

Section 4.6 – Surfaces Integrals

We have defined curves in the plane in three different ways:

Explicit form: $y = f(x)$

Implicit form: $F(x, y) = 0$

Parametric vector form: $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} \quad a \leq t \leq b$

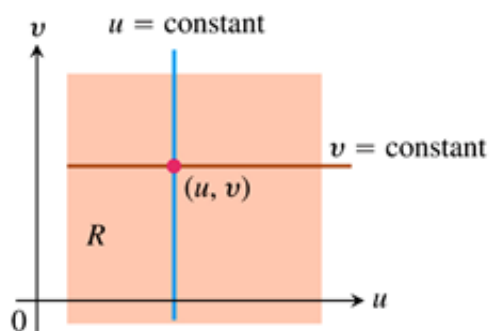
And

Explicit form: $z = f(x, y)$

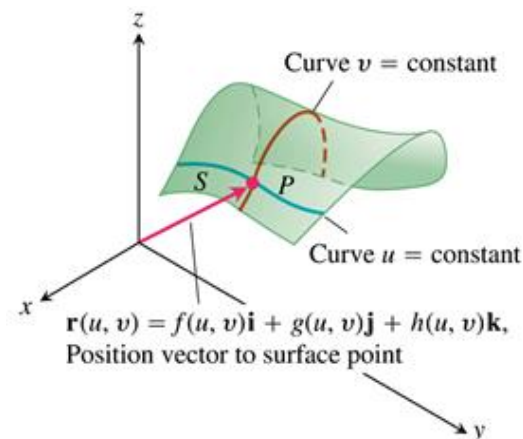
Implicit form: $F(x, y, z) = 0$

Parameterizations of Surfaces

Suppose:



Parameterization



We call the range of \mathbf{r} the **surface** S defined or traced by \mathbf{r} .

u and v : variable parameters

R : parameter domain

Example

Find a parameterization of the cone $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 1$

Solution

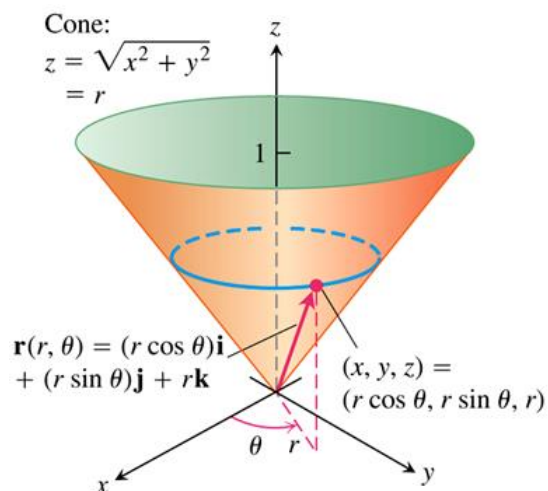
$$x = r \cos \theta, \quad y = r \sin \theta$$

$$z = \sqrt{x^2 + y^2} = r$$

Assume $u = r$ and $v = \theta$

$$\vec{r}(r, \theta) = (r \cos \theta)\hat{i} + (r \sin \theta)\hat{j} + r\hat{k}$$

$$0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi$$



Example

Find a parameterization of the cone $x^2 + y^2 + z^2 = a^2$

Solution

A typical point (x, y, z) on the sphere has

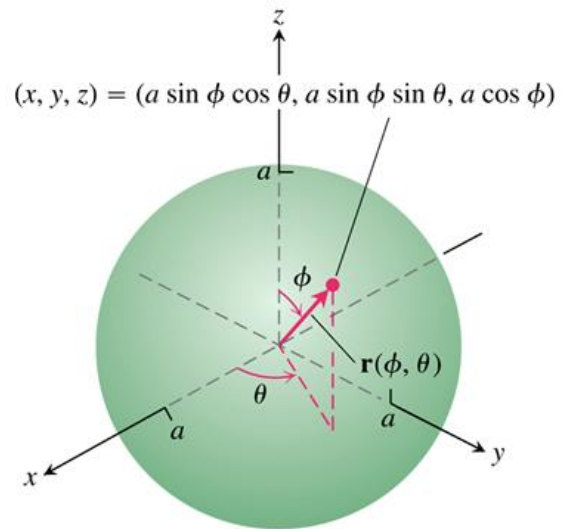
$$x = a \sin \phi \cos \theta, \quad y = a \sin \phi \sin \theta, \quad z = a \cos \phi$$

$$0 \leq \phi \leq 2\pi, \quad 0 \leq \theta \leq 2\pi$$

Taking $u = \phi$ and $v = \theta$

$$\vec{r}(\phi, \theta) = (a \sin \phi \cos \theta) \hat{i} + (a \sin \phi \sin \theta) \hat{j} + (a \cos \phi) \hat{k}$$

The parameterization is one-to-one on the interior of the domain R , though not on its boundary “poles” where $\phi = 0$ or $\theta = \pi$



