Solution   *Section* **2.3 – Mathematical Induction**

Exercise

Find all positive integers n for which the given statement is not true

$a)$ $3^n > 6n$ $b)$ $3^n > 2n+1$ $c)$ $2^n > n^2$ $d)$ $n! > 2n$

Solution

$a)$ $n = 1$ $3 < 6$

$n = 2$ $3^2 < 18$

$n = 3$, $27 > 18$

The statement is true for all $n \geq 3$ $3^n > 6n$

The statement is not true for $\boxed{n = 1, \ 2}$

$b)$ $n = 1$; $3 = 3$

$n = 2$; $9 > 5$

The statement is true for all $n \geq 2$ $3^n > 2n+1$

The statement is not true for $\boxed{n = 1}$

$c)$ $n = 1$; $2 < 4$

$n = 2$; $4 = 4$

$n = 3$; $8 < 9$

$n = 4$; $16 = 16$

$n = 5$; $32 > 25$

The statement is true for all $n \geq 5$; $2^n > n^2$

The statement is not true for $\boxed{n = 1, \ 2, \ 3, \ 4}$

$d)$ $n = 1$; $1 < 2$

$n = 2$; $2 < 4$

$n = 3$; $6 = 6$

$n = 4$; $12 > 8$

The statement is true for all $n \geq 4$; $n! > 2n$

The statement is not true for $\boxed{n = 1, \ 2, \ 3}$

Exercise

Prove that the statement is true for every positive integer n. $2+4+6+\ldots+2n=n(n+1)$

Solution

(1) For $n=1 \Rightarrow 2=1(1+1)=2$; hence P_1 is true.

(2) Assume $2+4+6+\ldots+2k=k(k+1)$ is true

$$\Rightarrow 2+4+6+\ldots+2k+2(k+1)=(k+1)(k+1+1)?$$

$$2+4+6+\ldots+2k+2(k+1)=2+4+6+\ldots+2k+2(k+1)$$
$$=k(k+1)+2(k+1) \qquad \textit{Factor } (k+1)$$
$$=(k+1)(k+2)$$
$$=(k+1)(k+1+1) \; \checkmark \qquad \text{Hence } P_{k+1} \text{ is also true.}$$

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1+3+5+\ldots+(2n-1)=n^2$

Solution

(1) For $n=1 \Rightarrow 1\overset{?}{=}1^2=1$; hence P_1 is true.

(2) Assume $1+3+5+\ldots+(2k-1)=k^2$ is true

$$\Rightarrow 1+3+5+\ldots+(2(k+1)-1)=(k+1)^2?$$

$$1+3+5+\ldots+(2k-1)+(2(k+1)-1)=1+3+5+\ldots+(2k-1)+(2k+2-1)$$
$$=k^2+(2k+1)$$
$$=k^2+2k+1$$
$$=(k+1)^2 \; \checkmark \qquad \text{Hence } P_{k+1} \text{ is also true.}$$

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $2+7+12+\ldots+(5n-3)=\frac{1}{2}n(5n-1)$

Solution

(1) For $n=1 \Rightarrow 2\overset{?}{=}\frac{1}{2}(1)(5(1)-1)=\frac{1}{2}(4)=2$; hence P_1 is true.

(2) Assume $2+7+12+\ldots+(5k-3)=\frac{1}{2}k(5k-1)$ is true

$$2+7+12+\ldots+(5(k+1)-3)=\frac{1}{2}(k+1)(5(k+1)-1)?$$

$$2+7+12+\ldots+(5k-3)+(5(k+1)-3)=2+7+12+\ldots+(5k-3)+(5k+5-3)$$

$$=\frac{1}{2}k(5k-1)+(5k+2)\frac{2}{2}$$

$$=\frac{1}{2}\left[5k^2-k+10k+4\right]$$

$$=\frac{1}{2}\left[5k^2-k+5k+5k+5-1\right]$$

$$=\frac{1}{2}\left[k(5k-1+5)+5k+5-1\right]$$

$$=\frac{1}{2}\left[(k+1)(5k+5-1)\right]$$

$$=\frac{1}{2}\left[(k+1)(5(k+1)-1)\right] \checkmark \qquad P_{k+1} \text{ is also true.}$$

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1+2.2+3.2^2+\ldots+n.2^{n-1}=1+(n-1).2^n$

Solution

(1) For $n=1 \Rightarrow 1\overset{?}{=}1+(1-1)2^1=1-0=1$; hence P_1 is true.

