

# Lecture One – Limits and Derivatives

## Section 1.1 – Idea of Limits

### Position Function

An object that is falling or vertically projected into the air has its height above the ground,  $s(t)$ , in feet, given by

$$s(t) = -16t^2 + v_0 t + s_0$$

$v_0$  is the original velocity (initial velocity) of the object, in *feet per second*

$t$  is the time that the object is in motion, in *second*

$s_0$  is the original height (initial height) of the object, in *feet*

The average rate is given by:  $\frac{\Delta s}{\Delta t}$

### Example

A rock breaks loose from the top of a tall cliff. What is its average speed

- a) During the first 2 sec of fall?
- b) During the 1-sec interval between second 1 and second 2?

### Solution

Since the rock falls free (*down*) without any initial velocity or height.  $\Rightarrow y(t) = 16t^2$

$$\begin{aligned} \text{a) For the first 2 sec: Average speed} &= \frac{\Delta y}{\Delta t} \\ &= \frac{y(2) - y(0)}{2 - 0} \\ &= \frac{16(2)^2 - 16(0)^2}{2} \\ &= \frac{64}{2} \\ &= 32 \text{ ft / sec} \end{aligned}$$

$$\begin{aligned} \text{b) From 1 sec to 2 sec: Average speed} &= \frac{y(2) - y(1)}{2 - 1} \\ &= \frac{16(2)^2 - 16(1)^2}{1} \\ &= 48 \text{ ft / sec} \end{aligned}$$

### ***Example***

Find the speed of a falling rock  $\left(y(t) = 16t^2\right)$  over a time interval  $\left[t_0, t_0 + h\right]$ . Then find the average speed at 1 sec and 2 sec.

### **Solution**

$$\begin{aligned}\frac{\Delta y}{\Delta t} &= \frac{16(t_0 + h)^2 - 16(t_0)^2}{(t_0 + h) - t_0} \\&= \frac{16(t_0^2 + 2ht_0 + h^2) - 16t_0^2}{t_0 + h - t_0} \\&= \frac{16t_0^2 + 32ht_0 + 16h^2 - 16t_0^2}{h} \\&= 32\frac{ht_0}{h} + 16\frac{h^2}{h} \\&= \underline{32t_0 + 16h} \quad | \end{aligned}$$

If  $t_0 = 1$

$$\begin{aligned}\frac{\Delta y}{\Delta t} &= 32(1) + 16h \\&= \underline{32 + 16h} \quad | \end{aligned}$$

The average speed has the limiting value 32 ft/sec as  $h$  approaches 0.

If  $t_0 = 2$

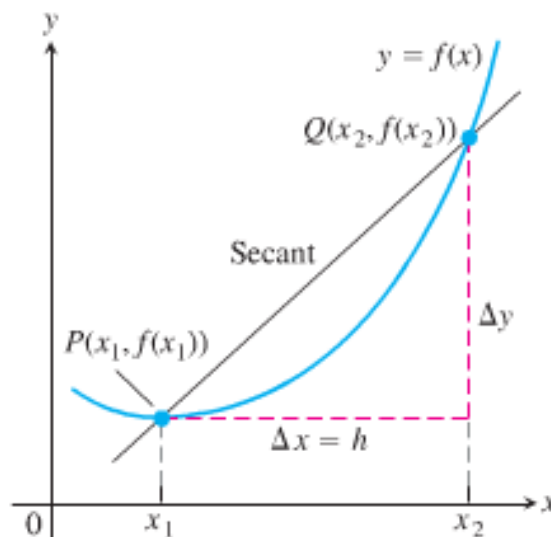
$$\begin{aligned}\frac{\Delta y}{\Delta t} &= 32(2) + 16h \\&= \underline{64 + 16h} \quad | \end{aligned}$$

The average speed has the limiting value 64 ft/sec as  $h$  approaches 0.

## Average Rates of Changes and Secant Lines

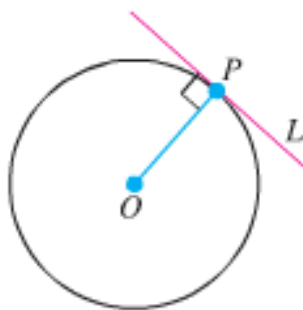
The average rate of change of  $y = f(x)$  with respect to  $x$  over the interval  $[x_1, x_2]$  is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ &= \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0\end{aligned}$$



## Defining the Slope of a Curve

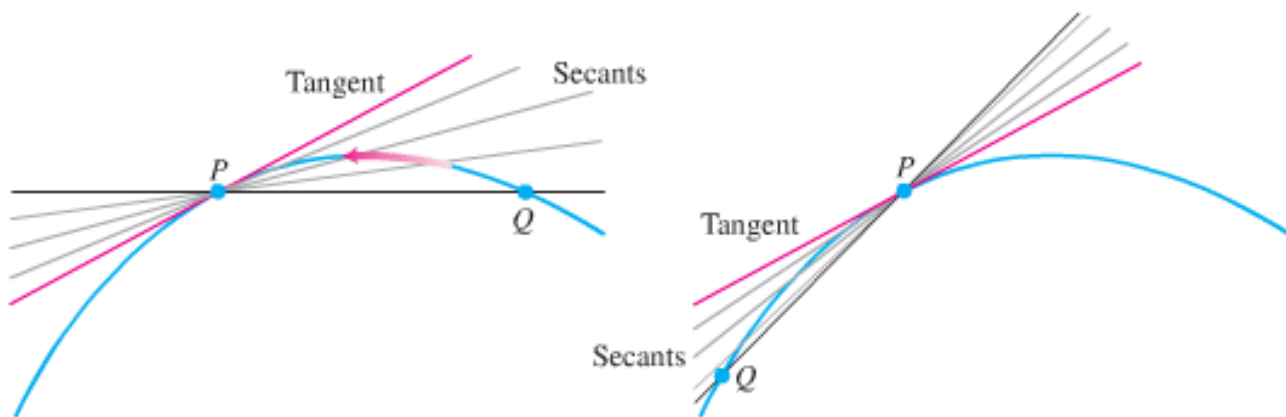
The slope of a line is the rate at which it rises or falls.



To define the tangency for general curves, we need an approach that makes the behavior of the secants through  $P$  and points  $Q$  as  $Q$  moves toward  $P$  along the curve:

1. Find the slope of the secant  $PQ$ .
2. Investigate the limiting value of the slope as  $Q$  approaches  $P$  along the curve.
3. If the limit exists, take it to be the slope of the curve at  $P$  and define the tangent to the curve at  $P$  to be the line through  $P$  with this slope.

$$m_{\text{tan}} = \lim_{t \rightarrow a} \frac{f(t) - f(a)}{t - a}$$

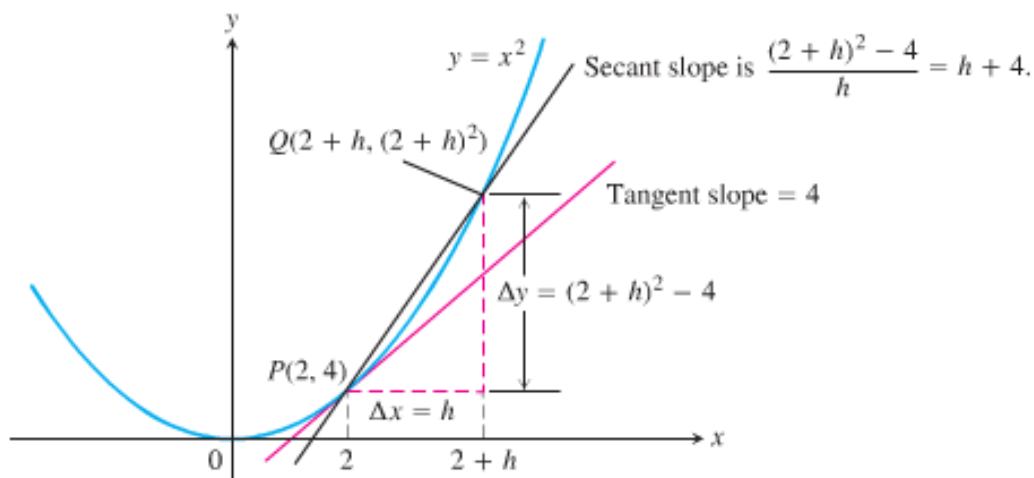


### Example

Find the slope of the parabola  $y = x^2$  at the point  $P(2, 4)$ . Write an equation for the tangent to the parabola at this point.

### Solution

$$\begin{aligned} \text{Secant slope} &= \frac{\Delta y}{\Delta x} = \frac{f(x_1 + h) - f(x_1)}{h} \\ &= \frac{f(2 + h) - f(2)}{h} \\ &= \frac{(2 + h)^2 - 2^2}{h} \\ &= \frac{4 + 4h + h^2 - 4}{h} \\ &= \frac{4h + h^2}{h} \\ &= 4 + h \end{aligned}$$



As  $Q$  approaches  $P$ ,  $h$  approaches 0. Then the secant slope  $h + 4 \rightarrow 4 = \text{slope}$

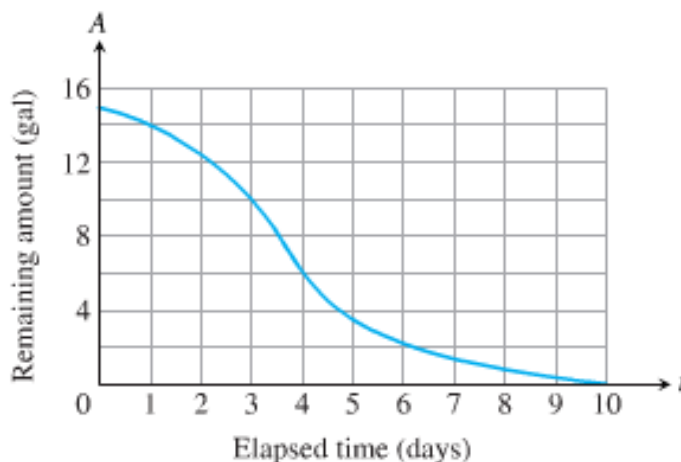
$$y = m(x - x_1) + y_1$$

$$y = 4(x - 2) + 4$$

$$\underline{y = 4x - 4}$$

## Exercises      Section 1.1 – Idea of Limits

- Find the average rate of change of the function  $f(x) = x^3 + 1$  over the interval  $[2, 3]$
- Find the average rate of change of the function  $f(x) = x^2$  over the interval  $[-1, 1]$
- Find the average rate of change of the function  $f(t) = 2 + \cos t$  over the interval  $[-\pi, \pi]$
- Find the slope of  $y = x^2 - 3$  at the point  $P(2, 1)$  and an equation of the tangent line at this  $P$ .
- Find the slope of  $y = x^2 - 2x - 3$  at the point  $P(2, -3)$  and an equation of the tangent line at this  $P$ .
- Find the slope of  $y = x^3$  at the point  $P(2, 8)$  and an equation of the tangent line at this  $P$ .
- Make a table of values for the function  $f(x) = \frac{x+2}{x-2}$  at the points  
 $x = 1.2, \quad x = \frac{11}{10}, \quad x = \frac{101}{100}, \quad x = \frac{1001}{1000}, \quad x = \frac{10001}{10000}, \quad \text{and } x = 1$ 
  - Find the average rate of change of  $f(x)$  over the intervals  $[1, x]$  for each  $x \neq 1$  in the table
  - Extending the table if necessary, try to determine the rate of change of  $f(x)$  at  $x = 1$ .
- The accompanying graph shows the total amount of gasoline  $A$  in the gas tank of an automobile after being driven for  $t$  days.



- Estimate the average rate of gasoline consumption over the time intervals  $[0, 3]$ ,  $[0, 5]$ , and  $[7, 10]$
- Estimate the instantaneous rate of gasoline consumption over the time  $t = 1$ ,  $t = 4$ , and  $t = 8$

## Section 1.2 – Definitions / Techniques of Limits

### Definition of the Limit of a Function

If  $f(x)$  becomes arbitrary close to a single number  $L$  as  $x$  approaches  $x_0$  from either side, then

$$\lim_{x \rightarrow x_0} f(x) = L$$

Which is read as “the limit of  $f(x)$  as  $x$  approaches  $x_0$  is  $L$ .”

Notation	Terminology
$x \rightarrow a^-$	$x$ approaches $a$ from the left (through values <i>less</i> than $a$ )
$x \rightarrow a^+$	$x$ approaches $a$ from the right (through values <i>greater</i> than $a$ )

### Example

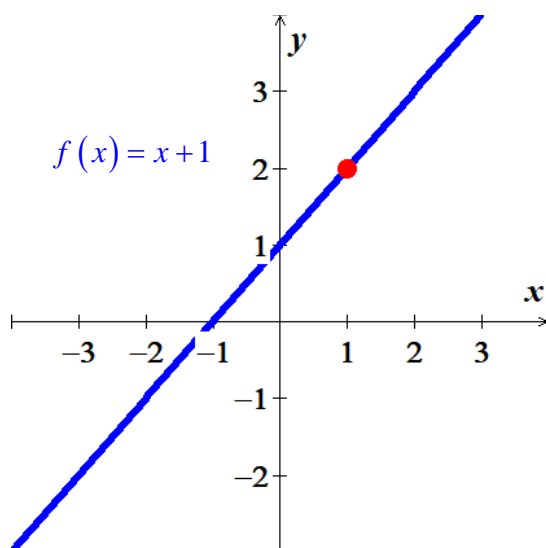
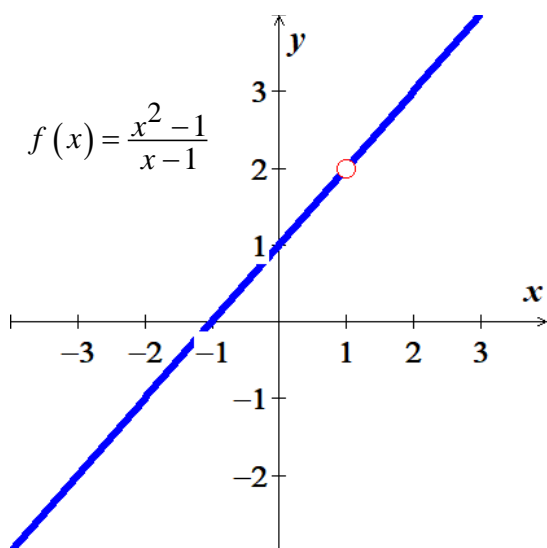
How does the function  $f(x) = \frac{x^2 - 1}{x - 1}$  behave near  $x = 1$ ?

### Solution

$$\begin{aligned} f(x) &= \frac{(x-1)(x+1)}{x-1} \\ &= x+1 \quad \text{for } x \neq 1 \end{aligned}$$

For  $x = 1$ :

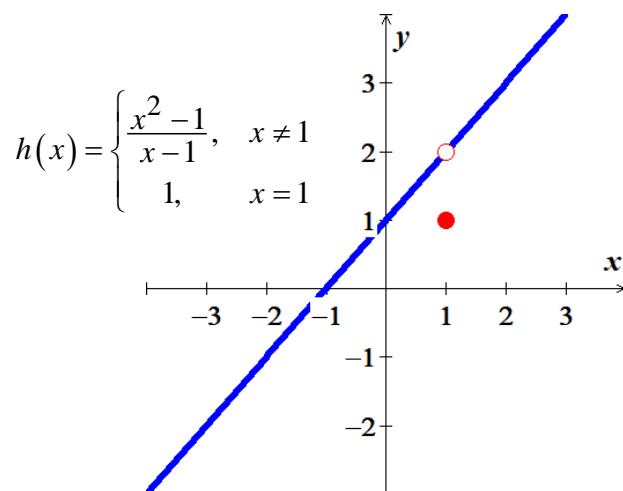
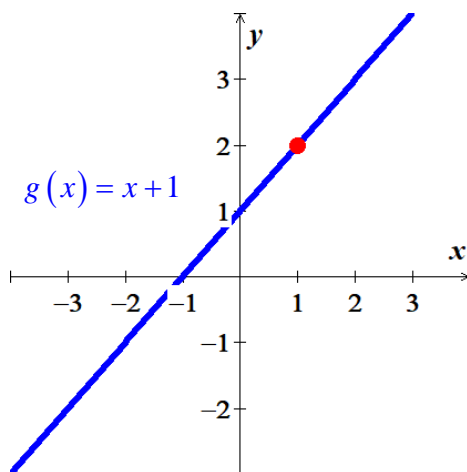
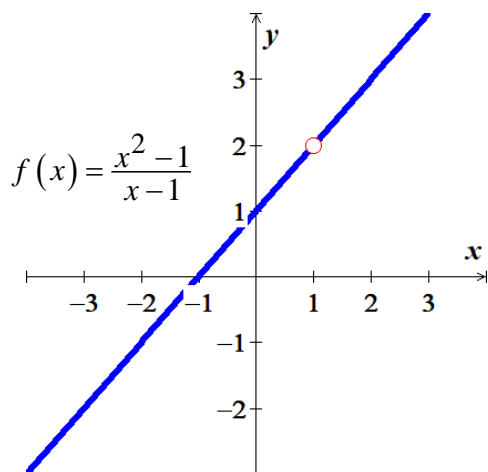
$$f(x=1) = 1+1 = 2$$



$x$	.9	.99	.999	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	2.001	2.01	2.1

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$$

$$= 2$$

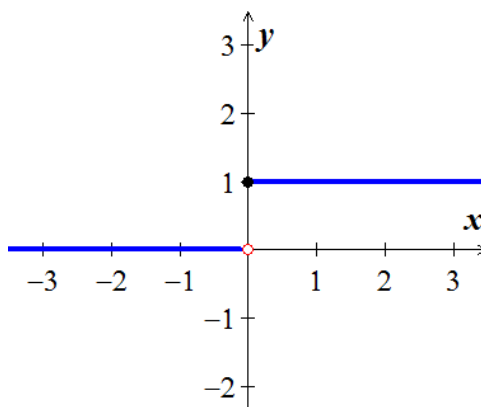


### Example

Discuss the behavior of the following function as  $x \rightarrow 0$ .

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

### Solution





The unit step function  $U(x)$  has no limit as  $x \rightarrow 0$ , it jumps, because the values jump at  $x = 0$ .

To the left of zero (*negative value*  $0^-$ )  $U(x) = 0$ . For the positive values of  $x$  close to zero ( $0^+$ )  $U(x) = 1$

## One-Sided Limits

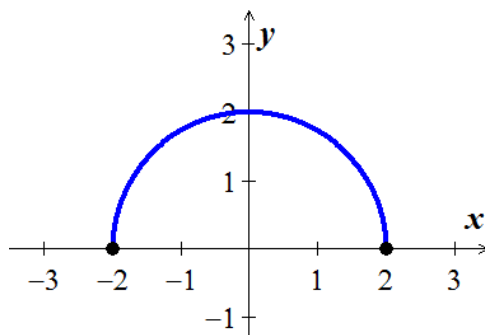
To have a limit  $L$  as  $x$  approaches  $c$ , a function  $f$  must be defined on **both sides** of  $c$  and its values  $f(x)$  must approach  $L$  as  $x$  approaches  $c$  from either side. Because of this, ordinary limits are called **two-sided**. If  $f$  fails to have two-sided limit at  $c$ , it may still have one-sided limit.

