Solution Section 3.1 – Integrals over Rectangular Regions

Exercise

Evaluate the iterated integral $\int_1^2 \int_0^4 2xy\ dydx$

Solution

$$\int_1^2 \int_0^4 2xy\ dydx = \int_1^2 x\left[y^2\right]_0^4 dx$$

$$= \int_1^2 16xdx$$

$$= 8\left[x^2\right]_1^2$$

$$= 8(4-1)$$

$$= 24$$

Exercise

Evaluate the iterated integral $\int_0^2 \int_{-1}^1 (x-y)\ dydx$

Solution

$$\int_0^2 \int_{-1}^1 (x-y)\ dydx = \int_0^2 \left[xy - \frac{1}{2}y^2\right]_{-1}^1 dx$$

$$= \int_0^2 \left[x - \frac{1}{2} - \left(-x - \frac{1}{2}\right)\right] dx$$

$$= \int_0^2 2x\ dx$$

$$= x^2 \Big|_0^2$$

$$= 4$$

Exercise

Evaluate the iterated integral $\displaystyle\int_0^1\int_0^1\left(1-\frac{x^2+y^2}{2}\right)dxdy$

Solution

$$\begin{aligned}\int_0^1\int_0^1\left(1-\frac{x^2+y^2}{2}\right)dxdy &= \int_0^1\left[x-\frac{1}{6}x^3-\frac{1}{2}y^2x\right]_0^1 dy\\ &= \int_0^1\left(1-\frac{1}{6}-\frac{1}{2}y^2\right)dy\\ &= \int_0^1\left(\frac{5}{6}-\frac{1}{2}y^2\right)dy\\ &= \left[\frac{5}{6}y-\frac{1}{6}y^3\right]_0^1\\ &= \frac{5}{6}-\frac{1}{6}\\ &= \frac{4}{6}\\ &= \boxed{\frac{2}{3}}\end{aligned}$$

Exercise

Evaluate the integral $\displaystyle\int_0^3\int_{-2}^0\left(x^2y-2xy\right)dydx$

Solution

$$\begin{aligned}\int_0^3\int_{-2}^0\left(x^2y-2xy\right)dydx &= \int_0^3\left[\frac{1}{2}x^2y^2-xy^2\right]_{-2}^0 dx\\ &= \int_0^3\left(-2x^2+4x\right)dx\\ &= \left[-\frac{2}{3}x^3+2x^2\right]_0^3\\ &= -18+18\\ &= \boxed{0}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 \int_0^1 \frac{y}{1+xy}\,dxdy$

Solution

$$\int_0^1 \int_0^1 \frac{y}{1+xy}\,dxdy = \int_0^1 \int_0^1 \frac{d(1+xy)}{1+xy}\,dy \qquad d(1+xy) = ydx$$

$$= \int_0^1 \left[\ln|1+xy| \right]_0^1 dy$$

$$= \int_0^1 \ln|1+y|\,dy \qquad d(1+y) = dy$$

$$= \left[(y+1)\ln|1+y| - (y+1) \right]_0^1 \qquad \int \ln u\,du = u\ln u - u$$

$$= 2\ln 2 - 2 + 1$$

$$= 2\ln 2 - 1$$

Exercise

Evaluate the integral $\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y}\,dydx$

Solution

$$\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y}\,dydx = \int_0^{\ln 2} e^{2x}dx \int_1^{\ln 5} e^{y}dy$$

$$= \left[\frac{1}{2} e^{2x} \right]_0^{\ln 2} \left[e^{y} \right]_1^{\ln 5}$$

$$= \frac{1}{2}\left(e^{2\ln 2} - 1 \right)\left(e^{\ln 5} - e \right)$$

$$= \frac{1}{2}(4-1)(5-e)$$

$$= \frac{15}{2} - \frac{3}{2}e$$

Exercise

Evaluate the integral $\int_0^1 \int_1^2 xye^{x}\,dydx$

Solution

$$\int_0^1 \int_1^2 xye^{x}\,dydx = \int_0^1 xe^{x} \left[\frac{1}{2} y^2 \right]_1^2 dx$$

$$= \frac{3}{2}\int_0^1 xe^x dx$$

$$= \frac{3}{2}\left[xe^x - e^x \right]_0^1$$

$$= \frac{3}{2}(e - e + 1)$$

$$= \frac{3}{2}$$

Exercise

Evaluate the double integral $\int_\pi^{2\pi}\int_0^\pi (\sin x + \cos y)\, dxdy$

