Solution Section 4.1 – Random Variable & probability

Exercise

Suppose a random sample of 2 light bulbs is selected from a group of 8 bulbs that contain 3 defective bulbs.

a) What is the expected value of the number of defective bulbs in the sample?

b) Probability Distribution Table

c) What is your expected return?

Solution

a) Let X represents the number of defective bulbs that occur on a given trial. X = {0, 1, 2}

$$P(x=0)=\frac{{}_3C_0 \cdot {}_5C_2}{{}_8C_2}=0.3571$$

$$P(x=1)=\frac{{}_3C_1 \cdot {}_5C_1}{{}_8C_2}=0.5357$$

$$P(x=2)=\frac{{}_3C_2 \cdot {}_5C_0}{{}_8C_2}=0.1071$$

b) Probability Distribution Table

x_i	0	1	2
P_i	.3571	.5357	.1071

c) $E[X]=0(.3571)+1(.5357)+2(.1071)\ \underline{=0.75}$

Exercise

Suppose 1000 raffle tickets are sold at a price of $10 each. Two first place tickets will be drawn, 5 second place tickets will be drawn and 10 third place tickets will be drawn. The first place prize is a $200 VCR, the second place prize is a $100 printer, and the third place prize is a $50 gift certificate.

Solution

Let X be the net gain to the ticketholder. Find the expected value of X.

X = {–10, 40. 90, 190} (you paid $10 to play)

$$P(x=-10)=\frac{983}{1000}=0.983 \qquad P(x=40)=\frac{10}{1000}=0.01$$

$$P(x=90)=\frac{5}{1000}=0.005 \qquad P(x=190)=\frac{2}{1000}=0.002$$

Payoff Table

x_i	-10	40	90	190
P_i	.983	.01	.005	.002

$$E[X] = -10(.983) + 40(.01) + 90(.005) + 190(.002)$$

$$\underline{= -\$8.60}$$

Exercise

Find the expected value of each random variable.

a)

x	2	3	4	5
$P(x)$	0.1	0.4	0.3	0.2

b)

x	4	6	8	10
$P(x)$	0.4	0.4	0.05	0.15

c)

x	9	12	15	18	21
$P(x)$	0.14	0.22	0.38	0.19	0.07

d)

x	30	32	36	38	44
$P(x)$	0.31	0.29	0.26	0.09	0.05

Solution

a) $E(x) = 2(.1) + 3(0.4) + 4(0.3) + 5(0.2)$

$\underline{= 3.6}$

b) $E(x) = 3(0.4) + 5(0.4) + 8(0.05) + 10(0.15)$

$\underline{= 5.9}$

c) $E(x) = 9(0.14) + 12(0.22) + 15(0.38) + 18(0.19) + 21(0.07)$

$\underline{= 14.49}$

d) $E(x) = 30(0.31) + 32(0.29) + 36(0.26) + 38(0.09) + 44(0.05)$

$\underline{= 33.56}$

Exercise

A delegation of 3 selected from a city council made up of 5 liberals and 6 conservatives.

a) What is the expected number of liberals in the delegation?

b) What is the expected number of conservatives in the delegation?

Solution

a) The probability of the delegation:

$$P(0)=\frac{C_{5,0}C_{6,3}}{C_{11,3}}=0.1212 \qquad P(1)=\frac{C_{5,1}C_{6,2}}{C_{11,3}}=0.4545$$

$$P(2)=\frac{C_{5,2}C_{6,1}}{C_{11,3}}=0.3636 \qquad P(3)=\frac{C_{5,3}C_{6,0}}{C_{11,3}}=0.0606$$

x	0	1	2	3
$P(x)$	0.1212	0.4545	0.3636	0.0606

$$E(x)=0(0.1212)+1(0.4545)+2(0.3636)+3(0.0606)$$

$$\approx 1.3636 \ \ \textit{liberals}$$

b) $$P(0)=\frac{C_{5,3}C_{6,0}}{C_{11,3}}=0.0606 \qquad P(1)=\frac{C_{5,2}C_{6,1}}{C_{11,3}}=0.3636$$

$$P(2)=\frac{C_{5,1}C_{6,2}}{C_{11,3}}=0.4545 \qquad P(3)=\frac{C_{5,0}C_{6,3}}{C_{11,3}}=0.1212$$

x	0	1	2	3
$P(x)$	0.0606	0.3636	0.4545	0.1212

$$E(x)=0(0.0606)+1(0.3636)+2(0.4545)+3(0.1212)$$

$$\approx 1.6364 \ \ \textit{conservatives}$$

Exercise

From a group of 3 women and 5 men, a delegation of 2 is selected. Find the expected number of women in the delegation.

Solution

$$P(0)=\frac{C_{3,0}C_{5,2}}{C_{8,2}}=0.357 \qquad P(1)=\frac{C_{3,1}C_{5,1}}{C_{8,2}}=0.5357 \qquad P(2)=\frac{C_{3,2}C_{5,0}}{C_{8,2}}=0.107$$

x	0	1	2
$P(x)$	0.357	0.536	0.107

$$E(x)=0(0.357)+1(0.536)+2(0.107)$$

$$\approx 0.75$$

Exercise

In a club with 20 senior and 10 junior members, what is the expected number of junior members on a 4-member committee?

Solution

$$P(0)=\frac{C_{10,0}C_{20,4}}{C_{30,4}}=0.1768 \qquad P(1)=\frac{C_{10,1}C_{20,3}}{C_{30,4}}=0.416 \qquad P(2)=\frac{C_{10,2}C_{20,2}}{C_{30,4}}=0.312$$

$$P(3)=\frac{C_{10,3}C_{20,1}}{C_{30,4}}=0.0547 \qquad P(4)=\frac{C_{10,4}C_{20,0}}{C_{30,4}}=0.0077$$

x	0	1	2	3	4
$P(x)$	0.1768	0.416	0.312	0.0547	.00766

$$E(x)=0(0.1768)+1(0.416)+2(0.312)+3(0.0547)+4(0.00766)$$

$$\underline{\approx 1.333}$$

Exercise

If 2 cards are drawn at one time from a deck of 52 cards, what is the expected number of diamonds?

