Solution Section 4.1 – Law of Sines

Exercise

In triangle ABC, $B = 110°$, $C = 40°$ and $b = 18\,in$. Find the length of side c.

Solution

$$A = 180° - (B + C)$$
$$= 180° - 110° - 40°$$
$$= 30°$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{a}{\sin 30°} = \frac{18}{\sin 110°}$$
$$a = \frac{18 \sin 30°}{\sin 110°}$$
$$\approx 9.6\ in$$

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$
$$\frac{c}{\sin 40°} = \frac{18}{\sin 110°}$$
$$c = \frac{18 \sin 40°}{\sin 110°}$$
$$\approx 12.3\ in$$

Exercise

In triangle ABC, $A = 110.4°$, $C = 21.8°$ and $c = 246\,in$. Find all the missing parts.

Solution

$$B = 180° - A - C$$
$$= 180° - 110.4° - 21.8°$$
$$= 47.8°$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$\frac{a}{\sin 110.4°} = \frac{246}{\sin 21.8°}$$
$$a = \frac{246 \sin 110.4°}{\sin 21.8°}$$
$$\approx 621\ in$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{b}{47.8} = \frac{246}{\sin 21.8°}$$
$$b = \frac{246 \sin 47.8°}{\sin 21.8°}$$
$$\approx 491\ in$$

Exercise

Find the missing parts of triangle ABC if $B = 34°$, $C = 82°$, and $a = 5.6\,cm$.

Solution

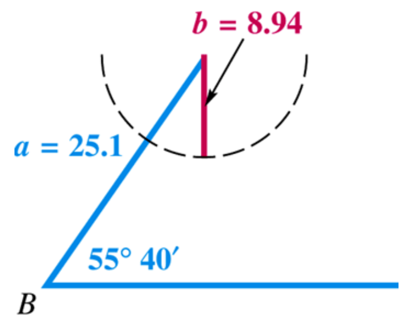
$$\begin{aligned} A &= 180° - (B + C) \\ &= 180° - (34° + 82°) \\ &= 180° - 116° \\ &= 64° \end{aligned}$$

$\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$ $b = \dfrac{a\sin B}{\sin A}$ $= \dfrac{5.6\sin 34°}{\sin 64°}$ $= 3.5\,cm$	$\dfrac{c}{\sin C} = \dfrac{a}{\sin A}$ $c = \dfrac{a\sin C}{\sin A}$ $= \dfrac{5.6\sin 82°}{\sin 64°}$ $= 6.2\,cm$

Exercise

Solve triangle ABC if $B = 55°40'$, $b = 8.94$ m, and $a = 25.1$ m.

Solution



$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin A}{25.1} = \frac{\sin\left(55° + \frac{40°}{60}\right)}{8.94}$$

$$\sin A = \frac{25.1\sin(55.667°)}{8.94} \approx 2.3184 > 1$$

Since $\sin A > 1$ is impossible, no such triangle exists.

Exercise

Solve triangle *ABC* if $A = 55.3°$, $a = 22.8\ ft$, and $b = 24.9$ ft

Solution

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

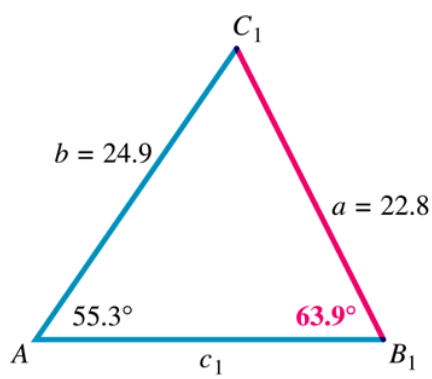
$$\sin B = \frac{24.9 \sin 55.3°}{22.8} \approx 0.89787$$

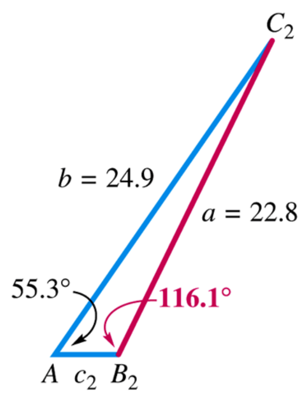
$$B = \sin^{-1}(0.89787)$$

$$B = 63.9° \quad \text{and} \quad B = 180° - 63.9° = 116.1°$$

$$C = 180° - A - B$$

$C = 180° - 55.3° - 63.9°$	$C = 180° - 55.3° - 116.1°$
$C = 60.8°$	$C = 8.6°$
$c = \frac{a \sin C}{\sin A}$	$c = \frac{a \sin C}{\sin A}$
$= \frac{22.8 \sin 60.8°}{\sin 55.3°}$	$= \frac{22.8 \sin 8.6°}{\sin 55.3°}$
$= 24.2\ ft$	$= 4.15\ ft$





Exercise

Solve triangle *ABC* given $A = 43.5°$, $a = 10.7$ in., and $c = 7.2$ in

Solution

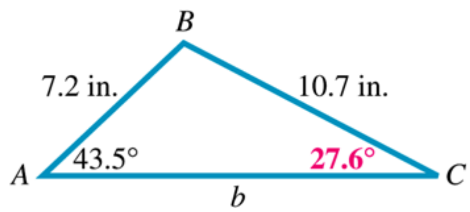
$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\sin C = \frac{7.2\sin 43.5°}{10.7} \approx 0.4632$$

$$C = \sin^{-1}(0.4632)$$

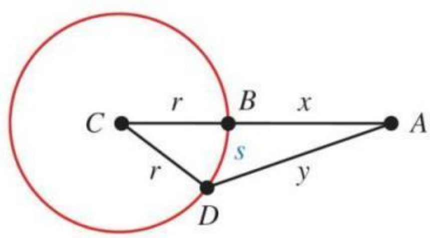
$$C = 27.6° \quad \text{and} \quad C = 180° - 27.6° = 152.4°$$

$$B = 180° - A - C$$



$B = 180° - 43.5° - 27.6°$ $B = 108.9°$ $b = \frac{a\sin B}{\sin A}$ $= \frac{10.7\sin 108.9°}{\sin 43.5°}$ $= 14.7\ in$	$B = 180° - 43.5° - 152.4°$ $B = -15.9°$ Is not possible

Exercise



If $A = 26°, s = 22, and\ \ r = 19$ find x

Solution

$$C = \theta = \frac{s}{r}\ rad = \frac{22}{19}\frac{180°}{\pi} \approx 66°$$

$$D = 180 - A - C = 180° - 26° - 66° = 88°$$

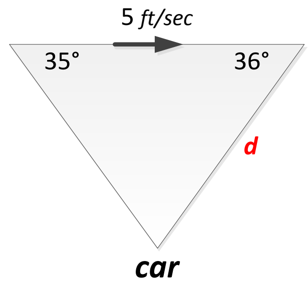
$$\frac{r+x}{\sin D} = \frac{r}{\sin A}$$

$$19 + x = \frac{19\sin 88°}{\sin 26°}$$

$$x = \frac{19\sin 88°}{\sin 26°} - 19 \approx 24$$

Exercise

A man flying in a hot-air balloon in a straight line at a constant rate of 5 feet per second, while keeping it at a constant altitude. As he approaches the parking lot of a market, he notices that the angle of depression from his balloon to a friend's car in the parking lot is 35°. A minute and a half later, after flying directly over this friend's car, he looks back to see his friend getting into the car and observes the angle of depression to be 36°. At that time, what is the distance between him and his friend?



Solution

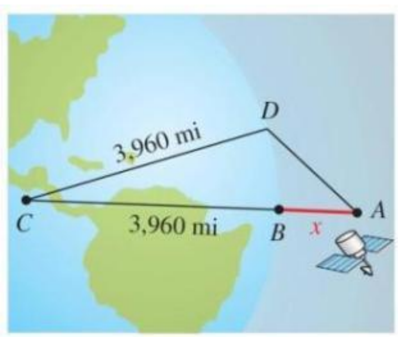
$$\angle car = 180° - 35° - 36° = 109°$$

$$\frac{d}{\sin 35°} = \frac{450}{\sin 109°}$$

$$\boxed{d = \frac{450 \sin 35°}{\sin 109°} \approx 273\ ft}$$

Exercise

A satellite is circling above the earth. When the satellite is directly above point *B*, angle *A* is 75.4°. If the distance between points *B* and *D* on the circumference of the earth is 910 miles and the radius of the earth is 3,960 miles, how far above the earth is the satellite?



Solution

$$\theta = \frac{s}{r}$$

$C = arc\ length\ BD$ divides by $radius$

$$C = \frac{910}{3960} rad$$

$$= \frac{910}{3960} \frac{180°}{\pi}$$

$$= 13.2°$$

$$D = 180° - (A + C)$$

$$= 180° - (75.4° + 13.2°)$$

$$= 91.4°$$

$$\frac{CA}{\sin D} = \frac{3960}{\sin A}$$

$$\frac{x+3960}{\sin 91.4°} = \frac{3960}{\sin 75.4°}$$

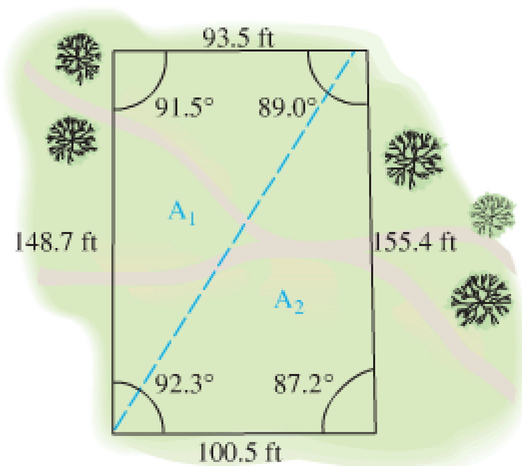
$$x+3960 = \frac{3960 \sin 91.4°}{\sin 75.4°}$$

$$x = \frac{3960 \sin 91.4°}{\sin 75.4°} - 3960$$

$$x = 130\ mi$$

Exercise

The dimensions of a land are given in the figure. Find the area of the property in square feet.



Solution

$$A_1 = \frac{1}{2}(148.7)(93.5)\sin 91.5° \approx 6949.3\ ft^2$$

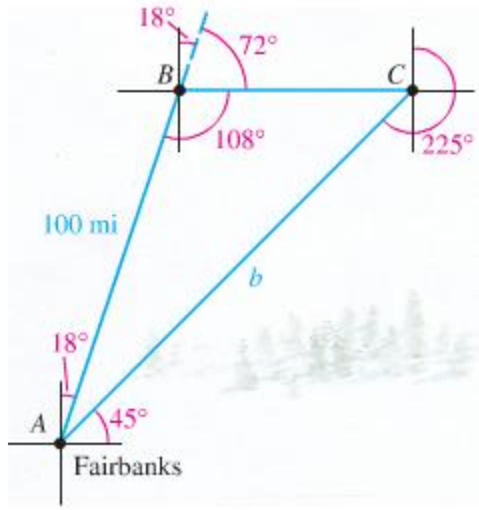
$$A_2 = \frac{1}{2}(100.5)(155.4)\sin 87.2° \approx 7799.5\ ft^2$$

The total area = $A_1 + A_2 = 6949.3 + 7799.5 \underline{= 14,748.8\ ft^2}$

Exercise

A pilot left Fairbanks in a light plane and flew 100 miles toward Fort in still air on a course with bearing of 18°. She then flew due east (bearing 90°) for some time drop supplies to a snowbound family. After the drop, her course to return to Fairbanks had bearing of 225°. What was her maximum distance from Fairbanks?

