Integration by Part

Evaluate $\int x^n e^{ax}\,dx$

		$\int e^{ax}$
$+$	x^n	$\frac{1}{a}e^{ax}$
$-$	nx^{n-1}	$\frac{1}{a^2}e^{ax}$
$+$	$n(n-1)x^{n-2}$	$\frac{1}{a^3}e^{ax}$
$-$	$n(n-1)(n-2)x^{n-3}$	$\frac{1}{a^4}e^{ax}$
	$\vdots\ \vdots$	$\vdots\ \vdots$

$$\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a^2}x^{n-1}e^{ax} + \frac{n(n-1)}{a^3}x^{n-2}e^{ax} - \frac{n(n-1)(n-2)}{a^4}x^{n-3}e^{ax} + \dots$$

$$= e^{ax}\sum_{k=0}^{n}(-1)^k \cdot \frac{n!}{(n-k)!}\cdot\frac{1}{a^{k+1}}\cdot x^{n-k}$$

Jose's Method

Evaluate $\displaystyle\int e^{ax}\cos bx\,dx$

$$\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx$$

$$\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{b^2}\left(b\sin bx + a\cos bx\right)$$

$$\int e^{ax}\cos bx\,dx = \boxed{\frac{e^{ax}}{a^2+b^2}\left(b\sin bx + a\cos bx\right) + C}$$

		$\int \cos bx\,dx$
+	e^{ax}	$\frac{1}{b}\sin bx$
-	ae^{ax}	$-\frac{1}{b^2}\cos bx$
+	a^2e^{ax}	$-\frac{1}{b^2}\int \cos bx\,dx$

Proof

Find $\displaystyle\int e^{ax}\cos bx\,dx$

Solution

Let: $u = e^{ax}$ $\quad dv = \cos bx dx$

$du = ae^{ax}dx \quad v = \int \cos bx dx = \frac{1}{b}\sin bx$

$$\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx dx \qquad \int u dv = uv - \int v du$$

Let: $u = e^{ax}$ $\quad dv = \sin bx dx$

$du = ae^{ax}dx \quad v = \int \sin bx dx = -\frac{1}{b}\cos bx$

$$\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left[-\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx\right]$$

$$= \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx$$

$$\int e^{ax}\cos bx\,dx + \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx + C_1$$

$$\frac{a^2+b^2}{b^2}\int e^{ax}\cos bx\,dx = \frac{1}{b^2}e^{ax}\left(b\sin bx + a\cos bx\right) + C_1$$

$$\int e^{ax}\cos bx\,dx = \boxed{\frac{e^{ax}}{a^2+b^2}\left(b\sin bx + a\cos bx\right) + C}$$