Solution Section 4.2 – Exponential and Logarithmic Integrals

Exercise

Find the integral $\int (2x+1)e^{x^2+x}dx$

Solution

$$u = x^2 + x \Rightarrow du = (2x+1)dx$$

$$\int (2x+1)e^{x^2+x}dx = \int e^u du$$

$$= e^u + C$$

$$= e^{x^2+x} + C$$

Exercise

Find the integral $\int \frac{1}{6x-5}dx$

Solution

$$u = 6x-5 \Rightarrow du = 6dx$$

$$\Rightarrow \frac{1}{6}du = dx$$

$$\int \frac{1}{6x-5}dx = \int \frac{1}{u}\frac{1}{6}du$$

$$= \frac{1}{6}\int \frac{1}{u}du$$

$$= \frac{1}{6}\ln|u| + C$$

$$= \frac{1}{6}\ln|6x-5| + C$$

Exercise

Find the integral $\displaystyle\int \frac{x^2+2x+3}{x^3+3x^2+9x+1}dx$

Solution

$$u = x^3+3x^2+9x+1 \;\; \Rightarrow du = (3x^2+6x+9)dx$$

$$\Rightarrow du = 3(x^2+2x+3)dx$$

$$\Rightarrow \tfrac{1}{3}du = (x^2+2x+3)dx$$

$$\int \frac{x^2+2x+3}{x^3+3x^2+9x+1}dx = \int \frac{1}{u}\frac{1}{3}du$$

$$= \frac{1}{3}\int \frac{1}{u}du$$

$$= \tfrac{1}{3}\ln|u| + C$$

$$= \tfrac{1}{3}\ln\left|x^3+3x^2+9x+1\right| + C$$

Exercise

Find the integral $\displaystyle\int \frac{1}{x(\ln x)^2}dx$

Solution

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$

$$\Rightarrow xdu = dx$$

$$\int \frac{1}{x(\ln x)^2}dx = \int \frac{1}{xu^2}xdu$$

$$= \int \frac{1}{u^2}du$$

$$= \int u^{-2}du$$

$$= \frac{u^{-1}}{-1} + C = -\frac{1}{u} + C$$

$$= -\frac{1}{\ln x} + C$$

Exercise

Find the integral $\displaystyle\int \frac{e^x}{1+e^x}dx$

Solution

$$u = 1+e^x \Rightarrow du = e^x dx$$

$$\int \frac{1}{u}du = \ln|u| + C$$

$$= \ln(1+e^x) + C$$

Exercise

Find the integral $\displaystyle\int \frac{1}{x^3} e^{1/4x^2} dx$

Solution

$$u = \frac{1}{4x^2} = \frac{1}{4}x^{-2} \Rightarrow du = \frac{1}{4}(-2x^{-3})dx$$

$$\Rightarrow du = -\frac{1}{2}x^{-3}dx$$

$$\Rightarrow -2du = \frac{1}{x^3}dx$$

$$\int e^u(-2)du = -2\int e^u du$$

$$= -2e^u + C$$

$$= -2e^{1/4x^2} + C$$

Exercise

Find the integral $\displaystyle\int \frac{e^{1/\sqrt{x}}}{x^{3/2}}dx$

Solution

$$u = \frac{1}{\sqrt{x}} = x^{-1/2} \Rightarrow du = -\frac{1}{2}x^{-3/2}dx$$

$$\Rightarrow -2du = \frac{1}{x^{3/2}}dx$$

$$\int \frac{e^{1/\sqrt{x}}}{x^{3/2}}dx = \int e^{u}(-2du)$$

$$= -2\int e^{u}du$$

$$= -2e^{u} + C$$

$$= -2e^{1/\sqrt{x}} + C$$

Exercise

Find the integral $\int \frac{-e^{3x}}{2-e^{3x}}dx$

Solution

$$u = 2 - e^{3x} \Rightarrow du = -3e^{3x}dx$$

$$\Rightarrow \frac{du}{-3e^{3x}} = dx$$

$$\int \frac{-e^{3x}}{2-e^{3x}}dx = \int \frac{-e^{3x}}{u}\frac{du}{-3e^{3x}}$$

$$= \frac{1}{3}\int \frac{1}{u}du$$

$$= \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln\left|2 - e^{3x}\right| + C$$

Exercise

Find the integral $\int (6x+e^{x})\sqrt{3x^{2}+e^{x}}\,dx$

Solution

$$u = 3x^{2} + e^{x} \Rightarrow du = (6x+e^{x})dx$$

$$\Rightarrow \frac{du}{6x+e^{x}} = dx$$

$$\int (6x+e^x)\sqrt{u}\,\frac{du}{6x+e^x} = \int u^{1/2}du$$

$$= \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}(3x^2 + e^x)^{3/2} + C$$

