Analytic Geometry

Section 4.6 – Circles and Parabolas

The *Distance* Formula

The distance, *d*, between the points (x_1, y_1) and (x_2, y_2) in the rectangular coordinate system is

$$d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

To complete the distance between two points. Find the square of the difference between the *x*-coordinate plus the square of the difference between the *y*-coordinates. The principal square root of this sum is the distance.

Midpoint Formula

Consider a line segment whose endpoints are (x_1, y_1) and (x_2, y_2). The coordinates of the segment's midpoint are

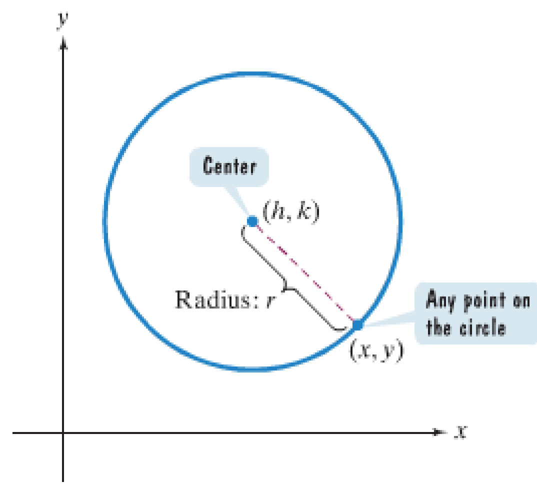
$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

To find the midpoint, take the average of the two *x*-coordinates and the average of the y-coordinates

The Standard Form of the Equation of a Circle

The *standard form of the equation of a circle* with center (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2$$

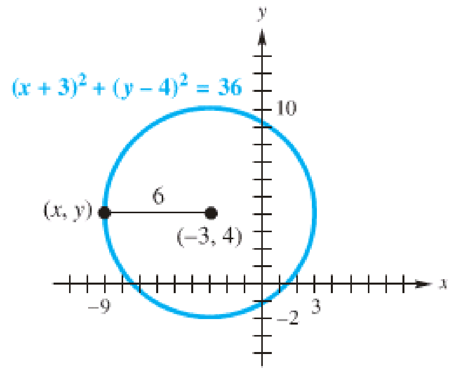


A circle with center (0, 0) and radius *r* has equation:
$x^2 + y^2 = r^2$

Example

Find the center-radius form of the equation of each circle.

a) Center at $(-3,4)$, radius 6

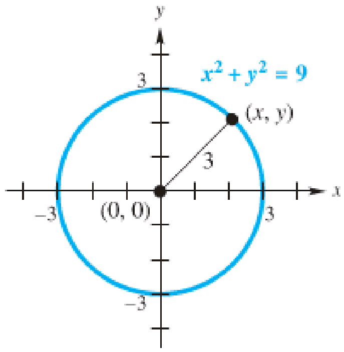


$$(x-h)^2+(y-k)^2=r^2$$

$$(x-(-3))^2+(y-4)^2=6^2$$

$$(x+3)^2+(y-4)^2=36$$

b) Center at $(0,0)$, radius 3



$$(x-0)^2+(y-0)^2=3^2$$

$$x^2+y^2=9$$

Example

Find the equation of a circle with center (–1, 4) that passes through (3, 7).

Solution

$$\begin{aligned} r=d&=\sqrt{(-1-3)^2+(4-7)^2} \\ &=\sqrt{(-4)^2+(-3)^2} \\ &=\sqrt{16+9} \\ &=\sqrt{25} \\ &=5 \end{aligned}$$

$$\boxed{(x+1)^2+(y-4)^2=25}$$

Example

Write in standard form: $x^2+y^2+4x-4y-1=0$

Solution

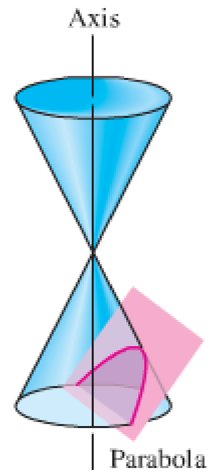
$$x^2+4x \quad +y^2-4y \quad =1$$

$$x^2+4x+\left(\frac{4}{2}\right)^2+y^2-4y+\left(\frac{-4}{2}\right)^2=1+\left(\frac{4}{2}\right)^2+\left(\frac{-4}{2}\right)^2$$

