Solution Section 3.2 – Double Integrals over General Regions

Exercise

Sketch the region of integration and evaluate the integral $\int_0^{\pi}\int_0^{x} x\sin y\, dydx$

Solution

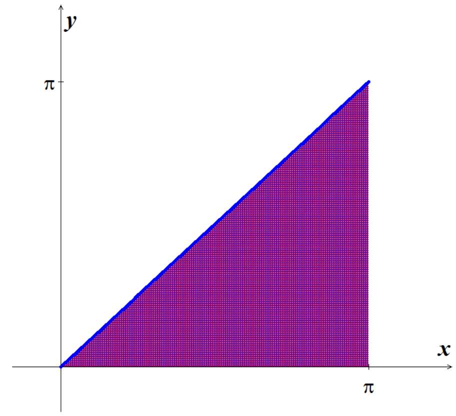
$$\int_0^{\pi}\int_0^{x} x\sin y\, dydx = \int_0^{\pi}\left(-x\cos y \,\Big|_0^x\right) dx$$

$$= \int_0^{\pi}(-x\cos x + x)\, dx$$

$$= -(x\sin x + \cos x) + \frac{1}{2}x^2 \,\Big|_0^{\pi}$$

$$= -(-1) + \frac{1}{2}\pi^2 - (-1)$$

$$= \frac{\pi^2}{2} + 2$$



		$\int \cos x$
+	x	$\sin x$
−	1	$-\cos x$

Exercise

Sketch the region of integration and evaluate the integral $\int_0^{\pi}\int_0^{\sin x} ydydx$

Solution

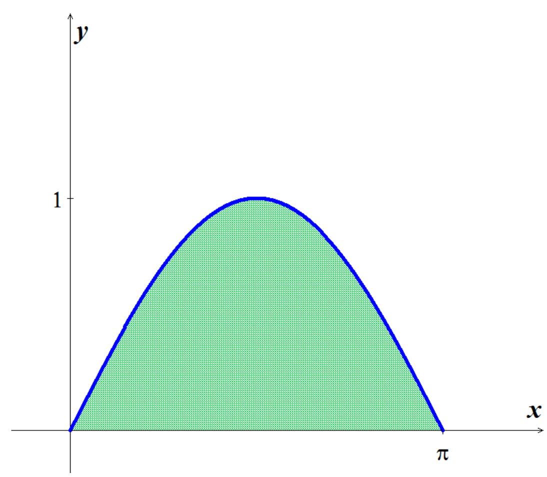
$$\int_0^{\pi}\int_0^{\sin x} ydydx = \int_0^{\pi}\left(\frac{1}{2}y^2 \,\Big|_0^{\sin x}\right) dx$$

$$= \int_0^{\pi}\frac{1}{2}\sin^2 x\, dx$$

$$= \frac{1}{4}\int_0^{\pi}(1-\cos 2x)\, dx$$

$$= \frac{1}{4}\left(x - \frac{1}{2}\sin 2x\right. \,\Big|_0^{\pi}$$

$$= \frac{\pi}{4}$$



Exercise

Sketch the region of integration and evaluate the integral $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dxdy$

Solution

$$\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dxdy = \int_1^{\ln 8}\left(x+y \Big|_0^{\ln y}\right) dy$$

$$= \int_1^{\ln 8}\left(e^{\ln y+y} - e^y\right) dy$$

$$= \int_1^{\ln 8}\left(e^{\ln y}e^y - e^y\right) dy$$

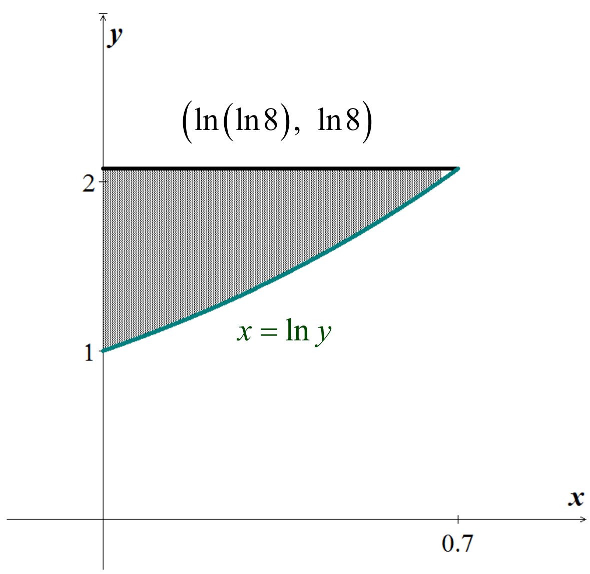
$$= \int_1^{\ln 8}\left(ye^y - e^y\right) dy$$

$$= ye^y - e^y - e^y \Big|_1^{\ln 8}$$

$$= (\ln 8)e^{\ln 8} - 2e^{\ln 8} - (e - 2e)$$

$$\underline{= 8\ln 8 - 16 - e}$$

		$\int e^y$
+	y	e^y
−	1	e^y

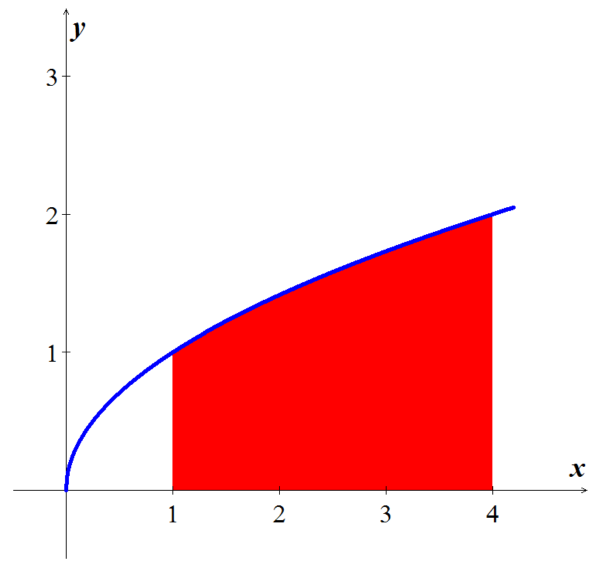


Exercise

Sketch the region of integration and evaluate the integral $\displaystyle\int_1^4\int_0^{\sqrt{x}} \frac{3}{2}e^{y/\sqrt{x}}\,dydx$

Solution

$$\begin{aligned}
\int_1^4\int_0^{\sqrt{x}} \frac{3}{2}e^{y/\sqrt{x}}\,dydx &= \frac{3}{2}\int_1^4 \left(\sqrt{x}e^{y/\sqrt{x}}\ \Big|_0^{\sqrt{x}}\right) dx \\
&= \frac{3}{2}\int_1^4 \sqrt{x}(e-1)\,dx \\
&= \frac{3}{2}(e-1)\int_1^4 x^{1/2}\,dx \\
&= \frac{3}{2}(e-1)\left(\frac{2}{3}x^{3/2}\ \Big|_1^4\right. \\
&= (e-1)\left(x^{3/2}\ \Big|_1^4\right. \\
&= (e-1)[8-1] \\
&= 7(e-1)
\end{aligned}$$



