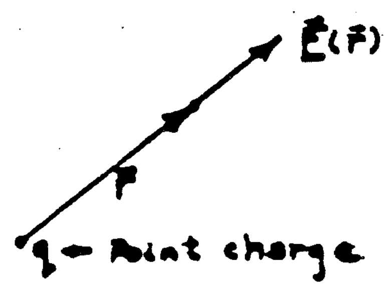
4. THE EQUATIONS OF MATHEMATICAL PHYSICS

a) ELECTROMAGNETICS

- Experimental results of Charles de Coulomb:

$$\vec{E} = \frac{q}{4\pi\epsilon}\,\frac{\hat{r}}{r^2}$$

force / unit positive charge



Permittivity $= \epsilon = 1/(36\pi) \cdot 10^{-9}$ F/m in air. The ELECTRIC FLUX DENSITY is denoted by $\vec{D} = \epsilon\vec{E}$. The electric flux emanating from a closed surface, S, is

OUTWARD POINTING UNIT NORMAL

$$\oiint_S \vec{D}\cdot\hat{n}\,ds = \frac{q}{4\pi}\oiint_S \frac{\hat{r}}{r^2}\cdot\hat{n}\,ds = \begin{cases} q & \text{if } S \text{ encloses charge} \\ 0 & \text{if } S \text{ does not enclose charge} \end{cases}$$

according to Gauss' theorem.

Suppose we have a continuous charge distribution of density, ρ. Then

$$\vec{D}(\vec{r}) = \iiint_V \frac{\rho(\vec{r}\,')}{4\pi} \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \, dv'$$

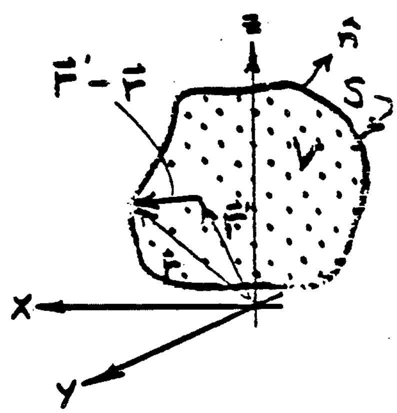
and the flux emanating from the charges within closed surface S is

$$\oint\!\!\oint_S \vec{D}(\vec{r}) \cdot \hat{n} \, ds$$

$$= \iiint_V \frac{\rho(\vec{r}\,')}{4\pi} \left\{ \oint\!\!\oint_S \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \cdot \hat{n} \, ds \right\} dv'$$

$$= \iiint_V \rho(\vec{r}\,') \, dv' = Q, \text{ the}$$

total charge within S.



This is the Gauss Law for electric field:

> The total electric flux emanating from any closed surface is just the total charge enclosed.

Applying the divergence theorem,

$$\oint\!\!\oint_S \vec{D} \cdot \hat{n} \, ds = \iiint_V \nabla \vec{D} \, dv = \iiint_V \rho \, dv .$$

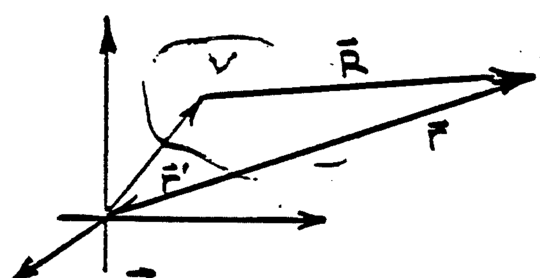
If V is allowed to shrink to a point, we obtain the "point form" of Gauss' law:

$$\nabla \cdot \vec{D} = \rho$$

- Experimental results of Jean-Baptiste Biot and Felix Savart:

The magnetic flux density due to a current with density $\vec{J}$ is

$$\vec{B}(\vec{r}) = \frac{\mu}{4\pi} \iiint_V \frac{\vec{J}(\vec{r}\,') \times \vec{R}}{R^3} dv'$$



$$\vec{R} = \vec{r} - \vec{r}\,'$$

Since $\vec{J}(\vec{r}\,')$ is independent of $\vec{r}$, curl $\vec{J}(\vec{r}\,') = 0$ and hence using the identities,

$$\nabla \times (\varphi \vec{J}) = \varphi \nabla \times \vec{J} + \vec{J} \times \nabla \varphi,$$

$$\nabla\left(\frac{1}{R}\right) = -\frac{\vec{R}}{R^3},$$

We write

$$\vec{B} = -\frac{\mu}{4\pi} \iiint_V \vec{J}(\vec{r}\,') \times \nabla\left(\frac{1}{R}\right) dv'.$$

Therefore

$$\vec{B} = \nabla \times \vec{A}, \quad \text{where}$$

$$\vec{A} = -\frac{\mu}{4\pi}\iiint_v \vec{J}(\vec{r}\,')\frac{1}{R}\,dv'$$

$$= \text{The Magnetic Vector Potential,}$$

where μ is the magnetic permeability which is $4\pi \cdot 10^{-7}$ H/m in air.

