

## Section 4.4 – Calculus in Polar Coordinates

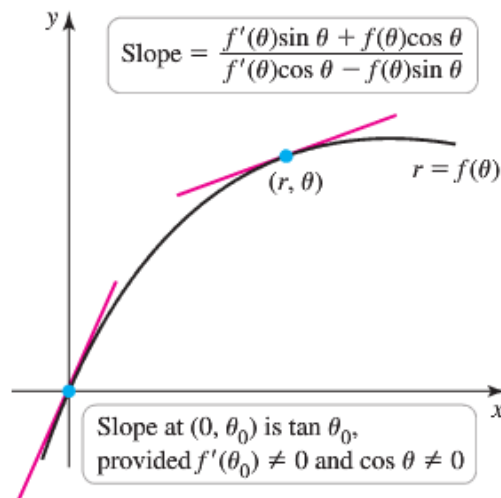
### Slope

The slope of a polar curve  $r = f(\theta)$  in the  $xy$ -plane is still given by  $\frac{dy}{dx}$ , which is not  $r' = \frac{df}{d\theta}$

$$x = r \cos \theta = f(\theta) \cos \theta, \quad y = r \sin \theta = f(\theta) \sin \theta$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\ &= \frac{\frac{d}{d\theta}(f(\theta) \cdot \sin \theta)}{\frac{d}{d\theta}(f(\theta) \cdot \cos \theta)} \\ &= \frac{\frac{df}{d\theta} \cdot \sin \theta + f(\theta) \cdot \cos \theta}{\frac{df}{d\theta} \cdot \cos \theta - f(\theta) \cdot \sin \theta} \end{aligned}$$

$$\left. \frac{dy}{dx} \right|_{(r, \theta)} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}$$

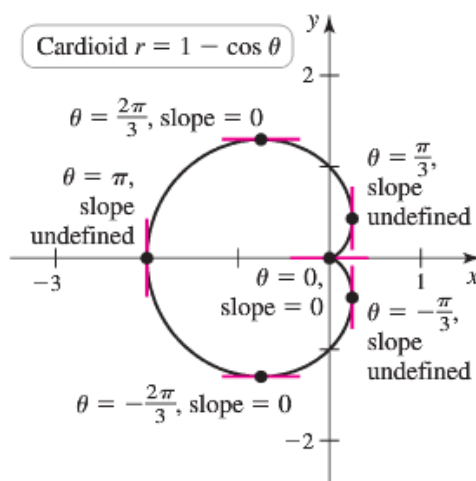


### Example

Find the points on the interval  $-\pi \leq \theta \leq \pi$  at which the cardioid  $r = f(\theta) = 1 - \cos \theta$  has a vertical or horizontal tangent line.

### Solution

$$\begin{aligned} \frac{dy}{dx} &= \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta} \\ &= \frac{\sin \theta \sin \theta + (1 - \cos \theta) \cos \theta}{\sin \theta \cos \theta - (1 - \cos \theta) \sin \theta} \\ &= \frac{\sin^2 \theta + \cos \theta - \cos^2 \theta}{\sin \theta \cos \theta - \sin \theta + \cos \theta \sin \theta} \\ &= \frac{1 - \cos^2 \theta + \cos \theta - \cos^2 \theta}{2 \sin \theta \cos \theta - \sin \theta} \\ &= \frac{-2 \cos^2 \theta + \cos \theta + 1}{\sin \theta (2 \cos \theta - 1)} \\ &= \frac{-(2 \cos \theta + 1)(\cos \theta - 1)}{\sin \theta (2 \cos \theta - 1)} \end{aligned}$$



The points with a horizontal tangent line:

$$\frac{dy}{dx} = 0 \Rightarrow (2 \cos \theta + 1)(\cos \theta - 1) = 0 \rightarrow \begin{cases} \cos \theta = -\frac{1}{2} \rightarrow \theta = \pm \frac{2\pi}{3} \\ \cos \theta = 1 \rightarrow \theta = 0 \end{cases} \quad \text{numerator is 0}$$