If the approach is from the *right*, the limit is a **right-hand limit**.  $\lim_{x \rightarrow c^+} f(x) = L$

If the approach is from the *left*, the limit is a **left-hand limit**.  $\lim_{x \rightarrow c^-} f(x) = M$

## Example

The domain of  $f(x) = \sqrt{4 - x^2}$  is  $[-2, 2]$ ; its graph is the semicircle.



We have:  $\lim_{x \rightarrow -2^+} \sqrt{4 - x^2} = 0$  and  $\lim_{x \rightarrow 2^-} \sqrt{4 - x^2} = 0$

The function doesn't have a left-hand limit at  $x = -2$  or a right-hand limit at  $x = 2$ .

It does not have ordinary two-sided limits at either  $-2$  or  $2$ .

## Theorem

A function  $f(x)$  has a limit as  $x$  approaches  $c$  if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \rightarrow c} f(x) = L \Leftrightarrow \lim_{x \rightarrow c^-} f(x) = L \text{ and } \lim_{x \rightarrow c^+} f(x) = L$$

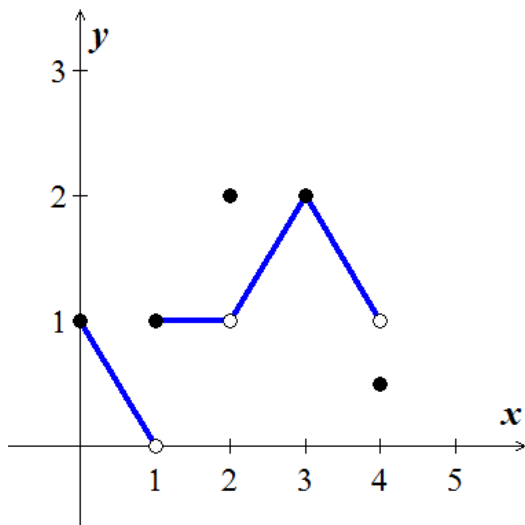
## Properties of Limits

**Constant function** ( $f(x) = k$ ):  $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} k = k$

**Identity function** ( $f(x) = x$ ):  $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} x = x_0$

## Example

Given the function graphed:



At  $x = 0$ :  $\lim_{x \rightarrow 0^+} f(x) = 1$

$\lim_{x \rightarrow 0^-} f(x)$  and  $\lim_{x \rightarrow 0} f(x)$  don't exist. The function is not defined to the left of  $x = 0$

At  $x = 1$ :  $\lim_{x \rightarrow 1^-} f(x) = 0$        $\lim_{x \rightarrow 1^+} f(x) = 1$

$\lim_{x \rightarrow 1} f(x)$  doesn't exist. The right-hand and left-hand limits are not equal.

At  $x = 2$ :  $\lim_{x \rightarrow 2^-} f(x) = 1$        $\lim_{x \rightarrow 2^+} f(x) = 1$   
 $\lim_{x \rightarrow 2} f(x) = 2$  even though  $f(2) = 2$

At  $x = 3$ :  $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3} f(x) = \underline{2}$

At  $x = 4$ :  $\lim_{x \rightarrow 4^-} f(x) = 1$  even though  $f(4) \neq 1$

$\lim_{x \rightarrow 4^+} f(x)$  and  $\lim_{x \rightarrow 4} f(x)$  do not exist.

The function is not defined to the right of  $x = 4$

## Definitions

We say that  $f(x)$  has right-hand limit  $L$  at  $x_0$  and  $\lim_{x \rightarrow x_0^+} f(x) = L$

If for every number  $\varepsilon > 0$  there exists a corresponding number  $\delta > 0$  such that for all  $x$

$$x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \varepsilon$$

We say that  $f(x)$  has left-hand limit  $L$  at  $x_0$  and  $\lim_{x \rightarrow x_0^-} f(x) = L$

If for every number  $\varepsilon > 0$  there exists a corresponding number  $\delta > 0$  such that for all  $x$

$$x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon$$

## Example

Prove that  $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$

### Solution

Let  $\varepsilon > 0$  be given.  $x_0 = 0$ ,  $L = 0$ , Find  $\delta > 0 \ni \forall x$

$$0 < x < \delta \Rightarrow |\sqrt{x} - 0| < \varepsilon$$

or  $0 < x < \delta \Rightarrow \sqrt{x} < \varepsilon$

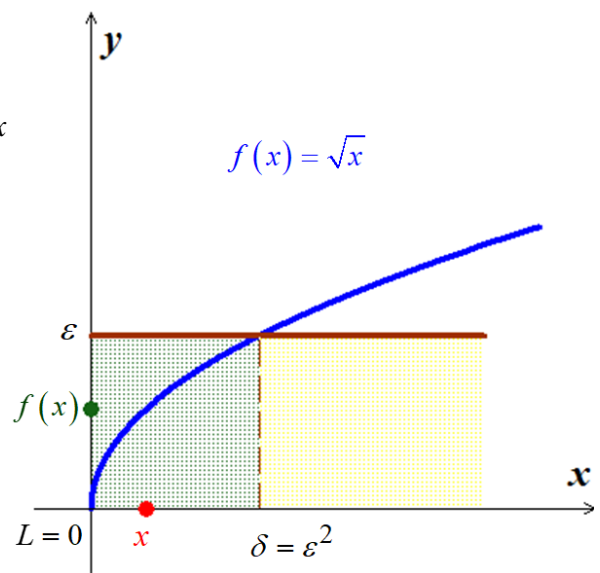
$$(\sqrt{x})^2 < \varepsilon^2$$

$$\Rightarrow x < \varepsilon^2 \text{ if } 0 < x < \delta$$

If we choose  $\delta = \varepsilon^2$ , we have

$$0 < x < \delta = \varepsilon^2 \Rightarrow \sqrt{x} < \varepsilon$$

According to the definition, this shows that  $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$



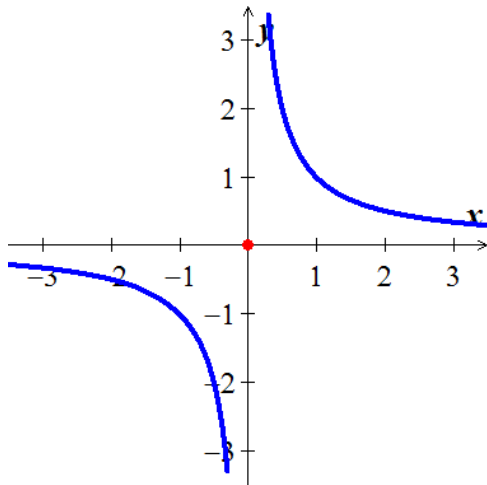
### Example

Discuss the behavior of the following function as  $x \rightarrow 0$ .

$$a) \quad g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad b) \quad f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0 \end{cases}$$

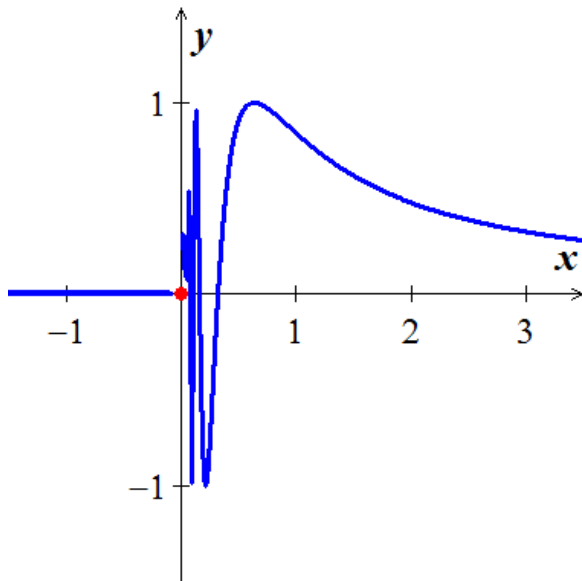
### Solution

a)



$g(x)$  has *no limit* as  $x \rightarrow 0$  because the values of  $g(x)$  grow arbitrary large (negative and positive) value as  $x \rightarrow 0$  and do not stay close.

b)



$f(x)$  has *no limit* as  $x \rightarrow 0$  because the function's values oscillate between  $-1$  and  $+1$  in every open interval containing  $0$ . The values do not stay close to any one number as  $x \rightarrow 0$ .

## Limit Laws

If  $\lim_{x \rightarrow c} f(x) = L$  and  $\lim_{x \rightarrow c} g(x) = M$

*Constant Multiple Rule:*  $\lim_{x \rightarrow c} [bf(x)] = b \lim_{x \rightarrow c} f(x) = \underline{bL}$

*Sum and Difference Rules:*  $\lim_{x \rightarrow c} [f(x) \pm g(x)] = \lim_{x \rightarrow c} f(x) \pm \lim_{x \rightarrow c} g(x) = \underline{L \pm M}$

*Product Rule:*  $\lim_{x \rightarrow c} [f(x) \cdot g(x)] = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x) = \underline{LM}$

*Quotient Rule:*  $\lim_{x \rightarrow c} \left( \frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} = \underline{\frac{L}{M}} \quad M \neq 0$

*Power Rule:*  $\lim_{x \rightarrow c} (f(x))^n = \left[ \lim_{x \rightarrow c} f(x) \right]^n = \underline{L^n}$

*Root Rule:*  $\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow c} f(x)} = \underline{\sqrt[n]{L}} \quad n > 0, \quad L > 0, \quad n \text{ is even}$

### Example

Find the following limits:

$$a) \lim_{x \rightarrow c} (x^3 + 4x^2 - 3)$$

$$b) \lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5}$$

$$c) \lim_{x \rightarrow -2} \sqrt{4x^2 - 3}$$

### Solution

$$\begin{aligned} a) \lim_{x \rightarrow c} (x^3 + 4x^2 - 3) &= \lim_{x \rightarrow c} x^3 + \lim_{x \rightarrow c} 4x^2 - \lim_{x \rightarrow c} (3) \\ &= \underline{c^3 + 4c^2 - 3} \end{aligned}$$

*Sum and Difference Rules*

$$\begin{aligned} b) \lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5} &= \frac{\lim_{x \rightarrow c} (x^4 + x^2 - 1)}{\lim_{x \rightarrow c} (x^2 + 5)} \\ &= \frac{\lim_{x \rightarrow c} x^4 + \lim_{x \rightarrow c} x^2 - \lim_{x \rightarrow c} 1}{\lim_{x \rightarrow c} x^2 + \lim_{x \rightarrow c} 5} \\ &= \underline{\frac{c^4 + c^2 - 1}{c^2 + 5}} \end{aligned}$$

*Quotient Rule*

*Sum and Difference Rules*

$$\begin{aligned} c) \lim_{x \rightarrow -2} \sqrt{4x^2 - 3} &= \sqrt{\lim_{x \rightarrow -2} (4x^2 - 3)} \\ &= \sqrt{\lim_{x \rightarrow -2} 4x^2 - \lim_{x \rightarrow -2} 3} \\ &= \sqrt{4(-2)^2 - 3} \\ &= \sqrt{16 - 3} \\ &= \underline{\sqrt{13}} \end{aligned}$$

*Root Rule*

*Difference Rule*

### ***Theorem – Limits of Polynomials***

If  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ , then  $\lim_{x \rightarrow c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0$

### ***Theorem – Limits of Rational Functions***

If  $P(x)$  and  $Q(x)$  are polynomials and  $Q(c) \neq 0$ , then  $\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$

### ***Example***

Find the limit:  $\lim_{x \rightarrow -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5}$

#### **Solution**

$$\begin{aligned} \lim_{x \rightarrow -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} &= \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} \\ &= \frac{0}{6} \\ &= 0 \end{aligned}$$

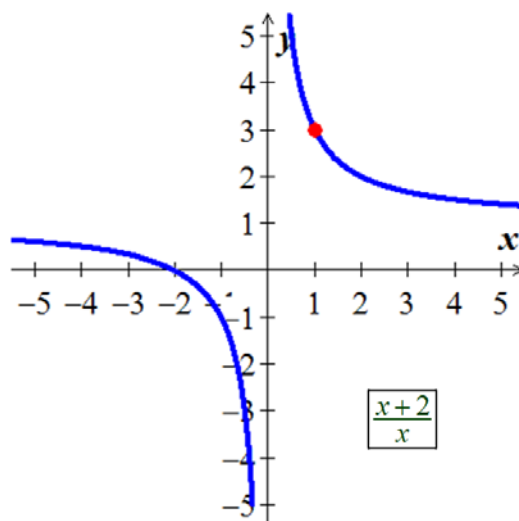
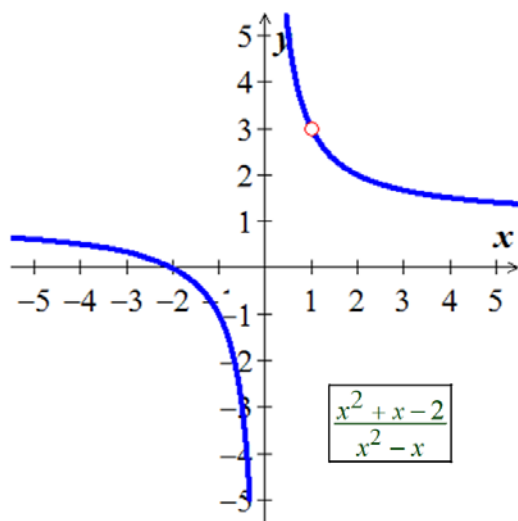
### ***Eliminating Zero Denominators Algebraically***

### ***Example***

Evaluate:  $\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}$

#### **Solution**

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} &= \frac{1^2 + 1 - 2}{1^2 - 1} = \frac{0}{0} \\ \lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} &= \lim_{x \rightarrow 1} \frac{(x-1)(x+2)}{x(x-1)} \\ &= \lim_{x \rightarrow 1} \frac{(x+2)}{x} \\ &= \frac{1+2}{1} \\ &= 3 \end{aligned}$$



### Example

Evaluate:  $\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$

### Solution

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} = \frac{\sqrt{0 + 100} - 10}{0} = \frac{0}{0}$$

$$\frac{\sqrt{x^2 + 100} - 10}{x^2} = \frac{\sqrt{x^2 + 100} - 10}{x^2} \cdot \frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10}$$

$$= \frac{x^2 + 100 - 100}{x^2(\sqrt{x^2 + 100} + 10)}$$

$$= \frac{x^2}{x^2(\sqrt{x^2 + 100} + 10)}$$

$$= \frac{1}{\sqrt{x^2 + 100} + 10}$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x^2 + 100} + 10}$$

$$= \frac{1}{\sqrt{0 + 100} + 10}$$

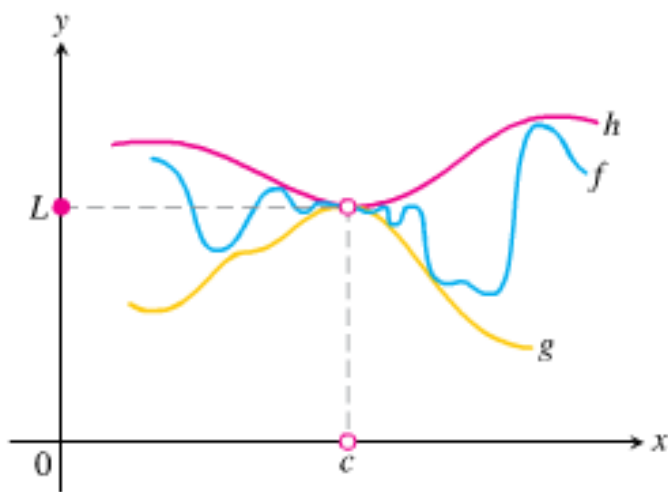
$$= \frac{1}{10 + 10}$$

$$= \frac{1}{20}$$

$$(a - b)(a + b) = a^2 - b^2; \quad (\sqrt{a})^2 = a$$



## The Sandwich (Squeeze) Theorem



Suppose that  $g(x) \leq f(x) \leq h(x)$  for all  $x$  in some open interval containing  $c$ , except possibly at  $x = c$  itself. Suppose also that

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L \quad \text{then} \quad \lim_{x \rightarrow c} f(x) = L$$

### Example

Given that  $1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$  for all  $x \neq 0$ , find the  $\lim_{x \rightarrow 0} u(x)$ , no matter how complicated  $u$  is.

### Solution

$$\lim_{x \rightarrow 0} \left( 1 - \frac{x^2}{4} \right) = 1 - \frac{0}{4}$$

$$\underline{= 1}$$

$$\lim_{x \rightarrow 0} \left( 1 + \frac{x^2}{2} \right) = 1$$

The Sandwich theorem implies that  $\lim_{x \rightarrow 0} u(x) = 1$

## Theorem

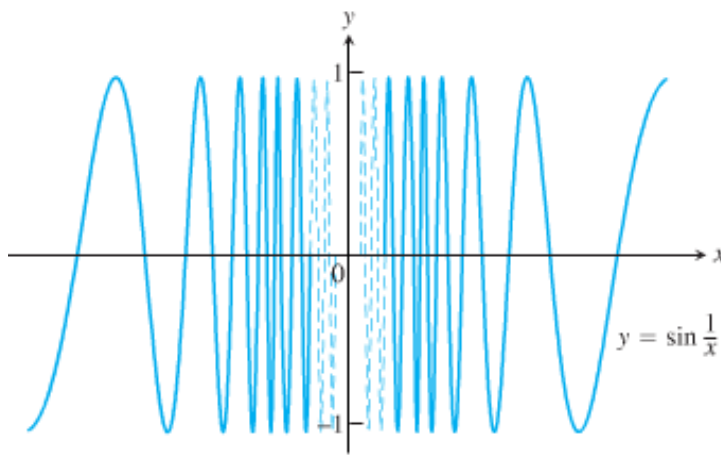
Suppose that  $f(x) \leq g(x)$  for all  $x$  in some open interval containing  $c$ , except possibly at  $x = c$  itself, and the limits of  $f$  and  $g$  both exist as  $x$  approaches  $c$ , then

$$\lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} g(x)$$

## Example

Show that  $y = \sin\left(\frac{1}{x}\right)$  has no limit as  $x$  approaches zero from either side.

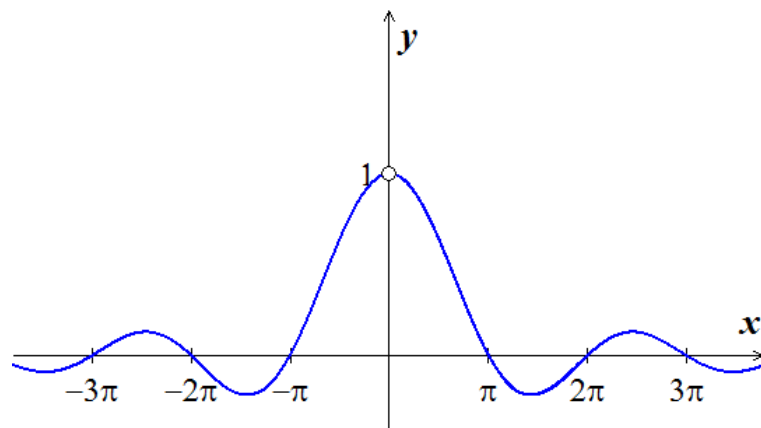
### Solution



As  $x$  approaches zero, its reciprocal,  $\frac{1}{x}$ , grows without bound and the values of  $\sin\left(\frac{1}{x}\right)$  cycle repeatedly from  $-1$  to  $1$ .

