Section 3.4 – Properties of Logarithms

Product Rule

$$\log_b MN = \log_b M + \log_b N \qquad \textbf{\textit{For } } M > 0 \textbf{\textit{ and }} N > 0$$

$$\begin{cases} \log_b M = x \;\Rightarrow\; M = b^x \\ \log_b N = y \;\Rightarrow\; N = b^y \end{cases} \Rightarrow \text{MN} = b^x\, b^y = b^{x+y}$$

Convert back to logarithmic form: $\log_b MN = x + y$

$$\log_b MN = \log_b M + \log_b N$$

Example

Use the product rule to expand the logarithmic expression

$$\log(100x) = \log 100 + \log x$$

Power Rule

$$\log_b M^{\,p} = p \log_b M$$

Example

Use the power rule to expand each logarithmic expression

$$\ln \sqrt[3]{x} = \ln(x)^{1/3} = \frac{1}{3}\ln x$$

Quotient Rule

$$\log_b \frac{M}{N} = \log_b M \;-\; \log_b N$$

Example

Use the quotient rule to expand the logarithmic expression

$$\ln\left(\frac{e^5}{11}\right) = \ln e^5 - \ln 11$$

$$= 5 - \ln 11$$

Example

Express each of the following in terms of sums and differences of logarithm: $\log_6 (7 \times 9)$

Solution

$$\log_6 (7 \times 9) = \log_6 7 + \log_6 9$$ *Product Rule*

Example

Express each of the following in terms of sums and differences of logarithm: $\log_9 \left(\frac{15}{7}\right)$

Solution

$$\log_9 \left(\frac{15}{7}\right) = \log_9 15 - \log_9 7$$ *Quotient Rule*

Example

Express each of the following in terms of sums and differences of logarithm: $\log_5 \sqrt{8}$

Solution

$$\log_5 \sqrt{8} = \log_5 \left(2^3\right)^{1/2}$$

$$= \log_5 2^{3/2}$$ *Power Rule*

$$= \frac{3}{2} \log_5 2$$

Example

Express each of the following in terms of sums and differences of logarithm: $\log_b \left(x^4 \sqrt[3]{y}\right)$

Solution

$$\log_b \left(x^4 \sqrt[3]{y}\right) = \log_b \left(x^4\right) + \log_b \left(\sqrt[3]{y}\right)$$ *Product Rule*

$$= \log_b \left(x^4\right) + \log_b \left(y^{1/3}\right)$$ *Power Rule*

$$= 4 \log_b x + \frac{1}{3} \log_b y$$

Example

Express each of the following in terms of sums and differences of logarithm: $\log_a\left(\frac{mnq}{p^2r^4}\right)$

Solution

$$\log_a\left(\frac{mnq}{p^2r^4}\right) = \log_a(mnq) - \log_a\left(p^2r^4\right) \qquad \textit{Quotient Rule}$$

$$= \log_a m + \log_a n + \log_a q - \left(\log_a p^2 + \log_a r^4\right) \qquad \textit{Product Rule}$$

$$= \log_a m + \log_a n + \log_a q - \log_a p^2 - \log_a r^4$$

$$\underline{= \log_a m + \log_a n + \log_a q - 2\log_a p - 4\log_a r} \qquad \textit{Power Rule}$$

Example

Express each of the following in terms of sums and differences of logarithm: $\log_5\left(\frac{\sqrt{x}}{25y^3}\right)$

Solution

$$\log_5\left(\frac{\sqrt{x}}{25y^3}\right) = \log_5\left(x^{1/2}\right) - \log_5\left(25y^3\right) \qquad \textit{Quotient Rule}$$

$$= \log_5\left(x^{1/2}\right) - \left[\log_5\left(5^2\right) + \log_5\left(y^3\right)\right] \qquad \textit{Product Rule}$$

$$= \log_5\left(x^{1/2}\right) - \log_5\left(5^2\right) - \log_5\left(y^3\right) \qquad \textit{I}\log_5\left(5^2\right) = 2$$

$$\underline{= \frac{1}{2}\log_5 x - 2 - 3\log_5 y}$$

Example

Write as a single logarithmic $\log(7x+6) - \log x$

Solution

$$\log(7x+6) - \log x = \log\frac{7x+6}{x}$$ *Quotient Rule*

Example

Write as a single logarithmic $\log_3(x+2) + \log_3 x - \log_3 2$

Solution

$$\log_3(x+2) + \log_3 x - \log_3 2 = \log_3 x(x+2) - \log_3 2$$ *Product Rule*

$$= \log_3 \frac{x(x+2)}{2}$$ *Quotient Rule*

Example

Write as a single logarithmic $2\ln x + \frac{1}{3}\ln(x+5)$

Solution

$$2\ln x + \frac{1}{3}\ln(x+5) = \ln x^2 + \ln(x+5)^{1/3}$$ *Power Rule*

$$= \ln x^2\,(x+5)^{1/3}$$ *Product Rule*

$$= \ln\left(x^2\sqrt[3]{x+5}\right)$$

Example

Write as a single logarithmic $2\log(x-3) - \log x$

Solution

$$2\log(x-3) - \log x = \log(x-3)^2 - \log x$$ *Power Rule*

$$= \log\frac{(x-3)^2}{x}$$ *Quotient Rule*

Exercises *Section* 3.4 – Properties of Logarithms

(**1 – 31**) Express the following in terms of sums and differences of logarithms

