SOLUTION *Section* 2.1 – Definitions and Examples

Exercise

Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous. $t^2 y'' = 4y' - \sin t$

Solution

$$y'' - \frac{4}{t^2} y' = -\frac{\sin t}{t^2}$$

$y'' + p(t)y' + q(t)y = g(t)$ It is linear and inhomogeneous

Exercise

Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous.

$$ty'' + (\sin t)y' = 4y - \cos 5t$$

Solution

$$y'' + \left(\frac{\sin t}{t}\right)y' - \frac{4}{t}y = -\frac{\cos 5t}{t}$$

$y'' + p(t)y' + q(t)y = g(t)$ It is linear and inhomogeneous $(g(t) \neq 0)$

Exercise $(g(t) \neq 0)$

Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous.

$$t^2 y'' + 4yy' = 0$$

Solution

It is nonlinear $(4yy')$

Exercise

Decide whether the equation is linear or nonlinear. For the linear equation, state whether the equation is homogenous or inhomogeneous.

$$y'' + 4y' + 7y = 3e^{-t}\sin t$$

Solution

Compare to $y'' + p(t)y' + q(t)y = g(t)$

$\Rightarrow p(t) = 4, \quad q(t) = 7, \quad g(t) = 3e^{-t}\sin t \qquad (g(t) \neq 0)$

Hence, the equation is linear and inhomogeneous.

Exercise

Show by direct substitution that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation. Then verify by direct substitution, that any linear combination $C_1y_1(t)+C_2y_2(t)$ of the 2 given solutions is also a solution.

$$y''+4y=0 \qquad y_1(t)=\cos 2t \qquad y_2(t)=\sin 2t$$

Solution

$$\begin{aligned} y &= C_1y_1+C_2y_2 \\ &= C_1\cos 2t+C_2\sin 2t \end{aligned}$$

$$y'=-2C_1\sin 2t+2C_2\cos 2t$$

$$y''=-4C_1\cos 2t-4C_2\sin 2t$$

If $C_1y_1(t)+C_2y_2(t)$, then

$$\begin{aligned} y''+4y &= -4C_1\cos 2t-4C_2\sin 2t+4\left(C_1\cos 2t+C_2\sin 2t\right) \\ &= -4C_1\cos 2t-4C_2\sin 2t+4C_1\cos 2t+4C_2\sin 2t \\ &= 0 \end{aligned}$$

Exercise

Show by direct substitution that the given functions $y_1(t)$ and $y_2(t)$ are solutions of the given differential equation. Then verify by direct substitution, that any linear combination $C_1y_1(t)+C_2y_2(t)$ of the 2 given solutions is also a solution.

$$y''-2y'+2y=0; \quad y_1(t)=e^t\cos t \quad y_2(t)=e^t\sin t$$

Solution

$$y_1(t)=e^t\cos t \quad \Rightarrow \quad y_1'(t)=e^t\cos t-e^t\sin t=e^t\left(\cos t-\sin t\right)$$

$$\begin{aligned} \Rightarrow \quad y_1''(t) &= e^t\left(\cos t-\sin t\right)+e^t\left(-\sin t-\cos t\right) \\ &= e^t\cos t-e^t\sin t-e^t\sin t-e^t\cos t \\ &= -2e^t\sin t \end{aligned}$$

$$\begin{aligned} y_1''-2y_1'+2y_1 &= -2e^t\sin t-2\left(e^t\cos t-e^t\sin t\right)+2e^t\cos t \\ &= -2e^t\sin t-2e^t\cos t+2e^t\sin t+2e^t\cos t \\ &= 0 \end{aligned}$$

$$y_2(t)=e^t\sin t \quad \Rightarrow \quad y_2'(t)=e^t\sin t+e^t\cos t$$

$$\begin{aligned} y_2''(t) &= e^t\sin t+e^t\cos t+e^t\cos t-e^t\sin t \\ &= 2e^t\cos t \end{aligned}$$

$$y_1'' - 2y_1' + 2y_1 = 2e^t \cos t - 2\left(e^t \cos t + e^t \sin t\right) + 2e^t \sin t$$

$$= 2e^t \cos t - 2e^t \cos t - 2e^t \sin t + 2e^t \sin t$$

$$\underline{= 0}$$

If $y(t) = C_1 e^t \cos t + C_2 e^t \sin t$

$$y'(t) = C_1 e^t \cos t - C_1 e^t \sin t + C_2 e^t \sin t + C_2 e^t \cos t$$

$$= \left(C_1 + C_2\right) e^t \cos t + \left(C_2 - C_1\right) e^t \sin t$$

$$y''(t) = \left(C_1 + C_2\right) e^t \cos t - \left(C_1 + C_2\right) e^t \sin t + \left(C_2 - C_1\right) e^t \sin t + \left(C_2 - C_1\right) e^t \cos t$$

$$= \left(C_1 + C_2 + C_2 - C_1\right) e^t \cos t + \left(C_2 - C_1 - C_1 - C_2\right) e^t \sin t$$

$$= 2C_2 e^t \cos t - 2C_1 e^t \sin t$$

$$y'' - 2y' + 2y = 2C_2 e^t \cos t - 2C_1 e^t \sin t - 2\left(\left(C_1 + C_2\right) e^t \cos t + \left(C_2 - C_1\right) e^t \sin t\right) + 2\left(C_1 e^t \cos t + C_2 e^t \sin t\right)$$

$$= 2C_2 e^t \cos t - 2C_1 e^t \sin t - 2C_1 e^t \cos t - 2C_2 e^t \cos t + 2C_2 e^t \sin t - 2C_1 e^t \sin t + 2C_1 e^t \cos t + 2C_2 e^t \sin t$$

$$\underline{= 0}$$

Exercise

Explain why $y_1(t)$ and $y_2(t)$ are linearly independent solutions. Calculate Wronskian and use it to explain the independence of the given solutions.

$$y'' + 9y = 0 \qquad y_1(t) = \cos 3t \qquad y_2(t) = \sin 3t$$

<u>Solution</u>

$$w(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix}$$

$$= 3\cos^2 3t + 3\sin^2 3t$$

$$= 3\left(\cos^2 3t + \sin^2 3t\right)$$

$$= 3 \neq 0$$

The solutions $y_1(t)\ \&\ y_2(t)$ are linearly independent.

Exercise

Show that $y_1(t) = e^t$ and $y_2(t) = e^{-3t}$ form a fundamental set of solutions for $y'' + 2y' - 3y = 0$, then find a solution satisfying $y(0) = 1$ and $y'(0) = -2$.

Solution

$$y_1(t) = e^t \Rightarrow y'' + 2y' - 3y = e^t + 2e^t - 3e^t = 0$$

$$y_2(t) = e^{-3t} \Rightarrow y'' + 2y' - 3y = 9e^{-3t} - 6e^{-3t} - 3e^{-3t} = 0$$

$$y(t) = C_1e^t + C_2e^{-3t} \qquad y(0) = C_1 + C_2 = 1$$

$$y'(t) = C_1e^t - 3C_2e^{-3t} \qquad y'(0) = C_1 - 3C_2 = -2$$

$$\Rightarrow C_1 = \tfrac{1}{4} \quad C_2 = \tfrac{3}{4}$$

$$y(t) = \tfrac{1}{4}e^t + \tfrac{3}{4}e^{-3t}$$

Exercise

Use the Wronskian to show that $\mathbf{f}_1 = 1, \ \mathbf{f}_2 = e^x, \ \mathbf{f}_3 = e^{2x}$ are linearly independence

Solution

The Wronskian is

$$W(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = e^x 4e^{2x} - 2e^{2x}e^x = 2e^{3x} \neq 0$$

Thus the functions are linearly independent.

Exercise

Determine whether $\{e^x, \ xe^x, \ (x+1)e^x\}$ is a set of linearly independent.

Solution

$$W = \begin{vmatrix} e^x & xe^x & (x+1)e^x \\ e^x & (x+1)e^x & (x+2)e^x \\ e^x & (x+2)e^x & (x+3)e^x \end{vmatrix}$$

$$= (x+1)(x+3)e^{3x} + x(x+2)e^{3x} + (x+1)(x+2)e^{3x} - (x+1)^2e^{3x} - (x+2)^2e^{3x} - x(x+3)e^{3x}$$

$$= \left(x^2 + 4x + 3 + x^2 + 2x + x^2 + 3x + 2 - x^2 - 2x - 1 - x^2 - 4x - 4 - x^2 - 3x\right)e^{3x}$$

$= 0$

Thus the set $\left\{e^x,\ xe^x,\ (x+1)e^x\right\}$ is linearly dependent.

Exercise

Show that the functions $y_1(x) = e^{-3x}, \quad y_2(x) = \cos 2x, \quad y_3(x) = \sin 2x$ are linearly independent.

Solution

$$W = \begin{vmatrix} e^{-3x} & \cos 2x & \sin 2x \\ -3e^{-3x} & -2\sin 2x & 2\cos 2x \\ 9e^{-3x} & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$= 8e^{-3x}\sin^2 2x + 18e^{-3x}\cos^2 2x + 12e^{-3x}\sin 2x\cos 2x$$

$$+ 18e^{-3x}\sin^2 2x + 8e^{-3x}\cos^2 2x - 12e^{-3x}\sin 2x\cos 2x$$

$$= 26e^{-3x} \neq 0$$

$\therefore\ y_1, y_2, \text{ and } y_3$ are linearly independent.

Solution Section 2.2 - Second-Order Equations and Systems

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation.

$$y'' + 2y' - 3y = 0$$

Solution

Let $v = y' \quad \Rightarrow \quad v' = y''$

$y'' = -2y' + 3y$

$v' = -2v + 3y$

The following system of the first-order equations: $\begin{cases} y' = v \\ v' = -2v + 3y \end{cases}$

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation.

$$y'' + 3y' + 4y = 2\cos 2t$$

Solution

Let $v = y' \quad \Rightarrow \quad v' = y''$

$y'' = -3y' - 4y + 2\cos 2t$

$v' = -3v - 4y + 2\cos 2t$

The following system of the first-order equations: $\begin{cases} y' = v \\ v' = -3v - 4y + 2\cos 2t \end{cases}$

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation.

$$y'' + 2y' + 2y = 2\sin 2\pi t$$

Solution

Let $v = y' \quad \Rightarrow \quad v' = y''$

$y'' = -2y' - 2y + 2\sin 2\pi t$

$v' = -2v - 2y + 2\sin 2\pi t$

The following system of the first-order equations: $\begin{cases} y' = v \\ v' = -2v - 2y + 2\sin 2\pi t \end{cases}$

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation.

$$y'' + \mu\left(t^2 - 1\right)y' + y = 0$$

Solution

Let $v = y' \quad \Rightarrow \quad v' = y''$

$$y'' = -\mu\left(t^2 - 1\right)y' - y$$

$$v' = -\mu\left(t^2 - 1\right)v - y$$

The following system of the first-order equations:

$$\begin{cases} y' = v \\ v' = -\mu\left(t^2 - 1\right)v - y \end{cases}$$

Exercise

Use the substitution $v = y'$ to write each second-order equation as a system of two first-order differential equation.

$$4y'' + 4y' + y = 0$$

Solution

Let $v = y' \quad \Rightarrow \quad v' = y''$

$$4y'' = -4y' - y$$

$$y'' = -y' - \frac{1}{4}y$$

$$v' = -v - \frac{1}{4}y$$

The following system of the first-order equations: $\begin{cases} y' = v \\ v' = -v - \frac{1}{4}y \end{cases}$

Exercise

Given the mass, damping, and spring constants of an undriven spring-mass system $my'' + \mu y' + ky = 0$

$$m = 1\ kg, \quad \mu = 0\ kg/s, \quad k = 4kg/s^2, \quad y(0) = -2\ m, \quad y'(0) = -2\ m/s$$

a) Provide separate plots of the position versus time (y vs. t) and the velocity versus time (v vs. t)

b) Provide a combined plot of both position and velocity versus time

c) Provide a plot of the velocity versus position (v vs. y) in the yv phase plane.

Solution

$$my'' = -\mu y' - ky$$

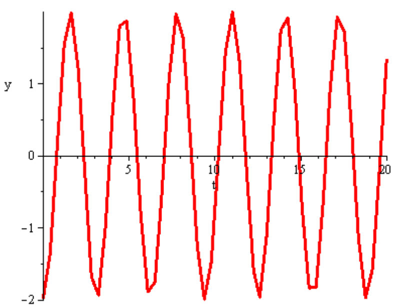
$$y'' = -\frac{\mu}{m} y' - \frac{k}{m} y$$

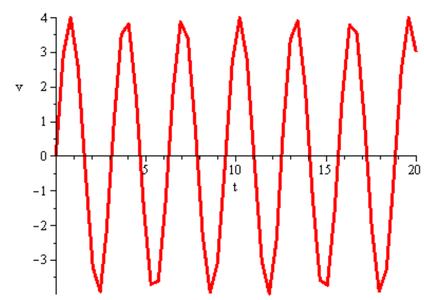
Let $\boxed{v = y'}$ $\Rightarrow v' = -\frac{\mu}{m} v - \frac{k}{m} y$

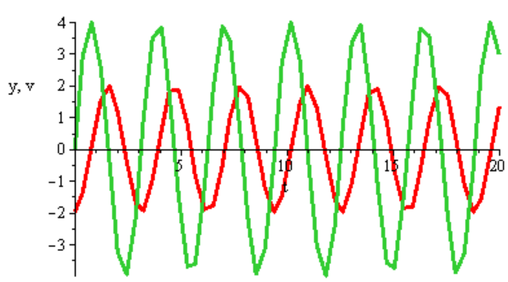
$$= -\frac{0}{1} v - \frac{4}{1} y$$

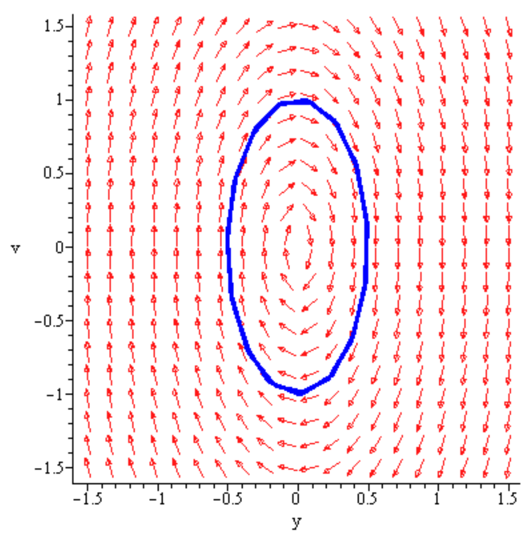
$$\boxed{v' = -4y}$$

$$y(0) = -2, \quad y'(0) = -2 = v(0)$$









Exercise

Given the mass, damping, and spring constants of an undriven spring-mass system $my'' + \mu y' + ky = 0$

$$m = 1\ kg, \quad \mu = 2\ kg/s, \quad k = 1 kg/s^2, \quad y(0) = -3\ m, \quad y'(0) = -2\ m/s$$

a) Provide separate plots of the position versus time (y vs. t) and the velocity versus time (v vs. t)

b) Provide a combined plot of both position and velocity versus time

c) Provide a plot of the velocity versus position (v vs. y) in the yv phase plane.

Solution

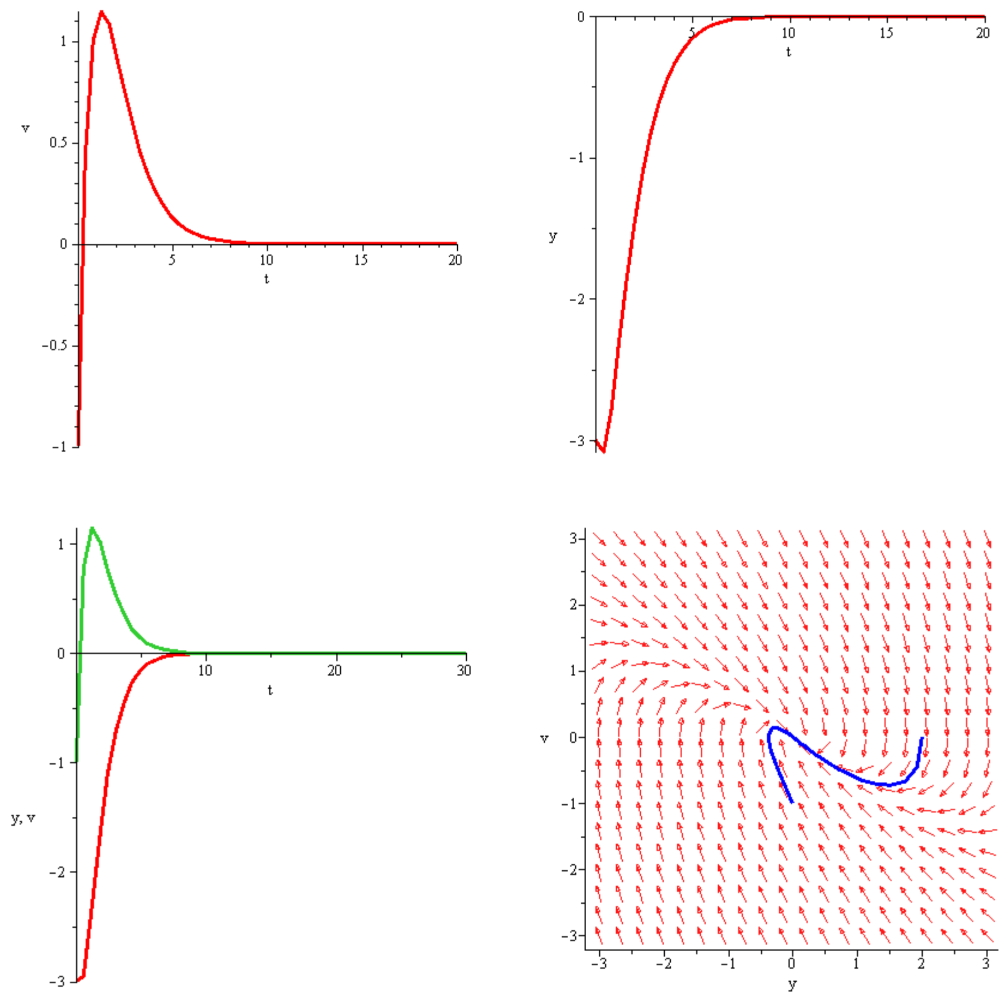
$$my'' = -\mu y' - ky \quad \Rightarrow \quad y'' = -\frac{\mu}{m}y' - \frac{k}{m}y$$

Let $v = y'$ $\Rightarrow v' = -\frac{\mu}{m}v - \frac{k}{m}y$

$$= -\frac{2}{1}v - \frac{1}{1}y$$

$$= -2v - y$$

The following system of the first-order equations: $\begin{cases} y' = v \\ v' = -2v - y \end{cases}$ *with* $\begin{array}{r} y(0) = -3 \\ y'(0) = -2 = v(0) \end{array}$



Solution   *Section* 2.3 - Linear, Homogeneous Equations with Constant Coefficients

Exercise

Find the general solution: $y'' - y' - 2y = 0$

Solution

The characteristic equation: $\lambda^2 - \lambda - 2 = 0$

$\Rightarrow \lambda_1 = 2;\ \lambda_2 = -1$

$$\boxed{y(t) = C_1 e^{2t} + C_2 e^{-t}}$$

Exercise

Find the general solution: $2y'' - 3y' - 2y = 0$

Solution

The characteristic equation: $2\lambda^2 - 3\lambda - 2 = 0$

$\Rightarrow \lambda_1 = 2;\ \lambda_2 = -\frac{1}{2}$

$$\boxed{y(t) = C_1 e^{2t} + C_2 e^{-t/2}}$$

Exercise

Find the general solution: $y'' + 5y' + 6y = 0$

Solution

The characteristic equation: $\lambda^2 + 5\lambda + 6 = 0$

$\Rightarrow \lambda_1 = -3;\ \lambda_2 = -2$

$$\boxed{y(t) = C_1 e^{-3t} + C_2 e^{-2t}}$$

Exercise

Find the general solution: $y'' + 2y' + 17y = 0$

Solution

The characteristic equation: $\lambda^2 + 2\lambda + 17 = 0 \qquad \Rightarrow \lambda_{1,2} = -1 \pm 4i$

$$y(t) = e^{-t}\left(C_1 \cos 4t + C_2 \sin 4t\right)$$

Exercise

Find the general solution: $y'' - 4y' + 4y = 0$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 4 = 0 \quad \Rightarrow \lambda_{1,2} = 2$

$$\boxed{y(t) = \left(C_1 + C_2 t\right)e^{2t}}$$

Exercise

Find the general solution: $y'' - 6y' + 9y = 0$

Solution

The characteristic equation: $\lambda^2 - 6\lambda + 9 = 0 \quad \Rightarrow \lambda_{1,2} = 3$

$$\boxed{y(t) = \left(C_1 + C_2 t\right)e^{3t}}$$

Exercise

Find the general solution: $6y'' + 5y' - 6y = 0$

Solution

The characteristic equation: $6\lambda^2 + 5\lambda - 6 = 0 \quad \Rightarrow \quad \lambda_{1,2} = -3,\ \frac{4}{3}$

$$\boxed{y(t) = C_1 e^{-3t} + C_2 e^{4t/3}}$$

Exercise

Find the general solution: $2y'' - 5y' - 3y = 0$

Solution

The characteristic equation: $2\lambda^2 - 5\lambda - 3 = 0 \quad \Rightarrow \quad \lambda_1 = -\frac{1}{2},\quad \lambda_2 = 3$

$$\boxed{y(t) = C_1 e^{-t/2} + C_2 e^{3t}}$$

Exercise

Find the general solution: $y'' - 10y' + 25y = 0$

Solution

The characteristic equation: $\lambda^2 - 10\lambda + 25 = (\lambda - 5)^2 = 0 \quad \Rightarrow \quad \lambda_{1,\,2} = 5$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{5x}}$$

Exercise

Find the general solution: $y'' + y' + y = 0$

Solution

The characteristic equation: $\lambda^2 + \lambda + 1 = 0 \Rightarrow \lambda_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$$\boxed{y(x) = e^{-x/2}\left(C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right)}$$

Exercise

Find the general solution: $3y'' - y' = 0$

Solution

The characteristic equation: $\lambda^2 + \lambda = \lambda(\lambda + 1) = 0 \Rightarrow \lambda_1 = 0, \quad \lambda_2 = -1$

$$\boxed{y(x) = C_1 + C_2 e^{-x}}$$

Exercise

Find the general solution: $2y'' + 5y' = 0$

Solution

The characteristic equation: $2\lambda^2 + 5\lambda = \lambda(2\lambda + 5) = 0 \Rightarrow \lambda_1 = 0, \quad \lambda_2 = -\frac{5}{2}$

$$\boxed{y(x) = C_1 + C_2 e^{-5x/2}}$$

Exercise

Find the general solution: $y'' - 3y' + 2y = 0$

Solution

The characteristic equation: $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda_1 = 1, \quad \lambda_2 = 2$

$$\boxed{y(x) = C_1 e^{x} + C_2 e^{2x}}$$

Exercise

Find the general solution: $y'' - y' - 6y = 0$

Solution

The characteristic equation: $\lambda^2 - \lambda - 6 = 0 \Rightarrow \lambda_1 = -2, \quad \lambda_2 = 3$

$$\boxed{y(x) = C_1 e^{-2x} + C_2 e^{3x}}$$

Exercise

Find the general solution: $3y'' + 2y' + y = 0$

Solution

The characteristic equation: $3\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{-2 \pm \sqrt{4-12}}{6} = -\frac{1}{3} \pm i\frac{\sqrt{2}}{3}$

$$\boxed{y(x) = e^{-x/3}\left(C_1 \cos\frac{\sqrt{2}}{3}x + C_2 \sin\frac{\sqrt{2}}{3}x\right)}$$

