Solution   Section 1.2 – Solutions to Separable Equations

Exercise

Find the general solution of the differential equation $y' = xy$

Solution

$$\frac{dy}{dx} = xy$$

$$\frac{dy}{y} = xdx$$

$$\int \frac{dy}{y} = \int xdx$$

$$\ln|y| = \frac{1}{2}x^2 + C$$

$$|y| = e^{x^2/2+C}$$

$$y(x) = \pm e^{x^2/2}e^C$$

$$\boxed{= Ae^{x^2/2}} \qquad \text{Where } A = \pm e^C$$

Exercise

Find the general solution of the differential equation $xy' = 2y$

Solution

$$x\frac{dy}{dx} = 2y$$

$$\frac{dy}{y} = 2\frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{2}{x}dx$$

$$\ln|y| = 2\ln|x| + C$$

$$= \ln x^2 + C$$

$$y(x) = \pm e^{\ln x^2 + C}$$

$$= \pm e^C x^2$$

$$\boxed{= Ax^2}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = e^{x-y}$

Solution

$$\frac{dy}{dx} = e^x e^{-y}$$

$$\frac{dy}{e^{-y}} = e^x dx$$

$$\int e^y dy = \int e^x dx$$

$$e^y = e^x + C$$

$$\boxed{y(x) = \ln\left(e^x + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \left(1 + y^2\right)e^x$

Solution

$$\frac{dy}{dx} = \left(1 + y^2\right)e^x$$

$$\int \frac{dy}{1 + y^2} = \int e^x dx$$

$$\tan^{-1} y = e^x + C$$

$$\boxed{y(x) = \tan\left(e^x + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = xy + y$

Solution

$$\frac{dy}{dx} = (x+1)y$$

$$\int \frac{dy}{y} = \int (x+1)dx$$

$$\ln y = \frac{1}{2}x^2 + x + C$$

$$\rightarrow \boxed{y(x) = e^{x^2/2 + x + C}}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$y' = ye^x - 2e^x + y - 2$

Solution

$$\frac{dy}{dx} = (y-2)e^x + y - 2$$

$$\frac{dy}{dx} = (y-2)\left(e^x + 1\right)$$

$$\frac{dy}{y-2} = \left(e^x + 1\right)dx$$

$$\int \frac{dy}{y-2} = \int \left(e^x + 1\right)dx$$

$$\ln|y-2| = e^x + x + C$$

$$y - 2 = \pm e^{e^x + x + C}$$

$$y - 2 = \pm e^C e^{e^x + x}$$

$$\boxed{y(x) = De^{e^x + x} + 2} \quad D = \pm e^C$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \dfrac{x}{y+2}$

Solution

$$\frac{dy}{dx} = \frac{x}{y+2}$$

$$(y+2)dy = xdx$$

$$\int (y+2)dy = \int xdx$$

$$\tfrac{1}{2}y^2 + 2y = \tfrac{1}{2}x^2 + C$$

$$y^2 + 4y = x^2 + 2C$$

$$y^2 + 4y - x^2 - D = 0, \quad (D = 2C)$$

$$y = \frac{-4 \pm \sqrt{16 - 4\left(-x^2 - D\right)}}{2} = \frac{-4 \pm \sqrt{16 + 4x^2 + 4D}}{2} = \frac{-4 \pm 2\sqrt{x^2 + (4+D)}}{2} = -2 \pm \sqrt{x^2 + E} \qquad E = 4 + D$$

$$\boxed{y(x) = -2 \pm \sqrt{x^2 + E}}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \dfrac{xy}{x-1}$

Solution

$$\frac{dy}{dx} = y\left(\frac{x}{x-1}\right)$$

$$\frac{dy}{y} = \left(\frac{x}{x-1}\right)dx$$

$$\int \frac{dy}{y} = \int \left(1 + \frac{1}{x-1}\right)dx$$

$$\ln|y| = x + \ln|x-1| + C$$

$$y(x) = \pm e^{x+\ln|x-1|+C}$$

$$= \pm e^{C} e^{x} e^{\ln|x-1|}$$

$$\boxed{= De^{x}|x-1|}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$y' = \frac{y^2 + ty + t^2}{t^2}$$

Solution

$$y' = \frac{y^2}{t^2} + \frac{y}{t} + 1 \qquad \text{Let } x = \frac{y}{t} \;\Rightarrow\; y = xt \;\rightarrow\; y' = x + tx'$$

$$y' = \frac{y^2}{t^2} + \frac{y}{t} + 1 = x^2 + x + 1$$

$$x + tx' = x^2 + x + 1$$

$$t\frac{dx}{dt} = x^2 + 1$$

$$\int \frac{dx}{x^2+1} = \int \frac{dt}{t}$$

$$\tan^{-1} x = \ln|t| + C$$

$$\tan^{-1}\frac{y}{t} = \ln|t| + C$$

$$\frac{y}{t} = \tan\left(\ln|t| + C\right)$$

$$\boxed{y(t) = t\tan\left(\ln|t| + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$$

Solution

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$$

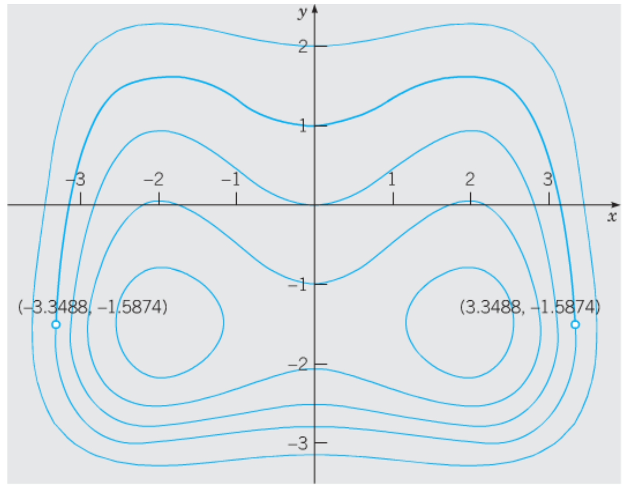
$$\left(4 + y^3\right) dy = \left(4x - x^3\right) dx$$

$$\int \left(4 + y^3\right) dy = \int \left(4x - x^3\right) dx$$

$$4y + \frac{1}{4} y^4 = 2x^2 - \frac{1}{4} x^4 + C_1$$

$$16y + y^4 = 8x^2 - x^4 + C$$

$$\boxed{y^4 + 16y + x^4 - 8x^2 = +C}$$



Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$y' = \frac{2xy + 2x}{x^2 - 1}$$

Solution

$$\frac{dy}{dx} = \frac{2x(y+1)}{x^2 - 1}$$

$$\frac{dy}{y+1} = \frac{2x}{x^2 - 1} dx \qquad\qquad d\left(x^2 - 1\right) = 2x\,dx$$

$$\int \frac{d(y+1)}{y+1} = \int \frac{d\left(x^2 - 1\right)}{x^2 - 1}$$

$$\ln|y+1| = \ln\left|x^2 - 1\right| + C$$

$$y + 1 = e^{\ln\left|x^2 - 1\right| + C}$$

$$y = e^C e^{\ln\left|x^2 - 1\right|} - 1$$

$$\boxed{y(x) = Ae^{\ln\left|x^2 - 1\right|} - 1}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = \sin 5x$

Solution

$$\int dy = \int \sin 5x dx$$

$$\boxed{y(x) = -\frac{1}{5}\cos 5x + C}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = (x+1)^2$

Solution

$$\int dy = \int \left(x^2 + 2x + 1\right) dx$$

$$\boxed{y(x) = \frac{1}{3}x^3 + x^2 + x + C}$$

Exercise

Find the general solution of the differential equation $dx + e^{3x} dy = 0$

Solution

$$\int dy = -\int e^{-3x} dx$$

$$\boxed{y(x) = \frac{1}{3}e^{-3x} + C}$$

Exercise

Find the general solution of the differential equation $dy - (y-1)^2\, dx = 0$

Solution

$$\int \frac{dy}{(y-1)^2} = \int dx$$

$$\int \frac{d(y-1)}{(y-1)^2} = \int dx$$

$$-\frac{1}{y-1} = x + C$$

$$\boxed{y(x) = 1 - \frac{1}{x+C}}$$

Exercise

Find the general solution of the differential equation $x\frac{dy}{dx} = 4y$

Solution

$$\int \frac{dy}{y} = 4\int \frac{dx}{x}$$

$$\ln y = 4\ln x + \ln C$$

$$\ln y = \ln Cx^4$$

$$\boxed{y(x) = Cx^4}$$

Exercise

Find the general solution of the differential equation $\frac{dx}{dy} = y^2 - 1$

Solution

$$\int dx = \int \left(y^2 - 1\right) dy$$

$$\boxed{x = \frac{1}{3}y^3 - y + C}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = e^{2y}$

