

Solution **Section 3.4 – Triple Integrals**

Exercise

Evaluate the integral $\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2) dz dy dx$

Solution

$$\begin{aligned} \int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2) dz dy dx &= \int_0^1 \int_0^1 \left[x^2 z + y^2 z + \frac{1}{3} z^3 \right]_0^1 dy dx \\ &= \int_0^1 \int_0^1 \left[x^2 + y^2 + \frac{1}{3} \right] dy dx \\ &= \int_0^1 \left[x^2 y + \frac{1}{3} y^3 + \frac{1}{3} y \right]_0^1 dx \\ &= \int_0^1 \left[x^2 + \frac{1}{3} + \frac{1}{3} \right] dx \\ &= \left[\frac{1}{3} x^3 + \frac{2}{3} x \right]_0^1 \\ &= \frac{1}{3} + \frac{2}{3} \\ &= \underline{1} \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz dx dy$

Solution

$$\begin{aligned} \int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz dx dy &= \int_0^{\sqrt{2}} \int_0^{3y} \left[8 - x^2 - y^2 - (x^2 + 3y^2) \right] dx dy \\ &= \int_0^{\sqrt{2}} \int_0^{3y} (8 - 2x^2 - 4y^2) dx dy \\ &= \int_0^{\sqrt{2}} \left[8x - \frac{2}{3} x^3 - 4y^2 x \right]_0^{3y} dy \end{aligned}$$

$$\begin{aligned}
&= \int_0^{\sqrt{2}} (24y - 18y^3 - 12y^3) dy \\
&= \int_0^{\sqrt{2}} (24y - 30y^3) dy \\
&= \left[12y^2 - \frac{15}{2}y^4 \right]_0^{\sqrt{2}} \\
&= 24 - 30 \\
&= \underline{-6}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/6} \int_0^1 \int_{-2}^3 y \sin z \, dx dy dz$

Solution

$$\begin{aligned}
\int_0^{\pi/6} \int_0^1 \int_{-2}^3 y \sin z \, dx dy dz &= \int_0^{\pi/6} \int_0^1 y \sin z \, [x]_{-2}^3 dy dz \\
&= 5 \int_0^{\pi/6} \int_0^1 y \sin z \, dy dz \\
&= 5 \int_0^{\pi/6} \sin z \, \left[\frac{1}{2} y^2 \right]_0^1 dz \\
&= \frac{5}{2} \int_0^{\pi/6} \sin z \, dz \\
&= -\frac{5}{2} [\cos z]_0^{\pi/6} \\
&= -\frac{5}{2} \left(\frac{\sqrt{3}}{2} - 1 \right) \\
&= \underline{\frac{5}{4} (2 - \sqrt{3})}
\end{aligned}$$

Exercise

Evaluate the integral $\int_{-1}^1 \int_0^1 \int_0^2 (x + y + z) dy dx dz$

Solution

$$\begin{aligned}
 \int_{-1}^1 \int_0^1 \int_0^2 (x + y + z) dy dx dz &= \int_{-1}^1 \int_0^1 \left[xy + \frac{1}{2} y^2 + zy \right]_0^2 dx dz \\
 &= \int_{-1}^1 \int_0^1 (2x + 2 + 2z) dx dz \\
 &= \int_{-1}^1 \left[x^2 + (2 + 2z)x \right]_0^1 dz \\
 &= \int_{-1}^1 (1 + 2 + 2z) dz \\
 &= \int_{-1}^1 (3 + 2z) dz \\
 &= \left[3z + z^2 \right]_{-1}^1 \\
 &= (3 + 1) - (-3 + 1) \\
 &= \underline{6}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz dy dx$

Solution

$$\begin{aligned}
 \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz dy dx &= \int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{9-x^2} dy dx \\
 &= \int_0^3 \sqrt{9-x^2} [y]_0^{\sqrt{9-x^2}} dx \\
 &= \int_0^3 (9-x^2) dx \\
 &= \left[9x - \frac{1}{3} x^3 \right]_0^3 \\
 &= \underline{18}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x dz dy dx$

Solution

$$\begin{aligned} \int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x dz dy dx &= \int_0^1 \int_0^{1-x^2} [xz]_3^{4-x^2-y} dy dx \\ &= \int_0^1 \int_0^{1-x^2} x(4-x^2-y-3) dy dx \\ &= \int_0^1 \int_0^{1-x^2} (x-x^3-xy) dy dx \\ &= \int_0^1 \left[(x-x^3)y - \frac{1}{2}xy^2 \right]_0^{1-x^2} dx \\ &= \int_0^1 \left[x(1-x^2)(1-x^2) - \frac{1}{2}x(1-x^2)^2 \right] dx \\ &= \int_0^1 (1-x^2)^2 \left(\frac{1}{2}x \right) dx & d(1-x^2) = -2x dx \\ &= -\frac{1}{4} \int_0^1 (1-x^2)^2 d(1-x^2) \\ &= -\frac{1}{12} \left[(1-x^2)^3 \right]_0^1 \\ &= -\frac{1}{12} (0-1) \\ &= \underline{\underline{\frac{1}{12}}} \end{aligned}$$

Exercise

Evaluate the integral $\int_0^\pi \int_0^\pi \int_0^\pi \cos(u+v+w) du dv dw$

Solution

$$\begin{aligned}\int_0^\pi \int_0^\pi \int_0^\pi \cos(u+v+w) du dv dw &= \int_0^\pi \int_0^\pi \left[\sin(u+v+w) \right]_0^\pi dv dw \\&= \int_0^\pi \int_0^\pi [\sin(v+w+\pi) - \sin(v+w)] dv dw \\&= \int_0^\pi [-\cos(v+w+\pi) + \cos(v+w)]_0^\pi dw \\&= \int_0^\pi [-\cos(w+2\pi) + \cos(w+\pi) + \cos(w+\pi) - \cos(w)] dw \\&= \int_0^\pi [-\cos(w+2\pi) + 2\cos(w+\pi) - \cos(w)] dw \\&= [-\sin(w+2\pi) + 2\sin(w+\pi) - \sin(w)]_0^\pi \\&= -\sin(3\pi) + 2\sin(2\pi) - \sin\pi - (-\sin(2\pi) + 2\sin(\pi) - \sin 0) \\&= \underline{0}\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\pi/4} \int_0^{\ln \sec v} \int_{-\infty}^{2t} e^x dx dt dv$

Solution

$$\begin{aligned}\int_0^{\pi/4} \int_0^{\ln \sec v} \int_{-\infty}^{2t} e^x dx dt dv &= \int_0^{\pi/4} \int_0^{\ln \sec v} \left[e^x \right]_{-\infty}^{2t} dt dv \\&= \int_0^{\pi/4} \int_0^{\ln \sec v} (e^{2t} - e^{-\infty}) dt dv \\&= \int_0^{\pi/4} \int_0^{\ln \sec v} (e^{2t} - e^{-\infty}) dt dv & e^{-\infty} = 0 \\&= \int_0^{\pi/4} \int_0^{\ln \sec v} e^{2t} dt dv\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int_0^{\pi/4} \left[e^{2t} \right]_0^{\ln \sec v} dv \\
&= \frac{1}{2} \int_0^{\pi/4} \left(e^{2 \ln \sec v} - 1 \right) dv & e^{2 \ln \sec v} = e^{\ln \sec^2 v} = \sec^2 v \\
&= \frac{1}{2} \int_0^{\pi/4} \left(\sec^2 v - 1 \right) dv \\
&= \frac{1}{2} [\tan v - v]_0^{\pi/4} \\
&= \frac{1}{2} \left(1 - \frac{\pi}{4} \right) \\
&= \frac{1}{2} - \frac{\pi}{8}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 \int_{-z}^z \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy dx dz$

Solution

$$\begin{aligned}
\int_0^1 \int_{-z}^z \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy dx dz &= \int_0^1 \int_{-z}^z y \bigg|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx dz \\
&= 2 \int_0^1 \int_{-z}^z \sqrt{1-x^2} dx dz & \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \\
&= 2 \int_0^1 \left[\frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x \right]_{-z}^z dz \\
&= 2 \int_0^1 \left(\frac{z}{2} \sqrt{1-z^2} + \frac{1}{2} \sin^{-1} z + \frac{z}{2} \sqrt{1-z^2} + \frac{1}{2} \sin^{-1} z \right) dz \\
&= 2 \int_0^1 \left(z \sqrt{1-z^2} + \sin^{-1} z \right) dz & \int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1-x^2} \\
&= - \int_0^1 (1-z^2)^{1/2} d(1-z^2) + 2 \int_0^1 (\sin^{-1} z) dz
\end{aligned}$$

$$\begin{aligned}
&= -\frac{2}{3}(1-z^2)^{3/2} + 2\left(z \sin^{-1} z + \sqrt{1-z^2}\right) \Big|_0^1 \\
&= 2 \sin^{-1} 1 + \frac{2}{3} - 2 \\
&= \pi - \frac{4}{3}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^\pi \int_0^y \int_0^{\sin x} dz dx dy$

Solution

$$\begin{aligned}
\int_0^\pi \int_0^y \int_0^{\sin x} dz dx dy &= \int_0^\pi \int_0^y z \Big|_0^{\sin x} dx dy \\
&= \int_0^\pi \int_0^y \sin x dx dy \\
&= - \int_0^\pi \cos x \Big|_0^y dy \\
&= - \int_0^\pi (\cos y - 1) dy \\
&= -(\sin y - y) \Big|_0^\pi \\
&= \pi
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^9 \int_0^1 \int_{2y}^2 \frac{4 \sin x^2}{\sqrt{z}} dx dy dz$

Solution

$$\begin{cases} 2y \leq x \leq y & \rightarrow & 0 \leq x \leq 2 \\ 0 \leq y \leq 1 & \rightarrow & 0 \leq y \leq \frac{x}{2} \end{cases}$$

$$\begin{aligned}
\int_0^9 \int_0^1 \int_{2y}^2 \frac{4 \sin x^2}{\sqrt{z}} dx dy dz &= \int_0^9 z^{-1/2} dz \int_0^2 \int_0^{x/2} 4 \sin x^2 dy dx \\
&= 8z^{1/2} \Big|_0^9 \int_0^2 \sin x^2 \left[y \right]_0^{x/2} dx \\
&= 4(3) \int_0^2 x \sin x^2 dx \\
&= 6 \int_0^2 \sin x^2 d(x^2) \\
&= -6 \cos x^2 \Big|_0^2 \\
&= -6(\cos 4 - 1) \\
&= \underline{6 - 6 \cos 4}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^\pi \int_0^\pi \int_0^\pi \cos(x + y + z) dx dy dz$

Solution

$$\begin{aligned}
\int_0^\pi \int_0^\pi \int_0^\pi \cos(x + y + z) dx dy dz &= \int_0^\pi \int_0^\pi \sin(x + y + z) \Big|_0^\pi dy dz \\
&= \int_0^\pi \int_0^\pi (\sin(\pi + y + z) - \sin(y + z)) dy dz \\
&= \int_0^\pi (-\cos(2\pi + z) + \cos(\pi + z) + \cos(\pi + z) - \cos(z)) dz \\
&\quad \cos(2\pi + z) = \cos z \quad \cos(\pi + z) = -\cos z \\
&= -4 \int_0^\pi \cos z dz \\
&= -4 \sin z \Big|_0^\pi \\
&= \underline{0}
\end{aligned}$$

Exercise

Evaluate the integral $\int_1^e \int_1^x \int_0^z \frac{2y}{z^3} dy dz dx$

Solution

$$\begin{aligned} \int_1^e \int_1^x \int_0^z \frac{2y}{z^3} dy dz dx &= \int_1^e \int_1^x \frac{1}{z^3} y^2 \Big|_0^z dz dx \\ &= \int_1^e \int_1^x \frac{1}{z} dz dx \\ &= \int_1^e \ln z \Big|_1^x dx \\ &= \int_1^e \ln x dx \end{aligned}$$

$u = \ln x \rightarrow du = \frac{dx}{x} \quad v = \int dx = x$

$$\int \ln x dx = x \ln x - \int dx$$
$$\begin{aligned} &= x \ln x - x \Big|_1^e \\ &= e - e + 1 \\ &= 1 \end{aligned}$$

Exercise

Evaluate the integral $\int_{\ln 6}^{\ln 7} \int_0^{\ln 2} \int_{\ln 4}^{\ln 5} e^{(x+y+z)} dz dy dx$

Solution

$$\begin{aligned} \int_{\ln 6}^{\ln 7} \int_0^{\ln 2} \int_{\ln 4}^{\ln 5} e^{(x+y+z)} dz dy dx &= \int_{\ln 6}^{\ln 7} \int_0^{\ln 2} \int_{\ln 4}^{\ln 5} e^x e^y e^z dz dy dx \\ &= \int_{\ln 6}^{\ln 7} e^x dx \int_0^{\ln 2} e^y dy \int_{\ln 4}^{\ln 5} e^z dz \\ &= e^x \Big|_{\ln 6}^{\ln 7} e^y \Big|_0^{\ln 2} e^z \Big|_{\ln 4}^{\ln 5} \\ &= (7-6)(2-1)(5-4) \end{aligned}$$

