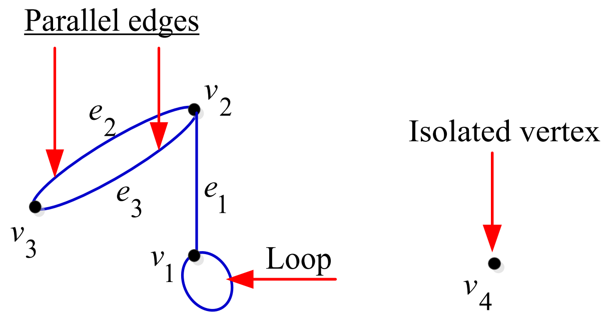
Section 4.6 – Graphs: Definitions and Basic Properties

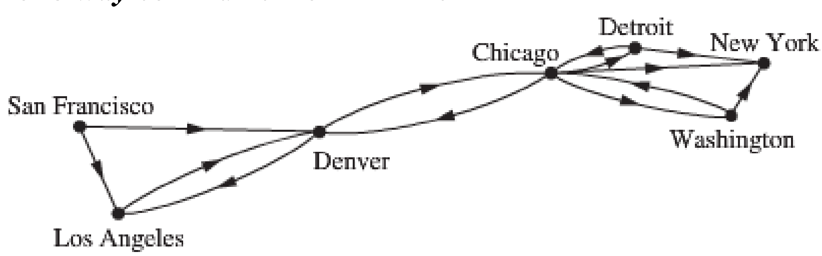
Definition

A graph $G=(V,\ E)$ consists of *V*, a nonempty set of vertices (or nodes) and *E*, a set of edges. Each edge has either one or two ***vertices*** (plural of ***vertex***) associated with it, called its ***endpoints***. An edge is said to connect its endpoints.
Visualize the graphs by using points to represent vertices and line segments, possibly curved, to represent edges, where the endpoints of a line segment representing an edge are the points representing the ***edge-endpoints***.



To model a computer network, we need graphs that have more than one edge connecting the same pair of vertices. Graphs that may have ***multiple edges*** connecting the same vertices are called ***multigraphs***.
Sometimes a communications link connects a data center with itself, a feedback loop for diagnostic purposes. Such edges are called ***loops***.

Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices or a vertex to itself, are sometimes called ***pseudographs***.
Sometimes we have a ***one-way*** communication link like



Basic Terminology

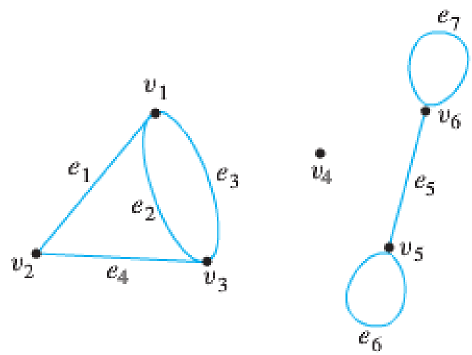
Definition

Two vertices *u* and *v* in an undirected graph *G* are called adjacent (or neighbors) in *G* if *u* and *v* are endpoints of an edge *e* of *G*. Such an edge *e* is called incident with the vertices *u* and *v* and *e* is said to connect *u* and *v*.

Graph Terminology			
Type	***Edges***	***Multiple Edges Allowed?***	***Loops Allowed?***
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

Example

Consider the following graph:



a) Write the vertex set and the edge set, and give a table showing the edge-point function.

b) Find all edges that are incident on v_1, all vertices that are adjacent to v_1, all edges that are adjacent to e_1, all loops, all parallel edges, all vertices that are adjacent to themselves, and all isolated vertices.

<u>Solution</u>

a) Vertex set = $\{v_1, v_2, v_3, v_4, v_5, v_6\}$

Edge set = $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

Edge-point function:

Edge	***Endpoints***
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_1, v_3\}$
e_4	$\{v_2, v_3\}$
e_5	$\{v_5, v_6\}$
e_6	$\{v_5\}$
e_7	$\{v_6\}$

b) e_1, e_2, and e_3 are incident on v_1

v_2 and v_3 are adjacent to v_1

e_2, e_3, and e_4 are adjacent to e_1

e_6 and e_7 are loops.