(2) $1+2.2+3.2^2+\ldots+k.2^{k-1}=1+(k-1).2^k$ is true

$$1+2.2+3.2^2+\ldots+k.2^{k-1}+(k+1).2^{(k+1)-1}=1+((k+1)-1).2^{k+1}?$$

$$1+2.2+3.2^2+\ldots+k.2^{k-1}+(k+1).2^{(k+1)-1}=1+(k-1).2^k+(k+1).2^{k+1-1}$$

$$=1+k.2^k-1.2^k+(k+1).2^k$$

$$=1+k.2^k-1.2^k+k.2^k+1.2^k$$

$$=1+2^1k.2^k$$

$$=1+(k+0).2^{k+1}$$

$$=1+((k+1)-1).2^{k+1} \checkmark \qquad P_{k+1} \text{ is also true.}$$

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Solution

(1) For $n = 1 \Rightarrow 1^2 \overset{?}{=} \dfrac{1(1+1)(2(1)+1)}{6} = \dfrac{1(2)(3)}{6} = \dfrac{6}{6} = 1$ ✔ ; hence P_1 is true.

(2) $1^2 + 2^2 + 3^2 + \ldots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$ is true

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \dfrac{(k+1)((k+1)+1)(2(k+1)+1)}{6} ?$$

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \dfrac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \dfrac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \dfrac{(k+1)\left[k(2k+1) + 6(k+1)\right]}{6}$$

$$= \dfrac{(k+1)\left[2k^2 + k + 6k + 6\right]}{6}$$

$$= \dfrac{(k+1)\left[2k^2 + 7k + 6\right]}{6}$$

$$= \dfrac{(k+1)\left((k+2)(2k+3)\right)}{6}$$

$$= \dfrac{(k+1)\left((k+1+1)(2k+2+1)\right)}{6}$$

$$= \dfrac{(k+1)\left((k+1)+1\right)\left(2(k+1)+1\right)}{6} \text{ ✔}$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \ldots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

Solution

(1) For $n = 1 \Rightarrow \dfrac{1}{1.2} \overset{?}{=} \dfrac{1}{1+1} = \dfrac{1}{2} = \dfrac{1}{1.2}$ ✔ ; hence P_1 is true.

(2) $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \ldots + \dfrac{1}{k(k+1)} = \dfrac{k}{k+1}$ is true

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{(k+1)+1}\overset{?}{}$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)(k+1)}{(k+1)(k+2)}$$

$$=\frac{k+1}{(k+1+1)}$$

$$=\frac{k+1}{(k+1)+1}\quad\checkmark$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\dfrac{1}{2}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+\ldots+\dfrac{1}{2^n}=1-\dfrac{1}{2^n}$

Solution

(1) For **n = 1** $\Rightarrow$ $\dfrac{1}{2}\overset{?}{=}1-\dfrac{1}{2}=\dfrac{1}{2}\ \checkmark$; P_1 is true.

(2) $\dfrac{1}{2}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+\ldots+\dfrac{1}{2^k}=1-\dfrac{1}{2^k}$ is true

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^{k+1}}\overset{?}{}$$

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^k}+\frac{1}{2^{k+1}}$$

$$=1-\frac{1}{2^k}+\frac{1}{2^k\cdot 2}$$

$$=\frac{2^{k+1}-2+1}{2^{k+1}}$$

$$=\frac{2^{k+1}-1}{2^{k+1}}$$

$$=\frac{2^{k+1}}{2^{k+1}}-\frac{1}{2^{k+1}}$$

$$= 1 - \frac{1}{2^{k+1}} \quad \checkmark$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\dfrac{1}{1 \cdot 4} + \dfrac{1}{4 \cdot 7} + \dfrac{1}{7 \cdot 10} + \cdots + \dfrac{1}{(3n-2) \cdot (3n+1)} = \dfrac{n}{3n+1}$

Solution

(1) For $n = 1 \Rightarrow \dfrac{1}{1 \cdot 4} \overset{?}{=} \dfrac{1}{3(1)+1} = \dfrac{1}{4} \; \checkmark$; P_1 is true.

