Work and Kinetic Energy (Chapter 7 Lecture 1)

7.1 Dot Product

The dot product of two vectors $\vec{A}$ & $\vec{B}$ is defined to be the product of the magnitude of vector $\vec{A}$ and the component of vector $\vec{B}$ in the direction of vector $\vec{A}$.

$$\vec{A}\cdot\vec{B} = |\vec{A}|B_{||}$$

And if the angle between $\vec{A}$ & $\vec{B}$ is θ then $B_{||} = B\cos\theta$

$$\therefore\ \boxed{\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta}$$

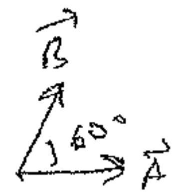
Some properties of a dot product

1. $\vec{A}\cdot\vec{A} = |\vec{A}|^2$ because $\theta = 0$
2. $\vec{A}\cdot\vec{B} = 0$ if $\vec{A}$ & $\vec{B}$ are perpendicular to each other.
3. Dot product is commutative $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$
4. Dot product is distributive over addition $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$
5. $\vec{A}\cdot C\vec{B} = C(\vec{A}\cdot\vec{B})$

Example:

a) Calculate the dot product of $\vec{A}$ & $\vec{B}$ if

$\vec{A} = 5m\ east$ & $\vec{B} = 2m\ 60°\ north\ of\ east$



Solution

$|\vec{A}| = 5m \quad |\vec{B}| = 2m \quad \theta = 60°$

$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

$= (5m)(2m)\cos 60° = 5m^2$

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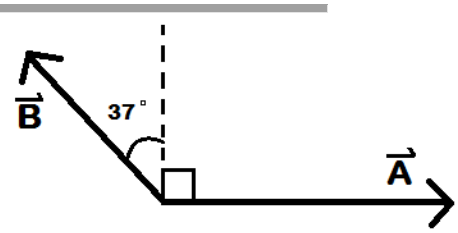
b) If $\vec{A} = 8m\ east$ & $\vec{B} = 4m\ 37°$

$|\vec{A}| = 3m \quad |\vec{B}| = 4m \quad \theta = 90° + 37° = 127°$

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos 127° = -7.2m^2$$



7.2 Dot Product in terms of x,y,z components

$$\vec{A}\cdot\vec{A} = |\vec{A}||\vec{A}|\cos 0 = |\vec{A}|^2$$

Since $|\hat{\imath}| = |\hat{\jmath}| = |\hat{k}| = 1$ it follows that

$$\boxed{\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1}$$

And since the unit vectors are perpendicular to each other

$$\boxed{\hat{\imath}\cdot\hat{\jmath} = \hat{\imath}\cdot\hat{k} = \hat{\jmath}\cdot\hat{k} = 0}$$

Now if $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$

& $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$

$$\vec{A}\cdot\vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k})\cdot(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$$

$$= A_x\hat{\imath}\cdot(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) + A_y\hat{\jmath}\cdot(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) + A_z\hat{k}\cdot(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$$

Since $\hat{\imath}\cdot\hat{\imath} = 1$ & $\hat{\imath}\cdot j = \hat{\imath}\cdot\hat{k} = 0$

$$= A_xB_x + A_yB_y + A_zB_z$$

$$\therefore \boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z}$$

The angle between two vectors may be obtained by equating the two different forms of a dot product.

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta = A_xB_x + A_yB_y + A_zB_z$$

$$\boxed{\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{|\vec{A}||\vec{B}|}}$$

Example: Given $\vec{A} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ and

$\vec{B} = -4\hat{\imath} + 4\hat{\jmath} - \hat{k}$

Calculate $\vec{A}\cdot\vec{B}$

Solution

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z = (2)(-4) + (3)(4) + (4)(-1) = 0$$

Example: Given the vectors

$\vec{A} = 2\hat{\imath} - 4\hat{\jmath}$ and $\vec{B} = \hat{\imath} + 3\hat{\jmath}$

