Section 3.7 – Phase Plane Portraits & Applications

Equilibrium Points

The dynamical behavior of a linear system is easier than non-linear system. We need to determine a set of points to satisfy the autonomous system $y' = 0\ \left(y' = f\left(y(t), t\right) \equiv 0 \right)$. These set of points are called ***equilibrium points***.

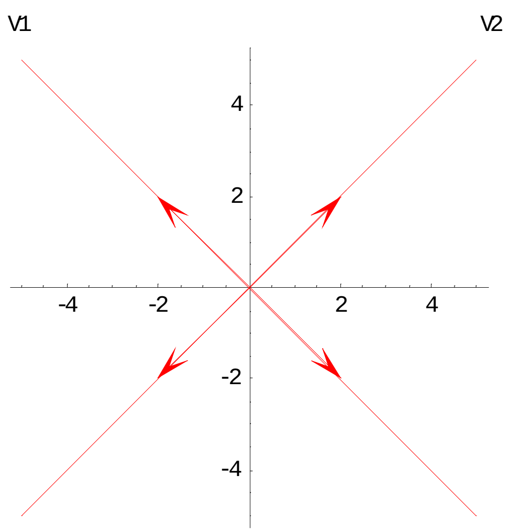
From these equilibrium points, we can determine the stability of the system.

The equilibrium point O_1 is the intersection of the eigenvectors, and we can plot those two lines by joining these points V_1O_1 and V_2O_1 together.

The general solution for the system is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 V_1 e^{\lambda_1 t} + C_2 V_2 e^{\lambda_2 t}$$

The behavior of the system or the solutions is depending on the value of λ_1 and λ_2, and if they are real or complex values.



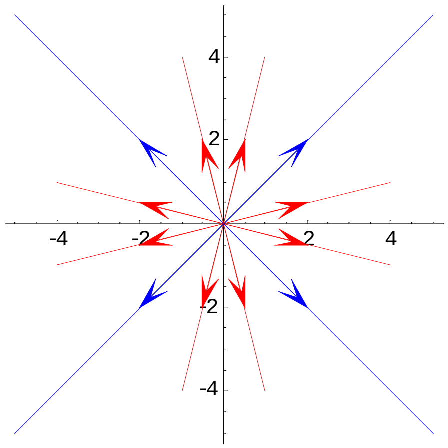
Eigenvectors V_1 and V_2 plot.

The family of all solution curves without the presence of the independent variable is **called** ***phase portrait***.

Stability of the equilibrium point condition

- An equilibrium point is *stable* if all nearby solutions stay nearby
- An equilibrium point is *asymptotically stable* if all nearby solutions not only stay nearby, but also tend the equilibrium point.

Case 1: If $\lambda_1 > 0$ and $\lambda_2 > 0$ are real values.



$\lambda_1 > 0$ and $\lambda_2 > 0$ *source* or *repel* (unstable at point (0, 0))

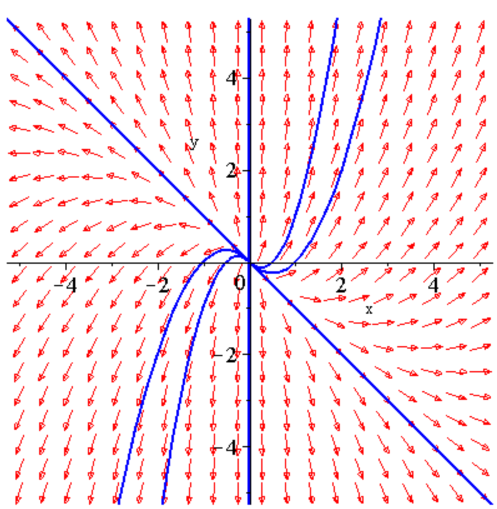
The system is unstable and the solution as the time go by, will diverge away from the equilibrium point

Example

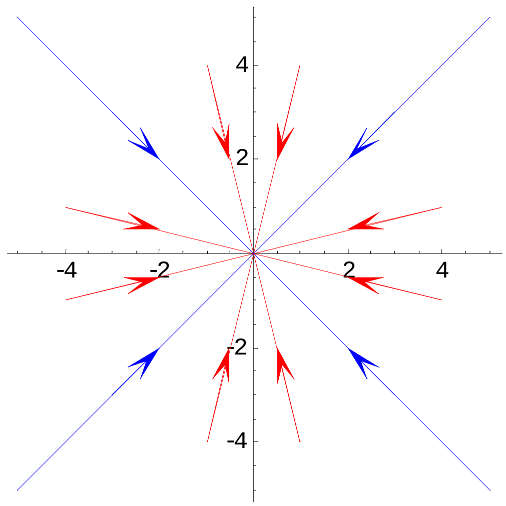
$$y' = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} y$$

Solution

$$\begin{cases} \lambda = 1 & \rightarrow \quad V = \begin{pmatrix} -1 & 1 \end{pmatrix}^T \\ \lambda = 2 & \rightarrow \quad V = \begin{pmatrix} 0 & 1 \end{pmatrix}^T \end{cases} \qquad y(t) = C_1 e^{t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



Case 2: If $\lambda_1 \,\&\, \lambda_2 < 0$



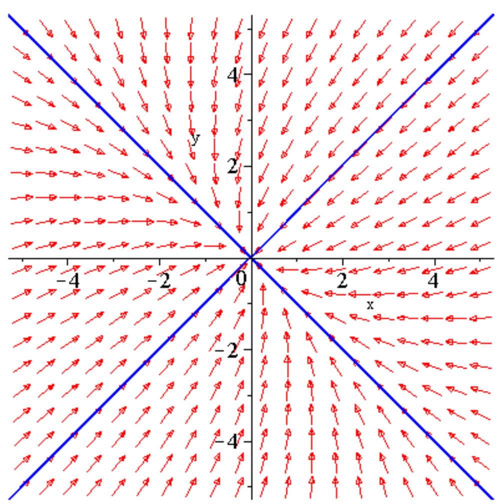
$\lambda_1 \,\&\, \lambda_2 < 0$ <u>*sink*</u> or <u>*attractor*</u> ((0, 0) is asymptotically stable) $\lambda_1 = \lambda_2 < 0$ <u>*proper node*</u>.

Example

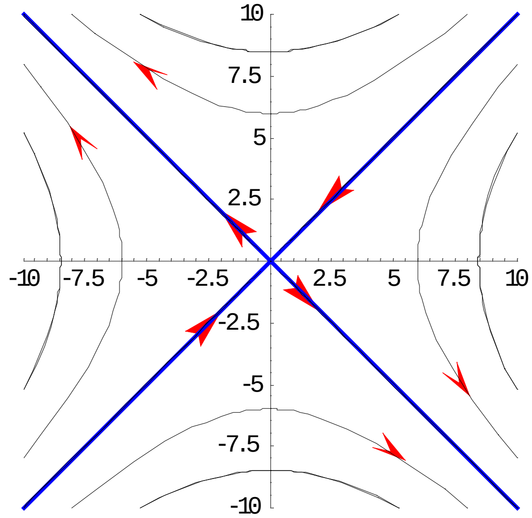
$$y' = \begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix} y$$

<u>***Solution***</u>

$$\begin{cases} \lambda = -4 & \rightarrow \quad V = \begin{pmatrix} 1 & 1 \end{pmatrix}^T \\ \lambda = -2 & \rightarrow \quad V = \begin{pmatrix} -1 & 1 \end{pmatrix}^T \end{cases} \qquad y(t) = C_1 e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



Case 3: If $\lambda_1 > 0$ & $\lambda_2 < 0$



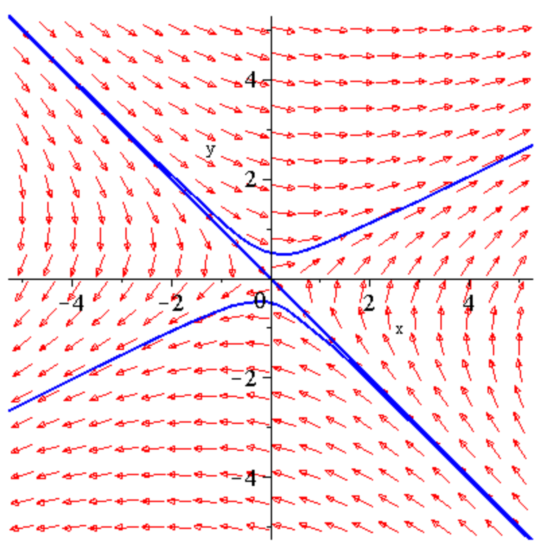
$\lambda_1 > 0$ & $\lambda_2 < 0$ A <u>*saddle point*</u>. ((0,0) is semi-stable)

