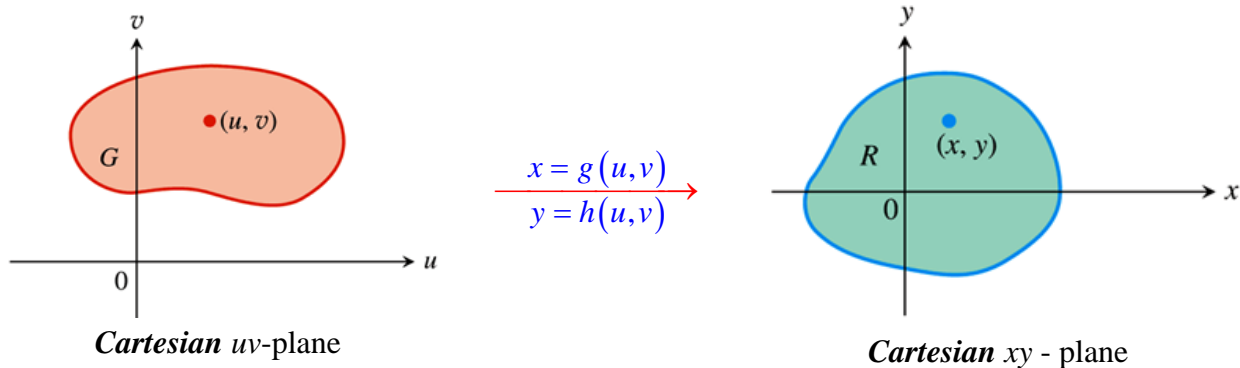


Section 3.7 – Change of variables in Multiple Integrals

Substitution in Double Integrals

Suppose that a region G in the uv -plane is transformed one-to-one into the region R in the xy -plane by equations of the form

$$x = g(u, v), \quad y = h(u, v)$$



R is the image of G under the transformation, and G the *preimage* of R .

$$\iint_R f(x, y) dx dy = \iint_G f(g(u, v), h(u, v)) |J(u, v)| du dv$$

Definition

The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

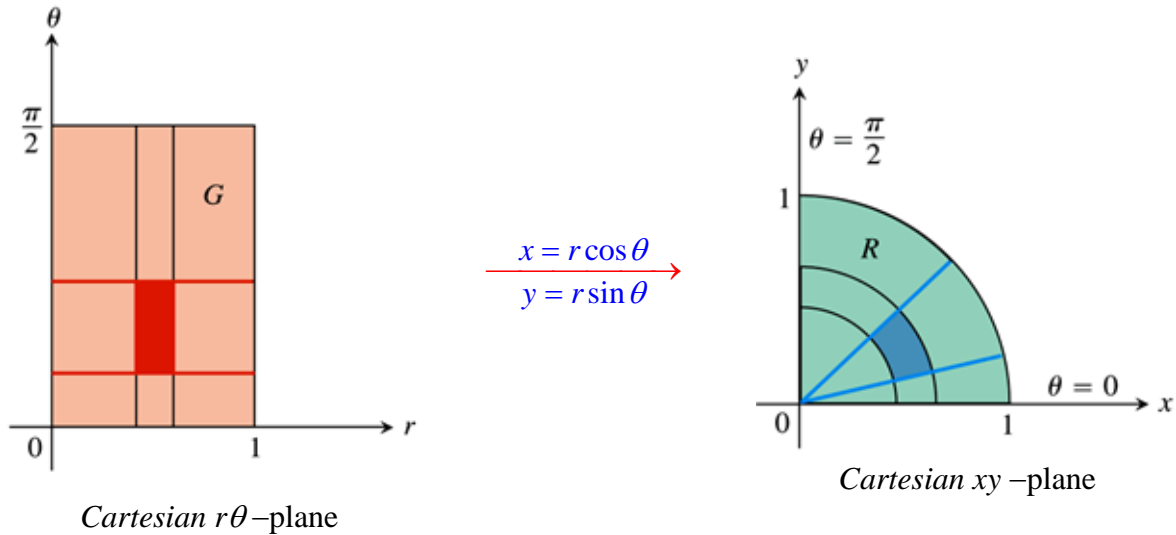
$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

Example

Find the Jacobian for the polar coordinate transformation $x = r \cos \theta$, $y = r \sin \theta$, write the Cartesian

integral $\iint_R f(x, y) dx dy$ as a polar integral.