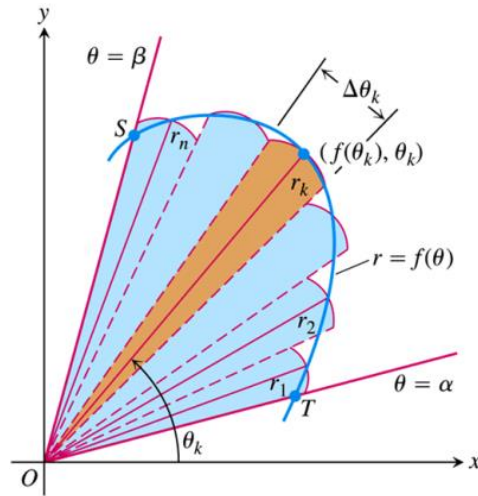
The points with a Vertical tangent line:

$$\sin \theta (2 \cos \theta - 1) = 0 \rightarrow \begin{cases} \sin \theta = 0 \rightarrow \theta = 0, \pm \pi \\ \cos \theta = \frac{1}{2} \rightarrow \theta = \pm \frac{\pi}{3} \end{cases} \quad \text{denominator is 0}$$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\theta \rightarrow 0^+} \left( \frac{-2 \cos^2 \theta + \cos \theta + 1}{2 \cos \theta \sin \theta - \sin \theta} \right) = \frac{0}{0} \\ &= \lim_{\theta \rightarrow 0^+} \left( \frac{4 \cos \theta \sin \theta - \sin \theta}{-2 \sin^2 \theta + 2 \cos^2 \theta - \cos \theta} \right) \\ &= \frac{0}{1} \\ &= 0 \end{aligned} \quad \text{Therefore, the curve has a slope of 0 at origin.}$$

## Area in the plane

The region OTS is bounded by the rays  $\theta = \alpha$  and  $\theta = \beta$  and the curve  $r = f(\theta)$ .



We approximate the region with  $n$  non-overlapping fan-shaped circular sectors based on a partition  $P$  of angle  $TOS$ . The typical sector has radius  $r_k = f(\theta_k)$  and central angle of radian measure  $\Delta \theta_k$ . Its area is

$\frac{\Delta \theta_k}{2\pi}$  times the area of a circle of radius  $r_k$ , or

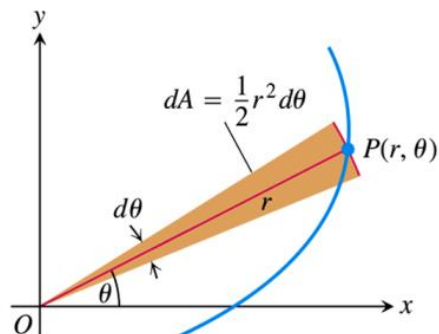
$$A_k = \frac{1}{2} r_k^2 \Delta \theta_k = \frac{1}{2} (f(\theta_k))^2 \Delta \theta_k$$

**Area of the Fan-Shaped Region between the Origin and the curve  $r = f(\theta)$ ,  $\alpha \leq \theta \leq \beta$**

$$A = \int_{\alpha}^{\beta} \frac{1}{2} (f(\theta))^2 d\theta$$

This is the integral of the *area differential*

$$dA = \frac{1}{2} r^2 d\theta = \frac{1}{2} (f(\theta))^2 d\theta$$



### Example

Find the area of the region in the plane enclosed by the cardioid  $r = 2(1 + \cos \theta)$

