

Solution **Section 4.5 – Inverse Trigonometric Functions**

Exercise

Find the exact value of the expression whenever it is defined: $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Solution

$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arccos\left(\frac{\sqrt{2}}{2}\right)$

Solution

$$\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arctan\left(-\frac{\sqrt{3}}{3}\right)$

Solution

$$\begin{aligned}\arctan\left(-\frac{\sqrt{3}}{3}\right) &= -\arctan\left(\frac{\sqrt{3}}{3}\right) \\ &= -\frac{\pi}{6}\end{aligned}$$

Exercise

Find the exact value of the expression whenever it is defined: $\sin\left(\arcsin\left(-\frac{3}{10}\right)\right)$

Solution

$$\alpha = \arcsin\left(-\frac{3}{10}\right)$$

$$\sin \alpha = -\frac{3}{10}$$

$$\sin\left(\arcsin\left(-\frac{3}{10}\right)\right) = -\frac{3}{10}$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan(\arctan(14))$

Solution

$$\underline{\tan(\arctan(14)) = 14}$$

Exercise

Find the exact value of the expression whenever it is defined: $\sin(\sin^{-1}(\frac{2}{3}))$

Solution

$$\underline{\sin(\sin^{-1}(\frac{2}{3})) = \frac{2}{3}}$$

Exercise

Find the exact value of the expression whenever it is defined: $\cos^{-1}(\cos(\frac{5\pi}{6}))$

Solution

$$\underline{\cos^{-1}(\cos(\frac{5\pi}{6})) = \frac{5\pi}{6}} \quad 0 \leq \frac{5\pi}{6} \leq \pi$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan^{-1}(\tan(-\frac{\pi}{6}))$

Solution

$$\underline{\tan^{-1}(\tan(-\frac{\pi}{6})) = -\frac{\pi}{6}} \quad -\frac{\pi}{2} \leq -\frac{\pi}{6} \leq \frac{\pi}{2}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arcsin(\sin(-\frac{\pi}{2}))$

Solution

$$\underline{\arcsin(\sin(-\frac{\pi}{2})) = -\frac{\pi}{2}} \quad -\frac{\pi}{2} \leq -\frac{\pi}{2} \leq \frac{\pi}{2}$$

Exercise

Find the exact value of the expression whenever it is defined: $\arccos(\cos(0))$

Solution

$$\underline{\arccos(\cos(0)) = 0} \quad 0 \leq 0 \leq \pi$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right)$

Solution

$$\underline{\tan^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right) = -\frac{\pi}{4}} \quad -\frac{\pi}{2} \leq -\frac{\pi}{4} \leq \frac{\pi}{2}$$

Exercise

Find the exact value of the expression whenever it is defined: $\sin\left(\arcsin\left(\frac{1}{2}\right) + \arccos 0\right)$

Solution

$$\begin{aligned} \sin\left(\arcsin\left(\frac{1}{2}\right) + \arccos 0\right) &= \sin\left(\frac{\pi}{6} + 0\right) \\ &= \sin\left(\frac{\pi}{6}\right) \\ &= \underline{\frac{1}{2}} \end{aligned}$$

Exercise

Find the exact value of the expression whenever it is defined: $\cos\left(\arctan\left(-\frac{3}{4}\right) - \arcsin\frac{4}{5}\right)$

Solution

$$\begin{aligned} \cos\left(\arctan\left(-\frac{3}{4}\right) - \arcsin\frac{4}{5}\right) &= \cos(\alpha - \beta) \\ &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \end{aligned}$$

$\alpha = \arctan\left(-\frac{3}{4}\right)$ $\tan \alpha = -\frac{3}{4}$ $r = \sqrt{3^2 + 4^2} = 5$ $\sin \alpha = -\frac{3}{5} \quad \cos \alpha = \frac{4}{5}$	$\beta = \arcsin \frac{4}{5}$ $\sin \beta = \frac{4}{5}$ $\Rightarrow \cos \beta = \frac{3}{5}$
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$$\cos\left(\arctan\left(-\frac{3}{4}\right) - \arcsin\frac{4}{5}\right) = \frac{4}{5}\frac{3}{5} + \left(-\frac{3}{5}\right)\frac{4}{5}$$

$$\underline{= 0}$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan\left(\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)\right)$