Exercise

Find the general solution: $2y'' + 2y' + y = 0$

Solution

The characteristic equation: $2\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{-2 \pm \sqrt{4-8}}{6} = -\frac{1}{3} \pm \frac{1}{3}i$

$$\boxed{y(x) = e^{-x/3}\left(C_1 \cos\frac{1}{3}x + C_2 \sin\frac{1}{3}x\right)}$$

Exercise

Find the general solution: $y'' + 14y' + 49y = 0$

Solution

The characteristic equation: $\lambda^2 + 14\lambda + 49 = (\lambda + 7)^2 = 0 \quad \Rightarrow \quad \lambda_{1,2} = -7$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{-7x}}$$

Exercise

Find the general solution of the given higher-order differential equation: $y''' + 3y'' + 3y' + y = 0$

Solution

$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 3)^3 = 0 \quad \Rightarrow \quad \lambda_{1,2,3} = -3$

$$\boxed{y(x) = \left(C_1 + C_2 x + C_3 x^2\right)e^{-3x}}$$

Exercise

Find the general solution of the given higher-order differential equation: $3y''' - 19y'' + 36y' - 10y = 0$

Solution

$3\lambda^3 - 19\lambda^2 + 36\lambda - 10 = 0$ ***Solve for λ***

$\lambda_1 = \frac{1}{3}, \quad \lambda_{2,3} = 3 \pm i$

$$\boxed{y(x) = C_1 e^{x/3} + e^{3x}\left(C_2 \cos x + C_3 \sin x\right)}$$

Rational Zero Theorem: $\pm\left\{\frac{10}{3}\right\} = \pm\left\{1, 2, 5, 10, \frac{1}{3}, \frac{2}{3}, \frac{5}{3}, \frac{10}{3}\right\}$

$$\lambda_1 = \frac{1}{3} \qquad \begin{array}{c|cccc} \frac{1}{3} & 3 & -19 & 36 & -10 \\ & & 1 & -6 & 10 \\ \hline & 3 & -18 & 30 & 0 \end{array}$$

$3\lambda^2 - 18\lambda + 30 = 0$

$\lambda_{1,2} = \frac{18 \pm \sqrt{324 - 360}}{6} = \frac{18 \pm i6}{6} = 3 \pm i$

Exercise

Find the general solution of the given higher-order differential equation: $y''' - 6y'' + 12y' - 8y = 0$

Solution

$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = (\lambda - 2)^3 = 0 \quad \Rightarrow \quad \lambda_{1,2,3} = 2$

$$\boxed{y(x) = \left(C_1 + C_2 x + C_3 x^2\right) e^{2x}}$$

Exercise

Find the general solution of the given higher–ODE: $y''' + 5y'' + 7y' + 3y = 0$

Solution

The characteristic equation:

$\lambda^3 + 5\lambda^2 + 7\lambda + 3 = 0 \quad \Rightarrow \boxed{\lambda_1 = -3}$ (*Rational Zero Theorem*)

$$\begin{array}{c|cccc} -3 & 1 & 5 & 7 & 3 \\ & & -3 & -6 & -3 \\ \hline & 1 & 2 & 1 & 0 \end{array} \quad \lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \lambda_{2,3} = -1$$

The general solution is: $\boxed{y(x) = C_1 e^{-3x} + C_2 e^{-x} + C_3 x e^{-x}}$

Exercise

Find the general solution of the given higher–ODE: $y^{(3)} + y' - 10y = 0$

Solution

The characteristic equation:

$$\lambda^3 + \lambda - 10 = 0 \quad \Rightarrow \boxed{\lambda_1 = 2} \quad (\textit{Rational Zero Theorem})$$

$$\begin{array}{r|rrrr} 2 & 1 & 0 & 1 & -10 \\ & & 2 & 4 & 10 \\ \hline & 1 & 2 & 5 & \boxed{0} \end{array} \quad \lambda^2 + 2\lambda + 5 = 0 \quad \Rightarrow \boxed{\lambda_{2,3} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i}$$

The general solution is: $\boxed{y(x) = C_1 e^{2x} + e^{-x}\left(C_2 \cos 2x + C_3 \sin 2x\right)}$

Exercise

Find the general solution of the given higher-order differential equation: $y^{(4)} + 2y'' + y = 0$

Solution

$$\lambda^4 + 2\lambda^2 + 1 = \left(\lambda^2 + 1\right)^2 = 0$$

$$\lambda^2 = -1 \quad \Rightarrow \quad \lambda = \pm i \quad \Rightarrow \lambda_{1,2} = -i,\ \lambda_{3,4} = i$$

$$\boxed{y(x) = \left(C_1 + C_2 x\right)e^{-ix} + \left(C_3 + C_4 x\right)e^{ix}}$$

OR $\boxed{y(x) = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x}$

Exercise

Find the general solution of the given higher-order differential equation: $y^{(4)} + y''' + y'' = 0$

Solution

$$\lambda^4 + \lambda^3 + \lambda^2 = \lambda^2\left(\lambda^2 + \lambda + 1\right) = 0$$

$$\lambda^2 = 0 \to \lambda_{1,2} = 0 \quad \Rightarrow \quad \lambda = \frac{-1 \pm \sqrt{1-4}}{2} \quad \Rightarrow \lambda_{3,4} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\boxed{y(x) = C_1 + C_2 x + e^{-x/2}\left(C_3 \cos\left(\frac{\sqrt{3}}{2}x\right) + C_4 \sin\left(\frac{\sqrt{3}}{2}x\right)\right)}$$

Exercise

Find the general solution of the given higher–ODE: $y^{(4)} + 4y = 0$

Solution

The characteristic equation: $\lambda^4 + 4 = 0 \quad \Rightarrow \lambda^2 = \pm 2i \quad \to \lambda_{1,2,3,4} = \pm\sqrt{\pm 2i}$

Since $i = e^{\frac{\pi}{2}i} \quad -i = e^{\frac{3\pi}{2}i}$

$$\sqrt{2i} = \left(2e^{i\pi/2}\right)^{1/2} = \sqrt{2}e^{i\pi/4} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1 + i$$

$$\sqrt{-2i} = \left(2e^{i3\pi/2}\right)^{1/2} = \sqrt{2}e^{i3\pi/4} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -1 + i$$

$$\lambda = \pm(\pm 1 + i) = \begin{cases} 1 \pm i \\ -1 \pm i \end{cases}$$

The general solution is: $\boxed{y(x) = e^{x}\left(C_1 \cos x + C_2 \sin x\right) + e^{-x}\left(C_3 \cos x + C_4 \sin x\right)}$

Exercise

Find the general solution of the given higher–ODE: $y^{(4)} + 2y''' + 9y'' - 2y' - 10y = 0$

Solution

The characteristic equation:

$\lambda^4 + 2\lambda^3 + 9\lambda^2 - 2\lambda - 10 = 0 \quad \Rightarrow \boxed{\lambda_1 = 1}$ (*Rational Zero Theorem*)

$$\begin{array}{r|rrrrr} 1 & 1 & 2 & 9 & -2 & -10 \\ & & 1 & 3 & 12 & 10 \\ \hline & 1 & 3 & 12 & 10 & \boxed{0} \end{array} \quad \lambda^3 + 3\lambda^2 + 12\lambda + 10 = 0 \quad \Rightarrow \boxed{\lambda_2 = -1}$$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 12 & 10 \\ & & -1 & -2 & -10 \\ \hline & 1 & 2 & 10 & \boxed{0} \end{array} \quad \lambda^2 + 2\lambda + 10 = 0 \quad \Rightarrow \boxed{\lambda_{3,4} = -1 \pm 3i}$$

The general solution is: $\boxed{y(x) = C_1 e^{x} + C_2 e^{-x} + e^{-x}\left(C_3 \cos 3x + C_4 \sin 3x\right)}$

Exercise

Find the general solution of the given higher-order differential equation: $y^{(5)} - 2y^{(4)} + 17y''' = 0$

Solution

$$\lambda^5 - 2\lambda^4 + 17\lambda^3 = \lambda^3\left(\lambda^2 - 2\lambda + 17\right) = 0$$

$$\lambda^3 = 0 \to \lambda_{1,2,3} = 0 \quad \Rightarrow \quad \lambda = \frac{2 \pm \sqrt{4 - 68}}{2} \quad \Rightarrow \lambda_{4,5} = 1 \pm 4i$$

$$\boxed{y(x) = C_1 + C_2 x + C_3 x^2 + e^{x}\left(C_4 \cos 4x + C_5 \sin 4x\right)}$$

Exercise

Find the general solution of the given higher-order differential equation: $\left(D^2+6D+13\right)^2 y=0$

Solution

The characteristic equation:

$$\left(\lambda^2+6\lambda+13\right)=0 \quad \Rightarrow \lambda=\frac{-6\pm\sqrt{-16}}{2}=-3\pm 2i \quad \textit{multiplicity } k=2$$

The general solution is: $\boxed{y(x)=e^{-3x}\left(C_1\cos 2x+C_2\sin 2x\right)+xe^{-3x}\left(C_3\cos 2x+C_4\sin 2x\right)}$

Exercise

Find the general solution of the given higher-order differential equation $\lambda^3(\lambda-1)(\lambda-2)^3\left(\lambda^2+9\right)=0$

Solution

$$\lambda^2+9=0 \quad \Rightarrow \quad \lambda^2=-9 \ \rightarrow \ \lambda=\pm 3i$$

The solution: $\lambda=0,\ 0,\ 0,\ 1,\ 2,\ 2,\ 2,\ \pm 3i$

$$\boxed{y(x)=C_1+C_2x+C_3x^2+C_4e^x+\left(C_5+C_6x+C_7x^2\right)e^{2x}+C_8\cos 3x+C_9\sin 3x}$$

Exercise

Find the general solution: $\quad y''-y'-2y=0; \quad y(0)=-1, \quad y'(0)=2$

Solution

The characteristic equation: $\lambda^2-\lambda-2=0 \qquad \Rightarrow \lambda_1=2;\ \lambda_2=-1$

$y(t)=C_1e^{2t}+C_2e^{-t} \qquad y(0)=C_1+C_2=-1$

$y'(t)=2C_1e^{2t}-C_2e^{-t} \qquad y'(0)=2C_1-C_2=2$

$C_1=\frac{1}{3} \quad C_2=-\frac{4}{3}$

$$\boxed{y(t)=\frac{1}{3}e^{2t}+-\frac{4}{3}e^{-t}}$$

Exercise

Find the general solution: $y'' - 2y' + 17y = 0; \quad y(0) = -2, \quad y'(0) = 3$

Solution

The characteristic equation: $\lambda^2 - 2\lambda + 17 = 0$

$\Rightarrow \lambda_{1,2} = 1 \pm 4i$

$y(t) = e^{t}\left(C_1 \cos 4t + C_2 \sin 4t\right)$

$$y(t) = e^{t}\left(C_1 \cos 4t + C_2 \sin 4t\right) \quad \rightarrow y(0) = \boxed{C_1 = -2}$$

$$y'(t) = e^{t}\left(C_1 \cos 4t + C_2 \sin 4t\right) + e^{t}\left(-4C_1 \sin 4t + 4C_2 \cos 4t\right)$$

$$\Rightarrow y'(0) = C_1 + 4C_2 = 3$$

$$\Rightarrow \boxed{C_2 = \frac{5}{4}}$$

$$\boxed{y(t) = e^{t}\left(-2\cos 4t + \frac{5}{4}\sin 4t\right)}$$

Exercise

Find the general solution: $y'' + 25y = 0; \quad y(0) = 1, \quad y'(0) = -1$

Solution

The characteristic equation: $\lambda^2 + 25 = 0$

$\Rightarrow \lambda_{1,2} = \pm 5i$

$y(t) = e^{0t}\left(C_1 \cos 5t + C_2 \sin 5t\right)$

$y(t) = C_1 \cos 5t + C_2 \sin 5t$

$$y(t) = C_1 \cos 5t + C_2 \sin 5t \quad \rightarrow y(0) = \boxed{C_1 = 1}$$

$$y'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t$$

$$\Rightarrow y'(0) = 5C_2 = -1 \rightarrow \boxed{C_2 = -\frac{1}{5}}$$

$$\boxed{y(t) = 5\cos 5t - \frac{1}{5}\sin 5t}$$

Exercise

Find the general solution: $y'' + 10y' + 25y = 0 \qquad y(0) = 2, \quad y'(0) = -1$

Solution

The characteristic equation: $\lambda^2 + 10\lambda + 25 = 0$

$\Rightarrow \lambda_{1,2} = -5$

$y(t) = \left(C_1 + C_2 t\right)e^{-5t}$

$$y(t) = \left(C_1 + C_2 t\right)e^{-5t} \qquad y(0) = \boxed{C_1 = 2}$$

$$y' = C_2 e^{-5t} - 5\left(C_1 + C_2 t\right)e^{-5t} \quad y'(0) = C_2 - 5C_1 = -1 \quad \Rightarrow \boxed{C_2 = 9}$$

$\boxed{y(t) = \left(2 + 9t\right)e^{-5t}}$

Exercise

Find the solution of the given initial value problem. $y'' - 2y' - 3y = 0; \quad y(0) = 2, \quad y'(0) = -3$

Solution

The characteristic equation: $\lambda^2 - 2\lambda - 3 = 0$

$\Rightarrow \lambda_1 = -1; \ \lambda_2 = 3$

$y(t) = C_1 e^{-t} + C_2 e^{3t}$

$$y(0) = C_1 + C_2 = 2$$

$y'(t) = -C_1 e^{-t} + 3C_2 e^{3t}$

$$y'(0) = -C_1 + 3C_2 = -3$$

$$\begin{aligned} C_1 + C_2 &= 2 \\ -C_1 + 3C_2 &= -3 \end{aligned} \quad \rightarrow \quad C_1 = \tfrac{9}{4} \quad C_2 = -\tfrac{1}{4}$$

$y(t) = \frac{9}{4}e^{-t} - \frac{1}{4}e^{3t}$

Exercise

Find the solution of the given initial value problem. $y'' - 4y' + 13y = 0; \quad y(0) = -1, \quad y'(0) = 2$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 13 = 0$

$$\Rightarrow \lambda_{1,\,2} = \frac{4 \pm \sqrt{16-52}}{2} = 2 \pm 3i$$

$$\boxed{y(x) = e^{2x}\left(C_1 \cos 3x + C_2 \sin 3x\right)}$$

$$y(0) = e^{0}\left(C_1 \cos(0) + C_2 \sin(0)\right)$$

$$\boxed{-1 = C_1}$$

$$y'(x) = 2e^{2x}\left(C_1 \cos 3x + C_2 \sin 3x\right) + e^{2x}\left(-3C_1 \sin 3x + 3C_2 \cos 3x\right)$$

$$y'(0) = 2C_1 + 3C_2 = 2 \quad \Rightarrow \quad \boxed{C_2 = \frac{2-2(-1)}{3} = \frac{4}{3}}$$

$$\boxed{y(x) = e^{2x}\left(-\cos 3x + \frac{4}{3}\sin 3x\right)}$$

Exercise

Find the solution of the given initial value problem. $y'' - 8y' + 17y = 0; \quad y(0) = 4, \quad y'(0) = -1$

Solution

The characteristic equation: $\lambda^2 - 8\lambda + 17 = 0$

$$\Rightarrow \lambda_{1,\,2} = \frac{8 \pm \sqrt{64-68}}{2} = \frac{8 \pm i2}{2} = 4 \pm i$$

$$\boxed{y(x) = e^{4x}\left(C_1 \cos x + C_2 \sin x\right)}$$

$$y(0) = e^{0}\left(C_1 \cos(0) + C_2 \sin(0)\right)$$

$$\boxed{4 = C_1}$$

$$y'(x) = 4e^{4x}\left(C_1 \cos x + C_2 \sin x\right) + e^{4x}\left(-C_1 \sin x + C_2 \cos x\right)$$

$$y'(0) = 4C_1 + C_2 = -1 \quad \Rightarrow \quad \boxed{C_2 = -1 - 16 = -17}$$

$$\boxed{y(x) = e^{4x}\left(4\cos x - 17\sin x\right)}$$

Exercise

Find the solution of the given initial value problem: $y'' - 4y' + 5y = 0;\quad y(0) = 1,\quad y'(0) = 5$

Solution

The characteristic equation: $\lambda^2 - 4\lambda + 5 = 0 \Rightarrow \lambda_{1,2} = 2 \pm i$

$$y(x) = e^{2x}\left(C_1 \cos x + C_2 \sin x\right)$$

$$y(x) = e^{2x}\left(C_1 \cos x + C_2 \sin x\right) \Rightarrow y(0) = \boxed{C_1 = 1}$$

$$y'(x) = 2e^{2x}\left(C_1 \cos x + C_2 \sin x\right) + e^{2x}\left(-C_1 \sin x + C_2 \cos x\right)$$

$$\Rightarrow y'(0) = 2C_1 + C_2 = 5$$

$$\Rightarrow \boxed{C_2 = 3}$$

$$y(x) = e^{2x}\left(\cos x + 3\sin x\right)$$

Exercise

The roots of the characteristic equation of a certain differential equation are:

$3, -5, 0, 0, 0, 0, -5, 2 \pm 3i$ and $2 \pm 3i$

Write a general solution of this homogeneous differential equation.

Solution

For $\lambda = 0, 0, 0, 0 \Rightarrow y_1 = C_1 + C_2 x + C_3 x^2 + C_4 x^3$

For $\lambda = 3 \Rightarrow y_2 = C_5 e^{3x}$

For $\lambda = -5, -5 \Rightarrow y_3 = C_6 e^{-5x} + C_7 x e^{-5x}$

For $\lambda = 2 \pm 3i, 2 \pm 3i \Rightarrow y_4 = e^{2x}\left(C_8 \cos 3x + C_9 \sin 3x\right) + x e^{-5x}\left(C_{10} \cos 3x + C_{11} \sin 3x\right)$

$$\begin{aligned} y(x) &= C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 e^{3x} + C_6 e^{-5x} + C_7 x e^{-5x} \\ &\quad + e^{2x}\left(C_8 \cos 3x + C_9 \sin 3x\right) + x e^{-5x}\left(C_{10} \cos 3x + C_{11} \sin 3x\right) \end{aligned}$$

Exercise

$y(x) = C_1 e^{2x} + C_2 e^{-2x} + C_3 \cos 2x + C_4 \sin 2x$ is the general solution of a homogeneous equation. What is the equation?

Solution

$\lambda_1 = 2, \quad \lambda_2 = -2, \quad \lambda_{3,4} = \pm 2i$

$(\lambda - 2)(\lambda + 2)(\lambda - 2i)(\lambda + 2i) = 0$

$$\left(\lambda^2 - 4\right)\left(\lambda^2 + 4\right) = 0$$

$$\lambda^4 - 16 = 0 \quad \Rightarrow \quad y^{(4)} - 16y = 0$$

Solution Section 2.4 - Harmonic Motion

Exercise

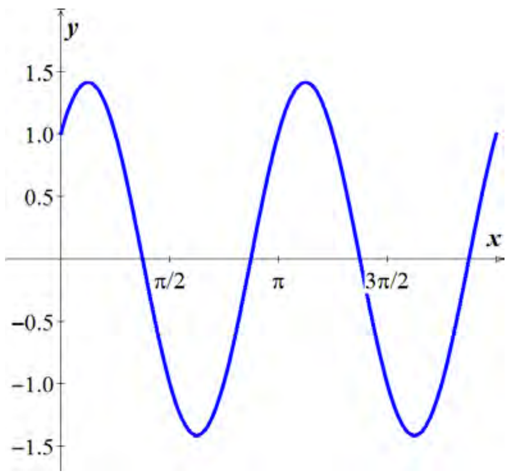
$y = \cos 2t + \sin 2t$

i. Plot the function

ii. Place the solution in the form $y = A\cos\left(\omega_0 t - \phi\right)$ and compare the graph with the plot in (***i***)

Solution

i. Plot the function



ii. Place the solution in the form $y = A\cos\left(\omega_0 t - \phi\right)$ and compare the graph with the plot in (***i***)

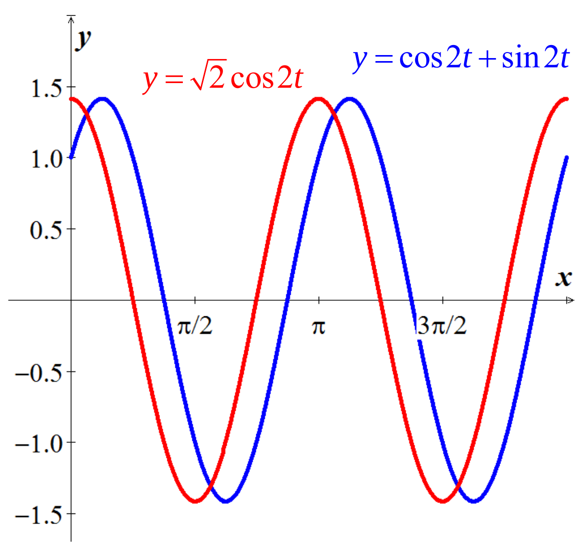
$y = 1.\cos 2t + 1.\sin 2t$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos 2t + \frac{1}{\sqrt{2}}\sin 2t\right)$ $\cos\phi = \frac{1}{\sqrt{2}} = \sin\phi$

$= \sqrt{2}\left(\cos\phi\cos 2t + \sin\phi\sin 2t\right)$

$= \sqrt{2}\cos\left(2t - \phi\right)$

$\boxed{= \sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)}$



Exercise

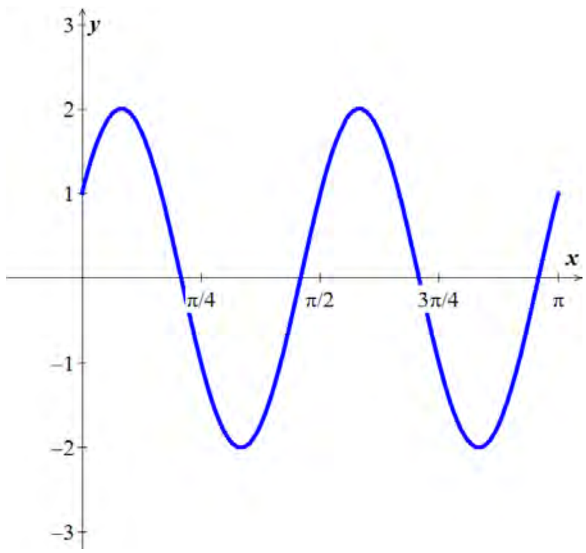
$y = \cos 4t + \sqrt{3}\sin 4t$

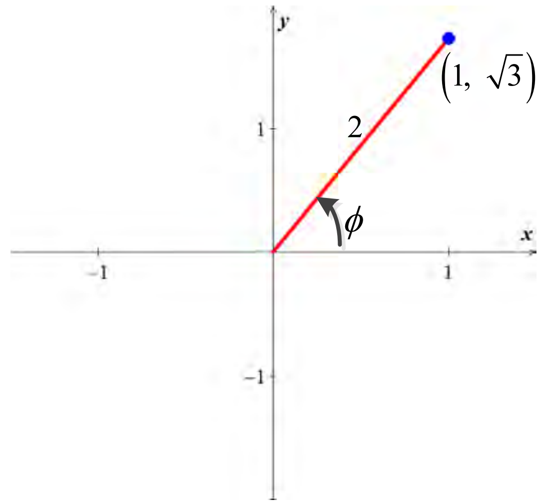
i. Plot the function

ii. Place the solution in the form $y = A\cos\left(\omega_0 t - \phi\right)$ and compare the graph with the plot in (***i***)

Solution

i. Plot the function

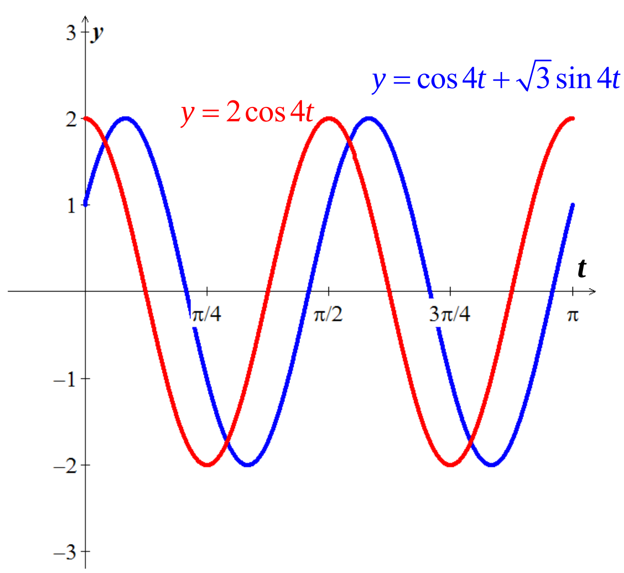




ii. Place the solution in the form $y = A\cos\left(\omega_0 t - \phi\right)$ and compare the graph with the plot in (***i***)

$$\begin{aligned}
y &= 1\cdot\cos 4t + \sqrt{3}\cdot\sin 4t \\
&= 2\left(\frac{1}{2}\cos 4t + \frac{\sqrt{3}}{2}\sin 4t\right) \\
&= 2\left(\cos\phi\cos 4t + \sin\phi\sin 4t\right) \\
&= 2\cos\left(4t - \phi\right) \\
&= 2\cos\left(4t - \frac{\pi}{3}\right) \\
&= 2\cos 4\left(t - \frac{\pi}{12}\right)
\end{aligned}$$

$\cos\phi = \frac{1}{2} \quad \sin\phi = \frac{\sqrt{3}}{2}$



Exercise

A 1-kg mass, when attached to a large spring, stretches the spring a distance of 4.9 *m*.

a) Calculate the spring constant.

b) The system is placed in a viscous medium that supplies a damping constant $\mu = 3\ kg/s$. The system is allowed to come to rest. Then the mass is displaced 1 *m* in the downward direction and given a sharp tap, imparting an instantaneous velocity of 1 *m/s* in the downward direction. Find the position of the mass as a function of time and plot the solution.