Solution



$x = r \cos \theta$, $y = r \sin \theta$ transform the rectangle G : $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$, into the quarter circle R bounded by $x^2 + y^2 = 1$ in QI .

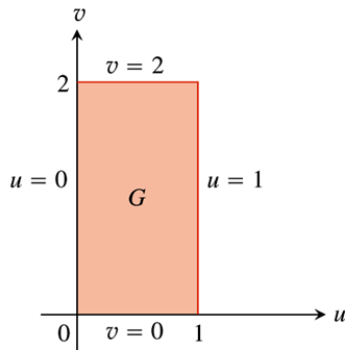
$$\begin{aligned} J(r, \theta) &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} \\ &= r \cos^2 \theta + r \sin^2 \theta \\ &= r (\cos^2 \theta + \sin^2 \theta) \\ &= \underline{r} \end{aligned}$$

Example

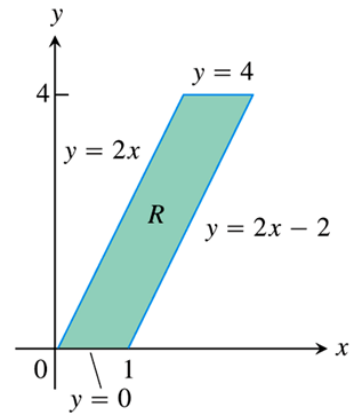
Evaluate $\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2} dx dy$ by applying the transformation $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$ and integrating over an appropriate region in the uv -plane.

Solution

$$\rightarrow \underline{y = 2v}, \quad 2u = 2x - y \Rightarrow \underline{x = \frac{2u+y}{2} = \frac{2u+2v}{2} = u+v}$$



$$\begin{array}{c} x = u + v \\ y = 2v \end{array}$$



xy-eqns for the boundary of R	Corresponding uv-eqns. for the boundary of G	Simplified uv-eqns.
$x = \frac{y}{2}$	$u + v = \frac{2v}{2} = v$	$u = 0$
$x = \frac{y}{2} + 1$	$u + v = \frac{2v}{2} + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$

$$\begin{aligned} J(u,v) &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial u}(u+v) & \frac{\partial}{\partial v}(u+v) \\ \frac{\partial}{\partial u}(2v) & \frac{\partial}{\partial v}(2v) \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \\ &= \underline{2} \end{aligned}$$

$$\begin{aligned} \int_0^4 \int_{y/2}^{(y/2)+1} \left(x - \frac{y}{2} \right) dx dy &= \int_0^{v=2} \int_{u=0}^{u=1} u |J(u,v)| du dv \\ &= \int_0^{v=2} \int_{u=0}^{u=1} (u)(2) du dv \end{aligned}$$

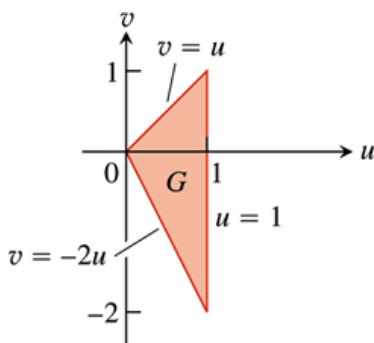
$$\begin{aligned}
 &= \int_0^{v=2} u^2 \Big|_0^1 dv \\
 &= \int_0^{v=2} dv \\
 &= v \Big|_0^2 \\
 &= \underline{2}
 \end{aligned}$$

Example

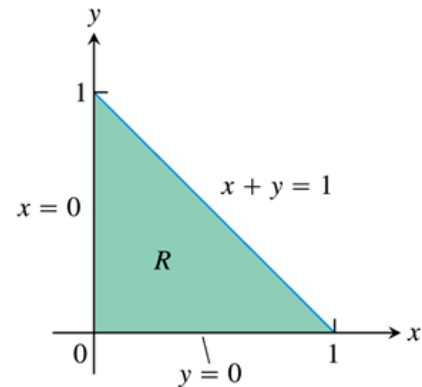
Evaluate $\int_0^1 \int_0^{1-x} \sqrt{x+y} (y-2x)^2 dy dx$

