Section 3.2 – Angle and Orthogonality in Inner Product Spaces

Cosine Formula

If u and v are nonzero vectors that implies $\Rightarrow$ $\cos\theta = \frac{u.v}{\|u\|.\|v\|}$ $\rightarrow$ $\boxed{\theta = \cos^{-1}\left(\frac{\langle \boldsymbol{u},\boldsymbol{v}\rangle}{\|\boldsymbol{u}\|.\|\boldsymbol{v}\|}\right)}$

$$-1 \le \frac{\boldsymbol{u}.\boldsymbol{v}}{\|\boldsymbol{u}\|.\|\boldsymbol{v}\|} \le 1$$

Example

Let R^4 have the Euclidean inner product. Find the cosine angle θ between the vectors $\boldsymbol{u} = (4,3,1,-2)$ and $\boldsymbol{v} = (-2,1,2,3)$.

Solution

$$\|\boldsymbol{u}\| = \sqrt{4^2 + 3^2 + 1^2 + (-2)^2} = \underline{\sqrt{30}}|$$

$$\|\boldsymbol{v}\| = \sqrt{(-2)^2 + 1^2 + 2^2 + 3^2} = \sqrt{18} = \underline{3\sqrt{2}}|$$

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle = 4(-2) + 3(1) + 1(2) - 2(3) = \underline{-9}|$$

$$\cos\theta = \frac{\langle \boldsymbol{u},\boldsymbol{v}\rangle}{\|\boldsymbol{u}\|.\|\boldsymbol{v}\|}$$

$$= -\frac{9}{3\sqrt{30}\sqrt{2}}$$

$$= -\frac{3}{\sqrt{60}}$$

$$\underline{= -\frac{3}{2\sqrt{15}}}|$$

Theorem – Cauchy-Schwarz Inequality

If v and w are vectors in a real inner product space *V*, then

$$\|\langle \boldsymbol{u},\boldsymbol{v}\rangle\| \le \|\boldsymbol{u}\|.\|\boldsymbol{v}\|$$

The following two alternative forms of the Cauchy-Schwarz inequality are useful to know:

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle^2 \le \langle \boldsymbol{u},\boldsymbol{u}\rangle\langle \boldsymbol{v},\boldsymbol{v}\rangle$$

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle^2 \le \|\boldsymbol{u}\|^2.\|\boldsymbol{v}\|^2$$

Theorem

If $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors in a real inner product space V, and if k is any scalar, then

a) $\|\boldsymbol{u}+\boldsymbol{v}\| \le \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ (***Triangle inequality for vectors***)

b) $d(\boldsymbol{u},\boldsymbol{v}) \le d(\boldsymbol{u},\boldsymbol{w}) + d(\boldsymbol{w},\boldsymbol{v})$ (***Triangle inequality for distances***)

Proof **(*a*)**

$$
\begin{aligned}
\|\boldsymbol{u}+\boldsymbol{v}\|^2 &= \langle \boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}+\boldsymbol{v} \rangle \\
&= \langle \boldsymbol{u},\boldsymbol{u} \rangle + 2\langle \boldsymbol{u},\boldsymbol{v} \rangle + \langle \boldsymbol{v},\boldsymbol{v} \rangle \\
&\le \langle \boldsymbol{u},\boldsymbol{u} \rangle + 2\left|\langle \boldsymbol{u},\boldsymbol{v} \rangle\right| + \langle \boldsymbol{v},\boldsymbol{v} \rangle \\
&\le \langle \boldsymbol{u},\boldsymbol{u} \rangle + 2\|\boldsymbol{u}\|\|\boldsymbol{v}\| + \langle \boldsymbol{v},\boldsymbol{v} \rangle \\
&= \|\boldsymbol{u}\|^2 + 2\|\boldsymbol{u}\|\|\boldsymbol{v}\| + \|\boldsymbol{v}\|^2 \\
&= \left(\|\boldsymbol{u}\| + \|\boldsymbol{v}\|\right)^2
\end{aligned}
$$

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 \le \left(\|\boldsymbol{u}\| + \|\boldsymbol{v}\|\right)^2$$

$$\|\boldsymbol{u}+\boldsymbol{v}\| \le \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$$

Definition

Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in an inner product space are called orthogonal if $\langle \boldsymbol{u},\boldsymbol{v} \rangle = 0$

