infty}{\infty}$$

$$= 3 \lim_{n \to \infty} \frac{\frac{1}{n}}{\frac{1}{n+1}}$$

$$= 3 \lim_{n \to \infty} \frac{n+1}{n}$$

$$= 3 > 1$$

By the Ratio Test, the given series diverges

Root Test

27) $$\sum_{n=1}^{\infty} \sin^n \frac{1}{\sqrt{n}}$$

$$\sqrt[n]{\sin^n \frac{1}{\sqrt{n}}} = \sin \frac{1}{\sqrt{n}}$$

$$\rho = \lim_{n\to\infty} \sin \frac{1}{\sqrt{n}}$$

$$= \sin 0$$

$$= \boxed{0 < 1}$$

By the Root Test, the given series converges

29) $$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$$

$$\sqrt[n]{\frac{e^{2n}}{n^n}} = \frac{e^2}{n}$$

$$\rho = \lim_{n\to\infty} \frac{e^2}{n}$$

$$= \boxed{0 < 1}$$

$\therefore$ By the Root Test, the given series converges.

3.6 Alternating Series Test

Theorem

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

1. u_n's all positive (rule #)

✓ $u_n > u_{n+1}$

✓ $u_n \to 0$

either (diverges)

Ex $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

$\frac{1}{n} > 0$

1. $n < n+1$

$\frac{1}{n} > \frac{1}{n+1}$

$u_n > u_{n+1}$ ✓

2. $\frac{1}{n} \to 0$ ✓ $\leftarrow \lim_{n\to\infty} \frac{1}{n} = 0$ ~~$\to$~~ $\longrightarrow$

By the alternating series, the given series converges

c/
$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2+5}{n^2+4}$$

~~$u_n \,?\, u_{n+1}$~~

$\frac{n^2+5}{n^2+4} \to 1 \neq 0$

By the alternating series, the given series diverges

Defn A series $\sum a_n$ converges absolutely (absolutely convergent) if $\sum |a_n|$ converges

Defn A series converges but doesn't converge absolutely converges conditionally

Ex $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$

$$\sum \left| (-1)^{n+1} \frac{1}{n^2} \right| = \sum \left| \frac{1}{n^2} \right|$$

$$= \sum \frac{1}{n^2} \qquad p=2 \text{ converges by p-series}$$

The given series converges because it converges absolutely.

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$$

$$\sum \left| \frac{\sin n}{n^2} \right| = |\sin 1| + \frac{|\sin 2|}{4} + \frac{|\sin 3|}{9} + \cdots$$

$-1 \le \sin n \le 1$

$0 \le |\sin n| \le 1$

$0 \le \dfrac{|\sin n|}{n^2} \le \dfrac{1}{n^2}$; $\dfrac{1}{n^2}$ converges p-series $(p=2)$

By the comparison, the given series converges absolutely.