Solution Section 2.3 – Partial Derivatives

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x,y) = 2x^2 - 3y - 4$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(2x^2 - 3y - 4\right)$$
$$= 4x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(2x^2 - 3y - 4\right)$$
$$= -3$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x,y) = x^2 - xy + y^2$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2 - xy + y^2\right)$$
$$= 2x - y$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2 - xy + y^2\right)$$
$$= -x + 2y$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x,y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(5xy - 7x^2 - y^2 + 3x - 6y + 2\right)$$
$$= 5y - 14x + 3$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(5xy - 7x^2 - y^2 + 3x - 6y + 2\right)$$
$$= 5x - 2y - 6$$

Exercise

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ $\quad f(x,y)=(xy-1)^2$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(xy-1)^2$$

$$=2y(xy-1)$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(xy-1)^2$$

$$=2x(xy-1)$$

Exercise

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ $\quad f(x,y)=\left(x^3+\frac{y}{2}\right)^{2/3}$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(x^3+\frac{y}{2}\right)^{2/3}$$

$$=\frac{2}{3}\left(x^3+\frac{y}{2}\right)^{-1/3}\left(3x^2\right)$$

$$=\frac{2x^2}{\sqrt[3]{x^3+\frac{y}{2}}}$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(x^3+\frac{y}{2}\right)^{2/3}$$

$$=\frac{2}{3}\left(x^3+\frac{y}{2}\right)^{-1/3}\left(\frac{1}{2}\right)$$

$$=\frac{1}{3\sqrt[3]{x^3+\frac{y}{2}}}$$

Exercise

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ $\quad f(x,y)=\dfrac{1}{x+y}$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{x+y}\right) \qquad\qquad \frac{\partial}{dx}\left(\frac{1}{u}\right) = -\frac{u'}{u^2}$$

$$= -\frac{1}{(x+y)^2}\frac{\partial}{\partial x}(x+y)$$

$$= -\frac{1}{(x+y)^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{x+y}\right)$$

$$= -\frac{1}{(x+y)^2}$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x,y) = \frac{x}{x^2+y^2}$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) \qquad\qquad \frac{\partial}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$$

$$= \frac{1\left(x^2+y^2\right) - x(2x)}{\left(x^2+y^2\right)^2}$$

$$= \frac{y^2 - x^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{(0)\left(x^2+y^2\right) - x(2y)}{\left(x^2+y^2\right)^2}$$

$$= -\frac{2xy}{\left(x^2+y^2\right)^2}$$

Exercise

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ $\qquad f(x,y)=\tan^{-1}\dfrac{y}{x}$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(\tan^{-1}\frac{y}{x}\right)=\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$=-\frac{y}{1+\frac{y^2}{x^2}}\left(\frac{1}{x^2}\right)$$

$$=-\frac{y}{\frac{x^2+y^2}{x^2}}\left(\frac{1}{x^2}\right)$$

$$=-\frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(\tan^{-1}\frac{y}{x}\right)=\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$=\frac{1}{\frac{x^2+y^2}{x^2}}\left(\frac{1}{x}\right)$$

$$=\frac{x}{x^2+y^2}$$

Exercise

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ $\qquad f(x,y)=e^{-x}\sin(x+y)$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(e^{-x}\sin(x+y)\right)$$

$$=\sin(x+y)\frac{\partial}{\partial x}\left(e^{-x}\right)+e^{-x}\frac{\partial}{\partial x}\left(\sin(x+y)\right)$$

$$=-e^{-x}\sin(x+y)+e^{-x}\cos(x+y)$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(e^{-x}\sin(x+y)\right)$$

$$=e^{-x}\cos(x+y)$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x,y)=e^{xy}\ln y$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(e^{xy}\ln y\right)$$

$$=ye^{xy}\ln y$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(e^{xy}\ln y\right)$$

$$=\ln y\frac{\partial}{\partial y}\left(e^{xy}\right)+e^{xy}\frac{\partial}{\partial y}(\ln y)$$

$$=xe^{xy}\ln y+\frac{1}{y}e^{xy}$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x,y)=\sin^2(x-3y)$

