Math 2318 – Linear Algebra **_Exam 1_** **_Review_**

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1. Solve the Solve the system by Gaussian elimination

$$\textbf{\textit{a}})\ \begin{cases} 2x_1 - 4x_2 + 3x_3 - 4x_4 - 11x_5 = 28 \\ x_1 + 2x_2 - x_3 + 2x_4 + 5x_5 = -13 \\ \qquad\qquad - 3x_3 + x_4 + 6x_5 = -10 \\ 3x_1 - 6x_2 + 10x_3 - 8x_4 - 28x_5 = 61 \end{cases}$$

$$\textbf{\textit{b}})\ \begin{cases} x_1 + x_3 + x_4 - 2x_5 = 1 \\ 2x_1 + x_2 + 3x_3 - x_4 + x_5 = 0 \\ 3x_1 - x_2 + 4x_3 + x_4 + x_5 = 1 \end{cases}$$

2. Given the matrices

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix},\quad B = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix},\quad C = \begin{bmatrix} 2 & 1 \\ 4 & 0 \\ 8 & -1 \\ 3 & 2 \end{bmatrix},\quad D = \begin{bmatrix} 2 & 1 & 3 & 6 \\ 2 & 0 & 0 & 4 \\ 1 & -1 & 1 & -1 \\ 1 & 3 & 1 & 2 \end{bmatrix}$$

a) $A - 3B$ ***b***) $3A + 4B$ ***c***) $D + C$ ***d***) AB ***e***) BA ***f***) CD ***g***) DC ***h***) CA ***i***) AC ***j***) CB

3. Find the inverse of the following matrices if they exist.

a) $A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$ ***b***) $B = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$ ***c***) $C = \begin{bmatrix} 2 & -4 \\ a & b \end{bmatrix}$

4. Evaluate the determinant

a) $\begin{vmatrix} 3 & 1 & 2 \\ -2 & 3 & 1 \\ 3 & 4 & -6 \end{vmatrix}$ ***b***) $\begin{vmatrix} x & 1 & -1 \\ x^2 & x & x \\ 0 & x & 1 \end{vmatrix}$ ***c***) $\begin{vmatrix} 1 & x & x \\ 2 & x^2 & 2x \\ x & 0 & -1 \end{vmatrix}$ ***d***) $\begin{vmatrix} a & c \\ -2 & -4 \end{vmatrix}$ ***e***) $\begin{vmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{vmatrix}$

5. Find $A^2,\ A^{-2},\ and\ A^{-k}$ by inspection

$a)\ A = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ $b)\ A = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

6. Find the components of the vector $\overrightarrow{P_1P_2}$ with initial point $P_1(2,\ -1,\ 4)$ and terminal point $P_2(7,\ 5,\ -8)$

7. Find $\boldsymbol{u} \times \boldsymbol{v}$, where $\boldsymbol{u} = (1,\ 2,\ -2)$ *and* $\boldsymbol{v} = (3,\ 0,\ 1)$ and show that $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ and to $\boldsymbol{v}$.

8. Calculate the scalar triple product $u \cdot (v \times w)$ of the vectors:

a) $\boldsymbol{u} = 3\boldsymbol{i} - 2\boldsymbol{j} - 5\boldsymbol{k} \quad \boldsymbol{v} = \boldsymbol{i} + 4\boldsymbol{j} - 4\boldsymbol{k} \quad \boldsymbol{w} = 3\boldsymbol{j} + 2\boldsymbol{k}$

b) $\boldsymbol{u} = (-2, 0, 6) \quad \boldsymbol{v} = (1, -3, 1) \quad \boldsymbol{w} = (-5, -1, 1)$

9. Given $\boldsymbol{u} = (3,\ 2,\ -1)$, $\boldsymbol{v} = (0,\ 2,\ -3)$, *and* $\boldsymbol{w} = (2,\ 6,\ 7)$ Compute the vectors

