Lecture One – Applications of Definite Integrals

Section 1.1 – Velocity and Net Change

Velocity, Position, and Displacement

Definitions

1. ***Position*** of an object at time *t*, denoted $s(t)$, is the location of the object relative to the origin.

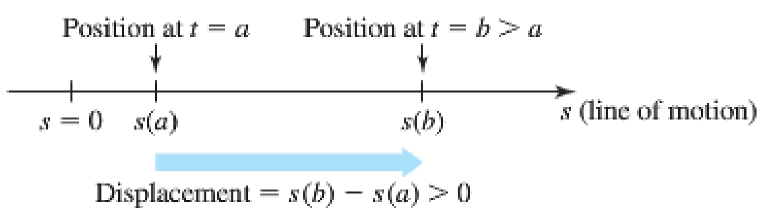
2. ***Velocity*** of an object at time t is $v(t) = s'(t)$

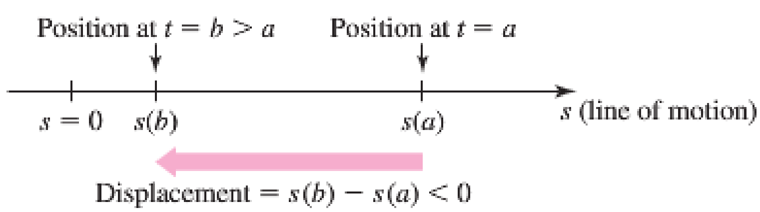
3. ***Displacement*** of the object between $t = a$ and $t = b > a$ is

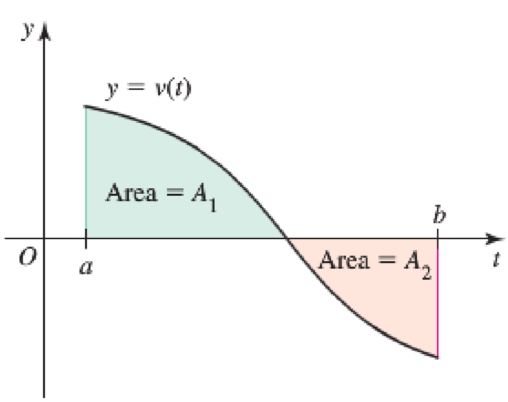
$$s(b) - s(a) = \int_a^b v(t)\,dt$$

4. ***Distance traveled*** by the object between $t = a$ and $t = b > a$ is

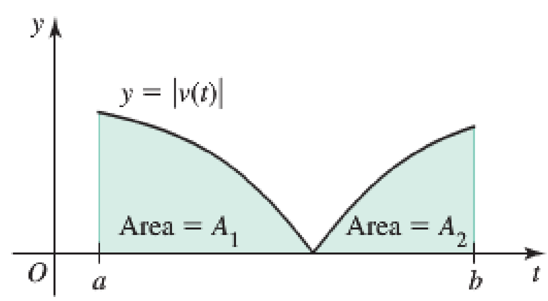
$$\int_a^b |v(t)|\,dt \qquad \text{where } |v(t)| \text{ is the speed of the object at time } t.$$







Displacement $= A_1 - A_2 = \int_a^b v(t)\,dt$



Distance traveled $= A_1 + A_2 = \int_a^b |v(t)|\,dt$

Example

A cyclist pedals along a straight road with velocity $v(t)=2t^2-8t+6 \ \ (mi/hr)$ for $0\le t\le 3$, where t is measured in hours.

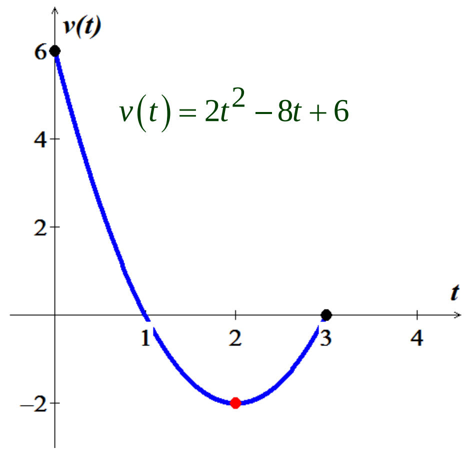
a) Graph the velocity function over the interval [0, 3]. Determine when the cyclist moves in the position direction and when she moves in the negative direction.

b) Find the displacement of the cyclist (in miles) on the time intervals [0, 1], [1, 3], and [0, 3]. Interpret these results.

c) Find the distance traveled over the interval [0, 3]

Solution

a) $v(t)=2t^2-8t+6=0 \quad \to \quad t=1,\ 3$



The velocity is zero at $t = 1$ and $t = 3$.

