Section 3.3 – Gram-Schmidt Process

Definition

A set of two or more vectors in a real inner product space is said to be ***orthogonal*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be ***orthonormal***.

Theorem

1. If $S=\left\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\right\}$ is an orthogonal basis for an inner product space V, and if $\vec{u}$ is any vector in V, then

$$\vec{u}=\frac{\left\langle \vec{u}, \vec{v}_1\right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1+\frac{\left\langle \vec{u}, \vec{v}_2\right\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2+\cdots+\frac{\left\langle \vec{u}, \vec{v}_n\right\rangle}{\left\|\vec{v}_n\right\|^2}\vec{v}_n$$

2. If $S=\left\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\right\}$ is an orthonormal basis for an inner product space V, and if $\vec{u}$ is any vector in V, then

$$\vec{u}=\left\langle \vec{u}, \vec{v}_1\right\rangle\vec{v}_1+\left\langle \vec{u}, \vec{v}_2\right\rangle\vec{v}_2+\cdots+\left\langle \vec{u}, \vec{v}_n\right\rangle\vec{v}_n$$

Proof

1. Since $S=\left\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\right\}$ is a basis for V, every vector $\vec{u}$ in V can be expressed in the form

$$\vec{u}=c_1\vec{v}_1+c_2\vec{v}_2+\cdots+c_n\vec{v}_n$$

Let show that $c_i=\dfrac{\left\langle \vec{u}, \vec{v}_i\right\rangle}{\left\|\vec{v}_i\right\|^2}$ *for* $i=1, 2, \dots, n$

$$\begin{aligned}\left\langle \vec{u}, \vec{v}_i\right\rangle &= \left\langle c_1\vec{v}_1+c_2\vec{v}_2+\cdots+c_n\vec{v}_n,\ \vec{v}_i\right\rangle\\ &= c_1\left\langle \vec{v}_1, \vec{v}_i\right\rangle+c_2\left\langle \vec{v}_2, \vec{v}_i\right\rangle+\cdots+c_n\left\langle \vec{v}_n, \vec{v}_i\right\rangle\end{aligned}$$

Since S is an orthogonal set, all of the inner products in the last equality are zero except the i^{th}, so we have

$$\begin{aligned}\left\langle \vec{u}, \vec{v}_i\right\rangle &= c_i\left\langle \vec{v}_i, \vec{v}_i\right\rangle\\ &= c_i\left\|\vec{v}_i\right\|^2\end{aligned}$$

The Gram-Schmidt Process

To convert a basis $\left\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r\right\}$ into an orthogonal basis $\left\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\right\}$, perform the following computations:

***Step* 1**: $\vec{v}_1 = \vec{u}_1$

***Step* 2**: $\vec{v}_2 = \vec{u}_2 - \dfrac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$

***Step* 3**: $\vec{v}_3 = \vec{u}_3 - \dfrac{\left\langle \vec{u}_3, \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 - \dfrac{\left\langle \vec{u}_3, \vec{v}_2 \right\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2$

***Step* 4**: $\vec{v}_4 = \vec{u}_4 - \dfrac{\left\langle \vec{u}_4, \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1 - \dfrac{\left\langle \vec{u}_4, \vec{v}_2 \right\rangle}{\left\|\vec{v}_2\right\|^2}\vec{v}_2 - \dfrac{\left\langle \vec{u}_4, \vec{v}_3 \right\rangle}{\left\|\vec{v}_3\right\|^2}\vec{v}_3$

To convert the orthogonal basis into an orthonormal basis $\left\{\vec{q}_1, \vec{q}_2, \vec{q}_3\right\}$, normalize the orthogonal basis vectors. $\boxed{\vec{q}_i = \dfrac{\vec{v}_i}{\left\|\vec{v}_i\right\|}}$

Example

Assume that the vector space $\mathbb{R}$ has the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors

$$\vec{u}_1 = (1,\ 1,\ 1) \quad \vec{u}_2 = (0,\ 1,\ 1) \quad \vec{u}_3 = (0,\ 0,\ 1)$$

Into the orthogonal basis $\left\{\vec{v}_1, \vec{v}_2, \vec{v}_3\right\}$, and then normalize the ***orthogonal*** basis vectors to obtain an orthonormal basis $\left\{\vec{q}_1, \vec{q}_2, \vec{q}_3\right\}$

