Section 2.9 – Rank and the Fundamental Matrix Spaces

The **R**educed **R**ow **E**chelon **F**orm (***rref***) is a matrix (R) with each pivot column has only one nonzero entry (the pivots which is always 1).

$$R = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{R = rref(A)}$$

Rank of a Matrix

The rank of a matrix A (***m*** by ***n***) is the number of ***nonzero rows*** in the row-reduced echelon form of A (it is the number of pivot). The common dimension of the row space and column space of a matrix A is called the ***rank*** of A and is denoted by

$$\boxed{rank(A) = r}$$

Note:

The rank of a matrix is well defined due to the uniqueness of the row-reduced echelon form. No matter what sequence of elementary row operations is performed to put the given matrix in row-reduced echelon form; there will always be the same number of nonzero rows.

Theorem

The row space and column space of a matrix A have the same dimension

The objective is to connect ***rank*** and ***dimension***.

- The ***rank*** of a matrix is the number of pivots.
- The ***dimension*** of a subspace is the number of vectors in a basis.

✓ ***A has full row rank if every row has a pivot:*** $r = m$***. No zero in R.***

✓ ***A has full column rank if every column has a pivot:*** $r = n$***. No free variables.***

Example

Find the rank of $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{bmatrix} \quad \begin{array}{l} \\ \\ R_3 - R_1 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & -2 & -2 & 4 \end{bmatrix} \quad \begin{array}{l} R_1 + R_2 \\ \\ R_3 + 2R_2 \end{array}$$

$$R = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix R has 2 nonzero rows, therefore the $rank(A) = 2$

Example

The columns of A are dependent. $A\vec{x} = \vec{0}$ has a nonzero solution.

$$Ax = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

$$-3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The rank of A is only $r = 2$.
Independent columns would give full column rank $r = n = 3$.

- The columns of A are independent exactly when the rank is $r = n$. There are $\boldsymbol{n}$ pivots and no free variables. Only $\vec{x} = \vec{0}$ is the nullspace.

Example

When all rows are multiplying of one pivot row, the rank is $r = 1$:

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad [6]$$

Solution

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix} \quad R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix} \quad 3R_2 - 5R_1$$

$$\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \quad \tfrac{1}{3}R_2$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad \begin{matrix} \tfrac{1}{5}R_1 \\ 5R_2 - 2R_1 \end{matrix} \quad \to \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The row-reduced echelon form $R = rref(A)$:

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad [1]$$

These matrices have only one pivot.

Dimension *Theorem* for Matrices

If A is a matrix with n columns, then $\boxed{rank(A) + nullity(A) = n}$

Theorem

If A is an $m \times n$ matrix, then

- $rank(A)$ = the number of leading variables in the general solution of $A\vec{x} = \vec{0}$
- $nullity(A)$ = the number of parameters in the general solution of $A\vec{x} = \vec{0}$

Theorem

If A is any matrix, then $rank(A) = rank\left(A^T\right)$

- ***$Ax = 0$ has $n - r$ free variables and special solutions: n columns minus r pivot columns. The null matrix N has $n - r$ columns (the special solutions).***
- ***The particular solution solves:*** $A\vec{x}_p = \vec{b}$
- ***Full column rank*** $R = \begin{bmatrix} n \text{ by } n \text{ identity matrix} \\ m - n \text{ rows of zeros} \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$

The reduced row echelon form looks like:

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{matrix} \boldsymbol{r} \text{ pivot rows} \\ \boldsymbol{m - r} \text{ zero rows} \end{matrix}$$

The pivot variables in the $n - r$ special columns come by changing F to $-F$:

Nullspace matrix: $$N = \begin{pmatrix} -F \\ I \end{pmatrix} \begin{matrix} \boldsymbol{r} \text{ pivot variables} \\ \boldsymbol{n - r} \text{ free variables} \end{matrix}$$

- Every matrix A with ***full column rank*** ($r = n$) has all these properties:
    1. All columns of A are pivot columns
    2. There are no free variables or special solutions.
    3. The nullspace $NS(A)$ contains only the zero vector $\vec{x} = \vec{0}$
    4. If $A\vec{x} = \vec{b}$ has a solution (might not) then it has only one solution.

Example

Suppose A is a square invertible matrix, $m = n = r$. What are $\vec{x}_p$ and $\vec{x}_n$ ***?***

Solution

The particular solution is the one and only solution $A^{-1}\vec{b}$.

