Formulas

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Derivative

Formula $$\left(U^m V^n W^p\right)' = U^{m-1}V^{n-1}W^{p-1}\left(mU'VW + nUV'W + pUVW'\right)$$

Proof

$$\begin{aligned}\left(U^m V^n W^p\right)' &= \left(U^m\right)' V^n W^p + U^m\left(V^n\right)' W^p + U^m V^n\left(W^p\right)' \\ &= mU^{m-1}U'V^nW^p + nU^mV^{n-1}V'W^p + pU^mV^nW^{p-1}W' \qquad \textit{factor}\ \ U^{m-1}V^{n-1}W^{p-1} \\ &= U^{m-1}V^{n-1}W^{p-1}\left(mU'VW + nUV'W + pUVW'\right)\end{aligned}$$

Derivative: Rational Function to Power 'n' in the form $\frac{ax^n+b}{cx^n+d}$

$$\left(\frac{ax^n+b}{cx^n+d}\right)' = \boxed{\frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}}$$

$$= \frac{n\begin{vmatrix} a & b \\ c & d \end{vmatrix} x^{n-1}}{\left(cx^n+d\right)^2}$$

Proof

$$u = ax^n + b \qquad v = cx^n + d$$

$$u' = nax^{n-1} \qquad v' = ncx^{n-1}$$

$$\left(\frac{ax^n+b}{cx^n+d}\right)' = \frac{nax^{n-1}\left(cx^n+d\right) - ncx^{n-1}\left(ax^n+b\right)}{\left(cx^n+d\right)^2}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

$$= \frac{nacx^{2n-1} + nadx^{n-1} - nacx^{2n-1} - nbcx^{n-1}}{\left(cx^n+d\right)^2}$$

$$= \frac{nadx^{n-1} - nbcx^{n-1}}{\left(cx^n+d\right)^2}$$

$$= \boxed{\frac{n(ad-bc)x^{n-1}}{\left(cx^n+d\right)^2}}$$

Example

Find $\left(\frac{x+2}{3x-2}\right)'$

Solution

$$\left(\frac{x+2}{3x-2}\right)' = \frac{-2-6}{(3x-2)^2}$$

$$= \boxed{\frac{-8}{(3x-2)^2}}$$

Derivative: Rational Function to Power 'n' in the form $\left(\frac{ax^n+b}{cx^n+d}\right)^m$

$$\frac{d}{dx}\left(\frac{ax^n+b}{cx^n+d}\right)^m = mn(ad-bc)x^{n-1}\frac{\left(ax^n+b\right)^{m-1}}{\left(cx^n+d\right)^{m+1}}$$

Proof

$u = ax^n + b \qquad v = cx^n + d$

$u' = nax^{n-1} \qquad v' = ncx^{n-1}$

$$\frac{d}{dx}\left(\frac{ax^n+b}{cx^n+d}\right)^m = m\frac{nax^{n-1}\left(cx^n+d\right)-ncx^{n-1}\left(ax^n+b\right)}{\left(cx^n+d\right)^2}\left(\frac{ax^n+b}{cx^n+d}\right)^{m-1} \qquad \left(\frac{u}{v}\right)' = \frac{u'v-v'u}{v^2}$$

$$= \frac{m\left(nacx^{2n-1}+nadx^{n-1}-nacx^{2n-1}-nbcx^{n-1}\right)\left(ax^n+b\right)^{m-1}}{\left(cx^n+d\right)^2\left(cx^n+d\right)^{m-1}}$$

$$= \frac{m\left(nadx^{n-1}-nbcx^{n-1}\right)\left(ax^n+b\right)^{m-1}}{\left(cx^n+d\right)^{m+1}}$$

$$= \frac{mn(ad-bc)x^{n-1}\left(ax^n+b\right)^{m-1}}{\left(cx^n+d\right)^{m+1}}$$

Example

Find $\qquad \frac{d}{dx}\left(\frac{5x^2-3}{2x^2-4}\right)^5$

Solution

$$\frac{d}{dx}\left(\frac{5x^2-3}{2x^2-4}\right)^5 = \frac{-140x\left(5x^2-3\right)^4}{\left(2x^2-4\right)^6}$$

Derivative: in the form $\qquad y=\dfrac{ax^2+bx+c}{dx^2+ex+f}$

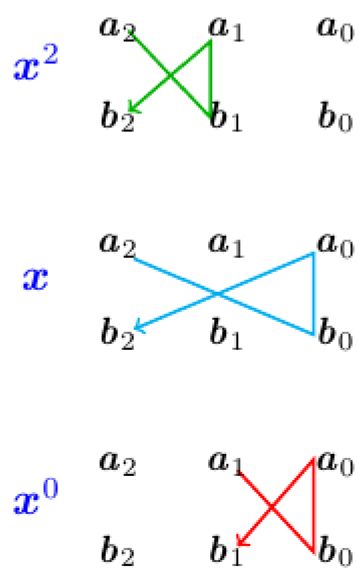
The numerator power of x is $2n-2$

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{dx^2+ex+f}\right)=\frac{(2ax+b)\left(dx^2+ex+f\right)-(2dx+e)\left(ax^2+bx+c\right)}{\left(dx^2+ex+f\right)^2}$$

$$=\frac{2adx^3+2aex^2+2afx+bdx^2+bex+bf-2adx^3-2bdx^2-2cdx-aex^2-bex-ce}{\left(dx^2+ex+f\right)^2}$$

$$=\frac{(ae-bd)x^2+2(af-cd)x+bf-ce}{\left(dx^2+ex+f\right)^2}$$

$$=\frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix}x^2+2\begin{vmatrix} a & c \\ d & f \end{vmatrix}x+\begin{vmatrix} b & c \\ e & f \end{vmatrix}}{\left(dx^2+ex+f\right)^2}$$



