Solution Section 4.2 – General Linear Transformations

Exercise

The matrix $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ gives a shearing transformation $T(x, y) = (x,\ 3x + y)$.

What happens to (1, 0) and (2, 0) on the x-axis.

What happens to the points on the vertical lines $x = 0$ and $x = a$?.

Solution

The points (1, 0) and (2, 0) on the x-axis transform by T to (1, 3) and (2, 6). The horizontal x-axis transforms to the straight line with slope 3 (going through (0, 0) of course). The points on the y-axis are not moved because $T(0, y) – (0, y)$. The y -axis is the line of eigenvectors of T with $\lambda = 1$.

The vertical line $x = a$ is moved up by $3a$, since $3a$ is added to the y component. This is ***shearing***. Vertical lines slide higher as you go from left to right.

Exercise

A nonlinear transformation T is invertible if every $\boldsymbol{b}$ in the output space comes from exactly one x in the input space. $T(\boldsymbol{x}) = \boldsymbol{b}$ always has exactly one solution. Which of these transformation (on real numbers $\boldsymbol{x}$ is invertible and what is T^{-1}? None are linear, not even T_3. When you solve $T(\boldsymbol{x}) = \boldsymbol{b}$, you are inverting T:

$$T_1(x) = x^2 \quad T_2(x) = x^3 \quad T_3(x) = x + 9 \quad T_4(x) = e^x \quad T_5(x) = \frac{1}{x} \quad \text{for nonzero } x\text{'s}$$

Solution

T_1 is not invertible because $x^2 = 1 \rightarrow x = \pm 1$ and $x^2 = -1$ has no solution.

T_4 is not invertible because $e^x = -1$ has no solution.

T_2 is invertible. The solutions to $x^3 = b \rightarrow x = b^{1/3} = T_2^{-1}(b)$

T_3 is invertible. The solutions to $x + 9 = b \rightarrow x = b - 9 = T_3^{-1}(b)$

T_5 is invertible. The solutions to $\frac{1}{x} = b \rightarrow x = \frac{1}{b} = T_5^{-1}(b)$

Exercise

If S and T are linear transformations, is $S(T(\boldsymbol{v}))$ linear or quadratic?

a) If $S(\boldsymbol{v}) = \boldsymbol{v}$ and $T(\boldsymbol{v}) = \boldsymbol{v}$, then $S(T(\boldsymbol{v})) = \boldsymbol{v}$ or $\boldsymbol{v}^2$?

b) $S(\boldsymbol{w}_1 + \boldsymbol{w}_2) = S(\boldsymbol{w}_1) + S(\boldsymbol{w}_2)$ and $T(\boldsymbol{v}_1 + \boldsymbol{v}_2) = T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2)$ combine into

$$S(T(\boldsymbol{v}_1 + \boldsymbol{v}_2)) = S(\_\_\_\_\_) = \_\_\_\_ + \_\_\_\_\_\_$$

Solution

a) $S(T(v)) = S(v) = v$ **since $T(v) = v$**

b)
$$S(T(\boldsymbol{v}_1 + \boldsymbol{v}_2)) = S(T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2))$$
$$= S(T(\boldsymbol{v}_1)) + S(T(\boldsymbol{v}_2))$$

It is quadratic.

Exercise

Find the range and kernel (like the column space and nullspace) of T:

a) $T(v_1, v_2) = (v_2, v_1)$

b) $T(v_1, v_2, v_3) = (v_1, v_2)$

c) $T(v_1, v_2) = (0,\ 0)$

d) $T(v_1, v_2) = (v_1, v_1)$

Solution

a) Range is the line $y = 0$, Kernel is the line $x = y$ in the xy plane.

b) Range is the xy plane, Kernel is the complementary line in $\mathbb{R}^3$.

c) Range is the point (0, 0), Kernel is plane

d) Range is the line $x = y$ in the xy plane, Kernel is the line $x = 0$.

Exercise

M is any 2 by 2 matrix and $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$. The transformation T is defined by $T(M)=AM$. What rules of matrix multiplication show that T is linear?

