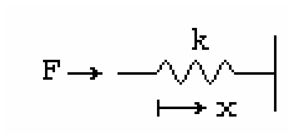
Section 2.3 – Harmonic Motion

Hooke's Law

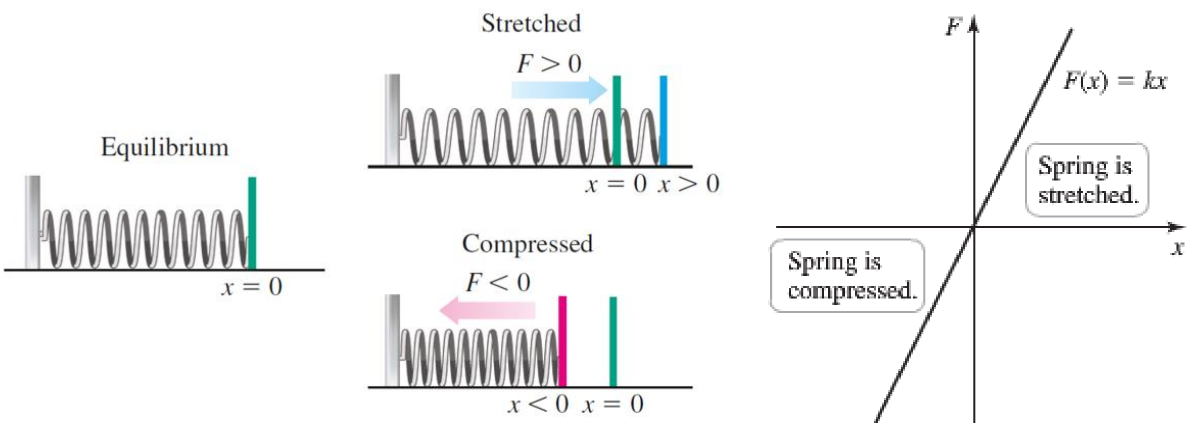
Hooke's Law says that the force required to hold a stretched or compressed spring *x* units from its natural (unstressed) length is proportional to *x*. In symbols

$$F = -ky, \quad k > 0 \quad (k : Spring\ constant)$$

$$F = kx$$



The constant *k*, measured in force units per unit length, is a characteristic of the spring, called ***the force constant*** (or ***spring constant***) of the spring.



- To stretch the spring to a position $x > 0$, a force $F > 0$ (in the ***positive*** direction) is required.
- To compress the spring to a position $x < 0$, a force $F < 0$ (in the ***negative*** direction) is required.

$1\ kg.m/s^2 = 1\ N \quad (Newton)$

Newton's Second Law

Force equals mass times acceleration

$$F = ma = m\frac{d^2y}{dt^2}$$

Mathematical model: $\quad m\dfrac{d^2y}{dt^2} = -ky$

$$m\frac{d^2y}{dt^2} + ky = 0$$

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

$$\frac{d^2y}{dt^2} + \omega^2 y = 0; \qquad \omega = \sqrt{\frac{k}{m}}$$

$\frac{\omega}{2\pi}$ is called natural frequency of the system

Damped, Free Vibrations

A resistance force R (e.g. friction) proportional to the velocity $v = y'$ and acting in a direction opposite to the motion

$$R = -cy', \quad c > 0$$

Force equation: $F = -ky(t) - cy'(t)$

Mathematical model: $my'' = -ky - cy$

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0 \qquad (c,\ m,\ k \text{ are constants})$$

The equation for the motion of a vibrating spring is given by

$$my'' + \mu y' + ky = F(t)$$

Where the constant coefficients are:

m	mass
μ	damping constant
k	spring constant
F(t)	external force

The differential equation that modeled simple *RLC* circuits is given by

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}$$

Comparing the 2 systems are almost identical.

Example

For a circuit without resistance $(R = 0)$ and no source voltage, then the equation simplifies to

$$L\frac{d^2I}{dt^2} + \frac{1}{C}I = 0 \qquad \textit{Divide by L}$$

$$\frac{d^2I}{dt^2} + \frac{1}{LC}I = 0$$

$$\lambda^2 + \frac{1}{LC} = 0$$

$$\lambda^2 = -\frac{1}{LC}$$

$\lambda = \pm i \frac{1}{\sqrt{LC}}$ $\lambda = a \pm ib \Rightarrow a = 0;\ \ b = \frac{1}{\sqrt{LC}}$

The general solution: $y(t) = e^{at}\left(A_1 \cos bt + A_2 \sin bt\right)$

$$\boxed{I(t) = C_1 \cos\left(\frac{t}{\sqrt{LC}}\right) + C_2 \sin\left(\frac{t}{\sqrt{LC}}\right)}$$

Combine the two systems:

$$y'' + \frac{\mu}{m} y' + \frac{k}{m} y = \frac{1}{m} F(t) \qquad (1)$$

$$\frac{d^2 I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC} I = \frac{1}{L}\frac{dE}{dt} \qquad (2)$$

If we let: $\begin{cases} c = \frac{\mu}{2m} & c = \frac{R}{2L} \\ \omega_0 = \sqrt{k/m} & \omega_0 = \sqrt{1/LC} \\ f(t) = \frac{1}{m} F(t) & f(t) = \frac{1}{L}\frac{dE}{dt} \\ x = y & x = I \end{cases}$

$$x'' + 2cx' + \omega_0^2 x = f(t)$$

Where $c \geq 0$ *and* $\omega_0 > 0$ are constants.

This equation called ***harmonic motion***.

c ***damping*** constant

f ***forcing term***

Simple Harmonic Motion

In the special case when there is no damping $(c=0)$ the motion is called ***simple harmonic motion***.

$$y'' + \omega_0^2 y = 0$$

The characteristic equation is:

$$\lambda^2 + \omega_0^2 = 0$$

The roots are $\lambda^2 = -\omega_0^2 \to \lambda = \pm i\omega_0$

$x(t) = a\cos\omega_0 t + b\sin\omega_0 t$

If we define $T = \dfrac{2\pi}{\omega_0} \Rightarrow T\omega_0 = 2\pi$

Then the periodic of the trigonometry functions implies that $x(t+T) = x(t)$ for all t.