Solution

$$\begin{cases} u = x + y \\ v = y - 2x \end{cases} \rightarrow \begin{cases} x = \frac{u}{3} - \frac{v}{3} \\ y = \frac{2u}{3} + \frac{v}{3} \end{cases}$$



$$\begin{aligned}
 x &= \frac{u}{3} - \frac{v}{3} \\
 y &= \frac{2u}{3} + \frac{v}{3}
 \end{aligned}$$



xy-eqns for the boundary of R	Corresponding uv -eqns. for the boundary of G	Simplified uv -eqns.
$x + y = 1$	$\left(\frac{u}{3} - \frac{v}{3}\right) + \left(\frac{2u}{3} + \frac{v}{3}\right) = 1$	$u = 1$
$x = 0$	$\frac{u}{3} - \frac{v}{3} = 0$	$v = u$
$y = 0$	$\frac{2u}{3} + \frac{v}{3} = 0$	$v = -2u$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \underline{\frac{1}{3}}$$

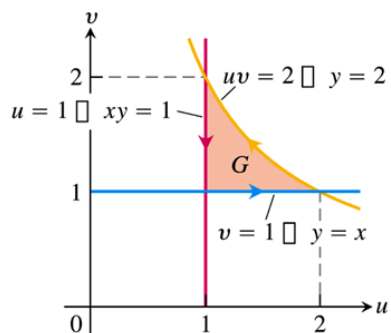
$$\begin{aligned}
\int_0^1 \int_0^{1-x} \sqrt{x+y} (y-2x)^2 dy dx &= \int_{u=0}^{u=1} \int_{v=-2u}^{v=u} u^{1/2} v^2 |J(u,v)| dv du \\
&= \int_0^1 \int_{-2u}^u u^{1/2} v^2 \left(\frac{1}{3}\right) dv du \\
&= \int_0^1 u^{1/2} \left[\frac{1}{9} v^3 \right]_{-2u}^u du \\
&= \frac{1}{9} \int_0^1 u^{1/2} (u^3 + 8u^3) du \\
&= \int_0^1 u^{7/2} du \\
&= \frac{2}{9} u^{9/2} \Big|_0^1 \\
&= \frac{2}{9}
\end{aligned}$$

Example

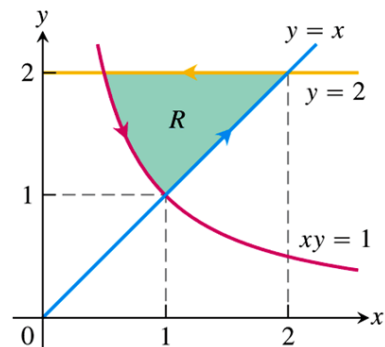
Evaluate $\int_1^2 \int_{1/y}^y \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dx dy$

Solution

$$\begin{cases} u = \sqrt{xy} \\ v = \sqrt{\frac{y}{x}} \end{cases} \rightarrow \begin{cases} u^2 = xy \\ v^2 = \frac{y}{x} \end{cases} \Rightarrow x = \frac{u}{v}, \quad y = uv$$



$$\begin{aligned}
x &= \frac{u}{v} \\
y &= uv
\end{aligned}$$



$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}$$

xy-eqns for the boundary of R	Corresponding uv-eqns. for the boundary of G	Simplified uv-eqns.
$x = y$	$\frac{u}{v} = uv$	$v = 1$
$x = \frac{1}{y}$	$\frac{u}{v} = \frac{1}{uv}$	$u = 1$
$y = 1$	$uv = 1$	
$y = 2$	$uv = 2$	$u = 2 \quad v = \frac{2}{u}$

$$\int_1^2 \int_{1/y}^y \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dx dy = \int_1^2 \int_1^{2/u} 2u e^u dv du$$