### Solution

$$A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos \theta)^2 d\theta$$

$$= \int_0^{2\pi} 2(1 + 2\cos \theta + \cos^2 \theta) d\theta$$

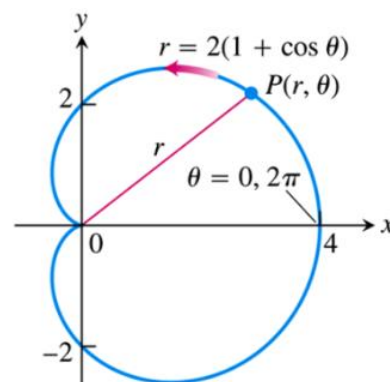
$$= \int_0^{2\pi} \left( 2 + 4\cos \theta + 2 \frac{1 + \cos 2\theta}{2} \right) d\theta$$

$$= \int_0^{2\pi} (3 + 4\cos \theta + \cos 2\theta) d\theta$$

$$= \left[ 3\theta + 4\sin \theta + \frac{1}{2} \sin 2\theta \right]_0^{2\pi}$$

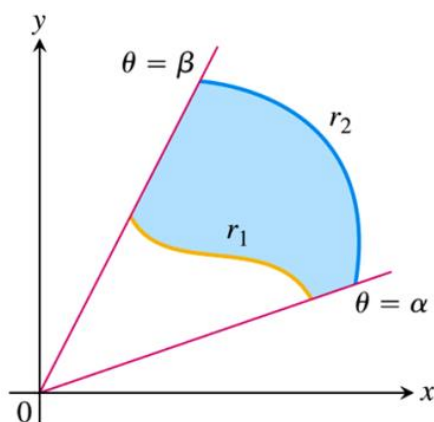
$$= \left[ 3(2\pi) + 4\sin(2\pi) + \frac{1}{2} \sin(2\pi) - \left( 3(0) + 4\sin(0) + \frac{1}{2} \sin(0) \right) \right]$$

$$= 6\pi$$



**Area of the Region**  $0 \leq r_1(\theta) \leq r \leq r_2(\theta), \quad \alpha \leq \theta \leq \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} (r_2^2 - r_1^2) d\theta$$

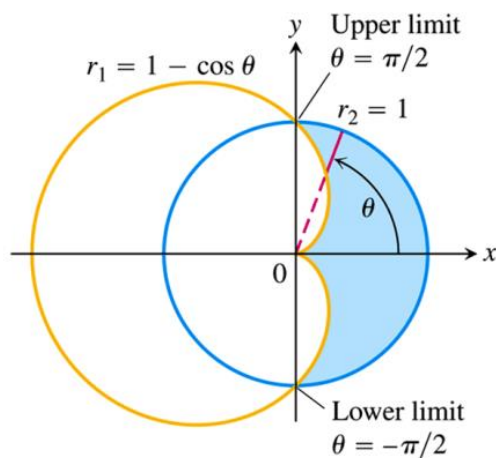


### Example

Find the area of the region that lies inside the circle  $r = 1$  and outside the cardioid  $r = 1 - \cos \theta$

### Solution

$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} \frac{1}{2} (r_2^2 - r_1^2) d\theta \\ &= 2 \int_0^{\pi/2} \frac{1}{2} (1^2 - (1 - \cos \theta)^2) d\theta \\ &= \int_0^{\pi/2} (1 - (1 - 2\cos \theta + \cos^2 \theta)) d\theta \\ &= \int_0^{\pi/2} \left( 2\cos \theta - \frac{1}{2} - \frac{1}{2} \cos 2\theta \right) d\theta \\ &= \left[ 2\sin \theta - \frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right]_0^{\pi/2} \\ &= \left[ 2\sin \frac{\pi}{2} - \frac{1}{2} \frac{\pi}{2} - \frac{1}{4} \sin 2\frac{\pi}{2} - 0 \right] \\ &= \underline{2 - \frac{\pi}{4} \text{ unit}^2} \end{aligned}$$



## Length of a Polar Curve

If  $r = f(\theta)$  has a continuous first derivative for  $\alpha \leq \theta \leq \beta$  and if the point  $P(r, \theta)$  traces the curve  $r = f(\theta)$  exactly once as  $\theta$  runs from  $\alpha$  to  $\beta$ , then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

