Lecture Two – Differentiation

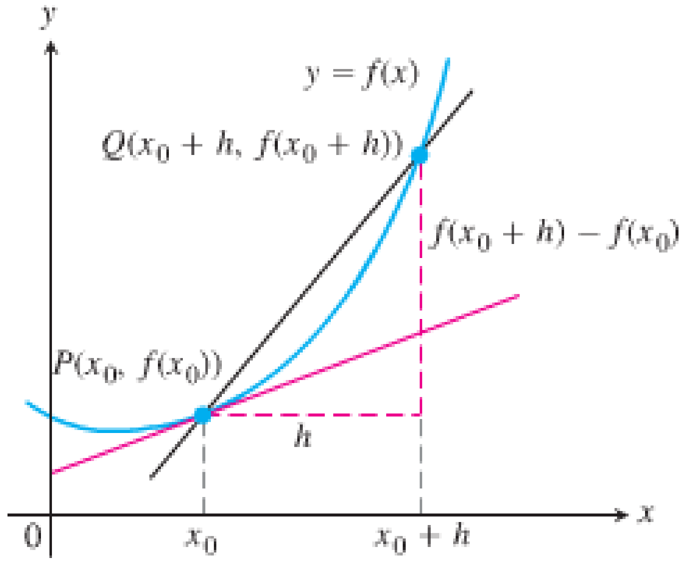
Section 2.1 –Tangents and the Derivative at a point

Definition

The slope of the curve $y = f(x)$ at the point $P\left(x_0, f\left(x_0\right)\right)$ is the number

$$m = \lim_{h \to 0} \frac{f\left(x_0 + h\right) - f\left(x_0\right)}{h}, \quad (\lim \exists)$$

The tangent line to the curve at P is the line through P with this slope.



Example

a) Find the slope of the curve $y = \frac{1}{x}$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

b) Where does the slope equal $-\frac{1}{4}$?

c) What happens to the tangent to the curve at the point $\left(a, \frac{1}{a}\right)$ as a changes?

Solution

a) The slope of $f(x) = \frac{1}{x}$ at $\left(a, \frac{1}{a}\right)$ is

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{a-(a+h)}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{a-a-h}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{-h}{a(a+h)}$$

$$= \lim_{h\to 0} \frac{-1}{a(a+h)}$$

$$= -\frac{1}{a^2}$$

The slope at $x=-1$ is: $= -\frac{1}{(-1)^2}$ $\underline{=-1}$

b) The slope equals to $x = -\frac{1}{4} \Rightarrow -\frac{1}{a^2} = -\frac{1}{4} \quad \Rightarrow \quad a^2 = 4 \to \boxed{a = \pm 2}$

$$\left.\begin{array}{l} x=-2 \Rightarrow y = -\frac{1}{2} \\ x = 2 \Rightarrow y = \frac{1}{2} \end{array}\right. \Rightarrow \left(-2,\ -\frac{1}{2}\right) \textit{ and } \left(2,\ \frac{1}{2}\right)$$

c) The slope $\left(-\frac{1}{a^2}\right)$ is always negative if $a \neq 0$

$\lim_{x\to\pm\infty}\left(-\frac{1}{a^2}\right) = 0$ The slope approaches 0 and the tangent becomes horizontal.

$\lim_{x\to 0^-}\left(-\frac{1}{a^2}\right) = -\infty$ The slope approaches $-\infty$ and the tangent increasingly steep.

Definition of the Derivative

The derivative of a function f at a point x_0, denoted $f'\left(x_0\right)$ is

$$f'\left(x_0\right) = \lim_{h \to 0} \frac{f\left(x_0 + h\right) - f\left(x_0\right)}{h}, \quad (\lim \exists)$$

Summary

The following are all interpretations for the limit of the difference quotient, $\lim\limits_{h \to 0} \dfrac{f\left(x_0 + h\right) - f\left(x_0\right)}{h}$

1. The slope of the graph of $y = f(x)$ @ $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ @ $x = x_0$
3. The rate of change of $f(x)$ with respect to x @ $x = x_0$
4. The derivative $f'\left(x_0\right)$ at a point

Exercises *Section* 2.1 –Tangents and the Derivative at a point

1. Find an equation for the tangent to the curve $y = 4 - x^2$ at the point $(-1,\ 3)$. Then sketch the curve and tangent together.

2. Find an equation for the tangent to the curve $y = \frac{1}{x^2}$ at the point $(-1,\ 1)$. Then sketch the curve and tangent together.

3. Find the slope of the function $f(x) = 2\sqrt{x}$ at the point $(1,\ 2)$. Then find an equation for the line tangent to the graph there.

4. Find the slope of the function $f(x) = x^3 + 3x$ at the point $(1,\ 4)$. Then find an equation for the line tangent to the graph there.

5. Find the slope of the curve $y = 1 - x^2$ at the point $x = 2$

6. Find the slope of the curve $y = \frac{1}{x-1}$ at the point $x = 3$

7. Find the slope of the curve $y = \frac{x-1}{x+1}$ at the point $x = 0$

8. Find equations of all lines having slope –1 that are tangent to the curve $y = \frac{1}{x-1}$

9. What is the rate of change of the area of a circle $\left(A = \pi r^2\right)$ with respect to the radius when the radius is $r = 3$?

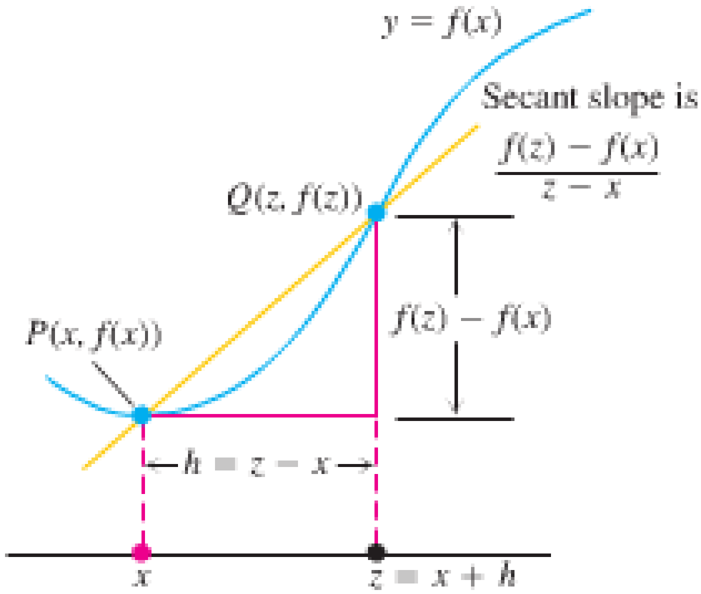
10. Find the slope of the tangent to the curve $y = \frac{1}{\sqrt{x}}$ at the point where $x = 4$

Section 2.2 –The Derivative as a Function

Definition

The derivative of the function $f(x)$ with respect to the variable *x* is the function f' whose value at *x* is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$



If f' exists at a particular *x*, we say that f is ***differentiable*** (has a ***derivative***) at *x*.
If f' exists at every point in the domain of f, we call f ***differentiable***

The process of finding derivatives is called ***differentiation***.

Notations

Some common alternative notations for the derivative are

$$f'(x), \quad f', \quad \frac{d}{dx}[f(x)], \quad \frac{d}{dx}f, \quad \frac{dy}{dx}, \quad y', \quad \dot{y}, \text{ and } D_x[y]$$

Example

Differentiate $f(x) = \dfrac{x}{x-1}$

Solution

$$f(x+h) = \frac{(x+h)}{(x+h)-1}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{x^2 - x + hx - h - x^2 - hx + x}{(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{-h}{(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)}$$

$$= \frac{-1}{(x-1)(x-1)}$$

$$= \frac{-1}{(x-1)^2}$$

Example

Find the derivative of $f(x) = x^2$

Solution

$$f(x+h) = (x+h)^2$$

$$= x^2 + 2hx + h^2$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{2hx + h^2}{h}$$

$$= \lim_{h\to 0} (2x+h)$$

$$= 2x$$

Example

a) Find the derivative of $f\left(x\right)=\sqrt{x}$ for $x>0$

b) Find the tangent line to the curve $y=\sqrt{x}$ *at* $x=4$

Solution

a)
$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\,\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\ &= \lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\ &= \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\ &= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \\ &= \frac{1}{\sqrt{x}+\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

b) The slope of the curve at $x = 4$ is: $f'\left(4\right)=\frac{1}{2\sqrt{4}}=\frac{1}{4}$

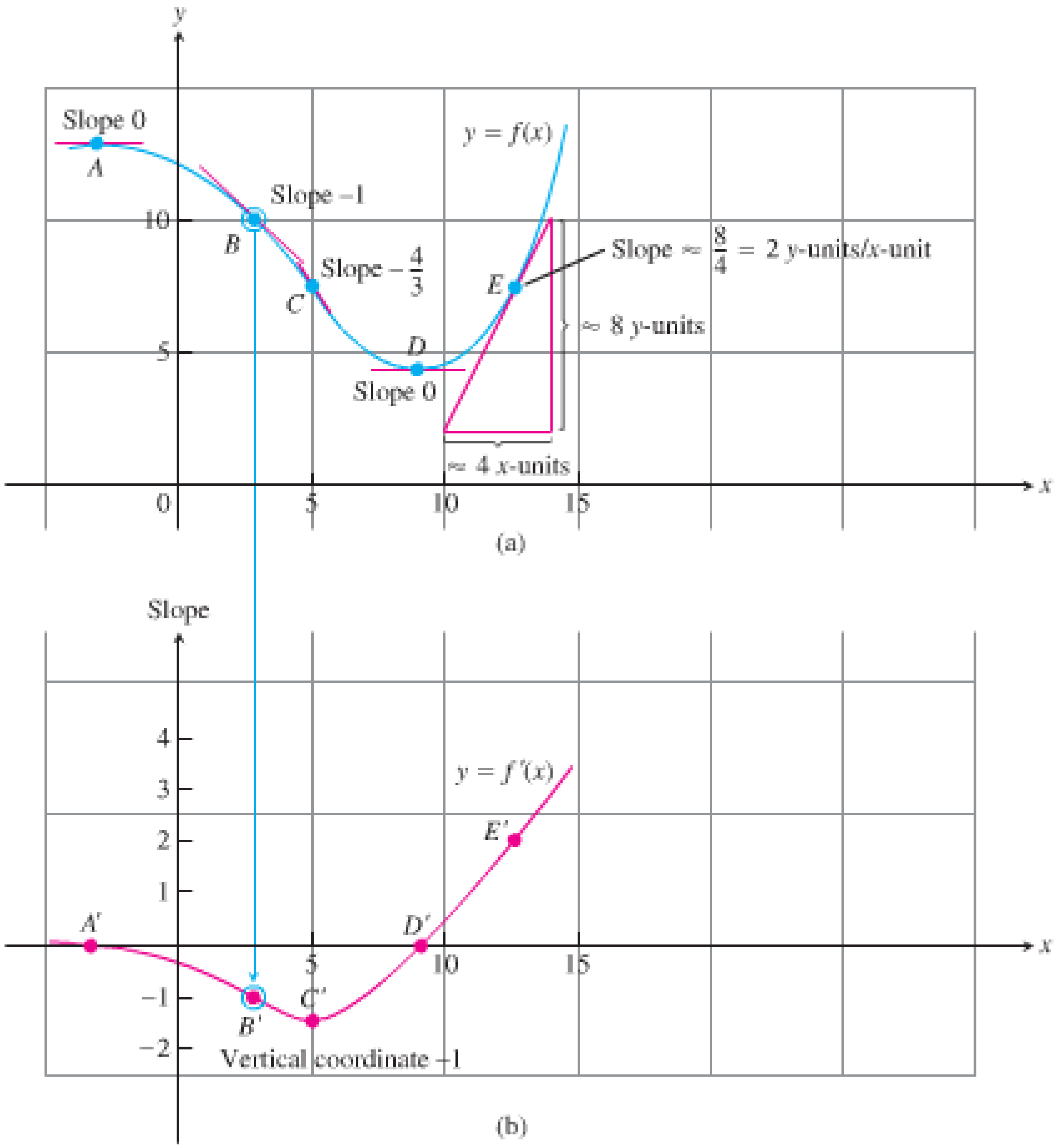
The tangent is the line through the point (4, 2) with slope $\frac{1}{4}$:

$$y-2=\tfrac{1}{4}\left(x-4\right)$$

$$y=\tfrac{1}{4}x-1+2$$

$$\boxed{y=\tfrac{1}{4}x+1}$$

Graphing



- ✓ The rate of change of f is positive, negative, or zero
- ✓ The rough size of the growth rate at any x and its size in relation to the size of $f(x)$
- ✓ Where the rate of change itself is increasing or decreasing.

Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

Theorem – Differentiability Implies Continuity

If f has a derivative at $x=c$, then f is continuous at $x=c$

Proof

Given that $f'(c)$ exists, we must show that $\lim\limits_{x\to c} f(x) = f(c)$, or equivalently, that $\lim\limits_{h\to 0} f(c+h) = f(c)$. If $h \neq 0$, then

$$\begin{aligned} f(c+h) &= f(c) + \big(f(c+h) - f(c)\big) \\ &= f(c) + \frac{f(c+h)-f(c)}{h} \cdot h \end{aligned}$$

Take the limits as $h \to 0$.

$$\begin{aligned} \lim_{h\to 0} f(c+h) &= \lim_{h\to 0} f(c) + \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \cdot \lim_{h\to 0} h \\ &= f(c) + f'(c) \cdot 0 \\ &= f(c) \end{aligned}$$

Exercises *Section* 2.2 –The Derivative as a Function

1. Fin the values of the derivatives of the function $f(x) = 4 - x^2$. Then find the values of $f'(-3),\ f'(0),\ f'(1)$

2. Fin the values of the derivatives of the function $r(s) = \sqrt{2s+1}$. Then find the values of $r'(0),\ r'\left(\frac{1}{2}\right),\ r'(1)$

3. Find the derivative of $f(x) = 3x^2 - 2x$

4. Find the derivative of y with the respect to t for the function $y = \frac{4}{t}$

5. Find the derivative of $\frac{dy}{dx}$ *if* $y = 2x^3$

6. Differentiate the function $y = \frac{x+3}{1-x}$ and find the slope of the tangent line at the given value of the independent variable.

