Lecture Four – Sequences, Series, and Analytic Geometry

Sequences and Series

Section 4.1 – Infinite Sequences and Summation Notation

An arbitrary *infinite sequence* may be denoted as follows:

$$a_1, \ a_2, \ a_3, \ \ldots, \ a_n, \ \ldots$$

An infinite sequence is a function whose domain is the set of positive integers.

Example

Find the first four terms and the tenth term of the sequence: $\left\{ \dfrac{n}{n+1} \right\}$

Solution

$$n = 1 \ \rightarrow \ \frac{1}{1+1} = \frac{1}{2}$$

$$n = 2 \ \rightarrow \ \frac{2}{2+1} = \frac{2}{3}$$

$$n = 3 \ \rightarrow \ \frac{3}{3+1} = \frac{3}{4}$$

$$n = 4 \ \rightarrow \ \frac{4}{4+1} = \frac{4}{5}$$

$$n = 10 \ \Rightarrow \ \frac{10}{11}$$

Example

Find the first four terms and the tenth term of the sequence: $\left\{ 2 + (0.1)^n \right\}$

Solution

$$n = 1 \ \rightarrow \quad 2 + 0.1 = 2.1$$

$$n = 2 \ \rightarrow \quad 2 + 0.1^2 = 2.01$$

$$n = 3 \ \rightarrow \quad 2 + 0.1^3 = 2.001$$

$$n = 4 \ \rightarrow \quad 2 + 0.1^4 = 2.0001$$

$$n = 10 \ \Rightarrow \quad 2.0000000001$$

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Example

Find the first four terms and the tenth term of the sequence: $\left\{(-1)^{n+1}\dfrac{n^2}{3n-1}\right\}$

Solution

$$n=1 \quad \rightarrow \quad (-1)^2\frac{1^2}{3(1)-1}=\frac{1}{2}$$

$$n=2 \quad \rightarrow \quad (-1)^3\frac{2^2}{3(2)-1}=-\frac{4}{5}$$

$$n=3 \quad \rightarrow \quad (-1)^4\frac{3^2}{3(3)-1}=\frac{9}{8}$$

$$n=4 \quad \rightarrow \quad (-1)^5\frac{4^2}{3(4)-1}=-\frac{16}{11}$$

$$n=10 \quad \Rightarrow \quad -\frac{100}{29}$$

Example

Find the first four terms and the tenth term of the sequence: $\{4\}$

Solution

$$n=1 \quad \rightarrow \quad 4$$
$$n=2 \quad \rightarrow \quad 4$$
$$n=3 \quad \rightarrow \quad 4$$
$$n=4 \quad \rightarrow \quad 4$$
$$n=10 \quad \Rightarrow \quad 4$$

Example

Find the first four terms of the recursively defined infinite sequence $a_1=3, \quad a_{n+1}=(n+1)a_n$

Solution

$$a_1=3$$

$$n=1 \quad a_2=(1+1)a_1=2(3)=6$$

$$n=2 \quad a_3=(2+1)a_2=3(6)=18$$

$$n=3 \quad a_4=(3+1)a_3=4(18)=72$$

Summation Notation

To find the sum of many terms of an infinite sequence, it is easy to express using summation notation.

$$\underbrace{\sum_{n=1}^{5}}_{\text{First value of } n} 2n+3 \quad \leftarrow \textit{Formula for each term}$$

Last value of n

First value of n

Example

Find the sum: $\displaystyle\sum_{k=1}^{4} k^2(k-3)$

Solution

$$\sum_{k=1}^{4} k^2(k-3) = 1^2(1-3) + 2^2(2-3) + 3^2(3-3) + 4^2(4-3)$$

$$= -2 - 4 + 0 + 16$$

$$= 10$$

Theorem on the Sum of a *Constant*

$$(1) \quad \sum_{k=1}^{n} c = nc \qquad (2) \quad \sum_{k=m}^{n} c = (n-m+1)c$$

Proof:

$$\sum_{k=1}^{n} c = \underbrace{c + c + \ldots + c}_{n} = nc$$

Example

Find the sum: $\displaystyle\sum_{k=10}^{20} 5$

Solution

$$\sum_{k=10}^{20} 5 = (20 - 10 + 1)5 = 55$$

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Theorem on Sums

If $a_1, a_2, a_3, \ldots, a_n, \ldots$ and $b_1, b_2, b_3, \ldots, b_n, \ldots$ are infinite sequences, then for every positive integer n,

(1) $\displaystyle\sum_{k=1}^{n}\left(a_k + b_k\right) = \sum_{k=1}^{n}a_k + \sum_{k=1}^{n}b_k$

(2) $\displaystyle\sum_{k=1}^{n}\left(a_k - b_k\right) = \sum_{k=1}^{n}a_k - \sum_{k=1}^{n}b_k$

(3) $\displaystyle\sum_{k=1}^{n}ca_k = c\left(\sum_{k=1}^{n}a_k\right)$

Proof

$$\sum_{k=1}^{n}\left(a_k + b_k\right) = \left(a_1 + b_1\right) + \left(a_2 + b_2\right) + \ldots + \left(a_n + b_n\right)$$

$$= \left(a_1 + a_2 + \ldots + a_n\right) + \left(b_1 + b_2 + \ldots + b_n\right)$$

$$= \sum_{k=1}^{n}a_k + \sum_{k=1}^{n}b_k$$

Example

Express the sum using summation notation $\quad 2^1 + 2^2 + 2^3 + \cdots + 2^{16}$

Solution

$$2^1 + 2^2 + 2^3 + \cdots + 2^{16} = \sum_{k=1}^{16}2^k$$

Exercises   *Section* 4.1 – Infinite Sequences and Summation Notation

Find the first four terms and the eight term of the sequence:

1. $\{12-3n\}$

2. $\left\{\dfrac{3n-2}{n^2+1}\right\}$

3. $\{9\}$

4. $\left\{(-1)^{n-1}\dfrac{n+7}{2n}\right\}$

5. $\left\{\dfrac{2^n}{n^2+2}\right\}$

6. $\left\{(-1)^{n-1}\dfrac{n}{2n-1}\right\}$

7. $\left\{\dfrac{2^n}{3^n+1}\right\}$

8. $\left\{\dfrac{n^2}{2^n}\right\}$

9. $\left\{\dfrac{n}{e^n}\right\}$

10. $\left\{c_n\right\}=\left\{(-1)^{n+1}n^2\right\}$

11. $\left\{c_n\right\}=\left\{\dfrac{(-1)^n}{(n+1)(n+2)}\right\}$

12. $\left\{c_n\right\}=\left\{\left(\dfrac{4}{3}\right)^n\right\}$

13. $\left\{b_n\right\}=\left\{\dfrac{3^n}{n}\right\}$

14. Graph the sequence $\left\{\dfrac{1}{\sqrt{n}}\right\}$

15. Find the first four terms of the sequence of partial sums for the given sequence. $\left\{3+\tfrac{1}{2}n\right\}$

Find the first five terms of the recursively defined infinite sequence

16. $a_1=2,\quad a_{k+1}=3a_k-5$

17. $a_1=-3,\quad a_{k+1}=a_k^2$

18. $a_1=5,\quad a_{k+1}=ka_k$

19. $a_1=2,\quad a_n=3+a_{n-1}$

20. $a_1=5,\quad a_n=2a_{n-1}$

21. $a_1=\sqrt{2},\quad a_n=\sqrt{2+a_{n-1}}$

22. $a_1=2,\quad a_{n+1}=7-2a_n$

23. $a_1=128,\quad a_{n+1}=\tfrac{1}{4}a_n$

24. $a_1=2,\quad a_{n+1}=\left(a_n\right)^n$

25. $a_1=A,\quad a_n=a_{n-1}+d$

26. $a_1=A,\quad a_n=ra_{n-1},\quad r\neq0$

27. $a_1=2,\quad a_2=2;\quad a_n=a_{n-1}\cdot a_{n-2}$

Express each sum using summation notation

28. $1+2+3+\ldots+20$

29. $1+2+3+\ldots+40$

30. $1^3+2^3+3^3+\ldots+8^3$

31. $1^2+2^2+3^2+\ldots+15^2$

32. $2^2+2^3+2^4+\ldots+2^{11}$

33. $\dfrac{1}{2}+\dfrac{2}{3}+\dfrac{3}{4}+\cdots+\dfrac{13}{14}$

34. $1-\dfrac{1}{3}+\dfrac{1}{9}-\dfrac{1}{27}+\cdots+(-1)^6\dfrac{1}{3^6}$

35. $\dfrac{2}{3}-\dfrac{4}{9}+\dfrac{8}{27}-\cdots+(-1)^{12}\left(\dfrac{2}{3}\right)^{11}$

36. $\dfrac{1}{2}+\dfrac{2}{3}+\dfrac{3}{4}+\cdots+\dfrac{14}{14+1}$

37. $\dfrac{1}{e}+\dfrac{2}{e^2}+\dfrac{3}{e^3}+\cdots+\dfrac{n}{e^n}$

Find the sum

38. $\displaystyle\sum_{k=1}^{5}(2k-7)$

43. $\displaystyle\sum_{k=1}^{40}k$

48. $\displaystyle\sum_{k=1}^{16}\left(k^2-4\right)$

39. $\displaystyle\sum_{k=0}^{5}k(k-2)$

44. $\displaystyle\sum_{k=1}^{5}(3k)$

49. $\displaystyle\sum_{k=1}^{6}(10-3k)$

40. $\displaystyle\sum_{k=1}^{5}(-3)^{k-1}$

45. $\displaystyle\sum_{k=1}^{10}\left(k^3+1\right)$

50. $\displaystyle\sum_{k=1}^{10}\left[1+(-1)^k\right]$

41. $\displaystyle\sum_{k=253}^{571}\left(\tfrac{1}{3}\right)$

46. $\displaystyle\sum_{k=1}^{24}\left(k^2-7k+2\right)$

51. $\displaystyle\sum_{k=1}^{6}\frac{3}{k+1}$

42. $\displaystyle\sum_{k=1}^{50}8$

47. $\displaystyle\sum_{k=6}^{20}\left(4k^2\right)$

52. $\displaystyle\sum_{k=137}^{428}2.1$

Write out each sum

53. $\displaystyle\sum_{k=1}^{n}(k+2)$

55. $\displaystyle\sum_{k=2}^{n}(-1)^k\ln k$

57. $\displaystyle\sum_{k=0}^{n}\frac{1}{3^k}$

54. $\displaystyle\sum_{k=1}^{n}k^2$

56. $\displaystyle\sum_{k=3}^{n}(-1)^{k+1}2^k$

58. Fred has a balance of \$3,000 on his card which charges 1% interest per month on any unpaid balance. Fred can afford to pay \$100 toward the balance each month. His balance each month after making a \$100 payment is given by the recursively defined sequence

$$B_0=\$3,000 \quad B_n=1.01B_{n-1}-100$$

Determine Fred's balance after making the first payment. That is, determine B_1

59. A pond currently has 2,000 trout in it. A fish hatchery decides to add an additional 20 trout each month. Is it also known that the trout population is grwoing at a rate of 3% per month. The size of the population after n months is given bu the recursively defined sequence

$$P_0=2,000 \quad P_n=1.03P_{n-1}+20$$

How many trout are in the pond after 2 months? That is, what is P_2?

60. Fred bought a car by taking out a loan for $18,500 at 0.5% interest per month. Fred's normal monthly payment is $434.47 per month, but he decides that he can afford to pay $100 extra toward the balance each month. His balance each month is given by the recursively defined sequence

$$B_0 = \$18,500 \quad B_n = 1.005B_{n-1} - 534.47$$

Determine Fred's balance after making the first payment. That is, determine B_1

61. The Environmental Protection Agency (EPA) determines that Maple Lake has 250 *tons* of pollutant as a result of industrial waste and that 10% of the pollutant present is neuttralized by solar oxidation every year. The EPA imposes new pollution control laws that result in 15 *tons* of new pollutant entering the lake each year. The amount of pollutant in the lake after n years is given by the recursively defined sequence

$$P_0 = 250 \quad P_n = 0.9P_{n-1} + 15$$

Determine the amount of pollutant in the lake after 2 years? That is, what is P_2 ?

62. Let $u_n = \dfrac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n \sqrt{5}}$

Define the nth term of a sequence

a) Show that $u_1 = 1$ and $u_2 = 1$

b) Show that $u_{n+2} = u_{n+1} + u_n$

c) Draw the conclusion that $\{u_n\}$ is a Fibonacci sequence

d) Find the first ten terms of the sequence from part (c)

Section **4.2 – Arithmetic and Geometric Sequences**

Arithmetic Sequence

Definition of Arithmetic Sequence

A sequence a_1, a_2, a_3, …, a_n, … is an arithmetic sequence if there is a real number d such that for every positive integer k,

$$a_{k+1} = a_k + d$$

The number $d = a_{k+1} - a_k$ is called the ***common difference*** of the sequence.

Example

Show that the sequence: 1, 4, 7, 10, … , $3n - 2$, … is arithmetic, and find the common difference.

Solution

If $a_n = 3n - 2$, then for every positive integer k,

$$a_{k+1} - a_k = [3(k+1) - 2] - (3k - 2)$$
$$= 3k + 3 - 2 - 3k + 2$$
$$= 3$$

Hence, the given sequence is arithmetic with common difference 3.

The *n*th Term of an Arithmetic Sequence: $\boxed{a_n = a_1 + (n-1)d}$

Example

The first three terms of an arithmetic sequence are 20, 16.5, and 13. Find the fifteenth term.

Solution

The common difference is: $a_2 - a_1 = 16.5 - 20 = -3.5$

Substituting $a_1 = 20$, $d = -3.5$, $n = 15$ in the formula:

$$a_{15} = a_1 + (15 - 1)d$$
$$= 20 + (15 - 1)(-3.5)$$
$$= -29$$

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Example

The fourth term of an arithmetic sequence is 5, and the ninth term is 20. Find the sixth term.

Solution

Given: $a_4 = 5 \quad a_9 = 20$

$$\begin{cases} a_4 = a_1 + (4-1)d \\ a_9 = a_1 + (9-1)d \end{cases} \Rightarrow \begin{cases} 5 = a_1 + 3d \\ 20 = a_1 + 8d \end{cases} \qquad \begin{aligned} a_{x1} &= y1 \\ a_{x2} &= y2 \end{aligned} \rightarrow d = \frac{y2 - y1}{x2 - x1}$$

$$\begin{aligned} 20 &= a_1 + 8d \\ -\ 5 &= a_1 + 3d \\ \hline 15 &= 5d \end{aligned} \qquad \Rightarrow \boxed{d = 3} \quad and \quad \boxed{a_1 = 5 - 3d = 5 - 9 = -4}$$

$$\begin{aligned} a_6 &= a_1 + (6-1)d \\ &= -4 + (5)3 \\ &\underline{= 11} \end{aligned}$$

Theorem

Formulas for S_n

If a_1, a_2, a_3, ..., a_n, ... is an arithmetic sequence with common difference d, then the nth partial sum S_n (that is, the sum of the first n terms) is given by either

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right] \quad or \quad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

Proof

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \ldots + a_n \\ &= a_1 + \left(a_1 + d\right) + \left(a_1 + 2d\right) + \ldots + \left(a_1 + (n-1)d\right) \\ &= a_1 + a_1 + \ldots + a_1 + \left[d + 2d + \ldots + (n-1)d\right] \\ &= na_1 + d\left[1 + 2 + \ldots + (n-1)\right] \end{aligned}$$

$$\begin{aligned} & \qquad\qquad\qquad\qquad\qquad\qquad \textit{Using the formula of sum: } S_n = \frac{n(n+1)}{2} \\ &= \frac{2na_1 + (n-1)nd}{2} \\ &= \frac{n}{2}\left[2a_1 + (n-1)d\right] \end{aligned}$$

Example

Find the sum of all even integers from 2 through 100.

Solution

The arithmetic sequence: 2, 4, 6, …, 2n, …

Substituting $n = 50, \quad a_1 = 2, \quad and \quad a_{50} = 100$ in the formula:

$$S_n = \frac{50}{2}(2+100) = \underline{2550}$$

Example

Express in terms of summation notation: $\frac{1}{4} + \frac{2}{9} + \frac{3}{14} + \frac{4}{19} + \frac{5}{24} + \frac{6}{29}$

Solution

Numerators : $1, 2, 3, 4, 5$ *common difference* 1

Denominators : $4, 9, 14, 19, 24, 29$ *common difference* 5

Using the formula for *n*th term:

$$a_n = a_1 + (n-1)d = 1 + (n-1)1 = n$$

$$a_n = a_1 + (n-1)d = 4 + (n-1)5 = 4 + 5n - 5 = 5n - 1$$

Hence the *n*th term is:

$$\frac{1}{4} + \frac{2}{9} + \frac{3}{14} + \frac{4}{19} + \frac{5}{24} + \frac{6}{29} = \sum_{n=1}^{6} \frac{n}{5n-1}$$

Geometric Sequence

Definition of *Geometric* Sequence

A sequence a_1, a_2, a_3, ..., a_n, ... is a geometric sequence if $a_1 \neq 0$ and if there is a real number $r \neq 0$ such that for every positive integer k.

$$a_{k+1} = a_k r$$

The number $r = \dfrac{a_{k+1}}{a_k}$ is called the **common ratio** of the sequence.

The formula for the nth Term of a Geometric Sequence: $\boxed{a_n = a_1 r^{n-1}}$

The common ratio for: 6, -12, 24, -48, ..., $(-2)^{n-1}(6)$, ... is $= \dfrac{-12}{6} = -2$

Example

A geometric sequence has first term 3 and common ratio $-\dfrac{1}{2}$. Find the first five terms and the tenth term.

Solution

$$a_1 = 3$$

$$a_n = a_1 r^{n-1}$$

$r = -\dfrac{1}{2}$ $\qquad a_2 = 3\left(-\dfrac{1}{2}\right) = -\dfrac{3}{2}$

$r^2 = \left(-\dfrac{1}{2}\right)^2 = \dfrac{1}{4}$ $\qquad a_3 = 3\left(\dfrac{1}{4}\right) = \dfrac{3}{4}$

$r^3 = \left(-\dfrac{1}{2}\right)^3 = -\dfrac{1}{8}$ $\qquad a_4 = 3\left(-\dfrac{1}{8}\right) = -\dfrac{3}{8}$

$r^4 = \left(-\dfrac{1}{2}\right)^4 = \dfrac{1}{16}$ $\qquad a_5 = 3\left(\dfrac{1}{16}\right) = \dfrac{3}{16}$

$r^9 = \left(-\dfrac{1}{2}\right)^9 = -\dfrac{1}{512}$ $\qquad a_{10} = 3\left(-\dfrac{1}{512}\right) = -\dfrac{3}{512}$

Example

The third term of a geometric is 5, and the sixth term is −40. Find the eighth term.

