Lecture 1 – Functions, Exponential & Logarithms

Section 1.1 – Functions

A ***set*** is a collection of objects of some type, and the objects are called ***elements*** of the set.

Notation or Terminology	***Meaning***	***Example***
$a \in S$	***a*** is an element of ***S***	$3 \in \mathbb{Z}$
$a \notin S$	***a*** is not an element of ***S***	$\frac{3}{2} \notin \mathbb{Z}$
$S \subset T$	*S* is a ***subset*** of *T* Every element of *S* is an element of *T*	$\mathbb{Z} \subset \mathbb{R}$
Constant	A letter or symbol that represents a specific element of a set.	$5,\ \sqrt{2},\ \pi$
Variable	A letter or symbol that represents any element of a set.	Let ***x*** denote any $\mathbb{R}$

Definition of a *Function*

A ***function*** is a relation between two variables such that to matches each element of a first set (called ***domain***) to an element of a second set (called ***range***) in such way that no element in the first set is assigned to two different elements in the second set.

The ***domain*** of the function is the set of all values of the independent variable for which the function is defined.

The ***range*** of the function is the set of all values taken on by the dependent variable.

The *Domain* of a Function

1. *Rational* function: $\dfrac{f(x)}{h(x)}$ $\Rightarrow$ ***Domain***: $h(x) \neq 0$

   Example: $f(x) = \dfrac{1}{x-3}$ ***Domain***: $x \neq 3$

2. *Irrational* function: $\sqrt{g(x)}$ $\Rightarrow$ ***Domain***: $g(x) \geq 0$

   Example: $g(x) = \sqrt{3-x} + 5$ ***Domain***: $x \leq 3$

3. Otherwise: *Domain* all real numbers

   Example: $f(x) = x^3 + |x|$ ***Domain***: All real numbers, $\mathbb{R}$, or $(-\infty,\ \infty)$

(1) & (2)→ Find the domain: $f(x)=\frac{x+1}{\sqrt{x-3}}$ $\Rightarrow$ ***Domain:*** $x>3$

$$\boxed{\begin{array}{l} ax^2+bx+c\ge 0 \;\rightarrow\; if\;\; a>0 \;\Rightarrow\; x\le x_1,\;\; x\ge x_2 \\ ax^2+bx+c\le 0 \;\rightarrow\; if\;\; a>0 \;\Rightarrow\; x_1\le x\le x_2 \end{array}}$$

Example

Let $g(x)=\dfrac{\sqrt{4+x}}{1-x}$. Find the domain of g.

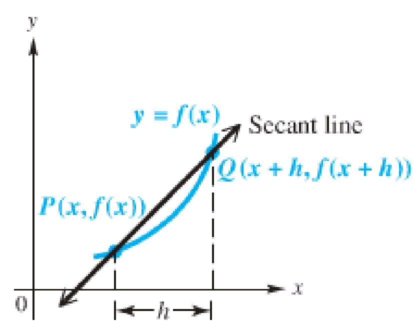
Solution

$$\begin{cases} 4+x\ge 0 \Rightarrow x\ge -4 \\ 1-x\ne 0 \Rightarrow x\ne 1 \end{cases} \quad \rightarrow \underline{[-4,\ 1)\cup(1,\ \infty)}$$

Difference Quotients

$$\frac{f(x+h)-f(x)}{(x+h)-x}$$

The difference quotient is given by: $\dfrac{f(x+h)-f(x)}{h}$



Example

For the function f given by $f(x)=2x^2-3x$, find the difference quotient $\frac{f(x+h)-f(x)}{h}$

Solution

$$\frac{f(x+h)-f(x)}{h}=\frac{\overbrace{2(x+h)^2-3(x+h)}^{f(x+h)}\;-\;\overbrace{(2x^2-3x)}^{f(x)}}{h}$$

$$=\frac{2x^2+4xh+2h^2-3x-3h-2x^2+3x}{h}$$

$$=\frac{4xh+2h^2-3h}{h}$$

$$=\frac{4xh}{h}+\frac{2h^2}{h}-\frac{3h}{h}$$

$$\underline{=4x+2h-3}$$

Piecewise-Defined Functions

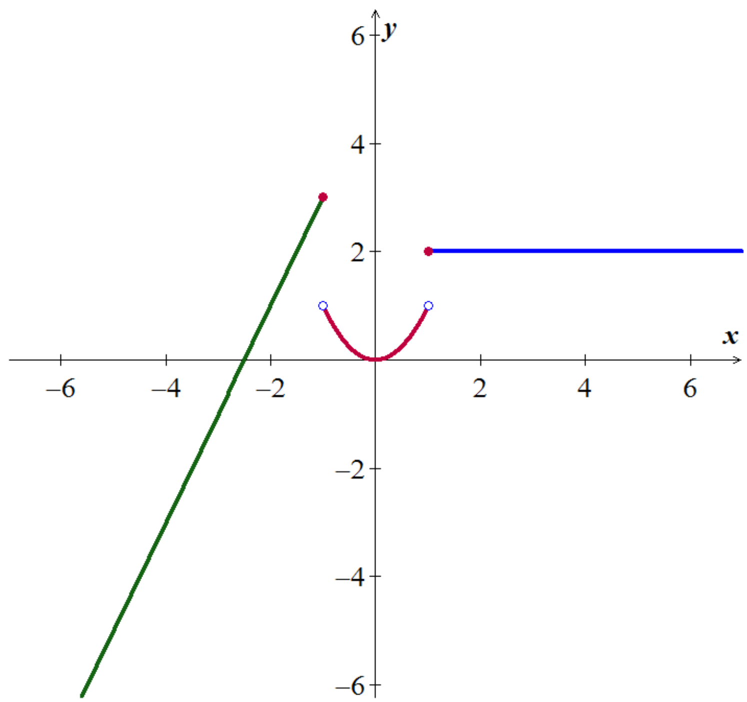
Function are sometimes described by more than one expression, we call such functions ***piecewise-defined functions***.

Example

Graph each function

$$f(x)=\begin{cases} 2x+5 & if \quad x\le -1 \\ x^2 & if \quad |x|<1 \\ 2 & if \quad x\ge 1 \end{cases}$$

Solution

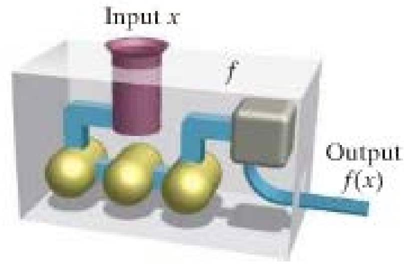


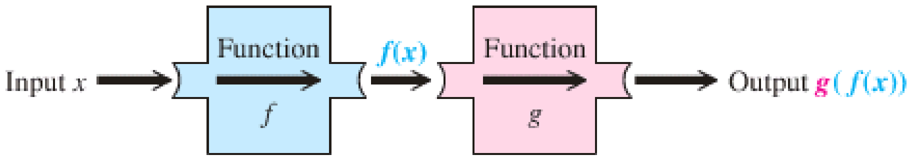
Composition of Functions

The composite function $f \circ g$, the composite of f and g, is defined as

$$\left(f \circ g\right)(x) = f\left(g(x)\right)$$

Where x is in the domain of g
and $g(x)$ is in the domain of f





Example

Let $f(x) = x^2 - 1$ *and* $g(x) = 3x + 5$

a) Find $\left(f \circ g\right)(x)$ and the domain of $f \circ g$

b) Find $\left(g \circ f\right)(x)$ and the domain of $g \circ f$

c) Find $\left(f\left(g\right)\right)(2)$ in two different ways: first using the functions f and g separately and second using the composite function $f \circ g$.

Solution

a) $\left(f \circ g\right)(x) = f\left(g(x)\right)$

$= f\left(3x + 5\right)$ **Domain**: $\left(3x+5\right) \to \mathbb{R}$

$= \left(\_\_\right)^2 - 1$

$= \left(3x + 5\right)^2 - 1$

$= 9x^2 + 30x + 25 - 1$

$= 9x^2 + 30x + 24$ **Domain**: $\left(9x^2 + 30x + 24\right) \to \mathbb{R}$

Domain of $f \circ g$: $\mathbb{R}$

b) $\left(g \circ f\right)(x) = g\left(f(x)\right)$

$= g\left(x^2 - 1\right)$ **Domain**: $\left(x^2 - 1\right) \to \mathbb{R}$

$= 3\left(x^2 - 1\right) + 5$

$= 3x^2 - 3 + 5$

$= 3x^2 + 2$ **Domain**: $\left(3x^2 + 2\right) \to \mathbb{R}$

Domain of $g \circ f$: $\mathbb{R}$

c) $g(2)=3(2)+5=11$

$$\begin{aligned}(f\circ g)(2) &= f\left(g(2)\right)\\ &= f(11)\\ &= 11^2-1\\ &= 120\end{aligned}$$

$$(f\circ g)(x)=9x^2+30x+24$$

$$(f\circ g)(2)=9(2)^2+30(2)+24\ \underline{=120}$$

Example

Let $f(x)=x^2-16$ *and* $g(x)=\sqrt{x}$

a) Find $(f\circ g)(x)$ and the domain of $f\circ g$

b) Find $(g\circ f)(x)$ and the domain of $g\circ f$

Solution

a) $(f\circ g)(x)=f\left(g(x)\right)$

$=f\left(\sqrt{x}\right)$ ***Domain*** $:\left(\sqrt{x}\right)\to \boldsymbol{x}\geq 0$

$=\left(\sqrt{x}\right)^2-16$

$=x-16$ ***Domain*** $:(x-16)\to \mathbb{R}$

Domain of $f\circ g$: $\boldsymbol{x}\geq 0$

b) $(g\circ f)(x)=g\left(f(x)\right)$

$=g\left(x^2-16\right)$ ***Domain*** $:\left(x^2-1\right)\to \mathbb{R}$

$=\sqrt{x^2-16}$ ***Domain*** $:\left(\sqrt{x^2-16}\right)\to |x|\geq 4$

Domain of $g\circ f$: $|x|\geq 4$ *or* $(-\infty,\ -4]\cup[4,\ \infty)$

Even and Odd Functions

Given the function $f(x)$ then find $f(-x)$ and simplify:

- If $f(-x) = f(x) \Rightarrow f$ is ***even***, or
- If $f(-x) = -f(x) \Rightarrow f$ is ***odd***
- ***Neither***

Example

Decide whether each function is even, odd, or neither

a) $f(x) = 8x^4 - 3x^2$

$$f(-x) = 8(-x)^4 - 3(-x)^2$$
$$= 8x^4 - 3x^2$$
$$= f(x)$$

Function is *Even*

b) $f(x) = 6x^3 - 9x$

$$f(-x) = 6(-x)^3 - 9(-x)$$
$$= -6x^3 + 9x$$
$$= -\left(6x^3 - 9x\right)$$
$$= -f(x)$$

Function is *Odd*

c) $f(x) = 3x^2 + 5x$

$$f(-x) = 3(-x)^2 + 5(-x)$$
$$= 3x^2 - 5x$$

Function is *Neither*

Exercises *Section* 1.1 –Functions

(**1 – 80**) Find the Domain

1. $f(x)=7x+4$

2. $f(x)=|3x-2|$

3. $f(x)=3x+\pi$

4. $f(x)=\sqrt{7}x+\frac{1}{2}$

5. $f(x)=-2x^2+3x-5$

6. $f(x)=x^3-2x^2+x-3$

7. $f(x)=x^2-2x-15$

8. $f(x)=4-\frac{2}{x}$

9. $f(x)=\frac{1}{x^4}$

10. $g(x)=\frac{3}{x-4}$

11. $y=\frac{2}{x-3}$

12. $y=\frac{-7}{x-5}$

13. $f(x)=\frac{x+5}{2-x}$

14. $f(x)=\frac{8}{x+4}$

15. $f(x)=\frac{1}{x+4}$

16. $f(x)=\frac{1}{x-4}$

17. $f(x)=\frac{3x}{x+2}$

18. $f(x)=x-\frac{2}{x-3}$

19. $f(x)=x+\frac{3}{x-5}$

20. $f(x)=\frac{1}{2}x-\frac{8}{x+7}$

21. $f(x)=\frac{1}{x-3}-\frac{8}{x+7}$

22. $f(x)=\frac{1}{x+4}-\frac{2x}{x-4}$

23. $f(x)=\frac{3x^2}{x+3}-\frac{4x}{x-2}$

24. $f(x)=\frac{1}{x^2-2x+1}$

25. $f(x)=\frac{x}{x^2+3x+2}$

26. $f(x)=\frac{x^2}{x^2-5x+4}$

27. $f(x)=\frac{1}{x^2-4x-5}$

28. $g(x)=\frac{2}{x^2+x-12}$

29. $h(x)=\frac{5}{\frac{4}{x}-1}$

30. $y=\sqrt{x}$

31. $f(x)=\sqrt{8-3x}$

32. $y=\sqrt{4x+1}$

33. $y=\sqrt{7-2x}$

34. $f(x)=\sqrt{8-x}$

35. $f(x)=\sqrt{3-2x}$

36. $f(x)=\sqrt{3+2x}$

37. $f(x)=\sqrt{5-x}$

38. $f(x)=\sqrt{x-5}$

39. $f(x)=\sqrt{6-3x}$

40. $f(x)=\sqrt{3x-6}$

41. $f(x)=\sqrt{2x+7}$

42. $f(x)=\sqrt{x^2-16}$

43. $f(x)=\sqrt{16-x^2}$

44. $f(x)=\sqrt{9-x^2}$

45. $f(x)=\sqrt{x^2-25}$

46. $f(x)=\sqrt{x^2-5x+4}$

47. $f(x)=\sqrt{x^2+5x+4}$

48. $f(x)=\sqrt{x^2+3x+2}$

49. $f(x)=\sqrt{x^2-3x+2}$

50. $f(x)=\sqrt{x-4}+\sqrt{x+1}$

51. $f(x)=\sqrt{3-x}+\sqrt{x-2}$

52. $f(x)=\sqrt{1-x}+\sqrt{4-x}$

53. $f(x)=\sqrt{1-x}-\sqrt{x-3}$

54. $f(x)=\sqrt{x+4}-\sqrt{x-1}$

55. $f(x)=\frac{\sqrt{x+1}}{x}$

56. $g(x)=\frac{\sqrt{x-3}}{x-6}$

57. $f(x)=\frac{\sqrt{x+4}}{\sqrt{x-1}}$

58. $f(x)=\frac{\sqrt{5-x}}{x}$

59. $f(x)=\frac{x}{\sqrt{5-x}}$

60. $f(x)=\dfrac{1}{x\sqrt{5-x}}$

61. $f(x)=\dfrac{x+1}{x^3-4x}$

62. $f(x)=\dfrac{\sqrt{x+5}}{x}$

63. $f(x)=\dfrac{x}{\sqrt{x+5}}$

64. $f(x)=\dfrac{1}{x\sqrt{x+5}}$

65. $f(x)=\dfrac{x+3}{\sqrt{x-3}}$

66. $f(x)=\dfrac{\sqrt{x+3}}{\sqrt{x-3}}$

67. $f(x)=\dfrac{\sqrt{x-2}}{\sqrt{x+2}}$

68. $f(x)=\dfrac{\sqrt{2-x}}{\sqrt{x+2}}$

69. $f(x)=\dfrac{x-4}{\sqrt{x-2}}$

70. $f(x)=\dfrac{1}{(x-3)\sqrt{x+3}}$

71. $f(x)=\sqrt{x+2}+\sqrt{2-x}$

72. $f(x)=\sqrt{(x-2)(x-6)}$

73. $f(x)=\sqrt{x+3}-\sqrt{4-x}$

74. $f(x)=\dfrac{\sqrt{4x-3}}{x^2-4}$

75. $f(x)=\dfrac{4x}{6x^2+13x-5}$

76. $f(x)=\dfrac{\sqrt{2x-3}}{x^2-5x+4}$

77. $f(x)=\dfrac{x^2}{\sqrt{x^2-5x+4}}$

78. $f(x)=\dfrac{x+2}{\sqrt{x^2+5x+4}}$

79. $f(x)=\dfrac{\sqrt{x+2}}{\sqrt{x^2+3x+2}}$

80. $f(x)=\dfrac{\sqrt{2x+3}}{x^2-6x+5}$

(**81 – 97**) Find and simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for the given function

