

## Section 2.5 – Numerical Integration

### Absolute and Relative Error

#### ***Definition***

Suppose  $c$  is a computed numerical solution to a problem having an exact solution  $x$ .

There are two common measures of the error in  $c$  as an approximation to  $x$ :

$$\text{absolute error} = |c - x| \quad \& \quad \text{relative error} = \frac{|c - x|}{|x|} \quad (\text{if } x \neq 0)$$

#### ***Example***

The ancient Greeks used  $\frac{22}{7}$  to approximate the value of  $\pi$ . Determine the absolute and relative error in this approximation to  $\pi$ .

#### **Solution**

$$\begin{aligned} \text{absolute error} &= \left| \frac{22}{7} - \pi \right| \\ &\approx 0.00126 \end{aligned}$$

$$\begin{aligned} \text{relative error} &= \frac{\left| \frac{22}{7} - \pi \right|}{\pi} \\ &\approx .000402 \\ &\approx .04\% \end{aligned}$$

## Midpoint Rule

### Definition

Suppose  $f$  is defined and integrable on  $[a, b]$ . The **midpoint Rule Approximation** to  $\int_a^b f(x)dx$  using  $n$  equally spaced subintervals on  $[a, b]$  is

$$\begin{aligned} M(n) &= f(m_1)\Delta x + f(m_2)\Delta x + \cdots + f(m_n)\Delta x \\ &= \sum_{k=1}^n f\left(\frac{x_{k-1} + x_k}{2}\right)\Delta x \end{aligned}$$

Where  $\Delta x = \frac{b-a}{n}$ ,

$$x_0 = a, \quad x_k = a + k\Delta x$$

$$m_k = \frac{x_{k-1} + x_k}{2} \text{ is the midpoint of } [x_{k-1}, x_k], \text{ for } k = 1, 2, \dots, n.$$

### Example

Approximate  $\int_2^4 x^2 dx$  using the Midpoint Rule with  $n = 4$  subinterval

### Solution

$$\text{With } a = 2, b = 4 \rightarrow \Delta x = \frac{4-2}{4} = 0.5$$

The grid points are:

$$x_0 = 2$$

$$x_1 = 2 + 0.5 = \underline{2.5}$$

$$x_2 = 2 + 2(.5) = \underline{3}$$

$$x_3 = 2 + 3(0.5) = \underline{3.5}$$

$$x_4 = 2 + 4(0.5) = \underline{4}$$

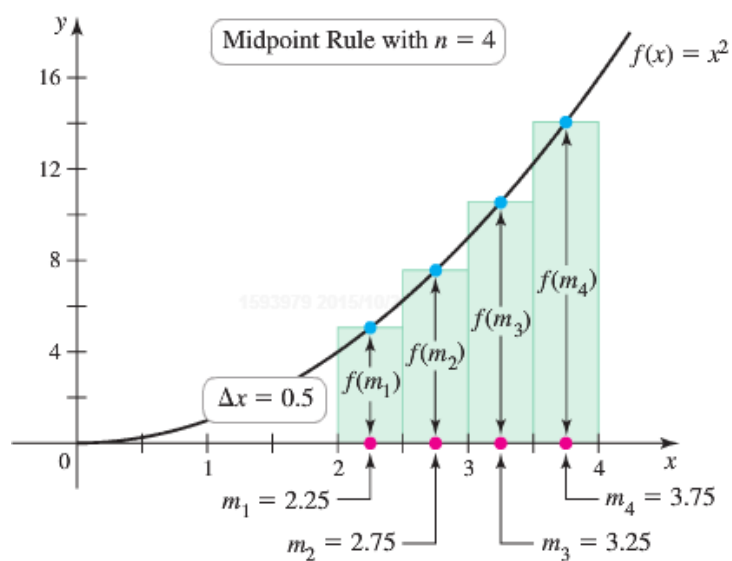
$$m_1 = \frac{2.5+2}{2} = \underline{2.25}$$

$$m_2 = \frac{3+2.5}{2} = \underline{2.75}$$

$$m_3 = \frac{2+3.5}{2} = \underline{3.25}$$

$$m_4 = \frac{3.5+4}{2} = \underline{3.75}$$

$$\begin{aligned}
 M(4) &= f(m_1)\Delta x + f(m_2)\Delta x + f(m_3)\Delta x + f(m_4)\Delta x \\
 &= (2.25^2 + 2.75^2 + 3.25^2 + 3.75^2)(0.5) \\
 &= 18.625
 \end{aligned}$$



