Solution   *Section* **2.4 – Cross Product**

Exercise

Prove when the cross product $\vec{u} \times \vec{v}$ is perpendicular to $\vec{u}$, then $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$

Solution

Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = (u_1, u_2, u_3) \cdot (u_2 v_3 - u_3 v_2, \ u_3 v_1 - u_1 v_3, \ u_1 v_2 - u_2 v_1)$$

$$= u_1 (u_2 v_3 - u_3 v_2) + u_2 (u_3 v_1 - u_1 v_3) + u_3 (u_1 v_2 - u_2 v_1)$$

$$= u_1 u_2 v_3 - u_1 u_3 v_2 + u_2 u_3 v_1 - u_2 u_1 v_3 + u_3 u_1 v_2 - u_3 u_2 v_1$$

$$= 0$$

Exercise

Find $\vec{u} \times \vec{v}$, where $\vec{u} = (1, 2, -2)$ *and* $\vec{v} = (3, 0, 1)$ and show that $\vec{u} \times \vec{v}$ is perpendicular to $\vec{u}$ and to $\vec{v}$.

Solution

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= \left(\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix}, \ -\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}, \ \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \right)$$

$$= (2, \ -7, \ -6)$$

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = (1, \ 2, \ -2) \cdot (2, \ -7, \ -6)$$

$$= 2 - 14 + 12$$

$$= 0$$

$$\vec{v} \cdot (\vec{u} \times \vec{v}) = (3, \ 0, \ 1) \cdot (2, \ -7, \ -6)$$

$$= 6 - 0 - 6$$

$$= 0$$

$\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

Exercise

Given $\vec{u} = (3,\ 2,\ -1)$, $\vec{v} = (0,\ 2,\ -3)$, and $\vec{w} = (2,\ 6,\ 7)$ Compute the vectors

a) $\vec{u} \times \vec{v}$

b) $\vec{v} \times \vec{w}$

c) $\vec{u} \times (\vec{v} \times \vec{w})$

d) $(\vec{u} \times \vec{v}) \times \vec{w}$

e) $\vec{u} \times (\vec{v} - 2\vec{w})$

Solution

a) $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ 0 & 2 & -3 \end{vmatrix}$

$$= \left(\begin{vmatrix} 2 & -1 \\ 2 & -3 \end{vmatrix},\ -\begin{vmatrix} 3 & -1 \\ 0 & -3 \end{vmatrix},\ \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} \right)$$

$$= (-4,\ 9,\ 6)$$

b) $\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & -3 \\ 2 & 6 & 7 \end{vmatrix}$

$$= \left(\begin{vmatrix} 2 & -3 \\ 6 & 7 \end{vmatrix},\ -\begin{vmatrix} 0 & -3 \\ 2 & 7 \end{vmatrix},\ \begin{vmatrix} 0 & 2 \\ 2 & 6 \end{vmatrix} \right)$$

$$= (32,\ -6,\ -4)$$

c) $u \times (v \times w) = (3,\ 2,\ -1) \times (32,\ -6,\ -4)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ 32 & -6 & -4 \end{vmatrix}$$

$$= \left(\begin{vmatrix} 2 & -1 \\ -6 & -4 \end{vmatrix},\ -\begin{vmatrix} 3 & -1 \\ 32 & -4 \end{vmatrix},\ \begin{vmatrix} 3 & 2 \\ 32 & -6 \end{vmatrix} \right)$$

$$= (-14,\ -20,\ -82)$$

d) $(u \times v) \times w = (-4,\ 9,\ 6) \times (2,\ 6,\ 7)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 9 & 6 \\ 2 & 6 & 7 \end{vmatrix}$$

$$= \left(\begin{vmatrix} 9 & 6 \\ 6 & 7 \end{vmatrix}, \ - \begin{vmatrix} -4 & 6 \\ 2 & 7 \end{vmatrix}, \ \begin{vmatrix} -4 & 9 \\ 2 & 6 \end{vmatrix} \right)$$

$$= (27, \ 40, \ -42)$$

e) $\quad u \times (v - 2w) = (3, \ 2, \ -1) \times \left[(0, \ 2, \ -3) - 2(2, \ 6, \ 7) \right]$

$$= (3, \ 2, \ -1) \times (-4, \ -10, \ -17)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ -4 & -1 & -17 \end{vmatrix}$$

