Section 1.4 – Inverse Matrices - Finding A^{-1}

Definition

The matrix A is invertible if there exists a matrix A^{-1} such that:

$$A^{-1}A = AA^{-1} = I$$

where A^{-1} *read as* "A ***inverse***" and A has to be a ***square matrix***.

Not all matrices have inverses.

1. The inverse exists *iff* elimination produces n pivots (row exchanges allow).
2. The matrix A cannot have two different inverses.
3. If A is invertible, the one and only one solution to $Ax = B$ is $x = A^{-1}B$

$$
\begin{aligned}
AX &= B \\
A^{-1}(AX) &= A^{-1}B && \textit{Multiply both side by } A^{-1} \\
\left(A^{-1}A\right)X &= A^{-1}B && \textit{Associate property} \\
IX &= A^{-1}B && \textit{Multiplicative inverse property} \\
X &= A^{-1}B && \textit{Identity property}
\end{aligned}
$$

4. Suppose there is a ***nonzero*** vector x such that $Ax = 0$. Then A cannot have an inverse
5. A 2 by 2 matrix is invertible iff $ad - bc$ is not zero.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \textit{Only for 2 by 2 matrices}$$

If $ad - bc = 0$ is the determinant, then A^{-1} doesn't exist

The Inverse of a Product AB

Theorem

If an $n \times n$ matrix has an inverse, that inverse is unique.

Proof

Suppose that A has an inverse A^{-1} and B is a matrix such that $BA = I$

$$\begin{aligned} B &= BI \\ &= B\left(AA^{-1}\right) \\ &= (BA)A^{-1} \\ &= IA^{-1} \\ &= A^{-1} \end{aligned}$$

Therefore, the inverse is unique

Theorem

If A and B are invertible then so is AB. The inverse of a product AB is $(AB)^{-1} = B^{-1}A^{-1}$

Proof

$$\begin{aligned} (AB)\left(B^{-1}A^{-1}\right) &= A\left(BB^{-1}\right)A^{-1} \\ &= (AI)A^{-1} \\ &= AA^{-1} \\ &= \underline{I} \end{aligned}$$

Reverse Order

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

$$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$$

Theorem

If A is invertible and n is a nonnegative integer, then:

a) A^{-1} is *invertible* and $\left(A^{-1}\right)^{-1} = A$

b) A^{n} is *invertible* and $\left(A^{n}\right)^{-1} = A^{-n} = \left(A^{-1}\right)^{n}$

c) kA is *invertible* for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$

Proof

$$
\begin{aligned}
(kA)\left(k^{-1}A^{-1}\right) &= k^{-1}(kA)A^{-1} \\
&= \left(k^{-1}k\right)AA^{-1} \\
&= (1)I \\
&= I
\end{aligned}
$$

$$
\begin{aligned}
\left(k^{-1}A^{-1}\right)(kA) &= k^{-1}\left(kA^{-1}\right)A \\
&= \left(k^{-1}k\right)A^{-1}A \\
&= (1)I \\
&= I
\end{aligned}
$$

Finding A^{-1} using Gauss-Jordan Elimination

$$\left[A \mid I\right] \rightarrow \left[I \mid A^{-1}\right]$$

Find A^{-1} *if* $A=\begin{bmatrix}1&0&1\\2&-2&-1\\3&0&0\end{bmatrix}$

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\2&-2&-1&0&1&0\\3&0&0&0&0&1\end{array}\right]\begin{array}{l}\\R_2-2R_1\\R_3-3R_1\end{array}$$

$$\begin{array}{cccccc}2&-2&-1&0&1&0\\-2&0&-2&-2&0&0\\\hline 0&-2&-3&-2&1&0\end{array}\qquad\begin{array}{cccccc}3&0&0&0&0&1\\-3&0&-3&-3&0&0\\\hline 0&0&-3&-3&0&1\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&-2&-3&-2&1&0\\0&0&-3&-3&0&1\end{array}\right]\ -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&\frac{3}{2}&1&-\frac{1}{2}&0\\0&0&-3&-3&0&1\end{array}\right]\ -\tfrac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&\frac{3}{2}&1&-\frac{1}{2}&0\\0&0&1&1&0&-\frac{1}{3}\end{array}\right]\begin{array}{l}R_1-R_3\\R_2-\frac{3}{2}R_3\\ \\ \end{array}$$

$$\begin{array}{cccccc}1&0&1&1&0&0\\0&0&-1&-1&0&\frac{1}{3}\\\hline 1&0&0&0&0&\frac{1}{3}\end{array}\qquad\begin{array}{cccccc}0&1&\frac{3}{2}&1&-\frac{1}{2}&0\\0&0&-\frac{3}{2}&-\frac{3}{2}&0&\frac{1}{2}\\\hline 0&1&0&-\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&0&0&0&\frac{1}{3}\\0&1&0&-\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\0&0&1&1&0&-\frac{1}{3}\end{array}\right]\qquad A^{-1}=\begin{bmatrix}0&0&\frac{1}{3}\\-\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\1&0&-\frac{1}{3}\end{bmatrix}$$

