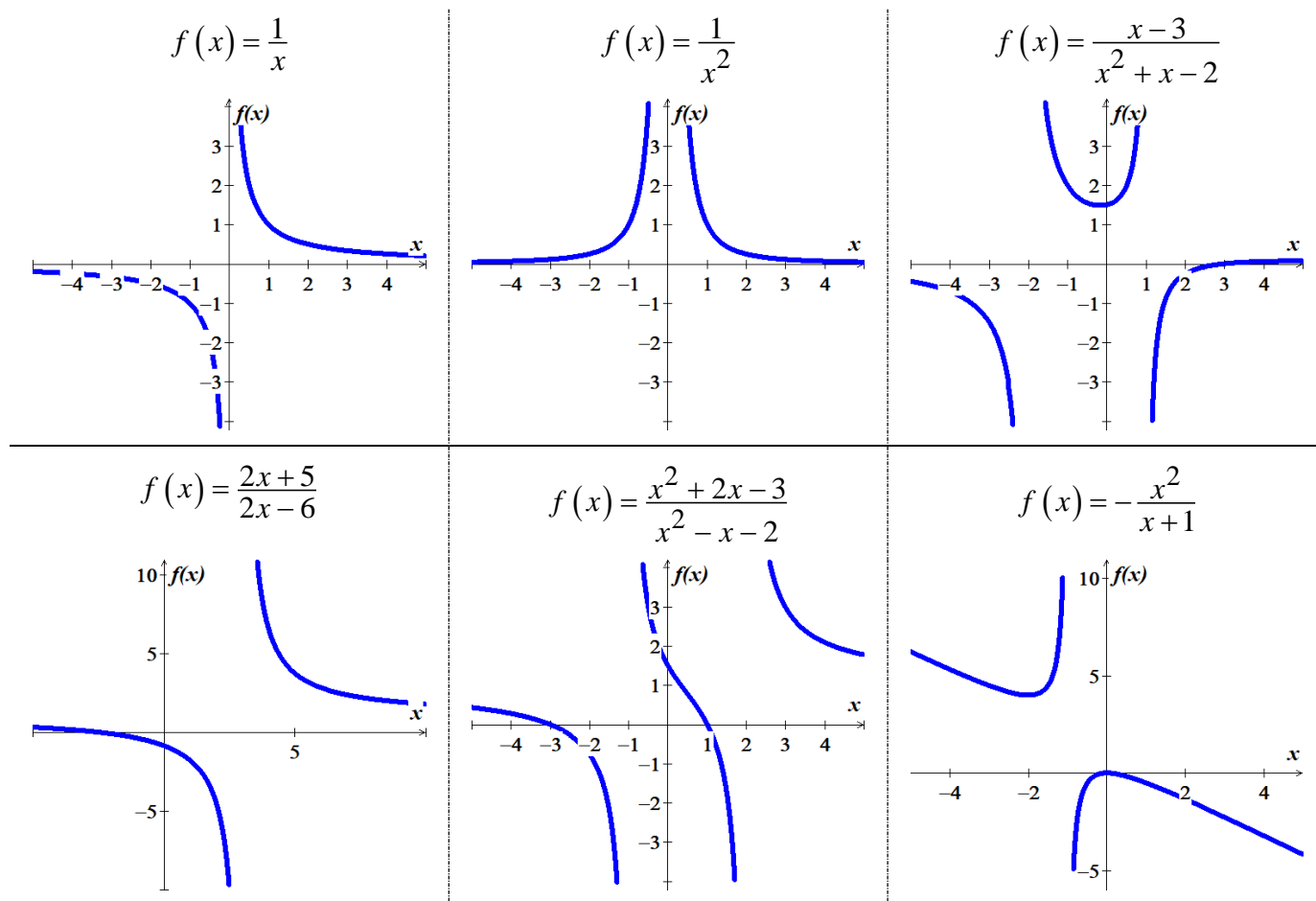


Section 2.7 – Rational Functions



Rational Function

A rational function is a function f that is a quotient of two polynomials, that is,

$$f(x) = \frac{g(x)}{h(x)}$$

Where $g(x)$ and $h(x)$ are polynomials. The domain of f consists of all real numbers *except* the zeros of the denominator $h(x)$.

