Exercise 1

Ox and Oy are bisector to 2 adjacent acute angles, $\widehat{AOB}$ and $\widehat{BOC}$ where the difference is $36°$ and $\widehat{AOC} = 90°$. Oz is the bisector of the angle $\widehat{xOy}$. Determine the angle $\widehat{BOz}$

Solution

$$\widehat{BOC} - \widehat{AOB} = 36°$$
$$\underline{\widehat{BOC} + \widehat{AOB} = 90°}$$
$$2\ \widehat{BOC} = 126°$$

$$\widehat{BOC} = 63°$$

$$\widehat{AOB} = 27°$$

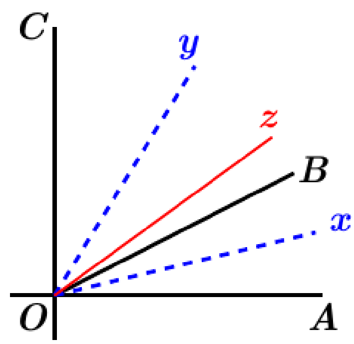
$$\widehat{xOB} = \frac{1}{2}\widehat{AOB} = \frac{27}{2}^{\circ}$$

$$\widehat{BOy} = \frac{63}{2}^{\circ}$$

$$\widehat{xOy} = \widehat{xOB} + \widehat{BOy} = \frac{1}{2}(63° + 27°) = 45°$$

$$\widehat{xOz} = \frac{45}{2}^{\circ}$$

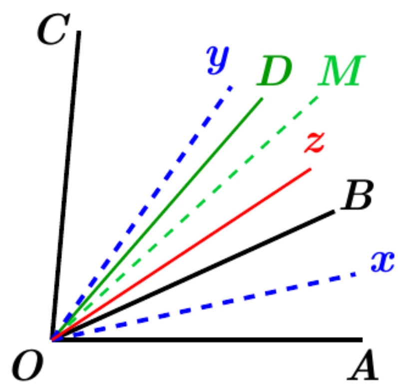
$$\widehat{BOZ} = \widehat{xOz} - \widehat{xOB} = \frac{1}{2}(45° - 27°) = 9°$$



Exercise 2

Ox and Oy are bisector to 2 adjacent acute angles, $\widehat{AOB}$ and $\widehat{BOC}$ where the difference is $36°$. Oz is the bisector of the angle $\widehat{xOy}$. Determine the angle $\widehat{BOz}$

Solution



Ox is the bisector $\widehat{AOB}$ (1)

OB is the bisector $\widehat{AOD}$ (2)

OM is the bisector $\widehat{AOC}$ (3)

Oz is the bisector $\widehat{xOy}$ (4)

Oy is the bisector $\widehat{BOC}$ (5)

$\widehat{BOC} - \widehat{AOB} = 36°$

$\widehat{BOC} - \widehat{BOD} = 36°$

$\boxed{\widehat{DOC} = 36°}$

$$(3) \rightarrow \quad \widehat{AOM} = \frac{1}{2}\widehat{AOC}$$
$$= \frac{1}{2}\left(2\widehat{AOB} + \widehat{DOC}\right)$$
$$= \frac{1}{2}\left(2\widehat{AOB} + 36°\right)$$
$$= \widehat{AOB} + 18°$$

$$\widehat{BOM} = \widehat{AOM} - \widehat{AOB}$$
$$= \widehat{AOB} + 18° - \widehat{AOB}$$
$$\boxed{= 18°}$$

$$(1) \rightarrow \quad \widehat{BOx} = \frac{1}{2}\widehat{AOB}$$

$$(4) \rightarrow \quad \widehat{BOy} = \frac{1}{2}\widehat{BOC}$$

$$(1)+(4) \rightarrow \quad \widehat{xOy} = \frac{1}{2}\widehat{AOC}$$

$$(3) \rightarrow \quad \widehat{AOM} = \frac{1}{2}\widehat{AOC} = \widehat{xOy}$$

$$\widehat{BOz} = \widehat{xOz} - \widehat{xOB}$$
$$= \frac{1}{2}\left(\widehat{xOy} - \widehat{AOB}\right)$$
$$= \frac{1}{2}\left(\widehat{AOM} - \widehat{AOB}\right)$$
$$= \frac{1}{2}\widehat{BOM}$$
$$\boxed{= 9°}$$

