Section 3.8 – Taylor and Maclaurin Series

The sum of a power series:

$$f(x)=\sum_{n=0}^{\infty} a_n (x-a)^n$$

$$= a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots$$

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1} + \cdots$$

$$f''(x) = 2a_2 + 2\cdot 3a_3(x-a) + 3\cdot 4a_4(x-a)^2 + \cdots + (n-1)\cdot na_n(x-a)^{n-2} + \cdots$$

$$f'''(x) = 1\cdot 2\cdot 3a_3 + 2\cdot 3\cdot 4a_4(x-a) + +3\cdot 4\cdot 5a_5(x-a)^2 \cdots + (n-2)\cdot(n-1)\cdot na_n(x-a)^{n-3} + \cdots$$

$$f^{(n)}(x) = n!a_n + \text{a sum of terms with } (x-a) \text{ as a factor}$$

In general: $\boxed{f^{(n)}(x) = n!a_n} \quad \Rightarrow \quad a_n = \dfrac{f^{(n)}(a)}{n!}$

If f has a series representation, then the series must be

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Taylor and Maclaurin Series

Definitions

Let f be a function with derivatives of all orders throughout some interval containing $\boldsymbol{a}$ as an interior point. Then the ***Taylor series generated by*** f at $x=a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The ***Maclaurin series generated by*** f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

The Taylor series generated by f at $x=0$.

Example

Find the Taylor series generated by $f(x)=\frac{1}{x}$ *at* $a=2$. Where, if anywhere, does the series converges to $\frac{1}{x}$.

Solution

$f(x)=x^{-1}$ $\qquad f(2)=2^{-1}=\frac{1}{2}$

$f'(x)=-x^{-2}$ $\qquad f'(2)=-\frac{1}{2^2}$

$f''(x)=2!x^{-3}$ $\qquad \frac{f''(2)}{2!}=2^{-3}=\frac{(-1)^3}{2^3}$

$f'''(x)=-3!x^{-4}$ $\qquad \frac{f^{(3)}(2)}{3!}=2^{-4}=\frac{(-1)^4}{2^4}$

$\vdots \quad \vdots$ $\qquad \vdots \quad \vdots$

$f^{(n)}(x)=(-1)^n n!x^{-(n+1)}$ $\qquad \frac{f^{(n)}(2)}{n!}=\frac{(-1)^n}{2^{n+1}}$

The Taylor series is:

$$f(2)+f'(2)(x-2)+\frac{f''(2)}{2!}(x-2)^2+\cdots+\frac{f^{(n)}(2)}{n!}(x-2)^n$$

$$=\frac{1}{2}-\frac{(x-2)}{2^2}+\frac{(x-2)^2}{2^3}+\cdots+(-1)^n\frac{(x-2)^n}{2^{n+1}}+\cdots$$

Taylor Polynomials

Definition

Let f be a function with derivatives of order *k* for $k=1,\ 2,\ \ldots,\ N$ in some interval containing a as an interior point. Then for any integer n from 0 through *N*, the Taylor polynomial of order *n* generated by f at $x=a$ is the polynomial

$$P_n(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Example

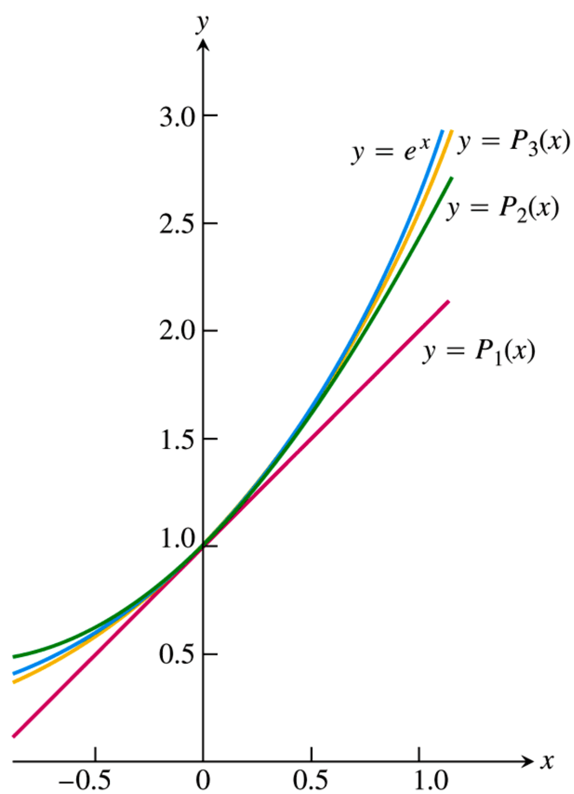
Find the Taylor series and the Taylor polynomials generated by $f(x)=e^x$ at $x=0$