$$= \left(\begin{vmatrix} 2 & -1 \\ -10 & -17 \end{vmatrix}, \ - \begin{vmatrix} 3 & -1 \\ -4 & -17 \end{vmatrix}, \ \begin{vmatrix} 3 & 2 \\ -4 & -10 \end{vmatrix} \right)$$

$$= (-44, \ 47, \ -22)$$

Exercise

Use the cross product to find a vector that is orthogonal to both

a) $\quad \vec{u} = (-6, \ 4, \ 2), \quad \vec{v} = (3, \ 1, \ 5)$

b) $\quad \vec{u} = (1, \ 1, \ -2), \quad \vec{v} = (2, \ -1, \ 2)$

c) $\quad \vec{u} = (-2, \ 1, \ 5), \quad \vec{v} = (3, \ 0, \ -3)$

Solution

a) $\quad \vec{u} \times \vec{v} = (-6, \ 4, \ 2) \times (3, \ 1, \ 5)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6 & 4 & 2 \\ 3 & 1 & 5 \end{vmatrix}$$

$$= \left(\begin{vmatrix} 4 & 2 \\ 1 & 5 \end{vmatrix}, \ - \begin{vmatrix} -6 & 2 \\ 3 & 5 \end{vmatrix}, \ \begin{vmatrix} -6 & 4 \\ 3 & 1 \end{vmatrix} \right)$$

$$= (18, \ 36, \ -18)$$

b) $\quad \vec{u} \times \vec{v} = (1, \ 1, \ -2) \times (2, \ -1, \ 2)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= \left(\begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix}, \ - \begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix}, \ \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \right)$$

$$= (0, \ -6, \ -3)$$

c) $\vec{u} \times \vec{v} = (-2, \ 1, \ 5) \times (3, \ 0, \ -3)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 5 \\ 3 & 0 & -3 \end{vmatrix}$$

$$= \left(\begin{vmatrix} 1 & 5 \\ 0 & -3 \end{vmatrix}, \ -\begin{vmatrix} -2 & 5 \\ 3 & -3 \end{vmatrix}, \ \begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} \right)$$

$$= (-3, \ 9, \ -3)$$

Exercise

Find the area of the parallelogram determined by the given vectors

a) $\vec{u} = (1, \ -1, \ 2)$ *and* $\vec{v} = (0, \ 3, \ 1)$

b) $\vec{u} = (3, \ -1, \ 4)$ *and* $\vec{v} = (6, \ -2, \ 8)$

c) $\vec{u} = (2, \ 3, \ 0)$ *and* $\vec{v} = (-1, \ 2, \ -2)$

Solution

a) $Area = \| \vec{u} \times \vec{v} \|$

$$= \begin{Vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 0 & 3 & 1 \end{Vmatrix}$$

$$= \left| \left(\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix}, \ -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}, \ \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} \right) \right|$$

$$= |(-7, \ -1, \ 3)|$$

$$= \sqrt{7^2 + 1^2 + 3^2}$$

$$= \sqrt{59} \quad \left(unit^2 \right)$$

b) $Area = \| \vec{u} \times \vec{v} \|$

$$= \begin{Vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 4 \\ 6 & -2 & 8 \end{Vmatrix}$$

$$= \left| \left(\begin{vmatrix} -1 & 4 \\ -2 & 8 \end{vmatrix}, \ -\begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix}, \ \begin{vmatrix} 3 & -1 \\ 6 & -2 \end{vmatrix} \right) \right|$$

$$= \left|(0,\ 0,\ 0)\right|$$

$$= 0$$

c) $Area = \left\|\vec{u} \times \vec{v}\right\|$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 0 \\ -1 & 2 & -2 \end{vmatrix}$$

$$= \left|\left(\begin{vmatrix} 3 & 0 \\ 2 & -2 \end{vmatrix},\ -\begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix},\ \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}\right)\right|$$

$$= \left|(-6,\ 4,\ 7)\right|$$

$$= \sqrt{(-6)^2 + 4^2 + 7^2}$$

$$= \sqrt{101} \quad \left(unit^2\right)$$

Exercise

Find the area of the parallelogram with the given vertices $P_1(3,2),\quad P_2(5,4),\quad P_3(9,4),\quad P_4(7,2)$

