Solution Section 2.1 – Vectors in 2-Space, 3-Space, and *n*-Space

Exercise

Sketch the following vectors with initial points located at the origin

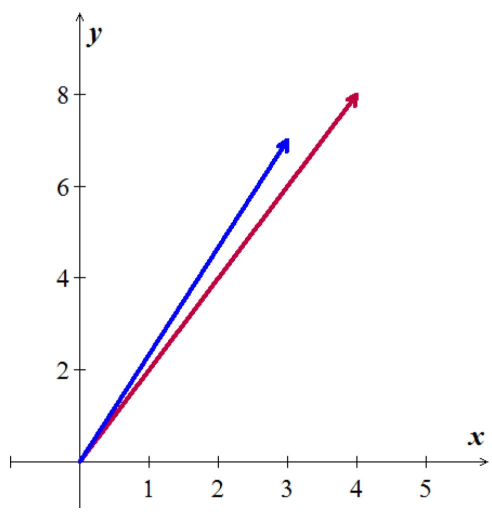
a) $P_1(4,\ 8)\quad P_2(3,\ 7)$

b) $P_1(-1,\ 0,\ 2)\quad P_2(0,\ -1,\ 0)$

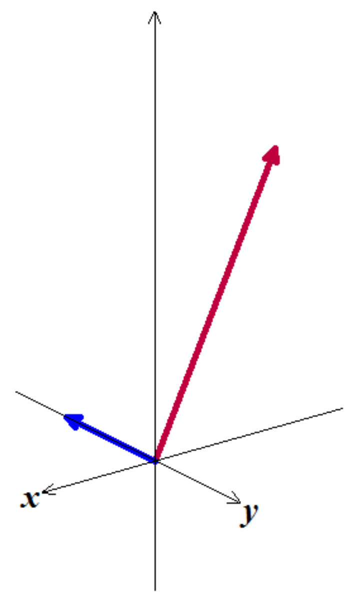
c) $P_1(3,\ -7,\ 2)\quad P_2(-2,\ 5,\ -4)$

Solution

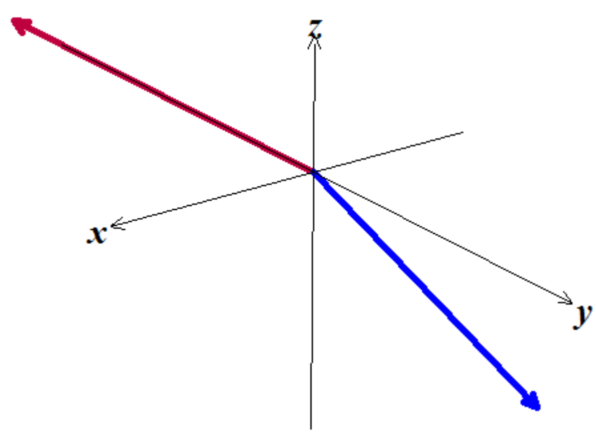
a)



b)



c)



Exercise

Find the components of the vector $\overrightarrow{P_1P_2}$

a) $P_1(3,\ 5)\quad P_2(2,\ 8)$

b) $P_1(5,\ -2,\ 1)\quad P_2(2,\ 4,\ 2)$

c) $P_1(0,\ 0,\ 0)\quad P_2(-1,\ 6,\ 1)$

Solution

a) $\overrightarrow{P_1P_2}=(2-3,\ 8-5)$

$\quad =(-1,\ 3)$

b) $\overrightarrow{P_1P_2}=(2-5,\ 4-(-2),\ 2-1)$

$\quad =(-3,\ 6,\ 1)$

c) $\overrightarrow{P_1P_2}=(-1-0,\ 6-0,\ 1-0)$

$\quad =(-1,\ 6,\ 1)$

Exercise

Find the terminal point of the vector that is equivalent to $\vec{u}=(1,\ 2)$ and whose initial point is $A(1,\ 1)$

Solution

The terminal point: $B(b_1,\ b_2)$

$(b_1-1,\ b_2-1)=(1,\ 2)$

$$\begin{cases} b_1-1=1 & \Rightarrow b_1=2 \\ b_2-1=2 & \Rightarrow b_2=3 \end{cases}$$

