Solution   *Section* 1.7 – Length of Curves

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j} + \sqrt{5}t\,\hat{k}; \quad 0 \le t \le \pi$$

Solution

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -(2\sin t)\hat{i} + (2\cos t)\hat{j} + \sqrt{5}\,\hat{k}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{4\sin^2 t + 4\cos^2 t + 5} \\ &= \sqrt{4+5} \\ &= 3 \end{aligned}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = -\frac{2\sin t}{3}\hat{i} + \frac{2\cos t}{3}\hat{j} + \frac{\sqrt{5}}{3}\hat{k}$$

$$\begin{aligned} Length:\ s &= \int_0^\pi |\vec{v}(t)|\, dt \\ &= \int_0^\pi 3\, dt \\ &= 3t\Big|_0^\pi \\ &= 3\pi \end{aligned}$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = t\hat{i} + \frac{2}{3}t^{3/2}\,\hat{k}; \quad 0 \le t \le 8$$

Solution

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \hat{i} + t^{1/2}\,\hat{k} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{1+t}$$

$$\vec{T} = \frac{1}{\sqrt{1+t}}\hat{i} + \frac{t^{1/2}}{\sqrt{1+t}}\hat{k} \qquad\qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$Length:\ s = \int_0^8 |\vec{v}(t)|\, dt$$

$$= \int_0^8 (1+t)^{1/2}\, dt \qquad\qquad = \int_0^8 (1+t)^{1/2}\, d(1+t)$$

$$= \frac{2}{3}(1+t)^{3/2}\Big|_0^8$$

$$= \frac{2}{3}\left[(9)^{3/2} - 1\right]$$

$$= \frac{2}{3}(27-1)$$

$$= \frac{52}{3}$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = (2+t)\hat{i} - (t+1)\hat{j} + t\,\hat{k}; \quad 0 \le t \le 3$$

Solution

$$\vec{v}(t) = \hat{i} - \hat{j} + \hat{k} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{1+1+1} = \sqrt{3}$$

$$\vec{T} = \frac{1}{\sqrt{3}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k} \qquad\qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

Length: $s = \int_0^3 |\vec{v}(t)|\, dt$

$$= \int_0^3 \sqrt{3}\, dt$$

$$= \sqrt{3}t\Big|_0^3$$

$$= 3\sqrt{3}$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = \left(\cos^3 t\right)\hat{i} + \left(\sin^3 t\right)\hat{k}; \quad 0 \le t \le \frac{\pi}{2}$$

Solution

$$\vec{v}(t) = -\left(3\cos^2 t \sin t\right)\hat{i} + \left(3\sin^2 t \cos t\right)\hat{k} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\left|\vec{v}\right| = \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t}$$

$$= 3\sqrt{\cos^2 t\sin^2 t\left(\cos^2 t + \sin^2 t\right)}$$

$$= 3\sqrt{\cos^2 t\sin^2 t}$$

$$= \underline{3\left|\cos t\sin t\right|}$$

$$\vec{T} = -\left(\frac{3\cos^2 t\sin t}{3\left|\cos t\sin t\right|}\right)\hat{i} + \left(\frac{3\sin^2 t\cos t}{3\left|\cos t\sin t\right|}\right)\hat{k} \qquad \vec{T} = \frac{\vec{v}}{\left|\vec{v}\right|}$$

$$= \underline{-(\cos t)\hat{i} + (\sin t)\hat{k}}$$

$$L = \int_0^{\pi/2} 3\cos t\sin t\ dt \qquad L = \int_a^b \left|\vec{v}(t)\right|\ dt$$

$$= \frac{3}{2}\int_0^{\pi/2} \sin 2t\ dt \qquad \sin 2t = 2\cos t\sin t$$

$$= \frac{3}{2}\left[-\frac{1}{2}\cos 2t\right]_0^{\pi/2}$$

$$= -\frac{3}{4}(\cos\pi - \cos 0)$$

$$= -\frac{3}{4}(-2)$$

$$= \underline{\frac{3}{2}}$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = (t\cos t)\hat{i} + (t\sin t)\hat{j} + \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\hat{k}; \quad 0 \le t \le \pi$$