Solution

$$\int_\pi^{2\pi}\int_0^\pi (\sin x + \cos y)\, dxdy = \int_\pi^{2\pi} \left[-\cos x + x\cos y \right]_0^\pi dy$$

$$= \int_\pi^{2\pi} (1 + \pi\cos y + 1)\, dy$$

$$= \left[2y + \pi \sin y \right]_\pi^{2\pi}$$

$$= 4\pi - 2\pi$$

$$= 2\pi$$

Exercise

Evaluate the double integral $\int_1^2\int_1^4 \frac{xy}{\left(x^2+y^2\right)^2} dxdy$

Solution

$$\int_1^2\int_1^4 \frac{xy}{\left(x^2+y^2\right)^2} dxdy = \frac{1}{2}\int_1^2\int_1^4 \frac{y}{\left(x^2+y^2\right)^2} d\left(x^2+y^2\right) dy$$

$$= -\frac{1}{2}\int_1^2 \left. \frac{y}{x^2+y^2} \right|_1^4 dy$$

$$= -\frac{1}{2}\int_1^2 \left(\frac{y}{16+y^2} - \frac{y}{1+y^2} \right) dy$$

$$= -\frac{1}{4}\int_1^2 \frac{d\left(16+y^2\right)}{16+y^2} + \frac{1}{4}\int_1^2 \frac{d\left(1+y^2\right)}{1+y^2}$$

$$= -\frac{1}{4}\ln\left(16+y^2\right) + \frac{1}{4}\ln\left(1+y^2\right)\Big|_1^2$$

$$= \frac{1}{4}\left(-\ln 20 + \ln 17 + \ln 5 - \ln 2\right)$$

$$= \frac{1}{4}\ln\left(\frac{17\times 5}{20\times 2}\right)$$

$$= \frac{1}{4}\ln\left(\frac{17}{8}\right)$$

Exercise

Evaluate the double integral $\int_1^3 \int_1^{e^x} \frac{x}{y}\, dydx$

Solution

$$\int_1^3 \int_1^{e^x} \frac{x}{y}\, dydx = \int_1^3 x \ln y \Big|_1^{e^x} dx$$

$$= \int_1^3 x(x)\, dx$$

$$= \frac{1}{3}x^3 \Big|_1^3$$

$$= \frac{26}{3}$$

Exercise

Evaluate the double integral $\int_1^2 \int_0^{\ln x} x^3 e^y\, dydx$

Solution

$$\int_1^2 \int_0^{\ln x} x^3 e^y\, dydx = \int_1^2 x^3 e^y \Big|_0^{\ln x} dx$$

$$= \int_1^2 x^3 (x-1)\, dx$$

$$= \int_1^2 \left(x^4 - x^3\right) dx$$

$$= \frac{1}{5}x^5 - \frac{1}{4}x^4 \Big|_1^2$$

$$= \frac{32}{5} - 4 - \frac{1}{5} + \frac{1}{4}$$

$$= \frac{31}{5} - \frac{15}{4}$$

$$\underline{= \frac{49}{20}}$$

Exercise

Evaluate the double integral $\int_1^{10} \int_0^{1/y} ye^{xy} dxdy$

Solution

$$\int_1^{10} \int_0^{1/y} ye^{xy} dxdy = \int_1^{10} \int_0^{1/y} e^{xy} d\left(e^{xy}\right) dy \qquad d\left(e^{xy}\right) = ye^{xy} dx$$

$$= \int_1^{10} e^{xy} \Big|_0^{1/y} dy$$

$$= \int_1^{10} (e-1)\ dy$$

$$= (e-1)y \Big|_1^{10}$$

$$= (e-1)(10-1)$$

$$\underline{= 9(e-1)}$$

Exercise

Evaluate the double integral $\int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} xy\ dxdy$

