

Section 3.4 - Concavity and the Second Derivative Test

Example

Find the second derivative of $f(x) = 4x(\ln x)$

Solution

$$\begin{aligned} f'(x) &= 4\ln x + 4x \frac{1}{x} \\ &= 4\ln x + 4 \end{aligned}$$

$$f''(x) = \frac{4}{x}$$

Velocity and acceleration

Example

Suppose a car is moving in a straight line, with its position from a starting point (in feet) at time t (in seconds) given by

$$s(t) = t^3 - 2t^2 - 7t + 9$$

a) Find the velocity at any time t

$$v(t) = s'(t) = 3t^2 - 4t - 7$$

b) Find the acceleration at any time t

$$a(t) = v'(t) = 6t - 4$$

c) Find the time intervals ($t \geq 0$) when the car is going forward or backing up

$$v(t) = 3t^2 - 4t - 7 = 0$$

$$t = \frac{7}{3} \quad t = -1$$

The car is backing up first $\left(0, \frac{7}{3}\right)$ and forward $\left(\frac{7}{3}, \infty\right)$

d) Find the time intervals ($t \geq 0$) when the car is speeding up or slowing down

$$a(t) = 6t - 4 = 0 \Rightarrow t = \frac{2}{3}$$

0	$\frac{2}{3}$	$\frac{7}{3}$	
	$v(0.5) < 0$	$v(1) < 0$	$v(3) > 0$
	—	—	+
	$a(0.5) < 0$	$a(1) > 0$	$a(3) > 0$
	—	+	+
	+	—	—

Concavity

Definition

Let f be differentiable on an open interval I . The graph of f is

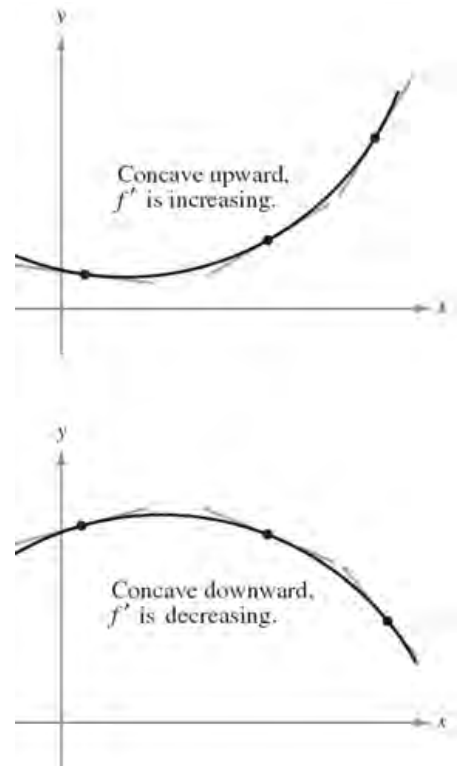
1. **Concave upward** on I if f' is increasing on the interval.
2. **Concave downward** on I if f' is decreasing on the interval.

Test for Concavity

Let f be function whose second derivative exists on an open interval I .

1. If $f''(x) > 0$ for all x in I , then f is **concave upward** on I .
2. If $f''(x) < 0$ for all x in I , then f is **concave downward** on I .

- 1) Locate the x values @ which $f''(x) = 0$ or undefined
- 2) Use these test x -value to determine the test intervals
- 3) Test the sign of $f''(x)$ in each interval



Find the second derivative of $f(x) = -2x^2$ and discuss the concavity of the graph

$$f'(x) = -4x$$

$$\Rightarrow f''(x) = -4 < 0 \text{ for all } x$$

f is concave downward for all x .

Example

Determine the intervals on which the graph of the function is concave upward or concave downward.

$$f(x) = x^4 - 8x^3 + 18x^2$$

Solution

$$f'(x) = 4x^3 - 24x^2 + 36x$$

$$f''(x) = 12x^2 - 48x + 36$$

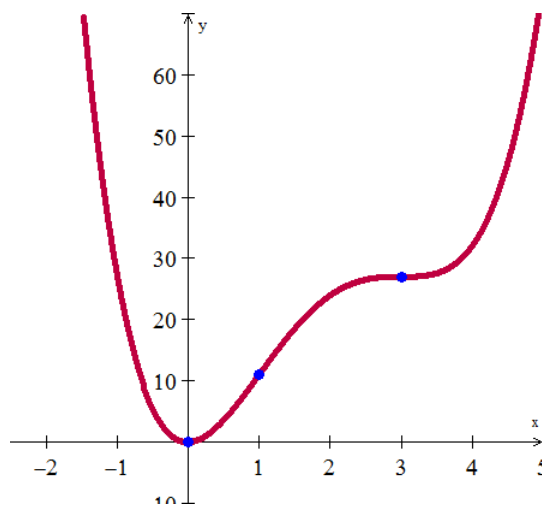
Solve for x :

$$x = 1 \quad x = 3$$

$-\infty$	1	3	∞
$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$	
<i>upward</i>	<i>downward</i>	<i>upward</i>	

f is concave upward on $(-\infty, 1)$ and $(3, \infty)$

f is concave downward on $(1, 3)$



Second-Derivative Test

Let $f'(c) = 0$ and let f'' exist (\exists)

1. If $f''(c) > 0 \Rightarrow f(c)$ is a relative Minimum
2. If $f''(c) < 0 \Rightarrow f(c)$ is a relative Maximum
3. If $f''(c) = 0 \Rightarrow$ Test fails \rightarrow use f' to determine Max, Min.