What is $\nabla \cdot \vec{B}$?

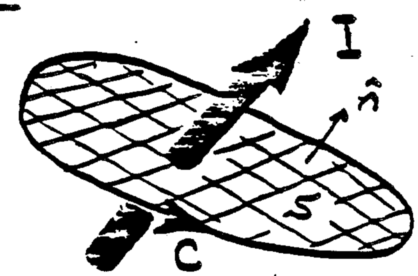
$$\nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = 0.$$

Therefore, $\vec{B}$ is <u>solenoidal</u> and hence Gauss' Law for magnetic fields is

The net magnetic flux penetrating any closed surface is zero.

From these results, we can derive Ampere's Circuital Law:

$$\oint_C \vec{H}\cdot d\vec{r} = \iint_S \vec{J}\cdot\hat{n}\,ds$$
$$= I$$



where I is the total current flowing through surface S bounded by closed curve, C, and $\vec{H}$ is the "magnetic field" (A/m) = $\vec{B}/\mu$.

In words

> The net circulation of the magnetic field around a closed loop is equal to the current flowing through the enclosed surface in the positive direction.

Applying Stoke's theorem, we obtain

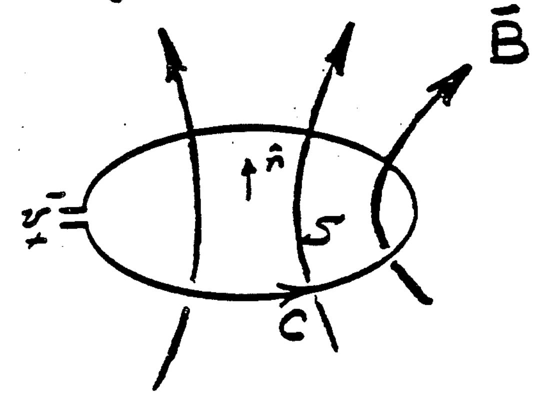
$$\oint_C \vec{H}\cdot d\vec{r} = \iint_S \nabla\times\vec{H}\cdot\hat{n}\,ds = \iint_S \vec{J}\cdot\hat{n}\,ds$$

Allowing S to shrink to a point, we obtain the "point form" of Ampere's Law:

$$\nabla\times\vec{H} = \vec{J}$$

● The experimental results of Michael Faraday:

Faraday found that a voltage was induced in a loop of wire when the magnetic field "linking" it was changed.



$$\mathcal{V} = -\oint_c \vec{E}\cdot d\vec{r} = \frac{d}{dt}\psi,$$

$$\psi = \iint_S \vec{B}\cdot\hat{n}\,ds =$$

The Magnetic Flux.

In words, Faraday's Law states that

> The voltage induced in a loop is the [redacted] time derivative of the magnetic flux linking it.

Applying Stoke's Theorem and letting the loop shrink to a point, we have the "point form" of Faraday's Law:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The CONTINUITY EQUATION is

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \qquad (0)$$

Collecting all of these equations, we have

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (1)$$

$$\nabla \times \vec{H} = \vec{J} \qquad (2)$$

$$\nabla \cdot \vec{B} = 0 \qquad (3)$$

$$\nabla \cdot \vec{D} = \rho \qquad (4)$$

Is there anything inconsistent between equations (0), (2), and (4)?

$$\nabla \cdot (\nabla \times \vec{H}) = 0 = \nabla \cdot \vec{J}$$

according to (2).

But

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = -\nabla \cdot \frac{\partial \vec{D}}{\partial t}$$

according to the continuity equation and Gauss' Law.

8)

James Clerk Maxwell resolved this inconsistency by postulating an additional current component in Ampere's Law. This is the "displacement current" given by

$$\frac{\partial \vec{D}}{\partial t}$$

He proposed the following set of equations which now bear his name:

MAXWELL'S EQUATIONS

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$\nabla \vec{B} = 0$$

$$\nabla \vec{D} = \rho$$

We will now consider several applications of these equations.