a) $\boldsymbol{u} \times \boldsymbol{v}$

b) $\boldsymbol{v} \times \boldsymbol{w}$

c) $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w})$

d) $(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w}$

e) $\boldsymbol{u} \times (\boldsymbol{v} - 2\,\boldsymbol{w})$

f) $\|\boldsymbol{u}\|$

g) Unit vector of $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$

h) Angle between $\boldsymbol{v}$, and $\boldsymbol{w}$

i) $\|3\boldsymbol{u} - 5\boldsymbol{v} + \boldsymbol{w}\|$

j) $\boldsymbol{u} \cdot \boldsymbol{v}$

k) $\boldsymbol{u} \cdot \boldsymbol{w}$

10. Determine whether the vectors form an orthogonal set

a) $\boldsymbol{v}_1 = (2,\ 3)$, $\boldsymbol{v}_2 = (-3,\ 2)$

b) $\boldsymbol{v}_1 = (-3,\ 4,\ -1)$, $\boldsymbol{v}_2 = (1,\ 2,\ 5)$, $\boldsymbol{v}_3 = (4,\ -3,\ 0)$

c) $\boldsymbol{v}_1 = (2, -2, 1)$, $\boldsymbol{v}_2 = (2, 1, -2)$, $\boldsymbol{v}_3 = (1, 2, 2)$

11. Find the vector component $\left(proj_{\boldsymbol{a}}\boldsymbol{u} = \dfrac{\boldsymbol{u} \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} \right)$ of $\boldsymbol{u}$ along $\boldsymbol{a}$ and the vector component of $\boldsymbol{u}$ orthogonal to $\boldsymbol{a}$.

a) $\boldsymbol{u} = (-1,\ -2)$, $\boldsymbol{a} = (-2,\ 3)$

b) $v = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

c) $\boldsymbol{u} = (1,\ 1,\ 1)$, $\boldsymbol{a} = (0,\ 2,\ -1)$

d) $\boldsymbol{u} = (2,\ 0,\ 1)$, $\boldsymbol{a} = (1,\ 2,\ 3)$

12. Find the area of the parallelogram determined by the given vectors $\boldsymbol{u} = (1,\ 1,\ 1)$, $\boldsymbol{v} = (3,\ 2,\ -5)$

13. Use the cross product to find a vector that is orthogonal to both $\boldsymbol{u} = (3,\ 3,\ 1)$, $\boldsymbol{v} = (0,\ 4,\ 2)$

14. Find the area of the triangle with the given vertices:

a) $A(2, 0) \quad B(3, 4) \quad C(-1, 2)$

b) $A(2, 6, -1) \quad B(1, 1, 1) \quad C(4, 6, 2)$

15. Find the volume of the parallelepiped with sides $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

$\boldsymbol{u} = (2, -6, 2)$, $\boldsymbol{v} = (0, 4, -2)$, $\boldsymbol{w} = (2, 2, -4)$

16. Express $\left((AB)^{-1}\right)^T$ in terms of $\left(A^{-1}\right)^T$ and $\left(B^{-1}\right)^T$

***Prove*:**

a) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

b) $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$

c) If A is invertible and $AB = AC$, prove that $B = C$

d) Prove if $A^T A = A$, then A is symmetric and $A = A^2$

e) $\det(A+B) \neq \det(A) + \det(B)$

f) $\det(AB) = \det(A)\det(B)$

g) $\det(kA) = k^n \det\left(A^T\right)$

h) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonzero vectors such that $\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$, then $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal.

i) Prove that $\|\boldsymbol{u}+\boldsymbol{v}\| = \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ *iff* $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel vectors.

j) Lagrange's identity: $\|\boldsymbol{u}\times\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 \; \|\boldsymbol{v}\|^2 - (\boldsymbol{u}.\boldsymbol{v})^2$

Solution

1. $a)\ \left(3+2x_2-2x_5,\ x_2,\ 2+x_5,\ -4-3x_5,\ x_5\right)$

$a)\ \left(3-\frac{7}{2}x_4+8x_5,\ \frac{1}{2}x_4+x_5,\ -2+\frac{5}{2}x_4-6x_5,\ x_4,\ x_5\right)$

2. $\boldsymbol{a})\ \begin{bmatrix}-1 & -3\\ -2 & -8\end{bmatrix}$ $\boldsymbol{b})\ \begin{bmatrix}10 & 17\\ 7 & 28\end{bmatrix}$ $\boldsymbol{c})$ *can't be determined* $\boldsymbol{d})\ \begin{bmatrix}5 & 16\\ 5 & 18\end{bmatrix}$

$\boldsymbol{e})\ \begin{bmatrix}4 & 11\\ 6 & 19\end{bmatrix}$ $\boldsymbol{f})$ *can't be determined* $\boldsymbol{g})\ \begin{bmatrix}48 & 11\\ 16 & 10\\ 3 & -2\\ 28 & 4\end{bmatrix}$ $\boldsymbol{h})\ \begin{bmatrix}5 & 10\\ 8 & 12\\ 15 & 20\\ 8 & 17\end{bmatrix}$