- ✓ Matrix ***A*** is ***symmetric*** across its main diagonal. So is A^{-1}
- ✓ Matrix ***A*** is ***tridiagonal*** (only three nonzero diagonals). But A^{-1} is a full matrix with no zeros. (another reason we don't compute A^{-1})

Singular *versus* Invertible

A^{-1} exists when A has a full set of n pivots. (Row exchanges allowed)

- With n pivots, elimination solves all the equations $Ax_i = b_i$. The columns x_i go into A^{-1}. Then $AA^{-1} = I$ is at least a ***right-inverse***.
- Elimination is really a sequence of multiplications.

Conclusion

- If A doesn't have n pivots, elimination will lead to a ***zero row***.
- Elimination steps are taken by an invertible M. So a row of MA is zero.
- If $AB = I$ then $MAB = M$. The zero row of MA, times B, gives a zero row of M.
- The invertible matrix M can't have a zero row! A must have n pivots if $AB = I$.

Elementary Matrices

Definition

Let ***e*** be an elementary row operation. Then the $n \times n$ ***elementary matrix*** E associated with ***e*** is the matrix obtained by applying ***e*** to the $n \times n$ identity matrix. Thus

$$E = eI$$

Example

a) $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \rightarrow$ *Multiply R_2 of I by -3*

b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow$ *Multiply the third row by -5*

c) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ *Interchange the first and second rows*

d) $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ *Add -3 times R_1 to R_2*

Theorem

Let e be an elementary operation and let E be the corresponding elementary matrix $E = e(I)$. Then for every $m \times n$ matrix A

$$e(A) = EA$$

That is, an elementary row operation can be performed on A by multiplying A on the left by the corresponding elementary matrix.

Example $m \times m$

Let $$A = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix} \quad M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$MA = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -4 & 6 & 2 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$

This result can be obtained from A by multiplying the first row by 2.

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -2 & 3 & 1 \\ -6 & 2 & 1 & 5 \\ 1 & 0 & 2 & 4 \end{bmatrix}$$

This result can be obtained from A by interchanging rows 2 and 3.

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ -6 & 2 & 1 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & 0 & 2 & 4 \\ 0 & -4 & 10 & 8 \end{bmatrix}$$

This result can be obtained from A by adding 3 times row 1 to row 3.

Uniqueness of Echelon Form

Two matrices A and B are row-equivalent if and only if they have the same reduced echelon form.

Proof

If A and B have the same reduced echelon form E, then A is row-equivalent to E and E is row-equivalent to B. It follows that A is row-equivalent to B.
Now Suppose A and B are row-equivalent. Let E_1 be a reduced echelon form of A and E_2 be a reduced echelon form of B. Then E_1 and E_2 are row equivalent.
Suppose $E_1 = IF_1 \quad and \quad E_2 = IF_2$. Since E_1 and E_2 are row equivalent, $E_2 = CE_1$ for some matrix C. This means $I = CI$ and $F_2 = CF_1$. But then $C = I \quad and \quad F_2 = F_1$.

Example

Show that the two matrices are row equivalent

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

Solution

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{matrix} R_1 + R_2 \\ R_2 - 2R_1 \end{matrix}$$

$$= \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

$$= B$$

Definition

A relationship ~ (equivalent) between elements of a set is called an equivalence relation if

- ✓ $A \sim A$ is always true,
- ✓ $A \sim B$ always implies $B \sim A$,
- ✓ $A \sim B$ and $B \sim C$ always implies $A \sim C$.

Exercises *Section* 1.4 – Inverse Matrices - Finding A^{-1}

1. Apply Gauss-Jordan method to find the inverse of this triangular "Pascal matrix"

Triangular Pascal matrix $$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

2. If A is invertible and $AB = AC$, prove that $B = C$

3. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find two matrices $B \neq C$ such that $AB = AC$

4. If A has ***row*** 1 + ***row*** 2 = ***row*** 3, show that A is not invertible

a) Explain why $Ax = (1,\ 0,\ 0)$ can't have a solution.

b) Which right sides (b_1, b_2, b_3) might allow a solution to $Ax = b$

c) What happens to ***row*** 3 in elimination?