Example

$$f(x)=\frac{x^2-6x+8}{x^2-2x+1}$$

$$f'(x)=\frac{\begin{vmatrix} 1 & -6 \\ 1 & -2 \end{vmatrix}x^2+2\begin{vmatrix} 1 & 8 \\ 1 & 1 \end{vmatrix}x+\begin{vmatrix} -6 & 8 \\ -2 & 1 \end{vmatrix}}{\left(x^2-2x+1\right)^2}$$

$$=\frac{4x^2-14x+10}{\left(x^2-2x+1\right)^2}$$

Derivative: in the form $\quad f(x)=\dfrac{a_3x^3+a_2x^2+a_1x+a_0}{b_3x^3+b_2x^2+b_1x+b_0}$

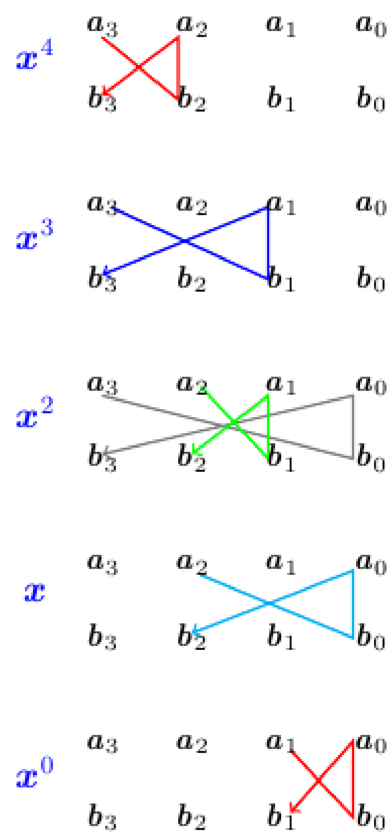
$$u=a_3x^3+a_2x^2+a_1x+a_0 \quad\rightarrow\quad u'=3a_3x^2+2a_2x+a_1$$
$$v=b_3x^3+b_2x^2+b_1x+b_0 \quad\rightarrow\quad v'=3b_3x^2+2b_2x+b_1$$

$$u'v-v'u=\left(3a_3x^2+2a_2x+a_1\right)\left(b_3x^3+b_2x^2+b_1x+b_0\right)$$
$$-\left(3b_3x^2+2b_2x+b_1\right)\left(a_3x^3+a_2x^2+a_1x+a_0\right)$$

x^5	x^4	x^3	x^2	x^1	x^0
$3a_3b_3$	$3a_3b_2$	$3a_3b_1$	$3a_3b_0$		
$-3a_3b_3$	$2a_2b_3$	$2a_2b_2$	$2a_2b_1$	$2a_2b_0$	
	$-3a_2b_3$	a_1b_3	a_1b_2	a_1b_1	a_1b_0
	$-2a_3b_2$	$-3a_1b_3$	$-3a_0b_3$		
		$-2a_2b_2$	$-2a_1b_2$	$-2a_0b_2$	
		$-a_3b_1$	$-a_2b_1$	$-a_1b_1$	$-a_0b_1$

$$f'(x)=\frac{\begin{gathered}\left(a_3b_2-a_2b_3\right)x^4+2\left(a_3b_1-a_1b_3\right)x^3+\left(\left(a_2b_1-a_1b_2\right)+3\left(a_3b_0-a_0b_3\right)\right)x^2\\+2\left(a_2b_0-a_0b_2\right)x+a_1b_0-a_0b_1\end{gathered}}{\left(b_3x^3+b_2x^2+b_1x+b_0\right)^2}$$

$$=\frac{\begin{vmatrix}a_3&a_2\\b_3&b_2\end{vmatrix}x^4+2\begin{vmatrix}a_3&a_1\\b_3&b_1\end{vmatrix}x^3+\left(\begin{vmatrix}a_2&a_1\\b_2&b_1\end{vmatrix}+3\begin{vmatrix}a_3&a_0\\b_3&b_0\end{vmatrix}\right)x^2+2\begin{vmatrix}a_2&a_0\\b_2&b_0\end{vmatrix}x+\begin{vmatrix}a_1&a_0\\b_1&b_0\end{vmatrix}}{\left(b_3x^3+b_2x^2+b_1x+b_0\right)^2}$$

