Solution   Section 2.10 – Applications

Exercise

Solve using the Laplace transform: $y' + y = te^t$, $y(0) = -2$

Solution

$$\mathcal{L}(y'+y) = \mathcal{L}\left(te^t\right)$$

$$\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}\left(te^t\right)$$

$$sY(s) - y(0) + Y(s) = \frac{1}{(s-1)^2} \qquad\qquad y(0) = -2$$

$$(s+1)Y(s) + 2 = \frac{1}{(s-1)^2}$$

$$(s+1)Y(s) = \frac{1}{(s-1)^2} - 2$$

$$Y(s) = \frac{1}{(s+1)(s-1)^2} - \frac{2}{s+1}$$

$$\frac{1}{(s+1)(s-1)^2} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$1 = (A+B)s^2 + (C-2A)s + A - B + C$$

$$\begin{cases} A+B=0 \\ C-2A=0 \\ A-B+C=1 \end{cases} \Rightarrow A = \frac{1}{4} \quad B = -\frac{1}{4} \quad C = \frac{1}{2}$$

$$Y(s) = \frac{1}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2} - \frac{2}{s+1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{7}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2}\right\}$$

$$y(t) = -\frac{7}{4}e^{-t} - \frac{1}{4}e^t + \frac{1}{2}te^t$$

Exercise

Solve using the Laplace transform: $y' - y = 2\cos 5t$, $y(0) = 0$

Solution

$$\mathcal{L}\{y' - y\} = \mathcal{L}\{2\cos 5t\}$$

$$sY(s) - y(0) - Y(s) = \frac{2s}{s^2+25} \qquad\qquad y(0) = 0$$

$$\left(s-1\right)Y(s)=\frac{2s}{s^2+25}$$

$$Y(s)=\frac{2s}{\left(s-1\right)\left(s^2+25\right)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+25}$$

$$2s=As^2+25A+Bs^2-Bs+Cs-C$$

$$\begin{cases} A+B=0 \\ -B+C=2 \\ 25A-C=0 \end{cases} \Rightarrow A=\frac{1}{13} \quad B=-\frac{1}{13} \quad C=\frac{25}{13}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{13}\frac{1}{s-1}-\frac{1}{13}\frac{s}{s^2+25}+\frac{25}{13}\frac{1}{5}\frac{5}{s^2+25}\right\}$$

$$\boxed{y(t)=\frac{1}{13}e^t-\frac{1}{13}\cos 4t+\frac{5}{13}\sin 5t}$$

Exercise

Solve using the Laplace transform: $y'-y=1+te^t$, $y(0)=0$

Solution

$$\mathcal{L}\{y'-y\}=\mathcal{L}\left\{1+te^t\right\}$$

$$sY(s)-y(0)-Y(s)=\frac{1}{s}+\frac{1}{\left(s-1\right)^2} \qquad y(0)=0$$

$$\left(s-1\right)Y(s)=\frac{s^2-s+1}{s\left(s-1\right)^2}$$

$$Y(s)=\frac{s^2-s+1}{s\left(s-1\right)^3}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{\left(s-1\right)^2}+\frac{D}{\left(s-1\right)^3}$$

$$s^2-s+1=As^3-3As^2+3As-A+Bs^3-2Bs^2+Bs+Cs^2+-Cs+Ds$$

$$\begin{array}{lll} s^3 & A+B=0 & \underline{B=1} \\ s^2 & -3A-2B+C=1 & \underline{C=0} \\ s & 3A+B-C+D=-1 & \underline{D=1} \\ s^0 & \underline{A=-1} & \end{array}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{1}{s}+\frac{1}{s-1}+\frac{1}{\left(s-1\right)^3}\right\}$$

$$\boxed{y(t)=-1+e^t+\frac{1}{2}t^2e^t}$$

Exercise

Solve using the Laplace transform: $y' + 3y = e^{2t}, \quad y(0) = -1$

Solution

$$\mathcal{L}(y' + 3y) = \mathcal{L}\left(e^{2t}\right)$$

$$\mathcal{L}(y') + 3\mathcal{L}(y) = \mathcal{L}\left(e^{2t}\right)$$

$$sY(s) - y(0) + 3Y(s) = \frac{1}{s-2}$$

$$(s+3)Y(s) + 1 = \frac{1}{s-2}$$

$$(s+3)Y(s) = \frac{1}{s-2} - 1$$

$$Y(s) = \frac{1}{(s-2)(s+3)} - \frac{1}{s+3}$$

$$\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$$

$$1 = (A+B)s + 3A - 2B$$

$$\begin{cases} A + B = 0 \\ 3A - 2B = 1 \end{cases} \Rightarrow A = \frac{1}{5} \quad B = -\frac{1}{5}$$

$$Y(s) = \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{1}{s+3} - \frac{1}{s+3}$$

$$y(t) = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{1}{5}e^{2t} - \frac{6}{5}e^{-3t}$$

Exercise

Solve using the Laplace transform: $y' + 4y = \cos t, \quad y(0) = 0$

Solution

$$\mathcal{L}(y' + 4y) = \mathcal{L}(\cos t)$$

$$sY(s) - y(0) + 4Y(s) = \frac{s}{s^2+1} \qquad y(0) = 0$$

$$(s+4)Y(s) = \frac{s}{s^2+1}$$

$$Y(s) = \frac{s}{(s+4)\left(s^2+1\right)}$$

$$\frac{s}{(s+4)\left(s^2+1\right)}=\frac{A}{s+4}+\frac{Bs+C}{s^2+1}$$

$$s=As^2+A+Bs^2+4Bs+Cs+4C$$

$$s=(A+B)s^2+(4B+C)s+A+4C$$

$$\begin{cases} A+B=0 \\ 4B+C=1 \Rightarrow A=-\frac{4}{17} \quad B=\frac{4}{17} \quad C=\frac{1}{17} \\ A+4C=0 \end{cases}$$

$$Y(s)=-\frac{4}{17}\frac{1}{s+4}+\frac{4}{17}\frac{s}{s^2+1}+\frac{1}{17}\frac{1}{s^2+1}$$

$$y(t)=-\frac{4}{17}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}+\frac{4}{17}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+\frac{1}{17}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$=-\frac{4}{17}e^{-4t}+\frac{4}{17}e^{-9t}\cos t+\frac{1}{17}\sin t$$

Exercise

Solve using the Laplace transform: $y'+4y=e^{-4t}, \quad y(0)=2$

Solution

$$\mathcal{L}\{y'+4y\}=\mathcal{L}\left\{e^{-4t}\right\}$$

$$sY(s)-y(0)+4Y(s)=\frac{1}{s+4} \qquad y(0)=2$$

$$(s+4)Y(s)=\frac{1}{s+4}+2$$

$$Y(s)=\frac{2s+9}{(s+4)^2}=\frac{A}{s+4}+\frac{B}{(s+4)^2}$$

$$2s+9=As+4A+B$$

$$s \qquad \underline{A=2}$$

$$s^0 \qquad 4A+B=9 \quad \underline{B=1}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{2}{s+4}+\frac{1}{(s+4)^2}\right\}$$

$$y(t)=2e^{-4t}+te^{-4t}$$

Exercise

Solve using the Laplace transform: $y'-4y=t^2e^{-2t}, \quad y(0)=1$

Solution

$$\mathcal{L}(y' - 4y) = \mathcal{L}\left(t^2 e^{-2t}\right)$$

$$sY(s) - y(0) - 4Y(s) = \frac{2!}{(s+2)^3} \qquad\qquad y(0) = 1$$

$$(s-4)Y(s) - 1 = \frac{2}{(s+2)^3}$$

$$(s-4)Y(s) = 1 + \frac{2}{(s+2)^3}$$

$$Y(s) = \frac{1}{s-4} + \frac{2}{(s-4)(s+2)^3}$$

$$\frac{2}{(s-4)(s+2)^3} = \frac{A}{s-4} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3}$$

$$2 = A\left(s^3 + 6s^2 + 12s + 8A\right) + B\left(s^2 + 4s + 4\right)(s-4) + C(s-4)(s+2) + D(s-4)$$

$$2 = As^3 + 6As^2 + 12As + 8A + Bs^3 + 4Bs^2 + 4Bs - 4Bs^2 - 16Bs - 16B$$
$$+ Cs^2 - 2Cs - 8C + Ds - 4D$$

$$2 = (A+B)s^3 + (6A+C)s^2 + (12A - 12B - 2C + D)s + 8A - 16B - 8C - 4D$$

$$\begin{cases} A + B = 0 \\ 6A + C = 0 \\ 12A - 12B - 2C + D = 0 \\ 8A - 16B - 8C - 4D = 2 \end{cases} \Rightarrow \begin{matrix} A = \frac{1}{108} & B = -\frac{1}{108} \\ C = -\frac{1}{18} & D = -\frac{1}{3} \end{matrix}$$

$$Y(s) = \frac{1}{s-4} + \frac{1}{108}\frac{1}{s-4} - \frac{1}{108}\frac{1}{s+2} - \frac{1}{18}\frac{1}{(s+2)^2} - \frac{1}{3}\frac{1}{(s+2)^3}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{109}{108}\frac{1}{s-4} - \frac{1}{108}\frac{1}{s+2} - \frac{1}{18}\frac{1}{(s+2)^2} - \frac{1}{3}\frac{1}{(s+2)^3}\right\}$$

$$\boxed{y(t) = \frac{109}{108}e^{4t} - \frac{1}{108}e^{-2t} - \frac{1}{18}te^{-2t} - \frac{1}{6}t^2e^{-2t}}$$

Exercise

Solve using the Laplace transform: $y' + 9y = e^{-t}$, $y(0) = 0$

Solution

$$\mathcal{L}(y' + 9y) = \mathcal{L}\left(e^{-t}\right) \qquad\qquad Y(s) = \mathcal{L}(y)(s)$$

$$\mathcal{L}(y') + 9\mathcal{L}(y) = \mathcal{L}\left(e^{-t}\right)$$

$$sY(s) - y(0) + 9Y(s) = \frac{1}{s+1}$$

$$(s+9)Y(s)=\frac{1}{s+1}$$

$$Y(s)=\frac{1}{(s+1)(s+9)}$$

$$\frac{1}{(s+1)(s+9)}=\frac{A}{s+1}+\frac{B}{s+9}$$

$$1=(A+B)s+9A+B$$

$$\begin{cases} A+B=0 \\ 9A+B=1 \end{cases} \Rightarrow A=\frac{1}{8} \quad B=-\frac{1}{8}$$

$$Y(s)=\frac{1}{8}\frac{1}{s+1}-\frac{1}{8}\frac{1}{s+9}$$

$$y(t)=\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+9}\right\}$$

$$=\frac{1}{8}e^{-t}-\frac{1}{8}e^{-9t}$$

Exercise

Solve using the Laplace transform: $y'+16y=\sin 3t, \quad y(0)=1$

Solution

$$\mathcal{L}(y'+16y)=\mathcal{L}(\sin 3t)$$

$$sY(s)-y(0)+16Y(s)=\frac{3}{s^2+9} \qquad y(0)=1$$

$$(s+16)Y(s)-1=\frac{3}{s^2+9}$$

$$(s+16)Y(s)=\frac{3}{s^2+9}+1$$

$$Y(s)=\frac{1}{s+16}+\frac{3}{(s+16)(s^2+9)}$$

$$\frac{3}{(s+16)(s^2+9)}=\frac{A}{s+16}+\frac{Bs+C}{s^2+9}$$

$$s=As^2+9A+Bs^2+16Bs+Cs+16C$$

$$s=(A+B)s^2+(16B+C)s+9A+16C$$

$$\begin{cases} A+B=0 \\ 16B+C=1 \\ 9A+16C=0 \end{cases} \Rightarrow A=\frac{3}{265} \quad B=-\frac{3}{265} \quad C=\frac{48}{265}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+16} + \frac{3}{265}\frac{1}{s+16} - \frac{3}{265}\frac{s}{s^2+9} + \frac{48}{265}\frac{1}{s^2+9}\right\}$$

$$\boxed{y(t) = \frac{268}{265}e^{-16t} - \frac{3}{265}\cos 3t + \frac{16}{265}\sin 3t}$$

Exercise

Solve using the Laplace transform: $y'' - y = e^{2t}$; $y(0) = 0$, $y'(0) = 1$

Solution

$$\mathcal{L}\{y'' - y\} = \mathcal{L}\left\{e^{2t}\right\}$$

$$s^2Y(s) - sy(0) - y'(0) - Y(s) = \frac{1}{s-2} \qquad y(0) = 0, \quad y'(0) = 1$$

$$\left(s^2 - 1\right)Y(s) - 1 = \frac{1}{s-2}$$

$$\left(s^2 - 1\right)Y(s) = \frac{1}{s-2} + 1$$

$$(s-1)(s+1)Y(s) = \frac{s-1}{s-2}$$

$$Y(s) = \frac{1}{(s+1)(s-2)} = \frac{A}{(s+1)} + \frac{B}{(s-2)}$$

$$(A+B)s + B - 2A = 1 \qquad \begin{cases} A + B = 0 \\ -2A + B = 1 \end{cases} \Rightarrow A = -\frac{1}{3};\ B = \frac{1}{3}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{-\frac{1}{(s+1)} + \frac{1}{(s-2)}\right\}$$

$$\boxed{y(t) = \frac{1}{3}\left(e^{2t} - e^{-t}\right)}$$

Exercise

Solve using the Laplace transform: $y'' - y = 2t$; $y(0) = 0$, $y'(0) = -1$

Solution

$$\mathcal{L}(y'' - y) = \mathcal{L}(2t)$$

$$s^2Y(s) - sy(0) - y'(0) - Y(s) = 2\frac{1}{s^2} \qquad y(0) = 0 \quad y'(0) = -1$$

$$\left(s^2 - 1\right)Y(s) + 1 = \frac{2}{s^2}$$

$$Y(s) = \frac{2}{s^2(s-1)(s+1)} - \frac{1}{(s-1)(s+1)}$$

$$\frac{2}{s^2(s-1)(s+1)} = \frac{A}{s^2} + \frac{B}{s-1} + \frac{C}{s+1}$$

$$2 = As^2 - A + Bs^3 + Bs^2 + Cs^3 - Cs^2$$

$$2 = (B+C)s^3 + (A+B-C)s^2 - A$$

$$\begin{cases} B + C = 0 \\ A + B - C = 0 \\ -A = 2 \end{cases} \Rightarrow A = -2 \quad B = 1 \quad C = -1$$

$$\frac{1}{(s-1)(s+1)} = \frac{D}{s-1} + \frac{E}{s+1}$$

$$\begin{cases} D + E = 0 \\ D - E = 1 \end{cases} \Rightarrow D = \frac{1}{2} \quad E = -\frac{1}{2}$$

$$Y(s) = -\frac{2}{s^2} + \frac{1}{s-1} - \frac{1}{s+1} - \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1}$$

$$= -\frac{2}{s^2} + \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}$$

$$y(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$y(t) = -2t + \frac{1}{2}e^t - \frac{1}{2}e^{-t}$$

Exercise

Solve using the Laplace transform: $y'' - y = t - 2$; $y(2) = 3, \quad y'(2) = 0$

Solution

Let: $w(t) = y(t+2) \quad \leftrightarrow \quad y(t) = w(t-2)$

$$\mathcal{L}\{y'' - y\} = \mathcal{L}\{t+2\}$$

$$\mathcal{L}\{w'' - w\} = \mathcal{L}\{t\}$$

$$s^2W(s) - sw(0) - w'(0) - W(s) = \frac{1}{s} \qquad y(2) = w(2-2) = w(0) = 3, \quad y'(2) = w'(0) = 0$$

$$\left(s^2 - 1\right)W(s) = \frac{1}{s} + 3s$$

$$W(s) = \frac{1+3s^2}{s\left(s^2-1\right)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$$

$$1 + 3s^2 = As^2 - A + Bs^2 + Bs + Cs^2 - Cs$$

$s^2 \quad A+C=3 \quad \underline{C=2}$

$s^1 \quad B-C=0 \quad \underline{B=2}$

$s^0 \quad \underline{A=-1}$

$$\mathcal{L}^{-1}\{W(s)\} = \mathcal{L}^{-1}\left\{-\frac{1}{s}+\frac{2}{s-1}+\frac{2}{s+1}\right\}$$

$$w(t) = -t + 2e^t + 2e^{-t}$$

$$y(t) = w(t-2) = -(t-2) + 2e^{t-2} + 2e^{-(t-2)}$$

$$\underline{= 2 - t + 2e^{t-2} + 2e^{-t+2}}$$

Exercise

Solve using the Laplace transform: $y'' + y = t$; $y(\pi) = y'(\pi) = 0$

Solution

Let: $w(t) = y(t+\pi) \quad \leftrightarrow \quad y(t) = w(t-\pi)$

$y'' + y = t \quad \to \quad w'' + w = t + \pi$

$\mathcal{L}\{w'' + w\} = \mathcal{L}\{t+\pi\}$

$s^2W(s) - sw(0) - w'(0) + W(s) = \frac{1}{s^2} + \frac{\pi}{s} \qquad y(\pi) = w(\pi-\pi) = w(0) = 0, \quad y'(\pi) = w'(0) = 0$

$(s^2+1)W(s) = \frac{1+\pi s}{s^2}$

$W(s) = \frac{1+\pi s}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$

$1 + \pi s = As^3 + As + Bs^2 + B + Cs^3 + Ds^2$

$s^3 \quad A+C=0 \quad \underline{C=-\pi}$

$s^2 \quad B+D=0 \quad \underline{D=-1}$

$s^1 \quad \underline{A=\pi}$

$s^0 \quad \underline{B=1}$

$$\mathcal{L}^{-1}\{W(s)\} = \mathcal{L}^{-1}\left\{\frac{\pi}{s}+\frac{1}{s^2}-\frac{\pi s}{s^2+1}-\frac{1}{s^2+1}\right\}$$

$$w(t) = \pi + t - \pi\cos t - \sin t$$

$$y(t) = w(t-\pi) = \pi + (t-\pi) - \pi\cos(t-\pi) - \sin(t-\pi)$$

$$= t - \pi(\cos t\cos\pi - \sin t\sin\pi) - (\cos t\sin\pi - \cos\pi\sin t)$$

$$\underline{= t + \pi\cos t + \sin t}$$

Exercise

Solve using the Laplace transform: $y'' - 2y' + 5y = -8e^{\pi - t}$; $y(\pi) = 2,\ y'(\pi) = 12$

Solution

Let: $w(t) = y(t+\pi) \leftrightarrow y(t) = w(t-\pi)$

$$y'' - 2y' + 5y = -8e^{\pi - t} \quad \to \quad w'' - 2w' + 5w = -8e^{-t}$$

$$\mathcal{L}\{w'' - 2w' + 5w\} = \mathcal{L}\{-8e^{-t}\}$$

$$s^2W(s) - sw(0) - w'(0) - 2sW(s) + 2w(0) + 5W(s) = -\frac{8}{s+1} \qquad y(\pi) = w(0) = 2, \quad y'(\pi) = w'(0) = 12$$

$$\left(s^2 - 2s + 5\right)W(s) = -\frac{8}{s+1} + 2s + 12 - 4$$

$$\left(s^2 - 2s + 5\right)W(s) = \frac{2s^2 + 10s}{s+1}$$

$$W(s) = \frac{2s^2 + 10s}{(s+1)\left((s-1)^2 + 4\right)} = \frac{A}{s+1} + \frac{B(s-1) + C}{(s-1)^2 + 4}$$

$$2s^2 + 10s = As^2 - 2As + 5A + Bs^2 - B + Cs + C$$

$$\begin{array}{ll} s^2 & A + B = 2 \\ s^1 & -2A + C = 10 \\ s^0 & 5A - B + C = 0 \end{array}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 5 & -1 & 1 \end{vmatrix} = 8 \quad \Delta_A = \begin{vmatrix} 2 & 1 & 0 \\ 10 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = -8$$

$$\underline{A = -1 \quad B = 3 \quad C = 8}$$

$$\mathcal{L}^{-1}\{W(s)\} = \mathcal{L}^{-1}\left\{-\frac{1}{s+1} + \frac{3(s-1)}{(s-1)^2 + 2^2} + \frac{4(2)}{(s-1)^2 + 2^2}\right\}$$

$$w(t) = -e^{-t} + 3e^t \cos 2t + 4e^t \sin 2t$$

$$y(t) = w(t-\pi) = -e^{-(t-\pi)} + 3e^{t-\pi}\cos 2(t-\pi) + 4e^{t-\pi}\sin 2(t-\pi)$$

$$\underline{= -e^{-t+\pi} + 3e^{t-\pi}\cos 2t + 4e^{t-\pi}\sin 2t}$$

Exercise

Solve using the Laplace transform: $y'' + y = t^2 + 2$; $y(0) = 1,\ y'(0) = -1$

Solution

$$\mathcal{L}\{y''+y\}=\mathcal{L}\left\{t^2+2\right\}$$

$$s^2Y(s)-sy(0)-y'(0)+Y(s)=\frac{2}{s^3}+\frac{2}{s} \qquad y(0)=1;\ \ y'(0)=-1$$

$$s^2Y(s)-s+1+Y(s)=\frac{2+2s^2}{s^3}$$

$$\left(s^2+1\right)Y(s)=\frac{2+2s^2}{s^3}+s-1$$

$$Y(s)=\frac{2+2s^2+s^4-s^3}{s^3\left(s^2+1\right)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{Ds+E}{s^2+1}$$

$$s^4-s^3+2s^2+2=As^4+As^2+Bs^3+Bs+Cs^2+C+Ds^4+Es^3$$

s^4 $\quad A+D=1 \quad \underline{D=1}$

s^3 $\quad B+E=-1 \quad \underline{E=-1}$

s^2 $\quad A+C=2 \quad \underline{A=0}$

s $\quad \underline{B=0}$

s^0 $\quad \underline{C=2}$

$$Y(s)=\frac{2}{s^3}+\frac{s}{s^2+1}-\frac{1}{s^2+1}$$

$$\underline{y(t)=t^2+\cos t-\sin t}$$

Exercise

Solve using the Laplace transform: $\quad y''+y=\sqrt{2}\sin\sqrt{2}t; \quad y(0)=10, \quad y'(0)=0$