### Example

Find the length of the cardioid  $r = 1 - \cos \theta$

#### Solution

$$r = 1 - \cos \theta \Rightarrow \frac{dr}{d\theta} = \sin \theta$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 - \cos \theta)^2 + \sin^2 \theta$$

$$\begin{aligned} &= 1 - 2\cos \theta + \cos^2 \theta + \sin^2 \theta \\ &= 2 - 2\cos \theta \end{aligned}$$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos \theta} d\theta$$

$$1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$$

$$= \int_0^{2\pi} \sqrt{4 \sin^2 \frac{\theta}{2}} d\theta$$

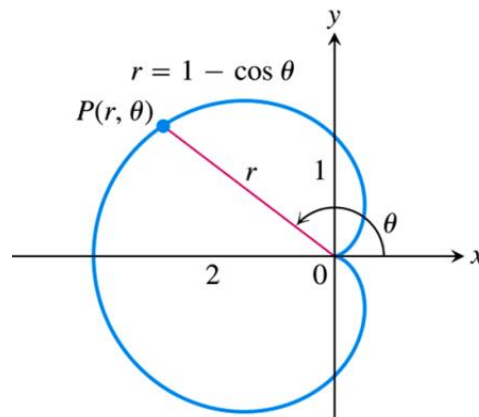
$$= \int_0^{2\pi} 2 \sin \frac{\theta}{2} d\theta$$

$$= \left[ -4 \cos \frac{\theta}{2} \right]_0^{2\pi}$$

$$= -4 \left( \cos \frac{2\pi}{2} - \cos 0 \right)$$

$$= -4(-1 - 1)$$

$$= \underline{8 \text{ unit}}$$



## Area of a surface of Revolution

### Theorem

Let  $f$  be a function whose derivative is continuous on an interval  $\alpha \leq \theta \leq \beta$ .

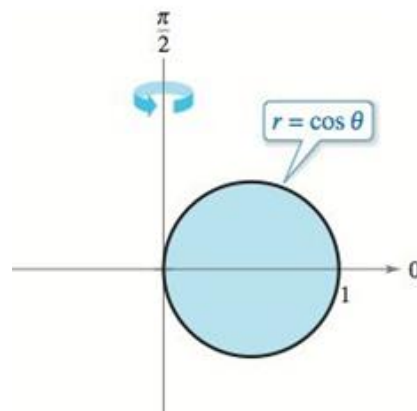
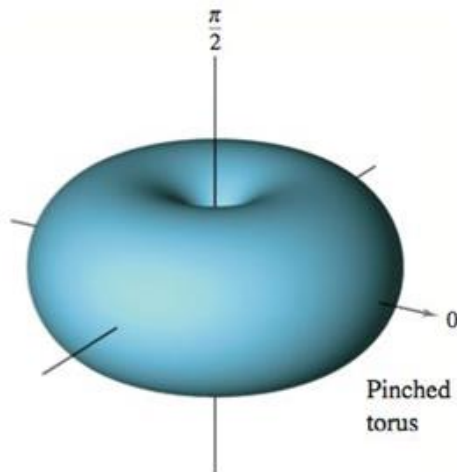
The area of the surface formed by revolving the graph of  $r = f(\theta)$  about the indicated line

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta) \sin \theta \sqrt{(f(\theta))^2 + (f'(\theta))^2} d\theta \quad \text{About the polar axis}$$

$$S = 2\pi \int_{\alpha}^{\beta} f(\theta) \cos \theta \sqrt{(f(\theta))^2 + (f'(\theta))^2} d\theta \quad \text{About the line } \theta = \frac{\pi}{2}$$

### Example

Find the area of the surface formed by revolving the circle  $f(\theta) = \cos \theta$  about the line  $\theta = \frac{\pi}{2}$