There is no single number  $L$  that the function's values stay increasingly close to as  $x$  approaches zero. The function has neither a right-hand limit nor a left-hand limit at  $x = 0$ .

## Limit Involving $\frac{\sin \theta}{\theta}$



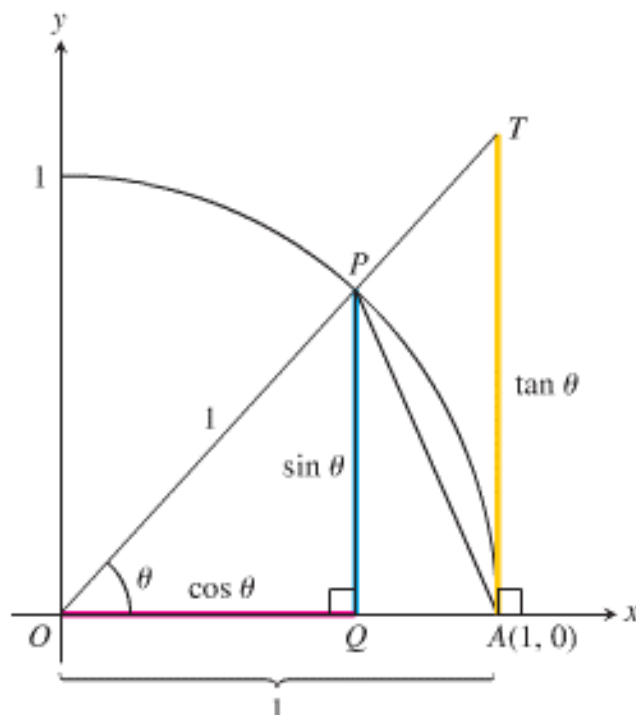
A central fact about  $\frac{\sin \theta}{\theta}$  is that in radian measure it limit as  $\theta \rightarrow 0$  is **1**.

### Theorem

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad (\theta \text{ in rad.})$$

### Proof

We need to show that the right-hand limit is 1,  $\theta < \frac{\pi}{2}$



Notice that:

$$\text{Area } \triangle OAP < \text{Area Sector } OAP < \text{Area } \triangle OAT$$

$$\text{Area } \triangle OAP = \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2}(1)(\sin \theta)$$

$$\text{Area Sector } \triangle OAP = \frac{1}{2}r^2 \times \theta = \frac{1}{2}(1)^2(\theta) = \frac{\theta}{2}$$

$$\text{Area } \triangle OAP = \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2}(1)(\tan \theta) = \frac{1}{2} \tan \theta$$

$$\Rightarrow \frac{1}{2} \sin \theta < \frac{1}{2} \theta < \frac{1}{2} \tan \theta$$

$$\frac{2}{\sin \theta} \frac{1}{2} \sin \theta < \frac{1}{2} \theta \frac{2}{\sin \theta} < \frac{1}{2} \frac{\sin \theta}{\cos \theta} \frac{2}{\sin \theta}$$

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta} \quad \text{Taking reciprocals reverses the inequalities}$$

$$1 > \frac{\sin \theta}{\theta} > \cos \theta$$

Since  $\lim_{\theta \rightarrow 0^+} \cos \theta = 1$ , then

$$\lim_{\theta \rightarrow 0^-} \frac{\sin \theta}{\theta} = 1 = \lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta}$$

$$\text{So } \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

### Example

Show that  $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$

### Solution

Using the half-angle formula:  $\cos x = 1 - 2 \sin^2 \left( \frac{x}{2} \right)$

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = \lim_{x \rightarrow 0} \frac{1 - 2 \sin^2 \left( \frac{x}{2} \right) - 1}{x}$$

$$= \lim_{x \rightarrow 0} \frac{-2 \sin^2 \left( \frac{x}{2} \right)}{x}$$

$$\text{Let } \theta = \frac{x}{2}$$

$$= - \lim_{\theta \rightarrow 0} \frac{2 \sin^2(\theta)}{2\theta}$$

$$= - \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \sin \theta$$

$$= -(1)(0)$$

$$\underline{= 0}$$

### Example

Show that  $\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} = \frac{2}{5}$

### Solution

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} = \lim_{x \rightarrow 0} \frac{\left(\frac{2}{5}\right) \sin 2x}{\left(\frac{2}{5}\right) 5x}$$

*Since we need  $2x$  in the denominator*

$$= \frac{2}{5} \lim_{x \rightarrow 0} \frac{\sin 2x}{2x}$$

$$= \frac{2}{5}(1)$$

$$\underline{= \frac{2}{5}}$$

### Example

Show that  $\lim_{x \rightarrow 0} \frac{\tan x \sec 2x}{3x} = \frac{1}{3}$

### Solution

$$\lim_{x \rightarrow 0} \frac{\tan x \sec 2x}{3x} = \frac{1}{3} \lim_{x \rightarrow 0} \frac{1}{x} \cdot \frac{\sin x}{\cos x} \cdot \frac{1}{\cos 2x}$$

$$= \frac{1}{3} \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} \cdot \frac{1}{\cos 2x}$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{1}{\cos x} = 1, \quad \lim_{x \rightarrow 0} \frac{1}{\cos 2x} = 1$$

$$= \frac{1}{3}(1)(1)(1)$$

$$\underline{= \frac{1}{3}}$$

# Exercises

## Section 1.2 – Definitions / Techniques of Limits

(1 – 121) Find the limit:

1.  $\lim_{x \rightarrow 3} (-1)$
2.  $\lim_{x \rightarrow -1} 3$
3.  $\lim_{x \rightarrow 1000} 18\pi^2$
4.  $\lim_{x \rightarrow 1} \sqrt{5x+6}$
5.  $\lim_{x \rightarrow 9} \sqrt{x}$
6.  $\lim_{x \rightarrow -3} (x^2 + 3x)$
7.  $\lim_{x \rightarrow -4} |x-4|$
8.  $\lim_{x \rightarrow 4} (x+2)$
9.  $\lim_{x \rightarrow 4} (x-4)$
10.  $\lim_{x \rightarrow 2} (5x-6)^{3/2}$
11.  $\lim_{x \rightarrow 9} \frac{x-9}{\sqrt{x}-3}$
12.  $\lim_{x \rightarrow 1} (2x+4)$
13.  $\lim_{x \rightarrow 1} \frac{x^2-4}{x-2}$
14.  $\lim_{x \rightarrow 2} \frac{x^2+4}{x-2}$
15.  $\lim_{x \rightarrow 0} \frac{|x|}{x}$
16.  $\lim_{x \rightarrow 3} \frac{x^2-x-1}{\sqrt{x+1}}$
17.  $\lim_{x \rightarrow 2} \frac{x^2+x-6}{x-2}$
18.  $\lim_{x \rightarrow 0} (3x-2)$
19.  $\lim_{x \rightarrow 1} (2x^2 - x + 4)$
20.  $\lim_{x \rightarrow -2} (x^3 - 2x^2 + 4x + 8)$
21.  $\lim_{x \rightarrow 2} \frac{x^2-4}{x-2}$
22.  $\lim_{x \rightarrow 2} \frac{x^3-8}{x-2}$
23.  $\lim_{x \rightarrow 3} \frac{x^2+x-12}{x-3}$
24.  $\lim_{x \rightarrow 0} \frac{\sqrt{x+4}-2}{x}$
25.  $\lim_{x \rightarrow -2} \frac{5}{x+2}$
26.  $\lim_{x \rightarrow 0} \frac{3}{\sqrt{3x+1}+1}$
27.  $\lim_{x \rightarrow 3} \frac{\sqrt{x+1}-1}{x}$
28.  $\lim_{x \rightarrow 1} \frac{x^2-1}{x-1}$
29.  $\lim_{x \rightarrow -2} \frac{|x+2|}{x+2}$
30.  $\lim_{x \rightarrow 0} (2z-8)^{1/3}$
31.  $\lim_{x \rightarrow 2} \frac{x^2-7x+10}{x-2}$
32.  $\lim_{x \rightarrow 0} \frac{5x^3+8x^2}{3x^4-16x^2}$
33.  $\lim_{x \rightarrow 1} \frac{1-1}{x-1}$
34.  $\lim_{u \rightarrow 1} \frac{u^4-1}{u^3-1}$
35.  $\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x+3}-2}$
36.  $\lim_{x \rightarrow -1} \frac{\sqrt{x^2+8}-3}{x+1}$
37.  $\lim_{x \rightarrow -3} \frac{2-\sqrt{x^2-5}}{x+3}$
38.  $\lim_{x \rightarrow 0} (2\sin x - 1)$
39.  $\lim_{x \rightarrow 0} \sin^2 x$
40.  $\lim_{x \rightarrow 0} \sec x$
41.  $\lim_{x \rightarrow 0} \frac{1+x+\sin x}{3\cos x}$
42.  $\lim_{x \rightarrow -\pi} \sqrt{x+4} \cos(x+\pi)$
43.  $\lim_{x \rightarrow -0.5^-} \sqrt{\frac{x+2}{x+1}}$
44.  $\lim_{x \rightarrow 1^+} \sqrt{\frac{x-1}{x+2}}$
45.  $\lim_{x \rightarrow -2^+} \left( \frac{x}{x+1} \right) \left( \frac{2x+5}{x^2+x} \right)$
46.  $\lim_{x \rightarrow 0^+} \frac{\sqrt{x^2+4x+5}-\sqrt{5}}{x}$
47.  $\lim_{x \rightarrow -2^+} (x+3) \frac{|x+2|}{x+2}$
48.  $\lim_{x \rightarrow 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|}$

49.  $\lim_{x \rightarrow 0^-} \frac{x}{\sin 3x}$
50.  $\lim_{\theta \rightarrow 0} \frac{\sin \sqrt{2} \cdot \theta}{\sqrt{2} \cdot \theta}$
51.  $\lim_{x \rightarrow 0} \frac{\sin 3x}{4x}$
52.  $\lim_{x \rightarrow 0} \frac{\tan 2x}{x}$
53.  $\lim_{x \rightarrow 0} 6x^2 (\cot x)(\csc 2x)$
54.  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta}$
55.  $\lim_{h \rightarrow 0} \frac{\sin(\sin h)}{\sin h}$
56.  $\lim_{\theta \rightarrow 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta}$
57.  $\lim_{\theta \rightarrow \pi/4} \frac{\sin^2 \theta - \cos^2 \theta}{\sin \theta - \cos \theta}$
58.  $\lim_{x \rightarrow \pi/2} \frac{\frac{1}{\sqrt{\sin x}} - 1}{x + \frac{\pi}{2}}$
59.  $\lim_{x \rightarrow 1} \frac{x^3 - 7x^2 + 12x}{4 - x}$
60.  $\lim_{x \rightarrow 4} \frac{x^3 - 7x^2 + 12x}{4 - x}$
61.  $\lim_{x \rightarrow 1} \frac{1 - x^2}{x^2 - 8x + 7}$
62.  $\lim_{x \rightarrow 3} \frac{\sqrt{3x+16} - 5}{x - 3}$
63.  $\lim_{x \rightarrow 3} \frac{1}{x-3} \left( \frac{1}{\sqrt{x+1}} - \frac{1}{2} \right)$
64.  $\lim_{x \rightarrow 1/3} \frac{x - \frac{1}{3}}{(3x-1)^2}$
65.  $\lim_{x \rightarrow 3} \frac{x^4 - 81}{x - 3}$
66.  $\lim_{x \rightarrow 1} \frac{x^5 - 1}{x - 1}$
67.  $\lim_{x \rightarrow 81} \frac{\sqrt[4]{x} - 3}{x - 81}$
68.  $\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{x - 1}$
69.  $\lim_{x \rightarrow 2} \frac{x^5 - 32}{x - 2}$
70.  $\lim_{x \rightarrow 1} \frac{x^6 - 1}{x - 1}$
71.  $\lim_{x \rightarrow -1} \frac{x^7 + 1}{x + 1}$
72.  $\lim_{x \rightarrow a} \frac{x^5 - a^5}{x - a}$
73.  $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} \quad n \in \mathbb{Z}^+$
74.  $\lim_{h \rightarrow 0} \frac{100}{(10h-1)^{11} + 2}$
75.  $\lim_{h \rightarrow 0} \frac{(5+h)^2 - 25}{h}$
76.  $\lim_{x \rightarrow 3} \frac{\frac{1}{x^2 + 2x} - \frac{1}{15}}{x - 3}$
77.  $\lim_{x \rightarrow 1} \frac{\sqrt{10x-9} - 1}{x - 1}$
78.  $\lim_{x \rightarrow 2} \left( \frac{1}{x-2} - \frac{2}{x^2 - 2x} \right)$
79.  $\lim_{x \rightarrow c} \frac{x^2 - 2cx + c^2}{x - c}$
80.  $\lim_{x \rightarrow -c} \frac{x^2 + 5cx + 4c^2}{x^2 + cx}$
81.  $\lim_{x \rightarrow 16} \frac{\sqrt[4]{x} - 2}{x - 16}$
82.  $\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x}-1}$
83.  $\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{4x+5}-3}$
84.  $\lim_{x \rightarrow 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}$
85.  $\lim_{x \rightarrow 0} \frac{x}{\sqrt{ax+1}-1} \quad (a \neq 0)$
86.  $\lim_{x \rightarrow \pi} \frac{\cos^2 x + 3\cos x + 2}{\cos x + 1}$
87.  $\lim_{x \rightarrow \frac{3\pi}{2}} \frac{\sin^2 x + 6\sin x + 5}{\sin^2 x - 1}$
88.  $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x - 1}{\sqrt{\sin x} - 1}$
89.  $\lim_{x \rightarrow 0} \frac{\frac{1}{2 + \sin x} - \frac{1}{2}}{\sin x}$
90.  $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{e^x - 1}$
91.  $\lim_{x \rightarrow \frac{\pi}{4}} \csc x$
92.  $\lim_{x \rightarrow 4} \frac{x-5}{(x^2 - 10x + 24)^2}$
93.  $\lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin^2 x}$
94.  $\lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{\sin x}$
95.  $\lim_{x \rightarrow 0} \frac{x^3 - 5x^2}{x^2}$
96.  $\lim_{x \rightarrow 5} \frac{4x^2 - 100}{x - 5}$

$$97. \lim_{x \rightarrow 3} \frac{\sqrt{9-6x+x^2}}{x-3}$$

$$98. \lim_{x \rightarrow 3} \frac{\sqrt{9+6x+x^2}}{x-3}$$

$$99. \lim_{x \rightarrow 3} \frac{\sqrt{x^2-9}}{x-3}$$

$$100. \lim_{x \rightarrow \frac{4\pi}{3}} \sin x$$

$$101. \lim_{x \rightarrow \frac{2\pi}{3}} \cos x$$

$$102. \lim_{x \rightarrow \frac{7\pi}{4}} \sin x$$

$$103. \lim_{x \rightarrow 1} \frac{\sin \sqrt{1-x}}{\sqrt{1-x^2}}$$

$$104. \lim_{x \rightarrow 2} \frac{\sin \sqrt{2-x}}{\sqrt{4-x^2}}$$

$$105. \lim_{x \rightarrow 0} \frac{\sin(\sqrt{5}x)}{\sin(\sqrt{3}x)}$$

$$106. \lim_{x \rightarrow 0} \frac{\sin(\sqrt{15}x)}{\sin(\sqrt{3}x)}$$

$$107. \lim_{x \rightarrow 0^+} \frac{x-\sqrt{x}}{\sqrt{\sin x}}$$

$$108. \lim_{x \rightarrow 1} \frac{x-\sqrt{x}}{\sqrt{\sin x}}$$

$$109. \lim_{x \rightarrow \pi} \frac{x-\sqrt{x}}{\sqrt{\sin x}}$$

$$110. \lim_{x \rightarrow 0} e^{x^3}$$

$$111. \lim_{x \rightarrow 1} e^{x^2}$$

$$112. \lim_{x \rightarrow 1} e^{x^3-1}$$

$$113. \lim_{x \rightarrow -1} e^{x^3-1}$$

$$114. \lim_{x \rightarrow 2} (e^{x^2} - \ln x)$$

$$115. \lim_{x \rightarrow 1} (e^{x^2} - \ln x)$$

$$116. \lim_{x \rightarrow e} \ln x$$

$$117. \lim_{x \rightarrow e} \ln x^2$$

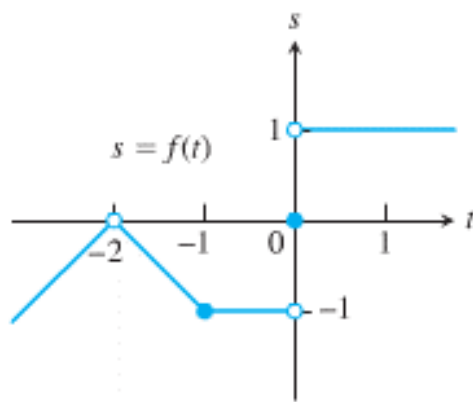
$$118. \lim_{x \rightarrow 0^+} \ln x$$

$$119. \lim_{x \rightarrow 1} \frac{1}{\ln x}$$

$$120. \lim_{x \rightarrow e} \ln e^{2x}$$

$$121. \lim_{x \rightarrow 1} \ln e^{x^2}$$

122. For the function  $f(t)$  graphed, find the following limits or explain why they do not exist.



$$a) \lim_{t \rightarrow -2} f(t) \quad b) \lim_{t \rightarrow -1} f(t) \quad c) \lim_{t \rightarrow 0} f(t) \quad d) \lim_{t \rightarrow -0.5} f(t)$$

123. Suppose  $\lim_{x \rightarrow c} f(x) = 5$  and  $\lim_{x \rightarrow c} g(x) = -2$ . Find

$$a) \lim_{x \rightarrow c} f(x)g(x)$$

$$c) \lim_{x \rightarrow c} (f(x) + 3g(x))$$

$$b) \lim_{x \rightarrow c} 2f(x)g(x)$$

$$d) \lim_{x \rightarrow c} \frac{f(x)}{f(x) - g(x)}$$



124. Explain why the limits do not exist for  $\lim_{x \rightarrow 0} \frac{x}{|x|}$

(125 – 126) Evaluate the limit using the form  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$  for

125.  $f(x) = x^2, \quad x = 1$

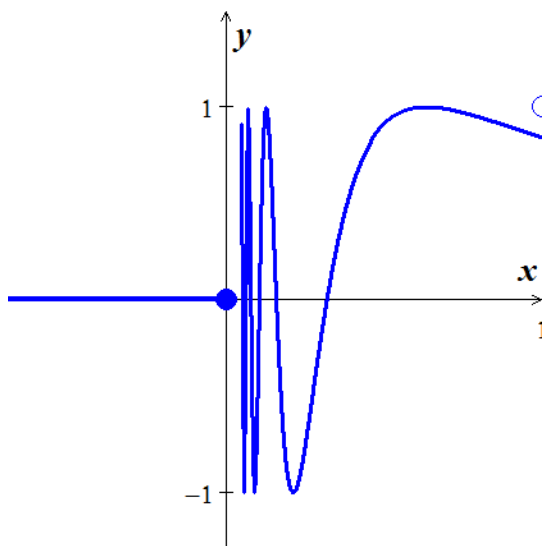
126.  $f(x) = \sqrt{3x+1}, \quad x = 0$

127. If  $\lim_{x \rightarrow 4} \frac{f(x) - 5}{x - 2} = 1$ , find  $\lim_{x \rightarrow 4} f(x)$

128. If  $\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 1$ , find  $\lim_{x \rightarrow 0} f(x)$  and  $\lim_{x \rightarrow 0} \frac{f(x)}{x}$

129. If  $x^4 \leq f(x) \leq x^2 \quad -1 \leq x \leq 1$  and  $x^2 \leq f(x) \leq x^4 \quad x < -1$  and  $x > 1$ . At what points  $c$  do you automatically know  $\lim_{x \rightarrow c} f(x)$ ? What can you say about the value of the limits at these points?

