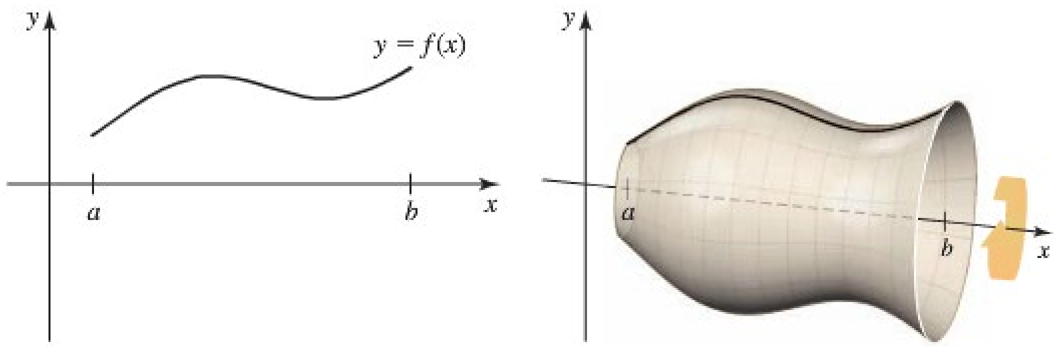
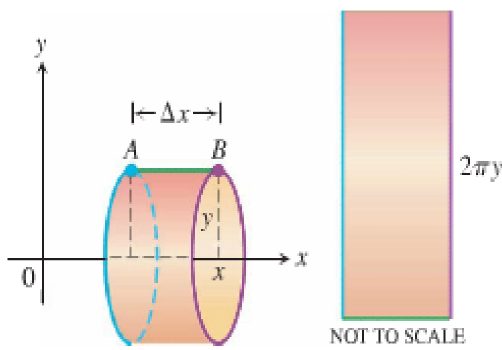
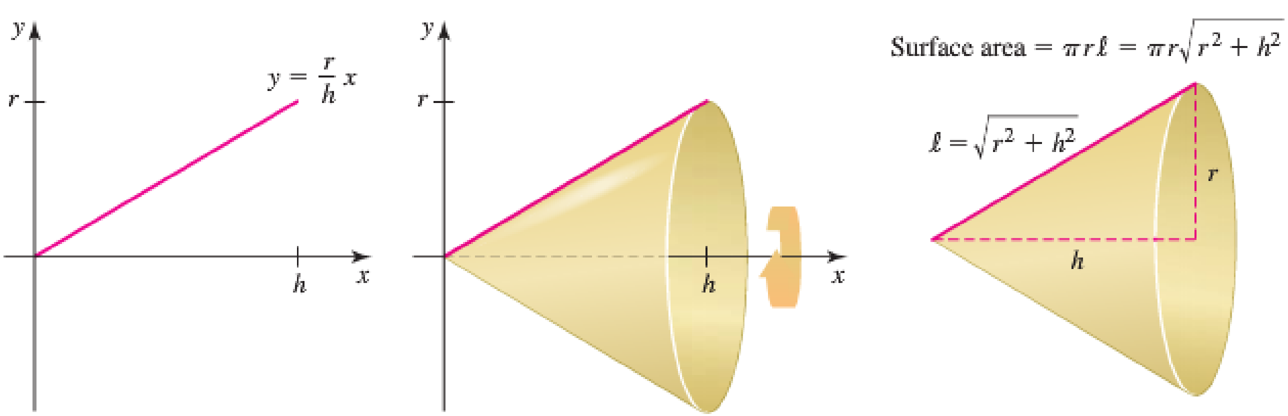
Section 1.6 – Surface Area

Consider a curve $y = f(x)$ on an interval $[a, b]$, where f is a nonnegative function with a continuous first derivative on $[a, b]$. Revolving the curve about the x-axis to generate a surface of revolution.



