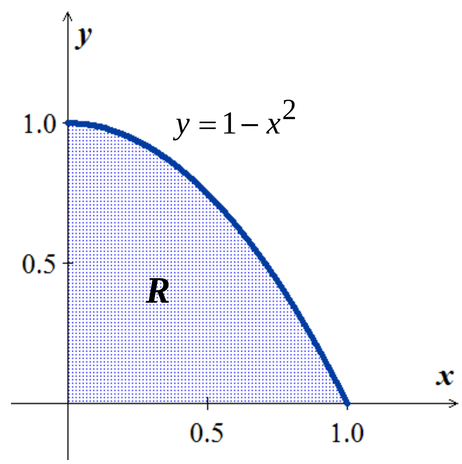
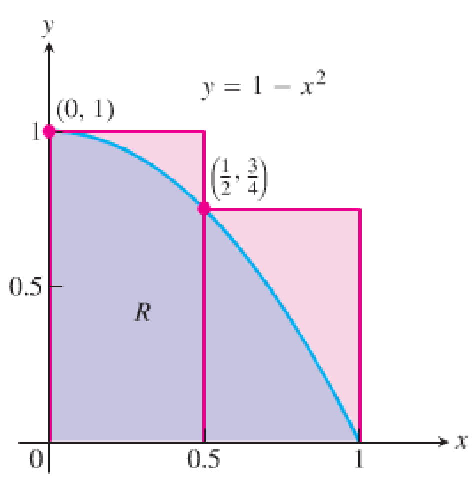
# *Section* 4.2 – Area under Curves

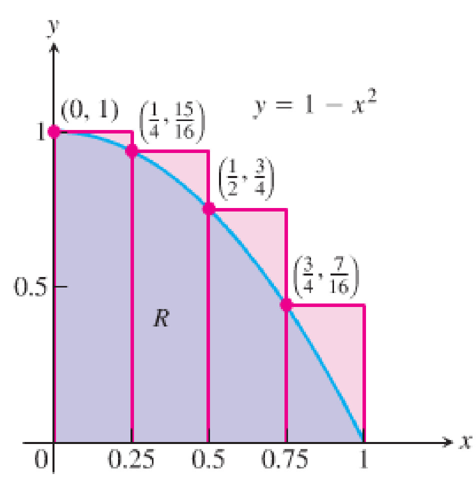
The ***definite integral*** is the key tool in calculus for defining and calculating quantities important to mathematics and science, such as areas, volumes, lengths, and more…

## *Area*

To find the area of the shaded region $R$ that lies above the $x$-axis, below the graph of $y = 1 - x^2$ and between the vertical lines $x = 0$ and $x = 1$.

