Lecture One – Vectors and Vector-Values Functions

Solution *Section* 1.1 – Vectors

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + z^2 = 4, \quad y = 0$

Solution

The circle $x^2 + z^2 = 4$ in the ***xz***-plane

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + y^2 = 4, \quad z = -2$

Solution

The circle $x^2 + y^2 = 4$ in the plane $z = -2$

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + y^2 + z^2 = 1, \quad x = 0$

Solution

The circle $y^2 + z^2 = 1$ in the ***yz***-plane

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2 + (y-1)^2 + z^2 = 4, \quad y = 0$

Solution

$x^2 + (0-1)^2 + z^2 = 4 \quad \Rightarrow x^2 + z^2 = 3$

The circle $x^2 + z^2 = 3$ in the ***xz***-plane

Exercise

Give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations $x^2+y^2+z^2=4, \quad y=x$

Solution

The circle formed by the intersection of the sphere $x^2+y^2+z^2=4$ and the plane $y=x$

Exercise

Find the distance between points $P_1(1,\ 1,\ 1), \quad P_2(3,\ 3,\ 0)$

Solution

$$
\begin{aligned}
\left|\overrightarrow{P_1P_2}\right| &= \sqrt{(3-1)^2+(3-1)^2+(0-1)^2} \\
&= \sqrt{4+4+1} \\
&= \sqrt{9} \\
&= 3
\end{aligned}
$$

Exercise

Find the distance between points $P_1(-1,\ 1,\ 5), \quad P_2(2,\ 5,\ 0)$

Solution

$$
\begin{aligned}
\left|\overrightarrow{P_1P_2}\right| &= \sqrt{(2+1)^2+(5-1)^2+(0-5)^2} \\
&= \sqrt{9+16+25} \\
&= \sqrt{50} \\
&= 5\sqrt{2}
\end{aligned}
$$

Exercise

Find the distance between points $P_1(1,\ 4,\ 5), \quad P_2(4,\ -2,\ 7)$

Solution

$$
\begin{aligned}
\left|\overrightarrow{P_1P_2}\right| &= \sqrt{(4-1)^2+(-2-4)^2+(7-5)^2} \\
&= \sqrt{9+36+4} \\
&= 7
\end{aligned}
$$

Exercise

Find the distance between points $P_1(3,\ 4,\ 5),\quad P_2(2,\ 3,\ 4)$

Solution

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(2-3)^2 + (3-4)^2 + (4-5)^2}$$

$$= \sqrt{1+1+1}$$

$$= \sqrt{3}$$

Exercise

Find the center and radii of the spheres $\quad x^2 + y^2 + z^2 + 4x - 4z = 0$

Solution

$$\left(x^2 + 4x\right) + y^2 + \left(z^2 - 4z\right) = 0$$

$$\left(x^2 + 4x + 4\right) + y^2 + \left(z^2 - 4z + 4\right) = 4 + 4$$

$$(x+2)^2 + y^2 + (z-2)^2 = 8$$

The center is at $(-2,\ 0,\ 2)$ and the radius is $\sqrt{8} = 2\sqrt{2}$

Exercise

Find the center and radii of the spheres $\quad x^2 + y^2 + z^2 - 6y + 8z = 0$

Solution

$$x^2 + \left(y^2 - 6y\right) + \left(z^2 + 8z\right) = 0$$

$$x^2 + \left(y^2 - 6y + \left(-\frac{6}{2}\right)^2\right) + \left(z^2 + 8z + \left(\frac{8}{2}\right)^2\right) = 9 + 16$$

$$x^2 + (y-3)^2 + (z+4)^2 = 25$$

The center is at $(0,\ 3,\ -4)$ and the radius is 5

Exercise

Find the center and radii of the spheres $\qquad 2x^2+2y^2+2z^2+x+y+z=9$

Solution

$$x^2+y^2+z^2+\frac{1}{2}x+\frac{1}{2}y+\frac{1}{2}z=\frac{9}{2}$$

$$\left(x^2+\frac{1}{2}x+\left(\frac{1}{2}\frac{1}{2}\right)^2\right)+\left(y^2+\frac{1}{2}y+\left(\frac{1}{4}\right)^2\right)+\left(z^2+\frac{1}{2}z+\left(\frac{1}{4}\right)^2\right)=\frac{9}{2}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}$$

$$\left(x+\frac{1}{4}\right)^2+\left(y+\frac{1}{4}\right)^2+\left(z+\frac{1}{4}\right)^2=\frac{9}{2}+\frac{3}{16}$$

$$\left(x+\frac{1}{4}\right)^2+\left(y+\frac{1}{4}\right)^2+\left(z+\frac{1}{4}\right)^2=\frac{75}{16}$$

The center is at $\left(-\frac{1}{4},\ -\frac{1}{4},\ -\frac{1}{4}\right)$ and the radius is $\boxed{\frac{5\sqrt{3}}{4}}$

Exercise

Find a formula for the distance from the point $P(x,\ y,\ z)$ to *x*-axis

Solution

The distance between $(x,\ y,\ z)$ and $(x,\ 0,\ 0)$ is:

$$d=\sqrt{(x-x)^2+(y-0)^2+(z-0)^2}$$

$$\boxed{=\sqrt{y^2+z^2}}$$

Exercise

Find a formula for the distance from the point $P(x,\ y,\ z)$ to *xy*-plane

Solution

The distance between $(x,\ y,\ z)$ and $(x,\ 0,\ z)$ is:

$$d=\sqrt{(x-x)^2+(y-0)^2+(z-z)^2}$$

$$\boxed{=y}$$

Exercise

Let $\vec{u} = \langle -3,\ 4\rangle$ *and* $\vec{v} = \langle 2,\ -5\rangle$. Find the component form and the magnitude if the vector

a) $3\vec{u} - 4\vec{v}$ *b)* $-2\vec{u}$ *c)* $\vec{u} + \vec{v}$

Solution

a) $3\vec{u} - 4\vec{v} = 3\langle -3,\ 4\rangle - 4\langle 2,\ -5\rangle$

$$= \langle -17,\ 32\rangle$$

b) $-2\vec{u} = -2\langle -3,\ 4\rangle$

$$= \langle 6,\ -8\rangle$$

c) $\vec{u} + \vec{v} = \langle -3,\ 4\rangle + \langle 2,\ -5\rangle$

$$= \langle -1,\ -1\rangle$$

Exercise

Let $\vec{u} = \langle 3,\ -2\rangle$ *and* $\vec{v} = \langle -2,\ 5\rangle$. Find the component form and the magnitude if the vector

a) $3\vec{u}$ *b)* $\vec{u} - \vec{v}$ *c)* $2\vec{u} - 3\vec{v}$ *d)* $-2\vec{u} + 5\vec{v}$ *e)* $-\frac{5}{13}\vec{u} + \frac{12}{13}\vec{v}$

Solution

a) $3\vec{u} = 3\langle 3,\ -2\rangle$

$$= \langle 9,\ -6\rangle$$

b) $\vec{u} - \vec{v} = \langle 3,\ -2\rangle - \langle -2,\ 5\rangle$

$$= \langle 5,\ -7\rangle$$

c) $2\vec{u} - 3\vec{v} = 2\langle 3,\ -2\rangle - 3\langle -2,\ 5\rangle$

$$= \langle 6,\ -4\rangle - \langle -6,\ 15\rangle$$

$$= \langle 12,\ -19\rangle$$

d) $-2\vec{u} + 5\vec{v} = -2\langle 3,\ -2\rangle + 5\langle -2,\ 5\rangle$

$$= \langle -6,\ 4\rangle + \langle -10,\ 25\rangle$$

$$= \langle -14,\ 29\rangle$$

e) $-\frac{5}{13}\vec{u} + \frac{12}{13}\vec{v} = -\frac{5}{13}\langle 3,\ -2\rangle + \frac{12}{13}\langle -2,\ 5\rangle$

$$= \langle -6,\ 4\rangle - \langle -10,\ 25\rangle$$

$$= \langle 4,\ -21\rangle$$

Exercise

Find scalars $a,\ b$, and c such that $\langle 2,\ 2,\ 2\rangle = a\langle 1,\ 1,\ 0\rangle + b\langle 0,\ 1,\ 1\rangle + c\langle 1,\ 0,\ 1\rangle$

Solution

$$\langle 2,\ 2,\ 2\rangle = a\langle 1,\ 1,\ 0\rangle + b\langle 0,\ 1,\ 1\rangle + c\langle 1,\ 0,\ 1\rangle$$

$$= \langle a+c,\ a+b,\ b+c\rangle$$

$$\begin{cases} a+c=2 \\ a+b=2 \\ b+c=2 \end{cases} \quad \begin{cases} c=2-a \\ b=2-a \\ \end{cases} \quad \begin{cases} \\ \\ 2a-4=2 \end{cases}$$

$$\underline{a=b=c=1}$$

Exercise

Find the component form of the vector: The sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ where

$$A=(1,\ -1),\quad B=(2,\ 0),\quad C=(-1,\ 3),\quad \textit{and}\ \ D=(-2,\ 2)$$

Solution

$$\overrightarrow{AB} = \langle 2-1,\ 0-(-1)\rangle$$

$$\underline{= \langle 1,\ 1\rangle}$$

$$\overrightarrow{CD} = \langle -2-(-1),\ 2-3\rangle$$

$$\underline{= \langle -1,\ -1\rangle}$$

$$\overrightarrow{AB} + \overrightarrow{CD} = \langle 1,\ 1\rangle + \langle -1,\ -1\rangle$$

$$\underline{= \langle 0,\ 0\rangle}$$

Exercise

Find the component form of the vector: The unit vector that makes an angle $\theta = \frac{2\pi}{3}$ with the positive x-axis

Solution

$$\left\langle \cos\frac{2\pi}{3},\ \sin\frac{2\pi}{3}\right\rangle = \underline{\left\langle -\frac{1}{2},\ \frac{\sqrt{3}}{2}\right\rangle}$$

Exercise

Find the component form of the vector: The unit vector obtained by rotating the vector $\langle 0,\ 1\rangle$ $120°$ counterclockwise about the origin

Solution

The angle of unit vector $\langle 0,\ 1\rangle$ is $90°$, this unit vector rotates $120°$ which makes an angle of $90° + 120° = 210°$ with the positive x-axis

$$\langle \cos 210°,\ \sin 210°\rangle = \boxed{\left\langle -\frac{\sqrt{3}}{2},\ -\frac{1}{2}\right\rangle}$$

Exercise

Find the component form of the vector: The unit vector obtained by rotating the vector $\langle 1,\ 0\rangle$ $135°$ counterclockwise about the origin

Solution

The angle of unit vector $\langle 1,\ 0\rangle$ is $0°$, this unit vector rotates $135°$ which makes an angle of $0° + 135° = 135°$ with the positive x-axis

$$\langle \cos 135°,\ \sin 135°\rangle = \boxed{\left\langle -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right\rangle}$$

Exercise

Find the component form of the vector: The unit vector that makes an angle $\theta = \frac{\pi}{6}$ with the positive x-axis

Solution

$$\boxed{\left\langle \cos\frac{\pi}{6},\ \sin\frac{\pi}{6}\right\rangle = \left\langle \frac{\sqrt{3}}{2},\ \frac{1}{2}\right\rangle}$$

Exercise

Find the component form of the vector: The vector 5 units long in the direction opposite to the direction of $\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$

Solution

$$-5\left(\frac{1}{\sqrt{\frac{9}{25}+\frac{16}{25}}}\right)\left(\frac{3}{5}\hat{i}+\frac{4}{5}\hat{j}\right)=-5(1)\left(\frac{3}{5}\hat{i}+\frac{4}{5}\hat{j}\right)$$

$$=-3\hat{i}-4\hat{j}$$

Exercise

Express the velocity vector $\vec{v}=\left(e^t\cos t-e^t\sin t\right)\hat{i}+\left(e^t\cos t+e^t\sin t\right)\hat{j}$ *when* $t=\ln 2$ in terms of its length and direction.

Solution

$$\vec{v}(t=\ln 2)=\left(e^{\ln 2}\cos(\ln 2)-e^{\ln 2}\sin(\ln 2)\right)\hat{i}+\left(e^{\ln 2}\cos(\ln 2)+e^{\ln 2}\sin(\ln 2)\right)\hat{j}$$

$$=\left(2\cos(\ln 2)-2\sin(\ln 2)\right)\hat{i}+\left(2\cos(\ln 2)+2\sin(\ln 2)\right)\hat{j}$$

$$Length=|\vec{v}|$$

$$=\sqrt{\left(2\cos(\ln 2)-2\sin(\ln 2)\right)^2+\left(2\cos(\ln 2)+2\sin(\ln 2)\right)^2}$$

$$=2\sqrt{\begin{aligned}&\cos^2(\ln 2)-2\cos(\ln 2)\sin(\ln 2)+\sin^2(\ln 2)\\&\quad+\cos^2(\ln 2)+2\cos(\ln 2)\sin(\ln 2)+\sin^2(\ln 2)\end{aligned}}$$

$$=2\sqrt{2}$$

$$Direction\ =\frac{\vec{v}}{|\vec{v}|}$$

$$=\frac{2\left(\left(\cos(\ln 2)-\sin(\ln 2)\right)\hat{i}+\left(\cos(\ln 2)+\sin(\ln 2)\right)\hat{j}\right)}{2\sqrt{2}}$$

$$=\frac{\left(\cos(\ln 2)-\sin(\ln 2)\right)}{\sqrt{2}}\hat{i}+\frac{\left(\cos(\ln 2)+\sin(\ln 2)\right)}{\sqrt{2}}\hat{j}$$

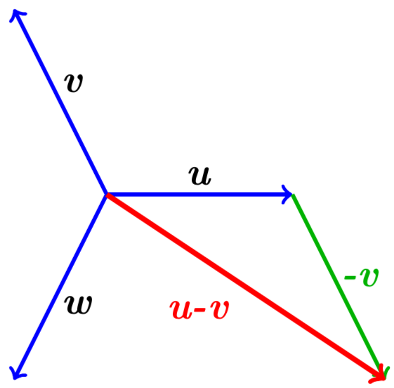
Exercise

Sketch the indicated vector

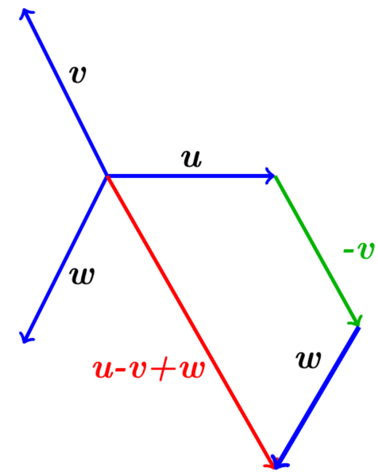
a) $\vec{u}-\vec{v}$ *b)* $2\vec{u}-\vec{v}$ *c)* $\vec{u}-\vec{v}+\vec{w}$

Solution

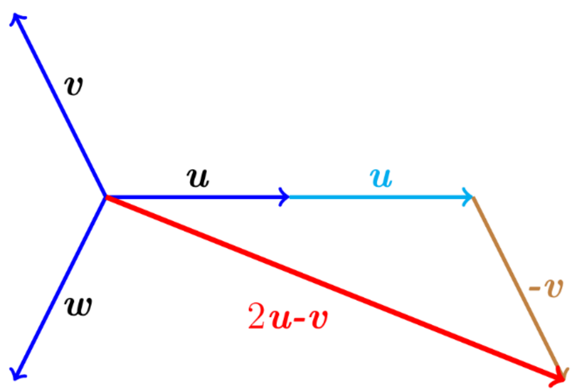
a)



b)



c)



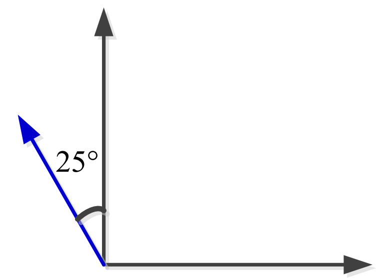
Exercise

An Airplane is flying in the direction 25° west of north at 800 *km/h*. Find the component form of the velocity of the airplane, assuming that the positive *x*-axis represents due east and the positive *y*-axis represents due north.

Solution

25° west of north is 25° + 90° = 115° north of east

$$800\left\langle \cos 115^\circ,\ \sin 115^\circ \right\rangle \approx \left\langle -338.095,\ 725.046 \right\rangle$$



Exercise

A jet airliner, flying due east at 500 *mph* in still air, encounters a 70-*mph* tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What speed and direction should the jetliner have in order for the resultant vector to be 500 *mph* due east?

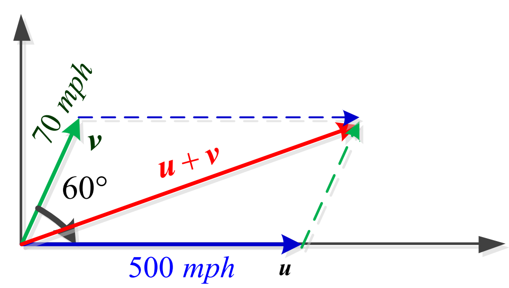
Solution

$\vec{u} = \left\langle x,\ y \right\rangle =$ the velocity of the airplane;

$\vec{v}\ =$ the velocity of the tailwind

$$\vec{v} = \left\langle 70\cos 60^\circ,\ 70\sin 60^\circ \right\rangle$$

$$= \left\langle 35,\ 35\sqrt{3} \right\rangle$$



$$\vec{u} + \vec{v} = \left\langle 500,\ 0 \right\rangle$$

$$\left\langle x,\ y \right\rangle + \left\langle 35,\ 35\sqrt{3} \right\rangle = \left\langle 500,\ 0 \right\rangle$$

$$\left\langle x,\ y \right\rangle = \left\langle 500,\ 0 \right\rangle - \left\langle 35,\ 35\sqrt{3} \right\rangle$$

$$= \left\langle 765,\ -35\sqrt{3} \right\rangle$$

$$\boxed{\vec{u} = \left\langle 765,\ -35\sqrt{3} \right\rangle}$$

$$\left|\vec{u}\right| = \sqrt{465^2 + \left(-35\sqrt{3}\right)^2}\ \underline{\approx 468.9\ mph}$$

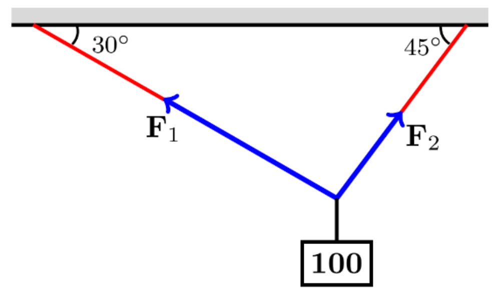
$$\underline{|\theta} = \tan^{-1}\frac{-35\sqrt{3}}{465}\ \underline{\approx -7.4^\circ}$$

The direction is 7.4° *south* of *east*

Exercise

Consider a 100-N weight suspended by two wires. Find the magnitudes and components of the force vectors $\vec{F}_1$ *and* $\vec{F}_2$

Solution



$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos 30°,\ \left|\vec{F}_1\right|\sin 30°\right\rangle$$

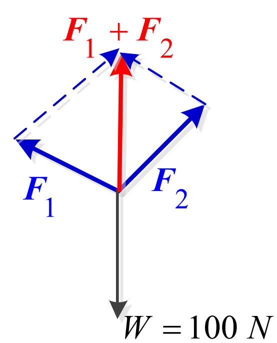
$$= \left\langle -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right|,\ \frac{1}{2}\left|\vec{F}_1\right|\right\rangle$$

$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos 45°,\ \left|\vec{F}_2\right|\sin 45°\right\rangle$$

$$= \left\langle \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|\right\rangle$$

$$\vec{F}_1 + \vec{F}_2 = \langle 0,\ 100\rangle$$

$$\left\langle -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right|,\ \frac{1}{2}\left|\vec{F}_1\right|\right\rangle + \left\langle \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|\right\rangle = \langle 0,\ 100\rangle$$



$$\left\langle -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|,\ \frac{1}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|\right\rangle = \langle 0,\ 100\rangle$$

$$\begin{cases} -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| = 0 \\ \frac{1}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| = 100 \end{cases}$$

$$\Rightarrow \ \boxed{\left|\vec{F}_1\right| \approx 73.205\ \ N} \quad \boxed{\left|\vec{F}_2\right| \approx 89.658\ \ N}$$

$$\Rightarrow \begin{cases} \left|\vec{F}_1\right| = \dfrac{200\sqrt{2}}{\sqrt{6}+\sqrt{2}} & \approx 73.205\ N \\ \left|\vec{F}_2\right| = \dfrac{200\sqrt{3}}{\sqrt{6}+\sqrt{2}} & \approx 89.658\ N \end{cases}$$

$$\vec{F}_1 = \left\langle -\frac{100\sqrt{6}}{\sqrt{6}+\sqrt{2}},\ \frac{100\sqrt{2}}{\sqrt{6}+\sqrt{2}}\right\rangle$$

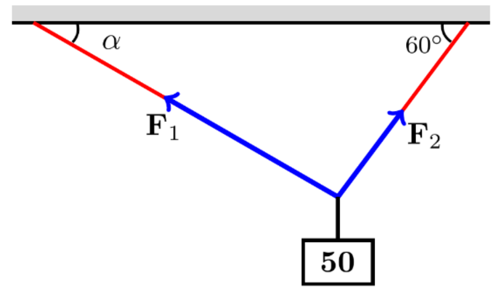
$$\boxed{\approx \langle -63.397,\ 36.603\rangle}$$

$$\vec{F}_2 = \left\langle \frac{100\sqrt{6}}{\sqrt{6}+\sqrt{2}},\ \frac{100\sqrt{6}}{\sqrt{6}+\sqrt{2}}\right\rangle$$

$$\boxed{\approx \langle 63.397,\ 63.397\rangle}$$

Exercise

Consider a 50-N weight suspended by two wires, If the magnitude of vector $\vec{F}_1 = 35\ N$, find the angle α and the magnitude of vector $\vec{F}_2$



Solution

$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos\alpha,\ \left|\vec{F}_1\right|\sin\alpha \right\rangle$$

$$= \langle -35\cos\alpha,\ 35\sin\alpha \rangle$$

$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos 60°,\ \left|\vec{F}_2\right|\sin 60° \right\rangle$$

$$= \left\langle \frac{1}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| \right\rangle$$

$$w = \langle 0,\ -50 \rangle \quad \Rightarrow \quad \vec{F}_1 + \vec{F}_2 = \langle 0,\ 50 \rangle$$

$$\langle -35\cos\alpha,\ 35\sin\alpha \rangle + \left\langle \frac{1}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| \right\rangle = \langle 0,\ 50 \rangle$$

$$\left\langle -35\cos\alpha + \frac{1}{2}\left|\vec{F}_2\right|,\ 35\sin\alpha + \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| \right\rangle = \langle 0,\ 50 \rangle$$

$$\to \begin{cases} -35\cos\alpha + \frac{1}{2}\left|\vec{F}_2\right| = 0 \\ 35\sin\alpha + \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| = 50 \end{cases} \qquad \to \begin{cases} \left|\vec{F}_2\right| = 70\cos\alpha \\ \end{cases}$$

$$35\sin\alpha + \frac{\sqrt{3}}{2}(70\cos\alpha) = 50$$

$$35\sqrt{3}\cos\alpha = 50 - 35\sin\alpha$$

$$\sqrt{3}\cos\alpha = \frac{10}{7} - \sin\alpha$$

$$\left(\sqrt{3}\cos\alpha\right)^2 = \left(\frac{10}{7} - \sin\alpha\right)^2$$

$$3\cos^2\alpha = \frac{100}{49} - \frac{20}{7}\sin\alpha + \sin^2\alpha$$

$$3\left(1 - \sin^2\alpha\right) = \frac{100}{49} - \frac{20}{7}\sin\alpha + \sin^2\alpha$$

$$3 - 3\sin^2\alpha - \frac{100}{49} + \frac{20}{7}\sin\alpha - \sin^2\alpha = 0$$

$$-4\sin^2\alpha + \frac{20}{7}\sin\alpha + \frac{47}{49} = 0$$

$$-196\sin^2\alpha + 140\sin\alpha + 47 = 0 \quad \Rightarrow \quad \sin\alpha = \frac{5 \pm 6\sqrt{2}}{14}$$

Since $\alpha > 0 \Rightarrow \sin\alpha > 0$

$$\to \sin\alpha = \frac{5+6\sqrt{2}}{14} \approx 0.963$$

$$\underline{|\alpha} \approx \sin^{-1}(0.963)$$

$$\underline{\approx 74.42^\circ\ |}$$

$$\left|\vec{F}_2\right| = 70\cos\alpha$$

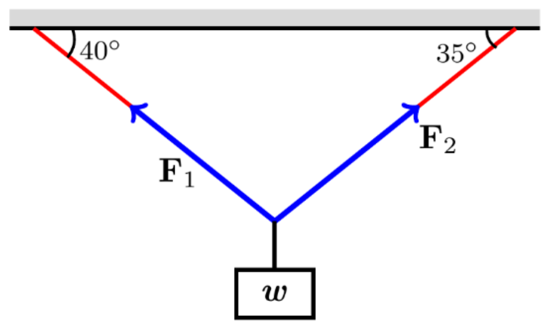
$$= 70\cos 74.42^\circ$$

$$\underline{\approx 18.81\ N\ |}$$

Exercise

Consider a ***w***-N weight suspended by two wires, If the magnitude of vector $\vec{F}_2 = 100\ N$, find ***w*** and the magnitude of vector $\vec{F}_1$

Solution



$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos 40^\circ,\ \left|\vec{F}_1\right|\sin 40^\circ \right\rangle$$

$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos 35^\circ,\ \left|\vec{F}_2\right|\sin 35^\circ \right\rangle$$

$$= \langle 100(0.819),\ 100(0.5736)\rangle$$

$$= \langle 81.915,\ 57.358\rangle$$

$$\vec{F}_1 + \vec{F}_2 = \langle 0,\ w\rangle$$

$$\left\langle -\left|\vec{F}_1\right|\cos 40^\circ,\ \left|\vec{F}_1\right|\sin 40^\circ \right\rangle + \langle 81.915,\ 57.358\rangle = \langle 0,\ w\rangle$$

$$\left\langle -\left|\vec{F}_1\right|\cos 40^\circ + 81.915,\ \left|\vec{F}_1\right|\sin 40^\circ + 57.358 \right\rangle = \langle 0,\ w\rangle$$

$$-\left|\vec{F}_1\right|\cos 40^\circ + 81.915 = 0$$

$$\left|\vec{F}_1\right|\cos 40^\circ = 81.915$$

$$\left|\vec{F}_1\right| = \frac{81.915}{\cos 40^\circ}$$

$$\underline{\approx 106.933\ N\ |}$$

$$w = \left|\vec{F}_1\right|\sin 40^\circ + 57.358$$

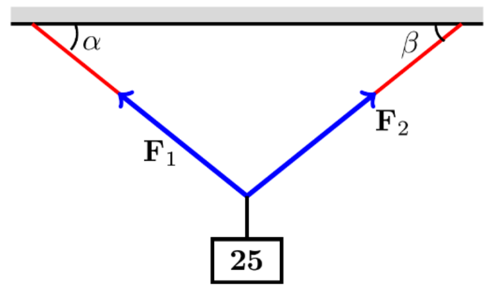
$$= 106.933\sin 40^\circ + 57.358$$

$$\underline{\approx 126.093\ N\ |}$$

Exercise

Consider a 25-N weight suspended by two wires, If the magnitude of vector $\vec{F}_1$ *and* $\vec{F}_2$ are both 75 N, then angles α and β are equal. Find α.

Solution



$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos\alpha,\ \left|\vec{F}_1\right|\sin\alpha \right\rangle$$

$$= \langle -75\cos\alpha,\ 75\sin\alpha \rangle$$

$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos\beta,\ \left|\vec{F}_2\right|\sin\beta \right\rangle$$

$$= \langle 75\cos\beta,\ 75\sin\beta \rangle$$

$$w = \langle 0,\ -25 \rangle \quad \Rightarrow \quad F_1 + F_2 = \langle 0,\ 25 \rangle$$

$$\langle -75\cos\alpha,\ 75\sin\alpha \rangle + \langle 75\cos\beta,\ 75\sin\beta \rangle = \langle 0,\ 25 \rangle \qquad \text{since} \quad \alpha = \beta$$

$$\langle -75\cos\alpha + 75\cos\alpha,\ 75\sin\alpha + 75\sin\alpha \rangle = \langle 0,\ 25 \rangle$$

$$-75\cos\alpha + 75\cos\beta = 0 \quad \Rightarrow \quad \cos\alpha = \cos\beta$$

$$150\sin\alpha = 25$$

$$\sin\alpha = \frac{25}{150}$$

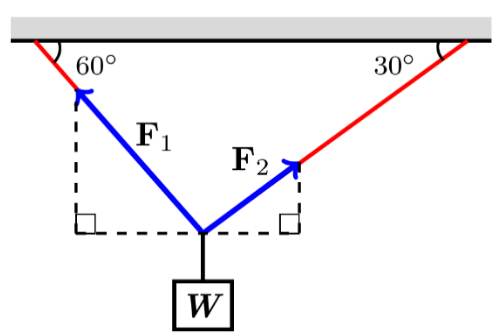
$$\alpha = \sin^{-1}\frac{25}{150}$$

$$\approx 9.59^\circ$$

Exercise

Consider a $W = 100\ N$ weight suspended by two wires. Find the magnitudes and components of the force vectors $\vec{F}_1$ *and* $\vec{F}_2$

Solution



$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos 60^\circ,\ \left|\vec{F}_1\right|\sin 60^\circ \right\rangle$$

$$= \left\langle -\frac{1}{2}\left|\vec{F}_1\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_1\right| \right\rangle$$

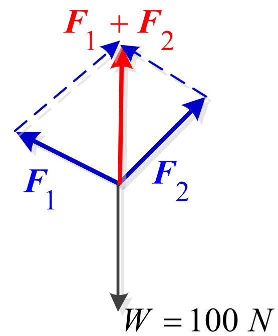
$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos 30^\circ,\ \left|\vec{F}_2\right|\sin 30^\circ \right\rangle$$

$$= \left\langle \frac{\sqrt{3}}{2}\left|\vec{F}_2\right|,\ \frac{1}{2}\left|\vec{F}_2\right| \right\rangle$$

$$\vec{F}_1 + \vec{F}_2 = \langle 0,\ 100\rangle$$

$$\left\langle -\frac{1}{2}\left|\vec{F}_1\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_1\right|\right\rangle + \left\langle \frac{\sqrt{3}}{2}\left|\vec{F}_2\right|,\ \frac{1}{2}\left|\vec{F}_2\right|\right\rangle = \langle 0,\ 100\rangle$$

$$\left\langle -\frac{1}{2}\left|\vec{F}_1\right| + \frac{\sqrt{3}}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_1\right| + \frac{1}{2}\left|\vec{F}_2\right|\right\rangle = \langle 0,\ 100\rangle$$



$$\begin{cases} -\frac{1}{2}\left|\vec{F}_1\right| + \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| = 0 \\ \frac{\sqrt{3}}{2}\left|\vec{F}_1\right| + \frac{1}{2}\left|\vec{F}_2\right| = 100 \end{cases}$$

$$\Delta = \begin{vmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix} = -1 \quad \Delta_1 = \begin{vmatrix} 0 & \frac{\sqrt{3}}{2} \\ 100 & \frac{1}{2} \end{vmatrix} = -50\sqrt{3} \quad \Delta = \begin{vmatrix} -\frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 100 \end{vmatrix} = -50$$

$$\Rightarrow \begin{cases} \boxed{\left|\vec{F}_1\right| = 50\sqrt{3}\ \ N} \\ \boxed{\left|\vec{F}_2\right| = 50\ \ N} \end{cases}$$

$$\vec{F}_1 = \left\langle -\frac{1}{2}\left(50\sqrt{3}\right),\ \frac{\sqrt{3}}{2}\left(50\sqrt{3}\right)\right\rangle$$

$$\boxed{= \left\langle -25\sqrt{3},\ 75\right\rangle}$$

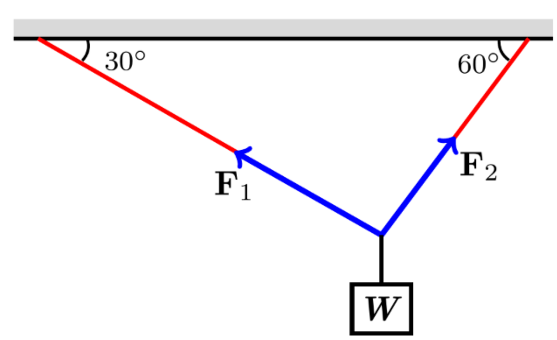
$$\vec{F}_2 = \left\langle \frac{\sqrt{3}}{2}(50),\ \frac{1}{2}(50)\right\rangle$$

$$\boxed{= \left\langle 25\sqrt{3},\ 25\right\rangle}$$

Exercise

Consider a $W = 50\ N$ weight suspended by two wires. Find the magnitudes and components of the force vectors $\vec{F}_1$ *and* $\vec{F}_2$

Solution



$$\vec{F}_1 = \left\langle -\left|\vec{F}_1\right|\cos 30°,\ \left|\vec{F}_1\right|\sin 30°\right\rangle$$

$$= \left\langle -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right|,\ \frac{1}{2}\left|\vec{F}_1\right|\right\rangle$$

$$\vec{F}_2 = \left\langle \left|\vec{F}_2\right|\cos 60°,\ \left|\vec{F}_2\right|\sin 60°\right\rangle$$

$$=\left\langle \frac{1}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| \right\rangle$$

$$\vec{F}_1+\vec{F}_2=\langle 0,\ 50\rangle$$

$$\left\langle -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right|,\ \frac{1}{2}\left|\vec{F}_1\right| \right\rangle+\left\langle \frac{1}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{3}}{2}\left|\vec{F}_2\right| \right\rangle=\langle 0,\ 50\rangle$$

$$\left\langle -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right|+\frac{1}{2}\left|\vec{F}_2\right|,\ \frac{1}{2}\left|\vec{F}_1\right|+\frac{\sqrt{3}}{2}\left|\vec{F}_2\right| \right\rangle=\langle 0,\ 50\rangle$$

$$\begin{cases} -\frac{\sqrt{3}}{2}\left|\vec{F}_1\right|+\frac{1}{2}\left|\vec{F}_2\right|=0 \\ \frac{1}{2}\left|\vec{F}_1\right|+\frac{\sqrt{3}}{2}\left|\vec{F}_2\right|=50 \end{cases}$$

$$\Delta=\begin{vmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{vmatrix}=-1 \quad \Delta_1=\begin{vmatrix} 0 & \frac{1}{2} \\ 50 & \frac{\sqrt{3}}{2} \end{vmatrix}=-25 \quad \Delta=\begin{vmatrix} -\frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 50 \end{vmatrix}=-25\sqrt{3}$$

$$\Rightarrow \begin{cases} \boxed{\left|\vec{F}_1\right|=25\ N} \\ \boxed{\left|\vec{F}_2\right|=25\sqrt{3}\ N} \end{cases}$$

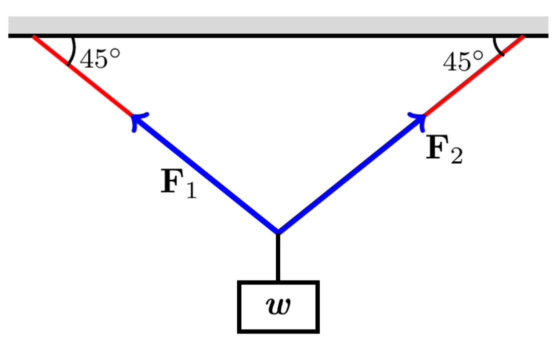
$$\boxed{\vec{F}_1=\left\langle -\frac{25\sqrt{3}}{2},\ \frac{25}{2} \right\rangle}$$

$$\boxed{\vec{F}_2=\left\langle \frac{25\sqrt{3}}{2},\ \frac{75}{2} \right\rangle}$$

Exercise

Consider a $W=100\ N$ weight suspended by two wires. Find the magnitudes and components of the force vectors $\vec{F}_1$ *and* $\vec{F}_2$

Solution



$$\vec{F}_1=\left\langle -\left|\vec{F}_1\right|\cos 45°,\ \left|\vec{F}_1\right|\sin 45° \right\rangle$$

$$=\left\langle -\frac{\sqrt{2}}{2}\left|\vec{F}_1\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_1\right| \right\rangle$$

$$\vec{F}_2=\left\langle \left|\vec{F}_2\right|\cos 45°,\ \left|\vec{F}_2\right|\sin 45° \right\rangle$$

$$=\left\langle \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| \right\rangle$$

$$\vec{F}_1 + \vec{F}_2 = \langle 0,\ 100\rangle$$

$$\left\langle -\frac{\sqrt{2}}{2}\left|\vec{F}_1\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_1\right|\right\rangle + \left\langle \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|\right\rangle = \langle 0,\ 100\rangle$$

$$\left\langle -\frac{\sqrt{2}}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|,\ \frac{\sqrt{2}}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right|\right\rangle = \langle 0,\ 100\rangle$$

$$\begin{cases} -\frac{\sqrt{2}}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| = 0 \\ \frac{\sqrt{2}}{2}\left|\vec{F}_1\right| + \frac{\sqrt{2}}{2}\left|\vec{F}_2\right| = 100 \end{cases}$$

$$\Delta = \begin{vmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{vmatrix} = -1 \quad \Delta_1 = \begin{vmatrix} 0 & \frac{\sqrt{2}}{2} \\ 100 & \frac{\sqrt{2}}{2} \end{vmatrix} = -50\sqrt{2} \quad \Delta = \begin{vmatrix} -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 100 \end{vmatrix} = -50\sqrt{2}$$

$$\Rightarrow \begin{cases} \boxed{\left|\vec{F}_1\right| = 50\sqrt{2}\ \ N} \\ \boxed{\left|\vec{F}_2\right| = 50\sqrt{2}\ \ N} \end{cases}$$

$$\vec{F}_1 = \left\langle -\frac{\sqrt{2}}{2}\left(50\sqrt{2}\right),\ \frac{\sqrt{2}}{2}\left(50\sqrt{2}\right)\right\rangle$$

$$\boxed{= \langle -50,\ \ 50\rangle}$$

$$\vec{F}_2 = \left\langle \frac{\sqrt{2}}{2}\left(50\sqrt{2}\right),\ \frac{\sqrt{2}}{2}\left(50\sqrt{2}\right)\right\rangle$$

$$\boxed{= \langle 50,\ \ 50\rangle}$$

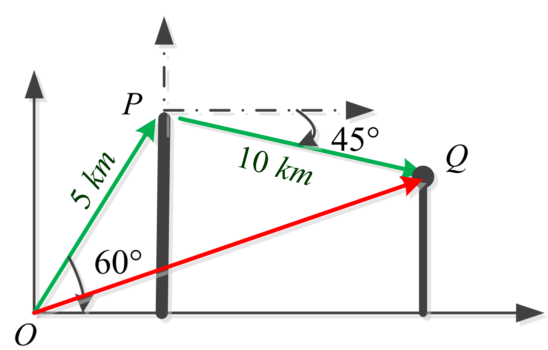
Exercise

A bird flies from its nest 5 km in the direction 60° north east, where it stops to rest on a tree. It then flies 10 km in the direction due southeast and lands atop a telephone pole. Place an xy-coordinate system so that the origin is the bird's nest, the x-axis points east, and the y-axis points north.

a) At what point is the tree located?

b) At what point is the telephone pole?

Solution



a) $\overrightarrow{OP} = \left(5\cos 60°\right)\hat{i} + \left(5\sin 60°\right)\hat{j}$

$$= \frac{5}{2}\hat{i} + \frac{5\sqrt{3}}{2}\hat{j}$$

The tree is located at the point

$$P = \left(\frac{5}{2},\ \frac{5\sqrt{3}}{2}\right)$$

b) $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$

$$= \frac{5}{2}\hat{i} + \frac{5\sqrt{3}}{2}\hat{j} + \left(10\cos 315°\right)\hat{i} + \left(10\sin 315°\right)\hat{j}$$

$$= \frac{5}{2}\hat{i} + \frac{5\sqrt{3}}{2}\hat{j} + \left(10\frac{\sqrt{2}}{2}\right)\hat{i} + \left(10\left(-\frac{\sqrt{2}}{2}\right)\right)\hat{j}$$

$$= \left(\frac{5}{2} + 5\sqrt{2}\right)\hat{i} + \left(\frac{5\sqrt{3}}{2} - \frac{10\sqrt{2}}{2}\right)\hat{j}$$

$$= \left(\frac{5+10\sqrt{2}}{2}\right)\hat{i} + \left(\frac{5\sqrt{3}-10\sqrt{2}}{2}\right)\hat{j}$$

The pole is located at the point $Q = \left(\frac{5+10\sqrt{2}}{2},\ \frac{5\sqrt{3}-10\sqrt{2}}{2}\right)$

Exercise

Suppose that A, B, and C are the corner points of the thin triangular plate of constant density.

a) Find the vector from C to the midpoint M of side AB.

b) Find the vector from C to the point that lies two-thirds of the way from C to M on the median CM.

c) Find the coordinates of the point in which the medians of $\triangle ABC$ intersect (this point is the plate's center of mass).

Solution

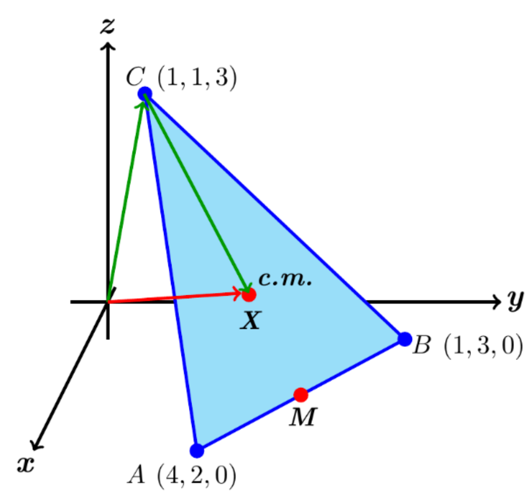
a) The midpoint of AB is:

$$M = \left(\frac{4+1}{2}, \frac{2+3}{2}, 0\right)$$

$$= \left(\frac{5}{2},\ \frac{5}{2},\ 0\right)$$

$$\overrightarrow{CM} = \left(\frac{5}{2} - 1\right)\hat{i} + \left(\frac{5}{2} - 1\right)\hat{j} + (0-3)\hat{k}$$

$$= \frac{3}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$$



b) The desired vector is

$$\overrightarrow{CX} = \frac{2}{3}\overrightarrow{CM}$$

$$= \frac{2}{3}\left(\frac{3}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}\right)$$

$$= \hat{i} + \hat{j} - 2\hat{k}$$

c) The vector whose sum is the vector from the origin to C and the result of part (b) will terminate at the center of mass.

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OC} + \overrightarrow{CX} \\ &= \hat{i} + \hat{j} + 3\hat{k} + \hat{i} + \hat{j} - 2\hat{k} \\ &= \boxed{2\hat{i} + 2\hat{j} + \hat{k}}\end{aligned}$$

Therefore; the center of mass point is $\boxed{(2,\ 2,\ 1)}$

Exercise

Show that a unit vector in the plane can be expressed as $\vec{u} = (\cos\theta)\,\hat{i} + (\sin\theta)\,\hat{j}$, obtained by rotating $\hat{i}$ through an angle θ in the counterclockwise direction. Explain why this form gives ***every*** *unit vector* in the plane.

Solution

Let $\vec{u}$ be any unit vector in the plane.

If $\vec{u}$ is positioned so that its initial point and terminal point is at (x, y), then $\vec{u}$ makes an angle θ with $\hat{i}$, measured in the *ccw* direction.

Since $|\vec{u}| = 1 \quad \Rightarrow \quad x = \cos\theta \quad$ *and* $\quad y = \sin\theta$

That implies to: $\vec{u} = (\cos\theta)\hat{i} + (\sin\theta)\hat{j}$

Since $\vec{u}$ is any unit vector in the plane; this holds for every unit vector in the plane.

Exercise

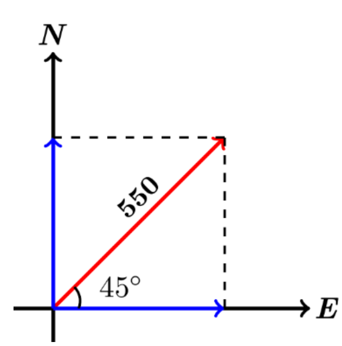
Assume the positive *x-axis* points east and the positive *y-axis* points north.

a) An airliner flies northeast at a constant altitude at 550 *mi/hr* in calm air. Find a and b such that its velocity may be expressed in the form $\vec{v} = a\,\hat{i} + b\,\hat{j}$

b) An airliner flies northeast at a constant altitude at 550 *mi/hr* relative to the air in a southerly crosswind $\vec{w} = \langle 0,\ 40\rangle$. Find the velocity of the airliner relative to the ground.

Solution

a) $$\begin{aligned}\vec{v} &= 550\langle -\cos 45°,\ \ \sin 45°\rangle \\ &= 550\left\langle -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right\rangle \\ &= \boxed{\left\langle -275\sqrt{2},\ 275\sqrt{2}\right\rangle}\end{aligned}$$



b) $\vec{v} = 550\left\langle -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right\rangle + \langle 0,\ \ 40\rangle$

$$= \left\langle -275\sqrt{2},\ 275\sqrt{2} \right\rangle$$

Exercise

Let $\overrightarrow{PQ}$ extended from $P(2,\ 0,\ 6)$ to $Q(2,\ -8,\ 5)$

a) Find the position vector equal to $\overrightarrow{PQ}$.

b) Find the midpoint M of the line segment PQ. Then find the magnitude of $\overrightarrow{PM}$.

c) Find a vector of length 8 with direction opposite that of $\overrightarrow{PQ}$.

Solution

a) $\overrightarrow{PQ} = \langle 2-2,\ -8-0,\ 5-6 \rangle$

$$= \langle 0,\ -8,\ -1 \rangle$$

b) $M = \left(\frac{2+2}{2},\ \frac{0-8}{2},\ \frac{6+5}{2} \right)$

$$= \left(2,\ -4,\ \frac{11}{2} \right)$$

$$\overrightarrow{PM} = \left\langle 0,\ -4,\ -\frac{1}{2} \right\rangle$$

$$\left| \overrightarrow{PM} \right| = \sqrt{16 + \frac{1}{4}}$$

$$= \frac{1}{2}\sqrt{65}$$

c) $\left| \overrightarrow{PQ} \right| = \sqrt{64+1}$

$$= \sqrt{65}$$

$$vector = \frac{-8}{\sqrt{65}} \langle 0,\ -8,\ -1 \rangle$$

$$= \frac{8}{\sqrt{65}} \langle 0,\ 8,\ 1 \rangle$$

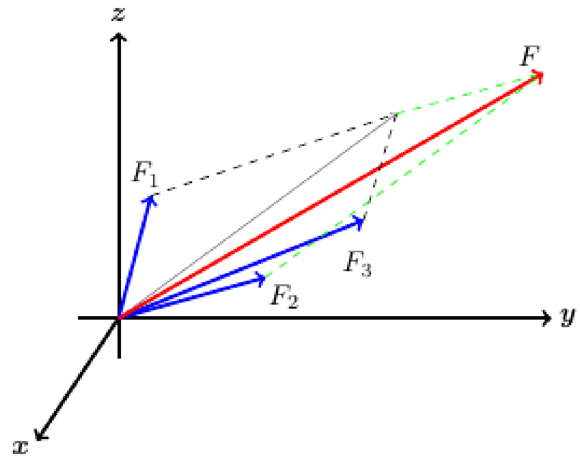
Exercise

An object at the origin is acted on by the forces $\vec{F}_1 = -10\hat{i} + 20\hat{k}$, $\vec{F}_2 = 40\hat{j} + 10\hat{k}$, and $\vec{F}_3 = -50\hat{i} + 20\hat{j}$. Find the magnitude of the combined force and use a sketch to illustrate the direction of the combined force.