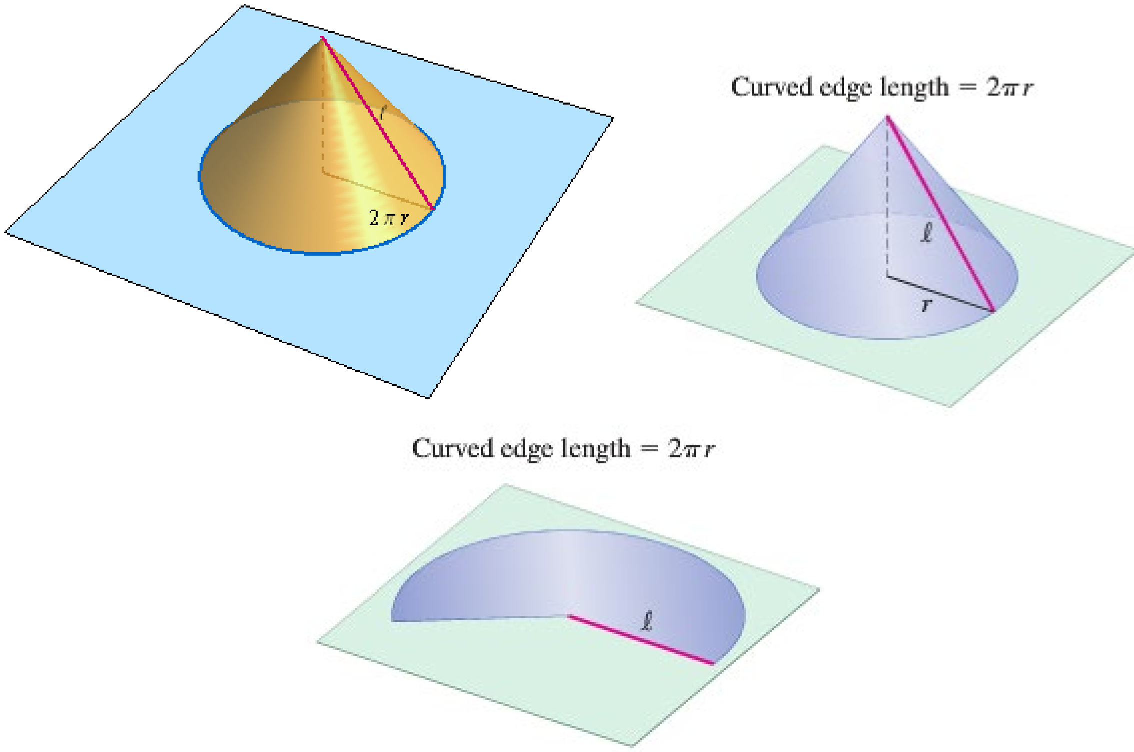


Consider the graph of $f(x) = \frac{r}{h}x$ on the interval $[0, h]$, where $h > 0$ and $r > 0$. When this line segment is revolved about the x-axis, it generates the surface of a cone of radius r and height h,

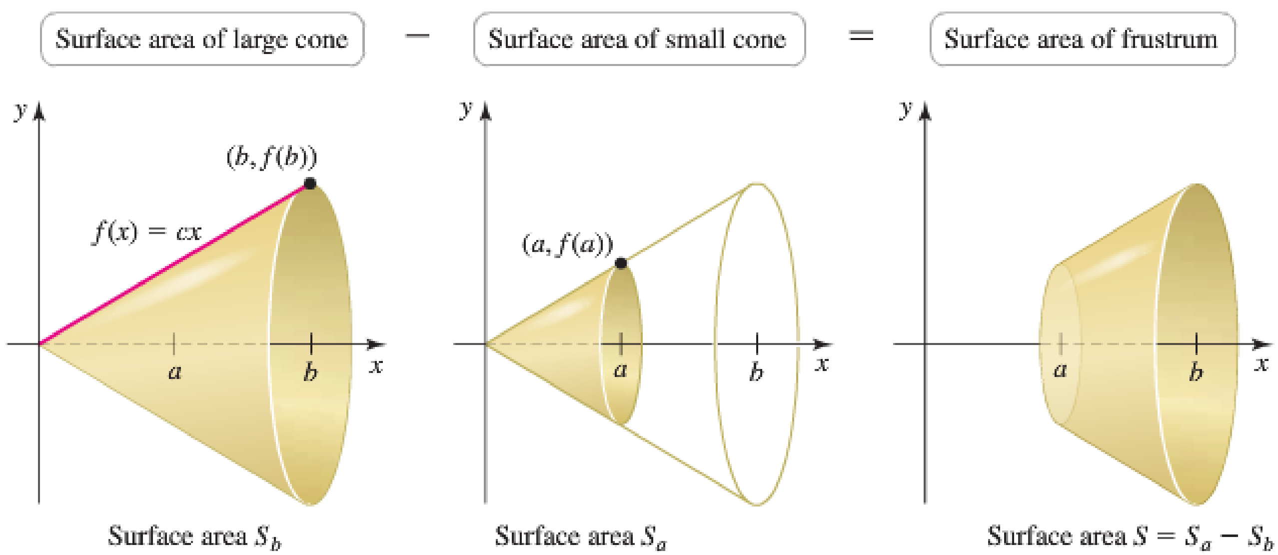


The surface area of a right circular cone, excluding the base, is $\pi r\sqrt{r^2 + h^2} = \pi r\ell$

One way to derive the formula for the surface area of a cone to cut the cone on a line from its base to its vertex. When the cone is unfolded it forms a sector of a circular disk of radius ℓ. So the area of the sector, which is also the surface area of the cone, is $\pi\ell^2 \frac{r}{\ell} = \pi r\ell$



Curved edge length = 2πr
ℓ
r
2πr
Curved edge length = 2πr
ℓ



Surface area of large cone
−
Surface area of small cone
=
Surface area of frustrum
y
(b, f(b))
f(x) = cx
a
b
x
(a, f(a))
Surface area S_b
Surface area S_a
Surface area S = S_a − S_b

Definition

If the function $f(x) \geq 0$ is continuously differentiable on [*a, b*], the area of the surface generated by revolving the graph of $y = f(x)$ about the *x-axis* is

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_a^b f(x) \sqrt{1 + \left(f'(x)\right)^2}\, dx$$

Example

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}, \quad 1 \leq x \leq 3$, about the *x-axis*.