Solution

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(\sin^2(x-3y)\right)$$

$$=2\sin(x-3y)\frac{\partial}{\partial x}\sin(x-3y)$$

$$=2\sin(x-3y)\cos(x-3y)\frac{\partial}{\partial x}(x-3y)$$

$$=2\sin(x-3y)\cos(x-3y)$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(\sin^2(x-3y)\right)$$

$$=2\sin(x-3y)\frac{\partial}{\partial y}\sin(x-3y)$$

$$=2\sin(x-3y)\cos(x-3y)\frac{\partial}{\partial y}(x-3y)$$

$$=-6\sin(x-3y)\cos(x-3y)$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x,y)=\cos^2\left(3x-y^2\right)$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\cos^2\left(3x - y^2\right)\right)$$

$$= 2\cos\left(3x - y^2\right)\frac{\partial}{\partial x}\left(\cos\left(3x - y^2\right)\right)$$

$$= -2\cos\left(3x - y^2\right)\sin\left(3x - y^2\right)\frac{\partial}{\partial x}\left(3x - y^2\right)$$

$$= -6\cos\left(3x - y^2\right)\sin\left(3x - y^2\right)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\cos^2\left(3x - y^2\right)\right)$$

$$= 2\cos\left(3x - y^2\right)\frac{\partial}{\partial y}\left(\cos\left(3x - y^2\right)\right)$$

$$= -2\cos\left(3x - y^2\right)\sin\left(3x - y^2\right)\frac{\partial}{\partial y}\left(3x - y^2\right)$$

$$= 4y\cos\left(3x - y^2\right)\sin\left(3x - y^2\right)$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x, y) = x^y$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^y\right)$$

$$= yx^{y-1}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^y\right)$$

$$= x^y \ln x$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x,\ y) = 3x^2y^5$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(3x^2y^5\right)$$

$$= 6xy^5$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(3x^2y^5\right)$$

$$= 15x^2y^4$$

Exercise

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ $\qquad f(x, y) = x\cos y - y\sin x$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y - y\sin x)$$

$$= \cos y - y\cos x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y - y\sin x)$$

$$= -x\sin y - \sin x$$

Exercise

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ $\qquad f(x, y) = \dfrac{x^2}{x^2 + y^2}$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^2}{x^2 + y^2}\right)$$

$$= \frac{2x\left(x^2 + y^2\right) - 2x^3}{\left(x^2 + y^2\right)^2}$$

$$= \frac{2xy^2}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2}{x^2 + y^2}\right)$$

$$= -\frac{2x^2y}{\left(x^2 + y^2\right)^2}$$

Exercise

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ $\quad f(x, y) = xye^{xy}$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(xye^{xy}\right)$$

$$= \left(y + xy^2\right)e^{xy}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xye^{xy}\right)$$

$$= \left(x + x^2 y\right)e^{xy}$$

Exercise

Find $f_x, f_y,$ *and* f_z $\quad f(x,y,z) = 1 + xy^2 - 2z^2$

Solution

$$f_x = y^2 \qquad f_y = 2xy \qquad f_z = -4z$$

Exercise

Find $f_x, f_y,$ *and* f_z $\quad f(x,y,z) = xy + yz + xz$

Solution

$$f_x = y + z \qquad f_y = x + y \qquad f_z = y + x$$

Exercise

Find $f_x, f_y,$ *and* f_z $\quad f(x,y,z) = x - \sqrt{y^2 + z^2}$

Solution

$$f_x = 1$$

$$f_y = -\frac{1}{2}\left(y^2 + z^2\right)^{-1/2} \frac{\partial}{\partial y}\left(y^2 + z^2\right)$$

$$= -\frac{1}{2}\left(y^2 + z^2\right)^{-1/2}(2y)$$

$$= -\frac{y}{\sqrt{y^2+z^2}}$$

$$f_z = -\frac{1}{2}\left(y^2+z^2\right)^{-1/2}\frac{\partial}{\partial z}\left(y^2+z^2\right)$$