The velocity is positive on $0\le t<1$, which means the cyclist moves in the positive s direction.

The velocity is negative on $1<t<3$, which means the cyclist moves in the negative s direction.

b) Displacement over [0, 1]

$$\begin{aligned}
s(1)-s(0) &= \int_0^1 v(t)\,dt \\
&= \int_0^1 \left(2t^2-8t+6\right)dt \\
&= \left[\frac{2}{3}t^3-4t^2+6t\right]_0^1 \\
&= \frac{2}{3}-4+6 \\
&= \frac{8}{3}
\end{aligned}$$

Displacement over [1, 3]

$$s(3)-s(1)=\int_1^3 \left(2t^2-8t+6\right)dt$$

$$= \left[\frac{2}{3}t^3 - 4t^2 + 6t\right]_1^3$$

$$= 18 - 36 + 18 - \frac{2}{3} + 4 - 6$$

$$\boxed{= -\frac{8}{3}}$$

Displacement over [0, 3]

$$s(3) - s(0) = \int_0^3 \left(2t^2 - 8t + 6\right) dt$$

$$= \left[\frac{2}{3}t^3 - 4t^2 + 6t\right]_0^3$$

$$= 18 - 36 + 18$$

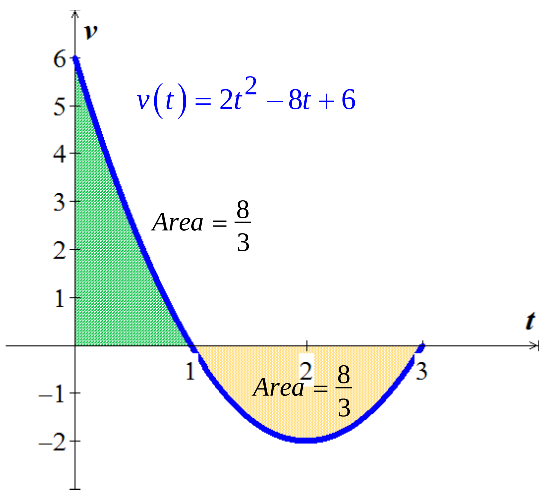
$\boxed{= 0}$ The cyclist returns to the starting point after 3 hours.

c) $\text{Distance} = \int_0^3 |v(t)|\,dt = \int_0^1 \left(2t^2 - 8t + 6\right) dt - \int_1^3 \left(2t^2 - 8t + 6\right) dt$

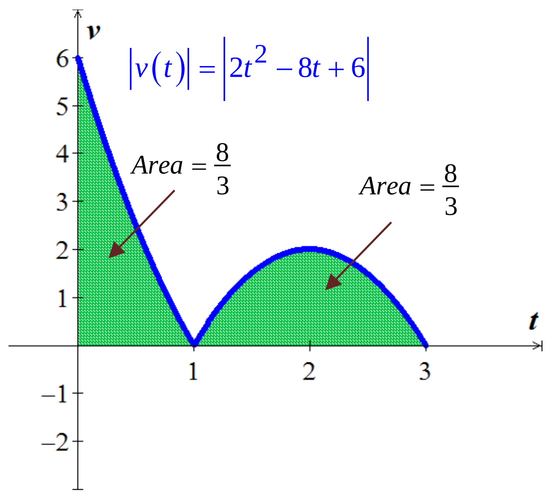
$$= \left[\frac{2}{3}t^3 - 4t^2 + 6t\right]_0^1 - \left[\frac{2}{3}t^3 - 4t^2 + 6t\right]_1^3$$

$$= \frac{8}{3} + \frac{8}{3}$$

$$\boxed{= \frac{16}{3}}$$



Displacement from $t = 0$ to $t = 3$ is 0



$\text{Distance} = \int_0^3 |v(t)|\,dt = \frac{8}{3} + \frac{8}{3} = \frac{16}{3}$