Example

$$y' = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} y$$

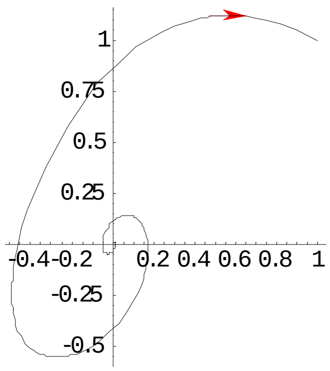
<u>Solution</u>

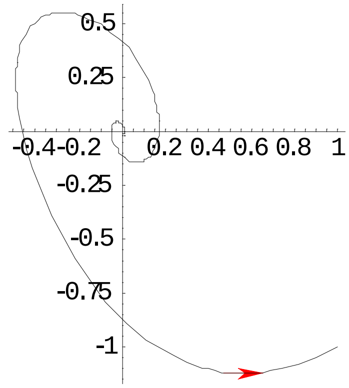
$$\begin{cases} \lambda = 3 & \rightarrow \quad V = \begin{pmatrix} 2 & 1 \end{pmatrix}^T \\ \lambda = -3 & \rightarrow \quad V = \begin{pmatrix} -1 & 1 \end{pmatrix}^T \end{cases}$$



Case 4: If λ_1 & λ_2 are complex values: $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$

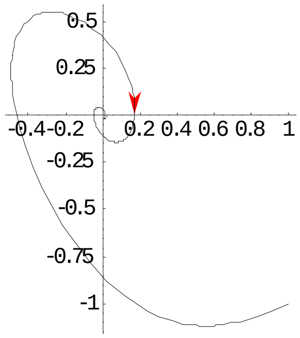
If $b > 0$, the behavior of the system is spiral clockwise (*cw*), then otherwise is *ccw*.

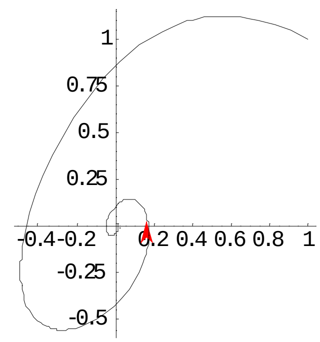




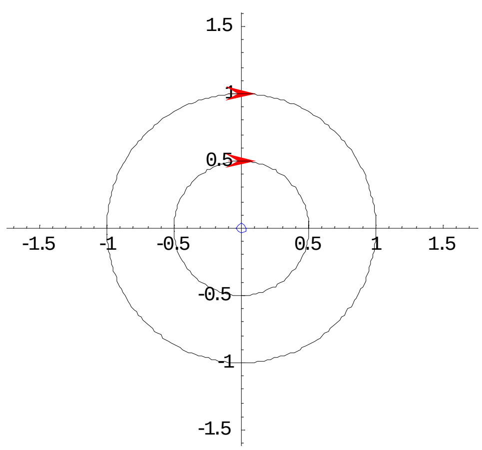
spiral out. (unstable at (0,0) point)

or





$a < 0$ *spiral in*. (asymptotically stable at (0,0) point)



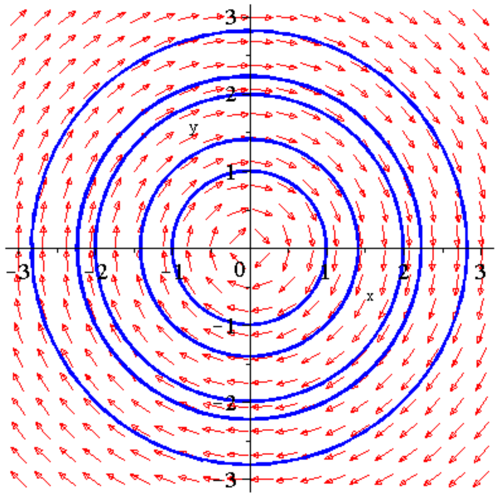
$a = 0 \quad \lambda_{1,2} = \pm\, ib$ ‘circle’ periodic solution- (0, 0) is a *center* stable.

Example

$$y' = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} y$$

Solution

$$\begin{cases} \lambda = 2i & \to \quad V = \begin{pmatrix} -i & 1 \end{pmatrix}^T \\ \lambda = -2i & \to \quad V = \begin{pmatrix} i & 1 \end{pmatrix}^T \end{cases} \qquad y(t) = C_1 \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + C_2 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}$$



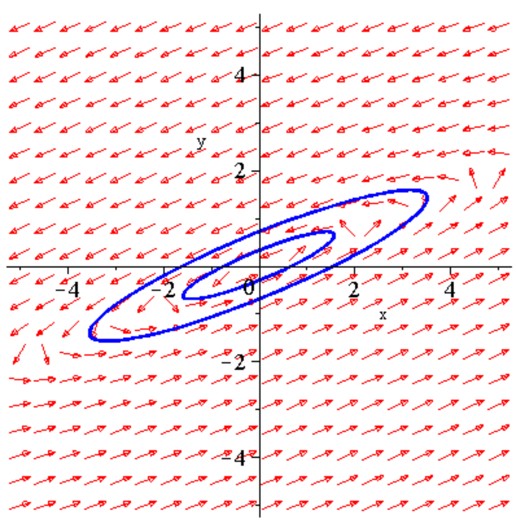
The equilibrium point is the center, but the solution curves are circles.

Example

$$y' = \begin{pmatrix} 4 & -10 \\ 2 & -4 \end{pmatrix} y$$

Solution

$$\begin{cases} \lambda = 2i & \to \quad V = \begin{pmatrix} 2+i & 1 \end{pmatrix}^T \\ \lambda = -2i & \to \quad V = \begin{pmatrix} 2-i & 1 \end{pmatrix}^T \end{cases}$$



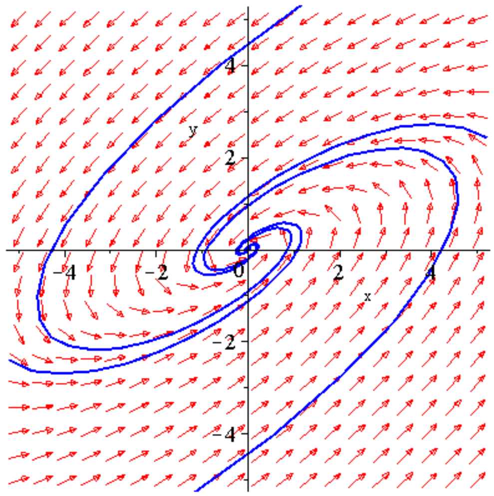
The equilibrium point is the center, but the solution curves are ellipses.

Example

$$y' = \begin{pmatrix} 1 & -4 \\ 2 & -3 \end{pmatrix} y$$

Solution

$$\begin{cases} \lambda = -1 + 2i & \rightarrow \quad V = \begin{pmatrix} 1+i & 1 \end{pmatrix}^T \\ \lambda = -1 - 2i & \rightarrow \quad V = \begin{pmatrix} 1-i & 1 \end{pmatrix}^T \end{cases}$$



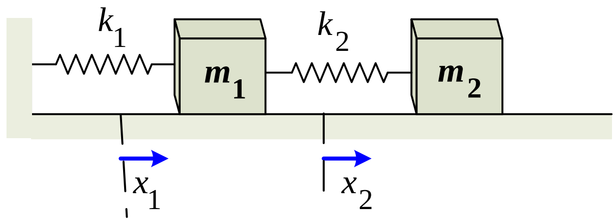
The behavior of the system at the equilibrium point center is an asymptotically stable and spiral in.

Stability properties of linear systems (in 2-dimensions)

Eigenvalues	Type of critical point	Stability
$\lambda_1 > \lambda_2 > 0$	*Improper node*	*Unstable.*
$\lambda_1 < \lambda_2 < 0$	*Improper node*	*Asymptotically stable*
$\lambda_2 < 0 < \lambda_1$	*Saddle point*	*Unstable.*
$\lambda_1 = \lambda_2 > 0$	*Proper/improper node*	*Unstable*
$\lambda_1 = \lambda_2 < 0$	*Proper/improper node*	*Asymptotically stable*
$\lambda_{1,2} = a \pm ib$	*Spiral point*	
a > 0	*spiral out*	*Unstable*
a < 0	*spiral in*	*Asymptotically stable*
$\lambda_{1,2} = \pm ib$	*Center*	*Stable*

Example

Consider the mass-and-spring system.