Solution

$$\tan\left(\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)\right) = \tan\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \tan\left(\frac{\pi}{6}\right)$$

$$\underline{= \frac{1}{\sqrt{3}}}$$

Exercise

Find the exact value of the expression whenever it is defined: $\sin\left(2\arccos\left(-\frac{3}{5}\right)\right)$

Solution

$$\sin\left(2\arccos\left(-\frac{3}{5}\right)\right) = \sin 2\alpha$$

$$= 2\sin\alpha\cos\alpha$$

$$\alpha = \arccos\left(-\frac{3}{5}\right) \rightarrow \cos\alpha = -\frac{3}{5}$$

$$\sin\alpha = \frac{4}{5}$$

$$\sin\left(2\arccos\left(-\frac{3}{5}\right)\right) = 2\left(-\frac{3}{5}\right)\left(\frac{4}{5}\right)$$

$$\underline{= -\frac{24}{25}}$$

Exercise

Find the exact value of the expression whenever it is defined: $\cos\left(2\sin^{-1}\left(\frac{15}{17}\right)\right)$

Solution

$$\cos\left(2\sin^{-1}\left(\frac{15}{17}\right)\right) = \cos 2\alpha$$

$$= 1 - 2\sin^2\alpha$$

$$\alpha = \sin^{-1}\left(\frac{15}{17}\right)$$

$$\sin \alpha = \frac{15}{17}$$

$$\begin{aligned}\cos\left(2\sin^{-1}\left(\frac{15}{17}\right)\right) &= 1 - 2\left(\frac{15}{17}\right)^2 \\ &= 1 - \frac{450}{289} \\ &= -\frac{161}{289}\end{aligned}$$

Exercise

Find the exact value of the expression whenever it is defined: $\tan\left(2\tan^{-1}\left(\frac{3}{4}\right)\right)$

Solution

$$\tan \tan\left(2\tan^{-1}\left(\frac{3}{4}\right)\right) = \tan 2\alpha$$

$$\alpha = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\tan \alpha = \frac{3}{4}$$

$$\begin{aligned}\tan\left(2\tan^{-1}\left(\frac{3}{4}\right)\right) &= \frac{2\tan \alpha}{1 - \tan^2 \alpha} \\ &= \frac{2\left(\frac{3}{4}\right)}{1 - \left(\frac{3}{4}\right)^2} \\ &= \frac{\frac{3}{2}}{1 - \frac{9}{16}} \\ &= \frac{3}{2} \frac{16}{7} \\ &= \frac{24}{7}\end{aligned}$$

Exercise

Find the exact value of the expression whenever it is defined: $\cos\left(\frac{1}{2}\tan^{-1}\left(\frac{8}{15}\right)\right)$

Solution

$$\cos\left(\frac{1}{2}\tan^{-1}\left(\frac{8}{15}\right)\right) = \cos\left(\frac{1}{2}\alpha\right)$$

$$\alpha = \tan^{-1}\left(\frac{8}{15}\right)$$

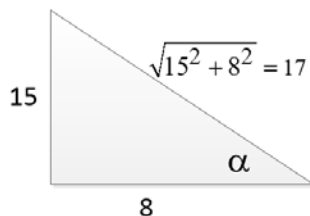
$$\tan \alpha = \frac{8}{15}$$

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1}{2}(1 + \cos \alpha)}$$

$$= \sqrt{\frac{1}{2}\left(1 + \frac{8}{17}\right)}$$

$$= \sqrt{\frac{25}{34}}$$

$$= \frac{5}{\sqrt{34}} \quad \text{or} \quad \frac{5\sqrt{34}}{34}$$



Exercise

Evaluate without using a calculator: $\cos\left(\cos^{-1}\frac{3}{5}\right)$

Solution

$$\cos\left(\cos^{-1}\frac{3}{5}\right) = \frac{3}{5}$$

Exercise

Evaluate without using a calculator: $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