Calculate the angle formed between $\vec{A}$ & $\vec{B}$

Solution

$$\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{|\vec{A}||\vec{B}|}$$

$A_x = 2 \quad A_y = -4 \quad A_z = 0$

$B_x = 1 \quad B_y = 3 \quad B_z = 0$

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2} = \sqrt{2^2 + (-4)^2} = \sqrt{20}$$

$$|\vec{B}|\sqrt{B_x^2 + B_y^2} = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$\cos\theta = \frac{(2)(1)+(-4)(3)}{\sqrt{20}\sqrt{10}} = -\frac{10}{\sqrt{20}\sqrt{10}} = -\frac{10}{10\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

$$\theta = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = 135°$$

Example:

Given the vectors $\vec{A} = 3\hat{\imath} - 6\hat{\jmath} + \hat{k}$

$\vec{B} = 2\hat{\imath} - 5\hat{\jmath}$

$\vec{C} = -\hat{\imath} + 7\hat{\jmath}$

a) Calculate $\vec{A}\cdot(\vec{B} + \vec{C})$

Solution

$A_x = 3 \quad A_y = -6 \quad A_z = 1$

$B_x = 2 \quad B_y = -5 \quad B_z = 0$

$C_x = -1 \quad C_y = 7 \quad C_z = 0$

$$\begin{aligned}
&\vec{A}\cdot(\vec{B} + \vec{C})\\
&= \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}\\
&= A_xB_x + A_yB_y + A_zB_z + A_xC_x + A_yC_y + A_zC_z\\
&= (3)(2) + (-6)(-5) + 0 + (3)(-1) + (-6)(7) + 0\\
&= 6 + 30 - 3 - 42\\
&\underline{\vec{A}\cdot(\vec{B} + \vec{C}) = -9}
\end{aligned}$$

b) Calculate $\vec{B}\cdot(2\vec{A} - 3\vec{C})$

Solution

$$\begin{aligned}
&\vec{B}\cdot(2\vec{A} - 3\vec{C}) = 2\vec{B}\cdot\vec{A} - 3\vec{B}\cdot\vec{C}\\
&= (A_xB_x + A_yB_y + A_zB_z) - 3(B_xC_x + B_yC_y + B_zC_z)\\
&= 2[(3)(2) + (-6)(-5) + 0] - 3[(2)(-1) + (-5)(7) + 0]\\
&= 2[36 - 3(-37)] = 72 + 107
\end{aligned}$$

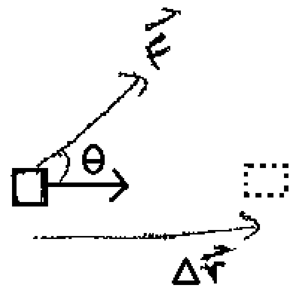
$$\vec{B}\cdot\left(2\vec{A}-3\vec{C}\right)=179$$

7.3 Work Done by a Constant Force

The work done by a force $\vec{F}$ in displacing an object through a displacement $\Delta\vec{r}$, is defined to be the dot product between the force and the displacement.

$$\boxed{w=\vec{F}\cdot\Delta\vec{r}}$$

w = work
$\vec{F}$ = Force
$\Delta\vec{r}$ = displacement



If the angle formed between the force & displacement is θ, then

$$w=\left|\vec{F}\right|\left|\Delta\vec{r}\right|\cos\theta$$

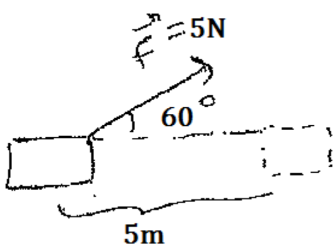
and with $\left|\vec{F}\right|=F$ and $\left|\Delta\vec{r}\right|=d$

$$\boxed{w=Fd\cos\theta}$$

The SI unit of measurement of work is $\underline{Nm}$, which is defined to be the Joule (J)

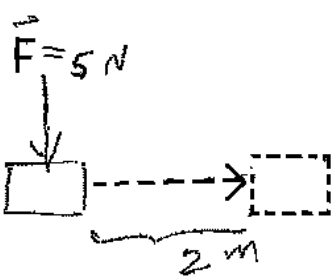
Example: In each of the following calculate the work done

a)



