Solution   *Section* 2.7 – First-Order Linear Equations

Exercise

Write an equivalent first-order differential equation and initial condition for y. $y=\int_1^x \frac{1}{t}\,dt$

Solution

$$\int_1^x \frac{1}{t}\,dt \quad\Rightarrow\quad \frac{dy}{dx}=\frac{1}{x}$$

$$y(1)=\int_1^1 \frac{1}{t}\,dt \qquad\qquad \int_a^a f(x)dx=0$$

$$=\ln t\,\Big|_1^1$$

$$=\ln 1-\ln 1$$

$$\underline{=0}$$

$$\boxed{\frac{dy}{dx}=\frac{1}{x};\quad y(1)=0}$$

Exercise

Write an equivalent first-order differential equation and initial condition for $y=2-\int_0^x \big(1+y(t)\big)\sin t\,dt$

Solution

$$y=2-\int_0^x \big(1+y(t)\big)\sin t\,dt$$

$$\frac{dy}{dx}=-\big(1+y(x)\big)\sin x$$

$$y(0)=2-\int_0^0 \big(1+y(t)\big)\sin t\,dt \qquad\qquad \int_a^a f(x)dx=0$$

$$\underline{=2}$$

$$\boxed{\frac{dy}{dx}=-\big(1+y(x)\big)\sin x;\quad y(0)=2}$$

Exercise

Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

$y' = 1 - \frac{y}{x}, \quad y(2) = -1, \quad dx = 0.5$

Solution

$$y_1 = y_0 + \left(1 - \frac{y_0}{x_0}\right) dx$$

$$= -1 + \left(1 - \frac{-1}{2}\right)(0.5)$$

$$= -0.25$$

$$y_2 = y_1 + \left(1 - \frac{y_1}{x_1}\right) dx$$

$$= -0.25 + \left(1 - \frac{-0.25}{2.5}\right)(0.5)$$

$$= 0.3$$

$$y_3 = y_2 + \left(1 - \frac{y_2}{x_2}\right) dx$$

$$= 0.3 + \left(1 - \frac{0.3}{3}\right)(0.5)$$

$$= 0.75$$

$$y' + \frac{1}{x} y = 1 \qquad P(x) = \frac{1}{x}, \quad Q(x) = 1$$

$$y_h = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$\int (1) e^{\int \frac{1}{x} dx} dx = \int x \, dx = \frac{1}{2} x^2$$

$$y(x) = \frac{1}{x}\left(\frac{1}{2} x^2 + C\right)$$

$$= \frac{1}{2} x + \frac{C}{x}$$

$$y(2) = \frac{1}{2}(2) + \frac{C}{2} = -1$$

$$1 + \frac{C}{2} = -1$$

$$\frac{C}{2} = -2$$

$$C = -4$$

$$y(x) = \frac{x}{2} - \frac{4}{x}$$

$$y(3.5) = \frac{3.5}{2} - \frac{4}{3.5}$$
$$\approx 0.6071$$

Exercise

Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

$y' = x(1-y), \quad y(1) = 0, \quad dx = 0.2$

Solution

$$y_1 = y_0 + x_0\left(1 - y_0\right)dx$$
$$= 0 + 1(1-0)(0.2)$$
$$= 0.2$$

$$y_2 = y_1 + x_1\left(1 - y_1\right)dx$$
$$= 0.2 + 1.2(1-0.2)(0.2)$$
$$= 0.392$$

$$y_3 = y_2 + x_2\left(1 - y_2\right)dx$$
$$= 0.392 + 1.4(1-0.392)(0.2)$$
$$= 0.5622$$

$$\frac{y'}{1-y} = x\,dx$$

$$\int \frac{dy}{1-y} = \int x\,dx$$

$$\ln|1-y| = \frac{1}{2}x^2 + C$$

$$1 - y = e^{\frac{1}{2}x^2 + C}$$

$$y = 1 - e^{\frac{1}{2}x^2 + C}$$

$$y(1) = 1 - e^{\frac{1}{2}1^2 + C} = 0$$

$$e^{\frac{1}{2} + C} = 1$$

$$\frac{1}{2} + C = 0$$

$$C = -\frac{1}{2}$$

$$\boxed{y(x) = 1 - e^{\frac{1}{2}\left(x^2 - 1\right)}}$$

$$y(1.6) = 1 - e^{\frac{1}{2}\left(1.6^2 - 1\right)}$$

$$\approx \boxed{0.5416}$$

Exercise

Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

$y' = y^2(1+2x), \quad y(-1) = 1, \quad dx = 0.5$

Solution

$$y_1 = y_0 + y_0^2\left(1 + 2x_0\right)dx$$

$$= 1 + 1^2(1 + 2(-1))(0.5)$$

$$= \boxed{.5}$$

$$y_2 = y_1 + y_1^2\left(1 + 2x_1\right)dx$$

$$= 0.5 + 0.5^2(1 + 2(-0.5))(0.5)$$

$$= \boxed{.5}$$

$$y_3 = y_2 + y_2^2\left(1 + 2x_2\right)dx$$

$$= .5 + .5^2(1 + 2(0))(0.5)$$

$$= \boxed{.625}$$

$$\frac{dy}{y^2} = (1+2x)dx$$

$$\int \frac{dy}{y^2} = \int (1+2x)dx$$

$$-\frac{1}{y} = x + x^2 + C$$

$$y = -\frac{1}{x + x^2 + C}$$

$$y(-1) = -\frac{1}{-1 + (-1)^2 + C}$$

$$1 = -\frac{1}{C}$$

$$\boxed{C = -1}$$

$$y(x) = -\frac{1}{x + x^2 - 1}$$

$$= \frac{1}{1 - x - x^2}$$

$$y(.5) = \frac{1}{1 - .5 - .5^2}$$

$$= 4$$

Exercise

Use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round the results to four decimals

$y' = ye^x, \quad y(0) = 2, \quad dx = 0.5$

Solution

$$y_1 = y_0 + \left(y_0 e^{x_0} \right) dx$$

$$= 2 + \left(2e^0\right)(0.5)$$

$$= 3$$

$$y_2 = y_1 + \left(y_1 e^{x_1} \right) dx$$

$$= 3 + \left(3e^{0.5}\right)(0.5)$$

$$= 5.47308$$

$$y_3 = y_2 + \left(y_2 e^{x_2} \right) dx$$

$$= 5.47308 + \left(5.47308e^1\right)(0.5)$$

$$= 12.9118$$

$$\frac{dy}{dx} = ye^x$$

$$\int \frac{dy}{y} = \int e^x dx$$

$$\ln y = e^x + C$$

$$\ln 2 = e^0 + C$$

$$C = \ln 2 - 1$$

$$\ln|y| = e^x + \ln 2 - 1$$

$$y(x) = e^{e^{x}+\ln 2-1}$$

$$= e^{\ln 2} e^{e^{x}-1}$$

$$\underline{= 2e^{e^{x}-1}}$$

$$y(1.5) = 2e^{e^{1.5}-1}$$

$$\underline{\approx 65.0292}$$

Exercise

Use the Euler method with $dx = 0.2$ to estimate $y(2)$ if $y' = \frac{y}{x}$ and $y(1) = 2$. What is the exact value of $y(2)$?

Solution

$$y_1(1) = y_0 + \left(\frac{y_0}{x_0}\right)dx$$

$$= 2 + \left(\tfrac{2}{1}\right)(0.2)$$

$$\underline{= 2.4}$$

$$y_2(1.2) = y_1 + \left(\frac{y_1}{x_1}\right)dx$$

$$= 2.4 + \left(\tfrac{2.4}{1.2}\right)(0.2)$$

$$\underline{= 2.8}$$

$$y_3 = y_2 + \left(\frac{y_2}{x_2}\right)dx$$

$$= 2.8 + \left(\tfrac{2.8}{1.4}\right)(0.2)$$

$$\underline{= 3.2}$$

$$y_4 = y_3 + \left(\frac{y_3}{x_3}\right)dx$$

$$= 3.2 + \left(\tfrac{3.2}{1.6}\right)(0.2)$$

$$\underline{= 3.6}$$

$$y_5 = y_4 + \left(\frac{y_4}{x_4}\right)dx$$

$$= 3.6 + \left(\tfrac{3.6}{1.8}\right)(0.2)$$

$$\underline{= 4}$$

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + C$$

$$\ln 2 = \ln 1 + C$$

$$\underline{C = \ln 2}$$

$$\ln y = \ln x + \ln 2$$

$$= \ln 2x$$

$$\underline{y(x) = 2x}$$

$$y(2) = 2(2)$$

$$\underline{= 4}$$

Exercise

Use Euler's Method to solve $y' = 1 + y, \quad y(0) = 1$ on the interval $0 \le x \le 1$ and taking $dx = 0.05$. Compare the approximation to the values of the exact solution.

Solution

y(t) = 2*exp(t) – 1

Euler Method

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.05	1.10000000	1.10254219	0.00254219
0.10	1.20500000	1.21034184	0.00534184
0.15	1.31525000	1.32366849	0.00841849
0.20	1.43101250	1.44280552	0.01179302
0.25	1.55256313	1.56805083	0.01548771
0.30	1.68019128	1.69971762	0.01952633
0.35	1.81420085	1.83813510	0.02393425
0.40	1.95491089	1.98364940	0.02873851
0.45	2.10265643	2.13662437	0.03396794
0.50	2.25778925	2.29744254	0.03965329
0.55	2.42067872	2.46650604	0.04582732
0.60	2.59171265	2.64423760	0.05252495
0.65	2.77129828	2.83108166	0.05978337
0.70	2.95986320	3.02750541	0.06764222
0.75	3.15785636	3.23400003	0.07614367
0.80	3.36574918	3.45108186	0.08533268

0.85	3.58403664	3.67929370	0.09525707
0.90	3.81323847	3.91920622	0.10596775
0.95	4.05390039	4.17141932	0.11751893
1.00	4.30659541	4.43656366	0.12996825

Exercise

Use Euler's Method to solve $y' = 2xy + 2y, \quad y(0) = 3$ on the interval $0 \le x \le 1$ and taking $dx = 0.1$. Compare the approximation to the values of the exact solution.

Solution

$$\frac{dy}{y} = (2x+2)dx$$

$$\ln y = x^2 + 2x + C_1$$

$$y = Ce^{x^2+2x} \quad \to 3 = C$$

$$y(x) = 3e^{x^2+2x}$$

t	***Euler Method***	***Exact***	***Difference***
0.000	3.0	3.0	0.0
0.100	3.6	3.70103418	0.10103418
0.200	4.392	4.65812166	0.26612166
0.300	5.44608	5.9811466	0.5350666
0.400	6.8620608	7.83508942	0.97302862
0.500	8.78343782	10.47102887	1.68759105
0.600	11.41846917	14.27646374	2.85799457
0.700	15.07237931	19.85810604	4.78572673
0.800	20.19698827	28.17999386	7.98300559
0.900	27.46790405	40.79715256	13.32924851
1.000	37.90570759	60.25661077	22.35090318

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y = Ce^{-5t};\quad y'(t) + 5y = 0$

Solution

$$y = Ce^{-5t} \quad \Rightarrow \quad y' = -5Ce^{-5t} = -5y$$

$$y'(t) + 5y = -5y + 5y$$

$$= 0 \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y = Ct^{-3};\quad ty'(t) + 3y = 0$

Solution

$$y = Ct^{-3}$$

$$y' = -3Ct^{-4}$$

$$t\left(-3Ct^{-4}\right) + 3Ct^{-3} = -3Ct^{-3} + 3Ct^{-3}$$

$$\underline{= 0} \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y = C_1 \sin 4t + C_2 \cos 4t;\quad y''(t) + 16y = 0$

Solution

$$y' = 4C_1 \cos 4t - 4C_2 \sin 4t$$

$$y'' = -16C_1 \sin 4t - 16C_2 \cos 4t$$

$$y''(t) + 16y = -16C_1 \sin 4t - 16C_2 \cos 4t + 16C_1 \sin 4t + 16C_2 \cos 4t$$

$$\underline{= 0} \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y = C_1 e^{-x} + C_2 e^{x};\quad y''(x) - y = 0$

Solution

$$y' = -C_1 e^{-x} + C_2 e^{x}$$

$$y'' = C_1 e^{-x} + C_2 e^{x}$$

$$y''(x) - y = C_1 e^{-x} + C_2 e^{x} - C_1 e^{-x} - C_2 e^{x}$$

$$\underline{= 0} \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y' + 4y = \cos t, \quad y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t + Ce^{-4t}, \quad y(0) = -1$

Solution

$$y(0) = \frac{4}{17}\cos(0) + \frac{1}{17}\sin(0) + Ce^{-4(0)}$$

$$-1 = \frac{4}{17} + C$$

$$C = -1 - \frac{4}{17} = -\frac{21}{17}$$

$$\boxed{y(t) = \frac{4}{17}\cos t + \frac{1}{17}\sin t - \frac{21}{17}e^{-4t}}$$

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad ty' + (t+1)y = 2te^{-t}, \quad y(t) = e^{-t}\left(t + \frac{C}{t}\right), \quad y(1) = \frac{1}{e}$

Solution

$$y(1) = \frac{1}{e} = e^{-1}$$

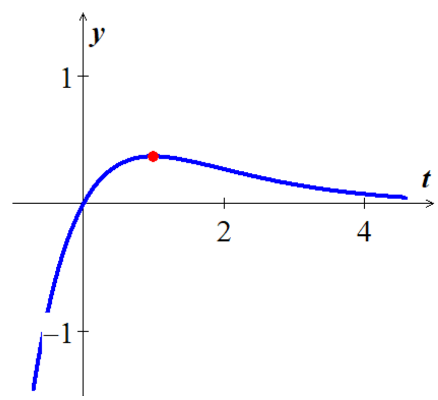
$$y(1) = e^{-1}\left(1 + \frac{C}{1}\right)$$

$$e^{-1} = e^{-1}(1 + C)$$

$$\Rightarrow\ 1 = 1 + C$$

Hence, $C = 0$

The solution is: $y(t) = te^{-t}$



This function is defined and differentiable on the whole real line. Hence, the interval of existence is the whole real line.

Exercise

Verify that the given function y is a solution of the differential equation that follows it. Assume that $C,\ C_1,\ and\ C_2$ are arbitrary constants. $\quad y' = y(2+y), \quad y(t) = \frac{2}{-1 + Ce^{-2t}}, \quad y(0) = -3$

Solution

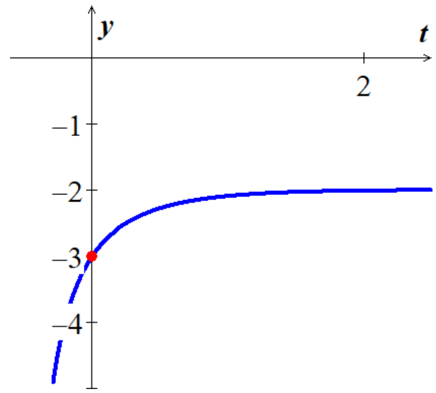
$$y(0) = \frac{2}{-1 + Ce^{-2(0)}}$$

$$-3 = \frac{2}{-1 + C}$$

$$3-3C=2$$
$$-3C=-1$$
$$\boxed{C=\frac{1}{3}}$$

The solution is:

$$y(t)=\frac{2}{-1+\frac{1}{3}e^{-2t}}$$
$$\boxed{=\frac{6}{-3+e^{-2t}}}$$



Exercise

Verify that the given function y is a solution of the initial value problem that follows it.

$$y=16e^{2t}-10; \quad y'-2y=20, \quad y(0)=6$$

Solution

$$y(0)=6$$
$$y(0)=16-10=6 \quad \checkmark$$

$$y=16e^{2t}-10$$
$$y'=32e^{2t}$$

$$y'-2y=32e^{2t}-32e^{2t}+20$$
$$\boxed{=20} \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the initial value problem that follows it.

$$y=8t^{6}-3; \quad ty'-6y=18, \quad y(1)=5$$

Solution

$$y=8t^{6}-3$$
$$y(1)=8-3$$
$$\boxed{=5} \quad \checkmark$$

$$y'=48t^{5}$$

$$ty'-6y=48t^{6}-48t^{6}+18$$
$$\boxed{=18} \quad \checkmark$$

Exercise

Verify that the given function y is a solution of the initial value problem that follows it.

$$y = -3\cos 3t; \quad y'' + 9y = 0, \quad y(0) = -3, \; y'(0) = 0$$

Solution

$$y = -3\cos 3t$$

$$y(0) = -3\cos 0$$

$$= -3 \quad \checkmark$$

$$y' = 9\sin 3t$$

$$y(0) = 0 \quad \checkmark$$

$$y'' = 27\cos 3t$$

$$y'' + 9y = 27\cos 3t - 27\cos 3t$$

$$= 0$$

Exercise

Verify that the given function y is a solution of the initial value problem that follows it.

$$y = \frac{1}{4}\left(e^{2x} - e^{-2x}\right); \quad y'' - 4y = 0, \quad y(0) = 0, \; y'(0) = 1$$