Example

Find a parameterization of the cone $x^2 + (y - 3)^2 = 9, \quad 0 \leq z \leq 5$

Solution

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

$$x^2 + y^2 - 6y + 9 = 9$$

$$x^2 + y^2 - 6y = 0$$

$$r^2 - 6r \sin \theta = 0$$

$$r(r - 6 \sin \theta) = 0$$

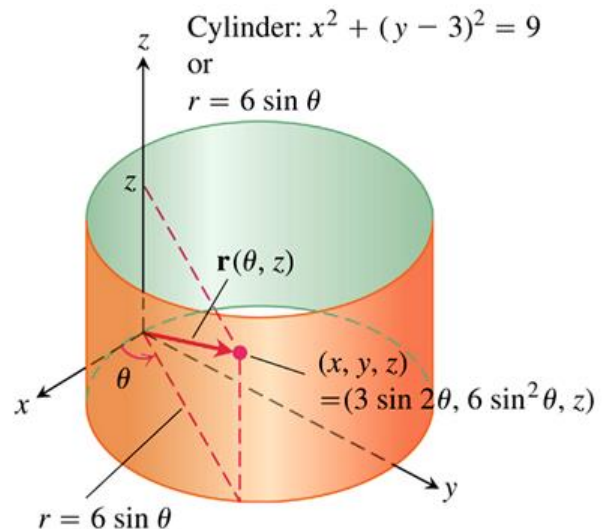
$$r = 6 \sin \theta, \quad 0 \leq \theta \leq \pi$$

A typical point on the cylinder has

$$\begin{cases} x = r \cos \theta = 6 \sin \theta \cos \theta = 3 \sin 2\theta \\ y = r \sin \theta = 6 \sin^2 \theta \\ z = z \end{cases}$$

Taking $u = \theta$ and $v = z$

$$\vec{r}(\theta, z) = (3 \sin 2\theta) \hat{i} + (6 \sin^2 \theta) \hat{j} + z \hat{k} \quad 0 \leq \theta \leq \pi, \quad 0 \leq z \leq 5$$



Surface Area

Calculating the area of a curved surface S based on the parameterization

$$\vec{r}(u, v) = f(u, v)\hat{i} + g(u, v)\hat{j} + h(u, v)\hat{k}, \quad a \leq u \leq b, \quad c \leq v \leq d$$

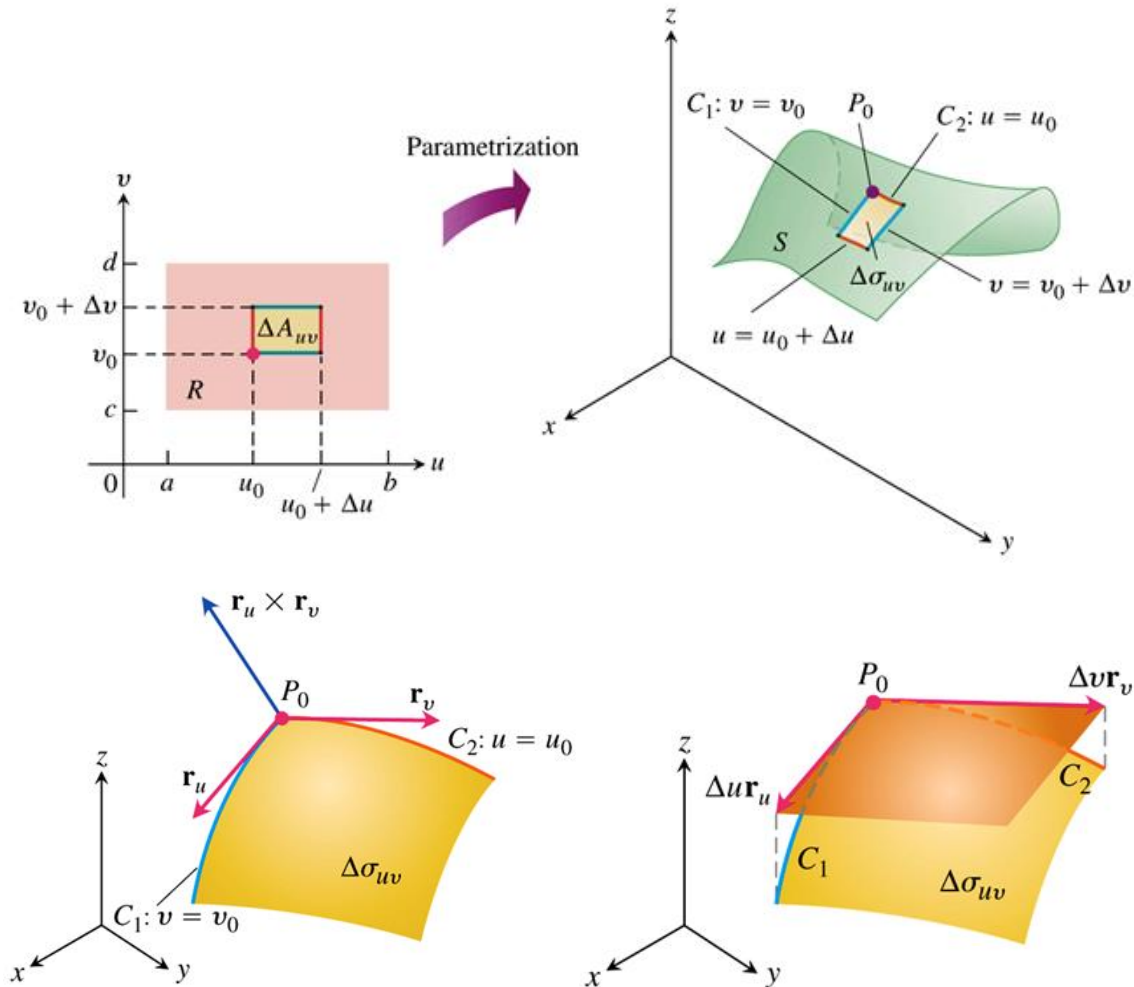
The definition of smoothness involves the partial derivatives of r with respect to u and v :