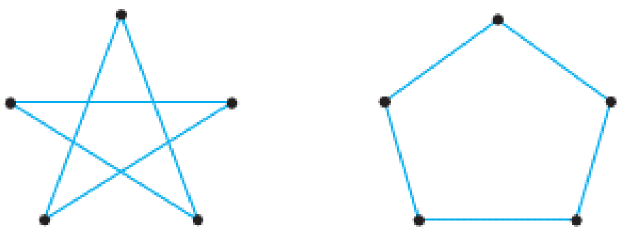
e_2 and e_3 are parallel.

v_5 and v_6 are adjacent to themselves.

v_4 is an isolated vertex.

Example

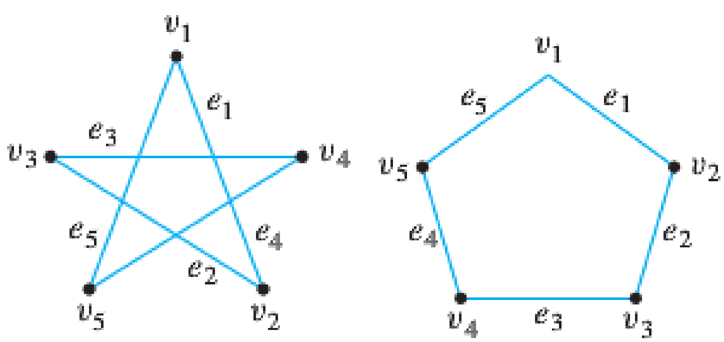
Consider the two drawing shown below.



(a) (b)

Label vertices and edges in such a way that both drawings represent the same graph

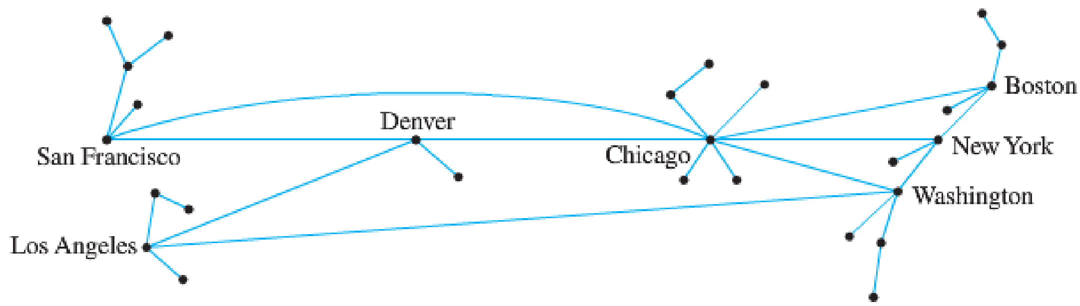
Solution



Edge	***Endpoints***
e_1	$\{v_1, v_2\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_3, v_4\}$
e_4	$\{v_4, v_5\}$
e_5	$\{v_5, v_1\}$

Definition

A ***directed graph*** (or ***digraph***) (V, E) consists of a nonempty set of vertices V and a set of ***directed edges*** (or ***arcs***) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to *start* at u and *end* at v.



Special Graphs

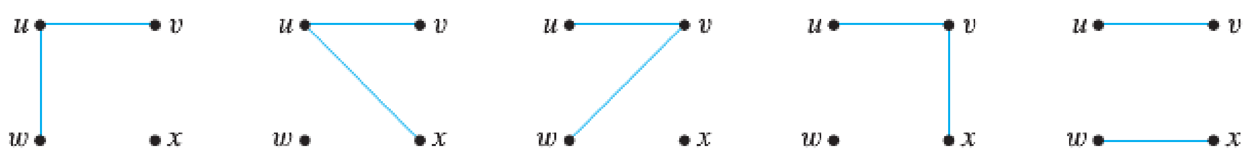
Definition

A ***simple graph*** is a graph that does not have any loops or parallel edges. In a simple graph, an edge with endpoints v and w is denoted $\{u, v\}$