$$= -\frac{1}{2}\left(y^2+z^2\right)^{-1/2}(2z)$$

$$= -\frac{z}{\sqrt{y^2+z^2}}$$

Exercise

Find $f_x, f_y,$ *and* f_z $\quad f(x,y,z) = \left(x^2+y^2+z^2\right)^{-1/2}$

Solution

$$f_x = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2x)$$

$$= -x\left(x^2+y^2+z^2\right)^{-3/2}$$

$$f_y = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2y)$$

$$= -y\left(x^2+y^2+z^2\right)^{-3/2}$$

$$f_z = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2z)$$

$$= -z\left(x^2+y^2+z^2\right)^{-3/2}$$

Exercise

Find $f_x, f_y,$ *and* f_z $\quad f(x,y,z) = \sec^{-1}(x+yz)$

Solution

$$f_x = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}\frac{\partial}{\partial x}(x+yz)$$

$$= \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}$$

$$f_y = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}} \frac{\partial}{\partial y}(x+yz)$$

$$= \boxed{\frac{z}{|x+yz|\sqrt{(x+yz)^2-1}}}$$

$$f_z = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}} \frac{\partial}{\partial z}(x+yz)$$

$$= \boxed{\frac{y}{|x+yz|\sqrt{(x+yz)^2-1}}}$$

Exercise

Find $f_x, f_y,$ *and* f_z $\quad f(x,y,z) = \ln(x+2y+3z)$

Solution

$$f_x = \frac{1}{x+2y+3z} \cdot \frac{\partial}{\partial x}(x+2y+3z)$$

$$= \boxed{\frac{1}{x+2y+3z}}$$

$$f_y = \frac{1}{x+2y+3z} \cdot \frac{\partial}{\partial y}(x+2y+3z)$$

$$= \boxed{\frac{2}{x+2y+3z}}$$

$$f_z = \frac{1}{x+2y+3z} \cdot \frac{\partial}{\partial z}(x+2y+3z)$$

$$= \boxed{\frac{3}{x+2y+3z}}$$

Exercise

Find $f_x, f_y,$ *and* f_z $\quad f(x,y,z) = e^{-\left(x^2+y^2+z^2\right)}$

Solution

$$f_x = e^{-\left(x^2+y^2+z^2\right)} \frac{\partial}{\partial x}\left(-\left(x^2+y^2+z^2\right)\right)$$

$$= \boxed{-2xe^{-\left(x^2+y^2+z^2\right)}}$$

$$f_y = e^{-\left(x^2+y^2+z^2\right)} \frac{\partial}{\partial y}\left(-\left(x^2+y^2+z^2\right)\right)$$

$$\boxed{= -2ye^{-\left(x^2+y^2+z^2\right)}}$$

$$f_z = e^{-\left(x^2+y^2+z^2\right)} \frac{\partial}{\partial z}\left(-\left(x^2+y^2+z^2\right)\right)$$

$$\boxed{= -2ze^{-\left(x^2+y^2+z^2\right)}}$$

Exercise

Find $f_x, f_y,$ *and* f_z $\quad f(x,y,z) = \tanh(x+2y+3z)$

Solution

$$\boxed{f_x = \operatorname{sech}^2(x+2y+3z)}$$

$$\boxed{f_y = 2\operatorname{sech}^2(x+2y+3z)}$$

$$\boxed{f_z = 3\operatorname{sech}^2(x+2y+3z)}$$

Exercise

Find $f_x, f_y,$ *and* f_z $\quad f(x,y,z) = \sinh\left(xy-z^2\right)$

Solution

$$f_x = \cosh\left(xy-z^2\right)\frac{\partial}{\partial x}\left(xy-z^2\right)$$

$$\boxed{= y\cosh\left(xy-z^2\right)}$$

$$\boxed{f_y = x\cosh\left(xy-z^2\right)}$$

$$\boxed{f_z = -2z\cosh\left(xy-z^2\right)}$$

Exercise

Find $f_x, f_y,$ *and* f_z $\quad f(x,y,z) = 4xyz^2 - \frac{3x}{y}$

Solution

$$\boxed{f_x = 4yz^2 - \frac{3}{y}}$$

$$\boxed{f_y = 4xz^2 + \frac{3x}{y^2}}$$

$$\boxed{f_z = 8xyz}$$

Exercise

Find $f_x,\, f_y,\, and\ f_z$ $\qquad f(x, y, z) = \frac{xyz}{x+y}$

Solution

$$\boxed{f_x = \frac{y^2 z}{(x+y)^2}} \qquad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

