3.3 Gram-Schmidt Process

Given $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$

orthogonal Basis : $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$

orthonormal Basis $\{\vec{q}_1, \vec{q}_2, \dots \vec{q}_3\}$

1. $\vec{v}_1 = \vec{u}_1$ $\qquad \vec{q}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$

2. $\vec{v}_2 = \vec{u}_2 - \dfrac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$ $\qquad \vec{q}_2 = \dfrac{\vec{v}_2}{\|\vec{v}_2\|}$

3. $\vec{v}_3 = u_3 - \dfrac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \dfrac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$

$$\vec{q}_i = \frac{\vec{v}_i}{\|\vec{v}_i\|}$$

Ex Given: $\vec{u}_1 = (1, 1, 1)$

$\vec{u}_2 = (0, 1, 1)$

$\vec{u}_3 = (0, 0, 1)$

Soln

$$\vec{v}_1 = \vec{u}_1 = \boxed{(1, 1, 1)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|v_1\|} = \frac{(1, 1, 1)}{\sqrt{1+1+1}} = \boxed{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$= (0, 1, 1) - \frac{(0, 1, 1) \cdot (1, 1, 1)}{1+1+1}(1, 1, 1)$$

$$= (0, 1, 1) - \frac{2}{3}(1, 1, 1)$$

$$\boxed{\vec{v}_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

$$= (0,0,1) - \frac{(0,0,1)\cdot(1,1,1)}{3}(1,1,1)$$

$$- \frac{(0,0,1)\cdot\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)}{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1}{3}\left(\frac{9}{6}\right)\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= (0,0,1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - \left(-\frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$$

$$= \boxed{\left(0, -\frac{1}{2}, \frac{1}{2}\right)}$$

$$\vec{q}_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{\sqrt{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}}}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= \boxed{\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}}}\left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

$$= \boxed{\left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} \qquad = \frac{\sqrt{2}}{2}$$

$$Q = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Gram - Schmidt (orthonormal)

$\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$i > 1$

$$\vec{q}_i = \frac{\vec{w}_i}{\|w_i\|}$$

$$\vec{w}_2 = \vec{v}_2 - \langle \vec{v}_2, \vec{q}_1 \rangle \vec{q}_1$$

$$= \vec{v}_2 - (\vec{v}_2 \cdot \vec{q}_1)\vec{q}_1$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{q}_1)\vec{q}_1 - (\vec{v}_3 \cdot \vec{q}_2)\vec{q}_2$$

Ex orthonormal basis for subspaces

$\vec{v}_1 = (1, 1, 0, 0)$ $\vec{v}_2 = (0, 1, 1, 0)$

$\vec{v}_3 = (1, 0, 1, 1)$

Soln

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{(1, 1, 0, 0)}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{q}_1)\,\vec{q}_1$$

$$= (0,1,1,0) - \left[(0,1,1,0)\cdot\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right)\right]\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right)$$

$$= (0,1,1,0) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right)$$

$$\boxed{= \left(-\frac{1}{2},\frac{1}{2},1,0\right)}$$

$$\vec{q}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|} = \frac{}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}\left(-\frac{1}{2},\frac{1}{2},1,0\right)$$

$$= \frac{1}{\sqrt{3}}\left(-\frac{1}{2},\frac{1}{2},1,0\right)$$

$$= \frac{2}{\sqrt{6}}\left(-\frac{1}{2},\frac{1}{2},1,0\right)$$

$$\boxed{= \left(-\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},0\right)}$$

$$\vec{w}_3 = \vec{v}_3 - \underbrace{(\vec{v}_3\cdot\vec{q}_1)\,\vec{q}_1}_{1} - \underbrace{(\vec{v}_3\cdot\vec{q}_2)}_{2}\,\vec{q}_2$$

$$(\vec{v}_3\cdot\vec{q}_1)\,\vec{q}_1 = \left[(1,0,1,1)\cdot\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right)\right]\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right)$$

$$= \left(\frac{1}{2},\frac{1}{2},0,0\right)$$

$$(\vec{v}_3 \cdot \vec{q}_2)\,\vec{q}_2 = \left[(1,0,1,1)\cdot\left(-\frac{1}{\sqrt6},\frac{1}{\sqrt6},\frac{2}{\sqrt6},0\right)\right]\left(\frac{-1}{\sqrt6},\frac{1}{\sqrt6},\frac{2}{\sqrt6},0\right)$$

$$= \left(-\frac{1}{\sqrt6}+\frac{2}{\sqrt6}\right)\left(-\frac{1}{\sqrt6},\frac{1}{\sqrt6},\frac{2}{\sqrt6},0\right)$$

$$= \left(-\frac{1}{6},\frac{1}{6},\frac{1}{3},0\right)$$

$$\vec{w}_3 = (1,0,1,1) - \left(\frac{1}{2},\frac{1}{2},0,0\right) - \left(\frac{-1}{6},\frac{1}{6},\frac{1}{3},0\right)$$

$$= \left(\frac{2}{3},-\frac{2}{3},\frac{2}{3},1\right)$$

$$\vec{q}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|} = \frac{1}{\sqrt{\frac{4}{9}+\frac{4}{9}+\frac{4}{9}+1}}\left(\frac{2}{3},\frac{-2}{3},\frac{2}{3},1\right)$$

$$= \frac{1}{\frac{\sqrt{21}}{3}}\left(\frac{2}{3},\frac{-2}{3},\frac{2}{3},1\right)$$

$$= \left(\frac{2}{\sqrt{21}},\frac{-2}{\sqrt{21}},\frac{2}{\sqrt{21}},\frac{3}{\sqrt{21}}\right)$$

orthonormal basis:

$$\left\{\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2},0,0\right),\left(-\frac{1}{\sqrt6},\frac{1}{\sqrt6},\frac{2}{\sqrt6},0\right),\left(\frac{2}{\sqrt{21}},\frac{-2}{\sqrt{21}},\frac{2}{\sqrt{21}},\frac{3}{\sqrt{21}}\right)\right\}$$

