Solution *Section* 3.1 – Integrals over Rectangular Regions

Exercise

Evaluate the iterated integral $\int_1^2 \int_0^4 2xy\ dydx$

Solution

$$\begin{aligned}\int_1^2 \int_0^4 2xy\ dydx &= \int_1^2 x\left[y^2\right]_0^4 dx \\ &= \int_1^2 16x dx \\ &= 8\left[x^2\right]_1^2 \\ &= 8(4-1) \\ &= 24\end{aligned}$$

Exercise

Evaluate the iterated integral $\int_0^2 \int_{-1}^1 (x-y)\ dydx$

Solution

$$\begin{aligned}\int_0^2 \int_{-1}^1 (x-y)\ dydx &= \int_0^2 \left[xy - \frac{1}{2}y^2\right]_{-1}^1 dx \\ &= \int_0^2 \left[x - \frac{1}{2} - \left(-x - \frac{1}{2}\right)\right] dx \\ &= \int_0^2 2x\ dx \\ &= x^2 \Big|_0^2 \\ &= 4\end{aligned}$$

Exercise

Evaluate the iterated integral $\int_0^1 \int_0^1 \left(1 - \frac{x^2 + y^2}{2}\right) dxdy$

Solution

$$\begin{aligned}
\int_0^1 \int_0^1 \left(1 - \frac{x^2 + y^2}{2}\right) dxdy &= \int_0^1 \left[x - \frac{1}{6}x^3 - \frac{1}{2}y^2 x\right]_0^1 dy \\
&= \int_0^1 \left(1 - \frac{1}{6} - \frac{1}{2}y^2\right) dy \\
&= \int_0^1 \left(\frac{5}{6} - \frac{1}{2}y^2\right) dy \\
&= \left[\frac{5}{6}y - \frac{1}{6}y^3\right]_0^1 \\
&= \frac{5}{6} - \frac{1}{6} \\
&= \frac{4}{6} \\
&= \frac{2}{3}
\end{aligned}$$

Exercise

Evaluate the iterated integral $\int_0^3 \int_{-2}^0 \left(x^2 y - 2xy\right) dydx$

Solution

$$\begin{aligned}
\int_0^3 \int_{-2}^0 \left(x^2 y - 2xy\right) dydx &= \int_0^3 \left[\frac{1}{2}x^2 y^2 - xy^2\right]_{-2}^0 dx \\
&= \int_0^3 \left(-2x^2 + 4x\right) dx \\
&= \left[-\frac{2}{3}x^3 + 2x^2\right]_0^3 \\
&= -18 + 18 \\
&= 0
\end{aligned}$$

Exercise

Evaluate the iterated integral $\int_0^1 \int_0^1 \frac{y}{1+xy} dxdy$

Solution

$$\int_0^1 \int_0^1 \frac{y}{1+xy} dxdy = \int_0^1 \int_0^1 \frac{d(1+xy)}{1+xy} dy \qquad d(1+xy) = ydx$$

$$= \int_0^1 \left[\ln|1+xy| \right]_0^1 dy$$

$$= \int_0^1 \ln|1+y| dy \qquad d(1+y) = dy$$

$$= \left[(y+1)\ln|1+y| - (y+1) \right]_0^1 \qquad \int \ln u \, du = u \ln u - u$$

$$= 2\ln 2 - 2 + 1$$

$$= 2\ln 2 - 1$$

Exercise

Evaluate the iterated integral $\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y} dydx$

Solution

$$\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y} dydx = \int_0^{\ln 2} e^{2x} dx \int_1^{\ln 5} e^{y} dy$$

$$= \left[\frac{1}{2} e^{2x} \right]_0^{\ln 2} \left[e^{y} \right]_1^{\ln 5}$$

$$= \frac{1}{2}\left(e^{2\ln 2} - 1 \right)\left(e^{\ln 5} - e \right)$$

$$= \frac{1}{2}(4-1)(5-e)$$

$$= \frac{15}{2} - \frac{3}{2}e$$

Exercise

Evaluate the iterated integral $\int_0^1 \int_1^2 xye^{x} dydx$

Solution

$$\int_0^1 \int_1^2 xye^{x} dydx = \int_0^1 xe^{x} \left[\frac{1}{2} y^2 \right]_1^2 dx$$

$$= \frac{3}{2}\int_0^1 xe^x dx$$

$$= \frac{3}{2}\left[xe^x - e^x \right]_0^1$$

$$= \frac{3}{2}(e - e + 1)$$

$$= \frac{3}{2}$$

Exercise

Evaluate the iterated integral $\int_\pi^{2\pi}\int_0^\pi (\sin x + \cos y)\, dxdy$

Solution

$$\int_\pi^{2\pi}\int_0^\pi (\sin x + \cos y)\, dxdy = \int_\pi^{2\pi} \left[-\cos x + x\cos y\right]_0^\pi \, dy$$

$$= \int_\pi^{2\pi} (1 + \pi\cos y + 1)\, dy$$

$$= \left[2y + \pi\sin y\right]_\pi^{2\pi}$$

$$= 4\pi - 2\pi$$

$$= 2\pi$$

Exercise

Evaluate the double integral over the given region R $\iint_R \left(6y^2 - 2x\right) dA \quad R: \ 0 \le x \le 1, \quad 0 \le y \le 2$

Solution

$$\iint_R \left(6y^2 - 2x\right) dA = \int_0^1\int_0^2 \left(6y^2 - 2x\right) dydx$$

$$= \int_0^1 \left[2y^3 - 2xy\right]_0^2 dx$$

$$= \int_0^1 (16 - 4x)\, dx$$

$$= \left[16x - 2x^2\right]_0^1$$

$$= 14$$

Exercise

Evaluate the double integral over the given region R $\iint\limits_R \left(\frac{\sqrt{x}}{y^2}\right) dA \quad R: \ 0 \le x \le 4, \quad 1 \le y \le 2$

Solution

$$\iint\limits_R \left(\frac{\sqrt{x}}{y^2}\right) dA = \int_0^4 \int_1^2 \left(\frac{\sqrt{x}}{y^2}\right) dydx$$

$$= \int_0^4 \left[-\frac{\sqrt{x}}{y}\right]_1^2 dx$$

$$= \int_0^4 -\sqrt{x}\left(\frac{1}{2} - 1\right) dx$$

$$= \frac{1}{2}\int_0^4 x^{1/2} dx$$

$$= \frac{1}{3}\left[x^{3/2}\right]_0^4$$

$$= \frac{8}{3}$$

Exercise

Evaluate the double integral over the given region R $\iint\limits_R y\sin(x+y)\,dA \quad R: \ -\pi \le x \le 0, \quad 0 \le y \le \pi$

Solution

$$\iint\limits_R y\sin(x+y)\,dA = \int_{-\pi}^0 \int_0^\pi y\sin(x+y)\,dxdy$$

		$\int \sin(x+y)$
+	y	$-\cos(x+y)$
−	1	$-\sin(x+y)$

$$= \int_{-\pi}^0 \left[-y\cos(x+y) + \sin(x+y)\right]_0^\pi dx$$

$$= \int_{-\pi}^0 \left[\sin(x+\pi) - \pi\cos(x+\pi) - \sin x\right] dx$$

$$= \left[-\cos(x+\pi) - \pi\sin(x+\pi) + \cos x\right]_{-\pi}^0$$

$$= -(-1) + 1 - (-1-1)$$

$$= 4$$

Exercise

Evaluate the double integral over the given region R. $\iint_R e^{x-y}dA \quad R: \quad 0 \le x \le \ln 2, \quad 0 \le y \le \ln 2$

Solution

$$\iint_R e^{x-y}dA = \int_0^{\ln 2}\int_0^{\ln 2} e^{x-y}dydx$$

$$= \int_0^{\ln 2}\left[-e^{x-y}\right]_0^{\ln 2} dx$$

$$= \int_0^{\ln 2}\left(-e^{x-\ln 2}+e^x\right)dx$$

$$= \left[-e^{x-\ln 2}+e^x\right]_0^{\ln 2}$$

$$= -1+e^{\ln 2}+e^{-\ln 2}-1 \qquad e^{-\ln 2}=e^{\ln 2^{-1}}=2^{-1}=\frac{1}{2}$$

$$= -2+2+\frac{1}{2}$$

$$= \frac{1}{2}$$

Exercise

Evaluate the double integral over the given region R. $\iint_R \frac{y}{x^2y^2+1}dA \quad R: \quad 0 \le x \le 1, \quad 0 \le y \le 1$

Solution

$$\iint_R \frac{y}{x^2y^2+1}dA = \int_0^1\int_0^1 \frac{y}{(xy)^2+1}dxdy \qquad \int \frac{du}{a^2+u^2}=\frac{1}{a}\tan^{-1}\frac{u}{a} \quad u=xy \to du=ydx$$

$$= \int_0^1\left[\tan^{-1}(xy)\right]_0^1 dy$$

$$= \int_0^1 \tan^{-1}y\ dy \qquad \int \tan^{-1}ax\,dx = x\tan^{-1}ax-\frac{1}{2a}\ln\left(1+a^2x^2\right)$$

$$= \left[y\tan^{-1}y-\frac{1}{2}\ln\left|1+y^2\right|\right]_0^1$$

$$= \tan^{-1}1-\frac{1}{2}\ln 2$$

$$= \frac{\pi}{4}-\frac{1}{2}\ln 2$$

Exercise

Integrate $f(x,y)=\frac{1}{xy}$ over the ***square*** $1 \le x \le 2, \quad 1 \le y \le 2$

Solution

$$\int_1^2 \int_1^2 \frac{1}{xy}\,dydx = \int_1^2 \frac{1}{x}\left[\ln y\right]_1^2\,dx$$

$$= \int_1^2 \frac{1}{x}\left[\ln 2 - \ln 1\right]dx$$

$$= \ln 2 \int_1^2 \frac{1}{x}dx$$

$$= \ln 2\ \left[\ln x\right]_1^2$$

$$= \ln 2 \cdot \ln 2$$

$$= (\ln 2)^2$$

Exercise

Integrate $f(x,y)=y\cos xy$ over the ***rectangle*** $0 \le x \le \pi, \quad 0 \le y \le 1$

Solution

$$\int_0^1 \int_0^\pi y\cos(xy)\,dxdy = \int_0^1 \left[\sin xy\right]_0^\pi\,dy$$

$$= \int_0^1 \sin(\pi y)\,dy$$

$$= -\frac{1}{\pi}\cos \pi y\,\Big|_0^1$$

$$= -\frac{1}{\pi}[-1-1]$$

$$= \frac{2}{\pi}$$

Exercise

Find the volume of the region bounded above the paraboloid $z = x^2 + y^2$ and below by the square $R: \quad -1 \le x \le 1, \quad -1 \le y \le 1$

Solution

$$V = \int_{-1}^{1}\int_{-1}^{1}\left(x^2 + y^2\right)dydx$$

$$= \int_{-1}^{1}\left[x^2 y + \frac{1}{3}y^3\right]_{-1}^{1} dx$$

$$= \int_{-1}^{1}\left[x^2 + \frac{1}{3} - \left(-x^2 - \frac{1}{3}\right)\right]dx$$

$$= \int_{-1}^{1}\left(2x^2 + \frac{2}{3}\right)dx$$

$$= \left[\frac{2}{3}x^3 + \frac{2}{3}x\right]_{-1}^{1}$$

$$= \frac{2}{3} + \frac{2}{3} - \left(-\frac{2}{3} - \frac{2}{3}\right)$$

$$= \frac{8}{3} \quad unit^3$$

Exercise

Find the volume of the region bounded above the plane $z = \frac{y}{2}$ and below by the rectangle

$R: \quad 0 \le x \le 4, \quad 0 \le y \le 2$

Solution

$$V = \int_{0}^{4}\int_{0}^{2}\frac{y}{2}dydx$$

$$= \int_{0}^{4}\left[\frac{1}{4}y^2\right]_{0}^{2} dx$$

$$= \int_{0}^{4}(1)dx$$

$$= x\Big|_{0}^{4}$$

$$= 4 \quad unit^3$$

Exercise

Find the volume of the region bounded above the surface $z = 4 - y^2$ and below by the rectangle $R: \ 0 \le x \le 1, \quad 0 \le y \le 2$

Solution

$$V = \int_0^1 \int_0^2 \left(4 - y^2\right) dy dx$$

$$= \int_0^1 \left[4y - \frac{1}{3}y^3\right]_0^2 dx$$

$$= \int_0^1 \left(8 - \frac{8}{3}\right) dx$$

$$= \int_0^1 \frac{16}{3} dx$$

$$= \left[\frac{16}{3}x\right]_0^1$$

$$= \frac{16}{3} \quad unit^3$$

Exercise

Find the volume of the region bounded above the ellipitical paraboloid $z = 16 - x^2 - y^2$ and below by the square $R: \ 0 \le x \le 2, \quad 0 \le y \le 2$

Solution

$$V = \int_0^2 \int_0^2 \left(16 - x^2 - y^2\right) dy dx$$

$$= \int_0^2 \left[16y - x^2 y - \frac{1}{3}y^3\right]_0^2 dx$$

$$= \int_0^2 \left(32 - 2x^2 - \frac{8}{3}\right) dx$$

$$= \int_0^2 \left(\frac{88}{3} - 2x^2\right) dx$$

$$= \left[\frac{88}{3}x - \frac{2}{3}x^3\right]_0^2 \quad = \frac{176}{3} - \frac{16}{3}$$

$$= \frac{160}{3} \quad unit^3$$

Solution   *Section* 3.2 – Double Integrals over General Regions

Exercise

Sketch the region of integration and evaluate the integral $\int_0^{\pi}\int_0^{x} x\sin y\, dydx$

Solution

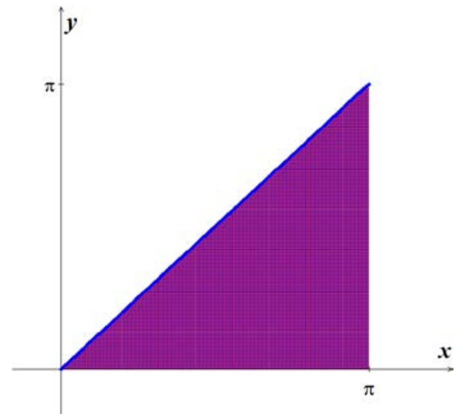
$$\int_0^{\pi}\int_0^{x} x\sin y\, dydx = \int_0^{\pi}\left[-x\cos y\right]_0^x dx$$

$$= \int_0^{\pi}\left[-x\cos x + x\right] dx$$

$$= \left[-\left(x\sin x + \cos x\right) + \frac{1}{2}x^2\right]_0^{\pi}$$

$$= -(-1) + \frac{1}{2}\pi^2 - (-1)$$

$$= \frac{\pi^2}{2} + 2$$



		$\int \cos x$
+	x	$\sin x$
–	1	$-\cos x$

Exercise

Sketch the region of integration and evaluate the integral $\int_0^{\pi}\int_0^{\sin x} ydydx$

Solution

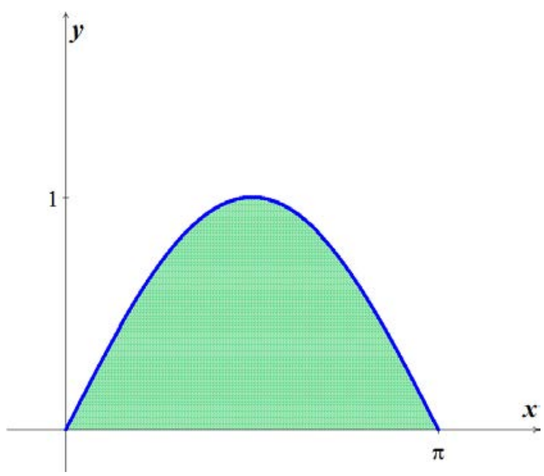
$$\int_0^{\pi}\int_0^{\sin x} ydydx = \int_0^{\pi}\left[\frac{1}{2}y^2\right]_0^{\sin x} dx$$

$$= \int_0^{\pi}\frac{1}{2}\sin^2 x\, dx$$

$$= \frac{1}{4}\int_0^{\pi}(1-\cos 2x)\, dx$$

$$= \frac{1}{4}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}$$

$$= \frac{\pi}{4}$$



Exercise

Sketch the region of integration and evaluate the integral $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}dxdy$

Solution

$$\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}dxdy = \int_1^{\ln 8}\left[e^{x+y}\right]_0^{\ln y} dy$$

$$= \int_1^{\ln 8}\left(e^{\ln y+y} - e^y\right)dy$$

$$= \int_1^{\ln 8}\left(e^{\ln y}e^y - e^y\right)dy$$

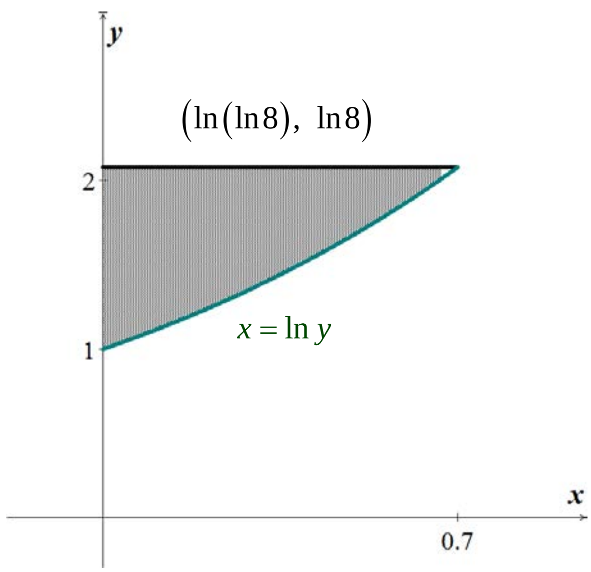
$$= \int_1^{\ln 8}\left(ye^y - e^y\right)dy$$

$$= \left[ye^y - e^y - e^y\right]_1^{\ln 8}$$

$$= (\ln 8)e^{\ln 8} - 2e^{\ln 8} - (e - 2e)$$

$$= 8\ln 8 - 16 - e$$

		$\int e^y$
+	y	e^y
−	1	e^y



Exercise

Sketch the region of integration and evaluate the integral $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}} dydx$

Solution

$$\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}} dydx = \frac{3}{2} \int_1^4 \left[\sqrt{x} e^{y/\sqrt{x}} \right]_0^{\sqrt{x}} dx$$

$$= \frac{3}{2} \int_1^4 \sqrt{x}(e-1)dx$$

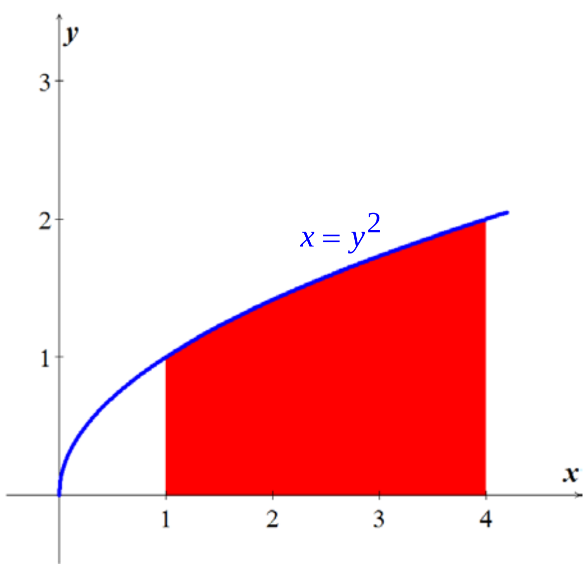
$$= \frac{3}{2}(e-1) \int_1^4 x^{1/2} dx$$

$$= \frac{3}{2}(e-1) \left[\frac{2}{3} x^{3/2} \right]_1^4$$

$$= (e-1) \left[x^{3/2} \right]_1^4$$

$$= (e-1)[8-1]$$

$$= 7(e-1)$$



Exercise

Integrate $f(x,y)=\frac{x}{y}$ over the region in the first quadrant bounded by the lines

$y=x, \quad y=2x, \quad x=1, \quad \textit{and} \quad x=2$

Solution

$$\begin{aligned}\int_1^2\int_x^{2x}\frac{x}{y}\,dydx &= \int_1^2\left[x\ln y\right]_x^{2x}dx\\ &= \int_1^2 x\left(\ln 2x-\ln x\right)dx\\ &= \int_1^2 x\left(\ln\frac{2x}{x}\right)dx \qquad\qquad \textbf{\textit{Quotient Rule}}:\ \ln M-\ln P=\ln\frac{M}{P}\\ &= \ln 2\int_1^2 x\,dx\\ &= \ln 2\left[\frac{1}{2}x^2\right]_1^2\\ &= \ln 2\left[\frac{1}{2}(4-1)\right]\\ &= \frac{3}{2}\ln 2\end{aligned}$$

Exercise

Integrate $f(x,y)=x^2+y^2$ over the triangular region with vertices $(0,0), \quad (1,0) \quad \textit{and} \quad (0,1)$

Solution

$$\begin{aligned}\int_0^1\int_0^{1-x}\left(x^2+y^2\right)dydx &= \int_0^1\left[x^2y+\frac{1}{3}y^3\right]_0^{1-x}dx\\ &= \int_0^1\left[x^2(1-x)+\frac{1}{3}(1-x)^3\right]dx\\ &= \int_0^1\left[x^2-x^3+\frac{1}{3}(1-x)^3\right]dx\\ &= \left[\frac{1}{3}x^3-\frac{1}{4}x^4-\frac{1}{12}(1-x)^4\right]_0^1\\ &= \frac{1}{3}-\frac{1}{4}-0-\left(0-0-\frac{1}{12}\right)\\ &= \frac{1}{6}\end{aligned}$$

Exercise

Integrate $f(s,t) = e^s \ln t$ over the region in the first quadrant of the st-plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$.

Solution

$$\int_1^2 \int_0^{\ln t} e^s \ln t \, ds dt = \int_1^2 \left[e^s \ln t \right]_0^{\ln t} dt$$

$$= \int_1^2 (t \ln t - \ln t) dt$$

$$\begin{array}{ll} u = \ln t & dv = dt \\ du = \frac{1}{t} dt & v = t \end{array} \quad \rightarrow \quad \int \ln t = t \ln t - \int t \frac{1}{t} dt = t \ln t - t$$

$$\int t \ln t = \frac{1}{2} t^2 \ln t - \frac{1}{4} t^2$$

$$= \left[\frac{1}{2} t^2 \ln t - \frac{1}{4} t^2 - t \ln t + t \right]_1^2$$

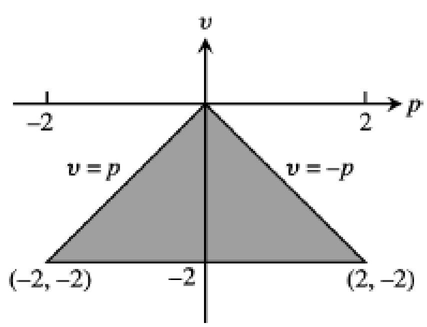
$$= 2 \ln 2 - 1 - 2 \ln 2 + 2 - \left(0 - \frac{1}{4} - 0 + 1 \right)$$

$$= \frac{1}{4}$$

Exercise

Evaluate $\displaystyle\int_{-2}^{0} \int_{v}^{-v} 2 dp dv$

Solution



$$\int_{-2}^{0} \int_{v}^{-v} 2 dp dv = 2 \int_{-2}^{0} [p]_v^{-v} dv$$

$$= -4 \int_{-2}^{0} v \, dv$$

$$= -2 \left[v^2 \right]_{-2}^{0}$$

$$= 8$$

Exercise

Evaluate $\displaystyle\int_{-\pi/3}^{\pi/3}\int_{0}^{\sec t} 3\cos t \ du dt$

Solution

$$\int_{-\pi/3}^{\pi/3}\int_{0}^{\sec t} 3\cos t \ du dt = \int_{-\pi/3}^{\pi/3} (3\cos t)\left[u\right]_0^{\sec t} \ dt$$

$$= \int_{-\pi/3}^{\pi/3} (3\cos t \sec t) dt$$

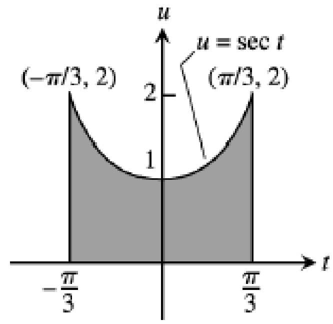
$\cos t \sec t = \cos t \frac{1}{\cos t} = 1$

$$= \int_{-\pi/3}^{\pi/3} 3dt$$

$$= 3t \Big|_{-\pi/3}^{\pi/3}$$

$$= 3\frac{2\pi}{3}$$

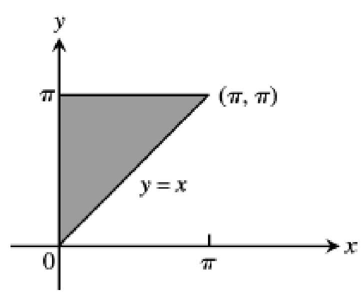
$$= 2\pi$$



Exercise

Sketch the region of integration, reverse the order of integration, and evaluate the integral

$$\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y} dydx$$



Solution

$$\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y} dydx = \int_0^{\pi}\int_0^{y} \frac{\sin y}{y} dxdy$$

$$= \int_0^{\pi} \frac{\sin y}{y}\left[x\right]_0^y \ dy$$

$$= \int_0^{\pi} \frac{\sin y}{y}(y) dy$$

$$= \int_0^{\pi} \sin y \ dy$$

$$= -\cos y \Big|_0^{\pi}$$

$$= -(-1-1)$$

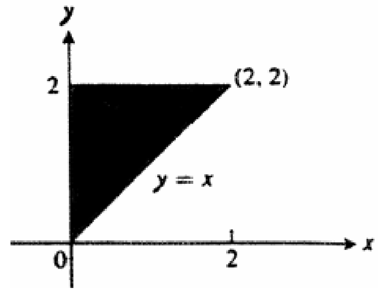
$$= 2$$

Exercise

Sketch the region of integration, reverse the order of integration, and evaluate the integral

$$\int_0^2 \int_x^2 2y^2 \sin xy \ dydx$$

Solution



$$\begin{aligned}\int_0^2 \int_x^2 2y^2 \sin xy \ dydx &= \int_0^2 \int_0^y 2y^2 \sin xy \ dxdy \\ &= -2\int_0^2 \left[y\cos xy\right]_0^y \ dy \\ &= -2\int_0^2 \left(y\cos y^2 - y\right)dy \qquad u = y^2 \ \Rightarrow \ du = 2ydy \\ &= -\int_0^2 \cos u du + \int_0^2 2ydy \\ &= \left[-\sin y^2 + y^2\right]_0^2 \\ &= -\sin 4 + 4\end{aligned}$$

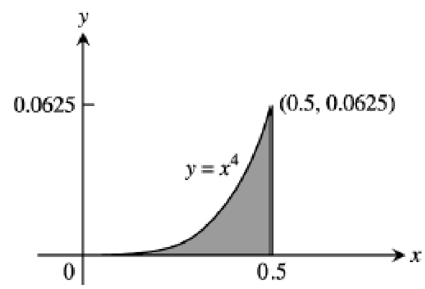
Exercise

Sketch the region of integration, reverse the order of integration, and evaluate the integral

$$\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos\left(16\pi x^5\right)dxdy$$

Solution

$$x = y^{1/4} \quad \Rightarrow \quad y = x^4$$



$$\begin{aligned}\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos\left(16\pi x^5\right)dxdy &= \int_0^{1/2} \int_0^{x^4} \cos\left(16\pi x^5\right)dydx \\ &= \int_0^{1/2} \cos\left(16\pi x^5\right)\left[y\right]_0^{x^4} \ dx \\ &= \int_0^{1/2} x^4 \cos\left(16\pi x^5\right)dx \qquad u=16\pi x^5 \ \rightarrow \ du=80\pi x^4 dx\end{aligned}$$

$$= \frac{1}{80\pi} \int_0^{1/2} \cos u \ du$$

$$= \frac{1}{80\pi} \left[\sin 16\pi x^5 \right]_0^{1/2}$$

$$= \frac{1}{80\pi} \left(\sin \frac{16\pi}{32} - 0 \right)$$

$$= \frac{1}{80\pi}$$

Exercise

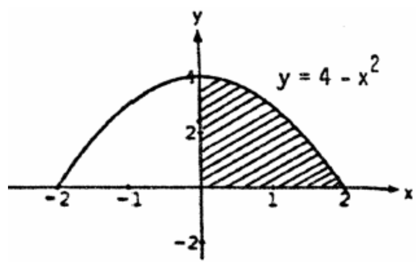
Sketch the region of integration, reverse the order of integration, and evaluate the integral

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} dydx$$

Solution

$$y = 4 - x^2 \quad \Rightarrow \quad x^2 = 4 - y \to x = \sqrt{4-y}$$

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} dydx = \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y} dxdy$$



$$= \int_0^4 \frac{e^{2y}}{4-y} \left[\frac{1}{2} x^2 \right]_0^{\sqrt{4-y}} dy$$

$$= \frac{1}{2} \int_0^4 \frac{e^{2y}}{4-y} (4-y) dy$$

$$= \frac{1}{2} \int_0^4 e^{2y} dy$$

$$= \frac{1}{4} \left[e^{2y} \right]_0^4$$

$$= \frac{1}{4} \left(e^8 - 1 \right)$$

Exercise

Find the volume of the region bounded above the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the *xy*-plane

Solution

$$V=\int_0^1\int_x^{2-x}\left(x^2+y^2\right)dydx$$

$y=x \qquad x+y=2\to y=2-x$

$x=0 \qquad y=x\to x+x=2\Rightarrow x=1$

$$=\int_0^1\left[x^2y+\frac{1}{3}y^3\right]_x^{2-x}dx$$

$$=\int_0^1\left(x^2(2-x)+\frac{1}{3}(2-x)^3-x^3-\frac{1}{3}x^3\right)dx$$

$$=\int_0^1\left(2x^2-x^3+\frac{1}{3}(2-x)^3-\frac{4}{3}x^3\right)dx$$

$$=\int_0^1\left(2x^2-\frac{7}{3}x^3\right)dx+\int_0^1\frac{1}{3}(2-x)^3\left(-d(2-x)\right)$$

$$=\left[\frac{2}{3}x^3-\frac{7}{12}x^4-\frac{1}{12}(2-x)^4\right]_0^1$$

$$=\left(\frac{2}{3}-\frac{7}{12}-\frac{1}{12}\right)-\left(-\frac{16}{12}\right)$$

$$=\frac{4}{3}$$

Exercise

Find the volume of the solid that is bounded above the cylinder $z=x^2$ and below by the region enclosed by the parabola $y=2-x^2$ and the line $y=x$ in the *xy*-plane

Solution

$$V=\int_{-2}^1\int_x^{2-x^2}x^2dydx$$

$$=\int_{-2}^1x^2[y]_x^{2-x^2}dx$$

$$=\int_{-2}^1x^2\left(2-x^2-x\right)dx$$

$$=\int_{-2}^1\left(2x^2-x^4-x^3\right)dx$$

$$=\left[\frac{2}{3}x^3-\frac{1}{5}x^5-\frac{1}{4}x^4\right]_{-2}^1 \qquad =\frac{2}{3}-\frac{1}{5}-\frac{1}{4}-\left(-\frac{15}{3}+\frac{32}{5}-\frac{16}{4}\right)$$

$$=\frac{63}{20}$$

Exercise

Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2+y^2=4$ and the plane $z+y=3$

Solution

$$V=\int_0^2\int_0^{\sqrt{4-x^2}}(3-y)\,dydx$$

$$=\int_0^2\left[3y-\frac{1}{2}y^2\right]_0^{\sqrt{4-x^2}}dx$$

$$=\int_0^2\left[3\sqrt{4-x^2}-\frac{1}{2}\left(4-x^2\right)\right]dx$$

$$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$=\left[\frac{3}{2}x\sqrt{4-x^2}+6\sin^{-1}\left(\frac{x}{2}\right)-2x+\frac{1}{6}x^3\right]_0^2$$

$$=0+6\frac{\pi}{2}-4+\frac{8}{6}-(0)$$

$$=3\pi-\frac{8}{3}$$

Exercise

Find the volume of the solid that is bounded on the front and back by the planes $x=2$, and $x=1$, on the sides by the cylinders $y=\pm\frac{1}{x}$ and above and below the planes $z=x+1$ and $z=0$.

