

Lecture Four – Parametric Equations and Polar Coordinates

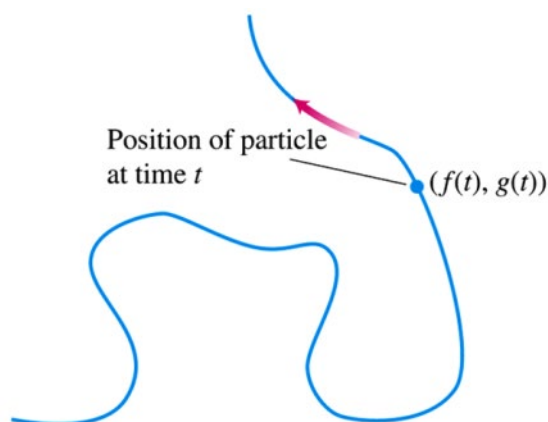
Section 4.1 – Parameterizations of Plane Curves

Definition

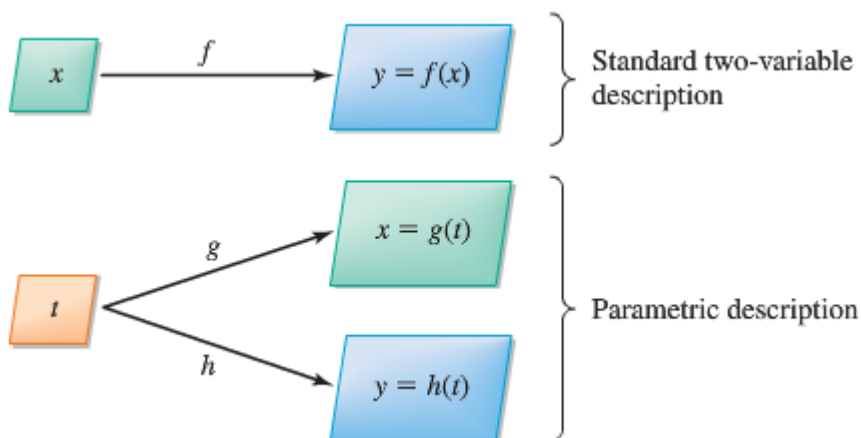
If x and y are given as functions

$$x = f(t), \quad y = g(t)$$

Over an interval I as t -values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve



The variable t is a parameter for the curve, and its domain I is the parameter interval.



Definition

The direction in which a parametric curve is generated as the parameter increases is called the forward or positive orientation of the curve.

Example

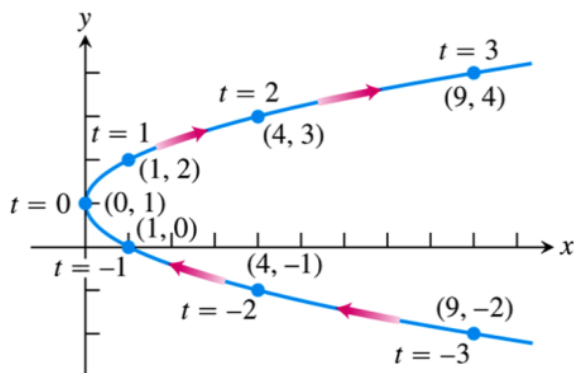
Sketch the curve defined by the parametric equations

$$x = t^2, \quad y = t + 1, \quad -\infty < t < \infty$$

Then obtain an algebraic equation in x and y .

Solution

t	$x = t^2$	$y = t + 1$
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3
3	9	4



$$y = t + 1 \Rightarrow t = y - 1$$

$$x = t^2 = (y - 1)^2 = y^2 - 2y + 1 \quad \text{represents a parabola}$$

Example

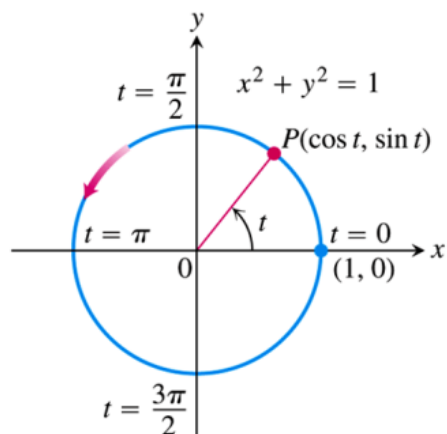
Graph the parametric curve $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$

Solution

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

This parametric curve lies along the unit circle $x^2 + y^2 = 1$.