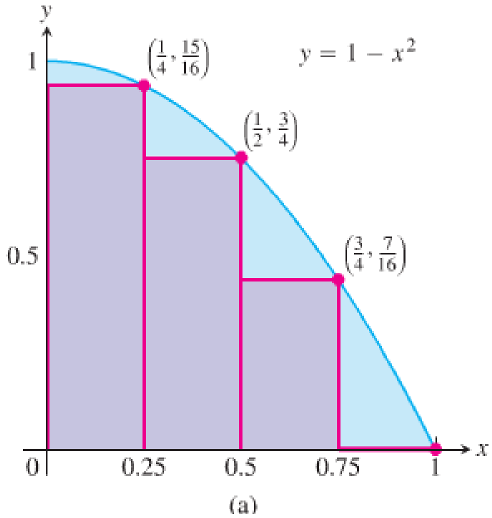


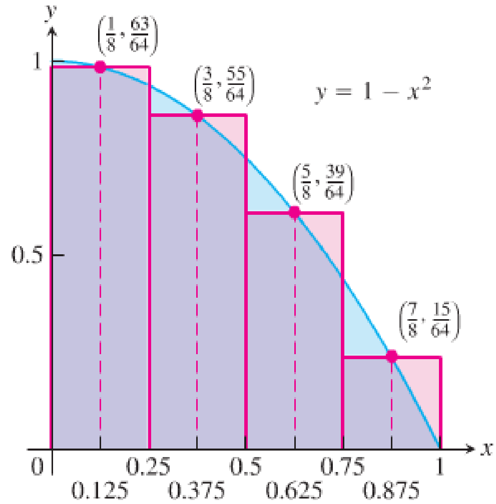



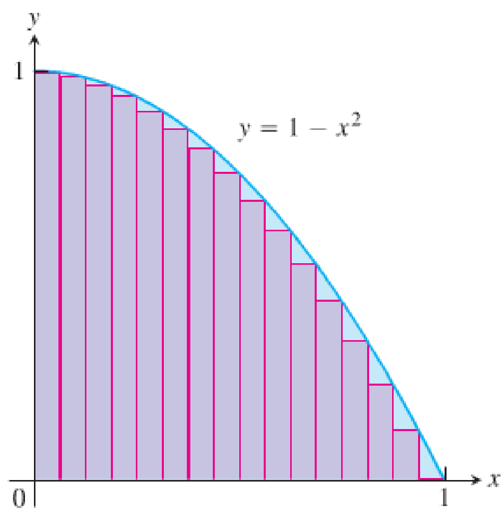

$$Area \approx 1 \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = 0.875$$

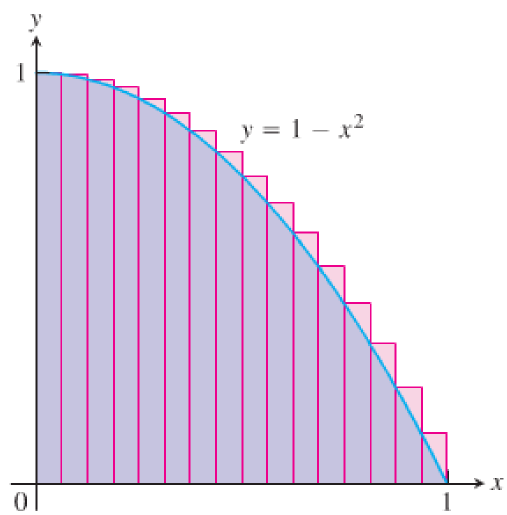
$$Area \approx 1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} = 0.78125$$




$Area \approx \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = 0.53125$ $\quad$ $Area \approx \frac{63}{64} \cdot \frac{1}{4} + \frac{55}{64} \cdot \frac{1}{4} + \frac{39}{64} \cdot \frac{1}{4} + \frac{15}{64} \cdot \frac{1}{4} = 0.671875$




In each case of the computations, the interval $[a, b]$ over which the function $f$ is defined was subdivided into $n$ equal subintervals (also called ***length***) $\Delta x = \frac{b-a}{n}$, and f was evaluated at a point in each subinterval. The finite sums can be given by the form:

$$f\left(c_1\right)\Delta x + f\left(c_2\right)\Delta x + f\left(c_3\right)\Delta x + \cdots + f\left(c_n\right)\Delta x$$

## Distance Traveled

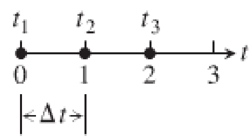
The distance formula is given by: $distance = velocity \times time$

### *Example*

The velocity function of a projectile fired straight up into the air is $f(t) = 160 - 9.8t\ m/\sec$. Use the summation technique to estimate how far the projectile rises during the first 3 *sec*. How close do the sums come to the exact value of 435.9 *m*?

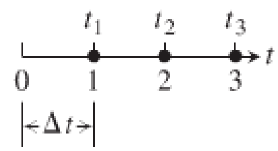
***Solution***

***i.*** $\Delta t = 1 \text{ sec} \quad \to t = 0, 1, 2$




$$D \approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t$$
$$= f(0)\Delta t + f(1)\Delta t + f(2)\Delta t$$
$$= (160 - 9.8(0))(1) + (160 - 9.8(1))(1) + (160 - 9.8(2))(1)$$
$$\underline{= 450.6|}$$

***ii.*** $\Delta t = 1 \text{ sec} \quad \to t = 1, 2, 3$




$$D \approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t$$
$$= (160 - 9.8(1))(1) + (160 - 9.8(2))(1) + (160 - 9.8(3))(1)$$
$$\underline{= 421.2|}$$