Solution

Given: $a_3 = 5 \quad a_6 = -40$

$a_n = a_1 r^{n-1} \quad \begin{cases} a_3 = a_1 r^{3-1} \\ a_6 = a_1 r^{6-1} \end{cases} \rightarrow \begin{cases} 5 = a_1 r^2 \\ -40 = a_1 r^5 \end{cases}$

$\dfrac{a_1 r^5}{a_1 r^2} = \dfrac{-40}{5} \Rightarrow r^3 = -8 \rightarrow \boxed{r = -2}$

$\begin{cases} a_{x1} = y1 \\ a_{x2} = y2 \end{cases} \rightarrow \boxed{r = \left(\dfrac{y2}{y1}\right)^{\frac{1}{x2-x1}}}$

$\boxed{a_1 = \dfrac{5}{r^2} = \dfrac{5}{(-2)^2} = \dfrac{5}{4}}$

$\boxed{a_8 = \dfrac{5}{4}(-2)^7 = -160}$

Theorem: Formula for S_n

The nth partial sum S_n of a geometric sequence with first term a_1 and common ratio $r \neq 1$ is

$$S_n = a_1 \frac{1 - r^n}{1 - r}$$

Proof

By definition, the nth partial sum S_n of a geometric sequence is:

$S_n = a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-2} + a_1 r^{n-1}$

$- \quad rS_n = a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-1} + a_1 r^n$

$\overline{\qquad S_n - rS_n = a_1 - a_1 r^n \qquad}$

$(1 - r)S_n = a_1\left(1 - r^n\right) \quad \rightarrow \quad \underline{S_n = a_1 \dfrac{1 - r^n}{1 - r}}$

Example

If the sequence 1, 0.3, 0.09, .0027, … is a geometric sequence, find the sum of the first five terms.

Solution

Given: $a_1 = 1, \quad r = \dfrac{0.3}{1} = 0.3, \quad n = 5$

$S_5 = a_1 \dfrac{1 - r^5}{1 - r} = 1\dfrac{1 - (0.3)^5}{1 - 0.3} = \underline{1.4251}$

Theorem on the Sum of an Infinite Geometric Series

If $|r| < 1$, then the infinite geometric series $a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1} + \ldots$ has the sum

$$S = \frac{a_1}{1-r}$$

Example

Find the sum S of the alternating infinite geometric series: to $\displaystyle\sum_{n=1}^{\infty} 3\left(-\frac{2}{3}\right)^{n-1}$

Solution

$$\sum_{n=1}^{\infty} 3\left(-\frac{2}{3}\right)^{n-1} = 3 - 2 + \frac{4}{3} - \frac{8}{9} + \ldots + 3\left(-\frac{2}{3}\right)^{n-1} + \ldots$$

$$S = \frac{a_1}{1-r} = \frac{3}{1-\left(-\frac{2}{3}\right)} = \frac{3}{\frac{5}{3}}$$

$$= \frac{9}{5}$$

Example

Find a rational number that corresponds to $5.4\overline{27}$

Solution

$$5.4\overline{27} = 5.427272727\ldots = 5.4 + 0.027 + 0.00027 + .0000027 + \ldots$$

$$a_1 = 0.027, \quad r = \frac{.00027}{.027} = 0.01$$

$$S = 5.4 + \frac{a_1}{1-r}$$

$$= \frac{54}{10} + \frac{.027}{1 - .01}$$

$$= \frac{54}{10} + \frac{.027}{.990}$$

$$= \frac{54}{10} + \frac{27}{990}$$

$$= \frac{54}{10} + \frac{3}{110}$$

$$= \frac{597}{110}$$

Example

Initially, a pendulum swings through an arc of 18 *inches*. On each successive swing, the length of the arc is 0.98 of the previous length.

    a) What is the length of the arc of the 10th swing?

    b) On which swing is the length of the arc first less than 12 inches?

    c) After 15 swings, what total distance will the pendulum have swung?

    d) When it stops, what total distance will the pendulum have swung?

Solution

    a) The length of the first swing: $a_1 = 18$

        The length of the second swing: $a_2 = 0.98a_1 = 0.98(18)$

    $a_3 = 0.98a_2 = 0.98^2(18)$

        The length of the arc of the 10th swing is: $a_{10} = 0.98^9(18) \approx \underline{15.007 \ in}$

    b) $a_n = 18(0.98)^{n-1}$

        $18(0.98)^{n-1} = 12 \quad \rightarrow \quad (0.98)^{n-1} = \frac{12}{18} = \frac{2}{3}$

        $n - 1 = \log_{0.98}\left(\frac{2}{3}\right)$

        $n = \log_{0.98}\left(\frac{2}{3}\right) + 1 \approx \underline{21.07}$

        The length of the arc of the pendulum exceeds 12 inches on the 21st swing and the first less than 12 inches on the 22nd swing.

    c) $L = 18 \cdot \dfrac{1 - 0.98^{15}}{1 - 0.98} \approx \underline{235.3 \ in}$ $S_n = a_1 \dfrac{1 - r^n}{1 - r}$

    d) $T = \dfrac{18}{1 - 0.98} = \underline{900}$ $S_n = \dfrac{a_1}{1 - r}$

        The pendulum will have swung a total of 900 inches when it finally stops.

Exercises   *Section* **4.2 – Arithmetic and Geometric Sequences**

1.   Show that the sequence $-6, \ -2, \ 2, \ \ldots, \ 4n-10, \ \ldots$ is arithmetic, and find the common difference.

Find the *n*th term, and the tenth term of the arithmetic sequence:

2.   2, 6, 10, 14, …

3.   3, 2.7, 2.4, 2.1, …

4.   −6, − 4.5, − 3, −1.5, …

5. $\ln 3, \ \ln 9, \ \ln 27, \ \ln 81, \ \ldots$

6. $a_1 = 2, \quad d = 3$

7. $a_1 = 5, \quad d = -3$

8. $a_1 = 1, \quad d = -\dfrac{1}{2}$

9. $a_1 = -2, \quad d = 4$

10. $a_1 = \sqrt{2}, \quad d = \sqrt{2}$

11. $a_1 = 0, \quad d = \pi$

12. $a_1 = 13, \quad d = 4$

13. $a_1 = -40, \quad d = 5$

14. $a_1 = -32, \quad d = 4$

Find the common difference for the arithmetic sequence with the specified terms:

15. $a_4 = 14, \quad a_{11} = 35$

16. $a_{12}; \quad a_1 = 9.1, \quad a_2 = 7.5$

17. $a_1; \quad a_8 = 47, \quad a_9 = 53$

18. $a_{10}; \quad a_2 = 1, \quad a_{18} = 49$

19. $a_{10}; \quad a_8 = 8, \quad a_{20} = 44$

20. $a_{12}; \quad a_8 = 4, \quad a_{18} = -96$

21. $a_8; \quad a_{15} = 0, \quad a_{40} = -50$

22. $a_{20}; \quad a_9 = -5, \quad a_{15} = 31$

23. $a_n; \quad a_8 = 8, \quad a_{20} = 44$

24. $a_n; \quad a_8 = 4, \quad a_{18} = -96$

25. $a_n; \quad a_{14} = -1, \quad a_{15} = 31$

26. $a_n; \quad a_9 = -5, \quad a_{15} = 31$

Find the sum S_n of the arithmetic sequence that satisfies the conditions:

27. $a_1 = 40, \quad d = -3, \quad n = 30$

28. $a_7 = \dfrac{7}{3}, \quad d = -\dfrac{2}{3}, \quad n = 15$

29.   Find the number of terms in the arithmetic sequence with the given conditions:

$a_1 = -2, \quad d = \dfrac{1}{4}, \quad S = 21$

30.   Find the number of integers between 32 and 390 that are divisible by 6, find their sum.

Find each arithmetic sum

31. $2 + 11 + 20 + \ldots + 16{,}058$

32. $60 + 64 + 68 + 72 + \cdots + 120$

33. $1 + 3 + 5 + \cdots + (2n - 1)$

34. $2 + 4 + 6 + \cdots + 2n$

35. $2 + 5 + 8 + \cdots + 41$

36. $7 + 12 + 17 + \cdots + (2 + 5n)$

37. $73 + 78 + 83 + 88 + \cdots + 558$

38. $7 + 1 - 5 - 11 - \cdots - 299$

39. $-1 + 2 + 7 + \cdots + (4n - 5)$

40. $5 + 9 + 13 + \cdots + 49$

41. $2 + 4 + 6 + \cdots + 70$

42. $1 + 3 + 5 + \cdots + 59$

43. $4 + 4.5 + 5 + 5.5 + \cdots + 100$

44. $8 + 8\dfrac{1}{4} + 8\dfrac{1}{2} + 8\dfrac{3}{4} + 9 + \cdots + 50$

45. Show that the given sequence is geometric, and find the common ratio

$$5, \ -\frac{5}{4}, \ \frac{5}{16}, \ \ldots, \ 5\left(-\frac{1}{4}\right)^{n-1}, \ \ldots$$

Find the nth term, the fifth term, and the eighth term of the geometric sequence

46. $8, \ 4, \ 2, \ 1, \ \ldots$

47. $300, \ -30, \ 3, \ -0.3, \ \ldots$

48. $1, \ -\sqrt{3}, \ 3, \ -3\sqrt{3}, \ \ldots$

49. $4, \ -6, \ 9, \ -13.5, \ \ldots$

50. $1, \ -x^2, \ x^4, \ -x^6, \ \ldots$

51. $10, \ 10^{2x-1}, \ 10^{4x-3}, \ 10^{6x-5}, \ \ldots$

52. $a_1 = 2, \quad r = 3$

53. $a_1 = 1, \quad r = -\dfrac{1}{2}$

54. $a_1 = -2, \quad r = 4$

55. $a_1 = \sqrt{2}, \quad r = \sqrt{2}$

56. $a_1 = 0, \quad r = \pi$

57. $\{s_n\} = \{3^n\}$

58. $\{s_n\} = \{(-5)^n\}$

59. $\{s_n\} = \left\{-3\left(\dfrac{1}{2}\right)^n\right\}$

60. $\{u_n\} = \left\{\dfrac{3^{n-1}}{2^n}\right\}$

61. $\{u_n\} = \left\{\dfrac{2^n}{3^{n-1}}\right\}$

62. Find all possible values of r for a geometric sequence with the two given terms $a_4 = 3, \quad a_6 = 9$

63. Find the sixth term of the geometric sequence whose first two terms are 4 and 6

Find the specified term of the geometric sequence that that has 2 given terms

64. $a_{10}; \ a_4 = 4, \ a_7 = 12$

65. $a_6; \ a_1 = 4, \ a_2 = 6$

66. $a_7; \ a_2 = 3, \ a_3 = -\sqrt{3}$

67. $a_6; \ a_2 = 3, \ a_3 = -\sqrt{2}$

68. $a_5; \ a_1 = 4, \ a_2 = 7$

69. $a_9; \ a_2 = 3, \ a_5 = -81$

70. $a_7; \ a_1 = -4, \ a_3 = -1$

71. $a_8; \ a_2 = 3, \ a_4 = 6$

Express the sum in terms of summation notation (Answers are not unique.)

72. $4 + 11 + 18 + 25 + 32$

73. $4 + 11 + 18 + \ldots + 466$

74. $2 + 4 + 8 + 16 + 32 + 64 + 128$

75. $2 - 4 + 8 - 16 + 32 - 64$

76. $3 + 8 + 13 + 18 + 23$

77. $256 + 192 + 144 + 108 + \cdots$

78. $\dfrac{5}{13} + \dfrac{10}{11} + \dfrac{15}{9} + \dfrac{20}{7}$

79. $\dfrac{1}{4} - \dfrac{1}{12} + \dfrac{1}{36} - \dfrac{1}{108}$

80. $3 + \dfrac{3}{5} + \dfrac{3}{25} + \dfrac{3}{125} + \dfrac{3}{625}$

81. $\dfrac{3}{7} + \dfrac{6}{11} + \dfrac{9}{15} + \dfrac{12}{19} + \dfrac{15}{23} + \dfrac{18}{27}$

82. $\dfrac{x}{3} + \dfrac{x^2}{9} + \dfrac{x^3}{27} + \cdots, \quad |x| < 3$

83. $2x + 4x^2 + 8x^3 + \cdots, \quad |x| < \dfrac{1}{2}$

Find the sum of the infinite geometric series if it exists:

84. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$

85. $1.5 + 0.015 + 0.00015 + \cdots$

86. $\sqrt{2} - 2 + \sqrt{8} - 4 + \cdots$

87. $256 + 192 + 144 + 108 + \cdots$

88. $\frac{1}{4} + \frac{2}{4} + \frac{2^2}{4} + \frac{2^3}{4} + \cdots + \frac{2^{n-1}}{4}$

89. $\frac{3}{9} + \frac{3^2}{9} + \frac{3^3}{9} + \cdots + \frac{3^n}{9}$

90. $-1 - 2 - 4 - 8 - \cdots - 2^{n-1}$

91. $2 + \frac{6}{5} + \frac{18}{25} + \cdots + 2\left(\frac{3}{5}\right)^{n-1}$

92. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

93. $2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \cdots$

94. $2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \cdots$

95. $9 + 12 + 16 + \frac{64}{3} + \cdots$

96. $8 + 12 + 18 + 27 + \cdots$

97. $6 + 2 + \frac{2}{3} + \frac{2}{9} + \cdots$

Find the sum:

98. $\displaystyle\sum_{k=1}^{20}(3k - 5)$

99. $\displaystyle\sum_{k=1}^{18}\left(\frac{1}{2}k + 7\right)$

100. $\displaystyle\sum_{k=1}^{80}(2k - 5)$

101. $\displaystyle\sum_{n=1}^{90}(3 - 2n)$

102. $\displaystyle\sum_{n=1}^{100}\left(6 - \frac{1}{2}n\right)$

103. $\displaystyle\sum_{n=1}^{80}\left(\frac{1}{3}n + \frac{1}{2}\right)$

104. $\displaystyle\sum_{k=1}^{10}3^k$

105. $\displaystyle\sum_{k=1}^{9}\left(-\sqrt{5}\right)^k$

106. $\displaystyle\sum_{k=0}^{9}\left(-\frac{1}{2}\right)^{k+1}$

107. $\displaystyle\sum_{n=1}^{\infty}2\left(\frac{2}{3}\right)^{n-1}$

108. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n}$

109. $\displaystyle\sum_{n=1}^{\infty}3\left(\frac{3}{2}\right)^{n}$

110. $\displaystyle\sum_{n=1}^{\infty}5\left(\frac{1}{4}\right)^{n-1}$

111. $\displaystyle\sum_{n=1}^{\infty}8\left(\frac{1}{3}\right)^{n-1}$

112. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{2}\cdot 3^{k-1}$

113. $\displaystyle\sum_{k=1}^{\infty}6\left(-\frac{2}{3}\right)^{k-1}$

114. $\displaystyle\sum_{k=1}^{\infty}4\left(-\frac{1}{2}\right)^{k-1}$

115. $\displaystyle\sum_{k=8}^{14}\left(3^{k-7} + 2j^2\right)$

116. The sum of the first 120 terms of 14, 16, 18, 20, $\cdots$

117. The sum of the first 46 terms of 2, -1, -4, -7, $\cdots$

Find the rational number represented by the repeating decimal

118. $0.\overline{23}$

119. $0.0\overline{71}$

120. $2.4\overline{17}$

121. $10.\overline{5}$

122. $5.1\overline{46}$

123. $3.2\overline{394}$

124. $1.\overline{6124}$

125. Find x so that $x + 3$, $2x + 1$, and $5x + 2$ are consecutive terms of an arithmetic sequence.

126. Find x so that $2x$, $3x + 2$, and $5x + 3$ are consecutive terms of an arithmetic sequence.

127. Find x so that x, $x + 2$, and $x + 3$ are consecutive terms of a geometric sequence.

128. Find x so that $x-1$, x and $x+2$ are consecutive terms of a geometric sequence.

129. How many terms must be added in an arithmetic sequence whose first term is 11 and whose common difference is 3 to obtain a sum of 1092?

130. How many terms must be added in an arithmetic sequence whose first term is 78 and whose common difference is –4 to obtain a sum of 702?

131. The first ten rows of seating in a certain section of a stadium have 30 seats, 32 seats, 34 seats, and so on. The eleventh through the twentieth rows each contain 50 seats. Find the total number of seats in the section.

132. The corner section of a football stadium has 15 seats in the first row and 40 rows in all. Each successive row contains two additional seats. How many seats are in this section?
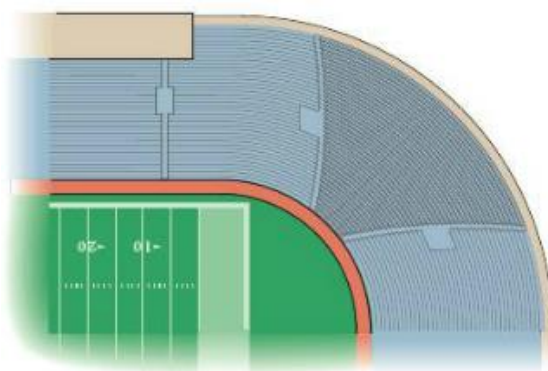
133. A gain bin is to be constructed in the shape of a frustum of a cone.
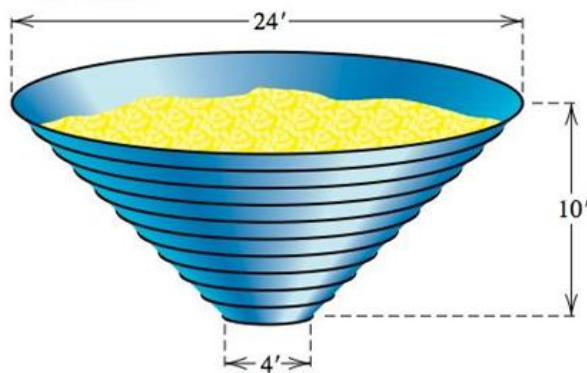
The bin is to be 10 *feet* tall with 11 metal rings positioned uniformly around it, from the 4-foot opening at the bottom to the 24-foot opening at the top. Find the total length of metal needed to make the rings.

134. A bicycle rider coasts downhill, traveling 4 *feet* the first second. In each succeeding second, the rider travels 5 *feet* farther than in the preceding second. If the rider reaches the bottom of the hill in 11 *seconds*, find the total distance traveled.

135. A contest will have five each prizes totaling $5,000, and there will be a $100 difference between successive prices. Find the first prize.

136. A Company is to distribute $46,000 in bonuses to its top ten salespeople. The tenth salesperson on the list will receive $1,000, and the difference in bonus money between successively ranked salesperson is to be constant. Find the bonus for each salesperson.

