

## ***Solution***      **Section 2.4 – Cross Product**

### ***Exercise***

Prove when the cross product  $\vec{u} \times \vec{v}$  is perpendicular to  $\vec{u}$ , then  $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$

### **Solution**

Let  $\vec{u} = (u_1, u_2, u_3)$  and  $\vec{v} = (v_1, v_2, v_3)$

$$\begin{aligned}\vec{u} \cdot (\vec{u} \times \vec{v}) &= (u_1, u_2, u_3) \cdot (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1) \\ &= u_1(u_2 v_3 - u_3 v_2) + u_2(u_3 v_1 - u_1 v_3) + u_3(u_1 v_2 - u_2 v_1) \\ &= u_1 u_2 v_3 - u_1 u_3 v_2 + u_2 u_3 v_1 - u_2 u_1 v_3 + u_3 u_1 v_2 - u_3 u_2 v_1 \\ &= 0\end{aligned}$$

### ***Exercise***

Find  $\vec{u} \times \vec{v}$ , where  $\vec{u} = (1, 2, -2)$  and  $\vec{v} = (3, 0, 1)$  and show that  $\vec{u} \times \vec{v}$  is perpendicular to  $\vec{u}$  and to  $\vec{v}$ .

### **Solution**

$$\begin{aligned}\vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} \\ &= \left( \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \right) \\ &= (2, -7, -6)\end{aligned}$$

$$\begin{aligned}\vec{u} \cdot (\vec{u} \times \vec{v}) &= (1, 2, -2) \cdot (2, -7, -6) \\ &= 2 - 14 + 12 \\ &= 0\end{aligned}$$

$$\begin{aligned}\vec{v} \cdot (\vec{u} \times \vec{v}) &= (3, 0, 1) \cdot (2, -7, -6) \\ &= 6 - 0 - 6 \\ &= 0\end{aligned}$$

$\vec{u} \times \vec{v}$  is orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

### Exercise

Given  $\vec{u} = (3, 2, -1)$ ,  $\vec{v} = (0, 2, -3)$ , and  $\vec{w} = (2, 6, 7)$  Compute the vectors

a)  $\vec{u} \times \vec{v}$

b)  $\vec{v} \times \vec{w}$

c)  $\vec{u} \times (\vec{v} \times \vec{w})$

d)  $(\vec{u} \times \vec{v}) \times \vec{w}$

e)  $\vec{u} \times (\vec{v} - 2\vec{w})$

### Solution

$$\begin{aligned} \text{a) } \vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ 0 & 2 & -3 \end{vmatrix} \\ &= \left( \begin{vmatrix} 2 & -1 \\ 2 & -3 \end{vmatrix}, -\begin{vmatrix} 3 & -1 \\ 0 & -3 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} \right) \\ &= \underline{(-4, 9, 6)} \end{aligned}$$

$$\begin{aligned} \text{b) } \vec{v} \times \vec{w} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & -3 \\ 2 & 6 & 7 \end{vmatrix} \\ &= \left( \begin{vmatrix} 2 & -3 \\ 6 & 7 \end{vmatrix}, -\begin{vmatrix} 0 & -3 \\ 2 & 7 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 2 & 6 \end{vmatrix} \right) \\ &= \underline{(32, -6, -4)} \end{aligned}$$

$$\begin{aligned} \text{c) } \vec{u} \times (\vec{v} \times \vec{w}) &= (3, 2, -1) \times (32, -6, -4) \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ 32 & -6 & -4 \end{vmatrix} \\ &= \left( \begin{vmatrix} 2 & -1 \\ -6 & -4 \end{vmatrix}, -\begin{vmatrix} 3 & -1 \\ 32 & -4 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ 32 & -6 \end{vmatrix} \right) \\ &= \underline{(-14, -20, -82)} \end{aligned}$$

$$\begin{aligned} \text{d) } (\vec{u} \times \vec{v}) \times \vec{w} &= (-4, 9, 6) \times (2, 6, 7) \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 9 & 6 \\ 2 & 6 & 7 \end{vmatrix} \end{aligned}$$

$$= \left( \begin{vmatrix} 9 & 6 \\ 6 & 7 \end{vmatrix}, -\begin{vmatrix} -4 & 6 \\ 2 & 7 \end{vmatrix}, \begin{vmatrix} -4 & 9 \\ 2 & 6 \end{vmatrix} \right) \\ = \underline{(27, 40, -42)}$$