$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u} = \frac{\partial f}{\partial u} \hat{i} + \frac{\partial g}{\partial u} \hat{j} + \frac{\partial h}{\partial u} \hat{k}$$

$$\vec{r}_v = \frac{\partial \vec{r}}{\partial v} = \frac{\partial f}{\partial v} \hat{i} + \frac{\partial g}{\partial v} \hat{j} + \frac{\partial h}{\partial v} \hat{k}$$

Definition

A **parameterized** surface $\vec{r}(u, v) = f(u, v)\hat{i} + g(u, v)\hat{j} + h(u, v)\hat{k}$ is smooth if \vec{r}_u and \vec{r}_v are continuous and $\vec{r}_u \times \vec{r}_v$ is never zero on the interior of the parameter domain.



Definition

The area of the smooth surface

$$\vec{r}(u, v) = f(u, v)\hat{i} + g(u, v)\hat{j} + h(u, v)\hat{k}, \quad a \leq u \leq b, \quad c \leq v \leq d$$

is

$$Area = \iint_R |\vec{r}_u \times \vec{r}_v| dA = \int_c^d \int_a^b |\vec{r}_u \times \vec{r}_v| du dv$$

Surface area Differential for a Parameterized Surface

$$d\sigma = |\vec{r}_u \times \vec{r}_v| du dv \quad \iint_S d\sigma$$

Surface area differential *Differential formula for surface area*

Example

Find the surface area of the cone $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 1$

Solution

$$x = r \cos \theta, \quad y = r \sin \theta, \quad \text{and} \quad z = \sqrt{x^2 + y^2} = r$$

$$\vec{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, r \rangle$$

$$\vec{r}_r = \langle \cos \theta, \sin \theta, 1 \rangle$$

$$\vec{r}_\theta = \langle -r \sin \theta, r \cos \theta, 0 \rangle$$

$$0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi$$

$$\begin{aligned} \vec{r}_r \times \vec{r}_\theta &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \theta & \sin \theta & 1 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} \\ &= -(r \cos \theta)\hat{i} - (r \sin \theta)\hat{j} + (r \cos^2 \theta + r \sin^2 \theta)\hat{k} \\ &= \langle -r \cos \theta, -r \sin \theta, r \rangle \end{aligned}$$

$$\begin{aligned} |\vec{r}_r \times \vec{r}_\theta| &= \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta + r^2} \\ &= \sqrt{r^2 + r^2} \\ &= r\sqrt{2} \end{aligned}$$

$$\begin{aligned}
A &= \int_0^{2\pi} \int_0^1 \left| \vec{r}_r \times \vec{r}_\theta \right| dr d\theta \\
&= \int_0^{2\pi} d\theta \int_0^1 \sqrt{2} r dr \\
&= \frac{\sqrt{2}}{2} (2\pi) \left(r^2 \right) \Big|_0^1 \\
&= \pi \sqrt{2} \text{ units}^2
\end{aligned}$$

Example

Find the surface area of a sphere of radius a .

Solution

$$\vec{r}(\phi, \theta) = (a \sin \phi \cos \theta) \hat{i} + (a \sin \phi \sin \theta) \hat{j} + (a \cos \phi) \hat{k}$$

$$\vec{r}_\phi = \langle a \cos \phi \cos \theta, a \cos \phi \sin \theta, -a \sin \phi \rangle$$

$$\vec{r}_\theta = \langle -a \sin \phi \sin \theta, a \sin \phi \cos \theta, 0 \rangle$$

$$0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi$$

$$\begin{aligned}
\vec{r}_\phi \times \vec{r}_\theta &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a \cos \phi \cos \theta & a \cos \phi \sin \theta & -a \sin \phi \\ -a \sin \phi \sin \theta & a \sin \phi \cos \theta & 0 \end{vmatrix} \\
&= (a^2 \sin^2 \phi \cos \theta) \hat{i} + (a^2 \sin^2 \phi \sin^2 \theta) \hat{j} + (a^2 \cos \phi \sin \phi \cos^2 \theta + a^2 \cos \phi \sin \phi \sin^2 \theta) \hat{k} \\
&= (a^2 \sin^2 \phi \cos \theta) \hat{i} + (a^2 \sin^2 \phi \sin \theta) \hat{j} + (a^2 \cos \phi \sin \phi) \hat{k}
\end{aligned}$$

$$\begin{aligned}
\left| \vec{r}_\phi \times \vec{r}_\theta \right| &= \sqrt{a^4 \sin^4 \phi \cos^2 \theta + a^4 \sin^4 \phi \sin^2 \theta + a^4 \cos^2 \phi \sin^2 \phi} \\
&= \sqrt{a^4 \sin^4 \phi (\cos^2 \theta + \sin^2 \theta) + a^4 \cos^2 \phi \sin^2 \phi} \\
&= \sqrt{a^4 \sin^4 \phi + a^4 \cos^2 \phi \sin^2 \phi} \\
&= \sqrt{a^4 \sin^2 \phi (\sin^2 \phi + \cos^2 \phi)} \\
&= a^2 \sin \phi
\end{aligned}$$

$$A = \int_0^{2\pi} \int_0^\pi a^2 \sin \phi d\phi d\theta$$

$$= a^2 (-\cos \phi) \Big|_0^\pi \int_0^{2\pi} d\theta$$

$$= \underline{4\pi a^2 \text{ unit}^2}$$

Example

Let S be the “football” surface formed by rotating the curve $x = \cos z$, $y = 0$, $-\frac{\pi}{2} \leq z \leq \frac{\pi}{2}$ around the z -axis. Find the parameterization for S and compute its surface area.

