

Formulas



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Quadratic equation

$$ax^2 + bx + c = 0$$

$$\text{If } a + b + c = 0 \Rightarrow x = 1, \frac{c}{a}$$

Proof

$$\begin{aligned}x_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\&= \frac{-(-a-c) \pm \sqrt{(-a-c)^2 - 4ac}}{2a} \\&= \frac{a+c \pm \sqrt{a^2 + 2ac + c^2 - 4ac}}{2a} \\&= \frac{a+c \pm \sqrt{a^2 - 2ac + c^2}}{2a} \\&= \frac{a+c \pm \sqrt{(a-c)^2}}{2a} \\&= \frac{a+c \pm (a-c)}{2a}\end{aligned}$$

$$\begin{aligned}\boxed{x_1} &= \frac{a+c+(a-c)}{2a} \\&= \frac{a+c+a-c}{2a} \\&= \frac{2a}{2a} \\&= \underline{1}\end{aligned}$$

$$a+b+c=0 \rightarrow b=-a-c$$

$$\begin{aligned}\boxed{x_2} &= \frac{a+c-(a-c)}{2a} \\&= \frac{a+c-a+c}{2a} \\&= \frac{2c}{2a} \\&= \underline{\frac{c}{a}}\end{aligned}$$

Example

$$2x^2 + x - 3 = 0$$

$$2+1-3=0$$

$$\Rightarrow \underline{x=1, -\frac{3}{2}}$$

Quadratic equation

$$ax^2 + bx + c = 0$$

$$\text{If } a - b + c = 0 \Rightarrow x = -1, -\frac{c}{a}$$

Proof

$$\begin{aligned}x_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\&= \frac{-(a+c) \pm \sqrt{(a+c)^2 - 4ac}}{2a} \\&= \frac{-a-c \pm \sqrt{a^2 + 2ac + c^2 - 4ac}}{2a} \\&= \frac{-a-c \pm \sqrt{a^2 - 2ac + c^2}}{2a} \\&= \frac{-a-c \pm \sqrt{(a-c)^2}}{2a} \\&= \frac{-a-c \pm (a-c)}{2a}\end{aligned}$$

$$a - b + c = 0 \rightarrow b = a + c$$

$$\begin{cases} \boxed{x_1 = \frac{-a-c+(a-c)}{2a} = \frac{-a-c+a-c}{2a} = \frac{2c}{2a} = -\frac{c}{a}} \\ \boxed{x_2 = \frac{-a-c-(a-c)}{2a} = \frac{-a-c-a+c}{2a} = \frac{-2a}{2a} = -1} \end{cases}$$

Example

$$2x^2 - x - 3 = 0$$

$$2 - (-1) - 3 = 0$$

$$\Rightarrow \boxed{x = -1, \frac{3}{2}}$$

Solving Exponential Function

$$a^{mx+n} = b^{px+q} \Rightarrow x = \frac{q \ln b - n \ln a}{m \ln a - p \ln b} \quad \text{coefficient } \frac{\text{no } x's}{x's}$$

Numerator: multiply q with $\ln b$ **minus** multiply n with $\ln a$

Denominator: multiply m with $\ln a$ **minus** multiply p with $\ln b$

Proof

$$\ln a^{mx+n} = \ln b^{px+q}$$

$$(mx+n) \ln a = (px+q) \ln b$$

$$mx \ln a + n \ln a = px \ln b + q \ln b$$

$$mx \ln a - px \ln b = q \ln b - n \ln a$$

$$x(m \ln a - p \ln b) = q \ln b - n \ln a$$

$$x = \frac{q \ln b - n \ln a}{m \ln a - p \ln b}$$

$$mx \ln a + n \ln a = px \ln b + q \ln b$$

Example

Solve: $3^{2x-1} = 7^{x+1}$

Solution

$$x = \frac{\ln 3 + \ln 7}{2 \ln 3 - \ln 7}$$

Example

Solve: $4^{x+3} = 3^{-x}$

Solution

$$x = \frac{-3 \ln 4}{\ln 4 + \ln 3}$$

$$\ln 4^{x+3} = \ln 3^{-x}$$

$$(x+3) \ln 4 = -x \ln 3$$

Example

Solve: $4^{-x} = 3^{x+3}$

Solution

$$x = \frac{3 \ln 3}{\ln 3 - \ln 4}$$

Growth & Decay Formula

$$A = A_0 e^{kt} \Rightarrow \underline{kT = \ln \frac{A}{A_0}}$$

Proof

$$A = A_0 e^{kt}$$

$$\frac{A}{A_0} = e^{kt}$$

$$\ln \frac{A}{A_0} = \ln e^{kt}$$

$$\boxed{\ln \frac{A}{A_0} = kt} \quad \checkmark$$



Inverse Functions

$$f(x) = \frac{ax+b}{cx+d} \Rightarrow f^{-1}(x) = \frac{-dx+b}{cx-a}$$

Proof

$$y = \frac{ax+b}{cx+d}$$

$$x = \frac{ay+b}{cy+d}$$

$$cxy + dx = ay + b$$

$$cxy - ay = -dx + b$$

$$(cx - a)y = -dx + b$$

$$y = \frac{-dx + b}{cx - a}$$

$f^{-1}(x) = \frac{-dx+b}{cx-a}$

 ✓

Interchange **a** and **d** and change there signs.

