Solution Section 2.3 – Derivatives of Products and Quotients

Exercise

Find the first derivative $y=(x+1)\left(\sqrt{x}+2\right)$

Solution

$$\begin{aligned}
y' &= (1)\left(x^{1/2}+2\right)+(x+1)\left(\tfrac{1}{2}x^{-1/2}\right) \\
&= x^{1/2}+2+\tfrac{1}{2}x^{1/2}+\tfrac{1}{2}x^{-1/2} \\
&= \tfrac{3}{2}x^{1/2}+\tfrac{1}{2}x^{-1/2}+2
\end{aligned}$$

Exercise

Find the first derivative $y=\left(4x+3x^2\right)(6-3x)$

Solution

$y=24x+6x^2-9x^3$

$$\begin{aligned}
y' &= \left(4x+3x^2\right)\frac{d}{dx}(6-3x)+(6-3x)\frac{d}{dx}\left(4x+3x^2\right) \\
&= \left(4x+3x^2\right)(-3)+(6-3x)(4+6x) \\
&= -12x-9x^2+24+36x-12x-18x^2 \\
&= -27x^2+12x+24
\end{aligned}$$

Exercise

Find the first derivative $y=\left(\frac{1}{x}+1\right)(2x+1)$

Solution

$$\begin{aligned}
y' &= \left(x^{-1}+1\right)\frac{d}{dx}(2x+1)+(2x+1)\frac{d}{dx}\left(x^{-1}+1\right) \\
&= \left(x^{-1}+1\right)(2)+(2x+1)\left(-x^{-2}\right) \\
&= \frac{2}{x}+2+(2x+1)\left(-\frac{1}{x^2}\right)
\end{aligned}$$

$$= \frac{2}{x} + 2 - \frac{2x}{x^2} - \frac{1}{x^2}$$

$$= \frac{2}{x} + 2 - \frac{2}{x} - \frac{1}{x^2}$$

$$= 2 - \frac{1}{x^2}$$

$$\boxed{= \frac{2x^2 - 1}{x^2}}$$

Exercise

Find the first derivative $y = 3x\left(2x^2 + 5x\right)$

Solution

$$y = 6x^3 + 15x^2$$

$$\Rightarrow \boxed{y' = 18x^2 + 30x}$$

Exercise

Find the first derivative $y = 3\left(2x^2 + 5x\right)$

Solution

$$y = 6x^2 + 15x$$

$$\Rightarrow y' = 12x + 15$$

Exercise

Find the derivative of $y = \dfrac{x^2 + 4x}{5}$

Solution

$$y = \tfrac{1}{5}\left[x^2 + 4x\right]$$

$$y' = \tfrac{1}{5}(2x + 4)$$

Exercise

Find the first derivative $y = \dfrac{3x^4}{5}$

Solution

$$y = \frac{3}{5}x^4$$

$$y' = \boxed{\frac{12}{5}x^3}$$

Exercise

Find the first derivative $y = \dfrac{3 - \frac{2}{x}}{x+4}$

Solution

$$y = \frac{\frac{3x-2}{x}}{x+4}$$

$$= \frac{3x-2}{x} \cdot \frac{1}{x+4}$$

$$= \frac{3x-2}{x^2+4x}$$

$$y' = \frac{\left(x^2+4x\right)(3) - (3x-2)(2x+4)}{\left[x(x+4)\right]^2}$$

$$= \frac{3x^2+12x-6x^2-12x+4x+8}{x^2(x+4)^2}$$

$$= \boxed{\frac{-3x^2+4x+8}{x^2(x+4)^2}}$$

Exercise

Find the first derivative: $f(x)=\dfrac{(3-4x)(5x+1)}{7x-9}$

Solution

$$D_x\left[\frac{(3-4x)(5x+1)}{7x-9}\right]=\frac{\left[(-4)(5x+1)+(3-4x)(5)\right](7x-9)-(3-4x)(5x+1)(7)}{(7x-9)^2}$$

$$=\frac{[-20x-4+15-20x](7x-9)-\left(15x+3-20x^2-4x\right)(7)}{(7x-9)^2}$$

$$=\frac{(-40x+11)(7x-9)-7\left(-20x^2+11x+3\right)}{(7x-9)^2}$$

$$=\frac{-280x^2+360x+77x-99+140x^2-77x-21}{(7x-9)^2}$$

$$=\frac{-140x^2+360x-120}{(7x-9)^2}$$

Exercise

Find the derivative $g(x)=\dfrac{x^2-4x+2}{x^2+3}$

Solution

$$g'=\frac{(2x-4)\left(x^2+3\right)-\left(x^2-4x+2\right)(2x)}{\left(x^2+3\right)^2}$$