7. Find the equation of the tangent line to $f(x) = x^2 + 1$ that is parallel to $2x + y = 0$

8. Use the definition of limits to find the derivative: $f(x) = \frac{3}{\sqrt{x}}$

9. Use the definition of limits to find the derivative: $f(x) = \sqrt{x+2}$

Section 2.3 – Differentiation Rules

Notations for the Derivative

The derivative of $y = f(x)$ may be written in any of the following ways:

1st derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}\left[f(x)\right]$	$D_x[y]$

Derivative of a *constant Function*

If f has the constant value $f\left(x\right) = c$

$$\frac{d}{dx}[c] = f'(c) = 0 \qquad c \text{ is constant}$$

Proof

Let $f(x) = c$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{c-c}{h}$$

$$= 0$$

So, $\frac{d}{dx}[c] = 0$

Example

Find the derivative

a) $f(x) = 9$

$f' = 0$

b) $h(t) = \pi$

$D_t\left[h(t)\right] = 0$

Power Rule

$$f(x) = x^{n} \quad \Rightarrow \quad f'(x) = nx^{n-1} \qquad \boldsymbol{n} \text{ is any real number}$$

Proof

Let $f(x) = x^{n}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^{n} - x^{n}}{h}$$

$$= \lim_{h\to 0} \frac{x^{n} + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^{2} + \ldots + h^{n} - x^{n}}{h}$$

$$= \lim_{h\to 0} \; nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h + \ldots + h^{n-1}$$

$$= nx^{n-1}$$

Example

Find the derivative of: $a)\ x^{3}$ $\quad b)\ x^{2/3}$ $\quad c)\ \frac{1}{x^{4}}$ $\quad d)\ x^{\sqrt{2}}$ $\quad e)\ \sqrt{x^{2+\pi}}$

Solution

a) $y = x^{3}$

$$\frac{dy}{dx} = 3x^{3-1}$$

$$\boxed{= 3x^{2}}$$

b) $y = x^{2/3}$

$$y' = \frac{2}{3}x^{2/3-1}$$

$$\boxed{= \frac{2}{3}x^{-1/3}}$$

c) $y = \frac{1}{x^{4}}$

$$y = x^{-4}$$

$$= -4x^{-4-1}$$

$$= -4x^{-5}$$

$$\boxed{= -\frac{4}{x^{5}}}$$

d) $D_x\left(x^{\sqrt{2}}\right) = \boxed{\sqrt{2}x^{\sqrt{2}-1}}$

e) $y = \sqrt{x^{2+\pi}}$

$$y = \left(x^{2+\pi}\right)^{1/2}$$

$$= x^{(2+\pi)/2}$$

$$y' = \left(\frac{2+\pi}{2}\right)x^{1+\pi/2-1}$$

$$= \frac{1}{2}(2+\pi)\sqrt{x^{\pi}}$$

Derivative Constant Multiple Rule

If f is a differentiable function of x, and c is a real number (constant), then

$$\frac{d}{dx}(cf) = c\frac{df}{dx}$$

In particular, if n is any real number, then

$$\frac{d}{dx}\left(cx^n\right) = cnx^{n-1}$$

Proof

$$\frac{d}{dx}(cf) = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}$$

$$= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad \textit{Factor } c$$

$$= c\frac{df}{dx}$$

Example

If $y = 8x^4$, find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = 8\left(4x^3\right)$$

$$\boxed{= 32x^3}$$

Example

If $y = -\frac{3}{4}x^{12}$, find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = -\frac{3}{4}\left(12x^{11}\right)$$

$$\underline{= -9x^{11}}$$

Sum or Difference Rule

The derivative of the sum or difference of two differentiable functions is the sum or difference of their derivatives.

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} \qquad\qquad \frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$$

$$= u' + v' \qquad\qquad = u' - v'$$

Proof

$$f(x) = u(x) + v(x)$$

$$\frac{d}{dx}\left[u(x)+v(x)\right] = \lim_{h\to 0}\frac{\left[u(x+h)+v(x+h)\right]-\left[u(x)+v(x)\right]}{h}$$

$$= \lim_{h\to 0}\frac{u(x+h)+v(x+h)-u(x)-v(x)}{h}$$

$$= \lim_{h\to 0}\left[\frac{u(x+h)-u(x)}{h} + \frac{v(x+h)-v(x)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{u(x+h)-u(x)}{h}\right] + \lim_{h\to 0}\left[\frac{v(x+h)-v(x)}{h}\right]$$

$$= \frac{du}{dx} + \frac{dv}{dx}$$

Example

Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$

$$= 3x^2 + \frac{8}{3}x - 5 + 0$$

$$\underline{= 3x^2 + \frac{8}{3}x - 5}$$

Example

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

Solution

$$y' = 4x^3 - 4x$$

$$y' = 0 \quad \Rightarrow \quad 4x^3 - 4x = 0$$

$$4x\left(x^2 - 1\right) = 0$$

$$\boxed{x = 0} \quad x^2 - 1 = 0 \Rightarrow \boxed{x = \pm 1}$$

The curve has horizontal tangents at $x = 0,\ 1,\ and\ \ -1$.

The corresponding points on the curve are; (0, 2), (1, 1) and (–1, 1)

Product Rule

The derivative of the product of two differentiable functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$(f.g)' = f'.g + g'.f$$

$$\frac{d}{dx}[f(x)g(x)h(x)] = f'gh + fg'h + fgh'$$

Example

Find the derivative of $f(x) = (2x+3)\left(3x^2\right)$

Solution

$$f' = (2x+3)\left(3x^2\right)' + (2x+3)'\left(3x^2\right) \qquad f(x) = 6x^3 + 9x^2$$

$$= (2x+3)(6x) + (2)\left(3x^2\right)$$

$$= 12x^2 + 18x + 6x^2$$

$$= 18x^2 + 18x$$

$$\boxed{= 18x(x+1)}$$

Proof of the Derivative Product Rule

$$\frac{d}{dx}(uv) = \lim_{h\to 0} \frac{u(x+h)v(x+h)-u(x)v(x)}{h}$$

$$= \lim_{h\to 0} \frac{u(x+h)v(x+h)-u(x+h)v(x)+u(x+h)v(x)-u(x)v(x)}{h}$$

$$= \lim_{h\to 0} \left[\frac{u(x+h)v(x+h)-u(x+h)v(x)}{h} + \frac{u(x+h)v(x)-u(x)v(x)}{h}\right]$$

$$= \lim_{h\to 0} \left[u(x+h)\frac{v(x+h)-v(x)}{h} + v(x)\frac{u(x+h)-u(x)}{h}\right]$$

$$= \lim_{h\to 0} u(x+h)\cdot \lim_{h\to 0} \frac{v(x+h)-v(x)}{h} + v(x)\cdot \lim_{h\to 0} \frac{u(x+h)-u(x)}{h}$$

$$= u\cdot\frac{dv}{dx} + v\cdot\frac{du}{dx}$$

Example

Find the derivative of $y = (3x^2+1)(x^3+3)$

Solution

$u = 3x^2+1 \quad v = x^3+3$

$u' = 6x \qquad v' = 3x^2$

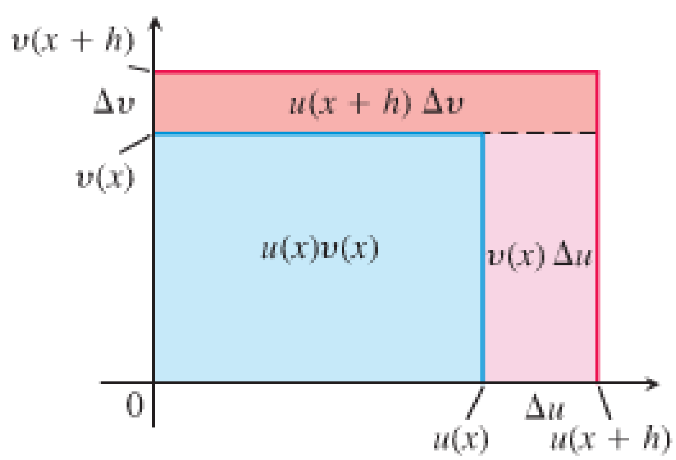
$y' = (6x)(x^3+3) + (3x^2)(3x^2+1)$

$= 6x^4 + 18x + 9x^4 + 3x^2$

$\underline{= 15x^4 + 3x^2 + 18x}$

$y = 3x^5 + 9x^2 + x^3 + 3$

$y' = 15x^4 + 18x + 3x^2$



Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)g(x)-g'(x)f(x)}{[g(x)]^2}$$

$$=\frac{f'g-g'f}{g^2}$$

Example

Find $f'(x)$ if $f(x)=\frac{2x-1}{4x+3}$

Solution

$$f'=\frac{(2x-1)'(4x+3)-(2x-1)(4x+3)'}{(4x+3)^2}$$

$u=2x-1 \quad v=4x+3$

$u'=2 \qquad v'=4$

$$=\frac{(2)(4x+3)-(2x-1)(4)}{(4x+3)^2}$$

$$=\frac{8x+6-8x+4}{(4x+3)^2}$$

$$=\boxed{\frac{10}{(4x+3)^2}}$$

Example

Find the derivative of $\quad y=\frac{(x-1)\left(x^2-2x\right)}{x^4}$

Solution

$$y=\frac{x^3-2x^2-x^2+2x}{x^4}$$

$$=\frac{x^3-3x^2+2x}{x^4}$$

$$=\frac{x^3}{x^4}-\frac{3x^2}{x^4}+\frac{2x}{x^4}$$

$$=x^{-1}-3x^{-2}+2x^{-3}$$

$$y'=-x^{-2}+6x^{-3}-6x^{-4}$$

$$\boxed{=-\frac{1}{x^2}+\frac{6}{x^3}-\frac{6}{x^4}}$$

Combining the product and Quotient Rules

Example

Find the derivative of $y = \frac{(1+x)(2x-1)}{x-1}$

Solution

$$y' = \frac{(x-1)\frac{d}{dx}\left[(1+x)(2x-1)\right] - (1+x)(2x-1)\frac{d}{dx}[x-1]}{(x-1)^2}$$

$$= \frac{(x-1)\left[(1)(2x-1) + 2(1+x)\right] - (1+x)(2x-1)(1)}{(x-1)^2}$$

$$= \frac{(x-1)(2x-1+2+2x) - (2x-1+2x^2-x)}{(x-1)^2}$$

$$= \frac{(x-1)(4x+1) - 2x+1-2x^2+x}{(x-1)^2}$$

$$= \frac{4x^2+x-4x-1-2x+1-2x^2+x}{(x-1)^2}$$

$$= \boxed{\frac{2x^2-4x}{(x-1)^2}}$$

Or

$$y = \frac{(1+x)(2x-1)}{x-1}$$

$$= \frac{2x-1+2x^2-x}{x-1}$$

$$= \frac{2x^2+x-1}{x-1}$$

$$y' = \frac{(x-1)(4x+1) - (2x^2+x-1)(1)}{(x-1)^2}$$

$$= \frac{4x^2+x-4x-1-2x^2-x+1}{(x-1)^2}$$

$$= \frac{2x^2+4x}{(x-1)^2}$$

$$= \boxed{\frac{2x(x+2)}{(x-1)^2}}$$

Second– and *Higher*–Order Derivatives

Notation for Higher-Order Derivatives							
1.	**1st derivative**	y'	***y*** prime	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$	$D_x[y]$
2.	**2^{nd} derivative**	y''	***y*** double prime	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}[f(x)]$	$D_x^2[y]$
3.	**3^{rd} derivative**	y'''	***y*** triple prime	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}[f(x)]$	$D_x^3[y]$
4.	**4^{th} derivative**	$y^{(4)}$		$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}[f(x)]$	$D_x^4[y]$
5.	**n^{th} derivative**	$y^{(n)}$		$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}[f(x)]$	$D_x^n[y]$

Example

Find the first four derivatives of $y = x^3 - 3x^2 + 2$

Solution

$y' = 3x^2 - 6x$

$y'' = 6x - 6$

$y''' = 6$

$y^{(4)} = 0$

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad \Rightarrow \boxed{f^{(n)}(x) = n!a_n}$$

