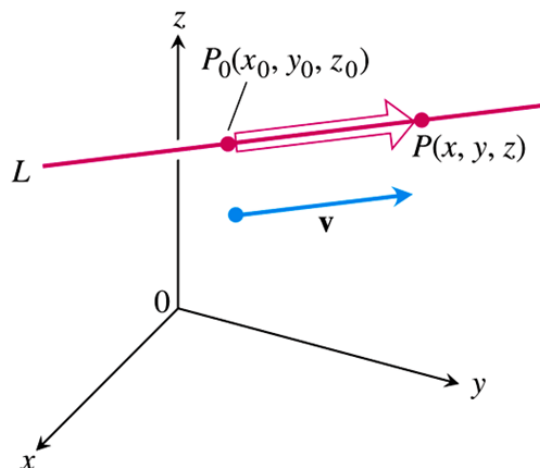


Section 1.4 – Lines and Curves in Space

Lines and Line Segments in Space



The expanded form of the equation $\overrightarrow{P_0P} = tv$ is

$$(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k} = t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

Vector Equation for a Line

A **vector equation for the line** L through $P_0(x_0, y_0, z_0)$ parallel to \mathbf{v} is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad -\infty < t < \infty$$

Where \mathbf{r} is the position vector of a point $P(x, y, z)$ on L and \mathbf{r}_0 is the position vector of $P_0(x_0, y_0, z_0)$.

Parametric Equations for a Line

A **standard parametrization** of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

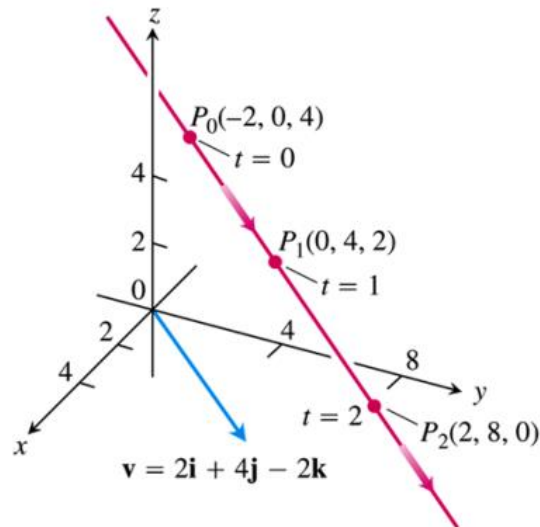
$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty$$

Example

Find the parametric equations for the line through $(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

Solution

$$x = -2 + 2t, \quad y = 4t, \quad z = 4 - 2t$$



Example

Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$

Solution

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t$$

The point $(x, y, z) = (-3 + 4t, 2 - 3t, -3 + 7t)$

On the line passes through P at $t = 0$ and Q at $t = 1$.

That implies the restriction $0 \leq t \leq 1$ to parameterize the segment

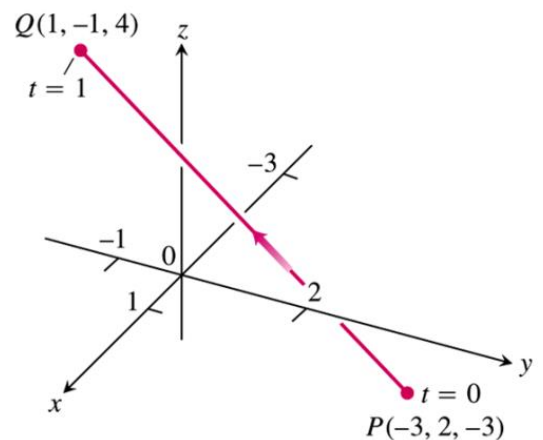
$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t, \quad 0 \leq t \leq 1$$

The position of a particle at time t is written:

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

$$= \mathbf{r}_0 + t|\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|}$$

Initial position
Time
Speed
Direction



Example

A helicopter is to fly directly from a helipad at the origin in the direction of the point (1, 1, 1) at a speed of 60 ft/sec. What is the position of the helicopter after 10 sec?

