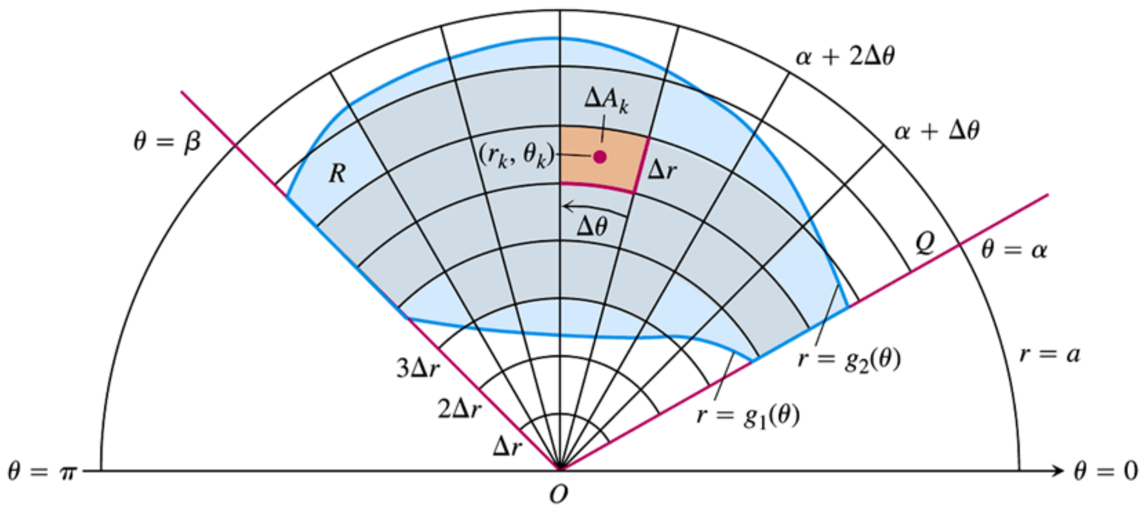
Section 3.3 – Double Integrals in Polar Coordinates

Integrals in Polar Coordinates



$$S_n = \sum_{k=1}^{n} f\left(r_k, \theta_k\right) \Delta A_k$$

If f is continuous throughout R, this sum will approach a limit as Δr and $\Delta\theta$ go to zero. The limit is called the double integral of f over R.

$$\lim_{n\to\infty} S_n = \iint_R f\left(r, \theta\right)\, dA$$

However, the area of a wedge-shaped sector of a circle having radius r and angle θ is

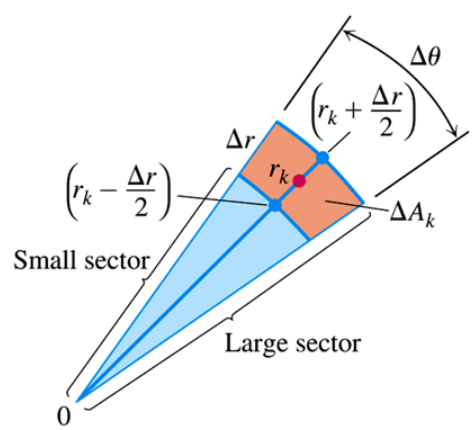
$$A = \frac{1}{2}\theta \cdot r^2$$

Inner radius: $\quad \frac{1}{2}\left(r_k - \frac{\Delta r}{2}\right)^2 \cdot \Delta\theta$

outer radius: $\quad \frac{1}{2}\left(r_k + \frac{\Delta r}{2}\right)^2 \cdot \Delta\theta$

$$\Delta A_k = \left(\begin{matrix} \textit{area of} \\ \textit{large sector} \end{matrix}\right) - \left(\begin{matrix} \textit{area of} \\ \textit{small sector} \end{matrix}\right)$$