Solution

$$y = \frac{1}{4}\left(e^{2x} - e^{-2x}\right)$$

$$y(0) = \frac{1}{4}(1-1)$$

$$= 0 \quad \checkmark$$

$$y' = \frac{1}{2}\left(e^{2x} + e^{-2x}\right)$$

$$y'(0) = \frac{1}{2}(1+1)$$

$$= 1 \quad \checkmark$$

$$y'' = e^{2x} - e^{-2x}$$

$$y'' - 4y = e^{2x} - e^{-2x} - e^{2x} + e^{-2x}$$

$$= 0 \quad \checkmark$$

Exercise

Find the general solution of the differential equation $y' = xy$

Solution

$$\frac{dy}{dx} = xy$$

$$\frac{dy}{y} = xdx$$

$$\int \frac{dy}{y} = \int x\,dx$$

$$\ln|y| = \frac{1}{2}x^2 + C$$

$$|y| = e^{x^2/2+C}$$

$$y(x) = \pm e^{x^2/2}e^C$$

$$\underline{= Ae^{x^2/2}} \qquad \text{Where } A = \pm e^C$$

Exercise

Find the general solution of the differential equation $xy' = 2y$

Solution

$$x\frac{dy}{dx} = 2y$$

$$\frac{dy}{y} = 2\frac{dx}{x}$$

$$\int \frac{dy}{y} = \int \frac{2}{x}dx$$

$$\ln|y| = 2\ln|x| + C$$

$$= \ln x^2 + C$$

$$y(x) = \pm e^{\ln x^2 + C}$$

$$= \pm e^C x^2$$

$$\underline{= Ax^2}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = e^{x-y}$

Solution

$$\frac{dy}{dx} = e^x e^{-y}$$

$$\frac{dy}{e^{-y}} = e^x dx$$

$$\int e^y dy = \int e^x dx$$

$$e^y = e^x + C$$

$$\boxed{y(x) = \ln\left(e^x + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \left(1 + y^2\right)e^x$

Solution

$$\frac{dy}{dx} = \left(1 + y^2\right)e^x$$

$$\frac{dy}{1 + y^2} = e^x dx$$

$$\int \frac{dy}{1 + y^2} = \int e^x dx$$

$$\tan^{-1} y = e^x + C$$

$$\boxed{y(x) = \tan\left(e^x + C\right)}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = xy + y$

Solution

$$\frac{dy}{dx} = (x+1)y$$

$$\frac{dy}{y} = (x+1)dx$$

$$\int \frac{dy}{y} = \int (x+1)dx$$

$$\ln y = \frac{1}{2}x^2 + x + C$$

$$\boxed{y = e^{x^2/2 + x + C}}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution

$$y' = ye^x - 2e^x + y - 2$$

Solution

$$\frac{dy}{dx} = (y-2)e^x + y - 2$$

$$\frac{dy}{dx} = (y-2)\left(e^x + 1\right)$$

$$\frac{dy}{y-2} = \left(e^x + 1\right)dx$$

$$\int \frac{dy}{y-2} = \int \left(e^x + 1\right)dx$$

$$\ln|y-2| = e^x + x + C$$

$$y - 2 = \pm e^{e^x + x + C}$$

$$y - 2 = \pm e^C e^{e^x + x}$$

$$\boxed{y(x) = De^{e^x + x} + 2} \quad D = \pm e^C$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \dfrac{x}{y+2}$

Solution

$$\frac{dy}{dx} = \frac{x}{y+2}$$

$$(y+2)\,dy = xdx$$

$$\int (y+2)\,dy = \int x\,dx$$

$$\tfrac{1}{2}y^2 + 2y = \tfrac{1}{2}x^2 + C$$

$$\boxed{y^2 + 4y = x^2 + 2C}$$

$$y^2 + 4y - x^2 - D = 0, \quad (D = 2C)$$

$$y = \frac{-4 \pm \sqrt{16 - 4\left(-x^2 - D\right)}}{2} = \frac{-4 \pm \sqrt{16 + 4x^2 + 4D}}{2}$$

$$= \frac{-4 \pm 2\sqrt{x^2 + (4+D)}}{2} \qquad E = 4 + D$$

$$= -2 \pm \sqrt{x^2 + E}$$

$$\boxed{y(x) = -2 \pm \sqrt{x^2 + E}}$$

Exercise

Find the general solution of the differential equation. If possible, find an explicit solution $y' = \dfrac{xy}{x-1}$

Solution

$$\frac{dy}{dx} = y\left(\frac{x}{x-1}\right)$$

$$\frac{dy}{y} = \left(\frac{x}{x-1}\right)dx$$

$$\int \frac{dy}{y} = \int \left(1 + \frac{1}{x-1}\right) dx$$

$$\ln|y| = x + \ln|x-1| + C$$

$$y(x) = \pm e^{x + \ln|x-1| + C}$$

$$= \pm e^C e^x e^{\ln|x-1|}$$

$$\boxed{= De^x |x-1|}$$

Exercise

Solve the differential equations: $\quad x\dfrac{dy}{dx} + y = e^x, \quad x > 0$

Solution

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{e^x}{x}$$

$$y_h = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int \frac{e^x}{x} e^{\int \frac{1}{x}dx} dx = \int x\frac{e^x}{x} dx$$

$$= \int e^x dx$$

$$= e^x$$

$$\boxed{y(x) = \frac{1}{x}\left(e^x + C\right)}, \quad x > 0$$

Exercise

Solve the differential equations: $y' + (\tan x)y = \cos^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

Solution

$$y' + (\tan x)y = \cos^2 x$$

$$y_h = e^{\int \tan x\, dx} = e^{\ln(\cos x)^{-1}} \qquad \int \tan x dx = -\ln|\cos x| = \ln(\cos x)^{-1}$$

$$= (\cos x)^{-1}$$

$$\int \cos^2 x (\cos x)^{-1} dx = \int \cos x\, dx$$

$$= \sin x$$

$$y(x) = \frac{1}{(\cos x)^{-1}}(\sin x + C)$$

$$y(x) = \cos x(\sin x + C)$$

$$\boxed{y(x) = \cos x \sin x + C\cos x}$$

Exercise

Solve the differential equations: $x\frac{dy}{dx} + 2y = 1 - \frac{1}{x}, \quad x > 0$

Solution

$$y' + \frac{2}{x}y = \frac{1}{x} - \frac{1}{x^2}$$

$$y_h = e^{\int \frac{2}{x} dx}$$

$$= e^{2\ln x}$$

$$= e^{\ln x^2}$$

$$= x^2$$

$$\int \left(\frac{1}{x} - \frac{1}{x^2}\right)x^2\, dx = \int (x-1)\, dx$$

$$= \frac{1}{2}x^2 - x$$

$$y(x) = \frac{1}{x^2}\left(\frac{1}{2}x^2 - x + C\right)$$

$$\boxed{y(x) = \frac{1}{2} - \frac{1}{x} + \frac{C}{x^2}, \quad x > 0}$$

Exercise

Solve the differential equations: $\quad (1+x)\,y' + y = \sqrt{x}$

Solution

$$y' + \frac{1}{1+x}\,y = \frac{\sqrt{x}}{1+x}$$

$$e^{\int \frac{1}{1+x}dx} = e^{\ln(1+x)}$$

$$= 1+x$$

$$\int \frac{\sqrt{x}}{1+x}(1+x)\,dx = \int x^{1/2}\,dx$$

$$= \frac{2}{3}x^{3/2}$$

$$y(x) = \frac{1}{1+x}\left(\frac{2}{3}x^{3/2} + C\right)$$

$$= \frac{2x^{3/2}}{3(1+x)} + \frac{C}{1+x}$$

Exercise

Solve the differential equations: $\quad e^{2x}y' + 2e^{2x}y = 2x$

Solution

$$y' + 2y = 2xe^{-2x}$$

$$e^{\int 2dx} = e^{2x}$$

$$\int 2xe^{-2x}\left(e^{2x}\right)dx = 2\int x\,dx$$

$$= x^2$$

$$y(x) = \frac{1}{e^{2x}}\left(x^2 + C\right)$$

$$= x^2e^{-2x} + Ce^{-2x}$$

Exercise

Solve the differential equations: $\quad (t+1)\frac{ds}{dt} + 2s = 3(t+1) + \frac{1}{(t+1)^2}, \quad t > -1$

Solution

$$s' + \frac{2}{t+1}s = 3 + \frac{1}{(t+1)^3}$$

$$e^{\int \frac{2}{t+1}dt} = e^{2\ln(t+1)}$$

$$= e^{\ln(t+1)^2}$$

$$= (t+1)^2$$

$$\int \left(3 + \frac{1}{(t+1)^3}\right)(t+1)^2\, dt = \int \left(3(t+1)^2 + \frac{1}{t+1}\right)dt \qquad d(t+1) = dt$$

$$= 3\int (t+1)^2\, d(t+1) + \int \frac{1}{t+1} d(t+1)$$

$$= (t+1)^3 + \ln(t+1)$$

$$s(t) = \frac{1}{(t+1)^2}\left((t+1)^3 + \ln(t+1) + C\right)$$

$$\boxed{= t + 1 + \frac{\ln(t+1)}{(t+1)^2} + \frac{C}{(t+1)^2}, \quad t > -1}$$

Exercise

Solve the differential equations: $\tan\theta \frac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \frac{\pi}{2}$

Solution

$$\frac{dr}{d\theta} + \frac{1}{\tan\theta}r = \frac{\sin^2\theta}{\tan\theta}$$

$$\frac{dr}{d\theta} + \frac{1}{\tan\theta}r = \sin^2\theta \frac{\cos\theta}{\sin\theta}$$

$$\frac{dr}{d\theta} + \frac{1}{\tan\theta}r = \sin\theta\cos\theta$$

$$e^{\int \cot\theta d\theta} = e^{\ln|\sin\theta|}$$

$$= \sin\theta, \quad 0 < \theta < \frac{\pi}{2}$$

$$\int (\sin\theta\cos\theta)(\sin\theta)\, d\theta = \int \left(\sin^2\theta\cos\theta\right) d\theta \qquad d(\sin\theta) = \cos\theta d\theta$$

$$= \int \sin^2\theta d(\sin\theta)$$

$$= \frac{1}{3}\sin^3\theta$$

$$r(\theta) = \frac{1}{\sin\theta}\left(\frac{1}{3}\sin^3\theta + C\right)$$

$$= \frac{1}{3}\sin^2\theta + \frac{C}{\sin\theta}$$

Exercise

Find the general solution of $y' = \cos x - y\sec x$

Solution

$$y' + (\sec x)\,y = \cos x$$

$$e^{\int \sec x dx} = e^{\ln|\sec x + \tan x|}$$

$$= \sec x + \tan x$$

$$\int \cos x(\sec x + \tan x)\,dx = \int \cos x\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)dx$$

$$= \int (1 + \sin x)\,dx$$

$$= x - \cos x$$

$$y(x) = \frac{1}{\sec x + \tan x}(x - \cos x + C)$$

Exercise

Find the general solution of $\left(1 + x^3\right)y' = 3x^2 y + x^2 + x^5$

Solution

$$y' - \frac{3x^2}{1+x^3}y = \frac{x^2\left(1+x^3\right)}{1+x^3}$$

$$y' - \frac{3x^2}{1+x^3}y = x^2$$

$$e^{\int -\frac{3x^2}{1+x^3}dx} = e^{-\int \frac{d\left(1+x^3\right)}{1+x^3}}$$

$$= e^{-\ln\left(1+x^3\right)}$$

$$= e^{\ln\left(1+x^3\right)^{-1}}$$

$$= \frac{1}{1+x^3}$$

$$\int \frac{1}{1+x^3}\left(x^2\right) dx = \frac{1}{3}\int \frac{d\left(1+x^3\right)}{1+x^3}$$

$$= \frac{1}{3}\ln\left(1+x^3\right)$$

$$y(x) = \left(1+x^3\right)\left(\frac{1}{3}\ln\left(1+x^3\right)+C\right)$$

$$= \frac{1}{3}\left(1+x^3\right)\ln\left(1+x^3\right)+C\left(1+x^3\right)$$

Exercise

Find the general solution of $\frac{dy}{dt} - 2y = 4 - t$

Solution

$$e^{\int -2dt} = e^{-2t}$$

$$\int (4-t)e^{-2t}\, dt = \int \left(4e^{-2t} - te^{-2t}\right) dt$$

$$= -2e^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t}$$

$$= -\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t}$$

		$\int e^{-2t}$
+	t	$-\frac{1}{2}e^{-2t}$
−	1	$\frac{1}{4}e^{-2t}$

$$y(t) = \frac{1}{e^{-2t}}\left(-\frac{7}{4}e^{-2t} + \frac{1}{2}te^{-2t} + C\right)$$

$$y(t) = \frac{1}{2}t - \frac{7}{4} + Ce^{2t}$$

Exercise

Find the general solution of $y' + y = \frac{1}{1+e^t}$

Solution

$$e^{\int dt} = e^t$$

$$\int \frac{1}{1+e^t}e^t\, dt = \int \frac{1}{1+e^t}d\left(1+e^t\right)$$

$$= \ln\left(1+e^t\right)$$

$$y(t) = \frac{1}{e^t}\left(\ln\left(1+e^t\right)+C\right)$$

$$\boxed{y(t) = e^{-t}\ln\left(1+e^t\right)+Ce^{-t}}$$

Exercise

Solve the differential equation $\quad y' = 3y - 4$

Solution

$$y' - 3y = -4$$

$$e^{\int -3dx} = e^{-3x}$$

$$\int -4e^{-3x}\,dx = \frac{4}{3}e^{-3x}$$

$$y(x) = \frac{1}{e^{-3x}}\left(\frac{4}{3}e^{-3x}+C\right)$$

$$\boxed{= \frac{4}{3} + Ce^{3x}}$$

Exercise

Solve the differential equation $\quad y' = -2y - 4$

Solution

$$y' + 2y = -4$$

$$e^{\int 2dx} = e^{2x}$$

$$\int -4e^{2x}dx = 2e^{2x}$$

$$y(x) = \frac{1}{e^{2x}}\left(2e^{2x}+C\right)$$

$$\boxed{= 2 + Ce^{-2x}}$$

Exercise

Solve the differential equation $\quad y' = -y + 2$

Solution

$$y' + y = 2$$

$$e^{\int dx} = e^x$$

$$\int 2e^x dx = 2e^x$$

$$y(x) = \frac{1}{e^x}\left(2e^x + C\right)$$

$$\underline{= 2 + Ce^{-x}}$$

Exercise

Solve the differential equation $\quad y' = 2y + 6$

Solution

$$y' - 2y = 6$$

$$e^{\int -2dx} = e^{-2x}$$

$$\int 6e^{-2x}\, dx = -3e^{-2x}$$

$$y(x) = e^{2x}\left(-3e^{-2x} + C\right)$$

$$\underline{= -3 + Ce^{2x}}$$

Exercise

Solve the differential equation $\quad x(x-1)dy - ydx = 0$

Solution

$$x(x-1)dy = ydx$$

$$\frac{dy}{y} = \frac{dx}{x(x-1)}$$

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$Ax - A + Bx = 1$$

$$\begin{cases} x & A + B = 0 \quad \rightarrow \underline{B = 1} \\ x^0 & -A = 1 \quad \rightarrow \underline{A = -1} \end{cases}$$

$$\int \frac{dy}{y} = \int \left(-\frac{1}{x} + \frac{1}{x-1}\right) dx$$

$$\ln y = -\ln|x| + \ln|x-1| + \ln C$$

$$\ln y = \ln\left|\frac{C(x-1)}{x}\right|$$

$$\boxed{y(x) = C\frac{(x-1)}{x}}$$

Exercise

Solve the differential equation $\quad xy' + 2y = 1 - x^{-1}$

Solution

$$xy' + 2y = 1 - \frac{1}{x}$$

$$y' + \frac{2}{x}y = \frac{1}{x} - \frac{1}{x^2}$$

$$e^{\int \frac{2}{x}dx} = e^{2\ln|x|}$$

$$= e^{\ln x^2}$$

$$= x^2$$

$$\int \left(\frac{1}{x} - \frac{1}{x^2}\right)x^2\, dx = \int (x-1)\, dx$$

$$= \frac{1}{2}x^2 - x$$

$$y(x) = \frac{1}{x^2}\left(\frac{1}{2}x^2 - x + C\right)$$

$$\boxed{= \frac{1}{2} - \frac{1}{x} + \frac{C}{x^2}}$$

Exercise

Solve the differential equation $\quad xy' - y = 2x\ln x$

Solution

$$y' - \frac{1}{x}y = 2\ln x$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln|x|}$$

$$= e^{\ln x^{-1}}$$

$$= \frac{1}{x}$$

$$\int (2\ln x)\frac{1}{x}dx = 2\int \ln x \, d(\ln x)$$

$$= (\ln x)^2$$

$$y(x) = x\left(\ln^2 x + C\right)$$

$$\boxed{= x(\ln x)^2 + Cx}$$

Exercise

Solve the differential equation $\left(1+e^x\right)dy + \left(ye^x + e^{-x}\right)dx = 0$

