Lecture One – Limits and Derivatives

Section 1.1 – Idea of Limits

Position Function

An object that is falling or vertically projected into the air has its height above the ground, $s(t)$, in feet, given by

$$s(t) = -16t^2 + v_0 t + s_0$$

v_0 is the original velocity (initial velocity) of the object, in *feet* per *second*

t is the time that the object is in motion, in *second*

s_0 is the original height (initial height) of the object, in *feet*

The average rate is given by: $\dfrac{\Delta s}{\Delta t}$

Example

A rock breaks loose from the top of a tall cliff. What is its average speed

a) During the first 2 *sec* of fall?

b) During the 1-*sec* interval between second 1 and *second* 2?

Solution

Since the rock falls free (*down*) without any initial velocity or height. $\Rightarrow y(t) = 16t^2$

a) For the first 2 *sec*: *Average speed* $= \dfrac{\Delta y}{\Delta t}$

$$= \frac{y(2) - y(0)}{2 - 0}$$

$$= \frac{16(2)^2 - 16(0)^2}{2}$$

$$= \frac{64}{2}$$

$$= 32 \ \ ft/\sec$$

b) From 1 *sec* to 2 *sec*: *Average speed* $= \dfrac{y(2) - y(1)}{2 - 1}$

$$= \frac{16(2)^2 - 16(1)^2}{1}$$

$$= 48 \ \ ft/\sec$$

Example

Find the speed of a falling rock $\left(y(t)=16t^2\right)$ over a time interval $\left[t_0,\ t_0+h\right]$. Then find the average speed at 1 *sec* and 2 *sec*.

Solution

$$\frac{\Delta y}{\Delta t}=\frac{16\left(t_0+h\right)^2-16\left(t_0\right)^2}{\left(t_0+h\right)-t_0}$$

$$=\frac{16\left(t_0^2+2ht_0+h^2\right)-16t_0^2}{t_0+h-t_0}$$

$$=\frac{16t_0^2+32ht_0+16h^2-16t_0^2}{h}$$

$$=32\frac{ht_0}{h}+16\frac{h^2}{h}$$

$$=32t_0+16h$$

If $t_0=1$

$$\frac{\Delta y}{\Delta t}=32(1)+16h$$

$$=32+16h$$

The average speed has the limiting value 32 *ft/sec* as h approaches 0.

If $t_0=2$

$$\frac{\Delta y}{\Delta t}=32(2)+16h$$

$$=64+16h$$

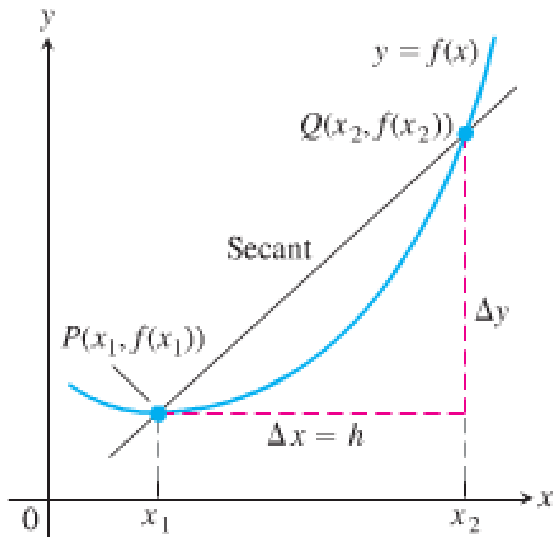
The average speed has the limiting value 64 *ft/sec* as h approaches 0.

Average Rates of Changes and Secant Lines

The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

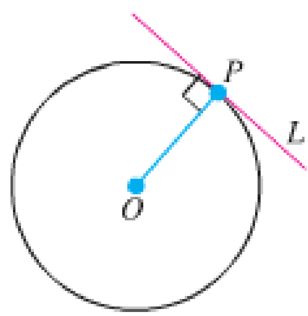
$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0$$



Defining the Slope of a Curve

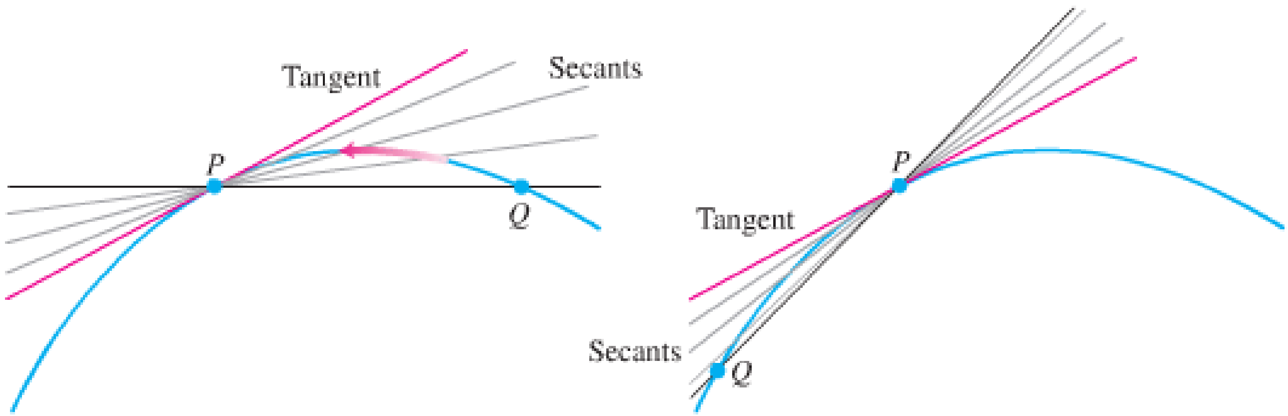
The slope of a line is the rate at which it rises or falls.



To define the tangency for general curves, we need an approach that makes the behavior of the secants through P and points Q as Q moves toward P along the curve:

1. Find the slope of the secant PQ.
2. Investigate the limiting value of the slope as Q approaches P along the curve.
3. If the limit exists, take it to be the slope of the curve at P and define the tangent to the curve at P to be the line through P with this slope.

$$m_{\tan} = \lim_{t \to a} \frac{f(t) - f(a)}{t - a}$$



Example

Find the slope of the parabola $y = x^2$ at the point $P(2,\ 4)$. Write an equation for the tangent to the parabola at this point.

Solution

$$\textbf{\textit{Secant slope}} = \frac{\Delta y}{\Delta x} = \frac{f(x_1 + h) - f(x_1)}{h}$$

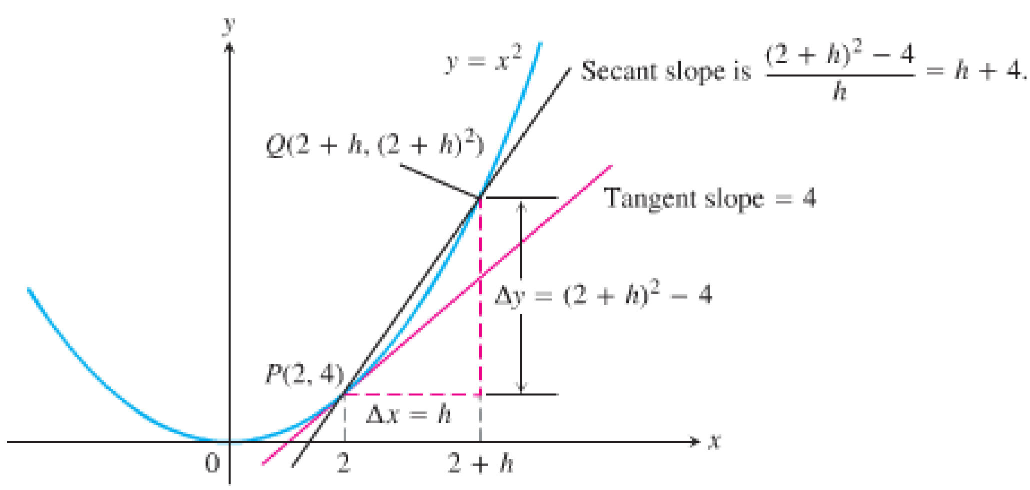
$$= \frac{f(2+h) - f(2)}{h}$$

$$= \frac{(2+h)^2 - 2^2}{h}$$

$$= \frac{4 + 4h + h^2 - 4}{h}$$

$$= \frac{4h}{h} + \frac{h^2}{h}$$

$$\underline{= 4 + h}$$



As Q approaches P, h approaches 0. Then the secant slope $h+4 \to 4 = slope$

$$y = m\left(x - x_1\right) + y_1$$

$$y = 4(x-2)+4$$

$$\underline{y = 4x - 4}$$

Exercises Section 1.1 – Idea of Limits

1. Find the average rate of change of the function $f(x) = x^3 + 1$ over the interval $[2,\ 3]$

2. Find the average rate of change of the function $f(x) = x^2$ over the interval $[-1,\ 1]$

3. Find the average rate of change of the function $f(t) = 2 + \cos t$ over the interval $[-\pi,\ \pi]$

4. Find the slope of $y = x^2 - 3$ at the point $P(2,\ 1)$ and an equation of the tangent line at this *P*.

5. Find the slope of $y = x^2 - 2x - 3$ at the point $P(2,\ -3)$ and an equation of the tangent line at this *P*.

6. Find the slope of $y = x^3$ at the point $P(2,\ 8)$ and an equation of the tangent line at this *P*.

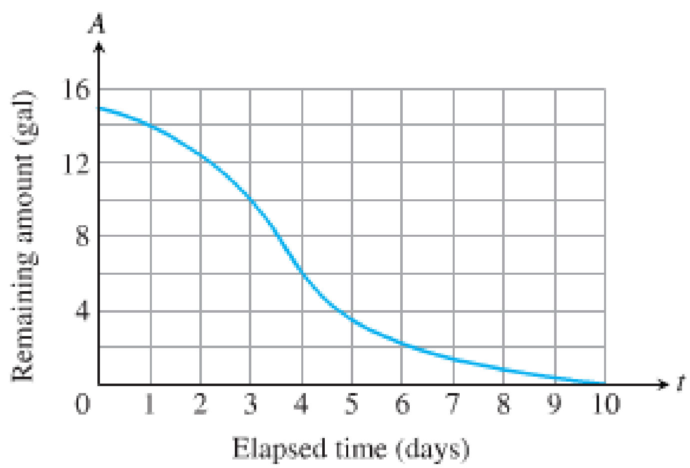
7. Make a table of values for the function $f(x) = \frac{x+2}{x-2}$ at the points

$x = 1.2,\quad x = \frac{11}{10},\quad x = \frac{101}{100},\quad x = \frac{1001}{1000},\quad x = \frac{10001}{10000},\quad \textit{and } x = 1$

a) Find the average rate of change of $f(x)$ over the intervals $[1,\ x]$ for each $x \ne 1$ in the table

b) Extending the table if necessary, try to determine the rate of change of $f(x)$ at $x = 1$.

8. The accompanying graph shows the total amount of gasoline A in the gas tank of an automobile after being driven for *t* days.



a) Estimate the average rate of gasoline consumption over the time intervals

$[0,\ 3],\ [0,\ 5],\ \textit{and } [7,\ 10]$

b) Estimate the instantaneous rate of gasoline consumption over the time $t = 1,\ t = 4,\ \textit{and } t = 8$

Section 1.2 – Definitions / Techniques of Limits

Definition of the Limit of a Function

If $f(x)$ becomes arbitrary close to a single number L as x approaches x_0 from either side, then

$$\lim_{x \to x_0} f(x) = L$$

Which is read as "the limit of $f(x)$ as x approaches x_0 is L."

Notation	***Terminology***
$x \to a^-$	***x*** approaches ***a*** from the left (through values ***less*** than a)
$x \to a^+$	***x*** approaches ***a*** from the right (through values ***greater*** than a)

Example

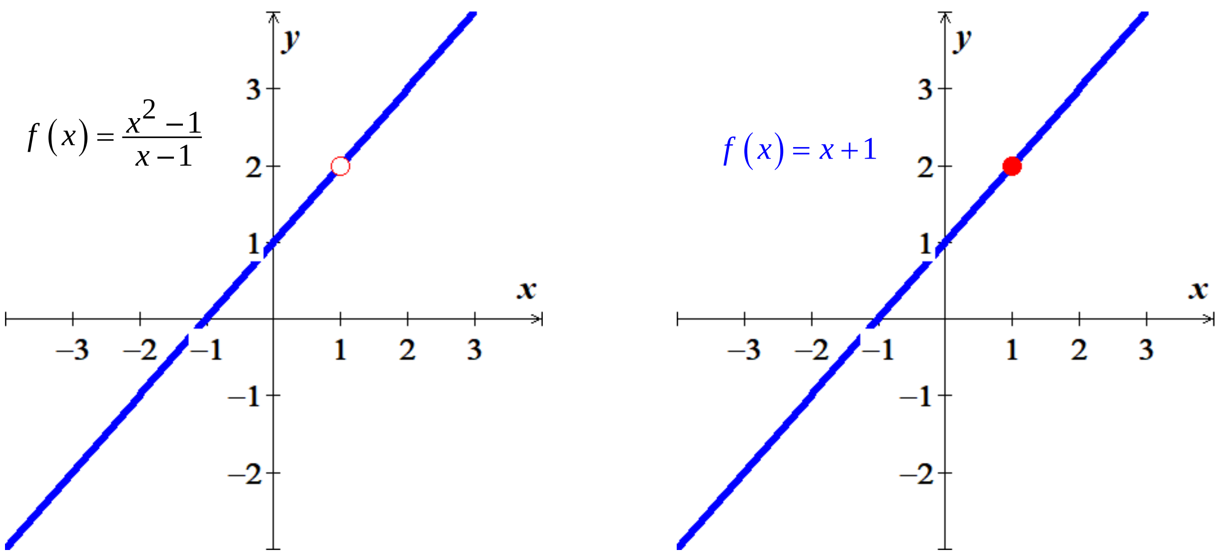
How does the function $f(x) = \dfrac{x^2 - 1}{x - 1}$ behave near $x = 1$?

Solution

$$\begin{aligned} f(x) &= \frac{(x-1)(x+1)}{x-1} \\ &= x + 1 \quad \textit{for} \quad x \neq 1 \end{aligned}$$

For $x = 1$:

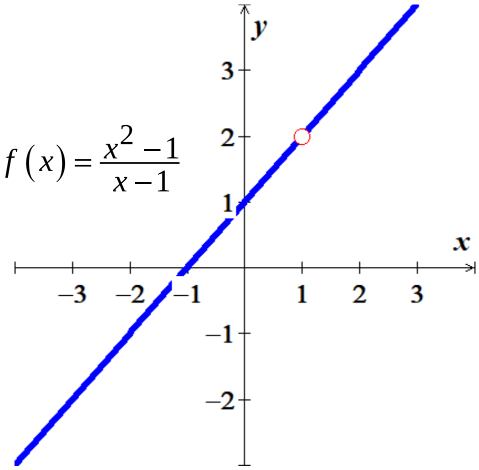
$$f(x = 1) = 1 + 1 = 2$$

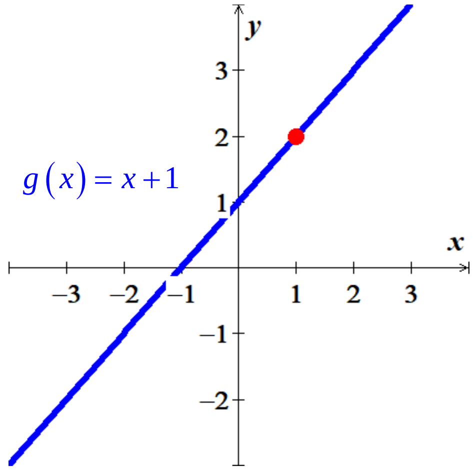


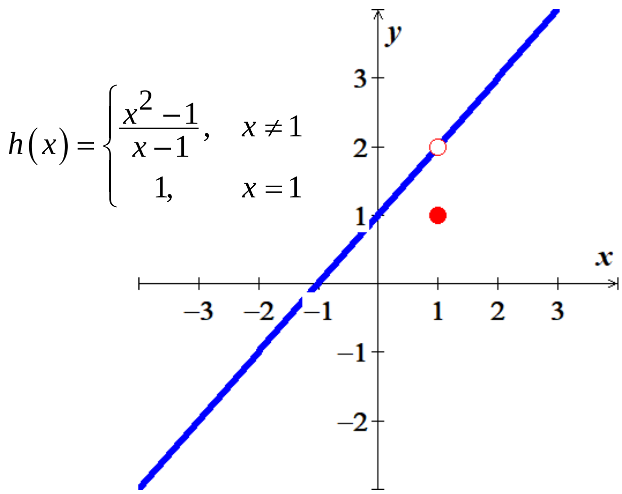
x	.9	.99	.999	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	2.001	2.01	2.1

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1}$$

$$\underline{= 2 \rfloor}$$





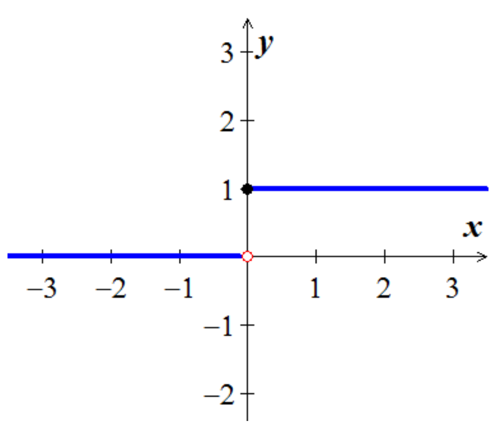


Example

Discuss the behavior of the following function as $x \to 0$.

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Solution



The unit step function $U(x)$ has no limit as $x \to 0$, it jumps, because the values jump at $x = 0$.