Solution

$$P(0)=\frac{C_{13,0}C_{39,2}}{C_{52,2}}=0.5588 \qquad P(1)=\frac{C_{13,1}C_{39,1}}{C_{52,2}}=0.3823 \qquad P(2)=\frac{C_{13,2}C_{39,0}}{C_{52,2}}=0.0588$$

x	0	1	2
$P(x)$	0.5588	0.3823	0.0588

$$E(x)=0(0.5588)+1(0.3823)+2(0.0588)$$

$$\underline{\approx 0.5}$$

Exercise

A local used-car dealer gets complaints about his car as shown in the table below. Find the expected number of complaints per day

Number of Complaints per day	0	1	2	3	4	5	6
Probability	0.02	0.06	0.16	0.25	0.32	0.13	0.06

Solution

$$E(x)=0(0.02)+1(0.06)+2(0.16)+3(0.25)+4(0.32)+5(0.13)+6(0.06)$$

$$\underline{\approx 3.42 \ \textit{complaints per day}}$$

Exercise

An insurance company has written 100 policies for \$100,000, 500 policies for \$50,000, and 1000 policies for \$10,000 for people of age 20. If experience shows that the probability that a person will die at age 20 is 0.0012, how much can the company expect to pay out during the ear policies were written?

Solution

For a single \$100,000 policy, we have:

	Pay	***Don't pay***
Outcome	\$100,000	\$100,000
Probability	0.0012	0.9998

$$E(x) = 100,000(0.0012) + 100,000(0.9998)$$

$$\boxed{\approx 120}$$

For all 100 such policies, the company can expect to pay out

$$100(120) = \boxed{\$12,000}$$

For a single \$50,000 policy, we have:

	Pay	***Don't pay***
Outcome	\$50,000	\$50,000
Probability	0.0012	0.9998

$$E(x) = 50,000(0.0012) + 50,000(0.9998)$$

$$\boxed{\approx 60}$$

For all 500 such policies, the company can expect to pay out

$$500(60) = \boxed{\$30,000}$$

For a single \$10,000 policy, we have:

	Pay	***Don't pay***
Outcome	\$10,000	\$10,000
Probability	0.0012	0.9998

$$E(x) = 10,000(0.0012) + 10,000(0.9998)$$

$$\boxed{\approx 12}$$

For all 1,000 such policies, the company can expect to pay out

$$1,000(12) = \boxed{\$12,000}$$

The total amount the company can expect to out is:

$$\$12,000 + \$30,000 + \$12,000 = \boxed{\$54,000}$$

Exercise

An insurance policy on an electrical device pays a benefit of $4,000 if the device fails during the first year. The amount of the benefit decreases by $1,000 each successive year until it reaches 0. If the device has not failed by the beginning of any given year, the probability of failure that year is 0.4. What the expected benefit under the policy? (Choose the appropriate)

a. $2,234 *b.* $2,400 *c.* $2,500 *d.* $2,667 *e.* $2,694

Solution

$P(3,000) = 0.6 * 0.4 = \underline{0.24}$ $P(2,000) = 0.6^2 * 0.4 = \underline{0.144}$

$P(1,000) = 0.6^3 * 0.4 = \underline{0.0864}$ $P(0) = 1 - (0.4 + 0.24 + 0.144 + 0.0864) = \underline{0.1296}$

x	4,000	3,000	2,000	1,000	0
$P(x)$	0.4	0.24	0.144	0.0864	0.1296

$$E(x) = 4,000(0.4) + 3,000(0.24) + 2,000(0.144) + 1,000(0.0864) + 0(0.1296)$$

$$= \underline{2694.4}$$

The answer is ***e***.

Exercise

A tour operator has a bus that can accommodate 20 tourists. The operator knows that tourists may not show up, so he sells 21 tickets. The probability that an individual tourist will not show up is 0.02, independent of all other tourists. Each ticket costs $50, and is non-refundable if a tourist fails to show up. If a tourist shows up and a seat is not available, the tour operator has to pay $100 (ticket cost + $50 penalty) to the tourist. What is the expected revenue of the tour operator? (Choose the appropriate)

a. $935 *b.* $950 *c.* $967 *d.* $976 *e.* $985

Solution

Total earning: $21 * 50 = \$1,050$ (if 1 or more tourists do not show up).

The tour operator earns: $1,050 - 100 = \$950$

The probability that all tourist show up: $(1 - 0.02)^{21} \approx 0.6543$

The expected revenue is

$$1050(1 - 0.6453) + 650(0.6453) = \underline{984.57}$$

The answer is ***e***.

Solution   Section 4.2 – Frequency Distribution; Measure of Central Tendency

Exercise

Find the mean: 9.4, 11.3, 10.5, 7.4, 9.1, 8.4, 9.7, 5.2, 1.1, 4.7

Solution

$$\overline{x} = \frac{9.4 + 11.3 + 10.5 + 7.4 + 9.1 + 8.4 + 9.7 + 5.2 + 1.1 + 4.7}{10}$$

$$= 7.68$$

Exercise

Find the median: 28.4, 9.1, 3.4, 27.6, 59.8, 32.1, 47.6, 29.8

Solution

$$\{3.4,\ 9.1,\ 27.6,\ 28.4,\ 29.8,\ 32.1,\ 47.6,\ 59.8\}$$

$$\textbf{Median} = \frac{28.4 + 29.8}{2} = 29.1$$

Exercise

Find the mode: 16, 15, 13, 15, 14, 13, 11, 15, 14

Solution

Mode: 15

Exercise

Find the mean and median: 8, 10, 16, 21, 25

Solution

Mean = 16   ***Median*** = 16

Exercise

Find the mean and median: 67, 89, 78, 86, 100, 96

Solution

Mean = 86 $\textbf{Median} = \frac{86 + 89}{2} = 87.5$

Exercise

Find the mean and median: 30,200; 23,700; 33,320; 29,410; 24,600; 27,750; 27,300; 32,680

Solution

$\boldsymbol{Mean} = 28,8456$ $\qquad\qquad \boldsymbol{Median} = \dfrac{27,750 + 29,410}{2} \underline{= 28,580}$

Exercise

Find the mean and median: 15.3, 27.2, 14.8, 16.5, 31.8, 40.1, 18.9, 28.4, 26.3, 35.3

Solution

$\boldsymbol{Mean} = 25.43$ $\qquad\qquad \boldsymbol{Median} = \dfrac{26.3 + 27.2}{2} \underline{= 26.75}$

Exercise

The number of nations participating in the winter Olympic games, is given below.