$$=\frac{2x^3+6x-4x^2-12-2x^3+8x^2-4x}{\left(x^2+3\right)^2}$$

$$=\frac{4x^2+2x-12}{\left(x^2+3\right)^2}$$

Exercise

Find the derivative of $y=\dfrac{x+4}{5x-2}$

Solution

$$y'=\dfrac{(5x-2)\dfrac{d}{dx}[(x+4)]-(x+4)\dfrac{d}{dx}[(5x-2)]}{(5x-2)^2}$$

$$=\dfrac{(5x-2)(1)-(x+4)(5)}{(5x-2)^2}$$

$$=\dfrac{5x-2-5x-20}{(5x-2)^2}$$

$$=-\dfrac{22}{(5x-2)^2}$$

Exercise

Find the derivative of $f(x)=x\left(1-\dfrac{2}{x+1}\right)$

Solution

$$f(x)=x-\dfrac{2x}{x+1}$$

$$\left(\dfrac{2x}{x+1}\right)' \Rightarrow \qquad \begin{array}{ll} f=2x & f'=2 \\ g=x+1 & g'=1 \end{array}$$

$$f'(x)=1-\dfrac{2(x+1)-2x}{(x+1)^2}$$

$$=1-\dfrac{2x+2-2x}{(x+1)^2}$$

$$=1-\dfrac{2}{(x+1)^2}$$

Exercise

Find the derivative of $g(s)=\dfrac{s^2-2s+5}{\sqrt{s}}$

Solution

$$g(s)=\frac{s^2}{s^{1/2}}-2\frac{s}{s^{1/2}}+\frac{5}{s^{1/2}}$$

$$=s^{3/2}-2s^{1/2}+5s^{-1/2}$$

$$g'(s)=\tfrac{3}{2}s^{1/2}-2\tfrac{1}{2}s^{-1/2}+5\left(-\tfrac{1}{2}\right)s^{-3/2}$$

$$=\tfrac{3}{2}s^{1/2}-s^{-1/2}-\tfrac{5}{2}s^{-3/2}$$

$$=\frac{3}{2}\sqrt{s}-\frac{1}{\sqrt{s}}-\frac{5}{2s^{3/2}}$$

$$=\frac{3}{2}\sqrt{s}-\frac{1}{\sqrt{s}}-\frac{5}{2s\sqrt{s}}$$

Exercise

Find the derivative of $f(x)=\dfrac{x+1}{\sqrt{x}}$

Solution

$$f(x)=\frac{x}{x^{1/2}}+\frac{1}{x^{1/2}}$$

$$=x^{1/2}+x^{-1/2}$$

$$f'(x)=\tfrac{1}{2}x^{-1/2}-\tfrac{1}{2}x^{-3/2}$$

$$=\frac{1}{2x^{1/2}}-\frac{1}{2x^{3/2}}$$

Exercise

Find the derivative $f(x)=\dfrac{x^2}{2x+1}$

Solution

$u=x^2 \quad v=2x+1$

$u'=2x \quad v'=2$

$$f'(x)=\frac{2x(2x+1)-x^2(2)}{(2x+1)^2}$$

$$=\frac{4x^2+2x-2x^2}{(2x+1)^2}$$

$$=\boxed{\frac{2x^2+2x}{(2x+1)^2}}$$

Exercise

Find the derivative $f(x)=\dfrac{x^2-x}{x^3+1}$

Solution

$u=x^2-x \quad v=x^3+1$

$u'=2x-1 \quad v'=3x^2$

$$f'(x)=\frac{(2x-1)\left(x^3+1\right)-\left(x^2-x\right)\left(3x^2\right)}{\left(x^3+1\right)^2}$$

$$=\frac{2x^4+2x-x^3-1-3x^4+3x^3}{\left(x^3+1\right)^2}$$

$$=\boxed{\frac{-x^4+2x^3+2x-1}{\left(x^3+1\right)^2}}$$

Exercise

Find the derivative $f(x)=\dfrac{2x}{x^2+3}$

Solution

$$f'(x)=\frac{2\left(x^2+3\right)-2x(2x)}{\left(x^2+3\right)^2}$$

$u=2x \quad v=x^2+3$

$u'=2 \quad v'=2x$

$$=\frac{2x^2+6-4x^2}{\left(x^2+3\right)^2}$$

$$=\boxed{\frac{-2x^2+6}{\left(x^2+3\right)^2}}$$

Exercise

Find the derivative $y=\dfrac{t^3-3t}{t^2-4}$

Solution

$$y'=\frac{\left(3t^2-3\right)\left(t^2-4\right)-\left(t^3-3t\right)(2t)}{\left(t^2-4\right)^2}$$