Thus, the solution x is periodic with period T.

ω_0 is called the ***natural frequency***.

Amplitude and Phase Angle

$y(t) = a\cos\omega_0 t + b\sin\omega_0 t$

Consider the point (a,b), we can rewrite this in polar coordinates with a length of A.

$$a = A\cos\phi \quad b = A\sin\phi$$

$$\begin{aligned} y(t) &= a\cos\omega_0 t + b\sin\omega_0 t \\ &= A\cos\phi\cos\omega_0 t + A\sin\phi\sin\omega_0 t \\ &= A\cos\left(\omega_0 t - \phi\right) \end{aligned}$$

Where A ***amplitude*** of the oscillation $\qquad A = \sqrt{a^2 + b^2}$

ϕ ***Phase*** of the oscillation $\qquad \tan\phi = \dfrac{b}{a} \qquad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Period: $T = \dfrac{2\pi}{\omega_0}$

Frequency: $\nu = \dfrac{1}{T}$

Time lag of the motion is: $\delta = \dfrac{\varphi}{\omega_0}$

Example

A mass of 4 kg is attached to a spring with a spring constant of $k = 169\ kg/s^2$. It is then stretched 10 cm from the spring mass equilibrium and set to oscillating with an initial velocity is 130 cm/s. Assuming it oscillates without damping, find the frequency, amplitude, and phase of the vibration.

Solution

$$my'' + \mu y' + ky = F(t)$$

$$4y'' + 169y = 0 \qquad \textit{Divide by 4}$$

$$y'' + 42.25y = 0$$

The natural frequency: $\omega_0 = \sqrt{42.25} = 6.5$

$$y(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

$$= C_1 \cos 6.5t + C_2 \sin 6.5t$$

Stretched 10 cm $\rightarrow y(0) = 10\ cm = .1\ m$

$$y(0) = C_1 \cos 6.5(0) + C_2 \sin 6.5(0) \quad \Rightarrow \quad \boxed{0.1 = C_1}$$

Initial velocity is 130 cm/s $\rightarrow y'(0) = 1.3\ m/s$

$$y'(t) = -6.5C_1 \sin 6.5t + 6.5C_2 \cos 6.5t$$

$$y'(0) = -6.5C_1 \sin 6.5(0) + 6.5C_2 \cos 6.5(0)$$

$$1.3 = 6.5C_2 \quad \Rightarrow \quad \boxed{C_2 = 0.2}$$

$$y(t) = 0.1\cos 6.5t + 0.2\sin 6.5t$$

$$A = \sqrt{.1^2 + .2^2} \approx 0.2236\ m \qquad \& \qquad \phi = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{.2}{.1} \approx \boxed{1.1071}$$

$$\boxed{y(t) = 0.2236\cos(6.5t - 1.1071)}$$

Damped Harmonic Motion

In this case, $c > 0$.

$$x'' + 2cx' + \omega_0^2 x = 0$$

The characteristic equation is: $\lambda^2 + 2c\lambda + \omega_0^2 = 0$

The roots are $\lambda = -c \pm \sqrt{c^2 - \omega_0^2}$

There are 3 cases to consider damping and depend on the sign of the discriminant $c^2 - \omega_0^2$

1. $c^2 - \omega_0^2 < 0 \Rightarrow c < \omega_0$. This is the ***underdamped*** case. The roots are distinct complex numbers.

   The general solution is

   $$x(t) = e^{-ct}\left(C_1 \cos \omega t + C_2 \sin \omega t\right) \quad \text{Where } \omega = \sqrt{\omega_0^2 - c^2}$$

2. $c^2 - \omega_0^2 > 0 \Rightarrow c > \omega_0$. This is the ***overdamped*** case. The roots are distinct and real numbers.

   The general solution is

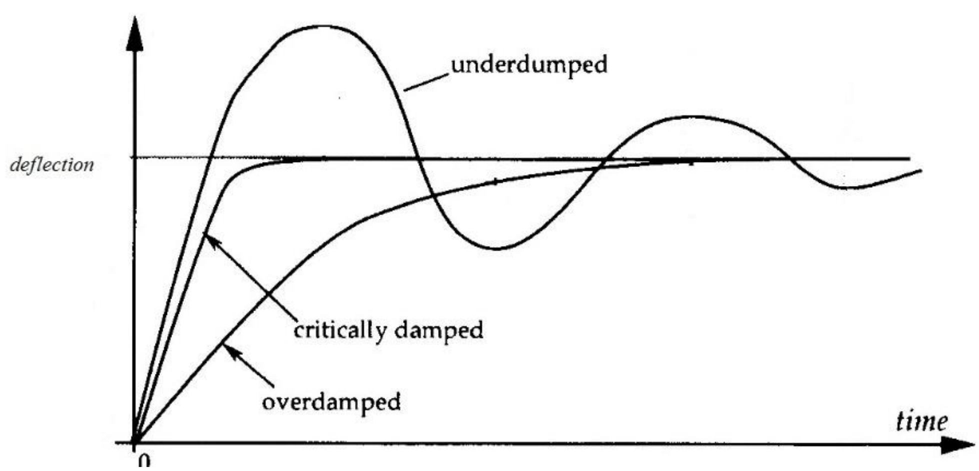
   $$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

   Where $\sqrt{\omega_0^2 - c^2} < \sqrt{c^2} < c \qquad \lambda_1 < \lambda_2 < 0$

3. $c^2 - \omega_0^2 = 0 \Rightarrow c = \omega_0$. This is the ***damped*** case. The root is a double root.