Solution

$$\int e^{-2y} dy = \int dx$$

$$-\frac{1}{2}e^{-2y} = x + C$$

$$e^{-2y} = -2x + C_1$$

$$-2y = \ln\left(C_1 - 2x\right)$$

$$\boxed{y(x) = -\frac{1}{2}\ln\left(C_1 - 2x\right)}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} + 2xy^2 = 0$

Solution

$$\frac{dy}{dx} = -2xy^2$$

$$-\int \frac{dy}{y^2} = \int 2x\,dx$$

$$\frac{1}{y} = x^2 + C$$

$$\boxed{y(x) = \frac{1}{x^2 + C}}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = e^{3x+2y}$

Solution

$$\frac{dy}{dx} = e^{3x}e^{2y}$$

$$\int e^{-2y}dy = \int e^{3x}dx$$

$$-\frac{1}{2}e^{-2y} = \frac{1}{3}e^{3x} + C$$

$$e^{-2y} = C_1 - \frac{2}{3}e^{3x}$$

$$-2y = \ln\left(C_1 - \frac{2}{3}e^{3x}\right)$$

$$\boxed{y(x) = -\frac{1}{2}\ln\left(C_1 - \frac{2}{3}e^{3x}\right)}$$

Exercise

Find the general solution of the differential equation $e^x y\frac{dy}{dx} = e^{-y} + e^{-2x-y}$

Solution

$$e^x y\frac{dy}{dx} = e^{-y}\left(1 + e^{-2x}\right)$$

$$ye^y dy = e^{-x}\left(1 + e^{-2x}\right)dx$$

$$\int ye^y dy = \int \left(e^{-x} + e^{-3x}\right)dx$$

$$\boxed{(y-1)e^y = -e^{-x} - \frac{1}{3}e^{-3x} + C}$$

		$\int e^y$
+	y	e^y
−	1	e^y

Exercise

Find the general solution of the differential equation $y\ln x\frac{dx}{dy} = \left(\frac{y+1}{x}\right)^2$

Solution

$$x^2 \ln x dx = \frac{1}{y}\left(y^2 + 2y + 1\right)dy$$

$$\int x^2 \ln x dx = \int \left(y + 2 + \frac{1}{y}\right)dy$$

$u = \ln x \quad dv = x^2 dx$

$du = \frac{dx}{x} \quad v = \frac{1}{3}x^3$

$$\frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx = \frac{1}{2}y^2 + 2y + \ln|y| + C$$

$$\boxed{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 = \frac{1}{2}y^2 + 2y + \ln|y| + C}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = \left(\frac{2y+3}{4x+5}\right)^2$

Solution

$$\int \frac{dy}{(2y+3)^2} = \int \frac{dx}{(4x+5)^2}$$

$$\frac{1}{2}\int \frac{d(2y+3)}{(2y+3)^2} = \frac{1}{4}\int \frac{d(4x+5)}{(4x+5)^2}$$

$$\frac{1}{2}\frac{-1}{2y+3} = \frac{1}{4}\frac{-1}{4x+5} + C$$

$$\boxed{\frac{2}{2y+3} = \frac{1}{4x+5} + C}$$

Exercise

Find the general solution of the differential equation $\csc y dx + \sec^2 x dy = 0$

Solution

$$\csc y dx = -\sec^2 x dy$$

$$\frac{dy}{\csc y} = -\frac{dx}{\sec^2 x}$$

$$\sin y dy = -\cos^2 x dx$$

$$\int \sin y \, dy = -\frac{1}{2}\int (1 + \cos 2x) dx$$

$$-\cos y = -\frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$

$$\boxed{\cos y = \frac{1}{2}x + \frac{1}{4}\sin 2x + C}$$

Exercise

Find the general solution of the differential equation $\sin 3x dx + 2y\cos^3 3x dy = 0$

Solution

$$\sin 3x dx = -2y\cos^3 3x dy$$

$$\int \frac{\sin 3x}{\cos^3 3x} dx = -\int 2y dy$$

$$-\frac{1}{3}\int \cos^{-3} 3x \, d(\cos 3x) = -\int 2y dy$$

$$-\frac{1}{6}\cos^{-2} 3x + C = y^2$$

$$\boxed{y^2 = -\frac{1}{6}\sec^2 3x + C}$$

Exercise

Find the general solution of the differential equation $\left(e^y+1\right)^2 e^{-y}dx + \left(e^x+1\right)^3 e^{-x}dy = 0$

Solution

$$\left(e^y+1\right)^2 e^{-y}dx = -\left(e^x+1\right)^3 e^{-x}dy$$

$$\frac{e^x}{\left(e^x+1\right)^3}dx = -\frac{e^y}{\left(e^y+1\right)^2}dy$$

$$\int \left(e^x+1\right)^{-3} d\left(e^x+1\right) = -\int \frac{1}{\left(e^y+1\right)^2} d\left(e^y+1\right)$$

$$\boxed{-\frac{1}{2}\frac{1}{\left(e^x+1\right)^2} + C = \frac{1}{e^y+1}}$$

Exercise

Find the general solution of the differential equation $x\left(1+y^2\right)^{1/2} dx = y\left(1+x^2\right)^{1/2} dy$

Solution

$$\int x\left(1+x^2\right)^{-1/2} dx = \int y\left(1+y^2\right)^{-1/2} dy$$

$$\frac{1}{2}\int \left(1+x^2\right)^{-1/2} d\left(1+x^2\right) = \frac{1}{2}\int \left(1+y^2\right)^{-1/2} d\left(1+y^2\right)$$

$$2\left(1+y^2\right)^{1/2}=2\left(1+x^2\right)^{1/2}+C$$

$$\boxed{\left(1+y^2\right)^{1/2}=\left(1+x^2\right)^{1/2}+C}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx}=y\sin x$

Solution

$$\int\frac{dy}{y}=\int\sin x\ dx$$

$$\ln|y|=-\cos x+C$$

$$y=e^{-\cos x+C}$$

$$\boxed{=Ae^{-\cos x}}$$

Exercise

Find the general solution of the differential equation. $(1+x)\frac{dy}{dx}=4y$

Solution

$$\int\frac{dy}{y}=\int\frac{4}{1+x}dx$$

$$\ln|y|=4\ln|1+x|+\ln C$$

$$=\ln(1+x)^4+\ln C$$

$$=\ln C(1+x)^4$$

$$\boxed{y(x)=C(1+x)^4}$$

Exercise

Find the general solution of the differential equation. $2\sqrt{x}\frac{dy}{dx}=\sqrt{1-y^2}$

Solution

$$\int\frac{dy}{\sqrt{1-y^2}}=\frac{1}{2}\int x^{-1/2}\ dx$$

$$\boxed{\arcsin y=\sqrt{x}+C}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = 3\sqrt{xy}$

Solution

$$\int y^{1/2} dy = 3\int x^{1/2} dx$$

$$\frac{2}{3} y^{3/2} = 2x^{3/2} + C$$

$$\underline{y^{3/2} = 3x^{3/2} + C_1}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = (64xy)^{1/3}$

Solution

$$\int y^{-1/3} dy = \int 4x^{1/3} dx$$

$$\frac{3}{2} y^{2/3} = 3x^{4/3} + C_1$$

$$\underline{y^{2/3} = 2x^{4/3} + C}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = 2x\sec y$

Solution

$$\int \cos y \, dy = \int 2x \, dx$$

$$\underline{\sin y = x^2 + C}$$

Exercise

Find the general solution of the differential equation. $\left(1 - x^2\right)\frac{dy}{dx} = 2y$

Solution

$$\int \frac{dy}{y} = 2\int \frac{1}{1 - x^2} dx$$

$$\int \frac{dy}{y} = \int \left(\frac{1}{1 + x} + \frac{1}{1 - x}\right) dx$$

$$\ln|y| = \ln|1 + x| - \ln|1 - x| + \ln C$$

$$\ln|y| = \ln C\left|\frac{1+x}{1-x}\right|$$

$$\boxed{y(x) = C\frac{1+x}{1-x}}$$

Exercise

Find the general solution of the differential equation. $\quad (1+x)^2\frac{dy}{dx} = (1+y)^2$

Solution

$$\int\frac{dy}{(1+y)^2} = \int\frac{1}{(1+x)^2}dx$$

$$-\frac{1}{1+y} = -\frac{1}{1+x} + C$$

$$\frac{1}{1+y} = \frac{1+C+Cx}{1+x}$$

$$y+1 = \frac{1+x}{C_1+Cx}$$

$$y = \frac{1+x}{C_1+Cx} - 1$$

$$= \frac{1+x-C_1-Cx}{C_1+Cx} \qquad A = 1-C_1 \quad B = 1-C$$

$$\boxed{= \frac{A+Bx}{C_1+Cx}}$$

Exercise

Find the general solution of the differential equation. $\quad \frac{dy}{dx} = xy^3$