Where $m_1 = 2, \quad m_2 = 1, \quad k_1 = 100, \quad k_2 = 50$ and $M\vec{x}'' = K\vec{x}$

Solution

$$\begin{cases} m_1 x_1'' = -k_1 x_1 + k_2\left(x_2 - x_1\right) \\ m_2 x_2'' = -k_2\left(x_2 - x_1\right) \end{cases} \rightarrow \begin{cases} m_1 x_1'' = \left(-k_1 - k_2\right)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - k_2 x_2 \end{cases}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} x'' = \begin{pmatrix} -150 & 50 \\ 50 & -50 \end{pmatrix}\vec{x} \qquad M = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow M^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$x'' = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -150 & 50 \\ 50 & -50 \end{pmatrix}\vec{x} \qquad M^{-1}M\vec{x}'' = M^{-1}K\vec{x}$$

$$= \begin{pmatrix} -75 & 25 \\ 50 & -50 \end{pmatrix}\vec{x} \qquad \vec{x}'' = \boldsymbol{A}\vec{x}$$

$$A = \begin{pmatrix} -75 & 25 \\ 50 & -50 \end{pmatrix}$$

$$\left|A - \lambda I\right| = \begin{vmatrix} -75-\lambda & 25 \\ 50 & -50-\lambda \end{vmatrix}$$

$$= \left(-75-\lambda\right)\left(-50-\lambda\right) - 1250$$

$$= \lambda^2 + 125\lambda + 2500 = 0$$

The eigenvalues are: $\lambda_1 = -100, \quad \lambda_2 = -25$

By the theorem, the natural frequencies: $\omega_1 = 10$ *and* $\omega_2 = 5$

For $\lambda_1 = -100 \quad \Rightarrow \quad \left(A + 100I\right)V_1 = 0$

$$\begin{pmatrix} 25 & 25 \\ 50 & 50 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = -b \qquad \rightarrow V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = -25 \quad \Rightarrow \quad \left(A + 25I\right)V_2 = 0$

$$\begin{pmatrix} -50 & 25 \\ 50 & -25 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2a = b \qquad \rightarrow V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The free oscillation of the mass-and-spring system, follows by:

$$\vec{x}(t) = \left(a_1 \cos 10t + b_1 \sin 10t\right)V_1 + \left(a_2 \cos 5t + b_2 \sin 5t\right)V_2$$

The natural mode:

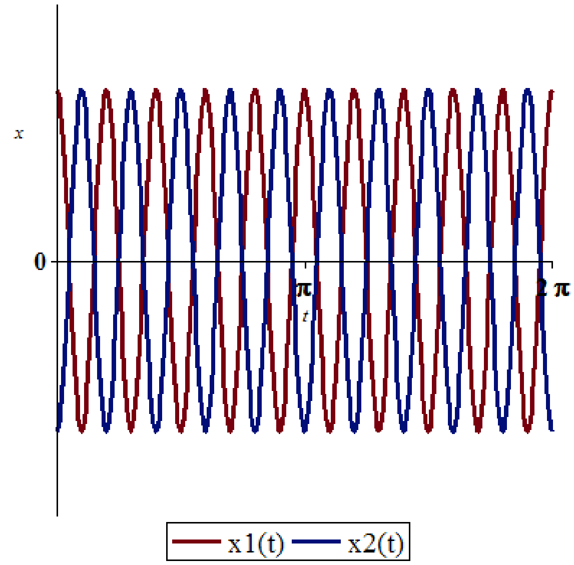
$$\vec{x}_1(t) = \left(a_1 \cos 10t + b_1 \sin 10t\right)V_1$$
$$= c_1 \cos\left(10t - \alpha_1\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Where $c_1 = \sqrt{a_1^2 + b_1^2}$;

$$\cos\alpha_1 = \frac{a_1}{c_1} \quad \sin\alpha_1 = \frac{b_1}{c_1}$$

Which has the scalar equations:

$$\begin{cases} x_1(t) = c_1 \cos\left(10t - \alpha_1\right) \\ x_2(t) = -c_1 \cos\left(10t - \alpha_1\right) \end{cases}$$



The ***second*** part:

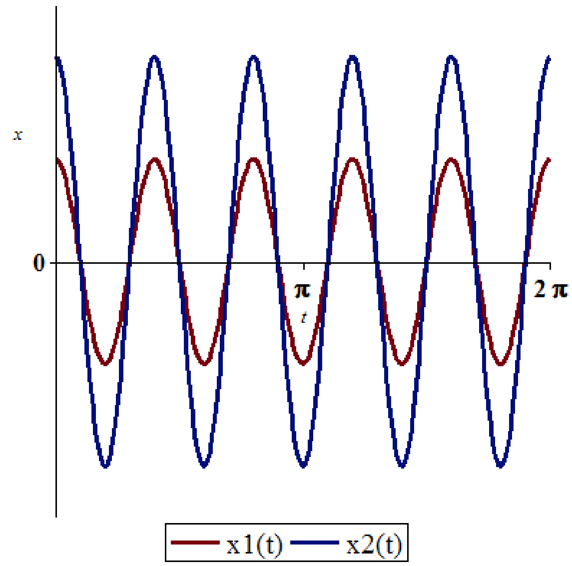
$$\vec{x}_2(t) = \left(a_2 \cos 5t + b_2 \sin 5t\right)V_2$$
$$= c_2 \cos\left(5t - \alpha_2\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Where $c_2 = \sqrt{a_2^2 + b_2^2}$;

$$\cos\alpha_2 = \frac{a_2}{c_2} \quad \sin\alpha_2 = \frac{b_2}{c_2}$$

Which has the scalar equations:

$$\begin{cases} x_1(t) = c_2 \cos\left(5t - \alpha_2\right) \\ x_2(t) = 2c_2 \cos\left(5t - \alpha_2\right) \end{cases}$$



Exercises *Section* 3.7 – Phase Plane Portraits & Applications

Sketch a rough approximation of a solution in each region determined by the half-line solutions. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

1. $y(t) = C_1 e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

2. $y(t) = C_1 e^{t}\begin{pmatrix} -1 \\ -2 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

3. $y(t) = C_1 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

4. $y(t) = C_1 e^{-t}\begin{pmatrix} -5 \\ 2 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

Sketch a rough approximation of the given system. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

5. $y' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} y$

6. $y' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} y$

Calculate the eigenvalues to determine the behavior of the system whether the equilibrium point at the origin is the center, a spiral sink or a source. Calculate and sketch the vector generated by the right-hand side of the system at the point (1, 0). Use this to help sketch the solution trajectory for the system passing through the point (1, 0).

7. $y' = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} y$

8. $y' = \begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix} y$

9. $y' = \begin{pmatrix} 7 & -10 \\ 4 & -5 \end{pmatrix} y$

10. $y' = \begin{pmatrix} -4 & 8 \\ -4 & 4 \end{pmatrix} y$

11. $y' = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} y$

12. For the given system $y' = \begin{pmatrix} -1 & 6 \\ -3 & 8 \end{pmatrix} y$

a) Sketch a rough approximation of the given system. Use arrows to indicate the direction of motion on all solutions. Determine the behavior of the equilibrium point and the stability.

b) Find the solution of the initial-value problem $y(0) = (0,\ 1)^T$

Find the general solution of the given system. Graph and construct a direction field and typical solution curves for the given system.

13. $x_1' = x_1 + 2x_2, \quad x_2' = 2x_1 + x_2$

14. $x_1' = 2x_1 + 3x_2, \quad x_2' = 2x_1 + x_2$

15. $x_1' = 6x_1 - 7x_2, \quad x_2' = x_1 - 2x_2$

16. $x_1' = -3x_1 + 4x_2, \quad x_2' = 6x_1 - 5x_2$

17. $x_1' = x_1 - 5x_2, \quad x_2' = x_1 - x_2$

18. $x_1' = -3x_1 - 2x_2, \quad x_2' = 9x_1 + 3x_2$

19. $x_1' = x_1 - 5x_2, \quad x_2' = x_1 + 3x_2$

20. $x_1' = 5x_1 - 9x_2, \quad x_2' = 2x_1 - x_2$

21. $x_1' = 3x_1 + 4x_2, \quad x_2' = 3x_1 + 2x_2; \quad x_1(0) = x_2(0) = 1$

22. $x_1' = 9x_1 + 5x_2, \quad x_2' = -6x_1 - 2x_2; \quad x_1(0) = 1,\ x_2(0) = 0$

23. $x_1' = 2x_1 - 5x_2, \quad x_2' = 4x_1 - 2x_2; \quad x_1(0) = 2,\ x_2(0) = 3$

24. $x_1' = x_1 - 2x_2, \quad x_2' = 2x_1 + x_2; \quad x_1(0) = 0,\ x_2(0) = 4$

25. $x_1' = x_1 - 2x_2, \quad x_2' = 3x_1 - 4x_2; \quad x_1(0) = -1,\ x_2(0) = 2$

26. $x_1' = -0.5x_1 + 2x_2, \quad x_2' = -2x_1 - 0.5x_2; \quad x_1(0) = -2,\ x_2(0) = 2$

27. $x_1' = 1.25x_1 + 0.75x_2, \quad x_2' = 0.75x_1 + 1.25x_2; \quad x_1(0) = -2,\ x_2(0) = 1$

Find the general solution of the given system.