To the left of zero $\left(\textit{negative value } \mathbf{0^{-}}\right)$ $U(x)=0$. For the positive values of x close to zero $\left(\mathbf{0^{+}}\right)$ $U(x)=1$

One-Sided Limits

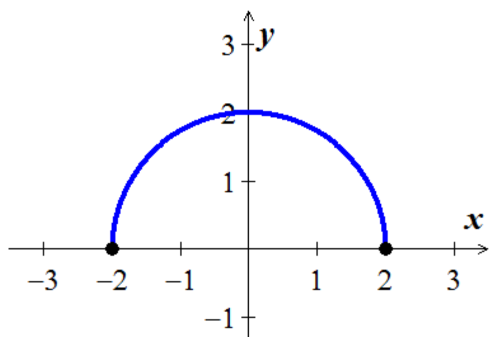
To have a limit L as x approaches c, a function f must be defined on ***both sides*** of c and its values $f(x)$ must approach L as x approaches c from either side. Because of this, ordinary limits are called ***two-sided***. If f fails to have two-sided limit at c, it may still have one-sided limit.

If the approach is from the *right*, the limit is a ***right-hand limit***. $\lim\limits_{x\to c^{+}} f(x)=L$

If the approach is from the *left*, the limit is a ***left-hand limit***. $\lim\limits_{x\to c^{-}} f(x)=M$

Example

The domain of $f(x)=\sqrt{4-x^2}$ is $[-2,\ 2]$; its graph is the semicircle.



We have: $\lim\limits_{x\to -2^{+}} \sqrt{4-x^2}=0$ *and* $\lim\limits_{x\to 2^{-}} \sqrt{4-x^2}=0$

The function doesn't have a left-hand limit at $x=-2$ or a right-hand limit at $x=2$.

It does not have ordinary two-sided limits at either –2 or 2.

Theorem

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \textit{and} \quad \lim_{x\to c^+} f(x) = L$$

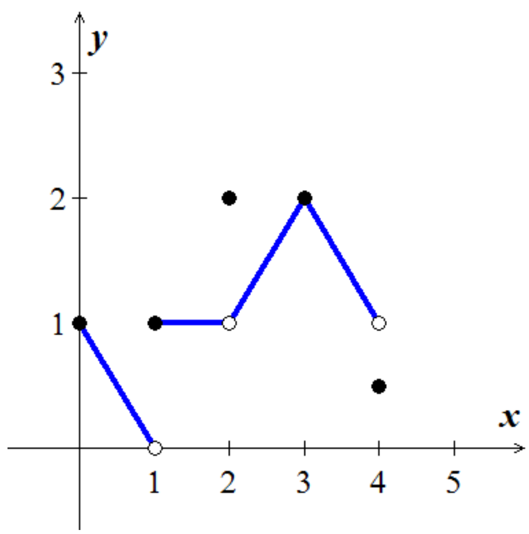
Properties of Limits

Constant function $\left(f(x) = k \right)$: $\qquad \lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} k = k$

Identity function $\left(f(x) = x \right)$: $\qquad \lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} x = x_0$

Example

Given the function graphed:



At $x = 0$: $\quad \lim\limits_{x\to 0^+} f(x) = 1$

$\lim\limits_{x\to 0^-} f(x)$ *and* $\lim\limits_{x\to 0} f(x)$ don't exist. The function is not defined to the left of $x = 0$

At $x = 1$: $\quad \lim\limits_{x\to 1^-} f(x) = 0 \qquad \lim\limits_{x\to 1^+} f(x) = 1$

$\lim\limits_{x\to 1} f(x)$ doesn't exist. The right-hand and left-hand limits are not equal.

At $x = 2$: $\quad \lim\limits_{x\to 2^-} f(x) = 1 \qquad \lim\limits_{x\to 2^+} f(x) = 1$

$\lim\limits_{x\to 2} f(x) = 2$ even though $f(2) = 2$

At $x = 3$: $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3} f(x) \; \underline{= 2}$

At $x = 4$: $\lim\limits_{x\to 4^-} f(x) = 1$ even though $f(4) \ne 1$

$\lim\limits_{x\to 4^+} f(x)$ *and* $\lim\limits_{x\to 4} f(x)$ do not exist.

The function is not defined to the right of $x = 4$

Definitions

We say that $f(x)$ has right-hand limit L at x_0 and $\lim\limits_{x\to x_0^+} f(x) = L$

If for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

We say that $f(x)$ has left-hand limit L at x_0 and $\lim\limits_{x\to x_0^-} f(x) = L$

If for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

Example

Prove that $\lim\limits_{x\to 0^+} \sqrt{x} = 0$

Solution

Let $\varepsilon > 0$ be given. $x_0 = 0, \quad L = 0, \quad Find\ \delta > 0 \ni \forall x$

$$0 < x < \delta \quad \Rightarrow \quad \left|\sqrt{x} - 0\right| < \varepsilon$$

or $\qquad 0 < x < \delta \quad \Rightarrow \quad \sqrt{x} < \varepsilon$

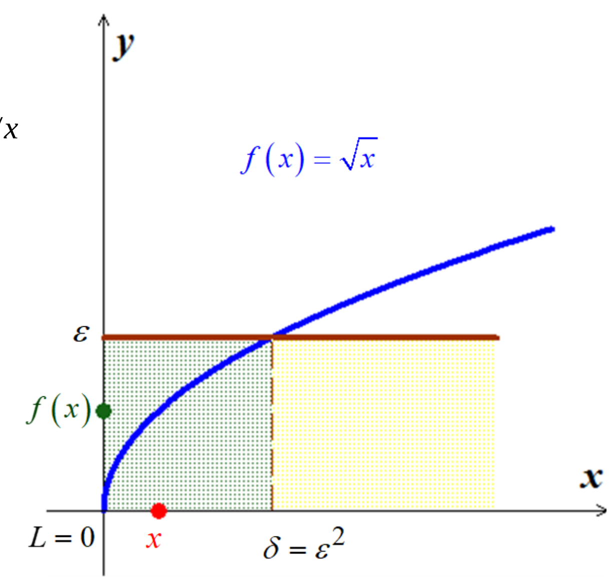
$(\sqrt{x})^2 < \varepsilon^2$

$\Rightarrow x < \varepsilon^2 \quad if \quad 0 < x < \delta$

If we choose $\delta = \varepsilon^2$, we have

$$0 < x < \delta = \varepsilon^2 \quad \Rightarrow \quad \sqrt{x} < \varepsilon$$

According to the definition, this shows that $\lim\limits_{x\to 0^+} \sqrt{x} = 0$



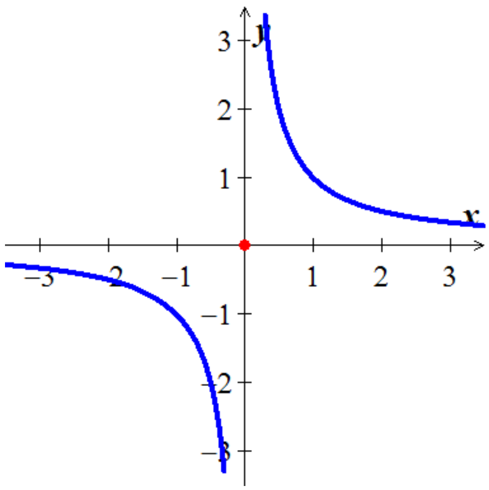
Example

Discuss the behavior of the following function as $x \to 0$.

$a) \quad g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ $\qquad b) \quad f(x) = \begin{cases} 0, & x \leq 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$

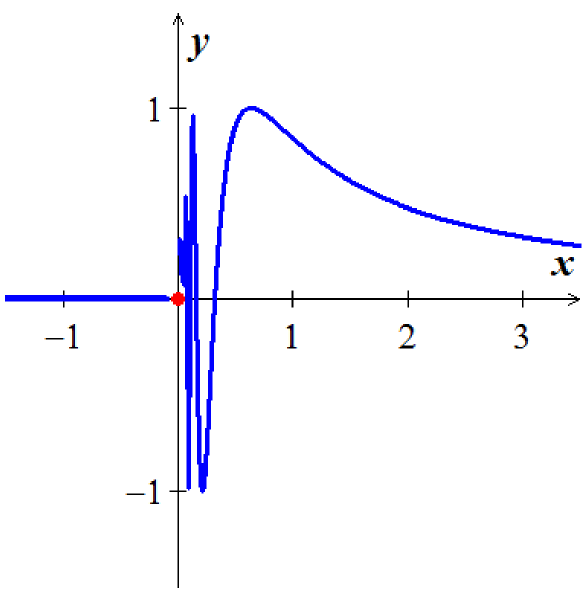
Solution

a)



$g(x)$ has *no limit* as $x \to 0$ because the values of $g(x)$ grow arbitrary large (negative and positive) value as $x \to 0$ and do not stay close.

b)



$f(x)$ has *no limit* as $x \to 0$ because the function's values oscillate between -1 and $+1$ in every open interval containing 0. The values do not stay close to any one number as $x \to 0$.

Limit Laws

If $\lim\limits_{x\to c} f(x) = L$ *and* $\lim\limits_{x\to c} g(x) = M$

Constant Multiple Rule: $\lim\limits_{x\to c} [bf(x)] = b \lim\limits_{x\to c} f(x) \; \underline{= bL}$

Sum and Difference Rules: $\lim\limits_{x\to c} [f(x) \pm g(x)] = \lim\limits_{x\to c} f(x) \pm \lim\limits_{x\to c} g(x) \; \underline{= L \pm M}$

Product Rule: $\lim\limits_{x\to c} [f(x) \cdot g(x)] = \lim\limits_{x\to c} f(x) \cdot \lim\limits_{x\to c} g(x) \; \underline{= L.M}$

Quotient Rule: $\lim\limits_{x\to c} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)} \; \underline{= \dfrac{L}{M}}$ $\quad M \neq 0$

Power Rule: $\lim\limits_{x\to c} (f(x))^n = \left[\lim\limits_{x\to c} f(x)\right]^n \; \underline{= L^n}$

Root Rule: $\lim\limits_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to c} f(x)} \; \underline{= \sqrt[n]{L}}$ $\quad n > 0, \quad L > 0,$ *n is even*

Example

Find the following limits:

$a)\ \lim\limits_{x\to c}\left(x^3+4x^2-3\right)$ $\qquad b)\ \lim\limits_{x\to c}\dfrac{x^4+x^2-1}{x^2+5}$ $\qquad c)\ \lim\limits_{x\to -2}\sqrt{4x^2-3}$

Solution

a) $$\lim_{x\to c}\left(x^3+4x^2-3\right)=\lim_{x\to c}x^3+\lim_{x\to c}4x^2-\lim_{x\to c}(3)$$ ***Sum and Difference Rules***

$$=c^3+4c^2-3$$

b) $$\lim_{x\to c}\frac{x^4+x^2-1}{x^2+5}=\frac{\lim\limits_{x\to c}\left(x^4+x^2-1\right)}{\lim\limits_{x\to c}\left(x^2+5\right)}$$ ***Quotient Rule***

$$=\frac{\lim\limits_{x\to c}x^4+\lim\limits_{x\to c}x^2-\lim\limits_{x\to c}1}{\lim\limits_{x\to c}x^2+\lim\limits_{x\to c}5}$$ ***Sum and Difference Rules***

$$=\frac{c^4+c^2-1}{c^2+5}$$

c) $$\lim_{x\to -2}\sqrt{4x^2-3}=\sqrt{\lim_{x\to -2}\left(4x^2-3\right)}$$ ***Root Rule***

$$=\sqrt{\lim_{x\to -2}4x^2-\lim_{x\to -2}3}$$ ***Difference Rule***

$$=\sqrt{4(-2)^2-3}$$

$$=\sqrt{16-3}$$

$$=\sqrt{13}$$

Theorem – Limits of Polynomials

If $P(x)=a_n x^n + a_{n-1}x^{n-1}+\cdots+a_1 x + a_0$, then $\lim\limits_{x\to c} P(x)=P(c)=a_n c^n + a_{n-1}c^{n-1}+\cdots+a_1 c + a_0$

Theorem – Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c)\neq 0$, then $\lim\limits_{x\to c}\dfrac{P(x)}{Q(x)}=\dfrac{P(c)}{Q(c)}$

Example

Find the limit: $\lim\limits_{x\to -1}\dfrac{x^3+4x^2-3}{x^2+5}$

Solution

$$\lim_{x\to -1}\frac{x^3+4x^2-3}{x^2+5}=\frac{(-1)^3+4(-1)^2-3}{(-1)^2+5}$$

$$=\frac{0}{6}$$

$$=0$$

Eliminating Zero Denominators Algebraically

Example

Evaluate: $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{x^2-x}$

Solution

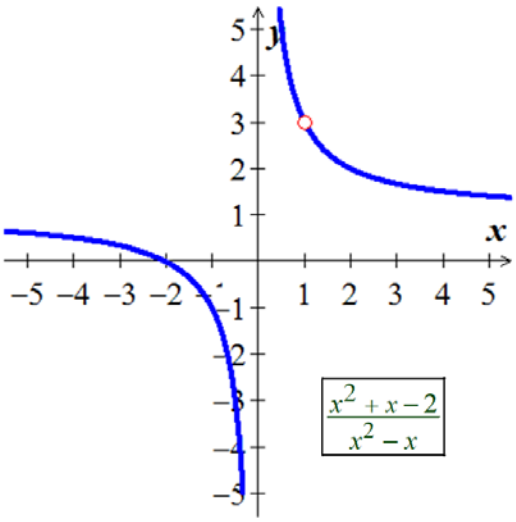
$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}=\frac{1^2+1-2}{1^2-1}=\frac{0}{0}$$

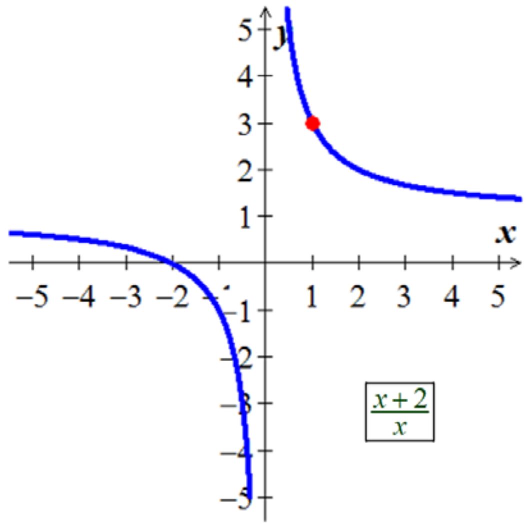
$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}=\lim_{x\to 1}\frac{(x-1)(x+2)}{x(x-1)}$$

$$=\lim_{x\to 1}\frac{(x+2)}{x}$$

$$=\frac{1+2}{1}$$

$$=3$$





Example

Evaluate: $\lim\limits_{x\to 0} \dfrac{\sqrt{x^2+100}-10}{x^2}$

Solution

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} = \frac{\sqrt{0+100}-10}{0} = \frac{0}{0}$$

$$\frac{\sqrt{x^2+100}-10}{x^2} = \frac{\sqrt{x^2+100}-10}{x^2} \cdot \frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$$

$(a-b)(a+b) = a^2 - b^2; \quad \left(\sqrt{a}\right)^2 = a$

$$= \frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$= \frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$= \frac{1}{\sqrt{x^2+100}+10}$$

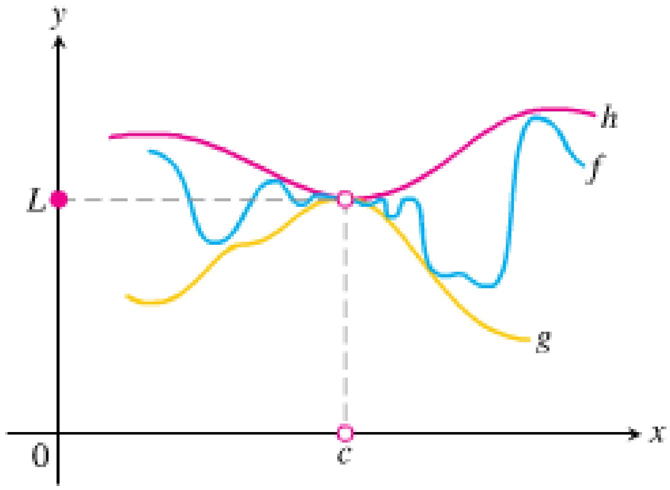
$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x\to 0} \frac{1}{\sqrt{x^2+100}+10}$$

$$= \frac{1}{\sqrt{0+100}+10}$$

$$= \frac{1}{10+10}$$

$$= \frac{1}{20}$$

The Sandwich (Squeeze) Theorem



Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L \quad \textit{then} \quad \lim_{x\to c} f(x) = L$$

Example

Given that $1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2}$ *for all* $x \ne 0$, find the $\lim_{x\to 0} u(x)$, no matter how complicated u is.

