Solution *Section* 1.5 – Length of Curves

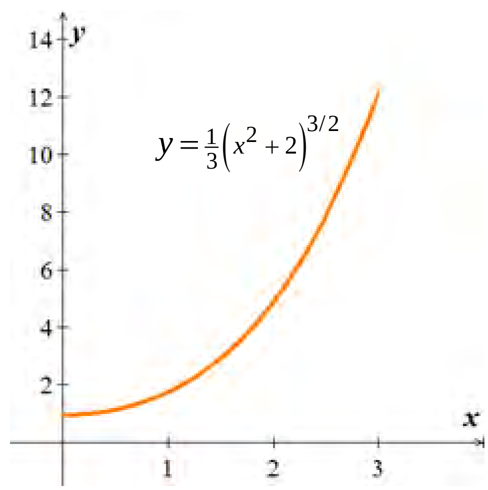
Exercise

Find the length of the curve $y=\frac{1}{3}\left(x^2+2\right)^{3/2}$ from $x = 0$ to $x = 3$.

Solution

$$\frac{dy}{dx}=\frac{1}{3}\frac{3}{2}\left(x^2+2\right)^{1/2}(2x)$$

$$=x\left(x^2+2\right)^{1/2}$$



$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+x^2\left(x^2+2\right)}$$

$$=\sqrt{1+x^4+2x^2}$$

$$=\sqrt{\left(x^2+1\right)^2}$$

$$=x^2+1$$

$$L=\int_0^3\left(x^2+1\right)dx$$

$$=\frac{1}{3}x^3+x\,\Big|_0^3$$

$$=9+3$$

$$=12\quad unit$$

Exercise

Find the length of the curve $y=(x)^{3/2}$ from $x = 0$ to $x = 4$.

Solution

$$\frac{dy}{dx}=\frac{3}{2}x^{1/2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{9}{4}x}$$

$$=\sqrt{\frac{4+9x}{4}}$$

$$=\frac{1}{2}\sqrt{4+9x}$$

$$L = \int_0^4 \frac{1}{2}(4+9x)^{1/2}\, dx$$

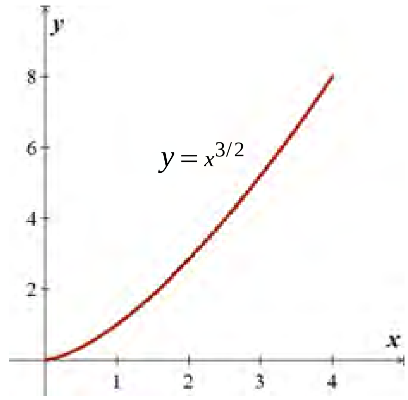
$$= \frac{1}{18}\int_0^4 (4+9x)^{1/2}\, d(4+9x)$$

$$= \frac{1}{18}\ (4+9x)^{27}\ \Big|_0^4$$

$$= \frac{1}{27}\left(40^{3/2} - 4^{3/2}\right)$$

$$= \frac{1}{27}\left(80\sqrt{10} - 8\right)$$

$$= \frac{8}{27}\left(10\sqrt{10} - 1\right) \quad \textit{unit}$$



Exercise

Find the length of the curve $x = \dfrac{y^{3/2}}{3} - y^{1/2}$ from $y = 1$ to $y = 9$.

Solution

$$a = \frac{1}{3}, \quad m = \frac{3}{2}, \quad b = -1, \quad n = \frac{1}{2}$$

1. $m + n = \frac{3}{2} + \frac{1}{2} = 2$ ✓

2. $abmn = \frac{1}{3}(-1)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(\frac{1}{3}y^{3/2} + y^{1/2}\ \Big|_1^9\right.$$

$$= 9 + 3 - \frac{4}{3}$$

$$= \frac{32}{3} \quad \textit{unit}$$

$$\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}$$

$$= \frac{1}{2}\left(y^{1/2} - \frac{1}{y^{1/2}}\right)$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\frac{1}{4}\left(y^{1/2} - \frac{1}{y^{1/2}}\right)^2}$$

$$= \sqrt{1+\frac{1}{4}\left(y - 2 + \frac{1}{y}\right)}$$

$$= \sqrt{1 + \frac{1}{4}y - \frac{1}{2} + \frac{1}{4y}}$$

$$= \sqrt{\frac{1}{4}y + \frac{1}{2} + \frac{1}{4y}}$$

$$= \sqrt{\frac{1}{4}\left(y + 2 + \frac{1}{y}\right)}$$

$$= \frac{1}{2}\sqrt{\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)^2}$$

$$= \frac{1}{2}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)$$

$$L = \frac{1}{2}\int_1^9 \left(y^{1/2} + y^{-1/2}\right)dy$$

$$= \frac{1}{2}\left(\frac{2}{3}y^{3/2} + 2y^{1/2}\right|_1^9$$

$$= \frac{1}{3}y^{3/2} + y^{1/2}\Big|_1^9$$

$$= \frac{1}{3}9^{3/2} + 3 - \left(\frac{1}{3} + 1\right)$$

$$= 9 + 3 - \frac{4}{3}$$

$$= \frac{32}{3} \quad \text{unit}$$

Exercise

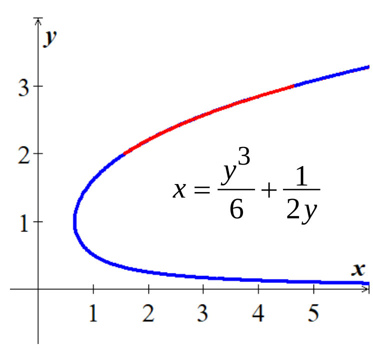
Find the length of the curve $x = \frac{y^3}{6} + \frac{1}{2y}$ from $y = 2$ to $y = 3$.