Solution

$$\vec{v}(t) = (\cos t - t\sin t)\hat{i} + (\sin t + t\cos t)\hat{j} + \left(\sqrt{2}\ t^{1/2}\right)\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\left|\vec{v}(t)\right| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 2t}$$

$$= \sqrt{\cos^2 t - 2t\cos t + \sin^2 t + \sin^2 t + 2t\cos t + \cos^2 t + 2t}$$

$$= \sqrt{2 + 2t}$$

$$\vec{T} = \frac{1}{\sqrt{2}}\left(\frac{\cos t - t\sin t}{\sqrt{1+t}}\right)\hat{i} + \frac{1}{\sqrt{2}}\left(\frac{\cos t + t\sin t}{\sqrt{1+t}}\right)\hat{j} + \left(\sqrt{\frac{t}{1+t}}\ t^{1/2}\right)\hat{k} \qquad \vec{T} = \frac{\vec{v}}{\left|\vec{v}\right|}$$

$$L = \int_0^{\pi} \sqrt{2}\sqrt{1+t}\ dt \qquad\qquad L = \int_a^b \left|\vec{v}(t)\right|\ dt$$

$$= \sqrt{2}\int_0^{\pi} \sqrt{1+t}\ d(1+t)$$

$$= \sqrt{2}\int_0^{\pi} \sqrt{1+t}\ d(1+t)$$

$$= \sqrt{2}\left(\sqrt{1+\pi}-1\right)$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve
$\boldsymbol{r}(t) = (t\sin t + \cos t)\boldsymbol{i} + (t\cos t - \sin t)\boldsymbol{j}; \quad \sqrt{2} \le t \le 2$

Solution

$$\vec{v}(t) = (\sin t + t\cos t - \sin t)\hat{i} + (\cos t - t\sin t - \cos t)\hat{j} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$= (t\cos t)\hat{i} - (t\sin t)\hat{j}$$

$$|\vec{v}| = \sqrt{t^2\cos^2 t + t^2\sin^2 t}$$

$$= \sqrt{t^2\left(\cos^2 t + \sin^2 t\right)}$$

$$= \sqrt{t^2}$$

$$= |t|$$

$$= t \quad \textit{because} \ \ \sqrt{2} \le t \le 2$$

$$\boldsymbol{T} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \left(\frac{t\cos t}{t}\right)\boldsymbol{i} - \left(\frac{t\sin t}{t}\right)\boldsymbol{j}$$

$$= (\cos t)\boldsymbol{i} - (\sin t)\boldsymbol{j}$$

$$L = \int_{\sqrt{2}}^{2} t\, dt \qquad\qquad L = \int_a^b \left|\vec{v}(t)\right|\ dt$$

$$= \frac{1}{2}t^2\Big|_{\sqrt{2}}^{2}$$

$$= \frac{1}{2}(4-2)$$

$$= 1$$

Exercise

Find the point on the curve $\vec{r}(t) = (5\sin t)\hat{i} + (5\cos t)\hat{j} + 12t\,\hat{k}$ at a distance 26π units along the curve from the point (0, 5, 0) in the direction of increasing arc length.

Solution

$\vec{v} = (5\cos t)\hat{i} - (5\sin t)\hat{j} + 12\,\hat{k}$ $\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$$\begin{aligned} |\vec{v}| &= \sqrt{25\cos^2 t + 25\sin^2 t + 144} \\ &= \sqrt{25\left(\cos^2 t + \sin^2 t\right) + 144} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

$s = \int_0^{t_0} 13\,dt$ $\qquad s = \int_0^{t_0} |\vec{v}(t)|\,dt$

$= 13t_0$

$s = 26\pi = 13t_0$

$t_0 = 2\pi$

$$\begin{aligned} \vec{r}(t = 2\pi) &= (5\sin 2\pi)\hat{i} + (5\cos 2\pi)\hat{j} + 12(2\pi)\,\hat{k} \\ &= 0\hat{i} + 5\hat{j} + 24\pi\,\hat{k} \end{aligned}$$

The point is: $(0,\ 5,\ 24\pi)$

Exercise

Find the arc length parameter along the curve from the point where $t = 0$. Also, find the length of the indicated portion of the curve. $\vec{r}(t) = (4\cos t)\hat{i} + (4\sin t)\hat{j} + 3t\,\hat{k}; \quad 0 \le t \le \frac{\pi}{2}$

