

Lecture Three – Exponential and Logarithmic Functions

Section 3.1 – Inverse Functions

Inverse Relations

Interchanging the first and second coordinates of each ordered pair in a relation produces the inverse relation.

If a relation is defined by an equation, interchanging the variables produces an equation of the inverse relation

Given the relation: $\{(Zambia, 4.2), (Columbia, 4.5), (Poland, 3.3), (Italy, 3.3), (US, 2.5)\}$

Inverse Relation: $\{(4.2, Zambia), (4.5, Columbia), (3.3, Poland), (3.3, Italy), (2.5, US)\}$

Example

Consider the relation g given by: $G = \{(2, 4), (-1, 3), (-2, 0)\}$

Solution

The inverse relation: $G = \{(4, 2), (3, -1), (0, -2)\}$

Example

Consider the relation given by: $F = \{(-2, 2), (-1, 1), (0, 0), (1, 3), (2, 5)\}$

Solution

The inverse relation: $G = \{(2, -2), (1, -1), (0, 0), (3, 1), (5, 2)\}$

One-to-One Functions

A function f is one-to-one (1 – 1) if different inputs have different outputs that is,

$$\text{if } a \neq b, \quad \text{then } f(a) \neq f(b)$$

A function f is one-to-one (1 – 1) if different outputs the same, the inputs are the same – that is,

$$\text{if } f(a) = f(b), \quad \text{then } a = b$$

Example

Given the function f described by $f(x) = 2x - 3$, prove that f is one-to-one.

Solution

$$f(a) = f(b)$$

$$2a - 3 = 2b - 3 \quad \text{Add 3 on both sides}$$

$$2a = 2b \quad \text{Divide by 2}$$

$$a = b$$

$\therefore f$ is one-to-one

Example

Given the function f described by $f(x) = -4x + 12$, prove that f is one-to-one.

Solution

$$f(a) = f(b)$$

$$-4a + 12 = -4b + 12 \quad \text{Subtract 12 from both sides}$$

$$-4a = -4b \quad \text{Divide by -4}$$

$$a = b$$

$\therefore f$ is one-to-one

Example

Given the function f described by $f(x) = x^2$, prove that f is one-to-one.

Solution

$$-1 \neq 1$$

$$\begin{cases} f(-1) = 1 \\ f(1) = 1 \end{cases} \Rightarrow f(-1) = f(1)$$

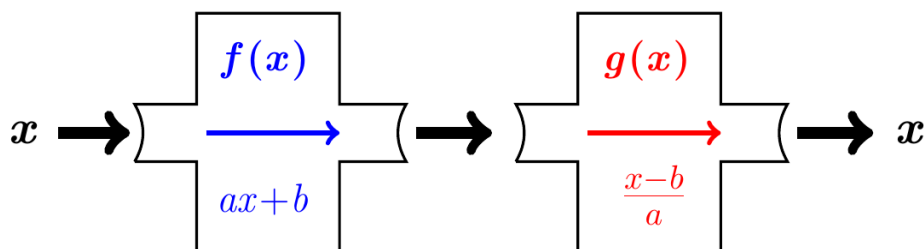
$\therefore f$ is **not** one-to-one

Definition of the Inverse of a Function

Let f and g be two functions such that

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x$$

$$\begin{array}{ccc} x & \xrightarrow{f} & f(x) \\ & \xleftarrow{g=f^{-1}} & \end{array} \quad g(f(x)) = f^{-1}(f(x)) = x$$



If the inverse of a function f is also a function, it is named f^{-1} read “ f – inverse”

The **-1** in f^{-1} is not an exponent! And is not equal to ~~$\frac{1}{f(x)}$~~

Domain and Range of f and f^{-1}

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

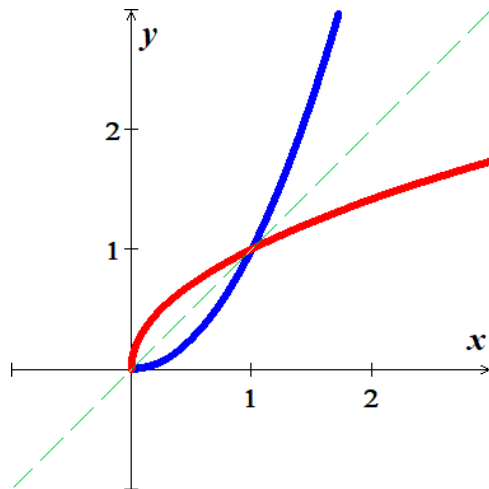
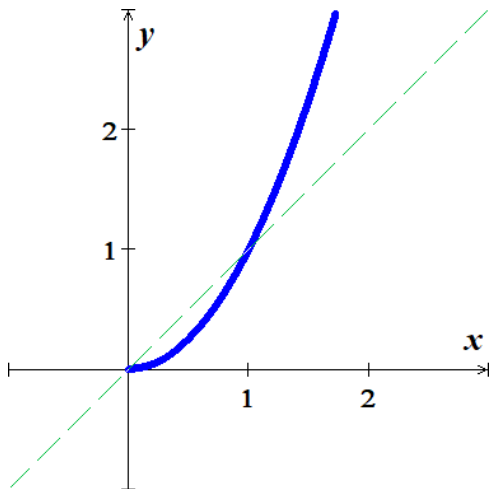
If a function f is one-to-one, then f^{-1} is the unique function such that each of the following holds.