Solution

$$a = \frac{1}{6}, \quad m = 3, \quad b = \frac{1}{2}, \quad n = -1$$

1. $m + n = 3 - 1 = 2$ ✓

2. $abmn = \frac{1}{6}\left(\frac{1}{2}\right)(3)(-1) = -\frac{1}{4}$ ✓



$$L = \left(\frac{y^3}{6} - \frac{1}{2y}\right|_2^3$$

$$= \frac{1}{2}\left[\frac{27}{2} - \frac{1}{3} - \left(\frac{8}{3} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left(\frac{26}{3} - \frac{13}{6}\right)$$

$$=\frac{13}{4}\quad unit$$

$$\frac{dx}{dy}=\frac{1}{2}y^2-\frac{1}{2y^2}$$

$$=\frac{1}{2}\left(y^2-y^{-2}\right)$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2}=\sqrt{1+\frac{1}{4}\left(y^2-y^{-2}\right)^2}$$

$$=\frac{1}{2}\sqrt{4+\left(y^4-2+y^{-4}\right)}$$

$$=\frac{1}{2}\sqrt{y^4+2+y^{-4}}$$

$$=\frac{1}{2}\sqrt{\left(y^2+y^{-2}\right)^2}$$

$$=\frac{1}{2}\left(y^2+y^{-2}\right)$$

$$L=\frac{1}{2}\int_2^3\left(y^2+y^{-2}\right)dy$$

$$=\left(\frac{y^3}{6}-\frac{1}{2y}\right|_2^3$$

$$=\frac{1}{2}\left[9-\frac{1}{3}-\left(\frac{8}{3}-\frac{1}{2}\right)\right]$$

$$=\frac{1}{2}\left(\frac{26}{3}-\frac{13}{6}\right)$$

$$=\frac{13}{4}\quad unit$$

Exercise

Find the length of the curve $f(x)=x^3+\frac{1}{12x}\quad for\quad \frac{1}{2}\le x\le 2$

Solution

$a=1,\quad m=3,\quad b=\frac{1}{12},\quad n=-1$

1. $m+n=2$ ✓ **2.** $abmn=-\frac{1}{4}$ ✓

$$L=\left(x^3-\frac{1}{12x}\right|_{1/2}^{2}$$

$$=8-\frac{1}{24}-\frac{1}{8}+\frac{1}{6}$$

$$=8\quad unit$$

Exercise

Find the length of the curve of $f(x)=\frac{1}{5}x^5+\frac{1}{12x^3} \quad 1\le x\le 2$

Solution

$$a=\frac{1}{5}, \quad m=5, \quad b=\frac{1}{12}, \quad n=-3$$

1. $m+n=5-3=2$ ✓ **2.** $abmn=\frac{1}{5}\left(\frac{1}{12}\right)(5)(-3)=-\frac{1}{4}$ ✓

$$L=\frac{1}{5}x^5-\frac{1}{12x^3}\Big|_1^2$$
$$=\frac{32}{5}-\frac{1}{96}-\frac{1}{5}+\frac{1}{12}$$
$$=\frac{31}{5}+\frac{7}{96}$$
$$=\frac{3011}{480} \quad unit$$

Exercise

Find the length of the curve of $y=\frac{1}{3}x^{1/2}-x^{3/2}, \quad 0\le x\le \frac{1}{3}$

Solution

$$a=\frac{1}{3}, \quad m=\frac{1}{2}, \quad b=-1, \quad n=\frac{3}{2}$$

1. $m+n=\frac{1}{2}+\frac{3}{2}=2$ ✓ **2.** $abmn=\frac{1}{3}(-1)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)=-\frac{1}{4}$ ✓

$$L=\frac{1}{3}x^{1/2}+x^{3/2}\Big|_0^{1/3}$$
$$=\frac{1}{3\sqrt{3}}+\frac{1}{3\sqrt{3}}$$
$$=\frac{2}{3\sqrt{3}} \quad unit \qquad =\frac{2\sqrt{3}}{9}$$

Exercise

Find the length of the curve of $y=\frac{1}{3}x^3+\frac{1}{4x}, \quad 1\le x\le 2$

Solution

$$a=\frac{1}{3}, \quad m=3, \quad b=\frac{1}{4}, \quad n=-1$$

1. $m+n=3-1=2$ ✓ **2.** $abmn=\frac{1}{3}\left(\frac{1}{4}\right)(3)(-1)=-\frac{1}{4}$ ✓

$$L=\frac{1}{3}x^3-\frac{1}{4x}\Big|_1^2$$

$$= \frac{8}{3} - \frac{1}{8} - \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{3} + \frac{1}{8}$$

$$= \frac{59}{24} \quad unit$$

Exercise

Find the length of the curve of $y = 2e^{x} + \frac{1}{8}e^{-x} \quad 0 \le x \le \ln 2$

Solution

$a = 2, \quad m = 1, \quad b = \frac{1}{8}, \quad n = -1$

1. $m = -n = 1$ ✓ **2.** $abmn = 2\left(\frac{1}{8}\right)(1)(-1) = -\frac{1}{4}$ ✓

$$L = 2e^{x} - \frac{1}{8}e^{-x} \Big|_{0}^{\ln 2}$$

$$= 2e^{\ln 2} - \frac{1}{8}e^{-\ln 2} - 2 + \frac{1}{8}$$

$$= 4 - \frac{1}{16} - \frac{15}{8}$$

$$= \frac{33}{16} \quad unit$$

Exercise

Find the length of the curve of $y = e^{2x} + \frac{1}{16}e^{-2x}, \quad 0 \le x \le \ln 3$

Solution

$a = 1, \quad m = 2, \quad b = \frac{1}{16}, \quad n = -2$

1. $m = -n = 2$ ✓ **2.** $abmn = 1\left(\frac{1}{16}\right)(2)(-2) = -\frac{1}{4}$ ✓

$$L = e^{2x} - \frac{1}{16}e^{-2x} \Big|_{0}^{\ln 3}$$

$$= e^{2\ln 3} - \frac{1}{16}e^{-2\ln 3} - 1 + \frac{1}{16}$$

$$= 9 - \frac{1}{16}\left(\frac{1}{9}\right) - \frac{15}{16}$$

$$= \frac{1{,}160}{144}$$

$$= \frac{145}{18} \quad unit$$

Exercise

Find the length of the curve $y = \ln(\cos x) \quad 0 \le x \le \frac{\pi}{4}$

Solution

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$$

$$L = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\ dx$$

$$= \int_0^{\pi/4} \sqrt{\sec^2 x}\ dx$$

$$= \int_0^{\pi/4} \sec x\ dx$$

$$= \ln\left|\sec x + \tan x\right| \Big|_0^{\pi/4}$$

$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln\left|\sec 0 + \tan 0\right|$$

$$= \ln\left|\sqrt{2} + 1\right| - \ln\left|1 + 0\right|$$

$$= \ln\left|\sqrt{2} + 1\right| - 0$$

$$= \ln\left(\sqrt{2} + 1\right) \quad \textit{unit}$$

Exercise

Find the length of the curve $f(y) = 2e^{\sqrt{2}y} + \frac{1}{16}e^{-\sqrt{2}y} \quad \textit{for} \quad 0 \le y \le \frac{\ln 2}{\sqrt{2}}$