Solution

$$\lim_{x\to 0}\left(1 - \frac{x^2}{4}\right) = 1 - \frac{0}{4}$$

$$= 1$$

$$\lim_{x\to 0}\left(1 + \frac{x^2}{2}\right) = 1$$

The Sandwich theorem implies that $\lim_{x\to 0} u(x) = 1$

Theorem

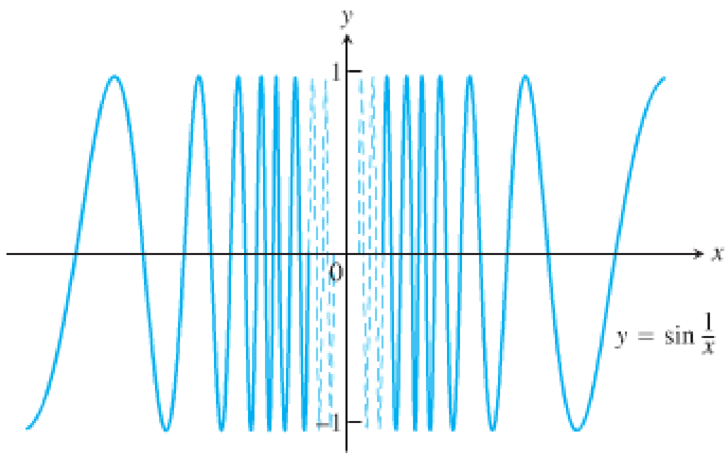
Suppose that $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x)$$

Example

Show that $y = \sin\left(\frac{1}{x}\right)$ has no limit as x approaches zero from either side.

Solution

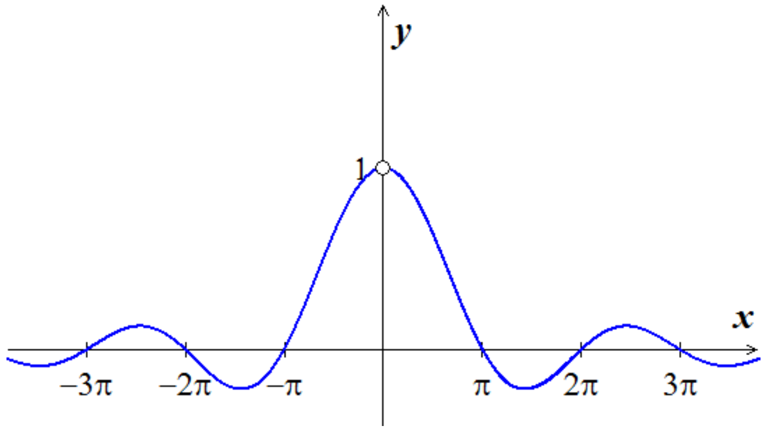


As x approaches zero, its reciprocal, $\frac{1}{x}$, grows without bound and the values of $\sin\left(\frac{1}{x}\right)$ cycle repeatedly from -1 to 1.

There is no single number L that the function's values stay increasingly close to as x approaches zero.

The function has neither a right-hand limit nor a left-hand limit at $x = 0$.

Limit Involving $\frac{\sin\theta}{\theta}$



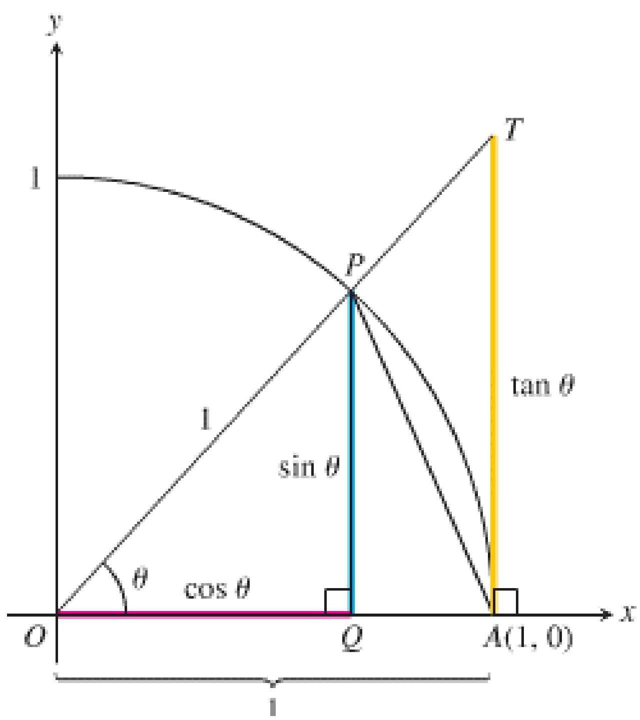
A central fact about $\frac{\sin\theta}{\theta}$ is that in radian measure it limit as $\theta \to 0$ is **1**.

Theorem

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \quad (\theta \text{ in rad.})$$

Proof

We need to show that the right-hand limit is 1, $\theta < \frac{\pi}{2}$



Notice that:

$Area\ \Delta OAP\ < Area\ Sector\ OAP\ < Area\ \Delta OAT$

$Area\ \Delta OAP = \frac{1}{2} base \times height = \frac{1}{2}(1)(\sin\theta)$

$$Area\ Sector\ \Delta OAP = \tfrac{1}{2}r^2 \times \theta = \tfrac{1}{2}(1)^2(\theta) = \frac{\theta}{2}$$

$$Area\ \Delta OAP = \tfrac{1}{2} base \times height = \tfrac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta$$

$$\Rightarrow \tfrac{1}{2}\sin\theta < \tfrac{1}{2}\theta < \tfrac{1}{2}\tan\theta$$

$$\frac{2}{\sin\theta}\frac{1}{2}\sin\theta < \frac{1}{2}\theta\frac{2}{\sin\theta} < \frac{1}{2}\frac{\sin\theta}{\cos\theta}\frac{2}{\sin\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$ ***Taking reciprocals reverses the inequalities***

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Since $\lim\limits_{\theta\to 0^+} \cos\theta = 1$, then

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta}$$

So $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$

Example

Show that $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x} = 0$

<u>Solution</u>

Using the half-angle formula: $\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right)$

$$\begin{aligned}
\lim_{x\to 0}\frac{\cos x - 1}{x} &= \lim_{x\to 0}\frac{1-2\sin^2\left(\frac{x}{2}\right)-1}{x} \\
&= \lim_{x\to 0}\frac{-2\sin^2\left(\frac{x}{2}\right)}{x} \qquad \textit{Let } \theta = \frac{x}{2} \\
&= -\lim_{\theta\to 0}\frac{2\sin^2(\theta)}{2\theta} \\
&= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta \\
&= -(1)(0) \\
&= 0
\end{aligned}$$

Example

Show that $\lim\limits_{x\to 0} \dfrac{\sin 2x}{5x} = \dfrac{2}{5}$

Solution

$$\lim_{x\to 0} \frac{\sin 2x}{5x} = \lim_{x\to 0} \frac{\left(\frac{2}{5}\right)\sin 2x}{\left(\frac{2}{5}\right)5x}$$

Since we need 2x in the denominator

$$= \frac{2}{5} \lim_{x\to 0} \frac{\sin 2x}{2x}$$

$$= \frac{2}{5}(1)$$

$$= \frac{2}{5}$$

Example

Show that $\lim\limits_{x\to 0} \dfrac{\tan x \ \sec 2x}{3x} = \dfrac{1}{3}$

Solution

$$\lim_{x\to 0} \frac{\tan x \ \sec 2x}{3x} = \frac{1}{3} \lim_{x\to 0} \frac{1}{x} \cdot \frac{\sin x}{\cos x} \cdot \frac{1}{\cos 2x}$$

$$= \frac{1}{3} \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} \cdot \frac{1}{\cos 2x}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \lim_{x\to 0} \frac{1}{\cos x} = 1, \quad \lim_{x\to 0} \frac{1}{\cos 2x} = 1$$

$$= \frac{1}{3}(1)(1)(1)$$

$$= \frac{1}{3}$$

Exercises *Section* 1.2 – Definitions / Techniques of Limits

(1 – 121) Find the limit:

1. $\lim\limits_{x\to 3} (-1)$

2. $\lim\limits_{x\to -1} 3$

3. $\lim\limits_{x\to 1000} 18\pi^2$

4. $\lim\limits_{x\to 1} \sqrt{5x+6}$

5. $\lim\limits_{x\to 9} \sqrt{x}$

6. $\lim\limits_{x\to -3} \left(x^2+3x\right)$

7. $\lim\limits_{x\to -4} |x-4|$

8. $\lim\limits_{x\to 4} (x+2)$

9. $\lim\limits_{x\to 4} (x-4)$

10. $\lim\limits_{x\to 2} (5x-6)^{3/2}$

11. $\lim\limits_{x\to 9} \dfrac{x-9}{\sqrt{x}-3}$

12. $\lim\limits_{x\to 1} (2x+4)$

13. $\lim\limits_{x\to 1} \dfrac{x^2-4}{x-2}$

14. $\lim\limits_{x\to 2} \dfrac{x^2+4}{x-2}$

15. $\lim\limits_{x\to 0} \dfrac{|x|}{x}$

16. $\lim\limits_{x\to 3} \dfrac{x^2-x-1}{\sqrt{x+1}}$

17. $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x-2}$

18. $\lim\limits_{x\to 0} (3x-2)$

19. $\lim\limits_{x\to 1} (2x^2-x+4)$

20. $\lim\limits_{x\to -2} \left(x^3-2x^2+4x+8\right)$

21. $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$

22. $\lim\limits_{x\to 2} \dfrac{x^3-8}{x-2}$

23. $\lim\limits_{x\to 3} \dfrac{x^2+x-12}{x-3}$

24. $\lim\limits_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$

25. $\lim\limits_{x\to -2} \dfrac{5}{x+2}$

26. $\lim\limits_{x\to 0} \dfrac{3}{\sqrt{3x+1}+1}$

27. $\lim\limits_{x\to 3} \dfrac{\sqrt{x+1}-1}{x}$

28. $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$

29. $\lim\limits_{x\to -2} \dfrac{|x+2|}{x+2}$

30. $\lim\limits_{x\to 0} (2z-8)^{1/3}$

31. $\lim\limits_{x\to 2} \dfrac{x^2-7x+10}{x-2}$

32. $\lim\limits_{x\to 0} \dfrac{5x^3+8x^2}{3x^4-16x^2}$

33. $\lim\limits_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1}$

34. $\lim\limits_{u\to 1} \dfrac{u^4-1}{u^3-1}$

35. $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x+3}-2}$

36. $\lim\limits_{x\to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$

37. $\lim\limits_{x\to -3} \dfrac{2-\sqrt{x^2-5}}{x+3}$

38. $\lim\limits_{x\to 0} (2\sin x-1)$

39. $\lim\limits_{x\to 0} \sin^2 x$

40. $\lim\limits_{x\to 0} \sec x$

41. $\lim\limits_{x\to 0} \dfrac{1+x+\sin x}{3\cos x}$

42. $\lim\limits_{x\to -\pi} \sqrt{x+4}\cos(x+\pi)$

43. $\lim\limits_{x\to -0.5^-} \sqrt{\dfrac{x+2}{x+1}}$

44. $\lim\limits_{x\to 1^+} \sqrt{\dfrac{x-1}{x+2}}$

45. $\lim\limits_{x\to -2^+} \left(\dfrac{x}{x+1}\right)\left(\dfrac{2x+5}{x^2+x}\right)$

46. $\lim\limits_{x\to 0^+} \dfrac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}$

47. $\lim\limits_{x\to -2^+} (x+3)\dfrac{|x+2|}{x+2}$

48. $\lim\limits_{x\to 1^+} \dfrac{\sqrt{2x}(x-1)}{|x-1|}$

49. $\lim\limits_{x\to 0^-} \dfrac{x}{\sin 3x}$

50. $\lim\limits_{\theta\to 0} \dfrac{\sin\sqrt{2}.\theta}{\sqrt{2}.\theta}$

51. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{4x}$

52. $\lim\limits_{x\to 0} \dfrac{\tan 2x}{x}$

53. $\lim\limits_{x\to 0} 6x^2(\cot x)(\csc 2x)$

54. $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\sin 2\theta}$

55. $\lim\limits_{h\to 0} \dfrac{\sin(\sin h)}{\sin h}$

56. $\lim\limits_{\theta\to 0} \dfrac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}$

57. $\lim\limits_{\theta\to \pi/4} \dfrac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta}$

58. $\lim\limits_{x\to \pi/2} \dfrac{\dfrac{1}{\sqrt{\sin x}}-1}{x+\dfrac{\pi}{2}}$

59. $\lim\limits_{x\to 1} \dfrac{x^3-7x^2+12x}{4-x}$

60. $\lim\limits_{x\to 4} \dfrac{x^3-7x^2+12x}{4-x}$

61. $\lim\limits_{x\to 1} \dfrac{1-x^2}{x^2-8x+7}$

62. $\lim\limits_{x\to 3} \dfrac{\sqrt{3x+16}-5}{x-3}$

63. $\lim\limits_{x\to 3} \dfrac{1}{x-3}\left(\dfrac{1}{\sqrt{x+1}}-\dfrac{1}{2}\right)$

64. $\lim\limits_{x\to 1/3} \dfrac{x-\dfrac{1}{3}}{(3x-1)^2}$

65. $\lim\limits_{x\to 3} \dfrac{x^4-81}{x-3}$

66. $\lim\limits_{x\to 1} \dfrac{x^5-1}{x-1}$

67. $\lim\limits_{x\to 81} \dfrac{\sqrt[4]{x}-3}{x-81}$

68. $\lim\limits_{x\to 1} \dfrac{\sqrt[3]{x}-1}{x-1}$

69. $\lim\limits_{x\to 2} \dfrac{x^5-32}{x-2}$

70. $\lim\limits_{x\to 1} \dfrac{x^6-1}{x-1}$

71. $\lim\limits_{x\to -1} \dfrac{x^7+1}{x+1}$

72. $\lim\limits_{x\to a} \dfrac{x^5-a^5}{x-a}$

73. $\lim\limits_{x\to a} \dfrac{x^n-a^n}{x-a} \quad n\in\mathbb{Z}^+$

74. $\lim\limits_{h\to 0} \dfrac{100}{(10h-1)^{11}+2}$

75. $\lim\limits_{h\to 0} \dfrac{(5+h)^2-25}{h}$

76. $\lim\limits_{x\to 3} \dfrac{\dfrac{1}{x^2+2x}-\dfrac{1}{15}}{x-3}$

77. $\lim\limits_{x\to 1} \dfrac{\sqrt{10x-9}-1}{x-1}$

78. $\lim\limits_{x\to 2} \left(\dfrac{1}{x-2}-\dfrac{2}{x^2-2x}\right)$

79. $\lim\limits_{x\to c} \dfrac{x^2-2cx+c^2}{x-c}$

80. $\lim\limits_{x\to -c} \dfrac{x^2+5cx+4c^2}{x^2+cx}$

81. $\lim\limits_{x\to 16} \dfrac{\sqrt[4]{x}-2}{x-16}$

82. $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$

83. $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{4x+5}-3}$

84. $\lim\limits_{x\to 4} \dfrac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}$

85. $\lim\limits_{x\to 0} \dfrac{x}{\sqrt{ax+1}-1} \quad (a\neq 0)$

86. $\lim\limits_{x\to \pi} \dfrac{\cos^2 x+3\cos x+2}{\cos x+1}$

87. $\lim\limits_{x\to \frac{3\pi}{2}} \dfrac{\sin^2 x+6\sin x+5}{\sin^2 x-1}$

88. $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\sin x-1}{\sqrt{\sin x}-1}$

89. $\lim\limits_{x\to 0} \dfrac{\dfrac{1}{2+\sin x}-\dfrac{1}{2}}{\sin x}$

90. $\lim\limits_{x\to 0} \dfrac{e^{2x}-1}{e^x-1}$

91. $\lim\limits_{x\to \frac{\pi}{4}} \csc x$

92. $\lim\limits_{x\to 4} \dfrac{x-5}{\left(x^2-10x+24\right)^2}$

93. $\lim\limits_{x\to 0} \dfrac{\cos x-1}{\sin^2 x}$

94. $\lim\limits_{x\to 0} \dfrac{1-\cos^2 x}{\sin x}$

95. $\lim\limits_{x\to 0} \dfrac{x^3-5x^2}{x^2}$

96. $\lim\limits_{x\to 5} \dfrac{4x^2-100}{x-5}$

97. $\lim_{x\to 3} \dfrac{\sqrt{9-6x+x^2}}{x-3}$

98. $\lim_{x\to 3} \dfrac{\sqrt{9+6x+x^2}}{x-3}$

99. $\lim_{x\to 3} \dfrac{\sqrt{x^2-9}}{x-3}$

100. $\lim_{x\to \frac{4\pi}{3}} \sin x$

101. $\lim_{x\to \frac{2\pi}{3}} \cos x$

102. $\lim_{x\to \frac{7\pi}{4}} \sin x$

103. $\lim_{x\to 1} \dfrac{\sin\sqrt{1-x}}{\sqrt{1-x^2}}$

104. $\lim_{x\to 2} \dfrac{\sin\sqrt{2-x}}{\sqrt{4-x^2}}$

105. $\lim_{x\to 0} \dfrac{\sin\left(\sqrt{5}\,x\right)}{\sin\left(\sqrt{3}\,x\right)}$

106. $\lim_{x\to 0} \dfrac{\sin\left(\sqrt{15}\,x\right)}{\sin\left(\sqrt{3}\,x\right)}$