$$\boxed{f_y = \frac{x^2 z}{(x+y)^2}}$$

$$\boxed{f_z = \frac{xy}{x+y}}$$

Exercise

Find $f_x,\, f_y,\, and\ f_z$ $\qquad f(x, y, z) = e^{x+2y+3z}$

Solution

$$\boxed{f_x = e^{x+2y+3z}}$$

$$\boxed{f_y = 2e^{x+2y+3z}}$$

$$\boxed{f_z = 3e^{x+2y+3z}}$$

Exercise

Find $f_x,\, f_y,\, and\ f_z$ $\qquad f(x, y, z) = x^2\sqrt{y+z}$

Solution

$$\boxed{f_x = 2x\sqrt{y+z}}$$

$$\boxed{f_y = \frac{1}{2}\frac{x^2}{\sqrt{y+z}}}$$

$$\boxed{f_z = \frac{1}{2}\frac{x^2}{\sqrt{y+z}}}$$

Exercise

Find partial derivatives of the function with respect to each variable $g(r,\theta) = r\cos\theta + r\sin\theta$

Solution

$$\boxed{g_r = \cos\theta + \sin\theta}$$

$$\boxed{g_\theta = -r\sin\theta + r\cos\theta}$$

Exercise

Find partial derivatives of the function with respect to each variable

$$f(x,y) = \frac{1}{2}\ln\left(x^2+y^2\right) + \tan^{-1}\frac{y}{x}$$

Solution

$$f_x = \frac{x}{x^2+y^2} - \frac{y}{x^2}\frac{1}{1+\frac{y^2}{x^2}}$$

$$= \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}$$

$$\boxed{= \frac{x-y}{x^2+y^2}}$$

$$f_y = \frac{y}{x^2+y^2} + \frac{1}{x}\frac{1}{1+\frac{y^2}{x^2}}$$

$$= \frac{y}{x^2+y^2} + \frac{x}{x^2+y^2}$$

$$\boxed{= \frac{x+}{x^2+y^2}}$$

Exercise

Find partial derivatives of the function with respect to each variable $h(x,y,z)=\sin(2\pi x+y-3z)$

Solution

$$\boxed{h_x = 2\pi\cos(2\pi x+y-3z)}$$

$$\boxed{h_y = \cos(2\pi x+y-3z)}$$

$$\boxed{h(x,y,z) = -3\cos(2\pi x+y-3z)}$$

Exercise

Find partial derivatives of the function with respect to each variable $f(r,l,T,w)=\frac{1}{2rl}\sqrt{\frac{T}{\pi w}}$

Solution

$$\boxed{f_r = -\frac{1}{2r^2 l}\sqrt{\frac{T}{\pi w}}}$$

$$\boxed{f_l = -\frac{1}{2rl^2}\sqrt{\frac{T}{\pi w}}}$$

$$\begin{aligned} f_T &= \frac{1}{4\pi rlw}\left(\frac{T}{\pi w}\right)^{-1/2} \\ &= \frac{1}{4\pi rlw}\sqrt{\frac{\pi w}{T}} \\ &= \boxed{\frac{1}{4rl}\sqrt{\frac{1}{\pi wT}}} \end{aligned}$$

$$\begin{aligned} f_w &= \frac{1}{4rl}\frac{-T}{\pi w^2}\left(\frac{T}{\pi w}\right)^{-1/2} \\ &= -\frac{T}{4\pi rlw^2}\sqrt{\frac{\pi w}{T}} \\ &= \boxed{-\frac{1}{4rlw}\sqrt{\frac{T}{\pi w}}} \end{aligned}$$