Solution

$$\int y^{-3}dy = \int xdx$$

$$-\frac{1}{2y^2} = \frac{1}{2}x^2 + C_1$$

$$\frac{1}{y^2} = -x^2 + C$$

$$\boxed{y^2 = \frac{1}{-x^2+C}}$$

Exercise

Find the general solution of the differential equation. $y\frac{dy}{dx} = x\left(y^2+1\right)$

Solution

$$\int \frac{y}{y^2+1}dy = \int xdx$$

$$\frac{1}{2}\int \frac{1}{y^2+1}d\left(y^2+1\right) = \frac{1}{2}x^2 + C$$

$$\ln\left(y^2+1\right) = x^2 + C$$

$$y^2+1 = e^{x^2+C}$$

$$\boxed{y^2 = Ae^{x^2}-1}$$

Exercise

Find the general solution of the differential equation. $y^3\frac{dy}{dx} = \left(y^4+1\right)\cos x$

Solution

$$\int \frac{y^3}{y^4+1}dy = \int \cos x dx$$

$$\frac{1}{4}\ln\left(y^4+1\right) = \sin x + C$$

$$\ln\left(y^4+1\right) = 4\sin x + C$$

$$y^4+1 = e^{4\sin x + C}$$

$$\boxed{y^4 = A\,e^{4\sin x}-1}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = \frac{1+\sqrt{x}}{1+\sqrt{y}}$

Solution

$$\int \left(1+y^{1/2}\right)dy = \int \left(1+x^{1/2}\right)dx$$

$$\boxed{y + \frac{2}{3}y^{3/2} = x + \frac{2}{3}x^{3/2} + C}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = \frac{(x-1)y^5}{x^2(2y^3-y)}$

Solution

$$\left(\frac{2y^3-y}{y^5}\right)dy = \left(\frac{x-1}{x^2}\right)dx$$

$$\int\left(2\frac{1}{y^2}-\frac{1}{y^4}\right)dy = \int\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$$

$$-\frac{2}{y}+\frac{1}{3y^3} = \ln|x| + \frac{1}{x} + C$$

$$\boxed{\frac{1-6y^2}{3y^3} = \ln|x| + \frac{1}{x} + C}$$

Exercise

Find the general solution of the differential equation. $\left(x^2+1\right)(\tan y)y' = x$

Solution

$$\int \tan y\ dy = \int \frac{x}{x^2+1}dx$$

$$\ln|\sec y| = \frac{1}{2}\ln\left(x^2+1\right) + \ln C$$

$$= \ln C\sqrt{x^2+1}$$

$$\boxed{\sec y = C\sqrt{x^2+1}}$$

Exercise

Find the general solution of the differential equation. $x^2y' = 1-x^2+y^2-x^2y^2$

Solution

$$x^2y' = 1-x^2+\left(1-x^2\right)y^2$$

$$x^2y' = \left(1-x^2\right)\left(1+y^2\right)$$

$$\int\frac{1}{1+y^2}dy = \int\frac{1-x^2}{x^2}dx$$

$$\int \frac{1}{1+y^2}\,dy = \int \left(\frac{1}{x^2}-1\right)\,dx$$

$$\boxed{\arctan y = -\frac{1}{x} - x + C}$$

Exercise

Find the general solution of the differential equation. $xy' + 4y = 0$

Solution

$$x\frac{dy}{dx} = -4y$$

$$\int \frac{dy}{y} = -4\int \frac{dx}{x}$$

$$\ln|y| = -4\ln|x| + C$$

$$\ln|y| = \ln x^{-4} + C$$

$$y(x) = e^{\ln x^{-4} + C}$$

$$= e^C e^{\ln x^{-4}}$$

$$\boxed{= Ax^{-4}}$$

Exercise

Find the general solution of the differential equation. $\left(x^2+1\right)y' + 2xy = 0$

Solution

$$\left(x^2+1\right)\frac{dy}{dx} = -2xy$$

$$\int \frac{dy}{y} = -\int \frac{2x}{x^2+1}\,dx$$

$$\ln|y| = -\ln\left(x^2+1\right) + \ln C$$

$$\ln|y| = \ln\frac{C}{x^2+1}$$

$$\boxed{y(x) = \frac{C}{x^2+1}}$$

Exercise

Find the general solution of the differential equation. $\dfrac{y'}{\left(x^2+1\right)y}=3$

Solution

$$\int \frac{1}{y}dy = \int \left(3x^2+3\right)dx$$

$$\ln|y| = x^3+3x+C$$

$$\boxed{y(x) = e^{x^3+3x+C}}$$

Exercise

Find the general solution of the differential equation. $y+e^x y'=0$

Solution

$$e^x\frac{dy}{dx} = -y$$

$$\int \frac{dy}{y} = -\int e^{-x}dx$$

$$\ln|y| = e^{-x}+C$$

$$\boxed{y(x) = e^{e^{-x}+C}}$$

Exercise

Find the general solution of the differential equation. $\dfrac{dx}{dt}=3xt^2$

Solution

$$\int \frac{dx}{x} = \int 3t^2 dt$$

$$\ln|x| = t^3+C$$

$$\boxed{x(t) = e^{t^3+C} = Ae^{t^3}}$$

Exercise

Find the general solution of the differential equation. $x\dfrac{dy}{dx}=\dfrac{1}{y^3}$

Solution

$$\int y^3 dy = \int \frac{1}{x}dx$$

$$\frac{1}{4}y^4 = \ln|x| + C_1$$

$$y^4 = 4\ln|x| + C$$

$$\underline{y^4 = \ln x^4 + C}$$

Exercise

Find the general solution of the differential equation. $\dfrac{dy}{dx} = \dfrac{x}{y^2\sqrt{x+1}}$

Solution

$$\int y^2 dy = \int \frac{x}{\sqrt{x+1}}dx$$

Let $u = x+1 \quad \to x = u-1 \quad \to du = dx$

$$\frac{1}{3}y^3 = \int \frac{u-1}{u^{1/2}}du$$

$$\frac{1}{3}y^3 = \int \left(u^{1/2} - u^{-1/2}\right)du$$

$$\frac{1}{3}y^3 = \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2} + C_1$$

$$\underline{y^3 = 2(x+1)^{3/2} - 6(x+1)^{1/2} + C}$$

Exercise

Find the general solution of the differential equation. $\dfrac{dx}{dt} - x^3 = x$

Solution

$$\frac{dx}{dt} = x^3 + x$$

$$\int \frac{dx}{x\left(x^2+1\right)} = \int dt$$

$$\frac{1}{x\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$Ax^2 + A + Bx^2 + Cx = 1$$

$$\begin{cases} x^2 & A+B=0 \\ x & \underline{C=0} \quad \to \underline{B=-1} \\ x^0 & \underline{A=1} \end{cases}$$

$$\int \frac{dx}{x} - \int \frac{dx}{x^2+1} = t + K$$

$$\underline{\ln|x| - \arctan x = t + K}$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx}=\frac{x}{ye^{x+2y}}$

Solution

$$\frac{dy}{dx}=\frac{x}{ye^{2y}e^{x}}$$

$$\int ye^{2y}dy=\int xe^{-x}dx$$

$$\frac{1}{2}ye^{2y}-\frac{1}{4}e^{2y}=-xe^{-x}-e^{-x}+C_1$$

$$(2y-1)e^{2y}=-4(x+1)e^{-x}+C$$

		$\int e^{2y}$
+	y	$\frac{1}{2}e^{2y}$
–	1	$\frac{1}{4}e^{2y}$

		$\int e^{-x}$
+	x	$-e^{-x}$
–	1	e^{-x}

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx}=\frac{\sec^2 y}{1+x^2}$

Solution

$$\int \cos^2 y\ dy=\int \frac{dx}{1+x^2}$$

$$\frac{1}{2}\int (1+\cos 2y)dy=\arctan x+C$$

$$\frac{1}{2}\left(y+\frac{1}{2}\sin 2y\right)=\arctan x+C$$

Exercise

Find the general solution of the differential equation. $x\frac{dv}{dx}=\frac{1-4v^2}{3v}$

Solution

$$\int \frac{3v}{1-4v^2}dv=\int \frac{dx}{x}$$

$$-\frac{3}{8}\int \frac{1}{1-4v^2}d\left(1-4v^2\right)=\int \frac{dx}{x}$$

$$-\frac{3}{8}\ln\left|1-4v^2\right|=\ln|x|+\ln C$$

$$\ln\left(\left|1-4v^2\right|\right)^{-3/8}=\ln|Cx|$$

$$\left(1-4v^2\right)^{-3/8}=Cx$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = 3x^2\left(1+y^2\right)^{3/2}$

Solution

$$\int\left(1+y^2\right)^{-3/2} dy = \int 3x^2 dx$$

$y = \tan\theta \qquad \sqrt{1+y^2} = \sec\theta$

$dy = \sec^2\theta\ d\theta$

$$\int \sec^{-3}\theta\ \sec^2\theta\ d\theta = x^3 + C$$

$$\int \sec\theta\ d\theta = x^3 + C$$

$$\ln\left|\sec\theta + \tan\theta\right| = x^3 + C$$

$$\boxed{\frac{1}{\sqrt{1+y^2}} + y = C_1 e^{x^3}}$$

Exercise

Find the general solution of the differential equation. $\frac{1}{y}dy + ye^{\cos x}\sin x dx = 0$

