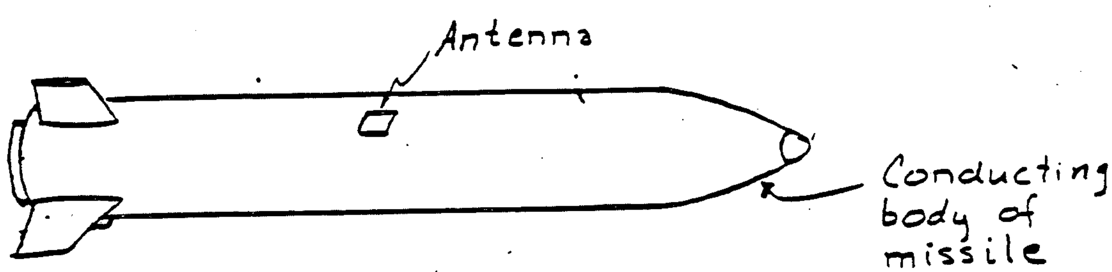
# 6. Curvilinear coordinates

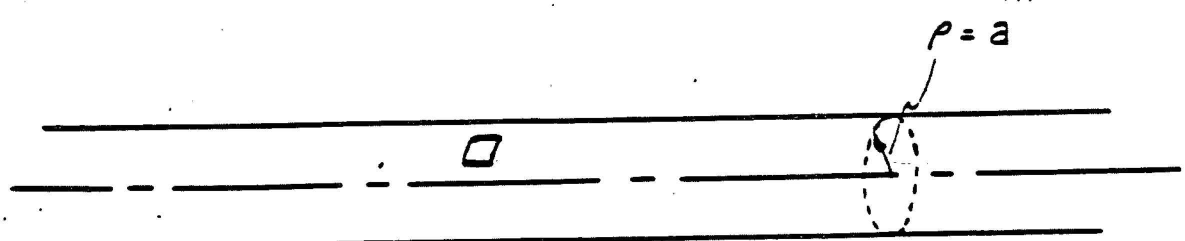
## a) Coordinate system

Why not use rectangular coordinates for any physical problem?

Most often, physical problems of interest involve some associated physical structure.

For example, suppose we wish to find the antenna radiation pattern produced by an antenna physically mounted on the side of a missile.




Model of missile: Conducting cylinder described by the simple equation, $\rho = a$. This is a constant "coordinate surface."

The electromagnetic field MUST obey certain contitions at the surface of the missile (<u>boundary conditions</u>).

It is <u>MUCH easier</u> to specify the shape of the missile (at least approximately) in cylindrical coordinates than in rectangular coordinates.

In general, whenever a physical boundary can be specified by a constant coordinate surface in some coordinate system, it is <u>much easier</u> to attack the problem using that coordinate system than the rectangular system.

# WHAT IS A COORDINATE SYSTEM?

If we have three functions $u_1$, $u_2$, and $u_3$, which map RXRXR into $U_1$, $U_2$, and $U_3$, respectively, in a 1:1 manner (except at a finite number of singular points), then we say that

$$(u_1, u_2, u_3) \quad \text{for } u_k \in U_k, \quad k = 1, 2, 3$$

forms a coordinate system.

Eg: $u_1 = \sqrt{x^2 + y^2}$, $\quad U_1 = \{\rho \mid \rho > 0\}$

$u_2 = \tan^{-1} \frac{y}{x}$, $\quad U_2 = \{\theta \mid 0 \le \theta < 2\pi\}$

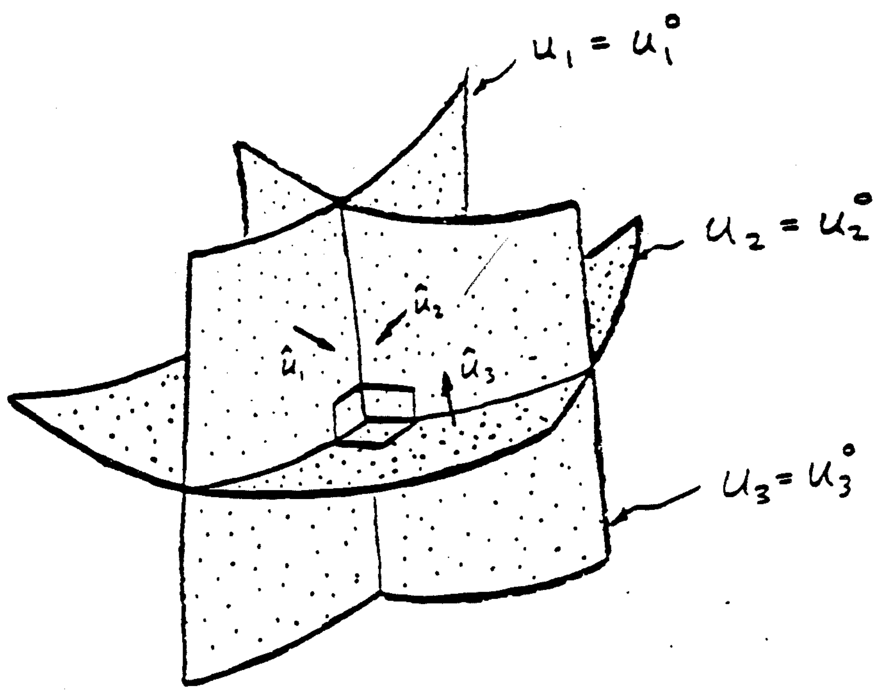
$u_3 = z$ $\quad U_3 = R$ = set of real numbers

forms a cylindrical coordinate system.