Solution

a) By Hooke's law:

$$\begin{aligned} k &= \frac{F}{y} \\ &= \frac{mg}{y} \\ &= \frac{(1kg)\left(9.8m/s^2\right)}{4.9m} \\ &= 2N/m \end{aligned}$$

b) ***Given:*** $m = 1;\ \mu = 3;\ k = 2;\ y(0) = 1;\ y'(0) = 1$

$$y'' + 3y' + 2y = 0$$

The characteristic equation is: $\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda_{1,2} = -1, -2$

The general solution: $\boxed{y(t) = C_1 e^{-t} + C_2 e^{-2t}}$

Exercise

The undamped system $\frac{2}{5}x'' + kx = 0,\quad x(0) = 2\quad x'(0) = v_0$ is observed to have period $\frac{\pi}{2}$ and amplitude 2. Find k and v_0

Solution

$$x'' + \frac{5}{2}kx = 0 \quad \Leftrightarrow \quad x'' + \omega_0^2 x = 0 \quad \left(\omega_0^2 = \frac{5k}{2}\right)$$

The characteristic equation is: $\lambda^2 + \omega_0^2 = 0 \quad \Rightarrow \lambda = \pm\omega_0 i$

It is a complex root, thus we have a complex solution:

$$z(t) = e^{i\omega_0 t} = \cos\omega_0 t + i\sin\omega_0 t = \text{cis}\,\omega_0 t$$

The general solution: $x(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$

This solution is periodic with period $T = \frac{2\pi}{\omega_0} = \frac{\pi}{2}$ (since the period is $\frac{\pi}{2}$ given)

$$\omega_0 = 4 \quad \Rightarrow \quad \boxed{k = \frac{2\omega_0^2}{5} = \frac{32}{5}}$$

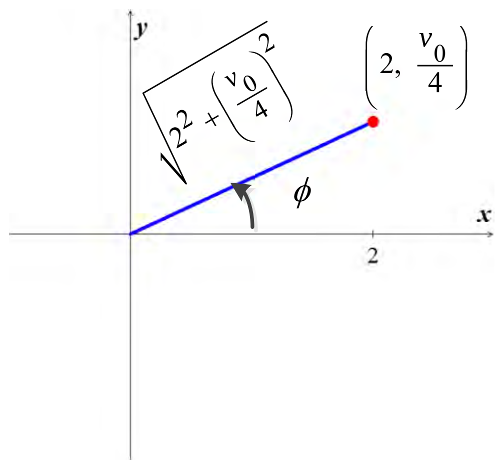
$$x(0) = C_1 \cos\omega_0 0 + C_2 \sin\omega_0 0 \quad \Rightarrow \quad \boxed{C_1 = 2}$$

$$x'(t) = -C_1\omega_0 \sin\omega_0 t + C_2\omega_0 \cos\omega_0 t$$

$$x'(0) = -C_1\omega_0 \sin\omega_0 0 + C_2\omega_0 \cos\omega_0 0 \quad \Rightarrow \quad \boxed{C_2 = \frac{v_0}{\omega_0}}$$

$$x(t) = 2\cos 4t + \frac{v_0}{4}\sin 4t$$

$$x(t) = \sqrt{4 + \frac{v_0^2}{16}}\ \cos(4t - \phi)$$



$$\tan\phi = \frac{v_0}{8}$$

The amplitude is 2, therefore:

$$\sqrt{4 + \frac{v_0^2}{16}} = 2$$

$$4 + \frac{v_0^2}{16} = 4$$

$$\frac{v_0^2}{16} = 0$$

$$\boxed{v_0 = 0}$$

Exercise

A body with mass $m = 0.5\ kg$ is attached to the end of a spring that is stretched 2 m by a force of 100 N. It is set in motion with initial position $x_0 = 1m$ and initial velocity $v_0 = -5m/s$. (Note that these initial conditions indicate that he body is displaced to the right and is moving to the left at time $t = 0$.) Find the position function of the body as well as the amplitude, frequency, period of oscillation, and time lag of its motion.

Solution

Spring constant: $\boxed{k = \frac{100N}{2m} = 50\ N/m}$

$my'' + \mu y' + ky = F(t)$

The differential equation can be written as:

$$0.5x'' + 50x = 0 \quad \Rightarrow \quad x'' + 100x = 0$$

$$\omega_0 = \sqrt{100} = 10 \ rad/s$$

Period: $T = \frac{2\pi}{\omega_0} = \frac{2\pi}{10} = \frac{\pi}{5} \approx 0.6283 \text{ sec}$

Frequency: $\upsilon = \frac{1}{T} = \frac{5}{\pi} \approx 1.5915 Hz$

Given: $x(0) = 1, \quad x'(0) = -5$

$$x(t) = A\cos 10t + B\sin 10t \quad \rightarrow \quad x(0) = A = 1$$

$$x'(t) = -10A\sin 10t + 10B\cos 10t \quad \rightarrow \quad x'(0) = 10B = -5 \Rightarrow B = -\frac{1}{2}$$

$$x(t) = \cos 10t - \frac{1}{2}\sin 10t$$

Amplitude of motion is: $A = \sqrt{1^2 + \left(-\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{2} m$

Time lag?

$$x(t) = \frac{\sqrt{5}}{2}\left(\frac{2}{\sqrt{5}}\cos 10t - \frac{1}{\sqrt{5}}\sin 10t\right)$$
$$= \frac{\sqrt{5}}{2}\cos(10t - \varphi)$$

Phase angle φ: $\hat{\varphi} = \tan^{-1}\left(\frac{\frac{1}{\sqrt{5}}}{\frac{2}{\sqrt{5}}}\right) \approx 0.46365$

Since $\cos\varphi = \frac{2}{\sqrt{5}} > 0 \quad and \quad \sin\varphi = -\frac{1}{\sqrt{5}} < 0 \quad \Rightarrow \quad \varphi = 2\pi - 0.46365 = 5.8195$

Time lag of the motion is: $\delta = \frac{\varphi}{\omega_0} \approx \frac{5.8195}{10} \approx 0.58195 \text{ sec}$

$$x(t) = \frac{\sqrt{5}}{2}\cos(10t - 5.8195)$$

Solution Section 2.5 - Inhomogeneous Equations; the Method of Undetermined Coefficients

Exercise

Show that the 3 solutions $y_1 = x, \quad y_2 = x\ln x, \quad y_3 = x^2$ of the 3rd order equation $x^3 y''' - x^2 y'' + 2xy' - 2y = 0$ are linearly independent on an open interval $x > 0$. Then find a particular solution that satisfies the initial conditions $y(1) = 3, \quad y'(1) = 2, \quad y''(1) = 1$

Solution

$$W = \begin{vmatrix} x & x\ln x & x^2 \\ 1 & 1+\ln x & 2x \\ 0 & \frac{1}{x} & 2 \end{vmatrix}$$

$$= 2x + 2x\ln x + x - 2x - 2x\ln x$$

$$\boxed{= x \neq 0} \quad \text{since} \quad x > 0$$

$\therefore y_1, y_2, y_3$ are linearly independent.

$$y(x) = C_1 x + C_2 x\ln x + C_3 x^2 \qquad y(1) = C_1 + C_3 = 3$$

$$y'(x) = C_1 + C_2(1 + \ln x) + 2C_3 x \qquad y'(1) = C_1 + C_2 + 2C_3 = 2$$

$$y''(x) = C_2 \frac{1}{x} + 2C_3 \qquad y''(1) = C_2 + 2C_3 = 1$$

$$\Rightarrow \quad C_1 = 1, \quad C_2 = -3, \quad \text{and} \quad C_3 = 2$$

$$\boxed{y(x) = x - 3x\ln x + 2x^2}$$

Exercise

Find the particular solution for $y'' + 3y' + 2y = 4e^{-3t}$

Solution

$$y(t) = Ae^{-3t} \qquad \Rightarrow y' = -3Ae^{-3t}$$

$$y'' = 9Ae^{-3t}$$

$$y'' + 3y' + 2y = 4e^{-3t}$$

$$9Ae^{-3t} + 3\left(-3Ae^{-3t}\right) + 2Ae^{-3t} = 4e^{-3t}$$

$$2Ae^{-3t} = 4e^{-3t}$$

$$2A = 4$$

$$A = 2$$

The particular solution: $\boxed{y(t) = 2e^{-3t}}$

Exercise

Find the particular solution for $y'' + 6y' + 8y = -3e^{-t}$

Solution

$$y(t) = Ae^{-t}$$

$$y' = -Ae^{-t}$$

$$y'' = Ae^{-t}$$

$$y'' + 6y' + 8y = -3e^{-t}$$

$$Ae^{-t} - 6Ae^{-t} + 8Ae^{-t} = -3e^{-t}$$

$$A - 6A + 8A = -3$$

$$3A = -3 \quad \Rightarrow \quad \boxed{A = -1}$$

Therefore, the particular solution is: $\boxed{y(t) = -e^{-t}}$

Exercise

Find the particular solution for $y'' + 2y' + 5y = 12e^{-t}$

Solution

$$y(t) = Ae^{-t}$$

$$y' = -Ae^{-t}$$

$$y'' = Ae^{-t}$$

$$y'' + 2y' + 5y = 12e^{-t}$$

$$Ae^{-t} + 2\left(-Ae^{-t}\right) + 5Ae^{-t} = 12e^{-t}$$

$$4Ae^{-t} = 12e^{-t}$$

$$4A = 12$$

$$A = 3$$

The particular solution: $\boxed{y(t) = 3e^{-t}}$

Exercise

Find the particular solution for the given differential equation $y'' + 3y' - 18y = 18e^{2t}$

Solution

$$y(t) = Ae^{2t}$$

$$y' = 2Ae^{2t}$$

$$y'' = 4Ae^{2t}$$

$$y'' + 3y' - 18y = 18e^{2t}$$

$$4Ae^{2t} + 3\left(2Ae^{2t}\right) - 18Ae^{2t} = 18e^{2t}$$

$$4Ae^{2t} + 6Ae^{2t} - 18Ae^{2t} = 18e^{2t}$$

$$-8Ae^{2t} = 18e^{2t}$$

$$-8A = 18$$

$$A = -\frac{18}{8} = -\frac{9}{4}$$

The particular solution: $y(t) = -\frac{9}{4}e^{2t}$

Exercise

Use $y(t) = a\cos\omega t + b\sin\omega t$ to find the particular solution for $y'' + 4y = \cos 3t$

Solution

The particular solution: $y(t) = a\cos 3t + b\sin 3t$

$$y' = -3a\sin 3t + 3b\cos 3t$$

$$y'' = -9a\cos 3t - 9b\sin 3t$$

$$y'' + 4y = \cos 3t$$

$$-9a\cos 3t - 9b\sin 3t + 4\left(a\cos 3t + b\sin 3t\right) = \cos 3t$$

$$-9a\cos 3t - 9b\sin 3t + 4a\cos 3t + 4b\sin 3t = \cos 3t$$

$$-5a\cos 3t - 5b\sin 3t = \cos 3t$$

$$a = -\frac{1}{5} \qquad b = 0$$

The particular solution: $\boxed{y(t) = -\frac{1}{5}\cos 3t}$

Exercise

Use $y(t) = a\cos\omega t + b\sin\omega t$ to find the particular solution for $y'' + 7y' + 6y = 3\sin 2t$

Solution

The particular solution: $y(t) = a\cos 2t + b\sin 2t$

$$y' = -2a\sin 2t + 2b\cos 2t$$
$$y'' = -4a\cos 2t - 4b\sin 2t$$

$$y'' + 7y' + 6y = 3\sin 2t$$

$$-4a\cos 2t - 4b\sin 2t + 7\left(-2a\sin 2t + 2b\cos 2t\right) + 6\left(a\cos 2t + b\sin 2t\right) = 3\sin 2t$$

$$-4a\cos 2t - 4b\sin 2t - 14a\sin 2t + 14b\cos 2t + 6a\cos 2t + 6b\sin 2t = 3\sin 2t$$

$$\left(14b + 2a\right)\cos 2t + \left(2b - 14a\right)\sin 2t = 3\sin 2t$$

$$\begin{cases} 14b + 2a = 0 \\ 2b - 14a = 3 \end{cases} \Rightarrow a = -\frac{21}{100} \quad b = \frac{3}{100}$$

The particular solution: $\boxed{y(t) = -\frac{21}{100}\cos 2t + \frac{3}{100}\sin 2t}$

Exercise

Find the particular solution for $y'' + 5y' + 4y = 2 + 3t$

Solution

The particular solution: $y(t) = at + b$

$$y' = a$$
$$y'' = 0$$

$$y'' + 5y' + 4y = 2 + 3t$$

$$0 + 5a + 4\left(at + b\right) = 2 + 3t$$

$$5a + 4b + 4at = 2 + 3t$$

$$\begin{cases} 5a + 4b = 2 \\ 4a = 3 \end{cases}$$

$$\Rightarrow \begin{cases} b = -\frac{7}{16} \\ a = \frac{3}{4} \end{cases}$$

The particular solution: $\boxed{y(t) = \frac{3}{4}t - \frac{7}{16}}$

Exercise

Find the particular solution for $y'' + 6y' + 8y = 2t - 3$

Solution

The particular solution: $y(t) = at + b$

$$y'' + 6y' + 8y = 2t - 3$$
$$0 + 6a + 8\left(at + b\right) = 2t - 3$$
$$6a + 8at + 8b = 2t - 3$$
$$8at + 6a + 8b = 2t - 3$$

$$\begin{cases} 6a + 8b = -3 \\ 8a = 2 \end{cases}$$

$$\Rightarrow \begin{cases} a = \frac{1}{4} \\ b = -\frac{9}{16} \end{cases}$$

The particular solution: $\underline{y(t) = \frac{1}{4}t - \frac{9}{16}}$

Exercise

Find the particular solution for $y'' + 3y' + 4y = t^3$

Solution

The particular solution: $y(t) = at^3 + bt^2 + ct + d$

$$y' = 3at^2 + 2bt + c$$
$$y'' = 6at + 2b$$

$$y'' + 3y' + 4y = t^3$$
$$6at + 2b + 3\left(3at^2 + 2bt + c\right) + 4\left(at^3 + bt^2 + ct + d\right) = t^3$$
$$6at + 2b + 9at^2 + 6bt + 3c + 4at^3 + 4bt^2 + 4ct + 4d = t^3$$
$$4at^3 + (9a + 4b)t^2 + (6a + 6b + 4c)t + 2b + 3c + 4d = t^3$$

$$\begin{cases} 4a = 1 \\ 9a + 4b = 0 \\ 6a + 6b + 4c = 0 \\ 2b + 3c + 4d = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{4} \\ b = -\frac{9}{4}a = -\frac{9}{16} \\ c = -\frac{6}{4}a - \frac{6}{4}b = -\frac{3}{2}\frac{1}{4} + \frac{3}{2}\frac{9}{16} = \frac{15}{32} \\ d = -\frac{1}{2}b - \frac{1}{2}d = -\frac{9}{128} \end{cases}$$

The particular solution: $\underline{y(t) = \frac{1}{4}t^3 - \frac{9}{16}t^2 + \frac{15}{32}t - \frac{9}{128}}$

Exercise

Find the particular solution for $y'' + 2y' + 2y = 2 + \cos 2t$

Solution

$y'' + 2y' + 2y = 2$ when $y = 1$

The particular solution: $z = Ae^{i2t}$

$$z' = (2i)Ae^{i2t}$$

$$z'' = (2i)^2 Ae^{i2t}$$

$$z'' + 2z' + 2z = e^{i2t}$$

$$(2i)^2 Ae^{i2t} + 2(2i)Ae^{i2t} + 2Ae^{i2t} = e^{i2t}$$

$$(2i)^2 A + 2(2i)A + 2A = 1$$

$$(-4 + 4i + 2)A = 1$$

$$A = \frac{1}{-2+4i} \cdot \frac{-2-4i}{-2-4i}$$

$$= -\frac{2}{10} - \frac{4}{10}i$$

$$= -\frac{1}{10} - \frac{1}{5}i$$

$$z_p = \left(-\frac{1}{10} - \frac{1}{5}i\right)e^{i2t}$$

$$= \left(-\frac{1}{10} - \frac{1}{5}i\right)(\cos 2t + i\sin 2t)$$

$$= -\frac{1}{10}\cos 2t + \frac{1}{5}\sin 2t + i\left(-\frac{1}{5}\cos 2t - \frac{1}{10}\sin 2t\right)$$

The general solution: $\boxed{y(t) = 1 - \frac{1}{10}\cos 2t + \frac{1}{5}\sin 2t}$

Exercise

Find the particular solution for $y'' - y = t - e^{-t}$

Solution

The characteristic eq.: $\lambda^2 - 1 = 0 \quad \Rightarrow \quad \lambda_1 = -1, \quad \lambda_2 = 1$

The particular solution: $y = at + b$

$$y' = a$$

$$y'' = 0$$

$$y'' - y = t$$

$$-at - b = t \quad \Rightarrow \quad \begin{cases} \boxed{a = -1}, & \boxed{b = 0} \end{cases}$$

$$y(t) = -t$$

The homogenous solution: $y_h = C_1 e^{-t} + C_2 e^{t}$

Because the inhomogeneous part of $y'' - y = e^{-t}$ is also the solution.