Future Value of the Position Function

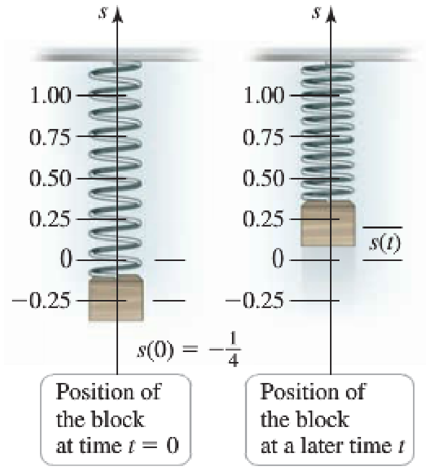
Theorem

Given the velocity $v(t)$ of an object moving along a line and its initial position $s(0)$, the position function of the object for future times $t \ge 0$ is

$$\underbrace{s(t)}_{\substack{\text{position}\\ \text{at time } t}} = \underbrace{s(0)}_{\substack{\text{initial}\\ \text{position}}} + \underbrace{\int_0^t v(x)\,dx}_{\substack{\text{displacement}\\ \text{over } [0,\ t]}}$$

Example

A block hangs at rest from a massless spring at the origin ($s = 0$). At $t = 0$, the block is pulled downward $\frac{1}{4}m$ to its initial position $s(0) = -\frac{1}{4}$ and released. Its velocity is given by $v(t) = \frac{1}{4}\sin t \ \ (m/s)$ for $t \ge 0$. Assume that the upward direction is positive.



a) Find the position of the block for $t \ge 0$

b) Graph the position function for $0 \le t \le 3\pi$.

c) When does the block move through the origin for the first time?

d) When does the block reach its highest point for the first time and what is its position at that time?

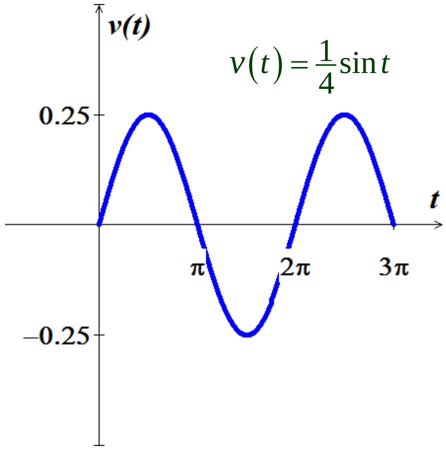
e) When does the block return to its lowest point?

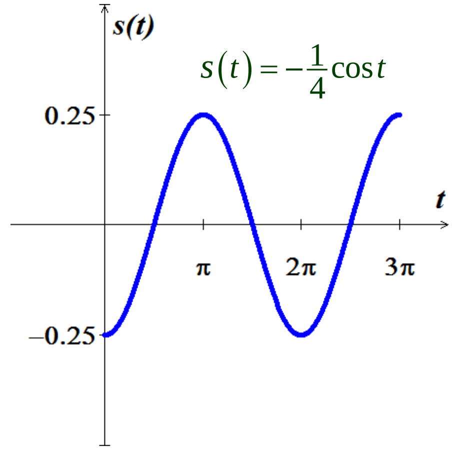
Solution

a)

1st method	**2nd method**
$s(t) = \int v(t)\,dt$	$s(t) = s(0) + \int_0^t v(x)\,dx$
$= \int \frac{1}{4}\sin t\ dt$	$= -\frac{1}{4} + \int_0^t \frac{1}{4}\sin x\ dx$
$= -\frac{1}{4}\cos t + C$	$= -\frac{1}{4} - \frac{1}{4}\left[\cos x\right]_0^t$
Since $s(0) = -\frac{1}{4}$, then	$= -\frac{1}{4} - \frac{1}{4}(\cos t - 1)$
$-\frac{1}{4} = -\frac{1}{4}\cos(0) + C \ \rightarrow\ \underline{C = 0}$	$= -\frac{1}{4}\cos t$
$s(t) = -\frac{1}{4}\cos t$	