The terminal point: $B(2,\ 3)$

Exercise

Find the initial point of the vector that is equivalent to $\vec{u}=(1,\ 1,\ 3)$ and whose terminal point is $B(-1,\ -1,\ 2)$

Solution

The initial point: $A(x, y, z)$

$(-1-x,\ -1-y,\ 2-z)=(1,\ 1,\ 3)$

$$\begin{cases} -1-x=1 & \Rightarrow x=-2 \\ -1-y=1 & \Rightarrow y=-2 \\ 2-z=3 & \Rightarrow z=-1 \end{cases}$$

The initial point: $\underline{A(-2,\ -2,\ -1)}$

Exercise

Find a nonzero vector $\vec{u}$ with initial point P (–1, 3, –5) such that

a) $\vec{u}$ has the same direction as $\vec{v} = (6, 7, -3)$

b) $\vec{u}$ is oppositely directed as $\vec{v} = (6, 7, -3)$

Solution

a) $\vec{u}$ has the same direction as $\vec{v}$

$\boldsymbol{u} = \vec{v} = (6, 7, -3)$

The initial point $P(-1, 3, -5)$ then the terminal point:

$(-1+6,\ 3+7,\ -5-3) = \underline{(5,\ 10,\ -8)}$

b) $\vec{u}$ is oppositely directed as $\vec{v} = (6, 7, -3)$

$\vec{u} = -\vec{v} = (-6, -7, 3)$

The initial point $P(-1, 3, -5)$ then the terminal point:

$(-1-6,\ 3-7,\ -5+3) = \underline{(-7,\ -4,\ -2)}$

Exercise

Let $\vec{u} = (-3, 1, 2)$, $\vec{v} = (4, 0, -8)$, and $\vec{w} = (6, -1, -4)$. Find the components

a) $\vec{v}-\vec{w}$

b) $6\vec{u}+2\vec{v}$

c) $5(\vec{v}-4\vec{u})$

d) $-3(\vec{v}-8\vec{w})$

e) $(2\vec{u}-7\vec{w})-(8\vec{v}+\vec{u})$

f) $-\vec{u}+(\vec{v}-4\vec{w})$

Solution

a) $\vec{v}-\vec{w} = (4-6,\ 0-(-1),\ -8-(-4))$

$\underline{=(-2,\ 1,\ -4)}$

b) $6\vec{u}+2\vec{v} = (-18,6,12)+(8,0,-16)$

$\underline{=(-10,\ 6,\ -4)}$

c) $5(\vec{v}-4\vec{u}) = 5(4-(-12),\ 0-4,\ -8-8)$

$= 5(16,\ -4,\ -16)$

$$=\boxed{(80,\ -20,\ -80)}$$

d) $-3(\vec{v}-8\vec{w})=-3(4-48,\ 0-(-8),\ -8-(-32))$

$$=-3(-44,\ 8,\ 24)$$

$$=\boxed{(32,\ -24,\ -72)}$$

e) $(2\vec{u}-7\vec{w})-(8\vec{v}+\vec{u})=\left[(-6,2,\ 4)-(42,\ -7,\ -28)\right]-\left[(32,\ 0,\ -64)+(-3,\ 1,\ 2)\right]$

$$=(-48,\ 9,\ 32)-(29,\ 1,\ -62)$$

$$=\boxed{(-77,\ 8,\ 94)}$$

f) $-u+(v-4w)=(3,-1,-2)+\left[(4,0,-8)-(24,-4,-16)\right]$

$$=(3,\ -1,\ -2)+(-20,\ 4,\ 8)$$

$$=\boxed{(-17,\ 3,\ 6)}$$

Exercise

Let $\vec{u}$ = (2, 1, 0, 1, –1) and $\vec{v}$ = (–2, 3, 1, 0, 2). Find scalars a and b so that $a\vec{u}+b\vec{v}=(-8,\ 8,\ 3,\ -1,\ 7)$

Solution

$$a\vec{u}+b\vec{v}=a(2,\ 1,\ 0,\ 1,\ -1)+b(-2,\ 3,\ 1,\ 0,\ 2)$$

$$=(a-2b,\ a+3b,\ b,\ a,\ -a+2b)$$

$$=\boxed{(-8,\ 8,\ 3,\ -1,\ 7)}$$

$$\begin{cases} a-2b=-8 \\ a+3b=8 \\ b=3 \\ a\quad=-1 \\ -a+2b=7 \end{cases}$$

$\rightarrow\quad a=-1\quad b=3$ ***Unique solution***

Exercise

Find all scalars $c_1,\ c_2,\ and\ c_3$ such that $c_1(1,\ 2,\ 0)+c_2(2,\ 1,\ 1)+c_3(0,\ 3,\ 1)=(0,\ 0,\ 0)$