$$= 2 \int_1^2 u e^u [v]_1^{2/u} du$$

$$= 2 \int_1^2 u e^u \left(\frac{2}{u} - 1 \right) du$$

$$= 2 \int_1^2 (2 - u) e^u du$$

$$= 2 \left[(2 - u + 1) e^u \right]_1^2$$

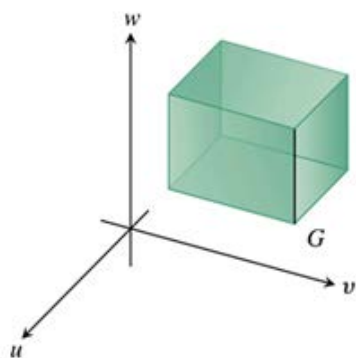
$$= 2 \left[(1) e^2 - 2e \right]$$

$$= \underline{2e(e - 2)}$$

	e^u	
(+)	$2 - u$	e^u
(-)	-1	e^u
	0	

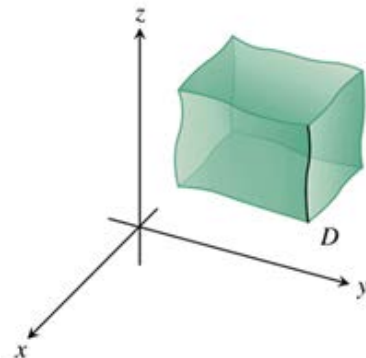
Substitutions in Triple Integrals

$$x = g(u, v, w), \quad y = h(u, v, w), \quad z = k(u, v, w)$$



Cartesian uvw - plane

$$\begin{aligned} x &= g(u, v, w) \\ y &= h(u, v, w) \\ z &= k(u, v, w) \end{aligned} \rightarrow$$



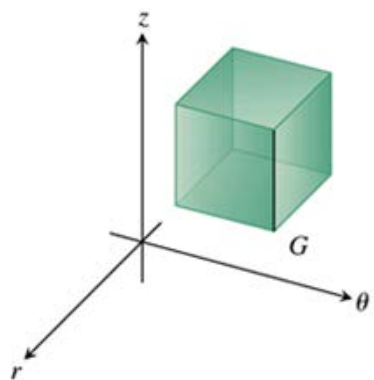
Cartesian xyz - plane

$$\iiint_R f(x, y, z) dx dy dz = \iiint_R H(u, v, w) |J(u, v, w)| du dv dw$$

The *Jacobian determinant* is

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x, y, z)}{\partial(u, v, w)}$$

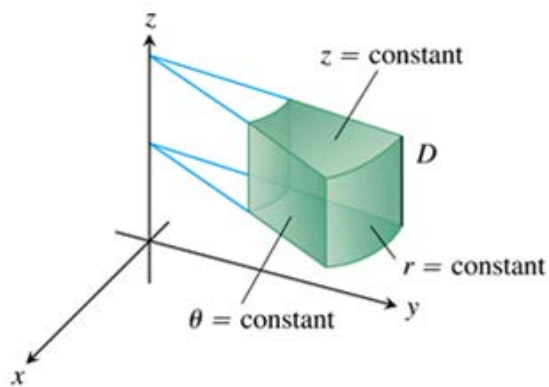
Cube with sides parallel to the axes



Cartesian rtheta z - plane

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ z &= z \end{aligned} \rightarrow$$

Cube with sides parallel to the axes



Cartesian xyz - plane

$$J(r, \theta, z) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = \underline{r}$$

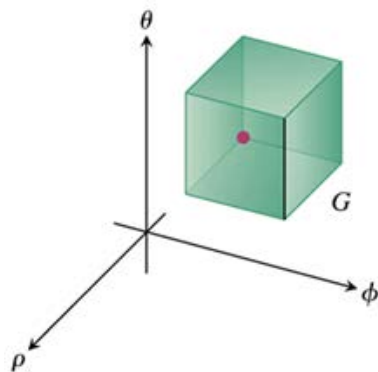
For spherical coordinates, ρ , ϕ , and θ take the place of u , v , and w . The transformation from Cartesian $\rho\phi\theta$ -space to Cartesian xyz -space is given by