Exercise 3

Four consecutive half-lines (segments): OA, OB, OC, and OD formed angles such as

$$\widehat{DOA} = \widehat{COB} = 2\widehat{AOB} \qquad \text{and} \qquad \widehat{COD} = 3\widehat{AOB}$$

Calculate the angles to demonstrate that the bisectors of $\widehat{AOB}$ and $\widehat{COD}$ are in a straight line.

Solution

$$\widehat{AOB} + \widehat{BOC} + \widehat{COD} + \widehat{DOA} = 360°$$

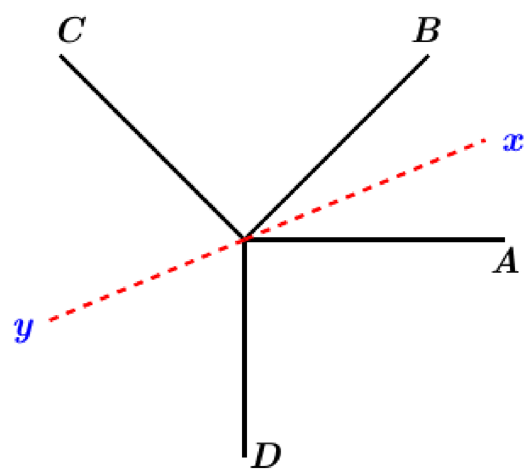
$$\widehat{AOB} + 2\widehat{AOB} + 3\widehat{AOB} + 2\widehat{AOB} = 360°$$

$$8\widehat{AOB} = 360°$$

$$\boxed{\widehat{AOB} = 45°}$$

$$\boxed{\widehat{DOA} = \widehat{COB} = 90°}$$

$$\boxed{\widehat{COD} = 135°}$$



Let:

Ox is the bisector $\widehat{AOB}$

Oy is the bisector $\widehat{COD}$

$$\widehat{xOy} = \widehat{xOB} + \widehat{BOC} + \widehat{COy}$$

$$= \frac{1}{2}\widehat{AOB} + 90° + \frac{1}{2}\widehat{COD}$$

$$= \frac{1}{2}(45° + 135°) + 90°$$

$$\boxed{= 180°}$$

Therefore; the bisectors of $\widehat{AOB}$ and $\widehat{COD}$ are in a straight line

Exercise 4

The segments OA and OB formed with OX the angles α and β.

a) Demonstrate that the bisector OC of the angle $\widehat{AOB}$ made with OX an angle $\dfrac{\alpha+\beta}{2}$.

b) Examine the cases where

i. $\alpha+\beta=90°$

ii. $\alpha+\beta=180°$

Solution

Given:

$$\widehat{XOA}=\alpha \quad \& \quad \widehat{XOB}=\beta$$

$$\widehat{AOC}=\frac{1}{2}\widehat{AOB}$$

$$=\frac{\beta-\alpha}{2}$$

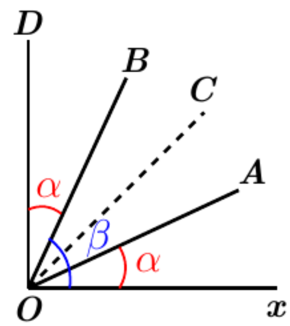
a)