Solution

$$\left(1+e^x\right)\frac{dy}{dx} + ye^x + e^{-x} = 0$$

$$\left(1+e^x\right)y' + e^x y = -\frac{1}{e^x}$$

$$y' + \frac{e^x}{1+e^x}y = -\frac{1}{e^x\left(1+e^x\right)}$$

$$e^{\int \frac{e^x}{1+e^x}dx} = e^{\int \frac{1}{1+e^x}d\left(1+e^x\right)}$$

$$= e^{\ln\left(1+e^x\right)}$$

$$= 1+e^x$$

$$\int \frac{-1}{e^x\left(1+e^x\right)}\left(1+e^x\right)dx = -\int e^{-x}\,dx$$

$$= e^{-x}$$

$$y(x) = \frac{1}{1+e^x}\left(e^{-x} + C\right)$$

$$\boxed{= \frac{e^{-x}+C}{1+e^x}}$$

Exercise

Solve the differential equation $\left(x+3y^2\right)dy + ydx = 0$

Solution

$$xdy + 3y^2dy + ydx = 0$$

$$xdy + ydx = -3y^2dy$$

$$d(xy) = -3y^2dy$$

$$\int d(xy) = -3\int y^2dy$$

$$\boxed{xy = -y^3 + C}$$

Exercise

Solve the differential equation $\qquad y' = \dfrac{y^2 + ty + t^2}{t^2}$

Solution

$$y' = \frac{y^2}{t^2} + \frac{y}{t} + 1 \qquad\qquad \text{Let } x = \frac{y}{t} \;\Rightarrow\; y = xt \;\rightarrow\; y' = x + tx'$$

$$y' = \frac{y^2}{t^2} + \frac{y}{t} + 1 = x^2 + x + 1$$

$$x + tx' = x^2 + x + 1$$

$$t\frac{dx}{dt} = x^2 + 1$$

$$\int \frac{dx}{x^2+1} = \int \frac{dt}{t}$$

$$\tan^{-1} x = \ln|t| + C$$

$$\tan^{-1}\frac{y}{t} = \ln|t| + C$$

$$\frac{y}{t} = \tan\left(\ln|t| + C\right)$$

$$\boxed{y(t) = t\tan\left(\ln|t| + C\right)}$$

Exercise

Solve the differential equation $\qquad \dfrac{dy}{dx} = \dfrac{4x - x^3}{4 + y^3}$

Solution

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$$

$$\left(4+y^3\right)dy=\left(4x-x^3\right)dx$$

$$\int\left(4+y^3\right)dy=\int\left(4x-x^3\right)dx$$

$$4y+\frac{1}{4}y^4=2x^2-\frac{1}{4}x^4+C_1$$

$$16y+y^4=8x^2-x^4+C$$

$$\boxed{y^4+16y+x^4-8x^2=C}$$

Exercise

Solve the differential equation $\qquad y'=\dfrac{2xy+2x}{x^2-1}$

Solution

$$\frac{dy}{dx}=\frac{2x(y+1)}{x^2-1}$$

$$\frac{dy}{y+1}=\frac{2x}{x^2-1}dx \qquad\qquad d\left(x^2-1\right)=2xdx$$

$$\int\frac{d(y+1)}{y+1}=\int\frac{d\left(x^2-1\right)}{x^2-1}$$

$$\ln|y+1|=\ln\left|x^2-1\right|+C$$

$$y+1=e^{\ln\left|x^2-1\right|+C}$$

$$y=e^Ce^{\ln\left|x^2-1\right|}-1$$

$$\boxed{y(x)=Ae^{\ln\left|x^2-1\right|}-1}$$

Exercise

Find the general solution of the differential equation $\qquad \dfrac{dy}{dx}=\sin 5x$

Solution

$$\int dy=\int\sin 5x\,dx$$

$$\boxed{y(x)=-\frac{1}{5}\cos 5x+C}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = (x+1)^2$

Solution

$$\int dy = \int \left(x^2 + 2x + 1\right) dx$$

$$\boxed{y(x) = \frac{1}{3}x^3 + x^2 + x + C}$$

Exercise

Find the general solution of the differential equation $dx + e^{3x} dy = 0$

Solution

$$\int dy = -\int e^{-3x}\, dx$$

$$\boxed{y(x) = \frac{1}{3}e^{-3x} + C}$$

Exercise

Find the general solution of the differential equation $dy - (y-1)^2\, dx = 0$

Solution

$$\int \frac{dy}{(y-1)^2} = \int dx$$

$$\int \frac{d(y-1)}{(y-1)^2} = \int dx$$

$$-\frac{1}{y-1} = x + C$$

$$\boxed{y(x) = 1 - \frac{1}{x+C}}$$

Exercise

Find the general solution of the differential equation $x\frac{dy}{dx} = 4y$

Solution

$$\int \frac{dy}{y} = 4\int \frac{dx}{x}$$

$$\ln y = 4\ln x + \ln C$$

$$\ln y = \ln Cx^4$$

$$\boxed{y(x) = Cx^4}$$

Exercise

Find the general solution of the differential equation $\frac{dx}{dy} = y^2 - 1$

Solution

$$\int dx = \int \left(y^2 - 1\right) dy$$

$$\boxed{x(y) = \frac{1}{3}y^3 - y + C}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} = e^{2y}$

Solution

$$\int e^{-2y} dy = \int dx$$

$$-\frac{1}{2}e^{-2y} = x + C$$

$$e^{-2y} = -2x + C_1$$

$$-2y = \ln\left(C_1 - 2x\right)$$

$$\boxed{y(x) = -\frac{1}{2}\ln\left(C_1 - 2x\right)}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx} + 2xy^2 = 0$

Solution

$$\frac{dy}{dx} = -2xy^2$$

$$-\int \frac{dy}{y^2} = \int 2x\, dx$$

$$\frac{1}{y} = x^2 + C$$

$$\boxed{y(x)=\frac{1}{x^2+C}}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx}=e^{3x+2y}$

Solution

$$\frac{dy}{dx}=e^{3x}e^{2y}$$

$$\int e^{-2y}\,dy=\int e^{3x}\,dx$$

$$-\frac{1}{2}e^{-2y}=\frac{1}{3}e^{3x}+C$$

$$e^{-2y}=C_1-\frac{2}{3}e^{3x}$$

$$-2y=\ln\left(C_1-\frac{2}{3}e^{3x}\right)$$

$$\boxed{y(x)=-\frac{1}{2}\ln\left(C_1-\frac{2}{3}e^{3x}\right)}$$

Exercise

Find the general solution of the differential equation $e^x y\frac{dy}{dx}=e^{-y}+e^{-2x-y}$

Solution

$$e^x y\frac{dy}{dx}=e^{-y}\left(1+e^{-2x}\right)$$

$$ye^y dy=e^{-x}\left(1+e^{-2x}\right)dx$$

$$\int ye^y dy=\int\left(e^{-x}+e^{-3x}\right)dx$$

		$\int e^y\,dy$
+	y	e^y
−	1	e^y

$$\boxed{(y-1)e^y=-e^{-x}-\frac{1}{3}e^{-3x}+C}$$

Exercise

Find the general solution of the differential equation $y\ln x\frac{dx}{dy}=\left(\frac{y+1}{x}\right)^2$

Solution

$$x^2\ln x\,dx=\frac{1}{y}\left(y^2+2y+1\right)dy$$

$$\int x^2\ln x\,dx=\int\left(y+2+\frac{1}{y}\right)dy$$

$$u=\ln x \qquad dv=x^2dx$$

$$du=\frac{dx}{x} \qquad v=\frac{1}{3}x^3$$

$$\frac{1}{3}x^3\ln x-\frac{1}{3}\int x^2\,dx=\frac{1}{2}y^2+2y+\ln|y|+C$$

$$\boxed{\frac{1}{3}x^3\ln x-\frac{1}{9}x^3=\frac{1}{2}y^2+2y+\ln|y|+C}$$

Exercise

Find the general solution of the differential equation $\frac{dy}{dx}=\left(\frac{2y+3}{4x+5}\right)^2$

Solution

$$\int\frac{dy}{(2y+3)^2}=\int\frac{dx}{(4x+5)^2}$$

$$\frac{1}{2}\int\frac{d(2y+3)}{(2y+3)^2}=\frac{1}{4}\int\frac{d(4x+5)}{(4x+5)^2}$$

$$\frac{1}{2}\frac{-1}{2y+3}=\frac{1}{4}\frac{-1}{4x+5}+C$$

$$\boxed{\frac{2}{2y+3}=\frac{1}{4x+5}+C}$$

Exercise

Find the general solution of the differential equation $\csc y\,dx+\sec^2 x\,dy=0$

Solution

$$\csc y\,dx=-\sec^2 x\,dy$$

$$\frac{dy}{\csc y}=-\frac{dx}{\sec^2 x}$$

$$\sin y\,dy = -\cos^2 x\,dx$$

$$\int \sin y\ dy = -\frac{1}{2}\int (1+\cos 2x)\,dx$$

$$-\cos y = -\frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$

$$\cos y = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

Exercise

Find the general solution of the differential equation $\sin 3x\,dx + 2y\cos^3 3x\,dy = 0$

Solution

$$\sin 3x\,dx = -2y\cos^3 3x\,dy$$

$$\int \frac{\sin 3x}{\cos^3 3x}dx = -\int 2y\,dy$$

$$-\frac{1}{3}\int \cos^{-3} 3x\ d(\cos 3x) = -\int 2y\,dy$$

$$-\frac{1}{6}\cos^{-2} 3x + C = y^2$$

$$y^2 = -\frac{1}{6}\sec^2 3x + C$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = (64xy)^{1/3}$

Solution

$$\int y^{-1/3}dy = \int 4x^{1/3}dx$$

$$\frac{3}{2}y^{2/3} = 3x^{4/3} + C_1$$

$$y^{2/3} = 2x^{4/3} + C$$

Exercise

Find the general solution of the differential equation. $\frac{dy}{dx} = 2x\sec y$

Solution

$$\int \cos y \, dy = \int 2x \, dx$$

$$\boxed{\sin y = x^2 + C}$$

Exercise

Find the general solution of the differential equation. $\dfrac{dy}{dx} = \dfrac{x}{ye^{x+2y}}$

Solution

$$\frac{dy}{dx} = \frac{x}{ye^{2y}e^{x}}$$

$$\int ye^{2y} dy = \int x\, e^{-x} dx$$

		$\int e^{2y}$
+	y	$\frac{1}{2}e^{2y}$
−	1	$\frac{1}{4}e^{2y}$

		$\int e^{-x}$
+	x	$-e^{-x}$
−	1	e^{-x}

$$\frac{1}{2}ye^{2y} - \frac{1}{4}e^{2y} = -xe^{-x} - e^{-x} + C_1$$

$$\boxed{(2y-1)e^{2y} = -4(x+1)e^{-x} + C}$$

Exercise

Find the general solution of $y' - y = 3e^{t}$

Solution

$$y_h = e^{\int -dt} = e^{-t}$$

$$\int 3e^{t}e^{-t} dt = \int 3\, dt = 3t$$

$$y(t) = \frac{1}{e^{-t}}(3t + C)$$

$$\boxed{y(t) = 3te^{t} + Ce^{t}}$$

Exercise

Find the general solution of $y' - y = e^{2t} - 1$

Solution

$$e^{-\int dt} = e^{-t}$$

$$\int \left(e^{2t} - 1\right) e^{-t}\, dt = \int \left(e^{t} - e^{-t}\right) dt$$

$$= e^{t} + e^{-t}$$

$$y(t) = \frac{1}{e^{-t}}\left(e^{t} + e^{-t} + C\right)$$

$$\underline{= e^{2t} + 1 + Ce^{t}}$$

Exercise

Find the general solution of $y' + y = te^{-t} + 1$

Solution

$$e^{\int dt} = e^{t}$$

$$\int \left(te^{-t} + 1\right) e^{t}\, dt = \int \left(t + e^{t}\right) dt$$

$$= t + e^{t}$$

$$y(t) = \frac{1}{e^{t}}\left(t + e^{t} + C\right)$$

$$\underline{= te^{-t} + 1 + Ce^{-t}}$$

Exercise

Find the general solution of $y' + y = 1 + e^{-x}\cos 2x$

Solution

$$e^{\int dx} = e^{x}$$

$$\int \left(1 + e^{-x}\cos 2x\right) e^{x} dx = \int \left(e^{x} + \cos 2x\right) dx$$

$$= e^{x} + \frac{1}{2}\sin 2x$$

$$y(x) = e^{-x}\left(e^x + \frac{1}{2}\sin 2x + C\right)$$

$$= e^x + \frac{1}{2}e^{-x}\sin 2x + Ce^{-x}$$

Exercise

Solve the differential equation: $y' + y\cot x = \cos x$

Solution

$$e^{\int \cot x dx} = e^{\ln|\sin x|}$$

$$= \sin x$$

$$\int \cos x \sin x \, dx = \int \sin x \, d(\sin x)$$

$$= \frac{1}{2}\sin^2 x$$

$$y(x) = \frac{1}{\sin x}\left(\frac{1}{2}\sin^2 x + C\right)$$

$$= \frac{1}{2}\sin x + \frac{C}{\sin x}$$

Exercise

Solve the differential equation: $y' + y\sin t = \sin t$

Solution

$$e^{\int \sin t dt} = e^{-\cos t}$$

$$\int (\sin t)e^{-\cos t} \, dt = \int e^{-\cos t} \, d(-\cos t)$$

$$= e^{-\cos t}$$

$$y(t) = \frac{1}{e^{-\cos t}}\left(e^{-\cos t} + C\right)$$

$$= 1 + Ce^{\cos t}$$

Exercise

Solve the differential equation: $y' + (\cot t)y = 2t\csc t$

Solution

$$e^{\int \cot t\, dt} = e^{\ln|\sin t|}$$
$$= \sin t$$

$$\int 2t \csc t \sin t\, dt = \int 2t\, dt$$
$$= t^2$$

$$y(t) = \frac{1}{\sin t}\left(t^2 + C\right)$$
$$\underline{= \left(t^2 + C\right)\csc t}$$

Exercise

Solve the differential equation: $y' + (1 + \sin t)y = 0$

Solution

$$e^{\int (1+\sin t)dt} = e^{t-\cos t}$$

$$y(t) = \frac{C}{e^{t-\cos t}}$$
$$\underline{= C\, e^{\cos t - t}}$$

Exercise

Find the general solution of $y' + \left(\frac{1}{2}\cos x\right)y = -\frac{3}{2}\cos x$

Solution

$$e^{\int \frac{1}{2}\cos x dx} = e^{\frac{1}{2}\sin x}$$

$$\int \left(-\frac{3}{2}\cos x\right)e^{\frac{1}{2}\sin x}dx = -3\int e^{\frac{1}{2}\sin x}d\left(\frac{1}{2}\sin x\right)$$
$$= -3e^{\frac{1}{2}\sin x}$$

$$y(x) = e^{-\frac{1}{2}\sin x}\left(-3e^{\frac{1}{2}\sin x} + C\right)$$
$$\underline{= -3 + Ce^{-\frac{1}{2}\sin x}}$$

Exercise

Solve the differential equation: $\dfrac{dy}{dx} + y = e^{3x}$

Solution

$$e^{\int dx} = e^{x}$$

$$\int e^{x} e^{3x} dx = \int e^{4x} dx$$

$$= \frac{1}{4} e^{4x}$$

$$y(x) = \frac{1}{e^{x}}\left(\frac{1}{4}e^{4x} + C\right)$$

$$= \frac{1}{4}e^{3x} + Ce^{-x}$$

Exercise

Solve the differential equation: $y' - ty = t$

Solution

$$e^{\int -t\,dt} = e^{-\frac{1}{2}t^2}$$

$$\int te^{-\frac{1}{2}t^2} dt = -\int e^{-\frac{1}{2}t^2} d\left(-\frac{1}{2}t^2\right)$$

$$= -e^{-\frac{1}{2}t^2}$$

$$y(t) = e^{\frac{1}{2}t^2}\left(e^{-\frac{1}{2}t^2} + C\right)$$

$$= 1 + Ce^{\frac{1}{2}t^2}$$

Exercise

Solve the differential equation: $y' = 2y + x^2 + 5$

Solution

$$y' - 2y = x^2 + 5$$

$$e^{\int -2dx} = e^{-2x}$$

$$\int \left(x^2+5\right)e^{-2x}\,dx = \left(-\frac{1}{2}x^2 - \frac{5}{2} - \frac{1}{2}x - \frac{1}{4}\right)e^{-2x}$$
$$= \left(-\frac{1}{2}x^2 - \frac{1}{2}x - \frac{11}{4}\right)e^{-2x}$$
$$= -\frac{1}{4}\left(2x^2 + 2x + 11\right)e^{-2x}$$

		$\int e^{-2x}$
+	x^2+5	$-\frac{1}{2}e^{-2x}$
−	$2x$	$\frac{1}{4}e^{-2x}$
+	2	$-\frac{1}{8}e^{-2x}$

$$y(x) = e^{2x}\left(-\frac{1}{4}\left(2x^2+2x+11\right)e^{-2x} + C\right)$$
$$= -\frac{1}{4}\left(2x^2+2x+11\right) + Ce^{2x}$$