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi$$

The Jacobian of the transformation

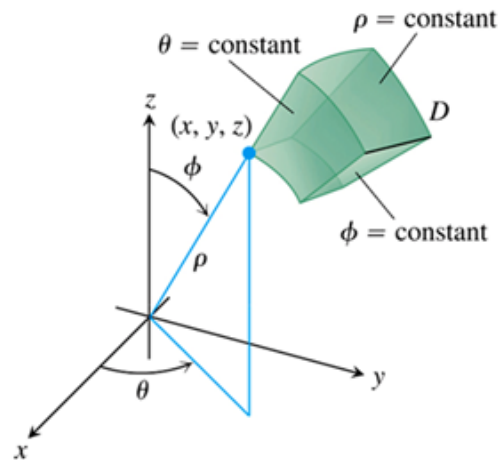
$$\begin{aligned} J(\rho, \phi, \theta) &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix} \\ &= \rho^2 \cos^2 \phi \sin \phi \cos^2 \theta + \rho^2 \sin^3 \phi \sin^2 \theta + \rho^2 \sin \phi \cos^2 \phi \sin^2 \theta + \rho^2 \sin^3 \phi \cos^2 \theta \\ &= \rho^2 \cos^2 \phi \sin \phi (\cos^2 \theta + \sin^2 \theta) + \rho^2 \sin^3 \phi (\sin^2 \theta + \cos^2 \theta) \\ &= \rho^2 \cos^2 \phi \sin \phi + \rho^2 \sin^3 \phi \\ &= \rho^2 \sin \phi (\cos^2 \phi + \sin^2 \phi) \\ &= \rho^2 \sin \phi \end{aligned}$$

$$\iiint_D F(x, y, z) dx dy dz = \iiint_G H(\rho, \phi, \theta) |\rho^2 \sin \phi| d\rho d\phi d\theta$$



Cartesian $\rho\phi\theta$ - plane

$$\begin{aligned} x &= \rho \sin \phi \cos \theta \\ y &= \rho \sin \phi \sin \theta \\ z &= \rho \cos \phi \end{aligned}$$



Cartesian xyz - plane

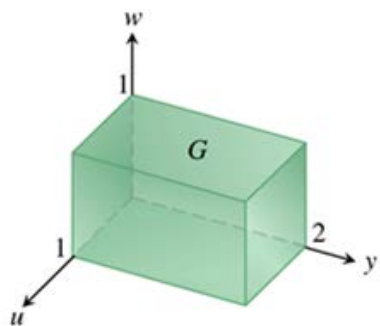
Example

Evaluate $\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx dy dz$ by applying the transformation

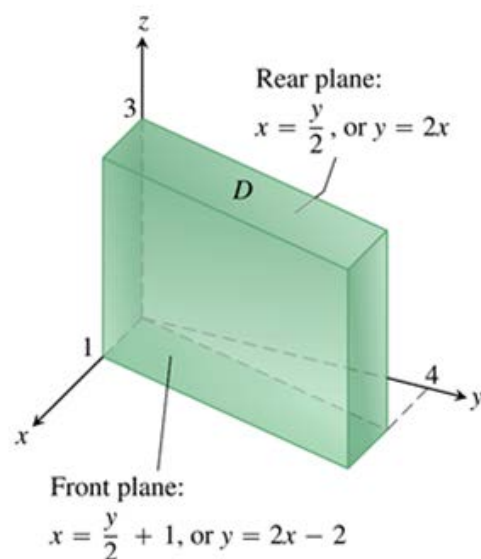
$u = \frac{2x-y}{2}$, $v = \frac{y}{2}$, $w = \frac{z}{3}$ and integrating over an appropriate region in the uvw -plane.

Solution

$$\rightarrow \begin{cases} u = \frac{2x-y}{2} \rightarrow x = u + \frac{y}{2} = u + v \\ v = \frac{y}{2} \rightarrow y = 2v \\ w = \frac{z}{3} \rightarrow z = 3w \end{cases}$$



$$\begin{aligned} x &= u + v \\ y &= 2v \\ z &= 3w \end{aligned} \rightarrow$$



<i>xyz-eqns</i> for the boundary of D	Corresponding <i>uvw-eqns.</i> for the boundary of G	Simplified <i>uvw-eqns.</i>
$x = \frac{y}{2}$	$u + v = v$	$u = 0$
$x = \frac{y}{2} + 1$	$u + v = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

$$\begin{aligned}
 J(u, v, w) &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \\
 &= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} \\
 &= 6
 \end{aligned}$$

$$\begin{aligned}
 \int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx dy dz &= \int_0^1 \int_0^2 \int_0^1 (u+w) |J(u, v, w)| \, dudvdw \\
 &= 6 \int_0^1 \int_0^2 \int_0^1 (u+w) \, dudvdw \\
 &= 6 \int_0^1 \int_0^2 \left[\frac{u^2}{2} + wu \right]_0^1 dv dw \\
 &= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + w \right) dv dw \\
 &= 6 \int_0^1 \left[\frac{1}{2}v + wv \right]_0^2 dw \\
 &= 6 \int_0^1 (1 + 2w) dw \\
 &= 6 \left[w + w^2 \right]_0^1 \\
 &= 6(1+1) \\
 &= 12
 \end{aligned}$$

