Solution *Section* 1.1 – Idea of Limits

Exercise

Find the average rate of change of the function $f(x) = x^3 + 1$ over the interval $[2,\ 3]$

Solution

$$\frac{\Delta f}{\Delta x} = \frac{f(3) - f(2)}{3 - 2}$$

$$= \frac{3^3 + 1 - \left(2^3 + 1\right)}{1}$$

$$= 27 + 1 - (8 + 1)$$

$$\underline{= 19}$$

Exercise

Find the average rate of change of the function $f(x) = x^2$ over the interval $[-1,\ 1]$

Solution

$$\frac{\Delta f}{\Delta x} = \frac{f(1) - f(-1)}{1 - (-1)}$$

$$= \frac{1^2 - (-1)^2}{2}$$

$$= \frac{0}{2}$$

$$\underline{= 0}$$

Exercise

Find the average rate of change of the function $f(t) = 2 + \cos t$ over the interval $[-\pi,\ \pi]$

Solution

$$\frac{\Delta f}{\Delta x} = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)}$$

$$= \frac{2 + \cos\pi - \left(2 + \cos(-\pi)\right)}{2\pi}$$

$$= \frac{2 - 1 - (2 - 1)}{2}$$

$$\underline{= 0}$$

Exercise

Find the slope of $y = x^2 - 3$ at the point $P(2,\ 1)$ and an equation of the tangent line at this *P*.

Solution

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1 + h) - f(x_1)}{h}$$

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{f(2+h) - f(2)}{h} \\
&= \frac{(2+h)^2 - 3 - \left(2^2 - 3\right)}{h} \\
&= \frac{4 + 4h + h^2 - 3 - (4-3)}{h} \\
&= \frac{4h + h^2}{h} \\
&= 4 + h
\end{aligned}$$

As *h* approaches 0. Then the secant slope $h + 4 \to 4 = slope$

$$y = m\left(x - x_1\right) + y_1$$

$$y = 4(x-2) + 1$$

$$y - 1 + 1 = 4x - 8 + 1$$

$$\boxed{y = 4x - 7}$$

Exercise

Find the slope of $y = x^2 - 2x - 3$ at the point $P(2,\ -3)$ and an equation of the tangent line at this *P*.

Solution

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{f(2+h) - f(2)}{h} \\
&= \frac{(2+h)^2 - 2(2+h) - 3 - \left(2^2 - 2(2) - 3\right)}{h} \\
&= \frac{4 + 4h + h^2 - 4 - 2h - 3 - (-3)}{h} \\
&= \frac{2h + h^2}{h} \\
&= 2 + h
\end{aligned}$$

As *h* approaches 0. Then the secant slope $2 + h \to 2 = slope$

$$y = m\left(x - x_1\right) + y_1$$

$$y + 3 = 2(x-2)$$

$$y = 2x - 4 - 3$$

$$\boxed{y = 2x - 7}$$

Exercise

Find the slope of $y = x^3$ at the point $P(2,\ 8)$ and an equation of the tangent line at this P.

Solution

$$\frac{\Delta y}{\Delta x} = \frac{f(2+h) - f(2)}{h}$$

$$= \frac{(2+h)^3 - 2^3}{h}$$

$$= \frac{8 + 12h + 6h^2 + h^3 - 8}{h}$$

$$\left. \underline{= 12 + 6h + h^2} \right|$$

As h approaches 0. Then $slope = 12$

$$y - 8 = 12(x - 2) \qquad\qquad y = m(x - x_1) + y_1$$

$$y = 12x - 24 + 8$$

$$\boxed{y = 12x - 16}$$

Exercise

Make a table of values for the function $f(x) = \dfrac{x+2}{x-2}$ at the points

$$x = 1.2, \quad x = \frac{11}{10}, \quad x = \frac{101}{100}, \quad x = \frac{1001}{1000}, \quad x = \frac{10001}{10000}, \quad and\ \ x = 1$$

a) Find the average rate of change of $f(x)$ over the intervals $[1,\ x]$ for each $x \neq 1$ in the table

b) Extending the table if necessary, try to determine the rate of change of $f(x)$ at $x = 1$.

Solution

a)

x	1.2	1.1	1.01	1.001	1.0001	1
$f(x)$	-4.0	$-3.\overline{4}$	$-3.\overline{04}$	$-3.\overline{004}$	$-3.\overline{004}$	-3

$$\frac{\Delta y}{\Delta x} = \frac{-4 - (-3)}{1.2 - 1} = -5.0$$

$$\frac{\Delta y}{\Delta x} = \frac{-3.\overline{4} - (-3)}{1.1 - 1} = -4.\overline{4}$$

$$\frac{\Delta y}{\Delta x} = \frac{-3.\overline{04} - (-3)}{1.01 - 1} = -4.\overline{04}$$

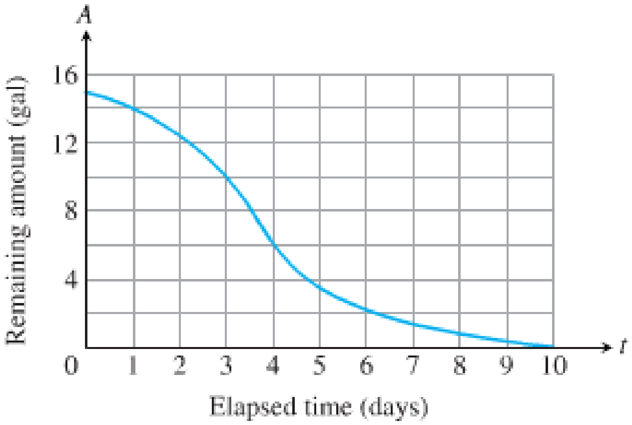
$$\frac{\Delta y}{\Delta x} = \frac{-3.\overline{004} - (-3)}{1.001 - 1} = -4.\overline{004}$$

$$\frac{\Delta y}{\Delta x} = \frac{-3.\overline{0004} - (-3)}{1.0001 - 1} = -4.\overline{0004}$$

b) The rate of change of $f(x)$ at $x = 1$ is $\left. \underline{-4} \right|$

Exercise

The accompanying graph shows the total amount of gasoline A in the gas tank of an automobile after being driven for t days.



a) Estimate the average rate of gasoline consumption over the time intervals

$$[0,\ 3],\ [0,\ 5],\ and\ [7,\ 10]$$

b) Estimate the instantaneous rate of gasoline consumption over the time $t=1,\ t=4,\ and\ t=8$

Solution

a) Average rate of gasoline consumption over the time intervals:

$$[0,\ 3] \quad \Rightarrow \quad Average\ Rate = \frac{10-15}{3-0} \approx \underline{= -1.67\ gal\ /\ day}|$$

$$[0,\ 5] \quad \Rightarrow \quad Average\ Rate = \frac{3.9-15}{3-0}\ \underline{\approx -2.2\ gal\ /\ day}|$$

$$[7,\ 10] \quad \Rightarrow \quad Average\ Rate = \frac{0-1.4}{10-7}\ \underline{\approx -0.5\ gal\ /\ day}|$$

b) At $t=1 \quad \to P(1,\ 14)$

At $t=4 \quad \to P(4,\ 6)$

At $t=8 \quad \to P(8,\ 1)$

Solution *Section* 1.2 – Definitions / Techniques of Limits

Exercise

Find the limit: $\lim\limits_{x \to 3} (-1)$

Solution

$$\lim_{x \to 3} (-1) \boxed{= -1}$$

Exercise

Find the limit: $\lim\limits_{x \to -1} (3)$

Solution

$$\lim_{x \to -1} (3) \boxed{= 3}$$

Exercise

Find the limit: $\lim\limits_{x \to 1000} 18\pi^2$

Solution

$$\lim_{x \to 1000} 18\pi^2 \boxed{= 18\pi^2}$$

Exercise

Find the limit: $\lim\limits_{x \to 1} \sqrt{5x+6}$

Solution

$$\lim_{x \to 1} \sqrt{5x+6} \boxed{= \sqrt{11}}$$

Exercise

Find the limit: $\lim\limits_{x \to 9} \sqrt{x}$

Solution

$$\lim_{x \to 9} \sqrt{x} = \sqrt{9} \boxed{= 3}$$

Exercise

Find the limit: $\lim\limits_{x\to -3} \left(x^2+3x\right)$

Solution

$$\lim\limits_{x\to -3} \left(x^2+3x\right) = (-3)^2 + 3(-3) = 9-9 \ \underline{=0}\big|$$

Exercise

Find the limit: $\lim\limits_{x\to -4} |x-4|$

Solution

$$\lim\limits_{x\to -4} |x-4| = |-4-4| = |-8| \ \underline{=8}\big|$$

Exercise

Find the limit: $\lim\limits_{x\to 4} (x+2)$

Solution

$$\lim\limits_{x\to 4} (x+2) = 4+2 \ \underline{=6}\big|$$

Exercise

Find the limit: $\lim\limits_{x\to 4} (x-4)$

Solution

$$\lim\limits_{x\to 4} (x-4) = 4-4 \ \underline{=0}\big|$$

Exercise

Find the limit: $\lim\limits_{x\to 2} (5x-6)^{3/2}$

Solution

$$\begin{aligned}\lim\limits_{x\to 2} (5x-6)^{3/2} &= (10-6)^{3/2} \\ &= \sqrt{4^3} \\ &\underline{=8}\ \big|\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to 9} \dfrac{x-9}{\sqrt{x}-3}$

Solution

$$\lim\limits_{x\to 9} \frac{x-9}{\sqrt{x}-3} = \frac{9-9}{3-3} = \frac{0}{0}$$

$$= \lim\limits_{x\to 9} \frac{\left(\sqrt{x}-3\right)\left(\sqrt{x}+3\right)}{\sqrt{x}-3}$$

$$= \lim\limits_{x\to 9} \left(\sqrt{x}+3\right)$$

$$\underline{= 6}|$$

Exercise

Find the limit: $\lim\limits_{x\to 1} (2x+4)$

Solution

$$\lim\limits_{x\to 1} (2x+4) = 2(1)+4 \ \underline{= 6}|$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{x^2-4}{x-2}$

Solution

$$\lim\limits_{x\to 1} \frac{x^2-4}{x-2} = \frac{1^2-4}{1-2}$$

$$= \frac{-3}{-1}$$

$$\underline{= 3}|$$

Exercise

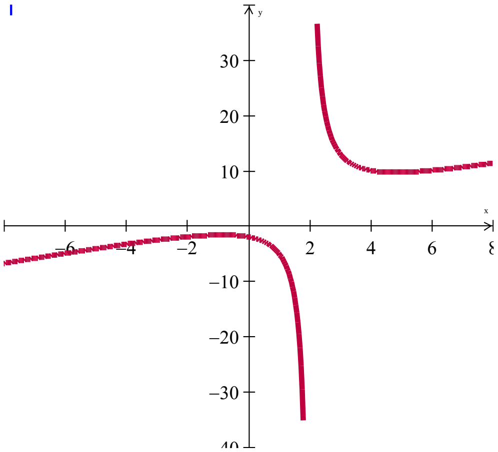
Find the limit: $\lim\limits_{x\to 2} \dfrac{x^2+4}{x-2}$

Solution

$$\lim\limits_{x\to 2} \frac{x^2+4}{x-2} = \frac{2^2+4}{2-2}$$

$$= \frac{8}{0}$$

$\underline{= \infty}|$ (***Doesn't exist***)



Exercise

Find the limit: $\lim\limits_{x\to 0} \frac{|x|}{x}$

Solution

$$\lim_{x\to 0} \frac{|x|}{x} = \frac{0}{0}$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \frac{x}{-x} = -1 \qquad \lim_{x\to 0^+} \frac{|x|}{x} = \frac{x}{x} = 1$$

Doesn't exist

Exercise

Find: $\lim\limits_{x\to 3} \frac{x^2 - x - 1}{\sqrt{x+1}}$

Solution

$$\lim_{x\to 3} \frac{x^2 - x - 1}{\sqrt{x+1}} = \frac{3^2 - 3 - 1}{\sqrt{3+1}}$$

$$= \frac{5}{2}$$

Exercise

Find: $\lim\limits_{x\to 2} \frac{x^2 + x - 6}{x - 2}$

Solution

$$\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2} = \frac{2^2 + 2 - 6}{2 - 2} = \frac{0}{0}$$

$$= \lim_{x\to 2} \frac{(x+3)(x-2)}{x-2}$$

$$= \lim_{x\to 2} (x+3)$$

$$= 5$$

Exercise

Find the limit: $\lim\limits_{x\to 0} (3x - 2)$

Solution

$$\lim_{x\to 0} (3x - 2) = 3(0) - 2$$

$$= -2$$

Exercise

Find the limit: $\lim\limits_{x\to 1}(2x^2 - x + 4)$

Solution

$$\begin{aligned}\lim_{x\to 1}(2x^2 - x + 4) &= 2(1)^2 - (1) + 4\\ &\underline{= 5\,}\big|\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to -2}\left(x^3 - 2x^2 + 4x + 8\right)$

Solution

$$\begin{aligned}\lim_{x\to -2}\left(x^3 - 2x^2 + 4x + 8\right) &= (-2)^3 - 2(-2)^2 + 4(-2) + 8\\ &\underline{= -16}\big|\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to 2}\dfrac{x^2 - 4}{x - 2}$

Solution

$$\begin{aligned}\lim_{x\to 2}\frac{x^2-4}{x-2} &= \frac{2^2-4}{2-2} = \frac{0}{0}\\ &= \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}\\ &= \lim_{x\to 2}(x+2)\\ &\underline{= 4\,}\big|\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to 2}\dfrac{x^3-8}{x-2}$

Solution

$$\lim_{x\to 2}\frac{x^3-8}{x-2} = \frac{0}{0}$$

$$\begin{aligned}\lim_{x\to 2}\frac{x^3-8}{x-2} &= \lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{x-2}\\ &= \lim_{x\to 2} x^2 + 2x + 4\\ &= 2^2 + 2(2) + 4\\ &\underline{= 12\,}\big|\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to 3}\frac{x^2+x-12}{x-3}$

Solution

$$\lim_{x\to 3}\frac{x^2+x-12}{x-3}=\frac{0}{0}$$

$$\lim_{x\to 3}\frac{(x-3)(x+4)}{x-3}=\lim_{x\to 3}(x+4)$$

$$\boxed{=7}$$

Exercise

Find the limit: $\lim\limits_{x\to 0}\frac{\sqrt{x+4}-2}{x}$

Solution

$$\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}=\frac{\sqrt{4}-2}{0}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}\,\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$$

$$=\lim_{x\to 0}\frac{x+4-4}{x\left(\sqrt{x+4}+2\right)}$$

$$=\lim_{x\to 0}\frac{x}{x\left(\sqrt{x+4}+2\right)}$$

$$=\lim_{x\to 0}\frac{1}{\sqrt{x+4}+2}$$

$$=\frac{1}{\sqrt{4}+2}$$

$$\boxed{=\frac{1}{4}}$$

Exercise

Find the limit: $\lim\limits_{x\to 0}\frac{3}{\sqrt{3x+1}+1}$

Solution

$$\lim_{x\to 0}\frac{3}{\sqrt{3x+1}+1}=\frac{3}{\sqrt{3(0)+1}+1}$$

$$=\frac{3}{1+1}$$

$$\boxed{=\frac{3}{2}}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} f(x)$ $\qquad f(x) = \begin{cases} x^2+1 & x<0 \\ 2x+1 & x>0 \end{cases}$

Solution

$$\lim_{x\to 0^-} x^2+1 = 1$$

$$\lim_{x\to 0+} 2x+1 = 1$$

$$\lim_{x\to 0} f(x) \underline{=1}\rfloor$$

Exercise

Find the limit: $\lim\limits_{x\to -2} \dfrac{5}{x+2}$

Solution

$$\lim_{x\to -2} \frac{5}{x+2} = \frac{5}{0}$$

$$\underline{=\infty}\rfloor$$

Exercise

Find the limit: $\lim\limits_{x\to 3} \dfrac{\sqrt{x+1}-1}{x}$

Solution

$$\lim_{x\to 3} \frac{\sqrt{x+1}-1}{x} = \frac{\sqrt{3+1}-1}{3} = \frac{2-1}{3}$$

$$\underline{=\frac{1}{3}}\rfloor$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$

Solution

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x\to 1} (x+1)$$

$$\underline{=2}\rfloor$$

Exercise

Find the limit: $\lim\limits_{x\to -2} \dfrac{|x+2|}{x+2}$

Solution

$$\lim_{x\to -2} \frac{|x+2|}{x+2} = \frac{|-2+2|}{-2+2} = \frac{0}{0}$$

$$\lim_{x\to -2^+} \frac{|x+2|}{x+2} = \frac{(x+2)}{(x+2)} = 1$$

$$\lim_{x\to -2^-} \frac{|x+2|}{x+2} = \frac{(x+2)}{-(x+2)} = -1$$

Doesn't exist

Exercise

Find the limit: $\lim\limits_{x\to 0} (2x-8)^{1/3}$

Solution

$$\begin{aligned}\lim_{x\to 0} (2x-8)^{1/3} &= (2(0)-8)^{1/3} \\ &= (-8)^{1/3} \\ &\underline{= -2}\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to 2} \dfrac{x^2-7x+10}{x-2}$

Solution

$$\lim_{x\to 2} \frac{x^2-7x+10}{x-2} = \frac{2^2-7(2)+10}{2-2} = \frac{0}{0}$$

$$\begin{aligned}\lim_{x\to 2} \frac{x^2-7x+10}{x-2} &= \lim_{x\to 2} \frac{(x-2)(x-5)}{x-2} \\ &= \lim_{x\to 2} (x-5) \\ &= 2-5 \\ &\underline{= -3}\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to 0}\dfrac{5x^3+8x^2}{3x^4-16x^2}$

Solution

$$\lim_{x\to 0}\frac{5x^3+8x^2}{3x^4-16x^2}=\frac{0}{0}$$

$$\begin{aligned}\lim_{x\to 0}\frac{5x^3+8x^2}{3x^4-16x^2}&=\lim_{x\to 0}\frac{x^2(5x+8)}{x^2\left(3x^2-16\right)}\\&=\lim_{x\to 0}\frac{5x+8}{3x^2-16}\\&=\frac{8}{-16}\\&=-\frac{1}{2}\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to 1}\dfrac{\frac{1}{x}-1}{x-1}$

Solution

$$\lim_{x\to 1}\frac{\frac{1}{x}-1}{x-1}=\frac{1-1}{1-1}=\frac{0}{0}$$

$$\begin{aligned}\lim_{x\to 1}\frac{\frac{1}{x}-1}{x-1}&=\lim_{x\to 1}\frac{\frac{1-x}{x}}{x-1}\\&=\lim_{x\to 1}\left(\frac{1-x}{x}\right)\left(\frac{1}{x-1}\right)\\&=\lim_{x\to 1}\left(\frac{-(x-1)}{x}\right)\left(\frac{1}{x-1}\right)\\&=\lim_{x\to 1}\frac{-1}{x}\\&=-1\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{u\to 1}\dfrac{u^4-1}{u^3-1}$