107. $\lim_{x\to 0^+} \dfrac{x-\sqrt{x}}{\sqrt{\sin x}}$

108. $\lim_{x\to 1} \dfrac{x-\sqrt{x}}{\sqrt{\sin x}}$

109. $\lim_{x\to \pi} \dfrac{x-\sqrt{x}}{\sqrt{\sin x}}$

110. $\lim_{x\to 0} e^{x^3}$

111. $\lim_{x\to 1} e^{x^2}$

112. $\lim_{x\to 1} e^{x^3-1}$

113. $\lim_{x\to -1} e^{x^3-1}$

114. $\lim_{x\to 2} \left(e^{x^2}-\ln x\right)$

115. $\lim_{x\to 1} \left(e^{x^2}-\ln x\right)$

116. $\lim_{x\to e} \ln x$

117. $\lim_{x\to e} \ln x^2$

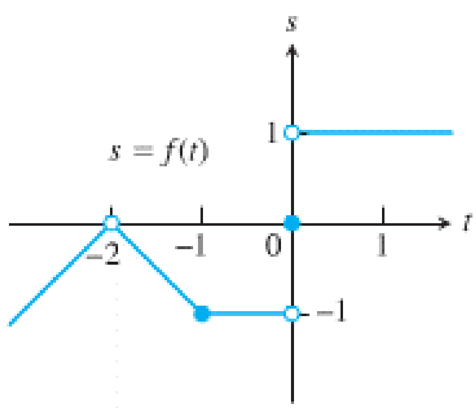
118. $\lim_{x\to 0^+} \ln x$

119. $\lim_{x\to 1} \dfrac{1}{\ln x}$

120. $\lim_{x\to e} \ln e^{2x}$

121. $\lim_{x\to 1} \ln e^{x^2}$

122. For the function $f(t)$ graphed, find the following limits or explain why they do not exist.



a) $\lim_{t\to -2} f(t)$ *b)* $\lim_{t\to -1} f(t)$ *c)* $\lim_{t\to 0} f(t)$ *d)* $\lim_{t\to -0.5} f(t)$

123. Suppose $\lim_{x\to c} f(x)=5$ *and* $\lim_{x\to c} g(x)=-2$. Find

a) $\lim_{x\to c} f(x)g(x)$

b) $\lim_{x\to c} 2f(x)g(x)$

c) $\lim_{x\to c} \left(f(x)+3g(x)\right)$

d) $\lim_{x\to c} \dfrac{f(x)}{f(x)-g(x)}$

124. Explain why the limits do not exist for $\lim\limits_{x\to 0}\frac{x}{|x|}$

(**125 – 126**) Evaluate the limit using the form $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$ for

125. $f(x)=x^2, \quad x=1$

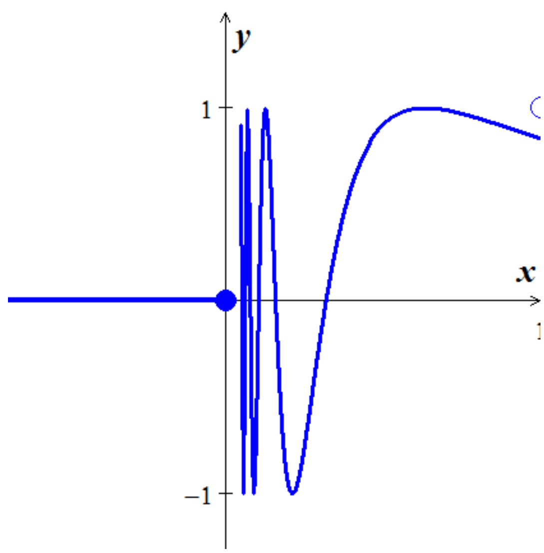
126. $f(x)=\sqrt{3x+1}, \quad x=0$

127. If $\lim\limits_{x\to 4}\frac{f(x)-5}{x-2}=1$, find $\lim\limits_{x\to 4} f(x)$

128. If $\lim\limits_{x\to 0}\frac{f(x)}{x^2}=1$, find $\lim\limits_{x\to 0} f(x)$ *and* $\lim\limits_{x\to 0}\frac{f(x)}{x}$

129. If $x^4 \le f(x) \le x^2 \quad -1\le x\le 1$ and $x^2 \le f(x) \le x^4 \quad x<-1$ *and* $x>1$. At what points c do you automatically know $\lim\limits_{x\to c} f(x)$? What can you say about the value of the limits at these points?

130. Let $f(x)=\begin{cases} 0, & x\le 0 \\ \sin\frac{1}{x}, & x>0 \end{cases}$

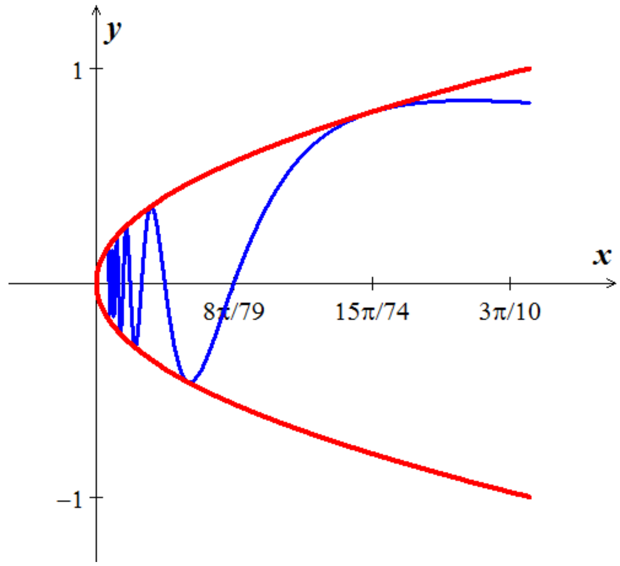


a) Does $\lim\limits_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

131. Let $g(x) = \sqrt{x}\,\sin\dfrac{1}{x}$

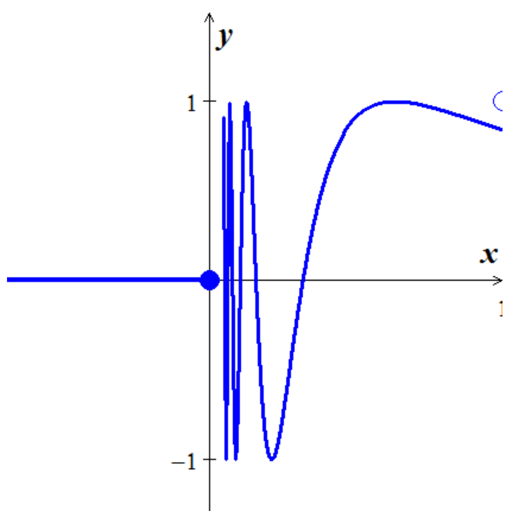


a) Does $\lim\limits_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

132. Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin\dfrac{1}{x}, & x > 0 \end{cases}$



a) Does $\lim\limits_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

133. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

a) $\lim\limits_{x \to -1^+} f(x) = 1$

b) $\lim\limits_{x \to 0^-} f(x) = 0$

c) $\lim\limits_{x \to 0^-} f(x) = 1$

d) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x)$

e) $\lim\limits_{x \to 0} f(x)$ *exists*

f) $\lim\limits_{x \to 0} f(x) = 0$

g) $\lim\limits_{x \to 0} f(x) = 1$

h) $\lim\limits_{x \to 1} f(x) = 1$

i) $\lim\limits_{x \to 1} f(x) = 0$

j) $\lim\limits_{x \to 2^-} f(x) = 2$

k) $\lim\limits_{x \to -1^-} f(x) = 0$ *does not exist*

l) $\lim\limits_{x \to 2^+} f(x) = 0$

Section 1.3 – Infinite Limits

Definitions

We say that $f(x)$ has the **limit L as x approaches infinity** and write $\lim\limits_{x\to\infty} f(x) = L$

If, $\forall \varepsilon > 0\ \exists N \ni \forall x, \quad x > M \quad \Rightarrow \quad |f(x) - L| < \varepsilon$

We say that $f(x)$ has the **limit L as x approaches *minus* infinity** and write $\lim\limits_{x\to-\infty} f(x) = L$

If, $\forall \varepsilon > 0\ \exists N \ni \forall x, \quad x < M \quad \Rightarrow \quad |f(x) - L| < \varepsilon$

***Basic Facts*:** $\lim\limits_{x\to\pm\infty} k = k$ *and* $\lim\limits_{x\to\pm\infty} \frac{1}{x} = 0$

Example

Find $\lim\limits_{x\to\infty} \dfrac{5x^2 + 8x - 3}{3x^2 + 2}$

Solution

$$\lim_{x\to\infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim_{x\to\infty} \frac{5 + \frac{8}{x} - \frac{3}{x^2}}{3 + \frac{2}{x^2}}$$

Divide by x^2

$$= \frac{5 + 0 - 0}{3 + 0}$$

$\lim\limits_{x\to\pm\infty} \frac{1}{x} = 0$

$$\boxed{= \frac{5}{3}}$$

Example

Find $\lim\limits_{x\to\infty} \dfrac{11x + 2}{2x^3 - 1}$

Solution

$$\lim_{x\to\infty} \frac{11x + 2}{2x^3 - 1} = \lim_{x\to\infty} \frac{\frac{11}{x^2} + \frac{2}{x^3}}{2 - \frac{1}{x^3}}$$

$\lim\limits_{x\to\pm\infty} \frac{1}{x} = 0$

$$= \frac{0 + 0}{2 - 0}$$

$$\boxed{= 0}$$

Vertical Asymptote (*VA*) - *Think Domain*

The line $x = a$ is a ***vertical asymptote*** for the graph of a function f if

$$\lim_{x\to a^+} f(x) \to \pm\infty \quad or \quad \lim_{x\to a^-} f(x) \to \pm\infty$$

As $\boldsymbol{x}$ approaches $\boldsymbol{a}$ from either the left or the right

$$\lim_{x\to 0^+} \frac{1}{x} \to \infty \quad or \quad \lim_{x\to 0^-} \frac{1}{x} \to -\infty$$

Example

Find $\lim_{x\to 3^+} \frac{2-5x}{x-3}$ *and* $\lim_{x\to 3^-} \frac{2-5x}{x-3}$

Solution

$$\lim_{x\to 3^+} \frac{2-5x}{x-3} = \frac{2-5(3)}{3^+ - 3} \begin{matrix} \to -13 \\ \to \textit{positive and approaches } 0 \end{matrix}$$

$$\underline{= -\infty}$$

$$\lim_{x\to 3^-} \frac{2-5x}{x-3} = \frac{2-5(3)}{3^- - 3} \begin{matrix} \to -13 \\ \to \textit{negative and approaches } 0 \end{matrix}$$

$$\underline{= \infty}$$

Example

Find $\lim_{x\to -4^+} \frac{-x^3+5x^2-6x}{-x^3-4x^2}$

Solution

$$\lim_{x\to -4^+} \frac{-x^3+5x^2-6x}{-x^3-4x^2} = \frac{168}{0}$$

$$\frac{-x^3+5x^2-6x}{-x^3-4x^2} = \frac{(x-2)(x-3)}{x(x+4)} \begin{matrix} \to \textit{positive} \\ \to \textit{negative and approaches } 0 \end{matrix}$$

$$\underline{= -\infty}$$

Example

Let $f(x)=\dfrac{x^2-4x+3}{x^2-1}$, determine the following limits and find the vertical asymptotes of *f*.

a) $\lim\limits_{x\to 1} f(x)$ *b)* $\lim\limits_{x\to -1^-} f(x)$ *c)* $\lim\limits_{x\to -1^+} f(x)$

Solution

a) $\lim\limits_{x\to 1}\dfrac{x^2-4x+3}{x^2-1}=\dfrac{0}{0}=\lim\limits_{x\to 1}\dfrac{(x-1)(x-3)}{(x-1)(x+1)}$

$$=\lim_{x\to 1}\frac{x-3}{x+1}$$

$$=-1$$

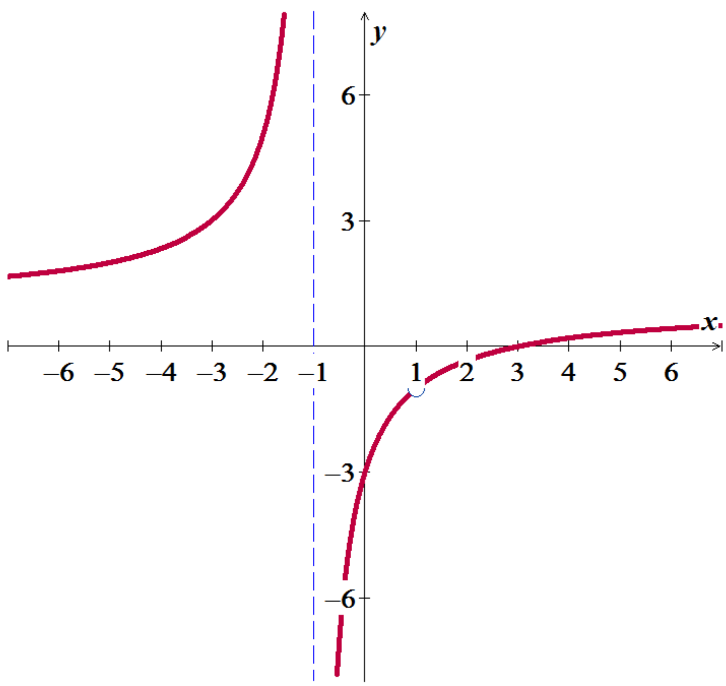
The vertical asymptote: $x=-1$, while the hole is $(1,\ -1)$

b) $\lim\limits_{x\to -1^-} f(x)=\lim\limits_{x\to -1^-}\dfrac{x-3}{x+1}$ $\begin{matrix}\to \textit{negative} \\ \to \textit{negative and approaches } 0\end{matrix}$

$$=\infty$$

c) $\lim\limits_{x\to -1^+} f(x)=\lim\limits_{x\to -1^+}\dfrac{x-3}{x+1}$ $\begin{matrix}\to \textit{negative} \\ \to \textit{positive and approaches } 0\end{matrix}$

$$=-\infty$$



Example

Find $\lim\limits_{\theta \to 0^+} \cot\theta$ *and* $\lim\limits_{\theta \to 0^-} \cot\theta$

Solution

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\cot 0 = \frac{1}{0}$$

As $\theta \to 0^+$ $\cos\theta > 0;$ $\sin\theta > 0$

$$\lim_{\theta \to 0^+} \cot\theta = \infty$$

As $\theta \to 0^-$ $\cos\theta > 0;$ $\sin\theta < 0$

$$\lim_{\theta \to 0^+} \cot\theta = -\infty$$

Exercises   *Section* 1.3 – Infinite Limits

(**1 – 50**) ***Find*** the limit

1. $\lim\limits_{x\to 5}\dfrac{x-7}{x(x-5)^2}$
2. $\lim\limits_{x\to -5^+}\dfrac{x-5}{x+5}$
3. $\lim\limits_{x\to 3^-}\dfrac{x-4}{x^2-3x}$
4. $\lim\limits_{x\to 0^+}\dfrac{1}{3x}$
5. $\lim\limits_{x\to -5^-}\dfrac{3x}{2x+10}$
6. $\lim\limits_{x\to 0}\dfrac{1}{x^{2/3}}$
7. $\lim\limits_{x\to 0^-}\dfrac{1}{3x^{1/3}}$
8. $\lim\limits_{x\to\left(-\frac{\pi}{2}\right)^+}\sec x$
9. $\lim\limits_{\theta\to 0^-}(1+\csc\theta)$
10. $\lim\limits_{\theta\to 0^+}\csc\theta$
11. $\lim\limits_{x\to 0^+}(-10\cot x)$
12. $\lim\limits_{\theta\to \frac{\pi}{2}^+}\frac{1}{3}\tan\theta$
13. $\lim\limits_{x\to 2^+}\dfrac{1}{x-2}$
14. $\lim\limits_{x\to 2^-}\dfrac{1}{x-2}$
15. $\lim\limits_{x\to 2}\dfrac{1}{x-2}$
16. $\lim\limits_{x\to 3^+}\dfrac{2}{(x-3)^3}$
17. $\lim\limits_{x\to 3^-}\dfrac{2}{(x-3)^3}$
18. $\lim\limits_{x\to 3}\dfrac{2}{(x-3)^3}$
19. $\lim\limits_{x\to 4^+}\dfrac{x-5}{(x-4)^2}$
20. $\lim\limits_{x\to 4^-}\dfrac{x-5}{(x-4)^2}$
21. $\lim\limits_{x\to 4}\dfrac{x-5}{(x-4)^2}$
22. $\lim\limits_{x\to 1^+}\dfrac{x-2}{(x-1)^3}$
23. $\lim\limits_{x\to 1^-}\dfrac{x-2}{(x-1)^3}$
24. $\lim\limits_{x\to 1}\dfrac{x-2}{(x-1)^3}$
25. $\lim\limits_{x\to 3^+}\dfrac{(x-1)(x-2)}{x-3}$
26. $\lim\limits_{x\to 3^-}\dfrac{(x-1)(x-2)}{x-3}$
27. $\lim\limits_{x\to 3}\dfrac{(x-1)(x-2)}{x-3}$
28. $\lim\limits_{x\to 2^+}\dfrac{x-4}{x(x+2)}$
29. $\lim\limits_{x\to 2^-}\dfrac{x-4}{x(x+2)}$
30. $\lim\limits_{x\to 2}\dfrac{x-4}{x(x+2)}$
31. $\lim\limits_{x\to 2^+}\dfrac{x^2-4x+3}{(x-2)^2}$
32. $\lim\limits_{x\to 2^-}\dfrac{x^2-4x+3}{(x-2)^2}$
33. $\lim\limits_{x\to 2}\dfrac{x^2-4x+3}{(x-2)^2}$
34. $\lim\limits_{x\to -2^+}\dfrac{x^3-5x^2+6x}{x^4-4x^2}$
35. $\lim\limits_{x\to -2^-}\dfrac{x^3-5x^2+6x}{x^4-4x^2}$
36. $\lim\limits_{x\to -2}\dfrac{x^3-5x^2+6x}{x^4-4x^2}$
37. $\lim\limits_{u\to 0^+}\dfrac{u-1}{\sin u}$
38. $\lim\limits_{x\to 0^-}\dfrac{2}{\tan x}$
39. $\lim\limits_{x\to 1^+}\dfrac{x^2-5x+6}{x-1}$
40. $\lim\limits_{x\to 4}\dfrac{x-5}{\left(x^2-10x+24\right)^2}$
41. $\lim\limits_{x\to 2\pi^-}\csc x$
42. $\lim\limits_{x\to 0^+}e^{\sqrt{x}}$
43. $\lim\limits_{x\to \frac{\pi}{2}^-}\dfrac{1+\sin x}{\cos x}$
44. $\lim\limits_{x\to \frac{\pi}{2}^+}\dfrac{1+\sin x}{\cos x}$
45. $\lim\limits_{x\to 0^-}\dfrac{e^x}{1-e^x}$
46. $\lim\limits_{x\to 0^+}\dfrac{e^x}{1-e^x}$
47. $\lim\limits_{x\to 1^-}\dfrac{x}{\ln x}$
48. $\lim\limits_{x\to 0^+}\dfrac{x}{\ln x}$
49. $\lim\limits_{x\to 0^-}\dfrac{2e^x+5e^{3x}}{e^{2x}-e^{3x}}$
50. $\lim\limits_{x\to 0^+}\dfrac{2e^x+5e^{3x}}{e^{2x}-e^{3x}}$

51. Let $f(x)=\dfrac{x^2-7x+12}{x-a}$

a) For what values of a, if any, does $\lim\limits_{x\to a^+} f(x)$ equal a finite number?

b) For what values of a, if any, does $\lim\limits_{x\to a^+} f(x)=\infty$?

c) For what values of a, if any, does $\lim\limits_{x\to a^+} f(x)=-\infty$?