### Solution

$$\sqrt{r^2 + (r')^2} = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$$

$$S = 2\pi \int_0^{\pi} \cos^2 \theta d\theta$$

$$= \pi \int_0^{\pi} (1 + \cos 2\theta) d\theta$$

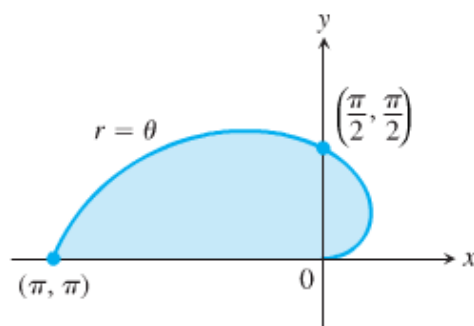
$$= \pi \left( \theta + \frac{1}{2} \sin 2\theta \right) \Big|_0^{\pi}$$

$$= \pi^2 \text{ unit}^2$$

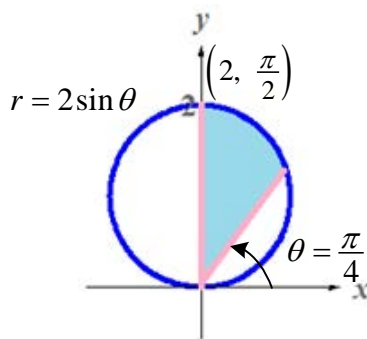
## Exercises      Section 4.4 – Calculus in Polar Coordinates

Find the slopes of the curves at the given points. Sketch the curves along with their tangents at these points.

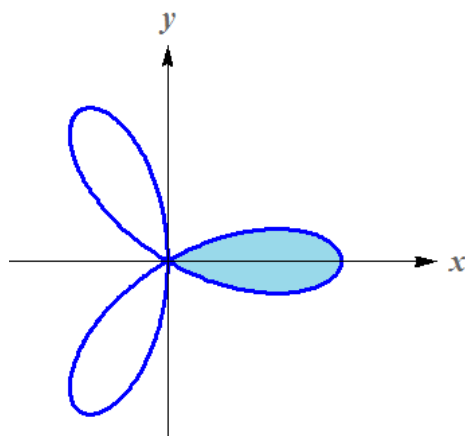
1. Cardioid  $r = -1 + \cos \theta$ ;  $\theta = \pm \frac{\pi}{2}$
2. Cardioid  $r = -1 + \sin \theta$ ;  $\theta = 0, \pi$
3. Four-leaved rose  $r = \sin 2\theta$ ;  $\theta = \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}$
4. Four-leaved rose  $r = \cos 2\theta$ ;  $\theta = 0, \pm \frac{\pi}{2}, \pi$
5. Find the area of the region bounded by the spiral  $r = \theta$  for  $0 \leq \theta \leq \pi$



6. Find the area of the region bounded by the circle  $r = 2\sin \theta$  for  $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$