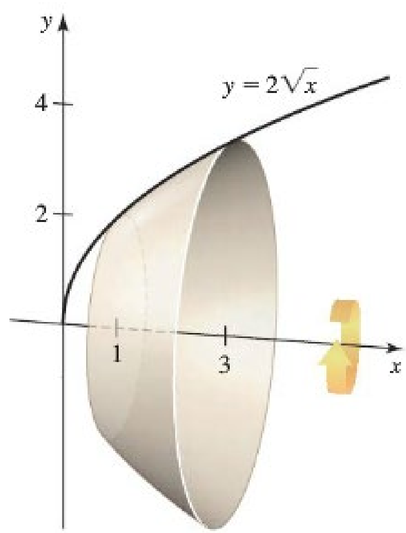
Solution

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}, \quad a = 1, \quad b = 3$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1 + \frac{1}{x}}$$

$$= \sqrt{\frac{x+1}{x}}$$



$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 4\pi \int_1^3 \left(\sqrt{x}\right) \frac{\sqrt{x+1}}{\sqrt{x}}\, dx$$

$$= 4\pi \int_1^3 (x+1)^{1/2}\, dx$$

$$= \frac{8\pi}{3} (x+1)^{3/2} \Big|_1^3$$

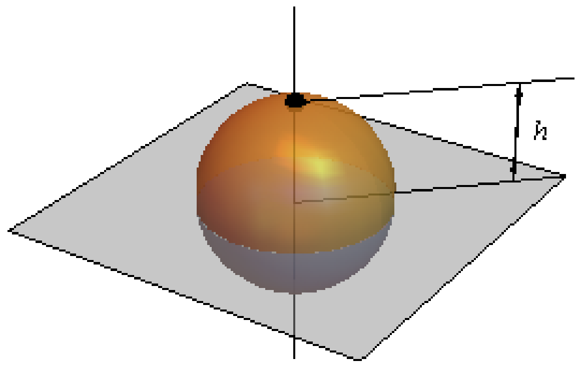
$$= \frac{8\pi}{3}\left(4^{3/2} - 2^{3/2}\right)$$

$$= \frac{8\pi}{3}\left(8 - 2\sqrt{2}\right)$$

$$\boxed{= \frac{16\pi}{3}\left(4 - \sqrt{2}\right) \quad unit^2}$$

Example

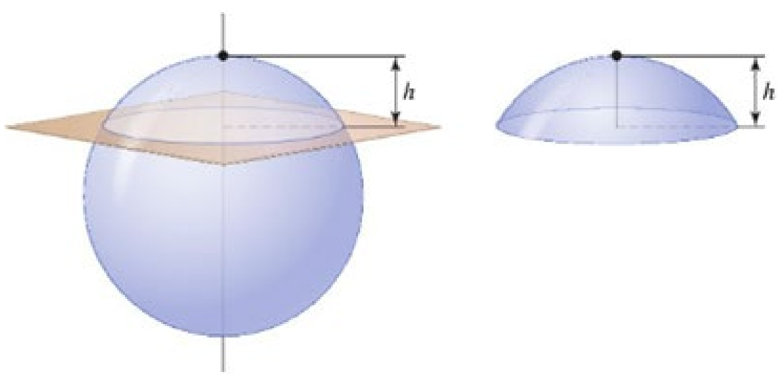
A spherical cap is produced when a sphere of radius a is sliced by a horizontal plane that is a vertical distance h below the north pole of the sphere, where $0 \le h \le 2a$. We take the spherical cap to be that part of the sphere above the plane, so that h is the depth of the cap.



Show that the area of a spherical cap of depth h cut from sphere of radius a is $2\pi ah$.

Solution

To generate the spherical surface, we revolved the curve $f(x) = \sqrt{a^2 - x^2}$ on the interval $[-a,\ a]$ about the $x\text{-axis}$.

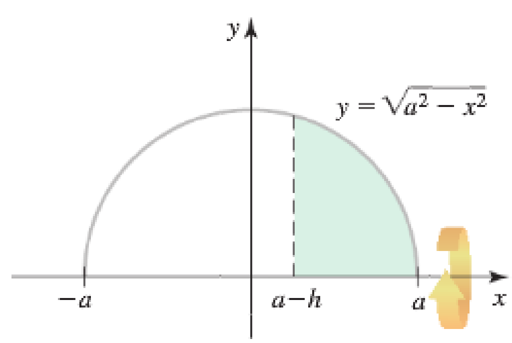


The spherical cap of height h corresponds to that part of the sphere on the interval $[-a + h,\ a]$ for $0 \le h \le 2a$

