Solution *Section* 1.2 – Definitions / Techniques of Limits

Exercise

Find the limit: $\lim\limits_{x\to 3}(-1)$

Solution

$$\lim_{x\to 3}(-1)\ \underline{=-1}$$

Exercise

Find the limit: $\lim\limits_{x\to -1}(3)$

Solution

$$\lim_{x\to -1}(3)\ \underline{=3}$$

Exercise

Find the limit: $\lim\limits_{x\to 1000} 18\pi^2$

Solution

$$\lim_{x\to 1000} 18\pi^2\ \underline{=18\pi^2}$$

Exercise

Find the limit: $\lim\limits_{x\to 1}\sqrt{5x+6}$

Solution

$$\lim_{x\to 1}\sqrt{5x+6}\ \underline{=\sqrt{11}}$$

Exercise

Find the limit: $\lim\limits_{x\to 9}\sqrt{x}$

Solution

$$\lim_{x\to 9}\sqrt{x}=\sqrt{9}$$

$$\underline{=3}$$

Exercise

Find the limit: $\lim\limits_{x \to -3} \left(x^2 + 3x\right)$

Solution

$$\lim_{x \to -3} \left(x^2 + 3x\right) = (-3)^2 + 3(-3)$$
$$= 9 - 9$$
$$\underline{= 0}$$

Exercise

Find the limit: $\lim\limits_{x \to -4} |x - 4|$

Solution

$$\lim_{x \to -4} |x - 4| = |-4 - 4|$$
$$= |-8|$$
$$\underline{= 8}$$

Exercise

Find the limit: $\lim\limits_{x \to 4} (x + 2)$

Solution

$$\lim_{x \to 4} (x + 2) = 4 + 2$$
$$\underline{= 6}$$

Exercise

Find the limit: $\lim\limits_{x \to 4} (x - 4)$

Solution

$$\lim_{x \to 4} (x - 4) = 4 - 4$$
$$\underline{= 0}$$

Exercise

Find the limit: $\lim\limits_{x\to 2} (5x-6)^{3/2}$

Solution

$$\lim_{x\to 2} (5x-6)^{3/2} = (10-6)^{3/2}$$
$$= \sqrt{4^3}$$
$$\underline{= 8}$$

Exercise

Find the limit: $\lim\limits_{x\to 9} \dfrac{x-9}{\sqrt{x}-3}$

Solution

$$\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} = \frac{9-9}{3-3} = \frac{0}{0}$$
$$= \lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}-3}$$
$$= \lim_{x\to 9} (\sqrt{x}+3)$$
$$\underline{= 6}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} (2x+4)$

Solution

$$\lim_{x\to 1} (2x+4) = 2(1)+4$$
$$\underline{= 6}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{x^2-4}{x-2}$

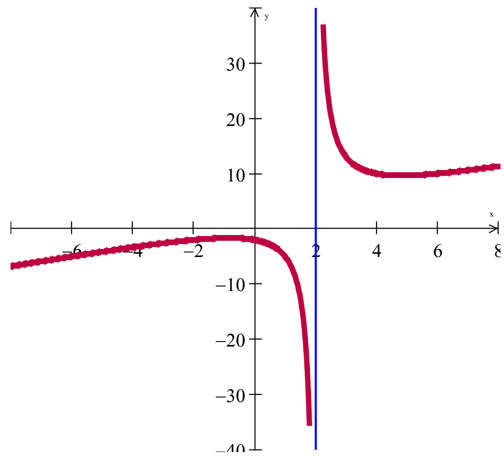
Solution

$$\lim_{x\to 1} \frac{x^2-4}{x-2} = \frac{1^2-4}{1-2}$$
$$= \frac{-3}{-1}$$

$\boxed{= 3}$

Exercise

Find the limit: $\lim\limits_{x \to 2} \dfrac{x^2 + 4}{x - 2}$



Solution

$$\lim_{x \to 2} \frac{x^2 + 4}{x - 2} = \frac{2^2 + 4}{2 - 2}$$

$$= \frac{8}{0}$$

$$\boxed{= \infty}$$ (***Doesn't exist***)

Exercise

Find the limit: $\lim\limits_{x \to 0} \dfrac{|x|}{x}$

Solution

$$\lim_{x \to 0} \frac{|x|}{x} = \frac{0}{0}$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \frac{x}{-x} = -1$$

$$\lim_{x \to 0^+} \frac{|x|}{x} = \frac{x}{x} = 1$$

Doesn't exist

Exercise

Find: $\lim\limits_{x \to 3} \dfrac{x^2 - x - 1}{\sqrt{x + 1}}$

Solution

$$\lim_{x \to 3} \frac{x^2 - x - 1}{\sqrt{x + 1}} = \frac{3^2 - 3 - 1}{\sqrt{3 + 1}}$$

$$\boxed{= \frac{5}{2}}$$

Exercise

Find: $\lim\limits_{x \to 2} \dfrac{x^2 + x - 6}{x - 2}$

Solution

$$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \frac{2^2+2-6}{2-2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(x+3)(x-2)}{x-2}$$

$$= \lim_{x\to 2} (x+3)$$

$$= 5$$

Exercise

Find the limit: $\lim\limits_{x\to 0} (3x-2)$

Solution

$$\lim_{x\to 0} (3x-2) = 3(0)-2$$

$$= -2$$

Exercise

Find the limit: $\lim\limits_{x\to 1} (2x^2-x+4)$

Solution

$$\lim_{x\to 1} (2x^2-x+4) = 2(1)^2-(1)+4$$

$$= 5$$

Exercise

Find the limit: $\lim\limits_{x\to -2} \left(x^3-2x^2+4x+8\right)$

Solution

$$\lim_{x\to -2} \left(x^3-2x^2+4x+8\right) = (-2)^3-2(-2)^2+4(-2)+8$$

$$= -16$$

Exercise

Find the limit: $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$

Solution

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\frac{2^2-4}{2-2}=\frac{0}{0}$$

$$=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}$$

$$=\lim_{x\to 2}(x+2)$$

$$\underline{=4}$$

Exercise

Find the limit: $\lim_{x\to 2}\frac{x^3-8}{x-2}$

Solution

$$\lim_{x\to 2}\frac{x^3-8}{x-2}=\frac{0}{0}$$

$$=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{x-2}$$

$$=\lim_{x\to 2}x^2+2x+4$$

$$=2^2+2(2)+4$$

$$\underline{=12}$$

Exercise

Find the limit: $\lim_{x\to 3}\frac{x^2+x-12}{x-3}$

Solution

$$\lim_{x\to 3}\frac{x^2+x-12}{x-3}=\frac{0}{0}$$

$$=\lim_{x\to 3}\frac{(x-3)(x+4)}{x-3}$$

$$=\lim_{x\to 3}(x+4)$$

$$\underline{=7}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$

Solution

$$\lim\limits_{x\to 0} \frac{\sqrt{x+4}-2}{x} = \frac{\sqrt{4}-2}{0} = \frac{0}{0}$$

$$= \lim\limits_{x\to 0} \frac{\sqrt{x+4}-2}{x} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$$

$$= \lim\limits_{x\to 0} \frac{x+4-4}{x\left(\sqrt{x+4}+2\right)}$$

$$= \lim\limits_{x\to 0} \frac{x}{x\left(\sqrt{x+4}+2\right)}$$

$$= \lim\limits_{x\to 0} \frac{1}{\sqrt{x+4}+2}$$

$$= \frac{1}{\sqrt{4}+2}$$

$$= \frac{1}{4}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{3}{\sqrt{3x+1}+1}$

Solution

$$\lim\limits_{x\to 0} \frac{3}{\sqrt{3x+1}+1} = \frac{3}{\sqrt{3(0)+1}+1}$$

$$= \frac{3}{1+1}$$

$$= \frac{3}{2}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} f(x) \qquad f(x) = \begin{cases} x^2+1 & x<0 \\ 2x+1 & x>0 \end{cases}$