## b) Orthogonal coordinate system

### (1-) Definition

When the COORDINATE SURFACES intersect one another (for every non-singular point) at right angles, then we have an ORTHOGONAL SYSTEM.




$$\hat{u}_1 = \frac{\nabla u_1}{|\nabla u_1|}$$

$$\hat{u}_2 = \frac{\nabla u_2}{|\nabla u_2|}$$

$$\hat{u}_3 = \frac{\nabla u_3}{|\nabla u_3|}$$

$$\left.\begin{aligned} \hat{u}_1 &= \nabla u_1 / |\nabla u_1| \\ \hat{u}_2 &= \nabla u_2 / |\nabla u_2| \\ u_3 &= \nabla u_3 / |\nabla u_3| \end{aligned}\right\}$$ In general are not constant but depend on $(u_1, u_2, u_3)$.

If $\hat{u}_1 \times \hat{u}_2 = \hat{u}_3$, then this is a right handed system.

## (2) Test for orthogonality

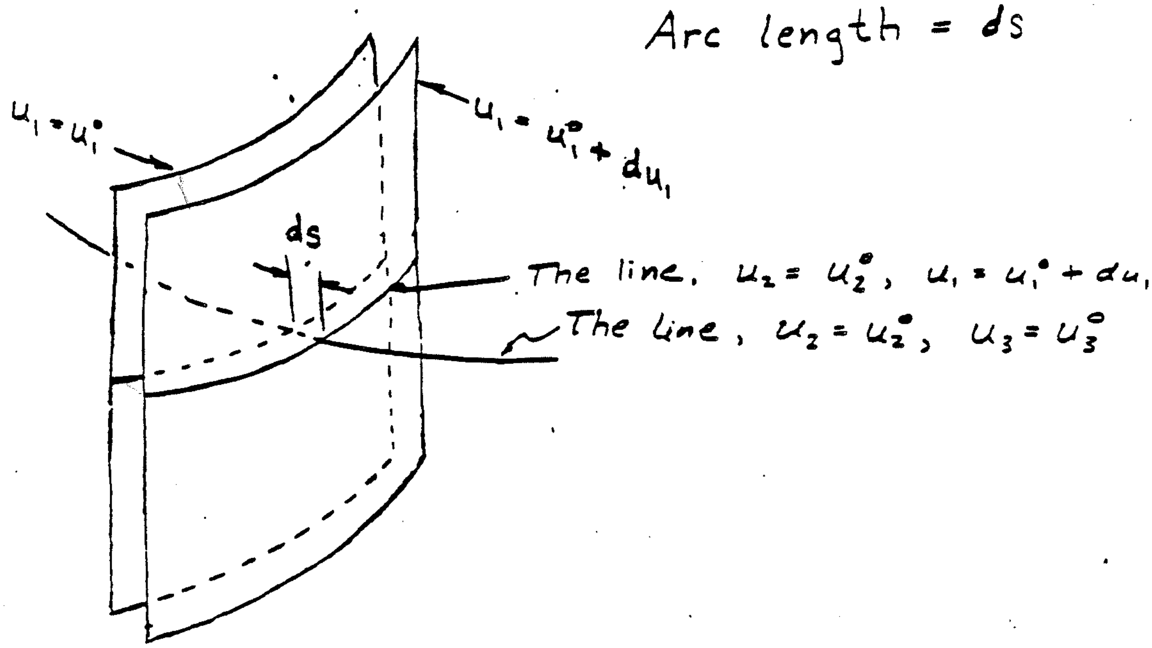
From the diagram above it is clear that a system is orthogonal if and only if the gradients of the three functions, $\nabla u_1$, $\nabla u_2$, and $\nabla u_3$ are mutually perpendicular for every point in the systém.

$$\nabla u_i \cdot \nabla u_j = 0 \quad \text{for } i \neq j.$$

## c) Arc length and scale factors

The arc length between the coordinate surface $u_1 = u_1^0$ and $u_1 = u_1^0 + du_1$ is <u>**NOT**</u> $du_1$.

Arc length = $ds$



This is so since $u_1$ might be an angle or some other non-linear measure. (Its units may not even be units of distance!)

We can find what the distance between these two surfaces is by computing the directional derivative of $u_1$ in the grad($u_1$) direction.