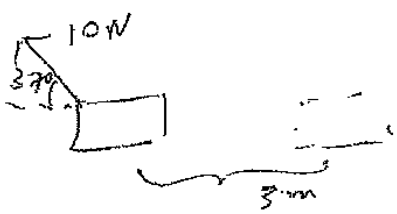
$F=5N; d=5m\ ; \theta=60°$
$w=Fd\cos\theta=(5)(5)\cos 60°$
$=12.5\,J$

b)



$F=5N; d=2m\ ; \theta=90°$
$w=Fd\cos\theta=(5)(2)\cos 90°$
$=0$

c)



$F=10N; d=3m\ ; \theta=180-37=143°$
$w=Fd\cos\theta=(10)(3)\cos 143°$
$=-24\,J$

Example: A particle is displaced from the point (-2,4) m to the point (3,7) m under the influence of a force $\vec{F}=(2\hat{\imath}+3\hat{\jmath})N$

Calculate the work done

Solution

$$\vec{r}_i = (-2\hat{\imath} + 4\hat{\jmath})m \qquad \vec{r}_f = (3\hat{\imath} + 7\hat{\jmath})m \qquad \vec{F} = (2\hat{\imath} + 3\hat{\jmath})N$$

$$w = \vec{F}\cdot\Delta\vec{r} = \vec{F}\cdot\left(\vec{r}_f - \vec{r}_i\right)$$
$$\vec{r}_f - \vec{r}_i = (3\hat{\imath} + 7\hat{\jmath})m - (-2\hat{\imath} + 4\hat{\jmath})m$$
$$= (5\hat{\imath} + 3\hat{\jmath})m$$

$$w = \vec{F}\cdot\Delta\vec{r} = [2\hat{\imath} + 3\hat{\jmath}]\cdot[5\hat{\imath} + 3\hat{\jmath}]$$
$$= (2*5 + 3*3)J = \underline{19J}$$

7.4 Work Done by a Variable Force

The work done by a force $\vec{F}$ in displacing an object through an infinite small displacement $d\vec{r}$ is given by

$$dw = \vec{F}\cdot d\vec{r}$$

And the work done in displacing an object from a position vector $\vec{r}_i$ to a position vector $\vec{r}_f$ is obtained by interpreting this equation.

$$w = \int dw = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}\cdot d\vec{r} = \int_{x_i,y_i,z_i}^{x_f,y_f,z_f} [F_x dx + F_y dy + F_z dz\,]$$

For a straight line displacement (say along the x-axis)

$$d\vec{r} = dx\hat{\imath}$$
$$\&\ \ d\vec{w} = \vec{F}\cdot d\vec{r} = (F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k})\cdot dx\hat{\imath}$$
$$= F_x dx$$

$$\boxed{w = \int_{x_i}^{x_f} F_x dx}$$

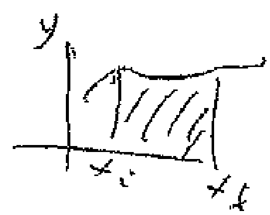
Example: A particle is displaced along the x-axis from x=2m to x=5m under the influence of a variable force that varies with x according to the formula $F = 2x^2$. Calculate the work done.

Solution

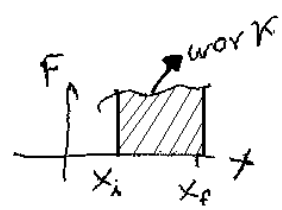
$$w = \int_{x_i}^{x_f} F_x dx = \int_2^5 (2x^2)\,dx = 2\frac{x^3}{3}\Big|_2^5$$
$$= 2\left[\frac{5^3}{3} - \frac{2^3}{3}\right] = 2\left[\frac{117}{3}\right] = \underline{78\ J}$$

7.5 Obtaining work from a graph force versus displacement

The geometrical meaning of the integral $I = \int_{x_i}^{x_f} y\,dx$ is the area enclosed between the y(x) curve and the vertical lines $x = x_i$ & $x = x_f$

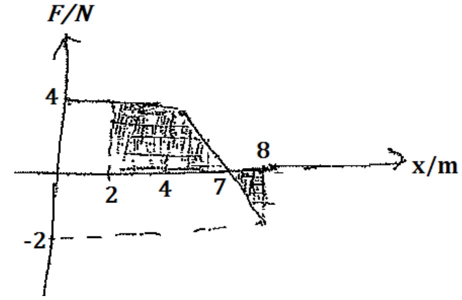


Since $w = \int_{x_i}^{x_f} F_x dx$, work done can be obtained from a graph of force versus displacement (x) as the area enclosed between the F versus x curve and the x-axis as well as between the vertical lines $x = x_i$ & $x = x_f$



Example: The following is a graph of force versus displacement for a certain particle. Calculate the work done in displacing the particle from x = 2 to x = 8

Solution
w = area of the shaded region shown
Note- Areas above the x-axis are taken to be positive, while areas below the x-axis are taken to be negative.