There are no special solutions or free variables. $R = I$ has no zero rows.

The only vector in the null space is $\vec{x}_n = \vec{0}$.

The complete solution is

$$\begin{aligned}\vec{x} &= \vec{x}_p + \vec{x}_n \\ &= A^{-1}\vec{b} + \vec{0} \\ &= A^{-1}\vec{b}\end{aligned}$$

Example

Compute $N(A)$ for $A:\ \mathbb{R}^2 \to \mathbb{R}^3$ given by $A = (x + y,\ x,\ 2x - y)$

Solution

To find $N(A)$, we must solve the equation $A(x,\ y) = (0,\ 0,\ 0)$

$$\begin{pmatrix} x+y \\ x \\ 2x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} x + y = 0 \Rightarrow \boxed{y = 0} \\ \boxed{x = 0} \end{array}$$

Thus $NS(A) = \{0\}$, the set that consists solely of the zero vector.

- If $A\vec{x} = \vec{0}$ has more unknowns than equations (more columns than rows) then it has nonzero solutions. There must be free columns, without pivots.

Definition

If W is a subspace of $\mathbb{R}^n$ that are orthogonal to every vector in W is called orthogonal complement of W and is denoted nu the symbol $W^{\perp}$. $N(A)^{\perp}$ is exactly the row space $C\left(A^T\right)$

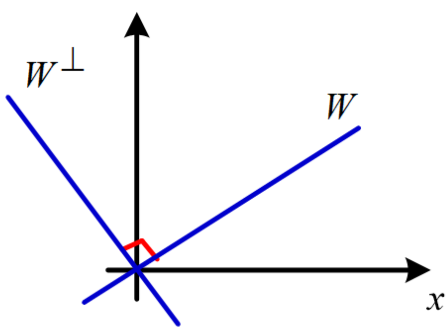
Fundamental Theorem of Linear Algebra

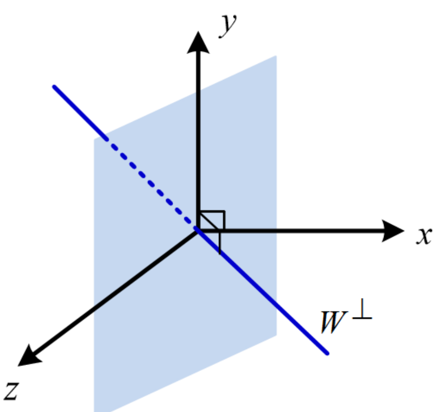
The nullspace is the orthogonal complement of the row space $\left(in\ \mathbb{R}^n\right)$.

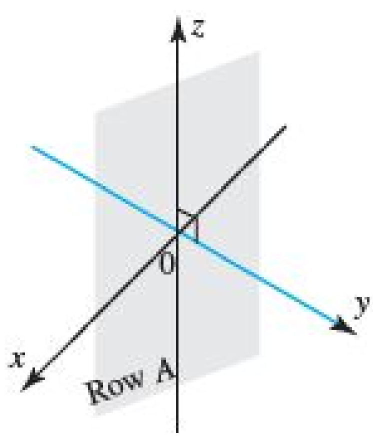
The left nullspace is the orthogonal complement of the column space $\left(in\ \mathbb{R}^m\right)$.

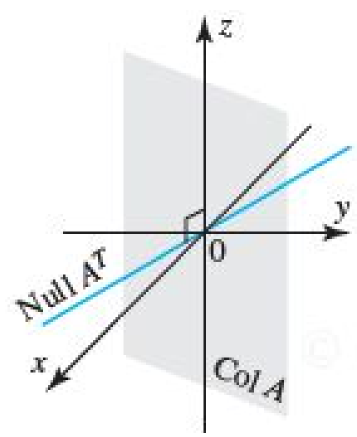
If W is a subspace of $\mathbb{R}^n$

- $W^{\perp}$ is a subspace of $\mathbb{R}^n$
- The only vector common to W and $W^{\perp}$ is 0.
- The orthogonal complement of $W^{\perp}$ is W.