Solution

$$\lim_{u\to 1}\frac{u^4-1}{u^3-1}=\lim_{u\to 1}\frac{\left(u^2-1\right)\left(u^2+1\right)}{(u-1)\left(u^2+u+1\right)}$$

$$=\lim_{u\to 1}\frac{(u-1)(u+1)\left(u^2+1\right)}{(u-1)\left(u^2+u+1\right)}$$

$$=\lim_{u\to 1}\frac{(u+1)\left(u^2+1\right)}{u^2+u+1}$$

$$=\frac{(1+1)\left(1^2+1\right)}{1^2+1+1}$$

$$\boxed{=\frac{4}{3}}$$

Exercise

Find the limit: $\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2}$

Solution

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2}=\frac{1-1}{\sqrt{1+3}-2}=\frac{0}{\sqrt{4}-2}=\frac{0}{0}$$

$$=\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2}\bullet\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$$

$$=\lim_{x\to 1}\frac{(x-1)\left(\sqrt{x+3}+2\right)}{x+3-4}$$

$$=\lim_{x\to 1}\frac{(x-1)\left(\sqrt{x+3}+2\right)}{x-1}$$

$$=\lim_{x\to 1}\left(\sqrt{x+3}+2\right)$$

$$=\sqrt{1+3}+2$$

$$\boxed{=4}$$

Exercise

Find the limit: $\lim_{x\to -1}\frac{\sqrt{x^2+8}-3}{x+1}$

Solution

$$\lim_{x\to -1}\frac{\sqrt{x^2+8}-3}{x+1}=\frac{\sqrt{(-1)^2+8}-3}{-1+1}=\frac{\sqrt{9}-3}{0}=\frac{0}{0}$$

$$= \lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} \bullet \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$$

$$= \lim_{x\to -1} \frac{x^2+8-9}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to -1} \frac{x^2-1}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to -1} \frac{(x-1)(x+1)}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

$$= \lim_{x\to -1} \frac{(x-1)}{\sqrt{x^2+8}+3}$$

$$= \frac{-2}{\sqrt{9}+3} \quad = \quad \frac{-2}{6}$$

$$\boxed{= -\frac{1}{3}}$$

Exercise

Find the limit: $\lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3}$

Solution

$$\lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3} = \frac{2-\sqrt{(-3)^2-5}}{-3+3} = \frac{2-\sqrt{9-5}}{0} = \frac{2-\sqrt{4}}{0} = \frac{0}{0}$$

$$= \lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3} \bullet \frac{2+\sqrt{x^2-5}}{2+\sqrt{x^2-5}}$$

$$= \lim_{x\to -3} \frac{4-\left(x^2-5\right)}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{4-x^2+5}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{9-x^2}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{(x-3)(x+3)}{(x+3)\left(2+\sqrt{x^2-5}\right)}$$

$$= \lim_{x\to -3} \frac{(x-3)}{2+\sqrt{x^2-5}}$$

$$= \frac{-6}{2+\sqrt{9-5}}$$

$$= \frac{-6}{2+\sqrt{4}}$$

$$\boxed{= -\frac{3}{2}}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} (2\sin x - 1)$

Solution

$$\lim_{x\to 0} (2\sin x - 1) = 2\sin(0) - 1$$

$$= 0 - 1$$

$$\boxed{= -1}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \sin^2 x$

Solution

$$\lim_{x\to 0} \sin^2 x = \sin^2(0)$$

$$\boxed{= 0}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \sec x$

Solution

$$\lim_{x\to 0} \sec x = \sec(0)$$

$$= \frac{1}{\cos(0)}$$

$$\boxed{= 1}$$

Exercise

Find the limit: $\lim\limits_{x \to 0} \dfrac{1+x+\sin x}{3\cos x}$

Solution

$$\lim_{x \to 0} \frac{1+x+\sin x}{3\cos x} = \frac{1+0+\sin(0)}{3\cos(0)}$$

$$\boxed{=\frac{1}{3}}$$

Exercise

Find the limit: $\lim\limits_{x \to -\pi} \sqrt{x+4}\cos(x+\pi)$

Solution

$$\lim_{x \to -\pi} \sqrt{x+4}\cos(x+\pi) = \sqrt{-\pi+4}\cos(-\pi+\pi)$$

$$= \sqrt{-\pi+4}\cos(0)$$

$$\boxed{=\sqrt{4-\pi}}$$

Exercise

Find $\lim\limits_{x \to -0.5^-} \sqrt{\dfrac{x+2}{x+1}}$

Solution

$$\lim_{x \to -0.5^-} \sqrt{\frac{x+2}{x+1}} = \sqrt{\frac{-0.5+2}{-0.5+1}}$$

$$= \sqrt{\frac{1.5}{0.5}}$$

$$\boxed{=\sqrt{3}}$$

Exercise

Find $\lim\limits_{x \to 1^+} \sqrt{\dfrac{x-1}{x+2}}$

Solution

$$\lim_{x \to 1^+} \sqrt{\frac{x-1}{x+2}} = \sqrt{\frac{1-1}{1+2}}$$

$$\boxed{=0}$$

Exercise

Find $\lim\limits_{x\to -2^+}\left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right)$

Solution

$$\lim\limits_{x\to -2^+}\left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right)=\left(\frac{-2}{-2+1}\right)\left(\frac{2(-2)+5}{(-2)^2+(-2)}\right)$$

$$=\left(\frac{-2}{-1}\right)\left(\frac{1}{2}\right)$$

$$\boxed{=1}$$

Exercise

Find $\lim\limits_{x\to 0^+}\frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}$

Solution

$$\lim\limits_{x\to 0^+}\frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}=\frac{\sqrt{5}-\sqrt{5}}{0}=\frac{0}{0}$$

$$=\lim\limits_{x\to 0^+}\frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}\,\frac{\sqrt{x^2+4x+5}+\sqrt{5}}{\sqrt{x^2+4x+5}+\sqrt{5}}$$

$$=\lim\limits_{x\to 0^+}\frac{x^2+4x+5-5}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$=\lim\limits_{x\to 0^+}\frac{x^2+4x}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$=\lim\limits_{x\to 0^+}\frac{x(x+4)}{x\left(\sqrt{x^2+4x+5}+\sqrt{5}\right)}$$

$$=\lim\limits_{x\to 0^+}\frac{x+4}{\sqrt{x^2+4x+5}+\sqrt{5}}$$

$$=\frac{0+4}{\sqrt{0^2+4(0)+5}+\sqrt{5}}$$

$$=\frac{4}{\sqrt{5}+\sqrt{5}}$$

$$=\frac{4}{2\sqrt{5}}$$

$$\boxed{=\frac{2}{\sqrt{5}}}$$

Exercise

Find $\displaystyle\lim_{x\to -2^+}(x+3)\frac{|x+2|}{x+2}$

Solution

$$\lim_{x\to -2^+}(x+3)\frac{|x+2|}{x+2}=(x+3)\frac{|-2+2|}{-2+2}=\frac{0}{0}$$

Since $x\to -2^+ \quad\Rightarrow\quad x>-2 \Rightarrow \quad |x+2|=(x+2)$

$$\begin{aligned}\lim_{x\to -2^+}(x+3)\frac{|x+2|}{x+2}&=\lim_{x\to -2^+}(x+3)\frac{x+2}{x+2}\\&=\lim_{x\to -2^+}(x+3)\\&=-2+3\\&=1\end{aligned}$$

Exercise

Find $\displaystyle\lim_{x\to 1^+}\frac{\sqrt{2x}(x-1)}{|x-1|}$

Solution

$$\lim_{x\to 1^+}\frac{\sqrt{2x}(x-1)}{|x-1|}=\frac{\sqrt{2(1)}(1-1)}{|1-1|}=\frac{0}{0}$$

Since $x\to 1^+ \quad\Rightarrow\quad x>1 \Rightarrow \quad |x-1|=x-1$

$$\begin{aligned}\lim_{x\to 1^+}\frac{\sqrt{2x}(x-1)}{|x-1|}&=\lim_{x\to 1^+}\frac{\sqrt{2x}(x-1)}{x-1}\\&=\lim_{x\to 1^+}\sqrt{2x}\\&=\sqrt{2}\end{aligned}$$

Exercise

Find $\displaystyle\lim_{\theta\to 0}\frac{\sin\sqrt{2}.\theta}{\sqrt{2}.\theta}$

Solution

Let: $\sqrt{2}\theta=x$

$$\lim_{\theta\to 0}\frac{\sin\sqrt{2}.\theta}{\sqrt{2}.\theta}=\lim_{x\to 0}\frac{\sin x}{x}=1$$

Exercise

Find $\lim\limits_{x\to 0} \dfrac{\sin 3x}{4x}$

Solution

$$\lim_{x\to 0} \frac{\sin 3x}{4x} = \lim_{x\to 0} \frac{\sin 3x}{4x}\frac{3}{3}$$

$$= \frac{3}{4}\lim_{x\to 0} \frac{\sin 3x}{3x} \qquad \text{Let: } 3x = u$$

$$= \frac{3}{4}\lim_{u\to 0} \frac{\sin u}{u} \qquad \textit{By definition: } \lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\boxed{= \frac{3}{4}}$$

Exercise

Find $\lim\limits_{x\to 0^-} \dfrac{x}{\sin 3x}$

Solution

$$\lim_{x\to 0^-} \frac{x}{\sin 3x} = \lim_{x\to 0^-} \frac{x}{\sin 3x}\left(\frac{3}{3}\right)$$

$$= \frac{1}{3}\lim_{x\to 0^-} \frac{3x}{\sin 3x}$$

$$= \frac{1}{3}\lim_{x\to 0^-} \frac{1}{\frac{\sin 3x}{3x}} \qquad \textit{By definition: } \lim_{u\to 0} \frac{\sin u}{u} = 1$$

$$\boxed{= \frac{1}{3}}$$

Exercise

Find $\lim\limits_{x\to 0} \dfrac{\tan 2x}{x}$

Solution

$$\lim_{x\to 0} \frac{\tan 2x}{x} = \lim_{x\to 0} \frac{\frac{\sin 2x}{\cos 2x}}{x}$$

$$= \lim_{x\to 0}\left(\frac{\sin 2x}{x}\bullet\frac{1}{\cos 2x}\right)$$

$$= \lim_{x\to 0}\left(2\frac{\sin 2x}{2x}\right)\lim_{x\to 0}\left(\frac{1}{\cos 2x}\right)$$

$$= 2\bullet\frac{1}{\cos 0}$$

$$\boxed{= 2}$$

Exercise

Find $\lim\limits_{x\to 0} 6x^2(\cot x)(\csc 2x)$

Solution

$$\lim_{x\to 0} 6x^2(\cot x)(\csc 2x) = \lim_{x\to 0} 6x^2\left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin 2x}\right)$$

$$= \lim_{x\to 0} 3\cos x\left(\frac{x}{\sin x}\right)\left(\frac{2x}{\sin 2x}\right)$$

$$= 3\lim_{x\to 0}(\cos x)\bullet \lim_{x\to 0}\left(\frac{x}{\sin x}\right)\bullet \lim_{x\to 0}\left(\frac{2x}{\sin 2x}\right) \quad = 3\bullet 1\bullet 1\bullet 1$$

$$\boxed{=3}$$

Exercise

Find $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\sin 2\theta}$

Solution

$$\lim_{\theta\to 0}\frac{\sin\theta}{\sin 2\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\sin 2\theta}\,\frac{2\theta}{2\theta}$$

$$= \frac{1}{2}\lim_{\theta\to 0}\left(\frac{2\theta}{\sin 2\theta}\bullet\frac{\sin\theta}{\theta}\right) \quad = \frac{1}{2}\bullet 1\bullet 1$$

$$\boxed{=\frac{1}{2}}$$

Exercise

Find $\lim\limits_{h\to 0}\dfrac{\sin(\sin h)}{\sin h}$

Solution

Let: $\sin h = \theta$ $\qquad\qquad \theta = \sin h \xrightarrow[h\to 0]{} 0$

$$\lim_{h\to 0}\frac{\sin(\sin h)}{\sin h} = \lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}$$

$$\boxed{=1}$$

Exercise

Find $\lim\limits_{\theta\to 0}\dfrac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}$

Solution

$$\lim_{\theta\to 0}\frac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}=\lim_{\theta\to 0}\frac{\theta\frac{\cos 4\theta}{\sin 4\theta}}{\sin^2\theta\frac{\cos^2 2\theta}{\sin^2 2\theta}}$$

$$=\lim_{\theta\to 0}\ \theta\frac{\cos 4\theta}{2\sin 2\theta\cos 2\theta}\frac{\sin^2 2\theta}{\sin^2\theta\cos^2 2\theta}$$

$$=\lim_{\theta\to 0}\left(\frac{1}{2}\cdot\theta\cdot\cos 4\theta\bullet\frac{2\sin\theta\cos\theta}{\sin^2\theta}\cdot\frac{1}{\cos^3 2\theta}\right)$$

$$=\lim_{\theta\to 0}\left(\cos 4\theta\bullet\frac{\theta}{\sin\theta}\cdot\cos\theta\cdot\frac{1}{\cos^3 2\theta}\right)$$

$$=\lim_{\theta\to 0}(\cos 4\theta)\bullet\lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right)\bullet\lim_{\theta\to 0}\left(\frac{\cos\theta}{\cos^3 2\theta}\right)$$

$$=1\bullet 1\bullet 1$$

$$\underline{=1|}$$

Exercise

Find $\lim_{\theta\to\pi/4}\frac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta}$

Solution

$$\lim_{\theta\to\pi/4}\frac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta}=\frac{\frac{1}{2}-\frac{1}{2}}{\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}}=\frac{0}{0}$$

$$=\lim_{\theta\to\pi/4}\frac{(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)}{\sin\theta-\cos\theta}$$

$$=\lim_{\theta\to\pi/4}(\sin\theta+\cos\theta)$$

$$=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}$$

$$\underline{=\sqrt{2}|}$$

Exercise

Find $\lim_{x\to\pi/2}\frac{\frac{1}{\sqrt{\sin x}}-1}{x+\frac{\pi}{2}}$

Solution

$$\lim_{x\to\pi/2}\frac{\frac{1}{\sqrt{\sin x}}-1}{x+\frac{\pi}{2}}=\frac{1-1}{\frac{\pi}{2}+\frac{\pi}{2}}\ \underline{=0|}$$

Exercise

Find $\lim\limits_{x\to 1} \dfrac{x^3-7x^2+12x}{4-x}$

Solution

$$\lim_{x\to 1} \frac{x^3-7x^2+12x}{4-x} = \frac{1-7+12}{4-1} \boxed{=2}$$

Exercise

Find $\lim\limits_{x\to 4} \dfrac{x^3-7x^2+12x}{4-x}$

Solution

$$\lim_{x\to 4} \frac{x^3-7x^2+12x}{4-x} = \frac{64-112+48}{4-4} = \frac{0}{0}$$

$$= \lim_{x\to 4} \frac{x(x-3)(x-4)}{4-x}$$

$$= \lim_{x\to 4} -x(x-3)$$

$$\boxed{=-4}$$

Exercise

Find $\lim\limits_{x\to 1} \dfrac{1-x^2}{x^2-8x+7}$

Solution

$$\lim_{x\to 1} \frac{1-x^2}{x^2-8x+7} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{(1-x)(1+x)}{(x-1)(x-7)}$$

$$= \lim_{x\to 1} \frac{1+x}{x-7}$$

$$\boxed{=-\frac{1}{3}}$$

Exercise

Find $\lim\limits_{x\to 3} \dfrac{\sqrt{3x+16}-5}{x-3}$

Solution

$$\lim_{x\to 3} \frac{\sqrt{3x+16}-5}{x-3} = \frac{\sqrt{9+16}-5}{3-3} = \frac{5-5}{0} = \frac{0}{0}$$

$$= \lim_{x \to 3} \frac{\sqrt{3x+16}-5}{x-3} \frac{\sqrt{3x+16}+5}{\sqrt{3x+16}+5}$$

$$= \lim_{x \to 3} \frac{3x+16-25}{(x-3)\left(\sqrt{3x+16}+5\right)}$$

$$= \lim_{x \to 3} \frac{3(x-3)}{(x-3)\left(\sqrt{3x+16}+5\right)}$$

$$= \lim_{x \to 3} \frac{3}{\sqrt{3x+16}+5}$$

$$= \frac{3}{5+5}$$

$$\boxed{= \frac{3}{10}}$$

Exercise

Find $\lim\limits_{x \to 3} \frac{1}{x-3}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{2}\right)$

Solution

$$\lim_{x \to 3} \frac{1}{x-3}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{2}\right) = \frac{1}{0}\left(\frac{1}{2} - \frac{1}{2}\right) = \frac{0}{0}$$

$$= \lim_{x \to 3} \frac{1}{x-3}\left(\frac{2-\sqrt{x+1}}{\sqrt{x+1}}\right)\left(\frac{2+\sqrt{x+1}}{2+\sqrt{x+1}}\right)$$

$$= \lim_{x \to 3} \frac{1}{x-3}\left(\frac{4-x-1}{2\sqrt{x+1}+x+1}\right)$$

$$= \lim_{x \to 3} \frac{x-3}{x-3}\left(\frac{-1}{2\sqrt{x+1}+x+1}\right)$$

$$= \lim_{x \to 3} \frac{-1}{2\sqrt{x+1}+x+1}$$

$$\boxed{= -\frac{1}{8}}$$

Exercise

Find $\lim\limits_{x \to 1/3} \frac{x-\frac{1}{3}}{(3x-1)^2}$

Solution

$$\lim_{x \to 1/3} \frac{x-\frac{1}{3}}{(3x-1)^2} = \frac{\frac{1}{3}-\frac{1}{3}}{\left(3\frac{1}{3}-1\right)^2} = \frac{0}{0}$$

$$= \lim_{x \to 1/3} \frac{x - \frac{1}{3}}{9\left(x - \frac{1}{3}\right)^2}$$

$$= \lim_{x \to 1/3} \frac{1}{9\left(x - \frac{1}{3}\right)} = \frac{1}{0}$$

$$\underline{= \infty \rfloor}$$

Exercise

Find $\lim_{x \to 3} \frac{x^4 - 81}{x - 3}$

Solution

$$\lim_{x \to 3} \frac{x^4 - 81}{x - 3} = \frac{81 - 81}{3 - 3} = \frac{0}{0}$$

$$= \lim_{x \to 3} \frac{(x-3)(x+3)\left(x^2+9\right)}{x-3} \qquad a^4 - b^4 = \left(a^2 - b^2\right)\left(a^2 + b^2\right) = (a-b)(a+b)\left(a^2+b^2\right)$$

$$= \lim_{x \to 3} (x+3)\left(x^2+9\right) = 6(18)$$

$$\underline{= 108 \rfloor}$$

Exercise

Find $\lim_{x \to 1} \frac{x^5 - 1}{x - 1}$

Solution

$$\lim_{x \to 1} \frac{x^5 - 1}{x - 1} = \frac{1-1}{1-1} = \frac{0}{0} \qquad \left(a^5 - b^5\right) = (a-b)\left(a^4 + a^3b + a^2b^2 + ab^3 + b^4\right)$$

$$= \lim_{x \to 1} \frac{(x-1)\left(x^4 + x^3 + x^2 + x + 1\right)}{x-1}$$

$$= \lim_{x \to 1} \left(x^4 + x^3 + x^2 + x + 1\right)$$

$$\underline{= 5 \rfloor}$$

Exercise

Find $\lim_{x \to 81} \frac{\sqrt[4]{x} - 3}{x - 81}$

Solution

$$\lim_{x\to 81}\frac{\sqrt[4]{x}-3}{x-81}=\frac{3-3}{81-81}=\frac{0}{0}$$

$$=\lim_{x\to 81}\frac{\sqrt[4]{x}-3}{\left(\sqrt{x}+9\right)\left(\sqrt{x}-9\right)}$$

$$=\lim_{x\to 81}\frac{\sqrt[4]{x}-3}{\left(\sqrt{x}+9\right)\left(\sqrt[4]{x}+3\right)\left(\sqrt[4]{x}-3\right)}$$

$$=\lim_{x\to 81}\frac{1}{\left(\sqrt{x}+9\right)\left(\sqrt[4]{x}+3\right)}$$

$$=\frac{1}{(18)(6)}$$

$$\boxed{=\frac{1}{108}}$$

Exercise

Find the limit: $\displaystyle\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}$

Solution

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}=\frac{0}{0}$$

$$=\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\left(\sqrt[3]{x}\right)^3-1^3}$$

$$=\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\left(\sqrt[3]{x}-1\right)\left(x^{2/3}+\sqrt[3]{x}+1\right)}$$

$$=\lim_{x\to 1}\frac{1}{x^{2/3}+\sqrt[3]{x}+1}$$

$$\boxed{=\frac{1}{3}}$$

Exercise

Find the limit: $\displaystyle\lim_{x\to 2}\frac{x^5-32}{x-2}$

Solution

$$\lim_{x\to 2}\frac{x^5-32}{x-2}=\frac{2^5-32}{2-2}=\frac{0}{0}$$

$$\begin{array}{r|rrrrrr} 2 & 1 & 0 & 0 & 0 & 0 & -32 \\ & & 2 & 4 & 8 & 16 & 32 \\ \hline & 1 & 2 & 4 & 8 & 16 & 0 \end{array}$$

$$=\lim_{x\to 2}\frac{(x-2)\left(x^4+2x^3+4x^2+8x+16\right)}{x-2}$$

$$\begin{aligned}
&= \lim_{x\to 2}\left(x^4+2x^3+4x^2+8x+16\right)\\
&= 16+16+16+16+16\\
&= 80
\end{aligned}$$