Solution

$$V=\int_1^2\int_{-1/x}^{1/x}(x+1)\,dydx$$

$$=\int_1^2(x+1)\left[y\right]_{-1/x}^{1/x}dx$$

$$=\int_1^2(x+1)\left(\frac{2}{x}\right)dx$$

$$=2\int_1^2\left(1+\frac{1}{x}\right)dx$$

$$=2\left[x+\ln x\right]_1^2$$

$$=2\left[2+\ln 2-1\right]$$

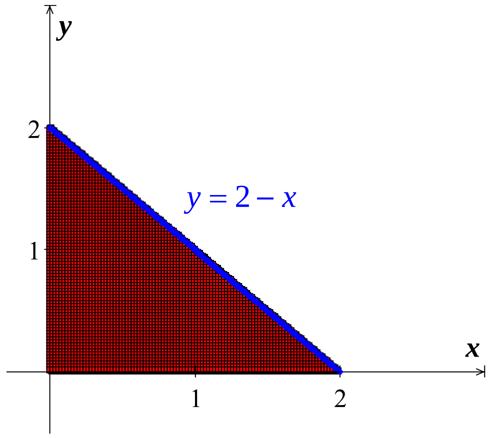
$$=2\left(1+\ln 2\right)$$

Exercise

Find the area of the region enclosed by the coordinate axes and the line $x + y = 2$.

Solution

$$\begin{aligned}\int_0^2 \int_0^{2-x} dydx &= \int_0^2 [y]_0^{2-x} dx \\ &= \int_0^2 (2-x)dx \\ &= \left[2x - \frac{1}{2}x^2\right]_0^2 \\ &= 4 - \frac{1}{2}(4) \\ &= 2\end{aligned}$$

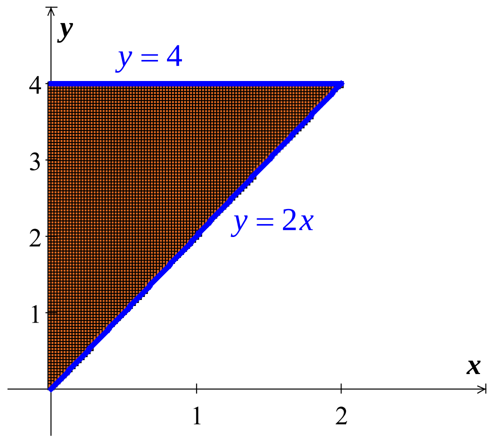


Exercise

Find the area of the region enclosed by the lines $x = 0$, $y = 2x$, and $y = 4$

Solution

$$\begin{aligned}\int_0^2 \int_{2x}^{4} dydx &= \int_0^2 [y]_{2x}^{4} dx \\ &= \int_0^2 (4-2x)dx \\ &= \left[4x - x^2\right]_0^2 \\ &= 4\end{aligned}$$



Exercise

Find the area of the region enclosed by the parabola $x = y - y^2$ and the line $y = -x$.

Solution

$$x = y - y^2 = -y \quad \to \quad 2y - y^2 = 0 \Rightarrow \boxed{y = 0,\ 2}$$

$$\int_0^2 \int_{-y}^{y-y^2} dxdy = \int_0^2 [x]_{-y}^{y-y^2} dy$$

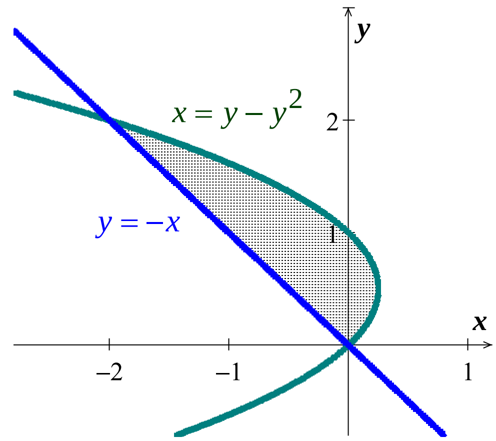
$$= \int_0^2 \left(y - y^2 + y\right) dy$$

$$= \int_0^2 \left(2y - y^2\right) dy$$

$$= \left[y^2 - \frac{1}{3}y^3\right]_0^2$$

$$= 4 - \frac{8}{3}$$

$$= \frac{4}{3}$$



Exercise

Find the area of the region enclosed by the curve $y = e^x$ and the lines $y = 0$, $x = 0$ and $x = \ln 2$.

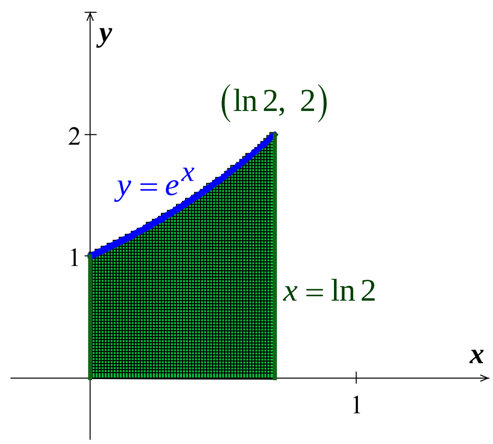
Solution

$$\int_0^{\ln 2} \int_0^{e^x} dy dx = \int_0^{\ln 2} \left[y\right]_0^{e^x} dx$$

$$= \int_0^{\ln 2} e^x dx$$

$$= \left[e^x\right]_0^{\ln 2} \quad = 2 - 1$$

$$= 1$$



Exercise

Find the area of the region enclosed by the curve $y = \ln x$ and $y = 2\ln x$ and the lines $x = e$ in the first quadrant.

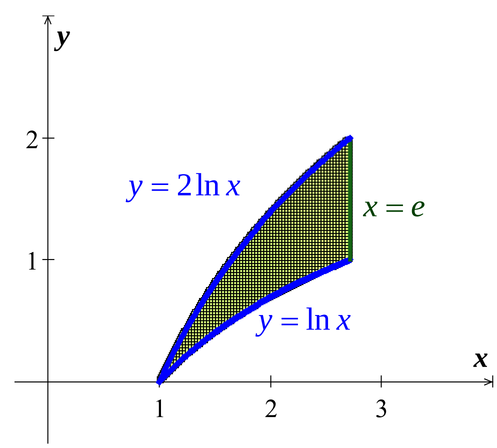
Solution

$$\int_1^e \int_{\ln x}^{2\ln x} dy dx = \int_1^e \left[y\right]_{\ln x}^{2\ln x} dx$$

$$= \int_0^{\ln 2} \ln x \, dx$$

$$= \left[x\ln x - x\right]_1^e \quad = e - e - (0 - 1)$$

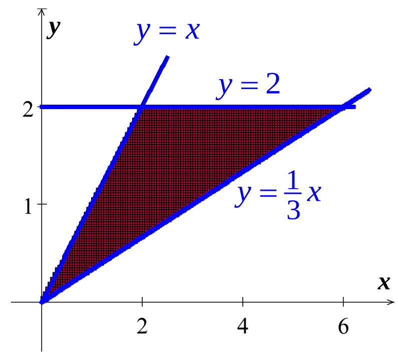
$$= 1$$



Exercise

Find the area of the region enclosed by the lines $y = x$, $y = \frac{x}{3}$, and $y = 2$

Solution



$$\int_0^2 \int_y^{3y} dxdy = \int_0^2 x\Big|_y^{3y} dy$$

$$= \int_0^2 (2y)dy$$

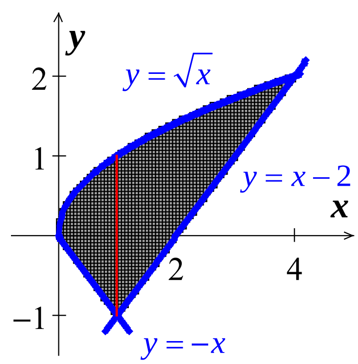
$$= y^2\Big|_0^2$$

$$\underline{= 4}$$

Exercise

Find the area of the region enclosed by the lines $y = x - 2$ and $y = -x$ and the curve $y = \sqrt{x}$

Solution



$$\int_0^1 \int_{-x}^{\sqrt{x}} dydx + \int_1^4 \int_{x-2}^{\sqrt{x}} dydx = \int_0^1 y\Big|_{-x}^{\sqrt{x}} dx + \int_1^4 y\Big|_{x-2}^{\sqrt{x}} dx$$

$$= \int_0^1 \left(\sqrt{x} - x\right)dx + \int_1^4 \left(\sqrt{x} - x + 2\right)dx$$

$$= \left[\frac{2}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^1 + \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right]_1^4$$

$$= \frac{2}{3} + \frac{1}{2} + \frac{2}{3}4^{3/2} - 2 + 8 - \frac{2}{3} - \frac{1}{2} + 2$$

$$\underline{= \frac{13}{3}}$$

Exercise

Find the area of the region enclosed by the parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$

Solution

$$\int_{-1}^1 \int_{2y^2-2}^{y^2-1} dxdy = \int_{-1}^1 [y]_{2y^2-2}^{y^2-1} dy$$

$$= \int_{-1}^{1} \left(y^2 - 1 - 2y^2 + 2\right) dy$$

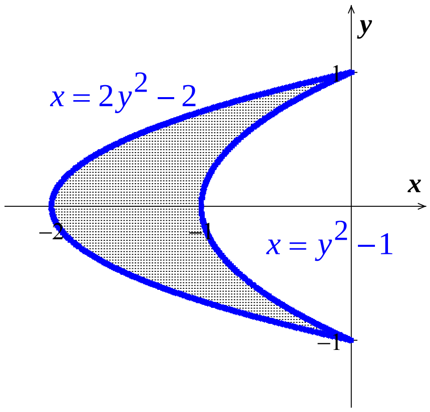
$$= \int_{-1}^{1} \left(1 - y^2\right) dy$$

$$= \left[y - \frac{1}{3} y^3 \right]_{-1}^{1}$$

$$= 1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)$$

$$= 2 - \frac{2}{3}$$

$$= \frac{4}{3}$$



Exercise

Find the area of the region $\int_0^6 \int_{y^2/3}^{2y} dxdy$

Solution

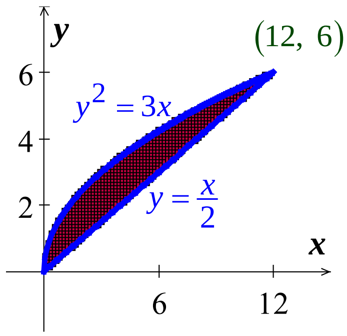
$$\int_0^6 \int_{y^2/3}^{2y} dxdy = \int_0^6 [x]_{y^2/3}^{2y} dy$$

$$= \int_0^6 \left(2y - \frac{1}{3} y^2\right) dy$$

$$= \left[y^2 - \frac{1}{9} y^3 \right]_0^6$$

$$= 36 - \frac{1}{9}(216)$$

$$= 12$$



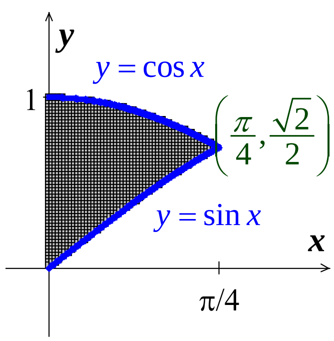
Exercise

Find the area of the region $\int_0^{\pi/4} \int_{\sin x}^{\cos x} dydx$

Solution

$$\int_0^{\pi/4} \int_{\sin x}^{\cos x} dydx = \int_0^{\pi/4} [y]_{\sin x}^{\cos x} dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x) dx$$



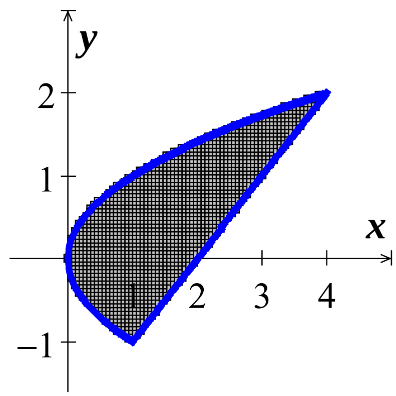
$$= \left[\sin x + \cos x\right]_0^{\pi/4}$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1)$$

$$\underline{= \sqrt{2} - 1}$$

Exercise

Find the area of the region $\int_{-1}^{2}\int_{y^2}^{y+2} dxdy$



Solution

$$\int_{-1}^{2}\int_{y^2}^{y+2} dxdy = \int_{-1}^{2}\left(y+2-y^2\right)dy$$

$$= \left[\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\right]_{-1}^{2}$$

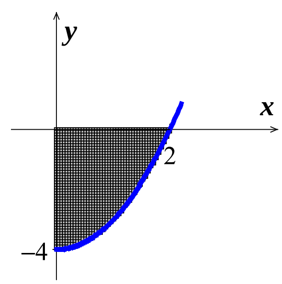
$$= \left(2+4-\frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

$$\underline{= \frac{9}{2}}$$

Exercise

Find the area of the region $\int_{0}^{2}\int_{x^2-4}^{0} dydx + \int_{0}^{4}\int_{0}^{\sqrt{x}} dydx$

Solution



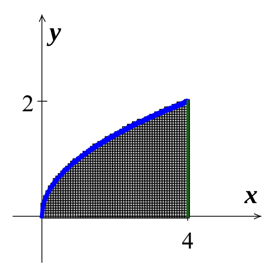
$$\int_{0}^{2}\int_{x^2-4}^{0} dydx + \int_{0}^{4}\int_{0}^{\sqrt{x}} dydx = \int_{0}^{2}\left(4-x^2\right)dx + \int_{0}^{4}\sqrt{x}\, dx$$

$$= \left[4x - \frac{1}{3}x^3\right]_0^2 + \frac{2}{3}\left[x^{3/2}\right]_0^4$$

$$= \left(8 - \frac{8}{3}\right) + \frac{2}{3}\left(4^{3/2}\right)$$

$$= \frac{16}{3} + \frac{16}{3}$$

$$\underline{= \frac{32}{3}}$$



Exercise

Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2, \quad 0 \le y \le 2$

Solution

$$\text{Average height } = \frac{1}{4}\int_0^2 \int_0^2 \left(x^2 + y^2\right) dydx$$

$$= \frac{1}{4}\int_0^2 \left[x^2 y + \frac{1}{3}y^3\right]_0^2 dx$$

$$= \frac{1}{4}\int_0^2 \left(2x^2 + \frac{8}{3}\right) dx$$

$$= \frac{1}{4}\left[\frac{2}{3}x^3 + \frac{8}{3}x\right]_0^2$$

$$= \frac{1}{4}\left[\frac{2}{3}(8) + \frac{8}{3}(2)\right]$$

$$= \frac{1}{4}\left[\frac{16}{3} + \frac{16}{3}\right]$$

$$= \frac{8}{3}$$

Exercise

Find the average height of $f(x,y) = \frac{1}{xy}$ over the square $\ln 2 \le x \le 2\ln 2, \quad \ln 2 \le y \le 2\ln 2$

Solution

$$\text{Average height } = \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2} \int_{\ln 2}^{2\ln 2} \frac{1}{xy} dydx$$

$$= \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2} \frac{1}{x}\left[\ln y\right]_{\ln 2}^{2\ln 2} dx$$

$$= \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2} \frac{1}{x}\left(2\ln 2 - \ln 2\right) dx$$

$$= \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2} \frac{1}{x}\left(\ln 2\right) dx$$

$$= \frac{1}{\ln 2}\left[\ln x\right]_{\ln 2}^{2\ln 2}$$

$$= \frac{1}{\ln 2}\left(2\ln 2 - \ln 2\right)$$

$$= 1$$

Solution Section 3.3 – Double Integrals in Polar Coordinates

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dydx$$

Solution

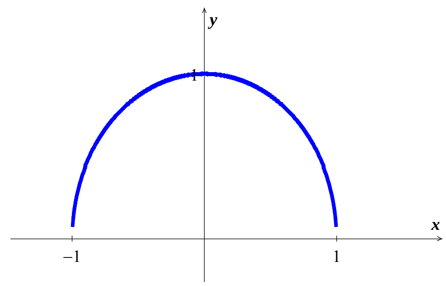
$$y=\sqrt{1-x^2} \quad \Rightarrow \quad y^2=1-x^2 \to x^2+y^2=1=r^2$$

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dydx=\int_{0}^{\pi}\int_{0}^{1} rdrd\theta$$

$$=\int_{0}^{\pi}\frac{1}{2}\left[r^2\right]_0^1 d\theta$$

$$=\frac{1}{2}\int_{0}^{\pi} d\theta \qquad =\frac{1}{2}[\theta]_0^{\pi}$$

$$=\frac{\pi}{2}$$

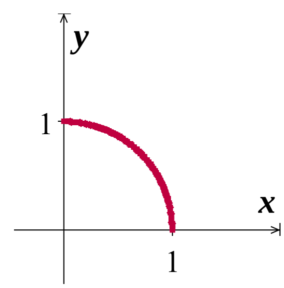


Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}}\left(x^2+y^2\right)dxdy$$

Solution



$$\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}}\left(x^2+y^2\right)dxdy=\int_{0}^{\pi/2}\int_{0}^{1} r^2\ rdrd\theta$$

$$=\frac{1}{4}\int_{0}^{\pi/2}\left[r^4\right]_0^1 d\theta$$

$$=\frac{1}{4}\int_{0}^{\pi/2} d\theta \qquad =\frac{1}{4}\left(\frac{\pi}{2}\right)$$

$$=\frac{\pi}{8}$$

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx$$

Solution

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx = \int_{0}^{2\pi}\int_{0}^{a} rdrd\theta$$

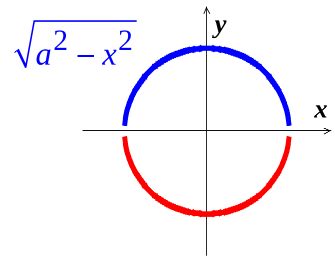
$-\sqrt{a^2-x^2}$

$$= \frac{1}{2}\int_{0}^{2\pi}\left[r^2\right]_0^a d\theta$$

$$= \frac{a^2}{2}\int_{0}^{2\pi} d\theta$$

$$= \frac{a^2}{2}\left[\theta\right]_0^{2\pi}$$

$$\underline{= \pi a^2}$$



Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{0}^{6}\int_{0}^{y} xdxdy$$

Solution

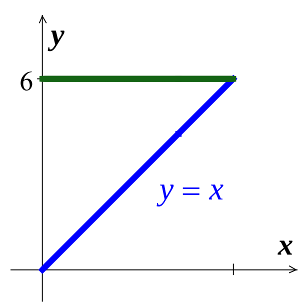
$\theta \quad x = r\cos\theta, \quad \sin\theta = \frac{6}{r} \to r = \frac{6}{\sin\theta} = 6\csc\theta \quad \frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$$\int_{0}^{6}\int_{0}^{y} xdxdy = \int_{\pi/4}^{\pi/2}\int_{0}^{6\csc\theta} r^2\cos\theta\, drd\theta$$

$$= \frac{1}{3}\int_{\pi/4}^{\pi/2} \cos\theta\left[r^3\right]_0^{6\csc\theta} d\theta$$

$$= \frac{216}{3}\int_{\pi/4}^{\pi/2} \cos\theta\csc^3\theta\, d\theta$$

$$= 72\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta\, d\theta \qquad d(\cot\theta) = -\csc^2\theta\, d\theta$$



$$= -72\int_{\pi/4}^{\pi/2} \cot\theta \; d(\cot\theta)$$

$$= -36\left[\cot^2\theta\right]_{\pi/4}^{\pi/2}$$

$$= -36(0-1)$$

$$= 36$$

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\, dydx$$

Solution

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\, dydx = \int_{\pi}^{3\pi/2}\int_{0}^{1} \frac{2}{1+r} r\; drd\theta$$

$$= 2\int_{\pi}^{3\pi/2}\int_{0}^{1} \left(1-\frac{1}{1+r}\right)\, drd\theta$$

$$= 2\int_{\pi}^{3\pi/2} \left[1-\ln(1+r)\right]_0^1 \, d\theta$$

$$= 2\int_{\pi}^{3\pi/2} (1-\ln 2)\, d\theta$$

$$= 2(1-\ln 2)\left[\theta\right]_{\pi}^{3\pi/2}$$

$$= 2(1-\ln 2)\left(\frac{3\pi}{2}-\pi\right)$$

$$= (1-\ln 2)\pi$$

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_0^{\ln 2}\int_0^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\,dxdy$$

Solution

$$\int_0^{\ln 2}\int_0^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\,dxdy = \int_0^{\pi/2}\int_0^{\ln 2} e^r r\, drd\theta$$

		$\int e^r$
+	r	e^r
−	1	e^r

$$= \int_0^{\pi/2}\left[re^r - e^r\right]_0^{\ln 2} d\theta$$

$$= \int_0^{\pi/2}\left(\ln 2e^{\ln 2} - e^{\ln 2} + 1\right)d\theta$$

$$= \int_0^{\pi/2}(2\ln 2 - 2 + 1)\,d\theta$$

$$= \int_0^{\pi/2}(2\ln 2 - 1)\,d\theta$$

$$= (2\ln 2 - 1)\left(\frac{\pi}{2} - 0\right)$$

$$\boxed{= \frac{\pi}{2}(2\ln 2 - 1)}$$

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dxdy$$

Solution

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dxdy = \int_0^{2\pi}\int_0^1 \ln\left(r^2+1\right)rdrd\theta$$

$$= 4\int_0^{\pi/2}\int_0^1 \ln\left(r^2+1\right)\tfrac{1}{2}d\left(r^2+1\right)d\theta$$

$$= 2\int_0^{\pi/2}\left[\left(\ln\left(r^2+1\right)\right)^2\right]_0^1 d\theta \qquad \int \ln ax dx = x\ln ax - x$$

$$= 2\int_0^{\pi/2}(\ln 4 - 1)d\theta$$

$$= 2(\ln 4 - 1)[\theta]_0^{\pi/2}$$

$$= 2(\ln 4 - 1)\left(\frac{\pi}{2} - 0\right)$$

$$\underline{= \pi(\ln 4 - 1)}$$

Exercise

Change the Cartesian integral into an equivalent polar integral. Then integrate the polar integral

$$\int_1^2\int_0^{\sqrt{2x-x^2}} \frac{1}{\left(x^2+y^2\right)^2}dydx$$

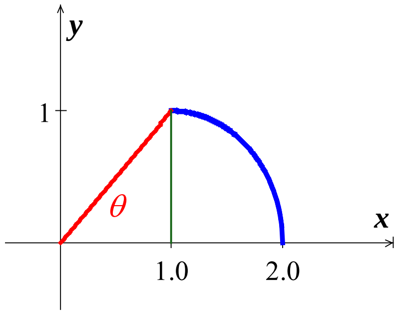
Solution

$$y^2 = 2x - x^2 \Rightarrow x^2 - 2x + 1 - 1 + y^2 = 0 \quad (x-1)^2 + y^2 = 1$$

$$r = \frac{x}{\cos\theta} = \frac{1}{\cos\theta} = \sec\theta$$

$$y = \sqrt{2x - x^2} \quad \rightarrow \quad y^2 = 2x - x^2 \Rightarrow x^2 + y^2 = 2x$$

$$r^2 = 2r\cos\theta \rightarrow r = 2\cos\theta$$



$$\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{\left(x^2+y^2\right)^2} dydx = \int_0^{\pi/4} \int_{\sec\theta}^{2\cos\theta} \frac{1}{r^4} rdrd\theta$$

$$= \int_0^{\pi/4} \int_{\sec\theta}^{2\cos\theta} r^{-3} drd\theta$$

$$= \int_0^{\pi/4} \left[-\frac{1}{2r^2}\right]_{\sec\theta}^{2\cos\theta} d\theta$$

$$= \int_0^{\pi/4} \left(-\frac{1}{8\cos^2\theta} + \frac{1}{2\sec^2\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \left(-\frac{1}{8}\sec^2\theta + \frac{1}{2}\cos^2\theta\right) d\theta \qquad \int \cos^2 axdx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$= \left[-\frac{1}{8}\tan\theta + \frac{1}{2}\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\right]_0^{\pi/4}$$

$$= \left[\frac{1}{4}\theta + \frac{1}{8}\sin 2\theta - \frac{1}{8}\tan\theta\right]_0^{\pi/4}$$

$$= \frac{1}{4}\frac{\pi}{4} + \frac{1}{8} - \frac{1}{8} - (0)$$

$$= \frac{\pi}{16}$$

Exercise

Find the area of the region cut from the first quadrant by the curve $r = 2(2 - \sin 2\theta)^{1/2}$

Solution

$$\int_0^{\pi/2} \int_0^{2\sqrt{2-\sin 2\theta}} rdrd\theta = \frac{1}{2}\int_0^{\pi/2} \left[r^2\right]_0^{2\sqrt{2-\sin 2\theta}} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 4(2-\sin 2\theta) d\theta$$

$$= 2\left[2\theta + \frac{1}{2}\cos 2\theta\right]_0^{\pi/2}$$

$$= 2\left[\pi - \frac{1}{2} - \left(\frac{1}{2}\right)\right]$$

$$= 2(\pi - 1)$$

Exercise

Find the area of the region lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$

Solution

$$A = 2\int_0^{\pi/2}\int_1^{1+\cos\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[r^2\right]_1^{1+\cos\theta} d\theta$$

$$= \int_0^{\pi/2}\left[(1+\cos\theta)^2 - 1\right]d\theta$$

$$= \int_0^{\pi/2}\left(1 + 2\cos\theta + \cos^2\theta - 1\right)d\theta$$

$$= \int_0^{\pi/2}\left(2\cos\theta + \cos^2\theta\right)d\theta \qquad \int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$= \left[2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= 2 + \frac{\pi}{4}$$

Exercise

Find the area enclosed by one leaf of the rose $r = 12\cos 3\theta$

Solution

$$A = 2\int_0^{\pi/6}\int_0^{12\cos 3\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/6}\left[r^2\right]_0^{12\cos 3\theta} d\theta$$

$$= 144\int_0^{\pi/6}\cos^2 3\theta\,d\theta \qquad \int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$= 144\left[\frac{\theta}{2} + \frac{\sin 6\theta}{12}\right]_0^{\pi/6}$$

$$= 144\left(\frac{\pi}{12}\right)$$

$$= 12\pi$$

Exercise

Find the area of the region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$

Solution

$$A = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[r^2\right]_0^{1-\cos\theta} d\theta$$

$$= 2\int_0^{\pi/2}(1-\cos\theta)^2\,d\theta$$

$$= 2\int_0^{\pi/2}\left(1 - 2\cos\theta + \cos^2\theta\right)d\theta \qquad \int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$= 2\left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= 2\left(\frac{\pi}{2} - 2 + \frac{\pi}{4}\right)$$

$$= \frac{3\pi}{2} - 4$$

Exercise

Integrate $f(x,y) = \dfrac{\ln\left(x^2+y^2\right)}{\sqrt{x^2+y^2}}$ over the region $1 \le x^2 + y^2 \le e$

Solution

$$\int_0^{2\pi}\int_1^{\sqrt{e}}\left(\frac{\ln r^2}{r}\right) r\,dr\,d\theta = \int_0^{2\pi}\int_1^{\sqrt{e}} 2\ln r\ dr\,d\theta$$

$$= 2\int_0^{2\pi}\left[r\ln r - r\right]_1^{\sqrt{e}}\ d\theta$$

$$= 2\int_0^{2\pi}\left[\sqrt{e}\ln e^{1/2} - \sqrt{e} - (0-1)\right]d\theta$$

$$= 2\int_0^{2\pi}\left[\frac{1}{2}\sqrt{e} - \sqrt{e} + 1\right]d\theta$$

$$= 2\left(-\frac{1}{2}\sqrt{e} + 1\right)\left[\theta\right]_0^{2\pi}$$

$$= 2\pi\left(2 - \sqrt{e}\right)$$

Exercise

Evaluate the integral $\int_0^\infty \int_0^\infty \frac{1}{\left(1+x^2+y^2\right)^2} dxdy$

Solution

$$\int_0^\infty \int_0^\infty \frac{1}{\left(1+x^2+y^2\right)^2} dxdy = \int_0^{\pi/2} \int_0^\infty \frac{1}{\left(1+r^2\right)^2} rdrd\theta$$

$$= \int_0^\infty \int_0^{\pi/2} d\theta \frac{rdr}{\left(1+r^2\right)^2}$$

$$= \int_0^\infty \left[\theta\right]_0^{\pi/2} \frac{rdr}{\left(1+r^2\right)^2} \qquad d\left(1+r^2\right) = 2rdr$$

$$= \frac{\pi}{2} \int_0^\infty \left(1+r^2\right)^{-2} \frac{1}{2} d\left(1+r^2\right)$$

$$= \frac{\pi}{4}\left[-\frac{1}{1+r^2}\right]_0^\infty \qquad \frac{1}{\infty} = 0$$

$$= -\frac{\pi}{4}(0-1)$$

$$= \frac{\pi}{4}$$

Exercise

The region enclosed by the lemniscates $r^2 = 2\cos 2\theta$ is the base of a solid right cylinder whose top is bounded by the sphere $z = \sqrt{2-r^2}$. Find the cylinder's volume.