$$f'(x) = -x\left(a^2 - x^2\right)^{-1/2}$$

$$1 + f'(x)^2 = 1 + \frac{x^2}{a^2 - x^2}$$

$$= \frac{a^2}{a^2 - x^2}$$



$$S = 2\pi \int_a^b f(x)\, \sqrt{1 + f'(x)^2}\, dx$$

$$= 2\pi \int_{a-h}^{a} \sqrt{a^2 - x^2}\; \frac{a}{\sqrt{a^2 - x^2}}\, dx$$

$$= 2\pi \int_{a-h}^{a} a\, dx$$

$$= 2\pi a x \Big|_{a-h}^{a}$$

$$\underline{= 2\pi a h \quad unit^2}$$

Surface Area for revolution about the y-axis

If $x = g(y) \geq 0$ is continuously differentiable on [*c, d*], the area of the surface generated by revolving the graph of $x = g(y)$ about the *y*-axis is

$$S = 2\pi \int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

$$= 2\pi \int_c^d g(y)\sqrt{1+\left(g'(y)\right)^2}\ dy$$

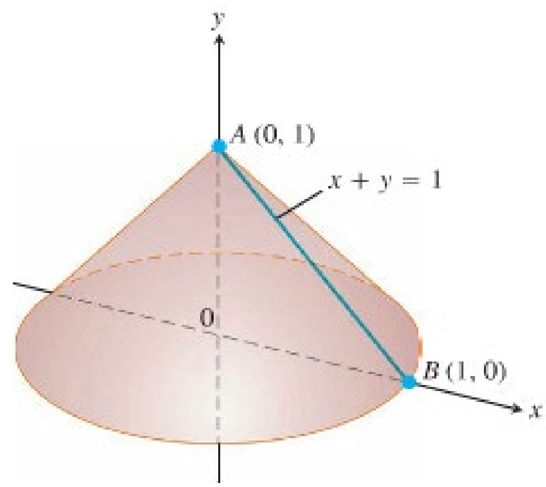
Example

The line segment $x = 1 - y, \quad 0 \leq y \leq 1$, is revolved about the *y*-axis to generate the cone. Find its lateral surface area (which excludes the base area)

Solution

$$Lateral\ Surface\ Area = \frac{base\ circumference}{2} \times slant\ height$$

$$\boxed{= \pi\sqrt{2}}$$



$$x = 1 - y \quad \frac{dx}{dy} = -1, \quad c = 0, \quad d = 1$$

$$S = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

$$= \int_0^1 2\pi(1-y)\sqrt{1+(-1)^2}\ dy$$

$$= 2\pi \int_0^1 (1-y)\sqrt{2}\ dy$$

$$= 2\pi\sqrt{2}\left(y - \frac{y^2}{2}\right)\Big|_0^1$$

$$= 2\pi\sqrt{2}\left(1-\frac{1}{2}\right)$$

$$\boxed{= \pi\sqrt{2} \quad unit^2}$$

Formula

***Surface* of a curve** $y = f(x)$ is given by the formula:

$$S = 2\pi \int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx$$

If $f(x) = ax^m + bx^n$, then

$$\sqrt{1+\left(f'(x)\right)^2} = \overline{f'(x)}$$

$\overline{f'(x)}$: is the conjugate of $f'(x)$

Iff $f(x)$ satisfies these 2 conditions:

1. $m + n = 2$
2. $abmn = -\frac{1}{4}$

Proof

$$f'(x) = max^{m-1} + nbx^{n-1}$$

$$1+\left(f'\right)^2 = 1+\left(max^{m-1} + nbx^{n-1}\right)^2$$

$$= 1 + m^2a^2x^{2m-2} + 2abmnx^{m+n-2} + n^2b^2x^{2n-2}$$

We need to combined to a perfect square $\quad a^2 - 2ab + b^2 = (a-b)^2$

- If $x^{m+n-2} = 1 = x^0 \rightarrow \boxed{m+n=2}$

$$= m^2a^2x^{2m-2} + (1+2abmn) + n^2b^2x^{2n-2} \qquad a^2 - 2ab + b^2 = (a-b)^2$$

- Let $1 + 2abmn = -2abmn \rightarrow \boxed{abmn = -\frac{1}{4}}$

$$= m^2a^2x^{2m-2} - 2abmn + n^2b^2x^{2n-2} \qquad x^{2(m+n-2)} = 1$$

$$= \left(max^{m-1} - nbx^{n-1}\right)^2$$

$$\sqrt{\left(max^{m-1} - nbx^{n-1}\right)^2} = max^{m-1} - nbx^{n-1} \qquad \checkmark$$

$$f'(x) = max^{m-1} + nbx^{n-1}$$

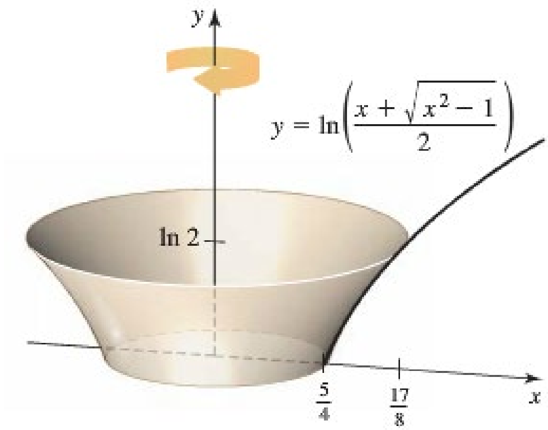
$$\sqrt{1+\left(f'(x)\right)^2} = max^{m-1} - nbx^{n-1} = \overline{f'(x)}$$

Example

Consider the function $y=\ln\left(\dfrac{x+\sqrt{x^2-1}}{2}\right)$

Find the area of the surface generated when the part of the curve between the points $\left(\frac{5}{4},\ 0\right)$ and $\left(\frac{17}{8},\ \ln 2\right)$ is revolved about y-*axis*.