Solution

The distribution law and the association law for multiplication give the linearity

$$\begin{aligned}A(cM+dN) &= A(cM)+A(dN)\\ &= (Ac)M+(Ad)N\\ &= cA(M)+dA(N)\end{aligned}$$

Exercise

Which of these transformations satisfy $T(v+w)=T(v)+T(w)$ and which satisfy $T(cv)=cT(v)$?

a) $T(v)=\dfrac{v}{\|v\|}$

b) $T(v)=v_1+v_2+v_3$

c) $T(v)=(v_1, 2v_2, 3v_3)$

d) $T(v)=$ largest component of ***v***.

Solution

a) This is scaling the vector into a normal vector. This it is impossible that we get additivity, because the sums of normal vectors don't have to be normal. For example *T*(0, 1) and *T*(1, 0) for instance. However, true to its name this does have the scaling property. For ***c*** value, this value will be canceled from ***v*** and $\|\boldsymbol{v}\|$.

b) This satisfies both. One immediate way to see that it is matrix multiplication by [1, 1, 1], which is a linear operation and thus satisfies both properties.

c) This satisfies both. This a matrix multiplication by $\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix}$

d) Doesn't satisfy additivity [(0, 1) and (1, 0) still work]. Scaling doesn't work either, if we scale by -1 we now pick out the negative of the smallest component, which doesn't have to be related in any way to the largest component.

Exercise

Consider the basis $S=\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ for R^3, where $\boldsymbol{v}_1=(1,\ 1,\ 1)$ $\boldsymbol{v}_2=(1,\ 1,\ 0)$ $\boldsymbol{v}_3=(1,\ 0,\ 0)$ and let $T: R^3 \to R^3$ be the linear transformation for which

$$T(\boldsymbol{v}_1)=(2,\ -1,\ 4),\quad T(\boldsymbol{v}_2)=(3,\ 0,\ 1),\quad T(\boldsymbol{v}_3)=(-1,\ 5,\ 1)$$

Find a formula for $T(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$, and then use that formula to compute $T(2,\ 4,\ -1)$

Solution

$$\boldsymbol{u}=c_1\boldsymbol{v}_1+c_2\boldsymbol{v}_2+c_3\boldsymbol{v}_3$$

$$\begin{aligned}(x_1, x_2, x_3)&=c_1(1,\ 1,\ 1)+c_2(1,\ 1,\ 0)+c_3(1,\ 0,\ 0)\\ &=(c_1+c_2+c_3,\ c_1+c_2,\ c_1)\end{aligned}$$

$$\begin{cases} c_1+c_2+c_3=x_1 \\ c_1+c_2 \qquad\ \ =x_2 \\ c_1 \qquad\qquad\ \ =x_3 \end{cases} \rightarrow \begin{cases} c_3=x_1-x_2 \\ c_2=x_2-x_3 \\ c_1=x_3 \end{cases}$$

$$\begin{aligned}T(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)&=x_3T(\boldsymbol{v}_1)+(x_2-x_3)T(\boldsymbol{v}_2)+(x_1-x_2)T(\boldsymbol{v}_3)\\ &=x_3(2,-1,4)+(x_2-x_3)(3,0,1)+(x_1-x_2)(-1,5,1)\\ &=(2x_3+3x_2-3x_3-x_1+x_2,\ -x_3+5x_1-5x_2,\ 4x_3+x_2-x_3+x_1-x_2)\\ &=(-x_1+4x_2-x_3,\ 5x_1-5x_2-x_3,\ x_1+3x_3)\end{aligned}$$

$$\begin{aligned}T(2,\ 4,\ -1)&=(-2+16+1,\ 10-20+1,\ 2-3)\\ &=\underline{(15,\ -9,\ -1)}\end{aligned}$$

Exercise

Consider the basis $S=\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ for R^3, where $\boldsymbol{v}_1=(1,\ 2,\ 1)$ $\boldsymbol{v}_2=(2,\ 9,\ 0)$ $\boldsymbol{v}_3=(3,\ 3,\ 4)$ and let $T: R^3 \to R^2$ be the linear transformation for which

$$T(\boldsymbol{v}_1)=(1,\ 0),\quad T(\boldsymbol{v}_2)=(-1,\ 1),\quad T(\boldsymbol{v}_3)=(0,\ 1)$$