Solution

$$\mathcal{L}\{y''+y\}=\mathcal{L}\left\{\sqrt{2}\sin\sqrt{2}t\right\}$$

$$s^2Y(s)-sy(0)-y'(0)+Y(s)=\sqrt{2}\frac{\sqrt{2}}{s^2+2} \qquad y(0)=10;\ \ y'(0)=0$$

$$\left(s^2+1\right)Y(s)-10s=\frac{2}{s^2+2}$$

$$\left(s^2+1\right)Y(s)=\frac{2}{s^2+2}+10s$$

$$\left(s^2+1\right)Y(s)=\frac{10s^3+20s+2}{s^2+2}$$

$$Y(s)=\frac{10s^3+20s+2}{\left(s^2+1\right)\left(s^2+2\right)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+2}$$

$$As^3+2As+Bs^2+2B+Cs^3+Cs+Ds^2+D=10s^3+20s+2$$

$$\begin{array}{ll} s^3 & A+C=10 \\ s^2 & B+D=0 \\ s^1 & 2A+C=20 \\ s^0 & 2B+D=2 \end{array} \rightarrow \begin{array}{l} \begin{cases} A+C=10 \\ 2A+C=20 \end{cases} \rightarrow \underline{A=10,\ C=0|} \\ \begin{cases} B+D=0 \\ 2B+D=2 \end{cases} \rightarrow \underline{B=2,\ D=-2|} \end{array}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{10s}{s^2+1}+\frac{2}{s^2+1}-\sqrt{2}\frac{\sqrt{2}}{s^2+2}\right\}$$

$$\underline{y(t)=10\cos t+2\sin t-\sqrt{2}\sin\sqrt{2}t|}$$

Exercise

Solve using the Laplace transform: $y''+y=-2\cos 2t$; $y(0)=1,\quad y'(0)=-1$

Solution

$$\mathcal{L}(y''+y)(s)=\mathcal{L}(-2\cos 2t)(s)$$

$$s^2Y(s)-sy(0)-y'(0)+Y(s)=-2\frac{s}{s^2+4} \qquad y(0)=1 \quad y'(0)=-1$$

$$s^2Y(s)-s+1+Y(s)=-2\frac{s}{s^2+4}$$

$$\left(s^2+1\right)Y(s)=\frac{-2s}{s^2+4}+s-1$$

$$Y(s)=\frac{-2s}{\left(s^2+4\right)\left(s^2+1\right)}+\frac{s}{s^2+1}-\frac{1}{s^2+1}$$

$$=\frac{As}{s^2+4}+\frac{Bs}{s^2+1}+\frac{s}{s^2+1}-\frac{1}{s^2+1}$$

$$\frac{-2s}{\left(s^2+4\right)\left(s^2+1\right)}=\frac{As}{s^2+4}+\frac{Bs}{s^2+1}$$

$$\Rightarrow \quad -2s=As\left(s^2+1\right)+Bs\left(s^2+4\right)$$

$$-2s=As^3+As+Bs^3+4Bs$$

$$-2s=(A+B)s^3+(A+4B)s$$

$$\begin{cases} A+B=0 \\ A+4B=-2 \end{cases} \Rightarrow \boxed{A=\frac{2}{3}} \quad \boxed{B=-\frac{2}{3}}$$

$$Y(s)=\frac{2}{3}\frac{s}{s^2+4}-\frac{2}{3}\frac{s}{s^2+1}+\frac{s}{s^2+1}-\frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{2}{3}\frac{s}{s^2+4}+\frac{1}{3}\frac{s}{s^2+1}-\frac{1}{s^2+1}\right\}$$

$$\boxed{y(t)=\frac{2}{3}\cos 2t+\frac{1}{3}\cos t-\sin t}$$

Exercise

Solve using the Laplace transform: $y''-y'=e^t\cos t$; $y(0)=0,\quad y'(0)=0$

Solution

$$\mathcal{L}\{y''-y'\}=\mathcal{L}\left\{e^t\cos t\right\}$$

$$s^2Y(s)-sy(0)-y'(0)+sY(s)-y(0)=\frac{s-1}{(s-1)^2+1}\qquad y(0)=0;\ y'(0)=0$$

$$\left(s^2+s\right)Y(s)=\frac{s-1}{s^2-2s+2}$$

$$Y(s)=\frac{s-1}{\left(s^2+s\right)\left(s^2-2s+2\right)}\quad=\frac{A}{s}+\frac{B}{s+1}+\frac{Cs+D}{s^2-2s+2}$$

$$As^3-As^2+2A+Bs^3-2Bs^2+2Bs+Cs^3+Cs^2+Ds^2+Ds=s-1$$

$$\begin{array}{ll} s^3 & A+B+C+D=0 \\ s^2 & -A-2B+C+D=0 \\ s^1 & 2B+D=1 \\ s^0 & 2A=-1\to \boxed{A=-\frac{1}{2}} \end{array}\quad\to\begin{cases} B+C+D=\frac{1}{2} \\ -2B+C+D=-\frac{1}{2} \\ 2B+D=1 \end{cases}\Rightarrow \boxed{B=\frac{1}{3}\quad C=-\frac{1}{6}\quad D=\frac{1}{3}}$$

$$Y(s)=-\frac{1}{2}\frac{1}{s}+\frac{1}{3}\frac{1}{s+1}-\frac{1}{6}\frac{s-2}{(s-1)^2+1}$$

$$=-\frac{1}{2}\frac{1}{s}+\frac{1}{3}\frac{1}{s+1}-\frac{1}{6}\frac{s-1-1}{(s-1)^2+1}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{1}{2}\frac{1}{s}+\frac{1}{3}\frac{1}{s+1}-\frac{1}{6}\frac{s-1}{(s-1)^2+1}+\frac{1}{6}\frac{1}{(s-1)^2+1}\right\}$$

$$\boxed{y(t)=-\frac{1}{2}t+\frac{1}{3}e^{-t}-\frac{1}{6}e^t\cos t+\frac{1}{6}e^t\sin t}$$

Exercise

Solve using the Laplace transform: $y'' + y' - y = t^3$; $y(0)=1, \quad y'(0)=0$

Solution

$$\mathcal{L}\left(y''-y'-y\right)=\mathcal{L}\left(t^3\right)$$

$$s^2Y(s)-sy(0)-y'(0)+sY(s)-y(0)-Y(s)=\frac{6}{s^4} \qquad y(0)=1 \quad y'(0)=0$$

$$\left(s^2+s-1\right)Y(s)-s-1=\frac{6}{s^4}$$

$$\left(s^2+s-1\right)Y(s)=\frac{6}{s^4}+s+1$$

$$Y(s)=\frac{s^5+s^4+6}{s^4\left(s^2+s-1\right)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s^4}+\frac{E}{s+\frac{1}{2}-\frac{\sqrt{5}}{2}}+\frac{F}{s+\frac{1}{2}+\frac{\sqrt{5}}{2}}$$

s^5	$A+E+F=1$	$E+F=19$
s^4	$A+B+\left(\frac{1+\sqrt{5}}{2}\right)E+\left(\frac{1-\sqrt{5}}{2}\right)F=1$	$\left(\frac{1+\sqrt{5}}{2}\right)E+\left(\frac{1-\sqrt{5}}{2}\right)F=31$
s^3	$-A+B+C=0$	$\underline{A=-18}$
s^2	$-B+C+D=0$	$\underline{B=-12}$
s^1	$-C+D=0$	$\underline{C=-6}$
s^0	$-D=6$	$\underline{D=-6}$

$$F=\frac{19}{2}-\frac{43\sqrt{5}}{10} \quad E=-\frac{19}{2}+\frac{43\sqrt{5}}{10}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{18}{s}-\frac{12}{s^2}-\frac{6}{s^3}-\frac{6}{s^4}+\frac{-\frac{19}{2}+\frac{43\sqrt{5}}{10}}{s+\frac{1}{2}-\frac{\sqrt{5}}{2}}+\frac{\frac{19}{2}-\frac{43\sqrt{5}}{10}}{s+\frac{1}{2}+\frac{\sqrt{5}}{2}}\right\}$$

$$y(t)=-18-12t-3t^2-t^3+\left(-\frac{19}{2}+\frac{43\sqrt{5}}{10}\right)e^{\left(\frac{-1+\sqrt{5}}{2}\right)t}+\left(\frac{19}{2}-\frac{43\sqrt{5}}{10}\right)e^{-\left(\frac{1+\sqrt{5}}{2}\right)t}$$

Exercise

Solve using the Laplace transform: $y''-y'-2y=4t^2, \quad y(0)=1, \quad y'(0)=4$

Solution

$$\mathcal{L}\{y''-y'-2y\}(s)=\mathcal{L}\left\{4t^2\right\}(s) \qquad \mathcal{L}\left\{t^n\right\}(s)=\frac{n!}{s^{n+1}}$$

$$s^2Y(s)-sy(0)-y'(0)-sY(s)+y(0)-2Y(s)=\frac{8}{s^3} \qquad y(0)=1, \quad y'(0)=4$$

$$\left(s^2-s-2\right)Y(s)-s-4+1=\frac{8}{s^3}$$

$$(s+1)(s-2)Y(s)=\frac{8}{s^3}+s+3$$

$$Y(s)=\frac{s^4+3s^3+8}{s^3(s+1)(s-2)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s+1}+\frac{E}{s-2}$$

$$s^4+3s^3+8=As^2\left(s^2-s-2\right)+Bs\left(s^2-s-2\right)+Cs^2-Cs-2C+Ds^3(s-2)+Es^3(s+1)$$

$$=As^4-As^3-2As^2+Bs^3-Bs^2-2Bs+Cs^2-Cs-2C+Ds^4-2Ds^3+Es^4+Es^3$$

$$\begin{cases} s^4 & A+D+E=1 & D+E=4 \\ s^3 & -A+B-2D+E=3 & -2D+E=-2 \\ s^2 & -2A-B+C=0 & \rightarrow \underline{A=-3} \\ s & -2B-C=0 & \rightarrow \underline{B=2} \\ s^0 & -2C=8 & \rightarrow \underline{C=-4} \end{cases} \quad \Rightarrow \underline{D=2,\ \ E=2}$$

$$\mathcal{L}^{-1}\{Y(s)\}(t)=\mathcal{L}^{-1}\left\{-\frac{3}{s}+\frac{2}{s^2}-\frac{4}{s^3}+\frac{2}{s+1}+\frac{2}{s-2}\right\}(t)$$

$$\underline{y(t)=-3+2t-2t^2+2e^{-t}+2e^{2t}}$$

Exercise

Solve using the Laplace transform: $y''-y'-2y=e^{2t}$; $y(0)=-1,\quad y'(0)=0$

Solution

$$\mathcal{L}(y''-y'-2y)=\mathcal{L}\left(e^{2t}\right)$$

$$s^2Y(s)-sy(0)-y'(0)-\big(sY(s)-y(0)\big)-2Y(s)=\frac{1}{s-2} \qquad y(0)=-1 \quad y'(0)=0$$

$$s^2Y(s)+s-sY(s)-1-2Y(s)=\frac{1}{s-2}$$

$$\left(s^2-s-2\right)Y(s)=\frac{1}{s-2}-s+1$$

$$(s+1)(s-2)(Y(s)=\frac{1}{s-2}-s+1$$

$$Y(s)=\frac{1}{(s+1)(s-2)^2}-\frac{s-1}{(s+1)(s-2)}$$

$$=\frac{1-(s-1)(s-2)}{(s+1)(s-2)^2}$$

$$Y(s)=\frac{-s^2+3s-1}{(s+1)(s-2)^2}=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}$$

$$-s^2+3s-1=As^2-4As+4A+Bs^2-Bs-2B+Cs+C$$

$$-s^2+3s-1=(A+B)s^2+(-4A-B+C)s+4A-2B+C$$

$$\begin{cases} A+B=-1 \\ -4A-B+C=3 \\ 4A-2B+C=-1 \end{cases} \Rightarrow A=-\frac{5}{9} \quad B=-\frac{4}{9} \quad C=\frac{1}{3}$$

$$Y(s)=-\frac{5}{9}\frac{1}{s+1}-\frac{4}{9}\frac{1}{s-2}+\frac{1}{3}\frac{1}{(s-2)^2}$$

$$y(t)=-\frac{5}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-\frac{4}{9}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}+\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$\boxed{y(t)=-\frac{5}{9}e^{-t}-\frac{4}{9}e^{2t}+\frac{1}{3}te^{2t}}$$

Exercise

Solve using the Laplace transform: $y''-y'-2y=0, \quad y(0)=-2, \quad y'(0)=5$

Solution

$$\mathcal{L}(y''-y'-2y)=0$$

$$s^2Y(s)-sy(0)-y'(0)-sY(s)+y(0)-2Y(s)=0 \qquad y(0)=-2 \quad y'(0)=5$$

$$\left(s^2-s-2\right)Y(s)=7-2s$$

$$Y(s)=\frac{7-2s}{s^2-s-2}=\frac{A}{s+1}+\frac{B}{s-2}$$

$$\begin{matrix} s & A+B=-2 \\ s^0 & -2A+B=7 \end{matrix} \quad \rightarrow \quad \boxed{A=-3, \quad B=1}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{-3}{s+1}+\frac{1}{s-2}\right\}$$

$$\boxed{y(t)=e^{2t}-3e^{-t}}$$

Exercise

Solve using the Laplace transform: $y''-y'-2y=-8\cos t-2\sin t\ ; \quad y\left(\frac{\pi}{2}\right)=1, \quad y'\left(\frac{\pi}{2}\right)=0$

Solution

Let: $w(t)=y\left(t+\frac{\pi}{2}\right) \quad \leftrightarrow \quad y(t)=w\left(t-\frac{\pi}{2}\right)$

$$y''-y'-2y=-8\cos t-2\sin t \quad \rightarrow \quad w''-w'-2w=-8\cos\left(t+\frac{\pi}{2}\right)-2\sin\left(t+\frac{\pi}{2}\right)$$

$$w''-w'-2w=-8\left(\cos t\cos\frac{\pi}{2}-\sin t\sin\frac{\pi}{2}\right)-2\left(\sin t\cos\frac{\pi}{2}+\cos t\sin\frac{\pi}{2}\right)$$

$$\mathcal{L}\{w''-w'-2w\}=\mathcal{L}\{8\sin t-2\cos t\}$$

$$s^2W(s)-sw(0)-w'(0)-sW(s)+w(0)-2W(s)=\frac{8}{s^2+1}-\frac{2s}{s^2+1}$$

$$y\left(\frac{\pi}{2}\right)=w(0)=1, \quad y'\left(\frac{\pi}{2}\right)=w'(0)=0$$

$$\left(s^2-s-2\right)W(s)=\frac{8-2s}{s^2+1}+s-1$$

$$W(s)=\frac{s^3-s^2-s+7}{(s+1)(s-2)\left(s^2+1\right)}=\frac{A}{s+1}+\frac{B}{s-2}+\frac{Cs+D}{s^2+1}$$

$$s^3-s^2-s+7=As^3-2As^2+As-2A+Bs^3+Bs^2+Bs+B+Cs^3-Cs^2-2Cs+Ds^2-Ds-2D$$

$$\begin{cases} s^3 & A+B+C=1 \\ s^2 & -2A+B-C+D=-1 \\ s^1 & A+B-2C-D=-1 \\ s^0 & -2A+B-2D=7 \end{cases} \quad \rightarrow \quad \boxed{A=-1,\ B=\frac{3}{5},\ C=\frac{7}{5},\ D=-\frac{11}{5}}$$

$$\mathcal{L}^{-1}\{W(s)\}=\mathcal{L}^{-1}\left\{-\frac{13}{3}\frac{1}{s+1}+\frac{74}{15}\frac{1}{s-2}+\frac{7}{5}\frac{s}{s^2+1}-\frac{11}{5}\frac{1}{s^2+1}\right\}$$

$$w(t)=-e^{-t}+\frac{3}{5}e^{2t}+\frac{7}{5}\cos t-\frac{11}{5}\sin t$$

$$y(t)=w\left(t-\frac{\pi}{2}\right)=-e^{-t+\frac{\pi}{2}}+\frac{3}{5}e^{2\left(t-\frac{\pi}{2}\right)}+\frac{7}{5}\cos\left(t-\frac{\pi}{2}\right)-\frac{11}{5}\sin\left(t-\frac{\pi}{2}\right)$$

$$=-e^{-t+\frac{\pi}{2}}+\frac{3}{5}e^{2t-\pi}+\frac{7}{5}\left(\cos t\cos\frac{\pi}{2}+\sin t\sin\frac{\pi}{2}\right)-\frac{11}{5}\left(\sin t\cos\frac{\pi}{2}-\cos t\sin\frac{\pi}{2}\right)$$

$$\boxed{=\frac{3}{5}e^{2t-\pi}-e^{-t+\frac{\pi}{2}}+\frac{7}{5}\sin t+\frac{11}{5}\cos t}$$

Exercise

Solve using the Laplace transform: $\quad x''-x'-6x=0\ ; \quad x(0)=2, \quad x'(0)=-1$

Solution

$$\mathcal{L}\{x''-x'-6x\}=0$$

$$s^2X(s) - sx(0) - x'(0) - sX(s) + x(0) - 6X(s) = 0 \qquad x(0) = 2, \ x'(0) = -1$$

$$\left(s^2 - s - 6\right)X(s) - 2s + 1 + 2 = 0$$

$$\left(s^2 - s - 6\right)X(s) = 2s - 3$$

$$X(s) = \frac{2s-3}{s^2 - s - 6} = \frac{A}{s-3} + \frac{B}{s+2}$$

$$As + 2A + Bs - 3B = 2s - 3$$

$$\begin{cases} A + B = 2 \\ 2A - 3B = -3 \end{cases} \qquad \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5 \quad \begin{vmatrix} 2 & 1 \\ -3 & -3 \end{vmatrix} = -3 \quad \rightarrow A = \frac{3}{5}, \ B = \frac{7}{5}$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{5}\frac{1}{s-3} + \frac{7}{5}\frac{1}{s+2}\right\}$$

$$\boxed{x(t) = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}}$$

Exercise

Solve using the Laplace transform: $y'' + 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 1$

Solution

$$\mathcal{L}\{y'' + 2y' + y\} = 0$$

$$s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + Y(s) = 0 \qquad y(0) = 1 \quad y'(0) = 1$$

$$\left(s^2 + 2s + 1\right)Y(s) - s - 1 - 2 = 0$$

$$Y(s) = \frac{s+3}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

$$s + 3 = As + A + B$$

$$s \qquad \boxed{A = 1}$$

$$s^0 \qquad A + B = 3 \quad \boxed{B = 2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} + \frac{2}{(s+1)^2}\right\}$$

$$\boxed{y(t) = e^{-t} + 2te^{-t}}$$

Exercise

Solve using the Laplace transform: $y'' + 2y' + y = t\ ,\quad y(0) = -3,\quad y(1) = -1$

Solution

$$\mathcal{L}\{y'' + 2y' + y\}(s) = \mathcal{L}\{t\}(s)$$

$$s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + Y(s) = \frac{1}{s^2} \qquad y(0) = -3$$

$$\left(s^2 + 2s + 1\right)Y(s) + 3s - y'(0) + 6 = \frac{1}{s^2}$$

$$(s+1)^2 Y(s) = \frac{1}{s^2} - 3s - 6 + y'(0)$$

$$Y(s) = \frac{-3s^3 + \left(y'(0) - 6\right)s^2 + 1}{s^2(s+1)^2} \qquad \text{Let} \quad k = y'(0) - 6$$

$$= \frac{-3s^3 + ks^2 + 1}{s^2(s+1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2}$$

$$-3s^3 + ks^2 + 1 = As^3 + 2As^2 + As + Bs^2 + 2Bs + B + Cs^3 + Cs^2 + Ds^2$$

$$\begin{cases} s^3 & A + C = -3 & \to \underline{C = -1} \\ s^2 & 2A + B + C + D = k & \to \underline{D = k + 4} \\ s & A + 2B = 0 & \to \underline{A = -2} \\ s^0 & \underline{B = 1} & \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{2}{s} + \frac{1}{s^2} - \frac{1}{s+1} + \frac{k+4}{(s+1)^2}\right\}$$

$$y(t) = -2 + t - e^{-t} + (k+4)te^{-t} \qquad y(1) = -1$$

$$-1 = -1 - e^{-1} + (k+4)e^{-1}$$

$$(k+4)e^{-1} = e^{-1}$$

$$k + 4 = 1 \quad \to \quad \underline{k = -3}$$

$$\boxed{y(t) = -2 + t - e^{-t} + te^{-t}}$$

Exercise

Solve using the Laplace transform: $y'' - 2y' - y = e^{2t} - e^{t}\ ;\quad y(0) = 1,\quad y'(0) = 3$

Solution

$$\mathcal{L}\{y''-2y'-y\}=\mathcal{L}\left\{e^{2t}-e^{t}\right\}$$

$$s^2Y(s)-sy(0)-y'(0)-2sY(s)+2y(0)-Y(s)=\frac{1}{s-2}-\frac{1}{s-1} \qquad y(0)=1 \quad y'(0)=3$$

$$\left(s^2-2s-1\right)Y(s)=\frac{1}{(s-2)(s-1)}+s+1$$

$$Y(s)=\frac{s^3-2s^2-s+3}{(s-2)(s-1)\left(s-1-\sqrt{2}\right)\left(s-1+\sqrt{2}\right)}=\frac{A}{s-2}+\frac{B}{s-1}+\frac{C}{s-1-\sqrt{2}}+\frac{D}{s-1+\sqrt{2}}$$

$$s^3-2s^2-s+3=A(s-1)\left(s^2-2s-1\right)+B(s-2)\left(s^2-2s-1\right)$$
$$+C\left(s-1+\sqrt{2}\right)\left(s^2-3s+2\right)+D\left(s-1-\sqrt{2}\right)\left(s^2-3s+2\right)$$

s^3 $\qquad A+B+C+D=1$

s^2 $\qquad -3A-4B+\left(-4+\sqrt{2}\right)C+\left(-4-\sqrt{2}\right)D=-2$

s^1 $\qquad A+3B+\left(5-3\sqrt{2}\right)C+\left(5+3\sqrt{2}\right)D=-1$

s^0 $\qquad A+2B+\left(-2+2\sqrt{2}\right)C-2\left(1+\sqrt{2}\right)D=3$

$$\boxed{A=-1 \quad B=\tfrac{1}{2} \quad C=\tfrac{3}{4}\left(1+\sqrt{2}\right) \quad D=\tfrac{3}{4}\left(1-\sqrt{2}\right)}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{-1}{s-2}+\frac{1}{2}\frac{1}{s-1}+\frac{3}{4}\left(1+\sqrt{2}\right)\frac{1}{s-1-\sqrt{2}}+\frac{3}{4}\left(1-\sqrt{2}\right)\frac{1}{s-1+\sqrt{2}}\right\}$$

$$\boxed{y(t)=-e^{2t}+\frac{1}{2}e^{t}+\left(\frac{3}{4}+\frac{3\sqrt{2}}{4}\right)e^{\left(1+\sqrt{2}\right)t}+\left(\frac{3}{4}-\frac{3\sqrt{2}}{4}\right)e^{\left(1-\sqrt{2}\right)t}}$$

Exercise

Solve using the Laplace transform: $\quad y''-2y'+y=6t-2$; $\quad y(-1)=3, \quad y'(-1)=7$