Exercise

Find the limit: $\displaystyle\lim_{x\to 1}\frac{x^6-1}{x-1}$

Solution

$$\begin{array}{r|rrrrrrr}
1 & 1 & 0 & 0 & 0 & 0 & 0 & -1\\
 & & 1 & 1 & 1 & 1 & 1 & 1\\
\hline
 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}$$

$$\begin{aligned}
\lim_{x\to 1}\frac{x^6-1}{x-1} &= \frac{0}{0}\\
&= \lim_{x\to 1}\frac{(x-1)\left(x^5+x^4+x^3+x^2+x+1\right)}{x-1}\\
&= \lim_{x\to 1}\left(x^5+x^4+x^3+x^2+x+1\right)\\
&= 6
\end{aligned}$$

Exercise

Find the limit: $\displaystyle\lim_{x\to -1}\frac{x^7+1}{x+1}$

Solution

$$\begin{array}{r|rrrrrrrr}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
 & & -1 & 1 & -1 & 1 & -1 & 1 & -1\\
\hline
 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 0
\end{array}$$

$$\begin{aligned}
\lim_{x\to -1}\frac{x^7+1}{x+1} &= \frac{-1+1}{-1+1} = \frac{0}{0}\\
&= \lim_{x\to 1}\frac{(x+1)\left(x^6-x^5+x^4-x^3+x^2-x+1\right)}{x+1}\\
&= \lim_{x\to 1}\left(x^6-x^5+x^4-x^3+x^2-x+1\right)\\
&= 1
\end{aligned}$$

Exercise

Find the limit: $\displaystyle\lim_{x\to a}\frac{x^5-a^5}{x-a}$

Solution

$$\begin{array}{r|rrrrrr}
a & 1 & 0 & 0 & 0 & 0 & -a^5\\
 & & a & a^2 & a^3 & a^4 & a^5\\
\hline
 & 1 & a & a^2 & a^3 & a^4 & 0
\end{array}$$

$$\lim_{x\to a}\frac{x^5-a^5}{x-a} = \frac{a^5-a^5}{a-a} = \frac{0}{0}$$

$$= \lim_{x\to a} \frac{(x-a)\left(x^4+ax^3+a^2x^2+a^3x+a^4\right)}{x-a}$$

$$= \lim_{x\to a} \left(x^4+ax^3+a^2x^2+a^3x+a^4\right)$$

$$= a^4+a^4+a^4+a^4+a^4$$

$$\boxed{= 5a^4}$$

Exercise

Find the limit: $\lim\limits_{x\to a} \dfrac{x^n-a^n}{x-a} \quad n\in\mathbb{Z}^+$

Solution

$$\begin{array}{c|ccccccc} a & 1 & 0 & 0 & 0 & \cdots & 0 & -a^n \\ & & a & a^2 & a^3 & \cdots & a^{n-1} & a^n \\ \hline & 1 & a & a^2 & a^3 & \cdots & a^{n-1} & 0 \end{array}$$

$$\lim_{x\to a} \frac{x^n-a^n}{x-a} = \frac{a^n-a^n}{a-a} = \frac{0}{0}$$

$$= \lim_{x\to a} \frac{(x-a)\left(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots+a^{n-2}x+a^{n-1}\right)}{x-a}$$

$$= \lim_{x\to a} \left(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots+a^{n-2}x+a^{n-1}\right)$$

$$= a^{n-1}+a^{n-1}+a^{n-1}+\ldots+a^{n-1}+a^{n-1}$$

$$\boxed{= na^{n-1}}$$

Exercise

Find the limit: $\lim\limits_{h\to 0} \dfrac{100}{(10h-1)^{11}+2}$

Solution

$$\lim_{h\to 0} \frac{100}{(10h-1)^{11}+2} = \frac{100}{(-1)^{11}+2}$$

$$= \frac{100}{-1+2}$$

$$\boxed{= 100}$$

Exercise

Find the limit: $\lim\limits_{h\to 0} \dfrac{(5+h)^2-25}{h}$

Solution

$$\lim_{h\to 0}\frac{(5+h)^2-25}{h}=\frac{5^2-25}{0}=\frac{0}{0}$$

$$\begin{aligned}
&=\lim_{h\to 0}\frac{((5+h)-5)((5+h)+5)}{h}\\
&=\lim_{h\to 0}\frac{h(h+10)}{h}\\
&=\lim_{h\to 0}\ (h+10)\\
&\boxed{=10}
\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to 3}\dfrac{\frac{1}{x^2+2x}-\frac{1}{15}}{x-3}$

Solution

$$\lim_{x\to 3}\frac{\frac{1}{x^2+2x}-\frac{1}{15}}{x-3}=\frac{\frac{1}{15}-\frac{1}{15}}{0}=\frac{0}{0}$$

$$\begin{aligned}
&=\lim_{x\to 3}\frac{1}{x-3}\left(\frac{1}{x(x+2)}-\frac{1}{15}\right)\\
&=\lim_{x\to 3}\frac{1}{x-3}\left(\frac{15-x^2-2x}{15x(x+2)}\right)\\
&=\lim_{x\to 3}\frac{-(x-3)(x+5)}{15x(x+2)(x-3)}\\
&=\lim_{x\to 3}\frac{-(x+5)}{15x(x+2)}\\
&=-\frac{8}{15(3)(5)}\\
&\boxed{=-\frac{8}{225}}
\end{aligned}$$

Exercise

Find the limit: $\lim\limits_{x\to 1}\dfrac{\sqrt{10x-9}-1}{x-1}$

Solution

$$\lim_{x\to 1}\frac{\sqrt{10x-9}-1}{x-1}=\frac{1-1}{0}=\frac{0}{0}$$

$$= \lim_{x\to 1} \frac{\sqrt{10x-9}-1}{x-1} \cdot \frac{\sqrt{10x-9}+1}{\sqrt{10x-9}+1}$$

$$= \lim_{x\to 1} \frac{10x-9-1}{(x-1)\left(\sqrt{10x-9}+1\right)}$$

$$= \lim_{x\to 1} \frac{10(x-1)}{(x-1)\left(\sqrt{10x-9}+1\right)}$$

$$= \lim_{x\to 1} \frac{10}{\sqrt{10x-9}+1}$$

$$= \frac{10}{2}$$

$$\boxed{= 5}$$

Exercise

Find the limit: $\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2x}\right)$

Solution

$$\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2x}\right) = \frac{1}{0} - \frac{2}{0} = \infty - \infty$$

$$= \lim_{x\to 2} \left(\frac{1}{x-2} - \frac{2}{x(x-2)}\right)$$

$$= \lim_{x\to 2} \frac{x-2}{x(x-2)}$$

$$= \lim_{x\to 2} \frac{1}{x}$$

$$\boxed{= \frac{1}{2}}$$

Exercise

Find the limit: $\lim_{x\to c} \frac{x^2-2cx+c^2}{x-c}$

Solution

$$\lim_{x\to c} \frac{x^2-2cx+c^2}{x-c} = \frac{c^2-2c^2+c^2}{0} = \frac{0}{0}$$

$$= \lim_{x\to c} \frac{(x-c)^2}{x-c}$$

$$= \lim_{x\to c} (x-c)$$

$$\boxed{= 0}$$

Exercise

Find the limit: $\lim\limits_{x\to -c} \dfrac{x^2+5cx+4c^2}{x^2+cx}$

Solution

$$\lim_{x\to -c} \frac{x^2+5cx+4c^2}{x^2+cx} = \frac{c^2-5c^2+4c^2}{c^2-c^2} = \frac{0}{0}$$

$$= \lim_{x\to -c} \frac{(x+c)(x+4c)}{x(x+c)}$$

$$= \lim_{x\to -c} \frac{x+4c}{x}$$

$$= \frac{-c+4c}{-c}$$

$$= \frac{3c}{-c}$$

$$\boxed{= -3}$$

Exercise

Find the limit: $\lim\limits_{x\to 16} \dfrac{\sqrt[4]{x}-2}{x-16}$

Solution

$$\lim_{x\to 16} \frac{\sqrt[4]{x}-2}{x-16} = \frac{\sqrt[4]{16}-2}{16-16} = \frac{2-2}{0} = \frac{0}{0}$$

$$\lim_{x\to 16} \frac{\sqrt[4]{x}-2}{\left(\sqrt[4]{x}\right)^4-2^4}$$

$$a^4-b^4=\left(a^2+b^2\right)(a-b)(a+b)$$

$$= \lim_{x\to 16} \frac{\sqrt[4]{x}-2}{\left(\sqrt{x}+2^2\right)\left(\sqrt[4]{x}+2\right)\left(\sqrt[4]{x}-2\right)}$$

$$= \lim_{x\to 16} \frac{1}{\left(\sqrt{x}+4\right)\left(\sqrt[4]{x}+2\right)}$$

$$= \frac{1}{\left(\sqrt{16}+4\right)\left(\sqrt[4]{16}+2\right)}$$

$$= \frac{1}{(4+4)(2+2)}$$

$$= \frac{1}{(8)(4)}$$

$$\boxed{= \frac{1}{32}}$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$

Solution

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{\sqrt{x}-1}$$

$$= \lim_{x\to 1} \left(\sqrt{x}+1\right)$$

$$= 2$$

Exercise

Find the limit: $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{4x+5}-3}$

Solution

$$\lim_{x\to 1} \frac{x-1}{\sqrt{4x+5}-3} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{x-1}{\sqrt{4x+5}-3} \bullet \frac{\sqrt{4x+5}+3}{\sqrt{4x+5}+3}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{4x+5}+3\right)}{4x+5-9}$$

$$= \lim_{x\to 1} \frac{(x-1)\left(\sqrt{4x+5}+3\right)}{4(x-1)}$$

$$= \frac{1}{5} \lim_{x\to 1} \left(\sqrt{4x+5}+3\right)$$

$$= \frac{6}{5}$$

Exercise

Find the limit: $\lim\limits_{x\to 4} \dfrac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}$

Solution

$$\lim_{x\to 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}} = \frac{0}{3-3} = \frac{0}{0}$$

$$= \lim_{x\to 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}} \bullet \frac{3+\sqrt{x+5}}{3+\sqrt{x+5}}$$

$$=3\lim_{x\to 4}\frac{(x-4)\left(3+\sqrt{x+5}\right)\sqrt{x+5}}{9-(x+5)}$$

$$=3\lim_{x\to 4}\frac{(x-4)\left(3+\sqrt{x+5}\right)\sqrt{x+5}}{4-x}$$

$$=-3\lim_{x\to 4}\left(3+\sqrt{x+5}\right)\sqrt{x+5}$$

$$=-3\,(6)(3)$$

$$\boxed{=-54}$$

Exercise

Find the limit: $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{ax+1}-1}\quad (a\neq 0)$

Solution

$$\lim_{x\to 0}\frac{x}{\sqrt{ax+1}-1}=\frac{0}{0}$$

$$=\lim_{x\to 0}\frac{x}{\sqrt{ax+1}-1}\bullet\frac{\sqrt{ax+1}+1}{\sqrt{ax+1}+1}$$

$$=\lim_{x\to 0}\frac{x\left(\sqrt{ax+1}+1\right)}{ax+1-1}$$

$$=\lim_{x\to 0}\frac{x\left(\sqrt{ax+1}+1\right)}{ax}$$

$$=\frac{1}{a}\lim_{x\to 0}\left(\sqrt{ax+1}+1\right)$$

$$\boxed{=\frac{2}{a}}$$

Exercise

Find the limit: $\lim\limits_{x\to\pi}\dfrac{\cos^2 x+3\cos x+2}{\cos x+1}$

Solution

$$\lim_{x\to\pi}\frac{\cos^2 x+3\cos x+2}{\cos x+1}=\frac{1-3+2}{-1+1}=\frac{0}{0}$$

$$=\lim_{x\to\pi}\frac{(\cos x+1)(\cos x+2)}{\cos x+1}$$

$$=\lim_{x\to\pi}(\cos x+2)$$

$$=-1+2$$

$$\boxed{=1}$$

Exercise

Find the limit: $\lim\limits_{x \to \frac{3\pi}{2}} \dfrac{\sin^2 x + 6\sin x + 5}{\sin^2 x - 1}$

Solution

$$
\begin{aligned}
\lim_{x \to \frac{3\pi}{2}} \frac{\sin^2 x + 6\sin x + 5}{\sin^2 x - 1} &= \frac{1 - 6 + 5}{1 - 1} = \frac{0}{0} \\
&= \lim_{x \to \frac{3\pi}{2}} \frac{(\sin x + 1)(\sin x + 5)}{(\sin x - 1)(\sin x + 1)} \\
&= \lim_{x \to \frac{3\pi}{2}} \frac{\sin x + 5}{\sin x - 1} \\
&= \frac{-1 + 5}{-1 - 1} \\
&\boxed{= -2}
\end{aligned}
$$

Exercise

Find the limit: $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\sin x - 1}{\sqrt{\sin x} - 1}$

Solution

$$
\begin{aligned}
\lim_{x \to \frac{\pi}{2}} \frac{\sin x - 1}{\sqrt{\sin x} - 1} &= \frac{1 - 1}{1 - 1} = \frac{0}{0} \\
&= \lim_{x \to \frac{\pi}{2}} \frac{\left(\sqrt{\sin x} - 1\right)\left(\sqrt{\sin x} + 1\right)}{\sqrt{\sin x} - 1} \\
&= \lim_{x \to \frac{\pi}{2}} \left(\sqrt{\sin x} + 1\right) \\
&\boxed{= 2}
\end{aligned}
$$

Exercise

Find the limit: $\lim\limits_{x \to 0} \dfrac{\frac{1}{2 + \sin x} - \frac{1}{2}}{\sin x}$

Solution

$$
\begin{aligned}
\lim_{x \to 0} \frac{\frac{1}{2 + \sin x} - \frac{1}{2}}{\sin x} &= \frac{\frac{1}{2} - \frac{1}{2}}{0} = \frac{0}{0} \\
&= \lim_{x \to 0} \frac{1}{\sin x} \bullet \frac{2 - \sin x - 2}{2(2 + \sin x)} \\
&= \frac{1}{2} \lim_{x \to 0} \frac{1}{\sin x} \bullet \frac{-\sin x}{(2 + \sin x)}
\end{aligned}
$$

$$= -\frac{1}{2} \lim_{x\to 0} \frac{1}{2+\sin x}$$

$$= -\frac{1}{2}\left(\frac{1}{2}\right)$$

$$\boxed{= -\frac{1}{4}}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{e^{2x}-1}{e^{x}-1}$

Solution

$$\lim_{x\to 0} \frac{e^{2x}-1}{e^{x}-1} = \frac{1-1}{1-1} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\left(e^{x}-1\right)\left(e^{x}+1\right)}{e^{x}-1}$$

$$= \lim_{x\to 0} \left(e^{x}+1\right)$$

$$\boxed{= 2}$$

Exercise

Find the limit: $\lim\limits_{x\to \frac{\pi}{4}} \csc x$

Solution

$$\lim_{x\to \frac{\pi}{4}} \csc x = \csc\frac{\pi}{4}$$

$$= \frac{1}{\cos\frac{\pi}{4}}$$

$$\boxed{= \sqrt{2}}$$

Exercise

Find the limit: $\lim\limits_{x\to 4} \dfrac{x-5}{\left(x^2-10x+24\right)^2}$

Solution

$$\lim_{x\to 4} \frac{x-5}{\left(x^2-10x+24\right)^2} = \frac{-1}{(16-41+24)^2}$$

$$\boxed{= -1}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{\sin^2 x}$

Solution

$$\lim\limits_{x\to 0} \frac{\cos x - 1}{\sin^2 x} = \frac{0}{0}$$

$$= \lim\limits_{x\to 0} \frac{\cos x - 1}{(1-\cos x)(1+\cos x)}$$

$$= -\lim\limits_{x\to 0} \frac{1}{1+\cos x}$$

$$\boxed{= -\frac{1}{2}}$$

Exercise

Find the limit: $\lim\limits_{x\to 0} \dfrac{1-\cos^2 x}{\sin x}$

Solution

$$\lim\limits_{x\to 0} \frac{1-\cos^2 x}{\sin x} = \frac{0}{0}$$

$$= \lim\limits_{x\to 0} \frac{\sin^2 x}{\sin x}$$

$$= \lim\limits_{x\to 0} \sin x$$

$$\boxed{= 0}$$

Exercise

Find $\lim\limits_{x\to 0} \dfrac{x^3 - 5x^2}{x^2}$

Solution

$$\lim\limits_{x\to 0} \frac{x^3 - 5x^2}{x^2} = \frac{0}{0}$$

$$= \lim\limits_{x\to 0} (x-5)$$

$$\boxed{= -5}$$

Exercise

Find $\lim\limits_{x\to 5} \dfrac{4x^2-100}{x-5}$

Solution

$$\lim_{x\to 5} \frac{4x^2-100}{x-5} = \frac{0}{0}$$

$$= \lim_{x\to 5} \frac{4(x-5)(x+5)}{x-5}$$

$$= \lim_{x\to 5} 4(x+5)$$

$$\boxed{= 40}$$

Exercise

For the function $f(t)$ graphed, find the following limits or explain why they do not exist.

