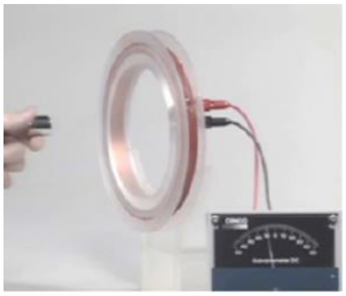
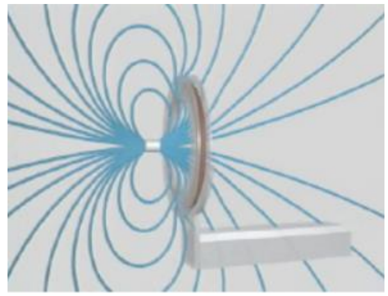
4.3 – Faraday's Law

Faraday's Law states that whenever the magnetic flux crossing a loop changes with time, there will be an induced voltage which is equal to the negative rate of change of magnetic flux





$$V_{ind} = -\frac{d\phi_B}{dt}$$

$V_{ind} \rightarrow$ Induced voltage

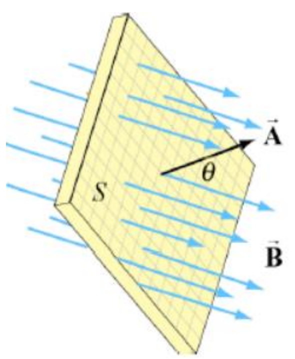
$\phi_B \rightarrow$ Magnetic flux

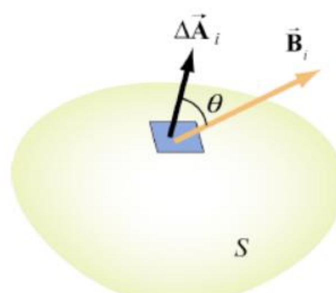
Where $\phi_B = \int_{surface} \vec{B} \bullet d\vec{A} = \int_{surface} B\,dA\cos\theta$

Michael Faraday

$\theta \rightarrow$ is the angle formed between $\vec{B}$ & $d\vec{A}$, if the magnetic field and the angle are constants over the surface, then the magnetic flux simplifies to

$$\boxed{\phi_B = \vec{B} \bullet \vec{A} = BA\cos\theta}$$





Remember direction of area is perpendicular to the plane of the loop. When fingers of the right hand are wrapped around the loop in a counterclockwise direction, thumb gives the direction of the area)

The average induced voltage number over a time interval Δt can be obtained by integrating the induced voltage over time & dividing by Δt

$$\bar{V}_{ind} = \frac{1}{\Delta t}\int_0^{\Delta t} V_{ind}\,dt \qquad but \quad V_{ind} = -\frac{d\phi_B}{dt}$$

$$= \frac{1}{\Delta t}\int_0^{\Delta t} -\frac{d\phi_B}{dt}\,dt$$

$$= -\frac{1}{\Delta t}\int_0^{\Delta t} d\phi_B$$

$$= -\frac{1}{\Delta t}\left[\phi_B(\Delta t) - \phi_B(0)\right]$$

$$= -\frac{1}{\Delta t}\Delta\phi_B$$

$$\boxed{V_{ind} = -\frac{\Delta\phi_B}{\Delta t} = -\frac{\phi_{Bf} - \phi_{Bi}}{\Delta t}}$$

V_{ind} → Average induced voltage

ϕ_{Bf} → Final magnetic flux

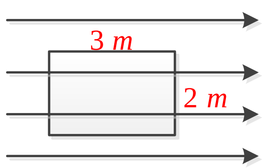
ϕ_{Bi} → Initial magnetic flux

Positive induced voltage means the direction of the induced current is counter clockwise and negative induced voltage means m the direction of the induced current is clockwise.

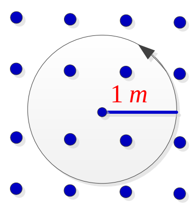
Example

For each of the following, calculate the magnetic flux crossing the loop. Assume the strength of the field to be $4T$ in each of them

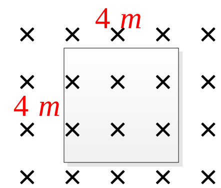
a)



b)



c)



Solution

Given: $B = 4T$

a) $A = (2)(3) = 6\ m^2$

$\theta = 90°$ (Since the direction of area is perpendicularly out of the paper & field is east)

$\phi_B = BA\cos\theta$

$= (4)(6)\cos 90°$

$\underline{= 0\ \ Tm^2|}$

b) $A = \pi r^2 = \pi(1)^2 = \pi\ m^2$

$\theta = 0°$ (Since both $\vec{B}$ & $\vec{A}$ are perpendicularly out)

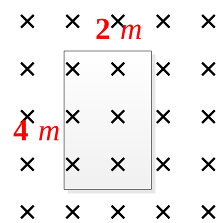
$$\phi_B = BA\cos\theta$$
$$= (4)(\pi)\cos 0°$$
$$\boxed{= 4\pi \ \ Tm^2}$$

c) $A = (4)(4) = 16\ m^2$

$\theta = 180°$ (Since area is perpendicularly out & field is perpendicularly in)

$$\phi_B = BA\cos\theta$$
$$= (4)(16)\cos 180°$$
$$\boxed{= -64 \ \ Tm^2}$$

Example



The rectangular loop shown is placed in a magnetic field as shown.
The strength of the field is 5*T*. If the strength of the field is reduced to 3*T* in 0.2 *sec*, calculate the average induced voltage during this time interval.

