Solution   *Section* 4.6 - De Moivre's Theorem

Exercise

Find $(1+i)^8$ and express the result in rectangular form.

Solution

$$1+i \Rightarrow \begin{cases} r = \sqrt{1^2+1^2} = \sqrt{2} \\ \theta = \tan^{-1} 1 = \frac{\pi}{4} \end{cases}$$

$$1+i = \sqrt{2} cis \frac{\pi}{4}$$

$$\begin{aligned}(1+i)^8 &= \left(\sqrt{2} cis \frac{\pi}{4}\right)^8 \\ &= \left(\sqrt{2}\right)^8 cis\left[8\left(\frac{\pi}{4}\right)\right] \\ &= 16 cis 2\pi \\ &= 16(\cos 2\pi + i \sin 2\pi) \\ &= 16(1+i0) \\ &= 16\end{aligned}$$

Exercise

Find $(1+i)^{10}$ and express the result in rectangular form.

Solution

$$\begin{aligned}(1+i)^{10} &= \left(\sqrt{2} cis \frac{\pi}{4}\right)^{10} \\ &= \left(\sqrt{2}\right)^{10} cis\left[10\left(\frac{\pi}{4}\right)\right] \\ &= 32 cis \frac{5\pi}{2} \\ &= 32 cis \frac{\pi}{2} \\ &= 32\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) \\ &= 32(0+i) \\ &= 32i\end{aligned}$$

Exercise

Find fifth roots of $z = 1 + i\sqrt{3}$ and express the result in rectangular form.

Solution

$$1 + i\sqrt{3} \Rightarrow \begin{cases} r = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2 \\ \theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60^\circ \end{cases}$$

$$\begin{aligned} \left(1 + i\sqrt{3}\right)^{1/5} &= \left(2cis60^\circ\right)^{1/5} \\ &= \sqrt[5]{2}\left(cis\frac{60^\circ}{5} + \frac{360^\circ k}{5}\right) \\ &= \sqrt[5]{2}\ cis\left(12^\circ + 72^\circ k\right) \end{aligned}$$

$If\ \ k = 0 \Rightarrow \sqrt[5]{2}\ cis\left(12^\circ + 72^\circ 0\right) = \underline{\sqrt[5]{2}\ cis12^\circ}$

$If\ \ k = 1 \Rightarrow \sqrt[5]{2}\ cis\left(12^\circ + 72^\circ.(1)\right) = \underline{\sqrt[5]{2}\ cis84^\circ}$

$If\ \ k = 2 \Rightarrow \sqrt[5]{2}\ cis\left(12^\circ + 72^\circ.(2)\right) = \underline{\sqrt[5]{2}\ cis156^\circ}$

$If\ \ k = 3 \Rightarrow \sqrt[5]{2}\ cis\left(12^\circ + 72^\circ.(3)\right) = \underline{\sqrt[5]{2}\ cis228^\circ}$

$If\ \ k = 4 \Rightarrow \sqrt[5]{2}\ cis\left(12^\circ + 72^\circ.(4)\right) = \underline{\sqrt[5]{2}\ cis300^\circ}$

Exercise

Find the fourth roots of $z = 16cis60^\circ$

Solution

$$\begin{aligned} \sqrt[4]{z} &= \sqrt[4]{16}\ cis\left(\frac{60^\circ}{4} + \frac{360^\circ}{4}k\right) \\ &= 2cis\left(15^\circ + 90^\circ k\right) \end{aligned}$$

$If\ \ k = 0 \Rightarrow 2\ cis\left(15^\circ + 90^\circ(0)\right) = \underline{2cis15^\circ}$

$If\ \ k = 1 \Rightarrow 2\ cis\left(15^\circ + 90^\circ(1)\right) = \underline{2cis105^\circ}$

$If\ \ k = 2 \Rightarrow 2\ cis\left(15^\circ + 90^\circ(2)\right) = \underline{2cis195^\circ}$

$If\ \ k = 3 \Rightarrow 2\ cis\left(15^\circ + 90^\circ(3)\right) = \underline{2cis285^\circ}$

Exercise

Find the cube roots of 27.

Solution

$$\begin{aligned}\sqrt[3]{27} &= (27 cis 0°)^{1/3}\\ &= \sqrt[3]{27}\ cis\left(\frac{0°}{3}+\frac{360°}{3}k\right)\\ &= 3\ cis(0°+120°k)\end{aligned}$$

$$If\ \ k=0 \Rightarrow z = 3\ cis(0°+120°(0)) = \underline{2cis0°}$$

$$If\ \ k=1 \Rightarrow z = 3\ cis(0°+120°(1)) = \underline{2cis120°}$$

$$If\ \ k=2 \Rightarrow z = 3\ cis(0°+120°(2)) = \underline{2cis240°}$$

Exercise

Find all complex number solutions of $x^3+1=0$.

Solution

$$x^3+1=0 \Rightarrow x^3=-1$$

$$-1 \Rightarrow \begin{cases} r=\sqrt{(-1)^2+0^2}=1 \\ \theta=\tan^{-1}\left(\frac{0}{-1}\right)=\pi \end{cases}$$

$$x^3=-1=1\ cis\pi$$

$$\begin{aligned}x &= (1\ cis\pi)^{1/3}\\ &= (1)^{1/3}\ \ cis\left(\frac{\pi}{3}+\frac{2\pi}{3}k\right)\\ &= cis\left(\frac{\pi}{3}+\frac{2\pi}{3}k\right)\end{aligned}$$

$$If\ \ k=0 \Rightarrow x = cis\left(\frac{\pi}{3}+\frac{2\pi}{3}(0)\right) = \underline{cis\,\frac{\pi}{3}} \qquad x=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

$$If\ \ k=1 \Rightarrow x = cis\left(\frac{\pi}{3}+\frac{2\pi}{3}(1)\right) = cis\left(\frac{3\pi}{3}\right) = \underline{cis\pi} \qquad x=\cos\pi+i\sin\pi\ \underline{=-1}$$

$$If\ \ k=2 \Rightarrow x = cis\left(\frac{\pi}{3}+\frac{2\pi}{3}(2)\right) = \underline{cis\,\frac{5\pi}{3}} \qquad x=\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}=\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

Exercise

Find $(2cis30°)^5$

Solution

$$(2cis30°)^5 = 2^5 cis(5(30°))$$

$$= 32(\cos 150° + i \sin 150°)$$

$$= 32\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$= -16\sqrt{3} + 16i$$