Leads to the formula: $\Delta A_k = r_k \Delta r \Delta\theta$

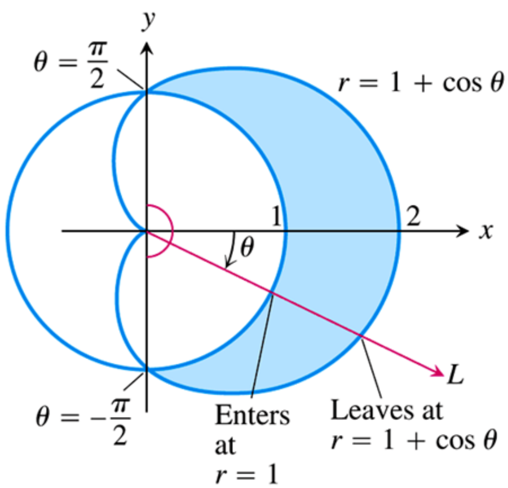


Example

Find the limits of integration for integrating $f(r,\theta)$ over the region *R* that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution

The sketch of the region:



From the graph, we can find the *r - limits of integration*. A typical ray from the origin enters *R* where $r = 1$ and leaves where $r = 1 + \cos\theta$

θ *- limits of integration*: The rays from the origin that intersects *R* run from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$. The integral is

$$\int_{-\pi 2}^{\pi/2} \int_{1}^{1+\cos\theta} f(r,\theta) r\, dr\, d\theta$$

Area in Polar Coordinates

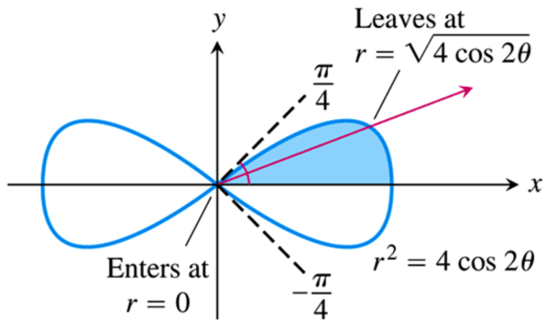
The area of a closed and bounded region *R* in the polar coordinate plane is

$$A = \iint_R r\, dr\, d\theta$$

Example

Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$

Solution



From the graph, we can determine the lemniscate limits of integration, and the total area is 4 times the first-quadrant portion, since it has a form of symmetry.

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4}(2\cos 2\theta)\,d\theta$$

$$= 4\int_0^{\pi/4}\cos 2\theta\; d(2\theta)$$

$$= 4\sin 2\theta\Big|_0^{\pi/4}$$

$$= 4\sin\frac{\pi}{2}$$

$$= 4 \quad unit^2$$

Changing Cartesian Integrals into Polar Integrals

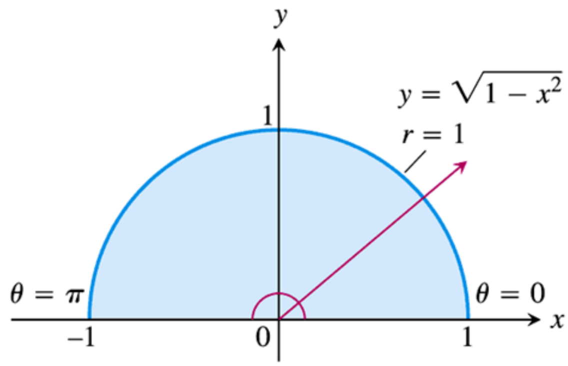
$$\iint_R f(x,y)\,dxdy = \iint_G f(r\cos\theta, r\sin\theta)\,r\;drd\theta$$

Example

Evaluate $\iint_R e^{x^2+y^2}\,dydx$

Where R is the semicircular region bounded by the x-axis and the curve $y=\sqrt{1-x^2}$