Solution

$$\overrightarrow{P_1P_2} = (5-3, 4-2) = (2,\ 2)$$

$$\overrightarrow{P_4P_3} = (9-7, 4-2) = (2,\ 2)$$

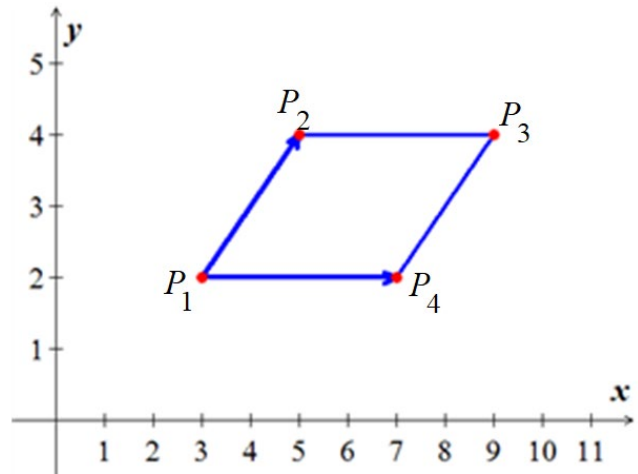$$\overrightarrow{P_1P_4} = (7-3, 2-2) = (4,\ 0)$$

$$\overrightarrow{P_2P_3} = (9-5, 4-4) = (4,\ 0)$$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_2} = (2,2) \times (4,0)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 0 \\ 4 & 0 & 0 \end{vmatrix}$$

$$= \left(\begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix},\ -\begin{vmatrix} 2 & 0 \\ 4 & 0 \end{vmatrix},\ \begin{vmatrix} 2 & 2 \\ 4 & 0 \end{vmatrix}\right)$$

$$= (0,\ 0,\ -8)$$

The area of the parallelogram is

$$\left\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_2}\right\| = \sqrt{0+0+(-8)^2}$$

$$= 8$$

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Exercise

Find the area of the triangle with the given vertices:

  a) $A(2, 0)$ $B(3, 4)$ $C(-1, 2)$

  b) $A(1, 1)$ $B(2, 2)$ $C(3, -3)$

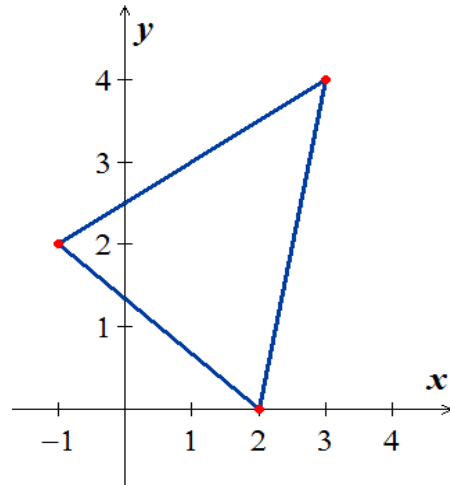 *c)* $P(2, 6, -1)$ $Q(1, 1, 1)$ $R = (4, 6, 2)$

Solution

  a) $\overrightarrow{AB} = (1, 4)$ $\overrightarrow{AC} = (-3, 2)$

  $\overrightarrow{AB} \times \overrightarrow{AC} = (1, 4, 0) \times (-3, 2, 0)$

$$= \left(\begin{vmatrix} 4 & 0 \\ 2 & 0 \end{vmatrix}, \ -\begin{vmatrix} 1 & 0 \\ -3 & 0 \end{vmatrix}, \ \begin{vmatrix} 1 & 4 \\ -3 & 2 \end{vmatrix} \right)$$

$$= (0, 0, 14)$$

$$\left\| \overrightarrow{AB} \times \overrightarrow{AC} \right\| = \sqrt{0 + 0 + 14^2}$$

$$= 14$$

The area of the triangle is

$$\tfrac{1}{2} \left\| \overrightarrow{AB} \times \overrightarrow{AC} \right\| = \tfrac{1}{2} 14$$

$$= 7$$

  b) $\overrightarrow{AB} = (1, 1)$ $\overrightarrow{AC} = (2, -4)$

  $\overrightarrow{AB} \times \overrightarrow{AC} = (1, 1, 0) \times (2, -4, 0)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 2 & -4 & 0 \end{vmatrix}$$