b)





c) The block move through the origin for the first time when $s=0$

$$s(t)=-\frac{1}{4}\cos t=0 \quad\Rightarrow\quad \boxed{t=\frac{\pi}{2}}$$

d) The block moves in the positive direction and reaches its high point for the first time when $t=\pi$

$$s(\pi)=-\frac{1}{4}\cos\pi \;\boxed{=\frac{1}{4}\;\; m}$$

e) The block return to the lowest point at $t=2\pi$. This motion repeats every 2π seconds

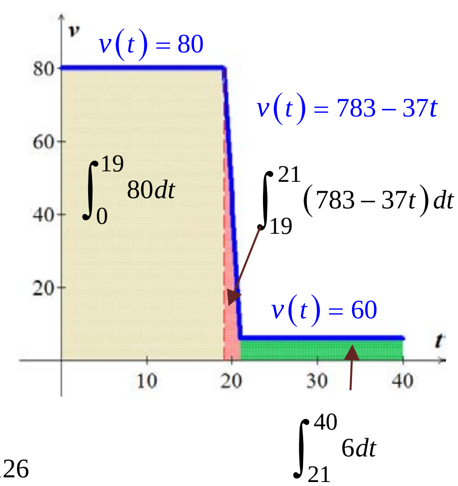
Example

Suppose a skydiver leaps from a hovering helicopter and fall in a straight line. He falls at a terminal velocity of 80 *m/s* for 19 *sec*, at which time he opens his parachute. The velocity decreases linearly to 6 *m/s* over two-second period and then remains constant until he reaches the ground at *t* = 40 *s*. The motion is described by the velocity function

$$v(t)=\begin{cases} 80 & \text{if } \ 0\le t<19 \\ 783-37t & \text{if } \ 19\le t<21 \\ 6 & \text{if } \ 21\le t\le 40 \end{cases}$$

Determine the altitude from which the skydiver jumper.

Solution



$$d=\int_0^{40}|v(t)|\,dt$$

$$=\int_0^{19}80dt+\int_{19}^{21}(783-37t)\,dt+\int_{21}^{40}6dt$$

$$=80t\Big|_0^{19}+\left(783\,t-\frac{37}{2}t^2\right)\Big|_{19}^{21}+6t\Big|_{21}^{40}$$

$$=152+783(21)-\frac{1}{2}(37)(21)^2-783(19)+\frac{1}{2}(37)(19)^2+240-126$$

$$\boxed{=1720}$$

The skydiver jumped from 1720 *m* above the ground.

Acceleration

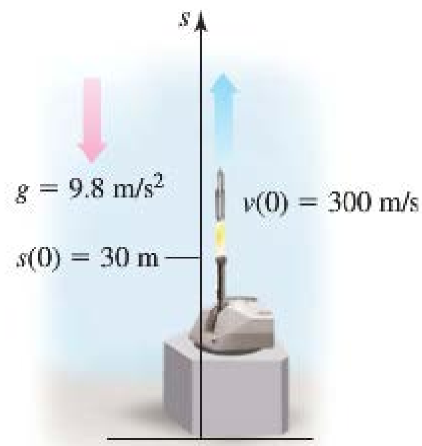
Theorem (velocity from Acceleration)

Given the acceleration $a(t)$ of an object moving along a line and its initial velocity $v(0)$, the velocity of the object for future times $t \ge 0$ is

$$v(t) = v(0) + \int_0^t a(x)\,dx$$

Example

An artillery shell is fired directly upward with an initial velocity of 300 *m/s* from a point 30 *m* above the ground. Assume that only the force of gravity acts on the shell and it produces an acceleration of $9.8\ m/s^2$. Find the velocity of the shell for $t \ge 0$