Solution

$$a = 2, \quad m = \sqrt{2}, \quad b = \frac{1}{16}, \quad n = -\sqrt{2}$$

1. $m = -n$ ✓ **2.** $abmn = 2\left(\sqrt{2}\right)\left(\frac{1}{16}\right)\left(-\sqrt{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(2e^{\sqrt{2}y} + \frac{1}{16}e^{-\sqrt{2}y}\right|_0^{\ln 2/\sqrt{2}}$$

$$= 2e^{\ln 2} + \frac{1}{16}e^{-\ln 2} - 2 - \frac{1}{16}$$

$$= 4 + \frac{1}{32} - \frac{33}{16}$$

$$= \frac{63}{32} \quad \textit{unit}$$

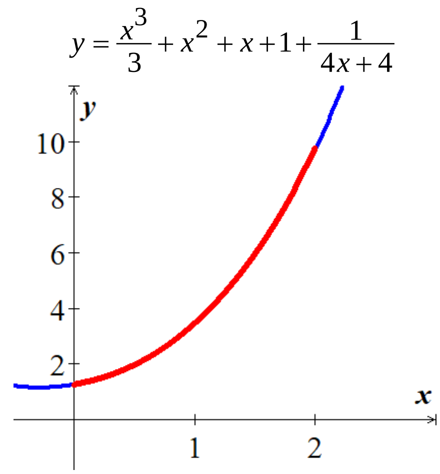
Exercise

Find the length of the curve $y = \frac{x^3}{3} + x^2 + x + 1 + \frac{1}{4x+4} \quad 0 \le x \le 2$

Solution

$$\frac{dy}{dx} = x^2 + 2x + 1 - \frac{1}{4}\frac{1}{(x+1)^2}$$

$$= (x+1)^2 - \frac{1}{4}\frac{1}{(x+1)^2}$$



$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left((x+1)^2 - \frac{1}{4}\frac{1}{(x+1)^2}\right)^2}$$

$$= \sqrt{1 + (x+1)^4 - \frac{1}{2} - \frac{1}{16}\frac{1}{(x+1)^4}}$$

$$= \sqrt{(x+1)^4 + \frac{1}{2} - \frac{1}{16}\frac{1}{(x+1)^4}}$$

$$= \sqrt{\left((x+1)^2 + \frac{1}{4}\frac{1}{(x+1)^2}\right)^2}$$

$$= (x+1)^2 + \frac{1}{4}(x+1)^{-2}$$

$$L = \int_0^2 \left((x+1)^2 + \frac{1}{4}(x+1)^{-2}\right) dx$$

$$= \int_0^2 \left((x+1)^2 + \frac{1}{4}(x+1)^{-2}\right) d(x+1)$$

$$= \frac{1}{3}(x+1)^3 - \frac{1}{4}\frac{1}{x+1} \Big|_0^2$$

$$= 9 - \frac{1}{12} - \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{53}{6} \quad \textit{unit}$$

Exercise

Find the length of the curve $y = \frac{x^3}{3} + x^2 + x + 1 + \frac{1}{4x+4} \quad 0 \le x \le 4$

Solution

$$\frac{dy}{dx} = x^2 + 2x + 1 - \frac{1}{4}\frac{1}{(x+1)^2}$$

$$= (x+1)^2 - \frac{1}{4}\frac{1}{(x+1)^2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left((x+1)^2 - \frac{1}{4}\frac{1}{(x+1)^2}\right)^2}$$

$$= \sqrt{1+(x+1)^4 - \frac{1}{2} - \frac{1}{16}\frac{1}{(x+1)^4}}$$

$$= \sqrt{(x+1)^4 + \frac{1}{2} - \frac{1}{16}\frac{1}{(x+1)^4}}$$

$$= \sqrt{\left((x+1)^2 + \frac{1}{4}\frac{1}{(x+1)^2}\right)^2}$$

$$= (x+1)^2 + \frac{1}{4}(x+1)^{-2}$$

$$L = \int_0^4 \left((x+1)^2 + \frac{1}{4}(x+1)^{-2}\right)dx$$

$$= \int_0^4 \left((x+1)^2 + \frac{1}{4}(x+1)^{-2}\right)d(x+1)$$

$$= \left(\frac{1}{3}(x+1)^3 - \frac{1}{4}(x+1)^{-1} \right|_0^4$$

$$= \frac{125}{3} - \frac{1}{20} - \frac{1}{3} + \frac{1}{4}$$

$$= \frac{124}{3} + \frac{1}{5}$$

$$= \frac{623}{15} \quad unit$$

Exercise

Find the length of the curve $y = \ln\left(e^x - 1\right) - \ln\left(e^x + 1\right) \quad \ln 2 \le x \le \ln 3$