(2) $\dfrac{1}{1 \cdot 4} + \dfrac{1}{4 \cdot 7} + \cdots + \dfrac{1}{(3k-2)(3k+1)} = \dfrac{k}{3k+1}$ is true

$$\dfrac{1}{1 \cdot 4} + \dfrac{1}{4 \cdot 7} + \cdots + \dfrac{1}{(3k-2)(3k+1)} + \dfrac{1}{(3(k+1)-2)(3(k+1)+1)} \overset{?}{=} \dfrac{k+1}{3(k+1)+1}$$

$$\dfrac{1}{1 \cdot 4} + \dfrac{1}{4 \cdot 7} + \cdots + \dfrac{1}{(3k-2)(3k+1)} + \dfrac{1}{(3(k+1)-2)(3(k+1)+1)} = \dfrac{k}{3k+1} + \dfrac{1}{(3k+1)(3k+4)}$$

$$= \dfrac{3k^2 + 4k + 1}{(3k+1)(3k+4)}$$

$$= \dfrac{(3k+1)(k+1)}{(3k+1)(3k+3+1)}$$

$$= \dfrac{k+1}{3(k+1)+1} \quad \checkmark$$

P_{k+1} is also true

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\dfrac{4}{5} + \dfrac{4}{5^2} + \dfrac{4}{5^3} + \cdots + \dfrac{4}{5^n} = 1 - \dfrac{1}{5^n}$

Solution

(1) For $n = 1 \Rightarrow \dfrac{4}{5} \overset{?}{=} 1 - \dfrac{1}{5} = \dfrac{4}{5} \; \checkmark$; P_1 is true.

(2) $\dfrac{4}{5} + \dfrac{4}{5^2} + \cdots + \dfrac{4}{5^k} = 1 - \dfrac{1}{5^k}$ is true

$$\dfrac{4}{5} + \dfrac{4}{5^2} + \cdots + \dfrac{4}{5^k} + \dfrac{4}{5^{k+1}} \overset{?}{=} 1 - \dfrac{1}{5^{k+1}}$$

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$$\frac{4}{5} + \frac{4}{5^2} + \cdots + \frac{4}{5^k} + \frac{4}{5^{k+1}} = 1 - \frac{1}{5^k} + \frac{4}{5^{k+1}}$$

$$= 1 - \left(\frac{1}{5^k} - \frac{4}{5^{k+1}} \right)$$

$$= 1 - \frac{5-4}{5^{k+1}}$$

$$= 1 - \frac{1}{5^{k+1}} \quad \checkmark$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\quad 1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

Solution

(1) For $\boldsymbol{n = 1} \Rightarrow 1^3 \stackrel{?}{=} \dfrac{1^2(1+1)^2}{4} = \dfrac{4}{4} = 1 \;\; \checkmark \quad ;\; P_1$ is true.

(2) $\dfrac{4}{5} + \dfrac{4}{5^2} + \cdots + \dfrac{4}{5^k} = 1 - \dfrac{1}{5^k}$ is true

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 \stackrel{?}{=} \frac{(k+1)^2\left((k+1)+1\right)^2}{4}$$

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{(k+1)^2\left[k^2 + 4(k+1)\right]}{4}$$

$$= \frac{(k+1)^2\left(k^2 + 4k + 4\right)}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \frac{(k+1)^2\left((k+1)+1\right)^2}{4} \quad \checkmark$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $3 + 3^2 + 3^3 + \ldots + 3^n = \frac{3}{2}\left(3^n - 1\right)$

Solution

(1) For $\boldsymbol{n = 1} \Rightarrow 3 \overset{?}{=} \frac{3}{2}\left(3^1 - 1\right) = \frac{3}{2} 2 = 3$ ✓ ; P_1 is true.