$e^{\ln u} = u$

$$= 1$$

Exercise

Evaluate the integral $\int_0^1 \int_0^{x^2} \int_0^{x+y} (2x - y - z) dz dy dx$

Solution

$$\begin{aligned} \int_0^1 \int_0^{x^2} \int_0^{x+y} (2x - y - z) dz dy dx &= \int_0^1 \int_0^{x^2} \left((2x - y)z - \frac{1}{2}z^2 \right) \Big|_0^{x+y} dy dx \\ &= \int_0^1 \int_0^{x^2} \left((2x - y)(x + y) - \frac{1}{2}(x + y)^2 \right) dy dx \\ &= \int_0^1 \int_0^{x^2} \left(2x^2 + xy - y^2 - \frac{1}{2}x^2 - xy - \frac{1}{2}y^2 \right) dy dx \\ &= \int_0^1 \int_0^{x^2} \left(\frac{3}{2}x^2 - \frac{3}{2}y^2 \right) dy dx \\ &= \int_0^1 \left(\frac{3}{2}x^2y - \frac{1}{2}y^3 \right) \Big|_0^{x^2} dx \\ &= \int_0^1 \left(\frac{3}{2}x^4 - \frac{1}{2}x^6 \right) dx \\ &= \frac{3}{10}x^5 - \frac{1}{14}x^7 \Big|_0^1 \\ &= \frac{3}{10} - \frac{1}{14} \\ &= \frac{32}{140} \\ &= \frac{8}{35} \end{aligned}$$

Exercise

Evaluate the integral $\int_{-2}^2 \int_3^6 \int_0^2 dx dy dz$

Solution

$$\begin{aligned} \int_{-2}^2 \int_3^6 \int_0^2 dx dy dz &= \int_{-2}^2 dz \int_3^6 dy \int_0^2 dx \\ &= z \Big|_{-2}^2 y \Big|_3^6 x \Big|_0^2 \end{aligned}$$

$$= (2+2)(6-3)(2-0)$$

$$= 24$$

Exercise

Evaluate the integral $\int_{-1}^1 \int_{-1}^2 \int_0^1 6xyz \, dydx dz$

Solution

$$\begin{aligned} \int_{-1}^1 \int_{-1}^2 \int_0^1 6xyz \, dydx dz &= 6 \int_{-1}^1 z \, dz \int_{-1}^2 x \, dx \int_0^1 y \, dy \\ &= 6 \left(\frac{1}{2} z^2 \right) \Big|_{-1}^1 \left(\frac{1}{2} x^2 \right) \Big|_{-1}^2 \left(\frac{1}{2} y^2 \right) \Big|_0^1 \\ &= \frac{3}{4} (1-1)(4-1)(1-0) \\ &= 0 \end{aligned}$$

Exercise

Evaluate the integral $\int_{-2}^2 \int_1^2 \int_1^e \frac{xy^2}{z} dz dx dy$

Solution

$$\begin{aligned} \int_{-2}^2 \int_1^2 \int_1^e \frac{xy^2}{z} dz dx dy &= \int_{-2}^2 y^2 dy \int_1^2 x dx \int_1^e \frac{dz}{z} \\ &= \frac{1}{3} y^3 \Big|_{-2}^2 \left(\frac{1}{2} x^2 \right) \Big|_1^2 \ln z \Big|_1^e \\ &= \frac{1}{6} (8+8)(4-1)(1-0) \\ &= 8 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\ln 4} \int_0^{\ln 3} \int_0^{\ln 2} e^{-x+y+z} dx dy dz$

Solution

$$\int_0^{\ln 4} \int_0^{\ln 3} \int_0^{\ln 2} e^{-x+y+z} dx dy dz = \int_0^{\ln 4} e^z dz \int_0^{\ln 3} e^y dy \int_0^{\ln 2} e^{-x} dx$$

$$\begin{aligned}
&= e^z \left|_0^{\ln 4} e^y \right|_0^{\ln 3} \left(-e^{-x} \right) \left|_0^{\ln 2} \right. \\
&= -\left(e^{\ln 4} - e^0 \right) \left(e^{\ln 3} - e^0 \right) \left(e^{\ln 2} - e^0 \right) \\
&= -(4-1)(3-1)\left(\frac{1}{2}-1\right) \\
&= \underline{3}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\frac{\pi}{2}} \sin \pi x \cos y \sin 2z \, dy dx dz$

Solution

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\frac{\pi}{2}} \sin \pi x \cos y \sin 2z \, dy dx dz &= \int_0^{\frac{\pi}{2}} \sin 2z \, dz \int_0^1 \sin \pi x \, dx \int_0^{\frac{\pi}{2}} \cos y \, dy \\
&= -\frac{1}{2} \cos 2z \left|_0^{\frac{\pi}{2}} \left[-\frac{1}{\pi} \cos \pi x \right] \right|_0^1 \sin y \left|_0^{\frac{\pi}{2}} \right. \\
&= \frac{1}{2\pi} (-1-1)(-1-1)(1-0) \\
&= \underline{\frac{2}{\pi}}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^2 \int_1^2 \int_0^1 yz e^x \, dx dz dy$

Solution

$$\begin{aligned}
\int_0^2 \int_1^2 \int_0^1 yz e^x \, dx dz dy &= \int_0^2 y dy \int_1^2 z dz \int_0^1 e^x dx \\
&= \frac{1}{2} y^2 \left|_0^2 \frac{1}{2} z^2 \right|_1^2 e^x \left|_0^1 \right. \\
&= \frac{1}{4} (4)(4-1)(e-1) \\
&= \underline{3(e-1)}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz dy dx$

Solution

$$\begin{aligned}
 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz dy dx &= \int_0^1 \int_0^{\sqrt{1-x^2}} z \Big|_0^{\sqrt{1-x^2}} dy dx \\
 &= \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2} dy dx \\
 &= \int_0^1 \sqrt{1-x^2} y \Big|_0^{\sqrt{1-x^2}} dx \\
 &= \int_0^1 (1-x^2) dx \\
 &= \left(x - \frac{1}{3} x^3 \right) \Big|_0^1 \\
 &= \frac{2}{3}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 2xz \, dz dy dx$

Solution

$$\begin{aligned}
 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} 2xz \, dz dy dx &= \int_0^1 \int_0^{\sqrt{1-x^2}} xz^2 \Big|_0^{\sqrt{1-x^2-y^2}} dy dx \\
 &= \int_0^1 \int_0^{\sqrt{1-x^2}} x(1-x^2-y^2) dy dx \\
 &= \int_0^1 \left(xy - x^3 y - \frac{1}{3} xy^3 \right) \Big|_0^{\sqrt{1-x^2}} dx \\
 &= \int_0^1 \left(x(1-x^2)^{1/2} - x^3(1-x^2)^{1/2} - \frac{1}{3} x(1-x^2)^{3/2} \right) dx
 \end{aligned}$$

Switching $dydx$ to $dx dy$

$$\begin{aligned}
 &= \int_0^1 \int_0^{\sqrt{1-y^2}} (x - x^3 - xy^2) dx dy \\
 &= \int_0^1 \left(\frac{1}{2}x^2 - \frac{1}{4}x^4 - \frac{1}{2}x^2 y^2 \right) \bigg|_0^{\sqrt{1-y^2}} dy \\
 &= \frac{1}{4} \int_0^1 \left(2(1-y^2) - (1-y^2)^2 - 2(1-y^2)y^2 \right) dy \\
 &= \frac{1}{4} \int_0^1 (2 - 2y^2 - 1 + 2y^2 - y^4 - 2y^2 + 2y^4) dy \\
 &= \frac{1}{4} \int_0^1 (y^4 - 2y^2 + 1) dy \\
 &= \frac{1}{4} \left(\frac{1}{5}y^5 - \frac{2}{3}y^3 + y \right) \bigg|_0^1 \\
 &= \frac{1}{4} \left(\frac{1}{5} - \frac{2}{3} + 1 \right) \\
 &= \frac{1}{4} \left(\frac{8}{15} \right) \\
 &= \underline{\frac{2}{15}}
 \end{aligned}$$

Exercise

Evaluate the integral $\int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \int_0^{16-\frac{1}{4}x^2-y^2} dz dx dy$

Solution

$$\begin{aligned}
 \int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \int_0^{16-\frac{1}{4}x^2-y^2} dz dx dy &= \int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} z \bigg|_0^{16-\frac{1}{4}x^2-y^2} dx dy \\
 &= \int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \left(16 - \frac{1}{4}x^2 - y^2 \right) dx dy \\
 &= \int_0^4 \left(16x - \frac{1}{12}x^3 - xy^2 \right) \bigg|_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} dy
 \end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^4 \left(16x - \frac{1}{12}x^3 - xy^2 \right) \Big|_0^{2\sqrt{16-y^2}} dy \\
&= 2 \int_0^4 \left(32\sqrt{16-y^2} - \frac{2}{3}(16-y^2)^{3/2} - 2y^2\sqrt{16-y^2} \right) dy
\end{aligned}$$

$$y = 4 \sin \theta \rightarrow dy = 4 \cos \theta d\theta \quad \sqrt{16-y^2} = 4 \cos \theta$$

$$\begin{aligned}
\int \sqrt{16-y^2} dy &= 16 \int \cos^2 \theta d\theta \\
&= 8 \int (1 + \cos 2\theta) d\theta \\
&= 8 \left(\theta + \frac{1}{2} \sin 2\theta \right) \\
&= \underline{8 \sin^{-1} \left(\frac{y}{4} \right) + \frac{1}{2} y \sqrt{16-y^2}}
\end{aligned}$$

$$\begin{aligned}
\int (16-y^2)^{3/2} dy &= \int (16 \cos^2 \theta)^{3/2} 4 \cos \theta d\theta \\
&= 256 \int \cos^4 \theta d\theta \\
&= 64 \int (1 + \cos 2\theta)^2 d\theta \\
&= 64 \int (1 + 2 \cos 2\theta + \cos^2 2\theta) d\theta \\
&= 64 \int \left(\frac{3}{2} + 2 \cos 2\theta + \frac{1}{2} \cos 4\theta \right) d\theta \\
&= 64 \left(\frac{3}{2} \theta + \sin 2\theta + \frac{1}{8} \sin 4\theta \right) \\
&= 64 \left(\frac{3}{2} \theta + 2 \sin \theta \cos \theta + \frac{1}{4} \sin 2\theta \cos 2\theta \right) \\
&= 64 \left(\frac{3}{2} \theta + 2 \sin \theta \cos \theta + \frac{1}{2} \sin \theta \cos \theta (1 - 2 \sin^2 \theta) \right) \\
&= 64 \left(\frac{3}{2} \sin^{-1} \left(\frac{y}{4} \right) + 2 \frac{y}{4} \frac{\sqrt{16-y^2}}{4} + \frac{1}{2} \frac{y}{4} \frac{\sqrt{16-y^2}}{4} \left(1 - \frac{y^2}{8} \right) \right) \\
&= 64 \left(\frac{3}{2} \sin^{-1} \left(\frac{y}{4} \right) + \frac{1}{8} y \sqrt{16-y^2} + \frac{1}{256} y (8-y^2) \sqrt{16-y^2} \right)
\end{aligned}$$

$$\begin{aligned}
&= 64 \left(\frac{3}{2} \sin^{-1} \left(\frac{y}{4} \right) + \frac{1}{8} y \sqrt{16 - y^2} + \frac{1}{32} y \sqrt{16 - y^2} - \frac{1}{256} y^3 \sqrt{16 - y^2} \right) \\
&= \left(96 \sin^{-1} \left(\frac{y}{4} \right) + 10 y \sqrt{16 - y^2} - \frac{1}{4} y^3 \sqrt{16 - y^2} \right)
\end{aligned}$$

$$\begin{aligned}
\int y^2 \sqrt{16 - y^2} \, dy &= 4^4 \int \sin^2 \theta \cos^2 \theta \, d\theta \\
&= 64 \int (1 - \cos 2\theta)(1 + \cos 2\theta) \, d\theta \\
&= 64 \int (1 - \cos^2 2\theta) \, d\theta \\
&= 64 \int \left(\frac{1}{2} - \frac{1}{2} \cos 4\theta \right) \, d\theta \\
&= 32 \left(\theta - \frac{1}{4} \sin 4\theta \right) \\
&= 32 \left(\theta - \frac{1}{2} \sin 2\theta \cos 2\theta \right) \\
&= 32 \left(\theta - \sin \theta \cos \theta (1 - 2 \sin^2 \theta) \right) \\
&= 32 \left(\sin^{-1} \left(\frac{y}{4} \right) - \frac{y}{4} \frac{\sqrt{16 - y^2}}{4} \left(1 - \frac{y^2}{8} \right) \right) \\
&= 32 \left(\sin^{-1} \left(\frac{y}{4} \right) - \frac{1}{128} y (8 - y^2) \sqrt{16 - y^2} \right) \\
&= 32 \sin^{-1} \left(\frac{y}{4} \right) - 2 y \sqrt{16 - y^2} + \frac{1}{4} y^3 \sqrt{16 - y^2}
\end{aligned}$$