***iii.*** $\Delta t = 0.5 \text{ sec} \quad \to t = 0,\ 0.5,\ 1,\ 1.5,\ 2,\ 2.5$

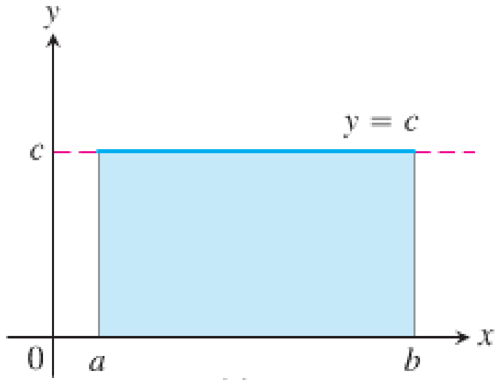
$$D \approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t + f(t_4)\Delta t + f(t_5)\Delta t + f(t_6)\Delta t$$
$$= (160 - 9.8(0))(1) + (160 - 9.8(0.5))(1) + (160 - 9.8(1))(1) + (160 - 9.8(1.5))(1)$$
$$+ (160 - 9.8(2))(1) + (160 - 9.8(2.5))(1)$$
$$\underline{\approx 443.25|}$$

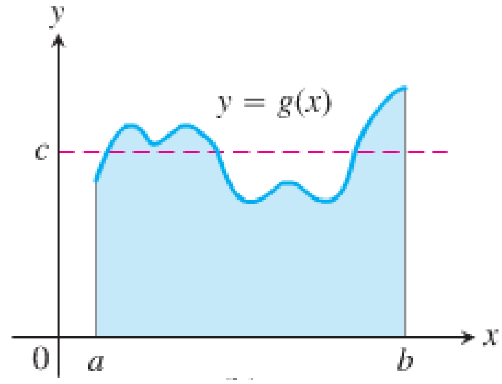
***iv.*** $\Delta t = 0.5 \text{ sec} \quad \to t = 0.5,\ 1,\ 1.5,\ 2,\ 2.5,\ 3$

$$D \approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t + f(t_4)\Delta t + f(t_5)\Delta t + f(t_6)\Delta t$$
$$= (160 - 9.8(0.5))(1) + (160 - 9.8(1))(1) + (160 - 9.8(1.5))(1) + (160 - 9.8(2))(1)$$
$$+ (160 - 9.8(2.5))(1) + (160 - 9.8(3))(1)$$
$$\underline{\approx 428.55|}$$

The true value is 435.9 if you use more subintervals $\Delta t = 0.25 \text{ sec}$, the interval 436.13 & 435.67
The projectile rose about 436 m during the first 3 sec of flight.

## Average Value of a Nonnegative Continuous Function



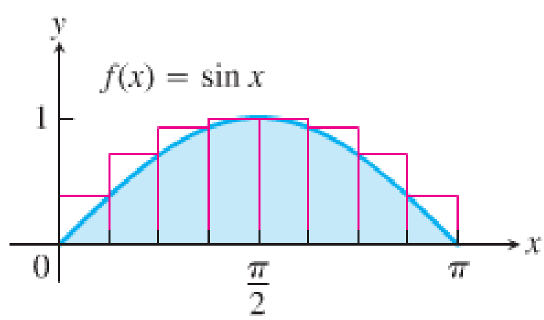




The average value of a collection of *n* numbers $x_1, x_2, \cdots, x_n$ is obtained by adding them together and dividing by *n*.

### *Example*

Estimate the average value of the function $f(x) = \sin x$ on the interval [0, π].

***Solution***




To get the upper sum approximation with 8 rectangles of equal width $\Delta x = \frac{\pi}{8}$.

$$A \approx \left(\sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \sin\frac{3\pi}{8} + \sin\frac{\pi}{2} + \sin\frac{\pi}{2} + \sin\frac{5\pi}{8} + \sin\frac{3\pi}{4} + \sin\frac{7\pi}{8}\right)\cdot\frac{\pi}{8}$$

$$\approx 2.365$$

## Finite Sums and Sigma Notation

***Sigma notation*** enables us to write a sum with many terms in the compact form

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

The Greek letter $\sum$ (capital ***sigma***, corresponding to our letter *S*)

