Section 4.6 – Orthogonal Diagonalization

Definition

A square matrix $\boldsymbol{A}$ is called orthogonally diagonalizable if there is an orthogonal matrix P such that $P^{-1}AP\left(=P^TAP\right)$ is diagonal; the matrix P is said to orthogonally diagonalize A.

$$P^TAP=D$$

We say that A is orthogonally diagonalizable and that P orthogonally diagonalizes A.

Theorem

If $\boldsymbol{A}$ is an n x n matrix, then the following are equivalent.

a) $\boldsymbol{A}$ is orthogonally diagonalizable

b) $\boldsymbol{A}$ has an orthonormal set of n eigenvectors.

c) $\boldsymbol{A}$ is symmetric.

Theorem

If $\boldsymbol{A}$ is symmetric matrix, then:

a) The eigenvalues of $\boldsymbol{A}$ are all real numbers.

b) Eigenvectors from different eigenspaces are orthogonal.

Example

Find an orthogonal matrix P that diagonalizes

$$A=\begin{pmatrix}4&2&2\\2&4&2\\2&2&4\end{pmatrix}$$

Solution

$$\begin{aligned}\det(A-\lambda I)&=\begin{pmatrix}4-\lambda&2&2\\2&4-\lambda&2\\2&2&4-\lambda\end{pmatrix}\\&=(4-\lambda)^3+8+8-4(4-\lambda)-4(4-\lambda)-4(4-\lambda)\\&=64-48\lambda+12\lambda^2-\lambda^3+16-12(4-\lambda)\\&=64-48\lambda+12\lambda^2-\lambda^3+16-48+12\lambda\\&=-\lambda^3+12\lambda^2-36\lambda+32=0\end{aligned}$$

The eigenvalues are: $\boxed{\lambda_{1,2} = 2 \quad and \quad \lambda_3 = 8}$

For $\lambda_{1,2} = 2$, we have: $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow 2x_1 + 2y_1 + 2z_1 = 0$$

$$\Rightarrow \boxed{x_1 + y_1 + z_1 = 0}$$

If $z_1 = 0 \quad \Rightarrow \quad \boxed{x_1 = -y_1}$

Therefore, the eigenvector $\boxed{V_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$

If $y_1 = 0 \quad \Rightarrow \quad \boxed{x_1 = -z_1}$

Therefore, the eigenvector $\boxed{V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$

For $\lambda_3 = 8$, we have: $(A - \lambda_3 I)V_3 = 0$

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -4x_3 + 2y_3 + 2z_3 = 0 \\ 2x_3 - 4y_3 + 2z_3 = 0 \\ 2x_3 + 2y_3 - 4z_3 = 0 \end{cases}$$

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \begin{matrix} \\ 2R_2 + R_1 \\ 2R_3 + R_1 \end{matrix}$$

$$\begin{pmatrix} -4 & 2 & 2 \\ 0 & -6 & 6 \\ 0 & 6 & -6 \end{pmatrix} \begin{matrix} 3R_1 + R_2 \\ \\ R_3 + R_2 \end{matrix}$$

$$\begin{pmatrix} -12 & 0 & 12 \\ 0 & -6 & 6 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} -\frac{1}{12}R_1 \\ -\frac{1}{6}R_2 \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \to x_3 - z_3 = 0 \\ \to y_3 - z_3 = 0 \end{array}$$

$$\Rightarrow \underline{x_3 = y_3 = z_3}$$

Therefore the eigenvector $V_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(-1,\ 1,\ 0)}{\sqrt{(-1)^2 + 1^2 + 0}}$$

$$= \frac{(-1,\ 1,\ 0)}{\sqrt{2}}$$

$$= \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$\vec{w}_2 = v_2 - \left(\vec{v}_2 \cdot \vec{u}_1\right)\vec{u}_1$$

$$= (-1,\ 0,\ 1) - \left[(-1,\ 0,\ 1) \cdot \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)\right]\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= (-1,\ 0,\ 1) - \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0\right)$$

$$= \left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)}{\sqrt{\frac{1}{4} + \frac{1}{4} + 1}}$$

$$= \frac{\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)}{\sqrt{\frac{6}{4}}}$$