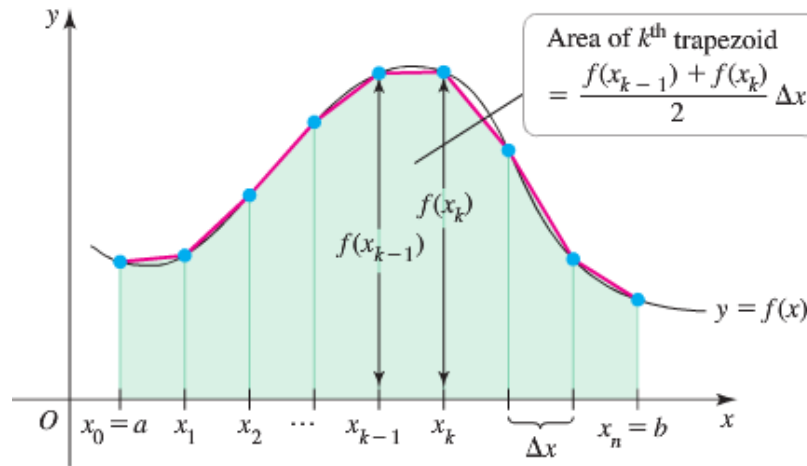
$$\begin{aligned}
 \text{Exact} &= \int_2^4 x^2 dx \\
 &= \left. \frac{1}{3}x^3 \right|_2^4 \\
 &= \frac{56}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{absolute error} &= \left| 18.625 - \frac{56}{3} \right| \\
 &\approx 0.0417
 \end{aligned}$$

$$\begin{aligned}
 \text{relative error} &= \frac{\left| 18.625 - \frac{56}{3} \right|}{\frac{56}{3}} \\
 &\approx .00223 = .223\%
 \end{aligned}$$

## Trapezoid Approximations

The **Trapezoid Rule** for the value of a definite integral is based on approximating the region between a curve and the  $x$ -axis with trapezoids instead of rectangles.



The length of each subinterval is  $\Delta x = \frac{b-a}{n}$  is called the **step size** or **mesh size**.

The area of a trapezoid:  $\Delta x \cdot \left( \frac{y_{i-1} + y_i}{2} \right)$

The area is the approximation by adding the areas of all trapezoids:

$$\begin{aligned} T &= \frac{1}{2}(y_0 + y_1)\Delta x + \frac{1}{2}(y_1 + y_2)\Delta x + \cdots + \frac{1}{2}(y_{n-2} + y_{n-1})\Delta x + \frac{1}{2}(y_{n-1} + y_n)\Delta x \\ &= \frac{1}{2}\Delta x(y_0 + y_1 + y_1 + y_2 + \cdots + y_{n-2} + y_{n-1} + y_{n-1} + y_n) \\ &= \frac{1}{2}\Delta x(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-2} + 2y_{n-1} + y_n) \end{aligned}$$

### The Trapezoid Rule

If  $f$  is continuous on  $[a, b]$  and if a regular partition of  $[a, b]$  is determined by the numbers  $a = x_0, x_1, \dots, x_n = b$ , then

$$\begin{aligned} \int_a^b f(x) dx &\approx \frac{b-a}{2n} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)] \\ T(n) &= \left( \frac{1}{2}f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2}f(x_n) \right) \Delta x \end{aligned}$$

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \dots, \quad x_{n-1} = a + (n-1)\Delta x, \quad x_n = b$$

$$\text{Where } \Delta x = \frac{b-a}{n} \text{ and } x_0 = a, \quad x_k = a + k\Delta x$$

## Error Estimate for the Trapezoidal Rule

If  $M$  is a positive real number such that  $|f''(x)| \leq M$  for all  $x$  in  $[a, b]$ , then the error involved in using the Trapezoidal Rule is not greater than  $\frac{M(b-a)^3}{12n^2}$

### Example

Use the Trapezoid Rule with  $n = 4$  to estimate  $\int_1^2 x^2 dx$ . Compare the estimate with the exact value.