Exercise

Solve the differential equation: $xy' + 2y = 3$

Solution

$$y' + \frac{2}{x}y = \frac{3}{x}$$
$$e^{\int \frac{2}{x}dx} = e^{2\ln x}$$
$$= x^2$$
$$\int x^2 \frac{3}{x}\,dx = \int 3x\,dx$$
$$= \frac{3}{2}x^2$$
$$y(x) = \frac{1}{x^2}\left(\frac{3}{2}x^2 + C\right)$$
$$= \frac{3}{2} + \frac{C}{x^2}$$

Exercise

Solve the differential equation: $y' + 2y = 1$

Solution

$$e^{\int 2dx} = e^{2x}$$
$$\int e^{2x}\,dx = \frac{1}{2}e^{2x}$$
$$y(x) = \frac{1}{e^{2x}}\left(\frac{1}{2}e^{2x} + C\right)$$

$$=\boxed{\frac{1}{2}+Ce^{-2x}}$$

Exercise

Solve the differential equation: $\quad y'+2y=e^{-t}$

Solution

$$e^{\int 2dt}=e^{2t}$$

$$\int e^{-t}e^{2t}\ dt=\int e^{t}\ dt$$

$$=e^{t}$$

$$y(t)=\frac{1}{e^{2t}}\left(e^{t}+C\right)$$

$$=\boxed{e^{-t}+Ce^{-2t}}$$

Exercise

Solve the differential equation: $\quad y'+2y=e^{-2t}$

Solution

$$e^{\int 2dt}=e^{2t}$$

$$\int e^{-2t}e^{2t}\ dt=t$$

$$\boxed{y(t)=(t+C)e^{-2t}}$$

Exercise

Find the general solution of $\quad y'-2y=e^{3t}$

Solution

$$e^{\int -2dt}=e^{-2t}$$

$$\int e^{3t}e^{-2t}dt=e^{t}$$

$$y(t)=e^{2t}\left(e^{t}+C\right)$$

$$= e^{3t} + Ce^{2t}$$

Exercise

Find the general solution of $y' + 2y = e^{-x} + x + 1$

Solution

$$e^{\int 2dx} = e^{2x}$$

$$\int \left(e^{-x} + x + 1\right)e^{2x}dx = \int \left(e^{x} + (x+1)e^{2x}\right)dx$$

$$= e^{x} + \left(\frac{1}{2}x + \frac{1}{2} - \frac{1}{4}\right)e^{2x}$$

$$= e^{x} + \left(\frac{1}{2}x + \frac{1}{4}\right)e^{2x}$$

		$\int e^{2x}\,dx$
+	$x+1$	$\frac{1}{2}e^{2x}$
–	1	$\frac{1}{4}e^{2x}$

$$y(x) = e^{-2x}\left(e^{x} + \left(\frac{1}{2}x + \frac{1}{4}\right)e^{2x} + C\right)$$

$$= e^{-x} + \frac{1}{2}x + \frac{1}{4} + Ce^{-2x}$$

Exercise

Solve the differential equation: $y' + 2xy = x$

Solution

$$e^{\int 2xdx} = e^{x^2}$$

$$\int xe^{x^2}\,dx = \frac{1}{2}\int e^{x^2}\,d\left(x^2\right)$$

$$= \frac{1}{2}e^{x^2}$$

$$y(x) = \frac{1}{e^{x^2}}\left(\frac{1}{2}e^{x^2} + C\right)$$

$$= \frac{1}{2} + Ce^{-x^2}$$

Exercise

Solve the differential equation: $y' - 2ty = t$

Solution

$$e^{\int -2tdt} = e^{-t^2}$$

$$\int te^{-t^2}\, dt = -\frac{1}{2}\int e^{-t^2}\, d\left(-t^2\right)$$

$$= -\frac{1}{2}e^{-t^2}$$

$$y(t) = \frac{1}{e^{-t^2}}\left(-\frac{1}{2}e^{-t^2} + C\right)$$

$$= Ce^{t^2} - \frac{1}{2}$$

Exercise

Find the general solution of $y' + 2ty = 5t$

Solution

$$u(t) = e^{\int 2tdt} = e^{t^2}$$

$$e^{t^2}y' + 2te^{t^2}y = 5te^{t^2}$$

$$\left(e^{t^2}y\right)' = 5te^{t^2}$$

$$e^{t^2}y = \int 5t\, e^{t^2}\, dt \qquad de^{t^2} = 2te^{t^2}\, dt$$

$$= 5\int \frac{1}{2}\, d\left(e^{t^2}\right)$$

$$= \frac{5}{2}e^{t^2} + C$$

$$y(t) = \frac{5}{2} + Ce^{-t^2}$$

Exercise

Solve the differential equation: $y' - 2xy = e^{x^2}$

Solution

$$e^{\int -2xdx} = e^{-x^2}$$

$$\int e^{x^2}e^{-x^2}\, dx = \int dx = x$$

$$y(x) = e^{x^2}(x + C)$$

Exercise

Solve the differential equation: $y' + 2xy = x^3$

Solution

$$e^{\int 2x\,dx} = e^{x^2}$$

$$\begin{aligned}\int x^3 e^{x^2}\,dx &= \frac{1}{2}\int x^2 e^{x^2}\,d\left(x^2\right)\\ &= \frac{1}{2}\int ue^{u}\,d(u)\\ &= \frac{1}{2}\left(x^2-1\right)e^{x^2}\end{aligned}$$

		$\int e^u\,du$
+	u	e^u
−	1	e^u

$$y(x) = e^{-x^2}\left(\frac{1}{2}\left(x^2-1\right)e^{x^2} + C\right)$$

$$= \frac{1}{2}\left(x^2-1\right) + Ce^{-x^2}$$

Exercise

Solve the differential equation: $y' - 2y = t^2 e^{2t}$

Solution

$$e^{\int -2dt} = e^{-2t}$$

$$\begin{aligned}\int e^{-2t} t^2 e^{2t} dt &= \int t^2 dt\\ &= \frac{1}{3}t^3\end{aligned}$$

$$y(t) = \frac{1}{e^{-2t}}\left(\frac{1}{3}t^3 + C\right)$$

$$= e^{2t}\left(\frac{1}{3}t^3 + C\right)$$

Exercise

Find the general solution of $x' - 2\frac{x}{t+1} = (t+1)^2$

Solution

$$e^{\int -\frac{2}{t+1}dt} = e^{-2\ln(t+1)}$$

$$= e^{\ln(t+1)^{-2}}$$
$$= (t+1)^{-2}$$
$$\int (t+1)^2 (t+1)^{-2}\, dt = \int dt$$
$$= t$$
$$x(t) = \frac{1}{(t+1)^{-2}}(t+C)$$
$$= (t+1)^2 (t+C)$$
$$\boxed{= t(t+1)^2 + C(t+1)^2}$$

Exercise

Find the general solution of $\quad y' + \frac{2}{t} y = \frac{\cos t}{t^2}$

Solution

$$e^{\int \frac{2}{t} dt} = e^{2\ln t}$$
$$= e^{\ln t^2}$$
$$= t^2$$
$$\int \frac{\cos t}{t^2} t^2\, dt = \int \cos t\, dt$$
$$= \sin t$$
$$\boxed{y(t) = \frac{1}{t^2}(\sin t + C)}$$

Exercise

Solve the differential equation: $\quad y' - 2(\cos 2t) y = 0$

Solution

$$e^{\int -2\cos 2t\, dt} = e^{-\sin 2t}$$
$$\boxed{y(t) = C\, e^{\sin 2t}}$$

Exercise

Find the general solution of $y' + 2y = \cos 3t$

Solution

$$e^{\int 2dt} = e^{2t}$$

$$\int (\cos 3t) e^{2t} dt = \left(\frac{1}{3}\sin 3t + \frac{1}{18}\cos 3t\right) e^{2t} - \frac{1}{36}\int (\cos 3t) e^{2t} dt$$

		$\int \cos 3t\, dt$
+	e^{2t}	$\frac{1}{3}\sin 3t$
−	$\frac{1}{2}e^{2t}$	$-\frac{1}{9}\cos 3t$
+	$\frac{1}{4}e^{2t}$	

$$\frac{37}{36}\int (\cos 3t) e^{2t} dt = \frac{1}{18}(6\sin 3t + \cos 3t) e^{2t}$$

$$\int (\cos 3t) e^{2t} dt = \frac{2}{37}(6\sin 3t + \cos 3t) e^{2t}$$

$$y(t) = e^{-2t}\left(\frac{2}{37}(6\sin 3t + \cos 3t) e^{2t} + C\right)$$

$$= \frac{2}{37}(6\sin 3t + \cos 3t) + Ce^{-2t}$$

Exercise

Find the general solution of $y' - 3y = 5$

Solution

$$u(t) = e^{-\int 3dt}$$

$$= e^{-3t}$$

$$e^{-3t} y' - 3e^{-3t} y = 5e^{-3t}$$

$$\left(e^{-3t} y\right)' = 5e^{-3t}$$

$$e^{-3t} y = \int 5e^{-3t}\, dt$$

$$e^{-3t} y = -\frac{5}{3}e^{-3t} + C$$

$$y(t) = -\frac{5}{3} + Ce^{3t}$$

Exercise

Solve the differential equation: $y' + 3y = 2xe^{-3x}$

Solution

$$e^{\int 3dx} = e^{3x}$$

$$\int 2xe^{-3x}e^{3x}\ dx = \int 2x\ dx$$

$$= x^2$$

$$\boxed{y(x) = \frac{1}{e^{3x}}\left(x^2 + C\right)}$$

Exercise

Solve the differential equation: $\quad y' + 3x^2 y = x^2$

Solution

$$e^{\int 3x^2 dx} = e^{x^3}$$

$$\int x^2 e^{x^3}\ dx = \frac{1}{3}\int e^{x^3}\ d\left(e^{x^3}\right)$$

$$= \frac{1}{3}e^{x^3}$$

$$y(x) = \frac{1}{e^{x^3}}\left(\frac{1}{3}e^{x^3} + C\right)$$

$$\boxed{= \frac{1}{3} + Ce^{-x^3}}$$

Exercise

Find the general solution of $\ y' + \frac{3}{t}y = \frac{\sin t}{t^3}, \quad (t \neq 0)$

Solution

$$e^{\int \frac{3}{t}dt} = e^{3\ln t} = e^{\ln t^3} = t^3$$

$$\int \frac{\sin t}{t^3}t^3\ dt = \int \sin t\ dt$$

$$= -\cos t$$

$$y(t) = \frac{1}{t^3}(-\cos t + C)$$

$$\boxed{= \frac{C}{t^3} - \frac{\cos t}{t^3}}$$

Exercise

Find the general solution of $y' + \frac{3}{x}y = 1 + \frac{1}{x}$

Solution

$$e^{\int \frac{3}{x}dx} = e^{3\ln x}$$
$$= x^3$$

$$\int \left(1+\frac{1}{x}\right)x^3\, dx = \int \left(x^3 + x^2\right)dx$$
$$= \frac{1}{4}x^4 + \frac{1}{3}x^3$$

$$y(x) = \frac{1}{x^3}\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 + C\right)$$
$$\boxed{= \frac{1}{4}x + \frac{1}{3}x + \frac{C}{x^3}}$$

Exercise

Find the general solution of $y' + \frac{3}{2}y = \frac{1}{2}e^x$

Solution

$$e^{\int \frac{3}{2}dx} = e^{3x/2}$$

$$\int \left(\frac{1}{2}e^x\right)e^{3x/2}dx = \frac{1}{2}\int e^{5x/2}dx$$
$$= \frac{1}{5}e^{5x/2}$$

$$y(x) = e^{-3x/2}\left(\frac{1}{5}e^{5x/2} + C\right)$$
$$\boxed{= \frac{1}{5}e^x + Ce^{-3x/2}}$$

Exercise

Find the general solution of $y' + 5y = t + 1$

Solution

$$e^{\int 5dt} = e^{5t}$$

$$\int (t+1)e^{5t}dt = \left(\frac{1}{5}t + \frac{1}{5} + \frac{1}{25}\right)e^{5t}$$

$$= \frac{1}{5}\left(t + \frac{6}{5}\right)e^{5t}$$

		$\int e^{5t}\, dt$
+	$t+1$	$\frac{1}{5}e^{5t}$
−	1	$\frac{1}{25}e^{5t}$

$$y(t) = \frac{1}{e^{5t}}\left(\frac{1}{5}\left(t+\frac{6}{5}\right)e^{5t} + C\right)$$

$$= \frac{1}{5}\left(t+\frac{6}{5}\right) + Ce^{-5t}$$

Exercise

Solve the differential equation: $xy' - y = x^2 \sin x$

Solution

$$y' - \frac{1}{x}y = x\sin x$$

$$e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int x^2 \sin x\, dx = -x^2\cos x + 2x\sin x + 2\cos x$$

		$\int \sin x\, dx$
+	x^2	$-\cos x$
−	$2x$	$-\sin x$
+	2	$\cos x$

$$y(x) = \frac{1}{x}\left(-x^2\cos x + 2x\sin x + 2\cos x + C\right)$$

$$= -x\cos x + 2\sin x + \frac{2}{x}\cos x + \frac{C}{x}$$

Exercise

Find the exact solution of the initial value problem. $y' = \frac{1-2t}{y}, \quad y(1) = -2$

Solution

$$y\frac{dy}{dt} = 1 - 2t$$

$$\int y dy = \int (1-2t)dt$$

$$\frac{1}{2}y^2 = t - t^2 + C_1$$

$$y^2 = 2t - 2t^2 + C$$

$$(-2)^2 = 2(1) - 2(1)^2 + C \quad \Rightarrow \quad \boxed{C=4}$$

$$y(t) = -\sqrt{2t - 2t^2 + 4}$$

The negative value is taken to satisfy the initial condition.

Exercise

Find the exact solution of the initial value problem. $y' = y^2 - 4, \quad y(0) = 0$

Solution

$$\frac{dy}{dt} = y^2 - 4$$

$$\frac{dy}{y^2 - 4} = dt$$

$$\frac{1}{y^2-4} = \frac{A}{y-2} + \frac{B}{y+2}$$

$$\frac{1}{y^2-4} = \frac{(A+B)y + 2A - 2B}{y-2}$$

$$\rightarrow \begin{cases} A + B = 0 \\ 2A - 2B = 1 \end{cases} \Rightarrow A = \frac{1}{4} \quad B = -\frac{1}{4}$$

$$\left(\frac{1}{4(y-2)} - \frac{1}{4(y+2)}\right) dy = dt$$

$$\int \left(\frac{1}{4(y-2)} - \frac{1}{4(y+2)}\right) dy = \int dt$$

$$\frac{1}{4}\left(\ln|y-2| - \ln|y+2|\right) = t + C$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4t + C$$

$$\frac{y-2}{y+2} = \pm e^{4t+C}$$

$$\frac{y-2}{y+2} = \pm e^{C} e^{4t} = ke^{4t}$$

$$y - 2 = ke^{4t} y + 2ke^{4t}$$

$$y - ke^{4t} y = 2 + 2ke^{4t}$$

$$y\left(1 - ke^{4t}\right) = 2 + 2ke^{4t}$$

$$y = \frac{2 + 2ke^{4t}}{1 - ke^{4t}}$$

$$0 = \frac{2 + 2ke^{4(0)}}{1 - ke^{4(0)}}$$

$$0 = 2 + 2k \quad \Rightarrow \quad \boxed{k = -1}$$

$$y(t) = \frac{2 - 2e^{4t}}{1 + e^{4t}}$$

Exercise

Find the exact solution of the initial value problem. $\dfrac{dy}{dx}=\dfrac{3x^2+4x+2}{2(y-1)}, \quad y(0)=-1$

Solution

$$\frac{dy}{dx}=\frac{3x^2+4x+2}{2y-2}$$

$$(2y-2)\,dy=\left(3x^2+4x+2\right)dx$$

$$\int(2y-2)\,dy=\int\left(3x^2+4x+2\right)dx$$

$$y^2-2y=x^3+2x^2+2x+C$$

$$y(0)=-1$$

$$(-1)^2-2(-1)=(0)^3+2(0)^2+2(0)+C$$

$$\Rightarrow \quad \underline{C=3}$$

$$\underline{y^2-2y=x^3+2x^2+2x+3}$$

Exercise

Find the exact solution of the initial value problem. $y'=\dfrac{x}{1+2y}, \quad y(-1)=0$

Solution

$$\frac{dy}{dx}=\frac{x}{1+2y}$$

$$\int(1+2y)\,dy=\int x\,dx$$

$$y+y^2=\frac{1}{2}x^2+C \qquad y(-1)=0$$

$$0=\frac{1}{2}(-1)^2+C$$

$$\underline{C=-\frac{1}{2}}$$

$$\underline{y+y^2=\frac{1}{2}x^2-\frac{1}{2}}$$

Exercise

Find the exact solution of the initial value problem $\left(e^{2y}-y\right)\cos x\dfrac{dy}{dx}=e^{y}\sin 2x, \quad y(0)=0$

Solution

$$\frac{e^{2y}-y}{e^y}dy = \frac{2\sin x\cos x}{\cos x}dx$$

$$\int\left(e^y - ye^{-y}\right)dy = \int 2\sin x\,dx$$

		$\int e^{-y}\,dy$
+	y	$-e^{-y}$
−	1	e^{-y}

$$e^y + ye^{-y} + e^{-y} = -2\cos x + C$$

$y(0)=0 \quad 1+1=-2+C$

$\rightarrow \ \underline{C=4}$

$$\boxed{e^y + ye^{-y} + e^{-y} = 4 - 2\cos x}$$

Exercise

Find the exact solution of the initial value problem $\dfrac{dy}{dx} = e^{-x^2}, \quad y(3)=5$