$$\begin{aligned} e) \quad u \times (v - 2w) &= (3, 2, -1) \times [(0, 2, -3) - 2(2, 6, 7)] \\ &= (3, 2, -1) \times (-4, -10, -17) \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ -4 & -10 & -17 \end{vmatrix} \\ &= \left( \begin{vmatrix} 2 & -1 \\ -10 & -17 \end{vmatrix}, -\begin{vmatrix} 3 & -1 \\ -4 & -17 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ -4 & -10 \end{vmatrix} \right) \\ &= \underline{(-44, 47, -22)} \end{aligned}$$

### Exercise

Use the cross product to find a vector that is orthogonal to both

- a)  $\vec{u} = (-6, 4, 2), \quad \vec{v} = (3, 1, 5)$
- b)  $\vec{u} = (1, 1, -2), \quad \vec{v} = (2, -1, 2)$
- c)  $\vec{u} = (-2, 1, 5), \quad \vec{v} = (3, 0, -3)$

### Solution

$$\begin{aligned} a) \quad \vec{u} \times \vec{v} &= (-6, 4, 2) \times (3, 1, 5) \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6 & 4 & 2 \\ 3 & 1 & 5 \end{vmatrix} \\ &= \left( \begin{vmatrix} 4 & 2 \\ 1 & 5 \end{vmatrix}, -\begin{vmatrix} -6 & 2 \\ 3 & 5 \end{vmatrix}, \begin{vmatrix} -6 & 4 \\ 3 & 1 \end{vmatrix} \right) \\ &= \underline{(18, 36, -18)} \end{aligned}$$

$$\begin{aligned} b) \quad \vec{u} \times \vec{v} &= (1, 1, -2) \times (2, -1, 2) \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ 2 & -1 & 2 \end{vmatrix} \\ &= \left( \begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix}, -\begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \right) \end{aligned}$$

$$= \underline{(0, -6, -3)}$$

$$c) \vec{u} \times \vec{v} = (-2, 1, 5) \times (3, 0, -3)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 5 \\ 3 & 0 & -3 \end{vmatrix}$$

$$= \left( \begin{vmatrix} 1 & 5 \\ 0 & -3 \end{vmatrix}, -\begin{vmatrix} -2 & 5 \\ 3 & -3 \end{vmatrix}, \begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} \right)$$

$$= \underline{(-3, 9, -3)}$$

### Exercise

Find the area of the parallelogram determined by the given vectors

$$a) \vec{u} = (1, -1, 2) \text{ and } \vec{v} = (0, 3, 1)$$

$$b) \vec{u} = (3, -1, 4) \text{ and } \vec{v} = (6, -2, 8)$$

$$c) \vec{u} = (2, 3, 0) \text{ and } \vec{v} = (-1, 2, -2)$$

### Solution

$$a) \text{ Area} = \|\vec{u} \times \vec{v}\|$$

$$= \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 0 & 3 & 1 \end{vmatrix} \right\|$$

$$= \left| \left( \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} \right) \right|$$

$$= |(-7, -1, 3)|$$

$$= \sqrt{7^2 + 1^2 + 3^2}$$

$$= \underline{\sqrt{59}} \quad (\text{unit}^2)$$

$$b) \text{ Area} = \|\vec{u} \times \vec{v}\|$$

$$= \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 4 \\ 6 & -2 & 8 \end{vmatrix} \right\|$$

$$= \left| \left( \begin{vmatrix} -1 & 4 \\ -2 & 8 \end{vmatrix}, -\begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix}, \begin{vmatrix} 3 & -1 \\ 6 & -2 \end{vmatrix} \right) \right|$$

$$= |(0, 0, 0)|$$

$$= 0$$

$$\begin{aligned} \text{c) Area} &= \|\vec{u} \times \vec{v}\| \\ &= \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 0 \\ -1 & 2 & -2 \end{vmatrix} \right\| \\ &= \left| \left( \begin{vmatrix} 3 & 0 \\ 2 & -2 \end{vmatrix}, -\begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} \right) \right| \\ &= |(-6, 4, 7)| \\ &= \sqrt{(-6)^2 + 4^2 + 7^2} \\ &= \sqrt{101} \quad (\text{unit}^2) \end{aligned}$$

