Solution   *Section* 1.4 – Volumes by Shells

Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*: $y = x$

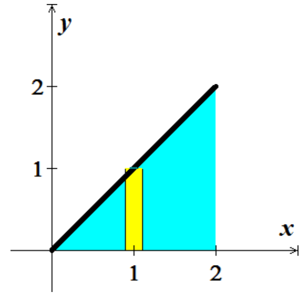
Solution

$$V = 2\pi \int_0^2 x(x)\,dx \qquad\qquad V = 2\pi \int_a^b x\,f(x)\,dx$$

$$= 2\pi \int_0^2 x^2\,dx$$

$$= \frac{2\pi}{3} x^3 \Big|_0^2$$

$$= \boxed{\frac{16\pi}{3}}$$



Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis* $y = 1 - x$

Solution

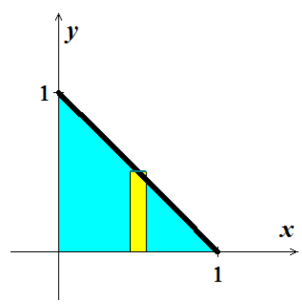
$$V = 2\pi \int_0^1 x(1-x)\,dx \qquad\qquad V = 2\pi \int_a^b x\,f(x)\,dx$$

$$= 2\pi \int_0^1 \left(x - x^2\right)\,dx$$

$$= 2\pi \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1$$

$$= 2\pi \left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \boxed{\frac{\pi}{3}}$$



Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis* $y = \sqrt{x}$

Solution

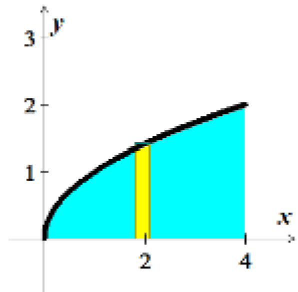
$$V = 2\pi \int_0^4 x\sqrt{x}\,dx \qquad\qquad V = 2\pi \int_a^b x\,f(x)\,dx$$

$$= 2\pi \int_0^4 x^{3/2}\, dx$$

$$= 2\pi \left(\frac{2}{5} x^{5/2}\right)\Big|_0^4$$

$$= \frac{4\pi}{5}\left(2^2\right)^{5/2}$$

$$= \frac{128\pi}{5}$$



Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis* $\quad y = \frac{1}{2}x^2 + 1$

Solution

$$f(x) = 3 - \left(\frac{1}{2}x^2 + 1\right) = 2 - \frac{1}{2}x^2$$

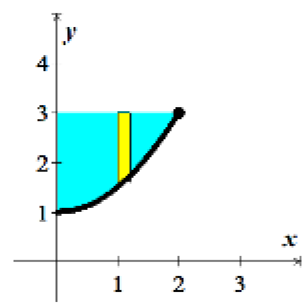
$$V = 2\pi \int_0^2 x\left(2 - \frac{1}{2}x^2\right)\, dx \qquad V = 2\pi \int_a^b x\, f(x)\, dx$$

$$= 2\pi \int_0^2 \left(2x - \frac{1}{2}x^3\right)\, dx$$

$$= 2\pi \left(x^2 - \frac{1}{8}x^4\right)\Big|_0^2$$

$$= 2\pi(4-2)$$

$$= 4\pi$$



Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y = \frac{1}{4}x^2, \quad y = 0, \quad x = 4$$

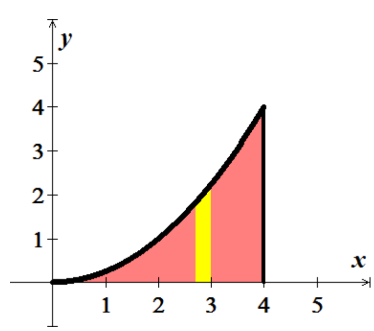
Solution

$$V = 2\pi \int_0^4 x\left(\frac{1}{4}x^2\right) dx \qquad V = 2\pi \int_a^b x\left(f(x) - g(x)\right) dx$$

$$= \frac{\pi}{2}\int_0^4 x^3 dx$$

$$= \frac{\pi}{8}x^4\Big|_0^4$$

$$= 32\pi$$



Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y=\frac{1}{2}x^3, \quad y=0, \quad x=3$$

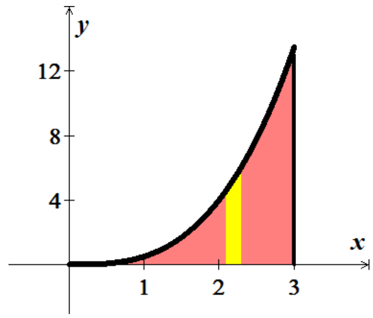
Solution

$$V=2\pi\int_0^3 x\left(\frac{1}{2}x^3\right)dx \qquad\qquad V=2\pi\int_a^b x\left(f(x)-g(x)\right)dx$$

$$=\pi\int_0^3 x^4 dx$$

$$=\frac{\pi}{5}x^5\Big|_0^3$$

$$=\frac{243\pi}{5}$$



Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y=x^2, \quad y=4x-x^2$$

Solution

$$y=4x-x^2=x^2 \ \Rightarrow\ 2x^2-4x=0 \ \rightarrow\ x=0,\ 2$$

$$f(x)=4x-x^2, \quad g(x)=x^2$$

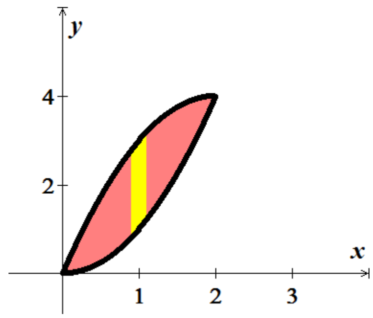
$$V=2\pi\int_0^2 x\left(4x-x^2-x^2\right)dx \qquad\qquad V=2\pi\int_a^b x\left(f(x)-g(x)\right)dx$$

$$=4\pi\int_0^2\left(2x^2-x^3\right)dx$$

$$=4\pi\left(\frac{2}{3}x^3-\frac{1}{4}x^4\right)\Big|_0^2$$

$$=4\pi\left(\frac{16}{3}-4\right)$$

$$=\frac{16\pi}{3}$$



Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

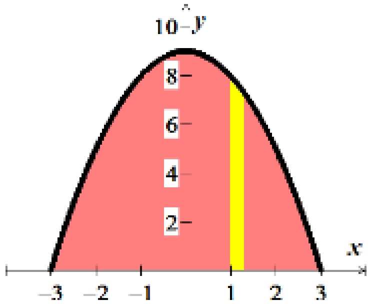
$$y = 9 - x^2, \quad y = 0$$

Solution

$y = 9 - x^2 = 0 \quad \rightarrow \quad \underline{x = \pm 3}$

$f(x) = 9 - x^2, \quad g(x) = 0$

$$V = 2\pi \int_0^3 x\left(9 - x^2\right)dx \qquad\qquad V = 2\pi \int_a^b x\left(f(x) - g(x)\right)dx$$



$$= 2\pi \int_0^3 \left(9x - x^3\right)dx$$

$$= 2\pi \left(\frac{9}{2}x^2 - \frac{1}{4}x^4\right)\Big|_0^3$$

$$= 2\pi \left(\frac{81}{2} - \frac{81}{4}\right)$$

$$= \frac{81\pi}{2}$$

Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

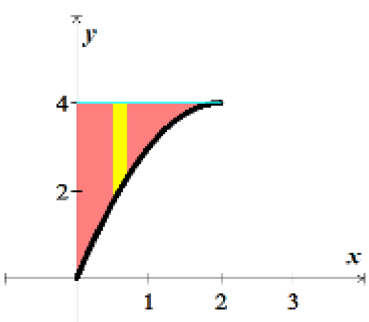
$$y = 4x - x^2, \quad x = 0, \quad y = 4$$

Solution

$y = 4x - x^2 = 4 \Rightarrow x^2 - 4x + 4 \quad \rightarrow \quad \underline{x = 2}$

$f(x) = 4, \quad g(x) = 4x - x^2$

$$V = 2\pi \int_0^2 x\left(4 - 4x + x^2\right)dx \qquad\qquad V = 2\pi \int_a^b x\left(f(x) - g(x)\right)dx$$



$$= 2\pi \int_0^2 \left(4x - 4x^2 + x^3\right)dx$$

$$= 2\pi \left(2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4\right)\Big|_0^2$$

$$= 2\pi \left(8 - \frac{32}{3} + 4\right)$$

$$= \frac{8\pi}{3}$$

Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

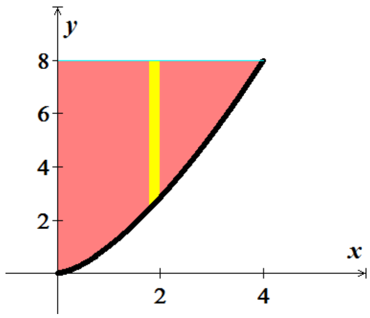
$$y = x^{3/2}, \quad y = 8, \quad x = 0$$