The index $k$ ends at $k = n$

The summation symbol $\rightarrow \displaystyle\sum_{k=1}^{n} a_k \rightarrow$ $a_k$ is a formula for the $k^{th}$ term

The index $k$ starts at $k = 1$

### *Example*

| *Sigma Notation* | *Written* | *Value of the Sum* |
|---|---|---|
| $\displaystyle\sum_{k=1}^{5} k$ | $1+2+3+4+5$ | $15$ |
| $\displaystyle\sum_{k=1}^{4} (-1)^k \cdot k$ | $(-1)^1 \cdot 1 + (-1)^2 \cdot 2 + (-1)^3 \cdot 3 + (-1)^4 \cdot 4$ | $-1+2-3+4 = 2$ |
| $\displaystyle\sum_{k=1}^{3} \frac{k}{k+1}$ | $\frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1}$ | $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \frac{23}{12}$ |
| $\displaystyle\sum_{k=4}^{5} \frac{k^2}{k-1}$ | $\frac{4^2}{4-1} + \frac{5^2}{5-1}$ | $\frac{16}{3} + \frac{25}{4} = \frac{139}{12}$ |

### *Example*

We can write:

$$1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2+10^2=\sum_{k=1}^{10} k^2$$

### *Example*

Express the sum 1+ 3 +5 +7 + 9 in sigma notation.

***Solution***

Starting with $k$ = 0: $1+3+5+7+9=\sum_{k=0}^{4}(2k+1)$

Starting with $k$ = 1: $1+3+5+7+9=\sum_{k=1}^{5}(2k-1)$

### *Theorem on Sums*

If $a_1, a_2, a_3, \dots, a_n, \dots$ and $b_1, b_2, b_3, \dots, b_n, \dots$ are infinite sequences, then for every positive integer ***n***,

(1) $\sum_{k=1}^{n}(a_k+b_k)=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k$

(2) $\sum_{k=1}^{n}(a_k-b_k)=\sum_{k=1}^{n}a_k-\sum_{k=1}^{n}b_k$

(3) $\sum_{k=1}^{n}ca_k=c\left(\sum_{k=1}^{n}a_k\right)$

(4) $\sum_{k=1}^{n}c=n\cdot c$

***Proof***

$$\sum_{k=1}^{n}\left(a_k+b_k\right)=\left(a_1+b_1\right)+\left(a_2+b_2\right)+\ldots+\left(a_n+b_n\right)$$

$$=\left(a_1+a_2+\ldots+a_n\right)+\left(b_1+b_2+\ldots+b_n\right)$$

$$=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k$$

## *Example*

(1) $\displaystyle\sum_{k=1}^{n}(k+4)=\sum_{k=1}^{n}k+\sum_{k=1}^{n}4=\sum_{k=1}^{n}k+4\cdot n$

(2) $\displaystyle\sum_{k=1}^{n}\left(3k-k^2\right)=3\sum_{k=1}^{n}k-\sum_{k=1}^{n}k^2$

(3) $\displaystyle\sum_{k=1}^{n}\left(-a_k\right)=\sum_{k=1}^{n}(-1)\cdot\left(a_k\right)=(-1)\cdot\sum_{k=1}^{n}\left(a_k\right)=-\sum_{k=1}^{n}a_k$

(4) $\displaystyle\sum_{k=1}^{n}\frac{1}{n}=n\cdot\frac{1}{n}=1$

## *Example*

Show that the sum of the first $n$ integers is $\displaystyle\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$

***Solution***

The sum of the first 4 integers is: $\displaystyle\sum_{k=1}^{4}k=\frac{4(5)}{2}=10$

To prove the formula in general:

$$\begin{array}{ccccccccc} 1 & + & 2 & + & 3 & + & \cdots & + & n \\ n & + & (n-1) & + & (n-2) & + & \cdots & + & 1 \\ \hline n+1 & + & n+1 & + & n+1 & + & \cdots & + & n+1 \end{array} \rightarrow n(n+1)$$

Since it is twice the desired quantity, the sum of the first $n$ integers is $\dfrac{n(n+1)}{2}$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} = (1+2+3+\cdots+n)^2$$

## Limits of Finite Sums

### *Example*

Find the limiting value of lower sum approximations to the area of the region $R$ below the graph of $y = 1 - x^2$ and above the interval [0, 1] on the $x$-axis using equal-width rectangles whose width approach zero and whose number approaches infinity.