$$= \frac{2}{\sqrt{6}}\left(-\frac{1}{2},\ -\frac{1}{2},\ 1\right)$$

$$= \left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)$$

$$\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \bullet \vec{u}_1\right)\vec{u}_1 - \left(\vec{v}_3 \bullet \vec{u}_2\right)\vec{u}_2$$

$$= (1,\ 1,\ 1) - (0)\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) - (0)\left(-\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}}\right)$$

$$= (1,\ 1,\ 1)$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$

$$= \frac{1}{\sqrt{1^2+1^2+1^2}}(1,\ 1,\ 1)$$

$$= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \qquad (\textit{Orthogonal})$$

$$P^T AP = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} & 0 \\ -\frac{2}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{4}{\sqrt{6}} \\ \frac{8}{\sqrt{3}} & \frac{8}{\sqrt{3}} & \frac{8}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

Spectral Decomposition

The spectral decomposition of A is:

$$A = \lambda_1 \vec{u}_1 \vec{u}_1^T + \lambda_2 \vec{u}_2 \vec{u}_2^T + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T$$

Example

The matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$

Solution

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(-2-\lambda) - 4$$

$$= \lambda^2 + \lambda - 6 = 0$$

The eigenvalues are: $\lambda_1 = -3 \quad and \quad \lambda_2 = 2$

For $\lambda_1 = -3$: $\quad (A - \lambda_1 I)\vec{v}_1 = 0$

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad 2x_1 = -y_1$$

Therefore, the eigenvector $\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

For $\lambda_2 = 2$: $\quad (A - \lambda_2 I)\vec{v}_2 = 0$

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad x_1 = 2y_1$$

Therefore, the eigenvector $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The corresponding eigenvectors are: $\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad and \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ -2)}{\sqrt{1^2 + (-2)^2}}$$

$$= \left(\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}} \right)$$

$$\vec{w}_2 = \vec{v}_2 - \left(\vec{v}_2 \bullet \vec{u}_1\right)\vec{u}_1$$

$$= (2,\ 1) - \left[(2,\ 1) \bullet \left(\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}}\right)\right]\left(\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}}\right)$$

$$= (2,\ 1) - (0)\left(\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}}\right)$$

$$= (2,\ 1)$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\left\|\vec{w}_2\right\|}$$

$$= \frac{(2,\ 1)}{\sqrt{5}}$$

$$= \left(\frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right)$$

$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} = \lambda_1 \vec{u}_1 \vec{u}_1^T + \lambda_2 \vec{u}_2 \vec{u}_2^T$$

$$= -3\begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} + 2\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$= -3\begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{pmatrix} + 2\begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

The spectral decomposition about the image of the vector $\vec{x} = (1,\ 1)$

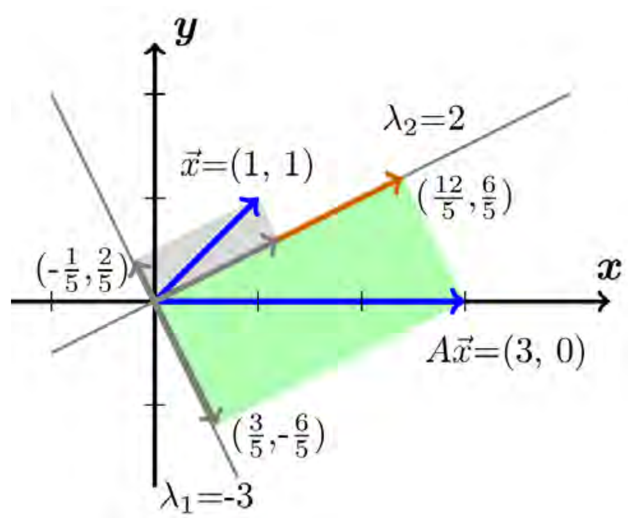
$$A\vec{x} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= -3\begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= -3\begin{pmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{pmatrix} + 2\begin{pmatrix} \frac{6}{5} \\ \frac{3}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{5} \\ -\frac{6}{5} \end{pmatrix} + \begin{pmatrix} \frac{12}{5} \\ \frac{6}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$



Example

Consider a 2 by 2 symmetric matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