Solution



$$y=\ln\left(\frac{x+\sqrt{x^2-1}}{2}\right)$$

$$e^y=\frac{x+\sqrt{x^2-1}}{2}$$

$$\left(2e^y-x\right)^2=\left(\sqrt{x^2-1}\right)^2$$

$$4e^{2y}-4xe^y+x^2=x^2-1$$

$$4xe^y=4e^{2y}+1$$

$$x=e^y+\frac{1}{4}e^{-y}=g(y)$$

$$\boxed{g'(y)=e^y-\frac{1}{4}e^{-y}}$$

$$a=\frac{1}{2},\quad m=1,\quad b=\frac{1}{2},\quad n=-1$$

1. $m=-n$ ✓

2. $abmn=-\frac{1}{4}$ ✓

$$S=2\pi\int_0^{\ln 2}\left(e^y+\frac{1}{4}e^{-y}\right)\left(e^y+\frac{1}{4}e^{-y}\right)dy$$

$$=2\pi\int_0^{\ln 2}\left(e^{2y}+\frac{1}{2}+\frac{1}{16}e^{-2y}\right)dy$$

$$=2\pi\left(\frac{1}{2}e^{2y}+\frac{y}{2}-\frac{1}{32}e^{-2y}\right.\Bigg|_0^{\ln 2}$$

$$=2\pi\left(2+\frac{\ln 2}{2}-\frac{1}{128}-\frac{1}{2}+\frac{1}{32}\right)$$

$$\boxed{=\pi\left(\frac{195}{64}+\ln 2\right)\quad unit^2}$$

OR

$$\sqrt{1+g'(y)^2} = \sqrt{1+\left(e^y - \frac{1}{4}e^{-y}\right)^2}$$

$$= \sqrt{1+e^{2y} - \frac{1}{2} + \frac{1}{16}e^{-2y}}$$

$$= \sqrt{e^{2y} + \frac{1}{2} + \frac{1}{16}e^{-2y}}$$

$$= \sqrt{\left(e^y + \frac{1}{4}e^{-y}\right)^2}$$

$$= e^y + \frac{1}{4}e^{-y}$$

$$S = 2\pi \int_0^{\ln 2} \left(e^y + \frac{1}{4}e^{-y}\right)^2 \, dy$$

$$= 2\pi \int_0^{\ln 2} \left(e^{2y} + \frac{1}{2} + \frac{1}{16}e^{-2y}\right) dy$$

$$= 2\pi \left(\frac{1}{2}e^{2y} + \frac{y}{2} - \frac{1}{32}e^{-2y}\right)\Big|_0^{\ln 2}$$

$$= 2\pi \left(2 + \frac{\ln 2}{2} - \frac{1}{128} - \frac{1}{2} + \frac{1}{32}\right)$$

$$\boxed{= \pi\left(\frac{195}{64} + \ln 2\right) \quad unit^2}$$

Exercises *Section* 1.6 – Surface Area

1. Find the lateral (side) surface area of the cone generated by revolving the line segment $y=\frac{x}{2}, \quad 0\le x\le 4$, about the x-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2}\times \text{base circumference}\times \text{slant height}$$

2. Find the lateral surface area of the cone generated by revolving the line segment $y=\frac{x}{2}, \quad 0\le x\le 4$, about the y-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2}\times \text{base circumference}\times \text{slant height}$$

3. Find the lateral surface area of the cone frustum generated by revolving the line segment $y=\frac{x}{2}+\frac{1}{2}, \quad 1\le x\le 3$, about the x-axis. Check your answer with the geometry formula

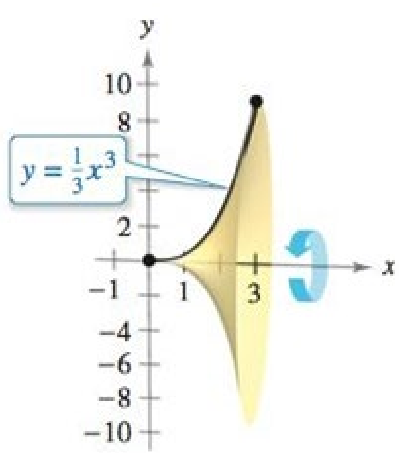
$$\text{Frustum surface area} = \pi\left(r_1+r_2\right)\times \text{slant height}$$

4. Find the lateral surface area of the cone frustum generated by revolving the line segment $y=\frac{x}{2}+\frac{1}{2}, \quad 1\le x\le 3$, about the y-axis. Check your answer with the geometry formula

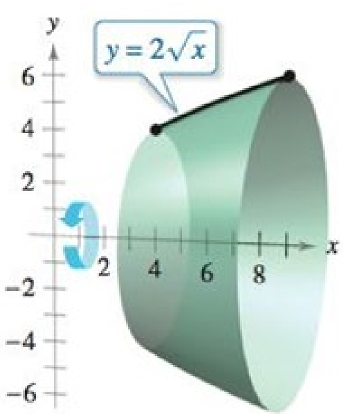
$$\text{Frustum surface area} = \pi\left(r_1+r_2\right)\times \text{slant height}$$

(5 – 22) Find the area of the surface generated by revolving the curve about the x-*axis*

5.