Solution

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$
$$= -10\hat{i} + 20\hat{k} + 40\hat{j} + 10\hat{k} - 50\hat{i} + 20\hat{j}$$
$$= -60\hat{i} + 60\hat{j} + 30\hat{k}$$

$$\left|\vec{F}\right| = \sqrt{3600 + 3600 + 900}$$
$$= \sqrt{8100}$$
$$= 90$$



Exercise

A remote sensing probe falls vertically with a terminal of 60 *m/s* when it encounters a horizontal crosswind blowing north at 4 *m/s* and an updraft blowing vertically at 10 *m/s*. find the magnitude and direction of the resulting velocity relative to the ground.

Solution

The velocity relative to the ground is:

$$\langle 0,\ 4,\ 10-60\rangle = \langle 0,\ 4,\ -50\rangle$$

$$\textit{Magnitude}: \sqrt{16+2500} = \sqrt{2516}$$
$$= 2\sqrt{629} \quad \approx 50.16\ \ m/s$$

$$\textit{Direction} = \cos^{-1}\frac{4}{\sqrt{2516}}$$
$$\approx 85.4^\circ$$

Below the horizontal in the northerly horizontal direction.

Exercise

A small plane is flying north in calm air at 250 *mi/hr* when it is hit by a horizontal crosswind blowing northeast at 40 *mi/hr* and a 25 *mi/hr* downdraft. Find the resulting velocity and speed of the plane.

Solution

$$\textit{Velocity vector} = \langle 250,\ 0,\ 0\rangle$$

$$\textit{Crosswind} = \langle 40\cos 45^\circ,\ 40\sin 45^\circ,\ 0\rangle$$

$$= \left\langle 20\sqrt{2},\ 20\sqrt{2},\ 0 \right\rangle$$

$$Downdraft = \langle 0,\ 0,\ -25 \rangle$$

$$Resulting\ velocity = \langle 250,\ 0,\ 0 \rangle + \left\langle 20\sqrt{2},\ 20\sqrt{2},\ 0 \right\rangle + \langle 0,\ 0,\ -25 \rangle$$

$$= \left\langle 250 + 20\sqrt{2},\ 20\sqrt{2},\ -25 \right\rangle$$

$$Speed = \sqrt{\left(250 + 20\sqrt{2}\right)^2 + 800 + 625}$$

$$= \sqrt{62500 + 10^4\sqrt{2} + 800 + 1,425}$$

$$= \sqrt{64,725 + 10^4\sqrt{2}}$$

$$= 5\sqrt{2,589 + 400\sqrt{2}}$$

$$\underline{= 280.83\ mph}$$

Solution Section 1.2 – Dot Products

Exercise

Find for $\vec{v} = 2\hat{i} - 4\hat{j} + \sqrt{5}\hat{k}, \quad \vec{u} = -2\hat{i} + 4\hat{j} - \sqrt{5}\hat{k}$

a) $\vec{v} \bullet \vec{u}, \ |\vec{v}|, \ |\vec{u}|$

b) The cosine of the angle between $\vec{v}$ and $\vec{u}$

c) The scalar component of $\vec{u}$ in the direction of $\vec{v}$

d) The vector $proj_{\vec{v}}\vec{u}$

Solution

a)
$$\begin{aligned}\vec{v} \bullet \vec{u} &= \left(2\hat{i} - 4\hat{j} + \sqrt{5}\hat{k}\right) \cdot \left(-2\hat{i} + 4\hat{j} - \sqrt{5}\hat{k}\right) \\ &= -4 - 16 - 5 \\ &= -25\end{aligned}$$

$$\begin{aligned}|\vec{v}| &= \sqrt{2^2 + (-4)^2 + \left(\sqrt{5}\right)^2} \\ &= \sqrt{4 + 16 + 5} \\ &= \sqrt{25} \\ &= 5\end{aligned}$$

$$\begin{aligned}|\vec{u}| &= \sqrt{(-2)^2 + 4^2 + \left(-\sqrt{5}\right)^2} \\ &= \sqrt{25} \\ &= 5\end{aligned}$$

b)
$$\begin{aligned}\cos\theta &= \frac{-25}{(5)(5)} \\ &= -1\end{aligned}$$

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{|\vec{u}|\ |\vec{v}|}$$

c)
$$\begin{aligned}|\vec{u}|\cos\theta &= (5)(-1) \\ &= -5\end{aligned}$$

d)
$$\begin{aligned}proj_{\vec{v}}\vec{u} &= \left(\frac{\vec{u} \bullet \vec{v}}{|\vec{v}|^2}\right)\vec{v} \\ &= \left(\frac{-25}{5^2}\right)\left(2\hat{i} - 4\hat{j} + \sqrt{5}\hat{k}\right) \\ &= -\left(2\hat{i} - 4\hat{j} + \sqrt{5}\hat{k}\right) \\ &= -2\hat{i} + 4\hat{j} - \sqrt{5}\hat{k}\end{aligned}$$

Exercise

Find for $\vec{v} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{k}$, $\vec{u} = 5\hat{i} + 12\hat{j}$

a) $\vec{v} \cdot \vec{u}$, $|\vec{v}|$, $|\vec{u}|$

b) The cosine of the angle between $\vec{v}$ and $\vec{u}$

c) The scalar component of $\vec{u}$ in the direction of $\vec{v}$

d) The vector $proj_{\vec{v}}\vec{u}$

Solution

a) $\vec{v} \cdot \vec{u} = \left(\frac{3}{5}\hat{i} + \frac{4}{5}\hat{k}\right) \cdot \left(5\hat{i} + 12\hat{j}\right)$

$$= 3$$

$$|\vec{v}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$

$$= \sqrt{\frac{9}{25} + \frac{16}{25}}$$

$$= \sqrt{\frac{25}{25}}$$

$$= 1$$

$$|\vec{u}| = \sqrt{5^2 + 12^2}$$

$$= 13$$

b) $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}|\ |\vec{v}|}$

$$= \frac{3}{(1)(13)}$$

$$= \frac{3}{13}$$

c) $|\vec{u}|\cos\theta = (13)\left(\frac{3}{13}\right) = 3$

d) $proj_{\vec{v}}\vec{u} = \left(\dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right)\vec{v}$

$$= \left(\frac{3}{1^2}\right)\left(\frac{3}{5}\hat{i} + \frac{4}{5}\hat{k}\right)$$

$$= \frac{9}{5}\hat{i} + \frac{12}{5}\hat{k}$$

Exercise

Find for $\vec{v} = 2\hat{i} + 10\hat{j} - 11\hat{k}$, $\vec{u} = 2\hat{i} + 2\hat{j} + \hat{k}$

a) $\vec{v} \cdot \vec{u}$, $|\vec{v}|$, $|\vec{u}|$

b) The cosine of the angle between $\vec{v}$ and $\vec{u}$

c) The scalar component of $\vec{u}$ in the direction of $\vec{v}$

d) The vector $proj_{\vec{v}}\vec{u}$

Solution

a)
$$\vec{v} \cdot \vec{u} = \left(2\hat{i} + 10\hat{j} - 11\hat{k}\right) \cdot \left(2\hat{i} + 2\hat{j} + \hat{k}\right)$$
$$= 4 + 20 - 11$$
$$= 13$$

$$|\vec{v}| = \sqrt{2^2 + 10^2 + (-11)^2}$$
$$= \sqrt{4 + 100 + 121}$$
$$= \sqrt{225}$$
$$= 15$$

$$|\vec{u}| = \sqrt{2^2 + 2^2 + 1^2}$$
$$= 3$$

b)
$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\ |\vec{v}|}$$
$$= \frac{13}{(3)(15)}$$
$$= \frac{13}{45}$$

c)
$$|\vec{u}|\ \cos\theta = (3)\left(\frac{13}{45}\right)$$
$$= \frac{13}{15}$$

d)
$$proj_{\vec{v}}\vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right)\vec{v}$$
$$= \left(\frac{13}{15^2}\right)\left(2\hat{i} + 10\hat{j} - 11\hat{k}\right)$$
$$= \frac{13}{225}\left(2\hat{i} + 10\hat{j} - 11\hat{k}\right)$$

Exercise

Find for $\vec{v} = -\hat{i} + \hat{j}, \quad \vec{u} = 2\hat{i} + \sqrt{17}\,\hat{j}$

a) $\vec{v} \bullet \vec{u}, \ |\vec{v}|, \ |\vec{u}|$

b) The cosine of the angle between $\vec{v}$ and $\vec{u}$

c) The scalar component of $\vec{u}$ in the direction of $\vec{v}$

d) The vector $proj_{\vec{v}}\vec{u}$

Solution

a) $\vec{v} \bullet \vec{u} = \left(5\hat{i} + \hat{j}\right) \bullet \left(2\hat{i} + \sqrt{17}\,\hat{j}\right)$

$$= 10 + \sqrt{17}$$

$$|\vec{v}| = \sqrt{25 + 1}$$

$$= \sqrt{26}$$

$$|\vec{u}| = \sqrt{4 + 17}$$

$$= \sqrt{21}$$

b) $\cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{|\vec{u}|\ |\vec{v}|}$

$$= \frac{10 + \sqrt{17}}{\sqrt{21}\sqrt{26}}$$

$$= \frac{10 + \sqrt{17}}{\sqrt{546}}$$

c) $|\vec{u}|\cos\theta = \left(\sqrt{21}\right)\left(\dfrac{10 + \sqrt{17}}{\sqrt{546}}\right)$

$$= \frac{10 + \sqrt{17}}{\sqrt{26}}$$

d) $proj_{\vec{v}}\vec{u} = \left(\dfrac{\vec{u} \bullet \vec{v}}{|\vec{v}|^2}\right)\vec{v}$

$$= \left(\frac{10 + \sqrt{17}}{26}\right)\left(5\hat{i} + \hat{j}\right)$$

Exercise

Find for $\vec{v} = \left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}} \right\rangle$, $\vec{u} = \left\langle \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{3}} \right\rangle$

a) $\vec{v} \bullet \vec{u},\ |\vec{v}|,\ |\vec{u}|$

b) The cosine of the angle between $\vec{v}$ and $\vec{u}$

c) The scalar component of $\vec{u}$ in the direction of $\vec{v}$

d) The vector $proj_{\vec{v}}\vec{u}$

Solution

a) $\vec{v} \bullet \vec{u} = \left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}} \right\rangle \bullet \left\langle \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{3}} \right\rangle$

$$= \frac{1}{2} - \frac{1}{3}$$

$$\boxed{= \frac{1}{6}}$$

$$|\vec{v}| = \sqrt{\frac{1}{2} + \frac{1}{3}}$$

$$= \frac{\sqrt{5}}{\sqrt{6}} \frac{\sqrt{6}}{\sqrt{6}}$$

$$\boxed{= \frac{\sqrt{30}}{6}}$$

$$|\vec{u}| = \sqrt{\frac{1}{2} + \frac{1}{3}}$$

$$= \frac{\sqrt{5}}{\sqrt{6}} \frac{\sqrt{6}}{\sqrt{6}}$$

$$\boxed{= \frac{\sqrt{30}}{6}}$$

b) $\cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{|\vec{u}|\ |\vec{v}|}$

$$= \frac{\frac{1}{6}}{\frac{\sqrt{30}}{6}\frac{\sqrt{30}}{6}}$$

$$= \frac{1}{6}\left(\frac{36}{30}\right)$$

$$\boxed{= \frac{1}{5}}$$

c) $|\vec{u}|\cos\theta = \left(\frac{\sqrt{30}}{6}\right)\left(\frac{1}{5}\right)$

$$= \frac{\sqrt{30}}{30}$$

$$= \frac{1}{\sqrt{30}}$$

d) $proj_{\vec{v}}\vec{u} = \left(\frac{\vec{u} \bullet \vec{v}}{|\vec{v}|^2}\right)\vec{v}$

$$= \frac{1}{6}\left(\frac{36}{30}\right)\left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}} \right\rangle$$

$$= \frac{1}{5}\left\langle \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{3}} \right\rangle$$

Exercise

Find the angles between the vectors $\vec{u} = 2\hat{i} + \hat{j}, \quad \vec{v} = \hat{i} + 2\hat{j} - \hat{k}$

Solution

$$\theta = \cos^{-1}\left(\frac{2+2+0}{\sqrt{4+1}\sqrt{1+4+1}}\right) \qquad \cos\theta = \frac{\vec{u} \bullet \vec{v}}{|\vec{u}|\ |\vec{v}|}$$

$$= \cos^{-1}\left(\frac{4}{\sqrt{5}\sqrt{6}}\right)$$

$$= \cos^{-1}\left(\frac{4}{\sqrt{30}}\right)$$

$$\approx 0.84\ \ rad$$

Exercise

Find the angles between the vectors $\vec{u} = \sqrt{3}\hat{i} - 7\hat{j}, \quad \vec{v} = \sqrt{3}\hat{i} + \hat{j} + \hat{k}$

Solution

$$\theta = \cos^{-1}\left(\frac{3-7+0}{\sqrt{3+49}\sqrt{3+1+1}}\right) \qquad \cos\theta = \frac{\vec{u} \bullet \vec{v}}{|\vec{u}|\ |\vec{v}|}$$

$$= \cos^{-1}\left(\frac{-4}{\sqrt{52}\sqrt{5}}\right)$$

$$= \cos^{-1}\left(-\frac{4}{\sqrt{260}}\right)$$

$$\approx 1.82\ \ rad$$

Exercise

Find the angles between the vectors $\vec{u} = \hat{i} + \sqrt{2}\hat{j} - \sqrt{2}\hat{k}, \quad \vec{v} = -\hat{i} + \hat{j} + \hat{k}$

Solution

$$\theta = \cos^{-1}\left(\frac{-1+\sqrt{2}-\sqrt{2}}{\sqrt{1+2+2}\sqrt{1+1+1}}\right) \qquad \cos\theta = \frac{\vec{u}\bullet\vec{v}}{|\vec{u}|\ |\vec{v}|}$$

$$= \cos^{-1}\left(\frac{-1}{\sqrt{5}\sqrt{3}}\right)$$

$$= \cos^{-1}\left(-\frac{1}{\sqrt{15}}\right)$$

$$\approx 1.83\ rad$$

Exercise

Consider $\vec{u} = -3\hat{j} + 4\hat{k}$, $\vec{v} = -4\hat{i} + \hat{j} + 5\hat{k}$

a) Find the angle between $\vec{u}$ and $\vec{v}$.

b) Compute $proj_{\vec{v}}\vec{u}$ and $scal_{\vec{v}}\vec{u}$

c) Compute $proj_{\vec{u}}\vec{v}$ and $scal_{\vec{u}}\vec{v}$

Solution

a) $$\theta = \cos^{-1}\frac{\left(-3\hat{j}+4\hat{k}\right)\bullet\left(-4\hat{i}+\hat{j}+5\hat{k}\right)}{\sqrt{9+16}\ \sqrt{16+1+25}} \qquad \cos\theta = \frac{\vec{u}\bullet\vec{v}}{|\vec{u}|\ |\vec{v}|}$$

$$= \cos^{-1}\frac{-3+20}{\sqrt{25}\ \sqrt{42}}$$

$$= \cos^{-1}\frac{17}{5\sqrt{42}}$$

$$\approx 1.02\ rad$$

b) $$proj_{\vec{v}}\vec{u} = \frac{17}{42}\left(-4\hat{i}+\hat{j}+5\hat{k}\right) \qquad proj_{\vec{v}}\vec{u} = \frac{\vec{u}\bullet\vec{v}}{|\vec{v}|^2}\vec{v}$$

$$= \frac{17}{42}\langle -4,\ 1,\ 5\rangle$$

$$scal_{\vec{v}}\vec{u} = \frac{17}{\sqrt{42}} \qquad scal_{\vec{v}}\vec{u} = \frac{\vec{u}\bullet\vec{v}}{|\vec{v}|}$$

c) $$proj_{\vec{u}}\vec{v} = \frac{17}{25}\left(-3\hat{j}+4\hat{k}\right) \qquad proj_{\vec{u}}\vec{v} = \frac{\vec{u}\bullet\vec{v}}{|\vec{u}|^2}\vec{u}$$

$$= \frac{17}{25}\langle 0,\ -3,\ 4\rangle$$

$$\boxed{scal_{\vec{u}}\,\vec{v} = \frac{17}{5}} \qquad scal_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$$

Exercise

Consider $\vec{u} = -\hat{i} + 2\hat{j} + 2\hat{k}$, $\vec{v} = 3\hat{i} + 6\hat{j} + 6\hat{k}$

a) Find the angle between $\vec{u}$ and $\vec{v}$.

b) Compute $proj_{\vec{v}}\vec{u}$ and $scal_{\vec{v}}\vec{u}$

c) Compute $proj_{\vec{u}}\vec{v}$ and $scal_{\vec{u}}\vec{v}$

Solution

a)

$$\begin{aligned}\theta &= \cos^{-1}\frac{\left(-\hat{i} + 2\hat{j} + 2\hat{k}\right)\cdot\left(3\hat{i} + 6\hat{j} + 6\hat{k}\right)}{\sqrt{1+4+4}\ \sqrt{9+36+36}} \\ &= \cos^{-1}\frac{-3+12+12}{3\ (9)} \\ &= \cos^{-1}\frac{21}{27} \\ &= \boxed{\cos^{-1}\frac{7}{9}} \\ &\boxed{\approx 0.68\ \ rad}\end{aligned}$$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\ |\vec{v}|}$$

b)

$$\begin{aligned}proj_{\vec{v}}\,\vec{u} &= \frac{21}{81}\langle 3,\ 6,\ 6\rangle \\ &= \boxed{\frac{7}{9}\langle 1,\ 2,\ 2\rangle}\end{aligned} \qquad proj_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}$$

$$\begin{aligned}scal_{\vec{v}}\,\vec{u} &= \frac{21}{9} \\ &= \boxed{\frac{7}{3}}\end{aligned} \qquad scal_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$$

c)

$$\begin{aligned}proj_{\vec{u}}\,\vec{v} &= \frac{21}{9}\langle -1,\ 2,\ 2\rangle \\ &= \boxed{\frac{7}{3}\langle -1,\ 2,\ 2\rangle}\end{aligned} \qquad proj_{\vec{u}}\,\vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2}\vec{u}$$

$$\begin{aligned}scal_{\vec{u}}\,\vec{v} &= \frac{21}{3} \\ &= \boxed{7}\end{aligned} \qquad scal_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$$

Exercise

The direction angles α, β, and γ of a vector $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$ are defined as follows:

is the angle between $\vec{v}$ and the positive x-axis $(0 \le \alpha \le \pi)$

is the angle between $\vec{v}$ and the positive y-axis $(0 \le \beta \le \pi)$

is the angle between $\vec{v}$ and the positive z-axis $(0 \le \gamma \le \pi)$

a) Show that $\cos\alpha = \dfrac{a}{|\vec{v}|}$, $\cos\beta = \dfrac{b}{|\vec{v}|}$, $\cos\gamma = \dfrac{c}{|\vec{v}|}$, and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. These cosines are called the direction cosines of $\vec{v}$.

b) Show that if $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$ is a unit vector, then a, b, and c are the direction cosines of $\vec{v}$.

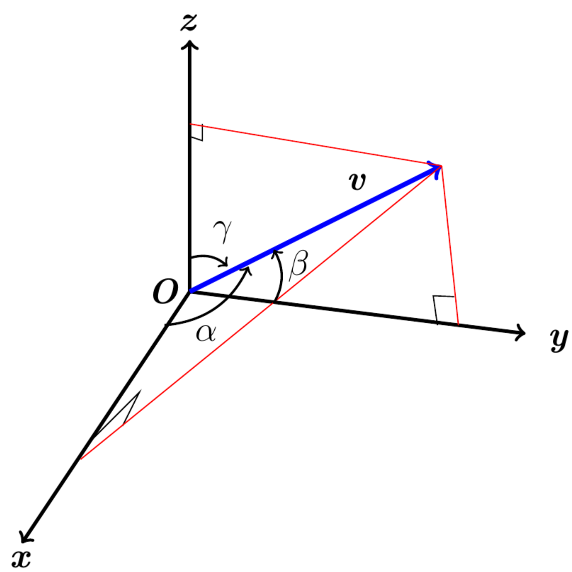
Solution

a)

$$\cos\alpha = \frac{\hat{i}\bullet\vec{v}}{|\hat{i}||\vec{v}|}$$

$$= \frac{\hat{i}\bullet\left(a\hat{i} + b\hat{j} + c\hat{k}\right)}{|\vec{v}|}$$

$$\boxed{= \frac{a}{|\vec{v}|}}$$



$$\cos\beta = \frac{\hat{j}\bullet\vec{v}}{|\hat{j}||\vec{v}|}$$

$$= \frac{\hat{j}\bullet\left(a\hat{i} + b\hat{j} + c\hat{k}\right)}{|\vec{v}|}$$

$$\boxed{= \frac{b}{|\vec{v}|}}$$

$$\cos\gamma = \frac{\hat{k}\bullet\vec{v}}{|\hat{k}||\vec{v}|}$$

$$= \frac{\hat{k}\bullet\left(a\hat{i} + b\hat{j} + c\hat{k}\right)}{|\vec{v}|}$$

$$\boxed{= \frac{c}{|\vec{v}|}}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{a}{|\vec{v}|}\right)^2 + \left(\frac{b}{|\vec{v}|}\right)^2 + \left(\frac{c}{|\vec{v}|}\right)^2$$

$$= \frac{a^2}{|\vec{v}|^2} + \frac{b^2}{|\vec{v}|^2} + \frac{c^2}{|\vec{v}|^2}$$

$$= \frac{a^2 + b^2 + c^2}{|\vec{v}|^2}$$

$$= \frac{a^2 + b^2 + c^2}{a^2 + b^2 + c^2}$$

$$= 1$$

b) If $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$ is a unit vector $\Rightarrow |\vec{v}| = 1$

$\cos\alpha = \frac{a}{|\vec{v}|} = a, \quad \cos\beta = \frac{b}{|\vec{v}|} = b, \quad \cos\gamma = \frac{c}{|\vec{v}|} = c$ are the direction cosines of $\vec{v}$.

Exercise

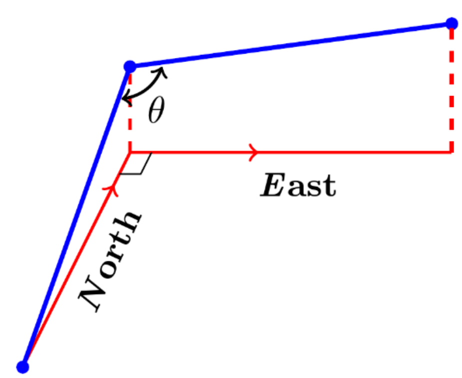
A water main is to be constructed with 20% grade in the north direction and a 10% grade in the east direction. Determine the angle θ required in the water main for the turn from north to east.

Solution

20% grade in the north direction

$z\hat{k} = 20\% x\hat{i} = .2x\hat{i}$

$\rightarrow If \ x = 10 \quad z = 2$



Let $\vec{u} = 10\hat{i} + 2\hat{k}$ be parallel to the pipe in the north direction.

$\vec{v} = 10\hat{j} + \hat{k}$ be parallel to the pipe in the east direction.

$$\theta = \cos^{-1} \frac{0 + 0 + 2}{\sqrt{100 + 4}\sqrt{100 + 1}} \qquad \theta = \cos^{-1} \frac{\vec{u} \bullet \vec{v}}{|\vec{u}||\vec{v}|}$$

$$= \cos^{-1} \frac{2}{\sqrt{104}\sqrt{101}}$$

$$\approx 88.88^\circ$$

Exercise

A gun with muzzle velocity of 1200 *ft/sec* is fired at an angle of 8° above the horizontal. Find the horizontal and vertical components of the velocity.

Solution

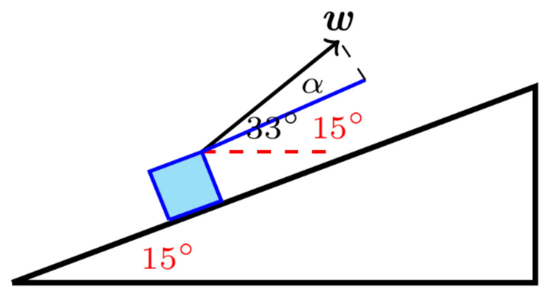
Horizontal component: $1200\cos 8^\circ \approx 1188 \ \ ft/s$

Vertical component: $1200\sin 8^\circ \approx 167 \ \ ft/s$

Exercise

Suppose that a box is being towed up an inclined plane. Find the force ***w*** needed to make the component of the force parallel to the indicated plane equal to 2.5 *lb*.

Solution



$$2.5 = |w|\cos\alpha$$

$$|\vec{w}| = \frac{2.5}{\cos(33° - 15°)}$$

$$= \frac{2.5}{\cos 18°}$$

$$\vec{w} = \frac{2.5}{\cos 18°}\langle \cos 33°,\ \sin 33°\rangle$$

$$= \langle 2.205,\ 1.432\rangle$$

Exercise

Find the work done by a force $\vec{F} = 5\hat{i}$ (magnitude 5 *N*) in moving an object along the line from the origin to the point (1, 1) (distance in meters)

Solution

$$P(1,\ 1) \Rightarrow \overrightarrow{OP} = \hat{i} + \hat{j}$$

$$W = F \bullet \overrightarrow{OP}$$

$$= 5\hat{i} \bullet (\hat{i} + \hat{j})$$

$$= 5\ J$$

Exercise

How much work does it take to slide a crate 20 *m* along a loading dock by pulling on it with a 200 *N* force at an angle of 30° from the horizontal?

Solution

$$W = |F|\left|\overrightarrow{PQ}\right|\cos\theta$$

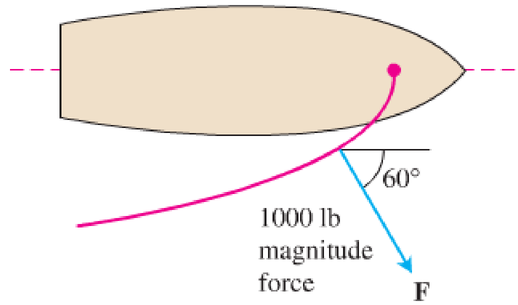
$$= (200)(20)\cos 30°$$

$$= 3464.10\ J$$

Exercise

The wind passing over a boat's sail exerted a 1000-*lb* magnitude force *F*. How much work did the wind perform in moving the boat forward 1 *mi*? Answer in foot-pounds.

Solution



$$W = |F|\left|\overrightarrow{PQ}\right|\cos\theta$$

$$= (1000N)\left(1\ mi\,\frac{5280\,ft}{1\ mi}\right)\cos 60^\circ$$

$$\underline{= 2,640,000 \quad ft\cdot lb}$$

Exercise

Use a dot product to find an equation of the line in the *xy*-plane passing through the point (x_0, y_0) perpendicular to the vector $\langle a,\ b\rangle$.

Solution

$$\langle x-x_0,\ y-y_0\rangle \cdot \langle a,\ b\rangle = 0$$

$$\underline{a(x-x_0)+b(y-y_0)=0}$$

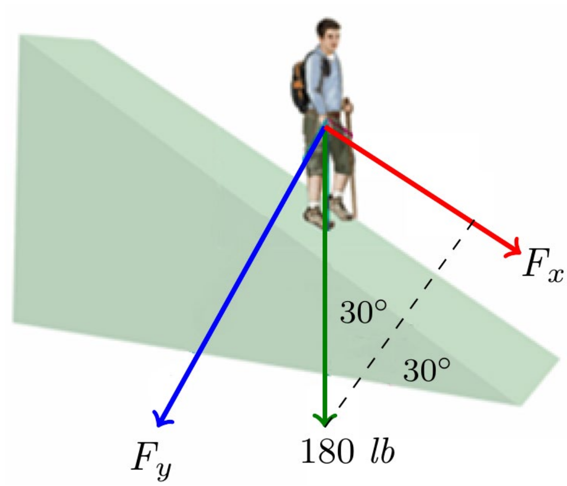
Exercise

A 180-*lb* man stands on a hillside that makes an angle of 30° with the horizontal, producing a force of $W = \langle 0,\ -180\rangle$ *lbs*.

a) Find the component of his weight in the downward direction perpendicular to the hillside and in the downward parallel to the hillside.

b) How much work is done when the man moves 10 *ft* up the hillside?

Solution



a) $|F_\perp| = |F_y| = 180\cos 30^\circ$

$$= 180\left(\frac{\sqrt{3}}{2}\right)$$

$$\underline{= 90\sqrt{3}\ \ lb}$$

$$|F_{/\!/}| = |F_x| = 180\sin 30^\circ$$

$$= 180\left(\frac{1}{2}\right)$$

$$\underline{= 90\ \ lb}$$

b) $Work = d \cdot F_x$

$= 10(90)$

$\boxed{= 900 \ \ ft\text{-}lbs}$

Solution Section 1.3 – Cross Products

Exercise

Find the length and direction of $\vec{u}\times\vec{v}$ *and* $\vec{v}\times\vec{u}$: $\vec{u}=2\hat{i}-2\hat{j}-\hat{k},\quad \vec{v}=\hat{i}-\hat{k}$

Solution

$$\vec{u}\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -2 & -1\\ 1 & 0 & -1\end{vmatrix}$$

$$=2\hat{i}-\hat{j}+2\hat{k}$$

Length:

$$|\vec{u}\times\vec{v}|=\sqrt{4+1+4}$$

$$\vec{u}=\hat{i}-\hat{k},\quad \vec{v}=\hat{j}+\hat{k}$$

Direction: $\frac{1}{3}\left(2\hat{i}-\hat{j}+2\hat{k}\right)$

$$\vec{v}\times\vec{u}=-\left(\vec{u}\times\vec{v}\right)$$

$$=-2\hat{i}+\hat{j}-2\hat{k}$$

Length:

$$|\vec{v}\times\vec{u}|=\sqrt{4+1+4}$$

$$=3$$

Direction: $\frac{1}{3}\left(-2\hat{i}+\hat{j}-2\hat{k}\right)$

Exercise

Find the length and direction of $\vec{u}\times\vec{v}$ *and* $\vec{v}\times\vec{u}$: $\vec{u}=\hat{i}+\hat{j}-\hat{k},\quad \vec{v}=0$

Solution

$$\vec{u}\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & -1\\ 0 & 0 & 0\end{vmatrix}$$

$$=0$$

Length: $=0$

Direction: ***No direction***

$$\vec{v}\times\vec{u} = -\left(\vec{u}\times\vec{v}\right) \quad = 0$$

Length: $= 0$

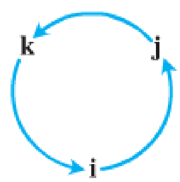
Direction: ***No direction***

Exercise

Find the length and direction of $\vec{u}\times\vec{v}$ *and* $\vec{v}\times\vec{u}$: $\quad \vec{u} = \hat{i}\times\hat{j}, \quad \vec{v} = \hat{j}\times\hat{k}$

Solution

$$\begin{aligned}\vec{u}\times\vec{v} &= \left(\hat{i}\times\hat{j}\right)\times\left(\hat{j}\times\hat{k}\right)\\ &= \hat{k}\times\hat{i}\\ &= \hat{j}\end{aligned}$$



Length: $= 1$

Direction: $= \hat{j}$

$$\begin{aligned}\vec{v}\times\vec{u} &= -\left(\vec{u}\times\vec{v}\right)\\ &= -\hat{j}\end{aligned}$$

Length: $= 1$

Direction: $= -\hat{j}$

Exercise

Find the length and direction of $\vec{u}\times\vec{v}$ *and* $\vec{v}\times\vec{u}$: $\quad \vec{u} = -8\hat{i} - 2\hat{j} - 4\hat{k}, \quad \vec{v} = 2\hat{i} + 2\hat{j} + \hat{k}$

Solution

$$\begin{aligned}\vec{u}\times\vec{v} &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -8 & -2 & -4\\ 2 & 2 & 1\end{vmatrix}\\ &= 6\hat{i} - 12\hat{k}\end{aligned}$$

Length:

$$\begin{aligned}\left|\vec{u}\times\vec{v}\right| &= \sqrt{36+144}\\ &= \sqrt{180}\\ &= 6\sqrt{5}\end{aligned}$$

$$\textit{Direction}: \ = \frac{1}{6\sqrt{5}}\left(6\hat{i} - 12\hat{k}\right)$$

$$= \frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{k}$$

$$\vec{v}\times\vec{u} = -\left(\vec{u}\times\vec{v}\right)$$

$$= -6\hat{i} + 12\hat{k}$$

$$\textit{Length}: \quad \left|\vec{v}\times\vec{u}\right| = 6\sqrt{5}$$

$$\textit{Direction}: \ -\frac{1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{k}$$

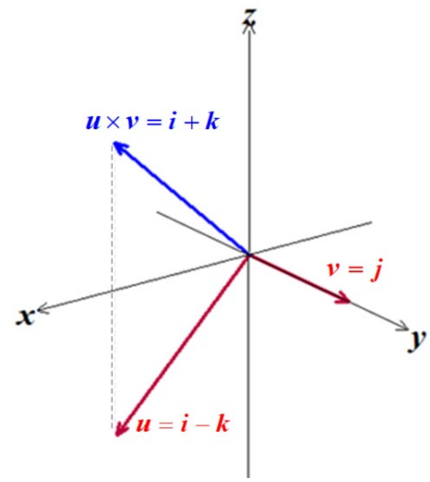
Exercise

Sketch the coordinate axes and then include the vectors $\vec{u}$, $\vec{v}$, and $\vec{u}\times\vec{v}$ as vectors starting origin for $\vec{u} = \hat{i} - \hat{k}, \quad \vec{v} = \hat{j}$

Solution

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \hat{i} + \hat{k}$$



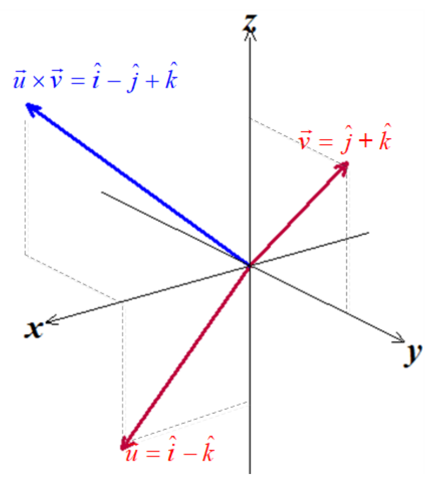
Exercise

Sketch the coordinate axes and then include the vectors $\vec{u}$, $\vec{v}$, and $\vec{u}\times\vec{v}$ as vectors starting origin for $\vec{u} = \hat{i} - \hat{k}, \quad \vec{v} = \hat{j} + \hat{k}$

Solution

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= \hat{i} - \hat{j} + \hat{k}$$



Exercise

Find the area of the triangle determined by the points P, Q, and R, and then find a unit vector perpendicular to plane PQR. $P(1,\ -1,\ 2),\quad Q(2,\ 0,\ -1),\quad and\ \ R(0,\ 2,\ 1)$

Solution

$$\overrightarrow{PQ}=(2-1)\hat{i}+(0+1)\hat{j}+(-1-2)\hat{k}$$
$$=\hat{i}+\hat{j}-3\hat{k}$$
$$\overrightarrow{PR}=(0-1)\hat{i}+(2+1)\hat{j}+(1-2)\hat{k}$$
$$=-\hat{i}+3\hat{j}-\hat{k}$$
$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & -3\\ -1 & 3 & -1\end{vmatrix}$$
$$=8\hat{i}+4\hat{j}+4\hat{k}$$
$$Area=\frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|$$
$$=\frac{1}{2}\sqrt{8^2+4^2+4^2}$$
$$=\frac{1}{2}\sqrt{96}$$
$$=2\sqrt{6}$$
$$\vec{u}=\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|}$$
$$=\frac{1}{4\sqrt{6}}\left(8\hat{i}+4\hat{j}+4\hat{k}\right)$$
$$=\frac{1}{\sqrt{6}}\left(2\hat{i}+\hat{j}+\hat{k}\right)$$

Exercise

Find the area of the triangle determined by the points P, Q, and R, and then find a unit vector perpendicular to plane PQR. $P(1,\ 1,\ 1),\quad Q(2,\ 1,\ 3),\quad and\ \ R(3,\ -1,\ 1)$

Solution

$$\overrightarrow{PQ}=(2-1)\hat{i}+(1-1)\hat{j}+(3-1)\hat{k}$$
$$=\hat{i}+2\hat{k}$$
$$\overrightarrow{PR}=(3-1)\hat{i}+(-1-1)\hat{j}+(1-1)\hat{k}$$

$$= 2\hat{i} - 2\hat{j}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{vmatrix}$$

$$= 4\hat{i} + 4\hat{j} - 2\hat{k}$$

$$Area = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$$

$$= \frac{1}{2}\sqrt{16+16+4}$$

$$= \frac{1}{2}\sqrt{36}$$

$$= 3$$

$$\vec{u} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|}$$

$$= \frac{1}{6}\left(4\hat{i} + 4\hat{j} - 2\hat{k}\right)$$

$$= \frac{1}{3}\left(2\hat{i} + 2\hat{j} - \hat{k}\right)$$

Exercise

Find the area of the triangle determined by the points P, Q, and R, and then find a unit vector perpendicular to plane PQR. $P(-2,\ 2,\ 0),\quad Q(0,\ 1,\ -1),\quad and\ \ R(-1,\ 2,\ -2)$

Solution

$$\overrightarrow{PQ} = 2\hat{i} - \hat{j} - \hat{k}$$

$$\overrightarrow{PR} = \hat{i} - 2\hat{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -1 \\ 1 & 0 & -2 \end{vmatrix}$$

$$= 2\hat{i} + 3\hat{j} + \hat{k}$$

$$Area = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$$

$$= \frac{1}{2}\sqrt{4+9+1}$$

$$= \frac{\sqrt{14}}{2}$$

$$\vec{u} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|}$$

$$= \frac{1}{\sqrt{14}}\left(2\hat{i} + 3\hat{j} + \hat{k}\right)$$

Exercise

Verify that $(\vec{u} \times \vec{v}) \bullet \vec{w} = (\vec{v} \times \vec{w}) \bullet \vec{u} = (\vec{w} \times \vec{u}) \bullet \vec{v}$ and find the volume of the parallelepiped determined by $\vec{u} = 2\hat{i}, \quad \vec{v} = 2\hat{j}, \quad and \quad \vec{w} = 2\hat{k}$

Solution

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle, \quad \vec{v} = \langle v_1, v_2, v_3 \rangle, \quad \vec{w} = \langle w_1, w_2, w_3 \rangle$

$$(\vec{u} \times \vec{v}) \bullet \vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$(\vec{v} \times \vec{w}) \bullet \vec{u} = \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$(\vec{w} \times \vec{u}) \bullet \vec{v} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Which all have the same absolute value, by interchanging the rows the determinant does not change its absolute value.

$$Volume = (\vec{u} \times \vec{v}) \bullet \vec{w}$$

$$= \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 8 \ \ unit^3$$

Exercise

Find $|\vec{v}|$, $|\vec{u}|$, $\vec{v}\bullet\vec{u}$, $\vec{u}\bullet\vec{v}$, $\vec{v}\times\vec{u}$, $\vec{u}\times\vec{v}$, $|\vec{v}\times\vec{u}|$, the angle between $\vec{v}$ and $\vec{u}$, the scalar component of $\vec{u}$ in the direction of $\vec{v}$, and the vector $proj_{\vec{v}}\vec{u}$

$$\vec{v}=\hat{i}+\hat{j}+2\hat{k}, \quad \vec{u}=-\hat{i}-\hat{k}$$

Solution

$$|\vec{v}|=\sqrt{1+1+4}$$

$$=\sqrt{6}$$

$$|\vec{u}|=\sqrt{1+1}$$

$$=\sqrt{2}$$

$$\vec{v}\bullet\vec{u}=-1+0-2=-3$$

$$\vec{u}\bullet\vec{v}=-3$$

$$\vec{v}\times\vec{u}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 2\\ -1 & 0 & -1\end{vmatrix}$$

$$=-\hat{i}-\hat{j}+\hat{k}$$

$$\vec{u}\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 0 & -1\\ 1 & 1 & 2\end{vmatrix}$$

$$=\hat{i}+\hat{j}-\hat{k}$$

$$|\vec{v}\times\vec{u}|=\left|-\hat{i}-\hat{j}+\hat{k}\right|$$

$$=\sqrt{1+1+1}$$

$$=\sqrt{3}$$

$$\theta=\cos^{-1}\left(-\frac{3}{\sqrt{6}\sqrt{2}}\right) \qquad \theta=\cos^{-1}\frac{\vec{v}\bullet\vec{u}}{|\vec{v}||\vec{u}|}$$

$$=\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$=\frac{5\pi}{6}$$

$$proj_{\vec{v}}\vec{u}=\frac{-3}{6}\left(\hat{i}+\hat{j}+2\hat{k}\right) \qquad proj_{\vec{v}}\vec{u}=\left(\frac{\vec{v}\bullet\vec{u}}{|\vec{v}|^2}\right)\vec{v}$$

$$=-\frac{1}{2}\left(\hat{i}+\hat{j}+2\hat{k}\right)$$

Exercise

Find $|\vec{v}|$, $|\vec{u}|$, $\vec{v}\bullet\vec{u}$, $\vec{u}\bullet\vec{v}$, $\vec{v}\times\vec{u}$, $\vec{u}\times\vec{v}$, $|\vec{v}\times\vec{u}|$, the angle between $\vec{v}$ and $\vec{u}$, the scalar component of $\vec{u}$ in the direction of $\vec{v}$, and the vector $proj_{\vec{v}}\vec{u}$

$$\vec{v} = 2\hat{i} + \hat{j} - \hat{k}, \quad \vec{u} = \hat{i} + \hat{j} - 5\hat{k}$$

Solution

$$|\vec{v}| = \sqrt{4+1+1}$$

$$= \sqrt{6}$$

$$|\vec{u}| = \sqrt{1+1+25}$$

$$= \sqrt{27}$$

$$= 3\sqrt{3}$$

$$\vec{v}\bullet\vec{u} = 2+1+5 = 8$$

$$\vec{u}\bullet\vec{v} = 8$$

$$\vec{v}\times\vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 1 & -5 \end{vmatrix}$$

$$= -4\hat{i} + 9\hat{j} + \hat{k}$$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -5 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= 4\hat{i} - 9\hat{j} - \hat{k}$$

$$|\vec{v}\times\vec{u}| = \sqrt{16+81+1}$$

$$= \sqrt{98}$$

$$= 7\sqrt{2}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{3}\sqrt{6}}\right) \qquad \theta = \cos^{-1}\frac{\vec{v}\bullet\vec{u}}{|\vec{v}||\vec{u}|}$$

$$= \cos^{-1}\left(\frac{9}{8\sqrt{2}}\right)$$

$$= 0.651$$

$$proj_{\vec{v}}\vec{u} = \frac{8}{6}\left(2\hat{i} + \hat{j} - \hat{k}\right) \qquad proj_{\vec{v}}\vec{u} = \left(\frac{\vec{v}\bullet\vec{u}}{|\vec{v}|^2}\right)\vec{v}$$

$$= \frac{4}{3}\left(2\hat{i} + \hat{j} - \hat{k}\right)$$

Exercise

Find the area of the parallelogram determined by vectors $\vec{u}$ and $\vec{v}$, then the volume of the parallelepiped determined by vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$.

$$\vec{u} = \hat{i} + \hat{j} - \hat{k}, \quad \vec{v} = 2\hat{i} + \hat{j} + \hat{k}, \quad \vec{w} = -\hat{i} - 2\hat{j} + 3\hat{k}$$

Solution

$$Area = |\vec{u} \times \vec{v}| = abs\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= \left|2\hat{i} - 3\hat{j} - \hat{k}\right|$$

$$= \sqrt{4+9+1}$$

$$= \sqrt{14} \quad unit^2$$

$$Volume = (\vec{u} \times \vec{v}) \bullet \vec{w} = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ -1 & -2 & 3 \end{vmatrix}$$

$$= 1 \quad unit^3$$

Exercise

Find the area of the parallelogram determined by vectors $\vec{u}$ and $\vec{v}$, then the volume of the parallelepiped determined by vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$.

$$\vec{u} = \hat{i} + \hat{j}, \quad \vec{v} = \hat{j}, \quad \vec{w} = \hat{i} + \hat{j} + \hat{k}$$

Solution

$$Area = |\vec{u} \times \vec{v}| = \left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{matrix}\right\|$$

$$= \left|\hat{k}\right|$$

$$= 1 \quad unit^2$$

$$Volume = (\vec{u} \times \vec{v}) \bullet \vec{w} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1 \quad unit^3$$

Exercise

Find the volume of the parallelepiped determined by

$$\vec{u} = \hat{i} - \hat{j} + \hat{k}, \quad \vec{v} = 2\hat{i} + \hat{j} - \hat{k}, \quad and \quad \vec{w} = -\hat{i} + 2\hat{j} - \hat{k}$$

Solution

$$Volume = (\vec{u} \times \vec{v}) \bullet \vec{w}$$

$$= abs\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 2 & -1 \end{vmatrix}$$

$$= 3 \ \ unit^3$$

Exercise

Find the volume of the parallelepiped determined by

$$\vec{u} = \hat{i} + \hat{j} - 2\hat{k}, \quad \vec{v} = -\hat{i} - \hat{k}, \quad and \quad \vec{w} = 2\hat{i} + 4\hat{j} - 2\hat{k}$$

Solution

$$Volume = (\vec{u} \times \vec{v}) \bullet \vec{w}$$

$$= abs\begin{vmatrix} 1 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix}$$

$$= 8 \ \ unit^3$$

Exercise

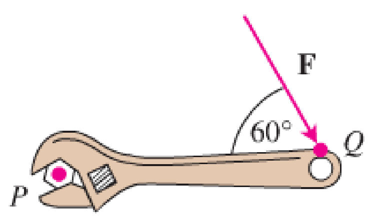
Find the magnitude of the torque force exerted by $\vec{F}$ on the bolt at P if $\left|\overrightarrow{PQ}\right| = 8\ in.$ and $\left|\vec{F}\right| = 30\ lb.$

Solution

$$\left|\overrightarrow{PQ} \times \vec{F}\right| = \left|\overrightarrow{PQ}\right| \ \left|\vec{F}\right| \ \sin 60°$$

$$= \frac{8}{12}(30)\frac{\sqrt{3}}{2}$$

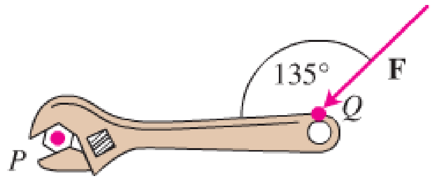
$$= 10\sqrt{3} \ \ ft.lb$$



Exercise

Find the magnitude of the torque force exerted by $\vec{F}$ on the bolt at P if $\left|\overrightarrow{PQ}\right| = 8\ in.$ and $\left|\vec{F}\right| = 30\ lb.$

Solution



$$\left|\overrightarrow{PQ}\times\vec{F}\right| = \left|\overrightarrow{PQ}\right|\ \left|\vec{F}\right|\ \sin 135°$$

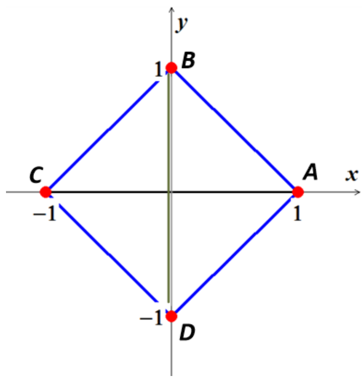
$$= \frac{8}{12}(30)\frac{\sqrt{2}}{2}$$

$$= 10\sqrt{2}\ \ ft.lb$$

Exercise

Find the area of the parallelogram whose vertices are: $A(1,\ 0),\quad B(0,\ 1),\quad C(-1,\ 0),\quad D(0,\ -1)$