Year	*Participating*
1968	37
1972	35
1976	37
1980	37
1984	49
1988	57
1992	64
1994	67
1998	72
2002	77
2006	85

Find: Mean, Media, and Mode

Solution

a. Mean: $\overline{x} = \dfrac{37+35+37+37+49+57+64+67+72+77+85}{11} \underline{\approx 56.1}$

b. Media is **57** $\qquad 35, 37, 37, 37, 49, 57, 64, 67, 72, 77, 85$

c. Mode: **37**

Exercise

Compute the mean for the grouped sample data listed in below table.

Class Interval	**Frequency**
0.5 – 5.5	6
5.5 – 10.5	20
10.5 – 15.5	18
15.5 – 20.5	4

Solution

Class Interval	x_i	**Frequency**
0.5 – 5.5	3	6
5.5 – 10.5	8	20
10.5 – 15.5	13	18
15.5 – 20.5	18	4
		48

$$\bar{X} = \frac{3(6)+8(20)+13(18)+18(4)}{48} \approx 10.08$$

Exercise

Compute the mode(s), median, and mean for each data set:

a) 2, 1, 2, 1, 1, 5, 1, 9, 4

b) 2, 5, 1, 4, 9, 8, 7

c) 8, 2 , 6, 8, 3, 3, 1, 5, 1, 8, 3

Solution

a) 1, 1, 1, 1, 2, 2, 4, 5, 9

Mode: 1

Median: 2

Mean: $\frac{1+1+1+1+2+2+4+5+9}{9} \approx 2.89$

b) 1, 2, 4, 5, 7, 8, 9

Mode: None

Median: 5

Mean: $\frac{1+2+4+5+7+8+9}{7} \approx 5.14$

c) 1, 1, 2, 3, 3, 3, 5, 6, 8, 8, 8

Mode: 3, 8

Median: 3

Mean: $\frac{1+1+2+3+3+3+5+6+8+8+8}{11} \approx 4.36$

Exercise

U.S. wheat prices and production figures for a recent decade are given in the following table.

Year	***Price* (*$ per bushel*)**	***Production* (*millions of bushels*)**
1996	4.30	2277
1997	3.38	2481
1998	2.65	2547
1999	2.48	2296
2000	2.62	2228
2001	2.78	1947
2002	3.56	1606
2003	3.40	2345
2004	3.40	2158
2005	3.45	2105

Find the mean and median for the following

a) Price per bushel of wheat

b) Wheat production

Solution

a) ***Mean*** = 32.02 $\quad$ ***Median*** $= \dfrac{3.38+3.40}{2} = 3.39$

b) ***Mean*** = 2199 $\quad$ ***Median*** $= \dfrac{2228+2277}{2} = 2252.5$

Solution   *Section* 4.3 – Measures of Variation / Dispersion

Exercise

Find the range and standard deviation for: $\{3,\ 7,\ 4,\ 12,\ 15,\ 18,\ 19,\ 27,\ 24,\ 11\}$

Solution

Range = 27-3 = 24

$$s = \sqrt{\frac{2554 - 10(14)^2}{9}} \approx 8.1$$

$$\boxed{\approx 8.1}$$

Exercise

Find the range and standard deviation for: S = {1.2, 1.4, 1.7, 1.3, 1.5}

Solution

$$Mean:\ \overline{X} = \frac{1.2+1.3+1.4+1.5+1.7}{5} = 1.42$$

Standard Variance:

$$s = \sqrt{\frac{(1.2-1.42)^2 + (1.3-1.42)^2 + (1.4-1.42)^2 + (1.5-1.42)^2 + (1.7-1.42)^2}{5-1}}$$

$$\boxed{\approx .19}$$

Exercise

Find the range and standard deviation for: 72, 61, 57, 83, 52, 66, 85

Solution

Range: 85 – 52 = 33   ***Standard deviation*** $\boxed{\approx 12.6}$

Exercise

Find the range and standard deviation for: 241, 248, 251, 257, 252, 287

Solution

Range: 287 – 241 = 46   ***Standard deviation*** $\boxed{\approx 16.1}$

Exercise

Find the range and standard deviation for: 122, 132, 141, 158, 162, 169, 180

Solution

Range: 180 – 122 = 58 ***Standard deviation*** ≈ 20.9

Exercise

Find the standard deviation for the following data

Interval	*Frequency*
30 – 39	1
40 – 49	6
50 – 59	13
60 – 69	22
70 – 79	17
80 – 89	13
90 - 99	8

Solution

Interval	f	x	xf	x^2	fx^2
30-39	1	34.5	34.5	1190.25	1190.25
40-49	6	44.5	267.0	1980.25	11,881.50
50-59	13	54.5	708.5	2970.25	38,613.25
60-69	22	64.5	1419.0	4160.24	91,525.50
70-79	17	74.5	1266.5	5550.25	94,354.25
80-89	13	84.5	1098.5	7140.25	92,823.25
90-99	8	94.5	756.0	8930.25	71,442.00
Totals:	80		5550.0		401,830.00

$$\bar{x} = \frac{\sum xf}{n}$$

$$= \frac{5550}{80}$$

$$= 69.375$$

$$s = \sqrt{\frac{401830 - 80(69.375)^2}{79}}$$

$$\approx 14.6$$

Exercise

Find the standard deviation for the following data

Interval	*Frequency*
0 – 24	4
25 – 49	8
50 – 74	5
75 – 99	10
100 – 124	4
125 – 149	5

Solution

Interval	*x*	*f*
0 – 24	12	4
25 – 49	37	8
50 – 74	62	5
75 – 99	87	10
100 – 124	112	4
125 – 149	137	5

Standard deviation ≈ 39.4

Exercise

Forever Power Company analysis conducted tests on the life of its batteries and those of a competitor (Brand X). They found that their batteries has a mean life (in hours) of 26.2, with a standard deviation of 4.1. Their results for a sample of 10 Brand X were as follows: 15, 18, 19, 23, 25, 25, 28, 30, 34, 38.

a) Find the mean and standard deviation for the sample of Brand X batteries.

b) Which batteries have a more uniform life in hours?

c) Which batteries have the highest average life in hours?