6.



7. $y=\frac{x^3}{9}, \quad 0\le x\le 2$

8. $y=\sqrt{x+1}, \quad 1\le x\le 5$

9. $y=\sqrt{2x-x^2}, \quad 0.5\le x\le 1.5$

10. $y=3x+4, \quad 0\le x\le 6$

11. $y=12-3x, \quad 1\le x\le 3$

12. $y=x^{3/2}-\frac{1}{3}x^{1/2}, \quad 1\le x\le 2$

13. $y=\sqrt{4x+6}, \quad 0\le x\le 5$

14. $y=\frac{1}{4}\left(e^{2x}+e^{-2x}\right), \quad -2\le x\le 2$

15. $y=\frac{1}{8}x^4+\frac{1}{4x^2}, \quad 1\le x\le 2$

16. $y = 8\sqrt{x}, \quad 9 \le x \le 20$

17. $y = x^3, \quad 0 \le x \le 1$

18. $y = \frac{1}{3}x^3 + \frac{1}{4x}, \quad \frac{1}{2} \le x \le 2$

19. $y = \sqrt{5x - x^2}, \quad 1 \le x \le 4$

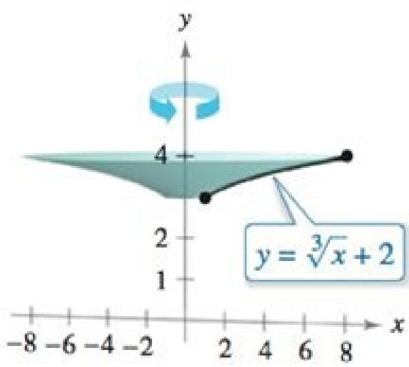
20. $y = \frac{1}{6}x^3 + \frac{1}{2x}, \quad 1 \le x \le 2$

21. $y = \sqrt{4 - x^2}, \quad -1 \le x \le 1$

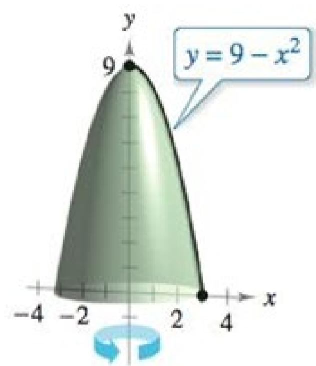
22. $y = \sqrt{9 - x^2}, \quad -2 \le x \le 2$

(23 – 29) Find the area of the surface generated by revolving the curve about the *y-axis*

23.



24.



25. $y = (3x)^{1/3}; \quad 0 \le x \le \frac{8}{3}$

26. $x = \sqrt{12y - y^2}; \quad 2 \le y \le 10$

27. $x = 4y^{3/2} - \frac{1}{12}y^{1/2}; \quad 1 \le y \le 4$

28. $y = 1 - \frac{1}{4}x^2, \quad 0 \le x \le 2$

29. $y = \frac{1}{2}x + 3, \quad 1 \le x \le 5$

30. A right circular cone is generated by revolving the region bounded by $y = \frac{3}{4}x$, $y = 3$, and $x = 0$ about the *y-axis*. Find the lateral surface area of the cone.

31. A right circular cone is generated by revolving the region bounded by $y = \frac{h}{r}x$, $y = h$, and $x = 0$ about the *y-axis*. Verify that the lateral surface area of the cone is $S = \pi r\sqrt{r^2 + h^2}$

32. Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{9 - x^2}, \quad 0 \le x \le 2$, about the *y-axis*

33. Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{r^2 - x^2}, \quad 0 \le x \le a$, about the *y-axis*. Assume that $a < r$.

34. Find the area of the surface generated by part of the curve $y = 4x - 1$ between the points $(1,\ 3)$ and $(4,\ 15)$ *about y-axis*

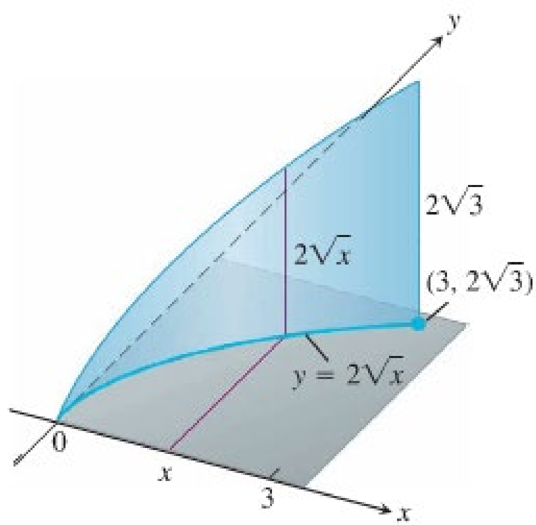
35. Find the area of the surface generated by part of the curve $y=\frac{1}{2}\ln\left(2x+\sqrt{4x^2-1}\right)$ between the points $\left(\frac{1}{2},\ 0\right)$ and $\left(\frac{17}{16},\ \ln 2\right)$ *about y-axis*