Solution

$$f^{(n)}(x)=e^x \quad \to \quad f^{(n)}(0)=1$$

$$\begin{aligned} P_n(x) &= f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots \\ &= 1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots+\frac{1}{n!}x^n+\cdots \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}x^k \end{aligned}$$

This is also the Maclaurin series of e^x



The Taylor polynomial of order n at $x = 0$ is

$$P_n(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots+\frac{x^n}{n!}$$

Example

Find the Taylor series and the Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$

Solution

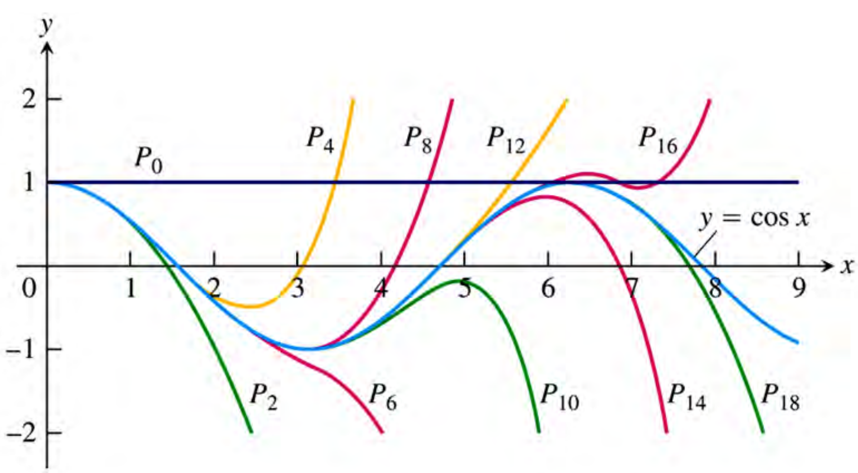
$$f(x) = \cos x, \qquad f'(x) = -\sin x,$$
$$f''(x) = -\cos x, \qquad f'''(x) = \sin x,$$
$$\vdots \qquad\qquad \vdots$$
$$f^{(2n)}(x) = (-1)^n \cos x, \qquad f^{(2n+1)}(x) = (-1)^{2n+1} \sin x$$

$$f^{(2n)}(0) = (-1)^n, \qquad f^{(2n+1)}(0) = 0$$

The Taylor series generated by f at $x = 0$ is

$$\begin{aligned} & f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ &= 1 + 0\cdot x - \frac{x^2}{2!} + 0\cdot x^3 + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \end{aligned}$$

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$$



Example

Find the Taylor series for $\cos x$ about $\frac{\pi}{3}$. Where is the series valid?

Solution

$$\cos x = \cos\left(x - \frac{\pi}{3} + \frac{\pi}{3}\right)$$

$$= \cos\left(x - \frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(x - \frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{2}\left[1 - \frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{3}\right)^4 - \cdots\right] - \frac{\sqrt{3}}{2}\left[\left(x - \frac{\pi}{3}\right) - \frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{5!}\left(x - \frac{\pi}{3}\right)^5 - \cdots\right]$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{2}\frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2 - \frac{\sqrt{3}}{2}\frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{2}\frac{1}{4!}\left(x - \frac{\pi}{3}\right)^4 + \frac{1}{5!}\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)^5 - \cdots$$

This series representation is valid for all *x*.