28. $x_1' = 4x_1 + x_2 + 4x_3, \quad x_2' = x_1 + 7x_2 + x_3, \quad x_3' = 4x_1 + x_2 + 4x_3$

29. $x_1' = x_1 + 2x_2 + 2x_3, \quad x_2' = 2x_1 + 7x_2 + x_3, \quad x_3' = 2x_1 + x_2 + 7x_3$

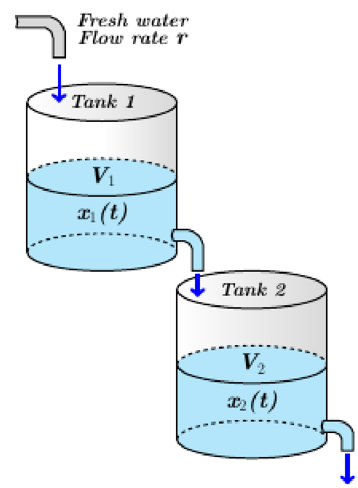
30. $x_1' = 4x_1 + x_2 + x_3, \quad x_2' = x_1 + 4x_2 + x_3, \quad x_3' = x_1 + x_2 + 4x_3$

31. $x_1' = 5x_1 + x_2 + 3x_3, \quad x_2' = x_1 + 7x_2 + x_3, \quad x_3' = 3x_1 + x_2 + 5x_3$

32. $x_1' = 5x_1 - 6x_3, \quad x_2' = 2x_1 - x_2 - 2x_3, \quad x_3' = 4x_1 - 2x_2 - 4x_3$

33. $x_1' = 3x_1 + 2x_2 + 2x_3, \quad x_2' = -5x_1 - 4x_2 - 2x_3, \quad x_3' = 5x_1 + 5x_2 + 3x_3$

Find the amount $x_1(t),\ x_2(t)$ of salt in each tank at time $t \ge 0$, with $x_1(0) = 15\ lb \quad x_2(0) = 0$. If



34. $V_1 = 50\ gal, \quad V_2 = 25\ gal, \quad r = 10\ gal/\min$

35. $V_1 = 25\ gal, \quad V_2 = 40\ gal, \quad r = 10\ gal/\min$

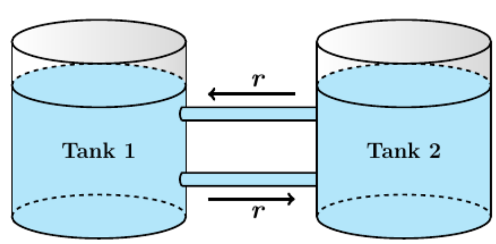
36. $V_1 = 50\ gal, \quad V_2 = 25\ gal, \quad r = 5\ gal/\min$

37. $V_1 = 25\ gal, \quad V_2 = 40\ gal, \quad r = 5\ gal/min$

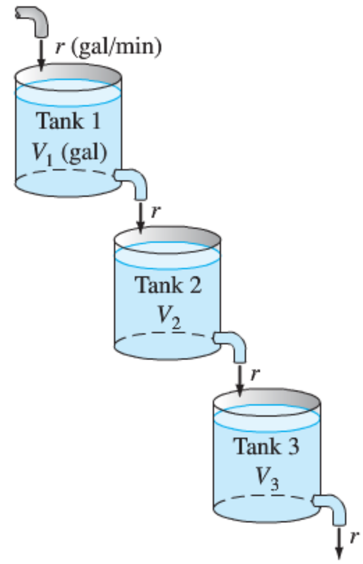
Find the amount $x_1(t),\ x_2(t)$ of salt in each tank at time $t \ge 0$, with $x_1(0) = 15\ lb \quad x_2(0) = 0$. If

38. $V_1 = 50\ gal, \quad V_2 = 25\ gal, \quad r = 10\ gal/\min$

39. $V_1 = 25\ gal, \quad V_2 = 40\ gal, \quad r = 10\ gal/\min$



Find the amount $x_1(t),\ x_2(t),\ x_3(t)$ of salt in each tank at time $t \ge 0$, if



40. $V_1 = 30\ gal,\quad V_2 = 15\ gal,\quad V_3 = 10\ gal,\quad r = 30\ gal/\min$

$x_1(0) = 27\ lb \quad x_2(0) = x_3(0) = 0$

41. $V_1 = 20\ gal,\quad V_2 = 30\ gal,\quad V_3 = 60\ gal,\quad r = 60\ gal/\min$

$x_1(0) = 45\ lb \quad x_2(0) = x_3(0) = 0$

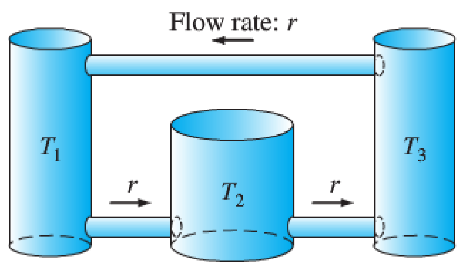
42. $V_1 = 15\ gal,\quad V_2 = 10\ gal,\quad V_3 = 30\ gal,\quad r = 60\ gal/\min$

$x_1(0) = 45\ lb \quad x_2(0) = x_3(0) = 0$

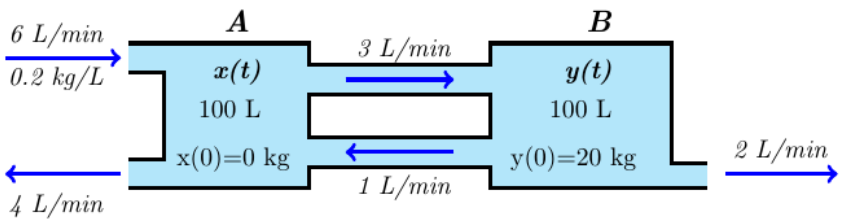
43. $V_1 = 20\ gal,\quad V_2 = 40\ gal,\quad V_3 = 50\ gal,\quad r = 10\ gal/\min$

$x_1(0) = 15 \quad x_2(0) = x_3(0) = 0$

44. If $V_1 = 50\ gal,\quad V_2 = 25\ gal,\quad V_3 = 50\ gal,\quad r = 10\ gal/\min$, find the amount $x_1(t),\ x_2(t),\ x_3(t)$ of salt in each tank at time $t \ge 0$

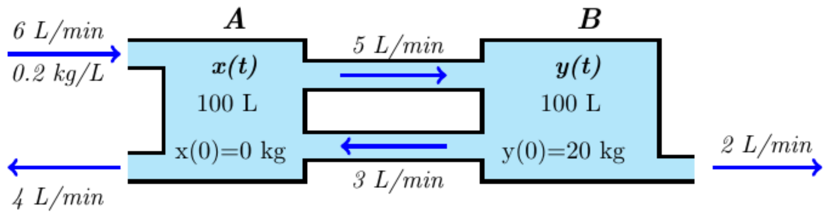


45. Two large tanks, each holding 100 *L* of liquid, are interconnected by pipes, with the liquid following from tank *A* into tank *B* at a rate of 3 *L/min* and from *B* to into *A* at a rate of 1 *L/min*.



