Solution Section 2.7 – Coordinates, Basis and Dimension

Exercise

Suppose $\vec{v}_1,\ \dots,\ \vec{v}_n$ is a basis for R^n and the *n* by *n* matrix *A* is invertible. Show that $A\vec{v}_1,\ \dots,\ A\vec{v}_n$ is also a basis for R^n.

Solution

Put the basis vectors $\vec{v}_1,\ \dots,\ \vec{v}_n$ in the columns of an invertible matrix **V**. then $A\vec{v}_1,\ \dots,\ A\vec{v}_n$ are the columns of ***AV***. Since ***A*** is invertible, so is ***AV*** and its column give a basis.

Suppose $c_1A\vec{v}_1 + \ \dots + c_nA\vec{v}_n = 0$. This is $A\vec{v} = 0$ with $\vec{v} = c_1\vec{v}_1 + \ \dots + c_n\vec{v}_n$. Multiply by A^{-1} to get $\vec{v} = 0$. By linear independence of $\vec{v}'s$, all $c_i = 0$. So, the $Av's$ are independent.

Exercise

Consider the matrix $A = \begin{pmatrix} 1 & 0 & a \\ 2 & -1 & b \\ 1 & 1 & c \\ -2 & 1 & d \end{pmatrix}$

a) Which vectors $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ will make the columns of ***A*** linearly dependent?

b) Which vectors $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ will make the columns of ***A*** a basis for $\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : y + w = 0 \right\}$?

c) For $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$, compute a basis for the four subspaces.

Solution

a) All linear combination of $\begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

b) To satisfy $b + d = 0$. For example, $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$A\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + B\begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix} + C\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} \quad ; \quad A \neq 0$$

c) $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix} + 4\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \\ 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & 5 \\ -2 & 1 & 2 \end{pmatrix} \qquad \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -4 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{pmatrix} \qquad \begin{matrix} \\ \\ R_3 + R_2 \\ R_4 + R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} x_1 + x_3 &= 0 \\ -x_2 - 4x_3 &= 0 \end{aligned}$$

The first 2 columns span the column space $C(\boldsymbol{A})$.

If $x_3 = 1$ that implies that the nullspace

$$N(\boldsymbol{A}): \left\{ \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} \right\}$$

Rank($\boldsymbol{A}$) = 2 and $\begin{bmatrix}-1 & -4 & 1\end{bmatrix}^T$ is a basis for the one-dimensional $N(\boldsymbol{A})$.

$$\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \quad \begin{pmatrix}-3\\1\\1\\0\end{pmatrix}$$

Exercise

Find a basis for $x-2y+3z=0$ in $\mathbb{R}^3$.

Find a basis for the intersection of that plane with *xy* plane. Then find a basis for all vectors perpendicular to the plane.

Solution

This plane is the nullspace of the matrix

$$A=\begin{pmatrix}1 & -2 & 3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

The special solutions: $s_1=\begin{pmatrix}2\\1\\0\end{pmatrix}$ $s_2=\begin{pmatrix}-3\\0\\1\end{pmatrix}$ give a basis for the nullspace, and for the plane.

The intersection of this plane with the *xy*-plane is a line $(x,\ -2x,\ 3x)$ and the vector $(1,\ -2,\ 3)^T$ lies in the *xy*-plane.

The vector $v_3=\begin{pmatrix}1\\-2\\3\end{pmatrix}$ is perpendicular to both vectors s_1 *and* s_2 : the space vectors perpendicular to a plane $\mathbb{R}^3$ is one-dimensional, it gives a basis.

Exercise

U comes from ***A*** by subtracting row 1 from row 3:

$$A=\begin{bmatrix}1&3&2\\0&1&1\\1&3&2\end{bmatrix}\quad \textit{and}\quad U=\begin{bmatrix}1&3&2\\0&1&1\\0&0&0\end{bmatrix}$$

Find the bases for the two column spaces. Find the bases for the two row spaces. Find bases for the two nullspaces.

Solution

$$\begin{bmatrix}1&3&2\\0&1&1\\0&0&0\end{bmatrix}\qquad R_1-3R_2$$

$$\begin{bmatrix}1&0&-1\\0&1&1\\0&0&0\end{bmatrix}\qquad \begin{aligned}x_1-x_3&=0\\x_2+x_3&=0\end{aligned}$$

a) The pivots are in the first two columns, so one possible basis for $C(\boldsymbol{A})$ is $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}3\\1\\3\end{bmatrix}\right\}$ and for

$C(\boldsymbol{U})$ is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}3\\1\\0\end{bmatrix}\right\}$

b) Both ***A*** and ***U*** have the same nullspace $N(\boldsymbol{A})=N(\boldsymbol{U})$,

with basis $\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$

c) Both ***A*** and ***U*** have the same row space

$$C\left(A^T\right)=C\left(U^T\right),\quad \textit{with basis}\ \left\{\begin{bmatrix}1\\3\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$$

Exercise

Write a 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives $c_1P_1 + \ldots + c_5P_5 = 0$, and check entries to prove c_i is zero.) The five permutation matrices are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

Solution

Assume:

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad P_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad P_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad P_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$P_1 + P_2 + P_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and $P_4 + P_5 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$$P_1 + P_2 + P_3 - P_4 - P_5 = I$$

$$c_1P_1 + c_2P_2 + c_3P_3 + c_4P_4 + c_5P_5 = \begin{pmatrix} c_3 & c_1 + c_4 & c_2 + c_5 \\ c_1 + c_5 & c_2 & c_3 + c_4 \\ c_2 + c_4 & c_3 + c_5 & c_1 \end{pmatrix} = 0$$

$$c_1 = c_2 = c_3 = 0 \quad (\textit{diagonal})$$

$$\begin{pmatrix} 0 & 0 + c_4 & 0 + c_5 \\ 0 + c_5 & 0 & 0 + c_4 \\ 0 + c_4 & 0 + c_5 & 0 \end{pmatrix} = 0$$

$$\boxed{c_4 = c_5 = 0}$$

Exercise

Choose three independent columns of $A = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix}$. Then choose a different three independent columns. Explain whether either of these choices forms a basis for $C(A)$.