Solution

The unit vector: $\frac{(1, 1, 1)}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$

Therefore; the position of the helicopter at any time t is

$$\begin{aligned} r(t) &= r_0 + t\mathbf{u} \\ &= 0 + t(60)\left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right) \\ &= 20\sqrt{3} t(\hat{i} + \hat{j} + \hat{k}) \end{aligned}$$

The position after 10 sec:

$$\begin{aligned} r(10) &= 20\sqrt{3} (10)(\hat{i} + \hat{j} + \hat{k}) \\ &= 200\sqrt{3}(\hat{i} + \hat{j} + \hat{k}) \end{aligned}$$

The distance is traveled:

$$\begin{aligned} |r(10)| &= 200\sqrt{3}\sqrt{1^2 + 1^2 + 1^2} \\ &= \underline{600 \text{ ft}} \end{aligned}$$

Distance from a Point S to a Line through P parallel to \mathbf{v}

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$

Example

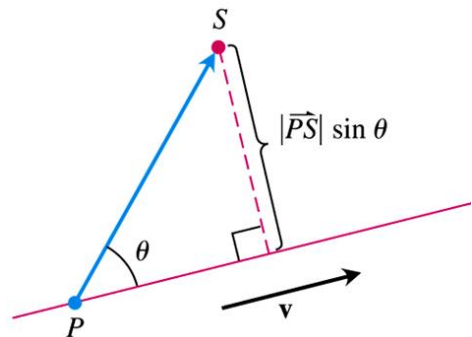
Find the distance from the point $S(1, 1, 5)$ to the line

$$L: x = 1 + t, \quad y = 3 - t, \quad z = 2t$$

Solution

At $t = 0$, the equations for L passes through $P(1, 3, 0)$ parallel to $\mathbf{v} = \hat{i} - \hat{j} + 2\hat{k}$

$$\begin{aligned} \overrightarrow{PS} &= (1-1)\hat{i} + (1-3)\hat{j} + (5-0)\hat{k} \\ &= -2\hat{j} + 5\hat{k} \end{aligned}$$

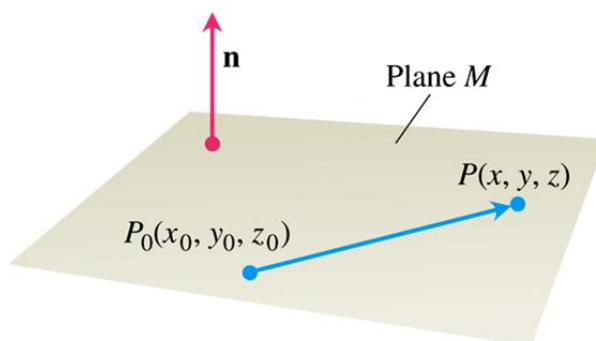


$$\begin{aligned}\overrightarrow{PS} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} \\ &= \hat{i} + 5\hat{j} + 2\hat{k}\end{aligned}$$

$$\begin{aligned}d &= \frac{|\overrightarrow{PS} \times \vec{v}|}{|\vec{v}|} \\ &= \frac{\sqrt{1+25+4}}{\sqrt{1+1+4}} \\ &= \frac{\sqrt{30}}{\sqrt{6}} \\ &= \sqrt{5}\end{aligned}$$

An Equation for a Plane in Space

A plane in space is determined by knowing a point on the plane and its “tilt” or orientation. This “tilt” is defined by specifying a vector that is perpendicular or normal to the plane.



The dot product $\vec{n} \cdot \overrightarrow{P_0P} = 0$, since $\overrightarrow{P_0P}$ is orthogonal to $\vec{n} = A\hat{i} + B\hat{j} + C\hat{k}$.

$$\vec{n} \cdot \overrightarrow{P_0P} = 0 \Leftrightarrow (A\hat{i} + B\hat{j} + C\hat{k}) \cdot ((x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k}) = 0$$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\vec{n} = A\hat{i} + B\hat{j} + C\hat{k}$ has

Vector equation: $\vec{n} \cdot \overrightarrow{P_0P} = 0$

Component equation: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Component equation simplified: $Ax + By + Cz = D$ where $D = Ax_0 + By_0 + Cz_0$

Example

Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\vec{n} = 5\hat{i} + 2\hat{j} - \hat{k}$

Solution

The component equation is

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0$$

$$5(x + 3) + 2y - z + 7 = 0$$

$$5x + 15 + 2y - z + 7 = 0$$

$$\underline{5x + 2y - z = -22}$$

Example

Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, $C(0, 3, 0)$.