Solution

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$= \frac{5\pi}{6}$$

Exercise

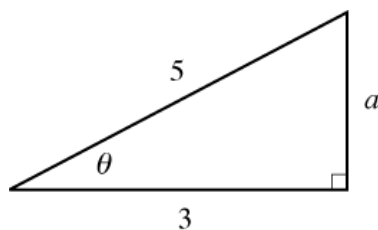
Evaluate without using a calculator: $\tan\left(\cos^{-1} \frac{3}{5}\right)$

Solution

$$\tan\left(\cos^{-1} \frac{3}{5}\right)$$

$$5^2 = 3^2 + a^2 \rightarrow \underline{a = 4}$$

$$\tan\left(\cos^{-1} \frac{3}{5}\right) = \tan \theta$$
$$\underline{= \frac{4}{3}}$$



Exercise

Evaluate without using a calculator: $\sin\left(\cos^{-1} \frac{1}{\sqrt{5}}\right)$

Solution

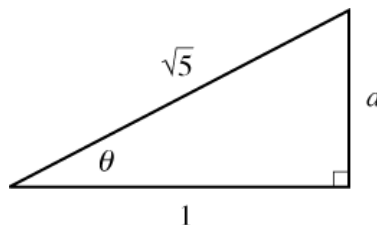
$$\sin\left(\cos^{-1} \frac{1}{\sqrt{5}}\right)$$

$$(\sqrt{5})^2 = 1^2 + a^2$$

$$a^2 = 5 - 1$$

$$\underline{a = 2}$$

$$\sin\left(\cos^{-1} \frac{1}{\sqrt{5}}\right) = \sin \theta$$
$$\underline{= \frac{2}{\sqrt{5}}}$$



Exercise

Evaluate without using a calculator: $\cos\left(\sin^{-1} \frac{1}{2}\right)$

Solution

$$\cos\left(\sin^{-1} \frac{1}{2}\right)$$

$$\sin \frac{\pi}{6} = \frac{1}{2}$$

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\cos\left(\sin^{-1} \frac{1}{2}\right) = \cos \frac{\pi}{6}$$

$$\underline{= \frac{\sqrt{3}}{2}}$$

Exercise

Evaluate without using a calculator: $\sin\left(\sin^{-1} \frac{3}{5}\right)$

Solution

$$\underline{\sin\left(\sin^{-1} \frac{3}{5}\right) = \frac{3}{5}}$$

Exercise

Evaluate without using a calculator: $\cos\left(\tan^{-1} \frac{3}{4}\right)$

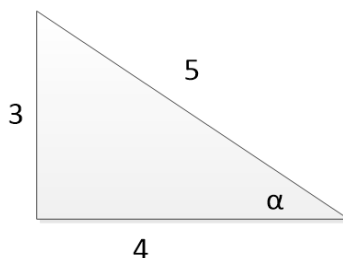
Solution

$$\alpha = \tan^{-1} \frac{3}{4}$$

$$\tan \alpha = \frac{3}{4}$$

$$r = \sqrt{3^2 + 4^2} = 5$$

$$\underline{\cos\left(\tan^{-1} \frac{3}{4}\right) = \frac{4}{5}}$$



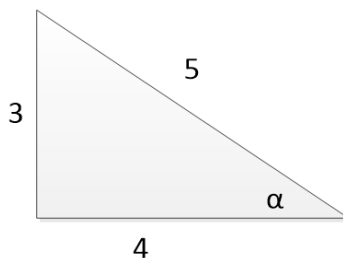
Exercise

Evaluate without using a calculator: $\tan\left(\sin^{-1} \frac{3}{5}\right)$

Solution

$$\sin \alpha = \frac{3}{5}$$

$$\underline{\tan\left(\sin^{-1} \frac{3}{5}\right) = \frac{3}{4}}$$



Exercise

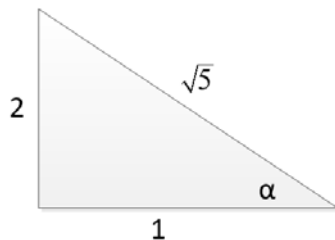
Evaluate without using a calculator: $\sec\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$