Solution

$$\begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix} \quad \begin{array}{l} \\ R_2 - 2R_1 \\ \\ \\ \end{array}$$

$$\begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix} \quad \begin{array}{l} 2R_2 - R_2 \\ \\ \\ R_4 - R_2 \end{array}$$

$$\begin{pmatrix} 4 & 0 & 1 & 2 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 0 & 1 & 2 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \\ \\ \frac{1}{9}R_3 \\ \\ \end{array}$$

$$\begin{pmatrix} 4 & 0 & 1 & 2 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} R_1 - 2R_3 \\ \\ \\ \\ \end{array}$$

$$\begin{pmatrix} 4 & 0 & 1 & 0 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \frac{1}{4}R_1 \\ \frac{1}{6}R_2 \\ \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{7}{6} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank(***A***) = 3, the columns space is 3 which form a basis of $C(A)$. The variable is x_3

If $x_3 = 1$

$$\begin{pmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{7}{6} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} x_1 + \tfrac{1}{4}x_3 &= 0 \\ x_2 + \tfrac{7}{6}x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

$$\underline{x_1 = -\tfrac{1}{4} \quad x_2 = -\tfrac{7}{6} \quad x_4 = 0}$$

$N(A)$ is spanned by $x_n = \begin{pmatrix} -\frac{1}{4} \\ -\frac{7}{6} \\ 1 \\ 0 \end{pmatrix}$, which gives the relation of the columns.

The special solution x_n gives a relation $-\frac{1}{4}\vec{v}_1 - \frac{7}{6}\vec{v}_2 + \vec{v}_3 = 0$. If we take any two columns from the first three columns and the column 4, they will span a three-dimensional space since there will be no relation among them. Hence, they form a basis of $C(A)$.

Exercise

Which of the following sets of vectors are bases for $\mathbb{R}^2$?

a) $\{(2,\ 1),\ (3,\ 0)\}$

b) $\{(0,\ 0),\ (1,\ 3)\}$

Solution

a) $k_1(2,\ 1) + k_2(3,\ 0) = (0,\ 0)$

$k_1(2,\ 1) + k_2(3,\ 0) = (b_1,\ b_2)$

$$\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} = -3 \neq 0$$

Therefore, the vectors $\{(2,\ 1),\ (3,\ 0)\}$ are linearly independent and span $\mathbb{R}^2$, so they form a basis for $\mathbb{R}^2$.

b) $k_1(0,\ 0)+k_2(1,\ 3)=(0,\ 0)$

$k_1(0,\ 0)+k_2(1,\ 3)=(b_1,\ b_2)$

$$\begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} = 0$$

Therefore; the vectors $\{(0,\ 0),\ (1,\ 3)\}$ are linearly dependent, so they don't form a basis for $\mathbb{R}^2$.

Exercise

Which of the following sets of vectors are bases for $\mathbb{R}^3$?

a) $\{(1,\ 0,\ 0),\ (2,\ 2,\ 0),\ (3,\ 3,\ 3)\}$

b) $\{(3,\ 1,\ -4),\ (2,\ 5,\ 6),\ (1,\ 4,\ 8)\}$

c) $\{(2,\ -3,\ 1),\ (4,\ 1,\ 1),\ (0,\ -7,\ 1)\}$

d) $\{(1,\ 6,\ 4),\ (2,\ 4,-1),\ (-1,\ 2,\ 5)\}$

Solution

a) $$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix} = 6 \neq 0$$

Therefore, the set of vectors are linearly independent.

The set form a basis for $\mathbb{R}^3$.

b) $$\begin{vmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{vmatrix} = 26 \neq 0$$

Therefore, the set of vectors are linearly independent.

The set form a basis for $\mathbb{R}^3$.

c) $$\begin{vmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Therefore, the set of vectors are linearly dependent.

The set don't form a basis for $\mathbb{R}^3$.

d) $\begin{vmatrix} 1 & 2 & -1 \\ 6 & 4 & 2 \\ 4 & -1 & 5 \end{vmatrix} \underline{=0|}$

Therefore; the set of vectors are linearly dependent.

The set don't form a basis for $\mathbb{R}^3$.

Exercise

Let *V* be the space spanned by $\vec{v}_1 = \cos^2 x, \quad \vec{v}_2 = \sin^2 x, \quad \vec{v}_3 = \cos 2x$

a) Show that $S = \left\{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \right\}$ is not a basis for *V*.

b) Find a basis for *V*.