As t increases from 0 to 2π , the point $(x, y) = (\cos t, \sin t)$ starts at $(1, 0)$ and traces the entire circle once counterclockwise.



Example

Graph the parametric curve $x = a \cos t$, $y = a \sin t$, $0 \leq t \leq 2\pi$

Solution

$$x^2 + y^2 = a^2 \cos^2 t + a^2 \sin^2 t = a^2 (\cos^2 t + \sin^2 t) = a^2$$

This parametric curve lies along the circle $x^2 + y^2 = a^2$.

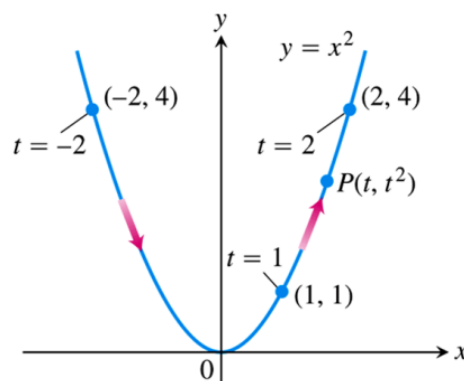
As t increases from 0 to 2π , the point $(x, y) = (a \cos t, a \sin t)$ starts at $(a, 0)$ with a radius $r = a$ and traces the entire circle once counterclockwise.

Example

Graph the parametric curve $x = t$, $y = t^2$, $-\infty < t < \infty$

Solution

$$y = x^2$$



Example

Find a parameterization for the line through the point (a, b) having slope m .

Solution

A Cartesian equation of the line is $y - b = m(x - a)$

$$\text{If } t = x - a \Rightarrow x = t + a$$

$$y - b = mt \Rightarrow y = mt + b$$

That is, $x = t + a$ $y = mt + b$, $-\infty < t < \infty$

Example

Sketch the curve defined by the parametric equations $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$, $t > 0$

Then obtain an algebraic equation in x and y .

Solution

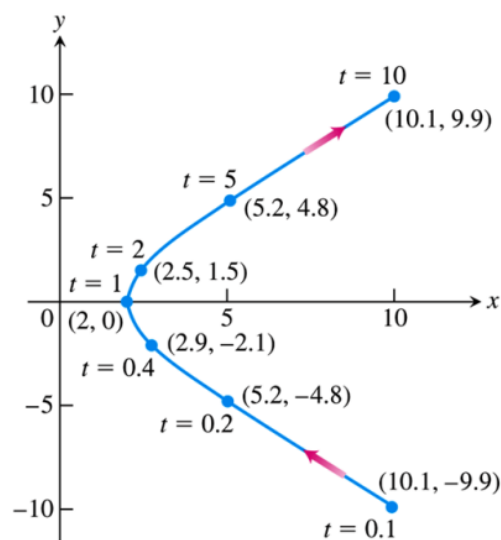
t	$x = t + \frac{1}{t}$	$y = t - \frac{1}{t}$
.2	5.2	-4.8
.5	2.5	-1.5
1	2	0
2	2.5	1.5
5	5.2	4.8

$$x + y = t + \frac{1}{t} + t - \frac{1}{t} = 2t$$

$$\frac{x - y}{2} = t + \frac{1}{t} - t + \frac{1}{t} = \frac{2}{t}$$

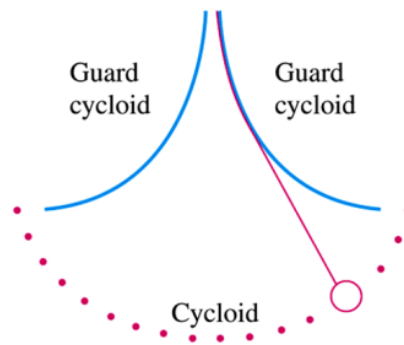
$$(x + y)(x - y) = 2t \left(\frac{2}{t} \right) = 4$$