Solution

The eigenvalues are:

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix}$$

$$= (a-\lambda)(c-\lambda) - b^2$$

$$= \lambda^2 - (a+c)\lambda + ac - b^2 = 0$$

$$\lambda = \frac{(a+c) \pm \sqrt{(a+c)^2 - 4\left(ac-b^2\right)}}{2} \qquad \therefore (a+c)^2 - 4\left(ac-b^2\right) > 0$$

The eigenvectors are:

For $\lambda_1 \Rightarrow \left(A - \lambda_1 I\right)V_1 = 0$

$$\begin{pmatrix} a-\lambda_1 & b \\ b & c-\lambda_1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} \left(a-\lambda_1\right)x_1 + by_1 = 0 & (1) \\ bx_1 + \left(c-\lambda_1\right)y_1 = 0 \end{cases}$$

$$(1) \Rightarrow by_1 = \left(\lambda_1 - a\right)x_1$$

$$V_1 = \begin{pmatrix} b \\ \lambda_1 - a \end{pmatrix}$$

For $\lambda_2 \Rightarrow \left(A - \lambda_2 I\right)V_2 = 0$

$$\begin{pmatrix} a-\lambda_2 & b \\ b & c-\lambda_2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{cases} \left(a-\lambda_2\right)x_2 + by_2 = 0 \\ bx_2 + \left(c-\lambda_2\right)y_2 = 0 & (2) \end{cases}$$

$$(2) \Rightarrow bx_2 = \left(\lambda_2 - c\right)y_2$$

$$V_2 = \begin{pmatrix} \lambda_2 - c \\ b \end{pmatrix}$$

$$\lambda_1 + \lambda_2 = \frac{(a+c) - \sqrt{(a+c)^2 - 4\left(ac - b^2\right)} + (a+c) + \sqrt{(a+c)^2 - 4\left(ac - b^2\right)}}{2}$$

$$= \frac{2(a+c)}{2}$$

$$\underline{= a + c}$$

$$V_1 \bullet V_2 = b\left(\lambda_1 - a\right) + b\left(\lambda_2 - c\right)$$

$$= b\left(\lambda_1 + \lambda_2 - a - c\right)$$

$$= b(a + c - a - c)$$

$$\underline{= 0}$$

Therefore, these eigenvectors are perpendicular.

Theorem

Orthogonal Eigenvectors: Eigenvectors of a real symmetric matrix (when they correspond to different λ′s) are always perpendicular.

Proof

Suppose $A\vec{x} = \lambda_1 \vec{x}$, $A\vec{y} = \lambda_2 \vec{y}$ and $A = A^T$.

The dot products of the first equation with *y* and the second with *x*:

$$
\begin{aligned}
\left(\lambda_1 \vec{x}\right)^T \vec{y} &= \left(A\vec{x}\right)^T \vec{y} \\
&= \vec{x}^T A^T \vec{y} \\
&= \vec{x}^T A\vec{y} \\
&= \vec{x}^T \lambda_2 \vec{y}
\end{aligned}
$$

$$\Rightarrow \quad \underline{\vec{x}^T \lambda_1 \vec{y} = \vec{x}^T \lambda_2 \vec{y}}$$

Since $\lambda_1 \ne \lambda_2$, this proves that $\vec{x}^T \vec{y} = 0$.

The eigenvector $\vec{x}$ $\left(\mathit{for}\ \lambda_1\right)$ is perpendicular to the eigenvector $\vec{y}$ $\left(\mathit{for}\ \lambda_2\right)$

Example

Find the λ′s and *v*′s for this symmetric matrix with trace zero: $A = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$

Solution

$$
\begin{aligned}
\det\left(A - \lambda I\right) &= \begin{vmatrix} -3-\lambda & 4 \\ 4 & 3-\lambda \end{vmatrix} \\
&= \left(-3-\lambda\right)\left(3-\lambda\right) - 16 \\
&= -9 + \lambda^2 - 16 \\
&= \underline{\lambda^2 - 25 = 0}
\end{aligned}
$$

The eigenvalues are: $\underline{\lambda_1 = -5 \quad \lambda_2 = 5}$

The eigenvectors are:

For $\lambda_1 = -5 \ \Rightarrow \ \left(A - \lambda_1 I\right)\vec{v}_1 = 0$

$$\begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \rightarrow \ 2x_1 + 4y_1 = 0$$

$$\Rightarrow \quad \underline{x_1 = -2y_1}$$

$$\Rightarrow \quad \boxed{\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$

For $\lambda_2 = 5 \ \Rightarrow \ (A - \lambda_2 I)\vec{v}_2 = 0$

$$\begin{pmatrix} -8 & 4 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \rightarrow \ 4x_2 - 2y_2 = 0$$

$$\Rightarrow \quad \boxed{2x_2 = y_2}$$

$$\Rightarrow \quad \boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

$$\begin{aligned} \vec{v}_1 \bullet \vec{v}_2 &= (-2)(1) + (1)(2) \\ &= -2 + 2 \\ &= 0 \end{aligned}$$

Thus, the eigenvectors are perpendicular.

The unit vector of the eigenvectors by dividing by their length $\sqrt{2^2 + 1^2} = \sqrt{5}$

The eigenvectors $\vec{v}_1$ *and* $\vec{v}_2$ are the columns of Q.

$$Q = \frac{\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}}{\sqrt{5}}$$

$$Q^{-1} = Q^T = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\begin{aligned} A &= QDQ^T \\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}\right)\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 10 & 5 \\ -5 & 10 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \quad \checkmark \end{aligned}$$

- Every symmetric matrix A has a complete set of orthogonal eigenvectors:

  $A = PDP^{-1} \quad \Rightarrow \quad A = QDQ^T$

Complex Eigenvalues of Real Matrices

For real matrices, complex λ's and x's come in "conjugate pairs"

$$\text{if} \quad Ax = \lambda x \quad \text{then} \quad A\bar{x} = \bar{\lambda}\bar{x}$$

Example

Given $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Solution

The eigenvalues of A:

$$\det(A - \lambda I) = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix}$$

$$= (\cos\theta - \lambda)(\cos\theta - \lambda) + \sin^2\theta$$

$$= \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta$$

$$= \lambda^2 - 2\lambda\cos\theta + 1 = 0$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2\cos\theta \pm \sqrt{4\left(\cos^2\theta - 1\right)}}{2} \qquad \cos^2\theta + \sin^2\theta = 1 \to \cos^2\theta - 1 = -\sin^2\theta$$

$$= \frac{2\cos\theta \pm 2\sqrt{-\sin^2\theta}}{2}$$

$$= \cos\theta \pm i\sin\theta$$

The *eigenvalues* are conjugate to each other.

For $\lambda_1 = \cos\theta + i\sin\theta$: $\quad (A - \lambda_1 I)\vec{v}_1 = 0$

$$\begin{pmatrix} \cos\theta - (\cos\theta + i\sin\theta) & -\sin\theta \\ \sin\theta & \cos\theta - (\cos\theta + i\sin\theta) \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \to \quad i\sin\theta x_1 = \sin\theta y_1$$

$$\Rightarrow \quad x_1 = iy_1$$

The eigenvectors:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\rightarrow \quad \boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}}$$

$$A\vec{v}_1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= (\cos\theta + i\sin\theta)\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$A\vec{v}_2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= (\cos\theta - i\sin\theta)\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$|\lambda| = \sqrt{\cos^2\theta + \sin^2\theta}$$

$$\boxed{= 1}$$

This fact holds for the eigenvalues of every orthogonal matrix.

Theorem – *Equivalent Statements*

If *A* is an $n\times n$ matrix, then the following statements are equivalent.

a) *A* is invertible

b) $A\vec{x}=\vec{0}$ has only the trivial solution

c) The reduced row echelon form of *A* is I_n

d) *A* is expressible as a product of elementary matrices

e) $A\vec{x}=\vec{b}$ is consistent for every $n\times 1$ matrix $\vec{b}$

f) $A\vec{x}=\vec{b}$ has exactly one solution for every $n\times 1$ matrix $\vec{b}$

g) $\det(A)\neq 0$

h) The column vectors of *A* are linearly independent

i) The row vectors of *A* are linearly independent

j) The column vectors of *A* span $\mathbb{R}^n$

k) The row vectors of *A* span $\mathbb{R}^n$

l) The column vectors of *A* form a basis for $\mathbb{R}^n$

m) The row vectors of *A* form a basis for $\mathbb{R}^n$

n) *A* has a rank *n*.