Solution

$$c_1(1,\ 2,\ 0)+c_2(2,\ 1,\ 1)+c_3(0,\ 3,\ 1)=(c_1+2c_2,\ 2c_1+c_2+3c_3,\ c_2+c_3)$$

$$=(0,\ 0,\ 0)$$

$$\begin{cases} c_1 + 2c_2 \qquad = 0 \\ 2c_1 + c_2 + 3c_3 = 0 \\ \qquad c_2 + c_3 = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad -\tfrac{1}{3}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \quad \tfrac{1}{2}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$c_1 = c_2 = c_3 = 0$$

Exercise

Find the distance between the given points $\begin{bmatrix} 5 & 1 & 8 & -1 & 2 & 9 \end{bmatrix}$, $\begin{bmatrix} 4 & 1 & 4 & 3 & 2 & 8 \end{bmatrix}$

Solution

$$\begin{aligned} d &= \sqrt{(4-5)^2 + (1-1)^2 + (4-8)^2 + (3+1)^2 + (2-2)^2 + (8-9)^2} \\ &= \sqrt{1+0+16+16+0+1} \\ &= \sqrt{34} \end{aligned}$$

Exercise

Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operation on $\vec{u}=(u_1,\ u_2)$ $\quad \vec{v}=(v_1,\ v_2)$

$$\vec{u}+\vec{v}=(u_1+v_1+1,\ \ u_2+v_2+1) \qquad k\vec{u}=(ku_1,\ ku_2)$$

a) Compute $\vec{u}+\vec{v}$ and $k\vec{u}$ for $\vec{u}$ = (0, 4), $\vec{v}$ = (1, –3), and k = 2.

b) Show that (0, 0) $\neq \vec{0}$.

c) Show that (–1, –1) = 0.

d) Show that $\vec{u}+(-\vec{u})=0$ for $\vec{u}=(u_1,\ u_2)$

e) Find two vector space axioms that fail to hold.

Solution

a) $\vec{u}+\vec{v}=(0+1+1,\ 4-3+1)$

$$=(2,\ 2)$$

$$k\vec{u}=(ku_1,\ ku_2)$$

$$=(2(0),\ 2(4))$$

$$=(0,\ 8)$$

b) $(0,\ 0)+(u_1,\ u_2)=(0+u_1+1,\ 0+u_2+1)$

$$=(u_1+1,\ u_2+1)$$

$$\neq(u_1,\ u_2)$$

Therefore (0, 0) is not the zero vector **0** required (by Axiom).

c) $(-1,\ -1)+(u_1,\ u_2)=(-1+u_1+1,\ -1+u_2+1)$

$$=(u_1,\ u_2)$$

$$(u_1,\ u_2)+(-1,\ -1)=(u_1-1+1,\ u_2-1+1)$$

$$=(u_1,\ u_2)$$

Therefore (–1, –1) = **0** holds.

