# *Section* 2.3 – Product and Quotient Rules

## *Product Rule*

The derivative of the product of two differentiable functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$(f.g)' = f'.g + g'.f$$

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'gh + fg'h + fgh'$$

### *Example*

Find the derivative of $f(x) = (2x+3)\left(3x^2\right)$

***Solution***

$$f(x) = 6x^3 + 9x^2$$

$$\begin{aligned} f' &= (2x+3)\left(3x^2\right)' + (2x+3)'\left(3x^2\right) \\ &= (2x+3)(6x) + (2)\left(3x^2\right) \\ &= 12x^2 + 18x + 6x^2 \\ &= 18x^2 + 18x \\ &= 18x(x+1) \end{aligned}$$

## *Proof of the Derivative Product Rule*

$$\begin{aligned} \frac{d}{dx}(uv) &= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\ &= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \\ &= \lim_{h\to 0}\left[\frac{u(x+h)v(x+h) - u(x+h)v(x)}{h} + \frac{u(x+h)v(x) - u(x)v(x)}{h}\right] \\ &= \lim_{h\to 0}\left[u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h}\right] \\ &= \lim_{h\to 0} u(x+h) \cdot \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} + v(x) \cdot \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \\ &= u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \end{aligned}$$

### ***Example***

Find the derivative of $y=\left(3x^2+1\right)\left(x^3+3\right)$

***Solution***

$u=3x^2+1 \quad v=x^3+3$

$u'=6x \qquad v'=3x^2$

$y'=(6x)\left(x^3+3\right)+\left(3x^2\right)\left(3x^2+1\right)$

$=6x^4+18x+9x^4+3x^2$

$\boxed{=15x^4+3x^2+18x}$

$y=3x^5+9x^2+x^3+3$

$y'=15x^4+18x+3x^2$

### ***Example***

Find the derivative of $y=\left(3x^3+2x+5\right)\left(x^2-2x+4\right)$

***Solution***

$y'=\underbrace{\left(9x^2+2\right)}_{u'}\underbrace{\left(x^2-2x+4\right)}_{v}+\underbrace{(2x-2)}_{v'}\underbrace{\left(3x^3+2x+5\right)}_{u}$

$=9x^4-18x^3+36x^2+2x^2-4x+8+6x^4+4x^2+10x-6x^3-4x-10$

$\boxed{=15x^4-24x^3+42x^2+2x-2}$

## ***Quotient Rule***

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)g(x)-g'(x)f(x)}{[g(x)]^2} \quad =\frac{f'g-g'f}{g^2}$$

$$\left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

$$\left(\frac{ax^n+b}{cx^n+d}\right)'=\frac{n(ad-bc)x^{n-1}}{(cx+d)^2}$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\frac{\begin{vmatrix}a & b\\ d & e\end{vmatrix}x^2+2\begin{vmatrix}a & c\\ d & f\end{vmatrix}x+\begin{vmatrix}b & c\\ e & f\end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

### ***Example***

Find $f'(x)$ if $\quad f(x)=\dfrac{2x-1}{4x+3}$

***Solution***

$$f'=\frac{(2x-1)'(4x+3)-(2x-1)(4x+3)'}{(4x+3)^2} \qquad u=2x-1 \quad v=4x+3 \quad u'=2 \quad v'=4$$

$$=\frac{(2)(4x+3)-(2x-1)(4)}{(4x+3)^2}$$

$$=\frac{8x+6-8x+4}{(4x+3)^2}$$

$$=\boxed{\frac{10}{(4x+3)^2}}$$

$$f'(x)=\frac{2(3)-(-1)(4)}{4x+3} \qquad \left(\frac{ax+b}{cx+d}\right)'=\frac{ad-bc}{(cx+d)^2}$$

$$=\boxed{\frac{10}{(4x+3)^2}}$$

### Example

Find the derivative of $y = \dfrac{(x-1)\left(x^2-2x\right)}{x^4}$

**Solution**

$$y = \frac{x^3 - 2x^2 - x^2 + 2x}{x^4}$$

$$= \frac{x^3 - 3x^2 + 2x}{x^4}$$

$$= \frac{x^3}{x^4} - \frac{3x^2}{x^4} + \frac{2x}{x^4}$$

$$= x^{-1} - 3x^{-2} + 2x^{-3}$$

$$y' = -x^{-2} + 6x^{-3} - 6x^{-4}$$

$$\boxed{= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}}$$

## Combining the product and Quotient Rules

### Example

Find the derivative of $y = \dfrac{(1+x)(2x-1)}{x-1}$

**Solution**

$$y = \frac{(1+x)(2x-1)}{x-1}$$

$$= \frac{2x^2 + x - 1}{x-1}$$

$$y' = \frac{\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} x^2 + 2\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} x + \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}}{x-1}$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right) = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix} x^2 + 2\begin{vmatrix} a & c \\ d & f \end{vmatrix} x + \begin{vmatrix} b & c \\ e & f \end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$

$$\boxed{= \frac{2x^2 - 4x}{(x-1)^2}}$$

*Or*

$$y' = \frac{(x-1)\frac{d}{dx}\left[(1+x)(2x-1)\right] - (1+x)(2x-1)\frac{d}{dx}[x-1]}{(x-1)^2}$$