$$\begin{aligned}
\int_0^4 \int_{-2\sqrt{16-y^2}}^{2\sqrt{16-y^2}} \int_0^{16-\frac{1}{4}x^2-y^2} dz \, dx \, dy &= \int_0^4 \left(64 \sqrt{16 - y^2} - \frac{4}{3} (16 - y^2)^{3/2} - 4 y^2 \sqrt{16 - y^2} \right) dy \\
&= 64 \left(8 \sin^{-1} \left(\frac{y}{4} \right) + \frac{1}{2} y \sqrt{16 - y^2} \right) \\
&\quad - \frac{4}{3} \left(96 \sin^{-1} \left(\frac{y}{4} \right) + 10 y \sqrt{16 - y^2} - \frac{1}{4} y^3 \sqrt{16 - y^2} \right) \\
&\quad - 4 \left(32 \sin^{-1} \left(\frac{y}{4} \right) - 2 y \sqrt{16 - y^2} + \frac{1}{4} y^3 \sqrt{16 - y^2} \right)
\end{aligned}$$

$$\begin{aligned}
&= 512 \sin^{-1}\left(\frac{y}{4}\right) + 32y\sqrt{16-y^2} \\
&\quad - 128 \sin^{-1}\left(\frac{y}{4}\right) - \frac{40}{3}y\sqrt{16-y^2} + \frac{1}{3}y^3\sqrt{16-y^2} \\
&\quad - 128 \sin^{-1}\left(\frac{y}{4}\right) + 8y\sqrt{16-y^2} - y^3\sqrt{16-y^2} \\
&= 256 \sin^{-1}\left(\frac{y}{4}\right) + \frac{80}{3}y\sqrt{16-y^2} - \frac{2}{3}y^3\sqrt{16-y^2} \Big|_0^4 \\
&= 256\left(\frac{\pi}{2}\right) \\
&= \underline{128\pi}
\end{aligned}$$

Exercise

Evaluate the integral $\int_1^6 \int_0^{4-\frac{2}{3}y} \int_0^{12-2y-3z} \frac{1}{y} dx dz dy$

Solution

$$\begin{aligned}
\int_1^6 \int_0^{4-\frac{2}{3}y} \int_0^{12-2y-3z} \frac{1}{y} dx dz dy &= \int_1^6 \int_0^{4-\frac{2}{3}y} \frac{1}{y} x \Big|_0^{12-2y-3z} dz dy \\
&= \int_1^6 \int_0^{4-\frac{2}{3}y} \frac{1}{y} (12-2y-3z) dz dy \\
&= \int_1^6 \left(12 \frac{1}{y} z - 2z - \frac{3}{2} \frac{1}{y} z^2 \right) \Big|_0^{4-\frac{2}{3}y} dy \\
&= \int_1^6 \left(12 \frac{1}{y} \left(4 - \frac{2}{3}y \right) - 2 \left(4 - \frac{2}{3}y \right) - \frac{3}{2} \frac{1}{y} \left(4 - \frac{2}{3}y \right)^2 \right) dy \\
&= \int_1^6 \left(\frac{48}{y} - 16 + \frac{4}{3}y - \frac{3}{2} \frac{1}{y} \left(16 - \frac{16}{3}y + \frac{4}{9}y^2 \right) \right) dy \\
&= \int_1^6 \left(\frac{48}{y} - 16 + \frac{4}{3}y - \frac{24}{y} + 8 - \frac{2}{3}y \right) dy \\
&= \int_1^6 \left(\frac{24}{y} - 8 + \frac{2}{3}y \right) dy \\
&= 24 \ln y - 8y + \frac{1}{3}y^2 \Big|_1^6
\end{aligned}$$

$$= 24 \ln 6 - 48 + 12 + 8 - \frac{1}{3}$$

$$\underline{= 24 \ln 6 - \frac{85}{3}}$$

Exercise

Evaluate the integral $\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{1+x^2+z^2}} dy dx dz$

Solution

$$\begin{aligned} \int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{1+x^2+z^2}} dy dx dz &= \int_0^3 \int_0^{\sqrt{9-z^2}} y \bigg|_0^{\sqrt{1+x^2+z^2}} dx dz \\ &= \int_0^3 \int_0^{\sqrt{9-z^2}} \sqrt{1+x^2+z^2} dx dz && \text{Let } x^2 + z^2 = r^2 \\ &= \int_0^{\frac{\pi}{2}} \int_0^3 \sqrt{1+r^2} r dr d\theta \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta \int_0^3 (1+r^2)^{1/2} d(1+r^2) \\ &= \frac{1}{2} \left(\frac{\pi}{2} \right) \frac{2}{3} (1+r^2)^{3/2} \bigg|_0^3 \\ &= \frac{\pi}{6} (10\sqrt{10} - 1) \end{aligned}$$

Exercise

Evaluate the integral $\int_0^\pi \int_0^\pi \int_0^{\sin x} \sin y dz dx dy$

Solution

$$\begin{aligned} \int_0^\pi \int_0^\pi \int_0^{\sin x} \sin y dz dx dy &= \int_0^\pi \int_0^\pi (\sin y) z \bigg|_0^{\sin x} dx dy \\ &= \int_0^\pi \int_0^\pi (\sin y \sin x) dx dy \\ &= - \int_0^\pi \sin y \cos x \bigg|_0^\pi dy \end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^{\pi} \sin y \, dy \\
&= -2 \cos y \Big|_0^{\pi} \\
&= \underline{4}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^{\ln 8} \int_1^{\sqrt{z}} \int_{\ln y}^{\ln 2y} e^{x+y^2-z} dx dy dz$

Solution

$$\begin{aligned}
\int_0^{\ln 8} \int_1^{\sqrt{z}} \int_{\ln y}^{\ln 2y} e^{x+y^2-z} dx dy dz &= \int_0^{\ln 8} \int_1^{\sqrt{z}} e^{y^2} e^{-z} e^x \Big|_{\ln y}^{\ln 2y} dy dz \\
&= \int_0^{\ln 8} \int_1^{\sqrt{z}} e^{y^2} e^{-z} (y) dy dz \\
&= \frac{1}{2} \int_0^{\ln 8} \int_1^{\sqrt{z}} e^{-z} e^{y^2} d(y^2) dz \\
&= \frac{1}{2} \int_0^{\ln 8} e^{-z} e^{y^2} \Big|_1^{\sqrt{z}} dz \\
&= \frac{1}{2} \int_0^{\ln 8} e^{-z} (e^z - e) dz \\
&= \frac{1}{2} \int_0^{\ln 8} (1 - e^{1-z}) dz \\
&= \frac{1}{2} (z + e^{1-z}) \Big|_0^{\ln 8} \\
&= \frac{1}{2} (\ln 8 + e^{1-\ln 8} - e) \\
&= \frac{1}{2} \left(\ln 8 + e \left(e^{\ln 8^{-1}} \right) - e \right) \\
&= \frac{1}{2} \left(\ln 8 + \frac{1}{8} e - e \right) \\
&= \underline{\frac{1}{2} \ln 8 - \frac{7}{16} e}
\end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{2-x} 4yz \, dz dy dx$

Solution

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{2-x} 4yz \, dz dy dx &= \int_0^1 \int_0^{\sqrt{1-x^2}} 2yz^2 \Big|_0^{2-x} dy dx \\ &= \int_0^1 \int_0^{\sqrt{1-x^2}} 2y(2-x)^2 dy dx \\ &= \int_0^1 \left(4 - 4x + x^2\right) y^2 \Big|_0^{\sqrt{1-x^2}} dx \\ &= \int_0^1 \left(4 - 4x + x^2\right) (1 - x^2) dx \\ &= \int_0^1 \left(4 - 4x - 3x^2 + 4x^3 - x^4\right) dx \\ &= \left(4x - 2x^2 - x^3 + x^4 - \frac{1}{5}x^5\right) \Big|_0^1 \\ &= \underline{\underline{\frac{9}{5}}} \end{aligned}$$

Exercise

Evaluate the integral $\int_0^2 \int_0^4 \int_{y^2}^4 \sqrt{x} \, dz dx dy$

Solution

$$\begin{aligned} \int_0^2 \int_0^4 \int_{y^2}^4 \sqrt{x} \, dz dx dy &= \int_0^2 \int_0^4 \sqrt{x} z \Big|_{y^2}^4 dx dy \\ &= \int_0^2 \left(4 - y^2\right) dy \int_0^4 x^{1/2} dx \\ &= \left(4y - \frac{1}{3}y^3\right) \Big|_0^2 \cdot \frac{2}{3}x^{3/2} \Big|_0^4 \\ &= \frac{2}{3} \left(8 - \frac{8}{3}\right) (8) \\ &= \underline{\underline{\frac{256}{9}}} \end{aligned}$$

Exercise

Evaluate the integral $\int_0^1 \int_y^{2-y} \int_0^{2-x-y} xy \, dz \, dx \, dy$

Solution

$$\begin{aligned} \int_0^1 \int_y^{2-y} \int_0^{2-x-y} xy \, dz \, dx \, dy &= \int_0^1 \int_y^{2-y} xyz \bigg|_0^{2-x-y} \, dx \, dy \\ &= \int_0^1 \int_y^{2-y} (2xy - x^2y - xy^2) \, dx \, dy \\ &= \int_0^1 \left(x^2y - \frac{1}{3}x^3y - \frac{1}{2}x^2y^2 \right) \bigg|_y^{2-y} \, dy \\ &= \int_0^1 \left((2-y)^2y - \frac{1}{3}(2-y)^3y - \frac{1}{2}(2-y)^2y^2 - y^3 + \frac{5}{6}y^4 \right) \, dy \\ &= \int_0^1 \left((4-4y+y^2) \left(y - \frac{2}{3}y + \frac{1}{3}y^2 - \frac{1}{2}y^2 \right) - y^3 + \frac{5}{6}y^4 \right) \, dy \\ &= \int_0^1 \left((4-4y+y^2) \left(\frac{1}{3}y - \frac{1}{6}y^2 \right) - y^3 + \frac{5}{6}y^4 \right) \, dy \\ &= \int_0^1 \left(\frac{4}{3}y - 2y^2 + y^3 - \frac{1}{6}y^4 - y^3 + \frac{5}{6}y^4 \right) \, dy \\ &= \int_0^1 \left(\frac{4}{3}y - 2y^2 + \frac{2}{3}y^4 \right) \, dy \\ &= \left(\frac{2}{3}y^2 - \frac{2}{3}y^3 + \frac{2}{15}y^5 \right) \bigg|_0^1 \\ &= \frac{2}{15} \end{aligned}$$

Exercise

Here is the region of integration of the integral

$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz \, dy \, dx$$

a) $dydzdx$ b) $dydx dz$ c) $dx dy dz$ d) $dx dz dy$ e) $dz dx dy$

Solution

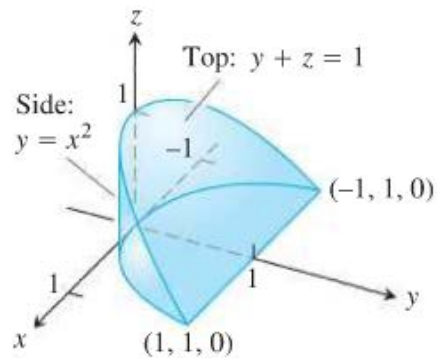
$$a) \int_{-1}^1 \int_0^{1-x^2} \int_{x^2}^{1-x} dydzdx$$

$$b) \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1-x} dydxdz$$

$$c) \int_0^1 \int_0^{1-x} \int_{-\sqrt{y}}^{\sqrt{y}} dx dy dz$$

$$d) \int_0^1 \int_0^{1-y} \int_{-\sqrt{y}}^{\sqrt{y}} dx dz dy$$

$$e) \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{1-y} dz dx dy$$



Exercise

Use another order to evaluate $\int_0^5 \int_{-1}^0 \int_0^{4x+4} dydxdz$

Solution

$$-1 \leq x \leq 0 \quad 0 \leq y \leq 4x+4 \quad 0 \leq z \leq 5$$

$$\begin{cases} x = -1 & \rightarrow y = 0 \\ x = 0 & \rightarrow y = 4 \end{cases}$$

$$y = 4x+4 \rightarrow x = \frac{y-4}{4}$$

$$\int_0^5 \int_{-1}^0 \int_0^{4x+4} dydxdz = \int_0^4 \int_{\frac{y-4}{4}}^0 \int_0^5 dzdxdy$$

$$= \int_0^4 x \Big|_{\frac{y-4}{4}}^0 dy \quad z \Big|_0^5$$

$$= \frac{5}{4} \int_0^4 (4-y) dy$$

$$= \frac{5}{4} \left(4y - \frac{1}{2} y^2 \right) \Big|_0^4$$

$$= 10$$

Exercise

Use another order to evaluate $\int_0^1 \int_{-2}^2 \int_0^{\sqrt{4-y^2}} dz dy dx$