Solution



From the triangle ABC:

$\angle ABC = 90° + 18° = 108°$

$\angle ACB = 360° - 225° - 90° = 45°$

$\angle BAC = 90° - 18° - 45° = 27°$

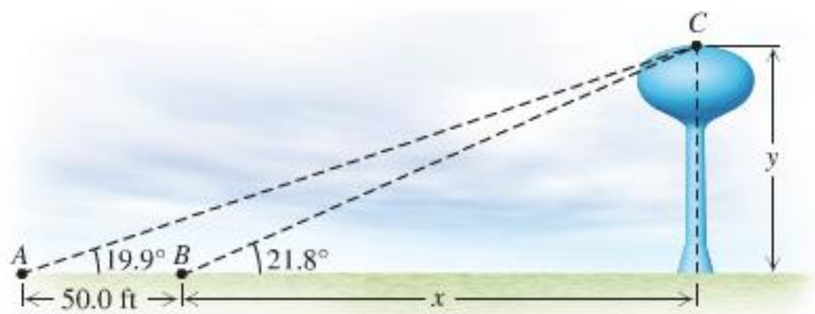
The length AC is the maximum distance from Fairbanks:

$$\frac{b}{\sin 108°} = \frac{100}{\sin 45°}$$

$$b = \frac{100 \sin 108°}{\sin 45°} \approx 134.5 \text{ miles}$$

Exercise

The angle of elevation of the top of a water tower from point A on the ground is 19.9°. From point B, 50.0 feet closer to the tower, the angle of elevation is 21.8°. What is the height of the tower?



Solution

$\angle ABC = 180° - 21.8° = 158.2°$

$\angle ACB = 180° - 19.9° - 158.2° = 1.9°$

Apply the law of sines in triangle *ABC*:

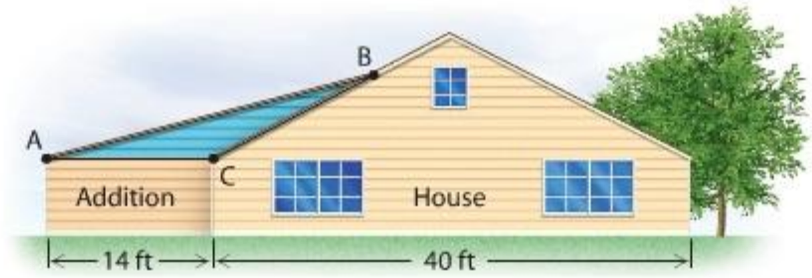
$$\frac{BC}{\sin 19.9°} = \frac{50}{\sin 1.9°} \Rightarrow BC = \frac{50 \sin 19.9°}{\sin 1.9°} \approx 513.3$$

Using the right triangle: $\sin 21.8° = \frac{y}{BC}$

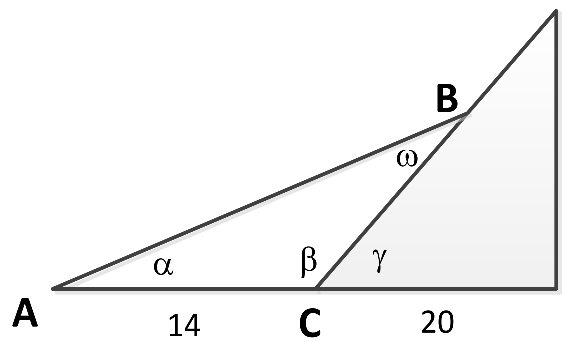
$$y = 513.3 \sin 21.8° \approx 191 \text{ ft}$$

Exercise

A 40-ft wide house has a roof with a 6-12 pitch (the roof rises 6 ft for a run of 12 ft). The owner plans a 14-ft wide addition that will have a 3-12 pitch to its roof. Find the lengths of $\overline{AB}$ and $\overline{BC}$.



Solution



$$\tan\gamma = \frac{6}{12} \Rightarrow \gamma = \tan^{-1}\left(\frac{6}{12}\right) = 26.565°$$

$$\tan\alpha = \frac{3}{12} \Rightarrow \alpha = \tan^{-1}\left(\frac{3}{12}\right) = 14.036°$$

$$\beta = 180° - \gamma = 180° - 26.565° = 153.435°$$

$$\omega = 180° - 14.036° - 153.435° = 12.529°$$

$$\frac{AB}{\sin 153.435°} = \frac{14}{\sin 12.529°}$$

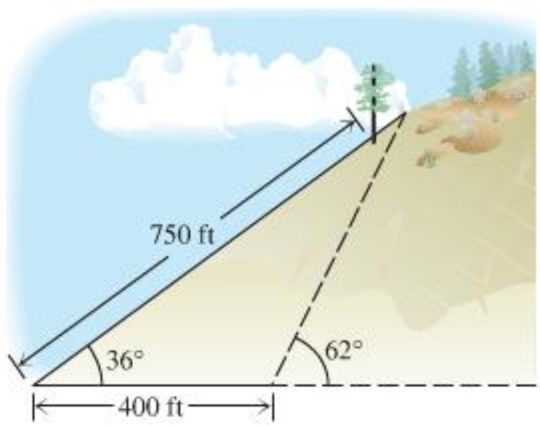
$$\Rightarrow \underline{|AB} = \frac{14\sin 153.435°}{\sin 12.529°} \underline{\approx 28.9\ ft|}$$

$$\frac{BC}{\sin 14.036°} = \frac{14}{\sin 12.529°}$$

$$\Rightarrow \underline{|BC} = \frac{14\sin 14.036°}{\sin 12.529°} \underline{\approx 15.7\ ft|}$$

Exercise

A hill has an angle of inclination of 36°. A study completed by a state's highway commission showed that the placement of a highway requires that 400 ft of the hill, measured horizontally, be removed. The engineers plan to leave a slope alongside the highway with an angle of inclination of 62°. Located 750 ft up the hill measured from the base is a tree containing the nest of an endangered hawk. Will this tree be removed in the excavation?



Solution

$\angle ACB = 180° - 62° = 118°$

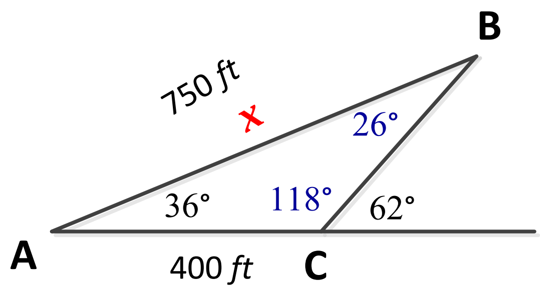
$\angle ABC = 180° - 118° - 36° = 26°$

Using the law of sines:

$$\frac{x}{\sin 118°} = \frac{400}{\sin 26°}$$

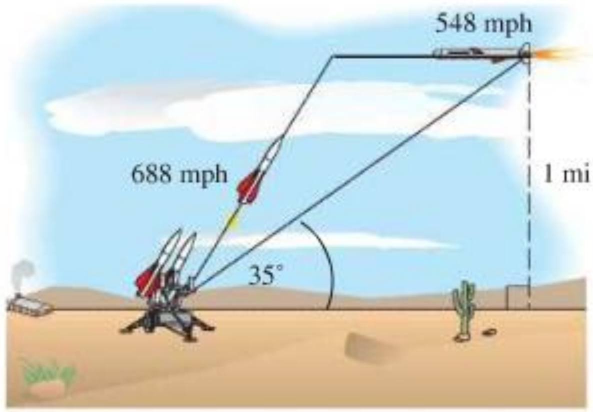
$$x = \frac{400 \sin 118°}{\sin 26°} \approx 805.7\ ft$$

Yes, the tree will have to be excavated.



Exercise

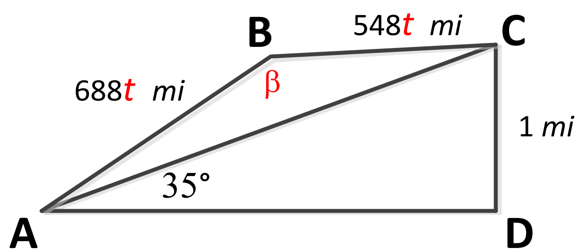
A cruise missile is traveling straight across the desert at 548 mph at an altitude of 1 mile. A gunner spots the missile coming in his direction and fires a projectile at the missile when the angle of elevation of the missile is 35°. If the speed of the projectile is 688 mph, then for what angle of elevation of the gun will the projectile hit the missile?



Solution

$\angle ACB = 35°$

$\angle BAC = 180° - 35° - \beta$



After ***t*** seconds;

The cruise missile distance: $548\dfrac{t}{3600}$ *miles*

The Projectile distance: $688\dfrac{t}{3600}$ *miles*

Using the law of sines:

$$\frac{\frac{548t}{3600}}{\sin(145° - \beta)} = \frac{\frac{688t}{3600}}{\sin 35°}$$

$$\frac{548t}{3600}\sin 35° = \frac{688t}{3600}\sin(145° - \beta)$$

$$548\sin 35° = 688\sin(145° - \beta)$$

$$\sin(145° - \beta) = \frac{548}{688}\sin 35°$$

$$145° - \beta = \sin^{-1}\left(\frac{548}{688}\sin 35°\right)$$

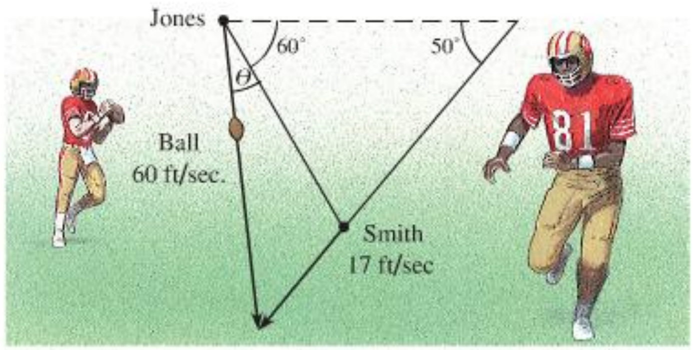
$$\boxed{\beta = 145° - \sin^{-1}\left(\frac{548}{688}\sin 35°\right) \approx 117.8°}$$

$$\Rightarrow \angle BAC = 180° - 35° - 117.8° = 27.2°$$

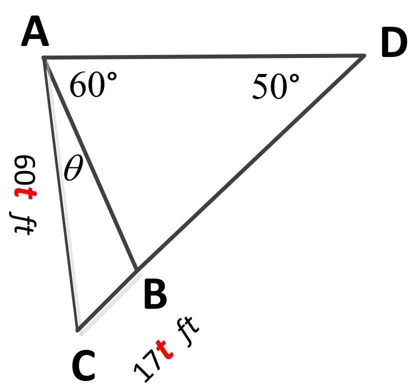
The angle of elevation of the projectile must be $(= 35° + 27.2°)$ $62.2°$

Exercise

When the ball is snapped, Smith starts running at a 50° angle to the line of scrimmage. At the moment when Smith is at a 60° angle from Jones, Smith is running at 17 ft/sec and Jones passes the ball at 60 ft/sec to Smith. However, to complete the pass, Jones must lead Smith by the angle θ. Find θ (find θ in his head. Note that θ can be found without knowing any distances.)



Solution



$$\angle ABD = 180° - 60° - 50° = 70°$$

$$\angle ABC = 180° - 70° = 110°$$

Using the law of sines:

$$\frac{17t}{\sin\theta} = \frac{60t}{\sin 110°}$$

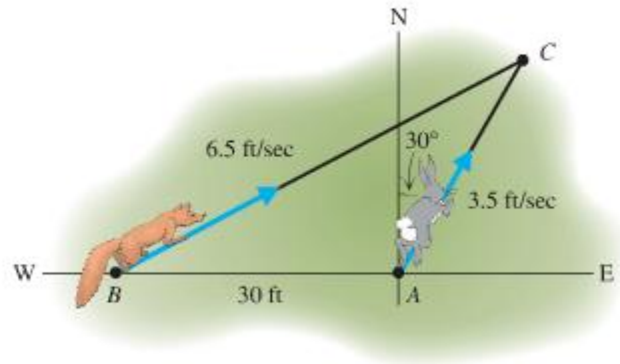
$$\frac{17}{\sin\theta} = \frac{60}{\sin 110°}$$

$$\sin\theta = \frac{17\sin 110°}{60}$$

$$\theta = \sin^{-1}\left(\frac{17\sin 110°}{60}\right) = 15.4°$$

Exercise

A rabbit starts running from point A in a straight line in the direction 30° from the north at 3.5 *ft/sec*. At the same time a fox starts running in a straight line from a position 30 *ft* to the west of the rabbit 6.5 *ft/sec*. The fox chooses his path so that he will catch the rabbit at point C. In how many seconds will the fox catch the rabbit?