Example

Find the inverse function of: $f(x) = \frac{1}{3x-2}$

Solution

$$f^{-1}(x) = \frac{2x+1}{3x}$$

$$f(x) = \frac{0x+1}{3x-2}$$

Example

Find the inverse function of: $f(x) = \frac{3x+2}{2x-5}$

Solution

$$f^{-1}(x) = \frac{5x+2}{2x-3}$$

$$f(x) = \frac{3x+2}{2x-5}$$

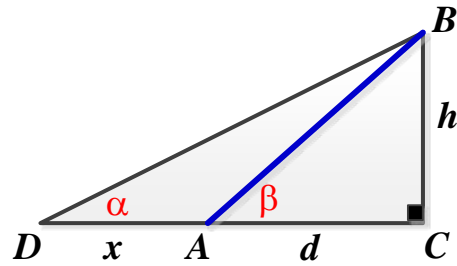
Example

Find the inverse function of: $f(x) = \frac{4x}{x+2}$

Solution

$$f^{-1}(x) = \frac{-2x}{x-4}$$

$$f(x) = \frac{4x}{x+2}$$



Proof

Triangle DCB : $\tan \alpha = \frac{h}{d+x} \Rightarrow h = (d+x) \tan \alpha$

Triangle ACB : $\tan \beta = \frac{h}{d} \Rightarrow h = d \tan \beta$

$$h = d \tan \beta = (d+x) \tan \alpha$$

$$d \tan \beta = d \tan \alpha + x \tan \alpha$$

$$d \tan \beta - d \tan \alpha = x \tan \alpha$$

$$d (\tan \beta - \tan \alpha) = x \tan \alpha$$

$$d = \frac{x \tan \alpha}{\tan \beta - \tan \alpha}$$

$$h = \frac{x \tan \alpha \tan \beta}{\tan \beta - \tan \alpha}$$

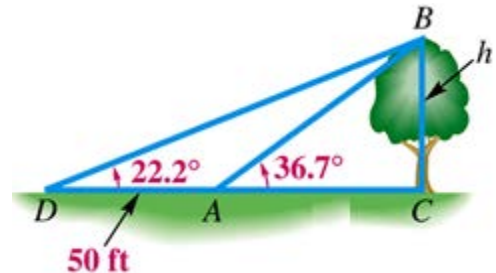
Height is equal to distance times (tan tan) divides by the (tan(larger angle) – tan) (difference between tangents)

Example

From a given point on the ground, the angle of elevation to the top of a tree is 36.7° . From a second point, 50 feet back, the angle of elevation to the top of the tree is 22.2° . Find the height of the tree to the nearest foot.

Solution

$$h = 50 \frac{\tan 22.2^\circ \tan 36.7^\circ}{\tan 36.7^\circ - \tan 22.2^\circ} \approx 45 \text{ ft}$$



Integration by Part

Evaluate $\int x^n e^{ax} dx$

| | | |
|---|----------------------|------------------------|
| | | $\int e^{ax}$ |
| + | x^n | $\frac{1}{a} e^{ax}$ |
| - | nx^{n-1} | $\frac{1}{a^2} e^{ax}$ |
| + | $n(n-1)x^{n-2}$ | $\frac{1}{a^3} e^{ax}$ |
| - | $n(n-1)(n-2)x^{n-3}$ | $\frac{1}{a^4} e^{ax}$ |
| | $\vdots \vdots$ | $\vdots \vdots$ |

$$\int x^n e^{ax} dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a^2} x^{n-1} e^{ax} + \frac{n(n-1)}{a^3} x^{n-2} e^{ax} - \frac{n(n-1)(n-2)}{a^4} x^{n-3} e^{ax} + \dots$$

$$= e^{ax} \sum_{k=0}^n (-1)^k \cdot \frac{n!}{(n-k)!} \cdot \frac{1}{a^{k+1}} \cdot x^{n-k}$$

Jose's Method

Evaluate $\int e^{ax} \cos bx \, dx$

$$\int e^{ax} \cos bx \, dx = \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx \, dx$$

$$\left(1 + \frac{a^2}{b^2}\right) \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{b^2} (b \sin bx + a \cos bx)$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (b \sin bx + a \cos bx) + C$$

| | | |
|---|--------------|-------------------------------------|
| | | $\int \cos bx \, dx$ |
| + | e^{ax} | $\frac{1}{b} \sin bx$ |
| - | ae^{ax} | $-\frac{1}{b^2} \cos bx$ |
| + | $a^2 e^{ax}$ | $-\frac{1}{b^2} \int \cos bx \, dx$ |

Proof

Find $\int e^{ax} \cos bx \, dx$

Solution

Let: $u = e^{ax} \quad dv = \cos bx \, dx$
 $du = ae^{ax} \, dx \quad v = \int \cos bx \, dx = \frac{1}{b} \sin bx$

$$\int e^{ax} \cos bx \, dx = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \int e^{ax} \sin bx \, dx \qquad \int u \, dv = uv - \int v \, du$$

Let: $u = e^{ax} \quad dv = \sin bx \, dx$
 $du = ae^{ax} \, dx \quad v = \int \sin bx \, dx = -\frac{1}{b} \cos bx$

$$\begin{aligned} \int e^{ax} \cos bx \, dx &= \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \left[-\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int e^{ax} \cos bx \, dx \right] \\ &= \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx \, dx \end{aligned}$$

$$\int e^{ax} \cos bx \, dx + \frac{a^2}{b^2} \int e^{ax} \cos bx \, dx = \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx + C_1$$

$$\frac{a^2 + b^2}{b^2} \int e^{ax} \cos bx \, dx = \frac{1}{b^2} e^{ax} (b \sin bx + a \cos bx) + C_1$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (b \sin bx + a \cos bx) + C$$

Length

Length of a curve $y = f(x)$ is given by the formula:

$$L = \int_c^d \sqrt{1 + [f'(x)]^2} \, dx = \int_c^d \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

If $f(x) = ax^m + bx^n$, then

$$L = \int_c^d \sqrt{1 + \left(\frac{df}{dx}\right)^2} \, dx = \left[ax^m - bx^n \right]_c^d$$

Iff $f(x)$ satisfies these 2 conditions:

1. $m + n = 2$
2. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = max^{m-1} + nbx^{n-1}$$

$$\begin{aligned} 1 + (f')^2 &= 1 + \left(max^{m-1} + nbx^{n-1} \right)^2 \\ &= 1 + m^2 a^2 x^{2m-2} + 2abmnx^{m+n-2} + n^2 b^2 x^{2n-2} \end{aligned}$$

We need to combined to a perfect square

$$a^2 - 2ab + b^2 = (a - b)^2$$

$$\rightarrow \text{ If } x^{m+n-2} = 1 = x^0 \rightarrow \boxed{m + n = 2}$$

$$= m^2 a^2 x^{2m-2} + (1 + 2abmn) + n^2 b^2 x^{2n-2} \quad a^2 - 2ab + b^2 = (a - b)^2$$

$$\rightarrow \text{ Let } 1 + 2abmn = -2abmn \rightarrow \boxed{abmn = -\frac{1}{4}}$$

$$\begin{aligned} &= m^2 a^2 x^{2m-2} - 2abmn + n^2 b^2 x^{2n-2} \quad x^{2(m+n-2)} = 1 \\ &= \left(max^{m-1} - nbx^{n-1} \right)^2 \end{aligned}$$

$$L = \int_c^d \sqrt{\left(max^{m-1} - nbx^{n-1} \right)^2} \, dx$$

$$= \int_c^d \left(max^{m-1} - nbx^{n-1} \right) \, dx$$

$$= \left[ax^m - bx^n \right]_c^d \quad \checkmark$$

Example

Find the length of the graph of $f(x) = \frac{x^3}{12} + \frac{1}{x}$, $1 \leq x \leq 4$