(2) $3 + 3^2 + \cdots + 3^k = \frac{3}{2}\left(3^k - 1\right)$ is true $\rightarrow$ Is $\quad 3 + 3^2 + \cdots + 3^k + 3^{k+1} \overset{?}{=} \frac{3}{2}\left(3^{k+1} - 1\right)$

$$3 + 3^2 + \cdots + 3^k + 3^{k+1} = \frac{3}{2}\left(3^k - 1\right) + 3^{k+1}$$
$$= \frac{1}{2} 3^{k+1} - \frac{3}{2} + 3^{k+1}$$
$$= \frac{3}{2} 3^{k+1} - \frac{3}{2}$$
$$= \frac{3}{2}\left(3^{k+1} - 1\right) \quad ✓$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\qquad x^{2n} + x^{2n-1}y + \cdots + xy^{2n-1} + y^{2n} = \dfrac{x^{2n+1} - y^{2n+1}}{x - y}$

Solution

(1) For $\boldsymbol{n = 1} \Rightarrow x^2 + xy + y^2 \overset{?}{=} \dfrac{x^3 - y^3}{x-y}$

$$= \frac{(x-y)\left(x^2 + xy + y^2\right)}{x-y}$$
$$= x^2 + xy + y^2 \quad ✓ \quad ; P_1 \text{ is true.}$$

(2) $x^{2k} + x^{2k-1}y + \cdots + xy^{2k-1} + y^{2k} = \dfrac{x^{2k+1} - y^{2k+1}}{x-y}$ is true

$$x^{2(k+1)} + x^{2(k+1)-1}y + \cdots + xy^{2(k+1)-1} + y^{2(k+1)} \overset{?}{=} \dfrac{x^{2(k+1)+1} - y^{2(k+1)+1}}{x-y}$$

$$x^{2k+2} + x^{2k+1}y + \cdots + xy^{2k+1} + y^{2k+2} = x^2\left(x^{2k} + x^{2k-1}y + \cdots + y^{2k}\right) + xy^{2k+1} + y^{2k+2}$$

$$= x^2\left(\frac{x^{2k+1} - y^{2k+1}}{x-y}\right) + xy^{2k+1} + y^{2k+2}$$

$$= \frac{x^{2k+3} - x^2 y^{2k+1} + x^2 y^{2k+1} + xy^{2k+2} - xy^{2k+2} - y^{2(k+1)+1}}{x-y}$$

$$= \frac{x^{2(k+1)+1} - y^{2(k+1)+1}}{x - y} \quad \checkmark$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\qquad 5 \cdot 6 + 5 \cdot 6^2 + 5 \cdot 6^3 + \cdots + 5 \cdot 6^n = 6\left(6^n - 1\right)$

Solution

(1) For $\boldsymbol{n = 1} \Rightarrow 5 \cdot 6 \overset{?}{=} 6\left(6^1 - 1\right) = 6(5) \checkmark$; P_1 is true.

(2) $5 \cdot 6 + 5 \cdot 6^2 + \cdots + 5 \cdot 6^k = 6\left(6^k - 1\right)$ is true

$$5 \cdot 6 + 5 \cdot 6^2 + \cdots + 5 \cdot 6^k + 5 \cdot 6^{k+1} \overset{?}{=} 6\left(6^{k+1} - 1\right)$$

$$5 \cdot 6 + 5 \cdot 6^2 + \cdots + 5 \cdot 6^k + 5 \cdot 6^{k+1} = 6\left(6^k - 1\right) + 5 \cdot 6^{k+1}$$

$$= 6^{k+1} - 6 + 5 \cdot 6^{k+1}$$

$$= 6^{k+1}(1 + 5) - 6$$

$$= 6 \cdot 6^{k+1} - 6$$

$$= 6\left(6^{k+1} - 1\right) \quad \checkmark$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\qquad 7 \cdot 8 + 7 \cdot 8^2 + 7 \cdot 8^3 + \cdots + 7 \cdot 8^n = 8\left(8^n - 1\right)$

Solution

(1) For $\boldsymbol{n = 1} \Rightarrow 7 \cdot 8 \overset{?}{=} 8\left(8^1 - 1\right) = 8(7) \checkmark$; P_1 is true.