Solution

$\angle BAC = 90° + 30° = 120°$

$\dfrac{6.5t}{\sin 120°} = \dfrac{3.5t}{\sin B}$

$\dfrac{6.5}{\sin 120°} = \dfrac{3.5}{\sin B}$

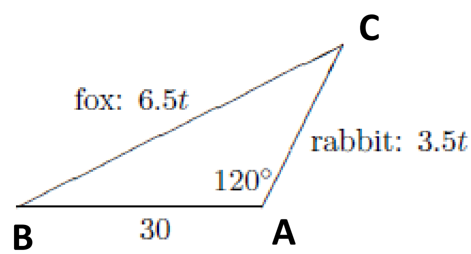
$\sin B = \dfrac{3.5 \sin 120°}{6.5}$

$B = \sin^{-1}\left(\dfrac{3.5 \sin 120°}{6.5}\right) \approx 28°$

$C = 180° - 120° - 28° = 32°$

$\dfrac{3.5t}{\sin 28} = \dfrac{30}{\sin 32°}$

$\underline{\left| t = \dfrac{30 \sin 28}{3.5 \sin 32°} \approx 7.6 \text{ sec} \right|}$

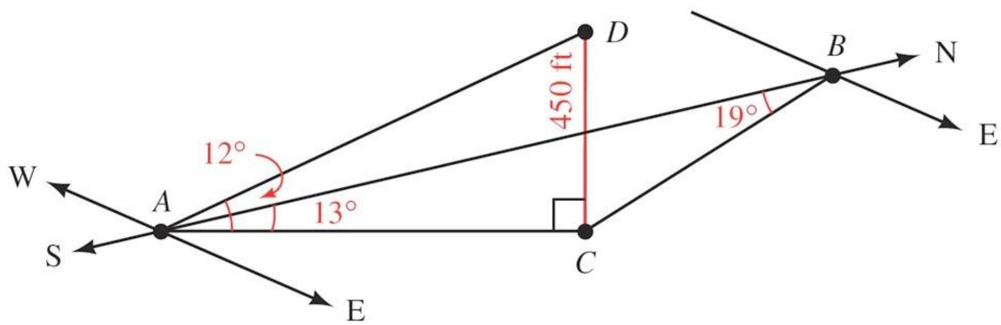


It will take 7.6 sec. to catch the rabbit.

Exercise

A hot-air balloon is flying over a dry lake when the wind stops blowing. The balloon comes to a stop 450 feet above the ground at point *D*. A jeep following the balloon runs out of gas at point *A*. The nearest service station is due north of the jeep at point *B*. The bearing of the balloon from the jeep at *A* is N 13° E, while the bearing of the balloon from the service station at *B* is S 19° E. If the angle of elevation of the balloon from *A* is 12°, how far will the people in the jeep have to walk to reach the service station at point *B*?

Solution



$$\tan 12° = \frac{DC}{AC}$$

$$AC = \frac{DC}{\tan 12°}$$

$$= \frac{450}{\tan 12°}$$

$$= 2{,}117\ ft$$

$$\angle ACB = 180° - (13° + 19°)$$

$$= 148°$$

Using triangle *ABC*

$$\frac{AB}{\sin 148°} = \frac{2117}{\sin 19°}$$

$$AB = \frac{2117 \sin 148°}{\sin 19°}$$

$$\boxed{= 3{,}400\ ft}$$

Solution Section 4.2 - Law of Cosines

Exercise

If $a = 13\ yd$, $b = 14\ yd$, and $c = 15\ yd$, find the largest angle.

Solution

$$C = \cos^{-1}\frac{a^2+b^2-c^2}{2ab}$$
$$= \cos^{-1}\left(\frac{13^2+14^2-15^2}{2(13)(14)}\right)$$
$$\approx 67^\circ$$

Exercise

Solve triangle ABC if $b = 63.4$ km, and $c = 75.2$ km, $A = 124^\circ\ 40'$

Solution

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$= (63.4)^2 + (75.2)^2 - 2(63.4)(75.2)\cos\left(124^\circ + \frac{40^\circ}{60}\right)$$
$$\approx 15098$$

$a \approx 122.9\ km$

$$\sin B = \frac{b\sin A}{a}$$
$$= \frac{63.4\sin 124.67^\circ}{122.9}$$

$$B = \sin^{-1}\left(\frac{63.4\sin 124.67^\circ}{122.9}\right)$$
$$\approx 25.1^\circ$$

$$C = 180^\circ - A - B$$
$$= 180^\circ - 124.67^\circ - 25.1^\circ$$
$$\approx 30.23^\circ$$

Exercise

Solve triangle *ABC* if a = 832 ft, b = 623 ft, and c = 345 ft

Solution

$$\underline{C} = \cos^{-1}\frac{a^2+b^2-c^2}{2ab}$$

$$= \cos^{-1}\left(\frac{832^2+623^2-345^2}{2(832)(623)}\right)$$

$$\underline{\approx 22^\circ}$$

$$\sin B = \frac{b\sin C}{c}$$

$$= \frac{623\sin 22^\circ}{345}$$

$$\underline{B} = \sin^{-1}\left(\frac{623\sin 22^\circ}{345}\right)$$

$$\underline{\approx 43^\circ}$$

$$\underline{A} = 180^\circ - 22^\circ - 43^\circ \ \underline{=115^\circ}$$

Exercise

Solve triangle *ABC* if $A = 42.3^\circ,\ b = 12.9m,\ and\ c = 15.4m$

Solution

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$= 12.9^2 + 15.4^2 - 2(12.9)(15.4)\cos 42.3^\circ$$

$$\approx 109.7$$

$$\boxed{a = 10.47\ m}$$

$$\sin B = \frac{b\sin A}{a} = \frac{12.9\sin 42.3^\circ}{10.47}$$

$$\underline{B} = \sin^{-1}\left(\frac{12.9\sin 42.3^\circ}{10.47}\right) \ \underline{\approx 56.0^\circ}$$

$$\underline{C} = 180^\circ - 42.3^\circ - 56^\circ \ \underline{=81.7^\circ}$$

Exercise

Solve triangle ABC if $a = 9.47\,ft,\ b = 15.9\,ft,\ and\ c = 21.1\,ft$

Solution

$$\underline{|C} = \cos^{-1}\frac{a^2+b^2-c^2}{2ab}$$

$$= \cos^{-1}\left(\frac{9.47^2+15.9^2-21.1^2}{2(9.47)(15.9)}\right)$$

$$\approx 109.9^\circ$$

$$\sin B = \frac{b\sin C}{c}$$

$$= \frac{15.9\sin 109.9^\circ}{21.1}$$

$$\underline{|B} = \sin^{-1}\left(\frac{15.9\sin 109.9^\circ}{21.1}\right)$$

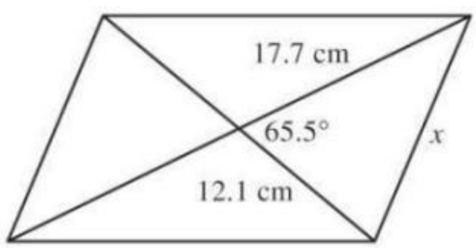
$$\approx 25.0^\circ$$

$$\underline{|A} = 180^\circ - 25^\circ - 109.9^\circ = 45.1^\circ$$

Exercise

The diagonals of a parallelogram are 24.2 cm and 35.4 cm and intersect at an angle of 65.5°. Find the length of the shorter side of the parallelogram

Solution



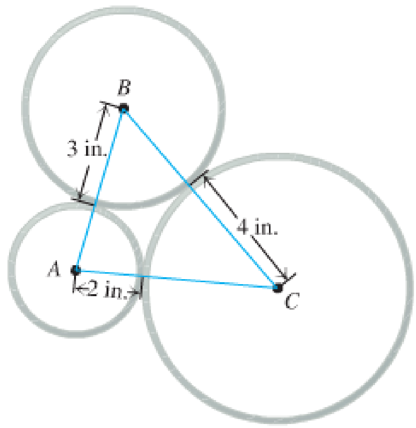
$$x^2 = 17.7^2 + 12.1^2 - 2(17.7)(12.1)\cos 65.5^\circ$$

$$= 282.07$$

$$\boxed{x = 16.8\ cm}$$

Exercise

An engineer wants to position three pipes at the vertices of a triangle. If the pipes A, B, and C have radii 2 in, 3 in, and 4 in, respectively, then what are the measures of the angles of the triangle ABC?



Solution

$$AC = 6 \quad AB = 5 \quad BC = 7$$

$$\underline{A} = \cos^{-1}\left(\frac{5^2+6^2-7^2}{2(5)(6)}\right) \underline{\approx 78.5^\circ}$$

$$\frac{6}{\sin B} = \frac{7}{\sin 78.5^\circ}$$

$$\sin B = \frac{6\sin 78.5^\circ}{7}$$

$$\underline{B} = \sin^{-1}\left(\frac{6\sin 78.5^\circ}{7}\right) \underline{\approx 57.1^\circ}$$

$$\underline{C} = 180^\circ - 78.5^\circ - 57.1^\circ \underline{= 44.4^\circ}$$

Exercise

Andrea and Steve left the airport at the same time. Andrea flew at 180 mph on a course with bearing 80°, and Steve flew at 240 mph on a course with bearing 210°. How far apart were they after 3 hr.?

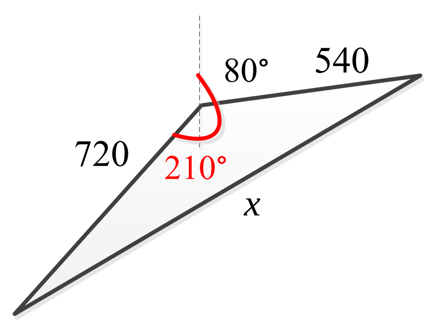
Solution

After 3 hrs., Steve flew: 3(240) = 720 mph
Andrea flew: 3(180) = 540 mph

$$x^2 = 720^2 + 540^2 - 2(720)(540)\cos 210^\circ$$

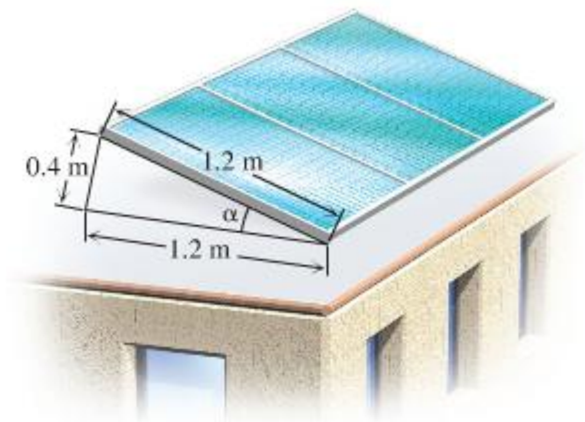
$$\underline{x} = \sqrt{720^2 + 540^2 - 2(720)(540)\cos 210^\circ}$$

$$\underline{\approx 1144.5 \text{ miles}}$$



Exercise

A solar panel with a width of 1.2 m is positioned on a flat roof. What is the angle of elevation α of the solar panel?



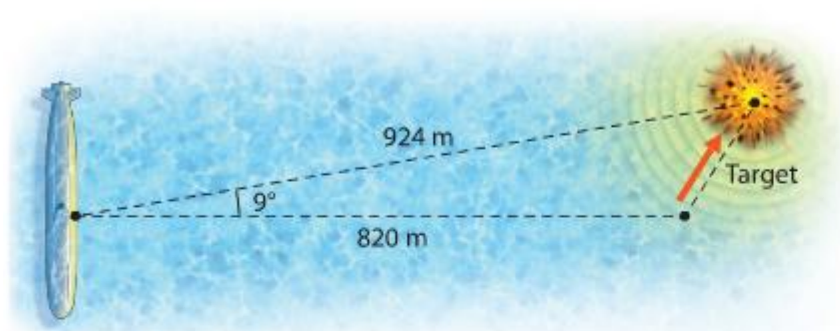
Solution

$$\alpha = \cos^{-1}\frac{1.2^2 + 1.2^2 - 0.4^2}{2(1.2)(1.2)} \approx 19.2^\circ$$

or $\quad \alpha = \frac{s}{r} = \frac{0.4}{1.2}$

Exercise

A submarine sights a moving target at a distance of 820 m. A torpedo is fired 9° ahead of the target, and travels 924 m in a straight line to hit the target. How far has the target moved from the time the torpedo is fired to the time of the hit?