Exercises *Section* 2.3 – Differentiation Rules

Find the derivative of each function

1. $y=\frac{1}{x^3}$

2. $D_x\left(x^{4/3}\right)$

3. $y=\sqrt{z}$

4. $D_t(-8t)$

5. $y=\frac{9}{4x^2}$

6. $y=6x^3+15x^2$

7. $y=3x^4-6x^3+\frac{x^2}{8}+5$

8. $p(t)=12t^4-6\sqrt{t}+\frac{5}{t}$

9. $f(x)=\frac{x^3+3\sqrt{x}}{x}$

10. $y=\frac{x^3-4x}{\sqrt{x}}$

11. $f(x)=\left(4x^2-3x\right)^2$

12. $y=(x+1)\left(\sqrt{x}+2\right)$

13. $y=\left(4x+3x^2\right)(6-3x)$

14. $y=\left(\frac{1}{x}+1\right)(2x+1)$

15. $y=3x\left(2x^2+5x\right)$

16. $y=3\left(2x^2+5x\right)$

17. $y=\frac{x^2+4x}{5}$

18. $y=\frac{3x^4}{5}$

19. $y=\frac{3-\frac{2}{x}}{x+4}$

20. $g(x)=\frac{x^2-4x+2}{x^2+3}$

21. $f(x)=\frac{(3-4x)(5x+1)}{7x-9}$

22. $f(x)=x\left(1-\frac{2}{x+1}\right)$

23. $g(s)=\frac{s^2-2s+5}{\sqrt{s}}$

24. $f(x)=\frac{x+1}{\sqrt{x}}$

25. $f(x)=\left(\sqrt{x}+3\right)\left(x^2-5x\right)$

26. $y=(2x+3)\left(5x^2-4x\right)$

27. $y=\left(x^2+1\right)\left(x+5+\frac{1}{x}\right)$

28. $y=\frac{x+4}{5x-2}$

29. $z=\frac{4-3x}{3x^2+x}$

30. $y=(2x-7)^{-1}(x+5)$

31. $f(x)=\frac{\sqrt{x}-1}{\sqrt{x}+1}$

32. $y=\frac{1}{\left(x^2-1\right)\left(x^2+x+1\right)}$

33. Find the first and second derivatives $y=-x^3+3$

34. Find the first and second derivatives $y=3x^7-7x^3+21x^2$

35. Find the first and second derivatives $y=6x^2-10x-\frac{1}{x}$

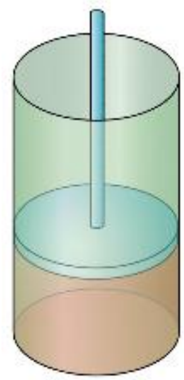
36. Find the first and second derivatives $y = \dfrac{x^2 + 5x - 1}{x^2}$

37. Find the first and second derivatives $y = \dfrac{x^2 + 3}{(x-1)^3 + (x+1)^3}$

38. Find an equation of the tangent line to the graph of $y = \dfrac{x^2 - 4}{2x + 5}$ when $x = 0$

39. Find an equation for the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point (2, 1).

40. If gas in a cylinder is maintained at a constant temperature T, the pressure P is related to the volume V by a formula of the form



$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2}$$

In which a, b, n, and R are constants. Find $\dfrac{dP}{dV}$

Section 2.4 – The Derivative as a Rate of Change

Definition

The ***instantaneous rate of change*** of f with respect to *x* at x_0 is the derivative

$$f'\left(x_0\right) = \lim_{x\to\infty} \frac{f\left(x_0+h\right)-f\left(x\right)}{h}$$

Provided the limit exists.

Example

The area A of the circle is related to its diameter by the equation $A = \frac{\pi}{4}D^2$

How fast does the area change with respect to the diameter when the diameter is 10 m?

Solution

The rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4}\cdot 2D = \frac{\pi D}{2}$$

When $D = 10$ m, the area is changing with respect to the diameter at the rate of

$$\frac{dA}{dD} = \frac{\pi(10)}{2} \approx 15.71\ m^2 / m$$

Motion along a Line: Displacement, Velocity, Speed, Acceleration, and *Jerk*

Suppose that an object is moving along a coordinate line (an *s*-axis), usually horizontal or vertical, so that we know its position *s* on that line as a function of time *t*:

$$s = f\left(t\right)$$

The ***displacement*** of the object over the time interval from t to $t + \Delta t$ is

$$\Delta s = f\left(t+\Delta t\right) - f\left(t\right)$$

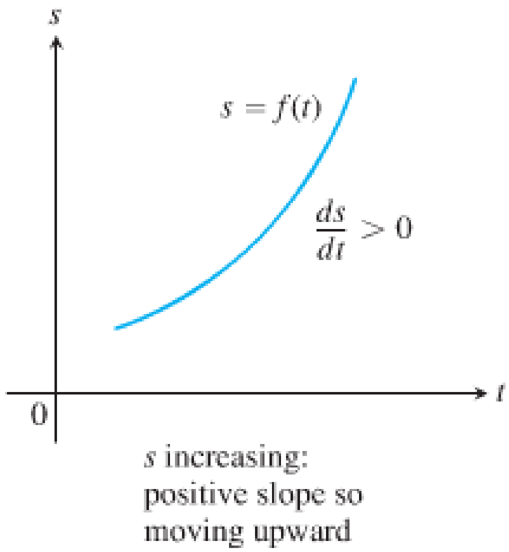
And the ***average velocity*** of the object over that time interval is

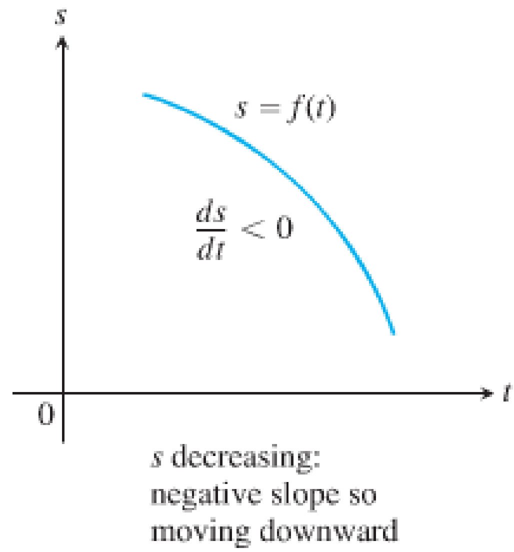
$$v_{avg} = \frac{displacement}{travel\ time} = \frac{\Delta s}{\Delta t} = \frac{f\left(t+\Delta t\right)-f\left(t\right)}{\Delta t}$$

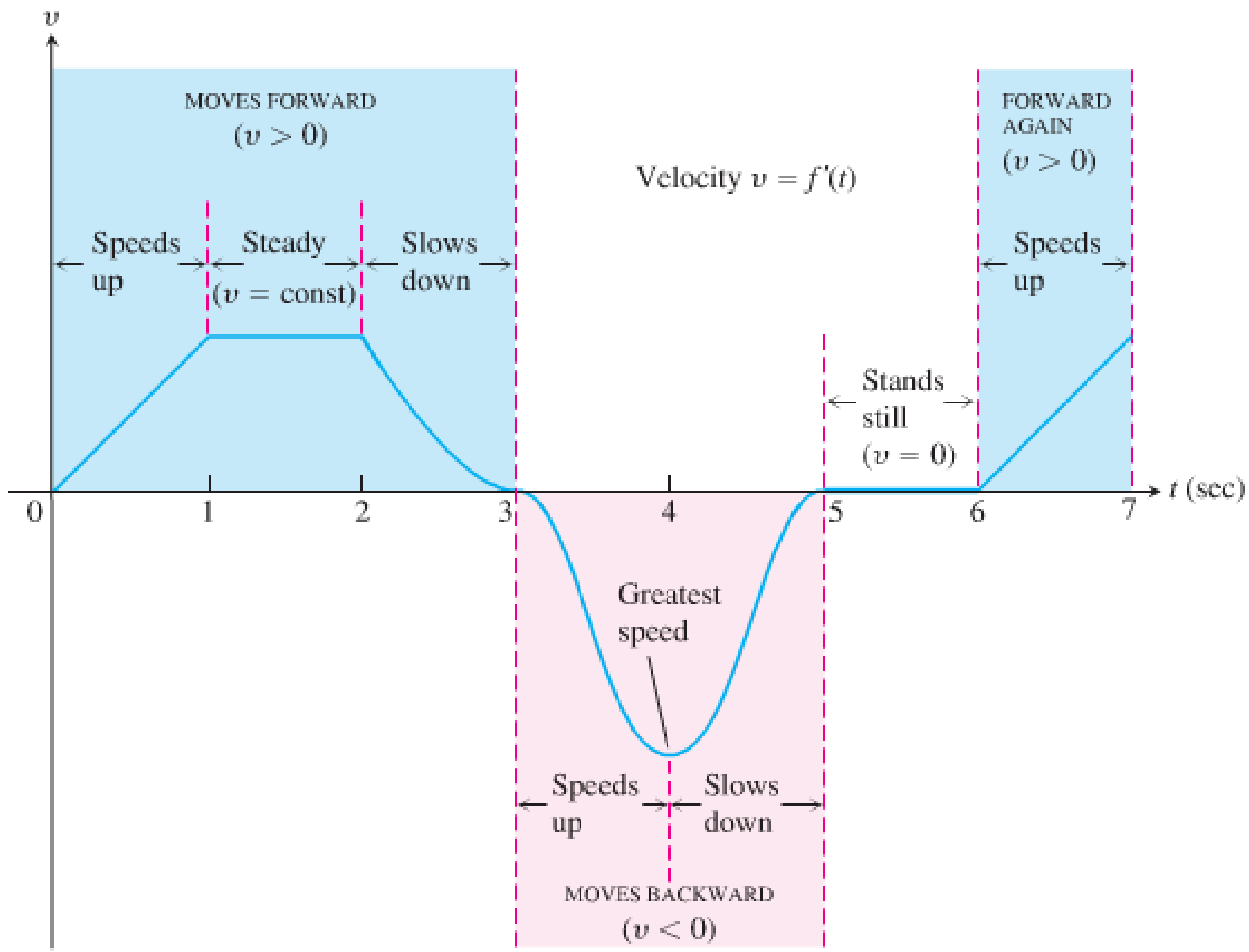
Definition

Speed is the absolute value of velocity

$$speed = \left|v\left(t\right)\right| = \left|\frac{ds}{dt}\right|$$


s
$s = f(t)$
$\frac{ds}{dt} > 0$
0
t
s increasing:
positive slope so
moving upward


s
$s = f(t)$
$\frac{ds}{dt} < 0$
0
t
s decreasing:
negative slope so
moving downward


v
MOVES FORWARD
($v > 0$)
Velocity $v = f'(t)$
FORWARD
AGAIN
($v > 0$)
Speeds
up
Steady
(v = const)
Slows
down
Speeds
up
Stands
still
($v = 0$)
0
1
2
3
4
5
6
7
t (sec)
Greatest
speed
Speeds
up
Slows
down
MOVES BACKWARD
($v < 0$)

Definition

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Jerk is the derivative of acceleration with the respect to time

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

When a ride in a car is jerky, it is not that the accelerations involved are necessarily large but that the changes in acceleration are abrupt.

Example

The free fall of a heavy ball bearing released from rest at time $t = 0$ *sec*.

a) How many meters does the ball fall in the first 2 *sec*?

b) What is its velocity, speed, and acceleration when $t = 2$?

Solution

a) The metric free-fall equation is $s = 4.9t^2$.

During the first 2 *sec*: $s(2) = 4.9(2)^2 = \underline{19.6\ m}$

b) At any time, the velocity is:

$$\begin{aligned} v &= \frac{ds}{dt} \\ &= \frac{d}{dt}\left(4.9t^2\right) \\ &= \underline{9.8t} \end{aligned}$$

At $t = 2$, velocity: $v = 9.8(2) = \underline{19.6\ m/\sec}$

$Speed = |v| = \underline{19.6\ m/\sec}$

Acceleration:

$a(t) = v'(t) = \underline{9.8\ m/\sec^2}$

Example

A dynamic blast blows a heavy rock straight up with a launch velocity of 160 *ft/sec* (about 109 *mph*).

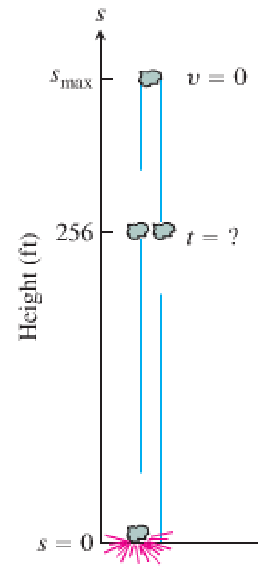
It reaches a height of $s = 160t - 16t^2$ after *t sec*.

a) How high does the rock go?

b) What are the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?

c) What is the acceleration of the rock at any time *t* during its flight (after the blast)?

d) When does the rock hit the ground again?



Solution

a) At any time *t* during the rock's motion, its velocity is

$$v = s' = 160 - 32t$$

The velocity is zero when it reaches maximum height:

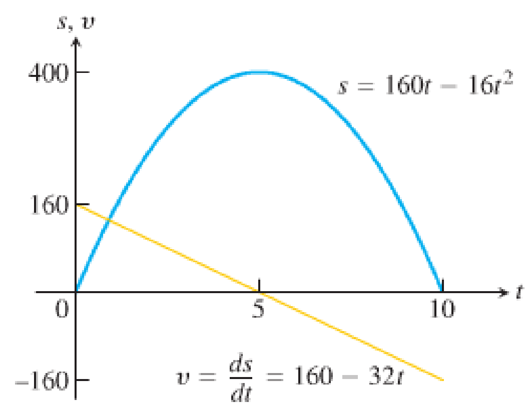
$$v = 160 - 32t = 0$$

$$160 = 32t$$

$$t = \frac{160}{32} = \underline{5 \text{ sec}}$$

The rock's height at $t = 5$ sec is

$$s(t=5) = 160(5) - 16(5)^2 = \underline{400 \; ft}$$



b) $s = 160t - 16t^2 = 256$

$$-16t^2 + 160t - 256 = 0 \quad \Rightarrow \quad t = 2\text{sec}, \quad t = 8\text{sec}$$

$$\begin{cases} t = 2\text{sec} & \Rightarrow \quad v = 160 - 32(2) = \underline{96 \; ft/\text{sec}} \\ t = 8\text{sec} & \quad v = 160 - 32(8) = \underline{-96 \; ft/\text{sec}} \end{cases}$$

The rock's speed is 96 *ft/sec*.