Exercise

Integrate $f(x,y)=\frac{x}{y}$ over the region in the first quadrant bounded by the lines

$y=x,\quad y=2x,\quad x=1,\quad and\quad x=2$

Solution

$$\begin{aligned}
\int_1^2\int_x^{2x} \frac{x}{y}\,dydx &= \int_1^2 \left(x\ln y\ \Big|_x^{2x}\right. dx \\
&= \int_1^2 x(\ln 2x-\ln x)\,dx \\
&= \int_1^2 x\left(\ln\frac{2x}{x}\right) dx
\end{aligned}$$

Quotient Rule: $\ln M-\ln P=\ln\frac{M}{P}$

$$= \ln 2 \int_1^2 x\, dx$$

$$= (\ln 2)\left(\tfrac{1}{2}x^2 \Big|_1^2\right.$$

$$= (\ln 2)\left[\tfrac{1}{2}(4-1)\right]$$

$$\boxed{= \tfrac{3}{2}\ln 2}$$

Exercise

Integrate $f(x,y) = x^2 + y^2$ over the triangular region with vertices $(0,\ 0)$ $(1,\ 0)$ *and* $(0,\ 1)$

Solution

$$\int_0^1 \int_0^{1-x} \left(x^2 + y^2\right) dy dx = \int_0^1 \left(x^2 y + \tfrac{1}{3}y^3 \Big|_0^{1-x} dx\right.$$

$$= \int_0^1 \left[x^2(1-x) + \tfrac{1}{3}(1-x)^3\right] dx$$

$$= \int_0^1 \left[x^2 - x^3 + \tfrac{1}{3}(1-x)^3\right] dx$$

$$= \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{12}(1-x)^4 \Big|_0^1$$

$$= \tfrac{1}{3} - \tfrac{1}{4} - \left(-\tfrac{1}{12}\right)$$

$$\boxed{= \tfrac{1}{6}}$$

Exercise

Integrate $f(s,t) = e^s \ln t$ over the region in the first quadrant of the *st*-plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$.

Solution

$$\int_1^2 \int_0^{\ln t} e^s \ln t\, ds dt = \int_1^2 \left(e^s \ln t \Big|_0^{\ln t} dt\right.$$

$$= \int_1^2 \left(t\ln t - \ln t\right) dt$$

$$\begin{aligned} u &= \ln t & dv &= dt \\ du &= \frac{1}{t}dt & v &= t \end{aligned} \quad \rightarrow \quad \int \ln t = t\ln t - \int t\frac{1}{t}dt = t\ln t - t$$

$$\int t\ln t = \frac{1}{2}t^2 \ln t - \frac{1}{4}t^2$$

$$= \frac{1}{2}t^2 \ln t - \frac{1}{4}t^2 - t\ln t + t \Big|_1^2$$

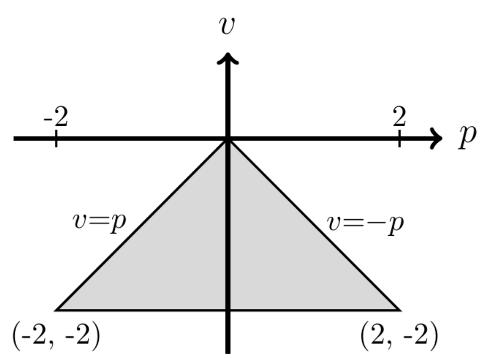
$$= 2\ln 2 - 1 - 2\ln 2 + 2 - \left(0 - \frac{1}{4} - 0 + 1\right)$$

$$= \frac{1}{4}$$

Exercise

Evaluate $\displaystyle\int_{-2}^{0}\int_{v}^{-v} 2\, dpdv$

Solution



$$\int_{-2}^{0}\int_{v}^{-v} 2dpdv = 2\int_{-2}^{0} \left(p \Big|_{v}^{-v} dv\right.$$

$$= -4\int_{-2}^{0} v\, dv$$

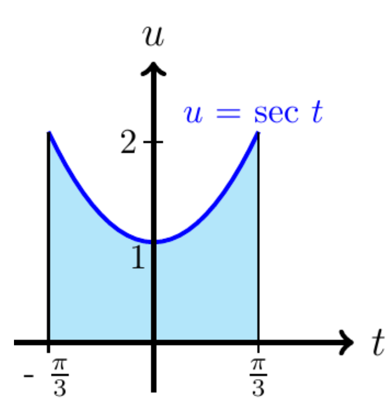
$$= -2\, v^2 \Big|_{-2}^{0}$$

$$= 8$$

Exercise

Evaluate $\displaystyle\int_{-\pi/3}^{\pi/3}\int_{0}^{\sec t} 3\cos t\; du\, dt$

Solution



$$\int_{-\pi/3}^{\pi/3}\int_{0}^{\sec t} 3\cos t\; dudt = \int_{-\pi/3}^{\pi/3} \left(3\cos t\right)\left(u \Big|_{0}^{\sec t} dt\right.$$

$$= \int_{-\pi/3}^{\pi/3} (3\cos t \sec t)\,dt \qquad \cos t \sec t = \cos t \frac{1}{\cos t} = 1$$

$$= \int_{-\pi/3}^{\pi/3} 3\,dt$$

$$= 3t \Big|_{-\pi/3}^{\pi/3}$$

$$= 3\frac{2\pi}{3}$$

$$= 2\pi$$

Exercise

Sketch the region of integration, reverse the order of integration, and evaluate the integral

$$\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y}\,dydx$$

Solution

$$\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y}\,dy\,dx = \int_0^{\pi} \int_0^{y} \frac{\sin y}{y}\,dxdy$$

$$= \int_0^{\pi} \frac{\sin y}{y}\left(x \Big|_0^y\right) dy$$

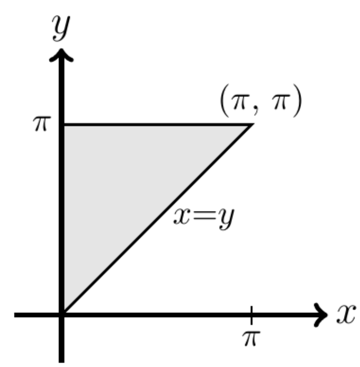
$$= \int_0^{\pi} \frac{\sin y}{y}(y)\,dy$$