52. Analyze $\lim\limits_{x\to 1^+} \sqrt{\dfrac{x-1}{x-3}}$ and $\lim\limits_{x\to 1^-} \sqrt{\dfrac{x-1}{x-3}}$

Section 1.4 – Limits at Infinity

Notation	***Terminology***
$f(x) \to \infty$	$f(x)$ increases without bound (can be made as large positive as desired)
$f(x) \to -\infty$	$f(x)$ decreases without bound (can be made as large negative as desired)

Horizontal Asymptote (*HA*)

The line $y = b$ is a ***horizontal asymptote*** for the graph of a function f if

$$\lim_{x\to\infty} f(x) = b \quad \textit{or} \quad \lim_{x\to-\infty} f(x) = b$$

Let $f(x) = \dfrac{p(x)}{q(x)}$

$$= \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}$$

$$= \frac{a_n x^n}{b_m x^m}$$

1. If the degree of numerator is less than of denominator $(n < m) \Rightarrow y = 0$

$$y = \frac{2x+1}{4x^2+5}$$

HA: $\boxed{y = 0}$

2. If the degree of numerator is equal of denominator $(n = m) \Rightarrow y = \dfrac{a_n}{b_m}$

$$y = \frac{2x^2+1}{4x^2+5}$$

HA: $y = \frac{2}{4} = \boxed{\frac{1}{2}}$

3. If the degree of numerator is greater than of denominator $(n > m) \Rightarrow$ No horizontal asymptote

$$y = \frac{2x^3+1}{4x^2+5}$$

$\Rightarrow$ *No HA*

Example

Find the horizontal asymptotes of the graph of $f(x) = \dfrac{x^3 - 2}{|x|^3 + 1}$

Solution

For $x \ge 0$

$$\lim_{x\to\infty} \frac{x^3-2}{|x|^3+1} = \lim_{x\to\infty} \frac{x^3}{x^3}$$

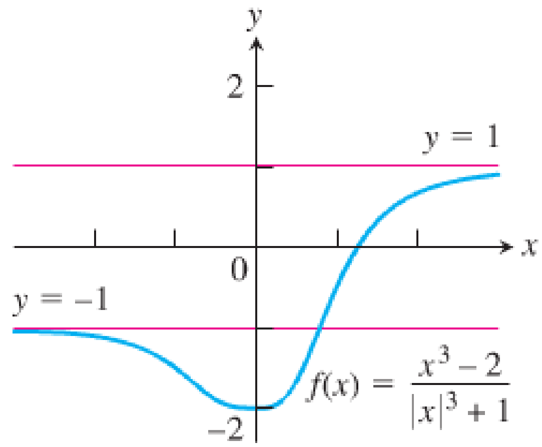
$$= 1$$

For $x \le 0$

$$\lim_{x\to\infty} \frac{x^3-2}{|x|^3+1} = \lim_{x\to-\infty} \frac{x^3}{(-x)^3}$$

$$= -1$$

The ***HA*** are $y = \pm 1$



Example

Find $\displaystyle \lim_{x\to\infty} \sin\left(\frac{1}{x}\right)$

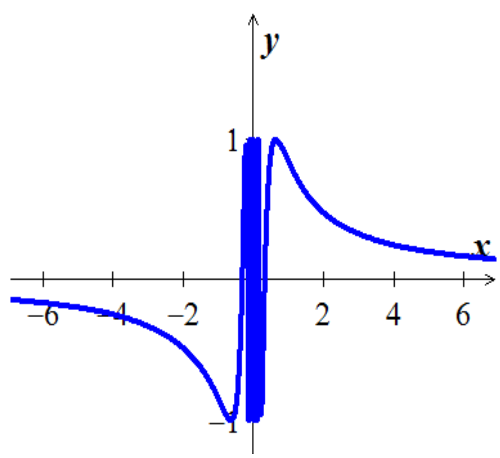
Solution

Let $t = \dfrac{1}{x}$

$\Rightarrow \quad t \to 0 \quad as \quad x \to \infty$

$$\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = \lim_{t\to 0} \sin t$$

$$= 0$$



Example

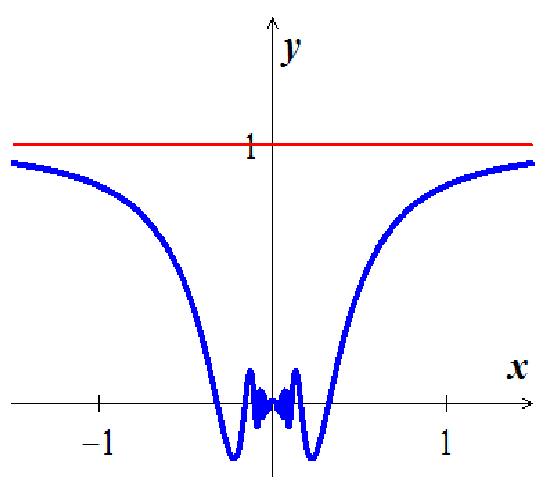
Find $\displaystyle \lim_{x\to\pm\infty} x \sin\left(\frac{1}{x}\right)$

Solution

Let $t = \dfrac{1}{x} \Rightarrow x = \dfrac{1}{t}$

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{t\to 0^+} \frac{\sin t}{t}$$

$$= 1$$

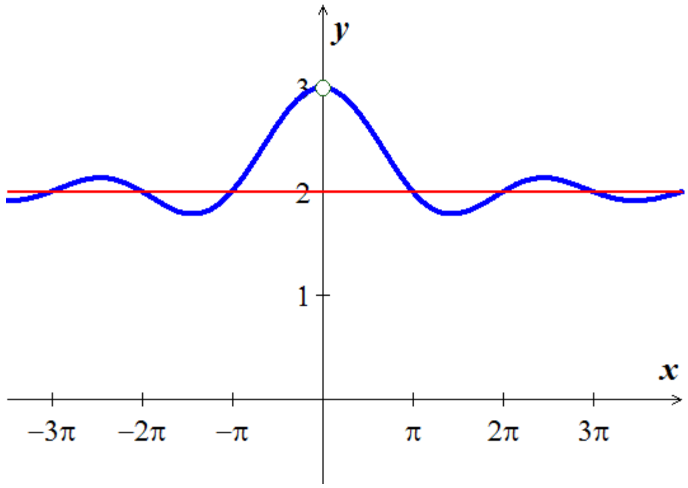


$$\lim_{x\to-\infty} x\sin\left(\frac{1}{x}\right) = \lim_{t\to0^-} \frac{\sin t}{t}$$

$$= 1$$

Example

Find the horizontal asymptote of $y = 2 + \dfrac{\sin x}{x}$



Solution

Since $0 \le \left|\dfrac{\sin x}{x}\right| \le \left|\dfrac{1}{x}\right|$

$$\lim_{x\to\pm\infty} \left|\frac{1}{x}\right| = 0$$

$$\lim_{x\to\pm\infty} \frac{\sin x}{x} = 0$$

$$\lim_{x\to\pm\infty} \left(2 + \frac{\sin x}{x}\right) = 2 + 0$$

$$= 2$$

The ***HA*** is $y = 2$

Example

Find $\lim\limits_{x\to\infty}\left(x-\sqrt{x^2+16}\right)$

Solution

$$\lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right)=\lim_{x\to\infty}\left(x-\sqrt{x^2+16}\right)\frac{x+\sqrt{x^2+16}}{x+\sqrt{x^2+16}} \qquad (a-b)(a+b)=a^2-b^2$$

$$=\lim_{x\to\infty}\frac{x^2-\left(x^2+16\right)}{x+\sqrt{x^2+16}}$$

$$=\lim_{x\to\infty}\frac{x^2-x^2-16}{x+\sqrt{x^2+16}}$$

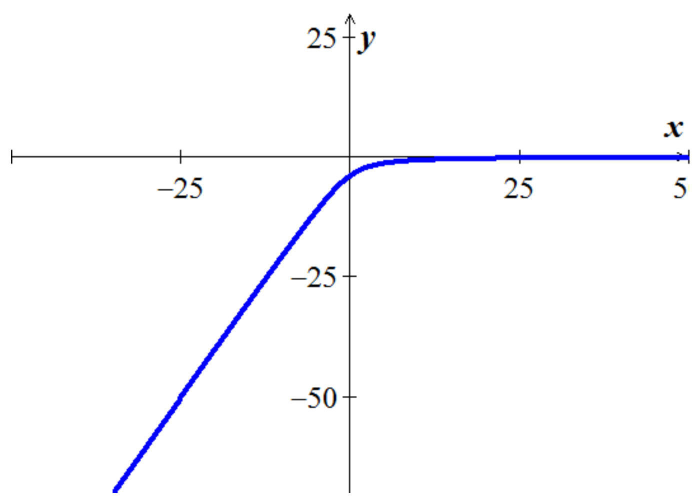
$$=\lim_{x\to\infty}\frac{-16}{x+\sqrt{x^2+16}}$$

$$=\lim_{x\to\infty}\frac{-\frac{16}{x}}{\frac{x}{x}+\sqrt{\frac{x^2}{x^2}+\frac{16}{x^2}}}$$

$$=\lim_{x\to\infty}\frac{-\frac{16}{x}}{1+\sqrt{1+\frac{16}{x^2}}}$$

$$=\frac{0}{1+\sqrt{1+0}}$$

$$\underline{=0}$$



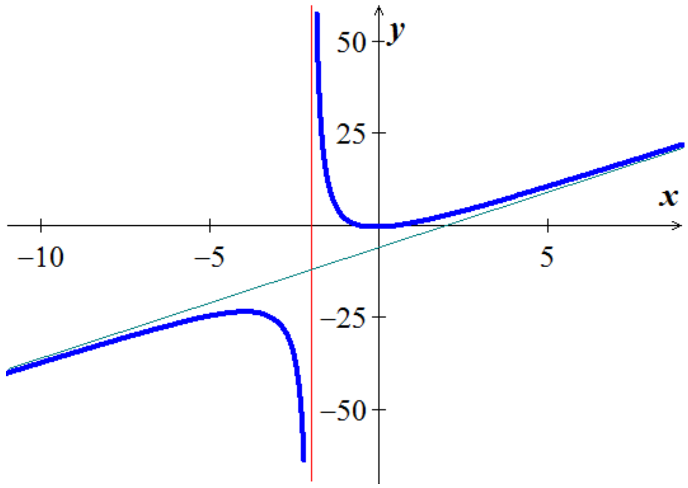
Slant or *Oblique* Asymptotes

When the degree of the numerator is one greater than the degree of the numerator, the graph has a ***slant*** or ***oblique*** asymptote and it is a line $y = ax + b, \quad a \ne 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y = \frac{3x^2 - 1}{x + 2}$$

$$\begin{array}{r|l} & \quad\ \ 3x - 6 \\ x + 2 & 3x^2 + 0x - 1 \\ & \underline{3x^2 + 6x} \\ & \qquad -6x - 1 \\ & \qquad \underline{-6x - 12} \\ & \qquad\quad R = 11 \end{array}$$

$$y = \frac{3x^2 - 1}{x + 2} \boxed{= (3x - 6) + \frac{11}{x + 2}}$$



The ***oblique asymptote*** is the line ***y* = 3*x* - 6**

Example

Find the horizontal and vertical asymptotes of the curve $y = \frac{x+3}{x+2}$

Solution

HA: $y \to \frac{x}{x} = 1 \quad \Rightarrow \quad \boxed{y = 1}$

VA: $x + 2 = 0 \quad \Rightarrow \quad \boxed{x = -2}$

Example

Find the horizontal and vertical asymptotes of the curve $f(x) = -\frac{8}{x^2 - 4}$

Solution

$$y \to \lim_{x \to \infty} -\frac{8}{x^2} = 0$$

HA: $\boxed{y = 0}$

VA: $x^2 - 4 = 0 \quad \Rightarrow \quad \boxed{x = \pm 2}$

$$\lim_{x \to 2^+} f(x) = -\infty \quad \textit{and} \quad \lim_{x \to 2^-} f(x) = \infty$$

Infinite Limits

The limit has a value of infinity or minus infinity, such a function $f(x) = \frac{1}{x}$. It is convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \to 0^+$.

Definition

We say $\lim_{x \to 0^+} f(x) = \infty$

That $\lim_{x \to 0^+} \frac{1}{x}$ doesn't exist because $\frac{1}{x}$ becomes arbitrary large and positive as $x \to 0^+$.

We say $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty$

That $\lim_{x \to 0^-} \frac{1}{x}$ doesn't exist because $\frac{1}{x}$ becomes arbitrary large and negative as $x \to 0^-$.

Example

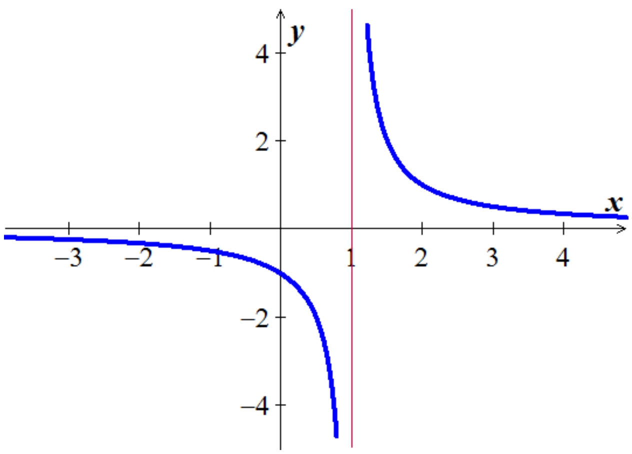
Find $\lim_{x \to 1^+} \frac{1}{x-1}$ *and* $\lim_{x \to 1^-} \frac{1}{x-1}$

Solution

As $x \to 1^+ \quad \Rightarrow \quad x - 1 \to 0^+$

$$\lim_{x \to 1^+} \frac{1}{x-1} = \infty$$

$$\lim_{x \to 1^-} \frac{1}{x-1} = -\infty$$



- $$\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)}$$
$$= \lim_{x\to 2} \frac{(x-2)}{(x+2)}$$
$$= \frac{0}{4}$$
$$= 0$$

- $$\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)}$$
$$= \lim_{x\to 2} \frac{1}{x+2}$$
$$= \frac{1}{4}$$

- $$\lim_{x\to 2^+} \frac{x-3}{x^2-4} = \lim_{x\to 2^+} \frac{x-3}{(x-2)(x+2)}$$
$$= -\infty$$

- $$\lim_{x\to 2^-} \frac{x-3}{x^2-4} = \lim_{x\to 2^-} \frac{x-3}{(x-2)(x+2)}$$
$$= \infty$$

- $$\lim_{x\to 2} \frac{x-3}{x^2-4} = \lim_{x\to 2} \frac{x-3}{(x-2)(x+2)}$$
$$= \text{doesn't exist} \qquad \nexists$$