Solution



$$\overrightarrow{AB} = -\hat{i} + \hat{j} \quad \overrightarrow{AD} = -\hat{i} - \hat{j}$$

$$Area(\triangle ABD) = Area(\triangle CBD)$$

$$Area = \left|\overrightarrow{AB}\times\overrightarrow{AD}\right|$$

$$= abs\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ -1 & -1 & 0 \end{vmatrix}$$

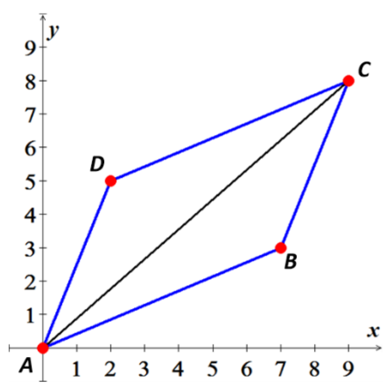
$$= abs\left|2\hat{k}\right|$$

$$= 2\ \ unit^2$$

Exercise

Find the area of the parallelogram whose vertices are: $A(0,\ 0),\quad B(7,\ 3),\quad C(9,\ 8),\quad D(2,\ 5)$

Solution



$$\overrightarrow{AB} = 7\hat{i} + 3\hat{j} \quad \overrightarrow{AC} = 9\hat{i} + 8\hat{j}$$

$$Area(\triangle ABC) = Area(\triangle ACD)$$

$$Area = \left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$$

$$= abs\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 3 & 0 \\ 9 & 8 & 0 \end{vmatrix}$$

$$= abs\left|29\hat{k}\right|$$

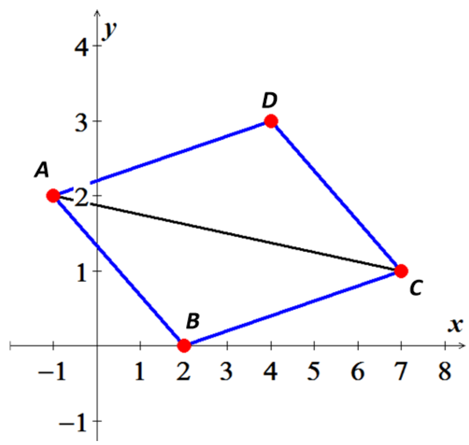
$$\underline{= 29 \ \ unit^2}$$

Exercise

Find the area of the parallelogram whose vertices are:

$A(-1,\ 2),\quad B(2,\ 0),\quad C(7,\ 1),\quad D(4,\ 3)$

Solution



$$\overrightarrow{AB} = 3\hat{i} - 2\hat{j} \quad \overrightarrow{AC} = 8\hat{i} - \hat{j}$$

$$Area(\triangle ABC) = Area(\triangle ACD)$$

$$Area = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= abs\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 0 \\ 8 & -1 & 0 \end{vmatrix}$$

$$= abs\left|13\hat{k}\right|$$

$$\underline{= 13\ unit^2}$$

Exercise

Find the area of the parallelogram whose vertices are:

$A(0,\ 0,\ 0),\quad B(3,\ 2,\ 4),\quad C(5,\ 1,\ 4),\quad D(2,\ -1,\ 0)$

Solution

$$\overrightarrow{AB} = 3\hat{i} + 2\hat{j} + 4\hat{k} \quad \overrightarrow{DC} = 3\hat{i} + 2\hat{j} + 4\hat{k}$$

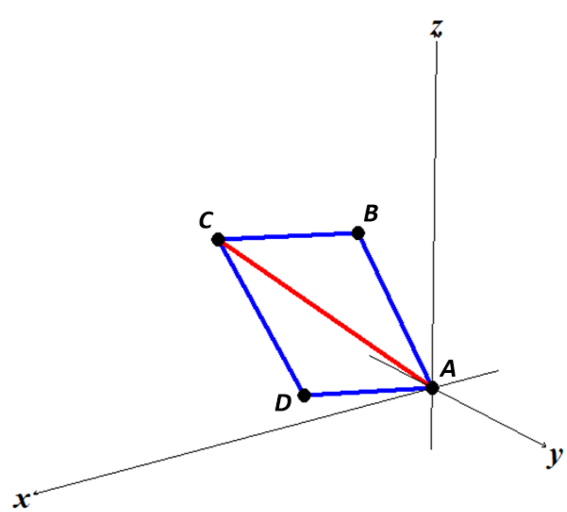
$\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$

$$\overrightarrow{AD} = 2\hat{i} - \hat{j} \quad \overrightarrow{BC} = 2\hat{i} - \hat{j}$$

$\overrightarrow{AD}$ is parallel to $\overrightarrow{BC}$

$$Area = \left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= abs\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 4 \\ 2 & -1 & 0 \end{vmatrix}$$



$$= abs\left|4\hat{i} + 8\hat{j} - 7\hat{k}\right|$$

$$= \sqrt{16 + 64 + 49}$$

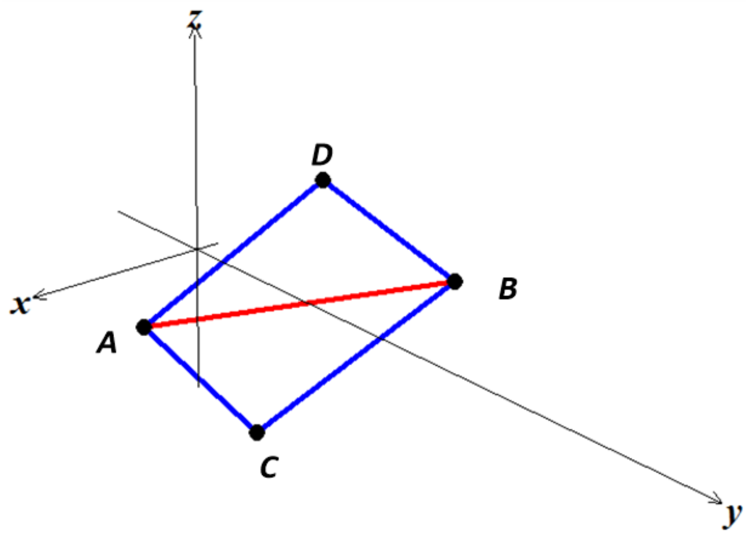
$$= \sqrt{129} \ \ unit^2$$

Exercise

Find the area of the parallelogram whose vertices are:

$$A(1,\ 0,\ -1),\quad B(1,\ 7,\ 2),\quad C(2,\ 4,\ -1),\quad D(0,\ 3,\ 2)$$

Solution



$$\overrightarrow{AC} = \hat{i} + 4\hat{j} \quad \overrightarrow{CB} = -\hat{i} + 3\hat{j} + 3\hat{k}$$

$$Area = \left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= abs\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 0 \\ -1 & 3 & 3 \end{vmatrix}$$

$$= abs\left|12\hat{i} - 3\hat{j} + 7\hat{k}\right|$$

$$= \sqrt{144 + 9 + 49}$$

$$= \sqrt{202} \ \ unit^2$$

Exercise

Find the area of the parallelogram with vertices $(1,\ 2,\ 3)$, $(1,\ 0,\ 6)$, and $(4,\ 2,\ 4)$

Solution

$$\langle 1,\ 0,\ 6\rangle - \langle 1,\ 2,\ 3\rangle = \langle 0,\ -2,\ 3\rangle$$

$$\langle 4,\ 2,\ 4\rangle - \langle 1,\ 2,\ 3\rangle = \langle 3,\ 0,\ 1\rangle$$

$$Area = \left|\langle 0,\ -2,\ 3\rangle \times \langle 3,\ 0,\ 1\rangle\right|$$

$$= \left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & 3 \\ 3 & 0 & 1 \end{matrix}\right\|$$

$$= \left|\langle -2,\ 9,\ 6\rangle\right|$$

$$= \sqrt{4 + 81 + 36}$$

$$= 11 \ \ unit^2$$

Exercise

Find the area of the parallelogram with vertices $(1,\ 0,\ 3)$, $(5,\ 0,\ -1)$, and $(0,\ 2,\ -2)$

Solution

$$\langle 5,\ 0,\ -1\rangle - \langle 1,\ 0,\ 3\rangle = \langle 4,\ 0,\ -4\rangle$$

$$\langle 0,\ 2,\ -2\rangle - \langle 1,\ 0,\ 3\rangle = \langle -1,\ 2,\ -5\rangle$$

$$Area = \left|\langle 4,\ 0,\ -4\rangle \times \langle -1,\ 2,\ -5\rangle\right|$$

$$= \left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 0 & 4 \\ -1 & 2 & -5 \end{matrix}\right\|$$

$$= \left|\langle 8,\ 24,\ 8\rangle\right|$$

$$= \sqrt{64 + 576 + 64}$$

$$= \sqrt{64(1 + 9 + 1)}$$

$$= 8\sqrt{11}\ \ unit^2$$

Exercise

Find the area of the triangle whose vertices are: $A(0,\ 0)$, $B(-2,\ 3)$, $C(3,\ 1)$

Solution

$$\overrightarrow{AB} = -2\hat{i} + 3\hat{j} \quad \overrightarrow{AC} = 3\hat{i} + \hat{j}$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= \left(\frac{1}{2}\right) abs \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 3 & 0 \\ 3 & 1 & 0 \end{vmatrix}$$

$$= \frac{1}{2}\left|-11\hat{k}\right|$$

$$= \frac{11}{2}\ unit^2$$

Exercise

Find the area of the triangle whose vertices are: $A(-1,\ -1),\quad B(3,\ 3),\quad C(2,\ 1)$

Solution

$$\overrightarrow{AB} = 4\hat{i} + 4\hat{j} \quad \overrightarrow{AC} = 3\hat{i} + 2\hat{j}$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= \frac{1}{2}\left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 3 & 2 & 0 \end{matrix}\right\|$$

$$= \frac{1}{2}\left|-4\hat{k}\right|$$

$$\underline{= 2\ unit^2}$$

Exercise

Find the area of the triangle whose vertices are: $A(1,\ 0,\ 0),\quad B(0,\ 0,\ 2),\quad C(0,\ 0,\ -1)$

Solution

$$\overrightarrow{AB} = -\hat{i} + 2\hat{k} \quad \overrightarrow{AC} = -\hat{i} - \hat{k}$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$= \frac{1}{2}\left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 2 \\ -1 & 0 & -1 \end{matrix}\right\|$$

$$= \frac{1}{2}\left|3\hat{k}\right|$$

$$\underline{= \frac{3}{2}\ unit^2}$$

Exercise

Find the area of the triangle whose vertices are: $A(0,\ 0,\ 0),\quad B(-1,\ 1,\ -1),\quad C(3,\ 0,\ 3)$

Solution

$$\overrightarrow{AB} = -\hat{i} + \hat{j} - \hat{k} \quad \overrightarrow{AC} = 3\hat{i} + 3\hat{k}$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$$

$$=\frac{1}{2}\left\|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -1 \\ 3 & 0 & 3 \end{vmatrix}\right\|$$

$$=\frac{1}{2}\left|3\hat{i}-3\hat{k}\right|$$

$$=\frac{1}{2}\sqrt{9+9}$$

$$\boxed{=\frac{3\sqrt{2}}{2}\ unit^2}$$

Exercise

Find the volume of the parallelepiped if four of its eight vertices are:

$A(0,\ 0,\ 0),\quad B(1,\ 2,\ 0),\quad C(0,\ -3,\ 2),\quad D(3,\ -4,\ 5)$

Solution

$$\overrightarrow{AB}=\hat{i}+2\hat{j}\quad \overrightarrow{AC}=-3\hat{j}+2\hat{k}\quad \overrightarrow{AD}=3\hat{i}-4\hat{j}+5\hat{k}$$

$$\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\bullet\overrightarrow{AD}=\begin{vmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 3 & -4 & 5 \end{vmatrix}$$

$$\boxed{=5}$$

$$Volume=\left|\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\bullet\overrightarrow{AD}\right|$$

$$\boxed{=5\ unit^3}$$

Exercise

Let $\vec{u}=\langle 2,\ 4,\ -5\rangle$ and $\vec{v}=\langle -6,\ 10,\ 2\rangle$

a) Compute $\vec{u}-3\vec{v}$

b) Compute $\left|\vec{u}+\vec{v}\right|$

c) Find the unit vector with the same direction as $\vec{u}$

d) Find a vector parallel to $\vec{v}$ with length 20.

e) Compute $\vec{u}\bullet\vec{v}$ and the angle between $\vec{u}$ and $\vec{v}$.

f) Compute $\vec{u}\times\vec{v}$, $\vec{v}\times\vec{u}$

g) Find the area of the triangle with vertices $(0,\ 0,\ 0)$, $(2,\ 4,\ -5)$, and $(-6,\ 10,\ 2)$

Solution

a) $\boldsymbol{u}-3\boldsymbol{v}=\langle 2,\ 4,\ -5\rangle-3\langle -6,\ 10,\ 2\rangle$

$$= \langle 2,\ 4,\ -5\rangle - \langle -18,\ 30,\ 6\rangle$$

$$\boxed{= \langle 20,\ -26,\ -11\rangle}$$

b) $|\boldsymbol{u}+\boldsymbol{v}| = |\langle 2,\ 4,\ -5\rangle + \langle -6,\ 10,\ 2\rangle|$

$$= |\langle -4,\ 14,\ -3\rangle|$$

$$= \sqrt{16+196+9}$$

$$\boxed{= \sqrt{221}}$$

c) unit vector of $\boldsymbol{u} = \langle 2,\ 4,\ -5\rangle$

$$\frac{\vec{u}}{|\vec{u}|} = \frac{\langle 2,\ 4,\ -5\rangle}{\sqrt{4+16+25}}$$

$$= \frac{\langle 2,\ 4,\ -5\rangle}{\sqrt{45}}$$

$$\boxed{= \frac{1}{3\sqrt{5}}\langle 2,\ 4,\ -5\rangle}$$

d) $|\vec{v}| = |\langle -6,\ 10,\ 2\rangle|$

$$= \sqrt{36+100+4}$$

$$= \sqrt{140}$$

$$\boxed{= 2\sqrt{35}}$$

The desired vector parallel to $\vec{v}$ with length 20 is:

$$= \frac{20}{2\sqrt{35}}\langle -6,\ 10,\ 2\rangle$$

$$\boxed{= \frac{20}{\sqrt{35}}\langle -3,\ 5,\ 1\rangle}$$

e) $\vec{u} \bullet \vec{v} = \langle 2,\ 4,\ -5\rangle \bullet \langle -6,\ 10,\ 2\rangle$

$$= -12 + 40 - 10$$

$$\boxed{= 18}$$

$$\theta = \cos^{-1}\frac{18}{3\sqrt{5}\ 2\sqrt{35}} \qquad \qquad \theta = \cos^{-1}\frac{\vec{u}\bullet\vec{v}}{|\vec{u}|\ |\vec{v}|}$$

$$\boxed{= \cos^{-1}\frac{3}{5\sqrt{7}}} \quad \boxed{\approx 76.9^\circ}$$

f) $\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & -5 \\ -6 & 10 & 2 \end{vmatrix}$

$$\boxed{= 58\hat{i} + 26\hat{j} + 44\hat{k}}$$

$$\boxed{\vec{v}\times\vec{u} = \langle -58,\ -26,\ -44\rangle}$$

g) Area of the triangle with vertices $(0,\ 0,\ 0)$, $(2,\ 4,\ -5)$, and $(-6,\ 10,\ 2)$

$$\begin{aligned} Area &= \frac{1}{2}|\vec{u}\times\vec{v}| \\ &= \frac{1}{2}\sqrt{58^2+26^2+44^2} \\ &= \frac{1}{2}\sqrt{5{,}976} \\ &= 3\sqrt{166}\ \ unit^2 \end{aligned}$$

Exercise

Find a unit vector normal to the vectors $\langle 2,\ -6,\ 9\rangle$ and $\langle -1,\ 0,\ 6\rangle$

Solution

$$\begin{aligned} \vec{N} &= \langle 2,\ -6,\ 9\rangle\times\langle -1,\ 0,\ 6\rangle \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -6 & 9 \\ -1 & 0 & 6 \end{vmatrix} \\ &= \langle -36,\ -21,\ -6\rangle \end{aligned}$$

$$\begin{aligned} |\vec{N}| &= \sqrt{36^2+21^2+6^2} \\ &= \sqrt{1{,}773} \\ &= 3\sqrt{197} \end{aligned}$$

$$\begin{aligned} \text{Unit normal } &= \frac{1}{3\sqrt{197}}\langle -36,\ -21,\ -6\rangle \\ &= -\frac{1}{\sqrt{197}}\langle 12,\ 7,\ 2\rangle \end{aligned}$$

Exercise

Find the angle between $\langle 2,\ 0,\ -2\rangle$ and $\langle 2,\ 2,\ 0\rangle$ using the dot product then the cross product.

Solution

$$\langle 2,\ 0,\ -2\rangle\cdot\langle 2,\ 2,\ 0\rangle = 4$$

$$|\langle 2,\ 0,\ -2\rangle| = \sqrt{4+4} = 2\sqrt{2}$$

$$|\langle 2,\ 2,\ 0\rangle| = 2\sqrt{2}$$

$$\theta = \cos^{-1}\frac{4}{(2\sqrt{2})(2\sqrt{2})} \qquad\qquad \theta = \cos^{-1}\frac{\vec{u}\bullet\vec{v}}{|\vec{u}|\ |\vec{v}|}$$

$$= \cos^{-1}\frac{1}{2}$$

$$\boxed{= \frac{\pi}{3}}$$

$$\langle 2,\ 0,\ -2\rangle \times \langle 2,\ 2,\ 0\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -2 \\ 2 & 2 & 0 \end{vmatrix}$$

$$= \langle 4,\ -4,\ 4\rangle$$

$$|\langle 4,\ -4,\ 4\rangle| = \sqrt{16+16+16}$$

$$= 4\sqrt{3}$$

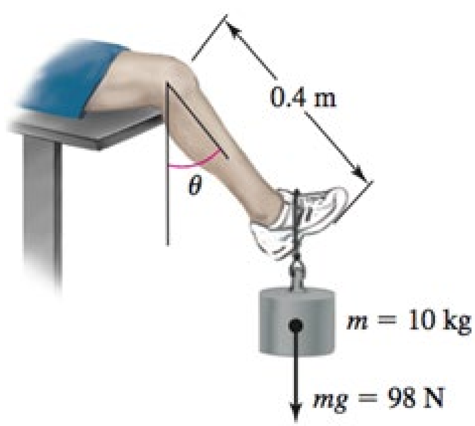
$$\theta = \sin^{-1}\frac{4\sqrt{3}}{\left(2\sqrt{2}\right)^2}$$

$$= \sin^{-1}\frac{\sqrt{3}}{2}$$

$$\boxed{= \frac{\pi}{3}}$$

Exercise

You do leg lifts with 10-*kg* weight attached to your foot, so the resulting force is $mg \approx 98\ N$ directed vertically downward. If the distance from your knee to the weight is $0.4m$ and her lower leg makes an angle of θ to the vertical, find the magnitude of the torque about your knee as your leg is lifted (as a function of θ).



a) What is the minimum and maximum magnitude of the torque?

b) Does the direction of the torque change as your leg is lifted?

Solution

a) $T(\theta) = (.4)(98)\sin\theta$ $\qquad\qquad T(\theta) = |r||F|\sin\theta$

$$= \frac{392}{10}\sin\theta$$

$$= \frac{196}{5}\sin\theta \quad (N-m)$$

The maximum torque is $\frac{196}{5} = 39.2$ when $\sin\theta = 1 \quad \Rightarrow \quad \theta = 90°$

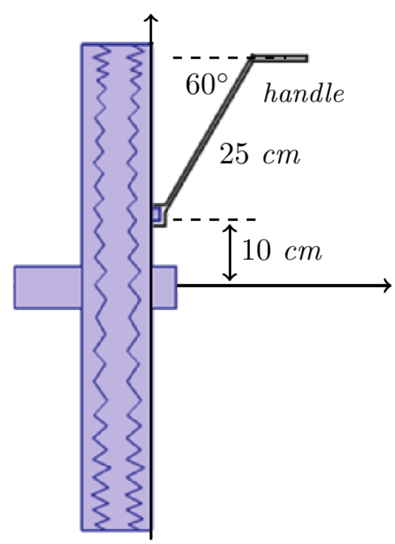
The minimum torque is 0 when $\sin\theta = 0 \quad \Rightarrow \quad \theta = 0°$

b) The direction of the torque does not change as the knee is lifted

Exercise

An automobile wheel has center at the origin and axle along the *y*-axis. One of the retaining nuts holding the wheel is at position $P_0(0,\ 0,\ 10)$. (Distances are measured in *cm*.) A bent tire wrench with arm 25 *cm* long and inclined at an angle of 60° to the direction of its handle is fitted to the nut in an upright direction. If the horizontal force $\vec{F} = 500\hat{i} \quad (N)$ is applied to the handle of the wrench, what is its torque on the nut? What part (component) of this torque is effective in trying to rotate the nut about its horizontal axis? What is the effective torque trying to rotate the wheel?

Solution



$$r_0 = 10\hat{k}$$

$$\vec{r} = r_y\hat{i} + r_z\hat{k}$$

$$= 25\cos 60°\hat{j} + (10 + 25\sin 60°)\hat{k}$$

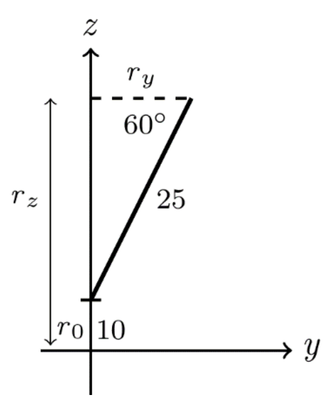
$$= \frac{25}{2}\hat{j} + \left(10 + \frac{25\sqrt{3}}{2}\right)\hat{k}$$

$$T = \left(r - r_0\right) \times \vec{F}$$

$$= \left(\frac{25}{2}\hat{j} + \frac{25\sqrt{3}}{2}\hat{k}\right) \times \left(500\hat{i}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \frac{25}{2} & \frac{25\sqrt{3}}{2} \\ 500 & 0 & 0 \end{vmatrix}$$

$$\underline{= 6,250\sqrt{3}\,\hat{j} - 6,250\hat{k}\,\big|}$$



Since the torque is effective in turning horizontally that implies $T = 6,250\sqrt{3}$ *N-cm*.

The effective torque $\left(r_0 = 0\right)$:

$$\left(10 + \frac{25\sqrt{3}}{2}\right)\hat{k} \times 500\hat{i} \quad \underline{= \left(5,000 + 6,250\sqrt{3}\right)\hat{j}\,\big|} \quad \textit{Joule}$$

Solution Section 1.4 – Lines and Curves in Space

Exercise

Find the parametric equation for the line through the point $P(3,\ -4,\ -1)$ parallel to the vector $\hat{i}+\hat{j}+\hat{k}$

Solution

$$\underline{x=3+t,\quad y=-4+t,\quad z=-1+t}$$

Exercise

Find the parametric equation for the line through the points $P(1,\ 2,\ -1)$ and $Q(-1,0,1)$

Solution

The direction: $\overrightarrow{PQ}=-2\hat{i}-2\hat{j}+2\hat{k}$ and $P(1,\ 2,\ -1)$

$$\underline{x=1-2t,\quad y=2-2t,\quad z=-1+2t}$$

Exercise

Find the parametric equation for the line through the points $P(-2,\ 0,\ 3)$ and $Q(3,\ 5,\ -2)$

Solution

The direction: $\overrightarrow{PQ}=5\hat{i}+5\hat{j}-5\hat{k}$ and $P(-2,\ 0,\ 3)$

$$\underline{x=-2+5t,\quad y=5t,\quad z=3-5t}$$

Exercise

Find the parametric equation for the line through the origin parallel to the vector $2\hat{j}+\hat{k}$

Solution

The direction: $2\hat{i}+\hat{k}$ and $P(0,\ 0,\ 0)$

$$\underline{x=0,\quad y=2t,\quad z=t}$$

Exercise

Find the parametric equation for the line through the point $P(3,\ -2,\ 1)$ parallel to the line

$x=1+2t,\quad y=2-t,\quad z=3t$

Solution

The direction: $2\hat{i} - \hat{j} + 3\hat{k}$ and $P(3,\ -2,\ 1)$

$x = 3 + 2t, \quad y = -2 - t, \quad z = 1 + 3t$

Exercise

Find the parametric equation for the line through $(2,\ 4,\ 5)$ perpendicular to the plane $3x + 7y - 5z = 21$

Solution

The direction: $3\hat{i} + 7\hat{j} - 5\hat{k}$ and $(2, 4, 5)$

$x = 2 + 3t, \quad y = 4 + 7t, \quad z = 5 - 5t$

Exercise

Find the parametric equation for the line through $(2,\ 3,\ 0)$ perpendicular to the vectors $\vec{u} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{v} = 3\hat{i} + 4\hat{j} + 5\hat{k}$

Solution

The direction:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= -2\hat{i} + 4\hat{j} - 2\hat{k}$$

The point on the line: $(2, 3, 0)$

$x = 2 - 2t, \quad y = 3 + 4t, \quad z = -2t$

Exercise

Find the parameterization for the line segment joining the points $(0,\ 0,\ 0)$, $\left(1,\ 1,\ \frac{3}{2}\right)$.

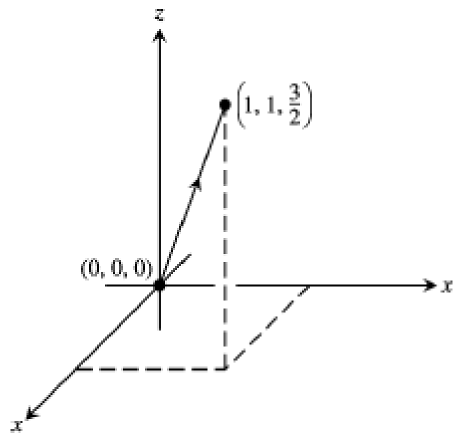
Draw coordinate axes and sketch the segment, indicate the direction on increasing t for the parameterization.

Solution

Let: $\quad P = (0,\ 0,\ 0) \quad Q = \left(1,\ 1,\ \frac{3}{2}\right)$

The direction: $\overrightarrow{PQ} = \hat{i} + \hat{j} + \frac{3}{2}\hat{k}$

The line is given by: $x = t, \quad y = t, \quad z = \frac{3}{2}t, \quad 0 \le t \le 1$



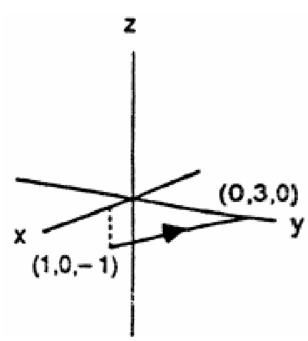
Exercise

Find the parameterization for the line segment joining the points $(1,\ 0,\ -1),\ \ (0,\ 3,\ 0)$. Draw coordinate axes and sketch the segment, indicate the direction on increasing *t* for the parameterization.

Solution

The direction: $\overrightarrow{PQ} = -\hat{i} + 3\hat{j} + \hat{k}$ and $(1,\ 0,\ -1)$

$x = 1 - t, \quad y = 3t, \quad z = -1 + t, \quad 0 \le t \le 1$



Exercise

Find equation for the plane through normal to $\vec{n} = 3\hat{i} - 2\hat{j} - \hat{k}$

Solution

$$3(x - 0) - 2(y - 2) - (z + 1) = 0$$

$$3x - 2y + 4 - z - 1 = 0$$

$$\underline{3x - 2y - z = -3}$$

Exercise

Find equation for the plane through $(1,\ -1,\ 3)$ parallel to the plane $3x+y+z=7$

Solution

$$3(x-1)+(y+1)+(z-3)=0$$

$$3x-3+y+1+z-3=0$$

$$\boxed{3x+y+z=5}$$

Exercise

Find equation for the plane through $(1,\ 1,\ -1)$, $(2,\ 0,\ 2)$ and $(0,\ -2,\ 1)$

Solution

$$\overrightarrow{PQ}=\hat{i}-\hat{j}+3\hat{k} \quad \overrightarrow{PS}=-\hat{i}-3\hat{j}+2\hat{k}$$

$$\overrightarrow{PQ}\times\overrightarrow{PS}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 3\\ -1 & -3 & 2\end{vmatrix}$$

$$=7\hat{i}-5\hat{j}-4\hat{k} \quad \text{is normal to the plane.}$$

$$7(x-2)-5(y+0)-4(z-2)=0$$

$$7x-14-5y-4z+8=0$$

$$\boxed{7x-5y-4z=6}$$

Exercise

Find equation for the plane through $P_0\ (2,\ 4,\ 5)$ perpendicular to the line $x=5+t, \quad y=1+3t, \quad z=4t$

Solution

$$x=5+t, \quad y=1+3t, \quad z=4t$$

$$\Rightarrow \quad \vec{n}=\hat{i}+3\hat{j}+4\hat{k}$$

$$1(x-2)+3(y-4)+4(z-5)=0$$

$$x-2+3y-12+4z-20=0$$

$$\boxed{x+3y+4z=34}$$

Exercise

Find equation for the plane through $A(1,\ -2,\ 1)$ perpendicular to the vector from the origin to A.

Solution

$$\Rightarrow \quad \vec{n} = \hat{i} - 2\hat{j} + \hat{k}$$

$$1(x-1) - 2(y+2) + 1(z-1) = 0$$

$$x - 1 - 2y - 4 + z - 1 = 0$$

$$\underline{x - 2y + z = 6}$$

Exercise

Find the point of intersection of the lines $x = 2t+1,\quad y = 3t+2,\quad z = 4t+3$ and $x = s+2,\quad y = 2s+4,\quad z = -4s-1$, and find the plane determined by these lines.

Solution

$$\begin{cases} x = 2t+1 = s+2 \\ y = 3t+2 = 2s+4 \\ z = 4t+3 = -4s-1 \end{cases}$$

$$\Rightarrow \quad \begin{cases} 2t - s = 1 \\ 3t - 2s = 2 \end{cases} \quad \rightarrow \underline{t = 0} \quad \underline{s = -1}$$

$$z = 4t + 3 = -4s - 1$$

$$4(0) + 3 = -4(-1) - 1$$

$$3 = 3 \qquad \checkmark \quad \text{(Satisfied)}$$

The lines intersect when $t = 0$ and $s = -1$

$\Rightarrow$ The point of intersection $x = 1,\quad y = 2,\quad z = 3$

Therefore; the point is $\underline{P(1,\ 2,\ 3)}$

The normal vectors: $\vec{n}_1 = 2\hat{i} + 3\hat{j} + 4\hat{k} \quad \vec{n}_2 = \hat{i} + 2\hat{j} - 4\hat{k}$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix}$$

$$= -20\hat{i} + 12\hat{j} + \hat{k}$$

$\boldsymbol{n}_1$ *and* $\boldsymbol{n}_2$ are directions of the lines.

The plane containing the lines is represented by

$$-20(x-1)+12(y-2)+1(z-3)=0$$

$$\Rightarrow \quad \underline{-20x+12y+z=7}$$

Exercise

Find the plane determined by the intersecting lines:

$$L_1 : x=-1+t, \quad y=2+t, \quad z=1-t; \quad -\infty < t < \infty$$

$$L_2 : x=1-4s, \quad y=1+2s, \quad z=2-2s; \quad -\infty < s < \infty$$

Solution

The normal vectors: $\vec{n}_1 = \hat{i}+\hat{j}-\hat{k} \quad \vec{n}_2 = -4\hat{i}+2\hat{j}-2\hat{k}$

$$\vec{n} = \vec{n}_1 \times \vec{n}_2$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{vmatrix}$$

$$= 6\hat{j}+6\hat{k}$$

Let $t=0$

$$L_1 : x=-1, \quad y=2, \quad z=1; \quad \Rightarrow \quad \underline{P(-1,\ 2,\ 1)}$$

Therefore; the desired plane is:

$$0(x+1)+6(y-2)+6(z-1)=0$$

$$6y-12+6z-6=0$$

$$6y+6z=18$$

$$\underline{y+z=3}$$

Exercise

Find a plane through $P_0(2,\ 1,\ -1)$ and perpendicular to the line of intersection of the planes

$2x+y-z=3, \quad x+2y+z=2$

Solution

The normal vectors: $\vec{n}_1 = 2\hat{i}+\hat{j}-\hat{k} \quad \vec{n}_2 = \hat{i}+2\hat{j}+\hat{k}$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= 3\hat{i} - 3\hat{j} + 3\hat{k}$$ is the vector in the direction of the line of intersection of the planes.

$$\Rightarrow 3(x-2) - 3(y-1) + 3(z+1) = 0$$

$$3x - 3y + 3z = 0$$

$\underline{x - y + z = 0}$ is the desired plane containing $P_0(2,\ 1,\ -1)$

Exercise

Find the distance from the point to the plane $(0,\ 0,\ 12), \quad x = 4t, \quad y = -2t, \quad z = 2t$

Solution

At $t = 0 \Rightarrow P(0,\ 0,\ 0)$ and let $S(0,\ 0,\ 12)$

$\overrightarrow{PS} = 12\hat{k}$ and $\vec{v} = 4\hat{i} - 2\hat{j} + 2\hat{k}$

$$\overrightarrow{PS} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 12 \\ 4 & -2 & 2 \end{vmatrix}$$

$$= 24\hat{i} + 48\hat{j}$$

$$d = \frac{\left|\overrightarrow{PS} \times \vec{v}\right|}{|\vec{v}|}$$

$$= \frac{\sqrt{24^2 + 48^2}}{\sqrt{16 + 4 + 4}}$$

$$= \frac{24\sqrt{5}}{\sqrt{24}}$$

$$= \sqrt{5}\sqrt{24}$$

$$\underline{= 2\sqrt{30}}$$

Exercise

Find the distance from the point to the plane $(2,\ 1,\ -1), \quad x = 2t, \quad y = 1 + 2t, \quad z = 2t$

Solution

At $t = 0 \Rightarrow P(0,\ 1,\ 0)$ and let $S(2,\ 1,\ -1)$

$\overrightarrow{PS} = 2\hat{i} - \hat{k}$ and $\vec{v} = 2\hat{i} + 2\hat{j} + 2\hat{k}$

$$\overrightarrow{PS}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -1 \\ 2 & 2 & 2 \end{vmatrix}$$

$$= 2\hat{i} - 6\hat{j} + 4\hat{k}$$

$$d = \frac{\sqrt{4+36+16}}{\sqrt{4+4+4}} \qquad\qquad d = \frac{\left|\overrightarrow{PS}\times\vec{v}\right|}{|\vec{v}|}$$

$$= \frac{\sqrt{56}}{\sqrt{12}}$$

$$= \frac{2\sqrt{14}}{2\sqrt{3}}$$

$$= \sqrt{\frac{14}{3}} \quad unit$$

Exercise

Find the distance from the point to the plane $(3,\ -1,\ 4)$, $x = 4 - t$, $y = 3 + 2t$, $z = -5 + 3t$

Solution

At $t = 0 \Rightarrow P(4,\ 3,\ -5)$ and let $S(3,\ -1,\ 4)$

$\overrightarrow{PS} = -\hat{i} - 4\hat{j} + 9\hat{k}$ and $\vec{v} = -\hat{i} + 2\hat{j} + 3\hat{k}$

$$\overrightarrow{PS}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 3 \\ -1 & 2 & 3 \end{vmatrix}$$

$$= -30\hat{i} - 6\hat{j} - 6\hat{k}$$

$$d = \frac{\sqrt{900+36+36}}{\sqrt{1+4+9}} \qquad\qquad d = \frac{\left|\overrightarrow{PS}\times\vec{v}\right|}{|\vec{v}|}$$

$$= \sqrt{\frac{972}{14}}$$

$$= \sqrt{\frac{486}{7}}$$

$$= \frac{9\sqrt{6}}{\sqrt{7}}\frac{\sqrt{7}}{\sqrt{7}}$$

$$= \frac{9\sqrt{42}}{7} \quad unit$$

Exercise

Find the distance from the point to the plane $(2,\ -3,\ 4),\quad x+2y+2z=13$

Solution

$\Rightarrow\ P(13,\ 0,\ 0)$ and let $S(2,\ -3,\ 4)$

$\overrightarrow{PS}=-11\hat{i}-3\hat{j}+4\hat{k}$ and $\vec{n}=\hat{i}+2\hat{j}+2\hat{k}$

$|\vec{n}|=\sqrt{1+4+4}\ \underline{=3}$

$d=\left|\left(-11\hat{i}-3\hat{j}+4\hat{k}\right)\bullet\left(\frac{1}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}\right)\right|\qquad d=\left|\overrightarrow{PS}\bullet\frac{\vec{n}}{|\vec{n}|}\right|$

$=\left|-\frac{11}{3}-\frac{6}{3}+\frac{8}{3}\right|$

$\underline{=3\ \ unit}$

Exercise

Find the distance from the point to the plane $(0,\ 0,\ 0),\quad 3x+2y+6z=6$

Solution

$3x+2y+6z=6$

$3x+2(0)+6(0)=6\ \rightarrow\ \underline{x=2}$

$\Rightarrow\ P(2,\ 0,\ 0)$ and let $S(0,\ 0,\ 0)$

$\overrightarrow{PS}=-2\hat{i}$ and $\vec{n}=3\hat{i}+2\hat{j}+6\hat{k}$

$\rightarrow |\boldsymbol{n}|=\sqrt{9+4+36}=\underline{7}$

$d=\left|\left(-2\hat{i}\right)\bullet\left(\frac{3}{7}\hat{i}+\frac{2}{7}\hat{j}+\frac{6}{7}\hat{k}\right)\right|\qquad d=\frac{\left|\overrightarrow{PS}\bullet\vec{n}\right|}{|\vec{n}|}$

$\underline{=\frac{6}{7}\ \ unit}$

Exercise

Find the distance from the point to the plane $(0,\ 1,\ 1),\quad 4y+3z=-12$

Solution

$\Rightarrow\ P(0,\ -3,\ 0)$ and let $S(0,\ 1,\ 1)$

$\overrightarrow{PS}=4\hat{j}+\hat{k}$ and $\vec{n}=4\hat{j}+3\hat{k}$

$\rightarrow |\vec{n}|=\sqrt{16+9}\ \underline{=5}$

$$d = \left|\left(4\hat{j} + \hat{k}\right)\cdot\left(\frac{4}{5}\hat{j} + \frac{3}{5}\hat{k}\right)\right| \qquad\qquad d = \frac{\left|\overrightarrow{PS} \bullet \vec{n}\right|}{|\vec{n}|}$$

$$= \left|\frac{16}{5} + \frac{3}{5}\right|$$

$$\underline{= \frac{19}{5} \quad unit}$$

Exercise

Find the distance from the point to the plane $(6,\ 0,\ -6),\quad x - y = 4$

Solution

Let $y = 0$, then the point P (4, 0 ,0) lies on the line $x - y = 4$

$\overrightarrow{PS} = 2\hat{i} - 6\hat{k}$ and $\vec{n} = \hat{i} - \hat{j}$

$$d = \frac{|2 + 0 + 0|}{\sqrt{1+1}} \qquad\qquad d = \frac{\left|\overrightarrow{PS} \bullet \vec{n}\right|}{|\vec{n}|}$$

$$= \frac{2}{\sqrt{2}}$$

$$\underline{= \sqrt{2} \quad unit}$$

Exercise

Find the distance from the point to the plane $(3,\ 0,\ 10),\quad 2x + 3y + z = 2$

Solution

Let $y = z = 0$, then the point P (1, 0 ,0) lies on the line $2x + 3y + z = 2$

$\overrightarrow{PS} = 2\hat{i} + 10\hat{k}$ and $\vec{n} = 2\hat{i} + 3\hat{j} + \hat{k}$

$$d = \frac{|4 + 0 + 10|}{\sqrt{4+9+1}} \qquad\qquad d = \frac{\left|\overrightarrow{PS} \bullet \vec{n}\right|}{|\vec{n}|}$$

$$= \frac{14}{\sqrt{14}}$$

$$\underline{= \sqrt{14} \quad unit}$$

Exercise

Find the distance from the point to the line $(2,\ 2,\ 0);\quad x = -t,\quad y = t,\quad z = -1 + t$

Solution

The line passes through the point $P(0, 0, -1)$ parallel to $\vec{v} = -\hat{i} + \hat{j} + \hat{k}$

$$\overrightarrow{PS} = 2\hat{i} + 2\hat{j} + \hat{k}$$

$$\overrightarrow{PS} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ -1 & 1 & 1 \end{vmatrix}$$

$$= \hat{i} - 3\hat{j} + 4\hat{k}$$

$$d = \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}} \qquad d = \frac{\left|\overrightarrow{PS} \times \vec{v}\right|}{|\vec{v}|}$$

$$= \frac{\sqrt{26}}{\sqrt{3}}$$

$$= \frac{\sqrt{78}}{3} \quad unit$$

Exercise

Find the distance from the point to the line $(0,\ 4,\ 1);\quad x = 2+t,\quad y = 2+t,\quad z = t$

Solution

The line passes through the point $P(2, 2, 0)$ parallel to $\vec{v} = \hat{i} + \hat{j} + \hat{k}$

$$\overrightarrow{PS} = -2\hat{i} + 2\hat{j} + \hat{k}$$

$$\overrightarrow{PS} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \hat{i} + 3\hat{j} - 4\hat{k}$$

$$d = \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}} \qquad d = \frac{\left|\overrightarrow{PS} \times \vec{v}\right|}{|\vec{v}|}$$

$$= \frac{\sqrt{26}}{\sqrt{3}}$$

$$= \frac{\sqrt{78}}{3} \quad unit$$

Exercise

Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$

Solution

$$x + 2y + 6z = 1 \quad \Rightarrow \quad P(1,\ 0,\ 0)$$

$$x + 2y + 6z = 10 \quad \Rightarrow \quad S(10,\ 0,\ 0)$$

$\overrightarrow{PS} = 9\hat{i}$ and $\vec{n} = \hat{i} + 2\hat{j} + 6\hat{k}$

$$\to |\vec{n}| = \sqrt{1+4+36} \ = \sqrt{41}$$

$$d = \left|\left(9\hat{i}\right) \bullet \frac{1}{\sqrt{41}}\left(\hat{i} + 2\hat{j} + 6\hat{k}\right)\right| \qquad d = \left|\overrightarrow{PS} \bullet \frac{\vec{n}}{|\vec{n}|}\right|$$

$$= \frac{1}{\sqrt{41}}|9|$$

$$= \frac{9}{\sqrt{41}} \quad unit$$

Exercise

Find the angle between the planes $x + y = 1, \quad 2x + y - 2z = 2$

Solution

The vectors: $\vec{n}_1 = \hat{i} + \hat{j}, \quad \vec{n}_2 = 2\hat{i} + \hat{j} - 2\hat{k}$ are normal to the planes.

The angle between them is:

$$\theta = \cos^{-1}\left(\frac{2+1}{\sqrt{1+1}\ \sqrt{4+1+4}}\right) \qquad \theta = \cos^{-1}\left(\frac{\vec{n}_1 \bullet \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)$$

$$= \cos^{-1}\left(\frac{3}{3\sqrt{2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi}{4}$$

Exercise

Find the angle between the planes $5x + y - z = 10, \quad x - 2y + 3z = -1$

Solution

The vectors: $\vec{n}_1 = 5\hat{i} + \hat{j} - \hat{k}, \quad \vec{n}_2 = \hat{i} - 2\hat{j} + 3\hat{k}$ are normal to the planes.

The angle between them is:

$$\theta = \cos^{-1}\left(\frac{5-2-3}{\sqrt{25+1+1}\sqrt{1+4+9}}\right) \qquad \theta = \cos^{-1}\left(\frac{\vec{n}_1 \bullet \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)$$

$$= \cos^{-1}(0)$$

$$= \frac{\pi}{2}$$

Exercise

Find the angle between the planes $x = 7, \quad x + y + \sqrt{2}z = -3$

Solution

The vectors: $\vec{n}_1 = \hat{i}, \quad \vec{n}_2 = \hat{i} + \hat{j} + \sqrt{2}\hat{k}$ are normal to the planes.

$$\theta = \cos^{-1}\left(\frac{1+0+0}{1\cdot\sqrt{1+1+2}}\right) \qquad \theta = \cos^{-1}\left(\frac{\vec{n}_1 \bullet \vec{n}_2}{\left|\vec{n}_1\right|\left|\vec{n}_2\right|}\right)$$

$$= \cos^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{3}$$

Exercise

Find the angle between the planes $x + y = 1, \quad y + z = 1$

Solution

The vectors: $\vec{n}_1 = \hat{i} + \hat{j}, \quad \vec{n}_2 = \hat{j} + \hat{k}$ are normal to the planes.

$$\theta = \cos^{-1}\left(\frac{0+1+1}{\sqrt{1+1}\cdot\sqrt{1+1}}\right) \qquad \theta = \cos^{-1}\left(\frac{\vec{n}_1 \bullet \vec{n}_2}{\left|\vec{n}_1\right|\left|\vec{n}_2\right|}\right)$$

$$= \cos^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{3}$$

Exercise

Find the point in which the line meets the plane $x = 1 - t, \quad y = 3t, \quad z = 1 + t; \quad 2x - y + 3z = 6$

Solution

$$2(1-t) - 3t + 3(1+t) = 6$$

$$2 - 2t - 3t + 3 + 3t = 6$$

$$-2t = 1$$

$$t = -\frac{1}{2}$$

$$x = 1 - \left(-\frac{1}{2}\right) = \frac{3}{2}, \quad y = 3\left(-\frac{1}{2}\right) = -\frac{3}{2}, \quad z = 1 + \left(-\frac{1}{2}\right) = \frac{1}{2}$$

$$\Rightarrow P\left(\frac{3}{2},\ -\frac{3}{2},\ \frac{1}{2}\right)$$

Exercise

Find the point in which the line meets the plane $x=2, \quad y=3+2t, \quad z=-2-2t; \quad 6x+3y-4z=-12$

Solution

$$12+3(3+2t)-4(-2-2t)=-12$$

$$12+9+6t+8+8t=-12$$

$$14t=-41$$

$$\boxed{t=-\frac{41}{14}}$$

$$x=2, \quad y=3+2\left(-\frac{41}{14}\right)=-\frac{20}{7}, \quad z=-2-2\left(-\frac{41}{14}\right)=\frac{27}{7}$$

$$\boxed{P\left(2, \ -\frac{20}{7}, \ \frac{27}{7}\right)}$$

Exercise

Find an equation of the line through the point $(0, \ 1, \ 1)$ and parallel to the line $\boldsymbol{R}(t)=\langle 1+2t, \ 3-5t, \ 7+6t\rangle$

Solution

Direction: $\vec{v}=\langle 2, \ -5, \ 6\rangle$

Line: $\langle 0, \ 1, \ 1\rangle+t\langle 2, \ -5, \ 6\rangle$

$$\boxed{=\langle 2t, \ 1-5t, \ 1+6t\rangle}$$

Exercise

Find an equation of the line through the point $(0, \ 1, \ 1)$ that is orthogonal to both $\langle 0, \ -1, \ 3\rangle$ and $\langle 2, \ -1, \ 2\rangle$

Solution

Direction: $\langle 0, \ -1, \ 3\rangle\times\langle 2, \ -1, \ 2\rangle=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & -1 & 3\\ 2 & -1 & 2\end{vmatrix}$

$$=\langle 1, \ 6, \ 2\rangle$$

Line through $(0, \ 1, \ 1)$: $\begin{cases}x=t\\ y=1+6t\\ z=1+2t\end{cases}$

Exercise

Find an equation of the line through the point $(0,\ 1,\ 1)$ that is orthogonal to the vector $\langle -2,\ 1,\ 7\rangle$ and the *y-axis*

Solution

$$(0,\ 1,\ 1)\ \perp\ \langle -2,\ 1,\ 7\rangle\ \&\ y-axis$$

$$\text{Direction: } \langle -2,\ 1,\ 7\rangle \times \langle 0,\ 1,\ 0\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 7 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \langle -7,\ 0,\ -2\rangle$$

$$\text{Line through } (0,\ 1,\ 1): \begin{cases} x = -7t \\ y = 1 \\ z = 1 - 2t \end{cases}$$

Exercise

Suppose that $\vec{n}$ is normal to a plane and that $\vec{v}$ is parallel to the plane. Describe how you would find a vector $\vec{n}$ that is both perpendicular to $\vec{v}$ and parallel to the plane.