Solution

$$\int \frac{1}{y^2}dy = -\int e^{\cos x}\sin x dx$$

$$-\frac{1}{y} = e^{\cos x} + C$$

$$\boxed{y(x) = \frac{-1}{e^{\cos x} + C}}$$

Exercise

Find the general solution of the differential equation. $\left(x + xy^2\right)dx + e^{x^2}ydy = 0$

Solution

$$x\left(1+y^2\right)dx = -e^{x^2}ydy$$

$$\int xe^{-x^2}dx = -\int \frac{y}{1+y^2}dy$$

$$-\frac{1}{2}\int e^{-x^2} d\left(e^{-x^2}\right) = -\frac{1}{2}\int \frac{1}{1+y^2} d\left(1+y^2\right)$$

$$\boxed{e^{-x^2} = \ln\left(1+y^2\right) + C}$$

Exercise

Find the exact solution of the initial value problem. $y' = \dfrac{y}{x}, \quad y(1) = -2$

Solution

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln|y| = \ln|x| + C$$

$$\begin{aligned} y &= \pm e^{\ln|x|+C} \\ &= \pm e^{C} e^{\ln|x|} \\ &= D|x| \\ &= Dx \end{aligned}$$

$$y = Dx \quad \Rightarrow \; D = \frac{y}{x} = \frac{-2}{1} = -2$$

$$\boxed{y(x) = -2x}$$

Exercise

Find the exact solution of the initial value problem. $y' = -\dfrac{2t\left(1+y^2\right)}{y}, \quad y(0) = 1$

Solution

$$\frac{dy}{dt} = -\frac{2t\left(1+y^2\right)}{y}$$

$$\int \frac{ydy}{1+y^2} = \int -2tdt$$

$$\frac{1}{2}\int \frac{1}{1+y^2} d\left(1+y^2\right) = -2\int tdt$$

$$\frac{1}{2}\ln\left(1+y^2\right) = -t^2 + C$$

$$\ln\left(1+y^2\right) = -2t^2 + 2C$$

$$1+y^2 = e^{-2t^2+2C}$$

$$1+y^2 = e^{2C}e^{-2t^2}$$

$$1+y^2 = De^{-2t^2}$$

$$1+1^2 = De^{-2(0)^2} \ \rightarrow \ \underline{2 = D}$$

$$y^2 = 2e^{-2t^2} - 1$$

$$y^2 = 2e^{-2t^2} - 1$$

$$y = \pm\sqrt{2e^{-2t^2} - 1}$$

$$\boxed{y(x) = \sqrt{2e^{-2t^2} - 1}}$$

$$2e^{-2t^2} - 1 > 0$$

$$2e^{-2t^2} > 1$$

$$e^{-2t^2} > \frac{1}{2}$$

$$-2t^2 > \ln\left(\frac{1}{2}\right)$$

$$t^2 < -\frac{1}{2}\ln\left(\frac{1}{2}\right) = \frac{1}{2}\ln 2$$

$$t^2 < \ln\sqrt{2}$$

$$t < \left|\ln\sqrt{2}\right|$$

The interval of existence: $\left(-\ln\sqrt{2},\ \ln\sqrt{2}\right)$

Exercise

Find the exact solution of the initial value problem. Indicate the interval of existence.

$$y' = \frac{\sin x}{y}, \qquad y\left(\frac{\pi}{2}\right) = 1$$

Solution

$$\frac{dy}{dx} = \frac{\sin x}{y}$$

$$y\,dy = \sin x\,dx$$

$$\int y\,dy = \int \sin x\,dx$$

$$\frac{1}{2}y^2 = -\cos x + C_1$$

$$y^2 = -2\cos x + C \quad \left(C = 2C_1\right)$$

$$y(x) = \pm\sqrt{-2\cos x + C}$$

$$y\left(\frac{\pi}{2}\right) = \sqrt{-2\cos\frac{\pi}{2} + C}$$

$1 = \sqrt{C} \quad \Rightarrow \quad \boxed{C = 1}$

$\boxed{y(x) = \sqrt{1 - 2\cos x}}$

The interval of existence will be the interval containing $\frac{\pi}{2}$ and $1 - 2\cos x > 0$

$\cos x < \frac{1}{2} \quad \Rightarrow \quad \boxed{\frac{\pi}{3} < x < \frac{5\pi}{3}}$

Exercise

Find the exact solution of the initial value problem. $4tdy = \left(y^2 + ty^2\right)dt, \quad y(1) = 1$

Solution

$$4tdy = y^2(1 + t)dt$$

$$4\int \frac{dy}{y^2} = \int \left(\frac{1}{t} + 1\right)dt \qquad\qquad \int \frac{dx}{x^2} = -\frac{1}{x}$$

$$-\frac{4}{y} = \ln|t| + t + C$$

$$y = \frac{-4}{\ln|t| + t + C}$$

$$1 = \frac{-4}{\ln|1| + 1 + C} \quad \Rightarrow \quad 1 = \frac{-4}{1 + C} \quad \to \quad 1 + C = -4 \quad \Rightarrow \quad \boxed{C = -5}$$

$$\boxed{y = \frac{-4}{\ln|t| + t - 5}}$$

Exercise

Find the exact solution of the initial value problem. $y' = \frac{1 - 2t}{y}, \quad y(1) = -2$

Solution

$$y\frac{dy}{dt} = 1 - 2t$$

$$\int ydy = \int (1 - 2t)dt$$

$$\frac{1}{2}y^2 = t - t^2 + C_1$$

$$y^2 = 2t - 2t^2 + C$$

$$(-2)^2 = 2(1) - 2(1)^2 + C \quad \Rightarrow \quad \boxed{C = 4}$$

$$\boxed{y = -\sqrt{2t - 2t^2 + 4}}$$

The negative value is taken to satisfy the initial condition.

Exercise

Find the exact solution of the initial value problem. $y' = y^2 - 4, \quad y(0) = 0$

Solution

$$\frac{dy}{dt} = y^2 - 4$$

$$\frac{dy}{y^2 - 4} = dt \qquad \frac{1}{y^2-4} = \frac{A}{y-2} + \frac{B}{y+2}$$

$$\frac{1}{y^2 - 4} = \frac{(A+B)y + 2A - 2B}{y-2} \qquad \to \begin{cases} A + B = 0 \\ 2A - 2B = 1 \end{cases} \Rightarrow A = \frac{1}{4} \quad B = -\frac{1}{4}$$

$$\left(\frac{1}{4(y-2)} - \frac{1}{4(y+2)}\right) dy = dt$$

$$\int \left(\frac{1}{4(y-2)} - \frac{1}{4(y+2)}\right) dy = \int dt$$

$$\frac{1}{4}\left(\ln|y-2| - \ln|y+2|\right) = t + C$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4t + C$$

$$\frac{y-2}{y+2} = \pm e^{4t+C}$$

$$\frac{y-2}{y+2} = \pm e^{C} e^{4t} = ke^{4t}$$

$$y - 2 = ke^{4t} y + 2ke^{4t}$$

$$y - ke^{4t} y = 2 + 2ke^{4t}$$

$$y\left(1 - ke^{4t}\right) = 2 + 2ke^{4t}$$

$$y = \frac{2 + 2ke^{4t}}{1 - ke^{4t}}$$

$$0 = \frac{2 + 2ke^{4(0)}}{1 - ke^{4(0)}}$$

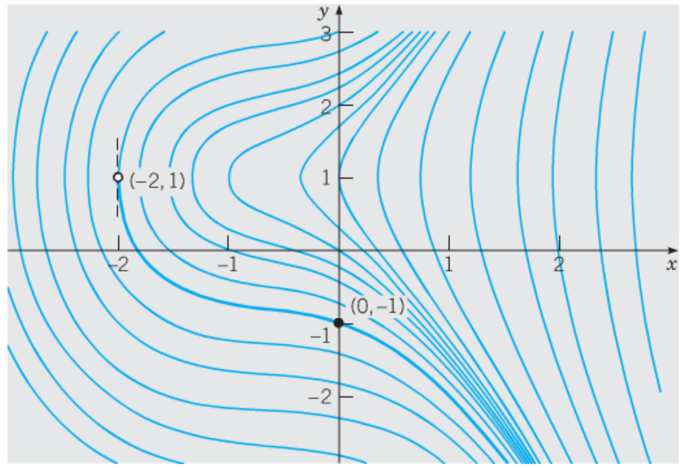
$$0 = 2 + 2k \quad \Rightarrow \quad \boxed{k = -1}$$

$$\boxed{y = \frac{2 - 2e^{4t}}{1 + e^{4t}}}$$

Exercise

Find the exact solution of the initial value problem. Indicate the interval of existence.