This is (by DEFINITION of the gradient)

$$\frac{du_1}{ds} = \nabla u_1 \cdot \frac{\nabla u_1}{|\nabla u_1|} = |\nabla u_1|.$$

Solving for ds, the arc length is

$$ds = \frac{1}{|\nabla u_1|}\, du_1.$$

The reciprocal magnitude of the gradient of $u_K$ is called the "scale factor" of the $u_K$ coordinate.

$$h_K = \frac{1}{|\nabla u_K|} = \frac{1}{\sqrt{\left(\frac{\partial u_K}{\partial x}\right)^2 + \left(\frac{\partial u_K}{\partial y}\right)^2 + \left(\frac{\partial u_K}{\partial z}\right)^2}}$$

$$ds_K = h_K\, du_K$$

Let's try an example. Suppose that we wish to compute the scale factors for cylindrical coordinates.

Usually, we write x, y, and z in terms of $\rho$, $\varphi$, and z, instead of the other way around.

$$x = \rho\cos\varphi \qquad u_1 = \rho$$
$$y = \rho\sin\varphi \qquad u_2 = \varphi$$
$$z = z \qquad u_3 = z$$

But according to the formula that we derived, we must take a gradient and so we need to write $\rho$, $\varphi$, and z in terms of x, y, and z.

$$\rho = \sqrt{x^2+y^2}$$
$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$

Then the gradients and scale factors are

$$\nabla\rho = \frac{x}{\sqrt{x^2+y^2}}\hat{x} + \frac{y}{\sqrt{x^2+y^2}}\hat{y} \Rightarrow h_\rho = \frac{1}{|\nabla\rho|} = \frac{1}{1} = 1$$
$$\nabla\varphi = \frac{-y\hat{x}}{x^2+y^2} + \frac{x\hat{y}}{x^2+y^2} \Rightarrow h_\varphi = \frac{1}{|\nabla\varphi|} = \sqrt{x^2+y^2}$$
$$\nabla z = \hat{z} \Rightarrow \qquad h_z = \frac{1}{|\nabla z|} = 1$$

But these scale factors are in terms of x, y, and z which is not what we wanted. We must now write this in terms of $\rho$, $\varphi$, and z by substituting the expressions for x, y, and z in terms of these variables

$$h_\rho = 1$$

$$h_\varphi = \sqrt{x^2+y^2} = \sqrt{\rho^2\cos^2\varphi + \rho^2\sin^2\varphi} = \rho$$

$$h_z = 1$$

This is a lot of work. Fortunately, there is an easier way to find the scale factors.

All we really need to do to find, say $h_1$, is to find the distance, ds, between the points $(u_1, u_2, u_3)$ and $(u_1+du_1, u_2, u_3)$ and to divide this distance by $du_1$.

Now, in rectangular coordinates, the point corresponding to point $(u_1, u_2, u_3)$ is the point $(x, y, z)$

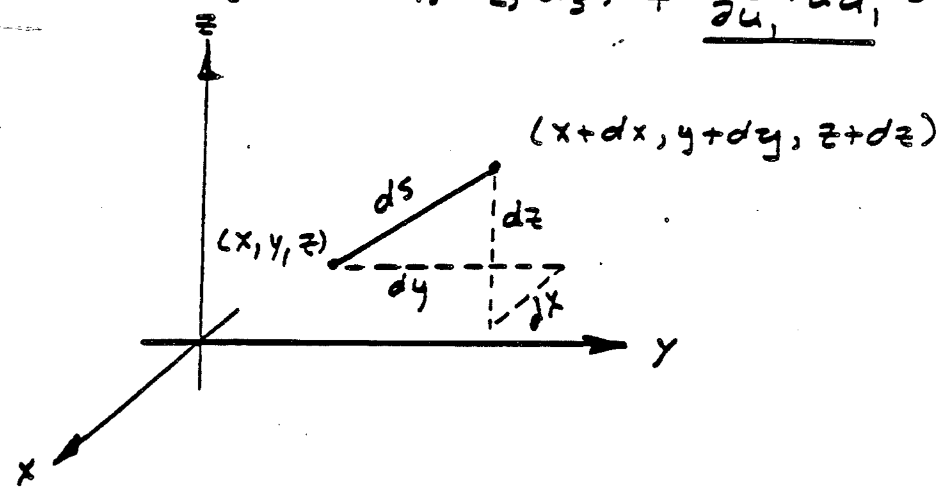
$$x = x(u_1, u_2, u_3)$$
$$y = y(u_1, u_2, u_3)$$
$$z = z(u_1, u_2, u_3)$$

The displaced point,
$(u_1 + du_1, u_2, u_3)$ is

$$x(u_1+du_1, u_2, u_3) = x(u_1, u_2, u_3) + \underline{\frac{\partial x}{\partial u_1} \cdot du_1} = x + \underline{dx}$$

$$y(u_1+du_1, u_2, u_3) = y(u_1, u_2, u_3) + \underline{\frac{\partial y}{\partial u_1} \cdot du_1} = y + \underline{dy}$$

$$z(u_1+du_1, u_2, u_3) = z(u_1, u_2, u_3) + \underline{\frac{\partial z}{\partial u_1} \cdot du_1} = z + \underline{dz}$$