Therefore: $\quad y_p = Ate^{-t}$

$$y_p' = -Ate^{-t} + Ae^{-t} = Ae^{-t}(1 - t)$$

$$y_p'' = -Ae^{-t} + Ate^{-t} - Ae^{-t} = Ae^{-t}(t - 2)$$

$$y'' - y = e^{-t}$$

$$Ate^{-t} - 2Ae^{-t} - Ate^{-t} = e^{-t}$$

$$-2Ae^{-t} = e^{-t}$$

$$-2A = 1 \quad \Rightarrow \quad \boxed{A = -\tfrac{1}{2}}$$

$$y_p = -\tfrac{1}{2} te^{-t}$$

Therefore,

$$y(t) = -t - \left(-\tfrac{1}{2} te^{-t}\right)$$

$$\boxed{y(t) = -t + \tfrac{1}{2} te^{-t}}$$

Exercise

Find the particular solution for $y'' - 2y' + y = 10e^{-2t}\cos t$

Solution

The particular solution: $y_p = e^{-2t}(A\cos t + B\sin t)$

$$y' = -2e^{-2t}(A\cos t + B\sin t) + e^{-2t}(-A\sin t + B\cos t)$$

$$= e^{-2t}((B-2A)\cos t - (A+2B)\sin t)$$

$$y'' = -2e^{-2t}((B-2A)\cos t - (A+2B)\sin t) + e^{-2t}((2A-B)\sin t - (A+2B)\cos t)$$

$$= e^{-2t}((3A-4B)\cos t + (4A+3B)\sin t)$$

$$y'' - 2y' + y = 10e^{-2t}\cos t$$

$$e^{-2t}((3A-4B)\cos t + (4A+3B)\sin t) - 2e^{-2t}((B-2A)\cos t - (A+2B)\sin t)$$
$$+ e^{-2t}(A\cos t + B\sin t) = 10e^{-2t}\cos t$$

$$((3A-4B-2B+4A+A)\cos t + (4A+3B+2A+4B+B)\sin t) = 10\cos t$$

$$\begin{cases} 8A-6B=10 \\ 6A+8B=0 \end{cases} \rightarrow A = \frac{80}{100} = \frac{4}{5} \quad B = -\frac{3}{5}$$

$$\boxed{y_p = e^{-2t}\left(\frac{4}{5}\cos t - \frac{3}{5}\sin t\right)}$$

Exercise

Find the particular solution for $y''' - 4y'' + 4y' = 5t^2 - 6t + 4t^2e^t + 3e^{5t}$

Solution

Characteristic equation: $\lambda^3 - 4\lambda^2 + 4\lambda = \lambda(\lambda-2)^2 = 0 \quad \Rightarrow \quad \lambda_1 = 0, \ \lambda_{2,3} = 2$

Homogeneous equation: $y_h = C_1 + (C_2 + C_3 t)e^{2t}$

The particular solution: $y_p = t(At^2 + Bt + C) + (Et^2 + Ft + G)e^t + He^{5t}$

$$y_p' = 3At^2 + 2Bt + C + (2Et + F)e^t + (Et^2 + Ft + G)e^t + 5He^{5t}$$

$$= 3At^2 + 2Bt + C + (Et^2 + (2E+F)t + F + G)e^t + 5He^{5t}$$

$$y_p'' = 6At + 2B + (2Et + 2E + F)e^t + (Et^2 + (2E+F)t + F + G)e^t + 25He^{5t}$$

$$= 6At + 2B + (Et^2 + (4E+F)t + 2E + 2F + G)e^t + 25He^{5t}$$

$$y''' = 6A + (2Et + 4E + F)e^t + (Et^2 + (4E+F)t + 2E + 2F + G)e^t + 125He^{5t}$$

$$= 6A + \left(Et^2 + (6E+F)t + 6E + 3F + G\right)e^t + 125He^{5t}$$

$$\begin{aligned} y''' - 4y'' + 4y' = \ & 6A + \left(Et^2 + (6E+F)t + 6E+3F+G\right)e^t + 125He^{5t} \\ & -24At - 8B - 4\left(Et^2 + (4E+F)t + 2E + 2F + G\right)e^t - 100He^{5t} \\ & +12At^2 + 8Bt + 4C + 4\left(Et^2 + (2E+F)t + F + G\right)e^t + 20He^{5t} \end{aligned}$$

$$= 12At^2 + (8B - 24A)t + 6A - 8B + 4C + \left(Et^2 + (-2E+F)t - 2E - F + G\right)e^t + 45He^{5t}$$

$$= 5t^2 - 6t + 4t^2e^t + 3e^{5t}$$

$$\begin{cases} 12A = 5 \\ 8B - 24A = -6 \\ 6A - 8B + 4C = 0 \end{cases} \rightarrow \begin{cases} A = \frac{5}{12} \\ B = \frac{1}{2} \\ C = \frac{3}{8} \end{cases} \qquad \begin{cases} E = 4 & \\ F - 2E = 0 & \Rightarrow \quad F = 8 \\ -2E - F + G = 0 & \Rightarrow \quad G = 16 \end{cases} \qquad H = \frac{3}{45} = \frac{1}{15}$$

$$\boxed{y_P = \frac{5}{12}t^3 + \frac{1}{2}t^2 + \frac{3}{8}t + \left(4t^2 + 8t + 16\right)e^t + \frac{1}{15}e^{5t}}$$

Exercise

Use the complex method to find the particular solution for $y'' + 4y' + 3y = \cos 2t + 3\sin 2t$

Solution

The characteristic eq.: $\lambda^2 + 4\lambda + 3 = 0 \Rightarrow \lambda_1 = -3, \quad \lambda_2 = -1$

The homogenous solution: $y_h = e^{-t}\left(C_1 \cos t + C_2 \sin t\right)$

The particular solution: $z = Ae^{i2t}$

$$z' = (2i)Ae^{i2t}$$

$$z'' = (2i)^2 Ae^{i2t}$$

$$z'' + 4z' + 3z = e^{i2t}$$

$$(2i)^2 Ae^{i2t} + 4(2i)Ae^{i2t} + 3Ae^{i2t} = e^{i2t}$$

$$(-4 + 8i + 3)A = 1$$

$$(-1 + 8i)A = 1$$

$$\begin{aligned} A &= \frac{1}{-1+8i} \cdot \frac{-1-8i}{-1-8i} \\ &= \frac{-1-8i}{65} \\ &= -\frac{1}{65} - i\frac{8}{65} \end{aligned}$$

This gives the particular solution:

$$z = \left(-\frac{1}{65} - i\frac{8}{65}\right)e^{i2t}$$

$$= \left(-\frac{1}{65} - i\frac{8}{65}\right)(\cos 2t + i\sin 2t)$$

$$= \left(-\frac{1}{65}\cos 2t + \frac{8}{65}\sin 2t\right) + i\left(-\frac{8}{65}\cos 2t - \frac{1}{65}\sin 2t\right)$$

$y = -\frac{1}{65}\cos 2t + \frac{8}{65}\sin 2t$ is a solution of $y'' + 4y' + 3y = \cos 2t$

$y = -\frac{8}{65}\cos 2t - \frac{1}{65}\sin 2t$ is a solution of $y'' + 4y' + 3y = \sin 2t$

Therefore;

$$y(t) = -\frac{1}{65}\cos 2t + \frac{8}{65}\sin 2t + 3\left(-\frac{8}{65}\cos 2t - \frac{1}{65}\sin 2t\right)$$

$$= -\frac{1}{65}\cos 2t + \frac{8}{65}\sin 2t - \frac{24}{65}\cos 2t - \frac{3}{65}\sin 2t$$

$$= -\frac{25}{65}\cos 2t + \frac{5}{65}\sin 2t$$

$$= -\frac{5}{13}\cos 2t + \frac{1}{13}\sin 2t$$

Exercise

Use the complex method to find the particular solution for $y'' + 4y = \cos 3t$

Solution

The particular solution: $z = Ae^{i3t}$

$$z' = (3i)Ae^{i3t}$$

$$z'' = (3i)^2 Ae^{i3t}$$

$$z'' + 4z = \cos 3t = e^{i3t}$$

$$(3i)^2 Ae^{i3t} + 4Ae^{i3t} = e^{i3t}$$

$$(-9+4)A = 1 \quad \rightarrow \quad A = -\frac{1}{5}$$

$$z = -\frac{1}{5}e^{i3t} \Rightarrow \boxed{y = -\frac{1}{5}\cos 3t}$$

Exercise

Find the general solution for the given differential equation $\quad y''+3y'+2y=4x^2$

Solution

The characteristic equation: $\lambda^2+3\lambda+2=0 \quad \Rightarrow \quad \lambda_1=-1,\ \lambda_2=-2$

$$y_h=C_1e^{-x}+C_2e^{-2x}$$

The particular equation: $\quad y_p=ax^2+bx+c$

$$y'_p=2ax+b$$

$$y''_p=2a$$

$$y''_p+3y'_p+2y_p=4x^2$$

$$2a+6ax+3b+2ax^2+2bx+2c=4x^2$$

$$\begin{cases} 2a=4 & \rightarrow a=2 \\ 6a+2b=0 & \rightarrow b=-6 \\ 2a+3b+2c=0 & \rightarrow c=7 \end{cases}$$

$$\underline{y(x)=C_1e^{-x}+C_2e^{-2x}+2x^2-6x+7}|$$

Exercise

Find the general solution for the given differential equation $\quad y''-3y'=8e^{3x}+4\sin x$

Solution

The characteristic equation: $\lambda^2-3\lambda=0 \quad \Rightarrow \quad \lambda_1=0,\ \lambda_2=3$

$$y_h=C_1+C_2e^{3x}$$

The particular equation: $\quad y_p=Ae^{3x}+B\cos x+C\sin x$

$$y'_p=3Ae^{3x}-B\sin x+C\cos x$$

$$y''_p=9Ae^{3x}-B\cos x-C\sin x$$

$$y''-3y'=8e^{3x}+4\sin x$$

$$\cancel{9Ae^{3x}}-B\cos x-C\sin x\cancel{-9Ae^{3x}}+3B\sin x-3C\cos x=8e^{3x}+4\sin x$$

The particular equation: $\quad y_p=Axe^{3x}+B\cos x+C\sin x$

$$y'_p=3Axe^{3x}+Ae^{3x}-B\sin x+C\cos x$$

$$y''_p=3Ae^{3x}+9Axe^{3x}+3Ae^{3x}-B\cos x-C\sin x$$

$$6Ae^{3x}+9Axe^{3x}-B\cos x-C\sin x-9Axe^{3x}-3Ae^{3x}+3B\sin x-3C\cos x=8e^{3x}+4\sin x$$

$$3Ae^{3x}-(B+3C)\cos x+(3B-C)\sin x=8e^{3x}+4\sin x$$

$$\begin{cases} 3A=8 & \to A=\frac{8}{3} \\ -B-3C=0 & B=\frac{6}{5} \\ 3B-C=4 & C=-\frac{2}{5} \end{cases}$$

$$\boxed{y(x)=C_1+C_2e^{3x}+\frac{8}{3}xe^{3x}+\frac{6}{5}\cos x-\frac{2}{5}\sin x}$$

Exercise

Find the general solution for the given differential equation $\quad y''+8y=5x+2e^{-x}$

Solution

The characteristic equation: $\lambda^2+8=0 \quad \Rightarrow \quad \lambda_{1,2}=\pm 2i\sqrt{2}$

$$y_h=C_1\cos\left(2\sqrt{2}x\right)+C_2\sin\left(2\sqrt{2}x\right)$$

The particular equation: $\quad y_p=A+Bx+Ce^{-x}$

$$y'_p=B-Ce^{-x}$$

$$y''_p=Ce^{-x}$$

$$y''+8y=5x+2e^{-x}$$

$$Ce^{-x}+8A+8Bx+8Ce^{-x}=5x+2e^{-x}$$

$$\begin{cases} 8A=0 & \to A=0 \\ 8B=5 & \to B=\frac{5}{8} \\ 9C=2 & C=\frac{2}{9} \end{cases}$$

$$\boxed{y(x)=C_1\cos\left(2\sqrt{2}x\right)+C_2\sin\left(2\sqrt{2}x\right)+\frac{5}{8}x+\frac{2}{9}e^{-x}}$$

Exercise

Find the general solution for the given differential equation $y'' + y = x\cos x - \cos x$

Solution

The characteristic equation: $\lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm i$

$$y_h = C_1 \cos x + C_2 \sin x$$

The particular equation: $y_p = (Ax + B)\cos x + (Cx + E)\sin x$

$$y'_p = A\cos x - (Ax + B)\sin x + C\sin x + (Cx + E)\cos x$$

$$= (Cx + A + E)\cos x - (Ax + B + C)\sin x$$

$$y''_p = C\cos x - (Cx + A + E)\sin x - A\sin x - (Ax + B + C)\cos x$$

$$-(Ax + B)\cos x - (Cx + 2A + E)\sin x + (Ax + B)\cos x + (Cx + E)\sin x = x\cos x - \cos x$$

$$\cancel{-2A\sin x = x\cos x - \cos x}$$

The particular equation: $y_p = \left(Ax^2 + Bx + C\right)\cos x + \left(Dx^2 + Ex + F\right)\sin x$

$$y'_p = (2Ax + B)\cos x - \left(Ax^2 + Bx + C\right)\sin x + (2Dx + E)\sin x + \left(Dx^2 + Ex + F\right)\cos x$$

$$= \left(Dx^2 + (2A + E)x + B + F\right)\cos x - \left(Ax^2 + (B - 2D)x + C - E\right)\sin x$$

$$y''_p = (2Dx + 2A + E)\cos x - \left(Dx^2 + (2A + E)x + B + F\right)\sin x$$

$$-(2Ax + B - 2D)\sin x - \left(Ax^2 + (B - 2D)x + C - E\right)\cos x$$

$$-\left(Ax^2 + (B - 4D)x - 2A + C - 2E\right)\cos x - \left(Dx^2 + (4A + E)x + 2B - 2D + F\right)\sin x$$

$$+\left(Ax^2 + Bx + C\right)\cos x + \left(Dx^2 + Ex + F\right)\sin x = x\cos x - \cos x$$

$$4Dx\cos x + (2A + 2E)\cos x - 4Ax\sin x + (2D - 2B)\sin x = x\cos x - \cos x$$

$$\begin{cases} 4D = 1 & \to D = \frac{1}{4} \\ 2A + 2E = -1 & E = -\frac{1}{2} \\ -4A = 0 & \to A = 0 \\ 2D - 2B = 0 & B = \frac{1}{4} \end{cases}$$

$$y(x) = C_1 \cos x + C_2 \sin x + \frac{1}{4}x\cos x + \frac{1}{4}x^2 \sin x - \frac{1}{2}x\sin x + C\cos x + F\sin x$$

$$y(x) = C_1 \cos x + C_2 \sin x + \frac{1}{4}x\cos x + \frac{1}{4}x^2 \sin x - \frac{1}{2}x\sin x$$

Exercise

Find the general solution for the given differential equation $\qquad y'' + 25y = 20\sin 5x$

Solution

The characteristic equation: $\lambda^2 + 25 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 5i$

$$y_h = C_1 \cos 5x + C_2 \sin 5x$$

The particular equation: $\quad y_p = A\cos 5x + B\sin 5x$

$$y_p' = -5A\sin 5x + 5B\cos 5x$$

$$y_p'' = -25A\cos 5x - 25B\sin 5x$$

$$-25A\cos x - 25B\sin x + 25A\cos x + 25B\sin x = 20\sin 5x$$

$$\cancel{0 = 20\sin 5x}$$

The particular equation: $\quad y_p = Ax\cos 5x + Bx\sin 5x$

$$y_p' = A\cos 5x - 5Ax\sin 5x + B\sin 5x + 5Bx\cos 5x = (A + 5Bx)\cos 5x + (B - 5Ax)\sin 5x$$

$$\begin{aligned} y_p'' &= 5B\cos 5x - 5(A + 5Bx)\sin 5x - 5A\sin 5x + 5(B - 5Ax)\cos 5x \\ &= 10B\cos 5x - 25Ax\cos 5x - 10A\sin 5x - 25Bx\sin 5x \end{aligned}$$

$$10B\cos 5x - 25Ax\cos 5x - 10A\sin 5x - 25Bx\sin 5x + 25Ax\cos 5x + 25Bx\cos 5x = 20\sin 5x$$

$$10B\cos 5x - 10A\sin 5x = 20\sin 5x$$

$$\begin{cases} 10B = 0 & \rightarrow B = 0 \\ -10A = 20 & \rightarrow A = -2 \end{cases}$$

$$\boxed{y(x) = C_1 \cos 5x + C_2 \sin 5x - 2x\cos 5x}$$

Exercise

Find the general solution for the given differential equation $\qquad y''' + 8y'' = -6x^2 + 9x + 2$

Solution

The characteristic equation: $\lambda^3 + 8\lambda^2 = \lambda^2(\lambda + 8) = 0 \quad \Rightarrow \quad \lambda_{1,2} = 0,\ \lambda_3 = -8$

$$y_h = \left(C_1 + C_2 x\right)e^0 + e^{-8x} = C_1 + C_2 x + e^{-8x}$$

The particular equation: $\quad y_p = Ax^4 + Bx^3 + Cx^2$

$$y_p' = 4Ax^3 + 3Bx^2 + 2Cx$$

$$y_p'' = 12Ax^2 + 6Bx + 2C$$

$$y_p'' = 24Ax + 6B$$

$$y''' + 8y'' = -6x^2 + 9x + 2$$

$$24Ax+6B+96Ax^2+48Bx+16C=-6x^2+9x+2$$

$$\begin{cases} 96A=-6 & \rightarrow A=-\frac{1}{16} \\ 24A+48B=9 & \rightarrow B=\frac{1}{96}\left(9+\frac{3}{2}\right)=\frac{21}{108}=\frac{7}{32} \\ 6B+16C=2 & \rightarrow C=\frac{1}{16}\left(2-\frac{21}{16}\right)=\frac{11}{256} \end{cases}$$

$$\underline{y(x)=C_1+C_2x+e^{-8x}-\frac{1}{16}x^4+\frac{7}{32}x^3+\frac{11}{256}x^2}$$

Exercise

Find the general solution for the given differential equation $\quad y'''-3y''+3y'-y=e^x-x+16$

Solution

The characteristic equation: $\lambda^3-3\lambda^2+3\lambda-1=(\lambda-1)^3=0 \quad \Rightarrow \quad \lambda_{1,2,3}=1$

$$y_h=\left(C_1+C_2x+C_3x^2\right)e^x$$

The particular equation: $\quad y_p=A+Bx+\left(Cx^3+Ex^4\right)e^x$

$$y_p'=B+\left(3Cx^2+4Ex^3+Cx^3+Ex^4\right)e^x=B+\left(3Cx^2+(4E+C)x^3+Ex^4\right)e^x$$

$$y_p''=\left(6Cx+3(4E+C)x^2+4Ex^3\right)e^x+\left(3Cx^2+(4E+C)x^3+Ex^4\right)e^x$$

$$=\left(6Cx+(12E+6C)x^2+(8E+C)x^3+Ex^4\right)e^x$$

$$y_p'''=\left(6C+(24E+12C)x+(24E+3C)x^2+4Ex^3\right)e^x$$

$$+\left(6Cx+(12E+6C)x^2+(8E+C)x^3+Ex^4\right)e^x$$

$$=\left(6C+(24E+18C)x+(36E+9C)x^2+(12E+C)x^3+Ex^4\right)e^x$$

$$\left(6C+(24E+18C)x+(36E+9C)x^2+(12E+C)x^3+Ex^4\right)e^x-3\left(6Cx+(12E+6C)x^2+(8E+C)x^3\right.$$

$$\left.+Ex^4\right)e^x+3B+3\left(3Cx^2+(4E+C)x^3+Ex^4\right)e^x-A-Bx-\left(Cx^3+Ex^4\right)e^x \ \underline{=e^x-x+16}$$

$$(6C+24Ex)e^x-A+3B-Bx \ \underline{=e^x-x+16}$$

$$\begin{cases} 6C=1 & \rightarrow C=\frac{1}{6} \\ 24E=0 & \rightarrow E=0 \\ -B=-1 & \rightarrow B=1 \\ 3B-A=16 & \rightarrow A=-13 \end{cases}$$

$$\underline{y(x)=\left(C_1+C_2x+C_3x^2\right)e^x+\frac{1}{6}x^3e^x+x-13}$$

Exercise

Find a solution to the homogeneous equation; then find a particular solution to form a general solution. Then find the solution satisfying the given initial conditions

$$y'' - 4y' - 5y = 4e^{-2t}; \quad y(0) = 0, \quad y'(0) = -1$$

Solution

The *homogeneous* eq.: $y'' - 4y' - 5y = 0$

The *characteristic* eq.: $\lambda^2 - 4\lambda - 5 = 0 \ \Rightarrow \ \lambda_1 = 5, \lambda_2 = -1$

$$y_h = C_1 e^{5t} + C_2 e^{-t}$$

$$y_p = Ae^{-2t}$$

$$y' = -2Ae^{-2t}$$

$$y'' = 4Ae^{-2t}$$

$$y'' - 4y' - 5y = 4e^{-2t}$$

$$4Ae^{-2t} + 8Ae^{-2t} - 5Ae^{-2t} = 4e^{-2t}$$

$$7A = 4$$

$$A = \frac{4}{7}$$

$$y_p = \frac{4}{7}e^{-2t}$$

$$y = C_1 e^{5t} + C_2 e^{-t} + \frac{4}{7}e^{-2t}$$

$$y(0) = C_1 e^{5(0)} + C_2 e^{-(0)} + \frac{4}{7}e^{-2(0)}$$

$$0 = C_1 + C_2 + \frac{4}{7}$$

$$C_1 + C_2 = -\frac{4}{7} \qquad \textbf{(1)}$$

$$y' = 5C_1 e^{5t} - C_2 e^{-t} - \frac{8}{7}e^{-2t}$$

$$y'(0) = 5C_1 e^{5(0)} - C_2 e^{-(0)} - \frac{8}{7}e^{-2(0)}$$

$$-1 = 5C_1 - C_2 - \frac{8}{7}$$

$$5C_1 - C_2 = \frac{1}{7} \qquad \textbf{(2)}$$

$$C_1 = -\frac{1}{14} \text{ and } C_2 = -\frac{1}{2}$$

$$\boxed{y(t) = -\frac{1}{14}e^{5t} - \frac{1}{2}e^{-t} + \frac{4}{7}e^{-2t}}$$

Exercise

Find a solution to the homogeneous equation; then find a particular solution to form a general solution. Then find the solution satisfying the given initial conditions

$$y'' + 2y' + 2y = 2\cos 2t; \quad y(0) = -2, \quad y'(0) = 0$$

Solution

The characteristic eq.: $\lambda^2 + 2\lambda + 2 = 0 \quad \Rightarrow \ \lambda_1 = -1 + i, \quad \lambda_2 = -1 - i$

The homogenous solution: $y_h = e^{-t}\left(C_1 \cos t + C_2 \sin t\right)$

The particular solution: $z = Ae^{i2t}$

$$z' = (2i)Ae^{i2t}$$

$$z'' = (2i)^2 Ae^{i2t}$$

$z'' + 2z' + 2z = 2e^{i2t}$

$(2i)^2 Ae^{i2t} + 2(2i)Ae^{i2t} + 2Ae^{i2t} = 2e^{i2t}$

$(-4 + 4i + 2)A = 2$

$(-2 + 4i)A = 2$

$A = \frac{2}{-2+4i} \cdot \frac{-2-4i}{-2-4i}$

$= \frac{-4-8i}{20}$

$= \frac{-4}{20} - \frac{8i}{20}$

$= -\frac{1}{5} - \frac{2}{5}i$

This gives the particular solution:

$z = \left(-\frac{1}{5} - \frac{2}{5}i\right)e^{i2t}$

$= \left(-\frac{1}{5} - \frac{2}{5}i\right)(\cos 2t + i\sin 2t)$

$= -\frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t - i\left(\frac{1}{5}\sin 2t + \frac{2}{5}\cos 2t\right)$

The real part of this solution $\left(-\frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t\right)$ is the particular solution of the system.

