Section 3.2 – Infinite Series

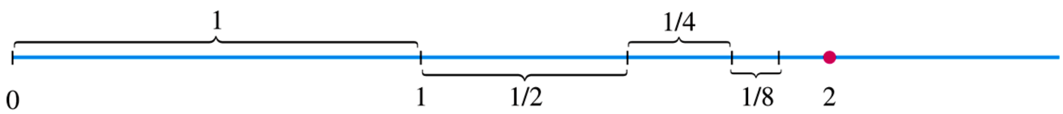
An ***infinite series*** is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

The sum of the first *n*th terms

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n$$

Partial Sum		***Value***	***Suggestive Expression For Partial Sum***
First:	$s_1 = 1$	1	$2 - 1$
Second	$s_2 = 1 + \frac{1}{2}$	$\frac{3}{2}$	$2 - \frac{1}{2}$
Third:	$s_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$\frac{7}{4}$	$2 - \frac{1}{4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n^{th}	$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$	$\frac{2^n - 1}{2^{n-1}}$	$2 - \frac{1}{2^{n-1}}$



Definition

Given a sequence of numbers $\{a_n\}$, an expression of the form $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is an ***infinite series***. The number a_n is the **nth term** of the series. The sequence $\{s_n\}$ is defined by

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$\vdots$$

$$s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k$$

$$\vdots$$

Is the ***sequence of partial sums*** of the series, the number $\{s_n\}$ being the ***nth* partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is *L*. In this case, we also write

$$a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums of the series does not converge, we say that the series diverges.

Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots = \sum_{n=1}^{\infty} ar^{n-1}$$

In which a and r are fixed real numbers, and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$

Definition of *Geometric* Sequence

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a geometric sequence if $a_1 \neq 0$ and if there is a real number $r \neq 0$ such that for every positive integer k.

$$a_{k+1} = a_k r$$

The number $r = \dfrac{a_{k+1}}{a_k}$ is called the ***common ratio*** of the sequence.

The formula for the nth Term of a Geometric Sequence: $\boxed{a_n = a_1 r^{n-1}}$

Theorem: Formula for S_n

The *n*th partial sum S_n of a geometric sequence with first term a_1 and common ratio $r \neq 1$ is

$$\boxed{S_n = a\frac{1-r^n}{1-r}}$$

Proof

By definition, the n^{th} partial sum S_n of a geometric sequence is:

$$\begin{array}{rl} & S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \\ - & rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \\ \hline & S_n - rS_n = a - ar^n \end{array}$$

$$(1-r)S_n = a\left(1-r^n\right)$$

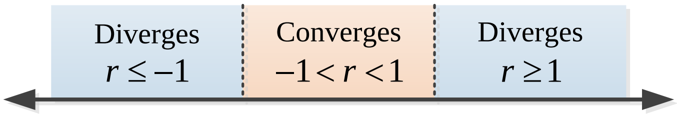
$$S_n = a\frac{1-r^n}{1-r}$$

Definition

If $|r| < 1$, the geometric series $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$ converges to $\dfrac{a}{1-r}$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1$$

If $|r| \ge 1$, the series diverges.



- The sequence may converge to a single value, which is the limit of the sequence.
- The sequence terms may increase in magnitude without bound (either with one sign or with mixed signs), in which case the sequence diverges.
- The sequence terms may remain bounded but settle into an oscillating pattern in which the terms approach two or more values, then the sequence diverges.
- The terms of a sequence may remain bounded, but wander chaotic forever without pattern, then the sequence diverges in this case.

Example

Find the geometric series with $a = \frac{1}{9}$ *and* $r = \frac{1}{3}$

Solution

$$\frac{1}{9} + \frac{1}{9}\frac{1}{3} + \frac{1}{9}\left(\frac{1}{3}\right)^2 + \cdots = \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1}$$

$$= \frac{1/9}{1-1/3}$$

$$= \frac{1}{6}$$

Example

The series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$ is geometric series with $a = 5$ *and* $r = -\frac{1}{4}$