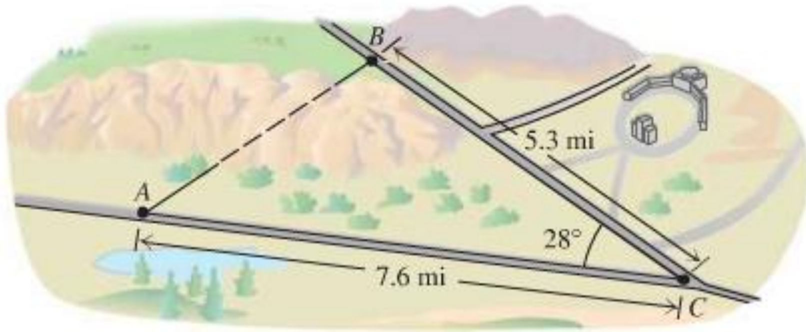


Solution

$$x = \sqrt{820^2 + 924^2 - 2(820)(924)\cos 9^\circ} \approx 171.7 \text{ m}$$

Exercise

A tunnel is planned through a mountain to connect points A and B on two existing roads. If the angle between the roads at point C is 28°, what is the distance from point A to B? Find $\angle CBA$ and $\angle CAB$ to the nearest tenth of a degree.



Solution

By the cosine law,

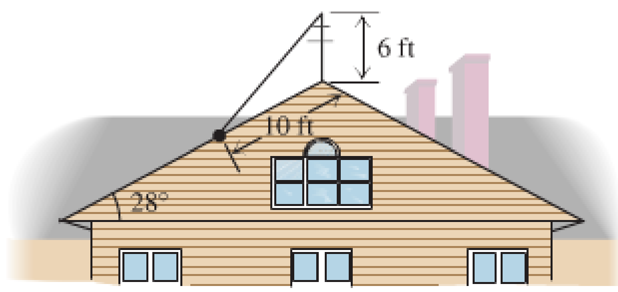
$$\underline{AB} = \sqrt{5.3^2 + 7.6^2 - 2(5.3)(7.6)\cos 28^\circ} \approx 3.8$$

$$\underline{\angle CBA} = \cos^{-1}\frac{3.8^2 + 5.3^2 - 7.6^2}{2(3.8)(5.3)} \approx 112^\circ$$

$$\underline{\angle BAC} = 180^\circ - 112^\circ - 28^\circ \approx 40^\circ$$

Exercise

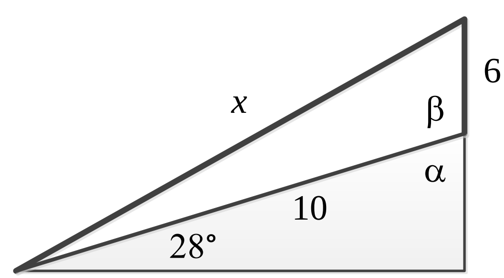
A 6-ft antenna is installed at the top of a roof. A guy wire is to be attached to the top of the antenna and to a point 10 ft down the roof. If the angle of elevation of the roof is 28°, then what length guy wire is needed?



Solution

$\alpha = 90^\circ - 28^\circ = 62^\circ$

$\beta = 180^\circ - 62^\circ = 118^\circ$

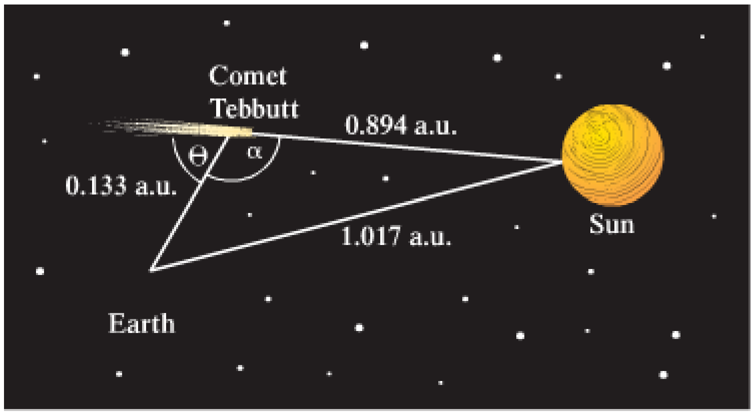


By the cosine law,

$$\underline{x} = \sqrt{6^2 + 100^2 - 2(6)(10)\cos 118^\circ} \approx 13.9\ ft$$

Exercise

On June 30, 1861, Comet Tebutt, one of the greatest comets, was visible even before sunset. One of the factors that causes a comet to be extra bright is a small scattering angle θ. When Comet Tebutt was at its brightest, it was 0.133 a.u. from the earth, 0.894 a.u. from the sun, and the earth was 1.017 a.u. from the sun. Find the phase angle α and the scattering angle θ for Comet Tebutt on June 30, 1861. (One astronomical unit (a.u) is the average distance between the earth and the sub.)



Solution

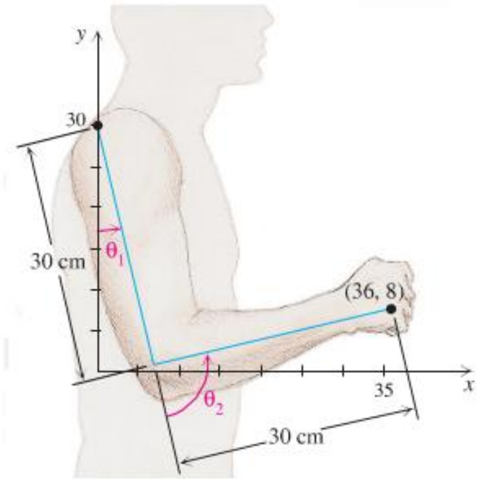
By the cosine law:

$$\alpha = \cos^{-1} \frac{0.133^2 + 0.894^2 - 1.017^2}{2(0.133)(0.891)} \approx 156^\circ$$

$$\theta = 180^\circ - \alpha = 180^\circ - 156^\circ = 24^\circ$$

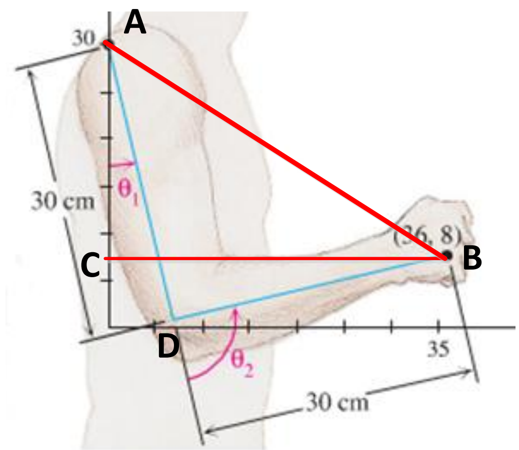
Exercise

A human arm consists of an upper arm of 30 cm and a lower arm of 30 cm. To move the hand to the point (36, 8), the human brain chooses angle θ_1 *and* θ_2 to the nearest tenth of a degree.



Solution

$AC = 30 - 8 = 22 \qquad\qquad BC = 36$



$$\begin{aligned} AB &= \sqrt{AC^2 + CB^2} \\ &= \sqrt{22^2 + 36^2} \\ &\approx 42.19 \end{aligned}$$

By the cosine law:

$$\angle ADB = \cos^{-1} \frac{AD^2 + DB^2 - AB^2}{2(AD)(DB)}$$

$$\angle ADB = \cos^{-1} \frac{30^2 + 30^2 - 42.19^2}{2(30)(30)} \quad \angle ADB \approx 89.4^\circ$$

$$\theta_2 = 180^\circ - 89.4^\circ = 90.6^\circ$$

$$\tan(\angle CAB) = \frac{BC}{AC} = \frac{36}{22}$$

$$\Rightarrow \angle CAB = \tan^{-1} \frac{36}{22} \approx 58.57^\circ$$

$$\frac{\sin DAB}{30} = \frac{\sin 89.4^\circ}{42.19} \Rightarrow \sin DAB = \frac{30 \sin 89.4^\circ}{42.19}$$

$$\angle DAB = \sin^{-1} \frac{30 \sin 89.4^\circ}{42.19} \approx 45.32^\circ$$

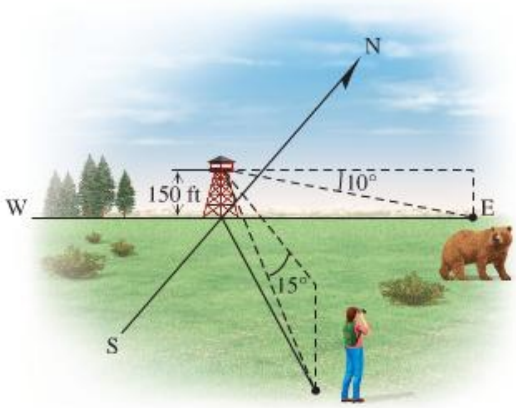
$$\theta_1 = \angle CAB - \angle DAB$$

$= 58.57° - 45.32°$

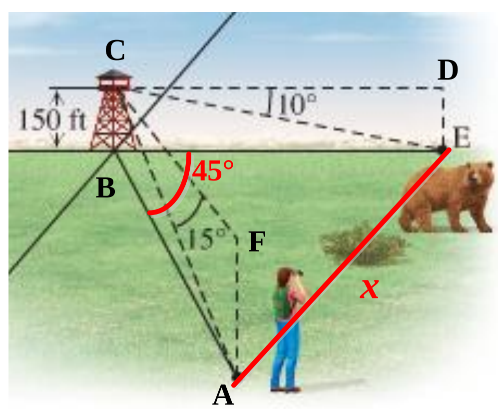
$\underline{= 13.25°}$

Exercise

A forest ranger is 150 ft above the ground in a fire tower when she spots an angry grizzly bear east of the tower with an angle of depression of 10°. Southeast of the tower she spots a hiker with an angle of depression of 15°. Find the distance between the hiker and the angry bear.



Solution



$\angle BEC = \angle ECD = 10°$

From triangle EBC:

$\tan 10° = \dfrac{150}{BE}$

$\Rightarrow BE = \dfrac{150}{\tan 10°} \approx 850.692$

$\angle BAC = \angle ACF = 15°$

From triangle ABC:

$\tan 15° = \dfrac{150}{AB}$

$\Rightarrow AB = \dfrac{150}{\tan 15°} \approx 559.808$

$x = \sqrt{AB^2 + BE^2 - 2(AB)(BE)\cos 45°}$

$= \sqrt{559.808^2 + 850.692^2 - 2(559.808)(850.692)\cos 45°}$

$\underline{\approx 603\, ft}$

Solution Section 4.3 – Vectors and Dot Product

Exercise

An arrow is shot into the air so that its horizontal velocity is 25 feet per second and its vertical is 15 feet per second. Find the velocity of the arrow.

Solution

The magnitude of the velocity is:

$$|V| = \sqrt{25^2 + 15^2}$$
$$= 29 \ ft/\sec$$

$$\tan\theta = \frac{|V_y|}{|V_x|} = \frac{15}{25} = 0.6$$

$$\underline{|\theta} = \tan^{-1} 0.6 \underline{\approx 31^\circ|}$$

Exercise

A boat travels 72 miles on a course of bearing N 27° E and then changes its course to travel 37 miles at N 55° E. How far north and how far east has the boat traveled on this 109-mile trip?