Solution

a) $\cos 2x = \cos^2 x - \sin^2 x$

$k_1 \cos^2 x + k_2 \sin^2 x + k_3 \cos 2x = 0$

$k_1 \cos^2 x + k_2 \sin^2 x + k_3 \left(\cos^2 x - \sin^2 x \right) = 0$

$\left(k_1 + k_3 \right) \cos^2 x + \left(k_2 - k_3 \right) \sin^2 x = 0$

$$\begin{cases} k_1 + k_3 = 0 & \rightarrow k_1 = -k_3 \\ k_2 - k_3 = 0 & \rightarrow k_2 = k_3 \end{cases}$$

If $k_3 = -1 \quad \Rightarrow k_1 = 1, \quad k_2 = -1$

$(1) \cos^2 x + (-1) \sin^2 x + (-1) \cos 2x = 0$

This shows that $\left\{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \right\}$ is linearly dependent, therefore it is ***not*** a basis for *V*.

b) For $c_1 \cos^2 x + c_2 \sin^2 x = 0$ to hold for all real *x* values, we must have $c_1 = 0 \quad (x = 0)$ and $c_2 = 0 \quad \left(x = \frac{\pi}{2} \right)$.

Therefore, the vectors $\vec{v}_1 = \cos^2 x \quad \vec{v}_2 = \sin^2 x$ are linearly independent.

$\boldsymbol{v} = k_1 \cos^2 x + k_2 \sin^2 x + k_3 \cos 2x$

$\quad = \left(k_1 + k_3 \right) \cos^2 x + \left(k_2 - k_3 \right) \sin^2 x$

This proves that the vectors $\vec{v}_1 = \cos^2 x \quad and \quad \vec{v}_2 = \sin^2 x$ span *V*.

We can conclude that $\vec{v}_1 = \cos^2 x \quad and \quad \vec{v}_2 = \sin^2 x$ can form a basis for *V*.

Exercise

Find the coordinate vector of $\vec{w}$ relative to the basis $S=\{\vec{u}_1, \vec{u}_2\}$ for $\mathbb{R}^2$

a) $\vec{u}_1=(1,\ 0),\ \ \vec{u}_2=(0,\ 1),\ \ \vec{w}=(3,\ -7)$

b) $\vec{u}_1=(2,\ -4),\ \ \vec{u}_2=(3,\ 8),\ \ \vec{w}=(1,\ 1)$

c) $\vec{u}_1=(1,\ 1),\ \ \vec{u}_2=(0,\ 2),\ \ \vec{w}=(a,\ b)$

d) $\vec{u}_1=(1,\ -1),\ \ \vec{u}_2=(1,\ 1),\ \ \vec{w}=(0,\ 1)$

e) $\vec{u}_1=(1,\ -1),\ \ \vec{u}_2=(1,\ 1),\ \ \vec{w}=(1,\ 1)$

Solution

a) $\vec{u}_1=(1,\ 0),\ \ \vec{u}_2=(0,\ 1),\ \ \vec{w}=(3,\ -7)$

We must first express $\vec{w}$ as a linear combination of the vectors in *S;* $\vec{w}=c_1\vec{u}_1+c_2\vec{u}_2$

$$\left(\begin{array}{cc|c}1 & 0 & 3\\ 0 & 1 & -7\end{array}\right) \qquad \begin{array}{l} c_1=3\\ c_2=-7\end{array}$$

$$\begin{aligned}(3,\ -7)&=3(1,\ 0)-7(0,\ 1)\\ &=3u_1-7u_2\end{aligned}$$

Therefore, $\boxed{(\vec{w})_S=(3,-7)}$

b) $\vec{u}_1=(2,\ -4),\ \ \vec{u}_2=(3,\ 8),\ \ \vec{w}=(1,\ 1)$

Solve: $\vec{w}=c_1\vec{u}_1+c_2\vec{u}_2$

$c_1(2,\ -4)+c_2(3,\ 8)=(1,\ 1)$

$$\begin{cases} 2c_1+3c_2=1\\ -4c_1+8c_2=1\end{cases}$$

$$\left[\begin{array}{cc|c}2 & 3 & 1\\ -4 & 8 & 1\end{array}\right] \quad R_2+2R_1$$

$$\left[\begin{array}{cc|c}2 & 3 & 1\\ 0 & 14 & 3\end{array}\right] \quad \tfrac{1}{14}R_2$$

$$\left[\begin{array}{cc|c}2 & 3 & 1\\ 0 & 1 & \frac{3}{14}\end{array}\right] \quad R_1-3R_2$$

$$\left[\begin{array}{cc|c}2 & 0 & \frac{5}{14}\\ 0 & 1 & \frac{3}{14}\end{array}\right] \quad \tfrac{1}{2}R_1$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{5}{28} \\ 0 & 1 & \frac{3}{14} \end{array}\right]$$

$$\frac{5}{28}(2,\ -4)+\frac{3}{14}(3,\ 8)=(1,\ 1)$$

Therefore, $\underline{(\vec{w})_S=\left(\frac{5}{28},\ \frac{3}{14}\right)}$

c) $\vec{u}_1=(1,\ 1),\quad \vec{u}_2=(0,\ 2),\quad \vec{w}=(a,\ b)$

Solve: $\vec{w}=c_1\vec{u}_1+c_2\vec{u}_2$

$$c_1(1,\ 1)+c_2(0,\ 2)=(a,\ b)$$

$$\begin{cases} \underline{c_1=a} \\ c_1+2c_2=b \end{cases} \quad \Rightarrow \quad \underline{c_2=\frac{b-a}{2}}$$

$$a(1,\ 1)+\frac{b-a}{2}(0,\ 2)=(a,\ b)$$

Therefore, $\underline{(\vec{w})_S=\left(a,\ \frac{b-a}{2}\right)}$

d) $\vec{u}_1=(1,\ -1),\quad \vec{u}_2=(1,\ 1),\quad \vec{w}=(0,\ 1)$

Solve: $\vec{w}=c_1\vec{u}_1+c_2\vec{u}_2$

$$c_1(1,\ -1)+c_2(1,\ 1)=(0,\ 1)$$

$$\begin{cases} c_1+c_2=0 \\ -c_1+c_2=1 \end{cases}$$

$$c_1=\frac{\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}}=\underline{-\frac{1}{2}} \qquad\qquad c_2=\frac{\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}}{2}=\underline{\frac{1}{2}}$$