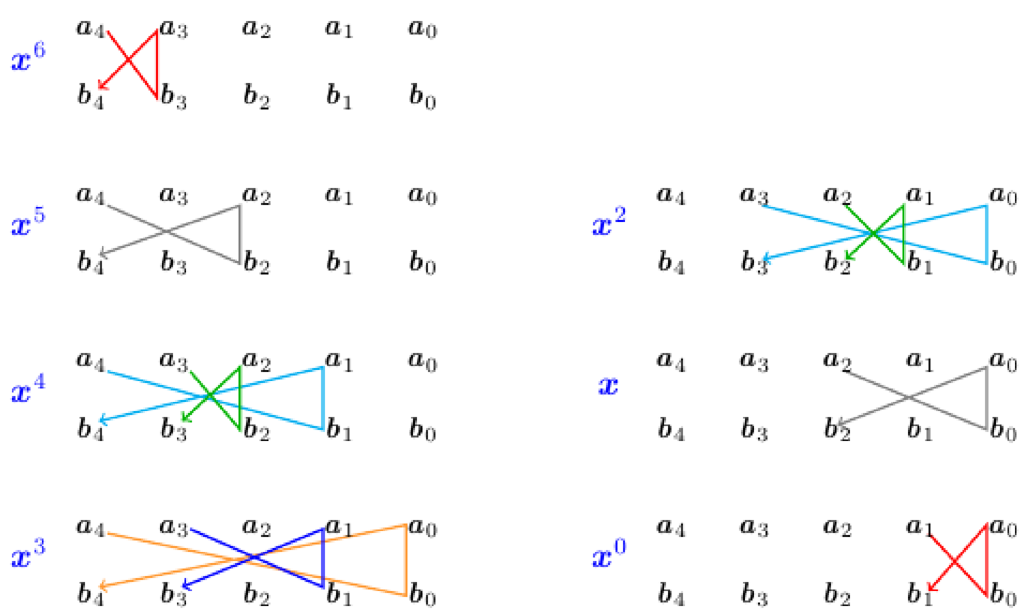

x^4
a_3 a_2 a_1 a_0
b_3 b_2 b_1 b_0
x^3
a_3 a_2 a_1 a_0
b_3 b_2 b_1 b_0
x^2
a_3 a_2 a_1 a_0
b_3 b_2 b_1 b_0
x
a_3 a_2 a_1 a_0
b_3 b_2 b_1 b_0
x^0
a_3 a_2 a_1 a_0
b_3 b_2 b_1 b_0

Derivative: in the form $\qquad f(x)=\dfrac{a_4x^4+a_3x^3+a_2x^2+a_1x+a_0}{b_4x^4+b_3x^3+b_2x^2+b_1x+b_0}$

$$\begin{aligned}u'v-v'u &= \left(4a_4x^3+3a_3x^2+2a_2x+a_1\right)\left(b_4x^4+b_3x^3+b_2x^2+b_1x+b_0\right)\\ &\quad -\left(4b_4x^3+3b_3x^2+2b_2x+b_1\right)\left(a_4x^4+a_3x^3+a_2x^2+a_1x+a_0\right)\end{aligned}$$

$$\begin{array}{ll}
x^7 & 4a_4b_4-4a_4b_4\\
x^6 & 4a_4b_3+3a_3b_4-4a_3b_4-3a_4b_3\\
x^5 & 4a_4b_2+3a_3b_3+2a_2b_4-4a_2b_4-3a_3b_3-2a_4b_2\\
x^4 & 4a_4b_1+3a_3b_2+2a_2b_3+a_1b_4-4a_1b_4-3a_2b_3-2a_3b_2-a_4b_1\\
x^3 & 4a_4b_0+3a_3b_1+2a_2b_2+a_1b_3-4a_0b_4-3a_1b_3-2a2b_2-a_3b_1\\
x^2 & 3a_3b_0+2a_2b_1+a_1b_2-3a_0b_3-2a_1b_2-a_2b_1\\
x^1 & 2a_2b_0+a_1b_1-2a_0b_2-a_1b_1\\
x^0 & a_1b_0-a_0b_1
\end{array}$$

$$f'(x)=\frac{\begin{array}{l}\left(a_4b_3-a_3b_4\right)x^6+2\left(a_4b_2-a_2b_4\right)x^5+\left(3\left(a_4b_1-a_1b_4\right)+\left(a_3b_2-a_2b_3\right)\right)x^4\\ +\left(4\left(a_4b_0-a_0b_4\right)+2\left(a_3b_1-a_1b_3\right)\right)x^3\\ +\left(\left(a_2b_1-a_1b_2\right)+3\left(a_3b_0-a_0b_3\right)\right)x^2+2\left(a_2b_0-a_0b_2\right)x+a_1b_0-a_0b_1\end{array}}{\left(b_4x^4+b_3x^3+b_2x^2+b_1x+b_0\right)^2}$$



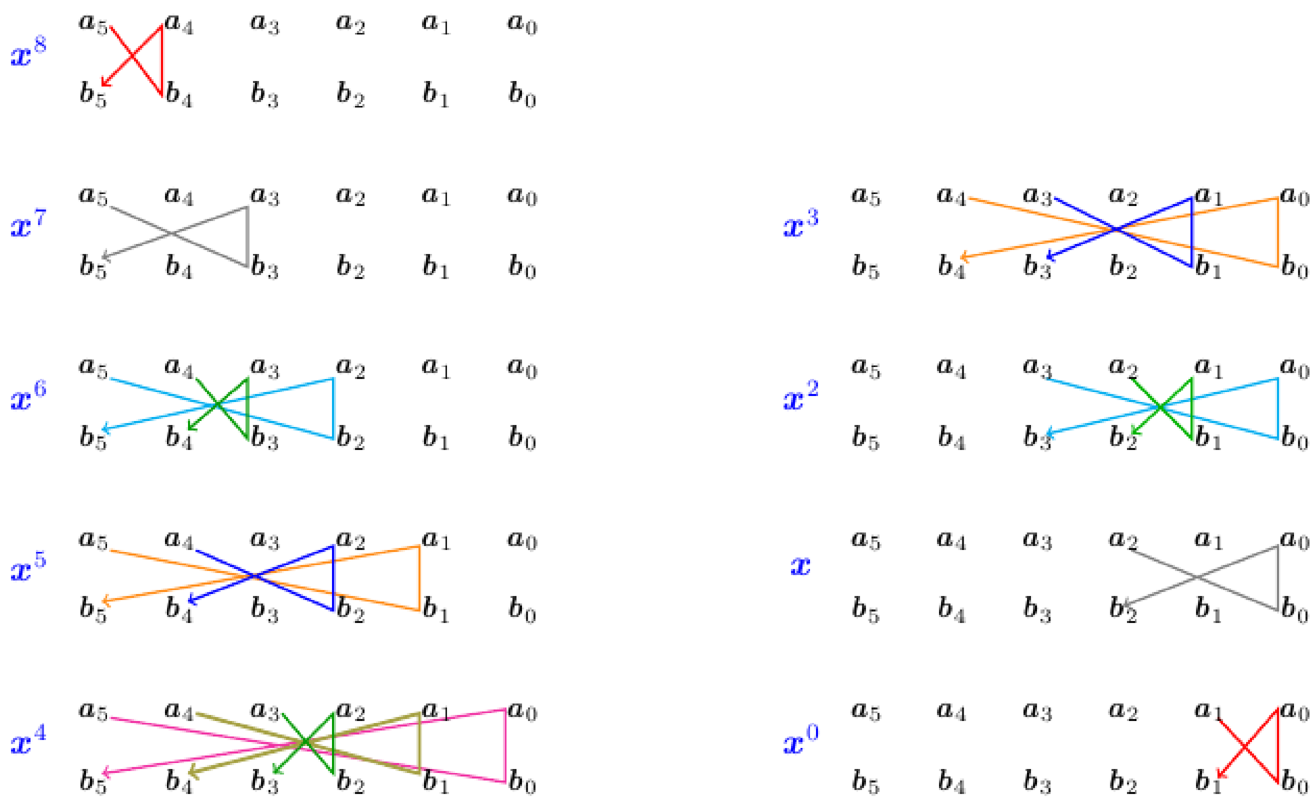
Derivative: in the form $\qquad f(x)=\dfrac{a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0}{ba_5x^5+b_4x^4+b_3x^3+b_2x^2+b_1x+b_0}$