$$\widehat{XOC}=\widehat{XOA}+\widehat{AOC}$$

$$=\alpha+\frac{\beta-\alpha}{2}$$

$$=\frac{\alpha+\beta}{2}$$

b) ***i.*** If $\alpha+\beta=90°$, then



$$\widehat{XOC}=45°$$

Let: $\widehat{XOD}=90°$ that implies OC is the bisector of $\widehat{XOD}$

Since OC is the bisector of $\widehat{AOB}$, then

$$\widehat{BOD}=90°-\beta \qquad \beta=90°-\alpha$$

$$=90°-90°+\alpha$$

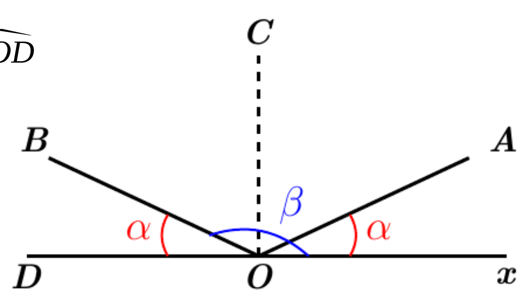
$$=\alpha$$

ii. If $\alpha+\beta=180°$, then

$$\widehat{XOC}=90°$$

Let: $\widehat{XOD}=180°$ that implies OC is the bisector of $\widehat{XOD}$

Since OC is the bisector of $\widehat{AOB}$, then



$$\widehat{BOD}=180°-\beta \qquad \beta=180°-\alpha$$

$$=180°-180°+\alpha$$

$$=\alpha$$

Exercise 5

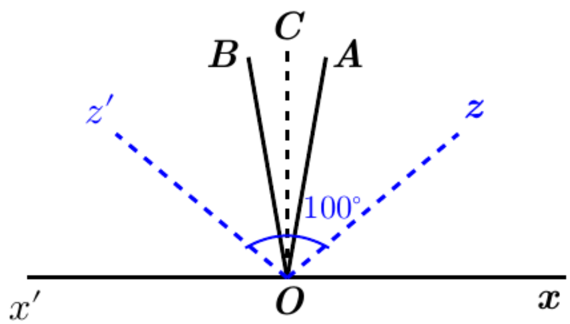
A point O takes on an infinite right $x'Ox$ be conducted the same side half-lines OA and OB, as well as the bisectors of angles $\widehat{xOA}$, $\widehat{AOB}$, and $\widehat{BOx'}$.

Calculate the angles of the figure such that the bisector of the angle $\widehat{AOB}$ is perpendicular to $x'Ox$ and the bisectors of the extreme angles formed an angle of $100°$.

Solution

Given: $\widehat{zOz'} = 100°$

$\widehat{xOC} = 90°$



OC is the bisector $\widehat{AOB}$

$\widehat{AOC} = \widehat{COB}$

Oz is the bisector $\widehat{xOA}$

$\widehat{xOz} = \widehat{zOA}$

Oz' is the bisector $\widehat{x'OB}$

$\widehat{x'Oz'} = \widehat{z'OB}$

$$\widehat{xOz} = \frac{180° - 100°}{2}$$

$$= 40°$$

$$\widehat{AOB} = 2\widehat{AOC}$$

$$= 2\left(90° - 2\widehat{xOz}\right)$$

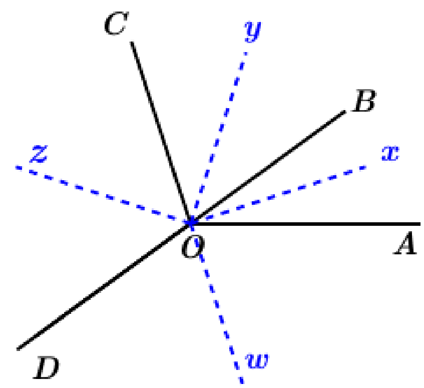
$$= 2\left(90° - 80°\right)$$

$$= 20°$$

Exercise 6

Four consecutive half-lines *OA*, *OB*, *OC*, and *OD* formed four adjacent consecutive angles which are between them like 1, 2, 3, 4.
Calculate the angles and the adjacent consecutive angles formed by their bisectors.