Solution

Let: $w(t)=y(t-1) \quad \leftrightarrow \quad y(t)=w(t+1)$

$$w''-2w'+w=6(t-1)-2=6t-8$$

$$\mathcal{L}\{w''-2w'+w\}=\mathcal{L}\{6t-8\}$$

$$s^2W(s)-sw(0)-w'(0)-2sW(s)+2w(0)+W(s)=\frac{6}{s^2}-\frac{8}{s} \qquad y(-1)=w(0)=3, \quad y'(-1)=w'(0)=7$$

$$\left(s^2-2s+1\right)W(s)=\frac{6-8s}{s^2}+3s+1$$

$$W(s)=\frac{3s^3+s^2-8s+6}{s^2(s-1)^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-1}+\frac{D}{(s-1)^2}$$

$$3s^3+s^2-8s+6=As\left(s^2-2s+1\right)+B\left(s^2-2s+1\right)+Cs^2(s-1)+Ds^2$$

$s^3 \quad A+C=3 \quad \boxed{C=-1}$

$s^2 \quad -2A+B-C+D=1 \quad \boxed{D=2}$

$s^1 \quad A-2B=-8 \quad \boxed{A=4}$

$s^0 \quad \boxed{B=6}$

$$\mathcal{L}^{-1}\{W(s)\}=\mathcal{L}^{-1}\left\{\frac{4}{s}+\frac{6}{s^2}-\frac{1}{s-1}+\frac{2}{(s-1)^2}\right\}$$

$$\boxed{w(t)=4+6t-e^t+2te^t}$$

$$y(t)=w(t+1)=4+6(t+1)-e^{t+1}+2(t+1)e^{t+1}$$

$$\boxed{=6t+10+2te^{t+1}+e^{t+1}}$$

Exercise

Solve using the Laplace transform: $y''-2y'+y=\cos t-\sin t$; $y(0)=1,\quad y'(0)=3$

Solution

$$\mathcal{L}\{y''-2y'+y\}=\mathcal{L}\{\cos t-\sin t\}$$

$$s^2Y(s)-sy(0)-y'(0)-2sY(s)+2y(0)+Y(s)=\frac{s}{s^2+1}-\frac{1}{s^2+1} \qquad y(0)=1,\quad y'(0)=3$$

$$\left(s^2-2s+1\right)Y(s)=\frac{s-1}{s^2+1}+s+1$$

$$Y(s)=\frac{s^3+s^2+2s}{\left(s^2+1\right)(s-1)^2}=\frac{As+B}{s^2+1}+\frac{C}{s-1}+\frac{D}{(s-1)^2}$$

$$s^3+s^2+2s=(As+B)\left(s^2-2s+1\right)+C\left(s^2+1\right)(s-1)+D\left(s^2+1\right)$$

$s^3 \quad A+C=1$

$s^2 \quad -2A+B-C+D=1 \qquad \boxed{A=-\frac{1}{2} \quad B=-\frac{1}{2} \quad C=\frac{3}{2} \quad D=2}$

$s^1 \quad A-2B+C=2$

$s^0 \quad B-C+D=0$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{1}{2}\frac{s}{s^2+1}-\frac{1}{2}\frac{1}{s^2+1}+\frac{3}{2}\frac{1}{s-1}+\frac{2}{(s-1)^2}\right\}$$

$$\boxed{y(t)=-\frac{1}{2}\cos t-\frac{1}{2}\sin t+\frac{3}{2}e^t+2te^t}$$

Exercise

Solve using the Laplace transform: $y'' - 2y' + 5y = 0,\quad y(0) = 2,\quad y'(0) = 4$

Solution

$$\mathcal{L}\{y'' - 2y' + 5y\} = 0$$

$$s^2Y(s) - sy(0) - y'(0) - 2sY(s) + 2y(0) + 5Y(s) = 0 \qquad y(0) = 2 \quad y'(0) = 4$$

$$(s^2 - 2s + 5)Y(s) = 2s$$

$$Y(s) = \frac{2s}{(s-1)^2 + 4}$$

$$= \frac{2(s-1) + 2}{(s-1)^2 + 4}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2(s-1)}{(s-1)^2 + 4} + \frac{2}{(s-1)^2 + 4}\right\}$$

$$\boxed{y(t) = 2e^t \cos 2t + e^t \sin 2t}$$

Exercise

Solve using the Laplace transform: $y'' - 2y' + 5y = 1 + t\ ,\quad y(0) = 0,\quad y'(0) = 0$

Solution

$$\mathcal{L}\{y'' - 2y' + 5y\} = \mathcal{L}\{1 + t\}$$

$$s^2Y(s) - sy(0) - y'(0) - 2sY(s) + 2y(0) + 5Y(s) = \frac{1}{s} + \frac{1}{s^2} \qquad y(0) = 0 \quad y'(0) = 0$$

$$(s^2 - 2s + 5)Y(s) = \frac{s+1}{s^2}$$

$$Y(s) = \frac{s+1}{s^2\left((s-1)^2 + 4\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C(s-1) + D}{(s-1)^2 + 4}$$

$$s + 1 = As^3 - 2As^2 + 5As + Bs^2 - 2Bs + 5B + Cs^3 - Cs^2 + Ds^2$$

$$\begin{array}{lll} s^3 & A + C = 0 & \boxed{C = -\frac{7}{25}} \\ s^2 & -2A + B - C + D = 0 & \boxed{D = \frac{2}{25}} \\ s^1 & 5A - 2B = 1 & \boxed{A = \frac{7}{25}} \\ s^0 & 5B = 1 & \boxed{B = \frac{1}{5}} \end{array}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{7}{25}\frac{1}{s} + \frac{1}{5}\frac{1}{s^2} - \frac{7}{25}\frac{s-1}{(s-1)^2+2^2} + \frac{1}{25}\frac{2}{(s-1)^2+2^2}\right\}$$

$$\boxed{y(t) = \frac{7}{25} + \frac{1}{5}t - \frac{7}{25}e^t\cos 2t + \frac{1}{25}e^t\sin 2t}$$

Exercise

Solve using the Laplace transform: $y'' + 3y' = -3t$; $y(0) = -1, \quad y'(0) = 1$

Solution

$$\mathcal{L}(y'' + 3y') = \mathcal{L}(-3t)$$

$$s^2Y(s) - sy(0) - y'(0) + 3\big(sY(s) - y(0)\big) = -3\frac{1}{s^2} \qquad y(0) = -1 \quad y'(0) = 1$$

$$s^2Y(s) + s - 1 + 3sY(s) + 3 = -\frac{3}{s^2}$$

$$\left(s^2 + 3s\right)Y(s) = -\frac{3}{s^2} - s - 2$$

$$Y(s) = -\frac{3}{s^3(s+3)} - \frac{s+2}{s(s+3)}$$

$$\frac{3}{s^3(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+3}$$

$$3 = As^2(s+3) + Bs(s+3) + C(s+3) + Ds^3$$

$$3 = (A+D)s^3 + (3A+B)s^2 + (3B+C) + 3C$$

$$\begin{cases} A + D = 0 \\ 3A + B = 0 \\ 3B + C = 0 \\ 3C = 3 \end{cases} \Rightarrow \begin{matrix} A = \frac{1}{9} & B = -\frac{1}{3} \\ C = 1 & D = -\frac{1}{9} \end{matrix}$$

$$\frac{s+2}{s(s+3)} = \frac{E}{s} + \frac{F}{s+3}$$

$$\begin{cases} E + F = 1 \\ 3E = 2 \end{cases} \Rightarrow E = \frac{2}{3} \quad F = \frac{1}{3}$$

$$Y(s) = -\left(\frac{1}{9}\frac{1}{s} - \frac{1}{3}\frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{9}\frac{1}{s+3}\right) - \left(\frac{2}{3}\frac{1}{s} + \frac{1}{3}\frac{1}{s+3}\right)$$

$$= -\frac{1}{9}\frac{1}{s} + \frac{1}{3}\frac{1}{s^2} - \frac{1}{s^3} + \frac{1}{9}\frac{1}{s+3} - \frac{2}{3}\frac{1}{s} - \frac{1}{3}\frac{1}{s+3}$$

$$= -\frac{7}{9}\frac{1}{s} + \frac{1}{3}\frac{1}{s^2} - \frac{1}{s^3} - \frac{2}{9}\frac{1}{s+3}$$

$$y(t) = -\frac{7}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{2!}\frac{2!}{s^3}\right\} - \frac{2}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$\boxed{y(t) = -\frac{7}{9} + \frac{1}{3}t - \frac{1}{2}t^2 - \frac{2}{9}e^{-3t}}$$

Exercise

Solve using the Laplace transform: $y'' + 3y = t^3$; $y(0) = 0,$ $y'(0) = 0$

Solution

$$\mathcal{L}\{y'' + 3y'\} = \mathcal{L}\{t^3\}$$

$$s^2Y(s) - sy(0) - y'(0) + 3Y(s) = \frac{6}{s^4} \qquad y(0) = 0 \quad y'(0) = 0$$

$$\left(s^2 + 3\right)Y(s) = \frac{6}{s^4}$$

$$Y(s) = \frac{6}{s^4\left(s^2+3\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s^4} + \frac{Es+F}{s^2+3}$$

$$6 = As^5 + 3As^3 + Bs^4 + 3Bs^2 + Cs^3 + 3Cs + Ds^2 + 3D + Es^5 + Fs^4$$

s^5 $A + E = 0$ $\boxed{E = 0}$

s^4 $B + F = 0$ $\boxed{F = \frac{2}{3}}$

s^3 $3A + C = 0$ $\boxed{A = 0}$

s^2 $3B + D = 0$ $\boxed{B = -\frac{2}{3}}$

s^1 $3C = 0$ $\boxed{C = 0}$

s^0 $3D = 6$ $\boxed{D = 2}$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{2}{3}\frac{1}{s^2} + \frac{2}{s^4} + \frac{2}{3}\frac{1}{s^2 + \left(\sqrt{3}\right)^2}\right\}$$

$$y(t) = -\frac{2}{3}t + 2\frac{1}{3!}t^3 + \frac{2}{3}\frac{1}{\sqrt{3}}\sin\left(\sqrt{3}t\right)$$

$$\boxed{= -\frac{2}{3}t + \frac{1}{3}t^3 + \frac{2\sqrt{3}}{9}\sin\left(\sqrt{3}t\right)}$$

Exercise

Solve using the Laplace transform: $y'' - 3y' + 2y = e^{-t}$, $y(1) = 0,$ $y'(1) = 0$

Solution

Let $v = t - 1 \quad \rightarrow \quad t = v + 1$

$x(v) = y(t) = y(v+1) \qquad\qquad y(1) = x(0) = 0 \quad y'(1) = x'(0) = 0$

$y''(t) - 3y'(t) + 2y(t) = e^{-t}$

$\rightarrow \quad y''(v+1) - 3y'(v+1) + 2y(v+1) = e^{-(v+1)}$

$x''(v) - 3x'(v) + 2x(v) = e^{-(v+1)}$

$\mathcal{L}\{x'' - 3x' + 2x\} = \mathcal{L}\left\{e^{-1}e^{-v}\right\}$

$s^2X(s) - sx(0) - x'(0) - 3sX(s) + 3x(0) + 2X(s) = \dfrac{e^{-1}}{s+1}$

$\left(s^2 - 3s + 2\right)X(s) = \dfrac{e^{-1}}{s+1}$

$X(s) = e^{-1}\dfrac{1}{(s+1)(s-1)(s-2)}$

$$\frac{1}{(s+1)(s-1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s-2}$$

$1 = As^2 - 3As + 2A + Bs^2 - Bs - 2B + Cs^2 - C$

$$\begin{cases} s^2 & A + B + C = 0 \\ s & -3A - B = 0 \\ s^0 & 2A - 2B - C = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ -3 & -1 & 0 \\ 2 & -2 & -1 \end{vmatrix} = 6 \quad \Delta_A = \begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & -2 & -1 \end{vmatrix} = 1 \quad \Delta_B = \begin{vmatrix} 1 & 0 & 1 \\ -3 & 0 & 0 \\ 2 & 1 & -1 \end{vmatrix} = -3$$

$\boxed{A = \frac{1}{6}, \quad B = -\frac{1}{2}, \quad C = \frac{1}{3}}$

$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{e^{-1}\left(\frac{1}{6}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s-1} + \frac{1}{3}\frac{1}{s-2}\right)\right\}$

$x(v) = e^{-1}\left(\frac{1}{6}e^{-v} - \frac{1}{2}e^{v} + \frac{1}{3}e^{2v}\right) \qquad v = t - 1$

$y(t) = e^{-1}\left(\frac{1}{6}e^{-t+1} - \frac{1}{2}e^{t-1} + \frac{1}{3}e^{2(t-1)}\right)$

$\boxed{= \frac{1}{6}e^{-t} - \frac{1}{2}e^{t-2} + \frac{1}{3}e^{2t-3}}$

Exercise

Solve using the Laplace transform: $y'' - 3y' + 2y = \cos t$; $y(0) = 0,\quad y'(0) = -1$

Solution

$$\mathcal{L}\left\{y'' - 3y' + 2y\right\} = \mathcal{L}\left\{\cos t\right\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3sY(s) + 3y(0) + 2Y(s) = \frac{s}{s^2+1} \qquad y(0)=0 \quad y'(0)=-1$$

$$\left(s^2 - 3s + 2\right)Y(s) = \frac{s}{s^2+1} - 1$$

$$Y(s) = \frac{-s^2 + s - 1}{(s-1)(s-2)\left(s^2+1\right)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{Cs+D}{s^2+1}$$

$$\begin{cases} s^3 & A + B + C = 0 \\ s^2 & -2A - B - 3C + D = -1 \\ s^1 & A + B + 2C - 3D = 1 \\ s^0 & -2A - B + 2D = -1 \end{cases} \rightarrow \boxed{A = \frac{1}{2},\ B = -\frac{3}{5},\ C = \frac{1}{10},\ D = -\frac{3}{10}}$$

$$\mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s-1} - \frac{3}{5}\frac{1}{s-2} + \frac{1}{10}\frac{s}{s^2+1} - \frac{3}{10}\frac{1}{s^2+1}\right\}$$

$$\boxed{y(t) = \frac{1}{2}e^t - \frac{3}{5}e^{2t} + \frac{1}{10}\cos t - \frac{3}{10}\sin t}$$

Exercise

Solve using the Laplace transform: $y'' - 4y = e^{-t}$; $y(0) = -1,\quad y'(0) = 0$

Solution

$$\mathcal{L}\left(y'' - 4y\right) = \mathcal{L}\left(e^{-t}\right)$$

$$s^2Y(s) - sy(0) - y'(0) - 4Y(s) = \frac{1}{s+1} \qquad y(0)=-1 \quad y'(0)=0$$

$$\left(s^2 - 4\right)Y(s) + s = \frac{1}{s+1}$$

$$Y(s) = \frac{1}{(s+1)\left(s^2-4\right)} - \frac{s}{s^2-4}$$

$$\frac{1}{(s+1)\left(s^2-4\right)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s+2}$$

$$1 = As^2 - 4A + Bs^2 + 3Bs + 2B + Cs^2 - Cs - 2C$$

$$1 = (A + B + C)s^2 + (3B - C)s + 2B - 4A - 2C$$

$$\begin{cases} A+B+C=0 \\ 3B-C=0 \\ -4A+2B-2C=1 \end{cases} \Rightarrow A=-\frac{1}{3} \quad B=\frac{1}{12} \quad C=\frac{1}{4}$$

$$\frac{s}{s^2-4}=\frac{D}{s-2}+\frac{E}{s+2}$$

$$s=(D+E)s+2D-2E$$

$$\begin{cases} D+E=1 \\ 2D-2E=0 \end{cases} \Rightarrow D=\frac{1}{2} \quad E=\frac{1}{2}$$

$$Y(s)=-\frac{1}{3}\frac{1}{s+1}+\frac{1}{12}\frac{1}{s-2}+\frac{1}{4}\frac{1}{s+2}-\frac{1}{2}\frac{1}{s-2}-\frac{1}{2}\frac{1}{s+2}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{1}{3}\frac{1}{s+1}-\frac{5}{12}\frac{1}{s-2}-\frac{1}{4}\frac{1}{s+2}\right\}$$

$$\boxed{y(t)=-\frac{1}{3}e^{-t}-\frac{5}{12}e^{2t}-\frac{1}{4}e^{-2t}}$$

Exercise

Solve using the Laplace transform: $y''-4y'=6e^{3t}-3e^{-t}, \quad y(0)=1 \quad y'(0)=-1$

Solution

$$\mathcal{L}\{y''-4y'\}=\mathcal{L}\left\{6e^{3t}-3e^{-t}\right\}$$

$$s^2Y(s)-sy(0)-y'(0)-4sY(s)+4y(0)=0 \qquad y(0)=1; \ y'(0)=-1$$

$$\left(s^2-4s\right)Y(s)-s+1+4=0$$

$$Y(s)=\frac{s-5}{s(s-4)}=\frac{A}{s}+\frac{B}{s-4}$$

$$As-4A+Bs=s-5$$

$$\begin{cases} A+B=1 \\ -4A=-5 \end{cases} \rightarrow \boxed{A=\frac{5}{4}, \ B=-\frac{1}{4}}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{5}{4}\frac{1}{s}-\frac{1}{4}\frac{1}{s-4}\right\}$$

$$\boxed{y(t)=\frac{5}{4}t-\frac{1}{4}e^{4t}}$$

Exercise

Solve using the Laplace transform: $y''-4y'+4y=t^3e^{2t} \ ; \quad y(0)=0, \quad y'(0)=0$

Solution

$$\mathcal{L}\{y''-4y'+4y\}=\mathcal{L}\{t^3e^{2t}\}$$

$$s^2Y(s)-sy(0)-y'(0)-4sY(s)+4y(0)+4Y(s)=\frac{6}{(s-2)^4} \qquad y(0)=0;\ \ y'(0)=0$$

$$\left(s^2-4s+4\right)Y(s)=\frac{6}{(s-2)^4}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\}$$

$$\boxed{y(t)=\frac{1}{20}t^5e^{2t}}$$

Exercise

Solve using the Laplace transform: $\quad y''-4y'+4y=t^3, \quad y(0)=1, \quad y'(0)=0$

Solution

$$\mathcal{L}\{y''-4y'+4y\}=\mathcal{L}\{t^3\}$$

$$s^2Y(s)-sy(0)-y'(0)-4sY(s)+4y(0)+4Y(s)=\frac{6}{s^4} \qquad y(0)=1;\ \ y'(0)=0$$

$$\left(s^2-4s+4\right)Y(s)=\frac{6}{s^4}+s-4$$

$$Y(s)=\frac{s^5-4s^4+6}{s^4(s-2)^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s^4}+\frac{E}{s-2}+\frac{F}{(s-2)^2}$$

s^5	$A+E=1$	$\boxed{E=-\frac{1}{4}}$
s^4	$-4A+B-2E+F=-4$	$\boxed{F=-\frac{13}{8}}$
s^3	$4A-4B+C=0$	$\boxed{A=\frac{3}{4}}$
s^2	$4B-4C+D=0$	$\boxed{B=\frac{9}{8}}$
s^1	$4C-4D=0$	$\boxed{C=\frac{3}{2}}$
s^0	$4D=6$	$\boxed{D=\frac{3}{2}}$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{3}{4}\frac{1}{s}+\frac{9}{8}\frac{1}{s^2}+\frac{3}{2}\frac{1}{s^3}+\frac{3}{2}\frac{1}{s^4}+\frac{1}{4}\frac{1}{s-2}-\frac{13}{8}\frac{1}{(s-2)^2}\right\}$$

$$\boxed{y(t)=\frac{3}{4}+\frac{9}{8}t+\frac{3}{2}t^2+\frac{3}{2}t^3+\frac{1}{4}e^{2t}-\frac{13}{8}te^{2t}}$$

Exercise

Solve using the Laplace transform: $x'' + 4x' + 4x = t^2; \quad x(0) = x'(0) = 0$

Solution

$$\mathcal{L}\{x'' + 4x' + 4x\} = \mathcal{L}\{t^2\}$$

$$s^2X(s) - sx(0) - x'(0) + 4sX(s) - 4x(0) + 4X(s) = \frac{2}{s^3} \qquad x(0) = x'(0) = 0$$

$$\left(s^2 + 4s + 4\right)X(s) = \frac{2}{s^3}$$

$$X(s) = \frac{2}{s^3(s+2)^2} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s+2} + \frac{E}{(s+2)^2}$$

$$As^2 + 4As + 4A + Bs^3 + 4Bs^2 + 4Bs + Cs^4 + 4Cs^3 + 4Cs^2 + Ds^4 + 2Ds^3 + Es^3 = 3$$

$s^4 \quad C + D = 0 \quad \rightarrow D = -\frac{9}{16}$

$s^3 \quad B + 4C + 2D + E = 0 \quad \rightarrow E = -\frac{3}{8}$

$s^2 \quad A + 4B + 4C = 0 \quad \rightarrow C = \frac{9}{16}$

$s^1 \quad 4A + 4B = 0 \quad \rightarrow B = -\frac{3}{4}$

$s^0 \quad 4A = 3 \rightarrow A = \frac{3}{4}$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{4}\frac{1}{s^3} - \frac{3}{4}\frac{1}{s^2} + \frac{9}{16}\frac{1}{s} - \frac{9}{16}\frac{1}{s+2} - \frac{3}{8}\frac{1}{(s+2)^2}\right\}$$

$$x(t) = \frac{3}{8}t^2 - \frac{3}{4}t + \frac{9}{16} - \frac{9}{16}e^{-2t} - \frac{3}{8}te^{-2t}$$

Exercise

Solve using the Laplace transform: $y'' + 4y = 4t^2 - 4t + 10; \quad y(0) = 0, \quad y'(0) = 3$

Solution

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{4t^2 - 4t + 10\}$$

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{8}{s^3} - \frac{4}{s^2} + \frac{10}{s} \qquad y(0) = 0 \ \ y'(0) = 3$$

$$\left(s^2 + 4\right)Y(s) = \frac{8 - 4s + 10s^2}{s^3} + 3$$

$$Y(s) = \frac{3s^3 + 10s^2 - 4s + 8}{s^3\left(s^2 + 4\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds + E}{s^2 + 4}$$

$$\begin{array}{lll} s^4 & A+D=0 & \underline{D=-2} \\ s^3 & B+E=3 & \underline{E=4} \\ s^2 & 4A+C=10 & \underline{A=2} \\ s^1 & 4B=-4 & \underline{B=-1} \\ s^0 & 4C=8 & \underline{C=2} \end{array}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{1}{s^2} + \frac{2}{s^3} - \frac{2s}{s^2+2^2} + \frac{4}{s^2+2^2}\right\}$$

$$\underline{y(t) = 2 - t + t^2 - 2\cos 2t + 2\sin 2t}$$

Exercise

Solve using the Laplace transform: $y'' - 4y = 4t - 8e^{-2t}$; $y(0) = 0$, $y'(0) = 5$