$$-\frac{1}{2}(1,\ -1)+\frac{1}{2}(1,\ 1)=(0,\ 1)$$

Therefore, $\underline{(\vec{w})_S=\left(-\frac{1}{2},\ \frac{1}{2}\right)}$

e) $\vec{u}_1=(1,\ -1),\quad \vec{u}_2=(1,\ 1),\quad \vec{w}=(1,\ 1)$

Solve: $\vec{w}=c_1\vec{u}_1+c_2\vec{u}_2$

$$c_1(1,\ -1)+c_2(1,\ 1)=(1,\ 1)$$

$$\begin{cases} c_1 + c_2 = 1 \\ -c_1 + c_2 = 1 \end{cases}$$

$$c_1 = \frac{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}} \underline{=0} \qquad\qquad c_2 = \frac{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}}{2} \underline{=1}$$

$0(1,\ -1) + 1(1,\ 1) = (1,\ 1)$

Therefore, $\underline{(\vec{w})_S = (0,\ 1)}$

Exercise

Find the coordinate vector of $\vec{v}$ relative to the basis $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

a) $\vec{v} = (2,\ -1,\ 3),\quad \vec{v}_1 = (1,\ 0,\ 0),\quad \vec{v}_2 = (2,\ 2,\ 0),\quad \vec{v}_3 = (3,\ 3,\ 3)$

b) $\vec{v} = (5,\ -12,\ 3),\quad \vec{v}_1 = (1,\ 2,\ 3),\quad \vec{v}_2 = (-4,\ 5,\ 6),\quad \vec{v}_3 = (7,\ -8,\ 9)$

Solution

a) $\vec{v} = (2,\ -1,\ 3),\quad \vec{v}_1 = (1,\ 0,\ 0),\quad \vec{v}_2 = (2,\ 2,\ 0),\quad \vec{v}_3 = (3,\ 3,\ 3)$

Solve: $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{v}$

$c_1(1,\ 0,\ 0) + c_2(2,\ 2,\ 0) + c_2(3,\ 3,\ 3) = (2,\ -1,\ 3)$

$$\begin{cases} c_1 + 2c_2 + 3c_3 = 2 & \rightarrow\ c_1 = 2 - 2c_2 - 3c_3 \ \underline{=3} \\ \quad 2c_2 + 3c_3 = -1 & \rightarrow\ c_2 = \frac{-3c_3 - 1}{2} \ \underline{=-2} \\ \qquad 3c_3 = 3 & \rightarrow\ \underline{c_3 = 1} \end{cases}$$

$3(1,\ 0,\ 0) - 2(2,\ 2,\ 0) + 1(3,\ 3,\ 3) = (2,\ -1,\ 3)$

Therefore, $\underline{(\vec{v})_S = (3,\ -2,\ 1)}$

b) $\vec{v} = (5,\ -12,\ 3),\quad \vec{v}_1 = (1,\ 2,\ 3),\quad \vec{v}_2 = (-4,\ 5,\ 6),\quad \vec{v}_3 = (7,\ -8,\ 9)$

Solve: $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{v}$

$c_1(1,\ 2,\ 3) + c_2(-4,\ 5,\ 6) + c_2(7,\ -8,\ 9) = (5,\ -12,\ 3)$

$$\begin{cases} c_1 - 4c_2 + 7c_3 = 5 \\ 2c_1 + 5c_2 - 8c_3 = -12 \\ 3c_1 + 6c_2 + 9c_3 = 3 \end{cases}$$

$$c_1 = \frac{\begin{vmatrix} 5 & -4 & 7 \\ -12 & 5 & -8 \\ 3 & 6 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & -4 & 7 \\ 2 & 5 & -8 \\ 3 & 6 & 9 \end{vmatrix}} = \frac{-480}{240} \underline{= -2}$$

$$c_2 = \frac{\begin{vmatrix} 1 & 5 & 7 \\ 2 & -12 & -8 \\ 3 & 3 & 9 \end{vmatrix}}{240} = \frac{0}{240} \underline{= 0}$$

$$c_2 = \frac{\begin{vmatrix} 1 & -4 & 5 \\ 2 & 5 & -12 \\ 3 & 6 & 3 \end{vmatrix}}{240} = \frac{240}{240} \underline{= 1}$$

$$-2(1,\ 2,\ 3) + 0(-4,\ 5,\ 6) + 1(7,\ -8,\ 9) = (5,\ -12,\ 3)$$

Therefore, $\underline{(\vec{v})_S = (-2,\ 0,\ 1)}$

Exercise

Show that $\{A_1, A_2, A_3, A_4\}$ is a basis for M_{22}, and express A as a linear combination of the basis vectors

a) $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $A = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$

b) $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

c) $A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$, $A_1 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Solution

a) Matrices $\{A_1, A_2, A_3, A_4\}$ are linearly independent if the equation

$$k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4 = \mathbf{0}$$

$$k_1\begin{pmatrix}1&0\\1&0\end{pmatrix}+k_2\begin{pmatrix}1&1\\0&0\end{pmatrix}+k_3\begin{pmatrix}1&0\\0&1\end{pmatrix}+k_4\begin{pmatrix}0&0\\1&0\end{pmatrix}=\mathbf{0}$$

Has only the trivial solution.