(1) ELECTROSTATICS AND "QUASI-STATICS"

If the fields are changing slowly enough (or are constant) then

$$\frac{\partial \vec{B}}{\partial t} \approx 0, \quad \frac{\partial \vec{D}}{\partial t} \approx 0.$$

Under these conditions, Ampere's Law is correct and Faraday's Law becomes Kirchoff's Voltage Law:

$$\oint \vec{E} \cdot d\vec{r} = 0.$$

The static electric field is conservative and can be represented as the gradient of a scalar potential. This potential is called the VOLTAGE:

$$v = -\int_{P_0}^{P} \vec{E} \cdot d\vec{r}.$$

We have

$$\vec{E} = -\nabla v.$$

Moreover,

$$\frac{\nabla \vec{D}}{\epsilon} = \nabla \vec{E} = \boxed{-\nabla^2 v = +\frac{\rho}{\epsilon}}, \quad \text{(for constant } \epsilon\text{).}$$

This equation is called <u>Poisson's equation</u> after Simeon Denis Poisson who did formative work in the theory of electrostatics.

In a region of space containing no sources (i.e., where there is no net charge), then the potential satisfies Laplace's equation,

$$\nabla^2 v = 0.$$

In order to solve these equations UNIQUELY, ADDITIONAL CONDITIONS on the values of the potential or its derivatives on the boundary of the region over which they hold MUST BE SPECIFIED. These conditions are called BOUNDARY CONDITIONS.

Partial differential equations together with their boundary conditions constitute a BOUNDARY VALUE PROBLEM.

(2) ELECTRODYNAMICS:

Let's ask some questions about Faraday's Law.

$$-dq \cdot \oint \vec{E} \cdot d\vec{r} = dq \cdot \frac{d\Psi}{d\varphi}.$$

The left hand side of the equation above is the amount of work done in moving a small positive charge dq around a closed loop. Since that work is not zero, where did the energy we expended go?

This energy is accounted for by an increase in energy stored in the electric and magnetic fields and energy that has become "detached" and radiated into space.

The electrodynamic case is more difficult than the static case. Let $\vec{A}$ be the magnetic vector potential:

$$\vec{H} = \nabla \times \vec{A}$$

Then substituting this into Maxwell's equations,

$$\nabla \times \vec{H} = \nabla \times (\nabla \times \vec{A})$$

$$= \nabla(\nabla \vec{A}) - \nabla^2 \vec{A} = \epsilon \frac{\partial \vec{E}}{\partial t} + \vec{J}$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} = -\mu \left(\nabla \times \frac{\partial \vec{A}}{\partial t}\right)$$

$$\therefore \nabla \times \left[\vec{E} + \mu \frac{\partial \vec{A}}{\partial t}\right] = 0$$

Thus

$$\vec{E} + \mu \frac{\partial \vec{A}}{\partial t}$$

is a conservative vector field and must be the gradient of some scalar, φ.

$$\vec{E} + \mu \frac{\partial \vec{A}}{\partial t} = \nabla \varphi \longrightarrow$$

$$\vec{E} = -\mu \frac{\partial \vec{A}}{\partial t} + \nabla \varphi .$$

Then

$$\epsilon \frac{\partial \vec{E}}{\partial t} = -\mu \epsilon \frac{\partial^2 \vec{A}}{\partial t} + \epsilon \frac{\partial \nabla \varphi}{\partial t} .$$

Therefore

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = -\mu \epsilon \frac{\partial^2 \vec{A}}{\partial t^2} + \epsilon \frac{\partial \nabla \varphi}{\partial t} + \vec{J}$$

Since a vector potential is unique only up to an additive conservative field, we can choose

$$\nabla \cdot \vec{A} = \epsilon \frac{\partial \varphi}{\partial t}$$

Then

$$\boxed{\nabla^2 \vec{A} - \mu \epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\vec{J}}$$

This is the inhomogeneous vector wave equation. (The process of choosing a particular conservative part of the potential $\vec{A}$ is called a "guage transformation.")