5. True or false (with a counterexample if false and a reason if true):

a) A 4 by 4 matrix with a row of zeros is not invertible.

b) A matrix with 1's down the main diagonal is invertible.

c) If A is invertible then A^{-1} is invertible.

d) If A is invertible then A^2 is invertible.

6. Do there exist 2 by 2 matrices A and B with real entries such that $AB - BA = I$, where I is the identity matrix?

7. If B is the inverse of A^2, show that AB is the inverse of A.

8. Find and check the inverses (assuming they exist) of these block matrices.

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}$$

9. For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

10. Find A^{-1} and B^{-1} (if they exist) by elimination.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

11. Find A^{-1} using the Gauss-Jordan method, which has a remarkable inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find the inverse, if exists, of

12. $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$

14. $\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$

15. $A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

16. $A = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$

17. $A = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$

18. $A = \begin{bmatrix} a & b \\ 3 & 3 \end{bmatrix}$

19. $A = \begin{bmatrix} -2 & a \\ 4 & a \end{bmatrix}$

20. $A = \begin{bmatrix} 4 & 4 \\ b & a \end{bmatrix}$

21. $A = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$

22. $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

23. $A = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$

24. $A = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$

25. $A = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$

26. $A = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$

27. $A = \begin{pmatrix} -6 & 9 \\ 2 & -3 \end{pmatrix}$

28. $A = \begin{pmatrix} -2 & 7 \\ 0 & 2 \end{pmatrix}$

29. $A = \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}$

30. $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

31. $A = \begin{pmatrix} 2 & 1 \\ a & a \end{pmatrix}$

32. $A = \begin{pmatrix} b & 3 \\ b & 2 \end{pmatrix}$

33. $A = \begin{pmatrix} 1 & a \\ 3 & a \end{pmatrix}$

34. $A = \begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix}$

35. $A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$

36. $A = \begin{pmatrix} -3 & \frac{1}{2} \\ 6 & -1 \end{pmatrix}$

37. $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

38. $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

39. $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

40. $A = \begin{bmatrix} -2 & 5 & 3 \\ 4 & -1 & 3 \\ 7 & -2 & 5 \end{bmatrix}$

41. $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{pmatrix}$

42. $A=\begin{pmatrix}1 & -1 & 1\\ 0 & -2 & 1\\ -2 & -3 & 0\end{pmatrix}$

43. $A=\begin{pmatrix}1 & 0 & 2\\ -1 & 2 & 3\\ 1 & -1 & 0\end{pmatrix}$

44. $A=\begin{pmatrix}1 & 1 & 1\\ 3 & 2 & -1\\ 3 & 1 & 2\end{pmatrix}$

45. $A=\begin{pmatrix}3 & 3 & 1\\ 1 & 2 & 1\\ 2 & -1 & 1\end{pmatrix}$

46. $A=\begin{pmatrix}-3 & 1 & -1\\ 1 & -4 & -7\\ 1 & 2 & 5\end{pmatrix}$

47. $A=\begin{pmatrix}1 & 1 & -3\\ 2 & -4 & 1\\ -5 & 7 & 1\end{pmatrix}$

48. $A=\begin{pmatrix}1 & 2 & -1\\ 3 & 5 & 3\\ 2 & 4 & 3\end{pmatrix}$

49. $A=\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 0\end{pmatrix}$

50. $A=\begin{pmatrix}\frac{1}{5} & \frac{1}{5} & -\frac{2}{5}\\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10}\\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10}\end{pmatrix}$

51. $A=\begin{pmatrix}\sqrt{2} & 3\sqrt{2} & 0\\ -4\sqrt{2} & \sqrt{2} & 0\\ 0 & 0 & 1\end{pmatrix}$

52. $A=\begin{pmatrix}-8 & 17 & 2 & \frac{1}{3}\\ 4 & 0 & \frac{2}{5} & -9\\ 0 & 0 & 0 & 0\\ -1 & 13 & 4 & 2\end{pmatrix}$

53. $A=\begin{bmatrix}-2 & -3 & 4 & 1\\ 0 & 1 & 1 & 0\\ 0 & 4 & -6 & 1\\ -2 & -2 & 5 & 1\end{bmatrix}$

54. $A=\begin{bmatrix}1 & -14 & 7 & 38\\ -1 & 2 & 1 & -2\\ 1 & 2 & -1 & -6\\ 1 & -2 & 3 & 6\end{bmatrix}$