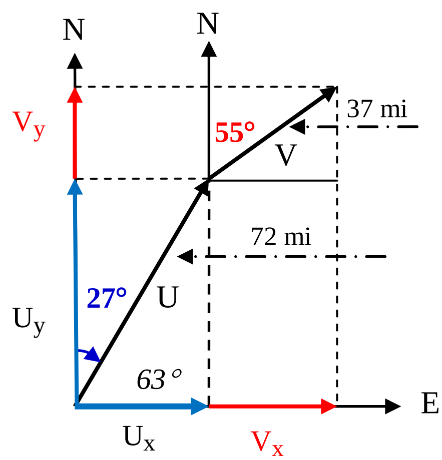
Solution

Total distance traveled east $= |U_x| + |V_x|$

$$= 72\cos 63^\circ + 37\cos 35^\circ$$
$$= 63 \ mi$$

Total distance traveled North $= |U_y| + |V_y|$

$$= 72\sin 63^\circ + 37\sin 35^\circ$$
$$= 85 \ mi$$



Exercise

Let **u** = $\langle -2, 1\rangle$ and **v** = $\langle 4, 3\rangle$. Find the following.

a) $\mathbf{u}+\mathbf{v}$
b) $-2\mathbf{u}$
c) $4\mathbf{u}-3\mathbf{v}$

Solution

a) $\mathbf{u}+\mathbf{v} = \langle -2, 1\rangle + \langle 4, 3\rangle$

$= \langle 2, 4\rangle$

b) $-2\mathbf{u} = -2\langle -2, 1\rangle$

$= \langle 4, -2\rangle$

c) $4\mathbf{u}-3\mathbf{v} = 4\langle -2, 1\rangle - 3\langle 4, 3\rangle$

$= \langle -8, 4\rangle - \langle 12, 9\rangle$

$= \langle -20, -5\rangle$

Exercise

Given: $|V| = 13.8,\ \theta = 24.2°$, find the magnitudes of the horizontal and vertical vector components of V, V_x *and* V_y, respectively

Solution

$$\left|V_x\right| = |V|\cos\theta = 13.8\cos 24.2° \approx 12.6$$

$$\left|V_y\right| = |V|\sin\theta = 13.8\sin 24.2° \approx 5.7$$

Exercise

Find the angle θ between the two vectors **u** = $\langle 3, 4\rangle$ and **v** = $\langle 2, 1\rangle$.

Solution

$$\theta = \cos^{-1}\frac{u.v}{|u|.|v|}$$

$$= \cos^{-1}\left(\frac{(3)(2)+(4)(1)}{\sqrt{3^2+4^2}\sqrt{2^2+1^2}}\right)$$

$$\approx 26.57°$$

Exercise

A bullet is fired into the air with an initial velocity of 1,800 feet per second at an angle of 60° from the horizontal. Find the magnitude of the horizontal and vertical vector component as of the velocity vector.

Solution

$$\left|V_x\right| = 1800\cos 60° = 900\ ft/\sec$$

$$\left|V_y\right| = 1800\sin 60° = 1600\ ft/\sec$$

Exercise

A bullet is fired into the air with an initial velocity of 1,200 feet per second at an angle of 45° from the horizontal.

a) Find the magnitude of the horizontal and vertical vector component as of the velocity vector.

b) Find the horizontal distance traveled by the bullet in 3 seconds. (Neglect the resistance of air on the bullet).

Solution

a) $\left|V_x\right| = 1200\cos 45° = 848.53 \approx 850\ ft/\sec$

$\left|V_y\right| = 1200\sin 45° \approx 850\ ft/\sec$

b) $x = 850*3 = 2{,}550\ ft$ $\quad x = \left|V_x\right|.t = 850*3 = 2{,}550\ ft$

Exercise

A ship travels 130 km on a bearing of S 42° E. How far east and how far south has it traveled?

Solution

$$V_{east} = 130\cos 42° = 96.6\ km$$

$$V_{south} = 130\sin 42° \approx 87\ km$$

Exercise

An arrow is shot into the air with so that its horizontal velocity is 15.0 ft./sec and its vertical velocity is 25.0 ft./sec. Find the velocity of the arrow?

Solution

$$|V| = \sqrt{V_x^2 + V_y^2}$$

$$V = \sqrt{15^2 + 25^2}$$

$$V \approx 29.2\, ft/\sec$$

$$\tan\theta = \frac{25}{15}$$

$$\theta = \tan^{-1}\left(\frac{25}{15}\right)$$

$$= 60°$$

Exercise

A plane travels 170 miles on a bearing of N 18 E and then changes its course to N 49 E and travels another 120 miles. Find the total distance traveled north and the total distance traveled east.

Solution

Total distance traveled east $= |U_x| + |V_x|$

$$= 170\cos 72° + 120\cos 41°$$

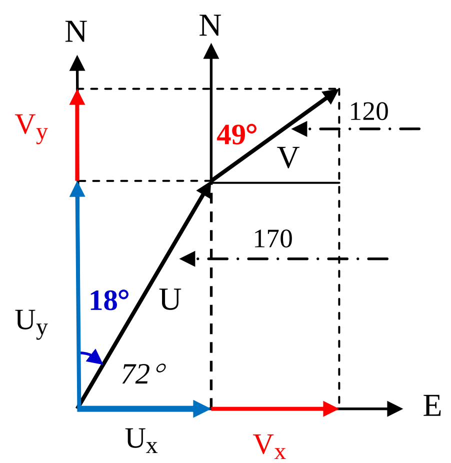
$$= 143.1\ mi$$

$$= 140\ mi\text{ - }East$$

Total distance traveled North $= |U_y| + |V_y|$

$$= 170\sin 72° + 120\sin 41°$$

$$= 240\ mi\text{ - }South$$

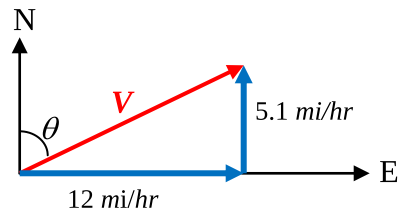


Exercise

A boat is crossing a river that run due north. The boat is pointed due east and is moving through the water at 12 miles per hour. If the current of the river is a constant 5.1 miles per hour, find the actual course of the boat through the water to two significant digits.

Solution

$$\tan\theta = \frac{12}{5.1}$$
$$= 2.3529$$



$$\theta = \tan^{-1} 2.3529$$
$$= 67°$$

$$|V| = \sqrt{12^2 + 5.1^2} \qquad \textbf{(\textit{using Pythagorean Theorem})}$$
$$= 13$$

Exercise

Two forces of 15 and 22 Newtons act on a point in the plane. (A **Newton** is a unit of force that equals .225 lb.) If the angle between the forces is 100°, find the magnitude of the resultant vector.

Solution

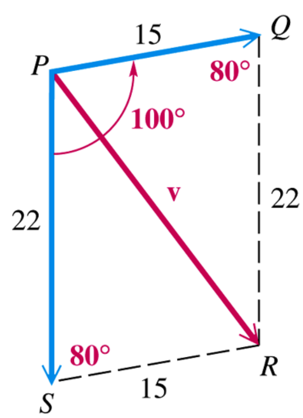
$$P + Q + R + S = 360°$$

$$2P + 2Q = 360°$$

$$2Q = 360° - 2(100°)$$

$$2Q = 160°$$

$$\boxed{Q = 80°}$$



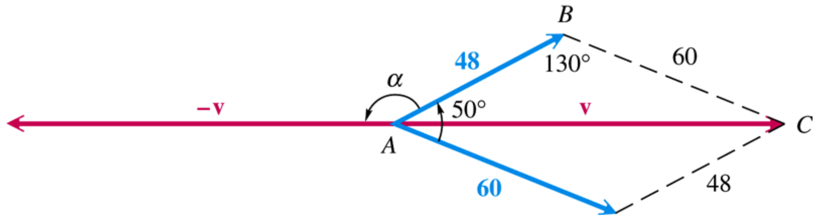
$$|V|^2 = 15^2 + 22^2 - 2(15)(22)\cos 80°$$
$$\approx 594.39$$

$$\boxed{|V| \approx 24.4\ N}$$

Exercise

Find the magnitude of the equilibrant of forces of 48 N and 60 N acting on a point *A*, if the angle between the forces is 50°. Then find the angle between the equilibrant and the 48-newton force.

Solution



The equilibrant is $-\mathbf{v}$.

$$2A + 2B = 360°$$
$$2B = 360° - 2(50°)$$
$$2B = 260°$$
$$B = 130°$$

$$|V|^2 = 48^2 + 60^2 - 2(48)(60)\cos 130°$$

$$\boxed{|V| \approx 98\ N}$$

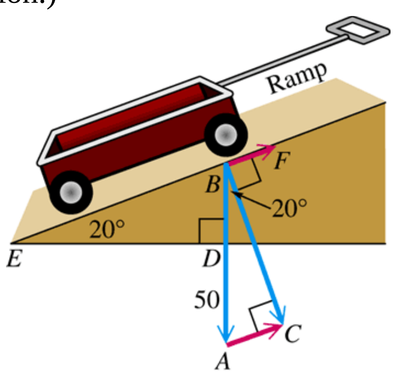
$$\frac{\sin BAC}{60} = \frac{\sin 130°}{98} \quad \Rightarrow \sin BAC = \frac{60 \sin 130°}{98}$$

$$\angle BAC = \sin^{-1}\left(\frac{60 \sin 130°}{98}\right) \approx 28°$$

$$\alpha = 180° - 28° = 152°$$

Exercise

Find the force required to keep a 50-lb wagon from sliding down a ramp inclined at 20° to the horizontal. (Assume there is no friction.)



Solution

Vector **BC** represents the force with which the weight pushes against the ramp.
Vector **BF** represents the force that would pull the weight up the ramp.
The vertical force **BA** represents the force of gravity.

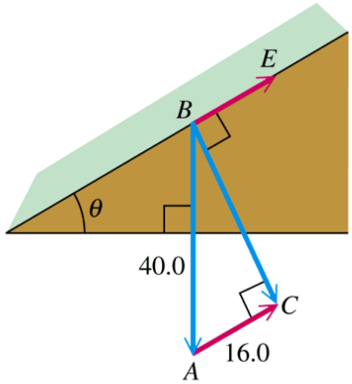
$$\sin 20° = \frac{|AC|}{50} \quad \Rightarrow \quad |AC| = 50 \sin 20° \approx 17$$

A force of approximately 17 lbs. will keep the wagon from sliding down the ramp.

Exercise

A force of 16.0 lbs. is required to hold a 40.0 lbs. lawn mower on an incline. What angle does the incline make with the horizontal?

Solution



Vector **BE** represents the force required to hold the mower on the incline.

$\sin B = \dfrac{16}{40}$

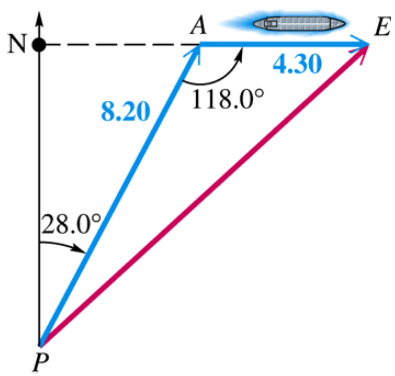
$\boxed{B = \sin^{-1}\left(\dfrac{16}{40}\right) \approx 23.6^\circ}$

The hill makes an angle of about 23.6° with the horizontal.

Exercise

A ship leaves port on a bearing of 28.0° and travels 8.20 mi. The ship then turns due east and travels 4.30 mi. How far is the ship from port? What is its bearing from port?

Solution



$\angle NAP = 90^\circ - 28^\circ = 62^\circ$

$\angle PAE = 180^\circ - 62^\circ = 118^\circ$

$$|PE|^2 = 8.20^2 + 4.30^2 - 2(8.2)(4.3)\cos 118^\circ$$
$$\approx 120.6$$

$|PE| = 10.9$

The ship is about 10.9 miles from port.