Solution

$$V = 4\int_0^{\pi/4} \int_0^{\sqrt{2\cos 2\theta}} r\sqrt{2-r^2}\, drd\theta \qquad d\left(2-r^2\right) = -2rdr$$

$$= -2\int_0^{\pi/4} \int_0^{\sqrt{2\cos 2\theta}} \left(2-r^2\right)^{1/2} d\left(2-r^2\right) d\theta$$

$$= -2\int_0^{\pi/4} \left[\frac{2}{3}\left(2-r^2\right)^{3/2}\right]_0^{\sqrt{2\cos 2\theta}} d\theta$$

$$= -\frac{4}{3}\int_0^{\pi/4}\left[\left(2-2\cos 2\theta\right)^{3/2} - 2^{3/2}\right]d\theta$$

$$= -\frac{4}{3}\int_0^{\pi/4}\left[2^{3/2}\left(1-\cos 2\theta\right)^{3/2}\right]d\theta + \frac{4}{3}\int_0^{\pi/4} 2^{3/2}\,d\theta$$

$$= -\frac{4}{3}2\sqrt{2}\int_0^{\pi/4}\left(2\sin^2\theta\right)^{3/2}d\theta + \frac{4}{3}2\sqrt{2}\left[\theta\right]_0^{\pi/4}$$

$$= -\frac{8\sqrt{2}}{3}\int_0^{\pi/4} 2\sqrt{2}\sin^3\theta\,d\theta + \frac{8}{3}\sqrt{2}\left(\frac{\pi}{4}\right)$$

$$= -\frac{32}{3}\int_0^{\pi/4}\sin^2\theta\sin\theta\,d\theta + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\int_0^{\pi/4}\left(1-\cos^2\theta\right)d\left(\cos\theta\right) + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\left[\cos\theta - \frac{1}{3}\cos^3\theta\right]_0^{\pi/4} + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\left[\frac{\sqrt{2}}{2} - \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3 - \left(1-\frac{1}{3}\right)\right] + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12} - \frac{2}{3}\right) + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{32}{3}\left(\frac{5\sqrt{2}-8}{12}\right) + \frac{2\pi\sqrt{2}}{3}$$

$$= 8\left(\frac{5\sqrt{2}-8}{9}\right) + \frac{2\pi\sqrt{2}}{3}$$

$$= \frac{40\sqrt{2}-64+6\pi\sqrt{2}}{9}$$

Solution Section 3.4 – Triple Integrals

Exercise

Evaluate the integral $\int_0^1 \int_0^1 \int_0^1 \left(x^2+y^2+z^2\right) dzdydx$

Solution

$$\begin{aligned}
\int_0^1 \int_0^1 \int_0^1 \left(x^2+y^2+z^2\right) dzdydx &= \int_0^1 \int_0^1 \left[x^2 z + y^2 z + \tfrac{1}{3} z^3\right]_0^1 dydx \\
&= \int_0^1 \int_0^1 \left[x^2 + y^2 + \frac{1}{3}\right] dydx \\
&= \int_0^1 \left[x^2 y + \tfrac{1}{3} y^3 + \tfrac{1}{3} y\right]_0^1 dx \\
&= \int_0^1 \left[x^2 + \frac{1}{3} + \frac{1}{3}\right] dx \\
&= \left[\frac{1}{3} x^3 + \frac{2}{3} x\right]_0^1 \\
&= \frac{1}{3} + \frac{2}{3} \\
&= 1
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dzdxdy$

Solution

$$\begin{aligned}
\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dzdxdy &= \int_0^{\sqrt{2}} \int_0^{3y} \left[8 - x^2 - y^2 - \left(x^2 + 3y^2\right)\right] dxdy \\
&= \int_0^{\sqrt{2}} \int_0^{3y} \left(8 - 2x^2 - 4y^2\right) dxdy \\
&= \int_0^{\sqrt{2}} \left[8x - \tfrac{2}{3} x^3 - 4y^2 x\right]_0^{3y} dy
\end{aligned}$$

$$= \int_0^{\sqrt{2}} \left(24y - 18y^3 - 12y^3\right) dy$$

$$= \int_0^{\sqrt{2}} \left(24y - 30y^3\right) dy$$

$$= \left[12y^2 - \frac{15}{2}y^4\right]_0^{\sqrt{2}}$$

$$= 24 - 30$$

$$\underline{= -6}$$

Exercise

Evaluate the integral $\int_0^{\pi/6} \int_0^1 \int_{-2}^3 y \sin z \; dxdydz$

Solution

$$\int_0^{\pi/6} \int_0^1 \int_{-2}^3 y \sin z \; dxdydz = \int_0^{\pi/6} \int_0^1 y \sin z \; [x]_{-2}^{3} \; dydz$$

$$= 5 \int_0^{\pi/6} \int_0^1 y \sin z \; dydz$$

$$= 5 \int_0^{\pi/6} \sin z \; \left[\frac{1}{2}y^2\right]_0^1 dz$$

$$= \frac{5}{2} \int_0^{\pi/6} \sin z \; dz$$

$$= -\frac{5}{2} [\cos z]_0^{\pi/6}$$

$$= -\frac{5}{2} \left(\frac{\sqrt{3}}{2} - 1\right)$$

$$\underline{= \frac{5}{4}\left(2 - \sqrt{3}\right)}$$

Exercise

Evaluate the integral $\int_{-1}^{1}\int_{0}^{1}\int_{0}^{2}(x+y+z)\,dydxdz$

Solution

$$\begin{aligned}
\int_{-1}^{1}\int_{0}^{1}\int_{0}^{2}(x+y+z)\,dydxdz &= \int_{-1}^{1}\int_{0}^{1}\left[xy+\frac{1}{2}y^2+zy\right]_0^2 dxdz \\
&= \int_{-1}^{1}\int_{0}^{1}(2x+2+2z)\,dxdz \\
&= \int_{-1}^{1}\left[x^2+(2+2z)x\right]_0^1 dz \\
&= \int_{-1}^{1}(1+2+2z)\,dz \\
&= \int_{-1}^{1}(3+2z)\,dz \\
&= \left[3z+z^2\right]_{-1}^{1} \\
&= (3+1)-(-3+1) \\
&= 6
\end{aligned}$$

Exercise

Evaluate the integral $\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2}} dzdydx$

Solution

$$\begin{aligned}
\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2}} dzdydx &= \int_{0}^{3}\int_{0}^{\sqrt{9-x^2}} \sqrt{9-x^2}\,dydx \\
&= \int_{0}^{3}\sqrt{9-x^2}\,\left[y\right]_0^{\sqrt{9-x^2}} dx \\
&= \int_{0}^{3}\left(9-x^2\right)dx \\
&= \left[9x-\frac{1}{3}x^3\right]_0^3 \\
&= 18
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} xdzdydx$

Solution

$$\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} xdzdydx = \int_0^1 \int_0^{1-x^2} \left[xz\right]_3^{4-x^2-y} dydx$$

$$= \int_0^1 \int_0^{1-x^2} x\left(4-x^2-y-3\right)dydx$$

$$= \int_0^1 \int_0^{1-x^2} \left(x-x^3-xy\right)dydx$$

$$= \int_0^1 \left[\left(x-x^3\right)y-\frac{1}{2}xy^2\right]_0^{1-x^2} dx$$

$$= \int_0^1 \left[x\left(1-x^2\right)\left(1-x^2\right)-\frac{1}{2}x\left(1-x^2\right)^2\right]dx$$

$$= \int_0^1 \left(1-x^2\right)^2\left(\frac{1}{2}x\right)dx \qquad d\left(1-x^2\right)=-2xdx$$

$$= -\frac{1}{4}\int_0^1 \left(1-x^2\right)^2 d\left(1-x^2\right)$$

$$= -\frac{1}{12}\left[\left(1-x^2\right)^3\right]_0^1$$

$$= -\frac{1}{12}(0-1)$$

$$\boxed{= \frac{1}{12}}$$

Exercise

Evaluate the integral $\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\cos(u+v+w)\,dudvdw$

Solution

$$\begin{aligned}
\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\cos(u+v+w)\,dudvdw &= \int_0^{\pi}\int_0^{\pi}\left[\sin(u+v+w)\right]_0^{\pi}\,dvdw \\
&= \int_0^{\pi}\int_0^{\pi}\left[\sin(v+w+\pi)-\sin(v+w)\right]dvdw \\
&= \int_0^{\pi}\left[-\cos(v+w+\pi)+\cos(v+w)\right]_0^{\pi}\,dw \\
&= \int_0^{\pi}\left[-\cos(w+2\pi)+\cos(w+\pi)+\cos(w+\pi)-\cos(w)\right]dw \\
&= \int_0^{\pi}\left[-\cos(w+2\pi)+2\cos(w+\pi)-\cos(w)\right]dw \\
&= \left[-\sin(w+2\pi)+2\sin(w+\pi)-\sin(w)\right]_0^{\pi} \\
&= -\sin(3\pi)+2\sin(2\pi)-\sin\pi-\left(-\sin(2\pi)+2\sin(\pi)-\sin 0\right) \\
&= 0
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/4}\int_0^{\ln\sec v}\int_{-\infty}^{2t}e^x\,dxdtdv$

Solution

$$\begin{aligned}
\int_0^{\pi/4}\int_0^{\ln\sec v}\int_{-\infty}^{2t}e^x\,dxdtdv &= \int_0^{\pi/4}\int_0^{\ln\sec v}\left[e^x\right]_{-\infty}^{2t}\,dtdv \\
&= \int_0^{\pi/4}\int_0^{\ln\sec v}\left(e^{2t}-e^{-\infty}\right)dtdv \\
&= \int_0^{\pi/4}\int_0^{\ln\sec v}\left(e^{2t}-e^{-\infty}\right)dtdv \qquad e^{-\infty}=0 \\
&= \int_0^{\pi/4}\int_0^{\ln\sec v}e^{2t}\,dtdv
\end{aligned}$$

$$= \frac{1}{2}\int_0^{\pi/4}\left[e^{2t}\right]_0^{\ln\sec v} dv$$

$$= \frac{1}{2}\int_0^{\pi/4}\left(e^{2\ln\sec v} - 1\right) dv \qquad\qquad e^{2\ln\sec v} = e^{\ln\sec^2 v} = \sec^2 v$$

$$= \frac{1}{2}\int_0^{\pi/4}\left(\sec^2 v - 1\right) dv$$

$$= \frac{1}{2}\left[\tan v - v\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left(1 - \frac{\pi}{4}\right)$$

$$= \frac{1}{2} - \frac{\pi}{8}$$

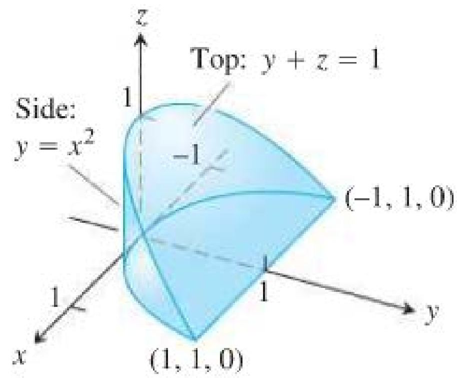
Exercise

Here is the region of integration of the integral

$$\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} dzdydx$$

a) *dydzdx* ***b***) *dydxdz* ***c***) *dxdydz*

d) *dxdzdy* ***e***) *dzdxdy*



Solution

a) $\displaystyle\int_{-1}^{1}\int_0^{1-x^2}\int_{x^2}^{1-x} dydzdx$

b) $\displaystyle\int_0^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-x} dydxdz$

c) $\displaystyle\int_0^{1}\int_0^{1-x}\int_{-\sqrt{y}}^{\sqrt{y}} dxdydz$

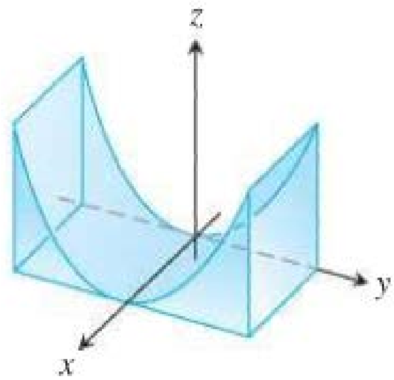
d) $\displaystyle\int_0^{1}\int_0^{1-y}\int_{-\sqrt{y}}^{\sqrt{y}} dxdzdy$

e) $\displaystyle\int_0^{1}\int_{-\sqrt{y}}^{\sqrt{y}}\int_0^{1-y} dzdxdy$

Exercise

Find the volumes of the region between the cylinder $z = y^2$ and the *xy*-plane that is bounded by the planes $x = 0, \quad x = 1, \quad y = -1, \quad y = 1$

Solution



$$V = \int_0^1 \int_{-1}^1 \int_0^{y^2} dzdydx$$

$$= \int_0^1 \int_{-1}^1 [z]_0^{y^2} \, dydx$$

$$= \int_0^1 \int_{-1}^1 y^2 dydx$$

$$= \frac{1}{3}\int_0^1 \left[y^3\right]_{-1}^1 dx$$

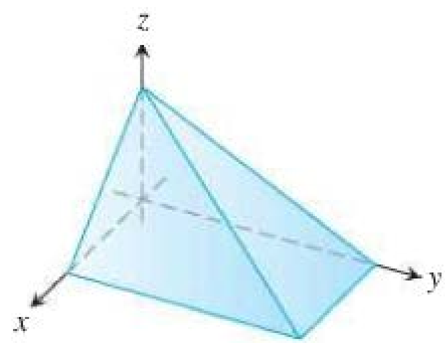
$$= \frac{2}{3}\int_0^1 dx$$

$$= \frac{2}{3}$$

Exercise

Find the volumes of the region in the first octant bounded by the coordinate planes and the planes $x + z = 1, \quad y + 2z = 2$

Solution



$$V = \int_0^1 \int_0^{1-x} \int_0^{2-2z} dydzdx$$

$$= \int_0^1 \int_0^{1-x} (2 - 2z) dzdx$$

$$= \int_0^1 \left[2z - z^2\right]_0^{1-x} dx$$

$$= \int_0^1 \left[2(1-x) - (1-x)^2\right] dx$$

$$= \int_0^1 (1-x)(2-1+x)\,dx$$

$$= \int_0^1 (1-x)(1+x)\,dx$$

$$= \int_0^1 \left(1-x^2\right)dx$$

$$= \left[x - \frac{1}{3}x^3\right]_0^1$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$

Exercise

Find the volumes of the region in the first octant bounded by the coordinate planes and the plane $y+z=2$, and the cylinder $x=4-y^2$

Solution

$$V = \int_0^4 \int_0^{\sqrt{4-x}} \int_0^{2-y} dzdydx$$

$$= \int_0^4 \int_0^{\sqrt{4-x}} (2-y)\,dydx$$

$$= \int_0^4 \left[2y - \frac{1}{2}y^2\right]_0^{\sqrt{4-x}} dydx$$

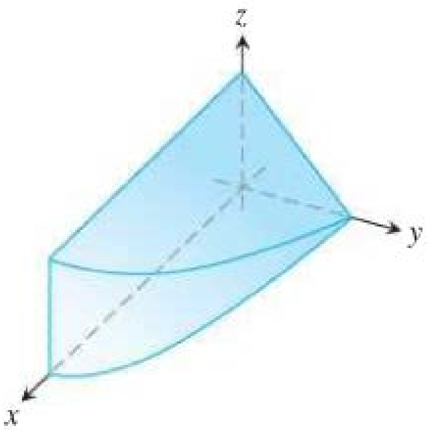
$$= \int_0^4 \left[2\sqrt{4-x} - \frac{1}{2}(4-x)\right]dx$$

$$= -\int_0^4 \left[2(4-x)^{1/2} - \frac{1}{2}(4-x)\right]d(4-x)$$

$$= -\left[\frac{4}{3}(4-x)^{3/2} - \frac{1}{4}(4-x)^2\right]_0^4$$

$$= -\left[0 - \left(\frac{4}{3}4^{3/2} - \frac{1}{4}4^2\right)\right]$$

$$= \frac{20}{3}$$



Exercise

Find the volumes of the wedge cut from the cylinder $x^2+y^2=1$ by the planes $z=-y, \quad z=0$

Solution

$$V=2\int_0^1\int_{-\sqrt{1-x^2}}^{0}\int_0^{-y} dzdydx$$

$$=-2\int_0^1\int_{-\sqrt{1-x^2}}^{0} ydydx$$

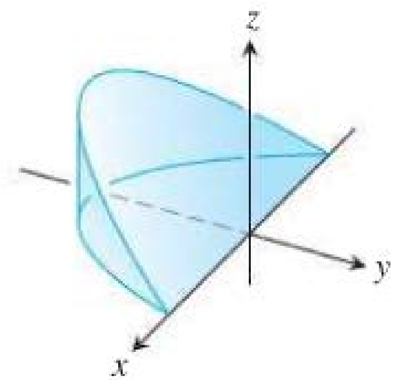
$$=-2\int_0^1\left[\frac{1}{2}y^2\right]_{-\sqrt{1-x^2}}^{0} dx$$

$$=\int_0^1\left(1-x^2\right)dx$$

$$=\left[x-\frac{1}{3}x^3\right]_0^1$$

$$=1-\frac{1}{3}$$

$$=\frac{2}{3}$$



Exercise

Find the volumes of the region cut from the cylinder $x^2+y^2=4$ by the plane $z=0$ and the plane $x+z=3$

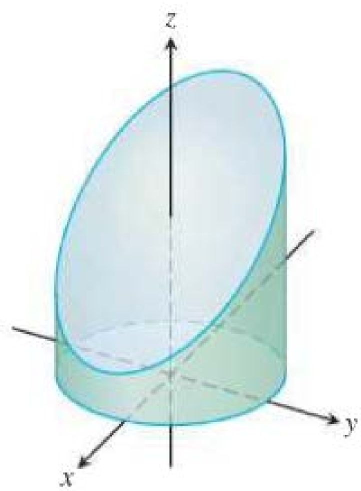
Solution

$$V=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{3-x} dzdydx$$

$$=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(3-x)dydx$$

$$=2\int_{-2}^{2}(3-x)\sqrt{4-x^2}dx$$

$$=6\int_{-2}^{2}\sqrt{4-x^2}dx-2\int_{-2}^{2}x\sqrt{4-x^2}dx \qquad d\left(4-x^2\right)=-2xdx$$



$$= 6\int_{-2}^{2}\sqrt{4-x^2}\,dx + \int_{-2}^{2}\left(4-x^2\right)^{1/2} d\left(4-x^2\right)$$

$$= 3\left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_{-2}^{2} + \frac{2}{3}\left[\left(4-x^2\right)^{3/2}\right]_{-2}^{2}$$

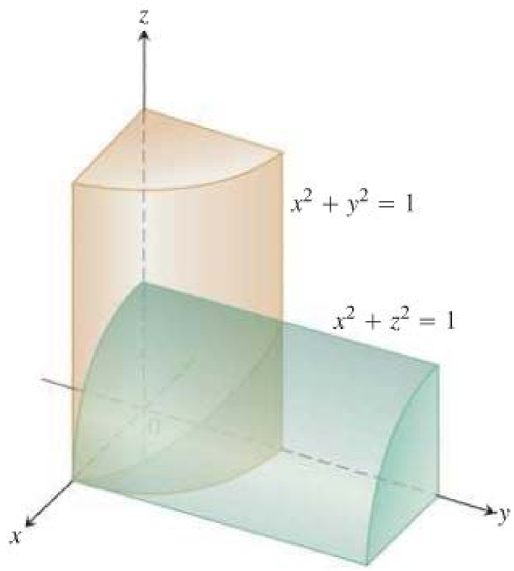
$$= 3\left[4\sin^{-1}1 - 4\sin^{-1}(-1)\right] + \frac{2}{3}(0)$$

$$= 12\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$\underline{= 12\pi}$$

Exercise

Find the volumes of the region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown below



Solution

$$V = 8\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dzdydx$$

$$= 8\int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dydx$$

$$= 8\int_0^1 \sqrt{1-x^2}\,[y]_0^{\sqrt{1-x^2}}\,dx$$

$$= 8\int_0^1 \left(1-x^2\right)dx$$

$$= 8\left[x - \frac{1}{3}x^3\right]_0^1$$

$$\underline{= \frac{16}{3}}$$

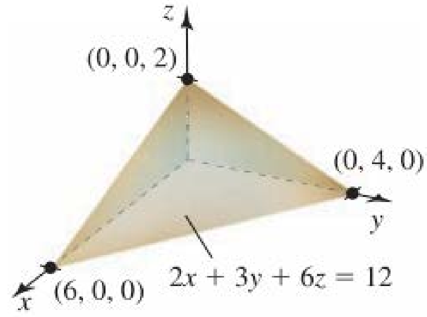
Exercise

Find the volume of the solid in the first octant bounded by the plane $2x + 3y + 6z = 12$ and the coordinate planes

Solution

$$z = \frac{12-2x-3y}{6} = 2 - \frac{x}{3} - \frac{y}{2} \qquad z = 0 \rightarrow \quad 2x + 3y = 12 \rightarrow y = 4 - \frac{2x}{3}$$

$$0 \le z \le 2 - \frac{x}{3} - \frac{y}{2}; \quad 0 \le y \le 4 - \frac{2x}{3}; \quad 0 \le x \le 6$$



$$V = \int_0^6 \int_0^{4-\frac{2x}{3}} \int_0^{2-\frac{x}{3}-\frac{y}{2}} 1\, dzdydx$$

$$= \int_0^6 \int_0^{4-\frac{2x}{3}} z \Big|_0^{2-\frac{x}{3}-\frac{y}{2}} dydx$$

$$= \int_0^6 \int_0^{4-\frac{2x}{3}} \left(2 - \frac{x}{3} - \frac{y}{2}\right) dydx$$

$$= \int_0^6 \left(2y - \frac{x}{3}y - \frac{1}{4}y^2\right) \Big|_0^{4-\frac{2x}{3}} dx$$

$$= \int_0^6 \left(8 - \frac{4}{3}x - \frac{4}{3}x + \frac{2}{9}x^2 - \frac{1}{4}\left(16 - \frac{16}{3}x + \frac{4}{9}x^2\right)\right) dx$$

$$= \int_0^6 \left(4 - \frac{4}{3}x + \frac{1}{9}x^2\right) dx$$

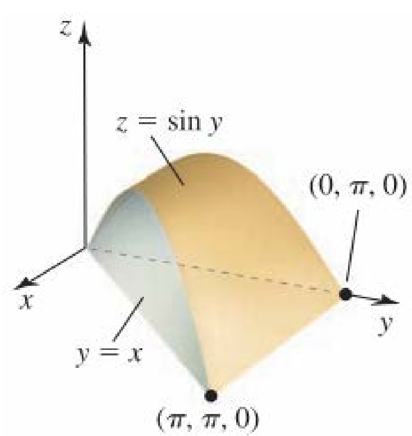
$$= 4x - \frac{2}{3}x^2 + \frac{1}{27}x^3 \Big|_0^6$$

$$= 8 \quad unit^3$$

Exercise

Find the volume of the solid in the first octant formed when the cylinder $z = \sin y$, for $0 \le y \le \pi$, is sliced by the planes $y = x$ and $x = 0$

Solution



$$V = \int_0^{\pi} \int_x^{\pi} \int_0^{\sin y} 1\, dzdydx$$

$$= \int_0^{\pi} \int_x^{\pi} z \Big|_0^{\sin y} dydx$$

$$= \int_0^{\pi} \int_x^{\pi} \sin y\, dydx$$

$$= -\int_0^{\pi} \cos y \Big|_x^{\pi} dx$$

$$= -\int_0^{\pi} (-1 - \cos x)\, dx$$

$$= (x + \sin x)\Big|_0^{\pi}$$

$$\underline{= \pi|} \quad unit^3$$

Exercise

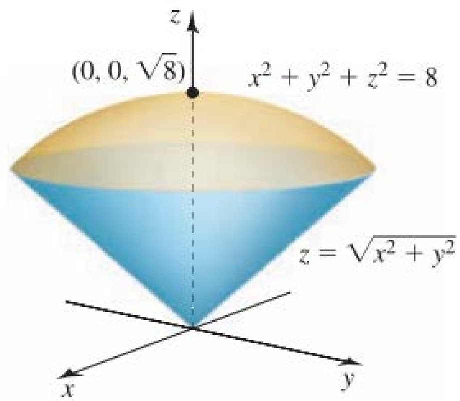
Find the volume of the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and bounded above the sphere $x^2 + y^2 + z^2 = 8$

Solution

$$z = \sqrt{x^2 + y^2} \quad z = \sqrt{8 - x^2 - y^2}$$

$$x^2 + y^2 + \left(\sqrt{x^2 + y^2}\right)^2 = 8 \Rightarrow x^2 + y^2 = 4 \rightarrow \underline{y = \pm\sqrt{4 - x^2}|}$$

$$(y = 0) \rightarrow x^2 = 4 \quad \underline{x = \pm 2|}$$



$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} 1\, dzdydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} z \Big|_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} dydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(\sqrt{8 - x^2 - y^2} - \sqrt{x^2 + y^2}\right) dydx$$

Convert to ***Polar coordinates***

$$= \int_0^{2\pi} \int_0^{2} \left(\sqrt{8 - r^2} - r\right) r\, drd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{2} \left(r\sqrt{8 - r^2} - r^2\right) dr$$

$$= (2\pi)\left(\int_0^{2} \frac{-1}{2}\left(8 - r^2\right)^{1/2} d\left(8 - r^2\right) - \left(\frac{1}{3}r^3\right)\Big|_0^2\right)$$

$$= (\pi)\left(-\frac{2}{3}\left(8 - r^2\right)^{3/2}\Big|_0^2 - \frac{8}{3}\right)$$

$$= \pi\left(-\frac{2}{3}\left(8-16\sqrt{2}\right)-\frac{8}{3}\right)$$

$$= \frac{32\pi}{3}\left(\sqrt{2}-1\right) \quad unit^3$$

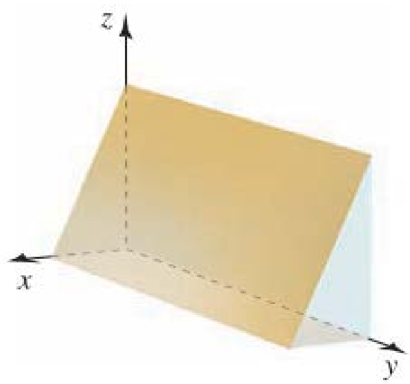
Exercise

Find the volume of the prism in the first octant bounded below by $z = 2-4x$ and $y = 8$

Solution

$$z = 2-4x = 0 \quad \Rightarrow \quad x = \frac{1}{2}$$

$$V = \int_0^{1/2}\int_0^{8}\int_0^{2-4x} 1\ dzdydx$$



$$= \int_0^{1/2}\int_0^{8} (2-4x)\,dydx$$

$$= \int_0^{1/2} (2-4x)\,y\Big|_0^8\ dx$$

$$= 16\int_0^{1/2} (1-2x)\,dx$$

$$= 16\left(x-x^2\right)\Big|_0^{1/2}$$

$$= 16\left(\frac{1}{2}-\frac{1}{4}\right)$$

$$= 4 \quad unit^3$$

Exercise

Find the volume of the wedge above the *xy*-plane formed when the cylinder $x^2+y^2=4$ is cut by the planes $z=0$ and $y=-z$

Solution

$$0 \le z \le -y \ (y<0); \quad -\sqrt{4-x^2} \le y \le 0; \quad y=0 \to x^2 = 4 \ \Rightarrow \ -2 \le x \le 2$$

$$V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{0}\int_0^{-y} 1\ dzdydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{0} (-y)\ dydx$$