For these matrices to span M_{22}, it must be expressed every matrix $A=\begin{pmatrix}a_1&a_2\\a_3&a_4\end{pmatrix}$ as

$$k_1A_1+k_2A_2+k_3A_3+k_4A_4=\mathbf{A}$$

$$k_1\begin{pmatrix}1&0\\1&0\end{pmatrix}+k_2\begin{pmatrix}1&1\\0&0\end{pmatrix}+k_3\begin{pmatrix}1&0\\0&1\end{pmatrix}+k_4\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}6&2\\5&3\end{pmatrix}$$

The 2 equations can be written as linear systems

$$\begin{matrix}k_1+k_2+k_3 & =0\\ k_2 & =0\\ k_1+k_4 & =0\\ k_3 & =0\end{matrix}\quad \textit{and} \quad \begin{matrix}k_1+k_2+k_3 & =a_1\\ k_2 & =a_2\\ k_1+k_4 & =a_3\\ k_3 & =a_4\end{matrix}$$

$$\begin{vmatrix}1&1&1&0\\0&1&0&0\\1&0&0&1\\0&0&1&0\end{vmatrix}=-1\neq 0,$$

That the homogeneous system has only the trivial solution.

$\{A_1,A_2,A_3,A_4\}$ span M_{22}

$$\to\begin{cases}k_1+k_2+k_3 & =6\\ k_2 & =2\\ k_1+k_4 & =5\\ k_3 & =3\end{cases}$$

$$\left[\begin{array}{cccc|c}1&1&1&0&6\\0&1&0&0&2\\1&0&0&1&5\\0&0&1&0&3\end{array}\right]\quad R_3-R_1$$

$$\left[\begin{array}{cccc|c}1&1&1&0&6\\0&1&0&0&2\\0&-1&-1&1&-1\\0&0&1&0&3\end{array}\right]\quad \begin{matrix}R_1-R_2\\ \\ R_3+R_2\\ \\ \end{matrix}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right] \quad \begin{array}{l} R_1 + R_3 \\ \\ \\ R_4 + R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad \begin{array}{l} \\ \\ -R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad \begin{array}{l} R_1 - R_4 \\ \\ R_3 + R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad \begin{array}{l} k_1 = 1 \\ k_2 = 2 \\ k_3 = 3 \\ k_4 = 4 \end{array}$$

$$\underline{\mathbf{A} = A_1 + 2A_2 + 3A_3 + 4A_4}$$

b) Matrices $\{A_1, A_2, A_3, A_4\}$ are linearly independent if the equation

$$k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4 = \mathbf{0}$$

$$k_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + k_3 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + k_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{0}$$

Has only the trivial solution.

For these matrices to span M_{22}, it must be expressed every matrix $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ as

$$k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4 = \mathbf{A}$$

$$k_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + k_3 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + k_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

The 2 equations can be written as linear systems

$$\begin{array}{ll} k_1 = 0 & k_1 = a_1 \\ k_1 + k_2 = 0 & k_1 + k_2 = a_2 \\ k_1 + k_2 + k_3 = 0 \quad \textit{and} & k_1 + k_2 + k_3 = a_3 \\ k_1 + k_2 + k_3 + k_4 = 0 & k_1 + k_2 + k_3 + k_4 = a_4 \end{array}$$

$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$, that the homogeneous system has only the trivial solution.

$\{A_1, A_2, A_3, A_4\}$ span M_{22}

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 \end{array}\right] \quad \begin{array}{l} \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} \\ \\ \\ R_4 - R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \quad \begin{array}{l} k_1 = 1 \\ k_2 = -1 \\ k_3 = 1 \\ k_4 = -1 \end{array}$$

$\mathbf{A} = A_1 - A_2 + A_3 - A_4$

c) $k_1\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + k_2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + k_3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + k_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \quad \begin{array}{l} \\ R_2 + R_1 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \qquad \tfrac{1}{2}R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \qquad \begin{aligned} k_1 &= 1 \\ k_2 &= 1 \\ k_3 &= -1 \\ k_4 &= 3 \end{aligned}$$

$$\mathbf{A} = A_1 + A_2 - A_3 + 3A_4$$

Exercise

Consider the eight vectors

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

a) List all of the one-element. Linearly dependent sets formed from these.

b) What are the two-element, linearly dependent sets?

c) Find a three-element set spanning a subspace of dimension three, and dimension of two? One? Zero?

d) Which four-element sets are linearly dependent? Explain why.

Solution

a) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ zero vector is the only linearly dependent.

b) The set that contains zero vector and any other vector.

c) 2-dimension:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad or \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad or \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

1-dimensional subspace if we allow duplicates (zero vector) $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$

d) All four-element sets are linearly dependent in three-dimensional space.

Exercise

Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space

a) $\begin{cases} x_1 + x_2 - x_3 = 0 \\ -2x_1 - x_2 + 2x_3 = 0 \\ -x_1 \qquad + x_3 = 0 \end{cases}$

b) $\begin{cases} 3x_1 + x_2 + x_3 + x_4 = 0 \\ 5x_1 - x_2 + x_3 - x_4 = 0 \end{cases}$

c) $\begin{cases} x_1 - 3x_2 + x_3 = 0 \\ 2x_1 - 6x_2 + 2x_3 = 0 \\ 3x_1 - 9x_2 + 3x_3 = 0 \end{cases}$

d) $\begin{cases} x + y + z = 0 \\ 3x + 2y - 2z = 0 \\ 4x + 3y - z = 0 \\ 6x + 5y + z = 0 \end{cases}$

e) $\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ x_1 \qquad + 5x_3 = 0 \\ \qquad x_2 + x_3 = 0 \end{cases}$