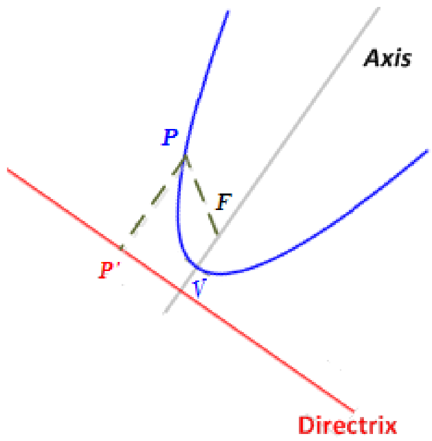
$$x^2+4x+2^2+y^2-4y+2^2=1+4+4$$

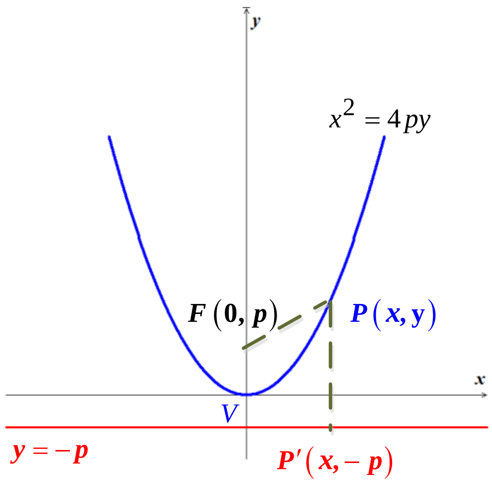
$$\boxed{(x+2)^2+(y-2)^2=9}$$

Definition of a Parabola



A ***parabola*** is the set of all points in a plane equidistant from a fixed point F (the ***focus***) and a fixed line l (the ***directrix***) that lie in the plane



