

Section 4.3 – Mathematical Induction

If n is a positive integer and we let P_n denote the mathematical statement $(xy)^n = x^n y^n$, we obtained the following *infinite sequence* of statements:

$$\text{Statement } P_1 : (xy)^1 = x^1 y^1$$

$$\text{Statement } P_2 : (xy)^2 = x^2 y^2$$

$$\text{Statement } P_3 : (xy)^3 = x^3 y^3$$

$$\vdots \qquad \qquad \qquad \vdots$$

$$\text{Statement } P_n : (xy)^n = x^n y^n$$

$$\vdots \qquad \qquad \qquad \vdots$$

Principle of Mathematical Induction

If with each positive integer n there is associated a statement P_n then all the statements P_n are true, provided the following two conditions are satisfied.

- 1) P_1 is true.
- 2) Whenever k is a positive integer such that P_k is true, then P_{k+1} is also true.

Steps in Applying the Principle of Mathematical Induction

- 1) Show that P_1 is true.
- 2) Assume that P_k is true, and then prove that P_{k+1} is true.

Example

Use the mathematical induction to prove that for every positive integer n , the sum of the first n positive integers is:

$$\frac{n(n+1)}{2}$$

Solution

(1) For $n = 1 \Rightarrow \frac{1(1+1)}{2} = 1$; hence P_1 is true.

(2) Assume that P_k is true. Thus the induction hypothesis is: $1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$

For $k + 1$:

$$1 + 2 + 3 + \dots + k + (k+1) \stackrel{?}{=} \frac{(k+1)((k+1)+1)}{2}$$

$$1 + 2 + 3 + \dots + k + (k+1) = (1 + 2 + 3 + \dots + k) + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1)$$

Induction hypothesis

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

Factor out k + 1

$$= \frac{(k+1)((k+1)+1)}{2}$$

Change form of k + 2

This shows that P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed

Example

Prove that for every positive integer n ,

$$1^2 + 3^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Solution

$$(1) \text{ For } n = 1 \Rightarrow 1^2 = \frac{1(2(1)-1)(2(1)+1)}{3}$$

$$1 = \frac{3}{3}$$

$$1 = 1 \quad \checkmark \quad \text{hence } P_1 \text{ is true.}$$

$$(2) \quad 1^2 + 3^2 + \dots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$$

For $k+1$:

$$1^2 + 3^2 + \dots + (2k-1)^2 + (2(k+1)-1)^2 \stackrel{?}{=} \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

$$1^2 + 3^2 + \dots + (2k-1)^2 + (2(k+1)-1)^2 \stackrel{?}{=} \frac{(k+1)(2k+1)(2k+3)}{3}$$

$$1^2 + 3^2 + \dots + (2k-1)^2 + [2k+2-1]^2 = \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3}$$

$$= \frac{(2k+1)[k(2k-1) + 3(2k+1)]}{3}$$

$$= \frac{(2k+1)(2k^2 - k + 6k + 3)}{3}$$

$$= \frac{(2k+1)(2k^2 + 5k + 3)}{3}$$

$$= \frac{(2k+1)(k+1)(2k+3)}{3} \quad \checkmark$$

This shows that P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed

Example

Prove that 2 is a factor of $n^2 + 5n$ for every positive integer n ,

Solution

$$\begin{aligned} \text{(1) For } n = 1 \Rightarrow n^2 + 5n &= 1^2 + 5(1) \\ &= 6 \\ &= 2 \cdot 3 \quad \checkmark \end{aligned}$$

Thus, 2 is a factor of $n^2 + 5n$ for $n = 1$; hence P_1 is true.

$$\begin{aligned} \text{(2) } 2 \text{ is a factor of } k^2 + 5k &\Leftrightarrow k^2 + 5k = 2p \\ \text{is 2 a factor of } (k+1)^2 + 5(k+1)? \end{aligned}$$

$$\begin{aligned} (k+1)^2 + 5(k+1) &= k^2 + 2k + 1 + 5k + 5 \\ &= k^2 + 5k + 2k + 6 \\ &= (k^2 + 5k) + 2(k+3) \\ &= 2p + 2(k+3) \\ &= 2 \cdot (p + k + 3) \quad \checkmark \end{aligned}$$

Thus, 2 is a factor of the last expression; hence P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed

Steps in Applying the Extended Principle of Mathematical Induction

1. Show that P_1 is true.
2. Assume that P_k is true with $k \geq j$, and then prove that P_{k+1} is true.

Example

Let a be a nonzero real number such that $a > -1$. Prove that $(1+a)^n > 1+na$ for every integer $n \geq 2$.

Solution

For $n = 1 \Rightarrow (1+a)^1 > 1+(1)a \Rightarrow P_1$ is false.