Solution

$$\int_3^x \frac{dy}{dt}dt = \int_3^x e^{-t^2}dt$$

$$y(x)-y(3) = \int_3^x e^{-t^2}dt$$

$$\boxed{y(x) = 5 + \int_3^x e^{-t^2}dt}$$

Exercise

Find the exact solution of the initial value problem. $\dfrac{dy}{dx} + 2y = 1, \quad y(0)=\dfrac{5}{2}$

Solution

$$\frac{dy}{dx} = 1-2y$$

$$\frac{dy}{1-2y} = dx$$

$$-\frac{1}{2}\int\frac{d(1-2y)}{1-2y} = \int dx$$

$$-\frac{1}{2}\ln|1-2y| = x + C$$

$$\ln|1-2y| = -2x + C \qquad y(0)=\frac{5}{2}$$

$\ln|1-5| = C$

$\boxed{C = \ln 4}$

$1-2y = e^{-2x+\ln 4}$

$1-2y = e^{-2x}e^{\ln 4}$

$\boxed{y(x) = \frac{1}{2} - 2e^{-2x}}$

Exercise

Find the exact solution of the initial value problem. $\sqrt{1-y^2}\,dx - \sqrt{1-x^2}\,dy = 0, \quad y(0) = \frac{\sqrt{3}}{2}$

Solution

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{dy}{\sqrt{1-y^2}} \qquad \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$\sin^{-1} x + C = \sin^{-1} y \qquad y(0) = \frac{\sqrt{3}}{2}$

$\sin^{-1} 0 + C = \sin^{-1}\frac{\sqrt{3}}{2} \quad \Rightarrow \quad \boxed{C = \frac{\pi}{3}}$

$\sin^{-1} y = \sin^{-1} x + \frac{\pi}{3}$

$y = \sin\left(\sin^{-1} x + \frac{\pi}{3}\right)$

$\quad = \sin\left(\sin^{-1} x\right)\cos\frac{\pi}{3} + \cos\left(\sin^{-1} x\right)\sin\frac{\pi}{3} \qquad \alpha = \sin^{-1} x \to \sin\alpha = x \quad \cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-x^2}$

$\boxed{y(x) = \frac{x}{2} + \frac{\sqrt{3}}{2}\sqrt{1-x^2}}$

Exercise

Find the exact solution of the initial value problem. $\left(1+x^4\right)dy + x\left(1+4y^2\right)dx = 0, \quad y(1) = 0$

Solution

$$\int \frac{1}{1+(2y)^2}\,dy = -\int \frac{x}{1+\left(x^2\right)^2}\,dx$$

$$\int \frac{1}{1+(2y)^2}\,dy = -\frac{1}{2}\int \frac{1}{1+\left(x^2\right)^2}\,d\left(x^2\right) \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$\frac{1}{2}\tan^{-1} 2y = -\frac{1}{2}\tan^{-1} x^2 + C$

$$\tan^{-1}2y+\tan^{-1}x^2=C_1$$

$$y(1)=0$$

$$\tan^{-1}0+\tan^{-1}1=C_1$$

$$\Rightarrow\ \boxed{C_1=\frac{\pi}{4}}$$

$$\boxed{\tan^{-1}2y+\tan^{-1}x^2=\frac{\pi}{4}}$$

$$2y=\tan\left(\frac{\pi}{4}-\tan^{-1}x^2\right)$$

$$=\frac{\tan\frac{\pi}{4}-\tan\left(\tan^{-1}x^2\right)}{1+\tan\left(\frac{\pi}{4}\right)\tan\left(\tan^{-1}x^2\right)}$$

$$\boxed{y(x)=\frac{1}{2}\frac{1-x^2}{1+x^2}}$$

Exercise

Find the exact solution of the initial value problem. $e^{-2t}\frac{dy}{dt}=\frac{1+e^{-2t}}{y},\quad y(0)=0$

Solution

$$ydy=\left(1+e^{-2t}\right)e^{2t}dt$$

$$\int ydy=\int\left(e^{2t}+1\right)dt$$

$$\frac{1}{2}y^2=\frac{1}{2}e^{2t}+t+C_1$$

$$y^2=e^{2t}+2t+C \qquad y(0)=0$$

$$0=1+C\quad\to\quad\boxed{C=-1}$$

$$\boxed{y^2=e^{2t}+2t-1}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt}=\frac{t+2}{y},\quad y(0)=2$

Solution

$$\int y\,dy=\int(t+2)\,dt$$

$$\frac{1}{2}y^2 = \frac{1}{2}t^2 + 2t + C_1$$

$$y^2 = t^2 + 4t + C$$

$$y(0) = 2$$

$$\underline{C = 4}$$

$$\boxed{y(t) = \sqrt{t^2 + 4t + 4}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{1}{t^2}\frac{dy}{dt} = y, \quad y(0) = 1$

Solution

$$\int \frac{1}{y}dy = \int t^2\, dt$$

$$\ln|y| = \frac{1}{3}t^3 + C$$

$$y(0) = 1$$

$$\ln|1| = C \quad \to \quad \underline{C = 0}$$

$$\ln|y| = \frac{1}{3}t^3$$

$$\boxed{y(t) = e^{t^3/3}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt} = -y^2 e^{2t}; \quad y(0) = 1$

Solution

$$-\int \frac{1}{y^2}dy = \int e^{2t} dt$$

$$\frac{1}{y} = \frac{1}{2}e^{2t} + C$$

$$y(0) = 1$$

$$1 = \frac{1}{2} + C \quad \to \quad \underline{C = \frac{1}{2}}$$

$$\frac{1}{y} = \frac{1}{2}\left(e^{2t} + 1\right)$$

$$\boxed{y(t) = \frac{2}{e^{2t} + 1}}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt}-(2t+1)y=0;\quad y(0)=2$

Solution

$$\frac{dy}{dt}=(2t+1)y$$

$$\int\frac{dy}{y}=\int(2t+1)\,dt$$

$$\ln|y|=t^2+t+C$$

$$y(0)=2$$

$$\ln 2=C$$

$$\ln|y|=t^2+t+\ln 2$$

$$y(t)=e^{\ln 2}e^{t^2+t}$$

$$=2e^{t^2+t}$$

Exercise

Find the exact solution of the initial value problem. $\frac{dy}{dt}+4ty^2=0;\quad y(0)=1$

Solution

$$-\int\frac{dy}{y^2}=\int 4t\,dt$$

$$\frac{1}{y}=2t^2+C$$

$$y(0)=1$$

$$1=C$$

$$\frac{1}{y}=2t^2+1$$

$$y(t)=\frac{1}{2t^2+1}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx}=ye^x;\quad y(0)=2e$

Solution

$$\int\frac{dy}{y}=\int e^x dx$$

$$\ln|y| = e^x + \ln C$$

$$y(0) = 2e \quad \to \quad \ln|2e| = 1 + \ln C$$

$$\ln 2 + 1 = 1 + \ln C \quad \Rightarrow \quad \boxed{C = 2}$$

$$\ln|y| = e^x + \ln 2$$

$$y(x) = e^{e^x + \ln 2}$$

$$= e^{e^x} e^{\ln 2}$$

$$\boxed{= 2e^{e^x}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 3x^2\left(y^2+1\right); \quad y(0) = 1$

Solution

$$\int \frac{1}{y^2+1}\, dy = \int 3x^2\, dx$$

$$\arctan y = x^3 + C$$

$$y(0) = 1$$

$$\arctan 1 = C$$

$$\boxed{C = \frac{\pi}{4}}$$

$$\boxed{y(x) = \tan\left(x^3 + \frac{\pi}{4}\right)}$$

Exercise

Find the exact solution of the initial value problem $2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}}; \quad y(5) = 2$

Solution

$$\int 2y\, dy = \int \frac{x}{\sqrt{x^2-16}}\, dx$$

$$y^2 = \frac{1}{2}\int \left(x^2-16\right)^{-1/2} d\left(x^2-16\right)$$

$$y^2 = \left(x^2-16\right)^{1/2} + C$$

$$y(5) = 2$$

$$4 = (9)^{1/2} + C$$

$$C = 4 - 3 = 1$$

$$\boxed{y^2 = 1 + \sqrt{x^2 - 16}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = 4x^3 y - y; \quad y(1) = -3$

Solution

$$\frac{dy}{dx} = \left(4x^3 - 1\right)y$$

$$\int \frac{dy}{y} = \int \left(4x^3 - 1\right)dx$$

$$\ln|y| = x^4 - x + C$$

$$y = Ce^{x^4 - x}$$

$$y(1) = -3 \quad \rightarrow \quad -3 = C$$

$$\boxed{y(x) = -3e^{x^4 - x}}$$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} + 1 = 2y; \quad y(1) = 1$

Solution

$$\int \frac{dy}{2y - 1} = \int dx$$

$$\frac{1}{2}\ln(2y - 1) = x + C$$

$$\ln(2y - 1) = 2x + C$$

$$2y - 1 = e^{2x + C}$$

$$y(x) = Ae^{2x} + \frac{1}{2}$$

$$y(1) = 1$$

$$1 = Ae^2 + 1 \quad \rightarrow \quad A = e^{-2}$$

$$\boxed{y(x) = e^{2x-2} + \frac{1}{2}}$$

Exercise

Find the exact solution of the initial value problem $(\tan x)\frac{dy}{dx} = y;\quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

Solution

$$\int \frac{dy}{y} = \int \frac{dx}{\tan x}$$

$$= \int \frac{\cos x\,dx}{\sin x}$$

$$\ln y = \ln(\sin x) + \ln C$$

$$y(x) = C\sin x$$

$$y\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \quad \rightarrow \quad \frac{\pi}{2} = C$$

$$y(x) = \frac{\pi}{2}\sin x$$

Exercise

Find the exact solution of the initial value problem $x\frac{dy}{dx} - y = 2x^2 y;\quad y(1) = 1$

Solution

$$x\frac{dy}{dx} = 2x^2 y + y$$

$$x\frac{dy}{dx} = \left(2x^2 + 1\right)y$$

$$\int \frac{dy}{y} = \int \left(2x + \frac{1}{x}\right)dx$$

$$\ln y = x^2 + \ln x + \ln C$$

$$y(x) = e^{x^2 + \ln x + \ln C}$$

$$= Cxe^{x^2}$$

$$y(1) = 1$$

$$1 = Ce \;\Rightarrow\; C = e^{-1}$$

$$y(x) = xe^{x^2 - 1}$$

Exercise

Find the exact solution of the initial value problem $\dfrac{dy}{dx} = 2xy^2 + 3x^2y^2; \quad y(1) = -1$

Solution

$$\frac{dy}{dx} = \left(2x + 3x^2\right)y^2$$

$$\int \frac{dy}{y^2} = \int \left(2x + 3x^2\right)dx$$

$$-\frac{1}{y} = x^2 + x^3 + C$$

$$y(x) = \frac{-1}{x^2 + x^3 + C}$$

$$y(1) = -1$$

$$-1 = \frac{-1}{C} \quad \Rightarrow \quad \underline{C = 1}$$

$$\boxed{y(x) = \frac{-1}{x^2 + x^3 + 1}}$$

Exercise

Find the exact solution of the initial value problem $\dfrac{dy}{dx} = 6e^{2x-y}; \quad y(0) = 0$

Solution

$$\int e^y dy = \int 6e^{2x} dx$$

$$e^y = 3e^{2x} + C$$

$$y(x) = \ln\left(3e^{2x} + C\right)$$

$$y(0) = 0 \quad \to \quad 0 = \ln(3 + C)$$

$$3 + C = 1 \to \underline{C = -2}$$

$$\boxed{y(x) = \ln\left(3e^{2x} - 2\right)}$$

Exercise

Find the exact solution of the initial value problem $2\sqrt{x}\dfrac{dy}{dx} = \cos^2 y; \quad y(4) = \frac{\pi}{4}$

Solution

$$\frac{dy}{\cos^2 y} = \frac{1}{2}x^{-1/2}dx$$

$$\int \sec^2 y\, dy = \int \frac{1}{2}x^{-1/2}\, dx$$

$$\tan y = \sqrt{x} + C$$

$$y(x) = \tan^{-1}\left(\sqrt{x} + C\right)$$

$$y(4) = \frac{\pi}{4}$$

$$\frac{\pi}{4} = \arctan(2 + C)$$

$$2 + C = 1 \quad \Rightarrow \quad \underline{C = -1}$$

$$\boxed{y(x) = \tan^{-1}\left(\sqrt{x} - 1\right)}$$

Exercise

Find the exact solution of the initial value problem $y' + 3y = 0;\quad y(0) = -3$

Solution

$$\frac{dy}{dx} = -3y$$

$$\int \frac{dy}{y} = -3\int dx$$

$$\ln|y| = -3x + C$$

$$y(x) = e^{-3x+C}$$

$$= Ae^{-3x}$$

$$y(0) = -3 \quad \to \quad \underline{A = -3}$$

$$\boxed{y(x) = -3e^{-3x}}$$

Exercise

Find the exact solution of the initial value problem $2y' - y = 0;\quad y(-1) = 2$

Solution

$$2\frac{dy}{dx} = y$$

$$\int \frac{dy}{y} = \frac{1}{2}\int dx$$

$$\ln|y| = \frac{1}{2}x + C$$

$$y(x) = e^{x/2+C}$$

$= Ae^{x/2}$

$y(-1) = 2$

$2 = Ae^{-1/2}$

$\underline{A = 2e^{1/2}}$

$y(x) = 2e^{1/2}e^{x/2}$

$\underline{= 2e^{(x+1)/2}}$

Exercise

Find the exact solution of the initial value problem $\sqrt{y}dx + (1+x)dy = 0; \quad y(0) = 1$

Solution

$$\int y^{-1/2}dy = -\int \frac{1}{x+1}dx$$

$2\sqrt{y} = -\ln|x+1| + C$

$y(0) = 1 \quad \rightarrow \underline{2 = C}$

$\underline{2\sqrt{y} = -\ln|x+1| + 2}$

Exercise

Find the exact solution of the initial value problem $\dfrac{dy}{dx} = 6y^2x, \quad y(1) = \dfrac{1}{25}$

Solution

$$\int \frac{dy}{y^2} = \int 6xdx$$

$-\dfrac{1}{y} = 3x^2 + C$

$y(1) = \dfrac{1}{25}$

$-25 = 3 + C$

$\underline{C = -28}$

$-\dfrac{1}{y} = 3x^2 - 28$

$\underline{y(x) = \dfrac{1}{28 - 3x^2}}$

Exercise

Find the exact solution of the initial value problem $\frac{dy}{dx} = \frac{3x^2+4x-4}{2y-4}, \quad y(1)=3$

Solution

$$\int (2y-4)\,dy = \int \left(3x^2+4x-4\right)dx$$

$$y^2-4y = x^3+2x^2-4x+C$$

$$y(1)=3$$

$$9-12 = 1+2-4+C$$

$$\underline{C=-2\,}|$$

$$\underline{y^2-4y = x^3+2x^2-4x-2\,}|$$

Exercise

Find the general solution of $y'-3y=4$; $y(0)=2$

Solution

$$e^{-\int 3dt} = e^{-3t}$$

$$\int 4e^{-3t}dt = -\frac{4}{3}e^{-3t}$$

$$y(t) = e^{3t}\left(-\frac{4}{3}e^{-3t}+C\right)$$

$$\underline{= -\frac{4}{3}+Ce^{3t}\,}|$$

$$y(0) = -\frac{4}{3}+Ce^{3(0)}$$

$$2 = -\frac{4}{3}+C$$

$$C = \frac{4}{3}+2 = \underline{\frac{10}{3}\,}|$$

$$\underline{y(t) = -\frac{4}{3}+\frac{10}{3}e^{3t}\,}|$$

Exercise

Find the general solution of $y' = y+2xe^{2x}$; $y(0)=3$

Solution

$$y'-y = 2xe^{2x}$$

$$e^{\int -1\,dx} = e^{-x}$$

$$\int 2xe^{2x}\left(e^{-x}\right)dx = 2\int xe^{x}dx$$

$$= 2\left(xe^{x} - e^{x}\right)$$

$$y(x) = \frac{1}{e^{-x}}\left(2xe^{x} - 2e^{x} + C\right)$$

$$= e^{x}\left(2xe^{x} - 2e^{x} + C\right)$$

$$\underline{= 2xe^{2x} - 2e^{2x} + Ce^{x}}$$

$$y(x=0) = 2(0)e^{2(0)} - 2e^{2(0)} + Ce^{(0)}$$

$$3 = -2 + C$$

$$\rightarrow \ \underline{C = 5}$$

$$\underline{y(x) = 2xe^{2x} - 2e^{2x} + 5e^{x}}$$

Exercise

Find the solution of the initial value problem $\left(1+t^2\right)y' + 4ty = \left(1+t^2\right)^{-2}, \quad y(1) = 0$