Solution

$$\mathcal{L}\{y'' - 4y\} = \mathcal{L}\left\{4t - 8e^{-2t}\right\}$$

$$s^2Y(s) - sy(0) - y'(0) - 4Y(s) = \frac{4}{s^2} - \frac{8}{s+2} \qquad y(0)=0 \qquad y'(0)=5$$

$$\left(s^2-4\right)Y(s) = \frac{-8s^2+4s+8}{s^2(s+2)} + 5$$

$$Y(s) = \frac{5s^3+2s^2+4s+8}{s^2(s+2)^2(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{(s+2)^2} + \frac{E}{s-2}$$

$$5s^3+2s^2+4s+8 = A\left(s^2+2s\right)\left(s^2-4\right) + B(s+2)\left(s^2-4\right) + C\left(s^4-4s^2\right) + D\left(s^3-2s^2\right) + Es^2\left(s^2+4s+4\right)$$

$$\begin{cases} s^4 & A+C+E=0 & & C+E=0 \\ s^3 & 2A+B+D+4E=5 & & D+4E=6 \\ s^2 & -4A+2B-4C-2D+4E=2 & & -4C-2D+4E=4 \\ s^1 & -8A-4B=4 & \underline{A=0} & \\ s^0 & -8B=8 & \underline{B=-1} & \end{cases}$$

$$\underline{C=-1 \quad D=2 \quad E=1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{1}{s^2} - \frac{1}{s+2} + \frac{2}{(s+2)^2} + \frac{1}{s-2}\right\}$$

$$\underline{y(t) = -t - e^{-2t} + 2te^{-2t} + e^{2t}}$$

Exercise

Solve using the Laplace transform: $y'' + 4y' = \cos(t-3) + 4t$, $y(3) = 0,$ $y'(3) = 7$

Solution

$$y''(t) + 4y'(t) = \cos(t-3) + 4t$$

Let $v = t - 3 \quad \to \quad t = v + 3$

$$x(v) = y(t) = y(v+3) \qquad\qquad y(3) = x(0) = 0 \quad y'(3) = x'(0) = 7$$

$$x''(v) + 4x'(v) = \cos v + 4v + 12$$

$$\mathcal{L}\{x'' + 4x'\} = \mathcal{L}\{\cos v + 4v + 12\}$$

$$s^2 X(s) - sx(0) - x'(0) + 4sX(s) - 4x(0) = \frac{s}{s^2+1} + \frac{4}{s^2} + \frac{12}{s}$$

$$\left(s^2 + 4s\right) X(s) = \frac{s}{s^2+1} + \frac{4+12s}{s^2} + 7$$

$$s(s+4) X(s) = \frac{s}{s^2+1} + \frac{7s^2+12s+4}{s^2}$$

$$X(s) = \frac{1}{(s+4)\left(s^2+1\right)} + \frac{7s^2+12s+4}{s^3(s+4)}$$

$$\frac{1}{(s+4)\left(s^2+1\right)} = \frac{A_1}{s+4} + \frac{A_2 s + A_3}{s^2+1}$$

$$1 = A_1 s^2 + A_1 + A_2 s^2 + 4A_2 s + A_3 s + 4A_3$$

$$\begin{cases} s^2 & A_1 + A_2 = 0 & \to A_1 = -A_2 & \boxed{A_1 = \frac{1}{17}} \\ s & 4A_2 + A_3 = 0 & \to A_3 = -4A_2 & \boxed{A_3 = \frac{4}{17}} \\ s^0 & A_1 + 4A_3 = 1 & \Rightarrow -A_2 - 16A_2 = 1 & \to \boxed{A_2 = -\frac{1}{17}} \end{cases}$$

$$\frac{1}{(s+4)\left(s^2+1\right)} = \frac{1}{17}\frac{1}{s+4} + \frac{1}{17}\frac{-s+4}{s^2+1}$$

$$\frac{7s^2+12s+4}{s^3(s+4)} = \frac{B_1}{s} + \frac{B_2}{s^2} + \frac{B_3}{s^3} + \frac{B_4}{s+4}$$

$$7s^2 + 12s + 4 = B_1 s^3 + 4B_1 s^2 + B_2 s^2 + 4B_2 s + B_3 s + 4B_3 + B_4 s^3$$

$$\begin{cases} s^3 & B_1 + B_4 = 0 & \rightarrow \boxed{B_4 = -\frac{17}{16}} \\ s^2 & 4B_1 + B_2 = 7 & \rightarrow \boxed{B_1 = \frac{17}{16}} \\ s & 4B_2 + B_3 = 12 & \rightarrow \boxed{B_2 = \frac{11}{4}} \\ s^0 & 4B_3 = 4 & \rightarrow \boxed{B_3 = 1} \end{cases}$$

$$\frac{7s^2 + 12s + 4}{s^3(s+4)} = \frac{17}{16}\frac{1}{s} + \frac{11}{4}\frac{1}{s^2} + \frac{1}{s^3} - \frac{17}{16}\frac{1}{s+4}$$

$$X(s) = \frac{1}{17}\frac{1}{s+4} + \frac{1}{17}\frac{-s+4}{s^2+1} + \frac{17}{16}\frac{1}{s} + \frac{11}{4}\frac{1}{s^2} + \frac{1}{s^3} - \frac{17}{16}\frac{1}{s+4}$$

$$= \frac{17}{16}\frac{1}{s} + \frac{11}{4}\frac{1}{s^2} + \frac{1}{s^3} - \frac{273}{272}\frac{1}{s+4} - \frac{1}{17}\frac{s}{s^2+1} + \frac{4}{17}\frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{17}{16}\frac{1}{s} + \frac{11}{4}\frac{1}{s^2} + \frac{1}{s^3} - \frac{273}{272}\frac{1}{s+4} - \frac{1}{17}\frac{s}{s^2+1} + \frac{4}{17}\frac{1}{s^2+1}\right\}$$

$$x(v) = \frac{17}{16} + \frac{11}{4}v + \frac{1}{2}v^2 - \frac{273}{272}e^{-4v} - \frac{1}{17}\cos v + \frac{4}{17}\sin v \qquad v = t-3$$

$$y(t) = \frac{17}{16} + \frac{11}{4}(t-3) + \frac{1}{2}(t-3)^2 - \frac{273}{272}e^{-4(t-3)} - \frac{1}{17}\cos(t-3) + \frac{4}{17}\sin(t-3)$$

$$= \frac{17}{16} + \frac{11}{4}t - \frac{33}{4} + \frac{1}{2}t^2 - 3t + \frac{9}{2} - \frac{273}{272}e^{-4(t-3)} + \frac{1}{17}\big(4\sin(t-3) - \cos(t-3)\big)$$

$$\boxed{= \frac{43}{16} - \frac{1}{4}t + \frac{1}{2}t^2 - \frac{273}{272}e^{-4(t-3)} + \frac{1}{17}\big(4\sin(t-3) - \cos(t-3)\big)}$$

Exercise

Solve using the Laplace transform: $y'' + 4y' + 8y = \sin t$, $y(0) = 1$, $y'(0) = 0$

Solution

$$\mathcal{L}\{y'' + 4y' + 8y\}(s) = \mathcal{L}\{\sin t\}(s)$$

$$s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 8Y(s) = \frac{1}{s^2+1} \qquad y(0) = 1, \quad y'(0) = 0$$

$$(s^2 + 4s + 8)Y(s) - s - 4 = \frac{1}{s^2+1}$$

$$(s^2 + 4s + 8)Y(s) = \frac{1}{s^2+1} + s + 4$$

$$Y(s) = \frac{s^3 + 4s^2 + s + 5}{(s^2+1)(s^2+4s+8)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4s+8}$$

$$s^3+4s^2+s+5 = As^3+4As^2+8As+Bs^2+4Bs+8B+Cs^3+Cs+Ds^2+D$$

$$\begin{cases} s^3 & A+C=1 \\ s^2 & 4A+B+D=4 \\ s & 8A+4B+C=1 \\ s^0 & 8B+D=5 \end{cases} \Rightarrow \begin{matrix} A=-\frac{4}{65} & B=\frac{7}{65} \\ C=\frac{69}{65} & D=\frac{269}{65} \end{matrix}$$

$$Y(s)=\frac{1}{65}\left(-4\frac{s}{s^2+1}+\frac{7}{s^2+1}+\frac{69(s+2)-138+269}{(s+2)^2+4}\right)$$

$$=\frac{1}{65}\left(-4\frac{s}{s^2+1}+\frac{7}{s^2+1}+\frac{69s}{(s+2)^2+4}+\frac{2}{2}\frac{131}{(s+2)^2+4}\right)$$

$$\mathcal{L}^{-1}\{Y(s)\}=\frac{1}{65}\mathcal{L}^{-1}\left\{-4\frac{s}{s^2+1}+\frac{7}{s^2+1}+\frac{69s}{(s+2)^2+4}+\frac{131}{2}\frac{2}{(s+2)^2+4}\right\}$$

$$\boxed{y(t)=-\frac{4}{65}\cos t+\frac{7}{65}\sin t+\frac{69}{65}e^{-2t}\cos 2t+\frac{131}{130}e^{-2t}\sin 2t}$$

Exercise

Solve using the Laplace transform: $y''+5y'-y=e^t-1$; $y(0)=1$, $y'(0)=1$

Solution

$$\mathcal{L}\{y''+5y'-y\}=\mathcal{L}\left\{e^t-1\right\}$$

$$s^2Y(s)-sy(0)-y'(0)+5sY(s)-5y(0)-Y(s)=\frac{1}{s-1}-\frac{1}{s} \qquad y(0)=1 \quad y'(0)=1$$

$$\left(s^2+5s-1\right)Y(s)=\frac{1}{s(s-1)}+s+6$$

$$Y(s)=\frac{s^3+5s^2-6s+1}{s(s-1)\left(s+\frac{5}{2}-\frac{\sqrt{29}}{2}\right)\left(s+\frac{5}{2}+\frac{\sqrt{29}}{2}\right)}$$

$$=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+\frac{5}{2}-\frac{\sqrt{29}}{2}}+\frac{D}{s+\frac{5}{2}+\frac{\sqrt{29}}{2}}$$

$$\begin{cases} s^3 & A+C+D=1 \\ s^2 & 4A+B+\left(\frac{3}{2}+\frac{\sqrt{29}}{2}\right)C+\left(\frac{3}{2}-\frac{\sqrt{29}}{2}\right)C=5 \\ s^1 & -6A+5B-\left(\frac{5}{2}+\frac{\sqrt{29}}{2}\right)C-\left(\frac{5}{2}-\frac{\sqrt{29}}{2}\right)D=-6 \\ s^0 & A-B=1 \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{1}{5}\frac{1}{s-1}+\left(-\frac{1}{10}+\frac{3}{10\sqrt{29}}\right)\frac{1}{s+\frac{5}{2}-\frac{\sqrt{29}}{2}}-\left(\frac{1}{10}+\frac{3}{10\sqrt{29}}\right)\frac{1}{s+\frac{5}{2}+\frac{\sqrt{29}}{2}}\right\}$$

$$\boxed{y(t)=1+\frac{1}{5}e^{t}+\left(-\frac{1}{10}+\frac{3\sqrt{29}}{290}\right)e^{\frac{-5+\sqrt{29}}{2}t}-\left(\frac{1}{10}+\frac{3\sqrt{29}}{290}\right)e^{\frac{-5-\sqrt{29}}{2}t}}$$

Exercise

Solve using the Laplace transform: $y''+5y'-6y=21e^{t-1}\quad y(1)=-1,\quad y'(1)=9$

Solution

Let: $w(t)=y(t+1)\quad\leftrightarrow\quad y(t)=w(t-1)$

$$\mathcal{L}\{w''+5w'-6w\}=\mathcal{L}\left\{21e^{t}\right\}$$

$$s^2W(s)-sw(0)-w'(0)+5sW(s)-5w(0)-6W(s)=21\frac{1}{s-1}\qquad y(1)=w(0)=-1,\quad y'(1)=w(0)=9$$

$$\left(s^2+5s-6\right)W(s)=\frac{21}{s-1}-s+4$$

$$W(s)=\frac{-s^2+5s+17}{(s+6)(s-1)^2}=\frac{A}{s+6}+\frac{B}{s-1}+\frac{C}{(s-1)^2}$$

$s^2\qquad A+B=-1$

$s^1\qquad -2A+5B+C=5$

$s^0\qquad A-6B+6C=17$

$$\Delta=\begin{vmatrix}1&1&0\\-2&5&1\\1&-6&6\end{vmatrix}=49\quad \Delta_A=\begin{vmatrix}-1&1&0\\5&5&1\\17&-6&6\end{vmatrix}=-49\quad \Delta_B=\begin{vmatrix}1&-1&0\\-2&5&1\\1&17&6\end{vmatrix}=0\quad \Delta_C=\begin{vmatrix}1&1&-1\\-2&5&5\\1&-6&17\end{vmatrix}=147$$

$$\boxed{A=-1,\quad B=0,\quad C=3}$$

$$\mathcal{L}^{-1}\{W(s)\}=\mathcal{L}^{-1}\left\{\frac{-1}{s+6}+\frac{3}{(s-1)^2}\right\}$$

$$w(t)=-e^{-6t}+3te^{t}$$

$$y(t)=w(t-1)\ \boxed{=-e^{-6(t-1)}+3(t-1)e^{(t-1)}}$$

Exercise

Solve using the Laplace transform: $y'' + 5y' + 4y = 0$; $y(0) = 1$, $y'(0) = 0$

Solution

$$\mathcal{L}\{y'' + 5y' + 4y\} = 0$$

$$s^2 Y(s) - sy(0) - y'(0) + 5sY(s) - 5y(0) + 4Y(s) = 0 \qquad y(0) = 1;\ y'(0) = 0$$

$$\left(s^2 + 5s + 4\right) Y(s) - s - 5 = 0$$

$$Y(s) = \frac{s+5}{s^2 + 5s + 4} = \frac{A}{s+1} + \frac{B}{s+4}$$

$$As + 4A + Bs + B = s + 5$$

$$\begin{cases} A + B = 1 \\ 4A + B = 5 \end{cases} \rightarrow \boxed{A = \frac{4}{3};\ B = -\frac{1}{3}}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{4}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4}\right\}$$

$$\boxed{y(t) = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}}$$

Exercise

Solve using the Laplace transform: $y'' + 6y = t^2 - 1$; $y(0) = 0$, $y'(0) = -1$

Solution

$$\mathcal{L}\{y'' + 6y\} = \mathcal{L}\left\{t^2 - 1\right\}$$

$$s^2 Y(s) - sy(0) - y'(0) + 6Y(s) = \frac{2}{s^3} - \frac{1}{s} \qquad y(0) = 0 \quad y'(0) = -1$$

$$\left(s^2 + 6\right) Y(s) = \frac{2 - s^2}{s^3} - 1$$

$$Y(s) = \frac{2 - s^2 - s^3}{s^3\left(s^2 + 6\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds + E}{s^2 + 6}$$

s^4 $\quad A + D = 0 \quad \boxed{D = \frac{2}{9}}$

s^3 $\quad B + E = -1 \quad \boxed{E = -1}$

s^2 $\quad 6A + C = -1 \quad \boxed{A = -\frac{2}{9}}$

s $\quad \boxed{B = 0}$

s^0 $\quad 6C = 2 \quad \boxed{C = \frac{1}{3}}$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{2}{9}\frac{1}{s} + \frac{1}{3}\frac{1}{s^3} + \frac{1}{9}\frac{s}{s^2+6} - \frac{1}{s^2+6}\right\}$$

$$\boxed{y(t) = -\frac{2}{9} + \frac{1}{6}t^2 + \frac{1}{9}\cos\sqrt{6}t - \frac{1}{\sqrt{6}}\sin\sqrt{6}t}$$

Exercise

Solve using the Laplace transform: $y'' - 6y' + 9y = t$; $y(0) = 0$, $y'(0) = 1$

Solution

$$\mathcal{L}\{y'' - 6y' + 9y\} = \mathcal{L}\{t\}$$

$$s^2Y(s) - sy(0) - y'(0) - 6sY(s) + 6y(0) + 9Y(s) = \frac{1}{s^2} \qquad y(0) = 0 \quad y'(0) = 1$$

$$\left(s^2 - 6s + 9\right)Y(s) = \frac{1}{s^2} + 1$$

$$Y(s) = \frac{s^2+1}{s^2(s-3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{(s-3)^2}$$

$$\begin{cases} s^3 & A + C = 0 & \boxed{C = -\frac{2}{27}} \\ s^2 & -6A + B - 3C + D = 1 & \boxed{D = \frac{10}{9}} \\ s & 9A - 6B = 0 & \boxed{A = \frac{2}{27}} \\ s^0 & 9B = 1 & \boxed{B = \frac{1}{9}} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{27}\frac{1}{s} + \frac{1}{9}\frac{1}{s^2} - \frac{2}{27}\frac{1}{s-3} + \frac{10}{9}\frac{1}{(s-3)^2}\right\}$$

$$\boxed{y(t) = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{10}{9}te^{3t}}$$

Exercise

Solve using the Laplace transform: $y'' - 6y' + 15y = 2\sin 3t$, $y(0) = -1$, $y'(0) = -4$

Solution

$$\mathcal{L}\{y'' - 6y' + 15y\}(s) = \mathcal{L}\{2\sin 3t\}(s)$$

$$s^2Y(s) - sy(0) - y'(0) - 6sY(s) + 6y(0) + 15Y(s) = \frac{6}{s^2+9} \qquad y(0) = -1, \quad y'(0) = -4$$

$$\left(s^2 - 6s + 15\right)Y(s) + s + 4 - 6 = \frac{6}{s^2+9}$$

$$\left(s^2-6s+15\right)Y(s)=\frac{6}{s^2+9}-s+2$$

$$Y(s)=\frac{-s^3+2s^2-9s+24}{\left(s^2+9\right)\left(s^2-6s+15\right)}=\frac{As+B}{s^2+9}+\frac{Cs+D}{s^2-6s+15}$$

$$-s^3+2s^2-9s+24=As^3-6As^2+15As+Bs^2-6Bs+15B+Cs^3+9Cs+Ds^2+9D$$

$$\begin{cases} s^3 & A+C=-1 \\ s^2 & -6A+B+D=2 \\ s & 15A-6B+9C=-9 \\ s^0 & 15B+9D=24 \end{cases} \quad\to\quad \begin{matrix} A=\frac{1}{10} & B=\frac{1}{10} \\ C=-\frac{11}{10} & D=\frac{5}{2} \end{matrix}$$

$$Y(s)=\frac{1}{10}\frac{s}{s^2+9}+\frac{1}{10}\frac{1}{s^2+9}+\frac{1}{10}\frac{-11(s-3)-33+25}{(s-3)^2-9+15}$$

$$=\frac{1}{10}\frac{s}{s^2+9}+\frac{1}{10}\frac{1}{s^2+9}\frac{3}{3}-\frac{11}{10}\frac{s-3}{(s-3)^2+6}-\frac{1}{10}\frac{8}{(s-3)^2+6}\frac{\sqrt{6}}{\sqrt{6}}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{10}\frac{s}{s^2+9}+\frac{1}{30}\frac{3}{s^2+9}-\frac{11}{10}\frac{s-3}{(s-3)^2+6}-\frac{8}{10\sqrt{6}}\frac{\sqrt{6}}{(s-3)^2+6}\right\}$$

$$\boxed{y(t)=\frac{1}{10}\cos 3t+\frac{1}{30}\sin 3t-\frac{11}{10}e^{3t}\cos\sqrt{6}t-\frac{8}{10\sqrt{6}}e^{3t}\sin\sqrt{6}t}$$

Exercise

Solve using the Laplace transform: $y''-6y'+13y=0\,;\quad y(0)=0,\quad y'(0)=-3$

Solution

$$\mathcal{L}\{y''-6y'+13y\}=0$$

$$s^2Y(s)-sy(0)-y'(0)-6sY(s)+6y(0)+13Y(s)=0 \qquad y(0)=0 \quad y'(0)=-3$$

$$\left(s^2-6s+13\right)Y(s)=-3$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{-3}{(s-3)^2+4}\right\}$$

$$\boxed{y(t)=-\frac{3}{2}e^{3t}\sin 2t}$$

Exercise

Solve using the Laplace transform: $y'' + 6y' + 9y = 0, \quad y(0) = -1, \quad y'(0) = 6$

Solution

$$\mathcal{L}\{y'' + 6y' + 9y\} = 0$$

$$s^2Y(s) - sy(0) - y'(0) + 6Y(s) - 6y(0) + 9Y(s) = 0 \qquad y(0) = -1 \quad y'(0) = 6$$

$$\left(s^2 + 6s + 9\right)Y(s) = -s$$

$$Y(s) = -\frac{s}{(s+3)^2} = \frac{A}{s+3} + \frac{B}{(s+3)^2}$$

$$\begin{cases} s & \underline{A = -1} \\ s^0 & 3A + B = 0 \quad \underline{B = 3} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{s+3} + \frac{3}{(s+3)^2}\right\}$$

$$\underline{y(t) = -e^{-3t} + 3te^{-3t}}$$

Exercise

Solve using the Laplace transform: $y'' + 6y' + 5y = 12e^t \;, \quad y(0) = -1, \quad y'(0) = 7$

Solution

$$\mathcal{L}\{y'' + 6y' + 5y\} = \mathcal{L}\left\{12e^t\right\}$$

$$s^2Y(s) - sy(0) - y'(0) + 6Y(s) - 6y(0) + 5Y(s) = \frac{12}{s-1} \qquad y(0) = -1 \quad y'(0) = 7$$

$$\left(s^2 + 6s + 5\right)Y(s) = \frac{12}{s-1} - s + 1$$

$$Y(s) = \frac{-s^2 + 2s + 11}{(s+1)(s+5)(s-1)} = \frac{A}{s+1} + \frac{B}{s+5} + \frac{C}{s-1}$$

$$\begin{cases} s^2 & A + B + C = -1 \\ s & 4A + 6C = 2 \\ s^0 & -5A - B + 5C = 11 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 0 & 6 \\ -5 & -1 & 5 \end{vmatrix} = -48 \quad \Delta_A = \begin{vmatrix} -1 & 1 & 1 \\ 2 & 0 & 6 \\ 11 & -1 & 5 \end{vmatrix} = 48 \quad \Delta_B = \begin{vmatrix} 1 & -1 & 1 \\ 4 & 2 & 6 \\ -5 & 11 & 5 \end{vmatrix} = 48$$

$$\underline{A = -1 \quad B = -1 \quad C = 1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{s+1} - \frac{1}{s+5} + \frac{1}{s-1}\right\}$$

$$\boxed{y(t) = -e^{-t} - e^{-5t} + e^{t}}$$

Exercise

Solve using the Laplace transform: $y'' - 7y' + 10y = 9\cos t + 7\sin t$; $y(0) = 5,\quad y'(0) = -4$