Solution

$$\lim\limits_{x\to 0^-} x^2+1 = 1$$

$$\lim\limits_{x\to 0+} 2x+1 = 1$$

$$\lim_{x\to 0} f(x) \underline{=1}$$

Exercise

Find the limit: $\displaystyle\lim_{x\to -2} \frac{5}{x+2}$

Solution

$$\lim_{x\to -2} \frac{5}{x+2} = \frac{5}{0}$$

$$\underline{=\infty}$$

Exercise

Find the limit: $\displaystyle\lim_{x\to 3} \frac{\sqrt{x+1}-1}{x}$

Solution

$$\lim_{x\to 3} \frac{\sqrt{x+1}-1}{x} = \frac{\sqrt{3+1}-1}{3} = \frac{2-1}{3}$$

$$\underline{=\frac{1}{3}}$$

Exercise

Find the limit: $\displaystyle\lim_{x\to 1} \frac{x^2-1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x\to 1} (x+1)$$

$$\underline{=2}$$

Exercise

Find the limit: $\lim\limits_{x\to -2} \dfrac{|x+2|}{x+2}$

Solution

$$\lim_{x\to -2} \frac{|x+2|}{x+2} = \frac{|-2+2|}{-2+2} = \frac{0}{0}$$

$$\lim_{x\to -2^+} \frac{|x+2|}{x+2} = \frac{(x+2)}{(x+2)}$$

$$\underline{=1}$$

$$\lim_{x\to -2^-} \frac{|x+2|}{x+2} = \frac{(x+2)}{-(x+2)}$$

$$\underline{=-1}$$

Doesn't exist

Exercise

Find the limit: $\lim\limits_{x\to 0} (2x-8)^{1/3}$

Solution

$$\lim_{x\to 0} (2x-8)^{1/3} = (2(0)-8)^{1/3}$$

$$=(-8)^{1/3}$$

$$\underline{=-2}$$

Exercise

Find the limit: $\lim\limits_{x\to 2} \dfrac{x^2-7x+10}{x-2}$

Solution

$$\lim_{x\to 2} \frac{x^2-7x+10}{x-2} = \frac{2^2-7(2)+10}{2-2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(x-2)(x-5)}{x-2}$$

$$= \lim_{x\to 2} (x-5)$$

$$= 2-5$$

$$\underline{=-3}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{5x^3+8x^2}{3x^4-16x^2}$

Solution

$$\lim_{x\to 0} \frac{5x^3+8x^2}{3x^4-16x^2} = \frac{0}{0}$$

$$\begin{aligned}\lim_{x\to 0} \frac{5x^3+8x^2}{3x^4-16x^2} &= \lim_{x\to 0} \frac{x^2(5x+8)}{x^2\left(3x^2-16\right)} \\ &= \lim_{x\to 0} \frac{5x+8}{3x^2-16} \\ &= \frac{8}{-16} \\ &= -\frac{1}{2}\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{\frac{1}{x}-1}{x-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$\begin{aligned}\lim_{x\to 1} \frac{\frac{1}{x}-1}{x-1} &= \lim_{x\to 1} \frac{\frac{1-x}{x}}{x-1} \\ &= \lim_{x\to 1} \left(\frac{1-x}{x}\right)\left(\frac{1}{x-1}\right) \\ &= \lim_{x\to 1} \left(\frac{-(x-1)}{x}\right)\left(\frac{1}{x-1}\right) \\ &= \lim_{x\to 1} \frac{-1}{x} \\ &= -1\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{u\to 1} \dfrac{u^4-1}{u^3-1}$

Solution

$$\lim_{u\to 1}\frac{u^4-1}{u^3-1}=\lim_{u\to 1}\frac{\left(u^2-1\right)\left(u^2+1\right)}{(u-1)\left(u^2+u+1\right)}$$

$$=\lim_{u\to 1}\frac{(u-1)(u+1)\left(u^2+1\right)}{(u-1)\left(u^2+u+1\right)}$$

$$=\lim_{u\to 1}\frac{(u+1)\left(u^2+1\right)}{u^2+u+1}$$

$$=\frac{(1+1)\left(1^2+1\right)}{1^2+1+1}$$

$$=\frac{4}{3}$$

Exercise

Find the limit: $\lim\limits_{x\to 1}\frac{x-1}{\sqrt{x+3}-2}$

Solution

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2}=\frac{1-1}{\sqrt{1+3}-2}$$

$$=\frac{0}{\sqrt{4}-2}=\frac{0}{0}$$

$$=\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2}\bullet\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$$

$$=\lim_{x\to 1}\frac{(x-1)\left(\sqrt{x+3}+2\right)}{x+3-4}$$

$$=\lim_{x\to 1}\frac{(x-1)\left(\sqrt{x+3}+2\right)}{x-1}$$

$$=\lim_{x\to 1}\left(\sqrt{x+3}+2\right)$$

$$=\sqrt{1+3}+2$$

$$=4$$

Exercise

Find the limit: $\lim\limits_{x\to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$

Solution

$$\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} = \frac{\sqrt{(-1)^2+8}-3}{-1+1} = \frac{\sqrt{9}-3}{0} = \frac{0}{0}$$

$$= \lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} \cdot \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$$

$$= \lim_{x\to -1} \frac{x^2+8-9}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to -1} \frac{x^2-1}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to -1} \frac{(x-1)(x+1)}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to -1} \frac{(x-1)}{\sqrt{x^2+8}+3}$$

$$= \frac{-2}{\sqrt{9}+3}$$

$$= \frac{-2}{6}$$

$$= -\frac{1}{3}$$

Exercise

Find the limit: $\lim\limits_{x\to -3} \dfrac{2-\sqrt{x^2-5}}{x+3}$

Solution

$$\lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3} = \frac{2-\sqrt{(-3)^2-5}}{-3+3}$$

$$= \frac{2-\sqrt{9-5}}{0}$$

$$= \frac{2-\sqrt{4}}{0} = \frac{0}{0}$$

$$= \lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3} \bullet \frac{2+\sqrt{x^2-5}}{2+\sqrt{x^2-5}}$$

$$= \lim_{x\to -3} \frac{4-\left(x^2-5\right)}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{4-x^2+5}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{9-x^2}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{(x-3)(x+3)}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{(x-3)}{2+\sqrt{x^2-5}}$$

$$= \frac{-6}{2+\sqrt{9-5}}$$

$$= \frac{-6}{2+\sqrt{4}}$$

$$= -\frac{6}{4}$$

$$= -\frac{3}{2}$$

Exercise

Find the limit: $\lim_{x\to 0} (2\sin x - 1)$

Solution

$$\lim_{x\to 0} (2\sin x - 1) = 2\sin(0) - 1$$

$$= 0 - 1$$

$$= -1$$

Exercise

Find the limit: $\lim_{x\to 0} \sin^2 x$

Solution

$$\lim_{x \to 0} \sin^2 x = \sin^2(0)$$

$$\underline{= 0}$$

Exercise

Find the limit: $\lim_{x \to 0} \sec x$

Solution

$$\lim_{x \to 0} \sec x = \sec(0)$$

$$= \frac{1}{\cos(0)}$$

$$\underline{= 1}$$

Exercise

Find the limit: $\lim_{x \to 0} \frac{1 + x + \sin x}{3\cos x}$

Solution

$$\lim_{x \to 0} \frac{1 + x + \sin x}{3\cos x} = \frac{1 + 0 + \sin(0)}{3\cos(0)}$$

$$\underline{= \frac{1}{3}}$$

Exercise

Find the limit: $\lim_{x \to -\pi} \sqrt{x+4}\ \cos(x+\pi)$

Solution

$$\lim_{x \to -\pi} \sqrt{x+4}\ \cos(x+\pi) = \sqrt{-\pi+4}\ \cos(-\pi+\pi)$$

$$= \sqrt{-\pi+4}\ \cos(0)$$

$$\underline{= \sqrt{4-\pi}}$$

Exercise

Find $\lim_{x \to -0.5^-} \sqrt{\frac{x+2}{x+1}}$

Solution

$$\lim_{x \to -0.5^-} \sqrt{\frac{x+2}{x+1}} = \sqrt{\frac{-0.5+2}{-0.5+1}}$$

$$= \sqrt{\frac{1.5}{0.5}}$$

$$= \sqrt{3}$$

Exercise

Find $\lim_{x \to 1^+} \sqrt{\frac{x-1}{x+2}}$

Solution

$$\lim_{x \to 1^+} \sqrt{\frac{x-1}{x+2}} = \sqrt{\frac{1-1}{1+2}}$$

$$= 0$$

Exercise

Find $\lim_{x \to -2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right)$

Solution

$$\lim_{x \to -2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right) = \left(\frac{-2}{-2+1}\right)\left(\frac{2(-2)+5}{(-2)^2+(-2)}\right)$$

$$= \left(\frac{-2}{-1}\right)\left(\frac{1}{2}\right)$$

$$= 1$$

Exercise

Find $\lim_{x \to 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}$

Solution

$$\lim_{x \to 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x} = \frac{\sqrt{5}-\sqrt{5}}{0} = \frac{0}{0}$$

$$= \lim_{x \to 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x} \frac{\sqrt{x^2+4x+5}+\sqrt{5}}{\sqrt{x^2+4x+5}+\sqrt{5}}$$

$$= \lim_{x\to 0^+} \frac{x^2+4x+5-5}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$= \lim_{x\to 0^+} \frac{x^2+4x}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$= \lim_{x\to 0^+} \frac{x(x+4)}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$= \lim_{x\to 0^+} \frac{x+4}{\sqrt{x^2+4x+5}+\sqrt{5}}$$

$$= \frac{0+4}{\sqrt{0^2+4(0)+5}+\sqrt{5}}$$

$$= \frac{4}{\sqrt{5}+\sqrt{5}}$$

$$= \frac{4}{2\sqrt{5}}$$

$$= \frac{2}{\sqrt{5}}$$

Exercise

Find $\lim\limits_{x\to -2^+} (x+3)\frac{|x+2|}{x+2}$

Solution

$$\lim_{x\to -2^+} (x+3)\frac{|x+2|}{x+2} = (x+3)\frac{|-2+2|}{-2+2} = \frac{0}{0}$$