$u=t^3-3t \quad v=t^2-4$

$u'=3t^2-3 \quad v'=2t$

$$=\frac{3t^4-15t^2+12-2t^4+2t^2}{\left(t^2-4\right)^2}$$

$$=\boxed{\frac{t^4-13t^2+12}{\left(t^2-4\right)^2}}$$

Exercise

Find the derivative $f(x)=5x^2\left(x^3+2\right)$

Solution

$$f(x)=5x^5+10x^2$$

$$f'(x)\ \boxed{=25x^4+20x}$$

Exercise

Find the derivative $f(x)=\dfrac{3x-4}{2x+3}$

Solution

$$f'(x)=\frac{3(2x+3)-2(3x-4)}{(2x+3)^2}$$

$$u=3x-4 \quad v=2x+3$$
$$u'=3 \quad v'=2$$

$$=\frac{6x+9-6x+8}{(2x+3)^2}$$

$$=\frac{17}{(2x+3)^2}$$

Exercise

Find the derivative $f(x)=\dfrac{3x+5}{x^2-3}$

Solution

$$f'(x)=\frac{3x^2-9-6x^2-10x}{\left(x^2-3\right)^2}$$

$$u=3x+5 \quad v=x^2-3$$
$$u'=3 \quad v'=2x$$

$$=\frac{-3x^2-10x-9}{\left(x^2-3\right)^2}$$

Exercise

Find the derivative $f(x)=\left(x^2-4\right)\left(x^2+5\right)$

Solution

$$u=x^2-4 \quad v=x^2+5$$
$$u'=2x \quad v'=2x$$

$$f'(x)=2x\left(x^2+5\right)+2x\left(x^2-4\right)$$

$$=2x^3+10x+2x^3-8x$$

$$=4x^3+2x$$

$$f(x)=x^4+5x^2-4x^2-20$$

$$=x^4+x^2-20$$

$$f'(x)=4x^3+2x$$

Exercise

A company that manufactures bicycles has determined that a new employee can assemble $M(d)$ bicycles per day after d days of on-the-job training, where

$$M(d) = \frac{100d^2}{3d^2+10}$$

a) Find the rate of change function for the number of bicycles assembled with respect to time.

b) Find and interpret $M'(2)$ and $M'(5)$

Solution

a) Find the rate of change function for the number of bicycles assembled with respect to time.

$$M' = \frac{(200d)\left(3d^2+10\right) - 100d^2(6d)}{\left(3d^2+10\right)^2}$$

$$= \frac{600d^3 + 2000d - 600d^3}{\left(3d^2+10\right)^2}$$

$$= \frac{2000d}{\left(3d^2+10\right)^2}$$

b) Find and interpret $M'(2)$ and $M'(5)$

$$M'(2) = \frac{2000(2)}{\left(3(2)^2+10\right)^2}$$

$$\underline{=8.3|}$$

After 2 days of training, employee can assemble about 8.3 bicycles per day.

$$M'(5) = \frac{2000(5)}{\left(3(5)^2+10\right)^2}$$

$$\underline{=1.4|}$$

After 4 days of training, employee can assemble about 1.4 bicycles per day.

Exercise

Find an equation of the tangent line to the graph of $y=\dfrac{x^2-4}{2x+5}$ when $x = 0$

Solution

$$y'=\frac{(2x+5)(2x)-(x^2-4)(2)}{(2x+5)^2}$$

$$=\frac{4x^2+10x-2x^2+8}{(2x+5)^2}$$

$$=\frac{2x^2+10x+8}{(2x+5)^2}$$

$$\Rightarrow x=0\to y'=\frac{8}{25}=m$$

$$x=0\to y=\frac{x^2-4}{2x+5}=-\frac{4}{5}$$

$$y-y_1=m\left(x-x_1\right)$$

$$\to y+\frac{4}{5}=\frac{8}{25}(x-0) \qquad \to y=\frac{8}{25}x-\frac{4}{5}$$

Exercise

A small business invests \$25,000.00 in a new product. In addition, the product will cost \$0.75 per unit to produce. Find the cost function and the average cost function. What is the limit of the average cost function as production increase?