$a)\ \lim\limits_{t\to -2} f(t) \quad b)\ \lim\limits_{t\to -1} f(t) \quad c)\ \lim\limits_{t\to 0} f(t) \quad d)\ \lim\limits_{t\to -0.5} f(t)$

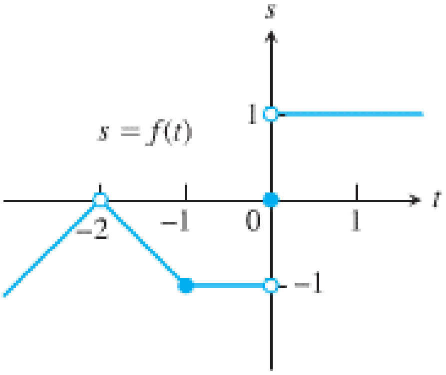
Solution

a) $\lim\limits_{t\to -2} f(t) = 0$

b) $\lim\limits_{t\to -1} f(t) = -1$

c) $\lim\limits_{t\to 0} f(t) = \text{doesn't exist}$

d) $\lim\limits_{t\to -.5} f(t) = -1$



Exercise

Suppose $\lim\limits_{x\to c} f(x) = 5$ *and* $\lim\limits_{x\to c} g(x) = -2$. Find

$a)\ \lim\limits_{x\to c} f(x)g(x)$

$b)\ \lim\limits_{x\to c} 2f(x)g(x)$

$c)\ \lim\limits_{x\to c} \left(f(x)+3g(x)\right)$

$d)\ \lim\limits_{x\to c} \dfrac{f(x)}{f(x)-g(x)}$

Solution

a) $\lim\limits_{x\to c} f(x)g(x) = \lim\limits_{x\to c} f(x) \bullet \lim\limits_{x\to c} g(x) = (5)(-2)\ \boxed{= -10}$

b) $\lim\limits_{x\to c} 2f(x)g(x) = 2\lim\limits_{x\to c} f(x) \bullet \lim\limits_{x\to c} g(x) = 2(-10)\ \boxed{= -20}$

c) $$\lim_{x\to c}\left(f(x)+3g(x)\right)=\lim_{x\to c} f(x)+3\lim_{x\to c} g(x)$$
$$=5+3(-2)$$
$$\underline{=-1|}$$

d) $$\lim_{x\to c}\frac{f(x)}{f(x)-g(x)}=\frac{\lim_{x\to c} f(x)}{\lim_{x\to c} f(x)-\lim_{x\to c} g(x)}$$
$$=\frac{5}{5-(-2)}$$
$$\underline{=\frac{5}{7}|}$$

Exercise

Explain why the limits do not exist for $\lim_{x\to 0}\frac{x}{|x|}$

Solution

$$\lim_{x\to 0}\frac{x}{|x|}=\frac{0}{0}$$

$$\lim_{x\to 0^-}\frac{x}{|x|}=\frac{-x}{x}=-1$$

Doesn't exist

$$\lim_{x\to 0^+}\frac{x}{|x|}=\frac{x}{x}=1$$

Exercise

Evaluate the limit using the form $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ for $f(x)=x^2, \quad x=1$

Solution

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$
$$=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$
$$=\lim_{h\to 0}\left(\frac{2xh}{h}+\frac{h^2}{h}\right)$$
$$=\lim_{h\to 0}(2x+h)$$
$$\underline{=2x|}$$

Exercise

Evaluate the limit using the form $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ for $f(x)=\sqrt{3x+1}, \quad x=0$

Solution

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{3x+3h+1}-\sqrt{3x+1}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{3x+3h+1}-\sqrt{3x+1}}{h}\bullet\frac{\sqrt{3x+3h+1}+\sqrt{3x+1}}{\sqrt{3x+3h+1}+\sqrt{3x+1}}$$

$$=\lim_{h\to 0}\frac{3x+3h+1-(3x+1)}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}$$

$$=\lim_{h\to 0}\frac{3x+3h+1-3x-1}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}$$

$$=\lim_{h\to 0}\frac{3h}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}$$

$$=\lim_{h\to 0}\frac{3}{\sqrt{3x+3h+1}+\sqrt{3x+1}}$$

$$=\frac{3}{\sqrt{3(0)+1}+\sqrt{3(0)+1}} \qquad \text{Given}: x=0$$

$$\boxed{=\frac{3}{2}}$$

Exercise

If $\lim\limits_{x\to 4}\dfrac{f(x)-5}{x-2}=1$, find $\lim\limits_{x\to 4}f(x)$

Solution

$$\lim_{x\to 4}\frac{f(x)-5}{x-2}=1$$

$$\frac{\lim\limits_{x\to 4}f(x)-5}{4-2}=1$$

$$\frac{\lim\limits_{x\to 4}f(x)-5}{2}=1 \qquad \textit{Multiply both sides by 2}$$

$$\lim_{x\to 4}f(x)-5=2 \qquad \textit{Add 5 on both sides}$$

$$\lim_{x\to 4}f(x)=7$$

Exercise

If $\lim\limits_{x\to 0}\dfrac{f(x)}{x^2}=1$, find $\lim\limits_{x\to 0} f(x)$ *and* $\lim\limits_{x\to 0}\dfrac{f(x)}{x}$

Solution

$$\lim_{x\to 0}\frac{f(x)}{x^2}=1$$

$$\frac{\lim\limits_{x\to 0} f(x)}{\lim\limits_{x\to 0} x^2}=1$$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0} x^2 \underline{=0}$$

$$\lim_{x\to 0}\frac{f(x)}{x}=\lim_{x\to 0}\left(\frac{f(x)}{x^2}\bullet x\right)$$

$$=\lim_{x\to 0}\frac{f(x)}{x^2}\bullet\lim_{x\to 0} x$$

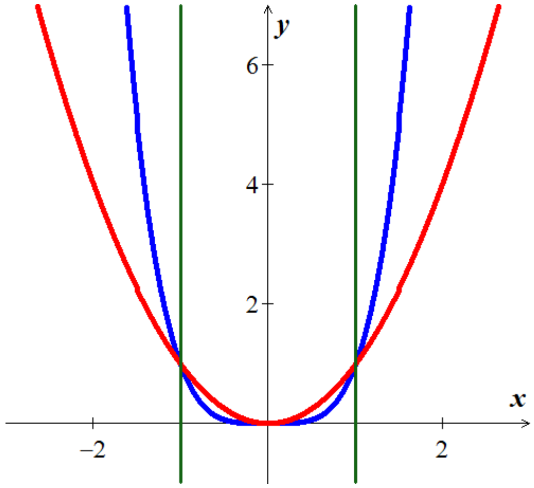
$$=1\bullet 0$$

$$\underline{=0}$$

Exercise

If $x^4\le f(x)\le x^2;\quad -1\le x\le 1$ and $x^2\le f(x)\le x^4;\quad x<-1$ *and* $x>1$. At what points c do you automatically know $\lim\limits_{x\to c} f(x)$? What can you say about the value of the limits at these points?

Solution



$$\lim_{x\to c} x^4=\lim_{x\to c} x^2 \quad\Rightarrow\quad c^4=c^2$$

$$c^4-c^2=0$$

$$c^2\left(c^2-1\right)=0$$

$$c^2=0 \qquad\qquad c^2-1=0$$

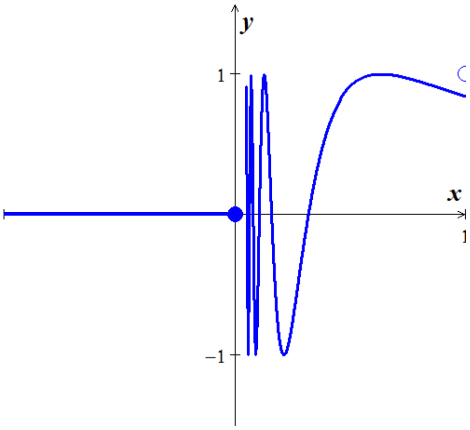
$$\boxed{c=0} \qquad\qquad \boxed{c=\pm 1}$$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0} x^2 \underline{=0}$$

$$\lim_{x\to -1} f(x)=\lim_{x\to 1} f(x) \underline{=1}$$

Exercise

Let $f(x)=\begin{cases} 0, & x\le 0 \\ \sin\frac{1}{x}, & x>0 \end{cases}$



a) Does $\lim\limits_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

Solution

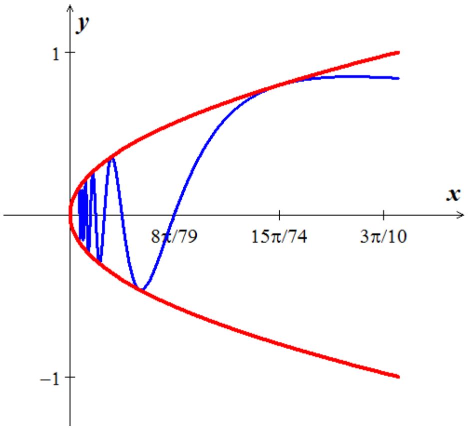
a) $\lim\limits_{x\to 0^+} f(x)$ doesn't exist, since $\sin\left(\frac{1}{x}\right)$ doesn't approach any single value as $x\to 0$

b) $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-} 0=0$

c) $\lim\limits_{x\to 0} f(x)$ doesn't exist, since $\lim\limits_{x\to 0^+} f(x)$ doesn't exist

Exercise

Let $g(x)=\sqrt{x}\,\sin\frac{1}{x}$



a) Does $\lim\limits_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim\limits_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim\limits_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

Solution

a) $\lim\limits_{x\to 0^+} g(x)$ exists, by the sandwich theorem $-\sqrt{x}\le g(x)\le\sqrt{x}$. *for* $x>0$

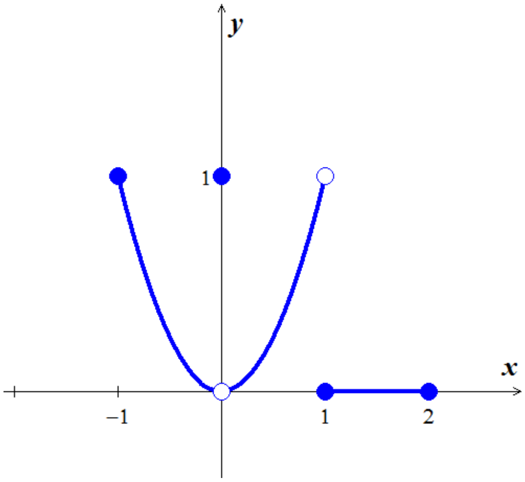
b) $\lim\limits_{x\to 0^-} g(x)$ doesn't exist, since $\sqrt{x}$ is not defined for $x<0$

c) $\lim\limits_{x\to 0} g(x)$ doesn't exist, since $\lim\limits_{x\to 0^-} g(x)$ doesn't exist.

Exercise

Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

Solution



a) $\lim\limits_{x\to -1^{+}} f(x) = 1$ ***True***

b) $\lim\limits_{x\to 0^{-}} f(x) = 0$ ***True***

c) $\lim\limits_{x\to 0^{-}} f(x) = 1$ ***False***

d) $\lim\limits_{x\to 0^{-}} f(x) = \lim\limits_{x\to 0^{+}} f(x)$ ***True***

e) $\lim\limits_{x\to 0} f(x)$ *exists* ***True***

f) $\lim\limits_{x\to 0} f(x) = 0$ ***True***

g) $\lim\limits_{x\to 0} f(x) = 1$ ***False***

h) $\lim\limits_{x\to 1} f(x) = 1$ ***False***

i) $\lim\limits_{x\to 1} f(x) = 0$ ***False***

j) $\lim\limits_{x\to 2^{-}} f(x) = 2$ ***False***

k) $\lim\limits_{x\to -1^{-}} f(x) = 0$ *does not exist* ***True***

l) $\lim\limits_{x\to 2^{+}} f(x) = 0$ ***False***

Solution Section 1.3 – Infinite Limits

Exercise

Find $\lim\limits_{x\to 5}\dfrac{x-7}{x(x-5)^2}$

Solution

$$\lim\limits_{x\to 5}\frac{x-7}{x(x-5)^2}=\frac{-2}{0}$$

$$\underline{=\infty|}$$

Exercise

Find $\lim\limits_{x\to -5^+}\dfrac{x-5}{x+5}$

Solution

$$\lim\limits_{x\to -5^+}\frac{x-5}{x+5}=\frac{-10}{0^+}$$

$$\underline{=-\infty|}$$

Exercise

Find $\lim\limits_{x\to 3^-}\dfrac{x-4}{x^2-3x}$

Solution

$$\lim\limits_{x\to 3^-}\frac{x-4}{x^2-3x}=\frac{-1}{0^-}$$

$$\underline{=\infty|}$$

Exercise

Find $\lim\limits_{x\to 0^+}\dfrac{1}{3x}$

Solution

$$\lim\limits_{x\to 0^+}\frac{1}{3x}=\frac{1}{0^+}$$

$$\underline{=\infty|}$$

Exercise

Find $\lim\limits_{x\to -5^-} \dfrac{3x}{2x+10}$

Solution

$$\lim_{x\to -5^-} \frac{3x}{2x+10} = \lim_{x\to -5^-} \frac{3}{2+\frac{10}{x}}$$

$$\underline{=\infty}\rfloor$$

Exercise

Find $\lim\limits_{x\to 0} \dfrac{1}{x^{2/3}}$

Solution

$$\lim_{x\to 0} \frac{1}{x^{2/3}} = \lim_{x\to 0} \frac{1}{\left(x^{1/3}\right)^2}$$

$$\underline{=\infty}\rfloor$$

Exercise

Find $\lim\limits_{x\to 0^-} \dfrac{1}{3x^{1/3}}$

Solution

$$\lim_{x\to 0^-} \frac{1}{3x^{1/3}} = \frac{1}{0^-}$$

$$\underline{=-\infty}\rfloor$$

Exercise

Find $\lim\limits_{x\to\left(-\frac{\pi}{2}\right)^+} \sec x$

Solution

$$\lim_{x\to\left(-\frac{\pi}{2}\right)^+} \sec x \; \underline{=\infty}\rfloor$$

Exercise

Find $\lim\limits_{\theta\to 0^-} (1+\csc\theta)$

Solution

$$\lim_{\theta\to 0^-}\left(1+\csc\theta\right)=\lim_{\theta\to 0^-}\left(1+\frac{1}{\sin\theta}\right)$$

$$\boxed{=-\infty}$$

Exercise

Find $\displaystyle\lim_{\theta\to 0^+}\csc\theta$

Solution

$$\lim_{\theta\to 0^+}\csc\theta=\lim_{\theta\to 0^+}\frac{1}{\sin\theta}$$

As $\theta\to 0^+$ $\sin\theta>0$

$$\boxed{=+\infty}$$

Exercise

Find $\displaystyle\lim_{x\to 0+}\left(-10\cot x\right)$

Solution

$$\lim_{x\to 0^+}\left(-10\cot x\right)=-10\lim_{x\to 0^+}\frac{\cos\theta}{\sin\theta}=-10\left(\frac{1}{0}\right)$$

As $x\to 0^+$ $\cos\theta>0;$ $\sin\theta>0$

$$\boxed{=-\infty}$$

Exercise

Find $\displaystyle\lim_{\theta\to \frac{\pi}{2}^+}\frac{1}{3}\tan\theta$

Solution

$$\lim_{\theta\to \frac{\pi}{2}^+}\frac{1}{3}\tan\theta=\frac{1}{3}\lim_{\theta\to \frac{\pi}{2}^+}\frac{\sin\theta}{\cos\theta}=\frac{1}{3}\left(-\frac{1}{0}\right)$$

As $\theta\to \frac{\pi}{2}^+$ $\cos\theta<0;$ $\sin\theta>0$

$$\boxed{=-\infty}$$

Exercise

Find $\displaystyle\lim_{x\to 2^+}\frac{1}{x-2}$

Solution

$$\lim_{x\to 2^+}\frac{1}{x-2}=\frac{1}{2^+-2}=\frac{1}{0^+}$$

$$\boxed{=\infty}$$

Exercise

Find $\lim\limits_{x\to 2^-} \frac{1}{x-2}$

Solution

$$\lim_{x\to 2^-} \frac{1}{x-2} = \frac{1}{2^- - 2} = \frac{1}{0^-}$$

$$\underline{= -\infty}\rfloor$$

Exercise

Find $\lim\limits_{x\to 2} \frac{1}{x-2}$

Solution

$$\lim_{x\to 2} \frac{1}{x-2} = \frac{1}{0}$$

$$\underline{= \infty}\rfloor$$

Exercise

Find $\lim\limits_{x\to 3^+} \frac{2}{(x-3)^3}$

Solution

$$\lim_{x\to 3^+} \frac{2}{(x-3)^3} = \frac{2}{0^+}$$

$$\underline{= \infty}\rfloor$$

Exercise

Find $\lim\limits_{x\to 3^-} \frac{2}{(x-3)^3}$

Solution

$$\lim_{x\to 3^-} \frac{2}{(x-3)^3} = \frac{2}{0^-}$$

$$\underline{= -\infty}\rfloor$$

Exercise

Find $\lim\limits_{x\to 3} \frac{2}{(x-3)^3}$

Solution

$$\lim_{x \to 3} \frac{2}{(x-3)^3} = \frac{2}{0}$$

$$\boxed{= \infty}$$

Exercise

Find $\lim_{x \to 4^+} \frac{x-5}{(x-4)^2}$

Solution

$$\lim_{x \to 4^+} \frac{x-5}{(x-4)^2} = \frac{-1}{0}$$

$$\boxed{= -\infty}$$

Exercise

Find $\lim_{x \to 4^-} \frac{x-5}{(x-4)^2}$

Solution

$$\lim_{x \to 4^-} \frac{x-5}{(x-4)^2} = \frac{-1}{0}$$

$$\boxed{= -\infty}$$

Exercise

Find $\lim_{x \to 4} \frac{x-5}{(x-4)^2}$

Solution

$$\lim_{x \to 4^-} \frac{x-5}{(x-4)^2} = \frac{-1}{0}$$

$$\boxed{= -\infty}$$

Exercise

Find $\lim_{x \to 1^+} \frac{x-2}{(x-1)^3}$

Solution

$$\lim_{x \to 1^+} \frac{x-2}{(x-1)^3} = \frac{-1}{0^+}$$

$$\boxed{= -\infty}$$

Exercise

Find $\lim\limits_{x\to1^-}\frac{x-2}{(x-1)^3}$

Solution

$$\lim_{x\to1^-}\frac{x-2}{(x-1)^3}=\frac{-1}{0^-}$$

$$=\infty$$

Exercise

Find $\lim\limits_{x\to1}\frac{x-2}{(x-1)^3}$

Solution

$$\lim_{x\to1}\frac{x-2}{(x-1)^3}=\frac{-1}{0^+}$$

$$=\nexists$$

Exercise

Find $\lim\limits_{x\to3^+}\frac{(x-1)(x-2)}{x-3}$

Solution

$$\lim_{x\to3^+}\frac{(x-1)(x-2)}{x-3}=\frac{2}{0}$$

$$=\infty$$

Exercise

Find $\lim\limits_{x\to3^-}\frac{(x-1)(x-2)}{x-3}$

Solution

$$\lim_{x\to3^-}\frac{(x-1)(x-2)}{x-3}=\frac{2}{0^-}$$

$$=-\infty$$

Exercise

Find $\displaystyle\lim_{x\to 3}\frac{(x-1)(x-2)}{x-3}$

Solution

$$\lim_{x\to 3^-}\frac{(x-1)(x-2)}{x-3}=\frac{2}{0^-}$$

$$\lim_{x\to 3^-}\frac{(x-1)(x-2)}{x-3}=-\infty \qquad \lim_{x\to 3^+}\frac{(x-1)(x-2)}{x-3}=\infty$$

$$\boxed{=\nexists}$$

Exercise

Find $\displaystyle\lim_{x\to -2^+}\frac{x-4}{x(x+2)}$

Solution

$$\lim_{x\to -2^+}\frac{x-4}{x(x+2)}=\frac{-6}{-0^+}$$

$$\boxed{=\infty}$$

Exercise

Find $\displaystyle\lim_{x\to -2^-}\frac{x-4}{x(x+2)}$

Solution

$$\lim_{x\to -2^-}\frac{x-4}{x(x+2)}=\frac{-6}{0^+}$$

$$\boxed{=-\infty}$$

Exercise

Find $\displaystyle\lim_{x\to -2}\frac{x-4}{x(x+2)}$

Solution

$$\lim_{x\to -2}\frac{x-4}{x(x+2)}\boxed{=\nexists}$$

$$\lim_{x\to -2^+}\frac{x-4}{x(x+2)}=\infty \qquad \lim_{x\to -2^-}\frac{x-4}{x(x+2)}=-\infty$$