Solution

a) ***Mean*** $= \frac{255}{10} = 25.5$ ***Standard deviation*** ≈ 7.2

b) Power Company has a smaller standard deviation of 4.1 hrs. as opposed to 7.2 hr. which indicates a more uniform life

c) Power Company has a higher mean of 26.2 hr. as opposed to 25.5 hr. which indicates a longer average life

Solution Section 4.4 – Bernoulli Trials & Binomial Distributions

Exercise

If a baseball player has a batting average of 0.350, what is the probability that the player will get the following number of hits in the next four times at bat?

a) Exactly 2 hits

b) At least 2 hits.

Solution

a) ***Given***: $p = .35 \to q = 1 - .35 = .65, \quad n = 4$

$$P(x=2) = C_{4,2}(.35)^2(.65)^2$$

$$\approx 0.311$$

b) $P(x \ge 2) = P(2) + P(3) + P(4)$

$$= C_{4,2}(.35)^2(.65)^2 + C_{4,3}(.35)^3(.65) + C_{4,4}(.35)^4(.65)^0$$

$$= .3105 + .1115 + .015$$

$$\approx 0.437$$

Exercise

If a true-false test with 10 questions is given, what is the probability of scoring

a) Exactly 70% just by guessing?

b) 70% or better just by guessing?

Solution

Given: $p = .5 \to q = 1 - .5 = .5, \quad n = 10$

a) $P(x=7) = C_{10,7}(.5)^7(.5)^3$

$$\approx 0.117$$

b) $P(x \ge 7) = P(7) + P(8) + P(9) + P(10)$

$$= C_{10,7}(.5)^7(.5)^3 + C_{10,8}(.5)^8(.5)^2 + C_{10,9}(.5)^9(.5)^1 + C_{10,10}(.5)^{10}(.5)^0$$

$$\approx 0.172$$

Exercise

If 60% of the electorate supports the mayor, what is the probability that in a random sample of 10 voters, fewer than half support her?

Solution

$$p = P(\text{electorate supports the mayor}) = .6 \rightarrow q = .4, \quad n = 10$$

$$P(x \le 4) = P(4) + P(3) + P(2) + P(1) + P(0)$$

$$= C_{10,4}(.6)^4(.4)^6 + C_{10,3}(.6)^3(.4)^7 + C_{10,2}(.6)^2(.4)^8 + C_{10,1}(.6)^1(.4)^9 + C_{10,0}(.4)^{10}$$

$$\approx 0.166$$

Exercise

Each year a company selects a number of employees for a management training program given by nearby university. On the average, 70% of those sent complete the program. Out of 7 people sent by the company, what is the probability that

a) Exactly 5 complete the program?

b) 5 or more complete the program?

Solution

a) $p = .7 \rightarrow q = 1 - .7 = .3, \quad n = 7$

$$P(x = 5) = C_{7,5}(.7)^5(.3)^2 = .318$$

$$= .318$$

b) $P(x \ge 5) = P(5) + P(6) + P(7)$

$$= .318 + C_{7,6}(.7)^6(.3) + C_{7,7}(.7)^7(.3)^0$$

$$= .318 + .2471 + .0824$$

$$\approx 0.647$$

Exercise

If the probability of a new employee in a fast-food chain still being with the company at the end of 1 year is 0.6, what is the probability that out of 8 newly hired people?

a) 5 will still be with the company after 1 year?

b) 5 or more will still be with the company after 1 year?

Solution

a) $p = .6 \rightarrow q = .4, \quad n = 8$

$$P(x = 5) = C_{8,5}(.6)^5(.4)^3$$

$$= .279$$

b) $P(x \geq 5) = P(5) + P(6) + P(7) + P(8)$

$$= C_{8,5}(.6)^5(.4)^3 + C_{8,6}(.6)^6(.4)^2 + C_{8,7}(.6)^7(.4)^1 + C_{8,8}(.6)^8$$

$$\approx 0.594$$

Exercise

A manufacturing process produces, on the average, 6 defective items out of 100. To control quality, each day a sample of 10 completed items is selected at random and inspected. If the sample produces more than 2 defective items, then the whole day's output is inspected and the manufacturing process is reviewed. What is the probability of this happening, assuming that the process is still producing 6% defective items?

Solution

$$p = P(defective) = .06 \rightarrow q = P(not\ defective) = .94 \qquad n = 10$$

$$P(x > 2) = 1 - P(x \leq 2) = 1 - [P(2) + P(1) + P(0)]$$

$$= 1 - \left[C_{10,2}(.06)^2(.94)^8 + C_{10,1}(.06)^1(.94)^9 + C_{10,0}(.06)^0(.94)^{10}\right]$$

$$\approx 1 - (.0988 + .3438 + .5386)$$

$$\approx 0.188$$

A day's output will be inspected with a probability of .0188

Exercise

A manufacturing process produces, on the average, 3% defective items. The company ships 10 items in each box and wishes to guarantee no more than 1 defective item per box. If this guarantee accompanies each box, what is the probability that the box will fail to satisfy the guarantee?

Solution

$$p = .03 \rightarrow q = .97 \quad n = 10$$

$$P(\text{ fail to satisfy}) = P(x \geq 2) = 1 - P(x < 2)$$

$$= 1 - P(x < 2)$$

$$= 1 - [P(0) + P(1)]$$

$$= 1 - C_{10,0}(.03)^0(.97)^{10} + C_{10,1}(.03)^1(.97)^9$$

$$\approx 0.035$$

Exercise

A manufacturing process produces, on the average, 5 defective items out of 100. To control quality, each day a random sample of 6 completed items is selected and inspected. If a success on a single trial (inspection of 1 item) is finding the item defective, then the inspection of each of the 6 items in the sample constitutes a binomial experiment, which has a binomial distribution.

a) Write the function defining the distribution

b) Construct a table and histogram for the distribution.

c) Compute the mean and standard deviation.