Since $v(t=2) > 0$, the rock is moving upward and *s* is increasing.

$v(t=8) < 0$, the rock is moving downward and *s* is decreasing.

c) Acceleration at any time is: $a = v' = \underline{-32 \; ft/\text{sec}^2}$

d) $s = 160t - 16t^2 = 0$

$$t(160 - 16t) = 0 \quad \Rightarrow \quad t = 0, \; t = 10.$$

At $t = 0$, the blast occurred and the rock was thrown upward, it took 10 *sec* to return to ground.

Derivatives in Economics

Example

Suppose that it costs $C(x) = x^3 - 6x^2 + 15x$ dollars to produce x radiators when 8 to 30 radiators are produced and that $R(x) = x^3 - 3x^2 + 12x$ gives the dollar revenue from selling x radiators.
Your shop currently produces 10 radiators a day. About how much extra will it cost to produce one more radiator a day, and what is your estimated increase in revenue for selling 11 radiators a day?

Solution

The cost of producing one more radiator a day when 10 are produced is about $C'(10)$:

$$C'(x) = 3x^2 - 12x + 15$$

$$C'(x = 10) = 3(10)^2 - 12(10) + 15 = \underline{195}$$

The additional cost will be about $195.00.

The marginal revenue is:

$$R'(x) = 3x^2 - 6x + 12$$

$$R'(x = 10) = 3(10)^2 - 6(10) + 12 = \underline{\$252.00}$$

If you increase sales to 11 radiators a day, the revenue is an additional of $252.00.

Exercises *Section* 2.4 – The Derivative as a Rate of Change

1. The position $s(t)=t^2-3t+2, \quad 0\le t\le 2$ of a body moving on a coordinate line, with s in meters and t in seconds.

a) Find the body's displacement and average velocity for the given time interval.
b) Find the body's speed and acceleration at the endpoints of the interval.
c) When, if ever, during the interval does the body change direction?

2. The position $s(t)=\frac{25}{t+5}, \quad -4\le t\le 0$ of a body moving on a coordinate line, with s in meters and t in seconds.

a) Find the body's displacement and average velocity for the given time interval.
b) Find the body's speed and acceleration at the endpoints of the interval.
c) When, if ever, during the interval does the body change direction?

3. At time t, the position of a body moving along the s-axis is $s=t^3-6t^2+9t$ ***m***.

a) Find the body's acceleration each time the velocity is zero.
b) Find the body's speed each time the acceleration is zero.
c) Find the total distance traveled by the body from $t=0$ to $t=2$.

4. A rock thrown vertically upward from the surface of the moon at a velocity of 24 *m/sec* (about 86 *km/h*) reaches a height of $s=24t-0.8t^2$ *m* in *t sec*.

a) Find the rock's velocity and acceleration at time t. (The acceleration in this case is the acceleration of gravity on the moon.)
b) How long does it take the rock to reach its highest point?
c) How high does the rock go?
d) How long does it take the rock to reach half its maximum height?
e) How long is the rock aloft?

5. Had Galileo dropped a cannonball from the Tower of Pisa, 179 *ft* above the ground, the ball's height above the ground *t sec* into the fall would have been $s=179-16t^2$.

a) What would have been the ball's velocity, speed, and acceleration at time t?
b) About how long would it have taken the ball to hit the ground?
c) What would have been the ball's velocity at the moment of impact?

Section 2.5 –Derivatives of Trigonometric Functions

Derivative of the *Sine* Function

If $f(x) = \sin x$, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$
$$= \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}$$
$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$
$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$
$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h}$$
$$= \sin x \underbrace{\lim_{h\to 0} \frac{\cos h - 1}{h}}_{0} + \cos x \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{1}$$
$$= \sin x \bullet (0) + \cos x \bullet (1)$$
$$= \cos x$$

$$\cos h = 1 - 2\sin^2\left(\frac{h}{2}\right)$$

$$\lim_{h\to 0} \frac{\cosh - 1}{h} = \lim_{h\to 0} \frac{1 - 2\sin^2\left(\frac{h}{2}\right) - 1}{h}$$
$$= \lim_{h\to 0} \frac{-2\sin^2\left(\frac{h}{2}\right)}{h} \qquad \textbf{Let } \theta = \frac{h}{2}$$
$$= -\lim_{\theta\to 0} \frac{2\sin^2(\theta)}{2\theta}$$
$$= -\lim_{\theta\to 0} \frac{\sin\theta}{\theta} \sin\theta$$
$$= -(1)(0)$$
$$= 0$$

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

Example

Find the derivative of $y = x^2 - \sin x$

Solution

$$y' = 2x - (\sin x)'$$
$$= 2x - \cos x$$

Example

Find the derivative of $y = x^2 \sin x$

Solution

$$y' = 2x\sin x + x^2 \cos x$$

Example

Find the derivative of $y = \dfrac{\sin x}{x}$

Solution

$$y' = \frac{x\cos x - \sin x \cdot (1)}{x^2}$$

$$= \boxed{\frac{x\cos x - \sin x}{x^2}}$$

Derivative of the *Cosine* Function

If $f(x) = \cos x$, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{(\cos x\cos h - \sin x\sin h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x(\cos h - 1) - \sin x\sin h}{h}$$

$$= \lim_{h \to 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h \to 0} \frac{\sin x\sin h}{h}$$

$$= \cos x \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{0} - \sin x \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{1} \qquad \lim_{h \to 0} \frac{\cosh - 1}{h} = 0$$

$$= \cos x \bullet (0) - \sin x \bullet (1)$$

$$= \boxed{-\sin x}$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

Example

Find the derivative of $y = 5x + \cos x$

Solution

$$y' = (5x)' + (\cos x)'$$

$$\boxed{y' = 5 - \sin x}$$

Example

Find the derivative of $y = \sin x \cos x$

Solution

$$\begin{aligned} y' &= (\sin x)' \cos x + (\cos x)' \sin x \\ &= (\cos x)\cos x + (-\sin x)\sin x \\ &= \boxed{\cos^2 x - \sin^2 x} \end{aligned}$$

Example

Find the derivative of $y = \dfrac{\cos x}{1 - \sin x}$

Solution

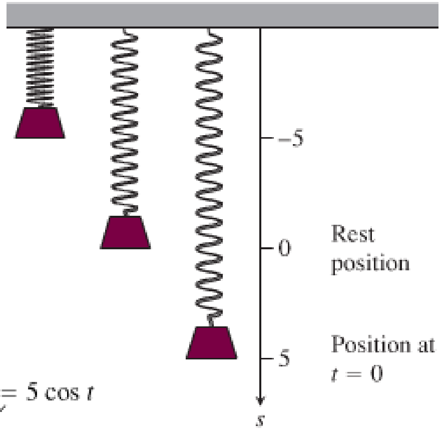
$$\begin{aligned} y' &= \frac{(1-\sin x)(\cos x)' - \cos x (1 - \sin x)'}{(1-\sin x)^2} \\ &= \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2} \\ &= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \qquad \sin^2 x + \cos^2 x = 1 \\ &= \boxed{\frac{1-\sin x}{(1-\sin x)^2}} \end{aligned}$$

Example

A weight hanging from a spring is stretched down 5 units beyond its rest position and released at time $t = 0$ to bob up and down. Its position at any later time t is:

$$s = 5\cos t$$

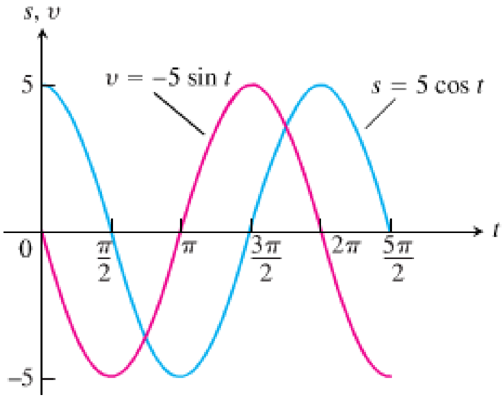
What are its velocity and acceleration at time t?



Solution

Velocity: $v = s' = -5\sin t$

Acceleration: $a = v' = -5\cos t$



Derivatives of the Other Trigonometric Functions

$$\begin{cases} (\tan x)' = \sec^2 x & (\cot x)' = -\csc^2 x \\ (\sec x)' = \sec x \tan x & (\csc x)' = -\csc x \cot x \end{cases}$$

Prove

Example

Find $\frac{d}{dx}(\tan x)$

Solution

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x}$$

Quotient Rule

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$\frac{1}{\cos x} = \sec x$$

$$\boxed{= \sec^2 x}$$

Example

Find y'' *if* $y = \sec x$

Solution

$$y' = \sec x \tan x$$

$$y'' = (\sec x)' \tan x + \sec x (\tan x)'$$

$$= (\sec x \tan x)\tan x + \sec x\left(\sec^2 x\right)$$

$$\boxed{= \sec x \tan^2 x + \sec^3 x}$$

Exercises Section 2.5 –Derivatives of Trigonometric Functions

Find the derivative of

1. $y = -10x + 3\cos x$

2. $y = \csc x - 4\sqrt{x} + 7$

3. $y = x^2 \cos x$

4. $y = \csc x \cot x$

5. $y = (\sin x + \cos x)\sec x$

6. $y = (\sec x + \tan x)(\sec x - \tan x)$

7. $y = \dfrac{\cos x}{x} + \dfrac{x}{\cos x}$

8. $y = x^2 \cos x - 2x\sin x - 2\cos x$

9. $y = (2 - x)\tan^2 x$

10. $y = t^2 - \sec t + 1$

11. $y = \dfrac{1 + \csc t}{1 - \csc t}$

12. $r = \theta \sin\theta + \cos\theta$

13. $y = \dfrac{3x + \tan x}{x\sec x}$

14. $p = \dfrac{\sin q + \cos q}{\cos q}$

15. $p = \dfrac{3q + \tan q}{q\sec q}$

16. Find $y^{(4)}$ *if* $y = 9\cos x$

17. Find $\dfrac{d^{999}}{dx^{999}}(\cos x)$

18. Find $\lim\limits_{x \to -\frac{\pi}{6}} \sqrt{1 + \cos(\pi \csc x)}$

19. A weight is attached to a spring and reaches its equilibrium position $(x = 0)$. It is then set in motion resulting in a displacement of

$$x = 10\cos t$$

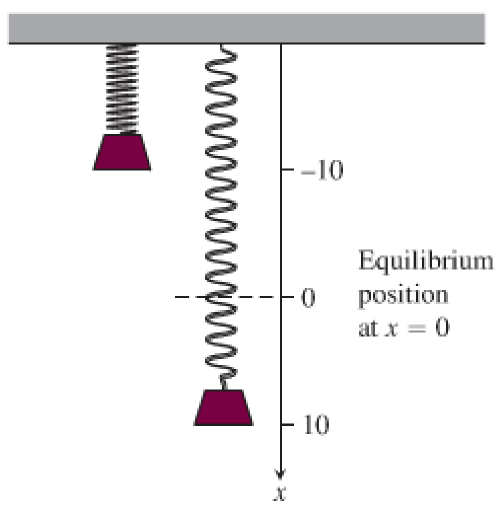
Where x is measured in centimeters and t is measured in seconds.

a) Find the spring's displacement when

$t = 0,\quad t = \dfrac{\pi}{3},\quad \textit{and}\quad t = \dfrac{3\pi}{4}$

b) Find the spring's velocity when

$t = 0,\quad t = \dfrac{\pi}{3},\quad \textit{and}\quad t = \dfrac{3\pi}{4}$



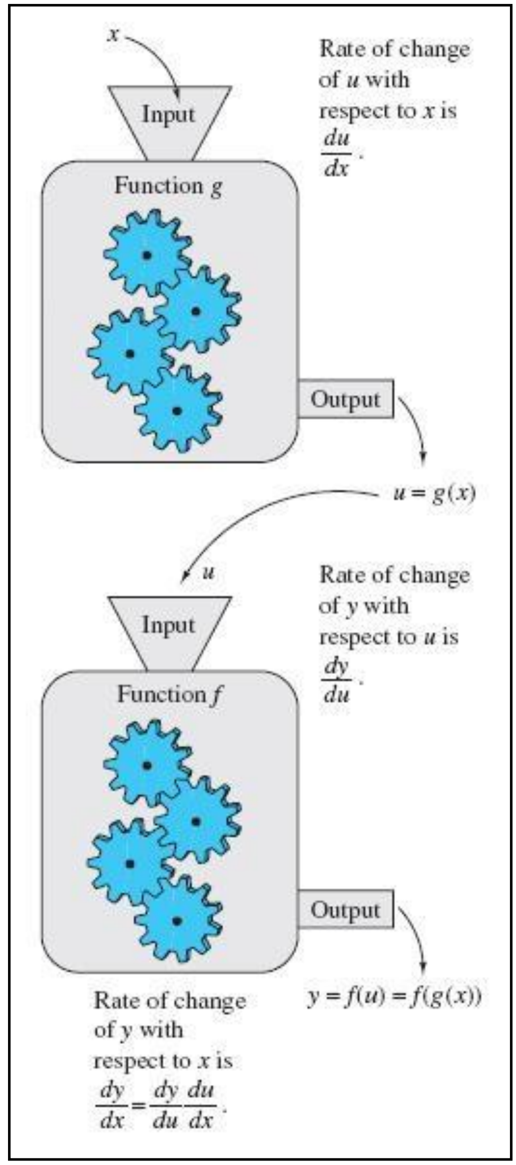
20. Assume that a particle's position on the x-axis is given by

$$x = 3\cos t + 4\sin t$$

a) Find the particle's position when $t = 0, \quad t = \frac{\pi}{2}, \quad \textit{and} \quad t = \pi$

b) Find the particle's velocity when $t = 0, \quad t = \frac{\pi}{2}, \quad \textit{and} \quad t = \pi$