Solution

$$\int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} xy\ dxdy = \frac{1}{2} \int_0^1 yx^2 \Big|_{\sqrt{y}}^{2-\sqrt{y}} dy$$

$$= \frac{1}{2}\int_0^1 y\left(\left(2-\sqrt{y}\right)^2 - y\right)dy$$

$$= \frac{1}{2}\int_0^1 y\left(4-4\sqrt{y}+y-y\right)dy$$

$$= 2\int_0^1 \left(y - y^{3/2}\right)dy$$

$$= 2\left(\frac{1}{2}y^{32} - \frac{2}{5}y^{5/2}\right)\Big|_0^1$$

$$= 2\left(\frac{1}{2}-\frac{2}{5}\right)$$

$$= \frac{1}{5}$$

Exercise

Evaluate the double integral $\int_0^1\int_{x^2}^{x}\sqrt{x}\ dydx$

Solution

$$\int_0^1\int_{x^2}^{x}\sqrt{x}\ dydx = \int_0^1 x^{1/2}y\Big|_{x^2}^{x}\ dx$$

$$= \int_0^1 x^{1/2}\left(x-x^2\right)\ dx$$

$$= \int_0^1 \left(x^{3/2}-x^{5/2}\right)\ dx$$

$$= \frac{2}{5}x^{5/2} - \frac{2}{7}x^{7/2}\Big|_0^1$$

$$= \frac{2}{5}-\frac{2}{7}$$

$$= \frac{4}{35}$$

Exercise

Evaluate the double integral $\int_0^{3/2}\int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y\,dxdy$

Solution

$$\int_0^{3/2}\int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y\,dxdy = \int_0^{3/2} yx\Big|_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}}\,dy$$

$$= 2\int_0^{3/2} y\sqrt{9-4y^2}\,dy$$

$$= -\frac{1}{4}\int_0^{3/2}\left(9-4y^2\right)^{1/2}\,d\left(9-4y^2\right)$$

$$= -\frac{1}{6}\left(9-4y^2\right)^{3/2}\Big|_0^{3/2}$$

$$= -\frac{1}{6}(-27)$$

$$= \frac{9}{2}$$

Exercise

Evaluate the double integral $\int_0^2\int_0^{4-x^2} 2x\,dydx$

Solution

$$\int_0^2\int_0^{4-x^2} 2x\,dydx = \int_0^2 2xy\Big|_0^{4-x^2}\,dx$$

$$= \int_0^2\left(8x-2x^3\right)dx$$

$$= 4x^2 - \frac{1}{2}x^4\Big|_0^2$$

$$= 16-8$$

$$= 8$$

Exercise

Evaluate the double integral $\int_0^1 \int_{2y}^2 4\cos\left(x^2\right)\, dxdy$

Solution

$$x = 2y \quad \rightarrow \quad y = \frac{x}{2}$$

$$\begin{aligned}
\int_0^1 \int_{2y}^2 4\cos\left(x^2\right)\, dxdy &= \int_0^2 \int_0^{x/2} 4\cos\left(x^2\right)\, dydx \\
&= \int_0^2 4\cos\left(x^2\right) y \Big|_0^{x/2}\, dx \\
&= \int_0^2 2x\cos\left(x^2\right) dx \\
&= \int_0^2 \cos x^2\, d\left(x^2\right) \\
&= \sin x^2 \Big|_0^2 \\
&= \sin 4
\end{aligned}$$

Exercise

Evaluate the double integral $\int_0^1 \int_{\sqrt[3]{y}}^1 \frac{2\pi \sin \pi x^2}{x^2}\, dxdy$