Solution



$$\iint_R e^{x^2+y^2}\,dydx = \int_0^{\pi}\int_0^1 e^{r^2}\,r\;drd\theta \qquad\qquad d\left(r^2\right)=2rdr$$

$$=\frac{1}{2}\int_0^{\pi}\int_0^1 e^{r^2}\,d\left(r^2\right)d\theta$$

$$=\frac{1}{2}\int_0^{\pi}\left[e^{r^2}\right]_0^1 d\theta$$

$$=\frac{1}{2}\int_0^{\pi}(e-1)\,d\theta$$

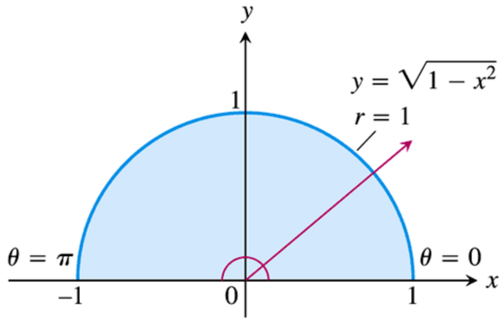
$$=\frac{1}{2}(e-1)\theta\Big|_0^{\pi}$$

$$=\frac{\pi}{2}(e-1)$$

Example

Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \left(x^2+y^2\right) dydx$

Solution



Since: $0 \le x \le 1 \quad \rightarrow \; \textit{interior of} \quad x^2+y^2=1 \quad \textit{and in QI}$

Let: $r^2 = x^2+y^2 \quad \textit{with} \quad 0 \le r \le 1$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \left(x^2+y^2\right) dydx = \int_0^{\pi/2} \int_0^1 \left(r^2\right) rdrd\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{4}r^4\right]_0^1 d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} d\theta$$

$$= \frac{1}{4}\theta\Big|_0^{\pi/2}$$

$$= \frac{\pi}{8}$$

- Or we can use the integral table to solve it

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \left(x^2+y^2\right) dydx = \int_0^1 \left[x^2\sqrt{1-x^2} + \frac{1}{3}\left(1-x^2\right)^3\right] dx$$

Example

Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the *xy*-plane.

Solution

The region of integration R is the unit circle: $x^2 + y^2 = 1 \quad \rightarrow \quad r = 1, \quad 0 \le \theta \le 2\pi$

$$Volume = \int_0^{2\pi}\int_0^1 \left(9 - r^2\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(9r - r^3\right) dr\,d\theta$$

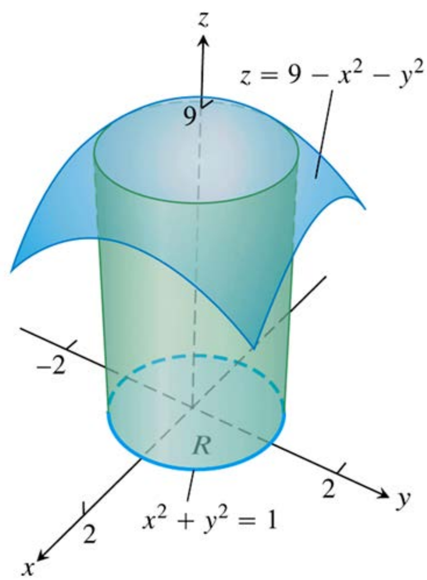
$$= \int_0^{2\pi} \left[\frac{9}{2}r^2 - \frac{1}{4}r^4\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{9}{2} - \frac{1}{4}\right) d\theta$$

$$= \frac{17}{4}\int_0^{2\pi} d\theta$$

$$= \frac{17}{4}\theta\Big|_0^{2\pi}$$

$$= \frac{17\pi}{2} \quad unit^3$$



Example

Using the polar integration, find the area of the region R in the *xy*-plane enclosed by the circle $x^2 + y^2 = 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.