Solution

$$y = \ln\left(e^x - 1\right) - \ln\left(e^x + 1\right) \quad \Rightarrow \frac{dy}{dx} = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1}$$

$$= \frac{e^{2x} + e^x - e^{2x} + e^x}{e^{2x} - 1}$$

$$= \frac{2e^x}{e^{2x}-1}$$

$$L = \int_{\ln 2}^{\ln 3} \sqrt{1+\left(\frac{2e^x}{e^{2x}-1}\right)^2}\ dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{1+\frac{4e^{2x}}{e^{4x}-2e^{2x}+1}}\ dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x}-2e^{2x}+1+4e^{2x}}{\left(e^{2x}-1\right)^2}}\ dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x}+2e^{2x}+1}{\left(e^{2x}-1\right)^2}}\ dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{\left(e^{2x}+1\right)^2}{\left(e^{2x}-1\right)^2}}\ dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{e^{2x}+1}{e^{2x}-1}\ dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\frac{e^{2x}+1}{e^x}}{\frac{e^{2x}-1}{e^x}}\ dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\frac{e^{2x}}{e^x}+\frac{1}{e^x}}{\frac{e^{2x}}{e^x}-\frac{1}{e^x}}\ dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{e^x+e^{-x}}{e^x-e^{-x}}\ dx \qquad \textit{or}\ \text{ Let } u = e^x - e^{-x} \quad \Rightarrow du = \left(e^x+e^{-x}\right)dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{1}{e^x-e^{-x}}\ d\left(e^x-e^{-x}\right) \qquad d\left(e^x-e^{-x}\right) = \left(e^x+e^{-x}\right)dx$$

$$= \ln\left|e^x-e^{-x}\right| \Big|_{\ln 2}^{\ln 3}$$

$$= \ln\left(3-\tfrac{1}{3}\right) - \ln\left(2-\tfrac{1}{2}\right)$$

$$= \ln\left(\frac{8}{3}\right) - \ln\left(\frac{3}{2}\right)$$

$$\boxed{= \ln\left(\frac{16}{9}\right) \quad unit}$$

Exercise

Find the length of the curve $f(x) = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2} \quad 1 \le x \le 4$

Solution

$$a = \frac{2}{3}, \quad m = \frac{3}{2}, \quad b = -\frac{1}{2}, \quad n = \frac{1}{2}$$

1. $m + n = \frac{3}{2} + \frac{1}{2} = 2$ ✓
2. $abmn = \frac{2}{3}\left(\frac{3}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$ ✓

$$L = \left(\frac{2}{3}x^{3/2} + \frac{1}{2}x^{1/2}\right|_1^4$$

$$= \frac{2}{3}4^{3/2} + 1 - \frac{2}{3} - \frac{1}{2}$$

$$= \frac{16}{3} - \frac{2}{3} + \frac{1}{2}$$

$$\boxed{= \frac{31}{6} \quad unit}$$

Exercise

Find the length of the curve $f(x) = x^3 + \frac{1}{12x} \quad 1 \le x \le 4$

Solution

$$a = 1, \quad m = 3, \quad b = \frac{1}{12}, \quad n = -1$$

1. $m + n = 3 - 1 = 2$ ✓
2. $abmn = (1)\left(\frac{1}{12}\right)(3)(-1) = -\frac{1}{4}$ ✓

$$L = \left(x^3 - \frac{1}{12x}\right|_1^4$$

$$= 4^3 - \frac{1}{48} - 1 + \frac{1}{12}$$

$$= 63 + \frac{3}{48}$$

$$\boxed{= \frac{3,027}{48} \quad unit}$$

Exercise

Find the length of the curve $f(x)=\frac{1}{8}x^4+\frac{1}{4x^2} \quad 1\le x\le 10$

Solution

$$a=\frac{1}{8},\quad m=4,\quad b=\frac{1}{4},\quad n=-2$$

1. $m+n=4-2=2$ ✓
2. $abmn=\left(\frac{1}{8}\right)\left(\frac{1}{4}\right)(4)(-2)=-\frac{1}{4}$ ✓

$$L=\left(\frac{1}{8}x^4-\frac{1}{4x^2}\right|_1^{10}$$

$$=\frac{10^4}{8}-\frac{1}{400}-\frac{1}{8}+\frac{1}{4}$$

$$=\frac{9,999}{8}+\frac{99}{400}$$

$$=\frac{9}{8}\left(1111+\frac{11}{50}\right)$$

$$=\frac{9}{8}\left(\frac{55,561}{50}\right)$$

$$=\frac{500,049}{400} \quad \textit{unit}$$

Exercise

Find the length of the curve $f(x)=\frac{1}{4}x^4+\frac{1}{8x^2} \quad 3\le x\le 8$

Solution

$$a=\frac{1}{4},\quad m=4,\quad b=\frac{1}{8},\quad n=-2$$

1. $m+n=4-2=2$ ✓
2. $abmn=\left(\frac{1}{4}\right)\left(\frac{1}{8}\right)(4)(-2)=-\frac{1}{4}$ ✓

$$L=\left(\frac{1}{4}x^4-\frac{1}{8x^2}\right|_3^{8}$$

$$=\frac{8^4}{4}-\frac{1}{8^3}-\frac{81}{4}+\frac{1}{72}$$

$$=\frac{4,015}{4}-\frac{1}{512}+\frac{1}{72}$$

$$=\frac{1}{4}\left(4,015-\frac{1}{128}+\frac{1}{18}\right)$$

$$=\frac{1}{4}\left(4,015+\frac{55}{1,152}\right)$$

$$\boxed{=\frac{4,625,335}{4,608}\quad unit}$$

Exercise

Find the length of the curve $f(x)=\frac{1}{10}x^5+\frac{1}{6x^3}\quad 1\le x\le 7$

Solution

$$a=\frac{1}{10},\quad m=5,\quad b=\frac{1}{6},\quad n=-3$$

1. $m+n=5-3=2$ ✓

2. $abmn=\left(\frac{1}{10}\right)\left(\frac{1}{6}\right)(5)(-3)=-\frac{1}{4}$ ✓

$$L=\left(\frac{1}{10}x^5-\frac{1}{6x^3}\right|_1^7$$

$$=\frac{7^5}{10}-\frac{1}{2,058}-\frac{1}{10}+\frac{1}{6}$$

$$=\frac{8,403}{5}+\frac{57}{343}$$

$$\boxed{=\frac{2,882,514}{1,715}\quad unit}$$

Exercise

Find the length of the curve $f(x)=\frac{3}{10}x^{1/3}-\frac{3}{2}x^{5/3}\quad 0\le x\le 12\text{b}$