Solution

$$x = \sqrt[3]{y} \quad \rightarrow \quad y = x^3$$

$$\begin{aligned}
\int_0^1 \int_{\sqrt[3]{y}}^1 \frac{2\pi \sin \pi x^2}{x^2}\, dxdy &= \int_0^1 \int_0^{x^3} \frac{2\pi \sin \pi x^2}{x^2}\, dydx \\
&= \int_0^1 \frac{2\pi \sin \pi x^2}{x^2}\, y \Big|_0^{x^3}\, dx \\
&= \int_0^1 2\pi x \sin \pi x^2\, dx \\
&= \int_0^1 \sin \pi x^2\, d\left(\pi x^2\right)
\end{aligned}$$

$$= -\cos \pi x^2 \Big|_0^1$$

$$= -(\cos \pi - \cos 0)$$

$$\underline{= 2}\,\rfloor$$

Exercise

Evaluate the double integral over the given region R $\displaystyle\iint_R \left(6y^2 - 2x\right) dA \quad R: \ 0 \le x \le 1, \quad 0 \le y \le 2$

Solution

$$\iint_R \left(6y^2 - 2x\right) dA = \int_0^1 \int_0^2 \left(6y^2 - 2x\right) dy\,dx$$

$$= \int_0^1 \left[2y^3 - 2xy\right]_0^2 dx$$

$$= \int_0^1 (16 - 4x)\,dx$$

$$= \left[16x - 2x^2\right]_0^1$$

$$\underline{= 14}\,\rfloor$$

Exercise

Evaluate the double integral over the given region R $\displaystyle\iint_R \left(\frac{\sqrt{x}}{y^2}\right) dA \quad R: \ 0 \le x \le 4, \quad 1 \le y \le 2$

Solution

$$\iint_R \left(\frac{\sqrt{x}}{y^2}\right) dA = \int_0^4 \int_1^2 \left(\frac{\sqrt{x}}{y^2}\right) dy\,dx$$

$$= \int_0^4 \left[-\frac{\sqrt{x}}{y}\right]_1^2 dx$$

$$= \int_0^4 -\sqrt{x}\left(\tfrac{1}{2} - 1\right) dx$$

$$= \frac{1}{2}\int_0^4 x^{1/2}\,dx$$

$$= \frac{1}{3}\left[x^{3/2} \right]_0^4$$

$$\boxed{= \frac{8}{3}}$$

Exercise

Evaluate the double integral over the given region R $\iint\limits_R y\sin(x+y)\,dA \quad R:\ -\pi \le x \le 0, \quad 0 \le y \le \pi$

Solution

$$\iint\limits_R y\sin(x+y)\,dA = \int_{-\pi}^{0}\int_0^{\pi} y\sin(x+y)\,dxdy$$

		$\int \sin(x+y)$
+	y	$-\cos(x+y)$
−	1	$-\sin(x+y)$

$$= \int_{-\pi}^{0} \left[-y\cos(x+y) + \sin(x+y) \right]_0^{\pi} dx$$

$$= \int_{-\pi}^{0} \left[\sin(x+\pi) - \pi\cos(x+\pi) - \sin x \right] dx$$

$$= \left[-\cos(x+\pi) - \pi\sin(x+\pi) + \cos x \right]_{-\pi}^{0}$$

$$= -(-1) + 1 - (-1-1)$$

$$\boxed{= 4}$$

Exercise

Evaluate the double integral over the given region R. $\iint\limits_R e^{x-y}\,dA \quad R:\ 0 \le x \le \ln 2, \quad 0 \le y \le \ln 2$