   The general solution is

   $$x(t) = C_1 e^{-ct} + C_2 t e^{-ct} \qquad \text{Where } \lambda = -c$$



Example

A mass of 4 kg is attached to a spring with a spring constant of $k = 169\ kg/s^2$ and damping constant $\mu = 12.8\ kg/s$. With initial values of $y(0) = 0.1\ m$ and $y'(0) = 1.3 m/s$. Find the frequency, amplitude, and phase of the vibration.

Solution

$$my'' + \mu y' + ky = F(t)$$
$$4y'' + 12.8y' + 169y = 0$$
$$y'' + 3.2y' + 42.25y = 0$$
$$\lambda^2 + 3.2\lambda + 42.25 = 0$$
$$\lambda = -1.6 \pm 6.3i$$

The general solution:

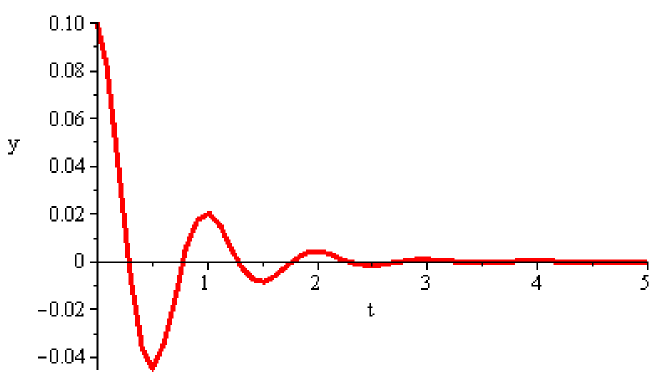
$$y(t) = e^{-1.6t}\left(C_1 \cos 6.3t + C_2 \sin 6.3t\right)$$

$$y(0) = e^{-1.6(0)}\left(C_1 \cos 6.3(0) + C_2 \sin 6.3(0)\right)$$
$$0.1 = C_1$$
$$y'(t) = -1.6e^{-1.6t}\left(C_1 \cos 6.3t + C_2 \sin 6.3t\right) + e^{-1.6t}\left(-6.3C_1 \sin 6.3t + 6.3C_2 \cos 6.3t\right)$$
$$y'(0) = -1.6e^{-1.6(0)}\left(C_1 \cos 6.3(0) + C_2 \sin 6.3(0)\right) + e^{-1.6(0)}\left(-6.3C_1 \sin 6.3(0) + 6.3C_2 \cos 6.3(0)\right)$$
$$1.3 = -1.6(0.1 + 0) + (1)\left(-0 + 6.3C_2\right)$$
$$1.3 = -0.16 + 6.3C_2$$
$$6.3C_2 = 1.46$$
$$C_2 \approx 0.2317$$

$$\boxed{y(t) = e^{-1.6t}\left(0.1\cos 6.3t + 0.2317\sin 6.3t\right)}$$

OR

$$A = \sqrt{.1^2 + .2317^2} \approx 0.2524\ m$$
$$\phi = \tan^{-1}\frac{b}{a}$$
$$= \tan^{-1}\frac{.2317}{.1}$$
$$\approx 1.1634$$
$$\boxed{y(t) = 0.2524\cos(6.3t - 1.1634)}$$



Example

A mass of 4 kg is attached to a spring with a spring constant of $k = 169\ kg/s^2$ and damping constant $\mu = 77.6\ kg/s$. With initial values of $y(0) = 0.1\ m$ and $y'(0) = 1.3 m/s$. Find the general solution.

Solution

$$my'' + \mu y' + ky = F(t)$$

$$4y'' + 77.6y' + 169y = 0$$

$$y'' + 19.4y' + 42.25y = 0$$

$$\lambda^2 + 19.4\lambda + 42.25 = 0$$

$$\lambda_1 = -16.9,\ \lambda_2 = -2.5$$

The general solution:

$$y(t) = C_1 e^{-16.9t} + C_2 e^{-2.5t}$$

$$0.1 = C_1 e^{-16.9(0)} + C_2 e^{-2.5(0)}$$

$$0.1 = C_1 + C_2$$

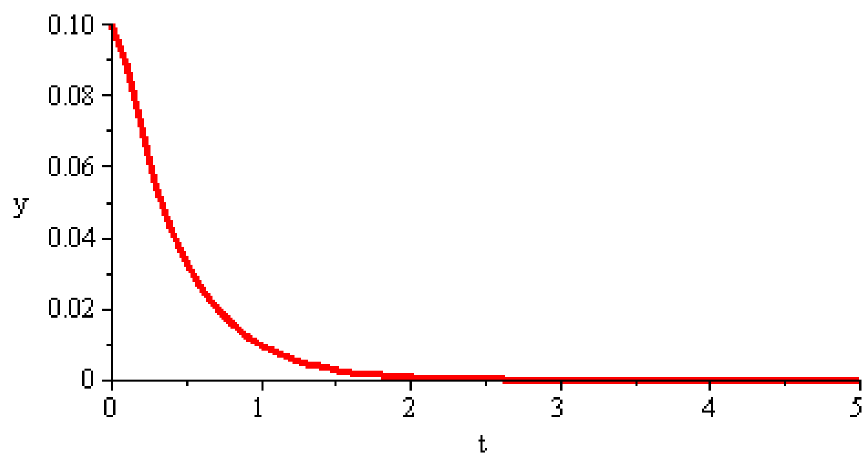
$$y' = -16.9C_1 e^{-16.9t} - 2.5C_2 e^{-2.5t}$$

$$1.3 = -16.9C_1 e^{-16.9(0)} - 2.5C_2 e^{-2.5(0)}$$

$$1.3 = -16.9C_1 - 2.5C_2$$

$$\left.\begin{array}{r} 0.1 = C_1 + C_2 \\ 1.3 = -16.9C_1 - 2.5C_2 \end{array}\right\} \rightarrow C_1 = -\frac{31}{288},\ C_2 = \frac{299}{1440}$$

$$\boxed{y(t) = -\frac{31}{288}e^{-16.9t} + \frac{299}{1440}e^{-2.5t}}$$



Example

A mass of 4 kg is attached to a spring with a spring constant of $k = 169\ kg/s^2$; with initial values of $y(0) = 0.1\ m$ and $y'(0) = 1.3 m/s$. Find the damping constant μ for which there is critical damping