Thus, the distance, ds, is just

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2}\, du$$

and the scale factor, $h_1$, is

$$h_1 = \frac{ds}{du_1} = \sqrt{\left(\frac{\partial x}{\partial u_1}\right)^2 + \left(\frac{\partial y}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_1}\right)^2}$$

Of course, in general,

$$h_k = \sqrt{\left(\frac{\partial x}{\partial u_k}\right)^2 + \left(\frac{\partial y}{\partial u_k}\right)^2 + \left(\frac{\partial z}{\partial u_k}\right)^2}$$

Let's use this formula to find the scale factors for cylindrical coordinates

$$x = \rho \cos\varphi$$
$$y = \rho \sin\varphi$$
$$z = z$$

$$\frac{\partial x}{\partial \rho} = \cos\varphi, \quad \frac{\partial y}{\partial \rho} = \sin\varphi, \quad \frac{\partial z}{\partial \rho} = 0$$

$$\frac{\partial x}{\partial \varphi} = -\rho \sin\varphi, \quad \frac{\partial y}{\partial \varphi} = \rho\cos\varphi, \quad \frac{\partial z}{\partial \varphi} = 0$$

$$\frac{\partial x}{\partial z} = 0, \quad \frac{\partial y}{\partial z} = 0, \quad \frac{\partial z}{\partial z} = 1$$

$$\therefore \quad h_\rho = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1, \quad h_\varphi = \sqrt{\rho^2 \sin^2\varphi + \rho^2\cos^2\varphi} = \rho,$$

$$h_z = 1.$$

A corollary to these results is the interesting result that for orthogonal coordinate systems,

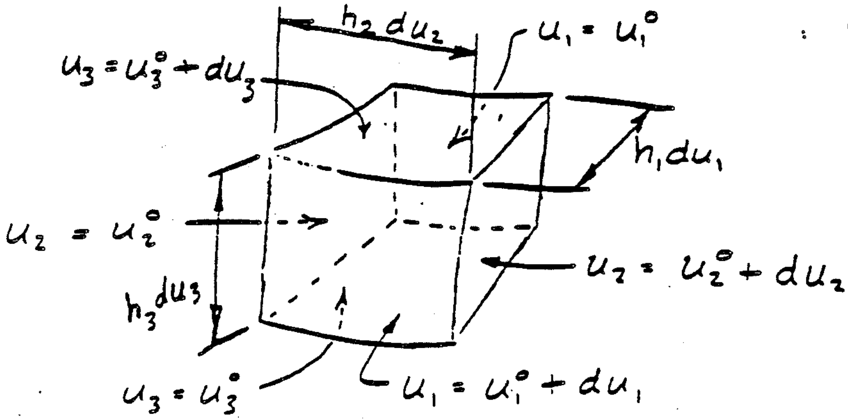
$$\sqrt{\left(\frac{\partial x}{\partial u_k}\right)^2 + \left(\frac{\partial y}{\partial u_k}\right)^2 + \left(\frac{\partial z}{\partial u_k}\right)^2}$$

$$= 1 \Big/ \sqrt{\left(\frac{\partial u_k}{\partial x}\right)^2 + \left(\frac{\partial u_k}{\partial y}\right)^2 + \left(\frac{\partial u_k}{\partial z}\right)^2}$$

## d) Volume

### (1) The volume element

Using the scale factors, it is easy to construct the volume element in the coordinate system.




$$\text{Volume} = dV$$
$$h_1 du_1 \cdot h_2 du_2 \cdot h_3 du_3$$
$$= h_1 h_2 h_3 \, du_1 du_2 du_3$$

### (2) The Jacobian

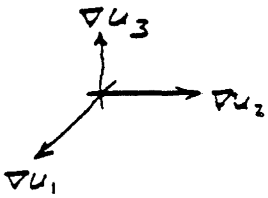
In terms of dx, dy, dz, the total differentials of $u_1$, $u_2$, and $u_3$, are

$$\left.\begin{aligned} du_1 &= \frac{\partial u_1}{\partial x}dx + \frac{\partial u_1}{\partial y}dy + \frac{\partial u_1}{\partial z}dz \\ du_2 &= \frac{\partial u_2}{\partial x}dx + \frac{\partial u_2}{\partial y}dy + \frac{\partial u_2}{\partial z}dz \\ du_3 &= \frac{\partial u_3}{\partial x}dx + \frac{\partial u_3}{\partial y}dy + \frac{\partial u_3}{\partial z}dz \end{aligned}\right\} \longleftrightarrow \begin{bmatrix} du_1 \\ du_2 \\ du_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