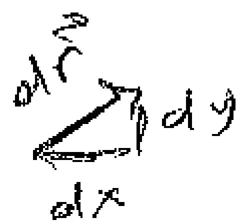


$$w = A(\underset{2}{\square}\,4) + A(4\,\underset{3}{\triangle}) - A(\overset{1}{\triangle}\,2)$$
$$= 8 + \frac{1}{2}(12) - \frac{1}{2}(2)$$
$$\underline{w = 13\,J}$$

Work Involving Two Dimensions

$$w = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} d\vec{r}$$

$$\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath}$$
$$d\vec{r} = dx_i + d_y\hat{\jmath}$$

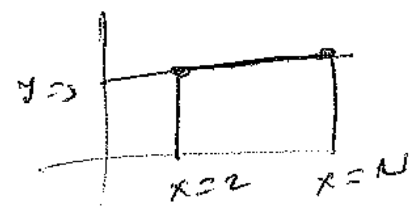


$$\vec{F} \cdot d\vec{r} = F_x dx + F_y dy$$

$$\boxed{w = \int_{(x_i,y_i)}^{(x_f,y_f)} [F_x dx + F_y dy]}$$

The path along which the work is used, can be used to express y in terms of x or vice versa.

Example: A particle is displaced by the force $\vec{F} = xy\hat{\imath} + y^2\hat{\jmath}$ from $x = 2$ to $x = 4$ along the $y = 5$ line. Calculate the work done.



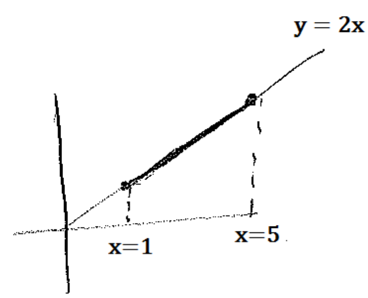
Solution

$y = 5 \implies dy = 0$
$d\vec{r} = dx\hat{\imath} + dy\hat{\jmath}$
$= dx\hat{\imath}$

$\vec{F} \cdot d\vec{r} = xydx\,\hat{\imath}\cdot\hat{\imath} = xydx$
On the path where the work is to be done $y = 5$

$$\therefore\ \vec{F} \cdot d\vec{r} = 5xdx$$
$$w = \int_{x=2}^{x=4} 5x\,dx = \frac{5x^2}{2}\Big|_{x=2}^{x=4} = 5\left(\frac{4^2}{2} - \frac{2^2}{2}\right)$$
$$= 10J$$

Example: A particle is displaced by the force $\vec{F} = x^3y^2\hat{\imath} + y^3x\,\hat{\jmath}$ along the path $y = 2x$ from $x = 1$ to $x = 5$. Calculate the work done.



Solution

$\vec{F} = x^3(2x)^2\hat{\imath} + (2x)^3x\hat{\jmath}$
$= 4x^5\hat{\imath} + 8x^4\hat{\jmath}$
$\vec{F} \cdot d\vec{r} = (4x^5\hat{\imath} + 8x^4\hat{\jmath}) \cdot (dx\hat{\imath} + dy\hat{\jmath})$
$= 4x^5dx + 8x^4dy$

But $y = 2x$
$\frac{dy}{dx} = 2$ or $dy = 2dx$

Substituting for dy

$$\vec{F} \cdot d\vec{r} = 4x^5 dx + 8x^4(2dx)$$
$$= 4x^5 dx + 16x^4 dx$$
$$= (4x^5 + 16x^4)dx$$
$$w = \int_{x=1}^{x=5} (4x^5 + 16x^4)dx = \left[\frac{4x^6}{6} + \frac{16x^5}{5}\right]\Big|_{x=1}^{x=5}$$
$$= 35412.8\,J$$