Solution

The cross product

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix}$$

$$= 3\hat{i} + 2\hat{j} + 6\hat{k}$$

Is the normal to the plane.

We substitute the components of this vector and the coordinates of $A(0, 0, 1)$ into the component form of the equation to obtain

$$3(x - 0) + 2(y - 0) + 6(z - 1) = 0$$

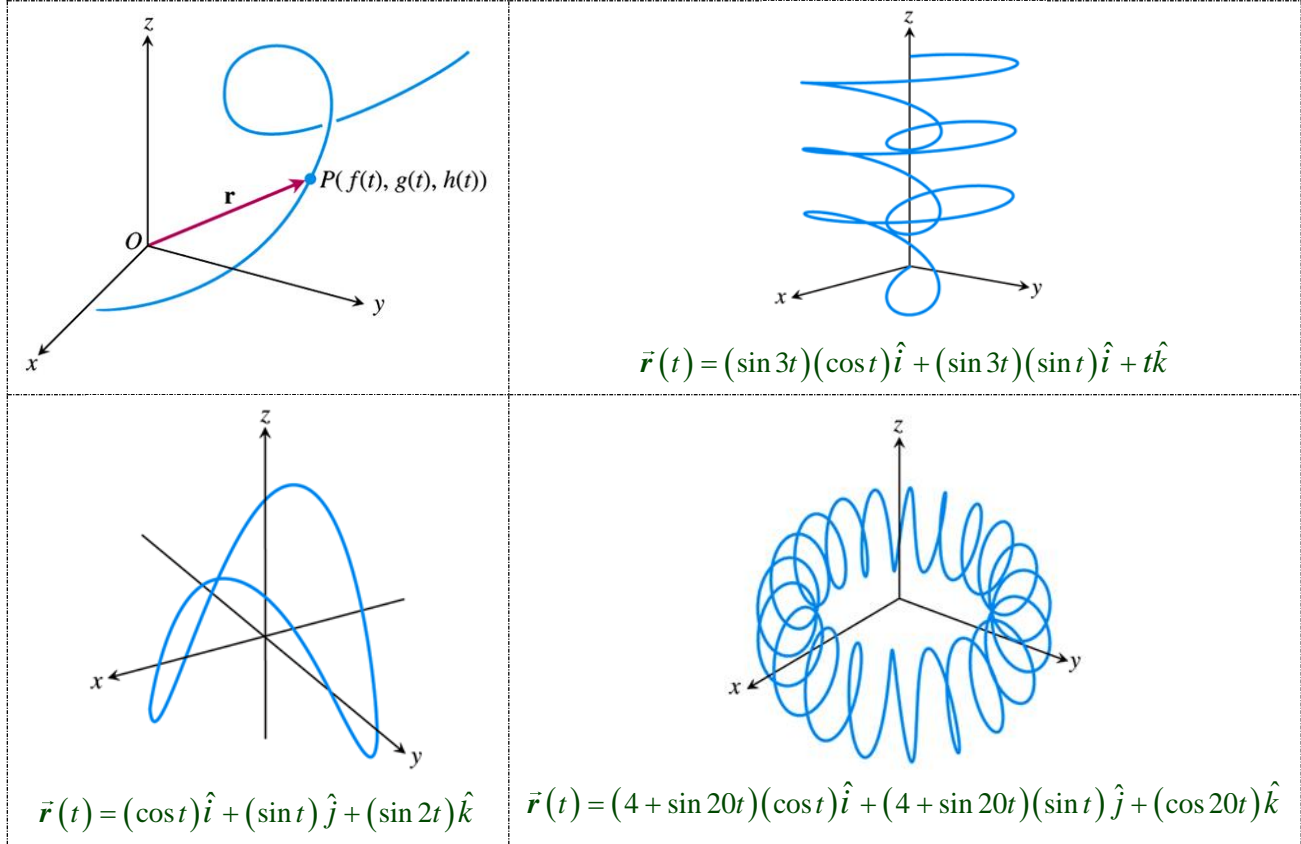
$$\underline{3x + 2y + 6z - 6 = 0} \quad \text{or} \quad \underline{3x + 2y + 6z = 6}$$