Solution

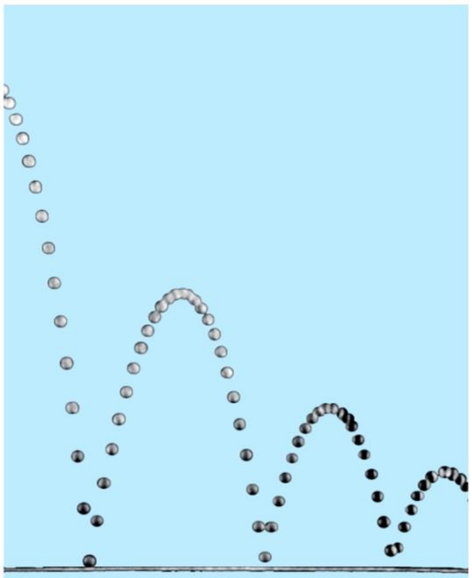
It converges to $\dfrac{a}{1-r} = \dfrac{5}{1-\left(-\frac{1}{4}\right)} = \dfrac{5}{\frac{5}{4}}$

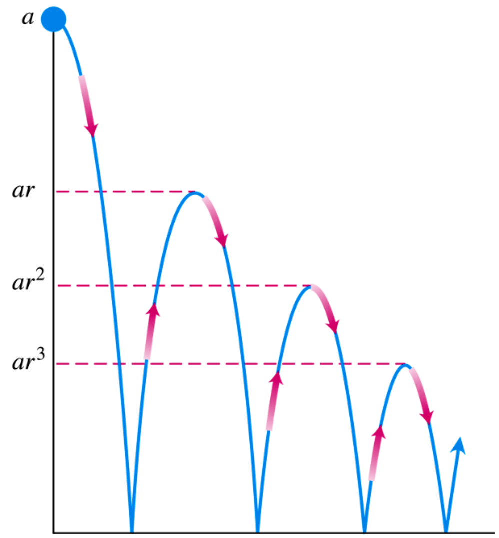
$$= 5 \cdot \frac{4}{5}$$

$$= 4$$

Example

Your drop a ball from *a* meters above a flat surface. Each time the ball hits the surface after falling a distance *h*, it rebounds a distance *rh*, where *r* is positive but less than 1. Find the total distance the ball travels up and down. $\left(a = 6\ m \quad and \quad r = \frac{2}{3}\right)$





Solution

The total distance is

$$\begin{aligned} s &= a + 2ar + 2ar^2 + 2ar^3 + \dots \\ &= a + \frac{2ar}{1-r} \\ &= a\left(1 + \frac{2r}{1-r}\right) \\ &= a\frac{1+r}{1-r} \end{aligned}$$

If $a = 6\ m \quad and \quad r = \frac{2}{3}$, the distance is:

$$s = 6\frac{1+\frac{2}{3}}{1-\frac{2}{3}} = \underline{30\ m}$$

Example

Express repeating decimal 5.232323… as the ratio of two integers.

Solution

$$\begin{aligned}
5.232323\cdots &= 5+\frac{23}{100}+\frac{23}{(100)^2}+\frac{23}{(100)^3}+\cdots \\
&= 5+\frac{23}{100}\left(1+\frac{1}{100}+\left(\frac{1}{100}\right)^2+\cdots\right)
\end{aligned}$$

$$1+\frac{1}{100}+\left(\frac{1}{100}\right)^2+\cdots=\frac{1}{1-\frac{1}{100}}=\frac{100}{99}$$

$$\begin{aligned}
&= 5+\frac{23}{100}\left(\frac{100}{99}\right) \\
&= 5+\frac{23}{99} \\
&= \boxed{\frac{518}{99}}
\end{aligned}$$

Example

Find the sum of the "telescoping" series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$.

Solution

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$\begin{aligned}
s_k &= \left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\ldots+\left(\cancel{\frac{1}{k}}-\frac{1}{k+1}\right) \\
&= 1-\frac{1}{k+1}
\end{aligned}$$

$$\lim_{k\to\infty}\frac{1}{k+1}=0$$

$$s_k = 1-0=1$$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\ \boxed{=1}$$

The nth-Term Test for a Divergent Series

Theorem

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$

The nth-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Example

a) The series $\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{n+1}{n} + \cdots$ diverges because each term is greater than 1, so the sum of the n terms is greater than n. $\frac{n+1}{n} \to 1$

b) $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$

c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty} (-1)^{n+1}$ doesn't exist.

d) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$

Theorem

If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

Sum Rule: $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$

Difference Rule: $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$

Constant Multiple Rule: $\sum k a_n = k \sum a_n = kA$

- Every nonzero constant multiple of a divergent series diverges.
- If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n + b_n)$ and $\sum (a_n - b_n)$ both diverge.