Exercises Section 3.7 – Change of Variables in Multiple Integrals

Let $S = \{0 \leq u \leq 1, 0 \leq v \leq 1\}$ be a unit square in the uv -plane. Find the image of S in the xy -plane under the following transformations.

1. $T: x = v, y = u$

3. $T: x = \frac{u+v}{2}, y = \frac{u-v}{2}$

2. $T: x = -v, y = u$

4. $T: x = u, y = 2v + 2$

5. a) Solve the system $u = x - y, v = 2x + y$ for x and y in terms of u and v . Then find the value of

the Jacobian $\frac{\partial(x, y)}{\partial(u, v)}$

b) Find the image under the transformation $u = x - y, v = 2x + y$ of the triangular region with vertices $(0, 0), (1, 1)$, and $(1, -2)$ in the xy -plane. Sketch the transformed region in the uv -plane.

6. Let R be the region in the first quadrant of the xy -plane bounded by the hyperbolas $xy = 1, xy = 9$ and the lines $y = x, y = 4x$. Use the transformation $x = \frac{u}{v}, y = uv$ with $u > 0$, and $v > 0$ to rewrite

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy} \right) dx dy$$

As an integral over an appropriate region G in the uv -plane. Then evaluate the uv -integral over G .

7. The area πab of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be found by integrating the function $f(x, y) = 1$ over the region bounded by the ellipse in the xy -plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x = au, y = bv$ and evaluate the transformed integral over the disk $G: u^2 + v^2 \leq 1$ in the uv -plane. Find the area this way.

8. Use the transformation $x = u + \frac{1}{2}v, y = v$ to evaluate the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3 (2x - y) e^{(2x-y)^2} dx dy$$

By first writing it as an integral over a region G in the uv -plane.

9. Use the transformation $x = \frac{u}{v}, y = uv$ to evaluate the integral

$$\int_1^2 \int_{1/y}^y (x^2 + y^2) dx dy + \int_2^4 \int_{y/4}^{4/y} (x^2 + y^2) dx dy$$

10. Find the Jacobian $\frac{\partial(x, y)}{\partial(u, v)}$ of the transformation

a) $x = u \cos v, \quad y = u \sin v$

b) $x = u \sin v, \quad y = u \cos v$

11. Find the Jacobian $\frac{\partial(x, y)}{\partial(u, v)}$ of the transformation

a) $x = u \cos v, \quad y = u \sin v, \quad z = w$

b) $x = 2u - 1, \quad y = 3v - 4, \quad z = \frac{1}{2}(w - 4)$

12. Evaluate the appropriate determinant to show that the Jacobian of the transformation from Cartesian $\rho\phi\theta$ -space to Cartesian xyz -space is $\rho^2 \sin \phi$
13. How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.

14. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

(*Hint:* Let $x = au$, $y = bv$, and $z = cw$. Then find the volume of an appropriate region in uvw -space)

15. Use the transformation $x = u^2 - v^2, \quad y = 2uv$ to evaluate the integral

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2} dy dx$$

(*Hint:* Show that the image of the triangular region G with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ in the uv -plane is the region of integration R in the xy -plane defined by the limits of integration.)