137. Assuming air resistance is negligible, a small object that is dropped from a hot air balloon falls 16 *feet* during the first second, 48 *feet* during the second second, 80 *feet* during the third second, 112 *feet* during the fourth second, and so on. Find an expression for the distance the object falls in *n* seconds.

138. A mosaic is designed in the shape of an equilateral triangle, 20 feet on each side. Each tile in the mosaic is in the shape of an equilateral triangle, 12 inches to a side. The tiles are to alternate in color as shown below.
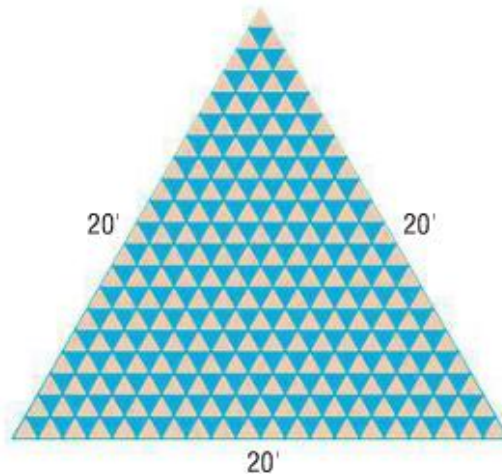
How many tiles of each color will be required?

139. A brick staircase has a total of 30 steps. The bottom step requires 100 bricks. Each successive step requires two fewer bricks than the prior step.
  a) How many bricks are required for the top step?
  b) How many bricks are required to build the staircase?

Section **4.3 – Mathematical Induction**

If n is a positive integer and we let P_n denote the mathematical statement $(xy)^n = x^n y^n$, we obtained the following **infinite sequence** of statements:

Statement P_1 : $(xy)^1 = x^1 y^1$

Statement P_2 : $(xy)^2 = x^2 y^2$

Statement P_3 : $(xy)^3 = x^3 y^3$

$$\vdots \qquad \qquad \vdots$$

Statement P_n : $(xy)^n = x^n y^n$

$$\vdots \qquad \qquad \vdots$$

Principle **of Mathematical Induction**

If with each positive integer n there is associated a statement P_n then all the statements P_n are true, provided the following two conditions are satisfied.

1) P_1 is true.

2) Whenever k is a positive integer such that P_k is true, then P_{k+1} is also true.

Steps in Applying the Principle of Mathematical Induction

1) Show that P_1 is true.

2) Assume that P_k is true, and then prove that P_{k+1} is true.

Example

Use the mathematical induction to prove that for every positive integer n, the sum of the first n positive integers is:

$$\frac{n(n+1)}{2}$$

Solution

(1) For $n = 1 \Rightarrow \frac{1(1+1)}{2} = 1$; hence P_1 is true.

(2) Assume that P_k is true. Thus the induction hypothesis is: $1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$

For $k + 1$:

$$1 + 2 + 3 + \ldots + k + (k+1) \overset{?}{=} \frac{(k+1)((k+1)+1)}{2}$$

$$1 + 2 + 3 + \ldots + k + (k+1) = (1 + 2 + 3 + \ldots + k) + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1) \qquad \text{*Induction hypothesis*}$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2} \qquad \text{*Factor out* \textbf{k + 1}}$$

$$= \frac{(k+1)((k+1)+1)}{2} \qquad \text{*Change form of* \textbf{k + 2}}$$

This shows that P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed

Example

Prove that for every positive integer n,

$$1^2 + 3^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Solution

(1) For $\boldsymbol{n = 1} \Rightarrow 1^2 = \frac{1(2(1)-1)(2(1)+1)}{3} = \frac{3}{3} = 1$; hence P_1 is true.

(2) $1^2 + 3^2 + \ldots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$

For $k + 1$:

$$1^2 + 3^2 + \ldots + (2k-1)^2 + (2(k+1)-1)^2 \overset{?}{=} \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

$$1^2 + 3^2 + \ldots + (2k-1)^2 + (2(k+1)-1)^2 \overset{?}{=} \frac{(k+1)(2k+1)(2k+3)}{3}$$

$$1^2 + 3^2 + \ldots + (2k-1)^2 + [2k+2-1]^2 = \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3}$$

$$= \frac{(2k+1)[k(2k-1) + 3(2k+1)]}{3}$$

$$= \frac{(2k+1)(2k^2 - k + 6k + 3)}{3}$$

$$= \frac{(2k+1)(2k^2 + 5k + 3)}{3}$$

$$= \frac{(2k+1)(k+1)(2k+3)}{3} \checkmark$$

This shows that P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed

Example

Prove that 2 is a factor of $n^2 + 5n$ for every positive integer n,

Solution

(1) For $n = 1 \Rightarrow n^2 + 5n = 1^2 + 5(1) = 6 = 2.3$

Thus, 2 is a factor of $n^2 + 5n$ for $n = 1$; hence P_1 is true.

(2) 2 is a factor of $k^2 + 5k \Leftrightarrow k^2 + 5k = 2p$

is 2 a factor of $(k+1)^2 + 5(k+1)$?

$$(k+1)^2 + 5(k+1) = k^2 + 2k + 1 + 5k + 5$$
$$= k^2 + 5k + 2k + 6$$
$$= \left(k^2 + 5k\right) + 2(k+3)$$
$$= 2p + 2(k+3)$$
$$= 2.(p + k + 3)\sqrt{}$$

Thus, 2 is a factor of the last expression; hence P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed

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Steps in Applying the Extended Principle of Mathematical Induction

1. Show that P_1 is true.

2. Assume that P_k is true with $k \geq j$, and then prove that P_{k+1} is true.

Example

Let a be a nonzero real number such that $a > -1$. Prove that $(1+a)^n > 1 + na$ for every integer $n \geq 2$.

Solution

For $n = 1 \Rightarrow (1+a)^1 > 1 + (1)a \Rightarrow P_1$ is false.

Step 1. For $n = 2 \Rightarrow (1+a)^2 \overset{?}{>} 1 + (2)a$

$$1 + 2a + a > 1 + a \quad \checkmark$$

$\Rightarrow P_2$ is true.

Step 2. Assume that P_k is true $(1+a)^k > 1 + ka$

We need to prove that P_{k+1} is true, that is $(1+a)^{k+1} > 1 + (k+1)a$

$$(1+a)^{k+1} = (1+a)^k (1+a)^1$$
$$> (1+ka)(1+a)$$

$$(1+ka)(1+a) = 1 + a + ka + ka^2$$
$$= 1 + (a + ka) + ka^2$$
$$= 1 + a(k+1) + ka^2$$
$$> 1 + (k+1)a$$

$$(1+a)^{k+1} > (1+ka)(1+a) > 1 + (k+1)a \quad \checkmark$$

Thus, P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed

Exercises   *Section* **4.3 – Mathematical Induction**

1. Find all positive integers n for which the given statement is not true

    a) $3^n > 6n$ b) $3^n > 2n+1$ c) $2^n > n^2$ d) $n! > 2n$

2. Prove that the statement is true for every positive integer n. $2+4+6+...+2n = n(n+1)$

3. Prove that the statement is true for every positive integer n. $1+3+5+...+(2n-1) = n^2$

4. Prove that the statement is true for every positive integer n.

    $2+7+12+...+(5n-3) = \frac{1}{2}n(5n-1)$

5. Prove that the statement is true: $1+2 \cdot 2 + 3 \cdot 2^2 +...+ n \cdot 2^{n-1} = 1+(n-1).2^n$

6. Prove that the statement is true: $1^2 + 2^2 + 3^2 +...+ n^2 = \frac{n(n+1)(2n+1)}{6}$

7. Prove that the statement is true: $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} +...+ \frac{1}{n(n+1)} = \frac{n}{n+1}$

8. Prove that the statement is true: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} +...+ \frac{1}{2^n} = 1 - \frac{1}{2^n}$

9. Prove that the statement is true: $\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + \cdots + \frac{1}{(3n-2) \cdot (3n+1)} = \frac{n}{3n+1}$

10. Prove that the statement is true: $\frac{4}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \cdots + \frac{4}{5^n} = 1 - \frac{1}{5^n}$

11. Prove that the statement is true: $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

12. Prove that the statement is true: $3 + 3^2 + 3^3 +...+ 3^n = \frac{3}{2}(3^n - 1)$

13. Prove that the statement is true: $x^{2n} + x^{2n-1}y + \cdots + xy^{2n-1} + y^{2n} = \frac{x^{2n+1} - y^{2n+1}}{x - y}$

14. Prove that the statement is true: $5 \cdot 6 + 5 \cdot 6^2 + 5 \cdot 6^3 + \cdots + 5 \cdot 6^n = 6(6^n - 1)$

15. Prove that the statement is true: $7 \cdot 8 + 7 \cdot 8^2 + 7 \cdot 8^3 + \cdots + 7 \cdot 8^n = 8(8^n - 1)$

16. Prove that the statement is true: $3 + 6 + 9 + \cdots + 3n = \frac{3n(n+1)}{2}$

17. Prove that the statement is true: $5 + 10 + 15 + \cdots + 5n = \frac{5n(n+1)}{2}$

18. Prove that the statement is true: $1 + 3 + 5 + \cdots + (2n-1) = n^2$

19. Prove that the statement is true: $4 + 7 + 10 + \cdots + (3n+1) = \frac{n(3n+5)}{2}$

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20. Prove that the statement is true for every positive integer n. $n < 2^n$

21. Prove that the statement is true for every positive integer n. 3 is a factor of $n^3 - n + 3$

22. Prove that the statement is true for every positive integer n. 4 is a factor of $5^n - 1$

23. Prove that the statement by mathematical induction: $\left(a^m\right)^n = a^{mn}$ (a and m are constant)

24. Prove that the statement by mathematical induction: $2^n > 2n$ *if* $n \geq 3$

25. Prove that the statement by mathematical induction: If $0 < a < 1$, then $a^n < a^{n-1}$

26. Prove that the statement by mathematical induction: If $n \geq 4$, then $n! > 2^n$

27. Prove that the statement by mathematical induction: $3^n > 2n + 1$ *if* $n \geq 2$

28. Prove that the statement by mathematical induction: $2^n > n^2$ *for* $n > 4$

29. Prove that the statement by mathematical induction: $4^n > n^4$ *for* $n \geq 5$

30. A pile of n rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must moved, one at a time, to a different peg with no ring ever placed on top of a smaller ring. Find the least number of moves that would be required. Prove your result by mathematical induction.
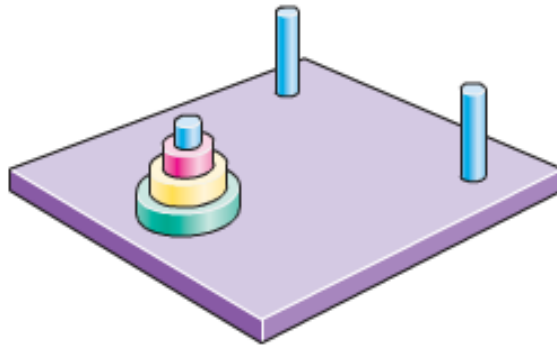
Section **4.4 – The Binomial Theorem**

A binomial is a sum $a + b$, where a and b represent numbers. If n is a positive integer, then a general formula for expanding $(a + b)^n$ is given by the **binomial theorem**.

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

The expansions of $(a + b)^n$ for n = 2, 3, 4, and 5 have the following properties:

✓ There are $n + 1$ terms, the first being a^n and the last b^n

✓ The power of a decreases by 1 and the power of b increases by 1. For each term, the sum of the exponents of a and b is n.

✓ Each term has the form $(c)a^{n-k}b^k$, where the coefficient c is an integer and $k = 0, 1, 2, \ldots, n$.

✓ The following formula is true for each of the first n terms of the expansion:

$$\frac{(coefficient\ of\ term).(exponent\ of\ a)}{number\ of\ term} = coefficient\ of\ next\ term$$

Coefficient of the $(k+1)st$ Term in the Expansion of $(a+b)^n$

$$\frac{n.(n-1).(n-2).(n-3)...(n-k+1)}{k.(k-1).....3.2.1}, \quad k = 1, 2, ..., n$$

Factorial Notation

Definition of $n!$ (n factorial)

$$\begin{cases} n! = n(n-1)(n-2)\cdots\cdots 1 & if\ \ n > 0 \\ 0! = 1 \end{cases}$$

Calculators: *Math $\rightarrow$ Prob $\rightarrow$ 4*

Illustration

$1! = 1$
$2! = 2.1 = 2$
$3! = 3.2.1 = 6$
$4! = 4.3.2.1 = 24$

Example

Simplify the quotient of factorial: $\dfrac{7!}{5!}$

Solution

$$\frac{7!}{5!} = \frac{7.6.\color{red}{5.4.3.2.1}}{\color{red}{5.4.3.2.1}} = 7.6 = 42$$

Example

Simplify the quotient of factorial: $\dfrac{10!}{6!}$

Solution

$$\frac{10!}{6!} = \frac{10.9.8.7.\color{red}{6!}}{\color{red}{6!}} = 10.9.8.7 = 5040$$

Coefficient of the $(k+1)\,st$ Term in the Expansion of $(a+b)^n$ (*Alternative Form*)

$$\boxed{\binom{n}{k} = C(n,\ k) = \frac{n!}{k!(n-k)!}}, \quad k = 0,1,2,...,n$$

Example

Find $\dbinom{5}{2}, \dbinom{5}{0}$

Solution

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{1.2.3.4.5}{(1.2)(1.2.3)} = \frac{20}{2} = 10$$

$$\binom{5}{0} = \frac{5!}{0!(5-0)!} = \frac{5!}{1(5!)} = 1$$

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Binomial Theorem

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{k}a^{n-k}b^k + \ldots + \binom{n}{n-1}ab^{n-1} + b^n$$

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \ldots + \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}a^{n-k}b^k + \ldots + nab^{n-1} + b^n$$

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

Example

Find the binomial expansion of $\left(2x + 3y^2\right)^4$

Solution

$$\left(2x+3y^2\right)^4 = (2x)^4 + \binom{4}{1}(2x)^3\left(3y^2\right)^1 + \binom{4}{2}(2x)^2\left(3y^2\right)^2 + \binom{4}{3}(2x)^1\left(3y^2\right)^3 + \left(3y^2\right)^4$$

$$= 16x^4 + 4\left(8x^3\right)\left(3y^2\right) + 6\left(4x^2\right)\left(9y^4\right) + 4(2x)\left(27y^6\right) + 81y^8$$

$$= 16x^4 + 96x^3y^2 + 216x^2y^4 + 216xy^6 + 81y^8$$

Example

Find the binomial expansion of $\left(\frac{1}{x} - 2\sqrt{x}\right)^5$

Solution

$$\left(\frac{1}{x} - 2\sqrt{x}\right)^5 = \left(\frac{1}{x}\right)^5 + 5\left(\frac{1}{x}\right)^4\left(-2\sqrt{x}\right)^1 + 10\left(\frac{1}{x}\right)^3\left(-2\sqrt{x}\right)^2 + 10\left(\frac{1}{x}\right)^2\left(-2\sqrt{x}\right)^3$$

$$+ 5\left(\frac{1}{x}\right)^1\left(-2\sqrt{x}\right)^4 + \left(-2\sqrt{x}\right)^5$$

$$= \frac{1}{x^5} - 10\frac{1}{x^4}\left(\sqrt{x}\right) + 10\frac{1}{x^3}(4x) - 10\frac{1}{x^2}\left(8x\sqrt{x}\right) + 5\left(\frac{1}{x}\right)\left(16x^2\right) - 32x^{5/2}$$

$$= \frac{1}{x^5} - 10\frac{1}{x^{7/2}} + 40\frac{1}{x^2} - 80\frac{1}{x^{1/2}} + 80x - 32x^{5/2}$$

Pascal's Triangle

```
                        1
                   1         1
              1         2         1
         1         3         3         1
    1         4         6         4         1
  1      5        10        10        5        1
 1      6       15        20       15        6       1
1     7      21       35       35      21      7      1
1    8     28      56       70      56     28     8    1
1   9    36     84     126    126     84    36    9    1
1  10   45    140    210    252    210   140   45   10   1
```
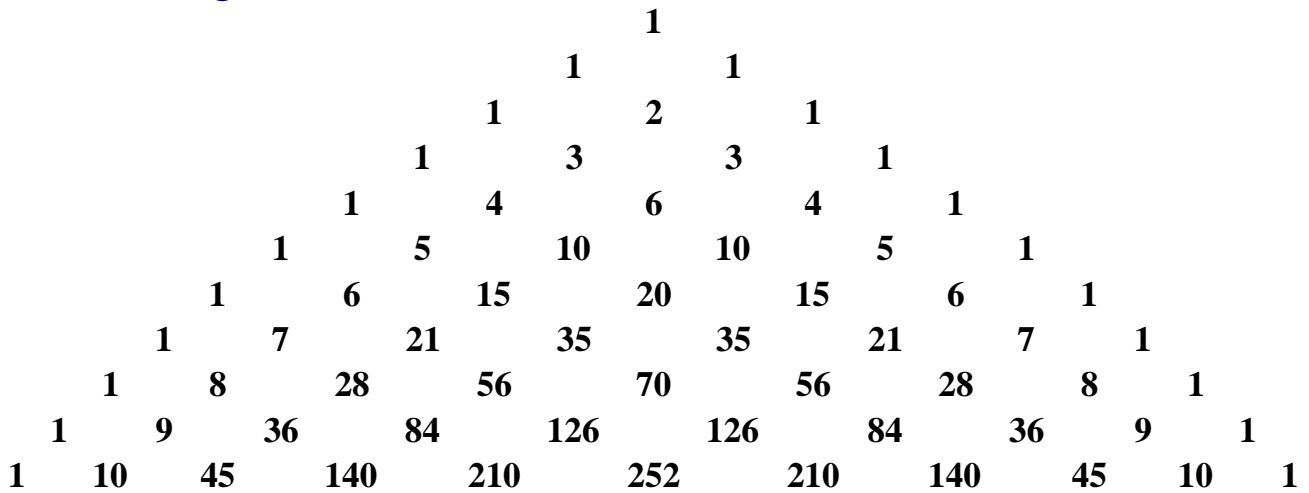
Example

Find the eighth row of the Pascal's triangle, and use it to expand $(a+b)^7$

Solution

```
  1      6      15      20      15      6      1
1      7      21      35      35      21      7      1
```
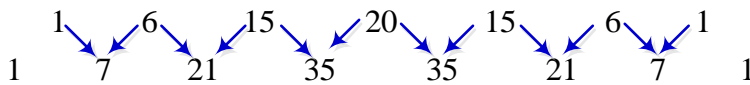
$$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

Exercises   *Section* **4.4 – The Binomial Theorem**

1. Find the fifth term in the expansion $\left(x^3 + \sqrt{y}\right)^{13}$

2. Find the term involving q^{10} in the binomial expansion $\left(\frac{1}{3}p + q^2\right)^{12}$

Expand and simplify:

3. $(4x - y)^3$

4. $(x + y)^6$

5. $(x - y)^7$

6. $(3t - 5x)^4$

7. $\left(\frac{1}{3}x + y^2\right)^5$

8. $\left(\frac{1}{x^2} + 3x\right)^6$

9. $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^5$

10. $(2y - 3)^4$

11. $(x + 2)^5$

12. $\left(x^2 - y^2\right)^6$

13. $(ax - by)^4$

14. $(ax + by)^5$

15. $\left(\sqrt{x} - \sqrt{3}\right)^4$

16. $\left(\sqrt{x} - \sqrt{2}\right)^6$

17. $(2x - 1)^{12}$

18. $\left(x - \frac{1}{x^2}\right)^9$

19. $\left(\frac{2}{x} - 3y\right)^5$

20. $\left(3\sqrt{x} + \sqrt[4]{x}\right)^4$

21. $(x + 1)^5$

22. $(x - 1)^5$

23. $(x - 2)^6$

24. $\left(\frac{1}{x^3} - 2x\right)^5$

Section **4.5 – Partial Fraction Decomposition**

1- Decompose $\frac{P}{Q}$, where Q has Only Non-repeated Linear Factor

Under the assumption that Q has only non-repeated linear factors, the polynomial Q has the form

$$Q(x) = (x - a_1)(x - a_2) \ \cdots \ (x - a_n)$$

Where no 2 of the number a_1, a_2,…, a_n are equal. In this case, the partial fraction decomposition of $\frac{P}{Q}$ is of the form

$$\frac{P}{Q} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} \ \cdots + \frac{A_n}{x - a_n}$$

Where the numbers A_1, A_2,…, A_n are to be determined.