***Solution***

The lower sum approximation using $n$ rectangles of equal width: $\Delta x = \frac{1-0}{n} = \frac{1}{n}$

By subdividing the interval [0, 1] into $n$ equal width subintervals:

$$[0,\ 1] = \left[0,\ \frac{1}{n}\right], \left[\frac{1}{n},\ \frac{2}{n}\right], \left[\frac{2}{n},\ \frac{3}{n}\right],\ \ldots,\ \left[\frac{n-1}{n},\ \frac{n}{n}\right] = \left[\frac{k-1}{n},\ \frac{k}{n}\right]$$

$$f\left(\frac{k}{n}\right) = 1 - \left(\frac{k}{n}\right)^2$$

$$\left[f\left(\frac{1}{n}\right)\right]\cdot\left(\frac{1}{n}\right) + \left[f\left(\frac{2}{n}\right)\right]\cdot\left(\frac{1}{n}\right) + \cdots + \left[f\left(\frac{n}{n}\right)\right]\cdot\left(\frac{1}{n}\right)$$

We can write this in sigma notation:

$$\begin{aligned}
\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\left(\frac{1}{n}\right) &= \sum_{k=1}^{n}\left[1 - \left(\frac{k}{n}\right)^2\right]\cdot\left(\frac{1}{n}\right) \\
&= \sum_{k=1}^{n}\left(\frac{1}{n} - \frac{k^2}{n^3}\right) \\
&= \sum_{k=1}^{n}\frac{1}{n} - \sum_{k=1}^{n}\frac{k^2}{n^3} \\
&= n\cdot\frac{1}{n} - \frac{1}{n^3}\sum_{k=1}^{n} k^2 \\
&= 1 - \frac{1}{n^3}\cdot\frac{n\cdot(n+1)\cdot(2n+1)}{6}
\end{aligned}$$

$$= 1 - \frac{2n^3 + 3n^2 + n}{6n^3}$$

$$= \frac{6n^3 - 2n^3 - 3n^2 - n}{6n^3}$$

$$= \frac{4n^3 - 3n^2 - n}{6n^3}$$

$$\lim_{x \to \infty} \left( \frac{4n^3 - 3n^2 - n}{6n^3} \right) = \frac{4}{6} = \boxed{\frac{2}{3}}$$

The lower sum approximation converge to $\frac{2}{3}$

The upper sum approximation also converge to $\frac{2}{3}$

# *Review*

## Definition of Arithmetic Sequence

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an arithmetic sequence if there is a real number ***d*** such that for every positive integer ***k***,

$$a_{k+1} = a_k + d$$

The number $d = a_{k+1} - a_k$ is called the ***common difference*** of the sequence.

***The nth Term of an Arithmetic Sequence***: $\boxed{a_n = a_1 + (n-1)d}$

## *Example*

Express the sum in terms of summation notation: $4 + 11 + 18 + 25 + 32$. (Answers are not unique)

***Solution***

Number of terms: $n = 5$

Difference in terms: $d = 11 - 4 = 7$

$a_n = a_1 + (n-1)d$

$\underline{|a_n} = 4 + (n-1)7 = 4 + 7n - 7 \ \underline{= 7n - 3|}$

$$\sum_{n=1}^{5} (7n - 3)$$

## *Theorem*

**Formulas for** $S_n$

If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an arithmetic sequence with common difference ***d***, then the *n*th partial sum $S_n$ (that is, the sum of the first ***n*** terms) is given by either

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right] \quad \textit{or} \quad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

## *Definition* of *Geometric* Sequence

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a geometric sequence if $a_1 \neq 0$ and if there is a real number $r \neq 0$ such that for every positive integer *k*.

$$a_{k+1} = a_k r$$

The number $r = \dfrac{a_{k+1}}{a_k}$ is called the ***common ratio*** of the sequence.