Example

Find all relative extrema for $f(x) = 4x^3 + 7x^2 - 10x + 8$

Solution

$$f'(x) = 12x^2 + 14x - 10 = 0$$

$$x = -\frac{5}{3} \quad x = \frac{1}{2}$$

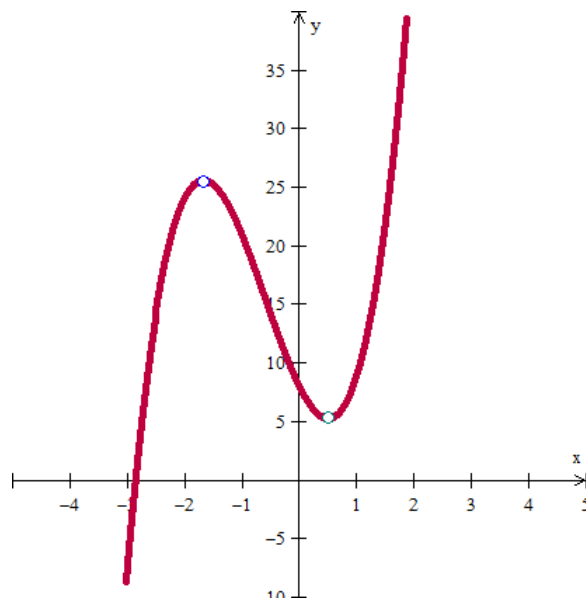
$$f''(x) = 24x + 14$$

$$f''\left(-\frac{5}{3}\right) = 24\left(-\frac{5}{3}\right) + 14 = -26 < 0 \quad \text{Leads to relative maximum}$$

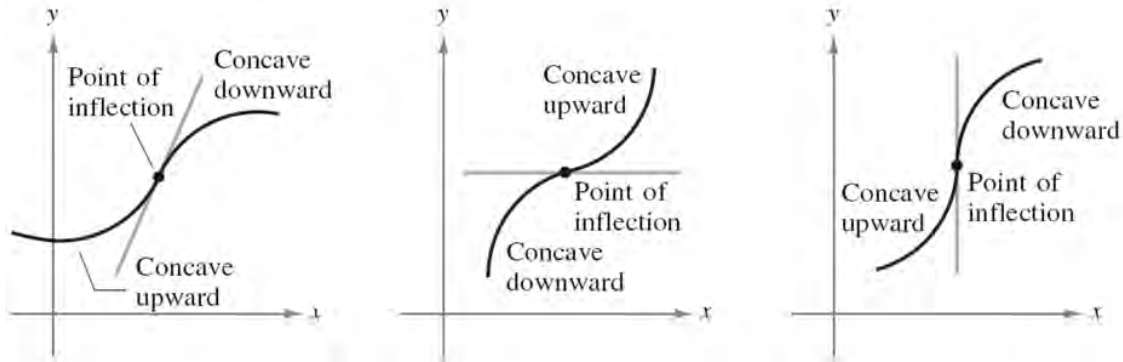
$$f''\left(\frac{1}{2}\right) = 24\left(\frac{1}{2}\right) + 14 = 26 > 0 \quad \text{Leads to relative minimum}$$