Example

Write the partial fraction decomposition of $\dfrac{x}{x^2 - 5x + 6}$

Solution

First factor the denominator, $x^2 - 5x + 6 = (x - 2)(x - 3)$

$$\frac{x}{x^2 - 5x + 6} = \frac{A}{x - 2} + \frac{B}{x - 3}$$

$$\frac{x}{x^2 - 5x + 6} = \frac{A(x - 3) + B(x - 2)}{(x - 2)(x - 3)}$$

$$x = A(x - 3) + B(x - 2)$$

$$x = Ax - 3A + Bx - 2B$$

$$x = (A + B)x - 3A - 2B \qquad\qquad 1x + 0 = (A + B)x - 3A - 2B$$

$$\rightarrow \begin{cases} 1 = A + B \\ 0 = -3A - 2B \end{cases}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ -3 & -2 & 0 \end{pmatrix} \xrightarrow{\ rref\ } \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \qquad \begin{array}{l} A+B=1 \\ A=-\frac{2}{3}B \end{array} \rightarrow \begin{array}{l} -\frac{2}{3}B+B=1 \Rightarrow \frac{1}{3}B=1 \Rightarrow B=3 \\ A=-\frac{2}{3}(3)=-2 \end{array}$$

Solve for A and B using any method, we get $A = -2 \quad B = 3$

Therefore; $\dfrac{x}{x^2 - 5x + 6} = \dfrac{-2}{x - 2} + \dfrac{3}{x - 3}$

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2- Decompose $\frac{P}{Q}$, where Q has Repeated Linear Factors

If a polynomial Q has a repeated linear factor, say $(x-a)^n$, $n \geq 2$ n is an integer, then in the partial fraction decomposition of $\frac{P}{Q}$, we allow for the terms

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} \; \cdots + \; \frac{A_n}{(x-a)^n}$$

Where the numbers A_1, A_2,..., A_n are to be determined.

Example

Write the partial fraction decomposition of $\dfrac{x+2}{x^3 - 2x^2 + x}$

Solution

First factor the denominator, $x^3 - 2x^2 + x = x(x-1)^2$

$$\frac{x+2}{x^3 - 2x^2 + x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$x + 2 = A(x-1)^2 + Bx(x-1) + Cx$$

$$= A\left(x^2 - 2x + 1\right) + Bx^2 - Bx + Cx$$

$$= Ax^2 - 2Ax + A + Bx^2 - Bx + Cx$$

$$= (A+B)x^2 + (-2A - B + C)x + A$$

$$\begin{cases} A + B = 0 \\ -2A - B + C = 1 \\ A = 2 \end{cases} \rightarrow \begin{cases} B = -A = -2 \\ C = 1 + 2A + B = 1 + 4 - 2 = 3 \end{cases}$$

$$\frac{x+2}{x^3 - 2x^2 + x} = \frac{2}{x} + \frac{-2}{x-1} + \frac{3}{(x-1)^2}$$

$$\frac{x+2}{x^3 - 2x^2 + x} = \frac{2}{x} - \frac{2}{x-1} + \frac{3}{(x-1)^2}$$

Example

Write the partial fraction decomposition of $\dfrac{x^3-8}{x^2(x-1)^3}$

Solution

$$\frac{x^3-8}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

$$x^3-8 = Ax(x-1)^3 + B(x-1)^3 + Cx^2(x-1)^2 + Dx^2(x-1) + Ex^2$$

Let $x=0 \rightarrow -8 = B(-1)^3 \Rightarrow B=8$

$$x^3-8 = Ax(x-1)^3 + 8(x-1)^3 + Cx^2(x-1)^2 + Dx^2(x-1) + Ex^2$$

Let $x=1 \rightarrow 1-8 = E \Rightarrow E=-7$

$$x^3-8 = Ax\left(x^3-3x^2+3x-1\right) + 8\left(x^3-3x^2+3x-1\right) + Cx^2\left(x^2-2x+1\right) + Dx^2(x-1) - 7x^2$$

$$x^3-8-8\left(x^3-3x^2+3x-1\right)+7x^2$$

$$= Ax^4-3Ax^3+3Ax^2-Ax+Cx^4-2Cx^3+Cx^2+Dx^3-Dx^2$$

$$x^3-8-8x^3+24x^2-24x+8+7x^2$$
$$= (A+C)x^4 + (-3A-2C+D)x^3 + (3A+C-D)x^2 - Ax$$

$$-7x^3+31x^2-24x = (A+C)x^4 + (-3A-2C+D)x^3 + (3A+C-D)x^2 - Ax$$

$$\rightarrow \begin{cases} A+C=0 \quad\quad C=-A=-24 \\ -3A-2C+D=-7 \\ 3A+C-D=31 \\ -A=-24 \quad\quad \rightarrow A=24 \end{cases}$$

$$D = -7+3A+2C = -7+72-48 = 17$$

$$\frac{x^3-8}{x^2(x-1)^3} = \frac{24}{x} + \frac{8}{x^2} - \frac{24}{x-1} + \frac{17}{(x-1)^2} - \frac{7}{(x-1)^3}$$

3- Decompose $\frac{P}{Q}$, where Q has a Non-repeated Irreducible Quadratic Factor

If Q contains a no-repeated irreducible quadratic factor of the form $ax^2 + bx + c$, then in the partial fraction decomposition of $\frac{P}{Q}$, we allow for the term

$$\frac{Ax + B}{ax^2 + bx + c}$$

Where the numbers A and B are to be determined.

Example

Write the partial fraction decomposition of $\dfrac{3x-5}{x^3-1}$

Solution

$$\frac{3x-5}{x^3-1} = \frac{3x-5}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$3x-5 = A(x^2+x+1) + (x-1)(Bx+C)$$

$$3x-5 = Ax^2 + Ax + A + Bx^2 + Cx - Bx - C$$

$$\begin{cases} A+B=0 \\ A-B+C=3 \\ A-C=-5 \end{cases} \quad \begin{pmatrix} 1 & 1 & 0 & | & 0 \\ 1 & -1 & 1 & | & 3 \\ 1 & 0 & -1 & | & -5 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 0 & | & -\frac{2}{3} \\ 0 & 1 & 0 & | & \frac{2}{3} \\ 0 & 0 & 1 & | & \frac{13}{3} \end{pmatrix}$$

$$A = -\frac{2}{3} \quad B = \frac{2}{3} \quad C = \frac{13}{3}$$

$$\frac{3x-5}{x^3-1} = \frac{-\frac{2}{3}}{x-1} + \frac{\frac{2}{3}x+\frac{13}{3}}{x^2+x+1}$$

$$= -\frac{2}{3}\frac{1}{x-1} + \frac{1}{3}\frac{2x+13}{x^2+x+1}$$

4- Decompose $\frac{P}{Q}$, where Q has a Repeated Irreducible Quadratic Factor

If Q contains a repeated irreducible quadratic factor of the form $\left(ax^2 + bx + c\right)^n$, $n \geq 2$, n an integer,

then in the partial fraction decomposition of $\frac{P}{Q}$, we allow for the terms

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{\left(ax^2 + bx + c\right)^2} + \cdots + \frac{A_n x + B_n}{\left(ax^2 + bx + c\right)^n}$$

Where the numbers A_1, B_1, A_2, B_2, ..., A_n, B_n are to be determined.

Example

Write the partial fraction decomposition of $\dfrac{x^3 + x^2}{\left(x^2 + 4\right)^2}$

Solution

$$\frac{x^3 + x^2}{\left(x^2 + 4\right)^2} = \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{\left(x^2 + 4\right)^2}$$

$$x^3 + x^2 = (Ax + B)\left(x^2 + 4\right) + Cx + D$$

$$= Ax^3 + 4Ax + Bx^2 + 4B + Cx + D$$

$$= Ax^3 + Bx^2 + (4A + C)x + 4B + D$$

$$\begin{cases} A = 1 \\ B = 1 \\ 4A + C = 0 \\ 4B + D = 0 \end{cases} \rightarrow \begin{array}{l} \\ \\ C = -4A = -4 \\ D = -4B = -4 \end{array}$$

$$A = 1, \quad B = 1, \quad C = -4, \quad D = -4$$

$$\frac{x^3 + x^2}{\left(x^2 + 4\right)^2} = \frac{x + 1}{x^2 + 4} + \frac{-4x - 4}{\left(x^2 + 4\right)^2}$$

Exercises   *Section* **4.5 – Partial Fraction Decomposition**

Write the partial fraction decomposition of each rational expression

1. $\dfrac{4}{x(x-1)}$

2. $\dfrac{3x}{(x+2)(x-1)}$

3. $\dfrac{1}{x\left(x^2+1\right)}$

4. $\dfrac{1}{(x+1)\left(x^2+4\right)}$

5. $\dfrac{x^2}{(x-1)^2(x+1)^2}$

6. $\dfrac{x+1}{x^2(x-2)^2}$

7. $\dfrac{x-3}{(x+2)(x+1)^2}$

8. $\dfrac{x^2+x}{(x+2)(x-1)^2}$

9. $\dfrac{10x^2+2x}{(x-1)^2\left(x^2+2\right)}$

10. $\dfrac{x^2+2x+3}{(x+1)\left(x^2+2x+4\right)}$

11. $\dfrac{x^2-11x-18}{x\left(x^2+3x+3\right)}$

12. $\dfrac{1}{(2x+3)(4x-1)}$

13. $\dfrac{x^2+2x+3}{\left(x^2+4\right)^2}$

14. $\dfrac{x^3+1}{\left(x^2+16\right)^2}$

15. $\dfrac{7x+3}{x^3-2x^2-3x}$

16. $\dfrac{x^2}{x^3-4x^2+5x-2}$

17. $\dfrac{x^3}{\left(x^2+16\right)^3}$

18. $\dfrac{4}{2x^2-5x-3}$

19. $\dfrac{2x+3}{x^4-9x^2}$

20. $\dfrac{x^2+9}{x^4-2x^2-8}$

21. $\dfrac{y}{y^2-2y-3}$

22. $\dfrac{x+3}{2x^3-8x}$

23. $\dfrac{x^2}{(x-1)\left(x^2+2x+1\right)}$

24. $\dfrac{3x^2+x+4}{x^3+x}$

25. $\dfrac{8x^2+8x+2}{\left(4x^2+1\right)^2}$

Analytic Geometry

Section **4.6 – Circles and Parabolas**

The *Distance* Formula

The distance, d, between the points (x_1, y_1) and (x_2, y_2) in the rectangular coordinate system is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

To complete the distance between two points. Find the square of the difference between the x-coordinate plus the square of the difference between the y-coordinates. The principal square root of this sum is the distance.

Midpoint **Formula**

Consider a line segment whose endpoints are (x_1, y_1) and (x_2, y_2). The coordinates of the segment's midpoint are

$$\left(\frac{x_1 + x_2}{2}, \ \frac{y_1 + y_2}{2} \right)$$

To find the midpoint, take the average of the two x-coordinates and the average of the y-coordinates

The Standard Form of the *Equation of a Circle*

The *standard form of the equation of a circle* with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

A circle with center (0, 0) and radius r has equation:
$$x^2 + y^2 = r^2$$
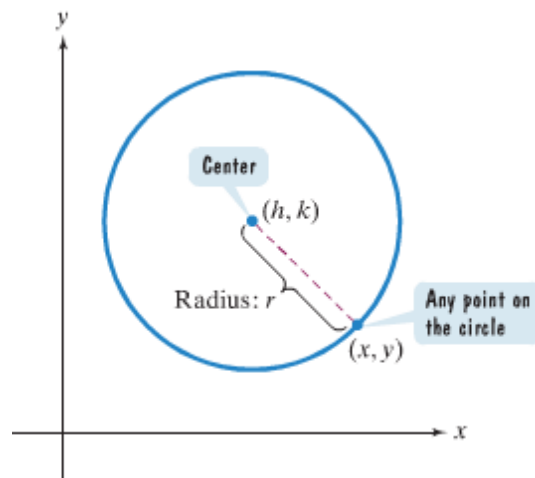
Example

Find the center-radius form of the equation of each circle.

a) Center at $(-3, 4)$, radius 6

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-(-3))^2 + (y-4)^2 = 6^2$$

$$(x+3)^2 + (y-4)^2 = 36$$
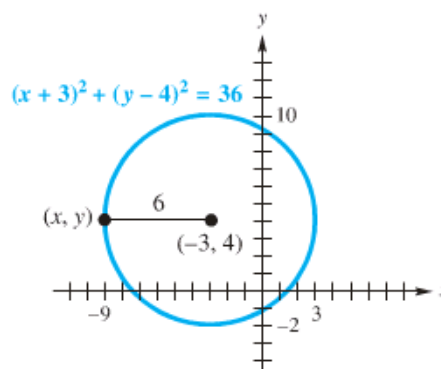
b) Center at $(0, 0)$, radius 3

$$(x-0)^2 + (y-0)^2 = 3^2$$

$$x^2 + y^2 = 9$$
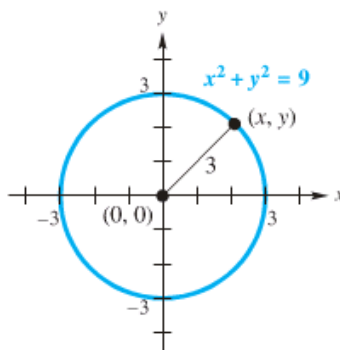
Example

Find the equation of a circle with center $(-1, 4)$ that passes through $(3, 7)$.

Solution

$$r = d = \sqrt{(-1-3)^2 + (4-7)^2}$$

$$= \sqrt{(-4)^2 + (-3)^2}$$

$$= \sqrt{16+9}$$

$$= \sqrt{25}$$

$$= 5$$

$$\underline{(x+1)^2 + (y-4)^2 = 25}$$

Example

Write in standard form: $x^2 + y^2 + 4x - 4y - 1 = 0$

Solution

$$x^2 + 4x \quad + y^2 - 4y \quad = 1$$

$$x^2 + 4x + \left(\frac{4}{2}\right)^2 + y^2 - 4y + \left(\frac{-4}{2}\right)^2 = 1 + \left(\frac{4}{2}\right)^2 + \left(\frac{-4}{2}\right)^2$$

$$x^2 + 4x + 2^2 + y^2 - 4y + 2^2 = 1 + 4 + 4$$

$$\underline{(x+2)^2 + (y-2)^2 = 9}$$

Definition of a **Parabola**

A *parabola* is the set of all points in a plane equidistant from a fixed point F (the *focus*) and a fixed line l (the *directrix*) that lie in the plane

Axis

Axis

$x^2 = 4py$

$F(0, p)$ $P(x, y)$

V

$y = -p$

$P'(x, -p)$

P

F

P'

V

Directrix
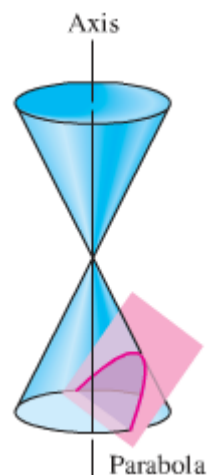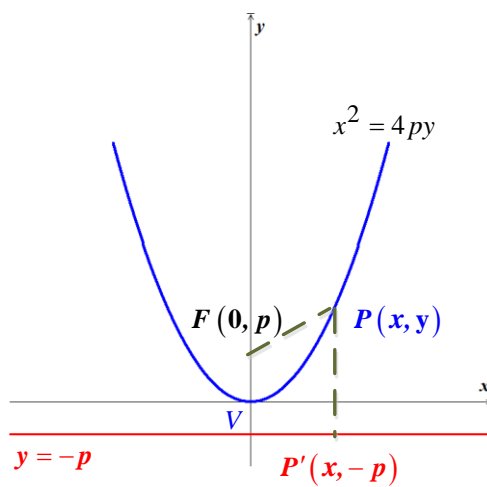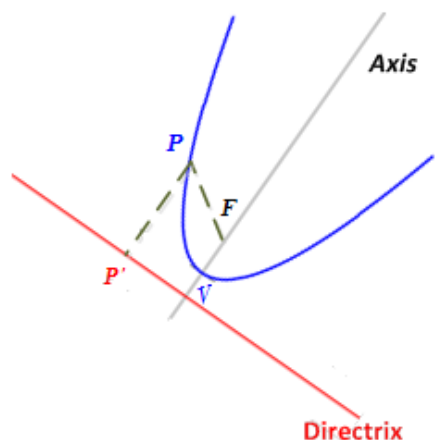
Parabola

$$y = \frac{1}{4p}x^2 \quad or \quad x^2 = 4py \quad \rightarrow \begin{cases} \textit{\textbf{Focus}}: \ F(0, \ p) \\ \textit{\textbf{Directrix}}: \ y = -p \end{cases}$$

$$x = \frac{1}{4p}y^2 \quad or \quad y^2 = 4px \quad \rightarrow \begin{cases} \textit{\textbf{Focus}}: \ F(p, \ 0) \\ \textit{\textbf{Directrix}}: \ x = -p \end{cases}$$

The standard equation $y = ax^2$ or $x = ay^2$ is a parabola with vertex $V(0, 0)$. Moreover,

$$a = \frac{1}{4p} \quad or \quad p = \frac{1}{4a}$$

Equation, focus, Directrix	Graph for $p > 0$	Graph for $p < 0$
$x^2 = 4py$ $\quad or \quad$ $y = \dfrac{1}{4p}x^2$ Focus: $F(0,\ p)$ Directrix: $y = -p$		
$y^2 = 4px$ $\quad or \quad$ $x = \dfrac{1}{4p}y^2$ Focus: $F(p,\ 0)$ Directrix: $x = -p$		
$(y-k)^2 = 4p(x-h)$ $x = ay^2 + by + c$ Focus: $F(h+p,\ k)$ Directrix: $x = h - p$		
$(x-h)^2 = 4p(y-k)$ $y = ax^2 + bx + c$ Focus: $F(h,\ k+p)$ Directrix: $y = k - p$		

Example

Find the focus and directrix of the parabola $y = -\frac{1}{6}x^2$.