Solution

Let (x, y, z) be an arbitrary point on the circle.

The parameters: $u = z$ and $v = \theta$.

We have:

$$\begin{cases} x = r \cos \theta = \cos u \cos v \\ y = r \sin \theta = \cos u \sin v \\ z = u \end{cases}$$

$$\vec{r}(u, v) = (\cos u \cos v) \hat{i} + (\cos u \sin v) \hat{j} + u \hat{k}$$

$$\vec{r}_u = (-\sin u \cos v) \hat{i} - (\sin u \sin v) \hat{j} + \hat{k}$$

$$\vec{r}_v = (-\cos u \sin v) \hat{i} + (\cos u \cos v) \hat{j}$$

$$-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}, \quad 0 \leq v \leq 2\pi$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin u \cos v & -\sin u \sin v & 1 \\ -\cos u \sin v & \cos u \cos v & 0 \end{vmatrix}$$

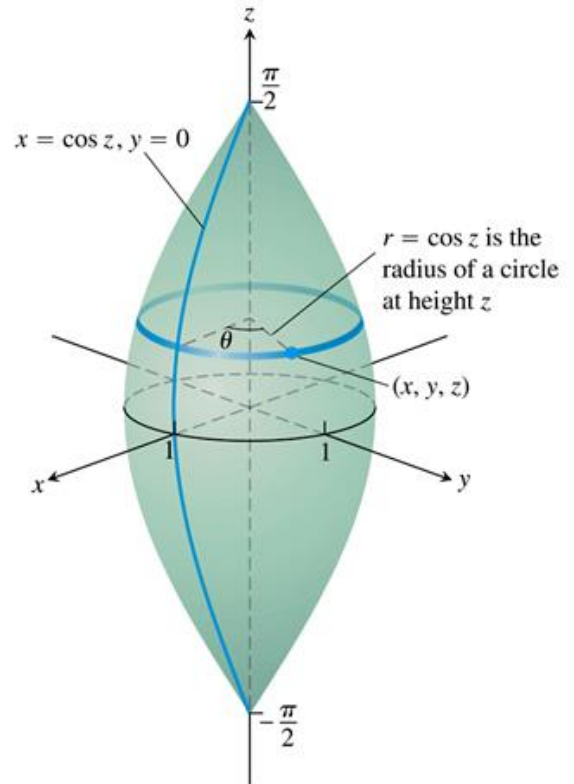
$$= (-\cos u \cos v) \hat{i} - (\cos u \sin v) \hat{j} - (\sin u \cos u \cos^2 v + \cos u \sin u \sin^2 v) \hat{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\cos^2 u \cos^2 v + \cos^2 u \sin^2 v + (\sin u \cos u (\cos^2 v + \sin^2 v))^2}$$

$$= \sqrt{\cos^2 u (\cos^2 v + \sin^2 v) + \sin^2 u \cos^2 u}$$

$$= \sqrt{\cos^2 u (1 + \sin^2 u)}$$

$$= \cos u \sqrt{1 + \sin^2 u}$$



$$A = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cos u \sqrt{1 + \sin^2 u} \, du dv$$

$$w = \sin u \Rightarrow dw = \cos u du \rightarrow \begin{cases} u = -\frac{\pi}{2} & \rightarrow w = -1 \\ u = \frac{\pi}{2} & \rightarrow w = 1 \end{cases}$$

$$= \int_0^{2\pi} \int_{-1}^1 \sqrt{1 + w^2} \, dw dv$$

$$= \int_0^{2\pi} \left[\frac{w}{2} \sqrt{1 + w^2} + \frac{1}{2} \ln \left(w + \sqrt{1 + w^2} \right) \right]_{-1}^1 dv$$

$$= \int_0^{2\pi} \left[\frac{1}{2} \sqrt{2} + \frac{1}{2} \ln(1 + \sqrt{2}) + \frac{1}{2} \sqrt{2} - \frac{1}{2} \ln(-1 + \sqrt{2}) \right] dv$$

$$\ln \left(-1 + \sqrt{2} \cdot \frac{1 + \sqrt{2}}{1 + \sqrt{2}} \right) = \ln \left(\frac{1}{1 + \sqrt{2}} \right) = -\ln(1 + \sqrt{2})$$

$$= \int_0^{2\pi} \left[\sqrt{2} + \frac{1}{2} \ln(1 + \sqrt{2}) + \frac{1}{2} \ln(1 + \sqrt{2}) \right] dv$$