81. $f(x)=9x+5$

82. $f(x)=6x+2$

83. $f(x)=4x+11$

84. $f(x)=3x-5$

85. $f(x)=-2x-3$

86. $f(x)=-4x+3$

87. $f(x)=3x-6$

88. $f(x)=-5x-7$

89. $f(x)=2x^2$

90. $f(x)=5x^2$

91. $f(x)=3x^2-4x$

92. $f(x)=2x^2-3x$

93. $f(x)=2x^2-x-3$

94. $f(x)=x^2-2x+5$

95. $f(x)=3x^2-2x+5$

96. $f(x)=-2x^2-3x+7$

97. $f(x)=\sqrt{x-3}$

98. Let $f(x)=4x-3$ and $g(x)=5x+7$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\dfrac{f}{g}\right)(x)$

99. Let $f(x)=2x^2+3$ and $g(x)=3x-4$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\dfrac{f}{g}\right)(x)$

100. Let $f(x)=x^2-2x-3$ and $g(x)=x^2+3x-2$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\dfrac{f}{g}\right)(x)$

101. Let $f(x)=\sqrt{4x-1}$ and $g(x)=\frac{1}{x}$. Find each of the following and give the domain

a) $(f+g)(x)$ *b)* $(f-g)(x)$ *c)* $(fg)(x)$ *d)* $\left(\frac{f}{g}\right)(x)$

102. Find $(f+g)(x),\ (f-g)(x),\ (f\cdot g)(x),\ \textit{and}\ (f/g)(x)$ and the domain of

$f(x)=\sqrt{3-2x},\quad g(x)=\sqrt{x+4}$

103. Find $(f+g)(x),\ (f-g)(x),\ (f\cdot g)(x),\ \textit{and}\ (f/g)(x)$ and the domain of

$f(x)=\frac{2x}{x-4},\quad g(x)=\frac{x}{x+5}$

104. Let $f(x)=\sqrt{4x-1}$ and $g(x)=\frac{1}{x}$. Find each of the following and give the domain

e) $(f+g)(x)$ *f)* $(f-g)(x)$ *g)* $(fg)(x)$ *h)* $\left(\frac{f}{g}\right)(x)$

105. Given that $f(x)=x+1$ and $g(x)=\sqrt{x+3}$

a) Find $(f+g)(x)$

b) Find the domain of $(f+g)(x)$

c) Find: $(f+g)(6)$

106. Given that $f(x)=x^2-4$ and $g(x)=x+2$

a) Find $(f+g)(x)$ and its domain

b) Find $(f/g)(x)$ and its domain

107. Find $(f\circ g)(x),\ (g\circ f)(x),\ f(g(-2))\ \textit{and}\ g(f(3))$

$f(x)=2x^2+3x-4,\quad g(x)=2x-1$

108. Find $(f\circ g)(x),\ (g\circ f)(x),\ f(g(-2))\ \textit{and}\ g(f(3))$

$f(x)=x^3+2x^2,\quad g(x)=3x$

109. Find $(f\circ g)(x),\ (g\circ f)(x),\ f(g(-2))\ \textit{and}\ g(f(3))$

$f(x)=|x|,\quad g(x)=-7$

(**110 – 139**) For the given function; find:

a) Find $(f \circ g)(x)$ and the ***domain*** of $f \circ g$

b) Find $(g \circ f)(x)$ and the ***domain*** of $g \circ f$

110. $f(x) = x - 3 \quad and \quad g(x) = x + 3$

111. $f(x) = \frac{2}{3}x \quad and \quad g(x) = \frac{3}{2}x$

112. $f(x) = x - 1 \quad and \quad g(x) = 3x^2 - 2x - 1$

113. $f(x) = 3x - 2 \quad and \quad g(x) = x^2 - 5$

114. $f(x) = x^2 - 2 \quad and \quad g(x) = 4x - 3$

115. $f(x) = 4x^2 - x + 10 \quad and \quad g(x) = 2x - 7$

116. $f(x) = \sqrt{x} \quad and \quad g(x) = x + 3$

117. $f(x) = \sqrt{x} \quad and \quad g(x) = 2 - 3x$

118. $f(x) = 3x + 2 \quad and \quad g(x) = \sqrt{x}$

119. $f(x) = x^4 \quad and \quad g(x) = \sqrt[4]{x}$

120. $f(x) = x^n \quad and \quad g(x) = \sqrt[n]{x}$

121. $f(x) = x^2 - 3x \quad and \quad g(x) = \sqrt{x+2}$

122. $f(x) = \sqrt{x-2} \quad and \quad g(x) = \sqrt{x+5}$

123. $f(x) = x^2 + 2 \quad and \quad g(x) = \sqrt{3-x}$

124. $f(x) = x^5 - 2 \quad and \quad g(x) = \sqrt[5]{x+2}$

125. $f(x) = 1 - x^2 \quad and \quad g(x) = \sqrt{x^2 - 25}$

126. $f(x) = 2x + 3 \quad and \quad g(x) = \frac{x-3}{2}$

127. $f(x) = 4x - 5 \quad and \quad g(x) = \frac{x+5}{4}$

128. $f(x) = \frac{4}{1-5x} \quad and \quad g(x) = \frac{1}{x}$

129. $f(x) = \frac{1}{x-2} \quad and \quad g(x) = \frac{x+2}{x}$

130. $f(x) = \frac{1}{1+x} \quad and \quad g(x) = \frac{1-x}{x}$

131. $f(x) = \frac{3x+5}{2} \quad and \quad g(x) = \frac{2x-5}{3}$

132. $f(x) = \frac{x-1}{x-2} \quad and \quad g(x) = \frac{x-3}{x-4}$

133. $f(x) = \frac{6}{x-3} \quad and \quad g(x) = \frac{1}{x}$

134. $f(x) = \frac{6}{x} \quad and \quad g(x) = \frac{1}{2x+1}$

135. $f(x) = 3x - 7 \quad and \quad g(x) = \frac{x+7}{3}$

136. $f(x) = \frac{2x+3}{x-4} \quad and \quad g(x) = \frac{4x+3}{x-2}$

137. $f(x) = \frac{2x+3}{x+4} \quad and \quad g(x) = \frac{-4x+3}{x-2}$

138. $f(x) = x + 1 \quad and \quad g(x) = x^3 - 5x^2 + 3x + 7$

139. $f(x) = x - 1 \quad and \quad g(x) = x^3 + 2x^2 - 3x - 9$

140. Given that $f(x) = 2x - 5$ and $g(x) = x^2 - 3x + 8$, find $(f \circ g)(x)$, $(g \circ f)(x)$ and their domain then find $(f \circ g)(7)$

141. Given that $f(x) = \sqrt{x}$ and $g(x) = x - 1$, find

a) $(f \circ g)(x) = f(g(x))$

b) $(g \circ f)(x) = g(f(x))$

c) $(f \circ g)(2) = f(g(2))$

142. Given that $f(x)=\frac{x}{x+5}$ and $g(x)=\frac{6}{x}$, find

a) $(f\circ g)(x)=f(g(x))$

b) $(g\circ f)(x)=g(f(x))$

c) $(f\circ g)(2)=f(g(2))$

(**143 – 167**) Determine whether f is even, odd, or neither

143. $f(x)=3x^4+2x^2-5$

144. $f(x)=8x^3-3x^2$

145. $f(x)=\sqrt{x^2+4}$

146. $f(x)=3x^2-5x+1$

147. $f(x)=\sqrt[3]{x^3-x}$

148. $f(x)=|x|-3$

149. $f(x)=x^3-\frac{1}{x}$

150. $f(x)=-x^3+2x$

151. $f(x)=x^5-2x^3$

152. $f(x)=.5x^4-2x^2+6$

153. $f(x)=.75x^2+|x|+4$

154. $f(x)=x^3-x+9$

155. $f(x)=x^4-5x+8$

156. $f(x)=x^3+x$

157. $g(x)=x^2-x$

158. $h(x)=2x^2+x^4$

159. $f(x)=2x^2+x^4+1$

160. $f(x)=\frac{1}{5}x^6-3x^2$

161. $f(x)=x\sqrt{1-x^2}$

162. $f(x)=x^2\sqrt{1-x^2}$

163. $f(x)=5x^7-6x^3-2x$

164. $f(x)=5x^6-3x^2-7$

165. $f(x)=x^2+6$

166. $f(x)=7x^3-x$

167. $h(x)=x^5+1$

168. $f(x)=\begin{cases} 2+x & if \quad x<-4 \\ -x & if \quad -4\le x\le 2 \\ 3x & if \quad x>2 \end{cases}$ Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

169. $f(x)=\begin{cases} -2x & if \quad x<-3 \\ 3x-1 & if \quad -3\le x\le 2 \\ -4x & if \quad x>2 \end{cases}$ Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

170. $f(x)=\begin{cases} x^3+3 & if \quad -2\le x\le 0 \\ x+3 & if \quad 0<x<1 \\ 4+x-x^2 & if \quad 1\le x\le 3 \end{cases}$ Find: $f(-5),\ f(-1),\ f(0),\ and\ f(3)$

171. $h(x)=\begin{cases} \dfrac{x^2-9}{x-3} & if \quad x\ne 3 \\ 6 & if \quad x=3 \end{cases}$ Find: $h(5),\ h(0),\ and\ h(3)$

172. Graph the piecewise function defined by $f(x)=\begin{cases} 3 & if \quad x\le -1 \\ x-2 & if \quad x>-1 \end{cases}$

173. Sketch the graph $f(x)=\begin{cases} x+2 & if \quad x\le -1 \\ x^3 & if \quad -1<x<1 \\ -x+3 & if \quad x\ge 1 \end{cases}$

174. Sketch the graph $f(x)=\begin{cases} x-3 & if \quad x\le -2 \\ -x^2 & if \quad -2<x<1 \\ -x+4 & if \quad x\ge 1 \end{cases}$

Section 1.2 – Polynomial Functions & Graphs

Polynomial Function

A *Polynomial function* $P(x)$ in x is a sum of the form is given by:

$$P(x)=a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

Where the coefficients $a_n,\ a_{n-1},\ \ldots, a_2,\ a_1, a_0$ are real numbers and the exponents are whole numbers.

Degree

$$\underline{a_n\ x^n}$$

Leading Coefficient ***Leading Term***

Degree of f	***Form of f(x)***	***Graph of f(x)***
0	$f(x)=a_0$	A horizontal line
1	$f(x)=a_1 x + a_0$	A line with slope a_1
2	$f(x)=a_2 x^2 + a_1 x + a_0$	A parabola with a vertical axis

All polynomial functions are ***continuous functions***.

End Behavior $\left(a_n x^n\right)$

If n (degree) is ***even***:

If $a_n<0 \ \rightarrow \ \begin{cases} x\to-\infty \ \Rightarrow \ f(x)\to-\infty \\ x\to\infty \ \Rightarrow \ f(x)\to-\infty \end{cases}$

If $a_n>0 \ \rightarrow \ \begin{cases} x\to-\infty \ \Rightarrow \ f(x)\to\infty \\ x\to\infty \ \Rightarrow \ f(x)\to\infty \end{cases}$

If n (degree) is ***odd***:

If $a_n<0 \ \rightarrow \ \begin{cases} x\to-\infty \ \Rightarrow \ f(x)\to\infty \\ x\to\infty \ \Rightarrow \ f(x)\to-\infty \end{cases}$

If $a_n>0 \ \rightarrow \ \begin{cases} x\to-\infty \ \Rightarrow \ f(x)\to-\infty \\ x\to\infty \ \Rightarrow \ f(x)\to\infty \end{cases}$

The intermediate value *Theorem*

For any polynomial function $f(x)$ with real coefficients and $f(a) \neq f(b)$ for $a < b$, then f takes on every value between $f(a)$ and $f(b)$ in the interval $[a,\ b]$.

$\therefore$ $f(a)$ and $f(b)$ are the opposite signs. Then the function has a real zero between a and b.

Example

Using the intermediate value theorem, determine, if possible, whether the function has a real zero between a and b.

a) $f(x) = x^3 + x^2 - 6x; \quad a = -4,\ b = -2$

b) $f(x) = x^3 + x^2 - 6x; \quad a = -1,\ b = 3$

Solution

a) $f(x) = x^3 + x^2 - 6x; \quad a = -4,\ b = -2$

$$f(-4) = (-4)^3 + (-4)^2 - 6(-4)$$
$$= -24$$

$$f(-2) = (-2)^3 + (-2)^2 - 6(-2)$$
$$= 8$$

$\therefore$ $f(x)$ has a zero between -4 and -2.

b) $f(x) = x^3 + x^2 - 6x; \quad a = -1,\ b = 3$

$f(-1) = (-1)^3 + (-1)^2 - 6(-1) \ = 6$

$f(3) = (3)^3 + (3)^2 - 6(3) \ = 18$ *Can't be determined.*

The Rational Zeros *Theorem*

If the polynomial $f(x)=a_n x^n + a_{n-1}x^{n-1}+\dots+a_1 x + a_0$ *has integer* coefficients, then

$$\textit{possible rational zeros} = \frac{\textit{possibilities for } a_0}{\textit{possibilities for } a_n}$$

Example

Find all rational solutions of the equation: $3x^4+14x^3+14x^2-8x-8=0$

Solution

Possibilities: $\pm\left\{\frac{8}{3}\right\} = \pm\left\{\frac{1,\ 2,\ 4,\ 8}{1,\ 3}\right\}$

$$=\pm\left\{1,\ 2,\ 4,\ 8,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3},\ \frac{8}{3}\right\}$$

The calculation will show that -2 is a zero.

$$\begin{array}{r|rrrrr}
-2 & 3 & 14 & 14 & -8 & -8 \\
 & & -6 & -16 & 4 & 8 \\
\hline
-\frac{2}{3} & 3 & 8 & -2 & -4 & 0 \\
 & & -2 & -4 & 4 & \\
\hline
 & 3 & 6 & -6 & 0 &
\end{array}$$

$\to 3x^3+8x^2-2x-4 \to \pm\left\{\frac{4}{3}\right\} = \pm\left\{1,\ 2,\ 4,\ \frac{1}{3},\ \frac{2}{3},\ \frac{4}{3}\right\}$

$\to 3x^2+6x-6=0 \Rightarrow \boxed{x=-1\pm\sqrt{3}}$

Hence, the polynomial has roots $\boxed{x=-2,\ -\frac{2}{3},\ -1\pm\sqrt{3}}$

Sketching

Example

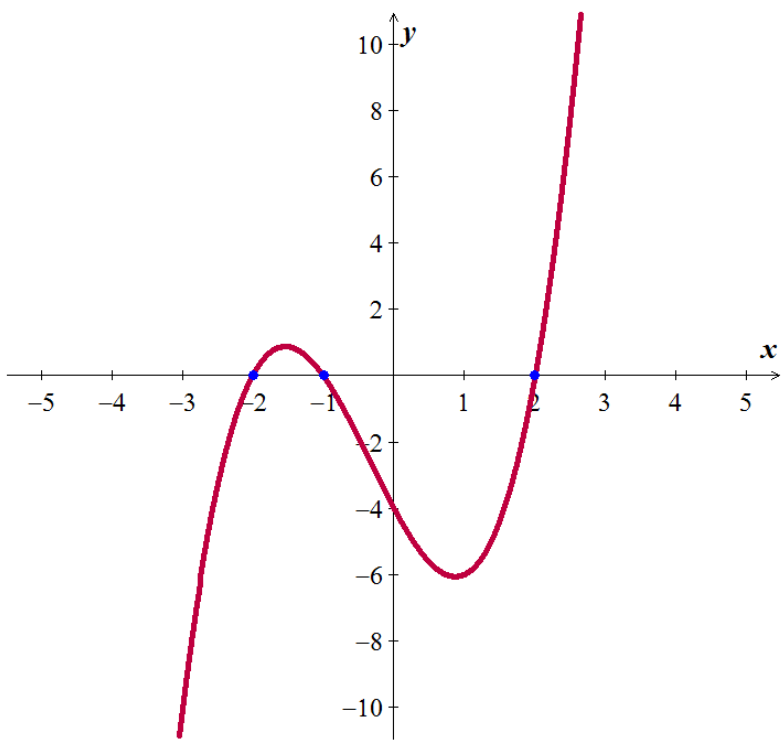
Let $f(x) = x^3 + x^2 - 4x - 4$. Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f.