Exercise

Find $\displaystyle\lim_{x\to 2^+}\frac{x^2-4x+3}{(x-2)^2}$

Solution

$$\lim_{x \to 2^+} \frac{x^2 - 4x + 3}{(x-2)^2} = \frac{-1}{0^+}$$

$$= -\infty$$

Exercise

Find $\displaystyle\lim_{x \to 2^-} \frac{x^2 - 4x + 3}{(x-2)^2}$

Solution

$$\lim_{x \to 2^-} \frac{x^2 - 4x + 3}{(x-2)^2} = \frac{-1}{0^+}$$

$$= -\infty$$

Exercise

Find $\displaystyle\lim_{x \to 2} \frac{x^2 - 4x + 3}{(x-2)^2}$

Solution

$$\lim_{x \to 2} \frac{x^2 - 4x + 3}{(x-2)^2} = \frac{-1}{0}$$

$$= -\infty$$

Exercise

Find $\displaystyle\lim_{x \to -2^+} \frac{x^3 - 5x^2 + 6x}{x^4 - 4x^2}$

Solution

$$\lim_{x \to -2^+} \frac{x^3 - 5x^2 + 6x}{x^4 - 4x^2} = \lim_{x \to -2^+} \frac{x(x-2)(x-3)}{x^2(x-2)(x+2)}$$

$$= \lim_{x \to -2^+} \frac{x-3}{x(x+2)} \qquad \frac{-}{-(+)}$$

$$= \infty$$

Exercise

Find $\displaystyle\lim_{x \to -2^-} \frac{x^3 - 5x^2 + 6x}{x^4 - 4x^2}$

Solution

$$\lim_{x \to -2^-} \frac{x^3 - 5x^2 + 6x}{x^4 - 4x^2} = \lim_{x \to -2^-} \frac{x(x-2)(x-3)}{x^2(x-2)(x+2)}$$

$$= \lim_{x \to -2^-} \frac{x-3}{x(x+2)} \qquad \frac{-}{-(-)}$$

$$= -\infty$$

Exercise

Find $\displaystyle \lim_{x \to -2} \frac{x^3 - 5x^2 + 6x}{x^4 - 4x^2}$

Solution

$$\lim_{x \to -2} \frac{x^3 - 5x^2 + 6x}{x^4 - 4x^2} = \frac{-8 - 20 - 12}{16 - 16}$$

$$= \frac{-40}{0}$$

$$= -\infty$$

Exercise

Find $\displaystyle \lim_{u \to 0^+} \frac{u-1}{\sin u}$

Solution

$$\lim_{u \to 0^+} \frac{u-1}{\sin u} = \frac{-1}{0^+}$$

$$= -\infty$$

Exercise

Find $\displaystyle \lim_{x \to 0^-} \frac{2}{\tan x}$

Solution

$$\lim_{x \to 0^-} \frac{2}{\tan x} = \frac{2}{0^-}$$

$$= -\infty$$

Exercise

Find $\displaystyle \lim_{x \to 1^+} \frac{x^2 - 5x + 6}{x-1}$

Solution

$$\lim_{x \to 1^+} \frac{x^2 - 5x + 6}{x-1} = \frac{2}{0^+}$$

$$= \infty$$

Exercise

Find $\lim\limits_{x\to 2\pi^-} \csc x$

Solution

$$\lim_{x\to 2\pi^-} \csc x = \frac{1}{\sin\left(2\pi^-\right)} = \frac{1}{0^-}$$

$$\underline{= -\infty\,}\rfloor$$

Exercise

Find $\lim\limits_{x\to 0^+} e^{\sqrt{x}}$

Solution

$$\lim_{x\to 0^+} e^{\sqrt{x}} \underline{=1}\rfloor$$

Exercise

Find $\lim\limits_{x\to \frac{\pi}{2}^-} \frac{1+\sin x}{\cos x}$

Solution

$$\lim_{x\to \frac{\pi}{2}^-} \frac{1+\sin x}{\cos x} = \frac{2}{0^+}$$

$$\underline{= \infty\,}\rfloor$$

Exercise

Find $\lim\limits_{x\to \frac{\pi}{2}^+} \frac{1+\sin x}{\cos x}$

Solution

$$\lim_{x\to \frac{\pi}{2}^+} \frac{1+\sin x}{\cos x} = \frac{2}{0^-}$$

$$\underline{= -\infty\,}\rfloor$$

Exercise

Find $\lim\limits_{x\to 0^-} \frac{e^x}{1+e^x}$

Solution

$$\lim_{x\to 0^-} \frac{e^x}{1-e^x} = \frac{1}{0^+}$$

$$\boxed{= \infty}$$

Exercise

Find $\displaystyle\lim_{x\to 0^+} \frac{e^x}{1-e^x}$

Solution

$$\lim_{x\to 0^+} \frac{e^x}{1-e^x} = \frac{1}{0^-}$$

$$\boxed{= -\infty}$$

Exercise

Find $\displaystyle\lim_{x\to 1^-} \frac{x}{\ln x}$

Solution

$$\lim_{x\to 1^-} \frac{x}{\ln x} = \frac{1}{0^-}$$

$$\boxed{= -\infty}$$

Exercise

Find $\displaystyle\lim_{x\to 0^+} \frac{x}{\ln x}$

Solution

$$\lim_{x\to 0^+} \frac{x}{\ln x} = \frac{0}{-\infty}$$

$$\boxed{= 0}$$

Exercise

Find $\displaystyle\lim_{x\to 0^-} \frac{2e^x+5e^{3x}}{e^{2x}-e^{3x}}$

Solution

$$\lim_{x\to 0^-} \frac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} = \lim_{x\to 0^-} \frac{2e^x+5e^{3x}}{e^{2x}\left(1-e^x\right)}$$

$$= \frac{7}{0}$$

$$\boxed{= \infty}$$

Exercise

Find $\displaystyle\lim_{x\to 0^+} \frac{2e^x+5e^{3x}}{e^{2x}-e^{3x}}$

Solution

$$\lim_{x\to 0^+} \frac{2e^x+5e^{3x}}{e^{2x}-e^{3x}} = \lim_{x\to 0^+} \frac{2e^x+5e^{3x}}{e^{2x}\left(1-e^x\right)}$$

$$= \frac{7}{0^-}$$

$$= -\infty$$

Exercise

Find $\displaystyle\lim_{x\to 1^-} \frac{\ln x}{\sin^{-1} x}$

Solution

$$\lim_{x\to 1^-} \frac{\ln x}{\sin^{-1} x} = \frac{\ln 1}{\sin^{-1} 1}$$

$$= \frac{0}{\frac{\pi}{2}}$$

$$= 0$$

Exercise

Let $f(x) = \dfrac{x^2-7x+12}{x-a}$

a) For what values of a, if any, does $\displaystyle\lim_{x\to a^+} f(x)$ equal a finite number?

b) For what values of a, if any, does $\displaystyle\lim_{x\to a^+} f(x) = \infty$?

c) For what values of a, if any, does $\displaystyle\lim_{x\to a^+} f(x) = -\infty$?

Solution

$$f(x) = \frac{x^2-7x+12}{x-a} = \frac{(x-3)(x-4)}{x-a}$$

a) If $a=3$, then $\displaystyle\lim_{x\to 3} \frac{(x-3)(x-4)}{x-3} = \lim_{x\to 3} (x-4) = -1$

If $a=4$, then $\displaystyle\lim_{x\to 4} \frac{(x-3)(x-4)}{x-4} = \lim_{x\to 4} (x-1) = 1$

b) $\lim\limits_{x\to a^+} f(x) = \infty$ for any number other than 3 or 4.

As $x \to a^+$, then $(x-a)$ is always positive.

$(x-3)(x-4) > 0 \quad \Rightarrow \quad (-\infty,\ 3) \cup (4,\ \infty)$

c) $\lim\limits_{x\to a^+} f(x) = -\infty$ for any number other than 3 or 4.

As $x \to a^+$, then $(x-a)$ is always positive, and $(3,\ 4)$

Exercise

Analyze $\lim\limits_{x\to 1^+} \sqrt{\dfrac{x-1}{x-3}}$ and $\lim\limits_{x\to 1^-} \sqrt{\dfrac{x-1}{x-3}}$

Solution

$$\lim_{x\to 1^+} \sqrt{\frac{x-1}{x-3}} = \sqrt{\frac{0^+}{-2}} \quad \nexists$$

$$\lim_{x\to 1^-} \sqrt{\frac{x-1}{x-3}} = \sqrt{\frac{0^-}{-2}} \ \underline{= 0}$$

Solution *Section* 1.4 – Limits at Infinity

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $h(x) = \dfrac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}}$

Solution

$$\lim_{x\to\infty} \frac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}} = \boxed{-\frac{5}{3}}$$

$$\lim_{x\to-\infty} \frac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}} = \boxed{-\frac{5}{3}}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{2x+3}{5x+7}$

Solution

$$\lim_{x\to\infty} \frac{2x+3}{5x+7} = \lim_{x\to\infty} \frac{2 + \frac{3}{x}}{5 + \frac{7}{x}} \boxed{= \frac{2}{5}}$$

$$\lim_{x\to-\infty} \frac{2x+3}{5x+7} = \lim_{x\to-\infty} \frac{2 + \frac{3}{x}}{5 + \frac{7}{x}} \boxed{= \frac{2}{5}}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{2x^3+7}{x^3 - x^2 + x + 7}$

Solution

$$\lim_{x\to\infty} \frac{2x^3+7}{x^3 - x^2 + x + 7} = \lim_{x\to\infty} \frac{2 + \frac{7}{x^3}}{1 - \frac{1}{x} + \frac{1}{x^2} + \frac{7}{x^3}} \boxed{= 2}$$

$$\lim_{x\to-\infty} \frac{2x^3+7}{x^3 - x^2 + x + 7} = \lim_{x\to-\infty} \frac{2 + \frac{7}{x^3}}{1 - \frac{1}{x} + \frac{1}{x^2} + \frac{7}{x^3}} \boxed{= 2}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{x+1}{x^2+3}$

Solution

$$\lim_{x\to\infty} \frac{x+1}{x^2+3} = \lim_{x\to\infty} \frac{\frac{x}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{3}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}} \boxed{=0}$$

$$\lim_{x\to-\infty} \frac{x+1}{x^2+3} = \lim_{x\to-\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}} \boxed{=0}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{7x^3}{x^3-3x^2+6x}$

Solution

$$\lim_{x\to\infty} \frac{7x^3}{x^3-3x^2+6x} = \lim_{x\to\infty} \frac{7}{1-\frac{3}{x}+\frac{6}{x^2}} \boxed{=7}$$

$$\lim_{x\to-\infty} \frac{7x^3}{x^3-3x^2+6x} = \lim_{x\to-\infty} \frac{7}{1-\frac{3}{x}+\frac{6}{x^2}} \boxed{=7}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{9x^4+x}{2x^4+5x^2-x+6}$

Solution

$$\lim_{x\to\infty} \frac{9x^4+x}{2x^4+5x^2-x+6} = \lim_{x\to\infty} \frac{\frac{9x^4}{x^4}+\frac{x}{x^4}}{\frac{2x^4}{x^4}+\frac{5x^2}{x^4}-\frac{x}{x^4}+\frac{6}{x^4}}$$

$$= \lim_{x\to\infty} \frac{9+\frac{1}{x^3}}{2+\frac{5}{x^2}-\frac{1}{x^3}+\frac{6}{x^4}}$$

$$\boxed{=\frac{9}{2}}$$

$$\lim_{x\to-\infty} \frac{9x^4+x}{2x^4+5x^2-x+6} = \lim_{x\to-\infty} \frac{9+\frac{1}{x^3}}{2+\frac{5}{x^2}-\frac{1}{x^3}+\frac{6}{x^4}} \boxed{=\frac{9}{2}}$$

Exercise

Find the limit as $x \to \infty$ and as $x \to -\infty$ of $f(x) = \dfrac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$

Solution

$$\lim_{x\to\infty} \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x} = \lim_{x\to\infty} \frac{-2 - \frac{2}{x^2} + \frac{3}{x^3}}{3 + \frac{3}{x} - \frac{5}{x^2}} \boxed{= -\frac{2}{3}}$$

$$\lim_{x\to-\infty} \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x} = \lim_{x\to-\infty} \frac{-2 - \frac{2}{x^2} + \frac{3}{x^3}}{3 + \frac{3}{x} - \frac{5}{x^2}} \boxed{= -\frac{2}{3}}$$

Exercise

Find $\lim\limits_{x\to\infty} x^{12}$

Solution

$$\lim_{x\to\infty} x^{12} \boxed{= \infty}$$

Exercise

Find $\lim\limits_{x\to-\infty} 3x^9$

Solution

$$\lim_{x\to-\infty} 3x^9 \boxed{= -\infty}$$

Exercise

Find $\lim\limits_{x\to-\infty} x^{-8}$

Solution

$$\lim_{x\to-\infty} x^{-8} = \frac{1}{(-\infty)^8} \boxed{= 0}$$

Exercise

Find $\lim\limits_{x\to-\infty} x^{-9}$

Solution

$$\lim_{x\to-\infty} x^{-9} = \frac{1}{(-\infty)^9} \boxed{= 0}$$

Exercise

Find $\lim\limits_{x\to-\infty} 2x^{-6}$

Solution

$$\lim_{x\to-\infty} 2x^{-6} = \frac{2}{\infty} \underline{=0}$$

Exercise

Find $\lim\limits_{x\to\infty} \left(3x^{12}-9x^{7}\right)$

Solution

$$\lim_{x\to\infty} \left(3x^{12}-9x^{7}\right) \underline{=\infty}$$

Exercise

Find $\lim\limits_{x\to-\infty} \left(3x^{7}+x^{2}\right)$

Solution

$$\lim_{x\to-\infty} \left(3x^{7}+x^{2}\right) = \lim_{x\to-\infty} x^{2}\left(3x^{5}+1\right)$$

$$\underline{=-\infty}$$

Exercise

Find $\lim\limits_{x\to-\infty} \left(-2x^{16}+2\right)$

Solution

$$\lim_{x\to-\infty} \left(-2x^{16}+2\right) \underline{=-\infty}$$

Exercise

Find $\lim\limits_{x\to-\infty} \left(2x^{-6}+4x^{5}\right)$

Solution

$$\lim_{x\to-\infty} \left(2x^{-6}+4x^{5}\right) = \lim_{x\to-\infty} x^{-6}\left(2+4x^{11}\right) \qquad +\infty(-\infty)$$

$$\underline{=-\infty}$$

Exercise

Find $\lim\limits_{x\to-\infty} \frac{\cos x}{3x}$

Solution

$$-\frac{1}{3x} \le \frac{\cos x}{3x} \le \frac{1}{3x}, \quad -1 \le \cos x \le 1$$

$$\lim_{x\to-\infty} \frac{\cos x}{3x} = 0 \qquad \textit{By the Sandwich Theorem}$$

Exercise

Find $\lim\limits_{x\to\infty} \frac{x+\sin x}{2x+7-5\sin x}$

Solution

$$\lim_{x\to\infty} \frac{x+\sin x}{2x+7-5\sin x} = \lim_{x\to\infty} \frac{1+\frac{\sin x}{x}}{2+\frac{7}{x}-\frac{5\sin x}{x}}$$

$$= \frac{1+0}{2+0-0}$$

$$\boxed{=\frac{1}{2}}$$

Exercise

Find $\lim\limits_{x\to\infty} \sqrt{\frac{8x^2-3}{2x^2+x}}$

Solution

$$\lim_{x\to\infty} \sqrt{\frac{8x^2-3}{2x^2+x}} = \lim_{x\to\infty} \sqrt{\frac{8-\frac{3}{x^2}}{2+\frac{1}{x}}}$$

$$= \sqrt{\frac{8}{2}}$$

$$\boxed{=2}$$

Exercise

Find $\lim\limits_{x\to-\infty} \left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3}$

Solution

$$\lim_{x\to-\infty} \left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3} = \lim_{x\to-\infty} \left(\frac{1+\frac{1}{x}-\frac{1}{x^2}}{8-\frac{3}{x^2}}\right)^{1/3}$$

$$=\left(\frac{1}{8}\right)^{1/3}$$

$$\boxed{=\frac{1}{2}}$$

Exercise

Find $\displaystyle\lim_{x\to\infty}\frac{2\sqrt{x}+x^{-1}}{3x-7}$

Solution

$$\lim_{x\to\infty}\frac{2\sqrt{x}+x^{-1}}{3x-7}=\lim_{x\to\infty}\frac{\frac{2\sqrt{x}}{x}+\frac{x^{-1}}{x}}{3-\frac{7}{x}}$$

$$=\lim_{x\to\infty}\frac{\frac{2}{x^{1/2}}+\frac{1}{x^2}}{3-\frac{7}{x}}$$

$$\boxed{=0}$$

Exercise

Find $\displaystyle\lim_{x\to\infty}\frac{x^{-1}+x^{-4}}{x^{-2}+x^{-3}}$

Solution

$$\lim_{x\to\infty}\frac{x^{-1}+x^{-4}}{x^{-2}+x^{-3}}=\lim_{x\to\infty}\frac{\frac{x^{-1}}{x^{-2}}+\frac{x^{-4}}{x^{-2}}}{\frac{x^{-2}}{x^{-2}}+\frac{x^{-3}}{x^{-2}}}$$

$$=\lim_{x\to\infty}\frac{x+\frac{1}{x^2}}{1+\frac{1}{x}}$$

$$\boxed{=\infty}$$

Exercise

Find $\displaystyle\lim_{x\to-\infty}\frac{4-3x^3}{\sqrt{x^6+9}}$

Solution

$$\lim_{x\to-\infty}\frac{4-3x^3}{\sqrt{x^6+9}}=\lim_{x\to-\infty}\frac{\frac{4-3x^3}{\sqrt{x^6}}}{\frac{\sqrt{x^6+9}}{\sqrt{x^6}}}$$

$$= \lim_{x\to -\infty} \frac{\frac{4-3x^3}{x^3}}{\sqrt{\frac{x^6+9}{x^6}}}$$

$$= \lim_{x\to -\infty} \frac{\frac{4}{x3}-3}{\sqrt{1+\frac{9}{x^6}}}$$

$$= \frac{-3}{\sqrt{1}}$$

$$\underline{=-3}|$$

Exercise

Find $\displaystyle \lim_{x\to -\infty}\left(\sqrt{x^2+3}+x\right)$

Solution

$$\lim_{x\to -\infty}\left(\sqrt{x^2+3}+x\right) = \lim_{x\to -\infty}\left(\sqrt{x^2+3}+x\right)\frac{\sqrt{x^2+3}-x}{\sqrt{x^2+3}-x}$$

$$= \lim_{x\to -\infty}\frac{x^2+3-x^2}{\sqrt{x^2+3}-x}$$

$$= \lim_{x\to -\infty}\frac{3}{\sqrt{x^2+3}-x}$$

$$= \lim_{x\to -\infty}\frac{\frac{3}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{3}{x^2}}-\frac{x}{x}}$$

$$= \lim_{x\to -\infty}\frac{\frac{3}{x}}{\sqrt{1+\frac{3}{x^2}}+1}$$

$$= \frac{0}{\sqrt{1}+1}$$

$$\underline{=0}|$$

Exercise

Find $\displaystyle \lim_{x\to \infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right)$

Solution

$$\lim_{x\to \infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right) = \lim_{x\to \infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right)\frac{\sqrt{x^2+3x}+\sqrt{x^2-2x}}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to\infty} \frac{\left(x^2+3x\right)-\left(x^2-2x\right)}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to\infty} \frac{x^2+3x-x^2+2x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to\infty} \frac{5x}{\sqrt{x^2+3x}+\sqrt{x^2-2x}}$$

$$= \lim_{x\to\infty} \frac{\frac{5x}{\sqrt{x^2}}}{\sqrt{\frac{x^2}{x^2}+\frac{3x}{x^2}}+\sqrt{\frac{x^2}{x^2}-\frac{2x}{x^2}}}$$

$$= \lim_{x\to\infty} \frac{5}{\sqrt{1+\frac{3}{x}}+\sqrt{1-\frac{2}{x}}}$$

$$= \frac{5}{\sqrt{1}+\sqrt{1}}$$

$$\boxed{=\frac{5}{2}}$$

Exercise

Find $\lim_{x\to\infty} \frac{2x-3}{4x+10}$

Solution

$$\lim_{x\to\infty} \frac{2x-3}{4x+10} \boxed{=\frac{1}{2}}$$

Exercise

Find $\lim_{x\to\infty} \frac{x^4-1}{x^5+2}$

Solution

$$\lim_{x\to\infty} \frac{x^4-1}{x^5+2} \boxed{=0}$$

Exercise

Find $\lim_{x\to-\infty} \left(-3x^3+5\right)$

Solution

$$\lim_{x\to-\infty} \left(-3x^3+5\right) \boxed{=\infty}$$

Exercise

Find $\lim\limits_{x\to\infty}\left(e^{-2x}+\frac{2}{x}\right)$

Solution

$$\lim_{x\to\infty}\left(e^{-2x}+\frac{2}{x}\right)=e^{-\infty}+0\ \boxed{=0}$$

Exercise

Find $\lim\limits_{x\to\infty}\frac{1}{\ln x+1}$

Solution

$$\lim_{x\to\infty}\frac{1}{\ln x+1}=\frac{1}{\infty}\ \boxed{=0}$$

Exercise

Find $\lim\limits_{x\to\infty}\left(3+\frac{10}{x^2}\right)$

Solution

$$\lim_{x\to\infty}\left(3+\frac{10}{x^2}\right)=3+0\ \boxed{=3}$$

Exercise

Find $\lim\limits_{x\to\infty}\left(5+\frac{1}{x}+\frac{10}{x^2}\right)$

Solution

$$\lim_{x\to\infty}\left(5+\frac{1}{x}+\frac{10}{x^2}\right)=5+0+0\ \boxed{=5}$$

Exercise

Find $\lim\limits_{x\to\infty}\frac{4x^2+2x+3}{x^2}$

Solution

$$\begin{aligned}\lim_{x\to\infty}\frac{4x^2+2x+3}{x^2}&=\lim_{x\to\infty}\frac{4x^2}{x^2}\\&\boxed{=4}\end{aligned}$$