Solution

C = 0.75x + 25000

$$\overline{C}=\frac{C}{x}=\frac{0.75x+25000}{x}$$

$$=\frac{0.75x}{x}+\frac{25000}{x}$$

$$=0.75+\frac{25000}{x}$$

$$\lim_{x\to\infty}\overline{C}=\lim_{x\to\infty}\left(0.75+\frac{25000}{x}\right)$$

$$=\lim_{x\to\infty}0.75+\lim_{x\to\infty}\frac{25000}{x}$$

$$=0.75+0$$

$$=\$0.75/unit$$

Exercise

A communications company has installed a new cable TV system in a city. The total number N (in thousands) of subscribers t months after the installation of the system is given by

$$N(t)=\frac{180t}{t+4}$$

a) Find $N'(t)$

b) Find $N(16)$ and $N'(16)$. Write a brief interpretation of these results.

c) Use the results from part (***b***) to estimate the total number of subscribers after 17 months.

Solution

a) $N'(t)=\dfrac{180(t+4)-180t}{(t+4)^2}$ $\qquad u=180t \quad v=t+4 \qquad \left(\dfrac{u}{v}\right)'=\dfrac{u'v-v'u}{v^2}$

$u'=180 \quad v'=1$

$$=\frac{180t+720-180t}{(t+4)^2}$$

$$=\boxed{\frac{720}{(t+4)^2}}$$

b) $N(16)=\dfrac{180(16)}{16+4}$ $\boxed{=144}$

$N'(16)=\dfrac{720}{(16+4)^2}$ $\boxed{=1.8}$

After 16 months, the total number of subscribers is 144,000 and is increasing at a rate of 1,800 subscribers per month.

c) The total subscribers after 17 months will be approximately 145,800

Exercise

One hour after a dose of x milligrams of a particular drug is administered to a person, the change in body temperature $T(x)$, in degrees Fahrenheit, is given approximately by

$$T(x)=x^2\left(1-\frac{x}{9}\right) \quad 0\le x\le 7$$

The rate $T'(x)$ at which T changes with respect to the size of the dosage x is called the sensitivity of the body to the dosage.

a) Find $T'(x)$

b) Find $T'(1)$, $T'(3)$, and $T'(6)$

Solution

a) $u = x^2 \quad v = 1 - \frac{x}{9}$

$u = 2x \quad v' = -\frac{1}{9}$

$$T'(x) = 2x\left(1 - \frac{x}{9}\right) + x^2\left(-\frac{1}{9}\right)$$

$$= 2x - \frac{2}{9}x^2 - \frac{1}{9}x^2$$

$$\boxed{= 2x - \frac{1}{3}x^2}$$

b) $T'(1) = 2(1) - \frac{1}{3}(1)^2 \ \boxed{= \frac{5}{3} \ \textit{per mg of drug}}$

$T'(3) = 2(3) - \frac{1}{3}(3)^2 \ \boxed{= 3 \ \textit{per mg of drug}}$

$T'(6) = 2(6) - \frac{1}{3}(6)^2 \ \boxed{= 0 \ \textit{per mg of drug}}$

Exercise

According to economic theory, the supply x of a quantity in a free market increases as the price p increases. Suppose that the number x of DVD players a retail chain is willing to sell per week at a price of \$$p$ is given by

$$x = \frac{100p}{0.1p + 1} \quad 10 \le p \le 70$$

a) Find $\frac{dx}{dp}$

b) Find the supply and the instantaneous rate of change of supply with respect to price is \$40. Write a brief interpretation of these results.

c) Use the results from part (***b***) to estimate the supply if the price is increased to \$41.

Solution

a) $u = 100p \quad v = 0.1p + 1$

$u' = 100 \quad v' = 0.1$

$$\frac{dx}{dp} = \frac{100(0.1p+1) - 100p(0.1)}{(0.1p+1)^2}$$

$$= \frac{10p + 100 - 10p}{(0.1p+1)^2}$$

$$\boxed{= \frac{100}{(0.1p+1)^2}}$$

b) $x(40) = \frac{100(40)}{0.1(40)+1} \ \boxed{= 800}$

$$\left.\frac{dx}{dp}\right|_{40} = \frac{100}{\left(0.1(40)+1\right)^2} = 4$$

At price level of $40, the supply is 800 DVD players and is increasing at the rate of 4 players per dollars.

c) At a price of $41, the demand will be approximately 800 + 4 = 804 DVD players