Solution

$$\begin{aligned} f(x) &= x^3 + x^2 - 4x - 4 \\ &= x^2(x+1) - 4(x+1) \\ &= (x+1)\left(x^2 - 4\right) \\ &= (x+1)(x+2)(x-2) \end{aligned}$$

The zeros of $f(x)$ (x-intercepts) are: $-2,\ -1,\ and\ 2$

<table>
<tr><th>Interval</th><th colspan="4">$-\infty$ -2 -1 0 2 ∞</th></tr>
<tr><td>Sign of $f(x)$</td><td>–</td><td>+</td><td>–</td><td>+</td></tr>
<tr><td>Position</td><td>Below x-axis</td><td>Above x-axis</td><td>Below x-axis</td><td>Above x-axis</td></tr>
</table>



We can conclude from the chart and the graph that:

$$f(x) > 0 \quad \text{if } x \text{ is in } (-2,\ -1) \cup (2,\ \infty)$$

$$f(x) < 0 \quad \text{if } x \text{ is in } (-\infty,\ -2) \cup (-1,\ 2)$$

Example

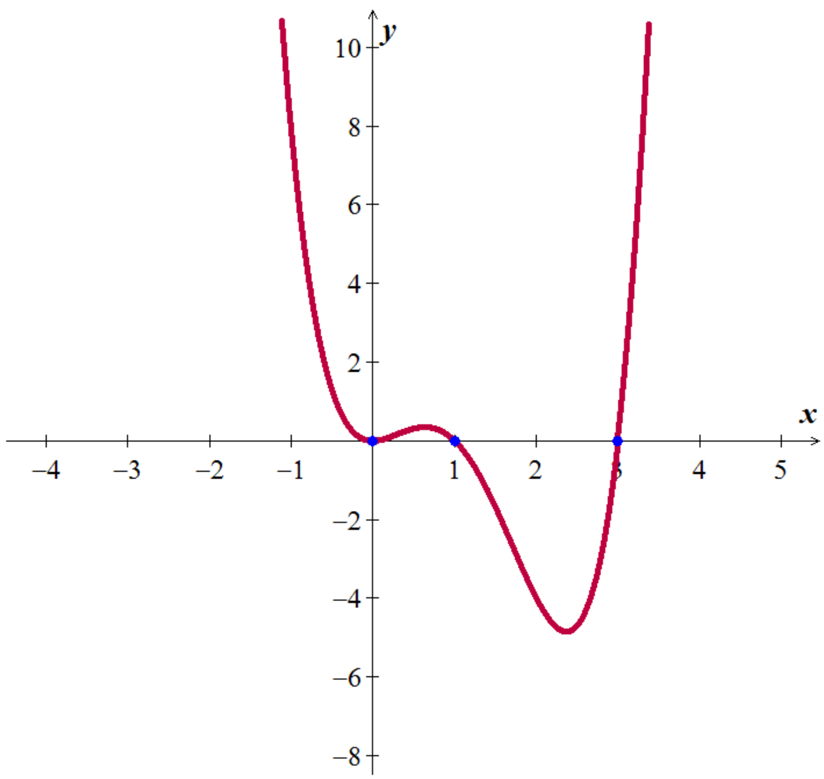
Let $f(x)=x^4-4x^3+3x^2$. Find all values of x such that $f(x)>0$ and all x such that $f(x)<0$, and then sketch the graph of f.

Solution

$$\begin{aligned} f(x) &= x^2\left(x^2-4x+3\right) \\ &= x^2(x-1)(x-3) \end{aligned}$$

The zeros are: 0, 1, 3. Since the factor x^2 is always positive, it has no factor

$-\infty$ 1	2	3 ∞
+	–	+



$$f(x)>0 \quad \Rightarrow \quad x \text{ is in } (-\infty,\ 0)\cup(0,\ 1)\cup(3,\ \infty)$$

$$f(x)<0 \quad \Rightarrow \quad x \text{ is in } (1,\ 3)$$

Exercises *Section* 1.2 – Polynomial Functions & Graphs

(**1 – 4**) Find the quotient and remainder if $f(x)$ is divided by $p(x)$

1. $f(x) = 2x^4 - x^3 + 7x - 12; \quad p(x) = x^2 - 3$

2. $f(x) = 3x^3 + 2x - 4; \quad p(x) = 2x^2 + 1$

3. $f(x) = 7x + 2; \quad p(x) = 2x^2 - x - 4$

4. $f(x) = 9x + 4; \quad p(x) = 2x - 5$

(**5 – 6**) Use the remainder theorem to find $f(c)$

5. $f(x) = x^4 - 6x^2 + 4x - 8; \quad c = -3$

6. $f(x) = x^4 + 3x^2 - 12; \quad c = -2$

7. Use the factor theorem to show that $x - c$ is a factor of $f(x)$: $f(x) = x^3 + x^2 - 2x + 12; \quad c = -3$

(**8 – 10**) Use the synthetic division to find the quotient and remainder if the first polynomial is divided by the second

8. $2x^3 - 3x^2 + 4x - 5; \quad x - 2$

9. $5x^3 - 6x^2 + 15; \quad x - 4$

10. $9x^3 - 6x^2 + 3x - 4; \quad x - \frac{1}{3}$

(**11 – 13**) Use the synthetic division to find $f(c)$

11. $f(x) = 2x^3 + 3x^2 - 4x + 4; \quad c = 3$

12. $f(x) = 8x^5 - 3x^2 + 7; \quad c = \frac{1}{2}$

13. $f(x) = x^3 - 3x^2 - 8; \quad c = 1 + \sqrt{2}$

14. Use the synthetic division to show that c is a zero of $f(x)$: $f(x) = 3x^4 + 8x^3 - 2x^2 - 10x + 4; \quad c = -2$

15. Use the synthetic division to show that c is a zero of $f(x)$: $f(x) = 27x^4 - 9x^3 + 3x^2 + 6x + 1; \quad c = -\frac{1}{3}$

(**16 – 18**) Find all values of k such that $f(x)$ is divisible by the given linear polynomial:

16. $f(x) = kx^3 + x^2 + k^2x + 3k^2 + 11; \quad x + 2$

17. $f(x) = x^3 + k^3x^2 + +2kx - 2k^4; \quad x - 1.6$

18. $f(x) = k^2x^3 - 4kx + 3; \quad x - 1$

(**19 – 30**) Find all solutions of the equation

19. $x^3 - x^2 - 10x - 8 = 0$

20. $x^3 + x^2 - 14x - 24 = 0$

21. $2x^3 - 3x^2 - 17x + 30 = 0$

22. $12x^3 + 8x^2 - 3x - 2 = 0$

23. $x^4 + 3x^3 - 30x^2 - 6x + 56 = 0$

24. $3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6 = 0$

25. $6x^5 + 19x^4 + x^3 - 6x^2 = 0$

26. $x^4 - x^3 - 9x^2 + 3x + 18 = 0$

27. $2x^4 - 9x^3 + 9x^2 + x - 3 = 0$

28. $8x^3 + 18x^2 + 45x + 27 = 0$

29. $3x^3 - x^2 + 11x - 20 = 0$

30. $6x^4 + 5x^3 - 17x^2 - 6x = 0$

31. If $f(x) = 3x^3 - kx^2 + x - 5k$, find a number k such that the graph of f contains the point (–1, 4).

32. If $f(x) = kx^3 + x^2 - kx + 2$, find a number k such that the graph of f contains the point (2, 12).

33. If one zero of $f(x) = x^3 - 2x^2 - 16x + 16k$ is 2, find two other zeros.

34. If one zero of $f(x) = x^3 - 3x^2 - kx + 12$ is –2, find two other zeros.

35. Find a polynomial $f(x)$ of degree 3 that has the zeros –1, 2, 3; and satisfies the given condition: $f(-2) = 80$

36. Find a polynomial $f(x)$ of degree 3 that has the zeros $-2i$, $2i$, 3; and satisfies the given condition: $f(1) = 20$

37. Find a polynomial $f(x)$ of degree 4 with leading coefficient 1 such that both –4 and 3 are zeros of multiplicity 2, and sketch the graph of f.

(**38 – 43**) Find the zeros of the following functions and state the multiplicity of each zero

38. $f(x) = x^2(3x+2)(2x-5)^3$

39. $f(x) = 4x^5 + 12x^4 + 9x^3$

40. $f(x) = \left(x^2 + x - 12\right)^3\left(x^2 - 9\right)^2$

41. $f(x) = \left(6x^2 + 7x - 5\right)^4\left(4x^2 - 1\right)^2$

42. $f(x) = x^4 + 7x^2 - 144$

43. $f(x) = x^4 + 21x^2 - 100$

(**44 – 102**) Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and then sketch the graph of f

44. $f(x) = x^4 - 4x^2$

45. $f(x) = x^4 + 3x^3 - 4x^2$

46. $f(x) = x^3 + 2x^2 - 4x - 8$

47. $f(x) = x^3 - 3x^2 - 9x + 27$

48. $f(x) = -x^4 + 12x^2 - 27$

49. $f(x) = x^2(x+2)(x-1)^2(x-2)$

50. $f(x) = 2x^3 + 11x^2 - 7x - 6$

51. $f(x) = x^3 + 2x^2 - 5x - 6$

52. $f(x) = x^3 + 8x^2 + 11x - 20$

53. $f(x) = x^4 + x^2 - 2$

54. $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$

55. $f(x) = 4x^5 - 8x^4 - x + 2$

56. $f(x) = 2x^4 - x^3 - 5x^2 + 2x + 2$

57. $f(x) = x^3 - x^2 - 10x - 8$

58. $f(x) = x^3 + x^2 - 14x - 24$

59. $f(x) = 2x^3 - 3x^2 - 17x + 30$

60. $f(x) = 12x^3 + 8x^2 - 3x - 2$

61. $f(x) = x^3 + x^2 - 6x - 8$

62. $f(x) = x^3 - 19x - 30$

63. $f(x) = 2x^3 + x^2 - 25x + 12$

64. $f(x) = 3x^3 + 11x^2 - 6x - 8$

65. $f(x) = 2x^3 + 9x^2 - 2x - 9$

66. $f(x) = x^3 + 3x^2 - 6x - 8$

67. $f(x) = 3x^3 - x^2 - 6x + 2$

68. $f(x) = x^3 - 8x^2 + 8x + 24$

69. $f(x) = x^3 - 7x^2 - 7x + 69$

70. $f(x) = x^3 - 3x - 2$

71. $f(x) = x^3 - 2x + 1$

72. $f(x) = x^3 - 2x^2 - 11x + 12$

73. $f(x) = x^3 - 2x^2 - 7x - 4$

74. $f(x) = x^3 - 10x - 12$

75. $f(x) = x^3 - 5x^2 + 17x - 13$

76. $f(x) = 6x^3 + 25x^2 - 24x + 5$

77. $f(x) = 8x^3 + 18x^2 + 45x + 27$

78. $f(x) = 3x^3 - x^2 + 11x - 20$

79. $f(x) = x^4 - x^3 - 9x^2 + 3x + 18$

80. $f(x) = 2x^4 - 9x^3 + 9x^2 + x - 3$

81. $f(x) = 6x^4 + 5x^3 - 17x^2 - 6x$

82. $f(x) = x^4 - 2x^2 - 16x - 15$

83. $f(x) = x^4 - 2x^3 - 5x^2 + 8x + 4$

84. $f(x) = 2x^4 - 17x^3 + 4x^2 + 35x - 24$

85. $f(x) = x^4 + x^3 - 3x^2 - 5x - 2$

86. $f(x) = 6x^4 - 17x^3 - 11x^2 + 42x$

87. $f(x) = x^4 - 5x^2 - 2x$

88. $f(x) = 3x^4 - 4x^3 - 11x^2 + 16x - 4$

89. $f(x) = 6x^4 + 23x^3 + 19x^2 - 8x - 4$

90. $f(x) = 4x^4 - 12x^3 + 3x^2 + 12x - 7$

91. $f(x) = 2x^4 - 9x^3 - 2x^2 + 27x - 12$

92. $f(x) = 2x^4 - 19x^3 + 51x^2 - 31x + 5$

93. $f(x) = 4x^4 - 35x^3 + 71x^2 - 4x - 6$

94. $f(x) = 2x^4 + 3x^3 - 4x^2 - 3x + 2$

95. $f(x) = x^4 + 3x^3 - 30x^2 - 6x + 56$

96. $f(x) = 3x^5 - 10x^4 - 6x^3 + 24x^2 + 11x - 6$

97. $f(x) = 6x^5 + 19x^4 + x^3 - 6x^2$

98. $f(x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

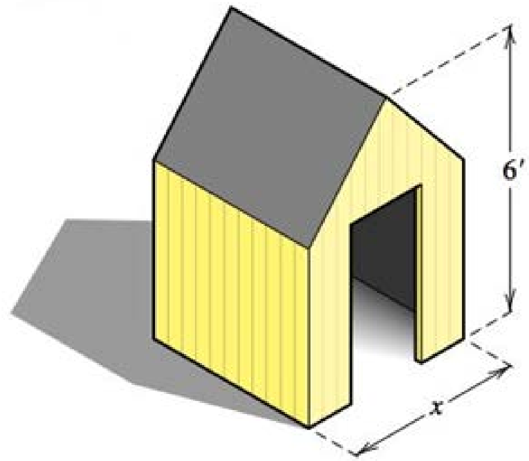
99. $f(x) = x^5 - x^4 - 7x^3 + 7x^2 + 12x - 12$

100. $f(x) = x^5 - 2x^3 - 8x$

101. $f(x) = x^5 - 7x^4 + 19x^3 - 37x^2 + 60x - 36$

102. $f(x) = 3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24$

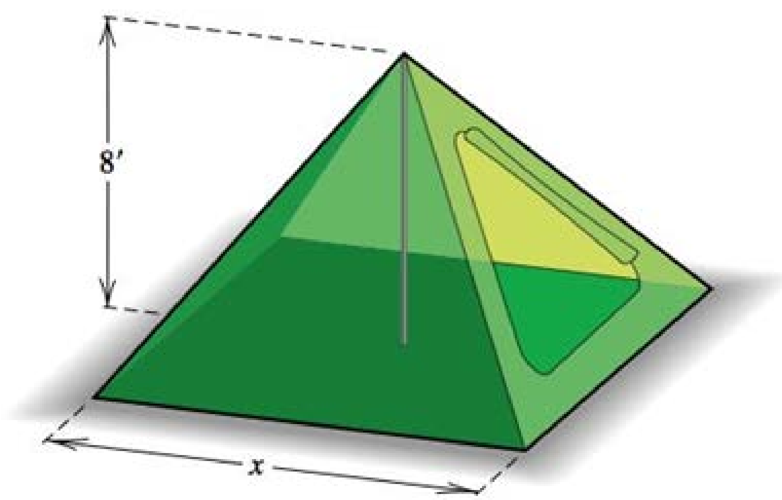
103. A storage shelter is to be constructed in the shape of a cube with a triangular prism forming the roof. The length x of a side of the cube is yet to be determined.