130. Let  $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0 \end{cases}$

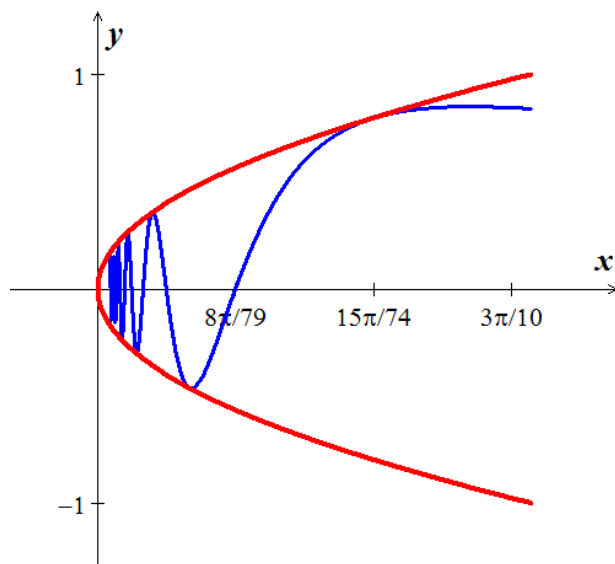


a) Does  $\lim_{x \rightarrow 0^+} f(x)$  exist? If so, what is it? If not, why not?

b) Does  $\lim_{x \rightarrow 0^-} f(x)$  exist? If so, what is it? If not, why not?

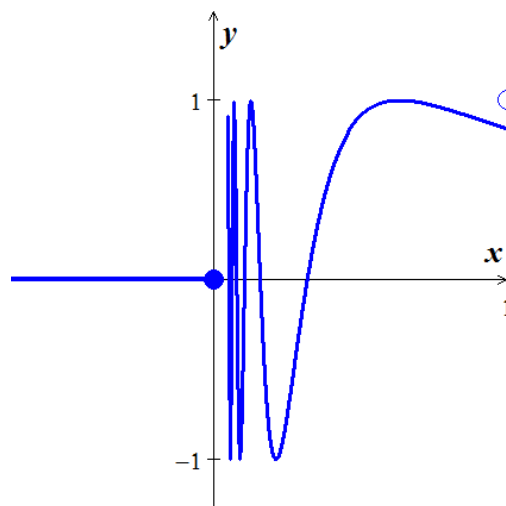
c) Does  $\lim_{x \rightarrow 0} f(x)$  exist? If so, what is it? If not, why not?

131. Let  $g(x) = \sqrt{x} \sin \frac{1}{x}$



- Does  $\lim_{x \rightarrow 0^+} g(x)$  exist? If so, what is it? If not, why not?
- Does  $\lim_{x \rightarrow 0^-} g(x)$  exist? If so, what is it? If not, why not?
- Does  $\lim_{x \rightarrow 0} g(x)$  exist? If so, what is it? If not, why not?

132. Let  $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0 \end{cases}$



- Does  $\lim_{x \rightarrow 0^+} f(x)$  exist? If so, what is it? If not, why not?
- Does  $\lim_{x \rightarrow 0^-} f(x)$  exist? If so, what is it? If not, why not?
- Does  $\lim_{x \rightarrow 0} f(x)$  exist? If so, what is it? If not, why not?

**133.** Which of the following statements about the function  $y = f(x)$  graphed here are true, and which are false?

a)  $\lim_{x \rightarrow -1^+} f(x) = 1$

g)  $\lim_{x \rightarrow 0} f(x) = 1$

b)  $\lim_{x \rightarrow 0^-} f(x) = 0$

h)  $\lim_{x \rightarrow 1} f(x) = 1$

c)  $\lim_{x \rightarrow 0^-} f(x) = 1$

i)  $\lim_{x \rightarrow 1} f(x) = 0$

d)  $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$

j)  $\lim_{x \rightarrow 2^-} f(x) = 2$

e)  $\lim_{x \rightarrow 0} f(x)$  exists

k)  $\lim_{x \rightarrow -1^-} f(x) = 0$  does not exist

f)  $\lim_{x \rightarrow 0} f(x) = 0$

l)  $\lim_{x \rightarrow 2^+} f(x) = 0$

## Section 1.3 – Infinite Limits

### Definitions

We say that  $f(x)$  has the **limit  $L$  as  $x$  approaches infinity** and write  $\lim_{x \rightarrow \infty} f(x) = L$

$$\text{If, } \forall \varepsilon > 0 \exists N \ni \forall x, \quad x > M \Rightarrow |f(x) - L| < \varepsilon$$

We say that  $f(x)$  has the **limit  $L$  as  $x$  approaches minus infinity** and write  $\lim_{x \rightarrow -\infty} f(x) = L$

$$\text{If, } \forall \varepsilon > 0 \exists N \ni \forall x, \quad x < M \Rightarrow |f(x) - L| < \varepsilon$$

**Basic Facts:**  $\lim_{x \rightarrow \pm\infty} k = k$  and  $\lim_{x \rightarrow \pm\infty} \frac{1}{x} = 0$

### Example

Find  $\lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2}$

### Solution

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} &= \lim_{x \rightarrow \infty} \frac{5 + \frac{8}{x} - \frac{3}{x^2}}{3 + \frac{2}{x^2}} \\ &= \frac{5 + 0 - 0}{3 + 0} \\ &= \underline{\underline{\frac{5}{3}}} \end{aligned}$$

*Divide by  $x^2$*

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x} = 0$$

### Example

Find  $\lim_{x \rightarrow \infty} \frac{11x + 2}{2x^3 - 1}$

### Solution

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{11x + 2}{2x^3 - 1} &= \lim_{x \rightarrow \infty} \frac{\frac{11}{x^2} + \frac{2}{x^3}}{2 - \frac{1}{x^3}} \\ &= \frac{0 + 0}{2 - 0} \\ &= \underline{\underline{0}} \end{aligned}$$

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x} = 0$$

## Vertical Asymptote (VA) - Think Domain

The line  $x = a$  is a **vertical asymptote** for the graph of a function  $f$  if

$$\lim_{x \rightarrow a^+} f(x) \rightarrow \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) \rightarrow \pm\infty$$

As  $x$  approaches  $a$  from either the left or the right

$$\lim_{x \rightarrow 0^+} \frac{1}{x} \rightarrow \infty \quad \text{or} \quad \lim_{x \rightarrow 0^-} \frac{1}{x} \rightarrow -\infty$$

### Example

Find  $\lim_{x \rightarrow 3^+} \frac{2-5x}{x-3}$  and  $\lim_{x \rightarrow 3^-} \frac{2-5x}{x-3}$

### Solution

$$\begin{aligned} \lim_{x \rightarrow 3^+} \frac{2-5x}{x-3} &= \frac{2-5(3)}{3^+ - 3} \rightarrow -13 \\ &\rightarrow \text{positive and approaches } 0 \\ &= -\infty \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 3^-} \frac{2-5x}{x-3} &= \frac{2-5(3)}{3^- - 3} \rightarrow -13 \\ &\rightarrow \text{negative and approaches } 0 \\ &= \infty \end{aligned}$$

### Example

Find  $\lim_{x \rightarrow -4^+} \frac{-x^3 + 5x^2 - 6x}{-x^3 - 4x^2}$

### Solution

$$\lim_{x \rightarrow -4^+} \frac{-x^3 + 5x^2 - 6x}{-x^3 - 4x^2} = \frac{168}{0}$$

$$\begin{aligned} \frac{-x^3 + 5x^2 - 6x}{-x^3 - 4x^2} &= \frac{(x-2)(x-3)}{x(x+4)} \rightarrow \text{positive} \\ &\rightarrow \text{negative and approaches } 0 \\ &= -\infty \end{aligned}$$

### Example

Let  $f(x) = \frac{x^2 - 4x + 3}{x^2 - 1}$ , determine the following limits and find the vertical asymptotes of  $f$ .

a)  $\lim_{x \rightarrow 1} f(x)$

b)  $\lim_{x \rightarrow -1^-} f(x)$

c)  $\lim_{x \rightarrow -1^+} f(x)$

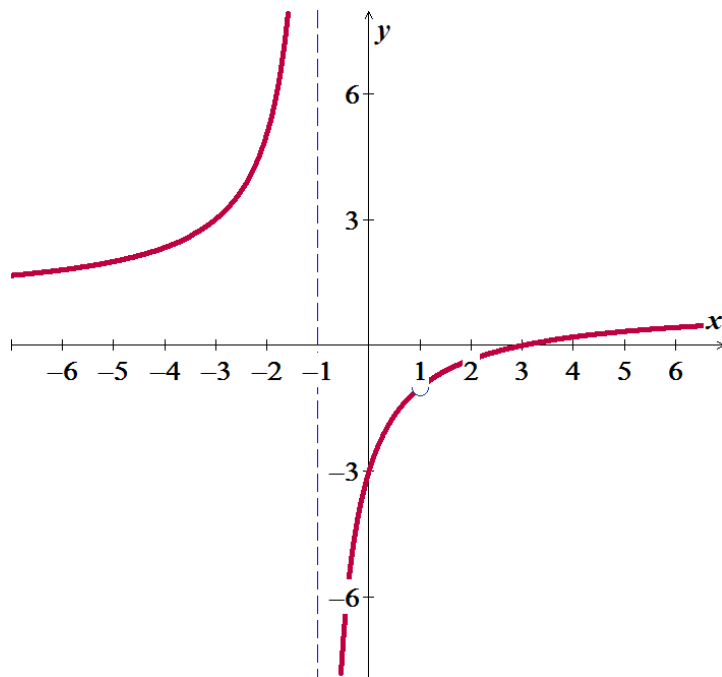
### Solution

$$\begin{aligned} \text{a) } \lim_{x \rightarrow 1} \frac{x^2 - 4x + 3}{x^2 - 1} &= \frac{0}{0} = \lim_{x \rightarrow 1} \frac{(x-1)(x-3)}{(x-1)(x+1)} \\ &= \lim_{x \rightarrow 1} \frac{x-3}{x+1} \\ &= -1 \end{aligned}$$

The vertical asymptote:  $x = -1$ , while the hole is  $(1, -1)$

$$\begin{aligned} \text{b) } \lim_{x \rightarrow -1^-} f(x) &= \lim_{x \rightarrow -1^-} \frac{x-3}{x+1} \quad \rightarrow \text{negative} \\ &\quad \rightarrow \text{negative and approaches 0} \\ &= \infty \end{aligned}$$

$$\begin{aligned} \text{c) } \lim_{x \rightarrow -1^+} f(x) &= \lim_{x \rightarrow -1^+} \frac{x-3}{x+1} \quad \rightarrow \text{negative} \\ &\quad \rightarrow \text{positive and approaches 0} \\ &= -\infty \end{aligned}$$



***Example***

Find  $\lim_{\theta \rightarrow 0^+} \cot \theta$  and  $\lim_{\theta \rightarrow 0^-} \cot \theta$

**Solution**

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$\cot 0 = \frac{1}{0}$$

As  $\theta \rightarrow 0^+$   $\cos \theta > 0$ ;  $\sin \theta > 0$

$$\lim_{\theta \rightarrow 0^+} \cot \theta = \underline{\infty}$$

As  $\theta \rightarrow 0^-$   $\cos \theta > 0$ ;  $\sin \theta < 0$

$$\lim_{\theta \rightarrow 0^-} \cot \theta = \underline{-\infty}$$

## Exercises Section 1.3 – Infinite Limits

(1 – 50) Find the limit

1.  $\lim_{x \rightarrow 5} \frac{x-7}{x(x-5)^2}$

2.  $\lim_{x \rightarrow -5^+} \frac{x-5}{x+5}$

3.  $\lim_{x \rightarrow 3^-} \frac{x-4}{x^2-3x}$

4.  $\lim_{x \rightarrow 0^+} \frac{1}{3x}$

5.  $\lim_{x \rightarrow -5^-} \frac{3x}{2x+10}$

6.  $\lim_{x \rightarrow 0} \frac{1}{x^{2/3}}$

7.  $\lim_{x \rightarrow 0^-} \frac{1}{3x^{1/3}}$

8.  $\lim_{x \rightarrow \left(-\frac{\pi}{2}\right)^+} \sec x$

9.  $\lim_{\theta \rightarrow 0^-} (1 + \csc \theta)$

10.  $\lim_{\theta \rightarrow 0^+} \csc \theta$

11.  $\lim_{x \rightarrow 0^+} (-10 \cot x)$

12.  $\lim_{\theta \rightarrow \frac{\pi}{2}^+} \frac{1}{3} \tan \theta$

13.  $\lim_{x \rightarrow 2^+} \frac{1}{x-2}$

14.  $\lim_{x \rightarrow 2^-} \frac{1}{x-2}$

15.  $\lim_{x \rightarrow 2} \frac{1}{x-2}$

16.  $\lim_{x \rightarrow 3^+} \frac{2}{(x-3)^3}$

17.  $\lim_{x \rightarrow 3^-} \frac{2}{(x-3)^3}$

18.  $\lim_{x \rightarrow 3} \frac{2}{(x-3)^3}$

19.  $\lim_{x \rightarrow 4^+} \frac{x-5}{(x-4)^2}$

20.  $\lim_{x \rightarrow 4^-} \frac{x-5}{(x-4)^2}$

21.  $\lim_{x \rightarrow 4} \frac{x-5}{(x-4)^2}$

22.  $\lim_{x \rightarrow 1^+} \frac{x-2}{(x-1)^3}$

23.  $\lim_{x \rightarrow 1^-} \frac{x-2}{(x-1)^3}$

24.  $\lim_{x \rightarrow 1} \frac{x-2}{(x-1)^3}$

25.  $\lim_{x \rightarrow 3^+} \frac{(x-1)(x-2)}{x-3}$

26.  $\lim_{x \rightarrow 3^-} \frac{(x-1)(x-2)}{x-3}$

27.  $\lim_{x \rightarrow 3} \frac{(x-1)(x-2)}{x-3}$

28.  $\lim_{x \rightarrow 2^+} \frac{x-4}{x(x+2)}$

29.  $\lim_{x \rightarrow 2^-} \frac{x-4}{x(x+2)}$

30.  $\lim_{x \rightarrow 2} \frac{x-4}{x(x+2)}$

31.  $\lim_{x \rightarrow 2^+} \frac{x^2-4x+3}{(x-2)^2}$

32.  $\lim_{x \rightarrow 2^-} \frac{x^2-4x+3}{(x-2)^2}$

33.  $\lim_{x \rightarrow 2} \frac{x^2-4x+3}{(x-2)^2}$

34.  $\lim_{x \rightarrow -2^+} \frac{x^3-5x^2+6x}{x^4-4x^2}$

35.  $\lim_{x \rightarrow -2^-} \frac{x^3-5x^2+6x}{x^4-4x^2}$

36.  $\lim_{x \rightarrow -2} \frac{x^3-5x^2+6x}{x^4-4x^2}$

37.  $\lim_{u \rightarrow 0^+} \frac{u-1}{\sin u}$

38.  $\lim_{x \rightarrow 0^-} \frac{2}{\tan x}$

39.  $\lim_{x \rightarrow 1^+} \frac{x^2-5x+6}{x-1}$

40.  $\lim_{x \rightarrow 4} \frac{x-5}{(x^2-10x+24)^2}$

41.  $\lim_{x \rightarrow 2\pi^-} \csc x$

42.  $\lim_{x \rightarrow 0^+} e^{\sqrt{x}}$

43.  $\lim_{x \rightarrow \frac{\pi}{2}^-} \frac{1+\sin x}{\cos x}$

44.  $\lim_{x \rightarrow \frac{\pi}{2}^+} \frac{1+\sin x}{\cos x}$

45.  $\lim_{x \rightarrow 0^-} \frac{e^x}{1-e^x}$

46.  $\lim_{x \rightarrow 0^+} \frac{e^x}{1-e^x}$

47.  $\lim_{x \rightarrow 1^-} \frac{x}{\ln x}$

48.  $\lim_{x \rightarrow 0^+} \frac{x}{\ln x}$

49.  $\lim_{x \rightarrow 0^-} \frac{2e^x+5e^{3x}}{e^{2x}-e^{3x}}$

50.  $\lim_{x \rightarrow 0^+} \frac{2e^x+5e^{3x}}{e^{2x}-e^{3x}}$



**51.** Let  $f(x) = \frac{x^2 - 7x + 12}{x - a}$

a) For what values of  $a$ , if any, does  $\lim_{x \rightarrow a^+} f(x)$  equal a finite number?

b) For what values of  $a$ , if any, does  $\lim_{x \rightarrow a^+} f(x) = \infty$ ?

c) For what values of  $a$ , if any, does  $\lim_{x \rightarrow a^+} f(x) = -\infty$ ?