$$\boxed{x^2 - y^2 = 4}$$



Cycloids

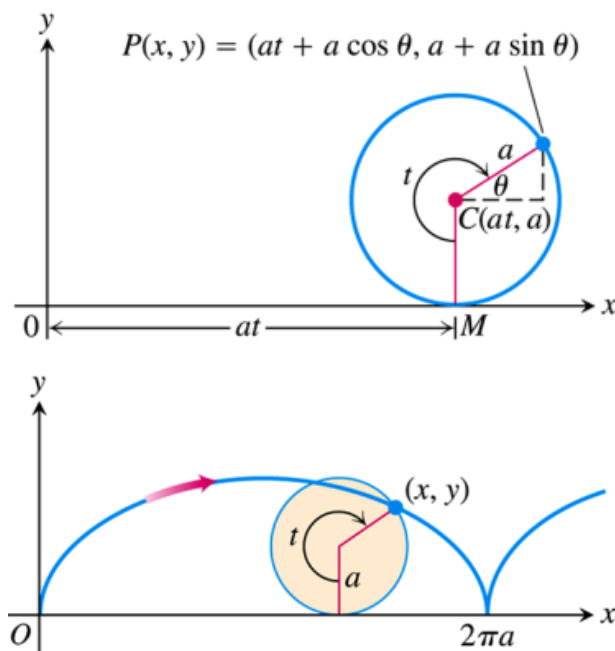
The problem with a pendulum clock whose bob swings in a circular arc is that the frequency of the swing depends on the amplitude of the swing.



Example

A wheel of a radius a rolls along a horizontal straight line. Find parametric equations for the path traced by a point P on the wheel's circumference. The path is called a cycloid.

Solution



We take the line to be on the x -axis, mark a point P on the wheel, start the wheel with P at the origin, and roll the wheel.

As parameter, we use the angle t through which the wheel turns, measured in radians.

The wheel's center C lies at (at, a) and the coordinates of P are

$$x = at + a \cos \theta, \quad y = a + a \sin \theta$$

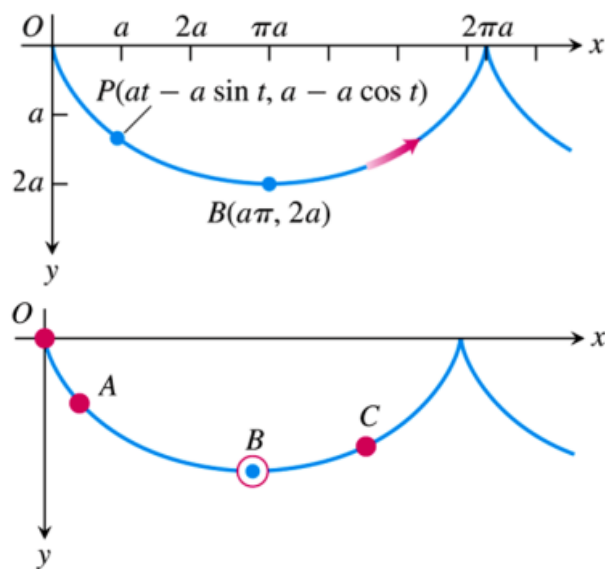
To express θ in terms of t , we observe that $t + \theta = \frac{3\pi}{2}$, so that $\theta = \frac{3\pi}{2} - t$

That makes $\cos \theta = \cos\left(\frac{3\pi}{2} - t\right) = -\sin t$

$$\sin \theta = \sin\left(\frac{3\pi}{2} - t\right) = -\cos t$$

$$x = at - a \sin t, \quad y = a - a \cos t$$

That implies to: $x = a(t - \sin t), \quad y = a(1 - \cos t)$



Exercises Section 4.1 – Parameterizations of Plane Curves

(1 – 28) Give parametric equations and parameter intervals for the motion of a particle in the xy -plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation.