Left Nullspace

A matrix A^T has m columns and has r ranks, so the number of free columns of A^T must be $m-r$.

$$\dim N\left(A^T\right) = m - r$$

The left nullspace is the collection of vectors $\vec{y}$ for which $A^T\vec{y} = \vec{0}$. Equivalently, $\vec{y}^T A = \vec{0}$, where $\vec{y}$ and $\vec{0}$ are row vectors. We can call "***left nullspace***" because $\vec{y}^T$ is on the left of matrix A in that equation.

To find a basis for the left nullspace we reduce an augmented type of A.

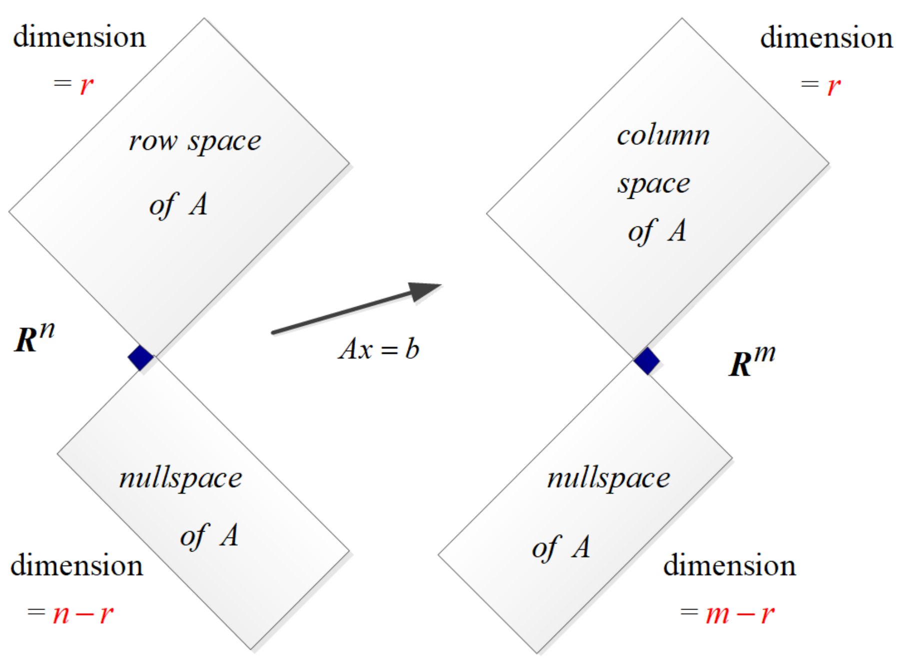
$$\left[A_{m\times n} \mid I_{m\times n}\right] \;\to\; \left[R_{m\times n} \mid E_{m\times n}\right]$$

Where matrix E can be found from $EA = R$.

If matrix A is a square matrix, then $E = A^{-1}$.

The Four Fundamental Subspaces

1. The ***row space*** is $C\left(A^T\right)$, a subspace of $\mathbb{R}^n$.
2. The ***column space*** is $C(A)$, a subspace of $\mathbb{R}^m$.
3. The ***null space*** is $N(A)$, a subspace of $\mathbb{R}^n$.
4. The ***left null space*** is $N\left(A^T\right)$, a subspace of $\mathbb{R}^m$.



Two pairs of orthogonal subspaces.

***For an m x n matrix of rank r*:**

Fundamental Space	***Subspace of***	***Dimension***
Nullspace	$\mathbb{R}^n$	$n-r$
Column Space	$\mathbb{R}^m$	r
Row space	$\mathbb{R}^n$	r
Left nullspace	$\mathbb{R}^m$	$m-r$

Example

Find a basis for each of the four subspaces associated with matrix A:

$$A=\begin{pmatrix}1&2&4\\2&4&8\end{pmatrix}$$

Solution

$$\begin{pmatrix}1&2&4\\2&4&8\end{pmatrix}\quad R_2-2R_1$$

$$\begin{pmatrix}1&2&4\\0&0&0\end{pmatrix}\quad x_1=-2x_2-4x_3 \leftarrow \textit{Row space}$$