Exercises *Section* 1.4 – Limits at Infinity

(1 – 8) Find the limit as $x \to \infty$ and as $x \to -\infty$ of

1. $h(x) = \dfrac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}}$

2. $f(x) = \dfrac{2x+3}{5x+7}$

3. $f(x) = \dfrac{2x^3+7}{x^3 - x^2 + x + 7}$

4. $f(x) = \dfrac{x+1}{x^2+3}$

5. $f(x) = \dfrac{7x^3}{x^3 - 3x^2 + 6x}$

6. $f(x) = \dfrac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

7. $f(x) = \dfrac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$

(8 – 60) Evaluate the limits

8. $\lim\limits_{x\to\infty} x^{12}$

9. $\lim\limits_{x\to-\infty} 3x^9$

10. $\lim\limits_{x\to-\infty} x^{-8}$

11. $\lim\limits_{x\to-\infty} x^{-9}$

12. $\lim\limits_{x\to-\infty} 2x^{-6}$

13. $\lim\limits_{x\to\infty} \left(3x^{12} - 9x^7\right)$

14. $\lim\limits_{x\to-\infty} \left(3x^7 + x^2\right)$

15. $\lim\limits_{x\to-\infty} \left(-2x^{16} + 2\right)$

16. $\lim\limits_{x\to-\infty} \left(2x^{-6} + 4x^5\right)$

17. $\lim\limits_{x\to-\infty} \dfrac{\cos x}{3x}$

18. $\lim\limits_{x\to\infty} \dfrac{x + \sin x}{2x + 7 - 5\sin x}$

19. $\lim\limits_{x\to\infty} \sqrt{\dfrac{8x^2-3}{2x^2+x}}$

20. $\lim\limits_{x\to-\infty} \left(\dfrac{x^2+x-1}{8x^2-3}\right)^{1/3}$

21. $\lim\limits_{x\to\infty} \dfrac{2\sqrt{x} + x^{-1}}{3x-7}$

22. $\lim\limits_{x\to\infty} \dfrac{x^{-1} + x^{-4}}{x^{-2} + x^{-3}}$

23. $\lim\limits_{x\to-\infty} \dfrac{4-3x^3}{\sqrt{x^6+9}}$

24. $\lim\limits_{x\to\infty} \left(\sqrt{x^2+3x} - \sqrt{x^2-2x}\right)$

25. $\lim\limits_{x\to-\infty} \left(\sqrt{x^2+3} + x\right)$

26. $\lim\limits_{x\to\infty} \dfrac{2x-3}{4x+10}$

27. $\lim\limits_{x\to\infty} \dfrac{x^4-1}{x^5+2}$

28. $\lim\limits_{x\to-\infty} \left(-3x^3 + 5\right)$

29. $\lim\limits_{x\to\infty} \left(e^{-2x} + \dfrac{2}{x}\right)$

30. $\lim\limits_{x\to\infty} \dfrac{1}{\ln x + 1}$

31. $\lim\limits_{x\to\infty}\left(3+\dfrac{10}{x^2}\right)$

32. $\lim\limits_{x\to\infty}\left(5+\dfrac{1}{x}+\dfrac{10}{x^2}\right)$

33. $\lim\limits_{x\to\infty}\dfrac{4x^2+2x+3}{x^2}$

34. $\lim\limits_{x\to\infty}\left(5+\dfrac{100}{x}+\dfrac{\sin^4 x^3}{x^2}\right)$

35. $\lim\limits_{\theta\to\infty}\dfrac{\cos\theta}{\theta^2}$

36. $\lim\limits_{\theta\to\infty}\dfrac{\cos\theta^5}{\sqrt{\theta}}$

37. $\lim\limits_{x\to\infty}\dfrac{4x}{20x+1}$

38. $\lim\limits_{x\to-\infty}\dfrac{4x}{20x+1}$

39. $\lim\limits_{x\to\infty}\dfrac{3x^2-7}{x^2+5x}$

40. $\lim\limits_{x\to-\infty}\dfrac{3x^2-7}{x^2+5x}$

41. $\lim\limits_{x\to\infty}\dfrac{6x^2-9x+8}{3x^2+2}$

42. $\lim\limits_{x\to-\infty}\dfrac{6x^2-9x+8}{3x^2+2}$

43. $\lim\limits_{x\to\infty}\dfrac{4x^2-7}{8x^2+5x+2}$

44. $\lim\limits_{x\to-\infty}\dfrac{4x^2-7}{8x^2+5x+2}$

45. $\lim\limits_{x\to\infty}\dfrac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

46. $\lim\limits_{x\to-\infty}\dfrac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

47. $\lim\limits_{x\to\infty}\dfrac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

48. $\lim\limits_{x\to-\infty}\dfrac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

49. $\lim\limits_{x\to\infty}16x^2\left(4x^2-\sqrt{16x^4+1}\right)$

50. $\lim\limits_{x\to-\infty}16x^2\left(4x^2-\sqrt{16x^4+1}\right)$

51. $\lim\limits_{x\to\infty}\dfrac{x-1}{x^{2/3}-1}$

52. $\lim\limits_{x\to-\infty}\dfrac{x-1}{x^{2/3}-1}$

53. $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2+2x+6}-3}{x-1}$

54. $\lim\limits_{x\to\infty}\dfrac{\left|1-x^2\right|}{x(x+1)}$

55. $\lim\limits_{x\to\infty}\left(\sqrt{|x|}-\sqrt{|x-1|}\right)$

56. $\lim\limits_{x\to\infty}\dfrac{\tan^{-1}x}{x}$

57. $\lim\limits_{x\to\infty}\dfrac{\cos x}{e^{3x}}$

58. $\lim\limits_{x\to0}\dfrac{2e^x+10e^{-x}}{e^x+e^{-x}}$

59. $\lim\limits_{x\to\infty}\dfrac{2e^x+10e^{-x}}{e^x+e^{-x}}$

60. $\lim\limits_{x\to-\infty}\dfrac{2e^x+10e^{-x}}{e^x+e^{-x}}$

(61 – 64) Graph the rational function and include the equations of the asymptotes

61. $y=\dfrac{1}{2x+4}$ **62.** $y=\dfrac{2x}{x+1}$ **63.** $y=\dfrac{x^2}{x-1}$ **64.** $y=\dfrac{x^3+1}{x^2}$

65. Let $f(x)=\dfrac{x^2-5x+6}{x^2-2x}$

a) Analyze $\lim\limits_{x\to 0^-} f(x)$, $\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 2^-} f(x)$, and $\lim\limits_{x\to 2^+} f(x)$

b) Does the graph of f have any vertical asymptotes? Explain?

(**66 – 85**) Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

66. $y=\dfrac{3x}{1-x}$

67. $y=\dfrac{x^2}{x^2+9}$

68. $y=\dfrac{x-2}{x^2-4x+3}$

69. $y=\dfrac{5x-1}{1-3x}$

70. $y=\dfrac{3}{x-5}$

71. $y=\dfrac{x^3-1}{x^2+1}$

72. $y=\dfrac{3x^2-27}{(x+3)(2x+1)}$

73. $y=\dfrac{x^3+3x^2-2}{x^2-4}$

74. $y=\dfrac{x-3}{x^2-9}$

75. $y=\dfrac{6}{\sqrt{x^2-4x}}$

76. $f(x)=\dfrac{4x^3+1}{1-x^3}$

77. $f(x)=\dfrac{x+1}{\sqrt{9x^2+x}}$

78. $f(x)=1-e^{-2x}$

79. $f(x)=\dfrac{1}{\ln x^2}$

80. $f(x)=\dfrac{1}{\tan^{-1} x}$

81. $f(x)=\dfrac{2x^2+6}{2x^2+3x-2}$

82. $f(x)=\dfrac{3x^2+2x-1}{4x+1}$

83. $f(x)=\dfrac{9x^2+4}{(2x-1)^2}$

84. $f(x)=\dfrac{1+x-2x^2-x^3}{x^2+1}$

85. $f(x)=\dfrac{x(x+2)^3}{3x^2-4x}$

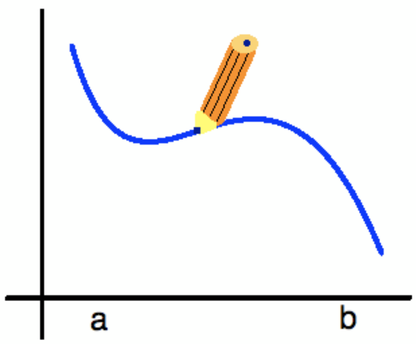
Section 1.5 – Continuity

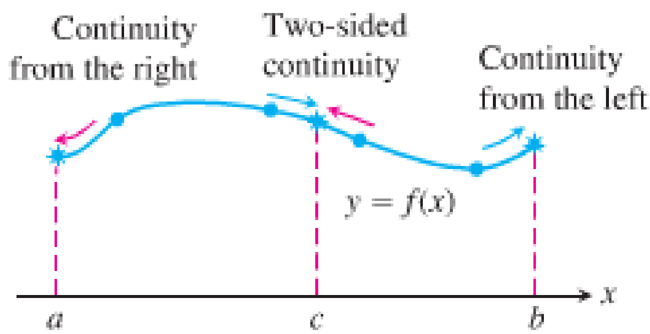
Definition of Continuity

Let c be a number in the interval (a, b), and let f be a function whose domain contains the interval (a, b). The function f is continuous at the point c if the following conditions are true.

1. $f(c)$ is defined
2. $\lim\limits_{x\to c} f(x)$ *exists*
3. $\lim\limits_{x\to c} f(x) = f(c)$

If f is continuous at every point in the interval (a, b), then it is continuous on an open interval (a, b)





Definition

Interior point: A function $y = f(x)$ is **continuous at an interior point *c*** of its domain if

$$\lim_{x\to c} f(x) = f(c)$$

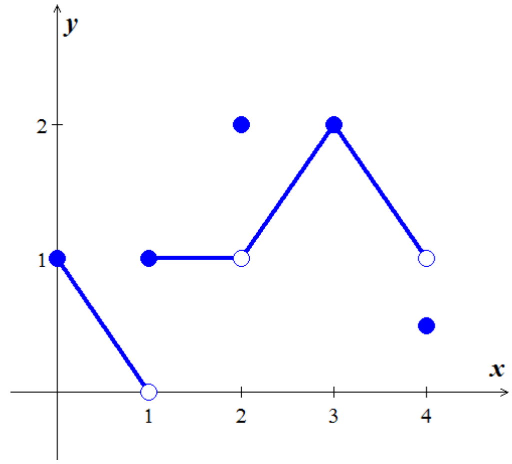
Endpoint: A function $y = f(x)$ is **continuous at a left point *a*** or is **continuous at a right point *b*** of its domain if

$$\lim_{x\to a^+} f(x) = f(a) \quad \textit{or} \quad \lim_{x\to b^-} f(x) = f(b), \quad \textit{respectively}$$

If a function f is not continuous at a point c, we say that f is ***discontinuous*** at c. (is a ***point of discontinuity***)

Example

Find the points at which the function f is continuous and the points at which f is not continuous



Solution

The function f is continuous at every point in its domain [0, 4] except at $x=1$, $x=2$, and $x=4$. At these points, there are breaks in the graph.

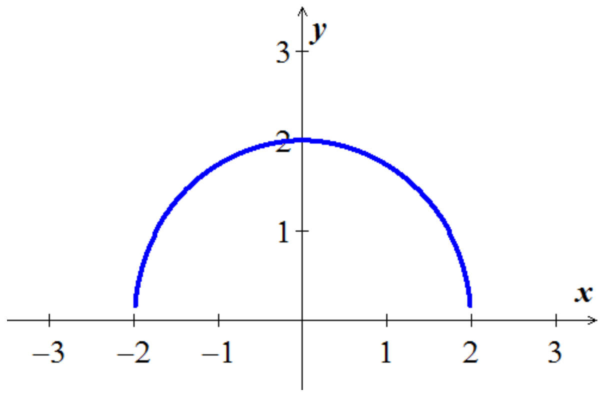
$x=0$	$\lim\limits_{x\to 0^+} f(x) = f(0) = 1$	f is continuous @ $x=0$
$x=1$	$\lim\limits_{x\to 1} f(x)$ *doesn't exist*	f is discontinuous @ $x=1$
$x=2$	$\lim\limits_{x\to 2} f(x) = 1,\ but\ \ 1 \neq f(2)$	f is discontinuous @ $x=2$
$x=3$	$\lim\limits_{x\to 3} f(x) = f(3) = 2$	f is continuous @ $x=3$
$x=4$	$\lim\limits_{x\to 4^-} f(x) = 1,\ but\ \ 1 \neq f(4)$	f is discontinuous @ $x=4$
$c<0,\ c>4$	These points are not in the domain of f.	f is discontinuous
$0<c<4,\ \ c \neq 1, 2$	$\lim\limits_{x\to c} f(x) = f(c)$	

Example

At what points the function $f(x)=\sqrt{4-x^2}$ is continuous?

Solution

The function is continuous at every point of its domain [–2, 2].
Including $x=-2$, where f is right-continuous, and $x=2$, where f is left-continuous.



Continuous Functions

A function is ***continuous on an interval*** iff it is continuous at every point of the interval. A ***continuous function*** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval.

Example

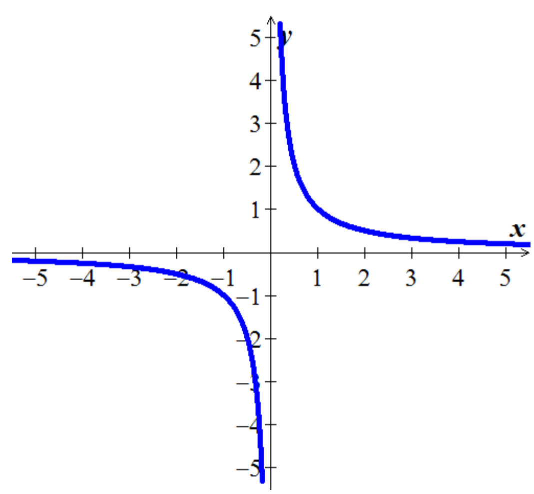
Determine at which points do the function $f(x)=\frac{1}{x}$ is continuous and discontinuous

Solution

The function $f(x)$ is a continuous function because it is continuous at every point of its domain.

It has a point of discontinuity at $x=0$, however, because it is not defined.

It is discontinuous on any interval containing $x=0$



Theorem – Properties of Continuous Functions

If the functions f *and* g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

Sums and Differences	$f \pm g$
Constant multiples	$k \cdot g$, for any number k.
Products	$f \cdot g$
Quotients	$\dfrac{f}{g}$
Powers	f^n ***n*** *a positive integer*
Roots	$\sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer

Proof

$$\lim_{x \to c} (f+g)(x) = \lim_{x \to c} \left(f(x) + g(x)\right)$$

$$= \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

$$= f(c) + g(c)$$

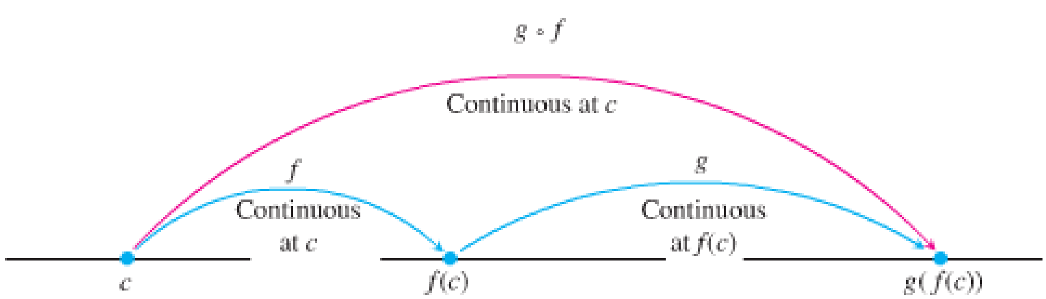
$$= (f+g)(c)$$

This shows that $f + g$ is continuous

Composites

All composites of continuous functions are continuous.

If $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is continuous at $x = c$



Example

Show that $y=\sqrt{x^2-2x-5}$ is continuous everywhere on its domain

Solution

Let $\begin{cases} f(x)=x^2-2x-5, & \mathit{Domain}:\mathbb{R} \\ g(x)=\sqrt{x} & \mathit{Domain}:[0,\ \infty) \end{cases}$

$\therefore$ The function y is continuous on $[0,\ \infty)$

Example

Show that $y=\left|\frac{x\sin x}{x^2+2}\right|$ is continuous everywhere on its domain

Solution

Let $\begin{cases} x\sin x & \mathit{Domain}:\mathbb{R} \\ x^2+2 & \mathit{Domain}:\mathbb{R} \end{cases}$

$\therefore$ The function is the composite of a quotient continuous functions with the continuous absolute value function.

Theorem

If g is continuous at the point b and $\lim\limits_{x\to c} f(x)=b$, then

$$\lim_{x\to c} g\big(f(x)\big)=g(b)=g\left(\lim_{x\to c} f(x)\right)$$

Proof

Let $\varepsilon>0$ be given. Since g is continuous at b, there exists a number $\delta_1>0$ such that

$$\left|g(y)-g(b)\right|<\varepsilon \quad \mathit{whenever} \quad 0<|y-b|<\delta_1$$

$\lim\limits_{x\to c} f(x)=b,\ \exists\ \delta>0 \ni \ \left|f(x)-b\right|<\delta_1 \quad \mathit{whenever} \quad 0<|x-c|<\delta$

If we let $y=f(x)$, we then have that $\quad |y-b|<\delta_1 \quad \mathit{whenever} \quad 0<|x-c|<\delta$

Which implies from the first statement that $\left|g(y)-g(b)\right|=\left|g\big(f(x)\big)-g(b)\right|<\varepsilon$ whenever $0<|x-c|<\delta$. From the definition of the limit, this proves that $\lim\limits_{x\to c} g\big(f(x)\big)=g(b)$

Example

Find the $\lim\limits_{x\to\frac{\pi}{2}} \cos\left(2x+\sin\left(\frac{3\pi}{2}+x\right)\right)$

Solution

$$\lim_{x\to\frac{\pi}{2}} \cos\left(2x+\sin\left(\frac{3\pi}{2}+x\right)\right) = \cos\left(\lim_{x\to\frac{\pi}{2}} 2x + \lim_{x\to\frac{\pi}{2}} \sin\left(\frac{3\pi}{2}+x\right)\right)$$

$$= \cos\left(\pi+\sin 2\pi\right)$$

$$= \cos\left(\pi+0\right)$$

$$= \cos\left(\pi\right)$$

$$= -1$$

Example

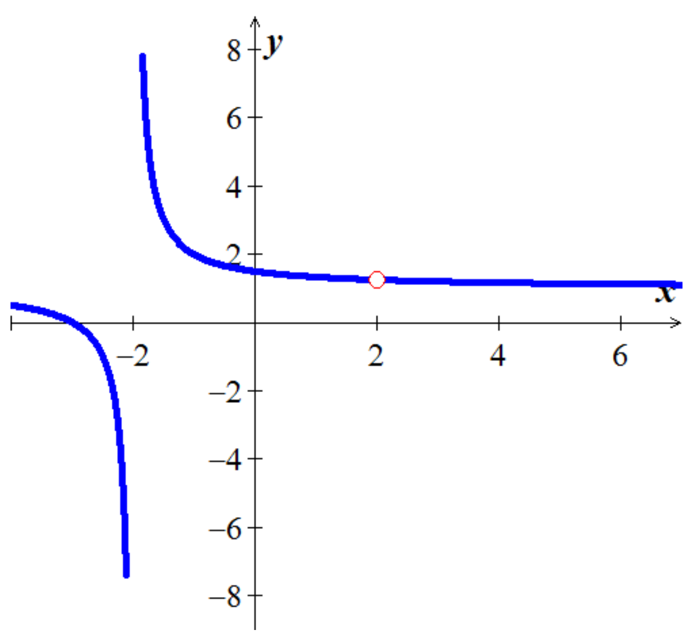
Show that $f(x)=\dfrac{x^2+x-6}{x^2-4}, \quad x\neq 2$ has a continuous extension to $x=2$, and find that extension.