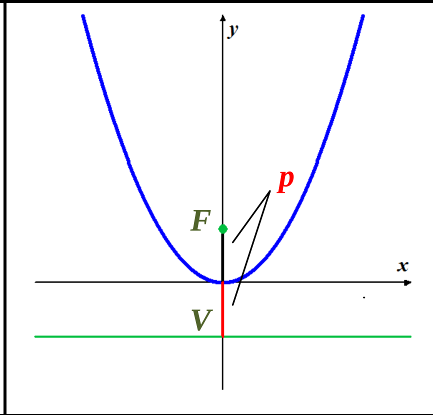
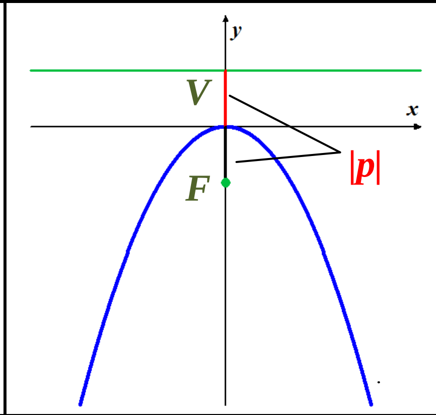
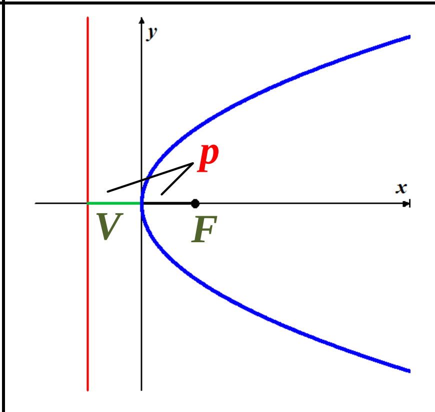
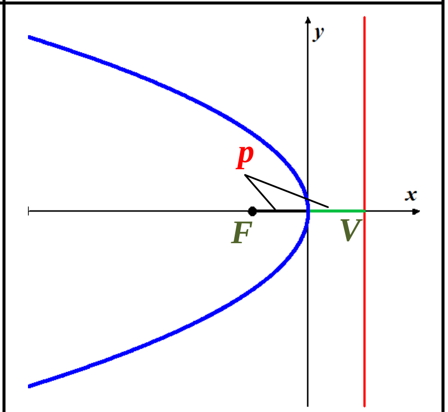
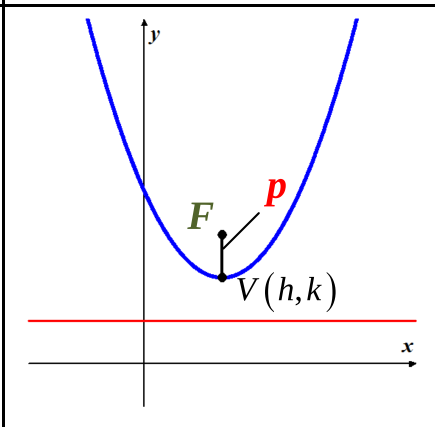
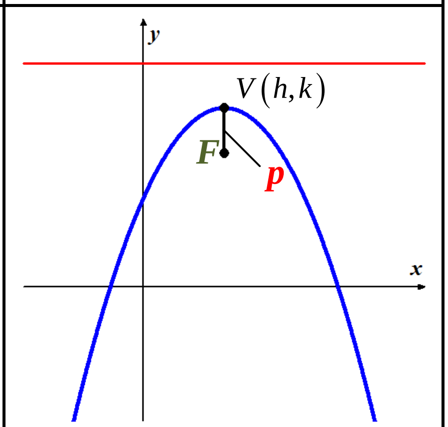
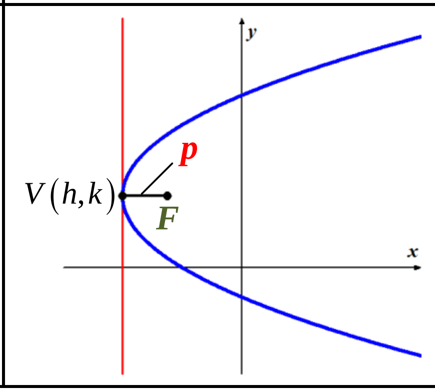
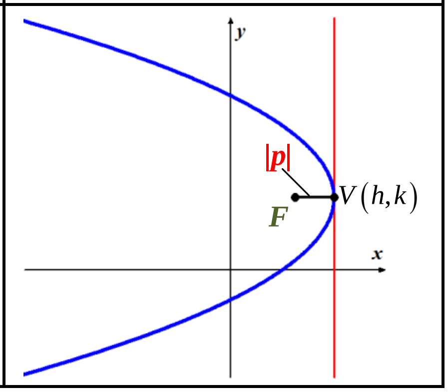


$$y = \frac{1}{4p}x^2 \quad or \quad x^2 = 4py \quad \rightarrow \begin{cases} \textit{Focus}: & F(0,\ p) \\ \textit{Directrix}: & y = -p \end{cases}$$

$$x = \frac{1}{4p}y^2 \quad or \quad y^2 = 4px \quad \rightarrow \begin{cases} \textit{Focus}: & F(p,\ 0) \\ \textit{Directrix}: & x = -p \end{cases}$$

The standard equation $y = ax^2$ *or* $x = ay^2$ is a parabola with vertex $V(0, 0)$. Moreover,

$$a = \frac{1}{4p} \quad or \quad p = \frac{1}{4a}$$

Equation, focus, Directrix	Graph for $p > 0$	Graph for $p < 0$
$x^2 = 4py \quad or \quad y = \frac{1}{4p}x^2$ Focus: $F(0,\ p)$ Directrix: $y = -p$		
$y^2 = 4px \quad or \quad x = \frac{1}{4p}y^2$ Focus: $F(p,\ 0)$ Directrix: $x = -p$		
$(y-k)^2 = 4p(x-h)$ $x = ay^2 + by + c$ Focus: $F(h+p,\ k)$ Directrix: $x = h - p$		
$(x-h)^2 = 4p(y-k)$ $y = ax^2 + bx + c$ Focus: $F(h,\ k+p)$ Directrix: $y = k - p$		

Example

Find the focus and directrix of the parabola $y=-\frac{1}{6}x^2$.

Solution

$$a=-\frac{1}{6} \quad \Rightarrow \quad \lfloor p=\frac{1}{4a}=\frac{1}{4\left(-\frac{1}{6}\right)}=-\frac{6}{4}=-\frac{3}{2} \rfloor$$

The parabola opens downward and has focus $F\left(0,\ -\frac{3}{2}\right)$.