**52.** Analyze  $\lim_{x \rightarrow 1^+} \sqrt{\frac{x-1}{x-3}}$  and  $\lim_{x \rightarrow 1^-} \sqrt{\frac{x-1}{x-3}}$

## Section 1.4 – Limits at Infinity

Notation	Terminology
$f(x) \rightarrow \infty$	$f(x)$ increases without bound (can be made as large positive as desired)
$f(x) \rightarrow -\infty$	$f(x)$ decreases without bound (can be made as large negative as desired)

### Horizontal Asymptote (HA)

The line  $y = b$  is a *horizontal asymptote* for the graph of a function  $f$  if

$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b$$

$$\begin{aligned} \text{Let } f(x) &= \frac{p(x)}{q(x)} \\ &= \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0} \\ &= \frac{a_n x^n}{b_m x^m} \end{aligned}$$

1. If the degree of numerator is less than of denominator ( $n < m$ )  $\Rightarrow y = 0$

$$y = \frac{2x+1}{4x^2+5}$$

$$\text{HA: } \underline{y = 0}$$

2. If the degree of numerator is equal of denominator ( $n = m$ )  $\Rightarrow y = \frac{a_n}{b_m}$

$$y = \frac{2x^2+1}{4x^2+5}$$

$$\text{HA: } y = \frac{2}{4} = \underline{\frac{1}{2}}$$

3. If the degree of numerator is greater than of denominator ( $n > m$ )  $\Rightarrow$  No horizontal asymptote

$$y = \frac{2x^3+1}{4x^2+5}$$

$$\Rightarrow \text{No HA}$$

### Example

Find the horizontal asymptotes of the graph of  $f(x) = \frac{x^3 - 2}{|x|^3 + 1}$

### Solution

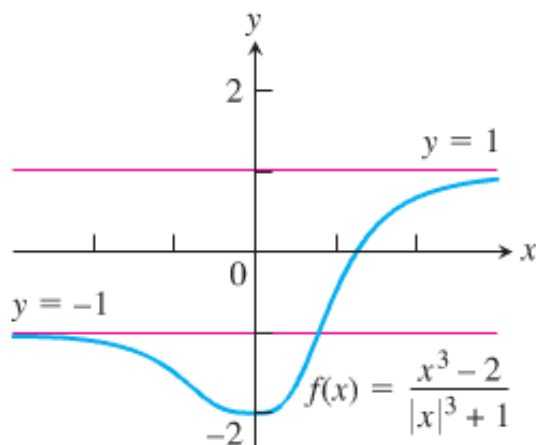
For  $x \geq 0$

$$\lim_{x \rightarrow \infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow \infty} \frac{x^3}{x^3} = 1$$

For  $x \leq 0$

$$\lim_{x \rightarrow -\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow -\infty} \frac{x^3}{(-x)^3} = -1$$

The **HA** are  $y = \pm 1$



### Example

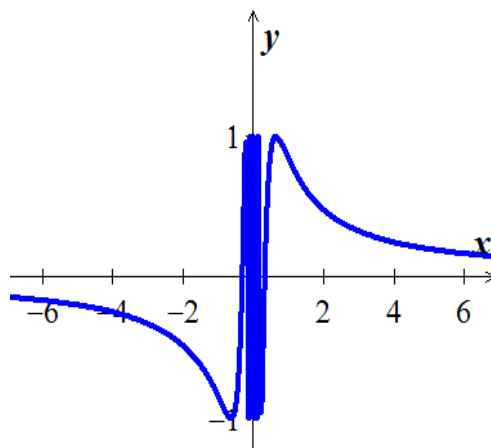
Find  $\lim_{x \rightarrow \infty} \sin\left(\frac{1}{x}\right)$

### Solution

Let  $t = \frac{1}{x}$

$\Rightarrow t \rightarrow 0$  as  $x \rightarrow \infty$

$$\lim_{x \rightarrow \infty} \sin\left(\frac{1}{x}\right) = \lim_{t \rightarrow 0} \sin t = 0$$



### Example

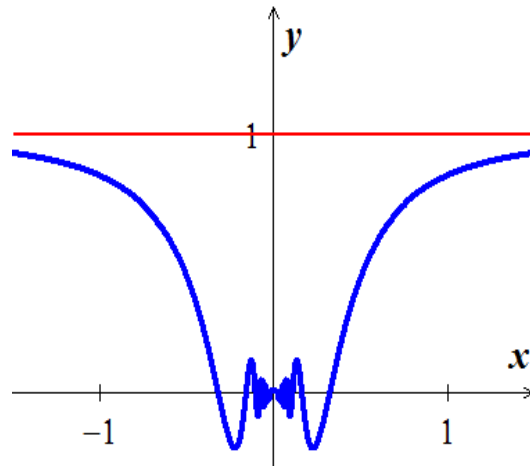
Find  $\lim_{x \rightarrow \pm\infty} x \sin\left(\frac{1}{x}\right)$

### Solution

$$\text{Let } t = \frac{1}{x} \Rightarrow x = \frac{1}{t}$$

$$\lim_{x \rightarrow \infty} x \sin\left(\frac{1}{x}\right) = \lim_{t \rightarrow 0^+} \frac{\sin t}{t} = 1$$

$$\lim_{x \rightarrow -\infty} x \sin\left(\frac{1}{x}\right) = \lim_{t \rightarrow 0^-} \frac{\sin t}{t} = 1$$



### Example

Find the horizontal asymptote of  $y = 2 + \frac{\sin x}{x}$

### Solution

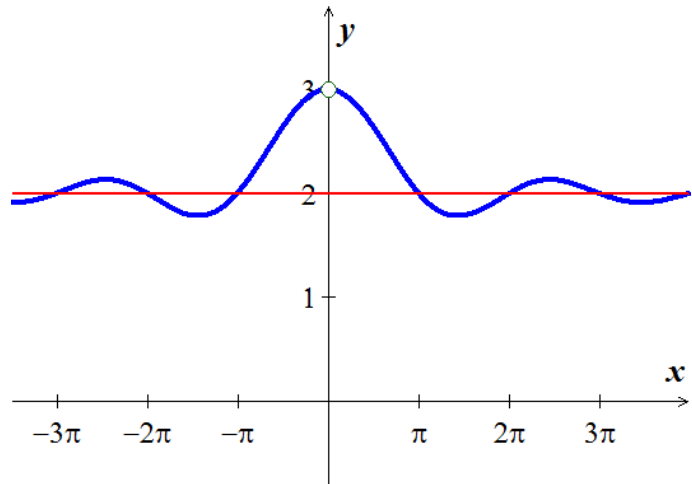
$$\text{Since } 0 \leq \left| \frac{\sin x}{x} \right| \leq \left| \frac{1}{x} \right|$$

$$\lim_{x \rightarrow \pm\infty} \left| \frac{1}{x} \right| = 0$$

$$\lim_{x \rightarrow \pm\infty} \frac{\sin x}{x} = 0$$

$$\lim_{x \rightarrow \pm\infty} \left( 2 + \frac{\sin x}{x} \right) = 2 + 0 = 2$$

The **HA** is  $y = 2$



### Example

Find  $\lim_{x \rightarrow \infty} \left( x - \sqrt{x^2 + 16} \right)$

### Solution

$$\lim_{x \rightarrow \infty} \left( x - \sqrt{x^2 + 16} \right) = \lim_{x \rightarrow \infty} \left( x - \sqrt{x^2 + 16} \right) \frac{x + \sqrt{x^2 + 16}}{x + \sqrt{x^2 + 16}}$$

$$(a - b)(a + b) = a^2 - b^2$$

$$= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + 16)}{x + \sqrt{x^2 + 16}}$$

$$= \lim_{x \rightarrow \infty} \frac{x^2 - x^2 - 16}{x + \sqrt{x^2 + 16}}$$

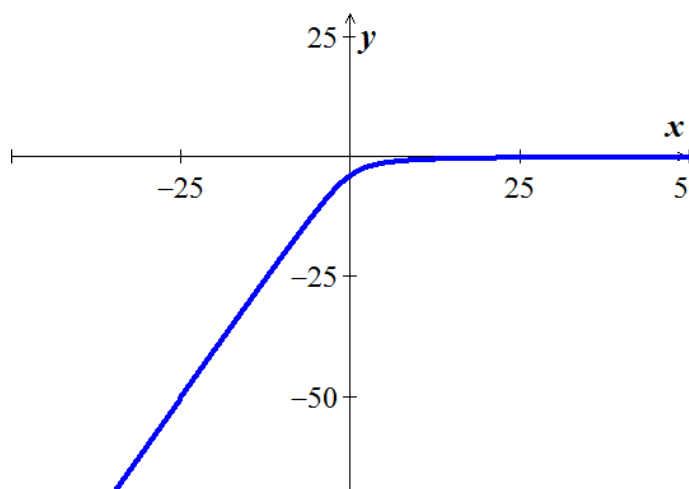
$$= \lim_{x \rightarrow \infty} \frac{-16}{x + \sqrt{x^2 + 16}}$$

$$= \lim_{x \rightarrow \infty} \frac{-\frac{16}{x}}{\frac{x}{x} + \sqrt{\frac{x^2}{x^2} + \frac{16}{x^2}}}$$

$$= \lim_{x \rightarrow \infty} \frac{-\frac{16}{x}}{1 + \sqrt{1 + \frac{16}{x^2}}}$$

$$= \frac{0}{1 + \sqrt{1 + 0}}$$

$$= 0$$



## Slant or Oblique Asymptotes

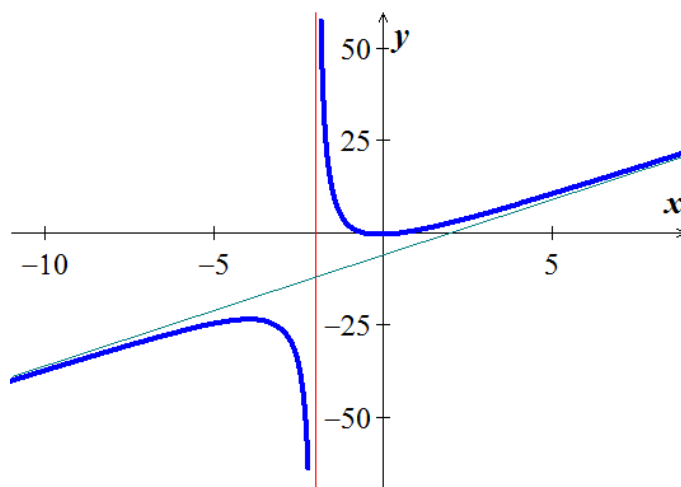
When the degree of the numerator is one greater than the degree of the denominator, the graph has a **slant** or **oblique** asymptote and it is a line  $y = ax + b$ ,  $a \neq 0$ . To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

$$y = \frac{3x^2 - 1}{x + 2}$$

$$\begin{array}{r} 3x - 6 \\ x + 2 \overline{) 3x^2 + 0x - 1} \\ \underline{3x^2 + 6x} \phantom{-1} \\ -6x - 1 \\ \underline{-6x - 12} \\ R = 11 \end{array}$$

$$y = \frac{3x^2 - 1}{x + 2} = (3x - 6) + \frac{11}{x + 2}$$

The **oblique asymptote** is the line  $y = 3x - 6$



### Example

Find the horizontal and vertical asymptotes of the curve  $y = \frac{x+3}{x+2}$

#### Solution

$$\text{HA: } y \rightarrow \frac{x}{x} = 1 \Rightarrow y = 1$$

$$\text{VA: } x + 2 = 0 \Rightarrow x = -2$$

### Example

Find the horizontal and vertical asymptotes of the curve  $f(x) = -\frac{8}{x^2 - 4}$

#### Solution

$$y \rightarrow \lim_{x \rightarrow \infty} -\frac{8}{x^2} = 0$$

$$\text{HA: } y = 0$$

$$\text{VA: } x^2 - 4 = 0 \Rightarrow x = \pm 2$$

$$\lim_{x \rightarrow 2^+} f(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow 2^-} f(x) = \infty$$

## Infinite Limits

The limit has a value of infinity or minus infinity, such a function  $f(x) = \frac{1}{x}$ . It is convenient to describe the behavior of  $f$  by saying that  $f(x)$  approaches  $\infty$  as  $x \rightarrow 0^+$ .

### Definition

We say  $\lim_{x \rightarrow 0^+} f(x) = \infty$

That  $\lim_{x \rightarrow 0^+} \frac{1}{x}$  doesn't exist because  $\frac{1}{x}$  becomes arbitrary large and positive as  $x \rightarrow 0^+$ .

We say  $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$

That  $\lim_{x \rightarrow 0^-} \frac{1}{x}$  doesn't exist because  $\frac{1}{x}$  becomes arbitrary large and negative as  $x \rightarrow 0^-$ .

### Example

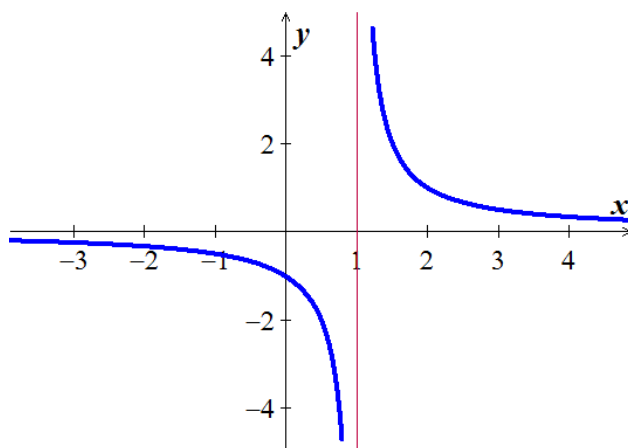
Find  $\lim_{x \rightarrow 1^+} \frac{1}{x-1}$  and  $\lim_{x \rightarrow 1^-} \frac{1}{x-1}$

### Solution

As  $x \rightarrow 1^+ \Rightarrow x-1 \rightarrow 0^+$

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1} = \infty$$

$$\lim_{x \rightarrow 1^-} \frac{1}{x-1} = -\infty$$



$$\begin{aligned}
 \text{➤ } \lim_{x \rightarrow 2} \frac{(x-2)^2}{x^2-4} &= \lim_{x \rightarrow 2} \frac{(x-2)^2}{(x-2)(x+2)} \\
 &= \lim_{x \rightarrow 2} \frac{(x-2)}{(x+2)} \\
 &= \frac{0}{4} \\
 &= \underline{0} \quad |
 \end{aligned}$$

$$\begin{aligned}
 \text{➤ } \lim_{x \rightarrow 2} \frac{x-2}{x^2-4} &= \lim_{x \rightarrow 2} \frac{x-2}{(x-2)(x+2)} \\
 &= \lim_{x \rightarrow 2} \frac{1}{x+2} \\
 &= \underline{\frac{1}{4}} \quad |
 \end{aligned}$$

$$\begin{aligned}
 \text{➤ } \lim_{x \rightarrow 2^+} \frac{x-3}{x^2-4} &= \lim_{x \rightarrow 2^+} \frac{x-3}{(x-2)(x+2)} \\
 &= \underline{-\infty} \quad |
 \end{aligned}$$

$$\begin{aligned}
 \text{➤ } \lim_{x \rightarrow 2^-} \frac{x-3}{x^2-4} &= \lim_{x \rightarrow 2^-} \frac{x-3}{(x-2)(x+2)} \\
 &= \underline{\infty} \quad |
 \end{aligned}$$

$$\begin{aligned}
 \text{➤ } \lim_{x \rightarrow 2} \frac{x-3}{x^2-4} &= \lim_{x \rightarrow 2} \frac{x-3}{(x-2)(x+2)} \\
 &= \underline{\text{doesn't exist}} \quad | \quad \underline{\text{DNE}} \quad |
 \end{aligned}$$



## Exercises      Section 1.4 – Limits at Infinity

(1 – 8)      Find the limit as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$  of

1.  $h(x) = \frac{-5 + \frac{7}{x}}{3 - \frac{1}{x^2}}$

2.  $f(x) = \frac{2x+3}{5x+7}$

3.  $f(x) = \frac{2x^3+7}{x^3-x^2+x+7}$

4.  $f(x) = \frac{x+1}{x^2+3}$

5.  $f(x) = \frac{7x^3}{x^3-3x^2+6x}$

6.  $f(x) = \frac{9x^4+x}{2x^4+5x^2-x+6}$

7.  $f(x) = \frac{-2x^3-2x+3}{3x^3+3x^2-5x}$

(8 – 60)      Evaluate the limits

8.  $\lim_{x \rightarrow \infty} x^{12}$

9.  $\lim_{x \rightarrow -\infty} 3x^9$

10.  $\lim_{x \rightarrow -\infty} x^{-8}$

11.  $\lim_{x \rightarrow -\infty} x^{-9}$

12.  $\lim_{x \rightarrow -\infty} 2x^{-6}$

13.  $\lim_{x \rightarrow \infty} (3x^{12} - 9x^7)$

14.  $\lim_{x \rightarrow -\infty} (3x^7 + x^2)$

15.  $\lim_{x \rightarrow -\infty} (-2x^{16} + 2)$

16.  $\lim_{x \rightarrow -\infty} (2x^{-6} + 4x^5)$

17.  $\lim_{x \rightarrow -\infty} \frac{\cos x}{3x}$

18.  $\lim_{x \rightarrow \infty} \frac{x + \sin x}{2x + 7 - 5 \sin x}$

19.  $\lim_{x \rightarrow \infty} \sqrt{\frac{8x^2-3}{2x^2+x}}$

20.  $\lim_{x \rightarrow -\infty} \left( \frac{x^2+x-1}{8x^2-3} \right)^{1/3}$

21.  $\lim_{x \rightarrow \infty} \frac{2\sqrt{x} + x^{-1}}{3x-7}$

22.  $\lim_{x \rightarrow \infty} \frac{x^{-1} + x^{-4}}{x^{-2} + x^{-3}}$

23.  $\lim_{x \rightarrow -\infty} \frac{4-3x^3}{\sqrt{x^6+9}}$

24.  $\lim_{x \rightarrow \infty} \left( \sqrt{x^2+3x} - \sqrt{x^2-2x} \right)$

25.  $\lim_{x \rightarrow -\infty} \left( \sqrt{x^2+3} + x \right)$

26.  $\lim_{x \rightarrow \infty} \frac{2x-3}{4x+10}$

27.  $\lim_{x \rightarrow \infty} \frac{x^4-1}{x^5+2}$

28.  $\lim_{x \rightarrow -\infty} (-3x^3+5)$

29.  $\lim_{x \rightarrow \infty} \left( e^{-2x} + \frac{2}{x} \right)$

30.  $\lim_{x \rightarrow \infty} \frac{1}{\ln x + 1}$

$$31. \lim_{x \rightarrow \infty} \left( 3 + \frac{10}{x^2} \right)$$

$$32. \lim_{x \rightarrow \infty} \left( 5 + \frac{1}{x} + \frac{10}{x^2} \right)$$

$$33. \lim_{x \rightarrow \infty} \frac{4x^2 + 2x + 3}{x^2}$$

$$34. \lim_{x \rightarrow \infty} \left( 5 + \frac{100}{x} + \frac{\sin^4 x^3}{x^2} \right)$$

$$35. \lim_{\theta \rightarrow \infty} \frac{\cos \theta}{\theta^2}$$

$$36. \lim_{\theta \rightarrow \infty} \frac{\cos \theta^5}{\sqrt{\theta}}$$

$$37. \lim_{x \rightarrow \infty} \frac{4x}{20x + 1}$$

$$38. \lim_{x \rightarrow -\infty} \frac{4x}{20x + 1}$$

$$39. \lim_{x \rightarrow \infty} \frac{3x^2 - 7}{x^2 + 5x}$$

$$40. \lim_{x \rightarrow -\infty} \frac{3x^2 - 7}{x^2 + 5x}$$

$$41. \lim_{x \rightarrow \infty} \frac{6x^2 - 9x + 8}{3x^2 + 2}$$

$$42. \lim_{x \rightarrow -\infty} \frac{6x^2 - 9x + 8}{3x^2 + 2}$$

$$43. \lim_{x \rightarrow \infty} \frac{4x^2 - 7}{8x^2 + 5x + 2}$$

$$44. \lim_{x \rightarrow -\infty} \frac{4x^2 - 7}{8x^2 + 5x + 2}$$

$$45. \lim_{x \rightarrow \infty} \frac{\sqrt{16x^4 + 64x^2 + x^2}}{2x^2 - 4}$$

$$46. \lim_{x \rightarrow -\infty} \frac{\sqrt{16x^4 + 64x^2 + x^2}}{2x^2 - 4}$$

$$47. \lim_{x \rightarrow \infty} \frac{3x^4 + 3x^3 - 36x^2}{x^4 - 25x^2 + 144}$$

$$48. \lim_{x \rightarrow -\infty} \frac{3x^4 + 3x^3 - 36x^2}{x^4 - 25x^2 + 144}$$

$$49. \lim_{x \rightarrow \infty} 16x^2 \left( 4x^2 - \sqrt{16x^4 + 1} \right)$$

$$50. \lim_{x \rightarrow -\infty} 16x^2 \left( 4x^2 - \sqrt{16x^4 + 1} \right)$$

$$51. \lim_{x \rightarrow \infty} \frac{x - 1}{x^{2/3} - 1}$$

$$52. \lim_{x \rightarrow -\infty} \frac{x - 1}{x^{2/3} - 1}$$

$$53. \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 2x + 6} - 3}{x - 1}$$

$$54. \lim_{x \rightarrow \infty} \frac{|1 - x^2|}{x(x + 1)}$$

$$55. \lim_{x \rightarrow \infty} \left( \sqrt{|x|} - \sqrt{|x - 1|} \right)$$

$$56. \lim_{x \rightarrow \infty} \frac{\tan^{-1} x}{x}$$

$$57. \lim_{x \rightarrow \infty} \frac{\cos x}{e^{3x}}$$

$$58. \lim_{x \rightarrow 0} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}}$$

$$59. \lim_{x \rightarrow \infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}}$$

$$60. \lim_{x \rightarrow -\infty} \frac{2e^x + 10e^{-x}}{e^x + e^{-x}}$$

(61 – 64) Graph the rational function and include the equations of the asymptotes

$$61. y = \frac{1}{2x + 4}$$

$$62. y = \frac{2x}{x + 1}$$

$$63. y = \frac{x^2}{x - 1}$$

$$64. y = \frac{x^3 + 1}{x^2}$$

65. Let  $f(x) = \frac{x^2 - 5x + 6}{x^2 - 2x}$

a) Analyze  $\lim_{x \rightarrow 0^-} f(x)$ ,  $\lim_{x \rightarrow 0^+} f(x)$ ,  $\lim_{x \rightarrow 2^-} f(x)$ , and  $\lim_{x \rightarrow 2^+} f(x)$

b) Does the graph of  $f$  have any vertical asymptotes? Explain?