$\boldsymbol{i})$ *can't be determined* $\boldsymbol{j})\ \begin{bmatrix}3 & 8\\ 4 & 8\\ 7 & 12\\ 5 & 14\end{bmatrix}$

3. $a)\ A^{-1}=\begin{bmatrix}1 & -2\\ 2 & -3\end{bmatrix}$ $b)\ B^{-1}$ Does not exist $c)\ C^{-1}=\begin{bmatrix}\frac{b}{2b+4a} & \frac{2}{b+2a}\\ -\frac{a}{2b+4a} & \frac{1}{b+2a}\end{bmatrix}$

4. $\boldsymbol{a})\ -109$ $\boldsymbol{b})\ -2x^3$ $\boldsymbol{c})\ -x^4+2x^3-x^2+2x$ $\boldsymbol{d})\ -4a+2c$ $\boldsymbol{e})\ 0$

5. $a)\ A^2=\begin{bmatrix}\frac{1}{16} & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & \frac{1}{4}\end{bmatrix}$ $A^{-2}=\begin{bmatrix}16 & 0 & 0\\ 0 & \frac{1}{9} & 0\\ 0 & 0 & 4\end{bmatrix}$ $A^{-k}=\begin{bmatrix}4^k & 0 & 0\\ 0 & 3^{-k} & 0\\ 0 & 0 & 2^k\end{bmatrix}$

$b)\ A^2=\begin{bmatrix}9 & 0 & 0 & 0\\ 0 & 36 & 0 & 0\\ 0 & 0 & 9 & 0\\ 0 & 0 & 0 & 4\end{bmatrix}$ $A^{-2}=\begin{bmatrix}\frac{1}{9} & 0 & 0 & 0\\ 0 & \frac{1}{36} & 0 & 0\\ 0 & 0 & \frac{1}{9} & 0\\ 0 & 0 & 0 & \frac{1}{4}\end{bmatrix}$ $A^{-k}=\begin{bmatrix}(-3)^{-k} & 0 & 0 & 0\\ 0 & (6)^{-k} & 0 & 0\\ 0 & 0 & (-3)^{-k} & 0\\ 0 & 0 & 0 & (-2)^{-k}\end{bmatrix}$

6. $(5, 6, -12)$

7. $(2, -7, -6)$, $\boldsymbol{u}\times\boldsymbol{v}$ is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$.

8. $a)$ 49 $b)$ -92

9. $a)\left(-4,\ 9,\ 6\right)$ $b)\ \left(32,\ -6,\ -4\right)$ $c)\ \left(-14,\ -20,\ -82\right)$

$d)\ \left(27,\ 40,\ -42\right)$ $e)\ \left(-44,\ 47,\ -22\right)$ $e)\ \sqrt{14}$

$g)$ $\left(\frac{3}{\sqrt{14}},\ \frac{2}{\sqrt{14}},\ -\frac{1}{\sqrt{14}}\right),\ \left(0,\ \frac{2}{\sqrt{13}},\ -\frac{3}{\sqrt{13}}\right),\ \left(\frac{2}{\sqrt{89}},\ \frac{6}{\sqrt{89}},\ \frac{7}{\sqrt{89}}\right)$

$h)$ $105.343°$ $i)$ 22.045 $j)$ 7 $k)$ 11

10. $a)$ *Yes* $b)$ *No* $c)$ *Yes*

11. $a)$ $\left(\frac{8}{13},\ -\frac{12}{13}\right)$ $\left(-\frac{21}{13},\ -\frac{14}{13}\right)$ $b)$ $(\cos\theta,\ 0)$ $(0,\ \sin\theta)$

$c)$ $\left(0,\ \frac{2}{5},\ \frac{-1}{5}\right)$ $\left(1,\ \frac{3}{5},\ \frac{6}{5}\right)$ $d)$ $\left(\frac{5}{14},\ \frac{5}{7},\ \frac{15}{14}\right)$ $\left(\frac{23}{14},\ -\frac{5}{7},\ -\frac{1}{14}\right)$

12. $\sqrt{114}$

13. $(2,\ -6,\ 12)$

14. $a)$ 7 $b)$ $\frac{\sqrt{374}}{2}$

15. 16

16. $\left(A^{-1}\right)^T\left(B^{-1}\right)^T$