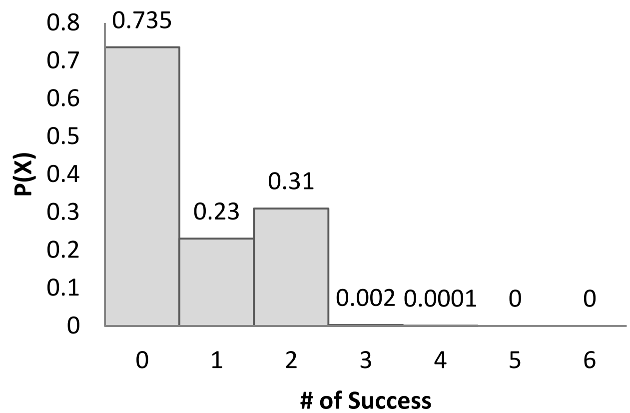
Solution

$$p = \frac{5}{100} = .05 \to q = .95 \qquad n = 6$$

a) $P(X) = C_{6,x}(.05)^{x}(.95)^{6-x} \quad , x = 0,1,2,3,4,5,6$

b) Table and histogram for the distribution.

x	*P(x)*
0	.735
1	.23
2	.31
3	.002
4	.0001
5	.000
6	.000



c) $\mu = np$

$= 6 \times .05$

$\underline{= 3}$

$\sigma = \sqrt{npq}$

$= \sqrt{6(.05)(.95)}$

$\underline{= .53}$

Exercise

Each year a company selects 5 employees for a management training program given at a nearly university. On the average, 40% of those sent complete the course in the top 10% of their class. If we consider an employee finishing in the top 10% of the class a success in a binomial experiment, then for the 5 employee entering the program there exists a binomial distribution involving P(x success out of 5).

a) Write the function defining the distribution
b) Construct a table and histogram for the distribution.
c) Compute the mean and standard deviation.

Solution

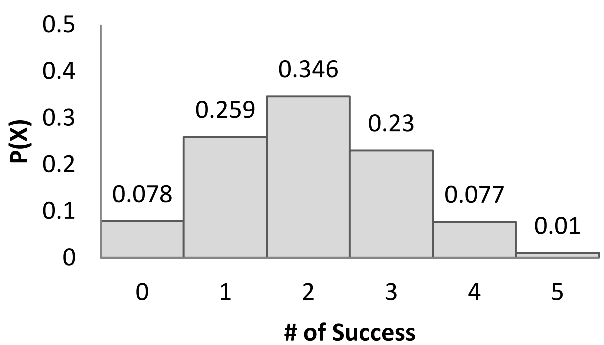
$$p = .4 \rightarrow q = .6 \qquad n = 5$$

a) $P(X) = C_{5,x}(.4)^{x}(.6)^{5-x}$, $x = 0, 1, 2, 3, 4, 5$

b) Table

x	$P(x)$
0	0.078
1	0.259
2	0.346
3	0.230
4	0.077
5	0.01

Histogram



c) $\mu = np$

$= 5 \times .4$

$\underline{= 2}$

$\sigma = \sqrt{npq}$

$= \sqrt{5(.4)(.6)}$

$\underline{\approx 1.095}$

Exercise

A person with tuberculosis is given a chest x-ray. Four tuberculosis x-ray specialists examine each x-ray independently. If each specialist can detect tuberculosis 80% of the time when it is present, what is the probability that at least 1 of the specialists will detect tuberculosis in this person?

Solution

$$p = .8 \rightarrow q = .2 \qquad n = 4$$

$$\begin{aligned} P(x \ge 1) &= 1 - P(x < 1) \\ &= 1 - P(x = 0) \\ &= 1 - C_{4,0}(.8)^4(.2)^4 \approx 0.998 \end{aligned}$$

Exercise

A pharmaceutical laboratory claims that a drug it produces causes serious side effects in 20 people out of 1,000 on the average. To check this claim, a hospital administers the drug to 10 randomly chosen patients and finds that 3 suffer from serious side effects. If the laboratory's claims are correct, what is the probability of the hospital obtaining these results?

Solution

$$p = \frac{20}{1000} = .02 \rightarrow q = .98 \qquad n = 10$$

$$\begin{aligned} P(x = 3) &= C_{10,3}(.02)^3(.98)^7 \\ &\approx 0.0008 \end{aligned}$$

Exercise

The probability that brown-eyed parents, both with the recessive gene for blue, will have a child with brown eye is .75. If such parents have 5 children, what is the probability that they will have

a) All blue-eyed children?

b) Exactly 3 children with brown eyes?

c) At least 3 children with brown eyes?

Solution

$$p = .75 \rightarrow q = .25 \qquad n = 5$$

a) $P(x = 0) = C_{5,0}(.75)^0(.25)^5 \approx 0.00098$

b) $P(x = 3) = C_{5,3}(.75)^3(.25)^2 \approx 0.264$

c) $$\begin{aligned} P(x \ge 3) &= P(3) + P(4) + P(5) \\ &= C_{5,3}(.75)^3(.25)^2 + C_{5,4}(.75)^4(.25)^1 + C_{5,5}(.75)^5(.25)^0 \\ &\approx .897 \end{aligned}$$

Exercise

The probability of gene mutation under a given level of radiation is 3×10^{-5}. What is the probability of the occurrence of at least 1 gene mutation if 10^5 genes are expected to this level of radiation?

Solution

$$p = 3\times10^{-5} \rightarrow q = 1-3\times10^{-5}$$

$$\begin{aligned} P(x\geq 1) &= 1-P(0) \\ &= 1-\left(1-3\times10^{-5}\right)^{10^5} \\ &\approx 0.95 \end{aligned}$$

Exercise

If the probability of a person contracting influenza on exposure is .6 consider the binomial distribution for a family of 6 that has been exposed.

a) Write the function defining the distribution.

b) Compute the mean and standard deviation.

Solution

$$p = 0.6 \rightarrow q = .4, \qquad n = 6$$

a) $P(X) = C_{6,x}(.6)^x(.4)^{6-x} \quad , x = 0,1,2,3,4,5,6$

b) $\mu = np = 6(.6) = 3.6$

$\sigma = \sqrt{npq} = \sqrt{6(.6)(.4)} = 1.2$

Exercise

The probability that a given drug will produce a serious side effect in a person using the drug is .02. In the binomial distribution for 450 people using the drug, what are the mean and standard deviation?

Solution

$$p = 0.02 \rightarrow q = .98, \qquad n = 450$$

$$\begin{aligned} \mu &= np \\ &= 450\times.02 \\ &= 9 \end{aligned}$$

$$\begin{aligned} \sigma &= \sqrt{npq} \\ &= \sqrt{450\times.02\times.98} \\ &\approx 2.97 \end{aligned}$$

Exercise

An opinion poll based on a small sample can be unrepresentative of the population. To see why, let us assume that 40% of the electorate favors a certain candidate. If a random sample of 7 is asked their preference, what is the probability that a majority will favor this candidate?