(66 – 85) Find the vertical, horizontal, hole, and oblique asymptotes (if any) of

66.  $y = \frac{3x}{1-x}$

73.  $y = \frac{x^3 + 3x^2 - 2}{x^2 - 4}$

80.  $f(x) = \frac{1}{\tan^{-1} x}$

67.  $y = \frac{x^2}{x^2 + 9}$

74.  $y = \frac{x-3}{x^2 - 9}$

81.  $f(x) = \frac{2x^2 + 6}{2x^2 + 3x - 2}$

68.  $y = \frac{x-2}{x^2 - 4x + 3}$

75.  $y = \frac{6}{\sqrt{x^2 - 4x}}$

82.  $f(x) = \frac{3x^2 + 2x - 1}{4x + 1}$

69.  $y = \frac{5x-1}{1-3x}$

76.  $f(x) = \frac{4x^3 + 1}{1 - x^3}$

83.  $f(x) = \frac{9x^2 + 4}{(2x-1)^2}$

70.  $y = \frac{3}{x-5}$

77.  $f(x) = \frac{x+1}{\sqrt{9x^2 + x}}$

84.  $f(x) = \frac{1+x-2x^2-x^3}{x^2+1}$

71.  $y = \frac{x^3-1}{x^2+1}$

78.  $f(x) = 1 - e^{-2x}$

85.  $f(x) = \frac{x(x+2)^3}{3x^2 - 4x}$

72.  $y = \frac{3x^2 - 27}{(x+3)(2x+1)}$

79.  $f(x) = \frac{1}{\ln x^2}$

(85 – 142) Find the limits

86.  $\lim_{x \rightarrow 0} \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x}$

91.  $\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x}$

96.  $\lim_{x \rightarrow \pi^-} \csc x$

87.  $\lim_{x \rightarrow 2} \frac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x}$

92.  $\lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x}$

97.  $\lim_{x \rightarrow \pi} \sin\left(\frac{x}{2} + \sin x\right)$

88.  $\lim_{x \rightarrow a} \frac{x^2 - a^2}{x^4 - a^4}$

93.  $\lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{\sqrt{x} - 1}$

98.  $\lim_{x \rightarrow \pi} \cos^2(x - \tan x)$

89.  $\lim_{x \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$

94.  $\lim_{x \rightarrow 64} \frac{x^{2/3} - 16}{\sqrt{x} - 8}$

99.  $\lim_{x \rightarrow 0} \frac{8x}{3 \sin x - x}$

90.  $\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$

95.  $\lim_{x \rightarrow 0} \frac{\tan(2x)}{\tan(\pi x)}$

100.  $\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\sin x}$

101.  $\lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$

$$102. \lim_{x \rightarrow -\infty} \frac{x^2 - 4x + 8}{3x^3}$$

$$103. \lim_{x \rightarrow -\infty} \frac{2x^2 + 3}{5x^2 + 7}$$

$$104. \lim_{x \rightarrow \infty} \frac{x^4 + x^3}{12x^3 + 128}$$

$$105. \lim_{x \rightarrow -\infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}}$$

$$106. \lim_{x \rightarrow \infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}}$$

$$107. \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$$

$$108. \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{1}{x^4}}{\frac{1}{x^2} - \frac{1}{x^3}}$$

$$109. \lim_{x \rightarrow \infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$$

$$110. \lim_{x \rightarrow 2^+} \ln(x - 2)$$

$$111. \lim_{x \rightarrow 1} x^2 \ln(2 - \sqrt{x})$$

$$112. \lim_{\theta \rightarrow 0^+} \sqrt{\theta} e^{\cos \frac{\pi}{\theta}}$$

$$113. \lim_{x \rightarrow \infty} \frac{2x - 3}{5x + 6}$$

$$114. \lim_{x \rightarrow \infty} \frac{2x^2 - 3}{5x^2 + 6}$$

$$115. \lim_{x \rightarrow \infty} \frac{2x - 3}{5x^3 + 6}$$

$$116. \lim_{x \rightarrow \infty} \frac{1}{5x^2 - 3x + 6}$$

$$117. \lim_{\theta \rightarrow 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta}$$

$$118. \lim_{x \rightarrow 0^+} \frac{\sqrt{x^2 + 4x + 5} - \sqrt{5}}{x}$$

$$119. \lim_{x \rightarrow 2} \frac{x^4 - 16}{x - 2}$$

$$120. \lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2}$$

$$121. \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$$

$$122. \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1}}{x + 1}$$

$$123. \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x + 1}$$

$$124. \lim_{x \rightarrow \infty} \frac{x - 3}{\sqrt{4x^2 + 25}}$$

$$125. \lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$$

$$126. \lim_{x \rightarrow \infty} \frac{x^4 - x}{15x^3 + 4}$$

$$127. \lim_{x \rightarrow \infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x}$$

$$128. \lim_{x \rightarrow \infty} \frac{x^{2/3} - x^{-1}}{x^{2/3} + \cos^2 x}$$

$$129. \lim_{x \rightarrow \infty} \frac{\sin 2x}{x}$$

$$130. \lim_{x \rightarrow 0} \frac{\sin 5x}{3x}$$

$$131. \lim_{x \rightarrow -\infty} \frac{\cos x}{2x}$$

$$132. \lim_{x \rightarrow -\infty} \left( \frac{x^2 + x - 1}{8x^2 - 3} \right)^{1/3}$$

$$133. \lim_{x \rightarrow -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}$$

$$134. \lim_{x \rightarrow -\infty} \left( \frac{1 - x^3}{x^2 + 7x} \right)^5$$

$$135. \lim_{x \rightarrow \infty} \sqrt{\frac{x^2 - 5x}{x^3 + x - 2}}$$

$$136. \lim_{x \rightarrow \infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7}$$

$$137. \lim_{x \rightarrow -5^-} \frac{3x}{2x + 10}$$

$$138. \lim_{x \rightarrow -8^+} \frac{3x}{x + 8}$$

$$139. \lim_{x \rightarrow 0} \frac{-1}{x^2(x + 1)}$$

$$140. \lim_{x \rightarrow 7} \frac{4}{(x - 7)^2}$$

$$141. \lim_{x \rightarrow 0} \frac{1}{x^{2/3}}$$

$$142. \lim_{x \rightarrow -\infty} \left( x + \sqrt{x^2 - 4x + 2} \right)$$

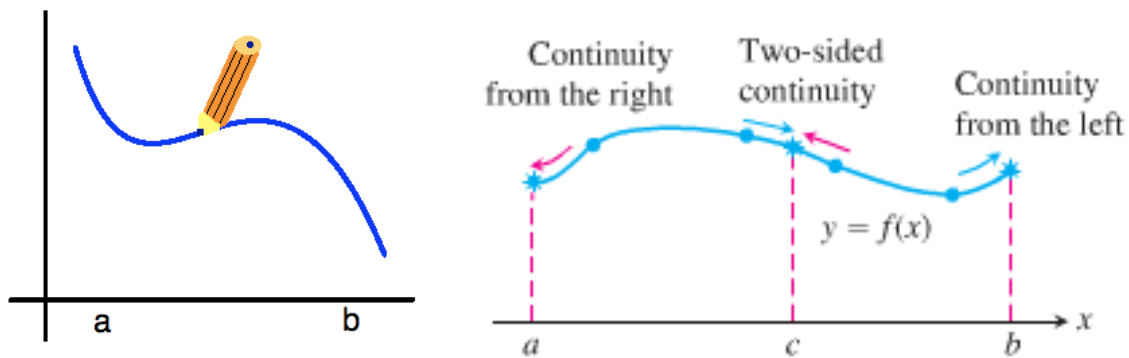
## Section 1.5 – Continuity

### Definition of Continuity

Let  $c$  be a number in the interval  $(a, b)$ , and let  $f$  be a function whose domain contains the interval  $(a, b)$ . The function  $f$  is continuous at the point  $c$  if the following conditions are true.

1.  $f(c)$  is defined
2.  $\lim_{x \rightarrow c} f(x)$  exists
3.  $\lim_{x \rightarrow c} f(x) = f(c)$

If  $f$  is continuous at every point in the interval  $(a, b)$ , then it is continuous on an open interval  $(a, b)$



### Definition

**Interior point:** A function  $y = f(x)$  is **continuous at an interior point  $c$**  of its domain if

$$\lim_{x \rightarrow c} f(x) = f(c)$$

**Endpoint:** A function  $y = f(x)$  is **continuous at a left point  $a$**  or is **continuous at a right point  $b$**  of its domain if

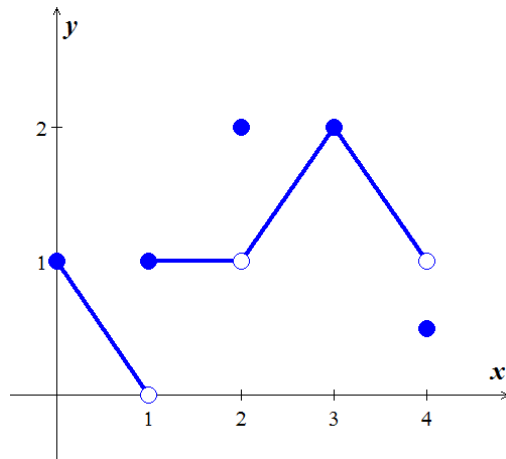
$$\lim_{x \rightarrow a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \rightarrow b^-} f(x) = f(b), \quad \text{respectively}$$



If a function  $f$  is not continuous at a point  $c$ , we say that  $f$  is **discontinuous** at  $c$ . (is a **point of discontinuity**)

### Example

Find the points at which the function  $f$  is continuous and the points at which  $f$  is not continuous



### Solution

The function  $f$  is continuous at every point in its domain  $[0, 4]$  except at  $x = 1$ ,  $x = 2$ , and  $x = 4$ . At these points, there are breaks in the graph.

$x = 0$	$\lim_{x \rightarrow 0^+} f(x) = f(0) = 1$	$f$ is continuous @ $x = 0$
$x = 1$	$\lim_{x \rightarrow 1} f(x)$ <i>doesn't exist</i>	$f$ is discontinuous @ $x = 1$
$x = 2$	$\lim_{x \rightarrow 2} f(x) = 1$ , but $1 \neq f(2)$	$f$ is discontinuous @ $x = 2$
$x = 3$	$\lim_{x \rightarrow 3} f(x) = f(3) = 2$	$f$ is continuous @ $x = 3$
$x = 4$	$\lim_{x \rightarrow 4^-} f(x) = 1$ , but $1 \neq f(4)$	$f$ is discontinuous @ $x = 4$
$c < 0, c > 4$	These points are not in the domain of $f$ .	$f$ is discontinuous
$0 < c < 4, c \neq 1, 2$	$\lim_{x \rightarrow c} f(x) = f(c)$	

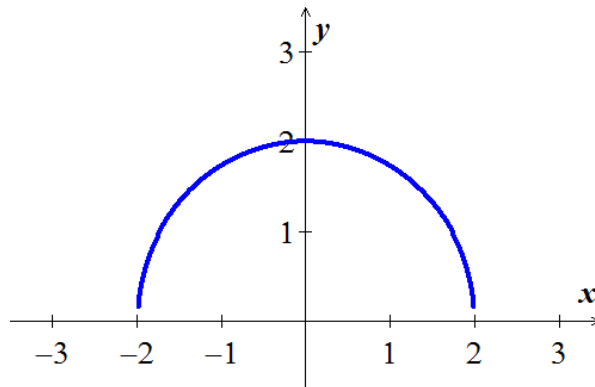
### Example

At what points the function  $f(x) = \sqrt{4 - x^2}$  is continuous?

### Solution

The function is continuous at every point of its domain  $[-2, 2]$ .

Including  $x = -2$ , where  $f$  is right-continuous, and  $x = 2$ , where  $f$  is left-continuous.



### Continuous Functions

A function is **continuous on an interval** iff it is continuous at every point of the interval. A **continuous function** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval.

### Example

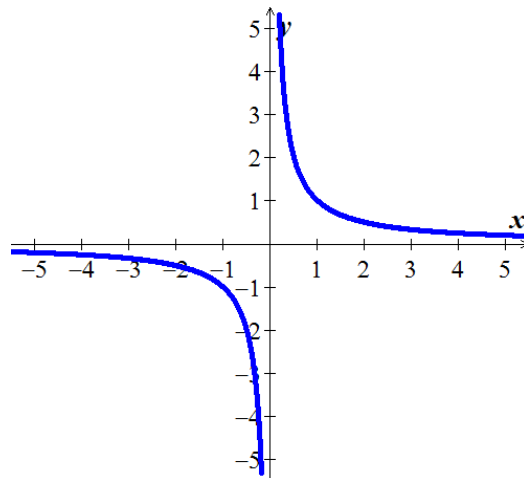
Determine at which points do the function  $f(x) = \frac{1}{x}$  is continuous and discontinuous

### Solution

The function  $f(x)$  is a continuous function because it is continuous at every point of its domain.

It has a point of discontinuity at  $x = 0$ , however, because it is not defined.

It is discontinuous on any interval containing  $x = 0$



## ***Theorem*** – Properties of Continuous Functions

If the functions  $f$  and  $g$  are continuous at  $x = c$ , then the following combinations are continuous at  $x = c$ .

*Sums and Differences*       $f \pm g$

*Constant multiples*       $k \cdot g$ , for any number  $k$ .

*Products*       $f \cdot g$

*Quotients*       $\frac{f}{g}$

*Powers*       $f^n$     ***n*** a positive integer

*Roots*       $\sqrt[n]{f}$ , provided it is defined on an open interval containing  $c$ , where  $n$  is a positive integer

### ***Proof***

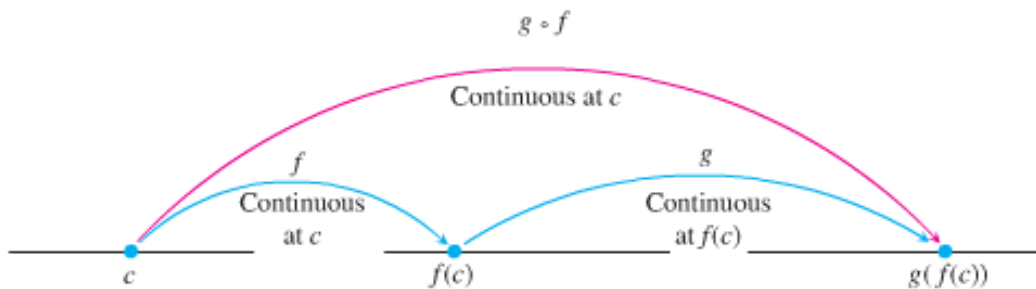
$$\begin{aligned}\lim_{x \rightarrow c} (f + g)(x) &= \lim_{x \rightarrow c} (f(x) + g(x)) \\ &= \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) \\ &= f(c) + g(c) \\ &= (f + g)(c)\end{aligned}$$

This shows that  $f + g$  is continuous

### ***Composites***

All composites of continuous functions are continuous.

If  $f(x)$  is continuous at  $x = c$  and  $g(x)$  is continuous at  $x = f(c)$ , then  $g \circ f$  is continuous at  $x = c$





### Example

Show that  $y = \sqrt{x^2 - 2x - 5}$  is continuous everywhere on its domain

### Solution

$$\text{Let } \begin{cases} f(x) = x^2 - 2x - 5, & \text{Domain : } \mathbb{R} \\ g(x) = \sqrt{x} & \text{Domain : } [0, \infty) \end{cases}$$

$\therefore$  The function  $y$  is continuous on  $[0, \infty)$

### Example

Show that  $y = \left| \frac{x \sin x}{x^2 + 2} \right|$  is continuous everywhere on its domain

### Solution

$$\text{Let } \begin{cases} x \sin x & \text{Domain : } \mathbb{R} \\ x^2 + 2 & \text{Domain : } \mathbb{R} \end{cases}$$

$\therefore$  The function is the composite of a quotient continuous functions with the continuous absolute value function.