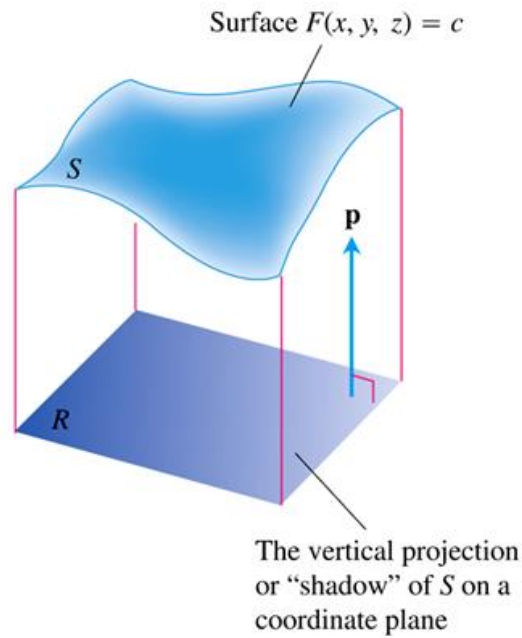
$$= \int_0^{2\pi} \left[\sqrt{2} + \ln(1 + \sqrt{2}) \right] dv$$

$$= \left(\sqrt{2} + \ln(1 + \sqrt{2}) \right) v \Big|_0^{2\pi}$$

$$= 2\pi \left[\sqrt{2} + \ln(1 + \sqrt{2}) \right] \text{ unit}^2$$

Implicit Surfaces

Surfaces are often presented as level sets of a function $F(x, y, z) = c$ for some constant c . Such a level surface does not come with an explicit parameterization, and is called an *implicit defined surface*.



The surface is defined by the equation $F(x, y, z) = c$ and \vec{p} is a unit vector normal to the plane region R .

$$\nabla F \cdot \vec{p} = \nabla F \cdot \hat{k} = F_z \neq 0$$

Define the parameters u and v by $u = x$ and $v = y$. Then $z = h(u, v)$ and

$$\vec{r}(u, v) = u\hat{i} + v\hat{j} + h(u, v)\hat{k}$$

Calculating the partial derivatives of \vec{r} ,

$$\vec{r}_u = \hat{i} + \frac{\partial h}{\partial u} \hat{k} \quad \text{and} \quad \vec{r}_v = \hat{j} + \frac{\partial h}{\partial v} \hat{k}$$

$$\frac{\partial h}{\partial u} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial h}{\partial v} = -\frac{F_y}{F_z}$$

$$\vec{r}_u = \hat{i} - \frac{F_x}{F_z} \hat{k} \quad \text{and} \quad \vec{r}_v = \hat{j} - \frac{F_y}{F_z} \hat{k}$$

$$\begin{aligned} \vec{r}_u \times \vec{r}_v &= \frac{F_x}{F_z} \hat{i} + \frac{F_y}{F_z} \hat{j} + \hat{k} \\ &= \frac{1}{F_z} (F_x \hat{i} + F_y \hat{j} + F_z \hat{k}) \\ &= \frac{\nabla F}{F_z} \end{aligned}$$

$$= \frac{\nabla F}{\nabla F \cdot \hat{\mathbf{k}}}$$

$$= \frac{\nabla F}{\nabla F \cdot \vec{p}}$$

Therefore, the surface area differential is given by

$$d\sigma = \left| \vec{r}_u \times \vec{r}_v \right| du dv = \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|} dx dy \quad u = x \text{ and } v = y$$

Formula for the Surface Area of an Implicit Surface

The area of the surface $F(x, y, z) = c$ over a closed and bounded plane region R is

$$\text{Surface area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|} dA$$

Where $\vec{p} = \hat{\mathbf{i}}, \hat{\mathbf{j}}, \text{ or } \hat{\mathbf{k}}$ is normal to R and $\nabla F \cdot \vec{p} \neq 0$

Example

Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 4$.

Solution

Let $F(x, y, z) = x^2 + y^2 - z = 0$ and R the disk $x^2 + y^2 \leq 4$

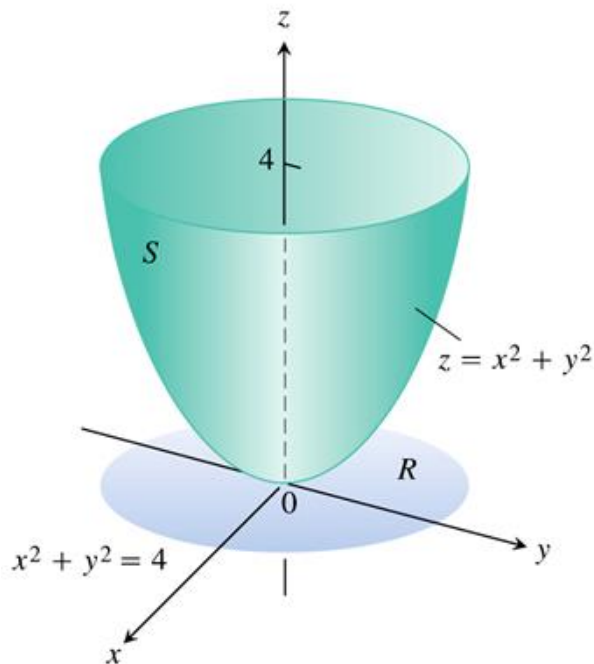
$$\nabla F = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\begin{aligned} |\nabla F| &= \sqrt{(2x)^2 + (2y)^2 + (-1)^2} \\ &= \sqrt{4x^2 + 4y^2 + 1} \end{aligned}$$

$$\begin{aligned} |\nabla F \cdot \vec{p}| &= |\nabla F \cdot \hat{k}| \\ &= |-1| \\ &= 1 \end{aligned}$$