QR-Decomposition

$$Q = \{\vec{q}_1, \vec{q}_2, \dots \vec{q}_n\}$$

$$R = \begin{bmatrix} \langle \vec{u}_1, \vec{q}_1 \rangle & \langle \vec{u}_2, \vec{q}_1 \rangle & \cdots & \langle \vec{u}_n, \vec{q}_1 \rangle \\ 0 & \langle \vec{u}_2, \vec{q}_2 \rangle & \cdots & \langle \vec{u}_n, \vec{q}_2 \rangle \\ \vdots & & \ddots & \\ 0 & 0 & & \langle \vec{u}_n, \vec{q}_n \rangle \end{bmatrix}$$

$$A = QR$$

Ex

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$\vec{u}_1 = (1, 1, 1)$ $\quad \vec{u}_2 = (0, 1, 1)$ $\quad \vec{u}_3 = (0, 0, 1)$

$\vec{q}_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ $\quad \vec{q}_2 = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$\vec{q}_3 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$\vec{u}_1 \cdot \vec{q}_1 = (1,1,1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$= \frac{3}{\sqrt{3}}$$

$$\vec{u}_2 \cdot \vec{q}_1 = (0,1,1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}$$

$$\vec{u}_2 \cdot \vec{q}_2 = (0,1,1) \cdot \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = \frac{2}{\sqrt{6}}$$

$$\vec{u}_3 \cdot \vec{q}_1 = (0,0,1) \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}$$

$$\vec{u}_3 \cdot \vec{q}_2 = (0,0,1) \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = \frac{1}{\sqrt{6}}$$

$$\vec{u}_3 \cdot \vec{q}_3 = (0,0,1) \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

$$R = \begin{pmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$A = QR$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}}_{R}$$

Calculus: Gram-Schmidt process

$$\langle p, q \rangle = \int_{-1}^{1} p(x)\,q(x)\,dx$$

Ex Basis $B = \{1, x, x^2\}$

$\vec{u}_1 = 1 \qquad \vec{u}_2 = x \qquad \vec{u}_3 = x^2$

$\vec{v}_1 = \vec{u}_1 = 1$

$$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1$$

$$\langle \vec{v}_1, \vec{v}_1 \rangle = \int_{-1}^{1} 1\,dx = x \Big|_{-1}^{1} = 2$$

$$\int_{-a}^{a} \text{odd} = 0$$

$$\langle \vec{u}_2, \vec{v}_1 \rangle = \int_{-1}^{1} x\,dx = \frac{1}{2}x^2 \Big|_{-1}^{1} = \frac{1}{2}(1-1) = 0$$

$$\vec{v}_2 = x - 0 = x$$

$$x - \frac{0}{2}(1) \qquad 2 \neq 0$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$\langle \vec{u}_3, \vec{v}_1 \rangle = \int_{-1}^{1} x^2\,dx$$

$$= \frac{1}{3} x^3 \Big|_{-1}^{1}$$

$$= \frac{2}{3}$$

$$\|\vec{v}_1\|^2 = 2$$

$$\langle \vec{u}_3, \vec{v}_2 \rangle = \int_{-1}^{1} x^3 x\,dx = 0$$

$$\vec{v}_3 = x^2 - \frac{2}{3}\,\frac{1}{2}(1) - 0$$

$$= x^2 - \frac{1}{3}$$

$$q_i = \frac{\vec{v}_i}{\|\vec{v}_i\|}$$

$$\vec{v}_1 = 1 \qquad \|\vec{v}_1\| = \cancel{2}\sqrt{2} \qquad \vec{q}_1 = \frac{1}{\sqrt{2}}$$

$$\langle \vec{v}_2, \vec{v}_2 \rangle = \int_{-1}^{1} x^2\,dx$$

$$= \frac{2}{3} x^3 \Big|_{0}^{1}$$

$$= \frac{2}{3}$$

$$\langle \vec{N}_3, \vec{N}_3 \rangle = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx$$

$$= \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx$$

$$= 2\left(\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{x}{9}\right|_0^1$$

$$= 2\left(\frac{1}{5} - \frac{2}{9} + \frac{1}{9}\right)$$

$$\frac{1}{5} - \frac{1}{9}$$

$$= \boxed{\frac{8}{45}}$$

$$\vec{q}_2 = \frac{x}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}\, x$$

$$\vec{q}_3 = \frac{1}{\frac{2\sqrt{2}}{3\sqrt{5}}}\left(x^2 - \frac{1}{3}\right)$$

$$= \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \frac{1}{3}\right)$$

orthonormal Basis set,

$$\left\{ \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}x, \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \frac{1}{3}\right) \right\}$$

$$\frac{3\sqrt{10}}{4}x^2 - \frac{\sqrt{10}}{4}$$