***The formula for the n<sup>th</sup> Term of a Geometric Sequence***: $\boxed{a_n = a_1 r^{n-1}}$

The common ratio for: $6,\ -12,\ 24,\ -48,\ \ldots,\ (-2)^{n-1}(6),\ \ldots$ is $= \frac{-12}{6} = -2$

### *Example*

Express the sum in terms of summation notation (Answers are not unique.)

$$\frac{1}{4} - \frac{1}{12} + \frac{1}{36} - \frac{1}{108}$$

***Solution***

$$\frac{1}{4} - \frac{1}{12} + \frac{1}{36} - \frac{1}{108} = \frac{1}{4} - \frac{1}{4}\frac{1}{3^1} + \frac{1}{4}\frac{1}{3^2} - \frac{1}{4}\frac{1}{3^3}$$

$$= \sum_{n=1}^{4} (-1)^{n+1} \frac{1}{4}\left(\frac{1}{3}\right)^{n-1}$$

## *Theorem*: Formula for $S_n$

The nth partial sum $S_n$ of a geometric sequence with first term $a_1$ and common ratio $r \neq 1$ is

$$\boxed{S_n = a_1 \frac{1 - r^n}{1 - r}}$$

## Riemann Sums

The theory of limits of finite approximations was made precise by the German mathematician ***Bernhard Riemann***.

We introduce the notion of a *Riemann sum*, which underlies the theory of the definite integral.

Let a closed interval [$a$, $b$] be partitioned by points $a < x_1 < x_2 < \cdots < x_{n-1} < b$

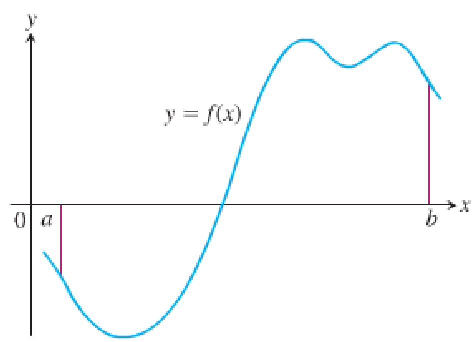
To make the notation consistent, so that

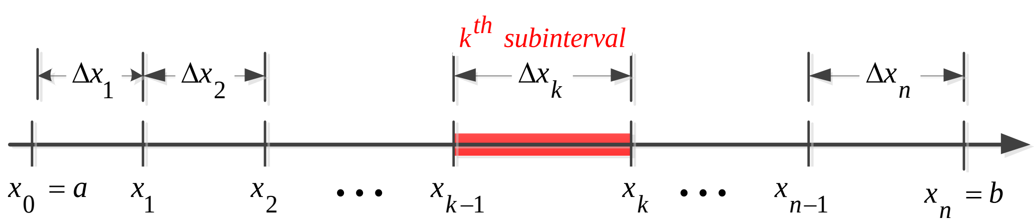
$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

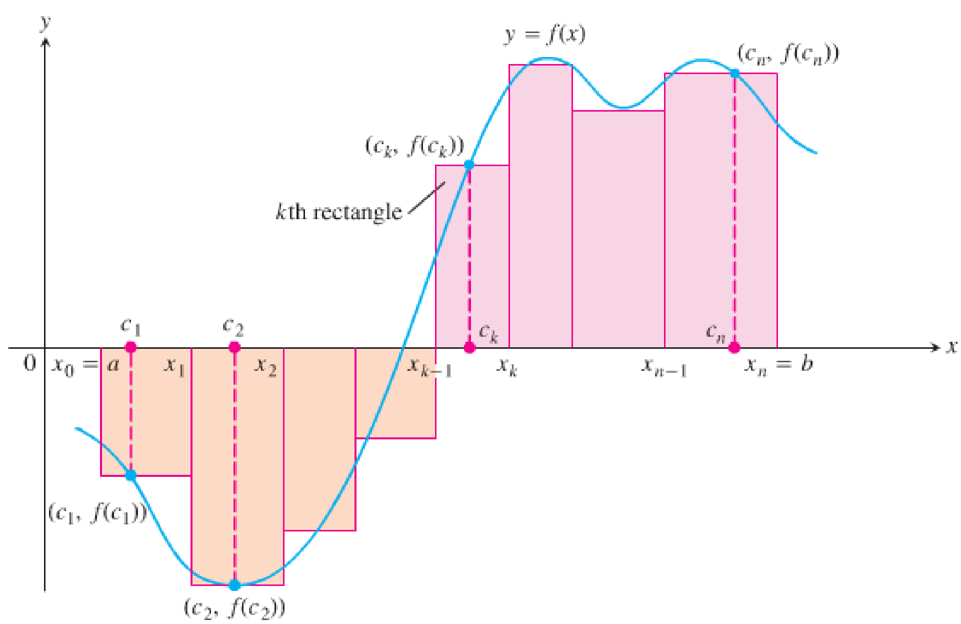
The set: $P = \{x_0, x_1, x_2, \cdots, x_{n-1}, x_n\}$ is called a partition of [$a$, $b$].