Exercise

Find all the second-order partial derivatives of $f(x,y)=x+y+xy$

Solution

$$\frac{\partial f}{\partial x}=1+y \quad \frac{\partial f}{\partial y}=1+x \quad \frac{\partial^2 f}{\partial x\partial y}=1$$

$$\frac{\partial^2 f}{\partial x^2} = 0 \quad \frac{\partial^2 f}{\partial y^2} = 0 \quad \frac{\partial^2 f}{\partial y \partial x} = 1$$

Exercise

Find all the second-order partial derivatives of $f(x,y) = \sin xy$

Solution

$$\boxed{\frac{\partial f}{\partial x} = y\cos xy} \qquad \boxed{\frac{\partial f}{\partial y} = x\cos xy}$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = -y^2 \sin xy} \qquad \boxed{\frac{\partial^2 f}{\partial y^2} = -x^2 \sin xy}$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \cos xy - xy\sin xy} \qquad \boxed{\frac{\partial^2 f}{\partial y \partial x} = \cos xy - xy\sin xy}$$

Exercise

Find all the second-order partial derivatives of $g(x,y) = x^2 y + \cos y + y\sin x$

Solution

$$\boxed{\frac{\partial g}{\partial x} = 2xy + y\cos x} \qquad \boxed{\frac{\partial g}{\partial y} = x^2 - \sin y + \sin x}$$

$$\boxed{\frac{\partial^2 g}{\partial x^2} = 2y - y\sin x} \qquad \boxed{\frac{\partial^2 g}{\partial y^2} = -\cos y}$$

$$\boxed{\frac{\partial^2 g}{\partial x \partial y} = 2x + \cos x} \qquad \boxed{\frac{\partial^2 g}{\partial y \partial x} = 2x + \cos x}$$

Exercise

Find all the second-order partial derivatives of $r(x,y) = \ln(x+y)$

Solution

$$\boxed{\frac{\partial r}{\partial x} = \frac{1}{x+y}} \qquad \boxed{\frac{\partial^2 r}{\partial x^2} = -\frac{1}{(x+y)^2}} \qquad \boxed{\frac{\partial^2 r}{\partial y \partial x} = -\frac{1}{(x+y)^2}}$$

$$\frac{\partial r}{\partial y} = \frac{1}{x+y} \qquad \boxed{\frac{\partial^2 r}{\partial y^2} = -\frac{1}{(x+y)^2}} \qquad \boxed{\frac{\partial^2 r}{\partial x \partial y} = -\frac{1}{(x+y)^2}}$$

Exercise

Find all the second-order partial derivatives of $w = x^2 \tan(xy)$

Solution

$$\frac{\partial w}{\partial x} = 2x\tan(xy) + x^2 y\sec^2(xy)$$

$$\frac{\partial^2 w}{\partial x^2} = 2\tan(xy) + 2xy\sec^2(xy) + 2xy\sec^2(xy) + 2x^2 y\sec(xy)\frac{\partial}{\partial x}\sec(xy)$$

$$= 2\tan(xy) + 4xy\sec^2(xy) + 2x^2 y\sec(xy)\sec(xy)\tan(xy)\frac{\partial}{\partial x}(xy)$$

$$= 2\tan(xy) + 4xy\sec^2(xy) + 2x^2 y^2\sec^2(xy)\tan(xy)$$

$$\frac{\partial w}{\partial y} = x^3\sec^2(xy)$$

$$\frac{\partial^2 w}{\partial y^2} = 2x^3\sec(xy)\left[x\sec(xy)\tan(xy)\right]$$

$$= 2x^4\sec^2(xy)\tan(xy)$$

$$\frac{\partial^2 w}{\partial y\partial x} = \frac{\partial^2 w}{\partial x\partial y} = 3x^2\sec^2(xy) + x^3\left(2\sec(xy)\sec(xy)\tan(xy)\cdot y\right)$$

