# *Section* 2.5 – Variation of Parameters

In this section, we will introduce a technique called ***variation of parameters***.

The inhomogeneous equation is given by: $y'' + p(t)y' + q(t)y = g(t)$

A fundamental set of solutions $y_1$ and $y_2$ to associated homogeneous equation $y'' + py' + qy = 0.$
Then the general solution to the inhomogeneous equation is given by

$$y_k = C_1 y_1 + C_2 y_2$$

$C_1$ *and* $C_2$ are arbitrary constants.

## *General Case*

A differential system can be written in a form:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

$y_1$ and $y_2$ fundamental set of solution to the homogenous equation, they are linearly independent Then the determinant will be recognized as the Wronskian:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

Which we can obtain:

$$v_1' = -\frac{y_2 g(t)}{y_1 y_2' - y_1' y_2} = -\frac{y_2}{W} g(t) \qquad \Rightarrow v_1(t) = -\int \frac{y_2 g(t)}{W} dt$$

$$v_2' = \frac{y_1 g(t)}{y_1 y_2' - y_1' y_2} = \frac{y_1}{W} g(t) \qquad \Rightarrow v_2(t) = \int \frac{y_1 g(t)}{W} dt$$

$$\boxed{y_p = v_1 y_1 + v_2 y_2}$$

### Example

$\left\{ y_1(x) = x^4,\ y_2(x) = x^2 \right\}$ is a fundamental set of solutions of $y'' - \frac{5}{x}y' + \frac{8}{x^2}y = 4x^3$.

Find a particular solution of the equation?

**Solution**

$$W = \begin{vmatrix} x^4 & x^2 \\ 4x^3 & 2x \end{vmatrix} = -2x^5 \neq 0$$

$$v_1(x) = -\int \frac{x^2\left(4x^3\right)}{-2x^5}dx = \int 2dx = \boxed{2x} \qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$v_2(x) = \int \frac{x^4\left(4x^3\right)}{-2x^5}dx = -2\int x^2 dx = \boxed{-\frac{2}{3}x^3} \qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

The particular solution:

$$\begin{aligned} y_p &= v_1 y_1 + v_2 y_2 \\ &= (2x)\left(x^4\right) - \frac{2}{3}x^3\left(x^2\right) \\ &= \boxed{\frac{4}{3}x^5} \end{aligned}$$

The general solution: $\boxed{y(x) = C_1 x^4 + C_2 x^2 + \frac{4}{3}x^5}$

### Example

$\left\{ y_1(x) = e^{2x},\ y_2(x) = xe^{2x} \right\}$ is a fundamental set of solutions of $y'' - 4y' + 4y = \frac{e^{2x}}{x}$.

Find a particular solution of the equation?

**Solution**

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x} + 2xe^{4x} - 2xe^{4x} = \boxed{e^{4x} \neq 0}$$

$$v_1(x) = -\int \frac{xe^{2x}}{e^{4x}}\frac{e^{2x}}{x}dx = -\int dx = \boxed{-x} \qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$v_2(x) = \int \frac{e^{2x}}{e^{4x}}\frac{e^{2x}}{x}dx = \int \frac{1}{x}dx = \boxed{\ln|x|} \qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

The particular solution:

$$y_p = -xe^{2x} + \ln|x|\left(xe^{2x}\right) \qquad y_p = v_1 y_1 + v_2 y_2$$

The general solution:

$$y(x) = C_1 e^{2x} + C_2 xe^{2x} - xe^{2x} + xe^{2x}\ln|x|$$
$$= C_1 e^{2x} + (C_2 - 1)xe^{2x} + xe^{2x}\ln|x|$$
$$\boxed{= C_1 e^{2x} + C_3 xe^{2x} + xe^{2x}\ln|x|}$$

## *Example*

Find the particular solution for $y'' + y = \tan t$

***Solution***

The homogeneous equation for the differential equation $\lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm i$