Solution

$$\vec{v}_1 = \vec{u}_1$$
$$= (1,\ 1,\ 1)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left\langle \vec{u}_2, \vec{v}_1 \right\rangle}{\left\|\vec{v}_1\right\|^2}\vec{v}_1$$

$$= (0,\ 1,\ 1) - \frac{0+1+1}{1^2+1^2+1^2}(1,\ 1,\ 1)$$

$$= (0,\ 1,\ 1) - \frac{2}{3}(1,\ 1,\ 1)$$

$$\boxed{= \left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}$$

$$\vec{v}_3 = \vec{u}_3 - proj_{\vec{v}_2}\vec{u}_3$$

$$= \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= (0,\ 0,\ 1) - \frac{0+0+1}{1^2+1^2+1^2}(1,\ 1,\ 1) - \frac{0+0+\frac{1}{3}}{\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$= (0,\ 0,\ 1) - \frac{1}{3}(1,\ 1,\ 1) - \frac{\frac{1}{3}}{\frac{2}{3}}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$= (0,\ 0,\ 1) - \frac{1}{3}(1,\ 1,\ 1) - \frac{1}{2}\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

$$\boxed{= \left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$= \frac{(1,\ 1,\ 1)}{\sqrt{3}}$$

$$\boxed{= \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

$$= \frac{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}{\sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2}}$$

$$= \frac{\left(-\frac{2}{3},\ \frac{1}{3},\ \frac{1}{3}\right)}{\frac{\sqrt{6}}{3}}$$

$$\boxed{= \left(-\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right)}$$

$$
\begin{aligned}
\vec{q}_3 &= \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|} \\
&= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\sqrt{0^2+\left(-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}} \\
&= \frac{\left(0,\ -\frac{1}{2},\ \frac{1}{2}\right)}{\frac{\sqrt{2}}{2}} \\
&= \left(0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)
\end{aligned}
$$

Gram-Schmidt **Process (*Orthonormal*)**

Suppose $\vec{v}_1, \ldots, \vec{v}_n$ linearly independent in $\mathbb{R}^n$, construct *n* ***orthonormal*** $\vec{u}_1, \ldots, \vec{u}_n$ that span the same space: span $\{\vec{u}_1, \ldots, \vec{u}_k\}$ = span $\{\vec{v}_1, \ldots, \vec{v}_k\}$

***Step* 1**: Since $\vec{v}_i$ are linearly independent ($\neq 0$), so $\|\vec{v}_1\| \neq 0$ (to create a normal vector)

Let $\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \vec{q}_1$, then $\|\vec{u}_1\| = 1$ since $\vec{u}_1$ is orthonormal and span $\{\vec{u}_1\} = span\{\vec{v}_1\}$

$\vec{w}_1 = \vec{v}_1 \Rightarrow \vec{v}_1 = \|\vec{w}_1\|\vec{u}_1$

***Step* 2**: $\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \bullet \vec{q}_1)\vec{q}_1$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \bullet \vec{u}_1}{\|\vec{v}_1\|}\vec{v}_1 \qquad (\vec{w}_2 \perp \vec{u}_1)$$

$$\vec{v}_2 = \|\vec{w}_2\|\vec{u}_2 + (\vec{v}_2 \bullet \vec{u}_1)\vec{u}_1 \qquad \vec{w}_2 = \|\vec{w}_2\|\vec{u}_2$$

$$\boxed{\vec{q}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}}$$

***Step* 3**: $\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \bullet \vec{q}_1)\vec{q}_1 - (\vec{v}_3 \bullet \vec{q}_2)\vec{q}_2$

$$\boxed{\vec{q}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}}$$

	$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$
$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \bullet \vec{q}_1)\vec{q}_1$	$\vec{q}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$
$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \bullet \vec{q}_1)\vec{q}_1 - (\vec{v} \bullet \vec{q}_2)\vec{q}_2$	$\vec{q}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}$
$\vec{w}_n = \vec{v}_n - (\vec{v}_n \bullet \vec{q}_1)\vec{q}_1 - (\vec{v}_n \bullet \vec{q}_2)\vec{q}_2 - \ldots - (\vec{v}_n \bullet \vec{q}_{n-1})\vec{q}_{n-1}$	$\vec{q}_n = \frac{\vec{w}_n}{\|\vec{w}_n\|}$

