Lecture One – Limits and Derivatives

Section 1.1 – Rates of Change and Tangents to Curves

Position Function

An object that is falling or vertically projected into the air has its height above the ground, $s(t)$, in feet, given by

$$s(t) = -16t^2 + v_0 t + s_0$$

v_0 is the original velocity (initial velocity) of the object, in feet per second

t is the time that the object is in motion, in second

s_0 is the original height (initial height) of the object, in feet

The average rate is given by: $\frac{\Delta s}{\Delta t}$

Example

A rock breaks loose from the top of a tall cliff. What is its average speed

a) During the first 2 *sec* of fall?

b) During the 1-*sec* interval between second 1 and second 2?

Solution

Since the rock falls free (*down*) without any initial velocity or height. $\Rightarrow y(t) = 16t^2$

a) For the first 2 *sec*:

$$\begin{aligned} \text{Average speed} &= \frac{\Delta y}{\Delta t} \\ &= \frac{y(2) - y(0)}{2 - 0} \\ &= \frac{16(2)^2 - 16(0)^2}{2} \\ &= \frac{64}{2} \\ &= 32 \ ft/\sec \end{aligned}$$

b) From 1 *sec* to 2 *sec*:

$$\begin{aligned} \text{Average speed} &= \frac{y(2) - y(1)}{2 - 1} \\ &= \frac{16(2)^2 - 16(1)^2}{1} \\ &= 48 \ ft/\sec \end{aligned}$$

Example

Find the speed of a falling rock $\left(y(t)=16t^2\right)$ over a time interval $\left[t_0,\ t_0+h\right]$. Then find the average speed at 1 *sec* and 2 *sec*.

Solution

$$\frac{\Delta y}{\Delta t}=\frac{16\left(t_0+h\right)^2-16\left(t_0\right)^2}{\left(t_0+h\right)-t_0}$$

$$=\frac{16\left(t_0^2+2ht_0+h^2\right)-16t_0^2}{t_0+h-t_0}$$

$$=\frac{16t_0^2+32ht_0+16h^2-16t_0^2}{h}$$

$$=32\frac{ht_0}{h}+16\frac{h^2}{h}$$

$$\boxed{=32t_0+16h}$$

If $t_0=1\Rightarrow\frac{\Delta y}{\Delta t}=32(1)+16h=\underline{32+16h}$

The average speed has the limiting value 32 *ft/sec* as h approaches 0.

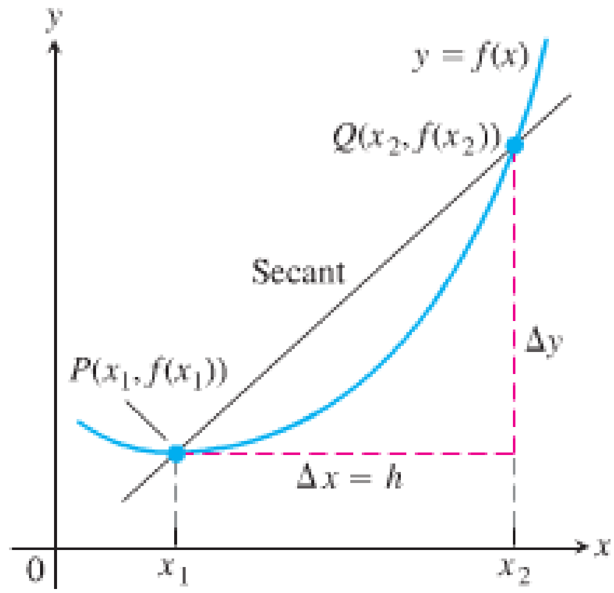
If $t_0=2\Rightarrow\frac{\Delta y}{\Delta t}=32(2)+16h=\underline{64+16h}$

The average speed has the limiting value 64 *ft/sec* as h approaches 0.

Average Rates of Changes and Secant Lines

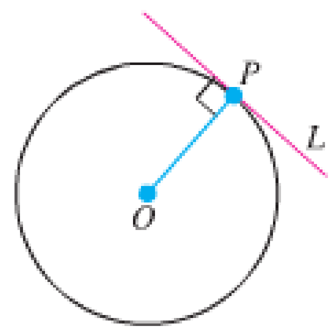
The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0$$



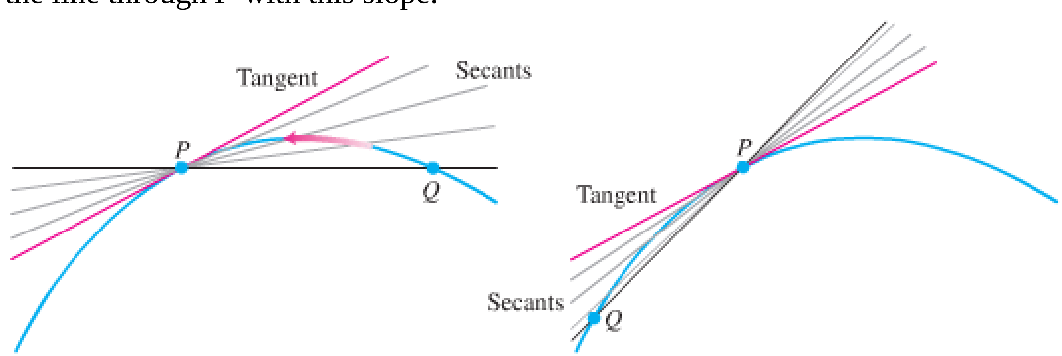
Defining the Slope of a Curve

The slope of a line is the rate at which it rises or falls.



To define the tangency for general curves, we need an approach that makes the behavior of the secants through P and points Q as Q moves toward P along the curve:

1. Find the slope of the secant PQ.
2. Investigate the limiting value of the slope as Q approaches P along the curve.
3. If the limit exists, take it to be the slope of the curve at P and define the tangent to the curve at P to be the line through P with this slope.



Example

Find the slope of the parabola $y = x^2$ at the point $P(2,\ 4)$. Write an equation for the tangent to the parabola at this point.

Solution

$$\textbf{\textit{Secant slope}} = \frac{\Delta y}{\Delta x} = \frac{f\left(x_1 + h\right) - f\left(x_1\right)}{h}$$

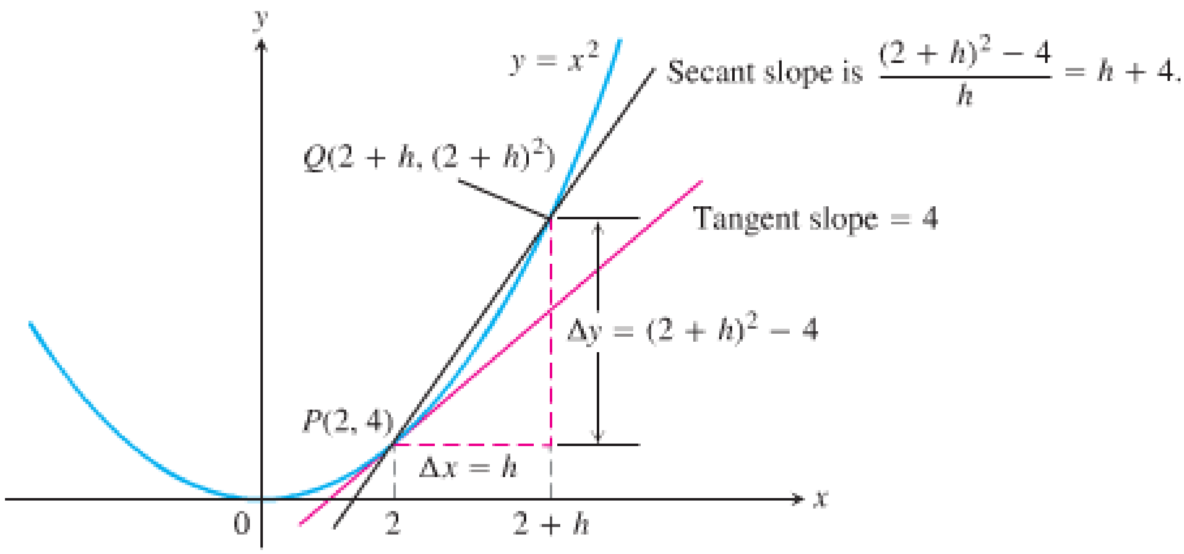
$$= \frac{f(2+h) - f(2)}{h}$$

$$= \frac{(2+h)^2 - 2^2}{h}$$

$$= \frac{4 + 4h + h^2 - 4}{h}$$

$$= \frac{4h}{h} + \frac{h^2}{h}$$

$$= 4 + h$$



As Q approaches P, h approaches 0. Then the secant slope $h + 4 \rightarrow 4 = slope$

$$y = m\left(x - x_1\right) + y_1$$

$$y = 4(x - 2) + 4$$

$$\boxed{y = 4x - 4}$$

Exercises *Section* 1.1 – Rates of Change and Tangents to Curves

1. Find the average rate of change of the function $f(x) = x^3 + 1$ over the interval $[2,\ 3]$

2. Find the average rate of change of the function $f(x) = x^2$ over the interval $[-1,\ 1]$

3. Find the average rate of change of the function $f(t) = 2 + \cos t$ over the interval $[-\pi,\ \pi]$

4. Find the slope of $y = x^2 - 3$ at the point $P(2,\ 1)$ and an equation of the tangent line at this *P*.

5. Find the slope of $y = x^2 - 2x - 3$ at the point $P(2,\ -3)$ and an equation of the tangent line at this *P*.

6. Find the slope of $y = x^3$ at the point $P(2,\ 8)$ and an equation of the tangent line at this *P*.

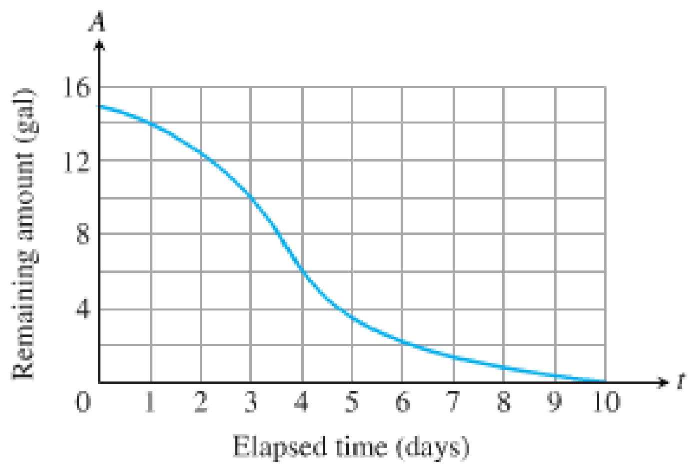
7. Make a table of values for the function $f(x) = \frac{x+2}{x-2}$ at the points

$x = 1.2,\quad x = \frac{11}{10},\quad x = \frac{101}{100},\quad x = \frac{1001}{1000},\quad x = \frac{10001}{10000},\quad \textit{and}\ x = 1$

a) Find the average rate of change of $f(x)$ over the intervals $[1,\ x]$ for each $x \neq 1$ in the table

b) Extending the table if necessary, try to determine the rate of change of $f(x)$ at $x = 1$.

8. The accompanying graph shows the total amount of gasoline A in the gas tank of an automobile after being driven for *t* days.



a) Estimate the average rate of gasoline consumption over the time intervals

$[0,\ 3],\ [0,\ 5],\ \textit{and}\ [7,\ 10]$

b) Estimate the instantaneous rate of gasoline consumption over the time $t = 1,\ t = 4,\ \textit{and}\ t = 8$

c) Estimate the maximum rate of gasoline consumption and the specific time at which it occurs.

Section 1.2 – Limit of a Function and Limit Laws

Definition of the Limit of a Function

If $f(x)$ becomes arbitrary close to a single number L as x approaches x_0 from either side, then

$$\lim_{x \to x_0} f(x) = L$$

Which is read as "the limit of $f(x)$ as x approaches x_0 is L."

Example

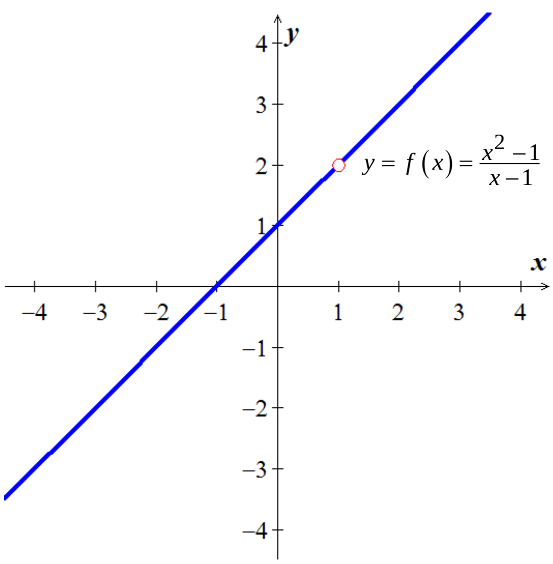
How does the function $f(x) = \dfrac{x^2 - 1}{x - 1}$ behave near $x = 1$?