Solution

$\vec{v} = -(4\sin t)\hat{i} + (4\cos t)\hat{j} + 3\,\hat{k}$ $\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$$\begin{aligned} |\vec{v}| &= \sqrt{16\sin^2 t + 16\cos^2 t + 9} \\ &= \sqrt{16 + 9} \\ &= 5 \end{aligned}$$

$$s = \int_0^t 5\, dt \qquad\qquad s = \int_0^t |\vec{v}(\tau)|\, d\tau$$

$$= 5t \,\Big|$$

$$s\left(\frac{\pi}{2}\right) = \frac{5\pi}{2}$$

Exercise

Find the arc length parameter along the curve from the point where $t = 0$. Also, find the length of the indicated portion of the curve. $\vec{r}(t) = \left(e^t \cos t\right)\hat{i} + \left(e^t \sin t\right)\hat{j} + e^t\, \hat{k}; \quad -\ln 4 \le t \le 0$

Solution

$$\vec{v}(t) = \left(e^t \cos t - e^t \sin t\right)\hat{i} + \left(e^t \sin t + e^t \cos t\right)\hat{j} + e^t\, \hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{\left(e^t \cos t - e^t \sin t\right)^2 + \left(e^t \sin t + e^t \cos t\right)^2 + e^{2t}}$$

$$= \sqrt{e^{2t}\cos^2 t - e^{2t}\cos t \sin t + e^{2t}\sin^2 t + e^{2t}\sin^2 t + e^{2t}\cos t \sin t + e^{2t}\cos^2 t + e^{2t}}$$

$$= \sqrt{2e^{2t}\left(\cos^2 t + \sin^2 t\right) + e^{2t}}$$

$$= \sqrt{3e^{2t}}$$

$$= \sqrt{3}\, e^t$$

$$s(t) = \int_0^t \sqrt{3}\, e^\tau\, d\tau \qquad s = \int_0^t |\vec{v}(\tau)|\, d\tau$$

$$= \sqrt{3}\, e^\tau \Big|_0^t$$

$$= \sqrt{3}\left(e^t - 1\right)$$

$$s(0) = \sqrt{3}\left(e^0 - 1\right) = 0$$

$$s(-\ln 4) = \sqrt{3}\left(e^{-\ln 4} - 1\right)$$

$$= \sqrt{3}\left(e^{\ln\frac{1}{4}} - 1\right)$$

$$= \sqrt{3}\left(\frac{1}{4} - 1\right)$$

$$= \sqrt{3}\left(-\frac{3}{4}\right)$$

$$= \frac{3\sqrt{3}}{4}$$

$$s(-\ln 4) - s(0) = \frac{3\sqrt{3}}{4}$$

Exercise

Find the arc length parameter along the curve from the point where $t = 0$. Also, find the length of the indicated portion of the curve. $\vec{r}(t) = (1+2t)\hat{i} + (1+3t)\hat{j} + (6-6t)\hat{k}; \quad -1 \le t \le 0$

Solution

$$\vec{v} = 2\hat{i} + 3\hat{j} - 6\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\mathbf{v}| = \sqrt{4+9+36} = 7$$

$$s(t) = \int_0^t 7\, d\tau \qquad s = \int_0^t |\vec{v}(\tau)|\, d\tau$$

$$= 7t$$

Length: $s(0) - s(-1) = 0 - (-7) = 7$

Exercise

Find the arc length of the curve $\vec{r}(t) = \left\langle 2t^{9/2},\ t^3 \right\rangle \quad for \quad 0 \le t \le 2$

Solution

$$\vec{v}(t) = \left\langle 9t^{7/2},\ 3t^2 \right\rangle \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$L = \int_0^2 \sqrt{\left(9t^{7/2}\right)^2 + \left(3t^2\right)^2}\, dt \qquad L = \int_0^{t_0} \left|\frac{d\vec{r}}{dt}\right| dt$$

$$= \int_0^2 \sqrt{81t^7 + 9t^4}\, dt$$

$$= \int_0^2 3t^2\sqrt{9t^3+1}\, dt$$

$$= \frac{1}{9}\int_0^2 \left(9t^3+1\right)^{1/2}\, d\left(9t^3+1\right)$$

$$= \frac{2}{27}\left(9t^3 + 1\right)^{3/2}\Bigg|_0^2$$

$$\underline{= \frac{2}{27}\left(73\sqrt{73} - 1\right)} \quad unit$$

Exercise

Find the arc length of the curve $\vec{r}(t) = \left\langle t^2,\ \frac{4\sqrt{2}}{3}t^{3/2},\ 2t \right\rangle \quad for \quad 1 \le t \le 3$