o) *A* has nullity 0.

p) The orthogonal complement of the null space of *A* is $\mathbb{R}^n$

q) The orthogonal complement of the row space of *A* is $\{0\}$

r) The range of T_A is $\mathbb{R}^n$

s) T_A is one-to-one.

t) $\lambda=0$ is not an eigenvalue of A.

u) $A^T A$ is invertible,

Exercises *Section* 4.6 – Orthogonal Diagonalization

(**1 – 10**) Determine whether the matrix *is* orthogonal

1. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

2. $\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

3. $\begin{pmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

4. $\begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$

5. $\begin{pmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{pmatrix}$

6. $\begin{pmatrix} -4 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{pmatrix}$

7. $\begin{pmatrix} 4 & -1 & -4 \\ -1 & 0 & -17 \\ 1 & 4 & -1 \end{pmatrix}$

8. $\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{pmatrix}$

9. $\begin{pmatrix} \frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{5}}{2} \\ 0 & \frac{2\sqrt{5}}{5} & 0 \\ -\frac{\sqrt{2}}{6} & -\frac{\sqrt{5}}{5} & \frac{1}{2} \end{pmatrix}$

10. $\begin{pmatrix} \frac{1}{8} & 0 & 0 & \frac{3\sqrt{7}}{8} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{3\sqrt{7}}{8} & 0 & 0 & \frac{1}{8} \end{pmatrix}$

(**11 – 24**)Find a matrix *P* that orthogonally diagonalizes *A*, and determine $P^{-1}AP$

11. $A=\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

12. $A=\begin{pmatrix} 6 & 2\sqrt{3} \\ 2\sqrt{3} & 7 \end{pmatrix}$

13. $A=\begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$

14. $A=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

15. $A=\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$

16. $A=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

17. $A=\begin{pmatrix} 0 & 10 & 10 \\ 10 & 5 & 0 \\ 10 & 0 & -5 \end{pmatrix}$

18. $A=\begin{pmatrix}0&3&0\\3&0&4\\0&4&0\end{pmatrix}$

19. $A=\begin{pmatrix}2&2&-2\\2&-1&4\\-2&4&-1\end{pmatrix}$

20. $A=\begin{pmatrix}-2&0&-36\\0&-3&0\\-36&0&-23\end{pmatrix}$

21. $A=\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}$

22. $A=\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}$

23. $A=\begin{pmatrix}-7&24&0&0\\24&7&0&0\\0&0&-7&24\\0&0&24&7\end{pmatrix}$

24. $A=\begin{pmatrix}3&1&0&0\\1&3&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$

25. Find the eigenvalues of A and B and check the Orthogonality of their first two eigenvectors. Graph these eigenvectors to see the discrete sines and cosines:

$$A=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}\qquad B=\begin{bmatrix}1&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&1\end{bmatrix}$$

The −1, 2, −1 pattern in both matrices is a "second derivative. Then $A\vec{x}=\lambda\vec{x}$ and $B\vec{x}=\lambda\vec{x}$ are like $\frac{d^2\vec{x}}{dt^2}=\lambda\vec{x}$ $\frac{d^2x}{dt^2}=\lambda x$. This has eigenvectors $x=\sin kt$ and $x=\cos kt$ that are the bases for Fourier series. The matrices lead to "discrete sines" and "discrete cosines" that are the bases for the discrete Fourier Transform. This DFT is absolutely central to all areas of digital signal processing.

26. Suppose $A\vec{x}=\lambda\vec{x}$ and $A\vec{y}=0\vec{y}$ and $\lambda\neq 0$. Then y is in the nullspace and $\vec{x}$ is in the column space. They are perpendicular because _______. (why are these subspaces orthogonal?) If the second eigenvalue is a nonzero number β, apply this argument to $A-\beta I$. The eigenvalue moves to zero and the eigenvectors stay the same – so they are perpendicular.