Solution

$$a = -\frac{1}{6} \implies \underline{\left\lfloor p = \frac{1}{4a} = \frac{1}{4\left(-\frac{1}{6}\right)} = -\frac{6}{4} = -\frac{3}{2}\right.}$$

The parabola opens downward and has focus $F\left(0, \ -\frac{3}{2}\right)$.

The directrix is the horizontal line $y = \frac{3}{2}$ which is a distance $\frac{3}{2}$ above *V*.

Example

a) Find an equation of a parabola that has vertex at the origin, open right, and passes through the point $P(7, -3)$.

b) Find the focus of the parabola.

Solution

a) An equation of a parabola with vertex at the origin that opens right has the form $x = ay^2$

$$7 = a(-3)^2$$
$$a = \frac{7}{9}$$

The equation is: $x = \frac{7}{9}y^2$

b) $\underline{\left\lfloor p = \frac{1}{4a} = \frac{1}{4\left(\frac{7}{9}\right)} = \frac{9}{28}\right.}$

Thus, the focus has coordinate $\left(\frac{9}{28}, \ 0\right)$

Example

Sketch the graph of $2x = y^2 + 8y + 22$

Solution

$$2x - 22 = y^2 + 8y$$

$$y^2 + 8y + \left(\tfrac{1}{2}8\right)^2 = 2x - 22 + \left(\tfrac{1}{2}8\right)^2$$

$$(y+4)^2 = 2x - 6$$

$$(y+4)^2 = 2(x-3)$$

The vertex is $V(h,k) = V(3, -4)$

The focus is $F(h+p,k) = F\left(3+\tfrac{1}{2},-4\right) = F\left(\tfrac{7}{2}, -4\right)$

The directrix is $x = h - p = 3 - \tfrac{1}{2} = \tfrac{5}{2}$
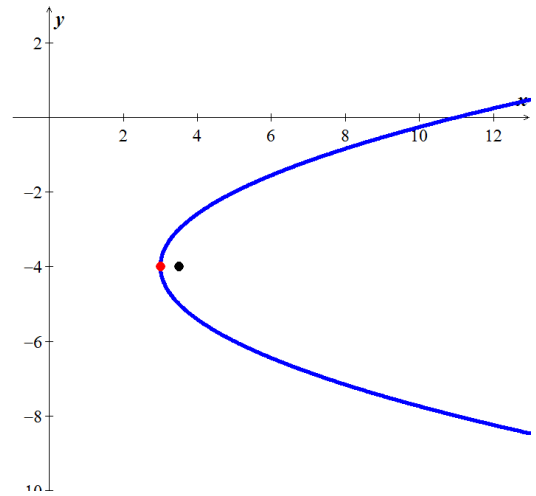
Example

A parabola has vertex $V(-4, 2)$ and directrix $y = 5$. Express the equation of the parabola in the form $y = ax^2 + bx + c$

Solution

Directrix: $y = k - p \Rightarrow \boxed{p = k - y = 2 - 5 = -3}$

$$(x-h)^2 = 4p(y-k)$$

$$(x+4)^2 = 4(-3)(y-2)$$

$$(x+4)^2 = -12(y-2)$$

$$x^2 + 8x + 16 = -12y + 24$$

$$x^2 + 8x + 16 - 24 = -12y + 24 - 24$$

$$-12y = x^2 + 8x - 8$$

$$y = -\tfrac{1}{12}x^2 - \tfrac{2}{3}x + \tfrac{2}{3}$$

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Example

The towers of the Golden Gate Bridge connecting San Francisco to Marin County are 1280 *meters* apart and rise 160 *meters* above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 200 *meters* from a tower?
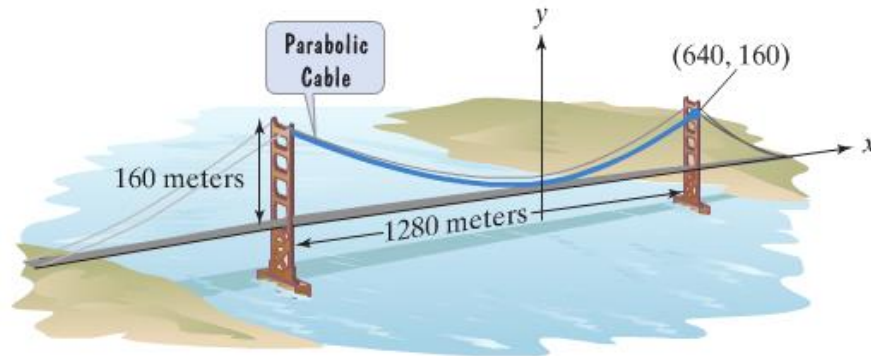
Solution

Given the point: $(640, \ 160)$

$$(640)^2 = 4p(160) \qquad\qquad x^2 = 4py$$

$$p = \frac{640^2}{640} = \underline{640|}$$

$$x = 640 - 200 = 440$$

$$(440)^2 = 4(640)y \qquad\qquad x^2 = 4py$$

$$y = \frac{440^2}{4(640)} \approx \underline{75.625|}$$

The height is 76 *meters*.

Exercises   *Section* **4.6 – Circles and Parabolas**

Find the center and the radius of

1. $x^2 + y^2 + 6x + 2y + 6 = 0$ **3.** $x^2 + y^2 - 10x - 6y - 30 = 0$

2. $x^2 + y^2 + 8x + 4y + 16 = 0$ **4.** $x^2 - 6x + y^2 + 10y + 25 = 0$

Find the vertex, focus, and directrix of the parabola. Sketch its graph.

5. $20x = y^2$ **10.** $y = x^2 - 4x + 2$ **15.** $(y+1)^2 = -4(x-2)$

6. $2y^2 = -3x$ **11.** $y^2 + 14y + 4x + 45 = 0$ **16.** $x^2 + 6x - 4y + 1 = 0$

7. $(x+2)^2 = -8(y-1)$ **12.** $x^2 + 20y = 10$ **17.** $y^2 + 2y - x = 0$

8. $(x-3)^2 = \frac{1}{2}(y+1)$ **13.** $x^2 = 16y$ **18.** $y^2 - 4y + 4x + 4 = 0$

     14. $x^2 = -\frac{1}{2}y$ **19.** $x^2 - 4x - 4y = 4$

9. $(y+1)^2 = -12(x+2)$

Find an equation of the parabola that satisfies the given conditions

20. *Focus* : $F(2,0)$ *directrix* : $x = -2$ **25.** *Vertex* : $V(-1,0)$ *focus* : $F(-4,0)$

21. *Focus* : $F(0,-40)$ *directrix* : $y = 4$ **26.** *Vertex* : $V(1,-2)$ *focus* : $F(1,0)$

22. *Focus* : $F(-3,-2)$ *directrix* : $y = 1$ **27.** *Vertex* : $V(0, 1)$ *focus* : $F(0, 2)$

23. *Vertex* : $V(3,-5)$ *directrix* : $x = 2$ **28.** *Vertex* : $V(3, 2)$ *focus* : $F(-1, 2)$

24. *Vertex* : $V(-2,3)$ *directrix* : $y = 5$

29. An arch in the shape of a parabola has the dimensions shown in the figure. How wide is the arch 9 *feet* up?

12 ft

12 ft
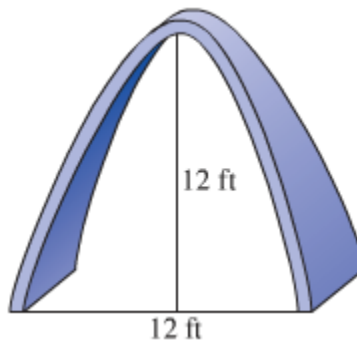
30. The cable in the center portion of a bridge is supported as shown in the figure to form a parabola. The center support is 10 *feet* high, the tallest supports are 210 *feet* high, and the distance between the two tallest supports is 400 *feet*. Find the height of the remaining supports if the supports are evenly spaced.

210

10

400
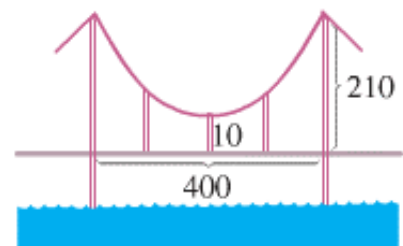
31. A headlight is being constructed in the shape of a paraboloid with depth 4 *inches* and diameter 5 *inches*. Determine the distance *d* that the bulb should be form the vertex in order to have the beam of light shine straight ahead.
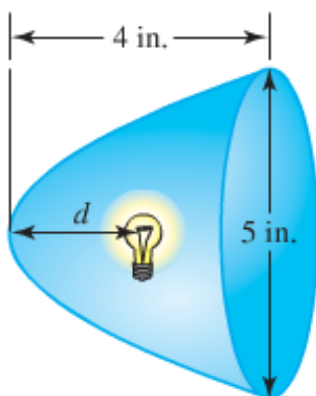
32. A satellite dish is shaped like a paraboloid of revolution. The signals that emanate from a satellite strike the surface of the dish and are reflected to a single point, where the receiver is located. If the dish is 8 *feet* across at its opening and 3 *feet* deep at its center, at what position should the receiver be placed? That is, where is the focus?
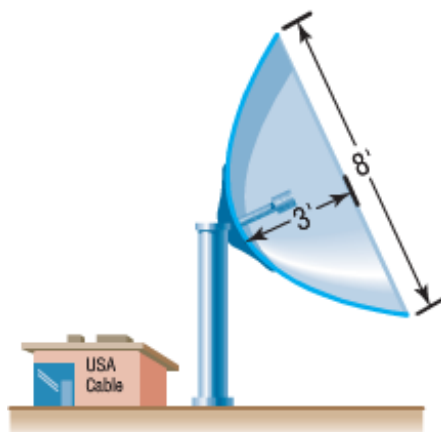
33. A cable TV receiving dish is in the shape of a paraboloid of revolution. Find the location of the receiver, which is placed at the focus, if the dish is 6 *feet* across at its opening and 2 *feet* deep.

34. The cables of a suspension bridge are in the shape of a parabola, as shown below. The towers supporting the cable are 600 *feet* apart and 80 *feet* high. If the cables touch the road surface midway between the towers, what is the height of the cable from the road at a point 150 feet from the center of the bridge?
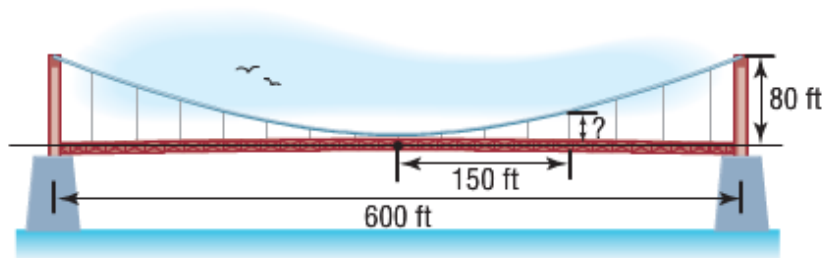
35. A bridge is built in the shape of a parabolic arch. The bridge has a span of 120 *feet* and a maximum height of 25 *feet*. Choose a suitable rectangular coordinate system and find the height of the arch at distances of 10, 30, and 50 *feet* from the center.

25 ft
120 ft
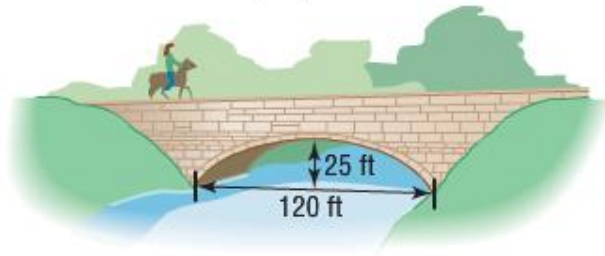
36. A mirror is shaped like a paraboloid of revolution and will be used to concentrate the rays of the sun at its focus, creating a heat source. If the mirror is 20 *feet* across at its opening and is 6 *feet* deep, where will the heat source be concentrated?

Sun's rays
20'
6'
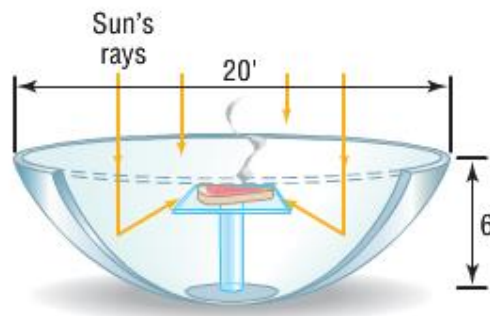
37. A reflecting telescope contains a mirror shaped a paraboloid of revolution. If the mirror is 4 *inches* across at its opening and is 3 *inches* deep, where will the collected light be concentrated?

38. Show that the graph of an equation of the form $Ax^2 + Dx + Ey + F = 0$ $A \neq 0$

   a) Is a parabola if $E \neq 0$

   b) Is a vertical line if $E = 0$ *and* $D^2 - 4AF = 0$

   c) Is two vertical lines if $E = 0$ *and* $D^2 - 4AF > 0$

   d) Contains no points if $E = 0$ *and* $D^2 - 4AF < 0$

39. The towers of a suspension bridge are 800 *feet* apart and rise 160 *feet* above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 100 *feet* from a tower?

Parabolic Cable
(400, 160)
Road
800 feet
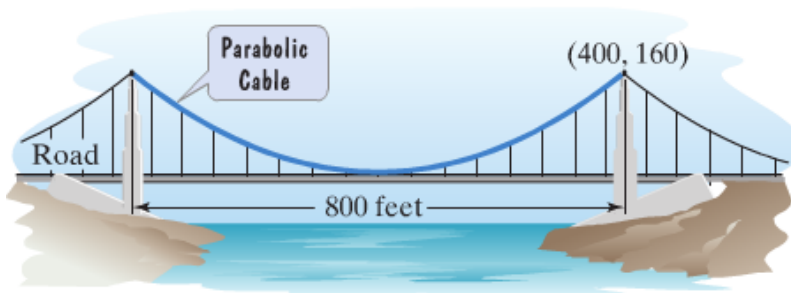
40. The cables of a suspension bridge are in the shape of a parabola. The towers supporting the cable are 400 *feet* apart and 100 *feet* high. If the cables are at a height of 10 *feet* midway between the towers, what is the height of the cable at a point 50 *feet* from the center of the bridge?

41. The focal length of the (finite) paraboloid is the distance *p* between its vertex and focus
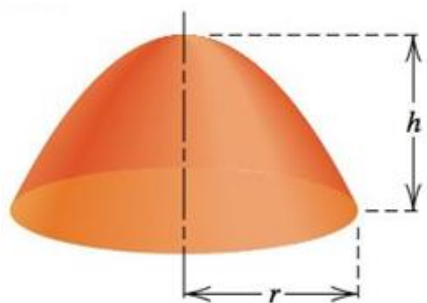
a) Express *p* in terms of *r* and *h*.
b) A reflector is to be constructed with a focal length of 10 *feet* and a depth of 5 *feet*. Find the radius of the reflector.