(2) $7 \cdot 8 + 7 \cdot 8^2 + \cdots + 7 \cdot 8^k = 8\left(8^k - 1\right)$ is true

$$7 \cdot 8 + 7 \cdot 8^2 + \cdots + 7 \cdot 8^k + 7 \cdot 8^{k+1} \overset{?}{=} 8\left(8^{k+1} - 1\right)$$

$$7 \cdot 8 + 7 \cdot 8^2 + \cdots + 7 \cdot 8^k + 7 \cdot 8^{k+1} = 8\left(8^k - 1\right) + 7 \cdot 8^{k+1}$$

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$$= 8^{k+1} - 8 + 7 \cdot 8^{k+1}$$
$$= 8^{k+1}(1+7) - 8$$
$$= 8 \cdot 8^{k+1} - 8$$
$$= 8\left(8^{k+1} - 1\right) \quad \checkmark$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\quad 3 + 6 + 9 + \cdots + 3n = \dfrac{3n(n+1)}{2}$

Solution

(1) For **n = 1** $\Rightarrow 3 \overset{?}{=} \dfrac{3(1)(1+1)}{2} = 3 \ \checkmark; \ P_1$ is true.

(2) $3 + 6 + 9 + \cdots + 3k = \dfrac{3k(k+1)}{2}$ is true

$$3 + 6 + 9 + \cdots + 3k + 3(k+1) \overset{?}{=} \dfrac{3(k+1)(k+2)}{2}$$

$$3 + 6 + 9 + \cdots + 3k + 3(k+1) = \dfrac{3k(k+1)}{2} + 3(k+1)$$
$$= \dfrac{3k(k+1) + 6(k+1)}{2}$$
$$= \dfrac{(k+1)(3k+6)}{2}$$
$$= \dfrac{3(k+1)(k+2)}{2} \quad \checkmark$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\quad 5 + 10 + 15 + \cdots + 5n = \dfrac{5n(n+1)}{2}$

Solution

(1) For **n = 1** $\Rightarrow 5 \overset{?}{=} \dfrac{5(1)(1+1)}{2} = 5 \ \checkmark; \ P_1$ is true.

(2) $5 + 10 + 15 + \cdots + 5k = \dfrac{5k(k+1)}{2}$ is true

$$5 + 10 + 15 + \cdots + 5k + 5(k+1) \overset{?}{=} \frac{5(k+1)(k+2)}{2}$$

$$5 + 10 + 15 + \cdots + 5k + 5(k+1) = \frac{5k(k+1)}{2} + 5(k+1)$$

$$= \frac{5k(k+1) + 10(k+1)}{2}$$

$$= \frac{(k+1)(5k+10)}{2}$$

$$= \frac{5(k+1)(k+2)}{2} \quad \checkmark$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true: $\qquad 1 + 3 + 5 + \cdots + (2n-1) = n^2$

Solution

(1) For **$n = 1$** $\Rightarrow$ $1 \overset{?}{=} 1^2 = 1$ $\checkmark$; P_1 is true.

(2) $1 + 3 + 5 + \cdots + (2k-1) = k^2$ is true

$$1 + 3 + 5 + \cdots + (2k-1) + (2(k+1)-1) \overset{?}{=} (k+1)^2$$

$$1 + 3 + 5 + \cdots + (2k-1) + (2(k+1)-1) = k^2 + 2k + 2 - 1$$

$$= k^2 + 2k + 1$$

$$= (k+1)^2 \quad \checkmark$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true:
$$4 + 7 + 10 + \cdots + (3n+1) = \frac{n(3n+5)}{2}$$

Solution

(1) For $n = 1 \Rightarrow 4 \overset{?}{=} \frac{1(3+5)}{2} = 4$ ✓ ; P_1 is true.

(2) $4 + 7 + 10 + \cdots + (3k+1) = \frac{k(3k+5)}{2}$ is true

$$4 + 7 + 10 + \cdots + (3k+1) + (3(k+1)+1) \overset{?}{=} \frac{(k+1)(3(k+1)+5)}{2} = \frac{(k+1)(3k+8)}{2}$$

$$
\begin{aligned}
4 + 7 + 10 + \cdots + (3k+1) + (3k+4) &= \frac{k(3k+5)}{2} + 3k + 4 \\
&= \frac{3k^2 + 5k + 6k + 8}{2} \\
&= \frac{3k^2 + 5k + 3k + 3k + 8}{2} \\
&= \frac{k(3k+8) + (3k+8)}{2} \\
&= \frac{(3k+8)(k+1)}{2} \quad ✓
\end{aligned}
$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $\left(a^m\right)^n = a^{mn}$ (a and m are constant)

Solution

➤ For $n = 1 \Rightarrow \left(a^m\right)^1 \overset{?}{=} a^{m(1)} \rightarrow a^m = a^m$ ✓ ; P_1 is true.