Solution

The desired vector is $\vec{n} \times \vec{v}$ or $\vec{v} \times \vec{n}$, since $\vec{n} \times \vec{v}$ is perpendicular to both $\vec{n}$ and $\vec{v}$, therefore, also parallel to the plane

Exercise

Given a point $(x_0,\ y_0,\ 0)$ and a vector $\boldsymbol{v} = \langle a,\ b,\ 0\rangle$ in $\mathbb{R}^3$, describe the set of points that satisfy the equation $\langle a,\ b,\ 0\rangle \times \langle x - x_0,\ y - y_0,\ 0\rangle = \mathbf{0}$. Use this result to determine an equation of a line in $\mathbb{R}^2$ passing through $(x_0,\ y_0)$ parallel to the vector $\langle a,\ b\rangle$

Solution

$$\langle a,\ b,\ 0\rangle \times \langle x - x_0,\ y - y_0,\ 0\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & 0 \\ x - x_0 & y - y_0 & 0 \end{vmatrix}$$

$$= \langle 0,\ 0,\ a(y - y_0) - b(x - x_0)\rangle$$

$$= \langle 0,\ 0,\ 0\rangle$$

$$a\left(y-y_0\right)-b\left(x-x_0\right)=0$$

$$a\left(y-y_0\right)=b\left(x-x_0\right)$$

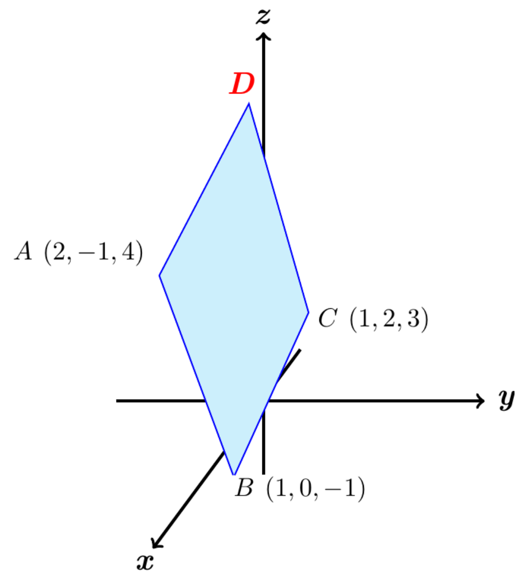
$$\frac{y-y_0}{x-x_0}=\frac{b}{a}=m \qquad (slope)$$

Equation of a line passing through $\left(x_0,\ y_0\right)$ parallel to the vector $\langle a,\ b\rangle$

$$\boxed{ay-bx=+ay_0-bx_0}$$

Exercise

The parallelogram has vertices at $A(2, -1, 4)$, $B(1, 0, -1)$, $C(1, 2, 3)$ and D. Find



a) The coordinates of D,

b) The cosine of the interior angle of B

c) The vector projection of $\overrightarrow{BA}$ onto $\overrightarrow{BC}$,

d) The area of the parallelogram,

e) An equation for the plane of the parallelogram,

f) The areas of the orthogonal projection of the parallelogram on the three coordinate planes.

Solution

a) $\overrightarrow{AB}=\langle 1-2,\ 0+1,\ -1-4\rangle$

$$=\langle -1,\ 1,\ -5\rangle$$

$$\overrightarrow{DC}=\langle 1-x,\ 2-y,\ 3-z\rangle$$

$$\overrightarrow{DC}=\overrightarrow{AB}$$

$$\langle 1-x,\ \ 2-y,\ \ 3-z\rangle = \langle -1,\ 1,\ \ -5\rangle$$

$$\begin{cases} 1-x=-1 & \rightarrow \quad x=2 \\ 2-y=1 & \rightarrow \quad y=1 \\ 3-z=-5 & \rightarrow \quad z=8 \end{cases}$$

$$\Rightarrow \quad \boxed{D = (2,\ 1,\ 8)}$$

b) $\overrightarrow{BA} = \langle 1,\ \ -1,\ 5\rangle$

$$\overrightarrow{BC} = \langle 1-1,\ 2-0,\ 3+1\rangle$$

$$= \langle 0,\ 2,\ 4\rangle$$

$$\cos B = \frac{\overrightarrow{BA} \bullet \overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\ \left|\overrightarrow{BC}\right|}$$

$$= \frac{\langle 1,\ \ -1,\ 5\rangle \bullet \langle 0,\ 2,\ 4\rangle}{\sqrt{1+1+25}\ \sqrt{4+16}}$$

$$= \frac{0-2+20}{\sqrt{27}\ \sqrt{20}}$$

$$= \frac{18}{3\sqrt{3}\ 2\sqrt{5}}$$

$$\boxed{= \frac{3}{\sqrt{15}}}$$

c) $proj_{\overrightarrow{BC}}\ \overrightarrow{BA} = \dfrac{\overrightarrow{BA} \bullet \overrightarrow{BC}}{\left|\overrightarrow{BC}\right|^2}\ \overrightarrow{BC}$

$$= \frac{\langle 1,\ \ -1,\ 5\rangle \bullet \langle 0,\ 2,\ 4\rangle}{4+16}\langle 0,\ 2,\ 4\rangle$$

$$= \frac{18}{20}\langle 0,\ 2,\ 4\rangle$$

$$= \frac{9}{10}\langle 0,\ 2,\ 4\rangle$$

$$\boxed{= \left\langle 0,\ \frac{9}{5},\ \frac{18}{5}\right\rangle}$$

d) $Area = \left|\overrightarrow{BA} \times \overrightarrow{BC}\right|$

$$= \left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 5 \\ 0 & 2 & 4 \end{matrix} \right\|$$

$$= \left|-14\hat{i} - 4\hat{j} + 2\hat{k}\right|$$

$$= \sqrt{196+16+4}$$

$$= \sqrt{216}$$

$$= 6\sqrt{6}$$

e) $\overrightarrow{BA} \times \overrightarrow{BC} = -14\hat{i} - 4\hat{j} + 2\hat{k} = \vec{n}$

$$-14(x-1) - 4y + 2(z+1) = 0$$

$$-14x + 14 - 4y + 2z + 2 = 0$$

$$-14x - 4y + 2z = -16$$

$$7x + 2y - z = 8$$

f) $\vec{n} = -14\hat{i} - 4\hat{j} + 2\hat{k}$

Area of the projection on $yz - plane$ $\left|\vec{n} \bullet \hat{i}\right| = 14$

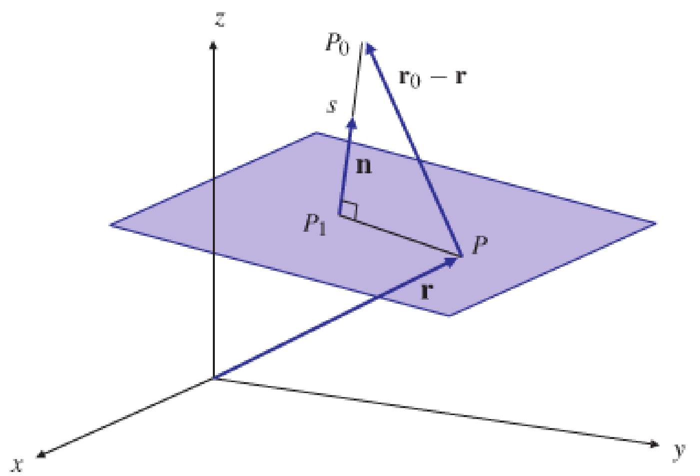
Area of the projection on $xz - plane$ $\left|\vec{n} \bullet \hat{j}\right| = 4$

Area of the projection on $xy - plane$ $\left|\vec{n} \bullet \hat{k}\right| = 2$

Exercise

a) Find the distance from the point $P_0\left(x_0, y_0, z_0\right)$ to the plane *P* having equation

$$Ax + By + Cz = D$$



b) What is the distance from $(2, \ -1, \ 3)$ to the plane $2x - 2y - z = 9$?

Solution

a) $\vec{n} = A\hat{i} + B\hat{j} + C\hat{k}$

$$|\vec{n}| = \sqrt{A^2 + B^2 + C^2}$$

Let $y = z = 0 \quad \rightarrow \quad P = \left(\frac{D}{A}, \ 0, \ 0\right)$

$$\overrightarrow{PP_0} = \left\langle x_0 - \frac{D}{A}, \ y_0, \ z_0 \right\rangle$$

$$d = \left(\left(x_0 - \frac{D}{A}\right)\hat{i} + y_0\hat{j} + z_0\hat{k}\right) \bullet \frac{A\hat{i} + B\hat{j} + C\hat{k}}{\sqrt{A^2 + B^2 + C^2}} \qquad d = \overrightarrow{PP_0} \bullet \frac{\vec{n}}{|\vec{n}|}$$

$$= \frac{1}{\sqrt{A^2 + B^2 + C^2}}\left(\left(x_0 - \frac{D}{A}\right)A + y_0 B + z_0 C\right)$$

$$\boxed{= \frac{Ax_0 + By_0 + Cz_0 - D}{\sqrt{A^2 + B^2 + C^2}}}$$

b) Distance $(2,\ -1,\ 3)$ to $2x - 2y - z = 9$

$$d = \frac{|2(2) - 2(-1) + (-1)(3) - 9|}{\sqrt{4 + 4 + 1}}$$

$$= \frac{|-6|}{3}$$

$$\boxed{= 2 \ \ units}$$

Solution Section 1.5 – Calculus of Vector-Valued Functions

Exercise

$\vec{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find an equation in *x* and *y* whose is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of *t*.

$\vec{r}(t) = (t+1)\hat{i} + \left(t^2 - 1\right)\hat{j}, \quad t = 1$

Solution

$$x = t+1, \quad y = t^2 - 1$$

$$y = (x-1)^2 - 1$$

$$\boxed{= x^2 - 2x}$$

$$\vec{v}(t) = \vec{r}\,' = \hat{i} + 2t\,\hat{j}$$

$$\boxed{\vec{v}(t=1) = \hat{i} + 2\hat{j}}$$

$$\vec{a} = \vec{v}\,' = 2\hat{j}$$

$$\boxed{\vec{a}(t=1) = 2\hat{j}}$$

Exercise

$\vec{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find an equation in *x* and *y* whose is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of *t*.

$\vec{r}(t) = \frac{t}{t+1}\hat{i} + \frac{1}{t}\hat{j}, \quad t = -\frac{1}{2}$

Solution

$$x = \frac{t}{t+1}, \quad y = \frac{1}{t} \to t = \frac{1}{y}$$

$$x = \frac{\frac{1}{y}}{\frac{1}{y}+1}$$

$$= \frac{1}{1+y}$$

$$1 + y = \frac{1}{x}$$

$$y = \frac{1}{x} - 1$$

$$\left(\frac{t}{t+1}\right)' = \frac{1}{(t+1)^2} \qquad \left(\frac{ax+b}{cx+d}\right)' = \frac{ad-bc}{(cx+d)^2}$$

$$\vec{v}(t) = \frac{1}{(t+1)^2}\hat{i} - \frac{1}{t^2}\hat{j}$$

$$\vec{v}\left(t=-\frac{1}{2}\right) = \frac{1}{\left(-\frac{1}{2}+1\right)^2}\hat{i} - \frac{1}{\frac{1}{4}}\hat{j}$$

$$= 4\hat{i} - 4\hat{j}$$

$$\left(\frac{1}{(t+1)^2}\right)' = \frac{-2}{(t+1)^3} \qquad \left(\frac{1}{U^n}\right)' = -\frac{nU'}{U^{n+1}}$$

$$\vec{a} = \vec{v}' = \frac{-2}{(t+1)^3}\hat{i} + \frac{2}{t^3}\hat{j}$$

$$\vec{a}\left(t=-\frac{1}{2}\right) = \frac{-2}{\left(-\frac{1}{2}+1\right)^3}\hat{i} + \frac{2}{\left(-\frac{1}{2}\right)^3}\hat{j}$$

$$= \frac{-2}{-\frac{1}{8}}\hat{i} + \frac{2}{-\frac{1}{8}}\hat{j}$$

$$= 16\hat{i} - 16\hat{j}$$

Exercise

$\vec{r}(t)$ is the position of a particle in the xy-plane at time t. Find an equation in x and y whose is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of t.

$$\vec{r}(t) = e^t\hat{i} + \frac{2}{9}e^{2t}\hat{j}, \quad t = \ln 3$$

Solution

$$x = e^t, \quad y = \frac{2}{9}e^{2t} = \frac{2}{9}\left(e^t\right)^2$$

$$y = \frac{2}{9}x^2$$

$$\vec{v}(t) = e^t\hat{i} + \frac{4}{9}e^{2t}\hat{j}$$

$$\vec{v}(t=\ln 3) = e^{\ln 3}\hat{i} + \frac{4}{9}e^{2\ln 3}\hat{j}$$

$$= 3\hat{i} + \frac{4}{9}e^{\ln 3^2}\hat{j} \qquad e^{\ln 3^2} = e^{\ln 9} = 9$$

$$= 3\hat{i} + 4\hat{j}$$

$$\vec{a}(t) = e^t\hat{i} + \frac{8}{9}e^{2t}\hat{j}$$

$$\vec{a}(t = \ln 3) = e^{\ln 3}\hat{i} + \frac{8}{9}e^{2\ln 3}\hat{j}$$

$$= 3\hat{i} + \frac{8}{9}e^{\ln 9}\hat{j}$$

$$\boxed{= 3\hat{i} + 8\hat{j}}$$

Exercise

$\vec{r}(t)$ is the position of a particle in the xy-plane at time t. Find an equation in x and y whose is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of t.

$$\vec{r}(t) = (\cos 2t)\hat{i} + (3\sin 2t)\hat{j}, \quad t = 0$$

Solution

$$x = \cos 2t, \quad y = 3\sin 2t \rightarrow \sin 2t = \frac{y}{3}$$

$$\cos^2 2t + \sin^2 2t = 1$$

$$x^2 + \frac{y^2}{9} = 1$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -(2\sin 2t)\hat{i} + (6\cos 2t)\hat{j}$$

$$\vec{v}(t = 0) = -(2\sin 0)\hat{i} + (6\cos 0)\hat{j}$$

$$\boxed{= 6\hat{j}}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = -(4\cos 2t)\hat{i} - (12\sin 2t)\hat{j}$$

$$\vec{a}(t = 0) = -(4\cos 2t)\hat{i} - (12\sin 2t)\hat{j}$$

$$\boxed{= -4\hat{i}}$$

Exercise

Give the position vectors of particles moving along various curves in the xy-plane. Find the particle's velocity and acceleration vectors at the stated times and sketch them as vectors on the curve

Motion on the circle $x^2 + y^2 = 1$ $\quad \vec{r}(t) = (\sin t)\hat{i} + (\cos t)\hat{j}, \quad t = \frac{\pi}{4}$ *and* $\frac{\pi}{2}$

Solution

$$\vec{v} = \frac{d\boldsymbol{r}}{dt} = (\cos t)\hat{i} - (\sin t)\hat{j}$$

$$\vec{v}\left(t = \frac{\pi}{4}\right) = \left(\cos\frac{\pi}{4}\right)\hat{i} - \left(\sin\frac{\pi}{4}\right)\hat{j}$$

$$= \boxed{\frac{\sqrt{2}}{2}\hat{i} - \frac{\sqrt{2}}{2}\hat{j}}$$

$$\vec{v}\left(t=\tfrac{\pi}{2}\right) = \left(\cos\tfrac{\pi}{2}\right)\hat{i} - \left(\sin\tfrac{\pi}{2}\right)\hat{j}$$

$$= \boxed{-\hat{j}}$$

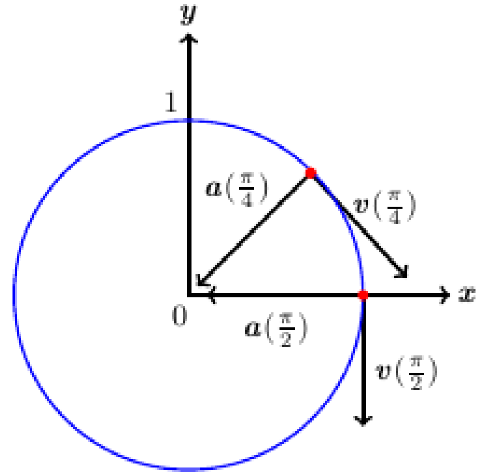
$$\vec{a} = \frac{d\vec{v}}{dt} = -(\sin t)\hat{i} - (\cos t)\hat{j}$$

$$\vec{a}\left(t=\tfrac{\pi}{4}\right) = -\left(\sin\tfrac{\pi}{4}\right)\hat{i} - \left(\cos\tfrac{\pi}{4}\right)\hat{j}$$

$$= \boxed{-\frac{\sqrt{2}}{2}\hat{i} - \frac{\sqrt{2}}{2}\hat{j}}$$

$$\vec{a}\left(t=\tfrac{\pi}{2}\right) = -\left(\sin\tfrac{\pi}{2}\right)\hat{i} - \left(\cos\tfrac{\pi}{2}\right)\hat{j}$$

$$= \boxed{-\hat{i}}$$



Exercise

Give the position vectors of particles moving along various curves in the *xy*-plane. Find the particle's velocity and acceleration vectors at the stated times and sketch them as vectors on the curve

Motion on the cycloid $x = t - \sin t, \quad y = 1 - \cos t$

$$\vec{r}(t) = (t - \sin t)\hat{i} + (1 - \cos t)\hat{j}; \quad t = \pi \ and \ \frac{3\pi}{2}$$

Solution

$$\vec{v} = \frac{d\vec{r}}{dt} = (1 - \cos t)\hat{i} + (\sin t)\hat{j}$$

$$\vec{v}(t=\pi) = (1 - \cos\pi)\hat{i} + (\sin\pi)\hat{j}$$

$$= \boxed{2\hat{i}}$$

$$\vec{v}\left(t=\tfrac{3\pi}{2}\right) = \left(1 - \cos\tfrac{3\pi}{2}\right)\hat{i} + \left(\sin\tfrac{3\pi}{2}\right)\hat{j}$$

$$= \boxed{\hat{i} - \hat{j}}$$

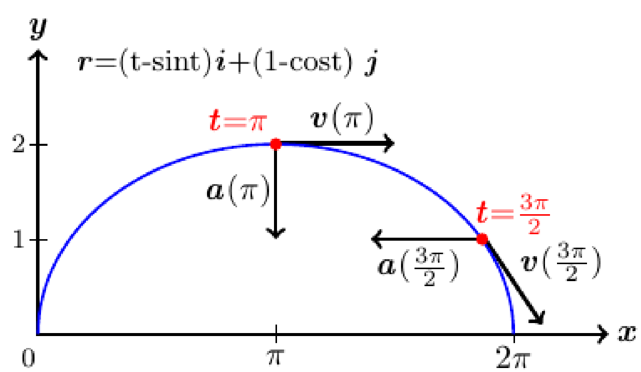
$$\vec{a} = \frac{d\vec{v}}{dt} = (\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\vec{a}(t=\pi) = (\sin\pi)\hat{i} + (\cos\pi)\hat{j}$$

$$= \boxed{-\hat{j}}$$

$$\vec{a}\left(t=\tfrac{3\pi}{2}\right) = \left(\sin\tfrac{3\pi}{2}\right)\hat{i} + \left(\cos\tfrac{3\pi}{2}\right)\hat{j}$$

$$= \boxed{-\hat{i}}$$



Exercise

$\vec{r}(t)$ is the position of a particle in the xy-plane at time t. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of t. Write the particle's velocity at that time as the product of its speed and direction.

$$\vec{r}(t) = (t+1)\hat{i} + \left(t^2 - 1\right)\hat{j} + 2t\,\hat{k}, \quad t = 1$$

Solution

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j} + 2\hat{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = 2\hat{j}$$

$$\vec{v}(t=1) = \hat{i} + 2\hat{j} + 2\hat{k}$$

Speed: $\left|\vec{v}(1)\right| = \sqrt{1+4+4}\ \underline{= 3}$

Direction: $\dfrac{\vec{v}(1)}{\left|\vec{v}(1)\right|} = \frac{1}{3}\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$

$$= \frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

$$\vec{v}(1) = 3\left(\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$$

Exercise

$\vec{r}(t)$ is the position of a particle in the xy-plane at time t. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of t. Write the particle's velocity at that time as the product of its speed and direction.

$$\vec{r}(t) = (t+1)\hat{i} + \frac{t^2}{\sqrt{2}}\hat{j} + \frac{t^3}{3}\hat{k}, \quad t = 1$$

Solution

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i} + \frac{2}{\sqrt{2}}t\,\hat{j} + t^2\hat{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{2}{\sqrt{2}}\hat{j} + 2t\,\hat{k}$$

$$\vec{v}(t=1) = \hat{i} + \frac{2}{\sqrt{2}}\hat{j} + \hat{k}$$

Speed: $\left|\vec{v}(1)\right| = \sqrt{1+2+1}\ \underline{= 2}$

Direction: $\dfrac{\vec{v}(1)}{\left|\vec{v}(1)\right|} = \frac{1}{2}\left(\hat{i} + \frac{2}{\sqrt{2}}\hat{j} + \hat{k}\right)$

$$= \frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}$$

$$\boxed{\vec{v}(1) = 2\left(\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}\right)}$$

Exercise

$\vec{r}(t)$ is the position of a particle in the xy-plane at time t. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of t. Write the particle's velocity at that time as the product of its speed and direction.

$$\vec{r}(t) = (2\cos t)\hat{i} + (3\sin t)\hat{j} + 4t\,\hat{k}, \quad t = \frac{\pi}{2}$$

Solution

$$\vec{v} = \frac{d\vec{r}}{dt} = -(2\sin t)\hat{i} + (3\cos t)\hat{j} + 4\hat{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = -(2\cos t)\hat{i} + (3\sin t)\hat{j}$$

$$\vec{v}\left(t = \frac{\pi}{2}\right) = -\left(2\sin\frac{\pi}{2}\right)\hat{i} + \left(3\cos\frac{\pi}{2}\right)\hat{j} + 4\hat{k}$$

$$= -2\hat{i} + 4\hat{k}$$

Speed: $\left|\vec{v}\left(\frac{\pi}{2}\right)\right| = \sqrt{4+16}$ $\boxed{= 2\sqrt{5}}$

Direction: $\dfrac{\vec{v}\left(\frac{\pi}{2}\right)}{\left|\vec{v}\left(\frac{\pi}{2}\right)\right|} = \dfrac{1}{2\sqrt{5}}\left(-2\hat{i} + 4\hat{k}\right)$

$$= -\frac{1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{k}$$

$$\boxed{\vec{v}\left(\frac{\pi}{2}\right) = 2\sqrt{5}\left(-\frac{1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{k}\right)}$$

Exercise

$\vec{r}(t)$ is the position of a particle in the xy-plane at time t. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of t. Write the particle's velocity at that time as the product of its speed and direction.

$$\vec{r}(t) = (2\ln(t+1))\hat{i} + t^2\hat{j} + \frac{t^2}{2}\hat{k}, \quad t = 1$$

Solution

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{2}{t+1}\hat{i} + 2t\,\hat{j} + t\,\hat{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{-2}{(t+1)^2}\hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{v}(t=1) = \hat{i} + 2\hat{j} + \hat{k}$$

Speed: $|\vec{v}(1)| = \sqrt{1+4+1} \ \underline{= \sqrt{6}}$

Direction: $\frac{\vec{v}(1)}{|\vec{v}(1)|} = \frac{1}{\sqrt{6}}(\hat{i} + 2\hat{j} + \hat{k})$

$$= \frac{1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} + \frac{1}{\sqrt{6}}\hat{k}$$

$$\underline{\vec{v}(1) = \sqrt{6}\left(\frac{1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} + \frac{1}{\sqrt{6}}\hat{k}\right)}$$

Exercise

$\vec{\boldsymbol{r}}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of *t*. Write the particle's velocity at that time as the product of its speed and direction.

$$\vec{\boldsymbol{r}}(t) = \left(e^{-t}\right)\hat{i} + (2\cos 3t)\hat{j} + (2\sin 3t)\hat{k}, \quad t = 0$$

Solution

$$\vec{v} = \frac{d\vec{\boldsymbol{r}}}{dt} = -e^{-t}\hat{i} - 6\sin 3t\,\hat{j} + 6\cos 3t\,\hat{k} \qquad \boldsymbol{v}(0) = -\boldsymbol{i} + 6\boldsymbol{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = e^{-t}\hat{i} - 18\cos 3t\,\hat{j} - 18\sin 3t\,\hat{k}$$

Speed: $|\vec{v}(0)| = 1 + 36 \ \underline{= \sqrt{37}}$

Direction: $\frac{\vec{v}(0)}{|\vec{v}(0)|} = \frac{1}{\sqrt{37}}(-\hat{i} + 6\hat{k})$

$$= -\frac{1}{\sqrt{37}}\hat{i} + \frac{6}{\sqrt{37}}\hat{k}$$

$$\underline{\vec{v}(1) = \sqrt{37}\left(-\frac{1}{\sqrt{37}}\hat{i} + \frac{6}{\sqrt{37}}\hat{k}\right)}$$

Exercise

Find all points on the ellipse $\vec{r}(t) = \langle 1,\ 8\sin t,\ \cos t\rangle$, for $0 \le t \le 2\pi$, at which $\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal.

Solution

$\vec{r}'(t) = \langle 0,\ 8\cos t,\ -\sin t \rangle$

$\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal that implies to $\vec{r}(t) \bullet \vec{r}'(t) = 0$

$$\begin{aligned}\vec{r}(t) \bullet \vec{r}'(t) &= \langle 1,\ 8\sin t,\ \cos t \rangle \bullet \langle 0,\ 8\cos t,\ -\sin t \rangle \\ &= 64\sin t\cos t - \cos t\sin t \\ &= 63\sin t\cos t = 0\end{aligned}$$

$$\to \begin{cases} \sin t = 0 & \Rightarrow\ t = 0,\ \pi,\ 2\pi \\ \cos t = 0 & \Rightarrow\ t = \frac{\pi}{2},\ \frac{3\pi}{2} \end{cases}$$

$t = 0,\ 2\pi \quad \to \quad \underline{(1,\ 0,\ 1)|}$

$t = \frac{\pi}{2} \quad \to \quad \underline{(1,\ 8,\ 0)|}$

$t = \pi \quad \to \quad \underline{(1,\ 0,\ -1)|}$

$t = \frac{3\pi}{2} \quad \to \quad \underline{(1,\ -8,\ 0)|}$

Solution   Section 1.6 – Motion in Space

Exercise

Evaluate the integral: $\int_0^1 \left(t^3\hat{i} + 7\hat{j} + (t+1)\hat{k}\right)dt$

Solution

$$\int_0^1 \left(t^3\hat{i} + 7\hat{j} + (t+1)\hat{k}\right)dt = \frac{1}{4}t^4\hat{i} + 7t\hat{j} + \left(\frac{1}{2}t^2 + t\right)\hat{k}\ \Big|_0^1$$

$$= \left(\frac{1}{4}\hat{i} + 7\hat{j} + \left(\frac{1}{2} + 1\right)\hat{k}\right) - 0$$

$$= \frac{1}{4}\hat{i} + 7\hat{j} + \frac{3}{2}\hat{k}$$

Exercise

Evaluate the integral: $\int_1^2 \left((6-6t)\hat{i} + 3\sqrt{t}\,\hat{j} + \frac{4}{t^2}\hat{k}\right)dt$

Solution

$$\int_1^2 \left((6-6t)\hat{i} + 3\sqrt{t}\,\hat{j} + \frac{4}{t^2}\hat{k}\right)dt = \left(6t - 3t^2\right)\hat{i} + 2t^{3/2}\hat{j} - \frac{4}{t}\hat{k}\ \Big|_1^2$$

$$= \left((12-12)\hat{i} + 2(2)^{3/2}\hat{j} - \frac{4}{2}\hat{k}\right) - \left((6-3)\hat{i} + 2\hat{j} - 4\hat{k}\right)$$

$$= 4\sqrt{2}\,\hat{j} - 2\hat{k} - 3\hat{i} - 2\hat{j} + 4\hat{k}$$

$$= -3\hat{i} + \left(4\sqrt{2} - 2\right)\hat{j} + 2\hat{k}$$

Exercise

Evaluate the integral: $\int_{-\pi/4}^{\pi/4} \left((\sin t)\hat{i} + (1+\cos t)\hat{j} + \left(\sec^2 t\right)\hat{k}\right)dt$

Solution

$$\int_{-\pi/4}^{\pi/4} \left((\sin t)\hat{i} + (1+\cos t)\hat{j} + \left(\sec^2 t\right)\hat{k}\right)dt = -(\cos t)\hat{i} + (t + \sin t)\hat{j} + (\tan t)\hat{k}\ \Big|_{-\pi/4}^{\pi/4}$$

$$=\left[-\left(\cos\frac{\pi}{4}\right)\hat{i}+\left(\frac{\pi}{4}+\sin\frac{\pi}{4}\right)\hat{j}+\left(\tan\frac{\pi}{4}\right)\hat{k}\right]$$

$$-\left[-\left(\cos\left(-\frac{\pi}{4}\right)\right)\hat{i}+\left(-\frac{\pi}{4}+\sin\left(-\frac{\pi}{4}\right)\right)\hat{j}+\left(\tan\left(-\frac{\pi}{4}\right)\right)\hat{k}\right]$$

$$=-\frac{\sqrt{2}}{2}\hat{i}+\left(\frac{\pi}{4}+\frac{\sqrt{2}}{2}\right)\hat{j}+\hat{k}+\frac{\sqrt{2}}{2}\hat{i}-\left(-\frac{\pi}{4}-\frac{\sqrt{2}}{2}\right)\hat{j}+\hat{k}$$

$$=2\left(\frac{\pi}{4}+\frac{\sqrt{2}}{2}\right)\hat{j}+2\hat{k}$$

$$=2\left(\frac{\pi+2\sqrt{2}}{4}\right)\hat{j}+2\hat{k}$$

$$\boxed{=\left(\frac{\pi+2\sqrt{2}}{2}\right)\hat{j}+2\hat{k}}$$

Exercise

Evaluate the integral: $\displaystyle\int_0^{\pi/3}\left((\sec t\tan t)\hat{i}+(\tan t)\hat{j}+(2\sin t\cos t)\hat{k}\right)dt$

Solution

$$\int_0^{\pi/3}\left((\sec t\tan t)\hat{i}+(\tan t)\hat{j}+(2\sin t\cos t)\hat{k}\right)dt=\int_0^{\pi/3}\left((\sec t\tan t)\hat{i}+(\tan t)\hat{j}+(\sin 2t)\hat{k}\right)dt$$

$$=(\sec t)\hat{i}+(-\ln(\cos t))\hat{j}-\left(\frac{1}{2}\cos 2t\right)\hat{k}\,\Big|_0^{\pi/3}$$

$$=\left[\left(\sec\frac{\pi}{3}\right)\hat{i}+\left(-\ln\left(\cos\frac{\pi}{3}\right)\right)\hat{j}-\left(\frac{1}{2}\cos\frac{2\pi}{3}\right)\hat{k}\right]$$

$$-\left[(\sec 0)\hat{i}+(-\ln(\cos 0))\hat{j}-\left(\frac{1}{2}\cos 0\right)\hat{k}\right]$$

$$=\left[2\hat{i}+\left(-\ln\frac{1}{2}\right)\hat{j}-\left(\frac{1}{2}\left(-\frac{1}{2}\right)\right)\hat{k}\right]-\left[\hat{i}+(-\ln(1))\hat{j}-\frac{1}{2}\hat{k}\right]$$

$$=2\hat{i}+\ln 2\,\hat{j}+\frac{1}{4}\hat{k}-\hat{i}+\frac{1}{2}\hat{k}$$

$$\boxed{=\hat{i}+(\ln 2)\,\hat{j}+\frac{3}{4}\hat{k}}$$

Exercise

Evaluate the integral: $\displaystyle\int_0^1 \left(\frac{2}{\sqrt{1-t^2}}\hat{i} + \frac{\sqrt{3}}{1+t^2}\hat{k} \right) dt$

Solution

$$\int_0^1 \left(\frac{2}{\sqrt{1-t^2}}\hat{i} + \frac{\sqrt{3}}{1+t^2}\hat{k} \right) dt = \left(2\sin^{-1} t\right)\hat{i} + \left(\sqrt{3}\tan^{-1} t\right)\hat{k} \Big|_0^1$$

$$= \left[\left(2\sin^{-1} 1\right)\hat{i} + \left(\sqrt{3}\tan^{-1} 1\right)\hat{k}\right] - \left[\left(2\sin^{-1} 0\right)\hat{i} + \left(\sqrt{3}\tan^{-1} 0\right)\hat{k}\right]$$

$$= \left[\left(2\tfrac{\pi}{2}\right)\hat{i} + \left(\sqrt{3}\tfrac{\pi}{4}\right)\hat{k}\right] - \left[(0)\hat{i} + (0)\hat{k}\right]$$

$$= \pi\hat{i} + \frac{\pi\sqrt{3}}{4}\hat{k}$$

Exercise

Evaluate the integral: $\displaystyle\int_1^{\ln 3} \left(te^t\hat{i} + e^t\hat{j} + (\ln t)\hat{k} \right) dt$

Solution

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = \int dx = x$

$$\int u\,dv = uv - \int v\,du$$

	e^t	
$(+)$	t	e^t
$(-)$	1	e^t
	0	

$$\int \ln x\,dx = x\ln x - \int x\frac{1}{x}dx = x\ln x - \int dx = x\ln x - x + C$$

$$\int_1^{\ln 3} \left(te^t\hat{i} + e^t\hat{j} + (\ln t)\hat{k} \right) dt = \left[\left(te^t - e^t\right)\hat{i} + e^t\hat{j} + (t\ln t - t)\hat{k}\right]_1^{\ln 3}$$

$$= \left[\left((\ln 3)e^{\ln 3} - e^{\ln 3}\right)\hat{i} + e^{\ln 3}\hat{j} + \left(\ln 3\ln(\ln 3) - \ln 3\right)\hat{k}\right]$$

$$-\left[(e-e)\hat{i} + e\hat{j} + \left(\ln(1) - 1\right)\hat{k}\right]$$

$$= (3\ln 3 - 3)\hat{i} + 3\hat{j} + \left(\ln 3\left(\ln(\ln 3) - 1\right)\right)\hat{k} - e\hat{j} + \hat{k}$$

$$= 3(\ln 3 - 1)\hat{i} + (3-e)\hat{j} + \left(\ln 3\left(\ln(\ln 3) - 1\right) + 1\right)\hat{k}$$

Exercise

Evaluate the integral: $\displaystyle\int_0^{\pi/2}\left(\cos t\,\hat{i} - \sin 2t\,\hat{j} + \sin^2 t\,\hat{k}\right)dt$

Solution

$$\int_0^{\pi/2}\left(\cos t\,\hat{i} - \sin 2t\,\hat{j} + \sin^2 t\,\hat{k}\right)dt = \int_0^{\pi/2}\left(\cos t\,\hat{i} - \sin 2t\,\hat{j} + \left(\frac{1}{2} - \frac{1}{2}\cos 2t\right)\hat{k}\right)dt$$

$$= \left[\sin t\,\hat{i} + \frac{1}{2}\cos 2t\,\hat{j} + \left(\frac{1}{2}t - \frac{1}{4}\sin 2t\right)\hat{k}\right]_0^{\pi/2}$$

$$= \left[\hat{i} + \frac{1}{2}(-1)\,\hat{j} + \frac{\pi}{4}\,\hat{k}\right] - \frac{1}{2}\,\hat{j}$$

$$= \hat{i} - \frac{1}{2}\,\hat{j} + \frac{\pi}{4}\,\hat{k} - \frac{1}{2}\,\hat{j}$$

$$= \hat{i} - \hat{j} + \frac{\pi}{4}\,\hat{k}$$

Exercise

Solve the initial value problem for $\boldsymbol{r}$ as a vector function of t.

$$\begin{cases} \textit{Differential equation}: & \dfrac{d\vec{r}}{dt} = -t\hat{i} - t\hat{j} - t\hat{k} \\ \textit{Initial condition}: & \vec{r}(0) = \hat{i} + 2\hat{j} + 3\hat{k} \end{cases}$$

Solution

$$\vec{r} = \int \frac{d\vec{r}}{dt}dt = \int\left(-t\hat{i} - t\hat{j} - t\hat{k}\right)dt$$

$$= -\frac{t^2}{2}\hat{i} - \frac{t^2}{2}\hat{j} - \frac{t^2}{2}\hat{k} + \vec{C}$$

$$\vec{r}(0) = -0\hat{i} - 0\hat{j} - 0\hat{k} + \vec{C}$$

$$\hat{i} + 2\hat{j} + 3\hat{k} = \vec{C}$$

$$\vec{r}(t) = -\frac{t^2}{2}\hat{i} - \frac{t^2}{2}\hat{j} - \frac{t^2}{2}\hat{k} + \hat{i} + 2\hat{j} + 3\hat{k}$$

$$= \left(-\frac{t^2}{2} + 1\right)\hat{i} + \left(2 - \frac{t^2}{2}\right)\hat{j} + \left(3 - \frac{t^2}{2}\right)\hat{k}$$

Exercise

Solve the initial value problem for $\boldsymbol{r}$ as a vector function of t.

$$\begin{cases} \textit{Differential equation}: & \dfrac{d\vec{\boldsymbol{r}}}{dt} = (180t)\,\hat{i} + \left(180t - 16t^2\right)\hat{j} \\ \textit{Initial condition}: & \vec{\boldsymbol{r}}(0) = 100\,\hat{j} \end{cases}$$

Solution

$$\vec{\boldsymbol{r}} = \int \left[(180t)\,\hat{i} + \left(180t - 16t^2\right)\hat{j}\right] dt$$

$$= \left(90t^2\right)\hat{i} + \left(90t^2 - \frac{16}{3}t^3\right)\hat{j} + \vec{C}$$

$$\vec{\boldsymbol{r}}(0) = 0\hat{i} + 0\,\hat{j} + \vec{C}$$

$$100\,\hat{j} = \vec{C}$$

$$\vec{\boldsymbol{r}}(t) = \left(90t^2\right)\hat{i} + \left(90t^2 - \frac{16}{3}t^3\right)\hat{j} + 100\,\hat{j}$$

$$\boxed{= \left(90t^2\right)\hat{i} + \left(90t^2 - \frac{16}{3}t^3 + 100\right)\hat{j}}$$

Exercise

Solve the initial value problem for $\boldsymbol{r}$ as a vector function of t.

$$\begin{cases} \textit{Differential equation}: & \dfrac{d\vec{\boldsymbol{r}}}{dt} = \frac{3}{2}(t+1)^{1/2}\,\hat{i} + e^{-t}\,\hat{j} + \dfrac{1}{t+1}\hat{k} \\ \textit{Initial condition}: & \vec{\boldsymbol{r}}(0) = \hat{k} \end{cases}$$

Solution

$$\vec{\boldsymbol{r}} = \int \left(\frac{3}{2}(t+1)^{1/2}\,\hat{i} + e^{-t}\,\hat{j} + \frac{1}{t+1}\hat{k}\right) dt$$

$$= (t+1)^{3/2}\,\hat{i} - e^{-t}\,\hat{j} + \ln(t+1)\,\hat{k} + \vec{C}$$

$$\vec{\boldsymbol{r}}(0) = \hat{i} - \hat{j} + \ln(1)\,\hat{k} + \vec{C}$$

$$\hat{k} = \hat{i} - \hat{j} + \vec{C}$$

$$\vec{C} = -\hat{i} + \hat{j} + \hat{k}$$

$$\vec{\boldsymbol{r}}(t) = (t+1)^{3/2}\,\hat{i} - e^{-t}\,\hat{j} + \ln(t+1)\,\hat{k} - \hat{i} + \hat{j} + \hat{k}$$

$$\boxed{= \left((t+1)^{3/2} - 1\right)\hat{i} + \left(1 - e^{-t}\right)\hat{j} + \left(\ln(t+1) + 1\right)\hat{k}}$$

Exercise

Solve the initial value problem for $\vec{r}$ as a vector function of t.

$$\textit{Differential equation}: \quad \frac{d^2\vec{r}}{dt^2} = -32\hat{k}$$

$$\textit{Initial condition}: \quad \vec{r}(0) = 100\hat{k}$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=0} = 8\hat{i} + 8\hat{j}$$

Solution

$$\frac{d\vec{r}}{dt} = \int \left(-32\hat{k}\right) dt$$

$$= -32t\,\hat{k} + \vec{C}_1$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=0} = 0\hat{k} + \vec{C}_1$$

$$8\hat{i} + 8\hat{j} = \vec{C}_1$$

$$\frac{d\vec{r}}{dt} = -32t\,\hat{k} + 8\hat{i} + 8\hat{j}$$

$$= 8\hat{i} + 8\hat{j} - 32t\,\hat{k}$$

$$\vec{r} = \int \left(8\hat{i} + 8\hat{j} - 32t\,\hat{k}\right) dt$$

$$= 8t\,\hat{i} + 8t\,\hat{j} - 16t^2\,\hat{k} + \vec{C}_2$$

$$\vec{r}(0) = 8(0)\,\hat{i} + 8(0)\,\hat{j} - 16(0)^2\,\hat{k} + \vec{C}_2$$

$$100\,\hat{k} = \vec{C}_2$$

$$\vec{r}(t) = 8t\,\hat{i} + 8t\,\hat{j} + \left(100 - 16t^2\right)\hat{k}$$

Exercise

Solve the initial value problem for $\vec{r}$ as a vector function of t.

$$\textit{Differential equation}: \quad \frac{d^2\vec{r}}{dt^2} = -\left(\hat{i} + \hat{j} + \hat{k}\right)$$

$$\textit{Initial condition}: \quad \vec{r}(0) = 10\hat{i} + 10\hat{j} + 10\hat{k}$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=0} = \vec{0}$$

Solution

$$\frac{d\vec{r}}{dt} = -\int \left(\hat{i} + \hat{j} + \hat{k}\right) dt$$

$$= -\left(t\hat{i} + t\hat{j} + t\hat{k}\right) + \vec{C}_1$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=0} = -\left(0\hat{i} + 0\hat{j} + 0\hat{k}\right) + \vec{C}_1$$

$$\underline{0 = \vec{C}_1 \,|}$$

$$\frac{d\vec{r}}{dt} = -\left(t\hat{i} + t\hat{j} + t\hat{k}\right)$$

$$\vec{r} = -\int \left(t\hat{i} + t\hat{j} + t\hat{k}\right) dt$$

$$= -\left(\frac{t^2}{2}\hat{i} + \frac{t^2}{2}\hat{j} + \frac{t^2}{2}\hat{k}\right) + \vec{C}_2$$

$$\vec{r}(0) = -\left(0\hat{i} + 0\hat{j} + 0\hat{k}\right) + \vec{C}_2$$

$$\underline{10\hat{i} + 10\hat{j} + 10\hat{k} = \vec{C}_2 \,|}$$

$$\vec{r}(t) = -\frac{t^2}{2}\hat{i} - \frac{t^2}{2}\hat{j} - \frac{t^2}{2}\hat{k} + 10\hat{i} + 10\hat{j} + 10\hat{k}$$

$$\underline{= \left(10 - \frac{t^2}{2}\right)\hat{i} + \left(10 - \frac{t^2}{2}\right)\hat{j} + \left(10 - \frac{t^2}{2}\right)\hat{k} \,|}$$

Exercise

Consider $\vec{r}(t) = \left\langle t+1,\ t^2 - 3\right\rangle$

a) Evaluate $\lim\limits_{t\to 0} \vec{r}(t)$ and $\lim\limits_{t\to\infty} \vec{r}(t)$, if each exists

b) Find $\vec{r}'(t)$ and evaluate $\vec{r}'(0)$

c) Find $\vec{r}''(t)$

d) Evaluate $\int \vec{r}(t)\,dt$

Solution

a) $\lim\limits_{t\to 0} \vec{r}(t) = \lim\limits_{t\to 0} \left\langle t+1,\ t^2 - 3\right\rangle$

$$\underline{= \langle 1,\ -3\rangle \,|}$$

$$\lim_{t\to\infty} \vec{r}(t) = \lim_{t\to\infty} \left\langle t+1,\ t^2 - 3\right\rangle$$

$$\boxed{= \nexists}$$

b) $\boxed{\vec{r}'(t) = \langle 1,\ 2t\rangle}$

$\boxed{\vec{r}'(0) = \langle 1,\ 0\rangle}$

c) $\boxed{\vec{r}''(t) = \langle 0,\ 2\rangle}$

d) $\displaystyle\int \vec{r}(t)\,dt = \int \left((t+1)\hat{i} + \left(t^2 - 3\right)\hat{j}\right)dt$

$$\boxed{= \left(\frac{1}{2}t^2 + t\right)\hat{i} + \left(\frac{1}{3}t^3 - 3t\right)\hat{j} + \vec{C}}$$

Exercise

Consider $\vec{r}(t) = \left\langle \dfrac{1}{2t+1},\ \dfrac{t}{t+1} \right\rangle$

a) Evaluate $\lim\limits_{t\to 0} \vec{r}(t)$ and $\lim\limits_{t\to\infty} \vec{r}(t)$, if each exists

b) Find $\vec{r}'(t)$ and evaluate $\vec{r}'(0)$

c) Find $\vec{r}''(t)$

d) Evaluate $\displaystyle\int \vec{r}(t)\,dt$

Solution

a) $\lim\limits_{t\to 0} \vec{r}(t) = \lim\limits_{t\to 0} \left\langle \dfrac{1}{2t+1},\ \dfrac{t}{t+1} \right\rangle$

$$\boxed{= \langle 1,\ 0\rangle}$$

$\lim\limits_{t\to \infty} \vec{r}(t) = \lim\limits_{t\to \infty} \left\langle \dfrac{1}{2t+1},\ \dfrac{t}{t+1} \right\rangle$

$$\boxed{= \langle 0,\ 1\rangle}$$

b) $\vec{r}'(t) = \left\langle \dfrac{-2}{(2t+1)^2},\ \dfrac{1}{(t+1)^2} \right\rangle$ $\qquad \left(\dfrac{ax+b}{cx+d}\right)' = \dfrac{ad-bc}{(cx+d)^2}$

$\boxed{\vec{r}'(0) = \langle -2,\ 1\rangle}$

c) $\vec{r}''(t) = \left\langle \dfrac{-8}{(2t+1)^3},\ \dfrac{-2}{(t+1)^3} \right\rangle$ $\qquad \left(\dfrac{1}{U^n}\right) = \dfrac{-nU'}{U^{n+1}}$

d) $\displaystyle\int \vec{r}(t)\,dt = \int \left(\frac{1}{2t+1}\hat{i} + \frac{t}{t+1}\hat{j} \right) dt$

$$= \frac{1}{2}\ln(2t+1)\hat{i} + \int \left(1 - \frac{1}{t+1}\right)\hat{j}\ dt$$

$$\boxed{= \frac{1}{2}\ln(2t+1)\hat{i} + \left(t - \ln(t+1)\right)\hat{j} + \vec{C}}$$

Exercise

Consider $\vec{r}(t) = \left\langle e^{-2t},\ te^{-t},\ \tan^{-1} t \right\rangle$

a) Evaluate $\lim\limits_{t\to 0} \vec{r}(t)$ and $\lim\limits_{t\to \infty} \vec{r}(t)$, if each exists

b) Find $\vec{r}'(t)$ and evaluate $\vec{r}'(0)$

c) Find $\vec{r}''(t)$

d) Evaluate $\displaystyle\int \vec{r}(t)\,dt$

Solution

a) $\lim\limits_{t\to 0} \vec{r}(t) = \lim\limits_{t\to 0} \left\langle e^{-2t},\ te^{-t},\ \tan^{-1} t \right\rangle$

$$\boxed{= \langle 1,\ 0,\ 0 \rangle}$$

$$\lim_{t\to \infty} te^{-t} = \lim_{t\to \infty} \frac{t}{e^t}$$

$$= \lim_{t\to \infty} \frac{1}{e^t}$$

$$\boxed{= 0}$$

$$\lim_{t\to \infty} \vec{r}(t) = \lim_{t\to \infty} \left\langle e^{-2t},\ te^{-t},\ \tan^{-1} t \right\rangle$$

$$\boxed{= \left\langle 0,\ 0,\ \frac{\pi}{2} \right\rangle}$$

b) $\vec{r}'(t) = \left\langle -2e^{-2t},\ (1-t)e^{-t},\ \dfrac{1}{1+t^2} \right\rangle$

$$\boxed{\vec{r}'(0) = \langle -2,\ 1,\ 1 \rangle}$$

c) $\vec{r}''(t) = \left\langle 4e^{-2t},\ (t-2)e^{-t},\ \dfrac{2t}{\left(1+t^2\right)^2} \right\rangle$ $\qquad \left(\dfrac{1}{U^n}\right)' = \dfrac{-nU'}{U^{n+1}}$

d) $\displaystyle\int \vec{r}(t)\,dt = \int \left(e^{-2t}\,\hat{i} + te^{-t}\,\hat{j} + \tan^{-1} t\,\hat{k}\right)dt$

$$= -\frac{1}{2}e^{-2t}\,\hat{i} - (t+1)e^{-t}\,\hat{j} + \left(t\tan^{-1} t - \frac{1}{2}\ln\left(t^2+1\right)\right)\hat{k} + \vec{C}$$

		$\int e^{-t}$
+	t	$-e^{-t}$
−	1	e^{-t}

Exercise

Consider $\vec{r}(t) = \langle \sin 2t,\ 3\cos 4t,\ t\rangle$

a) Evaluate $\lim\limits_{t\to 0} \vec{r}(t)$ and $\lim\limits_{t\to\infty} \vec{r}(t)$, if each exists

b) Find $\vec{r}\,'(t)$ and evaluate $\vec{r}\,'(0)$

c) Find $\vec{r}\,''(t)$

d) Evaluate $\displaystyle\int \vec{r}(t)\,dt$

Solution

a) $\lim\limits_{t\to 0} \vec{r}(t) = \lim\limits_{t\to 0} \langle \sin 2t,\ 3\cos 4t,\ t\rangle$

$$= \langle 0,\ 3,\ 0\rangle$$

$\lim\limits_{t\to\infty} \vec{r}(t) = \lim\limits_{t\to\infty} \langle \sin 2t,\ 3\cos 4t,\ t\rangle$

$$= \nexists$$

b) $\vec{r}\,'(t) = \langle 2\cos 2t,\ -12\sin 4t,\ 1\rangle$

$$\vec{r}\,'(0) = \langle 2,\ 0,\ 1\rangle$$

c) $\vec{r}\,''(t) = \langle -4\sin 2t,\ -48\cos 4t,\ 0\rangle$

d) $\displaystyle\int \vec{r}(t)\,dt = \int \left(\sin 2t\,\hat{i} + 3\cos 4t\,\hat{j} + t\,\hat{k}\right)dt$

$$= -\frac{1}{2}\cos 2t\,\hat{i} + \frac{3}{4}\sin 5t\,\hat{j} + \frac{1}{2}t^2\,\hat{k} + \vec{C}$$

Exercise

At time $t = 0$, a particle is located at the point $(1,\ 2,\ 3)$. It travels in a straight line to the point $(4,\ 1,\ 4)$, has speed 2 at $(1,\ 2,\ 3)$ and constant acceleration $3\hat{i} - \hat{j} + \hat{k}$. Find an equation for the position vector $\vec{r}(t)$ of the particle at time *t*.