In the region R , $dA = dxdy$. Therefore,

$$\begin{aligned} \text{Surface area} &= \iint_R \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|} dA \\ &= \iint_{x^2 + y^2 \leq 4} \sqrt{4(x^2 + y^2) + 1} dxdy \\ &= \int_0^{2\pi} d\theta \int_0^2 \sqrt{4r^2 + 1} r dr \\ &= 2\pi \left(\frac{1}{8} \right) \int_0^2 \sqrt{4r^2 + 1} d(4r^2 + 1) \\ &= (2\pi) \frac{1}{12} (4r^2 + 1)^{3/2} \Big|_0^2 \\ &= \frac{\pi}{6} (17^{3/2} - 1) \\ &= \frac{\pi}{6} (17\sqrt{17} - 1) \end{aligned}$$



Formula for the Surface Area of a Graph $z = f(x, y)$

For a graph $z = f(x, y)$ over the region R in the xy -plane, the surface area formula is

$$A = \iint_R \sqrt{f_x^2 + f_y^2 + 1} \, dxdy$$

Surface	Equation	Explicit Description	
		Normal Vector $\pm \langle -z_x, -z_y, 0 \rangle$	Magnitude $\left \langle -z_x, -z_y, 0 \rangle \right $
Cylinder	$x^2 + y^2 = a^2$ $0 \leq z \leq h$	$\langle x, y, 0 \rangle$	a
Cone	$z^2 = x^2 + y^2$ $0 \leq z \leq h$	$\left\langle \frac{x}{z}, \frac{y}{z}, -1 \right\rangle$	$\sqrt{2}$
Sphere	$x^2 + y^2 + z^2 = a^2$	$\left\langle \frac{x}{z}, \frac{y}{z}, 1 \right\rangle$	$\frac{a}{z}$
Paraboloid	$z = x^2 + y^2$ $0 \leq z \leq h$	$\langle 2x, 2y, -1 \rangle$	$\sqrt{1 + 4(x^2 + y^2)}$

Surface	Equation	Parametric Description	
		Normal Vector $\mathbf{t}_u \times \mathbf{t}_v$	Magnitude $\left \mathbf{t}_u \times \mathbf{t}_v \right $
Cylinder	$\mathbf{r} = \langle a \cos u, a \sin u, v \rangle$ $0 \leq u \leq 2\pi, 0 \leq v \leq h$	$\langle a \cos u, a \sin u, 0 \rangle$	a
Cone	$\mathbf{r} = \langle v \cos u, v \sin u, v \rangle$ $0 \leq u \leq 2\pi, 0 \leq v \leq h$	$\langle v \cos u, v \sin u, -v \rangle$	$\sqrt{2} \, v$
Sphere	$\mathbf{r} = \langle a \sin u \cos v, a \sin u \sin v, a \cos u \rangle$ $0 \leq u \leq \pi, 0 \leq v \leq 2\pi$	$\langle a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \sin^2 u \cos u \rangle$	$a^2 \sin u$
Paraboloid	$\mathbf{r} = \langle v \cos u, v \sin u, v^2 \rangle$ $0 \leq u \leq 2\pi, 0 \leq v \leq \sqrt{h}$	$\langle 2v^2 \cos u, 2v^2 \sin u, -v \rangle$	$v\sqrt{1 + 4v^2}$

Exercises Section 4.6 – Surfaces Integrals

(1–9) Find a parametrization of the surface:

1. The paraboloid $z = x^2 + y^2$, $z \leq 4$
2. The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 4$
3. The sphere $x^2 + y^2 + z^2 = 8$ cuts by the plane $z = -2$
4. The plane $2x - 4y + 3z = 16$
5. The cap of the sphere $x^2 + y^2 + z^2 = 16$ for $2\sqrt{2} \leq z \leq 4$
6. The frustum of the cone $z^2 = x^2 + y^2$ for $2 \leq z \leq 8$
7. The cone $z^2 = 4(x^2 + y^2)$ for $0 \leq z \leq 4$
8. The portion of the cylinder $x^2 + y^2 = 9$ in the first octant, for $0 \leq z \leq 3$
9. The cylinder $y^2 + z^2 = 36$ for $0 \leq x \leq 9$

(10–19) Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral of the portion of

10. A plane $y + 2z = 2$ inside the cylinder $x^2 + y^2 = 1$
11. A cone $z = \frac{\sqrt{x^2 + y^2}}{3}$ between the planes $z = 1$ and $z = \frac{4}{3}$
12. A cylinder $x^2 + z^2 = 10$ between the planes $y = -1$ and $y = 1$
13. Cap cut from the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 4$
14. The half cylinder $\{(r, \theta, z) : r = 4, 0 \leq \theta \leq \pi, 0 \leq z \leq 7\}$
15. The plane $z = 3 - x - 3y$ in the first octant
16. The plane $z = 10 - x - y$ above the square $|x| \leq 2, |y| \leq 2$
17. The hemisphere $x^2 + y^2 + z^2 = 100, z \geq 0$
18. A cone with base radius r and height h , where r and h are positive constants.
19. The cap of the sphere $x^2 + y^2 + z^2 = 4, 1 \leq z \leq 2$

(20–39) Use a surface integral to find the area of

20. Cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 2$.
21. Portion $x^2 - 2z = 0$ that lies above the triangle bounded by the lines $x = \sqrt{3}$, $y = 0$, and $y = x$ in the xy -plane.