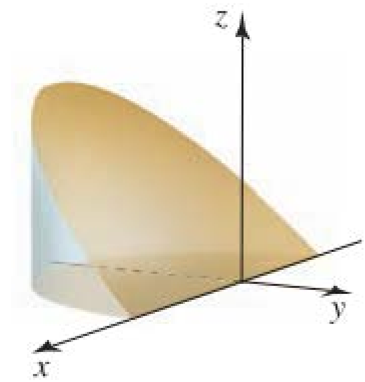
$$= -\frac{1}{2} \int_{-2}^{2} \left(y^2\right) \Big|_{-\sqrt{4-x^2}}^{0}\ dx$$

$$= \frac{1}{2} \int_{-2}^{2} \left(4 - x^2\right)\ dx$$

$$= \frac{1}{2}\left(4x - \frac{1}{3}x^3\right) \Big|_{-2}^{2}$$

$$= 8 - \frac{8}{3}$$

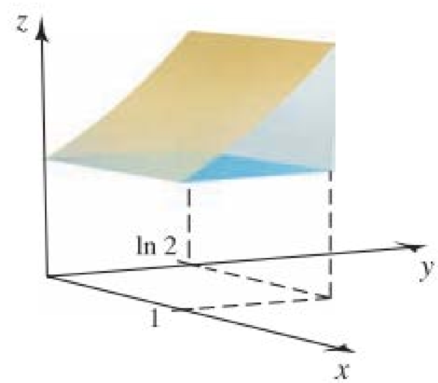
$$= \frac{16}{3}\ \ unit^3$$



Exercise

Find the volume of the solid bounded by the surfaces $z = e^y$ and $z = 1$ over the rectangle $\{(x, y):\ 0 \le x \le 1,\ 0 \le y \le \ln 2\}$

Solution



$$V = \int_0^1 \int_0^{\ln 2} \int_1^{e^y} 1\ dzdydx$$

$$= \int_0^1 dx \int_0^{\ln 2} \left(e^y - 1\right) dy$$

$$= x \Big|_0^1 \left(e^y - y\right) \Big|_0^{\ln 2}$$

$$= 1 - \ln 2\ \ unit^3$$

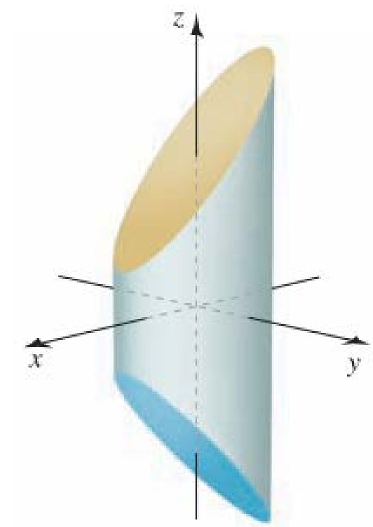
Exercise

Find the volume of the wedge of the cylinder $x^2+4y^2=4$ created by the planes $z=3-x$ and $z=x-3$

Solution

$$y^2=\frac{1}{4}\left(4-x^2\right) \quad \to \ y=\pm\frac{1}{2}\sqrt{4-x^2} \text{ v}$$

$$x^2=4 \quad \to \quad -2\le x\le 2$$



$$V=\int_{-2}^{2}\int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}}\int_{x-3}^{3-x} 1\, dzdydx$$

$$=\int_{-2}^{2}\int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}}(6-2x)\,dydx$$

$$=\int_{-2}^{2}(6-2x)\,y\Big|_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}}\,dx$$

$$=\int_{-2}^{2}(6-2x)\sqrt{4-x^2}\,dx$$

$$=\int_{-2}^{2}6\sqrt{4-x^2}\,dx+\int_{-2}^{2}\sqrt{4-x^2}\,d\left(4-x^2\right) \qquad \int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$=3x\sqrt{4-x^2}+12\sin^{-1}\frac{x}{2}+\frac{2}{3}\sqrt{4-x^2}\Big|_{-2}^{2}$$

$$=12\frac{\pi}{2}+12\frac{\pi}{2}$$

$$\underline{=12\pi} \quad unit^3$$

Exercise

Find the volume of the solid in the first octant bounded by the cone $z=1-\sqrt{x^2+y^2}$ and the plane $x+y+z=1$

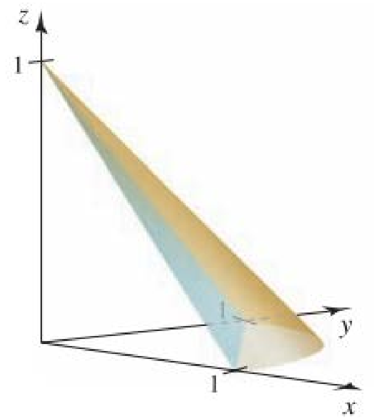
Solution

$$0\le z\le 1$$

$$z=1-\sqrt{x^2+y^2} \quad \to \ x^2+y^2=(1-z)^2 \Rightarrow \ x=\sqrt{(1-z)^2-y^2}$$

$$1-y-z \le x \le \sqrt{(1-z)^2 - y^2}$$

$$0 \le y \le 1-z$$



$$V = \int_0^1 \int_1^{1-z} \int_{1-y-z}^{\sqrt{(1-z)^2-y^2}} 1\, dxdydz$$

$$= \int_0^1 \int_1^{1-z} x \Big|_{1-y-z}^{\sqrt{(1-z)^2-y^2}} dydz$$

$$= \int_0^1 \int_1^{1-z} \left(\sqrt{(1-z)^2 - y^2} - 1 + y + z \right) dydz \qquad \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$= \int_0^1 \frac{y}{2}\sqrt{(1-z)^2 - y^2} + \frac{1}{2}(1-z)^2 \sin^{-1}\left(\frac{y}{1-z}\right) - y + \frac{1}{2}y^2 + zy \Big|_0^{1-z} dz$$

$$= \int_0^1 \left(\frac{1}{2}(1-z)^2 \sin^{-1}(1) + \frac{1}{2}(1-z)^2 - (1-z)^2 \right) dz$$

$$= \int_0^1 \left(\frac{\pi}{4}(1-z)^2 - \frac{1}{2}(1-z)^2 \right) dz$$

$$= \frac{\pi - 2}{4} \int_0^1 (1-z)^2 d(1-z)$$

$$= \frac{\pi - 2}{12}(1-z)^3 \Big|_0^1$$

$$= \frac{\pi - 2}{12} \quad unit^3$$

Exercise

Find the volume of the solid bounded by $x = 0$, $x = 1 - z^2$, $y = 0$, $z = 0$, and $z = 1 - y$

Solution

$$V = \int_0^1 \int_0^{1-z^2} \int_0^{1-z} 1\, dydxdz$$

$$= \int_0^1 \int_0^{1-z^2} (1-z)\, dxdz$$

$$= \int_0^1 (1-z)x \Big|_0^{1-z^2} dz$$

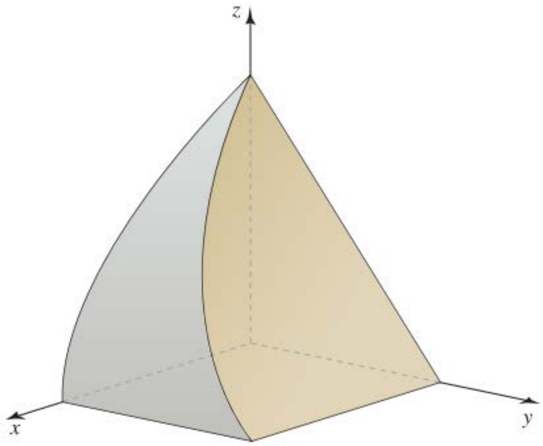
$$= \int_0^1 (1-z)\left(1-z^2\right) dz$$

$$= \int_0^1 \left(1-z^2-z+z^3\right) dz$$

$$= z - \frac{1}{3}z^3 - \frac{1}{2}z^2 + \frac{1}{4}z^4 \Big|_0^1$$

$$= 1 - \frac{1}{3} - \frac{1}{2} + \frac{1}{4}$$

$$= \frac{5}{12} \quad unit^3$$



Exercise

Find the volume of the solid bounded by $x=0$, $x=2$, $y=0$, $y=e^{-z}$, $z=0$, and $z=1$

Solution

$$V = \int_0^2 \int_0^1 \int_0^{e^{-z}} 1\, dydzdx$$

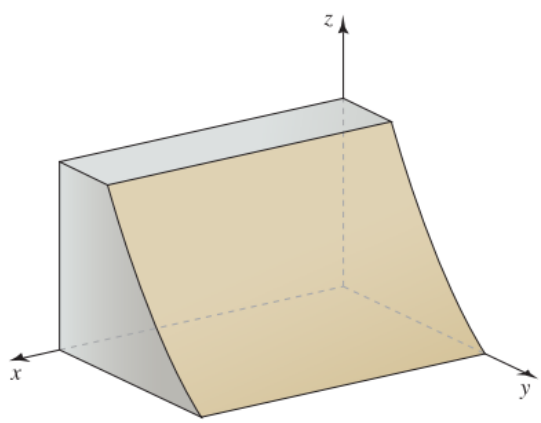
$$= \int_0^2 dx \int_0^1 y \Big|_0^{e^{-z}} dz$$

$$= 2\int_0^1 e^{-z}\, dz$$

$$= -2e^{-z} \Big|_0^1$$

$$= -2\left(e^{-1}-1\right)$$

$$= 2 - \frac{2}{e} \quad unit^3$$



Exercise

Find the volume of the solid bounded by $x=0$, $x=2$, $y=z$, $y=z+1$, $z=0$, and $z=4$

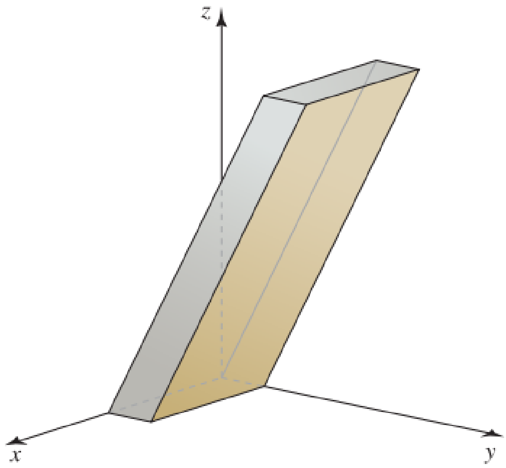
Solution

$$V=\int_0^2\int_0^4\int_z^{z+1} 1\ dydzdx$$

$$=\int_0^2\int_0^4 y\Big|_z^{z+1} dzdx$$

$$=\int_0^2 dx\int_0^4 dz \quad =(2)(4)$$

$$\underline{=8|} \quad unit^3$$



Exercise

Find the volume of the solid bounded by $x=0$, $y=z^2$, $z=0$, and $z=2-x-y$

Solution

$$y=2-x-z; \quad \underline{|x}=2-z-y\ \underline{=2-z-z^2|}$$

$$V=\int_0^1\int_0^{2-z-z^2}\int_{z^2}^{2-x-z} 1\ dydxdz$$

$$=\int_0^1\int_0^{2-z-z^2}\left(2-x-z-z^2\right)dxdz$$

$$=\int_0^1\left(\left(2-z-z^2\right)x-\frac{1}{2}x^2\right)\Big|_0^{2-z-z^2} dz$$

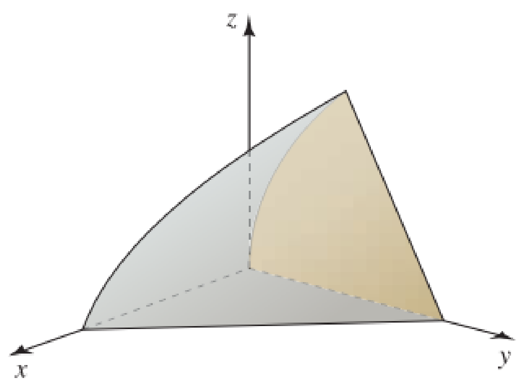
$$=\frac{1}{2}\int_0^1\left(2-z-z^2\right)^2 dz$$

$$=\frac{1}{2}\int_0^1\left(4-4z-3z^2+2z^3+z^4\right)dz$$

$$=\frac{1}{2}\left(4z-2z^2-z^3+\frac{1}{2}z^4+\frac{1}{5}z^5\right)\Big|_0^1$$

$$=\frac{1}{2}\left(4-2-1+\frac{1}{2}+\frac{1}{5}\right)$$

$$\underline{=\frac{17}{20}|} \quad unit^3$$



Exercise

Find the volume of the solid common to the cylinders $z = \sin x$ and $z = \sin y$ over the square

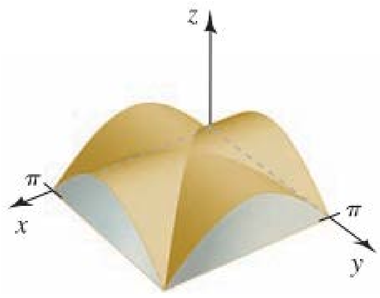
$$R = \{(x, y):\ 0 \le x \le \pi,\ 0 \le y \le \pi\}$$

Solution

$$z = \sin x = \sin y \quad \rightarrow \quad x = y \ \text{ or } \ y = \pi - x$$

$$V = 4\int_0^{\pi/2}\int_x^{\pi-x}\int_0^{\sin y} 1\,dzdydx$$

4: *by symmetry, volume – **4** times*



$$= 4\int_0^{\pi/2}\int_x^{\pi-x} \sin y\ dydx$$

$$= -4\int_0^{\pi/2} \cos y\Big|_x^{\pi-x}\ dx$$

$$= -4\int_0^{\pi/2} \big(\cos(\pi - x) - \cos x\big)\ dx$$

$$= -4\int_0^{\pi/2} (-2\cos x)\ dx$$

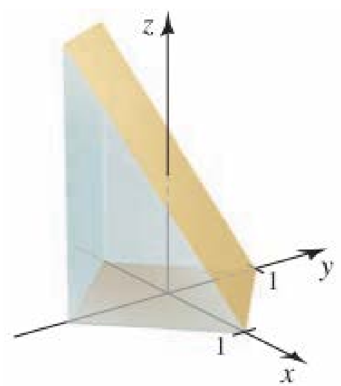
$$= 8\sin x\Big|_0^{\pi/2}$$

$$\underline{= 8|} \quad unit^3$$

Exercise

Find the volume of the wedge of the square column $|x| + |y| = 1$ created by the planes $z = 0$ and $x + y + z = 1$

Solution



$$0 \le z \le 1 - x - y$$

$$|x| + |y| = 1 \quad \rightarrow \quad \begin{cases} x + y = 1 \Rightarrow y = 1 - x \\ -x + y = 1 \Rightarrow y = 1 + x \\ x - y = 1 \Rightarrow y = x - 1 \\ -x - y = 1 \Rightarrow y = -x - 1 \end{cases}$$

$$\begin{cases} y = -x - 1 \\ y = x + 1 \end{cases} \Rightarrow -1 \le x \le 0 \qquad \begin{cases} y = x - 1 \\ y = -x + 1 \end{cases} \Rightarrow 0 \le x \le 1$$

$$V = \int_{-1}^{0}\int_{-x-1}^{x+1}\int_0^{1-x-y} 1dzdydx + \int_0^1\int_{x-1}^{-x+1}\int_0^{1-x-y} 1dzdydx$$

$$= \int_{-1}^{0}\int_{-x-1}^{x+1} (1-x-y)\ dydx + \int_{0}^{1}\int_{x-1}^{-x+1} (1-x-y)\ dydx$$

$$= \int_{-1}^{0} \left((1-x)y - \frac{1}{2}y^2\right)\Big|_{-x-1}^{x+1}\ dx + \int_{0}^{1} \left((1-x)y - \frac{1}{2}y^2\right)\Big|_{x-1}^{-x+1}\ dx$$

$$= \int_{-1}^{0} 2(1-x)(x+1)\ dx + \int_{0}^{1} 2(1-x)^2\ dx$$

$$= \int_{-1}^{0} 2\left(1-x^2\right)\ dx + 2\int_{0}^{1} \left(1-2x+x^2\right)\ dx$$

$$= 2\left(x - \frac{1}{3}x^3\right)\Big|_{-1}^{0} + 2\left(x - x^2 + \frac{1}{3}x^3\right)\Big|_{0}^{1}$$

$$= 2\left(1-\frac{1}{3}\right) + \frac{2}{3}$$

$$\underline{= 2}\quad unit^3$$

Exercise

Find the volume of a right circular cone with height h and base radius r.

Solution

The equation of a circle is centered at the origin with radius r: $x^2 + y^2 = r^2$

$$-\sqrt{r^2 - x^2} \le y \le \sqrt{r^2 - x^2} \quad \& \quad -r \le x \le r$$

$$z = a - b\sqrt{x^2 + y^2}\ \begin{cases} z = h & \underline{h = a} \\ z = 0 & 0 = a - br = h - br \ \Rightarrow b = \frac{h}{r} \end{cases}$$

The equation of a cone with height h: $z = h - \frac{h}{r}\sqrt{x^2 + y^2}$

$$V = \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}}\int_{0}^{h-\frac{h}{r}\sqrt{x^2+y^2}} 1dzdydx$$

$$= \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \left(h - \frac{h}{r}\sqrt{x^2+y^2}\right)dydx$$

Let $x^2 + y^2 = R^2$ (*Polar Coordinates*)

$$= \int_{0}^{2\pi}\int_{0}^{r} \left(h - \frac{h}{r}R\right) R\ dRd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^r \left(hR - \frac{h}{r}R^2\right) dR$$

$$= 2\pi \left(\frac{1}{2}hR^2 - \frac{h}{3r}R^3\right)\Big|_0^r$$

$$= 2\pi \left(\frac{1}{2}hr^2 - \frac{1}{3}hr^2\right)$$

$$= \frac{1}{3}\pi r^2 h \quad unit^3$$

Exercise

Find the volume of a tetrahedron whose vertices are located at $(0,\ 0,\ 0)$, $(a,\ 0,\ 0)$, $(0,\ b,\ 0)$, and $(0,\ 0,\ c)$

Solution

The equation of the plane through the vertices: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$0 \le z \le c\left(1 - \frac{x}{a} - \frac{y}{b}\right) \quad 0 \le y \le b\left(1 - \frac{x}{a}\right) \quad 0 \le x \le a$$

$$V = \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} 1 dzdydx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dydx$$

$$= c\int_0^a \left(\left(1 - \frac{x}{a}\right)y - \frac{1}{2b}y^2\right)\Big|_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= c\int_0^a \left(b\left(1 - \frac{x}{a}\right)^2 - \frac{1}{2}b\left(1 - \frac{x}{a}\right)^2\right) dx$$

$$= \frac{1}{2}bc\int_0^a \left(1 - \frac{2}{a}x + \frac{1}{a^2}x^2\right) dx$$

$$= \frac{1}{2}bc\left(x - \frac{1}{a}x^2 + \frac{1}{3a^2}x^3\right)\Big|_0^a$$

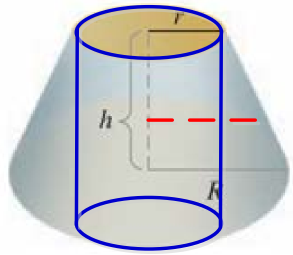
$$= \frac{1}{2}bc\left(a - a + \frac{1}{3}a\right)$$

$$= \frac{abc}{6}$$

Exercise

Find the volume of a truncated cone of height *h* whose ends have radii *r* and *R*.

Solution



There are 2 volumes to consider:

1. Volume of the cylinder: $V_1 = \pi r^2 h$
2. Volume V_2 that remains when cylinder is removed.

   V_2 is the annulus centered at the origin with inner radius r and outer radius R.

   Using Polar Coordinates: the equation of the frustum is: $z = \frac{h}{R-r}(R-a)$

$$V_2 = \int_0^{2\pi}\int_r^R\int_0^{\frac{h}{R-r}(R-a)} adzdad\theta$$

$$= \int_0^{2\pi}\int_r^R \frac{h}{R-r}(R-a)a\ dad\theta$$

$$= \frac{h}{R-r}\int_0^{2\pi} d\theta \int_r^R \left(Ra - a^2\right) da$$

$$= \frac{2\pi h}{R-r}\left(\frac{1}{2}Ra^2 - \frac{1}{3}a^3\right)\Big|_r^R$$

$$= \frac{2\pi h}{R-r}\left(\frac{1}{2}R^3 - \frac{1}{3}R^3 - \frac{1}{2}Rr^2 + \frac{1}{3}r^3\right)$$

$$= \frac{2\pi h}{R-r}\left(\frac{1}{6}R^3 - \frac{1}{2}Rr^2 + \frac{1}{3}r^3\right)$$

$$= \frac{1}{3}\frac{\pi h}{R-r}\left(R^3 - 3Rr^2 + 2r^3\right)$$

$$V_1 + V_1 = \pi r^2 h + \frac{1}{3}\frac{\pi h}{R-r}\left(R^3 - 3Rr^2 + 2r^3\right)$$

$$= \frac{1}{3}\frac{\pi h}{R-r}\left(3r^2(R-r) + R^3 - 3Rr^2 + 2r^3\right)$$

$$= \frac{1}{3}\frac{\pi h}{R-r}\left(R^3 - r^3\right)$$

$$= \frac{1}{3}\frac{\pi h}{R-r}(R-r)\left(R^3 + rR + r^2\right)$$

$$= \frac{1}{3}\pi h\left(R^3 + rR + r^2\right)$$

Solution   *Section* 3.5 – Triple Integrals in Cylindrical and Spherical Coordinates

Exercise

Evaluate the cylindrical coordinate integral $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\ rdr\ d\theta$

Solution

$$\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\ rdr\ d\theta = \int_0^{2\pi}\int_0^1\left(\sqrt{2-r^2}-r\right)rdr\ d\theta$$

$$= \int_0^{2\pi}\int_0^1\left(r\left(2-r^2\right)^{1/2}-r^2\right)drd\theta \qquad d\left(2-r^2\right)=-2rdr$$

$$= \int_0^{2\pi}\left(\int_0^1\left(-\tfrac{1}{2}\left(2-r^2\right)^{1/2}d\left(2-r^2\right)-r^2dr\right)\right)d\theta$$

$$= \int_0^{2\pi}\left[-\tfrac{1}{3}\left(2-r^2\right)^{3/2}-\tfrac{1}{3}r^3\right]_0^1 d\theta$$

$$= \int_0^{2\pi}\left[\left(-\tfrac{1}{3}-\tfrac{1}{3}\right)-\left(-\tfrac{1}{3}2^{3/2}\right)\right]d\theta$$

$$= \int_0^{2\pi}\left(-\frac{2}{3}+\frac{2^{3/2}}{3}\right)d\theta$$

$$= \frac{2\sqrt{2}-2}{3}\left[\theta\right]_0^{2\pi}$$

$$= 4\pi\frac{\sqrt{2}-1}{3}$$

Exercise

Evaluate the cylindrical coordinate integral $\int_0^{2\pi}\int_0^{\theta/(2\pi)}\int_0^{3+24r^2} dz\ rdr\ d\theta$

Solution

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\theta/(2\pi)}\int_0^{3+24r^2} dz\ rdr\ d\theta &= \int_0^{2\pi}\int_0^{\theta/(2\pi)}\left(3+24r^2\right)rdr\ d\theta \\
&= \int_0^{2\pi}\int_0^{\theta/(2\pi)}\left(3r+24r^3\right)dr\ d\theta \\
&= \int_0^{2\pi}\left[\frac{3}{2}r^2+6r^4\right]_0^{\theta/(2\pi)} d\theta \\
&= \int_0^{2\pi}\left[\frac{3}{8\pi^2}\theta^2+\frac{6}{16r^4}\theta^4\right]d\theta \\
&= \left[\frac{1}{8\pi^2}\theta^3+\frac{3}{8r^4}\frac{1}{5}\theta^5\right]_0^{2\pi} \\
&= \frac{1}{8\pi^2}8\pi^3+\frac{3}{8r^4}\frac{1}{5}32\pi^5 \\
&= \pi+\frac{12}{5}\pi \\
&= \underline{\frac{17}{5}\pi}
\end{aligned}$$

Exercise

Evaluate the cylindrical coordinate integral $\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} zdz\ rdr\ d\theta$

Solution

$$\begin{aligned}
\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} zdz\ rdr\ d\theta &= \int_0^{\pi}\int_0^{\theta/\pi}\left[\frac{1}{2}z^2\right]_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} rdr\ d\theta \\
&= \frac{1}{2}\int_0^{\pi}\int_0^{\theta/\pi}\left[9\left(4-r^2\right)-\left(4-r^2\right)\right]rdr\ d\theta \\
&= \frac{1}{2}\int_0^{\pi}\int_0^{\theta/\pi} 8\left(4-r^2\right)rdr\ d\theta
\end{aligned}$$

$$= 4\int_0^{\pi}\int_0^{\theta/\pi}\left(4r - r^3\right)dr\ d\theta$$

$$= 4\int_0^{\pi}\left[2r^2 - \frac{1}{4}r^4\right]_0^{\theta/\pi} d\theta$$

$$= 4\int_0^{\pi}\left(2\frac{\theta^2}{\pi^2} - \frac{1}{4}\frac{\theta^4}{\pi^4}\right)d\theta$$

$$= 4\left[\frac{2}{3}\frac{\theta^3}{\pi^2} - \frac{1}{20}\frac{\theta^5}{\pi^4}\right]_0^{\pi}$$

$$= 4\left[\frac{2}{3}\frac{\pi^3}{\pi^2} - \frac{1}{20}\frac{\pi^5}{\pi^4}\right]$$

$$= 4\left(\frac{2}{3}\pi - \frac{1}{20}\pi\right)$$

$$= 4\left(\frac{37}{60}\pi\right)$$

$$= \frac{37}{15}\pi$$

Exercise

Evaluate the cylindrical coordinate integral $\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}\left(r^2\sin^2\theta + z^2\right)dz\ rdr\ d\theta$

Solution

$$\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}\left(r^2\sin^2\theta + z^2\right)dz\ rdr\ d\theta = \int_0^{2\pi}\int_0^1\left[zr^2\sin^2\theta + \frac{1}{3}z^3\right]_{-1/2}^{1/2} rdr\ d\theta$$

$$= \int_0^{2\pi}\int_0^1\left[\frac{1}{2}r^2\sin^2\theta + \frac{1}{24} - \left(-\frac{1}{2}r^2\sin^2\theta - \frac{1}{24}\right)\right]rdrd\theta$$

$$= \int_0^{2\pi}\int_0^1\left(r^2\sin^2\theta + \frac{1}{12}\right)rdrd\theta$$

$$= \int_0^{2\pi}\int_0^1\left(r^3\sin^2\theta + \frac{1}{12}r\right)drd\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{4}r^4\sin^2\theta + \frac{1}{24}r^2\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{4}\sin^2\theta + \frac{1}{24}\right) d\theta \qquad \int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$= \left[\frac{1}{4}\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right) + \frac{1}{24}\theta\right]_0^{2\pi}$$

$$= \left[\frac{\theta}{8} - \frac{1}{16}\sin 2\theta + \frac{1}{24}\theta\right]_0^{2\pi}$$

$$= \frac{2\pi}{8} + \frac{1}{24}2\pi$$

$$= \frac{\pi}{3}$$

Exercise

Evaluate the integral $\displaystyle\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3 dr\ dz\ d\theta$

Solution

$$\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3 dr\ dz\ d\theta = \int_0^{2\pi}\int_0^3 \left[\frac{1}{4}r^4\right]_0^{z/3} dz\ d\theta$$

$$= \frac{1}{324}\int_0^{2\pi}\int_0^3 z^4\ dz\ d\theta$$

$$= \frac{1}{324}\int_0^{2\pi}\left[\frac{1}{5}z^5\right]_0^3 d\theta$$

$$= \frac{243}{1620}\int_0^{2\pi} d\theta$$

$$= \frac{3}{20}\left[\theta\right]_0^{2\pi}$$

$$= \frac{3\pi}{10}$$

Exercise

Evaluate the integral $\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} \left(r^2 \cos^2\theta + z^2\right) r \, d\theta \, drdz$

Solution

$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} \left(r^2 \cos^2\theta + z^2\right) r \, d\theta \, drdz &= \int_0^1 \int_0^{\sqrt{z}} \left[r^2 \left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta \right) + z^2\theta \right]_0^{2\pi} rdrdz \\
&= \int_0^1 \int_0^{\sqrt{z}} \left(\pi r^2 + 2\pi z^2\right) rdrdz \\
&= \int_0^1 \int_0^{\sqrt{z}} \left(\pi r^3 + 2\pi z^2 r\right) drdz \\
&= \int_0^1 \left[\frac{1}{4}\pi r^4 + \pi z^2 r^2 \right]_0^{\sqrt{z}} dz \\
&= \int_0^1 \left(\frac{1}{4}\pi z^2 + \pi z^3 \right) dz \\
&= \left[\frac{1}{12}\pi z^3 + \frac{1}{4}\pi z^4 \right]_0^1 \\
&= \frac{1}{12}\pi + \frac{1}{4}\pi \\
&= \frac{\pi}{3}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \int_0^{2\pi} (r\sin\theta + 1) r \, d\theta \, dz \, dr$

Solution

$$\begin{aligned}
\int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \int_0^{2\pi} (r\sin\theta + 1) r \, d\theta dz dr &= \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \left[-r\cos\theta + \theta \right]_0^{2\pi} r \, dzdr \\
&= \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \left(-r + 2\pi - (-r) \right) r \, dzdr \\
&= \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} 2\pi r \, dzdr
\end{aligned}$$

$$= 2\pi \int_0^2 r\left[z\right]_{r-2}^{\sqrt{4-r^2}} dr$$

$$= 2\pi \int_0^2 r\left[\left(4-r^2\right)^{1/2} - (r-2)\right]dr$$

$$= 2\pi \int_0^2 \left[r\left(4-r^2\right)^{1/2} - r^2 + 2r\right]dr \qquad d\left(4-r^2\right) = -2rdr$$

$$= 2\pi \left[-\frac{1}{3}\left(4-r^2\right)^{3/2} - \frac{1}{3}r^3 + r^2\right]_0^2$$

$$= 2\pi \left[-\frac{8}{3} + 4 - \left(-\frac{1}{3}(4)^{3/2}\right)\right]$$

$$= 2\pi \left(\frac{4}{3} + \frac{8}{3}\right)$$

$$= 8\pi$$

Exercise

Convert the integral $\int_{-1}^{1} \int_0^{\sqrt{1-y^2}} \int_0^x \left(x^2 + y^2\right) dzdxdy$ to an equivalent integral in cylindrical coordinates and evaluate the result.