Solution

$$\vec{a} = 3\hat{i} - \hat{j} + \hat{k} = \frac{d\vec{v}}{dt}$$

$$\vec{v} = \int \left(3\hat{i} - \hat{j} + \hat{k}\right) dt$$

$$= 3t\hat{i} - t\hat{j} + t\hat{k} + \vec{C}_1$$

Since the particle travels in a straight line in the direction of the vector:

$$(4-1)\hat{i} + (1-2)\hat{j} + (4-3)\hat{k} \ \underline{= 3\hat{i} - \hat{j} + \hat{k}|}$$

At $t = 0$, the particle has a speed of 2.

$$\vec{v}(0) = \frac{2}{\sqrt{9+1+1}}\left(3\hat{i} - \hat{j} + \hat{k}\right) = \vec{C}_1$$

$$\vec{C}_1 = \frac{6}{\sqrt{11}}\hat{i} - \frac{2}{\sqrt{11}}\hat{j} + \frac{2}{\sqrt{11}}\hat{k}$$

$$\vec{v} = 3t\hat{i} - t\hat{j} + t\hat{k} + \frac{6}{\sqrt{11}}\hat{i} - \frac{2}{\sqrt{11}}\hat{j} + \frac{2}{\sqrt{11}}\hat{k}$$

$$= \left(3t + \frac{6}{\sqrt{11}}\right)\hat{i} - \left(t + \frac{2}{\sqrt{11}}\right)\hat{j} + \left(t + \frac{2}{\sqrt{11}}\right)\hat{k}$$

$$\vec{r} = \int \left(\left(3t + \frac{6}{\sqrt{11}}\right)\hat{i} - \left(t + \frac{2}{\sqrt{11}}\right)\hat{j} + \left(t + \frac{2}{\sqrt{11}}\right)\hat{k}\right) dt$$

$$= \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t\right)\hat{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\hat{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\hat{k} + \vec{C}_2$$

At time $t = 0$, a particle is located at the point $(1,\ 2,\ 3)$ $\vec{r}(0) = \hat{i} + 2\hat{j} + 3\hat{k}$

$$\hat{i} + 2\hat{j} + 3\hat{k} = 0\hat{i} - 0\hat{j} + 0\hat{k} + \vec{C}_2$$

$$\vec{C}_2 = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{r}(t) = \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t\right)\hat{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\hat{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\hat{k} + \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\underline{= \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t + 1\right)\hat{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t - 2\right)\hat{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t + 3\right)\hat{k}|}$$

Exercise

A projectile is fired at a speed of 840 *m/sec* at an angle of 60°. How long will it take to get 21 *km* downrange?

Solution

$$x = \left(v_0 \cos\alpha\right)t$$

$$21\ km \frac{1000\ m}{1\ km} = \left(840\ \left(m/s\right)\ \cos 60°\right)t$$

$$t = \frac{21000}{840\cos 60°}$$

$$= 50\ \text{sec}$$

Exercise

Find the muzzle speed of a gun whose maximum range is 24.5 *km*.

Solution

$$R = \frac{v_0^2}{g}\sin 2\alpha$$

Maximum R occurs when sine equals to 1 $\rightarrow \quad \sin 2\alpha = 1 \quad \Rightarrow \quad 2\alpha = 90°$

$$24.5 = \frac{v_0^2}{9.8}\sin 90°$$

$$v_0^2 = (24.5)(9.8)$$

$$v_0 = \sqrt{(24.5)(9.8)}$$

$$= 490\ \ m/s$$

Exercise

A spring gun at ground level fires a golf ball at an angle of 45°. The ball lands 10 *m* away.

a) What was the ball's initial speed?

b) For the same initial speed, find the two firing angles that make the range 6 *m*.

Solution

a) $R = \dfrac{v_0^2}{g}\sin 2\alpha$

$$10 = \frac{v_0^2}{9.8}\sin(2\times 45°)$$

$$v_0^2 = \frac{98}{\sin 90°}$$

$$= 98$$

$$v_0 = \sqrt{98}$$

$$\underline{\approx 9.9\ \ m/s|}$$

b) $6 = \frac{98}{9.8}\sin 2\alpha$

$$\sin 2\alpha = 6\left(\frac{9.8}{98}\right) = 0.6$$

$$2\alpha = \sin^{-1}(0.6)$$

$$2\alpha \approx 36.87° \quad or \quad 2\alpha \approx 143.12°$$

$$\boxed{\alpha \approx 18.4°} \quad or \quad \boxed{\alpha \approx 71.6°}$$

Exercise

An electron in a TV tube is beamed horizontally at a speed of $5\times10^6\ m/\sec$ toward the face of the tube 40 *cm* away. About how far will the electron drop before it hits?

Solution

Given: $v_0 = 5\times10^6\ m/\sec, \quad x = 40cm = 0.4\ m$

$$x = \left(v_0 \cos\alpha\right)t$$

$$0.4 = \left(5\times10^6 \cos 0°\right)t \qquad\qquad \textit{Horizontal } \alpha = 0°$$

$$t = \frac{0.4}{5\times10^6} = .08\times10^{-6} = 8\times10^{-8}\ \sec$$

$$y = -\tfrac{1}{2}gt^2 + \left(v_0 \sin\alpha\right)t + y_0$$

$$= -\tfrac{1}{2}(9.8)\left(8\times10^{-8}\right)^2 + \left(5\times10^6 \sin 0°\right)\left(8\times10^{-8}\right) + 0$$

$$= -3.136\times10^{-14}\ m$$

Therefore, the electron drop $3.136\times10^{-12}\ cm$

Exercise

A golf ball is hit with an initial speed of 116 *ft/sec* at an angle of elevation of 45° from the tee to a green that is elevated 45 *feet* above the tee. Assuming that the pin, 369 *feet* downrange, does not get in the way, where will the ball land in relation to the pin?

Solution

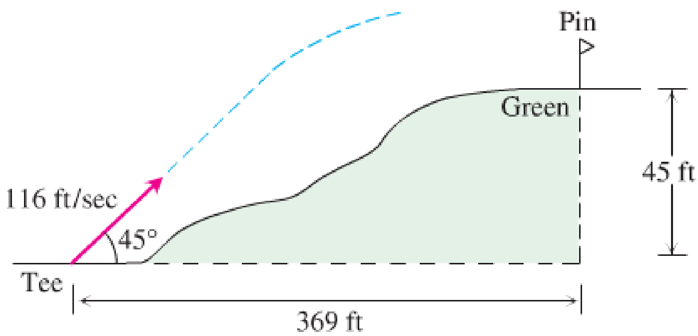
$v_0 = 116\ ft/\sec, \quad \alpha = 45°$

$x = \left(v_0 \cos\alpha\right)t$

$369 = \left(116\cos 45°\right)t$

$t = \dfrac{369}{116\cos 45°}$

$\approx 4.5\ \sec$



$y = -\frac{1}{2}gt^2 + \left(v_0 \sin\alpha\right)t + y_0$

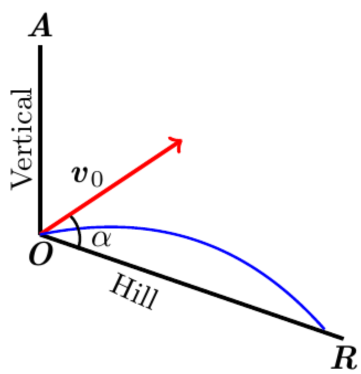
$= -\frac{1}{2}(32)(4.5)^2 + \left(116\sin 45°\right)t$

$\approx 45.11\ ft$

It will take the ball 4.5 *sec* to travel 369 *feet*. at the time the ball will be 45.11 *feet* in the air and will hit the green past the pin.

Exercise

An ideal projectile is launched straight down an inclined plane.



a) Show that the greatest downhill range is achieved when the initial velocity vector bisects angle *AOR*

b) If the projectile were fired uphill instead of down, what launch angle would maximize its range?

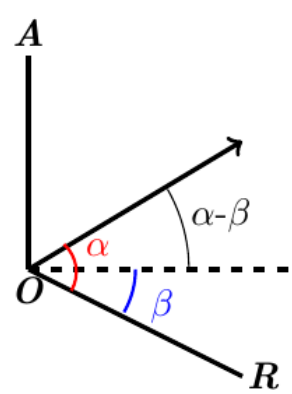
Solution

a) $x = \left(v_0 \cos(\alpha-\beta)\right)t, \quad y = \left(v_0 \sin(\alpha-\beta)\right)t - \frac{1}{2}gt^2$

$\tan\beta = \dfrac{y}{x}$

$= \dfrac{\left|\left(v_0 \sin(\alpha-\beta)\right)t - \frac{1}{2}gt^2\right|}{\left(v_0\cos(\alpha-\beta)\right)t}$

$= \dfrac{\left|v_0 \sin(\alpha-\beta) - \frac{1}{2}gt\right|}{v_0\cos(\alpha-\beta)}$



$$\tfrac{1}{2}gt - v_0 \sin(\alpha-\beta) = v_0 \cos(\alpha-\beta)\tan\beta$$

$$\tfrac{1}{2}gt = v_0 \cos(\alpha-\beta)\tan\beta + v_0 \sin(\alpha-\beta)$$

$$t = \frac{2v_0\left(\cos(\alpha-\beta)\tan\beta + \sin(\alpha-\beta)\right)}{g};$$

Which is time when the projectile hits the downhill slope.

$$x = v_0 \cos(\alpha-\beta)\frac{2v_0\left(\cos(\alpha-\beta)\tan\beta + \sin(\alpha-\beta)\right)}{g}$$

$$= \frac{2v_0^2}{g}\left(\cos^2(\alpha-\beta)\tan\beta + \cos(\alpha-\beta)\sin(\alpha-\beta)\right)$$

$$= \frac{2v_0^2}{g}\left(\cos^2(\alpha-\beta)\tan\beta + \tfrac{1}{2}\sin 2(\alpha-\beta)\right)$$

$$\frac{dx}{d\alpha} = \frac{2v_0^2}{g}\left(-2\cos(\alpha-\beta)\sin(\alpha-\beta)\tan\beta + \cos 2(\alpha-\beta)\right) = 0$$

$$-\sin 2(\alpha-\beta)\tan\beta + \cos 2(\alpha-\beta) = 0$$

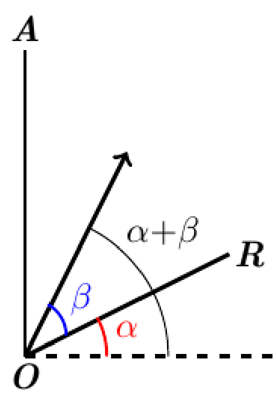
$$\sin 2(\alpha-\beta)\tan\beta = \cos 2(\alpha-\beta)$$

$$\tan\beta = \cot 2(\alpha-\beta) \quad\Rightarrow\quad 90^\circ - \beta = 2(\alpha-\beta)$$

$$\alpha - \beta = 45^\circ - \tfrac{1}{2}\beta$$

$$\alpha = \tfrac{1}{2}(90^\circ + \beta) \qquad \tfrac{1}{2}\angle AOR$$

b) $x = \left(v_0 \cos(\alpha+\beta)\right)t, \quad y = -\tfrac{1}{2}gt^2 + \left(v_0 \sin(\alpha+\beta)\right)t$



$$\tan\beta = \frac{y}{x}$$

$$= \frac{-\tfrac{1}{2}gt^2 + \left(v_0 \sin(\alpha+\beta)\right)t}{\left(v_0 \cos(\alpha+\beta)\right)t}$$

$$= \frac{-\tfrac{1}{2}gt + v_0 \sin(\alpha+\beta)}{v_0 \cos(\alpha+\beta)}$$

$$-\tfrac{1}{2}gt + v_0 \sin(\alpha+\beta) = v_0 \cos(\alpha+\beta)\tan\beta$$

$$\tfrac{1}{2}gt = v_0 \sin(\alpha+\beta) - v_0 \cos(\alpha+\beta)\tan\beta$$

$t = \frac{2v_0}{g}\left(v_0 \sin(\alpha+\beta) - \cos(\alpha+\beta)\tan\beta\right)$; which is time when the projectile hits the uphill slope.

$$x = \frac{2v_0^2}{g}\left(\cos(\alpha+\beta)\sin(\alpha+\beta) - \cos^2(\alpha+\beta)\tan\beta\right)$$

$$= \frac{2v_0^2}{g}\left(\frac{1}{2}\sin 2(\alpha+\beta) - \cos^2(\alpha+\beta)\tan\beta\right)$$

$$\frac{dx}{d\alpha} = \frac{2v_0^2}{g}\left(\cos 2(\alpha+\beta) + 2\cos(\alpha+\beta)\sin(\alpha+\beta)\tan\beta\right) = 0$$

$$\cos 2(\alpha+\beta) + \sin 2(\alpha+\beta)\tan\beta = 0$$

$$\sin 2(\alpha+\beta)\tan\beta = -\cos 2(\alpha+\beta)$$

$$\tan\beta = -\cot 2(\alpha+\beta)$$

$$\tan(-\beta) = \cot 2(\alpha+\beta) \quad \Rightarrow \quad 90^\circ + \beta = 2\alpha + 2\beta$$

$$\boxed{\alpha = \frac{1}{2}(90^\circ - \beta)} \qquad \frac{1}{2}\angle AOR$$

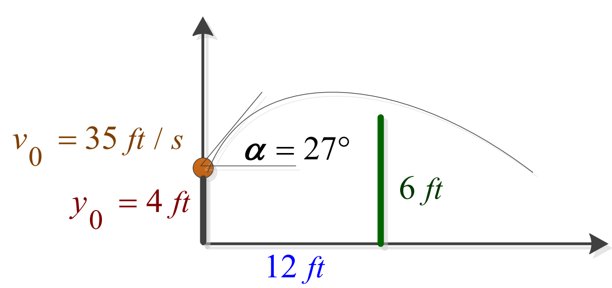
Exercise

A volleyball is hit when it is 4 *feet* above the ground and 12 *feet* from a 6-*foot*-high net. It leaves the point of impact with an initial velocity of 35 *ft/sec* at an angle of 27° and slips by the opposing team untouched.

a) Find a vector equation for the path of the volleyball.
b) How high does the volleyball go, and when does it reach maximum height?
c) Find its range and flight time.
d) When is the volleyball 7 *feet* above the ground? How far (ground distance) is the volleyball from where it will land?
e) Suppose that the net is raised to 8 *feet*. Does this change things? Explain.

Solution

Given: $y_0 = 4\ ft, \quad v_0 = 35\ ft/s, \quad \alpha = 27^\circ$



a) $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

$$x = (v_0 \cos\alpha)t = (35\cos 27^\circ)t$$

$$y = -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t + y_0$$

$$= -16t^2 + (35\sin 27^\circ)t + 4$$

$$\vec{r}(t) = (35\cos 27^\circ)t\ \boldsymbol{i} + \left(-16t^2 + (35\sin 27^\circ)t + 4\right)\boldsymbol{j}$$

b) $y_{max} = \dfrac{(v_0 \sin\alpha)^2}{2g} + y_0$

$= \dfrac{(35\sin 27°)^2}{2(32)} + 4$

$\approx 7.945\ ft$

$t = \dfrac{v_0 \sin\alpha}{g} = \dfrac{35\sin 27°}{32}$

$\approx 0.497\ \text{sec}$

c) $y = -16t^2 + (35\sin 27°)t + 4 = 0$ *Solve for **t***

$$t = \frac{-35\sin 27° - \sqrt{(35\sin 27°)^2 - 4(-16)(4)}}{2(-16)}$$

$\approx 1.201\ \text{sec}$

Range: $x = (35\cos 27°)(1.201)$

$\approx 37.453\ ft$

d) $y = -16t^2 + (35\sin 27°)t + 4 = 7$ *Solve for **t***

$-16t^2 + (35\sin 27°)t - 3 = 0$

$$t = \frac{-35\sin 27° \pm \sqrt{(-35\sin 27°)^2 - 4(-16)(-3)}}{2(-16)}$$

$\approx \begin{cases} 0.7396\ \text{sec} \\ 0.2535\ \text{sec} \end{cases}$

$x(t = 0.2535) = (35\cos 27°)(0.2535)$

$\approx 7.921\ ft$

$x(t = 0.74) = (35\cos 27°)(0.74)$

$\approx 23.077\ ft$

e) Since $y_{max} \approx 7.945\ ft$, the ball won't clear the 8 *ft* net, therefore, Yes, it changes things.

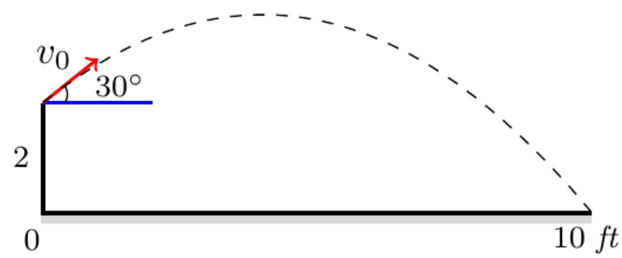
Exercise

A toddler on level ground throws a baseball into the air at an angle of 30° with the ground from a height of 2 *feet*. If the ball lands 10 *feet* from the child, determine the initial speed of the ball.

Solution

$$v_0 = \left\langle \left|v_0\right|\cos 30°, \quad \left|v_0\right|\sin 30° \right\rangle$$

$$= \left\langle \frac{\sqrt{3}}{2}\left|v_0\right|, \quad \frac{1}{2}\left|v_0\right| \right\rangle$$



$$\vec{r}(t) = v_{0x}t\,\hat{i} + \left(-\frac{1}{2}gt^2 + v_{0y}t + y_0\right)\hat{j}$$

$$= \frac{\sqrt{3}}{2}\left|v_0\right|t\,\hat{i} + \left(-16t^2 + \frac{1}{2}\left|v_0\right|t + 2\right)\hat{j}$$

At 10 *feet* $\rightarrow (x, y) = (10, 0)$

$$\begin{cases} x = \frac{\sqrt{3}}{2}\left|v_0\right|t = 10 \\ y = -16t^2 + \frac{1}{2}\left|v_0\right|t + 2 = 0 \end{cases}$$

$$\rightarrow \left|v_0\right|t = \frac{20}{\sqrt{3}}$$

$$-16t^2 + \frac{1}{2}\frac{20}{\sqrt{3}} + 2 = 0$$

$$16t^2 = \frac{10\sqrt{3}}{3} + 2$$

$$16t^2 = \frac{6 + 10\sqrt{3}}{3}$$

$$t = \sqrt{\frac{3 + 5\sqrt{3}}{24}}$$

$$\approx 0.697$$

$$\left|v_0\right| = \frac{20}{0.697\sqrt{3}}$$

$$\approx 16.6 \ ft/\sec$$

Exercise

A basketball player tosses a basketball into the air at an angle 45° with the ground from a height of 6 *feet* above the ground. If the ball goes through the basket 15 *feet* away and 10 *feet* above the ground, determine the initial velocity of the ball.

Solution

$$\begin{cases} x = \left|v_0\right| \cos 45^\circ\, t = 15 & \rightarrow \ \left|v_0\right| t = 15\sqrt{2} \quad (1) \\ y = -16t^2 + \left|v_0\right| \sin 45^\circ\, t + 6 = 10 & \quad (2) \end{cases}$$

$$(2) \rightarrow \ -16t^2 + 15\sqrt{2}\frac{1}{\sqrt{2}} + 6 = 10$$

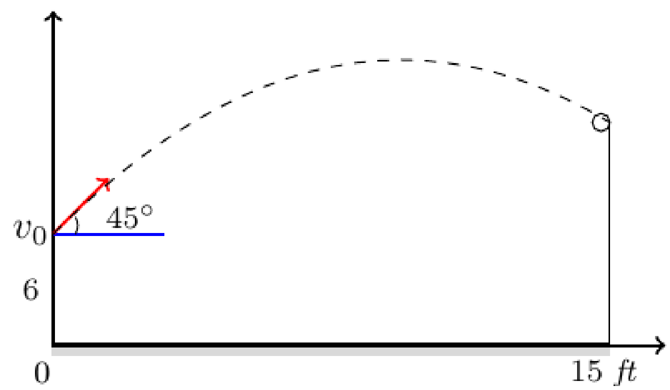
$$16t^2 = 11$$

$$\boxed{t = \frac{\sqrt{11}}{4}}$$

$$(2) \rightarrow \ \left|v_0\right| = 15\sqrt{2}\frac{4}{\sqrt{11}}$$

$$\left|v_0\right| = 60\sqrt{\frac{2}{11}}$$

$$\boxed{\approx 25.6 \ \ ft/\text{sec}}$$



Exercise

The position of a particle in the plane at time t is $\vec{r}(t) = \frac{1}{\sqrt{1+t^2}}\hat{i} + \frac{t}{\sqrt{1+t^2}}\hat{j}$. Find the particle's highest speed.

Solution

$$\vec{v}(t) = -\frac{t}{\left(1+t^2\right)^{3/2}}\hat{i} + \frac{1+t^2-t^2}{\left(1+t^2\right)^{3/2}}\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$= -\frac{t}{\left(1+t^2\right)^{3/2}}\hat{i} + \frac{1}{\left(1+t^2\right)^{3/2}}\hat{j}$$

$$|\vec{v}| = \sqrt{\frac{t^2}{\left(1+t^2\right)^3} + \frac{1}{\left(1+t^2\right)^3}}$$

$$= \sqrt{\frac{t^2+1}{\left(1+t^2\right)^3}}$$

$$= \frac{1}{t^2+1}$$

To maximize the speed $(|\vec{v}|)$:

$$\frac{d|\vec{v}|}{dt} = \frac{-2t}{\left(t^2+1\right)^2} = 0 \quad \Rightarrow \quad \underline{t = 0}$$

$$\underline{|\vec{v}|_{max}(0) = 1}$$

Exercise

A particle traveling in a straight line located at the point $(1,\ -1,\ 2)$ and has speed 2 at time $t = 0$. The particle moves toward the point $(3,\ 0,\ 3)$ with constant acceleration $2\hat{i} + \hat{j} + \hat{k}$. Find the position vector $\vec{r}(t)$ at time t.

Solution

$$\vec{a}(t) = 2\hat{i} + \hat{j} + \hat{k}$$

$$\begin{aligned}\vec{v}(t) &= \int \left(2\hat{i} + \hat{j} + \hat{k}\right) dt \\ &= \underline{2t\hat{i} + t\hat{j} + t\hat{k} + \vec{C}_1}\end{aligned}$$

The particle travels in the direction:

$$(3-1)\hat{i} + (0+1)\hat{j} + (3-2)\hat{k} = 2\hat{i} + \hat{j} + \hat{k}$$

At $t = 0 \quad \to \quad |\vec{v}| = 2$

$$\begin{aligned}\vec{v}(0) &= \frac{|\vec{v}(t=0)|}{|\vec{v}|}\left(2\hat{i} + \hat{j} + \hat{k}\right) \\ &= \frac{2}{\sqrt{4+1+1}}\left(2\hat{i} + \hat{j} + \hat{k}\right) \\ &= \underline{\frac{2}{\sqrt{6}}\left(2\hat{i} + \hat{j} + \hat{k}\right) = C_1}\end{aligned}$$

$$\vec{v}(t) = \left(2t + \frac{4}{\sqrt{6}}\right)\hat{i} + \left(t + \frac{2}{\sqrt{6}}\right)\hat{j} + \left(t + \frac{2}{\sqrt{6}}\right)\hat{k}$$

$$\begin{aligned}\vec{r}(t) &= \int \left(\left(2t + \frac{4}{\sqrt{6}}\right)\hat{i} + \left(t + \frac{2}{\sqrt{6}}\right)\hat{j} + \left(t + \frac{2}{\sqrt{6}}\right)\hat{k}\right) dt \\ &= \left(t^2 + \frac{4}{\sqrt{6}}t\right)\hat{i} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)\hat{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)\hat{k} + \vec{C}_2\end{aligned}$$

Given the starting point at $(1,\ -1,\ 2)$. Then, $\vec{r}_0 = \hat{i} - \hat{j} + 2\hat{k}$

$$\vec{r}(0) = \vec{0} + \vec{C}_2 = \hat{i} - \hat{j} + 2\hat{k}$$

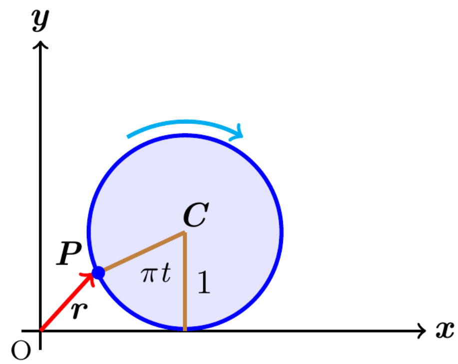
$$\vec{r}(t) = \left(t^2 + \frac{4}{\sqrt{6}}t\right)\hat{i} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)\hat{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)\hat{k} + \hat{i} - \hat{j} + 2\hat{k}$$

$$= \left(t^2 + \frac{4}{\sqrt{6}}t + 1\right)\hat{i} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t - 1\right)\hat{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t + 2\right)\hat{k}$$

Exercise

A circular wheel with radius 1 *foot* and center *C* rolls to the right along the *x*-axis at a half-run per second. At time *t* seconds, the position vector of the point *P* on the wheel's circumference is

$$\vec{r}(t) = (\pi t - \sin \pi t)\hat{i} + (1 - \cos \pi t)\hat{j}$$



a) Sketch the curve traced by *P* during the interval $0 \le t \le 3$

b) Find $\vec{v}$ and $\vec{a}$ at $t = 0$, 1, 2, and 3 and add these vectors to your sketch

c) At any given time, what is the forward speed of the topmost point of the wheel? Of *C*?

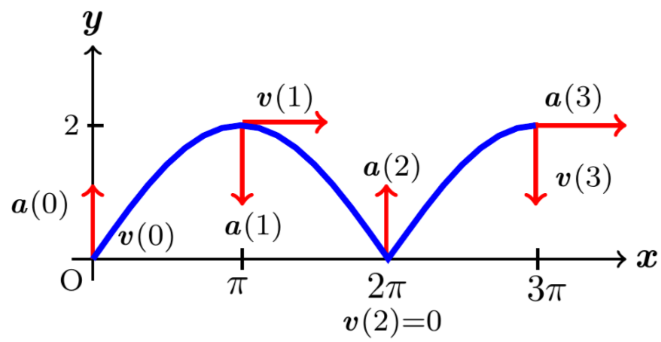
Solution

a) $x = \pi t - \sin \pi t \quad y = 1 - \cos \pi t$

t	***x***	***y***
0	0	0
$\frac{1}{2}$	$\frac{\pi}{2}$	1
1	π	2
2	2π	0
3	3π	2

b) $\vec{v}(t) = (\pi - \pi \cos \pi t)\hat{i} + (\pi \sin \pi t)\hat{j}$

$\vec{a}(t) = \left(\pi^2 \sin \pi t\right)\hat{i} + \left(\pi^2 \cos \pi t\right)\hat{j}$



t	$\vec{v}$	$\vec{a}$
0	0	$\pi^2\hat{j}$
1	$2\pi\hat{i}$	$-\pi^2\hat{j}$
2	0	$\pi^2\hat{j}$
3	$2\pi\hat{i}$	$-\pi^2\hat{j}$

c) Forward speed at the most point $\left|\vec{v}(1)\right| = \left|\vec{v}(3)\right| = 2\pi$

Since the circles makes $\frac{1}{2}$ *rev/sec*, the center moves π *ft* parallel to *x*-axis each second.

Forward speed of C is π *ft/sec*

Exercise

A shot leaves the thrower's hand 6.5 *ft* above the ground at a 45° angle at 44 *ft/sec*. Where is it 3 *sec* later?

Solution

Given: $r(0) = 6.5 = y_0, \quad \alpha = 45°, \quad \vec{v}(0) = 44$

$$y(t) = -\frac{1}{2}gt^2 + \left(v_0 \sin\alpha\right)t + y_0$$

$$= -16t^2 + \left(44 \text{in}\, 45°\right)t + 6.5$$

$$= -16t^2 + 22\sqrt{2}\,t + 6.5$$

$$y(3) = -144 + 66\sqrt{2} + \frac{13}{2}$$

$$= \frac{132\sqrt{2} - 275}{2} \qquad \approx -44.16$$

The shot is on the ground at $t = 3$ *sec*.

$$y = -16t^2 + 22\sqrt{2}\,t + 6.5 = 0$$

$$t = \frac{-22\sqrt{2} \pm \sqrt{968 + 416}}{-32}$$

$$= \frac{11\sqrt{2} \mp \sqrt{346}}{16}$$

$$\approx \begin{cases} 2.13 \\ -0.19 \end{cases}$$

$$\therefore \ t \approx 2.13$$

$$x = v_0 \cos\alpha t$$

$$\approx 22\sqrt{2}(2.13)$$

$$\approx 66.27 \ \ ft$$

Solution   *Section* 1.7 – Length of Curves

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j} + \sqrt{5}t\,\hat{k}; \quad 0 \le t \le \pi$$

Solution

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -(2\sin t)\hat{i} + (2\cos t)\hat{j} + \sqrt{5}\,\hat{k}$$

$$|\vec{v}| = \sqrt{4\sin^2 t + 4\cos^2 t + 5}$$

$$= \sqrt{4+5}$$

$$= 3$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = -\frac{2\sin t}{3}\hat{i} + \frac{2\cos t}{3}\hat{j} + \frac{\sqrt{5}}{3}\hat{k}$$

$$\text{Length: } s = \int_0^\pi |\vec{v}(t)|\,dt$$

$$= \int_0^\pi 3\,dt$$

$$= 3t\Big|_0^\pi$$

$$= 3\pi$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = t\hat{i} + \frac{2}{3}t^{3/2}\,\hat{k}; \quad 0 \le t \le 8$$

Solution

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \hat{i} + t^{1/2}\,\hat{k} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{1+t}$$

$$\vec{T} = \frac{1}{\sqrt{1+t}}\hat{i} + \frac{t^{1/2}}{\sqrt{1+t}}\hat{k} \qquad\qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\text{Length: } s = \int_0^8 |\vec{v}(t)|\,dt$$

$$= \int_0^8 (1+t)^{1/2}\, dt \qquad\qquad = \int_0^8 (1+t)^{1/2}\, d(1+t)$$

$$= \frac{2}{3}(1+t)^{3/2} \Big|_0^8$$

$$= \frac{2}{3}\left[(9)^{3/2} - 1\right]$$

$$= \frac{2}{3}(27-1)$$

$$= \frac{52}{3}$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = (2+t)\hat{i} - (t+1)\hat{j} + t\,\hat{k}; \quad 0 \le t \le 3$$

Solution

$$\vec{v}(t) = \hat{i} - \hat{j} + \hat{k} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{1+1+1} = \sqrt{3}$$

$$\vec{T} = \frac{1}{\sqrt{3}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k} \qquad\qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

Length: $s = \int_0^3 |\vec{v}(t)|\, dt$

$$= \int_0^3 \sqrt{3}\, dt$$

$$= \sqrt{3}t \Big|_0^3$$

$$= 3\sqrt{3}$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = \left(\cos^3 t\right)\hat{i} + \left(\sin^3 t\right)\hat{k}; \quad 0 \le t \le \frac{\pi}{2}$$

Solution

$$\vec{v}(t) = -\left(3\cos^2 t \sin t\right)\hat{i} + \left(3\sin^2 t \cos t\right)\hat{k} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\left|\vec{v}\right| = \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t}$$
$$= 3\sqrt{\cos^2 t\sin^2 t\left(\cos^2 t + \sin^2 t\right)}$$
$$= 3\sqrt{\cos^2 t\sin^2 t}$$
$$= \underline{3\left|\cos t\sin t\right|}$$

$$\vec{T} = -\left(\frac{3\cos^2 t\sin t}{3\left|\cos t\sin t\right|}\right)\hat{i} + \left(\frac{3\sin^2 t\cos t}{3\left|\cos t\sin t\right|}\right)\hat{k} \qquad \vec{T} = \frac{\vec{v}}{\left|\vec{v}\right|}$$
$$= \underline{-(\cos t)\hat{i} + (\sin t)\hat{k}}$$

$$L = \int_0^{\pi/2} 3\cos t\sin t\ dt \qquad L = \int_a^b \left|\vec{v}(t)\right|\ dt$$
$$= \frac{3}{2}\int_0^{\pi/2} \sin 2t\ dt \qquad \sin 2t = 2\cos t\sin t$$
$$= \frac{3}{2}\left[-\frac{1}{2}\cos 2t\right]_0^{\pi/2}$$
$$= -\frac{3}{4}(\cos\pi - \cos 0)$$
$$= -\frac{3}{4}(-2)$$
$$= \underline{\frac{3}{2}}$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$$\vec{r}(t) = (t\cos t)\hat{i} + (t\sin t)\hat{j} + \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\hat{k}; \quad 0 \le t \le \pi$$

Solution

$$\vec{v}(t) = (\cos t - t\sin t)\hat{i} + (\sin t + t\cos t)\hat{j} + \left(\sqrt{2}\ t^{1/2}\right)\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$
$$\left|\vec{v}(t)\right| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 2t}$$
$$= \sqrt{\cos^2 t - 2t\cos t + \sin^2 t + \sin^2 t + 2t\cos t + \cos^2 t + 2t}$$
$$= \sqrt{2 + 2t}$$

$$\vec{T} = \frac{1}{\sqrt{2}}\left(\frac{\cos t - t\sin t}{\sqrt{1+t}}\right)\hat{i} + \frac{1}{\sqrt{2}}\left(\frac{\cos t + t\sin t}{\sqrt{1+t}}\right)\hat{j} + \left(\sqrt{\frac{t}{1+t}}\ t^{1/2}\right)\hat{k} \qquad \vec{T} = \frac{\vec{v}}{\left|\vec{v}\right|}$$

$$L=\int_0^{\pi}\sqrt{2}\sqrt{1+t}\ dt \qquad\qquad L=\int_a^b\left|\vec{v}(t)\right|\ dt$$

$$=\sqrt{2}\int_0^{\pi}\sqrt{1+t}\ d(1+t)$$

$$=\sqrt{2}\int_0^{\pi}\sqrt{1+t}\ d(1+t)$$

$$=\sqrt{2}\left(\sqrt{1+\pi}-1\right)$$

Exercise

Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve

$\boldsymbol{r}(t)=(t\sin t+\cos t)\boldsymbol{i}+(t\cos t-\sin t)\boldsymbol{j};\quad \sqrt{2}\le t\le 2$

Solution

$$\vec{v}(t)=(\sin t+t\cos t-\sin t)\hat{i}+(\cos t-t\sin t-\cos t)\hat{j} \qquad\qquad \vec{v}(t)=\frac{d\vec{r}}{dt}$$

$$=(t\cos t)\hat{i}-(t\sin t)\hat{j}$$

$$|\vec{v}|=\sqrt{t^2\cos^2 t+t^2\sin^2 t}$$

$$=\sqrt{t^2\left(\cos^2 t+\sin^2 t\right)}$$

$$=\sqrt{t^2}$$

$$=|t|$$

$$=t \quad \textit{because}\ \ \sqrt{2}\le t\le 2$$

$$\boldsymbol{T}=\frac{\boldsymbol{v}}{|\boldsymbol{v}|}=\left(\frac{t\cos t}{t}\right)\boldsymbol{i}-\left(\frac{t\sin t}{t}\right)\boldsymbol{j}$$

$$=(\cos t)\boldsymbol{i}-(\sin t)\boldsymbol{j}$$

$$L=\int_{\sqrt{2}}^{2}t\,dt \qquad\qquad L=\int_a^b\left|\vec{v}(t)\right|\ dt$$

$$=\frac{1}{2}t^2\Big|_{\sqrt{2}}^{2}$$

$$=\frac{1}{2}(4-2)$$

$$=1$$

Exercise

Find the point on the curve $\vec{r}(t) = (5\sin t)\hat{i} + (5\cos t)\hat{j} + 12t\,\hat{k}$ at a distance 26π units along the curve from the point (0, 5, 0) in the direction of increasing arc length.

Solution

$$\vec{v} = (5\cos t)\hat{i} - (5\sin t)\hat{j} + 12\,\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{25\cos^2 t + 25\sin^2 t + 144} \\ &= \sqrt{25\left(\cos^2 t + \sin^2 t\right) + 144} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

$$s = \int_0^{t_0} 13\,dt \qquad s = \int_0^{t_0} |\vec{v}(t)|\,dt$$

$$= 13t_0$$

$$s = 26\pi = 13t_0$$

$$t_0 = 2\pi$$

$$\begin{aligned} \vec{r}(t = 2\pi) &= (5\sin 2\pi)\hat{i} + (5\cos 2\pi)\hat{j} + 12(2\pi)\,\hat{k} \\ &= 0\hat{i} + 5\hat{j} + 24\pi\,\hat{k} \end{aligned}$$

The point is: $(0,\ 5,\ 24\pi)$

Exercise

Find the arc length parameter along the curve from the point where $t = 0$. Also, find the length of the indicated portion of the curve. $\vec{r}(t) = (4\cos t)\hat{i} + (4\sin t)\hat{j} + 3t\,\hat{k};\quad 0 \le t \le \frac{\pi}{2}$

Solution

$$\vec{v} = -(4\sin t)\hat{i} + (4\cos t)\hat{j} + 3\,\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{16\sin^2 t + 16\cos^2 t + 9} \\ &= \sqrt{16 + 9} \\ &= 5 \end{aligned}$$

$$s = \int_0^t 5\, dt \qquad\qquad s = \int_0^t |\vec{v}(\tau)|\, d\tau$$

$$= 5t$$

$$s\left(\frac{\pi}{2}\right) = \frac{5\pi}{2}$$

Exercise

Find the arc length parameter along the curve from the point where $t = 0$. Also, find the length of the indicated portion of the curve. $\vec{r}(t) = \left(e^t \cos t\right)\hat{i} + \left(e^t \sin t\right)\hat{j} + e^t\, \hat{k}; \quad -\ln 4 \le t \le 0$

Solution

$$\vec{v}(t) = \left(e^t \cos t - e^t \sin t\right)\hat{i} + \left(e^t \sin t + e^t \cos t\right)\hat{j} + e^t\, \hat{k} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{\left(e^t \cos t - e^t \sin t\right)^2 + \left(e^t \sin t + e^t \cos t\right)^2 + e^{2t}}$$

$$= \sqrt{e^{2t}\cos^2 t - e^{2t}\cos t \sin t + e^{2t}\sin^2 t + e^{2t}\sin^2 t + e^{2t}\cos t \sin t + e^{2t}\cos^2 t + e^{2t}}$$

$$= \sqrt{2e^{2t}\left(\cos^2 t + \sin^2 t\right) + e^{2t}}$$

$$= \sqrt{3e^{2t}}$$

$$= \sqrt{3}\, e^t$$

$$s(t) = \int_0^t \sqrt{3}\, e^\tau\, d\tau \qquad\qquad s = \int_0^t |\vec{v}(\tau)|\, d\tau$$

$$= \sqrt{3}\, e^\tau \Big|_0^t$$

$$= \sqrt{3}\left(e^t - 1\right)$$

$$s(0) = \sqrt{3}\left(e^0 - 1\right) = 0$$

$$s(-\ln 4) = \sqrt{3}\left(e^{-\ln 4} - 1\right)$$

$$= \sqrt{3}\left(e^{\ln\frac{1}{4}} - 1\right)$$

$$= \sqrt{3}\left(\frac{1}{4} - 1\right)$$

$$= \sqrt{3}\left(-\tfrac{3}{4}\right)$$

$$= \frac{3\sqrt{3}}{4}$$

$$s(-\ln 4) - s(0) = \frac{3\sqrt{3}}{4}$$

Exercise

Find the arc length parameter along the curve from the point where $t = 0$. Also, find the length of the indicated portion of the curve. $\vec{r}(t) = (1+2t)\hat{i} + (1+3t)\hat{j} + (6-6t)\hat{k}; \quad -1 \le t \le 0$

Solution

$$\vec{v} = 2\hat{i} + 3\hat{j} - 6\hat{k} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|v| = \sqrt{4+9+36} = 7$$

$$s(t) = \int_0^t 7\, d\tau \qquad\qquad s = \int_0^t |\vec{v}(\tau)|\, d\tau$$

$$= 7t$$

Length: $s(0) - s(-1) = 0 - (-7) = 7$

Exercise

Find the arc length of the curve $\vec{r}(t) = \left\langle 2t^{9/2},\ t^3 \right\rangle \quad for \quad 0 \le t \le 2$

Solution

$$\vec{v}(t) = \left\langle 9t^{7/2},\ 3t^2 \right\rangle \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$L = \int_0^2 \sqrt{\left(9t^{7/2}\right)^2 + \left(3t^2\right)^2}\ dt \qquad\qquad L = \int_0^{t_0} \left|\frac{d\vec{r}}{dt}\right|\, dt$$

$$= \int_0^2 \sqrt{81t^7 + 9t^4}\ dt$$

$$= \int_0^2 3t^2\sqrt{9t^3+1}\ dt$$

$$= \frac{1}{9}\int_0^2 \left(9t^3+1\right)^{1/2}\ d\left(9t^3+1\right)$$

$$= \frac{2}{27}\left(9t^3 + 1\right)^{3/2} \Big|_0^2$$

$$\underline{= \frac{2}{27}\left(73\sqrt{73} - 1\right)} \quad unit$$

Exercise

Find the arc length of the curve $\vec{r}(t) = \left\langle t^2,\ \frac{4\sqrt{2}}{3}t^{3/2},\ 2t \right\rangle \quad for \quad 1 \le t \le 3$

Solution

$$\frac{d\vec{r}}{dt} = \left\langle 2t,\ 2\sqrt{2}\ t^{1/2},\ 2 \right\rangle$$

$$L = \int_1^3 \sqrt{4t^2 + 8t + 4}\ dt \qquad\qquad L = \int_a^b \left|\vec{r}'(t)\right|\ dt$$

$$= 2\int_1^3 \sqrt{(t+1)^2}\ dt$$

$$= 2\int_1^3 (t+1)\ dt$$

$$= 2\left(\frac{1}{2}t^2 + t\right)\Big|_1^3$$

$$= 2\left(\frac{9}{2} + 1 - \frac{1}{2} - 1\right)$$

$$\underline{= 12} \quad unit$$

Exercise

Find the arc length of the curve $\vec{r}(t) = \langle t,\ \ln \sec t,\ \ln(\sec t + \tan t) \rangle \quad for \quad 0 \le t \le \frac{\pi}{4}$

Solution

$$\frac{d\vec{r}}{dt} = \left\langle 1,\ \frac{\tan t \sec t}{\sec t},\ \frac{\tan t \sec t + \sec^2 t}{\sec t + \tan t} \right\rangle$$

$$= \langle 1,\ \tan t,\ \sec t \rangle$$

$$L = \int_0^{\pi/4} \sqrt{1 + \tan^2 t + \sec^2 t}\ dt \qquad\qquad L = \int_a^b \left|\vec{r}'(t)\right|\ dt$$

$$= \int_0^{\pi/4} \sqrt{2\sec^2 t}\ dt$$

$$= \sqrt{2}\int_0^{\pi/4} \sec t\ dt$$

$$= \sqrt{2}\ln(\sec t + \tan t)\Big|_0^{\pi/4}$$

$$\underline{= \sqrt{2}\ \ln\left(\sqrt{2}+1\right)}$$

Exercise

Find the lengths of the curves $\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j} + t^2\ \hat{k};\quad 0 \le t \le \frac{\pi}{4}$

Solution

$$\frac{d\vec{r}}{dt} = (-2\sin t)\hat{i} + (2\cos t)\hat{j} + 2t\ \hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{4\sin^2 t + 4\cos^2 t + 4t^2}$$

$$= \sqrt{4+4t^2}$$

$$= 2\sqrt{1+t^2}$$

$$L = 2\int_0^{\pi/4} \sqrt{1+t^2}\ dt \qquad\qquad L = \int_a^b \left|\vec{r}'(t)\right|\ dt$$

$$= t\sqrt{1+t^2} + \ln\left(t+\sqrt{1+t^2}\right)\Big|_0^{\pi/4}$$

$$= \frac{\pi}{4}\sqrt{1+\frac{\pi^2}{16}} + \ln\left(\frac{\pi}{4}+\sqrt{1+\frac{\pi^2}{16}}\right) - 0 - \ln 1$$

$$\underline{= \frac{\pi}{4}\sqrt{1+\frac{\pi^2}{16}} + \ln\left(\frac{\pi}{4}+\sqrt{1+\frac{\pi^2}{16}}\right)}$$

Exercise

Find the lengths of the curves $\vec{r}(t) = (3\cos t)\hat{i} + (3\sin t)\hat{j} + 2t^{3/2}\ \hat{k};\quad 0 \le t \le 3$

Solution

$$\frac{d\vec{r}}{dt} = (-3\sin t)\hat{i} + (3\cos t)\hat{j} + 3t^{1/2}\,\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{9\sin^2 t + 9\cos^2 t + 9t}$$

$$= 3\sqrt{1+t^2}$$

$$L = 3\int_0^3 \sqrt{1+t}\;dt \qquad\qquad L = \int_a^b \left|\vec{r}'(t)\right|\,dt$$

$$= 3\int_0^3 (1+t)^{1/2}\;d(1+t)$$

$$= 2(1+t)^{3/2}\Big|_0^3$$

$$= 2\left(4^{3/2} - 1\right)$$

$$= 2(8-1)$$

$$= 14$$

Exercise

The acceleration of a wayward firework is given by $\vec{a}(t) = \sqrt{2}\,\hat{j} + 2t\,\hat{k} \quad \text{for} \quad 0 \le t \le 3$. Suppose the initial velocity of the firework is $\vec{v}(0) = 1$.

a) Find the velocity of the firework, for $0 \le t \le 3$.

b) Find the length of the trajectory of the firework over the interval $0 \le t \le 3$

Solution

a) $\vec{v} = \int \left\langle 0,\ \sqrt{2},\ 2t \right\rangle\,dt$

$$= \left\langle 0,\ \sqrt{2}\,t,\ t^2 \right\rangle + \vec{C}$$

$$\vec{v}(0) = 1 = \langle 1,\ 0,\ 0\rangle$$

$$\langle 1,\ 0,\ 0\rangle = \langle 0,\ 0,\ 0\rangle + \vec{C}$$

$$\vec{C} = \langle 1,\ 0,\ 0\rangle$$

$$\vec{v}(t) = \left\langle 0,\ \sqrt{2}\,t,\ t^2 \right\rangle + \langle 1,\ 0,\ 0\rangle$$

$$= \left\langle 1,\ \sqrt{2}\,t,\ t^2 \right\rangle$$

b) $L = \int_0^3 \sqrt{1+2t^2+t^4}\ dt$ $\qquad$ $L = \int_a^b \left|\vec{r}'(t)\right|\ dt$

$$= \int_0^3 \sqrt{\left(1+t^2\right)^2}\ dt$$

$$= \int_0^3 \left(1+t^2\right)\ dt$$

$$= t + \frac{1}{3}t^3 \Big|_0^3$$

$$= 3+9$$

$$\underline{=12}\ \ unit$$

Exercise

If a string wound around a fixed circle in unwound while held taut in the plane of the circle, its end P traces an involute of the circle. The circle in question is the circle $x^2+y^2=1$ and the tracing point starts at (1, 0). The unwound portion of the string is tangent to the circle at Q, and t is the radian measure of the angle from the position x-axis to segment OQ. Derive the parametric equations

$x=\cos t + t\sin t, \quad y = \sin t - t\cos t, \quad t>0$ of the point $P(x, y)$ for the involute.