$$= \left(\begin{vmatrix} 1 & 0 \\ -4 & 0 \end{vmatrix}, \ -\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}, \ \begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} \right)$$

$$= (0, 0, -6)$$

$$\left\| \overrightarrow{AB} \times \overrightarrow{AC} \right\| = \sqrt{0 + 0 + (-6)^2}$$

$$= 6$$

The area of the triangle is

$$\tfrac{1}{2} \left\| \overrightarrow{AB} \times \overrightarrow{AC} \right\| = \tfrac{1}{2} (6)$$

$$= 3$$

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c) $\vec{PQ} = (-1,\ -5,\ 2)$ $\vec{PR} = (2,\ 0,\ 3)$

$\vec{PQ} \times \vec{PR} = (-1,\ -5,\ 2) \times (2,\ 0,\ 3)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -5 & 2 \\ 2 & 0 & 3 \end{vmatrix}$$

$$
\begin{array}{ccccc}
-1 & -5 & 2 & -1 & -5 \\
2 & 0 & 3 & 2 & 0
\end{array}
$$

$$= (-15,\ 7,\ 10)$$

$$\left\| \vec{PQ} \times \vec{PR} \right\| = \sqrt{(-15)^2 + 7^2 + 10^2}$$

$$= \sqrt{374}$$

The area of the triangle is

$$\tfrac{1}{2} \left\| \vec{PQ} \times \vec{PR} \right\| = \underline{\tfrac{1}{2}\sqrt{374}} \qquad unit^2$$

Exercise

a) Find the area of the parallelogram with edges $v = (3,\ 2)$ and $w = (1,\ 4)$

b) Find the area of the triangle with sides $\vec{v}$, $\vec{w}$, and $\vec{v} + \vec{w}$. Draw it.

c) Find the area of the triangle with sides $\vec{v}$, $\vec{w}$, and $\vec{v} - \vec{w}$. Draw it.

Solution

a) $Area = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix}$

$\underline{\ = 10\ }$

which is the parallelogram $OABC$

b) The area of the triangle with sides $\vec{v}$, $\vec{w}$, and $\vec{v} + \vec{w}$ is the triangle OCB or OAB which it is half the parallelogram (by definition).
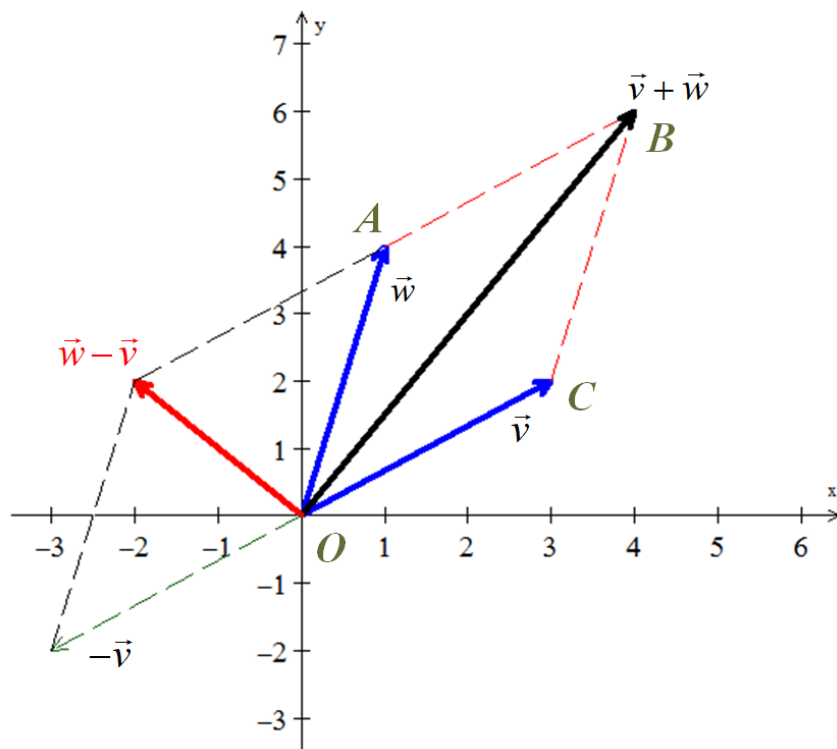
$$Area = 5$$