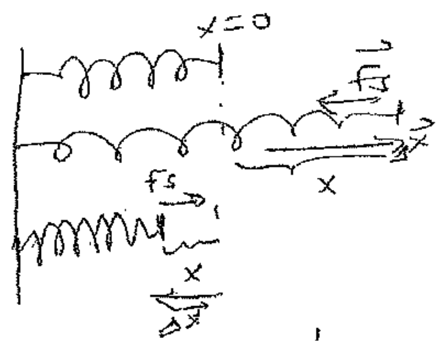
7.6 Work done by the force due to a spring
(Chapter 7 Lecture 2)

The dependence of the force due to a spring on displacement (extension or compression) is governed by Hook's Law

Hook's Law: States that the force due to a spring is directly proportional & opposite to the displacement (extension or compression) of the spring.

$$\boxed{F_s = -kx}$$

$F_s \rightarrow$ force due to a spring
$x \rightarrow$ displacement from relaxed position



k is a constant of proportionality called Hook's constant. It is constant for a given spring, but different for different springs. The unit of measurement for Hook's constant is N/m.

The work done by the force due to a spring in extending (or compressing) the spring from x_i to x_f is given by

$$w = \int_{x_i}^{x_f} F_s dx \quad \text{but} \quad F_s = -kx$$

$$\therefore \; w_s = \int_{x_i}^{x_f} (-kx)dx = -\frac{kx^2}{2}\Big|_{x_i}^{x_f} = -k\left(\frac{x_f^2}{2} - \frac{x_i^2}{2}\right)$$

$$\boxed{w_s = -\frac{k}{2}(x_f^2 - x_i^2)}$$

$w_s \rightarrow$ work done by a spring
$k \rightarrow$ Hook's Constant
$x_i(x_f) \rightarrow$ initial (final) position

Example: Calculate the work done by the force due to a spring of Hook's constant 100 N/m when

a) The spring is extended from its relaxed position by 2 cm

$k = 100$ N/m
$x_i = 0$ (relaxed position)
$x_f = 0.02$ m

$$w_s = -\frac{k}{2}(x_f^2 - x_i^2)$$
$$w_s = -\frac{100}{2}(.02^2 - 0^2)$$
$$\underline{= -2 \times 10^{-2} J}$$

b) The spring is compressed by 2 cm

$k = 100$ N/m
$x_i = 0$ (relaxed position)
$x_f = -0.02$ m

$$w_s = -\frac{k}{2}(x_f^2 - x_i^2)$$
$$w_s = -\frac{100}{2}((-.02)^2 - 0^2)$$
$$\underline{= -2 \times 10^{-2} J}$$

c) The spring is extended further from $x = 2\,cm$ to $x = 4\,cm$

$k = 100$ N/m
$x_i = 0.02$ (relaxed position)
$x_f = 0.04$ m

$$w_s = -\frac{k}{2}(x_f^2 - x_i^2)$$
$$w_s = -\frac{100}{2}(.04^2 - 0.02^2)$$
$$\underline{= -6 \times 10^{-2} J}$$

7.7 Work-Kinetic Energy Theorem:

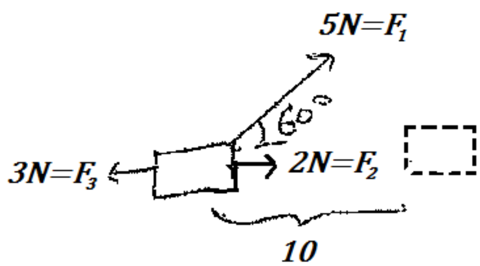
Net work done: on an object is defined to be the sum of all the work done by all the forces acting on the object or the work done by the net force acting on the object.