Solution

$$0 \leq x \leq 1 \quad -2 \leq y \leq 2 \quad 0 \leq z \leq \sqrt{4-y^2}$$

$$0 \leq z \leq 2$$

$$z = \sqrt{4-y^2} \rightarrow y = \pm\sqrt{4-z^2}$$

$$\begin{aligned} \int_0^1 \int_{-2}^2 \int_0^{\sqrt{4-y^2}} dz dy dx &= \int_0^1 \int_0^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dy dz dx \\ &= \int_0^1 dx \int_0^2 y \bigg|_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} dz \\ &= 2 \int_0^2 \sqrt{4-z^2} dz \\ &= 2 \left(\frac{z}{2} \sqrt{4-z^2} + 2 \sin^{-1} \frac{z}{2} \right) \bigg|_0^2 \\ &= 2 \left(2 \frac{\pi}{2} \right) \\ &= 2\pi \end{aligned}$$

Exercise

Use another order to evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dy dz dx$

Solution

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dy dz dx &= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz dy dx \\ &= \int_0^1 \int_0^{\sqrt{1-x^2}} z \bigg|_0^{\sqrt{1-x^2}} dy dx \\ &= \int_0^1 \sqrt{1-x^2} y \bigg|_0^{\sqrt{1-x^2}} dx \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 (1-x^2) dx \\
&= \left(x - \frac{1}{3}x^3 \right) \Big|_0^1 \\
&= \frac{2}{3}
\end{aligned}$$

Exercise

Use another order to evaluate

$$\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-z^2}} dy dz dx$$

Solution

$$0 \leq x \leq 4 \quad 0 \leq y \leq \sqrt{16-x^2-z^2} \quad 0 \leq z \leq \sqrt{16-x^2}$$

$$x^2 + y^2 + z^2 = 16$$

$$0 \leq x \leq \sqrt{16-y^2-z^2} \quad 0 \leq y \leq \sqrt{16-z^2} \quad 0 \leq z \leq 4$$

$$\begin{aligned}
\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{16-x^2-z^2}} dy dz dx &= \int_0^4 \int_0^{\sqrt{16-z^2}} \int_0^{\sqrt{16-y^2-z^2}} dx dy dz \\
&= \int_0^4 \int_0^{\sqrt{16-z^2}} x \Big|_0^{\sqrt{16-y^2-z^2}} dy dz \\
&= \int_0^4 \int_0^{\sqrt{16-z^2}} \sqrt{16-y^2-z^2} dy dz \quad \text{Let } y^2 + z^2 = r^2 \\
&= \int_0^{\frac{\pi}{2}} \int_0^4 \sqrt{16-r^2} r dr d\theta \\
&= -\frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta \int_0^4 (16-r^2)^{1/2} d(16-r^2) \\
&= -\frac{1}{2} \left(\frac{\pi}{2} \right) \left(\frac{2}{3} \right) (16-r^2)^{3/2} \Big|_0^4 \\
&= -\frac{\pi}{6} (-64) \\
&= \frac{32\pi}{3}
\end{aligned}$$

Exercise

Use another order to evaluate

$$\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin \sqrt{yz}}{x^{3/2}} dy dx dz$$

Solution

$$\begin{aligned} \int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin \sqrt{yz}}{x^{3/2}} dy dx dz &= \int_0^{\pi^2} \int_1^4 \int_z^{4z} x^{-3/2} \sin \sqrt{yz} dx dz dy \\ &= -2 \int_0^{\pi^2} \int_1^4 \sin \sqrt{yz} \left(x^{-1/2} \right) \Big|_z^{4z} dz dy \\ &= -2 \int_0^{\pi^2} \int_1^4 \sin \sqrt{yz} \left(\frac{1}{2\sqrt{z}} - \frac{1}{\sqrt{z}} \right) dz dy \\ &= \int_0^{\pi^2} \int_1^4 \frac{\sin \sqrt{yz}}{\sqrt{z}} dz dy \\ &= 2 \int_0^{\pi^2} \int_1^4 \frac{1}{\sqrt{y}} \sin \sqrt{yz} d(\sqrt{yz}) dy \quad d(\sqrt{yz}) = \frac{1}{2} \frac{y}{\sqrt{yz}} dz \\ &= -2 \int_0^{\pi^2} \frac{1}{\sqrt{y}} \cos \sqrt{yz} \Big|_1^4 dy \\ &= -2 \int_0^{\pi^2} \frac{1}{\sqrt{y}} (\cos(2\sqrt{y}) - \cos \sqrt{y}) dy \\ &= -4 \int_0^{\pi^2} (\cos(2\sqrt{y}) - \cos \sqrt{y}) d(\sqrt{y}) \\ &= -4 \left(\frac{1}{2} \cos(2\sqrt{y}) - \cos \sqrt{y} \right) \Big|_0^{\pi^2} \\ &= -4 \left(\frac{1}{2} + 1 - \frac{1}{2} - 1 \right) \\ &= 0 \end{aligned}$$

Exercise

Evaluate $\iiint_D (xy + xz + yz) dV$; $D = \{(x, y, z) : -1 \leq x \leq 1, -2 \leq y \leq 2, -3 \leq z \leq 3\}$

Solution

$$\begin{aligned}
\iiint_D (xy + xz + yz) dV &= \int_{-3}^3 \int_{-2}^2 \int_{-1}^1 (xy + xz + yz) dx dy dz \\
&= \int_{-3}^3 \int_{-2}^2 \left(\frac{1}{2} x^2 y + \frac{1}{2} x^2 z + xyz \right) \Big|_{-1}^1 dy dz \\
&= \int_{-3}^3 \int_{-2}^2 \left(\frac{1}{2} y + \frac{1}{2} z + yz - \frac{1}{2} y - \frac{1}{2} z + yz \right) dy dz \\
&= \int_{-3}^3 \int_{-2}^2 2yz dy dz \\
&= \int_{-3}^3 zy^2 \Big|_{-2}^2 dz \\
&= \underline{0}
\end{aligned}$$

Exercise

Evaluate $\iiint_D xyz e^{-x^2-y^2} dV$; $D = \{(x, y, z): 0 \leq x \leq \sqrt{\ln 2}, 0 \leq y \leq \sqrt{\ln 4}, 0 \leq z \leq 1\}$

Solution

$$\begin{aligned}
\iiint_D xyz e^{-x^2-y^2} dV &= \int_0^1 \int_0^{\sqrt{\ln 4}} \int_0^{\sqrt{\ln 2}} xyz e^{-x^2-y^2} dx dy dz \\
&= \int_0^1 z dz \int_0^{\sqrt{\ln 4}} y e^{-y^2} dy \int_0^{\sqrt{\ln 2}} x e^{-x^2} dx \\
&= \frac{1}{2} z^2 \Big|_0^1 \int_0^{\sqrt{\ln 4}} \left(-\frac{1}{2} \right) e^{-y^2} d(-y^2) \int_0^{\sqrt{\ln 2}} \left(-\frac{1}{2} \right) e^{-x^2} d(-x^2) \\
&= \frac{1}{8} e^{-y^2} \Big|_0^{\sqrt{\ln 4}} e^{-x^2} \Big|_0^{\sqrt{\ln 2}} \\
&= \frac{1}{8} \left(\frac{1}{4} - 1 \right) \left(\frac{1}{2} - 1 \right) \\
&= \underline{\frac{3}{64}}
\end{aligned}$$

Exercise

Let $D = \{(x, y, z): 0 \leq x \leq y^2, 0 \leq y \leq z^3, 0 \leq z \leq 2\}$

- Use a triple integral to find the volume of D .
- In theory, how many other possible orderings of the variables (besides the one used in part (a)) can be used to find the volume of D ? Verify the result of part (a) using one of these other ordering.
- What is the volume of the region $D = \{(x, y, z): 0 \leq x \leq y^p, 0 \leq y \leq z^q, 0 \leq z \leq 2\}$, where p and q are positive real numbers?

Solution

$$\begin{aligned} a) \quad V &= \int_0^2 \int_0^{z^3} \int_0^{y^2} dx dy dz \\ &= \int_0^2 \int_0^{z^3} x \Big|_0^{y^2} dy dz \\ &= \int_0^2 \int_0^{z^3} y^2 dy dz \\ &= \frac{1}{3} \int_0^2 y^3 \Big|_0^{z^3} dz \\ &= \frac{1}{3} \int_0^2 z^9 dz \\ &= \frac{1}{30} z^{10} \Big|_0^2 \\ &= \frac{512}{15} \text{ unit}^3 \end{aligned}$$

- b) There are total of 6: $dx dy dz$, $dx dz dy$, $dy dx dz$, $dy dz dx$, $dz dx dy$, $dz dy dx$

$$0 \leq x \leq y^2$$

$$z = 2 \rightarrow y = 2^3 = 8 \quad 0 \leq y \leq 8$$

$$y = z^3 \rightarrow z = \sqrt[3]{y} \quad \sqrt[3]{y} \leq z \leq 2$$

$$\begin{aligned} \int_0^8 \int_{\sqrt[3]{y}}^2 \int_0^{y^2} dx dz dy &= \int_0^8 \int_{\sqrt[3]{y}}^2 x \Big|_0^{y^2} dz dy \\ &= \int_0^8 \int_{\sqrt[3]{y}}^2 y^2 dz dy \end{aligned}$$

$$\begin{aligned}
&= \int_0^8 y^2 z \bigg|_{\sqrt[3]{y}}^2 dy \\
&= \int_0^8 y^2 (2 - y^{1/3}) dy \\
&= \int_0^8 (2y^2 - y^{7/3}) dy \\
&= \left(\frac{2}{3} y^3 - \frac{3}{10} y^{10/3} \right) \bigg|_0^{2^3} \\
&= \frac{2^{10}}{3} - \frac{3}{5} 2^9 \\
&= 2^9 \left(\frac{2}{3} - \frac{3}{5} \right) \\
&= \frac{2^9}{15} \quad \quad \quad = \frac{512}{15}
\end{aligned}$$

c) $D = \{(x, y, z) : 0 \leq x \leq y^p, 0 \leq y \leq z^q, 0 \leq z \leq 2\}, (p, q \in \mathbb{R})$

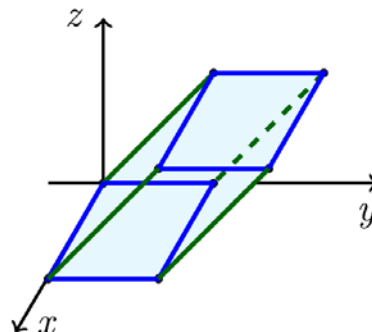
$$\begin{aligned}
V &= \int_0^2 \int_0^{z^q} \int_0^{y^p} dx dy dz \\
&= \int_0^2 \int_0^{z^q} x \bigg|_0^{y^p} dy dz \\
&= \int_0^2 \int_0^{z^q} y^p dy dz \\
&= \frac{1}{p+1} \int_0^2 y^{p+1} \bigg|_0^{z^q} dz \\
&= \frac{1}{p+1} \int_0^2 z^{q(p+1)} dz \\
&= \frac{1}{(p+1)(q(p+1)+1)} z^{q(p+1)+1} \bigg|_0^2 \\
&= \frac{2^{q(p+1)+1}}{(p+1)(q(p+1)+1)}
\end{aligned}$$

Exercise

Find the volume the parallelepiped (slanted box) with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, $(0, 1, 1)$, $(1, 1, 1)$, $(0, 2, 1)$, $(1, 2, 1)$

Solution

$$\begin{aligned} V &= \int_0^1 \int_z^{z+1} \int_0^1 dx dy dz \\ &= \int_0^1 \int_z^{z+1} x \Big|_0^1 dy dz \\ &= \int_0^1 \int_z^{z+1} dy dz \\ &= \int_0^1 y \Big|_z^{z+1} dz \\ &= \int_0^1 dz \\ &= z \Big|_0^1 \\ &= 1 \end{aligned}$$



Exercise

Find the volume the larger of two solids formed when the parallelepiped with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, $(2, 2, 0)$, $(0, 1, 1)$, $(2, 1, 1)$, $(0, 3, 1)$, $(2, 3, 1)$ is sliced by the plane $y = 2$.