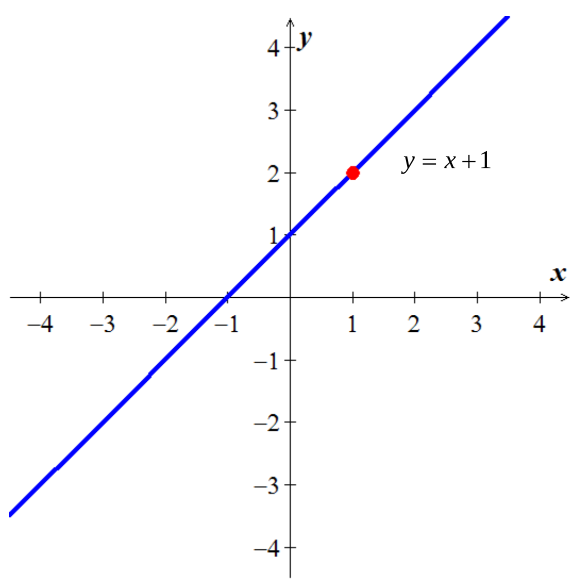
Solution

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1 \quad \textit{for} \quad x \neq 1$$

For $x = 1$:

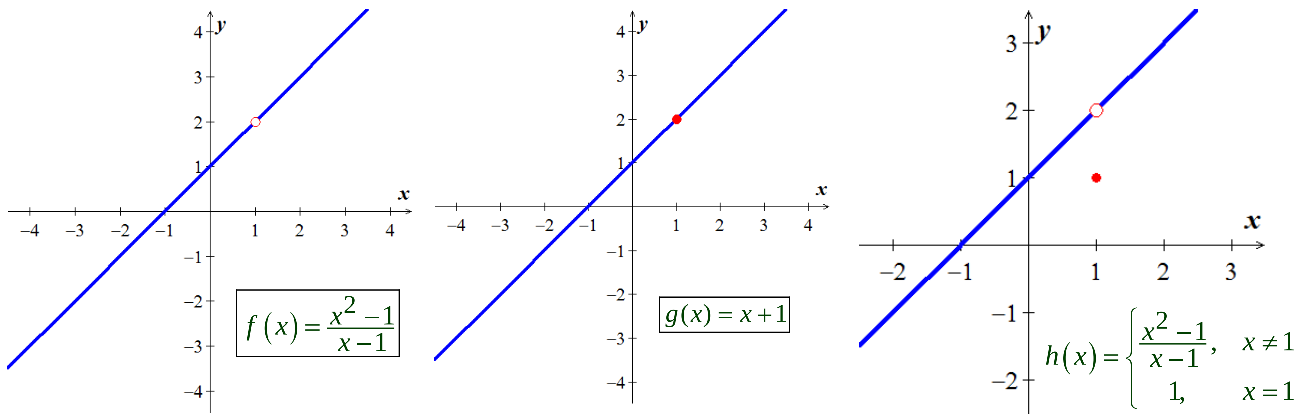
$$f(x = {\color{red}1}) = {\color{red}1} + 1 = 2$$





x	.9	.99	.999	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	2.001	2.01	2.1

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$



Properties of Limits

Constant function $\left(f(x)=k\right)$: $\quad \lim\limits_{x\to x_0} f(x)=\lim\limits_{x\to x_0} k=k$

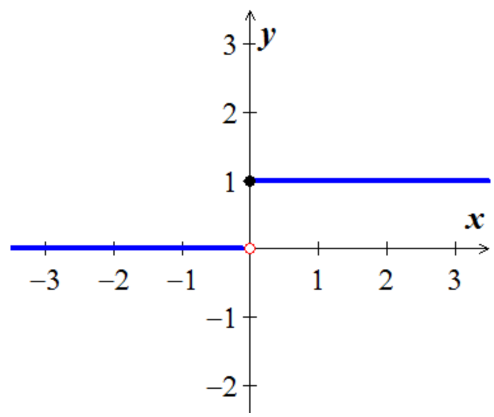
Identity function $\left(f(x)=x\right)$: $\quad \lim\limits_{x\to x_0} f(x)=\lim\limits_{x\to x_0} x=x_0$

Example

Discuss the behavior of the following function as $x\to 0$.

$$U(x)=\begin{cases}0, & x<0\\ 1, & x\geq 0\end{cases}$$

Solution



The unit step function $U(x)$ has no limit as $x\to 0$, it jumps, because the values jump at $x = 0$.

To the left of zero $\left(\text{negative value } \mathbf{0^-}\right)$ $U(x)=0$. For the positive values of x close to zero $\left(\mathbf{0^+}\right)$ $U(x)=1$

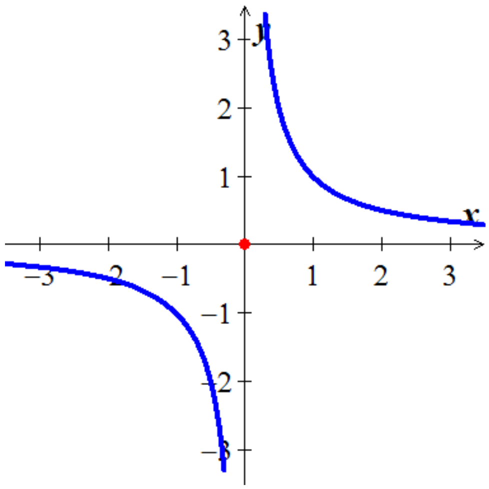
Example

Discuss the behavior of the following function as $x \to 0$.

a) $g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ *b)* $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$

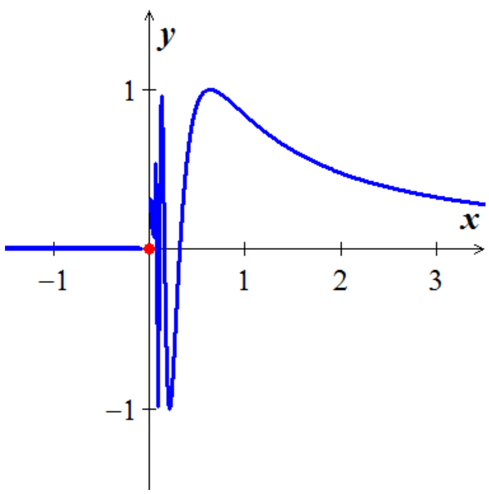
<u>Solution</u>

a)



$g(x)$ has *no limit* as $x \to 0$ because the values of $g(x)$ grow arbitrary large (negative and positive) value as $x \to 0$ and do not stay close.

b)



$f(x)$ has *no limit* as $x \to 0$ because the function's values oscillate between -1 and $+1$ in every open interval containing 0. The values do not stay close to any one number as $x \to 0$.

Limit Laws

If $\lim\limits_{x\to c} f(x) = L$ *and* $\lim\limits_{x\to c} g(x) = M$

Constant Multiple Rule: $\lim\limits_{x\to c}\left[bf(x)\right] = b\lim\limits_{x\to c} f(x) \boxed{= bL}$

Sum and Difference Rules: $\lim\limits_{x\to c}\left[f(x)\pm g(x)\right] = \lim\limits_{x\to c} f(x) \pm \lim\limits_{x\to c} g(x) \boxed{= L \pm M}$

Product Rule: $\lim\limits_{x\to c}\left[f(x)\cdot g(x)\right] = \lim\limits_{x\to c} f(x)\cdot \lim\limits_{x\to c} g(x) \boxed{= L.M}$

Quotient Rule: $\lim\limits_{x\to c}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)} \boxed{= \dfrac{L}{M}}$ $\quad M \ne 0$

Power Rule: $\lim\limits_{x\to c}\left[f(x)\right]^n = \left[\lim\limits_{x\to c} f(x)\right]^n \boxed{= L^n}$

Root Rule: $\lim\limits_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to c} f(x)} \boxed{= \sqrt[n]{L}}$ $\quad n > 0, \quad L > 0,$ *n is even*

Example

Find the following limits:

$a)\ \lim\limits_{x\to c}\left(x^3+4x^2-3\right)$ $\qquad b)\ \lim\limits_{x\to c}\dfrac{x^4+x^2-1}{x^2+5}$ $\qquad c)\ \lim\limits_{x\to -2}\sqrt{4x^2-3}$

Solution

a) $\lim\limits_{x\to c}\left(x^3+4x^2-3\right) = \lim\limits_{x\to c} x^3 + \lim\limits_{x\to c} 4x^2 - \lim\limits_{x\to c}(3)$ ***Sum and Difference Rules***

$$\boxed{= c^3 + 4c^2 - 3}$$

b) $\lim\limits_{x\to c}\dfrac{x^4+x^2-1}{x^2+5} = \dfrac{\lim\limits_{x\to c}\left(x^4+x^2-1\right)}{\lim\limits_{x\to c}\left(x^2+5\right)}$ ***Quotient Rule***

$$= \frac{\lim\limits_{x\to c} x^4 + \lim\limits_{x\to c} x^2 - \lim\limits_{x\to c} 1}{\lim\limits_{x\to c} x^2 + \lim\limits_{x\to c} 5}$$ ***Sum and Difference Rules***

$$\boxed{= \frac{c^4+c^2-1}{c^2+5}}$$

c) $\lim\limits_{x\to -2}\sqrt{4x^2-3} = \sqrt{\lim\limits_{x\to -2}\left(4x^2-3\right)}$ ***Root Rule***

$= \sqrt{\lim\limits_{x\to -2} 4x^2 - \lim\limits_{x\to -2} 3}$ ***Difference Rule***

$= \sqrt{4(-2)^2-3}$

$= \sqrt{16-3}$

$\boxed{= \sqrt{13}}$

Theorem – Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0$$

Theorem – Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \ne 0$, then

$$\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$$

Example

Find the limit: $\lim\limits_{x\to -1} \dfrac{x^3+4x^2-3}{x^2+5}$

Solution

$$\lim_{x\to -1} \frac{x^3+4x^2-3}{x^2+5} = \frac{(-1)^3+4(-1)^2-3}{(-1)^2+5}$$

$$= \frac{0}{6}$$

$$\boxed{= 0}$$

Eliminating Zero Denominators Algebraically

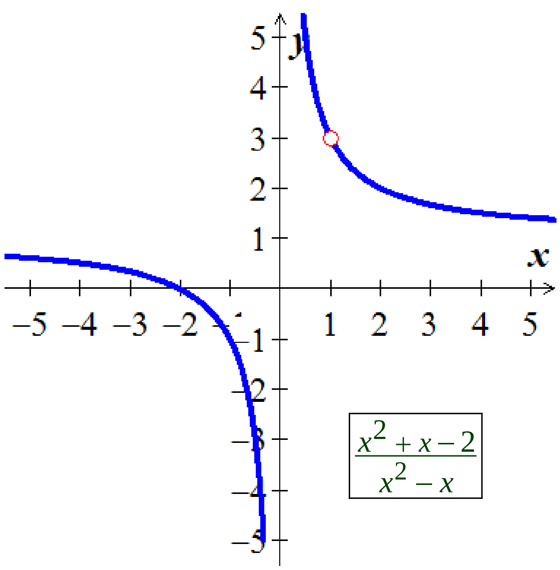
Example

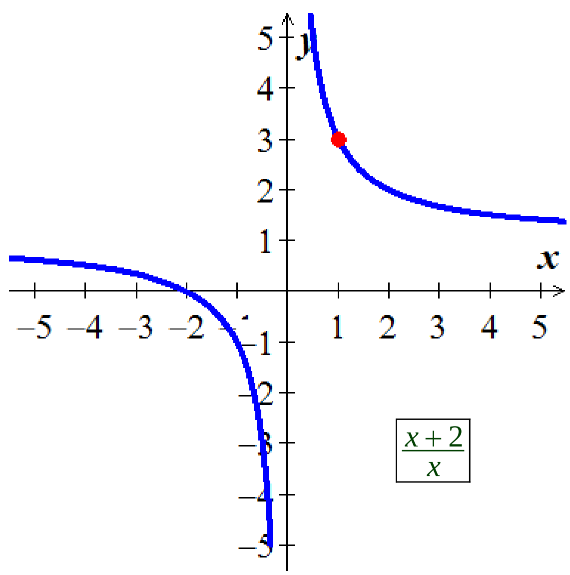
Evaluate: $\lim\limits_{x\to 1} \dfrac{x^2+x-2}{x^2-x}$

<u>Solution</u>

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \frac{1^2+1-2}{1^2-1} = \frac{0}{0}$$

$$\begin{aligned}\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} &= \lim_{x\to 1} \frac{(x-1)(x+2)}{x(x-1)} \\ &= \lim_{x\to 1} \frac{(x+2)}{x} \\ &= \frac{1+2}{1} \\ &= 3\end{aligned}$$





Example

Evaluate: $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+100}-10}{x^2}$

Solution

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}=\frac{\sqrt{0+100}-10}{0}=\frac{0}{0}$$

$$\frac{\sqrt{x^2+100}-10}{x^2}=\frac{\sqrt{x^2+100}-10}{x^2}\cdot\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$$

$(a-b)(a+b)=a^2-b^2;\quad \left(\sqrt{a}\right)^2=a$

$$=\frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$=\frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$=\frac{1}{\sqrt{x^2+100}+10}$$

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}=\lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10}$$

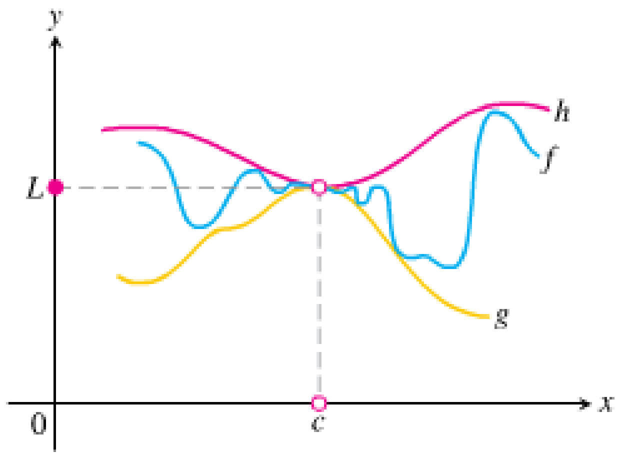
$$=\frac{1}{\sqrt{0+100}+10}$$

$$=\frac{1}{10+10}$$

$$=\frac{1}{20}$$

$$\underline{=0.05}$$

The Sandwich Theorem



Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L \quad \textit{then} \quad \lim_{x\to c} f(x) = L$$

Example

Given that $1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2} \quad \textit{for all } x \ne 0$, find the $\lim\limits_{x\to 0} u(x)$, no matter how complicated u is.