1. Basis for ***row space***: $\begin{pmatrix}1\\2\\4\end{pmatrix}$

2. Basis of the **column spaces**: $\begin{pmatrix}1\\2\end{pmatrix}$

Rank($\boldsymbol{A}$) = 1

Dimension of A = 1

The pivots variables are: x_1

The free variables are: $x_2,\ x_3$

Set $x_2=1 \quad x_3=0$

The special solution: $s_1=(-2,\ 1,\ 0)$

Set $x_2=0 \quad x_3=1$

The special solution: $s_2=(-4,\ 0,\ 1)$

3. Basis of the **Null space**: $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$

$$A^T = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{matrix}$$

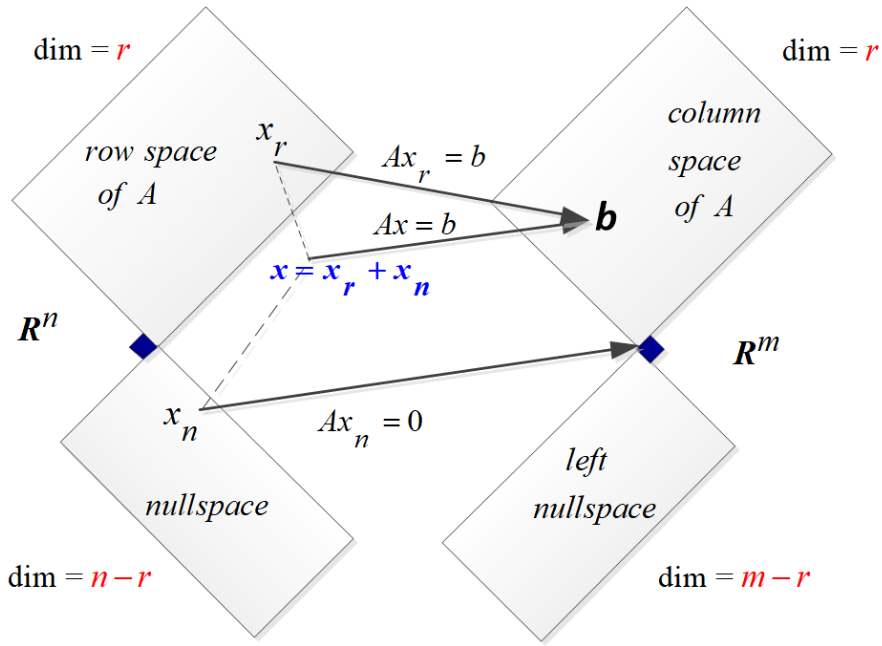
$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad y_1 = -2y_2$$

Set $y_2 = 1 \quad \Rightarrow \quad s^*_1 = (-2,\ 1)$

4. Basis of the **Left Nullspace**: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Combining Bases from Subspaces

- Any n linearly independent vectors in $\mathbb{R}^n$ must span $\mathbb{R}^n$. They are basis. Any n vectors that span $\mathbb{R}^n$ must be independent. They are a basis.
- If the n columns of $\boldsymbol{A}$ are independent, they span $\mathbb{R}^n$, So $A\vec{x}=\vec{b}$ is solvable,
- If the n columns span $\mathbb{R}^n$, they are independent. So $A\vec{x}=\vec{b}$ has only one solution.



- When the orthogonal complement of a subspace S is defined to be the subspace whose vectors pairs to zero with the vectors in S. The larger the S is, the more restriction $S^{\perp}$ has, and hence the smaller $S^{\perp}$ is.

Theorem – *Equivalent Statements*

If A is an $n \times n$ matrix, then the following statements are equivalent.

a) A is invertible

b) $A\vec{x} = \vec{0}$ has only the trivial solution

c) The reduced row echelon form of A is I_n

d) A is expressible as a product of elementary matrices

e) $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ matrix $\vec{b}$

f) $A\vec{x} = \vec{b}$ has exactly one solution for every $n \times 1$ matrix $\vec{b}$

g) $\det(A) \neq 0$

h) The column vectors of A are linearly independent

i) The row vectors of A are linearly independent

j) The column vectors of A span $\mathbb{R}^n$

k) The row vectors of A span $\mathbb{R}^n$

l) The column vectors of A form a basis for $\mathbb{R}^n$

m) The row vectors of A form a basis for $\mathbb{R}^n$

n) A has a rank n.

o) A has nullity 0.

p) The orthogonal complement of the null space of A is $\mathbb{R}^n$

q) The orthogonal complement of the row space of A is $\{0\}$

Exercises Section 2.9 – Rank and the Fundamental Matrix Spaces

1. Verify that $rank(A) = rank\left(A^T\right)$

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

2. Find the rank and nullity of the matrix; then verify that the values obtained satisfy $rank(A) + N(A) = n$

a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

3. If A is an $m \times n$ matrix, what is the largest possible value for its rank and the smallest possible value of the nullity of A.