Solution



$$\widehat{AOB}+\widehat{BOC}+\widehat{COD}+\widehat{DOA}=360°$$

$$\widehat{AOB}+2\widehat{AOB}+3\widehat{AOB}+4\widehat{AOB}=360°$$

$$10\widehat{AOB}=360°$$

$$\widehat{AOB}=36°$$

$$\widehat{BOC}=72°$$

$$\widehat{COD}=108°$$

$$\widehat{DOA}=144°$$

$$\widehat{xOy}=\frac{1}{2}\widehat{AOB}+\frac{1}{2}\widehat{BOC}$$
$$=\frac{1}{2}36°+\frac{1}{2}72°$$
$$=18°+36°$$
$$=54°$$

$$\widehat{yOz}=\frac{1}{2}\widehat{BOC}+\frac{1}{2}\widehat{COD}$$
$$=\frac{1}{2}72°+\frac{1}{2}108°$$
$$=36°+54°$$
$$=90°$$

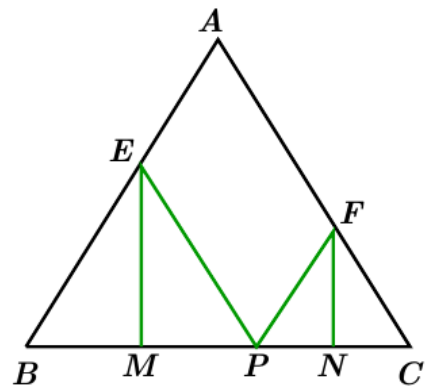
$$\widehat{zOw}=\frac{1}{2}\widehat{COD}+\frac{1}{2}\widehat{DOA}$$
$$=\frac{1}{2}108°+\frac{1}{2}144°$$
$$=54°+72°$$
$$=126°$$

$$\widehat{wOx}=\frac{1}{2}\widehat{DOA}+\frac{1}{2}\widehat{AOB}$$
$$=\frac{1}{2}144°+\frac{1}{2}36°$$
$$=72°+18°$$
$$=90°$$

Exercise 7

A point P is on the base BC of an isosceles triangle ABC. The two points M and N are the middle points of the segments PB and PC, respectively, which lead the perpendicular to the base BC; these perpendiculars meet AB in E, AC in F.
Demonstrate that the angle EPF is equal to A.

Solution



$$\widehat{BAC} = 180° - \widehat{ABC} - \widehat{ACB}$$

M is the middle of the segment BP and $EM \perp$ to BP, therefore

$$EB = EP \quad \& \quad \widehat{EBP} = \widehat{EPB}$$

N is the middle of the segment CP and $FN \perp$ to CP, therefore

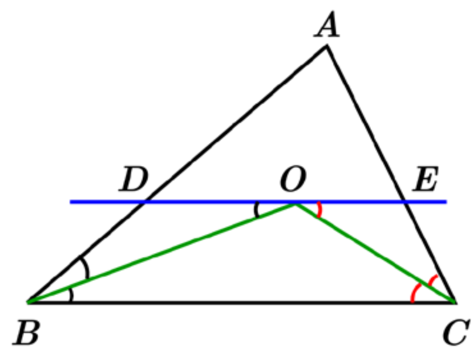
$$FP = FP \quad \& \quad \widehat{FPC} = \widehat{FCP}$$

$$\begin{aligned}\widehat{EPF} &= 180° - \widehat{CPF} - \widehat{BPE} \\ &= 180° - \widehat{PFC} - \widehat{PBE} \\ &= 180° - \widehat{ABC} - \widehat{ACB} \\ &= \widehat{A} \quad \checkmark\end{aligned}$$