Example

Find the Taylor series for $\ln x$ in powers of $x-2$. Where does the series converge to $\ln x$?

Solution

Let $t=\dfrac{x-2}{2}$, then

$$\ln x=\ln\left(2+(x-2)\right)$$
$$=\ln\left[2\left(1+\frac{x-2}{2}\right)\right]$$
$$=\ln 2+\ln(1+t)$$

$f(t)=\ln(1+t)$ $\qquad f(0)=\ln(1)=0$

$f'(t)=\dfrac{1}{1+t}$ $\qquad f'(0)=1$

$f''(t)=\dfrac{-1}{(1+t)^2}$ $\qquad f''(0)=-1$

$f'''(t)=\dfrac{2}{(1+t)^3}$ $\qquad f'''(0)=2$

$f^{(4)}(t)=\dfrac{-6}{(1+t)^4}$ $\qquad f'''(0)=-6$

$f^{(n)}(t)=(-1)^{n+1}\dfrac{(n-1)!}{(1+t)^n}$

$$\ln(1+t)=f(0)+f'(0)t+\frac{f''(0)}{2!}t^2+\frac{f'''(0)}{3!}t^3+\frac{f^{(IV)}(0)}{4!}t^4+\cdots$$
$$=t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\cdots$$

$$\ln x=\ln 2+\ln(1+t)$$
$$=\ln 2+t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\cdots$$
$$=\ln 2+\frac{x-2}{2}-\frac{(x-2)^2}{2\times 2^2}+\frac{(x-2)^3}{3\times 2^3}-\frac{(x-2)^4}{4\times 2^4}+\cdots$$
$$=\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\times 2^n}(x-2)^n$$

Since the series for $\ln(1+t)$ is valid for $-1<t\le 1$, this series for $\ln x$ is valid for $-1<\dfrac{x-2}{2}\le 1$

$-2<x-2\le 2 \quad\rightarrow\quad \underline{0<x\le 4}$

Exercises *Section* 3.8 – Taylor and Maclaurin Series

(**1 – 23**) Find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f *at* a

1. $f(x)=e^{2x}, \quad a=0$

2. $f(x)=\sin x, \quad a=0$

3. $f(x)=\ln(1+x), \quad a=0$

4. $f(x)=\dfrac{1}{x+2}, \quad a=0$

5. $f(x)=\sqrt{1-x}, \quad a=0$

6. $f(x)=x^3, \quad a=1$

7. $f(x)=8\sqrt{x}, \quad a=1$

8. $f(x)=\sin x, \quad a=\frac{\pi}{4}$

9. $f(x)=\cos x, \quad a=\frac{\pi}{6}$

10. $f(x)=\sqrt{x}, \quad a=9$

11. $f(x)=\sqrt[3]{x}, \quad a=8$

12. $f(x)=\ln x, \quad a=e$

13. $f(x)=\sqrt[4]{x}, \quad a=8$

14. $f(x)=\tan^{-1}x+x^2+1, \quad a=1$

15. $f(x)=e^{x}, \quad a=\ln 2$

16. $f(x)=e^{3x}; \quad a=0$

17. $f(x)=\frac{1}{x}; \quad a=1$

18. $f(x)=\cos x; \quad a=\frac{\pi}{2}$

19. $f(x)=\dfrac{1}{x+1}; \quad a=0$

20. $f(x)=\tan^{-1}4x; \quad a=0$

21. $f(x)=\sin 2x; \quad a=-\frac{\pi}{2}$

22. $f(x)=\cosh 3x; \quad a=0$

23. $f(x)=\dfrac{1}{4+x^2}; \quad a=0$

(**25 – 35**) Find the *n*th Maclaurin polynomial for the function

24. $f(x)=e^{4x}, \quad n=4$

25. $f(x)=e^{-x}, \quad n=5$

26. $f(x)=e^{-x/2}, \quad n=4$

27. $f(x)=e^{x/3}, \quad n=4$

28. $f(x)=\sin x, \quad n=5$

29. $f(x)=\cos \pi x, \quad n=4$

30. $f(x)=xe^{x}, \quad n=4$

31. $f(x)=x^2e^{-x}, \quad n=4$

32. $f(x)=\dfrac{1}{x+1}, \quad n=5$

33. $f(x)=\dfrac{x}{x+1}, \quad n=4$

34. $f(x)=\sec x, \quad n=2$

35. $f(x)=\tan x, \quad n=3$

(**36 – 39**) Find out the ***third*** term of the Maclaurin series for the following function.