a) If the total height of the structure is 6 *feet*, show that its volume V is given by $V = x^3 + \frac{1}{2}x^2(6 - x)$

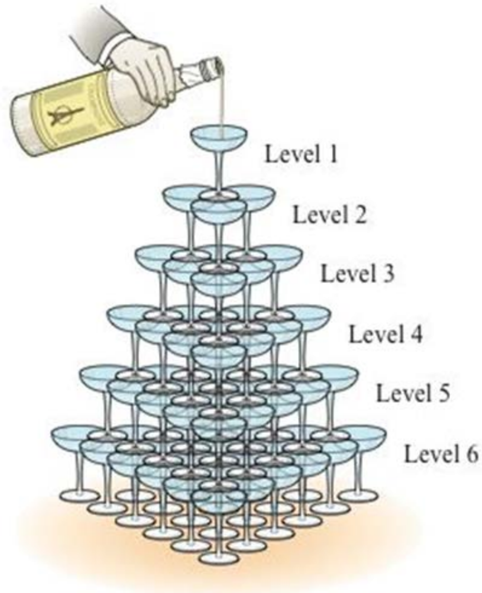
b) Determine x so that the volume is $80\ ft^3$

104. A canvas camping tent is to be constructed in the shape of a pyramid with a square base. An 8–*foot* pole will form the center support. Find the length x of a side of the base so that the total amount of canvas needed for the sides and bottom is $384\ ft^2$



105. Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

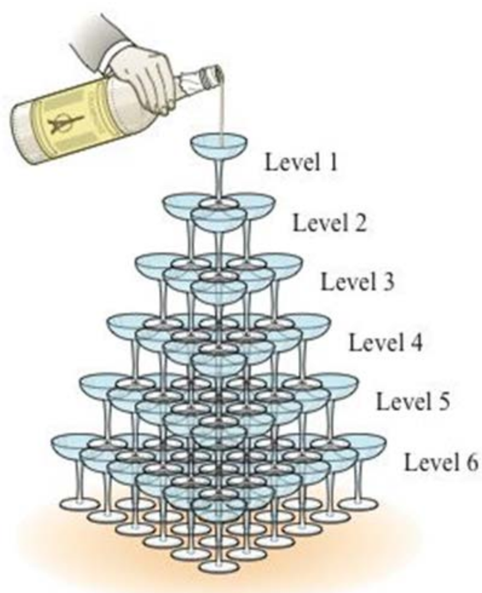
$$T(k)=\frac{1}{6}\left(k^3+3k^2+2k\right)$$



Where k is the number of levels in the pyramid. If 220 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

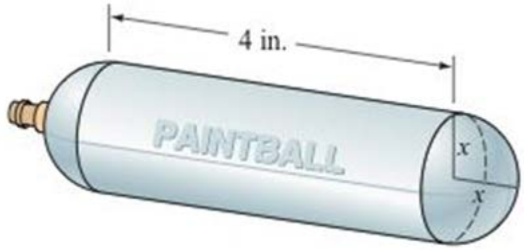
106. Glasses can be stacked to form a triangular pyramid. The total number of glasses in one of these pyramids is given by

$$T(k)=\frac{1}{6}\left(2k^3+3k^2+k\right)$$



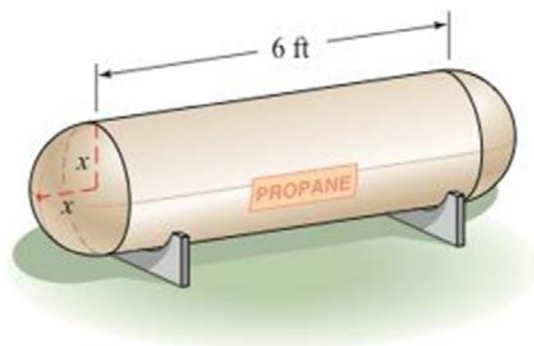
Where k is the number of levels in the pyramid. If 140 glasses are used to form a triangle pyramid, how many levels are in the pyramid?

107. A carbon dioxide cartridge for a paintball rifle has the shape of a right circular cylinder with a hemisphere at each end. The cylinder is 4 *inches* long, and the volume of the cartridge is $2\pi \ in^3$.

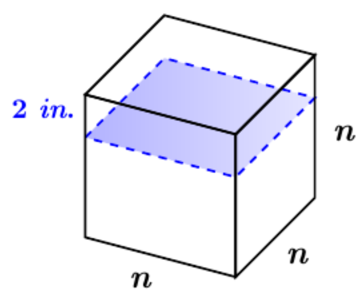


The common interior radius of the cylinder and the hemispheres is denoted by x. Estimate the length of the radius x.

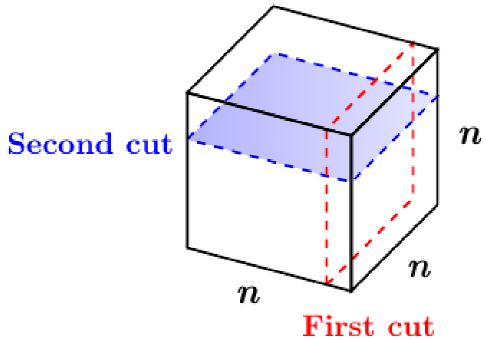
108. A propane tank has the shape of a circular cylinder with a hemisphere at each end. The cylinder is 6 *feet* long and the volume of the tank is $9\pi \ ft^3$. Find the length of the radius x.



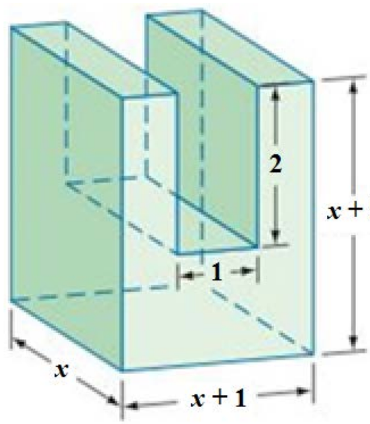
109. A cube measures n inches on each edge. If a slice 2 *inches* thick is cut from one face of the cube, the resulting solid has a volume of $567 \ in^3$. Find n.



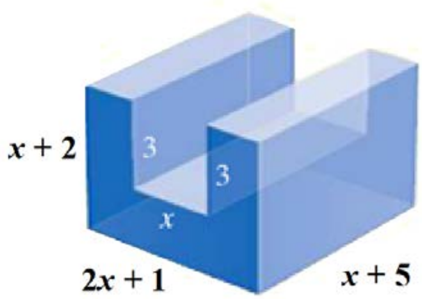
110. A cube measures n inches on each edge. If a slice 1 *inch* thick is cut from one face of the cube and then a slice 3 inches thick is cut from another face of the cube, the resulting solid has a volume of $1560 \ in^3$. Find the dimensions of the original cube.



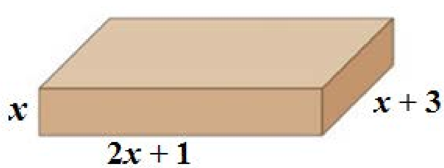
111. For what value of x will the volume of the following solid be $112\ in^3$



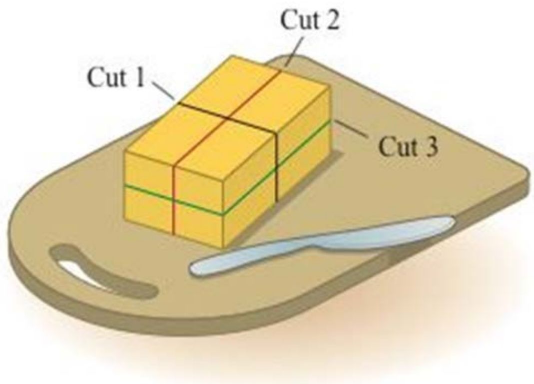
112. For what value of x will the volume of the following solid be $208\ in^3$



113. The length of rectangular box is 1 *inch* more than twice the height of the box, and the width is 3 *inches* more than the height. If the volume of the box is $126\ in^3$, find the dimensions of the box.



114. One straight cut through a thick piece of cheese produces two pieces. Two straight cuts can produce a maximum of four pieces. Two straight cuts can produce a maximum of four pieces. Three straight cuts can produce a maximum of eight pieces.



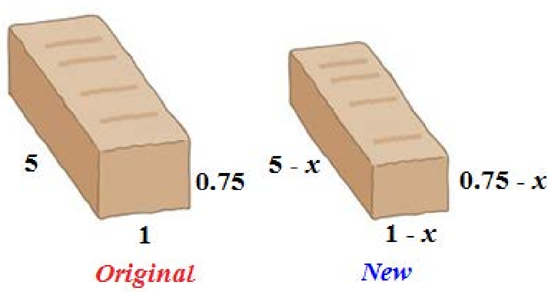
You might be inclined to think that every additional cut doubles number of pieces. However, for four straight cuts, you get a maximum of 15 pieces. The maximum number of pieces P that can be produced by n straight cuts is given by

$$P(n) = \frac{n^3 + 5n + 6}{6}$$

a) Determine number of pieces that can be produces by five straight cuts.
b) What is the fewest number of straight cuts that are needed to produce 64 pieces?

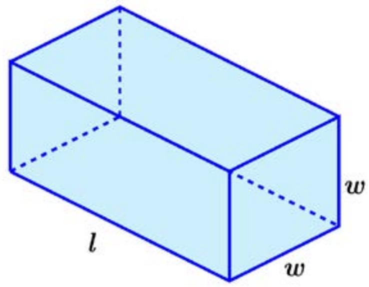
115. The number of ways one can select three cards from a group of n cards (the order of the selection matters), where $n \ge 3$, is given by $P(n) = n^3 - 3n^2 + 2n$. For a certain card trick, a magician has determined that there are exactly 504 *ways* to choose three cards from a given group. How many cards are in the group?

116. A nutrition bar in the shape of a rectangular solid measure 0.75 *in*. by 1 *in*. by 5 *inches*.



To reduce costs, the manufacturer has decided to decrease each of the dimensions of the nutrition bar by x *inches*, what value of x will produce a new bar with a volume that is $0.75\ in^3$ less than the present bar's volume.

117. A rectangular box is square on two ends and has length plus girth of 81 *inches*. (Girth: distance *around* the box). Determine the possible lengths $l\ (l > w)$ of the box if its volume is $4900\ \ in^3$.



Section 1.3 – Rational Functions

A function f is a ***rational function*** if $f(x)=\dfrac{g(x)}{h(x)}$,

Where $g(x)$ and $h(x)$ are polynomials. The domain of f consists of all real numbers ***except*** the zeros of the denominator $h(x)$.

Notation	***Terminology***
$x \to a^-$	x approaches a from the left (through values ***less*** than a)
$x \to a^+$	x approaches a from the right (through values ***greater*** than a)
$f(x) \to \infty$	$f(x)$ increases without bound (can be made as large positive as desired)
$f(x) \to -\infty$	$f(x)$ decreases without bound (can be made as large negative as desired)

The Domain of a Rational Function

Example

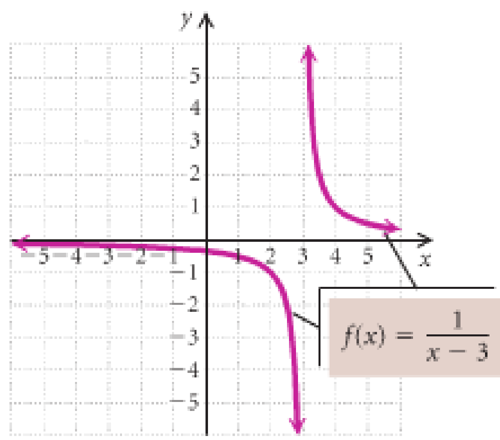
Consider: $f(x)=\dfrac{1}{x-3}$

Find the domain and graph *f*.

Solution

$x-3=0 \quad \Rightarrow \quad \boxed{x=3}$

Thus the domain is: $\{x \mid x \neq 3\}$ *or* $(-\infty,\ 3)\cup(3,\ \infty)$



Function	***Domain***	
$f(x)=\dfrac{1}{x}$	$\{x \mid x \neq 0\}$	$(-\infty,\ 0)\cup(0,\ \infty)$
$f(x)=\dfrac{1}{x^2}$	$\{x \mid x \neq 0\}$	$(-\infty,\ 0)\cup(0,\ \infty)$
$f(x)=\dfrac{x-3}{x^2+x-2}$	$\{x \mid x \neq -2 \text{ and } x \neq 1\}$	$(-\infty,\ -2)\cup(-2,\ 1)\cup(1,\ \infty)$
$f(x)=\dfrac{2x+5}{2x-6}=\dfrac{2x+5}{2(x-3)}$	$\{x \mid x \neq 3\}$	$(-\infty,\ 3)\cup(3,\ \infty)$

Asymptotes

Vertical Asymptote (*VA*) - *Think Domain*

The line $x = a$ is a ***vertical asymptote*** for the graph of a function f if

$$f(x) \to \infty \quad or \quad f(x) \to -\infty$$

As ***x*** approaches ***a*** from either the left or the right

When the denominator and the numerator have both 0, then both the numerator and denominator can be factored by using $(x - a)$ and can be cancelled out. This means there is a ***hole*** in the function at this point.

Horizontal Asymptote (*HA*)

The line $y = c$ is a ***horizontal asymptote*** for the graph of a function f if

$$f(x) \to c \quad as \quad x \to -\infty \quad or \quad x \to -\infty$$

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0} = \dfrac{a_n x^n}{b_m x^m}$ be a rational function.

1. If the degree of numerator is less than of denominator $(n < m) \Rightarrow y = 0$

   $$y = \frac{2x+1}{4x^2+5} \quad \Rightarrow \quad \underline{y = 0}$$

2. If the degree of numerator is equal of denominator $(n = m) \Rightarrow y = \dfrac{a_n}{b_m}$

   $$y = \frac{2x^2+1}{4x^2+5} \quad \Rightarrow \quad y = \frac{2}{4} = \frac{1}{2}$$

3. If the degree of numerator is greater than of denominator $(n > m) \Rightarrow$ No horizontal asymptote

   $$y = \frac{2x^3+1}{4x^2+5} \quad \Rightarrow No \;\; HA$$

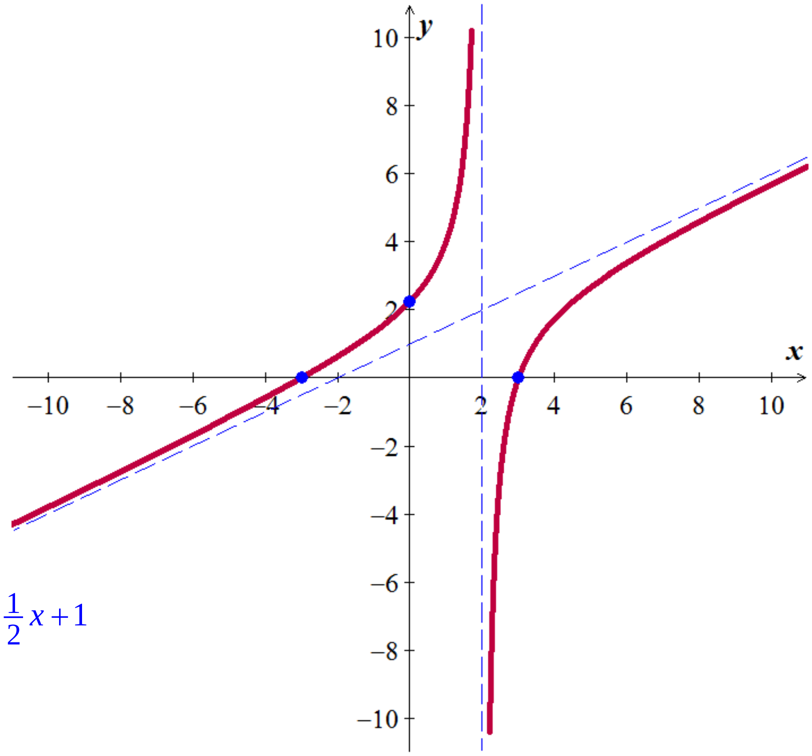
   If ***no*** Horizontal Asymptote, then there is an ***Oblique Asymptote.***

Slant or *Oblique* Asymptotes

When the degree of the numerator is one greater than the degree of the numerator, the graph has a slant or oblique asymptote and it is a line $y = ax + b, \quad a \neq 0$. To find the slant asymptote, divide the fraction using long division. The quotient (not remainder) is the slant asymptote.