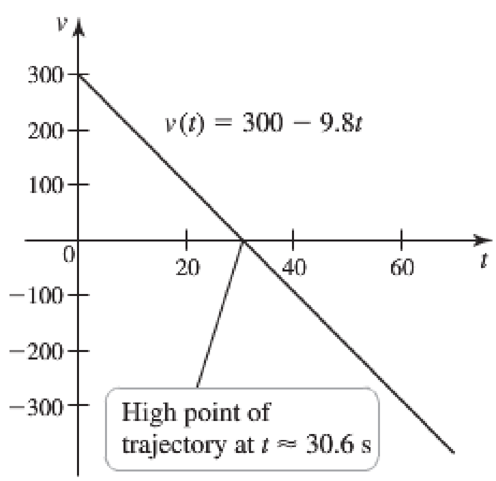


Solution

$$\begin{aligned} v(t) &= v(0) + \int_0^t a(x)\,dx \\ &= 300 + \int_0^t (-9.8)\,dx \qquad \textbf{\textit{Upward}} \\ &= 300 - 9.8t \end{aligned}$$

The velocity decreases from its initial value of 300 *m/s*, reaching zero at the high point of the trajectory when

$$v(t) = 300 - 9.8t = 0 \quad \rightarrow \quad t = \frac{300}{9.8} \approx 30.6\ \text{sec}$$



At this point the velocity becomes negative, and the shell begins its descent to Earth.

Net Change and Future Value

Theorem

Suppose a quantity Q changes over time at a known rate Q'. Then the net change in Q between $t = a$ and $t = b$ is

$$\int_a^b Q'(t)\,dt = Q(b) - Q(a) = \textit{net change in } Q \textit{ over } [a,\ b]$$

$$\int_0^t Q'(t)\,dt = Q(t) - Q(0)$$

Given the ***initial value*** $Q(0)$, the ***future value*** Q at future times $t \ge 0$ is

$$\underbrace{Q(t)}_{\substack{\textit{future}\\ \textit{value}}} = \underbrace{Q(0)}_{\substack{\textit{initial}\\ \textit{value}}} + \underbrace{\int_0^t Q'(t)\,dt}_{\substack{\textit{net charge}\\ \textit{over } [0,\ t]}}$$

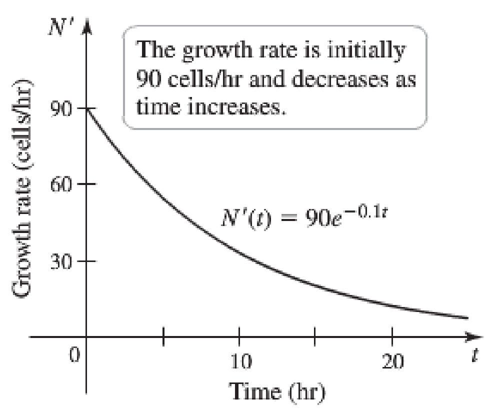
Velocity–Displacement Problems	***General Problems***
Position $s(t)$	Quantity $Q(t)$ (such as volume or population size)
Velocity: $s'(t) = v(t)$	Rate of change $Q'(t)$
Displacement: $s(b) - s(a) = \int_a^b v(t)\,dt$	Net change: $Q(b) - Q(a) = \int_a^b Q'(t)\,dt$
Future position: $s(t) = s(0) + \int_0^t v(x)\,dx$	Future value of Q: $Q(t) = Q(0) + \int_0^t Q'(x)\,dx$

Example

A culture of cells in a lab has a population of 100 cells when nutrients are added at time $t = 0$. Suppose the population $N(t)$ increases at a rate given by