36. Find the area of the surface generated by $y=1+\sqrt{1-x^2}$ between the points $(1,\ 1)$ and $\left(\frac{\sqrt{3}}{1},\ \frac{3}{2}\right)$ *about y-axis*

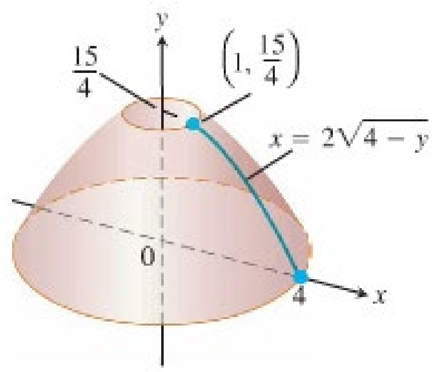
37. Find the area of the surface generated by $y=\frac{1}{3}x^3,\quad 0\le x\le 1,\quad x-axis$

38. Find the area of the surface generated by $x=\sqrt{4y-y^2},\quad 1\le y\le 2;\quad y-axis$

39. At points on the curve $y=2\sqrt{x}$, line segments of length $h=y$ are drawn perpendicular to the xy-plane. Find the area of the surface formed by these perpendiculars from $(0,\ 0)$ *to* $\left(3,\ 2\sqrt{3}\right)$

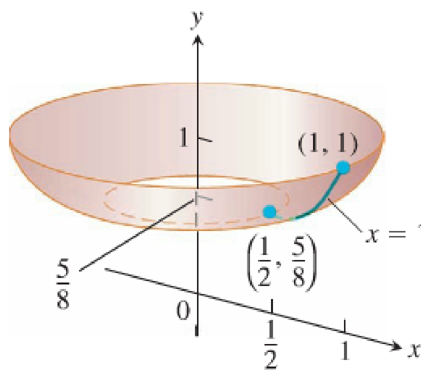


40. Find the area of the surface generated by $x=2\sqrt{4-y}\quad 0\le y\le \frac{15}{4},\quad y-axis$

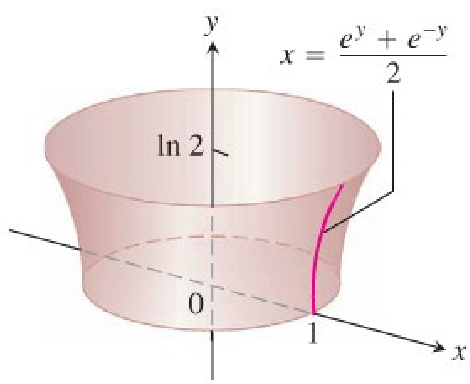


41. $y=\frac{1}{3}\left(x^2+2\right)^{3/2}$, $0\le x\le\sqrt{2}$; $y-axis$ (Hint: Express $ds=\sqrt{dx^2+dy^2}$ in terms of *dy*, and evaluate the integral $S=\int 2\pi y\ ds$ with appropriate limits.)

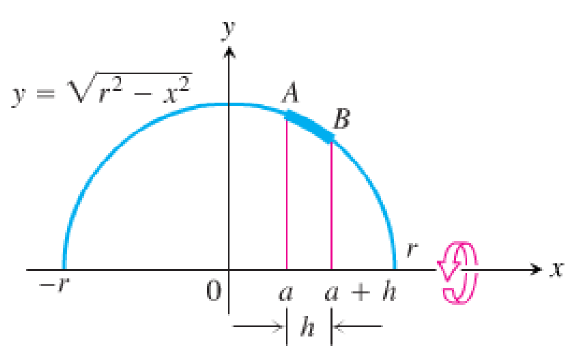
42. Find the area of the surface generated by $x=\sqrt{2y-1}$ $\frac{5}{8}\le y\le 1$, $y-axis$



43. Find the area of the surface generated by revolving the curve $x=\frac{1}{2}\left(e^{y}+e^{-y}\right)$, $0\le y\le \ln 2$, about *y*-axis



44. Did you know that if you can cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle $y=\sqrt{r^2-x^2}$ shown here is revolved about the *x*-axis to generate a sphere. Let *AB* be an arc of the semicircle that lies above an interval of length *h* on the *x*-axis. Show that the area swept out by *AB* does not depend on the location of the interval. (It does depend on the length of the interval.)



45. The curved surface of a funnel is generated by revolving the graph of $y = f(x) = x^3 + \frac{1}{12x}$ on the interval $[1,\ 2]$ about the *x-axis*. Approximately what volume of paint is needed to cover the outside of the funnel with a layer of paint 0.05 *cm* thick? Assume that *x* and *y* measured in centimeters.