Solution

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

$$\begin{aligned} 1 + [f'(x)]^2 &= 1 + \left(\frac{x^2}{4} - \frac{1}{x^2} \right)^2 \\ &= 1 + \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4} \\ &= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} \\ &= \left(\frac{x^2}{4} + \frac{1}{x^2} \right)^2 \end{aligned}$$

$$\begin{aligned} L &= \int_1^4 \sqrt{1 + (f'(x))^2} \, dx \\ &= \int_1^4 \sqrt{\left(\frac{x^2}{4} + \frac{1}{x^2} \right)^2} \, dx \\ &= \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2} \right) \, dx \\ &= \left(\frac{x^3}{12} - \frac{1}{x} \right)_1^4 \\ &= \left(\frac{4^3}{12} - \frac{1}{4} \right) - \left(\frac{1}{12} - \frac{1}{1} \right) \\ &= \frac{72}{12} \\ &= \underline{6 \text{ unit}} \end{aligned}$$

$$a = \frac{1}{12}, \quad m = 3, \quad b = 1, \quad n = -1$$

$$1. \quad m + n = 3 - 1 = 2 \quad \checkmark$$

$$2. \quad abmn = \frac{1}{12}(1)(3)(-1) = -\frac{1}{4} \quad \checkmark$$

$$L = \left(\frac{x^3}{12} - \frac{1}{x} \right)_1^4 = \underline{6 \text{ unit}}$$

Examples

$$f(x) = \frac{1}{3}x^{3/2} - x^{1/2} \rightarrow L = \frac{1}{3}x^{3/2} + x^{1/2} + C$$

$$f(x) = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2} \rightarrow L = \frac{2}{3}x^{3/2} + \frac{1}{2}x^{1/2}$$

$$f(x) = \frac{1}{6}x^3 + \frac{1}{2x} \rightarrow L = \frac{1}{6}x^3 - \frac{1}{2x} + C$$

$$f(x) = x^3 + \frac{1}{12x} \rightarrow L = x^3 - \frac{1}{12x} + C$$

$$f(x) = \frac{1}{6}x^3 + \frac{1}{2x} \rightarrow L = \frac{1}{6}x^3 - \frac{1}{2x}$$

$$f(x) = \frac{1}{8}x^4 + \frac{1}{4x^2} \rightarrow L = \frac{1}{8}x^4 - \frac{1}{4x^2}$$

$$f(x) = \frac{1}{4}x^4 + \frac{1}{8x^2} \rightarrow L = \frac{1}{4}x^4 - \frac{1}{8x^2}$$

$$f(x) = \frac{1}{10}x^5 + \frac{1}{6x^3} \rightarrow L = \frac{1}{10}x^5 - \frac{1}{6x^3}$$

$$f(x) = \frac{3}{10}x^{1/3} - \frac{3}{2}x^{5/3} \rightarrow L = \frac{3}{10}x^{1/3} + \frac{3}{2}x^{5/3}$$

$$f(x) = x^{1/2} - \frac{1}{3}x^{3/2} \rightarrow L = x^{1/2} + \frac{1}{3}x^{3/2}$$

If $f(x) = ae^{mx} + be^{nx}$, then

$$L = \int_c^d \sqrt{1 + \left(\frac{df}{dx}\right)^2} dx = \left[ae^{mx} - be^{nx} \right]_c^d$$

Iff $f(x)$ satisfies these 2 conditions:

1. $m = -n$
2. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = ame^{mx} + bne^{nx}$$

$$1 + (f')^2 = 1 + (ame^{mx} + bne^{nx})^2$$

$$= 1 + m^2 a^2 e^{2mx} + 2abmne^{(m+n)x} + n^2 b^2 e^{2nx}$$

$$\rightarrow \text{If } e^{(m+n)x} = 1 = e^{(x=0)} \rightarrow \boxed{m+n}$$

$$= m^2 a^2 x^{2m-2} + (1 + 2abmn) + n^2 b^2 x^{2n-2}$$

$$a^2 - 2ab + b^2 = (a-b)^2$$

$$\rightarrow \text{Let } 1 + 2abmn = -2abmn \rightarrow \boxed{abmn = -\frac{1}{4}}$$

$$= m^2 a^2 e^{2mx} - 2abmne^{(m+n)x} + n^2 b^2 e^{2nx}$$

$$x^{2(m+n-2)} = 1$$

$$= (ame^{mx} - bne^{nx})^2$$

$$(ame^{mx} - bne^{nx})^2 = m^2 a^2 e^{2mx} - 2abmne^{(m+n)x} + n^2 b^2 e^{2nx}$$

Example

$$f(x) = 2e^x + \frac{1}{8}e^{-x} \rightarrow L = 2e^x - \frac{1}{8}e^{-x}$$

$$f(x) = 2e^{\sqrt{2}x} + \frac{1}{16}e^{-\sqrt{2}x} \rightarrow L = 2e^{\sqrt{2}x} - \frac{1}{16}e^{-\sqrt{2}x}$$

$$y = ax^n + \frac{b}{x}$$

$$y' = nax^{n-1} - \frac{b}{x^2}$$

Surface

Surface of a curve $y = f(x)$ is given by the formula:

$$S = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx$$

If $f(x) = ax^m + bx^n$, then

$$\sqrt{1 + (f'(x))^2} = \overline{f'(x)}$$

$\overline{f'(x)}$: is the conjugate of $f'(x)$

Iff $f(x)$ satisfies these 2 conditions:

1. $m + n = 2$
2. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = max^{m-1} + nbx^{n-1}$$

$$\begin{aligned} 1 + (f')^2 &= 1 + (max^{m-1} + nbx^{n-1})^2 \\ &= 1 + m^2 a^2 x^{2m-2} + 2abmnx^{m+n-2} + n^2 b^2 x^{2n-2} \end{aligned}$$