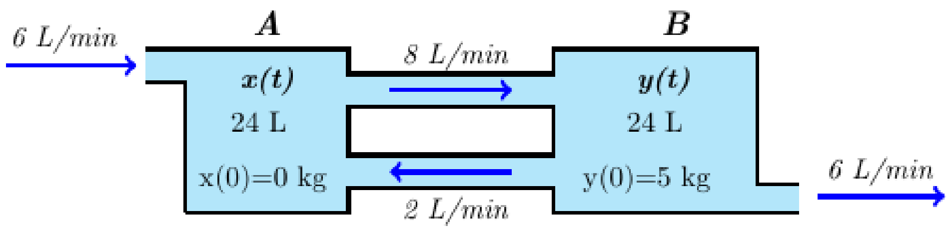
The liquid inside each tank is kept well stirred. A brine solution with a concentration of 0.2 *kg/L* of salt flows into tank *A* at a rate of 6 *L/min*. The diluted solution flows out the system from tank *A* at 4 *L/min* and from tank *B* at 2 *L/min*. If, initially, tank *A* contains pure water and tank *B* contains 20 *kg* of salt, determine the mass of salt in each tank at time $t \ge 0$.

46. Two large tanks, each holding 100 *L* of liquid, are interconnected by pipes, with the liquid following from tank *A* into tank *B* at a rate of 5 *L/min* and from *B* to into *A* at a rate of 3 *L/min*.



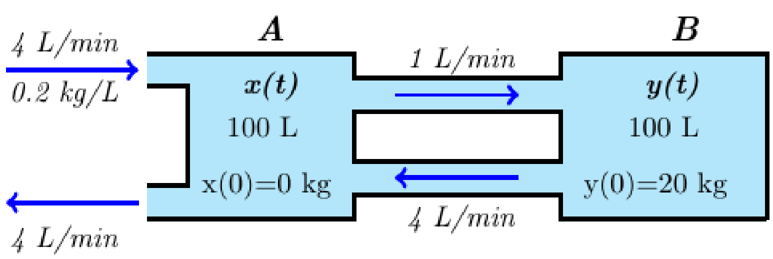
The liquid inside each tank is kept well stirred. A brine solution with a concentration of 0.2 *kg/L* of salt flows into tank *A* at a rate of 6 *L/min*. The diluted solution flows out the system from tank *A* at 4 *L/min* and from tank *B* at 2 *L/min*. If, initially, tank *A* contains pure water and tank *B* contains 20 *kg* of salt, determine the mass of salt in each tank at time $t \geq 0$.

47. Two large tanks, each holding 24 *L* of liquid, are interconnected by pipes, with the liquid following from tank *A* into tank *B* at a rate of 8 *L/min* and from *B* to into *A* at a rate of 2 *L/min*.



The liquid inside each tank is kept well stirred. A brine solution flows into tank *A* at a rate of 6 *L/min*. The diluted solution flows out the system from tank *B* at 6 *L/min*. If, initially, tank *A* contains pure water and tank *B* contains 5 *kg* of salt, determine the mass of salt in each tank at time $t \geq 0$.

48. Two large tanks, each holding 100 *L* of liquid, are interconnected by pipes, with the liquid following from tank *A* into tank *B* at a rate of 1 *L/min* and from *B* to into *A* at a rate of 4 *L/min*.

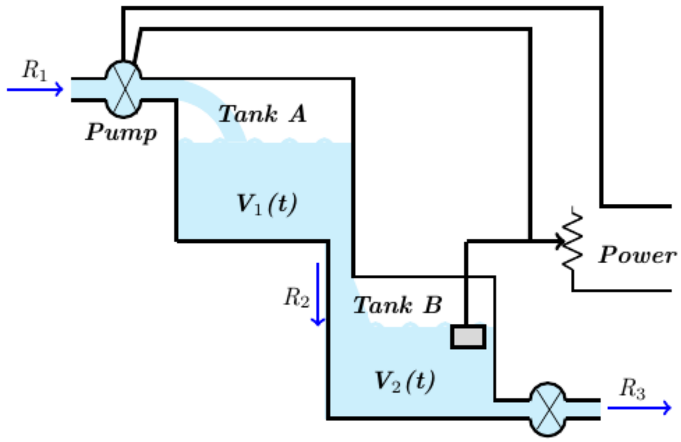


The liquid inside each tank is kept well stirred. A brine solution flows into tank *A* at a rate of 4 *L/min*. The diluted solution flows out the system from tank *A* at 4 *L/min*. If, initially, tank *A* contains pure water and tank *B* contains 20 *kg* of salt, determine the mass of salt in each tank at time $t \geq 0$.

49. Two 1,000 *liter* tanks are with salt water. Tank ***A*** contains 800 *liters* of water initially containing 20 *grams* of salt dissolved in it and Tank ***B*** contains 1,000 *liters* of water initially containing 80 *grams* of salt dissolved in it. Salt water with a concentration of $\frac{1}{2}$ *g/L* of salt enters Tank ***A*** at a rate of 4 *L/hr*. Fresh water enters Tank ***B*** at a rate of 7 *L/hr*. Through a connecting pipe water flows from Tank ***B*** into Tank ***A*** at a rate of 10 *L/hr*. Through a different connecting pipe 14 *L/hr* flows out of Tank ***A*** and 11 *L/hr* are drained out of the pipe (and hence out of the system completely) and only 3 *L/hr* flows back into Tank ***B***.

Find the amount of salt in each tank at any time.

50. Many physical and biological systems involve time delays. A pure time delay has its output the same as its input but shifted in time. A more common type of delay is pooling delay. Here the level of fluid in tank *B* determines the rate at which fluid enters tank *A*. Suppose this rate is given by $R_1(t)=\alpha\left(V-V_2(t)\right)$, where α and V are positive constants and $V_2(t)$ is the volume of fluid in tank *B* at time *t*.



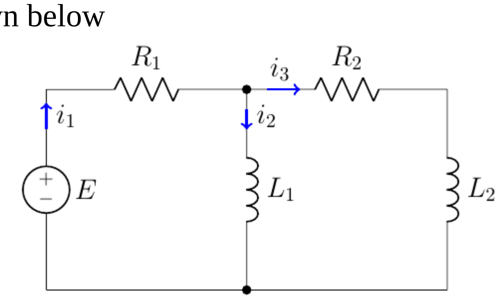
a) If the outflow rate R_3 from tank *B* is constant and the flow rate R_2 from tank *A* into tank *B* is $R_2(t)=KV_1(t)$ is the volume of fluid in tank *A* at time *t*, then show that this feedback system is governed by the system

$$\begin{cases} \dfrac{dV_1}{dt}=\alpha\left(V-V_2(t)\right)-KV_1(t) \\ \dfrac{dV_2}{dt}=KV_1(t)-R_3 \end{cases}$$

b) Find a general solution for the system in part (*a*) when $\alpha=5\ min^{-1}$, $V=20\ L$, $K=2\ min^{-1}$, and $R_3=10\ L/min$.

c) Using the general solution obtained in part (*b*), what can be said about the volume of fluid in each of the tanks as $t\to+\infty$?

51. The electrical network shown below

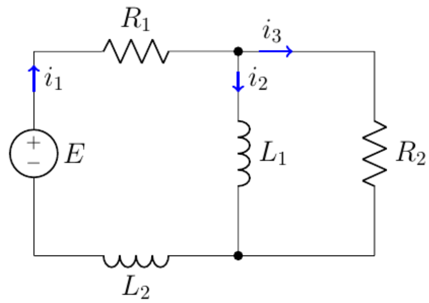


a) Find the system equations for the currents $i_2(t)$ and $i_3(t)$

b) Solve the system for the given: $R_1=2\ \Omega$, $R_2=3\ \Omega$, $L_1=1\ h$, $L_2=1\ h$, $E=60\ V$, with the initial values $i_2(0)=0$ & $i_3(0)=0$

c) Determine the current $i_1(t)$

52. The electrical network shown below

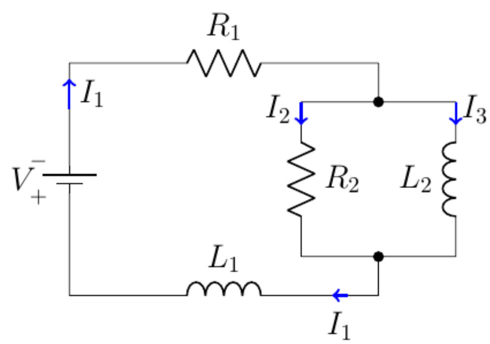


a) Find the system equations for the currents $i_1(t)$ and $i_2(t)$

b) Solve the system for the given: $R_1 = 8\ \Omega$, $R_2 = 3\ \Omega$, $L_1 = 1\ h$, $L_2 = 1\ h$, $E = 100\sin t\ V$, with the initial values $i_1(0) = 0$ & $i_2(0) = 0$

c) Determine the current $i_3(t)$

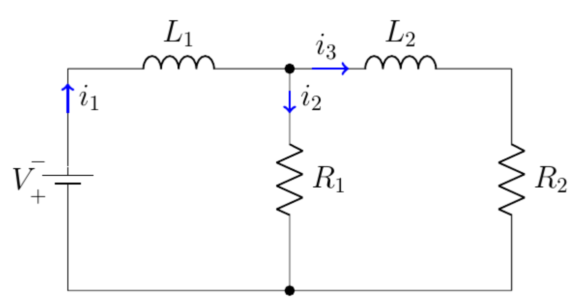
Find a system of differential equations and solve for the currents in the given network, with initial values: $I_1(0) = I_2(0) = I_3(0) = 0$