The directrix is the horizontal line $y=\frac{3}{2}$ which is a distance $\frac{3}{2}$ above *V*.

Example

a) Find an equation of a parabola that has vertex at the origin, open right, and passes through the point *P*(7, –3).

b) Find the focus of the parabola.

Solution

a) An equation of a parabola with vertex at the origin that opens right has the form $x=ay^2$

$$7=a(-3)^2$$

$$a=\frac{7}{9}$$

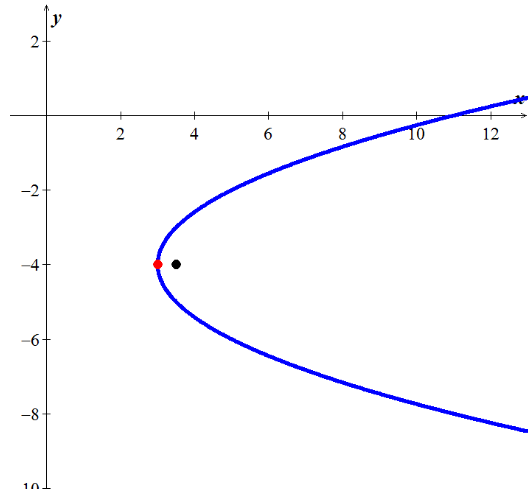
The equation is: $x=\frac{7}{9}y^2$

b) $\lfloor p=\frac{1}{4a}=\frac{1}{4\left(\frac{7}{9}\right)}=\frac{9}{28} \rfloor$

Thus, the focus has coordinate $\left(\frac{9}{28},\ 0\right)$

Example

Sketch the graph of $2x = y^2 + 8y + 22$



Solution

$$2x - 22 = y^2 + 8y$$

$$y^2 + 8y + \left(\tfrac{1}{2}8\right)^2 = 2x - 22 + \left(\tfrac{1}{2}8\right)^2$$

$$(y+4)^2 = 2x - 6$$

$$(y+4)^2 = 2(x-3)$$

The vertex is $V(h,k) = V(3,\ -4)$

The focus is $F(h+p,k) = F\left(3+\frac{1}{2},-4\right) = \boxed{F\left(\frac{7}{2},\ -4\right)}$

The directrix is $x = h - p = 3 - \frac{1}{2} = \boxed{\frac{5}{2}}$

Example

A parabola has vertex $V(-4, 2)$ and directrix $y = 5$. Express the equation of the parabola in the form $y = ax^2 + bx + c$

Solution

Directrix: $y = k - p \Rightarrow \boxed{p = k - y = 2 - 5 = -3}$

$$(x-h)^2 = 4p(y-k)$$

$$(x+4)^2 = 4(-3)(y-2)$$

$$(x+4)^2 = -12(y-2)$$

$$x^2 + 8x + 16 = -12y + 24$$

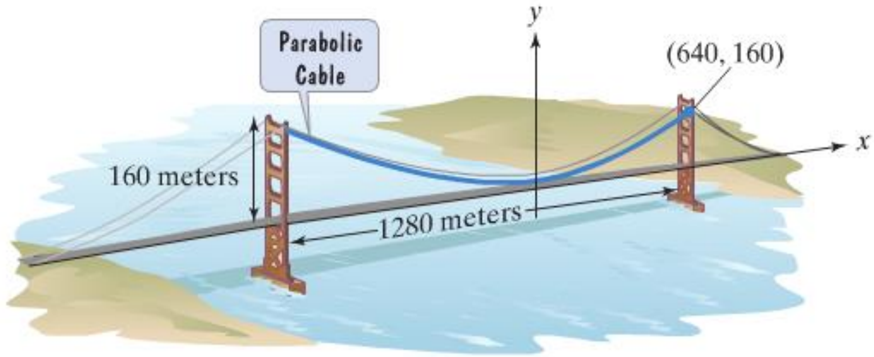
$$x^2 + 8x + 16 - 24 = -12y + 24 - 24$$

$$-12y = x^2 + 8x - 8$$

$$\boxed{y = -\frac{1}{12}x^2 - \frac{2}{3}x + \frac{2}{3}}$$

Example

The towers of the Golden Gate Bridge connecting San Francisco to Marin County are 1280 *meters* apart and rise 160 *meters* above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 200 *meters* from a tower?