### Exercise

Find the area of the parallelogram with the given vertices  $P_1(3,2)$ ,  $P_2(5,4)$ ,  $P_3(9,4)$ ,  $P_4(7,2)$

### Solution

$$\overrightarrow{P_1P_2} = (5-3, 4-2) = (2, 2)$$

$$\overrightarrow{P_4P_3} = (9-7, 4-2) = (2, 2)$$

$$\overrightarrow{P_1P_4} = (7-3, 2-2) = (4, 0)$$

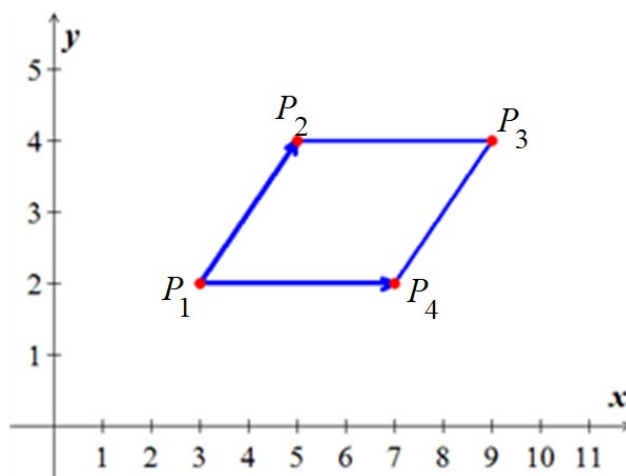
$$\overrightarrow{P_2P_3} = (9-5, 4-4) = (4, 0)$$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_4} = (2, 2) \times (4, 0)$$

$$\begin{aligned} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 0 \\ 4 & 0 & 0 \end{vmatrix} \\ &= \left( \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix}, -\begin{vmatrix} 2 & 0 \\ 4 & 0 \end{vmatrix}, \begin{vmatrix} 2 & 2 \\ 4 & 0 \end{vmatrix} \right) \\ &= (0, 0, -8) \end{aligned}$$

The area of the parallelogram is

$$\begin{aligned} \|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_4}\| &= \sqrt{0^2 + 0^2 + (-8)^2} \\ &= 8 \end{aligned}$$



### Exercise

Find the area of the triangle with the given vertices:

- a)  $A(2, 0)$   $B(3, 4)$   $C(-1, 2)$
- b)  $A(1, 1)$   $B(2, 2)$   $C(3, -3)$
- c)  $P(2, 6, -1)$   $Q(1, 1, 1)$   $R(4, 6, 2)$

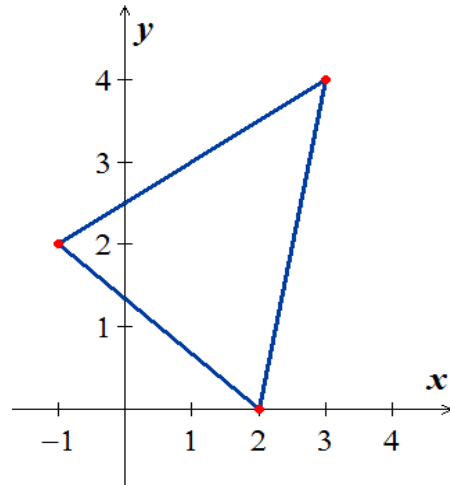
### Solution

$$\begin{aligned} \text{a) } \overrightarrow{AB} &= (1, 4) & \overrightarrow{AC} &= (-3, 2) \\ \overrightarrow{AB} \times \overrightarrow{AC} &= (1, 4, 0) \times (-3, 2, 0) \\ &= \left( \begin{vmatrix} 4 & 0 \\ 2 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ -3 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 4 \\ -3 & 2 \end{vmatrix} \right) \\ &= (0, 0, 14) \end{aligned}$$

$$\begin{aligned} \|\overrightarrow{AB} \times \overrightarrow{AC}\| &= \sqrt{0+0+14^2} \\ &= 14 \end{aligned}$$