Example

Draw all simple graphs with the four vertices $\{u, v, w, x\}$ and two edges, one of which is $\{u, v\}$

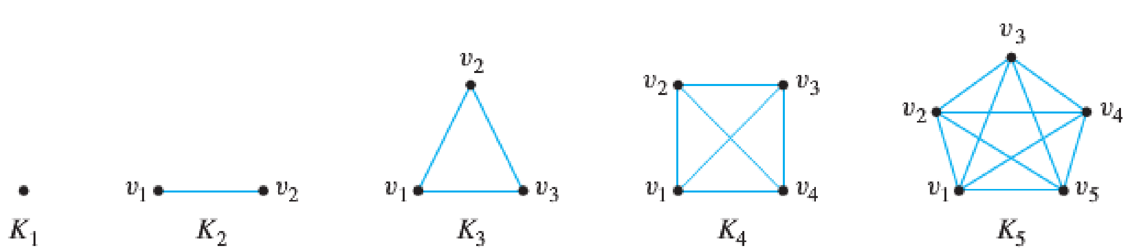
Solution



Definition

Let n be a positive integer. A ***complete graph*** on ***n vertices***, denoted $\boldsymbol{K}_n$ is a simple graph with n vertices and exactly one edge connecting each pair of distinct vertices.

The complete graphs $K_1, K_2, K_3, K_4,$ *and* K_5 can be drawn as follows



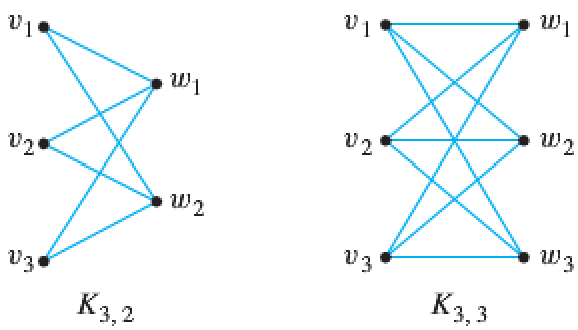
Definition

Let m and n be a positive integers. A ***complete bipartite graph*** on **(*m, n*)** ***vertices***, denoted $K_{m,n}$ is a simple graph with distinct vertices $v_1, v_2, \ldots, v_m$ *and* $w_1, w_2, \ldots, w_n$ that satisfies the following properties:

For all $i, k = 1, 2, \ldots, m$ and for all $j, l = 1, 2, \ldots, n$,

1. There is an edge from each vertex v_i to each vertex w_j
2. There is no edge from each vertex v_i to any other vertex v_k
3. There is no edge from each vertex w_j to any other vertex w_l

Example



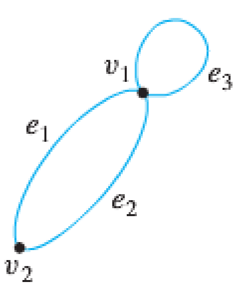
Definition

A graph H is said to be a subgraph of a graph if, and only if, every vertex in H is also a vertex in G, every edge in H is also an edge in G, and every edge in H has the same endpoints as it has in G.