Example

Use the Gram-Schmidt process to find an ***orthonormal*** basis for the subspaces of

$$\vec{v}_1 = (1,\ 1,\ 0,\ 0) \quad \vec{v}_2 = (0,\ 1,\ 1,\ 0) \quad \vec{v}_3 = (1,\ 0,\ 1,\ 1)$$

Solution

Step 1: $\vec{q}_1 = \dfrac{(1,\ 1,\ 0,\ 0)}{\sqrt{1^2 + 1^2 + 0 + 0}}$ $\qquad \vec{q}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$

$$= \frac{(1,\ 1,\ 0,\ 0)}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$$

Step 2: $\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{q}_1)\vec{q}_1$

$$= (0,\ 1,\ 1,\ 0) - \left[(0,\ 1,\ 1,\ 0) \cdot \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)\right]\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$$

$$= (0,\ 1,\ 1,\ 0) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$$

$$= (0,\ 1,\ 1,\ 0) - \left(\frac{1}{2},\ \frac{1}{2},\ 0,\ 0\right)$$

$$= \left(-\frac{1}{2},\ \frac{1}{2},\ 1,\ 0\right)$$

$$\|\vec{w}_2\| = \sqrt{\left(-\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + 1}$$

$$= \frac{\sqrt{3}}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{6}}{2}$$

$$\vec{q}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}$$

$$= \frac{\left(-\tfrac{1}{2},\ \tfrac{1}{2},\ 1,\ 0\right)}{\frac{\sqrt{6}}{2}}$$

$$= \frac{2}{\sqrt{6}}\left(-\frac{1}{2},\ \frac{1}{2},\ 1,\ 0\right)$$

$$= \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right)$$

***Step* 3**: $\vec{v}_3 \cdot \vec{q}_1 = (1,\ 0,\ 1,\ 1) \cdot \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right)$

$$= \frac{1}{\sqrt{2}}$$

$$\vec{v}_3 \cdot \vec{q}_2 = (1,\ 0,\ 1,\ 1) \cdot \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right)$$

$$= -\frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}}$$

$$\vec{w}_3 = \vec{v}_3 - \left(\vec{v}_3 \cdot \vec{q}_1\right)\vec{q}_1 - \left(\vec{v} \cdot \vec{q}_2\right)\vec{q}_2$$

$$= (1,\ 0,\ 1,\ 1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right) - \frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right)$$

$$= (1,\ 0,\ 1,\ 1) - \left(\frac{1}{2},\ \frac{1}{2},\ 0,\ 0\right) - \left(-\frac{1}{6},\ \frac{1}{6},\ \frac{2}{6},\ 0\right)$$

$$= \left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$\vec{q}_3 = \frac{\vec{w}_3}{\left\|\vec{w}_3\right\|}$$

$$= \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + 1^2}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \frac{1}{\sqrt{\frac{21}{9}}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \frac{3}{\sqrt{21}}\left(\frac{2}{3},\ -\frac{2}{3},\ \frac{2}{3},\ 1\right)$$

$$= \left(\frac{2}{\sqrt{21}},\ -\frac{2}{\sqrt{21}},\ \frac{2}{\sqrt{21}},\ \frac{3}{\sqrt{21}}\right)$$

The ***orthonormal*** basis:

$$\left\{ \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\ 0\right),\ \left(-\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{2}{\sqrt{6}},\ 0\right),\ \left(\frac{2}{\sqrt{21}},\ -\frac{2}{\sqrt{21}},\ \frac{2}{\sqrt{21}},\ \frac{3}{\sqrt{21}}\right) \right\}$$

QR–Decomposition

Problem

If A is an $m \times n$ matrix with linearly independent column vectors, and if Q is the matrix that results by applying the Gram-Schmidt process to the column vectors of A, what relationship, if any, exists between A and Q?