42. The parabolic arch is 50 *feet* above the water at the center and 200 *feet* wide at the base. Will a boat that is 30 *feet* tall clear the arch 30 *feet* from the center? *45.5 ft*
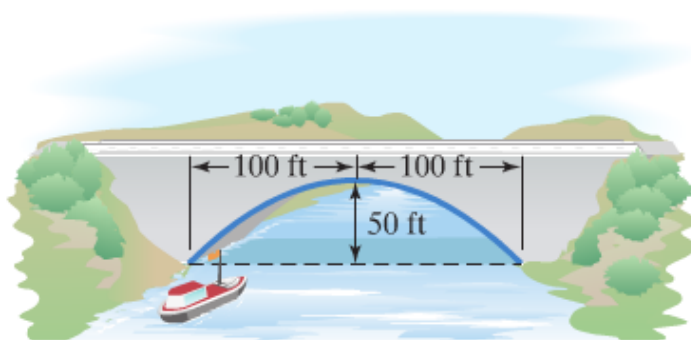
43. A satellite dish, as shown below, is in the shape of a parabolic surface. Signals coming from a satellite strike the surface of the dish and are reflected to the focus, where the receiver if located. The satellite dish shown has a diameter of 12 *feet* and a depth of 2 *feet*. How far from the base of the dish should the receiver be placed? *45.5 ft*
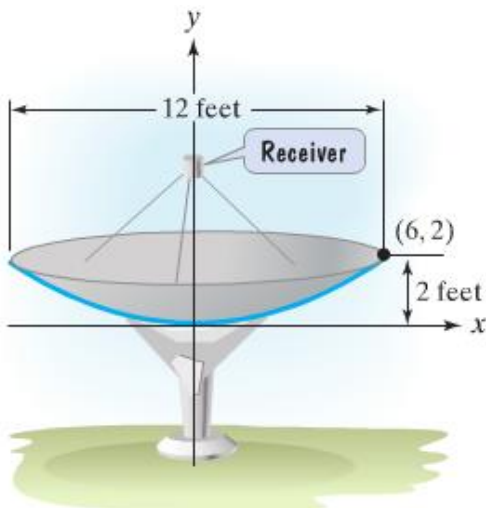
44. A searchlight is shaped like a paraboloid of revolution. If the light source is located 2 *feet* from the base along the axis of symmetry and the opening is 5 *feet* across, how deep should the searchlight be?   0.78125 *ft*

45. A searchlight is shaped like a paraboloid of revolution. If the light source is located 2 *feet* from the base along the axis of symmetry and the depth of the searchlight is 4 *feet* across, how deep should the opening be?   11.31 *ft*

46. A searchlight is shaped like a paraboloid, with the light source at the focus. If the reflector is 3 *feet* across at the opening and 1 *foot* deep, where is the focus? $\frac{9}{16}$ *ft*

47. A mirror for a reflecting telescope has the shape of a (finite) paraboloid of diameter 8 *inches* and depth 1 *inch*. How far from the center the mirror will the incoming light collect?   4 *in*
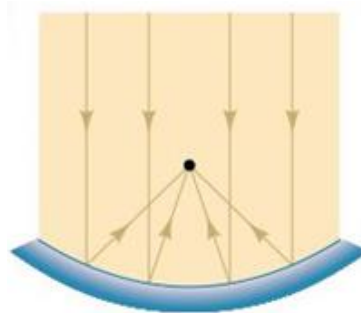
Section **4.7** *–* **Ellipses**

Definition **of an** *Ellipse*

An *ellipse* is the set of all points in a plane, the sum of whose distances from two fixed points (the *foci*) in the plane is a positive constant.
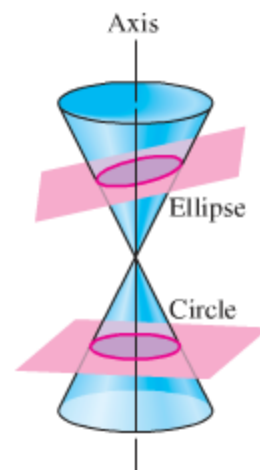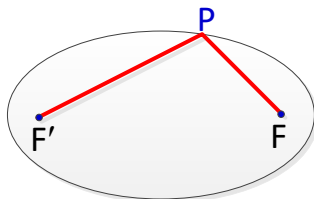
An ellipse is a curve that is the locus of all points in the plane the sum of the distances $d(P,F)$ and $d(P,F')$ from two fixed points $F'(-c,\ 0)$ and $F(c,\ 0)$ (the *foci*) separated by a distance of $2c$, with the center of the ellipse at the origin, is the distance length of the string and hence is constant. The constant of the distances of P from F and F' will be denoted by $2a$.
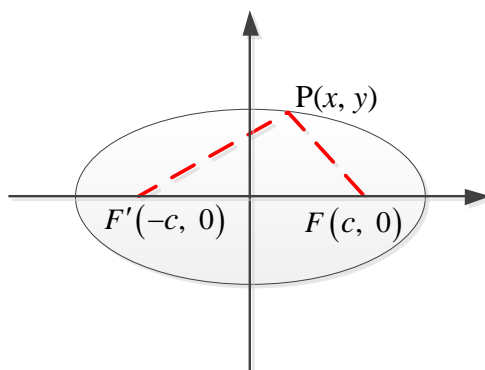
$$d(P,F) + d(P,F') = 2a$$

$$\sqrt{(x-c)^2 + (y-0)^2} + \sqrt{(x+c)^2 + (y-0)^2} = 2a$$

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

$$\left(\sqrt{(x-c)^2 + y^2}\right)^2 = \left(2a - \sqrt{(x+c)^2 + y^2}\right)^2$$

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

$$-2cx = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + 2cx$$

$$4a\sqrt{(x+c)^2 + y^2} = 4a^2 + 2cx + 2cx$$

$$\left(a\sqrt{(x+c)^2+y^2}\right)^2=\left(a^2+cx\right)^2$$

$$a^2\left(x^2+2cx+c^2+y^2\right)=a^4+2a^2cx+c^2x^2$$

$$a^2x^2+2a^2cx+a^2c^2+a^2y^2=a^4+2a^2cx+c^2x^2$$

$$a^2x^2-c^2x^2+a^2y^2=a^4-a^2c^2$$

$$x^2\left(a^2-c^2\right)+a^2y^2=a^2\left(a^2-c^2\right)$$

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$$

Since $a>c \Rightarrow a^2-c^2>0$, we let $b=\sqrt{a^2-c^2}\Rightarrow b^2=a^2-c^2$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The coordinates of every point (x, y) on the ellipse satisfy the equation.
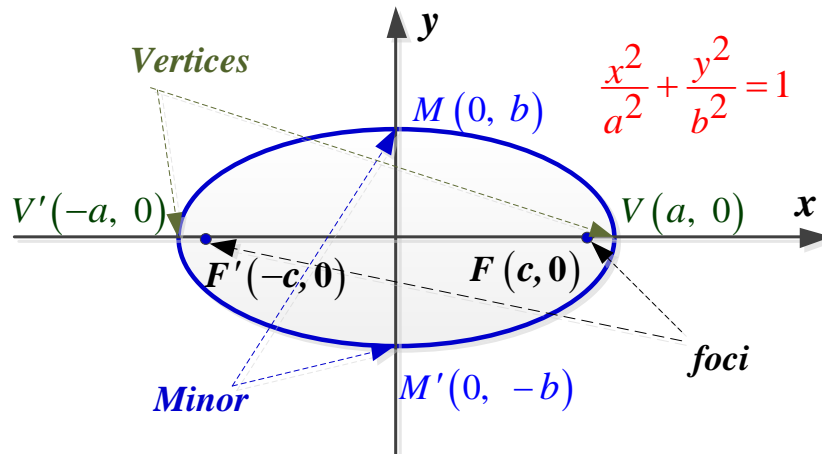
The **x-intercepts** are a and $-a$. The corresponding points $V\left(a,\ 0\right)$ and $V'\left(-a,\ 0\right)$ are called the **vertices** of the ellipse. The line segment $V'V$ is called the **major axis**.

The **y-intercepts** are b and $-b$. The corresponding points $M\left(0,\ b\right)$ and $M'\left(0,\ -b\right)$ are called the **minor axis** of the ellipse.

Standard **Equations of an** *Ellipse* **with Center at the Origin**

The graph of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ *or* $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$

Where $a > b > 0$, is an ellipse with center at the origin. The length of the major axis is $2a$, and the length of the minor axis is $2b$. The foci are the distance **c** from the origin where $c^2 = a^2 - b^2$

Example

Sketch the graph of $2x^2 + 9y^2 = 18$, and find the foci.

Solution

$$\frac{2x^2}{18} + \frac{9y^2}{18} = \frac{18}{18} \qquad \text{\textit{Divide each term by 18}}$$

$$\frac{x^2}{9} + \frac{y^2}{2} = 1$$

$$\begin{cases} a^2 = 9 \rightarrow a = 3 \\ b^2 = 2 \rightarrow b = \sqrt{2} \end{cases}$$

The **vertices** are: $V'(-3,0)$ and $V(3,0)$

The **minors** are: $M'(0,-\sqrt{2})$ and $M(0,\sqrt{2})$
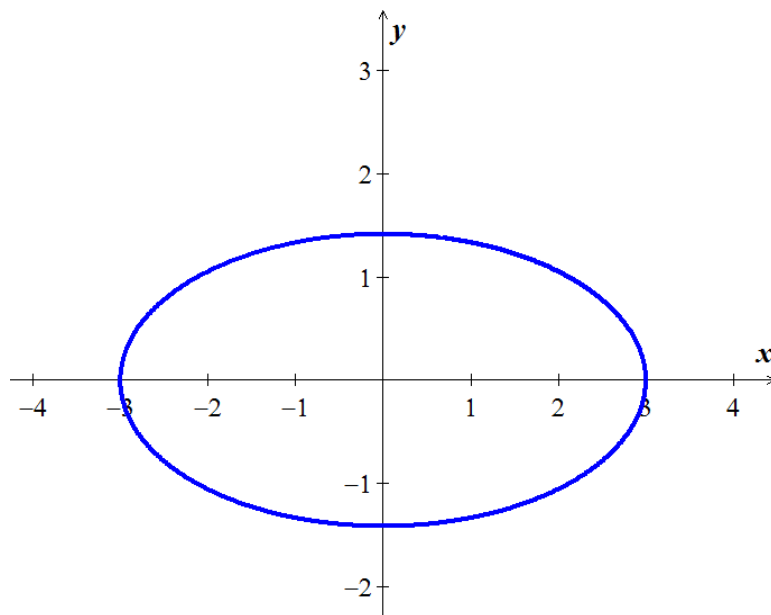
$$c = \sqrt{a^2 - b^2} = \sqrt{9-2} = \sqrt{7}$$

The **foci** are $F'\left(-\sqrt{7},0\right)$ and $F\left(\sqrt{7},0\right)$

Example

Sketch the graph of $9x^2 + 4y^2 = 25$, and find the foci.

Solution

$$\frac{9x^2}{25} + \frac{4y^2}{25} = \frac{25}{25} \qquad \textit{Divide each term by 25}$$

$$\frac{x^2}{\frac{25}{9}} + \frac{y^2}{\frac{25}{4}} = 1$$

Since $\frac{25}{4} > \frac{25}{9}$, the major axis and the foci are on the y-axis.

$$\begin{cases} a^2 = \frac{25}{4} \to a = \frac{5}{2} \\ b^2 = \frac{25}{9} \to b = \frac{5}{3} \end{cases}$$

The **vertices** are: $V'\left(0, -\frac{5}{2}\right)$ and $V\left(0, \frac{5}{2}\right)$

The **minors** are: $M'\left(-\frac{5}{3}, 0\right)$ and $M\left(\frac{5}{3}, 0\right)$
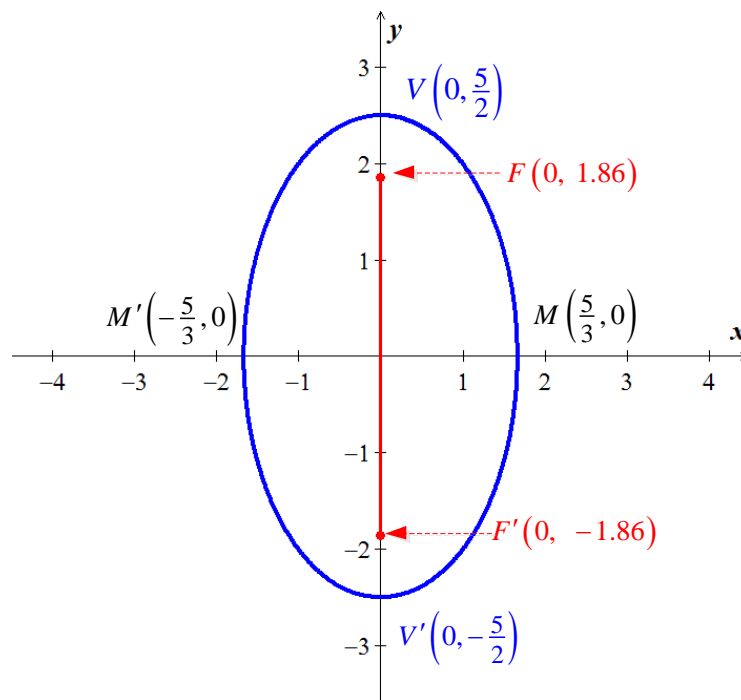
$$c = \sqrt{a^2 - b^2} = \sqrt{\frac{25}{4} - \frac{25}{9}} = 5\sqrt{\frac{1}{4} - \frac{1}{9}} = 5\sqrt{\frac{5}{36}} = \frac{5\sqrt{5}}{6}$$

The **foci** are $F'\left(0, -\frac{5\sqrt{5}}{6}\right)$ and $F\left(0, \frac{5\sqrt{5}}{6}\right)$

53

Example

Find an equation of the ellipse with vertices $(\pm 4, 0)$ and foci $(\pm 2, 0)$

Solution

Given: $a = 4, \quad c = 2$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$c^2 = a^2 - b^2 \qquad \left| b^2 = a^2 - c^2 = 4^2 - 2^2 \underline{= 12} \right.$$

$$\left. \frac{x^2}{16} + \frac{y^2}{12} = 1 \right|$$

Ellipse with **center (h, k)** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad \rightarrow \quad \dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

Example

Sketch the graph of the equation $16x^2 + 9y^2 + 64x - 18y - 71 = 0$

Solution

$$\left(16x^2 + 64x\right) + \left(9y^2 - 18y\right) = 71$$

$$16\left(x^2 + 4x + \underline{\quad}\right) + 9\left(y^2 - 2y + \underline{\quad}\right) = 71$$

$$16\left(x^2 + 4x + 4\right) + 9\left(y^2 - 2y + 1\right) = 71 + (16)4 + (9)1$$

$$16(x+2)^2 + 9(y-1)^2 = 144$$

$$\frac{16(x+2)^2}{144} + \frac{9(y-1)^2}{144} = \frac{144}{144}$$

$$\frac{(x+2)^2}{9} + \frac{(y-1)^2}{16} = 1$$
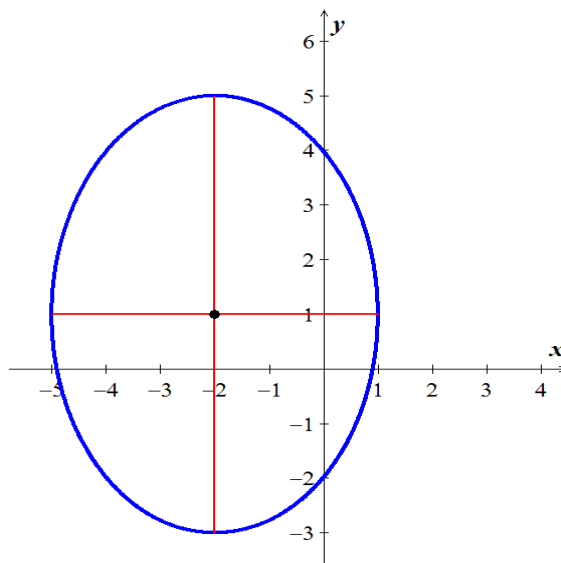
The center of the ellipse is $C(-2, 1)$ and major axis on the vertical line $x = -2$.

$a = 4, \quad b = 3$

The **vertices** are: $V'(-2, -3)$ and $V(-2, 5)$

The **minors** are: $M'(-5, -1)$ and $M(1, 1)$

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}$$

The **foci** are $\left. F\left(-2, \, 1 \pm \sqrt{7}\right) \right|$

Example

A semielliptical archway over a one-way road has a height of 10 *feet* and a width of 40 *feet*.
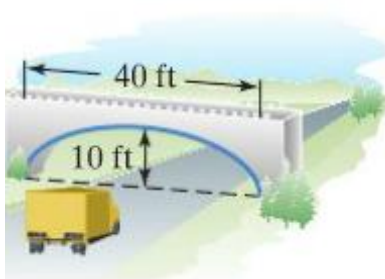
Your truck has a width of 10 *feet* and a height of 9 *feet*. Will your truck clear the opening of the archway?

Solution

Given: $a = \dfrac{40}{2} = 20, \quad b = 10$
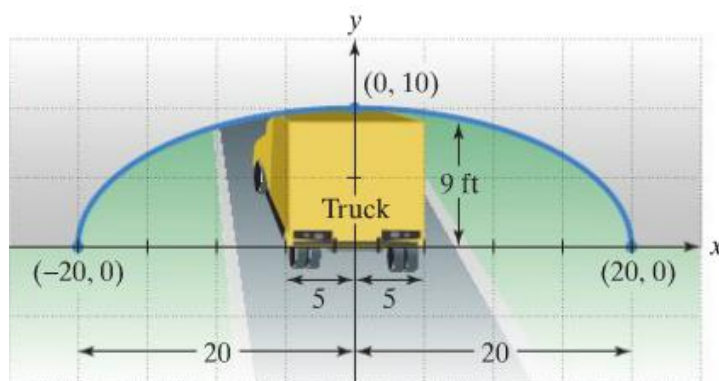
$$\frac{x^2}{40^2} + \frac{y^2}{10^2} = 1 \qquad\qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The edge of the 10-foot-truck corresponds to $x = 5$

$$\frac{5^2}{40^2} + \frac{y^2}{10^2} = 1$$

$$400\frac{25}{400} + 400\frac{y^2}{100} = 400$$

$$25 + 4y^2 = 400$$

$$y^2 = \frac{375}{4} \quad \rightarrow \quad \left\lfloor y = \sqrt{\frac{375}{4}} \approx 9.68 \ ft \right\rfloor$$

The truck will clear about 0.68 *feet* (8.16 *inches*)

Exercises　*Section* **4.7 – Ellipses**

Find the **center**, **vertices, minors** and **foci** of the ellipse, and then sketch the graph of

1.　$\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

2.　$\dfrac{x^2}{16} + \dfrac{y^2}{36} = 1$

3.　$\dfrac{x^2}{15} + \dfrac{y^2}{16} = 1$

4.　$\dfrac{25x^2}{36} + \dfrac{64y^2}{9} = 1$

5.　$12x^2 + 8y^2 = 96$

6.　$4x^2 + y^2 = 16$

7.　$4x^2 + 25y^2 = 1$

8.　$\dfrac{(x-3)^2}{16} + \dfrac{(y+4)^2}{9} = 1$

9.　$9x^2 + 4y^2 - 18x + 16y - 11 = 0$

10.　$\dfrac{(x+3)^2}{16} + \dfrac{(y-2)^2}{36} = 1$

11.　$\dfrac{(x+1)^2}{64} + \dfrac{(y-2)^2}{49} = 1$

12.　$4x^2 + 9y^2 - 32x - 36y + 64 = 0$

13.　$x^2 + 2y^2 + 2x - 20y + 43 = 0$

14.　$25x^2 + 4y^2 - 250x - 16y + 541 = 0$

15.　$4x^2 + y^2 = 2y$

16.　$2x^2 + 3y^2 - 8x + 6y + 5 = 0$

17.　$4x^2 + 3y^2 + 8x - 6y - 5 = 0$

18.　Find an equation for an ellipse with: $x - intercepts:\ \pm 4;\quad foci\ (-2,\ 0)\ and\ (2,\ 0)$

19.　Find an equation for an ellipse with: *Endpoints of major axis at* $(6,\ 0)\ and\ (-6,\ 0);\quad c = 4$

20.　Find an equation for an ellipse with: Center $(3, -2)$; $a = 5$; $c = 3$; major axis vertical

21.　Find an equation for an ellipse with: *major axis of length* 6;　*foci* $(0,\ 2)\ and\ (0,\ -2)$

22.　A patient's kidney stone is placed 12 *units* away from the source of the shock waves of a lithotripter. The lithotripter is based on an ellipse with a minor axis that measures 16 *units*. Find an equation of an ellipse that would satisfy this situation.