Exercise

Find the integral $\int \frac{2(e^x - e^{-x})}{(e^x + e^{-x})^2}dx$

Solution

$$u = e^x + e^{-x} \Rightarrow du = (e^x - e^{-x})dx$$

$$\int \frac{2(e^x - e^{-x})}{(e^x + e^{-x})^2}dx = 2\int \frac{1}{u^2}du$$

$$= 2\int u^{-2}du$$

$$= 2\frac{u^{-1}}{-1} + C$$

$$= -2\frac{1}{u} + C$$

$$= -\frac{2}{e^x + e^{-x}} + C$$

Exercise

Find the integral $\int \frac{x-3}{x+3}dx$

Solution

$$\int \frac{x-3}{x+3}dx = \int \left(1 - \frac{6}{x+3}\right)dx$$

$$= x - 6\ln|x+3| + C$$

Exercise

Find the integral $\int \frac{5}{e^{-5x}+7}dx$

Solution

$$\int \frac{5}{e^{-5x}+7}\frac{e^{5x}}{e^{5x}}dx = \int \frac{5e^{5x}}{1+7e^{5x}}dx$$

$$u = 1+7e^{5x} \Rightarrow du = 35e^{5x}dx$$

$$\Rightarrow \frac{du}{35e^{5x}} = dx$$

$$\int \frac{5e^{5x}}{1+7e^{5x}}dx = \int \frac{5e^{5x}}{u}\frac{du}{35e^{5x}}$$

$$= \frac{1}{7}\int \frac{1}{u}du$$

$$= \frac{1}{7}\ln|u| + C$$

$$= \frac{1}{7}\ln\left|1+7e^{5x}\right| + C$$

Exercise

Find the integral $\int \frac{4x^2-3x+2}{x^2}dx$

Solution

$$\int \frac{4x^2-3x+2}{x^2}dx = \int \left(\frac{4x^2}{x^2} - \frac{3x}{x^2} + \frac{2}{x^2}\right)dx$$

$$= \int \left(4 - \frac{3}{x} + 2x^{-2}\right)dx$$

$$= 4x - 3\ln|x| - 2x^{-1} + C$$

$$= 4x - 3\ln|x| - \frac{2}{x} + C$$

Exercise

Find the integral $\int \frac{2}{e^{-x}+1}dx$

Solution

$$\int \frac{2}{e^{-x}+1}dx = \int \frac{2}{e^{-x}+1}\frac{e^x}{e^x}dx$$

$$= 2\int \frac{e^x}{1+e^x}dx$$

$$= 2\int \frac{d(e^x+1)}{1+e^x}$$

$$= 2\ln(e^x+1)+C$$

Exercise

Find the integral $\int \frac{4x^2+2x+4}{x+1}dx$

Solution

$$\int \frac{4x^2+2x+4}{x+1}dx = \int \left(4x+2+\frac{6}{x+1}\right)dx$$

$$= \int (4x-2)dx + \int \frac{6}{x+1}dx$$

$$= \int (4x-2)dx + 6\int \frac{d(x+1)}{x+1} \qquad \int \frac{d(U)}{U} = \ln|U|$$

$$= 2x^2 - 2x + 6\ln|x+1| + C$$

Exercise

Find the indefinite integral. $\int 4xe^{x^2}\,dx$

Solution

Let $u = x^2 \rightarrow du = 2xdx$

$$\int 4xe^{x^2}\,dx = \int 2e^u (2xdx)$$

$$= \int 2e^u\,du$$

$$= 2e^u + C$$

$$= 2e^{x^2} + C$$

Exercise

Find the indefinite integral. $\int \frac{3x}{x^2+4}\,dx$

Solution

Let $u = x^2 + 4 \rightarrow du = 2xdx \rightarrow \frac{1}{2}du = xdx$

$$\int \frac{3x}{x^2+4}\,dx = \int \frac{3}{u}\frac{1}{2}\,du$$

$$= \frac{3}{2}\int \frac{1}{u}\,du$$

$$= \frac{3}{2}\ln|u| + C$$

$$= \frac{3}{2}\ln(x^2+4) + C$$

Exercise

Evaluate the integral $\int 12t^3 e^{-t^4}\,dt$

Solution

$$u = -t^4 \quad \rightarrow \quad du = -4t^3dt \Rightarrow \quad -\frac{du}{4} = t^3dt$$

$$\int 12t^3 e^{-t^4}\,dt = \int 12e^u\left(-\frac{du}{4}\right)$$

$$= -3\int e^u\,du$$

$$= -3e^u + C$$

$$= -3e^{-t^4} + C$$

$$\boxed{= -\frac{3}{e^{t^4}} + C}$$

Exercise

Evaluate the integral $\int \frac{7e^{7x}}{3+e^{7x}}dx$

Solution

Let: $u = 3 + e^{7x} \quad \rightarrow \quad du = 7e^{7x}dx$

$$\int \frac{7e^{7x}}{3+e^{7x}}dx = \int \frac{du}{u}$$

$$= \ln|u|$$

$$\boxed{= \ln\left(3 + e^{7x}\right) + C}$$

Exercise

Under certain conditions, the number of diseased cells $N(t)$ at time t increases at a rate $N'(t) = Ae^{kt}$, where A is the rate of increase at time 0 (in cells per day) and k is a constant.

a) Suppose $A = 60$, and at 4 days, the cells are growing at a rate of 180 per day. Find a formula for the number of cells after t days, given that 200 cells are present at $t = 0$.

b) Use the answer from part (a) to find the number of cells present after 9 days.

Solution

a)

$$N'(t) = Ae^{kt}$$

$$180 = 60e^{k(4)}$$

$$3 = e^{4k}$$

$$4k = \ln 3$$

$$k = \frac{\ln 3}{4} \approx 0.27465$$

$$N'(t) = 60e^{0.27465t}$$

$$N(t) = \int N'(t)\,dt$$

$$= \int 60e^{0.27465t}\,dt$$

$$= 218.5e^{0.27465t} + C$$

$$N(t=0) = 200$$

$$218.5e^{0.27465(0)} + C = 200$$

$$218.5 + C = 200$$

$$C = 200 - 218.5 = -18.5$$

$$N(t) = 218.5e^{0.27465t} - 18.5$$

b)

$$N(t=9) = 218.5e^{0.27465(9)} - 18.5 = 2,569 \text{ cells}$$