The area of the triangle is

$$\begin{aligned} \frac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AC}\| &= \frac{1}{2} 14 \\ &= 7 \end{aligned}$$



$$\begin{aligned} \text{b) } \overrightarrow{AB} &= (1, 1) & \overrightarrow{AC} &= (2, -4) \\ \overrightarrow{AB} \times \overrightarrow{AC} &= (1, 1, 0) \times (2, -4, 0) \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 2 & -4 & 0 \end{vmatrix} \\ &= \left( \begin{vmatrix} 1 & 0 \\ -4 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} \right) \\ &= (0, 0, -6) \end{aligned}$$

$$\begin{aligned} \|\overrightarrow{AB} \times \overrightarrow{AC}\| &= \sqrt{0+0+(-6)^2} \\ &= 6 \end{aligned}$$

The area of the triangle is

$$\begin{aligned} \frac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AC}\| &= \frac{1}{2} (6) \\ &= 3 \end{aligned}$$

$$c) \quad \overrightarrow{PQ} = (-1, -5, 2) \quad \overrightarrow{PR} = (2, 0, 3)$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (-1, -5, 2) \times (2, 0, 3)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -5 & 2 \\ 2 & 0 & 3 \end{vmatrix} \quad \begin{array}{ccccc} -1 & -5 & 2 & -1 & -5 \\ 2 & 0 & 3 & 2 & 0 \end{array}$$

$$= (-15, 7, 10)$$

$$\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \sqrt{(-15)^2 + 7^2 + 10^2}$$

$$= \sqrt{374}$$

The area of the triangle is

$$\frac{1}{2} \|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \frac{1}{2} \sqrt{374} \quad \text{unit}^2$$

### Exercise

a) Find the area of the parallelogram with edges  $v = (3, 2)$  and  $w = (1, 4)$

b) Find the area of the triangle with sides  $\vec{v}$ ,  $\vec{w}$ , and  $\vec{v} + \vec{w}$ . Draw it.

c) Find the area of the triangle with sides  $\vec{v}$ ,  $\vec{w}$ , and  $\vec{v} - \vec{w}$ . Draw it.

### Solution

$$a) \quad \text{Area} = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} \\ = 10$$

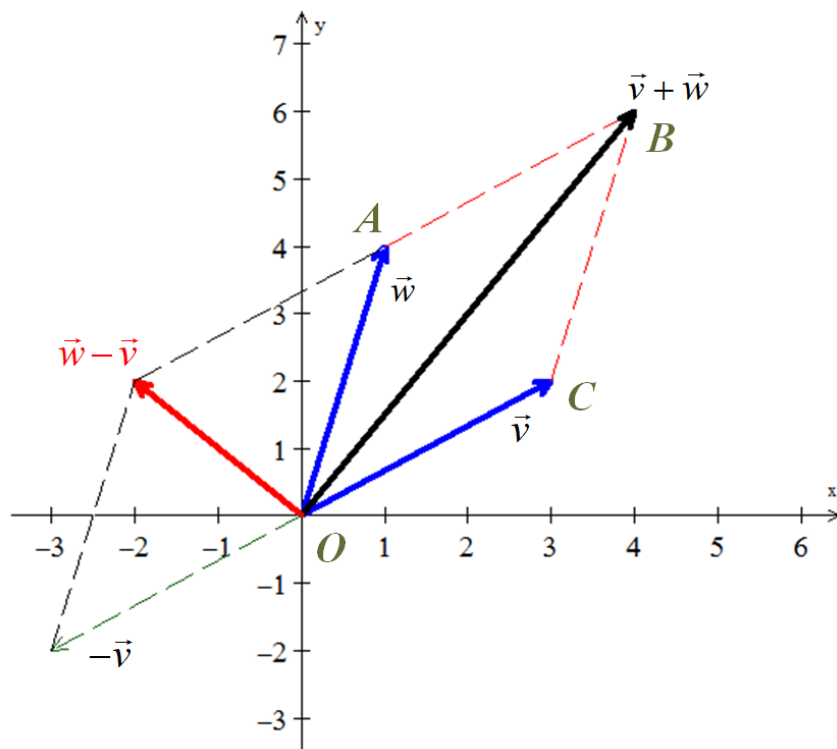
which is the parallelogram  $OABC$

b) The area of the triangle with sides  $\vec{v}$ ,  $\vec{w}$ , and  $\vec{v} + \vec{w}$  is the triangle  $OCB$  or  $OAB$  which it is half the parallelogram (by definition).