d) Let $\vec{u}=(u_1,\ u_2)$ &

$$-\vec{u}=(-2-u_1,\ -2-u_2)$$

$$\vec{u}+(-\vec{u})=(u_1+(-2-u_1)+1,\ u_2+(-2-u_2)+1)$$

$= (-1,\ -1)$

$\boxed{= \vec{0}}$

$\vec{u} + (-\vec{u}) = 0$ holds

e) Axiom 7: $k(\vec{u}+\vec{v}) \overset{?}{=} k\vec{u} + k\vec{v}$

$k(\vec{u}+\vec{v}) = k(u_1 + v_1 + 1,\ u_2 + v_2 + 1)$

$= (ku_1 + kv_1 + k,\ ku_2 + kv_2 + k)$

$k\vec{u} + k\vec{v} = (ku_1,\ ku_2) + (kv_1,\ kv_2)$

$= (ku_1 + kv_1 + 1,\ ku_2 + kv_2 + 1)$

Therefore, $k(\vec{u}+\vec{v}) \neq k\vec{u} + k\vec{v}$; Axiom 7 fails to hold

Axiom 8: $(k+m)\vec{u} \overset{?}{=} k\vec{u} + m\vec{u}$

$(k+m)\vec{u} = ((k+m)u_1,\ (k+m)u_2)$

$= (ku_1 + mu_1,\ ku_2 + mu_2)$

$k\vec{u} + m\vec{u} = (ku_1,\ ku_2) + (mu_1,\ mu_2)$

$= (ku_1 + mu_1 + 1,\ ku_2 + mu_2 + 1)$

Therefore, $(k+m)\vec{u} \neq k\vec{u} + m\vec{u}$; Axiom 8 fails to hold

Exercise

Find $\vec{w}$ given that $10\vec{u} + 3\vec{w} = 4\vec{v} - 2\vec{w}$, $\vec{u} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix} -20 \\ 5 \end{pmatrix}$.

Solution

$-10\vec{u} + 10\vec{u} + 3\vec{w} + 2\vec{w} = -10\vec{u} + 4\vec{v} - 2\vec{w} + 2\vec{w}$

$5\vec{w} = -10\vec{u} + 4\vec{v}$

$\vec{w} = -2\vec{u} + \frac{4}{5}\vec{v}$

$= -2\begin{pmatrix} 1 \\ -6 \end{pmatrix} + \frac{4}{5}\begin{pmatrix} -20 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 12 \end{pmatrix} + \begin{pmatrix} -16 \\ 4 \end{pmatrix}$

$$= \begin{pmatrix} -18 \\ 16 \end{pmatrix}$$

Exercise

Find $\vec{w}$ given that $\vec{u} + 3\vec{v} - 2\vec{w} = 5\vec{u} + \vec{v} - 4\vec{w}$, $\vec{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

Solution

$$\vec{u} - \vec{u} + 3\vec{v} - 3\vec{v} - 2\vec{w} + 4\vec{w} = 5\vec{u} - \vec{u} + \vec{v} - 3\vec{v} - 4\vec{w} + 4\vec{w}$$

$$2\vec{w} = 4\vec{u} - 2\vec{v}$$

$$\vec{w} = 2\vec{u} - \vec{v}$$

$$= 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -2 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Exercise

Find $\vec{w}$ given that $2\vec{u} + \vec{v} - 3\vec{w} = 5\vec{u} + 7\vec{v} + 3\vec{w}$, $\vec{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ *and* $\vec{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

Solution

$$2\vec{u} - 2\vec{u} + \vec{v} - \vec{v} - 3\vec{w} - 3\vec{w} = 5\vec{u} - 2\vec{u} + 7\vec{v} - \vec{v} + 3\vec{w} - 3\vec{w}$$

$$-6\vec{w} = 3\vec{u} + 6\vec{v}$$

$$\vec{w} = -\frac{1}{2}\vec{u} - \vec{v}$$

$$= -\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2} \\ -\frac{5}{2} \end{pmatrix}$$

Exercise

Draw $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, and $\vec{u}+2\vec{v}$ $\qquad \vec{u}=\begin{pmatrix}1\\-1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}-2\\3\end{pmatrix}$

Solution

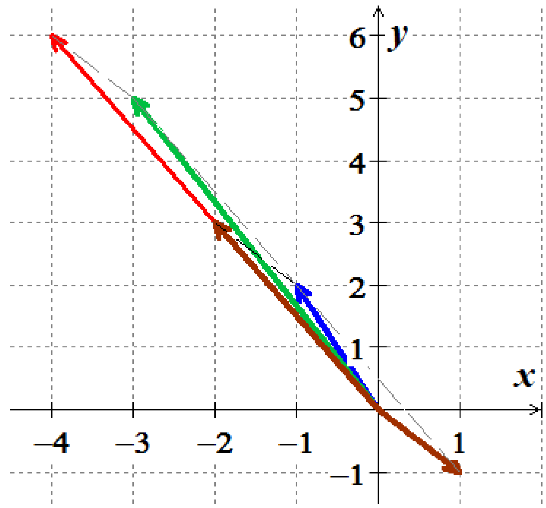
$$\vec{u}+\vec{v}=\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}-2\\3\end{pmatrix}$$

$$=\begin{pmatrix}-1\\2\end{pmatrix}$$

$$\vec{u}+2\vec{v}=\begin{pmatrix}1\\-1\end{pmatrix}+2\begin{pmatrix}-2\\3\end{pmatrix}$$

$$=\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}-4\\6\end{pmatrix}$$

$$=\begin{pmatrix}-3\\5\end{pmatrix}$$



Exercise

Draw $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, and $\vec{u}+2\vec{v}$ $\qquad \vec{u}=\begin{pmatrix}-1\\1\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}2\\1\end{pmatrix}$