22. Cap cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$.
23. Ellipse cut from the plane $z = cx$ (c a constant) by the cylinder $x^2 + y^2 = 1$.
24. From the nose of the paraboloid $x = 1 - y^2 - z^2$ by yz -plane.
25. First octant cut from the cylinder $y = \frac{2}{3}z^{3/2}$ by the planes $x = 1$ and $y = \frac{16}{3}$
26. Helicoid $\vec{r}(r, \theta) = (r \cos \theta)\hat{i} + (r \sin \theta)\hat{j} + \theta\hat{k}$, $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$
27. Surface $f(x, y) = \sqrt{2}xy$ above the origin $\{(r, \theta): 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$
28. Hemisphere $x^2 + y^2 + z^2 = 9$, for $z \geq 0$ (excluding the base)
29. Frustum of the cone $z^2 = x^2 + y^2$, for $2 \leq z \leq 4$ (excluding the bases)
30. Area of the plane $z = 6 - x - y$ above the square $|x| \leq 1, |y| \leq 1$
31. The cone $z^2 = 4(x^2 + y^2)$, $0 \leq z \leq 4$
32. The paraboloid $z = 2(x^2 + y^2)$, $0 \leq z \leq 8$
33. The trough $z = x^2$, $-2 \leq x \leq 2$, $0 \leq y \leq 4$
34. The part of the hyperbolic paraboloid $z = x^2 - y^2$ above the sector
 $R = \{(r, \theta): 0 \leq r \leq 4, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}\}$
35. $f(x, y, z) = xy$, where S is the plane $z = 2 - x - y$ in the first octant
36. $f(x, y, z) = x^2 + y^2$, where S is the paraboloid $z = x^2 + y^2$, $0 \leq z \leq 4$
37. $f(x, y, z) = 25 - x^2 - y^2$, where S is the hemisphere centered at the origin with radius 5, for $z \geq 0$
38. $f(x, y, z) = e^x$, where S is the plane $z = 8 - x - 2y$ in the first octant
39. $f(x, y, z) = e^z$, where S is the plane $z = 8 - x - 2y$ in the first octant

(40–46) Evaluate the surface integrals

40. $\iint_S (1 + yz) dS$; S is the plane $x + y + z = 2$ in the first octant.
41. $\iint_S \langle 0, y, z \rangle \cdot \vec{n} dS$; S is the curve surface of the cylinder $y^2 + z^2 = a^2$, $|x| \leq 8$ with outward normal vectors.

42. $\iint_S (x - y + z) dS$; S is the entire surface including the base of the hemisphere $x^2 + y^2 + z^2 = 4$,
for $z \geq 0$.

43. $\iint_S \nabla \ln |\vec{r}| \cdot \vec{n} dS$, where S is the hemisphere $x^2 + y^2 + z^2 = a^2$, for $z \geq 0$, and where
 $\vec{r} = \langle x, y, z \rangle$. Assume normal vectors point either outward or in the positive z -direction.

44. $\iint_S |\vec{r}| dS$, where S is the cylinder $x^2 + y^2 = 4$, for $0 \leq z \leq 8$, and where $\vec{r} = \langle x, y, z \rangle$. Assume
normal vectors point either outward or in the positive z -direction.

45. $\iint_S xyz dS$, where S is the part of the plane $z = 6 - y$ that lies on the cylinder $x^2 + y^2 = 4$.
Assume normal vectors point either outward or in the positive z -direction.

46. $\iint_S \frac{\langle x, 0, z \rangle}{\sqrt{x^2 + z^2}} \cdot \vec{n} dS$, where S is the cylinder $x^2 + y^2 = a^2$, $|y| \leq 2$. Assume normal vectors
point either outward or in the positive z -direction.

(47–50) Evaluate the surface integral $\iint_S f(x, y, z) dS$

47. $f(x, y, z) = x^2 + y^2$, where S is the hemisphere $x^2 + y^2 + z^2 = 36$, $z \geq 0$

48. $f(x, y, z) = y$, where S is the cylinder $x^2 + y^2 = 9$, $0 \leq z \leq 3$

49. $f(x, y, z) = x$, where S is the cylinder $x^2 + z^2 = 1$, $0 \leq y \leq 3$

50. $f(\rho, \varphi, \theta) = \cos \varphi$, where S is the part of the unit sphere in the first octant

(51–58) Find the flux of the vector fields across the given surface with the specified orientation

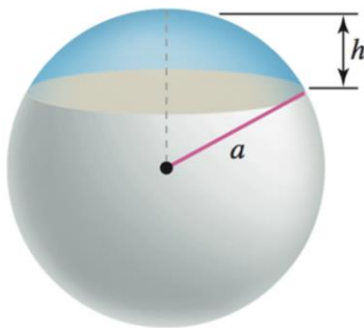
51. $\vec{F} = \frac{\vec{r}}{|\vec{r}|}$ across the sphere of radius a centered at the origin, where $\vec{r} = \langle x, y, z \rangle$. Assume the normal
vectors to the surface point outward.