55. $A=\begin{bmatrix}10 & 20 & -30 & 15\\ 3 & -7 & 14 & -8\\ -7 & -2 & -1 & 2\\ 4 & 4 & -3 & 1\end{bmatrix}$

56. Show that A is not invertible for any values of the entries

$$A=\begin{bmatrix}0 & a & 0 & 0 & 0\\ b & 0 & c & 0 & 0\\ 0 & d & 0 & e & 0\\ 0 & 0 & f & 0 & g\\ 0 & 0 & 0 & h & 0\end{bmatrix}$$

57. Prove that if A is an invertible matrix and B is row equivalent to A, then B is also invertible.

58. Determine if the given matrix has an inverse, and find the inverse if it exists. Check your answer by multiplying $A \cdot A^{-1} = I$

a) $\begin{bmatrix} 2 & 3 \\ -3 & -5 \end{bmatrix}$ *b)* $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}$

59. Show that the inverse of $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is $\begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$

60. If the product $C = AB$ is invertible (and *A* & *B* are square matrices), find a formula for A^{-1} that involves C^{-1} and *B*.

Hence, it is not possible to multiply a non-invertible matrix by another matric and obtain an invertible matrix as a result.

61. Prove that if *A* is an $m \times n$ matrix, there is an invertible matrix *C* such that CA is in reduced row-echelon form.

62. Prove that 2 $m \times n$ matrices *A* and *B* are row equivalent if and only if there exists a nonsingular matrix *P* such that $B = PA$

63. Let *A* and *B* be 2 $m \times n$ matrices. Suppose *A* is row equivalent to *B*. Prove that *A* is nonsingular if and only if *B* is nonsingular.

64. Show that if *A* and *B* are two $n \times n$ invertible matrices then *A* is row equivalent to *B*.

65. Prove that a square matrix *A* is nonsingular if and only if *A* is a product of elementary matrices.

66. Show that if $A \sim B$ (that is, if they are row equivalent), then $EA = B$ for some matrix *E* which is a product of elementary matrices.

67. Show that if $EA = B$ for some matrix *E* which is a product of elementary matrices, then $AC \sim BC$ for every $n \times n$ matrix *C*.

68. Let $A\vec{x} = 0$ be a homogeneous system of *n* linear equations in *n* unknowns that has only the trivial solution. Show that of *k* is any positive integer, then the system $A^k\vec{x} = 0$ also has only trivial solution.

69. Let $A\vec{x} = 0$ be a homogeneous system of *n* linear equations in *n* unknowns, and let *Q* be an invertible $n \times n$ matrix. Show that $A\vec{x} = 0$ has just trivial solution if and only if $(QA)\vec{x} = 0$ has just trivial solution.

70. Let $A\vec{x}=b$ be any consistent system of linear equations, and let $\vec{x}_1$ be a fixed solution. Show that every solution to the system can be written in the form $\vec{x}=\vec{x}_1+\vec{x}_0$ where $\vec{x}_0$ is a solution to $A\vec{x}=0$. Show also that every matrix of this form is a solution.

71. If A and B are $n\times n$ matrices satisfying $A^2=B^2=(AB)^2=I_n$. Prove that $AB=BA$.

72. Let $A=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix}$. Verify that $A^3=5I$, then find A^{-1} in term of A.

73. Consider $B(A,\ I)=(BA,\ B)$, thus if B is the inverse of A, then $(BA,\ B)$ becomes $\left(I,\ A^{-1}\right)$. On the other hand B is a product of elementary matrices since it is invertible. This indicates that the inverse of A can be obtained by applying elementary row operations to $(A,\ I)$ to get $\left(I,\ A^{-1}\right)$.

Find the inverses of

a) $A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix}$

b) $B=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{pmatrix}$

74. Let $A,\ B,\ C,\ X,\ Y,\ Z\in M_n(\mathbb{C})$, A and C are invertible. Find

a) $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}^{-1}$

b) $\begin{pmatrix} I & X & Y \\ 0 & I & Z \\ 0 & 0 & I \end{pmatrix}^{-1}$

75. Suppose that A, B, and $A-B$ are invertible $n\times n$ matrices. Show that

$$(A-B)^{-1}=A^{-1}+A^{-1}\left(B^{-1}-A^{-1}\right)^{-1}A^{-1}$$

76. Suppose P is invertible and $A=PBP^{-1}$. Solve for B in terms of A.

77. Suppose $(A-B)C=0$, where A and B are $m\times n$ matrices and C is invertible. Show that $A=B$.