To solve this problem, suppose that the column vectors of A are $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ and the orthonormal column vectors of Q are $\vec{q}_1, \vec{q}_2, \dots, \vec{q}_n$.

$$\begin{array}{l}
\vec{u}_1 = \left\langle \vec{u}_1, \vec{q}_1 \right\rangle \vec{q}_1 + \left\langle \vec{u}_1, \vec{q}_2 \right\rangle \vec{q}_2 + \cdots + \left\langle \vec{u}_1, \vec{q}_n \right\rangle \vec{q}_n \\
\vec{u}_2 = \left\langle \vec{u}_2, \vec{q}_1 \right\rangle \vec{q}_1 + \left\langle \vec{u}_2, \vec{q}_2 \right\rangle \vec{q}_2 + \cdots + \left\langle \vec{u}_2, \vec{q}_n \right\rangle \vec{q}_n \\
\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\quad \vdots \\
\vec{u}_n = \left\langle \vec{u}_n, \vec{q}_1 \right\rangle \vec{q}_1 + \left\langle \vec{u}_n, \vec{q}_2 \right\rangle \vec{q}_2 + \cdots + \left\langle \vec{u}_n, \vec{q}_n \right\rangle \vec{q}_n
\end{array}$$

$$R = \begin{bmatrix}
\left\langle \vec{u}_1, \vec{q}_1 \right\rangle & \left\langle \vec{u}_2, \vec{q}_1 \right\rangle & \cdots & \left\langle \vec{u}_n, \vec{q}_1 \right\rangle \\
0 & \left\langle \vec{u}_2, \vec{q}_2 \right\rangle & \cdots & \left\langle \vec{u}_n, \vec{q}_2 \right\rangle \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \left\langle \vec{u}_n, \vec{q}_n \right\rangle
\end{bmatrix}$$

The equation $A = QR$ is a factorization of A into the product of a matrix Q with orthonormal column vectors and an invertible upper triangular matrix R. We call it the ***QR-decomposition of A***.

Theorem

If A is an $m \times n$ matrix with linearly independent column vectors, then A can be factored as

$$A = QR$$

Where Q is an $m \times n$ matrix with orthonormal column vectors, and R is an $n \times n$ invertible upper triangular matrix.

Example

Find the QR-decomposition of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Solution

The column vectors of are

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

From the previous example

$$\vec{q}_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \qquad \vec{q}_2 = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \qquad \vec{q}_3 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$R = \begin{bmatrix} \left\langle \vec{u}_1, \vec{q}_1 \right\rangle & \left\langle \vec{u}_2, \vec{q}_1 \right\rangle & \left\langle \vec{u}_3, \vec{q}_1 \right\rangle \\ 0 & \left\langle \vec{u}_2, \vec{q}_2 \right\rangle & \left\langle \vec{u}_3, \vec{q}_2 \right\rangle \\ 0 & 0 & \left\langle \vec{u}_3, \vec{q}_3 \right\rangle \end{bmatrix}$$

$$= \begin{bmatrix} (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} & 0 + (1)\frac{1}{\sqrt{3}} + (1)\frac{1}{\sqrt{3}} & 0 + 0 + (1)\frac{1}{\sqrt{3}} \\ 0 & 0\left(\frac{-2}{\sqrt{6}}\right) + (1)\frac{1}{\sqrt{6}} + (1)\frac{1}{\sqrt{6}} & 0\left(\frac{-2}{\sqrt{6}}\right) + 0\frac{1}{\sqrt{6}} + (1)\frac{1}{\sqrt{6}} \\ 0 & 0 & 0 + (0)\frac{-1}{\sqrt{2}} + (1)\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{Q} \cdot \underbrace{\begin{pmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}}_{R}$$

Calculus: Applying the Gram-Schmidt Process

We can apply the Gram-Schmidt orthogonalization procedure to generate some polynomials that are orthonormal on the interval $x \in [-1,\ 1]$ with inner product

$$\langle p,\ q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

Example

Apply the Gram-Schmidt orthonormalization process to the basis $B = \{1,\ x,\ x^2\}$ in $\mathbb{P}_2$ using the inner product

Solution

$B = \{1,\ x,\ x^2\}$

Let $\vec{u}_1 = 1,\quad \vec{u}_2 = x,\quad \vec{u}_3 = x^2$

$\underline{\vec{v}_1 = \vec{u}_1 = 1}$

$$\begin{aligned}\langle \vec{v}_1,\ \vec{v}_1\rangle &= \int_{-1}^{1} dx \\ &= x\,\Big|_{-1}^{1} \\ &= 2\end{aligned}$$

$$\begin{aligned}\langle \vec{u}_2,\ \vec{v}_1\rangle &= \int_{-1}^{1} x\,dx \\ &= \frac{1}{2}x^2\,\Big|_{-1}^{1} \\ &= 0\end{aligned}$$