The bearing is given by:

$\dfrac{\sin APE}{4.3} = \dfrac{\sin 118}{10.9}$

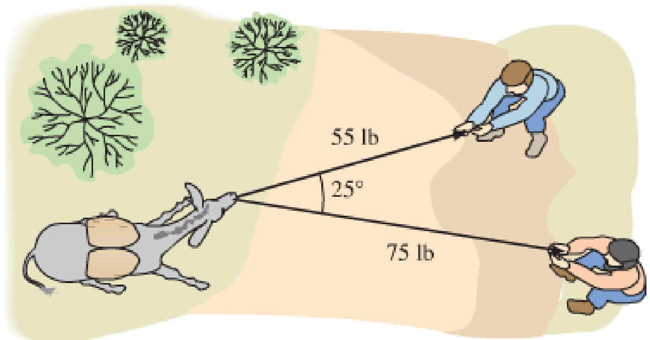
$\sin APE = \dfrac{4.3 \sin 118}{10.9}$

$\boxed{\angle APE = \sin^{-1}\left(\dfrac{4.3\sin 118}{10.9}\right) \approx 20.4^\circ}$

The bearing: $28^\circ + 20.4^\circ = 48.4^\circ$

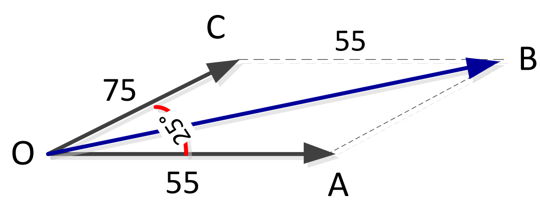
Exercise

Two prospectors are pulling on ropes attached around the neck of a donkey that does not want to move. One prospector pulls with a force of 55 lb, and the other pulls with a force of 75 lb. If the angle between the ropes is 25°, then how much force must the donkey use in order to stay put? (The donkey knows the proper direction in which to apply his force.)



Solution

$$\angle OCB = \angle OAB = \frac{1}{2}\left(360° - 2(25°)\right) = 155°$$



By the cosine law, the magnitude of the resultant force is:

$$OB = \sqrt{OC^2 + CB^2 - 2(OC)(OB)\cos(OCB)}$$

$$= \sqrt{75^2 + 55^2 - 2(75)(55)\cos(155°)}$$

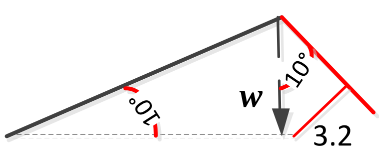
$$\approx 127\ lb$$

So the donkey must pull a force of 127 pounds in the direction opposite that of the resultant's.

Exercise

A solid steel ball is placed on a 10° incline. If a force of 3.2 lb in the direction of the incline is required to keep the ball in place, then what is the weight of the ball?

Solution

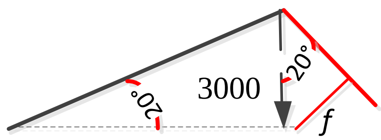


$$\sin 10° = \frac{f}{w} \to w = \frac{3.2}{\sin 10°} \approx 18.4\ lb$$

Exercise

Find the amount of force required for a winch to pull a 3000-lb car up a ramp that is inclined at 20°.

Solution

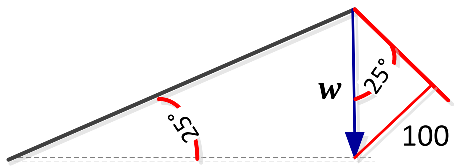


$$\sin 20° = \frac{f}{3000} \to f = 3000 \sin 20° \underline{\approx 1026.1\ lb}$$

Exercise

If the amount of force required to push a block of ice up an ice-covered driveway that is inclined at 25° is 100lb, then what is the weight of the block?

Solution

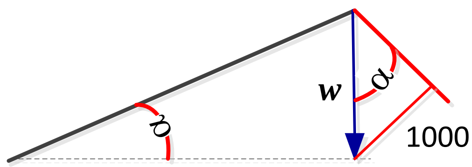


$$\sin 25° = \frac{100}{w} \to w = \frac{100}{\sin 25°} \underline{\approx 236.6\ lb}$$

Exercise

If superman exerts 1000 lb of force to prevent a 5000-lb boulder from rolling down a hill and crushing a bus full of children, then what is the angle of inclination of the hill?

Solution



$$\sin \alpha = \frac{1000}{5000} \to \alpha = \sin^{-1} \frac{1000}{5000} \underline{\approx 11.5°}$$

Exercise

If Sisyphus exerts a 500-lb force in rolling his 4000-lb spherical boulder uphill, then what is the angle of inclination of the hill?

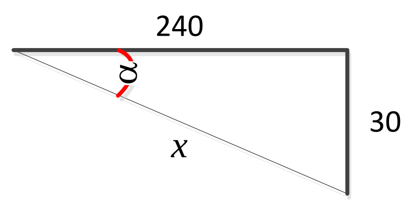
Solution

$$\sin\alpha = \frac{500}{4000} \rightarrow \alpha = \sin^{-1}\frac{500}{4000} \ \underline{\approx 7.2^\circ}$$

Exercise

A plane is headed due east with an air speed of 240 mph. The wind is from the north at 30 mph. Find the bearing for the course and the ground speed of the plane.

Solution



The bearing is: $\underline{\alpha} = \tan^{-1}\frac{30}{240} \underline{\approx 7.1^\circ}$

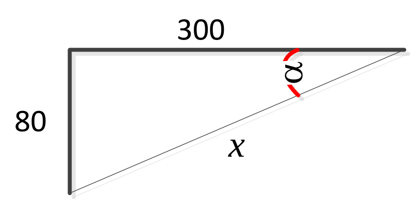
The ground speed can be found by using Pythagorean Theorem:

$$x = \sqrt{240^2 + 30^2} \ \underline{\approx 241.9\ mph}$$

Exercise

A plane is headed due west with an air speed of 300 mph. The wind is from the north at 80 mph. Find the bearing for the course and the ground speed of the plane.

Solution



$$\alpha = \tan^{-1}\frac{80}{300} \approx 14.9^\circ$$

The bearing is: $270^\circ - 14.9^\circ = 255.1^\circ$

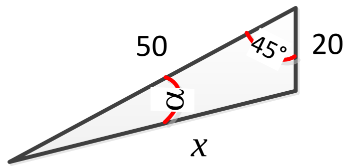
The ground speed can be found by using Pythagorean Theorem:

$$x = \sqrt{300^2 + 80^2} \ \underline{\approx 310.5\ mph}$$

Exercise

An ultralight is flying northeast at 50 mph. The wind is from the north at 20 mph. Find the bearing for the course and the ground speed of the ultralight.

Solution



Using the cosine law, the ground speed is:

$$x=\sqrt{50^2+20^2-2(50)(20)\cos 45^\circ}\ \boxed{\approx 38.5\ mph}$$

$$\frac{\sin\alpha}{20}=\frac{\sin 45^\circ}{38.5}\Rightarrow \sin\alpha=\frac{20\sin 45^\circ}{38.5}$$

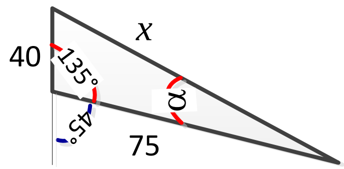
$$\alpha=\sin^{-1}\left(\frac{20\sin 45^\circ}{38.5}\right)\approx 21.5^\circ$$

The bearing is: $45^\circ+21.5^\circ=66.5^\circ$

Exercise

A superlight is flying northwest at 75 mph. The wind is from the south at 40 mph. Find the bearing for the course and the ground speed of the superlight.

Solution



Using the cosine law, the ground speed is:

$$x=\sqrt{75^2+40^2-2(75)(40)\cos 135^\circ}\ \boxed{\approx 107.1\ mph}$$

$$\frac{\sin\alpha}{40}=\frac{\sin 135^\circ}{107.1}\Rightarrow \sin\alpha=\frac{40\sin 135^\circ}{107.1}$$

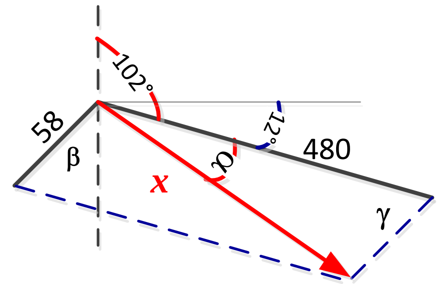
$$\alpha=\sin^{-1}\left(\frac{40\sin 135^\circ}{107.1}\right)\approx 15.3^\circ$$

The bearing is: $315^\circ+15.3^\circ=330.3^\circ$

Exercise

An airplane is heading on a bearing of 102° with an air speed of 480 mph. If the wind is out of the northeast (bearing 225°) at 58 mph, then what are the bearing of the course and the ground speed of the airplane?

Solution



x: the airplane's vector course and ground speed

$\beta = 225° - 180° = 45°$

From the parallelogram (from the vectors):

$\gamma = \frac{1}{2}\left(\left(90° - 12°\right) + 45°\right) = 57°$

By the cosine law:

$$x = \sqrt{480^2 + 58^2 - 2(480)(58)\cos 57°}$$
$$= 451\ mph$$

$$\frac{\sin\alpha}{58} = \frac{\sin 57°}{451}$$

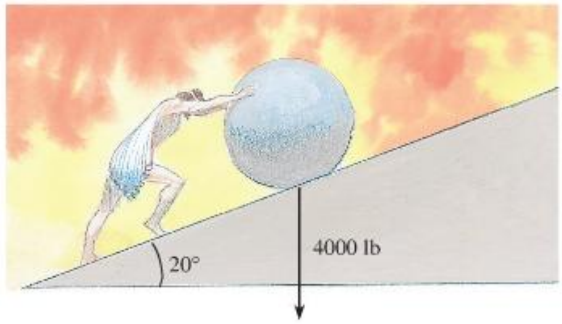
$$\sin\alpha = \frac{58\sin 57°}{451}$$

$$\alpha = \sin^{-1}\left(\frac{58\sin 57°}{451}\right) \approx 6.2°$$

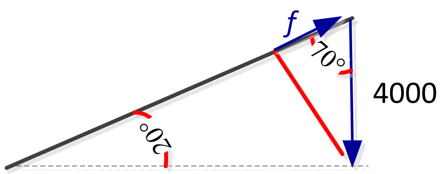
The bearing of the plane: $102° + 6.2° = 108.2°$

Exercise

In Roman mythology, Sisyphus revealed a secret of Zeus and thus incurred the god's wrath. As punishment, Zeus banished him to Hades, where he was doomed for eternity to roll a rock uphill, only to have it roll back on him. If Sisyphus stands in front of a 4000-lb spherical rock on a 20° incline, then what force applied in the direction of the incline would keep the rock from rolling down the incline?



Solution



$$\cos 70° = \frac{f}{4000}$$

$$f = 4000\cos 70° \approx 1368\ lb$$

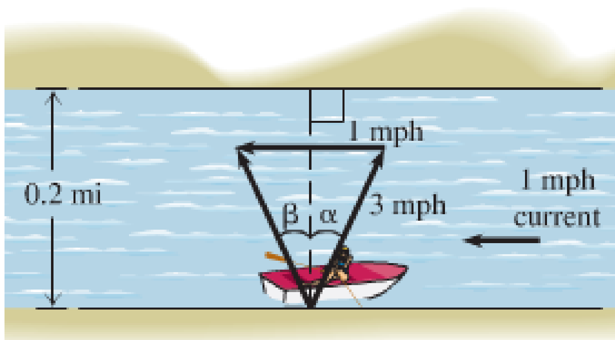
Exercise

A trigonometry student wants to cross a river that is 0.2 mi wide and has a current of 1 mph. The boat goes 3 mph in still water.