The determinant of this matrix is called the Jacobian, J, and (recall) is simply the triple scalar product of the vectors formed by the rows. Thus

$$\nabla u_1 \cdot \nabla u_2 \times \nabla u_3 = J.$$

Since, the coordinate system is orthogonal, $\nabla u_2 \times \nabla u_3$ is parallel to $\nabla u_1$ and hence




$$J = \nabla u_1 \cdot \nabla u_2 \times \nabla u_3 = |\nabla u_1||\nabla u_2||\nabla u_3| = \frac{1}{h_1 h_2 h_3}.$$

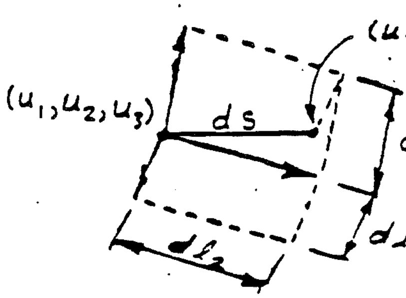
$$\therefore \; dx\,dy\,dz = \frac{du_1\,du_2\,du_3}{J}$$

e) Gradient

Using the DEFINITION of grad, we can now construct grad $w(u_1, u_2, u_3)$ in this system:

$$\frac{dw}{ds} = \frac{\partial w}{\partial u_1}\frac{du_1}{ds} + \frac{\partial w}{\partial u_2}\frac{du_2}{ds} + \frac{\partial w}{\partial u_3}\frac{du_3}{ds}.$$

$$dl_1 = h_1\,du_1$$
$$dl_2 = h_2\,du_2$$
$$dl_3 = h_3\,du_3$$




$$\therefore \frac{dw}{ds} = \frac{\partial w}{\partial u_1}\cdot\frac{1}{h_1}\frac{dl_1}{ds} + \frac{\partial w}{\partial u_2}\frac{1}{h_2}\frac{dl_2}{ds} + \frac{\partial w}{\partial u_3}\frac{1}{h_3}\frac{dl_3}{ds}$$

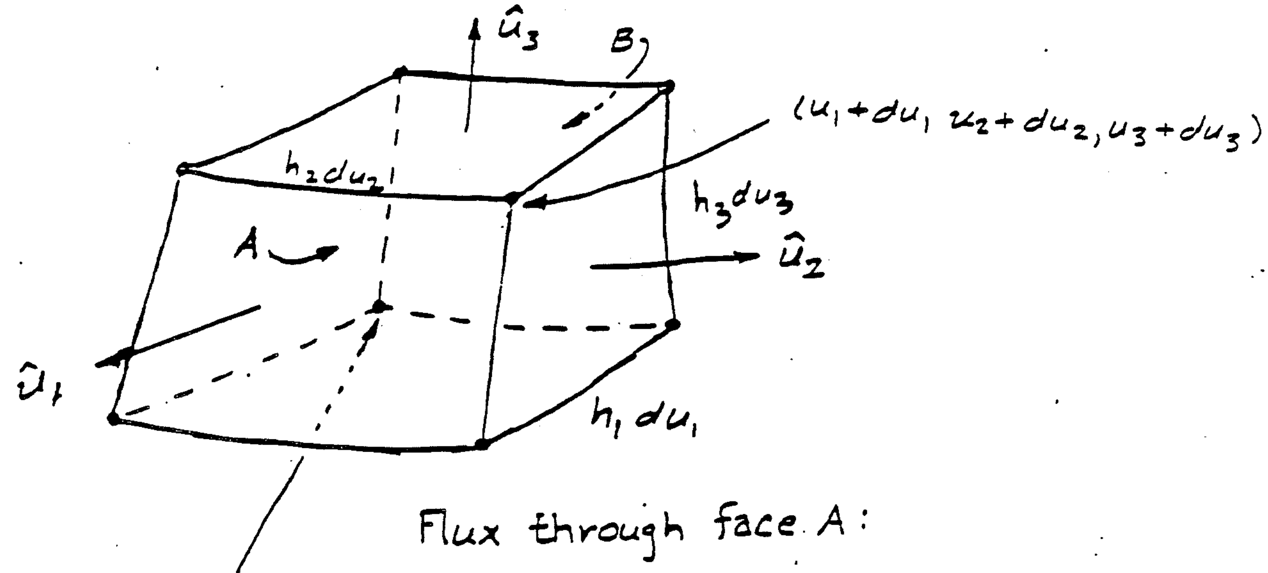
$$= \underbrace{\left[\frac{1}{h_1}\frac{\partial w}{\partial u_1}\hat{u}_1 + \frac{1}{h_2}\frac{\partial w}{\partial u_2}\hat{u}_2 + \frac{1}{h_3}\frac{\partial w}{\partial u_3}\hat{u}_3\right]}_{\text{grad } w} \cdot \underbrace{\left[\frac{dl_1}{ds}\hat{u}_1 + \frac{dl_2}{ds}\hat{u}_2 + \frac{dl_3}{ds}\hat{u}_3\right]}_{\text{unit vector}}$$