27. Which of these classes of matrices do A and B belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\qquad B=\frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$$

Which of these factorizations are possible for A and B: LU, QR, ADP^{-1}, QDQ^T?

28. *True* or *false*. Give a reason or a counterexample.

a) A matrix with real eigenvalues and eigenvectors is symmetric.
b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.
c) The inverse of a symmetric matrix is symmetric
d) The eigenvector matrix S of a symmetric matrix is symmetric.
e) A complex symmetric matrix has real eigenvalues.
f) If A is symmetric, then e^{iA} is symmetric.
g) If A is Hermitian, then e^{iA} is Hermitian.
h) An $n \times n$ matrix that is orthogonally diagonalizable must be symmetric.
i) If $A^T = A$ and if vectors $\vec{u}$ and $\vec{v}$ satisfy $A\vec{u} = 3\vec{u}$ and $A\vec{v} = 4\vec{v}$, then $\vec{u} \bullet \vec{v} = 0$
j) An $n \times n$ symmetric matrix has n distinct real eigenvalues.
k) For nonzero $\vec{v}$ in $\mathbb{R}^n$, the matrix $\vec{v}\vec{v}^T$ is called a projection matrix.
l) Every symmetric matrix is orthogonally diagonalizable
m) If $B = PDP^T$, where $P^T = P^{-1}$ and D is a diagonal matrix, then B is a symmetric matrix.
n) An orthogonal matrix is orthogonally diagonalizable.
o) The dimension of an eigenspace of a symmetric matrix equals the multiplicity of the corresponding eigenvalue.

29. Find a symmetric matrix $\begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$ that has a negative eigenvalue.

a) How do you know it must have a negative pivot?
b) How do you know it can't have two negative eigenvalues?

30. Prove that A is any $m \times n$ matrix, then $A^T A$ has an orthonormal set of n eigenvectors

31. Construct a 3 by 3 matrix A with no zero entries whose columns are mutually perpendicular. Compute $A^T A$. Why is it a diagonal matrix?

32. Assuming that $b \neq 0$, find a matrix that orthogonally diagonalizes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$

33. Suppose A is a symmetric $n \times n$ matrix and B is any $n \times m$ matrix. Show that $B^T AB$, $B^T B$, and BB^T are symmetric matrices.

34. Show that if A is an $n \times n$ symmetric matrix, then $(A\vec{x}) \bullet \vec{y} = \vec{x} \bullet (A\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

35. Suppose A is invertible and orthogonally diagonalizable. Explain why A^{-1} is also orthogonally diagonalizable.

36. Suppose A and B are both orthogonally diagonalizable and $AB = BA$. Explain why AB is also orthogonally diagonalizable.

37. Let $A = PDP^{-1}$, where P is orthogonal and D is diagonal, let λ be an eigenvalue of A of multiplicity k. Then λ appears k times on the diagonal of D. Explain why the dimension of the eigenspace for λ is k.

38. Suppose $A = PUP^{-1}$, where P is orthogonal and U is an upper triangular. Show that if A is symmetric, then U is symmetric and hence is actually a diagonal matrix.

39. Let $\vec{u}$ be a unit vector in $\mathbb{R}^n$, and let $B = \vec{u}\vec{u}^T$.

a) Given $\vec{x} \in \mathbb{R}^n$, compute $B\vec{x}$ and show that $B\vec{x}$ is the orthogonal projection of $\vec{x}$ onto $\vec{u}$.

b) Show that B is a symmetric matrix and $B^2 = B$.

c) Show that $\vec{u}$ is an eigenvector of B. What is the corresponding eigenvalue?

40. Let B be an $n \times n$ symmetric matrix such that $B^2 = B$. Any such matrix is called a ***projection matrix*** (or an *orthogonal projection matrix*). Given any $\vec{y} \in \mathbb{R}^n$, let $\hat{y} = B\vec{y}$ and $\vec{z} = \vec{y} - \hat{y}$.

a) Show that $\vec{z}$ is orthogonal to $\hat{y}$.

b) Let W be the column space of B. Show that $\vec{y}$ is the sum of a vector in W and a vector in $W^{\perp}$. Why does this prove that $B\vec{y}$ is the orthogonal projection of $\vec{y}$ onto the column space of B?