$$= \int_0^{\pi} \sin y\,dy$$

$$= -\cos y \Big|_0^{\pi}$$

$$= -(-1-1)$$

$$= 2$$

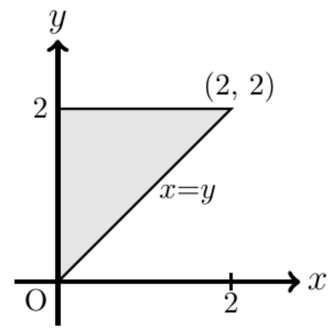


Exercise

Sketch the region of integration, reverse the order of integration, and evaluate the integral

$$\int_0^2 \int_x^2 2y^2 \sin xy \; dydx$$

Solution



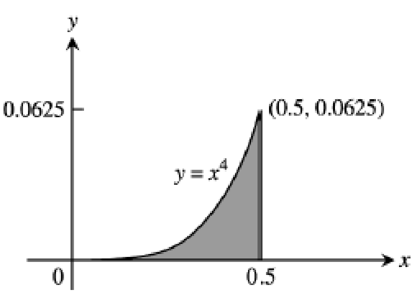
$$\begin{aligned}
\int_0^2 \int_x^2 2y^2 \sin xy \; dydx &= \int_0^2 \int_0^y 2y^2 \sin xy \; dxdy \\
&= -2\int_0^2 \left(y \cos xy \,\Big|_0^y \right) dy \\
&= -2\int_0^2 \left(y \cos y^2 - y \right) dy \qquad u = y^2 \;\Rightarrow\; du = 2y dy \\
&= -\int_0^2 \cos u \; du + \int_0^2 2y \; dy \\
&= -\sin y^2 + y^2 \,\Big|_0^2 \\
&= -\sin 4 + 4
\end{aligned}$$

Exercise

Sketch the region of integration, reverse the order of integration, and evaluate the integral

$$\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos\left(16\pi x^5\right) dxdy$$

Solution



$$x = y^{1/4} \quad \Rightarrow \quad y = x^4$$

$$\begin{aligned}
\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos\left(16\pi x^5\right) dxdy &= \int_0^{1/2} \int_0^{x^4} \cos\left(16\pi x^5\right) dydx \\
&= \int_0^{1/2} \cos\left(16\pi x^5\right) \left(y \,\Big|_0^{x^4} \right) dx \\
&= \int_0^{1/2} x^4 \cos\left(16\pi x^5\right) dx \qquad u=16\pi x^5 \;\to\; du=80\pi x^4 dx
\end{aligned}$$

$$= \frac{1}{80\pi} \int_0^{1/2} \cos u \, du$$

$$= \frac{1}{80\pi} \left(\sin 16\pi x^5 \right|_0^{1/2}$$

$$= \frac{1}{80\pi} \left(\sin \frac{16\pi}{32} - 0 \right)$$

$$= \frac{1}{80\pi}$$

Exercise

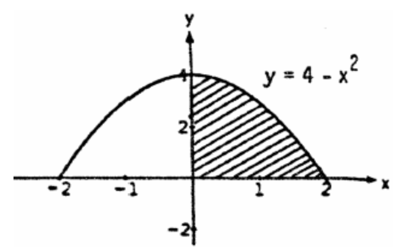
Sketch the region of integration, reverse the order of integration, and evaluate the integral

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} \, dydx$$

Solution

$$y = 4 - x^2 \quad \Rightarrow \quad x^2 = 4 - y \to x = \sqrt{4-y}$$

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} \, dydx = \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y} \, dxdy$$



$$= \int_0^4 \frac{e^{2y}}{4-y} \left(\frac{1}{2}x^2 \right|_0^{\sqrt{4-y}} \, dy$$

$$= \frac{1}{2} \int_0^4 \frac{e^{2y}}{4-y} (4-y) \, dy$$

$$= \frac{1}{2} \int_0^4 e^{2y} dy$$

$$= \frac{1}{4} e^{2y} \Big|_0^4$$

$$= \frac{1}{4}\left(e^8 - 1\right)$$

Exercise

Find the volume of the region bounded above the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the *xy*-plane

Solution

$$V = \int_0^1 \int_x^{2-x} \left(x^2 + y^2\right) dydx$$

$y = x \qquad x + y = 2 \rightarrow y = 2 - x$

$x = 0 \qquad y = x \rightarrow x + x = 2 \Rightarrow x = 1$

$$= \int_0^1 \left(x^2 y + \frac{1}{3}y^3 \Big|_x^{2-x} dx\right.$$

$$= \int_0^1 \left(x^2(2-x) + \frac{1}{3}(2-x)^3 - x^3 - \frac{1}{3}x^3\right) dx$$

$$= \int_0^1 \left(2x^2 - x^3 + \frac{1}{3}(2-x)^3 - \frac{4}{3}x^3\right) dx$$

$$= \int_0^1 \left(2x^2 - \frac{7}{3}x^3\right)dx + \int_0^1 \frac{1}{3}(2-x)^3\left(-d(2-x)\right)$$

$$= \frac{2}{3}x^3 - \frac{7}{12}x^4 - \frac{1}{12}(2-x)^4 \Big|_0^1$$

$$= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12}\right) - \left(-\frac{16}{12}\right)$$

$$= \frac{4}{3}$$

Exercise

Find the volume of the solid that is bounded above the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the *xy*-plane

Solution

$$V = \int_{-2}^1 \int_x^{2-x^2} x^2 \, dydx$$

$$= \int_{-2}^1 x^2 \left(y \Big|_x^{2-x^2} dx\right.$$

$$= \int_{-2}^1 x^2\left(2 - x^2 - x\right) dx$$

$$= \int_{-2}^{1} \left(2x^2 - x^4 - x^3\right) dx$$

$$= \frac{2}{3}x^3 - \frac{1}{5}x^5 - \frac{1}{4}x^4 \Big|_{-2}^{1}$$

$$= \frac{2}{3} - \frac{1}{5} - \frac{1}{4} - \left(-\frac{15}{3} + \frac{32}{5} - \frac{16}{4}\right)$$

$$= \frac{63}{20}$$

Exercise

Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$ and the plane $z + y = 3$

Solution

$$V = \int_0^2 \int_0^{\sqrt{4-x^2}} (3 - y)\, dydx$$

$$= \int_0^2 \left(3y - \frac{1}{2}y^2 \Big|_0^{\sqrt{4-x^2}}\right) dx$$

$$= \int_0^2 \left[3\sqrt{4 - x^2} - \frac{1}{2}\left(4 - x^2\right)\right] dx \qquad \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$= \frac{3}{2}x\sqrt{4 - x^2} + 6\sin^{-1}\left(\frac{x}{2}\right) - 2x + \frac{1}{6}x^3 \Big|_0^2$$

$$= 0 + 6\frac{\pi}{2} - 4 + \frac{8}{6} - (0)$$

$$= 3\pi - \frac{8}{3}$$

Exercise

Find the volume of the solid that is bounded on the front and back by the planes $x = 2$, and $x = 1$, on the sides by the cylinders $y = \pm\frac{1}{x}$ and above and below the planes $z = x + 1$ and $z = 0$.