Solution

$$y' + \frac{4t}{1+t^2}y = \frac{\left(1+t^2\right)^{-2}}{1+t^2}$$

$$y' + \frac{4t}{1+t^2}y = \left(1+t^2\right)^{-3}$$

$$e^{\int \frac{4t}{1+t^2}dt} = e^{2\int \frac{d\left(1+t^2\right)}{1+t^2}}$$

$$= e^{2\ln\left(1+t^2\right)}$$

$$= e^{\ln\left(1+t^2\right)^2}$$

$$= \left(1+t^2\right)^2$$

$$\int \left(1+t^2\right)^2\left(1+t^2\right)^{-3}dt = \int \frac{dt}{1+t^2}$$

$$\underline{= \tan^{-1}|t|}$$

$$y(t)=\frac{1}{\left(1+t^2\right)^2}\left(\tan^{-1}|t|+C\right)$$

Given $y(1)=0$, then

$$0=\frac{1}{\left(1+1^2\right)^2}\left(\tan^{-1}|1|+C\right)$$

$$0=\frac{\pi}{4}+C$$

$$\boxed{C=-\frac{\pi}{4}}$$

$$\boxed{y(t)=\frac{1}{\left(1+t^2\right)^2}\left(\tan^{-1}|t|-\frac{\pi}{4}\right)}$$

Exercise

Solve the initial value problem: $\quad y'+y=e^t, \quad y(0)=1$

Solution

$$e^{\int dt}=e^t$$

$$\int e^t e^t\, dt=\int e^{2t}\, dt$$

$$=\frac{1}{2}e^{2t}$$

$$y(t)=\frac{1}{e^t}\left(\frac{1}{2}e^{2t}+C\right)$$

$$\boxed{=\frac{1}{2}e^t+\frac{C}{e^t}}$$

$$y(0)=1$$

$$\frac{1}{2}+C=1$$

$$\boxed{C=\frac{1}{2}}$$

$$\boxed{y(t)=\frac{1}{2}\left(e^t+e^{-t}\right)}$$

Exercise

Solve the initial value problem: $y' + \frac{1}{2}y = t, \quad y(0) = 1$

Solution

$$e^{\int \frac{1}{2}dt} = e^{t/2}$$

$$\int te^{t/2}\, dt = (2t-4)e^{t/2}$$

		$\int e^{t/2}\, dt$
+	t	$2e^{t/2}$
−	1	$4e^{t/2}$

$$y(t) = \frac{1}{e^{t/2}}\left((2t-4)e^{t/2} + C\right)$$

$$= 2t - 4 + Ce^{-t/2}$$

$y(0) = 1$

$-4 + C = 1$

$C = 5$

$$y(t) = 2t - 4 + 5e^{-t/2}$$

Exercise

Solve the initial value problem: $y' = x + 5y, \quad y(0) = 3$

Solution

$$y' - 5y = x$$

$$e^{\int -5dx} = e^{-5x}$$

$$\int xe^{-5x}dx = \left(-\frac{1}{5}x - \frac{1}{25}\right)e^{-5x}$$

		$\int e^{-5x}dx$
+	x	$-\frac{1}{5}e^{-5x}$
−	1	$\frac{1}{25}e^{-5x}$

$$y(x) = e^{5x}\left(\left(-\frac{1}{5}x - \frac{1}{25}\right)e^{-5x} + C\right)$$

$$= -\frac{1}{5}x - \frac{1}{25} + Ce^{5x}$$

$y(0) = 3$

$3 = -\frac{1}{25} + C$

$C = \frac{76}{25}$

$$y(x) = -\frac{1}{5}x - \frac{1}{25} + \frac{76}{25}e^{5x}$$

Exercise

Solve the initial value problem: $y' = 2x - 3y, \quad y(0) = \frac{1}{3}$

Solution

$$y' + 3y = 2x$$

$$e^{\int 3dx} = e^{3x}$$

$$\int 2xe^{3x}dx = \left(\frac{2}{3}x - \frac{2}{9}\right)e^{3x}$$

		$\int e^{3x}\,dx$
+	$2x$	$\frac{1}{3}e^{3x}$
−	2	$\frac{1}{9}e^{3x}$

$$y(x) = e^{-3x}\left(\left(\frac{2}{3}x - \frac{2}{9}\right)e^{3x} + C\right)$$

$$= \frac{2}{3}x - \frac{2}{9} + Ce^{-3x}$$

$$y(0) = \frac{1}{3}$$

$$\frac{1}{3} = -\frac{2}{9} + C$$

$$C = \frac{5}{9}$$

$$y(x) = \frac{2}{3}x - \frac{2}{9} + \frac{5}{9}e^{-3x}$$

Exercise

Solve the initial value problem: $xy' + y = e^x, \quad y(1) = 2$

Solution

$$y' + \frac{1}{x}y = \frac{e^x}{x}$$

$$e^{\int \frac{1}{x}dx} = e^{\ln x}$$

$$= x$$

$$\int x\frac{e^x}{x}dx = \int e^x dx$$

$$= e^x$$

$$y(x) = \frac{1}{x}\left(e^x + C\right)$$

$$y(1) = 2 \quad 2 = e + C \quad \rightarrow \quad C = 2 - e$$

$$y(x) = \frac{1}{x}\left(e^x + 2 - e\right)$$

Exercise

Solve the initial value problem: $\quad y\dfrac{dx}{dy}-x=2y^2, \quad y(1)=5$

Solution

$$\frac{dx}{dy}-\frac{1}{y}x=2y$$

$$e^{\int -\frac{1}{y}dy}=e^{-\ln y}$$

$$=e^{\ln y^{-1}}$$

$$\boxed{=y^{-1}}$$

$$\int 2yy^{-1}\,dx=2\int dy$$

$$\boxed{=2y}$$

$$\boxed{x(y)=y(2y+C)}$$

$$y(1)=5$$

$$1=5(10+C)$$

$$\boxed{C=-\frac{49}{5}}$$

$$\boxed{x(y)=2y^2-\frac{49}{5}y}$$

Exercise

Solve the initial value problem: $\quad xy'+y=4x+1, \quad y(1)=8$

Solution

$$y'+\frac{1}{x}y=\frac{4x+1}{x}$$

$$e^{\int \frac{1}{x}dx}=e^{\ln x}$$

$$\boxed{=x}$$

$$\int x\frac{4x+1}{x}dx=\int (4x+1)\,dx$$

$$\boxed{=2x^2+x}$$

$$\boxed{y(x)=\frac{1}{x}\left(2x^2+x+C\right)}$$

$y(1)=8$

$8=3+C$

$\boxed{C=5}$

$$\boxed{y(x)=2x+1+\frac{5}{x}}$$

Exercise

Solve the initial value problem: $\quad y'+4xy=x^3e^{x^2}, \quad y(0)=-1$

Solution

$$e^{\int 4xdx}\ \boxed{=e^{2x^2}}$$

$$\int x^3e^{x^2}e^{2x^2}dx=\int x^3e^{3x^2}dx$$

$$=\frac{1}{6}\int x^2e^{3x^2}d\left(3x^2\right)$$

$$=\frac{1}{18}\int u\,e^u\,du$$

$$=\frac{1}{18}\left(3x^2-1\right)e^{3x^2}$$

	$u=3x^2$	$\int e^u\,du$
+	u	e^u
−	1	e^u

$$\boxed{y(x)=\frac{1}{e^{2x^2}}\left(\frac{1}{18}\left(3x^2-1\right)e^{3x^2}+C\right)}$$

$y(0)=-1$

$-1=-\frac{1}{18}+C$

$\boxed{C=-\frac{17}{18}}$

$$\boxed{y(x)=\frac{1}{18}\left(3x^2-1\right)e^{x^2}-\frac{17}{18}e^{-2x^2}}$$

Exercise

Solve the initial value problem: $\quad (x+1)y'+y=\ln x, \quad y(1)=10$

Solution

$$y'+\frac{1}{x+1}y=\frac{\ln x}{x+1}$$

$$e^{\int \frac{dx}{x+1}} = e^{\ln(x+1)}$$

$$\underline{= x+1 \,|}$$

$$\int \frac{\ln x}{x+1}(x+1)\,dx = \int \ln x\;dx$$

$$\underline{= x\ln x - x \,|}$$

$$\underline{y(x) = \frac{1}{x+1}(x\ln x - x + C) \,|}$$

$y(1) = 10$

$10 = \frac{1}{2}(-1+C)$

$\underline{C = 21 \,|}$

$$\underline{y(x) = \frac{1}{x+1}(x\ln x - x + 21) \,|}$$

Exercise

Solve the initial value problem: $\quad y' - (\sin x)y = 2\sin x, \quad y\left(\frac{\pi}{2}\right) = 1$

Solution

$$e^{\int -\sin x\;dx} \underline{= e^{\cos x} \,|}$$

$$\int 2\sin x e^{\cos x} dx = -2\int e^{\cos x}\;d(\cos x)$$

$$\underline{= -2e^{\cos x} \,|}$$

$$\underline{y(x) = \frac{1}{e^{\cos x}}\left(-2e^{\cos x} + C\right) \,|}$$

$y\left(\frac{\pi}{2}\right) = 1$

$1 = -2 + C$

$\underline{C = 3 \,|}$

$$\underline{y(x) = -2 + \frac{3}{e^{\cos x}} \,|}$$

Exercise

Solve the initial value problem: $y' + y = 2, \quad y(0) = 0$

Solution

$$e^{\int dx} = e^{x}$$

$$\int 2e^{x} dx = 2e^{x}$$

$$y(x) = \frac{1}{e^{x}}\left(2e^{x} + C\right)$$

$$= 2 + Ce^{-x}$$

$$y(0) = 0$$

$$0 = 2 + C$$

$$\Rightarrow C = -2$$

$$y(x) = 2 - 2e^{-x}$$

Exercise

Solve the initial value problem: $y' - 2y = 3e^{2x}, \quad y(0) = 0$

Solution

$$e^{\int -2dx} = e^{-2x}$$

$$\int 3e^{2x}e^{-2x} dx = 3x$$

$$y(x) = e^{2x}(3x + C)$$

$$y(0) = 0 \quad \rightarrow 0 = C$$

$$y(x) = 3xe^{2x}$$

Exercise

Solve the initial value problem: $xy' + 2y = 3x, \quad y(1) = 5$

Solution

$$y' + \frac{2}{x}y = 3$$

$$e^{\int \frac{2}{x}\,dx} = e^{2\ln x}$$
$$= x^2$$

$$\int 3x^2\ dx = x^3$$

$$y(x) = \frac{1}{x^2}\left(x^3 + C\right)$$
$$= x + \frac{C}{x^2}$$

$y(1) = 5$

$5 = 1 + C$

$\Rightarrow\ C = 4$

$$y(x) = x + \frac{4}{x^2}$$

Exercise

Solve the initial value problem: $\quad xy' + 5y = 7x^2, \quad y(2) = 5$

Solution

$$y' + \frac{5}{x}y = 7x$$

$$e^{\int \frac{5}{x}dx} = e^{5\ln x}$$
$$= x^5$$

$$\int 7x^2x^5\ dx = \frac{7}{8}x^8$$

$$y(x) = \frac{1}{x^5}\left(\frac{7}{8}x^8 + C\right)$$
$$= \frac{7}{8}x^3 + \frac{C}{x^5}$$

$y(2) = 5$

$5 = 7 + \frac{1}{32}C$

$\Rightarrow\ C = -64$

$$y(x) = \frac{7}{8}x^3 - \frac{64}{x^5}$$

Exercise

Solve the initial value problem: $xy' - y = x, \quad y(1) = 7$

Solution

$$y' - \frac{1}{x}y = 1$$

$$e^{\int \frac{-1}{x}\,dx} = e^{-\ln x}$$
$$= \frac{1}{x}$$

$$\int \frac{1}{x}\,dx = \ln x$$

$$y(x) = x(\ln x + C)$$
$$= x\ln x + Cx$$

$$y(1) = 7 \quad \rightarrow 7 = C$$

$$y(x) = x\ln x + 7x$$

Exercise

Solve the initial value problem: $xy' + y = 3xy, \quad y(1) = 0$

Solution

$$xy' + (1 - 3x)y = 0$$

$$y' + \left(\frac{1}{x} - 3\right)y = 0$$

$$e^{\int\left(\frac{1}{x}-3\right)dx} = e^{\ln x - 3x}$$
$$= e^{\ln x}e^{-3x}$$
$$= xe^{-3x}$$

$$y(x) = \frac{1}{xe^{-3x}}C$$
$$= \frac{Ce^{3x}}{x}$$

$$y(1) = 0 \quad \rightarrow 0 = C$$

$$y(x) = 0$$

Exercise

Solve the initial value problem: $xy' + 3y = 2x^5, \quad y(2) = 1$

Solution

$$y' + \frac{3}{x}y = 2x^4$$

$$e^{\int \frac{3}{x}dx} = e^{3\ln x}$$

$$= x^3$$

$$\int 2x^4 x^3 \, dx = \frac{1}{4}x^8$$

$$y(x) = \frac{1}{x^3}\left(\frac{1}{4}x^8 + C\right)$$

$$= \frac{1}{4}x^5 + Cx^{-3}$$

$$y(2) = 1$$

$$1 = 8 + \frac{C}{8}$$

$$\Rightarrow \underline{C = -56}$$

$$\boxed{y(x) = \frac{1}{4}x^5 - 56x^{-3}}$$

Exercise

Solve the initial value problem: $y' + y = e^x, \quad y(0) = 1$

Solution

$$e^{\int dx} = e^x$$

$$\int e^x e^x \, dx = \int e^{2x} \, dx$$

$$= \frac{1}{2}e^{2x}$$

$$y(x) = \frac{1}{e^x}\left(\frac{1}{2}e^{2x} + C\right)$$

$$= \frac{1}{2}e^x + Ce^{-x}$$

$$y(0) = 1$$

$$1 = \frac{1}{2} + C$$

$$\Rightarrow \ \boxed{C = \frac{1}{2}}$$

$$\boxed{y(x) = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}}$$

Exercise

Solve the initial value problem: $xy' - 3y = x^3, \quad y(1) = 10$

Solution

$$y' - \frac{3}{x}y = x^2$$

$$e^{\int -\frac{3}{x}dx} = e^{-3\ln x}$$
$$= x^{-3}$$

$$\int x^{-3}x^3 \ dx = \int dx$$
$$= x$$

$$y(x) = x^3(x + C)$$
$$\boxed{= x^4 + Cx^3}$$

$$y(1) = 10$$

$$10 = 1 + C$$

$$\Rightarrow \boxed{C = 9}$$

$$\boxed{y(x) = x^4 + 9x^3}$$

Exercise

Solve the initial value problem: $y' + 2xy = x, \quad y(0) = -2$

Solution

$$e^{\int 2x \ dx} = e^{x^2}$$

$$\int xe^{x^2} \ dx = \frac{1}{2}e^{x^2}$$

$$y(x) = e^{-x^2}\left(\frac{1}{2}e^{x^2} + C\right)$$

$$= \boxed{\frac{1}{2} + Ce^{-x^2}}$$

$$y(0) = -2$$

$$-2 = \frac{1}{2} + C$$

$$\Rightarrow \boxed{C = -\frac{5}{2}}$$

$$\boxed{y(x) = \frac{1}{2} - \frac{5}{2}e^{-x^2}}$$

Exercise

Solve the initial value problem: $\quad y' = (1-y)\cos x, \quad y(\pi) = 2$

Solution

$$y' + (\cos x)y = \cos x$$

$$e^{\int \cos x \, dx} = e^{\sin x}$$

$$\int \cos x \, e^{\sin x} \, dx = e^{\sin x}$$

$$y(x) = \frac{1}{e^{\sin x}}\left(e^{\sin x} + C\right)$$

$$= \boxed{1 + Ce^{-\sin x}}$$

$$y(\pi) = 2$$

$$2 = 1 + C$$

$$\Rightarrow \boxed{C = 1}$$

$$\boxed{y(x) = 1 + e^{-\sin x}}$$

Exercise

Solve the initial value problem: $\quad (1+x)y' + y = \cos x, \quad y(0) = 1$

Solution

$$y' + \frac{1}{x+1}y = \frac{\cos x}{1+x}$$

$$e^{\int \frac{1}{1+x} \, dx} = e^{\ln(1+x)}$$

$$= 1 + x$$

$$\int \frac{\cos x}{1+x}(1+x)\ dx = \sin x$$

$$\boxed{y(x) = \frac{1}{1+x}(\sin x + C)}$$

$$y(0) = 1 \quad \Rightarrow \boxed{C = 1}$$

$$\boxed{y(x) = \frac{1}{1+x}(\sin x + 1)}$$

Exercise

Solve the initial value problem: $\quad y' = 1 + x + y + xy, \quad y(0) = 0$

Solution

$$y' - (1+x)y = 1 + x$$

$$e^{-\int (1+x)dx} = e^{-x-\frac{1}{2}x^2}$$

$$\int (1+x)e^{-\left(x+x^2/2\right)}\ dx = -e^{-\left(x+x^2/2\right)}$$

$$y(x) = e^{x+\frac{1}{2}x^2}\left(-e^{-\left(x+\frac{1}{2}x^2\right)} + C\right)$$

$$\boxed{= -1 + Ce^{x+\frac{1}{2}x^2}}$$

$$y(0) = 0$$

$$0 = -1 + C$$

$$\Rightarrow \boxed{C = 1}$$

$$\boxed{y(x) = -1 + e^{x+\frac{1}{2}x^2}}$$

Exercise

Solve the initial value problem: $\quad xy' = 3y + x^4\cos x, \quad y(2\pi) = 0$

Solution

$$y' - \frac{3}{x}y = x^3\cos x$$

$$e^{-\int \frac{3}{x}dx} = e^{-3\ln x}$$

$$= x^{-3}$$

$$\int x^{-3}x^3 \cos x\, dx = \int \cos x\, dx$$

$$\underline{= \sin x}$$

$$y(x) = x^3(\sin x + C)$$

$$y(2\pi) = 0 \quad \rightarrow \quad \underline{0 = C}$$

$$\underline{y(x) = x^3 \sin x}$$

Exercise

Solve the initial value problem: $\quad y' = 2xy + 3x^2 e^{x^2}, \quad y(0) = 5$

Solution

$$y' - 2xy = 3x^2 e^{x^2}$$

$$e^{-\int 2x\, dx} = e^{-x^2}$$

$$\int 3x^2 e^{x^2} e^{-x^2}\, dx = \int 3x^2\, dx$$

$$= x^3$$

$$y(x) = e^{x^2}\left(x^3 + C\right)$$

$$y(0) = 5 \quad \rightarrow \quad \underline{5 = C}$$

$$\underline{y(x) = e^{x^2}\left(x^3 + 5\right)}$$

Exercise

Solve the initial value problem: $\quad \left(x^2 + 4\right)y' + 3xy = x, \quad y(0) = 1$