$$\text{Area} = 5$$

$$\vec{v} + \vec{w} = (3, 2) + (1, 4) \\ = (4, 6)$$

$$\text{Area} = \frac{1}{2} \begin{vmatrix} 3 & 4 \\ 2 & 6 \end{vmatrix} \\ = \frac{1}{2}(10) \\ = 5$$



- c) The area of the triangle with sides  $\vec{v}$ ,  $\vec{w}$ , and  $\vec{v} - \vec{w}$  is equivalent to the triangle  $OAC$  which it is half the parallelogram (by definition).

$$\begin{aligned}
 \text{Area} &= 5 \\
 \text{Area} &= \frac{1}{2} \begin{vmatrix} 2 & -2 \\ -3 & -2 \end{vmatrix} \\
 &= \frac{1}{2} |-10| \\
 &= \underline{5}
 \end{aligned}$$

### Exercise

Find the volume of the parallelepiped with sides  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$ .

a)  $\vec{u} = (2, -6, 2)$ ,  $\vec{v} = (0, 4, -2)$ ,  $\vec{w} = (2, 2, -4)$

b)  $\vec{u} = (3, 1, 2)$ ,  $\vec{v} = (4, 5, 1)$ ,  $\vec{w} = (1, 2, 4)$

### Solution

$$\begin{aligned}
 \text{a) } \vec{u} \cdot (\vec{v} \times \vec{w}) &= \begin{vmatrix} 2 & -6 & 2 \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{vmatrix} \\
 &= \underline{-16}
 \end{aligned}$$

The volume of the parallelepiped is  $|-16| = \underline{16} \text{ unit}^3$



$$b) \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 3 & 1 & 2 \\ 4 & 5 & 1 \\ 1 & 2 & 4 \end{vmatrix} \\ = \underline{45}$$

The volume of the parallelepiped is  $45 \text{ unit}^3$

### Exercise

Compute the scalar triple product  $\vec{u} \cdot (\vec{v} \times \vec{w})$

$$a) \quad \vec{u} = (-2, 0, 6), \quad \vec{v} = (1, -3, 1), \quad \vec{w} = (-5, -1, 1)$$

$$b) \quad \vec{u} = (-1, 2, 4), \quad \vec{v} = (3, 4, -2), \quad \vec{w} = (-1, 2, 5)$$

$$c) \quad \vec{u} = (a, 0, 0), \quad \vec{v} = (0, b, 0), \quad \vec{w} = (0, 0, c)$$

$$d) \quad \vec{u} = 3\hat{i} - 2\hat{j} - 5\hat{k}, \quad \vec{v} = \hat{i} + 4\hat{j} - 4\hat{k}, \quad \vec{w} = 3\hat{j} + 2\hat{k}$$

$$e) \quad \vec{u} = (3, -1, 6) \quad \vec{v} = (2, 4, 3) \quad \vec{w} = (5, -1, 2)$$

### Solution

$$a) \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} -2 & 0 & 6 \\ 1 & -3 & 1 \\ -5 & -1 & 1 \end{vmatrix} \\ = \underline{-92}$$

$$b) \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} -1 & 2 & 4 \\ 3 & 4 & -2 \\ -1 & 2 & 5 \end{vmatrix} \\ = \underline{-10}$$

$$c) \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \\ = \underline{abc}$$

$$d) \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} \\ = \underline{49}$$

$$e) \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 3 & -1 & 6 \\ 2 & 4 & 3 \\ 5 & -1 & 2 \end{vmatrix} \\ = -110 \quad |$$

### Exercise

Use the cross product to find the sine of the angle between the vectors  $\vec{u} = (2, 3, -6)$ ,  $\vec{v} = (2, 3, 6)$

### Solution

$$\vec{u} \times \vec{v} = (2, 3, -6) \times (2, 3, 6) \\ = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -6 \\ 2 & 3 & 6 \end{vmatrix} \\ = \left( \begin{vmatrix} 3 & -6 \\ 3 & 6 \end{vmatrix}, - \begin{vmatrix} 2 & -6 \\ 2 & 6 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} \right) \\ = (36, -24, 0) \quad |$$

$$\|\vec{u} \times \vec{v}\| = \sqrt{36^2 + (-24)^2 + 0} \\ = \sqrt{1872} \\ = 12\sqrt{13} \quad |$$

$$\sin \theta = \left( \frac{\|\vec{u} \times \vec{v}\|}{\|\vec{u}\| \|\vec{v}\|} \right) \\ = \frac{12\sqrt{13}}{\sqrt{2^2 + 3^2 + (-6)^2} \sqrt{2^2 + 3^2 + 6^2}} \\ = \frac{12\sqrt{13}}{(7)(7)} \\ = \frac{12}{49} \sqrt{13} \quad |$$