Solution

$$\mathcal{L}\{y'' - 7y' + 10y\} = \mathcal{L}\{9\cos t + 7\sin t\}$$

$$s^2Y(s) - sy(0) - y'(0) - 7sY(s) + 7y(0) + 10Y(s) = \frac{9s}{s^2+1} + \frac{7}{s^2+1} \qquad y(0) = 5,\quad y'(0) = -4$$

$$\left(s^2 - 7s + 10\right)Y(s) = \frac{9s+7}{s^2+1} + 5s - 39$$

$$Y(s) = \frac{5s^3 - 39s^2 + 14s - 32}{(s-2)(s-5)\left(s^2+1\right)} = \frac{A}{s-2} + \frac{B}{s-5} + \frac{Cs+D}{s^2+1}$$

$$\begin{cases} s^3 & A + B + C = 5 \\ s^2 & -5A - 2B - 7C + D = -39 \\ s & A + B + 10C - 7D = 14 \\ s^0 & -5A - 2B + 10D = -32 \end{cases} \rightarrow \boxed{A = 8,\ B = -4,\ C = 1,\ D = 0}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{8}{s-2} - \frac{4}{s-5} + \frac{s}{s^2+1}\right\}$$

$$\boxed{y(t) = 8e^{2t} - 4e^{5t} + \cos t}$$

Exercise

Solve using the Laplace transform: $y'' + 8y' + 25y = 0$, $y(\pi) = 0,\quad y'(\pi) = 6$

Solution

Let: $w(t) = y(t+\pi) \quad \leftrightarrow \quad y(t) = w(t-\pi)$

$$y''(t) + 8y'(t) + 25y(t) = 0$$

$$\mathcal{L}\{w'' + 8w' + 25w\} = 0$$

$$s^2W(s) - sw(0) - w'(0) + 8sW(s) - 8w(0) + 25W(s) = 0 \qquad y(\pi) = w(0) = 0,\quad y'(\pi) = w(0) = 6$$

$$\left(s^2 + 8s + 25\right)W(s) - 6 = 0$$

$$W(s)=\frac{6}{(s+4)^2-16+25}$$

$$\mathcal{L}^{-1}\{W(s)\}=\mathcal{L}^{-1}\left\{\frac{2\ (3)}{(s+4)^2+9}\right\}$$

$$w(t)=2e^{-4t}\sin 3t \qquad y(t)=w(t-\pi)$$

$$y(t)=2e^{-4(t-\pi)}\sin 3(t-\pi) \qquad \sin(3t-3\pi)=\sin 3t\cos 3\pi-\cos 3t\sin 3\pi=\sin 3t(-1)-0=-\sin 3t$$

$$\boxed{=-2e^{-4(t-\pi)}\sin 3t}$$

Exercise

Solve using the Laplace transform: $y''+9y=2\sin 2t;\quad y(0)=0,\quad y'(0)=-1$

Solution

$$\mathcal{L}(y''+9y)=\mathcal{L}(2\sin 2t)$$

$$s^2Y(s)-sy(0)-y'(0)+9Y(s)=2\frac{2}{s^2+2^2} \qquad y(0)=0 \quad y'(0)=-1$$

$$\left(s^2+9\right)Y(s)+1=\frac{4}{s^2+4}$$

$$Y(s)=\frac{4}{\left(s^2+9\right)\left(s^2+4\right)}-\frac{1}{s^2+9}$$

$$\frac{4}{\left(s^2+9\right)\left(s^2+4\right)}=\frac{A}{s^2+9}+\frac{B}{s^2+4}$$

$$4=(A+B)s^2+4A+9B$$

$$\begin{cases} A+B=0 \\ 4A+9B=4 \end{cases} \Rightarrow A=-\frac{4}{5} \quad B=\frac{4}{5}$$

$$Y(s)=-\frac{4}{5}\frac{1}{s^2+9}+\frac{4}{5}\frac{1}{s^2+4}-\frac{1}{s^2+9}$$

$$=\frac{4}{5}\frac{1}{s^2+4}-\frac{9}{5}\frac{1}{s^2+9}$$

$$=\frac{4}{5}\frac{1}{2}\frac{2}{s^2+4}-\frac{3}{5}\frac{3}{s^2+9}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{2}{5}\frac{2}{s^2+4}-\frac{3}{5}\frac{3}{s^2+9}\right\}$$

$$\boxed{y(t)=\frac{2}{5}\sin 2t-\frac{3}{5}\sin 3t}$$

Exercise

Solve using the Laplace transform: $y'' + 9y = 3\sin 2t;\quad y(0) = 0,\quad y'(0) = -1$

Solution

$$\mathcal{L}(y'' + 9y)(s) = \mathcal{L}(3\sin 2t)(s)$$

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = 3\frac{1}{s^2+4} \qquad y(0)=0 \quad y'(0)=-1$$

$$s^2Y(s) + 1 + Y(s) = \frac{3}{s^2+4}$$

$$\left(s^2+1\right)Y(s) = \frac{3}{s^2+4} - 1$$

$$\left(s^2+1\right)Y(s) = \frac{-s^2-1}{s^2+4}$$

$$Y(s) = \frac{-s^2-1}{\left(s^2+4\right)\left(s^2+1\right)} \quad = \frac{A}{s^2+4} + \frac{B}{s^2+1}$$

$$As^2 + A + Bs^2 + 4B = -s^2 - 1$$

$$\begin{cases} s^2 & \begin{cases} A + B = -1 \\ A + 4B = 1 \end{cases} \\ s^0 & \end{cases} \Rightarrow \begin{cases} A = \frac{2}{3} \\ B = -\frac{5}{3} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{3}\frac{1}{s^2+4} - \frac{5}{3}\frac{1}{s^2+1}\right\}$$

$$\boxed{y(t) = \frac{2}{3}\sin 2t - \frac{5}{3}\sin t}$$

Exercise

Solve using the Laplace transform: $y'' + 16y = 2\sin 4t\ ;\quad y(0) = -\frac{1}{2},\quad y'(0) = 0$

Solution

$$\mathcal{L}\{y'' + 16y\}(s) = \mathcal{L}\{2\sin 4t\}(s)$$

$$s^2Y(s) - sy(0) - y'(0) + 16Y(s) = \frac{8}{s^2+16}$$

$$\left(s^2+16\right)Y(s) + \frac{s}{2} = \frac{8}{s^2+16}$$

$$\left(s^2+16\right)Y(s) = \frac{8}{s^2+16} - \frac{s}{2}$$

$$Y(s) = \frac{8}{\left(s^2+16\right)^2} - \frac{1}{2}\frac{s}{s^2+16} \qquad \mathcal{L}^{-1}\frac{2a^3}{\left(s^2+a^2\right)^2} = \sin(at) - at\cos(at)$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{8}{128}\frac{128}{\left(s^2+4^2\right)^2} - \frac{1}{2}\frac{s}{s^2+16}\right\}$$

$$\boxed{y(t) = \frac{1}{16}(\sin 4t - 4t\cos 4t) - \frac{1}{2}\cos 4t}$$

Exercise

Solve using the Laplace transform: $y'' - 10y' + 9y = 5t$; $y(0) = -1,\quad y'(0) = 2$

Solution

$$\mathcal{L}\{y'' - 10y' + 9y\}(s) = \mathcal{L}\{5t\}(s)$$

$$s^2Y(s) - sy(0) - y'(0) - 10sY(s) + 10y(0) + 9Y(s) = \frac{5}{s^2}$$

$$\left(s^2 - 10s + 9\right)Y(s) = \frac{5}{s^2} - s + 12$$

$$Y(s) = \frac{-s^3 + 12s^2 + 5}{s^2(s-9)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

$$-s^3 + 12s^2 + 5 = As^3 - 10As^2 + 9As + Bs^2 - 10Bs + 9B + Cs^3 - Cs^2 + Ds^3 - 9Ds^2$$

$$\begin{cases} s^3 & A + C + D = -1 \\ s^2 & -10A + B - C - 9D = -12 \\ s^1 & 9A - 10B = 0 & \rightarrow \boxed{A = \frac{50}{81}} \\ s^0 & 9B = 5 & \rightarrow \boxed{B = \frac{5}{9}} \end{cases}$$

$$\begin{cases} C + D = -\frac{131}{81} \\ C - 9D = -\frac{517}{81} \end{cases} \rightarrow \boxed{C = \frac{31}{81}} \quad \boxed{D = -2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{50}{81}\frac{1}{s} + \frac{5}{9}\frac{1}{s^2} + \frac{31}{81}\frac{1}{s-9} - \frac{2}{s-1}\right\}$$

$$\boxed{y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}e^{9t} - 2e^t}$$

Exercise

Solve using the Laplace transform: $2y'' + 3y' - 2y = te^{-2t}$, $y(0) = 0$, $y'(0) = -2$

Solution

$$\mathcal{L}\{2y'' + 3y' - 2y\}(s) = \mathcal{L}\left\{te^{-2t}\right\}(s)$$

$$2s^2Y(s) - 2sy(0) - 2y'(0) + 3sY(s) - 3y(0) - 2Y(s) = \frac{1}{(s+2)^2}$$

$$\left(2s^2 + 3s - 2\right)Y(s) + 4 = \frac{1}{(s+2)^2}$$

$$(2s-1)(s+2)Y(s) = \frac{1}{(s+2)^2} - 4$$

$$Y(s) = \frac{-4s^2 - 16s - 15}{(2s-1)(s+2)^3} = \frac{A}{2s-1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3}$$

$$-4s^2 - 16s - 15 = As^3 + 6As^2 + 12As + 8A + (2Bs - B)\left(s^2 + 4s + 4\right) + 2Cs^2 + 2Cs - 2C + 2Ds - D$$

$$\begin{cases} s^3 & A + 2B = 0 \\ s^2 & 6A + 7B + 2C = -4 \\ s^1 & 12A + 4B + 3C + 2D = -16 \\ s^0 & 8A - 4B - 2C - D = -15 \end{cases}$$

$$\boxed{A = -\frac{192}{125} \quad B = \frac{96}{125} \quad C = -\frac{2}{25} \quad D = -\frac{1}{5}}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{192}{125}\frac{1}{2\left(s-\frac{1}{2}\right)} + \frac{96}{125}\frac{1}{s+2} - \frac{2}{25}\frac{1}{(s+2)^2} - \frac{1}{5}\frac{1}{(s+2)^3}\right\}$$

$$\boxed{y(t) = -\frac{96}{125}e^{t/2} + \frac{96}{125}e^{-2t} - \frac{2}{25}te^{-2t} - \frac{1}{5}t^2e^{-2t}}$$

Exercise

Solve using the Laplace transform: $2y'' + 20y' + 51y = 0$, $y(0) = 2$, $y'(0) = 0$

Solution

$$\mathcal{L}\{2y'' + 20y' + 51y\} = 0$$

$$2s^2Y(s) - 2sy(0) - 2y'(0) + 20sY(s) - 20y(0) + 51Y(s) = 0 \qquad y(0) = 2 \quad y'(0) = 0$$

$$\left(2s^2 + 20s + 51\right)Y(s) = 4s + 40$$

$$Y(s)=\frac{4s+40}{2\left(s^2+10s+\frac{51}{2}\right)}$$

$$=\frac{2s+20}{(s+5)^2+\frac{1}{2}}$$

$$=\frac{2s}{(s+5)^2+\left(\frac{1}{\sqrt{2}}\right)^2}+\frac{20}{\sqrt{2}}\frac{\frac{1}{\sqrt{2}}}{(s+5)^2+\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{2s}{(s+5)^2+\left(\frac{\sqrt{2}}{2}\right)^2}+10\sqrt{2}\frac{\frac{1}{\sqrt{2}}}{(s+5)^2+\left(\frac{\sqrt{2}}{2}\right)^2}\right\}$$

$$\boxed{y(t)=2e^{-5t}\cos\frac{\sqrt{2}t}{2}+10\sqrt{2}e^{-5t}\sin\frac{\sqrt{2}t}{2}}$$

Exercise

Solve using the Laplace transform: $y'''+y'=e^t\ ,\quad y(0)=y'(0)=y''(0)=0$

Solution

$$\mathcal{L}\{y'''+y'\}=\mathcal{L}\left\{e^t\right\}$$

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0)+sY(s)-y(0)=\frac{1}{s-1}$$

$$\left(s^3+s\right)Y(s)=\frac{1}{s-1}$$

$$Y(s)=\frac{1}{s(s-1)\left(s^2+1\right)}=\frac{A}{s}+\frac{B}{s-1}+\frac{Cs+D}{s^2+1}$$

$$1=As^3-As^2+As-A+Bs^3+Bs+Cs^3-Cs^2+Ds^2-Ds$$

$$\begin{cases} s^3 & A+B+C=0 & B+C=1 \\ s^2 & -A-C+D=0 & -C+D=-1 \\ s & A+B-D=0 & B-D=1 \\ s^0 & -A=1 & \rightarrow \boxed{A=-1} \end{cases} \Rightarrow B=\frac{1}{2}\quad C=\frac{1}{2}\quad D=-\frac{1}{2}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{1}{s}+\frac{1}{2}\frac{B}{s-1}+\frac{1}{2}\frac{s}{s^2+1}-\frac{1}{2}\frac{1}{s^2+1}\right\}$$

$$\boxed{y(t)=-1+\frac{1}{2}e^t+\frac{1}{2}\cos t-\frac{1}{2}\sin t}$$

Exercise

Solve using the Laplace transform: $2y^{(3)} + 3y'' - 3y' - 2y = e^{-t}; \quad y(0)=0, \quad y'(0)=0, \quad y''(0)=1$

Solution

$$\mathcal{L}\left\{2y^{(3)} + 3y'' - 3y' - 2y\right\} = \mathcal{L}\left\{e^{-t}\right\}$$

$$2s^3Y(s) - 2s^2y(0) - 2sy'(0) - 2y''(0) + 3s^2Y(s) - 3sy(0) - 3y'(0) - 3sY(s) + 3y(0) - 2Y(s) = \frac{1}{s+1}$$

$$\left(2s^3 + 3s^2 - 3s - 2\right)Y(s) - 2 = \frac{1}{s+1} \qquad y(0)=y'(0)=0, \; y''(0)=1$$

$$(s-1)(2s+1)(s+2)Y(s) = \frac{1}{s+1} + 2$$

$$Y(s) = \frac{2s+3}{(s-1)(2s+1)(s+2)(s+1)} = \frac{A}{s-1} + \frac{B}{2s+1} + \frac{C}{s+2} + \frac{D}{s+1}$$

$$A(2s+1)\left(s^2+3s+2\right) + B(s+2)\left(s^2-1\right) + C(2s+1)\left(s^2-1\right) + D(2s+1)\left(s^2+s-2\right) = 2s+3$$

$$\begin{cases} s^3 & 2A + B + 2C + 2D = 0 \\ s^2 & 7A + 2B + C + 3D = 0 \\ s^1 & 7A - B - 2C - 3D = 2 \\ s^0 & 2A - 2B - C - 2D = 3 \end{cases} \qquad \Rightarrow A = \frac{5}{18} \quad B = -\frac{16}{9} \quad C = \frac{1}{9} \quad D = \frac{1}{2}$$

$$Y(s) = \frac{5}{18}\frac{1}{s-1} - \frac{16}{9}\frac{1}{2s+1} + \frac{1}{9}\frac{1}{s+2} + \frac{1}{2}\frac{1}{s+1}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{5}{18}\frac{1}{s-1} - \frac{8}{9}\frac{1}{s+\frac{1}{2}} + \frac{1}{9}\frac{1}{s+2} + \frac{1}{2}\frac{1}{s+1}\right\}$$

$$\boxed{y(t) = \frac{5}{18}e^{t} - \frac{8}{9}e^{-t/2} + \frac{1}{9}e^{-2t} + \frac{1}{2}e^{-t}}$$

Exercise

Solve using the Laplace transform: $y^{(3)} + 2y'' - y' - 2y = \sin 3t; \quad y(0)=0, \quad y'(0)=0, \quad y''(0)=1$

Solution

$$\mathcal{L}\left\{y^{(3)} + 2y'' - y' - 2y\right\} = \mathcal{L}\{\sin 3t\}$$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) + 2s^2Y(s) - 2sy(0) - 2y'(0) - sY(s) + y(0) - 2Y(s) = \frac{3}{s^2+9}$$

$$\left(s^3 + 2s^2 - s - 2\right)Y(s) - 1 = \frac{3}{s^2+9} \qquad y(0)=y'(0)=0, \; y''(0)=1$$

$$(s-1)(s+1)(s+2)Y(s) = \frac{3}{s^2+9} + 1$$

$$Y(s)=\frac{s^2+12}{(s-1)(s+1)(s+2)\left(s^2+9\right)}=\frac{A}{s-1}+\frac{B}{s+1}+\frac{C}{s+2}+\frac{Ds+E}{s^2+9}$$

$$A\left(s^2+3s+2\right)\left(s^2+9\right)+B\left(s^2+s-2\right)\left(s^2+9\right)+C\left(s^2-1\right)\left(s^2+9\right)+(Ds+E)\left(s^3+2s^2-s-2\right)$$

$$\begin{cases} s^4 & A+B+C+D=0 \\ s^3 & 3A+B+2D+E=0 \\ s^2 & 11A+7B+8C-D+2E=1 \\ s^1 & 27A+9B-2D-E=0 \\ s^0 & 18A-18B-9C-2E=12 \end{cases} \rightarrow \begin{matrix} A=\frac{13}{60} & B=-\frac{13}{20} & C=\frac{16}{39} \\ D=\frac{3}{130} & E=-\frac{3}{65} \end{matrix}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{13}{60}\frac{1}{s-1}-\frac{13}{20}\frac{1}{s+1}+\frac{16}{39}\frac{1}{s+2}+\frac{3}{130}\frac{s}{s^2+9}-\frac{1}{65}\frac{3}{s^2+9}\right\}$$

$$y(t)=\frac{13}{60}e^t-\frac{13}{20}e^{-t}+\frac{16}{39}e^{-2t}+\frac{3}{130}\cos 3t-\frac{1}{65}\sin 3t$$

Exercise

Solve using the Laplace transform: $y^{(3)}-y''+y'-y=0$; $y(0)=1,\quad y'(0)=1,\quad y''(0)=3$

Solution

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0)-s^2Y(s)+sy(0)+y'(0)+sY(s)-y(0)-Y(s)=0$$

$$\left(s^3-s^2+s-1\right)Y(s)-s^2-s-3+s=0 \qquad y(0)=1 \quad y'(0)=1 \quad y''(0)=3$$

$$s^3-s^2+s-1=s^2(s-1)+(s-1)$$

$$Y(s)=\frac{s^2+3}{(s-1)\left(s^2+1\right)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+1}$$

$$\begin{cases} s^2 & A+B=1 \\ s & -B+C=0 \\ s^0 & A-C=3 \end{cases} \quad \begin{matrix} A+C=1 \\ A-C=3 \end{matrix} \quad A=2,\ C=-1,\ B=-1$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{2}{s-1}-\frac{s}{s^2+1}-\frac{1}{s^2+1}\right\}$$

$$y(t)=2e^t-\cos t-\sin t$$

Exercise

Solve using the Laplace transform: $y^{(3)}+4y''+y'-6y=-12\ ;\quad y(0)=1,\quad y'(0)=4,\quad y''(0)=-2$

Solution

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0)+4s^2Y(s)-4sy(0)-4y'(0)+sY(s)-y(0)-6Y(s)=-\frac{12}{s}$$

$$\left(s^3+4s^2+s-6\right)Y(s)=s^2-8s-15-\frac{12}{s}$$

$$s^3+4s^2+s-6=(s-1)\left(s^2+5s+6\right)$$

$$\begin{array}{c|cccc} 1 & 1 & 4 & 1 & -6 \\ & & 1 & 5 & 6 \\ \hline & 1 & 5 & 6 & 0 \end{array}$$

$$Y(s)=\frac{s^3-8s^2-15s-12}{s(s-1)(s+2)(s+3)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s+2}+\frac{D}{s+3}$$

$$\begin{cases} s^3 & A+B+C+D=1 \\ s^2 & 4A+5B+2C+D=-8 \\ s & A+6B-3C-2D=-15 \\ s^0 & -6A=-12 \end{cases} \qquad \underline{A=2,\quad B=1,\quad C=-3,\quad D=1}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{2}{s}+\frac{1}{s-1}-\frac{3}{s+2}+\frac{1}{s+3}\right\}$$

$$\boxed{y(t)=2+e^{t}-3e^{-2t}+e^{-3t}}$$

Exercise

Solve using the Laplace transform: $y^{(3)}+3y''+3y'+y=0\ ;\quad y(0)=-4,\quad y'(0)=4,\quad y''(0)=-2$

Solution

$$\mathcal{L}\left\{y^{(3)}+3y''+3y'+y\right\}(s)=0$$

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0)+3s^2Y(s)-3sy(0)-3y'(0)+3sY(s)-3y(0)+Y(s)=0$$

$$\left(s^3+3s^2+3s+1\right)Y(s)=-4s^2-8s-2$$

$$Y(s)=\frac{-4s^2-8s-2}{(s+1)^3}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{(s+1)^3}$$

$$\begin{cases} s^2 & \underline{A=-4} \\ s & 2A+B=-8 \quad \underline{B=0} \\ s^0 & A+B+C=-2 \quad \underline{C=2} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-4}{s+1} + \frac{2}{(s+1)^3}\right\}$$

$$\boxed{y(t) = -4e^{-t} + t^2e^{-t}}$$

Exercise

Solve using the Laplace transform: $y^{(3)} - 3y'' + 3y' - y = t^2e^t$, $y(0)=1$, $y'(0)=2$, $y''(0)=3$

Solution

$$\mathcal{L}\left\{y^{(3)} - 3y'' + 3y' - y\right\}(s) = \mathcal{L}\left\{t^2e^t\right\}(s) \qquad \mathcal{L}\left\{t^ne^{-at}\right\}(s) = \frac{n!}{(s+a)^{n+1}}$$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) - 3s^2Y(s) + 3sy(0) + 3y'(0) + 3sY(s) - 3y(0) - Y(s) = \frac{2}{(s-1)^3}$$

$$\left(s^3 - 3s^2 + 3s - 1\right)Y(s) - s^2 - 2s - 3 + 3s + 6 - 3 = \frac{2}{(s-1)^3}$$

$$(s-1)^3Y(s) = \frac{2}{(s-1)^3} + s^2 - s$$

$$Y(s) = \frac{2 + \left(s^2 - s\right)\left(s^3 - 3s^2 + 3s - 1\right)}{(s-1)^6}$$

$$= \frac{s^5 - 4s^4 + 6s^3 - 4s^2 + s + 2}{(s-1)^6} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{D}{(s-1)^4} + \frac{E}{(s-1)^5} + \frac{F}{(s-1)^6}$$

$$s^5 - 4s^4 + 6s^3 - 4s^2 + s + 2 = A(s-1)^5 + B(s-1)^4 + C(s-1)^3 + D(s-1)^2 + E(s-1) + F$$

$$(s-1)^5 = s^5 - 5s^4 + 10s^3 - 10s^2 + 5s - 1 \qquad (s-1)^4 = s^4 - 4s^3 + 6s^2 - 4s + 1$$

$$\begin{cases} s^4 & \underline{A=1} & \\ s^4 & -5A + B = -4 & \rightarrow \underline{B=1} \\ s^3 & 10A - 4B + C = 6 & \rightarrow \underline{C=0} \\ s^2 & -10A + 6B - 3C + D = -4 & \rightarrow \underline{D=0} \\ s^1 & 5A - 4B + 3C - 2D + E = 1 & \rightarrow \underline{E=0} \\ s^0 & -A + B - C + D - E + F = 2 & \rightarrow \underline{F=2} \end{cases}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1} + \frac{1}{(s-1)^2} + \frac{2}{(s-1)^6}\right\} \qquad \mathcal{L}^{-1}\left\{\frac{1}{(s+a)^{n+1}}\right\} = \frac{1}{n!}t^ne^{-at}$$

$$y(t) = e^t + te^t + \frac{2}{5!}t^5e^t \quad \boxed{= e^t\left(1 + t + \frac{1}{60}t^5\right)}$$