$$\vec{v} + \vec{w} = (3,\ 2) + (1,\ 4)$$

$$= (4,\ 6)$$

$$Area = \tfrac{1}{2} \begin{vmatrix} 3 & 4 \\ 2 & 6 \end{vmatrix}$$

$$= \tfrac{1}{2}(10)$$

$$\underline{\ = 5\ }$$

c) The area of the triangle with sides $\vec{v}$, $\vec{w}$, and $\vec{v} - \vec{w}$ is equivalent to the triangle OAC which it is half the parallelogram (by definition).

$$Area = 5$$

$$Area = \frac{1}{2}\begin{vmatrix} 2 & -2 \\ -3 & -2 \end{vmatrix}$$

$$= \frac{1}{2}\left|-10\right|$$

$$\underline{= 5}$$

Exercise

Find the volume of the parallelepiped with sides $\vec{u}$, $\vec{v}$, and $\vec{w}$.

a) $\vec{u} = (2, -6, 2)$, $\vec{v} = (0, 4, -2)$, $\vec{w} = (2, 2, -4)$

b) $\vec{u} = (3, 1, 2)$, $\vec{v} = (4, 5, 1)$, $\vec{w} = (1, 2, 4)$

Solution

a) $\vec{u} \bullet (\vec{v} \times \vec{w}) = \begin{vmatrix} 2 & -6 & 2 \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{vmatrix}$

$$\underline{= -16}$$

The volume of the parallelepiped is $\left|-16\right| = \underline{16}$ $unit^3$

b) $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 3 & 1 & 2 \\ 4 & 5 & 1 \\ 1 & 2 & 4 \end{vmatrix}$

$\underline{= 45}$

The volume of the parallelepiped is 45 $unit^3$

Exercise

Compute the scalar triple product $\vec{u} \cdot (\vec{v} \times \vec{w})$

a) $\vec{u} = (-2,\ 0,\ 6),\quad \vec{v} = (1,\ -3,\ 1),\quad \vec{w} = (-5,\ -1,\ 1)$

b) $\vec{u} = (-1,\ 2,\ 4),\quad \vec{v} = (3,\ 4,\ -2),\quad \vec{w} = (-1,\ 2,\ 5)$

c) $\vec{u} = (a,\ 0,\ 0),\quad \vec{v} = (0,\ b,\ 0),\quad \vec{w} = (0,\ 0,\ c)$

d) $\vec{u} = 3\hat{i} - 2\hat{j} - 5\hat{k},\quad \vec{v} = \hat{i} + 4\hat{j} - 4\hat{k},\quad \vec{w} = 3\hat{j} + 2\hat{k}$

e) $\vec{u} = (3,\ -1,\ 6)\quad \vec{v} = (2,\ 4,\ 3)\quad \vec{w} = (5,\ -1,\ 2)$

Solution

a) $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} -2 & 0 & 6 \\ 1 & -3 & 1 \\ -5 & -1 & 1 \end{vmatrix}$

$\underline{= -92}$

b) $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} -1 & 2 & 4 \\ 3 & 4 & -2 \\ -1 & 2 & 5 \end{vmatrix}$

$\underline{= -10}$

c) $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$

$\underline{= abc}$

d) $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix}$

$\underline{= 49}$

e) $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 3 & -1 & 6 \\ 2 & 4 & 3 \\ 5 & -1 & 2 \end{vmatrix}$

$\underline{= -110}$

Exercise

Use the cross product to find the sine of the angle between the vectors $\vec{u} = (2,\ 3,\ -6),\quad \vec{v} = (2,\ 3,\ 6)$

Solution

$\vec{u} \times \vec{v} = (2,\ 3,\ -6) \times (2,\ 3,\ 6)$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -6 \\ 2 & 3 & 6 \end{vmatrix}$

$= \left(\begin{vmatrix} 3 & -6 \\ 3 & 6 \end{vmatrix},\ -\begin{vmatrix} 2 & -6 \\ 2 & 6 \end{vmatrix},\ \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} \right)$