Solution

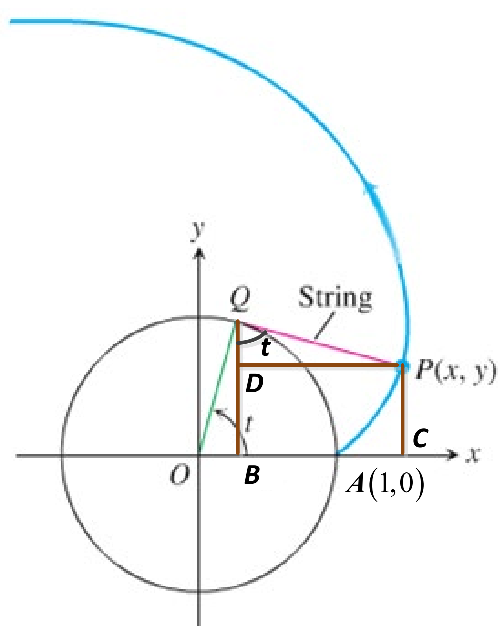
$\angle PQB = \angle QOB = t$

$PQ = arc(AQ) = t$

PQ = Length of the unwound string

$arc(AQ)$

$$\Delta PDQ: \begin{cases} \sin t = \frac{DP}{PQ} = \frac{DP}{t} \to \underline{DT = t\sin t} \\ \cos t = \frac{QD}{PQ} = \frac{QD}{t} \to \underline{QD = t\cos t} \end{cases}$$



$x = OB + BC$

$= OB + DP$

$\underline{= \cos t + t\sin t}$

$y = PC$

$= QB - QD$

$\underline{= \sin t - t\cos t}$

Solution Section 1.8 – Curvature and Normal Vectors

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the plane curves: $\vec{r}(t) = t\,\hat{i} + (\ln\cos t)\,\hat{j}, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

Solution

$$\vec{v}(t) = \hat{i} - \frac{\sin t}{\cos t}\hat{j} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$= \hat{i} - \tan t\,\hat{j}$$

$$|\vec{v}| = \sqrt{1+\tan^2 t}$$

$$= \sqrt{\sec^2 t}$$

$$= \sec t$$

$$\vec{T} = \frac{1}{\sec t}\hat{i} - \frac{\tan t}{\sec t}\hat{j} \qquad\qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \cos t\,\hat{i} - \cos t\frac{\sin t}{\cos t}\hat{j}$$

$$= \cos t\,\hat{i} - \sin t\,\hat{j}$$

$$\frac{d\vec{T}}{dt} = -(\sin t)\hat{i} - (\cos t)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\vec{N} = \frac{-(\sin t)\hat{i} - (\cos t)\hat{j}}{1} \qquad\qquad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$= -(\sin t)\hat{i} - (\cos t)\hat{j}$$

$$\kappa = \frac{1}{\sec t}(1) \qquad\qquad \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$= \cos t$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the plane curves: $\vec{r}(t) = (\ln\sec t)\hat{i} + t\,\hat{j}, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

Solution

$$\vec{v}(t) = \frac{\sec t\tan t}{\sec t}\hat{i} + \hat{j} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$= \tan t\,\hat{i} + \hat{j}$$

$$\left|\vec{v}\right| = \sqrt{\tan^2 t + 1}$$
$$= \sqrt{\sec^2 t}$$
$$= \sec t$$

$$\vec{T} = \frac{\tan t}{\sec t}\hat{i} + \frac{1}{\sec t}\hat{j} \qquad \vec{T} = \frac{\vec{v}}{\left|\vec{v}\right|}$$

$$= (\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\frac{d\vec{T}}{dt} = (\cos t)\hat{i} - (\sin t)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t}$$
$$= 1$$

$$\vec{N} = (\sin t)\hat{i} + (\cos t)\hat{j} \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$\kappa = \frac{1}{\sec t}(1) \qquad \kappa = \frac{1}{\left|\vec{v}\right|}\left|\frac{d\vec{T}}{dt}\right|$$
$$= \cos t$$

Exercise

Find ***T***, ***N***, and κ for the plane curves: $\vec{r}(t) = (2t+3)\hat{i} + \left(5 - t^2\right)\hat{j}$

Solution

$$\vec{v}(t) = 2\hat{i} - 2t\,\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\left|\vec{v}\right| = \sqrt{4 + 4t^2}$$
$$= 2\sqrt{1+t^2}$$

$$\vec{T} = \frac{2}{2\sqrt{1+t^2}}\hat{i} - \frac{2t}{2\sqrt{1+t^2}}\hat{j} \qquad \vec{T} = \frac{\vec{v}}{\left|\vec{v}\right|}$$

$$= \frac{1}{\sqrt{1+t^2}}\hat{i} - \frac{t}{\sqrt{1+t^2}}\hat{j}$$

$$\frac{d\vec{T}}{dt} = \frac{-2t}{2\left(1+t^2\right)^{3/2}}\hat{i} - \frac{\left(1+t^2\right)^{1/2} - \frac{1}{2}\left(1+t^2\right)^{-1/2}(2t)t}{\left(1+t^2\right)}\hat{j}$$

$$= \frac{-t}{\left(1+t^2\right)^{3/2}}\hat{i} - \frac{1+t^2-t^2}{\left(1+t^2\right)^{3/2}}\hat{j}$$

$$= \frac{-t}{\left(1+t^2\right)^{3/2}}\hat{i} - \frac{1}{\left(1+t^2\right)^{3/2}}\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{t^2}{\left(1+t^2\right)^3} + \frac{1}{\left(1+t^2\right)^3}}$$

$$= \sqrt{\frac{t^2+1}{\left(1+t^2\right)^3}}$$

$$= \sqrt{\frac{1}{\left(1+t^2\right)^2}}$$

$$= \frac{1}{\left(1+t^2\right)}$$

$$\vec{N} = \left(1+t^2\right)\left(\frac{-t}{\left(1+t^2\right)^{3/2}}\hat{i} - \frac{1}{\left(1+t^2\right)^{3/2}}\hat{j}\right) \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$\boxed{= \frac{-t}{\sqrt{1+t^2}}\hat{i} - \frac{1}{\sqrt{1+t^2}}\hat{j}}$$

$$\kappa = \frac{1}{2\sqrt{1+t^2}}\frac{1}{\left(1+t^2\right)} \qquad \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$\boxed{= \frac{1}{2\left(1+t^2\right)^{3/2}}}$$

Exercise

Find ***T***, ***N***, and κ for the plane curves: $\vec{r}(t) = (\cos t + t\sin t)\hat{i} + (\sin t - t\cos t)\hat{j}, \quad t > 0$

Solution

$$\vec{v} = (-\sin t + \sin t + t\cos t)\hat{i} + (\cos t - \cos t + t\sin t)\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$= (t\cos t)\hat{i} + (t\sin t)\hat{j}$$

$$|\vec{v}| = \sqrt{t^2\cos^2 t + t^2\sin^2 t}$$

$$= \sqrt{t^2\left(\cos^2 t + \sin^2 t\right)}$$

$$= \sqrt{t^2}$$

$$= |t|$$

$$= t$$

$$\vec{T} = \left(\frac{t\cos t}{t}\right)\hat{i} + \left(\frac{t\sin t}{t}\right)\hat{j} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$= (\cos t)\hat{i} + (\sin t)\hat{j}$$

$$\frac{d\vec{T}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t} \quad = 1$$

$$\vec{N} = (-\sin t)\hat{i} + (\cos t)\hat{j} \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$\kappa = \frac{1}{t} \qquad \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the space curves: $\vec{r}(t) = (3\sin t)\hat{i} + (3\cos t)\hat{j} + 4t\,\hat{k}$

Solution

$$\vec{v} = (3\cos t)\hat{i} - (3\sin t)\hat{j} + 4\,\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{9\cos^2 t + 9\sin^2 t + 16}$$

$$= \sqrt{9 + 16}$$

$$= 5$$

$$\vec{T} = \frac{3}{5}\cos t\ \hat{i} - \frac{3}{5}\sin t\ \hat{j} + \frac{4}{5}\ \hat{k} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\frac{d\vec{T}}{dt} = -\frac{3}{5}\sin t\ \hat{i} - \frac{3}{5}\cos t\ \hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{9}{25}\sin^2 t + \frac{9}{25}\cos^2 t}$$

$$= \sqrt{\frac{9}{25}}$$

$$= \frac{3}{5}$$

$$\vec{N} = \frac{5}{3}\left(-\frac{3}{5}\sin t\ \hat{i} - \frac{3}{5}\cos t\ \hat{j}\right) \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= (-\sin t)\hat{i} - (\cos t)\hat{j}$$

$$\kappa = \frac{1}{5}\frac{3}{5} \qquad \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$= \frac{3}{25}$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the space curves: $\boldsymbol{r}(t) = \left(e^t \cos t\right)\boldsymbol{i} + \left(e^t \sin t\right)\boldsymbol{j} + 2\boldsymbol{k}$

Solution

$$\vec{v}(t) = \left(e^t\cos t - e^t\sin t\right)\hat{i} + \left(e^t\sin t + e^t\cos t\right)\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{\left(e^t\cos t - e^t\sin t\right)^2 + \left(e^t\sin t + e^t\cos t\right)^2}$$

$$= \sqrt{e^{2t}\cos^2 t - e^{2t}\cos t\sin t + e^{2t}\sin^2 t + e^{2t}\sin^2 t + e^{2t}\cos t\sin t + e^{2t}\cos^2 t}$$

$$= \sqrt{2e^{2t}\left(\cos^2 t + \sin^2 t\right)}$$

$$= e^t\sqrt{2}$$

$$\vec{T} = \left(\frac{e^t\cos t - e^t\sin t}{\sqrt{2}e^t}\right)\hat{i} + \left(\frac{e^t\sin t + e^t\cos t}{\sqrt{2}e^t}\right)\hat{j} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\hat{i} + \left(\frac{\sin t + \cos t}{\sqrt{2}}\right)\hat{j}$$

$$\frac{d\vec{T}}{dt} = \left(\frac{-\sin t - \cos t}{\sqrt{2}}\right)\hat{i} + \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{(-\sin t - \cos t)^2}{2} + \frac{(\cos t - \sin t)^2}{2}}$$

$$= \frac{1}{\sqrt{2}}\sqrt{\sin^2 t + 2\sin t\cos t + \cos^2 t + \sin^2 t - 2\sin t\cos t + \cos^2 t}$$

$$= \frac{1}{\sqrt{2}}\sqrt{2\sin^2 t + 2\cos^2 t}$$

$$= \frac{1}{\sqrt{2}}\sqrt{2}$$

$$= 1$$

$$\vec{N} = \left(\frac{-\sin t - \cos t}{\sqrt{2}}\right)\hat{i} + \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\hat{j} \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$\kappa = \frac{1}{\sqrt{2}e^{t}} \qquad \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the space curves: $\vec{r}(t) = \frac{t^3}{3}\hat{i} + \frac{t^2}{2}\hat{j}, \quad t > 0$

Solution

$$\vec{v} = \left(t^2\right)\hat{i} + t\,\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{t^4 + t^2}$$

$$= |t|\sqrt{t^2 + 1}$$

$$= t\sqrt{t^2 + 1} \quad (t > 0)$$

$$\vec{T} = \left(\frac{t^2}{t\sqrt{t^2 + 1}}\right)\hat{i} + \left(\frac{t}{t\sqrt{t^2 + 1}}\right)\hat{j} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \left(\frac{t}{\sqrt{t^2 + 1}}\right)\hat{i} + \left(\frac{1}{\sqrt{t^2 + 1}}\right)\hat{j}$$

$$\frac{d\vec{T}}{dt} = \frac{\left(1 + t^2\right)^{1/2} - \frac{1}{2}\left(1 + t^2\right)^{-1/2}(2t)t}{\left(1 + t^2\right)}\hat{i} + \frac{-2t}{2\left(1 + t^2\right)^{3/2}}\hat{j}$$

$$= \frac{1 + t^2 - t^2}{\left(1 + t^2\right)^{3/2}}\hat{i} - \frac{t}{\left(1 + t^2\right)^{3/2}}\hat{j}$$

$$= \frac{1}{\left(1 + t^2\right)^{3/2}}\hat{i} - \frac{t}{\left(1 + t^2\right)^{3/2}}\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{1}{\left(1 + t^2\right)^3} + \frac{t^2}{\left(1 + t^2\right)^3}}$$

$$=\sqrt{\frac{t^2+1}{\left(1+t^2\right)^3}}$$

$$=\sqrt{\frac{1}{\left(1+t^2\right)^2}}$$

$$=\frac{1}{1+t^2}$$

$$\vec{N}=\left(1+t^2\right)\left(\frac{1}{\left(1+t^2\right)^{3/2}}\hat{i}-\frac{t}{\left(1+t^2\right)^{3/2}}\hat{j}\right) \qquad \vec{N}=\frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$=\frac{1}{\sqrt{1+t^2}}\hat{i}-\frac{t}{\sqrt{1+t^2}}\hat{j}$$

$$\kappa=\frac{1}{t\sqrt{t^2+1}}\frac{1}{1+t^2} \qquad \kappa=\frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$=\frac{t}{t\left(t^2+1\right)^{3/2}}$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, and κ for the space curves: $\vec{r}(t)=\left(\cos^3 t\right)\hat{i}+\left(\sin^3 t\right)\hat{j},\quad 0<t<\frac{\pi}{2}$

Solution

$$\vec{v}=-\left(3\cos^2 t\sin t\right)\hat{i}+\left(3\sin^2 t\cos t\right)\hat{j} \qquad \vec{v}(t)=\frac{d\vec{r}}{dt}$$

$$|\vec{v}|=\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}$$

$$=3\sqrt{\cos^2 t\sin^2 t\left(\cos^2 t+\sin^2 t\right)}$$

$$=3\sqrt{\cos^2 t\sin^2 t}$$

$$=3|\cos t\sin t|$$

$$=3\cos t\sin t$$

$$\vec{T}=-\left(\frac{3\cos^2 t\sin t}{3|\cos t\sin t|}\right)\hat{j}+\left(\frac{3\sin^2 t\cos t}{3|\cos t\sin t|}\right)\hat{j} \qquad \vec{T}=\frac{\vec{v}}{|\vec{v}|}$$

$$=-(\cos t)\hat{i}+(\sin t)\hat{j}$$

$$\frac{d\vec{T}}{dt} = (\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\boxed{\vec{N} = (\sin t)\hat{i} + (\cos t)\hat{j}} \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$\kappa = \frac{1}{3\cos t \sin t}(1) \qquad \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$\boxed{= \frac{t}{3\cos t \sin t}}$$

Exercise

Find ***T***, ***N***, and κ for the space curves: $\vec{r}(t) = (\cosh t)\hat{i} - (\sinh t)\hat{j} + t\,\hat{k}$

Solution

$$\vec{v} = (\sinh t)\hat{j} - (\cosh t)\hat{j} + \hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{\sinh^2 t + \cosh^2 t + 1} \qquad \cosh^2 t - \sinh^2 t = 1 \quad \Rightarrow \quad \cosh^2 t = 1 + \sinh^2 t$$

$$= \sqrt{\cosh^2 t + \cosh^2 t}$$

$$\boxed{= \sqrt{2}\cosh t}$$

$$\vec{T} = \left(\frac{1}{\sqrt{2}\cosh t}\sinh t\right)\hat{i} - \left(\frac{1}{\sqrt{2}\cosh t}\cosh t\right)\hat{j} + \frac{1}{\sqrt{2}\cosh t}\hat{k} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\boxed{= \left(\frac{1}{\sqrt{2}}\tanh t\right)\hat{i} - \frac{1}{\sqrt{2}}\hat{j} + \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right)\hat{k}}$$

$$\frac{d\vec{T}}{dt} = \left(\frac{1}{\sqrt{2}}\operatorname{sech}^2 t\right)\hat{i} - \left(\frac{1}{\sqrt{2}}\operatorname{sech} t \tanh t\right)\hat{k}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{1}{2}\operatorname{sech}^4 t + \frac{1}{2}\operatorname{sech}^2 t \tanh^2 t}$$

$$= \frac{1}{\sqrt{2}}\operatorname{sech} t\sqrt{\operatorname{sech}^2 t + \tanh^2 t}$$

$$= \frac{1}{\sqrt{2}}\operatorname{sech} t\sqrt{1}$$

$$\boxed{= \frac{1}{\sqrt{2}}\operatorname{sech} t}$$

$$\vec{N} = \frac{\sqrt{2}}{\operatorname{sech} t}\left(\left(\frac{1}{\sqrt{2}}\operatorname{sech}^2 t\right)\hat{i} - \left(\frac{1}{\sqrt{2}}\operatorname{sech} t \tanh t\right)\hat{k}\right) \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= (\operatorname{sech} t)\hat{i} - (\tanh t)\hat{k}$$

$$\kappa = \frac{1}{\sqrt{2}\cosh t}\left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right) \qquad \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$= \frac{1}{2}\operatorname{sech}^2 t$$

Exercise

Find an equation for the circle of curvature of the curve $\vec{r}(t) = t\,\hat{i} + (\sin t)\,\hat{j}$, at the point $\left(\frac{\pi}{2},\ 1\right)$. (The curve parametrizes the graph $y = \sin x$ in the *xy*-plane.)

Solution

$$\vec{v} = \hat{j} + (\cos t)\,\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{1+\cos^2 t}$$

$$\left|\vec{v}\left(\frac{\pi}{2}\right)\right| = \sqrt{1+\cos^2\left(\frac{\pi}{2}\right)}$$

$$= \sqrt{1+0}$$

$$= 1$$

$$\vec{T} = \frac{1}{\sqrt{1+\cos^2 t}}\hat{i} + \left(\frac{\cos t}{\sqrt{1+\cos^2 t}}\right)\hat{j} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\frac{d\vec{T}}{dt} = -\frac{1}{2}\frac{2\cos t(-\sin t)}{\left(1+\cos^2 t\right)^{3/2}}\hat{i} + \frac{-\sin t\left(1+\cos^2 t\right)^{1/2} - \cos t\left(\left(\frac{1}{2}\right)2\cos t(-\sin t)\right)\left(1+\cos^2 t\right)^{-1/2}}{\left(1+\cos^2 t\right)}\hat{j}$$

$$= \frac{\cos t \sin t}{\left(1+\cos^2 t\right)^{3/2}}\hat{i} + \frac{-\sin t\left(1+\cos^2 t\right) + \sin t\cos^2 t}{\left(1+\cos^2 t\right)^{3/2}}\hat{j}$$

$$= \frac{\cos t \sin t}{\left(1+\cos^2 t\right)^{3/2}}\hat{i} + \frac{-\sin t\left(1+\cos^2 t - \cos^2 t\right)}{\left(1+\cos^2 t\right)^{3/2}}\hat{j}$$

$$= \frac{\sin t \cos t}{\left(1+\cos^2 t\right)^{3/2}}\hat{i} - \frac{\sin t}{\left(1+\cos^2 t\right)^{3/2}}\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \left|\frac{(\sin t\cos t)\hat{i} - (\sin t)\hat{j}}{\left(1+\cos^2 t\right)^{3/2}}\right|$$

$$= \frac{\sqrt{\sin^2 t\cos^2 t + \sin^2 t}}{\left(1+\cos^2 t\right)^{3/2}}$$

$$= \frac{|\sin t|\sqrt{\cos^2 t + 1}}{\left(1+\cos^2 t\right)^{3/2}}$$

$$= \frac{|\sin t|}{1+\cos^2 t}$$

$$\left|\frac{d\vec{T}}{dt}\right|_{t=\frac{\pi}{2}} = \frac{\left|\sin\frac{\pi}{2}\right|}{1+\cos^2\frac{\pi}{2}}$$

$$\underline{=1\ |}$$

$\underline{\kappa = 1\ |}$ $\qquad\qquad$ $\kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$

The radius of curvature is: $\rho = \frac{1}{\kappa} = \frac{1}{1} = 1$

The center of the circle is $\left(\frac{\pi}{2},\ 0\right)$

The equation of the osculating circle is: $\left(x - \frac{\pi}{2}\right)^2 + y^2 = 1$

Exercise

Write $\vec{a}$ of the motion $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$ without finding $\boldsymbol{T}$ and $\boldsymbol{N}$. $\vec{r}(t) = (a\cos t)\hat{i} + (a\sin t)\hat{i} + bt\,\hat{k}$

Solution

$\vec{v} = (-a\sin t)\hat{i} + (a\cos t)\hat{j} + (b)\hat{k}$ $\qquad\qquad$ $\vec{v}(t) = \frac{d\vec{r}}{dt}$

$$|\vec{v}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}$$

$$= \sqrt{a^2 + b^2}$$

$$a_T = \frac{d}{dt}|\vec{v}| = 0$$

$$\vec{a} = \vec{v}' = (-a\cos t)\hat{i} - (a\sin t)\hat{j}$$

$$|\vec{a}| = \sqrt{a^2\cos^2 t + a^2\sin^2 t}$$

$= |a|$

$a_N = \sqrt{a^2 + 0}$ $\qquad a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$

$= |a|$

$\vec{a} = (0)\vec{T} + |a|\vec{N}$ $\qquad \vec{a} = a_T\vec{T} + a_N\vec{N}$

$= |a|\vec{N}$

Exercise

Write $\vec{a}$ of the motion $\vec{a} = a_T\vec{T} + a_N\vec{N}$ without finding ***T*** and ***N***. $\vec{r}(t) = (1+3t)\hat{i} + (t-2)\hat{j} - 3t\,\hat{k}$

Solution

$\vec{v} = 3\hat{i} + \hat{j} - 3\hat{k}$ $\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$|\vec{v}| = \sqrt{9+1+9}$

$= \sqrt{19}$

$a_T = \frac{d}{dt}|\vec{v}| = 0$

$\vec{a} = \vec{v}' = 0$

$a_N = 0$ $\qquad a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$

$\vec{a} = (0)\vec{T} + 0\vec{N}$ $\qquad \vec{a} = a_T\vec{T} + a_N\vec{N}$

$= \vec{0}$

Exercise

Write $\vec{a}$ of the motion $\vec{a} = a_T\vec{T} + a_N\vec{N}$ at the given value of t without finding ***T*** and ***N***.

$$\vec{r}(t) = (t+1)\hat{i} + 2t\,\hat{j} + t^2\hat{k}, \quad t = 1$$

Solution

$\vec{v} = \hat{i} + 2\hat{j} + 2t\,\hat{k}$ $\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$|\vec{v}| = \sqrt{1+4+4t^2}$

$= \sqrt{5+4t^2}$

$a_T = \frac{1}{2}(8t)\left(5+4t^2\right)^{-1/2}$ $\qquad a_T = \frac{d}{dt}|\vec{v}|$

$$= 4t\left(5+4t^2\right)^{-1/2}$$

$$a_T\Big|_{t=1} = 4(5+4)^{-1/2}$$

$$= 4(9)^{-1/2}$$

$$= \frac{4}{3}$$

$$\vec{a} = \vec{v}' = 2\hat{k}$$

$$|\vec{a}| = \sqrt{4} = 2$$

$$a_N = \sqrt{4-\frac{16}{9}} \qquad a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$$

$$= \sqrt{\frac{20}{9}}$$

$$= \frac{2\sqrt{5}}{3}$$

$$\vec{a} = \frac{4}{3}\vec{T} + \frac{2\sqrt{5}}{3}\vec{N} \qquad \vec{a} = a_T\vec{T} + a_N\vec{N}$$

Exercise

Write $\vec{a}$ of the motion $\vec{a} = a_T\vec{T} + a_N\vec{N}$ at the given value of t without finding $\boldsymbol{T}$ and $\boldsymbol{N}$.

$\vec{r}(t) = (t\cos t)\hat{i} + (t\sin t)\hat{j} + t^2\hat{k}, \quad t = 0$

Solution

$$\vec{v} = (\cos t - t\sin t)\hat{i} + (\sin t + t\cos t)\hat{j} + 2t\,\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 4t^2}$$

$$= \sqrt{\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + 4t^2}$$

$$= \sqrt{\cos^2 t + \sin^2 t + t^2\left(\sin^2 t + \cos^2 t\right) + 4t^2}$$

$$= \sqrt{1+5t^2}$$

$$a_T = \frac{d}{dt}|\vec{v}|$$

$$= \frac{1}{2}(10t)\left(1+5t^2\right)^{-1/2}$$

$$= 5t\left(1+5t^2\right)^{-1/2}$$

$$a_T\Big|_{t=0} = 0$$

$$\vec{a} = \vec{v}' = (-\sin t - \sin t - t\cos t)\hat{i} + (\cos t + \cos t - t\sin t)\hat{j} + 2\hat{k}$$

$$= (-2\sin t - t\cos t)\hat{i} + (2\cos t - t\sin t)\hat{j} + 2\hat{k}$$

$$\vec{a}\Big|_{t=0} = 2\hat{j} + 2\hat{k}$$

$$|\vec{a}|\Big|_{t=0} = \sqrt{4+4}$$

$$= 2\sqrt{2}$$

$$a_N = \sqrt{8-0} \qquad\qquad a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$$

$$= 2\sqrt{2}$$

$$\underline{\vec{a} = 2\sqrt{2}\vec{N}} \qquad\qquad \vec{a} = a_T\vec{T} + a_N\vec{N}$$

Exercise

Write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$ at the given value of t without finding $\boldsymbol{T}$ and $\boldsymbol{N}$.

$$\vec{r}(t) = \left(e^t \cos t\right)\hat{i} + \left(e^t \sin t\right)\hat{j} + \sqrt{2}e^t\hat{k}, \quad t = 0$$

Solution

$$\vec{v} = \left(e^t \cos t - e^t \sin t\right)\hat{i} + \left(e^t \sin t + e^t \cos t\right)\hat{j} + \sqrt{2}e^t\hat{k} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{\left(e^t \cos t - e^t \sin t\right)^2 + \left(e^t \sin t + e^t \cos t\right)^2 + 2e^{2t}}$$

$$= \sqrt{e^{2t}\left(\cos^2 t - 2\cos t\sin t + \sin^2 t\right) + e^{2t}\left(\cos^2 t + 2\cos t\sin t + \sin^2 t\right) + 2e^{2t}}$$

$$= e^t\sqrt{1 - 2\cos t\sin t + 1 + 2\cos t\sin t + 2}$$

$$= e^t\sqrt{4}$$

$$= 2e^t$$

$$a_T = \frac{d}{dt}|\vec{v}| = 2e^t$$

$$a_T\Big|_{t=0} = 2$$

$$\vec{a} = \vec{v}' = \left(e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t\right)\hat{i} + \left(e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t\right)\hat{j} + \sqrt{2}e^t\hat{k}$$

$$= \left(-2e^t \sin t\right)\hat{i} + \left(2e^t \cos t\right)\hat{j} + \sqrt{2}e^t\hat{k}$$

$$\vec{a}\Big|_{t=0} = 2\hat{j} + \sqrt{2}\hat{k}$$

$$\left|\vec{a}\right|\Big|_{t=0} = \sqrt{4+2}$$

$$= \sqrt{6}$$

$$a_N = \sqrt{6-4} \qquad\qquad a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$$

$$\underline{= \sqrt{2}|}$$

$$\underline{\vec{a} = 2\vec{T} + \sqrt{2}\vec{N}\,|} \qquad\qquad \vec{a} = a_T\vec{T} + a_N\vec{N}$$

Exercise

Write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$ without finding $\boldsymbol{T}$ and $\boldsymbol{N}$.

$$\vec{r}(t) = \left(2+3t+3t^2\right)\hat{i} + \left(4t+4t^2\right)\hat{j} - (6\cos t)\hat{k} \quad t = 0$$

Solution

$$\vec{v} = (3+6t)\hat{i} + (4+8t)\hat{j} + 6\sin t\,\hat{k} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{(3+6t)^2 + (4+8t)^2 + 36\sin^2 t}$$

$$= \sqrt{9(1+2t)^2 + 16(1+2t)^2 + 36\sin^2 t}$$

$$= \sqrt{(9+16)(1+2t)^2 + 36\sin^2 t}$$

$$= \sqrt{25(1+2t)^2 + 36\sin^2 t}$$

$$a_T = \frac{1}{2}\left(100(1+2t) + 72\sin t\cos t\right)\left(25(1+2t)^2 + 36\sin^2 t\right)^{-1/2} \qquad\qquad a_T = \frac{d}{dt}|\vec{v}|$$

$$= \frac{1}{2}\frac{100(1+2t) + 72\sin t\cos t}{\sqrt{25(1+2t)^2 + 36\sin^2 t}}$$

$$a_T\Big|_{t=0} = \frac{1}{2}\frac{100}{\sqrt{25}}$$

$$\underline{= 10\,|}$$

$$\vec{a} = 6\hat{i} + 8\hat{j} + 6\cos t\,\hat{k} \qquad\qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$\vec{a}\Big|_{t=0} = 6\hat{i} + 8\hat{j} + 6\hat{k}$$

$$\left|\vec{a}\right|\Big|_{t=0} = \sqrt{36+64+36}$$

$$= \sqrt{136}$$

$$a_N = \sqrt{133 - 100} \qquad\qquad a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$$

$$\underline{= 6|}$$

$$\underline{\vec{a} = 10\vec{T} + 6\vec{N}\,|} \qquad\qquad \vec{a} = a_T\vec{T} + a_N\vec{N}$$

Exercise

Write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$ without finding $\boldsymbol{T}$ and $\boldsymbol{N}$.

$$\vec{r}(t) = (2+t)\hat{i} + \left(t + 2t^2\right)\hat{j} + \left(1 + t^2\right)\hat{k} \quad t = 0$$

Solution

$$\vec{v} = \hat{i} + (1+4t)\hat{j} + 2t\,\hat{k} \qquad\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{1 + (1+4t)^2 + 4t^2}$$

$$= \sqrt{1 + 1 + 8t + 16t^2 + 4t^2}$$

$$= \sqrt{2 + 8t + 20t^2}$$

$$a_T = \frac{8 + 40t}{2\sqrt{2 + 8t + 20t^2}} \qquad\qquad a_T = \frac{d}{dt}|\vec{v}|$$

$$= \frac{4 + 20t}{\sqrt{2 + 8t + 20t^2}}$$

$$a_T\Big|_{t=0} = \frac{4}{\sqrt{2}}$$

$$\underline{= 2\sqrt{2}\,|}$$

$$\vec{a} = 4\hat{j} + 2\hat{k} \qquad\qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$|\vec{a}|\Big|_{t=0} = \sqrt{16 + 4}$$

$$= 2\sqrt{5}$$

$$a_N = \sqrt{20 - 8} \qquad\qquad a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2}$$

$$\underline{= 2\sqrt{3}\,|}$$

$$\underline{\vec{a} = 2\sqrt{2}\vec{T} + 2\sqrt{3}\vec{N}\,|} \qquad\qquad \vec{a} = a_T\vec{T} + a_N\vec{N}$$

Exercise

Graph the curves and sketch their velocity and acceleration vectors at the given values of t. Then write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T \boldsymbol{T} + a_N \boldsymbol{N}$ without finding $\boldsymbol{T}$ and $\boldsymbol{N}$, and find the value of κ at the given values of t.

$$\vec{r}(t) = (4\cos t)\hat{i} + \left(\sqrt{2}\sin t\right)\hat{j}, \quad t = 0 \text{ and } \frac{\pi}{4}$$

Solution

$$\begin{cases} x = 4\cos t & \rightarrow \cos t = \frac{x}{4} \\ y = \sqrt{2}\sin t & \rightarrow \sin t = \frac{y}{\sqrt{2}} \end{cases}$$

$$\cos^2 t + \sin^2 t = 1$$

$$\frac{x^2}{16} + \frac{y^2}{2} = 1 \quad \rightarrow \quad \textit{Ellipse}$$

$$\vec{v} = -4\sin t\,\hat{i} + \sqrt{2}\cos t\,\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\vec{a} = -4\cos t\,\hat{i} - \sqrt{2}\sin t\,\hat{j} \qquad \vec{a} = \frac{d\vec{v}}{dt}$$

$t = 0$

$$\vec{r}(0) = 4\hat{i}$$
$$\vec{v}(0) = \sqrt{2}\hat{j}$$
$$\vec{a}(0) = -4\hat{i}$$

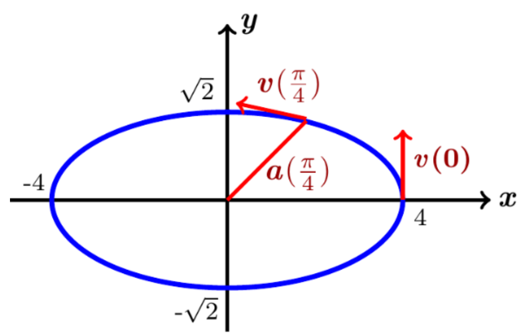
$t = \frac{\pi}{4}$

$$\vec{r}\left(\frac{\pi}{4}\right) = 2\sqrt{2}\,\hat{i} + \hat{j}$$
$$\vec{v}\left(\frac{\pi}{4}\right) = -2\sqrt{2}\,\hat{i} + \hat{j}$$
$$\vec{a}\left(\frac{\pi}{4}\right) = -2\sqrt{2}\,\hat{i} - \hat{j}$$

$$|\vec{v}| = \sqrt{16\sin^2 t + 2\cos^2 t}$$

$$a_T = \frac{32\sin t\cos t - 4\cos t\sin t}{2\sqrt{16\sin^2 t + 2\cos^2 t}} \qquad a_T = \frac{d}{dt}|\vec{v}|$$

$$= \frac{14\cos t\sin t}{\sqrt{16\sin^2 t + 2\cos^2 t}}$$



$t = 0$

$$a_T = 0$$

$$|\vec{a}| = \sqrt{16 + 0} \underline{= 4|}$$

$$a_N = \sqrt{4 - 0} \underline{= 4\,|}$$

$$\underline{\vec{a} = 0\vec{T} + 4\vec{N}\,|}$$

$$|\vec{v}(0)| = \sqrt{2}$$

$$\kappa = \frac{4}{2} \underline{= 2\,|} \qquad \kappa = \frac{a_N}{|\vec{v}|^2}$$

$t = \frac{\pi}{4}$

$$a_T = \frac{14\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}}{\sqrt{8+1}} = \frac{7}{3}$$

$$|\vec{a}| = \sqrt{8+1} \ \underline{= 3|}$$

$$a_N = \sqrt{9 - \frac{49}{9}}$$

$$= \frac{4\sqrt{2}}{3}$$

$$\vec{a} = \frac{7}{3}\vec{T} + \frac{4\sqrt{2}}{3}\vec{N}$$

$$\left|\vec{v}\left(\frac{\pi}{4}\right)\right| = \sqrt{8+1} \ \underline{= 3|}$$

$$\kappa = \frac{4\sqrt{2}}{3}\frac{1}{9} \qquad \kappa = \frac{a_N}{|\vec{v}|^2}$$

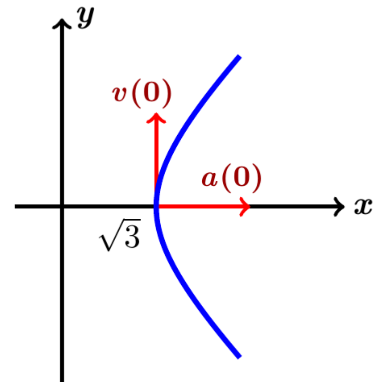
$$= \frac{4\sqrt{2}}{27}$$

Exercise

Graph the curves and sketch their velocity and acceleration vectors at the given values of t. Then write $\boldsymbol{a}$ of the motion $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$ without finding $\boldsymbol{T}$ and $\boldsymbol{N}$, and find the value of κ at the given values of t.

$$\vec{r}(t) = \left(\sqrt{3}\sec t\right)\hat{i} + \left(\sqrt{3}\tan t\right)\hat{j}, \quad t = 0$$

Solution



$$\begin{cases} x = \sqrt{3}\sec t & \rightarrow \sec t = \frac{x}{\sqrt{3}} \\ y = \sqrt{3}\tan t & \rightarrow \tan t = \frac{y}{\sqrt{3}} \end{cases}$$

$$\sec^2 t - \tan^2 t = 1$$

$$\frac{x^2}{3} - \frac{y^2}{3} = 1 \quad \rightarrow \quad \textit{Hyperbolic}$$

$$\vec{v} = \sqrt{3}\sec t\tan t\,\hat{i} + \sqrt{3}\sec^2 t\,\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\vec{a} = \sqrt{3}\left(\sec t\tan^2 t + \sec^3 t\right)\hat{i} + 2\sqrt{3}\sec^2 t\tan t\,\hat{j} \qquad \vec{a} = \frac{d\vec{v}}{dt}$$

At $t = 0$

$$\vec{r}(0) = \sqrt{3}\hat{i}, \quad \vec{v}(0) = \sqrt{3}\hat{j}, \quad \vec{a}(0) = \sqrt{3}\hat{i}$$

$$\left|\vec{v}(0)\right| = \sqrt{3}$$

$$\begin{aligned}
|\vec{v}| &= \sqrt{3\sec^2 t\tan^2 t + 3\sec^4 t} \\
&= \sqrt{3}\ \sqrt{\sec^2 t\left(\sec^2 t - 1\right) + \sec^4 t} \\
&= \sqrt{3}\sqrt{2\sec^4 t - \sec^2 t}
\end{aligned}$$

$$\begin{aligned}
a_T &= \frac{\sqrt{3}}{2}\frac{8\sec^4 t\tan t + 2\sec^2 t\tan t}{\sqrt{2\sec^4 t - \sec^2 t}} \qquad a_T = \frac{d}{dt}|\vec{v}| \\
&= \sqrt{3}\,\frac{\sec^2 t\tan t\left(4\sec^2 t + 1\right)}{\sec t\sqrt{2\sec^2 t - 1}} \\
&= \frac{\sqrt{3}\ \sec t\tan t\left(4\sec^2 t + 1\right)}{\sqrt{2\sec^2 t - 1}}.
\end{aligned}$$

$$a_T\Big|_{t=0} = 0$$

$$\begin{aligned}
a_N &= \sqrt{3 - 0} \qquad a_N = \sqrt{|\boldsymbol{a}|^2 - a_T^2} \\
&= \sqrt{3}
\end{aligned}$$

$$\vec{a} = 0\ \vec{T} + \sqrt{3}\ \vec{N} \qquad \vec{a} = a_T\vec{T} + a_N\vec{N}$$

$$\vec{a} = \sqrt{3}\ \vec{N}$$

$$\kappa = \frac{\sqrt{3}}{3} \qquad \kappa = \frac{a_N}{|\vec{v}|^2}$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$, τ, and κ at the given value of t for the plane curves

$$\vec{r}(t) = \tfrac{4}{9}(1+t)^{3/2}\,\hat{i} + \tfrac{4}{9}(1-t)^{3/2}\,\hat{j} + \tfrac{1}{3}t\,\hat{k}; \quad t = 0$$

Solution

$$\vec{v} = \tfrac{2}{3}(1+t)^{1/2}\,\hat{i} - \tfrac{2}{3}(1-t)^{1/2}\,\hat{j} + \frac{1}{3}\,\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\begin{aligned}
|\vec{v}| &= \sqrt{\tfrac{4}{9}(1+t) + \tfrac{4}{9}(1-t) + \tfrac{1}{9}} \\
&= \sqrt{\tfrac{4}{9}(1+t+1-t) + \tfrac{1}{9}} \\
&= \sqrt{\tfrac{8}{9} + \tfrac{1}{9}}
\end{aligned}$$

$$\boxed{=1}$$

$$\vec{T} = \tfrac{2}{3}(1+t)^{1/2}\,\hat{i} - \tfrac{2}{3}(1-t)^{1/2}\,\hat{j} + \tfrac{1}{3}\,\hat{k} \qquad\qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\boxed{\vec{T}(0) = \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}}$$

$$\frac{d\vec{T}}{dt} = \tfrac{1}{3}(1+t)^{-1/2}\,\hat{i} + \tfrac{1}{3}(1-t)^{-1/2}\,\hat{j}$$

$$\left.\frac{d\vec{T}}{dt}\right|_{t=0} = \tfrac{1}{3}\hat{i} + \tfrac{1}{3}\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}(0)\right| = \sqrt{\frac{1}{9}+\frac{1}{9}}$$

$$= \frac{\sqrt{2}}{3}$$

$$\vec{N}(0) = \frac{3}{\sqrt{2}}\left(\tfrac{1}{3}\hat{i} + \tfrac{1}{3}\hat{j}\right) \qquad\qquad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\boxed{= \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}}$$

$$\vec{B}(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{vmatrix} \qquad\qquad \vec{B} = \vec{T}\times\vec{N}$$

$$\boxed{= -\frac{1}{3\sqrt{2}}\hat{i} + \frac{1}{3\sqrt{2}}\hat{j} + \frac{4}{3\sqrt{2}}\hat{k}}$$

$$\vec{a}(t) = \tfrac{1}{3}(1+t)^{-1/2}\,\hat{i} + \tfrac{1}{3}(1-t)^{-1/2}\,\hat{j} \qquad\qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$\vec{a}(0) = \tfrac{1}{3}\hat{i} + \tfrac{1}{3}\hat{j}$$

$$\vec{v}(0)\times\vec{a}(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{vmatrix}$$

$$= -\tfrac{1}{9}\hat{i} + \tfrac{1}{9}\hat{j} + \tfrac{4}{9}\hat{k}$$

$$\vec{v}(0)\times\vec{a}(0) = \sqrt{\frac{1}{81}+\frac{1}{81}+\frac{16}{81}}$$

$$= \sqrt{\frac{18}{81}}$$

$$= \frac{3\sqrt{2}}{9}$$

$$= \frac{\sqrt{2}}{3}$$

$$\boxed{\kappa(0) = \frac{\sqrt{2}}{3}} \qquad \kappa = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|}$$

$$\vec{a}' = -\frac{1}{6}(1+t)^{-3/2}\,\hat{i} + \frac{1}{6}(1-t)^{-3/2}\,\hat{j}$$

$$\vec{a}'(0) = -\frac{1}{6}\hat{i} + \frac{1}{6}\hat{j}$$

$$\tau(0) = \frac{1}{\left(\frac{\sqrt{2}}{3}\right)^2}\begin{vmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 \end{vmatrix} \qquad \tau = \frac{1}{|\vec{v}\times\vec{a}|^2}\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}$$

$$= \frac{\frac{1}{3}\left(2\frac{1}{18}\right)}{\frac{2}{9}}$$

$$\boxed{= \frac{1}{6}}$$

Exercise

Find ***T***, ***N***, ***B***, τ, and κ at the given value of t for the plane curves

$$\vec{r}(t) = \left(e^t \sin 2t\right)\hat{i} + \left(e^t \cos 2t\right)\hat{j} + 2e^t\,\hat{k}; \quad t = 0$$