Exercise 8

Given the triangle ABC and the bisectors BO and CO of the angles of the base, where the point O is the intersection of the 2 bisectors. A line DOE passes through the point O parallel to base BC.
Prove that $DE = DB + CE$

Solution



CO is the bisector of $\widehat{BCE} \Rightarrow \widehat{BCO} = \widehat{OCE}$

$OE \parallel BC \Rightarrow \widehat{COE} = \widehat{BOC}$

$\therefore \widehat{EOC} = \widehat{OCE} \rightarrow \underline{OE = EC}$

Similar; BO is the bisector of $\widehat{DBC} \Rightarrow \widehat{DBO} = \widehat{OBC}$

$DO \parallel BC \Rightarrow \widehat{DOB} = \widehat{OBC}$

$\therefore \widehat{DOB} = \widehat{OBC} \rightarrow \underline{DO = DB}$

$DE = DO + OE$

$\underline{= DB + CE}$

Exercise 9

A right triangle *ABC* at *A* with a height *AH*. We drop perpendiculars *HE* and *HD* from *H* to sides *AB* and *AC* respectively.

a) Prove that $DE = AH$

b) Prove that *AM* is perpendicular to *DE*, where *M* is the middle point of *BC*.

c) Prove that *MN* (*N* is the middle point of *AB*) and the segment *Bx* (parallel to *DE*) are intersect on *AH*.

d) Prove that *AM* and *HD* are intersect on *Bx*.

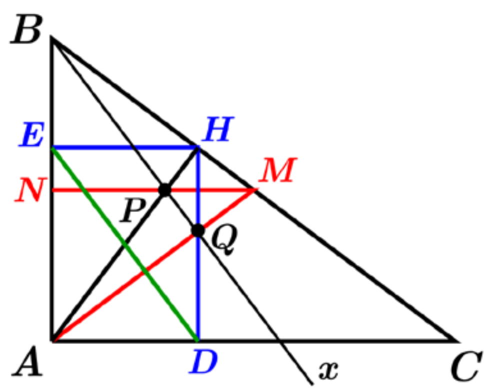
Solution

a) The triangles *AEH* and *ADH* are right triangles and angle *A* is right angle.
Then *AEHD* is a rectangle.
Therefore, $DE = AH$

b) A middle point of a hypotenuse of a right triangle is the center of the circle of that triangle.
Therefore, $MC = MA = MB$

That implies to: $\widehat{MAC} = \widehat{MCA}$

From the rectangle *ADHE*: $\widehat{EAH} = \widehat{EDH}$



$\widehat{EAH} + \widehat{HAM} + \widehat{MAC} = 90°$ $\qquad \widehat{HAM} + \widehat{MAC} = \widehat{HAC}$

$\widehat{EAH} + \widehat{HAC} = 90°$

$\widehat{EAH} + 90° - \widehat{MCA} = 90°$

$\widehat{EAH} = \widehat{MCA} = \widehat{EDH} = \widehat{MAC}$

$\widehat{ADE} + \widehat{EDH} = 90°$

$\widehat{ADE} + \widehat{MAD} = 90°$

Therefore, *AM* is perpendicular to *DE*.

c) *N* is the middle point of *AB* $\Rightarrow$ $NA = NB$

Bx parallel to *DE* $\Rightarrow$ $\widehat{ABx} = \widehat{AED} = \widehat{EDH} = \widehat{EAH}$

Let point *P* the intersection of B*x* and *AH*. Since $\widehat{ABP} = \widehat{BAP}$, then the triangle *BPA* is an isosceles. *PN* is the perpendicular to *AB* as well *MN*. Which gives us that points *M*, *P*, *N* are on the same line.
Therefore, segment *MN* and *AH* intersect at point *P*.

d) Let Point *Q* be the intersection of *AM* and *Bx*.

$\widehat{ABQ} = \widehat{BAH}$ $\quad$ & $\quad$ $\widehat{BAQ} = \widehat{ABH}$

Then, the triangles *BHA* and *BQA* are equivalent, therefore $AQ \perp BQ$ with hypotenuse *AB*.