Find a formula for $T(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$, and then use that formula to compute $T(7,13,7)$

Solution

$$\boldsymbol{u}=c_1\boldsymbol{v}_1+c_2\boldsymbol{v}_2+c_3\boldsymbol{v}_3$$

$$\begin{aligned}(x_1, x_2, x_3)&=c_1(1,\ 2,\ 1)+c_2(2,\ 9,\ 0)+c_3(3,\ 3,\ 4)\\ &=(c_1+2c_2+3c_3,\ 2c_1+9c_2+3c_3,\ c_1+4c_3)\end{aligned}$$

$$\begin{cases} c_1 + 2c_2 + 3c_3 = x_1 \\ 2c_1 + 9c_2 + 3c_3 = x_2 \\ c_1 \qquad\quad + 4c_3 = x_3 \end{cases} \rightarrow \begin{cases} c_1 + 7c_2 = x_2 - x_1 \\ \qquad\qquad 2c_1 + \frac{9}{7}x_2 - \frac{9}{7}x_1 - \frac{9}{7}c_1 + \frac{3}{4}x_3 - \frac{3}{4}c_1 = x_2 \\ c_1 + 4c_3 = x_3 \end{cases}$$

$$2c_1 - \frac{9}{7}c_1 - \frac{3}{4}c_1 = x_2 - \frac{9}{7}x_2 + \frac{9}{7}x_1 - \frac{3}{4}x_3$$

$$-\frac{1}{28}c_1 = \frac{9}{7}x_1 - \frac{2}{7}x_2 - \frac{3}{4}x_3$$

$$\underline{c_1 = -36x_1 + 8x_2 + 21x_3}$$

$$c_3 = \frac{1}{4}x_3 - \frac{1}{4}c_1 \quad \rightarrow \quad \underline{c_3 = 9x_1 - 2x_2 - 5x_3}$$

$$c_2 = \frac{1}{7}x_2 - \frac{1}{7}x_1 - \frac{1}{7}c_1 \quad \rightarrow \quad c_2 = \frac{1}{7}x_2 - \frac{1}{7}x_1 + \frac{36}{7}x_1 - \frac{8}{7}x_2 - 3x_3$$

$$\underline{c_2 = 5x_1 - x_2 - 3x_3}$$

$$\begin{aligned} T(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) &= (-36x_1 + 8x_2 + 21x_3)T(\boldsymbol{v}_1) + (5x_1 - x_2 - 3x_3)T(\boldsymbol{v}_2) \\ &\qquad + (9x_1 - 2x_2 - 5x_3)T(\boldsymbol{v}_3) \\ &= (-36x_1 + 8x_2 + 21x_3)(1,\ 0) + (5x_1 - x_2 - 3x_3)(-1,\ 1) + (9x_1 - 2x_2 - 5x_3)(0,\ 1) \\ &= (36x_1 - 8x_2 + 21x_3 - 5x_1 + x_2 + 3x_3,\ 5x_1 - x_2 - 3x_3 + 9x_1 - 2x_2 - 5x_3) \\ &= (41x_1 + 9x_2 + 24x_3,\ 14x_1 - 3x_2 - 8x_3) \end{aligned}$$

$$T(7,\ 13,\ 7) = (37(7) - 13(13) + 24(7),\ 8(7) + 3(13) - 8(7)) \ \underline{= (-2,\ 3)}$$

Exercise

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ be vectors in a vector space V, and let $T: V \rightarrow R^3$ be the linear transformation for which

$$T(\boldsymbol{v}_1) = (1,\ -1,\ 2), \quad T(\boldsymbol{v}_2) = (0,\ 3,\ 2), \quad T(\boldsymbol{v}_3) = (-3,\ 1,\ 2)$$

Find $T(2v_1 - 3v_2 + 4v_3)$

Solution

$$\begin{aligned} T(2v_1 - 3v_2 + 4v_3) &= 2T(v_1) - 3T(v_2) + 4T(v_3) \\ &= 2(1,\ -1,\ 2) - 3(0,\ 3,\ 2) + 4(-3,\ 1,\ 2) \\ &= (2,\ -2,\ 4) - (0,\ 9,\ 6) + (-12,\ 4,\ 8) \\ &\underline{= (-10,\ -7,\ 6)} \end{aligned}$$

