

Section 1.7 – Cramer's Rule

Cramer's Rule

Theorem

If $AX = B$ is a system of a linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(B_1)}{\det(A)} \quad x_2 = \frac{\det(B_2)}{\det(A)}, \quad \dots, \quad x_n = \frac{\det(B_n)}{\det(A)}$$

$$\text{Where } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\det(B_1) = \begin{bmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & & & \\ \vdots & & & \\ b_n & a_{n2} & & a_{nn} \end{bmatrix}$$

Example

Use Cramer's rule to solve

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ -2x_1 + x_2 &= 0 \\ -4x_1 + x_3 &= 0 \end{aligned}$$

Solution

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 7$$

$$|B_1| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$|B_2| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 2$$

$$|B_3| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 0 \end{vmatrix} = 4$$

$$x_1 = \frac{|B_1|}{|A|} = \frac{1}{7}$$

$$x_2 = \frac{|B_2|}{|A|} = \frac{2}{7}$$

$$x_3 = \frac{|B_3|}{|A|} = \frac{4}{7}$$

Solution: $\left(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right)$

Example

Use Cramer's Rule to solve.

$$\begin{aligned}x_1 + \quad + 2x_3 &= 6 \\-3x_1 + 4x_2 + 6x_3 &= 30 \\-x_1 - 2x_2 + 3x_3 &= 8\end{aligned}$$

Solution

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \Rightarrow \det(A) = 44$$

$$A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix} \Rightarrow \det(A_1) = -40$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix} \Rightarrow \det(A_2) = 72$$

$$A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix} \Rightarrow \det(A_3) = 152$$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = -\frac{10}{11}$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11}$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$

A Formula for A^{-1}

Theorem: Inverse of a matrix using its Adjoint

The i, j entry of A^{-1} is the cofactor C_{ji} (not C_{ij}) divided by $\det(A)$:

$$\text{Formula for } A^{-1}: \quad \left(A^{-1}\right)_{ij} = \frac{C_{ji}}{|A|} \quad \text{and} \quad A^{-1} = \frac{C^T}{|A|}$$

$$\boxed{A^{-1} = \frac{1}{\det(A)} \text{adj}(A)}$$

Example

Find the inverse matrix of $A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ using its adjoint.

Solution

$$C_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1; \quad C_{12} = -\begin{vmatrix} -2 & 0 \\ -4 & 1 \end{vmatrix} = 2; \quad C_{13} = \begin{vmatrix} -2 & 1 \\ -4 & 0 \end{vmatrix} = 4$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1; \quad C_{22} = \begin{vmatrix} 1 & 1 \\ -4 & 1 \end{vmatrix} = 5; \quad C_{23} = -\begin{vmatrix} 1 & 1 \\ -4 & 0 \end{vmatrix} = -4$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1; \quad C_{32} = -\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = -2; \quad C_{33} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3$$

$$C = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 5 & -2 \\ 4 & -4 & 3 \end{bmatrix} \quad \text{and} \quad \det(A) = \frac{1}{7} \quad \Rightarrow \quad A^{-1} = \frac{1}{7} \begin{bmatrix} 1 & -1 & -1 \\ 2 & 5 & -2 \\ 4 & -4 & 3 \end{bmatrix}$$

Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent

- a) A is invertible
- b) $Ax = 0$ has only the trivial solution
- c) The reduced row echelon form of A is I_n
- d) A can be expressed as a product of elementary matrices
- e) $Ax = b$ is consistent for every $n \times 1$ matrix b
- f) $\det(A) \neq 0$

Exercises Section 1.7 – Cramer's Rule

1. Use Cramer's Rule with ratios $\frac{\det B_j}{\det A}$ to solve $A\mathbf{x} = \mathbf{b}$. Also find the inverse matrix $A^{-1} = \frac{C^T}{\det A}$.

Why is the solution \mathbf{x} is the first part the same as column 3 of A^{-1} ? Which cofactors are involved in computing that column \mathbf{x} ?

$$Ax = b \quad \text{is} \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2. Verify that $\det(AB) = \det(BA)$ and determine whether the equality $\det(A+B) = \det(A) + \det(B)$ holds

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

3. Verify that $\det(kA) = k^n \det(A)$

$$a) \quad A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}, \quad k = 2$$

$$c) \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix}, \quad k = 3$$

$$b) \quad A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}, \quad k = -2$$

4. Solve by using Cramer's rule

$$a) \quad \begin{cases} 7x - 2y = 3 \\ 3x + y = 5 \end{cases}$$

$$b) \quad \begin{cases} 4x + 5y = 2 \\ 11x + y + 2z = 3 \\ x + 5y + 2z = 1 \end{cases}$$

$$c) \quad \begin{cases} x - 4y + z = 6 \\ 4x - y + 2z = -1 \\ 2x + 2y - 3z = -20 \end{cases}$$

$$d) \quad \begin{cases} -x_1 - 4x_2 + 2x_3 + x_4 = -32 \\ 2x_1 - x_2 + 7x_3 + 9x_4 = 14 \\ -x_1 + x_2 + 3x_3 + x_4 = 11 \\ -x_1 - 2x_2 + x_3 - 4x_4 = -4 \end{cases}$$

$$e) \quad \begin{cases} 2x - y + z = -1 \\ 3x + 4y - z = -1 \\ 4x - y + 2z = -1 \end{cases}$$

5. Show that the matrix A is invertible for all values of θ , then find A^{-1} using $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$

$$A = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$