Solution

$$y = x^{3/2} = 8 \Rightarrow x = \left(2^3\right)^{2/3} \quad \rightarrow \quad \underline{x = 4}$$

$$f(x) = 8, \quad g(x) = x^{3/2}$$

$$V = 2\pi \int_0^4 x\left(8 - x^{3/2}\right) dx \qquad\qquad V = 2\pi \int_a^b x\left(f(x) - g(x)\right) dx$$



$$= 2\pi \int_0^4 \left(8x - x^{5/2}\right) dx$$

$$= 2\pi \left(4x^2 - \frac{2}{7}x^{7/2}\right)\Big|_0^4$$

$$= 2\pi \left(64 - \frac{256}{7}\right)$$

$$\underline{= \frac{384\pi}{7}}$$

Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

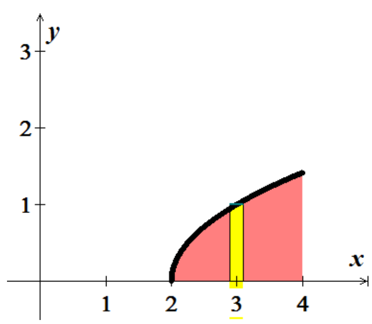
$$y = \sqrt{x-2}, \quad y = 0, \quad x = 4$$

Solution

$$y = \sqrt{x-2} = 0 \quad \rightarrow \quad \underline{x = 2}$$

$$f(x) = \sqrt{x-2}, \quad g(x) = 0$$

$$V = 2\pi \int_2^4 x\left(\sqrt{x-2}\right) dx \qquad\qquad V = 2\pi \int_a^b x\left(f(x) - g(x)\right) dx$$



$$= 2\pi \int_2^4 (u+2)u^{1/2}\, du \qquad\qquad u = x - 2 \quad x = u + 2$$

$$du = dx$$

$$= 2\pi \int_2^4 \left(u^{3/2} + 2u^{1/2}\right) du$$

$$= 2\pi \left(\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2}\right)\Big|_2^4$$

$$= 2\pi \left(\frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2}\right)\Big|_2^4$$

$$= 2\pi\left(\frac{8\sqrt{2}}{5} + \frac{8\sqrt{2}}{3}\right)$$

$$= 16\pi\sqrt{2}\left(\frac{1}{5} + \frac{1}{3}\right)$$

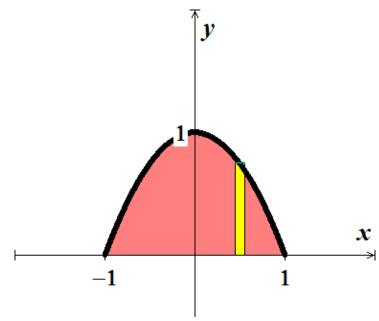
$$\boxed{= \frac{128\pi\sqrt{2}}{15}}$$

Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y = -x^2 + 1, \quad y = 0$$

Solution



$$y = -x^2 + 1 = 0 \quad \to \quad \boxed{x = \pm 1}$$

$$f(x) = -x^2 + 1, \quad g(x) = 0$$

$$V = 2\pi\int_0^1 x\left(-x^2 + 1\right)dx \qquad\qquad V = 2\pi\int_a^b x\left(f(x) - g(x)\right)dx$$

$$= 2\pi\int_0^1 \left(-x^3 + x\right)\, dx$$

$$= 2\pi\left(-\frac{1}{4}x^4 + \frac{1}{2}x^2\right)\Big|_0^1$$

$$= 2\pi\left(-\frac{1}{4} + \frac{1}{2}\right)$$

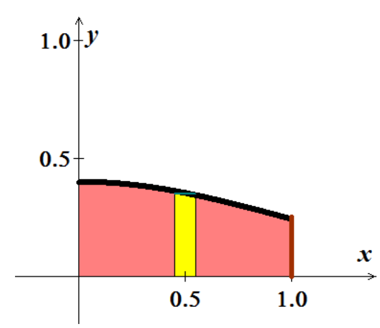
$$\boxed{= \frac{\pi}{2}}$$

Exercise

Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the *y-axis*

$$y = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \quad y = 0, \quad x = 0, \quad x = 1$$

Solution



$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \quad g(x) = 0$$

$$V = 2\pi\int_0^1 x\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right)dx \qquad\qquad V = 2\pi\int_a^b x\left(f(x) - g(x)\right)dx$$

$$= -\sqrt{2\pi}\int_0^1 e^{-x^2/2}\, d\left(-\frac{x^2}{2}\right)$$

$$= -\sqrt{2\pi}\left(e^{-x^2/2}\right)\Big|_0^1$$

$$= -\sqrt{2\pi}\left(e^{-1/2} - 1\right)$$

$$= \sqrt{2\pi}\left(1 - \frac{1}{\sqrt{e}}\right)$$

Exercise

Use the shell method to find the volume of the solid generated by revolving the shaded region about the indicated axis

Solution

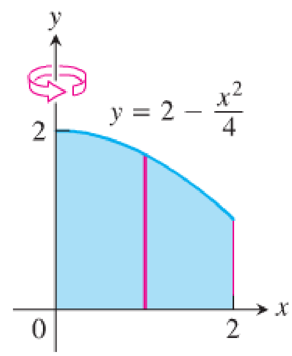
$$V = \int_0^2 2\pi(x)\left(2 - \frac{x^2}{4}\right)dx \qquad V = \int_a^b 2\pi\left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{height}\end{matrix}\right)dx$$

$$= 2\pi\int_0^2 \left(2x - \frac{x^3}{4}\right)dx$$

$$= 2\pi\left(x^2 - \frac{x^4}{16}\right)_0^2$$

$$= 2\pi\left[\left(2^2 - \frac{2^4}{16}\right) - 0\right]$$

$$= 6\pi \ \ unit^3$$



Exercise

Use the shell method to find the volume of the solid generated by revolving the shaded region about the indicated axis

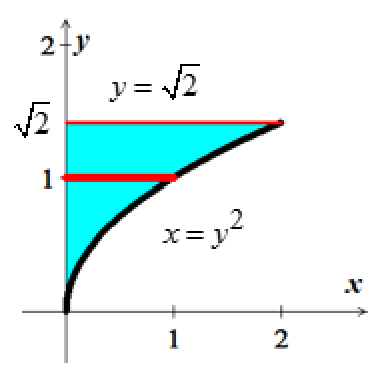
Solution

$$V = \int_0^{\sqrt{2}} 2\pi(y)\left(y^2\right)dy$$

$$= 2\pi\int_0^{\sqrt{2}} y^3 dy$$

$$= 2\pi\left(\frac{y^4}{4}\right)\Big|_0^{\sqrt{2}}$$

$$= 2\pi \ \ unit^3$$



Exercise

Use the shell method to find the volume of the solid generated by revolving the shaded region about the *y*-axis

Solution

$$V=\int_0^3 2\pi(x)\left(\frac{9x}{\sqrt{x^3+9}}\right)dx \qquad V=\int_a^b 2\pi\left(\begin{matrix}shell\\ radius\end{matrix}\right)\left(\begin{matrix}shell\\ height\end{matrix}\right)dx$$

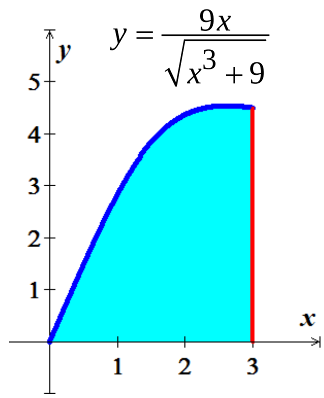
$$=2\pi\int_0^3\left(\frac{9x^2}{\sqrt{x^3+9}}\right)dx$$

$$=2\pi\int_0^3 3\left(x^3+9\right)^{1/2}d\left(x^3+9\right) \qquad d\left(x^3+9\right)=3x^2dx$$

$$=6\pi\left[2\left(x^3+9\right)^{1/2}\right]_0^3 \qquad =12\pi\left[\left(3^3+9\right)^{1/2}-(0+9)^{1/2}\right]$$

$$=12\pi[6-3]$$

$$=36\pi\ \ unit^3$$



Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y=x^2,\quad y=2-x,\quad x=0,\quad for\ x\ge 0$ about the *y*-axis.