Solution

Critical damping occurs when $c = \omega_0$

Since $c = \dfrac{\mu}{2m} = \omega_0$

$$\mu = 2m\omega_0$$

$$= 2m\sqrt{\frac{k}{m}}$$

$$= 2(4)\sqrt{\frac{169}{4}}$$

$$= 52\ \ kg/s$$

$$4y'' + 52y' + 169y = 0$$

$$\lambda^2 + 13\lambda + 42.25 = 0$$

$$y(t) = C_1 e^{-6.5t} + C_2 t e^{-6.5t}$$

$$0.1 = C_1$$

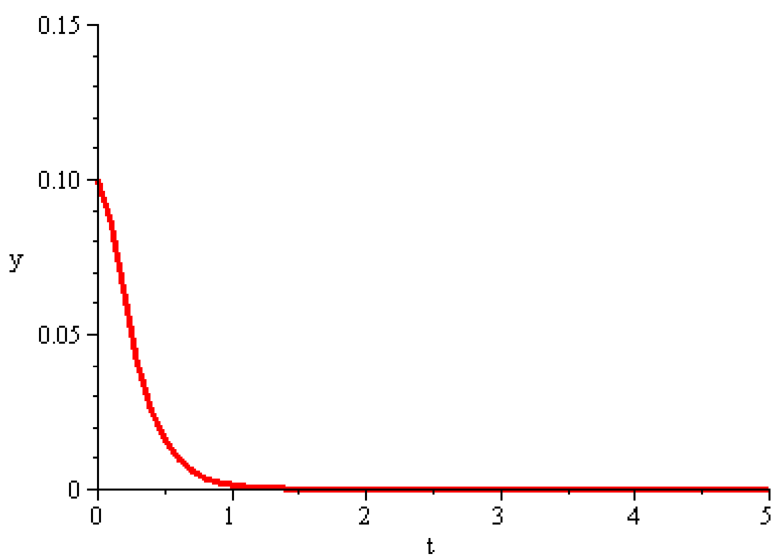
$$y' = -6.5C_1 e^{-6.5t} + C_2 e^{-6.5t} - 6.5C_2 t e^{-6.5t}$$

$$1.3 = -6.5C_1 + C_2$$

$$1.3 = -6.5(0.1) + C_2$$

$$C_2 = 1.95$$

$$\boxed{y(t) = 0.1e^{-6.5t} + 1.95te^{-6.5t}}$$



Important facts that the differential equations for electrical and mechanical (Translation and Rotational) are identical in some forms.

TABLE *A*: Relationships between the variables of the analog system components.

Electrical	***Mechanical Translation***	***Mechanical Rotational***
$i = C\dfrac{dv}{dt}$ $= Gv$ $= N\phi$ $= r^2\left(1-r^2\right)$ $= \dfrac{1}{L}\int vdt$	$f = M\dfrac{dv}{dt}$ $= Dv$ $= kx$ $= k\int vdt$	$T = J$ $= D\omega$ $= k\theta$ $= k\int \omega dt$

Engineers sometimes utilize the similarity by determining the properties of a proposed mechanical system with a simple electrical analog.

TABLE *B*: Analogous between electrical and mechanical systems.

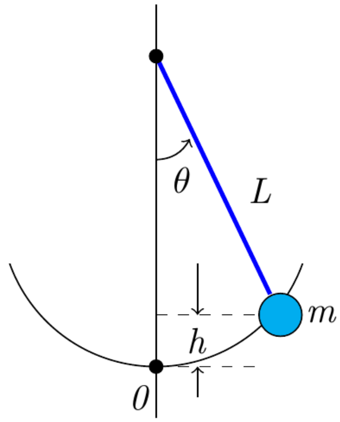
Electrical	***Mechanical Translation***	***Mechanical Rotational***
Current, ***i***	Force , ***f*** *N, lb*	Torque, ***T*** *N-m, lb-ft*
Voltage, ***V***	Velocity, ***v***	Angular velocity, ω
Flux linkages	Displacement	Angular displacement, $N\phi$,*xh* or θ *rad*
Capacitance. ***C***	Mass, **M** *kg, slug*	Moment of inertia, **J** *kg-m*^{2}, *lb-ft/sec*^{2}**.**
Conductance **G = 1/R**	Damping coefficient (of dash pot) **D** or **B** **N/m/sec**, *lb/ft/sec*	Rotational damping Coefficient friction: **D** or **B**
Inductance, ***L***	Compliance $\tau = \dfrac{1}{k}$ of spring	Torsional compliance $\tau = \dfrac{1}{k}$ of spring $k \to N \cdot m / rad$

Summary

	Abv.		***Unit***
Capacitor	*C*		Farad (**F**)
Current	*I*		Ampere (**A**)
Electric Charge	*q*		Coulomb (**C**)
Electromotive Force	*emf*		**Emf**
Inductor	*L*		Henry (**H**)
Resistor	*R*		Ohm (**Ω**)
Time	*t*		Second (**s**)
Voltage	*V*		Volt (**V**)

Pendulum

A simple Pendulum consists of a mass m swinging back and forth on the end if a string of length L.



We specify the position of the mass at time t by giving the counterclockwise angle $\theta = \theta(t)$ that the string or rod makes with the vertical at time t. To analyze the motion of the mass m, we apply the law of the conservation of mechanical energy, according to which the sum of the kinetic energy and the potential energy of m remains constant.