Solution

$$\int_{-\pi/2}^{\pi/2} \int_0^1 \int_0^{r\cos\theta} r^3 dzdrd\theta = \int_{-\pi/2}^{\pi/2} \int_0^1 r^3 \left[z\right]_0^{r\cos\theta} drd\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^1 r^3 r\cos\theta drd\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{1}{5}r^5 \cos\theta\right]_0^1 d\theta$$

$$= \frac{1}{5}\int_{-\pi/2}^{\pi/2} \cos\theta d\theta$$

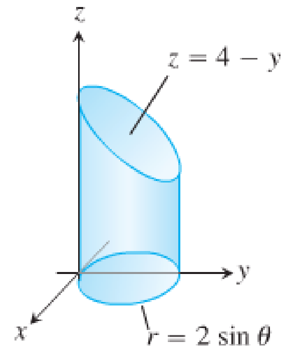
$$= \frac{1}{5}\left[\sin\theta\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{5}(1+1)$$

$$= \frac{2}{5}$$

Exercise

Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,dr\,d\theta$ over the region D that is the right circular cylinder whose base is the circle $r = 2\sin\theta$ in the *xy*-plane and whose top lies in the plane $z = 4 - y$



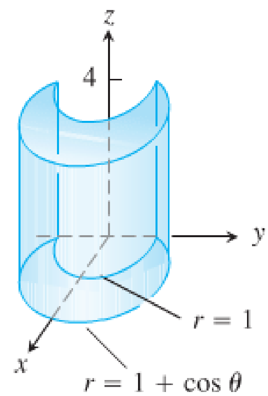
Solution

$$0 \le z \le 4 - y \Rightarrow 0 \le z \le 4 - r\sin\theta$$

$$\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{4-r\sin\theta} f(r,\theta,z)\,dz\;r\,dr\;d\theta$$

Exercise

Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,dr\,d\theta$ over the region D which is the solid right cylinder whose base is the region in the *xy*-plane that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$



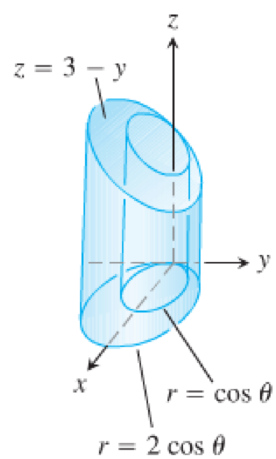
Solution

$$0 \le z \le 4 \quad 1 \le r \le 1 + \cos\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta}\int_0^{4} f(r,\theta,z)\,dz\;r\,dr\;d\theta$$

Exercise

Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,dr\,d\theta$ over the region D which is the solid right cylinder whose base is the region between the circles $r = \cos\theta$ and $r = 2\cos\theta$ and whose top lies in the plane $z = 3 - y$

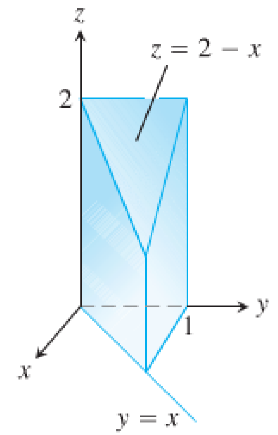


Solution

$$\int_{-\pi/2}^{\pi/2}\int_{\cos\theta}^{2\cos\theta}\int_0^{3-r\sin\theta} f(r,\theta,z)\,dz\;r\,dr\;d\theta$$

Exercise

Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,dr\,d\theta$ over the region D which is the prism whose base is the triangle in the xy-plane bounded by the y-axis and the lines $y = x$ and $y = 1$ and whose top lies in the plane $z = 2 - x$



Solution

$$0 \le z \le 2 - x \quad \to 0 \le z \le 2 - r\cos\theta$$

$$y = 1 \quad \to \quad r\sin\theta = 1 \to r = \frac{1}{\sin\theta} = \csc\theta$$

$$\int_{\pi/4}^{\pi/2} \int_0^{\csc\theta} \int_0^{2 - r\sin\theta} f(r,\theta,z)\,dz\ rdr\ d\theta$$

Exercise

Evaluate the spherical coordinate integral $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$

Solution

$$\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta = \frac{1}{3}\int_0^{\pi}\int_0^{\pi} \sin\phi \left[\rho^3\right]_0^{2\sin\phi} d\phi\ d\theta$$

$$= \frac{8}{3}\int_0^{\pi}\int_0^{\pi} \sin^4\phi\ d\phi\ d\theta$$

$$\int \sin^4 x\ dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right) dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right)$$

$$= \frac{8}{3}\int_0^{\pi} \left[\frac{3}{8}\phi - \frac{1}{4}\sin 2\phi + \frac{1}{32}\sin 4\phi\right]_0^{\pi} d\theta$$

$$=\frac{8}{3}\int_0^{\pi}\left[\frac{3\pi}{8}\right]d\theta$$

$$=\pi\left[\theta\right]_0^{\pi}$$

$$\underline{=\pi^2}$$

Exercise

Evaluate the spherical coordinate integral $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2}(\rho\cos\phi)\rho^2\sin\phi\ d\rho\ d\phi\ d\theta$

Solution

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2}(\rho\cos\phi)\rho^2\sin\phi\ d\rho d\phi d\theta &= \int_0^{2\pi}\int_0^{\pi/4}(\cos\phi\sin\phi)\left[\frac{1}{4}\rho^4\right]_0^2\ d\phi d\theta \\
&= 4\int_0^{2\pi}\int_0^{\pi/4}(\cos\phi\sin\phi)\ d\phi d\theta \\
&= 4\int_0^{2\pi}\int_0^{\pi/4}\sin\phi\ d(\sin\phi) d\theta \\
&= 2\int_0^{2\pi}\left[\sin^2\phi\right]_0^{\pi/4}\ d\theta \\
&= \int_0^{2\pi} d\theta \\
&\underline{= 2\pi}
\end{aligned}$$

Exercise

Evaluate the spherical coordinate integral $\int_0^{3\pi/2}\int_0^{\pi}\int_0^{1}5\rho^3\sin^3\phi\ d\rho\ d\phi\ d\theta$

Solution

$$\begin{aligned}
\int_0^{3\pi/2}\int_0^{\pi}\int_0^{1}5\rho^3\sin^3\phi\ d\rho d\phi d\theta &= \frac{5}{4}\int_0^{3\pi/2}\int_0^{\pi}\sin^3\phi\left[\rho^4\right]_0^1\ d\phi\ d\theta \\
&= \frac{5}{4}\int_0^{3\pi/2}\int_0^{\pi}\sin^2\phi\sin\phi\ d\phi d\theta \qquad d(\cos\phi) = -\sin\phi
\end{aligned}$$

$$= -\frac{5}{4}\int_0^{3\pi/2}\int_0^{\pi}\left(1-\cos^2\phi\right)\,d\left(\cos\phi\right)d\theta$$

$$= -\frac{5}{4}\int_0^{3\pi/2}\left[\cos\phi-\frac{1}{3}\cos^3\phi\right]_0^{\pi}\,d\theta$$

$$= -\frac{5}{4}\int_0^{3\pi/2}\left(-1+\frac{1}{3}-\left(1-\frac{1}{3}\right)\right)d\theta$$

$$= \frac{5}{3}\int_0^{3\pi/2}\,d\theta$$

$$= \frac{5}{3}\left(\frac{3\pi}{2}\right)$$

$$\underline{= \frac{5\pi}{2}}$$

Exercise

Evaluate the integral $\displaystyle\int_0^2\int_{-\pi}^0\int_{\pi/4}^{\pi/2}\rho^3\sin 2\phi\;d\phi\;d\theta\;d\rho$

Solution

$$\int_0^2\int_{-\pi}^0\int_{\pi/4}^{\pi/2}\rho^3\sin 2\phi\;d\phi d\theta d\rho = -\frac{1}{2}\int_0^2\int_{-\pi}^0\rho^3\left[\cos 2\phi\right]_{\pi/4}^{\pi/2}\,d\theta d\rho$$

$$= -\frac{1}{2}\int_0^2\int_{-\pi}^0\rho^3\left(-1-0\right)d\theta d\rho$$

$$= \frac{1}{2}\int_0^2\int_{-\pi}^0\rho^3 d\theta d\rho$$

$$= \frac{1}{2}\int_0^2\rho^3\left(\pi\right)d\rho$$

$$= \frac{\pi}{8}\left[\rho^4\right]_0^2$$

$$\underline{= 2\pi}$$

Exercise

Evaluate the integral $\displaystyle\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2} 5\rho^4 \sin^3\phi \, d\rho \, d\theta \, d\phi$

Solution

$$\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2} 5\rho^4 \sin^3\phi \, d\rho d\theta d\phi = \int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2} \sin^3\phi\left[\rho^5\right]_{\csc\phi}^{2} \, d\theta d\phi$$

$$= \int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2} \sin^3\phi\left(32 - \csc^5\phi\right) d\theta d\phi$$

$$= \int_{\pi/6}^{\pi/2} \left(32\sin^3\phi - \csc^2\phi\right)\left[\theta\right]_{-\pi/2}^{\pi/2} d\phi$$

$$= \pi\left(\int_{\pi/6}^{\pi/2} 32\sin^3\phi d\phi - \int_{\pi/6}^{\pi/2} \csc^2\phi d\phi\right)$$

$$= 32\pi\int_{\pi/6}^{\pi/2} \sin^2\phi \sin\phi d\phi - \pi\int_{\pi/6}^{\pi/2} \csc^2\phi d\phi$$

$$= 32\pi\int_{\pi/6}^{\pi/2} \left(1 - \cos^2\phi\right) \, d\left(\cos\phi\right) + \pi\left[\cot\phi\right]_{\pi/6}^{\pi/2}$$

$$= 32\pi\left[\cos\phi - \tfrac{1}{3}\cos^3\phi\right]_{\pi/6}^{\pi/2} + \pi\left(-\sqrt{3}\right)$$

$$= 32\pi\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}\right) - \pi\sqrt{3}$$

$$= 12\pi\sqrt{3} - \pi\sqrt{3}$$

$$\boxed{= 11\pi\sqrt{3}}$$

Exercise

Find the volume of the solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2,\ \ z \ge 0$

Solution

$$V = \int_0^{2\pi}\int_0^{\pi/2}\int_{\cos\phi}^{2} \rho^2 \sin\phi \; d\rho d\phi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi \left[\rho^3\right]_{\cos\phi}^{2} \; d\phi d\theta$$

$d(\cos\phi) = -\sin\phi$

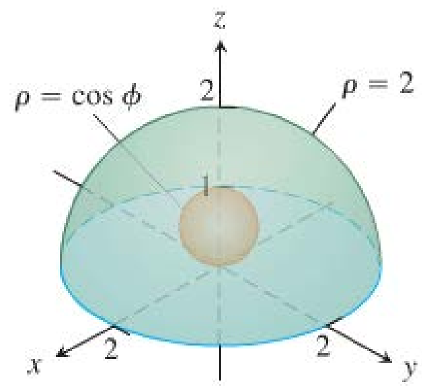
$$= -\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \left(8 - \cos^3\phi\right) \; d(\cos\phi) d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi} \left[8\cos\phi - \frac{1}{4}\cos^4\phi\right]_0^{\pi/2} \; d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi} \left(8 - \frac{1}{4}\right) d\theta$$

$$= \frac{31}{12}\left[\theta\right]_0^{2\pi}$$

$$= \frac{31\pi}{6}$$



Exercise

Find the volume of the solid bounded below by the hemisphere $\rho = 1,\ \ z \ge 0$, and above the cardioid of revolution $\rho = 1 + \cos\phi$

Solution

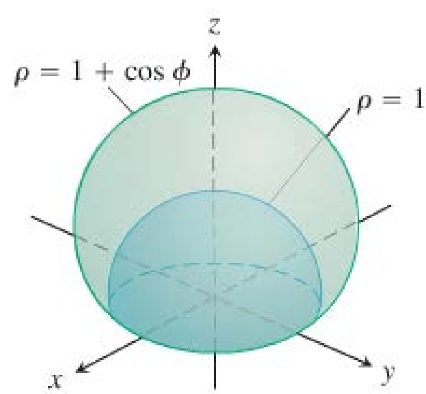
$$V = \int_0^{2\pi}\int_0^{\pi/2}\int_{1}^{1+\cos\phi} \rho^2 \sin\phi \; d\rho d\phi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi \left[\rho^3\right]_{1}^{1+\cos\phi} \; d\phi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \sin\phi \left[(1+\cos\phi)^3 - 1\right] d\phi d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} \left[(1+\cos\phi)^3 - 1\right] d(1+\cos\phi) d\theta$$

$$= -\frac{1}{3}\int_0^{2\pi} \left[\frac{1}{4}(1+\cos\phi)^4 - (1+\cos\phi)\right]_0^{\pi/2} d\theta$$



$$= -\frac{1}{3}\int_0^{2\pi}\left[\frac{1}{4}-1-\left(\frac{1}{4}(2)^4-(1+1)\right)\right]d\theta$$

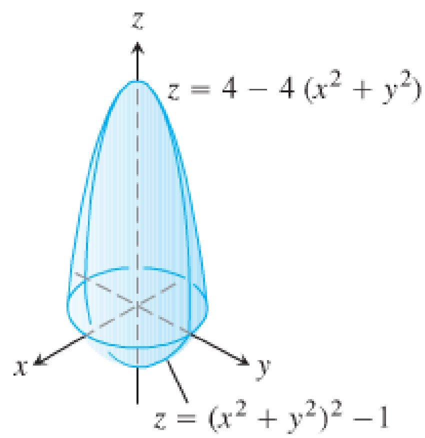
$$= \frac{11}{12}\int_0^{2\pi} d\theta$$

$$= \frac{11}{12}\left[\theta\right]_0^{2\pi}$$

$$= \frac{11\pi}{6}$$

Exercise

Find the volume of the solid



Solution

a) $\left(x^2+y^2\right)^2 - 1 \le z \le 4 - 4\left(x^2+y^2\right)$; $x^2+y^2=r^2$

$r^4 - 1 \le z \le 4 - 4r$

$4 - 4r = 0 \to r = 1 \qquad 0 \le r \le 1$

$0 \le \theta \le 2\pi \quad \to (4) \quad 0 \le \theta \le \frac{\pi}{2}$

$$V = 4\int_0^{\pi/2}\int_0^1\int_{r^4-1}^{4-4r^2} dz\, rdrd\theta$$

$$= 4\int_0^{\pi/2}\int_0^1\left[\left(4-4r^2\right)-\left(r^4-1\right)\right]rdrd\theta$$

$$= 4\int_0^{\pi/2}\int_0^1\left(5-4r^2-r^4\right)rdrd\theta$$

$$= 4\int_0^{\pi/2}\int_0^1\left(5r-4r^3-r^5\right)drd\theta$$

$$= 4\int_0^{\pi/2}\left[\frac{5}{2}r^2-r^4-\frac{1}{6}r^6\right]_0^1 d\theta$$

$$= 4\left(\frac{5}{2}-1-\frac{1}{6}\right)\int_0^{\pi/2} d\theta$$

$$= \frac{16}{3}\left[\theta\right]_0^{\pi/2}$$

$$= \frac{8\pi}{3}$$

b) $V = 4\int_0^{\pi/2}\int_0^1\int_{-\sqrt{1-r^2}}^{1-r} dz\, r dr d\theta$

$= 4\int_0^{\pi/2}\int_0^1\left[(1-r)+\sqrt{1-r^2}\right] r dr d\theta$

$= 4\int_0^{\pi/2}\int_0^1\left[r-r^2+r\left(1-r^2\right)^{1/2}\right] dr d\theta$

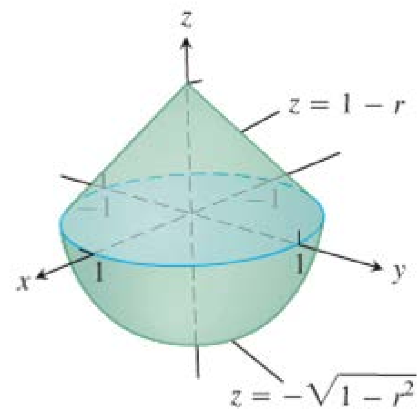
$= 4\int_0^{\pi/2}\left(\left[\frac{1}{2}r^2-\frac{1}{3}r^3\right]_0^1-\frac{1}{2}\int_0^1\left(1-r^2\right)^{1/2} d\left(1-r^2\right)\right) d\theta$

$= 4\int_0^{\pi/2}\left(\left(\frac{1}{2}-\frac{1}{3}\right)-\frac{1}{3}\left[\left(1-r^2\right)^{3/2}\right]_0^1\right) d\theta$

$= 4\left(\frac{1}{6}+\frac{1}{3}\right)\int_0^{\pi/2} d\theta$

$= 2[\theta]_0^{\pi/2}$

$= \pi$



c) $0 \le z \le \sqrt{1-x^2-y^2} = \sqrt{1-r^2}$; $x^2+y^2=r^2$

$V = \int_0^{\pi/2}\int_0^{\sin\theta}\int_0^{\sqrt{1-r^2}} dz\, r dr d\theta$

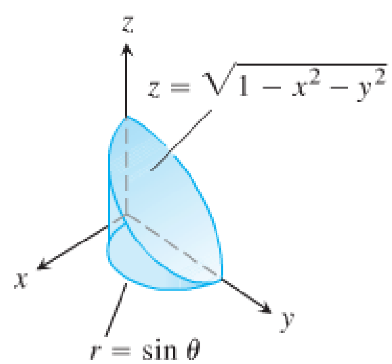
$= \int_0^{\pi/2}\int_0^{\sin\theta}\sqrt{1-r^2}\, r dr d\theta$

$d\left(1-r^2\right) = -2rdr$

$= -\frac{1}{2}\int_0^{\pi/2}\int_0^{\sin\theta}\left(1-r^2\right)^{1/2} d\left(1-r^2\right) d\theta$

$= -\frac{1}{2}\int_0^{\pi/2}\left[\frac{2}{3}\left(1-r^2\right)^{3/2}\right]_0^{\sin\theta} d\theta$

$= -\frac{1}{3}\int_0^{\pi/2}\left[\left(1-\sin^2\theta\right)^{3/2}-1\right] d\theta$



$$= -\frac{1}{3}\int_0^{\pi/2}\left[\left(\cos^2\theta\right)^{3/2} - 1\right]d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left(\cos^3\theta - 1\right)d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}\cos^2\theta\cos\theta d\theta + \frac{1}{3}\int_0^{\pi/2} d\theta \qquad d(\sin\theta) = \cos\theta d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left(1-\sin^2\theta\right)d(\sin\theta) + \frac{\pi}{6}$$

$$= -\frac{1}{3}\left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_0^{\pi/2} + \frac{\pi}{6}$$

$$= -\frac{1}{3}\left(1-\frac{1}{3}\right) + \frac{\pi}{6}$$

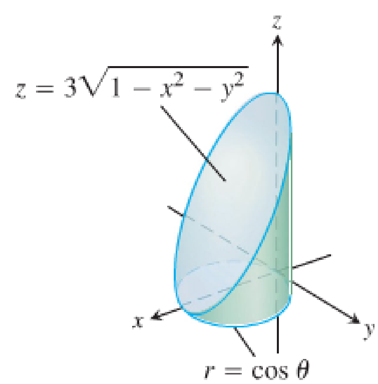
$$= -\frac{2}{9} + \frac{\pi}{6}$$

$$= \frac{3\pi - 4}{18}$$

d) $$V = \int_0^{\pi/2}\int_0^{\cos\theta}\int_0^{3\sqrt{1-r^2}} dz\ r dr d\theta$$

$$= \int_0^{\pi/2}\int_0^{\cos\theta} 3r\sqrt{1-r^2} dr d\theta \qquad d\left(1-r^2\right) = -2rdr$$

$$= -\frac{3}{2}\int_0^{\pi/2}\int_0^{\cos\theta}\left(1-r^2\right)^{1/2} dr d\theta$$



$$= -\int_0^{\pi/2}\left[\left(1-r^2\right)^{3/2}\right]_0^{\cos\theta} d\theta$$

$$= -\int_0^{\pi/2}\left[\left(1-\cos^2\theta\right)^{3/2} - 1\right]d\theta$$

$$= -\int_0^{\pi/2}\left(\sin^3\theta - 1\right)d\theta$$

$$= -\int_0^{\pi/2}\sin^2\theta\sin\theta d\theta + \int_0^{\pi/2} d\theta \qquad d(\cos\theta) = -\sin\theta d\theta$$

$$= \int_0^{\pi/2}\left(1-\cos^2\theta\right)d(\cos\theta) + [\theta]_0^{\pi/2}$$

$$=\left[\cos\theta-\frac{1}{3}\cos^3\theta\right]_0^{\pi/2}+\frac{\pi}{2}$$

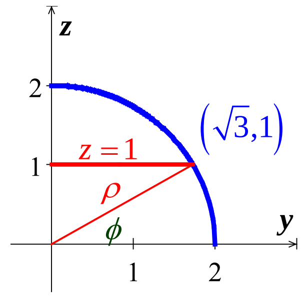
$$=-1+\frac{1}{3}+\frac{\pi}{2}$$

$$\boxed{=\frac{3\pi-4}{6}}$$

Exercise

Find the volume of the smaller region cut from the solid sphere $\rho\le 2$ by the plane $z=1$

Solution



$$\cos\phi=\frac{z}{\rho}\Rightarrow\rho=\frac{1}{\cos\phi}=\sec\phi$$

$$V=\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2}\rho^2\sin\phi\,d\rho d\phi d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3}\sin\phi\left[\rho^3\right]_{\sec\phi}^{2}\,d\phi d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3}\sin\phi\left[8-\sec^3\phi\right]d\phi d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3}\left(8\sin\phi-\tan\phi\sec^2\phi\right)d\phi d\theta \qquad d\left(\tan\phi\right)=\sec^2\phi d\phi$$

$$=\frac{1}{3}\int_0^{2\pi}\left(\int_0^{\pi/3}8\sin\phi\,d\phi-\int_0^{\pi/3}\tan\phi\,d\left(\tan\phi\right)\right)d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}\left[-8\cos\phi-\frac{1}{2}\tan^2\phi\right]_0^{\pi/3}d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}\left[-4-\frac{1}{2}(3)-(-8-0)\right]d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}\frac{5}{2}d\theta$$

$$=\frac{5}{6}[\theta]_0^{2\pi}$$

$$\boxed{=\frac{5\pi}{3}}$$

Exercise

Find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2 + 1$

Solution

$$x^2 + y^2 \le z \le x^2 + y^2 + 1 \quad \rightarrow \quad r^2 \le z \le r^2 + 1$$

$$x^2 + y^2 = 1 = r^2 \quad \rightarrow \quad 0 \le r \le 1$$

$$0 \le \theta \le 2\pi$$

$$V = 4\int_0^{\pi/2} \int_0^1 \int_{r^2}^{r^2+1} dz \; rdrd\theta$$

$$= 4\int_0^{\pi/2} \int_0^1 \left[r^2 + 1 - r^2\right] rdrd\theta$$

$$= 4\int_0^{\pi/2} \int_0^1 rdrd\theta$$

$$= 2\int_0^{\pi/2} \left[r^2\right]_0^1 d\theta$$

$$= 2\int_0^{\pi/2} d\theta$$

$$= 2[\theta]_0^{\pi/2}$$

$$= 2\left(\frac{\pi}{2}\right)$$

$$= \pi$$

Exercise

Find the volume of the region that lies inside the sphere $x^2+y^2+z^2=2$ and outside the cylinder $x^2+y^2=1$

Solution

$$V=8\int_0^{\pi/2}\int_1^{\sqrt{2}}\int_0^{\sqrt{2-r^2}} dz\ r dr d\theta$$

$$=8\int_0^{\pi/2}\int_1^{\sqrt{2}} r\left[z\right]_0^{\sqrt{2-r^2}}\ dr d\theta$$

$$=8\int_0^{\pi/2}\int_1^{\sqrt{2}} r\sqrt{2-r^2}\ dr d\theta \qquad d\left(1-r^2\right)=-2rdr$$

$$=-4\int_0^{\pi/2}\int_1^{\sqrt{2}} \left(2-r^2\right)^{1/2}\ d\left(2-r^2\right)d\theta$$

$$=-\frac{8}{3}\int_0^{\pi/2}\left[\left(2-r^2\right)^{3/2}\right]_1^{\sqrt{2}}\ d\theta$$

$$=-\frac{8}{3}\int_0^{\pi/2}(-1)d\theta$$

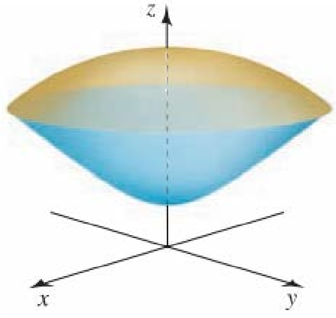
$$=\frac{8}{3}\left[\theta\right]_0^{\pi/2}$$

$$=\frac{8}{3}\left(\frac{\pi}{2}\right)$$

$$=\frac{4\pi}{3}$$

Exercise

Find the volume of the solid between the sphere $x^2 + y^2 + z^2 = 19$ and the hyperboloid $z^2 - x^2 - y^2 = 1$ for $z > 0$



Solution

$$z = \sqrt{19 - x^2 - y^2} \quad z = \sqrt{1 + x^2 + y^2}$$

$$19 - x^2 - y^2 = 1 + x^2 + y^2 \;\Rightarrow\; 2y^2 = 18 - 2x^2 \Rightarrow \quad y = \pm\sqrt{9 - x^2}$$

$$9 - x^2 = 0 \quad \to \quad -3 \le x \le 3$$

$$V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{1+x^2+y^2}}^{\sqrt{19-x^2-y^2}} 1\, dzdydx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left(\sqrt{19 - x^2 - y^2} - \sqrt{1 + x^2 + y^2} \right) dydx$$

*Convert to **Polar coordinates***

$$= \int_{0}^{2\pi} \int_{0}^{3} \left(\sqrt{19 - r^2} - \sqrt{1 + r^2} \right) r\, drd\theta$$

$$= \int_{0}^{2\pi} d\theta \left(-\frac{1}{2} \int_{0}^{3} \left(19 - r^2\right)^{1/2} d\left(19 - r^2\right) - \frac{1}{2} \int_{0}^{3} \left(1 + r^2\right)^{1/2} d\left(1 + r^2\right) \right)$$

$$= 2\pi \left(-\frac{1}{3}\left(19 - r^2\right)^{3/2} - \frac{1}{3}\left(1 + r^2\right)^{3/2} \right)_{0}^{3}$$

$$= -\frac{2}{3}\pi \left(10\sqrt{10} + 10\sqrt{10} - 19\sqrt{19} - 1\right)$$

$$= \frac{2\pi}{3}\left(1 + 19\sqrt{19} - 20\sqrt{10}\right)$$

Exercise

Evaluate the integral in cylindrical coordinates $\displaystyle\int_0^{2\pi}\int_0^1\int_{-1}^1 r\ dzdrd\theta$

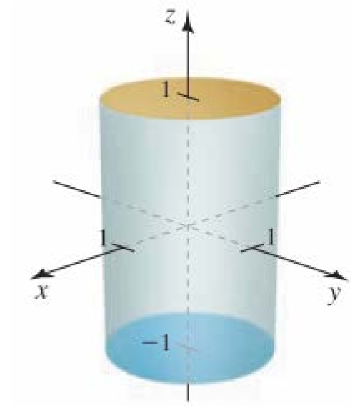
Solution

$$\int_0^{2\pi}\int_0^1\int_{-1}^1 r\ dzdrd\theta = \int_0^{2\pi} d\theta \int_0^1 r\ dr \int_{-1}^1 dz$$

$$= (2\pi)\left(\frac{1}{2}r^2\right)\Big|_0^1 \ z\Big|_{-1}^1$$

$$= (2\pi)\left(\frac{1}{2}\right)(2)$$

$$= 2\pi$$



Exercise

Evaluate the integral in cylindrical coordinates $\displaystyle\int_0^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_0^{9-3\sqrt{x^2+y^2}} dzdxdy$

Solution

$$\int_0^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_0^{9-3\sqrt{x^2+y^2}} dzdxdy = \int_0^{\pi}\int_0^3\int_0^{9-3r} r\ dzdrd\theta$$