If an object is under the influence of a number of force $\vec{F}_1, \vec{F}_2, \ldots$ the net work is

$$w_{net} = \left(\vec{F}_1 + \vec{F}_2 + \cdots\right) \cdot \Delta\vec{r} = \vec{F}_1 \Delta\vec{r} + \vec{F}_2 \Delta\vec{r} + \cdots$$
$$= w_1 + w_2 + \cdots$$

$$\therefore \boxed{w_{net} = \vec{F}_{net} \cdot \Delta\vec{r}} \text{ where } \vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \cdots$$

Example: Calculate the net work done on the object shown.



$$w_{net} = w_1 + w_2 + w_3$$
$$= F_1 d\cos\theta_1 + F_2 d\cos\theta_2 + F_3 d\cos\theta_3$$
$$= (5)(10)\cos 60° + (2)(10)\cos 0$$
$$+ (3)(10)\cos 180°$$
$$= 25 + 20 - 30$$
$$\underline{= 15J}$$

Example: A particle is displaced from the point (2,3) m to the point (5,7) m under the influence of the following forces:

$\vec{F}_1 = [2\hat{\imath} - 3\hat{\jmath}]N$	$\vec{F}_2 = [4\hat{\imath} + 6\hat{\jmath}]N$	$\vec{F}_3 = 6\hat{\imath}\ N$

Calculate the net work done on the object.

$w_{net} = \vec{F}_1\Delta\vec{r} + \vec{F}_2\Delta\vec{r} + \vec{F}_3\Delta\vec{r}$	$\Delta\vec{r} = \vec{r}_f - \vec{r}_i$	$\vec{r}_i = (2\hat{\imath} + 3\hat{\jmath})m$ $\vec{r}_f = (5\hat{\imath} + 7\hat{\jmath})m$

$\Delta\vec{r} = [5\hat{\imath} + 7\hat{\jmath}]m - [2\hat{\imath} + 3\hat{\jmath}]m$ $= [3\hat{\imath} + 4\hat{\jmath}]m$	$w_{net} = \vec{F}_1\Delta\vec{r} + \vec{F}_2\Delta\vec{r} + \vec{F}_3\Delta\vec{r}$ $= [2\hat{\imath} - 3\hat{\jmath}] \cdot [3\hat{\imath} + 4\hat{\jmath}] + [4\hat{\imath} + 6\hat{\jmath}] \cdot [3\hat{\imath} + 4\hat{\jmath}]$ $+[6\hat{\imath}] \cdot [3\hat{\imath} + 4\hat{\jmath}]$ $= (6 - 12) + (12 + 24) + 18$ $= 48J$

Relationship between net work done and kinetic energy

From Newton's 2nd Law

$F_{net} = ma$

$$w_{net} = \int_{x_i}^{x_f} F_{net}\,dx = \int_{x_i}^{x_f} ma\,dx \int_{x_i}^{x_f} m\frac{dv}{dt}dx$$

$$= \int_{v_i}^{v_f} m\,dv\frac{dx}{dt} \quad but\ \frac{dx}{dt} = v$$

$$w_{net} = \int_{v_i}^{v_f} m\,v\,dv = m\left(\frac{v_f^2}{2} - \frac{v_i^2}{2}\right)$$

$$\therefore \boxed{w_{net} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2}$$

The expression of $\frac{1}{2}mv^2$ is called the kinetic energy of an object of mass (m) and speed (v) and denoted by KE

$$\boxed{KE = \frac{1}{2}mv^2}$$

$$\therefore \quad KE_i = \frac{1}{2}mv_i^2 \quad \& \quad KE_f = \frac{1}{2}mv_f^2$$

$$\boxed{w_{net} = KE_f - KE_i = \Delta KE}$$

This is a mathematical statement of the work-kinetic energy theorem.

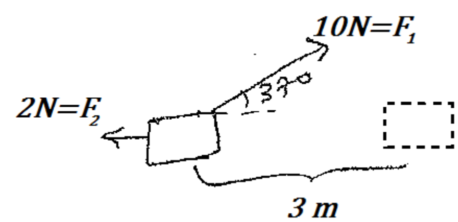
The work-kinetic energy theorem states that the net work done on an object is equal to the change in its kinetic energy.

Example: Under the influence of a number of forces, the speed of a 4kg object changed from 5m/s to 10 m/s. Calculate the net work done on the object.

$m = 4kg$
$v_i = 5$ m/s
$v_f = 10$ m/s

$$w_{net} = KE_f - KE_i = \Delta KE$$
$$= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$
$$= \frac{1}{2}(4)(10)^2 - \frac{1}{2}(4)(5)^2$$
$$= 200 - 50 = 150J$$

Example: An object of mass 6 kg is displaced by 8 m under the influence of the forces shown. If its initial speed is 2 m/s calculate its final speed.