$$u=a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0 \quad\rightarrow\quad u'=5a_5x^4+4a_4x^3+3a_3x^2+2a_2x+a_1$$
$$v=b_5x^5+b_4x^4+b_3x^3+b_2x^2+b_1x+b_0 \quad\rightarrow\quad v'=5b_5x^4+4b_4x^3+3b_3x^2+2b_2x+b_1$$

$$\begin{aligned}u'v-v'u &= \left(5a_5x^4+4a_4x^3+3a_3x^2+2a_2x+a_1\right)\left(b_5x^5+b_4x^4+b_3x^3+b_2x^2+b_1x+b_0\right)\\ &\quad-\left(5b_5x^4+4b_4x^3+3b_3x^2+2b_2x+b_1\right)\left(a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0\right)\end{aligned}$$

x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x^1	x^0
$5a_5b_5$	$5a_5b_4$	$5a_5b_3$	$5a_5b_2$	$5a_5b_1$	$5a_5b_0$				
$-5a_5b_5$	$4a_4b_5$	$4a_4b_4$	$4a_4b_3$	$4a_4b_2$	$4a_4b_1$	$4a_4b_0$			
	$-5a_4b_5$	$3a_3b_5$	$3a_3b_4$	$3a_3b_3$	$3a_3b_2$	$3a_3b_1$	$3a_3b_0$		
	$-4a_5b_4$	$-5a_3b_5$	$2a_2b_5$	$2a_2b_4$	$2a_2b_3$	$2a_2b_2$	$2a_2b_1$	$2a_2b_0$	
		$-4a_4b_4$	$-5a_2b_5$	a_1b_5	a_1b_4	a_1b_3	a_1b_2	a_1b_1	a_1b_0
		$-3a_5b_3$	$-4a_3b_4$	$-5a_1b_5$	$-5a_0b_5$	$-4a_0b_4$	$-3a_0b_3$	$-2a_0b_2$	$-a_0b_1$
			$-3a_4b_3$	$-4a_2b_4$	$-4a_1b_4$	$-3a_1b_3$	$-2a_1b_2$	$-a_1b_1$	
			$-2a_5b_2$	$-3a_3b_3$	$-3a_2b_3$	$-2a_2b_2$	$-a_2b_1$		
				$-2a_4b_2$	$-2a_3b_2$	$-a_3b_1$			
				$-a_5b_1$	$-a_4b_1$				

$$f'(x)=\frac{\begin{aligned}&\left(a_5b_4-a_4b_5\right)x^8 \quad +2\left(a_5b_3-a_3b_5\right)x^7\\ &+\left(3\left(a_5b_2-a_2b_5\right)+\left(a_4b_3-a_3b_4\right)\right)x^6\\ &+\left(4\left(a_5b_1-a_1b_5\right)+2\left(a_4b_2-a_2b_4\right)\right)x^5\\ &+\left(5\left(a_5b_0-a_0b_5\right)+3\left(a_4b_1-a_1b_4\right)+\left(a_3b_2-a_2b_3\right)\right)x^4\\ &+\left(4\left(a_4b_0-a_0b_4\right)+2\left(a_3b_1-a_1b_3\right)\right)x^3\\ &+\left(3\left(a_3b_0-a_0b_3\right)+\left(a_2b_1-a_1b_2\right)\right)x^2\\ &+2\left(a_2b_0-a_0b_2\right)x \quad +\left(a_1b_0-a_0b_1\right)\end{aligned}}{\left(b_5x^5+b_4x^4+b_3x^3+b_2x^2+b_1x+b_0\right)^2}$$



x^8 a_5 a_4 a_3 a_2 a_1 a_0 b_5 b_4 b_3 b_2 b_1 b_0
x^7 a_5 a_4 a_3 a_2 a_1 a_0 b_5 b_4 b_3 b_2 b_1 b_0
x^6 a_5 a_4 a_3 a_2 a_1 a_0 b_5 b_4 b_3 b_2 b_1 b_0
x^5 a_5 a_4 a_3 a_2 a_1 a_0 b_5 b_4 b_3 b_2 b_1 b_0
x^4 a_5 a_4 a_3 a_2 a_1 a_0 b_5 b_4 b_3 b_2 b_1 b_0
x^3 a_5 a_4 a_3 a_2 a_1 a_0 b_5 b_4 b_3 b_2 b_1 b_0
x^2 a_5 a_4 a_3 a_2 a_1 a_0 b_5 b_4 b_3 b_2 b_1 b_0
x a_5 a_4 a_3 a_2 a_1 a_0 b_5 b_4 b_3 b_2 b_1 b_0
x^0 a_5 a_4 a_3 a_2 a_1 a_0 b_5 b_4 b_3 b_2 b_1 b_0