The partition $P$ divides [$a$, $b$] into $n$ closed subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$$



These products are:

$$S_P = \sum_{k=1}^{n} f\left(c_k\right) \Delta x_k$$

The sum $S_P$ is called a ***Riemann sum*** *for f on the interval* [*a*, *b*], and $c_k$ in the subintervals.

If we choose *n* subintervals all having equal width $\Delta x = \frac{b-a}{n}$ to partition [*a*, *b*], then choose the point $c_k$ to be the right-hand endpoints of each subintervals when forming the Riemann sum. This choice leads to the Riemann sum formula

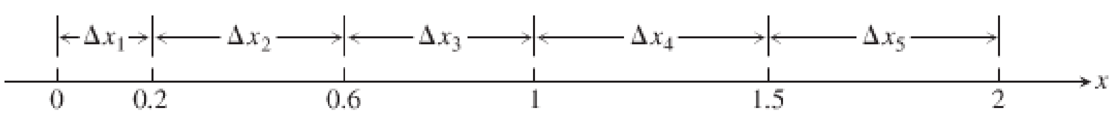
$$S_n = \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right) \cdot \left(\frac{b-a}{n}\right)$$

## *Example*

The set $P = \{0,\ 0.2,\ 0.6,\ 1,\ 1.5,\ 2\}$ is a partition of [0, 2]

***Solution***

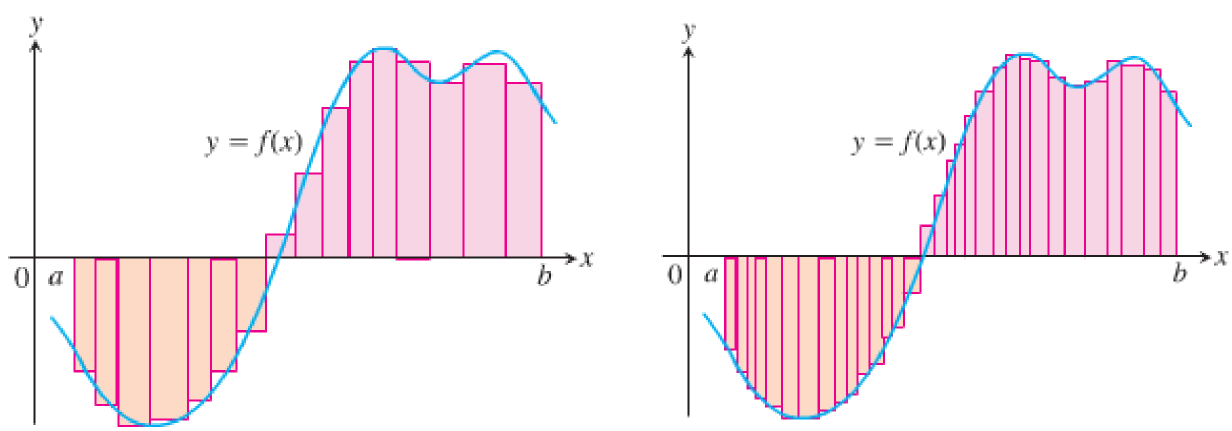
There are five subintervals of *P*: [0, 0.2], [0.2, 0.6], [0.6, 1], [1, 1.5], and [1.5, 2]




The lengths of the subintervals are:

$$\Delta x_1 = 0.2 \quad \Delta x_2 = 0.4 \quad \Delta x_3 = 0.4 \quad \Delta x_4 = 0.5 \quad \Delta x_5 = 0.5$$