Solution

Given the point: $(640,\ 160)$

$(640)^2 = 4p(160)$ $\qquad x^2 = 4py$

$p = \dfrac{640^2}{640} \underline{= 640|}$

$x = 640 - 200 = 440$

$(440)^2 = 4(640)y$ $\qquad x^2 = 4py$

$y = \dfrac{440^2}{4(640)} \underline{\approx 75.625|}$

The height is 76 *meters*.

Exercises *Section* 4.6 – Circles and Parabolas

Find the center and the radius of

1. $x^2+y^2+6x+2y+6=0$

2. $x^2+y^2+8x+4y+16=0$

3. $x^2+y^2-10x-6y-30=0$

4. $x^2-6x+y^2+10y+25=0$

Find the vertex, focus, and directrix of the parabola. Sketch its graph.

5. $20x=y^2$

6. $2y^2=-3x$

7. $(x+2)^2=-8(y-1)$

8. $(x-3)^2=\frac{1}{2}(y+1)$

9. $(y+1)^2=-12(x+2)$

10. $y=x^2-4x+2$

11. $y^2+14y+4x+45=0$

12. $x^2+20y=10$

13. $x^2=16y$

14. $x^2=-\frac{1}{2}y$

15. $(y+1)^2=-4(x-2)$

16. $x^2+6x-4y+1=0$

17. $y^2+2y-x=0$

18. $y^2-4y+4x+4=0$

19. $x^2-4x-4y=4$

Find an equation of the parabola that satisfies the given conditions

20. $Focus: F(2,0)$ $\quad directrix: x=-2$

21. $Focus: F(0,-40)$ $\quad directrix: y=4$

22. $Focus: F(-3,-2)$ $\quad directrix: y=1$

23. $Vertex: V(3,-5)$ $\quad directrix: x=2$

24. $Vertex: V(-2,3)$ $\quad directrix: y=5$

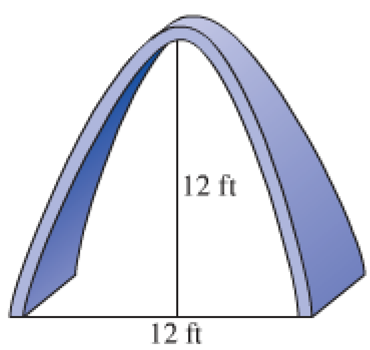
25. $Vertex: V(-1,0)$ $\quad focus: F(-4,0)$

26. $Vertex: V(1,-2)$ $\quad focus: F(1,0)$

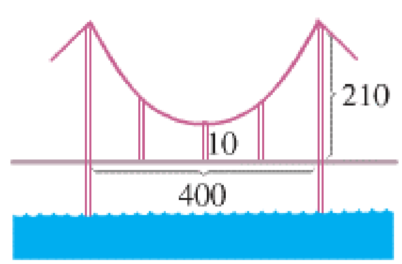
27. $Vertex: V(0,\ 1)$ $\quad focus: F(0,\ 2)$

28. $Vertex: V(3,\ 2)$ $\quad focus: F(-1,\ 2)$

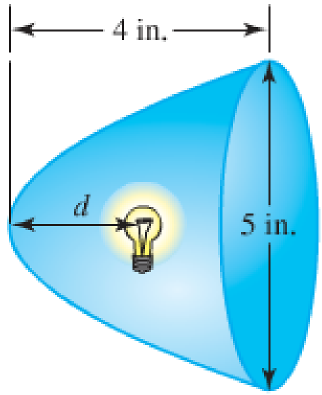
29. An arch in the shape of a parabola has the dimensions shown in the figure. How wide is the arch 9 *feet* up?



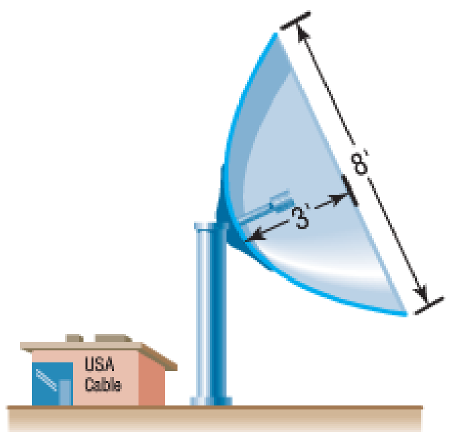
30. The cable in the center portion of a bridge is supported as shown in the figure to form a parabola. The center support is 10 *feet* high, the tallest supports are 210 *feet* high, and the distance between the two tallest supports is 400 *feet*. Find the height of the remaining supports if the supports are evenly spaced.