$$\text{RMAX: } \left(-\frac{5}{3}, \frac{691}{27}\right)$$

$$\text{RMIN: } \left(\frac{1}{2}, \frac{21}{4}\right)$$



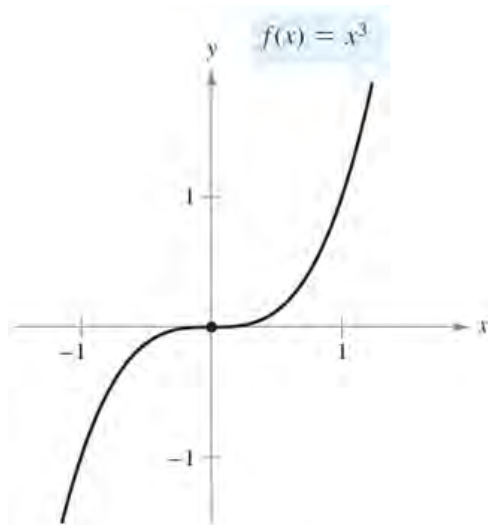
Point of Inflection



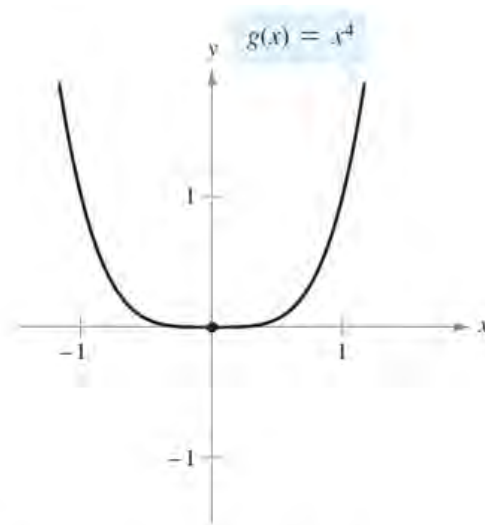
Definition

If the graph of a continuous function has a tangent line @ a point where its concavity changes from upward to downward (or down to upward) then the point is a point of inflection.

If $(c, f(c))$ is a point of inflection of a graph of $f \Rightarrow$ either $f''(c) = 0$ or undefined.



$f''(0) = 0$, and $(0, 0)$ is a point of inflection.



$g''(0) = 0$, but $(0, 0)$ is not a point of inflection.

Extended Applications: Diminishing Returns

$x \rightarrow y$	$y = f(x)$
input output	output input

Example

Find the point of diminishing returns for the model below, where R is the revenue (in thousands of dollars) and x is the advertising cost (in thousands of dollars).

$$R = \frac{1}{20,000}(450x^2 - x^3) \quad 0 \leq x \leq 300$$

Solution

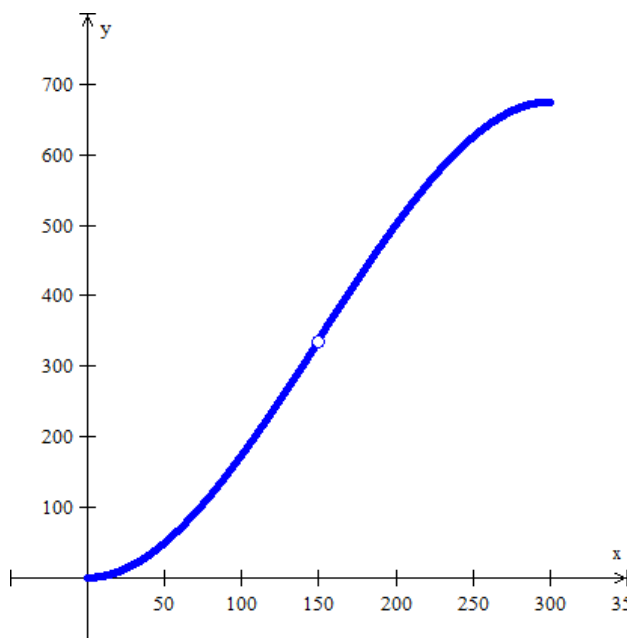
$$R' = \frac{1}{20,000}(900x - 3x^2)$$

$$R'' = \frac{1}{20,000}(900 - 6x) = 0$$

$$\Rightarrow x = \frac{900}{6} = 150$$

$x = 150$ (or \$150,000) is a ***diminishing point***

An increased investment beyond this point is usually considered a poor use of capital



Exercises Section 3.4 - Concavity and the Second Derivative Test

Determine the intervals on which the graph of the function is concave upward or concave downward.

1. $f(x) = \frac{x^2 - 1}{2x + 1}$

2. $f(x) = -4x^3 - 8x^2 + 32$

3. $f(x) = \frac{12}{x^2 + 4}$

4. Find the largest open interval where the function is concave upward $f(x) = 4x - 2e^{-x}$

5. Find the points of inflection. $f(x) = x^3 - 9x^2 + 24x - 18$

6. Find the second derivative of $f(x) = -2\sqrt{x}$ and discuss the concavity of the graph

7. Find the extrema using the second derivative test $f(x) = \frac{4}{x^2 + 1}$

8. Discuss the concavity of the graph of f and find its points of inflection. $f(x) = x^4 - 2x^3 + 1$

9. Find all relative extrema of $f(x) = x^4 - 4x^3 + 1$

10. The revenue R generated from sales of a certain product is related to the amount x spent on advertising by

$$R(x) = \frac{1}{15,000} (600x^2 - x^3), \quad 0 \leq x \leq 600$$

Where x and R are in thousands of dollars. Is there a point of diminishing returns for this function?

11. Find the point of diminishing returns (x, y) for the function

$$R(x) = -x^3 + 45x^2 + 400x + 8000, \quad 0 \leq x \leq 20$$

where $R(x)$ represents revenue in thousands of dollars and x represents the amount spent on advertising in tens of thousands of dollars.

12. The population of a certain species of fish introduced into a lake is described by the logistic equation

$$G(t) = \frac{12,000}{1 + 19e^{-1.2t}}$$

where $G(t)$ is the population after t years. Find the point at which the growth rate of this population begins to decline.