Therefore; $y_1 = \cos t \quad and \quad y_2 = \sin t$

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1 \neq 0$$

The system has a solution

$$v_1' = \sin t \ \tan t = \frac{\sin^2 t}{\cos t}$$

$$\boxed{v_1} = \int \frac{\sin^2 t}{\cos t} dt$$
$$\boxed{= -\ln|\sec t + \tan t| + \sin t}$$

$$v_2' = \cos t \ \tan t = \sin t$$

$$\boxed{v_2} = \int \sin t \ dt$$
$$\boxed{= -\cos t}$$

$$y_p = v_1 \cos t + v_2 \sin t$$
$$= \left(-\ln|\sec t + \tan t| + \sin t\right)\cos t + \left(-\cos t\right)\sin t$$
$$= -\cos t \ln|\sec t + \tan t| + \sin t \cos t - \cos t \sin t$$
$$\boxed{= -(\cos t) \ \ln|\sec t + \tan t|}$$

## ***Higher*-Order Equations**

The nonhomogeneous higher-order equation is given by:

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots + p_1(t)y' + p_0(t)y = g(t)$$

A fundamental set of solutions $y_1, y_2, \ldots, y_n$ to associated homogeneous equation

$$y_h = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$$

Then the particular solution is $y_p = u_1 y_1 + u_2 y_2 + \ldots + u_n y_n$

Where $u'_k$, $k = 1, 2, \ldots, n$ are determined by the $n$ equations:

$$\begin{aligned}
u'_1 y_1 + \quad u'_2 y_2 \quad + \ldots + \quad u'_n y_n &= 0 \\
u'_1 y'_1 + \quad u'_2 y'_2 \quad + \ldots + \quad u'_n y'_n &= 0 \\
\vdots \qquad \vdots \qquad \vdots \qquad\qquad & \vdots \\
u'_1 y_1^{(n-1)} + u'_2 y_2^{(n-1)} + \ldots + u'_n y_n^{(n-1)} &= g(t)
\end{aligned}$$

Using Cramer's Rule give: $\quad u'_k = \dfrac{W_k}{W}$

For the 3rd-order differential equation:

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y'_2 & y'_3 \\ g(t) & y''_2 & y''_3 \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y'_1 & 0 & y'_3 \\ y''_1 & g(t) & y''_3 \end{vmatrix} \qquad W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y'_1 & y'_2 & 0 \\ y''_1 & y''_2 & g(t) \end{vmatrix}$$

$$u_1 = \int \frac{W_1}{W} \quad u_2 = \int \frac{W_2}{W} \quad u_3 = \int \frac{W_3}{W}$$

# *Exercises* *Section* 2.5 – Variation of Parameters

**1.** $\left\{y_1(x)=e^{2x},\ y_2(x)=e^{-3x}\right\}$ is a fundamental set of solutions of $y''+y'-6y=3e^{2x}$.
Find a particular solution of the equation?

Find a particular solution to the given second-order differential equation (Use ***variation of parameters***):

**2.** $y''-y=t+3$

**3.** $y''-2y'+y=e^{t}$

**4.** $x''-4x'+4x=e^{2t}$

**5.** $x''+x=\tan^2 t$

**6.** $y''+25y=-2\tan(5x)$

**7.** $y''-6y'+9y=5e^{3x}$

**8.** $y''+4y=2\cos 2x$

**9.** $y''-5y'+6y=4e^{2x}+3$

**10.** Verify that $y_1(t)=t$ and $y_2(t)=t^{-3}$ are solution to the homogenous equation

$$t^2y''(t)+3ty'(t)-3y(t)=0$$

Use variation of parameters to find the general solution to

$$t^2y''(t)+3ty'(t)-3y(t)=\frac{1}{t}$$

Find the general solution to the given differential equation (Use ***variation of parameters***).