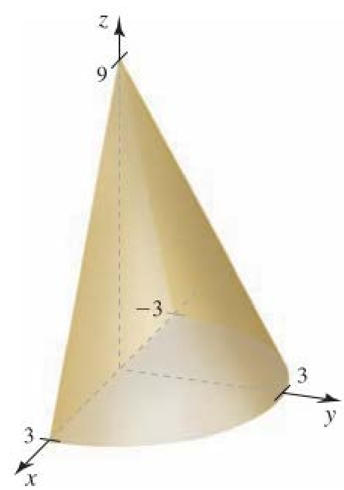
$$= \int_0^{\pi} d\theta \int_0^3 rz\Big|_0^{9-3r}\ dr$$

$$= \pi\int_0^3 \left(9r - 3r^2\right)\ dr$$

$$= \pi\left(\frac{9}{2}r^2 - r^3\right)\Big|_0^3$$

$$= \pi\left(\frac{81}{2} - 27\right)$$

$$= \frac{27}{2}\pi$$



Exercise

Evaluate the integral in cylindrical coordinates $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{-1}^{1}\left(x^2+y^2\right)^{3/2}\, dzdxdy$

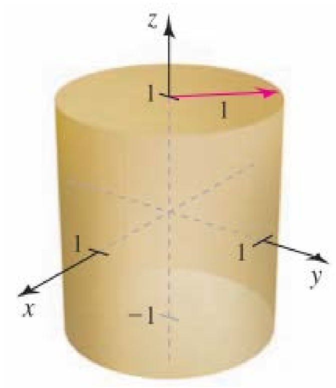
Solution

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{-1}^{1}\left(x^2+y^2\right)^{3/2}\, dzdxdy = \int_{0}^{2\pi}\int_{0}^{1}\int_{-1}^{1} r^3\, dz\, rdrd\theta$$

$$= \int_{0}^{2\pi} d\theta \int_{0}^{1} r^4 dr\ \left(z\right)\Big|_{-1}^{1}$$

$$= 2\pi\left(\frac{1}{5}r^5\right)\Big|_{0}^{1}(2)$$

$$= \boxed{\frac{4\pi}{5}}$$



Exercise

Evaluate the integral in cylindrical coordinates $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{2}\frac{1}{1+x^2+y^2}\, dzdydx$

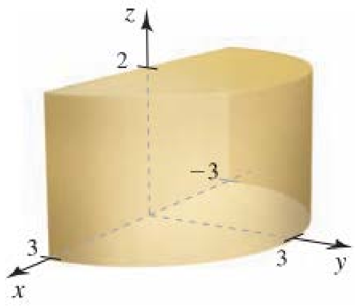
Solution

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{2}\frac{1}{1+x^2+y^2}\, dzdydx = \int_{0}^{\pi}\int_{0}^{3}\int_{0}^{2}\frac{1}{1+r^2}\, dz\, rdrd\theta$$

$$= \frac{1}{2}\int_{0}^{\pi} d\theta \int_{0}^{3}\frac{1}{1+r^2} d\left(1+r^2\right)\ \left[z\right]\Big|_{0}^{2}$$

$$= \pi\ln\left(1+r^2\right)\Big|_{0}^{3}$$

$$= \boxed{\pi\ln(10)}$$



Exercise

Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the plane $z=0$ and the hyperboloid $z=\sqrt{17}-\sqrt{1+x^2+y^2}$

Solution

$$z=\sqrt{17}-\sqrt{1+x^2+y^2}=0 \quad \rightarrow \quad 17=1+x^2+y^2$$

$$x^2+y^2=16=r^2$$

$$V=\int_0^{2\pi}\int_0^4\int_0^{\sqrt{17}-\sqrt{1+r^2}} 1dz\ rdrd\theta$$

$$=\int_0^{2\pi}d\theta\int_0^4 z\Big|_0^{\sqrt{17}-\sqrt{1+r^2}} rdr$$

$$=2\pi\int_0^4\left(\sqrt{17}-\sqrt{1+r^2}\right)rdr$$

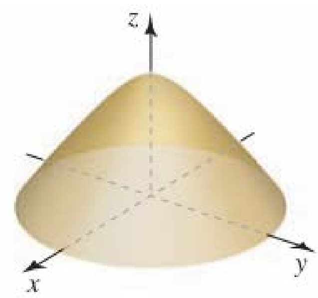
$$=2\pi\int_0^4\left(\sqrt{17}r-r\sqrt{1+r^2}\right)dr$$

$$=2\pi\left(\frac{1}{2}\sqrt{17}r^2\Big|_0^4-\frac{1}{2}\int_0^4\sqrt{1+r^2}\ d\left(1+r^2\right)\right)$$

$$=\pi\left(16\sqrt{17}-\frac{2}{3}\left(1+r^2\right)^{3/2}\Big|_0^4\right)$$

$$=\pi\left(16\sqrt{17}-\frac{2}{3}17\sqrt{17}+\frac{2}{3}\right)$$

$$=\pi\left(\frac{14\sqrt{17}+2}{3}\right) \quad unit^3$$



$z=\sqrt{17}-\sqrt{1+x^2+y^2}$

Exercise

Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the plane $z=25$ and the paraboloid $z=x^2+y^2$

Solution

$$z=x^2+y^2=r^2=25 \ \rightarrow \ r=5$$

$$V=\int_0^{2\pi}\int_0^5\int_{r^2}^{25} 1dz\ rdrd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^5 z\Big|_{r^2}^{25}\ rdr$$

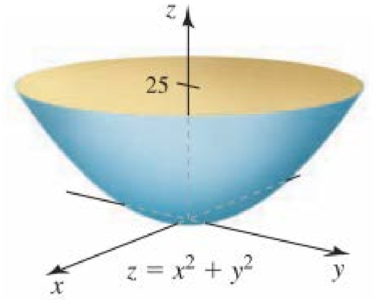
$$= 2\pi \int_0^5 \left(25 - r^2\right) r\ dr$$

$$= 2\pi \int_0^5 \left(25r - r^3\right)\ dr$$

$$= 2\pi \left(\frac{25}{2}r^2 - \frac{1}{4}r^4\right)\Big|_0^5$$

$$= 2\pi \left(\frac{1}{2}5^4 - \frac{1}{4}5^4\right)$$

$$= \frac{625\pi}{2}\ \ unit^3$$



Exercise

Evaluate the integral in cylindrical coordinates to find the volume of the solid bounded by the parabolic cylinders $z = y^2 + 1$ and $z = 2 - x^2$

Solution

$$2 - x^2 - \left(y^2 + 1\right) = 1 - \left(x^2 + y^2\right)$$

$$z = y^2 + 1 = 2 - x^2 \ \rightarrow\ x^2 + y^2 = 1 = r^2$$

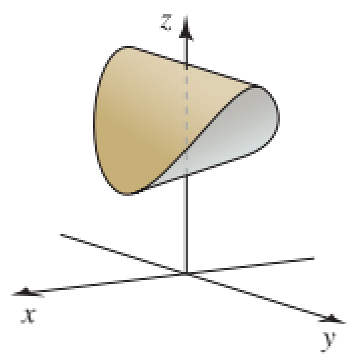
$$V = \int_0^{2\pi} \int_0^1 \left(1 - r^2\right)\ rdrd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 \left(r - r^3\right)\ dr$$

$$= 2\pi \left(\frac{1}{2}r^2 - \frac{1}{4}r^4\right)\Big|_0^1$$

$$= 2\pi \left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{\pi}{2}\ \ unit^3$$



Exercise

Evaluate the integral $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\sec\varphi}\rho^2\sin\varphi\, d\rho d\varphi d\theta$

Solution

$$\int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\sec\varphi}\rho^2\sin\varphi\, d\rho d\varphi d\theta=\int_0^{2\pi}d\theta\int_0^{\pi/3}\frac{1}{3}\sin\varphi\ \rho^3\Big|_0^{4\sec\varphi}\ d\varphi$$

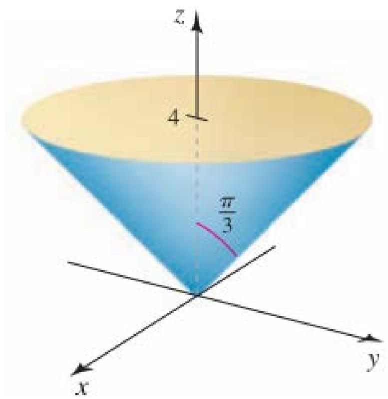
$$=\frac{128\pi}{3}\int_0^{\pi/3}\sin\varphi\ \sec^3\varphi\ d\varphi$$

$$=-\frac{128\pi}{3}\int_0^{\pi/3}\cos^{-3}\varphi\ d(\cos\varphi)$$

$$=\frac{64\pi}{3}\frac{1}{\cos^2\varphi}\Big|_0^{\pi/3}$$

$$=\frac{64\pi}{3}(4-1)$$

$$=64\pi$$



Exercise

Evaluate the integral $\displaystyle\int_0^{\pi}\int_0^{\pi/6}\int_{2\sec\varphi}^{4}\rho^2\sin\varphi\, d\rho d\varphi d\theta$

Solution

$$\int_0^{\pi}\int_0^{\pi/6}\int_{2\sec\varphi}^{4}\rho^2\sin\varphi\, d\rho d\varphi d\theta=\frac{1}{3}\int_0^{\pi}d\theta\int_0^{\pi/6}\sin\varphi\ \rho^3\Big|_{2\sec\varphi}^{4}\ d\varphi$$

$$=\frac{\pi}{3}\int_0^{\pi/6}\sin\varphi\left(64-8\sec^3\varphi\right)d\varphi$$

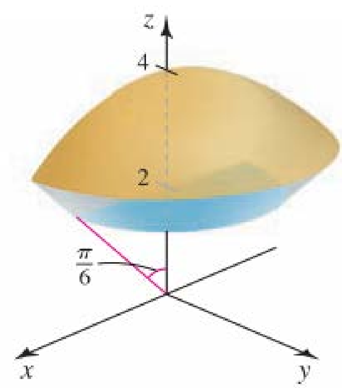
$$=\frac{8\pi}{3}\int_0^{\pi/6}\left(\cos^{-3}\varphi-8\right)d(\cos\varphi)$$

$$=\frac{8\pi}{3}\left(\frac{-1}{2\cos^2\varphi}-8\cos\varphi\right)\Big|_0^{\pi/6}$$

$$=\frac{8\pi}{3}\left(-\frac{2}{3}-4\sqrt{3}+\frac{1}{2}+8\right)$$

$$=\frac{8\pi}{3}\left(\frac{47}{3}-4\sqrt{3}\right)$$

$$=\left(\frac{188}{9}-\frac{32}{3}\sqrt{3}\right)\pi$$



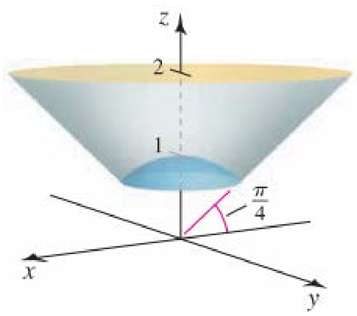
Exercise

Evaluate the integral $\int_0^{2\pi}\int_0^{\pi/4}\int_1^{2\sec\varphi}\left(\rho^{-3}\right)\rho^2\sin\varphi\, d\rho d\varphi d\theta$

Solution

$$\int_0^{2\pi}\int_0^{\pi/4}\int_1^{2\sec\varphi}\left(\rho^{-3}\right)\rho^2\sin\varphi\, d\rho d\varphi d\theta = \int_0^{2\pi}d\theta\int_0^{\pi/4}\int_1^{2\sec\varphi}\sin\varphi\left(\frac{1}{\rho}\,d\rho\right)d\varphi$$

$$=\int_0^{2\pi}d\theta\int_0^{\pi/4}\sin\varphi\ln(\rho)\Big|_1^{2\sec\varphi}\,d\varphi$$



$$=2\pi\int_0^{\pi/4}\sin\varphi\ \ln(2\sec\varphi)d\varphi$$

$$u=\ln(2\sec\varphi) \qquad dv=\sin\varphi d\varphi$$

$$du=\frac{2\sec\varphi\tan\varphi}{2\sec\varphi}=\tan\varphi \qquad v=-\cos\varphi$$

$$=2\pi\left[-\cos\varphi\ \ln(2\sec\varphi)\Big|_0^{\pi/4}+\int_0^{\pi/4}\sin\varphi d\varphi\right]$$

$$=2\pi\left(-\cos\varphi\ \ln(2\sec\varphi)-\cos\varphi\right)\Big|_0^{\pi/4}$$

$$=2\pi\left(-\frac{\sqrt{2}}{2}\ln\left(2\sqrt{2}\right)-\frac{\sqrt{2}}{2}+\ln 2+1\right)$$

$$=2\pi\left(\ln 2-\frac{\sqrt{2}}{2}\ln\left(2\sqrt{2}\right)+1-\frac{\sqrt{2}}{2}\right)$$

Exercise

Evaluate the integral $\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^{2\csc\varphi}\rho^2\sin\varphi\, d\rho d\varphi d\theta$

Solution

$$\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^{2\csc\varphi}\rho^2\sin\varphi\, d\rho d\varphi d\theta=\frac{1}{3}\int_0^{2\pi}d\theta\int_{\pi/6}^{\pi/3}\sin\varphi\left(\rho^3\right)\Big|_0^{2\csc\varphi}d\varphi$$

$$=\frac{16\pi}{3}\int_{\pi/6}^{\pi/3}\sin\varphi\csc^3\varphi\, d\varphi$$

$$=-\frac{16\pi}{3}\int_{\pi/6}^{\pi/3}\sin\varphi\csc\varphi\, d(\cot\varphi)$$

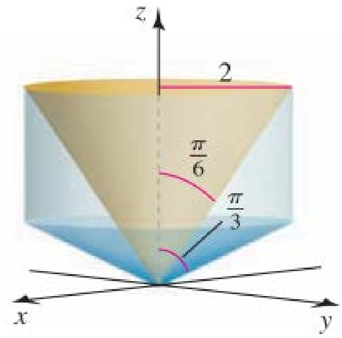
$$= -\frac{16\pi}{3}\int_{\pi/6}^{\pi/3} d(\cot\varphi)$$

$$= -\frac{16\pi}{3}\left(\cot\varphi\right)\Big|_{\pi/6}^{\pi/3}$$

$$= -\frac{16\pi}{3}\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right)$$

$$= \frac{32\pi}{3\sqrt{3}}$$

$$= \frac{32}{9}\pi\sqrt{3}$$



Exercise

Use the spherical coordinates to find the volume of a ball of radius $a > 0$

Solution

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2 \sin\varphi \; d\rho d\varphi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \; \left(\rho^3\right)\Big|_0^a$$

$$= \frac{2\pi}{3}a^3\left(-\cos\varphi\right)\Big|_0^{\pi}$$

$$= \frac{4}{3}\pi a^3 \quad unit^3$$

Exercise

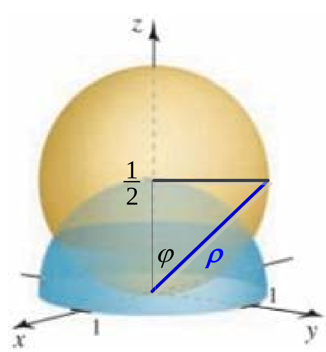
Use the spherical coordinates to find the volume of the solid bounded by the sphere $\rho = 2\cos\varphi$ and the hemisphere $\rho = 1,\ z \ge 0$

Solution

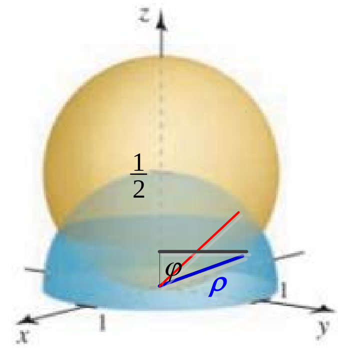
$$\rho = 2\cos\varphi = 1 \quad \to \quad \varphi = \frac{\pi}{3}$$

$$z = \frac{1}{2} \to \cos\varphi = \frac{1}{2}\frac{1}{\rho} \quad \to \quad \rho = \frac{1}{2}\sec\varphi$$

$$V = 2\int_0^{2\pi}\int_0^{\pi/3}\int_{\frac{1}{2}\sec\varphi}^{1} \rho^2 \sin\varphi \; d\rho d\varphi d\theta$$



$$= \frac{2}{3}\int_0^{2\pi} d\theta \int_0^{\pi/3} \sin\varphi \left(\rho^3\right)\Big|_{\frac{1}{2}\sec\varphi}^{1} \ d\varphi$$



$$= \frac{4\pi}{3}\int_0^{\pi/3} \sin\varphi \left(1-\frac{1}{8}\sec^3\varphi\right) \ d\varphi$$

$$= \frac{4\pi}{3}\left(\int_0^{\pi/3} \sin\varphi \ d\varphi + \frac{1}{8}\int_0^{\pi/3} \cos^{-3}\varphi \ d\left(\cos\varphi\right)\right)$$

$$= \frac{4\pi}{3}\left(-\cos\varphi - \frac{1}{16}\frac{1}{\cos^2\varphi}\right)\Big|_0^{\pi/3}$$

$$= \frac{4\pi}{3}\left(-\frac{1}{2} - \frac{1}{4} + 1 + \frac{1}{16}\right)$$

$$= \frac{5\pi}{12}$$

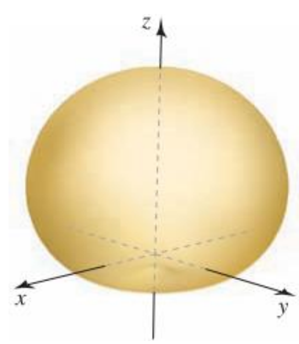
Exercise

Use the spherical coordinates to find the volume of the solid of revolution

$$D = \left\{\left(\rho,\varphi,\theta\right):\ 0 \le \rho \le 1+\cos\varphi,\ 0 \le \varphi \le \pi,\ 0 \le \theta \le 2\pi\right\}$$

Solution

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{1+\cos\varphi} \rho^2 \sin\varphi \ d\rho d\varphi d\theta$$



$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\rho^3\Big|_0^{1+\cos\varphi} \ d\varphi$$

$$= \frac{2\pi}{3}\int_0^{\pi} \sin\varphi\left(1+\cos\varphi\right)^3 \ d\varphi$$

$$= -\frac{2\pi}{3}\int_0^{\pi} \left(1+\cos\varphi\right)^3 \ d\left(1+\cos\varphi\right)$$

$$= -\frac{\pi}{6}\left(1+\cos\varphi\right)^4\Big|_0^{\pi}$$

$$= \frac{\pi}{6}2^4$$

$$= \frac{8}{3}\pi$$

Exercise

Use the spherical coordinates to find the volume of the solid outside the cone $\varphi = \frac{\pi}{4}$ and inside the sphere $\rho = 4\cos\varphi$

Solution

$$V = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{4\cos\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_{\pi/4}^{\pi/2} \sin\varphi \left(\rho^3\right)\Big|_0^{4\cos\varphi} d\varphi$$

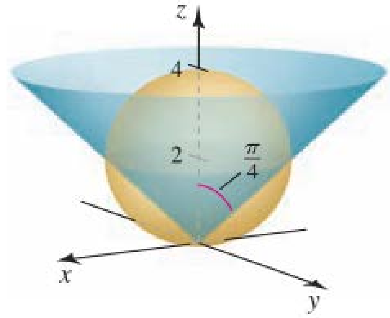
$$= \frac{128}{3}\pi \int_{\pi/4}^{\pi/2} \sin\varphi \left(\cos^3\varphi\right) d\varphi$$

$$= \frac{128}{3}\pi \int_{\pi/4}^{\pi/2} \left(-\cos^3\varphi\right) d\left(\cos\varphi\right)$$

$$= \frac{32}{3}\pi \left(-\cos^4\varphi\right)\Big|_{\pi/4}^{\pi/2}$$

$$= \frac{32}{3}\pi \left(\frac{1}{4}\right)$$

$$= \frac{8}{3}\pi$$



Exercise

Use the spherical coordinates to find the volume of the solid bounded by the cylinders $r = 1$ and $r = 2$, and the cone $\varphi = \frac{\pi}{6}$ and $\varphi = \frac{\pi}{3}$

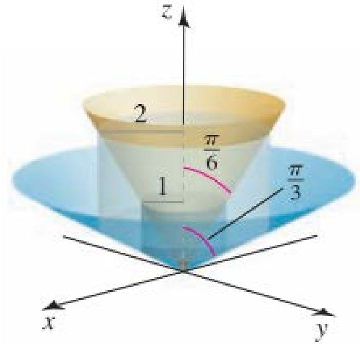
Solution

$$V = \int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_{\csc\varphi}^{2\csc\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_{\pi/6}^{\pi/3} \sin\varphi \left(\rho^3\right)\Big|_{\csc\varphi}^{2\csc\varphi} d\varphi$$

$$= \frac{14\pi}{3}\int_{\pi/6}^{\pi/3} \sin\varphi \left(\csc^3\varphi\right) d\varphi$$

$$= \frac{14\pi}{3}\int_{\pi/6}^{\pi/3} \csc^2\varphi \, d\varphi$$



$$= \frac{14\pi}{3}\left(-\cot\varphi\right)\Big|_{\pi/6}^{\pi/3}$$

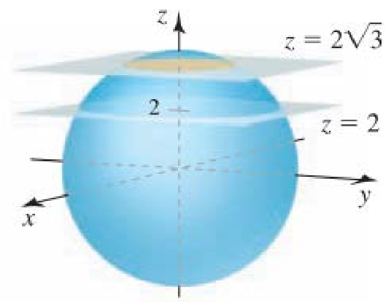
$$= \frac{14\pi}{3}\left(-\frac{1}{\sqrt{3}} + \sqrt{3}\right)$$

$$= \frac{14\pi}{3}\left(\frac{2}{\sqrt{3}}\right)$$

$$= \boxed{\frac{28}{9}\pi\sqrt{3}}$$

Exercise

Use the spherical coordinates to find the volume of the ball $\rho \le 4$ that lies between the planes $z = 2$ and $z = 2\sqrt{3}$



Solution

$$z = 2\sqrt{3} \quad \to \quad \cos\varphi = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \;\Rightarrow\; \varphi = \frac{\pi}{6}$$

$$z = 2 \quad \to \quad \cos\varphi = \frac{2}{4} = \frac{1}{2} \;\Rightarrow\; \varphi = \frac{\pi}{3}$$

$$V = \int_0^{2\pi}\int_0^{\pi/6}\int_{2\sqrt{3}\sec\varphi}^{4} \rho^2 \sin\varphi \; d\rho d\varphi d\theta - \int_0^{2\pi}\int_0^{\pi/3}\int_{2\sec\varphi}^{4} \rho^2 \sin\varphi \; d\rho d\varphi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\pi/6} \sin\varphi\left(\rho^3\right)\Big|_{2\sqrt{3}\sec\varphi}^{4} \; d\varphi - \frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\pi/3} \sin\varphi\left(\rho^3\right)\Big|_{2\sec\varphi}^{4} \; d\varphi$$

$$= \frac{2\pi}{3}\int_0^{\pi/6} \sin\varphi\left(64 - 24\sqrt{3}\sec^3\varphi\right) d\varphi - \frac{2\pi}{3}\int_0^{\pi/3} \sin\varphi\left(64 - 8\sec^3\varphi\right) d\varphi$$

$$= \frac{16\pi}{3}\int_0^{\pi/6} \left(3\sqrt{3}\cos^{-3}\varphi - 8\right) d\left(\cos\varphi\right) + \frac{16\pi}{3}\int_0^{\pi/3} \left(8 - \cos^{-3}\varphi\right) d\left(\cos\varphi\right)$$

$$= \frac{16\pi}{3}\left(-\frac{3\sqrt{3}}{2}\sec^2\varphi - 8\cos\varphi\right)\Big|_0^{\pi/6} + \frac{16\pi}{3}\left(8\cos\varphi + \frac{1}{2}\sec^2\varphi\right)\Big|_0^{\pi/3}$$

$$= \frac{16\pi}{3}\left(-2\sqrt{3} - 4\sqrt{3} + \frac{3\sqrt{3}}{2} + 8\right) + \frac{16\pi}{3}\left(4 + 2 - 8 - \frac{1}{2}\right)$$

$$= \frac{16\pi}{3}\left(-\frac{9\sqrt{3}}{2} + 8 - \frac{5}{2}\right)$$

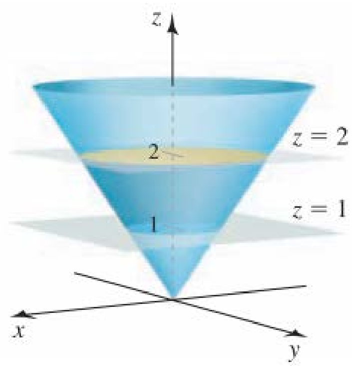
$$= \boxed{\left(9\sqrt{3} - 11\right)\frac{8\pi}{3}}$$

Exercise

Use the spherical coordinates to find the volume of the solid inside the cone $z = \left(x^2 + y^2\right)^{1/2}$ that lies between the planes $z = 1$ and $z = 2$

Solution

$$z = 2 \quad \rightarrow \quad x^2 + y^2 = 4 = r^2 \quad \Rightarrow \quad \varphi = \tan^{-1}\frac{2}{2} = \frac{\pi}{4}$$



$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_{\sec\varphi}^{2\sec\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\varphi \left(\rho^3\right)\Big|_{2\sec\varphi}^{\sec\varphi} \, d\varphi$$

$$= \frac{2\pi}{3}\int_0^{\pi/4} \left(-7\sec^3\varphi\right) \, d\left(\cos\varphi\right)$$

$$= \frac{7\pi}{3}\left(\frac{1}{\cos^2\varphi}\right)\Big|_0^{\pi/4}$$

$$= \frac{7\pi}{3}$$

Or: $Volume = \frac{1}{3}Ah = \frac{1}{3}\left(2^2\pi \times 2 - 1^2\pi \times 1\right) = \frac{7\pi}{3}$

Exercise

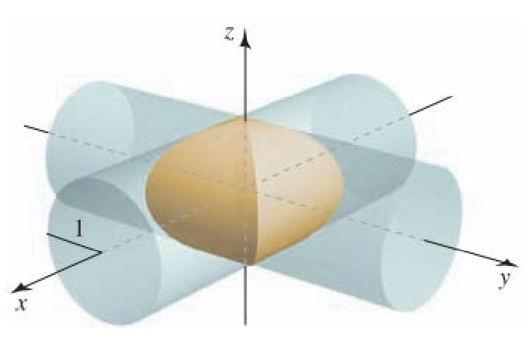
The x- and y-axes from the axes of two right circular cylinders with radius 1.
Find the volume of the solid that is common to the two cylinders.

Solution

Due to symmetry, this region is made up of *eight* identical pieces, one in each octant.

$$y = 0 \quad \rightarrow \quad x^2 + z^2 = 1 \quad \Rightarrow \quad x = \sqrt{1 - z^2}$$

$$// \text{ } x\text{-axis} \quad \rightarrow \quad y^2 + z^2 = 1 \quad \Rightarrow \quad y = \sqrt{1 - z^2}$$



$$V = 8\int_0^1\int_0^{\sqrt{1-z^2}}\int_0^{\sqrt{1-z^2}} 1 \, dydxdz$$

$$= 8\int_0^1\int_0^{\sqrt{1-z^2}} \sqrt{1 - z^2} \, dxdz$$

$$= 8\int_0^1 \sqrt{1-z^2}\; x\Big|_0^{\sqrt{1-z^2}}\; dz$$

$$= 8\int_0^1 \left(1-z^2\right) dz$$

$$= 8\left(z - \frac{1}{3}z^3\right)\Big|_0^1$$

$$\boxed{= \frac{16}{3}}$$

Solution Section 3.6 – Integrals for Mass Calculations

Exercise

Find the mass and center of mass of the thin rods with the following density functions.

$$\rho(x) = 1 + \sin x \quad for \quad 0 \le x \le \pi$$

Solution

$$M = \int_0^{\pi} (1 + \sin x)\, dx$$

$$= x - \cos x \Big|_0^{\pi}$$

$$= \pi + 2$$

$$M_{\bar{x}} = \int_0^{\pi} x(1 + \sin x)\, dx$$

$$= \int_0^{\pi} (x + x\sin x)\, dx$$

$$= \left[\frac{1}{2}x^2 - x\cos x + \sin x\right]_0^{\pi}$$

$$= \frac{1}{2}\pi^2 + \pi$$

		$\int \sin x$
+	x	$-\cos x$
–	1	$-\sin x$

Center of mass: $\bar{x} = \dfrac{M_{\bar{x}}}{M} = \dfrac{\frac{\pi^2 + 2\pi}{2}}{\pi + 2} = \dfrac{\pi}{2}$

Exercise

Find the mass and center of mass of the thin rods with the following density functions.

$$\rho(x) = 1 + x^3 \quad for \quad 0 \le x \le 1$$

Solution

$$M = \int_0^1 \left(1 + x^3\right) dx = \left[x + \frac{1}{4}x^4\right]_0^1 = \frac{5}{4}$$

$$M_{\bar{x}} = \int_0^1 x\left(1 + x^3\right) dx = \int_0^1 \left(x + x^4\right) dx = \left[\frac{1}{2}x^2 + \frac{1}{5}x^5\right]_0^1 = \frac{7}{10}$$

Center of mass: $\bar{x} = \dfrac{M_{\bar{x}}}{M} = \dfrac{\frac{7}{10}}{\frac{5}{4}} = \dfrac{14}{25}$

Exercise

Find the mass and center of mass of the thin rods with the following density functions.

$$\rho(x) = 2 - \frac{x^2}{16} \quad \textit{for} \quad 0 \le x \le 4$$

Solution

$$M = \int_0^4 \left(2 - \frac{1}{16}x^2\right)dx = \left[2x - \frac{1}{48}x^3\right]_0^4 = 8 - \frac{4}{3} = \frac{20}{3}$$

Center of mass:

$$\bar{x} = \frac{3}{20}\int_0^4 \left(2x - \frac{1}{16}x^3\right)dx \qquad \bar{x} = \frac{1}{M}\int_a^b x\rho(x)\,dx$$

$$= \frac{3}{20}\left(x^2 - \frac{1}{64}x^4\right)_0^4$$

$$= \frac{3}{20}(16 - 4)$$

$$= \frac{9}{5}$$

Exercise

Find the mass and center of mass of the thin rods with the following density functions.

$$\rho(x) = 2 + \cos x \quad \textit{for} \quad 0 \le x \le \pi$$

Solution

$$M = \int_0^\pi (2 + \cos x)\,dx = [2x + \sin x]_0^\pi = 2\pi$$

Center of mass:

$$\bar{x} = \frac{1}{2\pi}\int_0^\pi (2x + x\cos x)\,dx \qquad \bar{x} = \frac{1}{M}\int_a^b x\rho(x)\,dx$$

$$= \frac{1}{2\pi}\left(x^2 + x\sin x + \cos x\right)_0^\pi$$

$$= \frac{1}{2\pi}\left(\pi^2 - 2\right)$$

		$\int \cos x$
+	x	$\sin x$
–	1	$-\cos x$

Exercise

Find the mass and center of mass of the thin rods with the following density functions.

$$\rho(x) = \begin{cases} x^2 & \text{if} \quad 0 \le x \le 1 \\ 2x - x^2 & \text{if} \quad 1 \le x \le 2 \end{cases}$$

Solution

$$M = \int_0^1 x^2 dx + \int_1^2 \left(2x - x^2\right) dx$$

$$= \frac{1}{3}x^3 \Big|_0^1 + \left[x^2 - \frac{1}{3}x^3\right]_1^2$$

$$= \frac{1}{3} + 4 - \frac{8}{3} - 1 + \frac{1}{3}$$

$$= 1$$

Center of mass:

$$\overline{x} = 1\int_0^1 x^3 dx + 1\int_1^2 \left(2x^2 - x^3\right) dx \qquad \overline{x} = \frac{1}{M}\int_a^b x\rho(x)\,dx$$

$$= \frac{1}{4}x^4 \Big|_0^1 + \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_1^2$$

$$= \frac{1}{4} + \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4}$$

$$= \frac{7}{6}$$

Exercise

Find the mass and centroid (center of mass) of the following thin plates, assuming a constant density. Sketch the region corresponding to the plate and indicate the location of the center of mass. Use symmetry when possible to simplify your work.