### Theorem

If  $g$  is continuous at the point  $b$  and  $\lim_{x \rightarrow c} f(x) = b$ , then

$$\lim_{x \rightarrow c} g(f(x)) = g(b) = g\left(\lim_{x \rightarrow c} f(x)\right)$$

### Proof

Let  $\varepsilon > 0$  be given. Since  $g$  is continuous at  $b$ , there exists a number  $\delta_1 > 0$  such that

$$|g(y) - g(b)| < \varepsilon \quad \text{whenever} \quad 0 < |y - b| < \delta_1$$

$$\lim_{x \rightarrow c} f(x) = b, \exists \delta > 0 \ni |f(x) - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta$$

If we let  $y = f(x)$ , we then have that  $|y - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta$

Which implies from the first statement that  $|g(y) - g(b)| = |g(f(x)) - g(b)| < \varepsilon$  whenever

$0 < |x - c| < \delta$ . From the definition of the limit, this proves that  $\lim_{x \rightarrow c} g(f(x)) = g(b)$

### Example

Find the  $\lim_{x \rightarrow \frac{\pi}{2}} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)$

### Solution

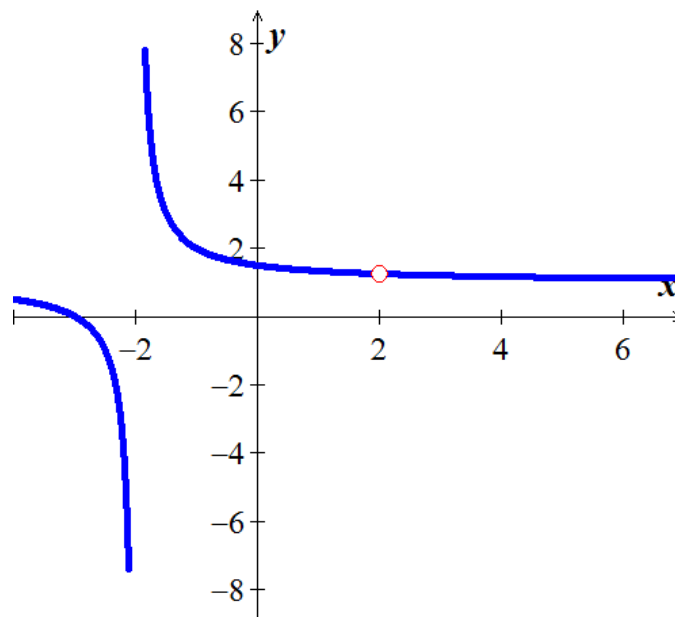
$$\begin{aligned}\lim_{x \rightarrow \frac{\pi}{2}} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) &= \cos\left(\lim_{x \rightarrow \frac{\pi}{2}} 2x + \lim_{x \rightarrow \frac{\pi}{2}} \sin\left(\frac{3\pi}{2} + x\right)\right) \\ &= \cos(\pi + \sin 2\pi) \\ &= \cos(\pi + 0) \\ &= \cos(\pi) \\ &= -1\end{aligned}$$

### Example

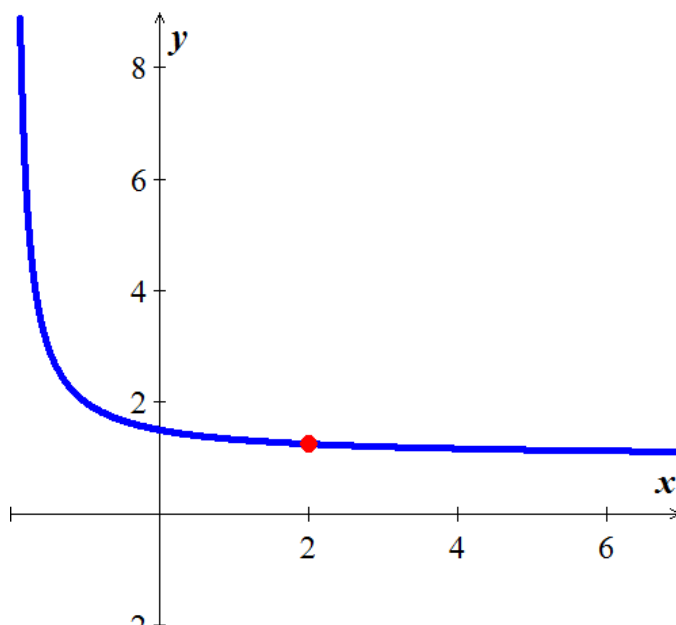
Show that  $f(x) = \frac{x^2 + x - 6}{x^2 - 4}$ ,  $x \neq 2$  has a continuous extension to  $x = 2$ , and find that extension.

### Solution

$$\begin{aligned}f(x) &= \frac{x^2 + x - 6}{x^2 - 4} \\ &= \frac{(x-2)(x+3)}{(x-2)(x+2)} \\ &= \frac{x+3}{x+2}\end{aligned}$$



After simplification the function is continuous at  $x = 2$



After simplification the function is continuous at  $x = 2$

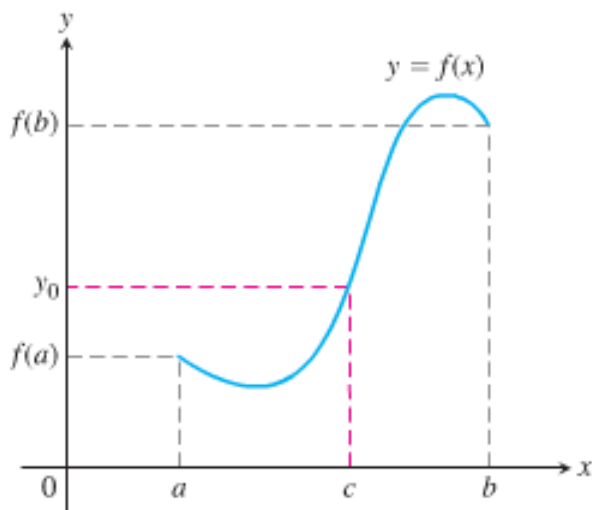
$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{x + 3}{x + 2}$$

$$\underline{= \frac{5}{4}}$$

The new function is the function  $f$  with its point of discontinuity at  $x = 2$  removed.

## ***Theorem*** – the Intermediate Value Theorem for Continuous Functions

If  $f$  is a continuous function on a closed interval  $[a, b]$ , and if  $y_0$  is any value between  $f(a)$  and  $f(b)$ , then  $y_0 = f(c)$  for some  $c$  in  $[a, b]$ .



### **A Consequence for Root Finding**

We call a solution of the equation  $f(x) = 0$  a **root** of the equation or zero of the function  $f$ . The Intermediate Value Theorem said that if  $f$  is continuous, then any interval on which  $f$  changes sign contains a zero of the function.

### ***Example***

Show that there is a root of the equation  $x^3 - x - 1$  between 1 and 2.

#### **Solution**

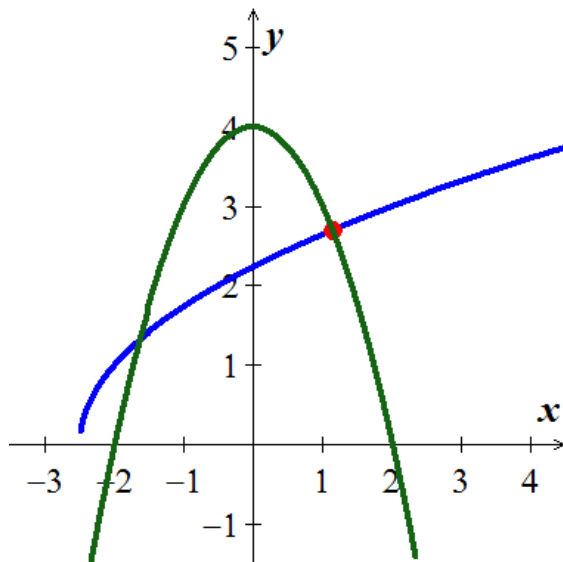
$$f(1) = 1^3 - 1 - 1 = -1 < 0$$

$$f(2) = 2^3 - 2 - 1 = 5 > 0$$

Since  $f$  is continuous, the Intermediate Value Theorem says there is a zero of  $f$  between 1 and 2.

### Example

Use the Intermediate Value Theorem to prove that the equation  $\sqrt{2x+5} = 4 - x^2$  has a solution.



### Solution

The function  $g(x) = \sqrt{2x+5}$  is continuous on the interval  $\left[-\frac{5}{2}, \infty\right)$  since it is the composite of the square root function with nonnegative linear function  $y = 2x + 5$ .

Then the function  $f(x) = \sqrt{2x+5} + x^2$  is the sum of the function  $g(x)$  and  $y = x^2$ .

It follows that  $f(x)$  is continuous on the interval  $\left[-\frac{5}{2}, \infty\right)$ .

By trial and error:

$$f(0) = \sqrt{2(0)+5} + 0^2 = \sqrt{5} > 0$$

$$f(2) = \sqrt{2(2)+5} + 2^2 = \sqrt{9} + 4 = 7 > 0$$

$f$  is continuous on the interval  $[0, 2] \subset \left[-\frac{5}{2}, \infty\right)$ .

Since the value  $y_0 = 4$  is between  $\sqrt{5}$  and 7, by the Intermediate Value Theorem there is a number  $c \in [0, 2] \ni f(c) = 4$ . That is, the number  $c$  solves the original equation.

## Exercises Section 1.5 – Continuity

1. Given the graphed function  $f(x)$

a) Does  $f(-1)$  exist?

b) Does  $\lim_{x \rightarrow -1^+} f(x)$  exist?

c) Does  $\lim_{x \rightarrow -1^+} f(x) = f(-1)$ ?

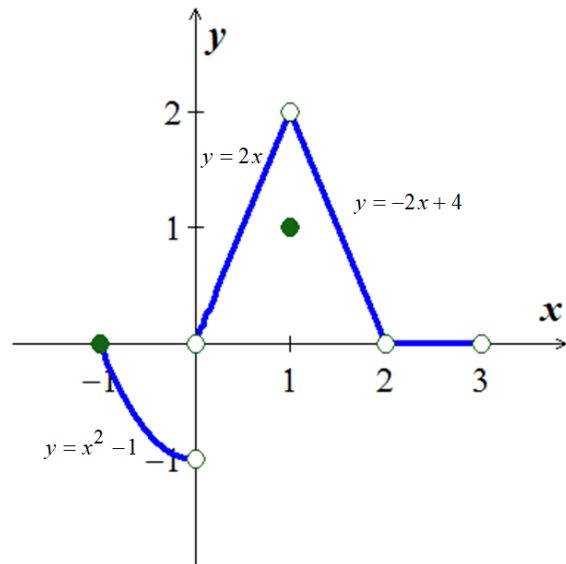
d) Is  $f$  continuous at  $x = -1$ ?

e) Does  $f(1)$  exist?

f) Does  $\lim_{x \rightarrow 1} f(x)$  exist?

g) Does  $\lim_{x \rightarrow 1} f(x) = f(1)$ ?

h) Is  $f$  continuous at  $x = 1$ ?



(2 – 11) At what point(s) is the given function continuous?

2.  $y = \frac{1}{x-2} - 3x$

6.  $y = \tan \frac{\pi x}{2}$

9.  $y = \sqrt{2x+3}$

3.  $y = \frac{x+3}{x^2-3x-10}$

7.  $y = \frac{x \tan x}{x^2+1}$

10.  $y = \sqrt[4]{3x-1}$

4.  $y = |x-1| + \sin x$

8.  $y = \frac{\sqrt{x^4+1}}{1+\sin^2 x}$

11.  $y = (2-x)^{1/5}$

5.  $y = \frac{x+2}{\cos x}$

12. Find  $\lim_{x \rightarrow \pi} \sin(x - \sin x)$ , then is the function continuous at the point being approached?

13. Find  $\lim_{x \rightarrow 0} \tan\left(\frac{\pi}{4} \cos(\sin x^{1/3})\right)$ , then is the function continuous at the point being approached?

14. Find  $\lim_{t \rightarrow 0} \cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right)$ , then is the function continuous at the point being approached?

15. Explain why the equation  $\cos x = x$  has at least one solution.

(16 – 19) Show that the equation has three solutions in the given interval

16.  $x^3 - 15x + 1 = 0$ ;  $[-4, 4]$

18.  $70x^3 - 87x^2 + 32x - 3 = 0$ ;  $(0, 1)$

17.  $x^3 + 10x^2 - 100x + 50 = 0$ ;  $(-20, 10)$

19.  $x^3 - 3x - 1 = 0$ ;  $[-2, 2]$

20. Show that the equation has six solutions in the given interval  $x^6 - 8x^4 + 10x^2 - 1 = 0$ ;  $[-3, 3]$
21. If functions  $f(x)$  and  $g(x)$  are continuous for  $0 \leq x \leq 1$ , could  $\frac{f(x)}{g(x)}$  possibly be discontinuous at a point of  $[0, 1]$ ? Give reason for your answer.
22. Suppose that a function  $f$  is continuous on the closed interval  $[0, 1]$  and that  $0 \leq f(x) \leq 1$  for every  $x$  in  $[0, 1]$ . Show that there must exist a number  $c$  in  $[0, 1]$  such that  $f(c) = c$  ( $c$  is called a **fixed point** of  $f$ ).
23. Use the Intermediate Value Theorem to show that the equation  $x^5 + 7x + 5 = 0$  has a solution in the interval  $(-1, 0)$ .
24. The amount of an antibiotic (in  $mg$ ) in the blood  $t$  hours after an intravenous line is opened is given by

$$m(t) = 100(e^{-0.1t} - e^{-0.3t})$$

- a) Use the Intermediate Value Theorem to show that the amount of drug is  $30\text{ mg}$  at some time in the interval  $[0, 5]$  and again at some time in the interval  $[5, 15]$
- b) Estimate the times at which  $m = 30\text{ mg}$
- c) Is the amount of drug in the blood ever  $50\text{ mg}$ ?

(25 – 27) Determine whether the following functions are continuous at  $a$ .

25.  $f(x) = \frac{1}{x-5}$ ;  $a = 5$

27.  $g(x) = \begin{cases} \frac{x^2-16}{x-4} & \text{if } x \neq 4; \\ 8 & \text{if } x = 4 \end{cases}$ ;  $a = 4$

26.  $h(x) = \sqrt{x^2 - 9}$ ;  $a = 3$

(28 – 31) Find the intervals on which the following functions are continuous. Specify right- or left-continuity at the endpoints

28.  $f(x) = \sqrt{x^2 - 5}$

29.  $f(x) = e^{\sqrt{x-2}}$

30.  $f(x) = \frac{2x}{x^3 - 25x}$

31.  $f(x) = \cos e^x$

32. Let  $g(x) = \begin{cases} 5x-2 & \text{if } x < 1 \\ a & \text{if } x = 1 \\ ax^2 + bx & \text{if } x > 1 \end{cases}$

Determine values of the constants  $a$  and  $b$  for which  $g(x)$  is continuous at  $x = 1$

## Section 1.6 – Precise Definition of a Limit

### Example

Consider the function  $y = 2x - 1$  near  $x_0 = 4$ . Intuitively it appears that  $y$  is close to 7 when  $x$  is close to 4, so  $\lim_{x \rightarrow 4} (2x - 1) = 7$ . However, how close to  $x_0 = 4$  does  $x$  have to be so that  $y = 2x - 1$  differs from 7 by, say less than 2 units?

### Solution

We need to find the values of  $x$  for  $|y - 7| < 2$ .

$$|y - 7| = |2x - 1 - 7| = |2x - 8|$$

$$|2x - 8| < 2$$

$$-2 < 2x - 8 < 2$$

$$-2 + 8 < 2x - 8 + 8 < 2 + 8$$

$$6 < 2x < 10$$

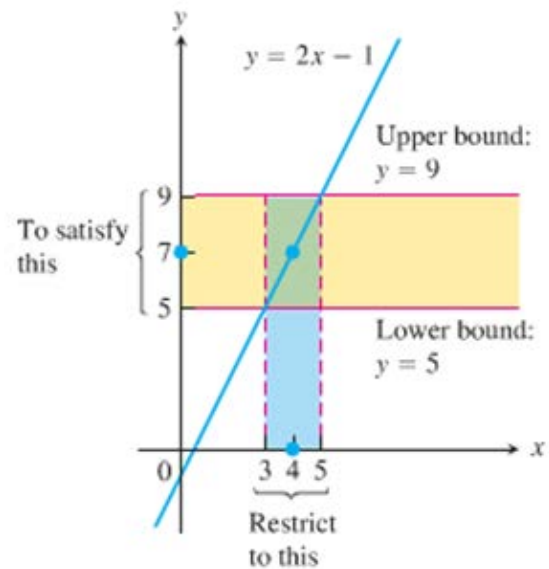
$$\frac{6}{2} < \frac{2x}{2} < \frac{10}{2}$$

$$3 < x < 5$$

$$3 - 4 < x - 4 < 5 - 4$$

$$-1 < x - 4 < 1$$

Keeping  $x$  within 1 unit of  $x_0 = 4$  will keep  $y$  within 2 units of  $y_0 = 7$



### Definition

Let  $f(x)$  be defined on an open interval about  $x_0$ , except possibly at  $x_0$  itself.