Thus, the general solution is:

$y = e^{-t}\left(C_1 \cos t + C_2 \sin t\right) - \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t$

The initial condition: $y(0) = -2$

$-2 = e^{-0}\left(C_1 \cos 0 + C_2 \sin 0\right) - \frac{1}{5}\cos 0 + \frac{2}{5}\sin 0$

$-2 = C_1 - \frac{1}{5} \quad \Rightarrow \quad \boxed{C_1 = -\frac{9}{5}}$

$$y' = e^{-t}\left(-C_1 \sin t + C_2 \cos t\right) - e^{-t}\left(C_1 \cos t + C_2 \sin t\right) + \frac{2}{5}\sin 2t + \frac{4}{5}\cos 2t$$
$$= -C_1 e^{-t}\sin t + C_2 e^{-t}\cos t - C_1 e^{-t}\cos t - C_2 e^{-t}\sin t + \frac{2}{5}\sin 2t + \frac{4}{5}\cos 2t$$
$$= \left(C_2 - C_1\right)e^{-t}\cos t - \left(C_1 + C_2\right)e^{-t}\sin t + \frac{2}{5}\sin 2t + \frac{4}{5}\cos 2t$$

The initial condition: $y'(0) = 0$

$$0 = \left(C_2 - C_1\right)e^{-0}\cos 0 - \left(C_1 + C_2\right)e^{-0}\sin 0 + \frac{2}{5}\sin 0 + \frac{4}{5}\cos 0$$
$$0 = C_2 - C_1 + \frac{4}{5}$$
$$\boxed{C_2 = C_1 - \frac{4}{5} = -\frac{9}{5} - \frac{4}{5} = -\frac{13}{5}}$$

The general solution: $\boxed{y(t) = e^{-t}\left(-\frac{9}{5}\cos t - \frac{13}{5}\sin t\right) - \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t}$

Exercise

Find a solution to the homogeneous equation; then find a particular solution to form a general solution. Then find the solution satisfying the given initial conditions

$$y'' - 2y' + y = t^3; \quad y(0) = 1, \quad y'(0) = 0$$

Solution

The homogeneous eq.: $y'' - 2y' + y = 0$

The characteristic eq.: $\lambda^2 - 2\lambda + 1 = 0 \;\Rightarrow\; \lambda_{1,2} = 1$

$$y_h = \left(C_1 + C_2 t\right)e^t$$
$$y_p = at^3 + bt^2 + ct + d$$
$$y_p' = 3at^2 + 2bt + c$$
$$y_p'' = 6at + 2b$$

$$y'' - 2y' + y = 6at + 2b - 6at^2 - 4bt - 2c + at^3 + bt^2 + ct + d$$
$$= at^3 + (b - 6a)t^2 + (6a - 4b + c)t + 2b - 2c + d$$

$$\begin{cases} a = 1 \\ b - 6a = 0 & \Rightarrow \quad b = 6 \\ 6a - 4b + c = 0 & \Rightarrow \quad c = 18 \\ 2b - 2c + d = 0 & \Rightarrow \quad d = 24 \end{cases}$$

The particular solution is: $y_p = t^3 + 6t^2 + 18t + 24$

The general solution: $y = \left(C_1 + C_2 t\right)e^t + t^3 + 6t^2 + 18t + 24$

$$y(0) = \left(C_1 + C_2(0)\right)e^{(0)} + (0)^3 + 6(0)^2 + 18(0) + 24$$

$$1 = C_1 + 24$$

$$y' = C_2 e^t \left(C_1 + C_2 t\right) e^t + 3t^2 + 12t + 18$$

$$y'(0) = C_2 e^{(0)} \left(C_1 + C_2 (0)\right) e^{(0)} + 3(0)^2 + 12(0) + 18$$

$$0 = C_2 + C_1 + 18$$

$$\boxed{C_1 = -23} \quad \boxed{C_2 = 5}$$

The general solution: $\underline{y(t) = (-23 + 5t)e^t + t^3 + 6t^2 + 18t + 24}$

Exercise

Find a solution to the homogeneous equation; then find a particular solution to form a general solution. Then find the solution satisfying the given initial conditions

$$y'' + 4y' + 4y = 4 - t; \quad y(0) = -1, \quad y'(0) = 0$$

Solution

The homogeneous *eqn.* : $y'' + 4y' + 4y = 0$

The characteristic *eqn.* : $\lambda^2 + 4\lambda + 4 = 0 \Rightarrow \lambda_{1,2} = -2 \qquad \Rightarrow y_h = \left(C_1 + C_2 t\right) e^{-2t}$

$$y_p = at + b$$

$$y'_p = a$$

$$y''_p = 0$$

$$y'' + 4y' + 4y = 4a + 4at + 4b = 4at + 4a + 4b$$

$$\begin{cases} 4a = -1 & \Rightarrow a = -\frac{1}{4} \\ 4a + 4b = 4 & \Rightarrow b = \frac{1}{4} \end{cases}$$

The particular solution is: $y_p = -\frac{1}{4}t + \frac{1}{4}$

The general solution is: $y(t) = \left(C_1 + C_2 t\right) e^{-2t} - \frac{1}{4}t + \frac{1}{4}$

$$y(0) = \left(C_1 + C_2 0\right) e^{-2(0)} - \frac{1}{4}0 + \frac{1}{4}$$

$$-1 = C_1 + \frac{1}{4} \Rightarrow \underline{C_1 = -\frac{5}{4}}$$

$$y'(t) = C_2 e^{-2t} - 2\left(C_1 + C_2 t\right) e^{-2t} - \frac{1}{4}$$

$$0 = C_2 - 2C_1 - \frac{1}{4}$$

$$C_2 = -\frac{10}{4} + \frac{1}{4} \Rightarrow \underline{C_2 = -\frac{9}{4}}$$

The general solution is: $\underline{y(t) = \left(C_1 + C_2 t\right) e^{-2t} - \frac{1}{4}t + \frac{1}{4}}$

Exercise

Find a solution to the homogeneous equation; then find a particular solution to form a general solution. Then find the solution satisfying the given initial conditions

$$y'' - 64y = 16; \quad y(0) = 1, \quad y'(0) = 0$$

Solution

The characteristic equation: $\lambda^2 - 64 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 8$

$$\boxed{y_h = C_1 e^{-8x} + C_2 e^{8x}}$$

The particular equation: $y_p = A \quad \Rightarrow \quad y'_p = y''_p = 0$

$$-64A = 16 \quad \Rightarrow \quad A = -\frac{1}{4} \Rightarrow \quad \boxed{y_p = -\frac{1}{4}}$$

$$y(x) = C_1 e^{-8x} + C_2 e^{8x} - \frac{1}{4}$$

$$y(0) = 1 \quad \to \quad C_1 + C_2 - \frac{1}{4} = 1$$

$$y' = -8C_1 e^{-8x} + 8C_2 e^{8x}$$

$$y'(0) = 0 \quad \to \quad -8C_1 + 8C_2 = 0$$

$$\begin{cases} C_1 + C_2 = \frac{5}{4} \\ -C_1 + C_2 = 0 \end{cases} \quad \to \quad C_2 = \frac{5}{8} = C_1$$

$$\boxed{y(x) = \frac{5}{8} e^{-8x} + \frac{5}{8} e^{8x} - \frac{1}{4}}$$

Exercise

Find a solution to the homogeneous equation; then find a particular solution to form a general solution. Then find the solution satisfying the given initial conditions

$$y'' - 5y' = t - 2 \quad y(0) = 0, \quad y'(0) = 2$$

Solution

The characteristic equation: $\lambda^2 - 5\lambda = 0 \quad \Rightarrow \quad \lambda_1 = 0, \ \lambda_2 = 5$

$$\boxed{y_h = C_1 + C_2 e^{5t}}$$

The particular equation: $y_p = At + Bt^2 \quad \Rightarrow \quad y'_p = A + 2Bt \ \to \ y''_p = 2B$

$$2B - 5A - 10Bt = t - 2$$

$$\begin{cases} -10B = 1 & \to B = -\frac{1}{10} \\ 2B - 5A = -2 & \to A = \frac{1}{5}\left(-\frac{1}{5} + 2\right) = \frac{9}{25} \end{cases}$$

$$\Rightarrow \quad \boxed{y_p = -\frac{1}{10} t^2 + \frac{9}{25} t}$$

$$y(t)=C_1+C_2e^{5t}-\frac{1}{10}t^2+\frac{9}{25}t$$

$$y(0)=0 \quad \Rightarrow \quad C_1+C_2=0 \Rightarrow \boxed{C_1=-C_2}$$

$$y'(t)=5C_2e^{5t}-\frac{1}{5}t+\frac{9}{25}$$

$$y'(0)=2 \quad \Rightarrow \quad 5C_2+\frac{9}{25}=2$$

$$\boxed{C_2=\frac{1}{5}\left(2-\frac{9}{25}\right)=\frac{41}{125}}$$

$$\boxed{C_1=-\frac{41}{125}}$$

$$\boxed{y(t)=-\frac{41}{125}+\frac{41}{125}e^{5t}-\frac{1}{10}t^2+\frac{9}{25}t}$$

Exercise

Find a solution to the homogeneous equation; then find a particular solution to form a general solution. Then find the solution satisfying the given initial conditions

$$y''+y=8\cos 2t-4\sin t \quad y\left(\frac{\pi}{2}\right)=-1, \quad y'\left(\frac{\pi}{2}\right)=0$$

Solution

The characteristic equation: $\lambda^2+1=0 \quad \Rightarrow \quad \lambda_{1,2}=\pm i$

$$\boxed{y_h=C_1\cos t+C_2\sin t}$$

The particular equation: $y_p=At\cos t+Bt\sin t+C\cos 2t+E\sin 2t$

$$y'_p=A\cos t-At\sin t+B\sin t+Bt\cos t-2C\sin 2t+2E\cos 2t$$

$$y''_p=-A\sin t-A\sin t-At\cos t+B\cos t+B\cos t-Bt\sin t-4C\cos 2t-4E\sin 2t$$

$$-2A\sin t+2B\cos t-At\cos t-Bt\sin t-4C\cos 2t-4E\sin 2t+At\cos t+Bt\sin t+C\cos 2t+E\sin 2t$$
$$=8\cos 2t-4\sin t$$

$$-2A\sin t+B\cos t-3C\cos 2t-3E\sin 2t=8\cos 2t-4\sin t$$

$$\begin{cases} -2A=-4 & \rightarrow A=2 \\ 2B=0 & \rightarrow B=0 \\ -3C=8 & \rightarrow C=-\frac{8}{3} \\ -3E=0 & \rightarrow E=0 \end{cases}$$

$$\boxed{y_p=2t\cos t-\frac{8}{3}\cos 2t}$$

$$\boxed{y(x)=C_1\cos t+C_2\sin t+2t\cos t-\frac{8}{3}\cos 2t}$$

$$y\left(\frac{\pi}{2}\right)=-1 \quad \rightarrow \quad C_2+\frac{8}{3}=-1 \Rightarrow \boxed{C_2=-\frac{11}{3}}$$

$$y'(x) = -C_1 \sin t + C_2 \cos t + 2\cos t - 2t\sin t + \frac{16}{3}\sin 2t$$

$$y'\left(\frac{\pi}{2}\right) = 0 \quad \rightarrow \quad -C_1 - 2\frac{\pi}{2} = 0 \Rightarrow \underline{C_1 = -\pi|}$$

$$\underline{y(x) = -\pi\cos t - \frac{11}{3}\sin t + 2t\cos t - \frac{8}{3}\cos 2t|}$$

Solution Section 2.6 - Variation of Parameters

Exercise

$\left\{y_1(x)=e^{2x},\ y_2(x)=e^{-3x}\right\}$ is a fundamental set of solutions of $y''+y'-6y=3e^{2x}$.

Find a particular solution of the equation?

Solution

$$W=\begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x}\end{vmatrix}=-3e^{-x}-2e^{-x}\ \underline{=-5e^{-x}\neq 0}$$

$$v_1(x)=-\int\frac{e^{-3x}\left(3e^{2x}\right)}{-5e^{-x}}dx=\frac{3}{5}\int dx\ \underline{=\frac{3}{5}x} \qquad v_1(x)=-\int\frac{y_2 g(x)}{W}dx$$

$$v_2(x)=\int\frac{e^{2x}\left(3e^{2x}\right)}{-5e^{-x}}dx=-\frac{3}{5}\int e^{5x}dx\ \underline{=-\frac{3}{25}e^{5x}} \qquad v_2(x)=\int\frac{y_1 g(x)}{W}dx$$

The particular solution:

$$\begin{aligned} y_p &= v_1y_1+v_2y_2 \\ &=\frac{3}{5}xe^{2x}-\frac{3}{25}e^{-3x}e^{5x} \\ &=\frac{3}{5}xe^{2x}-\frac{3}{25}e^{2x}\end{aligned}$$

The general solution:

$$\begin{aligned} y(x) &= C_1e^{2x}+C_2e^{-3x}+\frac{3}{5}xe^{2x}-\frac{3}{25}e^{2x} \\ &=\left(C_1-\frac{3}{25}\right)e^{2x}+C_2e^{-3x}+\frac{3}{5}xe^{2x} \\ &\underline{=C_3e^{2x}+C_2e^{-3x}+\frac{3}{5}xe^{2x}}\end{aligned}$$

Exercise

Find a particular solution to: $y''-y=t+3$

Solution

The homogeneous equation for the differential equation $y''-y=0$

$$\lambda^2-1=0 \qquad \textit{Solve for } \lambda$$

$$\lambda_1=-1 \quad \lambda_2=1$$

Therefore; $y_1=e^{-t}$ *and* $y_2=e^{t}$

$$W = \begin{vmatrix} e^{-t} & e^{t} \\ -e^{-t} & e^{t} \end{vmatrix} = 1+1 = 2 \neq 0$$

$$v'_1 = \frac{-y_2}{y_1 y'_2 - y'_1 y_2} g(t)$$

$$= -\frac{e^t}{2}(t+3)$$

$$v_1(t) = -\frac{1}{2}\int (t+3)e^t dt \qquad \begin{cases} u = t+3 & dv = e^t dt \\ du = dt & v = e^t \end{cases}$$

$$= -\frac{1}{2}\left[e^t(t+3) - \int e^t dt \right]$$

$$= -\frac{1}{2}\left(te^t + 3e^t - e^t \right)$$

$$= -\frac{1}{2}\left(te^t + 2e^t \right)$$

$$= -\left(\frac{1}{2}te^t + e^t \right)$$

$$v'_2 = \frac{y_1}{y_1 y'_2 - y'_1 y_2} g(t)$$

$$= \frac{e^{-t}}{2}(t+3)$$

$$v_1(t) = \frac{1}{2}\int (t+3)e^{-t} dt \qquad \begin{cases} u = t+3 & dv = e^{-t} dt \\ du = dt & v = -e^t \end{cases}$$

$$= \frac{1}{2}\left[-e^{-t}(t+3) + \int e^{-t} dt \right]$$

$$= \frac{1}{2}\left(-te^{-t} - 3e^{-t} - e^{-t} \right)$$

$$= -\frac{1}{2}\left(te^{-t} + 4e^{-t} \right)$$

$$= -\frac{1}{2}te^{-t} - 2e^{-t}$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$= -\left(\frac{1}{2}te^t + e^t \right)e^{-t} - \left(\frac{1}{2}te^{-t} + 2e^{-t} \right)e^t$$

$$= -\frac{1}{2}t - 1 - \frac{1}{2}t - 2$$

$$\underline{= -t - 3}$$

Exercise

Find a particular solution to: $y'' - 2y' + y = e^t$

Solution

The homogeneous equation for the differential equation $y'' - 2y' + y = 0$

$$\lambda^2 - 2\lambda + 1 = 0 \qquad \textit{Solve for } \lambda$$

$$\lambda_{1,2} = 1$$

Therefore; $y_1 = e^t$ *and* $y_2 = te^t$

$$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}$$

$$= e^{2t} + te^{2t} - te^{2t}$$

$$= e^{2t} \neq 0$$

$$v_1' = \frac{-y_2}{y_1y_2' - y_1'y_2} g(t)$$

$$= -\frac{te^t}{e^t\left(e^t + te^t\right) - e^t . te^t} e^t$$

$$= -\frac{te^{2t}}{e^{2t}}$$

$$= -t$$

$$v_1(t) = \int -t dt$$

$$= -\frac{1}{2}t^2$$

$$v_2' = \frac{y_1}{y_1y_2' - y_1'y_2} g(t)$$

$$= \frac{e^t}{e^{2t}} e^t$$

$$= 1$$

$$v_2(t) = \int 1\, dt$$

$$= t$$

$$y_p = v_1y_1 + v_2y_2$$

$$= -\frac{1}{2}t^2e^t + t^2e^t$$

$$= \frac{1}{2}t^2e^t$$

Exercise

Find a particular solution to: $x'' - 4x' + 4x = e^{2t}$

Solution

The homogeneous equation for the differential equation: $x'' - 4x' + 4x = 0$

$\lambda^2 - 4\lambda + 4 = 0$ ***Solve for λ***

$$\boxed{\lambda_{1,2} = 2}$$

Therefore; $x_1 = e^{2t}$ *and* $x_2 = te^{2t}$

$$W = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t} + 2te^{2t} \end{vmatrix}$$

$$= e^{4t} + 2te^{4t} - 2te^{4t}$$

$$= e^{4t} \neq 0$$

$$v_1' = \frac{-y_2}{W} g(t)$$

$$= -\frac{te^{2t}}{e^{4t}} e^{2t}$$

$$= -t$$

$$v_1(t) = \int -t\,dt$$

$$= -\frac{1}{2}t^2$$

$$v_2' = \frac{y_1}{W} g(t)$$

$$= \frac{e^{2t}}{e^{4t}} e^{2t}$$

$$= 1$$

$$v_2(t) = \int 1\,dt = t$$

$$y_p = v_1 y_1 + v_2 y_2$$

$$= -\frac{1}{2}t^2 e^{2t} + t^2 e^{2t}$$

$$\boxed{= \frac{1}{2}t^2 e^{2t}}$$

Exercise

Find a particular solution to: $x'' + x = \tan^2 t$

Solution

The homogeneous equation for the differential equation: $x'' + x = 0$

$x_1 = \cos t \quad \textit{and} \quad x_2 = \sin t$

$$W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$$

$$= 1$$

$x_p = v_1 x_1 + v_2 x_2$

$$v_1' = \frac{-\sin t}{1} \tan^2 t$$

$$= -\sin t\left(\sec^2 t - 1\right)$$

$$= -\sin t\left(\frac{1}{\cos^2 t} - 1\right)$$

$$= -\frac{\sin t}{\cos^2 t} + \sin t$$

$$= -\sec t \tan t + \sin t$$

$v_1 = -\sec t - \cos t$

$$v_2' = \frac{\cos t}{1} \tan^2 t$$

$$= \cos t\left(\sec^2 t - 1\right)$$

$$= \sec t - \cos t$$

$v_2 = \ln\left|\sec t + \tan t\right| - \sin t$

$$x_p = (-\sec t - \cos t)\cos t + \left(\ln\left|\sec t + \tan t\right| - \sin t\right)\sin t$$

$$= -\sec t \cos t - \cos^2 t + \sin t \ln\left|\sec t + \tan t\right| - \sin^2 t$$

$$= -\sec t \frac{1}{\sec t} - \left(\cos^2 t + \sin^2 t\right) + \sin t \ln\left|\sec t + \tan t\right|$$

$$\underline{= -2 + \sin t \ln\left|\sec t + \tan t\right|}$$

Exercise

Find a particular solution to the given second-order differential equation $y'' + 25y = -2\tan(5x)$

Solution

$$\lambda^2 + 25 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 5i$$

$$y_p = C_1 \cos 5x + C_2 \sin 5x$$

$$W = \begin{vmatrix} \cos 5x & \sin 5x \\ -5\sin 5x & 5\cos 5x \end{vmatrix} = 5\cos^2 5x + 5\sin^2 5x = 5 \neq 0$$

$$v_1(x) = -\int \frac{\sin 5x(-2\tan 5x)}{5} dx \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$= \frac{2}{5}\int \frac{\sin^2 5x}{\cos 5x} dx$$

$$= \frac{2}{5}\int \frac{1-\cos^2 5x}{\cos 5x} dx$$

$$= \frac{2}{5}\int (\sec 5x - \cos 5x) dx$$

$$= \frac{2}{5}\left[\frac{1}{5}\ln|\tan 5x + \sec 5x| - \frac{1}{5}\sin 5x\right]$$

$$= \frac{2}{25}\left(\ln|\tan 5x + \sec 5x| - \sin 5x\right)$$

$$v_2(x) = \int \frac{\cos 5x(-2\tan 5x)}{5} dx = -\frac{2}{5}\int \sin 5x dx = \frac{2}{25}\cos 5x \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

The particular solution:

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \frac{2}{25}\left(\ln|\tan 5x + \sec 5x| - \sin 5x\right)(\cos 5x) + \frac{2}{25}\cos 5x \sin 5x$$

$$= \frac{2}{25}\ln|\tan 5x + \sec 5x|$$

Exercise

Find a particular solution to the given second-order differential equation $y'' - 6y' + 9y = 5e^{3x}$

Solution

$$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \quad \Rightarrow \quad \lambda_{1,2} = 3$$

$$y_p = (C_1 + C_2 x)e^{3x}$$

$$W = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x} + 3xe^{6x} - 3xe^{6x} = e^{6x} \neq 0$$

$$v_1(x) = -\int \frac{xe^{3x}\left(5e^{3x}\right)}{e^{6x}}dx = -5\int xdx = \underline{-\frac{5}{2}x^2}\Big| \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$v_2(x) = \int \frac{e^{3x}\left(5e^{3x}\right)}{e^{6x}}dx = 5\int dx = \underline{5x}\Big| \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

The particular solution:

$$y_p = v_1 y_1 + v_2 y_2$$

$$= -\frac{5}{2}x^2 e^{3x} + 5x^2 e^{3x}$$

$$\boxed{= \frac{5}{2}x^2 e^{3x}}$$

Exercise

Find a particular solution to the given second-order differential equation $y'' + 4y = 2\cos 2x$

Solution

$$\lambda^2 + 4 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 2i$$

$$y_p = C_1 \cos 2x + C_2 \sin 2x$$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = \underline{2 \neq 0}\Big|$$

$$v_1(x) = -\int \frac{\sin 2x(2\cos 2x)}{2}dx = -\frac{1}{2}\int \sin 4x dx = \frac{1}{8}\cos 4x \qquad\qquad v_1(x) = -\int \frac{y_2 g(x)}{W}dx$$

$$v_2(x) = \int \frac{\cos 2x(2\cos 2x)}{2}dx \qquad\qquad v_2(x) = \int \frac{y_1 g(x)}{W}dx$$

$$= \int \cos^2 2x dx$$

$$= \frac{1}{2}\int (1 + \cos 4x)dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right)$$

The particular solution:

$$y_p = v_1 y_1 + v_2 y_2$$

$$\boxed{= \frac{1}{2}x\sin 2x}$$

Exercise

Find a particular solution to the given second-order differential equation $y'' - 5y' + 6y = 4e^{2x} + 3$

Solution

$$\lambda^2 - 5\lambda + 6 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{5 \pm \sqrt{1}}{2} = 2,\ 3$$

$$y_p = C_1 e^{2x} + C_2 e^{3x}$$

$$W = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x} \neq 0$$

$$v_1(x) = -\int \frac{e^{3x}\left(4e^{2x}+3\right)}{e^{5x}} dx = -\int \left(4 + 3e^{-2x}\right) dx = -4x + \frac{3}{2}e^{-2x} \qquad v_1(x) = -\int \frac{y_2 g(x)}{W} dx$$

$$v_2(x) = \int \frac{e^{2x}\left(4e^{2x}+3\right)}{e^{5x}} dx = \int \left(4e^{-x} + 3e^{-3x}\right) dx = -4e^{-x} - e^{-3x} \qquad v_2(x) = \int \frac{y_1 g(x)}{W} dx$$

The particular solution:

$$\begin{aligned} y_p &= v_1 y_1 + v_2 y_2 \\ &= \left(-4x + \tfrac{3}{2}e^{-2x}\right)e^{2x} - \left(4e^{-x} + e^{-3x}\right)e^{3x} \\ &= -4xe^{2x} + \tfrac{3}{2} - 4e^{2x} - 1 \\ &= -4xe^{2x} - 4e^{2x} + \tfrac{1}{2} \end{aligned}$$

Exercise

Verify that $y_1(t) = t$ and $y_2(t) = t^{-3}$ are solution to the homogenous equation

$$t^2 y''(t) + 3ty'(t) - 3y(t) = 0$$

Solution

The homogeneous equation for the differential equation: $y'' + \frac{3}{t}y' - \frac{3}{t^2}y = 0$

For $y_1 = t \quad \rightarrow y_1' = 1 \quad \rightarrow y_1'' = 0$

$$\begin{aligned} y'' + \frac{3}{t}y' - \frac{3}{t^2}y &= 0 + \frac{3}{t}(1) - \frac{3}{t^2}t \\ &= \frac{3}{t} - \frac{3}{t} \\ &= 0 \end{aligned}$$

$y_1(t)$ is a solution

For $y_2 = t^{-3} \quad \rightarrow y_1' = -3t^{-4} \quad \rightarrow y_1'' = 12t^{-5}$

$$y''+\frac{3}{t}y'-\frac{3}{t^2}y=12t^{-5}+\frac{3}{t}\left(-3t^{-4}\right)-\frac{3}{t^2}t^{-3}$$
$$=12t^{-5}-9t^{-5}-3t^{-5}$$
$$=0$$