Example

Find the sums of the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}$

Solution

$$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{3^{n-1}}{6^{n-1}} - \frac{1}{6^{n-1}} \right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$$

$$= \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{6}}$$

$$= 2 - \frac{6}{5}$$

$$= \frac{4}{5}$$

Example

Find the sums of the series $\displaystyle\sum_{n=0}^{\infty} \frac{4}{2^n}$

Solution

$$\sum_{n=0}^{\infty} \frac{4}{2^n} = 4\sum_{n=0}^{\infty} \frac{1}{2^n} = 4\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$= 4\frac{1}{1-\frac{1}{2}}$$

$$= 4(2)$$

$$= 8$$

Adding or Deleting Terms

We can add finite number of terms to a series or delete a finite number of terms without altering the series' convergence or divergence.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_{k-1} + \sum_{n=k}^{\infty} a_n$$

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \sum_{n=4}^{\infty} \frac{1}{5^n}$$

$$\sum_{n=4}^{\infty} \frac{1}{5^n} = \left(\sum_{n=1}^{\infty} \frac{1}{5^n} \right) - \frac{1}{5} - \frac{1}{25} - \frac{1}{125}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=5}^{\infty} \frac{1}{2^{n-5}} = \sum_{n=-4}^{\infty} \frac{1}{2^{n+4}}$$

Exercises Section 3.2 – Infinite Series

Find the limit of the following sequences or determine the limit does not exist

1. $a_n = \dfrac{n^3}{n^4+1}$

2. $a_n = n^{1/n}$

3. $\left\{\dfrac{n^{12}}{3n^{12}+4}\right\}$

4. $\left\{\dfrac{2e^{n+1}}{e^n}\right\}$

5. $\left\{\dfrac{\tan^{-1} n}{n}\right\}$

6. $\left\{\left(1+\dfrac{2}{n}\right)^n\right\}$

7. $\left\{\left(\dfrac{n}{n+5}\right)^n\right\}$

8. $\left\{\dfrac{\ln\left(\frac{1}{n}\right)}{n}\right\}$

9. $\{\ln \sin(1/n) + \ln n\}$

10. $a_n = \dfrac{n!}{n^n}$

Find a formula for the *n*th term partial sum of the series and use it to find the series' sum if the series converges

11. $2+\dfrac{2}{3}+\dfrac{2}{9}+\dfrac{2}{27}+\cdots+\dfrac{2}{3^{n-1}}+\cdots$

12. $\dfrac{9}{100}+\dfrac{9}{100^2}+\dfrac{9}{100^3}+\cdots+\dfrac{9}{100^n}+\cdots$

13. $1-2+4-8+\cdots+(-1)^{n-1}2^{n-1}+\cdots$

14. $\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\dfrac{1}{4\cdot 5}+\cdots+\dfrac{1}{(n+1)(n+2)}+\cdots$

15. $\dfrac{5}{1\cdot 2}+\dfrac{5}{2\cdot 3}+\dfrac{5}{3\cdot 4}+\cdots+\dfrac{5}{n(n+1)}+\cdots$

Write out the first few terms of each series to show how the series starts. Then find the sum of the series

16. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}$

17. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{4^n}$

18. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{5}{2^n}-\frac{1}{3^n}\right)$

19. $\displaystyle\sum_{n=0}^{\infty}\frac{2^{n+1}}{5^n}$

Determine if the geometric series converges or diverges. If a series converges, find its sum

20. $1+\left(\dfrac{2}{5}\right)+\left(\dfrac{2}{5}\right)^2+\left(\dfrac{2}{5}\right)^3+\left(\dfrac{2}{5}\right)^4+\cdots$

21. $1+(-3)+(-3)^2+(-3)^3+(-3)^4+\cdots$

22. $\left(-\dfrac{2}{3}\right)^2+\left(-\dfrac{2}{3}\right)^3+\left(-\dfrac{2}{3}\right)^4+\left(-\dfrac{2}{3}\right)^5+\cdots$

Express each of the numbers as the ratio of two integers (fraction)

23. $0.\overline{23} = 0.23\ 23\ 23\cdots$

24. $0.\overline{234} = 0.234\ 234\ 234\cdots$

25. $1.\overline{414} = 1.414\ 414\ 414\cdots$

26. $1.24\overline{123} = 1.24\ 123\ 123\ 123\cdots$

Determine if the series converges or diverges. Give reasons for your answers. If a series converges, find its sum.

27. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$

28. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}n$

29. $\displaystyle\sum_{n=0}^{\infty}\frac{\cos n\pi}{5^n}$

30. $\displaystyle\sum_{n=0}^{\infty}e^{-2n}$

31. $\displaystyle\sum_{n=1}^{\infty}\ln\frac{1}{3^n}$

32. $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{n!}$

33. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n+3^n}{4^n}$

34. $\displaystyle\sum_{n=0}^{\infty}\frac{e^{\pi n}}{\pi^{ne}}$

35. $\displaystyle\sum_{n=0}^{\infty}(1.075)^n$

36. $\displaystyle\sum_{n=0}^{\infty}\frac{3^n}{1000}$

37. $\displaystyle\sum_{n=1}^{\infty}\frac{n+10}{10n+1}$

38. $\displaystyle\sum_{n=1}^{\infty}\frac{4n+1}{3n-1}$

39. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)$

40. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$

41. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{n^3}$

42. $\displaystyle\sum_{n=0}^{\infty}\frac{3}{5^n}$

43. $\displaystyle\sum_{n=2}^{\infty}\frac{n}{\ln n}$

44. $\displaystyle\sum_{n=1}^{\infty}\ln\frac{1}{n}$

45. $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{k}{n}\right)^n$

46. $\displaystyle\sum_{n=1}^{\infty}e^{-n}$

47. $\displaystyle\sum_{n=1}^{\infty}\arctan n$

48. $\displaystyle\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right)$

Determine if the series converges or diverges and describe whether they do so monotonically or by oscillation. Give the limit when the sequence converges.

49. $a_n = 0.3^n$

50. $a_n = 1.3^n$

51. $a_n = (-0.6)^n$

52. $a_n = (-1.01)^n$

53. $a_n = 2^n 3^{-n}$

54. $a_n = (-0.003)^n$

Find the limit of the following sequences or state that they diverge

55. $a_n = \dfrac{\sin n}{2^n}$

56. $a_n = \dfrac{\cos\left(\frac{n\pi}{2}\right)}{\sqrt{n}}$

57. $a_n = \dfrac{2\tan^{-1} n}{n^3 + 4}$

58. $a_n = \dfrac{n \sin^3 n}{n+1}$

59. Many people take aspirin on a regular basis as a preventive measure for heart disease. Suppose a person take 80 mg of aspirin every 24 hr. Assume also that aspirin has a half-life of 24 hr; that is, every 24 hr half of the drug in the blood is eliminated.

a) Find a recurrence relation for the sequence $\{d_n\}$ that gives the amount of drug in the blood after the n^{th} dose, where $d_1 = 80$

b) Find the limit of $\{d_n\}$

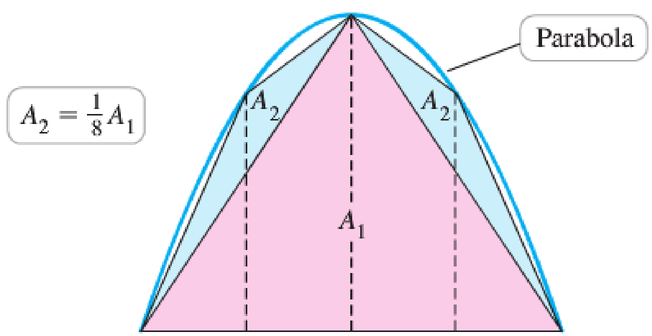
60. Suppose a tank is filled with 100L of a 40% alcohol solution (by volume). You repeatedly perform the following operation: Remove 2 L of the solution from the tank and replace them with 2 L of 10% alcohol solution

a) Let C_n be the concentration of the solution in the tank after the nth replacement, where $C_0 = 40\%$. Write the first five terms of the sequence $\{C_n\}$

b) After how many replacements does the alcohol concentration reach 15%?

c) Determine the limiting (steady-state) concentration of the solution that is approached after many replacements.

61. The Greeks solved several calculus problems almost 2000 years before the discovery of calculus. One example is Archimedes' calculation of the area of the region *R* bounded by a segment of a parabola, which he did using the "method of exhaustion".



The idea was to fill *R* with an infinite sequence of triangles. Archimedes began with an isosceles triangle inscribed in the parabola, with an area A_1, and proceeded in stages, with the number of new triangles doubling at each stage. He was able to show (the key to the solution) that at each stage, the area of a new triangle is $\frac{1}{8}$ of the area of a triangle at the previous stage; for example, $A_2 = \frac{1}{8}A_1$, and so forth. Show, as Archimedes did, that the area of *R* is $\frac{4}{3}$ times the area of A_1.