1. $\log_3(ab)$

2. $\log_7(7x)$

3. $\log\dfrac{x}{1000}$

4. $\log_5\left(\dfrac{125}{y}\right)$

5. $\log_b x^7$

6. $\ln\sqrt[7]{x}$

7. $\log_a \dfrac{x^2 y}{z^4}$

8. $\log_b \dfrac{x^2 y}{b^3}$

9. $\log_b\left(\dfrac{x^3 y}{z^2}\right)$

10. $\log_b\left(\dfrac{\sqrt[3]{x}\,y^4}{z^5}\right)$

11. $\log\left(\dfrac{100x^3\sqrt[3]{5-x}}{3(x+7)^2}\right)$

12. $\log_a \sqrt[4]{\dfrac{m^8 n^{12}}{a^3 b^5}}$

13. $\log_p \sqrt[3]{\dfrac{m^5 n^4}{t^2}}$

14. $\log_b \sqrt[n]{\dfrac{x^3 y^5}{z^m}}$

15. $\log_a \sqrt[3]{\dfrac{a^2 b}{c^5}}$

16. $\log_b\left(x^4\sqrt[3]{y}\right)$

17. $\log_5\left(\dfrac{\sqrt{x}}{25y^3}\right)$

18. $\log_a \dfrac{x^3 w}{y^2 z^4}$

19. $\log_a \dfrac{\sqrt{y}}{x^4\sqrt[3]{z}}$

20. $\ln\sqrt[4]{\dfrac{x^7}{y^5 z}}$

21. $\ln x\sqrt[3]{\dfrac{y^4}{z^5}}$

22. $\log_b \sqrt[5]{\dfrac{m^4 n^5}{x^2 a b^{10}}}$

23. $\log_b \dfrac{a^5 b^{10}}{c^2\sqrt[4]{d^3}}$

24. $\ln\left(x^2\sqrt{x^2+1}\right)$

25. $\ln\dfrac{x^2}{x^2+1}$

26. $\ln\left(\dfrac{x^2(x+1)^3}{(x+3)^{1/2}}\right)$

27. $\ln\sqrt{\dfrac{(x+1)^5}{(x+2)^{20}}}$

28. $\ln\dfrac{\left(x^2+1\right)^5}{\sqrt{1-x}}$

29. $\ln\left(\sqrt[3]{\dfrac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}}\right)$

30. $\ln\left(\sqrt{\dfrac{1}{x(x+1)}}\right)$

31. $\ln\left(\sqrt{\left(x^2+1\right)(x-1)^2}\right)$

(**32 – 55**) Write the expression as a single logarithm and simplify if necessary

32. $\log(x+5)+2\log x$

33. $3\log_b x-\frac{1}{3}\log_b y+4\log_b z$

34. $\frac{1}{2}\log_b(x+5)-5\log_b y$

35. $\ln\left(x^2-y^2\right)-\ln(x-y)$

36. $\ln(xz)-\ln\left(x\sqrt{y}\right)+2\ln\dfrac{y}{z}$

37. $\log\left(x^2 y\right)-\log z$

38. $\log\left(z^2\sqrt{y}\right)-\log z^{1/2}$

39. $2\log_a x+\frac{1}{3}\log_a(x-2)-5\log_a(2x+3)$

40. $5\log_a x-\frac{1}{2}\log_a(3x-4)-3\log_a(5x+1)$

41. $\log\left(x^3y^2\right)-2\log\left(x\sqrt[3]{y}\right)-3\log\left(\frac{x}{y}\right)$

42. $\ln y^3+\frac{1}{3}\ln\left(x^3y^6\right)-5\ln y$

43. $2\ln x-4\ln\left(\frac{1}{y}\right)-3\ln(xy)$

44. $4\ln x+7\ln y-3\ln z$

45. $\frac{1}{3}\left[5\ln(x+6)-\ln x-\ln(x^2-25)\right]$

46. $\frac{2}{3}\left[\ln\left(x^2-4\right)-\ln(x+2)\right]+\ln(x+y)$

47. $\frac{1}{2}\log_b m+\frac{3}{2}\log_b 2n-\log_b m^2n$

48. $\frac{1}{2}\log_y p^3q^4-\frac{2}{3}\log_y p^4q^3$

49. $\frac{1}{2}\log_a x+4\log_a y-3\log_a x$

50. $\frac{2}{3}\left[\ln\left(x^2-9\right)-\ln(x+3)\right]+\ln(x+y)$

51. $\frac{1}{4}\log_b x-2\log_b 5-10\log_b y$

52. $2\ln(x+4)-\ln x-\ln\left(x^2-3\right)$

53. $\ln x+\ln(y+3)+\ln(y+2)-\ln\left(y^2+5y+6\right)$

54. $\ln x+\ln(x+4)+\ln(x+1)-\ln\left(x^2+5x+4\right)$

55. $\ln\left(x^2-25\right)-2\ln(x+5)+\ln(x-5)$

56. Assume that $\log_{10} 2=.3010$. Find each logarithm $\log_{10} 4$, $\log_{10} 5$

57. Given that: $\log_a 2\approx 0.301, \log_a 7\approx 0.845$, and $\log_a 11\approx 1.041$ find each of the following:

a) $\log_a \frac{2}{11}$

b) $\log_a 14$

c) $\log_a 98$

d) $\log_a \frac{1}{7}$

e) $\log_a 9$

f) $\log_a \frac{77}{8}$