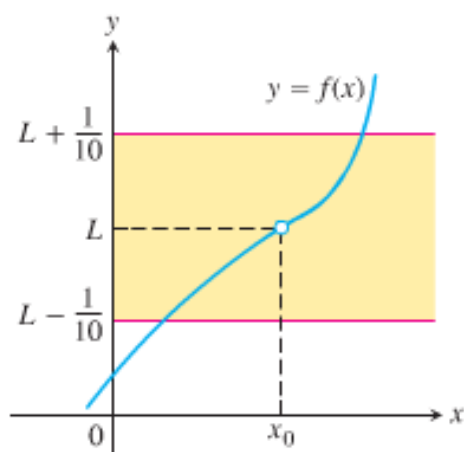
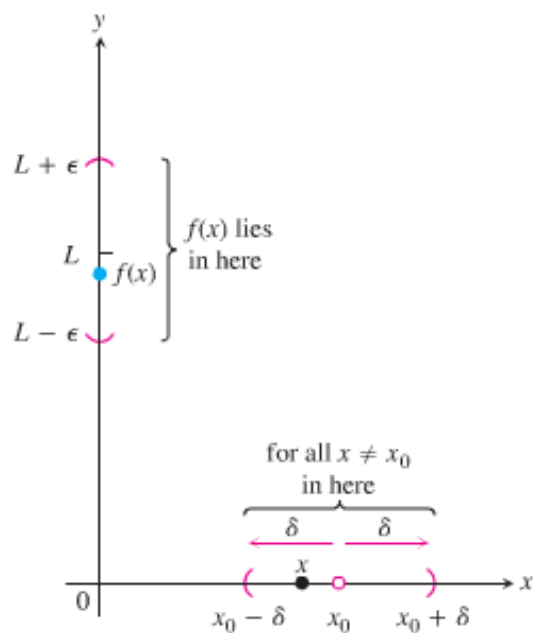
We say that **the limit of  $f(x)$  as  $x$  approaches  $x_0$  is the number  $L$** , and write

$$\lim_{x \rightarrow x_0} f(x) = L$$

If, for every number  $\varepsilon > 0$ , there exists a corresponding number  $\delta > 0$  such that for all  $x$ ,

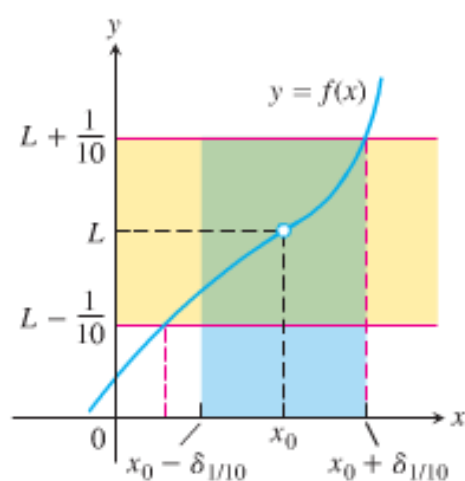
$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$





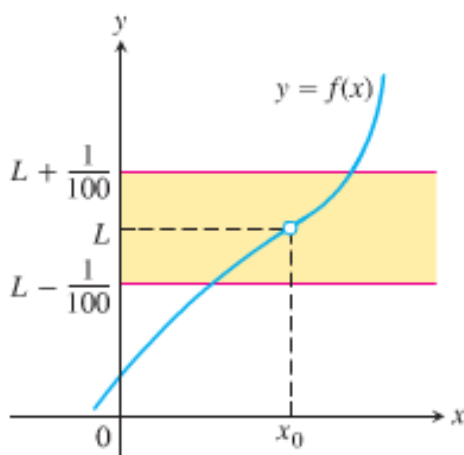
The challenge:

$$\text{Make } |f(x) - L| < \epsilon = \frac{1}{10}$$



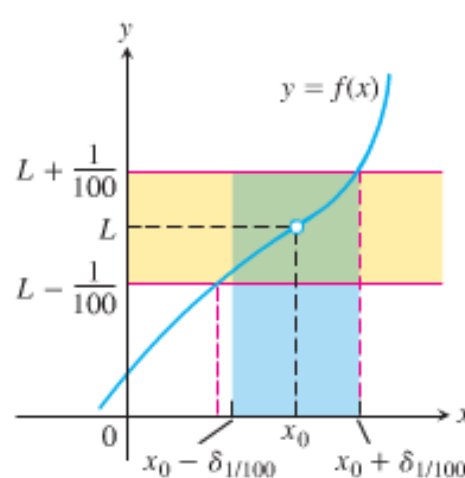
Response:

$$|x - x_0| < \delta_{1/10} \text{ (a number)}$$



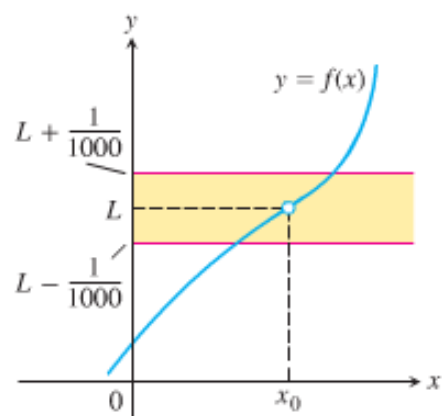
New challenge:

$$\text{Make } |f(x) - L| < \epsilon = \frac{1}{100}$$



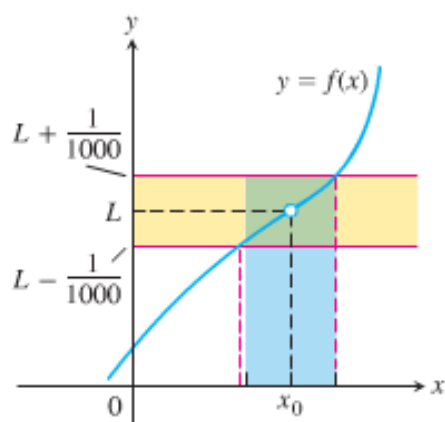
Response:

$$|x - x_0| < \delta_{1/100}$$



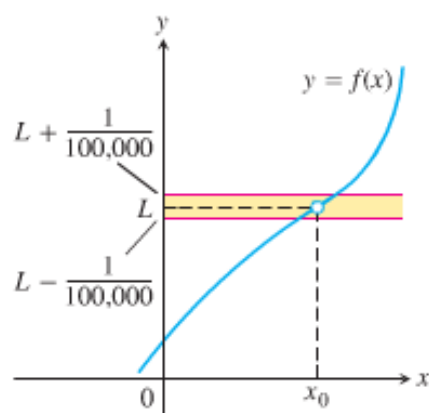
New challenge:

$$\epsilon = \frac{1}{1000}$$



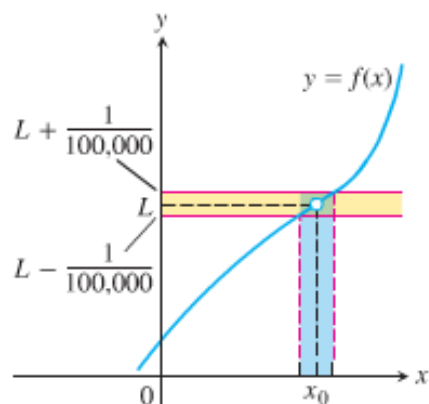
Response:

$$|x - x_0| < \delta_{1/1000}$$



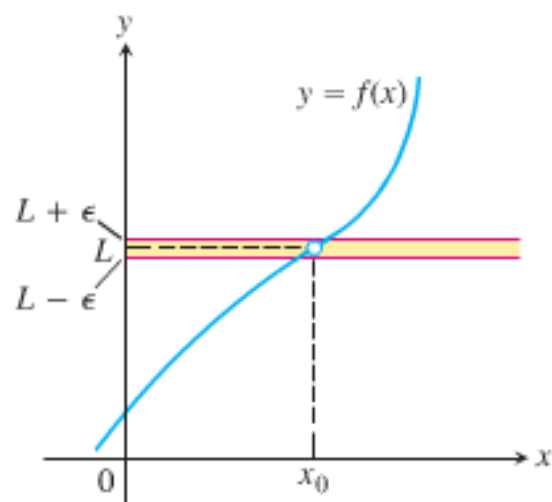
New challenge:

$$\epsilon = \frac{1}{100,000}$$



Response:

$$|x - x_0| < \delta_{1/100,000}$$



New challenge:

$$\epsilon = \dots$$

### Example

Show that  $\lim_{x \rightarrow 1} (5x - 3) = 2$

### Solution

Let  $x_0 = 1$ ,  $f(x) = 5x - 3$ , and  $L = 2$ .

For any given  $\varepsilon > 0$ , there exists a  $\delta > 0$  so that  $x \neq 1$  and  $x$  is within distance  $\delta$  of  $x_0 = 1$ , that is

$$0 < |x - 1| < \delta \Rightarrow |f(x) - 2| < \varepsilon$$

$$|(5x - 3) - 2| < \varepsilon$$

$$|5x - 5| < \varepsilon$$

$$5|x - 1| < \varepsilon$$

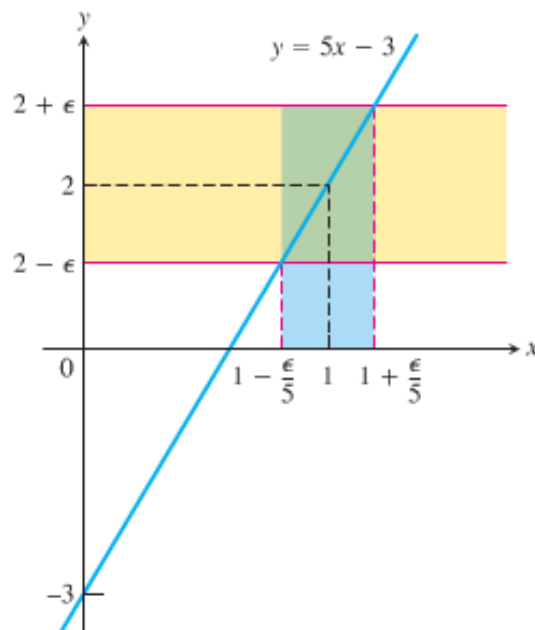
$$|x - 1| < \frac{\varepsilon}{5}$$

Thus, we can take:  $\delta = \frac{\varepsilon}{5}$

If  $0 < |x - 1| < \delta = \frac{\varepsilon}{5}$

$$|(5x - 3) - 2| = |5x - 5| = 5|x - 1| = 5 \frac{\varepsilon}{5} = \varepsilon$$

Which proves that  $\lim_{x \rightarrow 1} (5x - 3) = 2$



### Example

Prove the results presented graphically  $\lim_{x \rightarrow x_0} x = x_0$

### Solution

Let  $\varepsilon > 0$  be given, we must find  $\delta > 0$  such that for all  $x$

$$0 < |x - x_0| < \delta \Rightarrow |x - x_0| < \varepsilon$$

This implication will hold if  $\delta = \varepsilon$  or any smaller number.

### Example

For the limit  $\lim_{x \rightarrow 5} \sqrt{x-1} = 2$ , find a  $\delta > 0$  that works for  $\varepsilon = 1$ . That is, find a  $\delta > 0$  such that for all  $x$ :

$$0 < |x-5| < \delta \Rightarrow |\sqrt{x-1}-2| < 1$$

### Solution

$$|\sqrt{x-1}-2| < 1$$

$$-1 < \sqrt{x-1}-2 < 1$$

$$-1+2 < \sqrt{x-1}-2+2 < 1+2$$

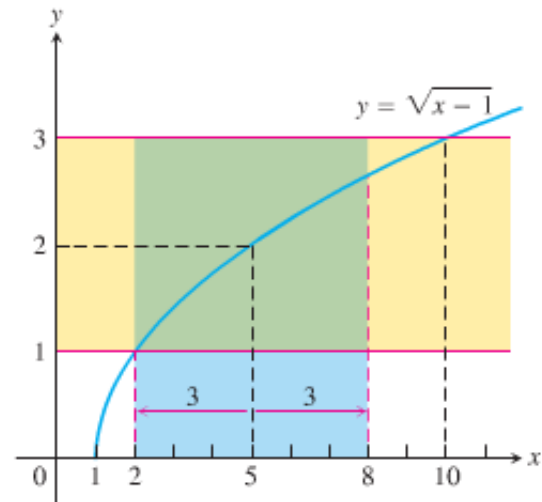
$$1 < \sqrt{x-1} < 3$$

*Square all sides*

$$1 < x-1 < 9$$

$$1+1 < x-1+1 < 9+1$$

$$2 < x < 10$$



The inequality holds for all  $x$  in the open interval  $(2, 10)$ .

So it holds for all  $x \neq 5$  in the interval as well.

Finding  $\delta$  value.

$$5 - \delta < x < 5 + \delta$$

Centered at  $x_0 = 5$  inside the interval  $(2, 10)$

$$\begin{cases} 5 - \delta = 2 \\ 5 + \delta < 10 \end{cases} \rightarrow \delta = 3 \text{ (to be centered)}$$



$$0 < |x-5| < 3 \Rightarrow |\sqrt{x-1}-2| < 1$$

### How to Find Algebraically a $\delta$ for a Given $f$ , $L$ , $x_0$ , and $\varepsilon > 0$

The process of finding a  $\delta > 0$  such that for all  $x$ :

$$0 < |x-x_0| < \delta \Rightarrow |f(x)-L| < \varepsilon$$

Can be accomplished in two steps

1. Solve the inequality  $|f(x)-L| < \varepsilon$  to find an open interval  $(a, b)$  containing  $x_0$  on which the inequality holds for all  $x \neq x_0$ .
2. Find a value of  $\delta > 0$  that places the open interval  $(x_0 - \delta, x_0 + \delta)$  centered at  $x_0$  inside the interval  $(a, b)$ . The inequality  $|f(x)-L| < \varepsilon$  will hold for all  $x \neq x_0$  in this  $\delta$ -interval.

### Example

Prove that  $\lim_{x \rightarrow 2} f(x) = 4$  if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

### Solution

We need to show that given  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for all  $x$ :

$$0 < |x - 2| < \delta \Rightarrow |f(x) - 4| < \varepsilon$$

1. Solve the inequality  $|f(x) - 4| < \varepsilon$  to find an open interval containing  $x_0 = 2$  on which the inequality holds for all  $x \neq x_0$ .

For  $x \neq x_0 = 2$ ,  $f(x) = x^2$ , and the inequality to solve is  $|x^2 - 4| < \varepsilon$ :

$$|x^2 - 4| < \varepsilon$$

$$-\varepsilon < x^2 - 4 < \varepsilon$$

*Add 4 to all sides*

$$4 - \varepsilon < x^2 < 4 + \varepsilon$$

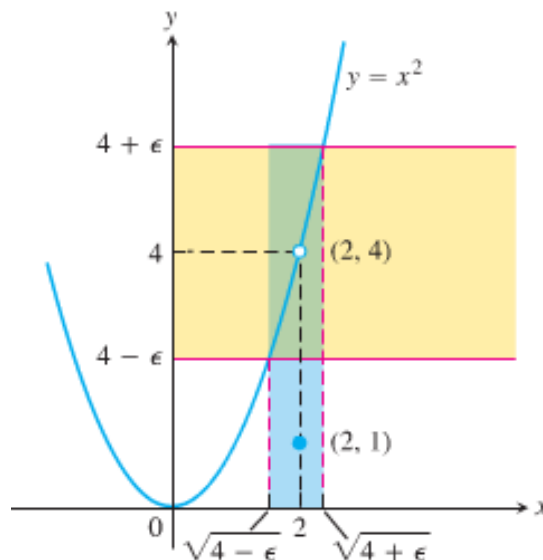
*Square root*

$$\sqrt{4 - \varepsilon} < |x| < \sqrt{4 + \varepsilon}$$

*Assume  $\varepsilon < 4$*

$$\sqrt{4 - \varepsilon} < x < \sqrt{4 + \varepsilon}$$

The inequality  $|f(x) - 4| < \varepsilon$  holds for all  $x \neq 2$  in the open interval  $(\sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon})$



2. Find a value of  $\delta > 0$  that places the open interval  $(2 - \delta, 2 + \delta)$  inside the interval  $(\sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon})$ .

Take  $\delta$  to be the distance from  $x_0 = 2$  to the nearer endpoint of  $(\sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon})$ .

$$\Rightarrow \delta = \min(2 - \sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon} - 2).$$

$$\begin{aligned}
0 &< |x-2| < \delta \\
-(2-\sqrt{4-\varepsilon}) &< x-2 < \sqrt{4+\varepsilon}-2 \\
-2+\sqrt{4-\varepsilon} &< x-2 < \sqrt{4+\varepsilon}-2 \\
\sqrt{4-\varepsilon} &< x < \sqrt{4+\varepsilon} \\
\therefore 0 &< |x-2| < \delta \Rightarrow |f(x)-4| < \varepsilon
\end{aligned}$$

### Example

Given that  $\lim_{x \rightarrow c} f(x) = L$  and  $\lim_{x \rightarrow c} g(x) = M$ , prove that  $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$

### Solution

We need to show that given  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for all  $x$ :

$$0 < |x-c| < \delta \Rightarrow |f(x) + g(x) - (L+M)| < \varepsilon$$

$$\begin{aligned}
|f(x) + g(x) - (L+M)| &= |f(x) + g(x) - L - M| \\
&= |(f(x) - L) + (g(x) - M)| \quad \text{Triangle Inequality } |a+b| \leq |a| + |b| \\
&\leq |(f(x) - L)| + |(g(x) - M)|
\end{aligned}$$

Since  $\lim_{x \rightarrow c} f(x) = L$ , there exists a number  $\delta_1 > 0$  such that for all  $x$ :

$$0 < |x-c| < \delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$$

Similarly, since  $\lim_{x \rightarrow c} g(x) = M$ , there exists a number  $\delta_2 > 0$  such that for all  $x$ :

$$0 < |x-c| < \delta_2 \Rightarrow |g(x) - M| < \frac{\varepsilon}{2}$$

Let  $\delta = \min\{\delta_1, \delta_2\}$ , the smaller of  $\delta_1$  and  $\delta_2$ . If  $0 < |x-c| < \delta$  then  $0 < |x-c| < \delta_1$ , so

$|f(x) - L| < \frac{\varepsilon}{2}$  and  $|x-c| < \delta_2$ , so  $|g(x) - M| < \frac{\varepsilon}{2}$ . Therefore

$$|f(x) + g(x) - (L+M)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This show that  $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$

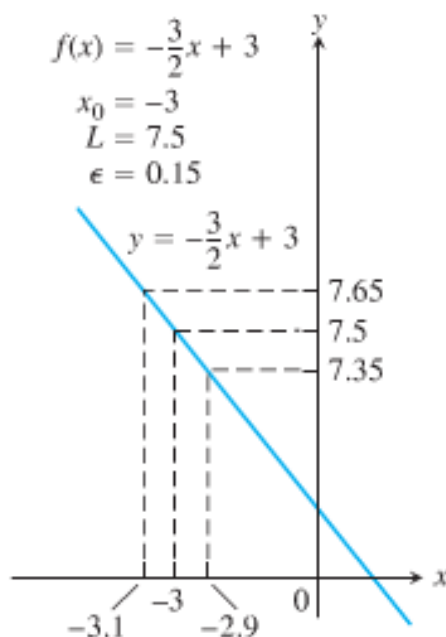
## Exercises      Section 1.6 – Precise Definition of Limits

(1 – 2) Sketch the interval  $(a, b)$  on the  $x$ -axis with the point  $x_0$  inside. Then find a value of  $\delta > 0$  such that for all  $x$ ,  $0 < |x - x_0| < \delta \Rightarrow a < x < b$  for

1.  $a = 1, \quad b = 7, \quad x_0 = 5$

2.  $a = -\frac{7}{2}, \quad b = -\frac{1}{2}, \quad x_0 = -\frac{3}{2}$

3. Use the graph to find a  $\delta > 0$  such that for all  $x$   $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$



(4 – 8) Find an open interval about  $x_0$  on which the inequality  $|f(x) - L| < \varepsilon$  holds. Then give a value for  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - x_0| < \delta$  the inequality  $|f(x) - L| < \varepsilon$  holds.

4.  $f(x) = x + 1, \quad L = 5, \quad x_0 = 4, \quad \varepsilon = 0.01$

5.  $f(x) = \sqrt{x+1}, \quad L = 1, \quad x_0 = 0, \quad \varepsilon = 0.1$

6.  $f(x) = \sqrt{x-7}, \quad L = 4, \quad x_0 = 23, \quad \varepsilon = 1$

7.  $f(x) = x^2, \quad L = 3, \quad x_0 = \sqrt{3}, \quad \varepsilon = 0.1$

8.  $f(x) = \frac{120}{x}, \quad L = 5, \quad x_0 = 24, \quad \varepsilon = 1$

(9 – 14) Give a formal proof that

9.  $\lim_{x \rightarrow 4} (9 - x) = 5$

10.  $\lim_{x \rightarrow 1} \frac{1}{x} = 1$

11.  $\lim_{x \rightarrow 5} \frac{x^2 - 25}{x - 5} = 10$

12.  $\lim_{x \rightarrow 0} f(x) = 0$  if  $f(x) = \begin{cases} 2x, & x < 0 \\ \frac{x}{2}, & x \geq 0 \end{cases}$

13.  $\lim_{x \rightarrow 1} (5x - 2) = 3$

14.  $\lim_{x \rightarrow 2} \frac{1}{(x - 2)^4} = \infty$

15. Prove that  $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$