Curves

The coordinates for a particle moving through space during a time interval I , are defined as function on I :

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I.$$

The points $(x, y, z) = (f(t), g(t), h(t))$, $t \in I$, make up the curve in space that we call the particle's path.



Example

Graph the vector function $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}$

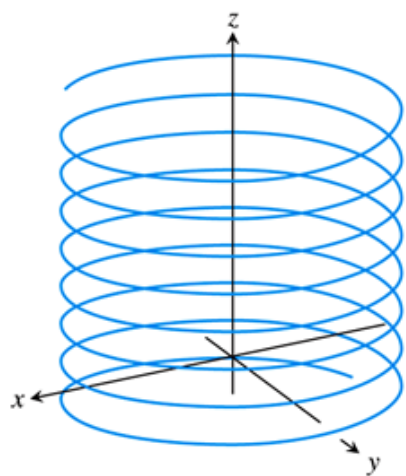
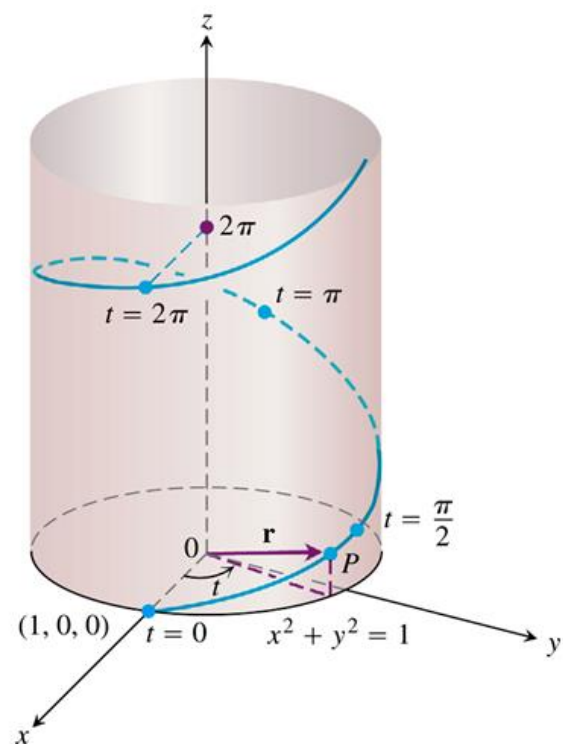
Solution

$$x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1$$

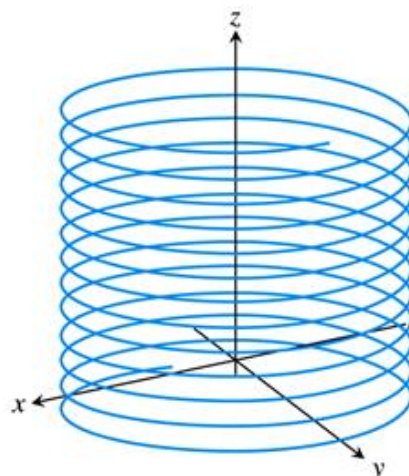
The curves traced by $\vec{r}(t)$ winds around a circular cylinder, satisfies the equation.