If

$$\vec{J}(\vec{r},t) = \vec{J}_o(\vec{r})\, e^{j\omega t},$$

then $\vec{A}$ must also be of the form

$$\vec{A} = \vec{A}_o(\vec{r})\, e^{j\omega t}.$$

Substituting,

$$\nabla^2 \vec{A}_o + k^2 \vec{A}_o = -\vec{J}_o$$

where

$$k = \omega\sqrt{\mu\epsilon}$$

and is called the "wave number." This is the inhomogeneous vector Helmholtz equation.

b) TELEGRAPH EQUATIONS

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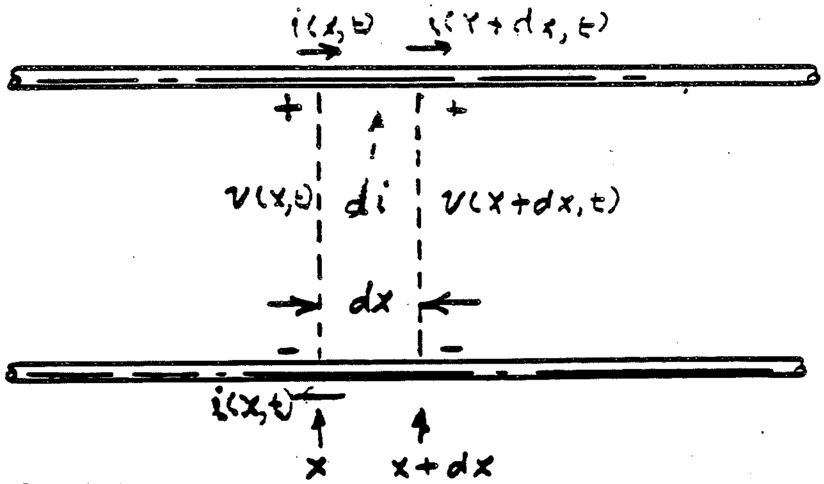
Consider a pair of long wires



The following quantities are defined:

R is the series resistance,
L is the series inductance,
G is the shunt conductance,
C is the shunt capacitance,
PER UNIT LENGTH.

Over the length dx, a small amount of current, dI, is shunted from the top wire to the bottom due to capacitive coupling and leakage conductance:

$$di = i(x+dx,t) - i(x,t)$$
$$= -\Big[\underbrace{G\cdot dx}_{\text{Total Conductance}}\cdot v(x,t) + \underbrace{C\cdot dx}_{\text{Total Capacitance}}\frac{\partial v(x,t)}{\partial t}\Big]$$

Therefore,

$$-\frac{\partial i(x,t)}{\partial x} = G\,v(x,t) + C\,\frac{\partial v(x,t)}{\partial t}.$$

The voltage drop over dx due to the wire resistance and inductance is

$$\begin{aligned} dv &= v(x+dx,t) - v(x,t) \\ &= -\Big[\underbrace{R\cdot dx}_{\text{Total Resistance}}\cdot i(x,t) + \underbrace{L\cdot dx}_{\text{Total Inductance}}\cdot \frac{\partial i(x,t)}{\partial t}\Big]. \end{aligned}$$

Therefore,

$$-\frac{\partial v(x,t)}{\partial x} = R\,i(x,t) + L\,\frac{\partial i(x,t)}{\partial t}.$$

Combining these two equations, we arrive at

$$\frac{\partial^2 v}{\partial x^2} = RGv + [RC + LG]\frac{\partial v}{\partial t} + LC\frac{\partial^2 v}{\partial t^2}.$$

Usually, the leakage conductance is exceedingly small so that $G \approx 0$ and

$$\boxed{\frac{\partial^2 v}{\partial x^2} = RC\,\frac{\partial v}{\partial t} + LC\,\frac{\partial^2 v}{\partial t^2}.}$$

This is the "telegraph equation."

c) HEAT FLOW

If T is the temperature distribution in a region, then T must satisfy the heat conduction equation

$$\boxed{\nabla^2 T = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}.}$$

where α^2 is called the thermal diffusivity.

d) SUMMARY

Many if not most physical phenomena are adequately characterized by one of the equations listed below:

Laplace's: $\nabla^2\varphi = 0$
Poisson's: $\nabla^2\varphi = \psi$
Helmholtz: $[\nabla^2 + k^2]\varphi = \psi$
Wave: $[\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}]\varphi = \psi$
Telegraph: $[\frac{\partial^2}{\partial x^2} + \lambda\frac{\partial^2}{\partial t^2} + \mu\frac{\partial}{\partial t}]\varphi = \psi$
Heat: $[\nabla^2 + \mu\frac{\partial}{\partial t}]\varphi = \psi.$

We will spend the remainder of the course learning methods of solving such equations.