Exponential Function

$$a^{mx+n}=b^{px+q} \quad \Rightarrow \quad x=\frac{q\ln b-n\ln a}{m\ln a-p\ln b} \qquad coefficient\,\frac{no\ x's}{x's}$$

Numerator: multiply q with $\ln b$ ***minus*** multiply n with $\ln a$

Denominator: multiply m with $\ln a$ ***minus*** multiply p with $\ln b$

Proof

$$\ln a^{mx+n}=\ln b^{px+q}$$

$$(mx+n)\ln a=(px+q)\ln b$$

$$mx\ln a+n\ln a=px\ln b+q\ln b$$

$$mx\ln a-px\ln b=q\ln b-n\ln a$$

$$x(m\ln a-p\ln b)=q\ln b-n\ln a$$

$$x=\frac{q\ln b-n\ln a}{m\ln a-p\ln b}$$

$$mx\ln a+n\ln a=px\ln b+q\ln b$$

Example

Solve: $3^{2x-1}=7^{x+1}$

Solution

$$x=\frac{\ln 3+\ln 7}{2\ln 3-\ln 7}$$

Example

Solve: $4^{x+3}=3^{-x}$

Solution

$$x=\frac{-3\ln 4}{\ln 4+\ln 3}$$

$$\ln 4^{\boxed{1}x+3} \quad -\ln 3^{\boxed{-1}x}$$

$-\ln$

Growth & Decay Formula

$$A = A_0 e^{kt} \quad \Rightarrow \quad kT = \ln \frac{A}{A_0}$$

Proof

$$A = A_0 e^{kt}$$

$$\frac{A}{A_0} = e^{kt}$$

$$\ln \frac{A}{A_0} = \ln e^{kt}$$

$$\boxed{\ln \frac{A}{A_0} = kt} \quad \surd$$

Inverse Functions

$$f(x)=\frac{ax+b}{cx+d} \quad \Rightarrow \quad f^{-1}(x)=\frac{-dx+b}{cx-a}$$

Proof

$$y=\frac{ax+b}{cx+d}$$

$$x=\frac{ay+b}{cy+d}$$

$$cxy+dx=ay+b$$

$$cxy-ay=-dx+b$$

$$(cx-a)y=-dx+b$$

$$y=\frac{-dx+b}{cx-a}$$

$$\boxed{f^{-1}(x)=\frac{-dx+b}{cx-a}} \quad \surd$$

Interchange ***a*** and ***d*** and change there signs.

Example

Find the inverse function of: $f(x)=\dfrac{1}{3x-2}$

Solution

$$f^{-1}(x)=\frac{2x+1}{3x} \qquad\qquad f(x)=\frac{0x+1}{3x-2}$$

Example

Find the inverse function of: $f(x)=\dfrac{3x+2}{2x-5}$

Solution

$$f^{-1}(x)=\frac{5x+2}{2x-3} \qquad\qquad f(x)=\frac{3x+2}{2x-5}$$

Example

Find the inverse function of: $f(x)=\dfrac{4x}{x+2}$

Solution

$$f^{-1}(x)=\frac{-2x}{x-4} \qquad\qquad f(x)=\frac{4x}{x+2}$$

Integration by Part

Evaluate $\int x^n e^{ax}\, dx$

		$\int e^{ax}$
+	x^n	$\frac{1}{a}e^{ax}$
−	nx^{n-1}	$\frac{1}{a^2}e^{ax}$
+	$n(n-1)x^{n-2}$	$\frac{1}{a^3}e^{ax}$
−	$n(n-1)(n-2)x^{n-3}$	$\frac{1}{a^4}e^{ax}$
	$\vdots \ \vdots$	$\vdots \ \vdots$

$$\int x^n e^{ax}\, dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a^2}x^{n-1}e^{ax} + \frac{n(n-1)}{a^3}x^{n-2}e^{ax} - \frac{n(n-1)(n-2)}{a^4}x^{n-3}e^{ax} + \dots$$

$$\boxed{= e^{ax}\sum_{k=0}^{n}(-1)^k \cdot \frac{n!}{(n-k)!}\cdot\frac{1}{a^{k+1}}\cdot x^{n-k}}$$

Jose's Method

Evaluate $\displaystyle\int e^{ax}\cos bx\, dx$

$$\int e^{ax}\cos bx\, dx = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\, dx$$

$$\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{b^2}\left(b\sin bx + a\cos bx\right)$$

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\left(b\sin bx + a\cos bx\right) + C$$

		$\int \cos bx\, dx$
$+$	e^{ax}	$\frac{1}{b}\sin bx$
$-$	ae^{ax}	$-\frac{1}{b^2}\cos bx$
$+$	a^2e^{ax}	$-\frac{1}{b^2}\int \cos bx\, dx$

Proof

Find $\displaystyle\int e^{ax}\cos bx\, dx$

Solution

Let: $u = e^{ax}$ $\qquad dv = \cos bx dx$

$du = ae^{ax}dx \qquad v = \int \cos bx dx = \frac{1}{b}\sin bx$

$$\int e^{ax}\cos bx\, dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx dx \qquad \int u dv = uv - \int v du$$