Solution

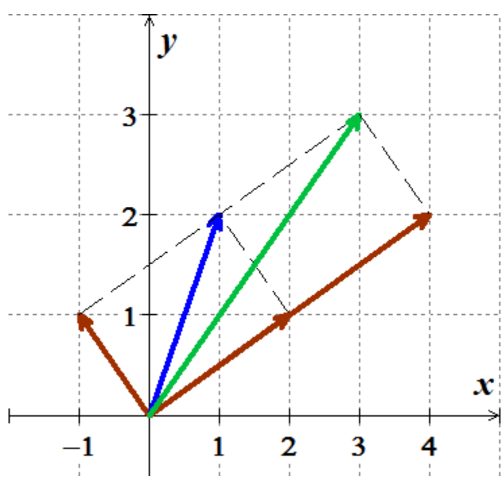
$$\vec{u}+\vec{v}=\begin{pmatrix}-1\\1\end{pmatrix}+\begin{pmatrix}2\\1\end{pmatrix}$$

$$=\begin{pmatrix}1\\2\end{pmatrix}$$

$$\vec{u}+2\vec{v}=\begin{pmatrix}-1\\1\end{pmatrix}+2\begin{pmatrix}2\\1\end{pmatrix}$$

$$=\begin{pmatrix}-1\\1\end{pmatrix}+\begin{pmatrix}4\\2\end{pmatrix}$$

$$=\begin{pmatrix}3\\3\end{pmatrix}$$



Exercise

Draw $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, and $\vec{u}+2\vec{v}$ $\qquad \vec{u}=\begin{pmatrix}1\\0\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}1\\2\end{pmatrix}$

Solution

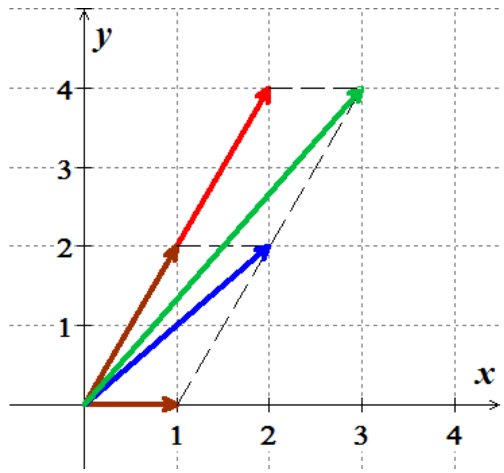
$$\vec{u}+\vec{v}=\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}1\\2\end{pmatrix}$$

$$\boxed{=\begin{pmatrix}2\\2\end{pmatrix}}$$

$$\vec{u}+2\vec{v}=\begin{pmatrix}1\\0\end{pmatrix}+2\begin{pmatrix}1\\2\end{pmatrix}$$

$$=\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}2\\4\end{pmatrix}$$

$$\boxed{=\begin{pmatrix}3\\4\end{pmatrix}}$$



Exercise

Draw $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, and $\vec{u}+2\vec{v}$ $\qquad \vec{u}=\begin{pmatrix}-1\\0\end{pmatrix}$ *and* $\vec{v}=\begin{pmatrix}1\\-1\end{pmatrix}$

Solution

$$\vec{u}+\vec{v}=\begin{pmatrix}-1\\0\end{pmatrix}+\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\boxed{=\begin{pmatrix}0\\-1\end{pmatrix}}$$

$$\vec{u}+2\vec{v}=\begin{pmatrix}-1\\0\end{pmatrix}+2\begin{pmatrix}1\\-1\end{pmatrix}$$

$$=\begin{pmatrix}-1\\0\end{pmatrix}+\begin{pmatrix}2\\-2\end{pmatrix}$$

$$\boxed{=\begin{pmatrix}1\\-2\end{pmatrix}}$$