### Exercise

Simplify  $(\vec{u} + \vec{v}) \times (\vec{u} - \vec{v})$

### Solution

$$\begin{aligned}(\vec{u} + \vec{v}) \times (\vec{u} - \vec{v}) &= (\vec{u} + \vec{v}) \times \vec{u} - (\vec{u} + \vec{v}) \times \vec{v} \\&= (\vec{u} \times \vec{u}) + (\vec{v} \times \vec{u}) - [(\vec{u} \times \vec{v}) + (\vec{v} \times \vec{v})] \\&= 0 + (\vec{v} \times \vec{u}) - [(\vec{u} \times \vec{v}) + 0] \\&= (\vec{v} \times \vec{u}) - (\vec{u} \times \vec{v}) \\&= (\vec{v} \times \vec{u}) - (-(\vec{v} \times \vec{u})) \\&= (\vec{v} \times \vec{u}) + (\vec{v} \times \vec{u}) \\&= \underline{2(\vec{v} \times \vec{u})} \quad | \end{aligned}$$

### Exercise

Prove Lagrange's identity:  $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$

### Solution

Let  $\vec{u} = (u_1, u_2, u_3)$  and  $\vec{v} = (v_1, v_2, v_3)$

$$\|\vec{u}\|^2 = u_1^2 + u_2^2 + u_3^2$$

$$\|\vec{v}\|^2 = v_1^2 + v_2^2 + v_3^2$$

$$(\vec{u} \cdot \vec{v})^2 = (u_1 v_1 + u_2 v_2 + u_3 v_3)^2$$

$$\|\vec{u} \times \vec{v}\|^2 = (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2$$

$$= u_2^2 v_3^2 - 2u_2 v_3 u_3 v_2 + u_3^2 v_2^2 + u_3^2 v_1^2 - 2u_3 v_1 u_1 v_3 + u_1^2 v_3^2 + u_1^2 v_2^2 - 2u_2 v_1 u_2 v_1 + u_2^2 v_1^2$$

$$\|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2 = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2$$

$$\begin{aligned}
&= u_1^2 v_1^2 + u_1^2 v_2^2 + u_1^2 v_3^2 + u_2^2 v_1^2 + u_2^2 v_2^2 + u_2^2 v_3^2 + u_3^2 v_1^2 + u_3^2 v_2^2 + u_3^2 v_3^2 \\
&\quad - u_1^2 v_1^2 - u_1 v_1 u_2 v_2 - u_1 v_1 u_3 v_3 \\
&\quad - u_2 v_2 u_1 v_1 - u_2^2 v_2^2 - u_2 v_2 u_3 v_3 \\
&\quad - u_1 v_1 u_3 v_3 - u_2 v_2 u_3 v_3 - u_3^2 v_3^2 \\
&= u_2^2 v_3^2 - 2u_2 v_2 u_3 v_3 + u_3^2 v_2^2 \\
&\quad + u_3^2 v_1^2 - 2u_1 v_1 u_3 v_3 + u_1^2 v_3^2 \\
&\quad + u_1^2 v_2^2 - 2u_1 v_1 u_2 v_2 + u_2^2 v_1^2
\end{aligned}$$

$$\Rightarrow \underline{\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2}$$

### Exercise

Polar coordinates satisfy  $x = r \cos \theta$  and  $y = r \sin \theta$ . Polar area  $J dr d\theta$  includes  $J$ :

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

The two columns are orthogonal. Their lengths are \_\_\_\_\_. Thus  $J =$  \_\_\_\_\_.

