Section 5.5 – Infinite Sequences and Summation Notation

An arbitrary ***infinite sequence*** may be denoted as follows:

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$$

An infinite sequence is a function whose domain is the set of positive integers.

Example

Find the first four terms and the tenth term of the sequence: $\left\{\dfrac{n}{n+1}\right\}$

Solution

$$n=1 \quad \rightarrow \quad \frac{1}{1+1}=\frac{1}{2}$$

$$n=2 \quad \rightarrow \quad \frac{2}{2+1}=\frac{2}{3}$$

$$n=3 \quad \rightarrow \quad \frac{3}{3+1}=\frac{3}{4}$$

$$n=4 \quad \rightarrow \quad \frac{4}{4+1}=\frac{4}{5}$$

$$n=10 \quad \Rightarrow \quad \frac{10}{11}$$

Example

Find the first four terms and the tenth term of the sequence: $\left\{2+(0.1)^n\right\}$

Solution

$$n=1 \quad \rightarrow \quad 2+0.1=2.1$$

$$n=2 \quad \rightarrow \quad 2+0.1^2=2.01$$

$$n=3 \quad \rightarrow \quad 2+0.1^3=2.001$$

$$n=4 \quad \rightarrow \quad 2+0.1^4=2.0001$$

$$n=10 \quad \Rightarrow \quad 2.0000000001$$

Example

Find the first four terms and the tenth term of the sequence: $\left\{(-1)^{n+1}\dfrac{n^2}{3n-1}\right\}$

Solution

$n=1 \quad \rightarrow \quad (-1)^2\dfrac{1^2}{3(1)-1}=\dfrac{1}{2}$

$n=2 \quad \rightarrow \quad (-1)^3\dfrac{2^2}{3(2)-1}=-\dfrac{4}{5}$

$n=3 \quad \rightarrow \quad (-1)^4\dfrac{3^2}{3(3)-1}=\dfrac{9}{8}$

$n=4 \quad \rightarrow \quad (-1)^5\dfrac{4^2}{3(4)-1}=-\dfrac{16}{11}$

$n=10 \quad \Rightarrow \quad -\dfrac{100}{29}$

Example

Find the first four terms and the tenth term of the sequence: $\{4\}$

Solution

$n=1 \quad \rightarrow \quad 4$

$n=2 \quad \rightarrow \quad 4$

$n=3 \quad \rightarrow \quad 4$

$n=4 \quad \rightarrow \quad 4$

$n=10 \quad \Rightarrow \quad 4$

Example

Find the first four terms of the recursively defined infinite sequence $a_1=3, \quad a_{n+1}=(n+1)a_n$

Solution

$a_1=3$

$n=1 \quad \rightarrow \quad a_2=(1+1)a_1=2(3)=6$

$n=2 \quad \rightarrow \quad a_3=(2+1)a_2=3(6)=18$

$n=3 \quad \rightarrow \quad a_4=(3+1)a_3=4(18)=72$

Summation Notation

To find the sum of many terms of an infinite sequence, it is easy to express using summation notation.

$$\overbrace{\sum_{\underbrace{n=1}_{\text{First value of } n}}^{5}}^{\text{Last value of } n} 2n+3 \quad \leftarrow \text{ Formula for each term}$$

Example

Find the sum: $\sum_{k=1}^{4} k^2(k-3)$

Solution

$$\begin{aligned}\sum_{k=1}^{4} k^2(k-3) &= 1^2(1-3)+2^2(2-3)+3^2(3-3)+4^2(4-3)\\ &= -2-4+0+16\\ &= 10\end{aligned}$$

Theorem on the Sum of a *Constant*

$$(1)\ \sum_{k=1}^{n} c = nc \qquad (2)\ \sum_{k=m}^{n} c = (n-m+1)c$$

Proof:

$$\sum_{k=1}^{n} c = \underbrace{c+c+\ldots+c}_{n} = nc$$

Example

Find the sum: $\sum_{k=10}^{20} 5$

Solution

$$\begin{aligned}\sum_{k=10}^{20} 5 &= (20-10+1)5\\ &= 55\end{aligned}$$

Theorem on Sums

If $a_1, a_2, a_3, \ldots, a_n, \ldots$ and $b_1, b_2, b_3, \ldots, b_n, \ldots$ are infinite sequences, then for every positive integer ***n***,