$$\frac{dy}{dx}=\frac{3x^2+4x+2}{2(y-1)}, \quad y(0)=-1$$



Solution

$$\frac{dy}{dx}=\frac{3x^2+4x+2}{2y-2}$$

$$(2y-2)dy=\left(3x^2+4x+2\right)dx$$

$$\int(2y-2)dy=\int\left(3x^2+4x+2\right)dx$$

$$y^2-2y=x^3+2x^2+2x+C$$

$$y(0)=-1$$

$$(-1)^2-2(-1)=(0)^3+2(0)^2+2(0)+C$$

$$\Rightarrow \boxed{C=3}$$

$$y^2-2y=x^3+2x^2+2x+3$$

Exercise

Find the exact solution of the initial value problem. $y'=\frac{x}{1+2y}, \quad y(-1)=0$

Solution

$$\frac{dy}{dx}=\frac{x}{1+2y}$$

$$\int(1+2y)dy=\int xdx$$

$$y+y^2=\frac{1}{2}x^2+C \qquad y(-1)=0$$

$$0=\frac{1}{2}(-1)^2+C \quad \Rightarrow \quad C=-\frac{1}{2}$$

$$y+y^2=\frac{1}{2}x^2-\frac{1}{2}$$

Exercise

Find the exact solution of the initial value problem $\left(e^{2y}-y\right)\cos x\frac{dy}{dx}=e^y\sin 2x, \quad y(0)=0$

Solution

$$\frac{e^{2y}-y}{e^y}dy=\frac{2\sin x\cos x}{\cos x}dx$$

$$\int \left(e^{y}-ye^{-y}\right)dy=\int 2\sin x\,dx$$

$$e^{y}+ye^{-y}+e^{-y}=-2\cos x+C$$

$$y(0)=0 \quad 1+1=-2+C \quad \rightarrow \underline{C=4}$$

$$\boxed{e^{y}+ye^{-y}+e^{-y}=4-2\cos x}$$

		$\int e^{-y}dy$
+	y	$-e^{-y}$
−	1	e^{-y}

Exercise

Find the exact solution of the initial value problem $\quad \dfrac{dy}{dx}=e^{-x^2}, \quad y(3)=5$

Solution

$$\int_3^x \frac{dy}{dt}dt=\int_3^x e^{-t^2}dt$$

$$y(x)-y(3)=\int_3^x e^{-t^2}dt$$

$$\boxed{y(x)=5+\int_3^x e^{-t^2}dt}$$

Exercise

Find the exact solution of the initial value problem. $\quad \dfrac{dy}{dx}+2y=1, \quad y(0)=\frac{5}{2}$

Solution

$$\frac{dy}{dx}=1-2y$$

$$\frac{dy}{1-2y}=dx$$

$$-\frac{1}{2}\int \frac{d(1-2y)}{1-2y}=\int dx$$

$$-\frac{1}{2}\ln|1-2y|=x+C$$

$$\ln|1-2y|=-2x+C \qquad y(0)=\frac{5}{2}$$

$$\ln|1-5|=C \quad \rightarrow \quad \underline{C=\ln 4}$$

$$1-2y=e^{-2x+\ln 4}$$

$$1-2y=e^{-2x}e^{\ln 4}$$

$$\boxed{y=\frac{1}{2}-2e^{-2x}}$$

Exercise

Find the exact solution of the initial value problem. $\sqrt{1-y^2}\,dx - \sqrt{1-x^2}\,dy = 0, \quad y(0) = \frac{\sqrt{3}}{2}$

Solution

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{dy}{\sqrt{1-y^2}} \qquad\qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$\sin^{-1}x + C = \sin^{-1}y \qquad y(0) = \frac{\sqrt{3}}{2}$$

$$\sin^{-1}0 + C = \sin^{-1}\frac{\sqrt{3}}{2} \quad \Rightarrow \quad \boxed{C = \frac{\pi}{3}}$$

$$\sin^{-1}y = \sin^{-1}x + \frac{\pi}{3}$$

$$y = \sin\left(\sin^{-1}x + \frac{\pi}{3}\right)$$

$$= \sin\left(\sin^{-1}x\right)\cos\frac{\pi}{3} + \cos\left(\sin^{-1}x\right)\sin\frac{\pi}{3} \qquad \alpha = \sin^{-1}x \to \sin\alpha = x \quad \cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-x^2}$$

$$\boxed{y(x) = \frac{x}{2} + \frac{\sqrt{3}}{2}\sqrt{1-x^2}}$$

Exercise

Find the exact solution of the initial value problem. $\left(1+x^4\right)dy + x\left(1+4y^2\right)dx = 0, \quad y(1) = 0$

Solution

$$\int \frac{1}{1+(2y)^2}\,dy = -\int \frac{x}{1+\left(x^2\right)^2}\,dx$$

$$\int \frac{1}{1+(2y)^2}\,dy = -\frac{1}{2}\int \frac{1}{1+\left(x^2\right)^2}\,d\left(x^2\right) \qquad\qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\frac{1}{2}\tan^{-1}2y = -\frac{1}{2}\tan^{-1}x^2 + C$$

$$\tan^{-1}2y + \tan^{-1}x^2 = C_1 \qquad y(1) = 0 \quad \tan^{-1}0 + \tan^{-1}1 = C_1 \quad \Rightarrow \boxed{C_1 = \frac{\pi}{4}}$$

$$\boxed{\tan^{-1}2y + \tan^{-1}x^2 = \frac{\pi}{4}}$$

$$2y = \tan\left(\frac{\pi}{4} - \tan^{-1}x^2\right)$$

$$= \frac{\tan\frac{\pi}{4} - \tan\left(\tan^{-1}x^2\right)}{1 + \tan\left(\frac{\pi}{4}\right)\tan\left(\tan^{-1}x^2\right)}$$

$$\boxed{y(x) = \frac{1}{2}\frac{1-x^2}{1+x^2}}$$

Exercise

Find the exact solution of the initial value problem. $e^{-2t}\frac{dy}{dt}=\frac{1+e^{-2t}}{y}, \quad y(0)=0$

Solution

$$ydy=\left(1+e^{-2t}\right)e^{2t}dt$$

$$\int ydy=\int\left(e^{2t}+1\right)dt$$

$$\frac{1}{2}y^2=\frac{1}{2}e^{2t}+t+C_1$$

$$y^2=e^{2t}+2t+C \qquad y(0)=0$$

$$0=1+C \quad \rightarrow \quad \boxed{C=-1}$$

$$\boxed{y^2=e^{2t}+2t-1}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt}=\frac{t+2}{y}, \quad y(0)=2$

Solution

$$\int ydy=\int(t+2)dt$$

$$\frac{1}{2}y^2=\frac{1}{2}t^2+2t+C_1$$

$$y^2=t^2+4t+C \qquad y(0)=2 \qquad \boxed{4=C}$$

$$\boxed{y=\sqrt{t^2+4t+4}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{1}{t^2}\frac{dy}{dt}=y, \quad y(0)=1$

Solution

$$\int\frac{1}{y}dy=\int t^2dt$$

$$\ln|y|=\frac{1}{3}t^3+C \qquad y(0)=1$$

$$\ln|1|=C\rightarrow \boxed{C=0}$$

$$\ln|y|=\frac{1}{3}t^3$$

$$\boxed{y(t)=e^{t^3/3}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt} = -y^2 e^{2t}; \quad y(0) = 1$

Solution

$$-\int \frac{1}{y^2} dy = \int e^{2t} dt$$

$$\frac{1}{y} = \frac{1}{2}e^{2t} + C \qquad y(0) = 1 \qquad 1 = \frac{1}{2} + C \rightarrow \boxed{C = \frac{1}{2}}$$

$$\frac{1}{y} = \frac{1}{2}\left(e^{2t} + 1\right)$$

$$\boxed{y(t) = \frac{2}{e^{2t} + 1}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt} - (2t+1)y = 0; \quad y(0) = 2$

Solution

$$\frac{dy}{dt} = (2t+1)y$$

$$\int \frac{dy}{y} = \int (2t+1)dt$$

$$\ln|y| = t^2 + t + C \qquad y(0) = 2$$

$$\boxed{\ln 2 = C}$$

$$\ln|y| = t^2 + t + \ln 2$$

$$y(t) = e^{\ln 2} e^{t^2 + t}$$

$$\boxed{= 2e^{t^2 + t}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt} + 4ty^2 = 0; \quad y(0) = 1$

Solution

$$-\int \frac{dy}{y^2} = \int 4t dt$$

$$\frac{1}{y} = 2t^2 + C \qquad y(0) = 1 \qquad \boxed{1 = C}$$

$$\frac{1}{y} = 2t^2 + 1$$

$$\boxed{y(t) = \frac{1}{2t^2+1}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = ye^x;\quad y(0) = 2e$