Solution

a) $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 + R_1 \end{array}$

$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \begin{array}{l} R_2 - R_2 \\ \\ R_3 - R_2 \end{array}$

$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad x_1 - x_3 = 0 \quad \rightarrow \underline{x_1 = x_3}$

$\underline{x_2 = 0}$

The solution: $\left(x_1,\ 0,\ x_1\right) = x_1\left(1,\ 0,\ 1\right)$

The solution space has dimension 1 and a basis $\underline{\left(1,\ 0,\ 1\right)}$

b) $\left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right] \quad 3R_2 - 5R_1$

$\left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 0 & -8 & -2 & -8 & 0 \end{array}\right] \quad 8R_1 + R_2$

$\left[\begin{array}{cccc|c} 24 & 0 & 6 & 0 & 0 \\ 0 & -8 & -2 & -8 & 0 \end{array}\right] \begin{array}{l} \frac{1}{24}R_1 \\ -\frac{1}{8}R_2 \end{array}$

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{array}\right] \quad \begin{array}{l} x_3 = s \\ x_4 = t \end{array} \quad \begin{array}{l} \underline{\lfloor x_1} = -\frac{1}{4}x_3 \ \underline{= s\rfloor} \\ \underline{\lfloor x_2} = -\frac{1}{4}x_3 - x_4 \ \underline{= -\frac{1}{4}s - t\rfloor} \end{array}$$

The solution:

$$\left(x_1,\ x_2,\ x_3,\ x_4\right) = \left(-\frac{1}{4}s,\ -\frac{1}{4}s - t,\ s,\ t\right)$$

$$= s\left(-\frac{1}{4},\ -\frac{1}{4},\ 1,\ 0\right) + t\left(0,\ -1,\ 0,\ 1\right)$$

The solution space has dimension 2 and a basis $\underline{\left(-\frac{1}{4},\ -\frac{1}{4},\ 1,\ 0\right),\ \left(0,\ -1,\ 0,\ 1\right)\rfloor}$

c) $\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$

$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad x_1 - 3x_2 + x_3 = 0 \ \rightarrow \ x_1 = 3x_2 - x_3$

The solution:

$$\left(x_1,\ x_2,\ x_3\right) = \left(3x_2 - x_3,\ x_2,\ x_3\right)$$

$$= x_2\left(3,\ 1,\ 0\right) + x_3\left(-1,\ 0,\ 1\right)$$

The solution space has dimension 2 and a basis $\underline{\left(3,\ 1,\ 0\right) \ \textit{and} \ \left(-1,\ 0,\ 1\right)\rfloor}$

d) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 2 & -2 & 0 \\ 4 & 3 & -1 & 0 \\ 6 & 5 & 1 & 0 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 4R_1 \\ R_4 - 6R_1 \end{array}$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right] \quad \begin{array}{l} R_1 + R_2 \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{array}$

$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \\ -R_2 \\ \\ \\ \end{array}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x = 4z \\ y = -5z \end{array}$$

The solution: $(x,\ y,\ z) = (4z,\ -5z,\ z) = z(4,\ -5,\ 1)$

The solution space has dimension 1 and a basis $(4,\ -5,\ 1)$

e) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{array}{l} \\ 2R_2 - R_1 \\ \\ \end{array}$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 7 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{array}{l} R_1 + R_2 \\ \\ R_3 + R_2 \end{array}$$

$$\begin{bmatrix} 2 & 0 & 10 \\ 0 & -1 & 7 \\ 0 & 0 & 8 \end{bmatrix} \qquad \begin{array}{l} \frac{1}{2}R_1 \\ -R_2 \\ \frac{1}{8}R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{array}{l} R_1 - 5R_3 \\ R_2 + 7R_3 \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

No basis and dimension = 0

Exercise

If $AS = SA$ for the shift matrix S. Show that $\boldsymbol{A}$ must have this special form:

$$\text{If } \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\text{then } A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

"The subspace of matrices that commute with the shift S has dimension ______."

Solution

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & a & b \\ 0 & d & e \\ 0 & g & h \end{bmatrix} = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \ d = g = h = 0 \quad b = f \quad a = e = i$$

$$A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

The subspace of matrices that commute with the shift S has dimension 3, because the matrix has only three variables.

Exercise

Find bases for the following subspaces of $\mathbb{R}^3$

a) All vectors of the form $(a, b, c, 0)$

b) All vectors of the form (a, b, c, d), where $d = a + b$ and $c = a - b$.

c) All vectors of the form (a, b, c, d), where $a = b = c = d$.

Solution

a) The subspace can be expressed as span $S = \{(1,\ 0,\ 0,\ 0),\ \ (0,\ 1,\ 0,\ 0),\ \ (0,\ 0,\ 1,\ 0)\}$ is a set of linearly independent vectors. Therefore; S forms a basis for the subspace, so its dimension is 3.

b) The subspace contains all vectors $(a,\ b,\ a+b,\ a-b) = a(1,\ 0,\ 1,\ 1) + b(0,\ 1,\ 1,\ -1)$, the set $S = \{(1,\ 0,\ 1,\ 1),\ \ (0,\ 1,\ 1,\ -1)\}$ is linearly independent vectors. Therefore; S forms a basis for the subspace, so its dimension is 2.

c) The subspace contains all vectors $(a,\ a,\ a,\ a) = a(1,\ 1,\ 1,\ 1)$, we can express the set $S = \{\ (1,\ 1,\ 1,\ 1)\ \}$ as span S and it is linearly independent. Therefore, S forms a basis for the subspace, so its dimension is 1.