Solution

$$V=\int_0^1 2\pi(x)\left((2-x)-x^2\right)dx \qquad V=\int_a^b 2\pi\left(\begin{matrix}shell\\ radius\end{matrix}\right)\left(\begin{matrix}shell\\ height\end{matrix}\right)dx$$

$$=2\pi\int_0^1 x\left(2-x-x^2\right)dx$$

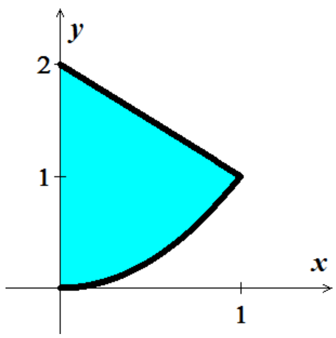
$$=2\pi\int_0^1\left(2x-x^2-x^3\right)dx$$

$$=2\pi\left[x^2-\tfrac{1}{3}x^3-\tfrac{1}{4}x^4\right]_0^1$$

$$=2\pi\left(1-\tfrac{1}{3}-\tfrac{1}{4}\right)$$

$$=12\pi\left(\tfrac{5}{12}\right)$$

$$=\frac{5\pi}{6}\ \ unit^3$$



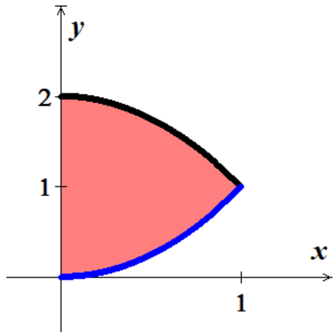
Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = 2 - x^2, \quad y = x^2, \quad x = 0$ about the *y*-axis.

Solution

$$y = 2 - x^2 = x^2 \quad \rightarrow \quad 2x^2 = 2 \Rightarrow x^2 = 1 \rightarrow \boxed{x = \pm 1}$$

Since *about* $y - axis,\quad a = x = 0 \quad b = 1$



$$V = \int_0^1 2\pi(x)\left(\left(2 - x^2\right) - x^2\right)dx$$

$$= 2\pi \int_0^1 x\left(2 - 2x^2\right)dx$$

$$= 4\pi \int_0^1 \left(x - x^3\right)dx$$

$$= 4\pi\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1$$

$$= 4\pi\left[\frac{1}{2} - \frac{1}{4}\right]_0^1$$

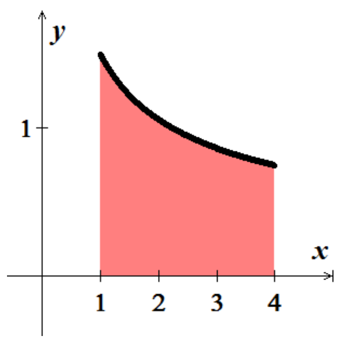
$$= 4\pi\left(\frac{1}{4}\right)$$

$$= \pi \ \ unit^3$$

Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = \frac{3}{2\sqrt{x}}, \quad y = 0, \quad x = 1, \quad x = 4$ about the *y*-axis.

Solution



$$V = \int_1^4 2\pi(x)\left(\frac{3}{2\sqrt{x}} - 0\right)dx$$

$$= \pi \int_1^4 x\left(3x^{-1/2}\right)dx$$

$$= 3\pi \int_1^4 x^{1/2}dx$$

$$= 3\pi\left[\frac{2}{3}x^{3/2}\right]_1^4$$

$$= 2\pi\left[4^{3/2} - 1^{3/2}\right]$$

$$= 2\pi(7)$$

$$\boxed{= 14\pi \ \ unit^3}$$

Exercise

Let $g(x) = \begin{cases} \dfrac{(\tan x)^2}{x} & 0 < x \le \frac{\pi}{4} \\ 0 & x = 0 \end{cases}$

a) Show that $x \cdot g(x) = (\tan x)^2, \quad 0 \le x \le \frac{\pi}{4}$

b) Find the volume of the solid generated by revolving the shaded region about the *y*-axis.

Solution

a) $x \cdot g(x) = \begin{cases} x \cdot \dfrac{(\tan x)^2}{x} & 0 < x \le \frac{\pi}{4} \\ x \cdot 0 & x = 0 \end{cases} \quad \Rightarrow x \cdot g(x) = \begin{cases} \tan^2 x & 0 < x \le \frac{\pi}{4} \\ 0 & x = 0 \end{cases}$

Since $x = 0 \quad \to \quad \tan x = 0 \qquad \Rightarrow x \cdot g(x) = \begin{cases} \tan^2 x & 0 < x \le \frac{\pi}{4} \\ \tan^2 x & x = 0 \end{cases}$

$$\Rightarrow \boxed{x \cdot g(x) = \tan^2 x \qquad 0 \le x \le \frac{\pi}{4}}$$

b) $V = 2\pi \displaystyle\int_0^{\pi/4} x \bullet g(x)\,dx \qquad\qquad V = \int_a^b 2\pi \begin{pmatrix} shell \\ radius \end{pmatrix} \begin{pmatrix} shell \\ height \end{pmatrix} dx$

$$= 2\pi \int_0^{\pi/4} \tan^2 x \; dx$$

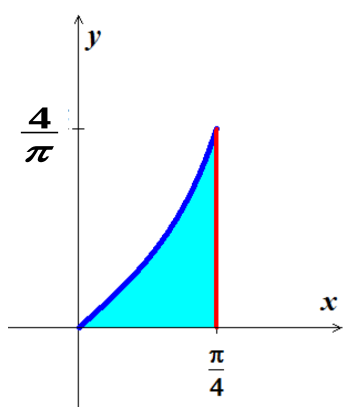
$$= 2\pi\left[\tan x - x\right]_0^{\pi/4}$$

$$= 2\pi\left[\left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - (\tan 0 - 0)\right]$$

$$= 2\pi\left(1 - \frac{\pi}{4}\right)$$

$$= 2\pi\left(\frac{4-\pi}{4}\right)$$

$$\boxed{= \frac{4\pi - \pi^2}{2} \ \ unit^3}$$

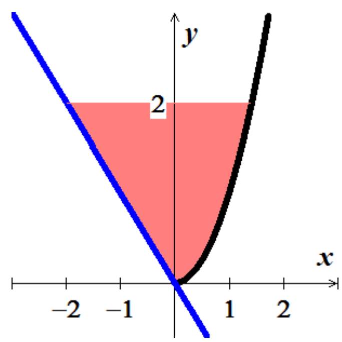


Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $x=\sqrt{y}, \quad x=-y, \quad y=2$ about the *x*-axis.

Solution

$$x=\sqrt{y}=-y \quad \rightarrow y=0=c$$



$$V=\int_0^2 2\pi(y)\left(\sqrt{y}-(-y)\right)dy$$

$$=2\pi\int_0^2\left(y^{3/2}+y^2\right)dy$$

$$=2\pi\left[\frac{2}{5}y^{5/2}+\frac{1}{3}y^3\right]_0^2$$

$$=2\pi\left[\frac{2}{5}(2)^{5/2}+\frac{1}{3}(2)^3\right]$$

$$=2\pi\left[\frac{8\sqrt{2}}{5}+\frac{8}{3}\right]$$

$$=16\pi\left(\frac{3\sqrt{2}+5}{15}\right)$$

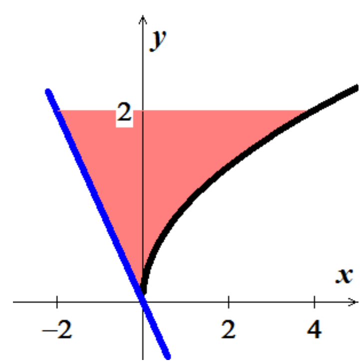
$$=\frac{16}{15}\pi\left(3\sqrt{2}+5\right) \ \ unit^3$$

Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $x=y^2, \quad x=-y, \quad y=2, \quad y\ge 0$ about the *x*-axis.