$HQ \,//\, AB$, line *HQ* has to be perpendicular to *AC*.

AM and *HD* are intersect on *Bx* at *Q*.

Exercise **10**

Given an isosceles triangle *ABC* with a peak at *A*. Extend base *BC* the length $CD = AB$, then extend *AB* of a length $BE = \frac{1}{2}BC$, at the end draw a line *EHF*, *H* is the middle point of *BC* and *F* is located on *AD*.

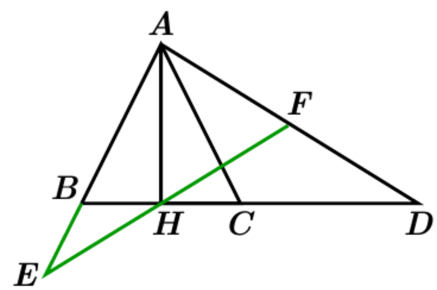
a) Prove that $\widehat{ADB} = \frac{1}{2}\widehat{ABC}$

b) Prove that $EA = HD$

c) Prove that $FA = FD = FH$

d) Calculate the value of the angles $\widehat{AFH}$ and $\widehat{ADB}$ where $\widehat{BAC} = 58°$.

Solution



a) Triangle *ABC* is isosceles, then $\widehat{ABC} = \widehat{ACB}$

Since, $CD = AB = AC$, then $\widehat{CAD} = \widehat{ADC}$

$$2\widehat{ADC} = 180° - \widehat{ACD}$$

$$2\widehat{ADC} = 180° - \left(180° - \widehat{ACB}\right)$$

$$2\widehat{ADC} = \widehat{ACB}$$

$$\widehat{ADB} = \frac{1}{2}\widehat{ACB}$$

$$= \frac{1}{2}\widehat{ABC}$$

b) $BE = \frac{1}{2}BC$ *H* the middle point of *BC*

$$= HC$$

$$CD = AB$$

$$HC + CD = BE + AB$$

$$EA = HD \quad \checkmark$$

c) $\widehat{ADH} = \frac{1}{2}\widehat{ABD}$

$$= \frac{1}{2}\left(180° - \widehat{HBE}\right)$$

$$= \frac{1}{2}\left(180° - 180° + 2\widehat{BHE}\right)$$

$$= \widehat{BHE}$$

$$\Rightarrow \quad FD = FH$$

$$\widehat{AHF} = 90° - \widehat{FHD}$$

$$= 90° - \widehat{ADH} \qquad (\Delta\ HDA)$$

$$= 90° - \left(90° - \widehat{HAF}\right)$$

$$= \widehat{HAF}$$

$\Rightarrow \quad \underline{FA = FH}$

$FA = FD = FH$ ✓

d) $\widehat{BAC} = 58°$

$$\widehat{ADB} = \frac{1}{2}\widehat{ACB}$$

$$= \frac{1}{2}\left(\frac{1}{2}\left(180° - \widehat{BAC}\right)\right)$$

$$= \frac{1}{4}(180° - 58°)$$

$$= \frac{122}{4}^{\circ}$$

$$\underline{= \frac{61}{2}^{\circ}} \qquad \underline{= 30.5°}$$

Triangle *AFH* is isosceles then,

$$\widehat{AFH} = 180° - \widehat{HFD}$$

$$= 180° - \left(180° - 2\widehat{FDH}\right)$$

$$= 2\widehat{FDH}$$

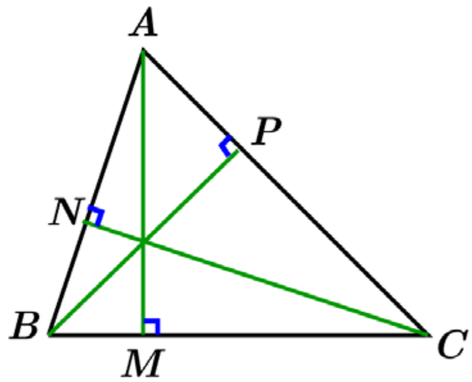
$$= 2\frac{61°}{2}$$

$$\underline{= 61°}$$

Exercise 11

Demonstrate that the heights of a triangle share the angles of triangle that equal to each other.