$y_2(t)$ is a solution

Wronskian: $W\left(t,t^{-3}\right)=\begin{vmatrix} t & t^{-3} \\ 1 & -3t^{-4} \end{vmatrix}=-4t^{-3}$

$$v_1' = -\frac{t^{-3}t^{-3}}{-4t^{-3}}=\frac{1}{4}t^{-3} \quad\Rightarrow\quad v_1=\int\left(\frac{1}{4}t^{-3}\right)dt=-\frac{1}{8}t^{-2}$$

$$v_2' = -\frac{t\cdot t^{-3}}{-4t^{-3}}=-\frac{1}{4}t \quad\Rightarrow\quad v_2=\int\left(-\frac{1}{4}t\right)dt=-\frac{1}{8}t^{2}$$

$$y_p=v_1y_1+v_2y_2$$
$$=-\frac{1}{8}t^{-2}t-\frac{1}{8}t^2t^{-3}$$
$$=-\frac{1}{8}t^{-1}-\frac{1}{8}t^{-1}$$
$$\boxed{=-\frac{1}{4}t^{-1}}$$

Thus, the general solution is: $\boxed{y(t)=C_1t+\frac{C_2}{t^3}-\frac{1}{4t}}$

Exercise

Find the general solution to the given differential equation. $y''-4y'+4y=(x+1)e^{2x}$

Solution

Characteristic Eqn.: $\lambda^2-4\lambda+4=(\lambda-2)^2=0 \quad\Rightarrow\quad \lambda_{1,2}=2$

The homogeneous Eqn.: $\boxed{y_h=C_1e^{2x}+C_2xe^{2x}}$

$$W=\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}+2xe^{2x} \end{vmatrix}=e^{4x}+2xe^{4x}-2xe^{4x}\ \boxed{=e^{4x}}$$

$$u_1' = -\frac{xe^{2x}(x+1)e^{2x}}{e^{4x}}=-x^2-x \quad\Rightarrow\quad u_1=\int\left(-x^2-x\right)dx=-\frac{1}{3}x^3-\frac{1}{2}x^2 \qquad u_1'=-\frac{y_2g(t)}{W}$$

$$u_2' = \frac{e^{2x}(x+1)e^{2x}}{e^{4x}}=x+ \quad\Rightarrow\quad u_2=\int(x+1)dx=\frac{1}{2}x^2+x \qquad u_2'=\frac{y_1g(t)}{W}$$

$$y_p=u_1y_1+u_2y_2=\left(-\frac{1}{3}x^3-\frac{1}{2}x^2\right)e^{2x}+\left(\frac{1}{2}x^2+x\right)xe^{2x}$$

$$=\left(-\frac{1}{3}x^3-\frac{1}{2}x^2+\frac{1}{2}x^3+x^2\right)e^{2x}$$

$$=\boxed{\left(\frac{1}{6}x^3+\frac{1}{2}x^2\right)e^{2x}}$$

$$\boxed{y(x)=C_1e^{2x}+C_2xe^{2x}+\left(\frac{1}{6}x^3+\frac{1}{2}x^2\right)e^{2x}}$$

Exercise

Find the general solution to the given differential equation. $\quad y''+9y=\csc 3x$

Solution

Characteristic Eqn.: $\lambda^2+9=0 \quad\Rightarrow\quad \lambda_{1,2}=3i$

The homogeneous Eqn.: $\boxed{y_h=C_1\cos 3x+C_2\sin 3x}$

$$W=\begin{vmatrix}\cos 3x & \sin 3x\\ -3\sin 3x & 3\cos 3x\end{vmatrix}=3\cos^2 3x+3\sin^2 3x\ \underline{=3}$$

$$u_1'=-\frac{(\sin 3x)(\csc 3x)}{3}=-\frac{1}{3}\quad\Rightarrow\quad u_1=\int\left(-\frac{1}{3}\right)dx=-\frac{1}{3}x \qquad u_1'=-\frac{y_2g(t)}{W}$$

$$u_2'=\frac{(\cos 3x)(\csc 3x)}{3}=\frac{1}{3}\frac{\cos 3x}{\sin 3x} \qquad u_2'=\frac{y_1g(t)}{W}$$

$$\Rightarrow\quad u_2=\int\left(\frac{1}{3}\frac{\cos 3x}{\sin 3x}\right)dx=\frac{1}{9}\int\frac{1}{\sin 3x}\,d(\sin 3x)=\frac{1}{9}\ln|\sin 3x|$$

$$y_p=-\frac{1}{3}x\cos 3x+\frac{1}{9}\sin 3x\ln|\sin 3x| \qquad y_p=u_1y_1+u_2y_2$$

$$\boxed{y(x)=C_1\cos 3x+C_2\sin 3x-\frac{1}{3}x\cos 3x+\frac{1}{9}\sin 3x\ln|\sin 3x|}$$

Exercise

Find the general solution to the given differential equation. $\quad y''-y=\frac{1}{x}$

Solution

Characteristic Eqn.: $\lambda^2-1=0 \quad\Rightarrow\quad \lambda_{1,2}=\pm 1$

The homogeneous Eqn.: $\boxed{y_h=C_1e^{-x}+C_2e^{x}}$

$$W=\begin{vmatrix}e^{-x} & e^{x}\\ -e^{-x} & e^{x}\end{vmatrix}=1+1\ \underline{=2}$$

$$u_1' = -\frac{e^x \frac{1}{x}}{2} = -\frac{1}{2}\frac{e^x}{x} \quad \Rightarrow \quad u_1 = -\frac{1}{2}\int \frac{e^x}{x} dx \qquad\qquad u_1' = -\frac{y_2 g(t)}{W}$$

$$u_2' = \frac{e^{-x} \frac{1}{x}}{2} = \frac{1}{2}\frac{e^{-x}}{x} \quad \Rightarrow \quad u_2 = \frac{1}{2}\int \frac{e^{-x}}{x} dx \qquad\qquad u_2' = \frac{y_1 g(t)}{W}$$

$$y_p = -\frac{1}{2}e^{-x}\int \frac{e^x}{x}dx + \frac{1}{2}e^x\int \frac{e^{-x}}{x}dx \qquad\qquad y_p = u_1 y_1 + u_2 y_2$$

$$\boxed{y(x) = C_1 e^{-x} + C_2 e^x - \frac{1}{2}e^{-x}\int \frac{e^x}{x}dx + \frac{1}{2}e^x\int \frac{e^{-x}}{x}dx}$$

Exercise

Find the general solution to the given differential equation. $y'' + y = \sin x$

Solution

Characteristic Eqn.: $\lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm i$

The homogeneous Eqn.: $\boxed{y_h = C_1 \cos x + C_2 \sin x}$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = \boxed{1}$$

$$u_1' = -\frac{(\sin x)(\sin x)}{1} = -\sin^2 x \quad \Rightarrow \quad u_1 = -\int \left(\sin^2 x\right)dx = -\frac{x}{2} + \frac{1}{4}\sin 2x \qquad u_1' = -\frac{y_2 g(t)}{W}$$

$$u_2' = \frac{(\cos x)(\sin x)}{1} = \cos x \sin x \quad \Rightarrow \quad u_2 = \int (\cos x \sin x)dx = \frac{1}{2}\sin^2 x \qquad u_2' = \frac{y_1 g(t)}{W}$$

$$y_p = \left(-\frac{x}{2} + \frac{1}{4}\sin 2x\right)\cos x + \frac{1}{2}\sin^2 x \sin x \qquad\qquad y_p = u_1 y_1 + u_2 y_2$$

$$= -\frac{1}{2}x\cos x + \frac{1}{4}\sin 2x \cos x + \frac{1}{2}\sin^3 x$$

$$= -\frac{1}{2}x\cos x + \frac{1}{2}\sin x \cos^2 x + \frac{1}{2}\sin^3 x$$

$$= -\frac{1}{2}x\cos x + \frac{1}{2}\sin x\left(\cos^2 x + \sin^2 x\right)$$

$$= -\frac{1}{2}x\cos x + \frac{1}{2}\sin x$$

$$\boxed{y(x) = C_1 \cos x + C_2 \sin x - \frac{1}{2}x\cos x + \frac{1}{2}\sin x}$$

Exercise

Find the general solution to the given differential equation. $y'' + y = \cos^2 x$

Solution

Characteristic Eqn.: $\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

The homogeneous Eqn.: $\underline{y_h = C_1 \cos x + C_2 \sin x}$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = \underline{1}$$

$$u_1' = -\frac{(\sin x)\left(\cos^2 x\right)}{1} = -\sin^2 x \quad \Rightarrow \quad u_1 = \int \left(\cos^2 x\right) d(\cos x) = \frac{1}{3}\cos^3 x \qquad u_1' = -\frac{y_2 g(t)}{W}$$

$$u_2' = \frac{(\cos x)\left(\cos^2 x\right)}{1} = \cos^3 x \qquad u_2' = \frac{y_1 g(t)}{W}$$

$$u_2 = \int \left(\cos^3 x\right) dx = \int \left(\cos^2 x\right) d(\sin x) = \int \left(1 - \sin^2 x\right) d(\sin x) = \sin x - \frac{1}{3}\sin^3 x$$

$$\begin{aligned} y_p &= \frac{1}{3}\cos^4 x + \sin^2 x - \frac{1}{3}\sin^4 x \qquad y_p = u_1 y_1 + u_2 y_2 \\ &= \frac{1}{3}\left(\cos^4 x - \sin^4 x\right) + \sin^2 x \\ &= \frac{1}{3}\left(\cos^2 x - \sin^2 x\right)\left(\cos^2 x + \sin^2 x\right) + \sin^2 x \\ &= \frac{1}{3}\cos 2x + \sin^2 x \end{aligned}$$

$$\underline{y(x) = C_1 \cos x + C_2 \sin x + \frac{1}{3}\cos 2x + \sin^2 x}$$

Exercise

Find the general solution to the given differential equation. $y'' - y = \cosh x$

Solution

Characteristic Eqn.: $\lambda^2 - 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 1$

The homogeneous Eqn.: $\boxed{y_h = A_1 e^{-x} + A_2 e^x}$

$$W = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix} = 1 + 1 \underline{= 2}$$

$$u_1' = -\frac{e^x \cosh x}{2} = -\frac{1}{2} e^x \frac{e^x + e^{-x}}{2} = -\frac{1}{4}\left(e^{2x} + 1\right) \qquad u_1' = -\frac{y_2 g(t)}{W}$$

$$\Rightarrow u_1 = -\frac{1}{4}\int \left(e^{2x} + 1\right) dx = -\frac{1}{4}\left(\frac{1}{2}e^{2x} + x\right) = -\frac{1}{8}e^{2x} - \frac{1}{4}x$$

$$u_2' = \frac{e^{-x} \cosh x}{2} = \frac{1}{2} e^{-x} \frac{e^x + e^{-x}}{2} = \frac{1}{4}\left(1 + e^{-2x}\right) \qquad u_2' = \frac{y_1 g(t)}{W}$$

$$\Rightarrow u_1 = \frac{1}{4}\int \left(1 + e^{-2x}\right) dx = \frac{1}{4}\left(x - \frac{1}{2}e^{-2x}\right) = \frac{1}{4}x - \frac{1}{8}e^{-2x}$$

$$y_p = \left(-\frac{1}{8}e^{2x} - \frac{1}{4}x\right)e^{-x} + \left(\frac{1}{4}x - \frac{1}{8}e^{-2x}\right)e^x \qquad y_p = u_1 y_1 + u_2 y_2$$

$$= -\frac{1}{8}e^x - \frac{1}{4}xe^{-x} + \frac{1}{4}xe^x - \frac{1}{8}e^{-x}$$

$$= -\frac{1}{8}e^x - \frac{1}{8}e^{-x} + \frac{1}{2}x\left(\frac{e^x - e^{-x}}{2}\right)$$

$$\boxed{= -\frac{1}{8}e^x - \frac{1}{8}e^{-x} + \frac{1}{2}x \sinh x}$$

$$y(x) = A_1 e^{-x} + A_2 e^x - \frac{1}{8}e^x - \frac{1}{8}e^{-x} + \frac{1}{2}x \sinh x$$

$$= \left(A_1 - \frac{1}{8}\right)e^{-x} + \left(A_2 - \frac{1}{8}\right)e^x + \frac{1}{2}x \sinh x$$

$$\boxed{y(x) = C_1 e^{-x} + C_2 e^x + \frac{1}{2}x \sinh x}$$

Or $\boxed{y(x) = C_1 e^{-x} + C_2 e^x - \frac{1}{4}xe^{-x} + \frac{1}{4}xe^x}$

Exercise

Find the general solution to the given differential equation. $y'' - 4y = \frac{e^x}{x}$

Solution

Characteristic Eqn.: $\lambda^2 - 4 = 0 \implies \lambda_{1,2} = \pm 2$

The homogeneous Eqn.: $\boxed{y_h = C_1 e^{-2x} + C_2 e^{2x}}$

$$W = \begin{vmatrix} e^{-2x} & e^{2x} \\ -2e^{-2x} & 2e^{2x} \end{vmatrix} = 2 + 2 = 4$$

$$u_1' = -\frac{e^{2x}\frac{e^{2x}}{x}}{4} = -\frac{1}{4}\frac{e^{4x}}{x} \implies u_1 = -\frac{1}{4}\int \frac{e^{4x}}{x}dx \qquad u_1' = -\frac{y_2 g(t)}{W}$$

$$u_2' = \frac{e^{-2x}\frac{e^{2x}}{x}}{4} = \frac{1}{4}\frac{1}{x} \implies u_2 = \frac{1}{4}\int \frac{1}{x}dx = \frac{1}{4}\ln|x| \qquad u_2' = \frac{y_1 g(t)}{W}$$

$$\boxed{y_p = -\frac{1}{4}e^{-2x}\int \frac{e^{4x}}{x}dx + \frac{1}{4}e^{2x}\ln|x|} \qquad y_p = u_1 y_1 + u_2 y_2$$

$$\boxed{y(x) = C_1 e^{-2x} + C_2 e^{2x} + \frac{1}{4}e^{2x}\ln|x| - \frac{1}{4}e^{-2x}\int \frac{e^{4x}}{x}dx}$$

Exercise

Find the general solution to the given differential equation. $y'' + 3y' + 2y = \sin e^x$

Solution

Characteristic Eqn.: $\lambda^2 + 3\lambda + 2 = 0 \implies \lambda_1 = -2,\ \lambda_2 = -1$

The homogeneous Eqn.: $\boxed{y_h = C_1 e^{-2x} + C_2 e^{-x}}$

$$W = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} = -e^{-3x} + 2e^{-3x} = e^{-3x}$$

$$u_1' = -\frac{e^{-x}\sin e^x}{e^{-3x}} = -e^{2x}\sin e^x \implies u_1 = -\int \left(e^{2x}\sin e^x\right)dx \qquad u_1' = -\frac{y_2 g(t)}{W}$$

$$u = e^x \qquad dv = e^x \sin e^x$$

$$du = e^x dx \qquad v = \int \sin e^x d\left(e^x\right) = -\cos e^x$$

$$\int \left(e^{2x}\sin e^x\right)dx = -e^x\cos e^x + \int e^x \cos e^x dx$$

$$= -e^x \cos e^x + \int \cos e^x d\left(e^x\right)$$

$$\boxed{= -e^x \cos e^x + \sin e^x}$$

$$u_1 = -\int \left(e^{2x} \sin e^x\right) dx = e^x \cos e^x - \sin e^x$$

$$u_2' = \frac{e^{-2x} \sin e^x}{e^{-3x}} = e^x \sin e^x \quad \Rightarrow \quad u_2 = \int \left(e^x \sin e^x\right) dx = \int \left(\sin e^x\right) d\left(e^x\right) = -\cos e^x \qquad u_2' = \frac{y_1 g(t)}{W}$$

$$y_p = e^{-2x}\left(e^x \cos e^x - \sin e^x\right) + e^{-x}\left(-\cos e^x\right) \qquad y_p = u_1 y_1 + u_2 y_2$$

$$= e^{-x} \cos e^x - e^{-2x} \sin e^x - e^{-x} \cos e^x$$

$$\boxed{= -e^{-2x} \sin e^x}$$

$$\boxed{y(x) = C_1 e^{-2x} + C_2 e^{-x} - e^{-2x} \sin e^x}$$

Exercise

Find the general solution to the given differential equation. $\quad y'' - 2y' + y = \dfrac{e^x}{1+x^2}$

Solution

Characteristic Eqn.: $\lambda^2 - 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = 1$

The homogeneous Eqn.: $\boxed{y_h = C_1 e^x + C_2 x e^x}$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} + xe^{2x} - xe^{2x} \; \boxed{= e^{2x}}$$

$$u_1' = -\frac{xe^x}{e^{2x}} \frac{e^x}{1+x^2} = -\frac{x}{1+x^2} \qquad u_1' = -\frac{y_2 g(t)}{W}$$

$$\Rightarrow u_1 = -\int \frac{x}{1+x^2} dx = -\frac{1}{2} \int \frac{1}{1+x^2} d\left(1+x^2\right) = -\frac{1}{2} \ln\left(1+x^2\right)$$

$$u_2' = \frac{e^x}{e^{2x}} \frac{e^x}{1+x^2} = \frac{1}{1+x^2} \quad \Rightarrow \quad u_2 = \int \left(\frac{1}{1+x^2}\right) dx = \tan^{-1} x \qquad u_2' = \frac{y_1 g(t)}{W}$$

$$\boxed{y_p = -\frac{1}{2} e^x \ln\left(1+x^2\right) + xe^x \tan^{-1} x} \qquad y_p = u_1 y_1 + u_2 y_2$$

$$\boxed{y(x) = C_1 e^x + C_2 x e^x - \frac{1}{2} e^x \ln\left(1+x^2\right) + xe^x \tan^{-1} x}$$

Exercise

Find the general solution to the given differential equation. $y'' + 2y' + y = e^{-x}\ln x$

Solution

Characteristic Eqn.: $\lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = -1$

The homogeneous Eqn.: $\boxed{y_h = C_1 e^{-x} + C_2 x e^{-x}}$

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-2x} - xe^{-2x} + xe^{-2x} \ \boxed{= e^{-2x}}$$

$$u_1' = -\frac{xe^{-x}}{e^{-2x}} e^{-x}\ln x = -x\ln x \ \Rightarrow \ u_1 = -\int x\ln x\,dx \qquad u_1' = -\frac{y_2 g(t)}{W}$$

$$u = \ln x \qquad dv = x$$

$$du = \frac{1}{x}dx \qquad v = \int x\,dx = \frac{1}{2}x^2$$

$$\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$$

$$u_1 = -\int x\ln x\,dx \ \boxed{= \frac{1}{4}x^2 - \frac{1}{2}x^2\ln x}$$

$$u_2' = \frac{e^{-x}}{e^{-2x}} e^{-x}\ln x = \ln x \ \Rightarrow \ u_1 = \int \ln x\,dx = x\ln x - x \qquad u_2' = \frac{y_1 g(t)}{W}$$

$$y_p = e^{-x}\left(\frac{1}{4}x^2 - \frac{1}{2}x^2\ln x\right) + xe^{-x}\left(x\ln x - x\right) \qquad y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{1}{4}x^2e^{-x} - \frac{1}{2}x^2e^{-x}\ln x + x^2e^{-x}\ln x - x^2e^{-x}$$

$$\boxed{= \frac{1}{2}x^2e^{-x}\ln x - \frac{3}{4}x^2e^{-x}}$$

$$\boxed{y(x) = C_1e^{-x} + C_2xe^{-x} + \frac{1}{2}x^2e^{-x}\ln x - \frac{3}{4}x^2e^{-x}}$$

Solution   Section 2.7 - Forced Harmonic Motion

Exercise

A 1-*kg* mass is attached to a spring $k = 4kg/s^2$ and the system is allowed to come to rest. The spring-mass system is attached to a machine that supplies external driving force $f(t) = 4\cos\omega t$ *Newtons*. The system is started from equilibrium; the mass is having neither initial displacement nor velocity. Ignore any damping forces.

a) Find the position of the mass as a function of time

b) Place your answer in the form $s(t) = A\sin\delta t\sin\overline{\omega}t$. Select an ω near the natural frequency of the system to demonstrate the "beating" of the system. Sketch a plot shows the "beats:" and include the envelope of the beating motion in your plot.

Solution

a) $mx'' + \omega_0^2 x = f(t)$

$$mx'' + kx = f(t) \qquad x(0) = x'(0) = 0$$

$$x'' + 4x = 4\cos\omega t$$

$$x(t) = \frac{A}{\left(\omega_0^2 - \omega^2\right)}\left(\cos\omega t - \cos\omega_0 t\right)$$

$$x(t) = \frac{4}{4-\omega^2}\left(\cos\omega t - \cos 2t\right)$$

b) $x(t) = \frac{4}{4-\omega^2}\left(\cos\omega t - \cos 2t\right)$ $\qquad \cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

$$= \frac{4}{4-\omega^2}\left[-2\sin\left(\frac{\omega+2}{2}t\right)\sin\left(\frac{\omega-2}{2}t\right)\right]$$

$$= \frac{4}{4-\omega^2}\left[2\sin\left(\frac{\omega+2}{2}t\right)\sin\left(\frac{2-\omega}{2}t\right)\right]$$

Mean frequency: $\overline{\omega} = \frac{\omega_0 + \omega}{2} = \frac{2+\omega}{2}$

$$2\overline{\omega} = 2 + \omega$$

Half difference: $\delta = \frac{\omega_0 - \omega}{2} = \frac{2-\omega}{2}$

$$2\delta = 2 - \omega$$

$$4 - \omega^2 = (2+\omega)(2-\omega) = 2\overline{\omega}2\delta$$

$$= 2\overline{\omega}2\delta$$

$$= 4\overline{\omega}\delta$$

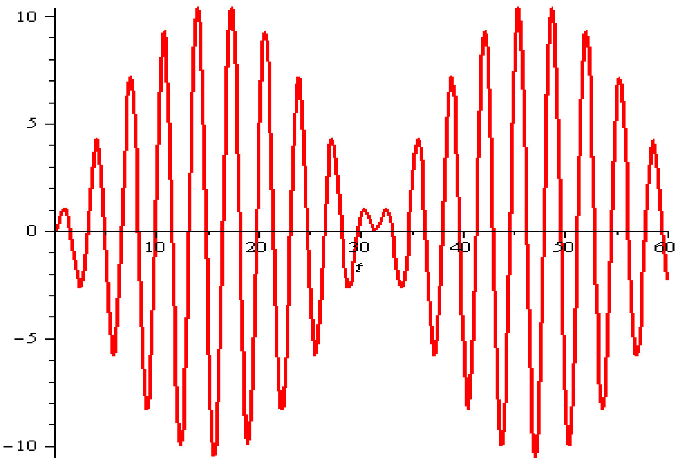
$$x(t)=\frac{8}{4\bar{\omega}\delta}\sin\varpi t\sin\delta t$$

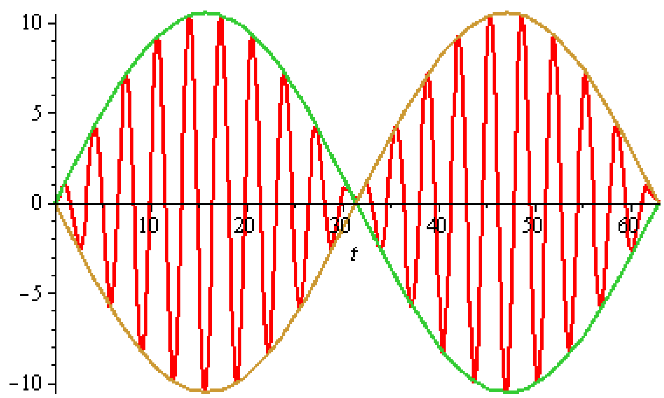
$$=\frac{2}{\bar{\omega}\delta}\sin\varpi t\sin\delta t$$