Solution

$$\frac{d\vec{r}}{dt} = \left\langle 2t,\ 2\sqrt{2}\ t^{1/2},\ 2 \right\rangle$$

$$L = \int_1^3 \sqrt{4t^2 + 8t + 4}\ dt \qquad\qquad L = \int_a^b \left|\vec{r}'(t)\right|\ dt$$

$$= 2\int_1^3 \sqrt{(t+1)^2}\ dt$$

$$= 2\int_1^3 (t+1)\ dt$$

$$= 2\left(\frac{1}{2}t^2 + t\right)\Bigg|_1^3$$

$$= 2\left(\frac{9}{2} + 1 - \frac{1}{2} - 1\right)$$

$$\underline{= 12} \quad unit$$

Exercise

Find the arc length of the curve $\vec{r}(t) = \langle t,\ \ln\sec t,\ \ln(\sec t + \tan t)\rangle \quad for \quad 0 \le t \le \frac{\pi}{4}$

Solution

$$\frac{d\vec{r}}{dt} = \left\langle 1,\ \frac{\tan t \sec t}{\sec t},\ \frac{\tan t \sec t + \sec^2 t}{\sec t + \tan t} \right\rangle$$

$$= \langle 1,\ \tan t,\ \sec t\rangle$$

$$L = \int_0^{\pi/4} \sqrt{1 + \tan^2 t + \sec^2 t}\ dt \qquad\qquad L = \int_a^b \left|\vec{r}'(t)\right|\ dt$$

$$= \int_0^{\pi/4} \sqrt{2\sec^2 t}\ dt$$

$$= \sqrt{2}\int_0^{\pi/4} \sec t\ dt$$

$$= \sqrt{2}\ln\left(\sec t + \tan t\right)\Big|_0^{\pi/4}$$

$$\boxed{= \sqrt{2}\ \ln\left(\sqrt{2}+1\right)}$$

Exercise

Find the lengths of the curves $\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j} + t^2\ \hat{k}; \quad 0 \le t \le \frac{\pi}{4}$

Solution

$$\frac{d\vec{r}}{dt} = (-2\sin t)\hat{i} + (2\cos t)\hat{j} + 2t\ \hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{4\sin^2 t + 4\cos^2 t + 4t^2}$$

$$= \sqrt{4+4t^2}$$

$$= 2\sqrt{1+t^2}$$

$$L = 2\int_0^{\pi/4} \sqrt{1+t^2}\ dt \qquad\qquad L = \int_a^b \left|\vec{r}'(t)\right|\ dt$$

$$= t\sqrt{1+t^2} + \ln\left(t+\sqrt{1+t^2}\right)\Big|_0^{\pi/4}$$

$$= \frac{\pi}{4}\sqrt{1+\frac{\pi^2}{16}} + \ln\left(\frac{\pi}{4}+\sqrt{1+\frac{\pi^2}{16}}\right) - 0 - \ln 1$$

$$\boxed{= \frac{\pi}{4}\sqrt{1+\frac{\pi^2}{16}} + \ln\left(\frac{\pi}{4}+\sqrt{1+\frac{\pi^2}{16}}\right)}$$

Exercise

Find the lengths of the curves $\vec{r}(t) = (3\cos t)\hat{i} + (3\sin t)\hat{j} + 2t^{3/2}\ \hat{k}; \quad 0 \le t \le 3$