$$= 3x^2\sec^2(xy) + 2x^3 y\sec^2(xy)\tan(xy)$$

Exercise

Find all the second-order partial derivatives of $w = ye^{x^2-y}$

Solution

$$\frac{\partial w}{\partial x} = 2xye^{x^2-y}$$

$$\frac{\partial^2 w}{\partial x^2} = 2ye^{x^2-y} + 4x^2 ye^{x^2-y}$$

$$= 2ye^{x^2-y}\left(1+2x^2\right)$$

$$\frac{\partial w}{\partial y} = e^{x^2-y} - ye^{x^2-y}$$

$$= e^{x^2-y}(1-y)$$

$$\frac{\partial^2 w}{\partial y^2} = -e^{x^2-y}(1-y) - e^{x^2-y}$$

$$= e^{x^2 - y}(-1 + y - 1)$$
$$\boxed{= (y-2)e^{x^2 - y}}$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}$$
$$= 2xe^{x^2-y} - 2xye^{x^2-y}$$
$$\boxed{= 2x(1-y)e^{x^2-y}}$$

Exercise

Find second-order partial derivatives of the function $g(x,y) = y + \frac{x}{y}$

Solution

$$\boxed{g_x = \frac{1}{y}} \qquad \boxed{g_y = 1 - \frac{x}{y^2}}$$

$$\boxed{g_{xx} = 0} \qquad \boxed{g_y = \frac{2x}{y^3}}$$

$$\boxed{g_{xy} = g_{yx} = -\frac{1}{y^2}}$$

Exercise

Find second-order partial derivatives of the function $g(x,y) = e^x + y\sin x$

Solution

$$\boxed{g_x = e^x + y\cos x} \qquad \boxed{g_y = \sin x}$$

$$\boxed{g_{xx} = e^x - y\sin x} \qquad \boxed{g_y = 0}$$

$$\boxed{g_{xy} = g_{yx} = \cos x}$$

Exercise

Find second-order partial derivatives of the function $f(x,y) = y^2 - 3xy + \cos y + 7e^y$

Solution

$$\boxed{f_x = -3y} \qquad \boxed{f_y = 2y - 3x - \sin y + 7e^y}$$

$\boxed{f_{xx} = 0}$ $\boxed{g_y = 2 - \cos y + 7e^y}$

$\boxed{f_{xy} = f_{yx} = -3}$

Exercise

Verify that the function satisfies Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$u(x, y) = y\left(3x^2 - y^2\right)$$

Solution

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(3x^2 y - y^3\right)$$

$$= 6xy$$

$$\frac{\partial^2 u}{\partial x^2} = 6y$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(3x^2 y - y^3\right)$$

$$= 3x^2 - 3y^2$$

$$\frac{\partial^2 u}{\partial y^2} = -6y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y - 6y$$

$$\boxed{= 0} \quad \checkmark$$

$\therefore$ The given function satisfies Laplace's equation

Exercise

Verify that the function satisfies Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$u(x, y) = \ln\left(x^2 + y^2\right)$$

Solution

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\ln\left(x^2 + y^2\right)$$

$$= \frac{2x}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right)$$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right) = \frac{(ae-bd)x^2+2(af-dd)x+(bf-ce)}{\left(dx^2+ex+f\right)^2}$$

$$= \frac{-2x^2+2y^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\ln\left(x^2+y^2\right)$$

$$= \frac{2y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right)$$

$$= \frac{2x^2-2y^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{-2x^2+2y^2}{\left(x^2+y^2\right)^2} + \frac{2x^2-2y^2}{\left(x^2+y^2\right)^2}$$

$$= 0 \quad \checkmark$$

$\therefore$ The given function satisfies Laplace's equation

Exercise

Let $f(x,y) = 2x+3y-4$. Find the slope of the line tangent to this surface at the point $(2,\ -1)$ and lying in the **a**. plane $x=2$ **b**. plane $y=-1$.