Solution

$$y' + \frac{3x}{x^2 + 4}y = \frac{x}{x^2 + 4}$$

$$e^{\int \frac{3x}{x^2+4} dx} = e^{\frac{3}{2}\int \frac{1}{x^2+4} d\left(x^2+4\right)}$$

$$= e^{\frac{3}{2}\ln\left(x^2+4\right)}$$

$$= \left(x^2 + 4\right)^{3/2}$$

$$\int \frac{x}{x^2+4}\left(x^2+4\right)^{3/2}\,dx = \frac{1}{2}\int \left(x^2+4\right)^{1/2}\,d\left(x^2+4\right)$$

$$= \frac{1}{3}\left(x^2+4\right)^{3/2}$$

$$y(x) = \left(x^2+4\right)^{-3/2}\left(\frac{1}{3}\left(x^2+4\right)^{3/2} + C\right)$$

$$= \frac{1}{3} + C\left(x^2+4\right)^{-3/2}$$

$$y(0)=1$$

$$1 = \frac{1}{3} + \frac{1}{8}C$$

$$\Rightarrow \boxed{C = \frac{16}{3}}$$

$$\boxed{y(x) = \frac{1}{3} + \frac{16}{3}\left(x^2+4\right)^{-3/2}}$$

Exercise

Solve the initial value problem: $\left(x^2+1\right)y' + 3x^3 y = 6xe^{-3x^2/2}$, $y(0)=1$

Solution

$$y' + \frac{3x^3}{x^2+1}y = \frac{6xe^{-3x^2/2}}{x^2+1}$$

$$y' + \left(3x - \frac{3x}{x^2+1}\right)y = \frac{6xe^{-3x^2/2}}{x^2+1}$$

$$e^{\int\left(3x - \frac{3x}{x^2+1}\right)dx} = e^{\frac{3}{2}x^2 - \frac{3}{2}\ln\left(x^2+1\right)}$$

$$= e^{\frac{3}{2}x^2}\, e^{\ln\left(x^2+1\right)^{-3/2}}$$

$$= e^{\frac{3}{2}x^2}\left(x^2+1\right)^{-3/2}$$

$$\int \frac{6xe^{-3x^2/2}}{x^2+1} e^{\frac{3}{2}x^2}\left(x^2+1\right)^{-3/2}\,dx = 3\int\left(x^2+1\right)^{-5/2} d\left(x^2+1\right)$$

$$= -2\left(x^2+1\right)^{-3/2}$$

$$y(x) = e^{-3x^2/2}\left(x^2+1\right)^{3/2}\left(-2\left(x^2+1\right)^{-3/2} + C\right)$$

$$= e^{-3x^2/2}\left(-2 + C\left(x^2+1\right)^{3/2}\right)$$

$y(0)=1$

$1 = -2 + C$

$\Rightarrow \ \boxed{C = 3}$

$$\boxed{y(x) = e^{-3x^2/2}\left(-2 + 3\left(x^2+1\right)^{3/2}\right)}$$

Exercise

Solve the initial value problem: $\quad y' - 2y = e^{3x}; \quad y(0) = 3$

Solution

$$e^{\int -2dx} = e^{-2x}$$

$$\int e^{3x}e^{-2x}dx = \int e^x dx$$

$$= e^x$$

$$y(x) = e^{2x}\left(e^x + C\right)$$

$$\boxed{= e^{3x} + Ce^{2x}}$$

$y(0)=1$

$1 = 1 + C$

$\Rightarrow \boxed{C = 0}$

$$\boxed{y(x) = e^{3x}}$$

Exercise

Solve the initial value problem: $\quad y' - 3y = 6; \quad y(0) = 1$

Solution

$$e^{\int -3dx} = e^{-3x}$$

$$\int 6e^{-3x}dx = -2e^{-3x}$$

$$y(x) = e^{3x}\left(-2e^{-3x} + C\right)$$

$$= -2 + Ce^{3x}$$

$$y(0) = 1$$

$$1 = -2 + C$$

$$\Rightarrow C = 3$$

$$y(x) = -2 + 3e^{3x}$$

Exercise

Solve the initial value problem: $2y' + 3y = e^x;\quad y(0) = 0$

Solution

$$y' + \frac{3}{2}y = \frac{1}{2}e^x$$

$$e^{\int \frac{3}{2}dx} = e^{3x/2}$$

$$\int e^x e^{3x/2}\, dx = \int e^{5x/2}\, dx$$

$$= \frac{2}{5}e^{5x/2}$$

$$y(x) = e^{-3x/2}\left(\frac{2}{5}e^{5x/2} + C\right)$$

$$= \frac{2}{5}e^x + Ce^{-3x/2}$$

$$y(0) = 0$$

$$0 = \frac{2}{5} + C$$

$$\Rightarrow C = -\frac{2}{5}$$

$$y(x) = \frac{2}{5}e^x - \frac{2}{5}e^{-3x/2}$$

Exercise

Solve the initial value problem: $y' + y = 1 + e^{-x}\cos 2x;\quad y\left(\frac{\pi}{2}\right) = 0$

Solution

$$e^{\int dx} = e^x$$

$$\int e^x\left(1 + e^{-x}\cos 2x\right)dx = \int \left(e^x + \cos 2x\right)dx$$

$$= e^{x} + \frac{1}{2}\sin 2x$$

$$y(x) = e^{-x}\left(e^{x} + \frac{1}{2}\sin 2x + C\right)$$

$$\boxed{= 1 + \frac{1}{2}e^{-x}\sin 2x + Ce^{-x}}$$

$$y\left(\frac{\pi}{2}\right) = 0$$

$$0 = 1 + Ce^{-\pi/2}$$

$$\Rightarrow \boxed{C = -e^{\pi/2}}$$

$$\boxed{y(x) = 1 + \frac{1}{2}e^{-x}\sin 2x - e^{-x+\pi/2}}$$

Exercise

Solve the initial value problem: $2y' + (\cos x)y = -3\cos x; \quad y(0) = -4$

Solution

$$y' + \left(\frac{1}{2}\cos x\right)y = -\frac{3}{2}\cos x$$

$$e^{\frac{1}{2}\int \cos x\,dx} = e^{\frac{1}{2}\sin x}$$

$$\int e^{\frac{1}{2}\sin x}(-3\cos x)\,dx = -6\int e^{\frac{1}{2}\sin x}\,d\left(\frac{1}{2}\sin x\right)$$

$$= -6e^{\frac{1}{2}\sin x}$$

$$y(x) = e^{-\frac{1}{2}\sin x}\left(-6e^{\frac{1}{2}\sin x} + C\right)$$

$$\boxed{= -6 + Ce^{-\frac{1}{2}\sin x}}$$

$$y(0) = -4$$

$$-4 = -6 + C$$

$$\Rightarrow \boxed{C = 2}$$

$$\boxed{y(x) = -6 + 2e^{-\frac{1}{2}\sin x}}$$

Exercise

Solve the initial value problem: $y' + 2y = e^{-x} + x + 1; \quad y(-1) = e$

Solution

$$e^{\int 2dx} = e^{2x}$$

$$\int \left(e^{-x} + x + 1\right)e^{2x}dx = \int \left(e^{x} + (x+1)e^{2x}\right)dx$$

$$= e^{x} + \left(\frac{1}{2}x + \frac{1}{4}\right)e^{2x}$$

		$\int e^{2x}\,dx$
+	$x+1$	$\frac{1}{2}e^{2x}$
−	1	$\frac{1}{4}e^{2x}$

$$y(x) = e^{-2x}\left(e^{x} + \left(\frac{1}{2}x + \frac{1}{4}\right)e^{2x} + C\right)$$

$$= e^{-x} + \frac{1}{2}x + \frac{1}{4} + Ce^{-2x}$$

$y(-1) = e$

$$e = e - \frac{1}{2} + \frac{1}{4} + Ce^{2}$$

$$\Rightarrow \; C = \frac{1}{4}e^{-2}$$

$$y(x) = e^{-x} + \frac{1}{2}x + \frac{1}{4} + \frac{1}{4}e^{-2x-2}$$

Exercise

Solve the initial value problem: $y' + \frac{y}{x} = xe^{-x}; \quad y(1) = e - 1$

Solution

$$e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$$

$$\int x^{2}e^{-x}\,dx = \left(-x^{2} - 2x - 2\right)e^{-x}$$

		$\int e^{-x}dx$
+	x^2	$-e^{-x}$
−	$2x$	e^{-x}
+	2	$-e^{-x}$

$$y(x) = \frac{1}{x}\left(-\left(x^{2} + 2x + 2\right)e^{-x} + C\right)$$

$y(1) = e - 1$

$$e - 1 = -5e^{-1} + C$$

$$\Rightarrow \; C = 5e^{-1} + e - 1$$

$$y(x) = \frac{1}{x}\left(-\left(x^{2} + 2x + 2\right)e^{-x} + 5e^{-1} + e - 1\right)$$

Exercise

Solve the initial value problem: $y' + 4y = e^{-x}; \quad y(1) = \frac{4}{3}$

Solution

$$e^{\int 4dx} = e^{4x}$$

$$\int e^{-x}e^{4x}dx = \int e^{3x}dx$$

$$= \frac{1}{3}e^{3x}$$

$$y(x) = e^{-4x}\left(\frac{1}{3}e^{3x} + C\right)$$

$$= \frac{1}{3}e^{-x} + Ce^{-4x}$$

$$y(1) = \frac{4}{3}$$

$$\frac{4}{3} = \frac{1}{3}e^{-1} + Ce^{-4}$$

$$\Rightarrow \underline{C = \frac{1}{3}\left(4e^4 - e^3\right)}$$

$$\underline{y(x) = \frac{1}{3}e^{-x} + \frac{1}{3}\left(4e^4 - e^3\right)e^{-4x}}$$

Exercise

Solve the initial value problem: $x^2y' + 3xy = x^4 \ln x + 1; \quad y(1) = 0$

Solution

$$y' + \frac{3}{x}y = x^2 \ln x + \frac{1}{x^2}$$

$$e^{\int \frac{3}{x}dx} = e^{3\ln x}$$

$$= x^3$$

$$\int \left(x^2 \ln x + \frac{1}{x^2}\right)x^3\, dx = \int \left(x^5 \ln x + x\right)dx$$

$$u = \ln x \qquad dv = x^5$$

$$du = \frac{1}{x} \qquad v = \frac{1}{6}x^6$$

$$= \frac{1}{6}x^6 \ln x - \frac{1}{6}\int x^5 dx + \frac{1}{2}x^2$$

$$=\frac{1}{6}x^6\ln x-\frac{1}{36}x^6+\frac{1}{2}x^2$$

$$y(x)=\frac{1}{x^3}\left(\frac{1}{6}x^6\ln x-\frac{1}{36}x^6+\frac{1}{2}x^2+C\right)$$

$$=\frac{1}{6}x^3\ln x-\frac{1}{36}x^3+\frac{1}{2x}+\frac{C}{x^3}$$

$y(1)=0$

$$0=-\frac{1}{36}+\frac{1}{2}+C$$

$$\Rightarrow\ C=-\frac{17}{36}$$

$$y(x)=\frac{1}{6}x^3\ln x-\frac{1}{36}x^3+\frac{1}{2x}-\frac{17}{36x^3}$$

Exercise

Find the solution of the initial value problem $\quad y'+\frac{3}{x}y=3x-2 \quad y(1)=1$

Solution

$$e^{\int\frac{3}{x}dx}=e^{3\ln x}$$

$$=x^3$$

$$\int(3x-2)x^3\,dx=\int\left(3x^4-2x^3\right)dx$$

$$=\frac{3}{5}x^5-\frac{1}{2}x^4$$

$$y(x)=\frac{1}{x^3}\left(\frac{3}{5}x^5-\frac{1}{2}x^4+C\right)$$

$$=\frac{3}{5}x^2-\frac{1}{2}x+\frac{C}{x^3}$$

$y(1)=1$

$$1=\frac{3}{5}-\frac{1}{2}+C$$

$$\Rightarrow\ C=\frac{9}{10}$$

$$y(x)=\frac{3}{5}x^2-\frac{1}{2}x+\frac{9}{10}x^{-3}$$

Exercise

Find the solution of the initial value problem $y' - (\sin x)y = 2\sin x, \quad y\left(\frac{\pi}{2}\right) = 1$

Solution

$$e^{\int -\sin x dx} = e^{\cos x}$$

$$\int (2\sin x)e^{\cos x}\, dx = -2\int e^{\cos x}\, d(\cos x)$$

$$= -2e^{\cos x}$$

$$y(x) = \frac{1}{e^{\cos x}}\left(-2e^{\cos x} + C\right)$$

$$= -2 + Ce^{-\cos x}$$

$$y\left(\frac{\pi}{2}\right) = 1$$

$$1 = -2 + C$$

$$\Rightarrow \quad C = 3$$

$$y(x) = Ce^{-\cos x} - 2$$

Exercise

Find the solution of the initial value problem $y' + (\tan x)y = \cos^2 x, \quad y(0) = -1$

Solution

$$e^{\int \tan x\, dx} = e^{\ln(\sec x)}$$

$$= \sec x$$

$$\int \cos^2 x(\sec x)dx = \int \cos x\, dx$$

$$= \sin x$$

$$y(x) = \frac{1}{\sec x}(\sin x + C)$$

$$= \sin x\cos x + C\cos x$$

$$y(0) = -1 \quad \rightarrow \quad -1 = C$$

$$y(x) = \sin x\cos x - \cos x$$

Exercise

Find the solution of the initial value problem $\quad t\,y' + 2y = t^2 - t + 1 \quad y(1) = \frac{1}{2}$

Solution

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

$$e^{\int \frac{2}{t}\,dt} = e^{2\ln|t|}$$

$$= t^2$$

$$\int \left(t - 1 + \frac{1}{t}\right)t^2\,dt = \int \left(t^3 - t^2 + t\right)dt$$

$$= \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2$$

$$y(t) = \frac{1}{t^2}\left(\frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + C\right)$$

$$\boxed{= \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{C}{t^2}}$$

$$y(1) = \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C$$

$$\Rightarrow \boxed{C = \frac{1}{12}}$$

$$\boxed{y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{2t^2}}$$

Exercise

Find the solution of the initial value problem $\quad y' + (\cos t)y = \cos t\ ; \quad y(\pi) = 2$

Solution

$$e^{\int \cos t\,dt} = e^{\sin t}$$

$$\int (\cos t)e^{\sin t}dt = \int e^{\sin t}d(\sin t)$$

$$= e^{\sin t}$$

$$y(t) = \frac{1}{e^{\sin t}}\left(e^{\sin t} + C\right)$$

$$\boxed{= 1 + Ce^{-\sin t}}$$

$$y(\pi)=2$$

$$1+C=2$$

$$\Rightarrow \underline{C=1}$$

$$\underline{y(t)=1+e^{-\sin t}}$$

Exercise

Find the solution of the initial value problem $\quad y'+2ty=2t\ ;\quad y(0)=1$

Solution

$$e^{\int 2t\,dt}=e^{t^2}$$

$$\int (2t)e^{t^2}\,dt=\int e^{t^2}\,d\left(t^2\right)$$

$$=e^{t^2}$$

$$y(t)=\frac{1}{e^{t^2}}\left(e^{t^2}+C\right)$$

$$\underline{=1+Ce^{-t^2}}$$

$$y(0)=1$$

$$1+C=1$$

$$\Rightarrow \underline{C=0}$$

$$\underline{y(t)=1}$$

Exercise

Find the solution of the initial value problem $\quad y'+y=\frac{e^{-t}}{t^2}\ ;\quad y(1)=0$