Section 2.6 – The Chain Rule

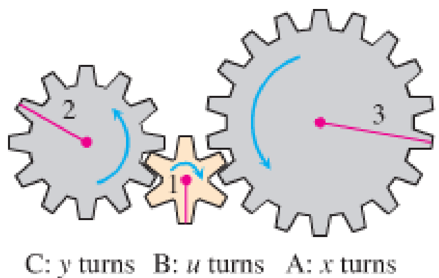


Derivative of a Composite Function

$$y = f(\color{red}{g(x)}) = f(\color{red}{u})$$

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

$$\frac{dy}{dx} = f'[g(x)] \bullet g'(x)$$



Example

Find the derivative of $y = \left(3x^2 + 1\right)^2$

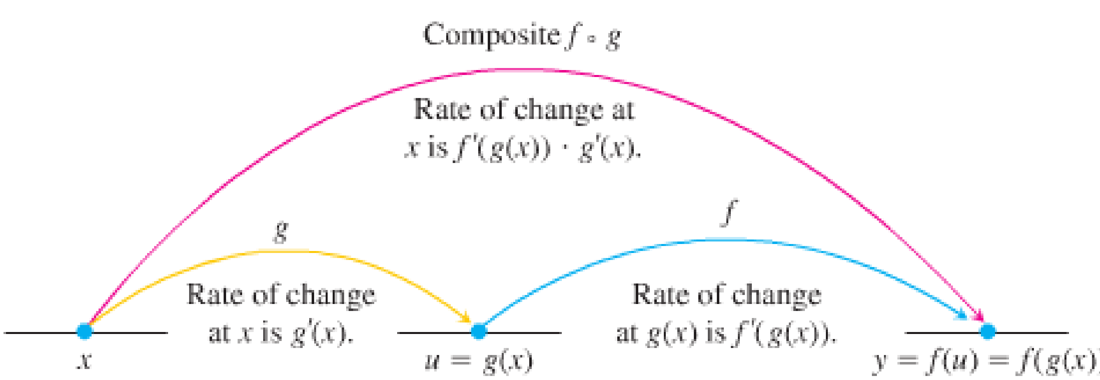
Solution

$$u = 3x^2 + 1 \quad \Rightarrow \quad (u)' = 6x$$

$$\begin{aligned} \frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2\left(3x^2 + 1\right) \cdot 6x \\ &= 36x^3 + 12x \end{aligned}$$

Calculating from the expand formula: $y = \left(3x^2 + 1\right)^2 = 9x^4 + 6x^2 + 1$

$$y' = 36x^3 + 12x$$



Intuitive "*Proof*" of the Chain Rule

Let Δu be the change in u when x changes by Δx, so that

$$\Delta u = g(x+\Delta x) - g(x)$$

Let Δy be the change in y when u changes by Δu, so that

$$\Delta y = f(u+\Delta u) - f(u)$$

If $\Delta u \neq 0 \Rightarrow \quad \dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u} \cdot \dfrac{\Delta u}{\Delta x}$

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \frac{dy}{du} \cdot \frac{du}{dx}
\end{aligned}$$

Example

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos\left(t^2+1\right)$. Find the velocity of the object as a function of t.

Solution

Let: $u = t^2 + 1 \quad \Rightarrow \quad u' = 2t$

$x = \cos(u) \quad \Rightarrow \quad x' = -\sin(u)$

By the Chain Rule:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{dx}{du} \cdot \frac{du}{dt} \\
&= -\sin(u) \cdot 2t \\
&= -2t\sin\left(t^2+1\right)
\end{aligned}$$

The General Power Rule

$$\frac{dy}{dx} = \frac{d}{dx}\left[u(x)^n\right]$$

$$= n\, u^{n\text{-}1}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[u^n\right] = \boxed{n\, u^{n\text{-}1} u'}$$

Example

Find the derivative of $\frac{d}{dx}\left(5x^3 - x^4\right)^7$

Solution

$$\frac{d}{dx}\overbrace{\left(5x^3 - x^4\right)^7}^{nu^n} = \overbrace{7\left(5x^3 - x^4\right)^6}^{nu^{n-1}}\ \overbrace{\left(15x^2 - 4x^3\right)}^{u'}$$

Example

Find the derivative of $\frac{d}{dx}\left(\frac{1}{3x-2}\right)$

Solution

$$\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1}$$

$$= -3(3x-2)^{-2}$$

$$\boxed{= -\frac{3}{(3x-2)^2}}$$

Example

Find the derivative of $\frac{d}{dx}\left(\sin^5 x\right)$

Solution

$$\frac{d}{dx}\left(\sin^5 x\right) = 5\sin^4 x\,(\sin x)'$$

$$\boxed{= 5\sin^4 x \cos x}$$

Example

Find the derivative of $g(t) = \tan(5 - \sin 2t)$

Solution

$$u = 5 - \sin 2t \quad (\tan u)' = \sec^2 u \cdot (u')$$

$$g'(t) = \sec^2(5 - \sin 2t) \cdot (5 - \sin 2t)'$$

$$= \sec^2(5 - \sin 2t) \cdot \left(0 - (\cos 2t)(2t)'\right)$$

$$= \sec^2(5 - \sin 2t) \cdot (-2\cos 2t)$$

$$= -2(\cos 2t)\sec^2(5 - \sin 2t)$$

Example

Show that the slope of every line tangent to the curve $y = \dfrac{1}{(1-2x)^3}$ is positive.

Solution

$$y = (1-2x)^{-3}$$

$$y' = -3(1-2x)^{-4}(-2)$$

$$= \frac{6}{(1-2x)^4}$$

At any point except $\left(x \neq \frac{1}{2}\right)$, the slope is $\dfrac{6}{(1-2x)^4}$ which is positive.

Exercises *Section* 2.6 – The Chain Rule

Find the derivative of

1. $y=\left(3x^4+1\right)^4\left(x^3+4\right)$

2. $p(t)=\dfrac{(2t+3)^3}{4t^2-1}$

3. $y=\left(x^3+1\right)^2$

4. $y=\left(x^2+3x\right)^4$

5. $y=\dfrac{4}{2x+1}$

6. $y=\dfrac{2}{(x-1)^3}$

7. $y=x^2\sqrt{x^2+1}$

8. $y=\left(\dfrac{x+1}{x-5}\right)^2$

9. $s(t)=\sqrt{2t^2+5t+2}$

10. $f(x)=\dfrac{1}{\left(x^2-3x\right)^2}$

11. $y=t^2\sqrt{t-2}$

12. $y=\left(\dfrac{6-5x}{x^2-1}\right)^2$

13. $y=4x(3x+5)^5$

14. $y=\left(3x^2-5x\right)^{1/2}$

15. $D_x\left(x^2+5x\right)^8$

16. $y=\dfrac{(3x+2)^7}{x-1}$

17. $y=\left(\dfrac{x^2}{8}+x-\dfrac{1}{x}\right)^4$

18. $y=\sqrt{3x^2-4x+6}$

19. $y=\cot\left(\pi-\dfrac{1}{x}\right)$

20. $y=5\cos^{-4}x$

21. $y=\sin\left(\dfrac{3\pi t}{2}\right)+\cos\left(\dfrac{3\pi t}{2}\right)$

22. $r=6(\sec\theta-\tan\theta)^{3/2}$

23. $g(x)=\dfrac{\tan 3x}{(x+7)^4}$

24. $f(\theta)=\left(\dfrac{\sin\theta}{1+\cos\theta}\right)^2$

25. $y=\sin^2(\pi t-2)$

26. $y=(t\tan t)^{10}$

27. $y=\cos\left(5\sin\left(\dfrac{t}{3}\right)\right)$

28. $y=4\sin\left(\sqrt{1+\sqrt{t}}\right)$

29. $y=\tan^2\left(\sin^3 x\right)$

30. Find the second derivatives of $y=\left(1+\dfrac{1}{x}\right)^3$

31. Find the second derivatives of $y=9\tan\left(\dfrac{x}{3}\right)$

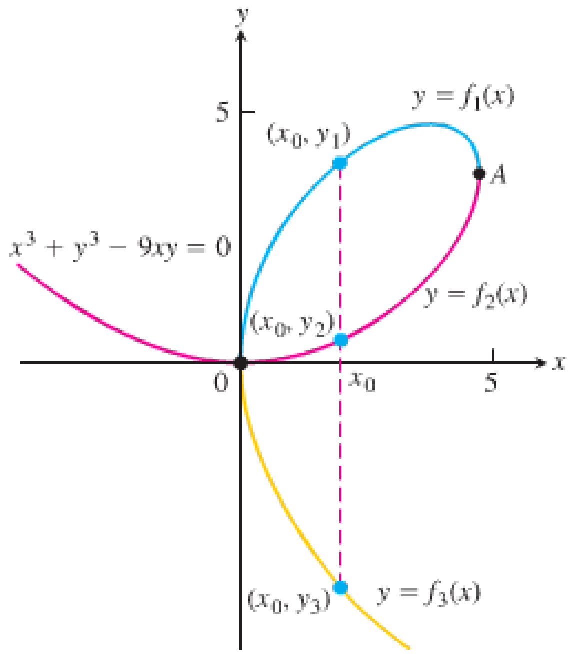
32. Find the tangent line to the graph of $y=\sqrt[3]{(x+4)^2}$ when $x = 4$.

Section 2.7 – Implicit Differentiation

Definition

A relation $F(x, y) = 0$ is said to define the function $y = f(x)$ implicitly if, for *x* in the domain of f $\rightarrow F(x, f(x)) = 0$

Example: $x^3 + y^3 - 9xy = 0, \quad x^2 + y^2 = 25$



Implicitly Defined Functions

It is always assumed that the given equation determines *y* implicitly as a differentiable function of *x* so that $\frac{dy}{dx}$ exists.

Example

Find $\frac{dy}{dx}$ if $y^2 = x$

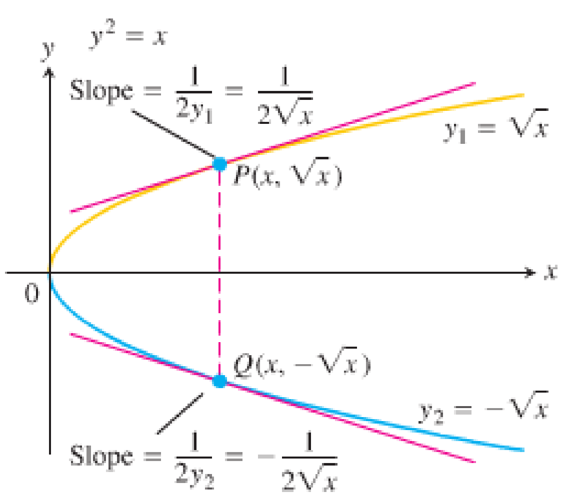
Solution

$$y^2 = x$$

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(x)$$

$$2y\frac{dy}{dx} = 1$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2y}}$$



Example

Find the slope of the circle $x^2 + y^2 = 25$ at the point (3, −4).

Solution

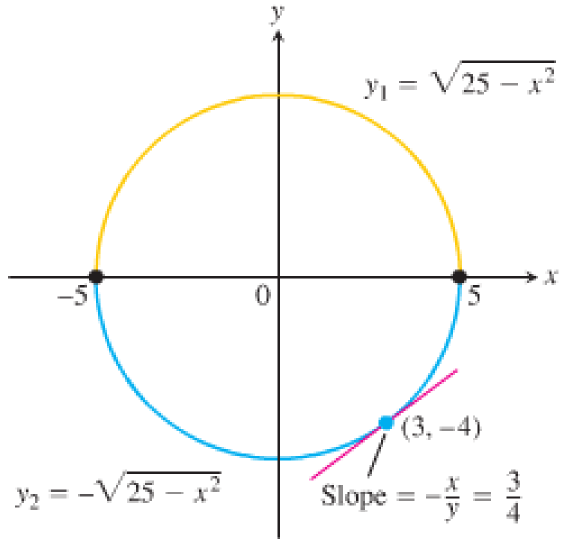
$$\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}\left(y^2\right)=\frac{d}{dx}(25)$$

$$2x+2y\frac{dy}{dx}=0$$

$$2y\frac{dy}{dx}=-2x$$

$$\boxed{\frac{dy}{dx}=-\frac{x}{y}}$$

The slope at (3, −4) is $\left.\frac{dy}{dx}\right|_{(3,-4)}=-\frac{3}{-4}=\frac{3}{4}$



Implicit Differentiation

- ✓ Differentiate both sides of the equation with respect to *x*, treating *y* as a differentiable function of *x*.
- ✓ Collect the terms with $\frac{dy}{dx}$ on one side of the equation and solve for $\frac{dy}{dx}$.