23.　A one-way road passes under an overpass in the form of half of an ellipse 15 *feet* high at the center and 20 *feet* wide. Assuming that a truck is 12 *feet* wide, what is the height of the tallest truck that can pass under the overpass?
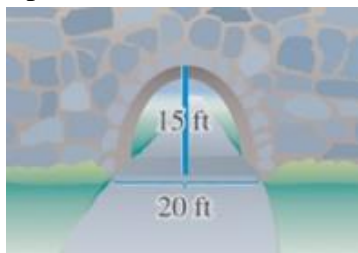

48.　The basic shape of an elliptical reflector is a hemi-ellipsoid of height h and diameter k. Waves emitted from focus F will reflect off the surface into focus F'.

　　a) Express the distance $d(V,\ F)$ and $d(V,\ F')$ in terms of h and k.

b) An elliptical reflector of height 17 *cm* is to be constructed so that waves emitted from *F* are reflected to a point *F′* that is 32 *cm* from *V*. Find the diameter of the reflector and the location of *F*.
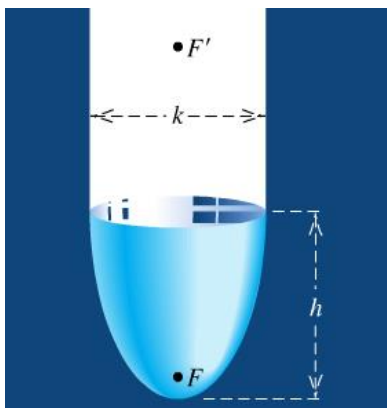

49. A lithotripter of height 15 *cm* and diameter 18 *cm* is to be constructed. High-energy underwater shock waves will be emitted from the focus *F* that is closest to the vertex *V*.
    a) Find the distance from *V* to *F*.
    b) How far from *V* (in the vertical direction) should a kidney stone located?
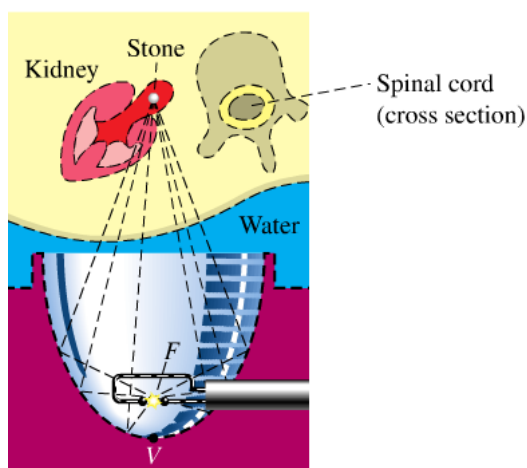
50. An Artist plans to create an elliptical design with major axis 60″ and minor axis 24″, centered on a door that measures 80″ by 36″. On a vertical line that dissects the door, approximately how far from each end of the door should the push-pins be inserted? How long should the string be?

57
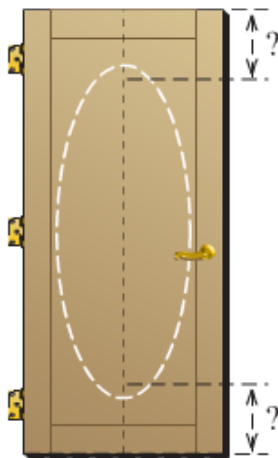
51. An arch of a bridge is semi-elliptical, with major axis horizontal. The base of the arch is 30 *feet.* across, and the highest part of the arch is 10 *feet.* above the horizontal roadway. Find the height of the arch 6 *feet.* from the center of the base.
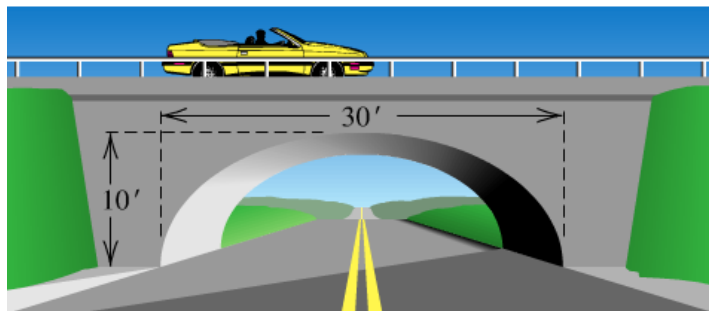
52. The whispering gallery in the Museum of Science and Industry in Chicago is 47.3 *feet* long. The distance from the center of the room to the foci is 20.3 *feet.* Find an equation that describes the shape of the room. How high is the room at its center?

53. An arch in the shape of the upper half of an ellipse is used to support a bridge that is to span a river 20 *meters* wide. The center of the arch is 6 *meters* above the center of the river. Write an equation for the ellipse in which the *x*-axis coincides with the water level and the *y*-axis passes through the center of the arch.
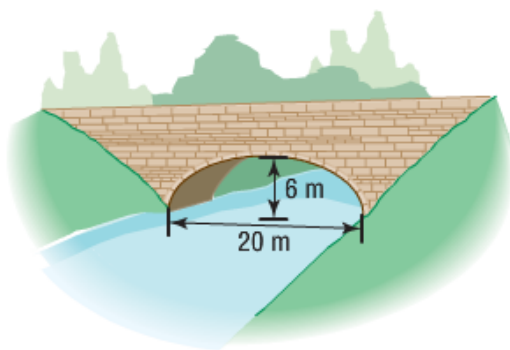
54. A bridge is built in the shape of a semielliptical arch. The bridge has a span of 120 *feet* and a maximum height of 25 *feet.* Choose a rectangular coordinate system and find the height of the arch at distances of 10, 30, and 50 feet from the center.

55. A bridge is built in the shape of a semielliptical arch. The bridge has a span of 100 *feet.* The height of the arch is 10 *feet.* Find the height of the arch at its center.

57. A racetrack is in the shape of an ellipse, 100 *feet* long and 50 feet wide. What is the width 10 *feet* from a vertex?
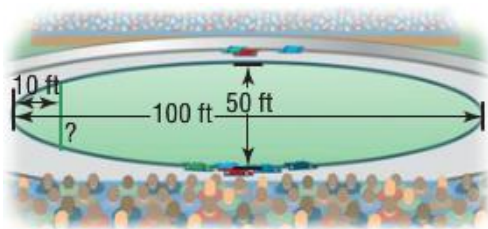

58. A homeowner is putting in a fireplace that has a 4-*inch* radius vent pipe. He needs to cut an elliptical hole in his roof to accommodate the pipe. If the pitch of his roof is $\frac{5}{4}$ (a rise of 5, run of 4) what are the dimensions of the hole?

59. A football is in the shape of a ***prolate spheroid***, which is simply a solid obtained by rotating an ellipse about its major axis. An inflated NFL football averages 11.125 inches in length and 28.25 inches in center circumference. If the volume of a prolate spheroid is $\frac{4}{3}\pi ab^2$, how much air does the football contain? (Neglect material thickness)

60. The fact that the orbit of a planet about the Sun is an ellipse with the Sun at one focus. The ***aphelion*** of a planet is its greatest distance from the Sun, and the ***perihelion*** is its shortest distance. The ***mean distance*** of a planet from the Sun is the length of the semi-major axis of the elliptical orbit.
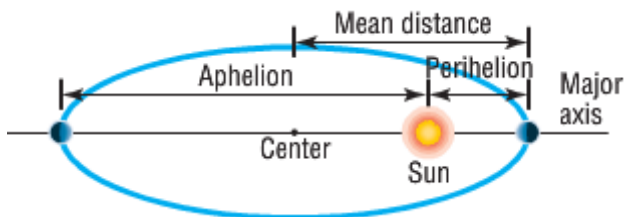

a) The mean distance of Earth from the Sun is 93 million *miles*. If the aphelion of Earth is 94.5 million miles, what is the perihelion? Write an equation for the orbit of Earth around the Sun.

b) The mean distance of Mars from the Sun is 142 million *miles*. If the perihelion of Mars is 128.5 million miles, what is the aphelion? Write an equation for the orbit of Mars about the Sun.

c) The aphelion of Jupiter is 507 million *miles*. If the distance from the center of it elliptical orbit to the Sun is 23.2 million *miles*, what is the perihelion? What is the mean distance? Write an equation for the orbit of Jupiter around the Sun.

d) The perihelion of Pluto is 4551 million *miles*, and the distance from the center of its elliptical orbit to the Sun is 897.5 million *miles*. Find the aphelion of Pluto. What is the mean distance of Pluto from the Sun? Write an equation for the orbit of Pluto about the Sun.

61. Will a truck that is 8 *feet* wide carrying a load that reaches 7 *feet* above the ground the semielliptical arch on the one-way road that passes under the bridge? $\approx 9.64\,ft$
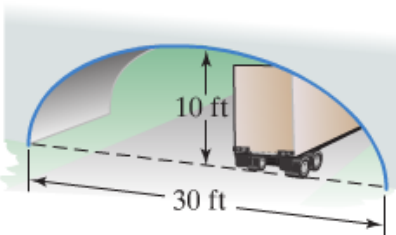
62. A semielliptic archway has a height of 20 *feet* and a width of 50 *feet* and a width of 50 *feet*. Can a truck 14 *feet* high and 10 *feet* wide drive under the archway without going into the other lane? 18.3 *ft*
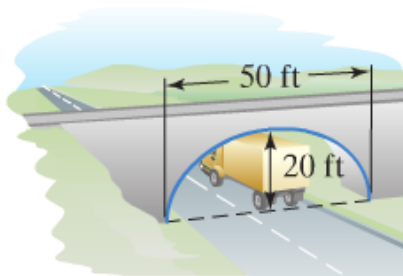
**63.** The elliptical ceiling in Statuary Hall is 96 *feet* long and 23 *feet* tall.
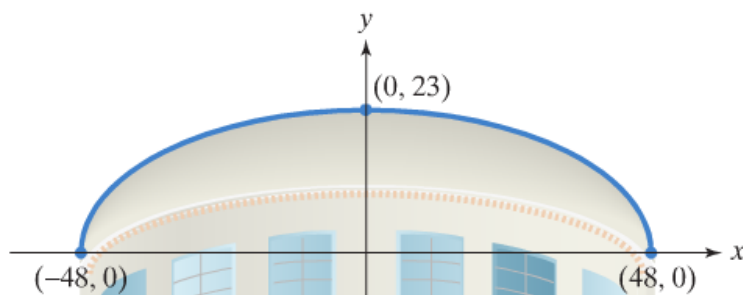
a) Using the rectangular coordinate system in the figure shown, write the standard form of the equation of the elliptical ceiling. $\dfrac{x^2}{2304}+\dfrac{y^2}{529}=1$

b) John Quincy Adams discovered that he could overhear the conversations of opposing party leaders near the left side of the chamber if he situated his desk at the focus at the right side of the chamber. How far from the center of the ellipse along the major axis did Adams situate his desk? ≈ 42.13

Section **4.8 – Hyperbolas**

Definition **of a Hyperbola**

A ***hyperbola*** is the set of all points in a plane, the difference of whose distances from two fixed points (the ***foci***) in the plane is a positive constant.
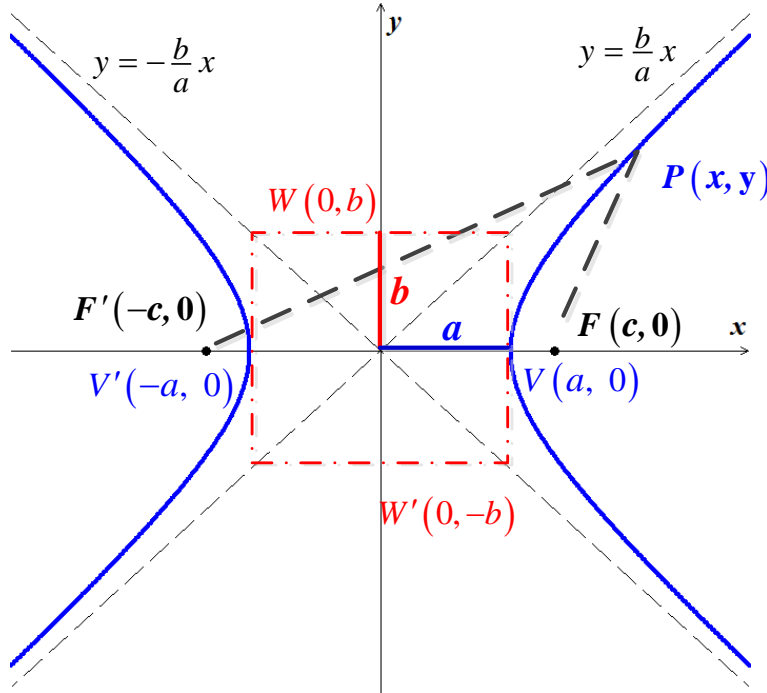
Let $P(x, y)$ be a point on the hyperbola and $F'(-c,\ 0)$ and $F(c,\ 0)$ (the ***foci***), where the midpoint of $F'F$ (the origin) is called the *center*. The following is true:

$$d(P,F') - d(P,F) = 2a \quad or \quad d(P,F) - d(P,F') = 2a$$

That implies to:

$$\left| d(P,F) - d(P,F') \right| = 2a$$

$$\left| \sqrt{(x-c)^2 + (y-0)^2} - \sqrt{(x+c)^2 + (y-0)^2} \right| = 2a$$

$$\left| \sqrt{x^2 - 2cx + c^2 + y^2} - \sqrt{x^2 + 2cx + c^2 + y^2} \right| = 2a$$

$$\sqrt{x^2 - 2cx + c^2 + y^2} = 2a + \sqrt{x^2 + 2cx + c^2 + y^2}$$

$$\left(\sqrt{x^2 - 2cx + c^2 + y^2} \right)^2 = \left(2a + \sqrt{x^2 + 2cx + c^2 + y^2} \right)^2$$

$$x^2 - 2cx + c^2 + y^2 = 4a^2 + 4a\sqrt{x^2 + 2cx + c^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

$$-2cx = 4a^2 + 4a\sqrt{x^2 + 2cx + c^2 + y^2} + 2cx$$

61

$$-4cx - 4a^2 = 4a\sqrt{x^2 + 2cx + c^2 + y^2}$$

$$\left(-cx - a^2\right)^2 = \left(a\sqrt{x^2 + 2cx + c^2 + y^2}\right)^2$$

$$c^2x^2 + 2a^2cx + a^4 = a^2\left(x^2 + 2cx + c^2 + y^2\right)$$

$$c^2x^2 + 2a^2cx + a^4 = a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2$$

$$c^2x^2 + a^4 = a^2x^2 + a^2c^2 + a^2y^2$$

$$c^2x^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$$

$$\left(c^2 - a^2\right)x^2 - a^2y^2 = a^2\left(c^2 - a^2\right)$$

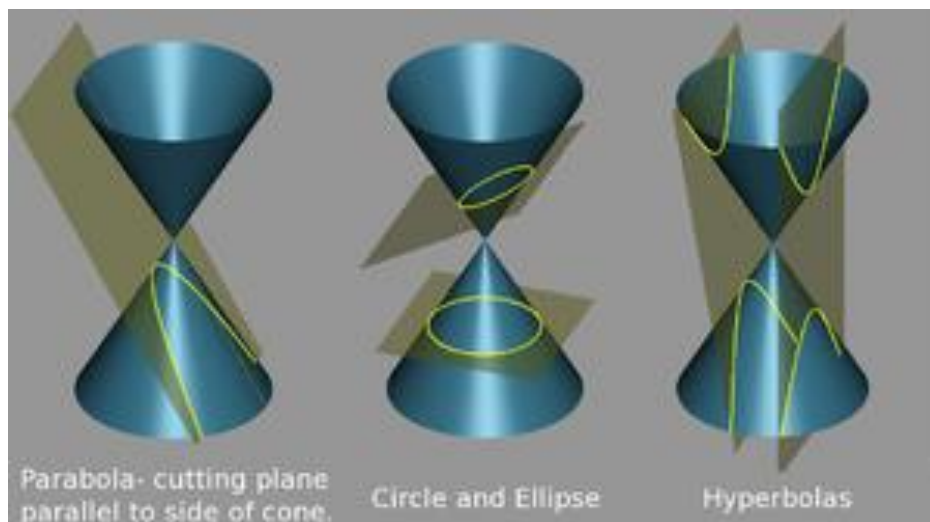$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

Finally, if we let $b^2 = c^2 - a^2;\quad b > 0$

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

Applying the tests for symmetry, we see that the hyperbola is symmetric with the respect to both axes and the origin.

The **x-intercepts** are a and $-a$. The corresponding points $V\left(a,\ 0\right)$ and $V'\left(-a,\ 0\right)$ are called the

vertices of the ellipse. The line segment $V'V$ is called the **transverse axis**.

The graph has no **y**-intercept, since $-\dfrac{y^2}{b^2} = 1$ has the *complex* solutions $y = \pm bi$. The points $W\left(0,b\right)$

and $W'\left(0,-b\right)$ are endpoints of the **conjugate axis** $W'W$ (there are not on the hyperbola)

Parabola- cutting plane parallel to side of cone.　　Circle and Ellipse　　Hyperbolas

Example

Sketch the graph of $9x^2 - 4y^2 = 36$. Find the foci and equations of the asymptotes.

Solution

$$\frac{9x^2}{36} - \frac{4y^2}{36} = \frac{36}{36}$$

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

$$\rightarrow \begin{cases} a^2 = 4 \rightarrow a = \pm 2 \\ b^2 = 9 \rightarrow b = \pm 3 \end{cases} \Rightarrow c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{4 + 9} = \pm\sqrt{13}$$
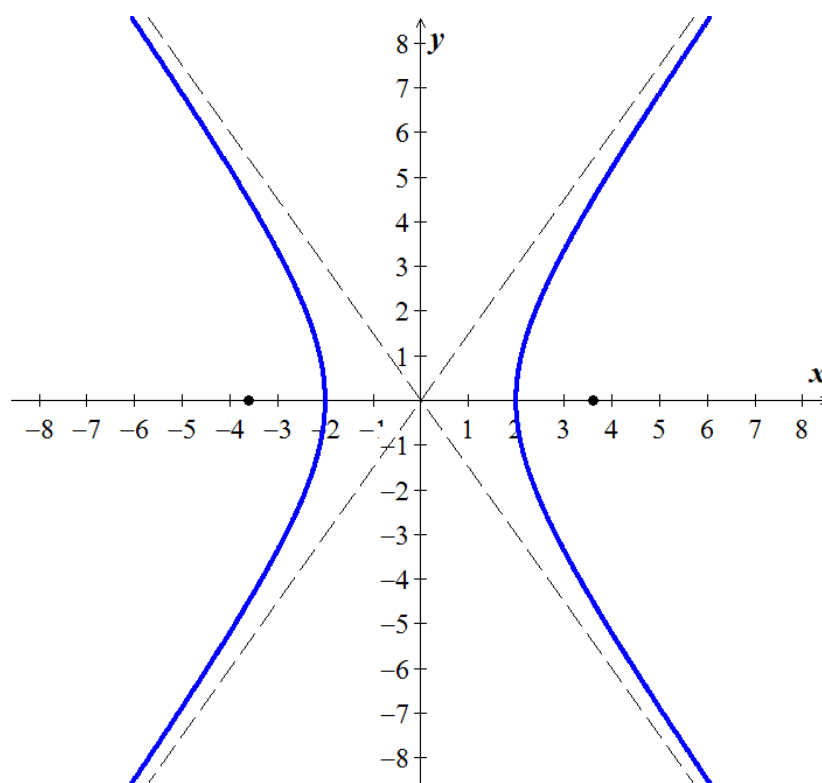
There are no y-intercepts.