Exercise

Find a basis for the null space of A.

a) $A=\begin{bmatrix}1&-1&3\\5&-4&-4\\7&-6&2\end{bmatrix}$

b) $A=\begin{bmatrix}1&4&5&2\\2&1&3&0\\-1&3&2&2\end{bmatrix}$

c) $A=\begin{bmatrix}1&4&5&6&9\\3&-2&1&4&-1\\-1&0&-1&-2&-1\\2&3&5&7&8\end{bmatrix}$

Solution

a) $A=\begin{bmatrix}1&-1&3\\5&-4&-4\\7&-6&2\end{bmatrix}\quad\begin{matrix}\\R_2-5R_1\\R_2-7R_1\end{matrix}$

$\begin{bmatrix}1&-1&3\\0&1&-19\\0&1&-19\end{bmatrix}\quad\begin{matrix}R_1+R_2\\ \\R_3-R_2\end{matrix}$

$\begin{bmatrix}1&0&-16\\0&1&-19\\0&0&0\end{bmatrix}\quad\begin{matrix}\to x_1=16x_3=16t\\ \to x_2=19x_3=19t\\ \\ \end{matrix}$

The general form of the solution of $A\vec{x}=\vec{0}$ is $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=t\begin{bmatrix}16\\19\\1\end{bmatrix}$

Therefore, the vector $\begin{bmatrix}16\\19\\1\end{bmatrix}$ forms a basis for the null space of A.

b) $A=\begin{bmatrix}1&4&5&2\\2&1&3&0\\-1&3&2&2\end{bmatrix}\quad\begin{matrix}\\R_2-2R_1\\R_3+R_1\end{matrix}$

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{bmatrix} \quad R_3 + R_2$$

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad -\tfrac{1}{7}R_2$$

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 - 4R_2$$

$$\begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \rightarrow\ x_1 = -x_3 + \frac{2}{7}x_4 \ \underline{= -t + \frac{2}{7}s} \\ \rightarrow\ x_2 = -x_3 - \frac{4}{7}x_4 \ \underline{= -t - \frac{4}{7}s} \\ \end{array}$$

The general form of the solution of $A\vec{x} = \vec{0}$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$

Therefore, the vectors $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

c) $$\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix} \quad \begin{array}{l} R_2 - 3R_1 \\ R_3 + R_1 \\ R_4 - 2R_1 \end{array}$$

$$\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & -14 & -14 & -14 & -28 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -5 & -5 & -5 & -10 \end{bmatrix} \quad \begin{array}{l} -\frac{1}{14}R_2 \\ \frac{1}{4}R_3 \\ -\frac{1}{5}R_4 \end{array}$$

$$\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix} \quad \begin{matrix} R_1 - 4R_2 \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \rightarrow \begin{cases} x_1 = -x_3 - 2x_4 - x_5 = -r - 2s - t \\ x_2 = -x_3 - x_4 - 2x_5 = -r - s - 2t \end{cases}$$

The general form of the solution of $A\vec{x} = \vec{0}$ is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Therefore, the vectors $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

Exercise

Find a basis for the subspace of $\mathbb{R}^4$ spanned by the given vectors

a) $(1, 1, -4, -3), \quad (2, 0, 2, -2), \quad (2, -1, 3, 2)$

b) $(-1, 1, -2, 0), \quad (3, 3, 6, 0), \quad (9, 0, 0, 3)$

Solution

a) $(1, 1, -4, -3), \quad (2, 0, 2, -2), \quad (2, -1, 3, 2)$

$$\begin{pmatrix} 1 & 1 & -4 & -3 \\ 2 & 0 & 2 & -2 \\ 2 & -1 & 3 & 2 \end{pmatrix} \quad \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & -4 & -3 \\ 0 & -2 & 10 & 4 \\ 0 & -3 & 11 & 8 \end{pmatrix} \quad -\tfrac{1}{2}R_2$$

$$\begin{pmatrix} 1 & 1 & -4 & -3 \\ 0 & 1 & -5 & -2 \\ 0 & -3 & 11 & 8 \end{pmatrix} \begin{array}{l} R_1 - R_2 \\ \\ R_3 + 3R_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & -4 & 2 \end{pmatrix} \begin{array}{l} \\ \\ -\frac{1}{4}R_3 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 1 & -\frac{1}{2} \end{pmatrix} \begin{array}{l} R_1 - R_3 \\ R_2 + 5R_3 \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{9}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{pmatrix}$$

A basis for the subspace is $\left(1,\ 0,\ 0,\ -\frac{1}{2}\right),\ \ \left(0,\ 1,\ 0,\ -\frac{9}{2}\right),\ \ \left(0,\ 0,\ 1,\ -\frac{1}{2}\right)$

b) $(-1,\ 1,\ -2,\ 0),\ \ (3,\ 3,\ 6,\ 0),\ \ (9,\ 0,\ 0,\ 3)$

$$\begin{pmatrix} -1 & 1 & -2 & 0 \\ 3 & 3 & 6 & 0 \\ 9 & 0 & 0 & 3 \end{pmatrix} \begin{array}{l} -R_1 \\ \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 3 & 3 & 6 & 0 \\ 9 & 0 & 0 & 3 \end{pmatrix} \begin{array}{l} \\ R_2 - 3R_1 \\ R_2 - 9R_1 \end{array}$$