## i) Divergence

Using the DEFINITION of div, we can now construct
div $\vec{F}(u_1, u_2, u_3)$ in this system:

$$\vec{F} = \hat{u}_1 F_1 + \hat{u}_2 F_2 + \hat{u}_3 F_3 .$$

$$\operatorname{div} \vec{F} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_{\Delta S} \vec{F} \cdot \hat{n}\, ds .$$




Flux through face A:

$F_1(u_1+du_1, u_2, u_3) h_2(u_1+du_1, u_2, u_3) h_3(u_1+du_1, u_2, u_3)\, du_2\, d$

Flux through face B:

$-F_1(u_1, u_2, u_3)\, h_2(u_1, u_2, u_3) h_3(u_1, u_2; u_3)\, du_2\, du_3$

In a similar way the flux through the remaining faces can be found.

Therefore, the total flux per unit volume is

$$\frac{1}{\Delta V} \oint\!\!\oint_{\Delta S} \vec{F} \cdot d\vec{S} =$$

$$\frac{1}{h_1 h_2 h_3 \, du_1 \, du_2 \, du_3} \cdot \Big\{$$

$$\Big[ F_1(u_1+du_1, u_2, u_3)\, h_2(u_1+du_1, u_2, u_3)\, h_3(u_1+du_1, u_2, u_3)$$
$$- F_1(u_1, u_2, u_3)\, h_2(u_1, u_2, u_3)\, h_3(u_1, u_2, u_3) \Big] du_2 \, du_3$$

$$+ \Big[ F_2(u_1, u_2+du_2, u_3)\, h_1(u_1, u_2+du_2, u_3)\, h_3(u_1, u_2+du_2, u_3)$$
$$- F_2(u_1, u_2, u_3)\, h_1(u_1, u_2, u_3)\, h_3(u_1, u_2, u_3) \Big] du_1 \, du_3$$

$$+ \Big[ F_3(u_1, u_2, u_3+du_3)\, h_1(u_1, u_2, u_3+du_3)\, h_2(u_1, u_2, u_3+du_3)$$
$$- F_3(u_1, u_2, u_3)\, h_1(u_1, u_2, u_3)\, h_2(u_1, u_2, u_3) \Big] du_1 \, du_2 \Big\}$$

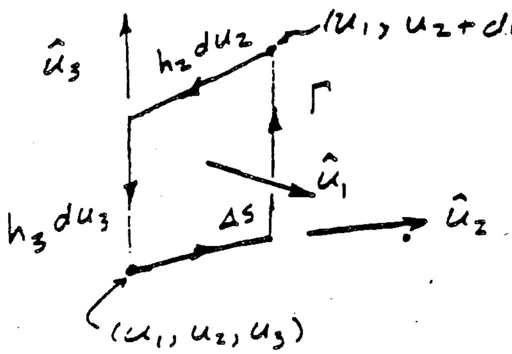
$$\therefore \; \operatorname{div} \vec{F} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial [h_2 h_3 F_1]}{\partial u_1} + \frac{\partial [h_1 h_3 F_2]}{\partial u_2} + \frac{\partial [h_1 h_2 F_3]}{\partial u_3} \right\}$$

Compare this with the formula in <u>rectangular</u> coordinates

$$\operatorname{div} \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

## g) Curl

Using the DEFINITION of curl, we can now construct
curl $\vec{F}(u_1, u_2, u_3)$ in this system:




The $\hat{u}_1$ component of curl $\vec{F}$ is defined as

$$\hat{u}_1 \cdot \text{curl}\,\vec{F} = \lim_{\Delta s \to 0} \frac{\oint_\Gamma \vec{F} \cdot d\vec{r}}{\Delta s}$$

The circulation of $\vec{F}$ about $\Gamma$ is

$$\begin{aligned}
&F_2(u_1, u_2, u_3)\, h_2(u_1, u_2, u_3)\, du_2 \\
&- F_2(u_1, u_2, u_3+du_3)\, h_2(u_1, u_2, u_3+du_3)\, du_2 \\
&+ F_3(u_1, u_2+du_2, u_3)\, h_3(u_1, u_2+du_2, u_3)\, du_3 \\
&- F_3(u_1, u_2, u_3)\, h_3(u_1, u_2, u_3)\, du_3 .
\end{aligned}$$