Example

Find all the asymptotes and sketch the graph of f if $f(x)=\dfrac{x^2-9}{2x-4}$



Solution

$$\begin{array}{r|l} & \tfrac{1}{2}x+1 \\ \hline 2x-4 & x^2 \qquad -9 \\ & \underline{x^2-2x} \\ & \qquad 2x-9 \\ & \qquad \underline{2x-4} \\ & \qquad\quad \boxed{-5} \end{array}$$

$$f(x)=\left(\tfrac{1}{2}x+1\right)-\frac{5}{2x-4}$$

VA: $x=2$ ***HA***: n/a

Hole: n/a ***Oblique asymptote***: $y=\frac{1}{2}x+1$

x	***y***
0	$\frac{9}{4}$
± 3	0

Example

Find the vertical asymptote of $f(x)=\dfrac{1}{x-2}$, and sketch the graph.

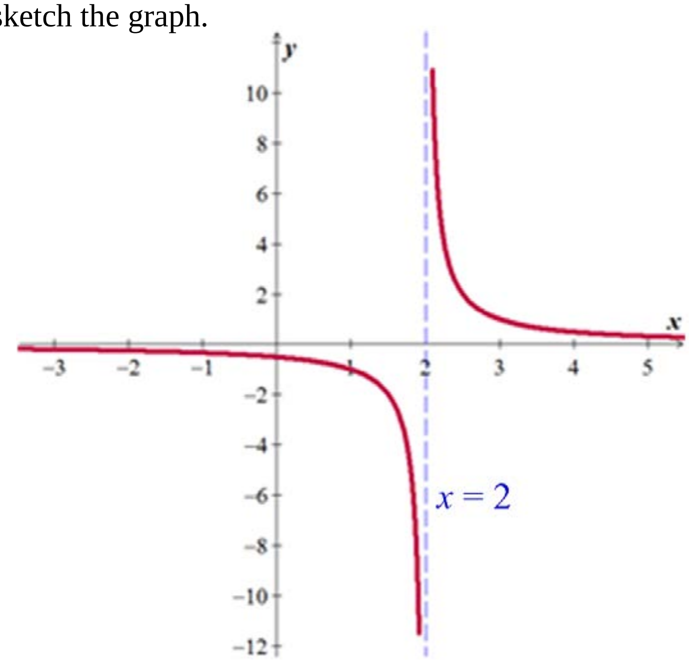
Solution

VA: $x=2$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

$f(x)\to\infty \quad as \quad x\to 2^+$

$f(x)\to-\infty \quad as \quad x\to 2^-$



Example

Sketch the graph of g if $g(x)=\dfrac{3x^2+x-4}{2x^2-7x+5}$

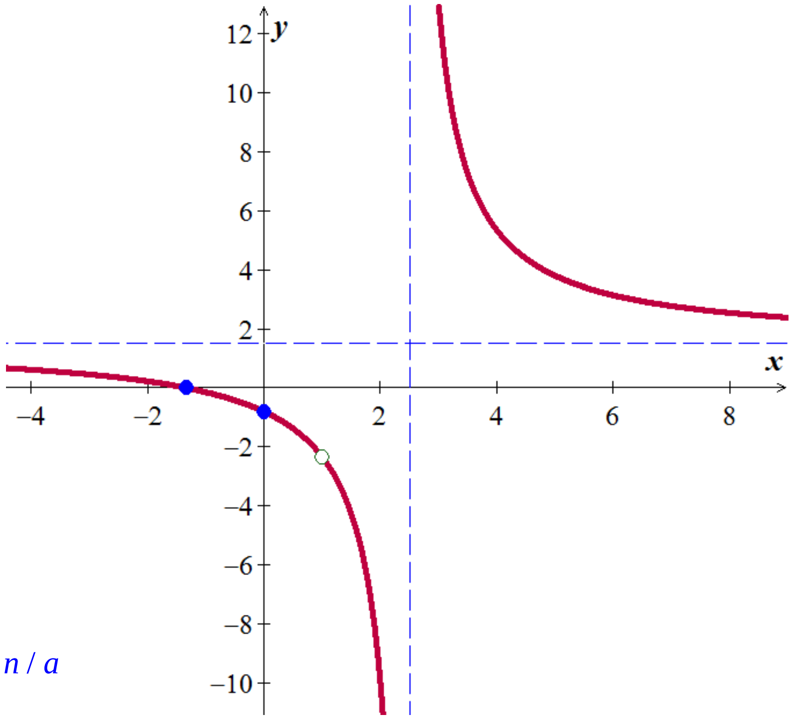
Solution

$$g(x)=\frac{(3x+4)(x-1)}{(2x-5)(x-1)}$$
$$=\frac{3x+4}{2x-5}$$

$$f(x)=\frac{3x+4}{2x-5}$$

g has a hole at $x=1$

$$f(1)=-\frac{7}{3}$$



VA: $x=\frac{5}{2}$ ***HA***: $y=0$

Hole: $\left(1,\ -\frac{7}{3}\right)$ ***Oblique asymptote***: n/a

Example

Find all asymptotes for the graph of f, if it exists

a) $f(x)=\dfrac{3x-1}{x^2-x-6}$ *b)* $f(x)=\dfrac{5x^2+1}{3x^2-4}$ *c)* $f(x)=\dfrac{2x^4-3x^2+5}{x^2+1}$

Solution

a) $f(x)=\dfrac{3x-1}{x^2-x-6}$

VA: $x=-2,\quad x=3$ ***HA***: $y=0$

Hole: n/a ***Oblique asymptote***: n/a

b) $f(x)=\dfrac{5x^2+1}{3x^2-4}$

$$3x^2-4=0\ \rightarrow\ 3x^2=4\ \rightarrow\ x^2=\frac{4}{3}\ \rightarrow\ \boxed{x=\pm\frac{2}{\sqrt{3}}}$$

VA: $x=\pm\frac{2}{\sqrt{3}}$ ***HA***: $y=\frac{5}{3}$

Hole: n/a ***Oblique asymptote***: n/a

c) $f(x)=\dfrac{2x^4-3x^2+5}{x^2+1}$

VA: n/a ***HA***: n/a

Hole: n/a ***Oblique asymptote***: $y=2x^2-5$

$$\begin{array}{r} 2x^2 \quad -5 \\ x^2+1\,\overline{)\,2x^4-3x^2+5} \\ \underline{-2x^4-2x^2} \quad\;\; \\ -5x^2+5 \end{array}$$

Example

Sketch the graph of f if $f(x)=\dfrac{3x+4}{2x-5}$

Solution

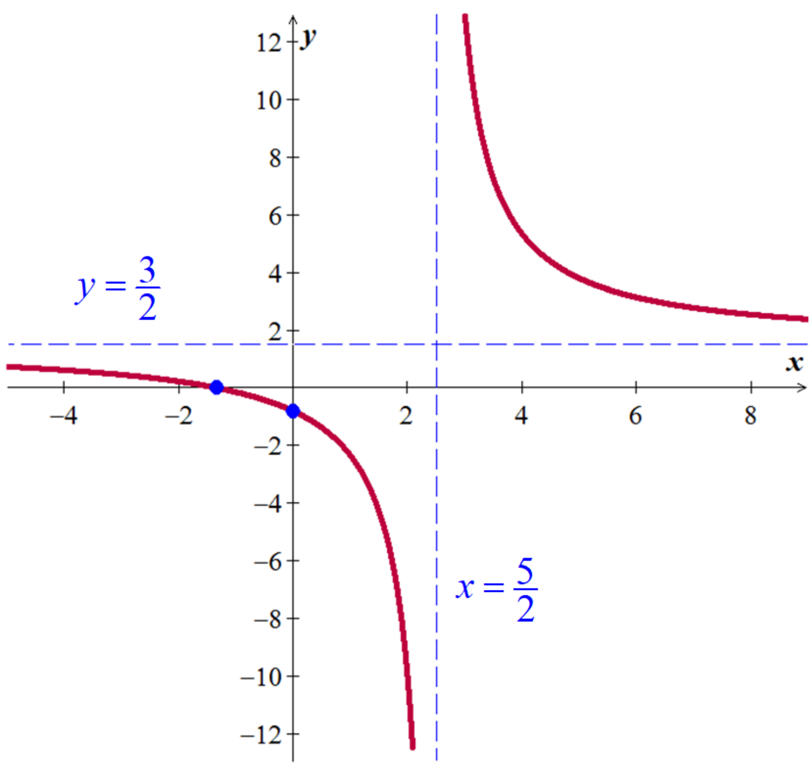
VA: $x=\frac{5}{2}$

HA: $y=-\frac{5}{3}$

Hole: n/a

Oblique asymptote: n/a

x	y
0	$-\frac{4}{5}$
$-\frac{4}{3}$	0
4	5.3



Example

Sketch the graph of f if $f(x) = \dfrac{x^2}{x^2 - x - 2}$

Solution

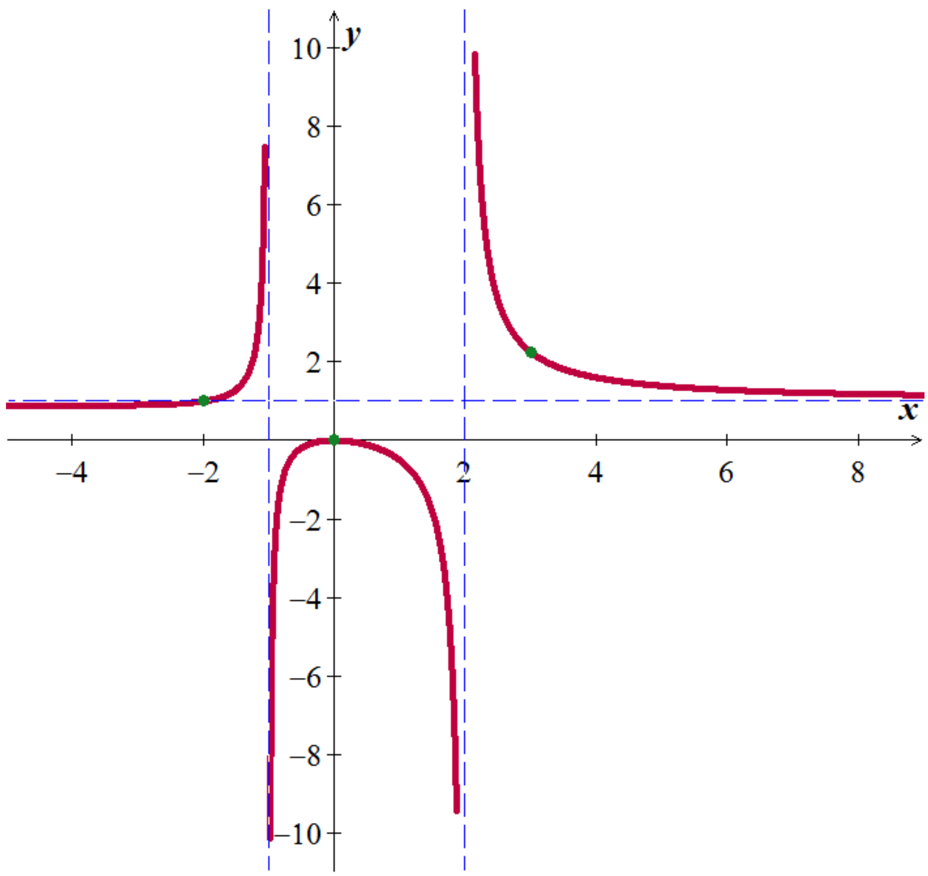
VA: $x = -1,\ 2$

HA: $y = 1$

Hole: n/a

Oblique asymptote: n/a

x	y
0	0
–4	0.88
–2	1
3	$\frac{9}{4}$



Example

Sketch the graph of f if $f(x)=\dfrac{x-1}{x^2-x-6}$

Solution

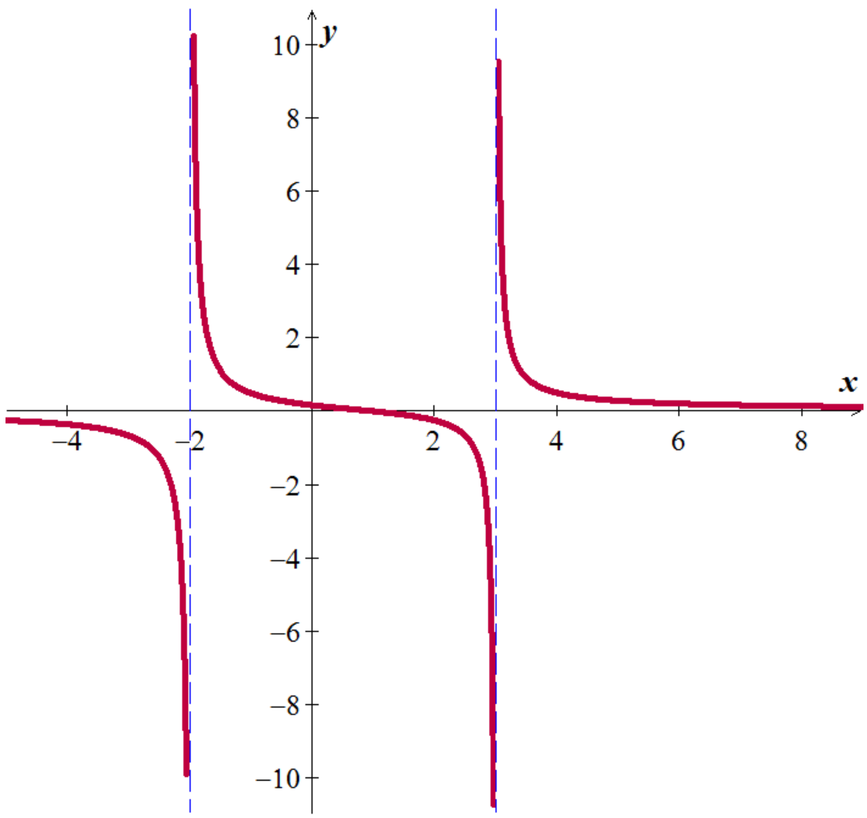
VA: $x=-2,\ 3$

HA: $y=0$

Hole: n/a

Oblique asymptote: n/a

x	***y***
–4	–.36
–3	–.67
0	$\frac{1}{6}$
1	0
4	.5
5	$\frac{2}{7}$



Exercises *Section* 1.3 – Rational Functions

(1 – 21) Determine all asymptotes of the function

1. $y=\dfrac{3x}{1-x}$

2. $y=\dfrac{x^2}{x^2+9}$

3. $y=\dfrac{x-2}{x^2-4x+3}$

4. $y=\dfrac{3}{x-5}$

5. $y=\dfrac{x^3-1}{x^2+1}$

6. $y=\dfrac{3x^2-27}{(x+3)(2x+1)}$

7. $y=\dfrac{x^3+3x^2-2}{x^2-4}$

8. $y=\dfrac{x-3}{x^2-9}$

9. $y=\dfrac{6}{\sqrt{x^2-4x}}$

10. $y=\dfrac{5x-1}{1-3x}$

11. $f(x)=\dfrac{2x-11}{x^2+2x-8}$

12. $f(x)=\dfrac{x^2-4x}{x^3-x}$

13. $f(x)=\dfrac{x-2}{x^3-5x}$

14. $f(x)=\dfrac{4x}{x^2+10x}$

15. $f(x)=\dfrac{3-x}{(x-4)(x+6)}$

16. $f(x)=\dfrac{x^3}{2x^3-x^2-3x}$

17. $f(x)=\dfrac{3x^2+5}{4x^2-3}$

18. $f(x)=\dfrac{x+6}{x^3+2x^2}$

19. $f(x)=\dfrac{x^2+4x-1}{x+3}$

20. $f(x)=\dfrac{x^2-6x}{x-5}$

21. $f(x)=\dfrac{x^3-x^2+x-4}{x^2+2x-1}$

(22 – 53) Determine all asymptotes (if any) (*Vertical Asymptote*, *Horizontal Asymptote*; *Hole*; *Oblique Asymptote*) and sketch the graph of