Example

Find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin xy$

Solution

$$\frac{d}{dx}\left(y^2\right)=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(\sin xy)$$

$$2y\frac{dy}{dx}=2x+\cos(xy)\frac{d}{dx}(xy)$$

$$2y\frac{dy}{dx}=2x+\cos(xy)\left(y+x\frac{dy}{dx}\right)$$

$$2y\frac{dy}{dx}=2x+y\cos(xy)+x\cos(xy)\frac{dy}{dx}$$

$$2y\frac{dy}{dx}-x\cos(xy)\frac{dy}{dx}=2x+y\cos(xy)$$

$$(2y-x\cos xy)\frac{dy}{dx}=2x+y\cos xy$$

$$\boxed{\frac{dy}{dx}=\frac{2x+y\cos xy}{2y-x\cos xy}}$$

$$\frac{d}{dx}\left(y^2\right)=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(\sin xy)$$

Let $y'=\frac{dy}{dx}$

$$2yy'=2x+\cos(xy)(xy)'$$

$$2yy'=2x+\cos(xy)(y+xy')$$

$$2yy'=2x+y\cos(xy)+x\cos(xy)y'$$

$$2yy'-x\cos(xy)y'=2x+y\cos(xy)$$

$$(2y-x\cos xy)y'=2x+y\cos xy$$

$$\boxed{y'=\frac{dy}{dx}=\frac{2x+y\cos xy}{2y-x\cos xy}}$$

Example

Find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 8$

Solution

Let $y' = \frac{dy}{dx}$

$$\frac{d}{dy}\left(2x^3 - 3y^2\right) = \frac{d}{dy}(8)$$

$$6x^2 - 6yy' = 0$$

$$6x^2 = 6yy'$$

$$\boxed{y' = \frac{6x^2}{6y} = \frac{x^2}{y}}$$

$$y'' = \left(\frac{x^2}{y}\right)'$$

$u = x^2 \qquad v = y$

$u' = 2x \qquad v' = y'$

$$y'' = \frac{2xy - x^2 y'}{y^2}$$

$$= \frac{2xy - x^2 \frac{x^2}{y}}{y^2}$$

$$= \frac{\frac{2xy^2 - x^4}{y}}{y^2}$$

$$\boxed{= \frac{2xy^2 - x^4}{y^3}}$$

Normal Lines

The normal is the line perpendicular to the tangent of the profile curve at the point of entry.

Example

Show that the point (2, 4) lies on the curve $x^3+y^3-9xy=0$. Then find the tangent and normal to the curve there.

Solution

$$\frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(y^3\right)-\frac{d}{dx}(9xy)=0$$

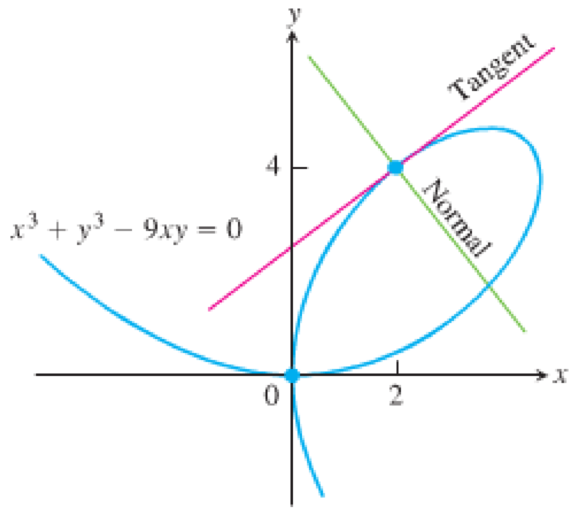
$$3x^2+3y^2y'-9\left(y+xy'\right)=0$$

$$3x^2+3y^2y'-9y-9xy'=0$$

$$3\left(y^2-3x\right)y'=9y-3x^2$$

$$y'=\frac{3\left(3y-x^2\right)}{3\left(y^2-3x\right)}$$

$$=\frac{3y-x^2}{y^2-3x}$$



The slope: $y'\Big|_{(2,4)}=\frac{3(4)-(2)^2}{(4)^2-3(2)}=\frac{8}{10}=\frac{4}{5}$

The tangent at (2, 4) is the line passes thru (2, 4) with slope $\frac{4}{5}$

$$y-4=\frac{4}{5}(x-2)$$

$$y-4=\frac{4}{5}x-\frac{8}{5}$$

$$y=\frac{4}{5}x-\frac{8}{5}+4$$

$$y=\frac{4}{5}x+\frac{12}{5}$$

The Normal to the curve at (2, 4) is the line perpendicular to the tangent there thru (2, 4) with slope $-\frac{5}{4}$

$$y-4=-\frac{5}{4}(x-2)$$

$$y-4=-\frac{5}{4}x+\frac{5}{2}$$

$$\boxed{y=-\frac{5}{4}x+\frac{13}{2}}$$

Exercises *Section* 2.7 – Implicit Differentiation

Find $\frac{dy}{dx}$

1. $y^2 + x^2 - 2y - 4x = 4$

2. $x^2 y^2 - 2x = 3$

3. $x + \sqrt{x}\sqrt{y} = y^2$

4. $x^2 y + xy^2 = 6$

5. $x^3 - xy + y^3 = 1$

6. $y^2 = \frac{x-1}{x+1}$

7. $(3xy+7)^2 = 6y$

8. $xy = \cot(xy)$

9. $x + \tan(xy) = 0$

10. $x\cos(2x+3y) = y\sin x$

11. Find $\frac{dr}{d\theta}$ $\quad r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4}$

12. Find $\frac{dr}{d\theta}$ $\quad \sin(r\theta) = \frac{1}{2}$

13. Find $\frac{d^2y}{dx^2}$ $\quad x^{2/3} + y^{2/3} = 1$

14. Find $\frac{d^2y}{dx^2}$ $\quad 2\sqrt{y} = x - y$

15. If $x^3 + y^3 = 16$, find the value of $\frac{d^2y}{dx^2}$ at the point (2, 2).

16. Find dy/dx: $x^2 - xy + y^2 = 4$ and evaluate the derivative at the given point $(0, -2)$

17. Find the slope of the curve $\left(x^2 + y^2\right)^2 = (x-y)^2$ at the point (–2, 1) and (–2, –1)

18. Find the slope of the tangent line to the circle $x^2 - 9y^2 = 16$ at the point (5, 1)

19. Find the equation of the tangent line to the circle $x^3 + y^3 = 9xy$ at the point (2, 4)

20. Find the lines that are (***a***) tangent and (***b***) normal to the curve $x^2 + xy - y^2 = 1$ at the point (2, 3).

21. Find the lines that are (***a***) tangent and (***b***) normal to the curve $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$ at the point (–1, 0).

22. Find the lines that are (***a***) tangent and (***b***) normal to the curve $x^2\cos^2 y - \sin y = 0$ at the point (0, π).

23. Suppose that *x* and *y* are both functions of *t*, which can be considered to represent time, and that *x* and *y* are related by the equation

$$xy^2 + y = x^2 + 17$$

Suppose further that when *x* = 2 and *y* = 3, then $\frac{dx}{dt} = 13$. Find the value of the $\frac{dy}{dt}$ at that moment.

Section 2.8 – Related Rates

The problem of finding a rate of change from other known rates of change is called a ***related rates problem***.

$$V=\frac{4}{3}\pi r^3 \quad\Rightarrow\quad \frac{dV}{dt}=\frac{dV}{dr}\frac{dr}{dt}=4\pi r^2\frac{dr}{dt}$$

Example

Water runs into a conical tank at the rate of 9 $ft^3/\min$. The tank stands point down and has a height of 10 *ft* and a base radius of 5 *ft*. How fast is the water level rising when the water is 6 *ft* deep?

Solution

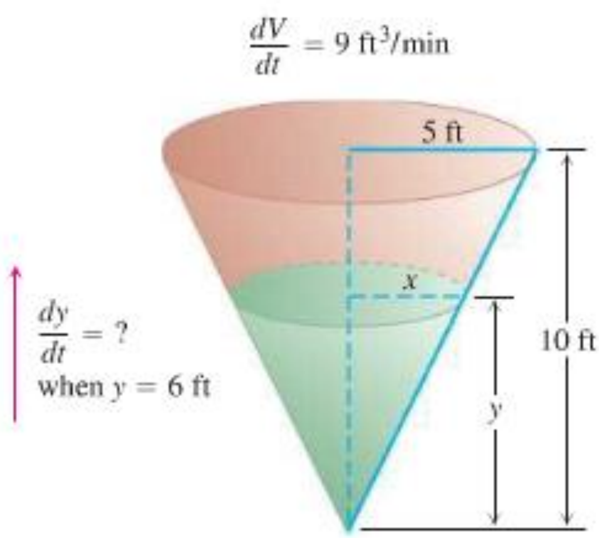
V = volume $\left(ft^3\right)$ of the water in the tank at time t (*min*)

x = radius (ft) of the surface of the water at time t

y = depth (ft) of the water in the tank at time t.

Given: $\dfrac{dV}{dt}=9\dfrac{ft^3}{min}$ $\quad y=6\,ft$

$h=10\,ft \qquad r=5\,ft$



The water forms a cone with volume:

$$V=\frac{1}{3}\pi x^2 y$$

From the triangles:

$$\frac{x}{y}=\frac{5}{10}\quad\Rightarrow\quad x=\frac{y}{2}$$

$$V=\frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y$$

$$=\frac{1}{3}\pi\left(\frac{y^2}{4}\right)y$$

$$=\frac{1}{12}\pi y^3$$

The derivative in function of time:

$$\frac{dV}{dt}=\frac{1}{12}\pi\left(3y^2\frac{dy}{dt}\right)$$

$$9=\frac{\pi}{4}(6)^2\frac{dy}{dt}$$

$$\frac{(9)(4)}{36\pi}=\frac{dy}{dt}$$

$$\frac{dy}{dt}=\frac{1}{\pi}\ \approx 0.3183$$

The water level is rising at about 0.32 ***ft/min***.

Related Rates Problem Strategy

1. Draw a picture and name the variables and constants. (use t for time).
2. Write down the given information (numerical).
3. Write down what you are asked to find.
4. Write an equation that relates the variables.
5. Differentiate with respect to t.
6. Evaluate.

Example

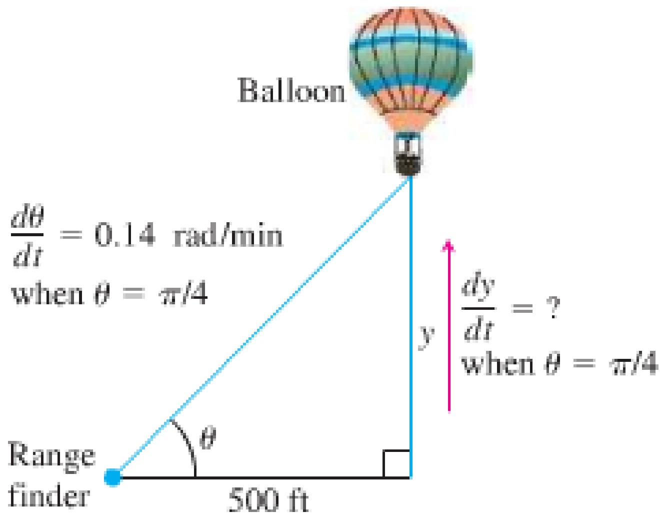
A hot air balloon rising straight up from a level filed is tracked by a range finder 500 *ft* from the liftoff point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.14 *rad/min*. How fast is the balloon rising at that moment?

Solution

θ = the angle in rad.
y = the height in feet of the balloon

Given: $\frac{d\theta}{dt} = 0.14\ rad\,/\,\text{min} \quad when \quad \theta = \frac{\pi}{4}$

$distance = 500\ ft$



$$\tan\theta = \frac{y}{500} \quad \Rightarrow \quad y = 500\tan\theta$$

$$\frac{dy}{dt} = 500\left(\sec^2\theta\right)\frac{d\theta}{dt}$$

$$= 500\left(\sec^2\frac{\pi}{4}\right)(0.14)$$

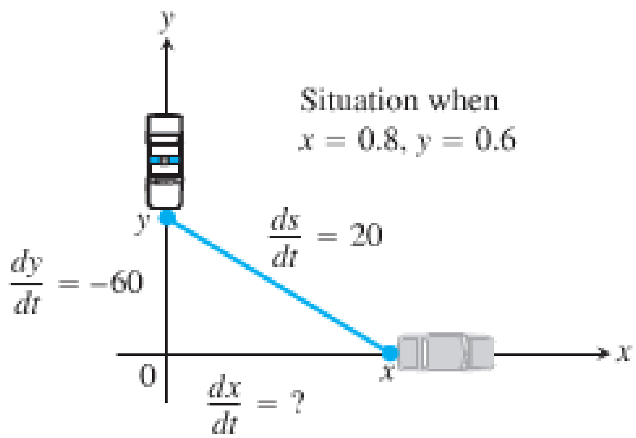
$$\underline{=140}$$

The balloon is rising at the rate of 140 *ft/min*.

Example

A police cruiser, approaching a right-angle intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 *mi* north of the intersection and the car is 0.8 *mi* to the east, the police determined with radar that the distance between them and the car is increasing at 20 *mph*. If the cruiser is moving at 60 *mph* at the instant of measurement, what is the speed of the car?

Solution



x = position of car at time t.

y = position of cruiser at time t.

s = distance between car and cruiser at time t.

Given: $x = 0.8\ mi \qquad y = 0.6\ mi$

$\frac{ds}{dt} = 20\ mph \qquad \frac{dy}{dt} = -60\ mph$

Using the Pythagorean Theorem to get the distance:

$$s^2 = x^2 + y^2$$

$$\underline{|s} = \sqrt{x^2 + y^2} = \sqrt{0.8^2 + 0.6^2}\ \underline{= 1|}$$

$$\frac{d}{dt}\left(s^2\right) = \frac{d}{dt}\left(x^2\right) + \frac{d}{dt}\left(y^2\right)$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$s\frac{ds}{dt} - y\frac{dy}{dt} = x\frac{dx}{dt}$$

$$x\frac{dx}{dt} = s\frac{ds}{dt} - y\frac{dy}{dt}$$

$$0.8\frac{dx}{dt} = (1)(20) - (0.6)(-60)$$

$$\frac{dx}{dt} = \frac{20 + 36}{0.8}$$

$$= \underline{70\ \ \boldsymbol{mph}|}$$

The car’s speed is 70 mph.