Solution

$$\begin{aligned} V &= \int_0^1 \int_z^2 \int_0^2 dx dy dz \\ &= \int_0^1 \int_z^2 x \Big|_0^2 dy dz \\ &= 2 \int_0^1 \int_z^2 dy dz \\ &= 2 \int_0^1 y \Big|_z^2 dz \end{aligned}$$

$$\begin{aligned}
 &= 2 \int_0^1 (2-z) dz \\
 &= 2 \left(2z - \frac{1}{2} z^2 \right) \Big|_0^1 \\
 &= 2 \left(2 - \frac{1}{2} \right) \\
 &= \underline{3}
 \end{aligned}$$

Exercise

Find the volume of the pyramid with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(2, 2, 0)$, $(0, 2, 0)$, $(0, 0, 4)$

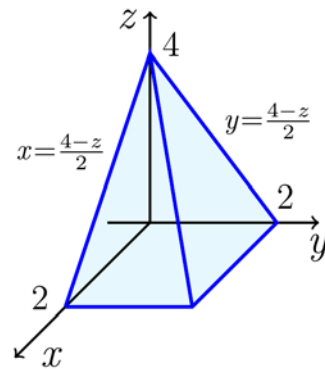
Solution

$$(2, 0) \quad \& \quad (0, 4) \rightarrow z = \frac{4}{-2}(x-2) = -2x + 4$$

$$\underline{x = \frac{4-z}{2}}$$

$$(2, 0) \quad \& \quad (0, 4) \rightarrow \underline{y = \frac{4-z}{2}}$$

$$\begin{aligned}
 V &= \int_0^4 \int_0^{\frac{4-z}{2}} \int_0^{\frac{4-z}{2}} dx dy dz \\
 &= \int_0^4 \int_0^{\frac{4-z}{2}} x \Big|_0^{\frac{4-z}{2}} dy dz \\
 &= \frac{1}{2} \int_0^4 \int_0^{\frac{4-z}{2}} (4-z) dy dz \\
 &= \frac{1}{2} \int_0^4 (4-z) y \Big|_0^{\frac{4-z}{2}} dz \\
 &= -\frac{1}{4} \int_0^4 (4-z)^2 d(4-z) \\
 &= -\frac{1}{12} (4-z)^3 \Big|_0^4 \\
 &= -\frac{1}{12} (-64) \\
 &= \underline{\frac{16}{3}}
 \end{aligned}$$



Exercise

Two different tetrahedrons fill the region in the first octant bounded by the coordinate planes and the plane $x + y + z = 4$. Both solids have densities that vary in the z -direction between $\rho = 4$ and $\rho = 8$, according to the functions $\rho_1 = 8 - z$ and $\rho_2 = 4 + z$. Find the mass of each solid

Solution

$$\begin{aligned} m_1 &= \int_0^4 \int_0^{4-x} \int_0^{4-x-y} (8-z) dz dy dx \\ &= \int_0^4 \int_0^{4-x} \left(8z - \frac{1}{2} z^2 \right) \Big|_0^{4-x-y} dy dx \\ &= \int_0^4 \int_0^{4-x} \left(32 - 8x - 8y - \frac{1}{2} (4-x-y)^2 \right) dy dx \\ &= \int_0^4 \int_0^{4-x} \left(24 - 4x - \frac{1}{2} x^2 - 4y - xy - \frac{1}{2} y^2 \right) dy dx \\ &= \int_0^4 \left(24y - 4xy - \frac{1}{2} x^2 y - 2y^2 - \frac{1}{2} xy^2 - \frac{1}{6} y^3 \right) \Big|_0^{4-x} dx \\ &= \int_0^4 \left(\frac{160}{3} - 24x + 2x^2 + \frac{1}{6} x^3 \right) dx \\ &= \left(\frac{160}{3} x - 12x^2 + \frac{2}{3} x^3 + \frac{1}{24} x^4 \right) \Big|_0^4 \\ &= \frac{224}{3} \end{aligned}$$

$$\begin{aligned} m_2 &= \int_0^4 \int_0^{4-x} \int_0^{4-x-y} (4+z) dz dy dx \\ &= \int_0^4 \int_0^{4-x} \left(4z + \frac{1}{2} z^2 \right) \Big|_0^{4-x-y} dy dx \\ &= \int_0^4 \int_0^{4-x} \left(24 - 8x - \frac{1}{2} x^2 - 8y + xy - \frac{1}{2} y^2 \right) dy dx \\ &= \int_0^4 \left(24y - 8xy - \frac{1}{2} x^2 y - 4y^2 + \frac{1}{2} xy^2 - \frac{1}{6} y^3 \right) \Big|_0^{4-x} dx \\ &= \int_0^4 \left(\frac{128}{3} - 24x + 4x^2 - \frac{1}{6} x^3 \right) dx \end{aligned}$$

$$= \left(\frac{128}{3}x - 12x^2 + \frac{4}{3}x^3 - \frac{1}{24}x^4 \right) \Big|_0^4$$

$$= \frac{160}{3}$$

Solid 1 has greater mass.

Exercise

Suppose a wedge of cheese fills the region in the first octant bounded by the planes $y = z$, $y = 4$ and $x = 4$. You could divide the wedge into two equal pieces (by volume) if you sliced the wedge with the plane $x = 2$. Instead find a with $0 < a < 1$ such that slicing the wedge with the plane $y = a$ divides the wedge into two pieces of equal volume

Solution

$$V = \int_0^4 \int_0^4 \int_0^y dz dy dx$$

$$= \int_0^4 dx \int_0^4 z \Big|_0^y dy$$

$$= x \Big|_0^4 \int_0^4 y dy$$

$$= 2y^2 \Big|_0^4$$

$$= 32$$

$$V = \frac{1}{2}(32) = \int_0^4 \int_0^a \int_0^y dz dy dx = 16$$

$$\int_0^4 \int_0^a \int_0^y dz dy dx = \int_0^4 dx \int_0^a z \Big|_0^y dy$$

$$= x \Big|_0^4 \int_0^a y dy$$

$$= 2y^2 \Big|_0^a$$

$$= 2a^2 = 16$$

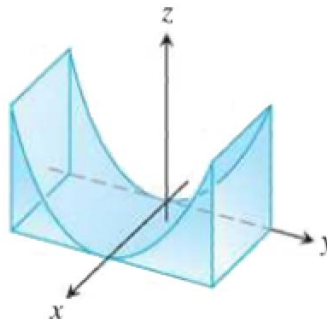
$$a = 2\sqrt{2}$$

Exercise

Find the volumes of the region between the cylinder $z = y^2$ and the xy -plane that is bounded by the planes $x = 0$, $x = 1$, $y = -1$, $y = 1$

Solution

$$\begin{aligned} V &= \int_0^1 \int_{-1}^1 \int_0^{y^2} dz dy dx \\ &= \int_0^1 \int_{-1}^1 [z]_0^{y^2} dy dx \\ &= \int_0^1 \int_{-1}^1 y^2 dy dx \\ &= \frac{1}{3} \int_0^1 [y^3]_{-1}^1 dx \\ &= \frac{2}{3} \int_0^1 dx \\ &= \frac{2}{3} \end{aligned}$$

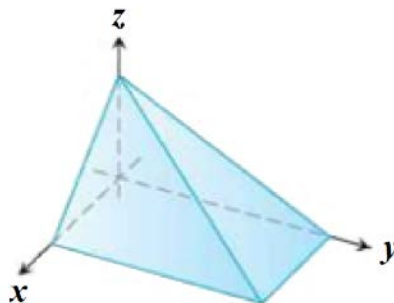


Exercise

Find the volumes of the region in the first octant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$

Solution

$$\begin{aligned} V &= \int_0^1 \int_0^{1-x} \int_0^{2-2z} dy dz dx \\ &= \int_0^1 \int_0^{1-x} (2-2z) dz dx \\ &= \int_0^1 [2z - z^2]_0^{1-x} dx \\ &= \int_0^1 [2(1-x) - (1-x)^2] dx \\ &= \int_0^1 (1-x)(2-1+x) dx \end{aligned}$$



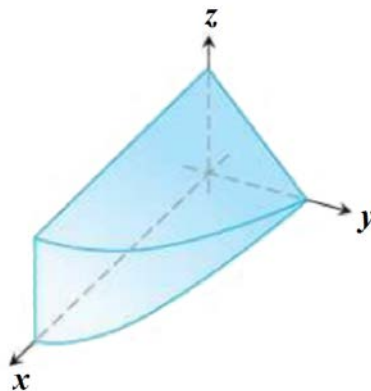
$$\begin{aligned}
&= \int_0^1 (1-x)(1+x) dx \\
&= \int_0^1 (1-x^2) dx \\
&= \left[x - \frac{1}{3}x^3 \right]_0^1 \\
&= 1 - \frac{1}{3} \\
&= \underline{\underline{\frac{2}{3}}}
\end{aligned}$$

Exercise

Find the volumes of the region in the first octant bounded by the coordinate planes and the plane $y + z = 2$, and the cylinder $x = 4 - y^2$

Solution

$$\begin{aligned}
V &= \int_0^4 \int_0^{\sqrt{4-x}} \int_0^{2-y} dz dy dx \\
&= \int_0^4 \int_0^{\sqrt{4-x}} (2-y) dy dx \\
&= \int_0^4 \left[2y - \frac{1}{2}y^2 \right]_0^{\sqrt{4-x}} dy dx \\
&= \int_0^4 \left[2\sqrt{4-x} - \frac{1}{2}(4-x) \right] dx \\
&= - \int_0^4 \left[2(4-x)^{1/2} - \frac{1}{2}(4-x) \right] d(4-x) \\
&= - \left[\frac{4}{3}(4-x)^{3/2} - \frac{1}{4}(4-x)^2 \right]_0^4 \\
&= - \left[0 - \left(\frac{4}{3}4^{3/2} - \frac{1}{4}4^2 \right) \right] \\
&= \underline{\underline{\frac{20}{3}}}
\end{aligned}$$

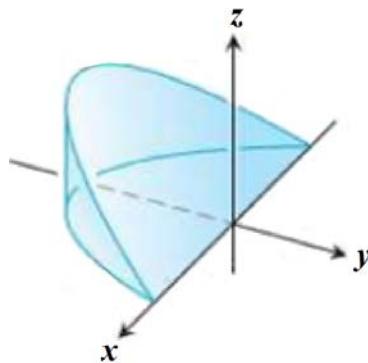


Exercise

Find the volumes of the wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y$, $z = 0$

Solution

$$\begin{aligned} V &= 2 \int_0^1 \int_{-\sqrt{1-x^2}}^0 \int_0^{-y} dz dy dx \\ &= -2 \int_0^1 \int_{-\sqrt{1-x^2}}^0 y dy dx \\ &= -2 \int_0^1 \left[\frac{1}{2} y^2 \right]_{-\sqrt{1-x^2}}^0 dx \\ &= \int_0^1 (1-x^2) dx \\ &= \left[x - \frac{1}{3} x^3 \right]_0^1 \\ &= 1 - \frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

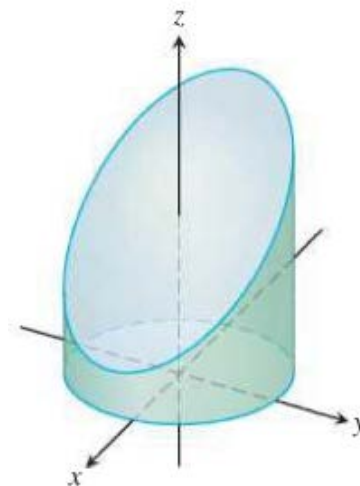


Exercise

Find the volumes of the region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$

Solution

$$\begin{aligned} V &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{3-x} dz dy dx \\ &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3-x) dy dx \\ &= 2 \int_{-2}^2 (3-x) \sqrt{4-x^2} dx \\ &= 6 \int_{-2}^2 \sqrt{4-x^2} dx - 2 \int_{-2}^2 x \sqrt{4-x^2} dx \quad d(4-x^2) = -2x dx \end{aligned}$$



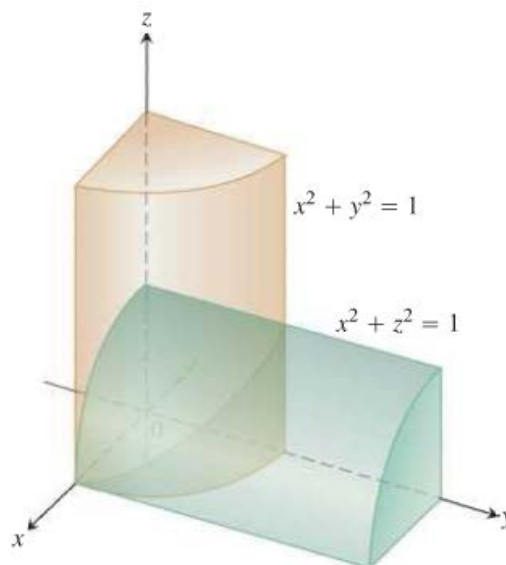
$$\begin{aligned}
&= 6 \int_{-2}^2 \sqrt{4-x^2} dx + \int_{-2}^2 (4-x^2)^{1/2} d(4-x^2) \\
&= 3 \left[x\sqrt{4-x^2} + 4 \sin^{-1} \frac{x}{2} \right]_{-2}^2 + \frac{2}{3} \left[(4-x^2)^{3/2} \right]_{-2}^2 \\
&= 3 \left[4 \sin^{-1} 1 - 4 \sin^{-1} (-1) \right] + \frac{2}{3} (0) \\
&= 12 \left(\frac{\pi}{2} + \frac{\pi}{2} \right) \\
&= 12\pi
\end{aligned}$$

Exercise

Find the volumes of the region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown below