Solution

$$p = 0.4 \to q = .6, \qquad n = 7$$

$$P(x \ge 4) = P(4) + P(5) + P(6) + P(7)$$

$$P(x \ge 4) = C_{7,4}(.4)^4(.6)^3 + C_{7,5}(.4)^5(.6)^2 + C_{7,6}(.4)^6(.6)^1 + C_{7,7}(.4)^7(.6)^0$$

$$\approx 0.29$$

(Better than one chance out of four)

Exercise

A multiple choice test is given with 5 choices only one is correct, for each of 5 questions. Answering each of the 5 questions by guessing constitutes a binomial experiment with an associated binomial distribution

a) Write the function defining the distribution.

b) Compute the mean and standard deviation.

Solution

$$p = \frac{1}{5} = 0.2 \to q = .8, \qquad n = 5$$

a) $P(X) = C_{5,x}(.2)^x(.8)^{5-x} \quad , x = 0,1,2,3,4,5$

b) $\mu = np = 5(.2) = 1$

$$\sigma = \sqrt{npq}$$

$$= \sqrt{5(.2)(.8)}$$

$$= .894$$

Exercise

Suppose a die is rolled 4 times.

a) Find the probability distribution for the number of times 1 is rolled.

b) What is the expected number of times 1 is rolled

Solution

a) $P(x = 0) = C_{4,0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^4 \approx 0.482$

$$P(x=1)=C_{4,1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^3 \approx 0.386$$

$$P(x=2)=C_{4,2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2 \approx 0.116$$

$$P(x=3)=C_{4,3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^1 \approx 0.0154$$

$$P(x=4)=C_{4,4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^0 \approx 0.00077$$

x	0	1	2	3	4
$P(x)$	0.482	0.386	0.116	0.0154	0.00077

b) $E(x)=0(0.482)+1(0.386)+2(0.116)+3(0.0154)+4(0.00077)$

≈ 0.667

Solution Section 4.5 – Normal Distribution

Exercise

A manufacturing process produces light bulbs with life expectancies that are normally distributed with a mean of 500 hours and a standard deviation of 100 hours. What percentage of the light bulbs can be expected to last 500 to 750 hours?

Solution

$\mu = 500,\ \sigma = 100 \qquad \Rightarrow 500\ \&\ 750$

$$z = \frac{x-\mu}{\sigma} = \frac{750-500}{100} = 2.5$$

$\Rightarrow$ **A = 49.38%**

Exercise

What is the probability of the light bulbs can be expected to last 400 to 500 hours?

Solution

$400 \rightarrow 500$

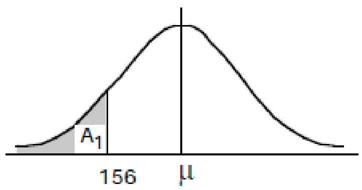
$$z = \frac{400-500}{100} = -1$$

$\Rightarrow$ **A = .3413**

Exercise

The average lifetime for a car battery of a certain brand is 170 weeks, with a standard deviation of 10 weeks. If the company guarantees the battery for 3 years, what percentage of the batteries sold would be expected to be returned before the end of the warranty period? Assume a normal distribution

Solution



$\mu = 170, \sigma = 10; \quad x \le 3(52) = 156\ weeks$

$$z = \frac{156-170}{10} = -1.4$$

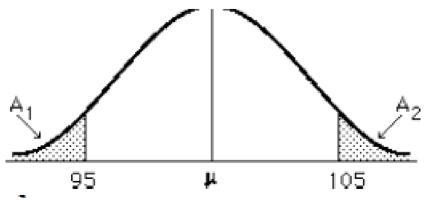
$$\begin{aligned} P(x \le 156) &= 0.5 - P(z = 1.4) \\ &= 0.5 - 0.4192 \\ &= 0.0808 \quad or \quad \boxed{8.08\%} \end{aligned}$$

Exercise

A manufacturing process produces a critical part of average length 100 millimeters, with a standard deviation of 2 millimeters. All parts deviating by more than 5 millimeters from the mean must be rejected. What percentage of the parts must be rejected, on the average? Assume a normal distribution.

Solution

$\mu = 100, \sigma = 2; \quad x = 105, x = 95$



$z(x=105) = \frac{105-100}{2} = 2.5 \quad \to A_1 = .4938$

$z(x=95) = \frac{95-100}{2} = -2.5 \quad \to A_2 = .4938$

$$\begin{aligned} Area &= 1-\left(A_1 + A_2\right) \\ &= 1-2\left(.4938\right) \\ &= .0124 \end{aligned}$$

Percentage of the parts to be rejected: **1.24%**

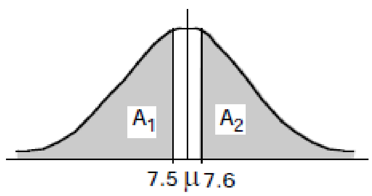
Exercise

An automated manufacturing process produces a component with an average width of 7.55 cm, with a standard deviation of 0.02 cm. All components deviating by more than 0.05 cm from the mean must be rejected. What percent of the parts must be rejected, on the average? Assume a normal distribution.

Solution

$\mu = 7.55, \ \sigma = 0.02;$

$x < \mu - 0.05, \ x > \mu + 0.05$



$P(x < \mu - 0.05) = P(x > \mu + 0.05)$

$z(x=7.6) = \frac{7.6-7.55}{0.02} = 2.5$

$\to A_1 = .4938$

$$\begin{aligned} P(\text{ parts rejected}) &= A_1 + A_2 \\ &= 2P(x > 7.6) \\ &= .5 - .4938 \\ &= .0062 \end{aligned}$$

$P = 2\left(.0062\right) = .0124$

Percentage of the parts to be rejected: **1.24%**

Exercise

A company claims that 60% of the households in a given community uses its product. A competitor surveys the community, using a random sample of 40 households, and finds only 15 households out of the 40 in the sample using the product. If the company's claim is correct, what is the probability of 15 or fewer households using the product in a sample of 40? Conclusion? Approximate a binomial distribution with a normal distribution.