### Solution

$$|\Delta x| = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$$

$$x_0 = 1$$

$$x_1 = 1 + \frac{1}{4} = \frac{5}{4}$$

$$x_2 = 1 + 2\left(\frac{1}{4}\right) = \frac{6}{4}$$

$$x_3 = 1 + 3\left(\frac{1}{4}\right) = \frac{7}{4}$$

$$x_4 = 2$$

$$T = \frac{1}{2} \Delta x (y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

$$= \frac{1}{2} \cdot \frac{1}{4} \left( 1^2 + 2\left(\frac{5}{4}\right)^2 + 2\left(\frac{6}{4}\right)^2 + 2\left(\frac{7}{4}\right)^2 + 2^2 \right)$$

$$= \frac{1}{8} \left( 1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4 \right)$$

$$= \frac{75}{32}$$

$$\approx 2.34375$$

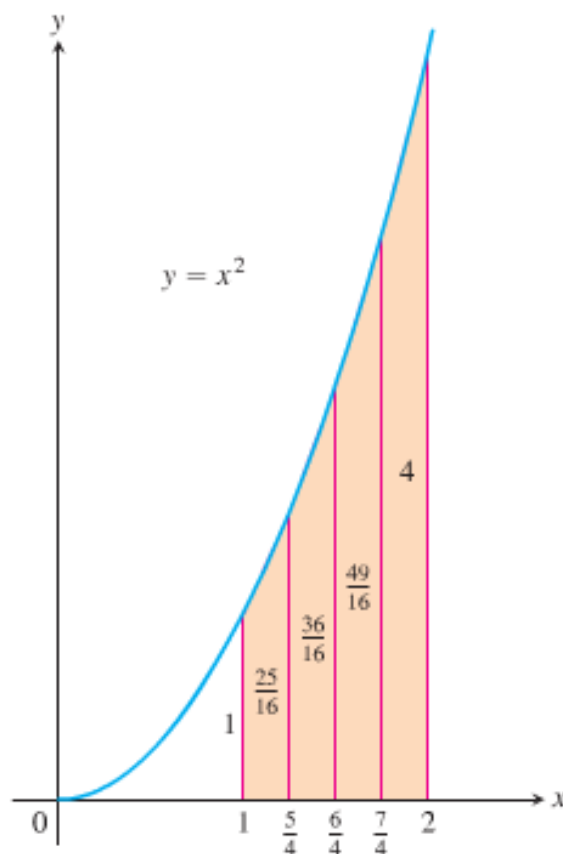
$$\int_1^2 x^2 dx = \frac{1}{3} x^3 \Big|_1^2$$

$$= \frac{1}{3} (2^3 - 1^3)$$

$$= \frac{7}{3} \approx 2.3333$$

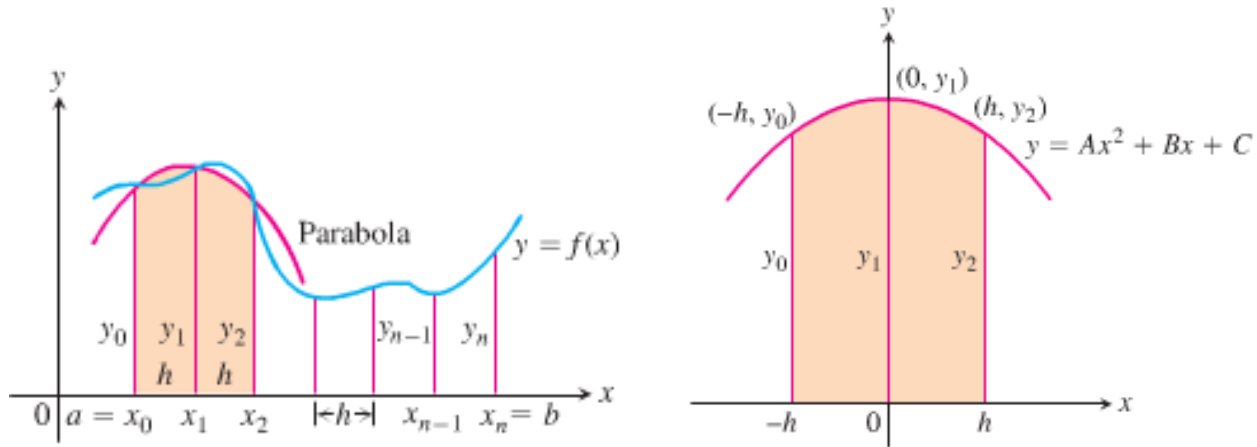
The difference:  $2.34375 - 2.3333 \approx 0.01042$

The percentage error:  $\frac{2.34375 - 2.3333}{2.3333} \approx 0.004466$  .446%



## Simpson's Rule: Approximations Using Parabolas

We partition the interval  $[a, b]$  into  $n$  subintervals of equal length  $h = \Delta x = \frac{b-a}{n}$   $n$ : even number