(1) $\displaystyle\sum_{k=1}^{n}\left(a_k + b_k\right) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(2) $\displaystyle\sum_{k=1}^{n}\left(a_k - b_k\right) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

(3) $\displaystyle\sum_{k=1}^{n} ca_k = c\left(\sum_{k=1}^{n} a_k\right)$

Proof

$$\begin{aligned}\sum_{k=1}^{n}\left(a_k + b_k\right) &= \left(a_1 + b_1\right) + \left(a_2 + b_2\right) + \ldots + \left(a_n + b_n\right)\\ &= \left(a_1 + a_2 + \ldots + a_n\right) + \left(b_1 + b_2 + \ldots + b_n\right)\\ &= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k\end{aligned}$$

Example

Express the sum using summation notation $2^1 + 2^2 + 2^3 + \cdots + 2^{16}$

Solution

$$2^1 + 2^2 + 2^3 + \cdots + 2^{16} = \sum_{k=1}^{16} 2^k$$

Exercises *Section* 5.5 – Infinite Sequences and Summation Notation

(1 – 13) Find the first four terms and the eight term of the sequence:

1. $\{12-3n\}$

2. $\left\{\dfrac{3n-2}{n^2+1}\right\}$

3. $\{9\}$

4. $\left\{(-1)^{n-1}\dfrac{n+7}{2n}\right\}$

5. $\left\{\dfrac{2^n}{n^2+2}\right\}$

6. $\left\{(-1)^{n-1}\dfrac{n}{2n-1}\right\}$

7. $\left\{\dfrac{2^n}{3^n+1}\right\}$

8. $\left\{\dfrac{n^2}{2^n}\right\}$

9. $\left\{\dfrac{n}{e^n}\right\}$

10. $\{c_n\}=\left\{(-1)^{n+1}n^2\right\}$

11. $\{c_n\}=\left\{\dfrac{(-1)^n}{(n+1)(n+2)}\right\}$

12. $\{c_n\}=\left\{\left(\frac{4}{3}\right)^n\right\}$

13. $\{b_n\}=\left\{\dfrac{3^n}{n}\right\}$

14. Graph the sequence $\left\{\dfrac{1}{\sqrt{n}}\right\}$

15. Find the first four terms of the sequence of partial sums for the given sequence. $\left\{3+\frac{1}{2}n\right\}$

(16 – 27) Find the first five terms of the recursively defined infinite sequence

16. $a_1=2, \quad a_{k+1}=3a_k-5$

17. $a_1=-3, \quad a_{k+1}=a_k^2$

18. $a_1=5, \quad a_{k+1}=ka_k$

19. $a_1=2, \quad a_n=3+a_{n-1}$

20. $a_1=5, \quad a_n=2a_{n-1}$

21. $a_1=\sqrt{2}, \quad a_n=\sqrt{2+a_{n-1}}$

22. $a_1=2, \quad a_{n+1}=7-2a_n$

23. $a_1=128, \quad a_{n+1}=\frac{1}{4}a_n$

24. $a_1=2, \quad a_{n+1}=\left(a_n\right)^n$

25. $a_1=A, \quad a_n=a_{n-1}+d$

26. $a_1=A, \quad a_n=ra_{n-1}, \quad r\neq 0$

27. $a_1=2, \quad a_2=2; \quad a_n=a_{n-1}\bullet a_{n-2}$

(28 – 37) Express each sum using summation notation

28. $1+2+3+\ldots+20$

29. $1+2+3+\ldots+40$

30. $1^3+2^3+3^3+\ldots+8^3$

31. $1^2+2^2+3^2+\ldots+15^2$

32. $2^2+2^3+2^4+\ldots+2^{11}$

33. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{13}{14}$

34. $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\cdots+(-1)^6\frac{1}{3^6}$