Solution

$$a=\frac{1}{10},\quad m=5,\quad b=\frac{1}{6},\quad n=-3$$

1. $m+n=5-3=2$ ✓

2. $abmn=\left(\frac{1}{10}\right)\left(\frac{1}{6}\right)(5)(-3)=-\frac{1}{4}$ ✓

$$L=\left(\frac{3}{10}x^{1/3}+\frac{3}{2}x^{5/3}\right|_0^{12}$$

$$=\frac{3}{10}\sqrt[3]{12}+\frac{3}{2}12\sqrt[3]{144}$$

$$\boxed{=\frac{3}{10}\sqrt[3]{12}+18\sqrt[3]{144}\quad unit}\qquad =\frac{3}{10}\sqrt[3]{12}\left(1+600\sqrt[3]{12}\right)$$

Exercise

Find the length of the curve $f(x)=x^{1/2}-\frac{1}{3}x^{3/2} \quad 2\le x\le 9$

Solution

$a=1, \quad m=\frac{1}{2}, \quad b=-\frac{1}{3}, \quad n=\frac{3}{2}$

1. $m+n=\frac{1}{2}+\frac{3}{2}=2$ ✓

2. $abmn=(1)\left(-\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)=-\frac{1}{4}$ ✓

$$L=\left(x^{1/2}+\frac{1}{3}x^{3/2}\right)\Big|_2^9$$

$$=3+9-\sqrt{2}-\frac{2\sqrt{2}}{3}$$

$$\boxed{=\frac{1}{3}\left(36-5\sqrt{2}\right) \quad unit}$$

Exercise

Find the length of the curve $y=x^{1/2}-\frac{1}{3}x^{3/2} \quad 1\le x\le 4$

Solution

$a=1, \quad m=\frac{1}{2}, \quad b=-\frac{1}{3}, \quad n=\frac{3}{2}$

1. $m+n=\frac{1}{2}+\frac{3}{2}=2$ ✓

2. $abmn=(1)\left(-\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)=-\frac{1}{4}$ ✓

$$L=\left(x^{1/2}+\frac{1}{3}x^{3/2}\right)\Big|_1^4$$

$$=2+\frac{8}{3}-1-\frac{1}{3}$$

$$=1+\frac{7}{3}$$

$$\boxed{=\frac{10}{3} \quad unit}$$

Exercise

Find the length of the curve $x=y^{2/3}, \quad 1\le y\le 8$

Solution

$$x'=\frac{2}{3}y^{-1/3}$$

$$(x')^2 = \frac{4}{9}y^{-2/3}$$

$$L = \int_1^8 \sqrt{1+\frac{4}{9y^{2/3}}}\ dy$$

$$= \int_1^8 \frac{1}{3y^{1/3}}\sqrt{9y^{2/3}+4}\ dy$$

$$= \frac{1}{3}\int_1^8 y^{-1/3}\sqrt{9y^{2/3}+4}\ dy$$

$$= \frac{1}{18}\int_1^8 \left(9y^{2/3}+4\right)^{1/2}\ d\left(9y^{2/3}+4\right)$$

$$= \frac{1}{27}\left(9y^{2/3}+4\right)^{3/2}\Big|_1^8$$

$$= \frac{1}{27}\left(\left(9\left(2^3\right)^{2/3}+4\right)^{3/2} - 13^{3/2}\right)$$

$$= \frac{1}{27}\left(40^{3/2} - 13^{3/2}\right) \quad unit$$

Exercise

Find the length of the curve $y = 2x + 4 \quad -2 \le x \le 2$

Solution

$$\sqrt{1+(y')^2} = \sqrt{1+2^2}$$

$$= \sqrt{5}$$

$$L = \int_{-2}^{2} \sqrt{5}\ dx$$

$$= \sqrt{5}\ x\Big|_{-2}^{2}$$

$$= 4\sqrt{5} \quad unit$$

Exercise

Find the length of the curve $y=\frac{x^3}{6}+\frac{1}{2x} \quad x\in[1,\ 2]$

Solution

$$a=\frac{1}{6},\quad m=3,\quad b=\frac{1}{2},\quad n=-1$$

1. $m+n=3-1=2$ ✓
2. $abmn=\left(\frac{1}{6}\right)(3)\left(\frac{1}{2}\right)(-1)=-\frac{1}{4}$ ✓

$$L=\frac{x^3}{6}-\frac{1}{2x}\Big|_1^2$$

$$=\frac{4}{3}-\frac{1}{4}-\frac{1}{6}+\frac{1}{2}$$

$$=\frac{7}{12}\quad unit$$

Exercise

Find the length of the curve $f(x)=x^{1/2}-\frac{1}{3}x^{3/2} \quad 1\le x\le 3$

Solution

$$a=1,\quad m=\frac{1}{2},\quad b=-\frac{1}{3},\quad n=\frac{3}{2}$$

1. $m+n=\frac{1}{2}+\frac{3}{2}=2$ ✓
2. $abmn=(1)\left(-\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)=-\frac{1}{4}$ ✓

$$L=\left(x^{1/2}+\frac{1}{3}x^{3/2}\right|_1^3$$

$$=\sqrt{3}+\sqrt{3}-1-\frac{1}{3}$$

$$=2\sqrt{3}-\frac{4}{3}\quad unit$$

Exercise

Find the length of the curve $y=\frac{3}{4}x^{4/3}-\frac{3}{8}x^{2/3}+5, \quad 1\le x\le 8$

Solution

$$a=\frac{3}{4},\quad m=\frac{4}{3},\quad b=-\frac{3}{8},\quad n=\frac{2}{3}$$

1. $m+n=\frac{4}{3}+\frac{2}{3}=2$ ✓

2. $abmn=\left(\frac{3}{4}\right)\left(\frac{4}{3}\right)\left(-\frac{3}{8}\right)\left(\frac{2}{3}\right)=-\frac{1}{4}$ ✓

$$L=\left(\frac{3}{4}x^{4/3}+\frac{3}{8}x^{2/3}\right|_1^8$$

$$=\frac{3}{4}\left(2^3\right)^{4/3}+\frac{3}{8}\left(2^3\right)^{2/3}-\frac{3}{4}-\frac{3}{8}$$