We need to combined to a perfect square

$$a^2 - 2ab + b^2 = (a - b)^2$$

$$\rightarrow \text{ If } x^{m+n-2} = 1 = x^0 \rightarrow \boxed{m + n = 2}$$

$$= m^2 a^2 x^{2m-2} + (1 + 2abmn) + n^2 b^2 x^{2n-2} \quad a^2 - 2ab + b^2 = (a - b)^2$$

$$\rightarrow \text{ Let } 1 + 2abmn = -2abmn \rightarrow \boxed{abmn = -\frac{1}{4}}$$

$$= m^2 a^2 x^{2m-2} - 2abmn + n^2 b^2 x^{2n-2} \quad x^{2(m+n-2)} = 1$$

$$= (max^{m-1} - nbx^{n-1})^2$$

$$\sqrt{(max^{m-1} - nbx^{n-1})^2} = max^{m-1} - nbx^{n-1} \quad \checkmark$$

$$f'(x) = max^{m-1} + nbx^{n-1} \Rightarrow \sqrt{1 + (f'(x))^2} = max^{m-1} - nbx^{n-1} = \overline{f'(x)}$$

Example

Find the surface of the graph of $f(x) = \frac{x^3}{12} + \frac{1}{x}$, $1 \leq x \leq 4$

Solution

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

$$\begin{aligned} 1 + [f'(x)]^2 &= 1 + \left(\frac{x^2}{4} - \frac{1}{x^2} \right)^2 \\ &= 1 + \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4} \\ &= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} \\ &= \left(\frac{x^2}{4} + \frac{1}{x^2} \right)^2 \end{aligned}$$

$$\begin{aligned} S &= 2\pi \int_1^4 f(x) \sqrt{1 + (f'(x))^2} \, dx \\ &= 2\pi \int_1^4 \left(\frac{x^3}{12} + \frac{1}{x} \right) \sqrt{\left(\frac{x^2}{4} + \frac{1}{x^2} \right)^2} \, dx \\ &= 2\pi \int_1^4 \left(\frac{x^3}{12} + \frac{1}{x} \right) \left(\frac{x^2}{4} + \frac{1}{x^2} \right) \, dx \\ &= 2\pi \int_1^4 \left(\frac{1}{48} x^5 + \frac{1}{12} x + \frac{1}{4} x + x^{-3} \right) \, dx \\ &= 2\pi \left(\frac{1}{288} x^6 + \frac{1}{6} x^2 - \frac{1}{2x^2} \right) \Big|_1^4 \\ &= \pi \left(\frac{256}{9} + \frac{16}{3} - \frac{1}{16} - \frac{1}{144} - \frac{1}{3} + 1 \right) \\ &= \frac{275}{8} \pi \text{ unit}^2 \end{aligned}$$

$$a = \frac{1}{12}, \quad m = 3, \quad b = 1, \quad n = -1$$

1. $m + n = 3 - 1 = 2$ ✓

2. $abmn = \frac{1}{12}(1)(3)(-1) = -\frac{1}{4}$ ✓

$$S = 2\pi \int_1^4 \left(\frac{x^3}{12} + \frac{1}{x} \right) \left(\frac{x^2}{4} + \frac{1}{x^2} \right) \, dx$$

If $f(x) = ae^{mx} + be^{nx}$, then

$$S = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx = 2\pi \int_a^b f(x) \overline{f'(x)} dx$$

Iff $f(x)$ satisfies these 2 conditions:

1. $m = -n$
2. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = ame^{mx} + bne^{nx}$$

$$1 + (f')^2 = 1 + (ame^{mx} + bne^{nx})^2$$

$$= 1 + m^2 a^2 e^{2mx} + 2abmne^{(m+n)x} + n^2 b^2 e^{2nx}$$

$$\rightarrow \text{If } e^{(m+n)x} = 1 = e^{(x=0)} \rightarrow \boxed{m = -n}$$

$$= m^2 a^2 x^{2m-2} + (1 + 2abmn) + n^2 b^2 x^{2n-2}$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

$$\rightarrow \text{Let } 1 + 2abmn = -2abmn \rightarrow \boxed{abmn = -\frac{1}{4}}$$

$$= m^2 a^2 e^{2mx} - 2abmne^{(m+n)x} + n^2 b^2 e^{2nx}$$

$$x^{2(m+n-2)} = 1$$

$$= (ame^{mx} - bne^{nx})^2$$

$$\sqrt{1 + (f')^2} = \sqrt{(ame^{mx} - bne^{nx})^2}$$

$$\sqrt{1 + f'(x)} = \overline{f'(x)} \quad \checkmark$$

Derivative

Formula $\left(U^m V^n W^p\right)' = U^{m-1} V^{n-1} W^{p-1} (mU'VW + nUV'W + pUVW')$

Proof

$$\begin{aligned}\left(U^m V^n W^p\right)' &= \left(U^m\right)' V^n W^p + U^m \left(V^n\right)' W^p + U^m V^n \left(W^p\right)' \\ &= mU^{m-1} U' V^n W^p + nU^m V^{n-1} V' W^p + pU^m V^n W^{p-1} W' \quad \text{factor } U^{m-1} V^{n-1} W^{p-1} \\ &= U^{m-1} V^{n-1} W^{p-1} (mU'VW + nUV'W + pUVW')\end{aligned}$$

Derivative: Rational Function to Power '***n***' in the form $\frac{ax^n + b}{cx^n + d}$

$$\left(\frac{ax^n + b}{cx^n + d} \right)' = \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2} = \frac{n \begin{vmatrix} a & b \\ c & d \end{vmatrix} x^{n-1}}{(cx^n + d)^2}$$

Proof

$$u = ax^n + b \quad v = cx^n + d$$

$$u' = nax^{n-1} \quad v' = ncx^{n-1}$$

$$\begin{aligned} \left(\frac{ax^n + b}{cx^n + d} \right)' &= \frac{nax^{n-1}(cx^n + d) - ncx^{n-1}(ax^n + b)}{(cx^n + d)^2} & \left(\frac{u}{v} \right)' &= \frac{u'v - v'u}{v^2} \\ &= \frac{nacx^{2n-1} + nadx^{n-1} - nacx^{2n-1} - nbcx^{n-1}}{(cx^n + d)^2} \\ &= \frac{nadx^{n-1} - nbcx^{n-1}}{(cx^n + d)^2} \\ &= \frac{n(ad - bc)x^{n-1}}{(cx^n + d)^2} \end{aligned}$$