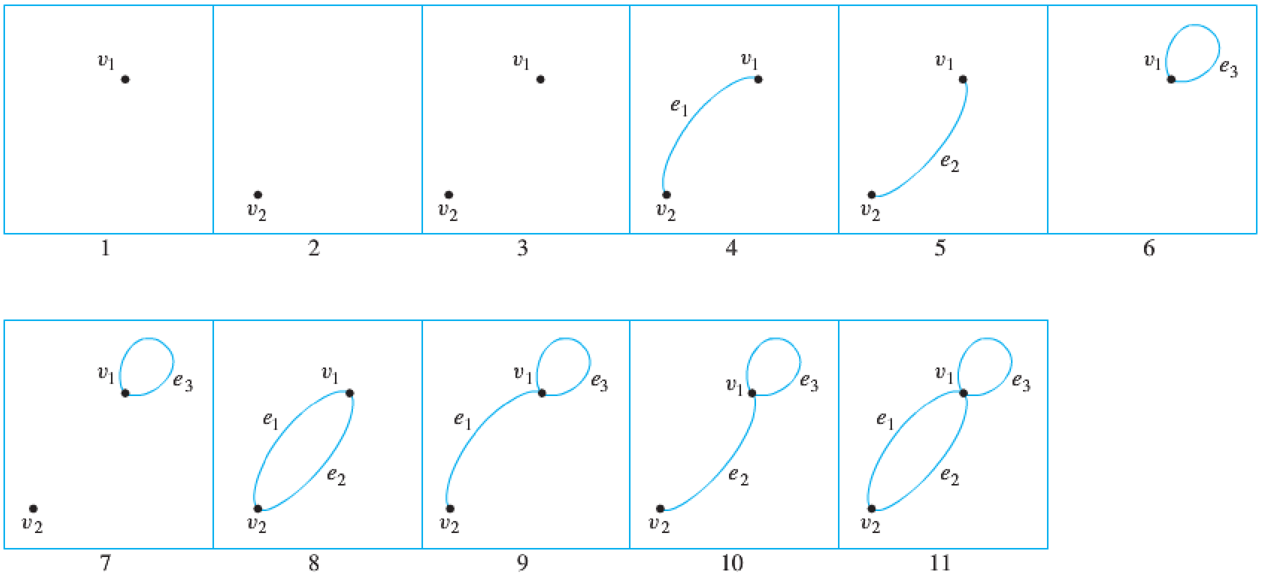
Example

List all subgraphs of the graph G with vertex set $\{v_1, v_2\}$ and edge set $\{e_1, e_2, e_3\}$ where the endpoints of e_1 are v_1 and v_2, the endpoints of e_2 are v_1 and v_2 and e_3 is a loop at v_1.

<u>Solution</u>



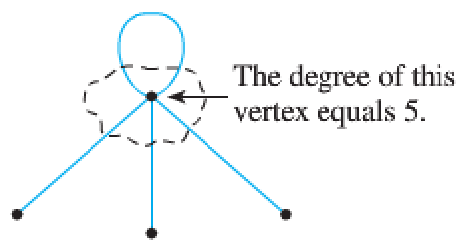
There are 11 subgraphs of G, which can be grouped according to those that do not have any edges, those that have one edge, those that have 2 edges, and those that have 3 edges.



The concept of Degree

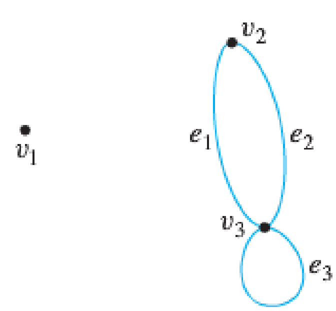
Definition

Let G be a graph and v a vertex of G. The ***degree of v***, denoted $\mathbf{deg}(v)$, equals the number of edges that are incident on v, with an edge that is a loop counted twice. The *total degree* of G is the sum of the degrees of all vertices of G.



Example

Find the degree of each vertex of the graph G shown below. Then find the total degree of G.



Solution

$\deg(v_1) = 0$ since no edge is incident on v_1 (v_1 *is isolated*)

$\deg(v_2) = 2$ since both e_1 and e_2 are incident on v_2

$\deg(v_3) = 4$ since both e_1 and e_2 are incident on v_3 and the loop e_3 is also incident on v_3 (contributes 2 to the degree of v_3)

Total degree of $G = \deg(v_1) + \deg(v_2) + \deg(v_3) = 0 + 2 + 4 = 6$

The Handshake *Theorem*

If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges of G. Specially, if the vertices of G are $v_1, v_2, \ldots, v_n$, where n is a nonegative integer, then

$$\begin{aligned}\text{The total degree of G } &= \deg(v_1) + \deg(v_2) + \ \ldots + \deg(v_n) \\ &= 2 \cdot (\textit{the number of edges of } G)\end{aligned}$$

Corollary

The total degree of a graph is even.