$v_i = 2$m/s
$m = 6$ kg
$v_f = ??$

$$w_{net} = w_1 + w_2$$
$$= F_1 d\cos\theta_1 + F_2 d\cos\theta_2$$
$$= 10(3)\cos 37° + 2(3)\cos 180°$$
$$= 24 - 6 = 18\,J$$

But $w_{net} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$

$$\therefore \quad 18J = \frac{1}{2}(6)v_f^2 - \frac{1}{2}(6)(2)^2$$
$$18 = 6v_f^2 - 12$$
$$30 = 6v_f^2$$
$$v_f = \sqrt{5} \approx \underline{2.236 \text{ m/s}}$$

Example: A particle of mass 2 kg is displaced from the point (1, 1) m to the point (3, 4) m under the influence of the forces $\vec{F}_1 = [4\hat{\imath} + 6\hat{\jmath}]N$ and the $\vec{F}_2$=$[2\hat{\imath} + 10\hat{\jmath}]N$. If its initial speed at (1, 1) m is 10 m/s, calculate its final speed at (3, 4) m.

$$w_{net} = \vec{F}_{net} \cdot \Delta\vec{r}$$

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2$$
$$= [4\hat{\imath} + 6\hat{\jmath}]N + [2\hat{\imath} + 10\hat{\jmath}]N$$
$$= [6\hat{\imath} + 16\hat{\jmath}]N$$

$$\Delta\vec{r} = \vec{r}_f + \vec{r}_i$$
$$\Delta\vec{r} = [(3\hat{\imath} + 4\hat{\jmath}) - (\hat{\imath} + \hat{\jmath})]m$$
$$= [2\hat{\imath} + 3\hat{\jmath}]m$$

$$\vec{r}_i = (\hat{\imath} + \hat{\jmath})m$$
$$\vec{r}_f = (3\hat{\imath} + 4\hat{\jmath})m$$

$$w_{net} = \vec{F}_{net} \cdot \Delta\vec{r} = [6\hat{\imath} + 16\hat{\jmath}] \cdot [2\hat{\imath} + 3\hat{\jmath}]J$$
$$= 12 + 48 = 60J$$

But $w_{net} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$ $\quad$ $v_i = 10$ m/s $\quad$ m= 2kg

$$60 = \frac{1}{2}(2)v_f^2 - \frac{1}{2}(2)10^2$$
$$60 = v_f^2 - 100$$
$$v_f^2 = 160$$
$$v_f = \sqrt{160}\text{m/s} = 4\sqrt{10} \text{ m/s}$$

Example A car of mass 3000 kg initially moving with a speed of 30 m/s was stopped in a distance of 200 m.

a) Calculate the work done by friction

Net work done = work done by friction (w_f)

$$w_f = w_{net} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$
$$= \cancel{\frac{1}{2}3000(0)^2} - \frac{1}{2}(3000)30^2$$
$$= -1.35 \times 10^6 N$$

b) Calculate the force of friction

$$w_f = fd\cos 180° = -1.35 \times 10^6$$
$$-f(200) = -1.35 \times 10^6$$
$$f = 6750N$$

Power (P)

The power of a force is defined to be the rate of doing work by the force.

Average Power $(\bar{P})$

Is defined to be work done per a unit time. If a force F does work w in a time interval Δt, then average power is defined as

$$\boxed{\bar{P} = \frac{w}{\Delta t}}$$

The unit of power is Joule/second which is defined to be the Watt (W).