Example

Find $\left(\frac{x+2}{3x-2}\right)'$

Solution

$$\begin{aligned}\left(\frac{x+2}{3x-2}\right)' &= \frac{-2-6}{(3x-2)^2} \\ &= \frac{-8}{(3x-2)^2}\end{aligned}$$

$$\begin{aligned}\left(\frac{x+2}{3x-2}\right)' &= \frac{3x-2-3(x+2)}{(3x-2)^2} \\ &= \frac{3x-2-3x-6}{(3x-2)^2} \\ &= \frac{-8}{(3x-2)^2}\end{aligned}$$

Example

Find $\left(\frac{5x^2-3}{2x^2-4}\right)'$

Solution

$$\begin{aligned}\left(\frac{5x^2-3}{2x^2-4}\right)' &= \frac{2(-20+6)x}{(2x^2-4)^2} \\ &= \frac{-28x}{(2x^2-4)^2}\end{aligned}$$

$$\begin{aligned}\left(\frac{5x^2-3}{2x^2-4}\right)' &= \frac{10x(2x^2-4)-4x(5x^2-3)}{(3x-2)^2} \\ &= \frac{20x^3-40x-20x^3+12x}{(3x-2)^2} \\ &= \frac{-28x}{(2x^2-4)^2}\end{aligned}$$

Derivative: Rational Function to Power ' n ' in the form $\left(\frac{ax^n+b}{cx^n+d}\right)^m$

$$\frac{d}{dx}\left(\frac{ax^n+b}{cx^n+d}\right)^m = mn(ad-bc)x^{n-1} \frac{(ax^n+b)^{m-1}}{(cx^n+d)^{m+1}}$$

Proof

$$u = ax^n + b \quad v = cx^n + d$$

$$u' = nax^{n-1} \quad v' = ncx^{n-1}$$

$$\begin{aligned} \frac{d}{dx}\left(\frac{ax^n+b}{cx^n+d}\right)^m &= m \frac{nax^{n-1}(cx^n+d) - ncx^{n-1}(ax^n+b)}{(cx^n+d)^2} \left(\frac{ax^n+b}{cx^n+d}\right)^{m-1} & \left(\frac{u}{v}\right)' &= \frac{u'v - v'u}{v^2} \\ &= \frac{m(nacx^{2n-1} + nadx^{n-1} - nacx^{2n-1} - nbcx^{n-1})(ax^n+b)^{m-1}}{(cx^n+d)^2 (cx^n+d)^{m-1}} \\ &= \frac{m(nadx^{n-1} - nbcx^{n-1})(ax^n+b)^{m-1}}{(cx^n+d)^{m+1}} \\ &= \frac{mn(ad-bc)x^{n-1}(ax^n+b)^{m-1}}{(cx^n+d)^{m+1}} \end{aligned}$$

Example

Find $\frac{d}{dx} \left(\frac{x+2}{3x-2} \right)^4$

Solution

$$\begin{aligned} \frac{d}{dx} \left(\frac{x+2}{3x-2} \right)^4 &= (1)(4)(-2-6) \frac{(x+2)^3}{(3x-2)^5} \\ &= -\frac{32(x+2)^3}{(3x-2)^5} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx} \left(\frac{x+2}{3x-2} \right)^4 &= 4 \left(\frac{x+2}{3x-2} \right)^3 \frac{d}{dx} \left(\frac{x+2}{3x-2} \right) \\ &= 4 \frac{(x+2)^3}{(3x-2)^3} \frac{3x-2-3(x+2)}{(3x-2)^2} \\ &= 4 \frac{(x+2)^3}{(3x-2)^3} \frac{3x-2-3x-6}{(3x-2)^2} \\ &= 4(-8) \frac{(x+2)^3}{(3x-2)^5} \\ &= -\frac{32(x+2)^3}{(3x-2)^5} \end{aligned}$$

Example

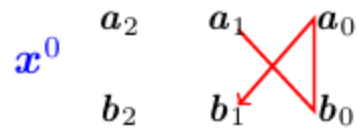
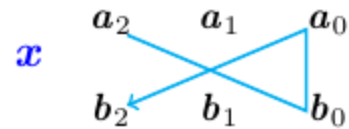
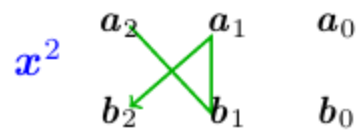
Find $\frac{d}{dx} \left(\frac{5x^2-3}{2x^2-4} \right)^5$

Solution

$$\frac{d}{dx} \left(\frac{5x^2-3}{2x^2-4} \right)^5 = \frac{-140x(5x^2-3)^4}{(2x^2-4)^6}$$

Derivative: in the form $y = \frac{ax^2 + bx + c}{dx^2 + ex + f}$

$$\begin{aligned} \frac{d}{dx} \left(\frac{ax^2 + bx + c}{dx^2 + ex + f} \right) &= \frac{(2ax + b)(dx^2 + ex + f) - (2dx + e)(ax^2 + bx + c)}{(dx^2 + ex + f)^2} \\ &= \frac{2adx^3 + 2aex^2 + 2afx + bdx^2 + bex + bf - 2adx^3 - 2bdx^2 - 2cdx - aex^2 - bex - ce}{(dx^2 + ex + f)^2} \\ &= \frac{(ae - bd)x^2 + 2(af - cd)x + bf - ce}{(dx^2 + ex + f)^2} \\ &= \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2 \begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{(dx^2 + ex + f)^2} \end{aligned}$$



Example

$$f(x) = \frac{x^2 - 6x + 8}{x^2 - 2x + 1}$$

Solution

$$f'(x) = \frac{\begin{vmatrix} 1 & -6 \\ 1 & -2 \end{vmatrix} x^2 + 2 \begin{vmatrix} 1 & 8 \\ 1 & 1 \end{vmatrix} x + \begin{vmatrix} -6 & 8 \\ -2 & 1 \end{vmatrix}}{(x^2 - 2x + 1)^2}$$

$$= \frac{4x^2 - 14x + 10}{(x^2 - 2x + 1)^2}$$

$$f'(x) = \frac{(2x - 6)(x^2 - 2x + 1) - (2x - 2)(x^2 - 6x + 8)}{(x^2 - 2x + 1)^2}$$

$$= \frac{2x^3 - 4x^2 + 2x - 6x^2 + 12x - 6 - (2x^3 - 12x^2 + 16x - 2x^2 + 12x - 16)}{(x^2 - 2x + 1)^2}$$