The distance along the circular arc from 0 to m is $s = L\theta$, so the velocity of the mass is

$$v = \frac{ds}{dt} = L\frac{d\theta}{dt}$$

Therefore, its kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2$$

Then its potential energy V is the product of its weight mg and its vertical height $h = L(1-\cos\theta)$ above O, so

$$V = mgL(1-\cos\theta)$$

The sum of T and V is constant C, therefore

$$\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + mgL(1-\cos\theta) = C$$

$$\frac{d}{dt}\left(\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + mgL(1-\cos\theta) = C\right)$$

$$mL^2\left(\frac{d\theta}{dt}\right)\frac{d^2\theta}{dt^2} + mgL\sin\theta\frac{d\theta}{dt} = 0$$

$$mL^2\frac{d^2\theta}{dt^2} + mgL\sin\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

Exercises Section 2.3 – Harmonic Motion

(*Exercises* 1 - 2)

i. Plot the function

ii. Place the solution in the form $y = A\cos\left(\omega_0 t - \phi\right)$ and compare the graph with the plot in (***i***)

1. $y = \cos 2t + \sin 2t$

2. $y = \cos 4t + \sqrt{3}\sin 4t$

3. A 1-*kg* mass, when attached to a large spring, stretches the spring a distance of 4.9 *m*.

a) Calculate the spring constant.

b) The system is placed in a viscous medium that supplies a damping constant $\mu = 3\ kg/s$. The system is allowed to come to rest. Then the mass is displaced 1 *m* in the downward direction and given a sharp tap, imparting an instantaneous velocity of 1 *m/s* in the downward direction. Find the position of the mass as a function of time and plot the solution.

4. The undamped system

$$\frac{2}{5}x'' + kx = 0, \quad x(0) = 2 \quad x'(0) = v_0$$

is observed to have period $\frac{\pi}{2}$ and amplitude 2. Find k and v_0

5. A body with mass *m* = 0.5 *kg* is attached to the end of a spring that is stretched 2 *m* by a force of 100 *N*. It is set in motion with initial position $x_0 = 1m$ and initial velocity $v_0 = -5m/s$. (Note that these initial conditions indicate that he body is displaced to the right and is moving to the left at time *t* = 0.) Find the position function of the body as well as the amplitude, frequency, period of oscillation, and time lag of its motion.

Given the mass, damping, and spring constants of an undriven spring-mass system $my'' + \mu y' + ky = 0$

a) Provide separate plots of the position versus time (*y* vs. *t*) and the velocity versus time (*v* vs. *t*)

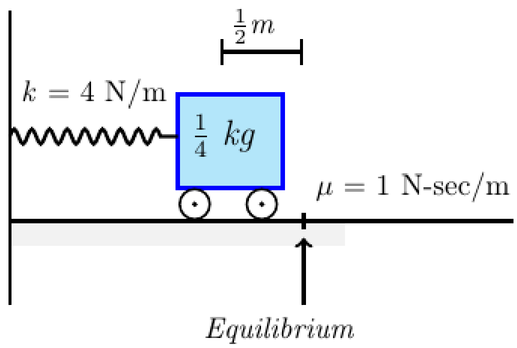
b) Provide a combined plot of both position and velocity versus time

c) Provide a plot of the velocity versus position (*v* vs. *y*) in the *yv* phase plane.

6. $m = 1\ kg, \quad \mu = 0\ kg/s, \quad k = 4kg/s^2, \quad y(0) = -2\ m, \quad y'(0) = -2\ m/s$

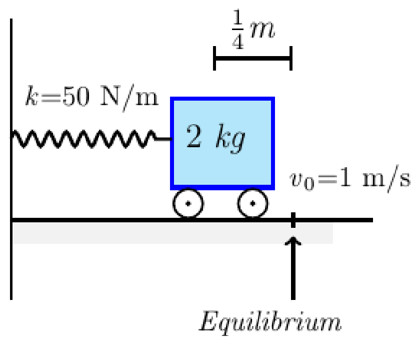
7. $m = 1\ kg, \quad \mu = 2\ kg/s, \quad k = 1kg/s^2, \quad y(0) = -3\ m, \quad y'(0) = -2\ m/s$

8. A $\frac{1}{4}$-*kg* mass is attached to a spring with a stiffness 4 *N/m*. The damping constant 1 *N - sec / m*. If the mass is displaced $x_0 = \frac{1}{2}\ m$ to the left and given an initial velocity of 1 *m/s* to the left.



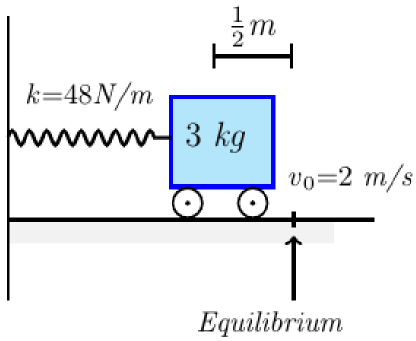
a) Find the equation of motion.
b) What is the maximum displacement that the mass will attain?

9. A 2-*kg* mass is attached to a spring with a stiffness $k = 50$ *N/m*. The mass is displaced $\frac{1}{4}$ *m* to the left of the equilibrium point and given a velocity of 1 *m/s* to the left. Neglecting the damping,



a) Find the equation of motion of the mass along with the amplitude, period, and frequency.
b) How long after release does the mass pass through the equilibrium position?

10. A 3-*kg* mass is attached to a spring with a stiffness $k = 48$ *N/m*. The mass is displaced $\frac{1}{2}$ *m* to the left of the equilibrium point and given a velocity of 2 *m/s* to the left. Neglecting the damping.



c) Find the equation of motion of the mass
d) Find the amplitude, period, and frequency.
e) How long after release does the mass pass through the equilibrium position?