Step 1. For $n = 2 \Rightarrow (1+a)^2 \overset{?}{>} 1+(2)a$

$$1+2a+a^2 > 1+a \quad \checkmark$$

$\Rightarrow P_2$ is true.

Step 2. Assume that P_k is true $(1+a)^k > 1+ka$

We need to prove that P_{k+1} is true, that is $(1+a)^{k+1} > 1+(k+1)a$

$$\begin{aligned}(1+a)^{k+1} &= (1+a)^k (1+a)^1 \\ &> (1+ka)(1+a)\end{aligned}$$

$$\begin{aligned}(1+ka)(1+a) &= 1+a+ka+ka^2 \\ &= 1+(a+ka)+ka^2 \\ &= 1+a(k+1)+ka^2 \\ &> 1+(k+1)a\end{aligned}$$

$$\begin{aligned}(1+a)^{k+1} &> (1+ka)(1+a) \\ &> 1+(k+1)a \quad \checkmark\end{aligned}$$

Thus, P_{k+1} is also true.

\therefore By the mathematical induction, the proof is completed

Exercises Section 4.3 – Mathematical Induction

1. Find all positive integers n for which the given statement is not true
 $a) \quad 3^n > 6n \qquad b) \quad 3^n > 2n+1 \qquad c) \quad 2^n > n^2 \qquad d) \quad n! > 2n$
2. Prove that the statement is true for every positive integer n . $2 + 4 + 6 + \dots + 2n = n(n+1)$
3. Prove that the statement is true for every positive integer n . $1 + 3 + 5 + \dots + (2n-1) = n^2$
4. Prove that the statement is true for every positive integer n .
 $2 + 7 + 12 + \dots + (5n-3) = \frac{1}{2}n(5n-1)$
5. Prove that the statement is true: $1 + 2 \cdot 2 + 3 \cdot 2^2 + \dots + n \cdot 2^{n-1} = 1 + (n-1) \cdot 2^n$
6. Prove that the statement is true: $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$
7. Prove that the statement is true: $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$
8. Prove that the statement is true: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$
9. Prove that the statement is true: $\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + \dots + \frac{1}{(3n-2) \cdot (3n+1)} = \frac{n}{3n+1}$
10. Prove that the statement is true: $\frac{4}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \dots + \frac{4}{5^n} = 1 - \frac{1}{5^n}$
11. Prove that the statement is true: $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$
12. Prove that the statement is true: $3 + 3^2 + 3^3 + \dots + 3^n = \frac{3}{2}(3^n - 1)$
13. Prove that the statement is true: $x^{2n} + x^{2n-1}y + \dots + xy^{2n-1} + y^{2n} = \frac{x^{2n+1} - y^{2n+1}}{x - y}$
14. Prove that the statement is true: $5 \cdot 6 + 5 \cdot 6^2 + 5 \cdot 6^3 + \dots + 5 \cdot 6^n = 6(6^n - 1)$
15. Prove that the statement is true: $7 \cdot 8 + 7 \cdot 8^2 + 7 \cdot 8^3 + \dots + 7 \cdot 8^n = 8(8^n - 1)$
16. Prove that the statement is true: $3 + 6 + 9 + \dots + 3n = \frac{3n(n+1)}{2}$
17. Prove that the statement is true: $5 + 10 + 15 + \dots + 5n = \frac{5n(n+1)}{2}$
18. Prove that the statement is true: $1 + 3 + 5 + \dots + (2n-1) = n^2$
19. Prove that the statement is true: $4 + 7 + 10 + \dots + (3n+1) = \frac{n(3n+5)}{2}$

20. Prove that the statement is true for every positive integer n . $n < 2^n$
21. Prove that the statement is true for every positive integer n . 3 is a factor of $n^3 - n + 3$
22. Prove that the statement is true for every positive integer n . 4 is a factor of $5^n - 1$
23. Prove that the statement by mathematical induction: $(a^m)^n = a^{mn}$ (a and m are constant)
24. Prove that the statement by mathematical induction: $2^n > 2n$ if $n \geq 3$
25. Prove that the statement by mathematical induction: If $0 < a < 1$, then $a^n < a^{n-1}$
26. Prove that the statement by mathematical induction: If $n \geq 4$, then $n! > 2^n$
27. Prove that the statement by mathematical induction: $3^n > 2n + 1$ if $n \geq 2$
28. Prove that the statement by mathematical induction: $2^n > n^2$ for $n > 4$
29. Prove that the statement by mathematical induction: $4^n > n^4$ for $n \geq 5$
30. A pile of n rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must moved one at a time, to a different peg with no ring ever placed on top of a smaller ring. Find the least number of moves that would be required. Prove your result by mathematical induction.