$$\boxed{(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x} \quad \text{for each } x \text{ in the domain of } f, \text{ and}$$

$$\boxed{(f \circ f^{-1})(x) = f(f^{-1}(x)) = x} \quad \text{for each } x \text{ in the domain of } f^{-1}$$

The condition that f is one-to-one in the definition of inverse function is important; otherwise, g will not define a function

Graphing



Example

Let $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$, is g the inverse function of f ?

Solution

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\sqrt[3]{x+1}\right) \\ &= \left(\sqrt[3]{x+1}\right)^3 - 1 \\ &= x + 1 - 1 \\ &= x\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g\left(x^3 - 1\right) \\ &= \sqrt[3]{x^3 - 1 + 1} \\ &= \sqrt[3]{x^3} \\ &= x\end{aligned}$$

g is the inverse function of f

Example

Show that each function is the inverse of the other: $f(x) = 4x - 7$ and $g(x) = \frac{x+7}{4}$

Solution

$$\begin{aligned}f(g(x)) &= f\left(\frac{x+7}{4}\right) \\ &= 4\left(\frac{x+7}{4}\right) - 7 \\ &= x + 7 - 7 \\ &= x\end{aligned}$$

$$\begin{aligned}g(f(x)) &= g(4x - 7) \\ &= \frac{4x - 7 + 7}{4} \\ &= \frac{4x}{4} \\ &= x\end{aligned}$$

Finding the *Inverse Function*

Example

Finding an Inverse Function

$$f(x) = 2x + 7$$

1. Replace $f(x)$ with y

$$y = 2x + 7$$

2. Interchange x and y

$$x = 2y + 7$$

3. Solve for y

$$x - 7 = 2y$$

$$\frac{x - 7}{2} = y$$

4. Replace y with $f^{-1}(x)$

$$f^{-1}(x) = \frac{x - 7}{2}$$

Example

Find the inverse of $f(x) = 4x^3 - 1$

Solution

$$y = 4x^3 - 1$$

$$x = 4y^3 - 1$$

$$x + 1 = 4y^3$$

$$\frac{x + 1}{4} = y^3$$

$$y = \left(\frac{x + 1}{4} \right)^{1/3}$$

$$\boxed{f^{-1}(x) = \sqrt[3]{\frac{x + 1}{4}}}$$

Example

Find a formula for the inverse $f(x) = \frac{5x - 3}{2x + 1}$

Solution

$$y = \frac{5x - 3}{2x + 1}$$

$$x = \frac{5y - 3}{2y + 1}$$

$$x(2y + 1) = 5y - 3$$

$$2xy + x = 5y - 3$$

$$2xy - 5y = -x - 3$$

$$y(2x - 5) = -x - 3$$

$$y = \frac{-x - 3}{2x - 5}$$

$$\underline{f^{-1}(x) = -\frac{x + 3}{2x - 5} \quad |}$$

Exercise Section 3.1 – Inverse Functions

(1 – 9) Find the inverse relation of the given sets:

1. $A = \{(-2, 2), (1, -1), (0, 4), (1, 3)\}$
2. $B = \{(1, -1), (2, -2), (3, -3), (4, -4)\}$
3. $C = \{(a, -a), (b, -b), (c, -c)\}$
4. $D = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4)\}$
5. $E = \{(-a, a), (-b, b), (-c, c), (-d, d)\}$

(6 – 14) Determine whether the function is one-to-one

- | | | |
|----------------------|----------------------------|------------------------------|
| 6. $f(x) = 3x - 7$ | 9. $f(x) = \sqrt[3]{x}$ | 12. $f(x) = (x - 2)^3$ |
| 7. $f(x) = x^2 - 9$ | 10. $f(x) = x $ | 13. $y = x^2 + 2$ |
| 8. $f(x) = \sqrt{x}$ | 11. $f(x) = \frac{2}{x+3}$ | 14. $f(x) = \frac{x+1}{x-3}$ |

15. Given that $f(x) = 5x + 8$, use composition of functions to show that $f^{-1}(x) = \frac{x-8}{5}$

16. Given the function $f(x) = (x+8)^3$
- a) Find $f^{-1}(x)$
 - b) Graph f and f^{-1} in the same rectangular coordinate system
 - c) Find the domain and the range of f and f^{-1}

(17 – 32) Prove that $f(x)$ and $g(x)$ are inverse functions of each other.