The ***endpoints***: $(0, \pm 3)$

The ***vertices***: $(\pm 2, 0)$

The ***foci*** are $F\left(\sqrt{13}, 0\right)$ *and* $F'\left(-\sqrt{13}, 0\right)$

The equations of the ***asymptotes*** are: $\left\lfloor y = \pm\dfrac{b}{a}x = \pm\dfrac{3}{2}x \right\rfloor$

63

Example

Sketch the graph of $4y^2 - 2x^2 = 1$. Find the foci and equations of the asymptotes.

Solution

$$\frac{y^2}{\frac{1}{4}} - \frac{x^2}{\frac{1}{2}} = 1$$

$$\rightarrow \begin{cases} a^2 = \frac{1}{4} \rightarrow a = \pm\frac{1}{2} \\ b^2 = \frac{1}{2} \rightarrow b = \pm\frac{1}{\sqrt{2}} \end{cases} \Rightarrow c = \pm\sqrt{a^2 + b^2} = \pm\sqrt{\frac{1}{4} + \frac{1}{2}} = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$$
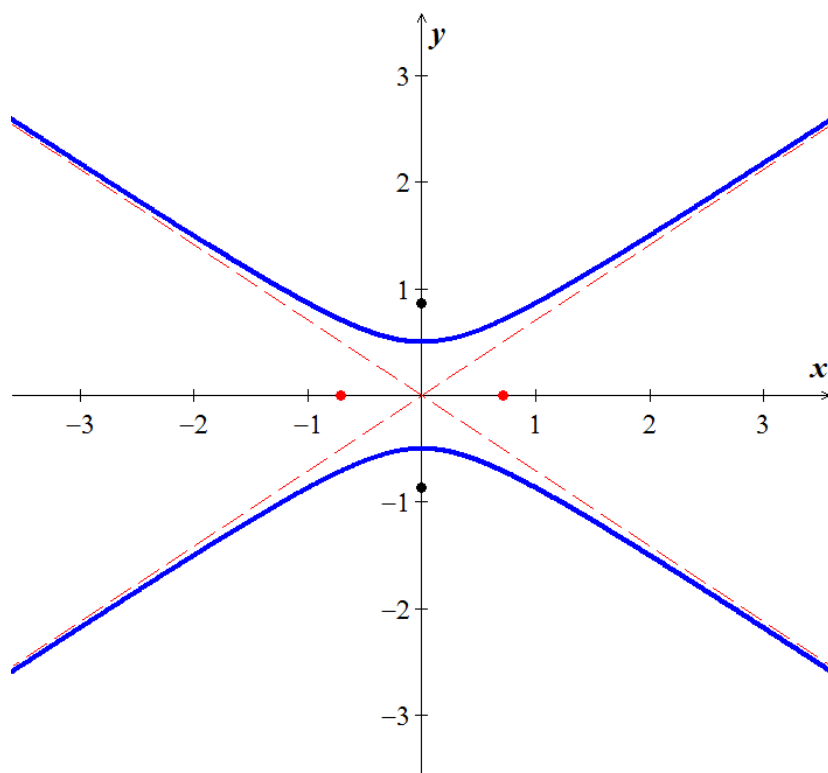
There are no *x*-intercepts.

The ***endpoints***: $\left(\pm\frac{1}{\sqrt{2}}, \ 0 \right)$

The ***vertices***: $\left(0, \ \pm\frac{1}{2} \right)$

The ***foci*** are $\left(0, \ \pm\frac{\sqrt{3}}{2} \right)$

The equations of the ***asymptotes*** are: $\left\lfloor y = \pm\frac{a}{b}x = \pm\frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}}x = \pm\frac{\sqrt{2}}{2}x \right\rfloor$

Example

A hyperbola has vertices $(\pm 3,\ 0)$ and passes through the point $P(5,\ 2)$. Find its equation, foci and asymptotes.

Solution

$$\frac{x^2}{3^2} - \frac{y^2}{b^2} = 1$$

Since $P(5,\ 2)$ in on the hyperbola $\Rightarrow \frac{5^2}{3^2} - \frac{2^2}{b^2} = 1$

$$-\frac{4}{b^2} = 1 - \frac{25}{9}$$

$$-\frac{4}{b^2} = -\frac{16}{9}$$
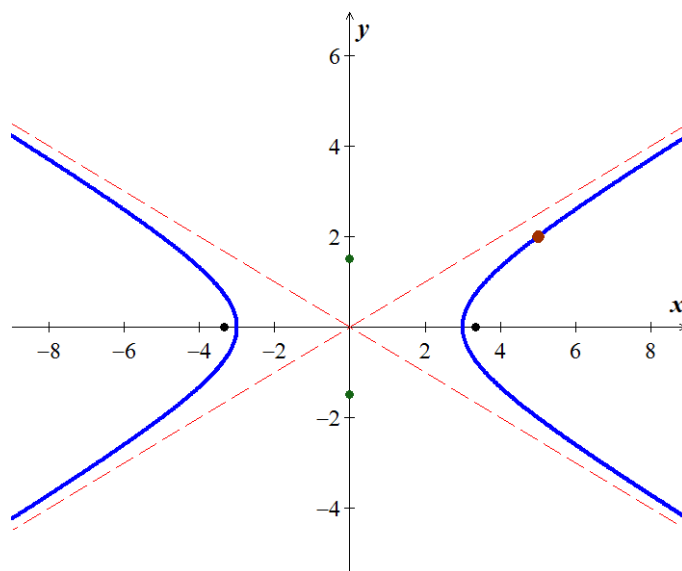
$$\frac{b^2}{4} = \frac{9}{16}$$

$$b^2 = \frac{9}{4}$$

$$\frac{x^2}{9} - \frac{y^2}{\frac{9}{4}} = 1 \qquad\qquad \frac{x^2}{9} - \frac{4y^2}{9} = 1 \rightarrow x^2 - 4y^2 = 9$$

$$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{9 + \frac{9}{4}} = \sqrt{\frac{45}{4}} = \frac{3\sqrt{5}}{2}$$

The *foci*: $\left(\pm\frac{3\sqrt{5}}{2},\ 0\right)$

The equations of the *asymptotes* are: $\underline{\left|y = \pm\frac{b}{a}x = \pm\frac{\frac{3}{2}}{3}x = \pm\frac{1}{2}x\right|}$

Example

Coast Guard station A is 200 *miles* directly east of another station B. A ship is sailing on a line parallel to and 50 *miles* north of the line through A and B. Radio signals are sent out from A and B at the rate of 980 $ft\,/\,\mu\sec$ (microsecond). If, at 1:00 PM, the signal from B reaches the ship 400 microseconds after the signal from A, locate the position of the ship at that time.
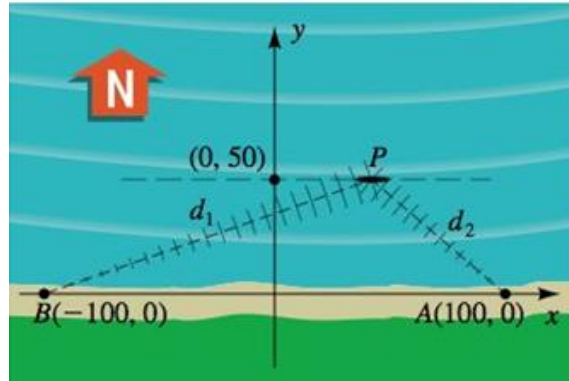
Solution

Given: $v = 980\ ft/\mu\sec \quad t = 400\ \mu\sec$

$$d_2 - d_1 = (980)(400) = 392{,}000\ ft$$

$$= 392{,}000\ ft\ \frac{1\ mi}{5280\ ft}$$

$$= 74.\overline{24}\ mi$$

Since $d_2 - d_1 = 2a \;\rightarrow\; a = \dfrac{74.\overline{24}}{2} = 37.\overline{12}$

$$a^2 = \left(37.\overline{12}\right)^2 \approx 1378$$

Distance from the origin to either focus is $c = 100$

Then, $b^2 = c^2 - a^2 \approx 10{,}000 - 1378 \approx 8622$

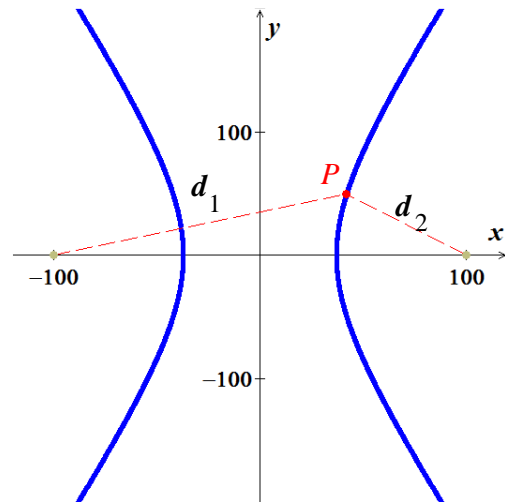$$\frac{x^2}{1378} - \frac{y^2}{8622} = 1 \qquad\qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Since $y_P = 50$

$$\frac{x^2}{1378} - \frac{50^2}{8622} = 1$$

$$x^2 = 1{,}378\left(1 + \frac{2{,}500}{8{,}622}\right)$$

$$x = \sqrt{1{,}378\left(\frac{11{,}122}{8{,}622}\right)} \approx 42.16$$

$$\therefore\ P\left(42,\ 50\right)$$

Exercises   *Section* 4.8 – Hyperbolas

Find the **center, vertices**, the **foci**, **endpoints** and the equations of the **asymptotes** of the hyperbola. Sketch its graph, showing the asymptotes and the foci.

1. $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$

2. $\dfrac{y^2}{9} - \dfrac{x^2}{4} = 1$

3. $x^2 - \dfrac{y^2}{24} = 1$

4. $y^2 - 4x^2 = 16$

5. $16x^2 - 36y^2 = 1$

6. $\dfrac{(y+2)^2}{9} - \dfrac{(x+2)^2}{4} = 1$

7. $\dfrac{(x-2)^2}{4} - \dfrac{(y+3)^2}{9} = 1$

8. $(y-2)^2 - 4(x+2)^2 = 4$

9. $(x+4)^2 - 9(y-3)^2 = 9$

10. $144x^2 - 25y^2 + 864x - 100y - 2404 = 0$

11. $4y^2 - x^2 + 40y - 4x + 60 = 0$

12. $4x^2 - 16x - 9y^2 + 36y = -16$

13. $2x^2 - y^2 + 4x + 4y = 4$

14. $2y^2 - x^2 + 2x + 8y + 3 = 0$

15. $2y^2 - 4x^2 - 16x - 2y - 19 = 0$

16. Suppose a hyperbola has center at the origin, foci at $F'(-c,\ 0)$ and $F(c,\ 0)$, and equation $d(P,\ F') - d(P,\ F) = 2a$. Let $b^2 = c^2 - a^2$, and show that an equation of the hyperbola is

$$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$$

17. A cooling tower is a hydraulic structure. Suppose its base diameter is 100 meters and its smallest diameter of 48 meters occurs 84 meters from the base. If the tower is 120 meters high approximate its diameter at the top.
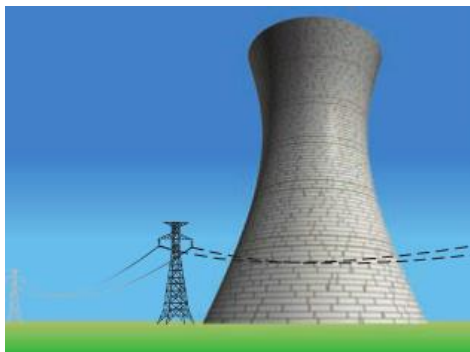

18. An airplane is flyting along the hyperbolic path. If an equation of the path is $2y^2 - x^2 = 8$, determine how close the airplane comes to town located at (3, 0). (Hunt: Let S denote the square of the distance from a point (x, y) on the path to (3, 0), and find the minimum value of S.)
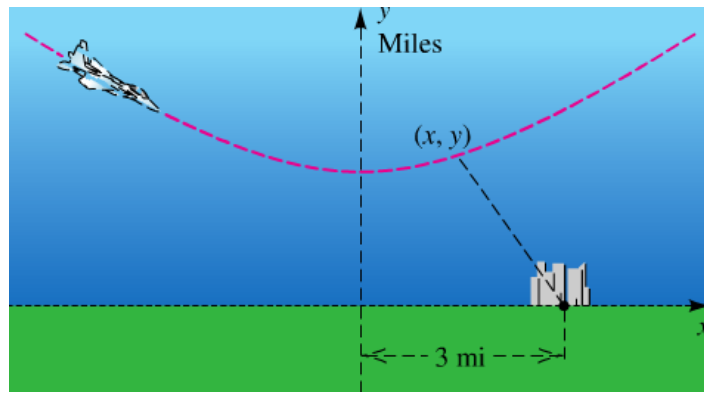
19. A ship is traveling a course that is 100 miles from, and parallel tom a straight shoreline. The ship sends out a distress signal that is received by two Coast Guard stations A and B, located 200 miles apart. By measuring the difference in signal reception times, it is determined that the ship is 160 miles closer to B than to A. Where is the ship?
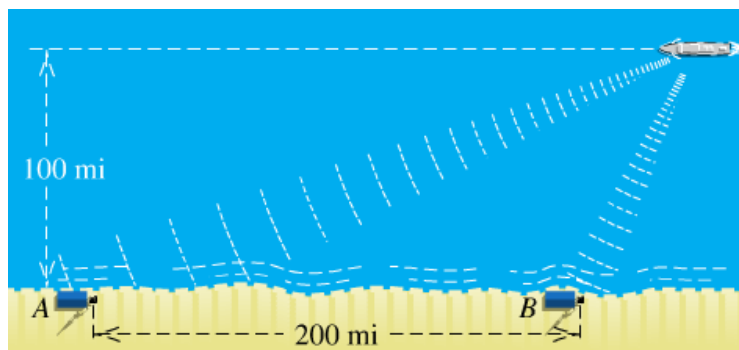

20. The Cassegrain telescope design (dating back to 1672) makes use of the reflective properties of both the parabola and the hyperbola. The figure shows a (split) parabolic mirror, with one focus at F_1 and axis along the line l, and a hyperbolic mirror, with one focus also at F_1 and transverse axis along l. Where do incoming light waves parallel to the common axis finally collect?
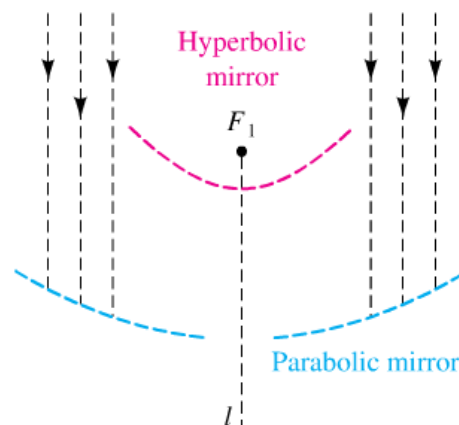
21. Suppose that two people standing 1 *mile* apart both see a flash of lightning. After a period of time, the person standing at point A hears the thunder. One second later, the person standing at point B hears the thunder. If the person at B is due west of the person at A and the lightning strike is known to occur due north of the person standing at point A, where did the lightning strike occur? (Sound travels at 1100 ft / sec and 1 *mile* $= 5280\, ft$)

22. Ernest Rutherford published a paper that he described the motion of alpha particles as they are shot at a piece of gold foil 0.00004 *cm* thick. Before conducting this experiment, Rutherford expected that the alpha particles would shoot through the foil just as a bullet would shoot through the foil just as a bullet would shoot through snow. Instead, a small fraction of the alpha particles bounced off the foil. This led to the conclusion that the nucleus of an atom is dense, while the remainder of the atom is sparse. Only the density of the nucleus could cause the alpha particles to deviate from their path. The figure shows a diagram from Rutherford's paper that indicates that the deflected alpha particles follow the path of one branch of a hyperbola.
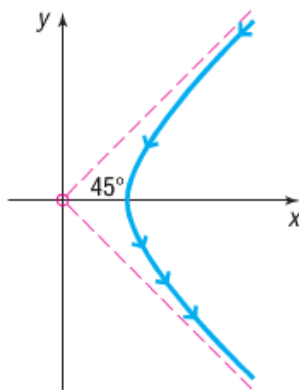
a) Find an equation of the asymptotes under this scenario.
b) If the vertex of the path of the alpha particles is 10 *cm* from the center of the hyperbola, find a model that describes the path of the particle.

23. Hyperbolas have interesting reflective properties that make them useful for lenses and mirrors. For example, if a ray of light strikes a convex hyperbolic mirror on a line that would (theoretically) pass through its rear focus, it is reflected through its rear focus, it is reflected through the front focus. This property and that of the parabola were used to develop the *Cassegrain* telescope in 1672. The focus of the parabolic mirror and the rear focus of the hyperbolic mirror are the same point. The rays are collected by the parabolic mirror, reflected toward the common focus, and thus are reflected by the hyperbolic mirror through the opening to its front focus, where the eyepiece is located. If the equation of the hyperbola is $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$ and the focal length (distance from the vertex to the focus) of the parabola is 6, find the equation of the parabola.

24. The ***eccentricity e*** of a hyperbola is defined as the number $\dfrac{c}{a}$, where a is the distance of a vertex from the center and c is the distance of a focus from the center. Because $c > a$, it follows that $e > 1$. Describe the general shape of a hyperbola whose eccentricity is close to 1. What is the shape if e is very large?

25. An explosive is recorded by two microphone that are 1 *mile* apart. Microphone M_1 received the sound 2 *seconds* before microphone M_2. Assuming sound travels at 1,100 *feet* per *second*, determine the possible locations of the explosion relative to the location of the microphones.

2640 *ft*

69

26. Radio towers A and B, 200 km apart, are situated along the coast, with A located due west of B. Simultaneous radio signals are sent from each tower to a ship, with the signal from B received 500 $\mu\sec$ before the signal from A.

    a) Assuming that the radio signals travel 300 $m/\mu\sec$, determine the equation of the hyperbola

       on which the ship is located. $\dfrac{x^2}{5625} - \dfrac{y^2}{4375} = 1$

    b) If the ship lies due north of tower B, how far out at sea is it? $\approx 58.3\ km$

27. An architect designs two houses that are shaped and positioned like a part of the branches of the hyperbola whose equation is $625y^2 - 400x^2 = 250,000$, where x and y are in yards. How far apart are the houses at their closest point? $40\ yd$