Let: $u = e^{ax}$ $\qquad dv = \sin bx dx$

$du = ae^{ax}dx \qquad v = \int \sin bx dx = -\frac{1}{b}\cos bx$

$$\int e^{ax}\cos bx\, dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left[-\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\, dx\right]$$

$$= \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\, dx$$

$$\int e^{ax}\cos bx\, dx + \frac{a^2}{b^2}\int e^{ax}\cos bx\, dx = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx + C_1$$

$$\frac{a^2+b^2}{b^2}\int e^{ax}\cos bx\, dx = \frac{1}{b^2}e^{ax}\left(b\sin bx + a\cos bx\right) + C_1$$

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\left(b\sin bx + a\cos bx\right) + C$$

Length

***Length* of a curve** $y = f(x)$ is given by the formula:

$$L = \int_c^d \sqrt{1+\left[f'(x)\right]^2}\, dx = \int_c^d \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

If $f(x) = ax^m + bx^n$, then

$$\boldsymbol{L} = \int_c^d \sqrt{1+\left(\frac{df}{dx}\right)^2}\, dx = \left(ax^m - bx^n\right. \Big|_c^d$$

Iff $f(x)$ satisfies these 2 conditions:

1. $m+n=2$
2. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = max^{m-1} + nbx^{n-1}$$

$$1+(f')^2 = 1+\left(max^{m-1}+nbx^{n-1}\right)^2$$

$$= 1+m^2a^2x^{2m-2} + 2abmnx^{m+n-2} + n^2b^2x^{2n-2}$$

We need to combined to a perfect square $\quad a^2 - 2ab + b^2 = (a-b)^2$

- If $x^{m+n-2} = 1 = x^0 \rightarrow \boxed{m+n=2}$

$$= m^2a^2x^{2m-2} + (1+2abmn) + n^2b^2x^{2n-2} \qquad a^2 - 2ab + b^2 = (a-b)^2$$

- Let $1+2abmn = -2abmn \rightarrow \boxed{abmn = -\frac{1}{4}}$

$$= m^2a^2x^{2m-2} - 2abmn + n^2b^2x^{2n-2} \qquad x^{2(m+n-2)} = 1$$

$$= \left(max^{m-1} - nbx^{n-1}\right)^2$$

$$\boldsymbol{L} = \int_c^d \sqrt{\left(max^{m-1} - nbx^{n-1}\right)^2}\, dx$$

$$= \int_c^d \left(max^{m-1} - nbx^{n-1}\right) dx$$

$$= \left(ax^m - bx^n\right. \Big|_c^d \quad \checkmark$$

Example

Find the length of the graph of $f(x)=\dfrac{x^3}{12}+\dfrac{1}{x}, \quad 1\le x\le 4$

Solution

$$f'(x)=\frac{x^2}{4}-\frac{1}{x^2}$$

$$\begin{aligned}1+\left[f'(x)\right]^2 &= 1+\left(\frac{x^2}{4}-\frac{1}{x^2}\right)^2\\ &= 1+\frac{x^4}{16}-\frac{1}{2}+\frac{1}{x^4}\\ &= \frac{x^4}{16}+\frac{1}{2}+\frac{1}{x^4}\\ &= \left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2\end{aligned}$$

$$\begin{aligned}L &= \int_1^4 \sqrt{1+\left(f'(x)\right)^2}\,dx\\ &= \int_1^4 \sqrt{\left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2}\,dx\\ &= \int_1^4 \left(\frac{x^2}{4}+\frac{1}{x^2}\right)dx\\ &= \left(\frac{x^3}{12}-\frac{1}{x}\right)\Bigg|_1^4\\ &= \left(\frac{4^3}{12}-\frac{1}{4}\right)-\left(\frac{1}{12}-\frac{1}{1}\right)\\ &= \frac{72}{12}\\ &= 6 \ \ unit\end{aligned}$$

$a=\frac{1}{12}, \quad m=3, \quad b=1, \quad n=-1$

1. $m+n=3-1=2$ $\checkmark$
2. $abmn=\frac{1}{12}(1)(3)(-1)=-\frac{1}{4}$ $\checkmark$

$$L=\left(\frac{x^3}{12}-\frac{1}{x}\right)\Bigg|_1^4 = 6 \ \ unit$$

Examples

$f(x)=\frac{1}{3}x^{3/2}-x^{1/2} \ \rightarrow \ L=\frac{1}{3}x^{3/2}+x^{1/2}+C$

$f(x)=\frac{2}{3}x^{3/2}-\frac{1}{2}x^{1/2} \ \rightarrow \ L=\frac{2}{3}x^{3/2}+\frac{1}{2}x^{1/2}$

$f(x)=\frac{1}{6}x^3+\frac{1}{2x} \ \rightarrow \ L=\frac{1}{6}x^3-\frac{1}{2x}+C$

$f(x)=x^3+\frac{1}{12x} \ \rightarrow \ L=x^3-\frac{1}{12x}+C$

$f(x)=\frac{1}{6}x^3+\frac{1}{2x} \ \rightarrow \ L=\frac{1}{6}x^3-\frac{1}{2x}+C$

$f(x)=\frac{1}{8}x^4+\frac{1}{4x^2} \ \rightarrow \ L=\frac{1}{8}x^4-\frac{1}{4x^2}+C$

$f(x)=\frac{1}{4}x^4+\frac{1}{8x^2} \ \rightarrow \ L=\frac{1}{4}x^4-\frac{1}{8x^2}+C$

$f(x)=x^{1/2}-\frac{1}{3}x^{3/2} \ \rightarrow \ L=x^{1/2}+\frac{1}{3}x^{3/2}$

If $f(x) = ae^{mx} + be^{nx}$, then

$$\boldsymbol{L} = \int_c^d \sqrt{1+\left(\frac{df}{dx}\right)^2}\, dx = \left(ae^{mx} - be^{nx}\right. \Big|_c^d$$