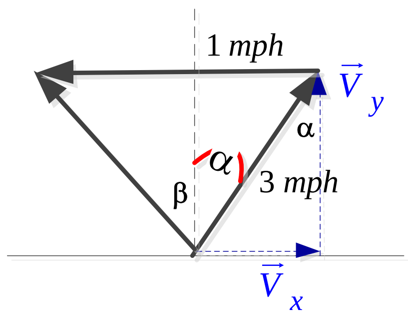
a) Write the distance the boats travels as a function of the angle β.

b) Write the actual speed of the boat as a function of α and β.

c) Write the time for the trip as a function of α. Find the angle α for which the student will cross the river in the shortest amount of time.



Solution



a) The vector in still water is given by:

$$V_x = V_S \cos\left(90° - \alpha\right) = 3\sin\alpha$$

$$V_y = 3\cos\alpha$$

$$\vec{V} = 3\sin\alpha\boldsymbol{i} + 3\cos\alpha\boldsymbol{j}$$

The vector of the current: $\vec{V}_c = -1\boldsymbol{i}$

Therefore, the actual direction and speed is determined by the vector:

$$v = \left(3\sin\alpha - 1\right)\boldsymbol{i} + 3\cos\alpha\boldsymbol{j}$$

The number ***t*** of hours it takes the boat to cross the river:

$$t = \frac{d}{v_y} = \frac{0.2}{3\cos\alpha}$$

The distance of the boat travels:

$$d = t\left|v\right|$$

$$= \frac{0.2}{3\cos\alpha}\sqrt{\left(3\sin\alpha - 1\right)^2 + \left(3\cos\alpha\right)^2}$$

$$= 0.2\sqrt{\frac{(3\sin\alpha - 1)^2}{(3\cos\alpha)^2} + \frac{(3\cos\alpha)^2}{(3\cos\alpha)^2}}$$

$$= 0.2\sqrt{\left(\frac{3\sin\alpha - 1}{3\cos\alpha}\right)^2 + 1} \qquad \tan\beta = \frac{v_x}{v_y} = \frac{3\sin\alpha - 1}{3\cos\alpha}$$

$$= 0.2\sqrt{(\tan\beta)^2 + 1}$$

$$= 0.2\sqrt{\tan^2\beta + 1} \qquad \sec^2\beta = \tan^2\beta + 1$$

$$= 0.2\sqrt{\sec^2\beta}$$

$$= 0.2|\sec\beta|$$

b) The actual speed $= \dfrac{d}{t}$

$$= \frac{0.2|\sec\beta|}{\frac{0.2}{3\cos\alpha}}$$

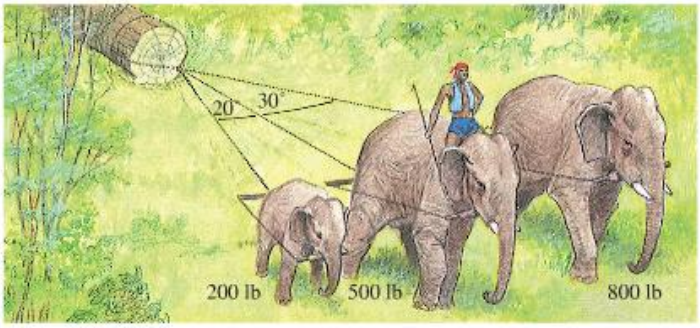
$$= 3\cos\alpha|\sec\beta|$$

c) The shortest time for the boat to cross river is straight ahead which $\alpha = 0$

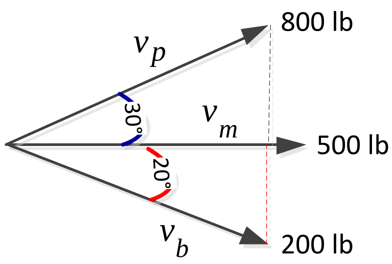
$$t = \frac{0.2}{3\cos 0°} = \frac{0.2}{3} = \frac{2}{30} \boxed{= \frac{1}{15}\text{ sec}}$$

Exercise

A male uses three elephants to pull a very large log out of the jungle. The papa elephant pulls with 800 lb. of force, the mama elephant pulls with 500 lb. of force, and the baby elephant pulls with 200 lb. force. The angles between the forces are shown in the figure. What is the magnitude of the resultant of all three forces? If mama is pulling due east, then in what direction will the log move?



Solution



$$v_p = 800\cos 30°\,\boldsymbol{i} + 800\sin 30°\,\boldsymbol{j}$$

$$v_m = 500\boldsymbol{i}$$

$$v_b = 200\cos 20°\,\boldsymbol{i} - 200\sin 20°\,\boldsymbol{j}$$

The total force is determined by:

$$\begin{aligned} F &= v_p + v_m + v_b \\ &= 800\cos 30°\,\boldsymbol{i} + 800\sin 30°\,\boldsymbol{j} + 500\boldsymbol{i} + 200\cos 20°\,\boldsymbol{i} - 200\sin 20°\,\boldsymbol{j} \\ &= (800\cos 30° + 500 + 200\cos 20°)\,\boldsymbol{i} + (800\sin 30° - 200\sin 20°)\,\boldsymbol{j} \\ &= 1380.76\boldsymbol{i} + 331.60\boldsymbol{j} \end{aligned}$$

The magnitude of the resultant:

$$|F| = \sqrt{1380.76^2 + 331.60^2} \ \underline{\approx 1420\ lbs}$$

The direction: $\tan^{-1}\left(\dfrac{331.6}{1380.76}\right) \ \underline{\approx 13.5°}$ $\quad \boxed{E13.5°N}$

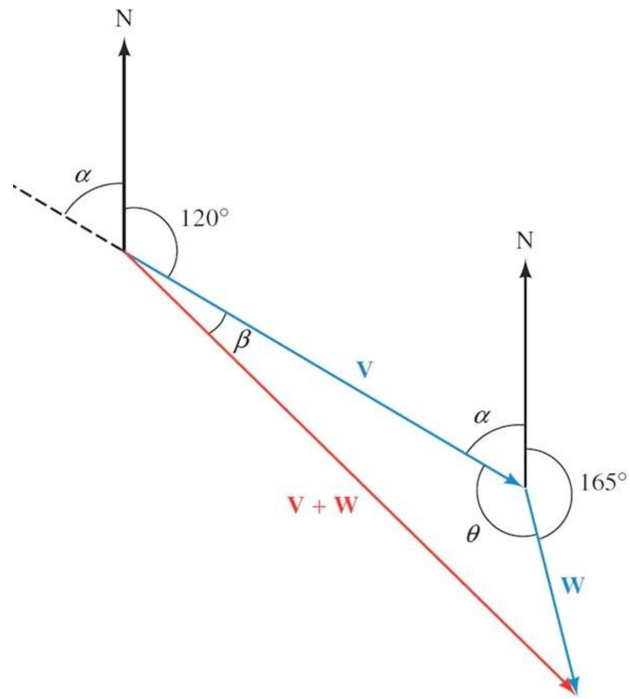
Exercise

A plane is flying with an airspeed of 185 miles per hour with heading 120°. The wind currents are running at a constant 32 miles per hour at 165° clockwise from due north. Find the true course and ground speed of the plane.

Solution

$$\begin{aligned}\alpha &= 180° - 120° \\ &= 60°\end{aligned}$$

$$\begin{aligned}\theta &= 360° - 165° - \alpha \\ &= 360° - 165° - 60° \\ &= 135°\end{aligned}$$



$$\begin{aligned}|V+W|^2 &= |V|^2 + |W|^2 - 2|V|\cdot|W|\cos\theta \\ &= 185^2 + 32^2 - 2(185)(32)\cos 135° \\ &= 43{,}621\end{aligned}$$

$$|V+W| = 210\ mph$$

$$\frac{\sin\beta}{32} = \frac{\sin\theta}{210}$$

$$\sin\beta = \frac{32\sin 135°}{210} = 0.1077$$

$$\beta = \sin^{-1}(0.1077) = 6°$$

The true course is:

$$120° + \beta = 120° + 6° \underline{= 126°}|.$$

The speed of the plane with respect to the ground is 210 mph.

Solution   Section 4.4 – Trigonometric Form of Complex Numbers

Exercise

Write $-\sqrt{3}+i$ in trigonometric form. (Use radian measure)

Solution

$$-\sqrt{3}+i \Rightarrow \begin{cases} x=-\sqrt{3} \\ y=1 \end{cases}$$

$$r=\sqrt{\left(-\sqrt{3}\right)^2+1^2}=2$$

$$\tan\theta=\frac{y}{x}=\frac{1}{-\sqrt{3}}=-\frac{\sqrt{3}}{3}$$

The reference angle for θ is $\frac{\pi}{6}$ and the angle is in quadrant II.

Therefore, $\theta=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$

$$\boxed{-\sqrt{3}+i=2\ cis\,\frac{5\pi}{6}}$$

Exercise

Write $3-4i$ in trigonometric form.

Solution

$$3-4i \Rightarrow \begin{cases} r=\sqrt{3^2+(-4)^2}=5 \\ \hat{\theta}=\tan^{-1}\left(\frac{4}{3}\right)\approx 53^\circ \end{cases}$$

The angle is in quadrant II; therefore, $\theta=180^\circ-53^\circ=127^\circ$

$$\boxed{3-4i=5\ cis127^\circ}$$

Exercise

Write $-21-20i$ in trigonometric form.

Solution

$$-21-20i \Rightarrow \begin{cases} r=\sqrt{(-21)^2+(-20)^2}=29 \\ \hat{\theta}=\tan^{-1}\left(\frac{20}{21}\right)\approx 43.6^\circ \end{cases}$$

The angle is in quadrant III; therefore, $\underline{\theta=}\ 180^\circ+43.6^\circ\ \underline{=223.6^\circ}$

$$\boxed{-21-20i=29\ cis223.6^\circ}$$

Exercise

Write $11+2i$ in trigonometric form.

Solution

$$11+2i \Rightarrow \begin{cases} r=\sqrt{11^2+2^2}=\sqrt{125}=5\sqrt{5} \\ \hat{\theta}=\tan^{-1}\left(\frac{2}{11}\right)\approx 10.3^\circ \end{cases}$$

The angle is in quadrant I; therefore, $\underline{\theta}\ \underline{=10.3^\circ}$

$$\boxed{11+2i=5\sqrt{5}\ cis10.3^\circ}$$

Exercise

Write $4(\cos 30^\circ + i\sin 30^\circ)$ in standard form.

Solution

$$\begin{aligned} 4(\cos 30^\circ+i\sin 30^\circ) &= 4\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right) \\ &= 2\sqrt{3}+2i \end{aligned}$$

Exercise

Write $\sqrt{2}\ cis\frac{7\pi}{4}$ in standard form.

Solution

$$\sqrt{2}\ cis\frac{7\pi}{4} = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$= 1 - i$$

Exercise

Find the quotient $\dfrac{20cis(75°)}{4cis(40°)}$. Write the result in rectangular form.

Solution

$$\frac{20cis(75°)}{4cis(40°)} = \frac{20}{4}cis(75° - 40°)$$

$$= 5cis(35°)$$

$$= 5(\cos 35° + i\sin 35°)$$

$$= 4.1 + 2.87i$$

Exercise

Divide $z_1 = 1 + i\sqrt{3}$ *by* $z_2 = \sqrt{3} + i$. Write the result in rectangular form.