$$\begin{aligned}\vec{v}_2 &= \vec{u}_2 - \frac{\langle \vec{u}_2,\ \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 \\ &= x - \frac{0}{2}(1) \\ &= x\end{aligned}$$

$$\langle \vec{v}_2,\ \vec{v}_2\rangle = \int_{-1}^{1} x^2\,dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$\boxed{= \frac{2}{3}}$$

$$\left\langle \vec{u}_3, \vec{v}_1 \right\rangle = \int_{-1}^{1} x^2 \, dx$$

$$= \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$\boxed{= \frac{2}{3}}$$

$$\left\langle \vec{u}_3, \vec{v}_2 \right\rangle = \int_{-1}^{1} x^3 \, dx$$

$$= \frac{1}{4}x^4 \Big|_{-1}^{1}$$

$$\boxed{= 0}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\left\langle \vec{u}_3, \vec{v}_1 \right\rangle}{\left\| \vec{v}_1 \right\|^2}\vec{v}_1 - \frac{\left\langle \vec{u}_3, \vec{v}_2 \right\rangle}{\left\| \vec{v}_2 \right\|^2}\vec{v}_2$$

$$= x^2 - \frac{3}{2}(x)(0) - \frac{1}{2}\frac{2}{3}$$

$$\boxed{= x^2 - \frac{1}{3}}$$

$$\left\langle \vec{v}_3, \vec{v}_3 \right\rangle = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 \, dx$$

$$= \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx$$

$$= \left(\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x\right) \Big|_{-1}^{1}$$

$$= \frac{1}{5} - \frac{2}{9} + \frac{1}{9} + \frac{1}{5} - \frac{2}{9} + \frac{1}{9}$$

$$\boxed{= \frac{8}{45}}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\left\| \vec{v}_1 \right\|}$$

$$\boxed{= \frac{1}{\sqrt{2}}}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\left\|\vec{v}_2\right\|}$$

$$= \frac{x}{\sqrt{2/3}}$$

$$\boxed{= \frac{\sqrt{3}}{\sqrt{2}}x}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\left\|\vec{v}_3\right\|}$$

$$= \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$

$$\boxed{= \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \frac{1}{3}\right)}$$

The ***orthonormal*** basis is $\left\{ \frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}x,\ \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right) \right\}$

Exercises *Section* 3.3 – Gram-Schmidt Process

(1 – 14) Use the Gram-Schmidt process to find an ***orthonormal*** basis for the subspaces of $\mathbb{R}^m$.

1. $\vec{u}_1 = (1,\ -3),\quad \vec{u}_2 = (2,\ 2)$

2. $\vec{u}_1 = (1,\ 0),\quad \vec{u}_2 = (3,\ -5)$

3. $\{(1,\ 1,\ 1),\ (-1,\ 1,\ 0),\ (1,\ 2,\ 1)\}$

4. $\{(1,\ 1,\ 1),\ (0,\ 1,\ 1),\ (0,\ 0,\ 1)\}$

5. $\{(1,\ 1,\ 1),\ (0,\ 2,\ 1),\ (1,\ 0,\ 3)\}$

6. $\{(2,\ 2,\ 2),\ (1,\ 0,\ -1),\ (0,\ 3,\ 1)\}$

7. $\{(1,\ -1,\ 0),\ (0,\ 1,\ 0),\ (2,\ 3,\ 1)\}$

8. $\{(3,\ 0,\ 4),\ (-1,\ 0,\ 7),\ (2,\ 9,\ 11)\}$

9. $\{(1,\ 1,\ 1,\ 1),\ (1,\ 2,\ 1,\ 0),\ (1,\ 3,\ 0,\ 0)\}$

10. $\{(0,\ 2,\ -1,\ 1),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 1,\ 1,\ -1)\}$

11. $\vec{u}_1 = (1,\ 0,\ 0),\quad \vec{u}_2 = (3,\ 7,\ -2),\quad \vec{u}_3 = (0,\ 4,\ 1)$

12. $\vec{u}_1 = (1, 1, 1, 1),\quad \vec{u}_2 = (1,\ 2,\ 4,\ 5),\quad \vec{u}_3 = (1,\ -3,\ -4,\ -2)$

13. $\vec{u}_1 = (1, 1, 1, 1),\quad \vec{u}_2 = (1,\ 1,\ 2,\ 4),\quad \vec{u}_3 = (1,\ 2,\ -4,\ -3)$

14. $\vec{u}_1 = (0,\ 2,\ 1,\ 0),\quad \vec{u}_2 = (1,\ -1,\ 0,\ 0),\quad \vec{u}_3 = (1,\ 2,\ 0,\ -1),\quad \vec{u}_4 = (1,\ 0,\ 0,\ 1)$

(15 – 26) Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of $\mathbb{R}^m$.