Solution

$n = 40,\ p = 0.6, q = 0.4$

$\mu = np = 40(.6) = 24$

$\sigma = \sqrt{npq} = \sqrt{40(.6)(.4)} \approx 3.1$

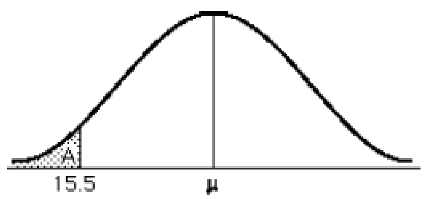
$z(x = 15.5) = \frac{15.5 - 24}{3.1} = -2.74$

$\to A = .4969$

$P(x \le 15) = .5 - A$

$= .5 - .4969$

$\boxed{= 0.0031}$



This is a rare event has occurred, e.g. the sample was not random, or the company's claim is false

Exercise

A union representative claims 60% of the union membership will vote in favor of a particular settlement. A random sample of 100 members is polled, and out of these, 47 favor the settlement. What is the approximate probability of 47 or fewer in a sample of 100 favoring the settlement when 60% of all the membership favor the settlement? Conclusion? Approximate a binomial distribution with a normal distribution.

Solution

$n = 100,\ p = 0.6,\ q = 0.4$

$\mu = np = 100(.6) = 60$

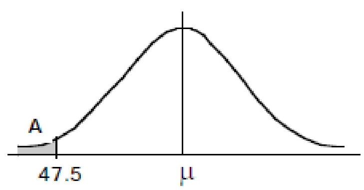
$\sigma = \sqrt{npq} = \sqrt{100(.6)(.4)} \approx 4.9$

$z(x = 47.5) = \frac{47.5 - 60}{4.9} = -2.55 \ \to A = .4946$

$P(x \le 47.5) = .5 - A$

$= .5 - .4946$

$\boxed{= 0.0054}$



Conclusion: Either a rare event has happened or the claim of 60% is false

Exercise

The average healing time of a certain type of incision is 240 hours, with standard deviation of 20 hours. What percentage of the people having this incision would heal in 8 days or less? Assume a normal distribution.

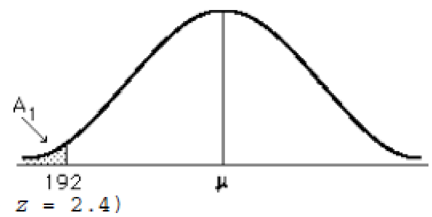
Solution

$\mu = 240, \sigma = 20; \quad x = 8\, days = 192\, hrs.$

$z(x = 192) = \dfrac{192-240}{20} = -2.4$

$\rightarrow A = .4918$

$P(x \leq 192) = .5 - .4918 \; \underline{= .0082}|$



The percentage of the people having this incision would heal is 0.82%

Exercise

The average height of a hay crop is 38 inches, with a standard deviation of 1.5 inches. What percentage of the crop will be 40 inches or more? Assume a normal distribution

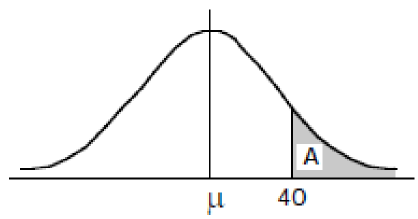
Solution

$\mu = 38, \sigma = 1.5$

$z(x = 40) = \dfrac{40-38}{1.5} = 1.33$

$\rightarrow A = .4082$

$P(x \geq 40) = .5 - .4082 = .0918 \;\; or \;\; \boxed{9.18\%}$



Exercise

In a family with 2 children, the probability that both children are girls is approximately .25. In a random sample of 1,000 families with 2 children, what is the approximate probability that 220 or fewer will have 2 girls? Approximate a binomial distribution with a normal distribution.

Solution

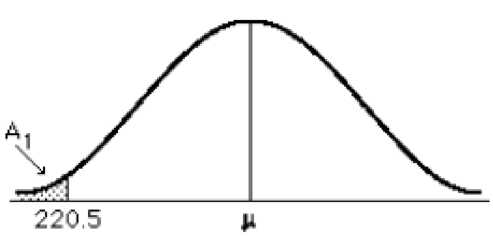
$n = 1000, \; p = 0.25, \;\; q = 0.75$

$\mu = np = 1000(.25) = 250$

$\sigma = \sqrt{npq} = \sqrt{1000(.25)(.75)} \approx 13.69$

$z(x = 220.5) = \dfrac{220.5-250}{13.69} = -2.15 \; \rightarrow A = .4842$

$P(x \leq 220.5) = .5 - .4842 \; \underline{= .0158}\,|$



Exercise

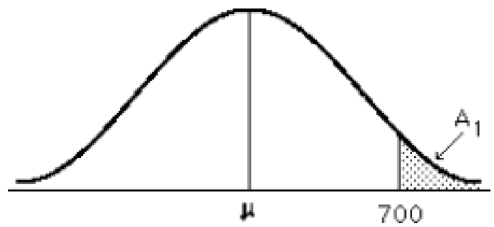
Aptitude Tests are scaled so that the mean score is 500 and the standard deviation is 100. What percentage of the students taking this test should score 700 or more? Assume a normal distribution

Solution

$\mu = 500, \sigma = 100$

$z(x = 500) = \dfrac{700-500}{100} = 2 \to A = .4722$

$P(x \geq 700) = .5 - .4722 = .0228 \ \ or \ \boxed{2.28\%}$



Exercise

Candidate Harkins claims a private poll shows that she will receive 52% of the vote for governor. Her opponent, Mankey, secures the services of another pollster, who finds that 470 out of a random sample of 1,000 registered voters favor Harkins. If Harkin's claim is correct, what is the probability that only 470 or fewer will favor her in a random sample of 1,000? Conclusion? Approximate a binomial distribution with a normal distribution.

Solution

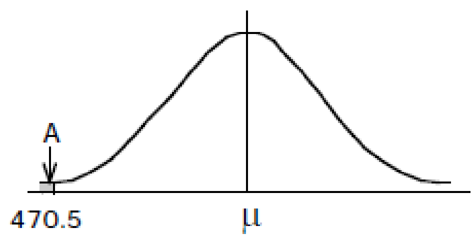
$n = 1000, \ p = 0.52, \ q = 0.48$

$\mu = np = 1000(.52) = 520$

$\sigma = \sqrt{npq} = \sqrt{1000(.52)(.48)} \approx 15.8$

$z(x = 470.5) = \dfrac{470.5-520}{15.8} = -3.13 \to A = .4991$

$P(x \leq 470.5) = .5 - .4991 = 0.0009$



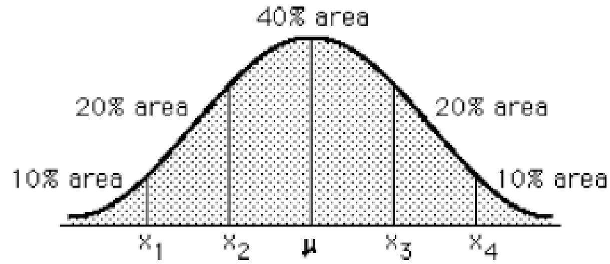
Conclusion: Either a rare event has happened or Harkin's claim is false

Exercise

An instructor grades on a curve by assuming the grades on a test are normally distributed. If the average grade is 70 and the standard deviation is 8, find the test scores for each grade interval if the instructor wishes to assign grades as follow: 10% A's, 20% B's, 40% C's, 20% D's, and 10% F's.