Solution

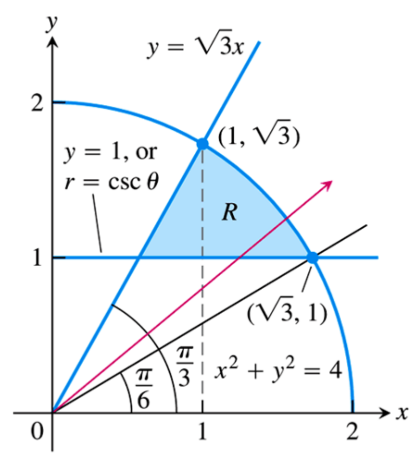
The $y = \sqrt{3}x$ has a slope of $\sqrt{3} = \tan\theta \Rightarrow \theta = \frac{\pi}{3}$

Line $y = 1$ intersects $x^2 + y^2 = 4$ when $x^2 + 1 = 4 \rightarrow x = \sqrt{3}$.

A line from origin to $\left(\sqrt{3}, 1\right)$ has a slope of

$$\frac{1}{\sqrt{3}} = \tan\theta \rightarrow \theta = \frac{\pi}{6}$$

$$\therefore \quad \boxed{\frac{\pi}{6} \le \theta \le \frac{\pi}{3}}$$



The polar coordinate r varies from the horizontal line $y = 1$ to the circle $x^2 + y^2 = 4$.

Substituting $r\sin\theta$ for y: $y=1 \to r\sin\theta = 1 \Rightarrow \underline{r = \frac{1}{\sin\theta} = \csc\theta}$ and the radius of the circle is 2.

$\therefore \quad \boxed{\csc\theta \le r \le 2}$

$$Area = \int_{\pi/6}^{\pi/3} \int_{\csc\theta}^{2} r\,dr\,d\theta$$

$$= \int_{\pi/6}^{\pi/3} \left[\frac{1}{2}r^2\right]_{\csc\theta}^{2} d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{\pi/3} \left(4 - \csc^2\theta\right) d\theta$$

$$= \frac{1}{2}\left[4\theta + \cot\theta\right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left[\frac{4\pi}{3} + \frac{1}{\sqrt{3}} - \left(\frac{4\pi}{6} + \sqrt{3}\right)\right]$$

$$= \frac{1}{2}\left(\frac{2\pi}{3} + \frac{\sqrt{3}}{3} - \sqrt{3}\right)$$

$$= \frac{1}{2}\left(\frac{2\pi - 2\sqrt{3}}{3}\right)$$

$$\underline{= \frac{\pi - \sqrt{3}}{3} \quad unit^2}$$

Exercises *Section* 3.3 – Double Integrals in Polar Coordinates

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

1. $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dydx$

2. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} \left(x^2+y^2\right)dxdy$

3. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx$

4. $\displaystyle\int_{0}^{6}\int_{0}^{y} xdxdy$

5. $\displaystyle\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}dydx$

6. $\displaystyle\int_{0}^{\ln 2}\int_{0}^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}dxdy$

7. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dxdy$

8. $\displaystyle\int_{1}^{2}\int_{0}^{\sqrt{2x-x^2}} \frac{1}{\left(x^2+y^2\right)^2}dydx$

9. Find the area of the region cut from the first quadrant by the curve $r=2(2-\sin 2\theta)^{1/2}$

10. Find the area of the region lies inside the cardioid $r=1+\cos\theta$ and outside the circle $r=1$

11. Find the area enclosed by one leaf of the rose $r=12\cos 3\theta$

12. Find the area of the region common to the interiors of the cardioids $r=1+\cos\theta$ and $r=1-\cos\theta$

13. Integrate $f(x,y)=\dfrac{\ln\left(x^2+y^2\right)}{\sqrt{x^2+y^2}}$ over the region $1\le x^2+y^2\le e$

14. Evaluate the integral $\displaystyle\int_{0}^{\infty}\int_{0}^{\infty} \frac{1}{\left(1+x^2+y^2\right)^2}dxdy$

15. The region enclosed by the lemniscates $r^2=2\cos 2\theta$ is the base of a solid right cylinder whose top is bounded by the sphere $z=\sqrt{2-r^2}$. Find the cylinder's volume.