$\therefore$ The net circulation per unit area is

$$\frac{1}{h_2 h_3\, du_2\, du_3}\cdot$$

$$\{du_3[F_3(u_1, u_2+du_2, u_3)\, h_3(u_1, u_2+du_2, u_3) - F_3(u_1, u_2, u_3)\, h_3(u_1, u_2, u$$

$$- du_2[F_2(u_1, u_2, u_3+du_3) h_2(u_1, u_2, u_3+du_3) - F_2(u_1, u_2, u_3)\, h_2(u_1, u_2, u$$

$$= \frac{1}{h_2 h_3}\left\{\frac{\partial}{\partial u_2}(h_3 F_3) - \frac{\partial}{\partial u_3}(h_2 F_2)\right\}$$

$$= \hat{u}_1 \cdot \text{curl}\ \vec{F}.$$

The other components are similarly found and

$$\text{curl}\ \vec{F} = \frac{1}{h_1 h_2 h_3}\cdot \left\{ \hat{u}_1 h_1\left[\frac{\partial(h_3 F_3)}{\partial u_2} - \frac{\partial(h_2 F_2)}{\partial u_3}\right] + \hat{u}_2 h_2\left[\frac{\partial(h_1 F_1)}{\partial u_3} - \frac{\partial(h_3 F_3)}{\partial u_1}\right] + \hat{u}_3 h_3\left[\frac{\partial(h_2 F_2)}{\partial u_1} - \frac{\partial(h_1 F_1)}{\partial u_2}\right]\right\}$$

## h) Laplacian

Using the results obtained above, we can now construct $\nabla^2 w(u_1, u_2, u_3)$ in this system:

$$\nabla^2 w = \nabla \cdot (\nabla w) = \nabla \left[ \frac{1}{h_1} \frac{\partial w}{\partial u_1} \hat{u}_1 + \frac{1}{h_2} \frac{\partial w}{\partial u_2} \hat{u}_2 + \frac{1}{h_3} \frac{\partial w}{\partial u_3} \hat{u}_3 \right]$$

$$= \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial w}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left( \frac{h_1 h_3}{h_2} \frac{\partial w}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial w}{\partial u} \right.\right.$$

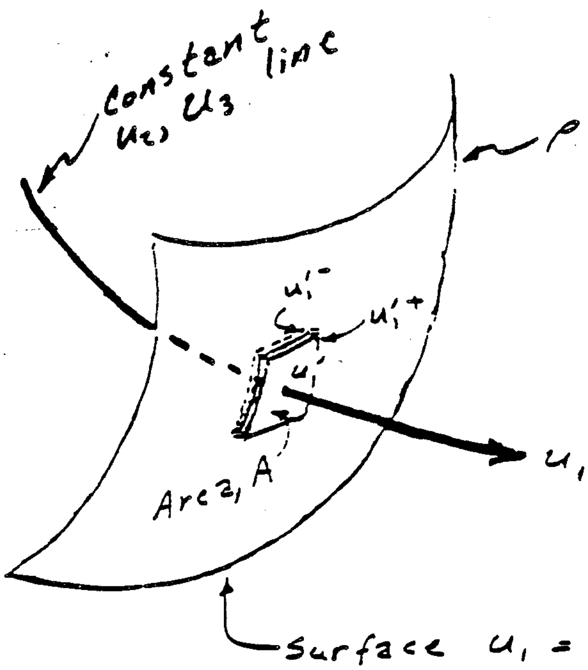
## i) Delta functions

We will now consider how to represent "point" sources in other coordinate systems.

### (1) Surface sources

Physical parameters such as charge and mass which are distributed in space are given in terms of a DENSITY.

A density is a quantity per unit VOLUME. If all the charge or mass is concentrated on a single coordinate surface, then the DENSITY is represented by




Surface density (eg. $C/m^2$)

$$\rho = \frac{1}{h_1}\delta(u_1 - u_1') \cdot 1$$

$$\iint_A \int_{u_1'^-}^{u_1'^+} \rho(u_1')\, \overbrace{dl_1\, dl_2\, dl_3}^{dV}$$

$$= A \int_{u_1'^-}^{u_1'^+} \rho(u_1')\, h_1\, du_1$$

$$= A \int_{u_1'^-}^{u_1'^+} \frac{1}{h_1}\delta(u_1 - u_1')\, h_1\, du_1$$

$$= 1 \cdot A$$

$u_1'^- = u_1' - \epsilon$

$u_1'^+ = u_1' + \epsilon$

$\epsilon > 0$

Without this factor of $\frac{1}{h_1}$, the integral of $\rho$ over this volume would be $h_1 \cdot A$ not $A$ as it should be.

(2) Line source

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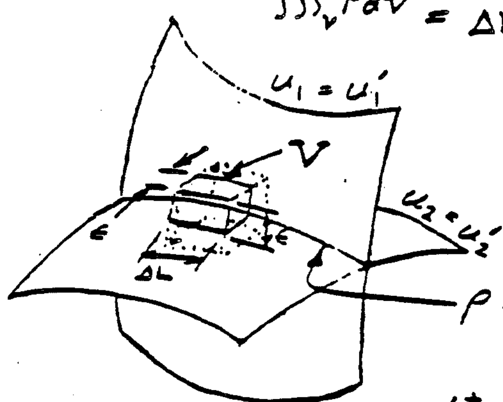
Suppose that all the charge is concentrated at the intersection of two coordinate surfaces.