Solution

$$\int \frac{dy}{y} = \int e^x dx$$

$$\ln|y| = e^x + \ln C$$

$$y(0) = 2e \quad \to \quad \ln|2e| = 1 + \ln C$$

$$\ln 2 + 1 = 1 + \ln C \quad \Rightarrow \quad \boxed{C = 2}$$

$$\ln|y| = e^x + \ln 2$$

$$y(x) = e^{e^x + \ln 2}$$

$$= e^{e^x} e^{\ln 2}$$

$$\boxed{= 2e^{e^x}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 3x^2\left(y^2+1\right);\quad y(0) = 1$

Solution

$$\int \frac{1}{y^2+1} dy = \int 3x^2 dx$$

$$\arctan y = x^3 + C$$

$$y(0) = 1 \quad \to \quad \arctan 1 = C \quad \Rightarrow \boxed{C = \frac{\pi}{4}}$$

$$\boxed{y(x) = \tan\left(x^3 + \frac{\pi}{4}\right)}$$

Exercise

Find the exact solution of the initial value problem $2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}};\quad y(5) = 2$

Solution

$$\int 2y dy = \int \frac{x}{\sqrt{x^2-16}} dx$$

$$y^2 = \frac{1}{2}\int \left(x^2 - 16\right)^{-1/2} d\left(x^2 - 16\right)$$

$$y^2 = \left(x^2 - 16\right)^{1/2} + C$$

$$y(5) = 2 \quad \to 4 = (9)^{1/2} + C \quad \Rightarrow C = 4 - 3 = \underline{1}$$

$$\boxed{y^2 = 1 + \sqrt{x^2 - 16}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 4x^3 y - y; \quad y(1) = -3$

Solution

$$\frac{dy}{dx} = \left(4x^3 - 1\right)y$$

$$\int \frac{dy}{y} = \int \left(4x^3 - 1\right)dx$$

$$\ln|y| = x^4 - x + C$$

$$y = Ce^{x^4 - x}$$

$$y(1) = -3 \quad \underline{-3 = C}$$

$$\boxed{y(x) = -3e^{x^4 - x}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} + 1 = 2y; \quad y(1) = 1$

Solution

$$\int \frac{dy}{2y - 1} = \int dx$$

$$\frac{1}{2}\ln(2y - 1) = x + C$$

$$\ln(2y - 1) = 2x + C$$

$$2y - 1 = e^{2x + C}$$

$$y(x) = Ae^{2x} + \frac{1}{2}$$

$$y(1) = 1 \quad 1 = Ae^2 + 1 \quad \to \quad \underline{A = e^{-2}}$$

$$\boxed{y(x) = e^{2x - 2} + \frac{1}{2}}$$

Exercise

Find the exact solution of the initial value problem $(\tan x)\dfrac{dy}{dx} = y; \quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

Solution

$$\int \frac{dy}{y} = \int \frac{dx}{\tan x} = \int \frac{\cos x dx}{\sin x}$$

$$\ln y = \ln(\sin x) + \ln C$$

$$y(x) = C \sin x$$

$$y\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \quad \to \underline{\frac{\pi}{2} = C}$$

$$\underline{y(x) = \frac{\pi}{2}\sin x}$$

Exercise

Find the exact solution of the initial value problem $x\dfrac{dy}{dx} - y = 2x^2 y; \quad y(1) = 1$

Solution

$$x\frac{dy}{dx} = 2x^2 y + y$$

$$x\frac{dy}{dx} = \left(2x^2 + 1\right)y$$

$$\int \frac{dy}{y} = \int \left(2x + \frac{1}{x}\right)dx$$

$$\ln y = x^2 + \ln x + \ln C$$

$$y(x) = e^{x^2 + \ln x + \ln C}$$

$$= Cxe^{x^2}$$

$$y(1) = 1 \quad \to 1 = Ce \quad \Rightarrow \underline{C = e^{-1}}$$

$$\underline{y(x) = xe^{x^2 - 1}}$$

Exercise

Find the exact solution of the initial value problem $\dfrac{dy}{dx} = 2xy^2 + 3x^2 y^2; \quad y(1) = -1$

Solution

$$\frac{dy}{dx} = \left(2x + 3x^2\right)y^2$$

$$\int \frac{dy}{y^2} = \int \left(2x + 3x^2\right)dx$$

$$-\frac{1}{y} = x^2 + x^3 + C$$

$$y(x) = \frac{-1}{x^2 + x^3 + C}$$

$$y(1) = -1 \quad \rightarrow \quad -1 = \frac{-1}{C} \quad \Rightarrow \quad \underline{C = 1}$$

$$\boxed{y(x) = \frac{-1}{x^2 + x^3 + 1}}$$

Exercise

Find the exact solution of the initial value problem $\quad \frac{dy}{dx} = 6e^{2x-y}; \quad y(0) = 0$

Solution

$$\int e^y dy = \int 6e^{2x} dx$$

$$e^y = 3e^{2x} + C$$

$$y(x) = \ln\left(3e^{2x} + C\right)$$

$$y(0) = 0 \quad \rightarrow \quad 0 = \ln(3 + C) \quad \Rightarrow 3 + C = 1 \rightarrow \underline{C = -2}$$

$$\boxed{y(x) = \ln\left(3e^{2x} - 2\right)}$$

Exercise

Find the exact solution of the initial value problem $\quad 2\sqrt{x}\frac{dy}{dx} = \cos^2 y; \quad y(4) = \frac{\pi}{4}$

Solution

$$\frac{dy}{\cos^2 y} = \frac{1}{2} x^{-1/2} dx$$

$$\int \sec^2 y \, dy = \int \frac{1}{2} x^{-1/2} \, dx$$

$$\tan y = \sqrt{x} + C$$

$$y(x) = \tan^{-1}\left(\sqrt{x} + C\right)$$

$$y(4) = \frac{\pi}{4} \quad \rightarrow \quad \frac{\pi}{4} = \arctan(2 + C) \quad \rightarrow 2 + C = 1 \Rightarrow \underline{C = -1}$$

$$\boxed{y(x) = \tan^{-1}\left(\sqrt{x} - 1\right)}$$

Exercise

Find the exact solution of the initial value problem $y' + 3y = 0; \quad y(0) = -3$

Solution

$$\frac{dy}{dx} = -3y$$

$$\int \frac{dy}{y} = -3\int dx$$

$$\ln|y| = -3x + C$$

$$y(x) = e^{-3x+C}$$

$$= Ae^{-3x}$$

$$y(0) = -3 \quad \rightarrow \quad \underline{-3 = A}$$

$$\boxed{y(x) = -3e^{-3x}}$$

Exercise

Find the exact solution of the initial value problem $2y' - y = 0; \quad y(-1) = 2$

Solution

$$2\frac{dy}{dx} = y$$

$$\int \frac{dy}{y} = \frac{1}{2}\int dx$$

$$\ln|y| = \frac{1}{2}x + C$$

$$y(x) = e^{x/2+C}$$

$$= Ae^{x/2}$$

$$y(-1) = 2 \quad \rightarrow \quad 2 = Ae^{-1/2} \quad \Rightarrow \quad \underline{A = 2e^{1/2}}$$

$$y(x) = 2e^{1/2}e^{x/2} \quad \boxed{= 2e^{(x+1)/2}}$$

Exercise

Find the exact solution of the initial value problem $2xy - y' = 0; \quad y(1) = 3$

Solution

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{dy}{y} = \int 2x\, dx$$

$$\ln|y| = x^2 + C$$

$$y(x) = e^{x^2+C}$$

$$= Ae^{x^2}$$

$$y(1) = 3 \quad \rightarrow \quad 3 = Ae \quad \Rightarrow \quad \boxed{A = 3e^{-1}}$$

$$\boxed{y(x) = \frac{3}{e}e^{x^2}}$$

Exercise

Find the exact solution of the initial value problem $\quad y\dfrac{dy}{dx} - \sin x = 0; \quad y\left(\frac{\pi}{2}\right) = -2$

Solution

$$\int y\ dy = \int \sin x\ dx$$

$$\frac{1}{2}y^2 = -\cos x + C$$

$$y\left(\frac{\pi}{2}\right) = -2 \quad \rightarrow \quad \boxed{2 = C}$$

$$\frac{1}{2}y^2 = -\cos x + 2$$

$$y^2 = 4 - 2\cos x$$

$$\boxed{y(x) = -\sqrt{4 - 2\cos x}} \quad \textit{Initial value is negative}$$

Exercise

Find the exact solution of the initial value problem $\quad \dfrac{dy}{dt} = \dfrac{1}{y^2}; \quad y(1) = 2$

Solution

$$\int y^2 dy = \int dt$$

$$\frac{1}{3}y^3 = t + C$$

$$y(1) = 2 \quad \rightarrow \quad \frac{8}{3} = 1 + C \quad \Rightarrow \quad \boxed{C = \frac{5}{3}}$$

$$\frac{1}{3}y^3 = t + \frac{5}{3}$$

$$y^3 = 3t + 5$$

$$\boxed{y(t) = (3t + 5)^{1/3}}$$

Exercise

Find the exact solution of the initial value problem $y' + \dfrac{1}{y+1} = 0; \quad y(1) = 0$