Since $x \to -2^+ \quad \Rightarrow \quad x > -2$

$\Rightarrow \quad |x+2| = (x+2)$

$$\lim_{x\to -2^+} (x+3)\frac{|x+2|}{x+2} = \lim_{x\to -2^+} (x+3)\frac{x+2}{x+2}$$

$$= \lim_{x\to -2^+} (x+3)$$

$$= -2+3$$

$$= 1$$

Exercise

Find $\lim\limits_{x\to 1^+} \dfrac{\sqrt{2x}\,(x-1)}{|x-1|}$

Solution

$$\lim_{x\to 1^+} \frac{\sqrt{2x}\,(x-1)}{|x-1|} = \frac{\sqrt{2(1)}\,(1-1)}{|1-1|} = \frac{0}{0}$$

Since $x \to 1^+ \;\Rightarrow\; x > 1$

$\Rightarrow\; |x-1| = x-1$

$$\lim_{x\to 1^+} \frac{\sqrt{2x}\,(x-1)}{|x-1|} = \lim_{x\to 1^+} \frac{\sqrt{2x}\,(x-1)}{x-1}$$

$$= \lim_{x\to 1^+} \sqrt{2x}$$

$$= \sqrt{2}$$

Exercise

Find $\lim\limits_{\theta\to 0} \dfrac{\sin\sqrt{2}.\theta}{\sqrt{2}.\theta}$

Solution

Let: $\sqrt{2}\theta = x \to 0$

$$\lim_{\theta\to 0} \frac{\sin\sqrt{2}.\theta}{\sqrt{2}.\theta} = \lim_{x\to 0} \frac{\sin x}{x}$$

$$= 1$$

Exercise

Find $\lim\limits_{x\to 0} \dfrac{\sin 3x}{4x}$

Solution

$$\lim_{x\to 0} \frac{\sin 3x}{4x} = \lim_{x\to 0} \frac{\sin 3x}{4x}\frac{3}{3}$$

$$= \frac{3}{4} \lim_{x\to 0} \frac{\sin 3x}{3x} \qquad \text{Let: } 3x = u$$

$$= \frac{3}{4} \lim_{u\to 0} \frac{\sin u}{u} \qquad \textit{By definition: } \lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$= \frac{3}{4}$$

Exercise

Find $\displaystyle\lim_{x\to 0^-} \frac{x}{\sin 3x}$

Solution

$$\lim_{x\to 0^-} \frac{x}{\sin 3x} = \lim_{x\to 0^-} \frac{x}{\sin 3x}\left(\frac{3}{3}\right)$$

$$= \frac{1}{3}\lim_{x\to 0^-} \frac{3x}{\sin 3x}$$

$$= \frac{1}{3}\lim_{x\to 0^-} \frac{1}{\frac{\sin 3x}{3x}}$$

By definition: $\displaystyle\lim_{u\to 0} \frac{\sin u}{u} = 1$

$$= \frac{1}{3}$$

Exercise

Find $\displaystyle\lim_{x\to 0} \frac{\tan 2x}{x}$

Solution

$$\lim_{x\to 0} \frac{\tan 2x}{x} = \lim_{x\to 0} \frac{\frac{\sin 2x}{\cos 2x}}{x}$$

$$= \lim_{x\to 0} \left(\frac{\sin 2x}{x} \bullet \frac{1}{\cos 2x}\right)$$

$$= \lim_{x\to 0} \left(2\frac{\sin 2x}{2x}\right) \lim_{x\to 0} \left(\frac{1}{\cos 2x}\right)$$

$$= 2\,\frac{1}{\cos 0}$$

$$= 2$$

Exercise

Find $\displaystyle\lim_{x\to 0} 6x^2(\cot x)(\csc 2x)$

Solution

$$\lim_{x\to 0} 6x^2(\cot x)(\csc 2x) = \lim_{x\to 0} 6x^2\left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin 2x}\right)$$

$$= \lim_{x\to 0} 3\cos x\left(\frac{x}{\sin x}\right)\left(\frac{2x}{\sin 2x}\right)$$

$$= 3\lim_{x\to 0} (\cos x) \bullet \lim_{x\to 0} \left(\frac{x}{\sin x}\right) \bullet \lim_{2x\to 0} \left(\frac{2x}{\sin 2x}\right)$$

$$= (3)(1)(1)(1)$$

$$= 3$$

Exercise

Find $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\sin 2\theta}$

Solution

$$\lim_{\theta\to 0} \frac{\sin\theta}{\sin 2\theta} = \lim_{\theta\to 0} \frac{\sin\theta}{\sin 2\theta}\frac{2\theta}{2\theta}$$

$$= \frac{1}{2}\lim_{\theta\to 0}\left(\frac{2\theta}{\sin 2\theta}\cdot\frac{\sin\theta}{\theta}\right)$$

$$= \frac{1}{2}(1)(1)$$

$$= \frac{1}{2}$$

Exercise

Find $\lim\limits_{h\to 0} \dfrac{\sin(\sin h)}{\sin h}$

Solution

Let: $\sin h = \theta$ $\qquad\qquad \theta = \sin h \xrightarrow[h\to 0]{} 0$

$$\lim_{h\to 0} \frac{\sin(\sin h)}{\sin h} = \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}$$

$$= 1$$

Exercise

Find $\lim\limits_{\theta\to 0} \dfrac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}$

Solution

$$\lim_{\theta\to 0} \frac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta} = \lim_{\theta\to 0} \frac{\theta\frac{\cos 4\theta}{\sin 4\theta}}{\sin^2\theta\frac{\cos^2 2\theta}{\sin^2 2\theta}}$$

$$= \lim_{\theta\to 0} \theta\frac{\cos 4\theta}{2\sin 2\theta\cos 2\theta}\frac{\sin^2 2\theta}{\sin^2\theta\cos^2 2\theta}$$

$$= \lim_{\theta\to 0}\left(\frac{1}{2}\cdot\theta\cdot\cos 4\theta\bullet\frac{2\sin\theta\cos\theta}{\sin^2\theta}\cdot\frac{1}{\cos^3 2\theta}\right)$$

$$= \lim_{\theta\to 0}\left(\cos 4\theta\bullet\frac{\theta}{\sin\theta}\cdot\cos\theta\cdot\frac{1}{\cos^3 2\theta}\right)$$

$$= \lim_{\theta \to 0} (\cos 4\theta) \quad \lim_{\theta \to 0} \left(\frac{\theta}{\sin\theta} \right) \quad \lim_{\theta \to 0} \left(\frac{\cos\theta}{\cos^3 2\theta} \right)$$

$$= (1)(1)(1)$$

$$\underline{= 1}$$

Exercise

Find $\displaystyle \lim_{\theta \to \pi/4} \frac{\sin^2\theta - \cos^2\theta}{\sin\theta - \cos\theta}$

Solution

$$\lim_{\theta \to \pi/4} \frac{\sin^2\theta - \cos^2\theta}{\sin\theta - \cos\theta} = \frac{\frac{1}{2} - \frac{1}{2}}{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}} = \frac{0}{0}$$

$$= \lim_{\theta \to \pi/4} \frac{(\sin\theta - \cos\theta)(\sin\theta + \cos\theta)}{\sin\theta - \cos\theta}$$

$$= \lim_{\theta \to \pi/4} (\sin\theta + \cos\theta)$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

$$\underline{= \sqrt{2}}$$

Exercise

Find $\displaystyle \lim_{x \to \pi/2} \frac{\frac{1}{\sqrt{\sin x}} - 1}{x + \frac{\pi}{2}}$

Solution

$$\lim_{x \to \pi/2} \frac{\frac{1}{\sqrt{\sin x}} - 1}{x + \frac{\pi}{2}} = \frac{1 - 1}{\frac{\pi}{2} + \frac{\pi}{2}}$$

$$\underline{= 0}$$

Exercise

Find $\displaystyle \lim_{x \to 1} \frac{x^3 - 7x^2 + 12x}{4 - x}$

Solution

$$\lim_{x \to 1} \frac{x^3 - 7x^2 + 12x}{4 - x} = \frac{1 - 7 + 12}{4 - 1}$$

$$\underline{= 2}$$

Exercise

Find $\lim\limits_{x\to 4} \dfrac{x^3-7x^2+12x}{4-x}$

Solution

$$\lim_{x\to 4} \frac{x^3-7x^2+12x}{4-x} = \frac{64-112+48}{4-4} = \frac{0}{0}$$

$$= \lim_{x\to 4} \frac{x(x-3)(x-4)}{4-x}$$

$$= \lim_{x\to 4} -x(x-3)$$

$$= -4$$

Exercise

Find $\lim\limits_{x\to 1} \dfrac{1-x^2}{x^2-8x+7}$

Solution

$$\lim_{x\to 1} \frac{1-x^2}{x^2-8x+7} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{(1-x)(1+x)}{(x-1)(x-7)}$$

$$= -\lim_{x\to 1} \frac{1+x}{x-7}$$

$$= \frac{1}{3}$$

Exercise

Find $\lim\limits_{x\to 3} \dfrac{\sqrt{3x+16}-5}{x-3}$

Solution

$$\lim_{x\to 3} \frac{\sqrt{3x+16}-5}{x-3} = \frac{\sqrt{9+16}-5}{3-3} = \frac{5-5}{0} = \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{\sqrt{3x+16}-5}{x-3} \frac{\sqrt{3x+16}+5}{\sqrt{3x+16}+5}$$

$$= \lim_{x\to 3} \frac{3x+16-25}{(x-3)\left(\sqrt{3x+16}+5\right)}$$

$$= \lim_{x\to 3} \frac{3(x-3)}{(x-3)\left(\sqrt{3x+16}+5\right)}$$

$$= \lim_{x\to 3} \frac{3}{\sqrt{3x+16}+5}$$

$$= \frac{3}{5+5}$$

$$= \frac{3}{10}$$

Exercise

Find $\displaystyle \lim_{x\to 3} \frac{1}{x-3}\left(\frac{1}{\sqrt{x+1}}-\frac{1}{2}\right)$