Exercise

Find $\lim\limits_{x\to\infty}\left(5+\dfrac{100}{x}+\dfrac{\sin^4 x^3}{x^2}\right)$

Solution

$$-1\le \sin\theta \le 1 \quad\Rightarrow\quad 0\le \sin^4\theta \le 1$$

$$0\le \frac{\sin^4\theta}{x^2}\le \frac{1}{x^2} \quad \to 0$$

$$\lim_{x\to\infty}\left(5+\frac{100}{x}+\frac{\sin^4 x^3}{x^2}\right) \underline{=5}\rfloor$$

Exercise

Find $\lim\limits_{\theta\to\infty}\dfrac{\cos\theta}{\theta^2}$

Solution

$$-1\le \cos\theta \le 1 \quad\Rightarrow\quad -\frac{1}{\theta^2}\le \frac{\cos\theta}{\theta^2}\le \frac{1}{\theta^2} \quad \to 0$$

$$\lim_{\theta\to\infty}\frac{\cos\theta}{\theta^2} \underline{=0}\rfloor$$

Exercise

Find $\lim\limits_{\theta\to\infty}\dfrac{\cos\theta^5}{\sqrt{\theta}}$

Solution

$$-1\le \cos\theta^5 \le 1 \quad\Rightarrow\quad -\frac{1}{\sqrt{\theta}}\le \frac{\cos\theta^5}{\sqrt{\theta}}\le \frac{1}{\sqrt{\theta}} \quad \to 0$$

$$\lim_{\theta\to\infty}\frac{\cos\theta^5}{\sqrt{\theta}} \underline{=0}\rfloor$$

Exercise

Find $\lim\limits_{x\to\infty}\dfrac{4x}{20x+1}$

Solution

$$\lim_{x\to\infty}\frac{4x}{20x+1}=\frac{4}{20} \underline{=\frac{1}{5}}\rfloor$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{4x}{20x+1}$

Solution

$$\lim_{x\to-\infty} \frac{4x}{20x+1} = \lim_{x\to-\infty} \frac{4x}{20x}$$

$$\boxed{=\frac{1}{5}}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{3x^2-7}{x^2+5x}$

Solution

$$\lim_{x\to\infty} \frac{3x^2-7}{x^2+5x} \boxed{=3}$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{3x^2-7}{x^2+5x}$

Solution

$$\lim_{x\to-\infty} \frac{3x^2-7}{x^2+5x} = \lim_{x\to-\infty} \frac{3x^2}{x^2}$$

$$\boxed{=3}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{6x^2-9x+8}{3x^2+2}$

Solution

$$\lim_{x\to\infty} \frac{6x^2-9x+8}{3x^2+2} = \lim_{x\to\infty} \frac{6x^2}{3x^2}$$

$$=\frac{6}{3}$$

$$\boxed{=2}$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{6x^2-9x+8}{3x^2+2}$

Solution

$$\lim_{x\to-\infty} \frac{6x^2-9x+8}{3x^2+2} = \lim_{x\to-\infty} \frac{6x^2}{3x^2}$$

$$= \frac{6}{3}$$

$$\boxed{= 2}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{4x^2-7}{8x^2+5x+2}$

Solution

$$\lim_{x\to\infty} \frac{4x^2-7}{8x^2+5x+2} = \lim_{x\to\infty} \frac{4x^2}{8x^2}$$

$$= \frac{4}{8}$$

$$\boxed{= \frac{1}{2}}$$

Exercise

Find $\lim\limits_{x\to-\infty} \dfrac{4x^2-7}{8x^2+5x+2}$

Solution

$$\lim_{x\to-\infty} \frac{4x^2-7}{8x^2+5x+2} = \lim_{x\to-\infty} \frac{4x^2}{8x^2}$$

$$= \frac{4}{8}$$

$$\boxed{= \frac{1}{2}}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

Solution

$$\lim_{x\to\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4} = \lim_{x\to\infty} \frac{\sqrt{16x^4}+x^2}{2x^2}$$

$$= \lim_{x\to\infty} \frac{4x^2+x^2}{2x^2}$$

$$= \lim_{x\to\infty} \frac{5x^2}{2x^2}$$

$$= \frac{5}{2}$$

Exercise

Find $\displaystyle \lim_{x\to-\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4}$

Solution

$$\lim_{x\to-\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4} = \lim_{x\to-\infty} \frac{\sqrt{16x^4}+x^2}{2x^2}$$

$$= \lim_{x\to-\infty} \frac{4x^2+x^2}{2x^2}$$

$$= \lim_{x\to-\infty} \frac{5x^2}{2x^2}$$

$$= \frac{5}{2}$$

Exercise

Find $\displaystyle \lim_{x\to\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

Solution

$$\lim_{x\to\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144} = \lim_{x\to\infty} \frac{3x^4}{x^4}$$

$$= 3$$

Exercise

Find $\displaystyle \lim_{x\to-\infty} \frac{3x^4+3x^3-36x^2}{x^4-25x^2+144}$

Solution

$$\lim_{x\to-\infty}\frac{3x^4+3x^3-36x^2}{x^4-25x^2+144}=\lim_{x\to-\infty}\frac{3x^4}{x^4}$$

$$=3$$

Exercise

Find $\displaystyle\lim_{x\to\infty}16x^2\left(4x^2-\sqrt{16x^4+1}\right)$

Solution

$$\lim_{x\to\infty}16x^2\left(4x^2-\sqrt{16x^4+1}\right)=\infty-\infty$$

$$=\lim_{x\to\infty}16x^2\left(4x^2-\sqrt{16x^4+1}\right)\frac{4x^2+\sqrt{16x^4+1}}{4x^2+\sqrt{16x^4+1}}$$

$$=\lim_{x\to\infty}16x^2\frac{16x^4-16x^4-1}{4x^2+\sqrt{16x^4+1}}$$

$$=\lim_{x\to\infty}16x^2\frac{-1}{4x^2+\sqrt{16x^4+1}}$$

$$=\lim_{x\to\infty}\frac{-16x^2}{4x^2+4x^2}$$

$$=\lim_{x\to\infty}\frac{-16x^2}{8x^2}$$

$$=-2$$

Exercise

Find $\displaystyle\lim_{x\to-\infty}16x^2\left(4x^2-\sqrt{16x^4+1}\right)$

Solution

$$\lim_{x\to-\infty}16x^2\left(4x^2-\sqrt{16x^4+1}\right)=\infty-\infty$$

$$=\lim_{x\to-\infty}16x^2\left(4x^2-\sqrt{16x^4+1}\right)\frac{4x^2+\sqrt{16x^4+1}}{4x^2+\sqrt{16x^4+1}}$$

$$=\lim_{x\to-\infty}16x^2\frac{16x^4-16x^4-1}{4x^2+\sqrt{16x^4+1}}$$

$$=\lim_{x\to-\infty}16x^2\frac{-1}{4x^2+\sqrt{16x^4+1}}$$

$$= \lim_{x\to-\infty} \frac{-16x^2}{4x^2 + 4x^2}$$

$$= \lim_{x\to-\infty} \frac{-16x^2}{8x^2}$$

$$\boxed{= -2}$$

Exercise

Find $\lim_{x\to\infty} \frac{x-1}{x^{2/3}-1}$

Solution

$$\lim_{x\to\infty} \frac{x-1}{x^{2/3}-1} = \lim_{x\to\infty} \frac{x}{x^{2/3}}$$

$$= \lim_{x\to\infty} x^{1/3}$$

$$\boxed{= \infty}$$

Exercise

Find $\lim_{x\to-\infty} \frac{x-1}{x^{2/3}-1}$

Solution

$$\lim_{x\to-\infty} \frac{x-1}{x^{2/3}-1} = \lim_{x\to-\infty} \frac{x}{x^{2/3}}$$

$$= \lim_{x\to-\infty} x^{1/3}$$

$$\boxed{= -\infty}$$

Exercise

Find $\lim_{x\to\infty} \frac{\sqrt{x^2+2x+6}-3}{x-1}$

Solution

$$\lim_{x\to\infty} \frac{\sqrt{x^2+2x+6}-3}{x-1} = \lim_{x\to\infty} \frac{\sqrt{x^2}}{x}$$

$$= \lim_{x\to\infty} \frac{x}{x}$$

$$\boxed{=1}$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{\left|1-x^2\right|}{x(x+1)}$

Solution

$$\lim_{x\to\infty} \frac{\left|1-x^2\right|}{x(x+1)} = \lim_{x\to\infty} \frac{x^2-1}{x^2+1}$$

$$= \lim_{x\to\infty} \frac{x^2}{x^2}$$

$$\underline{=1}\big|$$

Exercise

Find $\lim\limits_{x\to\infty} \left(\sqrt{|x|}-\sqrt{|x-1|}\right)$

Solution

$$\lim_{x\to\infty} \left(\sqrt{|x|}-\sqrt{|x-1|}\right) = \infty - \infty \qquad x \to \infty \Rightarrow |x| = x \;\; \& \;\; |x-1| = x-1$$

$$= \lim_{x\to\infty} \left(\sqrt{x}-\sqrt{x-1}\right)\frac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}}$$

$$= \lim_{x\to\infty} \frac{x-x+1}{\sqrt{x}+\sqrt{x-1}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x}+\sqrt{x-1}}$$

$$= \frac{1}{\infty}$$

$$\underline{=0}\big|$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{\tan^{-1}x}{x}$

Solution

$$-\frac{\pi}{2} \le \tan^{-1}x \le \frac{\pi}{2}$$

$$-\frac{\pi}{2x} \le \frac{\tan^{-1}x}{x} \le \frac{\pi}{2x} \quad \to \;0$$

$$\lim_{x\to\infty} \frac{\tan^{-1}x}{x} \; \underline{=0}\big|$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{\cos x}{e^{3x}}$

Solution

$$-1 \le \cos x \le 1$$

$$-\frac{1}{e^{3x}} \le \frac{\cos x}{e^{3x}} \le \frac{1}{e^{3x}} \quad \to \; 0$$

$$\lim_{x\to\infty} \frac{\cos x}{e^{3x}} = 0$$

Exercise

Find $\lim\limits_{x\to 0} \dfrac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}}$

Solution

$$\lim_{x\to 0} \frac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}} = \frac{2+10}{1+1}$$

$$= 6$$

Exercise

Find $\lim\limits_{x\to\infty} \dfrac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}}$

Solution

$$\lim_{x\to\infty} \frac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}} = \lim_{x\to\infty} \frac{2e^{x}}{e^{x}} \qquad \lim_{x\to\infty} e^{-x} = 0$$

$$= 2$$

Exercise

Find $\lim\limits_{x\to -\infty} \dfrac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}}$

Solution

$$\lim_{x\to\infty} \frac{2e^{x}+10e^{-x}}{e^{x}+e^{-x}} = \lim_{x\to\infty} \frac{10e^{-x}}{e^{-x}} \qquad \lim_{x\to -\infty} e^{x} = 0$$

$$= 10$$

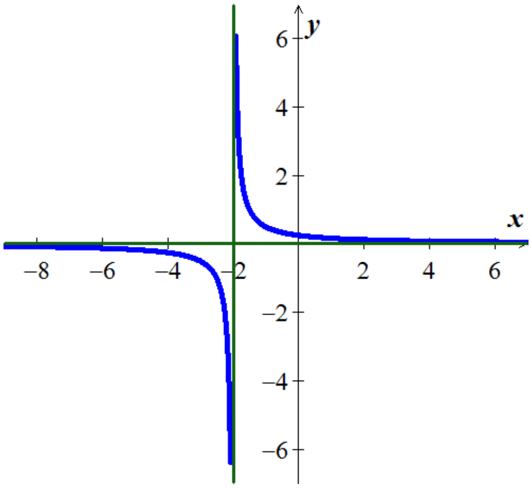
Exercise

Graph the rational function $y = \frac{1}{2x+4}$. Include the equations of the asymptotes.

Solution

VA: $2x = 4 = 0 \Rightarrow \boxed{x = -2}$

HA: $\underline{y = 0}|$



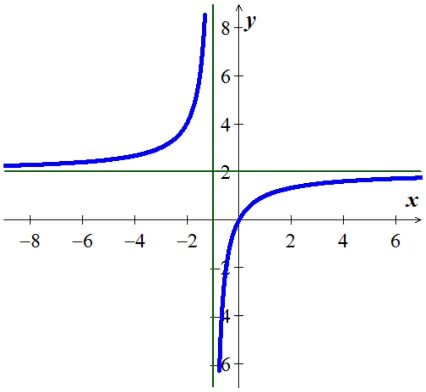
Exercise

Graph the rational function $y = \frac{2x}{x+1}$. Include the equations of the asymptotes.

Solution

VA: $\underline{x = -1}|$

HA: $\underline{y = 2}|$



Exercise

Graph the rational function $y = \frac{x^2}{x-1}$. Include the equations of the asymptotes.

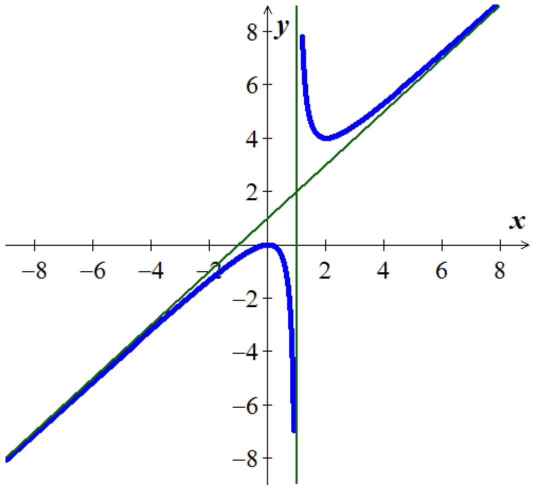
Solution

$$\begin{array}{r} x+1 \\ x-1\overline{)\; x^2 \qquad} \\ \underline{x^2 - x} \\ x \\ \underline{x-1} \\ \boxed{1} \end{array}$$

$$y = \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$$

VA: $\underline{x = 1}|$

Oblique Asymptote: $\underline{y = x + 1}|$



Exercise

Graph the rational function $y=\frac{x^3+1}{x^2}$. Include the equations of the asymptotes.

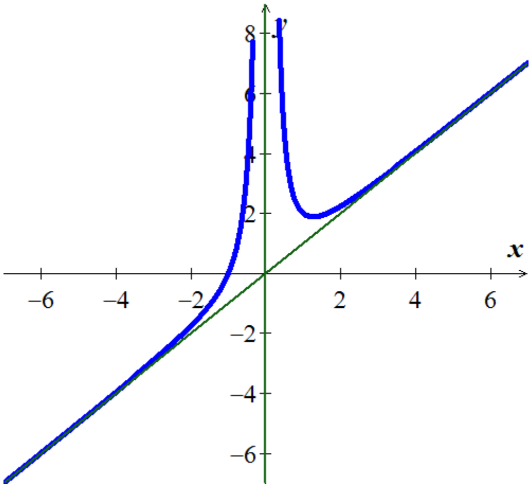
Solution

$$\begin{array}{r} x \\ x^2 \overline{\smash{)}\, x^3+1} \\ \underline{x^3} \\ \boxed{1} \end{array}$$

$$y=\frac{x^3+1}{x^2}=x+\frac{1}{x^2}$$

VA: $\boxed{x=0}$

Oblique Asymptote: $\boxed{y=x}$



Exercise

Let $\quad f(x)=\frac{x^2-5x+6}{x^2-2x}$

a) Analyze $\lim\limits_{x\to 0^-} f(x)$, $\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 2^-} f(x)$, and $\lim\limits_{x\to 2^+} f(x)$

b) Does the graph of f have any vertical asymptotes? Explain?

Solution

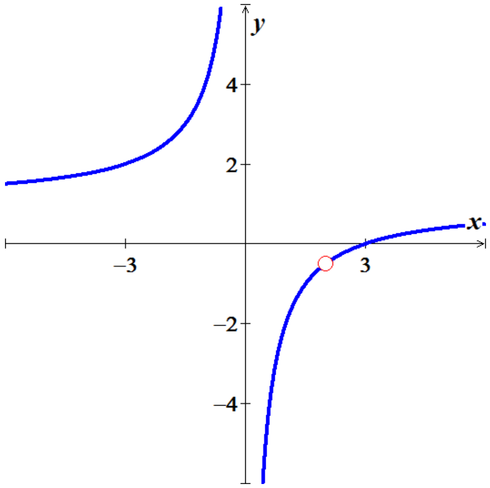
$$f(x)=\frac{x^2-5x+6}{x^2-2x}=\frac{(x-2)(x-3)}{x(x-2)}=\frac{x-3}{x}$$

a) $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\frac{x-3}{x}=\frac{-3}{0^-}\ \boxed{=\infty}$

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}\frac{x-3}{x}=\frac{-3}{0^+}\ \boxed{=-\infty}$

$\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^-}\frac{x-3}{x}=\frac{2-3}{2}\ \boxed{=-\frac{1}{2}}$

$\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+}\frac{x-3}{x}=\frac{2-3}{2}\ \boxed{=-\frac{1}{2}}$



b) ***VA***$: x=0$ $\qquad$ ***Hole***$:\ x=2\to f(2)=-\frac{1}{2}$

HA$: y=1$ $\qquad$ ***OA***$:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\frac{3x}{1-x}$

Solution

$\boldsymbol{VA}: x=1, \quad \boldsymbol{Hole}:\ n/a, \quad \boldsymbol{HA}: y=-3, \quad \boldsymbol{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\frac{x^2}{x^2+9}$

Solution

$\boldsymbol{VA}:\ n/a; \quad \boldsymbol{Hole}:\ n/a; \quad \boldsymbol{HA}: y=1; \quad \boldsymbol{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\frac{x-2}{x^2-4x+3}$

Solution

$\boldsymbol{VA}: x=1,\ 3; \quad \boldsymbol{Hole}:\ n/a; \quad \boldsymbol{HA}: y=0; \quad \boldsymbol{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\frac{5x-1}{1-3x}$

Solution

$\boldsymbol{VA}:\ x=\frac{1}{3}; \quad \boldsymbol{Hole}:\ n/a; \quad \boldsymbol{HA}:\ y=-\frac{5}{3}; \quad \boldsymbol{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\frac{3}{x-5}$

Solution

$\boldsymbol{VA}: x=5, \quad \boldsymbol{Hole}:\ n/a, \quad \boldsymbol{HA}: y=0, \quad \boldsymbol{OA}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\frac{x^3-1}{x^2+1}$

Solution

$$\begin{array}{r|l} & x \\ \hline x^2+1 & x^3-1 \\ & x^3+x \\ \hline & \boxed{-x-1} \end{array}$$

$$y=\frac{x^3-1}{x^2+1}=x+\frac{-x-1}{x^2+1}=x-\frac{x+1}{x^2+1}$$

$\textbf{\textit{VA}}:\ n/a,\quad \textbf{\textit{Hole}}:\ n/a,\quad \textbf{\textit{HA}}:\ n/a,\quad \textbf{\textit{OA}}:\ y=x$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{3x^2-27}{(x+3)(2x+1)}$

Solution

$\textbf{\textit{VA}}: x=-3,\ -\frac{1}{2};\quad \textbf{\textit{Hole}}:\ n/a;\quad \textbf{\textit{HA}}: y=\frac{3}{2};\quad \textbf{\textit{OA}}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{x^3+3x^2-2}{x^2-4}$