62.

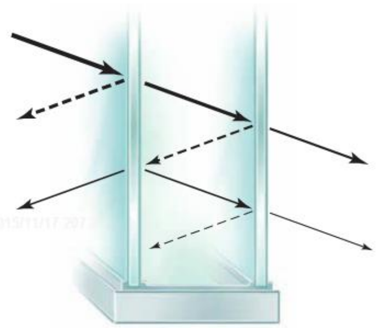
a) Evalute the series $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{\left(3^{k+1}-1\right)\left(3^k-1\right)}$

b) For what values of a does the the series converge, and in those cases, what is its value?

$$\sum_{k=1}^{\infty} \frac{a^k}{\left(a^{k+1}-1\right)\left(a^k-1\right)}$$

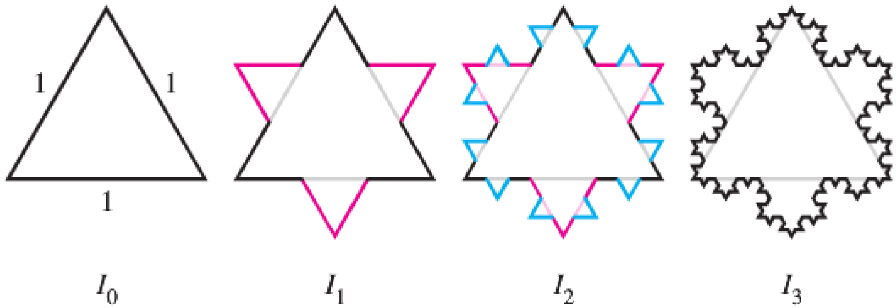
63. Suppose you borrow \$20,000 for a new car at a monthly interest rate of 0.75%. If you make payments of \$600/month, after how many months will the loan balance be zero? Estimate the answer by graphing the sequence of loan balances and then obtain an exact answer using infinite series.

64. An insulated windows consists of two parallel panes of glass with a small spacing between them. Suppose that each pane reflects a fraction p of the incoming light and transmits the remaining light. Considering all reflections of light between the panes, what fraction of the incoming light is ultimately transmitted by the windows? Assume the amount of incoming light is 1.



65. Suppose a rubber ball, when dropped from a given height, returns to a fraction p of that height. In the absence of air resistance, a ball dropped from a height h requires $\sqrt{\frac{2h}{g}}$ seconds to fall to the ground, where $g \approx 9.8\ m/s^2$ is the acceleration due to gravity. The time taken to bounce up to a given to fall from that height to the ground. How long does it take a ball dropped from 10 m to come to rest?

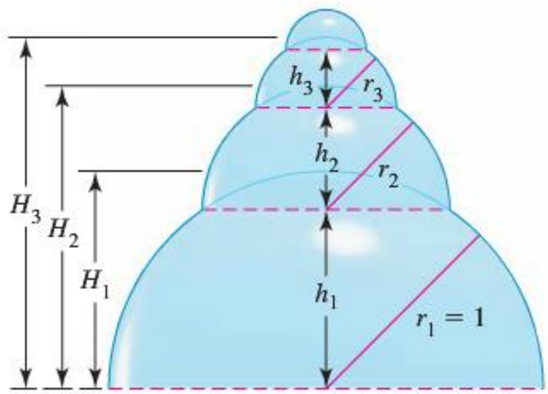
66. The fractal called the snowflake island (or Koch island) is constructed as flows: Let I_0 be an equilateral triangle with sides of length 1. The figure I_1 is obtained by replacing the middle third of each side of I_0 with a new outward equilateral triangle with sides of length $\frac{1}{3}$. The process is repeated where I_{n+1} is obtained by replacing the middle third of each side of I_n with a new outward equilateral triangle with sides of length of $\frac{1}{3^{n+1}}$. The limiting figure as $n \to \infty$ is called the snowflake island.



a) Let L_n be the perimeter of I_n. Show that $\lim_{n\to\infty} L_n = \infty$

b) Let A_n be the area of I_n. Find $\lim_{n\to\infty} A_n$. It exists!

67. Imagine a stack of hemispherical soap bubbles with decreasing radii $r_1 = 1, r_2, r_3, \ldots$ Let h_n be the distance between the diameters of bubble n and bubble $n+1$, and let H_n be the total height of the stack with n bubbles.



a) Use the Pythagorean theorem to show that in a stack with n bubbles $h_1^2 = r_1^2 - r_2^2$, $h_2^2 = r_2^2 - r_3^2$, and so forth. Note that for the last bubble $h_n = r_n$.