Solution

$$\iint\limits_R e^{x-y}\,dA = \int_0^{\ln 2}\int_0^{\ln 2} e^{x-y}\,dydx$$

$$= \int_0^{\ln 2} \left[-e^{x-y} \right]_0^{\ln 2} dx$$

$$= \int_0^{\ln 2} \left(-e^{x-\ln 2} + e^{x} \right) dx$$

$$= \left[-e^{x-\ln 2} + e^{x} \right]_0^{\ln 2}$$

$$= -1 + e^{\ln 2} + e^{-\ln 2} - 1 \qquad e^{-\ln 2} = e^{\ln 2^{-1}} = 2^{-1} = \frac{1}{2}$$

$$= -2 + 2 + \frac{1}{2}$$

$$\boxed{= \frac{1}{2}}$$

Exercise

Evaluate the double integral over the given region R. $\iint_R \frac{y}{x^2y^2+1}\,dA \quad R:\ 0 \le x \le 1,\quad 0 \le y \le 1$

Solution

$$\iint_R \frac{y}{x^2y^2+1}\,dA = \int_0^1 \int_0^1 \frac{y}{(xy)^2+1}\,dxdy \qquad \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} \quad u = xy \to du = ydx$$

$$= \int_0^1 \left[\tan^{-1}(xy)\right]_0^1 dy$$

$$= \int_0^1 \tan^{-1} y\ dy \qquad \int \tan^{-1} ax\,dx = x\tan^{-1} ax - \frac{1}{2a}\ln\left(1+a^2x^2\right)$$

$$= \left[y\tan^{-1} y - \frac{1}{2}\ln\left|1+y^2\right| \right]_0^1$$

$$= \tan^{-1} 1 - \frac{1}{2}\ln 2$$

$$\boxed{= \frac{\pi}{4} - \frac{1}{2}\ln 2}$$

Exercise

Evaluate $\iint_R x^{-1/2}e^y\,dA$; R is the region bounded by $x=1$, $x=4$, $y=\sqrt{x}$, and $y=0$

Solution

$$\iint_R x^{-1/2}e^y\,dA = \int_1^4 \int_0^{\sqrt{x}} x^{-1/2}e^y\ dydx$$

$$= \int_1^4 x^{-1/2}e^y \Big|_0^{\sqrt{x}}\ dx$$

$$= \int_1^4 x^{-1/2}\left(e^{\sqrt{x}} - 1\right)dx$$

$$= \int_1^4 \frac{1}{\sqrt{x}} e^{\sqrt{x}} dx - \int_1^4 x^{-1/2} dx$$

$$= 2\int_1^4 e^{\sqrt{x}} d\left(\sqrt{x}\right) - 2\sqrt{x}\,\Big|_1^4$$

$$= 2e^{\sqrt{x}}\Big|_1^4 - 2(2-1)$$

$$= 2\left(e^2 - e\right) - 2$$

$$\underline{= 2e^2 - 2e - 2}$$

Exercise

Evaluate $\iint\limits_R \left(x^2 + y^2\right) dA$; R is the region $\left\{(x, y):\ 0 \le x \le 2,\ 0 \le y \le x\right\}$

Solution

$$\iint\limits_R \left(x^2 + y^2\right) dA = \int_0^2 \int_0^x \left(x^2 + y^2\right) dy dx$$

$$= \int_0^2 \left(x^2 y + \frac{1}{3} y^3\right)\Big|_0^x dx$$

$$= \int_0^2 \left(x^3 + \frac{1}{3} x^3\right) dx$$

$$= \frac{4}{3} \int_0^2 x^3 dx$$

$$= \frac{1}{3} x^4 \Big|_0^2$$

$$\underline{= \frac{16}{3}}$$

Exercise

Evaluate $\displaystyle\iint_R \frac{2y}{\sqrt{x^4+1}}\,dA;$ *R is the region bounded by* $x=1, \quad x=2, \quad y=x^{3/2}, \quad y=0$