36. $f(x)=(1+x)^{1/3}$

37. $f(x)=(1+x)^{-1/2}$

38. $f(x)=\left(1+\frac{x}{2}\right)^{-3}$

39. $f(x)=(1+2x)^{-5}$

(40 – 55) Find the Maclaurin series for

40. xe^x

41. $5\cos \pi x$

42. $\dfrac{x^2}{x+1}$

43. e^{3x+1}

44. $\cos\left(2x^3\right)$

45. $\cos(2x-\pi)$

46. $x^2 \sin\left(\frac{x}{3}\right)$

47. $\cos^2\left(\frac{x}{2}\right)$

48. $\sin x \cos x$

49. $\tan^{-1}\left(5x^2\right)$

50. $\ln\left(2+x^2\right)$

51. $\dfrac{1+x^3}{1+x^2}$

52. $\ln\dfrac{1+x}{1-x}$

53. $\dfrac{e^{2x^2}-1}{x^2}$

54. $\cosh x - \cos x$

55. $\sinh x - \sin x$

(**56 – 59**) Finding Taylor and Maclaurin Series generated by f *at* $x = a$

56. $f(x) = x^3 - 2x + 4, \quad a = 2$

57. $f(x) = 2x^3 + x^2 + 3x - 8, \quad a = 1$

58. $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, \quad a = -1$

59. $f(x) = \cos\left(2x + \frac{\pi}{2}\right), \quad a = \frac{\pi}{4}$

(**60 – 68**) Find the Taylor series of the functions, where is each series representation valid?

60. $f(x) = e^{-2x}$ *about* -1

61. $f(x) = \sin x$ *about* $\frac{\pi}{2}$

62. $f(x) = \ln x$ *in powers of* $x-3$

63. $f(x) = \ln(2+x)$ *in powers of* $x-2$

64. $f(x) = e^{2x+3}$ *in powers of* $x+1$

65. $f(x) = \sin x - \cos x$ *about* $\frac{\pi}{4}$

66. $f(x) = \cos^2 x$ *about* $\frac{\pi}{8}$

67. $f(x) = \dfrac{x}{1+x}$ *in powers of* $x-1$

68. $f(x) = xe^x$ *in powers of* $x+2$

(**69 – 81**) Find the nth–order Taylor polynomial centered at c for the function

69. $f(x) = \dfrac{2}{x}, \quad n = 3, \quad c = 1$

70. $f(x) = \dfrac{1}{x^2}, \quad n = 4, \quad c = 2$

71. $f(x) = \sqrt{x}, \quad n = 3, \quad c = 4$

72. $f(x) = \sqrt[3]{x}, \quad n = 3, \quad c = 8$

73. $f(x) = \ln x, \quad n = 4, \quad c = 2$

74. $f(x) = x^2 \cos x, \quad n = 2, \quad c = \pi$

75. $f(x) = \sin 2x; \quad n = 3, \quad c = 0$

76. $f(x) = \cos x^2; \quad n = 2, \quad c = 0$

77. $f(x) = e^{-x}; \quad n = 2, \quad c = 0$

78. $f(x) = \cos x; \quad n = 2, \quad c = \frac{\pi}{4}$

79. $f(x) = \ln x; \quad n = 2, \quad c = 1$

80. $f(x) = \sinh 2x; \quad n = 4, \quad c = 0$

81. $f(x) = \cosh x; \quad n = 3, \quad c = \ln 2$

(82 – 84) Find the sums of the series

82. $1+x^2+\dfrac{x^4}{2!}+\dfrac{x^6}{3!}+\dfrac{x^8}{4!}+\cdots$

83. $1+\dfrac{x^2}{3!}+\dfrac{x^4}{5!}+\dfrac{x^6}{7!}+\dfrac{x^8}{9!}+\cdots$

84. $x^3-\dfrac{x^9}{3!\times 4}+\dfrac{x^{15}}{5!\times 16}-\dfrac{x^{21}}{7!\times 64}+\dfrac{x^{27}}{9!\times 256}-\cdots$