Solution

$$\alpha = \cos^{-1}\frac{1}{\sqrt{5}}$$

$$\cos \alpha = \frac{1}{\sqrt{5}}$$

$$\sec \alpha = \frac{1}{\frac{1}{\sqrt{5}}} \\ = \sqrt{5}$$



Exercise

Evaluate without using a calculator: $\cot\left(\tan^{-1}\frac{1}{2}\right)$

Solution

$$\alpha = \tan^{-1}\frac{1}{2}$$

$$\tan \alpha = \frac{1}{2}$$

$$\cot \alpha = \frac{1}{\tan \alpha} \\ = 2$$

Exercise

Write an equivalent expression that involves x only for $\cos\left(\cos^{-1}x\right)$

Solution

$$\alpha = \cos^{-1}x$$

$$\cos \alpha = x$$

$$\cos\left(\cos^{-1}x\right) = \cos \alpha \\ = x$$

Exercise

Write an equivalent expression that involves x only for $\tan(\cos^{-1} x)$

Solution

$$\alpha = \cos^{-1} x$$

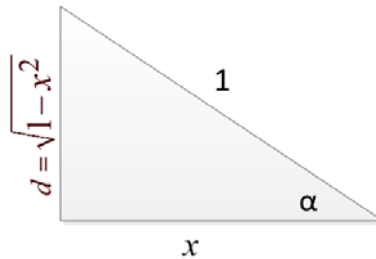
$$\cos \alpha = x = \frac{x}{1}$$

$$x^2 + d^2 = 1 \Rightarrow d^2 = 1 - x^2$$

$$d = \sqrt{1 - x^2}$$

$$\tan(\cos^{-1} x) = \tan \alpha$$

$$= \frac{\sqrt{1 - x^2}}{x}$$



Exercise

Write an equivalent expression that involves x only for $\csc(\sin^{-1} \frac{1}{x})$

Solution

$$\alpha = \sin^{-1} \frac{1}{x}$$

$$\sin \alpha = \frac{1}{x}$$

$$\csc(\sin^{-1} x) = \csc \alpha = \frac{1}{\sin \alpha}$$

$$= x$$

Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\sin(\tan^{-1} x)$

Solution

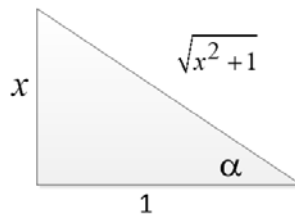
$$\sin(\tan^{-1} x) = \sin \alpha$$

$$\alpha = \tan^{-1} x$$

$$\tan \alpha = x$$

$$\sin(\tan^{-1} x) = \sin \alpha$$

$$= \frac{x}{\sqrt{x^2 + 1}}$$



Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\sec\left(\sin^{-1}\frac{x}{\sqrt{x^2+4}}\right)$

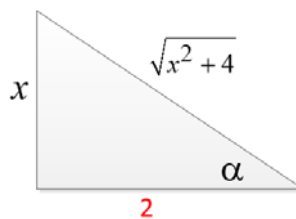
Solution

$$\alpha = \sin^{-1}\frac{x}{\sqrt{x^2+4}}$$

$$\sin \alpha = \frac{x}{\sqrt{x^2+4}}$$

$$\sqrt{\left(\sqrt{x^2+4}\right)^2 - x^2} = \sqrt{x^2+4-x^2} = \sqrt{4} = 2$$

$$\sec\left(\sin^{-1}\frac{x}{\sqrt{x^2+4}}\right) = \frac{1}{\cos \alpha} \\ = \frac{2}{\sqrt{x^2+4}}$$