The curve rises as the \mathbf{k} -components $z = t$ increases. Each time t increases by 2π , the curve completes one turn around the cylinder. The curve is called a helix (from an old Greek word for “spiral”). The equations

$$x = \cos t, \quad y = \sin t, \quad z = t$$



$$\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + 0.3t\hat{k}$$



$$\vec{r}(t) = (\cos 5t)\hat{i} + (\sin 5t)\hat{j} + t\hat{k}$$

Limits and Continuity

Definition

Let $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ be a vector function with domain D , and \mathbf{L} a vector. We say that \mathbf{r} has limit \mathbf{L} as t approaches t_0 and write

$$\lim_{t \rightarrow t_0} \vec{r}(t) = \mathbf{L}$$

If, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$

$$|\vec{r}(t) - \mathbf{L}| < \varepsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta$$

Example

Find the limit of $\vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}$ as t approaches $\frac{\pi}{4}$

Solution

$$\begin{aligned} \lim_{t \rightarrow \pi/4} \vec{r}(t) &= \left(\lim_{t \rightarrow \pi/4} \cos t \right) \hat{i} + \left(\lim_{t \rightarrow \pi/4} \sin t \right) \hat{j} + \left(\lim_{t \rightarrow \pi/4} t \right) \hat{k} \\ &= \frac{\sqrt{2}}{2} \hat{i} + \frac{\sqrt{2}}{2} \hat{j} + \frac{\pi}{4} \hat{k} \end{aligned}$$

Definition

A vector function $\vec{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t \rightarrow t_0} \vec{r}(t) = \vec{r}(t_0)$. The function

is continuous if it is continuous at every point in its domain.

Lines of Intersection

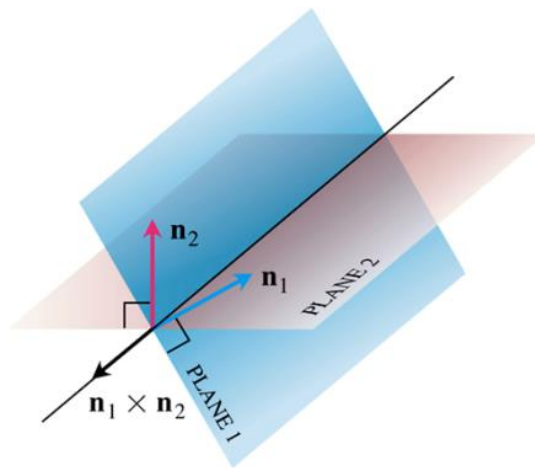
Example

Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$

Solution

The line of intersection of two planes is perpendicular to both planes' normal vectors \vec{n}_1 and \vec{n}_2 and therefore parallel to $\vec{n}_1 \times \vec{n}_2$.

$$\begin{aligned}\vec{n}_1 \times \vec{n}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} \\ &= \underline{14\hat{i} + 2\hat{j} + 15\hat{k}}\end{aligned}$$



Example

Find the point where the line $x = \frac{8}{3} + 2t$, $y = -2t$, $z = 1 + t$ intersects the plane $3x + 2y + 6z = 6$.

Solution

The point: $\left(\frac{8}{3} + 2t, -2t, 1 + t\right)$

lies in the plane if its coordinates satisfy the equation of the plane, that is, if

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) = 6$$

$$8 + 6t - 4t + 6 + 6t = 6$$

$$8t = -8$$

$$\underline{t = -1}$$

The point of intersection is: $\left(\frac{8}{3} + 2t, -2t, 1 + t\right)\bigg|_{t=-1} = \underline{\left(\frac{2}{3}, 2, 0\right)}$

The distance from a Point to a Plane

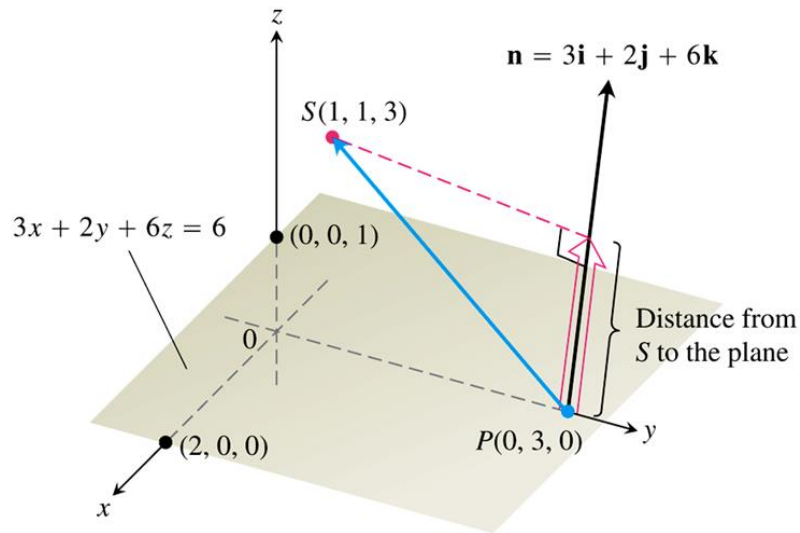
$$d = \left| \overrightarrow{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right|$$