(85 – 90) Use the geometric series $\displaystyle\sum_{k=0}^{\infty} x^k=\frac{1}{1-x}$, for $|x|<1$, to determine the Maclaurin series and the interval of convergence for the following functions.

85. $f(x)=\dfrac{1}{1-x^2}$

86. $f(x)=\dfrac{1}{1+x^3}$

87. $f(x)=\dfrac{1}{1+5x}$

88. $f(x)=\dfrac{10}{1+x}$

89. $f(x)=\dfrac{1}{(1-10x)^2}$

90. $f(x)=\ln(1-4x)$

91. The limit $\displaystyle\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi}\,n^{n+1/2}e^{-n}}=1$ that is the relative error in the approximation

$n!\approx\sqrt{2\pi}\,n^{n+1/2}e^{-n}$

Approaches zero as n increases. That is $n!$ grows at a rate comparable to $\sqrt{2\pi}\,n^{n+1/2}e^{-n}$. This result, known as Stirling's Formula, is often very useful in applied mathemmatics and statistics. Prove it by carrying out the following steps.

a) Use the identity $\ln(n!)=\displaystyle\sum_{j=1}^{n}\ln j$ and the increasing nature of ln to show that if $n\ge 1$,

$$\int_0^n \ln x\,dx < \ln(n!) < \int_1^{n+1}\ln x\,dx$$

And hence that $n\ln n-n<\ln(n!)<(n+1)\ln(n+1)-n$

b) If $c_n=\ln(n!)-\left(n+\frac{1}{2}\right)\ln n+n$, show that

$$c_n-c_{n+1}=\left(n+\frac{1}{2}\right)\ln\left(\frac{n+1}{n}\right)-1$$
$$=\left(n+\frac{1}{2}\right)\ln\left(\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}\right)-1$$

c) Use the Maclaurin series for $\ln\frac{1+t}{1-t}$ to show that

$$0 < c_n - c_{n+1} < \frac{1}{3}\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^3} + \cdots\right)$$
$$= \frac{1}{12}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

and therefore that $\{c_n\}$ is decreasing and $\left\{c_n - \frac{1}{12n}\right\}$ is increasing. Hence conclude that $\lim_{n\to\infty} c_n = c$ exists, and that

$$\lim_{n\to\infty} \frac{n!}{n^{n+1/2}e^{-n}} = \lim_{n\to\infty} e^{c_n} = e^c$$

92. Suppose you want to approximate $\sqrt[3]{128}$ to within 10^{-4} of the exact value.

a) Use a Taylor polynomial for $f(x) = (125+x)^{1/3}$ centered at 0.

b) Use a Taylor polynomial for $f(x) = x^{1/3}$ centered at 125.

c) Compare the two approaches. Are they equivalent?

93. Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

a) Use the definition of the derivative to show that $f'(0) = 0$

b) Assume the fact that $f^k(0) = 0$ for k = 1, 2, 3, …. (prove using the definition of the derivative.) Write the Taylor series for f centered at 0.

c) Explain why the Taylor series for f does not converge to f for $x \neq 0$

94. Teams *A* and *B* go into sudden death overtime after playing to a tie. The teams alternate possession of the ball and the first team to score wins. Each team has a $\frac{1}{6}$ chance of scoring when it has the ball, with Team *A* having the ball first.

a) The probability that Team *A* ultimately wins is $\sum_{k=0}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{2k}$. Evaluate this series.

b) The expected number of rounds (possessions by either team) required for the overtime to end is $\frac{1}{6}\sum_{k=1}^{\infty} k\left(\frac{5}{6}\right)^{k-1}$. Evaluate this series.