53. $R_1 = 2\ \Omega$, $R_2 = 1\ \Omega$, $L_1 = 0.2\ H$, $L_2 = 0.1\ H$, $V = 6\ V$

54. $R_1 = 2\ \Omega$, $R_2 = 1\ \Omega$, $L_1 = 0.1\ H$, $L_2 = 0.2\ H$, $V = 6\ V$

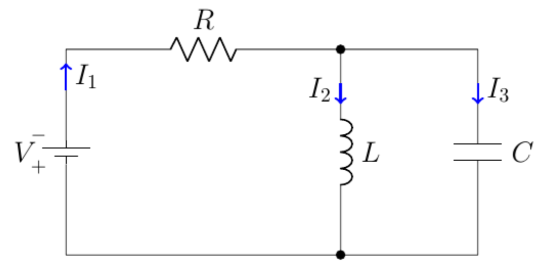
Find a system of differential equations and solve for the currents in the given network with initial values: $i_1(0) = i_2(0) = i_3(0) = 0$



55. $R_1 = 10\Omega$, $R_2 = 20\ \Omega$, $L_1 = 0.005\ H$, $L_2 = 0.01\ H$, $V = 50\ V$

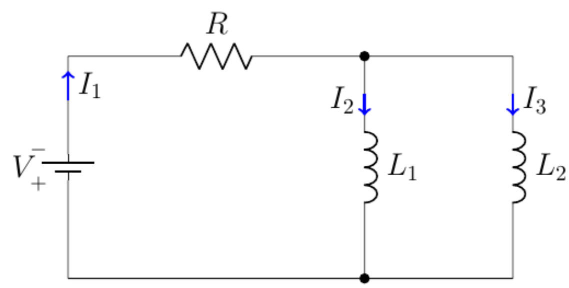
56. $R_1 = 10\ \Omega$, $R_2 = 40\ \Omega$, $L_1 = 10\ H$, $L_2 = 20\ H$, $V = 20\ V$

Find a system of differential equations and determine the charge on the capacitor and the currents in the given network with initial values: $I_1(0) = I_2(0) = I_3(0) = 0$



57. $R = 20\ \Omega,\quad L = 1\ H,\quad C = \frac{1}{160}\ F,\quad V = 5\ V,\quad q(0) = 2\ C$

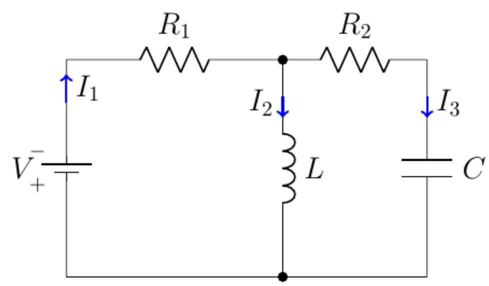
Find a system of differential equations and solve for the currents in the given network with initial values: $I_1(0) = I_2(0) = I_3(0) = 0$



58. $R = 10\ \Omega,\quad L_1 = 0.02\ H,\quad L_2 = 0.025\ H,\quad V = 10\ V$

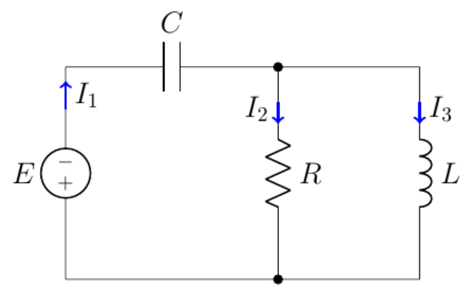
59. $R = 10\ \Omega,\quad L_1 = 2\ H,\quad L_2 = 25\ H,\quad V = 20\ V$

Find a system of differential equations and solve for the currents in the given network with initial values: $I_1(0) = I_2(0) = I_3(0) = 0$



60. $R_1 = 10\ \Omega,\quad R_2 = 5\ \Omega,\quad L = 20\ H,\quad C = \frac{1}{30}\ F,\quad V = 10\ V$

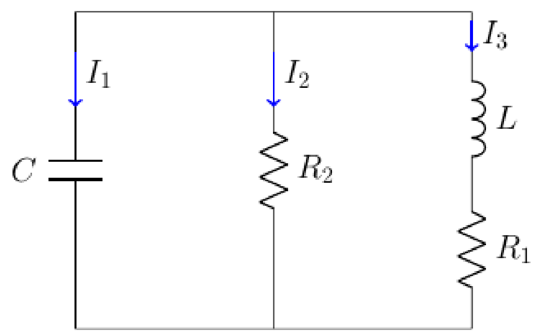
Find a system of differential equations and solve for the currents in the given network with initial values: $I_1(0) = I_2(0) = I_3(0) = 0$



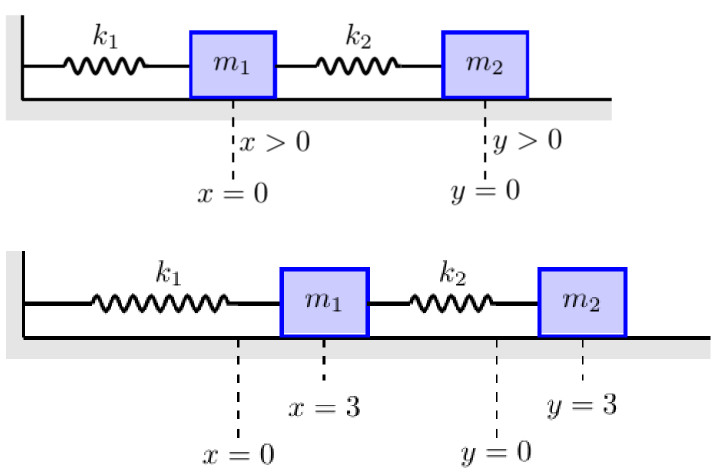
61. $R = 1\,\Omega,\quad L = 0.5\,H,\quad C = 0.5\,F,\quad E = \cos 3t\ V$

62. Derive three equations for the unknown currents $I_1, I_2,$ *and* I_3 with thegiven values of the given electric circuit shown below, then find the general solution

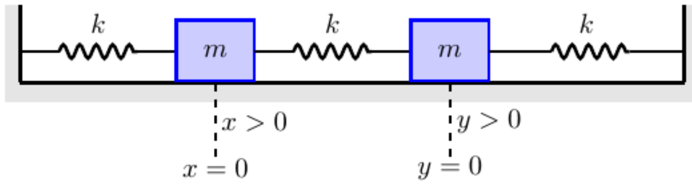
$R_1 = R_2 = 1\,\Omega$, $C = 1\,F$, and $L = 1\,H$.



63. On a smooth horizontal surface $m_1 = 2\ kg$ is attached to a fixed wall by a spring with spring constant $k_1 = 4\ N/m$. Another mass $m_2 = 1\ kg$ is attached to the first object by a spring with spring constant $k_2 = 2\ N/m$. The object are aligned horizontally so that the springs are their natural lengths. If both objects are displaced 3 m to the right of their equilibrium positions and then released, what are the equations of motion for the two objects?



64. Three identical springs with spring constant k and two identical masses m are attached in a straight line with the ends of the outside springs fixed.



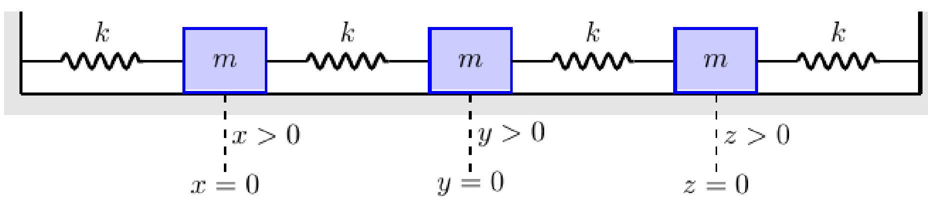
a) Determine and interpret the normal modes of the system.

b) Given the values $m = 2\ kg$, and $k = 2\ N/m$ with initial value
$x(0) = 1,\quad x'(0) = 0,\quad y(0) = 1,\quad y'(0) = 0$. what are the equations of motion for the two objects?

c) Given the values $m = 2\ kg$, and $k = 2\ N/m$ with initial value
$x(0) = 1,\quad x'(0) = 0,\quad y(0) = -1,\quad y'(0) = 0$. what are the equations of motion for the two objects?

d) Given the values $m = 2\ kg$, and $k = 2\ N/m$ with initial value
$x(0) = 1,\quad x'(0) = 0,\quad y(0) = 2,\quad y'(0) = 0$. what are the equations of motion for the two objects?