Solution

$$x=y^2=-y \quad \rightarrow y=0=c \quad d=2$$



$$V=\int_0^2 2\pi(y)\left(y^2-(-y)\right)dy$$

$$=2\pi\int_0^2\left(y^3+y^2\right)dy$$

$$=2\pi\left[\frac{1}{4}y^4+\frac{1}{3}y^3\right]_0^2$$

$$=2\pi\left(\frac{1}{4}(2)^4+\frac{1}{3}(2)^3\right)$$

$$=2\pi\left(4+\frac{8}{3}\right)$$

$$= 2\pi\left(\frac{20}{3}\right)$$

$$= \frac{40\pi}{3} \quad unit^3$$

Exercise

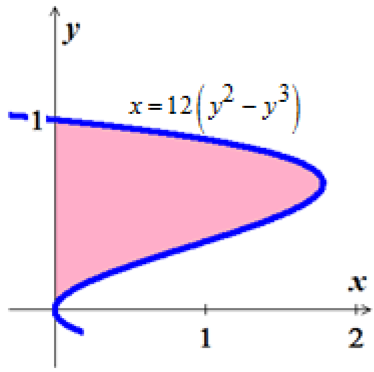
Use the shell method to find the volumes of the solids generated by revolving the shaded regions about the *indicated* axes.

a) The x-axis

b) The line $y = 1$

c) The line $y = \frac{8}{5}$

d) The line $y = -\frac{2}{5}$



Solution

a) $$V = \int_0^1 2\pi(y)\cdot\left[12\left(y^2 - y^3\right)\right]dy$$

$$= 24\pi\int_0^1 \left(y^3 - y^4\right)dy$$

$$= 24\pi\left[\frac{y^4}{4} - \frac{y^5}{5}\right]_0^1$$

$$= 24\pi\left(\frac{1}{4} - \frac{1}{5}\right)$$

$$= 24\pi\left(\frac{1}{20}\right)$$

$$= \frac{6\pi}{5} \quad unit^3$$

b) $$V = \int_0^1 2\pi(1-y)\cdot\left[12\left(y^2 - y^3\right)\right]dy$$

$$= 24\pi\int_0^1 \left(y^2 - y^3 - y^3 + y^4\right)dy$$

$$= 24\pi\int_0^1 \left(y^2 - 2y^3 + y^4\right)dy$$

$$= 24\pi\left[\frac{y^3}{3} - \frac{y^4}{2} + \frac{y^5}{5}\right]_0^1$$

$$= 24\pi\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right)$$

$$= 24\pi\left(\frac{1}{30}\right)$$

$$\underline{= \frac{4\pi}{5} \quad unit^3}$$

c) $$V = \int_c^d 2\pi\begin{pmatrix} shell \\ radius \end{pmatrix}\begin{pmatrix} shell \\ height \end{pmatrix} dy$$

$$= 2\pi\int_0^1 \left(\frac{8}{5} - y\right)\cdot\left[12\left(y^2 - y^3\right)\right] dy$$

$$= 24\pi\int_0^1 \left(\frac{8}{5}y^2 - \frac{8}{5}y^3 - y^3 + y^4\right) dy$$

$$= 24\pi\int_0^1 \left(\frac{8}{5}y^2 - \frac{13}{5}y^3 + y^4\right) dy$$

$$= 24\pi\left[\frac{8}{15}y^3 - \frac{13}{20}y^4 + \frac{y^5}{5}\right]_0^1$$

$$= 24\pi\left(\frac{8}{15} - \frac{13}{20} + \frac{1}{5}\right)$$

$$= 24\pi\left(\frac{5}{60}\right)$$

$$\underline{= 2\pi \quad unit^3}$$

d) $$V = \int_0^1 2\pi\left(y + \frac{2}{5}\right)\cdot\left[12\left(y^2 - y^3\right)\right] dy$$

$$= 24\pi\int_0^1 \left(y^3 - y^4 + \frac{2}{5}y^2 - \frac{2}{5}y^3\right) dy$$

$$= 24\pi\int_0^1 \left(\frac{3}{5}y^3 - y^4 + \frac{2}{5}y^2\right) dy$$

$$= 24\pi\left[\frac{3}{20}y^4 - \frac{1}{5}y^4 + \frac{2}{15}y^3\right]_0^1$$

$$= 24\pi\left(\frac{3}{20} - \frac{1}{5} + \frac{2}{15}\right)$$

$$= 24\pi\left(\frac{5}{60}\right)$$

$$\underline{= 2\pi \quad unit^3}$$

Exercise

Compute the volume of the solid generated by revolving the region bounded by the lines $y = x$ *and* $y = x^2$ about each coordinate axis using

a) The *shell* method

b) The *washer* method

Solution

$$y = x = x^2 \Rightarrow x^2 - x = 0 \to \boxed{x = 0,\ 1}$$

a) x-axis

$$V = \int_0^1 2\pi(y)\cdot\left[\sqrt{y} - y\right]dy \qquad V = \int_c^d 2\pi\begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix}dy$$

$$= 2\pi\int_0^1 \left(y^{3/2} - y^2\right)dy$$

$$= 2\pi\left[\frac{2}{5}y^{5/2} - \frac{1}{3}y^3\right]_0^1$$

$$= 2\pi\left(\frac{2}{5} - \frac{1}{3}\right)$$

$$= \frac{2\pi}{15}\ \text{unit}^3$$

y-axis

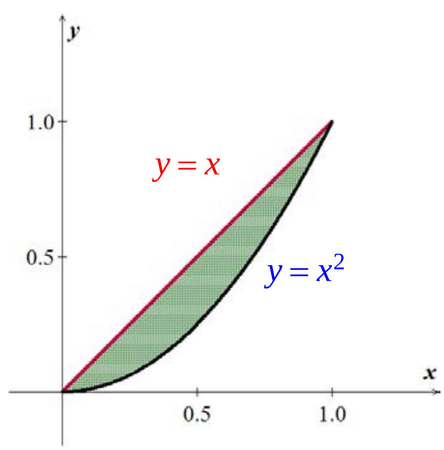
$$V = 2\pi\int_0^1 (x)\left(x - x^2\right)dx \qquad V = \int_a^b 2\pi\begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix}dx$$

$$= 2\pi\int_0^1 \left(x^2 - x^3\right)dx$$

$$= 2\pi\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1$$

$$= 2\pi\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{\pi}{6}\ \text{unit}^3$$



b) x-axis $\quad R(x) = x \quad$ *and* $\quad r(x) = x^2$

$$V = \int_a^b \pi\left[R(x)^2 - r(x)^2\right]dx$$

$$= \pi\int_0^1 \left(x^2 - x^4\right)dx$$

$$= \pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1$$

$$= \pi\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$\boxed{= \frac{2\pi}{15}\ \ unit^3}$$

y-axis $\quad R(y) = \sqrt{y}$ *and* $r(y) = y$

$$V = \int_c^d \pi\left[R(y)^2 - r(y)^2\right]dy$$

$$= \pi\int_0^1 \left(y - y^2\right)dy$$

$$= \pi\left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_0^1$$

$$= \pi\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$\boxed{= \frac{\pi}{6}\ \ unit^3}$$

Exercise

Use the *washer* method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = \sqrt{x}, \quad y = 2, \quad x = 0$ about

a) the *x*-axis

b) the *y*-axis

c) the line $x = 4$

d) the line $y = 1$

Solution

a) x-axis

$$V = \int_0^2 2\pi(y)\cdot\left(y^2 - 0\right)dy \qquad\qquad V = \int_c^d 2\pi\binom{shell}{radius}\binom{shell}{height}dy$$

$$= 2\pi\int_0^2 y^3 dy$$

$$= \frac{1}{2}\pi y^4\Big|_0^2$$

$$= \frac{1}{2}\pi(2)^4$$

$$\boxed{= 8\pi\ \ unit^3}$$

b) y-axis

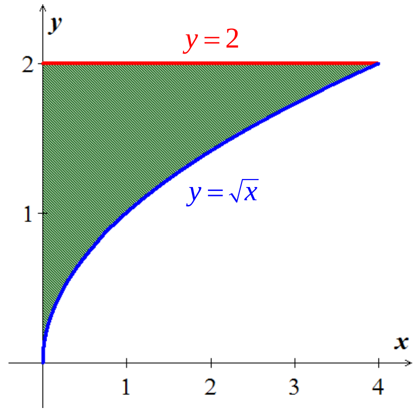
$$V = 2\pi \int_0^4 (x)\cdot\left(2-\sqrt{x}\right)dx \qquad V = \int_a^b 2\pi \left(\begin{array}{c} shell \\ radius \end{array}\right)\left(\begin{array}{c} shell \\ height \end{array}\right)dx$$