<u>Solution</u>

$$\lim_{x\to 0}\left(1 - \frac{x^2}{4}\right) = 1 - \frac{0}{4} = 1$$

$$\lim_{x\to 0}\left(1 + \frac{x^2}{2}\right) = 1$$

The Sandwich theorem implies that $\lim\limits_{x\to 0} u(x) = 1$

Theorem

Suppose that $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x)$$

Exercises *Section* 1.2 – Limit of a Function and Limit Laws

Find the limit:

1. $\lim\limits_{x\to 1}(2x+4)$

2. $\lim\limits_{x\to 1}\dfrac{x^2-4}{x-2}$

3. $\lim\limits_{x\to 2}\dfrac{x^2+4}{x-2}$

4. $\lim\limits_{x\to 0}\dfrac{|x|}{x}$

5. $\lim\limits_{x\to 3}\dfrac{x^2-x-1}{\sqrt{x+1}}$

6. $\lim\limits_{x\to 2}\dfrac{x^2+x-6}{x-2}$

7. $\lim\limits_{x\to 0}(3x-2)$

8. $\lim\limits_{x\to 1}(2x^2-x+4)$

9. $\lim\limits_{x\to -2}\left(x^3-2x^2+4x+8\right)$

10. $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}$

11. $\lim\limits_{x\to 2}\dfrac{x^3-8}{x-2}$

12. $\lim\limits_{x\to 3}\dfrac{x^2+x-12}{x-3}$

13. $\lim\limits_{x\to 0}\dfrac{\sqrt{x+4}-2}{x}$

14. $\lim\limits_{x\to 0}\dfrac{3}{\sqrt{3x+1}+1}$

15. $\lim\limits_{x\to 0} f(x)$ $\quad f(x)=\begin{cases} x^2+1 & x<0 \\ 2x+1 & x>0 \end{cases}$

16. $\lim\limits_{x\to -2}\dfrac{5}{x+2}$

17. $\lim\limits_{x\to 3}\dfrac{\sqrt{x+1}-1}{x}$

18. $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}$

19. $\lim\limits_{x\to -2}\dfrac{|x+2|}{x+2}$

20. $\lim\limits_{x\to 0}(2z-8)^{1/3}$

21. $\lim\limits_{x\to 2}\dfrac{x^2-7x+10}{x-2}$

22. $\lim\limits_{x\to 0}\dfrac{5x^3+8x^2}{3x^4-16x^2}$

23. $\lim\limits_{x\to 1}\dfrac{\frac{1}{x}-1}{x-1}$

24. $\lim\limits_{u\to 1}\dfrac{u^4-1}{u^3-1}$

25. $\lim\limits_{x\to 1}\dfrac{x-1}{\sqrt{x+3}-2}$

26. $\lim\limits_{x\to -1}\dfrac{\sqrt{x^2+8}-3}{x+1}$

27. $\lim\limits_{x\to -3}\dfrac{2-\sqrt{x^2-5}}{x+3}$

28. $\lim\limits_{x\to 0}(2\sin x-1)$

29. $\lim\limits_{x\to 0}\sin^2 x$

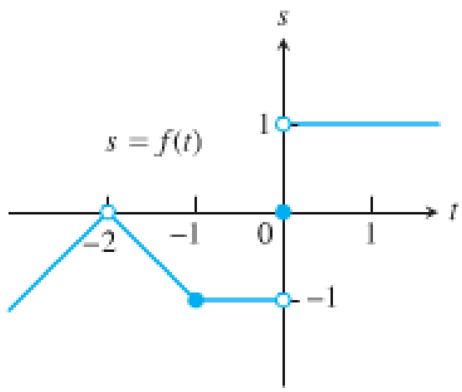
30. $\lim\limits_{x\to 0}\sec x$

31. $\lim\limits_{x\to 0}\dfrac{1+x+\sin x}{3\cos x}$

32. $\lim\limits_{x\to -\pi}\sqrt{x+4}\cos(x+\pi)$

33. For the function $f(t)$ graphed, find the following limits or explain why they do not exist.

$a)\ \lim\limits_{t\to -2} f(t) \quad b)\ \lim\limits_{t\to -1} f(t) \quad c)\ \lim\limits_{t\to 0} f(t) \quad d)\ \lim\limits_{t\to -0.5} f(t)$



34. Suppose $\lim\limits_{x\to c} f(x)=5$ *and* $\lim\limits_{x\to c} g(x)=-2$. Find

a) $\lim\limits_{x\to c} f(x)g(x)$

b) $\lim\limits_{x\to c} 2f(x)g(x)$

c) $\lim\limits_{x\to c} \left(f(x)+3g(x)\right)$

d) $\lim\limits_{x\to c} \dfrac{f(x)}{f(x)-g(x)}$

35. Explain why the limits do not exist for $\lim\limits_{x\to 0} \dfrac{x}{|x|}$

36. Evaluate the limit using the form $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ for $f(x)=x^2, \quad x=1$

37. Evaluate the limit using the form $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ for $f(x)=\sqrt{3x+1}, \quad x=0$

38. If $\lim\limits_{x\to 4} \dfrac{f(x)-5}{x-2}=1$, find $\lim\limits_{x\to 4} f(x)$

39. If $\lim\limits_{x\to 0} \dfrac{f(x)}{x^2}=1$, find $\lim\limits_{x\to 0} f(x)$ *and* $\lim\limits_{x\to 0} \dfrac{f(x)}{x}$

40. If $x^4 \le f(x) \le x^2 \quad -1\le x\le 1$ and $x^2 \le f(x) \le x^4 \quad x<-1$ *and* $x>1$. At what points c do you automatically know $\lim\limits_{x\to c} f(x)$? What can you say about the value of the limits at these points?

Section 1.3 – Precise Definition of a Limit

Example

Consider the function $y=2x-1$ near $x_0=4$. Intuitively it appears that *y* is close to 7 when *x* is close to 4, so $\lim\limits_{x\to 4}(2x-1)=7$. However, how close to $x_0=4$ does *x* have to be so that $y=2x-1$ differs from 7 by, say less than 2 units?

Solution

We need to find the values of *x* for $|y-7|<2$.

$|y-7|=|2x-1-7|=|2x-8|$

$|2x-8|<2$

$-2<2x-8<2$

$-2+8<2x-8+8<2+8$

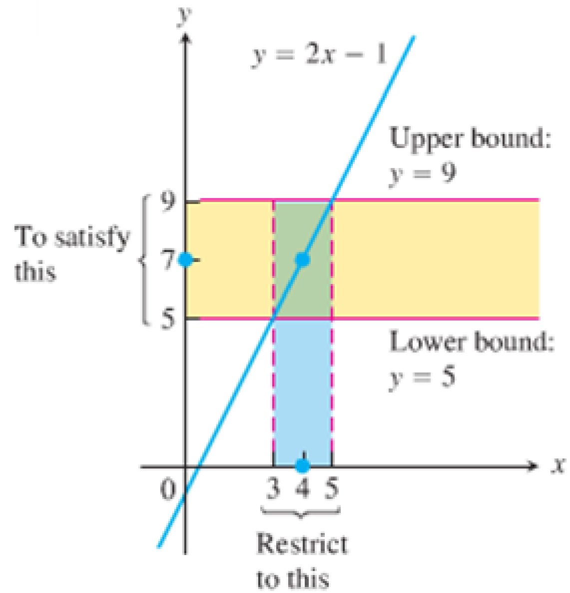
$6<2x<10$

$\frac{6}{2}<\frac{2x}{2}<\frac{10}{2}$

$3<x<5$

$3-4<x-4<5-4$

$-1<x-4<1$



Keeping *x* within 1 unit of $x_0=4$ will keep *y* within 2 units of $y_0=7$

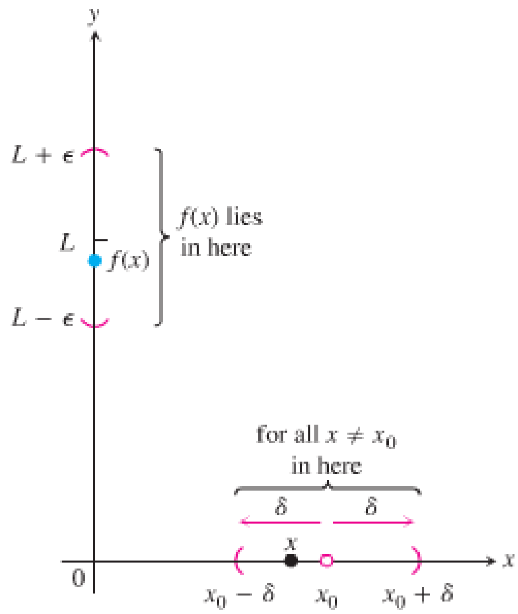
Definition

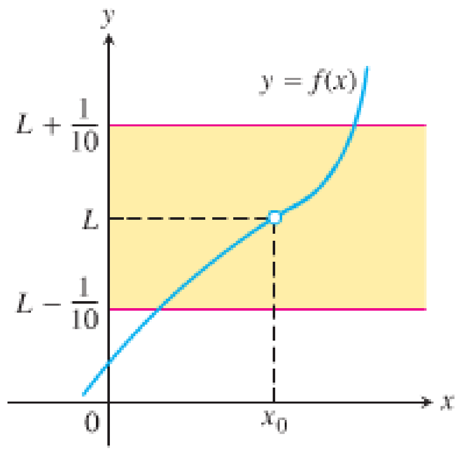
Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that **the limit of** $f(x)$ **as *x* approaches** x_0 **is the number *L***, and write

$$\lim_{x\to x_0} f(x)=L$$

If, for every number $\varepsilon>0$, there exists a corresponding number $\delta>0$ such that for all *x*,

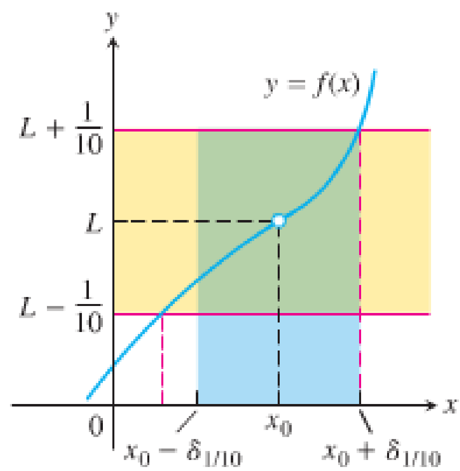
$$0<|x-x_0|<\delta \quad\Rightarrow\quad |f(x)-L|<\varepsilon$$





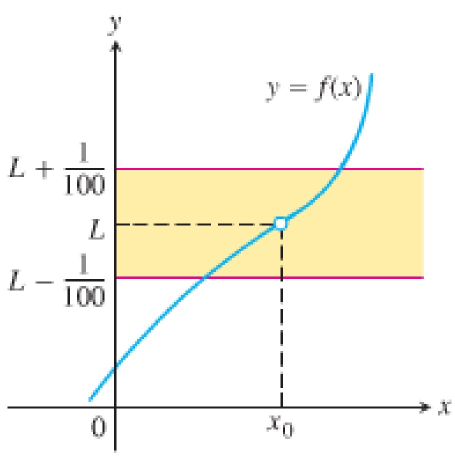
The challenge:

Make $\left|f(x) - L\right| < \epsilon = \frac{1}{10}$



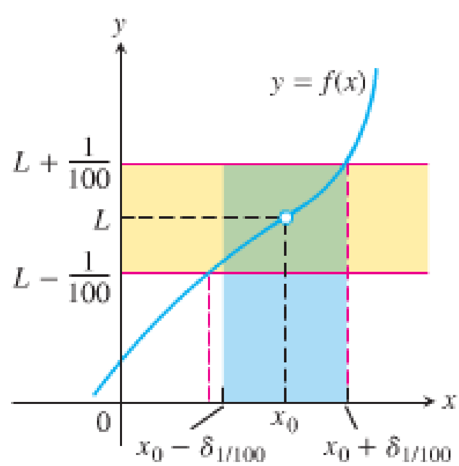
Response:

$\left|x - x_0\right| < \delta_{1/10}$ (a number)



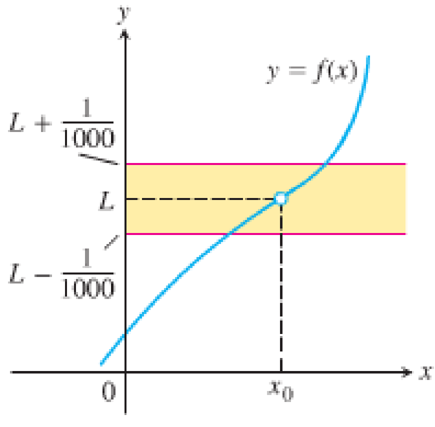
New challenge:

Make $\left|f(x) - L\right| < \epsilon = \frac{1}{100}$

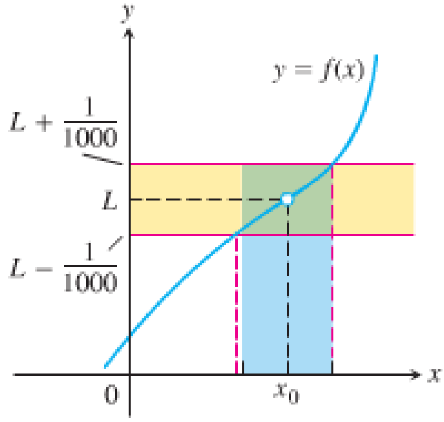


Response:

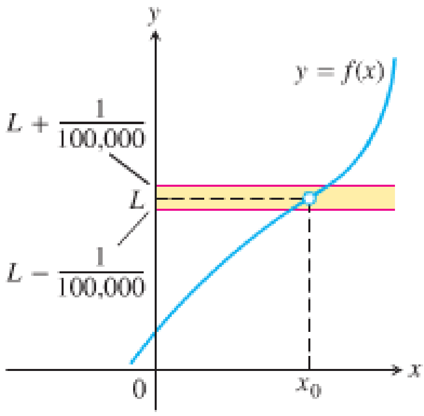
$\left|x - x_0\right| < \delta_{1/100}$



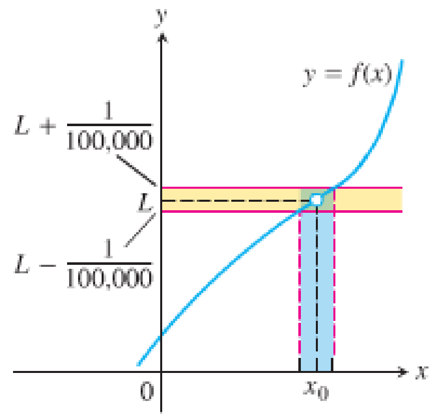
New challenge:
$\epsilon = \frac{1}{1000}$



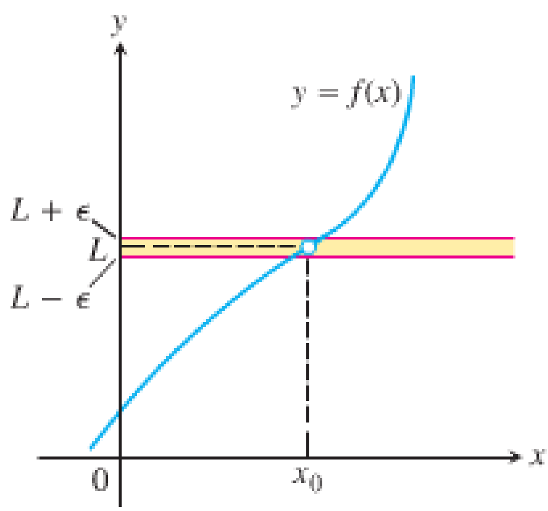
Response:
$|x - x_0| < \delta_{1/1000}$



New challenge:
$\epsilon = \frac{1}{100{,}000}$



Response:
$|x - x_0| < \delta_{1/100{,}000}$



New challenge:
$\epsilon = \cdots$

Example

Show that $\lim_{x \to 1} (5x - 3) = 2$

Solution

Let $x_0 = 1, \quad f(x) = 5x - 3, \quad and \quad L = 2$.

