Defn $\vec{u}=(u_1,u_2,u_3)$ $\quad \vec{v}=(v_1,v_2,v_3)$

$$\vec{u}\times\vec{v}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\u_1&u_2&u_3\\v_1&v_2&v_3\end{vmatrix}\begin{matrix}u_1&u_2\\v_1&v_2\end{matrix}$$

$$=(u_2v_3-v_2u_3)\hat{i}+(v_1u_3-v_3u_1)\hat{j}+(u_1v_2-v_1u_2)\hat{k}$$

$$=(\quad,\quad,\quad)$$

Ex $\vec{u}=(1,2,-2)$ $\quad \vec{v}=(3,0,1)$

$$\vec{u}\times\vec{v}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&2&-2\\3&0&1\end{vmatrix}\begin{matrix}1&2\\3&0\end{matrix}$$

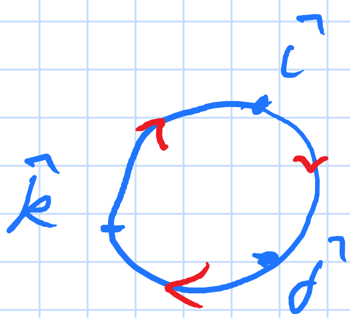
$$=2\hat{i}-7\hat{j}-6\hat{k}$$

$$=(2,-7,-6)$$

$\hat{k}\times\hat{k}=\vec{0}$

$\hat{i}\times\hat{i}=\vec{0}$

$\hat{j}\times\hat{j}=\vec{0}$



$$=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\u_1&u_2&0\\v_1&v_2&0\end{vmatrix}\begin{matrix}u_1&u_2\\v_1&v_2\end{matrix}$$

$$=(u_1v_2-u_2v_1)\hat{k}$$

Properties

$$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$

$\vec{u} \times \vec{v}$ is $\perp$ $\vec{u}$, $\vec{v}$ $\Rightarrow$ $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$, $\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$

$$\vec{u} \times \vec{u} = \vec{0}$$

Lagrange's: $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$

$$= \|\vec{u}\| \|\vec{v}\| \sin\theta$$

$$|\vec{u} \cdot \vec{v}| = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

$$\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$$

$$(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$$

$$k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$$

$$\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$$

Scalar Triple Product $\vec{u} \cdot (\vec{v} \times \vec{w})$

Ex

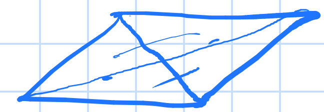
$$\vec{u} = -2\hat{i} + 6\hat{k}$$
$$\vec{v} = \hat{i} - 3\hat{j} + \hat{k}$$
$$\vec{w} = -5\hat{i} - \hat{j} + \hat{k}$$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} -2 & 0 & 6 \\ 1 & -3 & 1 \\ -5 & -1 & 1 \end{vmatrix}$$

$$= -92$$

Area of parallelogram $\Rightarrow$ $A = \|\vec{u} \times \vec{v}\|$

Area of a Δ $\Rightarrow$ $A = \frac{1}{2}\|\vec{u} \times \vec{v}\|$



Ex $P_1(2,2,0)$ $P_2(-1,0,2)$

$P_2(0,4,3)$ Area? $(P_1 P_2 P_3)\Delta$

$$\overrightarrow{P_1P_2} = (-3,-2,2)$$

$$\overrightarrow{P_1P_3} = (-2,2,3)$$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -2 & 2 \\ -2 & 2 & 3 \end{vmatrix}$$

$$= -10\hat{i} + 5\hat{j} - 10\hat{k}$$

square each component

$$\text{Area} = \frac{1}{2}\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\|$$

$$= \frac{1}{2}\sqrt{100+25+100}$$

$$= \frac{15}{2} \text{ unit}^2$$

Volume : Parallelepiped $|\vec{u} \cdot (\vec{v} \times \vec{w})|$

Ex $\vec{u} = (2,-6,2)$ $\vec{v} = (0,4,-2)$ $\vec{w} = (2,2,-4)$

$$\text{Volume} = \left| \begin{vmatrix} 2 & -6 & 2 \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{vmatrix} \right|$$