Solution

$$\frac{dy}{dx} = -\frac{1}{y+1}$$

$$\int (y+1)\,dy = -\int dx$$

$$\frac{1}{2}y^2 + y = -x + C$$

$$y(1) = 0 \quad \rightarrow \quad \boxed{C = 1}$$

$$\frac{1}{2}y^2 + y = -x + 1$$

$$y^2 + 2y + 2(x-1) = 0 \quad \rightarrow \quad y = \frac{-2 \pm 2\sqrt{1-2x+2}}{2}$$

$$\boxed{y(x) = -1 + \sqrt{3-2x}} \quad (\textit{initial condition} + \textit{sign})$$

Exercise

Find the exact solution of the initial value problem $y' + e^y t = e^y \sin t; \quad y(0) = 0$

Solution

$$\frac{dy}{dt} = (-t + \sin t)e^y$$

$$\int e^{-y}\,dy = \int (-t + \sin t)\,dt$$

$$-e^{-y} = -\frac{1}{2}t^2 - \cos t + C$$

$$e^{-y} = \frac{1}{2}t^2 + \cos t + C$$

$$y(0) = 0 \quad \rightarrow 1 = 1 + C \;\Rightarrow\; \boxed{C = 0}$$

$$e^{-y} = \frac{1}{2}t^2 + \cos t$$

$$-y = \ln\left(\frac{1}{2}t^2 + \cos t\right)$$

$$\boxed{y(x) = -\ln\left(\frac{1}{2}t^2 + \cos t\right)}$$

Exercise

Find the exact solution of the initial value problem $y' - 2ty^2 = 0; \quad y(0) = -1$

Solution

$$\frac{dy}{dt} = 2ty^2$$

$$\int \frac{dy}{y^2} = \int 2t\ dt$$

$$-\frac{1}{y} = t^2 + C$$

$$y(0) = -1 \quad \Rightarrow \quad \underline{C = 1|}$$

$$-\frac{1}{y} = t^2 + 1$$

$$\underline{y(x) = \frac{-1}{t+1}|}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 1 + y^2; \quad y\left(\frac{\pi}{4}\right) = -1$

Solution

$$\int \frac{dy}{1+y^2} = \int dx$$

$$\tan^{-1} y = x + C$$

$$y\left(\frac{\pi}{4}\right) = -1 \quad \rightarrow -\frac{\pi}{4} = \frac{\pi}{4} + C \ \Rightarrow \ \underline{C = -\frac{\pi}{2}|}$$

$$\tan^{-1} y = x - \frac{\pi}{2}$$

$$\underline{y(x) = \tan\left(x - \frac{\pi}{2}\right)|}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dt} = t - ty^2; \quad y(0) = \frac{1}{2}$

Solution

$$\frac{dy}{dt} = t\left(1 - y^2\right)$$

$$\int \frac{dy}{1-y^2} = \int t\ dt$$

$$\frac{1}{1-y^2} = \frac{A}{1-y} + \frac{B}{1+y}$$

$$1 = A + Ay + B - By$$

$$\begin{cases} A - B = 0 \\ A + B = 1 \end{cases} \quad \rightarrow \quad \underline{A = \frac{1}{2} = B|}$$

$$\int \left(\frac{1}{2}\frac{1}{1-y} + \frac{1}{2}\frac{1}{1+y}\right) dy = \int t\ dt$$

$$-\frac{1}{2}\ln|1-y|+\frac{1}{2}\ln|1+y|=\frac{1}{2}t+C$$

$$\ln|1+y|-\ln|1-y|=t+C$$

$$\ln\left|\frac{1+y}{1-y}\right|=t+C$$

$$\frac{1+y}{1-y}=Ae^{t}$$

$$y(0)=\frac{1}{2}\quad\rightarrow\frac{\frac{3}{2}}{\frac{1}{2}}=A\ \Rightarrow\ \underline{A=3}$$

$$\frac{1+y}{1-y}=3e^{t}$$

$$1+y=3e^{t}-3ye^{t}$$

$$y\left(1+3e^{t}\right)=3e^{t}-1$$

$$\underline{y(x)=\frac{3e^{t}-1}{1+3e^{t}}}$$

Exercise

Find the exact solution of the initial value problem $3y^2\frac{dy}{dt}+2t=1;\quad y(-1)=-1$

Solution

$$\int 3y^2dy=\int(1-2t)\,dt$$

$$y^3=t-t^2+C$$

$$y(-1)=-1\quad\rightarrow -1=-1-1+C\ \Rightarrow\ \underline{C=1}$$

$$y^3=t-t^2+1$$

$$\underline{y(t)=\left(t-t^2+1\right)^{1/3}}$$

Exercise

Find the exact solution of the initial value problem $e^xy'+(\cos y)^2=0;\quad y(0)=\frac{\pi}{4}$

Solution

$$e^x\frac{dy}{dx}=-(\cos y)^2$$

$$\int \sec^2 y\ dy=-\int e^{-x}dx$$

$$\tan y=e^{-x}+C$$

$$y(0)=\frac{\pi}{4} \quad \to \ 1=1+C \ \Rightarrow \ \underline{C=0}$$

$$\tan y = e^{-x}$$

$$\boxed{y(x)=\arctan\left(e^{-x}\right)}$$

Exercise

Find the exact solution of the initial value problem $\quad (2y-\sin y)y' + x = \sin x; \quad y(0)=0$

Solution

$$(2y-\sin y)\frac{dy}{dx} = -x+\sin x$$

$$\int (2y-\sin y)dy = \int (-x+\sin x)dx$$

$$y^2+\cos y = -\frac{1}{2}x^2 - \cos x + C$$

$$y(0)=0 \quad \to \ 1=-1+C \ \Rightarrow \ \underline{C=2}$$

$$\boxed{y^2+\cos y = -\frac{1}{2}x^2 - \cos x + 2}$$

Exercise

Find the exact solution of the initial value problem $\quad e^y y' + \frac{x}{y+1} = \frac{2}{y+1}; \quad y(1)=2$

Solution

$$e^y\frac{dy}{dx} = \frac{2-2x}{y+1}$$

$$\int (y+1)e^y dy = \int (2-2x)dx$$

$$ye^y = 2x - x^2 + C$$

$$y(1)=2 \quad \to \ 2e^2 = 2-1+C \ \Rightarrow \ \underline{C=2e^2-1}$$

$$\boxed{ye^y = 2x - x^2 + 2e^2 - 1}$$

Exercise

Find the exact solution of the initial value problem $\quad (\ln y)y' + x = 1; \quad y(3)=e$

Solution

$$(\ln y)\frac{dy}{dx} = 1-x$$

$$\int (\ln y)\,dy = \int (1-x)\,dx \qquad \begin{array}{ll} u = \ln y & dv = dy \\ du = \frac{1}{y}dy & v = y \end{array}$$

$$y\ln y - \int dy = x - \frac{1}{2}x^2 + C$$

$$y\ln y - y = x - \frac{1}{2}x^2 + C$$

$$y(3) = e \quad \to \quad e\ln e = 3 - \frac{9}{2} + C \;\Rightarrow\; \boxed{C = e + \frac{3}{2}}$$

$$\boxed{y\ln y - y = x - \frac{1}{2}x^2 + e + \frac{3}{2}}$$

Exercise

Find the exact solution of the initial value problem $\quad y' = x^3(1-y); \quad y(0) = 3$

Solution

$$\int \frac{dy}{1-y} = \int x^3 dx$$

$$-\ln|1-y| = \frac{1}{4}x^4 + C_1$$

$$\ln|1-y| = -\frac{1}{4}x^4 + C$$

$$y(0) = 3 \quad \to \quad \boxed{\ln 2 = C}$$

$$1 - y = e^{-\frac{1}{4}x^4 + C}$$

$$y = 1 - e^{\ln 2}e^{-\frac{1}{4}x^4}$$

$$\boxed{y(x) = 1 - 2e^{-x^4/4}}$$

Exercise

Find the exact solution of the initial value problem $\quad y' = \left(1+y^2\right)\tan x; \quad y(0) = \sqrt{3}$

Solution

$$\int \frac{1}{1+y^2}dy = \int \tan x \, dx$$

$$\tan^{-1} y = \ln|\sec x| + C$$

$$y(0) = \sqrt{3} \quad \to \quad \frac{\pi}{3} = \ln 1 + C \;\Rightarrow\; \boxed{C = \frac{\pi}{3}}$$

$$\boxed{y(x) = \tan\left(\ln|\sec x| + \frac{\pi}{3}\right)}$$

Exercise

Find the exact solution of the initial value problem $\frac{1}{2}\frac{dy}{dx}=\sqrt{1+y}\ \cos x;\quad y(\pi)=0$