Example

The vectors $\boldsymbol{u} = (1,1)$ and $\boldsymbol{v} = (1,-1)$ are orthogonal with respect to the Euclidean inner product on R^2, since

$$\boldsymbol{u} \bullet \boldsymbol{v} = 1(1) + 1(-1) = 0$$

They are not orthogonal with the respect to the weighted Euclidean inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 3u_1v_1 + 2u_2v_2$, since

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 3(1)(1) + 2(1)(-1) = 1 \ne 0$$

Example

$U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ *and* $V = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ are orthogonal, since

$$U \cdot V = 1(0) + 0(2) + 1(0) + 1(0) = \underline{0|}$$

Definition

If W is a subspace of an inner product space V, then the set of all vectors are orthogonal to every vector in W is called the ***orthogonal complement*** of W and is denoted by the symbol $W^{\perp}$

Theorem

If W is a subspace of an inner product space V, then:

a) $W^{\perp}$ is a subspace of V.

b) $W \cap W^{\perp} = \{0\}$

Proof

a) Let set $W^{\perp}$ contains at least the zero vector, since $\langle \mathbf{0},\ \boldsymbol{w} \rangle = 0$ for every vector $\boldsymbol{w}$ in W. We need to show that $W^{\perp}$ is closed under addition and scalar multiplication.

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $W^{\perp}$, so every vector $\boldsymbol{w}$ in W we have $\langle \boldsymbol{u},\ \boldsymbol{w} \rangle = 0$ and $\langle \boldsymbol{v},\ \boldsymbol{w} \rangle = 0$

$$\langle \boldsymbol{u} + \boldsymbol{v},\ \boldsymbol{w} \rangle = \langle \boldsymbol{u},\ \boldsymbol{w} \rangle + \langle \boldsymbol{v},\ \boldsymbol{w} \rangle = 0 + 0 = 0 \qquad \textit{Closed under addition}$$

$$\langle k\boldsymbol{u},\ \boldsymbol{w} \rangle = k\langle \boldsymbol{u},\ \boldsymbol{w} \rangle = k(0) = 0 \qquad \textit{Closed under scalar multiplication}$$

Which proves that $\boldsymbol{u} + \boldsymbol{w}$ and $k\boldsymbol{u}$ are in $W^{\perp}$

b) If $\boldsymbol{v}$ is any vector in both W and $W^{\perp}$, then $\boldsymbol{v}$ is orthogonal to itself; that is, $\langle \boldsymbol{v},\ \boldsymbol{v} \rangle = 0$. It follows from the positivity axiom for inner products that $\boldsymbol{v} = 0$

Theorem

If W is a subspace of a finite-dimensional inner product space V, then the orthogonal complement of $W^{\perp}$ is W; that is

$$\left(W^{\perp}\right)^{\perp} = W$$

Example

Let W be the subspace of R^6 spanned by the vectors

$$\boldsymbol{w}_1 = (1,3,-2,0,2,0), \qquad \boldsymbol{w}_2 = (2,6,-5,-2,4,-3)$$
$$\boldsymbol{w}_3 = (0,0,5,10,0,15), \qquad \boldsymbol{w}_4 = (2,6,0,8,4,18)$$

Find a basis for the orthogonal complement of W.

Solution

The Space W is the same as the row space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution

$$\left(x_1, x_2, x_3, x_4, x_5, x_6\right) = \left(-3x_2 - 4x_4 - 2x_5,\ x_2,\ -2x_4,\ x_4,\ x_5,\ 0\right)$$
$$= x_2(-3,1,0,0,0,0) + x_4(-4,0,-2,1,0,0) + x_5(-2,0,0,0,1,0)$$

$$v_1 = (-3,1,0,0,0,0), \quad v_2(-4,0,-2,1,0,0), \quad v_3(-2,0,0,0,1,0)$$

Definition

A collection of vectors in $\mathbb{R}^n$ (or inner space) is called orthogonal if any 2 are perpendicular.

$$v_i . v_j = v^T v = \begin{cases} 0 & for\ i \neq j \quad (orthogonal\ vectors) \\ 1 & for\ i = j \quad (unit\ vectors) \end{cases}$$

Theorem

If $v_1, \ldots, v_m$ are nonzero orthogonal vectors, then they are linearly independent.

Definition

A vector $\boldsymbol{v}$ is called normal if $\|v\| = 1$

A collection of vectors $v_1, \ldots, v_m$ is called orthonormal if they are orthogonal and each $\|v_i\| = 1$.

An orthonormal basis is a basis made up of orthonormal vectors.