$$= \frac{2x^3 - 4x^2 + 2x - 6x^2 + 12x - 6 - 2x^3 + 12x^2 - 16x + 2x^2 - 12x + 16}{(x^2 - 2x + 1)^2}$$

$$= \frac{4x^2 - 14x + 10}{(x^2 - 2x + 1)^2}$$

Example

$$f(x) = \frac{x + 4}{x^2 + x + 1}$$

$$\begin{matrix} 0 & 1 & 4 \\ 1 & 1 & 1 \end{matrix}$$

Solution

$$f'(x) = \frac{\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} x^2 + 2 \begin{vmatrix} 0 & 4 \\ 1 & 1 \end{vmatrix} x + \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix}}{(x^2 + x + 1)^2}$$

$$= \frac{-x^2 - 8x - 3}{(x^2 + x + 1)^2}$$

$$f'(x) = \frac{x^2 + x + 1 - (x + 4)(2x + 1)}{(x^2 + x + 1)^2}$$

$$= \frac{x^2 + x + 1 - 2x^2 - 9x - 4}{(x^2 + x + 1)^2}$$

$$= \frac{-x^2 - 8x - 3}{(x^2 + x + 1)^2}$$

Derivative: in the form $f(x) = \frac{a_3 x^3 + a_2 x^2 + a_1 x + a_0}{b_3 x^3 + b_2 x^2 + b_1 x + b_0}$

$$u = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \rightarrow u' = 3a_3 x^2 + 2a_2 x + a_1$$

$$v = b_3 x^3 + b_2 x^2 + b_1 x + b_0 \rightarrow v' = 3b_3 x^2 + 2b_2 x + b_1$$

$$u'v - v'u = (3a_3 x^2 + 2a_2 x + a_1)(b_3 x^3 + b_2 x^2 + b_1 x + b_0) - (3b_3 x^2 + 2b_2 x + b_1)(a_3 x^3 + a_2 x^2 + a_1 x + a_0)$$

| x^5 | x^4 | x^3 | x^2 | x^1 | x^0 |
|-------------|-------------|-------------|-------------|-------------|------------|
| $3a_3 b_3$ | $3a_3 b_2$ | $3a_3 b_1$ | $3a_3 b_0$ | | |
| $-3a_3 b_3$ | $2a_2 b_3$ | $2a_2 b_2$ | $2a_2 b_1$ | $2a_2 b_0$ | |
| | $-3a_2 b_3$ | $a_1 b_3$ | $a_1 b_2$ | $a_1 b_1$ | $a_1 b_0$ |
| | $-2a_3 b_2$ | $-3a_1 b_3$ | $-3a_0 b_3$ | | |
| | | $-2a_2 b_2$ | $-2a_1 b_2$ | $-2a_0 b_2$ | |
| | | $-a_3 b_1$ | $-a_2 b_1$ | $-a_1 b_1$ | $-a_0 b_1$ |

$$f'(x) = \frac{(a_3 b_2 - a_2 b_3)x^4 + 2(a_3 b_1 - a_1 b_3)x^3 + ((a_2 b_1 - a_1 b_2) + 3(a_3 b_0 - a_0 b_3))x^2 + 2(a_2 b_0 - a_0 b_2)x + a_1 b_0 - a_0 b_1}{(b_3 x^3 + b_2 x^2 + b_1 x + b_0)^2}$$

$$= \frac{\begin{vmatrix} a_3 & a_2 \\ b_3 & b_2 \end{vmatrix} x^4 + 2 \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} x^3 + \left(\begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix} + 3 \begin{vmatrix} a_3 & a_0 \\ b_3 & b_0 \end{vmatrix} \right) x^2 + 2 \begin{vmatrix} a_2 & a_0 \\ b_2 & b_0 \end{vmatrix} x + \begin{vmatrix} a_1 & a_0 \\ b_1 & b_0 \end{vmatrix}}{(b_3 x^3 + b_2 x^2 + b_1 x + b_0)^2}$$

Example

$$f(x) = \frac{x^3 + 2x^2 - 6x + 2}{2x^3 + x^2 - 2x + 1}$$

$$\begin{array}{cccc} 1 & 2 & -6 & 2 \\ 2 & 1 & -2 & 1 \end{array}$$

Solution

$$\begin{aligned} f'(x) &= \frac{(1-4)x^4 + 2(10)x^3 + ((-4+6) + 3(1-4))x^2 + 2(2-2)x + (-6+4)}{(2x^3 + x^2 - 2x + 1)^2} \\ &= \frac{-3x^4 + 20x^3 - 7x^2 - 2}{(2x^3 + x^2 - 2x + 1)^2} \end{aligned}$$

$$\begin{array}{cccc} \mathbf{x^4} & a_3 & a_2 & a_1 & a_0 \\ & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$\begin{array}{cccc} \mathbf{x^3} & a_3 & a_2 & a_1 & a_0 \\ & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$\begin{array}{cccc} \mathbf{x^2} & a_3 & a_2 & a_1 & a_0 \\ & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$\begin{array}{cccc} \mathbf{x} & a_3 & a_2 & a_1 & a_0 \\ & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$\begin{array}{cccc} \mathbf{x^0} & a_3 & a_2 & a_1 & a_0 \\ & b_3 & b_2 & b_1 & b_0 \end{array}$$

Derivative: in the form $f(x) = \frac{a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0}{b_4 x^4 + b_3 x^3 + b_2 x^2 + b_1 x + b_0}$

$$u'v - v'u = \left(4a_4 x^3 + 3a_3 x^2 + 2a_2 x + a_1\right) \left(b_4 x^4 + b_3 x^3 + b_2 x^2 + b_1 x + b_0\right) - \left(4b_4 x^3 + 3b_3 x^2 + 2b_2 x + b_1\right) \left(a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0\right)$$

$$x^7 \quad 4a_4 b_4 - 4a_4 b_4$$

$$x^6 \quad 4a_4 b_3 + 3a_3 b_4 - 4a_3 b_4 - 3a_4 b_3$$

$$x^5 \quad 4a_4 b_2 + 3a_3 b_3 + 2a_2 b_4 - 4a_2 b_4 - 3a_3 b_3 - 2a_4 b_2$$

$$x^4 \quad 4a_4 b_1 + 3a_3 b_2 + 2a_2 b_3 + a_1 b_4 - 4a_1 b_4 - 3a_2 b_3 - 2a_3 b_2 - a_4 b_1$$