Solution

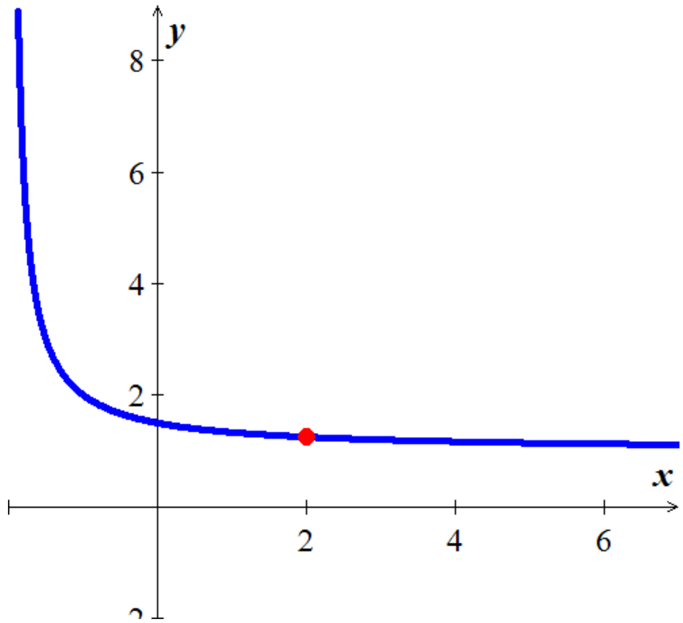
$$f(x)=\frac{x^2+x-6}{x^2-4}$$

$$=\frac{(x-2)(x+3)}{(x-2)(x+2)}$$

$$=\frac{x+3}{x+2}$$



After simplification the function is continuous at $x = 2$



After simplification the function is continuous at $x = 2$

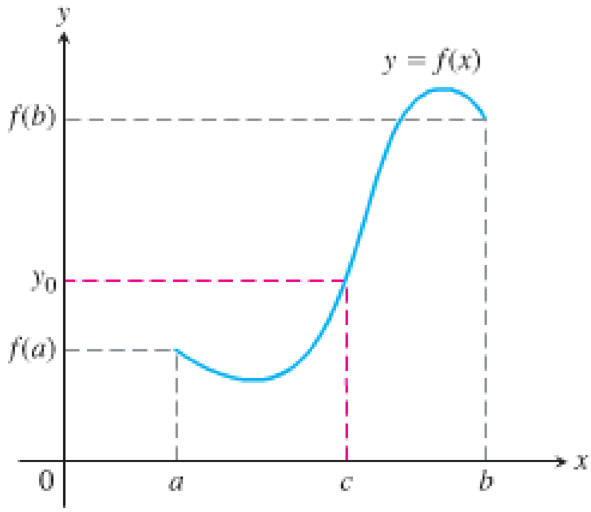
$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to 2} \frac{x+3}{x+2}$$

$$= \frac{5}{4}$$

The new function is the function *f* with its point of discontinuity at $x = 2$ removed.

Theorem – the Intermediate Value Theorem for Continuous Functions

If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



A Consequence for Root Finding

We call a solution of the equation $f(x)=0$ a ***root*** of the equation or zero of the function f. The Intermediate Value Theorem said that if f is continuous, then any interval on which f changes sign contains a zero of the function.

Example

Show that there is a root of the equation $x^3 - x - 1$ between 1 and 2.

Solution

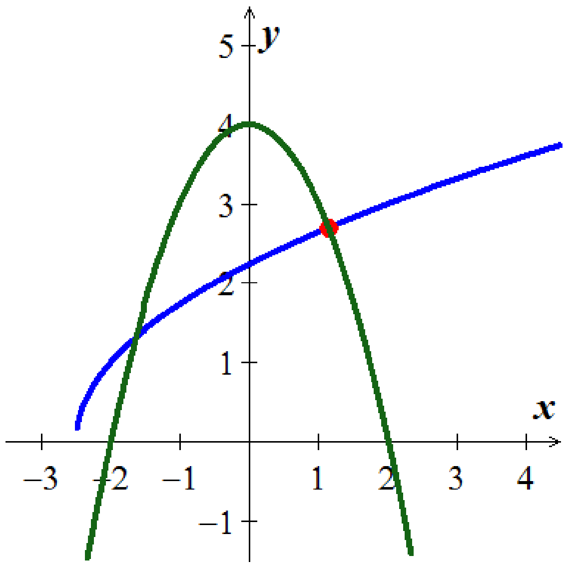
$$f(1) = 1^3 - 1 - 1 = -1 < 0$$

$$f(2) = 2^3 - 2 - 1 = 5 > 0$$

Since f is continuous, the Intermediate Value Theorem says there is a zero of f between 1 and 2.

Example

Use the Intermediate Value Theorem to prove that the equation $\sqrt{2x+5} = 4 - x^2$ has a solution.



Solution

The function $g(x) = \sqrt{2x+5}$ is continuous on the interval $\left[-\frac{5}{2}, \infty\right)$ since it is the composite of the square root function with nonnegative linear function $y = 2x+5$.

Then the function $f(x) = \sqrt{2x+5} + x^2$ is the sum of the function $g(x)$ and $y = x^2$.

It follows that $f(x)$ is continuous on the interval $\left[-\frac{5}{2}, \infty\right)$.

By trial and error:

$$f(0) = \sqrt{2(0)+5} + 0^2 = \sqrt{5} > 0$$

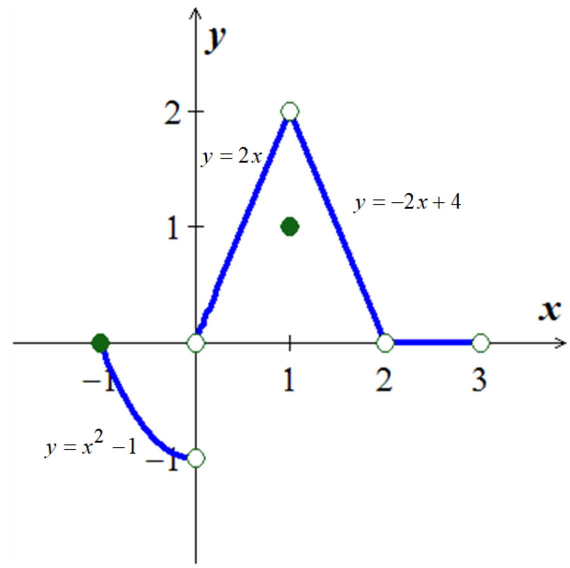
$$f(2) = \sqrt{2(2)+5} + 2^2 = \sqrt{9} + 4 = 7 > 0$$

f is continuous on the interval $[0, 2] \subset \left[-\frac{5}{2}, \infty\right)$.

Since the value $y_0 = 4$ is between $\sqrt{5}$ and 7, by the Intermediate Value Theorem there is a number $c \in [0, 2] \ni f(c) = 4$. That is, the number c solves the original equation.

Exercises Section 1.5 – Continuity

1. Given the graphed function $f(x)$



a) Does $f(-1)$ exist?

b) Does $\lim\limits_{x\to -1^+} f(x)$ exist?

c) Does $\lim\limits_{x\to -1^+} f(x) = f(-1)$?

d) Is f continuous at $x=-1$?

e) Does $f(1)$ exist?

f) Does $\lim\limits_{x\to 1} f(x)$ exist?

g) Does $\lim\limits_{x\to 1} f(x) = f(1)$?

h) Is f continuous at $x=1$?

(2 – 11) At what point(s) is the given function continuous?

2. $y = \dfrac{1}{x-2} - 3x$

3. $y = \dfrac{x+3}{x^2-3x-10}$

4. $y = |x-1| + \sin x$

5. $y = \dfrac{x+2}{\cos x}$

6. $y = \tan \dfrac{\pi x}{2}$

7. $y = \dfrac{x \tan x}{x^2+1}$

8. $y = \dfrac{\sqrt{x^4+1}}{1+\sin^2 x}$

9. $y = \sqrt{2x+3}$

10. $y = \sqrt[4]{3x-1}$

11. $y = (2-x)^{1/5}$

12. Find $\lim\limits_{x\to \pi} \sin(x - \sin x)$, then is the function continuous at the point being approached?

13. Find $\lim\limits_{x\to 0} \tan\left(\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right)$, then is the function continuous at the point being approached?

14. Find $\lim\limits_{t\to 0} \cos\left(\dfrac{\pi}{\sqrt{19-3\sec 2t}}\right)$, then is the function continuous at the point being approached?

15. Explain why the equation $\cos x = x$ has at least one solution.

(16 – 19) Show that the equation has three solutions in the given interval

16. $x^3 - 15x + 1 = 0;\quad [-4,\ 4]$

17. $x^3 + 10x^2 - 100x + 50 = 0;\quad (-20,\ 10)$

18. $70x^3 - 87x^2 + 32x - 3 = 0;\quad (0,\ 1)$

19. $x^3 - 3x - 1 = 0;\quad [-2,\ 2]$

20. Show that the equation has six solutions in the given interval $x^6-8x^4+10x^2-1=0;\quad [-3,\ 3]$

21. If functions $f(x)$ and $g(x)$ are continuous for $0 \le x \le 1$, could $\dfrac{f(x)}{g(x)}$ possibly be discontinuous at a point of [0, 1]? Give reason for your answer.

22. Suppose that a function f is continuous on the closed interval [0, 1] and that $0 \le f(x) \le 1$ for every x in [0, 1]. Show that there must exist a number c in [0, 1] such that $f(c)=c$ (***c*** is called a ***fixed point*** of f).

23. Use the Intermediate Value Theorem to show that the equation $x^5+7x+5=0$ has a solution in the interval $(-1,\ 0)$.

24. The amount of an antibiotic (in *mg*) in the blood *t* hours after an intravenous line is opened is given by

$$m(t)=100\left(e^{-0.1t}-e^{-0.3t}\right)$$

a) Use the Intermediate Value Theorem to show that the amount of drug is 30 *mg* at some time in the interval $[0,\ 5]$ and again at some time in the interval $[5,\ 15]$

b) Estimate the times at which $m=30$ *mg*

c) Is the amount of drug in the blood ever 50 *mg*?

(**25 – 27**) Determine whether the following functions are continuous at *a*.

25. $f(x)=\dfrac{1}{x-5};\quad a=5$

26. $h(x)=\sqrt{x^2-9};\quad a=3$

27. $g(x)=\begin{cases} \dfrac{x^2-16}{x-4} & if \quad x\ne 4 \\ 8 & if \quad x=4 \end{cases};\quad a=4$

(**28 – 31**) Find the intervals on which the following functions are continuous. Specify right- or left-continuity at the endpoints

28. $f(x)=\sqrt{x^2-5}$ **29.** $f(x)=e^{\sqrt{x-2}}$ **30.** $f(x)=\dfrac{2x}{x^3-25x}$ **31.** $f(x)=\cos e^x$

32. Let $g(x)=\begin{cases} 5x-2 & if \quad x<1 \\ a & if \quad x=1 \\ ax^2+bx & if \quad x>1 \end{cases}$

Determine values of the constants a and b for which $g(x)$ is continuous at $x=1$

Section 1.6 – Precise Definition of a Limit

Example

Consider the function $y = 2x - 1$ near $x_0 = 4$. Intuitively it appears that *y* is close to 7 when *x* is close to 4, so $\lim\limits_{x \to 4} (2x - 1) = 7$. However, how close to $x_0 = 4$ does *x* have to be so that $y = 2x - 1$ differs from 7 by, say less than 2 units?

Solution

We need to find the values of *x* for $|y - 7| < 2$.

$|y - 7| = |2x - 1 - 7| = |2x - 8|$

$$|2x - 8| < 2$$

$$-2 < 2x - 8 < 2$$

$$-2 + 8 < 2x - 8 + 8 < 2 + 8$$

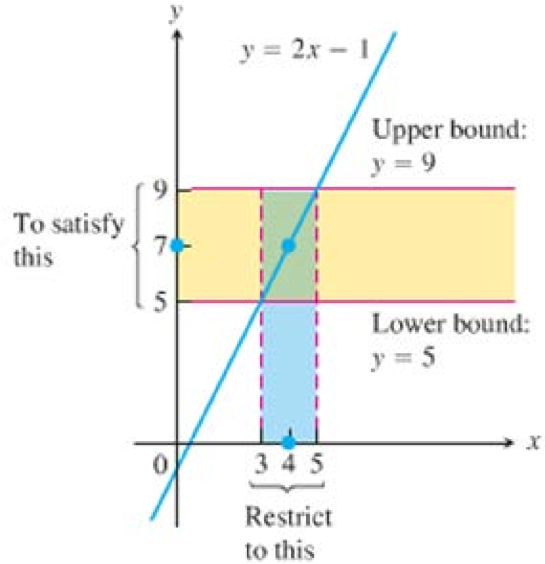
$$6 < 2x < 10$$

$$\frac{6}{2} < \frac{2x}{2} < \frac{10}{2}$$

$$3 < x < 5$$

$$3 - 4 < x - 4 < 5 - 4$$

$$-1 < x - 4 < 1$$



Keeping *x* within 1 unit of $x_0 = 4$ will keep *y* within 2 units of $y_0 = 7$

Definition

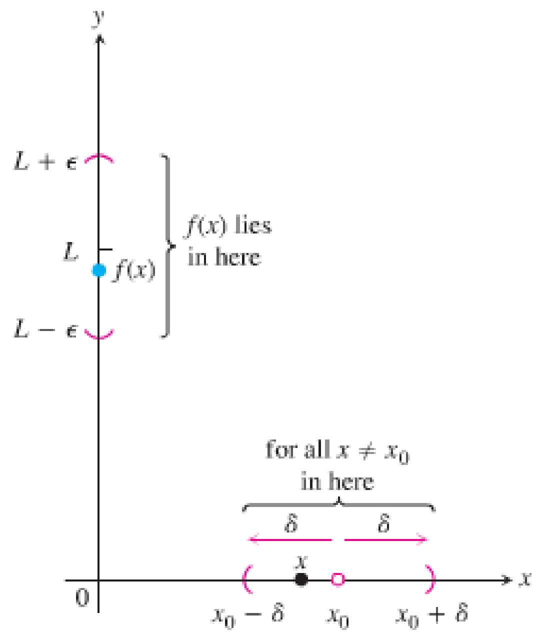
Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself.