$$= \frac{(x-1)\left[(1)(2x-1) + 2(1+x)\right] - (1+x)(2x-1)(1)}{(x-1)^2}$$

$$= \frac{(x-1)(2x-1+2+2x) - (2x-1+2x^2-x)}{(x-1)^2}$$

$$= \frac{(x-1)(4x+1) - 2x + 1 - 2x^2 + x}{(x-1)^2}$$

$$= \frac{4x^2 + x - 4x - 1 - 2x + 1 - 2x^2 + x}{(x-1)^2}$$

$$\boxed{= \frac{2x^2 - 4x}{(x-1)^2}}$$

# *Exercises* *Section* 2.3 – Product and Quotient Rules

Find the derivative of each function

**1.** $y=(x+1)\left(\sqrt{x}+2\right)$

**2.** $y=\left(4x+3x^2\right)(6-3x)$

**3.** $y=\left(\frac{1}{x}+1\right)(2x+1)$

**4.** $y=\dfrac{3-\frac{2}{x}}{x+4}$

**5.** $g(x)=\dfrac{x^2-4x+2}{x^2+3}$

**6.** $f(x)=\dfrac{(3-4x)(5x+1)}{7x-9}$

**7.** $f(x)=x\left(1-\dfrac{2}{x+1}\right)$

**8.** $f(x)=\left(\sqrt{x}+3\right)\left(x^2-5x\right)$

**9.** $y=(2x+3)\left(5x^2-4x\right)$

**10.** $y=\left(x^2+1\right)\left(x+5+\frac{1}{x}\right)$

**11.** $y=\dfrac{x+4}{5x-2}$

**12.** $z=\dfrac{4-3x}{3x^2+x}$

**13.** $y=(2x-7)^{-1}(x+5)$

**14.** $f(x)=\dfrac{\sqrt{x}-1}{\sqrt{x}+1}$

**15.** $y=\dfrac{1}{\left(x^2-1\right)\left(x^2+x+1\right)}$

**16.** $f(x)=\dfrac{x^{3/2}\left(x^2+1\right)}{x+1}$

**17.** $f(x)=\dfrac{x^3-4x^2+x}{x-2}$

**18.** $g(x)=\dfrac{x(3-x)}{2x^2}$

**19.** $y=\dfrac{2x^2}{3x+1}$

**20.** $f(x)=\dfrac{x^9+x^8+4x^5-7x}{x^4-3x^2+2x+1}$

**21.** $f(x)=\dfrac{x}{1+x^2}$

**22.** $y=\dfrac{x^2-2ax+a^2}{x-a}$

**23.** $f(x)=\dfrac{x^2+4x^{1/2}}{x^2}$

**24.** $f(x)=(2x+1)\left(3x^2+2\right)$

**25.** $f(x)=\dfrac{x^2-1}{x^2+1}$

**26.** $y=\dfrac{4x^3+3x+1}{2x^5}$

**27.** $y=\dfrac{4}{3-x}$

**28.** $y=\dfrac{2}{1-x^2}$

**29.** $f(x)=\dfrac{\pi}{2-\pi x}$

**30.** $y=\dfrac{x-4}{5x-2}$

**31.** $y=\dfrac{3x-4}{2x-1}$

**32.** $y=\dfrac{3x+4}{2x+1}$

**33.** $y=\dfrac{-3x+4}{2x+1}$

**34.** $y=\dfrac{-3x-4}{2x-1}$

**35.** $y=\dfrac{2x-3}{x+1}$

**36.** $y=\dfrac{3x}{3x-2}$

**37.** $y=\dfrac{x-3}{2x+5}$

**38.** $y=\dfrac{5x-3}{2x+5}$

**39.** $y=\dfrac{6x-8}{2x-3}$

**40.** $y=\dfrac{x^2-4}{5x^2-2}$

**41.** $y=\dfrac{3x^2-4}{2x^2-1}$

**42.** $y=\dfrac{3x^2+4}{2x^2+1}$

**43.** $y=\dfrac{2x^2-3}{x^2+1}$

**44.** $y=\dfrac{3x^2}{3x^2-2}$

**45.** $y=\dfrac{5x^2-3}{2x^2+5}$

**46.** $y=\dfrac{6x^2-8}{2x^2+1}$

**47.** $y=\dfrac{6x^3+8}{2x^3+1}$

**48.** $y=\dfrac{5x^3-3}{2x^3+5}$

**49.** $y=\dfrac{x^3}{3x^3-2}$

**50.** $y=\dfrac{2x^3-3}{2x^3+1}$

**51.** $y=\dfrac{2x^4-3}{2x^4+1}$

**52.** $y=\dfrac{x^2-4x+1}{5x^2-2x-1}$

**53.** $y=\dfrac{3x^2-4x+2}{2x^2+x-1}$

**54.** $y=\dfrac{3x^2+x-4}{2x^2+1}$

**55.** $y=\dfrac{2x^2-3}{x^2+5x+1}$

**56.** $y=\dfrac{3x^2}{3x^2+6x-8}$

**57.** $y=\dfrac{x^2+2x}{2x^2+x-5}$

**58.** $y=\dfrac{x^2+5x+1}{x^2}$

**59.** $y=\dfrac{x^2-3x+1}{x^2-8x+5}$

**60.** Find the first and second derivative $y=\dfrac{x^2+5x-1}{x^2}$

**61.** Find an equation of the tangent line to the graph of $y=\dfrac{x^2-4}{2x+5}$ when $x=0$

**62.** For what value(s) of $x$ is the line tangent to the curve $y=x\sqrt{6-x}$ horizontal? Vertical?

**63.** Find $y'$, $y''$, $y'''$: $\quad y=(x-3)\sqrt{x+2}$