Solution

$$V = \int_1^2 \int_{-1/x}^{1/x} (x + 1)\, dydx$$

$$= \int_1^2 (x+1)\left(y \,\Big|_{-1/x}^{1/x}\right) dx$$

$$= \int_1^2 (x+1)\left(\frac{2}{x}\right) dx$$

$$= 2\int_1^2 \left(1+\frac{1}{x}\right) dx$$

$$= 2\left(x + \ln x \,\Big|_1^2\right)$$

$$= 2(2+\ln 2 - 1)$$

$$= 2(1+\ln 2)$$

Exercise

Find the volume under the parabolic cylinder $z = x^2$ above the region enclosed by the parabola $y = 6 - x^2$ and the line $y = x$ in the *xy*-plane

Solution

$$y = 6 - x^2 = x$$

$$x^2 - x - 6 = 0 \quad \rightarrow \quad x = -3,\ 2$$

$$V = \int_{-3}^{2}\int_{x}^{6-x^2} z \; dydx$$

$$= \int_{-3}^{2}\int_{x}^{6-x^2} x^2 \; dydx$$

$$= \int_{-3}^{2} x^2 \, y \,\Big|_{x}^{6-x^2} \; dx$$

$$= \int_{-3}^{2} x^2\left(6 - x^2 - x\right) dx$$

$$= \int_{-3}^{2} \left(6x^2 - x^4 - x^3\right) dx$$

$$= 2x^3 - \frac{1}{5}x^5 - \frac{1}{4}x^4 \,\Big|_{-3}^{2}$$

$$= 16 - \frac{32}{5} - 4 - \left(-54 + \frac{3^5}{5} - \frac{81}{4} \right)$$

$$= \frac{125}{4} \ unit^3$$

Exercise

Find the area of the region enclosed by the line $y = 2x + 4$ and the parabola $y = 4 - x^2$ in the *xy*-plane.

Solution

$$y = 2x + 4 = 4 - x^2$$

$$x^2 + 2x = 0 \quad \to \quad x = 0, \ -2$$

$$A = \int_{-2}^{0} \int_{2x+4}^{4-x^2} dydx$$

$$= \int_{-2}^{0} y \Big|_{2x+4}^{4-x^2} dx$$

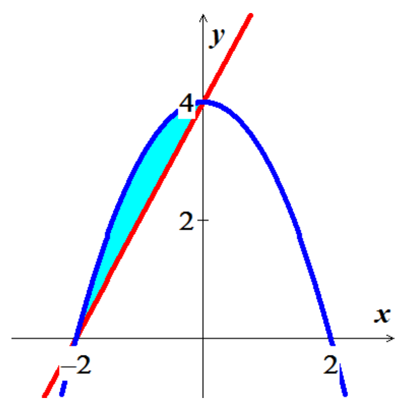
$$= \int_{-2}^{0} \left(4 - x^2 - 2x - 4\right) dx$$

$$= \int_{-2}^{0} \left(-x^2 - 2x\right) dx$$

$$= -\frac{1}{3}x^3 - x^2 \Big|_{-2}^{0}$$

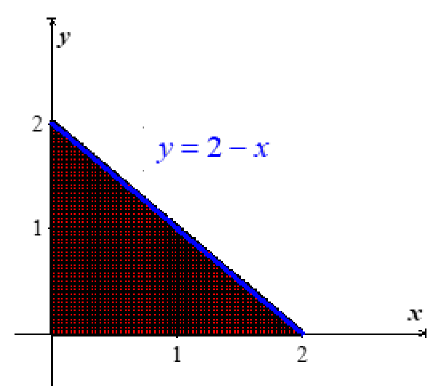
$$= -\frac{8}{3} + 4$$

$$= \frac{4}{3} \ unit^2$$



Exercise

Find the area of the region enclosed by the coordinate axes and the line $x + y = 2$.



Solution

$$\int_0^2 \int_0^{2-x} dydx = \int_0^2 \left(y \Big|_0^{2-x} dx\right.$$

$$= \int_0^2 (2 - x)\, dx$$

$$= 2x - \frac{1}{2}x^2 \Big|_0^2$$

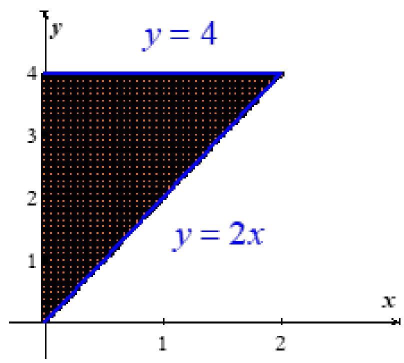
$$= 4 - \frac{1}{2}(4)$$

$$\underline{= 2}$$

Exercise

Find the area of the region enclosed by the lines $x = 0$, $y = 2x$, and $y = 4$

Solution



$$\int_0^2 \int_{2x}^4 dydx = \int_0^2 \left(y \Big|_{2x}^4\right) dx$$

$$= \int_0^2 (4 - 2x)\, dx$$

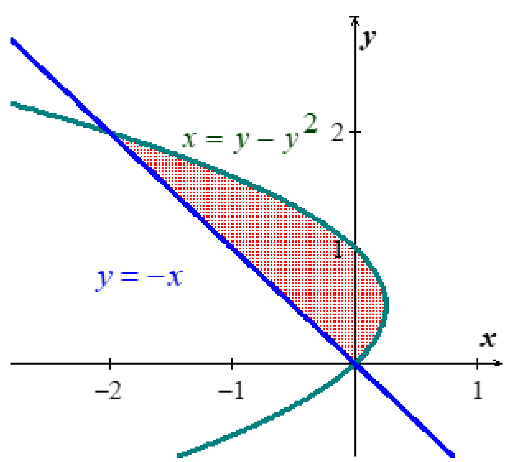
$$= 4x - x^2 \Big|_0^2$$

$$\underline{= 4}$$

Exercise

Find the area of the region enclosed by the parabola $x = y - y^2$ and the line $y = -x$.

Solution



$$x = y - y^2 = -y$$

$$2y - y^2 = 0$$

$$\underline{y = 0,\ 2}$$

$$\int_0^2 \int_{-y}^{y-y^2} dxdy = \int_0^2 \left(x \Big|_{-y}^{y-y^2}\right) dy$$

$$= \int_0^2 \left(y - y^2 + y\right) dy$$

$$= \int_0^2 \left(2y - y^2\right) dy$$

$$= y^2 - \frac{1}{3}y^3 \Big|_0^2$$

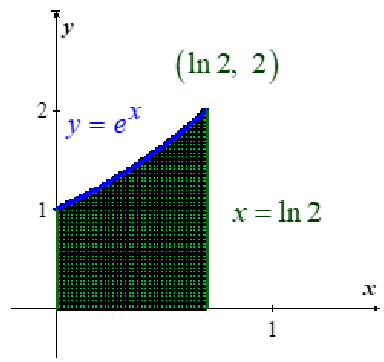
$$= 4 - \frac{8}{3}$$

$$= \frac{4}{3}$$

Exercise

Find the area of the region enclosed by the curve $y = e^x$ and the lines $y = 0$, $x = 0$ and $x = \ln 2$.