$\underline{= (36,\ -24,\ 0)}$

$\|\vec{u} \times \vec{v}\| = \sqrt{36^2 + (-24)^2 + 0}$

$= \sqrt{1872}$

$\underline{= 12\sqrt{13}}$

$\sin\theta = \left(\dfrac{\|\vec{u} \times \vec{v}\|}{\|\vec{u}\|\|\vec{v}\|} \right)$

$= \dfrac{12\sqrt{13}}{\sqrt{2^2 + 3^2 + (-6)^2}\sqrt{2^2 + 3^2 + 6^2}}$

$= \dfrac{12\sqrt{13}}{(7)(7)}$

$\underline{= \dfrac{12}{49}\sqrt{13}}$

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Exercise

Simplify $(\vec{u} + \vec{v}) \times (\vec{u} - \vec{v})$

Solution

$$(\vec{u} + \vec{v}) \times (\vec{u} - \vec{v}) = (\vec{u} + \vec{v}) \times \vec{u} - (\vec{u} + \vec{v}) \times \vec{v}$$

$$= (\vec{u} \times \vec{u}) + (\vec{v} \times \vec{u}) - \left[(\vec{u} \times \vec{v}) + (\vec{v} \times \vec{v}) \right]$$

$$= 0 + (\vec{v} \times \vec{u}) - \left[(\vec{u} \times \vec{v}) + 0 \right]$$

$$= (\vec{v} \times \vec{u}) - (\vec{u} \times \vec{v})$$

$$= (\vec{v} \times \vec{u}) - \left(-(\vec{v} \times \vec{u}) \right)$$

$$= (\vec{v} \times \vec{u}) + (\vec{v} \times \vec{u})$$

$$= 2(\vec{v} \times \vec{u})$$

Exercise

Prove Lagrange's identity: $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \ \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$

Solution

Let $\vec{u} = \left(u_1, \ u_2, \ u_3 \right)$ and $\vec{v} = \left(v_1, \ v_2, \ v_3 \right)$

$$\|\vec{u}\|^2 = u_1^2 + u_2^2 + u_3^2$$

$$\|\vec{v}\|^2 = v_1^2 + v_2^2 + v_3^2$$

$$(\vec{u} \cdot \vec{v})^2 = \left(u_1 v_1 + u_2 v_2 + u_3 v_3 \right)^2$$

$$\|\vec{u} \times \vec{v}\|^2 = \left(u_2 v_3 - u_3 v_2 \right)^2 + \left(u_3 v_1 - u_1 v_3 \right)^2 + \left(u_1 v_2 - u_2 v_1 \right)^2$$

$$= u_2^2 v_3^2 - 2 u_2 v_3 u_3 v_2 + u_3^2 v_2^2 + u_3^2 v_1^2 - 2 u_3 v_1 u_1 v_3 + u_1^2 v_3^2 + u_1^2 v_2^2 - 2 u_2 v_1 u_2 v_1 + u_2^2 v_1^2$$

$$\|\vec{u}\|^2 \ \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2 = \left(u_1^2 + u_2^2 + u_3^2 \right)\left(v_1^2 + v_2^2 + v_3^2 \right) - \left(u_1 v_1 + u_2 v_2 + u_3 v_3 \right)^2$$

$$= u_1^2 v_1^2 + u_1^2 v_2^2 + u_1^2 v_3^2 + u_2^2 v_1^2 + u_2^2 v_2^2 + u_2^2 v_3^2 + u_3^2 v_1^2 + u_3^2 v_2^2 + u_3^2 v_3^2$$

$$- u_1^2 v_1^2 - u_1 v_1 u_2 v_2 - u_1 v_1 u_3 v_3$$

$$- u_2 v_2 u_1 v_1 - u_2^2 v_2^2 - u_2 v_2 u_3 v_3$$

$$- u_1 v_1 u_3 v_3 - u_2 v_2 u_3 v_3 - u_3^2 v_3^2$$

$$= u_2^2 v_3^2 - 2 u_2 v_2 u_3 v_3 + u_3^2 v_2^2$$

$$+ u_3^2 v_1^2 - 2 u_1 v_1 u_3 v_3 + u_1^2 v_3^2$$

$$+ u_1^2 v_2^2 - 2 u_1 v_1 u_2 v_2 + u_2^2 v_1^2$$

$$\Rightarrow \underline{\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \; \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2}$$

Exercise

Polar coordinates satisfy $x = r \cos \theta$ *and* $y = \sin \theta$. Polar area $J \, dr \, d\theta$ includes J:

$$J = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

The two columns are orthogonal. Their lengths are _____. Thus $J =$ _____.