22. $f(x)=\dfrac{-3x}{x+2}$

23. $f(x)=\dfrac{x+1}{x^2+2x-3}$

24. $f(x)=\dfrac{2x^2-2x-4}{x^2+x-12}$

25. $f(x)=\dfrac{-2x^2+10x-12}{x^2+x}$

26. $f(x)=\dfrac{x^2-x-6}{x+1}$

27. $f(x)=\dfrac{x^3+1}{x-2}$

28. $f(x)=\dfrac{2x^2+x-6}{x^2+3x+2}$

29. $f(x)=\dfrac{x-1}{1-x^2}$

30. $f(x)=\dfrac{x^2+x-2}{x+2}$

31. $f(x)=\dfrac{x^3-2x^2-4x+8}{x-2}$

32. $f(x)=\dfrac{2x^2-3x-1}{x-2}$

33. $f(x)=\dfrac{2x+3}{3x^2+7x-6}$

34. $f(x)=\dfrac{x^2-1}{x^2+x-6}$

35. $f(x)=\dfrac{-2x^2-x+15}{x^2-x-12}$

36. $f(x)=\dfrac{1}{x-3}$

37. $f(x)=\dfrac{-2}{x+3}$

38. $f(x)=\dfrac{x}{x+2}$

39. $f(x)=\dfrac{x-5}{x+4}$

40. $f(x)=\dfrac{2x^2-2}{x^2-9}$

41. $f(x)=\dfrac{x^2-3}{x^2+4}$

42. $f(x)=\dfrac{x^2+4}{x^2-3}$

43. $f(x)=\dfrac{x^2}{x^2-6x+9}$

44. $f(x)=\dfrac{x^2+x+4}{x^2+2x-1}$

45. $f(x)=\dfrac{2x^2+14}{x^2-6x+5}$

46. $f(x)=\dfrac{x^2-4x-5}{2x+5}$

47. $f(x)=\dfrac{x-3}{x^2-3x+2}$

48. $f(x)=\dfrac{x^2+2}{x^2+3x+2}$

49. $f(x)=\dfrac{x-2}{x^2-3x+2}$

50. $f(x)=\dfrac{x^2+x}{x+1}$

51. $f(x)=\dfrac{x^2-2x}{x-2}$

52. $f(x)=\dfrac{x^2-3x}{x+3}$

53. $f(x)=\dfrac{x^3+3x^2-4x+6}{x+2}$

(**54 – 59**) Find an equation of a rational function f that satisfies the given conditions

54. $\begin{cases} \textit{vertical asymptote}:\ x=4 \\ \textit{horizontal asymptote}:\ y=-1 \\ x-\textit{intercept}:\ 3 \end{cases}$

55. $\begin{cases} \textit{vertical asymptote}:\ x=-4, x=5 \\ \textit{horizontal asymptote}:\ y=\dfrac{3}{2} \\ x-\textit{intercept}:\ -2 \end{cases}$

56. $\begin{cases} \textit{vertical asymptote}:\ x=5 \\ \textit{horizontal asymptote}:\ y=-1 \\ x-\textit{intercept}:\ 2 \end{cases}$

57. $\begin{cases} \textit{vertical asymptote}:\ x=-2,\ x=0 \\ \textit{horizontal asymptote}:\ y=0 \\ x-\textit{intercept}:\ 2,\quad f(3)=1 \end{cases}$

58. $\begin{cases} \textit{vertical asymptote}:\ x=-3,\ \ x=1 \\ \textit{horizontal asymptote}:\ y=0 \\ x-\textit{intercept}:\ -1,\ \ f(0)=-2 \\ \textit{hole at}\ \ x=2 \end{cases}$

59. $\begin{cases} \textit{vertical asymptote}:\ x=-1,\ \ x=3 \\ \textit{horizontal asymptote}:\ y=2 \\ x-\textit{intercept}:\ -2,\ \ 1 \\ \textit{hole}:\ \ x=0 \end{cases}$

Section 1.4 – Inverse, Exponential & Logarithmic Functions

One-to-One Function

A function f is one-to-one (1 – 1) if different inputs have different outputs that is,

$$\text{if } a \neq b, \quad \text{then} \quad f(a) \neq f(b)$$

Or $$\text{if } f(a) = f(b), \quad \text{then } a = b$$

Definition of Inverse Function

Let f be one-to-one function with domain *D* and range *R*. A function g with domain *R* and range *D* is the ***inverse function*** of f, provided the following condition is true for every *x* in *D* and every *y* in *R*:

$$y = f(x) \quad \text{iff} \quad x = g(y)$$

If the inverse of a function f is also a function, it is named f^{-1} read “ f – inverse”

The -1 in f^{-1} is not an exponent! And is not equal to $\cancel{\dfrac{1}{f(x)}}$

***Domain* and *Range* of f and f^{-1}**

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

Example

For the given function $f(x) = \dfrac{2x+3}{x+5}$

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) $f(a) = f(b)$

$$\frac{2a+3}{a+5} = \frac{2b+3}{b+5}$$

$$2ab + 10a + 3b + 15 = 2ab + 10b + 3a + 15$$

$$7a = 7b$$

$$a = b \quad \checkmark$$

$f(x)$ is $1-1$

b) $y=\dfrac{2x+3}{x+5}$

$xy+5y=2x+3$

$x(y-2)=3-5y$

$x=\dfrac{-5y+3}{y-2}$

$\boxed{f^{-1}(x)=\dfrac{-5x+3}{x-2}}$

c) Domain of $f(x)$ = Range of $f^{-1}(x)$: $\mathbb{R}-\{-5\}$

Range of $f(x)$ = Domain of $f^{-1}(x)$: $\mathbb{R}-\{2\}$

Definition (Exponential Functions)

The exponential function f with base $\boldsymbol{b}$ is defined by

$$f(x)=b^x \qquad \text{or} \qquad y=b^x$$

Base

where $b>0$, $b\neq 1$ and $\boldsymbol{x}$ is any real number.

Graphing Exponential

1. Define the Horizontal Asymptote $f(x)=b^x \pm d$

   $y=0\pm \text{d}$

   The exponential function always equals to 0

   $x\to\infty$ *or* $x\to-\infty \Rightarrow f(x)\to 0$

2. Define/Make a table

   (Force your exponential to = 0, then solve for x)

x	**f(x)**
$x-1$	
$\boldsymbol{x}$	
$x+1$	

Domain: $(-\infty,\infty)$

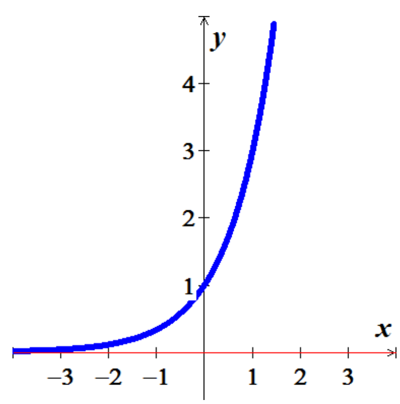
Range: (d,∞)

Example

$f(x)=3^x$

Asymptote: $y=0$

x	$f(x)$
-1	1/3
0	1
1	3



Example

Sketch $f(x)=3^{x-2}$

Solution

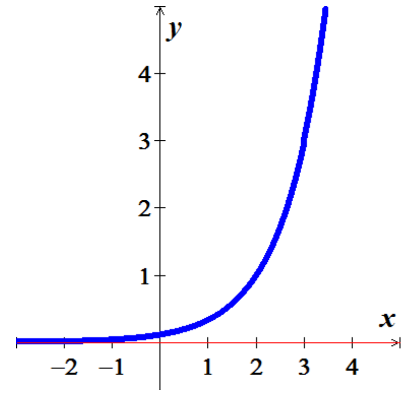
Shift right 2 unit

Asymptote: $y = 0$

Domain: $\mathbb{R}$

Range: $(0, \infty)$

x	$f(x)$
1	1/3
2	1
3	3



Example

Sketch the graph of $f(x)=2^{-x^2}$

Solution

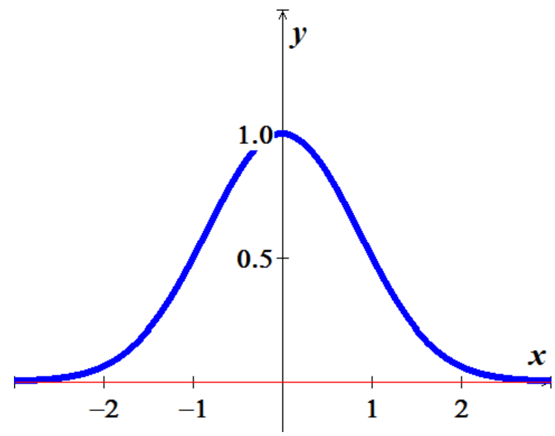
$f(x)=\dfrac{1}{2^{x^2}}$

Asymptote: $y = 0$

Domain: $\mathbb{R}$

Range: $(0, 1]$

x	$f(x)$
± 0	1
± 1	$\frac{1}{2}$
± 2	$\frac{1}{16}$



Natural Base *e*

The irrational number $e \approx 2.71828$ is called natural base

$f(x)=e^x$ is called natural exponential function

Example

Sketch $f(x)=e^{x+3}+1$

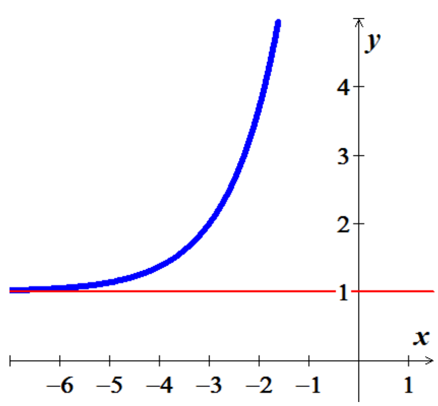
Solution

Asymptote: $y = 1$

Domain: $\mathbb{R}$

Range: $(1, \infty)$

x	*f(x)*
–4	1.4
–3	2
4	3.7



Logarithmic Function (*Definition*)

For $\boldsymbol{x} > \mathbf{0}$ and $b > 0$, $b \neq 1$

$$y = \log_b x \quad \text{is equivalent to} \quad x = b^y$$

$$y = \log_b x \iff x = b^y$$

Base

The function $f(x) = \log_b x$ is the logarithmic function with base b.

$\log_b x$: <u>*read*</u> log base b of x

log x ***means*** $\log_{10} x$

ln x ***means*** $\log_e x$ $\qquad \ln x$ *read* "**el en of *x***"

Example

Write the equation in its equivalent exponential form:

$$3 = \log_7 x \qquad \Rightarrow x = 7^3$$

Write the equation in its equivalent logarithmic form:

$$2^5 = x \qquad \Rightarrow 5 = \log_2 x$$

Basic Logarithmic Properties

$$\log_b b = 1 \quad \to \quad b = b^1 \qquad\qquad \log_b 1 = 0 \quad \to \quad 1 = b^0$$

Inverse Properties

$$\log_b b^x = x \qquad\qquad b^{\log_b x} = x$$

Change-of-Base Logarithmic

$$\log_b M = \frac{\log_a M}{\log_a b} \qquad\qquad \log_b M = \frac{\log M}{\log b} \quad \textbf{or} \quad \log_b M = \frac{\ln M}{\ln b}$$

Domain

The domain of a logarithmic function of the form $f(x) = \log_b x$ is the set of all positive real numbers. (*Inside* the log has to be > 0)

Range: $\mathbb{R}$

Example

Find the domain of

a) $f(x) = \log_4(x-5)$ ***Domain***: $x > 5$

b) $f(x) = \ln(4-x)$ ***Domain***: $x < 4$

c) $h(x) = \ln(x^2)$ ***Domain***: $\mathbb{R} - \{0\}$ or $\{x \mid x \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$

Graphs of *Logarithmic* Functions

Example

Graph $g(x) = \log(x-2) + 1$

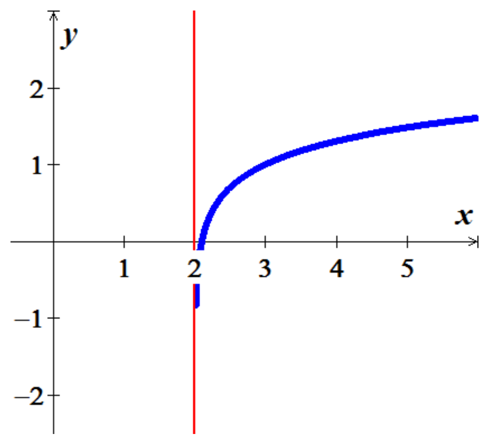
Solution

Asymptote: $x = 2$

Domain: $x > 2$

Range: $\mathbb{R}$

x	$g(x)$
2	
2.5	.7
3	1
4	1.3



Example

Graph $f(x) = \log_3 |x| \quad for\ x \neq 0$

Solution

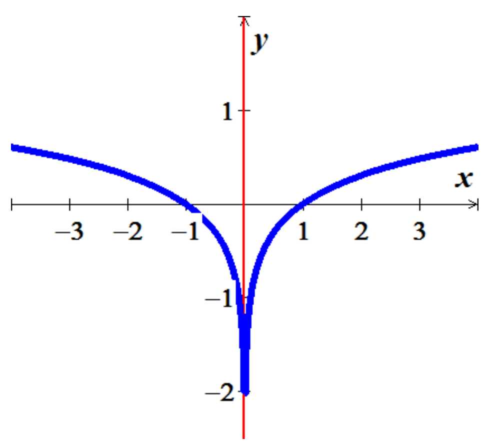
$f(-x) = \log_3 |-x| = \log_3 |x| = f(x)$

$\therefore$ The graph is symmetric with respect to the y-axis.