The region bounded by $y = \sin x$ and $y = 1 - \sin x$ between $x = \frac{\pi}{4}$ *and* $x = \frac{3\pi}{4}$

Solution

$$m = \int_{\pi/4}^{3\pi/4} (1 - \sin x - \sin x)\,dx$$

$$= \left[x + 2\cos x\right]_{\pi/4}^{3\pi/4}$$

$$= \frac{\pi}{2} - 2\sqrt{2}$$

Center of mass:

$$\overline{x} = \frac{2}{\pi - 4\sqrt{2}}\int_{\pi/4}^{3\pi/4} (x - 2x\sin x)\,dx \qquad \overline{x} = \frac{1}{M}\int_a^b x\rho(x)\,dx$$

$$=\frac{2}{\pi-4\sqrt{2}}\left[\frac{1}{2}x^2+2x\cos x+2\sin x\right]_{\pi/4}^{3\pi/4}$$

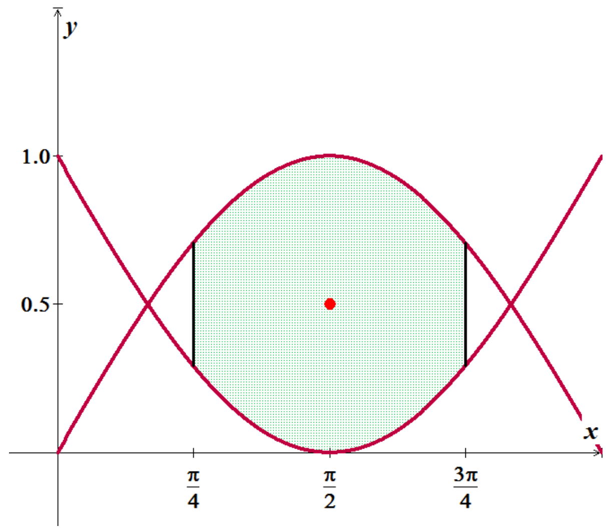
	$\int \sin x$
x	$-\cos x$
1	$-\sin x$

$$=\frac{2}{\pi-4\sqrt{2}}\left[\frac{9\pi^2}{32}-\frac{3\pi}{4}\sqrt{2}+\sqrt{2}-\frac{\pi^2}{32}-\frac{\pi}{4}\sqrt{2}-\sqrt{2}\right]$$

$$=\frac{2}{\pi-4\sqrt{2}}\left(\frac{\pi^2}{4}-\pi\sqrt{2}\right)$$

$$=\frac{\pi}{\pi-4\sqrt{2}}\left(\frac{\pi-4\sqrt{2}}{2}\right)$$

$$=\frac{\pi}{2}$$



$$y=1-\sin x\Big|_{x=\frac{\pi}{2}}=1;\quad y=\sin x\Big|_{x=\frac{\pi}{2}}=0$$

$$\bar{y}=\frac{1-0}{2}=\frac{1}{2}$$

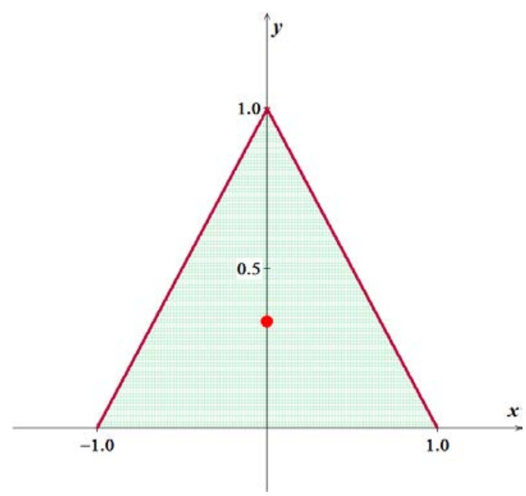
Centroid: $\left(\frac{\pi}{2},\ \frac{1}{2}\right)$

Exercise

Find the mass and centroid (center of mass) of the following thin plates, assuming a constant density. Sketch the region corresponding to the plate and indicate the location of the center of mass. Use symmetry when possible to simplify your work.

The region bounded by $y=1-|x|$ and the x-axis

Solution



By symmetry: $\bar{x}=0$

$$M=2\int_0^1(1-x)\,dx=2\left[x-\frac{1}{2}x^2\right]_0^1=1$$

Center of mass:

$$\bar{y}=\int_{-1}^{0}\int_0^{1+x}y\,dy\,dx+\int_0^1\int_0^{1-x}y\,dy\,dx$$

$$=\int_{-1}^{0}\frac{1}{2}(1+x)^2\,dx+\frac{1}{2}\int_0^1(1-x)^2\,dx$$

$$=\frac{1}{2}\int_{-1}^{0}(1+x)^2\,d(1+x)-\frac{1}{2}\int_0^1(1-x)^2\,d(1-x)$$

$$=\frac{1}{2}\left[\frac{1}{3}(1+x)^3\Big|_{-1}^{0}-\frac{1}{3}(1-x)^3\Big|_0^1\right]=\frac{1}{6}(1+1)$$

$$=\frac{1}{3}$$

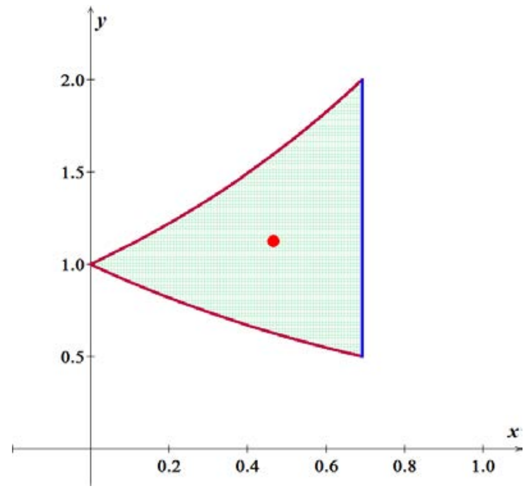
Exercise

Find the mass and centroid (center of mass) of the following thin plates, assuming a constant density. Sketch the region corresponding to the plate and indicate the location of the center of mass. Use symmetry when possible to simplify your work.

The region bounded by $y=e^{x}, \quad y=e^{-x}, \quad x=0, \quad and \quad x=\ln 2$

Solution

Assuming: $\rho = 1$



$$m=\int_{0}^{ln2}\int_{e^{-x}}^{e^{x}} 1dydx$$

$$=\int_{0}^{ln2}\left(e^{x}-e^{-x}\right)dx$$

$$=\left[e^{x}+e^{-x}\right]_{0}^{\ln 2}$$

$$=2+\frac{1}{2}-1-1$$

$$=\frac{1}{2}$$

$$\overline{x}=\frac{M_{y}}{m}=\frac{1}{\frac{1}{2}}\int_{0}^{\ln 2}\int_{e^{-x}}^{e^{x}} xdydx$$

$$=2\int_{0}^{\ln 2} x\left(e^{x}-e^{-x}\right)dx \qquad \int xe^{ax}dx=e^{ax}\left(\frac{x}{a}-\frac{1}{a^{2}}\right)$$

$$=2\left[e^{x}(x-1)-e^{-x}(-x-1)\right]_{0}^{\ln 2}$$

$$=2\left[2(\ln 2-1)-\frac{1}{2}(-\ln 2-1)+1-1\right]$$

$$=2\left(2\ln 2-2+\frac{1}{2}\ln 2+\frac{1}{2}\right)$$

$$=5\ln 2-3$$

$$\overline{x}=\frac{M_{x}}{m}=2\int_{0}^{\ln 2}\int_{e^{-x}}^{e^{x}} ydydx$$

$$=\int_{0}^{\ln 2}\left(e^{2x}-e^{-2x}\right)dx$$

$$=\frac{1}{2}\left[e^{2x}+e^{-2x}\right]_{0}^{\ln 2} \quad =\frac{1}{2}\left(4+\frac{1}{4}-1-1\right)$$

$$=\frac{9}{8}$$

So the center of mass is $\left(5\ln 2-3,\ \frac{9}{8}\right)$

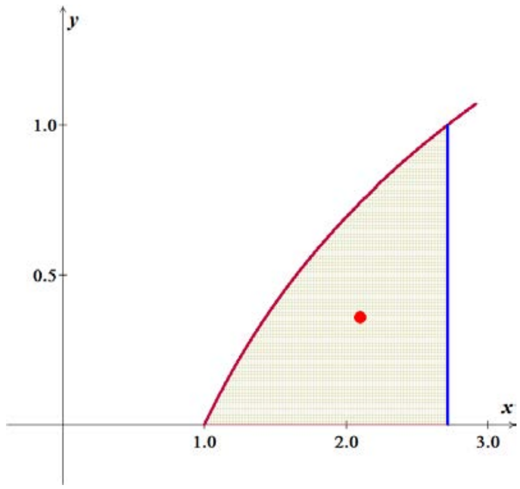
Exercise

Find the mass and centroid (center of mass) of the following thin plates, assuming a constant density. Sketch the region corresponding to the plate and indicate the location of the center of mass. Use symmetry when possible to simplify your work.

The region bounded by $y = \ln x, \quad x - axis, \quad and \quad x = e$

Solution

Assume: $\rho = 1$



$$m = \int_1^e \int_0^{\ln x} 1 dydx$$

$$= \int_1^e \ln x dx$$

$$= [x \ln x - x]_1^e = e - e - 0 + 1$$

$$\underline{= 1}$$

$$\bar{x} = \frac{M_y}{m} = \int_1^e \int_0^{\ln x} x dydx$$

$$= \int_1^e x \ln x dx$$

$$= \frac{1}{2}x^2\left(\ln x - \frac{1}{2}\right)\Big|_1^e$$

$$= \frac{1}{2}e^2\left(\frac{1}{2}\right) - \frac{1}{2}\left(-\frac{1}{2}\right)$$

$$\underline{= \frac{1}{4}\left(e^2 + 1\right)}$$

$$\int x \ln x dx = \frac{1}{2}x^2\left(\ln x - \frac{1}{2}\right)$$

$$u = \ln x \quad \rightarrow \quad x = e^u \quad \Rightarrow \quad dx = e^u du$$

$$\int x \ln x dx = \int ue^{2u} du = e^{2u}\left(\frac{1}{2}u - \frac{1}{4}\right)$$

$$= x^2\left(\frac{1}{2}\ln x - \frac{1}{4}\right)$$

$$\bar{y} = \frac{M_x}{m} = \int_1^e \int_0^{\ln x} y dydx$$

$$= \frac{1}{2}\int_1^e (\ln x)^2 dx$$

$$= \frac{1}{2}x\left((\ln x)^2 - 2\ln x + 2\right)\Big|_1^e$$

$$= \frac{1}{2}\left[e(1 - 2 + 2) - 2\right]$$

$$\underline{= \frac{1}{2}e - 1}$$

So the center of mass is $\left(\frac{1}{4}e^2 + \frac{1}{4}, \ \frac{1}{2}e - 1\right)$

Exercise

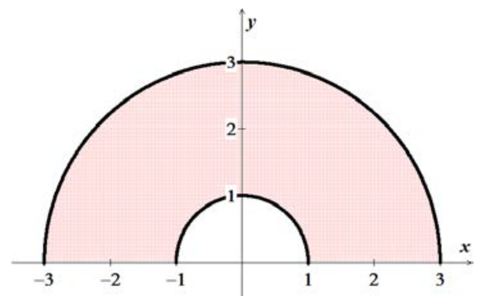
Find the mass and centroid (center of mass) of the following thin plates, assuming a constant density. Sketch the region corresponding to the plate and indicate the location of the center of mass. Use symmetry when possible to simplify your work.

The region bounded by $x^2 + y^2 = 1 \quad and \quad x^2 + y^2 = 9, \quad for \;\; y \ge 0$

Solution

Assume: $\rho = 1$

$x^2 + y^2 = 1 = r^2 \qquad x^2 + y^2 = 9 = r^2 \qquad \underline{1 \le r \le 3}$



$$m = \int_0^\pi \int_1^3 r\,dr\,d\theta$$

$$= [\theta]_0^\pi \left[\tfrac{1}{2}r^2\right]_1^3$$

$$\underline{= 4\pi}$$

By symmetry $\bar{x} = 0$ (*clearly*).

$$\bar{y} = \frac{M_x}{m} = \frac{1}{4\pi}\int_0^\pi \int_1^3 r^2 \sin\theta \, dr\,d\theta$$

$$= \frac{1}{4\pi}\int_0^\pi \sin\theta\, d\theta \int_1^3 r^2\, dr$$

$$= \frac{1}{4\pi}[-\cos\theta]_0^\pi \left[\tfrac{1}{3}r^3\right]_1^3 \qquad = \frac{1}{4\pi}(2)\left(\frac{26}{3}\right)$$

$$\underline{= \frac{13}{3\pi}}$$

$\therefore$ The center of mass is $\left(0, \;\; \frac{13}{3\pi}\right)$

Exercise

Find the coordinates of the center of mass of the following plane regions with variable density. Describe the distribution of mass in the region

$$R = \{(x, y):\ 0 \le x \le 4,\ 0 \le y \le 2\}; \quad \rho(x, y) = 1 + \frac{x}{2}$$

Solution

$$m = \int_0^4 \int_0^2 \left(1 + \frac{x}{2}\right) dydx$$

$$= 2\int_0^4 \left(1 + \frac{x}{2}\right) dx$$

$$= 2\left[x + \frac{1}{4}x^2\right]_0^4$$

$$= 16$$

$$\overline{x} = \frac{M_y}{m} = \frac{1}{16}\int_0^4 \int_0^2 \left(x + \frac{1}{2}x^2\right) dydx$$

$$= \frac{1}{8}\int_0^4 \left(x + \frac{1}{2}x^2\right) dx$$

$$= \frac{1}{8}\left[\frac{1}{2}x^2 + \frac{1}{6}x^3\right]_0^4$$

$$= \frac{1}{8}\left(8 + \frac{32}{3}\right)$$

$$= \frac{7}{3}$$

$$\overline{y} = \frac{M_x}{m} = \frac{1}{16}\int_0^4 \int_0^2 y\left(1 + \frac{x}{2}\right) dydx$$

$$= \frac{1}{16}\int_0^4 \left(1 + \frac{x}{2}\right)\left[\frac{1}{2}y^2\right]_0^2 dx$$

$$= \frac{1}{8}\int_0^4 \left(1 + \frac{x}{2}\right) dx$$

$$= \frac{1}{8}\left[x + \frac{1}{4}x^2\right]_0^4$$

$$= 1$$

$\therefore$ The center of mass is $\left(\frac{7}{3},\ 1\right)$

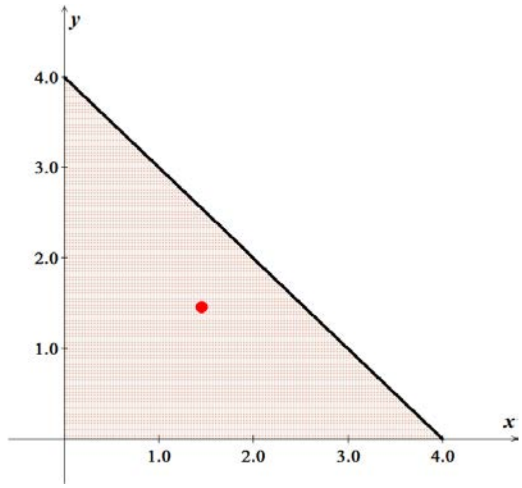
The density of the plate increases as you move toward the right.

Exercise

Find the coordinates of the center of mass of the following plane regions with variable density. Describe the distribution of mass in the region

The triangular plate in the first quadrant bounded by $x+y=4$ with $\rho(x,\ y)=1+x+y$

Solution



$$m=\int_0^4\int_0^{4-x}(1+x+y)\,dydx$$

$$=\int_0^4\left[y+xy+\frac{1}{2}y^2\right]_0^{4-x}dx$$

$$=\int_0^4\left[4-x+4x-x^2+\frac{1}{2}(4-x)^2\right]dx$$

$$=\int_0^4\left[4+3x-x^2+\frac{1}{2}\left(16-8x+x^2\right)\right]dx$$

$$=\int_0^4\left(12-x-\frac{1}{2}x^2\right)dx$$

$$=\left[12x-\frac{1}{2}x^2-\frac{1}{6}x^3\right]_0^4 \qquad =48-8-\frac{32}{3}$$

$$=\frac{88}{3}$$

By symmetry $\overline{x}=\overline{y}$

$$\overline{x}=\frac{M_y}{m}=\frac{3}{88}\int_0^4\int_0^{4-x}\left(x+x^2+xy\right)dydx$$

$$=\frac{3}{88}\int_0^4\left[xy+x^2y+\frac{1}{2}xy^y\right]_0^{4-x}dx$$

$$=\frac{3}{88}\int_0^4\left[4x-x^2+4x^2-x^3+\frac{1}{2}x\left(16-8x+x^2\right)\right]dx$$

$$=\frac{3}{88}\int_0^4\left(12x-x^2-\frac{1}{2}x^3\right)dx$$

$$=\frac{3}{88}\left[6x^2-\frac{1}{3}x^3-\frac{1}{8}x^4\right]_0^4 \qquad =\frac{3}{88}\left(96-\frac{64}{3}-32\right)$$

$$=\frac{16}{11}$$

$\therefore$ The center of mass is $\left(\frac{16}{3},\ \frac{16}{3}\right)$

Exercise

Find the coordinates of the center of mass of the following plane regions with variable density. Describe the distribution of mass in the region

The upper half $(y \geq 0)$ of the disk bounded by the circle $x^2 + y^2 = 4$ with $\rho(x, y) = 1 + \frac{y}{2}$

Solution

$$m = \int_0^{\pi} \int_0^2 \left(1 + \frac{r\sin\theta}{2}\right) rdrd\theta$$

$y = r\sin\theta$

$$= \int_0^{\pi} \int_0^2 \left(r + \frac{\sin\theta}{2} r^2\right) drd\theta$$

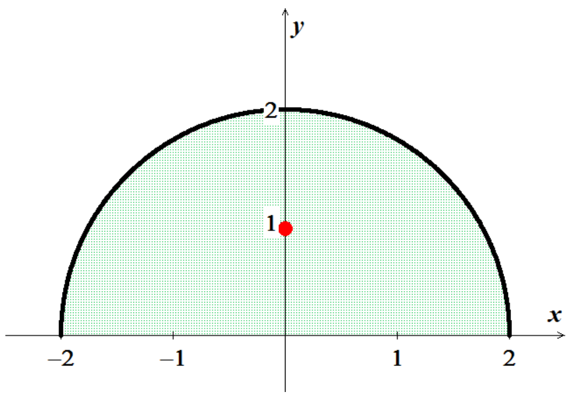
$$= \int_0^{\pi} \left[\frac{1}{2}r^2 + \frac{1}{6}(\sin\theta)r^3\right]_0^2 d\theta$$

$$= \int_0^{\pi} \left(2 + \frac{4}{3}\sin\theta\right) d\theta$$

$$= \left[2\theta - \frac{4}{3}\cos\theta\right]_0^{\pi}$$

$$= \left(2\pi + \frac{8}{3}\right)$$

$$= \frac{6\pi + 8}{3}$$



By symmetry $\bar{x} = 0$

$$\bar{y} = \frac{M_x}{m} = \frac{3}{6\pi + 8} \int_0^{\pi} \int_0^2 r\sin\theta\left(1 + \frac{r\sin\theta}{2}\right) rdrd\theta$$

$$= \frac{3}{6\pi + 8} \int_0^{\pi} \int_0^2 \left(r^2 \sin\theta + \frac{1}{2}r^3 \sin^2\theta\right) drd\theta$$

$$= \frac{3}{6\pi + 8} \int_0^{\pi} \left[\frac{1}{3}r^3 \sin\theta + \frac{1}{8}r^4 \sin^2\theta\right]_0^2 d\theta$$

$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$

$$= \frac{3}{6\pi + 8} \int_0^{\pi} \left(\frac{8}{3}\sin\theta + 1 - \cos 2\theta\right) d\theta$$

$$= \frac{3}{6\pi + 8} \left[-\frac{8}{3}\cos\theta + \theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi}$$

$$= \frac{3}{6\pi + 8} \left(\frac{8}{3} + \pi + \frac{8}{3}\right)$$

$$= \frac{3\pi + 16}{6\pi + 8}$$

$\therefore$ The center of mass is $\left(0, \ \frac{3\pi + 16}{6\pi + 8}\right)$

The density increases as the plate is moved up.

Exercise

Find the coordinates of the center of mass of the following plane regions with variable density. Describe the distribution of mass in the region

The upper half $(y \ge 0)$ of the disk bounded by the ellipse $x^2 + 9y^2 = 9$ with $\rho(x, y) = 1 + y$

Solution

$$m = \int_{-3}^{3}\int_{0}^{\frac{\sqrt{9-x^2}}{3}} (1+y)\ dydx$$

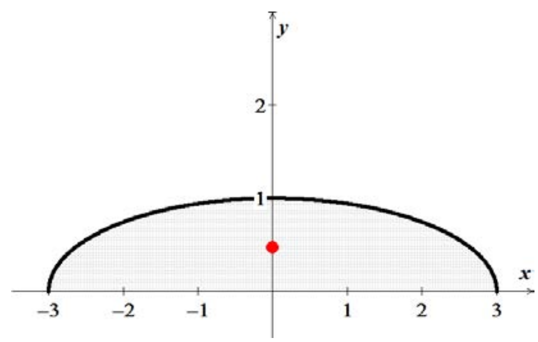
$$= \int_{-3}^{3} \left[y + \frac{1}{2}y^2 \right]_0^{\frac{\sqrt{9-x^2}}{3}} dx$$

$$= \int_{-3}^{3} \left(\frac{1}{3}\sqrt{9-x^2} + \frac{1}{2} - \frac{1}{18}x^2 \right) dx \qquad \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$= 2\left[\frac{x}{6}\sqrt{9-x^2} + \frac{9}{6}\sin^{-1}\frac{x}{3} + \frac{1}{2}x - \frac{1}{54}x^3 \right]_0^3$$

$$= 2\left(\frac{9}{6}\sin^{-1}1 + \frac{3}{2} - \frac{1}{2} \right)$$

$$= \frac{3\pi + 4}{2}$$



By symmetry $\overline{x} = 0$

$$\overline{y} = \frac{M_x}{m} = \frac{2}{3\pi + 4}\int_{-3}^{3}\int_{0}^{\frac{\sqrt{9-x^2}}{3}} \left(y + y^2\right)\ dydx$$

$$= \frac{2}{3\pi + 4}\int_{-3}^{3} \left[\frac{1}{2}y^2 + \frac{1}{3}y^3 \right]_0^{\frac{\sqrt{9-x^2}}{3}} dx$$

$$= \frac{2}{3\pi + 4}\int_{-3}^{3} \left[\frac{1}{18}\left(9 - x^2\right) + \frac{1}{81}\left(9 - x^2\right)^{3/2} \right] dx$$

$$= \frac{4}{3\pi+4}\left[\frac{1}{18}\left(9x - \frac{1}{3}x^3\right) + \frac{3}{8}\sin^{-1}\frac{x}{3} + \frac{1}{18}x\sqrt{9-x^2} + \frac{1}{36}x\sqrt{9-x^2}\left(\frac{9-2x^2}{9}\right) \right]_0^3$$

$$= \frac{4}{3\pi + 4}\left(\frac{1}{18}(27 - 9) + \frac{3}{8}\frac{\pi}{2} \right)$$

$$= \frac{4}{3\pi+4}\left(1 + \frac{3\pi}{16}\right)$$

$$= \frac{3\pi + 16}{12\pi + 16}$$

$x = 3\sin\theta \quad \to dx = 3\cos\theta\, d\theta$

$9 - x^2 = 9\cos^2\theta$

$$\int \left(9 - x^2\right)^{3/2} dx = \int (3\cos\theta)^3 (3\cos\theta)\, d\theta$$

$$= 81\int \left(\frac{1 + \cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{81}{4}\int \left(1 + 2\cos 2\theta + \cos^2 2\theta\right) d\theta$$

$$= \frac{81}{4}\int \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{81}{4}\left(\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right)$$

$$= \frac{81}{4}\left(\frac{3}{2}\sin^{-1}\frac{x}{3} + \frac{2x\sqrt{9-x^2}}{9} + \frac{x\sqrt{9-x^2}}{9}\left(1 - \frac{2x^2}{9}\right)\right)$$

$\therefore$ The center of mass is $\left(0,\ \frac{3\pi + 16}{12\pi + 16}\right)$, the density increases as the plate is moved up.

Exercise

Find the center of mass of the following solids, assuming a constant density of 1. Sketch the region and indicate the location of the centroid. Use symmetry when possible and choose a convenient coordinate system.

The upper half of the ball $\quad x^2+y^2+z^2 \le 16 \quad (\text{for } z \ge 0)$

Solution

Assume: $\rho = 1$

The mass is the volume of a half-sphere of radius 4: $\frac{1}{2}\frac{4\pi}{3}4^3 = \boxed{\frac{128\pi}{3} = m}$

In spherical coordinates $z = \rho\cos\phi$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{3}{128\pi}\int_0^{2\pi}\int_0^{\pi/2}\int_0^4 \rho\cos\phi\; \rho^2\sin\phi\; d\rho\, d\phi\, d\theta$$

$$= \frac{3}{128\pi}\int_0^{2\pi} d\theta \int_0^{\pi/2}\frac{1}{2}\sin 2\phi d\phi \int_0^a \rho^3\, d\rho \qquad\qquad 2\sin\phi\cos\phi = \sin 2\phi$$

$$= \left(\frac{3}{128\pi}\right)\ \theta\Big|_0^{2\pi}\ \left[-\frac{1}{4}\cos 2\phi\Big|_0^{\pi/2}\right. \ \left[\frac{1}{4}\rho^4\right]_0^4$$

$$= \left(\frac{3}{128\pi}\right)(2\pi)\left(\frac{1}{2}\right)(64)$$

$$= \boxed{\frac{3}{2}}$$

$\therefore$ The center of mass is $\left(0,\ \ 0,\ \ \frac{3}{2}\right)$

Exercise

Find the center of mass of the following solids, assuming a constant density of 1. Sketch the region and indicate the location of the centroid. Use symmetry when possible and choose a convenient coordinate system.

The region bounded by the paraboloid $\quad z = x^2 + y^2$ and the plane $z = 25$

Solution

Given: $\rho = 1$

$z = x^2 + y^2 = 25 = r^2 \quad \to \quad r = 5$

$$m = \int_0^{2\pi}\int_0^5\int_{r^2}^{25} r\, dz\, dr\, d\theta$$

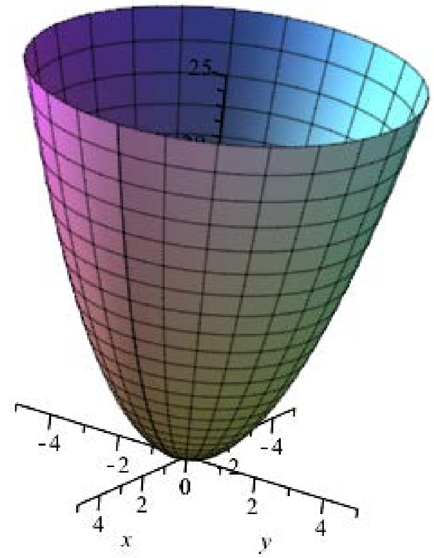
$$= \int_0^{2\pi}\int_0^5 rz\Big|_{r^2}^{25}\ dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^5 \left(25r - r^3\right) dr$$

$$= \left[\theta\right]_0^{2\pi} \left[\frac{25}{2}r^2 - \frac{1}{4}r^4\right]_0^5$$

$$= (2\pi)\left(5^4\right)\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{625\pi}{2}$$



By symmetry $\bar{x} = \bar{y} = 0$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{2}{625\pi} \int_0^{2\pi} \int_0^5 \int_{r^2}^{25} rz \, dz \, dr \, d\theta$$

$$= \frac{1}{625\pi} \int_0^{2\pi} \int_0^5 rz^2 \Big|_{r^2}^{25} dr \, d\theta$$

$$= \frac{1}{625\pi} \int_0^{2\pi} d\theta \int_0^5 \left(625r - r^5\right) dr$$

$$= \frac{1}{625\pi}(2\pi)\left[\frac{5^4}{2}r^2 - \frac{1}{6}r^6\right]_0^5 \qquad = \frac{2}{5^4}\left(5^6\right)\left(\frac{1}{2} - \frac{1}{6}\right)$$

$$= \frac{50}{3}$$

$\therefore$ The center of mass is $\left(0, \ 0, \ \frac{50}{3}\right)$

Exercise

Find the center of mass of the following solids, assuming a constant density of 1. Sketch the region and indicate the location of the centroid. Use symmetry when possible and choose a convenient coordinate system.