Exercise

Let $T: R^2 \to R^2$ be the linear operation given by the formula $T(x,y)=(2x-y,\ -8x+4y)$

Which of the following vectors are in $R(T)$

a) $(1,\ -4)$ ***b***) $(5,\ 0)$ ***c***) $(-3,\ 12)$

Solution

a) $T(x,y)=(2x-y,\ -8x+4y)=(1,\ -4)$

$$\begin{cases} 2x-y=1 \\ -8x+4y=-4 \end{cases} \Leftrightarrow \left[\begin{array}{cc|c} 2 & -1 & 1 \\ -8 & 4 & -4 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]$$

This is a consistent system, therefore (1, −4) is in $R(T)$

b) $T(x,y)=(2x-y,\ -8x+4y)=(5,\ 0)$

$$\begin{cases} 2x-y=5 \\ -8x+4y=0 \end{cases} \Leftrightarrow \left[\begin{array}{cc|c} 2 & -1 & 5 \\ -8 & 4 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{array}\right] \to 0=1$$

This is an inconsistent system, therefore (5, 0) is not in $R(T)$

c) $T(x,y)=(2x-y,\ -8x+4y)=(-3,\ 12)$

$$\begin{cases} 2x-y=-3 \\ -8x+4y=12 \end{cases} \Leftrightarrow \left[\begin{array}{cc|c} 2 & -1 & -3 \\ -8 & 4 & 12 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \end{array}\right]$$

This is a consistent system, therefore (−3, 12) is in $R(T)$

Exercise

Let $T: R^2 \to R^2$ be the linear operation given by the formula $T(x,\ y)=(2x-y,\ -8x+4y)$

Which of the following vectors are in $\ker(T)$

a) $(5,\ 10)$ ***b***) $(3,\ 2)$ ***c***) $(1,\ 1)$

Solution

a) $T(5,\ 10)=(10-10,\ -40+40)=(0,\ 0)$; therefore (5, 10) is in $\ker(T)$

b) $T(3,\ 2)=(6-2,\ -24+8)=(4,\ -16)$; therefore (3, 2) is not in $\ker(T)$

c) $T(1,\ 1)=(2-1,\ -8+4)=(1,\ -4)$; therefore (1, 1) is not in $\ker(T)$

Exercise

Let $T: R^4 \to R^3$ be the linear operation given by the formula

$T\left(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\right) = \left(4x_1 + x_2 - 2x_3 - 3x_4,\ 2x_1 + x_2 + x_3 - 4x_4,\ 6x_1 - 9x_3 + 9x_4\right)$

Which of the following vectors are in $R(T)$ **a**) $(0,\ 0,\ 6)$ **b**) $(1,\ 3,\ 0)$ **c**) $(2,\ 4,\ 1)$

Solution

a) $T\left(x_1, x_2, x_3, x_4\right) = \left(4x_1 + x_2 - 2x_3 - 3x_4,\ 2x_1 + x_2 + x_3 - 4x_4,\ 6x_1 - 9x_3 + 9x_4\right) = (0,\ 0,\ 6)$

$$\begin{cases} 4x_1 + x_2 - 2x_3 - 3x_4 = 0 \\ 2x_1 + x_2 + x_3 - 4x_4 = 0 \\ 6x_1 \qquad - 9x_3 + 9x_4 = 6 \end{cases}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 0 \\ 2 & 1 & 1 & -4 & 0 \\ 6 & 0 & -9 & 9 & 6 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & -\frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