Solution

$$e^{\int dt}=e^{t}$$

$$\int \left(e^{t}\right)\frac{e^{-t}}{t^2}\,dt=\int t^{-2}\,dt$$

$$=-\frac{1}{t}$$

$$\underline{y(t)=\frac{1}{e^{t}}\left(-\frac{1}{t}+C\right)}$$

$y(1) = 0$

$\frac{1}{e}(-1+C) = 0$

$\Rightarrow \underline{C = 1}$

$$\boxed{y(t) = \frac{1}{e^t}\left(-\frac{1}{t}+1\right)}$$

Exercise

Find the solution of the initial value problem $\quad ty' + 2y = \sin t \;; \quad y(\pi) = \frac{1}{\pi}$

Solution

$y' + \frac{2}{t}y = \frac{\sin t}{t}$

$$e^{\int \frac{2}{t}dt} = e^{2\ln t}$$
$$= e^{\ln t^2}$$
$$= t^2$$

$$\int (t^2)\frac{\sin t}{t}dt = \int (t\sin t)dt$$
$$= -t\cos t + \sin t$$

		$\int \sin t \; dt$
+	t	$-\cos t$
–	1	$-\sin t$

$$\boxed{y(t) = \frac{1}{t^2}(\sin t - t\cos t + C)}$$

$y(\pi) = \frac{1}{\pi}$

$\frac{1}{\pi^2}(\pi + C) = \frac{1}{\pi}$

$\Rightarrow \underline{C = 0}$

$$\boxed{y(t) = \frac{1}{t^2}(\sin t - t\cos t)}$$

Exercise

Solve the initial value problem: $\quad t\frac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(2) = 1$

Solution

$y' + \frac{2}{t}y = t^2$

$$e^{\int \frac{2}{t}dt} = e^{2\ln t}$$
$$= e^{\ln t^2}$$
$$= t^2$$

$$\int t^2 t^2 \, dt = \int t^4 \, dt$$
$$= \frac{1}{5}t^5$$

$$y(t) = \frac{1}{t^2}\left(\frac{1}{5}t^5 + C\right) = \frac{1}{5}t^3 + \frac{C}{t^2}$$

$$y(2) = \frac{1}{5}2^3 + \frac{C}{2^2}$$

$$1 = \frac{8}{5} + \frac{C}{4}$$

$$\frac{C}{4} = 1 - \frac{8}{5} = -\frac{3}{5}$$

$$\boxed{C = -\frac{12}{5}}$$

$$\boxed{y(t) = \frac{1}{5}t^3 - \frac{12}{5t^2}}$$

Exercise

Solve the initial value problem: $\theta \frac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0, \quad y\left(\frac{\pi}{2}\right) = 1$

Solution

$$y' + \frac{1}{\theta}y = \frac{\sin\theta}{\theta}$$

$$e^{\int \frac{1}{\theta}d\theta} = e^{\ln|\theta|}$$
$$= \theta \quad (> 0)$$

$$\int \frac{\sin\theta}{\theta}\theta \, d\theta = \int \sin\theta \, d\theta$$
$$= -\cos\theta$$

$$y(\theta) = \frac{1}{\theta}(-\cos\theta + C)$$

$$y\left(\frac{\pi}{2}\right) = \frac{2}{\pi}\left(-\cos\frac{\pi}{2} + C\right)$$

$$1 = \frac{2}{\pi}(0 + C)$$

$$1 = \frac{2}{\pi} C$$

$$\underline{C = \frac{\pi}{2}}$$

$$\boxed{y(\theta) = -\frac{\cos\theta}{\theta} + \frac{\pi}{2\theta}}$$

Exercise

Solve the initial value problem: $\frac{dy}{dx} + xy = x, \quad y(0) = -6$

Solution

$$y' + xy = x$$

$$e^{\int x\,dx} = e^{x^2/2}$$

$$\int xe^{x^2/2}\,dx = \int e^{x^2/2}\,d\left(\frac{x^2}{2}\right)$$

$$= e^{x^2/2}$$

$$y(x) = \frac{1}{e^{x^2/2}}\left(e^{x^2/2} + C\right)$$

$$y(0) = \frac{1}{e^{0^2/2}}\left(e^{0^2/2} + C\right)$$

$$-6 = 1(1+C)$$

$$-6 = 1 + C$$

$$\rightarrow \ \underline{C = -7}$$

$$y(x) = \frac{1}{e^{x^2/2}}\left(e^{x^2/2} - 7\right)$$

$$\boxed{= 1 - \frac{7}{e^{x^2/2}}}$$

Exercise

Solve the initial value problem $y' = \frac{y}{x}, \quad y(1) = -2$

Solution

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln|y| = \ln|x| + C$$

$$y = \pm e^{\ln|x|+C}$$

$$= \pm e^{C} e^{\ln|x|}$$

$$= Dx$$

$$y = Dx$$

$$D = \frac{y}{x} = \frac{-2}{1} = -2$$

$$\underline{y = -2x}$$

Exercise

Solve the initial value problem $\quad y' = \frac{\sin x}{y}, \quad y\left(\frac{\pi}{2}\right) = 1$

Solution

$$\frac{dy}{dx} = \frac{\sin x}{y}$$

$$y\,dy = \sin x\,dx$$

$$\int y\,dy = \int \sin x\,dx$$

$$\frac{1}{2}y^2 = -\cos x + C_1$$

$$y^2 = -2\cos x + C \quad \left(C = 2C_1\right)$$

$$y(x) = \pm\sqrt{-2\cos x + C}$$

$$y\left(\frac{\pi}{2}\right) = \sqrt{-2\cos\frac{\pi}{2} + C}$$

$$1 = \sqrt{C}$$

$$\underline{C = 1}$$

$$\underline{y(x) = \sqrt{1 - 2\cos x}}$$

The interval of existence will be the interval containing $\frac{\pi}{2}$ and $1 - 2\cos x > 0$

$$\cos x < \frac{1}{2} \quad \Rightarrow \quad \underline{\frac{\pi}{3} < x < \frac{5\pi}{3}}$$

Exercise

Find the general solution of $y' = y + 2xe^{2x}$; $y(0) = 3$

Solution

$$y' - y = 2xe^{2x}$$

$$e^{\int -1dx} = e^{-x}$$

$$\int 2xe^{2x}\left(e^{-x}\right)dx = 2\int xe^{x}dx$$

$$= 2\left(xe^{x} - e^{x}\right)$$

$$y(x) = \frac{1}{e^{-x}}\left(2xe^{x} - 2e^{x} + C\right)$$

$$= e^{x}\left(2xe^{x} - 2e^{x} + C\right)$$

$$= 2xe^{2x} - 2e^{2x} + Ce^{x}$$

$$y(x=0) = 2(0)e^{2(0)} - 2e^{2(0)} + Ce^{(0)}$$

$$3 = -2 + C$$

$$\rightarrow \underline{C = 5}$$

$$y(x) = 2xe^{2x} - 2e^{2x} + 5e^{x}$$

Exercise

Find the general solution of $\left(x^2 + 1\right)y' + 3xy = 6x$; $y(0) = -1$

Solution

$$y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

$$e^{\int \frac{3x}{x^2+1}dx} = e^{\frac{3}{2}\ln\left(x^2+1\right)}$$

$$= e^{\ln\left(x^2+1\right)^{\frac{3}{2}}}$$

$$= \left(x^2+1\right)^{\frac{3}{2}}$$

$$\int \left(x^2+1\right)^{\frac{3}{2}}\frac{6x}{x^2+1}dx = 3\int \left(x^2+1\right)^{\frac{1}{2}}d\left(x^2+1\right)$$

$$= 2\left(x^2+1\right)^{\frac{3}{2}}$$

$$y(x) = 2 + C\left(x^2+1\right)^{-\frac{3}{2}}$$

$$y(0) = 2 + C\left((0)^2+1\right)^{-\frac{3}{2}}$$

$$-1 = 2 + C(1)^{-\frac{3}{2}}$$

$$\rightarrow \underline{C = -3}$$

$$\underline{y(x) = 2 - 3\left(x^2+1\right)^{-\frac{3}{2}}}$$

Exercise

Solve the initial value problem $\quad y' = \left(4t^3+1\right)y, \quad y(0) = 4$

Solution

$$\frac{dy}{dt} = \left(4t^3+1\right)y$$

$$\int \frac{dy}{y} = \int \left(4t^3+1\right)dt$$

$$\ln y = t^4 + t + C$$

$$y(t) = e^{t^4+t+C}$$

$$\underline{= Ae^{t^4+t}}$$

$$y(0) = 4 \quad \underline{\rightarrow\ 4 = A}$$

$$\underline{y(t) = 4e^{t^4+t}}$$

Exercise

Solve the initial value problem $\quad y' = \dfrac{e^t}{2y}, \quad y(\ln 2) = 1$

Solution

$$\int 2y\,dy = \int e^t\,dt$$

$$y^2 = e^t + C$$

$$y(\ln 2) = 1$$
$$1 = 2 + C$$
$$\Rightarrow \underline{C = -1}$$

$$\underline{y^2 = e^t - 1}$$

Exercise

Solve the initial value problem $(\sec x)\, y' = y^3, \quad y(0) = 3$

Solution

$$\int y^{-3}\, dy = \int \frac{dx}{\sec x}$$
$$= \int \cos x\, dx$$

$$-\frac{1}{2}\frac{1}{y^2} = \sin x + C_1$$

$$y^2 = \frac{1}{-2\sin x + C}$$

$$y = \pm\sqrt{\frac{1}{-2\sin x + C}}$$ *Since the initial value is positive*

$$y = \frac{1}{\sqrt{-2\sin x + C}}$$

$$3 = \sqrt{\frac{1}{C}} \Rightarrow \underline{C = \frac{1}{9}}$$

$$y(x) = \frac{1}{\sqrt{-2\sin x + \frac{1}{9}}}$$
$$\underline{= \frac{3}{\sqrt{-2\sin x + 1}}}$$

Exercise

Solve the initial value problem $\dfrac{dy}{dx} = e^{x-y}, \quad y(0) = \ln 3$

Solution

$$dy = \left(e^x e^{-y}\right) dx$$

$$\int e^y\, dy = \int e^x\, dx$$

$$e^{y}=e^{x}+C$$

$$y=\ln\left(e^{x}+C\right)$$

$$y(0)=\ln 3$$

$$\ln 3=\ln(1+C)$$

$$1+C=3$$

$$\Rightarrow \underline{C=2}$$

$$\boxed{y(x)=\ln\left(e^{x}+2\right)}$$

Exercise

Solve the initial value problem $\quad y'=2e^{3y-t}, \quad y(0)=0$

Solution

$$\frac{dy}{dt}=2e^{3y}e^{-t}$$

$$\int e^{-3y}dy=\int 2e^{-t}\,dt$$

$$-\frac{1}{3}e^{-3y}=-2e^{-t}+C_1$$

$$e^{-3y}=6e^{-t}+C$$

$$y(0)=0$$

$$1=6+C$$

$$\Rightarrow \underline{C=-5}$$

$$e^{-3y}=6e^{-t}-5$$

$$-3y=\ln\left(6e^{-t}-5\right)$$

$$\boxed{y(t)=-\frac{1}{3}\ln\left(6e^{-t}-5\right)}$$

Exercise

Solve the initial value problem $\quad y'=3y-6, \quad y(0)=9$

Solution

$$y'-3y=-6$$

$$e^{\int -3dx}=e^{-3x}$$

$$\int -6e^{-3x}\,dx = 2e^{-3x}$$

$$y = \frac{1}{e^{-3x}}\left(2e^{-3x} + C\right)$$

$$= 2 + Ce^{3x}$$

$$y(0) = 9$$

$$9 = 2 + C$$

$$\to C = 7$$

$$y(x) = 7e^{3x} + 2$$

Exercise

Solve the initial value problem $\quad y' = -y + 2, \quad y(0) = -2$

Solution

$$y' + y = 2$$

$$e^{\int dx} = e^{x}$$

$$\int 2e^{x}\,dx = 2e^{x}$$

$$y = \frac{1}{e^{x}}\left(2e^{x} + C\right)$$

$$= 2 + Ce^{-x}$$

$$y(0) = -2$$

$$-2 = 2 + C$$

$$\to C = -4$$

$$y(x) = 2 - 4e^{-x}$$

Exercise

Solve the initial value problem $\quad y' = -2y - 4, \quad y(0) = 0$

Solution

$$y' + 2y = -4$$

$$e^{\int 2dx} = e^{2x}$$

$$\int -4e^{2x}dx = -2e^{2x}$$

$$y = \frac{1}{e^{2x}}\left(-2e^{2x} + C\right)$$

$$= -2 + Ce^{-2x}$$

$$y(0) = 0$$

$$0 = -2 + C$$

$$\rightarrow C = 2$$

$$y(x) = 2e^{-2x} - 2$$

Exercise

Solve the initial value problem $\frac{dy}{dx} + 3x^2 y = x^2, \quad y(0) = -1$

Solution

$$y' + 3x^2 y = x^2$$

$$e^{\int 3x^2 dx} = e^{x^3}$$

$$\int e^{x^3}\left(x^2\right)dx = \frac{1}{3}\int e^{x^3} d\left(x^3\right)$$

$$= \frac{1}{3}e^{x^3}$$

$$y(x) = \frac{1}{e^{x^3}}\left(\frac{1}{3}e^{x^3} + C\right)$$

$$= \frac{1}{3} + \frac{C}{e^{x^3}}$$

$$y(0) = \frac{1}{3} + \frac{C}{e^0} = -1$$

$$C = -1 - \frac{1}{3}$$

$$= -\frac{4}{3}$$

$$y(x) = \frac{1}{3} - \frac{4}{3e^{x^3}}$$

Exercise

Solve the initial value problem $xdy + (y - \cos x)\,dx = 0, \quad y\left(\frac{\pi}{2}\right) = 0$

Solution

$$xdy + (y - \cos x)\,dx = 0$$

$$x\frac{dy}{dx} + y - \cos x = 0$$

$$xy' + y = \cos x$$

$$y' + \frac{1}{x}y = \frac{\cos x}{x}$$

$$e^{\int \frac{1}{x}dx} = e^{\ln x}$$

$$= x$$

$$\int \frac{\cos x}{x}(x)\,dx = \int \cos x\ dx$$

$$= \sin x$$

$$y(x) = \frac{1}{x}(\sin x + C)$$

$$y\left(\frac{\pi}{2}\right) = \frac{2}{\pi}\left(\sin\frac{\pi}{2} + C\right) = 0$$

$$1 + C = 0$$

$$\underline{C = -1}|$$

$$\underline{y(x) = \frac{\sin x - 1}{x}}|$$

Exercise

Solve the initial value problem $\frac{dy}{dt} = \frac{t+1}{2ty}, \quad y(1) = 4$

Solution

$$\frac{dy}{dt} = \frac{t+1}{2ty}$$

$$\int 2y\,dy = \int \frac{t+1}{t}\,dt$$

$$\int 2y\,dy = \int \left(1 + \frac{1}{t}\right)dt$$

$$y^2 = t + \ln t + C$$

$$y(1) = 4$$

$$16 = 1 + \ln 1 + C$$
$$\underline{C = 15}$$

$$y^2 = t + \ln t + 15$$

$$\underline{y(t) = \sqrt{t + \ln t + 15}}$$ Since $y(t \geq 1)$

Exercise

Solve the initial value problem $\dfrac{dy}{dt} = \sqrt{y}\ \sin t, \quad y(0) = 4$

Solution

$$\frac{dy}{dt} = \sqrt{y}\ \sin t$$

$$\int y^{-1/2}\, dy = \int \sin t\ dt$$

$$2y^{1/2} = -\cos t + C$$

$$y(0) = 4$$

$$2\sqrt{4} = -\cos 0 + C$$

$$4 = -1 + C$$

$$\underline{C = 5}$$

$$2y^{1/2} = -\cos t + 5$$

$$y^{1/2} = -\frac{1}{2}\cos t + \frac{5}{2}$$

$$\underline{y(t) = \frac{1}{4}(5 - \cos t)^2}$$

Exercise

Solve the initial value problem $y'(t) + 3y = 0, \quad y(0) = 6$

Solution

$$\frac{dy}{dt} = -3y$$

$$\int \frac{dy}{y} = -3\int dt$$

$$\ln y = -3t + C_1$$

$$y(t) = e^{-3t + C_1}$$

$$= e^{-3t}e^{C_1}$$

$$= Ce^{-3t} \qquad\qquad C = e^{C_1}$$

$$y(0) = 6$$

$$\underline{C = 6}$$

$$\boxed{y(t) = 6e^{-3t}}$$

Exercise

Solve the initial value problem $\quad y'(t) = 2y + 4, \quad y(0) = 8$

Solution

$$y' - 2y = 4$$

$$e^{\int -2dt} = e^{-2t}$$

$$\int 4e^{-2t}dt = -2e^{-2t}$$

$$y(t) = \frac{1}{e^{-2t}}\left(-2e^{-2t} + C\right)$$

$$= -2 + Ce^{2t}$$

$$y(0) = 8$$

$$8 = -2 + C$$

$$\underline{C = 10}$$

$$\boxed{y(t) = 10e^{2t} - 2}$$