Solution

$$\lim_{x\to 3} \frac{1}{x-3}\left(\frac{1}{\sqrt{x+1}}-\frac{1}{2}\right) = \frac{1}{0}\left(\frac{1}{2}-\frac{1}{2}\right) = \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{1}{x-3}\left(\frac{2-\sqrt{x+1}}{\sqrt{x+1}}\right)\left(\frac{2+\sqrt{x+1}}{2+\sqrt{x+1}}\right)$$

$$= \lim_{x\to 3} \frac{1}{x-3}\left(\frac{4-x-1}{2\sqrt{x+1}+x+1}\right)$$

$$= \lim_{x\to 3} \frac{x-3}{x-3}\left(\frac{-1}{2\sqrt{x+1}+x+1}\right)$$

$$= \lim_{x\to 3} \frac{-1}{2\sqrt{x+1}+x+1}$$

$$= -\frac{1}{8}$$

Exercise

Find $\displaystyle \lim_{x\to 1/3} \frac{x-\frac{1}{3}}{(3x-1)^2}$

Solution

$$\lim_{x\to 1/3} \frac{x-\frac{1}{3}}{(3x-1)^2} = \frac{\frac{1}{3}-\frac{1}{3}}{\left(3\frac{1}{3}-1\right)^2} = \frac{0}{0}$$

$$= \lim_{x\to 1/3} \frac{x-\frac{1}{3}}{9\left(x-\frac{1}{3}\right)^2}$$

$$= \lim_{x\to 1/3} \frac{1}{9\left(x-\frac{1}{3}\right)} = \frac{1}{0}$$

$$= \infty$$

Exercise

Find $\lim\limits_{x\to 3} \dfrac{x^4-81}{x-3}$

Solution

$$\lim_{x\to 3} \frac{x^4-81}{x-3} = \frac{81-81}{3-3} = \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{(x-3)(x+3)\left(x^2+9\right)}{x-3} \qquad a^4-b^4 = \left(a^2-b^2\right)\left(a^2+b^2\right) = (a-b)(a+b)\left(a^2+b^2\right)$$

$$= \lim_{x\to 3} (x+3)\left(x^2+9\right) = 6(18)$$

$$\underline{= 108}$$

Exercise

Find $\lim\limits_{x\to 1} \dfrac{x^5-1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{x^5-1}{x-1} = \frac{1-1}{1-1} = \frac{0}{0} \qquad \left(a^5-b^5\right) = (a-b)\left(a^4+a^3b+a^2b^2+ab^3+b^4\right)$$

$$= \lim_{x\to 1} \frac{(x-1)\left(x^4+x^3+x^2+x+1\right)}{x-1}$$

$$= \lim_{x\to 1} \left(x^4+x^3+x^2+x+1\right)$$

$$\underline{= 5}$$

Exercise

Find $\lim\limits_{x\to 81} \dfrac{\sqrt[4]{x}-3}{x-81}$

Solution

$$\lim_{x\to 81} \frac{\sqrt[4]{x}-3}{x-81} = \frac{3-3}{81-81} = \frac{0}{0}$$

$$= \lim_{x\to 81} \frac{\sqrt[4]{x}-3}{\left(\sqrt{x}+9\right)\left(\sqrt{x}-9\right)}$$

$$= \lim_{x\to 81} \frac{\sqrt[4]{x}-3}{\left(\sqrt{x}+9\right)\left(\sqrt[4]{x}+3\right)\left(\sqrt[4]{x}-3\right)}$$

$$= \lim_{x \to 81} \frac{1}{(\sqrt{x}+9)(\sqrt[4]{x}+3)}$$

$$= \frac{1}{(18)(6)}$$

$$= \frac{1}{108}$$

Exercise

Find the limit: $\lim_{x \to 1} \frac{\sqrt[3]{x}-1}{x-1}$

Solution

$$\lim_{x \to 1} \frac{\sqrt[3]{x}-1}{x-1} = \frac{0}{0}$$

$$= \lim_{x \to 1} \frac{\sqrt[3]{x}-1}{(\sqrt[3]{x})^3 - 1^3}$$

$$= \lim_{x \to 1} \frac{\sqrt[3]{x}-1}{(\sqrt[3]{x}-1)(x^{2/3}+\sqrt[3]{x}+1)}$$

$$= \lim_{x \to 1} \frac{1}{x^{2/3}+\sqrt[3]{x}+1}$$

$$= \frac{1}{3}$$

Exercise

Find the limit: $\lim_{x \to 2} \frac{x^5-32}{x-2}$

Solution

2	1	0	0	0	0	−32
		2	4	8	16	32
	1	2	4	8	16	0

$$\lim_{x \to 2} \frac{x^5-32}{x-2} = \frac{2^5-32}{2-2} = \frac{0}{0}$$

$$= \lim_{x \to 2} \frac{(x-2)(x^4+2x^3+4x^2+8x+16)}{x-2}$$

$$= \lim_{x \to 2} (x^4+2x^3+4x^2+8x+16)$$

$$= 16+16+16+16+16$$

$$= 80$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{x^6-1}{x-1}$

Solution

$$\begin{array}{r|rrrrrrr} 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ & & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}$$

$$\lim_{x\to 1} \frac{x^6-1}{x-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(x^5+x^4+x^3+x^2+x+1\right)}{x-1}$$

$$= \lim_{x\to 1} \left(x^5+x^4+x^3+x^2+x+1\right)$$

$$= 6$$

Exercise

Find the limit: $\lim\limits_{x\to -1} \dfrac{x^7+1}{x+1}$

Solution

$$\begin{array}{r|rrrrrrrr} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 \end{array}$$

$$\lim_{x\to -1} \frac{x^7+1}{x+1} = \frac{-1+1}{-1+1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{(x+1)\left(x^6-x^5+x^4-x^3+x^2-x+1\right)}{x+1}$$

$$= \lim_{x\to 1} \left(x^6-x^5+x^4-x^3+x^2-x+1\right)$$

$$= 1$$

Exercise

Find the limit: $\lim\limits_{x\to a} \dfrac{x^5-a^5}{x-a}$

Solution

$$\begin{array}{c|cccccc} a & 1 & 0 & 0 & 0 & 0 & -a^5 \\ & & a & a^2 & a^3 & a^4 & a^5 \\ \hline & 1 & a & a^2 & a^3 & a^4 & 0 \end{array}$$

$$\lim_{x \to a} \frac{x^5 - a^5}{x - a} = \frac{a^5 - a^5}{a - a} = \frac{0}{0}$$

$$= \lim_{x \to a} \frac{(x-a)\left(x^4 + ax^3 + a^2x^2 + a^3x + a^4\right)}{x - a}$$

$$= \lim_{x \to a} \left(x^4 + ax^3 + a^2x^2 + a^3x + a^4\right)$$

$$= a^4 + a^4 + a^4 + a^4 + a^4$$

$$= 5a^4$$

Exercise

Find the limit: $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a}$ $\quad n \in \mathbb{Z}^+$

Solution

$$\begin{array}{c|ccccccc} a & 1 & 0 & 0 & 0 & \cdots & 0 & -a^n \\ & & a & a^2 & a^3 & \cdots & a^{n-1} & a^n \\ \hline & 1 & a & a^2 & a^3 & \cdots & a^{n-1} & 0 \end{array}$$