Solution



$$\int_0^{\ln 2} \int_0^{e^x} dydx = \int_0^{\ln 2} \left(y \Big|_0^{e^x}\right) dx$$

$$= \int_0^{\ln 2} e^x dx$$

$$= e^x \Big|_0^{\ln 2}$$

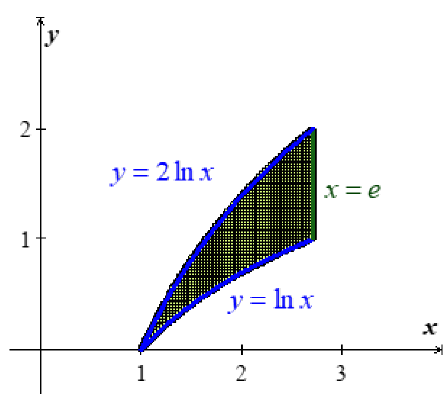
$$= 2 - 1$$

$$= 1$$

Exercise

Find the area of the region enclosed by the curve $y = \ln x$ and $y = 2\ln x$ and the lines $x = e$ in the first quadrant.

Solution



$$\int_1^e \int_{\ln x}^{2\ln x} dydx = \int_1^e \left(y \Big|_{\ln x}^{2\ln x}\right) dx$$

$$= \int_0^{\ln 2} \ln x\, dx$$

$$= x\ln x - x \Big|_1^e$$

$$= e - e - (0 - 1)$$

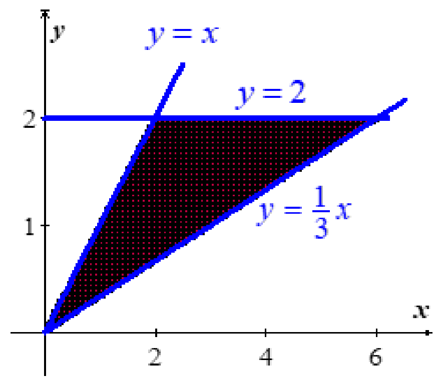
$$\underline{= 1}$$

Exercise

Find the area of the region enclosed by the lines $y = x$, $y = \frac{x}{3}$, and $y = 2$

Solution

$$\int_0^2 \int_y^{3y} dxdy = \int_0^2 x \Big|_y^{3y} dy$$



$$= \int_0^2 (2y)dy$$

$$= y^2 \Big|_0^2$$

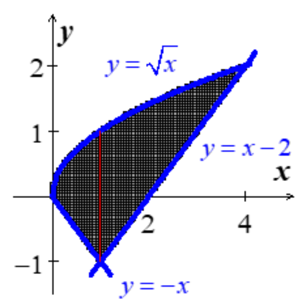
$$\underline{= 4}$$

Exercise

Find the area of the region enclosed by the lines $y = x - 2$ and $y = -x$ and the curve $y = \sqrt{x}$

Solution

$$A = \int_0^1 \int_{-x}^{\sqrt{x}} dydx + \int_1^4 \int_{x-2}^{\sqrt{x}} dydx$$



$$= \int_0^1 y \Big|_{-x}^{\sqrt{x}} dx + \int_1^4 y \Big|_{x-2}^{\sqrt{x}} dx$$

$$= \int_0^1 (\sqrt{x} - x)dx + \int_1^4 (\sqrt{x} - x + 2)dx$$

$$= \left(\frac{2}{3}x^{3/2} + \frac{1}{2}x^2\right) \Big|_0^1 + \left(\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right) \Big|_1^4$$

$$= \frac{2}{3} + \frac{1}{2} + \frac{2}{3}4^{3/2} - 2 + 8 - \frac{2}{3} - \frac{1}{2} + 2$$

$$\underline{= \frac{13}{3}}$$

Exercise

Find the area of the region enclosed by the parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$

Solution

$$\int_{-1}^{1}\int_{2y^2-2}^{y^2-1} dxdy = \int_{-1}^{1}\left(y \Big|_{2y^2-2}^{y^2-1} \; dy\right.$$

$$= \int_{-1}^{1}\left(y^2 - 1 - 2y^2 + 2\right)dy$$

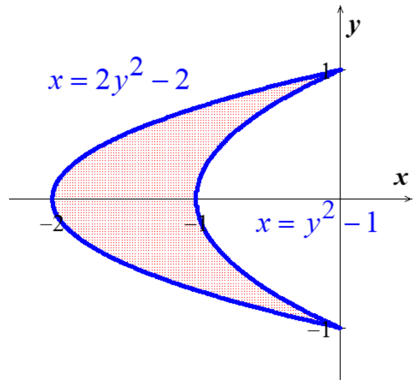
$$= \int_{-1}^{1}\left(1 - y^2\right)dy$$

$$= y - \frac{1}{3}y^3 \Big|_{-1}^{1}$$

$$= 1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)$$

$$= 2 - \frac{2}{3}$$

$$= \frac{4}{3}$$



Exercise

Find the area of the region bounded by the lines $y = -x - 4$, $y = x$, and $y = 2x - 4$. Make a sketch of the region.

Solution

$(y = -x - 4) \cap (y = x)$

$\rightarrow y = x = -x - 4$

$2x = -4 \quad \rightarrow x = -2 \quad (-2, \ -2)$

$(y = -x - 4) \cap (y = 2x - 4)$

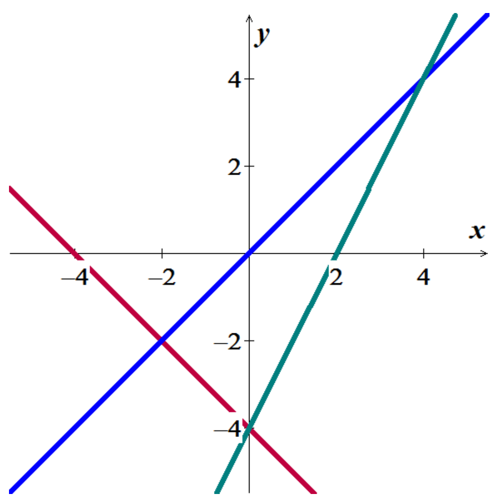
$\rightarrow y = 2x - 4 = -x - 4$

$x = 0 \quad \rightarrow \quad (0, \ -4)$

$(y = x) \cap (y = 2x - 4)$

$\rightarrow y = 2x - 4 = x$

$x = 4 \quad \rightarrow \quad (4, \ 4)$



$$A = \int_{-2}^{0}\int_{-x-4}^{x} dydx + \int_{0}^{4}\int_{2x-4}^{x} dydx$$

$$= \int_{-2}^{0} y \Big|_{-x-4}^{x} dx + \int_{0}^{4} y \Big|_{2x-4}^{x} dx$$

$$= \int_{-2}^{0} (2x+4)\,dx + \int_{0}^{4} (-x+4)\,dx$$

$$= \left(x^2+4x\right|_{-2}^{0} + \left(-\frac{1}{2}x^2+4x\right|_{0}^{4}$$

$$= (-4+8) + (-8+16)$$

$$\underline{= 12 \ \ unit^2}$$

Exercise

Find the area of the region bounded by the lines $y = |x|$ and $y = 20 - x^2$. Make a sketch of the region.