4. Discuss how the rank of A varies with t.

a) $A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{bmatrix}$

5. Are there values of r and s for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

Has rank 1? Has rank 2? If so, find those values.

6. Find the row reduced form R and the rank r of A (those depend on c).
Which are the pivot columns of A? Which variables are free? What are the special solutions and the nullspace matrix N (always depending on c)?

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & c \end{bmatrix} \quad and \quad A = \begin{bmatrix} c & c \\ c & c \end{bmatrix}$$

7. Find the row reduced form R and the rank r of A (those depend on c).
Which are the pivot columns of A? Which variables are free? What are the special solutions and the nullspace matrix N (always depending on c)?

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad and \quad A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}$$

8. If A has a rank r, then it has an r by r sub-matrix S that is invertible. Remove $m-r$ rows and $n-r$ columns to find an invertible sub-matrix S inside each A (you could keep the pivot rows and pivot columns of A).

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

9. Suppose that column 3 of 4 x 6 matrix is all zero. Then x_3 must be a ______ variable. Give one special solution for this matrix.

10. Fill in the missing numbers to make A rank 1, rank 2, rank 3. (your solution should be 3 matrices)

$$A = \begin{pmatrix} & -3 & \\ 1 & 3 & -1 \\ & 9 & -3 \end{pmatrix}$$

11. Fill out these matrices so that they have rank 1:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{pmatrix} \quad B = \begin{pmatrix} 2 & & \\ 1 & & \\ 2 & 6 & -3 \end{pmatrix} \quad M = \begin{pmatrix} a & b \\ c & \end{pmatrix}$$

12. Suppose A and B are n by n matrices, and $AB = I$. Prove from $rank(AB) \le rank(A)$ that the $rank(A) = n$. So A is invertible and B must be its two-sided inverse. Therefore $BA = I$ (which is not so obvious!).

13. Every m by n matrix of rank r reduces to (m by r) times (r by n):

$$A = \text{(pivot columns of } A\text{) (first } r \text{ rows of } R\text{)} = (COL)(ROW)^T$$

Write the 3 by 4 matrix $A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix}$ as the product of the 3 by 2 from the pivot columns and the 2 by 4 matrix from R.

14. Suppose R is m by n matrix of rank r, with pivot columns first: $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

a) What are the shapes of those 4 blocks?
b) Find the right-inverse B with $RB = I$ if $r = m$.
c) Find the right-inverse C with $CR = I$ if $r = n$.
d) What is the reduced row echelon form of R^T (with shapes)?
e) What is the reduced row echelon form of $R^T R$ (with shapes)?
Prove that $R^T R$ has the same nullspace as R. Then show that $A^T A$ always has the same nullspace as A (a value fact).
f) Suppose you allow elementary column operations on $\boldsymbol{A}$ as well as elementary row operations (which get to R). What is the "row-and-column reduced form" for an m by n matrix of rank $\boldsymbol{r}$?

15. True or False (check addition or give a counterexample)

a) The symmetric matrices in M $\left(with\ A^T = A\right)$ from a subspace.
b) The skew-symmetric matrices in M $\left(with\ A^T = -A\right)$ from a subspace.
c) The un-symmetric matrices in M $\left(with\ A^T \neq A\right)$ from a subspace.
d) Invertible matrices
e) Singular matrices

16. Let $A = \begin{pmatrix} 1 & 2 & -2 & 3 & 0 \\ 2 & 4 & -3 & 7 & 0 \\ 3 & 6 & -5 & 10 & -2 \\ 5 & 10 & -9 & 16 & 0 \end{pmatrix}$

a) Reduce A to row-reduced echelon form.
b) What is the rank of A?
c) What are the pivots?
d) What are the free variables?
e) Find the special solutions. What is the nullspace $N(A)$?
f) Exhibit an r x r submatrix of A which is invertible, where $r = rank(A)$. (An r x r submatrix of A is obtained by keeping r rows and r columns of A)

17. Let $A = \begin{pmatrix} -1 & 2 & 5 & 0 & 5 \\ 2 & 1 & 0 & 0 & -15 \\ 6 & -1 & -8 & -1 & -47 \\ 0 & 2 & 4 & 3 & 16 \end{pmatrix}$

a) Reduce A to row-reduced echelon form.
b) What is the rank of A?
c) What the pivots?
d) What are the free variables?
e) Find the special solutions. What is the nullspace $N(A)$?
f) Give the complete solution to $Ax = b,$ *where* $b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

18. Let $A = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

a) Reduce A to row-reduced echelon form.
b) What is the rank of A?
c) What the pivots?
d) What are the free variables?
e) Find the special solutions.
f) What is the nullspace $N(A)$?