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \frac{a^n - a^n}{a - a} = \frac{0}{0}$$

$$= \lim_{x \to a} \frac{(x-a)\left(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-2}x + a^{n-1}\right)}{x - a}$$

$$= \lim_{x \to a} \left(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-2}x + a^{n-1}\right)$$

$$= a^{n-1} + a^{n-1} + a^{n-1} + \ldots + a^{n-1} + a^{n-1}$$

$$= na^{n-1}$$

Exercise

Find the limit: $\lim\limits_{h \to 0} \dfrac{100}{(10h-1)^{11} + 2}$

Solution

$$\lim_{h\to 0} \frac{100}{(10h-1)^{11}+2} = \frac{100}{(-1)^{11}+2}$$

$$= \frac{100}{-1+2}$$

$$\boxed{= 100}$$

Exercise

Find the limit: $\lim_{h\to 0} \frac{(5+h)^2-25}{h}$

Solution

$$\lim_{h\to 0} \frac{(5+h)^2-25}{h} = \frac{5^2-25}{0} = \frac{0}{0}$$

$$= \lim_{h\to 0} \frac{\big((5+h)-5\big)\big((5+h)+5\big)}{h}$$

$$= \lim_{h\to 0} \frac{h(h+10)}{h}$$

$$= \lim_{h\to 0} (h+10)$$

$$\boxed{= 10}$$

Exercise

Find the limit: $\lim_{x\to 3} \frac{\frac{1}{x^2+2x}-\frac{1}{15}}{x-3}$

Solution

$$\lim_{x\to 3} \frac{\frac{1}{x^2+2x}-\frac{1}{15}}{x-3} = \frac{\frac{1}{15}-\frac{1}{15}}{0} = \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{1}{x-3}\left(\frac{1}{x(x+2)}-\frac{1}{15}\right)$$

$$= \lim_{x\to 3} \frac{1}{x-3}\left(\frac{15-x^2-2x}{15x(x+2)}\right)$$

$$= \lim_{x\to 3} \frac{-(x-3)(x+5)}{15x(x+2)(x-3)}$$

$$= \lim_{x\to 3} \frac{-(x+5)}{15x(x+2)}$$

$$= -\frac{8}{15(3)(5)}$$

$$= -\frac{8}{225}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \frac{\sqrt{10x-9}-1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{\sqrt{10x-9}-1}{x-1} = \frac{1-1}{0} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{\sqrt{10x-9}-1}{x-1} \cdot \frac{\sqrt{10x-9}+1}{\sqrt{10x-9}+1}$$

$$= \lim_{x\to 1} \frac{10x-9-1}{(x-1)\left(\sqrt{10x-9}+1\right)}$$

$$= \lim_{x\to 1} \frac{10(x-1)}{(x-1)\left(\sqrt{10x-9}+1\right)}$$

$$= \lim_{x\to 1} \frac{10}{\sqrt{10x-9}+1}$$

$$= \frac{10}{2}$$

$$= 5$$

Exercise

Find the limit: $\lim\limits_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2x}\right)$

Solution

$$\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2x}\right) = \frac{1}{0} - \frac{2}{0} = \infty - \infty$$

$$= \lim_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x(x-2)}\right)$$

$$= \lim_{x\to 2} \frac{x-2}{x(x-2)}$$

$$= \lim_{x\to 2} \frac{1}{x}$$

$$= \frac{1}{2}$$

Exercise

Find the limit: $\lim\limits_{x\to c} \dfrac{x^2-2cx+c^2}{x-c}$

Solution

$$\lim_{x\to c} \frac{x^2-2cx+c^2}{x-c} = \frac{c^2-2c^2+c^2}{0} = \frac{0}{0}$$

$$= \lim_{x\to c} \frac{(x-c)^2}{x-c}$$

$$= \lim_{x\to c} (x-c)$$

$$= 0$$

Exercise

Find the limit: $\lim\limits_{x\to -c} \dfrac{x^2+5cx+4c^2}{x^2+cx}$

Solution

$$\lim_{x\to -c} \frac{x^2+5cx+4c^2}{x^2+cx} = \frac{c^2-5c^2+4c^2}{c^2-c^2} = \frac{0}{0}$$

$$= \lim_{x\to -c} \frac{(x+c)(x+4c)}{x(x+c)}$$

$$= \lim_{x\to -c} \frac{x+4c}{x}$$

$$= \frac{-c+4c}{-c}$$

$$= \frac{3c}{-c}$$

$$= -3$$

Exercise

Find the limit: $\lim\limits_{x\to 16} \dfrac{\sqrt[4]{x}-2}{x-16}$

Solution

$$\lim_{x\to 16} \frac{\sqrt[4]{x}-2}{x-16} = \frac{\sqrt[4]{16}-2}{16-16} = \frac{2-2}{0} = \frac{0}{0}$$

$$\lim_{x\to 16} \frac{\sqrt[4]{x}-2}{\left(\sqrt[4]{x}\right)^4-2^4}$$

$$a^4-b^4=\left(a^2+b^2\right)(a-b)(a+b)$$

$$= \lim_{x\to16} \frac{\sqrt[4]{x}-2}{\left(\sqrt{x}+2^2\right)\left(\sqrt[4]{x}+2\right)\left(\sqrt[4]{x}-2\right)}$$

$$= \lim_{x\to16} \frac{1}{\left(\sqrt{x}+4\right)\left(\sqrt[4]{x}+2\right)}$$

$$= \frac{1}{\left(\sqrt{16}+4\right)\left(\sqrt[4]{16}+2\right)}$$

$$= \frac{1}{(4+4)(2+2)}$$

$$= \frac{1}{(8)(4)}$$

$$= \frac{1}{32}$$

Exercise

Find the limit: $\displaystyle \lim_{x\to1} \frac{x-1}{\sqrt{x}-1}$

Solution

$$\lim_{x\to1} \frac{x-1}{\sqrt{x}-1} = \frac{0}{0}$$

$$= \lim_{x\to1} \frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{\sqrt{x}-1}$$

$$= \lim_{x\to1} \left(\sqrt{x}+1\right)$$

$$= 2$$

Exercise

Find the limit: $\displaystyle \lim_{x\to1} \frac{x-1}{\sqrt{4x+5}-3}$

Solution

$$\lim_{x\to1} \frac{x-1}{\sqrt{4x+5}-3} = \frac{0}{0}$$

$$= \lim_{x\to1} \frac{x-1}{\sqrt{4x+5}-3} \cdot \frac{\sqrt{4x+5}+3}{\sqrt{4x+5}+3}$$

$$= \lim_{x\to1} \frac{(x-1)\left(\sqrt{4x+5}+3\right)}{4x+5-9}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{4x+5}+3\right)}{4(x-1)}$$

$$= \frac{1}{5} \lim_{x\to 1} \left(\sqrt{4x+5}+3\right)$$

$$\boxed{= \frac{6}{5}}$$

Exercise

Find the limit: $\displaystyle \lim_{x\to 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}$

Solution

$$\lim_{x\to 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}} = \frac{0}{3-3} = \frac{0}{0}$$

$$= \lim_{x\to 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}} \bullet \frac{3+\sqrt{x+5}}{3+\sqrt{x+5}}$$

$$= 3 \lim_{x\to 4} \frac{(x-4)\left(3+\sqrt{x+5}\right)\sqrt{x+5}}{9-(x+5)}$$

$$= 3 \lim_{x\to 4} \frac{(x-4)\left(3+\sqrt{x+5}\right)\sqrt{x+5}}{4-x}$$

$$= -3 \lim_{x\to 4} \left(3+\sqrt{x+5}\right)\sqrt{x+5}$$

$$= -3\ (6)(3)$$

$$\boxed{= -54}$$

Exercise

Find the limit: $\displaystyle \lim_{x\to 0} \frac{x}{\sqrt{ax+1}-1} \quad (a \neq 0)$

Solution

$$\lim_{x\to 0} \frac{x}{\sqrt{ax+1}-1} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{x}{\sqrt{ax+1}-1} \bullet \frac{\sqrt{ax+1}+1}{\sqrt{ax+1}+1}$$

$$= \lim_{x\to 0} \frac{x\left(\sqrt{ax+1}+1\right)}{ax+1-1}$$

$$= \lim_{x\to 0} \frac{x\left(\sqrt{ax+1}+1\right)}{ax}$$

$$= \frac{1}{a} \lim_{x\to 0} \left(\sqrt{ax+1}+1\right)$$

$$= \frac{2}{a}$$

Exercise

Find the limit: $\lim\limits_{x\to\pi} \frac{\cos^2 x + 3\cos x + 2}{\cos x + 1}$

Solution

$$\lim_{x\to\pi} \frac{\cos^2 x + 3\cos x + 2}{\cos x + 1} = \frac{1-3+2}{-1+1} = \frac{0}{0}$$

$$= \lim_{x\to\pi} \frac{(\cos x+1)(\cos x+2)}{\cos x+1}$$

$$= \lim_{x\to\pi} (\cos x + 2)$$

$$= -1+2$$

$$= 1$$

Exercise

Find the limit: $\lim\limits_{x\to\frac{3\pi}{2}} \frac{\sin^2 x + 6\sin x + 5}{\sin^2 x - 1}$