Example

Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$

Solution

The coefficients in the equation $3x + 2y + 6z = 6$ give $\vec{n} = 3\hat{i} + 2\hat{j} + 6\hat{k}$



$$\overrightarrow{PS} = \hat{i} - 2\hat{j} + 3\hat{k}$$

$$|\vec{n}| = \sqrt{3^2 + 2^2 + 6^2}$$

$$= 7$$

The distance from S to the plane is

$$d = \left| \overrightarrow{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right|$$

$$= \left| (\hat{i} - 2\hat{j} + 3\hat{k}) \cdot \left(\frac{3}{7}\hat{i} + \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k} \right) \right|$$

$$= \left| \frac{3}{7} - \frac{4}{7} + \frac{18}{7} \right|$$

$$= \frac{17}{7}$$

Angles Between Planes

Example

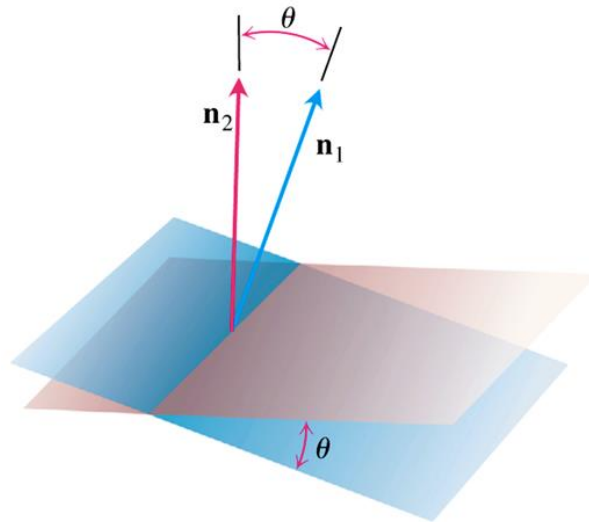
Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$

Solution

The vectors: $\vec{n}_1 = 3\hat{i} - 6\hat{j} - 2\hat{k}$, $\vec{n}_2 = 2\hat{i} + \hat{j} - 2\hat{k}$ are normal to the planes.

The angle between them is:

$$\begin{aligned}\theta &= \cos^{-1} \left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} \right) \\ &= \cos^{-1} \left(\frac{6 - 6 + 4}{\sqrt{9 + 36 + 4} \sqrt{4 + 1 + 4}} \right) \\ &= \cos^{-1} \left(\frac{4}{21} \right) \\ &\approx 1.38 \text{ rad}\end{aligned}$$