Solution

$$\vec{v} = e^t(\sin 2t + 2\cos 2t)\hat{i} + e^t(\cos 2t - 2\sin 2t)\hat{j} + 2e^t\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = e^t\sqrt{(\sin 2t + 2\cos 2t)^2 + (\cos 2t - 2\sin 2t)^2 + 4}$$

$$= e^t\sqrt{\sin^2 2t + 4\sin 2t\cos 2t + 4\cos^2 2t + \cos^2 2t - 4\sin 2t\cos 2t + 4\sin^2 2t + 4}$$

$$= e^t\sqrt{5\cos^2 2t + 5\sin^2 2t + 4}$$

$$= e^t\sqrt{5+4}$$

$$\boxed{= 3e^t}$$

$$\vec{T} = \frac{e^t(\sin 2t + 2\cos 2t)\hat{i} + e^t(\cos 2t - 2\sin 2t)\hat{j} + 2e^t\hat{k}}{3e^t} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{1}{3}\left((\sin 2t + 2\cos 2t)\hat{i} + (\cos 2t - 2\sin 2t)\hat{j} + 2\hat{k}\right)$$

$$\boxed{\vec{T}(0) = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}}$$

$$\frac{d\vec{T}}{dt} = \frac{1}{3}\left(2\cos 2t - 4\sin 2t\right)\hat{i} + \frac{1}{3}\left(-2\sin 2t - 4\sin 2t\right)\hat{j}$$

$$\left.\frac{d\vec{T}}{dt}\right|_{t=0} = \frac{2}{3}\hat{i} - \frac{4}{3}\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}(0)\right| = \sqrt{\frac{4}{9} + \frac{16}{9}}$$

$$= \boxed{\frac{2}{3}\sqrt{5}}$$

$$\vec{N}(0) = \frac{3}{2\sqrt{5}}\left(\frac{2}{3}\hat{i} - \frac{4}{3}\hat{j}\right) \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= \boxed{\frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}}$$

$$\vec{B}(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \end{vmatrix} \qquad \vec{B} = \vec{T} \times \vec{N}$$

$$= \boxed{\frac{4}{3\sqrt{5}}\hat{i} + \frac{2}{3\sqrt{5}}\hat{j} - \frac{5}{3\sqrt{5}}\hat{k}}$$

$$\vec{a}(t) = e^t\left(4\cos 2t - 3\sin 2t\right)\hat{i} + e^t\left(-4\sin 2t - 3\cos 2t\right)\hat{j} + 2e^t\hat{k} \qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$\vec{a}(0) = 4\hat{i} - 3\hat{j} + 2\hat{k}$$

$$\vec{v}(0) \times \vec{a}(0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 2 \\ 4 & -3 & 2 \end{vmatrix}$$

$$= 8\hat{i} + 4\hat{j} - 10\hat{k}$$

$$\vec{v}(0) \times \vec{a}(0) = \sqrt{64 + 16 + 100}$$

$$= \sqrt{180}$$

$$= 6\sqrt{5}$$

$$\kappa(0) = \frac{6\sqrt{5}}{3^2} \qquad \kappa = \frac{\left|\vec{v} \times \vec{a}\right|}{\left|\vec{v}\right|}$$

$$= \boxed{\frac{2\sqrt{5}}{3}}$$

$$\vec{a}' = e^t\left(4\cos 2t - 3\sin 2t - 8\sin 2t - 6\cos 2t\right)\hat{i}$$

$$+ e^t\left(-4\sin 2t - 3\cos 2t - 8\cos 2t + 6\sin 2t\right)\hat{j} + 2e^t\hat{k}$$

$$= e^t\left(-2\cos 2t - 11\sin 2t\right)\hat{i} + e^t\left(2\sin 2t - 11\cos 2t\right)\hat{j} + 2e^t\hat{k}$$

$\vec{a}'(0) = -2\hat{i} - 11\hat{j} + 2\hat{k}$

$$\tau(0) = \frac{1}{180}\begin{vmatrix} 2 & 1 & 2 \\ 4 & -3 & 2 \\ 2 & -11 & 2 \end{vmatrix}$$

$$\tau = \frac{1}{|\vec{v}\times\vec{a}|^2}\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}$$

$$= \frac{-80}{180}$$

$$\boxed{= -\frac{4}{9}}$$

Exercise

Find $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$, τ, and κ at the given value of t for the plane curves $\vec{r}(t) = t\,\hat{i} + \left(\frac{1}{2}e^{2t}\right)\hat{j}$; $t = \ln 2$

Solution

$$\vec{v} = \hat{i} + e^{2t}\,\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\vec{v}(\ln 2) = \hat{i} + e^{2\ln 2}\,\hat{j}$$

$$= \hat{i} + e^{\ln 4}\,\hat{j}$$

$$\boxed{= \hat{i} + 4\,\hat{j}}$$

$$|\vec{v}| = \sqrt{1+e^{4t}}\,\Big|_{t=\ln 2}$$

$$= \sqrt{1+e^{4\ln 2}}$$

$$= \sqrt{1+e^{\ln 2^4}}$$

$$= \sqrt{1+16}$$

$$\boxed{= \sqrt{17}}$$

$$\vec{T} = \frac{1}{\sqrt{1+e^{4t}}}\,\hat{i} + \frac{e^{2t}}{\sqrt{1+e^{4t}}}\,\hat{j} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\boxed{\vec{T}(\ln 2) = \frac{1}{\sqrt{17}}\,\hat{i} + \frac{4}{\sqrt{17}}\,\hat{j}}$$

$$\frac{d\vec{T}}{dt} = \frac{-2e^{4t}}{\left(1+e^{4t}\right)^{3/2}}\,\hat{i} + \frac{2e^{2t}\left(1+e^{4t}\right) - \left(2e^{4t}\right)e^{2t}}{\left(1+e^{4t}\right)^{3/2}}\,\hat{j}$$

$$= \frac{-2e^{4t}}{\left(1+e^{4t}\right)^{3/2}}\,\hat{i} + \frac{2e^{2t}}{\left(1+e^{4t}\right)^{3/2}}\,\hat{j}$$

$$\left.\frac{d\vec{T}}{dt}\right|_{t=\ln 2} = -\frac{2e^{4\ln 2}}{(1+16)^{3/2}}\hat{i} + \frac{2(4)}{17\sqrt{17}}\hat{j}$$

$$= -\frac{32}{17\sqrt{17}}\hat{i} + \frac{8}{17\sqrt{17}}\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}(0)\right| = \sqrt{\frac{32^2}{17^3} + \frac{64}{17^3}}$$

$$= \frac{8\sqrt{17}}{17\sqrt{17}}$$

$$= \boxed{\frac{8}{17}}$$

$$\vec{N}(\ln 2) = \frac{17}{8}\left(-\frac{32}{17\sqrt{17}}\hat{i} + \frac{8}{17\sqrt{17}}\hat{j}\right) \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= \boxed{-\frac{4}{\sqrt{17}}\hat{i} + \frac{1}{\sqrt{17}}\hat{j}}$$

$$\vec{B}(\ln 2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} & 0 \\ \frac{-4}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 \end{vmatrix} \qquad \vec{B} = \vec{T} \times \vec{N}$$

$$\boxed{= \hat{k}}$$

$$\vec{a}(t) = 2e^{2t}\hat{j} \qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$\vec{a}(\ln 2) = 8\hat{j}$$

$$(\vec{v} \times \vec{a})(\ln 2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 0 \\ 0 & 8 & 0 \end{vmatrix}$$

$$= 8\hat{k}$$

$$\left|(\vec{v} \times \vec{a})(\ln 2)\right| = 8$$

$$\boxed{\kappa(\ln 2) = \frac{8}{\sqrt{17}}} \qquad \kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$$

$$\vec{a}' = 4e^{2t}\hat{j}$$

$$\vec{a}'(\ln 2) = 16\hat{j}$$

$$\tau(\ln 2) = \frac{1}{180}\begin{vmatrix} 1 & 4 & 0 \\ 0 & 8 & 0 \\ 0 & 16 & 0 \end{vmatrix} \qquad \tau = \frac{1}{|\vec{v}\times\vec{a}|^2}\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}$$

$$= 0$$

Exercise

Find ***r***, ***T***, ***N***, and ***B*** at the given value of t. Then find equations for the osculating, normal, and rectifying planes at that value of t. $\boldsymbol{r}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} - \boldsymbol{k}, \quad t = \frac{\pi}{4}$

Solution

$$\vec{r}\left(t = \frac{\pi}{4}\right) = \left(\cos\frac{\pi}{4}\right)\hat{i} + \left(\sin\frac{\pi}{4}\right)\hat{j} - \hat{k}$$

$$= \frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j} - \hat{k}$$

$$\vec{v} = (-\sin t)\hat{i} + (\cos t)\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\vec{T} = (-\sin t)\hat{i} + (\cos t)\hat{j} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{T}\left(t = \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j}$$

$$\frac{d\vec{T}}{dt} = (-\cos t)\hat{i} + (-\sin t)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

$$\vec{N} = (-\cos t)\hat{i} + (-\sin t)\hat{j} \qquad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\vec{N}\left(t = \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\hat{i} - \frac{\sqrt{2}}{2}\hat{j}$$

$$\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \hat{k}$$

$$\vec{B}\left(t = \frac{\pi}{4}\right) = \hat{k}$$

The normal to the osculating plane $\boldsymbol{r} = \frac{\sqrt{2}}{2}\boldsymbol{i} + \frac{\sqrt{2}}{2}\boldsymbol{j} - \boldsymbol{k} \;\Rightarrow\; P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1\right)$ lies on the osculating plane (using ***B***):

$$0\left(x-\frac{\sqrt{2}}{2}\right)+0\left(y-\frac{\sqrt{2}}{2}\right)+\left(z-(-1)\right)=0$$

$z=-1$ is the osculating plane.

T is normal to the normal plane

$$-\frac{\sqrt{2}}{2}\left(x-\frac{\sqrt{2}}{2}\right)+\frac{\sqrt{2}}{2}y\left(x-\frac{\sqrt{2}}{2}\right)+0\left(z-(-1)\right)=0$$

$$-\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y=0$$

$-x+y=0$ is the normal plane

N is normal to the rectifying plane:

$$-\frac{\sqrt{2}}{2}\left(x-\frac{\sqrt{2}}{2}\right)-\frac{\sqrt{2}}{2}\left(y-\frac{\sqrt{2}}{2}\right)+0\left(z-(-1)\right)=0$$

$$-\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}y+\frac{1}{2}+\frac{1}{2}=0$$

$x+y=\sqrt{2}$ is the rectifying plane.

Exercise

Find ***r, T, N***, and ***B*** at the given value of *t*. Then find equations for the osculating, normal, and rectifying planes at that value of *t*. $\vec{r}(t)=(\cos t)\hat{i}+(\sin t)\hat{j}+t\,\hat{k},\quad t=0$

Solution

$$\vec{r}(t=0)=(\cos 0)\hat{i}+(\sin 0)\hat{j}+0\hat{k}$$

$$\underline{=\hat{i}\,|}$$

$$\vec{v}=(-\sin t)\hat{i}+(\cos t)\hat{j}+\hat{k} \qquad \vec{v}(t)=\frac{d\vec{r}}{dt}$$

$$|\vec{v}|=\sqrt{\sin^2 t+\cos^2 t+1}$$

$$=\sqrt{2}$$

$$\vec{T}=-\left(\frac{\sin t}{\sqrt{2}}\right)\hat{i}+\left(\frac{\cos t}{\sqrt{2}}\right)\hat{j}+\frac{1}{\sqrt{2}}\hat{k} \qquad \vec{T}=\frac{\vec{v}}{|\vec{v}|}$$

$$\underline{\vec{T}(t=0)=\frac{1}{\sqrt{2}}\hat{j}+\frac{1}{\sqrt{2}}\hat{k}\,|}$$

$$\frac{d\vec{T}}{dt}=\left(-\frac{1}{\sqrt{2}}\cos t\right)\hat{i}-\left(\frac{1}{\sqrt{2}}\sin t\right)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right|=\sqrt{\frac{1}{2}\cos^2 t+\frac{1}{2}\sin^2 t}$$

$$= \frac{1}{\sqrt{2}}$$

$$\vec{N} = \sqrt{2}\left(\left(-\frac{1}{\sqrt{2}}\cos t\right)\hat{i} - \left(\frac{1}{\sqrt{2}}\sin t\right)\hat{j}\right) \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= (-\cos t)\hat{i} - (\sin t)\hat{j}$$

$$\vec{N}(t=0) = -\hat{i}$$

$$\vec{B}(t=0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \end{vmatrix} \qquad \vec{B} = \vec{T} \times \vec{N}$$

$$= -\frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$$

The normal to the osculating plane $\vec{r}(t) = \hat{i} \quad \Rightarrow \quad P(1,\ 0,\ 0)$ lies on the osculating plane (using ***B***):

$$0(x-1) - \frac{1}{\sqrt{2}}(y-0) + \frac{1}{\sqrt{2}}(z-0) = 0$$

$$-\frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}z = 0$$

$y - z = 0$ is the osculating plane.

T is normal to the normal plane

$$0(x-1) + \frac{1}{\sqrt{2}}(y-0) + \frac{1}{\sqrt{2}}(z-0) = 0$$

$$\frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}z = 0$$

$y + z = 0$ is the normal plane

N is normal to the rectifying plane:

$$-(x-1) + 0(y-0) + 0(z-0) = 0$$

$$-x + 1 = 0$$

$x = 1$ is the rectifying plane.

Exercise

Find ***B*** and τ for: $\vec{r}(t) = (3\sin t)\hat{i} + (3\cos t)\hat{j} + 4t\,\hat{k}$

Solution

$$\vec{v} = (3\cos t)\hat{i} - (3\sin t)\hat{j} + 4\,\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{9\cos^2 t + 9\sin^2 t + 16}$$
$$= \sqrt{9+16}$$
$$\boxed{= 5}$$

$$\boxed{\vec{T} = \frac{3}{5}\cos t\,\hat{i} - \frac{3}{5}\sin t\,\hat{j} + \frac{4}{5}\,\hat{k}} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\frac{d\vec{T}}{dt} = -\frac{3}{5}\sin t\,\hat{i} - \frac{3}{5}\cos t\,\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{9}{25}\sin^2 t + \frac{9}{25}\cos^2 t}$$
$$= \sqrt{\frac{9}{25}}$$
$$= \frac{3}{5}$$

$$\vec{N} = \frac{5}{3}\left(-\frac{3}{5}\sin t\,\hat{i} - \frac{3}{5}\cos t\,\hat{j}\right) \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$
$$\boxed{= (-\sin t)\hat{i} - (\cos t)\,\hat{j}}$$

$$\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{3}{5}\cos t & -\frac{3}{5}\sin t & \frac{4}{5} \\ -\sin t & -\cos t & 0 \end{vmatrix} \qquad \vec{B} = \vec{T} \times \vec{N}$$
$$\boxed{= \left(\frac{4}{5}\cos t\right)\hat{i} - \left(\frac{4}{5}\sin t\right)\hat{j} - \frac{3}{5}\hat{k}}$$

$$\vec{a} = (-3\sin t)\hat{i} - (3\cos t)\,\hat{j} \qquad \vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \end{vmatrix}$$
$$= (12\cos t)\hat{i} - (12\sin t)\,\hat{j} - 9\hat{k}$$

$$|\vec{v} \times \vec{a}|^2 = 144\cos^2 t + 144\sin^2 t + 81$$
$$= 144 + 81$$
$$= 225$$

$$\tau = \frac{\begin{vmatrix} 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \\ -3\cos t & 3\sin t & 0 \end{vmatrix}}{225} \qquad \tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2}$$

$$= \frac{4\left(-9\sin^2 t - 9\cos^2 t\right)}{225}$$

$$= -\frac{36}{225}$$

$$= -\frac{4}{25}$$

Exercise

Find $\boldsymbol{B}$ and τ for: $\vec{r}(t) = (\cos t + t\sin t)\hat{i} + (\sin t - t\cos t)\hat{j} + 3\hat{k}$

Solution

$$\vec{v} = (-\sin t + \sin t + t\cos t)\hat{i} + (\cos t - \cos t + t\sin t)\hat{j} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$= (t\cos t)\hat{i} + (t\sin t)\hat{j}$$

$$|\vec{v}| = \sqrt{t^2\cos^2 t + t^2\sin^2 t}$$

$$= \sqrt{t^2\left(\cos^2 t + \sin^2 t\right)}$$

$$= \sqrt{t^2}$$

$$= |t|$$

$$= t$$

$$\vec{T} = \left(\frac{t\cos t}{t}\right)\hat{i} + \left(\frac{t\sin t}{t}\right)\hat{j} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$= (\cos t)\hat{i} + (\sin t)\hat{j}$$

$$\frac{d\vec{T}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t} \quad = 1$$

$$\vec{N} = (-\sin t)\hat{j} + (\cos t)\hat{j} \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$\boldsymbol{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & \sin t & 0 \\ -\sin t & -\cos t & 0 \end{vmatrix} \qquad \vec{B} = \vec{T} \times \vec{N}$$

$$= \left(\cos^2 t + \sin^2 t\right)\hat{k}$$

$$= \hat{k}$$

$$\vec{a}\left(\cos t - t\sin t\right)\hat{i} + \left(\sin t + t\cos t\right)\hat{j} \qquad \vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{v}\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t\cos t & t\sin t & 0 \\ \cos t - t\sin t & \sin t + t\cos t & 0 \end{vmatrix}$$

$$= \left(t\cos t\sin t + t^2\cos^2 t - t\sin t\cos t + t^2\sin^2 t\right)\hat{k}$$

$$= t^2\hat{k}$$

$$\left|\vec{v}\times\vec{a}\right|^2 = \left(t^2\right)^2 = t^4$$

$$\tau = \frac{\begin{vmatrix} t\cos t & t\sin t & 0 \\ \cos t - t\sin t & \sin t + t\cos t & 0 \\ -2\sin t - t\cos t & 2\cos t - t\sin t & 0 \end{vmatrix}}{t^4} \qquad \tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{\left|\vec{v}\times\vec{a}\right|^2}$$

$$= \frac{0}{t^4}$$

$$\underline{= 0\,|}$$

Exercise

Find $\boldsymbol{B}$ and τ for: $\vec{r}(t) = (6\sin 2t)\hat{i} + (6\cos 2t)\hat{j} + 5t\,\hat{k}$

Solution

$$\vec{v} = (12\cos 2t)\hat{i} - (12\sin 2t)\hat{j} + 5\,\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = \sqrt{144\cos^2 t + 144\sin^2 t + 25}$$

$$= \sqrt{144 + 25}$$

$$\underline{= 13\,|}$$

$$\underline{\vec{T} = \frac{12}{13}\cos 2t\;\hat{i} - \frac{12}{13}\sin 2t\,\hat{j} + \frac{5}{13}\;\hat{k}\;\Big|} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\frac{d\vec{T}}{dt} = -\frac{24}{13}\sin 2t\;\hat{i} - \frac{24}{13}\cos 2t\,\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{576}{169}\sin^2 t + \frac{576}{169}\cos^2 t}$$

$$= \sqrt{\frac{576}{169}}$$

$$= \frac{24}{13}$$

$$\vec{N} = \frac{13}{24}\left(-\frac{24}{13}\sin 2t\,\hat{i} - \frac{24}{13}\cos 2t\,\hat{j}\right) \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= \boxed{(-\sin 2t)\hat{i} - (\cos 2t)\hat{j}}$$

$$\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{12}{13}\cos 2t & -\frac{12}{13}\sin 2t & \frac{5}{13} \\ -\sin 2t & -\cos 2t & 0 \end{vmatrix}$$

$$= \boxed{\left(\frac{5}{13}\cos 2t\right)\hat{i} - \left(\frac{5}{13}\sin 2t\right)\hat{j} - \frac{12}{13}\hat{k}}$$

$$\vec{a} = (-24\sin 2t)\hat{i} - (24\cos 2t)\hat{j} \qquad \vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 12\cos 2t & -12\sin 2t & 5 \\ -24\sin 2t & -24\cos 2t & 0 \end{vmatrix}$$

$$= (120\cos 2t)\hat{i} - (120\sin 2t)\hat{j} - 288\hat{k}$$

$$|\vec{v} \times \vec{a}|^2 = 14400\cos^2 2t + 14400\sin^2 2t + 288^2$$

$$= 14400 + 82944 = 97344$$

$$\tau \frac{\begin{vmatrix} 12\cos 2t & -12\sin 2t & 5 \\ -24\sin 2t & -24\cos 2t & 0 \\ -48\cos 2t & 48\sin 2t & 0 \end{vmatrix}}{97344} \qquad \tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2}$$

$$= \frac{5\left(-1152\sin^2 2t - 1152\cos^2 2t\right)}{97344}$$

$$= -\frac{5760}{97344}$$

$$= \boxed{-\frac{10}{169}}$$

Exercise

The speedometer on your car reads a steady 35 *mph*. Could you be accelerating? Explain.

Solution

Yes.

If the car is moving along a curved path, then $\kappa \neq 0$ and $a_N = \kappa|\vec{v}|^2 \neq 0$

$\Rightarrow \vec{a} = a_T\vec{T} + a_N\vec{N} \neq 0$

Exercise

Can anything be said about the acceleration of a particle that is moving at a constant speed? Give reasons for your answer.

Solution

$|v|$ is constant $\Rightarrow \vec{a}_T = \frac{d\vec{v}}{dt} = 0$

$\vec{a} = a_T \vec{T} + a_N \vec{N}$

$\quad = a_N \vec{N}$ is orthogonal to $\vec{T}$.

$\therefore$ The acceleration is normal to the path.

Exercise

Find ***T***, ***N***, ***B***, τ and κ as functions of *t* for the plane curves: $\vec{r}(t) = (\sin t)\hat{i} + \left(\sqrt{2}\cos t\right)\hat{j} + (\sin t)\hat{k}$, then write $\vec{a}$ of the motion $\vec{a} = a_T \vec{T} + a_N \vec{N}$

Solution

$\vec{v} = (\cos t)\hat{i} - \left(\sqrt{2}\sin t\right)\hat{j} + (\cos t)\hat{k}$ $\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$|\vec{v}| = \sqrt{\cos^2 t + 2\sin^2 t + \cos^2 t}$

$\quad = \sqrt{2\cos^2 t + 2\sin^2 t}$

$\quad = \sqrt{2}$

$$\vec{T} = \left(\frac{\cos t}{\sqrt{2}}\right)\hat{i} - (\sin t)\hat{j} + \left(\frac{\cos t}{\sqrt{2}}\right)\hat{k}$$

$\frac{d\vec{T}}{dt} = -\frac{\sin t}{\sqrt{2}}\hat{i} - (\cos t)\hat{j} - \frac{\sin t}{\sqrt{2}}\hat{k}$

$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{1}{2}\sin^2 t + \cos^2 t + \frac{1}{2}\sin^2 t}$

$\quad = \sqrt{\sin^2 t + \cos^2 t}$

$\quad = 1$

$$\vec{N} = -\frac{\sin t}{\sqrt{2}}\hat{i} - (\cos t)\hat{j} - \frac{\sin t}{\sqrt{2}}\hat{k} \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$\boldsymbol{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\cos t}{\sqrt{2}} & -\sin t & \frac{\cos t}{\sqrt{2}} \\ -\frac{\sin t}{\sqrt{2}} & -\cos t & -\frac{\sin t}{\sqrt{2}} \end{vmatrix}$$

$$= \left(\frac{\sin^2 t}{\sqrt{2}} + \frac{\cos^2 t}{\sqrt{2}} \right)\hat{i} - \left(-\frac{\sin t \cos t}{\sqrt{2}} + \frac{\sin t \cos t}{\sqrt{2}} \right)\hat{j} + \left(-\frac{\cos^2 t}{\sqrt{2}} - \frac{\sin^2 t}{\sqrt{2}} \right)\hat{k}$$

$$= \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{k}$$

$$\kappa = \frac{1}{\sqrt{2}} \qquad\qquad \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

$$\vec{a} = (-\sin t)\hat{i} - \left(\sqrt{2}\cos t\right)\hat{j} - (\sin t)\,\hat{k}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & -\sqrt{2}\sin t & \cos t \\ -\sin t & -\sqrt{2}\cos t & -\sin t \end{vmatrix}$$

$$= \left(\sqrt{2}\sin^2 t + \sqrt{2}\cos^2 t\right)\hat{i} + \left(-\sqrt{2}\cos^2 t - \sqrt{2}\sin^2 t\right)\hat{k}$$

$$= \left(\sqrt{2}\right)\hat{i} - \left(\sqrt{2}\right)\hat{k}$$

$$|\vec{v} \times \vec{a}| = \sqrt{2+2} = 2$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2} = \frac{\begin{vmatrix} \cos t & -\sqrt{2}\sin t & \cos t \\ -\sin t & -\sqrt{2}\cos t & -\sin t \\ -\cos t & \sqrt{2}\sin t & -\cos t \end{vmatrix}}{4}$$

$$= \frac{\sqrt{2}\cos^3 t - \sqrt{2}\sin^2 t\cos t - \sqrt{2}\sin^2 t\cos t - \sqrt{2}\cos^3 t + \sqrt{2}\sin^2 t\cos t + \sqrt{2}\sin^2 t\cos t}{4}$$

$$= 0$$

Exercise

Consider the ellipse $\vec{r}(t) = \langle 3\cos t,\ 4\sin t\rangle$ *for* $0 \le t \le 2\pi$

a) Find the tangent vector $\vec{r}'$, the unit vector $\vec{T}$, and the principal unit normal vector $\vec{N}$ at all points on the curve.

b) At what points does $|\vec{r}'|$ have maximum and minimum values?

c) At what points does the curvature have maximum and minimum values? Interpret this result in light of part (*b*).

d) Find the points (if any) at which $\vec{r}$ and $\vec{N}$ are parallel.

Solution

a) $\underline{\vec{r}'(t) = \langle -3\sin t,\ 4\cos t\rangle}$

$$\vec{T} = \frac{\langle -3\sin t,\ 4\cos t\rangle}{\sqrt{9\sin^2 t + 16\cos^2 t}} \qquad \vec{T} = \frac{r'(t)}{|r'(t)|}$$

$$= \frac{1}{\sqrt{9\sin^2 t + 9\cos^2 t + 7\cos^2 t}}\langle -3\sin t,\ 4\cos t\rangle$$

$$= \left\langle -\frac{3\sin t}{\sqrt{9+7\cos^2 t}},\ \frac{4\cos t}{\sqrt{9+7\cos^2 t}}\right\rangle$$

$$\frac{d\vec{T}}{dt} = \left\langle -3\frac{\cos t\left(9+7\cos^2 t\right) + 7\cos t\sin^2 t}{\left(9+7\cos^2 t\right)^{3/2}},\ 4\frac{-\sin t\left(9+7\cos^2 t\right) + 7\cos^2 t\sin t}{\left(9+7\cos^2 t\right)^{3/2}}\right\rangle$$

$$= \left\langle -3\frac{9\cos t + 7\cos^3 t + 7\cos t\sin^2 t}{\left(9+7\cos^2 t\right)^{3/2}},\ 4\frac{-9\sin t - 7\cos^2 t\sin t + 7\cos^2 t\sin t}{\left(9+7\cos^2 t\right)^{3/2}}\right\rangle$$

$$= \left\langle -3\cos t\frac{9+7\cos^2 t + 7\sin^2 t}{\left(9+7\cos^2 t\right)^{3/2}},\ -\frac{36\sin t}{\left(9+7\cos^2 t\right)^{3/2}}\right\rangle$$

$$= \left\langle -\frac{48\cos t}{\left(9+7\cos^2 t\right)^{3/2}},\ -\frac{36\sin t}{\left(9+7\cos^2 t\right)^{3/2}}\right\rangle$$

$$= -\frac{12}{\left(9+7\cos^2 t\right)^{3/2}}\langle 4\cos t,\ 3\sin t\rangle$$

$$\left|\frac{d\vec{T}}{dt}\right| = \frac{12}{\left(9+7\cos^2 t\right)^{3/2}}\sqrt{16\cos^2 t + 9\sin^2 t}$$

$$= \frac{12}{\left(9+7\cos^2 t\right)^{3/2}}\sqrt{9+7\cos^2 t}$$

$$= \frac{12}{\sqrt{9+7\cos^2 t}}$$

$$\vec{N} = \frac{\sqrt{9+7\cos^2 t}}{12}\frac{-12}{\left(9+7\cos^2 t\right)^{3/2}}\langle 4\cos t,\ 3\sin t\rangle \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= \left\langle -\frac{4\cos t}{\sqrt{9+7\cos^2 t}},\ -\frac{3\sin t}{\sqrt{9+7\cos^2 t}}\right\rangle$$

b) $\left|\vec{r}'(t)\right| = \sqrt{9\sin^2 t + 16\cos^2 t}$

$$\frac{d}{dt}\left|\vec{r}'(t)\right| = \frac{18\sin t\cos t - 32\cos t\sin t}{2\sqrt{9\sin^2 t + 16\cos^2 t}}$$

$$= \frac{-7\cos t\sin t}{\sqrt{9\sin^2 t + 16\cos^2 t}} = 0$$

$$\begin{cases} \cos t = 0 & \to t = \frac{\pi}{2}, \frac{3\pi}{2} & \left|\vec{r}'(t)\right| = 3 & \textit{Minimum} \\ \sin t = 0 & \to t = 0,\ \pi & \left|\vec{r}'(t)\right| = 4 & \textit{Maximum} \end{cases}$$

c) $r''(t)\times\vec{r}'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\cos t & -4\sin t & 0 \\ -3\sin t & 4\cos t & 0 \end{vmatrix}$

$$= \left\langle 0,\ 0,\ -12\cos^2 t - 12\sin^2 t\right\rangle$$

$$= \langle 0,\ 0,\ -12\rangle$$

$$\tau = \frac{\left|\langle 0,\ 0,\ -12\rangle\right|}{\left(9+7\cos^2 t\right)^{3/2}} \qquad \kappa = \frac{\left|r''(t)\times\vec{r}'(t)\right|}{\left(\left|r'(t)\right|\right)^3}$$

$$\boxed{= \frac{12}{\left(9+7\cos^2 t\right)^{3/2}}}$$

For τ to be maximum the denominator has to be the smallest

$$\cos^2 t = 0 \quad \to t = \frac{\pi}{2}, \frac{3\pi}{2}$$

From part (b) result of t-values

$$\begin{cases} t = \frac{\pi}{2}, \ \frac{3\pi}{2} & \tau = \frac{12}{27} = \frac{4}{9} \quad \textit{Maximum} \\ t = 0, \ \pi & \tau = \frac{12}{64} = \frac{3}{16} \quad \textit{Minimum} \end{cases}$$

Velocity is maximized where curvature is minimal.
Velocity is maximized where curvature is minimal.

d) $\vec{r}(t) = \langle 3\cos t, \ 4\sin t \rangle \quad // \quad \vec{N} = \left\langle -\frac{4\cos t}{\sqrt{9+7\cos^2 t}}, \ -\frac{3\sin t}{\sqrt{9+7\cos^2 t}} \right\rangle$

$$3\cos t = -\frac{4\cos t}{\sqrt{9+7\cos^2 t}} \cdot m \quad \to \quad \cos t = 0$$

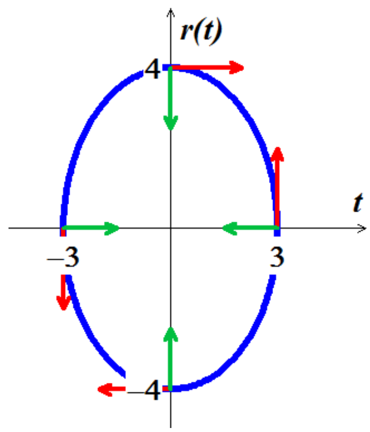
$$4\sin t = -\frac{3\sin t}{\sqrt{9+7\cos^2 t}} \cdot m \quad \to \quad \sin t = 0$$

$\cos t = 0 \quad \to \quad \vec{r}(t) = \langle 0, \ 4\sin t \rangle$

Points are: $(0, \ 4) \quad (0, \ -4)$

$\sin t = 0 \quad \to \quad \vec{r}(t) = \langle 3\cos t, \ 0 \rangle$

Points are: $(3, \ 0) \quad (-3, \ 0)$



Exercise

Find the following for all values of t for which the given curve is defined by

$$\vec{r}(t) = \langle 6\cos t, \ 3\sin t \rangle, \quad 0 \le t \le 2\pi$$

a) Find the tangent vector and the unit tangent vector
b) Find the curvature.
c) Find the principal unit normal vector.
d) Verify that $|\vec{N}| = 1 \quad \textit{and} \quad \vec{T} \bullet \vec{N} = 0$
e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

Solution

a) $\vec{v}(t) = \langle -6\sin t, \ 3\cos t \rangle \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$$\begin{aligned} |\vec{v}(t)| &= \sqrt{36\sin^2 t + 9\cos^2 t} \\ &= 3\sqrt{4\sin^2 t + \cos^2 t} \\ &= 3\sqrt{1 + 3\sin^2 t} \end{aligned}$$

$$\vec{T} = \frac{\langle -6\sin t,\ 3\cos t\rangle}{3\sqrt{1+3\sin^2 t}} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\boxed{= \frac{1}{\sqrt{1+3\sin^2 t}}\langle -2\sin t,\ \cos t\rangle}$$

b) $$r''(t)\times\vec{r}'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6\cos t & -3\sin t & 0 \\ -6\sin t & 3\cos t & 0 \end{vmatrix}$$

$$= \left\langle 0,\ 0,\ -18\cos^2 t - 18\sin^2 t \right\rangle$$

$$= \langle 0,\ 0,\ -18\rangle$$

$$\tau = \frac{|\langle 0,\ 0,\ -18\rangle|}{\left(3\sqrt{1+3\sin^2 t}\right)^3} \qquad \kappa = \frac{|r''(t)\times\vec{r}'(t)|}{\left(|r'(t)|\right)^3}$$

$$= \frac{18}{27\left(1+3\sin^2 t\right)^{3/2}}$$

$$\boxed{= \frac{2}{3\left(1+3\sin^2 t\right)^{3/2}}}$$

c) $$\frac{d\vec{T}}{dt} = \left\langle -2\frac{\cos t\left(1+3\sin^2 t\right) - 3\cos t\sin^2 t}{\left(1+3\sin^2 t\right)^{3/2}},\ \frac{-\sin t\left(1+3\sin^2 t\right) - 3\cos^2 t\sin t}{\left(1+3\sin^2 t\right)^{3/2}} \right\rangle$$

$$= \left\langle \frac{-2\cos t}{\left(1+3\sin^2 t\right)^{3/2}},\ -\frac{\sin t + 3\sin^3 t - 3\cos^2 t\sin t}{\left(1+3\sin^2 t\right)^{3/2}} \right\rangle$$

$$= \left\langle \frac{-2\cos t}{\left(1+3\sin^2 t\right)^{3/2}},\ -\sin t\frac{1+3\left(\sin^2 t + \cos^2 t\right)}{\left(1+3\sin^2 t\right)^{3/2}} \right\rangle$$

$$= \left\langle \frac{-2\cos t}{\left(1+3\sin^2 t\right)^{3/2}},\ \frac{-4\sin t}{\left(1+3\sin^2 t\right)^{3/2}} \right\rangle$$

$$= \frac{-2}{\left(1+3\sin^2 t\right)^{3/2}}\langle \cos t,\ 2\sin t\rangle$$

$$\left|\frac{d\vec{T}}{dt}\right| = \frac{2}{\left(1+3\sin^2 t\right)^{3/2}}\sqrt{\cos^2 t + 4\sin^2 t}$$

$$= \frac{2}{\left(1+3\sin^2 t\right)^{3/2}}\sqrt{1+3\sin^2 t}$$

$$= \frac{2}{\sqrt{1+3\sin^2 t}}$$

$$\vec{N} = \frac{\sqrt{1+3\sin^2 t}}{2} = \frac{-2}{\left(1+3\sin^2 t\right)^{3/2}}\langle \cos t,\ \ 2\sin t\rangle \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$\boxed{= \frac{1}{\sqrt{1+3\sin^2 t}}\langle -\cos t,\ -2\sin t\rangle}$$

d) $$\left|\vec{N}\right| = \frac{1}{\sqrt{1+3\sin^2 t}}\sqrt{\cos^2 t + 4\sin^2 t}$$

$$= \frac{1}{\sqrt{1+3\sin^2 t}}\sqrt{1+3\sin^2 t}$$

$$\boxed{=1} \quad \checkmark$$

$$\vec{T}\bullet\vec{N} = \frac{1}{\sqrt{1+3\sin^2 t}}\langle -2\sin t,\ \cos t\rangle \bullet \frac{1}{\sqrt{1+3\sin^2 t}}\langle -\cos t,\ -2\sin t\rangle$$

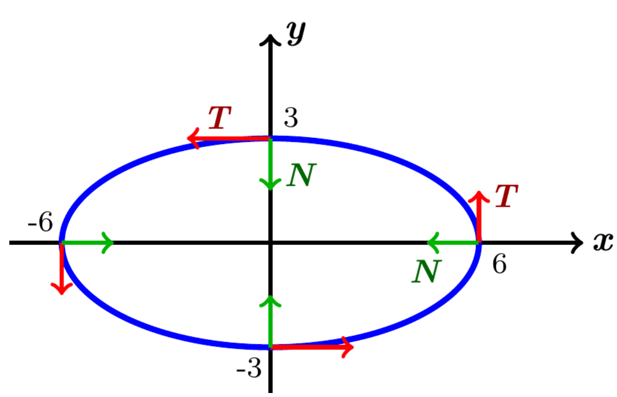
$$= \frac{2\sin t\cos t - 2\sin t\cos t}{1+3\sin^2 t}$$

$$\boxed{=0} \quad \checkmark$$

e) $x = 6\cos t \quad y = 3\sin t$

$\cos^2 t + \sin^2 t = 1$

$\frac{x^2}{36} + \frac{y^2}{9} = 1$



Exercise

Find the following for all values of t for which the given curve is defined by

$$\vec{r}(t) = (\cos t)\hat{i} + (2\sin t)\hat{j} + \hat{k}, \quad 0 \le t \le 2\pi$$

a) Find the tangent vector and the unit tangent vector

b) Find the curvature.

c) Find the principal unit normal vector.

d) Verify that $|\vec{N}| = 1 \quad and \quad \vec{T} \bullet \vec{N} = 0$

e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

Solution

a) $\vec{v}(t) = \langle -\sin t,\ 2\cos t,\ 0\rangle \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$$|\vec{v}(t)| = \sqrt{\sin^2 t + 4\cos^2 t}$$

$$= \sqrt{1 + 3\cos^2 t}$$

$$\boxed{\vec{T} = \frac{\langle -\sin t,\ 2\cos t,\ 0\rangle}{\sqrt{1+3\cos^2 t}}} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

b)

$$r''(t) \times \vec{r}'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\cos t & -2\sin t & 0 \\ -\sin t & 2\cos t & 0 \end{vmatrix}$$

$$= \left\langle 0,\ 0,\ -2\cos^2 t - 2\sin^2 t \right\rangle$$

$$= \langle 0,\ 0,\ -2\rangle$$

$$\kappa = \frac{|\langle 0,\ 0,\ -2\rangle|}{\left(\sqrt{1+3\cos^2 t}\right)^3} \qquad \kappa = \frac{|r''(t)\times\vec{r}'(t)|}{\left(|r'(t)|\right)^3}$$

$$\boxed{= \frac{2}{\left(1+3\cos^2 t\right)^{3/2}}}$$

c)

$$\frac{d\vec{T}}{dt} = \left\langle -\frac{\cos t\left(1+3\cos^2 t\right) + 3\cos t\sin^2 t}{\left(1+3\cos^2 t\right)^{3/2}},\ 2\frac{-\sin t\left(1+3\cos^2 t\right) + 3\cos^2 t\sin t}{\left(1+3\cos^2 t\right)^{3/2}},\ 0\right\rangle$$

$$= \frac{-1}{\left(1+3\cos^2 t\right)^{3/2}}\left\langle \cos t + 3\cos^3 t + 3\cos t\sin^2 t,\ \ 2\sin t,\ 0\right\rangle$$

$$= \frac{-2}{\left(1+3\cos^2 t\right)^{3/2}} \langle 2\cos t,\ \ \sin t,\ 0 \rangle$$

$$\left|\frac{d\vec{T}}{dt}\right| = \frac{2}{\left(1+3\cos^2 t\right)^{3/2}} \sqrt{4\cos^2 t + \sin^2 t}$$

$$= \frac{2}{\left(1+3\cos^2 t\right)^{3/2}} \sqrt{3\cos^2 t + 1}$$

$$= \frac{2}{\sqrt{1+3\cos^2 t}}$$

$$\vec{N} = \frac{\sqrt{1+3\cos^2 t}}{2} \cdot \frac{-2}{\left(1+3\cos^2 t\right)^{3/2}} \langle 2\cos t,\ \ \sin t,\ 0 \rangle \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= \frac{1}{\sqrt{1+3\cos^2 t}} \langle -2\cos t,\ -\sin t,\ 0 \rangle$$

d) $$\left|\vec{N}\right| = \frac{1}{\sqrt{1+3\cos^2 t}} \sqrt{4\cos^2 t + \sin^2 t}$$

$$= \frac{1}{\sqrt{1+3\cos^2 t}} \sqrt{1+3\cos^2 t}$$

$$= 1 \quad \checkmark$$

$$\vec{T} \bullet \vec{N} = \frac{\langle -\sin t,\ 2\cos t,\ 0 \rangle}{\sqrt{1+3\cos^2 t}} \bullet \frac{\langle -2\cos t,\ -\sin t,\ 0 \rangle}{\sqrt{1+3\cos^2 t}}$$

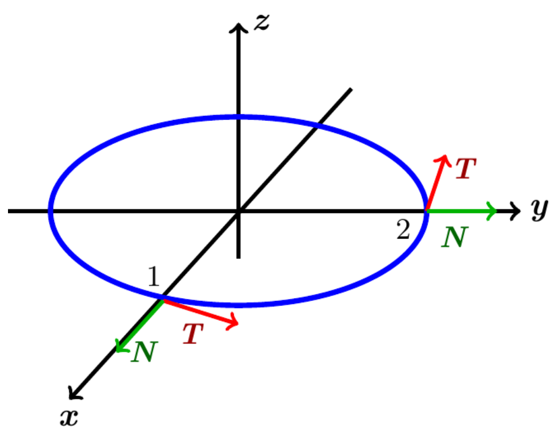
$$= \frac{2\sin t\cos t - 2\sin t\cos t}{1+3\cos^2 t}$$

$$= 0 \quad \checkmark$$

e) $x = \cos t \quad y = 2\sin t$

$\cos^2 t + \sin^2 t = 1$

$x^2 + \frac{y^2}{4} = 1$



Exercise

Find the following for all values of t for which the given curve is defined by

$$\vec{r}(t) = t\hat{i} + (2\cos t)\hat{j} + (2\sin t)\hat{k}, \quad 0 \le t \le 2\pi$$

a) Find the tangent vector and the unit tangent vector

b) Find the curvature.

c) Find the principal unit normal vector.

d) Verify that $\left|\vec{N}\right| = 1 \quad and \quad \vec{T} \bullet \vec{N} = 0$

e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

Solution

a) $\vec{v}(t) = \langle 1, \ -2\sin t, \ 2\cos t \rangle \qquad \vec{v}(t) = \dfrac{d\vec{r}}{dt}$

$$\left|\vec{v}(t)\right| = \sqrt{1 + 4\sin^2 t + 4\cos^2 t}$$

$$= \sqrt{5}$$

$$\boxed{\vec{T} = \frac{1}{\sqrt{5}}\langle 1, \ -2\sin t, \ 2\cos t \rangle} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

b) $r''(t) \times \vec{r}\,'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2\cos t & -2\sin t \\ 1 & -2\sin t & 2\cos t \end{vmatrix}$

$$= \langle -4, \ -2\sin t, \ 2\cos t \rangle$$

$$\tau = \frac{\left|\langle -4, \ -2\sin t, \ 2\cos t \rangle\right|}{\left(\sqrt{5}\right)^3} \qquad \kappa = \frac{\left|r''(t) \times \vec{r}\,'(t)\right|}{\left(\left|r'(t)\right|\right)^3}$$

$$= \frac{\sqrt{16 + 4\sin^2 t + 4\cos^2 t}}{5\sqrt{5}}$$

$$= \frac{\sqrt{20}}{5\sqrt{5}}$$

$$\boxed{= \frac{2}{5}}$$

c) $\dfrac{d\vec{T}}{dt} = \dfrac{1}{\sqrt{5}}\langle 0, \ -2\cos t, \ -2\sin t \rangle$

$$\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{\sqrt{5}}\sqrt{4\cos^2 t + 4\sin^2 t}$$

$$= \frac{2}{\sqrt{5}}$$

$$\vec{N} = \frac{\sqrt{5}}{2} \cdot \frac{1}{\sqrt{5}}\langle 0, \ -2\cos t, \ -2\sin t \rangle \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$\underline{=\langle 0,\ -\cos t,\ -\sin t\rangle}$$

d) $\left|\vec{N}\right| = \sqrt{\cos^2 t + \sin^2 t}$

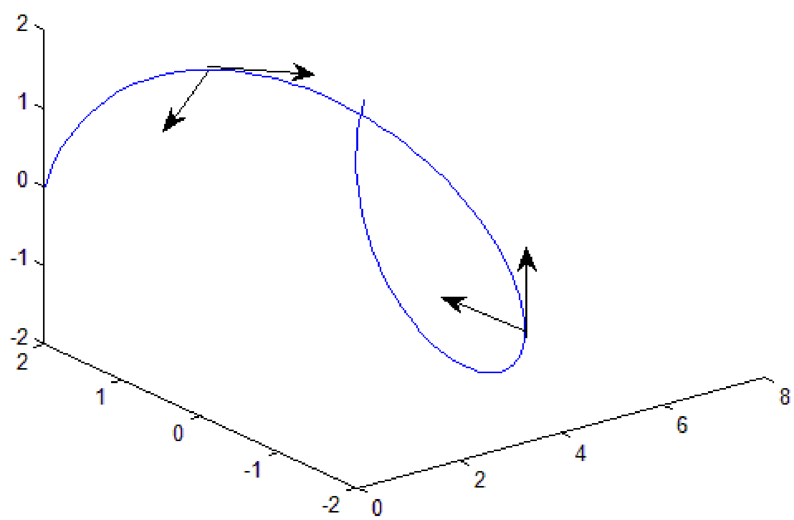
$$\underline{=1} \quad \checkmark$$

$$\vec{T}\bullet\vec{N} = \frac{1}{\sqrt{5}}\langle 1,\ -2\sin t,\ 2\cos t\rangle \bullet \langle 0,\ -\cos t,\ -\sin t\rangle$$

$$= \frac{1}{\sqrt{5}}\left(2\sin t\cos t - 2\sin t\cos t\right)$$

$$\underline{=0} \quad \checkmark$$

e)



Exercise

Find the following for all values of *t* for which the given curve is defined by

$\vec{r}(t) = (\cos t)\hat{i} + (2\cos t)\hat{j} + \left(\sqrt{5}\sin t\right)\hat{k}, \quad 0 \le t \le 2\pi$

a) Find the tangent vector and the unit tangent vector

b) Find the curvature.

c) Find the principal unit normal vector.

d) Verify that $\left|\vec{N}\right| = 1 \quad and \quad \vec{T}\bullet\vec{N} = 0$

e) Graph the curve and sketch $\vec{T}$ and $\vec{N}$ at two points.