Solution



Consider the 2 right triangles *APB* and *ANC*, which they have the same angle *A*.

Therefore, $\widehat{ABP} = \widehat{ACN}$.

Similar, consider the 2 right triangles *BPC* and *AMC*, which they have the same angle *C*.

Therefore, $\widehat{MAC} = \widehat{CBP}$.

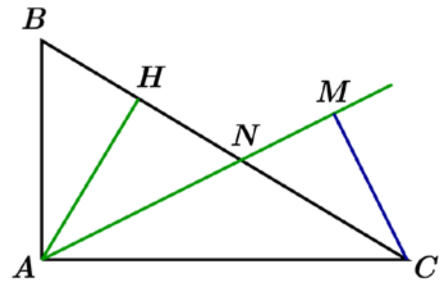
Similar, consider the 2 right triangles *BNC* and *AMB*, which they have the same angle *B*.

Therefore, $\widehat{BCN} = \widehat{BAM}$.

Exercise 12

A right triangle ABC at A where $AB < AC$, drop a perpendicular AH from A to the hypotenuse BC where $HN = HB$. From C drops a perpendicular CM at AN. Demonstrate that BC is the bisector of the angle $\widehat{ACM}$.

Solution



Consider the 2 right triangles ABC and ABH with a common angle B, then

$\widehat{BAH} = \widehat{ACB}$

Given: $HN = HB$, then $\widehat{HAN} = \widehat{BAH} = \widehat{ACB}$

$$\widehat{NAC} = 90° - \widehat{HAB} - \widehat{HAN}$$
$$= 90° - 2\widehat{HCA}$$

Consider the 2 right triangles AHN and CMC, where $\widehat{HNA} = \widehat{MNC}$

Therefore, $\widehat{HAN} = \widehat{NCM}$

Since $\widehat{HAN} = \widehat{ACB}$

Then $\widehat{ACB} = \widehat{MCB}$

Therefore, BC is the bisector of the angle $\widehat{ACM}$

Exercise 13

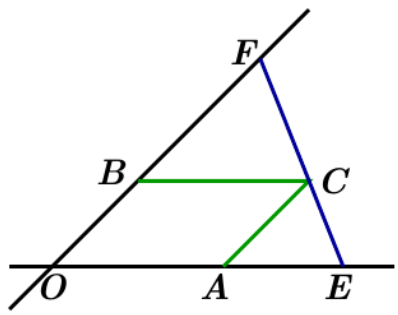
On the sides of an angle that it takes the length *OA* and *OB*, so that $OA+OB=\ell$ (is given) and construct a parallelogram *OABC*. What is the place of the summit *C* of parallelogram?

Solution

Let segment *OE* extension of segment *OA* such that $OE=\ell$
Let segment *OF* extension of segment *OB* such that $OF=\ell$

Then, the triangle *OEF* is an isosceles.

$$\widehat{OEF}=\widehat{OFE}=90°-\frac{1}{2}\widehat{EOF}$$



$$OA+OB=\ell$$

$$\begin{cases} OA+AC=\ell \\ OA+AE=\ell \end{cases} \Rightarrow \quad AC=AE$$

$$\begin{cases} OB+BC=\ell \\ OB+BF=\ell \end{cases} \Rightarrow \quad BC=BF$$

$$\widehat{OEF}=\widehat{OFE}=\widehat{FCB}=\widehat{ACE}$$

Therefore, the point *C*, *E*, and *F* are aligned.