Exercises **Section 1.4 – Lines and Curves in Space**

1. Find the parametric equation for the line through the point $P(3, -4, -1)$ parallel to the vector $\hat{i} + \hat{j} + \hat{k}$
2. Find the parametric equation for the line through the points $P(1, 2, -1)$ and $Q(-1, 0, 1)$
3. Find the parametric equation for the line through the points $P(-2, 0, 3)$ and $Q(3, 5, -2)$
4. Find the parametric equation for the line through the origin parallel to the vector $2\hat{j} + \hat{k}$
5. Find the parametric equation for the line through the point $P(3, -2, 1)$ parallel to the line $x = 1 + 2t, \quad y = 2 - t, \quad z = 3t$
6. Find the parametric equation for the line through $(2, 4, 5)$ perpendicular to the plane $3x + 7y - 5z = 21$
7. Find the parametric equation for the line through $(2, 3, 0)$ perpendicular to the vectors $\vec{u} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{v} = 3\hat{i} + 4\hat{j} + 5\hat{k}$
8. Find the parameterization for the line segment joining the points $(0, 0, 0), \left(1, 1, \frac{3}{2}\right)$. Draw coordinate axes and sketch the segment, indicate the direction on increasing t for the parametrization.
9. Find the parameterization for the line segment joining the points $(1, 0, -1), (0, 3, 0)$. Draw coordinate axes and sketch the segment, indicate the direction on increasing t for the parametrization.
10. Find equation for the plane through $P_0(0, 2, -1)$ normal to $\mathbf{n} = 3\hat{i} - 2\hat{j} - \hat{k}$
11. Find equation for the plane through $(1, -1, 3)$ parallel to the plane $3x + y + z = 7$
12. Find equation for the plane through $(1, 1, -1), (2, 0, 2)$ and $(0, -2, 1)$
13. Find equation for the plane through $P_0(2, 4, 5)$ perpendicular to the line $x = 5 + t, \quad y = 1 + 3t, \quad z = 4t$
14. Find equation for the plane through $A(1, -2, 1)$ perpendicular to the vector from the origin to A .
15. Find the point of intersection of the lines $x = 2t + 1, \quad y = 3t + 2, \quad z = 4t + 3$ and $x = s + 2, \quad y = 2s + 4, \quad z = -4s - 1$, and find the plane determined by these lines.

16. Find the plane determined by the intersecting lines:

$$L_1 : x = -1 + t, \quad y = 2 + t, \quad z = 1 - t; \quad -\infty < t < \infty$$

$$L_2 : x = 1 - 4s, \quad y = 1 + 2s, \quad z = 2 - 2s; \quad -\infty < s < \infty$$

17. Find a plane through $P_0(2, 1, -1)$ and perpendicular to the line of intersection of the planes

$$2x + y - z = 3, \quad x + 2y + z = 2$$

Find the distance from the point to the plane

18. $(0, 0, 12), \quad x = 4t, \quad y = -2t, \quad z = 2t$

19. $(2, 1, -1), \quad x = 2t, \quad y = 1 + 2t, \quad z = 2t$

20. $(3, -1, 4), \quad x = 4 - t, \quad y = 3 + 2t, \quad z = -5 + 3t$

21. $(2, -3, 4), \quad x + 2y + 2z = 13$

22. $(0, 0, 0), \quad 3x + 2y + 6z = 6$

23. $(0, 1, 1), \quad 4y + 3z = -12$

24. $(6, 0, -6), \quad x - y = 4$

25. $(3, 0, 10), \quad 2x + 3y + z = 2$

Find the distance from the point to the line

26. $(2, 2, 0); \quad x = -t, \quad y = t, \quad z = -1 + t$

27. $(0, 4, 1); \quad x = 2 + t, \quad y = 2 + t, \quad z = t$

28. Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$

Find the angle between the planes

29. $x + y = 1, \quad 2x + y - 2z = 2$

30. $5x + y - z = 10, \quad x - 2y + 3z = -1$

31. $x = 7, \quad x + y + \sqrt{2}z = -3$

32. $x + y = 1, \quad y + z = 1$

33. Find the point in which the line meets the plane $x = 1 - t, \quad y = 3t, \quad z = 1 + t; \quad 2x - y + 3z = 6$

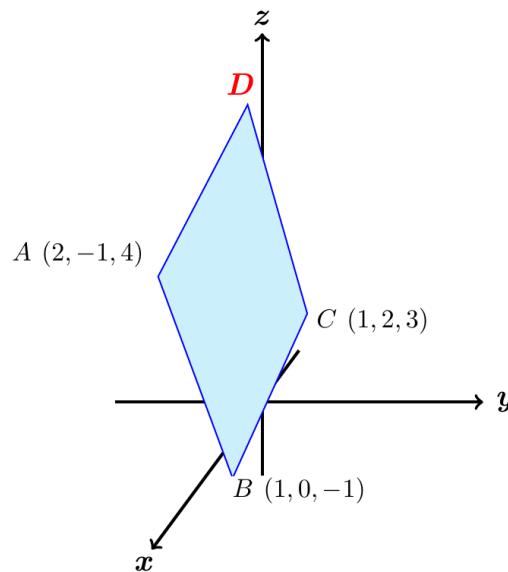
34. Find the point in which the line meets the plane