11. A 20-*kg* mass is attached to a spring with a stiffness $k = 200$ *N/m*. The damping constant $\mu = 140$ *N - sec / m*. If the mass is pulled 25 *cm* to the right of the equilibrium point and given an initial velocity of 1 *m/s*. Neglecting the damping,

a) Find the equation of motion.
b) When will it first return to its equilibrium position?

12. A $\frac{1}{4}$-*kg* mass is attached to a spring with a stiffness $k = 8$ *N/m*. The damping constant $\mu = \frac{1}{4}$ *N - sec / m*. If the mass is displaced $x_0 = 1$ *m* to the left of equilibrium and released, what is the maximum displacement to the right that the mass will attain?

13. A $\frac{1}{4}$-*kg* mass is attached to a spring with a stiffness $k = 8$ *N/m*. The damping constant $\mu = 2$ *N - sec / m*. If the mass is pushed 50 *cm* to the left of equilibrium and given a leftward velocity of 2 *m/sec*, when will the mass attain its maximum displacement to the left?

14. A 8-*lb* mass weight stretches a spring 2 *feet*. Assuming that a damping force numerically equal to 2 times the instantaneous velocity acts on the system, determine the equation if motion if the mass released from the equilibrium position with an upward velocity of 3 *ft/s*

15. A 8-*lb* mass weight is attached to the end of a spring, causing the spring to stretch a spring 6 *in*. beyond its natural length. The block is then pulled down 3 *in*. and released. Determine the motion of the block, assuming there are no damping or external applied force.

16. A 8-*lb* mass weight is attached to the end of a spring, causing the spring to stretch a spring 6 *in*. beyond its natural length. The block is then pulled down 3 *in*. and released. Determine the motion of the block, assuming there damping is present and that the damping coefficient is $\mu = 1$ *lb-sec/ft* and external applied force.

17. A 16-*lb* mass weight is attached to a 5-*foot* spring. At equilibrium, the spring measures 5.2 *feet*. If the mass is initially released from rest at a point $x_0 = 2$ *ft* above the equilibrium position, find the displacements $x(t)$ if it is further known that the surrounding medium offers a resistance numerically equal to the instantaneous velocity.

18. A 16-*lb* mass weight is attached to a spring, stretches $\frac{8}{9}$ *ft* by itself. There is no damping and no external forces acting on the system. The spring is initially displaced 6 *inches* upwards from its equilibrium position and given an initial velocity of 1 *ft/sec* downward. Find the displacement $y(t)$ at any time *t*.

19. A 16-*lb* mass weight is attached to a spring, stretches $\frac{8}{9}$ *ft* by itself. A damper to the mass that will exert of 12 *lbs*. when the velocity is 2 *ft/sec*. The spring is initially displaced 6 *inches* upwards from its equilibrium position and given an initial velocity of 1 *ft/sec* downward. Find the displacement $y(t)$ at any time *t*.

20. A 16-*lb* mass weight is attached to a spring, stretches $\frac{8}{9}$ *ft* by itself. A damper to the mass that will exert of 5 *lbs*. when the velocity is 2 *ft/sec*. The spring is initially displaced 6 *inches* upwards from its equilibrium position and given an initial velocity of 1 *ft/sec* downward. Find the displacement $y(t)$ at any time *t*.

21. A mass weighing 4-*lb* is attached to a spring whose spring constant is 16 *lb/ft*.

a) Find the equation of motion.
b) What is the period of simple harmonic motion?

22. A 20-*kg* mass is attached to a spring. If the frequency of simple harmonic motion is $\frac{2}{\pi}$ *cycles/s*.

a) What is the spring constant *k*?
b) Find the equation of motion.
c) What is the frequency of simple harmonic motion if the original mass is replaced with an 80-*kg* mass.?

23. A 24-*lb* mass weight is attached to the end of a spring, stretches it 4 *inches*. Initially, the mass is released from rest from a point 3 *inches* above the equilibrium position.

a) Find the equation of the motion.
b) If the mass is initially released from the equilibrium position with a downward velocity of 2 *ft/s*

24. The motion of a mass-spring system with damping is given by:

$$y'' + 4y' + ky = 0 \; ; \quad y(0) = 1, \quad y'(0) = 0$$

Find the equation of motion and sketch its graph for $k = 2,\ 4,\ 6,\ and\ 8$.

25. A 10-*lb* mass weight is attached to the end of a spring, stretches it 3 *inches*. This mass is removed and replaced with a mass of 1.6 *slugs*, which initially released from a point 4 inches above the equilibrium position with a downward velocity of. $\frac{5}{4}$ *ft/s*

a) Find the equation of the motion.
b) Find the amplitude, phase angle, period and the frequency.
c) Express the motion equation in amplitude and phase angle form.
d) Determine the times the mass attains a displacement below the equilibrium position numerically equal to $\frac{1}{2}$ the amplitude of motion.

26. A 64-*lb* mass weight is attached to the end of a spring, stretches it 0.32 *foot*. This mass is initially released from a point 8 *inches* above the equilibrium position with a downward velocity of 5 *ft/s*.

a) Find the equation of the motion.
b) Find the amplitude, phase angle, period and the frequency.
c) Write the motion equation with phase angle form.
d) How many complete cycles will the mass have completed at the end of 3π *sec*.
e) At what time does the mass pass through the equilibrium position heading downward for the second time?

f) At what times does the mass attain its extreme displacements on either side of the equilibrium position?

g) What is the position of the mass at $t = 3\ sec$?

h) What is the instantaneous velocity at $t = 3\ sec$?

i) What is the acceleration at $t = 3\ sec$?

j) What is the instantaneous velocity at the times when the mass passes through the equilibrium position?

k) At what times is the mass 5 inches below the equilibrium position?

l) At what times is the mass 5 inches below the equilibrium position heading in the upward direction?