7. Find the area of the region inside one leaf of the three-leaved rose  $r = \cos 3\theta$



Find the area of the region inside the

8. Oval limaçon  $r = 4 + 2\sin\theta$
9. Cardioid  $r = a(1 + \cos\theta)$ ,  $a > 0$
10. Six-leaved rose  $r^2 = 2\sin 3\theta$
11. Curve  $r = \sqrt{\cos\theta}$
12. Right lobe of  $r = \sqrt{\cos 2\theta}$
13. Cardioid  $r = 4 + 4\sin\theta$
14. Limaçon  $r = 2 + \cos\theta$
15. Circle  $r = 6$  above the line  $r = 3\csc\theta$
16. Inner loop  $r = \cos\theta - \frac{1}{2}$
17. One leave of  $r = \cos 3\theta$
18. Find the area of the region shared by the circles  $r = 2\cos\theta$  and  $r = 2\sin\theta$
19. Find the area of the region shared by the circle  $r = 2$  and the cardioid  $r = 2(1 - \cos\theta)$
20. Find the area of the region in the plane enclosed by the four-leaf rose  $r = f(\theta) = 2\cos 2\theta$
21. Find the area of the region that lies inside the circle  $r = 1$  and outside the cardioid  $r = 1 + \cos\theta$
22. Find the area of the region outside the circle  $r = \frac{1}{2}$  and inside the circle  $r = \cos\theta$
23. Find the area of the region outside the circle  $r = \frac{1}{\sqrt{2}}$  and inside the curve  $r = \sqrt{\cos\theta}$
24. Find the area of the region inside the circle  $r = \frac{1}{\sqrt{2}}$  in  $QI$  and inside the right lobe of  $r = \sqrt{\cos 2\theta}$
25. Find the area of the region inside the rose  $r = 4\sin 2\theta$  and inside the circle  $r = 2$
26. Find the area of the region inside the lemniscate  $r^2 = 2\sin 2\theta$  and outside the circle  $r = 1$
27. Find the area of the region inside all the leaves of the rose  $r = 3\sin 2\theta$
28. Find the area of the region inside one leaf of the rose  $r = \cos 5\theta$
29. Find the area of the region of a complete three-leaf rose  $r = 2\cos 3\theta$
30. Find the area of the region inside the rose  $r = 4\cos 2\theta$  and outside the circle  $r = 2$
31. Find the area of the region bounded by the lemniscate  $r^2 = 6\sin 2\theta$
32. Find the area of the region bounded by the limaçon  $r = 2 - 4\sin\theta$
33. Find the area of the region bounded by the limaçon  $r = 4 - 2\cos\theta$

Find the area of the given region

34. Inner loop of  $r = 1 + 2\cos\theta$
35. Inner loop of  $r = 2 - 4\cos\theta$
36. Inner loop of  $r = 1 + 2\sin\theta$
37. Inner loop of  $r = 4 - 6\sin\theta$
38. Between the loops  $r = 1 + 2\cos\theta$
39. Between the loops  $r = 2(1 + 2\sin\theta)$
40. Between the loops  $r = 3 - 6\sin\theta$
41. Between the loops  $r = \frac{1}{2} + \cos\theta$
42. Inside  $r = 2\cos\theta$  and outside  $r = 1$
43. Inside  $r = 3\sin\theta$  and outside  $r = 1 + \sin\theta$
44. Common interior of  $r = 4\sin 2\theta$  and  $r = 2$
45. Common interior of  $r = 4\sin\theta$  and  $r = 2$



Find the area of the given region

46. Common interior of  $r = 2\cos\theta$  and  $r = 2\sin\theta$
47. Common interior of  $r = 2(1 + \cos\theta)$  and  $r = 2(1 - \cos\theta)$
48. Common interior of  $r = 3 - 2\sin\theta$  and  $r = -3 + 2\sin\theta$
49. Common interior of  $r = 5 - 3\sin\theta$  and  $r = 5 - 3\cos\theta$
50. Inside  $r = a(1 + \cos\theta)$  and outside  $r = a\cos\theta$
51. Inside  $r = 2a\cos\theta$  and outside  $r = a$
52. Common interior of  $r = a(1 + \cos\theta)$  and  $r = a\sin\theta$
53. Common interior of  $r = a\cos\theta$  and  $r = a\sin\theta$ , where  $a > 0$

