

Section 4.5 – Partial Fraction Decomposition

1- Decompose $\frac{P}{Q}$, where Q has Only Non-repeated Linear Factor

Under the assumption that Q has only non-repeated linear factors, the polynomial Q has the form

$$Q(x) = (x - a_1)(x - a_2) \cdots (x - a_n)$$

Where no 2 of the number a_1, a_2, \dots, a_n are equal. In this case, the partial fraction decomposition of $\frac{P}{Q}$ is of the form

$$\frac{P}{Q} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} \cdots + \frac{A_n}{x - a_n}$$

Where the numbers A_1, A_2, \dots, A_n are to be determined.

Example

Write the partial fraction decomposition of $\frac{x}{x^2 - 5x + 6}$

Solution

First factor the denominator, $x^2 - 5x + 6 = (x - 2)(x - 3)$

$$\frac{x}{x^2 - 5x + 6} = \frac{A}{x - 2} + \frac{B}{x - 3}$$

$$\frac{x}{x^2 - 5x + 6} = \frac{A(x - 3) + B(x - 2)}{(x - 2)(x - 3)}$$

$$x = Ax - 3A + Bx - 2B$$

$$x = (A + B)x - 3A - 2B \qquad 1x + 0 = (A + B)x - 3A - 2B$$

$$x \quad A + B = 1$$

$$x^0 \quad -3A - 2B = 0$$

$$A = \begin{vmatrix} 1 & 1 \\ 0 & -2 \\ 1 & 1 \\ -3 & -2 \end{vmatrix} = \frac{-2}{1} = -2$$

$$B = 1 - (-2) = 3$$

$$\text{Therefore; } \frac{x}{x^2 - 5x + 6} = \frac{-2}{x - 2} + \frac{3}{x - 3}$$

2- Decompose $\frac{P}{Q}$, where Q has Repeated Linear Factors

If a polynomial Q has a repeated linear factor, say $(x-a)^n$, $n \geq 2$ n is an integer, then in the partial fraction decomposition of $\frac{P}{Q}$, we allow for the terms

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_n}{(x-a)^n}$$

Where the numbers A_1, A_2, \dots, A_n are to be determined.

Example

Write the partial fraction decomposition of $\frac{x+2}{x^3-2x^2+x}$

Solution

First factor the denominator, $x^3 - 2x^2 + x = x(x-1)^2$

$$\frac{x+2}{x^3-2x^2+x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\begin{aligned} x+2 &= A(x-1)^2 + Bx(x-1) + Cx \\ &= Ax^2 - 2Ax + A + Bx^2 - Bx + Cx \end{aligned}$$

$$\begin{array}{lcl} x^2 & A+B=0 & \rightarrow B=-A \underline{=-2} \\ x & -2A-B+C=1 & \rightarrow C=1+4-2 \underline{=3} \\ x^0 & A=2 & \end{array}$$

$$\frac{x+2}{x^3-2x^2+x} = \frac{2}{x} + \frac{-2}{x-1} + \frac{3}{(x-1)^2}$$

$$\frac{x+2}{x^3-2x^2+x} = \frac{2}{x} - \frac{2}{x-1} + \frac{3}{(x-1)^2}$$

Example

Write the partial fraction decomposition of $\frac{x^3-8}{x^2(x-1)^3}$

Solution

$$\frac{x^3-8}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

$$x^3-8 = Ax(x-1)^3 + B(x-1)^3 + Cx^2(x-1)^2 + Dx^2(x-1) + Ex^2$$

$$\text{Let } x=0 \rightarrow -8 = B(-1)^3 \Rightarrow B=8$$

$$x^3-8 = Ax(x-1)^3 + 8(x-1)^3 + Cx^2(x-1)^2 + Dx^2(x-1) + Ex^2$$

$$\text{Let } x=1 \rightarrow 1-8 = E \Rightarrow E=-7$$

$$x^3-8 = Ax(x^3-3x^2+3x-1) + 8(x^3-3x^2+3x-1) + Cx^2(x^2-2x+1) + Dx^2(x-1) - 7x^2$$

$$\begin{aligned} x^3-8-8(x^3-3x^2+3x-1) + 7x^2 \\ = Ax^4 - 3Ax^3 + 3Ax^2 - Ax + Cx^4 - 2Cx^3 + Cx^2 + Dx^3 - Dx^2 \end{aligned}$$

$$\begin{aligned} x^3-8-8x^3+24x^2-24x+8+7x^2 \\ = (A+C)x^4 + (-3A-2C+D)x^3 + (3A+C-D)x^2 - Ax \end{aligned}$$

$$-7x^3+31x^2-24x = (A+C)x^4 + (-3A-2C+D)x^3 + (3A+C-D)x^2 - Ax$$

$$\rightarrow \begin{cases} A+C=0 & C=-A=-24 \\ -3A-2C+D=-7 \\ 3A+C-D=31 \\ -A=-24 & \rightarrow A=24 \end{cases} \quad D = -7 + 3A + 2C = -7 + 72 - 48 = 17$$

$$\frac{x^3-8}{x^2(x-1)^3} = \frac{24}{x} + \frac{8}{x^2} - \frac{24}{x-1} + \frac{17}{(x-1)^2} - \frac{7}{(x-1)^3} \quad \Bigg|$$

3- Decompose $\frac{P}{Q}$, where Q has a Non-repeated Irreducible Quadratic Factor

If Q contains a no-repeated irreducible quadratic factor of the form $ax^2 + bx + c$, then in the partial fraction decomposition of $\frac{P}{Q}$, we allow for the term

$$\frac{Ax + B}{ax^2 + bx + c}$$

Where the numbers A and B are to be determined.