The parabola has an equation of the form:  $y = Ax^2 + Bx + C$

So the area under it from  $x = -h$  to  $x = h$  is

$$\begin{aligned}
 A_p &= \int_{-h}^h (Ax^2 + Bx + C) dx \\
 &= \left[ \frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx \right]_{-h}^h \\
 &= \frac{A}{3}h^3 + \frac{B}{2}h^2 + Ch - \left( \frac{A}{3}(-h)^3 + \frac{B}{2}(-h)^2 + C(-h) \right) \\
 &= \frac{A}{3}h^3 + \frac{B}{2}h^2 + Ch + \frac{A}{3}h^3 - \frac{B}{2}h^2 + Ch \\
 &= \frac{2}{3}Ah^3 + 2Ch \\
 &= \frac{h}{3}(2Ah^2 + 6C)
 \end{aligned}$$

Since the curve passes through the three points  $(-h, y_0)$ ,  $(0, y_1)$ , and  $(h, y_2)$

$$y_0 = Ah^2 - Bh + C \quad y_1 = C \quad y_2 = Ah^2 + Bh + C$$

$$C = y_1, \quad Ah^2 - Bh = y_0 - y_1$$

$$\underline{Ah^2 + Bh = y_2 - y_1}$$

$$2Ah^2 = y_0 - 2y_1 + y_2$$

$$\begin{aligned}
 A_p &= \frac{h}{3}(2Ah^2 + 6C) \\
 &= \frac{h}{3}(y_0 - 2y_1 + y_2 + 6y_1) \\
 &= \frac{h}{3}(y_0 + 4y_1 + y_2)
 \end{aligned}$$

Computing the areas under all the parabolas and adding the results gives the approximation

$$\begin{aligned}\int_a^b f(x)dx &\approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n) \\ &= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)\end{aligned}$$

### ***Simpson's Rule***

To approximate  $\int_a^b f(x)dx$ , use  $S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \dots, \quad x_{n-1} = a + (n-1)\Delta x, \quad x_n = b$$

$$\text{Where } \Delta x = \frac{b-a}{n}$$

$$\int_a^b f(x)dx \approx \frac{b-a}{3n} \left[ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]$$

### ***Error Estimate for the Trapezoidal Rule***

If  $M$  is a positive real number such that  $\left| f^{(4)}(x) \right| \leq M$  for all  $x$  in  $[a, b]$ , then the error involved in using the

Simpson's Rule is not greater than  $\frac{M(b-a)^5}{180n^4}$

### ***Example***

Use Simpson's Rule with  $n = 4$  to approximate  $\int_0^2 5x^4 dx$

#### **Solution**

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$$

$$x_0 = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 0 + 2\frac{1}{2} = 1, \quad x_3 = 0 + 3\frac{1}{2} = \frac{3}{2}, \quad x_4 = 2$$

$$\begin{aligned}S &= \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) \\ &= \frac{1}{3} \frac{1}{2} \left( 5(0)^4 + 4(5)\left(\frac{1}{2}\right)^4 + 2(5)(1)^4 + 4(5)\left(\frac{3}{2}\right)^4 + 5(2)^4 \right)\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{6} \left( 0 + \frac{5}{4} + 10 + \frac{405}{4} + 80 \right) \\
&= \frac{1}{6} \left( \frac{385}{2} \right) \\
&= \frac{385}{12}
\end{aligned}$$

$\approx 32.08333$

The exact value is 32.

### Example

The table lists rates of change  $s'(t)$  in global sea level  $s(t)$  in various years from 1995 ( $t = 0$ ) to 2011 ( $t = 16$ ), with rates of change reported in  $mm/yr$ .