$$=12+\frac{3}{2}-\frac{3}{4}-\frac{3}{8}$$

$$=\frac{96+12-6-3}{8}$$

$$=\frac{99}{8}\quad unit$$

Exercise

Find the length of the curve $y=\ln x-\frac{1}{8}x^2;\quad 1\le x\le 2$

Solution

$$\sqrt{1+\left(y'\right)^2}=\sqrt{1+\left(\frac{1}{x}-\frac{1}{4}x\right)^2}$$

$$=\sqrt{1+\frac{1}{x^2}-\frac{1}{2}+\frac{1}{16}x^2}$$

$$=\sqrt{\frac{1}{x^2}+\frac{1}{2}+\frac{1}{16}x^2}$$

$$=\sqrt{\left(\frac{1}{x}+\frac{1}{4}x\right)^2}$$

$$=\frac{1}{x}+\frac{1}{4}x$$

$$L=\int_1^2\left(\frac{1}{x}+\frac{1}{4}x\right)dx$$

$$=\ln x+\frac{1}{8}x^2\Big|_1^2$$

$$=\ln 2+\frac{1}{2}-\frac{1}{8}$$

$$=\ln 2+\frac{3}{8}\quad unit$$

Exercise

Find the length of the curve $y=\frac{1}{2}x^2-\frac{1}{4}\ln x;\quad 1\le x\le 3$

Solution

$$\sqrt{1+(y')^2}=\sqrt{1+\left(x-\frac{1}{4x}\right)^2}$$

$$=\sqrt{1+x^2-\frac{1}{2}+\frac{1}{16x^2}}$$

$$=\sqrt{x^2+\frac{1}{2}+\frac{1}{16x^2}}$$

$$=\sqrt{\left(x+\frac{1}{4x}\right)^2}$$

$$=x+\frac{1}{4x}$$

$$L=\int_1^3\left(x+\frac{1}{4x}\right)dx$$

$$=\frac{1}{2}x^2+\frac{1}{4}\ln x\ \Big|_1^3$$

$$=\frac{9}{2}+\frac{1}{4}\ln 3-\frac{1}{2}$$

$$=4+\frac{1}{4}\ln 2\quad \textit{unit}$$

Exercise

Find the length of the curve $y=\int_{-2}^{x}\sqrt{2t^4-1}\,dt\quad -2\le x\le -1$

Solution

$$\frac{dy}{dt}=\sqrt{2t^4-1}$$

$$\sqrt{1+\left(\frac{dy}{dt}\right)^2}=\sqrt{1+2t^4-1}$$

$$=\sqrt{2t^4}$$

$$=\sqrt{2}\,t^2$$

$$L=\sqrt{2}\int_{-2}^{-1}t^2\,dt$$

$$= \frac{\sqrt{2}}{3} t^3 \Big|_{-2}^{-1}$$

$$= \frac{\sqrt{2}}{3}(-1+8)$$

$$= \frac{7\sqrt{2}}{3} \quad unit$$

Exercise

Find the length of the curve $x = \int_0^y \sqrt{\sec^4 t - 1}\, dt \quad -\frac{\pi}{4} \le y \le \frac{\pi}{4}$

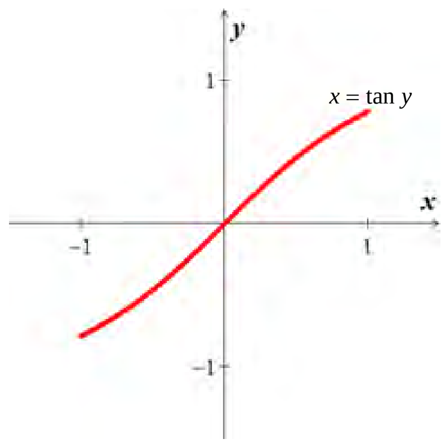
Solution

$$\frac{dx}{dy} = \sqrt{\sec^4 y - 1}$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\sec^4 y - 1}$$

$$= \sqrt{\sec^4 y}$$

$$= \sec^2 y$$



$$L = \int_{-\pi/4}^{\pi/4} \sec^2 y \; dy$$

$$= \tan y \Big|_{-\pi/4}^{\pi/4}$$

$$= 1 - (-1)$$

$$= 2 \quad unit$$

Exercise

Find the length of the curve $y = 3 - 2x \quad 0 \le x \le 2$. Check your answer by finding the length of the segment as the hypotenuse of a right triangle.

Solution

$$\frac{dy}{dx} = -2$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+4}$$

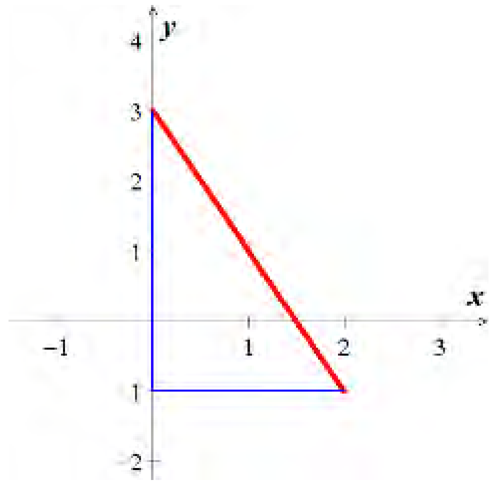
$$= \sqrt{5}$$

$$L = \int_0^2 \sqrt{5}\ dx$$

$$= \sqrt{5}\, x \Big|_0^2$$

$$\underline{= 2\sqrt{5} \quad unit}$$

$$\begin{cases} x = 0 & \rightarrow y = 3 \\ x = 2 & \rightarrow y = -1 \end{cases}$$



$$d = \sqrt{(2-0)^2 + (3+1)^2}$$

$$= \sqrt{4+16}$$

$$= \sqrt{20}$$

$$\underline{= 2\sqrt{5}}$$

Exercise

Find a curve through the origin in the *xy*-plane whose length from $x = 0$ to $x = 1$ is $L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x}\ dx$

Solution

$$L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x}\ dx$$

$$\frac{dy}{dx} = \frac{e^{x/2}}{2} \qquad \rightarrow dy = \frac{e^{x/2}}{2} dx$$

$$y = \int \frac{e^{x/2}}{2} dx$$

$$\underline{= e^{x/2} + C}$$

$$0 = e^{0/2} + C$$

$$0 = 1 + C \qquad \Rightarrow \underline{C = -1}$$

$$\underline{y = e^{x/2} - 1}$$

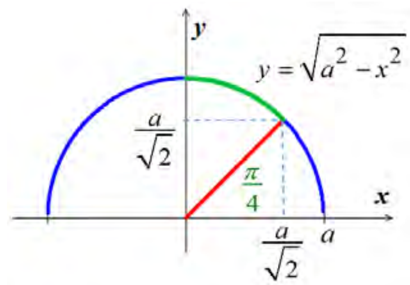
Exercise

Confirm that the circumference of a circle of radius a is $2\pi a$.