Solution

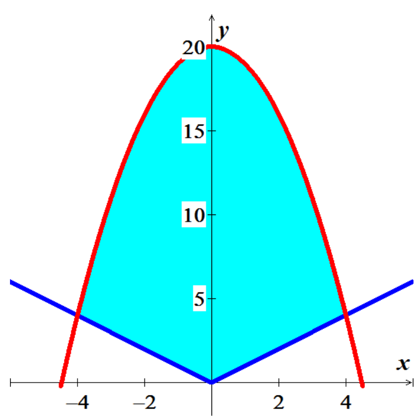
$$y = 20 - x^2 = x$$

$$x^2 + x - 20 = 0 \quad \rightarrow \quad x = \cancel{-5},\ 4$$

$$A = 2\int_{0}^{4}\int_{x}^{20-x^2} dydx$$

$$= 2\int_{0}^{4} y \Big|_{x}^{20-x^2} dx$$

$$= 2\int_{0}^{4} \left(20 - x^2 - x\right) dx$$



$$= 2\left(20x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right)\Big|_{0}^{4}$$

$$= 2\left(80 - \frac{64}{3} - 8\right)$$

$$\underline{= \frac{304}{3} \ \ unit^2}$$

Exercise

Find the area of the region bounded by the lines $y = x^2$ and $y = 1 + x - x^2$. Make a sketch of the region.

Solution

$$y = 1 + x - x^2 = x^2$$

$$2x^2 - x - 1 = 0 \quad \to \quad x = 1, \ -\frac{1}{2}$$

$$A = \int_{-\frac{1}{2}}^{1} \int_{x^2}^{1+x-x^2} dy\,dx$$

$$= \int_{-\frac{1}{2}}^{1} \left(1 + x - 2x^2\right) dx$$

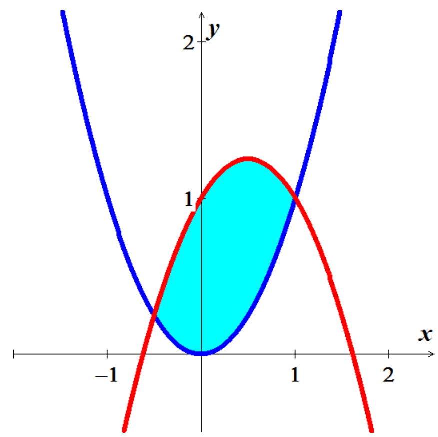
$$= x + \frac{1}{2}x^2 - \frac{2}{3}x^3 \Big|_{-\frac{1}{2}}^{1}$$

$$= 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{2} - \frac{1}{8} - \frac{1}{12}$$

$$= 2 - \frac{21}{24}$$

$$= \frac{27}{24}$$

$$= \frac{9}{8} \ unit^2$$



Exercise

Find the area of the region $\displaystyle\int_0^6 \int_{y^2/3}^{2y} dx\,dy$

Solution

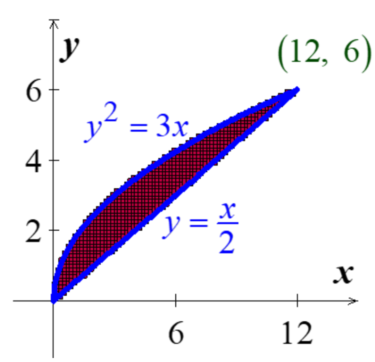
$$\int_0^6 \int_{y^2/3}^{2y} dx\,dy = \int_0^6 \left(x \Big|_{y^2/3}^{2y}\right) dy$$

$$= \int_0^6 \left(2y - \frac{1}{3}y^2\right) dy$$

$$= y^2 - \frac{1}{9}y^3 \Big|_0^6$$

$$= 36 - \frac{1}{9}(216)$$

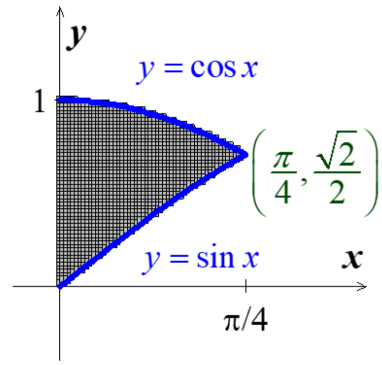
$$= 12$$



Exercise

Find the area of the region $\displaystyle\int_0^{\pi/4}\int_{\sin x}^{\cos x} dydx$

Solution



$$\int_0^{\pi/4}\int_{\sin x}^{\cos x} dydx = \int_0^{\pi/4} \left[y\right]_{\sin x}^{\cos x} dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x)\,dx$$

$$= \left[\sin x + \cos x\right]_0^{\pi/4}$$

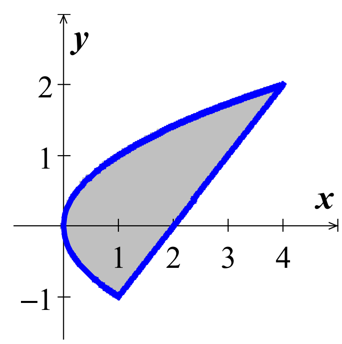
$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1)$$

$$= \sqrt{2} - 1$$

Exercise

Find the area of the region $\displaystyle\int_{-1}^{2}\int_{y^2}^{y+2} dxdy$

Solution



$$\int_{-1}^{2}\int_{y^2}^{y+2} dxdy = \int_{-1}^{2}\left(y + 2 - y^2\right)dy$$

$$= \left(\tfrac{1}{2}y^2 + 2y - \tfrac{1}{3}y^3\right\rvert_{-1}^{2}$$

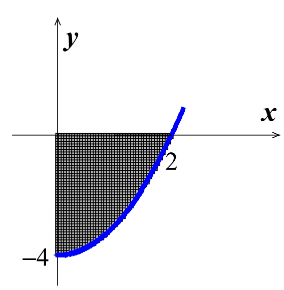
$$= \left(2 + 4 - \tfrac{8}{3}\right) - \left(\tfrac{1}{2} - 2 + \tfrac{1}{3}\right)$$

$$= \frac{9}{2}$$

Exercise

Find the area of the region $\displaystyle\int_0^2\int_{x^2-4}^{0} dydx + \int_0^4\int_0^{\sqrt{x}} dydx$