52. $\vec{F} = \langle x, y, z \rangle$ across the curved surface of the cylinder $x^2 + y^2 = 1$ for $|z| \leq 8$

53. $\vec{F} = \langle 0, 0, -1 \rangle$ across the slanted face of the tetrahedron $z = 4 - x - y$ in the first octant; normal
vectors point upward

54. $\vec{F} = \langle x, y, z \rangle$ across the slanted face of the tetrahedron $z = 10 - 2x - 5y$ in the first octant; normal vectors point upward
55. $\vec{F} = \langle x, y, z \rangle$ across the slanted face of the cone $z^2 = x^2 + y^2$ for $0 \leq z \leq 1$; normal vectors point upward
56. $\vec{F} = \langle e^{-y}, 2z, xy \rangle$ across the curved sides of the surface
 $S = \{(x, y, z) : z = \cos y, -\pi \leq y \leq \pi, 0 \leq x \leq 4\}$; normal vectors point upward.
57. $\vec{F} = \frac{\vec{r}}{|\vec{r}|^3}$ across the sphere of radius a centered at the origin, where $\vec{r} = \langle x, y, z \rangle$; normal vectors point outward
58. $\vec{F} = \langle -y, x, 1 \rangle$ across the cylinder $y = x^2$ for $0 \leq x \leq 1, 0 \leq z \leq 4$; normal vectors point in the general direction of the positive y -axis
59. Consider the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where a, b , and c are positive real numbers.
- Show that the surface is described by the parametric equations
 $\vec{r}(u, v) = \langle a \cos u \sin v, b \sin u \sin v, c \cos v \rangle$ for $0 \leq u \leq 2\pi, 0 \leq v \leq \pi$
 - Write an integral for the surface area of the ellipsoid.
60. The cone $z^2 = x^2 + y^2, z \geq 0$, cuts the sphere $x^2 + y^2 + z^2 = 16$ along a curve C .
- Find the surface area of the sphere below C , for $z \geq 0$
 - Find the surface area of the sphere above C .
 - Find the surface area of the cone below C , for $z \geq 0$
61. Consider the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $(x-1)^2 + y^2 = 1$ for $z \geq 0$.
- Find the surface area of the cylinder inside the sphere
 - Find the surface area of the sphere inside the cylinder.
62. Find the upward flux of the field $\vec{F} = \langle x, y, z \rangle$ across the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ in the first octant.
Show that the flux equals c times the area of the base of the origin.
63. Consider the field $\vec{F} = \langle x, y, z \rangle$ and the cone $z^2 = \frac{x^2 + y^2}{a^2}$, for $0 \leq z \leq 1$
- Show that when $a = 1$, the outward flux across the cone is zero.
 - Find the outward flux (away from the z -axis); for any $a > 0$.

64. A sphere of radius a is sliced parallel to the equatorial plane at a distance $a - h$ from the equatorial plane. Find the general formula for the surface area of the resulting spherical cap (excluding the base) with thickness h .



65. Consider the radial field $\vec{F} = \frac{\vec{r}}{|\vec{r}|^p}$, where $\vec{r} = \langle x, y, z \rangle$ and p is a real number. Let S be the sphere of radius a centered at the origin. Show that the outward flux of \vec{F} across the sphere is $\frac{4\pi}{a^{p-3}}$. It is instructive to do the calculation using both an explicit and parametric description of the sphere.

(66–68) The heat flow vector field for conducting objects is $\vec{F} = -k\nabla T$, where $T(x, y, z)$ is the temperature in the object and $k > 0$ is a constant that depends on the material. Compute the outward flux of \vec{F} across the following surfaces S for the given temperature distributions. Assume $k = 1$.

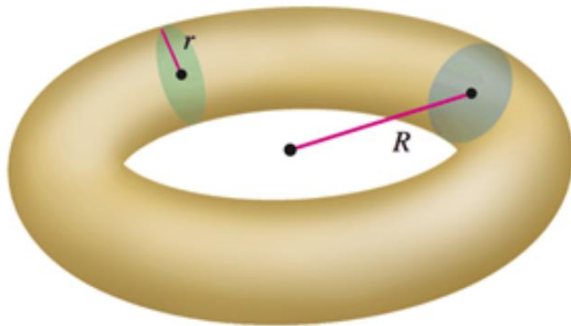
66. $T(x, y, z) = 100e^{-x-y}$; S consists of the faces of the cube $|x| \leq 1$, $|y| \leq 1$, $|z| \leq 1$

67. $T(x, y, z) = 100e^{-x^2-y^2-z^2}$; S is the sphere $x^2 + y^2 + z^2 = a^2$

68. $T(x, y, z) = -\ln(x^2 + y^2 + z^2)$; S is the sphere $x^2 + y^2 + z^2 = a^2$

69. Given: $\vec{r}(u, v) = \langle (R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u \rangle$

- a) Show that a torus with radii $R > r$ may be described parametrically by $\vec{r}(u, v)$ for $0 \leq u \leq 2\pi$, $0 \leq v \leq 2\pi$



- b) Show that the surface area of the torus is $4\pi^2 Rr$