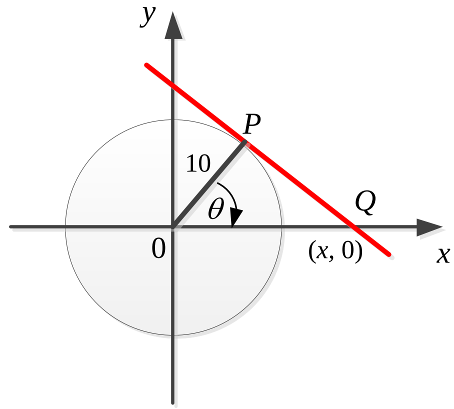
Example

A particle moves clockwise at a constant rate along a circle of radius 10 *ft* centered at the origin. The particles initial position is (0, 10) on the *y*-axis and its final destination is the point (10, 0) on the *x*-axis. Once the particle is in motion, the tangent line at P intersects the *x*-axis at a point Q (which moves over time). If it takes the particle 30 *sec* to travel from start to finish, how fast is the point Q moving along the *x*-axis when it is 20 *ft* from the center of the circle?

Solution

Since the particle travels 30 sec from start to finish of angle $90° = \frac{\pi}{2}$, the particle is traveling along the circle at a constant rate of $\frac{\frac{\pi}{2}\ rad}{30\ \sec} \cdot \frac{60\ \sec}{1\ \min} = \pi\ rad / \min$

That implies $\frac{d\theta}{dt} = -\pi$



$$x = 20\ ft \quad and \quad \frac{d\theta}{dt} = -\pi rad / \min$$

$$\cos\theta = \frac{10}{x} \quad \Leftrightarrow \quad x = \frac{10}{\cos\theta} = 10\sec\theta$$

$$20 = 10\sec\theta \quad \Rightarrow \quad \sec\theta = 2$$

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}(10\sec\theta)\\ &= 10\sec\theta\tan\theta\frac{d\theta}{dt}\\ &= 10\sec\theta\tan\theta(-\pi)\\ &= -10\pi\ \sec\theta\tan\theta\end{aligned}$$

$$20 = 10\sec\theta \quad \Rightarrow \quad \sec\theta = 2$$

$$\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{4-1} = \sqrt{3}$$

$$\begin{aligned}\frac{dx}{dt} &= -10\pi\ \sec\theta\tan\theta\\ &= -10\pi\ (2)\left(\sqrt{3}\right)\\ &\boxed{= -20\pi\sqrt{3}\ \ ft / \min}\end{aligned}$$

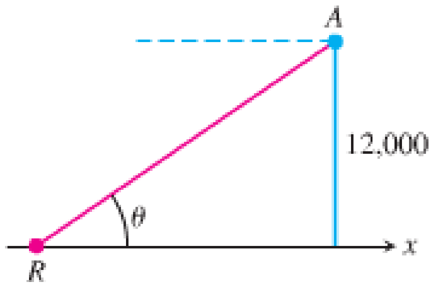
The point Q is moving towards the origin at the speed of $20\sqrt{3}\pi\ \approx 108.8\ ft / \min$

Example

A jet airliner is flying at a constant altitude of 12,000 *ft* above sea level as it approaches a Pacific island. The aircraft comes within the direct line of sight of a radar station located on the island, and the radar indicates the initial angle between sea level and its line of sight to the aircraft is 30°. How fast (in miles per hour) is the aircraft approaching the island when first detected by the radar instrument if it is turning upward (counterclockwise ***ccw***) at the rate of $\frac{2}{3}\ deg / sec$ in order to keep the aircraft within its direct line of sight?

Solution

From the triangle:



$$\tan\theta = \frac{12,000}{x} \quad \Rightarrow \quad x = \frac{12,000}{\tan\theta}$$

$$= \frac{12,000}{5280}\cot\theta \quad mi$$

$$\frac{dx}{dt} = -\frac{12,000}{5280}\csc^2\theta\,\frac{d\theta}{dt}$$

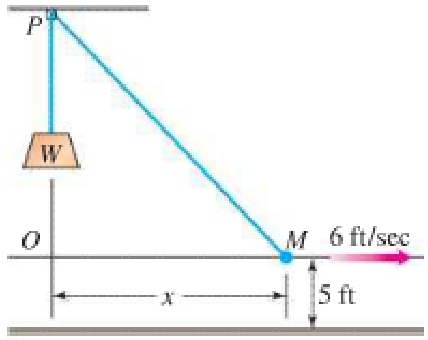
$$= -\frac{12,000}{5280}\csc^2\left(\frac{\pi}{6}\right)\cdot\ \left(\frac{2}{3}\right)\frac{deg}{sec}\ \frac{\pi}{180°}\ \frac{3600sec}{1hr}$$

$$\approx -380\ \ mi / hr$$

The negative appears because the distance x is decreasing, so the aircraft is approaching the island at a speed of approximately 380 ***mi/hr*** when first detected by the radar.

Example

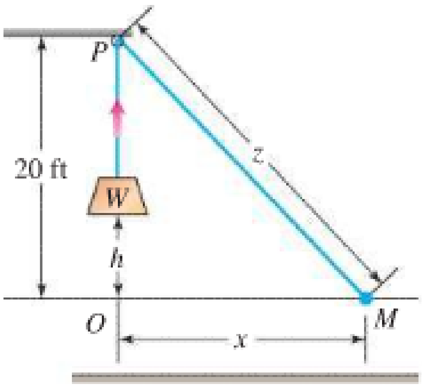
A rope is running through a pulley at P and bearing a weight W at one end. The other end is held 5 *ft* above the ground in the hand M of a worker. Suppose the pulley is 25 *ft* above ground, the rope is 45 *ft* long, and the worker is walking rapidly away from the vertical line PW at the rate of 6 *ft/sec*. How fast is the weight being raised when the worker's hand is 21 *ft* away from PW?



Solution

Given: $\quad when \;\; x = 21 \;\; \to \;\; \dfrac{dx}{dt} = 6$

$$45 = z + 20 - h \quad \Rightarrow \quad \boxed{z = 45 - 20 + h = 25 + h}$$



$$z^2 = 20^2 + x^2$$

$$(25+h)^2 = 20^2 + x^2$$

$$\frac{d}{dt}\left((25+h)^2\right) = \frac{d}{dt}\left(20^2 + x^2\right)$$

$$2(25+h)\frac{dh}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dh}{dt} = \frac{x}{25+h}\frac{dx}{dt}$$

$$when \;\; x = 21 \;\; \to \;\; (25+h)^2 = 20^2 + x^2 = 20^2 + 21^2 = 841$$

$$25 + h = \sqrt{841} = 29$$

$$\frac{dh}{dt} = \frac{21}{29}(6) \approx \boxed{4.3 \; ft / \sec}$$

As the rate 4.3 ***ft/sec*** at which the weight is being raised when $x = 21$ *ft*.

Exercises *Section* **2.8 – Related Rates**

1. If $y = x^2$ and $\frac{dx}{dt} = 3$, then what is $\frac{dy}{dt}$ *when* $x = -1$

2. If $x = y^3 - y$ and $\frac{dy}{dt} = 5$, then what is $\frac{dx}{dt}$ *when* $y = 2$

3. A cone-shaped icicle is dripping from the roof. The radius of the icicle is decreasing at a rate of 0.2 *cm* per *hour*, while the length is increasing at a rate of 0.8 *cm* per *hour*. If the icicle is currently 4 cm in radius and 20 *cm* long, is the volume of the icicle increasing or decreasing and at what rate?

4. A cube's surface area increases at the rate of 72 $in^2/\sec$. At what rate is the cube's volume changing when the edge length is x = 3 *in*?

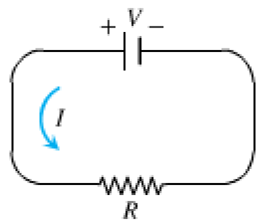
5. The radius r and height h of a right circular cone are related to the cone's volume V by the equation $V = \frac{1}{3}\pi r^2 h$.

a) How is $\frac{dV}{dt}$ related to $\frac{dh}{dt}$ if ***r*** is constant?

b) How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ if ***h*** is constant?

c) How is $\frac{dV}{dt}$ related to $\frac{dr}{dt}$ and $\frac{dh}{dt}$ if neither ***r*** nor ***h*** is constant?

6. The voltage ***V*** (volts), current ***I*** (amperes), and resistance ***R*** (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 *volt/sec* while I is decreasing at the rate of $\frac{1}{3}\, amp/\sec$. Let t denote time in seconds.



a) What is the value of $\frac{dV}{dt}$?

b) What is the value of $\frac{dI}{dt}$?

c) What equation relates $\frac{dR}{dt}$ to $\frac{dV}{dt}$ and $\frac{dI}{dt}$?

d) Find the rate at which R is changing when V = 12 volts and I = 2 amp. Is R increasing or decreasing?

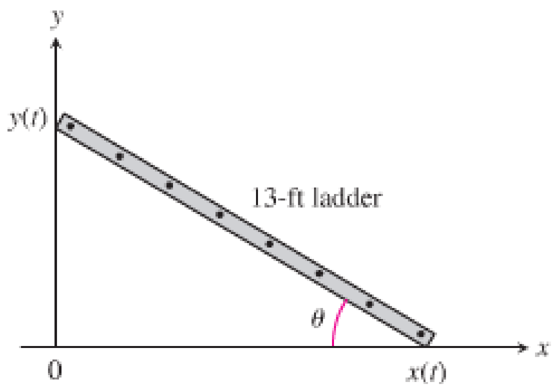
7. Let x and y be differentiable functions of t and let $s = \sqrt{x^2 + y^2}$ be the distance between the points $(x, 0)$ and $(0, y)$ in the xy–plane.

a) How is $\frac{ds}{dt}$ related to $\frac{dx}{dt}$ if y is constant?

b) How is $\frac{ds}{dt}$ related to $\frac{dx}{dt}$ and $\frac{dy}{dt}$ if neither x nor y is constant?

c) How is $\frac{dx}{dt}$ related to $\frac{dy}{dt}$ if s is constant?

8. A 13-*ft* ladder is leaning against a house when its base starts to slide away. By the time the base is 12 *ft* from the house, the base is moving at the rate of 5 *ft/sec*.



a) How fast is the top of the ladder sliding down the wall then?

b) At what rate is the area of the triangle formed by the ladder, wall, and the ground changing then?

c) At what rate is the angle θ between the ladder and the ground changing then?

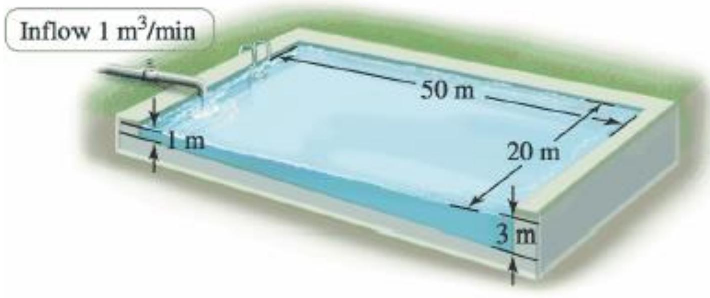
9. A 13-*ft* ladder is leaning against a vertical wall when he begins pulling the foot of the ladder away from the wall at a rate of 0.5 *ft/s*. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 5 *ft* from the wall?

10. A 12-*ft* ladder is leaning against a vertical wall when he begins pulling the foot of the ladder away from the wall at a rate of 0.2 *ft/s*. What is the configuration of the ladder at the instant that the vertical speed of the top of the ladder equals the horizontal speed of the foot of the ladder?

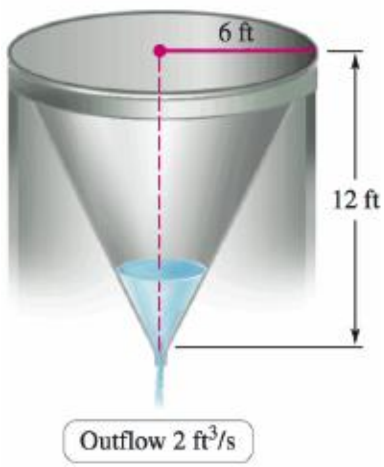
11. A swimming pool is 50 *m* long and 20 *m* wide. Its length decreases linearly along the length from 3 *m* to 1 *m*. It is initially empty and is filed at a rate of 1 m^3 / min.

a) How fast is the water level rising 250 min after the filling begins?

b) How long will it take to fill the pool?



12. An inverted conical water tank with a height of 12 *ft* and a radius of 6 *ft* is drained through a hole in the vertex at a rate of 2 ft^3 / sec. What is the rate of change of the water depth when the water depth is 3 *ft*? (*Hint*: Use similar triangles.)