Solution

$$\begin{array}{r|l} & x+3 \\ \hline x^2-4 & x^3+3x^2-2 \\ & x^3\qquad -4x \\ \hline & \quad 3x^2+4x-2 \end{array}$$

$$y=\frac{x^3+3x^2-2}{x^2-4}=x+3+\frac{4x+10}{x^2-4}$$

$\textbf{\textit{VA}}:\ x=\pm 2,\quad \textbf{\textit{Hole}}:\ n/a,\quad \textbf{\textit{HA}}:\ n/a,\quad \textbf{\textit{OA}}:\ y=x+3$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{x-3}{x^2-9}$

Solution

$\textbf{\textit{VA}}:\ x=-3;\quad \textbf{\textit{Hole}}:\ x=3;\quad \textbf{\textit{HA}}:\ y=0;\quad \textbf{\textit{OA}}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $y=\dfrac{6}{\sqrt{x^2-4x}}$

Solution

$\textbf{\textit{VA}}:\ x=0,\ 4;\quad \textbf{\textit{Hole}}:\ n/a;\quad \textbf{\textit{HA}}:\ y=0;\quad \textbf{\textit{OA}}:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\frac{4x^3+1}{1-x^3}$

Solution

$VA:\ x=1;\quad Hole:\ n/a;\quad HA:\ y=-4;\quad OA:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\frac{x+1}{\sqrt{9x^2+x}}$

Solution

$VA:\ x=0,\ -\frac{1}{9};\quad Hole:\ n/a;\quad HA:\ y=\frac{1}{3};\quad OA:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=1-e^{-2x}$

Solution

$VA:\ n/a;\quad Hole:\ n/a;\quad HA:\ y=1;\quad OA:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\frac{1}{\ln x^2}$

Solution

$VA:\ x=0;\quad Hole:\ n/a;\quad HA:\ y=0;\quad OA:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\frac{1}{\tan^{-1}x}$

Solution

$VA:\ x=0;\quad Hole:\ n/a;\quad HA:\ y=\frac{1}{\frac{\pi}{2}}=\frac{2}{\pi};\quad OA:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\frac{2x^2+6}{2x^2+3x-2}$

Solution

$VA:\ x=-2,\ \frac{1}{2};\quad Hole:\ n/a;\quad HA:\ y=1;\quad OA:\ n/a$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{3x^2+2x-1}{4x+1}$

Solution

$$\begin{array}{rl} & \frac{3}{4}x+\frac{5}{16} \\ 4x+1\big) & \overline{3x^2+2x-1} \\ & \underline{3x^2+\frac{3}{4}x} \\ & \qquad \frac{5}{4}x-1 \end{array}$$

$$\textbf{VA}:\ x=-\frac{1}{4};\quad \textbf{Hole}:\ n/a;\quad \textbf{HA}:\ n/a;\quad \textbf{OA}:\ y=\frac{3}{4}x+\frac{5}{16}$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{9x^2+4}{(2x-1)^2}$

Solution

$$\textbf{VA}:\ x=\frac{1}{2};\quad \textbf{Hole}:\ n/a;\quad \textbf{HA}:\ y=\frac{9}{4};\quad \textbf{OA}:\ n/a$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{1+x-2x^2-x^3}{x^2+1}$

Solution

$$\begin{array}{rl} & \qquad\qquad -x-2 \\ x^2+1\big) & \overline{-x^3-2x^2+x+1} \\ & \underline{-x^3\qquad\quad -x} \\ & \quad -2x^2+2x \end{array}$$

$$\textbf{VA}:\ n/a;\quad \textbf{Hole}:\ n/a;\quad \textbf{HA}:\ n/a;\quad \textbf{OA}:\ y=-x-2$$

Exercise

Find the vertical, horizontal, hole, and oblique asymptotes (if any) of $f(x)=\dfrac{x(x+2)^3}{3x^2-4x}$

Solution

$$f(x)=\frac{x\left(x^3+6x^2+12x+8\right)}{x(3x-4)}=\frac{x^3+6x^2+12x+8}{3x-4}$$

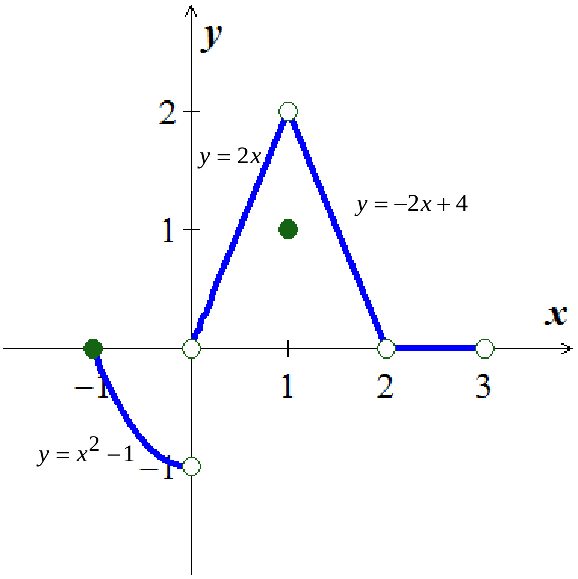
$$
\begin{array}{r|l}
 & \frac{1}{3}x^2+\frac{22}{9}x+\frac{196}{27} \\ \hline
3x-4 & x^3+6x^2+12x+8 \\
 & \underline{x^3-\frac{4}{3}x^2} \\
 & \quad \frac{22}{3}x^2+12x \\
 & \quad \underline{\frac{22}{3}x^2-\frac{88}{9}x} \\
 & \qquad \quad \frac{196}{9}x
\end{array}
$$

$\boldsymbol{VA}:\ x=\frac{4}{3};\quad \boldsymbol{Hole}:\ (0,\ -2);\quad \boldsymbol{HA}:\ n/a;\quad \boldsymbol{OA}:\ y=\frac{1}{3}x^2+\frac{22}{9}x+\frac{196}{27}$

Solution   *Section* 1.5 – Continuity

Exercise

Given the graphed function $f(x)$



a) Does $f(-1)$ exist?

b) Does $\lim\limits_{x\to -1^+} f(x)$ exist?

c) Does $\lim\limits_{x\to -1^+} f(x) = f(-1)$?

d) Is f continuous at $x = -1$?

e) Does $f(1)$ exist?

f) Does $\lim\limits_{x\to 1} f(x)$ exist?

g) Does $\lim\limits_{x\to 1} f(x) = f(1)$?

h) Is f continuous at $x = 1$?

Solution

a) Yes $f(-1) = 0$

b) Yes, $\lim\limits_{x\to -1^+} f(x) = 0$

c) Yes

d) Yes

e) Yes, $f(1) = 1$

f) Yes, $\lim\limits_{x\to 1} f(x) = 2$

g) No

h) No

Exercise

At what points is the function $y = \dfrac{1}{x-2} - 3x$ continuous?

Solution

The function is continuous everywhere except when $x - 2 = 0 \Rightarrow x = 2$

Exercise

At what points is the function $y = \dfrac{x+3}{x^2 - 3x - 10}$ continuous?

Solution

The function is continuous everywhere except when $x^2 - 3x - 10 = 0 \Rightarrow x = -2,\ 5$

Exercise

At what points is the function $y = |x - 1| + \sin x$ continuous?

Solution

The function is continuous everywhere

Exercise

At what points is the function $y = \dfrac{x+2}{\cos x}$ continuous?

Solution

The function is continuous everywhere except when $\cos x = 0 \Rightarrow x = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}$

Exercise

At what points is the function $y = \tan \dfrac{\pi x}{2}$ continuous?

Solution

The function is continuous everywhere except when $x = 2n - 1, \quad n \in \mathbb{Z}$

Exercise

At what points is the function $y = \dfrac{x \tan x}{x^2 + 1}$ continuous?

Solution

The function is continuous everywhere except when $x = (2n-1)\frac{\pi}{2}, \quad n \in \mathbb{Z}$

Exercise

At what points is the function $y = \dfrac{\sqrt{x^4 + 1}}{1 + \sin^2 x}$ continuous?

Solution

The function is continuous everywhere

Exercise

At what points is the function $y = \sqrt{2x+3}$ continuous?

Solution

The function is continuous on the interval $2x + 3 \ge 0 \quad \rightarrow \quad x \ge -\frac{3}{2} \Rightarrow \left[-\frac{3}{2},\ \infty\right)$, and discontinuous when $x < -\frac{3}{2}$

Exercise

At what points is the function $y = \sqrt[4]{3x-1}$ continuous?

Solution

The function is continuous on the interval $3x - 1 \ge 0 \rightarrow \left[\frac{1}{3},\ \infty\right)$, and discontinuous when $x < \frac{1}{3}$

Exercise

At what points is the function $y=(2-x)^{1/5}$ continuous?

Solution

The function is continuous everywhere $\forall x$

Exercise

Find $\lim\limits_{x\to\pi}\sin(x-\sin x)$, then is the function continuous at the point being approached?

Solution

$$\begin{aligned}\lim_{x\to\pi}\sin(x-\sin x)&=\sin(\pi-\sin\pi)\\&=\sin(\pi-0)\\&=\sin(\pi)\\&\underline{=0|}\end{aligned}$$

The function is continuous at $x=\pi$

Exercise

Find $\lim\limits_{x\to 0}\tan\left(\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right)$, then is the function continuous at the point being approached?

Solution

$$\begin{aligned}\lim_{x\to 0}\tan\left(\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right)&=\tan\left(\frac{\pi}{4}\cos\left(\sin(0)^{1/3}\right)\right)\\&=\tan\left(\frac{\pi}{4}\cos(0)\right)\\&=\tan\left(\frac{\pi}{4}\right)\\&\underline{=1|}\end{aligned}$$

The function is continuous at $x=0$

Exercise

Find $\lim\limits_{t\to 0}\cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right)$, then is the function continuous at the point being approached?

Solution

$$\begin{aligned}\lim_{t\to 0}\cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right)&=\cos\left(\frac{\pi}{\sqrt{19-3\sec 2(0)}}\right)\\&=\cos\left(\frac{\pi}{\sqrt{19-3}}\right)\\&=\cos\left(\frac{\pi}{\sqrt{16}}\right)\end{aligned}$$

$$= \cos\left(\frac{\pi}{4}\right)$$

$$\boxed{= \frac{\sqrt{2}}{2}} \qquad \text{The function is continuous at } t = 0$$

Exercise

Explain why the equation $\cos x = x$ has at least one solution.

Solution

$\cos x - x = 0$

$$\begin{cases} if \quad x = -\frac{\pi}{2} & \to \cos\left(-\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) > 0 \\ if \quad x = \frac{\pi}{2} & \to \cos\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right) < 0 \end{cases} \Rightarrow \cos x - x = 0 \text{ for some } x \text{ between } -\frac{\pi}{2} \text{ and } \frac{\pi}{2}$$

According to the Intermediate Value Theorem, and the function $\cos x = x$ is continuous and has at least one solution.

Exercise

Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval [–4, 4]

Solution

$f(-4) = (-4)^3 - 15(-4) + 1 = -3$

$f(-2) = (-2)^3 - 15(-2) + 1 = 23$

$f(-1) = (-1)^3 - 15(-1) + 1 = 15$

$f(1) = (1)^3 - 15(1) + 1 = -13$

$f(4) = (4)^3 - 15(4) + 1 = 5$

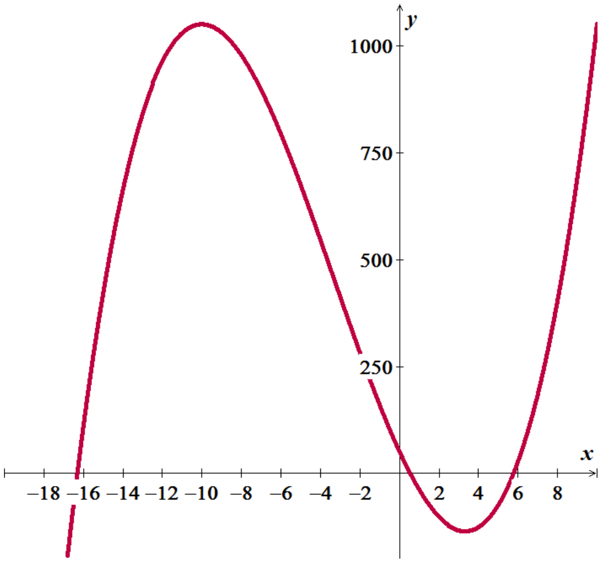
By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-4 < x < -1$, $-1 < x < 1$, and $1 < x < 4$. Thus, $x^3 - 15x + 1 = 0$ has three solutions in [–4, 4]. Since the polynomial of degree 3 can have at most 3 solutions, these are the solutions.

Exercise

Show that the equation has three solutions in the given interval $x^3+10x^2-100x+50=0;\quad (-20,\ 10)$

Solution

x	y
–19	–1299
–18	–742
–17	–273
–16	114
–15	425
–14	666
–13	962
–12	1029
–10	1050
–9	1031
–8	978
–7	897
–6	794
–5	675
–4	546
–3	413
–2	282
–1	159
0	50
1	–39
2	–102
3	–133
4	–126
5	–75
6	26



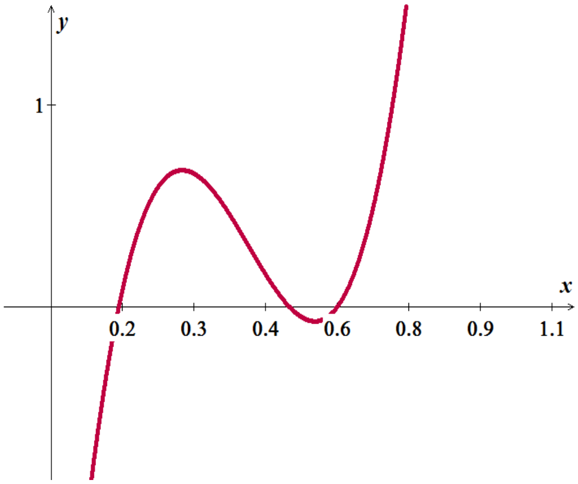
By the Intermediate Value Theorem, $f(x)=0$ for some x in each of the intervals $-17<x<-16$, $0<x<1$, and $5<x<6$.

Exercise

Show that the equation has three solutions in the given interval $70x^3 - 87x^2 + 32x - 3 = 0;\quad (0,\ 1)$

Solution

x	**y**
.05	–1.6
.1	–0.6
.15	0.08
.2	.48
.25	.656
.3	.66
.35	.543
.4	.36
.45	.161
.5	0
.55	–.07
.6	0
.65	.266
.7	.78
.75	1.6
.8	2.76
.85	4.33
.9	6.36
.95	8.9



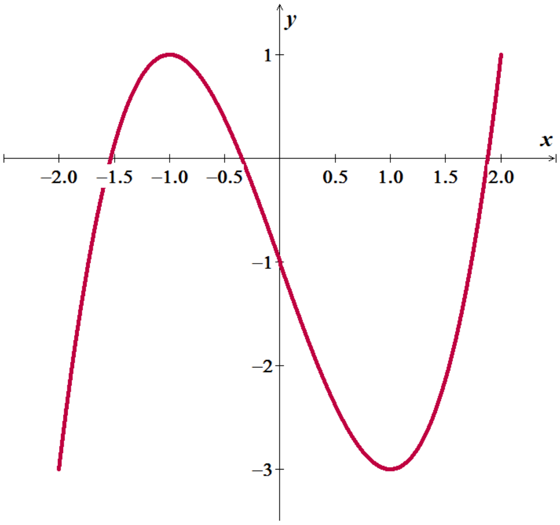
By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $0.1 < x < 0.15$, $0.5 < x < 0.55$, and $0.55 < x < 0.6$.

Exercise

Show that the equation has three solutions in the given interval $x^3 - 3x - 1 = 0;\quad [-2,\ 2]$

Solution

x	**y**
–2	–3.0
–1.75	–1.109
–1.5	0.125
–1.25	0.797
–1.0	1
–0.75	0.828
–0.5	0.375
–0.25	–0.266
0	–1.0
0.25	–1.73
0.5	–2.375
0.75	–2.828
1.0	–3.0
1.25	–2.797
1.5	–2.12
1.75	–0.89
2.	1.0



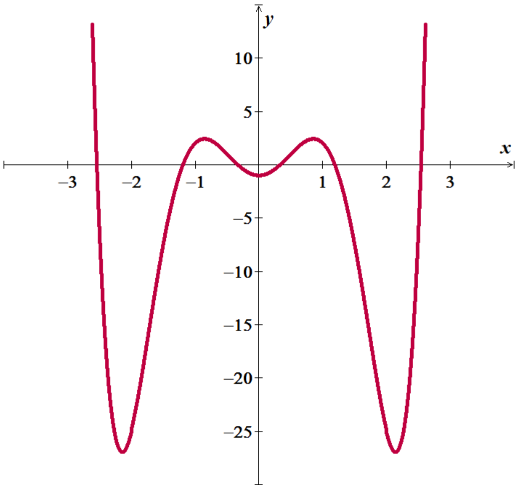
By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-1.75 < x < -1.5$, $-0.5 < x < -0.25$, and $1.75 < x < 2$.

Exercise

Show that the equation has six solutions in the given interval $x^6 - 8x^4 + 10x^2 - 1 = 0; \quad [-3,\ 3]$

Solution

x	**y**
–3.0	170.0
–2.5	–6.86
–2.0	–25.0
–1.5	–7.61
–1.0	2.0
–0.5	1.02
0.0	–1.0
0.5	1.01
1.0	2.0
1.5	–7.6
2.0	–25.0
2.5	–6.86
3.0	170.0



By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-3.0 < x < -2.5$, $-1.5 < x < -1.0$, $-0.5 \le x \le 0$, $-0.0 \le x \le 0.5$, $1.0 \le x \le 1.5$ and $2.5 < x < 3.0$.

Exercise

If functions $f(x)$ and $g(x)$ are continuous for $0 \le x \le 1$, could $\dfrac{f(x)}{g(x)}$ possibly be discontinuous at a point of [0, 1]? Give reason for your answer.

Solution

Yes, if we can get a value of $g(x)$ is between [0, 1], $x = \frac{1}{2}$ $\Rightarrow g(x) = 2x - 1$ and $f(x) = x$.

Then $\dfrac{f(x)}{g(x)} = \dfrac{x}{2x-1} \Rightarrow \dfrac{f(x)}{g(x)}$ is discontinuous at $x = \frac{1}{2}$

Exercise

Suppose that a function f is continuous on the closed interval [0, 1] and that $0 \le f(x) \le 1$ for every x in [0, 1]. Show that there must exist a number c in [0, 1] such that $f(c) = c$ (**c** is called a ***fixed point*** of f).

Solution

Let $f(x) = x \Rightarrow f(0) = 0 \quad or \quad f(1) = 1$. In these cases, c = 0 or c =1.

Let $f(0) = a > 0 \quad and \quad f(1) = b < 1$ because $0 \le f(x) \le 1$.

Define $g(x) = f(x) - x \Rightarrow g$ is continuous on [0, 1]. $\Rightarrow \begin{cases} g(0) = f(0) - 0 = a > 0 \\ g(1) = f(1) - 1 = b - 1 < 0 \end{cases}$

By the Intermediate Value Theorem there is a number c in [0, 1] such that $g(c) = 0 \Rightarrow f(c) - c = 0 \Rightarrow f(c) = c$

Exercise

Use the Intermediate Value Theorem to show that the equation $x^5 + 7x + 5 = 0$ has a solution in the interval $(-1,\ 0)$.

Solution

$f(-1) = -1 - 7 + 5 = -3 < 0$

$f(0) = 5 > 0$

By Intermediate value theorem, the function has a solution in $(-1,\ 0)$

Exercise

The amount of an antibiotic (in *mg*) in the blood *t* hours after an intravenous line is opened is given by

$$m(t) = 100\left(e^{-0.1t} - e^{-0.3t}\right)$$

a) Use the Intermediate Value Theorem to show that the amount of drug is 30 *mg* at some time in the interval $[0,\ 5]$ and again at some time in the interval $[5,\ 15]$

b) Estimate the times at which $m = 30\ mg$

c) Is the amount of drug in the blood ever 50 *mg*?