$$N'(t) = 90e^{-0.1t} \quad \textit{cells / hr}$$

Find $N(t)$ for $t \ge 0$



Solution

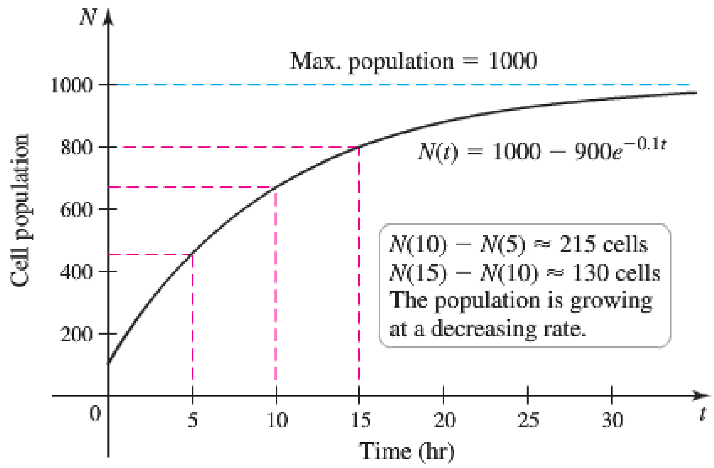
$$N(t) = N(0) + \int_0^t N'(x)\,dx$$

$$= 100 + \int_0^t 90e^{-0.1x}dx$$

$$= 100 - \frac{90}{0.1}\left[e^{-0.1x} \right]_0^t$$

$$= 100 - 900\left(e^{-0.1t} - 1\right)$$

$$\underline{= 1000 - 900e^{-0.1t}|}$$



The graph of the population function shows that the population increases, but at a decreasing rate. Note that the initial condition $N(0) = 100$ cells is satisfied and that population size approaches 1000 cells as $t \to \infty$.

Example

A book publisher estimates that the marginal cost of a particular title (in dollars/book) is given by

$$C'(x) = 12 - 0.0002x$$

Where $0 \le x \le 50,000$ is the number of books printed. What is the cost of producing the 12,001st through the 15,000 book?

Solution

$$C(15,000) - C(12,000) = \int_{12,000}^{15,000} C'(x)\,dx$$

$$= \int_{12,000}^{15,000} (12 - 0.0002x)\,dx$$

$$= \left[12x - 0.0001x^2 \right]_{12,000}^{15,000}$$

$$= 180 \times 10^3 - 225 \times 10^2 - 144 \times 10^3 + 144 \times 10^2$$

$$\underline{= \$27,900|}$$

Exercises *Section* 1.1 – Velocity and Net Change

Assume *t* is time measured in seconds and velocities have units of *m/s*.

a) Graph the velocity function over the given interval. Then determine when the motion is in the positive direction.

b) Find the displacement over the given interval.

c) Find the distance traveled over the given interval.

1. $v(t)=6-2t; \quad 0\le t\le 6$

2. $v(t)=10\sin 2t; \quad 0\le t\le 2\pi$

3. $v(t)=50e^{-2t}; \quad 0\le t\le 4$

Consider an object moving along a line with the following velocities and initial positions

a) Graph the velocity function on the given interval. Then determine when the object is moving in the positive direction and when it is moving in the negative direction.

b) Determine the position function for $t\ge 0$ using both the antiderivative method and the Fundamental Theorem of Calculus. Check for agreement between the two methods.

c) Graph the position function on the given interval.

4. $v(t)=6-2t \quad on\ [0,\ 5] \quad s(0)=0$

5. $v(t)=9-t^2 \quad on\ [0,\ 4] \quad s(0)=-2$

Find the position and velocity of an object moving along a straight line with the given acceleration, initial velocity, ansd initial position. Assume units of meters and seconds.

6. $a(t)=-9.8, \quad v(0)=20, \quad s(0)=0$

7. $a(t)=e^{-t}, \quad v(0)=60, \quad s(0)=40$

8. A mass hanging from a spring is set in motion and its ensuing velocity is given by $v(t)=2\pi\cos\pi t$ for $t\ge 0$. Assume that the position direction is upward and $s(0)=0$.

a) Determine the position function for $t\ge 0$.

b) Graph the position function on the interval [0, 3].

c) At what times does the mass reach its lowest point the first three times?

d) At what times does the mass reach its highest point the first three times?

9. The velocity of an airplane flying into a headwind is given by $v(t)=30\left(16-t^2\right)$ *mi/hr* for $0\le t\le 3\ hr$. Assume that $s(0)=0$

a) Determine and graph the position function for $0\le t\le 3$.

b) How far does the airplane travel in the first 2 *hr*.?

c) How far has the airplane traveled at the instant its velocity reaches 400 *mi/hr*.?

10. A car slows down with an acceleration of $a(t)=-15\ ft/s^2$. Assume that $v(0)=60\ ft/s$ and $s(0)=0$

a) Determine and graph the position function for $t\ge 0$.

b) How far does the car travel in the time it takes to come to rest?