For any given $\varepsilon > 0$, there exists a $\delta > 0$ so that $x \neq 1$ and x is within distance δ of $x_0 = 1$, that is

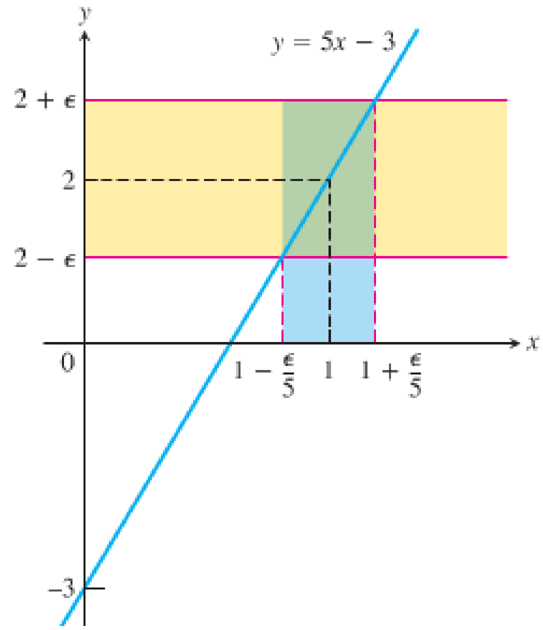
$$0 < |x - 1| < \delta \quad \Rightarrow \quad |f(x) - 2| < \varepsilon$$

$$|(5x - 3) - 2| < \varepsilon$$

$$|5x - 5| < \varepsilon$$

$$5|x - 1| < \varepsilon$$

$$|x - 1| < \frac{\varepsilon}{5}$$



Thus, we can take: $\delta = \frac{\varepsilon}{5}$

If $0 < |x - 1| < \delta = \frac{\varepsilon}{5}$

$$|(5x - 3) - 2| = |5x - 5| = 5|x - 1| = 5\frac{\varepsilon}{5} = \varepsilon$$

Which proves that $\lim_{x \to 1} (5x - 3) = 2$

Example

Prove the results presented graphically $\lim_{x \to x_0} x = x_0$

Solution

Let $\varepsilon > 0$ be given, we must find $\delta > 0$ such that for all x

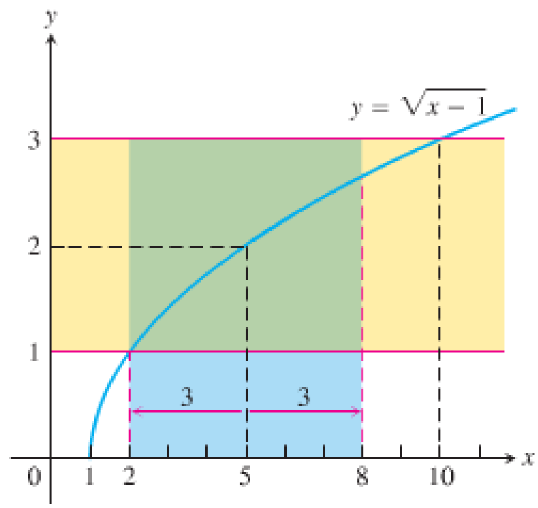
$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |x - x_0| < \varepsilon$$

This implication will hold if $\delta = \varepsilon$ or any smaller number.

Example

For the limit $\lim\limits_{x\to 5}\sqrt{x-1}=2$, find a $\delta > 0$ that works for $\varepsilon = 1$. That is, find a $\delta > 0$ such that for all x:

$0<|x-5|<\delta \quad \Rightarrow \quad \left|\sqrt{x-1}-2\right|<1$



Solution

$\left|\sqrt{x-1}-2\right|<1$

$$-1<\sqrt{x-1}-2<1$$

$$-1+2<\sqrt{x-1}-2+2<1+2$$

$$1<\sqrt{x-1}<3 \qquad \textit{Square all sides}$$

$$1<x-1<9$$

$$1+1<x-1+1<9+1$$

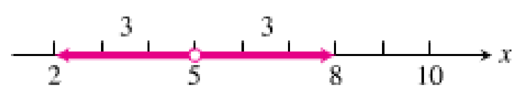
$$2<x<10$$

The inequality holds for all x in the open interval $(2,\ 10)$.
So it holds for all $x\neq 5$ in the interval as well.

Finding δ value.

$5-\delta<x<5+\delta$ Centered at $x_0=5$ inside the interval $(2,\ 10)$

$\begin{cases}5-\delta=2\\ 5+\delta<10\end{cases}\to \delta=3$ ***(to be centered)***



$0<|x-5|<3 \quad \Rightarrow \quad \left|\sqrt{x-1}-2\right|<1$

How to Find *Algebraically* a δ for a Given f, L, x_0, and $\varepsilon > 0$

The process of finding a $\delta > 0$ such that for all x:

$$0<|x-x_0|<\delta \quad \Rightarrow \quad |f(x)-L|<\varepsilon$$

Can be accomplished in two steps

1. Solve the inequality $|f(x)-L|<\varepsilon$ to find an open interval $(a,\ b)$ containing x_0 on which the inequality holds for all $x\neq x_0$.
2. Find a value of $\delta > 0$ that places the open interval $(x_0-\delta,\ x_0+\delta)$ centered at x_0 inside the interval $(a,\ b)$. The inequality $|f(x)-L|<\varepsilon$ will hold for all $x\neq x_0$ in this δ–interval.

Example

Prove that $\lim_{x \to 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

<u>Solution</u>

We need to show that given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x:

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |f(x) - 4| < \varepsilon$$

1. *Solve the inequality* $|f(x) - 4| < \varepsilon$ *to find an open interval containing* $x_0 = 2$ *on which the inequality holds for all* $x \neq x_0$.

For $x \neq x_0 = 2$, $f(x) = x^2$, and the inequality to solve is $|x^2 - 4| < \varepsilon$:

$$|x^2 - 4| < \varepsilon$$

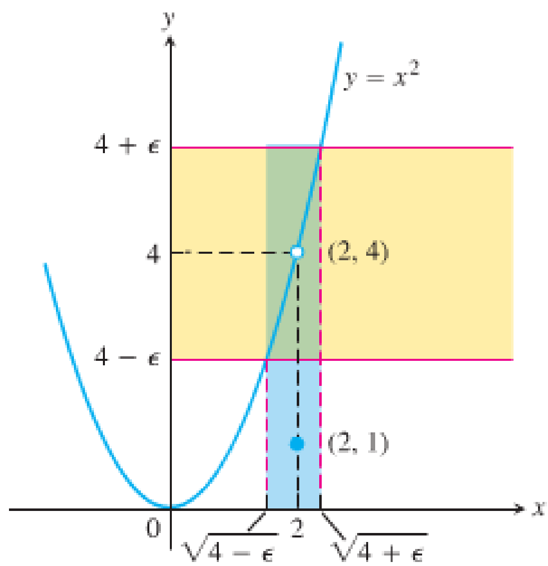
$$-\varepsilon < x^2 - 4 < \varepsilon \qquad \textit{Add 4 to all sides}$$

$$4 - \varepsilon < x^2 < 4 + \varepsilon \qquad \textit{Square root}$$

$$\sqrt{4 - \varepsilon} < |x| < \sqrt{4 + \varepsilon} \qquad \textit{Assume } \varepsilon < 4$$

$$\sqrt{4 - \varepsilon} < x < \sqrt{4 + \varepsilon}$$

The inequality $|f(x) - 4| < \varepsilon$ holds for all $x \neq 2$ in the open interval $\left(\sqrt{4 - \varepsilon},\ \sqrt{4 + \varepsilon}\right)$



2. *Find a value of* $\delta > 0$ *that places the open interval* $(2 - \delta,\ 2 + \delta)$ *inside the interval* $\left(\sqrt{4 - \varepsilon},\ \sqrt{4 + \varepsilon}\right)$.

Take δ to be the distance from $x_0 = 2$ to the nearer endpoint of $\left(\sqrt{4 - \varepsilon},\ \sqrt{4 + \varepsilon}\right)$.

$$\Rightarrow \delta = \min\left(2-\sqrt{4-\varepsilon},\ \sqrt{4+\varepsilon}-2\right).$$

$$0<|x-2|<\delta$$

$$-\left(2-\sqrt{4-\varepsilon}\right)<x-2<\sqrt{4+\varepsilon}-2$$

$$-2+\sqrt{4-\varepsilon}<x-2<\sqrt{4+\varepsilon}-2$$

$$\sqrt{4-\varepsilon}<x<\sqrt{4+\varepsilon}$$

$$\therefore\ \ 0<|x-2|<\delta \ \ \Rightarrow\ \ |f(x)-4|<\varepsilon$$

Example

Given that $\lim\limits_{x\to c} f(x)=L$ *and* $\lim\limits_{x\to c} g(x)=M$, prove that $\lim\limits_{x\to c}\left(f(x)+g(x)\right)=L+M$

Solution

We need to show that given $\varepsilon>0$ there exists a $\delta>0$ such that for all *x*:

$$0<|x-c|<\delta \ \ \Rightarrow\ \ |f(x)+g(x)-(L+M)|<\varepsilon$$

$$\begin{aligned} |f(x)+g(x)-(L+M)| &= |f(x)+g(x)-L-M| \\ &= |(f(x)-L)+(g(x)-M)| \qquad \textit{Triangle Inequality } |a+b|\le|a|+|b| \\ &\le |(f(x)-L)|+|(g(x)-M)| \end{aligned}$$

Since $\lim\limits_{x\to c} f(x)=L$, there exists a number $\delta_1>0$ such that for all *x*:

$$0<|x-c|<\delta_1 \ \ \Rightarrow\ \ |f(x)-L|<\frac{\varepsilon}{2}$$

Similarly, since $\lim\limits_{x\to c} g(x)=M$, there exists a number $\delta_2>0$ such that for all *x*:

$$0<|x-c|<\delta_2 \ \ \Rightarrow\ \ |g(x)-M|<\frac{\varepsilon}{2}$$

Let $\delta=\min\{\delta_1,\ \delta_2\}$, the smaller of δ_1 and δ_2. If $0<|x-c|<\delta$ then $0<|x-c|<\delta_1$, so $|f(x)-L|<\frac{\varepsilon}{2}$ and $|x-c|<\delta_2$, so $|g(x)-M|<\frac{\varepsilon}{2}$. Therefore

$$|f(x)+g(x)-(L+M)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

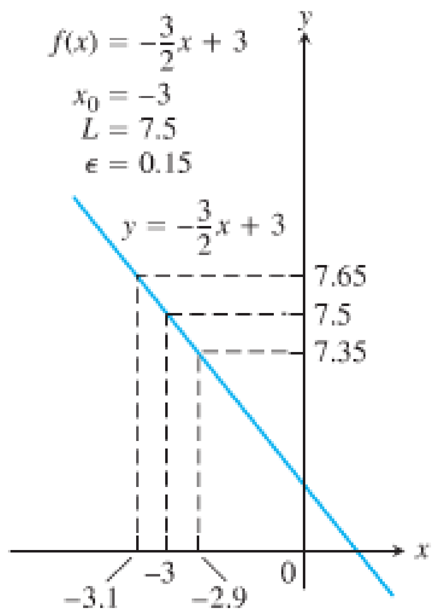
This show that $\lim\limits_{x\to c}\left(f(x)+g(x)\right)=L+M$

Exercises *Section* 1.3 – Precise Definition of a Limit

1. Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0<|x-x_0|<\delta \;\Rightarrow\; a<x<b$ for $a=1, \quad b=7, \quad x_0=5$

2. Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0<|x-x_0|<\delta \;\Rightarrow\; a<x<b$ for $a=-\frac{7}{2}, \quad b=-\frac{1}{2}, \quad x_0=-\frac{3}{2}$

3. Use the graph to find a $\delta > 0$ such that for all x $0<|x-x_0|<\delta \;\Rightarrow\; |f(x)-L|<\varepsilon$



4. Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=x+1, \quad L=5, \quad x_0=4, \quad \varepsilon=0.01$$

5. Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=\sqrt{x+1}, \quad L=1, \quad x_0=0, \quad \varepsilon=0.1$$

6. Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=\sqrt{x-7}, \quad L=4, \quad x_0=23, \quad \varepsilon=1$$

7. Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=x^2, \quad L=3, \quad x_0=\sqrt{3}, \quad \varepsilon=0.1$$

8. Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

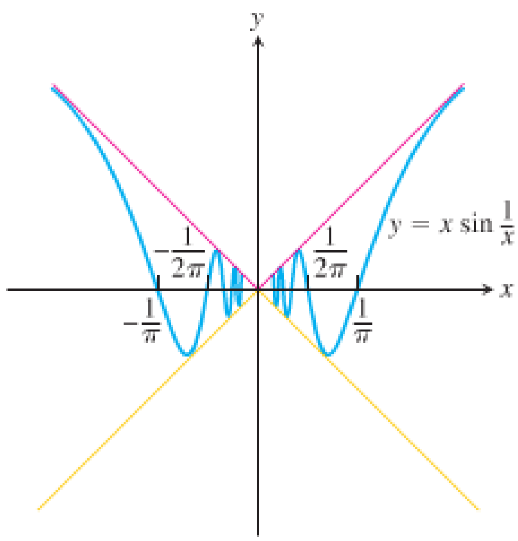
$$f(x)=\frac{120}{x}, \quad L=5, \quad x_0=24, \quad \varepsilon=1$$

9. Prove that $\lim\limits_{x\to 4}(9-x)=5$

10. Prove that $\lim\limits_{x\to 1}\frac{1}{x}=1$

11. Prove that $\lim\limits_{x\to 0} f(x)=0 \quad if \quad f(x)=\begin{cases} 2x, & x<0 \\ \frac{x}{2}, & x\geq 0 \end{cases}$

12. Prove that $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right)=0$



Section 1.4 – One-Sided Limits

Notation	***Terminology***
$x \to a^-$	***x*** approaches ***a*** from the left (through values ***less*** than a)
$x \to a^+$	***x*** approaches ***a*** from the right (through values ***greater*** than a)
$f(x) \to \infty$	$f(x)$ increases without bound (can be made as large positive as desired)
$f(x) \to -\infty$	$f(x)$ decreases without bound (can be made as large negative as desired)

One-Sided Limits

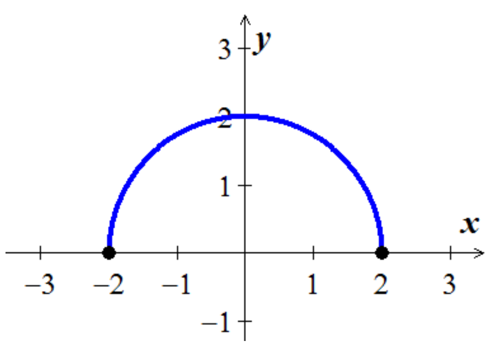
To have a limit L as x approaches c, a function f must be defined on ***both sides*** of c and its values $f(x)$ must approach L as x approaches c from either side. Because of this, ordinary limits are called ***two-sided***. If f fails to have two-sided limit at c, it may still have one-sided limit.