- | | |
|--|---|
| 1. $x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty$ | 15. $x = t - 1, \quad y = t^3; \quad -4 \leq t \leq 4$ |
| 2. $x = -\sqrt{t}, \quad y = t, \quad t \geq 0$ | 16. $x = e^{2t}, \quad y = e^t + 1; \quad 0 \leq t \leq 25$ |
| 3. $x = 3 - 3t, \quad y = 2t, \quad 0 \leq t \leq 1$ | 17. $x = 3 \cos t, \quad y = 3 \sin t; \quad \pi \leq t \leq 2\pi$ |
| 4. $x = \cos 2t, \quad y = \sin 2t, \quad 0 \leq t \leq \pi$ | 18. $x = -7 \cos 2t, \quad y = -7 \sin 2t; \quad 0 \leq t \leq \pi$ |
| 5. $x = \cos(\pi - t), \quad y = \sin(\pi - t), \quad 0 \leq t \leq \pi$ | 19. $x = 1 - 3 \sin 4\pi t, \quad y = 2 + 3 \cos 4\pi t; \quad 0 \leq t \leq \frac{1}{2}$ |
| 6. $x = 4 \sin t, \quad y = 5 \cos t, \quad 0 \leq t \leq 2\pi$ | 20. $x = 2t, \quad y = 3t - 4; \quad -10 \leq t \leq 10$ |
| 7. $x = 1 + \sin t, \quad y = \cos t - 2, \quad 0 \leq t \leq 2\pi$ | 21. $x = t^2 + 2, \quad y = 4t; \quad -4 \leq t \leq 4$ |
| 8. $x = t^2, \quad y = t^6 - 2t^4, \quad -\infty < t < \infty$ | 22. $x = -t + 6, \quad y = 3t - 3; \quad -5 \leq t \leq 5$ |
| 9. $x = \frac{t}{t-1}, \quad y = \frac{t-2}{t+1}, \quad -1 < t < 1$ | 23. $x = t^3 - 1, \quad y = 5t + 1; \quad -3 \leq t \leq 3$ |
| 10. $x = \sqrt{t+1}, \quad y = \sqrt{t}, \quad t \geq 0$ | 24. $x = \cos t, \quad y = \sin^2 t, \quad 0 \leq t \leq \pi$ |
| 11. $x = 2 \sinh t, \quad y = 2 \cosh t, \quad -\infty < t < \infty$ | 25. $x = 1 - \sin^2 t, \quad y = \cos t, \quad \pi \leq t \leq 2\pi$ |
| 12. $x = 4 \cos 2\pi t, \quad y = 4 \sin 2\pi t, \quad 0 \leq t \leq 1$ | 26. $x = \cos t, \quad y = 1 + \sin t; \quad 0 \leq t \leq 2\pi$ |
| 13. $x = \sqrt{t} + 4, \quad y = 3\sqrt{t}; \quad 0 \leq t \leq 16$ | 27. $x = 2 \sin t - 3, \quad y = 5 + \cos 2t; \quad 0 \leq t \leq 2\pi$ |
| 14. $x = (t+1)^2, \quad y = t + 2; \quad -10 \leq t \leq 10$ | 28. $x = 4 \cos t - 1, \quad y = 4 \sin t + 2; \quad 0 \leq t \leq 2\pi$ |

(29 – 32) For the given parametric equations:

- Plot the following curve, indicating the positive orientation.
- Eliminate the parameter to obtain an equation in x and y .
- Identify or briefly describe the curve.
- Evaluate $\frac{dy}{dx}$ at the specified point.

29. $x = t^2 + 4, \quad y = 6 - t; \quad -\infty \leq t \leq \infty$ (5, 5)
30. $x = e^t, \quad y = 3e^{-2t}; \quad -\infty \leq t \leq \infty$ (1, 3)
31. $x = 10 \sin 2t, \quad y = 16 \cos 2t; \quad 0 \leq t \leq \pi$ ($5\sqrt{3}$, 8).
32. $x = \ln t, \quad y = 8 \ln t^2; \quad 1 \leq t \leq e^2$ (1, 16)

(33 – 47) Find parametric equations:

33. For the left half of the parabola $y = x^2 + 1$, originating at $(0, 1)$
34. A line that passes through the points $(1, 1)$ and $(3, 5)$, oriented in the direction of increasing x .
35. Lower half of the circle centered at $(-2, 2)$ with radius 6, oriented in the counterclockwise direction.
36. Upper half of the parabola $x = y^2$, originating at $(0, 0)$
37. The circle centered at the origin with radius 4, generated counterclockwise.
38. The circle centered at the origin with radius 12, generated clockwise with initial point $(0, 12)$
39. The circle centered at $(2, 3)$ with radius 1, generated counterclockwise.
40. The circle centered at $(2, 0)$ with radius 3, generated clockwise.
41. The circle centered at $(-2, -3)$ with radius 8, generated clockwise.
42. The circle $x^2 + y^2 = 9$, generated clockwise.
43. The upper half of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, generated counterclockwise.
44. The right side of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, generated counterclockwise.
45. The line $y = 4x + 11$
46. The line segment from $P(-1, 0)$ to $Q(1, 1)$ and the line segment from Q to P .
47. The segment of the curve $f(x) = x^3 + 2x$ from $(0, 0)$ to $(2, 12)$

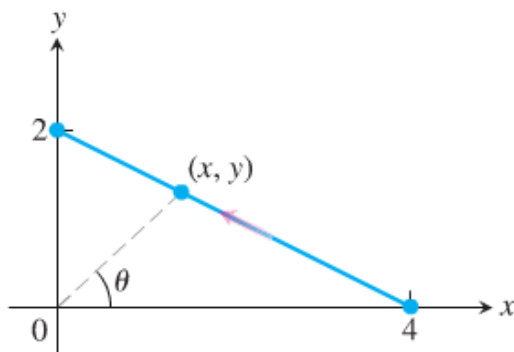
48. What is the relationship among a , b , c , and d such that the equations

$$\begin{cases} x(t) = a \cos t + b \sin t \\ y(t) = c \cos t + d \sin t \end{cases} \quad \text{describe a circle?}$$

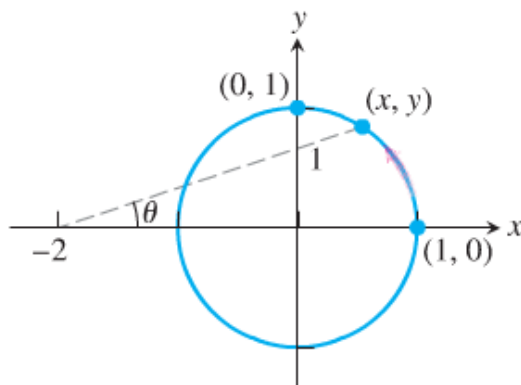
What is the radius of the circle?

49. Find parametric equations (not unique) of an ellipse centered at the origin with major axis of length 6 on the x -axis and minor axis of length 3 on the y -axis, generated counterclockwise. Graph the ellipse and find a description in terms of x and y .
50. Find parametric equations (not unique) of an ellipse centered at the origin with major axis of length 12 on the x -axis and minor axis of length 2 on the y -axis, generated clockwise. Graph the ellipse and find a description in terms of x and y .

51. Find parametric equations (not unique) of an ellipse centered at $(-2, -3)$ with major and minor axes of lengths 30 and 20, parallel to the x -axis and y -axis, respectively. Graph the ellipse and find a description in terms of x and y .
52. Find a parametric equation and a parameter interval for the motion of a particle starting at the point $(2, 0)$ and tracing the top half of the circle $x^2 + y^2 = 4$ four times.
53. Find a parametrization for the line segment joining points $(0, 2)$ and $(4, 0)$ using the angle θ in the accompanying figure as the parameter.



54. Find a parametrization for the circle $x^2 + y^2 = 1$ starting at $(1, 0)$ and moving counterclockwise to the terminal point $(0, 1)$, using the angle θ in the accompanying figure as the parameter.



55. A common task is to parameterize curves given either by either Cartesian equations or by graphs. Find a parametric representation of the following curves.
- The segment of the parabola $y = 9 - x^2$, for $-1 \leq x \leq 3$
 - The complete curve $x = (y - 5)^2 + \sqrt{y}$
 - The piecewise linear path that connects $P(-2, 0)$ to $Q(0, 3)$ to $R(4, 0)$ (in that order), where the parameter varies over the interval $0 \leq t \leq 2$

56. A projectile launched from the ground with an initial speed of 20 m/s and a launch angle θ follows a trajectory approximated by

$$x = (20 \cos \theta)t, \quad y = -4.9t^2 + (20 \sin \theta)t$$

Where x and y are the horizontal and vertical positions of the projectile relative to the launch point $(0, 0)$.