16. Evaluate $\iint_R y^4 dA$; R is the region bounded by the hyperbolas $xy = 1$ and $xy = 4$ and the lines

$$\frac{y}{x} = 1, \text{ and } \frac{y}{x} = 3$$

17. Evaluate $\iint_R (y^2 + xy - 2x^2) dA$; R is the region bounded by the lines $y = x$, $y = x - 3$,
 $y = -2x + 3$, and $y = -2x - 3$
18. Evaluate $\iiint_D x dV$; R is bounded by the planes $y - 2x = 0$, $y - 2x = 1$, $z - 3y = 0$, $z - 3y = 1$,
 $z - 4x = 0$ and $z - 4x = 3$
19. Let R be the region bounded by the lines $x + y = 1$; $x + y = 4$; $x - 2y = 0$; $x - 2y = -4$
 Evaluate the integral $\iint_R 3xy dA$
20. Let R be the region bounded by the square with vertices $(0, 1)$, $(1, 2)$, $(2, 1)$, & $(1, 0)$.
 Evaluate the integral $\iint_R (x + y)^2 \sin^2(x - y) dA$
21. Evaluate $\iiint_D yz dV$ D is bounded by the planes: $x + 2y = 1$, $x + 2y = 2$, $x - z = 0$, $x - z = 2$,
 $2y - z = 0$, and $2y - z = 3$
22. Evaluate $\iint_R xy dA$; R is the square with vertices $(0, 0)$, $(1, 1)$, $(2, 0)$, and $(1, -1)$
23. Evaluate $\iint_R x^2 y dA$; $R = \{(x, y): 0 \leq x \leq 2, x \leq y \leq x + 4\}$
24. Evaluate $\iint_R x^2 \sqrt{x + 2y} dA$; $R = \{(x, y): 0 \leq x \leq 2, -\frac{x}{2} \leq y \leq 1 - x\}$
25. Evaluate $\iint_R xy dA$; where R is bounded by the ellipse $9x^2 + 4y^2 = 36$.
26. Evaluate $\int_0^1 \int_y^{y+2} \sqrt{x - y} dx dy$

27. Evaluate $\iint_R \sqrt{y^2 - x^2} \, dA$; where R is the diamond bounded by $y - x = 0$, $y - x = 2$, $y + x = 0$, and $y + x = 2$
28. Evaluate $\iint_R \left(\frac{y-x}{y+2x+1} \right)^4 \, dA$; where R is the parallelogram bounded by $y - x = 1$, $y - x = 2$, $y + 2x = 0$, and $y + 2x = 4$
29. Evaluate $\iint_R e^{xy} \, dA$; where R is the region bounded by $xy = 1$, $xy = 4$, $\frac{y}{x} = 1$, and $\frac{y}{x} = 3$
30. Evaluate $\iint_R xy \, dA$; where R is the region bounded by the hyperbolas $xy = 1$, $xy = 4$, $y = 1$, and $y = 3$
31. Evaluate $\iint_R (x-y)\sqrt{x-2y} \, dA$; where R is the triangular region bounded by $y = 0$, $x - 2y = 0$, and $x - y = 1$
32. Evaluate $\iiint_D xy \, dV$; D is bounded by the planes: $y - x = 0$, $y - x = 2$, $z - y = 0$, $z - y = 2$, $z = 0$, and $z = 3$
33. Evaluate $\iiint_D dV$; D is bounded by the planes: $y - 2x = 0$, $y - 2x = 1$, $z - 3y = 0$, $z - 3y = 1$, $z - 4x = 0$, and $z - 4x = 3$
34. Evaluate $\iiint_D z \, dV$; D is bounded by the paraboloid $z = 16 - x^2 - 4y^2$ and the xy -plane.
35. Evaluate $\iiint_D dV$; D is bounded by the upper half of the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$ and the xy -plane.
36. Evaluate $\iiint_D xz \, dV$; D is bounded by the planes: $y = x$, $y = x + 2$, $x - z = 0$, $z = x + 3$, $z = 0$, and $z = 4$

(37 – 41) Let R be the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$ are real numbers.

37. Find the area of R .

38. Evaluates $\iint_R |xy| dA$

39. Find the center of mass of the upper half of R ($y \geq 0$) assuming it has a constant density.

40. Find the average square of the distance between points of R and the origin.

41. Find the average distance between points in the upper half of R and the x -axis.

(42 – 45) Let D be the region bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where $a > 0$, $b > 0$ and $c > 0$ are real numbers.

42. Find the Volume of D .

43. Evaluates $\iiint_D |xyz| dV$

44. Find the center of mass of the upper half of D ($z \geq 0$) assuming it has a constant density.

45. Find the average square of the distance between points of D and the origin.