If it is underdamped, write the position function in the form $x(t) = C_1 e^{-pt} \cos\left(\omega_1 t - \alpha_1\right)$.

Also, find the undamped position function $u(t) = C_0 \cos\left(\omega_0 t - \alpha_0\right)$ that would result if the mass on the spring were set in motion with the same initial position and velocity, but with the dashpot disconnected (so $c = 0$). Then, construct a figure that illustrates the effect of damping by comparing the graphs of $x(t)$ and $u(t)$

27. $m = \frac{1}{2}, \quad c = 3, \quad k = 4; \quad x_0 = 2, \quad v_0 = 0$

28. $m = 1, \quad c = 8, \quad k = 16; \quad x_0 = 5, \quad v_0 = -10$

29. $m = 1, \quad c = 10, \quad k = 125; \quad x_0 = 6, \quad v_0 = 50$

30. $m = 2, \quad c = 12, \quad k = 50; \quad x_0 = 0, \quad v_0 = -8$

31. $m = 2, \quad c = 16, \quad k = 40; \quad x_0 = 5, \quad v_0 = 4$

32. $m = 3, \quad c = 30, \quad k = 63; \quad x_0 = 2, \quad v_0 = 2$

33. $m = 4, \quad c = 20, \quad k = 169; \quad x_0 = 4, \quad v_0 = 16$

34. Suppose that the mass in a mass–spring–dashpot system with $m = 10,\ c = 9,\ \textit{and}\ k = 2$ is set in motion with $x(0) = 0$ and $x'(0) = 5$

a) Find the position function $x(t)$ and graph the function

b) Find how far the mass moves to the right before starting back toward the origin.

35. Suppose that the mass in a mass–spring–dashpot system with $m = 25,\ c = 10,\ \textit{and}\ k = 226$ is set in motion with $x(0) = 20$ and $x'(0) = 41$

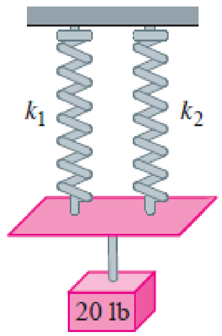
a) Find the position function $x(t)$ and graph the function

b) Find the pseudoperiod of the oscillations and the equations of the "envelope curves" that are dashed.

36. A mass of 1 *slug* is suspended from a spring, the spring constant is 9 *lb/ft*. The mass is initially released from a point 1 *foot* above the equilibrium position with an upward velocity of $\sqrt{3}$ *ft/s*. Find the times at which the mass is heading downward at a velocity of 3 *ft/s*

37. Two parallel springs, with constants k_1 and k_2, support a single mass, the effective spring constant of the system is given by $k=\dfrac{4k_1k_2}{k_1+k_2}$.

A mass weight 20 *pounds* stretches one spring 6 *inches* and another spring 2 *inches*. The springs are attached to a common rigid support and then to a metal plate. The mass is attached to the center of the plate in the double-spring constant arrangement.



a) Determine the effective spring constant of this system.
b) Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of 2 *ft/s*.

38. A 12–*lb* weight is attached both to a vertically suspended spring that it stretches 6 *in*. and to a dashpot that provides 3 *lb*. of resistance for every foot per second of velocity.

a) If the weight is pulled down 1 *foot*. below its static equilibrium position and then released from rest at time $t=0$, find its position function $x(t)$.
b) Find the frequency, time-varying amplitude, and phase angle of the motion.

39. A $\frac{1}{8}$-*kg* mass is attached to a spring with a spring constant $k=16$ *N/m*. The mass is displaced $\frac{1}{2}$ *m* to the right of the equilibrium point and given an outward velocity (to the right) of $\sqrt{2}$ *m/sec*. Neglecting any damping or external forces that may be present,

a) Determine the equation of motion of the mass
b) Determine the equation of motion amplitude, period, and natural frequency.
c) How long after release does the mass pass through the equilibrium position?

40. A 3-*kg* mass is attached to a spring with a spring constant 75 *N/m*. The mass is displaced $\frac{1}{4}$ *m* to the left and given a velocity of 1 *m/sec* to the right. The damping force is negligible.

a) Determine the equation of motion of the mass
b) Determine the equation of motion amplitude, period, and natural frequency.
c) How long after release does the mass pass through the equilibrium position?

41. A 3-*kg* mass is attached to a spring with a spring constant 300 *N/m*. The mass is pulled down 10 *cm* and released with downward velocity of 1 *m/sec*. The damping force is negligible.

a) Determine the equation of motion of the mass
b) Solve the equation to find the time when the maximum downward displacement of the mass from its equilibrium position is first achieved.

c) What is the maximum downward displacement?

42. A 10-*kg* mass is attached to the end of a spring hanging vertically, stretches the spring 0.03 *m*. The mass is pulled down another 7 *cm* and released (with no initial velocity).

a) Determine the spring constant *k*.

b) Determine the equation of motion of the mass

43. A 10-*kg* mass is attached to a spring with spring constant $k = 300$ *N/m*. At time $t = 0$, the mass is pulled down another 10 *cm* and released with a downward velocity of 100 *cm/sec*.

a) Determine the equation of motion.

b) What is the maximum downward displacement?