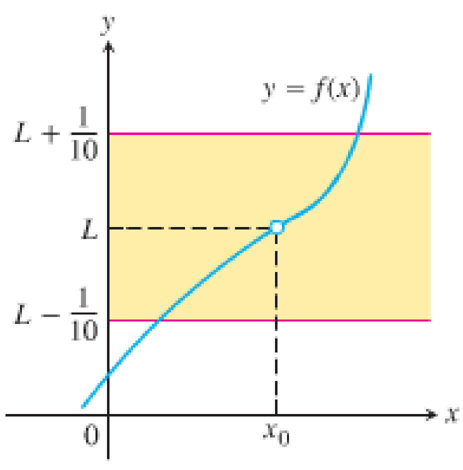
We say that **the limit of** $f(x)$ **as *x* approaches** x_0 **is the number *L***, and write

$$\lim_{x \to x_0} f(x) = L$$

If, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all *x*,

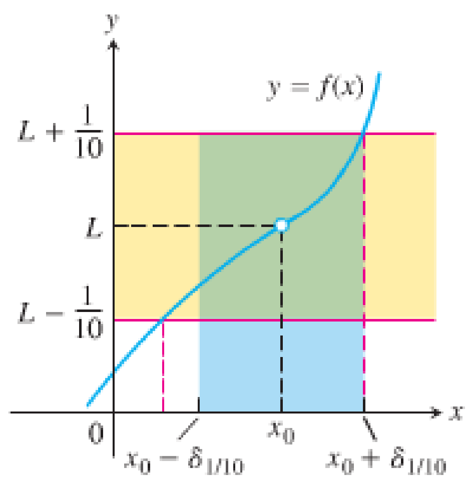
$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$





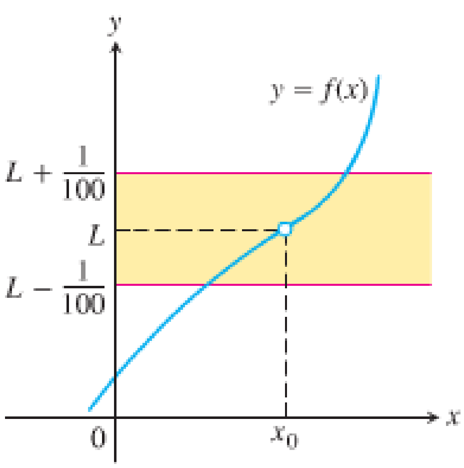
The challenge:

Make $\left|f(x) - L\right| < \epsilon = \frac{1}{10}$



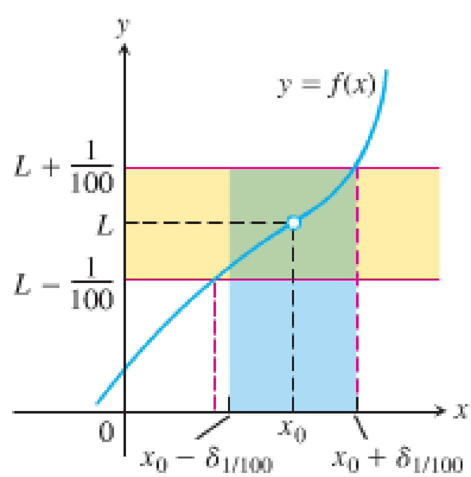
Response:

$\left|x - x_0\right| < \delta_{1/10}$ (a number)



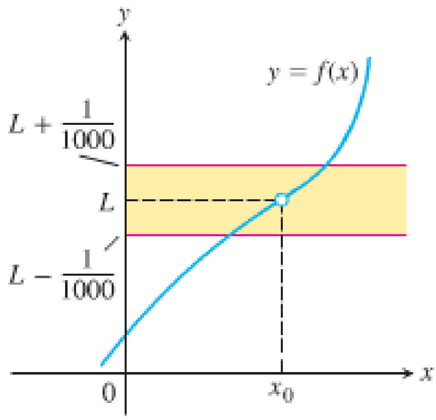
New challenge:

Make $\left|f(x) - L\right| < \epsilon = \frac{1}{100}$

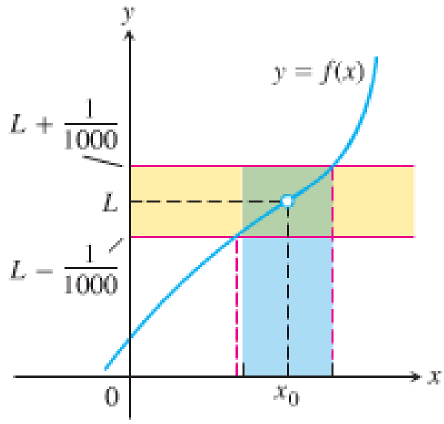


Response:

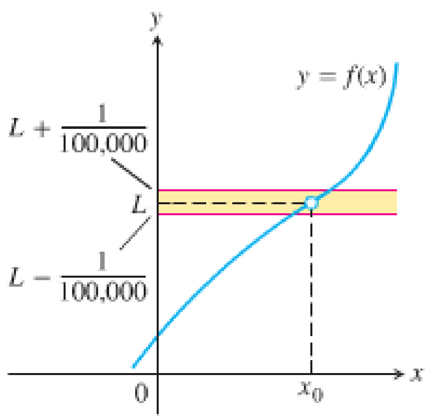
$\left|x - x_0\right| < \delta_{1/100}$



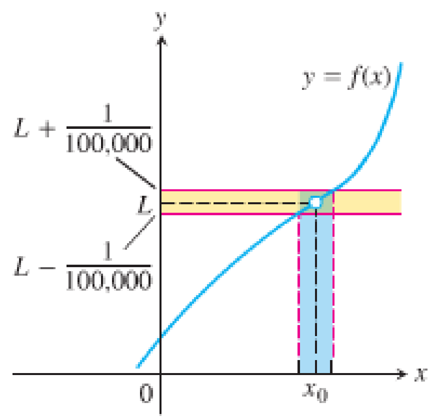
New challenge:
$\epsilon = \frac{1}{1000}$



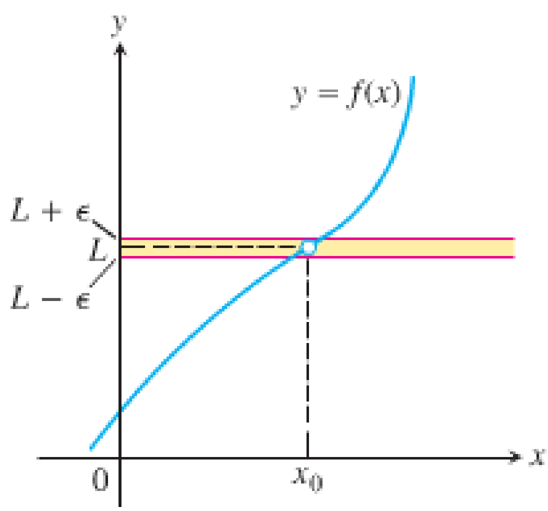
Response:
$|x - x_0| < \delta_{1/1000}$



New challenge:
$\epsilon = \frac{1}{100{,}000}$



Response:
$|x - x_0| < \delta_{1/100{,}000}$



New challenge:
$\epsilon = \cdots$

Example

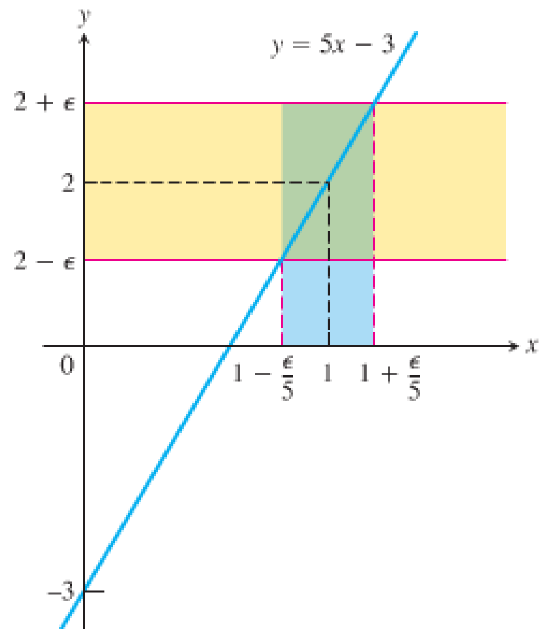
Show that $\lim_{x\to 1}(5x-3)=2$

Solution

Let $x_0 = 1, \quad f(x) = 5x-3, \quad and \quad L = 2$.

For any given $\varepsilon > 0$, there exists a $\delta > 0$ so that $x \neq 1$ and x is within distance δ of $x_0 = 1$, that is

$$0 < |x-1| < \delta \quad \Rightarrow \quad |f(x)-2| < \varepsilon$$



$$|(5x-3)-2| < \varepsilon$$
$$|5x-5| < \varepsilon$$
$$5|x-1| < \varepsilon$$
$$|x-1| < \frac{\varepsilon}{5}$$

Thus, we can take: $\delta = \frac{\varepsilon}{5}$

If $0 < |x-1| < \delta = \frac{\varepsilon}{5}$

$$|(5x-3)-2| = |5x-5| = 5|x-1| = 5\frac{\varepsilon}{5} = \varepsilon$$

Which proves that $\lim_{x\to 1}(5x-3)=2$

Example

Prove the results presented graphically $\lim_{x\to x_0} x = x_0$

Solution

Let $\varepsilon > 0$ be given, we must find $\delta > 0$ such that for all x

$$0 < |x-x_0| < \delta \quad \Rightarrow \quad |x-x_0| < \varepsilon$$

This implication will hold if $\delta = \varepsilon$ or any smaller number.

Example

For the limit $\lim\limits_{x\to 5}\sqrt{x-1}=2$, find a $\delta>0$ that works for $\varepsilon=1$. That is, find a $\delta>0$ such that for all *x*:

$0<|x-5|<\delta \quad \Rightarrow \quad \left|\sqrt{x-1}-2\right|<1$

Solution

$\left|\sqrt{x-1}-2\right|<1$

$$-1<\sqrt{x-1}-2<1$$

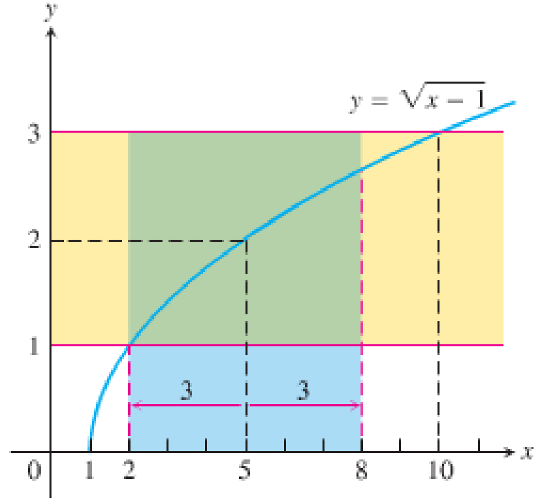
$$-1+2<\sqrt{x-1}-2+2<1+2$$

$$1<\sqrt{x-1}<3 \qquad \textit{Square all sides}$$

$$1<x-1<9$$

$$1+1<x-1+1<9+1$$

$$2<x<10$$

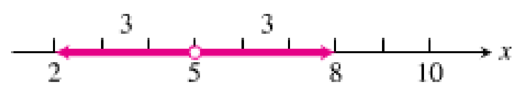


The inequality holds for all *x* in the open interval $(2,\ 10)$.
So it holds for all $x\neq 5$ in the interval as well.

Finding δ value.

$5-\delta<x<5+\delta$ Centered at $x_0=5$ inside the interval $(2,\ 10)$

$\begin{cases}5-\delta=2\\ 5+\delta<10\end{cases} \to \delta=3$ (*to be centered*)



$0<|x-5|<3 \quad \Rightarrow \quad \left|\sqrt{x-1}-2\right|<1$

How to Find *Algebraically* a δ for a Given *f*, *L*, x_0, and $\varepsilon>0$

The process of finding a $\delta>0$ such that for all *x*:

$$0<|x-x_0|<\delta \quad \Rightarrow \quad |f(x)-L|<\varepsilon$$

Can be accomplished in two steps

1. Solve the inequality $|f(x)-L|<\varepsilon$ to find an open interval $(a,\ b)$ containing x_0 on which the inequality holds for all $x\neq x_0$.
2. Find a value of $\delta>0$ that places the open interval $(x_0-\delta,\ x_0+\delta)$ centered at x_0 inside the interval $(a,\ b)$. The inequality $|f(x)-L|<\varepsilon$ will hold for all $x\neq x_0$ in this δ–interval.

Example

Prove that $\lim_{x\to 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

Solution

We need to show that given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x:

$$0 < |x-2| < \delta \quad \Rightarrow \quad |f(x) - 4| < \varepsilon$$

1. *Solve the inequality* $|f(x) - 4| < \varepsilon$ *to find an open interval containing* $x_0 = 2$ *on which the inequality holds for all* $x \neq x_0$.

For $x \neq x_0 = 2$, $f(x) = x^2$, and the inequality to solve is $|x^2 - 4| < \varepsilon$:

$|x^2 - 4| < \varepsilon$

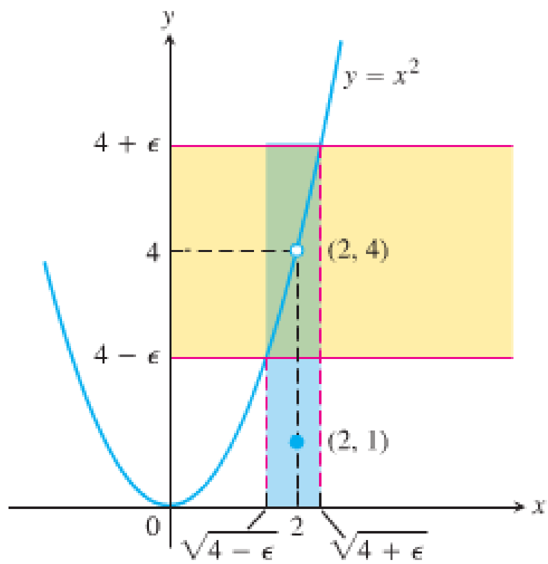
$-\varepsilon < x^2 - 4 < \varepsilon$ ***Add 4 to all sides***

$4 - \varepsilon < x^2 < 4 + \varepsilon$ ***Square root***

$\sqrt{4-\varepsilon} < |x| < \sqrt{4+\varepsilon}$ ***Assume*** $\varepsilon < 4$

$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$

The inequality $|f(x) - 4| < \varepsilon$ holds for all $x \neq 2$ in the open interval $\left(\sqrt{4-\varepsilon},\ \sqrt{4+\varepsilon}\right)$



2. *Find a value of* $\delta > 0$ *that places the open interval* $(2-\delta,\ 2+\delta)$ *inside the interval* $\left(\sqrt{4-\varepsilon},\ \sqrt{4+\varepsilon}\right)$.

Take δ to be the distance from $x_0 = 2$ to the nearer endpoint of $\left(\sqrt{4-\varepsilon},\ \sqrt{4+\varepsilon}\right)$.

$\Rightarrow \delta = \min\left(2 - \sqrt{4-\varepsilon},\ \sqrt{4+\varepsilon} - 2\right)$.

$$0 < |x-2| < \delta$$

$$-\left(2-\sqrt{4-\varepsilon}\right) < x-2 < \sqrt{4+\varepsilon}-2$$

$$-2+\sqrt{4-\varepsilon} < x-2 < \sqrt{4+\varepsilon}-2$$

$$\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$$

$$\therefore \quad 0 < |x-2| < \delta \quad \Rightarrow \quad |f(x)-4| < \varepsilon$$

Example

Given that $\lim\limits_{x\to c} f(x) = L$ *and* $\lim\limits_{x\to c} g(x) = M$, prove that $\lim\limits_{x\to c}\left(f(x)+g(x)\right) = L+M$

Solution

We need to show that given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x:

$$0 < |x-c| < \delta \quad \Rightarrow \quad \left|f(x)+g(x)-(L+M)\right| < \varepsilon$$

$$\begin{aligned} \left|f(x)+g(x)-(L+M)\right| &= \left|f(x)+g(x)-L-M\right| \\ &= \left|\left(f(x)-L\right)+\left(g(x)-M\right)\right| \\ &\le \left|\left(f(x)-L\right)\right| + \left|\left(g(x)-M\right)\right| \end{aligned}$$

Triangle Inequality $|a+b| \le |a|+|b|$

Since $\lim\limits_{x\to c} f(x) = L$, there exists a number $\delta_1 > 0$ such that for all x:

$$0 < |x-c| < \delta_1 \quad \Rightarrow \quad |f(x)-L| < \frac{\varepsilon}{2}$$

Similarly, since $\lim\limits_{x\to c} g(x) = M$, there exists a number $\delta_2 > 0$ such that for all x:

$$0 < |x-c| < \delta_2 \quad \Rightarrow \quad |g(x)-M| < \frac{\varepsilon}{2}$$

Let $\delta = \min\left\{\delta_1,\ \delta_2\right\}$, the smaller of δ_1 and δ_2. If $0 < |x-c| < \delta$ then $0 < |x-c| < \delta_1$, so $|f(x)-L| < \frac{\varepsilon}{2}$ and $|x-c| < \delta_2$, so $|g(x)-M| < \frac{\varepsilon}{2}$. Therefore

$$\left|f(x)+g(x)-(L+M)\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

This show that $\lim\limits_{x\to c}\left(f(x)+g(x)\right) = L+M$

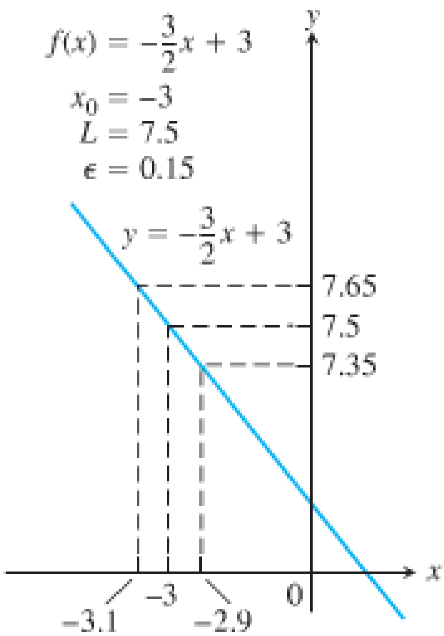
Exercises *Section* 1.6 – Precise Definition of Limits

(1 – 2) Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \Rightarrow a < x < b$ for

1. $a = 1, \quad b = 7, \quad x_0 = 5$

2. $a = -\frac{7}{2}, \quad b = -\frac{1}{2}, \quad x_0 = -\frac{3}{2}$

3. Use the graph to find a $\delta > 0$ such that for all x $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$



(4 – 8) Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

4. $f(x) = x + 1, \quad L = 5, \quad x_0 = 4, \quad \varepsilon = 0.01$

5. $f(x) = \sqrt{x+1}, \quad L = 1, \quad x_0 = 0, \quad \varepsilon = 0.1$

6. $f(x) = \sqrt{x-7}, \quad L = 4, \quad x_0 = 23, \quad \varepsilon = 1$

7. $f(x) = x^2, \quad L = 3, \quad x_0 = \sqrt{3}, \quad \varepsilon = 0.1$

8. $f(x) = \frac{120}{x}, \quad L = 5, \quad x_0 = 24, \quad \varepsilon = 1$

(9 – 14) Give a formal proof that

9. $\lim_{x\to4}(9-x)=5$

10. $\lim_{x\to1}\frac{1}{x}=1$

11. $\lim_{x\to5}\frac{x^2-25}{x-5}=10$

12. $\lim_{x\to0}f(x)=0$ *if* $f(x)=\begin{cases}2x, & x<0\\ \frac{x}{2}, & x\ge0\end{cases}$

13. $\lim_{x\to1}(5x-2)=3$

14. $\lim_{x\to2}\frac{1}{(x-2)^4}=\infty$

15. Prove that $\lim_{x\to0}x\sin\left(\frac{1}{x}\right)=0$