The Domain of a Rational Function

Example

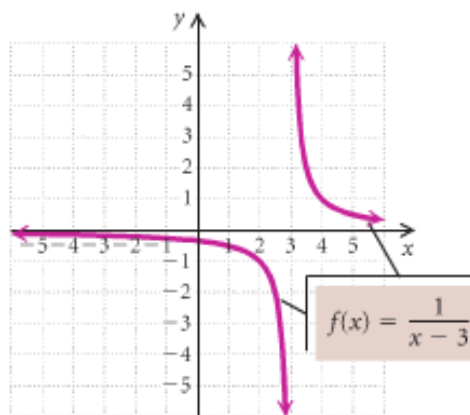
Consider: $f(x) = \frac{1}{x-3}$

Find the domain and graph f .

Solution

$$x - 3 = 0 \Rightarrow \boxed{x = 3}$$

Thus the domain is: $\{x | x \neq 3\}$ *or* $(-\infty, 3) \cup (3, \infty)$



Function	Domain	
$f(x) = \frac{1}{x}$	$\{x x \neq 0\}$	$(-\infty, 0) \cup (0, \infty)$
$f(x) = \frac{1}{x^2}$	$\{x x \neq 0\}$	$(-\infty, 0) \cup (0, \infty)$
$f(x) = \frac{x-3}{x^2 + x - 2}$	$\{x x \neq -2 \text{ and } x \neq 1\}$	$(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$
$f(x) = \frac{2x+5}{2x-6} = \frac{2x+5}{2(x-3)}$	$\{x x \neq 3\}$	$(-\infty, 3) \cup (3, \infty)$

Asymptotes

Vertical Asymptote (VA) - Think Domain

The line $x = a$ is a **vertical asymptote** for the graph of a function f if

$$f(x) \rightarrow \infty \text{ or } f(x) \rightarrow -\infty$$

As x approaches a from either the left or the right

Example

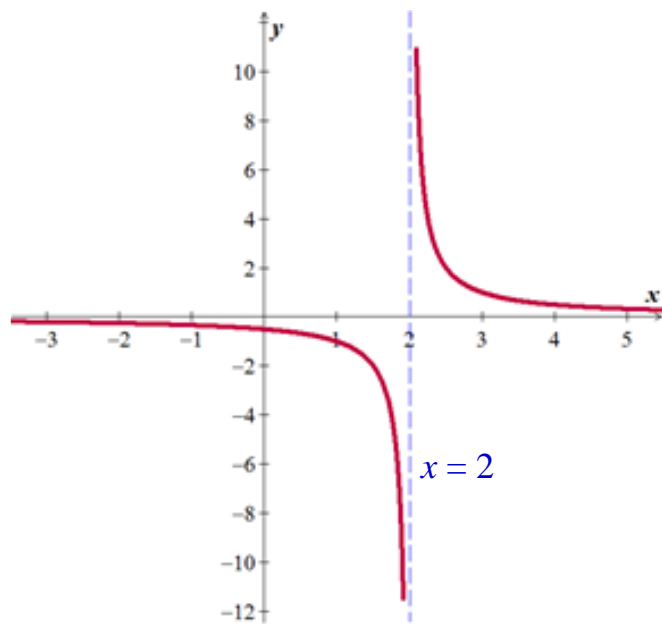
Find the vertical asymptote of $f(x) = \frac{1}{x-2}$, and sketch the graph.

Solution

VA: $x = 2$

$$f(x) \rightarrow \infty \text{ as } x \rightarrow 2^+$$

$$f(x) \rightarrow -\infty \text{ as } x \rightarrow 2^-$$



Horizontal Asymptote (**HA**)

The line $y = c$ is a **horizontal asymptote** for the graph of a function f if

$$f(x) \rightarrow c \text{ as } x \rightarrow -\infty \text{ or } x \rightarrow \infty$$

Let $f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0} = \frac{a_n x^n}{b_m x^m}$ be a rational function.

1. If the degree of numerator is less than of denominator ($n < m$) $\Rightarrow y = 0$

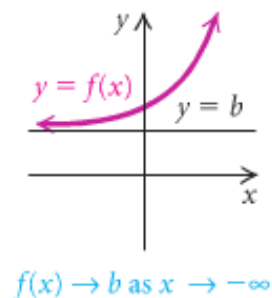
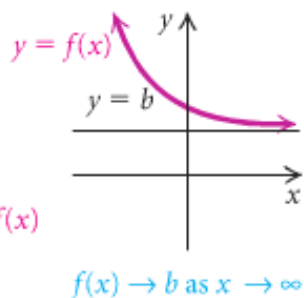
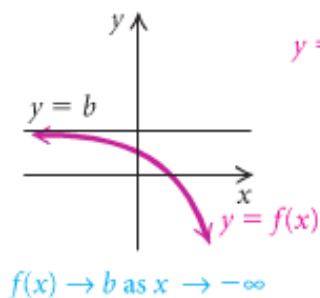
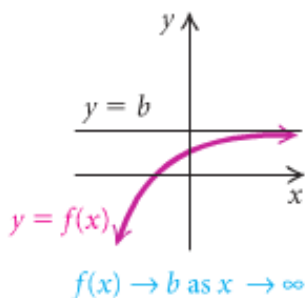
$$y = \frac{2x+1}{4x^2+5} \Rightarrow \boxed{y=0}$$