$$x^3 \quad 4a_4 b_0 + 3a_3 b_1 + 2a_2 b_2 + a_1 b_3 - 4a_0 b_4 - 3a_1 b_3 - 2a_2 b_2 - a_3 b_1$$

$$x^2 \quad 3a_3 b_0 + 2a_2 b_1 + a_1 b_2 - 3a_0 b_3 - 2a_1 b_2 - a_2 b_1$$

$$x^1 \quad 2a_2 b_0 + a_1 b_1 - 2a_0 b_2 - a_1 b_1$$

$$x^0 \quad a_1 b_0 - a_0 b_1$$

$$\left(a_4 b_3 - a_3 b_4\right) x^6 + 2\left(a_4 b_2 - a_2 b_4\right) x^5 + \left(3\left(a_4 b_1 - a_1 b_4\right) + \left(a_3 b_2 - a_2 b_3\right)\right) x^4 + \left(4\left(a_4 b_0 - a_0 b_4\right) + 2\left(a_3 b_1 - a_1 b_3\right)\right) x^3$$

$$f'(x) = \frac{+ \left(\left(a_2 b_1 - a_1 b_2\right) + 3\left(a_3 b_0 - a_0 b_3\right)\right) x^2 + 2\left(a_2 b_0 - a_0 b_2\right) x + a_1 b_0 - a_0 b_1}{\left(b_4 x^4 + b_3 x^3 + b_2 x^2 + b_1 x + b_0\right)^2}$$

$$x^6 \quad \begin{array}{ccccc} a_4 & a_3 & a_2 & a_1 & a_0 \\ b_4 & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$x^5 \quad \begin{array}{ccccc} a_4 & a_3 & a_2 & a_1 & a_0 \\ b_4 & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$x^4 \quad \begin{array}{ccccc} a_4 & a_3 & a_2 & a_1 & a_0 \\ b_4 & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$x^3 \quad \begin{array}{ccccc} a_4 & a_3 & a_2 & a_1 & a_0 \\ b_4 & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$x^2 \quad \begin{array}{ccccc} a_4 & a_3 & a_2 & a_1 & a_0 \\ b_4 & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$x \quad \begin{array}{ccccc} a_4 & a_3 & a_2 & a_1 & a_0 \\ b_4 & b_3 & b_2 & b_1 & b_0 \end{array}$$

$$x^0 \quad \begin{array}{ccccc} a_4 & a_3 & a_2 & a_1 & a_0 \\ b_4 & b_3 & b_2 & b_1 & b_0 \end{array}$$

Derivative: in the form $f(x) = \frac{a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0}{b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0}$

$$u = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \rightarrow u' = 5a_5x^4 + 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$$

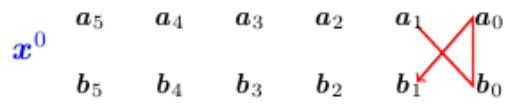
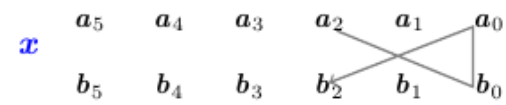
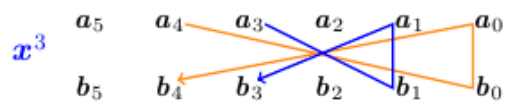
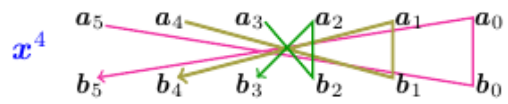
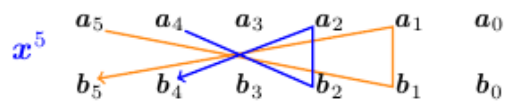
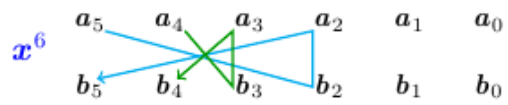
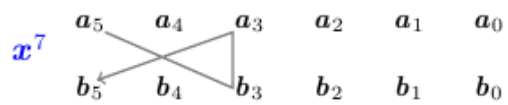
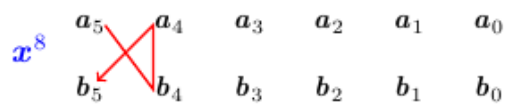
$$v = b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0 \rightarrow v' = 5b_5x^4 + 4b_4x^3 + 3b_3x^2 + 2b_2x + b_1$$

$$u'v - v'u = (5a_5x^4 + 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1)(b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0) - (5b_5x^4 + 4b_4x^3 + 3b_3x^2 + 2b_2x + b_1)(a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0)$$

| x^9 | x^8 | x^7 | x^6 | x^5 | x^4 | x^3 | x^2 | x^1 | x^0 |
|------------|------------|------------|------------|------------|------------|------------|------------|------------|-----------|
| $5a_5b_5$ | $5a_5b_4$ | $5a_5b_3$ | $5a_5b_2$ | $5a_5b_1$ | $5a_5b_0$ | | | | |
| $-5a_5b_5$ | $4a_4b_5$ | $4a_4b_4$ | $4a_4b_3$ | $4a_4b_2$ | $4a_4b_1$ | $4a_4b_0$ | | | |
| | $-5a_4b_5$ | $3a_3b_5$ | $3a_3b_4$ | $3a_3b_3$ | $3a_3b_2$ | $3a_3b_1$ | $3a_3b_0$ | | |
| | $-4a_5b_4$ | $-5a_3b_5$ | $2a_2b_5$ | $2a_2b_4$ | $2a_2b_3$ | $2a_2b_2$ | $2a_2b_1$ | $2a_2b_0$ | |
| | | $-4a_4b_4$ | $-5a_2b_5$ | a_1b_5 | a_1b_4 | a_1b_3 | a_1b_2 | a_1b_1 | a_1b_0 |
| | | $-3a_5b_3$ | $-4a_3b_4$ | $-5a_1b_5$ | $-5a_0b_5$ | $-4a_0b_4$ | $-3a_0b_3$ | $-2a_0b_2$ | $-a_0b_1$ |
| | | | $-3a_4b_3$ | $-4a_2b_4$ | $-4a_1b_4$ | $-3a_1b_3$ | $-2a_1b_2$ | $-a_1b_1$ | |
| | | | $-2a_5b_2$ | $-3a_3b_3$ | $-3a_2b_3$ | $-2a_2b_2$ | $-a_2b_1$ | | |
| | | | | $-2a_4b_2$ | $-2a_3b_2$ | $-a_3b_1$ | | | |
| | | | | $-a_5b_1$ | $-a_4b_1$ | | | | |