**11.** $y''-y=\frac{1}{x}$

**12.** $y''-y=\sinh 2x$

**13.** $y''-y=x$

**14.** $y''-y=\cosh x$

**15.** $y''-y=\sin x$

**16.** $y''-y=e^{x}$

**17.** $y''+y=\sec x$

**18.** $y''+y=\tan x$

**19.** $y''+y=\sin x$

**20.** $y''+y=\csc x$

**21.** $y''+y=\cos^2 x$

**22.** $y''+y=\csc^2 x$

**23.** $y''+y=\sec^2 x$

**24.** $y''+y=\sec x\tan x$

**25.** $y''+y'=x$

**26.** $y''-y'=e^{x}\cos x$

**27.** $y''+y'-2y=xe^{x}$

**28.** $y''+y'-2y=e^{3x}$

**29.** $y''+2y'+y=e^{-x}\ln x$

**30.** $y''-2y'+y=\dfrac{e^{x}}{1+x^2}$

**31.** $y''+2y'+y=e^{-x}$

**32.** $y''-2y'-8y=3e^{-2x}$

**33.** $y''+3y'+2y=\sin e^{x}$

**34.** $y''+3y'+2y=4e^{x}$

**35.** $y''+3y'+2y=\dfrac{1}{1+e^{x}}$

**36.** $y''-4y=\sinh 2x$

**37.** $y''+4y=\sec 2x$

**38.** $y''+4y=\cos 3x$

**39.** $y''+4y=\sin^2 x$

**40.** $y''+4y=\sin^2 2t$

**41.** $y''-4y=\dfrac{e^{x}}{x}$

**42.** $y'' - 4y = xe^{x}$

**43.** $y'' - 4y' + 4y = 2e^{2x}$

**44.** $y'' - 4y' + 4y = (x+1)e^{2x}$

**45.** $y'' + 4y' + 5y = 10$

**46.** $y'' - 9y = \dfrac{9x}{e^{3x}}$

**47.** $y'' + 9y = \csc 3x$

**48.** $y'' + 9y = 3\tan 3t$

**49.** $y'' + 9y = \sin 3x$

**50.** $y'' + 9y = 2\sec 3x$

**51.** $4y'' + 36y = \csc 3x$

**52.** $\left(D^2 + 5D + 6\right)y = x^2 + 2x$

**53.** $\left(D^2 - 3D + 2\right)y = \dfrac{1}{1+e^{-x}}$

**54.** $y''' + y' = \sec x$

**55.** $y''' - 3y'' + 2y' = \dfrac{e^{x}}{1+e^{-x}}$

**56.** $y''' - 6y'' + 11y' - 6y = e^{x}$

**57.** $x^3 y^{(3)} - 4x^2 y'' + 8xy' - 8y = 4\ln x$

Find the general solution by to ***variation of parameters*** with the given initial conditions.

**58.** $y'' + y = \sec t$ ; $y(0) = 1,\quad y'(0) = 2$

**59.** $y'' + y = \sec^3 t$ ; $y(0) = 1,\quad y'(0) = \frac{1}{2}$

**60.** $y'' - y = t + \sin t$ ; $y(0) = 2,\quad y'(0) = 3$

**61.** $y'' - 2y' + y = \dfrac{e^{x}}{x}$; $y(0) = 1,\quad y'(0) = 0$

**62.** $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$ ; $y(0) = 1,\quad y'(0) = 0$

**63.** $y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x$ ; $y(0) = 1,\quad y'(0) = 2$

**64.** $y'' + 4y = \sin^2 2t$ ; $y\left(\frac{\pi}{8}\right) = 0,\quad y'\left(\frac{\pi}{8}\right) = 0$

**65.** $y'' + 4y = \sin^2 2t$ ; $y(0) = 0,\quad y'(0) = 0$

**66.** $y'' - 4y' + 4y = \left(12x^2 - 6x\right)e^{2x}$ ; $y(0) = 1,\quad y'(0) = 0$

**67.** $2y'' + y' - y = x + 1$ ; $y(0) = 1,\quad y'(0) = 0$

**68.** $4y'' - y = xe^{x/2}$; $y(0) = 1,\quad y'(0) = 0$

**69.** $t^2 y'' - ty' + y = t$ ; $y(1) = 1,\quad y'(1) = 4$