Solution

a) $m(0) = 100(1-1) = 0$

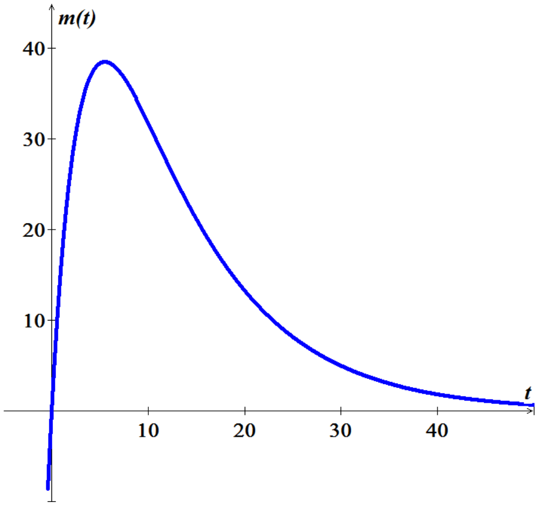
$m(5) \approx 38.34 > 30$

$m(15) \approx 21.2 < 30$

30 is an intermediate value between for both $[0,\ 5]$ and $[5,\ 15]$.

b) $m(t) = 100\left(e^{-0.1t} - e^{-0.3t}\right) = 30$

$$e^{-0.1t} - e^{-0.3t} = 0.3 \xrightarrow{\text{software}} \begin{cases} \boxed{t_1 \approx 2.4} \\ \boxed{t_2 \approx 10.8} \end{cases}$$



c) No, peak is 38.5 (using the graph)

Exercise

Determine whether the following functions are continuous at *a*. $f(x) = \dfrac{1}{x-5};\quad a = 5$

Solution

$f(5)\ \nexists$

The function is continuous everywhere except @ $x = 5$

Exercise

Determine whether the following functions are continuous at a. $h(x)=\sqrt{x^2-9}$; $a=3$

Solution

$$\lim_{x\to 3^-} h(x) \not\exists \quad \therefore\ h \text{ is discontinuous @ } 3$$

Exercise

Determine whether the following functions are continuous at a. $g(x)=\begin{cases}\dfrac{x^2-16}{x-4} & if \quad x\neq 4\\ 9 & if \quad x=4\end{cases}$; $a=4$

Solution

$$\lim_{x\to 4} g(x) = \lim_{x\to 4}\frac{(x-4)(x+4)}{x-4} = \lim_{x\to 4}(x+4) = 8 \neq 9 = g(4)$$

$\therefore$ g is discontinuous @ 4

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $f(x)=\sqrt{x^2-5}$

Solution

$\sqrt{x^2-5\geq 0} \quad \Rightarrow \quad x\leq -5$ & $x\geq 5$

The function is continuous at –5 to the left and right of $x=5$

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $f(x)=e^{\sqrt{x-2}}$

Solution

The function is continuous at and to the right of $x=2$

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $f(x)=\dfrac{2x}{x^3-25x}$

Solution

The function is continuous everywhere except at $x=0,\ \pm 5$

The function is continuous to the left of –5, then to the right of –5 to the left of 0, then to the right of 0 thru the left of 5 then to the tight of 5.

Exercise

Find the intervals on which the following functions are continuous. Specify right- or left- continuity at the endpoints $f(x) = \cos e^x$

Solution

The function is continuous everywhere.

Exercise

Let $g(x) = \begin{cases} 5x - 2 & if \quad x < 1 \\ a & if \quad x = 1 \\ ax^2 + bx & if \quad x > 1 \end{cases}$

Determine values of the constants a and b for which $g(x)$ is continuous at $x = 1$

Solution

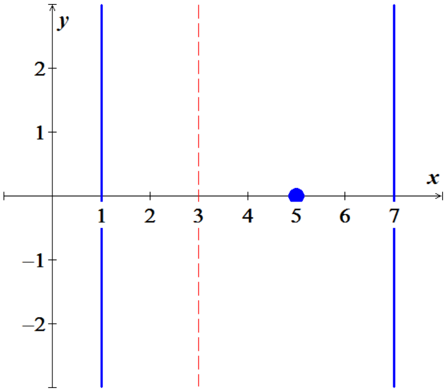
$$\lim_{x \to 1^-} g(x) = g(1) = 5 - 2 = \underline{3 = a|}$$

$$\lim_{x \to 1^-} g(x) = g(1) = a + b = 3 + b = 3 \quad \Rightarrow \quad \underline{b = 0|}$$

Solution Section 1.6 – Precise Definition of Limits

Exercise

Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0<\left|x-x_0\right|<\delta \;\Rightarrow\; a<x<b$ for $a=1,\;\; b=7,\;\; x_0=5$



Solution

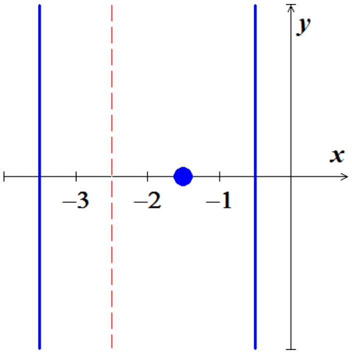
$$\left|x-5\right|<\delta \;\Rightarrow\; -\delta<x-5<\delta$$
$$-\delta+5<x<\delta+5$$

$$-\delta+5=1 \;\Rightarrow\; \boxed{\delta=4}$$

$$\delta+5=7 \;\Rightarrow\; \boxed{\delta=2}$$

Exercise

Sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0<\left|x-x_0\right|<\delta \;\Rightarrow\; a<x<b$ for $a=-\frac{7}{2},\;\; b=-\frac{1}{2},\;\; x_0=-\frac{3}{2}$



Solution

$$\left|x+\frac{3}{2}\right|<\delta \;\Rightarrow\; -\delta<x+\frac{3}{2}<\delta$$
$$-\delta-\frac{3}{2}<x<\delta-\frac{3}{2}$$

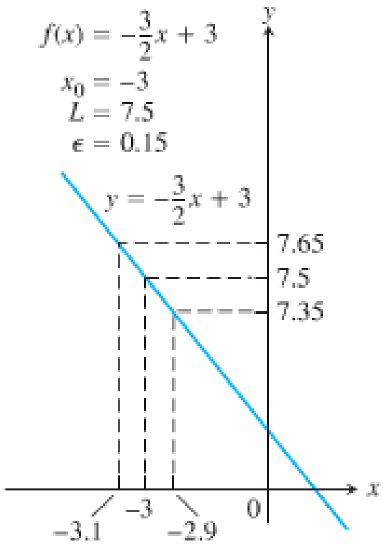
$$-\delta-\frac{3}{2}=-\frac{7}{2} \;\Rightarrow\; \boxed{\delta=\frac{7}{2}-\frac{3}{2}=2}$$

$$\delta-\frac{3}{2}=-\frac{1}{2} \;\Rightarrow\; \boxed{\delta=\frac{1}{2}-\frac{3}{2}=-1}$$

Exercise

Use the graph to find a $\delta > 0$ such that for all x

$$0<\left|x-x_0\right|<\delta \;\Rightarrow\; \left|f(x)-L\right|<\varepsilon$$



Solution

Given: $a=-3.1,\;\; b=-2.9,\;\; x_0=-3$

$$\left|x+3\right|<\delta \;\Rightarrow\; -\delta<x+3<\delta$$
$$-\delta-3<x<\delta-3$$

$$-\delta-3=-3.1 \;\Rightarrow\; \boxed{\delta=3.1-3=0.1}$$

$$\delta-3=-2.9 \;\Rightarrow\; \boxed{\delta=3-2.9=0.1}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = x + 1, \quad L = 5, \quad x_0 = 4, \quad \varepsilon = 0.01$$

Solution

$$|(x+1) - 5| < .01 \quad \Rightarrow \quad |x - 4| < .01$$

$$-.01 < x - 4 < .01$$

$$-.01 + 4 < x - 4 + 4 < .01 + 4$$

$$3.99 < x < 4.01$$

$$|x - 4| < \delta \quad \Rightarrow \quad -\delta < x - 4 < \delta$$

$$-\delta + 4 < x < \delta + 4$$

$$-\delta + 4 = 3.99 \;\Rightarrow\; \underline{\delta} = 4 - 3.99 = \underline{0.01}$$

$$\delta + 4 = 4.01 \;\Rightarrow\; \underline{\delta} = 4.01 - 4 = \underline{0.01}$$

$$\Rightarrow \underline{\delta = .01}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

$$f(x) = \sqrt{x+1}, \quad L = 1, \quad x_0 = 0, \quad \varepsilon = 0.1$$

Solution

$$\left|\sqrt{x+1} - 1\right| < 0.1 \quad \Rightarrow \quad -0.1 < \sqrt{x+1} - 1 < 0.1$$

$$-0.1 + 1 < \sqrt{x+1} - 1 + 1 < 0.1 + 1$$

$$.9 < \sqrt{x+1} < 1.1$$

$$(.9)^2 < \left(\sqrt{x+1}\right)^2 < (1.1)^2$$

$$.81 < x + 1 < 1.21$$

$$.81 - 1 < x + 1 - 1 < 1.21 - 1$$

$$-0.19 < x < 0.21$$

$$|x - 0| < \delta \quad \Rightarrow \quad -\delta < x < \delta$$

$$\left.\begin{aligned} -\delta = -0.19 \;\Rightarrow\; &\underline{\delta = 0.19} \\ &\underline{\delta = 0.21} \end{aligned}\right. \;\rightarrow\; \boxed{\delta = 0.19}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=\sqrt{x-7}, \quad L=4, \quad x_0=23, \quad \varepsilon=1$$

Solution

$$\left|\sqrt{x-7}-4\right|<1 \quad \Rightarrow \quad -1<\sqrt{x-7}-4<1$$

$$3<\sqrt{x-7}<5$$

$$(3)^2<\left(\sqrt{x-7}\right)^2<(5)^2$$

$$9<x-7<25$$

$$9+7<x-7+7<25+7$$

$$16<x<32$$

$$|x-23|<\delta \quad \Rightarrow \quad -\delta<x-23<\delta$$

$$-\delta+23<x<\delta+23$$

$$\left.\begin{array}{l} -\delta+23=16 \ \Rightarrow \ \delta=23-16=\underline{7} \\ \delta+23=32 \ \Rightarrow \ \delta=32-23=\underline{9} \end{array}\right\} \rightarrow \boxed{\delta=7}$$

Exercise

Find an open interval about x_0 on which the inequality $|f(x)-L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-x_0|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

$$f(x)=x^2, \quad L=3, \quad x_0=\sqrt{3}, \quad \varepsilon=0.1$$

Solution

$$\left|x^2-3\right|<0.1 \quad \Rightarrow \quad -0.1<x^2-3<0.1$$

$$2.9<x^2<3.1$$

$$\sqrt{2.9}<x<\sqrt{3.1}$$

$$\left|x-\sqrt{3}\right|<\delta \quad \Rightarrow \quad -\delta<x-\sqrt{3}<\delta$$

$$-\delta+\sqrt{3}<x<\delta+\sqrt{3}$$

$$\begin{array}{l} -\delta+\sqrt{3}=\sqrt{2.9} \ \Rightarrow \ \delta=\sqrt{3}-\sqrt{2.9}=\underline{.029} \\ \delta+\sqrt{3}=\sqrt{3.1} \ \Rightarrow \ \delta=\sqrt{3.1}-\sqrt{3}=\underline{.029} \end{array} \rightarrow \boxed{\delta=.029}$$

Exercise

Find an open interval about x_0 on which the inequality $\left|f(x)-L\right|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<\left|x-x_0\right|<\delta$ the inequality $\left|f(x)-L\right|<\varepsilon$ holds.

$$f(x)=\frac{120}{x}, \quad L=5, \quad x_0=24, \quad \varepsilon=1$$

Solution

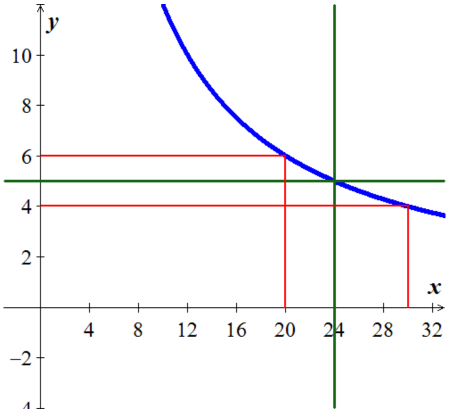
$$\left|\frac{120}{x}-5\right|<0.1 \quad \Rightarrow \quad -1<\frac{120}{x}-5<1$$

$$4<\frac{120}{x}<6$$

$$\frac{1}{6}<\frac{x}{120}<\frac{1}{4}$$

$$\frac{1}{6}(120)<x<\frac{1}{4}(120)$$

$$20<x<30$$



$$|x-24|<\delta \quad \Rightarrow \quad -\delta<x-24<\delta$$

$$-\delta+24<x<\delta+24$$

$$\left.\begin{aligned} -\delta+24=20 &\Rightarrow \lfloor \delta=24-20=4 \rfloor \\ \delta+24=30 &\Rightarrow \lfloor \delta=30-24=6 \rfloor \end{aligned}\right. \to \boxed{\delta=4}$$

Exercise

Prove that $\displaystyle\lim_{x\to 4}(9-x)=5$

Solution

$$\left|(9-x)-5\right|<\varepsilon \quad \Rightarrow \quad -\varepsilon<4-x<\varepsilon$$

$$-\varepsilon-4<-x<\varepsilon-4$$ ***divide by*** $(-)$.

$$\varepsilon+4>x>4-\varepsilon$$

$$4-\varepsilon<x<\varepsilon+4$$

$$|x-4|<\delta \quad \Rightarrow \quad -\delta<x-4<\delta$$

$$-\delta+4<x<\delta+4$$

$$\left.\begin{aligned} -\delta+4=4-\varepsilon &\Rightarrow -\delta=-\varepsilon \Rightarrow \delta=\varepsilon \\ \delta+4=\varepsilon+4 &\Rightarrow \delta=\varepsilon \end{aligned}\right. \to \boxed{\delta=\varepsilon}$$

Exercise

Prove that $\lim\limits_{x\to 1}\frac{1}{x}=1$

Solution

$$\left|\frac{1}{x}-1\right|<\varepsilon \quad\Rightarrow\quad -\varepsilon<\frac{1}{x}-1<\varepsilon$$

$$-\varepsilon+1<\frac{1}{x}<\varepsilon+1$$

$$\frac{1}{\varepsilon+1}>x>\frac{1}{-\varepsilon+1}$$

$$\frac{1}{1+\varepsilon}<x<\frac{1}{1-\varepsilon}$$

$$|x-1|<\delta \quad\Rightarrow\quad -\delta<x-1<\delta$$

$$1-\delta<x<1+\delta$$

$$\left.\begin{array}{l} 1-\delta=\frac{1}{1+\varepsilon} \quad\Rightarrow\quad \delta=1+\frac{1}{1+\varepsilon}=\frac{2+\varepsilon}{1+\varepsilon} \\ 1+\delta=\frac{1}{1-\varepsilon} \quad\Rightarrow\quad \delta=\frac{1}{1-\varepsilon}-1=\frac{\varepsilon}{1-\varepsilon} \end{array}\right. \to \textit{the smallest}: \boxed{\delta=\frac{\varepsilon}{1-\varepsilon}}$$

Exercise

Prove that $\lim\limits_{x\to 5}\frac{x^2-25}{x-5}=10$

Solution

$$\left|\frac{x^2-25}{x-5}-10\right|<\varepsilon \quad\Rightarrow\quad -\varepsilon<\frac{(x-5)(x+5)}{x-5}-10<\varepsilon$$

$$-\varepsilon+10<x+5<\varepsilon+10$$

$$-\varepsilon+5<x<\varepsilon+15$$

$$|x-10|<\delta \quad\Rightarrow\quad -\delta<x-10<\delta$$

$$10-\delta<x<10+\delta$$

$$\left.\begin{array}{l} 10-\delta=5-\varepsilon \quad\Rightarrow\quad \delta=5+\varepsilon \\ 10+\delta=\varepsilon+15 \quad\Rightarrow\quad \delta=\varepsilon+5 \end{array}\right. \to \textit{the smallest}: \boxed{\delta=\varepsilon+5}$$

Exercise

Prove that $\lim\limits_{x\to 0} f(x)=0 \quad \textit{if} \quad f(x)=\begin{cases} 2x, & x<0 \\ \frac{x}{2}, & x\ge 0 \end{cases}$

Solution

For $x<0$: $|2x-0|<\varepsilon \quad\Rightarrow\quad -\varepsilon<2x<0$

$$-\frac{\varepsilon}{2}<x<0$$

For $x\ge 0$: $\left|\frac{x}{2}-0\right|<\varepsilon \quad\Rightarrow\quad 0\le\frac{x}{2}<\varepsilon$

$$0 \le x < 2\varepsilon$$

$$|x-0| < \delta \quad \Rightarrow \quad -\delta < x < \delta$$

$$-\delta = -\tfrac{\varepsilon}{2} \;\Rightarrow\; \delta = \tfrac{\varepsilon}{2} \quad \to \textit{ the smallest}: \boxed{\delta = \frac{\varepsilon}{2}}$$

$$\delta = 2\varepsilon$$

Exercise

Prove that $\lim\limits_{x\to 1} (5x-2) = 3$

Solution

$$|(5x-2)-3| < \varepsilon \quad \Rightarrow \quad -\varepsilon < 5x-5 < \varepsilon$$

$$5-\varepsilon < 5x < \varepsilon + 5$$

$$1-\tfrac{1}{5}\varepsilon < x < 1+\tfrac{1}{5}\varepsilon$$

$$|x-3| < \delta \quad \Rightarrow \quad -\delta < x-3 < \delta$$

$$3-\delta < x < 3+\delta$$

$$3-\delta = 1-\tfrac{1}{5}\varepsilon \quad \Rightarrow \quad \delta = \tfrac{1}{5}\varepsilon + 2$$

$$3+\delta = 1+\tfrac{1}{5}\varepsilon \quad \Rightarrow \quad \delta = \tfrac{1}{5}\varepsilon - 2$$

$$\to \textit{ the smallest}: \boxed{\delta = \tfrac{1}{5}\varepsilon - 2}$$

Exercise

Prove that $\lim\limits_{x\to 2} \dfrac{1}{(x-2)^4} = \infty$

Solution

Let $N > 0$ and let $\delta = \dfrac{1}{\sqrt[4]{N}}$

Suppose that $0 < |x-2| < \delta$

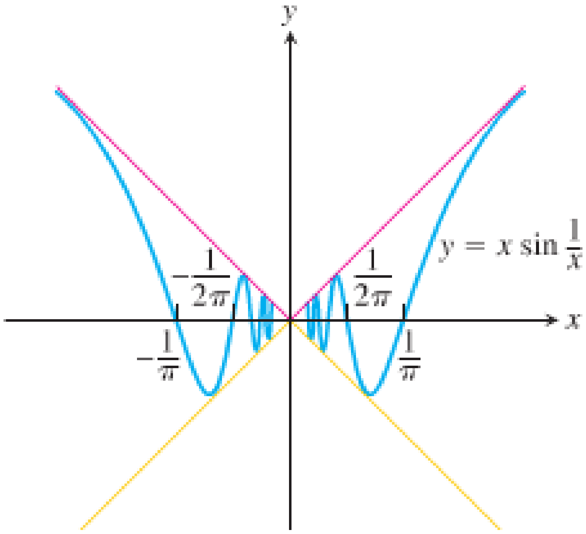
$$|x-2| < \delta = \frac{1}{\sqrt[4]{N}}$$

$$\frac{1}{|x-2|} > \sqrt[4]{N}$$

$$\boxed{\frac{1}{(x-2)^4} > N} \quad ✓$$

Exercise

Prove that $\lim\limits_{x\to 0} x\dfrac{1}{\sin x} = 0$



Solution

$$\left.\begin{array}{ll} -x \le x\sin\frac{1}{x} \le x, & \forall x > 0 \\ -x \ge x\sin\frac{1}{x} \ge x, & \forall x < 0 \end{array}\right\} \to \lim_{x\to 0}(-x) = \lim_{x\to 0}(x) = 0$$

Then by the sandwich theorem, $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$