31. A headlight is being constructed in the shape of a paraboloid with depth 4 *inches* and diameter 5 *inches*. Determine the distance d that the bulb should be form the vertex in order to have the beam of light shine straight ahead.

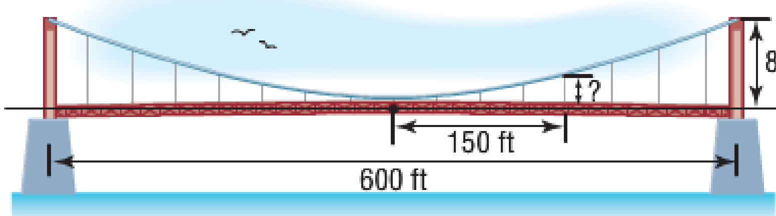


32. A satellite dish is shaped like a paraboloid of revolution. The signals that emanate from a satellite strike the surface of the dish and are reflected to a single point, where the receiver is located. If the dish is 8 *feet* across at its opening and 3 *feet* deep at its center, at what position should the receiver be placed? That is, where is the focus?



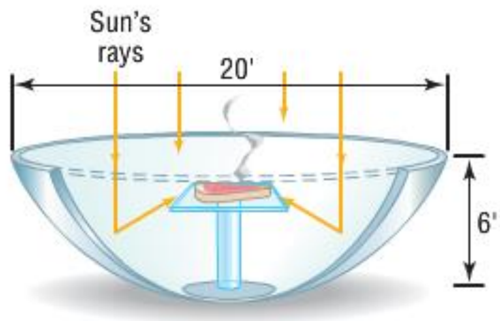
33. A cable TV receiving dish is in the shape of a paraboloid of revolution. Find the location of the receiver, which is placed at the focus, if the dish is 6 *feet* across at its opening and 2 *feet* deep.

34. The cables of a suspension bridge are in the shape of a parabola, as shown below. The towers supporting the cable are 600 *feet* apart and 80 *feet* high. If the cables touch the road surface midway between the towers, what is the height of the cable from the road at a point 150 feet from the center of the bridge?



35. A bridge is built in the shape of a parabolic arch. The bridge has a span of 120 *feet* and a maximum height of 25 *feet*. Choose a suitable rectangular coordinate system and find the height of the arch at distances of 10, 30, and 50 *feet* from the center.

36. A mirror is shaped like a paraboloid of revolution and will be used to concentrate the rays of the sun at its focus, creating a heat source. If the mirror is 20 *feet* across at its opening and is 6 *feet* deep, where will the heat source be concentrated?



37. A reflecting telescope contains a mirror shaped a paraboloid of revolution. If the mirror is 4 *inches* across at its opening and is 3 *inches* deep, where will the collected light be concentrated?

38. Show that the graph of an equation of the form $Ax^2 + Dx + Ey + F = 0 \qquad A \neq 0$

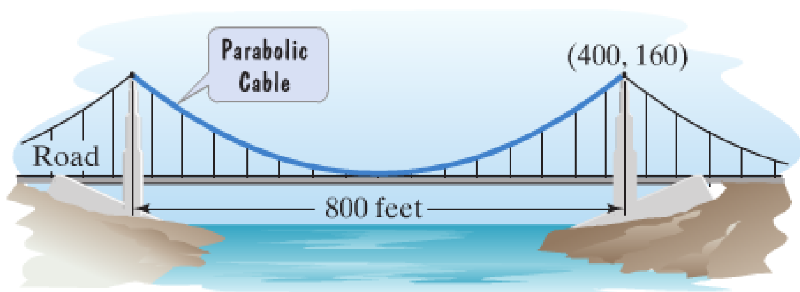
a) Is a parabola if $E \neq 0$

b) Is a vertical line if $E = 0$ *and* $D^2 - 4AF = 0$

c) Is two vertical lines if $E = 0$ *and* $D^2 - 4AF > 0$

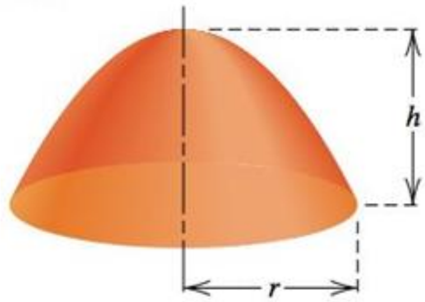
d) Contains no points if $E = 0$ *and* $D^2 - 4AF < 0$