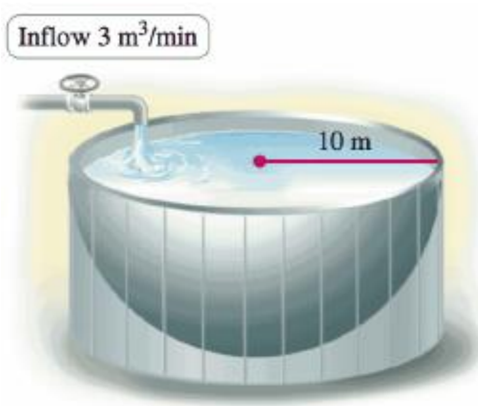


13. A hemispherical tank with a radius of 10 *m* is filled from an inflow pipe at a rate of $3\ m^3$ / min.

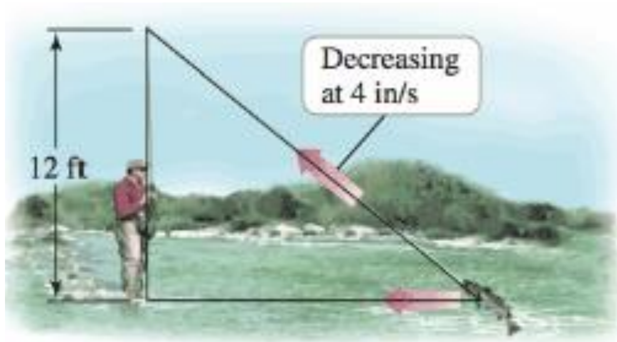
(Hint: The volume of a cap of thickness h sliced from a sphere of radius r is $\frac{\pi h^2(3r-h)}{3}$).

a) How fast is the water level rising when the water level is 5 *m* from the bottom of the tank?

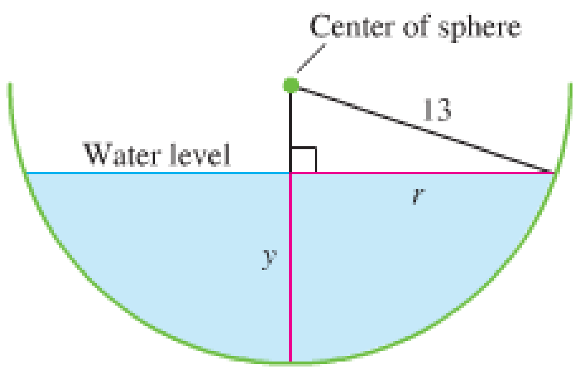
b) What is the rate of change of the surface area of the water when the water is 5 *m* deep?



14. A fisherman hooks a trout and reels in his line at 4 *in/sec*. Assume the trip of the fishing rod is 12 *ft* above the water directly above the fisherman and the fish is pulled horizontally directly towards the fisherman. Find the horizontal peed of the fish when it is 20 *ft* from the fisherman.



15. Water is flowing at the rate of 6 m^3 / min from a reservoir shaped like a hemispherical bowl of radius 13 *m*. Answer the following questions, given that the volume of water in a hemispherical bowl of radius *R* is $V = \frac{\pi}{3} y^2 (3R - y)$ when the water is y meters deep.



a) At what rate the water level changing when the water is 8 *m* deep?

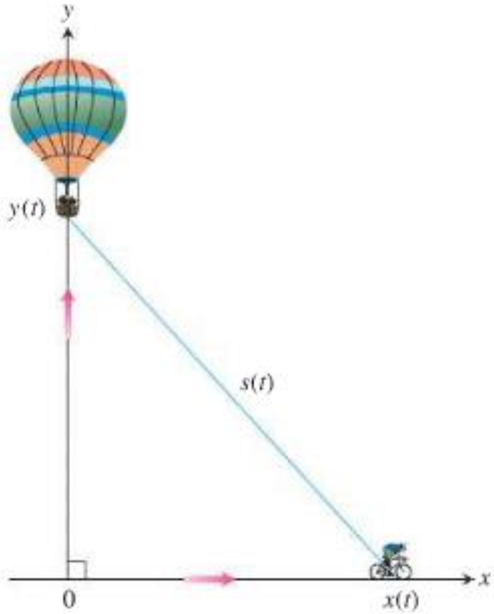
b) What is the radius *r* of the water's surface when the water is *y m* deep?

c) At what rate is the radius *r* changing when the water is 8 *m* deep?

16. A spherical balloon is inflated with helium at the rate of $100\pi\ ft^3$ / min . How fast is the balloon's radius increasing at the instant the radius is 5 *ft*? How fast the surface area increasing?

17. An observer stands 300 *ft* from the launch site of a hot-air balloon. The balloon is launched vertically and maintains a constant upward velocity of 20 *ft/sec*. what is the rate of change of the angle of elevation of the balloon when it is 400 *ft* from the ground? The angle of elevation is the angle θ between the observer's line of sight to the balloon and the ground.

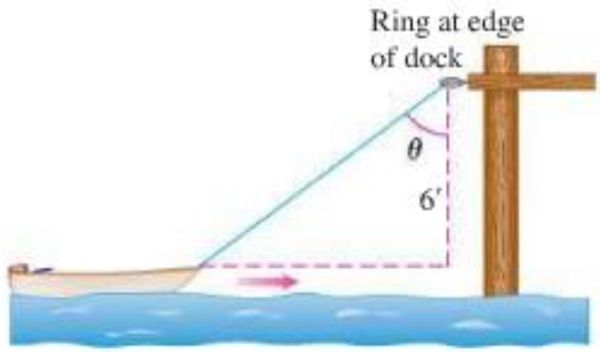
18. A balloon rising vertically above a level, straight road at a constant rate of 1 *ft/sec*. Just when the balloon is 65 *ft* above the ground, a bicycle moving at a constant rate of 17 *ft/sec* passes under it. How fast is the distance $s(t)$ between the bicycle and the balloon increasing 3 *sec* later?



19. A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 *ft* above the bow. The rope is hauled in at rate of 2 *ft/sec*.

a) How fast is the boat approaching the dock when 10 *ft* of rope are out?

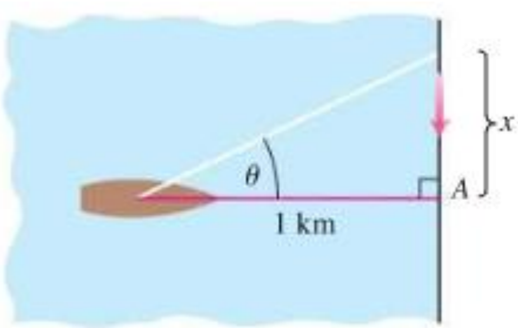
b) At what rate is the angle θ changing at this instant?



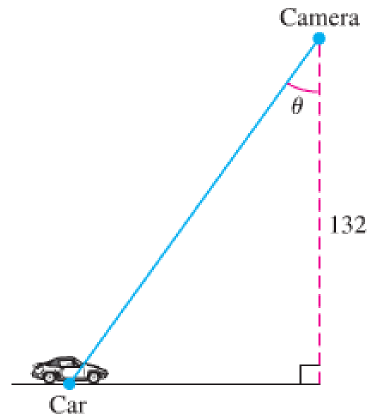
20. The figure shows a boat 1 km offshore, sweeping the shore with a searchlight. The light turns at a constant rate, $\frac{d\theta}{dt} = -0.6\ rad / \sec$.

a) How fast is the light moving along the shore when it reaches point A?

b) How many revolutions per minute is 0.3 rad/sec?



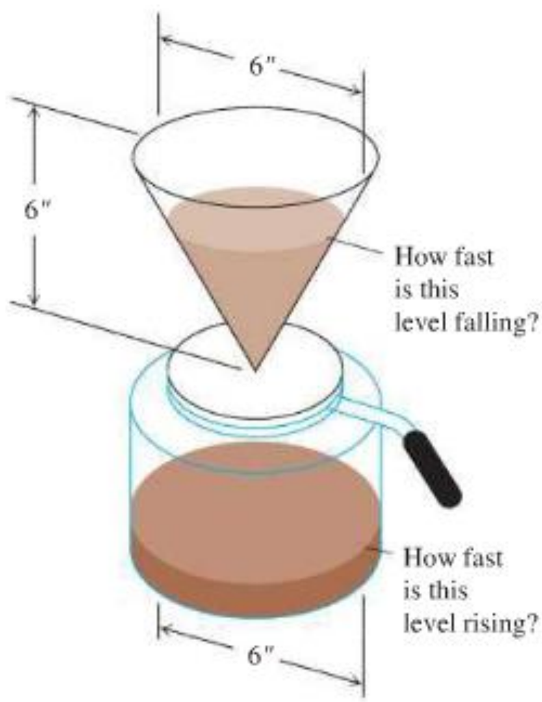
21. You are videotaping a race from a stand 132 *ft* from the track, following a car that is moving at 180 *mi/h* (264 *ft/sec*). How fast will your camera angle θ be changing when the car is right in front of you? A half second later?



22. The coordinates of a particle in the metric *xy*–plane are differentiable functions of time *t* with $\dfrac{dx}{dt} = -1\ m/\sec$ and $\dfrac{dy}{dt} = -5\ m/\sec$. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)?

23. A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its *x*-coordinate (measure in meters) increases at a steady 10 *m/sec*. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3m$?

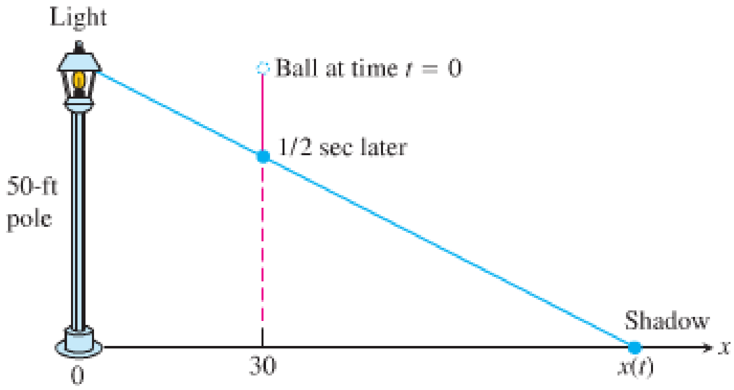
24. Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 $in^3 / \min$.



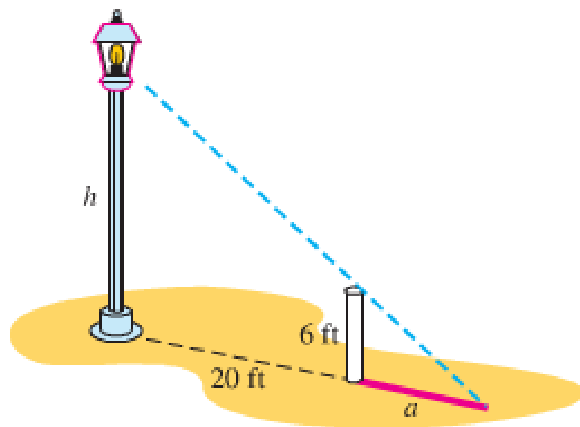
a) How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?

b) How fast is the level in the cone falling then?

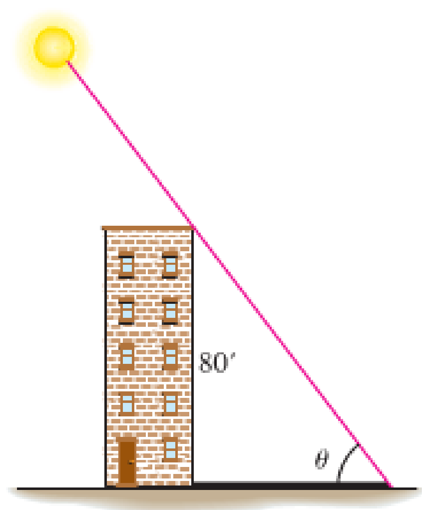
25. A light shines from the top of a pole 50 *ft* high. A ball is dropped from the same height from a point 30 ft away from the light. How fast is the shadow of the ball moving along the ground $\frac{1}{2}$ *sec* later? (Assume the ball falls a distance $s = 16t^2$ ft in t sec.)



26. To find the height of a lamppost, you stand a 6 *ft* pole 20 *ft* from the lamp and measure the length *a* of its shadow, finding it to be 15 *ft*, give or take an inch. Calculate the height of the lamppost using the value of *a* = 15 and estimate the possible error in the result.

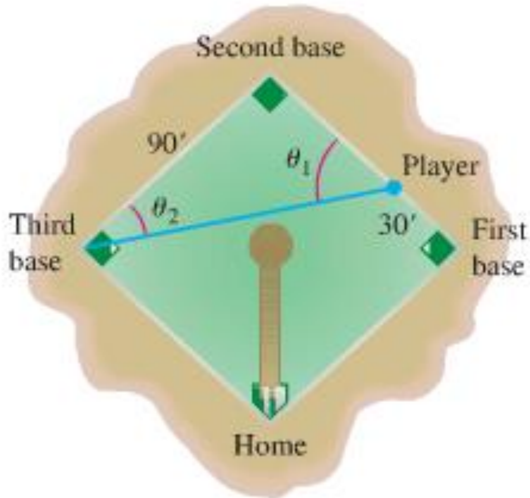


27. On a morning of a day when the sun will pass directly overhead, the shadow of an 80−*ft* building on level ground is 60 *ft* long. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of 0.27° / *min*. At what rate is the shadow decreasing?



28. A spherical iron ball 8 *in*. in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of 10 $in^3/\min$, how fast is the thickness of the ice decreasing when it is 2 *in*. thick? How fast is the outer surface area of ice decreasing?

29. A baseball diamond is a square 90 *ft* on a side. A player runs from first base to second at a rate of 16 *ft/sec*.



a) At what rate is the player's distance from third base changing when the player is 30 *ft* from first base?

b) At what rates are angles θ_1 *and* θ_2 changing at that time?

c) The player slides into second base at the rate of 15 *ft/sec*. At what rates are angles θ_1 *and* θ_2 changing as the player touches base?

30. Runners stand at first and second base in a baseball game. At the moment a ball is hit the runner at first base runs to second base at 18 *ft/s*; simultaneously the runner on second runs to third base at 20 *ft/s*. How fast is the distance between the runners changing 1 *s* after the ball is hit?

(***Hint***: The distance between consecutive bases I 90 *ft* and the bases lie at the corners of a square.)