This is a consistent system, therefore (0, 0, 6) is in $R(T)$

b) $T\left(x_1, x_2, x_3, x_4\right) = \left(4x_1 + x_2 - 2x_3 - 3x_4,\ 2x_1 + x_2 + x_3 - 4x_4,\ 6x_1 - 9x_3 + 9x_4\right) = (1,\ 3,\ 0)$

$$\begin{cases} 4x_1 + x_2 - 2x_3 - 3x_4 = 1 \\ 2x_1 + x_2 + x_3 - 4x_4 = 3 \\ 6x_1 \qquad - 9x_3 + 9x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 1 \\ 2 & 1 & 1 & -4 & 3 \\ 6 & 0 & -9 & 9 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & -\frac{3}{2} & 0 & -\frac{3}{2} \\ 0 & 1 & 4 & 0 & 10 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

This is a consistent system, therefore (1, 3, 0) is in $R(T)$

c) $T\left(x_1, x_2, x_3, x_4\right) = \left(4x_1 + x_2 - 2x_3 - 3x_4,\ 2x_1 + x_2 + x_3 - 4x_4,\ 6x_1 - 9x_3 + 9x_4\right) = (2,\ 4,\ 1)$

$$\begin{cases} 4x_1 + x_2 - 2x_3 - 3x_4 = 2 \\ 2x_1 + x_2 + x_3 - 4x_4 = 4 \\ 6x_1 \qquad - 9x_3 + 9x_4 = 1 \end{cases}$$

$$\left[\begin{array}{cccc|c} 4 & 1 & -2 & -3 & 2 \\ 2 & 1 & 1 & -4 & 4 \\ 6 & 0 & -9 & 9 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & -\frac{3}{2} & 0 & -\frac{19}{12} \\ 0 & 1 & 4 & 0 & \frac{71}{6} \\ 0 & 0 & 0 & 1 & \frac{7}{6} \end{array}\right]$$

This is a consistent system, therefore (2, 4, 1) is in $R(T)$

Exercise

Let $T: R^4 \to R^3$ be the linear operation given by the formula

$$T\left(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\right) = \left(4x_1 + x_2 - 2x_3 - 3x_4,\ 2x_1 + x_2 + x_3 - 4x_4,\ 6x_1 - 9x_3 + 9x_4\right)$$

Which of the following vectors are in $\ker(T)$ ***a***) $(3,\ -8,\ 2,\ 0)$ ***b***) $(0,\ 0,\ 0,\ 1)$ ***c***) $(0,\ -4,\ 1,\ 0)$

Solution

a) $T(3,\ -8,\ 2,\ 0) = (12 - 8 - 4,\ 6 - 8 + 2,\ 18 - 18) \ \underline{= (0,\ 0,\ 0)}$

Therefore, (3, –8, 2, 0) is in $\ker(T)$

b) $T(0,\ 0,\ 0,\ 1) \ \underline{= (-3,\ -4,\ 9)}$

Therefore, (0, 0, 0, 1) is ***not*** in $\ker(T)$

c) $T(0,\ -4,\ 1,\ 0) = (-4 - 2,\ -4 + 1,\ -9) \ \underline{= (-6,\ -3,\ -9)}$

Therefore, (0, –4, 1, 0) is ***not*** in $\ker(T)$

Exercise

Determine if the given function *T* is a linear transformation

$$T: M_{22} \to M_{22} \quad by \quad T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2ab & 3cd \\ 0 & 0 \end{bmatrix}$$

Solution

Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$$\begin{aligned}
T(A+B) &= T\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \\
&= \begin{bmatrix} 2\left(a_1 + a_2\right)\left(b_1 + b_2\right) & 3\left(c_1 + c_2\right)\left(d_1 + d_2\right) \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2a_1b_1 + 2a_1b_2 + 2a_2b_1 + 2a_2b_2 & 3c_1d_1 + 3c_1d_2 + 3c_2d_1 + 3c_2d_2 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2a_1b_1 & 3c_1d_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2a_2b_2 & 3c_2d_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2a_1b_2 + 2a_2b_1 & 3c_1d_2 + 3c_2d_1 \\ 0 & 0 \end{bmatrix} \\
&= T(A) + T(B) + \begin{bmatrix} 2a_1b_2 + 2a_2b_1 & 3c_1d_2 + 3c_2d_1 \\ 0 & 0 \end{bmatrix} \\
&\underline{\neq T(A) + T(B)}
\end{aligned}$$