2. If the degree of numerator is equal of denominator ($n = m$) $\Rightarrow y = \frac{a_n}{b_m}$

$$y = \frac{2x^2+1}{4x^2+5} \Rightarrow \boxed{y = \frac{2}{4} = \frac{1}{2}}$$

3. If the degree of numerator is greater than of denominator ($n > m$) \Rightarrow No horizontal asymptote

$$y = \frac{2x^3+1}{4x^2+5} \Rightarrow \text{No HA}$$



Example

Determine the horizontal asymptote of $f(x) = \frac{-7x^4 - 10x^2 + 1}{11x^4 + x - 2}$.

Solution

$$f(x) = \frac{-7x^4 - 10x^2 + 1}{11x^4 + x - 2} \rightarrow \frac{-7x^4}{11x^4} = -\frac{7}{11}$$

Therefore, the horizontal asymptote (**HA**) is: $\boxed{y = -\frac{7}{11}}$

Example

Find the vertical and the horizontal asymptote for the graph of f , if it exists

$$a) f(x) = \frac{3x-1}{x^2-x-6}$$

$$b) f(x) = \frac{5x^2+1}{3x^2-4}$$

$$c) f(x) = \frac{2x^4-3x^2+5}{x^2+1}$$

Solution

$$a) f(x) = \frac{3x-1}{x^2-x-6}$$

$$x^2 - x - 6 = 0 \rightarrow x = -2, 3$$

$$\text{VA: } x = -2, x = 3$$

$$\text{HA: } y = 0$$

$$b) f(x) = \frac{5x^2+1}{3x^2-4}$$

$$3x^2 - 4 = 0 \rightarrow 3x^2 = 4 \rightarrow x^2 = \frac{4}{3} \rightarrow \boxed{x = \pm \frac{2}{\sqrt{3}}}$$

$$\text{VA: } x = -\frac{2}{\sqrt{3}}, x = \frac{2}{\sqrt{3}}$$

$$\text{HA: } y = \frac{5}{3}$$

$$c) f(x) = \frac{2x^4-3x^2+5}{x^2+1}$$

$$x^2 + 1 = 0 \rightarrow x^2 = -1$$

$$\text{VA: } n/a$$

$$\text{HA: } n/a$$

Slant or Oblique Asymptotes

When the degree of the numerator is one greater than the degree of the denominator, the graph has a slant or oblique asymptote and it is a line $y = ax + b$, $a \neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y = \frac{3x^2 - 1}{x + 2}$$

$$\begin{array}{r} 3x - 6 \\ x + 2 \overline{) 3x^2 + 0x - 1} \end{array}$$

$$\begin{array}{r} 3x^2 + 6x \\ -6x - 1 \\ \hline -6x - 12 \\ \hline \end{array}$$

$$R = 11$$

$$y = \frac{3x^2 - 1}{x + 2} = 3x - 6 + \frac{11}{x + 2}$$

The **oblique asymptote** is the line $y = 3x - 6$

Example

Find all the asymptotes of $f(x) = \frac{2x^2 - 3x - 1}{x - 2}$

Solution

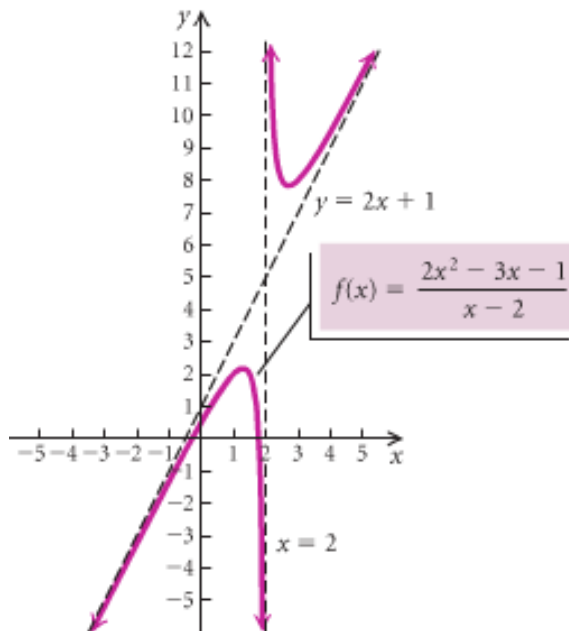
$$\begin{array}{r} 2x + 1 \\ x - 2 \overline{) 2x^2 - 3x - 1} \end{array}$$