44. A 10-*kg* mass is attached to the end of a spring hanging vertically at rest. The mass is pulled down another 7 *cm* and released (with no initial velocity).

a) Determine the spring constant *k*.

b) Determine the equation of motion of the mass

45. A 10-*kg* mass is attached to the end of a spring hanging vertically, stretches the spring 0.7 *m*. The mass is started in motion from the equilibrium position with an initial velocity 1 *m/sec* in the upward direction. IF the force due to air resistance is $-90y'$ *N*

c) Determine the spring constant *k*.

d) Determine the equation of motion of the mass

46. A $\frac{1}{4}$-*slug* mass is attached to the end of a spring hanging vertically, stretches the spring 1.28 *ft*. The mass is started in motion from the equilibrium position with an initial velocity 4 *ft/sec* in the downward direction. If the force due to air resistance is $-2y'$ *lb*

a) Determine the spring constant *k*.

b) Determine the equation of motion of the mass

47. A 20-*kg* mass is attached to the end of a spring hanging vertically at rest. When given an initial downward velocity of 2 *m/s* from its equilibrium position the mass was observed to attain a maximum displacement of 0.2 *m* from its equilibrium position.

a) Determine the spring constant *k*.

b) Determine the equation of motion of the mass

48. A steel ball weighing 128-*lb* is attached to the end of a spring, stretches 2 *ft* from its natural length. The ball is started in motion with no initial velocity by displacing it 6 *in* above the equilibrium position. Assuming no air resistance.

a) Determine the spring constant *k*.

c) Find the equation of the ball position at time *t*.

d) Find the position of the ball at $t = \frac{\pi}{12}$ *sec*

49. A 9-*lb* mass is attached to the end of a spring hanging vertically with spring constant $k = 32$ *lb/ft*, is perturbed from its equilibrium position with a certain upward initial velocity. The amplitude of the resulting vibrations is observed to be 4 *in*.

a) Determine the equation of motion.
b) What is the initial velocity?
c) Determine the period and frequency of the vibrations?

50. A 2-*kg* mass is suspended from a spring with a spring constant of 10 *N/m*. The mass is started in motion from the equilibrium position with an initial velocity 1.5 *m/sec*. Assuming no air resistance

a) Determine the equation of motion of the mass.
b) Determine the circular frequency, natural frequency, and period.

51. A $\frac{1}{4}$-*slug* mass is attached to a spring having a spring constant of 1 *lb/ft*. The mass is started in motion initially displacing it 2 *ft* in the downward direction with an initial velocity 2 *ft/sec* in the upward direction. If the force due to air resistance is $-1x'$ *lb*. Find the subsequent motion of the mass

52. A spring with a mass of 2-*kg* has natural length 0.5 *m*. A force of 25.6 *N* is required to maintain it stretched to a length of 0.7 *m*. If the spring is stretched to a length of 0.7 *m* and then released with initial velocity zero. Find the position of the mass at any time *t*.

53. A spring with a mass of 2-*kg* has natural length 0.5 *m*. A force of 25.6 *N* is required to maintain it stretched to a length of 0.7 *m*. The spring is immersed in a fluid with damping constant $c = 40$. If the spring is started from the equilibrium position and is given a push to start it with initial velocity 0.6 *m/s*. Find the position of the mass at any time *t*.

54. A spring with a mass of 3-*kg* is held stretched 0.6 *m* beyond its natural length by a force of 20 *N*. If the spring begins at its equilibrium and with initial velocity 1.2 *m/s*. Find the position of the mass at any time *t*.

55. A spring with a mass of 2-*kg* is held stretched 0.5 *m*, has damping constant 14, and a force of 6 *N*. If the spring is stretched 1 *m* beyond at its equilibrium and with no initial velocity.

a) Find the position of the mass at any time *t*.
b) Find the mass that would produce critical damping.

56. A spring has a mass of 1-*kg* and its spring constant $k = 100$. The spring is released at a point 0.1 *m* above its equilibrium position. Graph the position function for the following values of damping constant *c*: 10, 15, 20, 25, 30. What type of damping occurs each case

57. A 4-*kg* mass is attached to a spring and set in motion. A record of the displacements was made and found to be described by $y(t) = 25\cos\left(2t - \frac{\pi}{6}\right)$, with displacement measured in centimeters and time in seconds.

a) Determine the displacement y_0.

b) Determine the initial velocity y'_0 ?

c) Determine the spring constant k.

d) Determine the period and frequency of the vibrations?

58. A 10-kg mass is attached to a spring with a spring constant $k = 100$ N/m; the dashpot has damping constant 7 kg/sec. At time $t = 0$, the system is set into motion by pulling the mass down 0.5 m from its equilibrium rest position while simultaneously giving it an initial downward velocity of 1 m/s

a) Solve the equation of motion.

b) What is the $\lim\limits_{t \to \infty} y(t)$

c) Plot the solution.

d) How long it takes for the magnitude of the vibrations to be reduced to 0.1 m.

(Estimate the smallest time, τ, for which $|y(t)| \le 0.1$ m, $\tau \le t < \infty$)

59. A spring and dashpot system is to be designed for a 32-lb weight so that the overall system is critically damped

a) How must the damping constant c and the spring constant k be related?

b) Assume the system is to be designed so that the mass, when given initial velocity of 4 ft/sec from its rest position, will have a maximum displacement of 6 in. What values of damping constant c and spring constant k are required?

c) It is observed that the time interval between successive zero crossing is 20% larger for the damped vibration displacement than for the undamped vibration displacement. What is the damping constant c? (Spring constant k remains same from part (*b*)).

60. Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L = 0.25$ H, $R = 10$ Ω, $C = 0.001$ F, $E(t) = 0$, $q(0) = q_0$ C, and $i(0) = 0$.

61. Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit at $t = 0.01$ sec when $L = 0.05$ h, $R = 2$ Ω, $C = 0.01$ f, $E(t) = 0$, $q(0) = 5$ C, and $i(0) = 0$ A. Determine the first time at which the charge on the capacitor is equal to zero.

62. Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L = 0.25$ h, $R = 20$ Ω, $C = \frac{1}{300}$ f, $E(t) = 0$, $q(0) = 4$ C, and $i(0) = 0$ A. Is the charge on the capacitor ever equal to zero?

63. Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L = \frac{5}{3}$ h, $R = 10$ Ω, $C = \frac{1}{30}$ f, $E(t) = 0$, $q(0) = 0$ C, and $i(0) = 0$ A.