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 9 & -18 & 3 \end{pmatrix} \begin{array}{l} \\ \frac{1}{6}R_2 \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 9 & -18 & 3 \end{pmatrix} \begin{array}{l} R_1 + R_2 \\ \\ R_3 - 9R_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -18 & 3 \end{pmatrix} \begin{array}{l} \\ \\ -\frac{1}{18}R_3 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{6} \end{pmatrix} \qquad R_1 - 2R_3$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{6} \end{pmatrix}$$

A basis for the subspace is $(1,\ 0,\ 0,\ 0),\quad (0,\ 1,\ 0,\ 0),\quad \left(0,\ 0,\ 1,\ -\frac{1}{6}\right)$

Exercise

Determine whether the given vectors form a basis for the given vector space

a) $\vec{v}_1(3,\ -2,\ 1),\quad \vec{v}_2(2,\ 3,\ 1),\quad \vec{v}_3(2,\ 1,\ -3),\quad in\ \mathbb{R}^3$

b) $\vec{v}_1=(1,\ 1,\ 0,\ 0),\quad \vec{v}_2=(0,\ 1,\ 1,\ 0),\quad \vec{v}_3=(0,\ 0,\ 1,\ 1),\quad \vec{v}_4=(1,\ 0,\ 0,\ 1),\quad for\ \mathbb{R}^4$

c) $M_1=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\quad M_2=\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\quad M_3=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},\quad M_4=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\quad M_{22}$

Solution

a) $\vec{v}_1(3,\ -2,\ 1),\quad \vec{v}_2(2,\ 3,\ 1),\quad \vec{v}_3(2,\ 1,\ -3),\quad in\ \mathbb{R}^3$

$$\begin{vmatrix} 3 & 2 & 2 \\ -2 & 3 & 1 \\ 1 & 1 & -3 \end{vmatrix} = 14 \neq 0$$

The given vectors are linearly independent and span $\mathbb{R}^3$, so they form a basis for $\mathbb{R}^3$.

b) $\vec{v}_1=(1,\ 1,\ 0,\ 0),\quad \vec{v}_2=(0,\ 1,\ 1,\ 0),\quad \vec{v}_3=(0,\ 0,\ 1,\ 1),\quad \vec{v}_4=(1,\ 0,\ 0,\ 1),\quad for\ \mathbb{R}^4$

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 2 \neq 0$$

The given vectors are linearly independent and span $\boldsymbol{R}^4$, so they form a basis for $\boldsymbol{R}^4$.

c) $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad M_{22}$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

They form a basis for M_{22}.

Exercise

Find a basis for, and the dimension of, the null space of the given matrix $\begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & -2 & -2 & 1 \end{bmatrix}$

Solution

$$\begin{bmatrix} 2 & 1 & -1 & 1 \\ 4 & -2 & -2 & 1 \end{bmatrix} \quad R_2 - 2R_1$$

$$\begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & -4 & 0 & -1 \end{bmatrix} \quad 4R_1 + R_2$$

$$\begin{bmatrix} 8 & 0 & -4 & 3 \\ 0 & -4 & 0 & -1 \end{bmatrix} \quad \begin{matrix} \frac{1}{8}R_1 \\ -\frac{1}{4}R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & \frac{3}{8} \\ 0 & 1 & 0 & \frac{1}{4} \end{bmatrix} \quad \begin{aligned} x_1 &= -\frac{1}{2}x_3 - \frac{3}{8}x_4 \\ x_2 &= -\frac{1}{4}x_4 \end{aligned}$$

$$\boldsymbol{x} = x_3 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{3}{8} \\ \frac{1}{4} \\ 0 \\ 1 \end{bmatrix}$$

The *bases* are: $\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} -\frac{3}{8} \\ \frac{1}{4} \\ 0 \\ 1 \end{bmatrix}$

Dimension: 2

Exercise

Let $\mathbb{R}$ be the set of all real numbers and let $\mathbb{R}^+$ be the set of all positive real numbers. Show that $\mathbb{R}^+$ is a vector space over $\mathbb{R}$ under the addition

$$\alpha \oplus \beta = \alpha\beta \quad \alpha,\ \beta \in \mathbb{R}^+$$

And the scalar multiplication

$$a \odot \alpha = \alpha^a \quad \alpha \in \mathbb{R}^+,\ a \in \mathbb{R}$$

Find the dimension of the vector space. Is $\mathbb{R}^+$ also a vector space if the scalar multiplication is instead defined as

$$a \otimes \alpha = a^\alpha \quad \alpha \in \mathbb{R}^+,\ a \in \mathbb{R}?$$

Solution

$$\begin{aligned} ab \odot \alpha &= \alpha^{ab} \qquad \alpha \in \mathbb{R}^+,\ a,\ b \in \mathbb{R} \\ &= \left(\alpha^b\right)^a \\ &= a \odot \left(\alpha^b\right) \\ &= a \odot (b \odot \alpha) \end{aligned}$$

Since for $\alpha \in \mathbb{R}^+$, then

$\alpha = (\log \alpha) \odot 10$

Thus $\{10\}$ is a basis, therefore the dimension of the vector space is **1**.

$\mathbb{R}^+$ is not a vector space over $\mathbb{R}$ with respect to $\otimes$.

Since,

$$\begin{aligned} 2 \otimes (1 \oplus 1) &= 2 \otimes ((1)(1)) \\ &= 2 \otimes 1 \\ &= 2^1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} (2 \otimes 1) \oplus (2 \otimes 1) &= \left(2^1\right) \oplus \left(2^1\right) \\ &= 2 \oplus 2 \\ &= (2)(2) \\ &= 4 \end{aligned}$$

$2 \neq 4$

$2 \otimes (1 \oplus 1) \neq (2 \otimes 1) \oplus (2 \otimes 1)$