19. Let $A = \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}$

a) Reduce A to row-reduced echelon form.
b) What is the rank of A?
c) What the pivots?
d) What are the free variables?
e) Find the special solutions.
f) What is the nullspace $N(A)$?

20. The 3 by 3 matrix A has rank 2.

$$A\vec{x}=\vec{b} \quad is \quad \begin{aligned} x_1 + 2x_2 + 3x_3 + 5x_4 &= b_1 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 &= b_2 \\ 3x_1 + 6x_2 + 7x_3 + 13x_4 &= b_3 \end{aligned}$$

a) Reduce $\begin{bmatrix} A & \vec{b} \end{bmatrix}$ to $\begin{bmatrix} U & \vec{c} \end{bmatrix}$, so that $A\vec{x}=\vec{b}$ becomes triangular system $U\vec{x}=\vec{c}$.

b) Find the condition on (b_1, b_2, b_3) for $A\vec{x}=\vec{b}$ to have a solution

c) Describe the column space of A. Which plane in $\mathbb{R}^3$?

d) Describe the nullspace of A. Which special solutions in $\mathbb{R}^4$?

e) Find a particular solution to $A\vec{x}=(0,\ 6,\ -6)$ and then complete solution.

21. Find the special solutions and describe the complete solution to $A\vec{x}=\vec{0}$ for

A_1 = 3 by 4 zero matrix $\qquad A_2=\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \qquad A_3=\begin{bmatrix} A_1 & A_2 \end{bmatrix}$

Which are the pivot columns? Which are the free variables? What is the R (Reduced Row Echelon matrix) in each case?

22. Create a 3 by 4 matrix whose special solutions to $A\vec{x}=\vec{0}$ are $\vec{s}_1$ *and* $\vec{s}_2$:

$$\vec{s}_1=\begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad and \quad \vec{s}_2=\begin{pmatrix} -2 \\ 0 \\ -6 \\ 1 \end{pmatrix}$$

You could create the matrix A in row reduced form R. Then describe all possible matrices A with the required Nullspace $N(A)$ = all combinations of $\vec{s}_1$ *and* $\vec{s}_2$.

23. The plane $x-3y-z=12$ is parallel to the plane $x-3y-z=0$. One particular point on this plane is $(12,\ 0,\ 0)$. All points on the plane have the form (fill the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} \\ 0 \\ 0 \end{bmatrix}+y\begin{bmatrix} \\ 1 \\ 0 \end{bmatrix}+z\begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}$$

24. Construct a matrix whose column space contains (1, 1, 5) and (0, 3, 1) and whose Nullspace contains $(1,\ 1,\ 2)$.

25. Construct a matrix whose column space contains (1, 1, 0) and (0, 1, 1) and whose Nullspace contains (1, 0, 1) and (0, 0, 1).

26. Construct a matrix whose column space contains (1, 1, 1) and whose Nullspace contains $(1, 1, 1, 1)$.

27. How is the Nullspace N(*C*) related to the spaces N(*A*) and N(*B*), if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

28. Why does no 3 by 3 matrix have a nullspace that equals its column space?

29. If *AB* = 0 then the column space *B* is contained in the _______ of *A*. Give an example of *A* and *B*.

30. True or false (with reason if true or example to show it is false)

a) A square matrix has no free variables.
b) An invertible matrix has no free variables.
c) An *m* by *n* matrix has no more than ***n*** pivot variables.
d) An *m* by *n* matrix has no more than ***m*** pivot variables.

31. Suppose an *m* by *n* matrix has *r* pivots. The number of special solutions is ______.

The Nullspace contains only *x* = 0 when *r* = _______.

The column space is all of $\mathbb{R}^m$ when *r* = ______.