Solution

$$\lim_{x\to\frac{3\pi}{2}} \frac{\sin^2 x + 6\sin x + 5}{\sin^2 x - 1} = \frac{1-6+5}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to\frac{3\pi}{2}} \frac{(\sin x+1)(\sin x+5)}{(\sin x-1)(\sin x+1)}$$

$$= \lim_{x\to\frac{3\pi}{2}} \frac{\sin x+5}{\sin x-1}$$

$$= \frac{-1+5}{-1-1}$$

$$= -2$$

Exercise

Find the limit: $\lim\limits_{x\to\frac{\pi}{2}} \frac{\sin x - 1}{\sqrt{\sin x} - 1}$

Solution

$$\lim_{x\to\frac{\pi}{2}} \frac{\sin x - 1}{\sqrt{\sin x} - 1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\left(\sqrt{\sin x} - 1\right)\left(\sqrt{\sin x} + 1\right)}{\sqrt{\sin x} - 1}$$

$$= \lim_{x\to\frac{\pi}{2}} \left(\sqrt{\sin x} + 1\right)$$

$$= 2$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{\frac{1}{2+\sin x} - \frac{1}{2}}{\sin x}$

Solution

$$\lim_{x\to 0} \frac{\frac{1}{2+\sin x} - \frac{1}{2}}{\sin x} = \frac{\frac{1}{2} - \frac{1}{2}}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1}{\sin x} \cdot \frac{2 - \sin x - 2}{2(2+\sin x)}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{1}{\sin x} \frac{-\sin x}{(2+\sin x)}$$

$$= -\frac{1}{2} \lim_{x\to 0} \frac{1}{2+\sin x}$$

$$= -\frac{1}{2}\left(\frac{1}{2}\right)$$

$$= -\frac{1}{4}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{e^{2x} - 1}{e^x - 1}$

Solution

$$\lim_{x\to 0} \frac{e^{2x} - 1}{e^x - 1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\left(e^x - 1\right)\left(e^x + 1\right)}{e^x - 1}$$

$$= \lim_{x\to 0} \left(e^x + 1\right)$$

$$= 2$$

Exercise

Find the limit: $\lim\limits_{x \to \frac{\pi}{4}} \csc x$

Solution

$$\lim_{x \to \frac{\pi}{4}} \csc x = \csc \frac{\pi}{4}$$

$$= \frac{1}{\cos \frac{\pi}{4}}$$

$$= \sqrt{2}$$

Exercise

Find the limit: $\lim\limits_{x \to 4} \frac{x-5}{\left(x^2-10x+24\right)^2}$

Solution

$$\lim_{x \to 4} \frac{x-5}{\left(x^2-10x+24\right)^2} = \frac{-1}{(16-41+24)^2}$$

$$= -1$$

Exercise

Find the limit: $\lim\limits_{x \to 0} \frac{\cos x - 1}{\sin^2 x}$

Solution

$$\lim_{x \to 0} \frac{\cos x - 1}{\sin^2 x} = \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\cos x - 1}{(1-\cos x)(1+\cos x)}$$

$$= -\lim_{x \to 0} \frac{1}{1+\cos x}$$

$$= -\frac{1}{2}$$

Exercise

Find the limit: $\lim\limits_{x \to 0} \frac{1-\cos^2 x}{\sin x}$

Solution

$$\lim_{x\to 0} \frac{1-\cos^2 x}{\sin x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{\sin x}$$

$$= \lim_{x\to 0} \sin x$$

$$= 0$$

Exercise

Find $\lim_{x\to 0} \frac{x^3 - 5x^2}{x^2}$

Solution

$$\lim_{x\to 0} \frac{x^3 - 5x^2}{x^2} = \frac{0}{0}$$

$$= \lim_{x\to 0} (x-5)$$

$$= -5$$

Exercise

Find $\lim_{x\to 5} \frac{4x^2 - 100}{x-5}$

Solution

$$\lim_{x\to 5} \frac{4x^2 - 100}{x-5} = \frac{0}{0}$$

$$= \lim_{x\to 5} \frac{4(x-5)(x+5)}{x-5}$$

$$= \lim_{x\to 5} 4(x+5)$$

$$= 40$$

Exercise

Find $\lim_{x\to 3} \frac{\sqrt{9-6x+x^2}}{x-3}$

Solution

$$\lim_{x\to 3} \frac{\sqrt{9-6x+x^2}}{x-3} = \frac{\sqrt{9-18+9}}{3-3} = \frac{0}{0}$$

$$= \lim_{x \to 3} \frac{\sqrt{(x-3)^2}}{x-3}$$

$$= \lim_{x \to 3} \frac{x-3}{x-3}$$

$$\underline{= 1}$$

Exercise

Find $\quad \lim_{x \to 3} \frac{\sqrt{9+6x+x^2}}{x-3}$

Solution

$$\lim_{x \to 3} \frac{\sqrt{9+6x+x^2}}{x-3} = \frac{\sqrt{9+18+9}}{3-3}$$

$$= \frac{\sqrt{36}}{0}$$

$$\underline{= \infty}$$

Exercise

Find $\quad \lim_{x \to 3} \frac{\sqrt{x^2-9}}{x-3}$

Solution

$$\lim_{x \to 3} \frac{\sqrt{x^2-9}}{x-3} = \frac{\sqrt{9-9}}{3-3} = \frac{0}{0}$$

$$= \lim_{x \to 3} \frac{\sqrt{(x-3)(x+3)}}{x-3}$$

$$= \lim_{x \to 3} \sqrt{\frac{x+3}{x-3}}$$

$$= \sqrt{\frac{6}{0}}$$

$$\underline{= \infty}$$

Exercise

Find $\quad \lim_{x \to \frac{4\pi}{3}} \sin x$

Solution

$$\lim_{x\to\frac{4\pi}{3}} \sin x = \sin\frac{4\pi}{3}$$

$$\boxed{= -\frac{\sqrt{3}}{2}}$$

Exercise

Find $\displaystyle\lim_{x\to\frac{2\pi}{3}} \cos x$

Solution

$$\lim_{x\to\frac{2\pi}{3}} \cos x = \cos\frac{2\pi}{3}$$

$$\boxed{= -\frac{1}{2}}$$

Exercise

Find $\displaystyle\lim_{x\to\frac{7\pi}{4}} \sin x$

Solution

$$\lim_{x\to\frac{7\pi}{4}} \sin x = \sin\frac{7\pi}{4}$$

$$\boxed{= -\frac{\sqrt{2}}{2}}$$

Exercise

Find $\displaystyle\lim_{x\to 1} \frac{\sin\sqrt{1-x}}{\sqrt{1-x^2}}$

Solution

$$\lim_{x\to 1} \frac{\sin\sqrt{1-x}}{\sqrt{1-x^2}} = \frac{\sin 0}{0} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{\sin\sqrt{1-x}}{\sqrt{1-x}} \frac{1}{\sqrt{1+x}}$$

$$= \lim_{(1-x)\to 0} \frac{\sin\sqrt{1-x}}{\sqrt{1-x}} \lim_{x\to 1} \frac{1}{\sqrt{1+x}}$$

$$= 1\left(\frac{1}{\sqrt{2}}\right)$$

$$\boxed{= \frac{1}{\sqrt{2}}}$$

Exercise

Find $\displaystyle\lim_{x\to 2}\frac{\sin\sqrt{2-x}}{\sqrt{4-x^2}}$

Solution

$$\lim_{x\to 2}\frac{\sin\sqrt{2-x}}{\sqrt{4-x^2}}=\frac{\sin 0}{0}=\frac{0}{0}$$

$$=\lim_{x\to 2}\frac{\sin\sqrt{2-x}}{\sqrt{2-x}}\frac{1}{\sqrt{2+x}}$$

$$=\lim_{\sqrt{2-x}\to 0}\frac{\sin\sqrt{2-x}}{\sqrt{2-x}}\quad\lim_{x\to 2}\frac{1}{\sqrt{2+x}}$$

$$=1\left(\frac{1}{2}\right)$$

$$=\frac{1}{2}$$

Exercise

Find $\displaystyle\lim_{x\to 0}\frac{\sin\left(\sqrt{5}\,x\right)}{\sin\left(\sqrt{3}\,x\right)}$

Solution

$$\lim_{x\to 0}\frac{\sin\left(\sqrt{5}\,x\right)}{\sin\left(\sqrt{3}\,x\right)}=\frac{\sin 0}{\sin 0}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{\sqrt{5}\,x}{\sqrt{3}\,x}\frac{\sin\left(\sqrt{5}\,x\right)}{\sqrt{5}\,x}\cdot\frac{1}{\frac{\sin\left(\sqrt{3}\,x\right)}{\sqrt{3}\,x}}$$

$$=\frac{\sqrt{5}}{\sqrt{3}}\lim_{\sqrt{5}\,x\to 0}\frac{\sin\left(\sqrt{5}\,x\right)}{\sqrt{5}\,x}\cdot\frac{1}{\displaystyle\lim_{\sqrt{3}\,x\to 0}\frac{\sin\left(\sqrt{3}\,x\right)}{\sqrt{3}\,x}}$$

$$=\frac{\sqrt{5}}{\sqrt{3}}$$

Exercise

Find $\quad \lim\limits_{x\to 0} \dfrac{\sin\left(\sqrt{15}\,x\right)}{\sin\left(\sqrt{3}\,x\right)}$