➤ $\left(a^m\right)^k = a^{mk}$ is true

$$\left(a^m\right)^{(k+1)} \overset{?}{=} a^{m(k+1)}$$

$$
\begin{aligned}
\left(a^m\right)^{(k+1)} &= \left(a^m\right)^k a^m \\
&= a^{km} a^m \\
&= a^{km+m}
\end{aligned}
$$

$$= a^{m(k+1)} \checkmark$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. $n < 2^n$

Solution

Step 1. For $\boldsymbol{n = 1} \Rightarrow 1 \overset{?}{<} 2^1 \checkmark \Rightarrow P_1$ is true.

Step 2. Assume that P_k is true $k < 2^k$

We need to prove that P_{k+1} is true, that is $k + 1 < 2^{k+1}$

$$k + 1 < k + k = 2k$$
$$< 2 \cdot 2^k$$
$$= 2^{k+1} \checkmark$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. 3 is a factor of $n^3 - n + 3$

Solution

➤ For $\boldsymbol{n = 1} \Rightarrow 1^3 - 1 + 3 = 3 = 3(1) \checkmark \Rightarrow P_1$ is true.

➤ Assume that P_k is true 3 is a factor of $k^3 - k + 3$

We need to prove that P_{k+1} is true, that is $(k+1)^3 - (k+1) + 3$

$$(k+1)^3 - (k+1) + 3 = k^3 + 3k^2 + 3k + 1 - k - 1 + 3$$
$$= \left(k^3 - k + 3\right) + 3k^2 + 3k$$
$$= 3K + 3k^2 + 3k$$
$$= 3\left(K + k^2 + k\right) \checkmark$$

P_{k+1} is also true.

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement is true for every positive integer n. 4 is a factor of $5^n - 1$

Solution

➢ For $\boldsymbol{n = 1} \Rightarrow 5^1 - 1 = 4 = 4(1)$ ✓ $\Rightarrow P_1$ is true.

➢ Assume that P_k is true 4 is a factor of $5^k - 1$

We need to prove that P_{k+1} is true, that is $5^{k+1} - 1$

$$5^{k+1} - 1 = 5^k 5^1 - 5 + 4$$
$$= 5\left(5^k - 1\right) + 4$$
$$= 5\left(5^k - 1\right) + 4$$

By the induction hypothesis, 4 is a factor of $5^k - 1$ and 4 is a factor of 4, so 4 is a factor of the $(k+1)$ term. ✓

Thus, P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $\quad 2^n > 2n \quad if \quad n \geq 3$

Solution

➢ For $\boldsymbol{n = 3} \Rightarrow 2^3 \geq 2(3) \Rightarrow 8 \geq 6$ ✓ $\Rightarrow P_1$ is true.

➢ Assume that P_k is true: $2^k > 2k$;

we need to prove that $P_{k+1} : 2^{k+1} > 2(k+1)$ is true

$$2^k > 2k$$
$$2^k \cdot 2 > 2k \cdot 2$$
$$2^{k+1} > 4k = 2k + 2k \qquad k \geq 3$$
$$\qquad\quad > 2k + 2$$
$$\qquad\quad = 2(k+1) ✓$$

P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: If $0 < a < 1$, then $a^n < a^{n-1}$

- For $n = 1 \Rightarrow a^1 < a^{1-1} \Rightarrow a < 1$ ✓   since $0 < a < 1 \Rightarrow P_1$ is true.

- Assume that P_k is true: $a^k < a^{k-1}$;

  We need to prove that $P_{k+1} : a^{k+1} < a^k$ is true

  $$a^k < a^{k-1} \rightarrow a^k \cdot a < a^{k-1} \cdot a$$
  $$a^{k+1} < a^k \quad \checkmark$$

  P_{k+1} is also true.

  ∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: If $n \geq 4$, then $n! > 2^n$

- For $n = 4 \Rightarrow 4! > 2^4 \Rightarrow 24 > 16$ ✓ $\Rightarrow P_1$ is true.