39. The towers of a suspension bridge are 800 *feet* apart and rise 160 *feet* above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 100 *feet* from a tower?



40. The cables of a suspension bridge are in the shape of a parabola. The towers supporting the cable are 400 *feet* apart and 100 *feet* high. If the cables are at a height of 10 *feet* midway between the towers, what is the height of the cable at a point 50 *feet* from the center of the bridge?

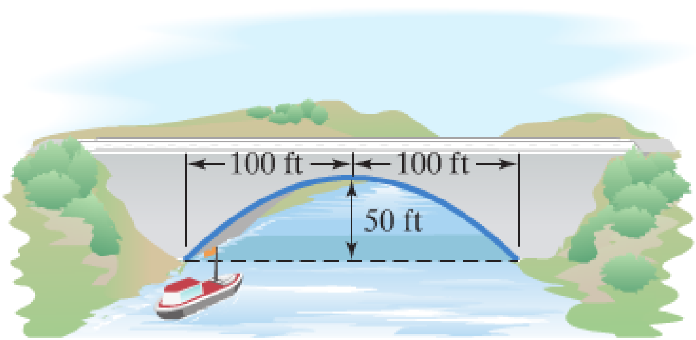
41. The focal length of the (finite) paraboloid is the distance p between its vertex and focus



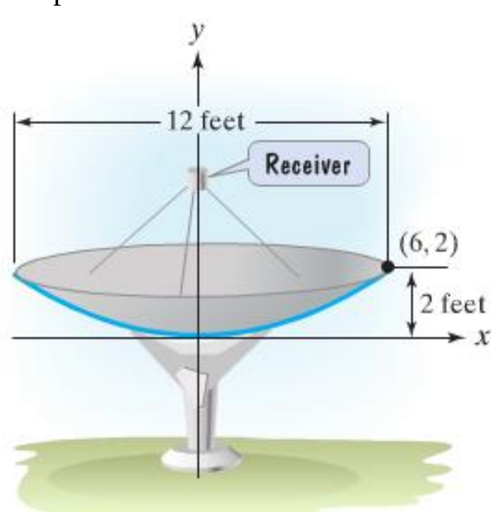
a) Express p in terms of r and h.

b) A reflector is to be constructed with a focal length of 10 *feet* and a depth of 5 *feet*. Find the radius of the reflector.

42. The parabolic arch is 50 *feet* above the water at the center and 200 *feet* wide at the base. Will a boat that is 30 *feet* tall clear the arch 30 *feet* from the center? 45.5 *ft*



43. A satellite dish, as shown below, is in the shape of a parabolic surface. Signals coming from a satellite strike the surface of the dish and are reflected to the focus, where the receiver if located. The satellite dish shown has a diameter of 12 *feet* and a depth of 2 *feet*. How far from the base of the dish should the receiver be placed? 45.5 *ft*



44. A searchlight is shaped like a paraboloid of revolution. If the light source is located 2 *feet* from the base along the axis of symmetry and the opening is 5 *feet* across, how deep should the searchlight be? 0.78125 *ft*

45. A searchlight is shaped like a paraboloid of revolution. If the light source is located 2 *feet* from the base along the axis of symmetry and the depth of the searchlight is 4 *feet* across, how deep should the opening be? 11.31 *ft*

46. A searchlight is shaped like a paraboloid, with the light source at the focus. If the reflector is 3 *feet* across at the opening and 1 *foot* deep, where is the focus? $\frac{9}{16}$ *ft*

47. A mirror for a reflecting telescope has the shape of a (finite) paraboloid of diameter 8 *inches* and depth 1 *inch*. How far from the center the mirror will the incoming light collect? 4 *in*