Solution

$$\begin{aligned}
V &= 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz dy dx \\
&= 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2} dy dx \\
&= 8 \int_0^1 \sqrt{1-x^2} [y]_0^{\sqrt{1-x^2}} dx \\
&= 8 \int_0^1 (1-x^2) dx \\
&= 8 \left[x - \frac{1}{3}x^3 \right]_0^1 \\
&= \frac{16}{3}
\end{aligned}$$



Exercise

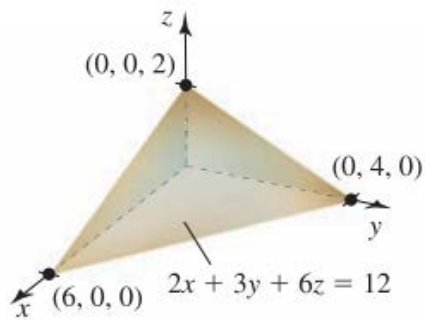
Find the volume of the solid in the first octant bounded by the plane $2x + 3y + 6z = 12$ and the coordinate planes

Solution

$$z = \frac{12 - 2x - 3y}{6} = 2 - \frac{x}{3} - \frac{y}{2} \quad z = 0 \rightarrow 2x + 3y = 12 \rightarrow y = 4 - \frac{2x}{3}$$

$$0 \leq z \leq 2 - \frac{x}{3} - \frac{y}{2}; \quad 0 \leq y \leq 4 - \frac{2x}{3}; \quad 0 \leq x \leq 6$$

$$\begin{aligned}
 V &= \int_0^6 \int_0^{4-\frac{2x}{3}} \int_0^{2-\frac{x}{3}-\frac{y}{2}} 1 \, dz \, dy \, dx \\
 &= \int_0^6 \int_0^{4-\frac{2x}{3}} z \Big|_0^{2-\frac{x}{3}-\frac{y}{2}} \, dy \, dx \\
 &= \int_0^6 \int_0^{4-\frac{2x}{3}} \left(2 - \frac{x}{3} - \frac{y}{2} \right) \, dy \, dx \\
 &= \int_0^6 \left(2y - \frac{x}{3}y - \frac{1}{4}y^2 \right) \Big|_0^{4-\frac{2x}{3}} \, dx \\
 &= \int_0^6 \left(8 - \frac{4}{3}x - \frac{4}{3}x + \frac{2}{9}x^2 - \frac{1}{4} \left(16 - \frac{16}{3}x + \frac{4}{9}x^2 \right) \right) \, dx \\
 &= \int_0^6 \left(4 - \frac{4}{3}x + \frac{1}{9}x^2 \right) \, dx \\
 &= 4x - \frac{2}{3}x^2 + \frac{1}{27}x^3 \Big|_0^6 \\
 &= 8 \text{ unit}^3
 \end{aligned}$$

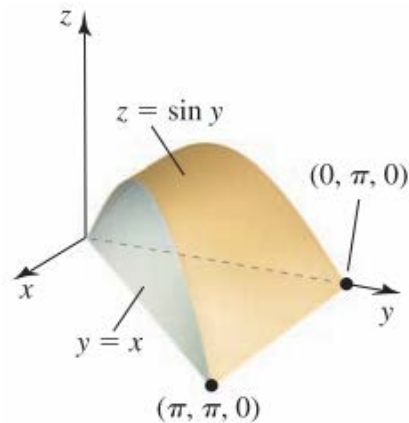


Exercise

Find the volume of the solid in the first octant formed when the cylinder $z = \sin y$, for $0 \leq y \leq \pi$, is sliced by the planes $y = x$ and $x = 0$

Solution

$$\begin{aligned}
 V &= \int_0^\pi \int_x^\pi \int_0^{\sin y} 1 \, dz \, dy \, dx \\
 &= \int_0^\pi \int_x^\pi z \Big|_0^{\sin y} \, dy \, dx \\
 &= \int_0^\pi \int_x^\pi \sin y \, dy \, dx \\
 &= - \int_0^\pi \cos y \Big|_x^\pi \, dx
 \end{aligned}$$



$$\begin{aligned}
&= - \int_0^{\pi} (-1 - \cos x) dx \\
&= (x + \sin x) \Big|_0^{\pi} \\
&= \pi \text{ unit}^3
\end{aligned}$$

Exercise

Find the volume of the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and bounded above the sphere $x^2 + y^2 + z^2 = 8$

Solution

$$z = \sqrt{x^2 + y^2} \quad z = \sqrt{8 - x^2 - y^2}$$

$$x^2 + y^2 + \left(\sqrt{x^2 + y^2}\right)^2 = 8 \Rightarrow x^2 + y^2 = 4 \rightarrow y = \pm\sqrt{4 - x^2}$$

$$(y = 0) \rightarrow x^2 = 4 \quad x = \pm 2$$

$$V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} 1 \, dz \, dy \, dx$$

$$= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} z \Big|_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} dy \, dx$$

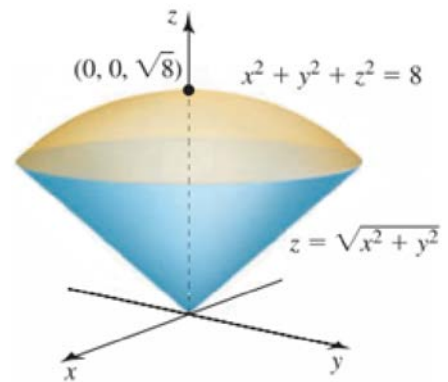
$$= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(\sqrt{8-x^2-y^2} - \sqrt{x^2+y^2} \right) dy \, dx$$

$$= \int_0^{2\pi} \int_0^2 \left(\sqrt{8-r^2} - r \right) r \, dr \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^2 \left(r\sqrt{8-r^2} - r^2 \right) dr$$

$$= (2\pi) \left(\int_0^2 \frac{-1}{2} (8-r^2)^{1/2} d(8-r^2) - \left(\frac{1}{3} r^3 \right) \Big|_0^2 \right)$$

$$= (\pi) \left(-\frac{2}{3} (8-r^2)^{3/2} \Big|_0^2 - \frac{8}{3} \right)$$



Convert to **Polar** coordinates

$$= \pi \left(-\frac{2}{3} (8 - 16\sqrt{2}) - \frac{8}{3} \right)$$

$$= \frac{32\pi}{3} (\sqrt{2} - 1) \Big| \text{ unit}^3$$

Exercise

The solid between the sphere $x^2 + y^2 + z^2 = 19$ and the hyperboloid $z^2 - x^2 - y^2 = 1$, for $z > 0$

Solution

$$z^2 = 1 + x^2 + y^2$$

The intersection of the sphere and hyperboloid:

$$x^2 + y^2 + z^2 = 19 \rightarrow x^2 + y^2 + 1 + x^2 + y^2 = 19$$

$$2x^2 + 2y^2 = 18 \rightarrow x^2 + y^2 = 9$$

$$V = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{1+x^2+y^2}}^{\sqrt{19-x^2-y^2}} dz dy dx$$

$$= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z \Big|_{\sqrt{1+x^2+y^2}}^{\sqrt{19-x^2-y^2}} dy dx$$

$$= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left(\sqrt{19-x^2-y^2} - \sqrt{1+x^2+y^2} \right) dy dx$$

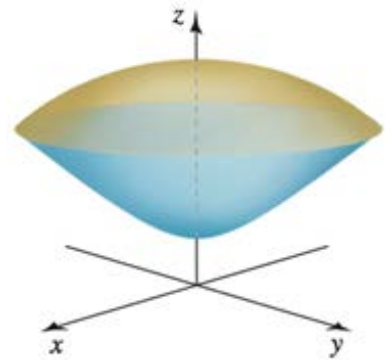
$$= \int_0^{2\pi} \int_0^3 \left(\sqrt{19-r^2} - \sqrt{1+r^2} \right) r dr d\theta$$

$$= \int_0^{2\pi} d\theta \left(-\frac{1}{2} \int_0^3 (19-r^2)^{1/2} d(19-r^2) - \frac{1}{2} \int_0^3 (1+r^2)^{1/2} d(1+r^2) \right)$$

$$= -\frac{2\pi}{3} \left((19-r^2)^{3/2} + (1+r^2)^{3/2} \right) \Big|_0^3$$

$$= -\frac{2\pi}{3} (10\sqrt{10} + 10\sqrt{10} - 19\sqrt{19} - 1)$$

$$= \frac{2\pi}{3} (1 - 20\sqrt{10} + 19\sqrt{19}) \Big|$$



Exercise

Find the volume of the prism in the first octant bounded below by $z = 2 - 4x$ and $y = 8$

Solution

$$z = 2 - 4x = 0 \Rightarrow x = \frac{1}{2}$$

$$V = \int_0^{1/2} \int_0^8 \int_0^{2-4x} 1 \, dz dy dx$$

$$= \int_0^{1/2} \int_0^8 (2 - 4x) \, dy dx$$

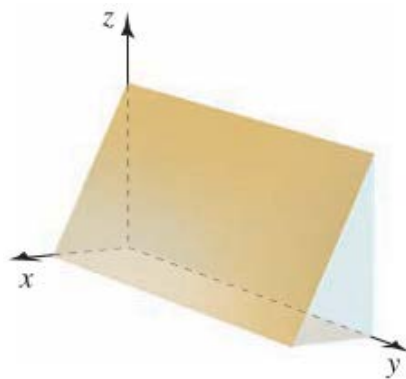
$$= \int_0^{1/2} (2 - 4x) y \Big|_0^8 \, dx$$

$$= 16 \int_0^{1/2} (1 - 2x) \, dx$$

$$= 16 \left(x - x^2 \right) \Big|_0^{1/2}$$

$$= 16 \left(\frac{1}{2} - \frac{1}{4} \right)$$

$$= \underline{4 \text{ unit}^3}$$



Exercise

Find the volume of the wedge above the xy -plane formed when the cylinder $x^2 + y^2 = 4$ is cut by the planes $z = 0$ and $y = -z$

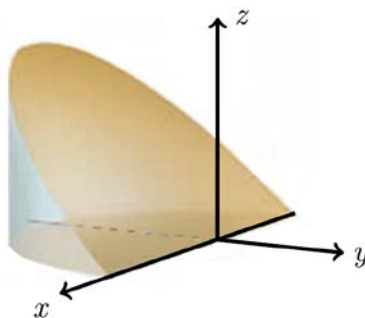
Solution

$$0 \leq z \leq -y \text{ (} y < 0 \text{)}; \quad -\sqrt{4-x^2} \leq y \leq 0; \quad y = 0 \rightarrow x^2 = 4 \Rightarrow -2 \leq x \leq 2$$

$$V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^0 \int_0^{-y} 1 \, dz dy dx$$

$$= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^0 (-y) \, dy dx$$

$$= -\frac{1}{2} \int_{-2}^2 \left(y^2 \right) \Big|_{-\sqrt{4-x^2}}^0 \, dx$$



$$\begin{aligned}
&= \frac{1}{2} \int_{-2}^2 (4 - x^2) dx \\
&= \frac{1}{2} \left(4x - \frac{1}{3} x^3 \right) \Big|_{-2}^2 \\
&= 8 - \frac{8}{3} \\
&= \frac{16}{3} \text{ unit}^3
\end{aligned}$$

Exercise

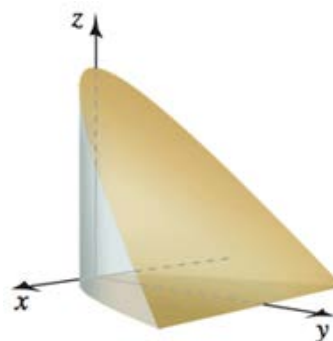
The wedge bounded by the parabolic cylinder $y = x^2$ and the planes $z = 3 - y$ and $z = 0$

Solution

$$z = 3 - y = 0 \rightarrow y = 3$$

$$y = x^2 = 3 \rightarrow x = \pm\sqrt{3}$$

$$\begin{aligned}
V &= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^3 \int_0^{3-y} dz dy dx \\
&= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^3 z \Big|_0^{3-y} dy dx \\
&= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{x^2}^3 (3 - y) dy dx \\
&= \int_{-\sqrt{3}}^{\sqrt{3}} \left(3y - \frac{1}{2} y^2 \right) \Big|_{x^2}^3 dx \\
&= \int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{9}{2} - 3x^2 + \frac{1}{2} x^4 \right) dx \\
&= \left(\frac{9}{2} x - x^3 + \frac{1}{10} x^5 \right) \Big|_{-\sqrt{3}}^{\sqrt{3}} \\
&= 2 \left(\frac{9}{2} \sqrt{3} - 3\sqrt{3} + \frac{9}{10} \sqrt{3} \right) \\
&= \frac{24\sqrt{3}}{5} \text{ unit}^3
\end{aligned}$$