32. Find the complete solution in the form $\vec{x}_p + \vec{x}_n$ to these full rank system:

$a)\ x + y + z = 4$

$b)\ \begin{cases} x + y + z = 4 \\ x - y + z = 4 \end{cases}$

33. Find the complete solution in the form $\vec{x}_p + \vec{x}_n$ to the system:

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

34. If ***A*** is 3 x 7 matrix, its largest possible rank is ________. In this case, there is a pivot in every ________ of ***U*** and ***R***, the solution to $A\vec{x} = \vec{b}$ _________ (always exists or is unique), and the column space of ***A*** is _________. Construct an example of such a matrix ***A***.

35. If ***A*** is 6 x 3 matrix, its largest possible rank is ________. In this case, there is a pivot in every ________ of ***U*** and *R*, the solution to $A\vec{x} = \vec{b}$ _________ (always exists or is unique), and the nullspace of ***A*** is _________. Construct an example of such a matrix ***A***.

36. Find the rank of $A, A^T A$ and AA^T for $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix}$

37. Explain why these are all false:

a) The complete solution is any linear combination of $\vec{x}_p$ *and* $\vec{x}_n$.

b) A system $A\vec{x} = \vec{b}$ has at most one particular solution.

c) The solution $\vec{x}_p$ with all free variables zero is the shortest solution (minimum length $\|\vec{x}\|$). Find a 2 by 2 counterexample.

d) If A is invertible there is no solution $\vec{x}_n$ in the null space.

38. Write down all known relation between r and m and n if $A\vec{x} = \vec{b}$ has

a) No solution for some $\vec{b}$.

b) Infinitely many solutions for every $\vec{b}$.

c) Exactly one solution for some $\vec{b}$, no solution for other $\vec{b}$.

d) Exactly one solution for every $\vec{b}$.

39. Find a basis for its row space, find a basis for its column space, and determine its rank

$$a)\ \begin{bmatrix} 0 & 2 & -3 & 4 & 1 & 2 & 1 & 7 \\ 0 & 0 & 3 & -2 & 0 & 4 & -5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 6 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad b)\ \begin{bmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{bmatrix}$$

40. Find a basis for the row space, find a basis for the null space, find $dim\,RS$, find $dim\,NS$, and verify $dim\,RS + dim\,NS = n$

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{bmatrix}$$

41. Determine if $\vec{b}$ lies in the column space of the given matrix. If it does, express $\vec{b}$ as linear combination of the column.

$$\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

42. Find the transition matrix from B to C and find $[\vec{x}]_c$

a) $B = \{(3, 1), (-1, -2)\}, \quad C = \{(1, -3), (5, 0)\}, \quad [x]_B = [-1 \quad -2]^T$

b) $B = \{(1, 1, 1),\ (-2, -1, 0),\ (2, 1, 2)\}, \quad C = \{(-6, -2, 1),\ (-1, 1, 5),\ (-1, -1, 1)\},$

$[\vec{x}]_B = [-3 \quad 2 \quad 4]^T$

43. Does A and A^T have the same number of pivots.

(**44 – 49**) For the given matrix A, which is given in row reduction echelon form

- ***a)*** What is the rank of A?
- ***b)*** What is the dimension of A?
- ***c)*** What are the pivots?
- ***d)*** What are the free variables?
- ***e)*** Find the special (homogeneous) solutions.
- ***f)*** What is the nullspace $N(A)$?
- ***g)*** Find the particular solution $Ax = b$
- ***h)*** Give the complete solution.

44. $A = \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad where \;\; b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

45. $A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad where \;\; b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

46. $A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad where \;\; b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

47. $A = \begin{pmatrix} 1 & 0 & 0 & \frac{13}{11} \\ 0 & 1 & 0 & -\frac{17}{11} \\ 0 & 0 & 1 & \frac{6}{11} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad where \;\; b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

48. $A = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad where \;\; b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

49. $A = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad where \;\; b = A\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

(**50 – 55**) Find a basis for each of the four subspaces associated with each given matrix

50. $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{pmatrix}$

51. $B = \begin{pmatrix} 1 & 3 & 0 & 5 \\ 2 & 6 & 1 & 16 \\ 5 & 15 & 0 & 25 \end{pmatrix}$

52. $C = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$

53. $D = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix}$

54. $M = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 3 & 1 & -3 & -1 \\ 5 & -3 & 5 & 1 \end{pmatrix}$

55. $N = \begin{pmatrix} 3 & 2 & -1 \\ 6 & 3 & 5 \\ -3 & -1 & -6 \\ 0 & -1 & 7 \end{pmatrix}$