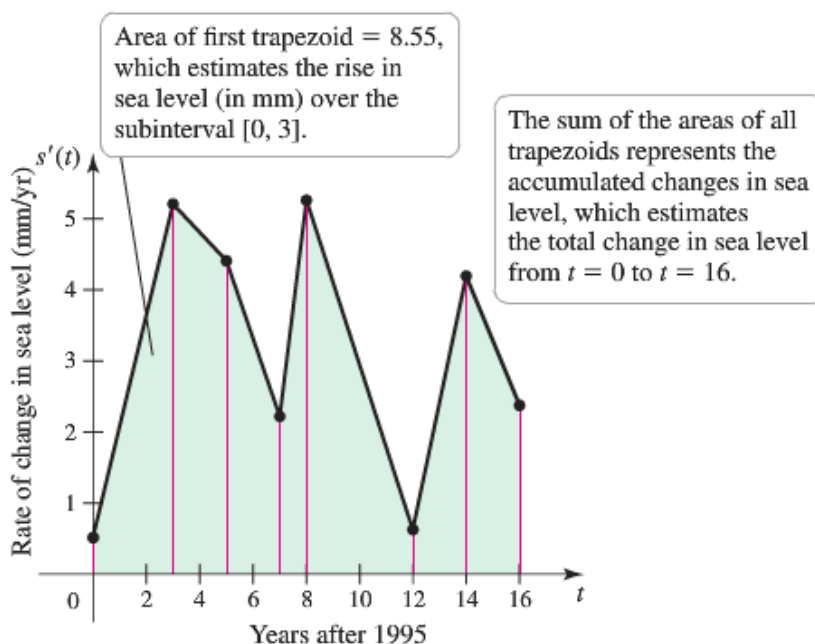
Years	1995	1998	2000	2002	2003	2007	2009	2011
$t$	0	3	5	7	8	12	14	16
$s'(t)$ (mm/yr)	0.51	5.19	4.39	2.21	5.24	0.63	4.19	2.38

- Assuming  $s'(t)$  is continuous on  $[0, 16]$ , explain how a definite integral can be used to find the net change in sea level from 1995 to 2011; then write the definite integral.
- Use the data in the table and generalize the trapezoid Rule to estimate the value of the integral from part (a).

### Solution

- The net change in any quantity  $Q$  over the interval  $[a, b]$  is  $Q(b) - Q(a)$

$$\text{Net change in } s(t) = S(b) - S(a) = \int_0^{16} s'(t) dt$$





b) From the figure the values accompanied by 7 trapezoids whose area approximates  $\int_0^{16} s'(t) dt$

**Area** of the **first** trapezoid:

$$\begin{aligned} T_1 &= \frac{1}{2}(s'(0) + s'(3)) \cdot 3 \\ &= \frac{1}{2}(0.51 + 5.19) \cdot 3 \\ &= \underline{8.55} \end{aligned}$$

$$\begin{aligned} T_2 &= \frac{1}{2}(s'(3) + s'(5)) \cdot 2 \\ &= \frac{1}{2}(5.19 + 4.39) \cdot 2 \\ &= \underline{9.58} \end{aligned}$$

$$\begin{aligned} T_3 &= \frac{1}{2}(s'(5) + s'(7)) \cdot 2 \\ &= \frac{1}{2}(4.39 + 2.21) \cdot 2 \\ &= \underline{6.6} \end{aligned}$$

$$\begin{aligned} T_4 &= \frac{1}{2}(s'(7) + s'(8)) \cdot 1 \\ &= \frac{1}{2}(2.21 + 5.24) \\ &= \underline{3.725} \end{aligned}$$

$$\begin{aligned} T_5 &= \frac{1}{2}(s'(8) + s'(12)) \cdot 4 \\ &= \frac{1}{2}(5.24 + 0.63) \cdot 4 \\ &= \underline{11.74} \end{aligned}$$

$$\begin{aligned} T_6 &= \frac{1}{2}(s'(12) + s'(14)) \cdot 2 \\ &= \frac{1}{2}(0.63 + 4.19) \cdot 2 \\ &= \underline{4.82} \end{aligned}$$

$$\begin{aligned} T_7 &= \frac{1}{2}(s'(14) + s'(16)) \cdot 2 \\ &= \frac{1}{2}(4.19 + 2.38) \cdot 2 \\ &= \underline{6.57} \end{aligned}$$

$$\begin{aligned} T(7) &= T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 \\ &= \underline{\approx 51.585 \text{ mm}} \end{aligned}$$

## Exercises      Section 2.5 – Numerical Integration

Find the *Midpoint* Rule approximations to

1.  $\int_0^1 \sin \pi x \, dx \quad n = 6 \text{ subintervals}$

3.  $\int_0^1 e^{-\sqrt{x}} \, dx \quad n = 6 \text{ subintervals}$

2.  $\int_0^\pi x^2 \sin x \, dx \quad n = 8 \text{ subintervals}$

4.  $\int_0^1 e^{-x} \, dx \quad n = 8 \text{ subintervals}$

Estimate the minimum number of subintervals to approximate the integrals with an error of magnitude of  $10^{-4}$  by (a) the *Trapezoid* Rule and (b) *Simpson's* Rule.