- Graph the trajectory for various of θ in the range $0 < \theta < \frac{\pi}{2}$.
 - Based on your observations, what value of θ gives the greatest range (the horizontal distance between the launch and landing points)?
57. Many fascinating curves are generated by points on rolling wheels. The path of a light on the rim of a rolling wheel is a cycloid, which has the parametric equations

$$\begin{cases} x(t) = a(t - \sin t) \\ y(t) = a(1 - \cos t) \end{cases} \quad \text{for } t \geq 0$$

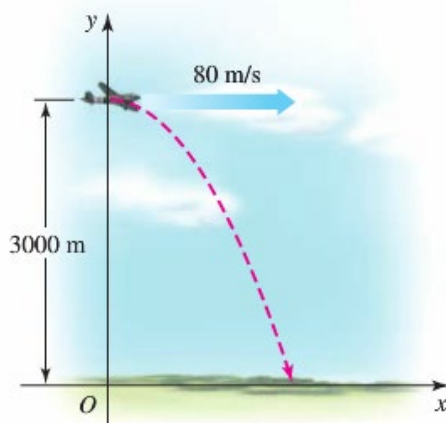


Where $a > 0$. Graph the cycloid with $a = 1$. On what interval does the parameter generate one arch of the cycloid?

- (58 – 60) Find parametric equations that describe the circular path of the objects. Assume (x, y) denotes the position of the object relative to the origin at the center of the circle.

58. A go-cart moves counterclockwise with constant speed around a circular track of radius 400 m , completing in 1.5 min .
59. The tip of the 15-in second hand of a clock completes one revolution in 60 sec .
60. A Ferris wheel has a radius of 20 m and completes a revolution in the clockwise direction at constant speed in 3 min . Assume that x and y measure the horizontal and vertical positions of a seat on the Ferris wheel relative to a coordinate system whose origin is at the low point of the wheel. Assume the seat begins moving at the origin.

61. A plane traveling horizontally at 80 m/s over flat ground at an elevation of 3000 m releases an emergency packet. The trajectory of the packet is given by



$$x = 80t, \quad y = -4.9t^2 + 3000, \quad \text{for } t \geq 0$$

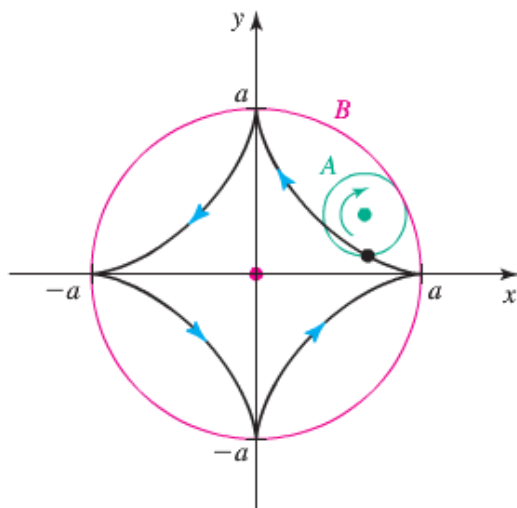
Where the origin is the point on the ground directly beneath the plane at the moment of the release. Graph the trajectory of the packet and find the coordinates of the point where the packet lands.

62. A plane traveling horizontally at 100 m/s over flat ground at an elevation of $4,000 \text{ m}$ must drop an emergency packet on a target on the ground. The trajectory of the packet is given by

$$x(t) = 100t, \quad y(t) = -4.9t^2 + 4,000, \quad \text{for } t \geq 0$$

Where the origin is the point on the ground directly beneath the plane at the moment of the release. How many horizontal meters before the target should the packet be released in order to hit the target?

63. The path of a point on circle A with radius $\frac{a}{4}$ that rolls on the inside of circle B with a radius a is an asteroïd or hypocycloid. Its parametric equations are



$$x = a \cos^3 t, \quad y = a \sin^3 t, \quad 0 \leq t \leq 2\pi$$

Graph the asteroïd with $a = 1$ and find its equation in terms of x and y .