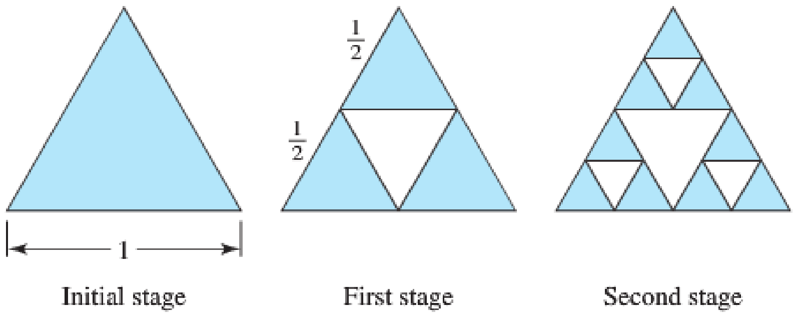
b) Use part (*a*) to show that the height of a stack with n bubbles is

$$H_n = \sqrt{r_1^2 - r_2^2} + \sqrt{r_2^2 - r_3^2} + \cdots + \sqrt{r_{n-1}^2 - r_n^2} + r_n$$

c) The height of a stack of bubbles depends on how the radii decrease. Suppose that $r_1 = 1,\ r_2 = a,\ r_3 = a^2,\ \ldots,\ r_n = a^{n-1}$ where $0 < a < 1$ is a fixed real number. In terms of *a*, find the height H_n of a stack with n bubbles.

d) Suppose the stack in part (*c*) is extended indefinitely $(n \to \infty)$. In terms of *a*, how high would the stack be?

68. The fractal called the *Sierpinski triangle* is the limit of a sequence of figures. Starting with the equilateral triangle with sides of length 1, an inverted equilateral triangle with sides of length $\frac{1}{2}$ is removed. Then, the three inverted equilateral triangles with sides of length $\frac{1}{4}$ are removed from this figure.



The process continues in this way. Let T_n be the total area of the removed triangles after stage n of the process. The area on equilateral triangle with side length L is $A = \frac{\sqrt{3}}{4}L^2$.

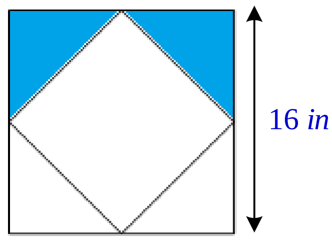
a) Find T_1 and T_2 the total area of the removed triangles after stages 1 and 2, respectively.

b) Find T_n for $n = 1, 2, 3, \ldots$

c) Find $\lim\limits_{n\to\infty} T_n$

d) What is the area of the original triangle that remains as $n \to \infty$?

69. The sides of a ***square*** are 16 *inches* in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the triangles outside the second square are shaded.

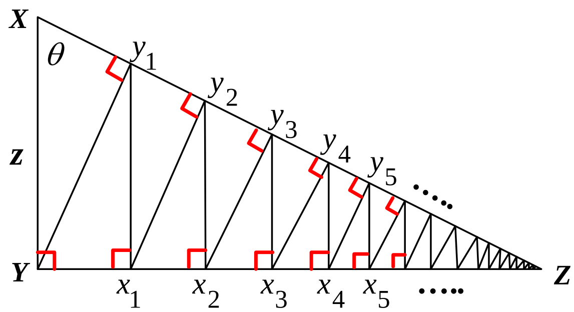


Determine the area of the shaded regions

a) When this process is continued five more times

b) When this pattern of shading is continued infinitely.

70. A right triangle XYZ is shown below where $|XY| = z$ and $\angle X = \theta$. Line segments are continually drawn to be perpendicular to the triangle.



a) Find the total length of the perpendicular line segments $|Yy_1| + |x_1y_1| + |x_1y_2| + \cdots$ in terms of z and θ.

b) Find the total length of the perpendicular line segments when $z = 1$ and $\theta = \frac{\pi}{6}$

71. The sphereflake is a computer–generated fractal that was created by Eric Haines. The radius of the large sphere is 1. To the large sphere, nine spheres of radius $\frac{1}{3}$ are attached. To each of these, nine spheres of radius $\frac{1}{9}$ are attached. This process is continued infinitely.

Prove that the sphereflake has an infinite surface area.