Exercise

Solve using the Laplace transform: $y^{(3)} + y'' + 3y' - 5y = 16e^{-t}$; $y(0)=0,\quad y'(0)=2,\quad y''(0)=-4$

Solution

$$\mathcal{L}\left\{y^{(3)} + y'' + 3y' - 5y\right\}(s) = \mathcal{L}\left\{16e^{-t}\right\}(s)$$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) + s^2Y(s) - sy(0) - y'(0) + 3sY(s) - 3y(0) - 5Y(s) = \frac{16}{s+1}$$

$$\left(s^3 + s^2 + 3s - 5\right)Y(s) = \frac{16}{s+1} + 2s - 2$$

$$s^3 + s^2 + 3s - 5 = (s-1)\left(s^2 + 2s + 5\right)$$

$$\begin{array}{c|cccc} 1 & 1 & 1 & 3 & -5 \\ & & 1 & 2 & 5 \\ \hline & 1 & 2 & 5 & 0 \end{array}$$

$$Y(s) = \frac{2s^2 + 14}{(s+1)(s-1)\left(s^2 + 2s + 5\right)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C(s+1) + D}{(s+1)^2 + 4}$$

$$\begin{cases} s^3 & A + B + C = 0 \\ s^2 & A + 3B + C + D = 2 \\ s & 3A + 7B - C = 0 \\ s^0 & -5A + 5B - C - D = 14 \end{cases}$$

$$\underline{A = -2,\quad B = 1,\quad C = 1,\quad D = 0}$$

$$\mathcal{L}^{-1}\left\{Y(s)\right\} = \mathcal{L}^{-1}\left\{\frac{-2}{s+1} + \frac{1}{s-1} + \frac{(s+1)}{(s+1)^2 + 4}\right\}$$

$$\underline{y(t) = -2e^{-t} + e^{t} + e^{-t}\cos 2t}$$

Exercise

Solve using the Laplace transform: $y''' + 4y'' + 5y' + 2y = 10\cos t$, $y(0) = y'(0) = 0,\quad y''(0) = 3$

Solution

$$\mathcal{L}\left\{y''' + 4y'' + 5y' + 2y\right\} = \mathcal{L}\left\{10\cos t\right\}$$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) + 4s^2Y(s) - 4sy(0) - 4y'(0) + 5sY(s) - 5y(0) + 2Y(s) = \frac{10s}{s^2+1}$$

$$\left(s^3 + 4s^2 + 5s + 2\right)Y(s) = \frac{10s}{s^2+1} + 3$$

$$\begin{array}{c|cccc} -1 & 1 & 4 & 5 & 2 \\ & & -1 & -3 & -2 \\ \hline & 1 & 3 & 2 & 0 \end{array} \quad \to \quad s^2 + 3s + 2 = 0 \qquad \underline{s = -1,\ -1,\ -2}$$

$$Y(s)=\frac{3s^2+10s+3}{(s+2)\left(s^2+1\right)(s+1)^2}=\frac{A}{s+2}+\frac{Bs+C}{s^2+1}+\frac{D}{s+1}+\frac{E}{(s+1)^2}$$

$$\begin{aligned}3s^2+10s+3=&\,A\left(s^2+1\right)\left(s^2+2s+1\right)+(Bs+C)(s+2)\left(s^2+2s+1\right)\\&+D(s+1)(s+2)\left(s^2+1\right)+E\left(s^2+1\right)(s+2)\\=&\,A\left(s^2+1\right)\left(s^2+2s+1\right)+\left(Bs^2+2Bs+Cs+2C\right)\left(s^2+2s+1\right)\\&+D\left(s^2+3s+2\right)\left(s^2+1\right)+Es^3+2Es^2+Es+2E\\=&\,As^4+2As^3+2As^2+2As+A+Bs^4+4Bs^3+5Bs^2+2Bs+Cs^3+4Cs^2\\&+5Cs+2C+Ds^4+3Ds^3+3Ds^2+3Ds+2D+Es^2+3Es+2E\end{aligned}$$

$$\begin{cases}s^4 & A+B+D=0\\ s^3 & 2A+4B+C+3D+E=0\\ s^2 & 2A+5B+4C+3D+2E=3\\ s^1 & 2A+2B+5C+3D+E=10\\ s^0 & A+2C+2D+2E=3\end{cases}\quad\rightarrow\quad \begin{array}{lll}A=-1 & B=-1 & C=2\\ D=2 & E=-2\end{array}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{-1}{s+2}-\frac{s}{s^2+1}+\frac{2}{s^2+1}+\frac{2}{s+1}-\frac{2}{(s+1)^2}\right\}$$

$$\boxed{y(t)=-e^{-2t}-\cos t+2\sin t+2e^{-t}-2te^{-t}}$$

Exercise

Solve using the Laplace transform: $y^{(4)}+2y''+y=4te^{t};\quad y(0)=y'(0)=y''(0)=y^{(3)}(0)=0$

Solution

$$\mathcal{L}\left\{y^{(4)}+2y''+y\right\}=\mathcal{L}\left\{4te^{t}\right\}$$

$$s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)+2s^2Y(s)-2sy(0)-2y'(0)+Y(s)=\frac{4}{(s-1)^2}$$

$$\left(s^4+2s^2+1\right)Y(s)=\frac{4}{(s-1)^2}\qquad y(0)=y'(0)=y''(0)=y^{(3)}(0)=0$$

$$\left(s^2+1\right)^2Y(s)=\frac{4}{(s-1)^2}$$

$$Y(s)=\frac{4}{(s-1)^2\left(s^2+1\right)^2}=\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{Cs+D}{s^2+1}+\frac{Es+F}{\left(s^2+1\right)^2}$$

$$A(s-1)\left(s^4+2s^2+1\right)+B\left(s^4+2s^2+1\right)+(Cs+D)\left(s^2-2s+1\right)\left(s^2+1\right)+(Es+F)\left(s^2-2s+1\right)=4$$

$$\begin{cases} s^5 & A+C=0 \\ s^4 & -A-2C+D=0 \\ s^3 & 2A+C-2D+E=0 \\ s^2 & -2A+2B-2C+2D-2E+F=0 \\ s^1 & A+C-2D+E-2F=0 \\ s^0 & -A+B+D+F=4 \end{cases} \rightarrow \begin{array}{lll} A=-\frac{16}{17} & B=\frac{28}{17} & C=\frac{16}{17} \\ D=\frac{16}{17} & E=\frac{48}{17} & F=\frac{8}{17} \end{array}$$

$$Y(s)=-\frac{16}{17}\frac{1}{s-1}+\frac{28}{17}\frac{1}{(s-1)^2}+\frac{16}{17}\frac{s}{s^2+1}+\frac{16}{17}\frac{1}{s^2+1}+\frac{48}{17}\frac{s}{\left(s^2+1\right)^2}+\frac{8}{17}\frac{1}{\left(s^2+1\right)^2}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{-\frac{16}{17}\frac{1}{s-1}+\frac{28}{17}\frac{1}{(s-1)^2}+\frac{16}{17}\frac{s}{s^2+1}+\frac{16}{17}\frac{1}{s^2+1}+\frac{24}{17}\frac{2s}{\left(s^2+1\right)^2}+\frac{4}{17}\frac{2}{\left(s^2+1\right)^2}\right\}$$

$$\boxed{y(t)=-\frac{16}{17}e^t+\frac{28}{17}te^t+\frac{16}{17}\cos t+\frac{16}{17}\sin t+\frac{27}{17}t\sin t+\frac{4}{17}\sin t-t\cos t}$$

Exercise

Solve using the Laplace transform: $y^{(4)}-y=0$; $y(0)=1,\quad y'(0)=0,\quad y''(0)=0,\quad y^{(3)}(0)=0$

Solution

$$\mathcal{L}\left\{y^{(4)}-y\right\}(s)=0$$

$$s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)-Y(s)=0$$

$$\left(s^4-1\right)Y(s)=s^3$$

$$Y(s)=\frac{s^3}{(s-1)(s+1)\left(s^2+1\right)}=\frac{A}{s-1}+\frac{B}{s+1}+\frac{Cs+D}{s^2+1}$$

$$s^3=As^3+As^2+As+A+Bs^3-Bs^2+Bs-B++Cs^3-Cs+Ds^2-D$$

$$\begin{cases} s^3 & A+B+C=1 \\ s^2 & A-B+D=0 \\ s & A+B-C=0 \\ s^0 & A-B-D=0 \end{cases} \quad \rightarrow \quad \begin{array}{ll} A=\frac{1}{4} & B=\frac{1}{4} \\ C=\frac{1}{2} & D=0 \end{array}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s-1} + \frac{1}{4}\frac{1}{s+1} + \frac{1}{2}\frac{s}{s^2+1}\right\}$$

$$\boxed{y(t) = \frac{1}{4}e^{t} + \frac{1}{4}e^{-t} + \frac{1}{2}\cos t}$$

Exercise

Solve using the Laplace transform: $y^{(4)} - 4y = 0$; $y(0)=1,\quad y'(0)=0,\quad y''(0)=-2,\quad y^{(3)}(0)=0$

Solution

$$\mathcal{L}\left\{y^{(4)} - 4y\right\} = 0$$

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - 4Y(s) = 0 \qquad y(0)=1,\ y'(0)=0,\ y''(0)=-2,\ y^{(3)}(0)=0$$

$$\left(s^4-4\right)Y(s) - s^3 + 2s = 0$$

$$Y(s) = \frac{s\left(s^2-2\right)}{\left(s^2-2\right)\left(s^2+2\right)}$$

$$= \frac{s}{s^2+2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\}$$

$$\boxed{y(t) = \cos\sqrt{2}t}$$

Exercise

Solve using the Laplace transform:

$$y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = 0\ ; \quad y(0)=0,\quad y'(0)=1,\quad y''(0)=0,\quad y^{(3)}(0)=1$$

Solution

$$\mathcal{L}\left\{y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = 0\right\} = 0$$

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - 4s^3Y(s) + 4s^2y(0)$$

$$+4sy'(0) + 4y''(0) + 6s^2Y(s) - 6sy(0) - 6y'(0) - 4sY(s) + 4y(0) + Y(s) = 0$$

$$\left(s^4 - 4s^3 + 6s^2 - 4s + 1\right)Y(s) - s^2 - 1 + 4s - 6 = 0$$

$$(s+1)^4 Y(s) = s^2 - 4s + 7$$

$$Y(s) = \frac{s^2 - 4s + 7}{(s+1)^4} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^4}$$

$$As^3 + 3As^2 + 3As + A + Bs^2 + 2Bs + B + Cs + C + D = s^2 - 4s + 7$$

$$\begin{cases} s^3 & A = 0 \\ s^2 & 3A + B = 1 \\ s^1 & 3A + 2B + C = -4 \\ s^0 & A + B + C + D = 7 \end{cases} \quad \Rightarrow B = 1 \quad C = -6 \quad D = 13$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2} - \frac{6}{(s+1)^3} + \frac{13}{(s+1)^4}\right\}$$

$$y(t) = te^t - 3t^2e^t + \frac{13}{6}t^3e^t$$

$$= \left(t - 3t^2 + \frac{13}{6}t^3\right)e^t$$

Exercise

Given: $y'' - 4y' + 3y = 0, \quad y(0) = 1 \quad y'(0) = -1$

a) Show that the general solution is: $y(t) = C_1e^{3t} + C_2e^t$ and find C_1 *and* C_2

b) Use Laplace transform to solve the system

Solution

a) $\lambda^2 - 4\lambda + 3 = 0 \quad \Rightarrow \quad \boxed{\lambda = 3,\ 1}$

That implies to the general solution: $y = C_1e^{3t} + C_2e^t$

$$1 = C_1e^{3(0)} + C_2e^{(0)}$$

$$1 = C_1 + C_2$$

$$y' = 3C_1e^{3t} + C_2e^t$$

$$-1 = 3C_1e^{3(0)} + C_2e^{(0)}$$

$$\underline{-1 = 3C_1 + C_2}$$

$$\begin{cases} C_1 + C_2 = 1 \\ 3C_1 + C_2 = -1 \end{cases} \Rightarrow \boxed{C_1 = -1} \quad \boxed{C_2 = 2}$$

Therefore; the general solution is: $\boxed{y = -e^{3t} + 2e^{t}}$

b) $\mathcal{L}\left(y'' - 4y' + 3y\right)(s) = 0$

$$s^2Y(s) - sy(0) - y'(0) - 4\left(sY(s) - y(0)\right) + 3Y(s) = 0$$

$$s^2Y(s) - s + 1 - 4\left(sY(s) - 1\right) + 3Y(s) = 0$$

$$s^2Y(s) - s + 1 - 4sY(s) + 4 + 3Y(s) = 0$$

$$\left(s^2 - 4s + 3\right)Y(s) = s - 5$$

$$Y(s) = \frac{s-5}{s^2 - 4s + 3}$$

$$= \frac{s-5}{(s-1)(s-3)}$$

$$= \frac{A}{s-1} + \frac{B}{s-3} = \frac{(A+B)s - 3A - B}{(s-1)(s-3)} \qquad \begin{cases} A + B = 1 \\ -3A - B = -5 \end{cases} \Rightarrow \boxed{A = 2} \quad \boxed{B = -1}$$

$$= \frac{2}{s-1} - \frac{1}{s-3}$$

That implies: $\boxed{y(t) = 2e^{t} - e^{3t}}$

Exercise

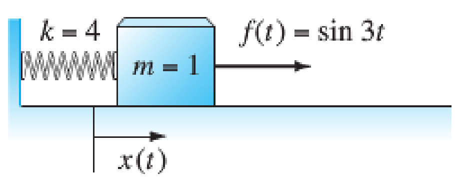
Solve the initial value problem $x'' + 4x = \sin 3t; \quad x(0) = x'(0) = 0.$

Such problem arises in the motion of a mass-and-spring system with external force as shown below.

Solution

$$\mathcal{L}\{x'' + 4x\} = \mathcal{L}\{\sin 3t\}$$

$$s^2X(s) - sx(0) - x'(0) + 4X(s) = \frac{3}{s^2 + 9}$$



$$\left(s^2 + 4\right)X(s) = \frac{3}{s^2 + 9}$$

$$X(s) = \frac{3}{\left(s^2 + 4\right)\left(s^2 + 9\right)} = \frac{As + B}{s^2 + 4} + \frac{Cs + D}{s^2 + 9}$$

$$As^3 + 9AS + Bs^2 + 9B + Cs^3 + 4CS + Ds^2 + 4C = 3$$

$$\begin{cases} s^3 & A+C=0 \\ s^2 & B+D=0 \\ s^1 & 9A+4C=0 \\ s^0 & 9B+4D=3 \end{cases} \rightarrow \ A=C=0 \quad 5B=3 \Rightarrow B=\frac{3}{5} \quad D=-\frac{3}{5}$$

$$X(s)=\frac{3}{5}\frac{1}{s^2+4}-\frac{3}{5}\frac{1}{s^2+9}$$

$$\mathcal{L}^{-1}\{X(s)\}=\mathcal{L}^{-1}\left\{\frac{3}{10}\frac{2}{s^2+4}-\frac{1}{5}\frac{3}{s^2+9}\right\}$$

$$\boxed{x(t)=\frac{3}{10}\sin 2t-\frac{1}{5}\sin 3t}$$

Exercise

Solve the system $\begin{cases} 2x''=-6x+2y \\ y''=2x-2y+40\sin 3t \end{cases}$

Subject to the initial conditions $x(0)=x'(0)=y(0)=y'(0)=0$

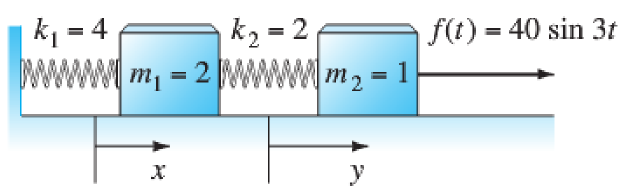
Thus the force $f(t)=40\sin 3t$ is applied to the second mass as shown below, beginning at time $t=0$ when the system is at rest in its equilibrium position.

Solution

$$\begin{cases} \mathcal{L}\{2x''\}=\mathcal{L}\{-6x+2y\} \\ \mathcal{L}\{y''\}=\mathcal{L}\{2x-2y+40\sin 3t\} \end{cases}$$

$$\begin{cases} 2s^2X(s)-2sx(0))-2x'(0)=-6X(s)+2Y(s) \\ s^2Y(s)-sy(0))-y'(0)=2X(s)-2Y(s)+\dfrac{120}{s^2+9} \end{cases}$$

Given: $x(0)=x'(0)=y(0)=y'(0)=0$



$$\begin{cases} 2s^2X(s)=-6X(s)+2Y(s) \\ s^2Y(s)=2X(s)-2Y(s)+\dfrac{120}{s^2+9} \end{cases}$$

$$\begin{cases} \left(s^2+3\right)X(s)-Y(s)=0 & (1) \\ -2X(s) \quad +\left(s^2+2\right)Y(s)=\dfrac{120}{s^2+9} & (2) \end{cases}$$

$$\begin{vmatrix} s^2+3 & -1 \\ -2 & s^2+2 \end{vmatrix}=s^4+5s^2+4=\left(s^2+1\right)\left(s^2+4\right)$$

$$\begin{vmatrix} 0 & -1 \\ \frac{120}{s^2+9} & s^2+2 \end{vmatrix} = \frac{120}{s^2+9} \quad \rightarrow \quad X(s) = \frac{120}{(s^2+1)(s^2+4)(s^2+9)}$$

$$\begin{vmatrix} s^2+3 & 0 \\ -2 & \frac{120}{s^2+9} \end{vmatrix} = 120\frac{s^2+3}{s^2+9} \quad \rightarrow \quad Y(s) = \frac{120(s^2+3)}{(s^2+1)(s^2+4)(s^2+9)}$$

$$X(s) = \frac{120}{(s^2+1)(s^2+4)(s^2+9)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} + \frac{Es+F}{s^2+9}$$

$$(As+B)(s^2+4)(s^2+9) + (Cs+D)(s^2+1)(s^2+9) + (Es+F)(s^2+1)(s^2+4) = 120$$

$$(As+B)(s^4+13s^2+36) + (Cs+D)(s^4+10s^2+9) + (Es+F)(s^4+5s^2+4) = 120$$

$$\begin{cases} s^5 & A+C+E=0 \\ s^4 & B+D+F=0 \\ s^3 & 13A+10C+5E=0 \\ s^2 & 13B+10D+5F=0 \\ s & 36A+9C+4E=0 \\ & 36B+9D+4F=120 \end{cases}$$

$$\Rightarrow \begin{cases} A+C+E=0 \\ 13A+10C+5E=0 \\ 36A+9C+4E=0 \end{cases} \rightarrow A+C+E=0$$

$$\Rightarrow \begin{cases} B+D+F=0 \\ 13B+10D+5F=0 \\ 36B+9D+4F=120 \end{cases} \rightarrow B=5;\ D=-8;\ F=3$$

$$Y(s) = \frac{120(s^2+3)}{(s^2+1)(s^2+4)(s^2+9)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} + \frac{Es+F}{s^2+9}$$

$$(As+B)(s^2+4)(s^2+9) + (Cs+D)(s^2+1)(s^2+9) + (Es+F)(s^2+1)(s^2+4) = 120s^2+360$$

$$\begin{cases} s^5 & A+C+E=0 \\ s^4 & B+D+F=0 \\ s^3 & 13A+10C+5E=0 \\ s^2 & 13B+10D+5F=120 \\ s & 36A+9C+4E=0 \\ & 36B+9D+4F=360 \end{cases}$$

$$\Rightarrow \begin{cases} A+C+E=0 \\ 13A+10C+5E=0 \\ 36A+9C+4E=0 \end{cases} \rightarrow A+C+E=0$$

$$\Rightarrow \begin{cases} B+D+F=0 \\ 13B+10D+5F=120 \\ 36B+9D+4F=360 \end{cases} \rightarrow B=10;\ D=8;\ F=-18$$

$$X(s) = \frac{5}{s^2+1} - \frac{8}{s^2+4} + \frac{3}{s^2+9}$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{5\frac{1}{s^2+1} - 4\frac{2}{s^2+4} + \frac{3}{s^2+9}\right\}$$

$$\underline{x(t) = 5\sin t - 4\sin 2t + \sin 3t}$$

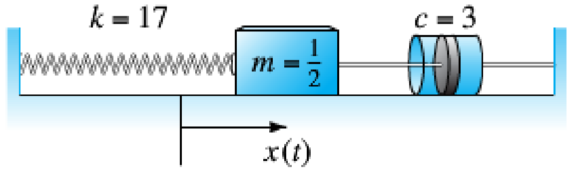
$$Y(s) = \frac{10}{s^2+1} + \frac{8}{s^2+4} - \frac{18}{s^2+9}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{10\frac{1}{s^2+1} + 4\frac{2}{s^2+4} - 6\frac{3}{s^2+9}\right\}$$

$$\underline{y(t) = 10\sin t + 4\sin 2t - 6\sin 3t}$$

Exercise

Consider a mass-spring system with $m = \frac{1}{2}, \quad k = 17, \quad and \ c = 3$.



Let $x(t)$ be the displacement of the mass m from its equilibrium position. If the mass is set in motion with $x(0) = 3 \ and \ x'(0) = 1$, find $x(t)$ for the resulting damped free oscillations.

Solution

$$\frac{1}{2}x'' + 3x' + 17x = 0 \qquad\qquad mx'' + cx' + kx = 0$$

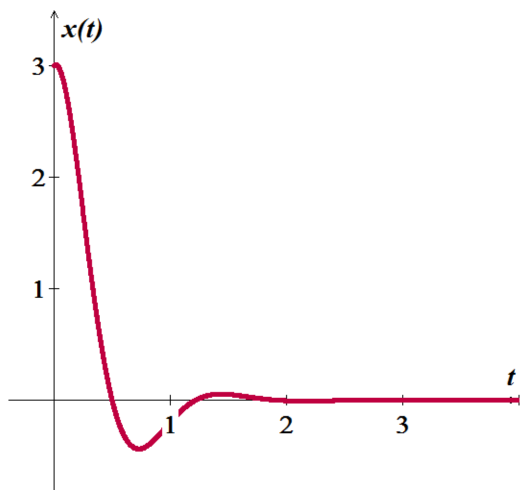
$$x'' + 6x' + 34x = 0 \qquad x(0) = 3; \ \ x'(0) = 1$$

$$\mathcal{L}\{x'' + 6x' + 34x\} = 0$$

$$s^2 X(s) - sx(0) - x'(0) + 6sX(s) - 6x(0) + 34X(s) = 0$$

$$s^2 X(s) - 3s - 1 + 6sX(s) - 18 + 34X(s) = 0$$

$$\left(s^2 + 6s + 34\right)X(s) = 3s + 19$$



$$\begin{aligned} X(s) &= \frac{3s + 19}{s^2 + 6s + 34} \\ &= \frac{3s+19}{(s+3)^2 + 25} \\ &= \frac{3s+9+10}{(s+3)^2+25} \\ &= \frac{3(s+3)}{(s+3)^2+25} + \frac{10}{(s+3)^2+25} \end{aligned}$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{3 \cdot \frac{(s+3)}{(s+3)^2+25} + 5 \cdot \frac{2}{(s+3)^2+25}\right\}$$

$$\boxed{x(t) = (3\cos 5t + 2\sin 5t)e^{-3t}}$$

Exercise

A $4-lb$ weight stretches a spring 2 *feet*. The weight is released from rest 18 *inches* above the equilibrium position, and the resulting motion takes place in a medium offering a damping force numerically equal to $\frac{7}{8}$ times the instantaneous velocity. Use the Laplace transform to find the equation of motion $x(t)$.