If we choose $\omega=1.8$ near to $\omega_0=2$

That implies to: $\bar{\omega}=\frac{2+1.8}{2}\approx 1.9$ and $\delta=\frac{2-1.8}{2}\approx 0.1$

$$x(t)=\frac{2}{0.19}\sin 0.1t\sin 1.9t$$





Exercise

Find a particular solution to the differential equation using undetermined coefficients. Find and plot the solution of the initial value problem. Superimpose the plots of the transient response and the steady state solution.

$$x'' + 7x' + 10x = 3\cos 3t \qquad x(0) = -1,\ \ x'(0) = 0$$

Solution

The particular solution: $x(t) = a\cos 3t + b\sin 3t$

$$x' = -3a\sin 3t + 3b\cos 3t$$
$$x'' = -9a\cos 3t - 9b\sin 3t$$

$$-9a\cos 3t - 9b\sin 3t + 7\left(-3a\sin 3t + 3b\cos 3t\right) + 10\left(a\cos 3t + b\sin 3t\right) = 3\cos 3t$$

$$-9a\cos 3t - 9b\sin 3t - 21a\sin 3t + 21b\cos 3t + 10a\cos 3t + 10b\sin 3t = 3\cos 3t$$

$$a\cos 3t - 21a\sin 3t + 21b\cos 3t + b\sin 3t = 3\cos 3t$$

$$\left(a + 21b\right)\cos 3t + \left(b - 21a\right)\sin 3t = 3\cos 3t$$

$$\begin{cases} a + 21b = 3 \\ -21a + b = 0 \end{cases} \Rightarrow a = \frac{3}{442} \quad b = \frac{63}{442}$$

The particular solution (***steady-state solution***):

$$\boxed{x_p(t) = \frac{3}{442}\cos 3t + \frac{63}{442}\sin 3t}$$

The homogeneous eq.: $x'' + 7x' + 10x = 0$

The characteristic eq.: $\lambda^2 + 7\lambda + 10 = 0 \ \Rightarrow \ \lambda_1 = -5,\ \lambda_2 = -2$

$$x_h(t) = C_1 e^{-5t} + C_2 e^{-2t}$$

$$x(t) = \frac{3}{442}\cos 3t + \frac{63}{442}\sin 3t + C_1 e^{-5t} + C_2 e^{-2t} \qquad x(0) = -1,\ \ x'(0) = 0$$

$$x(0) = \frac{3}{442}\cos 3(0) + \frac{63}{442}\sin 3(0) + C_1 e^{-5(0)} + C_2 e^{-2(0)}$$

$$-1 = \frac{3}{442} + C_1 + C_2$$

$$C_1 + C_2 = -\frac{445}{442}$$

$$x'(t) = -\frac{9}{442}\sin 3t + \frac{189}{442}\cos 3t - 5C_1 e^{-5t} - 2C_2 e^{-2t}$$

$$x'(0) = -\frac{9}{442}\sin 3(0) + \frac{189}{442}\cos 3(0) - 5C_1 e^{-5(0)} - 2C_2 e^{-2(0)}$$

$$0 = \frac{189}{442} - 5C_1 - 2C_2$$

$$5C_1 + 2C_2 = \frac{189}{442}$$

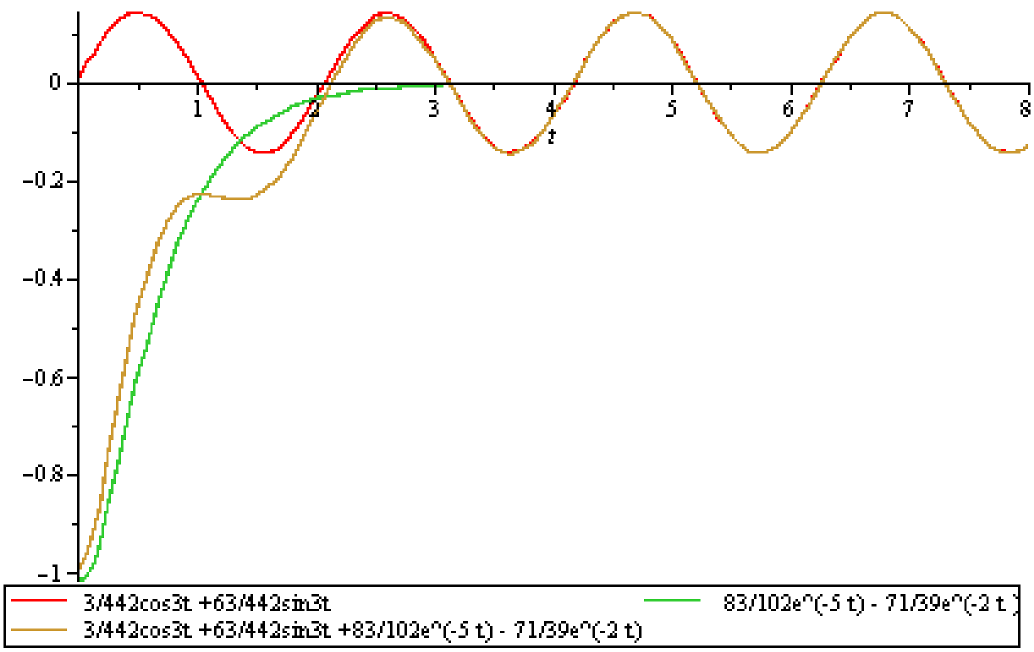
$$\left.\begin{aligned} C_1 + C_2 &= -\frac{445}{442} \\ 5C_1 + 2C_2 &= \frac{189}{442} \end{aligned}\right\} \rightarrow C_1 = \frac{83}{102} \quad C_2 = -\frac{71}{39}$$

Transient response solution:

$$\boxed{x_h(t) = \frac{83}{102}e^{-5t} - \frac{71}{39}e^{-2t}}$$

The general solution:

$$\boxed{x(t) = \frac{3}{442}\cos 3t + \frac{63}{442}\sin 3t + \frac{83}{102}e^{-5t} - \frac{71}{39}e^{-2t}}$$



Complex Method

$$x'' + 7x' + 10x = 3\cos 3t$$

The particular solution: $z = Ae^{i3t}$

$$z' = (3i)Ae^{i3t}$$

$$z'' = (3i)^2 Ae^{i3t}$$

$$z'' + 7z' + 10z = 3e^{i3t}$$

$$(3i)^2 Ae^{i3t} + 7(3i)Ae^{i3t} + 10Ae^{i3t} = 3e^{i3t}$$

$$(-9 + 21i + 10)A = 3$$

$$(1 + 21i)A = 3$$

$$\begin{aligned} A &= 3\frac{1}{1+21i}\cdot\frac{1-21i}{1-21i} \\ &= 3.\frac{1-21i}{1+441} \\ &= \frac{3}{442} - i\frac{63}{442} \end{aligned}$$

$$\begin{aligned} z &= \left(\frac{3}{442} - i\frac{63}{442}\right)e^{i3t} \\ &= \left(\frac{3}{442} - i\frac{63}{442}\right)(\cos 3t + i\sin 3t) \\ &= \frac{3}{442}\cos 3t + \frac{63}{442}\sin 3t + i\left(\frac{3}{442}\sin 3t - \frac{63}{442}\cos 3t\right) \end{aligned}$$

The particular solution (***steady-state solution***):

$$\boxed{x_p(t) = \frac{3}{442}\cos 3t + \frac{63}{442}\sin 3t}$$

Exercise

Find a particular solution to the differential equation using undetermined coefficients. Find and plot the solution of the initial value problem. Superimpose the plots of the transient response and the steady state solution.

$$x'' + 4x' + 5x = 3\sin t \qquad x(0) = 0, \;\; x'(0) = -3$$

Solution

$$x'' + 4x' + 5x = 3\sin t$$

The particular solution: $z = Ae^{it}$

$$z' = (i)Ae^{it}$$

$$z'' = (i)^2 Ae^{it} = -Ae^{it}$$

$$z'' + 4z' + 5z = 3e^{it}$$

$$-Ae^{i3t} + 4iAe^{i3t} + 5Ae^{i3t} = 3e^{it}$$

$$(-1 + 4i + 5)A = 3$$

$$(4 + 4i)A = 3$$

$$A = \frac{3}{4}\frac{1}{1+i}\cdot\frac{1-i}{1-i}$$

$$= \frac{3}{4}\left(\frac{1}{2} - i\frac{1}{2}\right)$$

$$= \frac{3}{8} - i\frac{3}{8}$$

$$z = \left(\frac{3}{8} - i\frac{3}{8}\right)e^{it}$$

$$= \left(\frac{3}{8} - i\frac{3}{8}\right)(\cos t + i\sin t)$$

$$= \frac{3}{8}\left[\cos t + \sin t + i(\sin t - \cos t)\right]$$

$$\boxed{x_p(t) = \boldsymbol{Im}(z) = \frac{3}{8}(\sin t - \cos t)}$$

The homogeneous eq.: $x'' + 4x' + 5x = 0$

The characteristic eq.: $\lambda^2 + 4\lambda + 5 = 0 \;\; \Rightarrow \;\; \boxed{\lambda = -2 \pm i}$

$$x_h(t) = e^{-2t}\left(C_1\cos t + C_2\sin t\right)$$

$$x(t) = \frac{3}{8}(\sin t - \cos t) + e^{-2t}\left(C_1\cos t + C_2\sin t\right) \qquad x(0) = 0, \;\; x'(0) = -3$$

$$x(0) = \frac{3}{8}(\sin 0 - \cos 0) + e^{-2(0)}\left(C_1\cos 0 + C_2\sin 0\right)$$

$$0 = -\frac{3}{8} + C_1$$

$$\boxed{C_1 = \frac{3}{8}}$$

$$x'(t) = \frac{3}{8}(\cos t + \sin t) - 2e^{-2t}\left(C_1 \cos t + C_2 \sin t\right) + e^{-2t}\left(-C_1 \sin t + C_2 \cos t\right)$$

$$x'(0) = \frac{3}{8}(\cos 0 + \sin 0) - 2e^{-2(0)}\left(C_1 \cos 0 + C_2 \sin 0\right) + e^{-2(0)}\left(-C_1 \sin 0 + C_2 \cos 0\right)$$

$$-3 = \frac{3}{8} - 2C_1 + C_2$$

$$\boxed{C_2 = -3 - \frac{3}{8} + 2\left(\frac{3}{8}\right) = -\frac{21}{8}}$$

$$x(t) = \frac{3}{8}(\sin t - \cos t) + e^{-2t}\left(\frac{3}{8}\cos t - \frac{21}{8}\sin t\right)$$

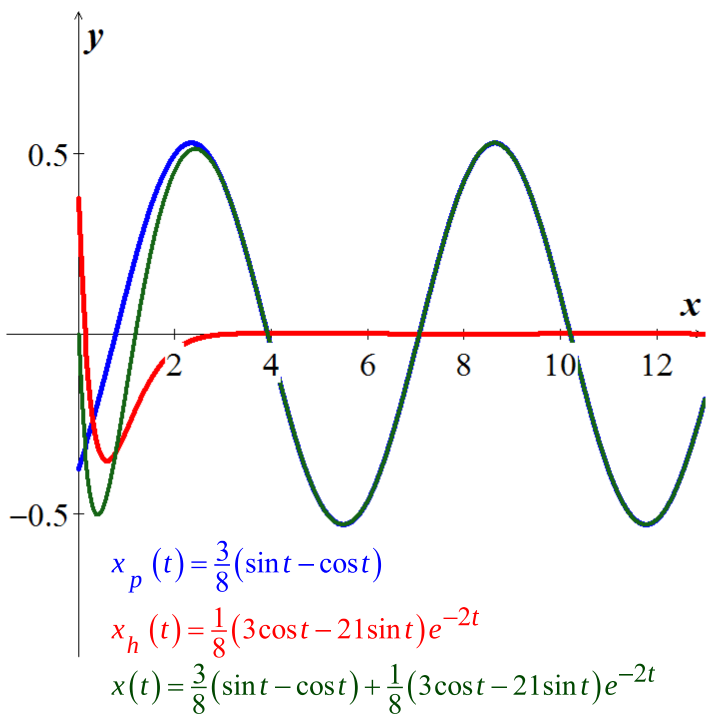
$$\boxed{x(t) = \frac{3}{8}(\sin t - \cos t) + \frac{1}{8}e^{-2t}(3\cos t - 21\sin t)}$$

The steady-state solution is the particular solution:

$$\boxed{x_p(t) = \frac{3}{8}(\sin t - \cos t)}$$

The transient response is:

$$\boxed{x_h(t) = \frac{1}{8}(3\cos t - 21\sin t)e^{-2t}}$$



Exercise

Find a particular solution of $y'' - 2y' + 5y = 2\cos 3x - 4\sin 3x + e^{2x}$ given the set $y_p = A\cos 3x + B\sin 3x + Ce^{2x}$ where *A*, *B*, *C* are to be determined

Solution

$$y_p = A\cos 3x + B\sin 3x + Ce^{2x}$$

$$y'_p = -3A\sin 3x + 3B\cos 3x + 2Ce^{2x}$$

$$y''_p = -9A\cos 3x - 9B\sin 3x + 4Ce^{2x}$$

$$\begin{aligned} y'' - 2y' + 5y &= -9A\cos 3x - 9B\sin 3x + 4Ce^{2x} + 6A\sin 3x - 6B\cos 3x - 4Ce^{2x} \\ &\quad + 5A\cos 3x + 5B\sin 3x + 5Ce^{2x} \\ &= (-4A - 6B)\cos 3x + (6A - 4B)\sin 3x + 5Ce^{2x} = 2\cos 3x - 4\sin 3x + e^{2x} \end{aligned}$$

$$\begin{cases} \left.\begin{matrix} -4A - 6B = 2 \\ 6A - 4B = -4 \end{matrix}\right\} \to & A = -\frac{8}{13},\ B = \frac{1}{13} \\ \quad 5C = 1 & \to C = \frac{1}{5} \end{cases}$$

The particular solution: $\boxed{y_p = -\frac{8}{13}\cos 3x + \frac{1}{13}\sin 3x + \frac{1}{5}e^{2x}}$

Solution   Section 2.8 - Euler's & Runge-Kutta Methods

Exercise

Calculate the first five iterations of Euler's method with step $h = 0.1$ of

$$y' = ty \quad y(0) = 1$$

Solution

t	y
0.1	1.00000000
0.2	1.01000000
0.3	1.03020000
0.4	1.06110600
0.5	1.10355024

$$\frac{dy}{dt} = ty$$

$$\int \frac{dy}{y} = \int t\,dt$$

$$\ln y = \frac{1}{2}t^2 + C$$

$$y(t) = e^{t^2/2+C}$$

$$y(0) = e^C = 1 \quad \to \quad \boxed{C = 0}$$

$$\underline{y(t) = e^{t^2/2}}$$

Exercise

Calculate the first five iterations of Euler's method with step $h = 0.1$ of

$$z' = x - 2z \quad z(0) = 1$$

Solution

x	z
0.0	1.00000000
0.1	0.80000000
0.2	0. 65000000
0.3	0.54000000
0.4	0.46200000
0.5	0.40960000

Exercise

Calculate the first five iterations of Euler's method with step $h = 0.1$ of: $z' = 5 - z \quad z(0) = 0$

Solution

$x_0 = 0; \quad z_0 = 0$

The *first* step:

$z_1 = z_0 + h(5 - z_0) = 0 + 0.1(5 - 0) = 0.5$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

The *second* step:

$z_2 = z_1 + h(5 - z_1) = 0.5 + 0.1(5 - 0.5) = 0.95$

$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

Euler Method

t	*Approx.*	*Exact*	*Difference*
0.00	0.00000000	0.00000000	0.00000000
0.10	0.50000000	0.47581291	-0.02418709
0.20	0.95000000	0.90634623	-0.04365377
0.30	1.35500000	1.29590890	-0.05909110
0.40	1.71950000	1.64839977	-0.07110023
0.50	2.04755000	1.96734670	-0.08020330

$\boxed{y(t) = 5 - 5e^{-t}}$

Exercise

Given: $y' + 2xy = x \quad y(0) = 8$

a) Use a computer and Euler's method to calculate three separate approximate solutions on the interval $[0, 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

Solution

x	y
0.0	8.00000000
0.2	8.00000000
0.4	7.40000000
0.6	6.29600000
0.8	4.90496000
1.0	3.49537280
x	y
0.0	8.00000000
0.1	8.00000000

0.2	7.85000000
0.3	7.55600000
0.4	7.13264000
0.5	6.60202880
0.6	5.99182592
0.7	5.33280681
0.8	4.65621386
0.9	3.99121964
1.0	3.36280010

x	*y*	*x*	*y*
0.0	8.00000000		
0.05	8.00000000	0.55	6.16870319
0.10	7.96250000	0.60	5.85692451
0.15	7.88787500	0.65	5.53550904
0.20	7.77705688	0.70	5.20820096
0.25	7.63151574	0.75	4.87862689
0.30	7.45322784	0.80	4.55022987
0.35	7.24463101	0.85	4.22621148
0.40	7.00856892	0.90	3.90948351
0.45	6.74822617	0.95	3.60262999
0.50	6.46705599	1.00	3.30788014

$$y(t) = \frac{15}{2}e^{-t^2} + \frac{1}{2}$$

t	***Approx.***	***Exact***	***Difference***
0.00	8.00000000	8.00000000	0.00000000
0.20	8.00000000	7.70592079	-0.29407921
0.40	7.40000000	6.89107842	-0.50892158
0.60	6.29600000	5.73257245	-0.56342755
0.80	4.90496000	4.45469318	-0.45026682
1.00	3.49537280	3.25909581	-0.23627699

t	***Approx.***	***Exact***	***Difference***
0.00	8.00000000	8.00000000	0.00000000
0.10	8.00000000	7.92537375	-0.07462625
0.20	7.85000000	7.70592079	-0.14407921
0.30	7.55600000	7.35448389	-0.20151611
0.40	7.13264000	6.89107842	-0.24156158
0.50	6.60202880	6.34100587	-0.26102293
0.60	5.99182592	5.73257245	-0.25925347
0.70	5.33280681	5.09469796	-0.23810885
0.80	4.65621386	4.45469318	-0.20152068
0.90	3.99121964	3.83643550	-0.15478414
1.00	3.36280010	3.25909581	-0.10370430

t	*Approx.*	*Exact*	*Difference*
0.00	8.00000000	8.00000000	0.00000000
0.05	8.00000000	7.98127342	-0.01872658
0.10	7.96250000	7.92537375	-0.03712625
0.15	7.88787500	7.83313428	-0.05474072
0.20	7.77705688	7.70592079	-0.07113608
0.25	7.63151574	7.54559797	-0.08591777
0.30	7.45322784	7.35448389	-0.09874395
0.35	7.24463101	7.13529429	-0.10933672
0.40	7.00856892	6.89107842	-0.11749051
0.45	6.74822617	6.62514862	-0.12307755
0.50	6.46705599	6.34100587	-0.12605012
0.55	6.16870319	6.04226366	-0.12643953
0.60	5.85692451	5.73257245	-0.12435207
0.65	5.53550904	5.41554691	-0.11996214
0.70	5.20820096	5.09469796	-0.11350300
0.75	4.87862689	4.77337119	-0.10525570
0.80	4.55022987	4.45469318	-0.09553669
0.85	4.22621148	4.14152671	-0.08468477
0.90	3.90948351	3.83643550	-0.07304801
0.95	3.60262999	3.54165879	-0.06097120
1.00	3.30788014	3.25909581	-0.04878433

Exercise

Given: $z' - 2z = xe^{2x} \quad z(0) = 1$

a) Use a computer and Euler's method to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a third with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

Solution

a)

Euler Method

t	*Approx.*	*Exact*	*Difference*
0.00	1.00000000	1.00000000	0.00000000
0.20	1.40000000	1.52166119	0.12166119
0.40	2.01967299	2.40358420	0.38391121
0.60	3.00558546	3.91773797	0.91215251
0.80	4.60623367	6.53800280	1.93176913
1.00	7.24121233	11.08358415	3.84237182

Euler Method

t	*Approx.*	*Exact*	*Difference*
0.00	1.00000000	1.00000000	0.00000000
0.10	1.20000000	1.22750977	0.02750977
0.20	1.45221403	1.52166119	0.06944716
0.30	1.77249333	1.90411415	0.13162082
0.40	2.18165556	2.40358420	0.22192865
0.50	2.70700830	3.05806706	0.35105875
0.60	3.38432406	3.91773797	0.53341391
0.70	4.26039588	5.04872396	0.78832807
0.80	5.39633906	6.53800280	1.14166374
0.90	6.87184946	8.49975469	1.62790522
1.00	8.79068763	11.08358415	2.29289652

Euler Method

t	*Approx.*	*Exact*	*Difference*
0.00	1.00000000	1.00000000	0.00000000
0.05	1.10000000	1.10655238	0.00655238
0.10	1.21276293	1.22750977	0.01474684
0.15	1.34014623	1.36504472	0.02489849
0.20	1.48428480	1.52166119	0.03737639
0.25	1.64763153	1.70024381	0.05261229
0.30	1.83300369	1.90411415	0.07111045
0.35	2.04363584	2.13709506	0.09345922
0.40	2.28324010	2.40358420	0.12034410
0.45	2.55607493	2.70863793	0.15256300
0.50	2.86702349	3.05806706	0.19104356
0.55	3.22168289	3.45854614	0.23686325

0.60	3.62646574	3.91773797	0.29127223
0.65	4.08871582	4.44443559	0.35571976
0.70	4.61683955	5.04872396	0.43188441
0.75	5.22045550	5.74216412	0.52170862
0.80	5.91056439	6.53800280	0.62743841
0.85	6.69974213	7.45141089	0.75166876
0.90	7.60235911	8.49975469	0.89739558
0.95	8.63482915	9.70290431	1.06807516
1.00	9.81589205	11.08358415	1.26769209

b) $z' - 2z = xe^{2x}$

$P(x) = -2, \quad Q(x) = xe^{2x}$

$$e^{\int -2dx} = e^{-2x}$$

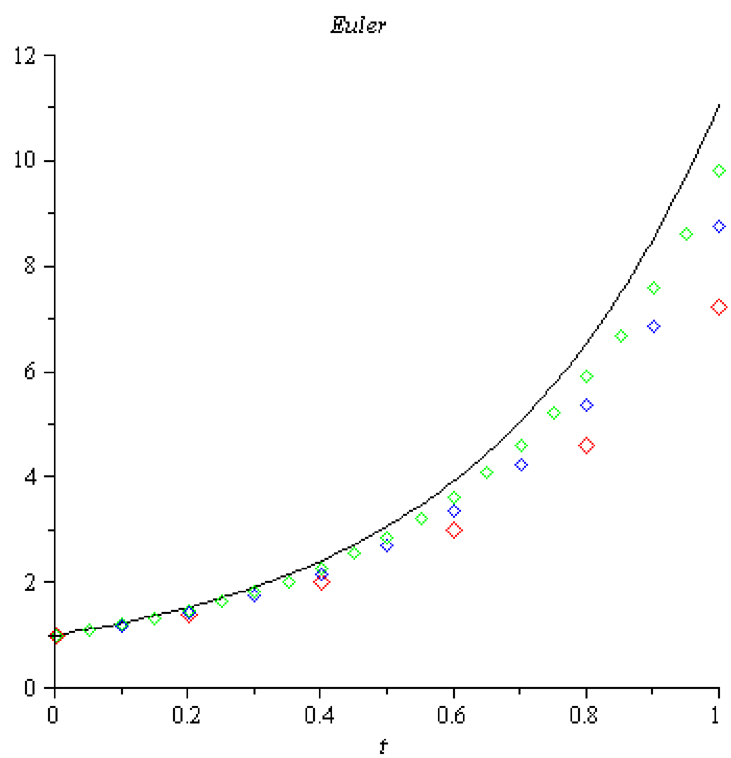
$$\int xe^{2x}e^{-2x}\,dx = \int xdx = \frac{1}{2}x^2$$

$$z(x) = \frac{1}{e^{-2x}}\left(\frac{1}{2}x^2 + C\right)$$

$$= \frac{1}{2}x^2e^{2x} + Ce^{2x}$$

$z(0) = \underline{C = 1}$

$$\boxed{z(x) = \frac{1}{2}x^2e^{2x} + e^{2x}}$$



Exercise

Consider the initial value problem $y' = 12y(4-y) \quad y(0)=1$

Use Euler's method with step size $h = 0.04$ to sketch solution on the interval $[0,\ 2]$