Solution



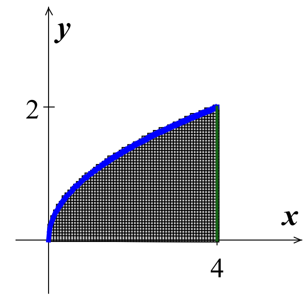
$$\int_0^2\int_{x^2-4}^{0} dydx + \int_0^4\int_0^{\sqrt{x}} dydx = \int_0^2\left(4 - x^2\right)dx + \int_0^4 \sqrt{x}\,dx$$

$$= \left(4x - \frac{1}{3}x^3 \right|_0^2 + \frac{2}{3}\left(x^{3/2} \right|_0^4$$

$$= \left(8 - \frac{8}{3}\right) + \frac{2}{3}\left(4^{3/2}\right)$$

$$= \frac{16}{3} + \frac{16}{3}$$

$$= \frac{32}{3}$$



Exercise

Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2, \quad 0 \le y \le 2$

Solution

$$Average\ height = \frac{1}{4}\int_0^2 \int_0^2 \left(x^2 + y^2\right) dydx$$

$$= \frac{1}{4}\int_0^2 \left(x^2 y + \frac{1}{3}y^3 \right|_0^2 dx$$

$$= \frac{1}{4}\int_0^2 \left(2x^2 + \frac{8}{3}\right) dx$$

$$= \frac{1}{4}\left(\frac{2}{3}x^3 + \frac{8}{3}x \right|_0^2$$

$$= \frac{1}{4}\left(\frac{2}{3}(8) + \frac{8}{3}(2)\right)$$

$$= \frac{1}{4}\left[\frac{16}{3} + \frac{16}{3}\right]$$

$$= \frac{8}{3}$$

Exercise

Find the average height of $f(x,y) = \frac{1}{xy}$ over the square $\ln 2 \le x \le 2\ln 2, \quad \ln 2 \le y \le 2\ln 2$

Solution

$$\text{Average height} = \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2} \int_{\ln 2}^{2\ln 2} \frac{1}{xy} dydx$$

$$= \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \frac{1}{x}\left(\ln y \Big|_{\ln 2}^{2\ln 2}\right) dx$$

$$= \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} \frac{1}{x}(2\ln 2 - \ln 2)\, dx$$

$$= \frac{1}{(\ln 2)^2} \int_{\ln 2}^{2\ln 2} (\ln 2)\frac{1}{x}\, dx$$

$$= \frac{1}{\ln 2}\left(\ln x \Big|_{\ln 2}^{2\ln 2}\right)$$

$$= \frac{1}{\ln 2}(2\ln 2 - \ln 2)$$

$$= 1$$

Exercise

Evaluate the integral over the given region

$$\iint_R y\,dA \qquad R = \left\{(x, y):\ \ 0 \le x \le \frac{\pi}{3},\quad 0 \le y \le \sec x\right\}$$

Solution

$$\iint_R y\,dA = \int_0^{\frac{\pi}{3}} \int_0^{\sec x} y\ dydx$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{3}} y^2 \Big|_0^{\sec x}\ dx$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{3}} \sec^2 x\, dx$$

$$= \frac{1}{2} \tan x \Big|_0^{\frac{\pi}{3}}$$

$$= \frac{\sqrt{3}}{2}$$

Exercise

Evaluate the integral over the given region

$$\iint_R (x+y)\,dA \quad R \text{ is the region bounded by } y=\frac{1}{x} \text{ and } y=\frac{5}{2}-x$$

Solution

$$y=\frac{1}{x}=\frac{5}{2}-x$$

$$2x^2-5x+2=0 \quad \to \quad \boxed{x=\frac{1}{2},\ 2}$$

$$\iint_R (x+y)\,dA=\int_{\frac{1}{2}}^{2}\int_{1/x}^{\frac{5}{2}-x}(x+y)\,dydx$$

$$=\int_{\frac{1}{2}}^{2}\left(xy+\frac{1}{2}y^2\ \Big|_{1/x}^{\frac{5}{2}-x}\right)dx$$

$$=\int_{\frac{1}{2}}^{2}\left(x\left(\frac{5}{2}-x\right)+\frac{1}{2}\left(\frac{5}{2}-x\right)^2-1-\frac{1}{2x^2}\right)dx$$

$$=\int_{\frac{1}{2}}^{2}\left(\frac{5}{2}x-x^2+\frac{25}{8}-\frac{5}{2}x+\frac{1}{2}x^2-1-\frac{1}{2x^2}\right)dx$$

$$=\int_{\frac{1}{2}}^{2}\left(-\frac{1}{2}x^2+\frac{17}{8}-\frac{1}{2x^2}\right)dx$$

$$=-\frac{1}{6}x^3+\frac{17}{8}x+\frac{1}{2x}\ \Big|_{\frac{1}{2}}^{2}$$

$$=-\frac{4}{3}+\frac{17}{4}+\frac{1}{4}+\frac{1}{48}-\frac{17}{16}-1$$

$$=\frac{27}{24}$$

$$\boxed{=\frac{9}{8}}$$

Exercise

Evaluate the integral over the given region

$$\iint_R \frac{xy}{1+x^2+y^2}\,dA \quad R=\{(x,\ y):\ \ 0\le y\le x,\quad 0\le x\le 2\}$$

Solution

$$\iint_R \frac{xy}{1+x^2+y^2}\,dA=\int_0^2\int_0^x \frac{xy}{1+x^2+y^2}\,dydx$$

$$=\frac{1}{2}\int_0^2\int_0^x \frac{x}{1+x^2+y^2}\,d\left(1+x^2+y^2\right)dx \qquad d\left(1+x^2+y^2\right)=2ydy$$

$$=\frac{1}{2}\int_0^2 x\ \ln\left(1+x^2+y^2\right)\Big|_0^x\ dx$$

$$=\frac{1}{2}\int_0^2 \left(x\ln\left(1+2x^2\right)-x\ln\left(1+x^2\right)\right)dx$$

$$=\frac{1}{2}\int_0^2 x\ln\left(1+2x^2\right)dx\ -\ \frac{1}{2}\int_0^2 x\ln\left(1+x^2\right)dx$$

$$=\frac{1}{8}\int_0^2 \ln\left(1+2x^2\right)d\left(1+2x^2\right)\ -\ \frac{1}{4}\int_0^2 \ln\left(1+x^2\right)d\left(1+x^2\right)$$

$$\int \ln y\ dy=y(\ln y-1)$$

$$=\frac{1}{8}\left(1+2x^2\right)\left(\ln\left(1+2x^2\right)-1\right)\Big|_0^2\ -\ \frac{1}{4}\left(1+x^2\right)\left(\ln\left(1+x^2\right)-1\right)\Big|_0^2$$

$$=\frac{1}{8}(9\ln 9-9+1)-\frac{1}{4}(5\ln 5-5+1)$$

$$=\frac{9}{8}\ln 9+1-\frac{5}{4}\ln 5-1$$

$$=\frac{9}{8}\ln 9-\frac{5}{4}\ln 5$$

Exercise

Evaluate the integral over the given region

$$\iint_R x\sec^2 y \, dA \quad R = \left\{(x, y):\ \ 0 \le y \le x^2,\ \ 0 \le x \le \frac{\sqrt{\pi}}{2}\right\}$$