$$\begin{array}{r} -2x^2 + 4x \\ \hline x - 1 \\ -x + 2 \\ \hline 1 \end{array}$$

$$f(x) = \frac{2x^2 - 3x - 1}{x - 2} = (2x + 1) + \frac{1}{x - 2}$$

The **oblique asymptote** is the line $y = 2x + 1$

VA:: $x = 2$



Graph That Has a *Hole*

Example

Sketch the graph of g if $g(x) = \frac{(3x+4)(x-1)}{(2x-5)(x-1)}$

Solution

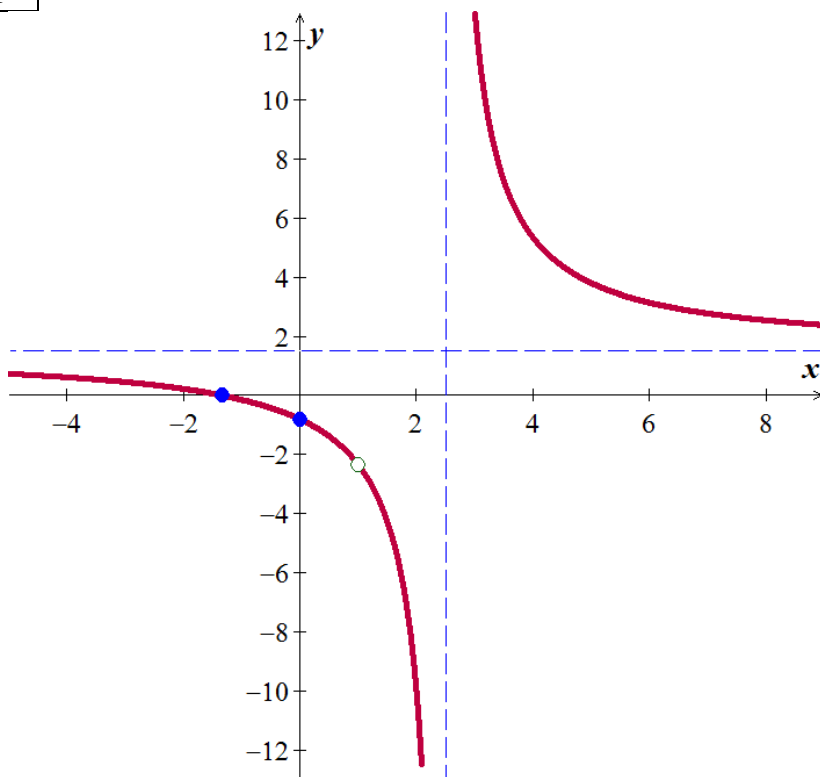
$$g(x) = \frac{(3x+4)(x-1)}{(2x-5)(x-1)} = \frac{3x+4}{2x-5} = f(x)$$

VA: $x = \frac{5}{2}$

HA: $y = \frac{3}{2}$

The only difference between the graphs that g has a *hole* at $x = 1 \rightarrow f(1) = -\frac{7}{3}$

x	y
-4	.6
1.3	0
0	-.8
4	5.3
6	3.1



Exercises Section 2.7 – Rational Functions

(1 – 21) Determine all asymptotes of the function

1. $y = \frac{3x}{1-x}$

2. $y = \frac{x^2}{x^2+9}$

3. $y = \frac{x-2}{x^2-4x+3}$

4. $y = \frac{3}{x-5}$

5. $y = \frac{x^3-1}{x^2+1}$

6. $y = \frac{3x^2-27}{(x+3)(2x+1)}$

7. $y = \frac{x^3+3x^2-2}{x^2-4}$

8. $y = \frac{x-3}{x^2-9}$

9. $y = \frac{6}{\sqrt{x^2-4x}}$

10. $y = \frac{5x-1}{1-3x}$

11. $f(x) = \frac{2x-11}{x^2+2x-8}$

12. $f(x) = \frac{x^2-4x}{x^3-x}$

13. $f(x) = \frac{x-2}{x^3-5x}$

14. $f(x) = \frac{4x}{x^2+10x}$

15. $f(x) = \frac{3-x}{(x-4)(x+6)}$

16. $f(x) = \frac{x^3}{2x^3-x^2-3x}$

17. $f(x) = \frac{3x^2+5}{4x^2-3}$

18. $f(x) = \frac{x+6}{x^3+2x^2}$

19. $f(x) = \frac{x^2+4x-1}{x+3}$

20. $f(x) = \frac{x^2-6x}{x-5}$

21. $f(x) = \frac{x^3-x^2+x-4}{x^2+2x-1}$

(22 – 53) Determine all asymptotes (if any) (*Vertical Asymptote, Horizontal Asymptote, Hole, Oblique Asymptote*) and sketch the graph of