46. When the circle $x^2 + (y-a)^2 = r^2$ on the interval $[-r,\ r]$ is revolved about the *x-axis*, the result is the surface of a torus, where $0 < r < a$. Show that the surface area of the torus is $S = 4\pi^2 ar$.

47. A 1.5–*mm* layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the curve $y = \sqrt{8x - x^2}$ on the interval $[1,\ 7]$ is revolved about the *x-axis*. Assume *x* and *y* are in *meters*.

48. A 1.5–*mm* layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the upper portion of the circle $x^2 + y^2 = 100$ on the interval $[-8,\ 8]$ is revolved about the *x-axis*. Assume *x* and *y* are in *meters*.

49. Find the surface area of a cone (excluding the base) with radius 4 and height 8 using integration and a surface area integral.

50. Let $f(x) = \frac{1}{3}x^3$ and let *R* be the region bounded by the graph of *f* and the *x-axis* on the interval $[0,\ 2]$

a) Find the area of the surface generated when the graph of *f* on $[0,\ 2]$ is revolved about the *x-axis*.

b) Find the volume of the solid generated when *R* is revolved about the *y-axis*.

c) Find the volume of the solid generated when *R* is revolved about the *x-axis*.

51. Let $f(x) = \sqrt{3x - x^2}$ and let *R* be the region bounded by the graph of *f* and the *x-axis* on the interval $[0,\ 3]$

a) Find the area of the surface generated when the graph of *f* on $[0,\ 3]$ is revolved about the *x-axis*.

b) Find the volume of the solid generated when *R* is revolved about the *x-axis*.

52. Let $f(x) = \frac{1}{2}x^4 + \frac{1}{16x^2}$ and let *R* be the region bounded by the graph of *f* and the *x-axis* on the interval $[1,\ 2]$

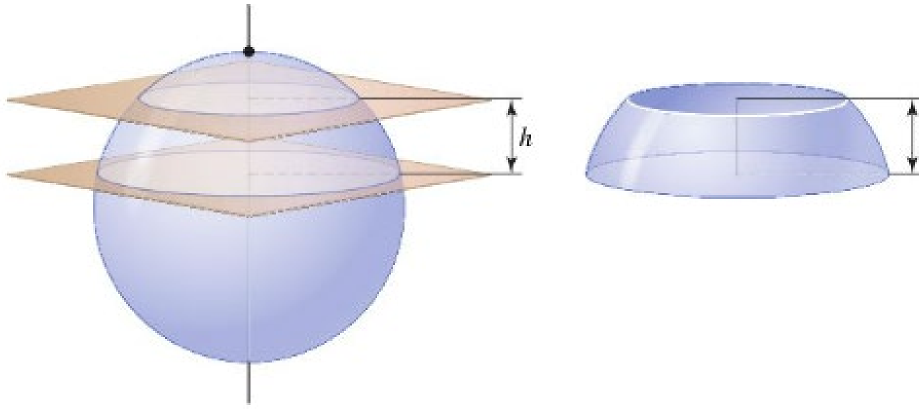
a) Find the area of the surface generated when the graph of *f* on $[1,\ 2]$ is revolved about the *x-axis*.

b) Find the length of the curve $y = f(x)$ on $[1,\ 2]$

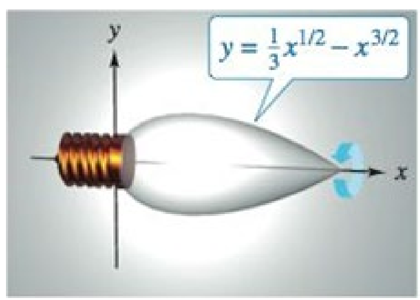
c) Find the volume of the solid generated when *R* is revolved about the *y-axis*.

d) Find the volume of the solid generated when *R* is revolved about the *x-axis*.

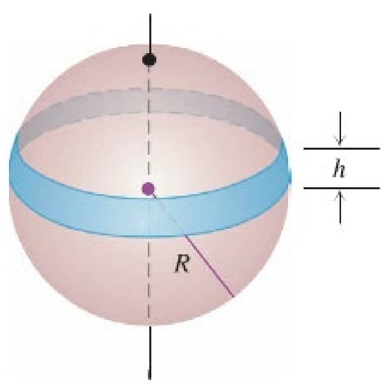
53. Suppose a sphere of radius r is sliced by two horizontal planes h units apart. Show that the surface area of the resulting zone on the sphere is $2\pi h$, independent of the location of the cutting planes.



54. An ornamental light bulb is designed by revolving the graph of $y = \frac{1}{3}x^{1/2} - x^{3/2}, \quad 0 \le x \le \frac{1}{3}$ about the x-*axis*, where x and y are mesured in *feet*. Find the surface area of the bulb and use the result to approximate the amount of glass needed to make the bulb.
(Assume that the glass is 0.015 *inch* thick)



55. The shaded band is cut from a sphere of radius R by parallel planes h units apart. Show that the surface area of the band is $2\pi Rh$



56. A drawing of a 90-*ft* dome is used by the National Weather Service. How much outside surface is there to paint (not counting the bottom)?