Solution

$$\iint_R x\sec^2 y \, dA = \int_0^{\frac{\sqrt{\pi}}{2}} \int_0^{x^2} x\sec^2 y \, dydx$$

$$= \int_0^{\frac{\sqrt{\pi}}{2}} x\tan y \Big|_0^{x^2} dx$$

$$= \int_0^{\frac{\sqrt{\pi}}{2}} x\tan x^2 \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\sqrt{\pi}}{2}} \tan x^2 \, d\left(x^2\right)$$

$$= \frac{1}{2}\ln\left|\sec x^2\right| \Big|_0^{\frac{\sqrt{\pi}}{2}}$$

$$= \frac{1}{2}\ln\left|\sec\frac{\pi}{4}\right|$$

$$= \frac{1}{2}\ln\sqrt{2}$$

$$= \frac{1}{4}\ln 2$$

Exercise

Consider the region $R = \{(x, y): \ |x| + |y| \le 1\}$

a) Use a double integral to show that the area of R is 2.

b) Find the volume of the square column whose base is R and whose upper surface is $z = 12 - 3x - 4y$.

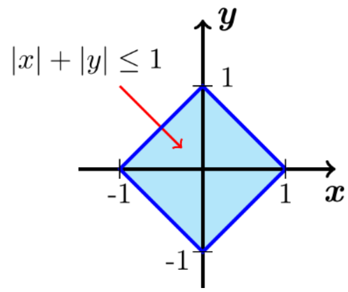
c) Find the volume of the solid above R and beneath the cylinder $x^2 + z^2 = 1$.

d) Find the volume of the pyramid whose base is R and whose vertex is on the z-axis at $(0,\ 0,\ 6)$

Solution

$$-1 \le x \le 0 \quad \to \quad \begin{cases} y = x + 1 \\ y = -x - 1 \end{cases}$$

$$0 \le x \le 1 \quad \to \quad \begin{cases} y = x - 1 \\ y = -x + 1 \end{cases}$$



a) $$A = \int_{-1}^{0} \int_{-x-1}^{x+1} dydx \ + \ \int_{0}^{1} \int_{x-1}^{1-x} dydx$$

$$= \int_{-1}^{0} y \Big|_{-x-1}^{x+1} dx \ + \ \int_{0}^{1} y \Big|_{x-1}^{1-x} dx$$

$$= \int_{-1}^{0} (2x + 2)\, dx \ + \ \int_{0}^{1} (2 - 2x)\, dx$$

$$= \left(x^2 + 2x \ \Big|_{-1}^{0} \ + \ \left(2x - x^2 \ \Big|_{0}^{1}\right.\right.$$

$$= -1 + 2 + 2 - 1$$

$$= 2\ |$$

b) $$V = \int_{-1}^{0} \int_{-x-1}^{x+1} (12 - 3x - 4y)\, dydx \ + \ \int_{0}^{1} \int_{x-1}^{1-x} (12 - 3x - 4y)\, dydx$$

$$= \int_{-1}^{0} \left(12y - 3xy - 2y^2 \ \Big|_{-x-1}^{x+1} dx \ + \ \int_{0}^{1} \left(12y - 3xy - 2y^2 \ \Big|_{x-1}^{1-x} dx\right.\right.$$

$$= \int_{-1}^{0} \left(12x + 12 - 3x(x+1) - 2(x+1)^2 - 12(-x-1) + 3x(x+1) + 2(-x-1)^2\right) dx$$

$$+ \int_{0}^{1} \left(12(1-x) - 3x(1-x) - 2(1-x)^2 - 12(x-1) + 3x(x-1) + 2(x-1)^2\right) dx$$

$$= \int_{-1}^{0} \left(18x + 24 - 6x^2\right) dx + \int_{0}^{1} \left(24 - 30x + 6x^2\right) dx$$

$$= \left(9x^2 + 24x - 2x^3\right. \Big|_{-1}^{0} + \left(24x - 15x^2 + 2x^3\right. \Big|_{0}^{1}$$

$$= -9 + 24 - 2 + 24 - 15 + 2$$

$$\underline{= 24}$$

c) $x^2 + z^2 = 1 \quad \rightarrow \quad z = \sqrt{1 - x^2}$

$$V = \int_{-1}^{0} \int_{-x-1}^{x+1} \sqrt{1-x^2}\ dydx + \int_{0}^{1} \int_{x-1}^{1-x} \sqrt{1-x^2}\ dydx \qquad \textit{due to the symmetry}$$

$$= 2\int_{0}^{1} \int_{x-1}^{1-x} \sqrt{1-x^2}\ dydx$$

$$= 2\int_{0}^{1} \sqrt{1-x^2}\ y \Big|_{x-1}^{1-x} dx$$

$$= 2\int_{0}^{1} (2-2x)\sqrt{1-x^2}\ dx$$

$$= 4\int_{0}^{1} \sqrt{1-x^2}\ dx - 4\int_{0}^{1} x\sqrt{1-x^2}\ dx$$

$$= \left(2x\sqrt{1-x^2} + 2\sin^{-1} x\right) \Big|_{0}^{1} + 2\int_{0}^{1} \left(1-x^2\right)^{1/2} d\left(1-x^2\right)$$

$$= 2\sin^{-1} 1 + \frac{4}{3}\left(1-x^2\right)^{3/2} \Big|_{0}^{1}$$

$$\underline{= \pi - \frac{4}{3}}$$

d) $(1,\ 0,\ 0) \quad (0,\ 1,\ 0) \quad (0,\ 0,\ 6)$

$z = 6(1 - x - y)$

Using symmetry

$$V = 4\int_{0}^{1} \int_{0}^{1-x} 6(1-x-y)\ dydx$$

$$= 24\int_{0}^{1} \left((1-x)y - \frac{1}{2}y^2\right) \Big|_{0}^{1-x} dx$$

$$= 24\int_0^1 \left((1-x)^2 - \frac{1}{2}(1-x)^2\right) dx$$

$$= -12\int_0^1 (1-x)^2 \, d(1-x)$$

$$= -4(1-x)^3 \Big|_0^1$$

$$= 4$$