### Solution

The length of the first column is:

$$\begin{aligned}
\ell_1 &= \sqrt{\cos^2 \theta + \sin^2 \theta} \\
&= 1
\end{aligned}$$

The length of the second column is:

$$\begin{aligned}
\ell_2 &= \sqrt{r^2 \sin^2 \theta + r^2 \cos^2 \theta} \\
&= \sqrt{r^2 (\sin^2 \theta + \cos^2 \theta)} \\
&= \sqrt{r^2} \\
&= r
\end{aligned}$$

So,  $J$  is the product  $1 \cdot r = r$ .

$$\begin{aligned}
 \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} &= r \cos^2 \theta + r \sin^2 \theta \\
 &= r (\cos^2 \theta + \sin^2 \theta) \\
 &= \underline{r}
 \end{aligned}$$

### Exercise

Prove that  $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$  if and only if  $\vec{u}$  and  $\vec{v}$  are parallel vectors.

### Solution

If  $\vec{u}$  and  $\vec{v}$  are parallel vectors, then  $\vec{u} \times \vec{v} = 0$

Which the two vectors are collinear, which implies that  $\vec{v} = a\vec{u}$

$$\begin{aligned}
 \|\vec{u} + \vec{v}\| &= \|\vec{u} + a\vec{u}\| \\
 &= \|(1+a)\vec{u}\| \\
 &= (1+a)\|\vec{u}\| \\
 &= \|\vec{u}\| + a\|\vec{u}\| \\
 &= \|\vec{u}\| + \|a\vec{u}\| \\
 &= \|\vec{u}\| + \|\vec{v}\| \quad \checkmark
 \end{aligned}$$

### Exercise

State the following statements as True or False

- The cross product of two nonzero vectors  $\vec{u}$  and  $\vec{v}$  is a nonzero vector if and only if  $\vec{u}$  and  $\vec{v}$  are not parallel.
- A normal vector to a plane can be obtained by taking the cross product of two nonzero and noncollinear vectors lying in the plane.
- The scalar triple product of  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  determines a vector whose length is equal to the volume of the parallelepiped determined by  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$ .
- If  $\vec{u}$  and  $\vec{v}$  are vectors in 3-space, then  $\|\vec{u} \times \vec{v}\|$  is equal to the area of the parallelogram determined by  $\vec{u}$  and  $\vec{v}$ .
- For all vectors  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  in  $R^3$ , the vectors  $(\vec{u} \times \vec{v}) \times \vec{w}$  and  $\vec{u} \times (\vec{v} \times \vec{w})$  are the same.
- If  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$  are vectors in  $R^3$ , where  $\vec{u}$  is nonzero and  $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$ , then  $\vec{v} = \vec{w}$

### Solution

a) True,

$\vec{u} \times \vec{v} = \|\vec{u}\| \|\vec{v}\| \sin \theta = 0$  if  $\theta = 0$  which the two vectors are parallel.

b) True;

The cross product of two nonzero and non collinear vectors will be perpendicular to both vectors, hence normal to the plane containing the vectors.

c) False;

The scalar triple product is a scalar, not a vector.

d) True;

e) False;

$$\text{Let } \vec{u} = \hat{i} \quad \vec{v} = \vec{w} = \hat{j}$$

$$\begin{aligned} (\vec{u} \times \vec{v}) \times \vec{w} &= (\hat{i} \times \hat{j}) \times \hat{j} \\ &= \hat{k} \times \hat{j} \\ &= -\hat{i} \end{aligned}$$

$$\begin{aligned} \vec{u} \times (\vec{v} \times \vec{w}) &= \hat{i} \times (\hat{j} \times \hat{j}) \\ &= \hat{i} \times \vec{0} \\ &= \vec{0} \end{aligned}$$

$$\text{Hence, } (\vec{u} \times \vec{v}) \times \vec{w} \neq (\vec{u} \times \vec{v}) \times \vec{w}$$

f) False;

$$\text{Let } \vec{u} = \hat{i} + \hat{j} \quad \vec{v} = \hat{i} + \hat{j} + \hat{k} \quad \vec{w} = -\hat{i} - \hat{j} + \hat{k}$$

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} \\ &= \hat{i} - \hat{j} \end{aligned}$$

$$\begin{aligned} \vec{u} \times \vec{w} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{vmatrix} \\ &= \hat{i} - \hat{j} \end{aligned}$$

$$\vec{u} \times \vec{v} = \vec{u} \times \vec{w}, \text{ but } \vec{v} \neq \vec{w}$$