Solution

$m=\frac{4}{32}=\frac{1}{8} \qquad (w=mg)$

$k=\frac{4}{2}=2 \qquad (xk=mg)$

$c=\frac{7}{8}$

$\frac{1}{8}x''+\frac{7}{8}x'+2x=0 \qquad mx''+cx'+kx=f(t)$

$x''+7x'+16x=0\ ; \quad x(0)=-\frac{18}{12}=-\frac{3}{2}, \quad x'(0)=0$

$\mathcal{L}\{x''+7x'+16x\}=0$

$s^2X(s)-sx(0)-x'(0)+7sX(s)-7x(0)+16X(s)=0$

$\left(s^2+7s+16\right)X(s)=-\frac{3}{2}s-\frac{21}{2}$

$X(s)=-\frac{3}{2}\frac{s+7}{\left(s+\frac{7}{2}\right)^2+\frac{15}{4}}$

$$\frac{s+7}{\left(s+\frac{7}{2}\right)^2+\frac{15}{4}}=\frac{A\left(s+\frac{7}{2}\right)}{\left(s+\frac{7}{2}\right)^2+\frac{15}{4}}+\frac{B}{\left(s+\frac{7}{2}\right)^2+\frac{15}{4}}$$

$$\begin{cases} s & A=1 \\ s^0 & \frac{7}{2}A+B=7 \quad B=\frac{7}{2} \end{cases}$$

$$\{X(s)\}=-\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{s+\frac{7}{2}}{\left(s+\frac{7}{2}\right)^2+\left(\frac{\sqrt{15}}{2}\right)^2}\right\}-\frac{3}{2}\frac{7}{2}\frac{2}{\sqrt{15}}\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{15}}{2}}{\left(s+\frac{7}{2}\right)^2+\left(\frac{\sqrt{15}}{2}\right)^2}\right\}$$

$$\boxed{x(t)=-\frac{3}{2}e^{7t/2}\cos\frac{\sqrt{15}}{2}t-\frac{7\sqrt{15}}{10}e^{7t/2}\sin\frac{\sqrt{15}}{2}t}$$

Exercise

Consider a mass-spring-dashpot system with $m=\frac{1}{2}, \quad k=17, \quad c=3, \quad and\ f(t)=15\sin 2t$ with initial conditions $x(0)=x'(0)=0$. Let $x(t)$ be the displacement of the mass m from its equilibrium position. Find the resulting transient motion and steady periodic motion of the mass..

Solution

$$\frac{1}{2}x''+3x'+17x=15\sin 2t \qquad mx''+cx'+kx=0$$

$$x''+6x'+34x=30\sin 2t \qquad x(0)=x'(0)=0$$

$$\mathcal{L}\{x''+6x'+34x\}=\mathcal{L}\{30\sin 2t\}$$

$$s^2X(s)-sx(0)-x'(0)+6sX(s)-6x(0)+34X(s)=\frac{60}{s^2+4}$$

$$\left(s^2+6s+34\right)X(s)=\frac{60}{s^2+4}$$

$$X(s)=\frac{60}{\left(s^2+4\right)\left((s+3)^2+25\right)}=\frac{As+B}{s^2+4}+\frac{Cs+D}{(s+3)^2+25}$$

$$As^3+6As^2+34sA+Bs^2+6sB+34B+Cs^3+4Cs+Ds^2+4D=60$$

$$\begin{cases} s^3 & A+C=0 \\ s^2 & 6A+B+D=0 \\ s^1 & 34A+6B+4C=0 \\ s^0 & 34B+4D=60 \end{cases}$$

$$\begin{cases} C=-A & 6A-\frac{15}{2}B=-15 \\ & 30A+6B=0 \\ D=15-\frac{17}{2}B & \end{cases} \rightarrow \begin{cases} -30A+\frac{75}{2}B=75 \\ 30A+6B=0 \end{cases} \quad \boxed{B=\frac{150}{87}=\frac{50}{29}}$$

$$A=-\frac{10}{29}; \quad B=\frac{50}{29}; \quad C=\frac{10}{29}; \quad D=\frac{10}{29}$$

$$X(s)=\frac{10}{29}\left(\frac{-s+5}{s^2+4}+\frac{s+1}{(s+3)^2+25}\right)$$

$$=\frac{10}{29}\left(\frac{5}{s^2+4}-\frac{s}{s^2+4}+\frac{s+3}{(s+3)^2+25}-\frac{2}{(s+3)^2+25}\right)$$

$$\mathcal{L}^{-1}\{X(s)\}=\frac{10}{29}\mathcal{L}^{-1}\left(\frac{5}{2}\cdot\frac{2}{s^2+4}-\frac{s}{s^2+4}+\frac{s+3}{(s+3)^2+25}-\frac{2}{5}\cdot\frac{5}{(s+3)^2+25}\right)$$

$$x(t)=\frac{10}{29}\left(\frac{5}{2}\sin 2t-\cos 2t+e^{-3t}\left(\cos 5t-\frac{2}{5}\sin 5t\right)\right)$$

$$= \frac{5}{29}(5\sin 2t - 2\cos 2t) + \frac{2}{29}e^{-3t}(5\cos 5t - 2\sin 5t)$$

Exercise

A $8\text{-}kg$ mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $3\ N\text{-}sec/m$ and the spring constant is $40\ N/m$. If the mass is driven by an external force equal to $f(t) = 2\sin 2t\cos 2t\ N$. Find the solution.

Solution

Given: $m = 8 \quad k = 40 \quad c = 3$

$$8y'' + 3y' + 40y = 2\sin 2t\cos 2t \qquad my'' + cy' + ky = F(t)$$

$$= \sin 4t$$

$$8y'' + 3y' + 40y = \sin 4t\ ; \quad y(0) = 0, \quad y'(0) = 0$$

$$\mathcal{L}\{8y'' + 3y' + 40y\} = \mathcal{L}\{\sin 4t\}$$

$$8s^2Y(s) - 8sy(0) - 8y'(0) + 3sY(s) - 3y(0) + 40Y(s) = \frac{4}{s^2+16}$$

$$\left(8s^2 + 3s + 40\right)Y(s) = \frac{4}{s^2+16}$$

$$Y(s) = \frac{4}{8\left(s^2 + \frac{3}{8}s + 5\right)\left(s^2+16\right)}$$

$$= \frac{\frac{1}{2}}{\left(\left(s+\frac{3}{16}\right)^2 + \frac{1271}{256}\right)\left(s^2+16\right)} = \frac{A\left(s+\frac{3}{16}\right) + B}{\left(s+\frac{3}{16}\right)^2 + \frac{1271}{256}} + \frac{Cs+D}{s^2+16}$$

$$\begin{array}{ll} s^3 & A + C = 0 \\ s^2 & \frac{3}{16}A + B + \frac{3}{8}C + D = 0 \\ s & 16A + 5C + \frac{3}{8}D = 0 \\ s^0 & 3A + 16B + 5D = \frac{1}{2} \end{array}$$

$$A = \frac{3}{1972} \quad B = \frac{1417}{31552} \quad C = -\frac{3}{1972} \quad D = -\frac{22}{493}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \left\{\frac{3}{1972}\frac{s+\frac{3}{16}}{\left(s+\frac{3}{16}\right)^2 + \frac{1271}{256}} + \frac{1417}{31552}\frac{1}{\left(s+\frac{3}{16}\right)^2 + \frac{1271}{256}} - \frac{3}{1972}\frac{s}{s^2+16} - \frac{22}{493}\frac{1}{s^2+16}\right\}$$

$$y(t) = \frac{3}{1972}e^{-3t/16}\cos\frac{\sqrt{1271}}{16}t + \frac{1417}{31552}\frac{16}{\sqrt{1271}}e^{-3t/16}\sin\frac{\sqrt{1271}}{16}t - \frac{3}{1972}\cos 4t - \frac{22}{493}\frac{1}{4}\sin 4t$$

$$= \frac{3}{1972}e^{-3t/16}\cos\frac{\sqrt{1271}}{16}t + \frac{1417}{1972\sqrt{1271}}e^{-3t/16}\sin\frac{\sqrt{1271}}{16}t - \frac{3}{1972}\cos 4t - \frac{11}{986}\sin 4t$$

Exercise

A 2-*kg* mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $c = 8$ *kg/sec* and the spring constant is $k = 80$ *N/m*. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20e^{-t}$ *N*. (t in *seconds*). Find the equation of motion.

Solution

$$2y'' + 8y' + 80y = 20e^{-t} \;; \quad y(0) = 0, \quad y'(0) = 0 \qquad\qquad my'' + cy' + ky = F(t)$$

$$\mathcal{L}^{-1}\{2y'' + 8y' + 80y\} = \mathcal{L}^{-1}\{20e^{-t}\}$$

$$2s^2Y(s) - 2sy(0) - 2y'(0) + 8sY(s) - 8y(0) + 80Y(s) = \frac{20}{s+1}$$

$$2\left(s^2 + 4s + 40\right)Y(s) = \frac{20}{s+1}$$

$$Y(s) = \frac{10}{(s+1)\left((s+2)^2 + 36\right)} = \frac{A}{s+1} + \frac{B(s+2) + C}{(s+2)^2 + 36}$$

$$\begin{cases} s^2 & A + B = 0 \\ s & 4A + 3B + C = 0 \\ s^0 & 40A + 2B + C = 10 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ 4 & 3 & 1 \\ 40 & 2 & 1 \end{vmatrix} = 37 \quad \Delta_A = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 3 & 1 \\ 10 & 2 & 1 \end{vmatrix} = 10$$

$$A = \frac{10}{37} \quad B = -\frac{10}{37} \quad C = -\frac{10}{37}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{10}{37}\frac{1}{s+1} - \frac{10}{37}\frac{(s+2)}{(s+2)^2 + 6^2} - \frac{10}{37}\frac{1}{6}\frac{6}{(s+2)^2 + 6^2}\right\}$$

$$y(t) = \frac{10}{37}e^{-t} - \left(\frac{10}{37}\cos 6t + \frac{5}{111}\sin 6t\right)e^{-2t}$$

Exercise

A 10-*kg* mass is attached to a spring having a spring constant of 140 *N/m*. The mass is started in motion initially from the equilibrium position with an initial velocity 1 *m/sec* in the upward direction and with an applied external force $F(t) = 5\sin t$. If the force due to air resistance is $-90y'$ *N*. Find the equation motion of the mass.

Solution

$$10y'' + 90y' + 140y = 5\sin t$$

$$y'' + 9y' + 14y = \frac{1}{2}\sin t\ ; \quad y(0) = 0, \quad y'(0) = -1$$

$$s^2Y(s) - sy(0) - y'(0) + 9sY(s) - 9y(0) + 14Y(s) = \frac{1}{2}\frac{1}{s^2+1}$$

$$\left(s^2 + 9s + 14\right)Y(s) = \frac{1}{2}\frac{1}{s^2+1} - 1$$

$$Y(s) = \frac{-s^2 - \frac{1}{2}}{(s+2)(s+7)\left(s^2+1\right)} = \frac{A}{s+2} + \frac{B}{s+7} + \frac{Cs+D}{s^2+1}$$

$$\begin{cases} s^3 & A + B + C = 0 \\ s^2 & 7A + 2B + 9C + D = -1 \\ s & A + B + 14C + 9D = 0 \\ s^0 & 7A + 2B + 14D = -\frac{1}{2} \end{cases} \qquad \underline{A = -\frac{9}{50} \quad B = \frac{99}{500} \quad C = -\frac{9}{500} \quad D = \frac{13}{500}}$$

$$\{Y(s)\} = \left\{-\frac{9}{50}\frac{1}{s+2} + \frac{99}{500}\frac{1}{s+7} - \frac{9}{500}\frac{s}{s^2+1} + \frac{13}{500}\frac{1}{s^2+1}\right\}$$

$$y(t) = -\frac{9}{50}e^{-2t} + \frac{99}{500}e^{-7t} - \frac{9}{500}\cos t + \frac{13}{500}\sin t$$

$$= \underline{\frac{1}{500}\left(99e^{-7t} - 90e^{-2t} + 13\sin t - 9\cos t\right)}$$

Exercise

A 128-*lb* weight is attached to a spring having a spring constant of 64 *lb/ft*. The weight is started in motion initially by displacing it 6 *in* above the equilibrium position with no initial velocity and with an applied external force $F(t) = 8\sin 4t$. Assume no air resistance. Find the equation motion of the mass.

Solution

$$m = \frac{128}{32} \ \underline{= 4}$$

$$4y'' + 64y = 8\sin 4t$$

$$y'' + 16y = 2\sin 4t\ ; \quad y(0) = -\frac{6}{12} = -\frac{1}{2}, \quad y'(0) = 0$$

$$s^2Y(s) - sy(0) - y'(0) + 16Y(s) = \frac{8}{s^2+16}$$

$$\left(s^2 + 16\right)Y(s) = \frac{8}{s^2+16} - \frac{1}{2}s$$

$$Y(s) = \frac{8}{\left(s^2+16\right)^2} - \frac{1}{2}\frac{s}{s^2+16}$$

$$\frac{8}{\left(s^2+16\right)^2}=\frac{As+B}{s^2+16}+\frac{C\left(s^2-16\right)}{\left(s^2+16\right)^2}+\frac{Ds}{\left(s^2+16\right)^2}$$

$s^3 \quad A=0$

$s^2 \quad B+C=0$

$s \quad 16A+D=0$

$s^0 \quad 16B-16C=8$

$$\boxed{A=0 \quad B=\frac{1}{4} \quad C=-\frac{1}{4} \quad D=0}$$

$$\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s^2+16}-\frac{1}{4}\frac{s^2-16}{\left(s^2+16\right)^2}-\frac{1}{2}\frac{s}{s^2+16}\right\}$$

$$\boxed{y(t)=\frac{1}{16}\sin 4t-\frac{1}{4}t\cos 4t-\frac{1}{2}\cos 4t}$$

Exercise

Find the motion of a damped mass-and-spring system with $m=1$, $c=2$, and $k=26$ under the influence of an external force $F(t)=82\cos 4t$ with $x(0)=6$ and $x'(0)=0$.

Solution

Given: $m=1$, $c=2$, $k=26$, and $F(t)=82\cos 4t$ $\quad x(0)=6;\ x'(0)=0$

$x''+2x'+26x=82\cos 4t \qquad mx''+cx'+kx=F(t)$

$\mathcal{L}\{x''+2x'+26x\}=\mathcal{L}\{82\cos 4t\}$

$$s^2X(s)-sx(0)-x'(0)+2sX(s)-2x(0)+26X(s)=\frac{82s}{s^2+16}$$

$$\left(s^2+2s+26\right)X(s)=\frac{82s}{s^2+16}+6s+12$$

$$X(s)=\frac{6s^3+12s^2+178s+192}{\left(s^2+16\right)\left((s+1)^2+25\right)}=\frac{As+B}{s^2+16}+\frac{C(s+1)+D}{(s+1)^2+25}$$

$$\begin{cases} s^3 & A+C=6 \\ s^2 & 2A+B+C+D=12 \\ s & 26A+2B+16C=178 \\ s^0 & 26B+16C+16D=192 \end{cases}$$

$$\boxed{A=5 \quad B=16 \quad C=1 \quad D=-15}$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{5s}{s^2+16} + \frac{16}{s^2+16} + \frac{s+1}{(s+1)^2+25} - \frac{15}{(s+1)^2+5^2}\right\}$$

$$\underline{x(t) = 5\cos 4t + 4\sin 4t + e^{-t}(\cos 5t - 3\sin 5t)}$$

Exercise

A spring with a mass of 2-*kg* has natural length $0.5\ m$. A force of $25.6\ N$ is required to maintain it stretched to a length of $0.7\ m$. The spring is immersed in a fluid with damping constant $c = 40$. If the spring is started from the equilibrium position and is given a push to start it with initial velocity $0.6\ m/s$. Find the position of the mass at any time *t*.

Solution

$$k = \frac{25.6}{0.7-0.5} \underline{= 128} \qquad\qquad k(x_2 - x_1) = F$$

$$2x'' + 40x + 128 = 0\ ; \quad x(0) = 0, \quad x'(0) = 0.6$$

$$\mathcal{L}\{x'' + 20x + 64\} = 0$$

$$s^2X(s) - sx(0) - x'(0) + 20sX(s) - 20x(0) + 64X(s) = 0$$

$$\left(s^2 + 20s + 64\right)X(s) = \frac{6}{10}$$

$$X(s) = \frac{3}{5}\frac{1}{(s+16)(s+4)} = \frac{3}{5}\left(\frac{A}{s+16} + \frac{B}{s+4}\right)$$

$$\begin{cases} s & A + B = 0 \\ s^0 & 4A + 16B = 1 \end{cases} \rightarrow \underline{A = -\frac{1}{12},\ B = \frac{1}{12}}$$

$$\mathcal{L}^{-1}\{X(s)\} = \frac{3}{5}\mathcal{L}^{-1}\left\{-\frac{1}{12}\frac{1}{s+16} + \frac{1}{12}\frac{1}{s+4}\right\}$$

$$x(t) = \frac{3}{5}\left(-\frac{1}{12}e^{-16t} + \frac{1}{12}e^{-t}\right)$$

$$\underline{= \frac{1}{20}e^{-4t} - \frac{1}{20}e^{-16t}}$$

Exercise

A spring with a mass of 3-*kg* is held stretched $0.6\ m$ beyond its natural length by a force of $20\ N$. If the spring begins at its equilibrium and with initial velocity $1.2\ m/s$. Find the position of the mass.

Solution

$$k = \frac{20}{0.6} = \frac{100}{3} \qquad\qquad kx = F$$

$$3x'' + \frac{100}{3}x = 0\ ; \quad x(0) = 0, \quad x'(0) = 1.2 = \frac{6}{5} \qquad mx'' + cx' + kx = F(t)$$

$$\mathcal{L}\{9x''+100x\}=0$$

$$9s^2X(s)-9sx(0)-9x'(0)+100X(s)=0$$

$$\left(9s^2+100\right)X(s)=\frac{36}{5}$$

$$\mathcal{L}^{-1}\{X(s)\}=\frac{6}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+\frac{100}{9}}\right\}$$

$$x(t)=\frac{6}{5}\frac{3}{10}\sin\frac{10}{3}t$$

$$=\frac{9}{25}\sin\frac{10}{3}t$$

Exercise

A spring with a mass of 2-*kg* is held stretched 0.5 *m*, has damping constant 14, and a force of 6 *N*. If the spring is stretched 1 *m* beyond at its equilibrium and with no initial velocity. Find the position of the mass at any time *t*.