$$= 2\pi \int_0^4 \left(2x - x^{3/2}\right)dx$$

$$= 2\pi \left[x^2 - \frac{2}{5}x^{5/2} \right]_0^4$$

$$= 2\pi \left(16 - \frac{64}{5}\right)$$

$$= \frac{32\pi}{5} \ unit^3$$



c) the line $x = 4$

$$V = \int_0^4 2\pi(4-x)\left(2-\sqrt{x}\right)dx \qquad V = \int_a^b 2\pi \left(\begin{array}{c} shell \\ radius \end{array}\right)\left(\begin{array}{c} shell \\ height \end{array}\right)dx$$

$$= 2\pi \int_0^4 \left(8 - 4x^{1/2} - 2x - x^{3/2}\right)dx$$

$$= 2\pi \left[8x - \frac{8}{3}x^{3/2} - x^2 - \frac{2}{5}x^{5/2} \right]_0^4$$

$$= 2\pi \left(32 - \frac{64}{3} - 16 + \frac{64}{5}\right)$$

$$= \frac{224\pi}{15} \ unit^3$$

d) the line $y = 1$

$$V = 2\pi \int_0^2 (2-y)\left(y^2\right)dy \qquad V = \int_c^d 2\pi \left(\begin{array}{c} shell \\ radius \end{array}\right)\left(\begin{array}{c} shell \\ height \end{array}\right)dy$$

$$= 2\pi \int_0^2 \left(2y^2 - y^3\right)dy$$

$$= 2\pi \left[\frac{2}{3}y^3 - \frac{1}{4}y^4 \right]_0^2$$

$$= 2\pi \left(\frac{16}{3} - \frac{16}{4}\right)$$

$$= \frac{32\pi}{12}$$

$$= \frac{8\pi}{3} \ unit^3$$

Exercise

The region bounded by the curve $y=\sqrt{x}$, the x-axis, and the line $x=4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution

$$V=2\pi\int_c^d\left(\begin{matrix}shell\\ radius\end{matrix}\right)\left(\begin{matrix}shell\\ height\end{matrix}\right)dy$$

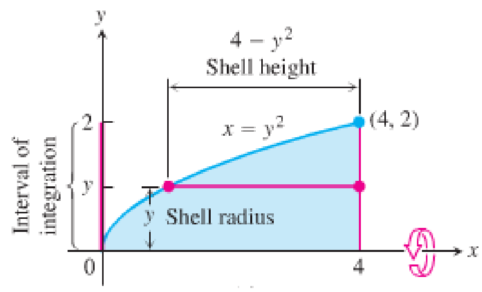
$$=2\pi\int_0^2 (y)\left(4-y^2\right)dy$$

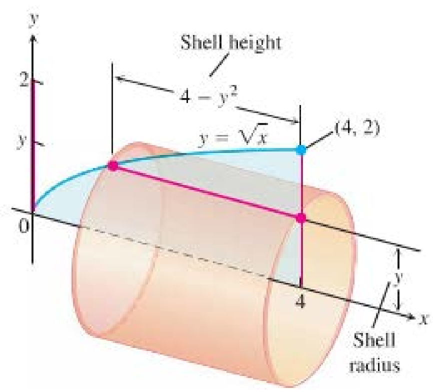
$$=2\pi\int_0^2 \left(4y-y^3\right)dy$$

$$=2\pi\left[2y^2-\frac{y^4}{4}\right]_0^2$$

$$=2\pi\left(2(2)^2-\frac{(2)^4}{4}\right)$$

$$\underline{=8\pi\ \ unit^3}$$





Exercise

The region bounded by the curve $y=\sqrt{x}$, the x-axis, and the line $x=4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution

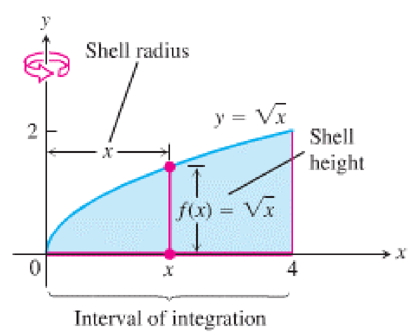
$$V=2\pi\int_0^4 (x)\left(\sqrt{x}\right)dx \qquad V=\int_a^b 2\pi\left(\begin{matrix}shell\\ radius\end{matrix}\right)\left(\begin{matrix}shell\\ height\end{matrix}\right)dx$$

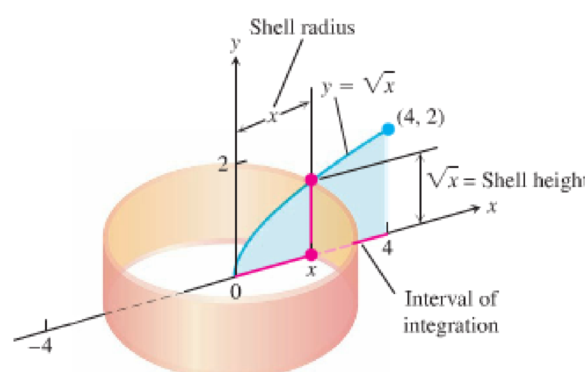
$$=2\pi\int_0^4 x^{3/2}dx$$

$$=2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4$$

$$=\frac{4}{5}\pi\left[4^{5/2}\right]$$

$$\underline{=\frac{128\pi}{5}\ \ unit^3}$$

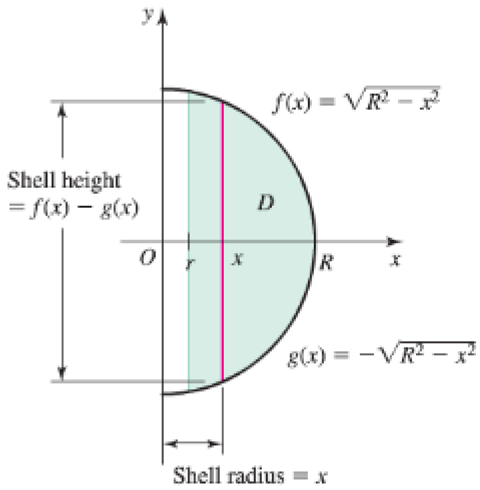


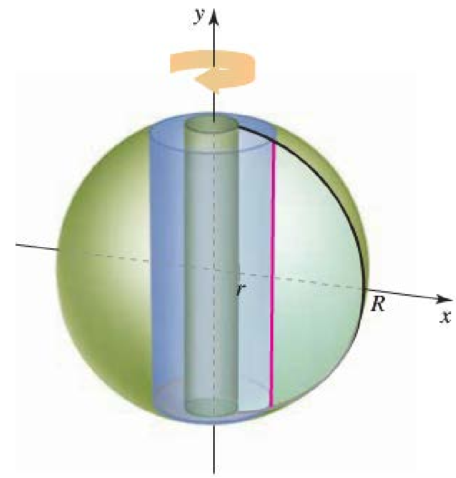


Exercise

A cylinder hole with radius r is drilled symmetrically through the center of a sphere with radius R, where $r \le R$. What is the volume of the remaining material?

Solution





Let D be the region in the xy-plane bounded above by $f(x) = \sqrt{R^2 - x^2}$, the upper half of the circle of radius R, and bounded below by $g(x) = -\sqrt{R^2 - x^2}$, the lower half of the circle of radius R, for $r \le x \le R$.

The radius of a typical shell is x. Height is $f(x) - g(x) = 2\sqrt{R^2 - x^2}$

$$V = 2\pi \int_r^R x\left(2\sqrt{R^2 - x^2}\right)dx$$

$$= -2\pi \int_r^R \left(R^2 - x^2\right)^{1/2} d\left(R^2 - x^2\right)$$

$$= -\frac{4}{3}\pi\left(R^2 - x^2\right)^{3/2}\Big|_r^R$$

$$= \frac{4}{3}\pi\left(R^2 - r^2\right)^{3/2} \quad unit^3$$

Exercise

Use the shell method to find the volume of the solid generated by revolving the region bounded by the curve and lines $y = \sqrt{x}, \quad y = 2 - x, \quad y = 0$ about the x-axis.