Example

Q rotates every vector in the plane through the angle θ.

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$Q^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \boxed{= Q^T} \qquad \cos^2\theta + \sin^2\theta = 1$$

The dot product $(\cos\theta\sin\theta - \sin\theta\cos\theta = 0)$, the columns are orthogonal.

They are unit vectors because $\cos^2\theta + \sin^2\theta = 1$. Those columns give an orthonormal basis for the plane $\boldsymbol{R}^2$.

We have: $QQ^T = I = Q^T Q$ (This type is called ***rotation***)

Exercises *Section* 3.2 – Angle and Orthogonality in Inner Product Spaces

1. Which of the following form orthonormal sets?

a) (1, 0), (0, 2) in $\boldsymbol{R}^2$

b) $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\boldsymbol{R}^2$

c) $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ in $\boldsymbol{R}^2$

d) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ in $\boldsymbol{R}^3$

e) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ in $\boldsymbol{R}^3$

f) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ in $\boldsymbol{R}^3$

2. Find the cosine of the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

a) $\boldsymbol{u}=(1,-3), \quad \boldsymbol{v}=(2,4)$

b) $\boldsymbol{u}=(-1,0), \quad \boldsymbol{v}=(3,8)$

c) $\boldsymbol{u}=(-1,5,2), \quad \boldsymbol{v}=(2,4,-9)$

d) $\boldsymbol{u}=(4,1,8), \quad \boldsymbol{v}=(1,0,-3)$

e) $\boldsymbol{u}=(1,0,1,0), \quad \boldsymbol{v}=(-3,-3,-3,-3)$

f) $\boldsymbol{u}=(2,1,7,-1), \quad \boldsymbol{v}=(4,0,0,0)$

3. Find the cosine of the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$.

a) $A=\begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix} \quad B=\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$

b) $A=\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} \quad B=\begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$

4. Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

a) $\boldsymbol{u}=(-1,3,2), \quad \boldsymbol{v}=(4,2,-1)$

b) $\boldsymbol{u}=(a,b), \quad \boldsymbol{v}=(-b,a)$

c) $\boldsymbol{u}=(-2,-2,-2), \quad \boldsymbol{v}=(1,1,1)$

d) $\boldsymbol{u}=(-4,\ 6,\ -10,\ 1), \quad \boldsymbol{v}=(2,\ 1,\ -2,\ 9)$

e) $\boldsymbol{u}=(-4,\ 6,\ -10,\ 1), \quad \boldsymbol{v}=(2,\ 1,\ -2,\ 9)$

5. Do there exist scalars k and l such that the vectors $\boldsymbol{u}=(2,k,6), \quad \boldsymbol{v}=(l,5,3), \quad \textit{and} \quad \boldsymbol{w}=(1,2,3)$ are mutually orthogonal with respect to the Euclidean inner product?

6. Let $\boldsymbol{R}^3$ have the Euclidean inner product. For which values of k are $\boldsymbol{u}$ and $\boldsymbol{v}$ orthogonal?

a) $\boldsymbol{u}=(2,1,3), \quad \boldsymbol{v}=(1,7,k)$

b) $\boldsymbol{u}=(k,k,1), \quad \boldsymbol{v}=(k,5,6)$

7. Let V be an inner product space. Show that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal unit vectors in V, then $\|\boldsymbol{u}-\boldsymbol{v}\|=\sqrt{2}$

8. Let **S** be a subspace of $\mathbb{R}^n$. Explain what $\left(\mathbf{S}^{\perp}\right)^{\perp} = \mathbf{S}$ means and why it is true.

9. The methane molecule CH_4 is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at $(0,\ 0,\ 0)$, $(1,\ 1,\ 0)$, $(1,\ 0,\ 1)$ and $(0,\ 1,\ 1)$ – (***note*** that all six edges have length $\sqrt{2}$, so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$ to the vertices?

10. Determine if the given vectors are orthogonal.

$\boldsymbol{x}_1 = (1,\ 0,\ 1,\ 0)$, $\boldsymbol{x}_2 = (0,\ 1,\ 0,\ 1)$, $\boldsymbol{x}_3 = (1,\ 0,\ -1,\ 0)$, $\boldsymbol{x}_4 = (1,\ 1,\ -1,\ -1)$

11. Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

a) $\left(\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)\ \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right)\ \left(-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}\right)$

b) $\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{1}{3}\right)\ \left(\frac{2}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)\ \left(\frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\right)$