determinant

absolute value

$$= |-16|$$

$$= 16 \text{ unit}^3$$

2.5 Subspaces, Span, Nullspaces

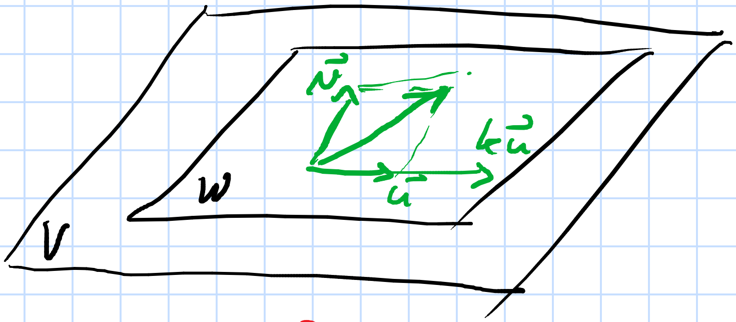
Defn subspaces

A subset of W of a vector space V is called a subspace

closed under addition → $\vec{u}+\vec{v}$ is in W } Axiom 1

$k\vec{v}$ subspace W } Axiom 6

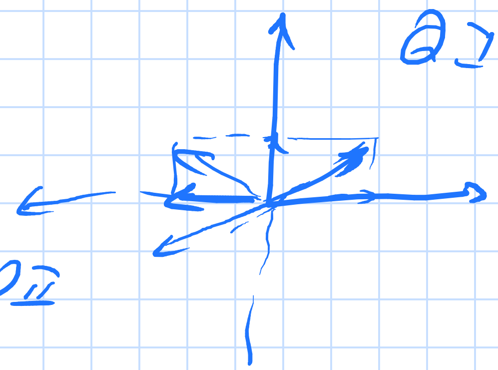
closed under scalar multiplication



Linear combination ⇒ $c_1\vec{u}+c_2\vec{v}$ in subspace in W

QI ⇒ $(-)_a\,\vec{u}$ in QI

QI is not subspace



QI & QIII $\quad \hat{j}+(-\hat{i}) \in QII$

Ex All upper triangular Matrices are subspace

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 0 & 2c \end{bmatrix}$$

$$k\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} ka & kb \\ 0 & kc \end{pmatrix}$$

Span Defn

$$S = \{w_1, w_2, \ldots, w_n\}$$

linear combinations of the vectors

$\text{span}\{w_1, \ldots, w_n\}$ $\qquad$ $\text{span}(S)$

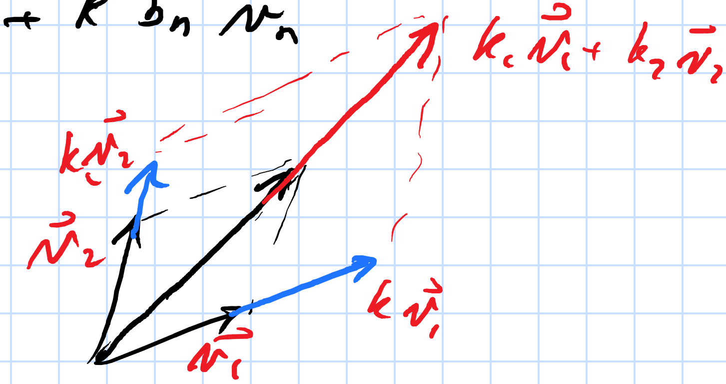
Theorem let $\vec{v}_1, \ldots, \vec{v}_n$ be vector space V
S be their span

S is subspace in V

$$\forall \vec{u}, \vec{v} \in S \quad \vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$$
$$\vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n$$

$$\vec{u} + \vec{v} = (a_1 + b_1)\vec{v}_1 + \cdots + (a_n + b_n)\vec{v}_n$$

$$k\vec{v} = k(b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n)$$
$$= kb_1\vec{v}_1 + \cdots + kb_n\vec{v}_n$$



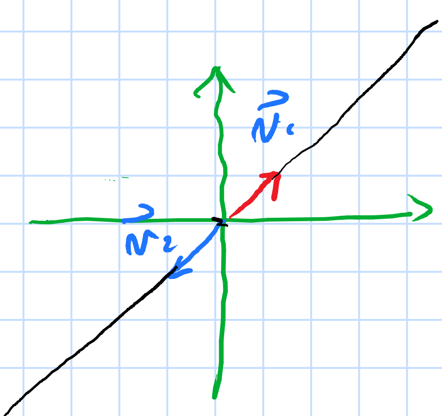
Ex $\vec{v}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad \vec{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}$

span full 2-dimensional $\mathbb{R}^2$

Ex $v_1 = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad \vec{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}$ $\quad \vec{v}_3 = \begin{pmatrix}4\\3\end{pmatrix}$ $\quad$ full space $\mathbb{R}^2$

Ex $\vec{v}_1 = \begin{pmatrix}1\\1\end{pmatrix}$ $\quad \vec{v}_2 = \begin{pmatrix}-1\\-1\end{pmatrix}$

span a line in $\mathbb{R}^2$



$\vec{v}_1, \vec{v}_2, \vec{v}_3$ find a linear combination $\to$ Space $\mathbb{R}^n$

$$|\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3| \neq 0$$

Ex $\vec{v}_1 = (1,1,2)$ $\quad \vec{v}_2 = (1,0,1)$ $\quad \vec{v}_3 = (2,1,3)$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$

$$\underbrace{= 0}$$

$\therefore \vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ do not span in $\mathbb{R}^3$

Solns Spaces of Homogeneous (Null Space)

$$A\vec{x} = \vec{0} \quad \text{in } \mathbb{R}^n$$

$\vec{x} = \vec{0}$ empty space $\qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$A\vec{x}_1 = \vec{0} \qquad A\vec{x}_2 = \vec{0}$

$$\begin{aligned} A(\vec{x}_1 + \vec{x}_2) &= A\vec{x}_1 + A\vec{x}_2 \\ &= \vec{0} + \vec{0} \\ &= \vec{0} \ \checkmark \end{aligned}$$

closed under addition

$$\begin{aligned} A(k\vec{x}_1) &= k(A\vec{x}_1) \\ &= k\vec{0} \\ &= \vec{0} \end{aligned}$$

$\to$ closed under scalar multiplication

Exer

$$\begin{vmatrix} a \\ b \\ c \end{vmatrix}$$