- | | |
|--|---|
| 17. $f(x) = 4x; \quad g(x) = \frac{x}{4}$ | 25. $f(x) = \frac{3x}{x-1}; \quad g(x) = \frac{x}{x-3}$ |
| 18. $f(x) = 2x; \quad g(x) = \frac{1}{2x}$ | 26. $f(x) = x^3 + 2; \quad g(x) = \sqrt[3]{x-2}$ |
| 19. $f(x) = 4x - 1; \quad g(x) = \frac{x+1}{4}$ | 27. $f(x) = x^3 - 1; \quad g(x) = \sqrt[3]{x+1}$ |
| 20. $f(x) = \frac{1}{2}x - \frac{3}{2}; \quad g(x) = 2x + 3$ | 28. $f(x) = (x+4)^3; \quad g(x) = \sqrt[3]{x} - 4$ |
| 21. $f(x) = -\frac{1}{2}x - \frac{1}{2}; \quad g(x) = -2x + 1$ | 29. $f(x) = x^3 - 1 \quad g(x) = \sqrt[3]{x+1}$ |
| 22. $f(x) = 3x + 2; \quad g(x) = \frac{1}{3}(x-2)$ | 30. $f(x) = 3x - 2 \quad g(x) = \frac{x+2}{3}$ |
| 23. $f(x) = \frac{5}{x+3}; \quad g(x) = \frac{5}{x} - 3$ | 31. $f(x) = x^2 + 5, x \leq 0 \quad g(x) = -\sqrt{x-5}, x \geq 5$ |
| 24. $f(x) = \frac{2x}{x+1}; \quad g(x) = \frac{-x}{x-2}$ | 32. $f(x) = x^3 - 4; \quad g(x) = \sqrt[3]{x+4}$ |

(33 – 35) Find the inverse of

33. $f(x) = (x - 2)^3$

34. $f(x) = \frac{x+1}{x-3}$

35. $f(x) = \frac{2x+1}{x-3}$

(36 – 38) Determine the domain and range of f^{-1} (Hint: first find the domain and range of f)

36. $f(x) = -\frac{2}{x-1}$

37. $f(x) = \frac{5}{x+3}$

38. $f(x) = \frac{4x+5}{3x-8}$

(39 – 66) For the given functions

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

39. $f(x) = \frac{2x}{x-1}$

48. $f(x) = \frac{3x-1}{x-2}$

58. $f(x) = 2 - 3x^2; \quad x \leq 0$

40. $f(x) = \frac{x}{x-2}$

49. $f(x) = \frac{3x-2}{x+4}$

59. $f(x) = 2x^3 - 5$

41. $f(x) = \frac{x+1}{x-1}$

50. $f(x) = \frac{-3x-2}{x+4}$

60. $f(x) = \sqrt{3-x}$

42. $f(x) = \frac{2x+1}{x+3}$

51. $f(x) = \sqrt{x-1} \quad x \geq 1$

61. $f(x) = \sqrt[3]{x} + 1$

43. $f(x) = \frac{3x-1}{x-2}$

52. $f(x) = \sqrt{2-x} \quad x \leq 2$

62. $f(x) = (x^3 + 1)^5$

44. $f(x) = \frac{2x}{x-1}$

53. $f(x) = x^2 + 4x \quad x \geq -2$

63. $f(x) = x^2 - 6x; \quad x \geq 3$

54. $f(x) = 3x + 5$

64. $f(x) = (x-2)^3$

45. $f(x) = \frac{x}{x-2}$

55. $f(x) = \frac{1}{3x-2}$

65. $f(x) = \frac{x+1}{x-3}$

46. $f(x) = \frac{x+1}{x-1}$

56. $f(x) = \frac{3x+2}{2x-5}$

66. $f(x) = \frac{2x+1}{x-3}$

47. $f(x) = \frac{2x+1}{x+3}$

57. $f(x) = \frac{4x}{x-2}$

67. The function $w(x) = 2x + 24$ can be used to convert a U.S. women's shoe size into an Italian women's shoe size. Determine the function $w^{-1}(x)$ that can use to convert an Italian women's shoe size to its equivalent U.S. shoe size.



68. The function $m(x) = 1.3x - 4.7$ can be used to convert a U.S. men's shoe size into an U.K. women's shoe size. Determine the function $m^{-1}(x)$ that can be used to convert an U.K. men's shoe size to its equivalent U.S. shoe size.
69. A catering service use the function $c(x) = \frac{300 + 12x}{x}$ to determine the amount, in *dollars*, it charges per person for a sit-down dinner, where x is the number of people in attendance.
- a) Find $c(30)$ and explain what it represents
 - b) Find $c^{-1}(x)$
 - c) Use $c^{-1}(x)$ to determine how many people attended a dinner for which the cost per person was \$15.00
70. A landscaping service use the function $c(x) = \frac{600 + 140x}{x}$ to determine the amount, in *dollars*, it charges per tree to deliver, where x is the number of trees.
- a) Find $c(5)$ and explain what it represents
 - b) Find $c^{-1}(x)$
 - c) Use $c^{-1}(x)$ to determine how many trees were delivered for which the cost per tree was \$160.00