The longest subinterval length is 0.5, so the norm of the partition is $\|P\| = 0.5$



# *Exercises*  *Section* 4.2 – Area under Curves

Use finite approximations to estimate the area under the graph of the function using

*a)* A lower sum with two rectangles of equal width

*b)* A lower sum with four rectangles of equal width

*c)* A upper sum with two rectangles of equal width

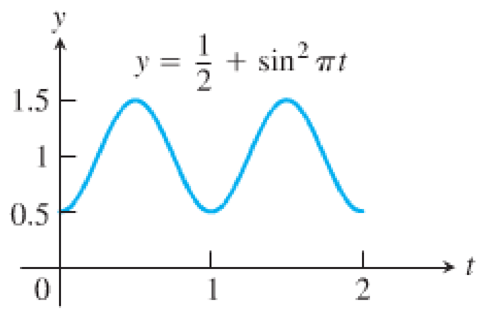
*d)* A upper sum with four rectangles of equal width

**1.** $f(x)=\frac{1}{x}$ *between* $x=1$ *and* $x=5$

**2.** $f(x)=4-x^2$ *between* $x=-2$ *and* $x=2$

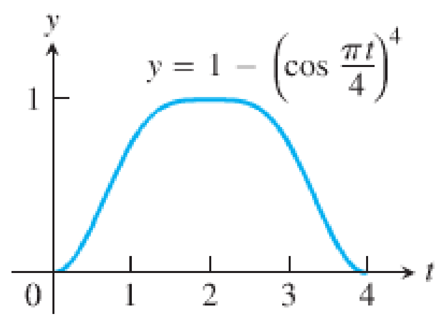
**3.** Use finite approximations to estimate the average value of *f* on the given interval by partitioning the interval into four subintervals of equal length and evaluating *f* at the subinterval midpoints.

$$f(t)=\frac{1}{2}+\sin^2 \pi t \quad on \quad [0,\ 2]$$




**4.** Use finite approximations to estimate the average value of *f* on the given interval by partitioning the interval into four subintervals of equal length and evaluating *f* at the subinterval midpoints.

$$f(t)=1-\left(\cos\frac{\pi t}{4}\right)^4 \quad on \quad [0,\ 4]$$




Write the sums without sigma notation. Then evaluate them:

**5.** $\displaystyle\sum_{k=1}^{2}\frac{6k}{k+1}$  **6.** $\displaystyle\sum_{k=1}^{3}\frac{k-1}{k}$  **7.** $\displaystyle\sum_{k=1}^{5}\sin k\pi$  **8.** $\displaystyle\sum_{k=1}^{4}(-1)^k\cos k\pi$

**9.** Write the following expression 1 + 2 + 4 + 8 + 16 + 32 in sigma notation

**10.** Write the following expression 1 – 2 + 4 – 8 + 16 – 32 in sigma notation

**11.** Write the following expression $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ in sigma notation

**12.** Write the following expression $-\frac{1}{5}+\frac{2}{5}-\frac{3}{5}+\frac{4}{5}-\frac{5}{5}$ in sigma notation

**13.** Suppose that $\sum_{k=1}^{n} a_k = -5$ *and* $\sum_{k=1}^{n} b_k = 6$. Find the value of $\sum_{k=1}^{n}\left(b_k - 2a_k\right)$

Evaluate the sums

**14.** $\sum_{k=1}^{10} k^3$

**15.** $\sum_{k=1}^{7} (-2k)$

**16.** $\sum_{k=1}^{5} \frac{\pi k}{15}$

**17.** $\sum_{k=1}^{5} k(3k+5)$

**18.** $\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$

**19.** $\sum_{k=1}^{500} 7$

**20.** $\sum_{k=18}^{71} k(k-1)$

**21.** $\sum_{k=1}^{n} \left(\frac{1}{n}+2n\right)$

**22.** Graph the function $f(x)=x^2-1$ over the given interval [0, 2]. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\sum_{k=1}^{4} f(c_k)\Delta x_k$, given $c_k$ is the

*a)* Left-hand endpoint
*b)* Right-hand endpoint
*c)* Midpoint of k$^{th}$ subinterval.

(Make a separate sketch for each set of rectangles.)