Solution

$$\lim_{x\to 0} \frac{\sin\left(\sqrt{15}\,x\right)}{\sin\left(\sqrt{3}\,x\right)} = \frac{\sin 0}{\sin 0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sqrt{15}\,x}{\sqrt{3}\,x}\ \frac{\sin\left(\sqrt{5}\,x\right)}{\sqrt{15}\,x}\cdot\frac{1}{\frac{\sin\left(\sqrt{3}\,x\right)}{\sqrt{3}\,x}}$$

$$= \sqrt{\frac{15}{3}}\ \lim_{\sqrt{15}\,x\to 0} \frac{\sin\left(\sqrt{15}\,x\right)}{\sqrt{15}\,x}\cdot\frac{1}{\lim\limits_{\sqrt{3}\,x\to 0} \frac{\sin\left(\sqrt{3}\,x\right)}{\sqrt{3}\,x}}$$

$$= \sqrt{3}$$

Exercise

Find $\quad \lim\limits_{x\to 0^+} \dfrac{x-\sqrt{x}}{\sqrt{\sin x}}$

Solution

$$\lim_{x\to 0^+} \frac{x-\sqrt{x}}{\sqrt{\sin x}} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\frac{x-\sqrt{x}}{\sqrt{\sin x}}}{\sqrt{x}}\cdot\frac{1}{\sqrt{x}}$$

$$= \lim_{x\to 0^+} \frac{1}{\sqrt{\frac{\sin x}{x}}}\ \lim_{x\to 0^+} \frac{x-\sqrt{x}}{\sqrt{x}}$$

$$= (1)\ \lim_{x\to 0^+} \left(\frac{x}{\sqrt{x}} - \frac{\sqrt{x}}{\sqrt{x}}\right)$$

$$= \lim_{x\to 0^+} \left(\sqrt{x}-1\right)$$

$$= -1$$

Exercise

Find $\displaystyle\lim_{x\to 1} \frac{x-\sqrt{x}}{\sqrt{\sin x}}$

Solution

$$\lim_{x\to 1} \frac{x-\sqrt{x}}{\sqrt{\sin x}} = \frac{0}{\sqrt{\sin 1}}$$

$$\underline{= 0}$$

Exercise

Find $\displaystyle\lim_{x\to \pi} \frac{x-\sqrt{x}}{\sqrt{\sin x}}$

Solution

$$\lim_{x\to \pi} \frac{x-\sqrt{x}}{\sqrt{\sin x}} = \frac{\pi-\sqrt{\pi}}{\sqrt{\sin \pi}}$$

$$= \frac{\pi-\sqrt{\pi}}{0}$$

$$\underline{= \infty}$$

Exercise

Find $\displaystyle\lim_{x\to 0} e^{x^3}$

Solution

$$\lim_{x\to 0} e^{x^3} = e^0$$

$$\underline{= 1}$$

Exercise

Find $\displaystyle\lim_{x\to 1} e^{x^2}$

Solution

$$\lim_{x\to 1} e^{x^2} = e^1$$

$$\underline{= e}$$

Exercise

Find $\lim\limits_{x\to 1} e^{x^3-1}$

Solution

$$\lim_{x\to 1} e^{x^3-1} = e^{1-1}$$
$$= e^0$$
$$= 1$$

Exercise

Find $\lim\limits_{x\to -1} e^{x^3-1}$

Solution

$$\lim_{x\to -1} e^{x^3-1} = e^{-1-1}$$
$$= e^{-2}$$
$$= \frac{1}{e^2}$$

Exercise

Find $\lim\limits_{x\to 2} \left(e^{x^2} - \ln x\right)$

Solution

$$\lim_{x\to 2} \left(e^{x^2} - \ln x\right) = e^4 - \ln 2$$

Exercise

Find $\lim\limits_{x\to 1} \left(e^{x^2} - \ln x\right)$

Solution

$$\lim_{x\to 1} \left(e^{x^2} - \ln x\right) = e - \ln 1$$
$$= e$$

Exercise

Find $\lim\limits_{x\to e} \ln x$

Solution

$$\lim_{x\to e} \ln x = \ln e$$

$$= 1$$

Exercise

Find $\lim\limits_{x\to e} \ln x^2$

Solution

$$\lim_{x\to e} \ln x^2 = \ln e^2$$

$$= 2\ln e$$

$$= 2$$

Exercise

Find $\lim\limits_{x\to 0^+} \ln x$

Solution

$$\lim_{x\to 0^+} \ln x = \ln 0^+$$

$$= -\infty$$

Exercise

Find $\lim\limits_{x\to 1} \frac{1}{\ln x}$

Solution

$$\lim_{x\to 1} \frac{1}{\ln x} = \frac{1}{\ln 1}$$

$$= \frac{1}{0}$$

$$= \infty$$

Exercise

Find $\lim\limits_{x \to e} \ln e^{2x}$

Solution

$$\lim_{x \to e} \ln e^{2x} = \ln e^{2e}$$

$$= 2e \ln e$$

$$\underline{= 2e}$$

Exercise

Find $\lim\limits_{x \to 1} \ln e^{x^2}$

Solution

$$\lim_{x \to 1} \ln e^{x^2} = \ln e$$

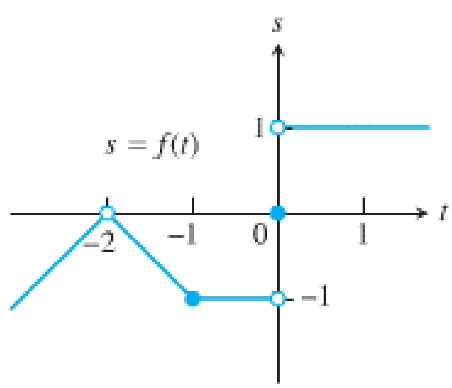
$$\underline{= 1}$$

Exercise

For the function $f(t)$ graphed, find the following limits or explain why they do not exist.

$a)\ \lim\limits_{t \to -2} f(t) \qquad b)\ \lim\limits_{t \to -1} f(t) \qquad c)\ \lim\limits_{t \to 0} f(t) \qquad d)\ \lim\limits_{t \to -0.5} f(t)$

Solution



a) $\lim\limits_{t \to -2} f(t) = 0$

b) $\lim\limits_{t \to -1} f(t) = -1$

c) $\lim\limits_{t \to 0} f(t) =$ *doesn't exist*

d) $\lim\limits_{t \to -.5} f(t) = -1$

Exercise

Suppose $\lim\limits_{x\to c} f(x)=5$ *and* $\lim\limits_{x\to c} g(x)=-2$. Find

a) $\lim\limits_{x\to c} f(x)g(x)$

b) $\lim\limits_{x\to c} 2f(x)g(x)$

c) $\lim\limits_{x\to c} \left(f(x)+3g(x)\right)$

d) $\lim\limits_{x\to c} \dfrac{f(x)}{f(x)-g(x)}$

Solution

a)
$$\lim_{x\to c} f(x)g(x)=\lim_{x\to c} f(x)\bullet \lim_{x\to c} g(x)$$
$$=(5)(-2)$$
$$=-10$$

b)
$$\lim_{x\to c} 2f(x)g(x)=2\lim_{x\to c} f(x)\bullet \lim_{x\to c} g(x)$$
$$=2(-10)$$
$$=-20$$

c)
$$\lim_{x\to c} \left(f(x)+3g(x)\right)=\lim_{x\to c} f(x)+3\lim_{x\to c} g(x)$$
$$=5+3(-2)$$
$$=-1$$

d)
$$\lim_{x\to c} \frac{f(x)}{f(x)-g(x)}=\frac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} f(x)-\lim\limits_{x\to c} g(x)}$$
$$=\frac{5}{5-(-2)}$$
$$=\frac{5}{7}$$

Exercise

Explain why the limits do not exist for $\lim\limits_{x\to 0} \dfrac{x}{|x|}$

Solution

$$\lim_{x\to 0} \frac{x}{|x|}=\frac{0}{0}$$

$$\lim_{x\to 0^-} \frac{x}{|x|}=\frac{-x}{x}=-1$$

$$\lim_{x\to 0^+} \frac{x}{|x|}=\frac{x}{x}=1$$

Doesn't exist

Exercise

Evaluate the limit using the form $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ for $f(x)=x^2, \quad x=1$

Solution

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

$$=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

$$=\lim_{h\to 0}\left(\frac{2xh}{h}+\frac{h^2}{h}\right)$$

$$=\lim_{h\to 0}(2x+h)$$

$$=2x$$

Exercise

Evaluate the limit using the form $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ for $f(x)=\sqrt{3x+1}, \quad x=0$