Solution

$$\frac{1}{2}\int(1+y)^{-1/2}\,dy=\int\cos x\,dx$$

$$\sqrt{1+y}=\sin x+C$$

$$y(\pi)=0\quad \underline{1=C}$$

$$\sqrt{1+y}=\sin x+1$$

$$\boxed{y(x)=(\sin x+1)^2-1}$$

Exercise

Find the exact solution of the initial value problem $x^2\frac{dy}{dx}=\frac{4x^2-x-2}{(x+1)(y+1)};\quad y(1)=1$

Solution

$$(y+1)\,dy=\frac{4x^2-x-2}{x^2(x+1)}\,dx$$

$$\frac{4x^2-x-2}{x^2(x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$$

$$4x^2-x-2=Ax^2+Ax+Bx+B+Cx^2$$

$$\begin{cases} x^2 & A+C=4 \quad \underline{C=3} \\ x & A+B=-1 \quad \underline{A=1} \\ x^0 & \underline{B=-2} \end{cases}$$

$$\int(y+1)\,dy=\int\frac{dx}{x}-\int\frac{2}{x^2}\,dx+3\int\frac{dx}{x+1}$$

$$\frac{1}{2}y^2+y=\ln|x|+\frac{2}{x}+3\ln|x+1|+C$$

$$y(1)=1\quad\rightarrow\frac{1}{2}+1=2+3\ln 2+C\quad\Rightarrow\quad \underline{C=-\frac{1}{2}-\ln 8}$$

$$\boxed{\frac{1}{2}y^2+y=\ln|x|+\frac{2}{x}+3\ln|x+1|-\frac{1}{2}-\ln 8}$$

Exercise

Find the exact solution of the initial value problem $\frac{1}{\theta}\frac{dy}{d\theta}=\frac{y\sin\theta}{y^2+1}\quad y(\pi)=1$

Solution

$$\int \frac{y^2+1}{y}dy = \int \theta \sin\theta \, d\theta$$

$$\int \left(y + \frac{1}{y} \right) dy = -\theta\cos\theta + \sin\theta + C$$

$$\frac{1}{2}y^2 + \ln|y| = -\theta\cos\theta + \sin\theta + C$$

$$y(\pi) = 1 \quad \frac{1}{2} = \pi + C \;\Rightarrow\; \underline{C = \frac{1}{2} - \pi}$$

$$\underline{\frac{1}{2}y^2 + \ln|y| = -\theta\cos\theta + \sin\theta + \frac{1}{2} - \pi}$$

Exercise

Find the exact solution of the initial value problem $\quad x^2dx + 2ydy = 0; \quad y(0) = 2$

Solution

$$\int 2ydy = -\int x^2dx$$

$$y^2 = -\frac{1}{3}x^3 + C$$

$$y(0) = 2 \quad \to \quad \underline{4 = C}$$

$$\underline{y^2 = -\frac{1}{3}x^3 + 4}$$

Exercise

Find the exact solution of the initial value problem $\quad \frac{1}{t}\frac{dy}{dt} = 2\cos^2 y; \quad y(0) = \frac{\pi}{4}$

Solution

$$\int \sec^2 y \, dy = \int 2t \, dt$$

$$\tan y = t^2 + C$$

$$y(0) = \frac{\pi}{4} \quad \to \quad \underline{1 = C}$$

$$\underline{y(t) = \tan^{-1}\left(t^2 + 1\right)}$$

Exercise

Find the exact solution of the initial value problem $\quad \frac{dy}{dx} = 8x^3e^{-2y}; \quad y(1) = 0$

Solution

$$\int e^{2y}dy = \int 8x^3 dx$$

$$\frac{1}{2}e^{2y} = 2x^4 + C$$

$$y(1) = 0 \quad \rightarrow \quad \boxed{\frac{1}{2} = C}$$

$$\frac{1}{2}e^{2y} = 2x^4 + \frac{1}{2}$$

$$e^{2y} = 4x^4 + 1$$

$$\boxed{y(x) = \frac{1}{2}\ln\left(4x^4 + 1\right)}$$

Exercise

Find the exact solution of the initial value problem $\quad \frac{dy}{dx} = x^2(1+y); \quad y(0) = 3$

Solution

$$\int \frac{1}{1+y}dy = \int x^2 dx$$

$$\ln|1+y| = \frac{1}{3}x^3 + C$$

$$y(0) = 3 \quad \rightarrow \quad \boxed{\ln 4 = C}$$

$$1 + y = e^{\frac{1}{3}x^3 + \ln 4}$$

$$\boxed{y(x) = 4e^{x^3/3} - 1}$$

Exercise

Find the exact solution of the initial value problem $\quad \sqrt{y}dx + (1+x)dy = 0; \quad y(0) = 1$

Solution

$$\int y^{-1/2}dy = -\int \frac{1}{x+1}dx$$

$$2\sqrt{y} = -\ln|x+1| + C$$

$$y(0) = 1 \quad \rightarrow \quad \boxed{2 = C}$$

$$\boxed{2\sqrt{y} = -\ln|x+1| + 2}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 6y^2x, \quad y(1) = \frac{1}{25}$

Solution

$$\int \frac{dy}{y^2} = \int 6x\,dx$$

$$-\frac{1}{y} = 3x^2 + C \qquad y(1) = \frac{1}{25}$$

$$-25 = 3 + C \quad \rightarrow \quad \underline{C = -28}$$

$$-\frac{1}{y} = 3x^2 - 28$$

$$\underline{y(x) = \frac{1}{28 - 3x^2}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = \frac{3x^2 + 4x - 4}{2y - 4}, \quad y(1) = 3$

Solution

$$\int (2y-4)\,dy = \int \left(3x^2 + 4x - 4\right)dx$$

$$y^2 - 4y = x^3 + 2x^2 - 4x + C \qquad y(1) = 3$$

$$9 - 12 = 1 + 2 - 4 + C \quad \rightarrow \quad \underline{C = -2}$$

$$\underline{y^2 - 4y = x^3 + 2x^2 - 4x - 2}$$

Exercise

Find the exact solution of the initial value problem $y' = e^{-y}(2x - 4) \quad y(5) = 0$

Solution

$$\int e^y\,dy = \int (2x-4)\,dx$$

$$e^y = x^2 - 4x + C \qquad y(5) = 0$$

$$e^0 = 25 - 20 + C \quad \rightarrow \quad \underline{C = -4}$$

$$e^y = x^2 - 4x - 4$$

$$\underline{y(x) = \ln\left|x^2 - 4x - 4\right|}$$

Exercise

Find the exact solution of the initial value problem $\dfrac{dr}{d\theta} = \dfrac{r^2}{\theta}, \quad r(1) = 2$

Solution

$$\int \frac{dr}{r^2} = \int \frac{d\theta}{\theta}$$

$$-\frac{1}{r} = \ln|\theta| + C \qquad\qquad r(1) = 2$$

$$\boxed{-\frac{1}{2} = C}$$

$$-\frac{1}{r} = \ln|\theta| - \frac{1}{2}$$

$$\frac{1}{r} = \frac{1 - 2\ln|\theta|}{2}$$

$$\boxed{r(\theta) = \frac{2}{1 - 2\ln|\theta|}}$$

Exercise

Find the exact solution of the initial value problem $\dfrac{dy}{dt} = e^{y-t}\left(1+t^2\right)\sec y, \quad y(0) = 0$

Solution

$$\frac{dy}{dt} = e^{-t}\left(1+t^2\right)e^y \sec y$$

$$\int \left(e^{-y}\cos y\right)dy = \int \left(1+t^2\right)e^{-t}dt$$

$$\int \left(e^{-y}\cos y\right)dy = e^{-y}(\sin y - \cos y) - \int \left(e^{-y}\cos y\right)dy$$

$$2\int \left(e^{-y}\cos y\right)dy = e^{-y}(\sin y - \cos y)$$

$$\int \left(e^{-y}\cos y\right)dy = \frac{1}{2}e^{-y}(\sin y - \cos y)$$

$$\int \left(1+t^2\right)e^{-t}dt = e^{-t}\left(-1 - t^2 - 2t - 2\right)$$

		$\int \cos y$		$\int e^{-t}$
+	e^{-y}	$\sin y$	$1+t^2$	$-e^{-t}$
−	$-e^{-y}$	$-\cos y$	$2t$	e^{-t}
+	e^{-y}		2	$-e^{-t}$

$$\frac{1}{2}e^{-y}(\sin y - \cos y) = -e^{-t}\left(t^2 + 2t + 3\right) + C \qquad y(0) = 0$$

$$-\frac{1}{2} = -3 + C \quad \to \quad \boxed{C = \frac{5}{2}}$$

$$\boxed{\frac{1}{2}e^{-y}(\sin y - \cos y) = -e^{-t}\left(t^2 + 2t + 3\right) + \frac{5}{2}}$$