Exercise

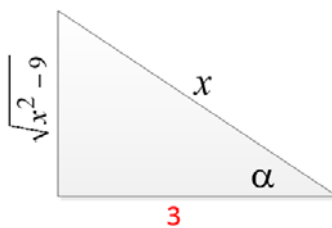
Write the expression as an algebraic expression in x for $x > 0$: $\cot\left(\sin^{-1}\frac{\sqrt{x^2-9}}{x}\right)$

Solution

$$\alpha = \sin^{-1}\frac{\sqrt{x^2-9}}{x}$$

$$\sin \alpha = \frac{\sqrt{x^2-9}}{x}$$

$$\cot\left(\sin^{-1}\frac{\sqrt{x^2-9}}{x}\right) = \cot \alpha \\ = \frac{3}{\sqrt{x^2-9}}$$



Exercise

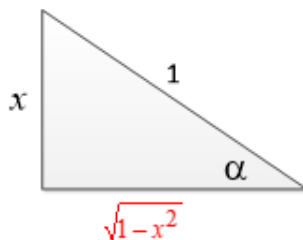
Write the expression as an algebraic expression in x for $x > 0$: $\sin\left(2\sin^{-1}x\right)$

Solution

$$\alpha = \sin^{-1}x$$

$$\sin \alpha = x$$

$$\begin{aligned}
 \sin\left(2 \sin^{-1} x\right) &= \sin 2\alpha \\
 &= 2 \sin \alpha \cos \alpha \\
 &= \underline{2x\sqrt{1-x^2}}
 \end{aligned}$$

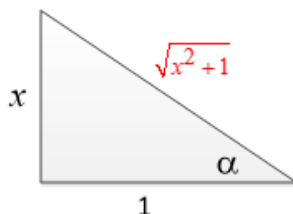


Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\cos\left(2 \tan^{-1} x\right)$

Solution

$$\begin{aligned}
 \alpha &= \tan^{-1} x \\
 \tan \alpha &= x \\
 \cos\left(2 \tan^{-1} x\right) &= \cos(2\alpha) \\
 &= 2 \cos^2 \alpha - 1 \\
 &= 2 \left(\frac{1}{\sqrt{x^2 + 1}} \right)^2 - 1 \\
 &= \frac{2}{x^2 + 1} - 1 \\
 &= \underline{\frac{-x^2 + 1}{x^2 + 1}}
 \end{aligned}$$



Exercise

Write the expression as an algebraic expression in x for $x > 0$: $\cos\left(\frac{1}{2} \arccos x\right)$

Solution

$$\begin{aligned}
 \alpha &= \arccos x \\
 \cos \alpha &= x \\
 \cos\left(\frac{1}{2} \arccos x\right) &= \cos\left(\frac{\alpha}{2}\right) \\
 &= \sqrt{\frac{1 + \cos \alpha}{2}} \\
 &= \underline{\sqrt{\frac{1 + x}{2}}}
 \end{aligned}$$

Exercise

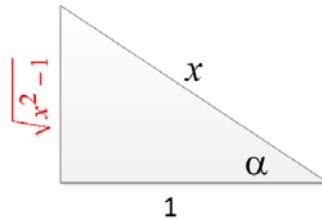
Write the expression as an algebraic expression in x for $x > 0$: $\tan\left(\frac{1}{2}\cos^{-1}\frac{1}{x}\right)$

Solution

$$\alpha = \cos^{-1}\frac{1}{x}$$

$$\cos \alpha = \frac{1}{x}$$

$$\begin{aligned}\tan\left(\frac{1}{2}\cos^{-1}\frac{1}{x}\right) &= \tan\left(\frac{\alpha}{2}\right) \\ &= \frac{1 - \cos \alpha}{\sin \alpha} \\ &= \frac{1 - \frac{1}{x}}{\frac{\sqrt{x^2 - 1}}{x}} \\ &= \frac{\frac{x-1}{x}}{\frac{\sqrt{x^2 - 1}}{x}} \\ &= \frac{x-1}{\sqrt{x^2 - 1}}\end{aligned}$$