- Assume that P_k is true: $k! > 2^k$;

  We need to prove that $P_{k+1} : (k+1)! > 2^{k+1}$ is true

  $$(k+1)! = k! \cdot (k+1)$$
  $$> 2^k (k+1) \qquad\qquad k \geq 4 \Rightarrow k+1 > 2$$
  $$> 2^k \cdot 2$$
  $$= 2^{k+1} \quad \checkmark$$

  Thus, P_{k+1} is also true.

  ∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $3^n > 2n+1$ $\;if\;\; n \geq 2$

- For $n = 2 \Rightarrow 3^2 > 2(2)+1 \Rightarrow 9 > 5$ ✓ $\Rightarrow P_1$ is true.

➤ Assume that P_k is true: $3^k > 2k+1$

We need to prove that P_{k+1}: $3^{k+1} > 2(k+1)+1$ is true

$$3^k > 2k+1 \implies 3^k \cdot 3 > (2k+1)\cdot 3$$
$$3^{k+1} > 6k+3$$
$$> 2k+2+1$$
$$= 2(k+1)+1 \quad \checkmark$$

Thus, P_{k+1} is also true.

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $\quad 2^n > n^2 \quad for \quad n > 4$

Solution

➤ For $n = 5 \implies 2^5 > 5^2 \implies 32 > 25$ $\checkmark$ $\implies P_1$ is true.

➤ Assume that P_k is true: $2^k > k^2$

We need to prove that P_{k+1}: $2^{k+1} > (k+1)^2$ is true

$$2^k > k^2 \implies 2^k \cdot 2 > k^2 \cdot 2$$
$$2^{k+1} > 2k^2 \qquad\qquad k < k+1 \implies k^2 > 2k+1$$
$$> (k+1)^2 \quad \checkmark$$

Thus, P_{k+1} is also true

∴ By the mathematical induction, the proof is completed.

Exercise

Prove that the statement by mathematical induction: $\quad 4^n > n^4 \quad for \quad n \geq 5$

Solution

➤ For $n = 5 \implies 4^5 > 5^4 \implies 1024 > 625$ $\checkmark$ $\implies P_1$ is true.

➤ Assume that P_k is true: $4^k > k^4$

We need to prove that P_{k+1}: $4^{k+1} > (k+1)^4$ is true

$$4^k > k^4 \quad \Rightarrow \quad 4^k \cdot 4 > k^4 \cdot 4$$
$$4^{k+1} > 4k^4 \qquad\qquad k < k+1 \quad \Rightarrow \quad k^2 > 2k+1$$
$$> (k+1)^4 \;\checkmark$$

Thus, P_{k+1} is also true

$\therefore$ By the mathematical induction, the proof is completed.

Exercise

A pile of n rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must moved, one at a time, to a different peg with no ring ever placed on top of a smaller ring. Find the least number of moves that would be required. Prove your result by mathematical induction.
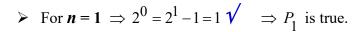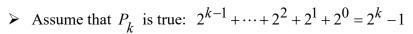
<u>Solution</u>

With 1 ring, 1 move is required.

With 2 rings, 3 moves are required $\Rightarrow \quad 3 = 2+1$

With 3 rings, 7 moves are required $\Rightarrow \quad 7 = 2^2 + 2 + 1$

With n rings, $2^{n-1} + \cdots + 2^2 + 2^1 + 2^0 = 2^n - 1$ moves are required

$\blacktriangleright$ For $\boldsymbol{n = 1} \Rightarrow 2^0 = 2^1 - 1 = 1 \;\checkmark \quad \Rightarrow P_1$ is true.

$\blacktriangleright$ Assume that P_k is true: $\quad 2^{k-1} + \cdots + 2^2 + 2^1 + 2^0 = 2^k - 1$

$$2^k + 2^{k-1} + \cdots + 2^2 + 2^1 + 1 \overset{?}{=} 2^{k+1} - 1$$

$$2^k + 2^{k-1} + \cdots + 2^2 + 2^1 + 1 = 2^k + 2^k - 1$$
$$= 2 \cdot 2^k - 1$$
$$= 2^{k+1} - 1 \;\checkmark$$

$\therefore$ By the mathematical induction, the proof is completed.