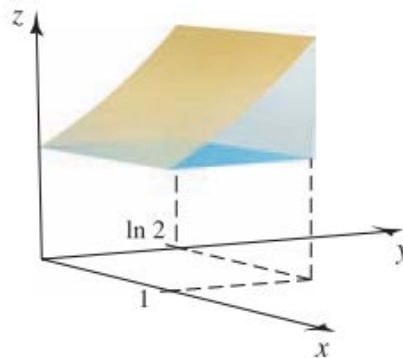


Exercise

Find the volume of the solid bounded by the surfaces $z = e^y$ and $z = 1$ over the rectangle $\{(x, y): 0 \leq x \leq 1, 0 \leq y \leq \ln 2\}$

Solution

$$\begin{aligned}
 V &= \int_0^1 \int_0^{\ln 2} \int_1^{e^y} 1 \, dz dy dx \\
 &= \int_0^1 dx \int_0^{\ln 2} (e^y - 1) dy \\
 &= x \Big|_0^1 (e^y - y) \Big|_0^{\ln 2} \\
 &= \underline{1 - \ln 2} \text{ unit}^3
 \end{aligned}$$



Exercise

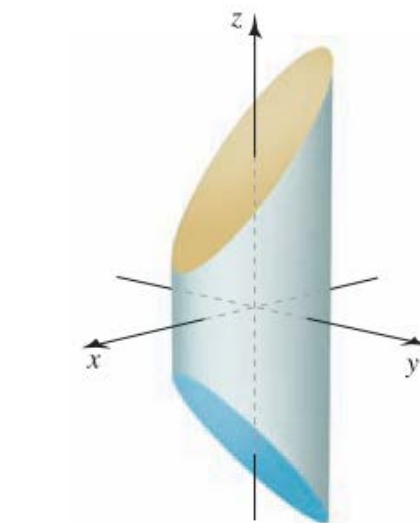
Find the volume of the wedge of the cylinder $x^2 + 4y^2 = 4$ created by the planes $z = 3 - x$ and $z = x - 3$

Solution

$$y^2 = \frac{1}{4}(4 - x^2) \rightarrow y = \pm \frac{1}{2}\sqrt{4 - x^2}$$

$$x^2 = 4 \rightarrow -2 \leq x \leq 2$$

$$\begin{aligned}
 V &= \int_{-2}^2 \int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} \int_{x-3}^{3-x} 1 \, dz dy dx \\
 &= \int_{-2}^2 \int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} (6 - 2x) dy dx \\
 &= \int_{-2}^2 (6 - 2x) y \Big|_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} dx \\
 &= \int_{-2}^2 (6 - 2x) \sqrt{4 - x^2} dx \\
 &= \int_{-2}^2 6\sqrt{4 - x^2} dx + \int_{-2}^2 \sqrt{4 - x^2} d(4 - x^2)
 \end{aligned}$$



$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$$

$$\begin{aligned}
&= 3x\sqrt{4-x^2} + 12\sin^{-1}\frac{x}{2} + \frac{2}{3}\sqrt{4-x^2} \Big|_{-2}^2 \\
&= 12\frac{\pi}{2} + 12\frac{\pi}{2} \\
&= \underline{12\pi \text{ unit}^3}
\end{aligned}$$

Exercise

Find the volume of the solid in the first octant bounded by the cone $z = 1 - \sqrt{x^2 + y^2}$ and the plane $x + y + z = 1$

Solution

$$0 \leq z \leq 1$$

$$z = 1 - \sqrt{x^2 + y^2} \rightarrow x^2 + y^2 = (1-z)^2 \Rightarrow x = \sqrt{(1-z)^2 - y^2}$$

$$1 - y - z \leq x \leq \sqrt{(1-z)^2 - y^2}$$

$$0 \leq y \leq 1 - z$$

$$V = \int_0^1 \int_0^{1-z} \int_{1-y-z}^{\sqrt{(1-z)^2 - y^2}} 1 \, dx \, dy \, dz$$

$$= \int_0^1 \int_0^{1-z} x \Big|_{1-y-z}^{\sqrt{(1-z)^2 - y^2}} dy \, dz$$

$$= \int_0^1 \int_0^{1-z} \left(\sqrt{(1-z)^2 - y^2} - 1 + y + z \right) dy \, dz$$

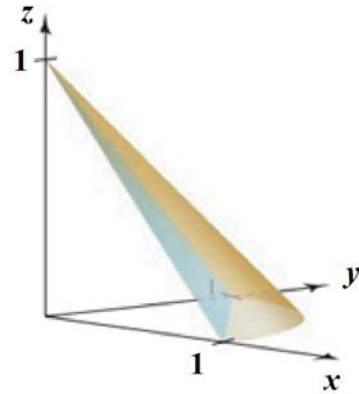
$$\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$$

$$= \int_0^1 \left[\frac{y}{2} \sqrt{(1-z)^2 - y^2} + \frac{1}{2} (1-z)^2 \sin^{-1} \left(\frac{y}{1-z} \right) - y + \frac{1}{2} y^2 + zy \right]_0^{1-z} dz$$

$$= \int_0^1 \left(\frac{1}{2} (1-z)^2 \sin^{-1}(1) + \frac{1}{2} (1-z)^2 - (1-z)^2 \right) dz$$

$$= \int_0^1 \left(\frac{\pi}{4} (1-z)^2 - \frac{1}{2} (1-z)^2 \right) dz$$

$$= \frac{\pi-2}{4} \int_0^1 (1-z)^2 \, d(1-z)$$



$$= \frac{\pi-2}{12} (1-z)^3 \Big|_0^1$$

$$= \frac{\pi-2}{12} \text{ unit}^3$$

Exercise

Find the volume of the solid bounded by $x = 0$, $x = 1 - z^2$, $y = 0$, $z = 0$, and $z = 1 - y$

Solution

$$V = \int_0^1 \int_0^{1-z^2} \int_0^{1-z} 1 \, dy \, dx \, dz$$

$$= \int_0^1 \int_0^{1-z^2} (1-z) \, dx \, dz$$

$$= \int_0^1 (1-z)x \Big|_0^{1-z^2} dz$$

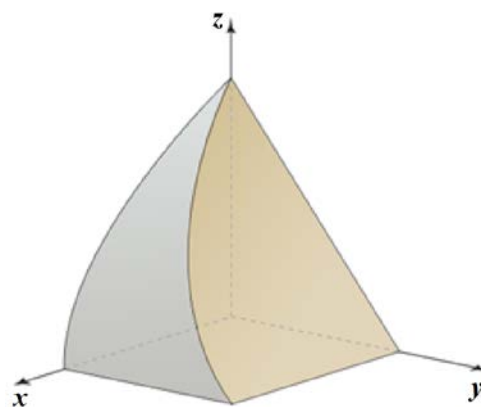
$$= \int_0^1 (1-z)(1-z^2) \, dz$$

$$= \int_0^1 (1 - z^2 - z + z^3) \, dz$$

$$= z - \frac{1}{3}z^3 - \frac{1}{2}z^2 + \frac{1}{4}z^4 \Big|_0^1$$

$$= 1 - \frac{1}{3} - \frac{1}{2} + \frac{1}{4}$$

$$= \frac{5}{12} \text{ unit}^3$$



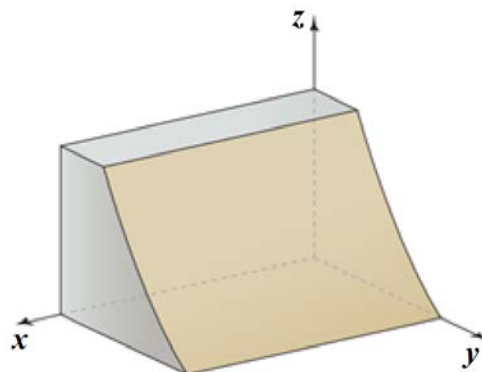
Exercise

Find the volume of the solid bounded by $x = 0$, $x = 2$, $y = 0$, $y = e^{-z}$, $z = 0$, and $z = 1$

Solution

$$V = \int_0^2 \int_0^1 \int_0^{e^{-z}} 1 \, dy \, dz \, dx$$

$$\begin{aligned}
&= \int_0^2 dx \int_0^1 y \left| e^{-z} \right|_0^1 dz \\
&= 2 \int_0^1 e^{-z} dz \\
&= -2e^{-z} \Big|_0^1 \\
&= -2(e^{-1} - 1) \\
&= \underline{2 - \frac{2}{e} \text{ unit}^3}
\end{aligned}$$

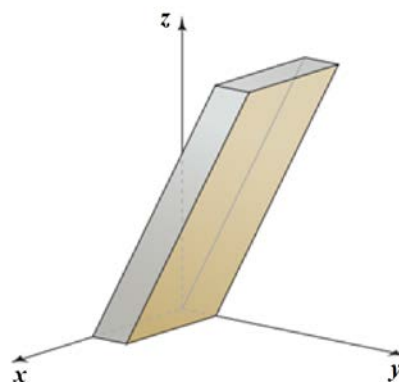


Exercise

Find the volume of the solid bounded by $x = 0$, $x = 2$, $y = z$, $y = z + 1$, $z = 0$, and $z = 4$

Solution

$$\begin{aligned}
V &= \int_0^2 \int_0^4 \int_z^{z+1} 1 \, dy \, dz \, dx \\
&= \int_0^2 \int_0^4 y \Big|_z^{z+1} dz \, dx \\
&= \int_0^2 dx \int_0^4 dz = (2)(4) \\
&= \underline{8} \text{ unit}^3
\end{aligned}$$

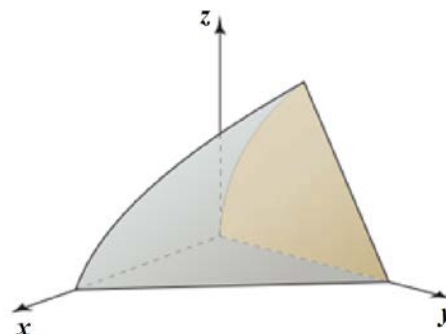


Exercise

Find the volume of the solid bounded by $x = 0$, $y = z^2$, $z = 0$, and $z = 2 - x - y$

Solution

$$\begin{aligned}
&y = 2 - x - z; \quad |x = 2 - z - y = 2 - z - z^2| \\
V &= \int_0^1 \int_0^{2-z-z^2} \int_{z^2}^{2-x-z} 1 \, dy \, dx \, dz \\
&= \int_0^1 \int_0^{2-z-z^2} (2 - x - z - z^2) \, dx \, dz
\end{aligned}$$



$$\begin{aligned}
&= \int_0^1 \left(\left((2-z-z^2)x - \frac{1}{2}x^2 \right) \right) \Big|_0^{2-z-z^2} dz \\
&= \frac{1}{2} \int_0^1 (2-z-z^2)^2 dz \\
&= \frac{1}{2} \int_0^1 (4-4z-3z^2+2z^3+z^4) dz \\
&= \frac{1}{2} \left(4z-2z^2-z^3+\frac{1}{2}z^4+\frac{1}{5}z^5 \right) \Big|_0^1 \\
&= \frac{1}{2} \left(4-2-1+\frac{1}{2}+\frac{1}{5} \right) \\
&= \underline{\underline{\frac{17}{20}}} \quad \text{unit}^3
\end{aligned}$$

Exercise

Find the volume of the solid common to the cylinders $z = \sin x$ and $z = \sin y$ over the square

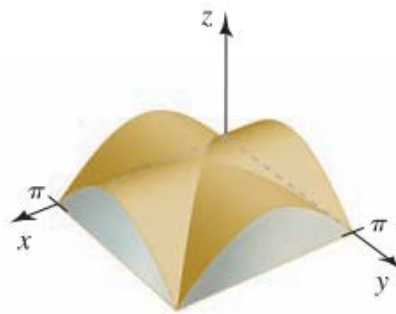
$$R = \{(x, y): 0 \leq x \leq \pi, 0 \leq y \leq \pi\}$$

Solution

$$z = \sin x = \sin y \rightarrow x = y \text{ or } y = \pi - x$$

$$\begin{aligned}
V &= 4 \int_0^{\pi/2} \int_x^{\pi-x} \int_0^{\sin y} 1 dz dy dx \\
&= 4 \int_0^{\pi/2} \int_x^{\pi-x} \sin y dy dx \\
&= -4 \int_0^{\pi/2} \cos y \Big|_x^{\pi-x} dx \\
&= -4 \int_0^{\pi/2} (\cos(\pi-x) - \cos x) dx \\
&= -4 \int_0^{\pi/2} (-2 \cos x) dx \\
&= 8 \sin x \Big|_0^{\pi/2} \\
&= \underline{\underline{8}} \quad \text{unit}^3
\end{aligned}$$

4: by symmetry, volume – 4 times



Exercise

Find the volume of the wedge of the square column $|x| + |y| = 1$ created by the planes $z = 0$ and $x + y + z = 1$