Solution

$$f(x)=\sqrt{a^2-x^2} \quad \text{for} \quad -a\le x\le a$$

$$f'(x)=-\frac{x}{\sqrt{a^2-x^2}} \qquad \text{but } \boxed{x\ne \pm a}$$



$$\sqrt{1+f'(x)^2}=\sqrt{1+\frac{x^2}{a^2-x^2}}$$

$$=\frac{a}{\sqrt{a^2-x^2}}$$

Let's compute the length of $\frac{1}{8}$ of the circle on $\left[0,\ \frac{a}{\sqrt{2}}\right]$

$$L=8a\int_0^{a/\sqrt{2}}\frac{dx}{\sqrt{a^2-x^2}}$$

$$=8a\ \sin^{-1}\left(\frac{x}{a}\right)\Big|_0^{a/\sqrt{2}}$$

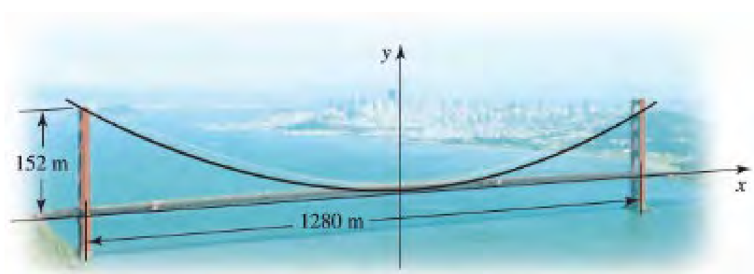
$$=8a\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$=8a\left(\frac{\pi}{4}\right)$$

$$\underline{=2\pi a \quad unit}$$

Exercise

The profile of the cables on a suspension bridge may be modeled by a parabola. The central span of the Golden Gate Bridge is 1280 *m* long and 152 *m* high. The parabola $y=0.00037x^2$ gives a good fit to the shape of the cables, where $|x|\le 640$, and x and y are measured in *meters*. Approximate the length of the cables that stretch between the tops of the two towers.



Solution

$$y'=0.00074x$$

$$L = \int_{-640}^{640} \sqrt{1+(.00074x)^2}\ dx \qquad \int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|$$

$$= \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\left|x+\sqrt{1+x^2}\right| \Bigg|_{-640}^{640}$$

$$= 320\sqrt{1+640^2} + \frac{1}{2}\ln\left|640+\sqrt{1+640^2}\right| + 320\sqrt{1+x^2} - \frac{1}{2}\ln\left|-640+\sqrt{1+640^2}\right|$$

$$\approx 1326.4 \quad m$$

Exercise

Electrical wires suspended between two towers form a caternary modeled by the equation

$$y = 20\cosh\frac{x}{20}, \quad -20 \le x \le 20$$

Where x and y are measured in *meters*. The towers are 40 *meters* apart. Find the length of the suspended cable.

Solution

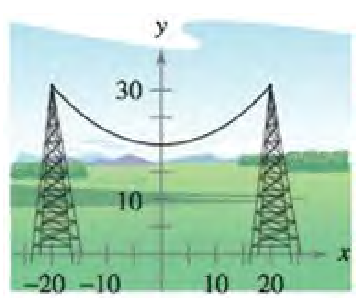
$$y = 20\cosh\frac{x}{20}$$

$$y' = \sinh\frac{x}{20}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\sinh^2\frac{x}{20}}$$

$$= \sqrt{\cosh^2\frac{x}{20}}$$

$$= \cosh\frac{x}{20}$$



$$L = \int_{-20}^{20} \cosh\frac{x}{20}\ dx$$

$$= 2(20)\ \sinh\frac{x}{20}\Bigg|_{0}^{20}$$

$$= 40(\sinh 1 - \sinh 0)$$

$$= 40\sinh 1 \quad unit$$

$$= 20\left(e - e^{-1}\right) \quad unit$$

Exercise

A barn is 100 *feet* long and 40 *feet* wide. A cross section of the roof is the inverted caternary $y = 31 - 10\left(e^{x/20} + e^{-x/20}\right)$. Find the number of ***square feet*** of roofing on the barn.

Solution

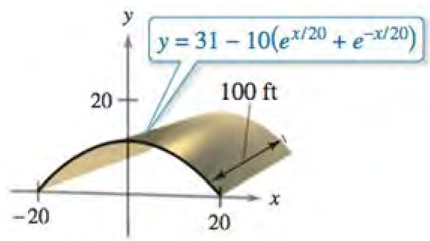
$$a = 10, \quad m = \frac{1}{20}, \quad b = 10, \quad n = -\frac{1}{20}$$

1. $m = -n$ ✓
2. $abmn = 10(10)\left(\frac{1}{20}\right)\left(-\frac{1}{20}\right) = -\frac{1}{4}$ ✓

$$L = 10\left(e^{x/20} - e^{-x/20}\right)\Big|_{-20}^{20}$$

$$= 10\left(e - \frac{1}{e} - \frac{1}{e} + e\right)$$

$$= 20\left(e - \frac{1}{e}\right)\ ft \quad \approx 47\ ft$$



∴ There are $100(47) = 4,700\ ft^2$ of roofing on the barn

Exercise

A cable for a suspension bridge has the shape of a parabola with equation $y = kx^2$. Let h represent the height of the cable from it lowest point to its highest point and let $2w$ represent the total span of the bridge.