Solution

$$y(t) = \frac{4}{e^{\log 3 - 48t} + 1}$$

t	***Approx.***	***Exact***	***Difference***
0.00	1.00000000	1.00000000	0.00000000
0.04	2.44000000	2.77812333	0.33812333
0.08	4.26707200	3.75770045	-0.50937155
0.12	3.72005658	3.96254078	0.24248419
0.16	4.21993115	3.99446397	-0.22546718
0.20	3.77444588	3.99918742	0.22474154
0.24	4.18308995	3.99988085	-0.18320910
0.28	3.81546672	3.99998253	0.18451582
0.32	4.15342541	3.99999744	-0.15342797
0.36	3.84754974	3.99999962	0.15244989
0.40	4.12909852	3.99999994	-0.12909858
0.44	3.87322947	3.99999999	0.12677052
0.48	4.10891492	4.00000000	-0.10891492
0.52	3.89410430	4.00000000	0.10589570
0.56	4.09204138	4.00000000	-0.09204138
0.60	3.91125556	4.00000000	0.08874444
0.64	4.07786461	4.00000000	-0.07786461
0.68	3.92545437	4.00000000	0.07454563
0.72	4.06591460	4.00000000	-0.06591460
0.76	3.93727310	4.00000000	0.06272690
0.80	4.05582011	4.00000000	-0.05582011
0.84	3.94714987	4.00000000	0.05285013
0.88	4.04728141	4.00000000	-0.04728141
0.92	3.95542805	4.00000000	0.04457195
0.96	4.04005260	4.00000000	-0.04005260
1.00	3.96238159	4.00000000	0.03761841
1.04	4.03392967	4.00000000	-0.03392967
1.08	3.96823212	4.00000000	0.03176788
1.12	4.02874204	4.00000000	-0.02874204
1.16	3.97316079	4.00000000	0.02683921
1.20	4.02434630	4.00000000	-0.02434630
1.24	3.97731688	4.00000000	0.02268312
1.28	4.02062150	4.00000000	-0.02062150
1.32	3.98082411	4.00000000	0.01917589
1.36	4.01746532	4.00000000	-0.01746532
1.40	3.98378549	4.00000000	0.01621451
1.44	4.01479115	4.00000000	-0.01479115
1.48	3.98628712	4.00000000	0.01371288
1.52	4.01252558	4.00000000	-0.01252558
1.56	3.98840115	4.00000000	0.01159885
1.60	4.01060636	4.00000000	-0.01060636
1.64	3.99018815	4.00000000	0.00981185
1.68	4.00898069	4.00000000	-0.00898069
1.72	3.99169905	4.00000000	0.00830095
1.76	4.00760380	4.00000000	-0.00760380
1.80	3.99297675	4.00000000	0.00702325
1.84	4.00643771	4.00000000	-0.00643771
1.88	3.99405741	4.00000000	0.00594259
1.92	4.00545023	4.00000000	-0.00545023
1.96	3.99497153	4.00000000	0.00502847
2.00	4.00461405	4.00000000	-0.00461405

Exercise

You've seen that the error in Euler's method varies directly as the first power of the step size $\left(i.e\ \ E_h \approx \lambda h\right)$. This makes Euler's method an order to halve the error? How does this affect the number of required iterations?

Solution

Because $E_h \approx \lambda h$ halving the step size should halve the error.

$$E \approx \lambda\left(\frac{1}{2}h\right) \approx \frac{1}{2}\lambda h \approx \frac{1}{2}E_h$$

The number of iterations is given by: $N = \dfrac{b-a}{h}$, therefore halving the step size should double the number of iterations.

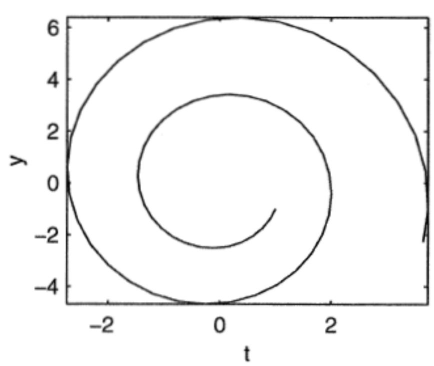
$$N = \frac{b-a}{\frac{1}{2}h} = 2\frac{b-a}{h} \approx 2N_h$$

Exercise

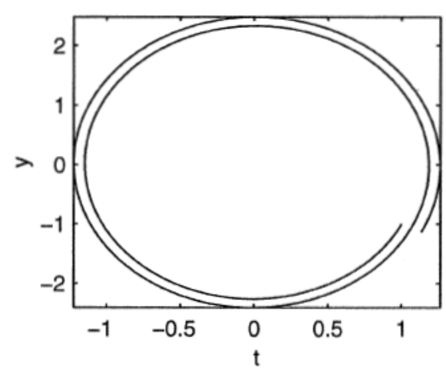
Use Euler's method to provide an approximate solution over the given time interval using the given steps sizes. Provide a plot of ***v*** versus ***y*** for each step size

$$y'' + 4y = 0, \quad y(0) = 4, \quad y'(0) = 0, \quad [0,\ 2\pi]; \quad h = 0.1,\ 0.01,\ 0.001$$

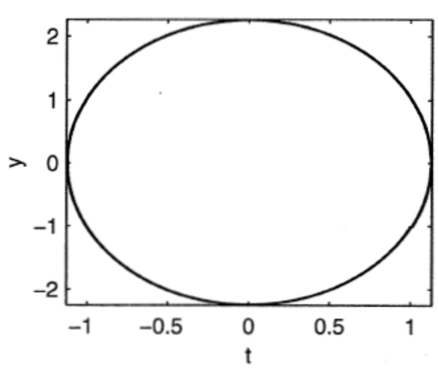
Solution



***h* = 0.1**



***h* = 0.01**



***h* = 0.001**

Exercise

$z' + z = \cos x \quad z(0) = 1$

a) Use a computer and Runge-Kutta method to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

Solution

$$z(x) = \frac{1}{2}e^{-x} + \frac{1}{2}\cos x + \frac{1}{2}\sin x$$

Runge-Kutta 2nd Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.20	0.99800666	0.99873333	0.00072667
0.40	0.98887689	0.99039969	0.00152281
0.60	0.96709749	0.96939486	0.00229738
0.80	0.92871746	0.93169588	0.00297842
1.00	0.87131508	0.87482637	0.00351128

Runge-Kutta 4th Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.20	0.99873272	0.99873333	0.00000061
0.40	0.99039822	0.99039969	0.00000147
0.60	0.96939245	0.96939486	0.00000241
0.80	0.93169258	0.93169588	0.00000330
1.00	0.87482232	0.87482637	0.00000405

Runge-Kutta 2nd Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.10	0.99975021	0.99983750	0.00008729
0.20	0.99855245	0.99873333	0.00018088
0.30	0.99555979	0.99583746	0.00027767
0.40	0.99002480	0.99039969	0.00037489
0.50	0.98129932	0.98176938	0.00047006
0.60	0.96883388	0.96939486	0.00056098
0.70	0.95217687	0.95282259	0.00064572
0.80	0.93097330	0.93169588	0.00072258
0.90	0.90496314	0.90575327	0.00079013
1.00	0.87397921	0.87482637	0.00084716

Runge-Kutta 4th Order

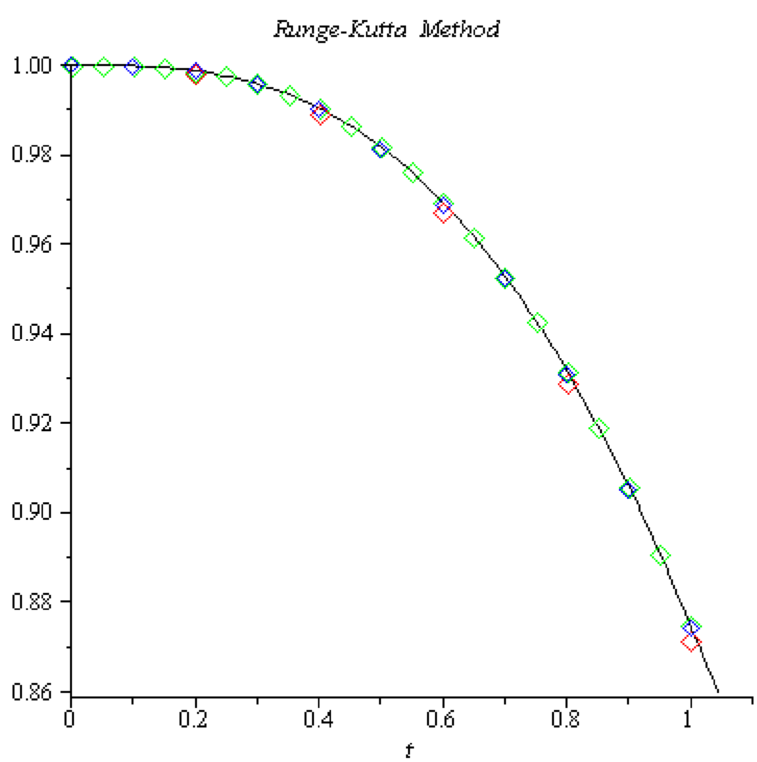
t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.10	0.99983748	0.99983750	0.00000002
0.20	0.99873329	0.99873333	0.00000004
0.30	0.99583739	0.99583746	0.00000007
0.40	0.99039960	0.99039969	0.00000009
0.50	0.98176926	0.98176938	0.00000012
0.60	0.96939471	0.96939486	0.00000015
0.70	0.95282241	0.95282259	0.00000018
0.80	0.93169568	0.93169588	0.00000020
0.90	0.90575304	0.90575327	0.00000023
1.00	0.87482612	0.87482637	0.00000025

Runge-Kutta 2nd Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.05	0.99996876	0.99997943	0.00001067
0.10	0.99981570	0.99983750	0.00002180
0.15	0.99942531	0.99945859	0.00003328
0.20	0.99868831	0.99873333	0.00004502
0.25	0.99750164	0.99755858	0.00005694
0.30	0.99576852	0.99583746	0.00006894
0.35	0.99339836	0.99347931	0.00008094
0.40	0.99030682	0.99039969	0.00009287
0.45	0.98641574	0.98652039	0.00010465
0.50	0.98165315	0.98176938	0.00011623
0.55	0.97595326	0.97608078	0.00012752
0.60	0.96925639	0.96939486	0.00013847
0.65	0.96150896	0.96165799	0.00014903
0.70	0.95266344	0.95282259	0.00015915
0.75	0.94267832	0.94284709	0.00016877
0.80	0.93151803	0.93169588	0.00017785
0.85	0.91915289	0.91933924	0.00018635
0.90	0.90555903	0.90575327	0.00019423
0.95	0.89071835	0.89091981	0.00020146
1.00	0.87461836	0.87482637	0.00020801

Runge-Kutta 4th Order

t	y	y(t)	Difference
0.00	1.00000000	1.00000000	0.00000000
0.05	0.99997943	0.99997943	0.00000000
0.10	0.99983750	0.99983750	0.00000000
0.15	0.99945859	0.99945859	0.00000000
0.20	0.99873333	0.99873333	0.00000000
0.25	0.99755858	0.99755858	0.00000000
0.30	0.99583745	0.99583746	0.00000000
0.35	0.99347930	0.99347931	0.00000001
0.40	0.99039969	0.99039969	0.00000001
0.45	0.98652039	0.98652039	0.00000001
0.50	0.98176937	0.98176938	0.00000001
0.55	0.97608077	0.97608078	0.00000001
0.60	0.96939485	0.96939486	0.00000001
0.65	0.96165798	0.96165799	0.00000001
0.70	0.95282258	0.95282259	0.00000001
0.75	0.94284708	0.94284709	0.00000001
0.80	0.93169587	0.93169588	0.00000001
0.85	0.91933923	0.91933924	0.00000001
0.90	0.90575325	0.90575327	0.00000001
0.95	0.89091979	0.89091981	0.00000001
1.00	0.87482635	0.87482637	0.00000002



Exercise

Given $x' = \dfrac{t}{x} \quad x(0) = 1$

a) Use a computer and Runge-Kutta method to calculate three separate approximate solutions on the interval $[0,\ 1]$, one with step size $h = 0.2$, a second with step size $h = 0.1$, a second with step size $h = 0.05$.

b) Use the appropriate analytic to compute the exact solution

c) Plot the exact solution and approximate solutions as discrete points.

Solution

a)

Runge-Kutta 2th Order

t	*Approx.*	*Exact*	*Difference*
0.00	1.00000000	1.00000000	0.00000000
0.20	1.01961161	1.01980390	0.00019229
0.40	1.07636229	1.07703296	0.00067067
0.60	1.16495094	1.16619038	0.00123944
0.80	1.27887002	1.28062485	0.00175483
1.00	1.41205020	1.41421356	0.00216336

Runge-Kutta 4th Order

t	*Approx.*	*Exact*	*Difference*
0.00	1.00000000	1.00000000	0.00000000
0.20	1.01980437	1.01980390	-0.00000046
0.40	1.07703431	1.07703296	-0.00000135
0.60	1.16619234	1.16619038	-0.00000196
0.80	1.28062701	1.28062485	-0.00000216
1.00	1.41421570	1.41421356	-0.00000214

Runge-Kutta 2th Order

t	*Approx.*	*Exact*	*Difference*
0.00	1.00000000	1.00000000	0.00000000
0.10	1.00497519	1.00498756	0.00001238
0.20	1.01975618	1.01980390	0.00004772
0.30	1.04392938	1.04403065	0.00010127
0.40	1.07686631	1.07703296	0.00016665
0.50	1.11779652	1.11803399	0.00023747
0.60	1.16588199	1.16619038	0.00030839
0.70	1.22027989	1.22065556	0.00037567
0.80	1.28018776	1.28062485	0.00043708
0.90	1.34487075	1.34536240	0.00049165
1.00	1.41367433	1.41421356	0.00053923

Runge-Kutta 4th Order

t	*Approx.*	*Exact*	*Difference*
0.00	1.00000000	1.00000000	0.00000000
0.10	1.00498757	1.00498756	-0.00000001

0.20	1.01980393	1.01980390	-0.00000003
0.30	1.04403071	1.04403065	-0.00000006
0.40	1.07703304	1.07703296	-0.00000008
0.50	1.11803409	1.11803399	-0.00000010
0.60	1.16619050	1.16619038	-0.00000012
0.70	1.22065569	1.22065556	-0.00000013
0.80	1.28062498	1.28062485	-0.00000013
0.90	1.34536254	1.34536240	-0.00000013
1.00	1.41421369	1.41421356	-0.00000013

Runge-Kutta 2th Order

t	***Approx.***	***Exact***	***Difference***
0.00	1.00000000	1.00000000	0.00000000
0.05	1.00124844	1.00124922	0.00000078
0.10	1.00498447	1.00498756	0.00000309
0.15	1.01118058	1.01118742	0.00000684
0.20	1.01979199	1.01980390	0.00001191
0.25	1.03075829	1.03077641	0.00001812
0.30	1.04400537	1.04403065	0.00002528
0.35	1.05944783	1.05948101	0.00003317
0.40	1.07699136	1.07703296	0.00004160
0.45	1.09653524	1.09658561	0.00005037
0.50	1.11797470	1.11803399	0.00005929
0.55	1.14120301	1.14127122	0.00006821
0.60	1.16611337	1.16619038	0.00007701
0.65	1.19260047	1.19268604	0.00008557
0.70	1.22056174	1.22065556	0.00009382
0.75	1.24989830	1.25000000	0.00010170
0.80	1.28051568	1.28062485	0.00010917
0.85	1.31232426	1.31244047	0.00011621
0.90	1.34523959	1.34536240	0.00012281
0.95	1.37918245	1.37931142	0.00012898
1.00	1.41407885	1.41421356	0.00013471

Runge-Kutta 4th Order

t	***Approx.***	***Exact***	***Difference***
0.00	1.00000000	1.00000000	0.00000000
0.05	1.00124922	1.00124922	-0.00000000
0.10	1.00498756	1.00498756	-0.00000000
0.15	1.01118742	1.01118742	-0.00000000
0.20	1.01980390	1.01980390	-0.00000000
0.25	1.03077641	1.03077641	-0.00000000
0.30	1.04403065	1.04403065	-0.00000000
0.35	1.05948101	1.05948101	-0.00000000
0.40	1.07703297	1.07703296	-0.00000001
0.45	1.09658562	1.09658561	-0.00000001
0.50	1.11803400	1.11803399	-0.00000001
0.55	1.14127123	1.14127122	-0.00000001
0.60	1.16619039	1.16619038	-0.00000001
0.65	1.19268605	1.19268604	-0.00000001
0.70	1.22065557	1.22065556	-0.00000001
0.75	1.25000001	1.25000000	-0.00000001
0.80	1.28062486	1.28062485	-0.00000001
0.85	1.31244048	1.31244047	-0.00000001
0.90	1.34536241	1.34536240	-0.00000001
0.95	1.37931143	1.37931142	-0.00000001
1.00	1.41421357	1.41421356	-0.00000001

b) The equation is separable:

$$xdx = tdt$$

$$\int xdx = \int tdt$$

$$\frac{1}{2}x^2 = \frac{1}{2}t^2 + C$$

$$x^2 = t^2 + 2C$$

$$x = \sqrt{t^2 + 2C}$$

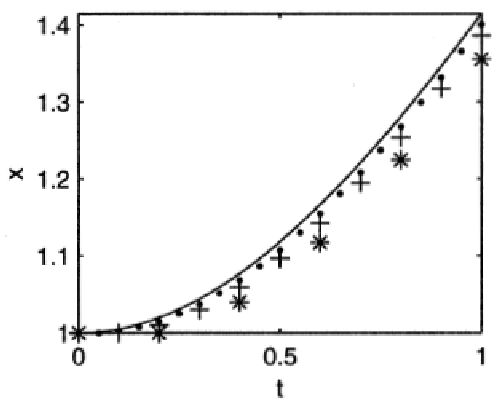
$$x(0) = \sqrt{2C}$$

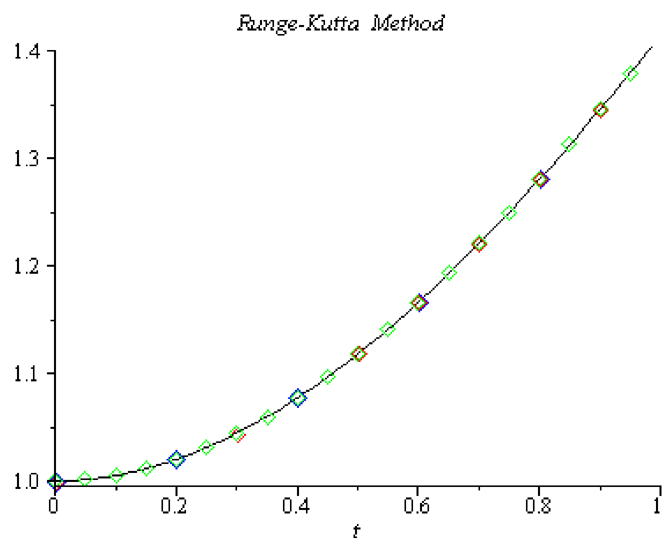
$$1 = \sqrt{2C}$$

$$1 = 2C$$

$$C = \frac{1}{2}$$

$$\boxed{x(t) = \sqrt{t^2 + 1}}$$





Exercise

Consider the initial value problem $y' = \frac{t}{y^2} \quad y(0) = 1$

Use Runge-Kutta method with step size $h = 0.04$ to sketch solution on the interval $[0,\ 2]$

Solution

$$y(t) = 3^{1/3}\left(\frac{1}{2}t^2 + \frac{1}{3}\right)^{1/3}$$

Runge-Kutta 4th Order

t	Approx.	Exact	Difference
0.00	1.00000000	1.00000000	0.00000000
0.04	1.00079936	1.00079936	-0.00000000
0.08	1.00318981	1.00318981	-0.00000000
0.12	1.00714877	1.00714877	-0.00000000
0.16	1.01263957	1.01263957	-0.00000000
0.20	1.01961283	1.01961282	-0.00000000
0.24	1.02800822	1.02800822	-0.00000000
0.28	1.03775651	1.03775651	-0.00000000
0.32	1.04878166	1.04878166	-0.00000001
0.36	1.06100297	1.06100297	-0.00000001
0.40	1.07433708	1.07433707	-0.00000001
0.44	1.08869975	1.08869974	-0.00000001
0.48	1.10400743	1.10400742	-0.00000001
0.52	1.12017855	1.12017854	-0.00000001
0.56	1.13713450	1.13713449	-0.00000001
0.60	1.15480036	1.15480035	-0.00000001
0.64	1.17310545	1.17310544	-0.00000001
0.68	1.19198361	1.19198360	-0.00000001
0.72	1.21137336	1.21137335	-0.00000001
0.76	1.23121787	1.23121787	-0.00000001
0.80	1.25146496	1.25146495	-0.00000001
0.84	1.27206683	1.27206682	-0.00000001
0.88	1.29297992	1.29297991	-0.00000001
0.92	1.31416464	1.31416463	-0.00000001
0.96	1.33558509	1.33558508	-0.00000001
1.00	1.35720882	1.35720881	-0.00000001
1.04	1.37900650	1.37900650	-0.00000001
1.08	1.40095174	1.40095173	-0.00000001
1.12	1.42302075	1.42302075	-0.00000001
1.16	1.44519217	1.44519216	-0.00000001
1.20	1.46744679	1.46744678	-0.00000001
1.24	1.48976740	1.48976739	-0.00000001
1.28	1.51213855	1.51213854	-0.00000001
1.32	1.53454641	1.53454640	-0.00000001
1.36	1.55697860	1.55697859	-0.00000001
1.40	1.57942403	1.57942403	-0.00000001
1.44	1.60187281	1.60187281	-0.00000001
1.48	1.62431609	1.62431608	-0.00000001
1.52	1.64674596	1.64674596	-0.00000001

1.56	1.66915540	1.66915539	-0.00000001
1.60	1.69153812	1.69153811	-0.00000001
1.64	1.71388854	1.71388853	-0.00000001
1.68	1.73620170	1.73620169	-0.00000001
1.72	1.75847320	1.75847319	-0.00000001
1.76	1.78069914	1.78069913	-0.00000001
1.80	1.80287607	1.80287606	-0.00000000
1.84	1.82500094	1.82500094	-0.00000000
1.88	1.84707109	1.84707109	-0.00000000
1.92	1.86908417	1.86908417	-0.00000000
1.96	1.89103813	1.89103813	-0.00000000
2.00	1.91293119	1.91293118	-0.00000000