Solution

$x = y^2$

$y = 2 - x^2 = 2 - y^2 \;\Rightarrow\; y^2 + y - 2 = 0 \to y = \cancel{-2}, 1$

Given: $y = 0$

$$V = 2\pi \int_0^1 y\left(2 - y - y^2\right) dy$$

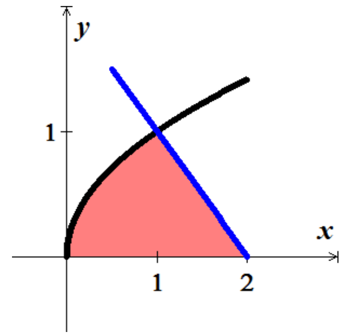
$$= 2\pi \int_0^1 \left(2y - y^2 - y^3\right) dy$$

$$= 2\pi \left(y^2 - \frac{1}{3}y^3 - \frac{1}{4}y^4\right)\Big|_0^1$$

$$= 2\pi \left(1 - \frac{1}{3} - \frac{1}{4}\right)$$

$$= 2\pi \left(\frac{5}{12}\right)$$

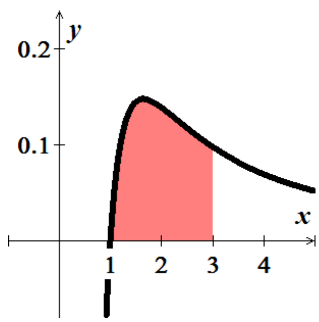
$$= \frac{5\pi}{6} \quad unit^3$$



Exercise

Find the volume of the region bounded by $y = \frac{\ln x}{x^2}$, $\quad y = 0, \quad x = 1, \quad and\ x = 3$ revolved about the *y-axis*

Solution



$$V = 2\pi \int_1^3 x \frac{\ln x}{x^2} dx$$

$$= 2\pi \int_1^3 \ln x \ d\left(\ln x\right)$$

$$= \pi \left(\ln x\right)^2 \Big|_1^3$$

$$= \pi \left(\ln 3\right)^2 \quad unit^3$$

Exercise

Find the volume of the region bounded by $y = \frac{e^x}{x}$, $\quad y = 0, \quad x = 1, \quad and\ x = 2$ revolved about the *y-axis*

Solution

$$V = 2\pi \int_1^2 x \frac{e^x}{x} dx$$

$$= 2\pi \int_1^2 e^x dx$$

$$= 2\pi e^{x}\Big|_{1}^{2}$$

$$\boxed{= 2\pi\left(e^{2} - e\right)\ unit^{3}}$$

Exercise

Find the volume of the region bounded by $y^2 = \ln x$, $y^2 = \ln x^3$, *and* $y = 2$ revolved about the *x-axis*

Solution

$$\begin{cases} y^2 = \ln x & \rightarrow \quad x = e^{y^2} \\ y^2 = \ln x^3 & \rightarrow \quad x = e^{y^2/3} \end{cases}$$

$$V = 2\pi \int_0^2 y\left(e^{y^2} - e^{y^2/3}\right) dy$$

$$= \pi \int_0^2 \left(e^{y^2} - e^{y^2/3}\right) d\left(y^2\right)$$

$$= \pi\left(e^{y^2} - 3e^{y^2/3}\right)\Big|_0^2$$

$$= \pi\left(e^4 - 3e^{4/3} - 1 + 3\right)$$

$$\boxed{= \pi\left(2 + e^4 - 3e^{4/3}\right)\ unit^3}$$

Exercise

Find the volume using both the disk/washer and shell methods of

$$y = (x-2)^3 - 2, \quad x = 0, \quad y = 25; \text{ revolved about the } y\text{-}axis$$

Solution

Using ***washers***:

$$(x-2)^3 = y + 2 \quad \rightarrow \quad x = 2 + \sqrt[3]{y+2}$$

$$x = 0 \ \Rightarrow\ y = (-2)^3 - 2 = -10$$

$$V = \pi \int_{-10}^{25} \left(2 + \sqrt[3]{y+2}\right)^2 dy \qquad\qquad V = \pi \int_c^d f(y)^2\, dy$$

$$= \pi \int_{-10}^{25} \left(4 + 4(y+2)^{1/3} + (y+2)^{2/3}\right) d(y+2)$$

$$= \pi\left(4(y+2)+3(y+2)^{4/3}+\frac{3}{5}(y+2)^{5/3}\right)\Big|_{-10}^{25}$$

$$= \pi\left(108+3(27)^{4/3}+\frac{3}{5}(27)^{5/3}-\left(-32+3(-8)^{4/3}+\frac{3}{5}(-8)^{5/3}\right)\right)$$

$$= \pi\left(108+243+\frac{729}{5}+32-48+\frac{96}{5}\right)$$

$$= \pi(335+165)$$

$$\underline{= 500\pi \ \ unit^3}$$

Using ***Shells***:

$$y = 25 \quad \to \ x = 2+\sqrt[3]{27} = 5$$

$$V = 2\pi\int_0^5 x\left(25-(x-2)^3+2\right)dx \qquad\qquad V = 2\pi\int_a^b x\left(f(x)-g(x)\right)dx$$

$$= 2\pi\int_0^5 x\left(27-x^3+6x^2-12x+8\right)dx$$

$$= 2\pi\int_0^5 \left(-x^4+6x^3-12x^2+35x\right)dx$$

$$= 2\pi\left(-\frac{1}{5}x^5+\frac{3}{2}x^4-4x^3+\frac{35}{2}x^2\right)\Big|_0^5$$

$$= 2\pi\left(-5^4+\frac{3}{2}5^4-4(5)^3+\frac{35}{2}(5)^2\right)$$

$$= 2\pi\left(-625+\frac{1875}{2}-500+\frac{875}{2}\right)$$

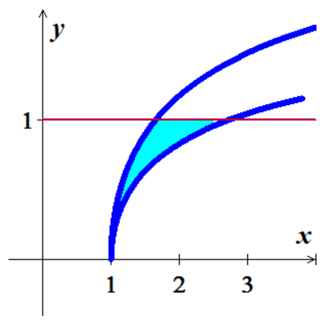
$$= 2\pi(250)$$

$$\underline{= 500\pi \ \ unit^3}$$

Exercise

Find the volume using both the disk/washer and shell methods of $y=\sqrt{\ln x}$, $\quad y=\sqrt{\ln x^2}$, $\quad y=1$; revolved about the *x-axis*

Solution



Using ***washers***:

$$y=\sqrt{\ln x}=\sqrt{\ln x^2} \quad \to \quad \ln x = \ln x^2$$

$$x = x^2 \ \Rightarrow \ \underline{x = 0,\ 1}$$

$$y = 1 = \sqrt{\ln x} \quad \Rightarrow \quad \underline{x = e}$$

$$y = 1 = \sqrt{\ln x^2} \quad \Rightarrow \quad x^2 = e \ \to \ \underline{x = \sqrt{e}}$$

$$V=\pi\int_1^{\sqrt{e}}\left(\ln x^2-\ln x\right)dx+\pi\int_{\sqrt{e}}^{e}(1-\ln x)\,dx \qquad V=\pi\int_a^b\left(f(x)^2-g(x)^2\right)dx$$

$$=\pi\int_1^{\sqrt{e}}(2\ln x-\ln x)\,dx+\pi\int_{\sqrt{e}}^{e}(1-\ln x)\,dx$$

$$=\pi\int_1^{\sqrt{e}}(\ln x)\,dx+\pi\int_{\sqrt{e}}^{e}(1-\ln x)\,dx \qquad \int \ln x\,dx=x\ln x-x$$

$$=\pi(x\ln x-x)\Big|_1^{\sqrt{e}}+\pi(2x-x\ln x)\Big|_{\sqrt{e}}^{e}$$

$$=\pi\left(\tfrac{1}{2}\sqrt{e}-\sqrt{e}+1\right)+\pi\left(2e-e-2\sqrt{e}+\tfrac{1}{2}\sqrt{e}\right)$$

$$=\pi\left(-\tfrac{1}{2}\sqrt{e}+1+e-2\sqrt{e}+\tfrac{1}{2}\sqrt{e}\right)$$

$$=\pi\left(e-2\sqrt{e}+1\right)$$

$$\underline{=\pi\left(\sqrt{e}-1\right)^2\quad unit^3}$$

Using ***Shells***:

$$y=\sqrt{\ln x}\quad\Rightarrow\quad \underline{x=e^{y^2}}$$

$$y=\sqrt{\ln x^2}\quad\Rightarrow\quad 2\ln x=y^2\ \rightarrow\ \underline{x=e^{y^2/2}}$$

$$V=2\pi\int_0^1 y\left(e^{y^2}-e^{y^2/2}\right)dy \qquad V=2\pi\int_c^d y\left(p(y)-q(y)\right)dy$$

$$=\pi\int_0^1 e^{y^2}d\left(y^2\right)-2\pi\int_0^1 e^{y^2/2}d\left(\tfrac{1}{2}y^2\right)$$

$$=\pi\left(e^{y^2}-2e^{y^2/2}\right)_0^1$$

$$=\pi\left(e-2e^{1/2}-1+2\right)$$

$$=\pi\left(e-2\sqrt{e}+1\right)$$

$$\underline{=\pi\left(\sqrt{e}-1\right)^2\quad unit^3}$$

Exercise

Find the volume using both the disk/washer and shell methods of $y=\frac{6}{x+3}$, $y=2-x$; revolved about the x-*axis*