22. $f(x) = \frac{-3x}{x+2}$

23. $f(x) = \frac{x+1}{x^2+2x-3}$

24. $f(x) = \frac{2x^2-2x-4}{x^2+x-12}$

25. $f(x) = \frac{-2x^2+10x-12}{x^2+x}$

26. $f(x) = \frac{x^2-x-6}{x+1}$

27. $f(x) = \frac{x^3+1}{x-2}$

28. $f(x) = \frac{2x^2+x-6}{x^2+3x+2}$

29. $f(x) = \frac{x-1}{1-x^2}$

30. $f(x) = \frac{x^2+x-2}{x+2}$

31. $f(x) = \frac{x^3-2x^2-4x+8}{x-2}$

32. $f(x) = \frac{2x^2-3x-1}{x-2}$

33. $f(x) = \frac{2x+3}{3x^2+7x-6}$

34. $f(x) = \frac{x^2-1}{x^2+x-6}$

35. $f(x) = \frac{-2x^2-x+15}{x^2-x-12}$

36. $f(x) = \frac{1}{x-3}$

37. $f(x) = \frac{-2}{x+3}$

38. $f(x) = \frac{x}{x+2}$

39. $f(x) = \frac{x-5}{x+4}$

40. $f(x) = \frac{2x^2-2}{x^2-9}$

41. $f(x) = \frac{x^2-3}{x^2+4}$

42. $f(x) = \frac{x^2+4}{x^2-3}$

$$43. \quad f(x) = \frac{x^2}{x^2 - 6x + 9}$$

$$44. \quad f(x) = \frac{x^2 + x + 4}{x^2 + 2x - 1}$$

$$45. \quad f(x) = \frac{2x^2 + 14}{x^2 - 6x + 5}$$

$$46. \quad f(x) = \frac{x^2 - 4x - 5}{2x + 5}$$

$$47. \quad f(x) = \frac{x - 3}{x^2 - 3x + 2}$$

$$48. \quad f(x) = \frac{x^2 + 2}{x^2 + 3x + 2}$$

$$49. \quad f(x) = \frac{x - 2}{x^2 - 3x + 2}$$

$$50. \quad f(x) = \frac{x^2 + x}{x + 1}$$

$$51. \quad f(x) = \frac{x^2 - 2x}{x - 2}$$

$$52. \quad f(x) = \frac{x^2 - 3x}{x + 3}$$

$$53. \quad f(x) = \frac{x^3 + 3x^2 - 4x + 6}{x + 2}$$

(54 – 59) Find an equation of a rational function f that satisfies the given conditions

$$54. \quad \begin{cases} \text{vertical asymptote : } x = 4 \\ \text{horizontal asymptote : } y = -1 \\ x\text{-intercept : } 3 \end{cases}$$

$$55. \quad \begin{cases} \text{vertical asymptote : } x = -4, x = 5 \\ \text{horizontal asymptote : } y = \frac{3}{2} \\ x\text{-intercept : } -2 \end{cases}$$

$$56. \quad \begin{cases} \text{vertical asymptote : } x = 5 \\ \text{horizontal asymptote : } y = -1 \\ x\text{-intercept : } 2 \end{cases}$$

$$57. \quad \begin{cases} \text{vertical asymptote : } x = -2, x = 0 \\ \text{horizontal asymptote : } y = 0 \\ x\text{-intercept : } 2, \quad f(3) = 1 \end{cases}$$

$$58. \quad \begin{cases} \text{vertical asymptote : } x = -3, x = 1 \\ \text{horizontal asymptote : } y = 0 \\ x\text{-intercept : } -1, \quad f(0) = -2 \\ \text{hole at } x = 2 \end{cases}$$

$$59. \quad \begin{cases} \text{vertical asymptote : } x = -1, x = 3 \\ \text{horizontal asymptote : } y = 2 \\ x\text{-intercept : } -2, 1 \\ \text{hole : } x = 0 \end{cases}$$