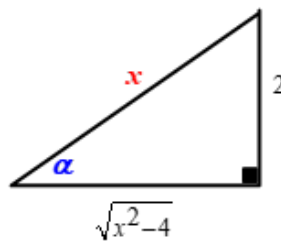
Exercise

Write the expression as an algebraic expression in x : $\sec\left(\tan^{-1}\frac{2}{\sqrt{x^2 - 4}}\right)$ $x > 0$

Solution

$$\tan \alpha = \frac{2}{\sqrt{x^2 - 4}}$$

$$\begin{aligned}\sec\left(\tan^{-1}\frac{2}{\sqrt{x^2 - 4}}\right) &= \sec \alpha \\ &= \frac{x}{\sqrt{x^2 - 4}}\end{aligned}$$



Exercise

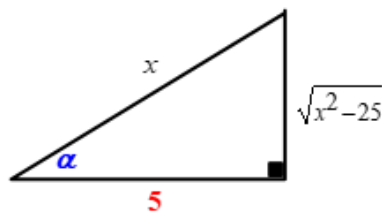
Write the expression as an algebraic expression in x : $\sec\left(\sin^{-1}\frac{\sqrt{x^2-25}}{x}\right) \quad x > 0$

Solution

$$\sin \alpha = \frac{\sqrt{x^2-25}}{x}$$

$$\sec\left(\sin^{-1}\frac{\sqrt{x^2-25}}{x}\right) = \sec \alpha$$

$$= \frac{x}{5}$$



Exercise

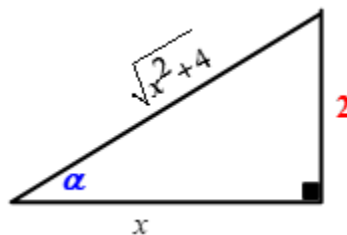
Write the expression as an algebraic expression in x : $\sin\left(\cos^{-1}\frac{x}{\sqrt{x^2+4}}\right) \quad x > 0$

Solution

$$\cos \alpha = \frac{x}{\sqrt{x^2+4}}$$

$$\sin\left(\cos^{-1}\frac{x}{\sqrt{x^2+4}}\right) = \sin \alpha$$

$$= \frac{2}{\sqrt{x^2+4}}$$



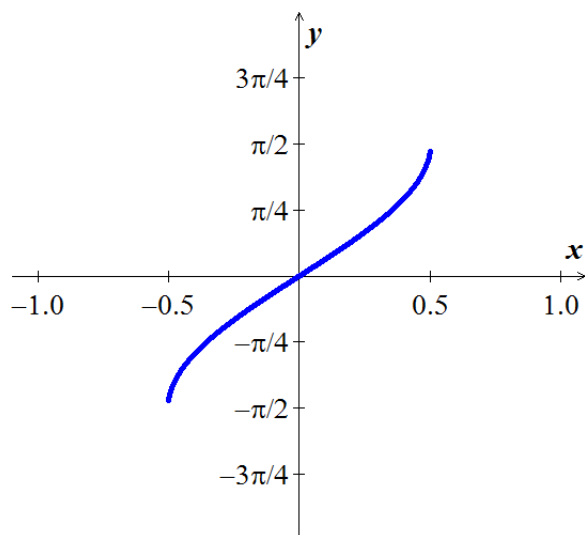
Exercise

Sketch the graph of the equation: $y = \sin^{-1} 2x$

Solution

$$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \quad \text{and} \quad -1 \leq 2x \leq 1$$