Solution

Using ***washers***:

$$y=\frac{6}{x+3}=2-x$$

$$-x^2-x+6=6$$

$$x(x+1)=0 \Rightarrow \underline{x=-1,\ 0}$$

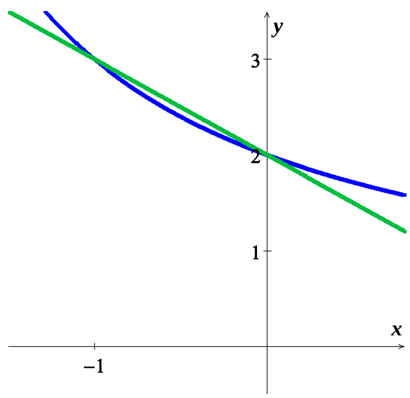
$$V=\pi\int_{-1}^{0}\left((2-x)^2-\frac{36}{(x+3)^2}\right)dx \qquad V=\pi\int_a^b\left(f(x)^2-g(x)^2\right)dx$$

$$=\pi\int_{-1}^{0}-(2-x)^2\,d(2-x)-\pi\int_{-1}^{0}\frac{36}{(x+3)^2}d(x+3)$$

$$=\pi\left(-\frac{1}{3}(2-x)^3+\frac{36}{x+3}\right)_{-1}^{0}$$

$$=\pi\left(-\frac{8}{3}+12+9-18\right)$$

$$=\frac{\pi}{3}\ unit^3$$



Using ***Shells***:

$$y=\frac{6}{x+3}\ \rightarrow\ x=\frac{6}{y}-3$$

$$y=2-x\ \rightarrow\ x=2-y$$

$$V=2\pi\int_2^3 y\left(2-y-\frac{6}{y}+3\right)dy \qquad V=2\pi\int_c^d y\left(p(y)-q(y)\right)dy$$

$$=2\pi\int_2^3\left(5y-y^2-6\right)dy$$

$$=2\pi\left(\frac{5}{2}y^2-\frac{1}{3}y^3-6y\right)_2^3$$

$$=2\pi\left(\frac{45}{2}-9-18-10+\frac{8}{3}+12\right)$$

$$=2\pi\left(\frac{151}{6}-25\right)$$

$$=\frac{\pi}{3}\ unit^3$$

Exercise

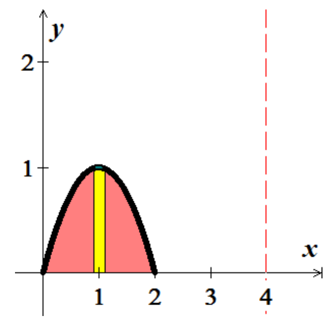
Use the shell method to find the volume of the solid generated by the revolving the plane region about the given line

$$y = 2x - x^2, \quad y = 0, \quad \text{about the line} \quad x = 4$$

Solution

$y = 2x - x^2 = 0 \quad \underline{x = 0,\ 2}$

$p(x) = 4 - x, \quad f(x) = 2x - x^2$



$$V = 2\pi \int_0^2 (4 - x)\left(2x - x^2\right) dx \qquad V = 2\pi \int_a^b p(x) f(x)\, dx$$

$$= 2\pi \int_0^2 \left(8x - 6x^2 + x^3\right) dx$$

$$= 2\pi \left(4x^2 - 2x^3 + \frac{1}{4}x^4\right)\Big|_0^2$$

$$= 2\pi (16 - 16 + 4)$$

$$\underline{= 8\pi}$$

Exercise

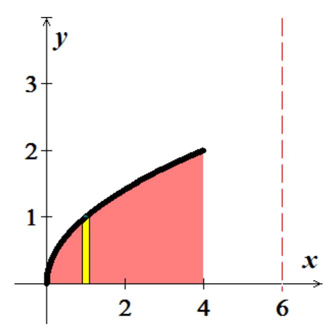
Use the shell method to find the volume of the solid generated by the revolving the plane region about the given line

$$y = \sqrt{x}, \quad y = 0, \quad x = 4, \quad \text{about the line} \quad x = 6$$

Solution

$y = \sqrt{x} = 0 \quad \underline{x = 0}$

$p(x) = 6 - x, \quad f(x) = \sqrt{x}$



$$V = 2\pi \int_0^4 (6 - x)\left(\sqrt{x}\right) dx \qquad V = 2\pi \int_a^b p(x) f(x)\, dx$$

$$= 2\pi \int_0^4 \left(6x^{1/2} - x^{3/2}\right) dx$$

$$= 2\pi \left(4x^{3/2} - \frac{2}{5}x^{5/2}\right)\Big|_0^4$$

$$= 2\pi \left(32 - \frac{64}{5}\right)$$

$$\underline{= \frac{192\pi}{5}}$$

Exercise

Use the shell method to find the volume of the solid generated by the revolving the plane region about the given line

$$y = x^2, \quad y = 4x - x^2, \quad \textit{about the line} \quad x = 4$$

Solution

$$y = x^2 = 4x - x^2 \Rightarrow 2x^2 - 4x = 0 \quad \underline{x = 0,\ 2}|$$

$$p(x) = 4 - x, \quad f(x) = 4x - x^2, \quad g(x) = x^2$$

$$V = 2\pi \int_0^2 (4 - x)\left(4x - x^2 - x^2\right) dx \qquad V = 2\pi \int_a^b p(x)\left(f(x) - g(x)\right) dx$$

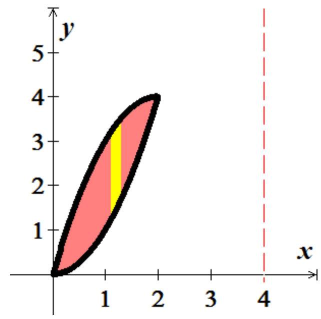
$$= 2\pi \int_0^2 (4 - x)\left(4x - 2x^2\right) dx$$

$$= 2\pi \int_0^2 \left(16x - 12x^2 + 2x^3\right) dx$$

$$= 2\pi \left(8x^2 - 4x^3 + \frac{1}{2}x^4\right)\Big|_0^2$$

$$= 2\pi(32 - 32 + 8)$$

$$\underline{= 16\pi}|$$



Exercise

Use the shell method to find the volume of the solid generated by the revolving the plane region about the given line

$$y = \frac{1}{3}x^3, \quad y = 6x - x^2, \quad \textit{about the line} \quad x = 3$$

Solution

$$y = \frac{1}{3}x^3 = 6x - x^2 \Rightarrow x\left(x^2 - 3x + 18\right) = 0 \quad \underline{x = 0,\ 3,\ \cancel{-6}}|$$

$$p(x) = 3 - x, \quad f(x) = 6x - x^2, \quad g(x) = \frac{1}{3}x^3$$

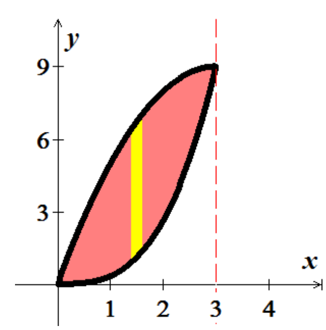
$$V = 2\pi \int_0^3 (3 - x)\left(3x - x^2 - \frac{1}{3}x^3\right) dx \qquad V = 2\pi \int_a^b p(x)\left(f(x) - g(x)\right) dx$$

$$= 2\pi \int_0^3 \left(18x - 9x^2 + \frac{1}{3}x^4\right) dx$$

$$= 2\pi \left(9x^2 - 3x^3 + \frac{1}{15}x^5\right)\Big|_0^3$$

$$= 2\pi \left(81 - 81 + \frac{81}{5}\right)$$

$$\underline{= \frac{162\pi}{5}}|$$



Exercise

Use the disk method or shell method to find the volume of the solid generated vy revolving the region bounded by the graph of the equations about the given lines.