Solution

The length of the first column is:

$$\ell_1 = \sqrt{\cos^2 \theta + \sin^2 \theta}$$

$$\underline{= 1}$$

The length of the second column is:

$$\ell_2 = \sqrt{r^2 \sin^2 \theta + r^2 \cos^2 \theta}$$

$$= \sqrt{r^2 \left(\sin^2 \theta + \cos^2 \theta \right)}$$

$$= \sqrt{r^2}$$

$$\underline{= r}$$

So, J is the product $1. \, r = r$.

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

$$= r\left(\cos^2\theta + \sin^2\theta\right)$$

$$\underline{= r}$$

Exercise

Prove that $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$ if and only if $\vec{u}$ and $\vec{v}$ are parallel vectors.

Solution

If $\vec{u}$ and $\vec{v}$ are parallel vectors, then $\vec{u} \times \vec{v} = 0$
Which the two vectors are collinear, which implies that $\vec{u} = a\vec{v}$

$$\|\vec{u} + \vec{v}\| = \|\vec{u} + a\vec{u}\|$$

$$= \|(1+a)\vec{u}\|$$

$$= (1+a)\|\vec{u}\|$$

$$= \|\vec{u}\| + a\|\vec{u}\|$$

$$= \|\vec{u}\| + \|a\vec{u}\|$$

$$= \|\vec{u}\| + \|\vec{v}\| \quad \checkmark$$

Exercise

State the following statements as True or False

a) The cross product of two nonzero vectors $\vec{u}$ and $\vec{v}$ is a nonzero vector if and only if $\vec{u}$ and $\vec{v}$ are not parallel.

b) A normal vector to a plane can be obtained by taking the cross product of two nonzero and noncollinear vectors lying in the plane.

c) The scalar triple product of $\vec{u}$, $\vec{v}$, and $\vec{w}$ determines a vector whose length is equal to the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$.

d) If $\vec{u}$ and $\vec{v}$ are vectors in 3-space, then $\|\vec{u} \times \vec{v}\|$ is equal to the area of the parallelogram determine by $\vec{u}$ and $\vec{v}$.

e) For all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in R^3, the vectors $(\vec{u} \times \vec{v}) \times \vec{w}$ and $\vec{u} \times (\vec{v} \times \vec{w})$ are the same.

f) If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in R^3, where $\vec{u}$ is nonzero and $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, then $\vec{v} = \vec{w}$

Solution

a) True,

$\vec{u} \times \vec{v} = \|\vec{u}\| \|\vec{v}\| \sin\theta = 0$ if $\theta = 0$ which the two vectors are parallel.

b) True;

The cross product of two nonzero and non collinear vectors will be perpendicular to both vectors, hence normal to the plane containing the vectors.

c) False;

The scalar triple product is a scalar, not a vector.

d) True;

e) False;

Let $\vec{u} = \hat{i} \quad \vec{v} = \vec{w} = \hat{j}$

$$(\vec{u} \times \vec{v}) \times \vec{w} = (\hat{i} \times \hat{j}) \times \hat{j}$$
$$= \hat{k} \times \hat{j}$$
$$\underline{\underline{= -\hat{i}}}$$

$$\vec{u} \times (\vec{v} \times \vec{w}) = \hat{i} \times (\hat{j} \times \hat{j})$$
$$= \hat{i} \times \vec{0}$$
$$\underline{\underline{= \vec{0}}}$$

Hence, $(\vec{u} \times \vec{v}) \times \vec{w} \neq (\vec{u} \times \vec{v}) \times \vec{w}$

f) False;

Let $\vec{u} = \hat{i} + \hat{j} \quad \vec{v} = \hat{i} + \hat{j} + \hat{k} \quad \vec{w} = -\hat{i} - \hat{j} + \hat{k}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$
$$\underline{\underline{= \hat{i} - \hat{j}}}$$

$$\vec{u} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{vmatrix}$$
$$\underline{\underline{= \hat{i} - \hat{j}}}$$

$\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, but $\vec{v} \neq \vec{w}$