11. The owners of an oil reserve begin extracting oil at $t=0$. Based on estimates of the reserves, suppose the projected extraction rate is given by $Q'(t)=3t^2(40-t)^2$, where $0\le t\le 40$, Q is measured in millions of barrels, and t is measured in years.

a) When does the peak extraction rate occur?

b) How much oil is extracted in the first 10, 20, and 30 years?

c) What is the total amount of oil extracted in 40 year?

d) Is one-fourth of the total oil extracted in the first one-fourth of the extraction period? Explain.

12. Starting with an initial value of $P(0)=55$, the population of a prairie dog community grows at a rate of $P'(t)=20-\frac{t}{5}$ (in units of prairie *dogs/month*), for $0\le t\le 200$.

a) What is the population 6 months later?

b) Find the population $P(t)$ for $0\le t\le 200$.

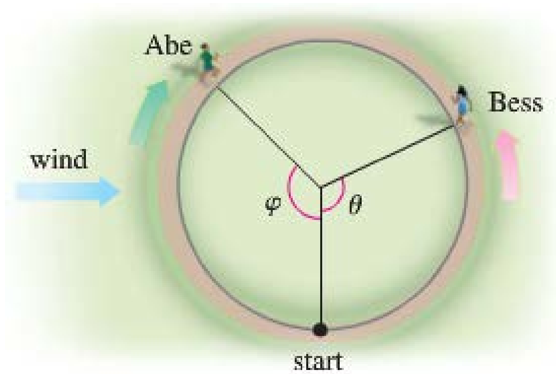
13. The population of a community of foxes is observed to fluctuate on a 10-year cycle due to variations in the availability of prey. When population measurements began (t = 0 years), the population was 35 foxes. The growth rate in units of *foxes/yr*. was observed to be

$$P'(t)=5+10\sin\left(\frac{\pi t}{5}\right)$$

a) What is the population 15 years later? 35 years later?

b) Find the population $P(t)$ at any time $t\ge 0$.

14. A strong west wind blows across a circular running track. Abe and Bess start at the south end of the track and at the same time, Abe starts running clockwise and Bess starts running counterclockwise. Abe runs with a speed (in units of *mi/hr*.) given by $u(\varphi)=3-2\cos\varphi$ and Bess runs with a speed given by $v(\theta)=3+2\cos\theta$, where φ and θ are the central angles of the runners



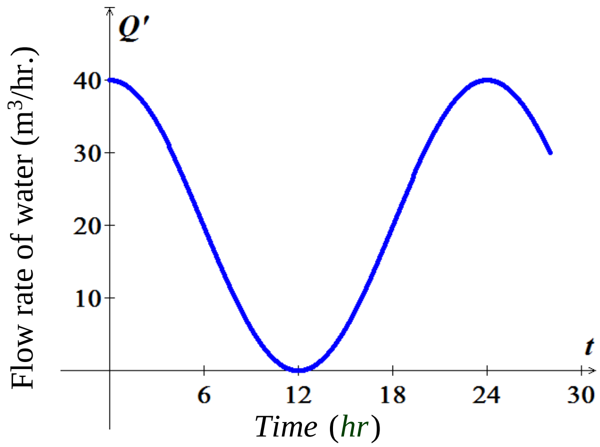
a) Graph the speed functions u and v, and explain why they describe the runners' speed (in light of the wind).

b) Which runner has the greater average speed for one lap?

c) If the track has a radius of $\frac{1}{10}$ *mi*, how long does it take each runner to complete one lap and who wins the race?

15. A reservoir with a capacity of 2500 m^3 is filled with a single inflow pipe. The reservoir is empty and the inflow pipe is opened at $t = 0$. Letting $Q(t)$ be the amount of water in the reservoir at time t, the flow rate of water into reservoir (in m^3/hr) oscillates on 24-*hr* cycle and is given by

$$Q'(t) = 20\left[1 + \cos\frac{\pi t}{12}\right]$$



a) How much water flows into the reservoir in the first 2 hr.?

b) Find and graph the function that gives the amount of water in the reservoir over the interval $[0, t]$ where $t \geq 0$.

c) When is the reservoir full?