Example

Draw a graph with the specified properties or show that no such graph exists.

a) A graph with four vertices of degrees 1, 1, 2, and 3

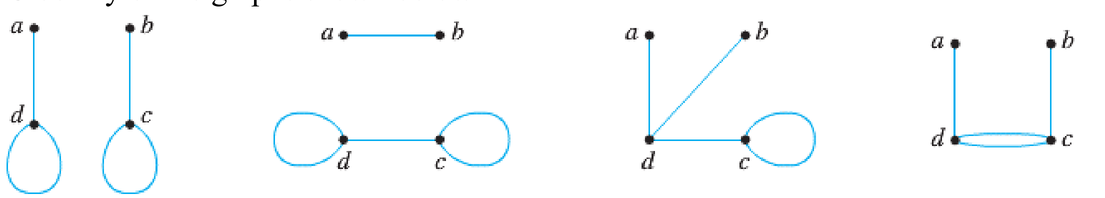
b) A graph with four vertices of degrees 1, 1, 3, and 3

c) A simple graph with four vertices of degrees 1, 1, 2, and 3

Solution

a) No such graph is possible. By Corollary, the total degree of a graph is even.
But a graph with four vertices of degrees 1, 1, 2, and 3 would have a total degree of $1+1+2+3=7$ which is odd.

b) Let G be any of the graphs shown below



In each case, no matter how the edges are labeled, $\deg(a) = \deg(b) = 1$ and $\deg(c) = \deg(d) = 3$

c) There is no simple graph with four vertices of degrees 1, 1, 3, and 3.

Example

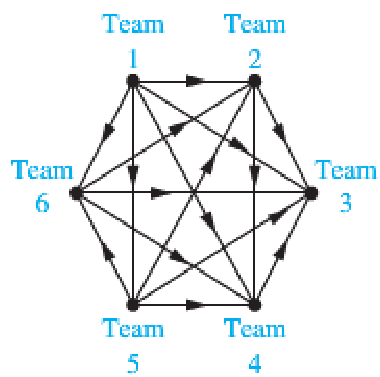
Is it possible in a group of 9 people for each to be friends with exactly five others?

Solution

Imagine constructing an "acquaintance graph" in which each of the nine people represented by a vertex and 2 vertices are joined by an edge if, and only if, the people they represent are friends. Suppose each of the people were friends with exactly five others. Then the degree of each of the 9 vertices of the graph would be 5, and so the total degree of the graph would be **45** (*odd*). Contradicts Corollary, which says that the total degree of a graph is even.
Therefore, the answer is no.

Example

A tournament where each team plays every other team exactly once and no ties are allowed is called a round-robin tournament. Such tournaments can be modeled using directed graphs where each team is represented by a vertex, Note that (*a, b*) is an edge if team *a* beats team *b*. This graph is a simple directed graph, containing no loops or multiple directed edges (because no 2 teams play each other more than once. Such a directed graph model is presented

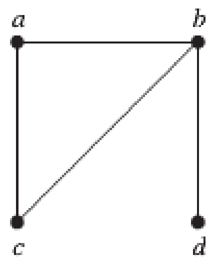


We see that team 1 is undefeated in this tournament, and Team 3 is winless.

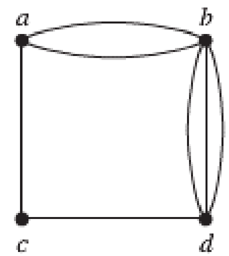
Exercises *Section* 4.6 – Graphs: Definitions and Basic Properties

1. Determine whether the graph shown has directed or undirected edges, whether it has multiple edges, and whether it has one or more loops.