Solution

$$\begin{aligned}
\iint_R \frac{2y}{\sqrt{x^4+1}}\,dA &= \int_1^2 \int_0^{x^{3/2}} \frac{2y}{\sqrt{x^4+1}}\,dydx \\
&= \int_1^2 \frac{1}{\sqrt{x^4+1}}\, y^2 \Big|_0^{x^{3/2}}\, dx \\
&= \int_1^2 \frac{x^3}{\sqrt{x^4+1}}\,dx \\
&= \frac{1}{4}\int_1^2 \left(x^4+1\right)^{-1/2} d\left(x^4+1\right) \\
&= \frac{1}{2}\sqrt{x^4+1}\,\Big|_1^2 \\
&= \frac{1}{2}\left(\sqrt{17}-\sqrt{2}\right)
\end{aligned}$$

Exercise

Integrate $f(x,y)=\frac{1}{xy}$ over the ***square*** $1\le x\le 2, \quad 1\le y\le 2$

Solution

$$\begin{aligned}
\int_1^2\int_1^2 \frac{1}{xy}\,dydx &= \int_1^2 \frac{1}{x}\left[\ln y\right]_1^2\,dx \\
&= \int_1^2 \frac{1}{x}\left[\ln 2-\ln 1\right]dx \\
&= \ln 2\int_1^2 \frac{1}{x}\,dx \\
&= \ln 2\,\left[\ln x\right]_1^2 \\
&= \ln 2\cdot\ln 2 \\
&= (\ln 2)^2
\end{aligned}$$

Exercise

Integrate $f(x,y) = y\cos xy$ over the ***rectangle*** $0 \le x \le \pi, \quad 0 \le y \le 1$

Solution

$$\int_0^1 \int_0^{\pi} y\cos(xy)\,dxdy = \int_0^1 \left[\sin xy\right]_0^{\pi} dy$$

$$= \int_0^1 \sin(\pi y)\,dy$$

$$= -\frac{1}{\pi}\cos \pi y \Big|_0^1$$

$$= -\frac{1}{\pi}[-1-1]$$

$$= \frac{2}{\pi}$$

Exercise

Find the volume of the region bounded above the paraboloid $z = x^2 + y^2$ and below by the square $R: \quad -1 \le x \le 1, \quad -1 \le y \le 1$

Solution

$$V = \int_{-1}^1 \int_{-1}^1 \left(x^2 + y^2\right) dydx$$

$$= \int_{-1}^1 \left[x^2 y + \frac{1}{3}y^3\right]_{-1}^1 dx$$

$$= \int_{-1}^1 \left[x^2 + \frac{1}{3} - \left(-x^2 - \frac{1}{3}\right)\right] dx$$

$$= \int_{-1}^1 \left(2x^2 + \frac{2}{3}\right) dx$$

$$= \left[\frac{2}{3}x^3 + \frac{2}{3}x\right]_{-1}^1$$

$$= \frac{2}{3} + \frac{2}{3} - \left(-\frac{2}{3} - \frac{2}{3}\right)$$

$$= \frac{8}{3} \ \ unit^3$$

Exercise

Find the volume of the region bounded above the plane $z = \frac{y}{2}$ and below by the rectangle

$R: \; 0 \le x \le 4, \quad 0 \le y \le 2$

Solution

$$V = \int_0^4 \int_0^2 \frac{y}{2}\, dydx$$

$$= \int_0^4 \left[\frac{1}{4}y^2\right]_0^2 dx$$

$$= \int_0^4 (1)\, dx$$

$$= x \Big|_0^4$$

$$\boxed{= 4 \quad unit^3}$$

Exercise

Find the volume of the region bounded above the surface $z = 4 - y^2$ and below by the rectangle

$R: \; 0 \le x \le 1, \quad 0 \le y \le 2$

Solution

$$V = \int_0^1 \int_0^2 \left(4 - y^2\right) dydx$$

$$= \int_0^1 \left[4y - \frac{1}{3}y^3\right]_0^2 dx$$

$$= \int_0^1 \left(8 - \frac{8}{3}\right) dx$$

$$= \int_0^1 \frac{16}{3}\, dx$$

$$= \left[\frac{16}{3}x\right]_0^1$$

$$\boxed{= \frac{16}{3} \quad unit^3}$$

Exercise

Find the volume of the region bounded above the ellipitical paraboloid $z = 16 - x^2 - y^2$ and below by the square $R: \ 0 \le x \le 2, \quad 0 \le y \le 2$