$$-\frac{1}{2} \leq x \leq \frac{1}{2}$$



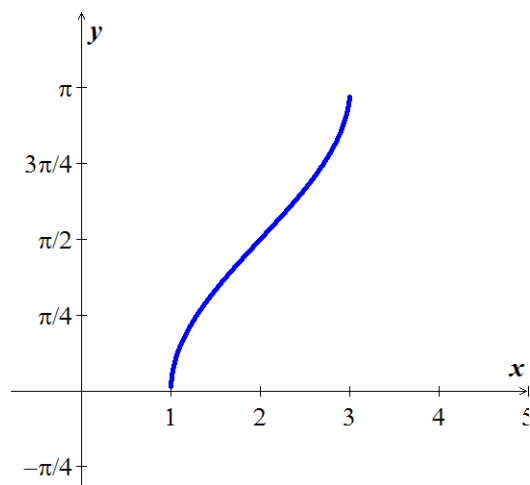
Exercise

Sketch the graph of the equation: $y = \sin^{-1}(x-2) + \frac{\pi}{2}$

Solution

$$-\frac{\pi}{2} + \frac{\pi}{2} \leq y \leq \frac{\pi}{2} + \frac{\pi}{2} \quad \text{and} \quad -1 \leq x-2 \leq 1$$

$$0 \leq y \leq \pi \quad \text{and} \quad 1 \leq x \leq 3$$



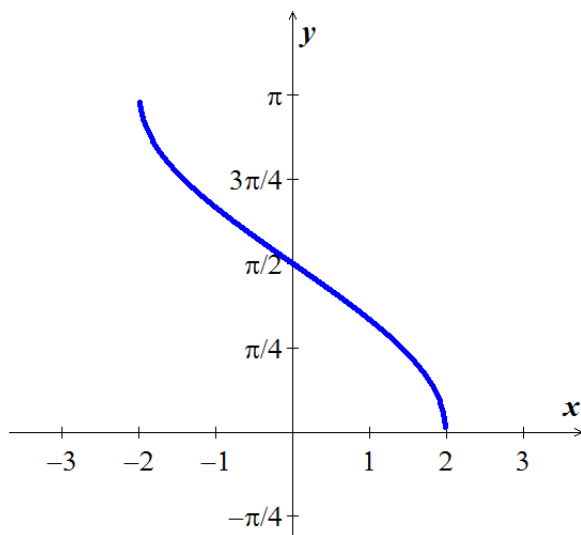
Exercise

Sketch the graph of the equation: $y = \cos^{-1} \frac{1}{2}x$

Solution

$$0 \leq y \leq \pi \quad \text{and} \quad -1 \leq \frac{1}{2}x \leq 1$$

$$-2 \leq x \leq 2$$



Exercise

Evaluate $\sin\left(\tan^{-1} \frac{3}{4}\right)$ without using a calculator

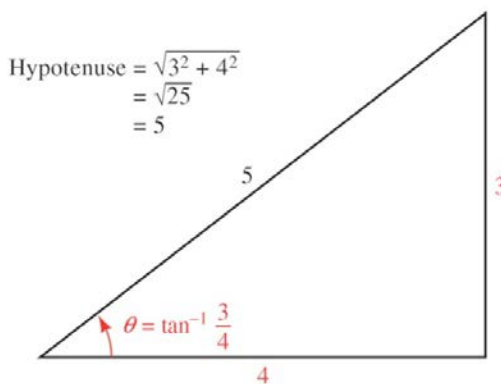
Solution

$$\theta = \tan^{-1} \frac{3}{4}$$

$$\tan \theta = \frac{3}{4} \rightarrow 0^\circ < \theta < 90^\circ$$

$$\sin\left(\tan^{-1} \frac{3}{4}\right) = \sin \theta$$

$$\boxed{= \frac{3}{5}}$$



Exercise

Evaluate $\sin(\cos^{-1} x)$ as an equivalent expression in x only

Solution

$$\begin{aligned}\sin \theta &= \frac{y}{r} \\ &= \frac{\sqrt{1-x^2}}{1} \\ &= \sqrt{1-x^2}\end{aligned}$$

$$\sin(\cos^{-1} x) = \sin \theta$$

$$= \sqrt{1-x^2}$$