If the approach is from the *right*, the limit is a ***right-hand limit***. $\lim\limits_{x \to c^+} f(x) = L$

If the approach is from the *left*, the limit is a ***left-hand limit***. $\lim\limits_{x \to c^-} f(x) = M$

Example

The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2,\ 2]$; its graph is the semicircle.



We have: $\lim\limits_{x \to -2^+} \sqrt{4 - x^2} = 0$ *and* $\lim\limits_{x \to 2^-} \sqrt{4 - x^2} = 0$

The function doesn't have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have ordinary two-sided limits at either -2 or 2.

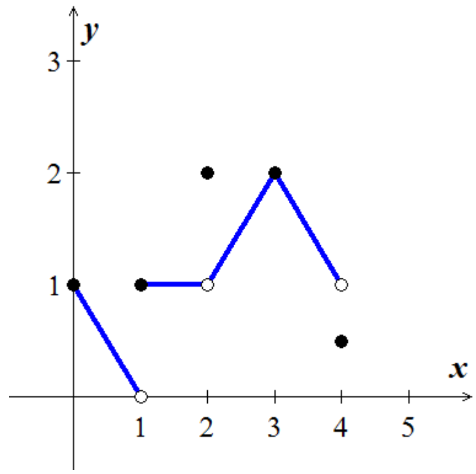
Theorem

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \textit{and} \quad \lim_{x\to c^+} f(x) = L$$

Example

Given the function graphed:



At $x = 0$: $\lim\limits_{x\to 0^+} f(x) = 1$

$\lim\limits_{x\to 0^-} f(x)$ *and* $\lim\limits_{x\to 0} f(x)$ don't exist. The function is not defined to the left of $x = 0$

At $x = 1$: $\lim\limits_{x\to 1^-} f(x) = 0 \qquad \lim\limits_{x\to 1^+} f(x) = 1$

$\lim\limits_{x\to 1} f(x)$ doesn't exist. The right-hand and left-hand limits are not equal.

At $x = 2$: $\lim\limits_{x\to 2^-} f(x) = 1 \qquad \lim\limits_{x\to 2^+} f(x) = 1$

$\lim\limits_{x\to 2} f(x) = 2$ even though $f(2) = 2$

At $x = 3$: $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3} f(x) = 2$

At $x = 4$: $\lim\limits_{x\to 4^-} f(x) = 1$ even though $f(4) \ne 1$

$\lim\limits_{x\to 4^+} f(x)$ *and* $\lim\limits_{x\to 4} f(x)$ do not exist.

The function is not defined to the right of $x = 4$

Definitions

We say that $f(x)$ has right-hand limit L at x_0 and $\lim\limits_{x \to x_0^+} f(x) = L$

If for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

We say that $f(x)$ has left-hand limit L at x_0 and $\lim\limits_{x \to x_0^-} f(x) = L$

If for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

Example

Prove that $\lim\limits_{x \to 0^+} \sqrt{x} = 0$

Solution

Let $\varepsilon > 0$ be given. $x_0 = 0$, $L = 0$, *Find* $\delta > 0 \ni \forall x$

$$0 < x < \delta \quad \Rightarrow \quad \left|\sqrt{x} - 0\right| < \varepsilon$$

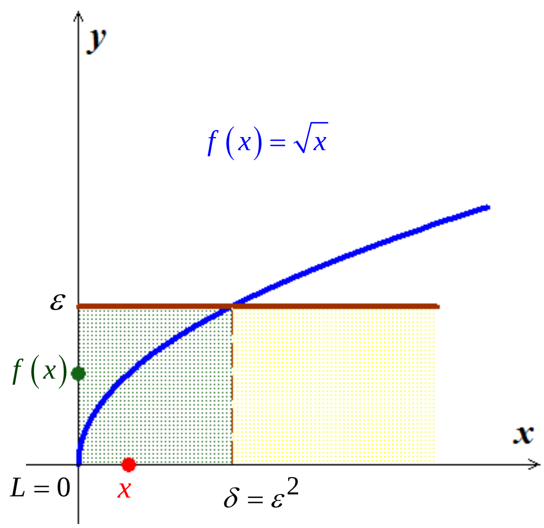
or $\quad 0 < x < \delta \quad \Rightarrow \quad \sqrt{x} < \varepsilon$

$$\left(\sqrt{x}\right)^2 < \varepsilon^2$$

$$\Rightarrow x < \varepsilon^2 \quad \text{if} \quad 0 < x < \delta$$

If we choose $\delta = \varepsilon^2$, we have

$$0 < x < \delta = \varepsilon^2 \quad \Rightarrow \quad \sqrt{x} < \varepsilon$$

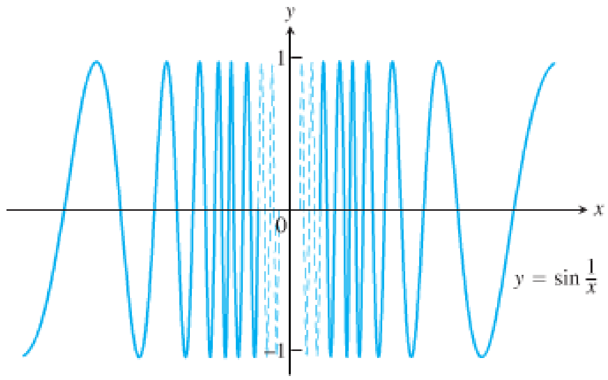


According to the definition, this shows that $\lim\limits_{x \to 0^+} \sqrt{x} = 0$

Example

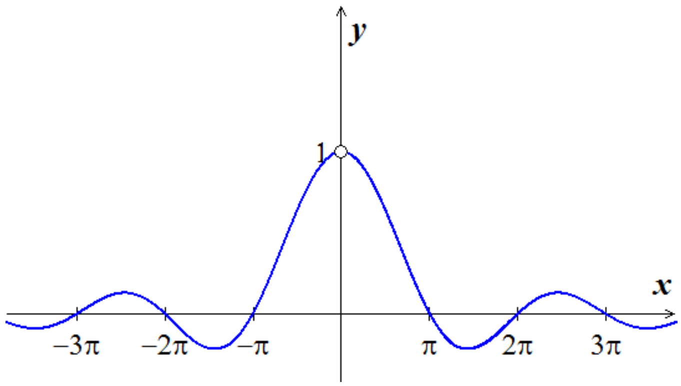
Show that $y = \sin\left(\frac{1}{x}\right)$ has no limit as x approaches zero from either side.

<u>Solution</u>



As x approaches zero, its reciprocal, $\frac{1}{x}$, grows without bound and the values of $\sin\left(\frac{1}{x}\right)$ cycle repeatedly from -1 to 1. There is no single number L that the function's values stay increasingly close to as x approaches zero.. The function has neither a right-hand limit nor a left-hand limit at $x = 0$.

Limit Involving $\frac{\sin\theta}{\theta}$



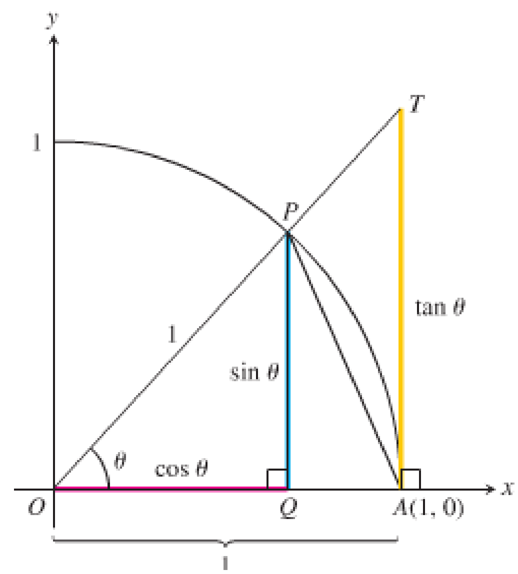
A central fact about $\frac{\sin\theta}{\theta}$ is that in radian measure it limit as $\theta \to 0$ is **1**.

Theorem

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \quad (\theta \text{ in rad.})$$

Proof

We need to show that the right-hand limit is 1, $\theta < \frac{\pi}{2}$



Notice that:

$$Area\ \Delta OAP \ < Area\ Sector\ OAP \ < Area\ \Delta OAT$$

$$Area\ \Delta OAP = \tfrac{1}{2} base \times height = \tfrac{1}{2}(1)(\sin\theta)$$

$$Area\ Sector\ \Delta OAP = \tfrac{1}{2} r^2 \times \theta = \tfrac{1}{2}(1)^2(\theta) = \frac{\theta}{2}$$

$$Area\ \Delta OAP = \tfrac{1}{2} base \times height = \tfrac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta$$

$$\Rightarrow \tfrac{1}{2}\sin\theta < \tfrac{1}{2}\theta < \tfrac{1}{2}\tan\theta$$

$$\frac{2}{\sin\theta}\frac{1}{2}\sin\theta < \frac{1}{2}\theta\frac{2}{\sin\theta} < \frac{1}{2}\frac{\sin\theta}{\cos\theta}\frac{2}{\sin\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

Taking reciprocals reverses the inequalities

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Since $\lim\limits_{\theta\to 0^+}\cos\theta = 1$, then $\lim\limits_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1 = \lim\limits_{\theta\to 0^+}\frac{\sin\theta}{\theta}$

So $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

Example

Show that $\lim\limits_{h\to 0}\dfrac{\cos h-1}{h}=0$

Solution

Using the half-angle formula: $\cos h=1-2\sin^2\left(\frac{h}{2}\right)$

$$\lim_{h\to 0}\frac{\cos h-1}{h}=\lim_{h\to 0}\frac{1-2\sin^2\left(\frac{h}{2}\right)-1}{h}$$

$$=\lim_{h\to 0}\frac{-2\sin^2\left(\frac{h}{2}\right)}{h}$$

Let $\theta=\frac{h}{2}$

$$=-\lim_{\theta\to 0}\frac{2\sin^2(\theta)}{2\theta}$$

$$=-\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta$$

$$=-(1)(0)$$

$$\boxed{=0}$$

Example

Show that $\lim\limits_{x\to 0}\dfrac{\sin 2x}{5x}=\dfrac{2}{5}$

Solution

$$\lim_{x\to 0}\frac{\sin 2x}{5x}=\lim_{x\to 0}\frac{\left(\frac{2}{5}\right)\sin 2x}{\left(\frac{2}{5}\right)5x}$$

Since we need 2x in the denominator

$$=\frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x}$$

$$=\frac{2}{5}(1)$$

$$\boxed{=\frac{2}{5}}$$

Example

Show that $\lim_{x\to 0} \dfrac{\tan x \ \sec 2x}{3x} = \dfrac{1}{3}$

Solution

$$\lim_{x\to 0} \frac{\tan x \ \sec 2x}{3x} = \frac{1}{3}\lim_{x\to 0} \frac{1}{x}\cdot\frac{\sin x}{\cos x}\cdot\frac{1}{\cos 2x}$$

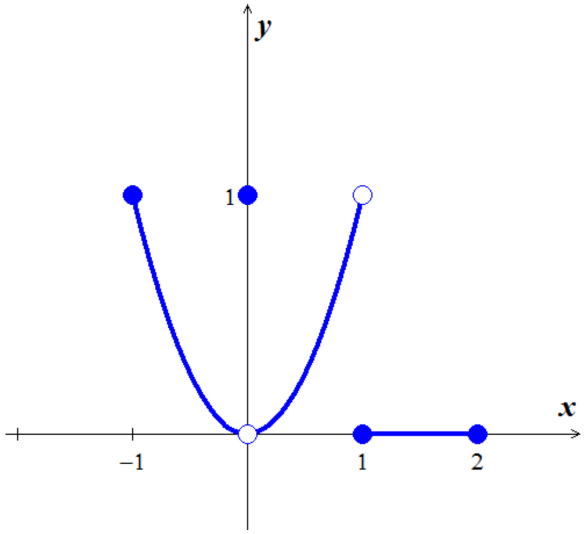
$$= \frac{1}{3}\lim_{x\to 0} \frac{\sin x}{x}\cdot\frac{1}{\cos x}\cdot\frac{1}{\cos 2x} \qquad \lim_{x\to 0}\frac{\sin x}{x} = 1, \quad \lim_{x\to 0}\frac{1}{\cos x} = 1, \quad \lim_{x\to 0}\frac{1}{\cos 2x} = 1$$

$$= \frac{1}{3}(1)(1)(1)$$

$$= \frac{1}{3}$$

Exercises *Section* 1.4 – One-Sided Limits

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a) $\lim\limits_{x\to -1^+} f(x) = 1$

b) $\lim\limits_{x\to 0^-} f(x) = 0$

c) $\lim\limits_{x\to 0^-} f(x) = 1$

d) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$

e) $\lim\limits_{x\to 0} f(x)$ *exists*

f) $\lim\limits_{x\to 0} f(x) = 0$

g) $\lim\limits_{x\to 0} f(x) = 1$

h) $\lim\limits_{x\to 1} f(x) = 1$

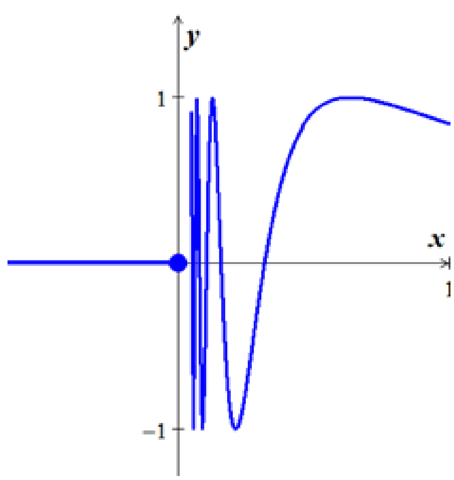
i) $\lim\limits_{x\to 1} f(x) = 0$

j) $\lim\limits_{x\to 2^-} f(x) = 2$

k) $\lim\limits_{x\to -1^-} f(x) = 0$ *does not exist*

l) $\lim\limits_{x\to 2^+} f(x) = 0$

2. Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$

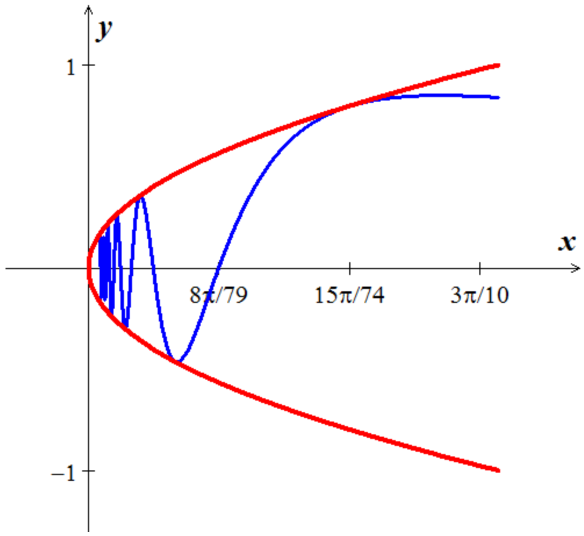


a) Does $\lim\limits_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

3. Let $g(x) = \sqrt{x}\ \sin\frac{1}{x}$



a) Does $\lim_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

Find

4. $\lim_{x\to -0.5^-} \sqrt{\dfrac{x+2}{x+1}}$

5. $\lim_{x\to 1^+} \sqrt{\dfrac{x-1}{x+2}}$

6. $\lim_{x\to -2^+} \left(\dfrac{x}{x+1}\right)\left(\dfrac{2x+5}{x^2+x}\right)$