The the density of charge is




$$\iiint_V \rho\, dV = \Delta L \int_{u_2'^-}^{u_2'^+}\int_{u_1'^-}^{u_1'^+} \rho\, dl_1\, dl_2 = \Delta L \int_{u_2'^-}^{u_2'^+}\int_{u_1'^-}^{u_1'^+} \rho\, h_1 h_2\, du_1 du_2$$

$$= \Delta L \int_{u_2'^-}^{u_2'^+}\int_{u_1'^-}^{u_1'^+} \frac{1}{h_1 h_2}\, \delta(u_1 - u_1')\,\delta(u_2 - u_2') \cdot h_1 h_2\, du_1 du_2 = 1 \cdot$$

$$\rho = \frac{1}{h_1 h_2}\, \delta(u_1 - u_1')\, \delta(u_2 - u_2')$$

$$u_1'^{\pm} \equiv u_1' \pm \epsilon$$
$$u_2'^{\pm} \equiv u_2' \pm \epsilon$$

(3) Point source

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Finally, the density of a point source which is located at the intersection of three coordinate surfaces is

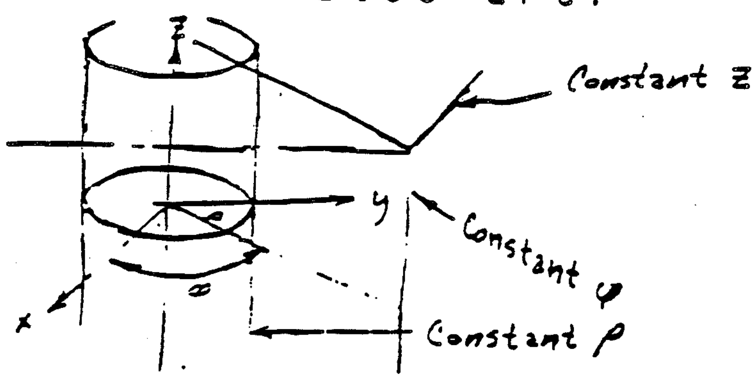
$$\rho = \frac{1}{h_1 h_2 h_3}\, \delta(u_1 - u_1')\, \delta(u_2 - u_2')\, \delta(u_3 - u_3')$$

$$\iiint \rho\, dV = \iiint \rho \cdot h_1 h_2 h_3\, du_1 du_2 du_3 = 1$$

## j) Applications

### (1) Cylindrical coordinates

The scale factors for cylindrical coordinates are:




$$h_\rho = 1$$
$$h_\varphi = \rho$$
$$h_z = 1$$

Therefore, the grad, div, curl, and Laplacian in cylindrical coordinates are:

$$\operatorname{grad} w = \frac{\partial w}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial w}{\partial \varphi}\hat{\varphi} + \frac{\partial w}{\partial z}\hat{z}$$

$$\operatorname{div} \bar{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \varphi}F_\varphi + \frac{\partial}{\partial z}F_z$$

$$\operatorname{curl} \bar{F} = \hat{\rho}\left[\frac{1}{\rho}\frac{\partial F_z}{\partial \varphi} - \frac{\partial F_\varphi}{\partial z}\right] + \hat{\varphi}\left[\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right] + \hat{z}\frac{1}{\rho}\left[\frac{\partial(\rho F_\varphi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \varphi}\right]$$

$$\nabla^2 w = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial w}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{\partial^2 w}{\partial z^2}.$$

(2) Spherical coordinates
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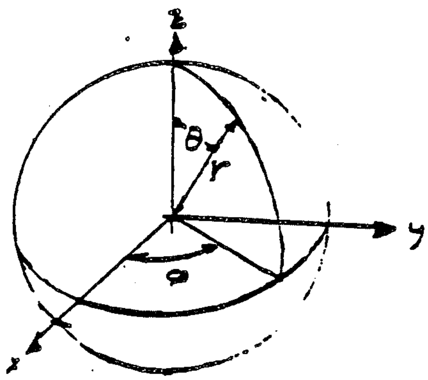
The scale factors for spherical coordinates are




$$h_1 = 1$$
$$h_2 = r$$
$$h_3 = r\sin\theta$$

Therefore, the grad, div, curl, and laplacian in cylindrical coordinates are:

$$\text{grad } w = \frac{\partial w}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial w}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial w}{\partial \varphi}\hat{\varphi}$$

$$\text{div } \vec{F} = \frac{1}{r^2}\frac{\partial (r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\varphi) + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial \varphi}$$

$$\text{curl } \vec{F} = \hat{r}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\varphi) - \frac{1}{r\sin\theta}\frac{\partial F_\theta}{\partial\varphi}\right]$$
$$+ \hat{\theta}\left[\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(r F_\varphi)\right]$$
$$+ \hat{\varphi}\left[\frac{1}{r}\frac{\partial (r F_\theta)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial\theta}\right]$$

$$\nabla^2 w = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$