Find the length of

54.  $r = \theta^2$ ,  $0 \leq \theta \leq \sqrt{5}$
55.  $r = \frac{e^\theta}{\sqrt{2}}$ ,  $0 \leq \theta \leq \pi$
56.  $r = a\sin^2\left(\frac{\theta}{2}\right)$ ,  $0 \leq \theta \leq \pi$ ,  $a > 0$
57.  $r = \frac{6}{1 + \cos\theta}$ ,  $0 \leq \theta \leq \frac{\pi}{2}$
58.  $r = \cos^3\left(\frac{\theta}{3}\right)$ ,  $0 \leq \theta \leq \frac{\pi}{4}$
59.  $r = \sqrt{1 + \sin 2\theta}$ ,  $0 \leq \theta \leq \pi\sqrt{2}$
60.  $r = 8$   $0 \leq \theta \leq 2\pi$
61.  $r = a$   $0 \leq \theta \leq 2\pi$
62.  $r = 4\sin\theta$   $0 \leq \theta \leq \pi$
63.  $r = 2a\cos\theta$   $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
64.  $r = 1 + \sin\theta$   $0 \leq \theta \leq 2\pi$
65.  $r = 8(1 + \cos\theta)$   $0 \leq \theta \leq 2\pi$
66.  $r = 2\theta$   $0 \leq \theta \leq \frac{\pi}{2}$
67.  $r = \sec\theta$   $0 \leq \theta \leq \frac{\pi}{3}$
68.  $r = \frac{1}{\theta}$   $\pi \leq \theta \leq 2\pi$
69.  $r = e^\theta$   $0 \leq \theta \leq \pi$
70.  $r = 5\cos\theta$   $\frac{\pi}{2} \leq \theta \leq \pi$
71.  $r = 3(1 - \cos\theta)$   $0 \leq \theta \leq \pi$

Find the surface area bounded by

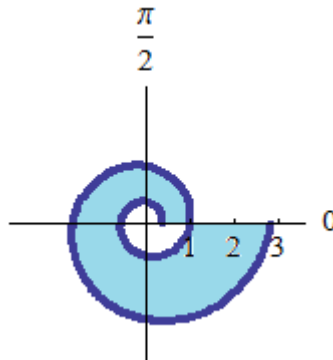
72.  $r = 6\cos\theta$   $0 \leq \theta \leq \frac{\pi}{2}$  revolving about Polar axis
73.  $r = a\cos\theta$   $0 \leq \theta \leq \frac{\pi}{2}$  revolving about  $\theta = \frac{\pi}{2}$
74.  $r = e^{a\theta}$   $0 \leq \theta \leq \frac{\pi}{2}$  revolving about  $\theta = \frac{\pi}{2}$
75.  $r = a(1 + \cos\theta)$   $0 \leq \theta \leq \pi$  revolving about polar axis
76.  $r = 1 + 4\cos\theta$   $0 \leq \theta \leq \frac{\pi}{2}$  revolving about Polar axis
77.  $r = 2\sin\theta$   $0 \leq \theta \leq \frac{\pi}{2}$  revolving about  $\theta = \frac{\pi}{2}$

78. Find the surface area of the torus generated by revolving the circle given by  $r = 2$  about the line  $r = 5 \sec \theta$
79. Find the surface area of the torus generated by revolving the circle given by  $r = a$  about the line  $r = b \sec \theta$ , where  $0 < a < b$

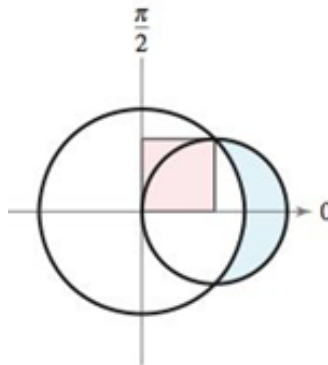
80. Let  $a$  and  $b$  be positive constants. Find the area of the region in the first quadrant bounded by the graph of the polar equation

$$r = \frac{ab}{a \sin \theta + b \cos \theta}, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

81. The curve represented by the equation  $r = ae^{b\theta}$ , where  $a$  and  $b$  are constants, is called a logarithmic spiral. The figure shows the graph of  $r = e^{\theta/6}$ ,  $-2\pi \leq \theta \leq 2\pi$ . Find the area of the shaded region.