7. $\lim_{x\to 0^+} \dfrac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}$

8. $\lim_{x\to -2^+} (x+3)\dfrac{|x+2|}{x+2}$

9. $\lim_{x\to 1^+} \dfrac{\sqrt{2x}(x-1)}{|x-1|}$

10. $\lim_{\theta\to 0} \dfrac{\sin\sqrt{2}.\theta}{\sqrt{2}.\theta}$

11. $\lim_{x\to 0} \dfrac{\sin 3x}{4x}$

12. $\lim_{x\to 0^-} \dfrac{x}{\sin 3x}$

13. $\lim_{x\to 0} \dfrac{\tan 2x}{x}$

14. $\lim_{x\to 0} 6x^2(\cot x)(\csc 2x)$

15. $\lim_{\theta\to 0} \dfrac{\sin\theta}{\sin 2\theta}$

16. $\lim_{h\to 0} \dfrac{\sin(\sin h)}{\sin h}$

17. $\lim_{\theta\to 0} \dfrac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}$

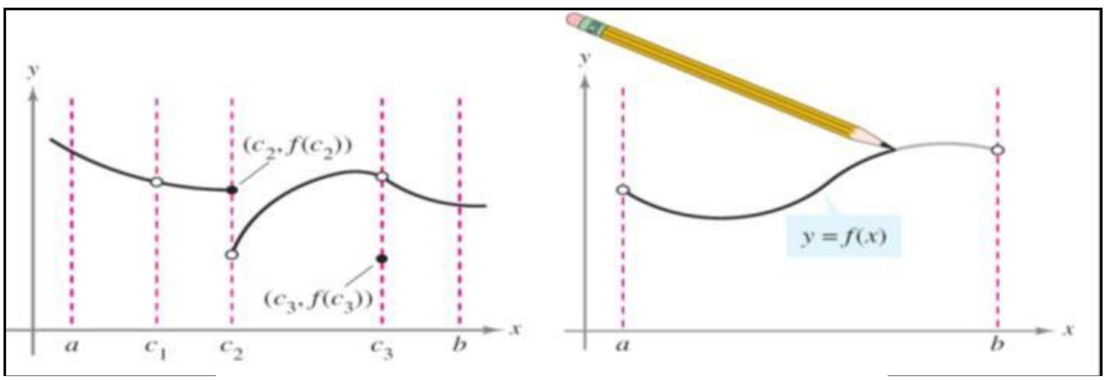
Section 1.5 – Continuity

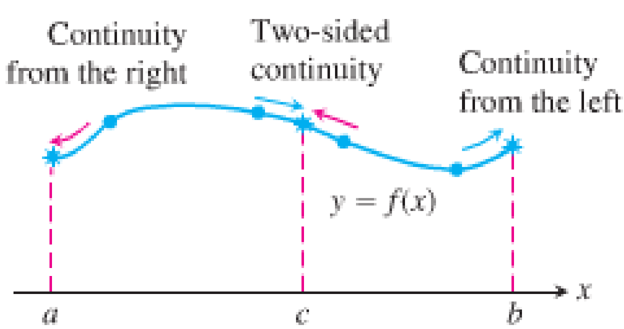
Definition of Continuity

Let c be a number in the interval (a, b), and let f be a function whose domain contains the interval (a, b). The function f is continuous at the point c if the following conditions are true.

1. $f(c)$ is defined
2. $\lim\limits_{x \to c} f(x)$ *exists*
3. $\lim\limits_{x \to c} f(x) = f(c)$

If f is continuous at every point in the interval (a, b), then it is continuous on an open interval (a, b)





Definition

Interior point: A function $y = f(x)$ is **continuous at an interior point *c*** of its domain if

$$\lim_{x \to c} f(x) = f(c)$$

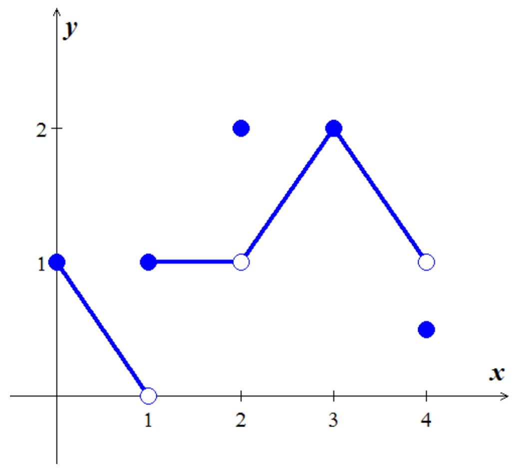
Endpoint: A function $y = f(x)$ is **continuous at a left point *a*** or is **continuous at a right point *b*** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad or \quad \lim_{x \to b^-} f(x) = f(b), \quad respectively$$

- If a function f is not continuous at a point c, we say that f is ***discontinuous*** at c. (is a ***point of discontinuity***)

Example

Find the points at which the function f is continuous and the points at which f is not continuous



Solution

The function f is continuous at every point in its domain [0, 4] except at $x=1,\ x=2,$ and $x=4$. At these points, there are breaks in the graph.

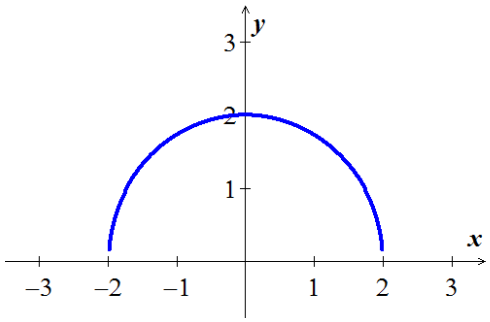
$x=0$	$\lim\limits_{x\to 0^+} f(x)=f(0)=1$	f is continuous $@\, x=0$
$x=1$	$\lim\limits_{x\to 1} f(x)$ doesn't exist	f is discontinuous $@\, x=1$
$x=2$	$\lim\limits_{x\to 2} f(x)=1,\ but\ \ 1\neq f(2)$	f is discontinuous $@\, x=2$
$x=3$	$\lim\limits_{x\to 3} f(x)=f(3)=2$	f is continuous $@\, x=3$
$x=4$	$\lim\limits_{x\to 4^-} f(x)=1,\ but\ \ 1\neq f(4)$	f is discontinuous $@\, x=4$
$c<0,\ c>4$	These points are not in the domain of f.	f is discontinuous
$0<c<4,\ \ c\neq 1,2$	$\lim\limits_{x\to c} f(x)=f(c)$	

Example

At what points the function $f(x)=\sqrt{4-x^2}$ is continuous?

Solution

The function is continuous at every point of its domain [–2, 2].
Including $x=-2$, where f is right-continuous, and $x=2$, where f is left-continuous.



Continuous Functions

A function is ***continuous on an interval*** iff it is continuous at every point of the interval. A ***continuous function*** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval.

Example

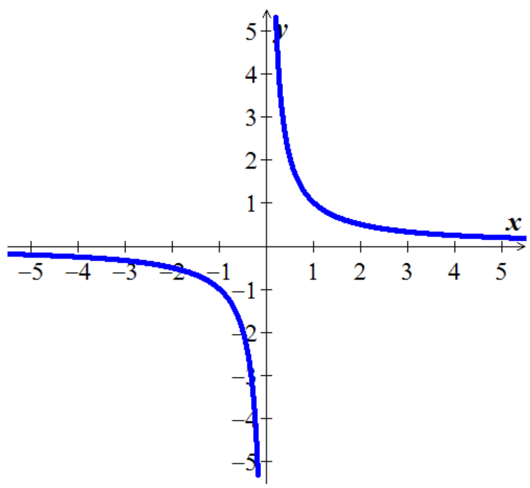
Determine at which points do the function $f(x)=\dfrac{1}{x}$ is continuous and discontinuous

Solution

The function $f(x)$ is a continuous function because it is continuous at every point of its domain.

It has a point of discontinuity at $x=0$, however, because it is not defined.

It is discontinuous on any interval containing $x=0$



Theorem – Properties of Continuous Functions

If the functions f *and* g are continuous at $x=c$, then the following combinations are continuous at $x=c$.

Sums and Differences	$f \pm g$
Constant multiples	$k \cdot g$, for any number *k*.
Products	$f \cdot g$
Quotients	$\dfrac{f}{g}$
Powers	f^n ***n*** *a positive integer*
Roots	$\sqrt[n]{f}$, provided it is defined on an open interval containing *c*, where *n* is a positive integer

Proof

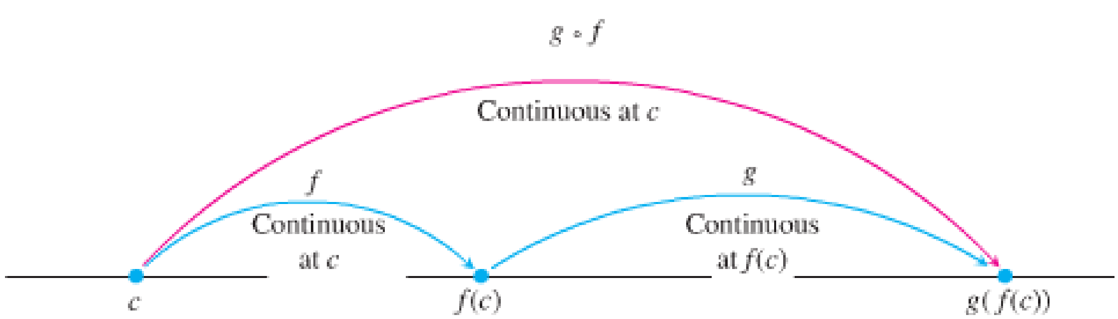
$$\begin{aligned}\lim_{x\to c}(f+g)(x) &= \lim_{x\to c}\left(f(x)+g(x)\right)\\ &= \lim_{x\to c} f(x) + \lim_{x\to c} g(x)\\ &= f(c)+g(c)\\ &= (f+g)(c)\end{aligned}$$

This shows that $f+g$ is continuous

Composites

All composites of continuous functions are continuous.

If $f(x)$ is continuous at $x=c$ and $g(x)$ is continuous at $x=f(c)$, then $g\circ f$ is continuous at $x=c$



Example

Show that $y = \sqrt{x^2 - 2x - 5}$ is continuous everywhere on its domain

Solution

Let $\begin{cases} f(x) = x^2 - 2x - 5, & Domain: \mathbb{R} \\ g(x) = \sqrt{x} & Domain: [0,\ \infty) \end{cases}$

$\therefore$ The function y is continuous on $[0,\ \infty)$

Example

Show that $y = \left| \dfrac{x \sin x}{x^2 + 2} \right|$ is continuous everywhere on its domain

Solution

Let $\begin{cases} x \sin x & Domain: \mathbb{R} \\ x^2 + 2 & Domain: \mathbb{R} \end{cases}$

$\therefore$ The function is the composite of a quotient continuous functions with the continuous absolute value function.