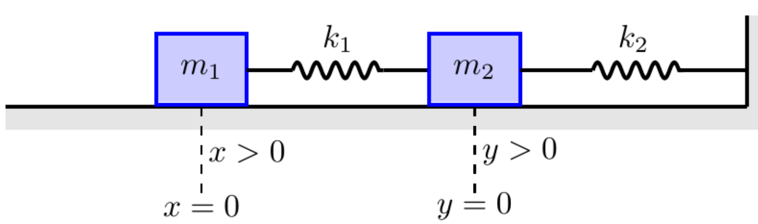
65. Four springs with the same spring constant and three equal masses are attached in a straight line on a horizontal frictionless surface.



a) What are the equations of motion for the three objects?

b) Determine the normal frequencies for the system, describe the three normal modes of vibration.

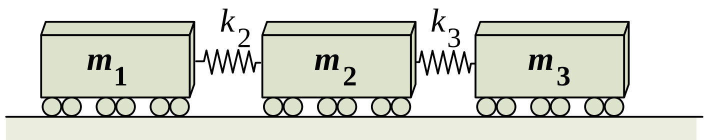
66. Two springs and two masses are attached in a straight line on a horizontal frictionless surface. The system is set in motion by holding the mass m_2 at it equilibrium position and pulling the mass m_1 to the left of its equilibrium position a distance 1 m and them releasing both masses.



a) Express Newton's law for the system and determine the equations of motion for the two masses if $m_1 = 1\ kg$, $m_2 = 2\ kg$, $k_1 = 4\ N/m$, and $k_2 = \frac{10}{3}\ N/m$

b) Express Newton's law for the system and determine the equations of motion for the two masses if $m_1 = 1\ kg$, $m_2 = 1\ kg$, $k_1 = 3\ N/m$, and $k_2 = 2\ N/m$

67. Three railway cars are connected by buffer springs that react when compressed, but disengage instead of stretching.

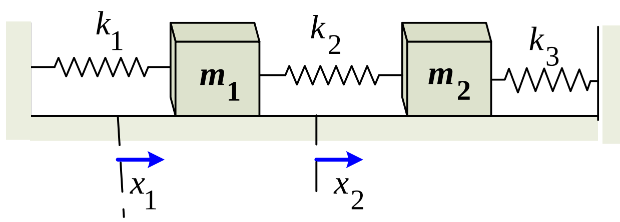


Given that $k_2 = k_3 = k = 3000\ lb/ft$ and $m_1 = m_3 = 750\ lbs$ *and* $m_2 = 500\ lbs$

Suppose that the leftmost car is moving to the right with velocity v_0 and at time $t = 0$ strikes the other 2 cars. The corresponding initial conditions are:

$$x_1(0) = x_2(0) = x_3(0) = 0$$
$$x_1'(0) = v_0 \quad x_2'(0) = x_3'(0) = 0$$

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

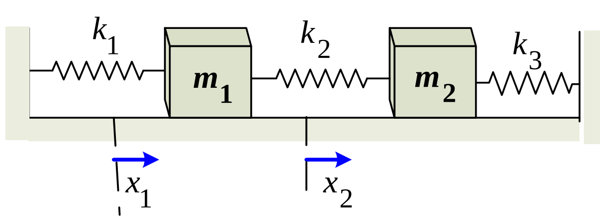
68. $m_1 = m_2 = 1; \quad k_1 = 0,\ k_2 = 2,\ k_3 = 0 \quad (no\ walls)$

69. $m_1 = m_2 = 1; \quad k_1 = 1,\ k_2 = 2,\ k_3 = 1$

70. $m_1 = m_2 = 1; \quad k_1 = 2,\ k_2 = 1,\ k_3 = 2$

71. $m_1 = 1,\ m_2 = 2; \quad k_1 = 2,\ k_2 = k_3 = 4$

Consider the mass-and-spring system shown below and with the given masses and spring constants values.



The mass-and-spring system is set in motion from rest $x_1'(0) = x_2'(0) = 0$ in its equilibrium position $x_1(0) = x_2(0) = 0$.

Find the 2 natural frequencies of the system and describe its natural modes of oscillation.

For the given external forces $F_1(t)$ and $F_2(t)$ acting on the masses m_1 and m_2, respectively.

Find the resulting motion of the system and describe it as a superposition of oscillations at three different frequencies.

72. $m_1 = m_2 = 1; \quad k_1 = 1,\ k_2 = 4,\ k_3 = 1 \quad F_1(t) = 96\cos 5t,\ F_2(t) = 0$

73. $m_1 = 1,\ m_2 = 2; \quad k_1 = 1,\ k_2 = k_3 = 2; \quad F_1(t) = 0,\ F_2(t) = 120\cos 3t$

74. $m_1 = m_2 = 1; \quad k_1 = 4,\ k_2 = 6,\ k_3 = 4; \quad F_1(t) = 30\cos t,\ \ F_2(t) = 60\cos t$

75. Consider a mass-and-spring system containing two masses $m_1 = m_2 = 1$ whose displacement functions $x(t)$ and $y(t)$ satisfy the differential equations

$$x'' = -40x + 8y$$
$$y'' = 12x - 60y$$

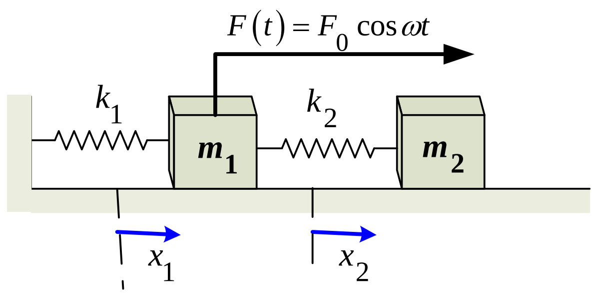
a) Describe the two fundamental modes of free oscillation of the system.

b) Assume that the two masses start in motion with the initial conditions

$$x(0) = 19,\ \ x'(0) = 12 \quad and \quad y(0) = 3,\ \ y'(0) = 6$$

And are acted on by the same force, $F_1(t) = F_2(t) = -195\cos 7t$. Describe the resulting motion as a superposition of oscillations at three different frequencies.

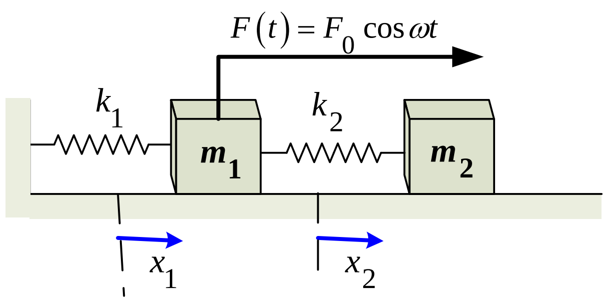
76. Consider a mass-and-spring system shown below. Assume that $m_1 = 1; \quad k_1 = 50; \quad F_0 = 5$ in mks units, and that $\omega = 10$. Then find m_2 so that in the resulting steady periodic oscillations, the mass m_1 will remain at rest (!).



Thus the effect of the second mass-and-spring pair will be to neutralize the effect of the force on the first mass. This is an example of a dynamic damper. It has an electrical analogy that some cable companies use to prevent your reception of certain cable channels.