15. $\{(1,\ 1,\ 0),\ (1,\ 2,\ 0),\ (0,\ 1,\ 2)\}$

16. $\{(1,\ -2,\ 2),\ (2,\ 2,\ 1),\ (2,\ -1,\ -2)\}$

17. $\{(1,\ 0,\ 0),\ (1,\ 1,\ 1),\ (1,\ 1,\ -1)\}$

18. $\{(4,\ -3,\ 0),\ (1,\ 2,\ 0),\ (0,\ 0,\ 4)\}$

19. $\{(0,\ 1,\ 2),\ (2,\ 0,\ 0),\ (1,\ 1,\ 1)\}$

20. $\{(0,\ 1,\ 1),\ (1,\ 1,\ 0),\ (1,\ 0,\ 1)\}$

21. $\{(1,\ 2,\ -2),\ (1,\ 0,\ -4),\ (5,\ 2,\ 0),\ (1,\ 1,\ -1)\}$

22. $\{(-3,\ 1,\ 2),\ (1,\ 1,\ 1),\ (2,\ 0,\ -1),\ (1,\ -3,\ 2)\}$

23. $\{(2,\ 1,\ 1),\ (0,\ 3,\ -1),\ (3,\ -4,\ -2),\ (-1,\ -1,\ 3)\}$

24. $\vec{u}_1 = (1, 1, 0, -1),\quad \vec{u}_2 = (1, 3, 0, 1),\quad \vec{u}_3 = (4, 2, 2, 0)$

25. $\vec{u}_1 = (1, 1, 1, 1),\quad \vec{u}_2 = (1, 1, 2, 4),\quad \vec{u}_3 = (1,\ 2,\ -4,\ -3)$

26. $\{(3,\ 4,\ 0,\ 0),\ (-1,\ 1,\ 0,\ 0),\ (2,\ 1,\ 0,\ -1),\ (0,\ 1,\ 1,\ 0)\}$

27. Find the ***QR***-decomposition of

a) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

b) $\begin{bmatrix} 3 & 5 \\ -4 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

28. Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product

$\vec{u} = (0,\ -2,\ 2,\ 1), \quad \vec{v} = (-1,\ -1,\ 1,\ 1)$

(**29 – 33**) Apply the Gram-Schmidt ***orthonormalization*** process in $\mathbb{C}^0\ [-1,\ 1]$ spanned by the functions, using the inner product

29. $f_1(x) = x + 2, \quad f_2(x) = x^2 - 3x + 4$

30. $f_1(x) = x, \quad f_2(x) = x^3, \quad f_3(x) = x^5$

31. $f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = \frac{1}{2}\left(3x^2 - 1\right)$

32. $f_1(x) = 1, \quad f_2(x) = \sin \pi x, \quad f_3(x) = \cos \pi x$

33. $f_1(x) = \sin \pi x, \quad f_2(x) = \sin 2\pi x, \quad f_3(x) = \sin 3\pi x$

34. For $\mathbb{P}_3[x]$, define the inner product over $\mathbb{R}$ as

$$\langle f,\ g \rangle = \int_{-1}^{1} f(x)g(x)\ dx$$

a) If $f(x) = 1$ is a unit vector in $\mathbb{P}_3[x]$?

b) Find an orthonormal basis for the subspace spanned by $x \text{ and } x^2$.

c) Complete the basis in part (*b*) to an orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product.

d) Is

$$[f,\ g] = \int_0^1 f(x)g(x)\ dx$$

Also, an inner product for $\mathbb{P}_3[x]$

e) Find a pair of vectors $\vec{v}$ and $\vec{w}$ such that

$$\langle \vec{v},\ \vec{w} \rangle = 0 \quad \textit{but} \quad [\vec{v},\ \vec{w}] \neq 0$$

f) Is the basis found in part (*c*) are orthonormal basis for $\mathbb{P}_3[x]$ with respect to the inner product in part (*d*)?