Function *T* is NOT a linear transformation

Exercise

Determine if the given function T is a linear transformation

$$T: M_{22} \to M_{22} \;\; by \;\; T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+d & 0 \\ 0 & b+c \end{bmatrix}$$

Solution

Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$$T(A+B) = T\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1+a_2+d_1+d_2 & 0 \\ 0 & b_1+b_2+c_1+c_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1+d_1 & 0 \\ 0 & b_1+c_1 \end{bmatrix} + \begin{bmatrix} a_2+d_2 & 0 \\ 0 & b_2+c_2 \end{bmatrix}$$

$$= T(A) + T(B) \quad \checkmark$$

$$T(kA) = T\left(k\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \right)$$

$$= T\left(\begin{bmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} ka_1+kd_1 & 0 \\ 0 & kb_1+kc_1 \end{bmatrix}$$

$$= \begin{bmatrix} k(a_1+d_1) & 0 \\ 0 & k(b_1+c_1) \end{bmatrix}$$

$$= k\begin{bmatrix} a_1+d_1 & 0 \\ 0 & b_1+c_1 \end{bmatrix}$$

$$= kT(A) \quad \checkmark$$

Since $T(A+B) = T(A) + T(B)$ and $T(kA) = kT(A)$, then function T is a linear transformation.

Exercise

Determine if the given function T is a linear transformation where A is fixed 2×3 matrix

$$T: M_{22} \to M_{23} \;\; by \;\; T(B) = BA$$

Solution

$$\begin{aligned} T(B+C) &= (B+C)A \\ &= BA + CA \\ &= T(B) + T(C) \end{aligned}$$

$$\begin{aligned} T(rB) &= (rB)A \\ &= r(BA) \\ &= rT(B) \end{aligned}$$

Function T is a linear transformation

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T = f_A$. $\quad T(x, y, z) = (2x+y,\, x-y+z)$

Solution

Let $\boldsymbol{u} = (x_1, y_1, z_1)$ *and* $\boldsymbol{v} = (x_2, y_2, z_2)$

$$\begin{aligned} T(\boldsymbol{u}+\boldsymbol{v}) &= T(x_1+x_2,\, y_1+y_2,\, z_1+z_2) \\ &= \left(2(x_1+x_2)+y_1+y_2,\; x_1+x_2-(y_1+y_2)+z_1+z_2\right) \\ &= \left(2x_1+y_1+2x_2+y_2,\; x_1-y_1+z_1+x_2-y_2+z_2\right) \\ &= \left(2x_1+y_1,\, x_1-y_1+z_1\right)+\left(2x_2+y_2,\, x_2-y_2+z_2\right) \\ &= T(x_1, y_1, z_1)+T(x_2, y_2, z_2) \\ &= T(\boldsymbol{u})+T(\boldsymbol{v}) \end{aligned}$$

$$\begin{aligned} T(r\boldsymbol{u}) &= T(rx_1,\, ry_1,\, rz_1) \\ &= \left(2rx_1+ry_1,\, rx_1-ry_1+rz_1\right) \\ &= r\left(2x_1+y_1,\, x_1-y_1+z_1\right) \\ &= rT(\boldsymbol{u}) \end{aligned}$$

Since $T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u})+T(\boldsymbol{v})$ and $T(r\boldsymbol{u}) = rT(\boldsymbol{u})$, then function T is a linear transformation.

Domain: $T: \mathbb{R}^3 \to \mathbb{R}^2$

$$T(x, y, z) = (2x + y,\ x - y + z) \quad \Rightarrow \begin{pmatrix} 2x + y \\ x - y + z \end{pmatrix}$$

$$A = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \end{array}$$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T.

$T(x, y) = \left(x^2, y\right)$

Solution

Let $\boldsymbol{u} = \left(x_1, y_1\right)$ *and* $\boldsymbol{v} = \left(x_2, y_2\right)$

$$\begin{aligned} T(\boldsymbol{u} + \boldsymbol{v}) &= T\left(x_1 + x_2, y_1 + y_2\right) \\ &= \left(\left(x_1 + x_2\right)^2,\ y_1 + y_2\right) \\ &= \left(x_1^2 + x_2^2 + 2x_1x_2,\ y_1 + y_2\right) \\ &= \left(x_1^2,\ y_1\right) + \left(x_2^2,\ y_2\right) + \left(2x_1x_2,\ 0\right) \\ &= T(\boldsymbol{u}) + T(\boldsymbol{v}) + \left(2x_1x_2,\ 0\right) \\ &\neq T(\boldsymbol{u}) + T(\boldsymbol{v}) \end{aligned}$$

The function T is ***not*** a linear transformation.