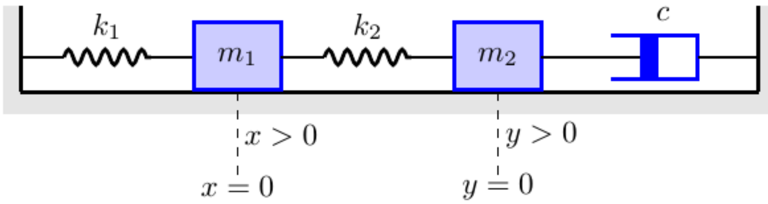
77. Consider a mass-and-spring system shown below. Assume that

$$m_1 = 2,\ m_2 = \frac{1}{2}; \quad k_1 = 75,\ k_2 = 25; \quad F_0 = 100 \quad and \ \ \omega = 10 \ \text{(in } \textbf{mks} \text{ units)}.$$



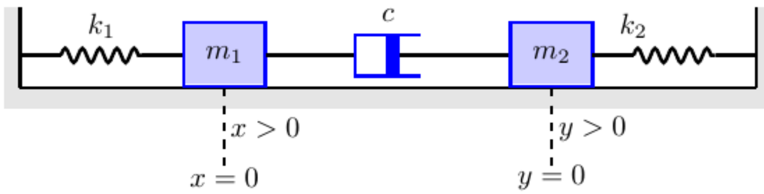
Find the solution of the system $M\,\vec{x}'' = K\,\vec{x} + F$ that satisfies the initial conditions $\vec{x}(0) = \vec{x}'(0) = \mathbf{0}$

78. Two springs, two masses, and a dashpot are attached in a straight line on a horizontal frictionless surface. The dashpot damping force on mass m_2, given by $F=-cy'$



Derive the system equation of differential equations for the displacements *x* and *y*.

79. Two springs, two masses, and a dashpot are attached in a straight line on a horizontal frictionless surface. The system is set in motion by holding the mass m_2 at equilibrium position and pushing the mass m_1 to the left of its equilibrium position a distance 2 *m* and then releasing both masses.



If $m_1 = m_2 = 1\ kg$ and $k_1 = k_2 = 1\ N/m$, and $c = 1\ N\text{-}sec$

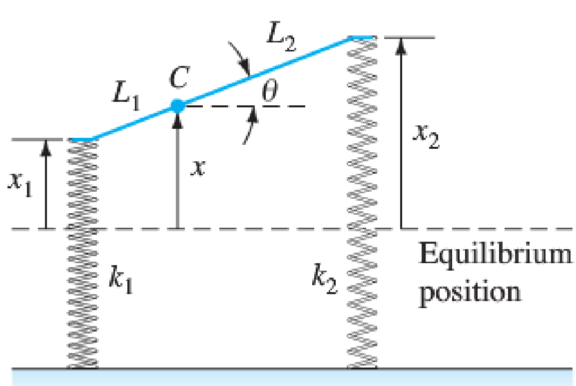
Determine the equations of motion for the two masses

80. A car with two axles and with separate front and rear suspension systems.

We assume that the car body acts as would a solid bar of mass *m* and length $L = L_1 + L_2$. It has moment of inertia I about its center of mass *C*, which is at distance L_1 from the front of the car. The car has front and back suspension springs with Hooke's constants k_1 and k_2, respectively. When the car is in motion, let $x(t)$ denote the vertical displacement of the center of mass of the car from equilibrium; let $\theta(t)$ denote its angular displacement (in radians) from the horizontal. Then Newton's laws of motion for linear and angular acceleration can be used to derive the equations.

$$mx'' = -(k_1 + k_2)x + (k_1L_1 - k_2L_2)\theta$$

$$I\theta'' = (k_1L_1 - k_2L_2)x - \left(k_1L_1^2 + k_2L_2^2\right)\theta$$



Suppose that $m = 75$ *slugs* (the car weighs 2400 *lb*), $L_1 = 7\,ft$, $L_2 = 3\,ft$ (it's a rear engine car), $k_1 = k_2 = 2000\ \ lb/\ ft$, and $I = 1000\ ft.lb.s^2$.

a) Find the two natural frequencies ω_1 and ω_2 of the car.

b) Now suppose that the car is driven at a speed of $v\ \ ft/\sec$ along a washboard surface shaped like a sine curve with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1\ \ or\ \ \omega = \omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

The system is taken as a model for an undamped car with the given parameters in *fps* units.

a) Find the two natural frequencies ω_1 and ω_2 of the car (in hertz).

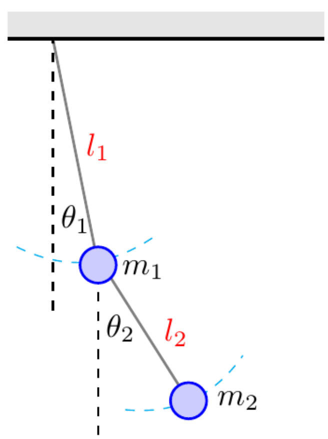
b) Assume that his car is driven along a sinusoidal washboard surface with a wavelength of 40 *ft*. The result is a periodic force on the car with frequency $\omega = \frac{2\pi}{40}v = \frac{\pi}{20}v$. Resonance occurs when $\omega = \omega_1\ \ or\ \ \omega = \omega_2$. Find the corresponding two critical speeds of the car (in *ft/sec*)

81. $m = 100;\quad I = 800;\quad L_1 = L_2 = 5;\quad k_1 = k_2 = 2000$

82. $m = 100;\quad I = 1000;\quad L_1 = 6,\ \ L_2 = 4;\quad k_1 = k_2 = 2000$

83. $m = 100;\quad I = 800;\quad L_1 = L_2 = 5;\quad k_1 = 1000,\ \ k_2 = 2000$

84. A double pendulum swinging in a vertical plane under the influence of gravity satisfies the system



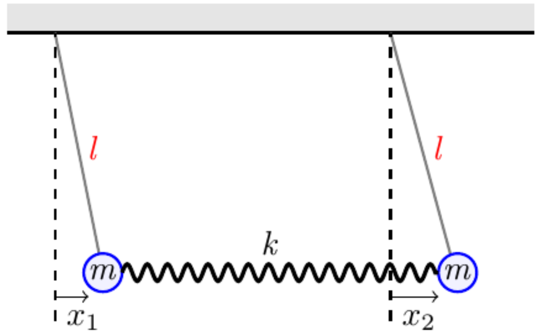
$$\begin{cases} \left(m_1 + m_2\right)\ell_1^2\theta_1'' + m_2\ell_1\ell_2\theta_2'' + \left(m_1 + m_2\right)\ell_1 g\theta_1 = 0 \\ m_2\ell_2^2\theta_2'' + m_2\ell_1\ell_2\theta_1'' + m_2\ell_2 g\theta_2 = 0 \end{cases}$$

Where θ_1 and θ_2 are small angles.

Solve the system when $m_1 = 3\ kg,\quad m_2 = 2\ kg,\quad \ell_1 = \ell_2 = 5\ m$

$$\theta_1(0) = \frac{\pi}{6},\quad \theta_2(0) = 0,\quad \theta_1'(0) = \theta_2'(0) = 0$$

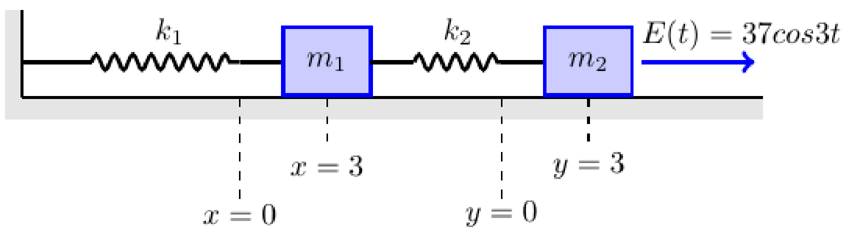
85. The motion of a pair of identical pendulums coupled by a spring is modeled by the system



$$\begin{cases} mx_1'' = -\dfrac{mg}{\ell}x_1 - k\left(x_1 - x_2\right) \\ mx_2'' = -\dfrac{mg}{\ell}x_2 + k\left(x_1 - x_2\right) \end{cases}$$

For small displacements. Determine the two normal frequencies for the system.

86. On a smooth horizontal surface $m_1 = 2\ kg$ is attached to a fixed wall by a spring with spring constant $k_1 = 4\ N/m$. Another mass $m_2 = 1\ kg$ is attached to the first object by a spring with spring constant $k_2 = 2\ N/m$. The object are aligned horizontally so that the springs are their natural lengths.



Suppose an external force $E(t) = 37\cos 3t$ is applied to the second object of mass 1 *kg*.

a) Find the general solution

b) Show that $x(t)$ satisfies the equation $x^{(4)}(t) + 5x''(t) + 4x(t) = 37\cos 3t$

c) Find a general solution $x(t)$ to equation in part (*b*).

d) Substitute $x(t)$ to obtain a formula for $y(t)$

e) If both masses are displaced 2 *m* to the right of their equilibrium positions and then released, find the displacement functions $x(t)$ and $y(t)$