Solution

$$V = \int_0^2 \int_0^2 \left(16 - x^2 - y^2\right) dy dx$$

$$= \int_0^2 \left[16y - x^2 y - \frac{1}{3}y^3\right]_0^2 dx$$

$$= \int_0^2 \left(32 - 2x^2 - \frac{8}{3}\right) dx$$

$$= \int_0^2 \left(\frac{88}{3} - 2x^2\right) dx$$

$$= \left[\frac{88}{3}x - \frac{2}{3}x^3\right]_0^2$$

$$= \frac{176}{3} - \frac{16}{3}$$

$$= \frac{160}{3} \ \ unit^3$$

Exercise

Evaluate $\displaystyle\int_0^{1/2} \left(\sin^{-1}[2x] - \sin^{-1} x\right) dx$ by converting it to a double integral.

Solution

$$0 \le x \le \frac{1}{2}$$

$$\begin{cases} \sin^{-1} 2x = y & \rightarrow \ 2x = \sin y \quad \Rightarrow x = \frac{1}{2}\sin y \\ \sin^{-1} x = y & \rightarrow \ x = \sin y \end{cases}$$

$$\begin{cases} x = 0 & \rightarrow y = 0 \\ x = \frac{1}{2} & \rightarrow \begin{cases} y = \sin^{-1} 1 = \frac{\pi}{2} \\ y = \sin^{-1} \frac{1}{2} = \frac{\pi}{6} \end{cases} \end{cases}$$

$$\int_0^{1/2}\left(\sin^{-1}(2x)-\sin^{-1}x\right)dx=\int_0^{\frac{\pi}{6}}\int_{\frac{1}{2}\sin y}^{\sin y}dxdy+\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_{\frac{1}{2}\sin y}^{\frac{1}{2}}dxdy$$

$$=\int_0^{\frac{\pi}{6}}x\Big|_{\frac{1}{2}\sin y}^{\sin y}dy+\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}x\Big|_{\frac{1}{2}\sin y}^{\frac{1}{2}}dy$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{6}}\sin y\,dy+\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(1-\sin y)\,dy$$

$$=-\frac{1}{2}\cos y\Big|_0^{\frac{\pi}{6}}+\frac{1}{2}(y+\cos y)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$=-\frac{1}{2}\left(\frac{\sqrt{3}}{2}-1\right)+\frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{6}-\frac{\sqrt{3}}{2}\right)$$

$$=\frac{1}{2}-\frac{\sqrt{3}}{4}+\frac{\pi}{6}-\frac{\sqrt{3}}{4}$$

$$=\frac{\pi}{6}+\frac{1}{2}-\frac{\sqrt{3}}{2}$$

Exercise

Find the volume of the solid beneath the cylinder $f(x, y)=e^{-x}$ and above the region $R=\{(x, y):\ 0\le x\le \ln 4,\ \ -2\le y\le 2\}$

Solution

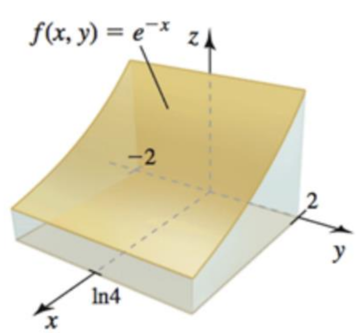
$$V=\int_{-2}^{2}\int_0^{\ln 4}e^{-x}dxdy$$

$$=-\int_{-2}^{2}e^{-x}\Big|_0^{\ln 4}dy$$

$$=-\int_{-2}^{2}\left(\frac{1}{4}-1\right)dy$$

$$=\frac{3}{4}y\Big|_{-2}^{2}$$

$$=3\ \ unit^3$$



Exercise

Find the volume of the solid beneath the plane $f(x, y) = 6 - x - 2y$ and above the region $R = \{(x, y): \ 0 \le x \le 2, \ \ 0 \le y \le 1\}$