35. $\frac{2}{3}-\frac{4}{9}+\frac{8}{27}-\cdots+(-1)^{12}\left(\frac{2}{3}\right)^{11}$

36. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{14}{14+1}$

37. $\frac{1}{e}+\frac{2}{e^2}+\frac{3}{e^3}+\cdots+\frac{n}{e^n}$

(38 – 52) Find the sum

38. $\sum_{k=1}^{5}(2k-7)$

39. $\sum_{k=0}^{5}k(k-2)$

40. $\sum_{k=1}^{5}(-3)^{k-1}$

41. $\sum_{k=253}^{571}\left(\frac{1}{3}\right)$

42. $\sum_{k=1}^{50}8$

43. $\sum_{k=1}^{40}k$

44. $\sum_{k=1}^{5}(3k)$

45. $\sum_{k=1}^{10}\left(k^3+1\right)$

46. $\sum_{k=1}^{24}\left(k^2-7k+2\right)$

47. $\sum_{k=6}^{20}\left(4k^2\right)$

48. $\sum_{k=1}^{16}\left(k^2-4\right)$

49. $\sum_{k=1}^{6}(10-3k)$

50. $\sum_{k=1}^{10}\left[1+(-1)^k\right]$

51. $\sum_{k=1}^{6}\frac{3}{k+1}$

52. $\sum_{k=137}^{428}2.1$

(53 – 56) Write out each sum

53. $\sum_{k=1}^{n}(k+2)$

54. $\sum_{k=1}^{n}k^2$

55. $\sum_{k=2}^{n}(-1)^k\ln k$

56. $\sum_{k=3}^{n}(-1)^{k+1}2^k$

57. $\sum_{k=0}^{n}\frac{1}{3^k}$

58. Fred has a balance of \$3,000 on his card which charges 1% interest per month on any unpaid balance. Fred can afford to pay \$100 toward the balance each month. His balance each month after making a \$100 payment is given by the recursively defined sequence

$$B_0=\$3{,}000 \quad B_n=1.01B_{n-1}-100$$

Determine Fred's balance after making the first payment. That is, determine B_1

59. A pond currently has 2,000 trout in it. A fish hatchery decides to add an additional 20 trout each month. Is it also known that the trout population is grwoing at a rate of 3% per month. The size of the population after n months is given bu the recursively defined sequence

$$P_0=2{,}000 \quad P_n=1.03P_{n-1}+20$$

How many trout are in the pond after 2 months? That is, what is P_2?

60. Fred bought a car by taking out a loan for \$18,500 at 0.5% interest per month. Fred's normal monthly payment is \$434.47 per month, but he decides that he can afford to pay \$100 extra toward the balance each month. His balance each month is given by the recursively defined sequence

$$B_0 = \$18,500 \quad B_n = 1.005B_{n-1} - 534.47$$

Determine Fred's balance after making the first payment. That is, determine B_1

61. The Environmental Protection Agency (EPA) determines that Maple Lake has 250 *tons* of pollutant as a result of industrial waste and that 10% of the pollutant present is neuttralized by solar oxidation every year. The EPA imposes new pollution control laws that result in 15 *tons* of new pollutant entering the lake each year. The amount of pollutant in the lake after *n* years is given by the recursively defined sequence

$$P_0 = 250 \quad P_n = 0.9P_{n-1} + 15$$

Determine the amount of pollutant in the lake after 2 years? That is, what is P_2 ?

62. Let $$u_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n \sqrt{5}}$$

Define the *n*th term of a sequence

a) Show that $u_1 = 1$ and $u_2 = 1$

b) Show that $u_{n+2} = u_{n+1} + u_n$

c) Draw the conclusion that $\{u_n\}$ is a Fibonacci sequence

d) Find the first ten terms of the sequence from part (*c*)