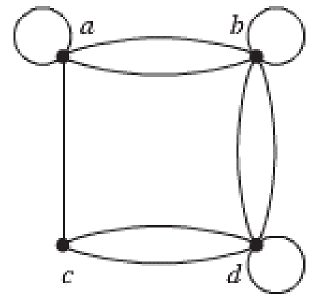
a)



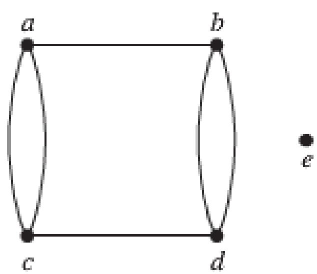
b)



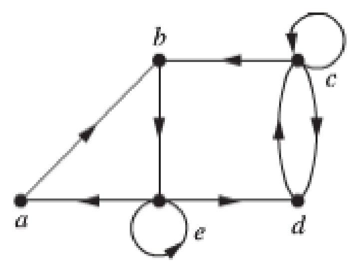
c)



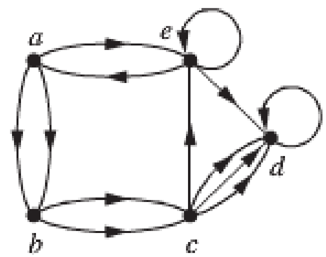
d)



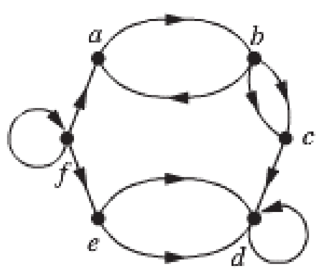
e)



f)

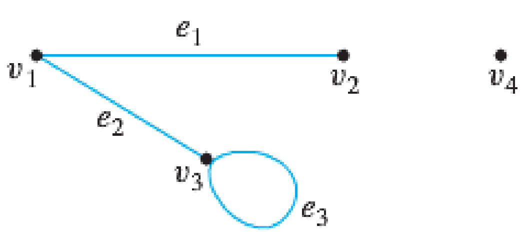


g)

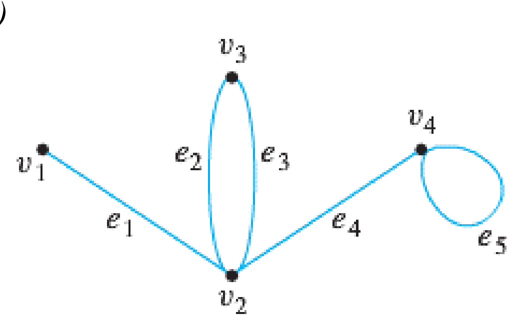


2. Define each graph formally by specifying its vertex set, its edge set, and a table giving the edge-endpoint function

a)



b)



3. Graph G has vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{e_1, e_2, e_3, e_4\}$, with edge-endpoint function as follow

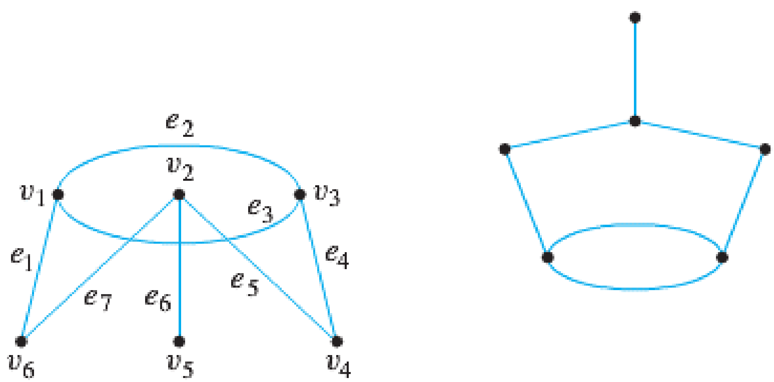
Edge	***Endpoints***
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_2\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_2\}$

4. Graph H has vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{e_1, e_2, e_3, e_4\}$, with edge-endpoint function as follow