Example

Write the partial fraction decomposition of $\frac{3x-5}{x^3-1}$

Solution

$$\begin{aligned}\frac{3x-5}{x^3-1} &= \frac{3x-5}{(x-1)(x^2+x+1)} \\ &= \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}\end{aligned}$$

$$3x-5 = Ax^2 + Ax + A + Bx^2 + Cx - Bx - C$$

$$x^2 \quad A + B = 0 \quad \rightarrow B = -A$$

$$x \quad A - B + C = 3 \quad (1)$$

$$x^0 \quad A - C = -5 \quad \rightarrow C = A + 5$$

$$(1) \rightarrow A + A + A + 5 = 3$$

$$3A = -2$$

$$A = -\frac{2}{3} \quad B = \frac{2}{3} \quad C = \frac{13}{3} \quad \Bigg|$$

$$\frac{3x-5}{x^3-1} = \frac{-\frac{2}{3}}{x-1} + \frac{\frac{2}{3}x + \frac{13}{3}}{x^2+x+1} \quad \Bigg|$$

$$= -\frac{2}{3} \frac{1}{x-1} + \frac{1}{3} \frac{2x+13}{x^2+x+1}$$

4- Decompose $\frac{P}{Q}$, where Q has a Repeated Irreducible Quadratic Factor

If Q contains a repeated irreducible quadratic factor of the form $(ax^2 + bx + c)^n$, $n \geq 2$, n an integer,

then in the partial fraction decomposition of $\frac{P}{Q}$, we allow for the terms

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \dots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}$$

Where the numbers $A_1, B_1, A_2, B_2, \dots, A_n, B_n$ are to be determined.

Example

Write the partial fraction decomposition of $\frac{x^3 + x^2}{(x^2 + 4)^2}$

Solution

$$\frac{x^3 + x^2}{(x^2 + 4)^2} = \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{(x^2 + 4)^2}$$

$$\begin{aligned} x^3 + x^2 &= (Ax + B)(x^2 + 4) + Cx + D \\ &= Ax^3 + 4Ax + Bx^2 + 4B + Cx + D \end{aligned}$$

$$x^3 \quad \underline{A=1}$$

$$x^2 \quad \underline{B=1}$$

$$x^1 \quad 4A + C = 0 \rightarrow C = -4A = \underline{-4}$$

$$x^0 \quad 4B + D = 0 \rightarrow D = -4B = \underline{-4}$$

$$\underline{\frac{x^3 + x^2}{(x^2 + 4)^2} = \frac{x+1}{x^2 + 4} + \frac{-4x-4}{(x^2 + 4)^2}}$$

Exercises

Section 4.5 – Partial Fraction Decomposition

Write the partial fraction decomposition of each rational expression

1. $\frac{4}{x(x-1)}$

2. $\frac{3x}{(x+2)(x-1)}$

3. $\frac{1}{x(x^2+1)}$

4. $\frac{1}{(x+1)(x^2+4)}$

5. $\frac{x^2}{(x-1)^2(x+1)^2}$

6. $\frac{x+1}{x^2(x-2)^2}$

7. $\frac{x-3}{(x+2)(x+1)^2}$

8. $\frac{x^2+x}{(x+2)(x-1)^2}$

9. $\frac{10x^2+2x}{(x-1)^2(x^2+2)}$

10. $\frac{x^2+2x+3}{(x+1)(x^2+2x+4)}$

11. $\frac{x^2-11x-18}{x(x^2+3x+3)}$

12. $\frac{1}{(2x+3)(4x-1)}$

13. $\frac{x^2+2x+3}{(x^2+4)^2}$

14. $\frac{x^3+1}{(x^2+16)^2}$

15. $\frac{7x+3}{x^3-2x^2-3x}$

16. $\frac{x^2}{x^3-4x^2+5x-2}$

17. $\frac{x^3}{(x^2+16)^3}$

18. $\frac{4}{2x^2-5x-3}$

19. $\frac{2x+3}{x^4-9x^2}$

20. $\frac{x^2+9}{x^4-2x^2-8}$

21. $\frac{y}{y^2-2y-3}$

22. $\frac{x+3}{2x^3-8x}$

23. $\frac{x^2}{(x-1)(x^2+2x+1)}$

24. $\frac{3x^2+x+4}{x^3+x}$

25. $\frac{8x^2+8x+2}{(4x^2+1)^2}$

26. $\frac{1}{x^2+2x}$

27. $\frac{2x+1}{x^2-7x+12}$

28. $\frac{x^2+x}{x^4-3x^2-4}$

29. $\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}$

30. $\frac{3x^2+7x-2}{x^3-x^2-2x}$

31. $\frac{3x^2+2x+5}{(x-1)(x^2-x-20)}$

32. $\frac{5x^2-3x+2}{x^3-2x^2}$

33. $\frac{7x^2-13x+13}{(x-2)(x^2-2x+3)}$

34. $\frac{1}{x^2-5x+6}$

35. $\frac{1}{x^2-5x+5}$

36. $\frac{5x^2+20x+6}{x^3+2x^2+x}$

37. $\frac{2x^3-4x-8}{(x^2-x)(x^2+4)}$

38. $\frac{8x^3+13x}{(x^2+2)^2}$

39. $\frac{1}{x^2-9}$

40. $\frac{2}{9x^2-1}$

41. $\frac{5}{x^2+3x-4}$

42. $\frac{3-x}{3x^2-2x-1}$

43. $\frac{x^2+12x+12}{x^3-4x}$

44. $\frac{5x-2}{(x-2)^2}$