5.  $\int_1^3 (2x-1) \, dx$

6.  $\int_{-1}^1 (x^2+1) \, dx$

7.  $\int_2^4 \frac{1}{(s-1)^2} \, ds$

Find the *Trapezoid* & *Simpson's* Rule approximations and error to

8.  $\int_0^1 \sin \pi x \, dx \quad n = 6 \text{ subintervals}$

15.  $\int_{\pi/2}^\pi \frac{\sin x}{x} \, dx \quad n = 6 \text{ subintervals}$

9.  $\int_0^1 e^{-x} \, dx \quad n = 8 \text{ subintervals}$

16.  $\int_0^{\pi/4} x \tan x \, dx \quad n = 6 \text{ subintervals}$

10.  $\int_1^5 (3x^2-2x) \, dx \quad n = 8 \text{ subintervals}$

17.  $\int_0^1 e^{-x^2} \, dx \quad n = 10 \text{ subintervals}$

11.  $\int_0^{\pi/4} 3 \sin 2x \, dx \quad n = 8 \text{ subintervals}$

18.  $\int_0^2 \frac{1}{\sqrt{1+x^2}} \, dx \quad n = 10 \text{ subintervals}$

12.  $\int_0^8 e^{-2x} \, dx \quad n = 8 \text{ subintervals}$

19.  $\int_0^{1/2} \sin(e^{x/2}) \, dx \quad n = 8 \text{ subintervals}$

13.  $\int_{-1}^1 \sqrt{x^2+1} \, dx \quad n = 8 \text{ subintervals}$

20.  $\int_2^3 \frac{1}{\ln x} \, dx \quad n = 10 \text{ subintervals}$

14.  $\int_0^{1/2} \sin(x^2) \, dx \quad n = 4 \text{ subintervals}$

21.  $\int_1^2 e^{1/x} \, dx \quad n = 4 \text{ subintervals}$

22.  $\int_0^1 \ln(1+e^x) dx \quad n=8 \text{ subintervals}$

25.  $\int_0^3 \frac{1}{1+x^5} dx \quad n=6 \text{ subintervals}$

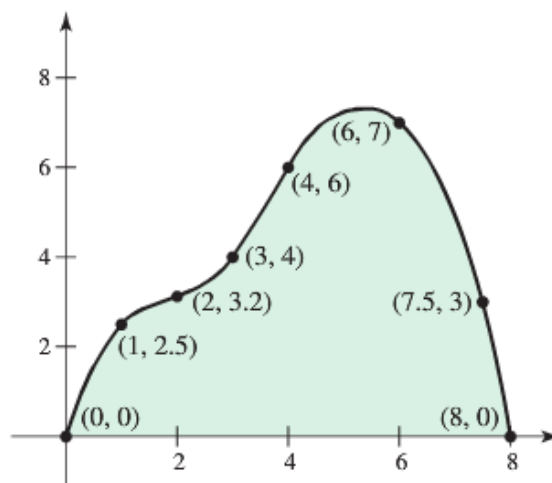
23.  $\int_0^1 x^5 e^x dx \quad n=10 \text{ subintervals}$

26.  $\int_1^4 \frac{e^x}{x} dx \quad n=10 \text{ subintervals}$

24.  $\int_0^4 \sqrt{x} \sin x dx \quad n=8 \text{ subintervals}$

27.  $\int_1^2 \frac{dx}{x} \quad n=10 \text{ subintervals}$

28. A piece of wood paneling must be cut in the shape shown below. The coordinates of several point on its curved surface are also shown (with units of inches).



- Estimate the surface area of the paneling using the Trapezoid Rule
  - Estimate the surface area of the paneling using a left Riemann sum.
  - Could two identical pieces be cut from a 9-in by 9-in piece of wood?
29. The region bounded by the curves  $y = \frac{1}{1+e^{-x}}$ ,  $x=0$  and  $x=10$  is rotated about  $x$ -axis. Use Simpson's Rule with  $n=10$  to estimate the volume of the resulting solid.
30. A pendulum with length  $L$  that makes a maximum angle  $\theta_0$  with the vertical. Using Newton's Second Law it can be shown that the period  $T$  (the time for one complete swing) is given by

$$T = 4 \sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2 \sin^2 x}}$$

Where  $k = \sin\left(\frac{1}{2}\theta_0\right)$  and  $g$  is the acceleration due to gravity. If  $L = 1 \text{ m}$  and  $\theta_0 = 42^\circ$ , use Simpson's Rule with  $n=10$  to find the period.