Solution

$$\frac{z_1}{z_2} = \frac{1+i\sqrt{3}}{\sqrt{3}+i}$$

$$= \frac{1+i\sqrt{3}}{\sqrt{3}+i}\,\frac{\sqrt{3}-i}{\sqrt{3}-i}$$

$$= \frac{\sqrt{3} - i + 3i - \sqrt{3}\,i^2}{3+1}$$

$$= \frac{2\sqrt{3}+2i}{4}$$

$$= \frac{2\sqrt{3}}{4} + \frac{2i}{4}$$

$$= \frac{\sqrt{3}}{2} + \frac{i}{2}$$

$\boxed{or}$

$$1+i\sqrt{3}: \begin{cases} r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} \\ \theta = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3} \end{cases}$$

$$\sqrt{3}+i: \begin{cases} r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} \\ \theta = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6} \end{cases}$$

$$\frac{z_1}{z_2} = \frac{2cis\frac{\pi}{3}}{2cis\frac{\pi}{6}}$$

$$= \frac{2}{2}cis\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \frac{2}{2}cis\left(\frac{\pi}{6}\right)$$

$$= cis\left(\frac{\pi}{6}\right)$$

$$= \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$$

Solution *Section* 4.5 – Polar Coordinates

Exercise

Convert to rectangular coordinates $(2,\ 60°)$

Solution

$$(2,\ 60°) = 2(\cos 60° + i\sin 60°)$$

$$= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= 1 + i\sqrt{3}$$

Exercise

Convert to rectangular coordinates $\left(\sqrt{2},\ -225°\right)$

Solution

$$\left(\sqrt{2},\ -225°\right) = \sqrt{2}\left(\cos(-225°) + i\sin(-225°)\right)$$

$$= \sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$= -1 + i$$

Exercise

Convert to rectangular coordinates $\left(4\sqrt{3},\ -\frac{\pi}{6}\right)$

Solution

$$\left(4\sqrt{3},\ -\frac{\pi}{6}\right) = 4\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$= 4\sqrt{3}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$$

$$= 6 - 2\sqrt{3}$$

Exercise

Convert to polar coordinates $\qquad (-3,\ -3) \quad r \geq 0 \quad 0° \leq \theta < 360°$

Solution

$$(-3,\ -3) \to \begin{cases} r = \sqrt{(-3)^2 + (-3)^2} = 3\sqrt{2} \\ \hat{\theta} = \tan^{-1}\left(\frac{3}{3}\right) = \tan^{-1}(1) = 45° \end{cases}$$

The angle is in quadrant III; therefore, $\underline{\theta} = 180° + 45° \ \underline{= 225°}$

$$\boxed{(-3,\ -3) \ = \ \left(3\sqrt{2},\ 225°\right)}$$

Exercise

Convert to polar coordinates $\qquad \left(2,\ -2\sqrt{3}\right) \quad r \geq 0 \quad 0° \leq \theta < 360°$

Solution

$$\left(2,\ -2\sqrt{3}\right) \to \begin{cases} r = \sqrt{2^2 + \left(-2\sqrt{3}\right)^2} = 4 \\ \hat{\theta} = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \tan^{-1}\left(\sqrt{3}\right) = 60° \end{cases}$$

The angle is in quadrant IV; therefore, $\underline{\theta} = 360° - 60° \ \underline{= 300°}$

$$\boxed{\left(2,\ -2\sqrt{3}\right) \ = \ (4,\ 300°)}$$

Exercise

Convert to polar coordinates $\qquad (-2,\ 0) \quad r \geq 0 \quad 0 \leq \theta < 2\pi$

Solution

$$(-2,\ 0) \to \begin{cases} r = \sqrt{(-2)^2 + 0^2} = 2 \\ \hat{\theta} = \tan^{-1}\left(\frac{0}{2}\right) = 0 \Rightarrow \theta = \pi \end{cases}$$

$$\boxed{(-2,\ 0) \ = \ (2,\ \pi)}$$

Exercise

Convert to polar coordinates $\left(-1,\ -\sqrt{3}\right) \quad r \geq 0 \quad 0 \leq \theta < 2\pi$

Solution

$$\left(-1,\ -\sqrt{3}\right) \rightarrow \begin{cases} r = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = 2 \\ \hat{\theta} = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3} \end{cases}$$

The angle is in quadrant III; therefore, $\underline{|\theta} = \pi + \frac{\pi}{3} \ \underline{= \frac{4\pi}{3}|}$

$$\boxed{\left(-1,\ -\sqrt{3}\right) \ = \ \left(2,\ \frac{4\pi}{3}\right)}$$

Exercise

Write the equation in rectangular coordinates $r^2 = 4$

Solution

$$r^2 = 4$$

$$x^2 + y^2 = 4$$

Exercise

Write the equation in rectangular coordinates $r = 6\cos\theta$

Solution

$$r = 6\cos\theta$$

$$r = 6\frac{x}{r}$$

$$r^2 = 6x$$

$$x^2 + y^2 = 6x$$

Exercise

Write the equation in rectangular coordinates $\quad r^2 = 4\cos 2\theta$

Solution

$$r^2 = 4\left(\cos^2\theta - \sin^2\theta\right) \qquad \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r}$$

$$= 4\left(\left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2\right)$$

$$= 4\left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right)$$

$$= 4\left(\frac{x^2 - y^2}{r^2}\right)$$

$$r^4 = 4\left(x^2 - y^2\right) \qquad r^2 = x^2 + y^2$$

$$\left(x^2 + y^2\right)^4 = 4x^2 - 4y^2$$

Exercise

Write the equation in rectangular coordinates $\quad r\left(\cos\theta - \sin\theta\right) = 2$

Solution

$$r\left(\cos\theta - \sin\theta\right) = 2 \qquad \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r}$$

$$r\left(\frac{x}{r} - \frac{y}{r}\right) = 2$$

$$r\left(\frac{x-y}{r}\right) = 2$$

$$x - y = 2$$

Exercise

Write the equation in polar coordinates $x+y=5$

Solution

$$r\cos\theta + r\sin\theta = 5 \qquad\qquad x = r\cos\theta \quad y = r\sin\theta$$

$$r(\cos\theta + \sin\theta) = 5$$

$$r = \frac{5}{\cos\theta + \sin\theta}$$

Exercise

Write the equation in polar coordinates $x^2 + y^2 = 9$

Solution

$$x^2 + y^2 = 9 \qquad\qquad r^2 = x^2 + y^2$$

$$r^2 = 9$$

Exercise

Write the equation in polar coordinates $x^2 + y^2 = 4x$

Solution

$$r^2 = 4r\cos\theta$$

$$\frac{r^2}{r} = \frac{4r\cos\theta}{r}$$

$$r = 4\cos\theta$$

Exercise

Write the equation in polar coordinates $y = -x$

Solution

$$y = -x \qquad\qquad x = r\cos\theta \quad y = r\sin\theta$$

$$r\sin\theta = -r\cos\theta$$

$$\sin\theta = -\cos\theta$$

Solution   *Section* 4.6 - De Moivre's Theorem

Exercise

Find $(1+i)^8$ and express the result in rectangular form.

Solution

$$1+i \Rightarrow \begin{cases} r=\sqrt{1^2+1^2}=\sqrt{2} \\ \theta=\tan^{-1}1=\frac{\pi}{4} \end{cases}$$

$$1+i=\sqrt{2}\,cis\frac{\pi}{4}$$

$$\begin{aligned}(1+i)^8 &= \left(\sqrt{2}\,cis\frac{\pi}{4}\right)^8 \\ &= \left(\sqrt{2}\right)^8 cis\left[8\left(\frac{\pi}{4}\right)\right] \\ &= 16 cis 2\pi \\ &= 16(\cos 2\pi + i\sin 2\pi) \\ &= 16(1+i0) \\ &= 16\end{aligned}$$

Exercise

Find $(1+i)^{10}$ and express the result in rectangular form.

Solution

$$\begin{aligned}(1+i)^{10} &= \left(\sqrt{2}\,cis\frac{\pi}{4}\right)^{10} \\ &= \left(\sqrt{2}\right)^{10} cis\left[10\left(\frac{\pi}{4}\right)\right] \\ &= 32 cis\frac{5\pi}{2} \\ &= 32 cis\frac{\pi}{2} \\ &= 32\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right) \\ &= 32(0+i) \\ &= 32i\end{aligned}$$

Exercise

Find fifth roots of $z = 1 + i\sqrt{3}$ and express the result in rectangular form.

Solution

$$1+i\sqrt{3} \Rightarrow \begin{cases} r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2 \\ \theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60° \end{cases}$$

$$\begin{aligned} \left(1+i\sqrt{3}\right)^{1/5} &= \left(2cis60°\right)^{1/5} \\ &= \sqrt[5]{2}\left(cis\,\frac{60°}{5} + \frac{360°k}{5}\right) \\ &= \sqrt[5]{2}\ cis\left(12° + 72°k\right) \end{aligned}$$

$If\ \ k = 0 \Rightarrow \sqrt[5]{2}\ cis\left(12° + 72°0\right) = \underline{\sqrt[5]{2}\ cis12°}$

$If\ \ k = 1 \Rightarrow \sqrt[5]{2}\ cis\left(12° + 72°.(1)\right) = \underline{\sqrt[5]{2}\ cis84°}$

$If\ \ k = 2 \Rightarrow \sqrt[5]{2}\ cis\left(12° + 72°.(2)\right) = \underline{\sqrt[5]{2}\ cis156°}$

$If\ \ k = 3 \Rightarrow \sqrt[5]{2}\ cis\left(12° + 72°.(3)\right) = \underline{\sqrt[5]{2}\ cis228°}$

$If\ \ k = 4 \Rightarrow \sqrt[5]{2}\ cis\left(12° + 72°.(4)\right) = \underline{\sqrt[5]{2}\ cis300°}$

Exercise

Find the fourth roots of $z = 16cis60°$

Solution

$$\begin{aligned} \sqrt[4]{z} &= \sqrt[4]{16}\ cis\left(\frac{60°}{4} + \frac{360°}{4}k\right) \\ &= 2cis\left(15° + 90°k\right) \end{aligned}$$

$If\ \ k = 0 \Rightarrow 2\ cis\left(15° + 90°(0)\right) = \underline{2cis15°}$

$If\ \ k = 1 \Rightarrow 2\ cis\left(15° + 90°(1)\right) = \underline{2cis105°}$

$If\ \ k = 2 \Rightarrow 2\ cis\left(15° + 90°(2)\right) = \underline{2cis195°}$

$If\ \ k = 3 \Rightarrow 2\ cis\left(15° + 90°(3)\right) = \underline{2cis285°}$

Exercise

Find the cube roots of 27.

Solution

$$\sqrt[3]{27} = (27cis0°)^{1/3}$$
$$= \sqrt[3]{27}\ cis\left(\frac{0°}{3} + \frac{360°}{3}k\right)$$
$$= 3\ cis(0° + 120°k)$$

$$If\ \ k = 0 \Rightarrow z = 3\ cis(0° + 120°(0)) = \underline{2cis0°}$$

$$If\ \ k = 1 \Rightarrow z = 3\ cis(0° + 120°(1)) = \underline{2cis120°}$$

$$If\ \ k = 2 \Rightarrow z = 3\ cis(0° + 120°(2)) = \underline{2cis240°}$$

Exercise

Find all complex number solutions of $x^3 + 1 = 0$.

Solution

$$x^3 + 1 = 0 \Rightarrow x^3 = -1$$

$$-1 \Rightarrow \begin{cases} r = \sqrt{(-1)^2 + 0^2} = 1 \\ \theta = \tan^{-1}\left(\frac{0}{-1}\right) = \pi \end{cases}$$

$$x^3 = -1 = 1\ cis\pi$$
$$x = (1\ cis\pi)^{1/3}$$
$$= (1)^{1/3}\ cis\left(\frac{\pi}{3} + \frac{2\pi}{3}k\right)$$
$$= cis\left(\frac{\pi}{3} + \frac{2\pi}{3}k\right)$$

$$If\ \ k = 0 \Rightarrow x = cis\left(\frac{\pi}{3} + \frac{2\pi}{3}(0)\right) = \underline{cis\frac{\pi}{3}} \qquad x = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$If\ \ k = 1 \Rightarrow x = cis\left(\frac{\pi}{3} + \frac{2\pi}{3}(1)\right) = cis\left(\frac{3\pi}{3}\right) = \underline{cis\pi} \qquad x = \cos\pi + i\sin\pi \underline{= -1}$$

$$If\ \ k = 2 \Rightarrow x = cis\left(\frac{\pi}{3} + \frac{2\pi}{3}(2)\right) = \underline{cis\frac{5\pi}{3}} \qquad x = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Exercise

Find $\left(2cis30°\right)^5$

Solution

$$\left(2cis30°\right)^5 = 2^5 cis\left(5(30°)\right)$$

$$= 32\left(\cos 150° + i\sin 150°\right)$$

$$= 32\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$= -16\sqrt{3} + 16i$$