The tetrahedron in the first octant bounded by $z = 1 - x - y$ and the coordinate planes

Solution

Given: $\rho = 1$

The mass is the volume of a pyramid: $m = V = \frac{1}{3}hA = \frac{1}{3}(1)\left(\frac{1}{2}\right) = \frac{1}{6}$

The region is symmetric with respect to the line $x = y = z \quad \to \quad \bar{x} = \bar{y} = \bar{z}$

$$\bar{z} = \frac{M_{xy}}{m} = 6\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx$$

$$= 3\int_0^1 \int_0^{1-x} z^2 \Big|_0^{1-x-y} dy \, dx$$

$$= 3\int_0^1 \int_0^{1-x} (1-x-y)^2 \, dy\, dx$$

$$= 3\int_0^1 \int_0^{1-x} \left(1 + x^2 + y^2 - 2x - 2y + 2xy\right) dy\, dx$$

$$= 3\int_0^1 \left(y + x^2 y + \frac{1}{3}y^3 - 2xy - y^2 + xy^2\right)\Big|_0^{1-x} dx$$

$$= 3\int_0^1 \left(1 - x + x^2 - x^3 + \frac{1}{3}\left(1 - 3x + 3x^2 - x^3\right) - 2x + 2x^2 - 1 + 2x - x^3 + x - 2x^2 + x^3\right) dx$$

$$= 3\int_0^1 \left(\frac{1}{3} - x + x^2 - \frac{1}{3}x^3\right) dx$$

$$= 3\left(\frac{1}{3}x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{12}x^4\right)\Big|_0^1 \qquad = 3\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{3} - \frac{1}{12}\right)$$

$$= \frac{1}{4}$$

$\therefore$ The center of mass is $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

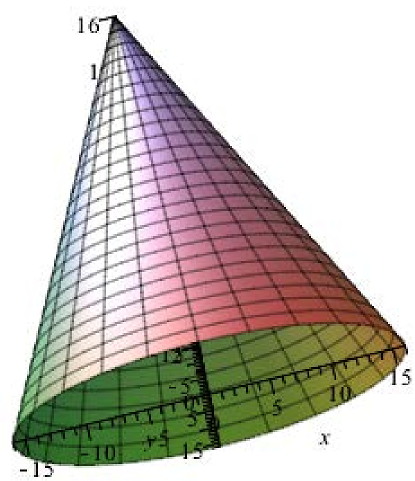
Exercise

Find the center of mass of the following solids, assuming a constant density of 1. Sketch the region and indicate the location of the centroid. Use symmetry when possible and choose a convenient coordinate system.

The solid bounded by the cone $z = 16 - r$ and the plane $z = 0$

Solution

Given: $\rho = 1$



$$m = \int_0^{2\pi} \int_0^{16} \int_0^{16-r} r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{16} rz \Big|_0^{16-r} dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{16} \left(16r - r^2\right) dr\, d\theta$$

$$= (2\pi)\left[8r^2 - \frac{1}{3}r^3\right]_0^{16}$$

$$= (2\pi)\left(2048 - \frac{4096}{3}\right)$$

$$= \frac{4096\pi}{3}$$

By symmetry $\overline{x} = \overline{y} = 0$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{3}{4096\pi}\int_0^{2\pi}\int_0^{16}\int_0^{16-r} rz\, dz\, dr\, d\theta$$

$$= \frac{3}{4096\pi}\int_0^{2\pi} d\theta \int_0^{16}\left.\frac{1}{2}rz^2\right]_0^{16-r} dr$$

$$= \frac{3}{4096}\int_0^{16}\left(256r - 32r^2 + r^3\right) dr$$

$$= \frac{3}{4096}\left[128r^2 - \frac{32}{3}r^3 + \frac{1}{4}r^3\right]_0^{16}$$

$$= 4$$

∴ The center of mass is $(0,\ 0,\ 4)$

Exercise

Consider the thin constant-density plate $\{(r,\ \theta):\ a \le r \le 1,\ 0 \le \theta \le \pi\}$ bounded by two semicircles and the *x*-axis.

a) Find the graph the *y*-coordinate of the center of mass of the plate as a function of a.

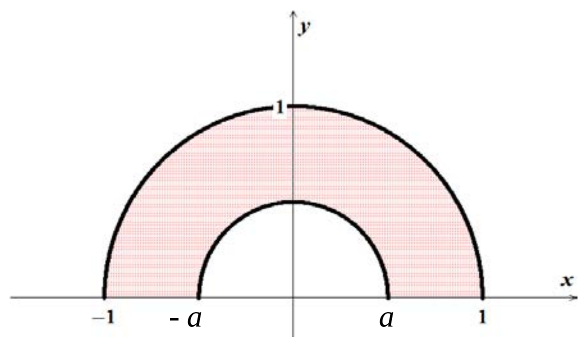
b) For what value of a is the center of mass on the edge of the plate?

Solution

a) $$m = \int_0^{\pi}\int_a^1 r dr d\theta$$

$$= [\theta]_0^{\pi}\ \left[\frac{1}{2}r^2\right]_a^1$$

$$= \frac{\pi}{2}\left(1 - a^2\right)$$



By symmetry $\bar{x} = 0$ (*clearly*).

$$\bar{y} = \frac{M_x}{m} = \frac{2}{\pi\left(1-a^2\right)}\int_0^{\pi}\int_a^1 r^2\sin\theta\, dr d\theta$$

$$= \frac{2}{\pi\left(1-a^2\right)}\int_0^{\pi}\sin\theta d\theta\int_a^1 r^2\, dr$$

$$= \frac{2}{\pi\left(1-a^2\right)}\left[-\cos\theta\right]_0^{\pi}\ \left[\frac{1}{3}r^3\right]_a^1$$

$$= \frac{4\left(1-a^3\right)}{3\pi\left(1-a^2\right)}$$

$$1 - a^3 = (1-a)\left(1 + a + a^2\right)$$

$$= \frac{4\left(1 + a + a^2\right)}{3\pi(1+a)}$$

b) Since the center of mass has $\bar{x}=0$, therefore it lies on *y*-axis on the edge of the plate exactly

when $\dfrac{4\left(1+a+a^2\right)}{3\pi(1+a)}=a \ \ or \ \ 1$

$$\frac{4\left(1+a+a^2\right)}{3\pi(1+a)}=a$$

$$4+4a+4a^2=3\pi a+3\pi a^2$$

$$(3\pi-4)a^2+(3\pi-4)a-4=0$$

$$a=\frac{-(3\pi-4)\pm\sqrt{(3\pi-4)^2+16(3\pi-4)}}{2(3\pi-4)}$$

$$=\frac{-3\pi+4\pm\sqrt{(3\pi-4)(3\pi+12)}}{2(3\pi-4)}$$

$$\boxed{\approx 0.49366} \quad \cancel{\approx -1.49}$$

$$\frac{4\left(1+a+a^2\right)}{3\pi(1+a)}=1$$

$$4+4a+4a^2=3\pi+3\pi a$$

$$4a^2+(4-3\pi)a+4-3\pi=0$$

$$a=\frac{-4+3\pi\pm\sqrt{(4-3\pi)^2-16(4-3\pi)}}{8}$$

$a\approx \cancel{\begin{cases}-.67\\ 2.02\end{cases}}$ outside the range $0\le a\le 1$

Exercise

Consider the thin constant-density plate $\left\{(\rho,\ \phi,\ \theta):\ 0<a\le\rho\le 1,\ 0\le\phi\le\frac{\pi}{2},\ 0\le\theta\le 2\pi\right\}$ bounded by two hemispheres and the *xy*-axis.

a) Find the graph the *z*-coordinate of the center of mass of the plate as a function of *a*.

b) For what value of *a* is the center of mass on the edge of the solid?

Solution

a) $m=\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\int_a^1 \rho^2\sin\phi\ d\rho\ d\phi\ d\theta$

$$=\frac{1}{3}\int_0^{2\pi}d\theta\ \int_0^{\pi/2}\sin\phi\ d\phi\ \left[\rho^3\right|_a^1$$

$$=\frac{2\pi}{3}\left[-\cos\phi\right]_0^{\pi/2}\left(1-a^3\right)$$

$$\boxed{=\frac{2\pi}{3}\left(1-a^3\right)}$$

By symmetry $\bar{x}=0$ (*clearly*).

$$\bar{z}=\frac{3}{2\pi}\frac{1}{1-a^3}\int_0^{2\pi}\int_0^{\pi/2}\int_a^1(\rho\cos\phi)\rho^2\sin\phi\ d\rho\ d\phi\ d\theta$$

$$=\frac{3}{2\pi}\frac{1}{1-a^3}\int_0^{2\pi}d\theta\ \int_0^{\pi/2}\frac{1}{2}\sin 2\phi\ d\phi\ \int_a^1\rho^3\ d\rho$$

$$=\frac{3}{2\pi}\frac{1}{1-a^3}\frac{1}{2}(2\pi)\left[-\frac{1}{2}\cos 2\phi\right]_0^{2\pi}\left(\frac{1}{4}\left(1-a^4\right)\right)$$

$$= \frac{3}{8}\frac{1-a^4}{1-a^3}$$

b) Since the center of mass has $\bar{x} = \bar{y} = 0$, therefore it lies on *z*-axis on the edge of the plate exactly when $\frac{3-3a^4}{8-8a^3} = a$ *or* 1

$$\frac{3-3a^4}{8-8a^3} = a$$

$$3-3a^4 = 8a - 8a^4$$

$$5a^4 - 8a + 3 = 0$$

$$a = \frac{\left(1450+450\sqrt{11}\right)^{2/3} - 5\left(1450+450\sqrt{11}\right)^{1/3} - 50}{15\left(1450+450\sqrt{11}\right)^{1/3}}$$

$$\approx 0.38936$$

$$\frac{3-3a^4}{8-8a^3} = 1$$

$$3-3a^4 = 8 - 8a^3$$

$$-3a^4 + 8a^3 - 5 = 0$$

outside the range $0 \le a \le 1$

Exercise

A cylindrical soda can has a radius of 4 *cm* and a height of 12 *cm*. When the can is full of soda, the center of mass of the contents of the can is 6 cm above the base on the axis of the can (halfway along the axis of the can). As the can is drained, the center of mass descends for a while. However, when the can is empty (filled only with air), the center of mass is once again 6 *cm* above the base on the axis of the can. Find the depth of soda in the can for which the center of mass is at its lowest point. Neglect the mass of the can, and assume the density of the soda is $1\ g/cm^3$ and the density of air is $0.001\ g/cm^3$.

Solution

Volume of a full soda can: $V = 2\pi\ \rho r^2 h = 16\pi h$

Volume of air in can: $V = 2\pi\ \rho_2 r^2 (12-h) = 16\pi(0.001)(12-h) = \frac{16}{1000}\pi(12-h)$

Mass: $m = 16\pi h + \frac{16\pi(12-h)}{1000} = 16\pi\left(\frac{999h+12}{1000}\right) = \frac{6\pi}{125}(333h+4)$

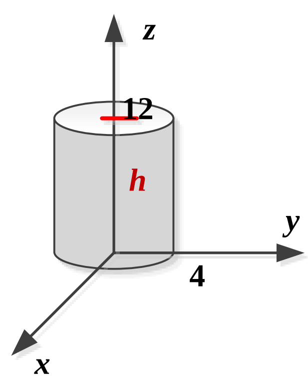
OR

$$m = \int_0^{2\pi}\int_0^4\left(\int_0^h \rho_1 dz + \int_h^{12}\rho_2 dz\right) r dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^4 r dr\left(\int_0^h dz + \frac{1}{1000}\int_h^{12} dz\right)$$

$$= (2\pi)\frac{1}{2}(16)\left(h + \frac{1}{1000}(12-h)\right)$$

$$= 16\pi h + \frac{16\pi(12-h)}{1000}$$



$$\bar{z} = \frac{125}{6\pi(333h+4)} \int_0^{2\pi} \int_0^4 \left(\int_0^h zdz + \frac{1}{1000} \int_h^{12} zdz \right) rdrd\theta$$

$$= \frac{125}{6\pi(333h+4)} \int_0^{2\pi} d\theta \int_0^4 rdr \left(\int_0^h zdz + \frac{1}{1000} \int_h^{12} zdz \right)$$

$$= \frac{125}{6\pi(333h+4)} (2\pi) \left[\frac{1}{2} r^2 \right]_0^4 \left(\left[\frac{1}{2} z^2 \right]_0^h + \frac{1}{1000} \left[\frac{1}{2} z^2 \right]_h^{12} \right)$$

$$= \frac{125}{3(333h+4)} (8) \left(\frac{1}{2} \right) \left(h^2 + \frac{144}{1000} - \frac{h^2}{1000} \right)$$

$$= \frac{125}{3} \cdot \frac{4}{1000} \cdot \frac{999h^2 144}{333h+4}$$

$$= \frac{333h^2+48}{666h+8}$$

For the lowest center of mass point when the derivative of the function is zero.

$$\left(\frac{333h^2+48}{666h+8} \right)' = \frac{666h(666h+8) - 666(333h^2+48)}{(666h+8)^2} = 0$$

$$666h^2 + 8h - 333h^2 - 48 = 0$$

$$333h^2 + 8h - 48 = 0$$

$$h = \frac{-8 + \sqrt{64 + 63936}}{666} \approx 0.367841$$

∴ The depth of soda in the can for which the center of mass is at its lowest point ≈ 0.367841

Solution   *Section* 3.7 – Change of Variables in Multiple Integrals

Exercise

a) Solve the system $u = x - y, \quad v = 2x + y$ for *x* and *y* in terms of *u* and *v*. Then find the value of the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$

b) Find the image under the transformation $u = x - y, \quad v = 2x + y$ of the triangular region with vertices (0, 0), (1, 1), and (1, −2) in the *xy*-plane. Sketch the transformed region in the *uv*-plane.

Solution

a) $\begin{array}{l} u = x - y \\ v = 2x + y \end{array} \quad \to \quad \begin{cases} x = \frac{1}{3}u + \frac{1}{3}v \\ y = -\frac{2}{3}u + \frac{1}{3}v \end{cases}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \underline{\frac{1}{3}}$$

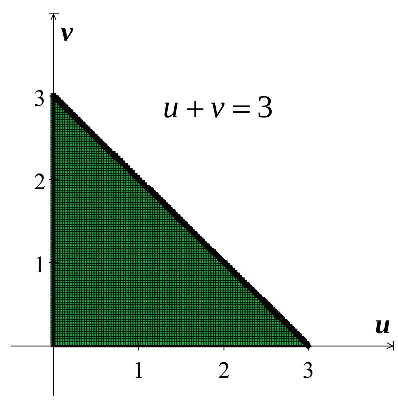
b) From (0, 0) to (1, 1) $\Rightarrow \ y = x \ \to u = x - y = 0$

From (0, 0) to (1, −2) $\Rightarrow \ y = -2x \ \to u = 2x + y = 0$

From (1, 1) to (1, −2) $\Rightarrow \ x = 1$

$$\to x = \frac{1}{3}u + \frac{1}{3}v = 1$$

$$u + v = 3$$



OR: $(0,\ 0) \ \to \ \begin{cases} u = 0 \\ v = 0 \end{cases}$

$(1,\ 1) \ \to \ \begin{cases} u = 0 \\ v = 3 \end{cases}$

$(1,\ -2) \ \to \ \begin{cases} u = 3 \\ v = 0 \end{cases}$

Exercise

Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy=1,\ xy=9$ and the lines $y=x,\ y=4x$. Use the transformation $x=\frac{u}{v},\ y=uv$ with $u>0,\ and\ v>0$ to rewrite

$$\iint_R \left(\sqrt{\frac{y}{x}}+\sqrt{xy}\right)dxdy$$

As an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G.

Solution

$$\begin{matrix} x=\frac{u}{v} \\ y=uv \end{matrix} \quad \to \quad \begin{matrix} u=xv \\ y=xv^2 \end{matrix} \quad \begin{cases} \frac{y}{x}=v^2 \\ xy=u^2 \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix}=\frac{u}{v}+\frac{u}{v}=\boxed{2\frac{u}{v}}$$

$$\begin{matrix} xy=1=u^2 \\ xy=9=u^2 \end{matrix} \quad \to \quad \begin{cases} u=1 \\ u=3 \end{cases} \qquad \begin{matrix} y=x \Rightarrow \frac{y}{x}=1=v^2 \\ y=4x \Rightarrow \frac{y}{x}=4=v^2 \end{matrix} \quad \to \quad \begin{cases} v=1 \\ v=2 \end{cases}$$

$$\iint_R \left(\sqrt{\frac{y}{x}}+\sqrt{xy}\right)dxdy=\int_1^3\int_1^2 (v+u)\frac{2u}{v}dvdu$$

$$=2\int_1^3\int_1^2\left(u+\frac{u^2}{v}\right)dvdu$$

$$=2\int_1^3\left[uv+u^2\ln v\right]_1^2 du$$

$$=2\int_1^3\left(2u+u^2\ln 2-u\right)du$$

$$=2\int_1^3\left(u+u^2\ln 2\right)du$$

$$=2\left[\frac{1}{2}u^2+\frac{1}{3}u^3\ln 2\right]_1^3$$

$$=2\left(\frac{9}{2}+9\ln 2-\frac{1}{2}-\frac{1}{3}\ln 2\right)$$

$$\boxed{=8+\frac{52}{3}\ln 2}$$

Exercise

The area πab of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ can be found by integrating the function $f(x,y)=1$ over the region bounded by the ellipse in the *xy*-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x=au,\ y=bv$ and evaluate the transformed integral over the disk *G*: $u^2+v^2\le 1$ in the *uv*-plane. Find the area this way.

Solution

$$x=au,\ y=bv$$

$$\frac{\partial(x,y)}{\partial(u,v)}=J(u,\ v)=\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}=\underline{ab}$$

$$u^2+v^2\le 1 \quad \to \quad -1\le u\le 1$$

$$u^2+v^2\le 1 \quad \to \quad v^2\le 1-u^2 \Rightarrow \quad -\sqrt{1-u^2}\le v\le \sqrt{1-u^2}$$

$$\iint_R dxdy=\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab\ dvdu$$

$$=ab\int_{-1}^{1}\left(\sqrt{1-u^2}+\sqrt{1-u^2}\right)du$$

$$=2ab\int_{-1}^{1}\left(1-u^2\right)^{1/2}du \qquad \int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$=2ab\left[\frac{u}{2}\sqrt{1-u^2}+\frac{1}{2}\sin^{-1}u\right]_{-1}^{1}$$

$$=2ab\left[\frac{1}{2}\sin^{-1}1-\left(\frac{1}{2}\sin^{-1}(-1)\right)\right]$$

$$=2ab\left[\frac{1}{2}\frac{\pi}{2}-\left(-\frac{1}{2}\frac{\pi}{2}\right)\right]$$

$$=2ab\left(\frac{\pi}{2}\right)$$

$$\underline{=ab\pi}$$

Exercise

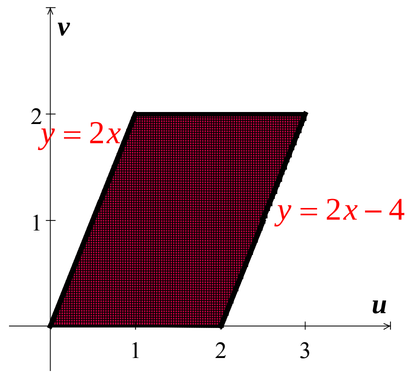
Use the transformation $x = u + \frac{1}{2}v, \;\; y = v$ to evaluate the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3 (2x-y) e^{(2x-y)^2} dxdy$$

By first writing it as an integral over a region *G* in the *uv*-plane.

Solution

$$\frac{\partial(x,y)}{\partial(u,v)} = J(u,\ v) = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = \underline{1}$$



$$\begin{aligned} x &= u + \tfrac{1}{2}v \\ y &= v \end{aligned} \quad \rightarrow \quad \begin{aligned} u &= x - \tfrac{1}{2}y \\ v &= y \end{aligned}$$

$x = \frac{y}{2} \rightarrow y = 2x$

$x = \frac{y+4}{2} \rightarrow y = 2x - 4$

$0 \le x \le 2$

$x = \frac{y}{2}$	$u = x - \frac{y}{2} = \frac{y}{2} - \frac{y}{2} = 0$	$u = 0$
$x = \frac{y}{2} + 2$	$u = x - \frac{y}{2} = \frac{y}{2} + 2 - \frac{y}{2} = 2$	$u = 2$
$y = 0$	$v = 0$	$v = 0$
$y = 2$	$v = 2$	$v = 2$

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3 (2x-y) e^{(2x-y)^2} dxdy = \int_0^2 \int_0^2 v^3 (2u) e^{4u^2} dudv \qquad d\left(4u^2\right) = 8udu$$

$$= \frac{1}{4} \int_0^2 \int_0^2 v^3 e^{4u^2} d\left(4u^2\right) dv$$

$$= \frac{1}{4} \int_0^2 v^3 \left[e^{4u^2} \right]_0^2 dv$$

$$= \frac{1}{4}\left(e^{16} - 1\right) \int_0^2 v^3 dv$$

$$= \frac{1}{4}\left(e^{16} - 1\right) \left[\frac{1}{4} v^4 \right]_0^2$$

$$\underline{= e^{16} - 1}$$

Exercise

Use the transformation $x=\frac{u}{v},\ \ y=uv$ to evaluate the integral

$$\int_1^2\int_{1/y}^{y}\left(x^2+y^2\right)dxdy+\int_2^4\int_{y/4}^{4/y}\left(x^2+y^2\right)dxdy$$

Solution

$$\begin{aligned} x&=\frac{u}{v} \\ y&=uv \end{aligned} \quad\to\quad \begin{aligned} u&=xv \\ y&=xv^2 \end{aligned} \quad \begin{cases} \frac{y}{x}=v^2 \\ xy=u^2 \end{cases}$$

$$J(u,v)=\begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix}=\frac{u}{v}+\frac{u}{v}=\frac{2u}{v}$$

$x=y$	$\frac{y}{x}=1=v^2$	$v=1$
$x=\frac{1}{y}$	$xy=1=u^2$	$u=1$
$x=\frac{4}{y}$	$xy=4=u^2$	$u=2$
$x=\frac{y}{4}$	$\frac{y}{x}=4=v^2$	$v=2$

$$\begin{aligned} \int_1^2\int_{1/y}^{y}\left(x^2+y^2\right)dxdy+\int_2^4\int_{y/4}^{4/y}\left(x^2+y^2\right)dxdy &= \int_1^2\int_1^2\left(\frac{u^2}{v^2}+u^2v^2\right)\left(\frac{2u}{v}\right)dudv \\ &=2\int_1^2\int_1^2\left(\frac{u^3}{v^3}+u^3v\right)dudv \\ &=2\int_1^2\left(\frac{1}{v^3}+v\right)\left[\frac{1}{4}u^4\right]_1^2 dv \\ &=\frac{1}{2}(16-1)\int_1^2\left(v^{-3}+v\right)dv \\ &=\frac{15}{2}\left[-\frac{1}{2}v^{-2}+\frac{1}{2}v^2\right]_1^2 \\ &=\frac{15}{4}\left[-\frac{1}{4}+4-(-1+1)\right] \\ &=\frac{15}{4}\left(\frac{15}{4}\right) \\ &=\frac{225}{16} \end{aligned}$$

Exercise

Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ of the transformation

a) $x = u\cos v, \quad y = u\sin v$

b) $x = u\sin v, \quad y = u\cos v$

Solution

a) $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} = u\cos^2 v + u\sin^2 v \ \boxed{= u}$$

b) $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sin v & u\cos v \\ \cos v & -u\sin v \end{vmatrix} = -u\sin^2 v - u\cos^2 v \ \boxed{= -u}$$

Exercise

Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ of the transformation

a) $x = u\cos v, \quad y = u\sin v, \quad z = w$

b) $x = 2u - 1, \quad y = 3v - 4, \quad z = \frac{1}{2}(w-4)$

Solution

a) $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u\cos^2 v + u\sin^2 v \ \boxed{= u}$$

b) $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} \ \boxed{= 3}$$

Exercise

Evaluate the appropriate determinant to show that the Jacobian of the transformation from Cartesian $\rho\phi\theta$–space to Cartesian *xyz*–space is $\rho^2 \sin\phi$

Solution

$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \begin{matrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta \\ \cos\phi & -\rho\sin\phi \end{matrix}$$

$$= \rho^2 \cos^2\phi \sin\phi \cos^2\theta + \rho^2 \sin^3\phi \sin^2\theta + \rho^2 \sin\phi \cos^2\phi \sin^2\theta + \rho^2 \sin^3\phi \cos^2\theta$$

$$= \rho^2 \cos^2\phi \sin\phi \left(\cos^2\theta + \sin^2\theta\right) + \rho^2 \sin^3\phi \left(\sin^2\theta + \cos^2\theta\right)$$

$$= \rho^2 \cos^2\phi \sin\phi + \rho^2 \sin^3\phi$$

$$= \rho^2 \sin\phi \left(\cos^2\phi + \sin^2\phi\right)$$

$$= \rho^2 \sin\phi$$

Exercise

How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.

Solution

Let $u = g(x) \quad \Rightarrow \quad J(x) = \frac{du}{dx} = g'(x)$

$$\int_a^b f(u)\,du = \int_{g(a)}^{g(b)} f\left(g(x)\right) g'(x)\,dx$$

$g'(x)$ represents the Jacobian of the transformation $u = g(x) \quad or \quad x = g^{-1}(u)$

Exercise

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

(***Hint***: Let $x = au,\ y = bv,\ and\ z = cw$. Then find the volume of an appropriate region in *uvw*-space)

Solution

$$J(u,v,w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \Rightarrow \frac{(au)^2}{a^2} + \frac{(bv)^2}{b^2} + \frac{(cw)^2}{c^2} \le 1 \quad \to \quad u^2 + v^2 + w^2 \le 1$$

$$u^2 + v^2 + w^2 \le 1 \quad \Rightarrow \quad V = \frac{4\pi}{3} = \iiint_G du\,dv\,dw$$

$$V = \iiint_R dx\,dy\,dz$$

$$= \iiint\limits_G abc\ dudvdw$$

$$= abc \iiint\limits_G dudvdw$$

$$= \frac{4\pi abc}{3}$$

Exercise

Use the transformation $x = u^2 - v^2,\ \ y = 2uv$ to evaluate the integral

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2}\ dydx$$

(Hint: Show that the image of the triangular region G with vertices (0, 0), (1, 0), (1, 1) in the uv-plane is the region of integration R in the xy-plane defined by the limits of integration.)

Solution

$x = u^2 - v^2,\ \ y = 2uv$

$J(u,v) = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 = \underline{4\left(u^2 + v^2\right)}$

$y = 2\sqrt{1-x}$	$2uv = 2\sqrt{1-u^2+v^2} \to u^2v^2 = 1-u^2+v^2$ $u^2v^2 + u^2 = 1+v^2 \Rightarrow u^2\left(v^2+1\right) = 1+v^2$	$u = \pm 1$
$y = 0$	$2uv = 0$	$u = 0,\ v = 0$
$x = 0$	$u^2 - v^2 = 0$	$u = \pm v$

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2}\ dydx = \int_0^1 \int_0^u \sqrt{\left(u^2-v^2\right)^2 + (2uv)^2} \cdot 4\left(u^2+v^2\right) dvdu$$

$$= 4\int_0^1 \int_0^u \sqrt{u^4 + v^4 - 2u^2v^2 + 4u^2v^2} \cdot \left(u^2+v^2\right) dvdu$$

$$= 4\int_0^1 \int_0^u \sqrt{u^4 + v^4 + 2u^2v^2} \cdot \left(u^2+v^2\right) dvdu$$

$$= 4\int_0^1 \int_0^u \sqrt{\left(u^2+v^2\right)^2} \cdot \left(u^2+v^2\right) dvdu$$

$$= 4\int_0^1 \int_0^u \left(u^2 + v^2\right)^2 dvdu$$

$$= 4\int_0^1 \int_0^u \left(u^4 + v^4 + 2u^2v^2\right) dvdu$$

$$= 4\int_0^1 \left[u^4v + \frac{1}{5}v^5 + \frac{2}{3}u^2v^3\right]_0^u du$$

$$= 4\int_0^1 \left(u^5 + \frac{1}{5}u^5 + \frac{2}{3}u^5\right) du$$

$$= \frac{112}{15}\int_0^1 u^5 du$$

$$= \frac{112}{15}\left[\frac{1}{6}u^6\right]_0^1$$

$$= \frac{56}{45}$$