Theorem

If g is continuous at the point b and $\lim\limits_{x \to c} f(x) = b$, then

$$\lim_{x \to c} g\left(f(x)\right) = g(b) = g\left(\lim_{x \to c} f(x)\right)$$

Proof

Let $\varepsilon > 0$ be given. Since g is continuous at b, there exists a number $\delta_1 > 0$ such that

$$\left|g(y) - g(b)\right| < \varepsilon \quad whenever \quad 0 < |y - b| < \delta_1$$

$\lim\limits_{x \to c} f(x) = b,\ \exists\ \delta > 0 \ni \ \left|f(x) - b\right| < \delta_1 \quad whenever \quad 0 < |x - c| < \delta$

If we let $y = f(x)$, we then have that $\quad |y - b| < \delta_1 \quad whenever \quad 0 < |x - c| < \delta$

Which implies from the first statement that $\left|g(y) - g(b)\right| = \left|g\left(f(x)\right) - g(b)\right| < \varepsilon$ whenever $0 < |x - c| < \delta$. From the definition of the limit, this proves that $\lim\limits_{x \to c} g\left(f(x)\right) = g(b)$

Example

Find the $\lim\limits_{x\to\frac{\pi}{2}} \cos\left(2x+\sin\left(\frac{3\pi}{2}+x\right)\right)$

Solution

$$\lim_{x\to\frac{\pi}{2}} \cos\left(2x+\sin\left(\frac{3\pi}{2}+x\right)\right) = \cos\left(\lim_{x\to\frac{\pi}{2}} 2x + \lim_{x\to\frac{\pi}{2}} \sin\left(\frac{3\pi}{2}+x\right)\right)$$

$$= \cos(\pi + \sin 2\pi)$$

$$= \cos(\pi + 0)$$

$$= \cos(\pi)$$

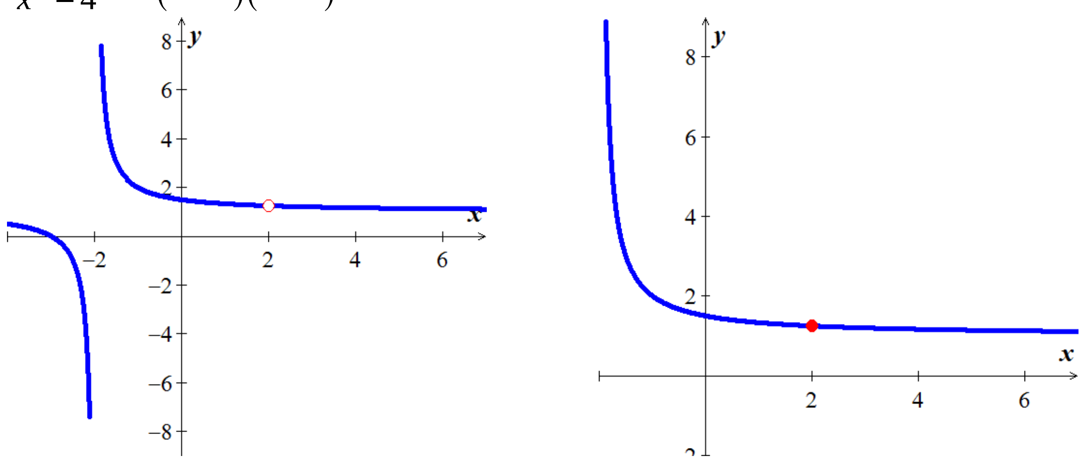
$$\boxed{= -1}$$

Example

Show that $f(x) = \dfrac{x^2+x-6}{x^2-4}, \quad x \neq 2$ has a continuous extension to $x=2$, and find that extension.

Solution

$$f(x) = \frac{x^2+x-6}{x^2-4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}$$



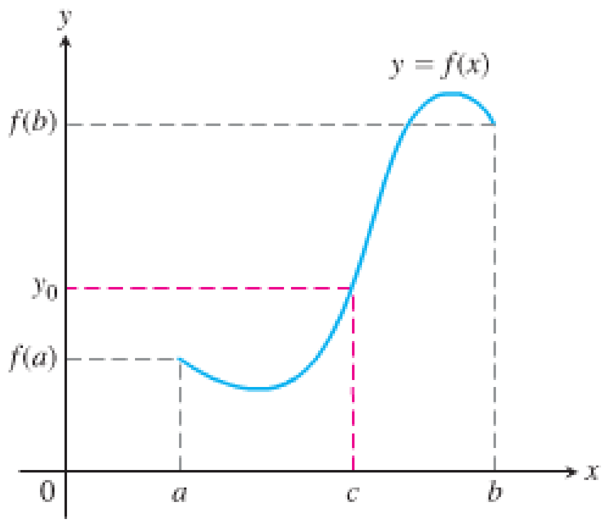
After simplification the function is continuous at $x=2$

$$\lim_{x\to 2} \frac{x^2+x-6}{x^2-4} = \lim_{x\to 2} \frac{x+3}{x+2} \boxed{= \frac{5}{4}}$$

The new function is the function *f* with its point of discontinuity at $x=2$ removed.

Theorem – the Intermediate Value Theorem for Continuous Functions

If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some $\boldsymbol{c}$ in $[a, b]$.



A Consequence for Root Finding

We call a solution of the equation $f(x) = 0$ a ***root*** of the equation or zero of the function f. The Intermediate Value Theorem said that if $\boldsymbol{f}$ is continuous, then any interval on which f changes sign contains a zero of the function.

Example

Show that there is a root of the equation $x^3 - x - 1$ between 1 and 2.

Solution

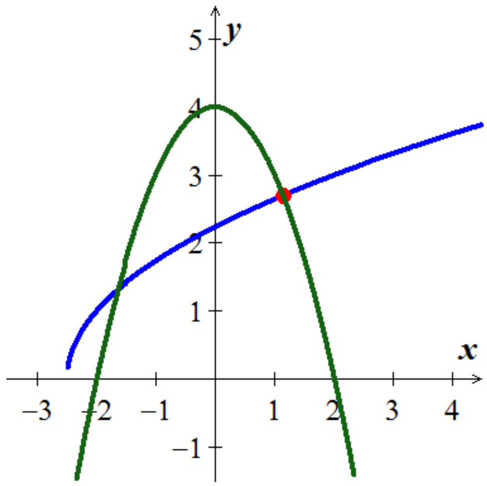
$$f(1) = 1^3 - 1 - 1 = -1 < 0$$

$$f(2) = 2^3 - 2 - 1 = 5 > 0$$

Since f is continuous, the Intermediate Value Theorem says there is a zero of f between 1 and 2.

Example

Use the Intermediate Value Theorem to prove that the equation $\sqrt{2x+5}=4-x^2$ has a solution.



Solution

The function $g(x)=\sqrt{2x+5}$ is continuous on the interval $\left[-\frac{5}{2},\ \infty\right)$ since it is the composite of the square root function with nonnegative linear function $y=2x+5$. Then the function $f(x)=\sqrt{2x+5}+x^2$ is the sum of the function $g(x)$ and $y=x^2$. It follows that $f(x)$ is continuous on the interval $\left[-\frac{5}{2},\ \infty\right)$.

By trial and error:

$$f(0)=\sqrt{2(0)+5}+0^2=\sqrt{5}>0$$

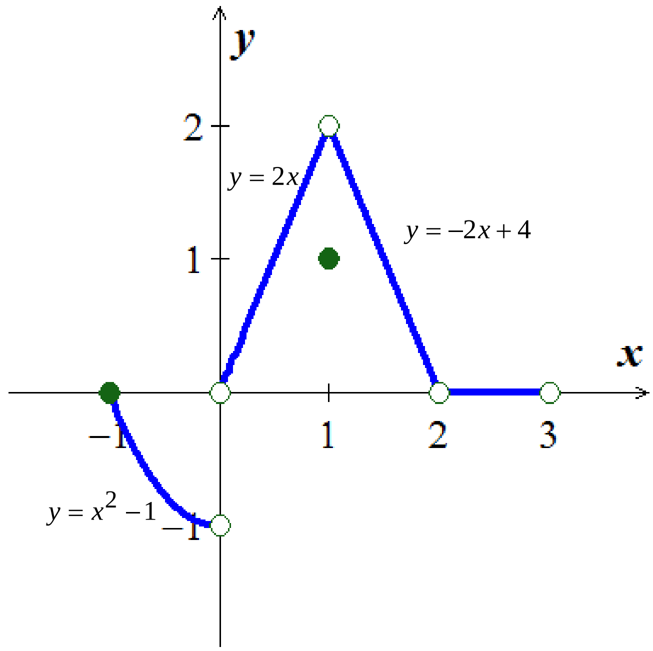
$$f(2)=\sqrt{2(2)+5}+2^2=\sqrt{9}+4=7>0$$

f is continuous on the interval $[0,\ 2]\subset\left[-\frac{5}{2},\ \infty\right)$.

Since the value $y_0=4$ is between $\sqrt{5}$ and 7, by the Intermediate Value Theorem there is a number $c\in[0,\ 2]\ \ni f(c)=4$. That is, the number c solves the original equation.

Exercises Section 1.5 – Continuity

1. Given the graphed function $f(x)$



a) Does $f(-1)$ exist?

b) Does $\lim\limits_{x \to -1^+} f(x)$ exist?

c) Does $\lim\limits_{x \to -1^+} f(x) = f(-1)$?

d) Is f continuous at $x = -1$?

e) Does $f(1)$ exist?

f) Does $\lim\limits_{x \to 1} f(x)$ exist?

g) Does $\lim\limits_{x \to 1} f(x) = f(1)$?

h) Is f continuous at $x = 1$?

13. At what points is the function $y = \dfrac{1}{x-2} - 3x$ continuous?

14. At what points is the function $y = \dfrac{x+3}{x^2 - 3x - 10}$ continuous?

15. At what points is the function $y = |x-1| + \sin x$ continuous?

16. At what points is the function $y = \dfrac{x+2}{\cos x}$ continuous?

17. At what points is the function $y = \tan \dfrac{\pi x}{2}$ continuous?

18. At what points is the function $y = \dfrac{x \tan x}{x^2 + 1}$ continuous?

19. At what points is the function $y = \dfrac{\sqrt{x^4 + 1}}{1 + \sin^2 x}$ continuous?

20. At what points is the function $y = \sqrt{2x+3}$ continuous?

21. At what points is the function $y = \sqrt[4]{3x-1}$ continuous?

22. At what points is the function $y = (2-x)^{1/5}$ continuous?

23. Find $\lim\limits_{x \to \pi} \sin(x - \sin x)$, then is the function continuous at the point being approached?

24. Find $\lim_{x\to 0} \tan\left(\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right)$, then is the function continuous at the point being approached?

25. Find $\lim_{t\to 0} \cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right)$, then is the function continuous at the point being approached?

26. Explain why the equation $\cos x = x$ has at least one solution.

27. Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval [−4, 4]

28. If functions $f(x)$ and $g(x)$ are continuous for $0 \le x \le 1$, could $\frac{f(x)}{g(x)}$ possibly be discontinuous at a point of [0, 1]? Give reason for your answer.

29. Suppose that a function f is continuous on the closed interval [0, 1] and that $0 \le f(x) \le 1$ for every x in [0, 1]. Show that there must exist a number c in [0, 1] such that $f(c) = c$ (***c*** is called a ***fixed point*** of f).

Section 1.6 – Limits Involving Infinity; Asymptotes of Graphs

Finite limits as $x \to \pm\infty$

Definitions

We say that $f(x)$ has the **limit *L* as *x* approaches infinity** and write $\lim\limits_{x\to\infty} f(x) = L$

If, $\forall \varepsilon > 0\ \exists N \ni \forall x, \qquad x > M \quad \Rightarrow \quad |f(x) - L| < \varepsilon$

We say that $f(x)$ has the **limit *L* as *x* approaches *minus* infinity** and write $\lim\limits_{x\to-\infty} f(x) = L$

If, $\forall \varepsilon > 0\ \exists N \ni \forall x, \qquad x < M \quad \Rightarrow \quad |f(x) - L| < \varepsilon$

Basic Facts: $\boxed{\lim\limits_{x\to\pm\infty} k = k \quad \text{and} \quad \lim\limits_{x\to\pm\infty} \frac{1}{x} = 0}$

Example

Find $\lim\limits_{x\to\infty} \dfrac{5x^2+8x-3}{3x^2+2}$

<u>Solution</u>

$$\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty} \frac{5+\frac{8}{x}-\frac{3}{x^2}}{3+\frac{2}{x^2}} \qquad \textit{Divide by } x^2$$

$$= \frac{5+0-0}{3+0} \qquad \lim_{x\to\pm\infty} \frac{1}{x} = 0$$

$$= \boxed{\frac{5}{3}}$$

Example

Find $\lim\limits_{x\to\infty} \dfrac{11x+2}{2x^3-1}$

<u>Solution</u>

$$\lim_{x\to\infty} \frac{11x+2}{2x^3-1} = \lim_{x\to\infty} \frac{\frac{11}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}} \qquad \textit{Divide by } x^3$$

$$= \frac{0+0}{2-0} \qquad \lim_{x\to\pm\infty} \frac{1}{x} = 0$$

$$= \boxed{0}$$

Horizontal Asymptote (*HA*)

The line $y=b$ is a ***horizontal asymptote*** for the graph of a function f if

$$\lim_{x\to\infty} f(x)=b \quad \text{or} \quad \lim_{x\to-\infty} f(x)=b$$

Let $f(x)=\dfrac{p(x)}{q(x)}=\dfrac{a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+\ldots+b_1x+b_0}=\dfrac{a_nx^n}{b_mx^m}$ be a rational function. (***Proof*** !)

1. If the degree of numerator is less than of denominator $(n<m)\Rightarrow y=0$

$$y=\frac{2x+1}{4x^2+5} \quad \Rightarrow \boxed{y=0}$$

2. If the degree of numerator is equal of denominator $(n=m)\Rightarrow y=\dfrac{a_n}{b_m}$

$$y=\frac{2x^2+1}{4x^2+5} \quad \Rightarrow y=\frac{2}{4}=\frac{1}{2}$$

3. If the degree of numerator is greater than of denominator $(n>m)\Rightarrow$ No horizontal asymptote

$$y=\frac{2x^3+1}{4x^2+5} \quad \Rightarrow \text{No } HA$$

Example

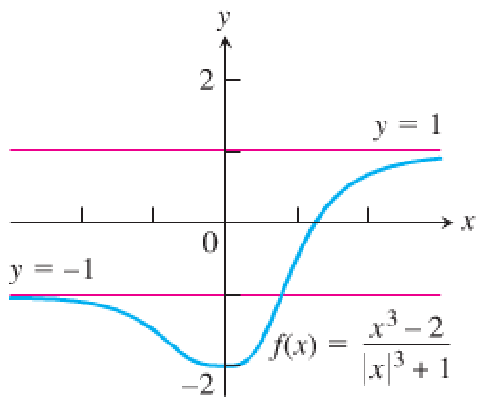
Find the horizontal asymptotes of the graph of $f(x)=\dfrac{x^3-2}{|x|^3+1}$

Solution

For $x\ge 0$ $\quad \displaystyle\lim_{x\to\infty}\frac{x^3-2}{|x|^3+1}=\lim_{x\to\infty}\frac{x^3}{x^3}=1$

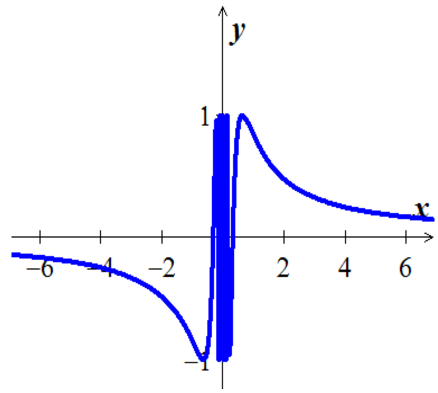
For $x\le 0$ $\quad \displaystyle\lim_{x\to\infty}\frac{x^3-2}{|x|^3+1}=\lim_{x\to-\infty}\frac{x^3}{(-x)^3}=-1$

The ***HA*** are $y=-1$ and $y=1$.