Show that the length C of the cable is given by $$C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\, dx$$

Solution

$$y' = 2kx$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + 4k^2x^2}$$

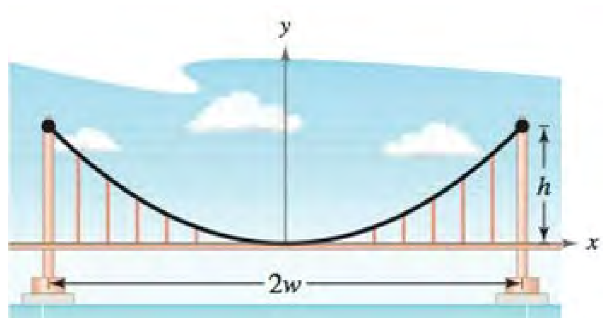
At $(w,\ h) \to h = kw^2$

$$\Rightarrow\ k = \frac{h}{w^2}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{4h^2}{w^4}x^2}$$

∴ By symmetry:

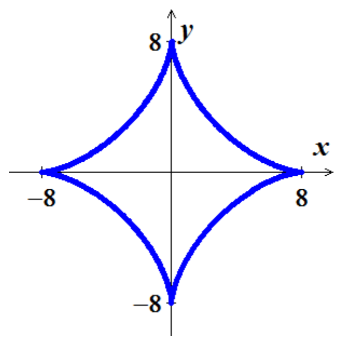
$$C = 2\int_0^w \sqrt{1 + \frac{4h^2}{w^4}x^2}\, dx$$



Exercise

Find the total length of the graph of the astroid $x^{2/3} + y^{2/3} = 4$

Solution



$$x^{2/3} + y^{2/3} = 4$$

$$y = \left(4 - x^{2/3}\right)^{3/2}$$

$$y' = \frac{3}{2}\left(-\frac{2}{3}x^{-1/3}\right)\left(4 - x^{2/3}\right)^{1/2}$$

$$= -\frac{1}{x^{1/3}}\left(4 - x^{2/3}\right)^{1/2}$$

$$1 + (y')^2 = 1 + \frac{1}{x^{2/3}}\left(4 - x^{2/3}\right)$$

$$= \frac{4}{x^{2/3}}$$

$$y = 0 \quad \to \quad x^{2/3} = 4$$

$$\underline{x = 4^{3/2} = 8}$$

$$L = 4\int_0^8 \sqrt{\frac{4}{x^{2/3}}}\, dx$$

$$= 8\int_0^8 x^{-1/3}\, dx$$

$$= 12\, x^{2/3} \Big|_0^8$$

$$= 12(4 - 0)$$

$$\underline{= 48 \quad unit}$$

Exercise

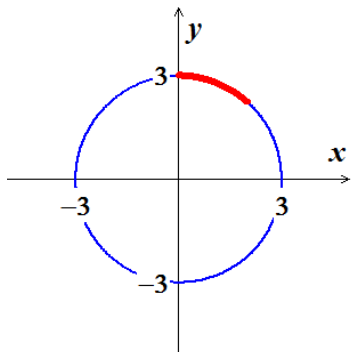
Find the arc length from $(0,\ 3)$ clockwise to $\left(2,\ \sqrt{5}\right)$ along the circle $x^2 + y^2 = 9$

Solution

$$y = \sqrt{9 - x^2}$$

$$y' = -\frac{x}{\sqrt{9 - x^2}}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{x^2}{9-x^2}}$$
$$= \sqrt{\frac{9}{9-x^2}}$$
$$= \frac{3}{\sqrt{9-x^2}}$$



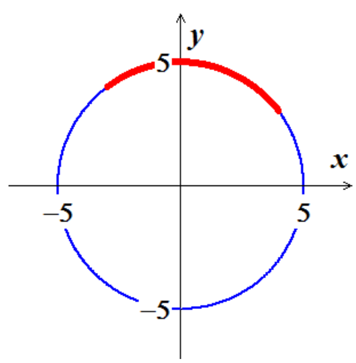
$$L = \int_0^2 \frac{3}{\sqrt{9-x^2}}\, dx$$
$$= 3 \arcsin\frac{x}{3}\Big|_0^2$$
$$= 3\arcsin\frac{2}{3} \quad \textit{unit} \qquad \approx 2.1892$$

Exercise

Find the arc length from $(-3,\ 4)$ clockwise to $(4,\ 3)$ along the circle $x^2+y^2=25$. Show that the result is one-fourth the circumference of the circle.

Solution

$$y = \sqrt{25-x^2}$$
$$y' = -\frac{x}{\sqrt{25-x^2}}$$



$$\sqrt{1+(y')^2} = \sqrt{1+\frac{x^2}{25-x^2}}$$
$$= \sqrt{\frac{25}{25-x^2}}$$
$$= \frac{5}{\sqrt{25-x^2}}$$

$$L = \int_{-3}^4 \frac{5}{\sqrt{25-x^2}}\, dx$$
$$= 5 \arcsin\frac{x}{5}\Big|_{-3}^4$$
$$= 5\left(\arcsin\frac{4}{5}+\arcsin\frac{3}{5}\right) \quad \textit{unit} \qquad \approx 7.854$$

Exercise

$y = \ln x$ between $x = 1$ and $x = b > 1$ that

$$\int \frac{\sqrt{x^2 + a^2}}{x}\,dx = \sqrt{x^2 + a^2} - a\ln\left(\frac{a + \sqrt{x^2 + a^2}}{x}\right) + C$$

Use any means to approximate the value of b for which the curve has length 2.

Solution

Given: $L = 2$

$y = \ln x \ \rightarrow \ y' = \dfrac{1}{x}$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{1}{x^2}}$$

$$= \frac{\sqrt{x^2 + 1}}{x}$$

$$L = \int_1^b \frac{\sqrt{x^2 + 1}}{x}\,dx$$

$$= \sqrt{x^2 + 1} - \ln\left(\frac{1 + \sqrt{x^2 + 1}}{x}\right)\Bigg|_1^b$$

$$= \sqrt{b^2 + 1} - \ln\left(\frac{1 + \sqrt{b^2 + 1}}{b}\right) - \sqrt{2} + \ln\left(1 + \sqrt{2}\right) \ = 2$$

Using Mapple:

$$fsolve\left(\sqrt{b^2 + 1} - \ln\left(\frac{1 + \sqrt{b^2 + 1}}{b}\right) - \sqrt{2} + \ln\left(1 + \sqrt{2}\right) = 2\ ,\ \ b\right)$$

$b = 2.714999998$

$\boxed{b \approx 2.715}$