Solution

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{3x+3h+1}-\sqrt{3x+1}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{3x+3h+1}-\sqrt{3x+1}}{h}\cdot\frac{\sqrt{3x+3h+1}+\sqrt{3x+1}}{\sqrt{3x+3h+1}+\sqrt{3x+1}}$$

$$=\lim_{h\to 0}\frac{3x+3h+1-(3x+1)}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}$$

$$=\lim_{h\to 0}\frac{3x+3h+1-3x-1}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}$$

$$=\lim_{h\to 0}\frac{3h}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}$$

$$=\lim_{h\to 0}\frac{3}{\sqrt{3x+3h+1}+\sqrt{3x+1}}$$

$$=\frac{3}{\sqrt{3(0)+1}+\sqrt{3(0)+1}} \qquad \textit{Given}: x=0$$

$$=\frac{3}{2}$$

Exercise

If $\lim\limits_{x\to 4}\dfrac{f(x)-5}{x-2}=1$, find $\lim\limits_{x\to 4} f(x)$

Solution

$$\lim_{x\to 4}\frac{f(x)-5}{x-2}=1$$

$$\frac{\lim\limits_{x\to 4} f(x)-5}{4-2}=1$$

$$\frac{\lim\limits_{x\to 4} f(x)-5}{2}=1$$ *Multiply both sides by 2*

$$\lim_{x\to 4} f(x)-5=2$$ *Add 5 on both sides*

$$\lim_{x\to 4} f(x)=7$$

Exercise

If $\lim\limits_{x\to 0}\dfrac{f(x)}{x^2}=1$, find $\lim\limits_{x\to 0} f(x)$ *and* $\lim\limits_{x\to 0}\dfrac{f(x)}{x}$

Solution

$$\lim_{x\to 0}\frac{f(x)}{x^2}=1$$

$$\frac{\lim\limits_{x\to 0} f(x)}{\lim\limits_{x\to 0} x^2}=1$$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0} x^2$$

$$\underline{=0}$$

$$\lim_{x\to 0}\frac{f(x)}{x}=\lim_{x\to 0}\left(\frac{f(x)}{x^2}\bullet x\right)$$

$$=\lim_{x\to 0}\frac{f(x)}{x^2}\bullet\lim_{x\to 0} x$$

$$=1\bullet 0$$

$$\underline{=0}$$

Exercise

If $x^4 \le f(x) \le x^2$; $-1 \le x \le 1$ and $x^2 \le f(x) \le x^4$; $x < -1$ *and* $x > 1$. At what points c do you automatically know $\lim\limits_{x\to c} f(x)$? What can you say about the value of the limits at these points?

Solution

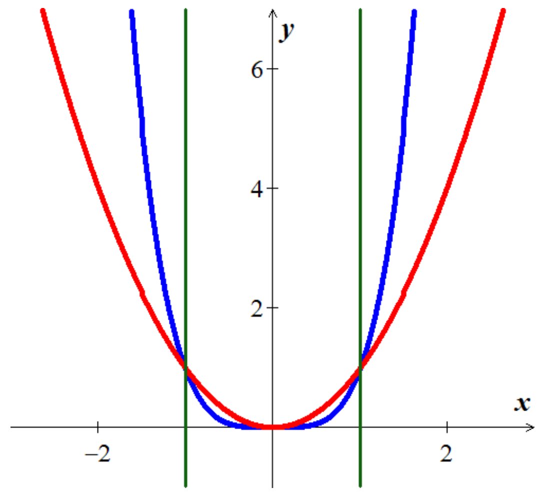
$$\lim_{x\to c} x^4 = \lim_{x\to c} x^2 \quad \Rightarrow \quad c^4 = c^2$$

$$c^4 - c^2 = 0$$

$$c^2\left(c^2 - 1\right) = 0$$

$$c^2 = 0 \qquad\qquad c^2 - 1 = 0$$

$$\boxed{c = 0} \qquad\qquad \boxed{c = \pm 1}$$



$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2$$

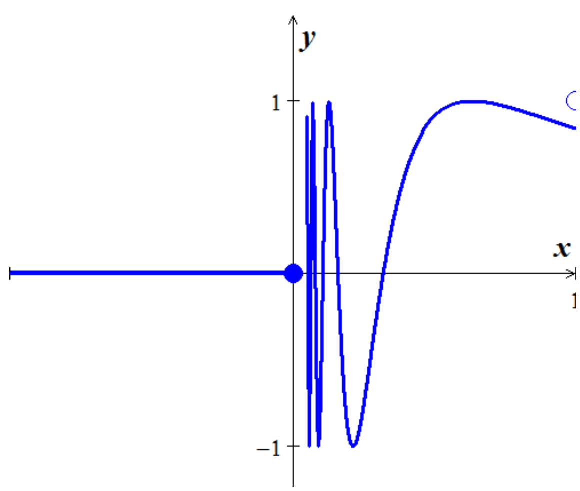
$$\underline{= 0}\,|$$

$$\lim_{x\to -1} f(x) = \lim_{x\to 1} f(x)$$

$$\underline{= 1}\,|$$

Exercise

Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$



a) Does $\lim\limits_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

Solution

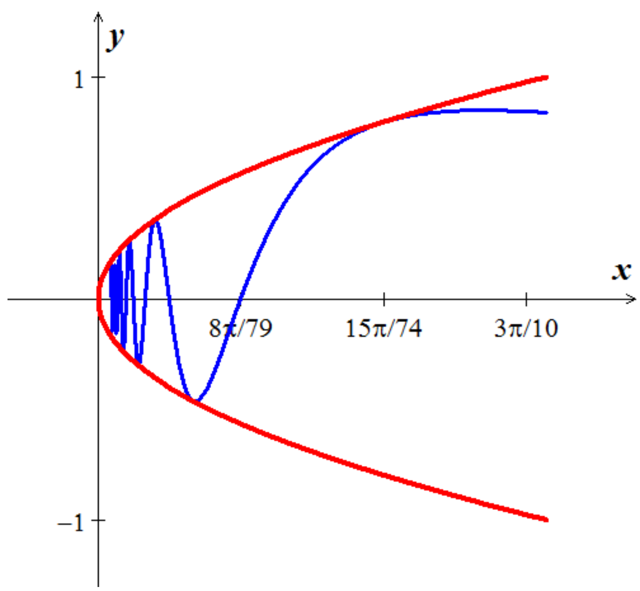
a) $\lim\limits_{x\to 0^+} f(x)$ doesn't exist, since $\sin\left(\frac{1}{x}\right)$ doesn't approach any single value as $x \to 0$

b) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} 0 = 0$

c) $\lim\limits_{x\to 0} f(x)$ doesn't exist, since $\lim\limits_{x\to 0^+} f(x)$ doesn't exist

Exercise

Let $g(x) = \sqrt{x}\ \sin\frac{1}{x}$



a) Does $\lim\limits_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

Solution

a) $\lim\limits_{x\to 0^+} g(x)$ exists, by the sandwich theorem $-\sqrt{x} \le g(x) \le \sqrt{x}.$ *for* $x > 0$

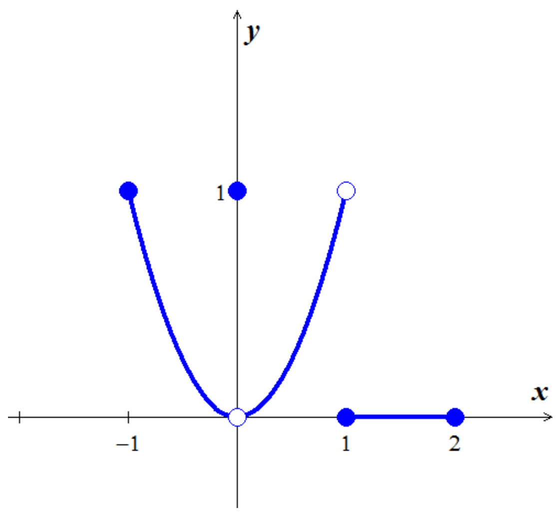
b) $\lim\limits_{x\to 0^-} g(x)$ doesn't exist, since $\sqrt{x}$ is not defined for $x < 0$

c) $\lim\limits_{x\to 0} g(x)$ doesn't exist, since $\lim\limits_{x\to 0^-} g(x)$ doesn't exist.

Exercise

Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

Solution



a) $\lim\limits_{x\to -1^+} f(x) = 1$ ***True***

b) $\lim\limits_{x\to 0^-} f(x) = 0$ ***True***

c) $\lim\limits_{x\to 0^-} f(x) = 1$ ***False***

d) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$ ***True***

e) $\lim\limits_{x\to 0} f(x)$ *exists* ***True***

f) $\lim\limits_{x\to 0} f(x) = 0$ ***True***

g) $\lim\limits_{x\to 0} f(x) = 1$ ***False***

h) $\lim\limits_{x\to 1} f(x) = 1$ ***False***

i) $\lim\limits_{x\to 1} f(x) = 0$ ***False***

j) $\lim\limits_{x\to 2^-} f(x) = 2$ ***False***

k) $\lim\limits_{x\to -1^-} f(x) = 0$ *does not exist* ***True***

l) $\lim\limits_{x\to 2^+} f(x) = 0$ ***False***