$$\begin{aligned}
& (a_5b_4 - a_4b_5)x^8 + 2(a_5b_3 - a_3b_5)x^7 \\
& + (3(a_5b_2 - a_2b_5) + (a_4b_3 - a_3b_4))x^6 \\
& + (4(a_5b_1 - a_1b_5) + 2(a_4b_2 - a_2b_4))x^5 \\
& + (5(a_5b_0 - a_0b_5) + 3(a_4b_1 - a_1b_4) + (a_3b_2 - a_2b_3))x^4 \\
& + (4(a_4b_0 - a_0b_4) + 2(a_3b_1 - a_1b_3))x^3 \\
& + (3(a_3b_0 - a_0b_3) + (a_2b_1 - a_1b_2))x^2 \\
& + 2(a_2b_0 - a_0b_2)x + (a_1b_0 - a_0b_1)
\end{aligned}$$

$$f'(x) = \frac{\text{above expression}}{(b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0)^2}$$



$\sqrt{3} \approx 1.732050807568877293$

| | | | |
|--|--|---|---|
| <div><div><div>1</div><div>2</div><div>3</div></div><div><div>1</div><div>2</div><div>3</div><div>4</div></div><div><div>1</div><div>2</div><div>3</div><div>4</div></div></div> | <div><div>1.732050807568877293</div><div><div>1</div><div>2</div><div>3</div><div>4</div></div></div> | <div><div>1 × 1 = 1</div></div> | <div><div>1 → 1 × 2 = 2</div></div> |
| <div><div><div>1</div><div>200</div></div></div> | <div><div><div>27</div><div>189</div></div><div><div>7</div><div>1100</div></div></div> | <div><div>27 × 7 = 189</div></div> | <div><div>2 → 17 × 2 = 34</div></div> |
| <div><div><div>189</div><div>1100</div></div></div> | <div><div><div>343</div><div>1029</div></div><div><div>3</div><div>7100</div></div></div> | <div><div>343 × 3 = 1029</div></div> | <div><div>3 → 173 × 2 = 346</div></div> |
| <div><div><div>1029</div><div>7100</div></div></div> | <div><div><div>3462</div><div>6924</div></div><div><div>2</div><div>1700</div></div></div> | <div><div>3462 × 2 = 6924</div></div> | <div><div>4 → 1732 × 2 = 3464</div></div> |
| <div><div><div>6924</div><div>17600</div><div>1560000</div></div></div> | <div><div><div>3464</div><div>34,64</div><div>>15,600</div></div><div><div>346405</div><div>×5=1,732,025</div></div></div> | <div><div>3464 × = 34,64 >15,600</div><div>346405×5=1,732,025</div></div> | <div><div>17320×2=34640</div><div>173205×2=346410</div></div> |
| <div><div><div>1732025</div><div>2797500</div><div>279750000</div></div></div> | <div><div><div>346410</div><div>× =?</div><div>346410 >2797500</div></div><div><div>34641008</div><div>×8=277128064</div></div></div> | <div><div>346410 × =? 346410 >2797500</div><div>34641008×8=277128064</div></div> | <div><div>1732050×2=3464100</div><div>17320508×2=34641016</div></div> |
| <div><div><div>277128064</div><div>262193600</div><div>26219360000</div></div></div> | <div><div><div>34641016</div><div>× =?</div><div>34641016 >262193600</div></div><div><div>346410167</div><div>×7=2424871169</div></div></div> | <div><div>34641016 × =? 34641016 >262193600</div><div>346410167×7=2424871169</div></div> | <div><div>173205080 × 2 = 346410160</div><div>1732050807 × 2 = 3464101614</div></div> |
| <div><div><div>2424871169</div><div>197064875100</div></div></div> | <div><div><div>3464101614</div><div>5×5=173205080725</div></div></div> | <div><div>34641016145×5=173205080725</div></div> | <div><div>17320508075×2=34641016150</div></div> |
| <div><div><div>173205080725</div><div>2385979437500</div></div></div> | <div><div><div>346410161506</div><div>×6=2078460969036</div></div></div> | <div><div>346410161506×6=2078460969036</div></div> | <div><div>173205080756×2=346410161512</div></div> |
| <div><div><div>2078460969036</div><div>30751846846400</div></div></div> | <div><div><div>3464101615128</div><div>×8=2771281291024</div></div></div> | <div><div>3464101615128×8=2771281291024</div></div> | <div><div>1732050807568×2=3464101615136</div></div> |
| <div><div><div>27712812921024</div><div>303903392537600</div></div></div> | <div><div><div>34641016151368</div><div>×8=27712812910944</div></div></div> | <div><div>34641016151368×8=27712812910944</div></div> | <div><div>17320508075688×2=34641016151376</div></div> |
| <div><div><div>277128129210944</div><div>2677526332665600</div></div></div> | <div><div><div>346410161513767</div><div>×7=2424871130596369</div></div></div> | <div><div>346410161513767×7=2424871130596369</div></div> | <div><div>173205080756887×2=346410161513774</div></div> |
| <div><div><div>2424871130596369</div><div>25265520206923100</div></div></div> | <div><div><div>3464101615137747</div><div>×7=24248711305964229</div></div></div> | <div><div>3464101615137747×7=24248711305964229</div></div> | <div><div>1732050807568877×2=3464101615137754</div></div> |
| <div><div><div>24248711305964229</div><div>101680890095887100</div></div></div> | <div><div><div>34641016151377542</div><div>×2=69282032302755084</div></div></div> | <div><div>34641016151377542×2=69282032302755084</div></div> | <div><div>17320508075688772×2=34641016151377544</div></div> |
| <div><div><div>69282032302755084</div><div>3239885779313201600</div></div></div> | <div><div><div>346410161513775449</div><div>×9=3117691453623979041</div></div></div> | <div><div>346410161513775449×9=3117691453623979041</div></div> | <div><div>173205080756887729×2=346410161513775459</div></div> |
| <div><div><div>3117691453623979041</div><div>12219432568922255900</div></div></div> | <div><div><div>3464101615137754593</div><div>×3=1039230845413263263779</div></div></div> | <div><div>3464101615137754593×3=1039230845413263263779</div></div> | |
| <div><div><div>1039230845413263263779</div><div>1827127723508992121</div></div></div> | | | |