Solution

$$\frac{d\vec{r}}{dt} = (-3\sin t)\hat{i} + (3\cos t)\hat{j} + 3t^{1/2}\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{9\sin^2 t + 9\cos^2 t + 9t}$$

$$= 3\sqrt{1+t^2}$$

$$L = 3\int_0^3 \sqrt{1+t}\ dt \qquad\qquad L = \int_a^b |\vec{r}'(t)|\ dt$$

$$= 3\int_0^3 (1+t)^{1/2}\ d(1+t)$$

$$= 2(1+t)^{3/2}\Big|_0^3$$

$$= 2\left(4^{3/2} - 1\right)$$

$$= 2(8-1)$$

$$\underline{= 14\ |}$$

Exercise

The acceleration of a wayward firework is given by $\vec{a}(t) = \sqrt{2}\,\hat{j} + 2t\,\hat{k}$ *for* $0 \le t \le 3$. Suppose the initial velocity of the firework is $\vec{v}(0) = 1$.

a) Find the velocity of the firework, for $0 \le t \le 3$.

b) Find the length of the trajectory of the firework over the interval $0 \le t \le 3$

Solution

a) $\vec{v} = \int \left\langle 0,\ \sqrt{2},\ 2t \right\rangle\ dt$

$$= \left\langle 0,\ \sqrt{2}\,t,\ t^2 \right\rangle + \vec{C}$$

$$\vec{v}(0) = 1 = \langle 1,\ 0,\ 0\rangle$$

$$\langle 1,\ 0,\ 0\rangle = \langle 0,\ 0,\ 0\rangle + \vec{C}$$

$$\vec{C} = \langle 1,\ 0,\ 0\rangle$$

$$\vec{v}(t) = \left\langle 0,\ \sqrt{2}\,t,\ t^2 \right\rangle + \langle 1,\ 0,\ 0\rangle$$

$$= \left\langle 1,\ \sqrt{2}\,t,\ t^2 \right\rangle$$

b) $L = \int_0^3 \sqrt{1+2t^2+t^4}\ dt$

$L = \int_a^b |\vec{r}'(t)|\ dt$

$= \int_0^3 \sqrt{(1+t^2)^2}\ dt$

$= \int_0^3 (1+t^2)\ dt$

$= t + \frac{1}{3}t^3 \Big|_0^3$

$= 3 + 9$

$\underline{= 12}$ *unit*

Exercise

If a string wound around a fixed circle in unwound while held taut in the plane of the circle, its end *P* traces an involute of the circle. The circle in question is the circle $x^2 + y^2 = 1$ and the tracing point starts at (1, 0). The unwound portion of the string is tangent to the circle at *Q*, and *t* is the radian measure of the angle from the position *x*-axis to segment *OQ*. Derive the parametric equations

$x = \cos t + t\sin t, \quad y = \sin t - t\cos t, \quad t > 0$ of the point $P(x, y)$ for the involute.

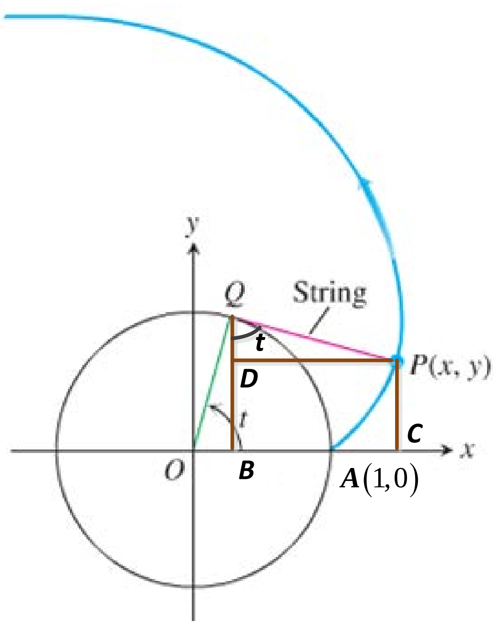
Solution

$\angle PQB = \angle QOB = t$

$PQ = arc(AQ) = t$

PQ = Length of the unwound string

$arc(AQ)$



$$\Delta PDQ: \begin{cases} \sin t = \frac{DP}{PQ} = \frac{DP}{t} \to \underline{DT = t\sin t} \\ \cos t = \frac{QD}{PQ} = \frac{QD}{t} \to \underline{QD = t\cos t} \end{cases}$$

$x = OB + BC$

$= OB + DP$

$\underline{= \cos t + t\sin t}$

$y = PC$

$= QB - QD$

$\underline{= \sin t - t\cos t}$