Asymptote: $x = 0$

Domain: $\mathbb{R} - \{0\}$

Range: $\mathbb{R}$



Exercises ***Section* 1.4 – Inverse, Exponential & Logarithmic Functions**

(**1 – 9**) Determine whether the function is *one*-to-*one*

1. $f(x)=3x-7$

2. $f(x)=x^2-9$

3. $f(x)=\sqrt{x}$

4. $f(x)=\sqrt[3]{x}$

5. $f(x)=|x|$

6. $f(x)=\dfrac{2}{x+3}$

7. $f(x)=(x-2)^3$

8. $y=x^2+2$

9. $f(x)=\dfrac{x+1}{x-3}$

10. Given the function $f(x)=(x+8)^3$

a) Find $f^{-1}(x)$

b) Graph f and f^{-1} in the same rectangular coordinate system

c) Find the domain and the range of f and f^{-1}

(**11 – 38**) For the given functions

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

11. $f(x)=\dfrac{2x}{x-1}$

12. $f(x)=\dfrac{x}{x-2}$

13. $f(x)=\dfrac{x+1}{x-1}$

14. $f(x)=\dfrac{2x+1}{x+3}$

15. $f(x)=\dfrac{3x-1}{x-2}$

16. $f(x)=\dfrac{2x}{x-1}$

17. $f(x)=\dfrac{x}{x-2}$

18. $f(x)=\dfrac{x+1}{x-1}$

19. $f(x)=\dfrac{2x+1}{x+3}$

20. $f(x)=\dfrac{3x-1}{x-2}$

21. $f(x)=\dfrac{3x-2}{x+4}$

22. $f(x)=\dfrac{-3x-2}{x+4}$

23. $f(x)=\sqrt{x-1} \quad x\ge 1$

24. $f(x)=\sqrt{2-x} \quad x\le 2$

25. $f(x)=x^2+4x \quad x\ge -2$

26. $f(x)=3x+5$

27. $f(x)=\dfrac{1}{3x-2}$

28. $f(x)=\dfrac{3x+2}{2x-5}$

29. $f(x)=\dfrac{4x}{x-2}$

30. $f(x)=2-3x^2; \quad x\le 0$

31. $f(x)=2x^3-5$

32. $f(x)=\sqrt{3-x}$

33. $f(x)=\sqrt[3]{x}+1$

34. $f(x)=\left(x^3+1\right)^5$

35. $f(x)=x^2-6x; \quad x\ge 3$

36. $f(x)=(x-2)^3$

37. $f(x)=\dfrac{x+1}{x-3}$

38. $f(x)=\dfrac{2x+1}{x-3}$

39. Simplify the expression $\dfrac{\left(e^{x}+e^{-x}\right)\left(e^{x}+e^{-x}\right)-\left(e^{x}-e^{-x}\right)\left(e^{x}-e^{-x}\right)}{\left(e^{x}+e^{-x}\right)^{2}}$

40. Simplify the expression $\dfrac{\left(e^{x}-e^{-x}\right)^{2}-\left(e^{x}+e^{-x}\right)^{2}}{\left(e^{x}+e^{-x}\right)^{2}}$

(**41 – 52**)Write the equation in its equivalent logarithmic form

41. $2^{6}=64$

42. $5^{4}=625$

43. $5^{-3}=\frac{1}{125}$

44. $\sqrt[3]{64}=4$

45. $b^{3}=343$

46. $8^{y}=300$

47. $\sqrt[n]{x}=y$

48. $\left(\frac{2}{3}\right)^{-3}=\frac{27}{8}$

49. $\left(\frac{1}{2}\right)^{-5}=32$

50. $e^{x-2}=2y$

51. $e=3x$

52. $\sqrt[3]{e^{2x}}=y$

(**53 – 64**) Write the equation in its equivalent exponential form

53. $\log_{5}125=y$

54. $\log_{4}16=x$

55. $\log_{5}\frac{1}{5}=x$

56. $\log_{2}\frac{1}{8}=x$

57. $\log_{6}\sqrt{6}=x$

58. $\log_{3}\frac{1}{\sqrt{3}}=x$

59. $6=\log_{2}64$

60. $2=\log_{9}x$

61. $\log_{\sqrt{3}}81=8$

62. $\log_{4}\frac{1}{64}=-3$

63. $\log_{4}26=y$

64. $\ln M=c$

(**65 – 71**) Evaluate the expression without using a calculator

65. $\log_{4}16$

66. $\log_{2}\frac{1}{8}$

67. $\log_{6}\sqrt{6}$

68. $\log_{3}\frac{1}{\sqrt{3}}$

69. $\log_{3}\sqrt[7]{3}$

70. $\log_{3}\sqrt{9}$

71. $\log_{\frac{1}{2}}\sqrt{\frac{1}{2}}$

(**72 – 80**) Simplify

72. $\log_{5}1$

73. $\log_{7}7^{2}$

74. $3^{\log_{3}8}$

75. $10^{\log 3}$

76. $e^{2+\ln 3}$

77. $\ln e^{-3}$

78. $\ln e^{x-5}$

79. $\log_{b}b^{n}$

80. $\ln e^{x^{2}+3x}$

(**81 – 108**) Find the domain of

81. $f(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$

82. $f(x)=\dfrac{e^{|x|}}{1+e^x}$

83. $f(x)=\sqrt{1-e^x}$

84. $f(x)=\sqrt{e^x-e^{-x}}$

85. $f(x)=\log_5(x+4)$

86. $f(x)=\log_5(x+6)$

87. $f(x)=\log(2-x)$

88. $f(x)=\log(7-x)$

89. $f(x)=\ln(x-2)^2$

90. $f(x)=\ln(x-7)^2$

91. $f(x)=\log\left(x^2-4x-12\right)$

92. $f(x)=\log\left(\frac{x-2}{x+5}\right)$

93. $f(x)=\log\left(\frac{3-x}{x-2}\right)$

94. $f(x)=\ln\left(\dfrac{x^2}{x-4}\right)$

95. $f(x)=\log_3\left(x^3-x\right)$

96. $f(x)=\log\sqrt{2x-5}$

97. $f(x)=3\ln(5x-6)$

98. $f(x)=\log\left(\frac{x}{x-2}\right)$

99. $f(x)=\ln\left(x^2+4\right)$

100. $f(x)=\ln|4x-8|$

101. $f(x)=\ln\left(x^2-9\right)$

102. $f(x)=\ln|5-x|$

103. $f(x)=\ln(x-4)^2$

104. $f(x)=\ln\left(x^2-4\right)$

105. $f(x)=\ln\left(x^2-4x+3\right)$

106. $f(x)=\ln\left(2x^2-5x+3\right)$

107. $f(x)=\log\left(x^2+4x+3\right)$

108. $f(x)=\ln\left(x^4-x^2\right)$

(**109 – 129**) Find the ***asymptote***, ***domain***, and ***range*** of the given functions. Then, sketch the graph

109. $f(x)=2^x+3$

110. $f(x)=2^{3-x}$

111. $f(x)=\left(\frac{2}{5}\right)^{-x}$

112. $f(x)=-\left(\frac{1}{2}\right)^x+4$

113. $f(x)=4^x$

114. $f(x)=2-4^x$

115. $f(x)=-3+4^{x-1}$

116. $f(x)=1+\left(\frac{1}{4}\right)^{x+1}$

117. $f(x)=e^{x-2}$

118. $f(x)=3-e^{x-2}$

119. $f(x)=e^{x+4}$

120. $f(x)=2+e^{x-1}$

121. $f(x)=\log_4(x-2)$

122. $f(x)=\log_4|x|$

123. $f(x)=\left(\log_4 x\right)-2$

124. $f(x)=\log(3-x)$

125. $f(x)=2-\log(x+2)$

126. $f(x)=\ln(x-2)$

127. $f(x)=\ln(3-x)$

128. $f(x)=2+\ln(x+1)$

129. $f(x)=1-\ln(x-2)$

130. On a study by psychologists Bornstein and Bornstein, it was found that the average walking speed *w*, in feet per second, of a person living in a city of population *P*, in ***thousands***, is given by the function:

$$w(P) = 0.37\ln P + 0.05$$

a) The population is 124,848. Find the average walking speed of people living in Hartford.
b) The population is 1,236,249. Find the average walking speed of people living in San Antonio.

131. The loudness of sounds is measured in a unit called a decibel. To measure with this unit, we first assign an intensity of I_0 to a very faint sound, called the threshold sound. If a particular sound has intensity *I*, then the decibel rating of this louder sound is

$$d = 10\log\frac{I}{I_0}$$

Find the exact decibel rating of a sound with intensity $10{,}000 I_0$

132. Students in an accounting class took a final exam and then took equivalent forms of the exam at monthly intervals thereafter. The average score $S(t)$, as a percent, after *t* months was found to be given by the function

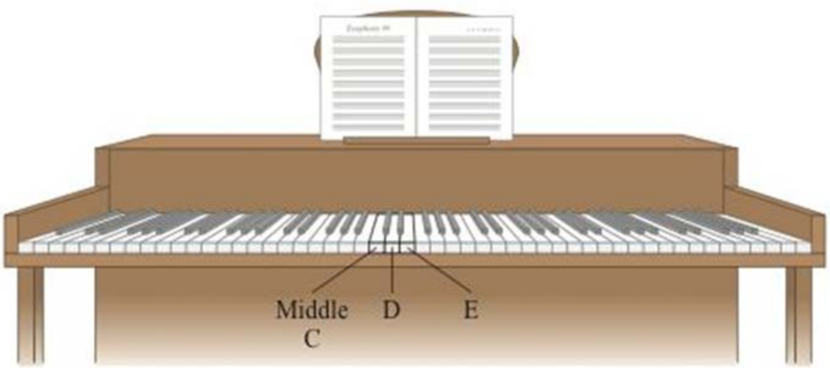
$$S(t) = 78 - 15\ \log(t+1); \quad t \ge 0$$

a) What was the average score when the students initially took the test, *t* = 0?

b) What was the average score after 4 *months*? 24 *months*?

133. Starting on the left side of a standard 88–*key* piano, the frequency, in *vibrations* per *second*, of the *n*th note is given by

$$f(n) = (27.5)\ 2^{\frac{n-1}{12}}$$



a) Determine the frequency of middle *C*, key number 40 on an 88–*key* piano.
b) Is the difference in frequency between middle *C* (key number 40) and *D* (key number 42) the same as the difference in frequency between *D* (key number 42) and *E* (key number 44)?

Section 1.5 – Exponential and Logarithmic Equations

Properties of Logarithms For $M > 0$ and $N > 0$

***Product* Rule** $\log_b MN = \log_b M + \log_b N$

***Power* Rule** $\log_b M^p = p\log_b M$

***Quotient* Rule** $\log_b \frac{M}{N} = \log_b M - \log_b N$

Example

Express $\log_a \frac{x^3\sqrt{y}}{z^2}$ in terms of logarithms of *x*, *y*, and *z*.

Solution

$$\log_a \frac{x^3\sqrt{y}}{z^2} = \log_a x^3 y^{1/2} - \log_a z^2 \qquad \textit{Quotient Rule}$$

$$= \log_a x^3 + \log_a y^{1/2} - \log_a z^2 \qquad \textit{Product Rule}$$

$$= 3\log_a x + \frac{1}{2}\log_a y - 2\log_a z \qquad \textit{Power Rule}$$

Example

Express as one logarithm: $\frac{1}{3}\log_a\left(x^2-1\right) - \log_a y - 4\log_a z$

Solution

$$\frac{1}{3}\log_a\left(x^2-1\right) - \log_a y - 4\log_a z = \log_a\left(x^2-1\right)^{1/3} - \log_a y - \log_a z^4 \qquad \textit{Power Rule}$$

$$= \log_a \sqrt[3]{x^2-1} - \left(\log_a y + \log_a z^4\right) \qquad \textit{Factor } (-)$$

$$= \log_a \sqrt[3]{x^2-1} - \left(\log_a yz^4\right) \qquad \textit{Product Rule}$$

$$= \log_a \frac{\sqrt[3]{x^2-1}}{yz^4} \qquad \textit{Quotient Rule}$$

Exponential Functions are One-to-One

$$b^{M} = b^{N} \quad \leftrightarrow \quad M = N \quad \text{for any b} > 0, \neq 1$$

Example

Solve $8^{x+2} = 4^{x-3}$

Solution

$$\left(2^3\right)^{x+2} = \left(2^2\right)^{x-3}$$

$$2^{3(x+2)} = 2^{2(x-3)}$$

$$3(x+2) = 2(x-3)$$

$$3x + 6 = 2x - 6$$

$$3x - 2x = -6 - 6$$

$$\boxed{x = -12}$$

Using *Natural Logarithms*

1. Isolate the exponential expression
2. Take the natural logarithm on both sides of the equation
3. Simplify using one of the following properties: $\ln b^{x} = x \ln b$ *or* $\ln e^{x} = x$
4. Solve for the variable

Example

Solve the equation $3^{x} = 21$

Solution

1st ***method***	2nd ***method***
$3^{x} = 21$ ***ln both sides*** $\ln 3^{x} = \ln 21$ $x \ln 3 = \ln 21$ $x = \dfrac{\ln 21}{\ln 3}$	$3^{x} = 21 \Rightarrow x = \log_{3} 21$ ***Convert to log form*** $x = \dfrac{\ln 21}{\ln 3}$ ***Change of base***

Example

Solve the equation $5^{2x+1} = 6^{x-2}$

Solution

$$\ln 5^{2x+1} = \ln 6^{x-2}$$

$$(2x+1)\ln 5 = (x-2)\ln 6$$

$$2x\ln 5 + \ln 5 = x\ln 6 - 2\ln 6$$

$$2x\ln 5 - x\ln 6 = -2\ln 6 - \ln 5$$

$$x(2\ln 5 - \ln 6) = -\ln 6^2 - \ln 5$$

$$x\left(\ln 5^2 - \ln 6\right) = -(\ln 36 + \ln 5)$$

$$x\left(\ln \frac{25}{6}\right) = -\ln(36 \times 5)$$

$$\underline{|x = -\frac{\ln(180)}{\ln \frac{25}{6}}} \quad \underline{\approx -3.64|}$$

Example

Solve the equation $\dfrac{5^x - 5^{-x}}{2} = 3$

Solution

$5^x - 5^{-x} = 6$ *Multiply by 2 both sides*

$5^x 5^x - 5^{-x} 5^x = 6 5^x$ *Multiply by* 5^x *both sides*

$$\left(5^x\right)^2 - 1 = 6\left(5^x\right)$$

$$\left(5^x\right)^2 - 6\left(5^x\right) - 1 = 0$$

$$5^x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-1)}}{2(1)} = \frac{6 \pm \sqrt{40}}{2} = \frac{6 \pm 2\sqrt{10}}{2} = \begin{cases} 3+\sqrt{10} \\ 3-\sqrt{10} < 0 \end{cases}$$

$$5^x = 3 + \sqrt{10}$$

$$\ln 5^x = \ln\left(3+\sqrt{10}\right)$$

$$x\ln 5 = \ln\left(3+\sqrt{10}\right)$$

$$\underline{x = \frac{\ln\left(3+\sqrt{10}\right)}{\ln 5}\Big|} \quad \underline{\approx 1.13|}$$

Logarithmic Equations

1. Express the equation in the form $\log_b M = c$
2. Use the definition of a logarithm to rewrite the equation in exponential form:

$$\log_b M = c \quad \Rightarrow \quad b^c = M$$

3. Solve for the variable
4. Check proposed solution in the original equation. Include only the set for $M > 0$

Example

Solve: $\log x + \log(x-3) = 1$

Solution

$\log[x(x-3)] = 1$ *Product Rule*

$x(x-3) = 10^1$ *Convert to exponential form*

$x^2 - 3x = 10$

$x^2 - 3x - 10 = 0$ *Solve for x*

$\Rightarrow x = -2, 5$

Check: $x = -2 \Rightarrow \cancel{\log(-2) + \log(x-3) = 1}$

$x = 5 \Rightarrow \log(5) + \log(5-3) = 1$

Example

Solve the equation $\log_2 x + \log_2(x+2) = 3$

Solution

$\log_2[x(x+2)] = 3$ *Product Rule*

$x(x+2) = 2^3$ *Change to exponential form*

$x^2 + 2x - 8 = 0$ *Solve for x*

$x = -4 \quad x = 2$

Check: $\log_2(-4) + \log_2(-4+2) = 3$ Not a solution (negative inside the log)

$\log_2(2) + \log_2(2+2) = 3$ Only solution

Property of Logarithmic Equality

The logarithmic function with base ***b*** is 1-1. Thus the following equivalent conditions are satisfied for positive real numbers M and N.