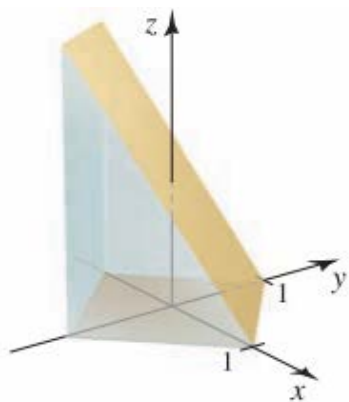
Solution

$$0 \leq z \leq 1 - x - y$$

$$|x| + |y| = 1 \rightarrow \begin{cases} x + y = 1 \Rightarrow y = 1 - x \\ -x + y = 1 \Rightarrow y = 1 + x \\ x - y = 1 \Rightarrow y = x - 1 \\ -x - y = 1 \Rightarrow y = -x - 1 \end{cases}$$

$$\begin{cases} y = -x - 1 \\ y = x + 1 \end{cases} \Rightarrow -1 \leq x \leq 0 \quad \begin{cases} y = x - 1 \\ y = -x + 1 \end{cases} \Rightarrow 0 \leq x \leq 1$$

$$\begin{aligned} V &= \int_{-1}^0 \int_{-x-1}^{x+1} \int_0^{1-x-y} 1 dz dy dx + \int_0^1 \int_{x-1}^{-x+1} \int_0^{1-x-y} 1 dz dy dx \\ &= \int_{-1}^0 \int_{-x-1}^{x+1} (1 - x - y) dy dx + \int_0^1 \int_{x-1}^{-x+1} (1 - x - y) dy dx \\ &= \int_{-1}^0 \left((1-x)y - \frac{1}{2}y^2 \right) \Big|_{-x-1}^{x+1} dx + \int_0^1 \left((1-x)y - \frac{1}{2}y^2 \right) \Big|_{x-1}^{-x+1} dx \\ &= \int_{-1}^0 2(1-x)(x+1) dx + \int_0^1 2(1-x)^2 dx \\ &= \int_{-1}^0 2(1-x^2) dx + 2 \int_0^1 (1-2x+x^2) dx \\ &= 2 \left(x - \frac{1}{3}x^3 \right) \Big|_{-1}^0 + 2 \left(x - x^2 + \frac{1}{3}x^3 \right) \Big|_0^1 \\ &= 2 \left(1 - \frac{1}{3} \right) + \frac{2}{3} \\ &= 2 \text{ unit}^3 \end{aligned}$$



Exercise

Find the volume of a right circular cone with height h and base radius r .

Solution

The equation of a circle is centered at the origin with radius r : $x^2 + y^2 = r^2$

$$-\sqrt{r^2 - x^2} \leq y \leq \sqrt{r^2 - x^2} \quad \& \quad -r \leq x \leq r$$

$$z = a - b\sqrt{x^2 + y^2} \begin{cases} z = h & \underline{h = a} \\ z = 0 & 0 = a - br = h - br \Rightarrow b = \frac{h}{r} \end{cases}$$

The equation of a cone with height h : $z = h - \frac{h}{r}\sqrt{x^2 + y^2}$

$$\begin{aligned} V &= \int_{-r}^r \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_0^{h-\frac{h}{r}\sqrt{x^2+y^2}} 1 dz dy dx \\ &= \int_{-r}^r \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \left(h - \frac{h}{r}\sqrt{x^2+y^2} \right) dy dx && \text{Let } x^2 + y^2 = R^2 \text{ (Polar Coordinates)} \\ &= \int_0^{2\pi} \int_0^r \left(h - \frac{h}{r}R \right) R dR d\theta \\ &= \int_0^{2\pi} d\theta \int_0^r \left(hR - \frac{h}{r}R^2 \right) dR \\ &= 2\pi \left(\frac{1}{2}hR^2 - \frac{h}{3r}R^3 \right) \Big|_0^r \\ &= 2\pi \left(\frac{1}{2}hr^2 - \frac{1}{3}hr^2 \right) \\ &= \frac{1}{3}\pi r^2 h \quad \text{unit}^3 \end{aligned}$$

Exercise

Find the volume of a tetrahedron whose vertices are located at $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$

Solution

The equation of the plane through the vertices: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$0 \leq z \leq c \left(1 - \frac{x}{a} - \frac{y}{b} \right) \quad 0 \leq y \leq b \left(1 - \frac{x}{a} \right) \quad 0 \leq x \leq a$$

$$\begin{aligned} V &= \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} 1 dz dy dx \\ &= \int_0^a \int_0^{b(1-\frac{x}{a})} c \left(1 - \frac{x}{a} - \frac{y}{b} \right) dy dx \end{aligned}$$

$$\begin{aligned}
&= c \int_0^a \left(\left(1 - \frac{x}{a}\right)y - \frac{1}{2b}y^2 \right) \Big|_0^{b\left(1-\frac{x}{a}\right)} dx \\
&= c \int_0^a \left(b\left(1 - \frac{x}{a}\right)^2 - \frac{1}{2}b\left(1 - \frac{x}{a}\right)^2 \right) dx \\
&= \frac{1}{2}bc \int_0^a \left(1 - \frac{2}{a}x + \frac{1}{a^2}x^2 \right) dx \\
&= \frac{1}{2}bc \left(x - \frac{1}{a}x^2 + \frac{1}{3a^2}x^3 \right) \Big|_0^a \\
&= \frac{1}{2}bc \left(a - a + \frac{1}{3}a \right) \\
&= \frac{abc}{6}
\end{aligned}$$

Exercise

Find the volume of a truncated cone of height h whose ends have radii r and R .

Solution

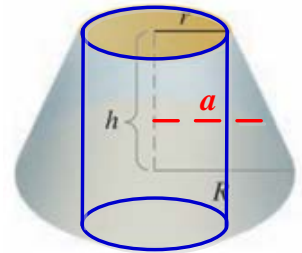
There are 2 volumes to consider:

1. Volume of the cylinder: $V_1 = \pi r^2 h$
2. Volume V_2 that remains when cylinder is removed.

V_2 is the annulus centered at the origin with inner radius r and outer radius R .

Using Polar Coordinates: the equation of the frustum is: $z = \frac{h}{R-r}(R-a)$

$$\begin{aligned}
V_2 &= \int_0^{2\pi} \int_r^R \int_0^{\frac{h}{R-r}(R-a)} a \, dz \, da \, d\theta \\
&= \int_0^{2\pi} \int_r^R \frac{h}{R-r}(R-a)a \, da \, d\theta \\
&= \frac{h}{R-r} \int_0^{2\pi} d\theta \int_r^R (Ra - a^2) \, da \\
&= \frac{2\pi h}{R-r} \left(\frac{1}{2}Ra^2 - \frac{1}{3}a^3 \right) \Big|_r^R \\
&= \frac{2\pi h}{R-r} \left(\frac{1}{2}R^3 - \frac{1}{3}R^3 - \frac{1}{2}Rr^2 + \frac{1}{3}r^3 \right)
\end{aligned}$$



$$\begin{aligned}
&= \frac{2\pi h}{R-r} \left(\frac{1}{6} R^3 - \frac{1}{2} Rr^2 + \frac{1}{3} r^3 \right) \\
&= \frac{1}{3} \frac{\pi h}{R-r} \left(R^3 - 3Rr^2 + 2r^3 \right) \\
V_1 + V_1 &= \pi r^2 h + \frac{1}{3} \frac{\pi h}{R-r} \left(R^3 - 3Rr^2 + 2r^3 \right) \\
&= \frac{1}{3} \frac{\pi h}{R-r} \left(3r^2 (R-r) + R^3 - 3Rr^2 + 2r^3 \right) \\
&= \frac{1}{3} \frac{\pi h}{R-r} \left(R^3 - r^3 \right) \\
&= \frac{1}{3} \frac{\pi h}{R-r} (R-r) \left(R^2 + rR + r^2 \right) \\
&= \frac{1}{3} \pi h \left(R^2 + rR + r^2 \right)
\end{aligned}$$

Exercise

Find the volume of the solid that is bounded above by the cylinder $z = 4 - x^2$, on the sides by the cylinder $x^2 + y^2 = 4$, and below by the xy -plane.

Solution

$$z = 4 - x^2 \rightarrow 0 \leq z \leq 4 - x^2$$

$$x^2 + y^2 = 4 \rightarrow -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}$$

$$\text{Since it is symmetric, then } 0 \leq y \leq \sqrt{4 - x^2}$$

$$y = 0 \rightarrow x = \pm 2 \quad 0 \leq x \leq 2$$

$$V = 4 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{4-x^2} dz dy dx$$

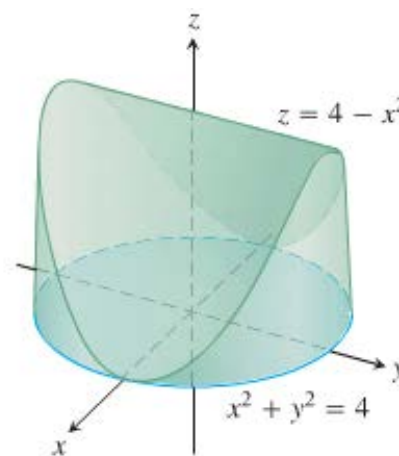
$$= 4 \int_0^2 \int_0^{\sqrt{4-x^2}} z \Big|_0^{4-x^2} dy dx$$

$$= 4 \int_0^2 \int_0^{\sqrt{4-x^2}} (4 - x^2) dy dx$$

$$= 4 \int_0^2 (4 - x^2) y \Big|_0^{\sqrt{4-x^2}} dx$$

$$= 4 \int_0^2 (4 - x^2)^{3/2} dx$$

$$x = 2 \sin \alpha \rightarrow 4 - x^2 = 4 \cos^2 \alpha$$



$$dx = 2 \cos \alpha d\alpha$$

$$\begin{cases} x = 2 & \rightarrow \alpha = \sin^{-1} 1 = \frac{\pi}{2} \\ x = 0 & \rightarrow \alpha = \sin^{-1} 0 = 0 \end{cases}$$

$$= 4 \int_0^{\pi/2} 16 \cos^4 \alpha d\alpha$$

$$= 64 \int_0^{\pi/2} \left(\frac{1 + \cos 2\alpha}{2} \right)^2 d\alpha$$

$$= 16 \int_0^{\pi/2} (1 + 2 \cos 2\alpha + \cos^2 2\alpha) d\alpha$$

$$= 16 \int_0^{\pi/2} \left(1 + 2 \cos 2\alpha + \frac{1}{2} + \frac{1}{2} \cos 4\alpha \right) d\alpha$$

$$= 16 \left(\frac{3}{2} \alpha + \sin 2\alpha + \frac{1}{8} \sin 4\alpha \right) \Big|_0^{\pi/2}$$

$$= 16 \left(\frac{3}{2} \alpha + \sin 2\alpha + \frac{1}{8} \sin 4\alpha \right) \Big|_0^{\pi/2}$$

$$= 16 \left(\frac{3\pi}{4} \right)$$

$$= 12\pi \text{ unit}^3$$

Exercise

Find the volume of the prism in the first octant bounded by the planes $y = 3 - 3x$ and $z = 2$

Solution

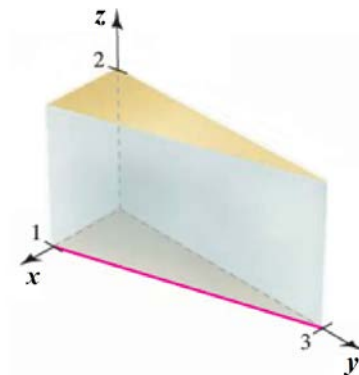
$$V = \int_0^1 \int_0^{3-3x} \int_0^2 dz dy dx$$

$$= \int_0^1 \int_0^{3-3x} z \Big|_0^2 dy dx$$

$$= 2 \int_0^1 \int_0^{3-3x} dy dx$$

$$= 2 \int_0^1 y \Big|_0^{3-3x} dx$$

$$= 2 \int_0^1 (3 - 3x) dx$$



$$\begin{aligned}
 &= 2 \left(3x - \frac{3}{2} x^2 \right) \Big|_0^1 \\
 &= 2 \left(3 - \frac{3}{2} \right) \\
 &= \underline{3 \text{ unit}^3}
 \end{aligned}$$

Exercise

Find the volume of the prism in the first octant bounded by the planes $x^2 + y^2 = 4$ and $x^2 + z^2 = 4$

Solution

$$x^2 + y^2 = 4 \rightarrow y^2 \leq 4 - x^2$$

$$-\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}$$

$$x^2 + z^2 = 4 \rightarrow z^2 \leq 4 - x^2$$

$$-\sqrt{4 - x^2} \leq z \leq \sqrt{4 - x^2}$$

$$x^2 \leq 4 \rightarrow -2 \leq x \leq 2$$

$$V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dz dy dx$$

$$= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} z \Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy dx$$

$$= 2 \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4-x^2} dy dx$$

$$= 2 \int_{-2}^2 \sqrt{4-x^2} y \Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= 4 \int_{-2}^2 (4 - x^2) dx$$

$$= 4 \left(4x - \frac{1}{3} x^3 \right) \Big|_{-2}^2$$

$$= 8 \left(8 - \frac{8}{3} \right)$$

$$= \underline{\underline{\frac{128}{3} \text{ unit}^3}}$$