$$x = 2, \quad y = 3 + 2t, \quad z = -2 - 2t; \quad 6x + 3y - 4z = -12$$

35. Find an equation of the line through the point $(0, 1, 1)$ and parallel to the line

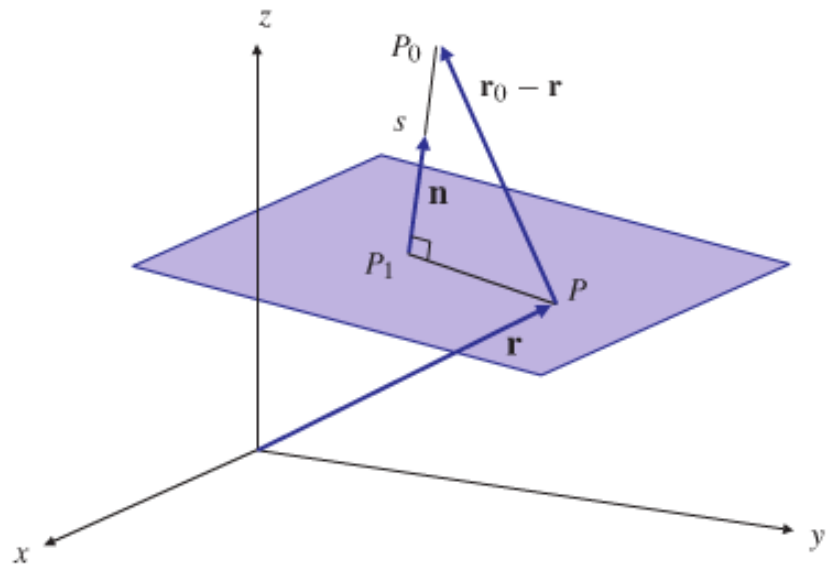
$$\mathbf{R}(t) = \langle 1 + 2t, 3 - 5t, 7 + 6t \rangle$$

36. Find an equation of the line through the point $(0, 1, 1)$ that is orthogonal to both $\langle 0, -1, 3 \rangle$ and $\langle 2, -1, 2 \rangle$
37. Find an equation of the line through the point $(0, 1, 1)$ that is orthogonal to the vector $\langle -2, 1, 7 \rangle$ and the y -axis
38. Suppose that \mathbf{n} is normal to a plane and that \mathbf{v} is parallel to the plane. Describe how you would find a vector \mathbf{n} that is both perpendicular to \mathbf{v} and parallel to the plane.
39. Given a point $(x_0, y_0, 0)$ and a vector $\mathbf{v} = \langle a, b, 0 \rangle$ in \mathbb{R}^3 , describe the set of points that satisfy the equation $\langle a, b, 0 \rangle \times \langle x - x_0, y - y_0, 0 \rangle = \mathbf{0}$. Use this result to determine an equation of a line in \mathbb{R}^2 passing through (x_0, y_0) parallel to the vector $\langle a, b \rangle$.
40. The parallelogram has vertices at $A(2, -1, 4)$, $B(1, 0, -1)$, $C(1, 2, 3)$ and D . Find



- The coordinates of D ,
 - The cosine of the interior angle of B
 - The vector projection of \overrightarrow{BA} onto \overrightarrow{BC} ,
 - The area of the parallelogram,
 - An equation for the plane of the parallelogram,
 - The areas of the orthogonal projection of the parallelogram on the three coordinate planes.
41. a) Find the distance from the point $P_0(x_0, y_0, z_0)$ to the plane P having equation

$$Ax + By + Cz = D$$



b) What is the distance from $(2, -1, 3)$ to the plane $2x - 2y - z = 9$?