Iff $f(x)$ satisfies these 2 conditions:

1. $m = -n$
2. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = ame^{mx} + bne^{nx}$$

$$1+(f')^2 = 1 + \left(ame^{mx} + bne^{nx}\right)^2$$

$$= 1 + m^2a^2e^{2mx} + 2abmne^{(m+n)x} + n^2b^2e^{2nx}$$

- If $e^{(m+n)x} = 1 = e^{(x=0)} \rightarrow \boxed{m-n}$

$$= m^2a^2x^{2m-2} + (1+2abmn) + n^2b^2x^{2n-2}$$

$a^2 - 2ab + b^2 = (a-b)^2$

- Let $1 + 2abmn = -2abmn \rightarrow \boxed{abmn = -\frac{1}{4}}$

$$= m^2a^2e^{2mx} - 2abmne^{(m+n)x} + n^2b^2e^{2nx}$$

$x^{2(m+n-2)} = 1$

$$= \left(ame^{mx} - bne^{nx}\right)^2$$

$$\left(ame^{mx} - bne^{nx}\right)^2 = m^2a^2e^{2mx} - 2abmne^{(m+n)x} + n^2b^2e^{2nx}$$

Example

$f(x) = 2e^{x} + \frac{1}{8}e^{-x} \rightarrow L = 2e^{x} - \frac{1}{8}e^{-x}$

$f(x) = 2e^{\sqrt{2}x} + \frac{1}{16}e^{-\sqrt{2}x} \rightarrow L = 2e^{\sqrt{2}x} - \frac{1}{16}e^{-\sqrt{2}x}$

Quadratic equation

$$ax^2+bx+c=0$$

If $\boldsymbol{a+b+c=0} \Rightarrow x=1,\ \frac{c}{a}$

Proof

$a+b+c=0 \ \rightarrow \ b=-a-c$

$$\begin{aligned}
x_{1,2} &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\
&= \frac{-(-a-c)\pm\sqrt{(-a-c)^2-4ac}}{2a} \\
&= \frac{a+c\pm\sqrt{a^2+2ac+c^2-4ac}}{2a} \\
&= \frac{a+c\pm\sqrt{a^2-2ac+c^2}}{2a} \\
&= \frac{a+c\pm\sqrt{(a-c)^2}}{2a} \\
&= \frac{a+c\pm(a-c)}{2a}
\end{aligned}$$

$$\begin{aligned}
x_1 &= \frac{a+c+(a-c)}{2a} \\
&= \frac{a+c+a-c}{2a} \\
&= \frac{2a}{2a} \\
&\boxed{=1}
\end{aligned}$$

$$\begin{aligned}
x_2 &= \frac{a+c-(a-c)}{2a} \\
&= \frac{a+c-a+c}{2a} \\
&\frac{2c}{2a} \\
&\boxed{=\frac{c}{a}}
\end{aligned}$$

Example

$$2x^2+x-3=0$$

$$2+1-3=0$$

$$\Rightarrow \boxed{x=1,\quad -\frac{3}{2}}$$

Quadratic equation

$$ax^2+bx+c=0$$

If $\boldsymbol{a - b + c = 0} \Rightarrow x=-1,\ -\frac{c}{a}$

Proof

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$a-b+c=0 \;\rightarrow\; b=a+c$

$$=\frac{-(a+c)\pm\sqrt{(a+c)^2-4ac}}{2a}$$

$$=\frac{-a-c\pm\sqrt{a^2+2ac+c^2-4ac}}{2a}$$

$$=\frac{-a-c\pm\sqrt{a^2-2ac+c^2}}{2a}$$

$$=\frac{-a-c\pm\sqrt{(a-c)^2}}{2a}$$

$$=\frac{-a-c\pm(a-c)}{2a}$$

$$x_1=\frac{-a-c+(a-c)}{2a}$$

$$=\frac{-a-c+a-c}{2a}$$

$$=\frac{2c}{2a}$$

$$\boxed{=-\frac{c}{a}}$$

$$x_2=\frac{-a-c-(a-c)}{2a}$$

$$=\frac{-a-c-a+c}{2a}$$

$$=\frac{-2a}{2a}$$

$$\boxed{=-1}$$

Example

$$2x^2-x-3=0$$

$$2-(-1)-3=0$$

$$\Rightarrow \boxed{x=-1,\ \frac{3}{2}}$$

Surface

***Surface* of a curve** $y = f(x)$ is given by the formula:

$$S = 2\pi \int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\ dx$$

If $f(x) = ax^m + bx^n$, then

$$\sqrt{1+\left(f'(x)\right)^2} = \overline{f'(x)}$$

$\overline{f'(x)}$: is the conjugate of $f'(x)$

Iff $f(x)$ satisfies these 2 conditions:

3. $m+n=2$
4. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = max^{m-1} + nbx^{n-1}$$

$$1+\left(f'\right)^2 = 1+\left(max^{m-1}+nbx^{n-1}\right)^2$$

$$= 1 + m^2a^2x^{2m-2} + 2abmnx^{m+n-2} + n^2b^2x^{2n-2}$$

We need to combined to a perfect square $\qquad a^2 - 2ab + b^2 = (a-b)^2$

- If $x^{m+n-2} = 1 = x^0 \quad \rightarrow \quad \boxed{m+n=2}$

$$= m^2a^2x^{2m-2} + (1+2abmn) + n^2b^2x^{2n-2} \qquad a^2 - 2ab + b^2 = (a-b)^2$$