Example

Find $\lim\limits_{x\to\infty} \sin\left(\frac{1}{x}\right)$



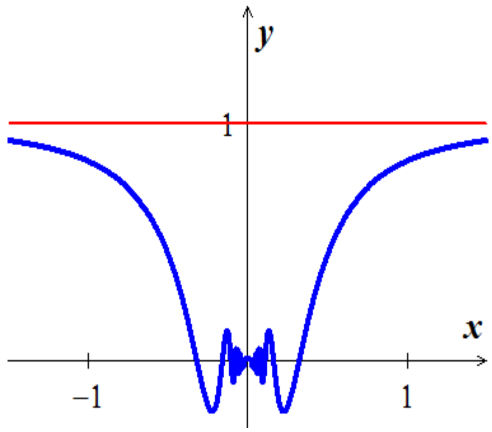
Solution

Let $t = \frac{1}{x} \Rightarrow t \to 0 \ \ as \ \ x \to \infty$

$$\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = \lim_{t\to 0} \sin t \ \underline{= 0|}$$

Example

Find $\lim\limits_{x\to\pm\infty} x\sin\left(\frac{1}{x}\right)$



Solution

Let $t = \frac{1}{x} \Rightarrow x = \frac{1}{t}$

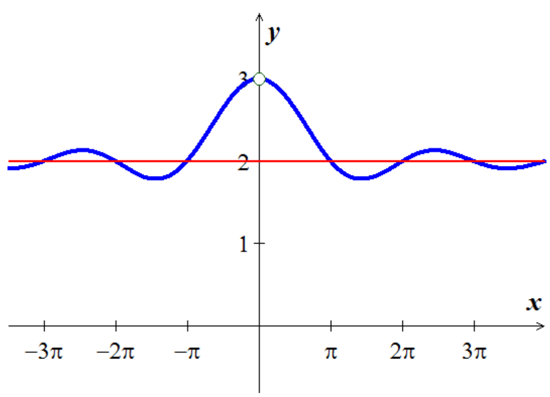
$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{t\to 0^+} \frac{\sin t}{t} \ \underline{= 1|}$$

$$\lim_{x\to-\infty} x\sin\left(\frac{1}{x}\right) = \lim_{t\to 0^-} \frac{\sin t}{t} \ \underline{= 1|}$$

Example

Find the horizontal asymptote of $y = 2 + \frac{\sin x}{x}$

Solution



Since $0 \le \left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right|$

$$\lim_{x\to\pm\infty} \left|\frac{1}{x}\right| = 0 \quad \Rightarrow \quad \lim_{x\to\pm\infty} \frac{\sin x}{x} = 0$$

$$\lim_{x\to\pm\infty} \left(2 + \frac{\sin x}{x}\right) = 2 + 0 \ \underline{= 2|}$$

The ***HA*** are $y = 2$

Example

Find $\lim\limits_{x\to\infty}\left(x-\sqrt{x^2+16}\right)$

Solution

$$\lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right)=\lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right)\frac{x+\sqrt{x^2+16}}{x+\sqrt{x^2+16}} \qquad (a-b)(a+b)=a^2-b^2$$

$$=\lim_{x\to\infty}\frac{x^2-\left(x^2+16\right)}{x+\sqrt{x^2+16}}$$

$$=\lim_{x\to\infty}\frac{x^2-x^2-16}{x+\sqrt{x^2+16}}$$

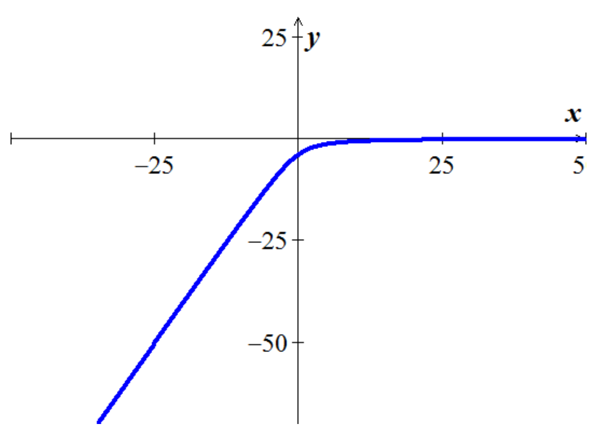
$$=\lim_{x\to\infty}\frac{-16}{x+\sqrt{x^2+16}}$$

$$=\lim_{x\to\infty}\frac{-\frac{16}{x}}{\frac{x}{x}+\sqrt{\frac{x^2}{x^2}+\frac{16}{x^2}}}$$

$$=\lim_{x\to\infty}\frac{-\frac{16}{x}}{1+\sqrt{1+\frac{16}{x^2}}}$$

$$=\frac{0}{1+\sqrt{1+0}}$$

$$\underline{=0}$$



Slant or *Oblique* Asymptotes

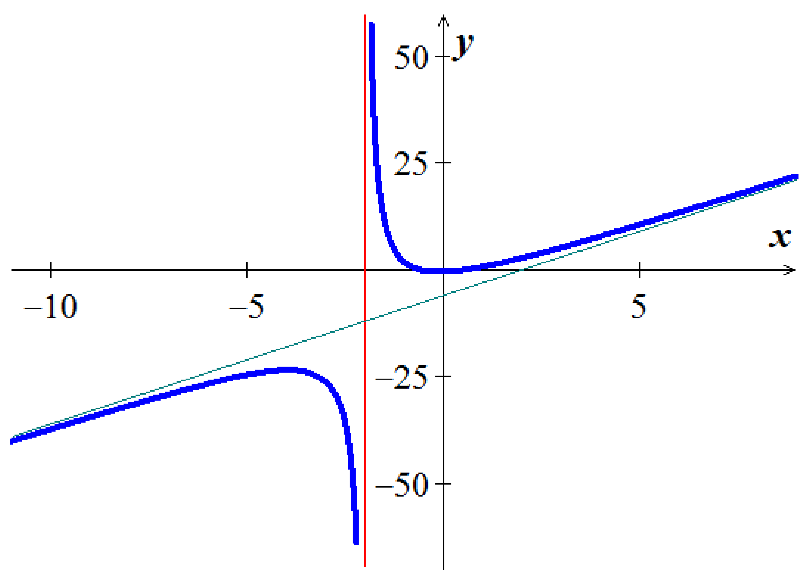
When the degree of the numerator is one greater than the degree of the numerator, the graph has a ***slant*** or ***oblique*** asymptote and it is a line $y = ax + b, \quad a \neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y = \frac{3x^2 - 1}{x + 2}$$

$$\begin{array}{r} 3x - 6 \\ x + 2 \overline{\smash{)}\, 3x^2 + 0x - 1} \\ \underline{3x^2 + 6x} \quad\quad \\ -6x - 1 \\ \underline{-6x - 12} \\ R = 11 \end{array}$$

$$y = \frac{3x^2 - 1}{x + 2} = (3x - 6) + \frac{11}{x + 2}$$

The ***oblique asymptote*** is the line ***y* = 3*x* - 6**



Infinite Limits

The limit has a value of infinity or minus infinity, such a function $f(x)=\frac{1}{x}$. It is convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x\to 0^+$.

Definition

We say $\lim\limits_{x\to 0^+} f(x)=\infty$

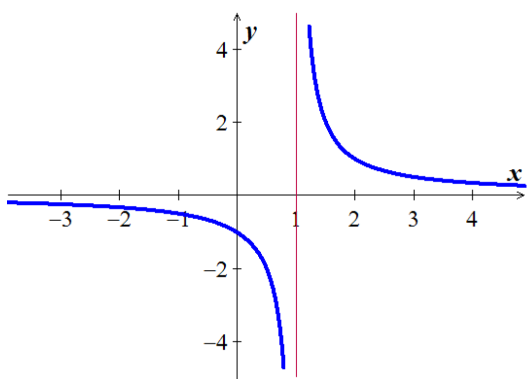
That $\lim\limits_{x\to 0^+}\frac{1}{x}$ doesn't exist because $\frac{1}{x}$ becomes arbitrary large and positive as $x\to 0^+$.

We say $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\frac{1}{x}=-\infty$

That $\lim\limits_{x\to 0^-}\frac{1}{x}$ doesn't exist because $\frac{1}{x}$ becomes arbitrary large and negative as $x\to 0^-$.

Example

Find $\lim\limits_{x\to 1^+}\frac{1}{x-1}$ *and* $\lim\limits_{x\to 1^-}\frac{1}{x-1}$



Solution

As $x\to 1^+ \quad\Rightarrow\quad x-1\to 0^+$

$$\lim_{x\to 1^+}\frac{1}{x-1}=\infty$$

$$\lim_{x\to 1^-}\frac{1}{x-1}=-\infty$$

- $\lim\limits_{x\to 2}\frac{(x-2)^2}{x^2-4}=\lim\limits_{x\to 2}\frac{(x-2)^2}{(x-2)(x+2)}=\lim\limits_{x\to 2}\frac{(x-2)}{(x+2)}=\frac{0}{4}\ \underline{=0|}$
- $\lim\limits_{x\to 2}\frac{x-2}{x^2-4}=\lim\limits_{x\to 2}\frac{x-2}{(x-2)(x+2)}=\lim\limits_{x\to 2}\frac{1}{x+2}\ \underline{=\frac{1}{4}|}$
- $\lim\limits_{x\to 2^+}\frac{x-3}{x^2-4}=\lim\limits_{x\to 2^+}\frac{x-3}{(x-2)(x+2)}\ \underline{=-\infty|}$
- $\lim\limits_{x\to 2^-}\frac{x-3}{x^2-4}=\lim\limits_{x\to 2^-}\frac{x-3}{(x-2)(x+2)}\ \underline{=\infty|}$
- $\lim\limits_{x\to 2}\frac{x-3}{x^2-4}=\lim\limits_{x\to 2}\frac{x-3}{(x-2)(x+2)}\ \underline{=\text{doesn't exist}|}$

Vertical Asymptote (*VA*) - ***Think Domain***

The line $x = a$ is a ***vertical asymptote*** for the graph of a function f if

$$\lim_{x\to a^+} f(x) \to \pm\infty \quad or \quad \lim_{x\to a^-} f(x) \to \pm\infty$$

As ***x*** approaches ***a*** from either the left or the right

$$\lim_{x\to 0^+} \frac{1}{x} \to \infty \quad or \quad \lim_{x\to 0^-} \frac{1}{x} \to -\infty$$

Example

Find the horizontal and vertical asymptotes of the curve $y = \frac{x+3}{x+2}$

Solution

HA: $y \to \frac{x}{x} = 1 \quad \Rightarrow \quad \boxed{y = 1}$

VA: $x + 2 = 0 \quad \Rightarrow \quad \boxed{x = -2}$

Example

Find the horizontal and vertical asymptotes of the curve $f(x) = -\frac{8}{x^2 - 4}$

Solution

HA: $y \to \lim_{x\to\infty} -\frac{8}{x^2} = 0 \quad \Rightarrow \quad \boxed{y = 0}$

VA: $x^2 - 4 = 0 \quad \Rightarrow \quad \boxed{x = \pm 2}$

$$\lim_{x\to 2^+} f(x) = -\infty \quad and \quad \lim_{x\to 2^-} f(x) = \infty$$

Exercises *Section* 1.6 – Limits Involving Infinity; Asymptotes

1. Find the limit as $x \to \infty$ and as $x \to -\infty$ of $h(x) = \dfrac{-5+\frac{7}{x}}{3-\frac{1}{x^2}}$

2. Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{2x+3}{5x+7}$

3. Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{2x^3+7}{x^3-x^2+x+7}$

4. Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{x+1}{x^2+3}$

5. Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{7x^3}{x^3-3x^2+6x}$

6. Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{9x^4+x}{2x^4+5x^2-x+6}$

7. Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{-2x^3-2x+3}{3x^3+3x^2-5x}$

Find

8. $\lim\limits_{x\to-\infty} \dfrac{\cos x}{3x}$

9. $\lim\limits_{x\to\infty} \dfrac{x+\sin x}{2x+7-5\sin x}$

10. $\lim\limits_{x\to\infty} \sqrt{\dfrac{8x^2-3}{2x^2+x}}$

11. $\lim\limits_{x\to-\infty} \left(\dfrac{x^2+x-1}{8x^2-3}\right)^{1/3}$

12. $\lim\limits_{x\to\infty} \dfrac{2\sqrt{x}+x^{-1}}{3x-7}$

13. $\lim\limits_{x\to\infty} \dfrac{x^{-1}+x^{-4}}{x^{-2}+x^{-3}}$

14. $\lim\limits_{x\to-\infty} \dfrac{4-3x^3}{\sqrt{x^6+9}}$

15. $\lim\limits_{x\to0^+} \dfrac{1}{3x}$

16. $\lim\limits_{x\to-5^-} \dfrac{3x}{2x+10}$

17. $\lim\limits_{x\to0} \dfrac{1}{x^{2/3}}$

18. $\lim\limits_{x\to0^-} \dfrac{1}{3x^{1/3}}$

19. $\lim\limits_{x\to\left(-\frac{\pi}{2}\right)^+} \sec x$

20. $\lim\limits_{\theta\to0^-} (1+\csc\theta)$

21. $\lim\limits_{x\to-\infty} \left(\sqrt{x^2+3}+x\right)$

22. $\lim\limits_{x\to\infty} \left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right)$

23. Graph the rational function $y = \frac{1}{2x+4}$. Include the equations of the asymptotes.

24. Graph the rational function $y = \frac{2x}{x+1}$. Include the equations of the asymptotes.

25. Graph the rational function $y = \frac{x^2}{x-1}$. Include the equations of the asymptotes.

26. Graph the rational function $y = \frac{x^3+1}{x^2}$. Include the equations of the asymptotes.

Find the vertical, horizontal and oblique asymptotes (if any) of

27. $y = \frac{3x}{1-x}$

28. $y = \frac{x^2}{x^2+9}$

29. $y = \frac{x-2}{x^2-4x+3}$

30. $y = \frac{3}{x-5}$

31. $y = \frac{x^3-1}{x^2+1}$

32. $y = \frac{3x^2-27}{(x+3)(2x+1)}$

33. $y = \frac{x^3+3x^2-2}{x^2-4}$

34. $y = \frac{x-3}{x^2-9}$

35. $y = \frac{6}{\sqrt{x^2-4x}}$

36. $y = \frac{5x-1}{1-3x}$