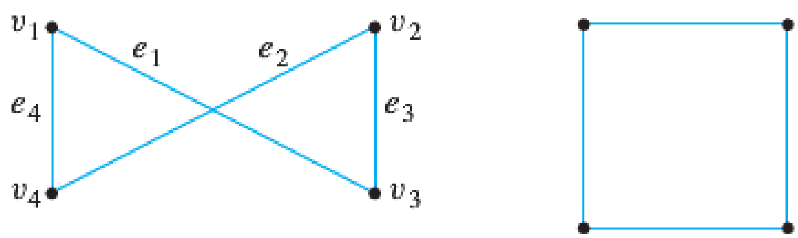
Edge	***Endpoints***
e_1	$\{v_1\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_1, v_5\}$

5. Show that the 2 drawings represent the same graph by labeling the vertices and edges of the right-hand drawing to correspond to those of the left-hand drawing.

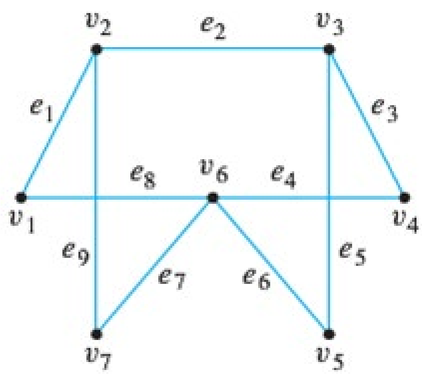
a)

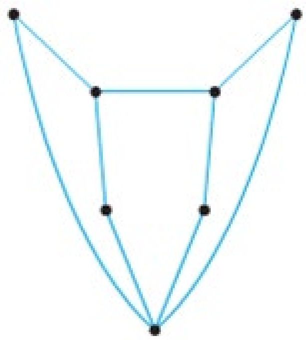


b)

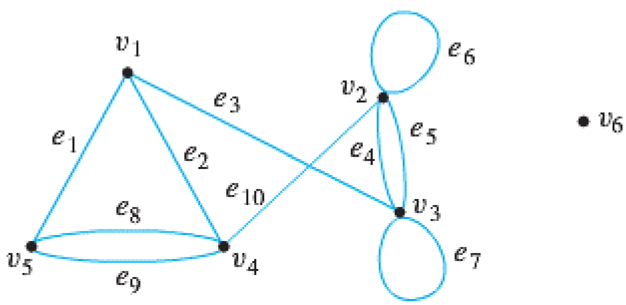


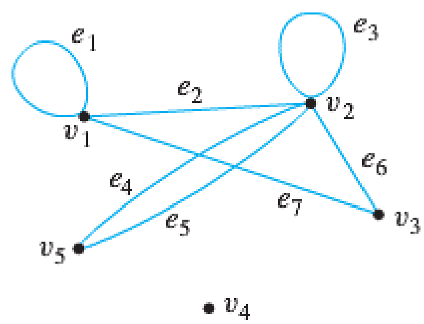
c)





6. For each of the graphs
   i. Find all edges that are incident on v_1
   ii. Find all vertices that are adjacent to v_3
   iii. Find all edges that are adjacent to e_1
   iv. Find all loops
   v. Find all parallel edges
   vi. Find all isolated vertices
   vii. Find the degree of v_3
   viii. Find the total degree of the graph





7. Let G be a simple graph. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, irreflexive relation on G.

8. Let G be an undirected graph with a loop at every vertex. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, reflexive relation on G.

9. Explain how graphs can be used to model electronic mail messages in a network. Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed? Describe a graph that models the electronic mail sent in a network in a particular week.

10. A bipartite graph G is a simple graph whose vertex set can be portioned into two disjoint nonempty subsets V_1 and V_2 such that vertices in V_1 may be connected to vertices in V_2, but no vertices in

V_1 are connected to other vertices in V_1 and no vertices in V_2 are connected to other vertices in V_2. For example, the graph *G* illustrated in (*i*) can be redrawn as shown in (*ii*). From the drawing in (*ii*), you can see that *G* is bipartite with mutually disjoint vertex set $V_1 = \{v_1, v_3, v_5\}$ and $V_1 = \{v_2, v_4, v_6\}$