For any $M > 0$, $N > 0$, $b > 0, \neq 1$

If $\log_b M = \log_b N \Rightarrow M = N$

If $M \neq N \Rightarrow \log_b M \neq \log_b N$

Example

Solve the equation $\log_6(4x-5) = \log_6(2x+1)$

Solution

$$\log_6(4x-5) = \log_6(2x+1)$$
$$4x-5 = 2x+1$$
$$4x-2x = 5+1$$
$$2x = 6$$
$$x = 3$$

Check: $\log_6(4(3)-5) = \log_6(2(3)+1)$

$\log_6(7) = \log_6(7)$ ***True statement***

$\boxed{x=3}$ is a solution

Example

Solve the equation $\ln(x+6) - \ln 10 = \ln(x-1) - \ln 2$

Solution

$$\ln(x+6) - \ln 10 = \ln(x-1) - \ln 2$$
$$\ln(x+6) - \ln(x-1) = \ln 10 - \ln 2$$
$$\ln\left(\frac{x+6}{x-1}\right) = \ln\frac{10}{2}$$
$$\frac{x+6}{x-1} = 5$$
$$x+6 = 5(x-1)$$
$$x+6 = 5x-5$$
$$x-5x = -5-6$$
$$-4x = -11$$

$$\boxed{x = \frac{11}{4}}$$

Check: $\ln\left(\frac{11}{4} + 6\right) - \ln 10 = \ln\left(\frac{11}{4} - 1\right) - \ln 2$

$$\ln\left(\frac{35}{4}\right) - \ln 10 = \ln\left(\frac{7}{4}\right) - \ln 2$$

$x = \frac{11}{4}$ is the solution

Example

Solve the equation $\log \sqrt[3]{x} = \sqrt{\log x}$ for x.

Solution

$$\log x^{1/3} = \sqrt{\log x}$$

$$\left(\frac{1}{3}\log x\right)^2 = \left(\sqrt{\log x}\right)^2$$

$$\frac{1}{9}(\log x)^2 = \log x$$

$$(\log x)^2 = 9\log x$$

$$(\log x)^2 - 9\log x = 0$$

$$\log x(\log x - 9) = 0$$

$\log x = 0$ $\quad$ $\log x - 9 = 0$

$\boxed{x = 1}$ $\quad$ $\log x = 9$

$\boxed{x = 10^9}$

Check: $x = 1 \quad \Rightarrow \quad \log \sqrt[3]{1} = \sqrt{\log 1} \to 0 = 0$

$x = 10^9 \quad \Rightarrow \quad \log \sqrt[3]{10^9} = \sqrt{\log 10^9} \to 3 = 3$

The equation has two solutions: $\boxed{x = 1,\ 10^9}$

Example (*hyperbolic secant function*)

Solve the equation $y = \dfrac{2}{e^x + e^{-x}}$ for x in terms of y.

Solution

$$y = \frac{2}{e^x + e^{-x}}$$

$$y\left(e^x + e^{-x}\right) = 2$$

$$ye^x + ye^{-x} = 2$$

$$ye^x e^x + ye^{-x} e^x = 2e^x$$

$$y\left(e^x\right)^2 - 2e^x + y = 0$$

$$\begin{aligned} e^x &= \frac{2 \pm \sqrt{4 - 4y^2}}{2y} \\ &= \frac{2 \pm \sqrt{4\left(1 - y^2\right)}}{2y} \\ &= \frac{2 \pm 2\sqrt{1 - y^2}}{2y} \\ &= \frac{1 \pm \sqrt{1 - y^2}}{y} \end{aligned}$$

$$\ln e^x = \ln\left(\frac{1 \pm \sqrt{1 - y^2}}{y}\right)$$

$$\boxed{x = \ln \frac{1 \pm \sqrt{1 - y^2}}{y}}$$

Exercises *Section* 1.5 – Exponential and Logarithmic Equations

(**1 – 31**) Express the following in terms of sums and differences of logarithms

1. $\log_3(ab)$

2. $\log_7(7x)$

3. $\log\frac{x}{1000}$

4. $\log_5\left(\frac{125}{y}\right)$

5. $\log_b x^7$

6. $\ln\sqrt[7]{x}$

7. $\log_a \frac{x^2 y}{z^4}$

8. $\log_b \frac{x^2 y}{b^3}$

9. $\log_b\left(\frac{x^3 y}{z^2}\right)$

10. $\log_b\left(\frac{\sqrt[3]{x}y^4}{z^5}\right)$

11. $\log\left(\frac{100x^3\sqrt[3]{5-x}}{3(x+7)^2}\right)$

12. $\log_a \sqrt[4]{\frac{m^8 n^{12}}{a^3 b^5}}$

13. $\log_p \sqrt[3]{\frac{m^5 n^4}{t^2}}$

14. $\log_b \sqrt[n]{\frac{x^3 y^5}{z^m}}$

15. $\log_a \sqrt[3]{\frac{a^2 b}{c^5}}$

16. $\log_b\left(x^4\sqrt[3]{y}\right)$

17. $\log_5\left(\frac{\sqrt{x}}{25y^3}\right)$

18. $\log_a \frac{x^3 w}{y^2 z^4}$

19. $\log_a \frac{\sqrt{y}}{x^4\sqrt[3]{z}}$

20. $\ln\sqrt[4]{\frac{x^7}{y^5 z}}$

21. $\ln x\sqrt[3]{\frac{y^4}{z^5}}$

22. $\log_b \sqrt[5]{\frac{m^4 n^5}{x^2 a b^{10}}}$

23. $\log_b \frac{a^5 b^{10}}{c^2\sqrt[4]{d^3}}$

24. $\ln\left(x^2\sqrt{x^2+1}\right)$

25. $\ln\frac{x^2}{x^2+1}$

26. $\ln\left(\frac{x^2(x+1)^3}{(x+3)^{1/2}}\right)$

27. $\ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$

28. $\ln\frac{\left(x^2+1\right)^5}{\sqrt{1-x}}$

29. $\ln\left(\sqrt[3]{\frac{x(x+1)(x-2)}{\left(x^2+1\right)(2x+3)}}\right)$

30. $\ln\left(\sqrt{\frac{1}{x(x+1)}}\right)$

31. $\ln\left(\sqrt{\left(x^2+1\right)(x-1)^2}\right)$

(**32 – 55**) Write the expression as a single logarithm and simplify if necessary

32. $\log(x+5)+2\log x$

33. $3\log_b x-\frac{1}{3}\log_b y+4\log_b z$

34. $\frac{1}{2}\log_b(x+5)-5\log_b y$

35. $\ln\left(x^2-y^2\right)-\ln(x-y)$

36. $\ln(xz)-\ln\left(x\sqrt{y}\right)+2\ln\frac{y}{z}$

37. $\log\left(x^2 y\right)-\log z$

38. $\log\left(z^2\sqrt{y}\right)-\log z^{1/2}$

39. $2\log_a x+\frac{1}{3}\log_a(x-2)-5\log_a(2x+3)$

40. $5\log_a x - \frac{1}{2}\log_a(3x-4) - 3\log_a(5x+1)$

41. $\log\left(x^3y^2\right) - 2\log\left(x\sqrt[3]{y}\right) - 3\log\left(\frac{x}{y}\right)$

42. $\ln y^3 + \frac{1}{3}\ln\left(x^3y^6\right) - 5\ln y$

43. $2\ln x - 4\ln\left(\frac{1}{y}\right) - 3\ln(xy)$

44. $4\ln x + 7\ln y - 3\ln z$

45. $\frac{1}{3}\left[5\ln(x+6) - \ln x - \ln(x^2-25)\right]$

46. $\frac{2}{3}\left[\ln\left(x^2-4\right) - \ln(x+2)\right] + \ln(x+y)$

47. $\frac{1}{2}\log_b m + \frac{3}{2}\log_b 2n - \log_b m^2n$

48. $\frac{1}{2}\log_y p^3q^4 - \frac{2}{3}\log_y p^4q^3$

49. $\frac{1}{2}\log_a x + 4\log_a y - 3\log_a x$

50. $\frac{2}{3}\left[\ln\left(x^2-9\right) - \ln(x+3)\right] + \ln(x+y)$

51. $\frac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y$

52. $2\ln(x+4) - \ln x - \ln\left(x^2-3\right)$

53. $\ln x + \ln(y+3) + \ln(y+2) - \ln\left(y^2+5y+6\right)$

54. $\ln x + \ln(x+4) + \ln(x+1) - \ln\left(x^2+5x+4\right)$

55. $\ln\left(x^2-25\right) - 2\ln(x+5) + \ln(x-5)$

(**56 – 169**) Solve the equations

56. $2^x = 128$

57. $3^x = 243$

58. $5^x = 70$

59. $6^x = 50$

60. $5^x = 134$

61. $7^x = 12$

62. $9^x = \frac{1}{\sqrt[3]{3}}$

63. $49^x = \frac{1}{343}$

64. $2^{5x+3} = \frac{1}{16}$

65. $\left(\frac{2}{5}\right)^x = \frac{8}{125}$

66. $2^{3x-7} = 32$

67. $4^{2x-1} = 64$

68. $3^{1-x} = \frac{1}{27}$

69. $2^{-x^2} = 5$

70. $2^{-x} = 8$

71. $\left(\frac{1}{3}\right)^x = 81$

72. $3^{-x} = 120$

73. $27 = 3^{5x}\,9^{x^2}$

74. $4^{x+3} = 3^{-x}$

75. $2^{x+4} = 8^{x-6}$

76. $8^{x+2} = 4^{x-3}$

77. $7^x = 12$

78. $5^{x+4} = 4^{x+5}$

79. $5^{x+2} = 4^{1-x}$

80. $3^{2x-1} = 0.4^{x+2}$

81. $4^{3x-5} = 16$

82. $4^{x+3} = 3^{-x}$

83. $7^{2x+1} = 3^{x+2}$

84. $3^{x-1} = 7^{2x+5}$

85. $4^{x-2} = 2^{3x+3}$

86. $3^{5x-8} = 9^{x+2}$

87. $3^{x+4} = 2^{1-3x}$

88. $3^{2-3x} = 4^{2x+1}$

89. $4^{x+3} = 3^{-x}$

90. $7^{x+6} = 7^{3x-4}$

91. $2^{-100x} = (0.5)^{x-4}$

92. $4^{x}\left(\frac{1}{2}\right)^{3-2x} = 8.\left(2^{x}\right)^{2}$

93. $5^{x} + 125\left(5^{-x}\right) = 30$

94. $4^{x} - 3\left(4^{-x}\right) = 8$

95. $5^{3x-6} = 125$

96. $e^{x} = 15$

97. $e^{x+1} = 20$

98. $9e^{x} = 107$

99. $e^{x\ln 3} = 27$

100. $e^{x^2} = e^{7x-12}$

101. $f(x) = xe^{x} + e^{x}$

102. $f(x) = x^3\left(4e^{4x}\right) + 3x^2e^{4x}$

103. $e^{2x} - 2e^{x} - 3 = 0$

104. $e^{0.08t} = 2500$

105. $e^{x^2} = 200$

106. $e^{2x+1} \cdot e^{-4x} = 3e$

107. $e^{2x} - 8e^{x} + 7 = 0$

108. $e^{2x} + 2e^{x} - 15 = 0$

109. $e^{x} + e^{-x} - 6 = 0$

110. $e^{1-3x} \cdot e^{5x} = 2e$

111. $6\ln(2x) = 30$

112. $\log_5(x-7) = 2$

113. $\log_4(5+x) = 3$

114. $\log(4x-18) = 1$

115. $\log_3 x = -2$

116. $\log\left(x^2+19\right) = 2$

117. $\ln\left(x^2-12\right) = \ln x$

118. $\log\left(2x^2+3x\right) = \log(10x+30)$

119. $\log_5(2x+3) = \log_5 11 + \log_5 3$

120. $\log_3 x - \log_9(x+42) = 0$

121. $\log_5 x + \log_5(4x-1) = 1$

122. $\log x - \log(x+3) = 1$

123. $\log x + \log(x-9) = 1$

124. $\log_2(x+1) + \log_2(x-1) = 3$

125. $\log_8(x+1) - \log_8 x = 2$

126. $\ln(x+8) + \ln(x-1) = 2\ln x$

127. $\ln(4x+6) - \ln(x+5) = \ln x$

128. $\ln(5+4x) - \ln(x+3) = \ln 3$

129. $\ln\sqrt[4]{x} = \sqrt{\ln x}$

130. $\sqrt{\ln x} = \ln\sqrt{x}$

131. $\log x^2 = (\log x)^2$

132. $\log x^3 = (\log x)^2$

133. $\log(\log x) = 1$

134. $\log(\log x) = 2$

135. $\ln(\ln x) = 2$

136. $\ln\left(e^{x^2}\right) = 64$

137. $e^{\ln(x-1)} = 4$

138. $10^{\log(2x+5)} = 9$

139. $\log\sqrt{x^3-9} = 2$

140. $\log\sqrt{x^3-17} = \frac{1}{2}$

141. $\log_4 x = \log_4 (8-x)$

142. $\log_7 (x-5) = \log_7 (6x)$

143. $\ln x^2 = \ln(12-x)$

144. $\log_2 (x+7) + \log_2 x = 3$

145. $\ln x = 1 - \ln(x+2)$

146. $\ln x = 1 + \ln(x+1)$

147. $\log_6 (2x-3) = \log_6 12 - \log_6 3$

148. $\log(3x+2) + \log(x-1) = 1$

149. $\log_5 (x+2) + \log_5 (x-2) = 1$

150. $\log_2 x + \log_2 (x-4) = 2$

151. $\log_3 x + \log_3 (x+6) = 3$

152. $\log_3 (x+3) + \log_3 (x+5) = 1$

153. $\ln x = \frac{1}{2}\ln\left(2x + \frac{5}{2}\right) + \frac{1}{2}\ln 2$

154. $\ln(-4-x) + \ln 3 = \ln(2-x)$

155. $\log_4 x + \log_4 (x-2) = \log_4 (15)$

156. $\ln(x-5) - \ln(x+4) = \ln(x-1) - \ln(x+2)$

157. $\ln(4-x) = \ln(x+8) + \ln(2x+13)$

158. $\log\left(x^2+4\right) - \log(x+2) = 2 + \log(x-2)$

159. $\log_3 (x-2) = \log_3 27 - \log_3 (x-4) - 5^{\log_5 1}$

160. $\log_2 (x+3) = \log_2 (x-3) + \log_3 9 + 4^{\log_4 3}$

161. $\dfrac{10^x - 10^{-x}}{2} = 20$

162. $\dfrac{10^x + 10^{-x}}{2} = 8$

163. $\dfrac{10^x + 10^{-x}}{10^x - 10^{-x}} = 5$

164. $\dfrac{10^x + 10^{-x}}{10^x - 10^{-x}} = 2$

165. $\dfrac{e^x + e^{-x}}{2} = 15$

166. $\dfrac{e^x - e^{-x}}{2} = 15$

167. $\dfrac{1}{e^x - e^{-x}} = 4$

168. $\dfrac{e^x + e^{-x}}{e^x - e^{-x}} = 3$

169. $\dfrac{e^x - e^{-x}}{e^x + e^{-x}} = 6$

(**170 – 173**) Use common logarithms to solve for $\boldsymbol{x}$ in terms of $\boldsymbol{y}$

170. $y = \dfrac{10^x + 10^{-x}}{2}$

171. $y = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$

172. $y = \dfrac{e^x - e^{-x}}{2}$

173. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

174. Solve for t using logarithms with base a: $2a^{t/3} = 5$

175. Solve for t using logarithms with base a: $K = H - Ca^t$