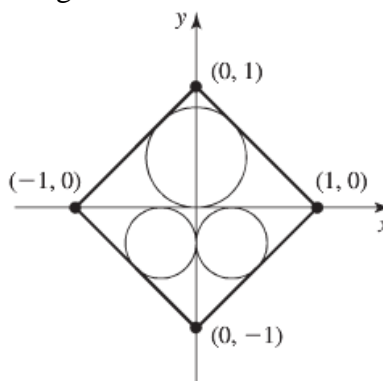


82. The larger circle in the figure is the graph of  $r = 1$ .



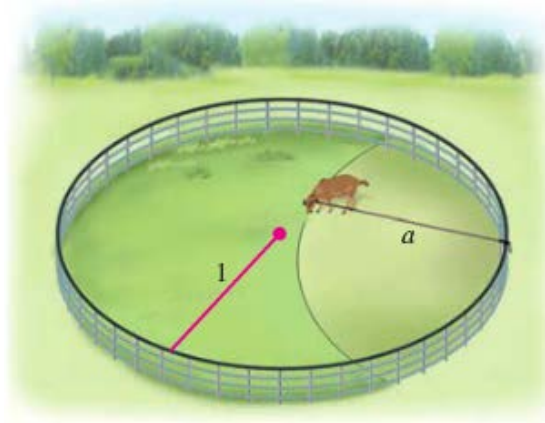
Find the polar equation of the smaller circle such that the shaded regions are equal.

83. Find equations of the circles in the figure.



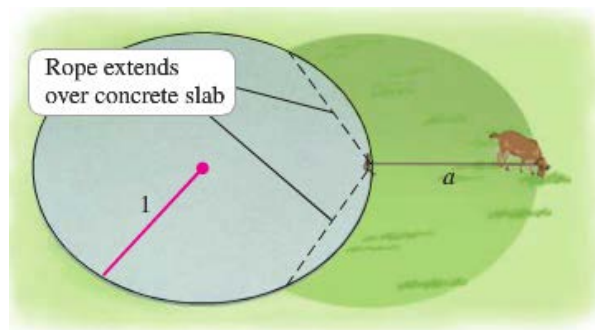
Determine whether the combined area of the circles is greater than or less than the area of the region inside the square but outside the circles.

84. A circular corral of unit radius is enclosed by a fence. A goat inside the corral is tied to the fence with a rope of length  $0 \leq a \leq 2$ .



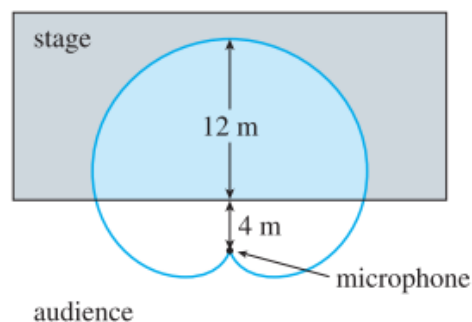
What is the area of the region (inside the corral) that the goat can graze? Check your answer with the special cases  $a = 0$  and  $a = 2$

85. A circular concrete slab of unit radius is surrounded by grass. A goat is tied to the edge of the slab with a rope of length  $0 \leq a \leq 2$ .



What is the area of the grassy region that the goat can graze? Note that the rope can extend over the concrete slab. Check your answer with the special cases  $a = 0$  and  $a = 2$

86. When recording live performance, sound engineers often use a microphone with a cardioid pickup pattern because it suppresses noise from the audience. Suppose the microphone is placed 4 m from the front of the stage and the boundary of the optimal pickup region is given by the cardioid  $r = 8 + 8\sin \theta$ , where  $r$  is measured in meters and the microphone is at the pole.



The musicians want to know the area they will have on stage within the optimal pickup range of the microphone, Answer their question.

**87.** The curve given by the parametric equations

$$x(t) = \frac{1-t^2}{1+t^2} \quad \text{and} \quad y(t) = \frac{t(1-t^2)}{1+t^2}$$

- a) Find the rectangular equation of the strophoid.
- b) Find a polar equation of the strophoid.
- c) Sketch a graph of the strophoid.
- d) Find the equations of the two tangent lines at the origin.
- e) Find the points on the graph at which the tangent lines are horizontal.