- Let $1+2abmn = -2abmn \quad \rightarrow \quad \boxed{abmn = -\frac{1}{4}}$

$$= m^2a^2x^{2m-2} - 2abmn + n^2b^2x^{2n-2} \qquad x^{2(m+n-2)} = 1$$

$$= \left(max^{m-1} - nbx^{n-1}\right)^2$$

$$\sqrt{\left(max^{m-1} - nbx^{n-1}\right)^2} = max^{m-1} - nbx^{n-1} \qquad \checkmark$$

$$f'(x) = max^{m-1} + nbx^{n-1} \quad \Rightarrow \quad \sqrt{1+\left(f'(x)\right)^2} = max^{m-1} - nbx^{n-1} = \overline{f'(x)}$$

Example

Find the surface of the graph of $f(x)=\frac{x^3}{12}+\frac{1}{x}, \quad 1\le x\le 4$

Solution

$$f'(x)=\frac{x^2}{4}-\frac{1}{x^2}$$

$$\begin{aligned}1+\left[f'(x)\right]^2 &= 1+\left(\frac{x^2}{4}-\frac{1}{x^2}\right)^2\\ &= 1+\frac{x^4}{16}-\frac{1}{2}+\frac{1}{x^4}\\ &= \frac{x^4}{16}+\frac{1}{2}+\frac{1}{x^4}\\ &= \left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2\end{aligned}$$

$a=\frac{1}{12}, \quad m=3, \quad b=1, \quad n=-1$

3. $m+n=3-1=2$ ✓

4. $abmn=\frac{1}{12}(1)(3)(-1)=-\frac{1}{4}$ ✓

$$\begin{aligned}S &= 2\pi\int_1^4 f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx\\ &= 2\pi\int_1^4\left(\frac{x^3}{12}+\frac{1}{x}\right)\sqrt{\left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2}\,dx\\ &= 2\pi\int_1^4\left(\frac{x^3}{12}+\frac{1}{x}\right)\left(\frac{x^2}{4}+\frac{1}{x^2}\right)dx\\ &= 2\pi\int_1^4\left(\frac{1}{48}x^5+\frac{1}{12}x+\frac{1}{4}x+x^{-3}\right)dx\\ &= 2\pi\left(\frac{1}{288}x^6+\frac{1}{6}x^2-\frac{1}{2x^2}\right)\Big|_1^4\\ &= \pi\left(\frac{256}{9}+\frac{16}{3}-\frac{1}{16}-\frac{1}{144}-\frac{1}{3}+1\right)\\ &= \frac{275}{8}\pi \quad unit^2\end{aligned}$$

$$S=2\pi\int_1^4\left(\frac{x^3}{12}+\frac{1}{x}\right)\left(\frac{x^2}{4}+\frac{1}{x^2}\right)dx$$

If $f(x) = ae^{mx} + be^{nx}$, then

$$S = 2\pi \int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\, dx \; = \; 2\pi \int_a^b f(x)\overline{f'(x)}\, dx$$

Iff $f(x)$ satisfies these 2 conditions:

3. $m = -n$

4. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = ame^{mx} + bne^{nx}$$

$$1 + (f')^2 = 1 + \left(ame^{mx} + bne^{nx}\right)^2$$

$$= 1 + m^2a^2e^{2mx} + 2abmne^{(m+n)x} + n^2b^2e^{2nx}$$

➤ If $e^{(m+n)x} = 1 = e^{(x=0)} \quad \rightarrow \quad \boxed{m-n}$

$$= m^2a^2x^{2m-2} + (1 + 2abmn) + n^2b^2x^{2n-2}$$

$a^2 - 2ab + b^2 = (a-b)^2$

➤ Let $\quad 1 + 2abmn = -2abmn \quad \rightarrow \quad \boxed{abmn = -\frac{1}{4}}$

$$= m^2a^2e^{2mx} - 2abmne^{(m+n)x} + n^2b^2e^{2nx}$$

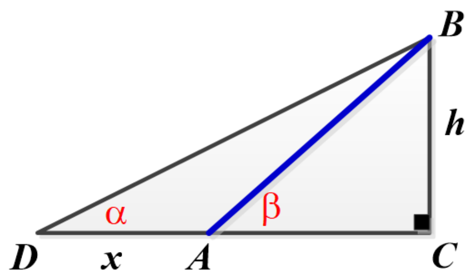
$x^{2(m+n-2)} = 1$

$$= \left(ame^{mx} - bne^{nx}\right)^2$$

$$\sqrt{1+(f')^2} = \sqrt{\left(ame^{mx} - bne^{nx}\right)^2}$$

$$\sqrt{1 + f'(x)} = \overline{f'(x)}$$ ✓

Trigonometry



Proof

Triangle DCB: $\tan\alpha = \dfrac{h}{50+x} \quad \Rightarrow \quad h = (50+x)\tan\alpha$

Triangle ACB: $\tan\beta = \dfrac{h}{x} \quad \Rightarrow \quad h = x\tan\beta$

$$x\tan\beta = (50+x)\tan\alpha$$

$$x\tan\beta = 50\tan\alpha + x\tan\alpha$$

$$x\tan\beta - x\tan\alpha = 50\tan\alpha$$

$$x(\tan\beta - \tan\alpha) = 50\tan\alpha$$

$$x = \frac{50\tan\alpha}{\tan\beta - \tan\alpha}$$

$$\boxed{h = x\,\frac{50\tan\alpha\tan\beta}{\tan\beta - \tan\alpha}}$$

Height is equal to distance times (tan tan) divides by the (tan(larger angle) − tan) (difference between tangents)

Example

From a given point on the ground, the angle of elevation to the top of a tree is 36.7°. From a second point, 50 *feet* back, the angle of elevation to the top of the tree is 22.2°. Find the height of the tree to the nearest foot.

Solution

$$h = 50\frac{\tan 22.2^\circ \ \tan 36.7^\circ}{\tan 36.7^\circ - \tan 22.2^\circ}$$

$$\approx 45\ ft$$