Solution

a) In the plane $x=2$

$$m = f_y\Big|_{(2,-1)} = 3$$

b) In the plane $y=-1$

$$m = f_z\Big|_{(2,-1)} = 2$$

Exercise

Let $w = f(x,y,z)$ be a function of three independent variables and writs the formal definition of the partial derivative $\frac{\partial f}{\partial y}$ at (x_0, y_0, z_0). Use this definition to find $\frac{\partial f}{\partial y}$ at $(-1,\ 0,\ 3)$ for $f(x,y,z) = -2xy^2 + yz^2$.

Solution

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x_0, y_0 + h, z_0) - f(x_0, y_0, z_0)}{h}$$

$$\begin{aligned}
f_y(-1,0,3) &= \lim_{h\to 0} \frac{f(-1,0+h,3) - f(-1,0,3)}{h} \\
&= \lim_{h\to 0} \frac{-2(-1)h^2 + h(3)^2 - (0+0)}{h} \\
&= \lim_{h\to 0} \frac{2h^2 + 9h}{h} \\
&= \lim_{h\to 0} (2h+9) \\
&= 9
\end{aligned}$$

Exercise

Find the value of $\frac{\partial x}{\partial z}$ at the point $(1,-1,-3)$ if the equation $xz + y\ln x - x^2 + 4 = 0$ defines x as a function of the two independent variables y and z and the partial derivative exists.

Solution

$$\frac{\partial x}{\partial z} z + x + y\left(\frac{1}{x}\right)\frac{\partial x}{\partial z} - 2x\frac{\partial x}{\partial z} = 0$$

$$\left(z + \frac{y}{x} - 2x\right)\frac{\partial x}{\partial z} = -x$$

$$\Rightarrow \frac{\partial x}{\partial z} = -\frac{x}{z + \frac{y}{x} - 2x}$$

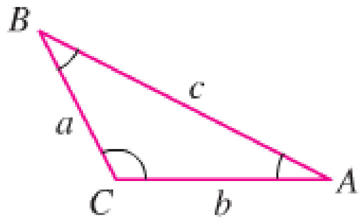
$$\left.\frac{\partial x}{\partial z}\right|_{(1,-1,-3)} = -\frac{1}{-3 + \frac{-1}{1} - 2}$$

$$= \frac{1}{6}$$

Exercise

Express A implicitly as a function of a, b, and c and calculate $\frac{\partial A}{\partial a}$ and $\frac{\partial A}{\partial b}$.

Solution



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\frac{\partial}{\partial a}\left(a^2 = b^2 + c^2 - 2bc\cos A\right)$$

$$2a = \left(2bc\sin A\right)\frac{\partial A}{\partial a} \quad \rightarrow \quad \boxed{\frac{\partial A}{\partial a} = \frac{a}{bc\sin A}}$$

$$\frac{\partial}{\partial b}\left(a^2 = b^2 + c^2 - 2bc\cos A\right)$$

$$0 = 2b - 2c\cos A + 2bc\sin A\left(\frac{\partial A}{\partial b}\right)$$

$$\left(\frac{\partial A}{\partial b}\right) = \frac{c\cos A - b}{bc\sin A}$$

Exercise

An important partial differential equation that describes the distribution of heat in a region at time t can be represented by the one-dimensional heat equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

Show that $u(x,t) = \sin(\alpha x)\cdot e^{-\beta t}$ satisfies the heat equation for constants α and β. What is the relationship between α and β for this function to be a solution?

Solution

$$u_t = -\beta\sin(\alpha x)\cdot e^{-\beta t}$$

$$u_x = \alpha\cos(\alpha x)\cdot e^{-\beta t}$$

$$u_{xx} = -\alpha^2\sin(\alpha x)\cdot e^{-\beta t}$$

$$\text{For } \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \quad \rightarrow \quad u_t = u_{xx}$$

$$-\beta\sin(\alpha x)\cdot e^{-\beta t} = -\alpha^2\sin(\alpha x)\cdot e^{-\beta t}$$

$$\Rightarrow \boxed{\beta = \alpha^2}$$