Solution

a) $\vec{v}(t) = \left\langle -\sin t,\ -2\sin t,\ \sqrt{5}\cos t\right\rangle \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$$\left|\vec{v}(t)\right| = \sqrt{\sin^2 t + 4\sin^2 t + 5\cos^2 t}$$

$$= \sqrt{5\sin^2 t + 5\cos^2 t}$$

$$= \sqrt{5}$$

$$\boxed{\vec{T} = \frac{1}{\sqrt{5}}\left\langle -\sin t,\ -2\sin t,\ \sqrt{5}\cos t\right\rangle} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

b) $\vec{r}''(t) = \left\langle -\cos t,\ -2\cos t,\ -\sqrt{5}\sin t\right\rangle$

$$r''(t)\times \vec{r}'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\cos t & -2\cos t & -\sqrt{5}\sin t \\ -\sin t & -2\sin t & \sqrt{5}\cos t \end{vmatrix}$$

$$= \left(-2\sqrt{5}\cos^2 t - 2\sqrt{5}\sin^2 t\right)\hat{i} - \left(-\sqrt{5}\cos^2 t - \sqrt{5}\sin^2 t\right)\hat{j}$$

$$+ \left(2\sin t\cos t - 2\sin t\cos t\right)\hat{k}$$

$$= -2\sqrt{5}\hat{i} + \sqrt{5}\hat{j}$$

$$\left|r''(t)\times \vec{r}'(t)\right| = \sqrt{20+5}$$

$$\boxed{= 5}$$

$$\kappa = \frac{5}{\left(\sqrt{5}\right)^3} \qquad \kappa = \frac{\left|r''(t)\times \vec{r}'(t)\right|}{\left(\left|r'(t)\right|\right)^3}$$

$$\boxed{= \frac{1}{\sqrt{5}}}$$

c) $\dfrac{d\vec{T}}{dt} = \dfrac{1}{\sqrt{5}}\left\langle -\cos t,\ -2\cos t,\ -\sqrt{5}\sin t\right\rangle$

$$\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{\sqrt{5}}\sqrt{\cos^2 t + 4\cos^2 t + 5\sin^2 t}$$

$$= \frac{1}{\sqrt{5}}\sqrt{5\cos^2 t + 5\sin^2 t}$$

$$= \frac{\sqrt{5}}{\sqrt{5}}$$

$$\boxed{= 1}$$

$$\boxed{\vec{N} = \frac{1}{\sqrt{5}}\left\langle -\cos t,\ -2\cos t,\ -\sqrt{5}\sin t\right\rangle} \qquad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

d) $\left|\vec{N}\right| = \dfrac{1}{\sqrt{5}}\sqrt{\cos^2 t + 4\cos^2 t + 5\sin^2 t}$

$$= \frac{1}{\sqrt{5}}\sqrt{5}$$

$$\boxed{= 1} \quad \checkmark$$

$$\vec{T}\bullet\vec{N}=\frac{\left\langle -\sin t,\ -2\sin t,\ \sqrt{5}\cos t\right\rangle}{\sqrt{5}}\bullet\frac{\left\langle -\cos t,\ -2\cos t,\ -\sqrt{5}\sin t\right\rangle}{\sqrt{5}}$$

$$=\frac{\sin t\cos t+4\sin t\cos t-5\cos t\sin t}{5}$$

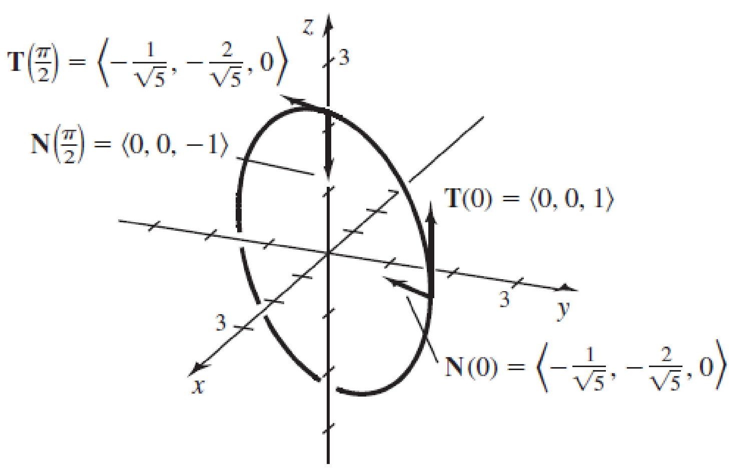
$$=0 \quad \checkmark$$

e) $\vec{r}(t)=(\cos t)\hat{i}+(2\cos t)\hat{j}+\left(\sqrt{5}\sin t\right)\hat{k}$

$$\vec{T}=\frac{1}{\sqrt{5}}\left\langle -\sin t,\ -2\sin t,\ \sqrt{5}\cos t\right\rangle$$

$$\vec{N}=\frac{1}{\sqrt{5}}\left\langle -\cos t,\ -2\cos t,\ -\sqrt{5}\sin t\right\rangle$$

t	$\vec{r}$	$\vec{T}$	$\vec{N}$
0	$(1,\ 1,\ 0)$	$(0,\ 0,\ 1)$	$\left(-\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}},\ 0\right)$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2},\ \sqrt{2},\ \sqrt{\frac{5}{2}}\right)$		
$\frac{\pi}{2}$	$\left(0,\ 0,\ \sqrt{5}\right)$	$\left(-\frac{1}{\sqrt{5}},\ -\frac{2}{\sqrt{5}},\ 0\right)$	$(0,\ 0,\ -1)$
π	$(-1,\ -1,\ 0)$		
$\frac{3\pi}{2}$	$\left(0,\ 0,\ -\sqrt{5}\right)$		
2π	$(1,\ 1,\ 0)$		



Exercise

Find equations for the osculating, normal and rectifying planes of the curve $\vec{r}(t)=t\hat{i}+t^2\hat{j}+t^3\hat{k}$ at the point (1, 1, 1).

Solution

$$\vec{v}(t) = \hat{i} + 2t\hat{j} + 3t^2\hat{k} \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\left|\vec{v}(t)\right| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\left|\vec{v}(1)\right| = \sqrt{1+4+9}$$

$$= \sqrt{14}$$

$$\vec{T}(t) = \frac{1}{\sqrt{14}}\left(\hat{i} + 2t\hat{j} + 3t^2\hat{k}\right) \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{T}(1) = \frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$$ (Normal to the normal plane).

$$\frac{1}{\sqrt{14}}(x-1) + \frac{2}{\sqrt{14}}(y-1) + \frac{3}{\sqrt{14}}(z-1) = 0$$

$$x - 1 + 2(y-1) + 3(z-1) = 0$$

$\boxed{x + 2y + 3z = 6}$ (equation of the normal plane).

$$\vec{a}(t) = 2\hat{j} + 6t\hat{k} \qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$\vec{a}(1) = 2\hat{j} + 6\hat{k}$$

$$(\vec{v} \times \vec{a})(1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix}$$

$$= 6\hat{i} - 6\hat{j} + 2\hat{k}$$

$$\left|\vec{v} \times \vec{a}\right| = \sqrt{36 + 36 + 4}$$

$$= \sqrt{76}$$

$$\kappa = \frac{\sqrt{76}}{\left(\sqrt{14}\right)^3} \qquad \kappa = \frac{|\vec{v} \times \vec{a}|}{\left(|\vec{v}|\right)^3}$$

$$= \frac{2\sqrt{19}}{14\sqrt{14}}$$

$$\boxed{= \frac{\sqrt{19}}{7\sqrt{14}}}$$

$$\frac{ds}{dt} = \left|\vec{v}(t)\right| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\frac{ds}{dt}(1) = \sqrt{14}$$

$$\frac{d^2s}{dt^2} = \left.\frac{4t + 18t^3}{\sqrt{1 + 4t^2 + 9t^4}}\right|_{t=1}$$

$$= \frac{22}{\sqrt{14}}$$

$$\vec{a} = \frac{d^2 s}{dt^2}\vec{T} + \kappa\left(\frac{ds}{dt}\right)^2 \vec{N}$$

$$2\hat{j} + 6\hat{k} = \frac{22}{\sqrt{14}}\left(\frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}\right) + \frac{\sqrt{19}}{7\sqrt{14}}\left(\sqrt{14}\right)^2 \vec{N}$$

$$2\hat{j} + 6\hat{k} = \frac{11}{7}\left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \frac{2\sqrt{19}}{\sqrt{14}}\vec{N}$$

$$\frac{2\sqrt{19}}{\sqrt{14}}\vec{N} = 2\hat{j} + 6\hat{k} - \frac{11}{7}\hat{i} - \frac{22}{7}\hat{j} - \frac{33}{7}\hat{k}$$

$$\frac{2\sqrt{19}}{\sqrt{14}}\vec{N} = -\frac{11}{7}\hat{i} - \frac{8}{7}\hat{j} + \frac{9}{7}\hat{k}$$

$$\boxed{\vec{N} = \frac{\sqrt{14}}{2\sqrt{19}}\left(-\frac{11}{7}\hat{i} - \frac{8}{7}\hat{j} + \frac{9}{7}\hat{k}\right)}$$

$$-\frac{11}{7}(x-1) - \frac{8}{7}(y-1) + \frac{9}{7}(z-1) = 0$$

$$-11x + 11 - 8y + 8 + 9z - 9 = 0$$

$$\boxed{11x + 8y - 9z = 10}$$

$$\vec{B}(1) = \vec{T}(1) \times \vec{N}(1)$$

$$= \frac{1}{\sqrt{14}} \cdot \frac{\sqrt{14}}{2\sqrt{19}} \cdot \frac{1}{7}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -11 & -8 & 9 \end{vmatrix}$$

$$= \frac{1}{14\sqrt{19}}\left(42\hat{i} - 42\hat{j} + 14\hat{k}\right)$$

$$\boxed{= \frac{1}{\sqrt{19}}\left(3\hat{i} - 3\hat{j} + \hat{k}\right)}$$

$$3(x-1) - 3(y-1) + (z-1) = 0$$

$$3x - 3 - 3y + 3 + z - 1 = 0$$

$$\boxed{3x - 3y + z = 1}$$

Exercise

Consider the position vector $\vec{r}(t) = \left(t^2 + 1\right)\hat{i} + (2t)\hat{j}, \quad t \ge 0$ of the moving objects

a) Find the normal and tangential components of the acceleration.

b) Graph the trajectory and sketch the normal and tangential components of the acceleration at two points on the trajectory. Show that their sum gives the total acceleration.

Solution

a) $\vec{v}(t) = 2t\hat{i} + 2\hat{j}$ $\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$$\left|\vec{v}(t)\right| = \sqrt{4t^2+4}$$

$$= 2\sqrt{t^2+1}$$

$$\vec{T}(t) = \frac{2t\hat{i}+2\hat{j}}{2\sqrt{t^2+1}} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{t}{\sqrt{t^2+1}}\hat{i} + \frac{1}{\sqrt{t^2+1}}\hat{j}$$

$$\frac{d\vec{T}}{dt} = \frac{t^2+1-t^2}{\left(t^2+1\right)^{3/2}}\hat{i} - \frac{t}{\left(t^2+1\right)^{3/2}}\hat{j}$$

$$= \frac{1}{\left(t^2+1\right)^{3/2}}\hat{i} - \frac{t}{\left(t^2+1\right)^{3/2}}\hat{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{1}{\left(t^2+1\right)^3} + \frac{t^2}{\left(t^2+1\right)^3}}$$

$$= \sqrt{\frac{1+t^2}{\left(t^2+1\right)^3}}$$

$$= \frac{1}{t^2+1}$$

$$\vec{N} = \left(t^2+1\right)\left(\frac{1}{\left(t^2+1\right)^{3/2}}\hat{i} - \frac{t}{\left(t^2+1\right)^{3/2}}\hat{j}\right)$$

$$= \frac{1}{\sqrt{t^2+1}}\hat{i} - \frac{t}{\sqrt{t^2+1}}\hat{j}$$

$$\vec{a}(t) = 2\hat{i} \qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$a_T = \frac{2t}{\sqrt{t^2+1}} \qquad a_T = \frac{d}{dt}|\vec{v}|$$

$$\vec{v}\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t & 2 & 0 \\ 2 & 0 & 0 \end{vmatrix}$$

$$= -4\hat{k}$$

$$\left|\vec{v}\times\vec{a}\right| = 4$$

$$\kappa = \frac{4}{8\left(\sqrt{t^2+1}\right)^3} \qquad \kappa = \frac{|\vec{v}\times\vec{a}|}{\left(|\vec{v}|\right)^3}$$

$$= \frac{1}{2\left(t^2+1\right)^{3/2}}$$

$$a_N = \frac{4}{2\sqrt{t^2+1}} \qquad a_N = \kappa|\vec{v}|^2 = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|}$$

$$= \frac{2}{\sqrt{t^2+1}}$$

$$\boxed{\vec{a} = \frac{2t}{\sqrt{t^2+1}}\vec{T} + \frac{2}{\sqrt{t^2+1}}\vec{N}}$$

b) $x = t^2+1 \quad y = 2t$

$t = \frac{1}{2}y \quad \rightarrow \quad \boxed{x = \frac{1}{4}y^2+1}$

At $t = 1$

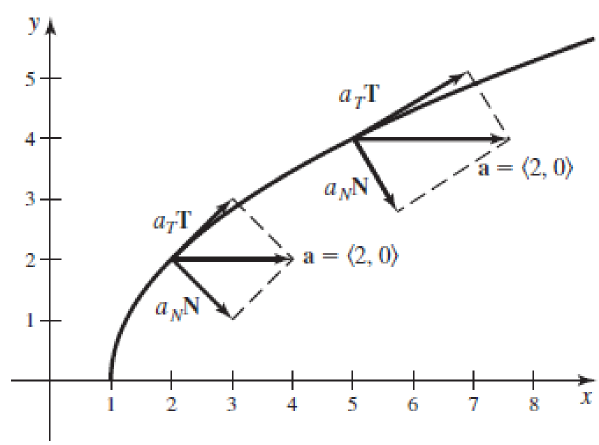
$$\vec{a} = \frac{2}{\sqrt{2}}\vec{T} + \frac{2}{\sqrt{2}}\vec{N}$$

$$= \frac{2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right) + \frac{2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}\right)$$

$$= \hat{i} + \hat{j} + \hat{i} - \hat{j}$$

$$\boxed{= 2\hat{i}}$$

$$= \langle 2,\ 0\rangle$$



At $t = 2$

$$\vec{a} = \frac{4}{\sqrt{5}}\left(\frac{2}{\sqrt{5}}\hat{i} + \frac{1}{\sqrt{5}}\hat{j}\right) + \frac{2}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}\right)$$

$$= \frac{8}{5}\hat{i} + \frac{4}{5}\hat{j} + \frac{2}{5}\hat{i} - \frac{4}{5}\hat{j}$$

$$\boxed{= 2\hat{i}} \quad = \langle 2,\ 0\rangle$$

At $t = 5$

$$\vec{a} = \frac{10}{\sqrt{26}}\left(\frac{5}{\sqrt{26}}\hat{i} + \frac{1}{\sqrt{26}}\hat{j}\right) + \frac{2}{\sqrt{26}}\left(\frac{1}{\sqrt{26}}\hat{i} - \frac{5}{\sqrt{26}}\hat{j}\right)$$

$$= \frac{50}{26}\hat{i} + \frac{10}{26}\hat{j} + \frac{2}{26}\hat{i} - \frac{10}{26}\hat{j}$$

$$= \frac{52}{26}\hat{i}$$

$$\boxed{= 2\hat{i}} \quad = \langle 2,\ 0\rangle$$

Exercise

Consider the position vector $\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j}, \quad 0 \le t \le 2\pi$ of the moving objects

c) Find the normal and tangential components of the acceleration.

d) Graph the trajectory and sketch the normal and tangential components of the acceleration at two points on the trajectory. Show that their sum gives the total acceleration.

Solution

a) $\vec{v}(t) = -2\sin t\,\hat{i} + 2\cos t\,\hat{j}$ $\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$\vec{a}(t) = -2\cos t\,\hat{i} - 2\sin t\,\hat{j}$ $\qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$

$$|\vec{v}| = \sqrt{4\cos^2 t + 4\sin^2 t}$$

$$= \sqrt{2}$$

$$a_T = \frac{d|\vec{v}|}{dt} = 0$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin t & 2\cos t & 0 \\ -2\cos t & -2\sin t & 0 \end{vmatrix}$$

$$= \left(4\sin^2 t + 4\cos^2 t\right)\hat{k}$$

$$= 4\hat{k}$$

$a_N = \frac{4}{2} = 2$ $\qquad a_N = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$

$$\vec{a} = 0\vec{T} + 2\vec{N}$$

b) $t = 0 \quad \rightarrow \quad \vec{a} = -2\,\hat{i}$

$$= 2\langle -1,\ 0\rangle$$

$$= 2\vec{N}$$

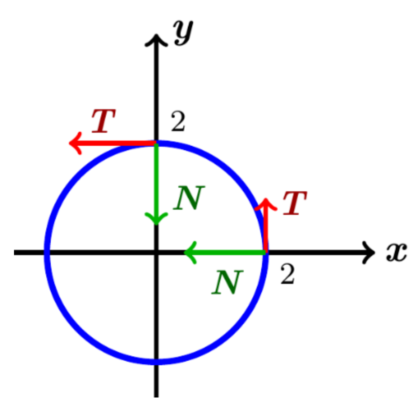
$t = \frac{\pi}{2} \quad \rightarrow \quad \vec{a} = \langle 0,\ -2\rangle$

$$= 2\langle 0,\ -1\rangle$$

$$= 2\vec{N}$$

$x = 2\cos t \quad y = 2\sin t$

$x^2 + y^2 = 4$



Exercise

Consider the position vector $\vec{r}(t) = 3t\,\hat{i} + (4-t)\,\hat{j} + t\,\hat{k}, \quad t \ge 0$ of the moving objects

Find the normal and tangential components of the acceleration.

Solution

a) $\vec{v}(t) = 3\hat{i} - \hat{j} + \hat{k}$ $\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$\vec{a}(t) = \vec{0}$ $\qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$

$$|\vec{v}| = \sqrt{9+1+1}$$

$$= \sqrt{11}$$

$$a_T = \frac{d|\vec{v}|}{dt} \underline{= 0}$$

$$a_N \underline{= 0}$$

$$\underline{\vec{a} = 0\vec{T} + 0\vec{N}}$$

Exercise

Consider the position vector $\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j} + (10t)\hat{k}, \quad 0 \le t \le 2\pi$ of the moving objects

a) Find the normal and tangential components of the acceleration.

b) Graph the trajectory and sketch the normal and tangential components of the acceleration at two points on the trajectory. Show that their sum gives the total acceleration.

Solution

a) $\vec{v}(t) = -2\sin t\,\hat{i} + 2\cos t\,\hat{j} + 10\hat{k}$ $\qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$

$\vec{a}(t) = -2\cos t\,\hat{i} - 2\sin t\,\hat{j}$ $\qquad \vec{a}(t) = \frac{d\vec{v}}{dt}$

$$|\vec{v}| = \sqrt{4\sin^2 t + 4\cos^2 t + 100}$$

$$= 2\sqrt{26}$$

$$a_T = \frac{d|\vec{v}|}{dt} \underline{= 0}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin t & 2\cos t & 10 \\ -2\cos t & -2\sin t & 0 \end{vmatrix}$$

$$= -20\sin t\,\hat{i} + 20\cos t\,\hat{j} - 4\,\hat{k}$$

$$a_N = \frac{\sqrt{400\sin^2 t + 400\cos^2 t + 16}}{2\sqrt{26}} \qquad a_N = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$$

$$= \frac{\sqrt{416}}{2\sqrt{26}}$$

$$= \frac{4\sqrt{26}}{2\sqrt{26}}$$

$$= 2 \rfloor$$

$$\vec{a} = 0\vec{T} + 2\vec{N} \rfloor$$

b) $t = 0 \quad \rightarrow \quad \vec{a} = \langle -2,\ 0,\ 0 \rangle$

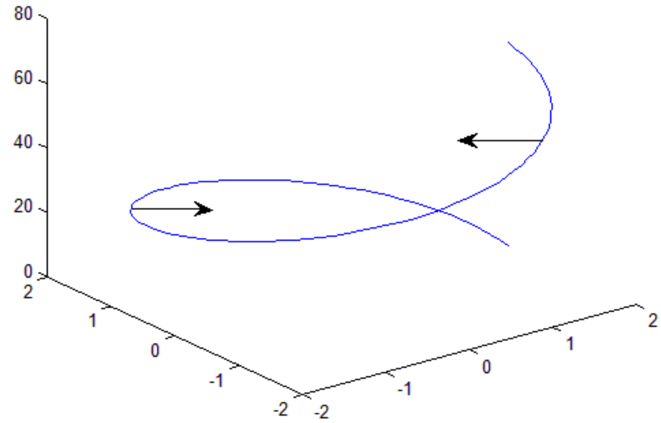
$$= 2\langle -1,\ 0,\ 0 \rangle$$

$$2\vec{N}$$

$t = \frac{\pi}{2} \quad \rightarrow \quad \vec{a} = \langle 0,\ -2,\ 0 \rangle$

$$= 2\langle 0,\ -1,\ 0 \rangle$$

$$2\vec{N}$$



Exercise

Compute the unit binormal vector ***B*** and the torsion of the curve $\vec{r}(t) = \left\langle t,\ t^2,\ t^3 \right\rangle$, $at\ t = 1$

Solution

$$\vec{v}(t) = \left\langle 1,\ 2t,\ 3t^2 \right\rangle \qquad \vec{v}(t) = \frac{d\vec{r}}{dt}$$

$$\vec{v}(t=1) = \langle 1,\ 2,\ 3 \rangle$$

$$|\vec{v}(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

$$|\vec{v}(t=1)| = \sqrt{1 + 4 + 9}$$

$$= \sqrt{14}$$

$$\vec{T} = \frac{1}{\sqrt{1 + 4t^2 + 9t^4}} \left\langle 1,\ 2t,\ 3t^2 \right\rangle \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{T}(t=1) = \frac{1}{\sqrt{14}} \langle 1,\ 2,\ 3 \rangle \rfloor$$

$$\frac{d\vec{T}}{dt} = \frac{1}{\left(1 + 4t^2 + 9t^4\right)^{3/2}} \left\langle -4t - 18t^3,\ 2 + 8t^2 + 18t^4 - 8t^2 - 36t^4,\ 3t\left(2 + 8t^2 + 18t^4 - 4t^2 - 18t^4\right) \right\rangle$$

$$= \frac{1}{\left(1 + 4t^2 + 9t^4\right)^{3/2}} \left\langle -4t - 18t^3,\ 2 - 18t^4,\ 6t + 12t^3 \right\rangle$$

$$= \frac{2}{\left(1+4t^2+9t^4\right)^{3/2}}\left\langle -2t-9t^3,\ 1-9t^4,\ 3t+6t^3 \right\rangle$$

$$\left|\frac{d\vec{T}}{dt}\right| = \frac{2}{\left(1+4t^2+9t^4\right)^{3/2}}\sqrt{\left(-2t-9t^3\right)^2+\left(1-9t^4\right)^2+\left(3t+6t^3\right)^2}$$

$$= \frac{2}{\left(1+4t^2+9t^4\right)^{3/2}}\sqrt{4t^2+36t^4+81t^6+1-18t^4+81t^8+9t^2+36t^4+36t^6}$$

$$= \frac{2}{\left(1+4t^2+9t^4\right)^{3/2}}\sqrt{1+13t^2+54t^4+117t^6+81t^8}$$

$$\vec{N} = \frac{\left(1+4t^2+9t^4\right)^{3/2}}{2\sqrt{1+13t^2+54t^4+117t^6+81t^8}}\frac{2}{\left(1+4t^2+9t^4\right)^{3/2}}\left\langle -2t-9t^3,\ 1-9t^4,\ 3t+6t^3 \right\rangle \quad \vec{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

$$= \frac{1}{\sqrt{1+13t^2+54t^4+117t^6+81t^8}}\left\langle -2t-9t^3,\ 1-9t^4,\ 3t+6t^3 \right\rangle$$

$$\vec{N}(t=1) = \frac{1}{\sqrt{1+13+54+117+81}}\langle -11,\ -8,\ 9\rangle$$

$$= \frac{1}{\sqrt{266}}\langle -11,\ -8,\ 9\rangle$$

$$\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ \frac{-11}{\sqrt{266}} & \frac{-8}{\sqrt{266}} & \frac{9}{\sqrt{266}} \end{vmatrix} \qquad \vec{B} = \vec{T}\times\vec{N}$$

$$= \left\langle \frac{42}{14\sqrt{19}},\ \frac{-42}{14\sqrt{19}},\ \frac{14}{14\sqrt{19}} \right\rangle$$

$$= \left\langle \frac{3}{\sqrt{19}},\ \frac{-3}{\sqrt{19}},\ \frac{1}{\sqrt{19}} \right\rangle$$

$$\vec{r}''(t) = \langle 0,\ 2,\ 6t\rangle$$

$$\vec{r}''\times\vec{r}' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 6t \\ 1 & 2t & 3t^2 \end{vmatrix}$$

$$= \left\langle -6t^2,\ 6t,\ -2 \right\rangle$$

$$\left|\vec{r}'' \times \vec{r}'\right| = \sqrt{36t^4 + 36t^2 + 4}\,\Big|_{t=1}$$

$$= \sqrt{76}$$

$$\vec{r}'''(t) = \langle 0,\ 0,\ 6\rangle$$

$$\tau = \frac{1}{76}\begin{vmatrix} 1 & 2t & 3t^2 \\ 0 & 2 & 6t \\ 0 & 0 & 6 \end{vmatrix} \qquad\qquad \tau = \frac{1}{\left|\vec{r}' \times \vec{r}''\right|^2}\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}$$

$$= \frac{12}{76}$$

$$\boxed{= \frac{3}{19}}$$

Exercise

At point P, the velocity and acceleration of a particle moving in the plane are $\vec{v} = 3\hat{i} + 4\hat{j}$ and $\vec{a} = 5\hat{i} + 15\hat{j}$. Find the curvature of the particle's path at P.

Solution

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ 5 & 15 & 0 \end{vmatrix}$$

$$\boxed{= 25\hat{k}}$$

$$\left|\vec{v} \times \vec{a}\right| = 25$$

$$\left|\vec{v}\right| = \sqrt{9+16}$$

$$\boxed{= 5}$$

$$\kappa = \frac{25}{5^3}$$

$$\boxed{= \frac{1}{5}}$$

Exercise

Consider the curve $C:\ \vec{r}(t) = \langle 3\sin t,\ 4\sin t,\ 5\cos t\rangle, \quad \mathit{for}\ \ 0 \le t \le 2\pi$

a) Find $\boldsymbol{T}(t)$ at all points of C.

b) Find $\boldsymbol{N}(t)$ and the curvature at all points of C.

c) Sketch the curve and show $\boldsymbol{T}(t)$ and $\boldsymbol{N}(t)$ at the points of C corresponding to $t = 0$ and $t = \frac{\pi}{2}$.

d) Are the results of parts (a) and (b) consistent with the graph?

e) Find $\boldsymbol{B}(t)$ at all points of C.

f) Describe three calculations that serve to check the accuracy of your results in part (a) – (*f*).

g) Compute the torsion at all points of C. Interpret this result.

Solution

a) $\vec{v}(t) = \langle 3\cos t,\ 4\cos t,\ -5\sin t\rangle \qquad \vec{v}(t) = \dfrac{d\vec{r}}{dt}$

$$\begin{aligned}|\vec{v}(t)| &= \sqrt{9\cos^2 t + 16\cos^2 t + 25\sin^2 t}\\ &= \sqrt{25\cos^2 t + 25\sin^2 t}\\ &= 5\end{aligned}$$

$$\boxed{\vec{T} = \frac{1}{5}\langle 3\cos t,\ 4\cos t,\ -5\sin t\rangle} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

b) $\dfrac{d\vec{T}}{dt} = \dfrac{1}{5}\langle -3\sin t,\ -4\sin t,\ -5\cos t\rangle$

$$\begin{aligned}\left|\frac{d\vec{T}}{dt}\right| &= \frac{1}{5}\sqrt{9\sin^2 t + 16\sin^2 t + 25\cos^2 t}\\ &= 1\end{aligned}$$

$$\boxed{\vec{N} = \frac{1}{5}\langle -3\sin t,\ -4\sin t,\ -5\cos t\rangle} \qquad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

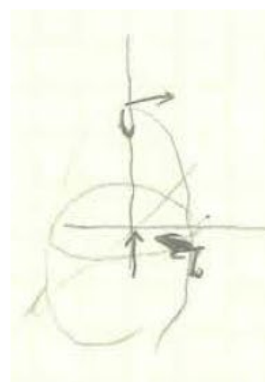
$$\boxed{\kappa = \frac{1}{25}\sqrt{9\sin^2 t + 16\sin^2 t + 25\cos^2 t}} \qquad \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

c) At $t = 0 \ \rightarrow \ \vec{T} = \left\langle \frac{3}{5},\ \frac{4}{5},\ 0\right\rangle \quad \vec{N} = \langle 0,\ 0,\ -1\rangle$

$t = \frac{\pi}{2} \ \rightarrow \ \vec{T} = \langle 0,\ 0,\ -1\rangle \quad \vec{N} = \left\langle -\frac{3}{5},\ -\frac{4}{5},\ 0\right\rangle$

$9\sin^2 t + 16\sin^2 t + 25\cos^2 t = 25$

$x^2 + y^2 + z^2 = 25$



d) Yes; the results of parts (a) and (b) consistent with the graph

e) $$\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{3}{5}\cos t & \frac{4}{5}\cos t & -\sin t \\ -\frac{3}{5}\sin t & -\frac{4}{5}\sin t & -\cos t \end{vmatrix} \qquad \vec{B} = \vec{T}\times\vec{N}$$

$$= \left\langle -\frac{4}{5}\cos^2 t - \frac{4}{5}\sin^2 t,\ \ \frac{3}{5}\sin^2 t + \frac{3}{5}\cos^2 t,\ \ 0\right\rangle$$

$$\boxed{= \left\langle -\frac{4}{5},\ \frac{3}{5},\ 0\right\rangle}$$

f) $|\vec{T}| = \frac{1}{5}\sqrt{9\cos^2 t + 16\cos^2 t + 25\sin^2 t}$

$$=\frac{1}{5}\sqrt{25\cos^2 t+25\sin^2 t}$$

$$=\frac{1}{5}\sqrt{25}$$

$$\underline{=1} \quad \checkmark$$

$$\left|\vec{N}\right|=\frac{1}{5}\sqrt{9\sin^2 t+16\sin^2 t+25\cos^2 t}$$

$$=\frac{1}{5}\sqrt{25\cos^2 t+25\sin^2 t}$$

$$\underline{=1} \quad \checkmark$$

$$\left|\vec{B}\right|=\sqrt{\frac{16}{25}+\frac{9}{25}}$$

$$=\sqrt{\frac{25}{25}}$$

$$\underline{=1} \quad \checkmark$$

$$\vec{T}\bullet\vec{N}=\frac{1}{5}\langle 3\cos t,\ 4\cos t,\ -5\sin t\rangle\bullet\frac{1}{5}\langle -3\sin t,\ -4\sin t,\ -5\cos t\rangle$$

$$=\frac{1}{25}\left(-9\cos t\sin t-16\cos t\sin t+25\cos t\sin t\right)$$

$$\underline{=0} \quad \checkmark$$

$$\vec{T}\bullet\vec{B}=\frac{1}{5}\langle 3\cos t,\ 4\cos t,\ -5\sin t\rangle\bullet\left\langle -\frac{4}{5},\ \frac{3}{5},\ 0\right\rangle$$

$$=\frac{1}{25}\left(-12\cos t+12\cos t\right)$$

$$\underline{=0} \quad \checkmark$$

$$\vec{B}\bullet\vec{N}=\left\langle -\frac{4}{5},\ \frac{3}{5},\ 0\right\rangle\bullet\frac{1}{5}\langle -3\sin t,\ -4\sin t,\ -5\cos t\rangle$$

$$=\frac{1}{25}\left(12\sin t-12\sin t+0\right)$$

$$\underline{=0} \quad \checkmark$$

g) Since $\vec{B}$ is constant, then $\underline{\tau=0}$

Exercise

Consider the curve $C:\ \vec{r}(t)=\langle 3\sin t,\ 3\cos t,\ 4t\rangle,\quad \textit{for}\ 0\le t\le 2\pi$

a) Find $\boldsymbol{T}(t)$ at all points of C.

b) Find $\boldsymbol{N}(t)$ and the curvature at all points of C.

c) Sketch the curve and show $\boldsymbol{T}(t)$ and $\boldsymbol{N}(t)$ at the points of C corresponding to $t=0$ and $t=\frac{\pi}{2}$.

d) Are the results of parts (a) and (b) consistent with the graph?

e) Find $\boldsymbol{B}(t)$ at all points of C.

f) Describe three calculations that serve to check the accuracy of your results in part (a) – (f).

g) Compute the torsion at all points of C. Interpret this result.

Solution

a) $\vec{v}(t) = \langle 3\cos t,\ -3\sin t,\ 4\rangle$ $\quad \vec{v}(t) = \dfrac{d\vec{r}}{dt}$

$$\begin{aligned} |\vec{v}(t)| &= \sqrt{9\cos^2 t + 9\sin^2 t + 16} \\ &= \sqrt{9+16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

$$\boxed{\vec{T} = \frac{1}{5}\langle 3\cos t,\ -3\sin t,\ 4\rangle} \qquad \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

b) $\dfrac{d\vec{T}}{dt} = \dfrac{1}{5}\langle -3\sin t,\ -3\cos t,\ 0\rangle$

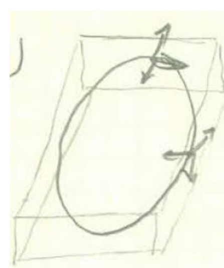
$$\begin{aligned} \left|\frac{d\vec{T}}{dt}\right| &= \frac{1}{5}\sqrt{9\sin^2 t + 9\cos^2 t} \\ &= \frac{3}{5} \end{aligned}$$

$$\boxed{\vec{N} = \langle -\sin t,\ -\cos t,\ 0\rangle} \qquad \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

$$\boxed{\kappa = \frac{3}{25}} \qquad \kappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

c) At $t = 0 \ \rightarrow\ \vec{T} = \left\langle \frac{3}{5},\ 0,\ \frac{4}{5}\right\rangle \quad \vec{N} = \langle 0,\ -1,\ .\rangle$

$t = \frac{\pi}{2} \ \rightarrow\ \vec{T} = \left\langle 0,\ -\frac{3}{5},\ \frac{4}{5}\right\rangle \quad \vec{N} = \langle -1,\ 0,\ 0\rangle$



d) Yes; the results of parts (a) and (b) consistent with the graph

e)
$$\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{3}{5}\cos t & -\frac{3}{5}\sin t & \frac{4}{5} \\ -\sin t & -\cos t & 0 \end{vmatrix} \qquad \vec{B} = \vec{T}\times\vec{N}$$

$$\boxed{= \left\langle \frac{4}{5}\cos t,\ -\frac{4}{5}\sin t,\ -\frac{3}{5}\right\rangle}$$

f)
$$\begin{aligned} |\vec{T}| &= \frac{1}{5}\sqrt{9\cos^2 t + 9\sin^2 t + 16} \\ &= \frac{1}{5}\sqrt{9+16} \\ &= \frac{1}{5}\sqrt{25} \end{aligned}$$

$$\underline{=1} \quad \checkmark$$

$$\left|\vec{N}\right| = \sqrt{\sin^2 t + \cos^2 t}$$

$$\underline{=1} \quad \checkmark$$

$$\left|\vec{B}\right| = \sqrt{\frac{16}{25}\cos^2 t + \frac{16}{25}\sin^2 t + \frac{9}{25}}$$

$$= \sqrt{\frac{16}{25} + \frac{9}{25}}$$

$$\underline{=1} \quad \checkmark$$

$$\vec{T} \cdot \vec{N} = \frac{1}{5}\langle 3\cos t,\ -3\sin t,\ 4\rangle \cdot \langle -\sin t,\ -\cos t,\ 0\rangle$$

$$= \frac{1}{5}\left(-3\cos t \sin t + 3\cos t \sin t + 0\right)$$

$$\underline{=0} \quad \checkmark$$

$$\vec{T} \cdot \vec{B} = \frac{1}{25}\langle 3\cos t,\ -3\sin t,\ 4\rangle \cdot \langle 4\cos t,\ -4\sin t,\ -3\rangle$$

$$= \frac{1}{25}\left(12\cos^2 t + 12\sin^2 t - 12\right)$$

$$= \frac{1}{25}(12 - 12)$$

$$\underline{=0} \quad \checkmark$$

$$\vec{B} \cdot \vec{N} = \frac{1}{5}\langle 4\cos t,\ -4\sin t,\ -3\rangle \cdot \langle -\sin t,\ -\cos t,\ 0\rangle$$

$$= \frac{1}{5}\left(-4\sin t\cos t + 4\sin t\cos t + 0\right)$$

$$\underline{=0} \quad \checkmark$$

g) $\dfrac{d\vec{B}}{dt} = \dfrac{1}{5}\langle -4\sin t,\ -4\cos t,\ 0\rangle$

$$\tau = \frac{1}{5}\langle -4\sin t,\ -4\cos t,\ 0\rangle \cdot \langle -\sin t,\ -\cos t,\ 0\rangle \qquad \tau = \frac{d\vec{B}}{dt} \cdot \vec{N}$$

$$= \frac{1}{5}\left(4\sin^2 t + 4\cos^2 t + 0\right)$$

$$\underline{= \frac{4}{5}}$$

Exercise

Suppose $\boldsymbol{r}(t)=\langle f(t),\ g(t),\ h(t)\rangle$, where *f, g*, and *h* are the quadratic functions $f(t)=a_1t^2+b_1t+c_1$, $g(t)=a_2t^2+b_2t+c_2$, and $h(t)=a_3t^2+b_3t+c_3$, and where at least one of the leading coefficients $a_1,\ a_2,\ or\ a_3$ is nonzero. Apart from a set of degenerate cases (for example $\boldsymbol{r}(t)=\langle t^2,\ t^2,\ t^2\rangle$, whose graph is a line), it can be shown that the graph of $\boldsymbol{r}(t)$ is a parabola that lies in a plane

a) Show by direct computation that $\boldsymbol{v}\times\boldsymbol{a}$ is constant. Then explain why the unit binormal vector is constant at all points on the curve. What does this result say about the torsion of the curve?

b) Compute $\boldsymbol{a}'(t)$ and explain why the torsion is zero at all points on the curve for which the torsion is defined.

Solution

a)
$$\vec{r}(t)=\left\langle a_1t^2+b_1t+c_1,\quad a_2t^2+b_2t+c_2,\quad a_3t^2+b_3t+c_3\right\rangle$$

$$\vec{v}(t)=\left\langle 2a_1t+b_1,\quad 2a_2t+b_2,\quad 2a_3t+b_3\right\rangle \qquad \vec{v}(t)=\frac{d\vec{r}}{dt}$$

$$\vec{a}(t)=\left\langle 2a_1,\quad 2a_2,\quad 2a_3\right\rangle \qquad \vec{a}(t)=\frac{d\vec{v}}{dt}$$

$$\vec{v}\times\vec{a}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2a_1t+b_1 & 2a_2t+b_2 & 2a_3t+b_3\\ 2a_1 & 2a_2 & 2a_3\end{vmatrix}$$

$$=2\left\langle 2a_2a_3t+a_3b_2-2a_2a_3t-a_2b_3,\ 2a_1a_3t+a_1b_3-2a_1a_3t-a_3b_1,\ 2a_1a_2t+a_2b_1-2a_1a_2t-a_1b_2\right\rangle$$

$$=2\left\langle a_3b_2-a_2b_3,\quad a_1b_3-a_3b_1,\quad a_2b_1-a_1b_2\right\rangle$$

$=$ ***Constant***

$$\left|\vec{v}\times\vec{a}\right|=2\sqrt{\left(a_3b_2-a_2b_3\right)^2+\left(a_1b_3-a_3b_1\right)^2+\left(a_2b_1-a_1b_2\right)^2}$$

$$\vec{B}=\frac{\vec{v}\times\vec{a}}{\left|\vec{v}\times\vec{a}\right|}=constant$$

$$\Rightarrow\quad \underline{\tau=0}$$

b)
$$\vec{a}'=\langle 0,\ 0,\ 0\rangle$$

$$\tau=\frac{(\vec{v}\times\vec{a})\times\vec{a}'}{\left|\vec{v}\times\vec{a}\right|^2}$$

$$\underline{=0}$$

Exercise

Let f and g be continuous on an interval I. consider the curve

$$C: \mathbf{r}(t) = \left\langle a_1 f(t) + a_2 g(t) + a_3,\ b_1 f(t) + b_2 g(t) + b_3,\ c_1 f(t) + c_2 g(t) + c_3 \right\rangle$$

For t in I, and where $a_i,\ b_i,\ and\ c_i$, for $i = 1,\ 2,\ and\ 3$, are real numbers

a) Show that, in general, C lies in a plane.

b) Explain why the torsion is zero at all points of C for which the torsion is defined.

Solution

a) $\vec{r}(t) = \left\langle a_1 f(t) + a_2 g(t) + a_3,\ b_1 f(t) + b_2 g(t) + b_3,\ c_1 f(t) + c_2 g(t) + c_3 \right\rangle$

$$\vec{r}(s) = \left\langle a_1 f(s) + a_2 g(s) + a_3,\ b_1 f(s) + b_2 g(s) + b_3,\ c_1 f(s) + c_2 g(s) + c_3 \right\rangle$$

$$\vec{r}(t) \times \vec{r}(s) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 f(t) + a_2 g(t) + a_3 & b_1 f(t) + b_2 g(t) + b_3 & c_1 f(t) + c_2 g(t) + c_3 \\ a_1 f(s) + a_2 g(s) + a_3 & b_1 f(s) + b_2 g(s) + b_3 & c_1 f(s) + c_2 g(s) + c_3 \end{vmatrix}$$

$$\begin{aligned} &= \Big[\big(b_1 f(t) + b_2 g(t) + b_3\big)\big(c_1 f(s) + c_2 g(s) + c_3\big) - \\ &\qquad \big(c_1 f(t) + c_2 g(t) + c_3\big)\big(b_1 f(s) + b_2 g(s) + b_3\big) \Big] \hat{\boldsymbol{i}} \\ &\quad + \Big[\big(c_1 f(t) + c_2 g(t) + c_3\big)\big(a_1 f(s) + a_2 g(s) + a_3\big) - \\ &\qquad \big(a_1 f(t) + a_2 g(t) + a_3\big)\big(c_1 f(s) + c_2 g(s) + c_3\big) \Big] \hat{\boldsymbol{j}} \\ &\quad + \Big[\big(a_1 f(t) + a_2 g(t) + a_3\big)\big(b_1 f(s) + b_2 g(s) + b_3\big) - \\ &\qquad \big(b_1 f(t) + b_2 g(t) + b_3\big)\big(a_1 f(s) + a_2 g(s) + a_3\big) \Big] \hat{\boldsymbol{k}} \end{aligned}$$

$$
\begin{aligned}
&= \Big[b_1c_1 f(t)f(s) + b_1c_2 f(t)g(s) + b_1c_3 f(t) + b_2c_1 g(t)f(s) \\
&\quad + b_2c_2 g(t)g(s) + b_2c_3 g(t) + b_3c_1 f(s) + b_3c_2 g(s) + b_3c_3 \\
&\quad - b_1c_1 f(t)f(s) - b_2c_1 f(t)g(s) - b_3c_1 f(t) - b_1c_2 g(t)f(s) \\
&\quad - b_2c_2 g(t)g(s) - b_3c_2 g(t) - b_1c_3 f(s) - b_2c_3 g(s) - b_3c_3 \Big] \hat{\boldsymbol{i}} \\
&+ \Big[a_1c_1 f(t)f(s) + a_2c_1 f(t)g(s) + a_3\,c_1 f(t) + a_1c_2 g(t)f(s) \\
&\quad + a_2c_2 g(t)g(s) + a_3c_2 g(t) + a_1c_3 f(s) + a_2c_3 g(s) + a_3c_3 \\
&\quad - a_1c_1 f(s)f(t) - a_2c_1 f(s)g(t) - a_3c_1 f(s) - a_1c_2 g(s)f(t) \\
&\quad - a_2c_2 g(s)g(t) - a_3c_2 g(s) - a_1c_3 f(t) - a_2c_3 g(t) - a_3c_3 \Big] \hat{\boldsymbol{j}} \\
&+ \Big[a_1b_1 f(s)f(t) + a_2b_1 f(s)g(t) + a_3b_1 f(s) + a_1b_2 g(s)f(t) \\
&\quad + a_2b_2 g(s)g(t) + a_3b_2 g(s) + a_1b_3 f(t) + a_2b_3 g(t) + a_3b_3 \\
&\quad - a_1b_1 f(t)f(s) - a_2b_1 f(t)g(s) - a_3b_1 f(t) - a_1b_2 g(t)f(s) \\
&\quad - a_2b_2 g(t)g(s) - a_3b_2 g(t) - a_1b_3 f(s) - a_2b_3 g(s) - a_3\,b_3 \Big] \hat{\boldsymbol{k}}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big[\left(b_1c_2 - b_2c_1\right) f(t)g(s) + \left(b_2c_1 - b_1c_2\right) f(s)g(t) + \left(b_1c_3 - b_3c_1\right) f(t) \\
&\qquad + \left(b_2c_3 - b_3c_2\right) g(t) + \left(b_3c_1 - b_1c_3\right) f(s) + \left(b_3c_2 - b_2c_3\right) g(s) \Big] \hat{\boldsymbol{i}} \\
&+ \Big[\left(a_2c_1 - a_1c_2\right) f(t)g(s) + \left(a_1c_2 - a_2c_1\right) f(s)g(t) + \left(a_3c_1 - a_1c_3\right) f(t) \\
&\qquad + \left(a_3c_2 - a_2c_3\right) g(t) + \left(a_1c_3 - a_3c_1\right) f(s) + \left(a_2c_3 - a_3c_2\right) g(s) \Big] \hat{\boldsymbol{j}} \\
&+ \Big[\left(a_2b_1 - a_1b_2\right) f(s)g(t) + \left(a_1b_2 - a_2b_1\right) f(t)g(s) + \left(a_1b_3 - a_3b_1\right) f(t) \\
&\qquad + \left(a_2b_3 - a_3b_2\right) g(t) \left(+a_3b_1 - a_1b_3\right) f(s) + \left(a_3b_2 - a_2b_3\right) g(s) \Big] \hat{\boldsymbol{k}}
\end{aligned}
$$

If $a_3 = b_3 = c_3 = 0$

$$
\begin{aligned}
\vec{r}(t) \times \vec{r}(s) &= \Big[\left(b_1c_2 - b_2c_1\right) f(t)g(s) + \left(b_2c_1 - b_1c_2\right) f(s)g(t) \Big] \hat{\boldsymbol{i}} \\
&\quad + \Big[\left(a_2c_1 - a_1c_2\right) f(t)g(s) + \left(a_1c_2 - a_2c_1\right) f(s)g(t) \Big] \hat{\boldsymbol{j}} \\
&\quad + \Big[\left(a_1b_2 - a_2b_1\right) f(t)g(s) + \left(a_2b_1 - a_1b_2\right) f(s)g(t) \Big] \hat{\boldsymbol{k}} \\
&= \Big[\left(b_1c_2 - b_2c_1\right)\left(f(t)g(s) - f(s)g(t)\right) \Big] \hat{\boldsymbol{i}} \\
&\quad + \Big[\left(a_2c_1 - a_1c_2\right)\left(f(t)g(s) - f(s)g(t)\right) \Big] \hat{\boldsymbol{j}} \\
&\quad + \Big[\left(a_1b_2 - a_2b_1\right)\left(f(t)g(s) - f(s)g(t)\right) \Big] \hat{\boldsymbol{k}} \\
&= \left(f(t)g(s) - f(s)g(t)\right) \left\langle b_1c_2 - b_2c_1,\ a_2c_1 - a_1c_2,\ a_1b_2 - a_2b_1 \right\rangle
\end{aligned}
$$

Which is orthogonal to the same vector, the vectors $\vec{r}(t)$ must all be in the same plane.

If a_3, b_3, & $c_3 \neq 0$

Consider $\vec{p}(t) = \vec{r}(t) - \langle a_3, b_3, c_3 \rangle$

$\vec{p}(t)$ is the form of $\vec{r}(t) \times \vec{r}(s)$ with $a_3 = b_3 = c_3 = 0$.

$\therefore$ $\vec{p}(t)$ lies in a plane where $\vec{r}(t) = \vec{p}(t) + \langle a_3, b_3, c_3 \rangle$ lies in a plane too.

b) If the curve lies in a plane, $\vec{B}$ is always normal to the plane with $|\vec{B}| = 1$.

Hence, $\vec{B}$ is constant, so $\tau = \dfrac{d\vec{B}}{dt} \bullet \vec{N}$ $\underline{= 0}$