Domain: $T: \ \mathbb{R}^2 \to \mathbb{R}^2$

Exercise

Determine if the given function T is a linear transformation. Also give the domain and range of T; if T is linear, find the A such $T = f_A$.

$T(x, y, z) = (z - x, z - y)$

Solution

Let $\boldsymbol{u} = \left(x_1, y_1, z_1\right)$ *and* $\boldsymbol{v} = \left(x_2, y_2, z_2\right)$

$$\begin{aligned} T(\boldsymbol{u} + \boldsymbol{v}) &= T\left(x_1 + x_2, y_1 + y_2, z_1 + z_2\right) \\ &= \left(z_1 + z_2 - \left(x_1 + x_2\right), z_1 + z_2 - \left(y_1 + y_2\right)\right) \end{aligned}$$

$$=\left(z_1+z_2-x_1-x_2,\ z_1+z_2-y_1-y_2\right)$$
$$=\left(z_1-x_1,\ z_1-y_1\right)+\left(z_2-x_2,\ z_2-y_2\right)$$
$$=T\left(x_1, y_1, z_1\right)+T\left(x_2, y_2, z_2\right)$$
$$=T(\boldsymbol{u})+T(\boldsymbol{v})$$

$$T(r\boldsymbol{u})=T\left(rx_1,\ ry_1,\ rz_1\right)$$
$$=\left(rz_1-rx_1,\ rz_1-ry_1\right)$$
$$=r\left(z_1-x_1,\ z_1-y_1\right)$$
$$=rT(\boldsymbol{u})$$

Since $T(\boldsymbol{u}+\boldsymbol{v})=T(\boldsymbol{u})+T(\boldsymbol{v})$ and $T(r\boldsymbol{u})=rT(\boldsymbol{u})$, then function T is a linear transformation.

Domain: $T:\ \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$T(x, y, z)=(z-x, z-y) \Rightarrow \begin{pmatrix} -x+z \\ -y+z \end{pmatrix}$$

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ A= & \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}$$

Exercise

Show that the function $T:\ \mathrm{R}^3 \rightarrow R^2$ given the formula $T\left(x_1, x_2, x_3\right)=\left(2x_1-x_2+x_3\ ,\ x_2-4x_3\right)$ is linear transformation

Solution

Let $\boldsymbol{u}=\left(x_1, x_2, x_3\right)$ *and* $\boldsymbol{v}=\left(y_1, y_2, y_3\right)$

$$T(\boldsymbol{u}+\boldsymbol{v})=T\left(x_1+y_1, x_2+y_2, x_3+y_3\right)$$
$$=\left(2\left(x_1+y_1\right)-x_2-y_2+x_3+y_3\ ,\ x_2+y_2-4\left(x_3+y_3\right)\right)$$
$$=\left(2x_1+2y_1-x_2-y_2+x_3+y_3\ ,\ x_2+y_2-4x_3-4y_3\right)$$
$$=\left(2x_1-x_2+x_3\ ,\ x_2-4x_3\right)+\left(2y_1-y_2+y_3\ ,\ y_2-4y_3\right)$$
$$=T(\boldsymbol{u})+T(\boldsymbol{v})$$

$$T(r\boldsymbol{u})=T\left(rx_1, rx_2, rx_3\right)$$
$$=\left(2rx_1-rx_2+rx_3\ ,\ rx_2-4rx_3\right)$$

$$= r\left(2x_1 - x_2 + x_3\ ,\ x_2 - 4x_3\right)$$
$$= rT(\boldsymbol{u})$$

Since $T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$ and $T(r\boldsymbol{u}) = rT(\boldsymbol{u})$, then function T is a linear transformation.