$y = x^3, \quad y = 0, \quad x = 2$

a) *the x-axis* *b)* *the y-axis* *c)* *the line* $x = 4$

Solution

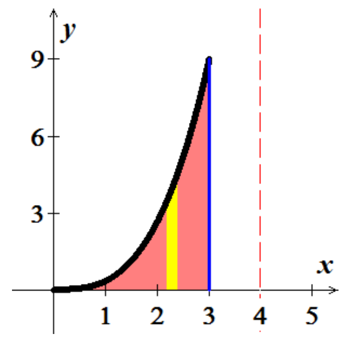
a) Using ***Disk method***:

$f(x) = x^3, \quad g(x) = 0$

$$V = \pi \int_0^2 x^6 dx \qquad V = \pi \int_a^b \left((f(x))^2 - (g(x))^2 \right) dx$$

$$= \frac{\pi}{7} x^7 \Big|_0^2$$

$$= \frac{128\pi}{7}$$



b) Using ***Shell method***:

$p(x) = x, \quad f(x) = x^3, \quad g(x) = 0$

$$V = 2\pi \int_0^2 x\left(x^3\right) dx \qquad V = 2\pi \int_a^b p(x)\left(f(x) - g(x) \right) dx$$

$$= 2\pi \int_0^2 x^4\, dx$$

$$= \frac{2\pi}{5} x^5 \Big|_0^2$$

$$= \frac{164\pi}{5}$$

c) Using ***Shell method***:

$p(x) = 4 - x, \quad f(x) = x^3, \quad g(x) = 0$

$$V = 2\pi \int_0^2 (4 - x)\left(x^3\right) dx \qquad V = 2\pi \int_a^b p(x)\left(f(x) - g(x) \right) dx$$

$$= 2\pi \int_0^2 \left(4x^3 - x^4\right) dx$$

$$= 2\pi \left(x^4 - \frac{1}{5} x^5 \right) \Big|_0^2$$

$$= 2\pi \left(16 - \frac{32}{5} \right)$$

$$= \frac{96\pi}{5}$$

Exercise

Use the disk method or shell method to find the volume of the solid generated vy revolving the region bounded by the graph of the equations about the given lines.

$y=\frac{10}{x^2}, \quad y=0, \quad x=1, \quad x=5$

a) the x-axis *b) the y-axis* *c) the line* $y=10$

Solution

a) Using ***Disk method***:

$R(x)=\frac{10}{x^2}, \quad r(x)=0$

$$V=\pi\int_1^5 100x^{-4}\,dx \qquad V=\pi\int_a^b\left((R(x))^2-(r(x))^2\right)dx$$

$$=-\frac{100}{3}\pi x^{-3}\Big|_1^5$$

$$=-\frac{100}{3}\pi\left(\frac{1}{125}-1\right)$$

$$=\frac{496\pi}{15}$$

b) Using ***Shell method***:

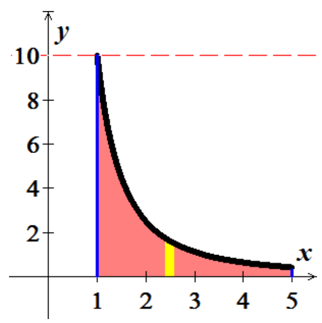
$p(x)=x, \quad f(x)=\frac{10}{x^2}, \quad g(x)=0$

$$V=2\pi\int_1^5 x\left(\frac{10}{x^2}\right)dx \qquad V=2\pi\int_a^b p(x)\left(f(x)-g(x)\right)\,dx$$

$$=20\pi\int_1^5\frac{1}{x}\,dx$$

$$=20\pi\ln x\Big|_1^5$$

$$=20\pi\ln 5$$



c) Using ***Disk method***:

$R(x)=10, \quad r(x)=10-\frac{10}{x^2}$

$$V=\pi\int_1^5\left(100-\left(10-10x^{-2}\right)^2\right)dx \qquad V=\pi\int_a^b\left((R(x))^2-(r(x))^2\right)dx$$

$$=\pi\int_1^5\left(200x^{-2}-100x^{-4}\right)dx$$

$$=100\pi\left(-\frac{2}{x}+\frac{1}{3x^3}\right)\Big|_1^5$$

$$=100\pi\left(-\frac{2}{5}+\frac{1}{375}+2-\frac{1}{3}\right)$$

$$=100\pi\left(2-\frac{274}{375}\right)$$

$$=100\pi\left(\frac{476}{375}\right)$$

$$\boxed{=\frac{1904\pi}{15}}$$

Exercise

Let V_1 and V_2 be the volumes of the solids that result when the plane region bounded by $y=\frac{1}{x}$, $y=0$, $x=\frac{1}{4}$, and $x=c$ $\left(where\ \ c>\frac{1}{4}\right)$ is revolved about the $x\text{-}axis$ and the $y\text{-}axis$, respectively. Find the value of c for which $V_1=V_2$

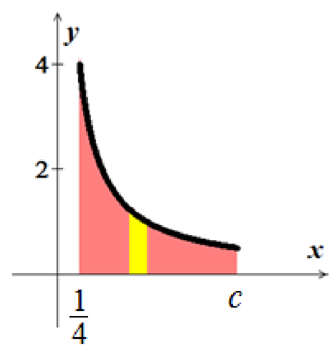
Solution

$$V_1=\pi\int_{1/4}^{c}\frac{1}{x^2}dx \qquad\qquad V=\pi\int_a^b\left(R(x)^2-r(x)^2\right)dx$$

$$=-\pi\frac{1}{x}\Big|_{1/4}^{c}$$

$$=-\pi\left(\frac{1}{c}-4\right)$$

$$\boxed{=\frac{4c-1}{c}\pi}$$



$$V_2=2\pi\int_{1/4}^{c}x\frac{1}{x}dx$$

$$=2\pi x\Big|_{1/4}^{c}$$

$$\boxed{=2\pi\left(c-\frac{1}{4}\right)}$$

Since $V_1=V_2$

$$\frac{4c-1}{c}\pi=2\pi\left(c-\frac{1}{4}\right)$$

$$4c-1=2c^2-\frac{1}{2}c$$

$$2c^2-\frac{9}{2}c+1=0$$

$$4c^2-9c+2=0 \quad\rightarrow\quad \boxed{c=2},\ \cancel{\tfrac{1}{4}} \qquad\qquad \left(\tfrac{1}{4}\text{ has no volume}\right)$$

Exercise

The region bounded by $y = r^2 - x^2$, $y = 0$, and $x = 0$ is revolved about the *y-axis* to form a paraboloid.

A hole, centered along the axis of revolution, is drilled through this solid. The hole has a radius k, $0 < k < r$. Find the volume of the resulting ring

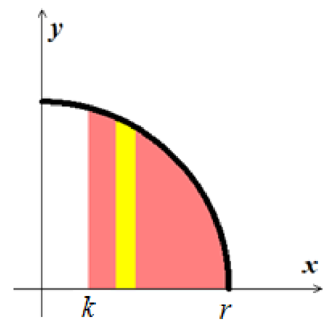
a) By integrating with respect to x

b) By integrating with respect to y.

Solution

a) $f(x) = r^2 - x^2, \quad g(x) = 0$

$$V = 2\pi \int_k^r x\left(r^2 - x^2\right)dx \qquad V = 2\pi \int_a^b x\left(f(x) - g(x)\right)dx \quad (\textbf{\textit{Shell Method}})$$



$$= 2\pi \int_k^r \left(r^2 x - x^3\right)dx$$

$$= 2\pi \left(\frac{1}{2}r^2x^2 - \frac{1}{4}x^4\right)\Big|_k^r$$

$$= \frac{1}{2}\pi\left(2r^4 - r^4 - 2r^2k^2 + k^4\right)$$

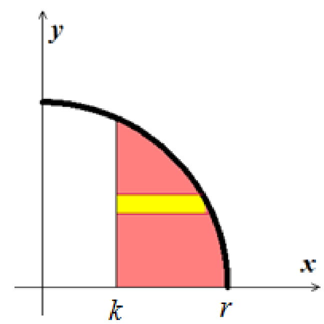
$$= \frac{1}{2}\pi\left(r^4 - 2r^2k^2 + k^4\right)$$

$$= \frac{1}{2}\pi\left(r^2 - k^2\right)^2$$

b) $y = r^2 - x^2 \quad \to \quad x = \sqrt{r^2 - y}$

$R(y) = \sqrt{r^2 - y}, \quad r(y) = k$

$$V = \pi \int_0^{r^2 - k} \left(r^2 - y - k^2\right)dy \qquad V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right)dy$$



$$= \pi\left(\left(r^2 - k^2\right)y - \frac{1}{2}y^2\right)\Big|_0^{r^2 - k}$$

$$= \pi\left(\left(r^2 - k^2\right)^2 - \frac{1}{2}\left(r^2 - k^2\right)^2\right)$$

$$= \frac{1}{2}\pi\left(r^2 - k^2\right)^2$$