For a constant force $w = \vec{F} \cdot \Delta\vec{r}$ &	$\bar{P} = \frac{\vec{F}\cdot\Delta\vec{r}}{\Delta t} = \vec{F} \cdot \frac{\Delta\vec{r}}{\Delta t}$ but $\frac{\Delta\vec{r}}{\Delta t}$ is equal to average velocity$(\bar{v})$

For a constant force

$$\boxed{\bar{P} = \vec{F} \cdot \bar{\vec{v}}}$$

Instantaneous Power (P): is power at a given instant of time. Instantaneous power can be obtained as the derivative of work with respect to time

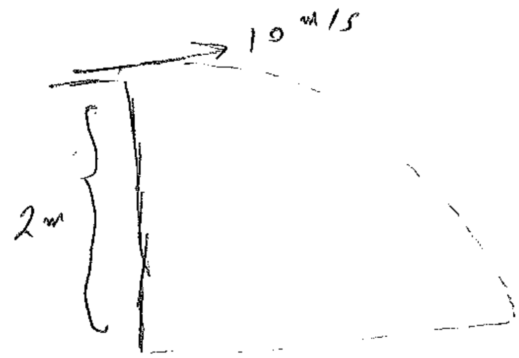
$$\boxed{P = \frac{dw}{dt}}$$

$$dw = \vec{F} \cdot d\vec{r} \quad \Longrightarrow \quad P = \frac{\vec{F} \cdot d\vec{r}}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt}$$

But $\frac{d\vec{r}}{dt}$ is instantaneous velocity $(\vec{v})$. Therefore instantaneous power at a given instant can be obtained from the dot product of the instantaneous force & instantaneous velocity at the given instant

$$\boxed{P = \vec{F} \cdot \vec{v}}$$

Example: An object of mass 0.2 kg is thrown horizontally from a 2m tall table with a speed of 10 m/s.



a) Calculate the average rate of doing work on the object by gravitational force by the time the object hits the ground.

Solution

$m = 0.2\ kg$

$v_i = 10$m/s

$\Delta y = -2m$

$\bar{P} = ??$

$$\bar{P} = \vec{F} \cdot \bar{\vec{v}} = \vec{F} \cdot \frac{\Delta \vec{r}}{\Delta t}$$

$$\vec{F} = -m|g|\hat{\jmath}$$
$$= -(0.2)(10)\hat{\jmath}$$
$$= -2N\ \hat{\jmath}$$

$$\frac{\Delta \vec{r}}{\Delta t} = \frac{\Delta x\hat{\imath} + \Delta y\hat{\jmath}}{\Delta t} = (-2\hat{\jmath}) \cdot \left(\frac{\Delta x\hat{\imath}}{\Delta t} + \frac{\Delta y\hat{\jmath}}{\Delta \mathrm{t}}\right) = -2\frac{\Delta y}{\Delta t}$$

Δt is the time taken to hit the ground

$$\Delta y = \frac{1}{2}gt^2 \quad \Rightarrow \quad -2 = \frac{1}{2}(-10)t^2$$

$$t^2 = \frac{2}{5} \approx 0.633\ seconds \quad \Rightarrow\Rightarrow$$

$$\bar{P} = \vec{F} \cdot \frac{\Delta \vec{r}}{\Delta t} = -2\frac{\Delta y}{\Delta t}$$
$$= \frac{-2(-2)}{.633}$$
$$\underline{\bar{P} = 6.3\,Watt}$$

b) Calculate the instantaneous rate of doing work on the object by gravitational force by the time the object hits the ground.

Solution

Let the velocity just before it hits the ground be $\vec{v}_f = v_{fx}\hat{\imath} + v_{fy}\hat{\jmath}$

$$P = \vec{F} \cdot \vec{v}_f \qquad \vec{F} = -m|g|\hat{\jmath} = -2(10)\hat{\jmath} = -2\hat{\jmath}\ N$$

$$\therefore \quad P = \vec{F} \cdot \vec{v}_f = (-2\hat{\jmath}) \cdot \left(v_{fx}\hat{\imath} + v_{fy}\hat{\jmath}\right)$$
$$= -2v_{fy}$$

v_{fy} may be calculated from the equations of projectile motion.

$$v_{fy}^2 = v_{iy}^2 + 2g\Delta y$$
$$= 2(-10)(-2)$$
$$v_{fy}^2 = 40$$
$$v_{fy} = -\sqrt{40}\ \text{m/s} \approx -6.32\ \text{m/s}$$

(negative b/c it's downwards)

$$\therefore \quad P = \vec{F} \cdot \vec{v} = -m|g|v_{fy}$$
$$= -2v_{fy} = -2(-6.32)$$
$$= -12.6\,Watt$$