Solution

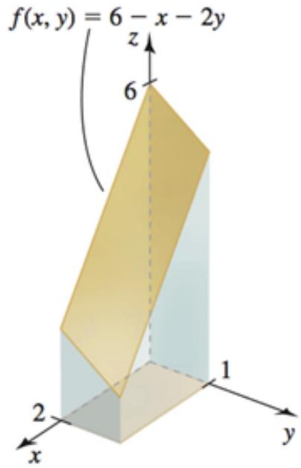
$$V = \int_0^1 \int_0^2 (6 - x - 2y)\,dxdy$$

$$= \int_0^1 \left(6x - \frac{1}{2}x^2 - 2yx\right)\Big|_0^2 \,dy$$

$$= \int_0^1 (10 - 4y)\,dy$$

$$= \left(10y - 2y^2\right)\Big|_0^1$$

$$= 8 \ \ unit^3$$



Exercise

Find the volume of the solid beneath the plane $f(x, y) = 24 - 3x - 4y$ and above the region $R = \{(x, y): \ -1 \le x \le 3, \ \ 0 \le y \le 2\}$

Solution

$$V = \int_{-1}^3 \int_0^2 (24 - 3x - 4y)\,dydx$$

$$= \int_{-1}^3 \left(24y - 3xy - 2y^2\right)\Big|_0^2 \,dx$$

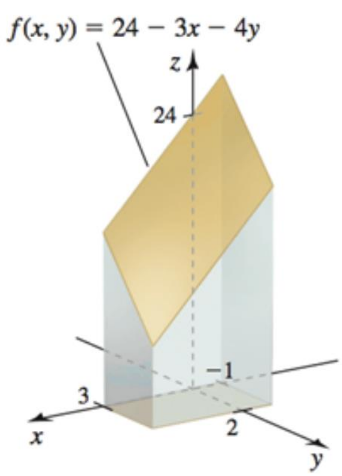
$$= \int_{-1}^3 (48 - 6x - 8)\,dx$$

$$= \int_{-1}^3 (40 - 6x)\,dx$$

$$= \left(40x - 3x^2\right)\Big|_{-1}^3$$

$$= 120 - 27 + 40 + 3$$

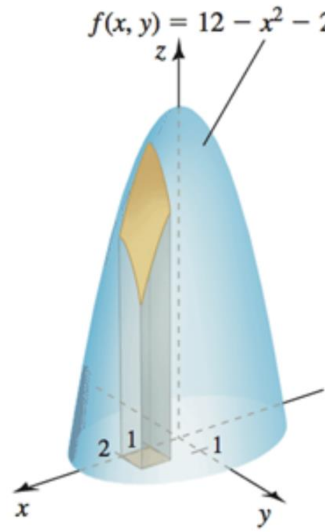
$$= 136 \ \ unit^3$$



Exercise

Find the volume of the solid beneath the paraboloid $f(x, y) = 12 - x^2 - 2y^2$ and above the region $R = \{(x, y): \ 1 \le x \le 2, \quad 0 \le y \le 1\}$

Solution



$$V = \int_1^2 \int_0^1 \left(12 - x^2 - 2y^2\right) dydx$$

$$= \int_1^2 \left(12y - x^2y - \frac{2}{3}y^3\right)\Big|_0^1 \ dx$$

$$= \int_1^2 \left(12 - x^2 - \frac{2}{3}\right) dx$$

$$= \int_1^2 \left(\frac{34}{3} - x^2\right) dx$$

$$= \left(\frac{34}{3}x - \frac{1}{3}x^3\right)\Big|_1^2$$

$$= \frac{68}{3} - \frac{8}{3} - \frac{34}{3} + \frac{1}{3}$$

$$= 9 \ \ unit^3$$