Solution

$$k=\frac{6}{.5}=12 \qquad kx=F$$

$$2x''+14x'+12x=0\ ; \quad x(0)=1, \quad x'(0)=0 \qquad mx''+cx'+kx=F(t)$$

$$\mathcal{L}\{x''+7x'+6x\}=0$$

$$s^2X(s)-sx(0)-x'(0)+7sX(s)-7x(0)+6X(s)=0$$

$$\left(s^2+7s+6\right)X(s)=s+7$$

$$X(s)=\frac{s+7}{(s+1)(s+6)}=\frac{A}{s+1}+\frac{B}{s+6}$$

$$s+7=As+6A+Bs+B$$

$$\begin{cases} s & A+B=1 \\ s^0 & 6A+B=7 \end{cases} \rightarrow \quad A=\frac{6}{5},\ B=-\frac{1}{5}$$

$$\mathcal{L}^{-1}\{X(s)\}=\mathcal{L}^{-1}\left\{\frac{6}{5}\frac{1}{s+1}-\frac{1}{5}\frac{1}{s+6}\right\}$$

$$x(t)=\frac{6}{5}e^{-t}-\frac{1}{5}e^{-6t}$$

Exercise

When a uniform beam is supported by an elastic foundation, the differential equation for its deflection $y(x)$ is

$$EI\frac{d^4y}{dx^4}+ky=w(x)$$

Where k is the modulus of the foundation and $-ky$ is the restoring force of the foundation that acts in the direction opposite to that of the load $w(x)$. For algebraic convenience, suppose that the differential equation is written as

$$\frac{d^4y}{dx^4}+4a^4y=\frac{w(x)}{EI}$$

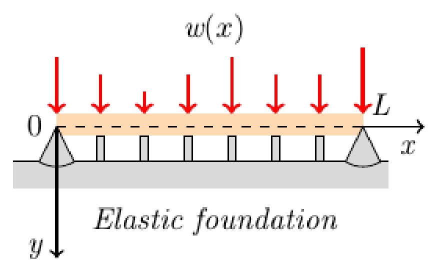
Where $a=\left(\frac{k}{4EI}\right)^{1/4}$. Assume $L=\pi$ and $a=1$. Find the deflection $y(x)$ of a beam that is supported on an elastic foundation when

a) The beam is simply supported at both ends and a constant load w_0 is uniformly distributed along its length,

b) The bean is embedded at both ends and $w(x)$ is concentrated load w_0 applied at $x=\frac{\pi}{2}$

Solution

a) $y(0)=y''(0)=0 \quad \& \quad y(\pi)=y''(\pi)=0$



Let: $y'(0)=c_1 \quad y'(0)=c_2$

$$\mathcal{L}\left\{\frac{d^4y}{dx^4}+4y\right\}=\mathcal{L}\left\{\frac{w(x)}{EI}\right\}$$

$$s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)+4Y(s)=\frac{w_0}{EI}\frac{1}{s}$$

$$\left(s^4+4\right)Y(s)=\frac{w_0}{EI}\frac{1}{s}+c_1s^2+c_2$$

$$Y(s)=\frac{w_0}{EI}\frac{1}{s\left(s^4+4\right)}+\frac{c_1s^2}{s^4+4}+\frac{c_2}{s^4+4}$$

$$\frac{1}{s\left(s^4+4\right)}=\frac{A}{s}+\frac{B(s-1)+C}{(s-1)^2+1}+\frac{D(s+1)+E}{(s+1)^2+1}$$

$s^4 \qquad A+B+D=0$

$s^3 \qquad B+C-D+E=0$

$s^2 \qquad 2C-2E=0$

$s \qquad -2B+2C+2D+2E=0$

$s^0 \qquad 4A=1$

$$A=\frac{1}{4},\ B=-\frac{1}{8},\ C=0,\ D=-\frac{1}{8},\ E=0$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{w_0}{EI}\left(\frac{1}{4}\frac{1}{s} - \frac{1}{8}\frac{s-1}{(s-1)^2+1} - \frac{1}{8}\frac{s+1}{(s+1)^2+1}\right) + \frac{c_1}{2}\frac{2s^2}{s^4+4} + \frac{c_2}{4}\frac{4}{s^4+4}\right\}$$

$$y(x) = \frac{w_0}{EI}\left(\frac{1}{4} - \frac{1}{8}e^x\cos x - \frac{1}{8}e^{-x}\cos x\right) + \frac{c_1}{2}(\sin x\cosh x + \cos x\sinh x)$$
$$+ \frac{c_2}{4}(\sin x\cosh x - \cos x\sinh x)$$

$$y(x) = \frac{w_0}{4EI}(1 - \cos x\cosh x) + \frac{c_1}{2}(\sin x\cosh x + \cos x\sinh x) + \frac{c_2}{4}(\sin x\cosh x - \cos x\sinh x)$$

$$y(\pi) = \frac{w_0}{4EI}(1 + \cosh\pi) - \frac{1}{2}c_1\sinh\pi + \frac{1}{4}c_2\sinh\pi = 0$$

$$\boxed{2c_1\sinh\pi - c_2\sinh\pi = \frac{w_0}{EI}(1 + \cosh\pi)}$$

$$y' = \frac{w_0}{4EI}(\sin x\cosh x - \cos x\sinh x) + c_1\cos x\cosh x + \frac{1}{2}c_2\sin x\sinh x$$

$$y'' = \frac{w_0}{2EI}\sin x\sinh x + c_1(-\sin x\cosh x + \cos x\sinh x) + \frac{1}{2}c_2(\cos x\sinh x + \sin x\cosh x)$$

$$y''(\pi) = -c_1\sinh\pi - \frac{1}{2}c_2\sinh\pi = 0$$

$$\boxed{c_1 = -\frac{1}{2}c_2}$$

$$2c_1\sinh\pi + 2c_1\sinh\pi = \frac{w_0}{EI}(1 + \cosh\pi)$$

$$\boxed{c_1 = \frac{w_0}{4EI}(1 + \cosh\pi)\operatorname{csch}\pi \quad c_2 = -\frac{w_0}{2EI}(1 + \cosh\pi)\operatorname{csch}\pi}$$

$$y(x) = \frac{w_0}{4EI}(1 - \cos x\cosh x) + \frac{w_0}{8EI}(1 + \cosh\pi)\operatorname{csch}\pi(\sin x\cosh x + \cos x\sinh x)$$
$$- \frac{w_0}{4EI}(1 + \cosh\pi)\operatorname{csch}\pi(\sin x\cosh x - \cos x\sinh x)$$

b) $\mathcal{L}\left\{\frac{d^4y}{dx^4} + 4y\right\} = \mathcal{L}\left\{\delta\left(t - \frac{\pi}{2}\right)\right\}$

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) + 4Y(s) = \frac{w_0}{EI}e^{-\pi s/2}$$

$$Y(s) = \frac{w_0}{4EI}\frac{4}{s^4+4}e^{-\pi s/2} + \frac{c_1}{2}\frac{2s^2}{s^4+4} + \frac{c_2}{4}\frac{4}{s^4+4}$$

$$y(x) = \frac{w_0}{4EI}\left(\sin\left(x - \frac{\pi}{2}\right)\cosh\left(x - \frac{\pi}{2}\right) - \cos\left(x - \frac{\pi}{2}\right)\sinh\left(x - \frac{\pi}{2}\right)\right)u\left(x - \frac{\pi}{2}\right)$$
$$+ \frac{c_1}{2}\sin x\sinh x + \frac{c_2}{4}(\sin x\cosh x - \cos x\sinh x)$$

$$y(\pi) = \frac{w_0}{4EI}\cosh\frac{\pi}{2} + \frac{c_2}{4}\sinh\pi = 0 \quad \to \quad \boxed{c_2 = -\frac{w_0}{EI}\frac{\cosh\frac{\pi}{2}}{\sinh\pi}}$$

$$\boxed{c_1 = \frac{w_0}{EI}\frac{\sinh\frac{\pi}{2}}{\sinh\pi}}$$

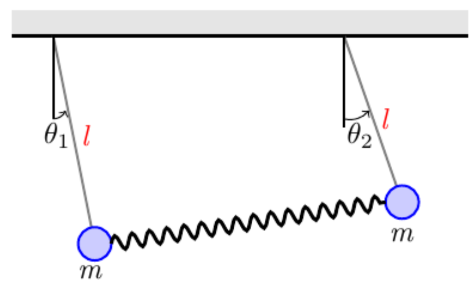
$$y(x) = \frac{w_0}{4EI}\left(\sin\left(x-\frac{\pi}{2}\right)\cosh\left(x-\frac{\pi}{2}\right) - \cos\left(x-\frac{\pi}{2}\right)\sinh\left(x-\frac{\pi}{2}\right)\right)u\left(x-\frac{\pi}{2}\right)$$

$$+\frac{w_0}{2EI}\frac{\sinh\frac{\pi}{2}}{\sinh\pi}\sin x\sinh x - \frac{w_0}{4EI}\frac{\cosh\frac{\pi}{2}}{\sinh\pi}(\sin x\cosh x - \cos x\sinh x)$$

Exercise

Suppose two identical pendulums are coupled by means of a spring with constant k. when the displacement angles $\theta_1(t)$ and $\theta_2(t)$ are small, the system of linear differential equations describing the motion is

$$\begin{cases} \theta_1'' + \frac{g}{l}\theta_1 = -\frac{k}{m}(\theta_1 - \theta_2) \\ \theta_2'' + \frac{g}{l}\theta_2 = \frac{k}{m}(\theta_1 - \theta_2) \end{cases}$$



a) Use Laplace transform to solve the system when

$$\theta_1'(0) = 0 \quad \theta_1(0) = \theta_0 \quad \theta_2'(0) = 0 \quad \theta_2(0) = \psi_0$$

Where θ_0 and ψ_0 constants. Let $\omega^2 = \frac{g}{l}$, $K = \frac{k}{m}$

b) Use the solution in part (*a*) to discuss the motion of the coupled pendulums in the special case when the initial conditions are $\theta_1'(0) = 0$, $\theta_1(0) = \theta_0$, $\theta_2'(0) = \theta_0$, $\theta_2(0) = 0$

c) Use the solution in part (*a*) to discuss the motion of the coupled pendulums in the special case when the initial conditions are $\theta_1'(0) = 0$, $\theta_1(0) = \theta_0$, $\theta_2'(0) = -\theta_0$, $\theta_2(0) = 0$

Solution

a) $$\begin{cases} \theta_1'' + \omega^2\theta_1 = -K\theta_1 + K\theta_2 \\ \theta_2'' + \omega^2\theta_2 = K\theta_1 - K\theta_2 \end{cases}$$

$$\begin{cases} s^2\theta_1(s) - s\theta_1(0) - \theta_1'(0) + \omega^2\theta_1(s) + K\theta_1(s) = +K\theta_2(s) \\ s^2\theta_2(s) - s\theta_2(0) - \theta_2'(0) + \omega^2\theta_2(s) + K\theta_2(s) = K\theta_1(s) \end{cases}$$

$$\begin{cases} \left(s^2+\omega^2+K\right)\theta_1(s)-K\theta_2(s)=s\theta_0 \\ -K\theta_1(s)+\left(s^2+\omega^2+K\right)\theta_2(s)=s\psi_0 \end{cases}$$

$$\Delta=\begin{vmatrix} s^2+\omega^2+K & -K \\ -K & s^2+\omega^2+K \end{vmatrix}=\left(s^2+\omega^2+K\right)^2-K^2=\left(s^2+\omega^2\right)\left(s^2+\omega^2+2K\right)$$

$$\Delta_1=\begin{vmatrix} s\theta_0 & -K \\ s\psi_0 & s^2+\omega^2+K \end{vmatrix}=s^3\theta_0+\left(\omega^2\theta_0+K\theta_0+K\psi_0\right)s$$

$$\Delta_2=\begin{vmatrix} s^2+\omega^2+K & s\theta_0 \\ -K & s\psi_0 \end{vmatrix}=s^3\psi_0+\left(\omega^2\psi_0+K\psi_0+K\theta_0\right)s$$

$$\theta_1(s)=\frac{s^3\theta_0+\left(\omega^2\theta_0+K\theta_0+K\psi_0\right)s}{\left(s^2+\omega^2\right)\left(s^2+\omega^2+2K\right)}=\frac{As+B}{s^2+\omega^2}+\frac{Cs+D}{s^2+\left(\omega^2+2K\right)}$$

$$\begin{cases} s^3 & A+C=\theta_0 \\ s^2 & B+D=0 \\ s & \left(\omega^2+2K\right)A+\omega^2C=\omega^2\theta_0+K\theta_0+K\psi_0 \\ s^0 & \left(\omega^2+2K\right)B+\omega^2D=0 \end{cases}$$

$$\left(\omega^2+2K\right)A+\omega^2\left(\theta_0-A\right)=\omega^2\theta_0+K\theta_0+K\psi_0$$

$$2KA=K\theta_0+K\psi_0 \quad\to\quad \boxed{A=\frac{1}{2}\left(\theta_0+\psi_0\right)}$$

$$C=\theta_0-A \quad\to\quad \boxed{C=\frac{1}{2}\left(\theta_0-\psi_0\right)}$$

$$\boxed{B=D=0}$$

$$\left\{\theta_1(s)\right\}=\frac{1}{2}\left(\theta_0+\psi_0\right)\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\}+\frac{1}{2}\left(\theta_0-\psi_0\right)\mathcal{L}^{-1}\frac{s}{s^2+\left(\sqrt{\omega^2+2K}\right)^2}$$

$$\boxed{\theta_1(t)=\frac{1}{2}\left(\theta_0+\psi_0\right)\cos\omega t+\frac{1}{2}\left(\theta_0-\psi_0\right)\cos\sqrt{\omega^2+2K}\,t}$$

$$\theta_2(s)=\frac{s^3\psi_0+\left(\omega^2\psi_0+K\psi_0+K\theta_0\right)s}{\left(s^2+\omega^2\right)\left(s^2+\omega^2+2K\right)}=\frac{as+b}{s^2+\omega^2}+\frac{cs+d}{s^2+\left(\omega^2+2K\right)}$$

$$\begin{cases} s^3 & a + c = \psi_0 \\ s^2 & b + d = 0 \\ s & \left(\omega^2 + 2K\right)a + \omega^2 c = \omega^2 \psi_0 + K\psi_0 + K\theta_0 \\ s^0 & \left(\omega^2 + 2K\right)b + \omega^2 d = 0 \end{cases}$$

$$\left(\omega^2 + 2K\right)a + \omega^2\left(\psi_0 - a\right) = \omega^2 \psi_0 + K\psi_0 + K\theta_0$$

$$2Ka = K\theta_0 + K\psi_0 \quad \rightarrow \quad \underline{a = \tfrac{1}{2}\left(\theta_0 + \psi_0\right)}$$

$$C = \psi_0 - A \quad \rightarrow \quad \underline{C = -\tfrac{1}{2}\left(\theta_0 - \psi_0\right)}$$

$$\underline{b = d = 0}$$

$$\underline{\theta_2(t) = \tfrac{1}{2}\left(\theta_0 + \psi_0\right)\cos\omega t - \tfrac{1}{2}\left(\theta_0 - \psi_0\right)\cos\sqrt{\omega^2 + 2K}\, t}$$

b) $\theta_1'(0) = 0, \quad \theta_1(0) = \theta_0, \quad \theta_2'(0) = \theta_0, \quad \theta_2(0) = 0$

$$\Rightarrow \quad \underline{\psi_0 = \theta_0}$$

$$\underline{\theta_1(t) = \theta_0 \cos\omega t} \qquad \& \qquad \underline{\theta_2(t) = \theta_0 \cos\omega t}$$

∴ This means that both pendulums swing in the same direction (free) and the spring exerts no influemce on the motion.

c) $\theta_1'(0) = 0, \quad \theta_1(0) = \theta_0, \quad \theta_2'(0) = -\theta_0, \quad \theta_2(0) = 0$

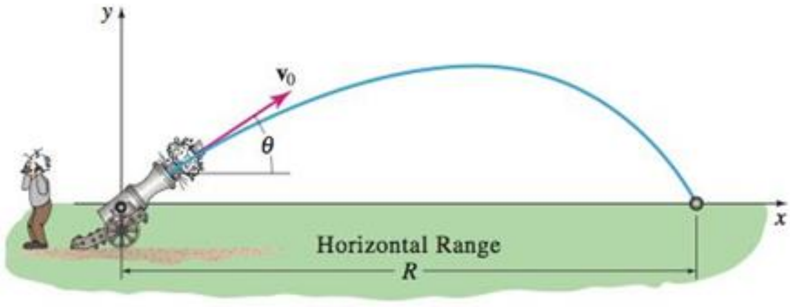
$$\Rightarrow \quad \underline{\psi_0 = -\theta_0}$$

$$\underline{\theta_1(t) = \theta_0 \cos\sqrt{\omega^2 + 2K}\, t} \qquad \& \qquad \underline{\theta_2(t) = -\theta_0 \cos\sqrt{\omega^2 + 2K}\, t}$$

∴ This means that both pendulums swing in the opposite directions, stretching and compressing the spring. The amplitude of both displacements is $\left|\theta_0\right|$. Which the psring is stretched to its maximum.

Exercise

A projectile, such as the canon ball, has weight $w = mg$ and initial velocity $\boldsymbol{v}_0$ that is tangent to its path of motion.



If air resistance and all other forces except its weight are ignored, that motion of the projectile is describe by the system of linear differential equations:

$$\begin{cases} m\dfrac{d^2x}{dt^2} = 0 \\ m\dfrac{d^2y}{dt^2} = -mg \end{cases}$$

a) Use Laplace transform to solve the system when

$$x(0) = 0 \quad x'(0) = v_0 \cos\theta \quad y(0) = 0 \quad y'(0) = v_0 \sin\theta$$

Where $v_0 = |v|$ is constant and θ is the constant angle of elevation.

The solutions $x(t)$ and $y(t)$ are parametric equations of the trajectory of the projectile.

b) Use $x(t)$ in part (*a*) to eliminate the parameter *t* in $y(t)$. Use the resulting equation for *y* to show that the horizontal range *R* of the projectile is given by

$$R = \frac{v_0^2}{g}\sin 2\theta$$

c) From the formula in part (*b*), we see that *R* is a maximum when $\sin 2\theta = 1$ or when $\theta = \frac{\pi}{4}$. Show that the same range – less than the maximum– can be obtained by firing the gun at either of two complementary angles θ and $\frac{\pi}{2} - \theta$. The only difference is that the smaller angle results in a low trajectory whereas the larger angle fives a high trajectory.

d) Suppose $g = 32$ *ft/s*^{2}, $\theta = 30°$, and $v_0 = 300$ *ft/s*. Use part (*b*) to find the horizontal range of the projectile.

e) Find the time when the projectile hits the ground.

f) Use the parametric equations $x(t)$ and $y(t)$ in part (*a*) along with the numerical data in part (*d*) to plot the ballistic curve of the projectile.

g) Repeat with $\theta = 52°$ and $v_0 = 300$ *ft/s*.

h) Superimpose both curves part (*f*) & (*g*) on the same coordinate system.

Solution

a) $$\begin{aligned}\frac{d^2x}{dt^2}&=0\\ \frac{d^2y}{dt^2}&=-g\end{aligned} \quad\rightarrow\quad \begin{cases} s^2X(s)-sx(0)-x'(0)=0 \\ s^2Y(s)-sy(0)-y'(0)=-\frac{g}{s}\end{cases}$$

$$x(0)=0 \quad x'(0)=v_0\cos\theta \quad y(0)=0 \quad y'(0)=v_0\sin\theta$$

$$X(s)=v_0\cos\theta\frac{1}{s^2}$$

$$\boxed{x(t)=\left(v_0\cos\theta\right)t}$$

$$Y(s)=v_0\sin\theta\frac{1}{s^2}-\frac{g}{s^3}$$

$$\boxed{y(t)=\left(v_0\sin\theta\right)t-\frac{1}{2}gt^2}$$

b) $t=\dfrac{x}{v_0\cos\theta}$

$$\begin{aligned}y(x)&=\left(v_0\sin\theta\right)t-\frac{1}{2}gt^2\\ &=\left(v_0\sin\theta\right)\frac{x}{v_0\cos\theta}-\frac{1}{2}g\frac{x^2}{v_0^2\cos^2\theta}\\ &=\frac{1}{2v_0^2\cos^2\theta}\left(2v_0^2\cos\theta\sin\theta x-gx^2\right)\\ &=\frac{1}{2v_0^2\cos^2\theta}\left(v_0^2\sin2\theta-gx\right)x=0\end{aligned}$$

At $x=y=0$, the projectile hits the ground.

$$v_0^2\sin2\theta-gx=0$$

$$x=\boxed{R(\theta)=\frac{1}{g}v_0^2\sin2\theta}$$

c) $$\begin{aligned}R\left(\frac{\pi}{2}-\theta\right)&=\frac{1}{g}v_0^2\sin\left(\pi-2\theta\right)\\ &=\frac{1}{g}v_0^2\sin2\theta\\ &=R(\theta)\end{aligned}$$

$$\sin\left(\pi-\alpha\right)=\sin\pi\cos\alpha-\cos\pi\sin\alpha$$

d) ***Given***: $g=32$ *ft/s*^{2}, $\theta=30°$, and $v_0=300$ *ft/s*

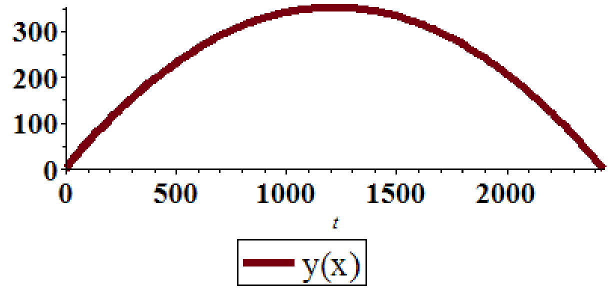
$$R(30°)=\frac{1}{32}(300)^2\sin60° \boxed{\approx 2,436\ ft}$$

e) $x=\left(v_0\cos\theta\right)t=2,436$

$$t=\frac{2,436}{300\cos30°} \boxed{\approx 9.38\ sec}$$

f) $y(x)=\dfrac{1}{2v_0^2\cos^2\theta}\left(\left(v_0^2\sin 2\theta\right)x-gx^2\right)$

$$\underline{=0.57735x-0.000237x^2}$$

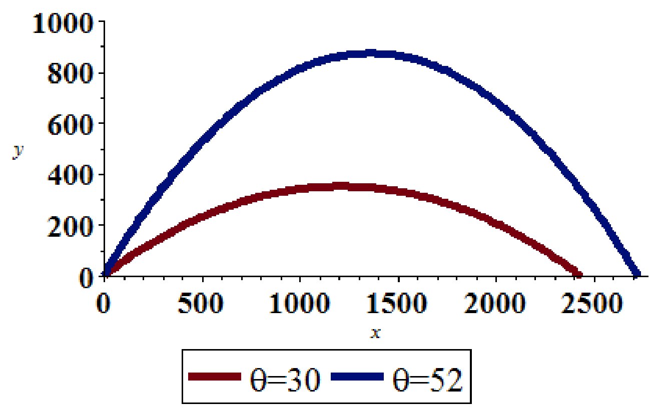


g) ***Given***: $g=32\ ft/s^2$, $\theta=52°$, and $v_0=300\ ft/s$

$$R(30°)=\frac{1}{32}(300)^2\sin 104° \underline{\approx 2729\ ft}$$

h) $y(x)=\dfrac{1}{2v_0^2\cos^2\theta}\left(\left(v_0^2\sin 2\theta\right)x-gx^2\right)$

$$\underline{=1.2799x-0.000469x^2}$$



Exercise

Derive the system of differential equations describing the straight-line vertical motion of the coupled springs. Use the Laplace transform to solve the system when

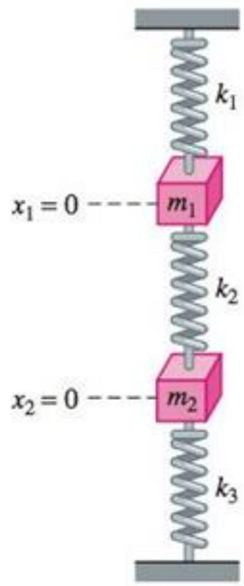
$$k_1=1,\ k_2=1,\ k_3=1,\ m_1=1,\ m_2=1 \ \text{ and }\ x_1(0)=0,\ \ x_1'(0)=-1,\ \ x_2(0)=0,\ \ x_2'(0)=1$$

Solution

$$x_1''+2x_1-x_2=0 \qquad\qquad m_1x_1''=-k_1x_1+k_2\left(x_2-x_1\right)$$

$$x_2''+2x_2-x_1=0 \qquad\qquad m_2x_2''=-k_2\left(x_2-x_1\right)-k_3x_2$$

$$\begin{cases} s^2X_1(s)-sx_1(0)-x_1'(0)+2X_1(s)-X_2(s)=0 \\ s^2X_2(s)-sx_2(0)-x_2'(0)+2X_2(s)-X_1(s)=0 \end{cases}$$



$$\begin{cases} (s^2+2)X_1(s)-X_2(s)=-1 \\ -X_1(s)+(s^2+2)X_2(s)=1 \end{cases}$$

$$\Delta=\begin{vmatrix} s^2+2 & -1 \\ -1 & s^2+2 \end{vmatrix}=(s^2+2)^2-1$$

$$\Delta_1=\begin{vmatrix} -1 & -1 \\ 1 & s^2+2 \end{vmatrix}=-s^2-1 \quad \Delta_2=\begin{vmatrix} s^2+2 & -1 \\ -1 & 1 \end{vmatrix}=s^2+1$$

$$X_1(s)=\frac{-(s^2+1)}{s^4+4s^2+3}$$

$$=-\frac{s^2+1}{(s^2+1)(s^2+3)}$$

$$\mathcal{L}^{-1}\{X_1(s)\}=\mathcal{L}^{-1}\left\{-\frac{1}{s^2+3}\right\}$$

$$\boxed{x_1(t)=-\sin\sqrt{3}t}$$

$$X_2(s)=\frac{s^2+1}{(s^2+1)(s^2+3)}=\frac{1}{s^2+3}$$

$$\mathcal{L}^{-1}\{X_1(s)\}=\mathcal{L}^{-1}\left\{-\frac{1}{s^2+3}\right\}$$

$$\boxed{x_1(t)=-\sin\sqrt{3}t}$$