Solution

$\mu = 70, \sigma = 8$



Area between μ *and* $x_3 = 0.2 \to z = 0.52$

$$z\left(x = x_3\right) = \frac{x_3 - 70}{8} = 0.52$$

$$8(.52) = x_3 - 70$$

$$\Rightarrow x_3 = 70 + 4.16 = 74.16$$

$$z\left(x = x_2\right) = \frac{70 - x_2}{8} = 0.52$$

$$4.16 = 70 - x_2$$

$$\Rightarrow x_2 = 70 - 4.16 = 65.84$$

Area between μ *and* $x_4 = 0.4 \to z = 1.28$

$$\frac{x_4 - 70}{8} = 1.28$$

$$8(1.28) = x_4 - 70$$

$$\Rightarrow x_4 = 70 + 10.24 = 80.24$$

$$\frac{70 - x_1}{8} = 1.28$$

$$10.24 = 70 - x_1$$

$$\Rightarrow x_1 = 70 - 10.2 = 59.8$$

A*: 80.2 or greater*

B*: 74.2 – 80.2*

C*: 65.8 – 74.2*

D*: 59.8 – 65.8*

F*: < 59.8*

Exercise

At the discount Market, the average weekly grocery bill is \$74.50, with a standard deviation of \$24.30. What are the largest and smallest amounts spent by the middle 50% of this market's customers?

Solution

Given: $\begin{cases} \mu = 74.5 \\ \sigma = 24.30 \end{cases}$

Middle 50% cutoffs at 25% below the mean $z = -0.67$

at 75% below the mean $z = 0.67$

$$\frac{x_1 - 74.5}{24.3} = -0.67 \qquad\qquad \frac{x_1 - 74.5}{24.3} = 0.67$$

$$x_1 - 74.5 = -16.281 \qquad\qquad x_1 - 74.5 = 16.281$$

$$x_1 \approx 58.22 \qquad\qquad x_2 \approx 90.78$$

The middle 50% of customers spend between \$58.22 and \$90.78

Exercise

A certain type of light bulb has an average life of 500 hours, with a standard deviation of 100 hours. The length of life of the bulb can be closely approximated by a normal curve. An amusement park buys and installs 10,000 such bulbs. Find the total number that can be expected to last for each period of time.

Solution

At least 500 hours

$$z = \frac{500 - 500}{100} = 0$$

$$P(x \ge 500) = 1 - P(z < 0) = 1 - 0.5 = 0.5$$

0.5 (10,000) = 5000 bulbs

Less than 500 hours

$$z = \frac{500 - 500}{100} = 0$$

$$P(z < 0) = 0.5$$

5000 bulbs

Between 680 and 780 hours

For $x = 680 \to z = \frac{680 - 500}{100} = 1.8$

For $x = 780 \to z = \frac{780 - 500}{100} = 2.8$

$$P(680 \le x \le 780) = P(1.8 \le z \le 2.8)$$

$$= P(z \le 2.8) - P(z \le 1.8)$$

$$= 0.9974 - 0.9641$$
$$= 0.0333$$

0.0333 (10,000) = 333 bulbs

Between 350 and 550 hours

For $x = 350 \to z = \frac{350-500}{100} = -1.5$

For $x = 550 \to z = \frac{550-500}{100} = 0.5$

$$P(350 \le x \le 550) = P(-1.5 \le z \le 0.5)$$
$$= P(z \le 0.5) - P(z \le -1.5)$$
$$= 0.6915 - 0.0668$$
$$= 0.6247$$

0.6247 (10,000) = 6247 bulbs

Less than 770 hours

For $x = 770 \to z = \frac{770-500}{100} = 2.7$

$$P(x < 770) = P(z < 2.7) = 0.9965$$

9965 bulbs

More than 440 hours

For $x = 440 \to z = \frac{440-500}{100} = -0.6$

$$P(x > 440) = P(z > -0.6)$$
$$= 1 - 0.2743$$
$$= 0.7257$$

7257 bulbs

Find the shortest and longest lengths of life for the middle 60% of the bulbs.

$P(z < z_1) = 0.2000$ (20%) $\Rightarrow z_1 = -0.84$

$P(z < z_2) = 0.8000$ (80%) $\Rightarrow z_2 = 0.84$

$\frac{x_1 - 500}{100} = -0.84$	$\frac{x_2 - 500}{100} = 0.84$
$x_1 - 500 = -84$	$x_2 - 500 = 84$
$x_1 = 416$	$x_2 = 584$

For the middle 60% of the bulbs: 416 hrs. and 584 hrs.

Exercise

A machine that fills quart milk cartons is set up to average 32.2 oz. per carton, with a standard deviation of 1.2 oz. What is the probability that a filled carton will contain less than 32 oz. of milk?

Solution

$$\text{Given: } \begin{cases} \mu = 32.2 \\ \sigma = 1.2 \end{cases}$$

$$x = 32 \to z = \frac{32-32.2}{1.2} = -0.17$$

$$P(x < 32) \underline{= 0.4325}|$$

Exercise

A machine produces bolts with an average diameter of 0.25 in. and a standard deviation of 0.02 in. What is the probability that a bolt will be produced with a diameter greater than 0.3 in.?

Solution

$$\text{Given: } \begin{cases} \mu = 0.25 \\ \sigma = 0.02 \end{cases}$$

$$x = 0.3 \to z = \frac{0.3-0.25}{0.02} = 2.5$$

$$\begin{aligned} P(x > 0.3) &= 1 - P(x < 0.3) \\ &= 1 - 0.9938 \\ &\underline{= 0.0062}| \end{aligned}$$