Solution

Given: $B_i = 5T, \quad B_f = 3T$

$$\theta_i = \theta_f = 180°$$

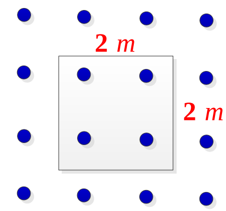
$$A_i = A_f = (2)(4) = 8\ m^2$$

$$\phi_{Bi} = B_i A_i \cos(\theta_i)$$
$$= (5)(8)\cos(180°)$$
$$\boxed{= -40\ Tm^2}$$

$$\phi_{Bf} = B_f A_f \cos(\theta_f)$$
$$= (3)(8)\cos(180°)$$
$$\boxed{= -24\ Tm^2}$$

$$V_{ind} = -\frac{\Delta\phi_B}{\Delta t} = -\frac{\phi_{Bf} - \phi_{Bi}}{\Delta t}$$
$$= -\frac{-24-(-40)}{0.2}$$
$$\boxed{= -80 \ \ V}$$

Example



The square loop shown is placed in a magnetic field as shown.
The strength of the magnetic field varies with time according to the equation

$$B(t) = 2t^2 + 1$$

a) Obtain an expression for the instantaneous voltage as a function of time
b) Calculate the induced voltage after 0.2 *sec*.

Solution

a) $\phi_B = B(t)A\cos\theta$

$$= \left(2t^2 + 1\right)(2\times 2)\cos(0)$$

$$= 8t^2 + 4$$

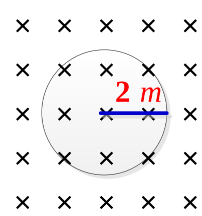
$$V_{ind}(t) = -\frac{d\phi_B}{dt}$$

$$= -\frac{d}{dt}\left(8t^2 + 4\right)$$

$$\underline{= -16t}$$

b) $V_{ind}(t = 0.2) = -16(0.2)\ \underline{= -3.2\ V}$

Example



The circular loop shown is placed in a magnetic field of strength 0.5*T*.
If the shape of the loop is changed to a square in 0.4 *sec*, calculate the average induced current in the loop if the resistance of the loop is 10Ω.

Solution

Given: $B_i = B_f = 0.5T,\quad \theta_i = \theta_f = 180°$

$$A_i = \pi r^2 = \pi(2)^2 = 4\pi\ m^2$$

If the shape of the circle is converted to a square, the length (circumference) of the loop should remain the same. Let the side of the square be *x*, then

$$2\pi r = 4x \quad\Rightarrow\quad x = \frac{\pi r}{2} = \frac{\pi(2)}{2} = \pi\ m$$

$$A_f = \pi^2\ m^2$$

$$\phi_{Bi} = B_i A_i \cos\left(\theta_i\right) = (0.5)(4\pi)\cos(180°)\underline{= -2\pi\ Tm^2}$$

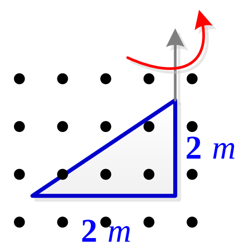
$$\phi_{Bf} = B_f A_f \cos\left(\theta_f\right) = (0.5)\left(\pi^2\right)\cos(180°)\underline{= -0.5\pi^2\ Tm^2}$$

$$\bar{V}_{ind} = -\frac{\phi_{Bf} - \phi_{Bi}}{\Delta t} = -\frac{-2\pi - \left(-0.5\pi^2\right)}{0.4} \underline{= -3.37\ V|}$$

$$\bar{I}_{ind} = \frac{\bar{V}_{ind}}{R} = \frac{-3.37}{10} \underline{= -0.337\ \Omega|}$$

Example

The triangular loop shown is placed in a magnetic field of strength 10*T*.



Now if the loop is rotated across the axis shown by 90° in 0.5 sec so that the plane of the loop is perpendicular to the plane of the paper. Calculate the average induced voltage.

<u>Solution</u>

Given: $B_i = B_f = 10T, \quad \theta_i = 0°, \quad \theta_f = 90°, \quad \Delta t = 0.5\text{sec}$

$$A_i = A_f = \frac{1}{2}(2)(2) \underline{= 2\ m^2|}$$

$$\phi_{Bi} = B_i A_i \cos\left(\theta_i\right) = (10)(2)\cos(0°) \underline{= 20\ Tm^2|}$$

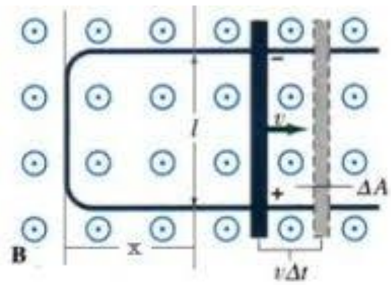
$$\phi_{Bf} = B_f A_f \cos\left(\theta_f\right) = (10)(2)\cos(90°) \underline{= 0\ Tm^2|}$$

$$\bar{V}_{ind} = -\frac{\phi_{Bf} - \phi_{Bi}}{\Delta t} = -\frac{0 - 20}{0.5} \underline{= -40\ V|}$$

$$\bar{I}_{ind} = \frac{\bar{V}_{ind}}{R} = \frac{-3.37}{10} \underline{= -0.337\ \Omega|}$$

Motional εmf

Motional εmf is induced voltage produced by the motion of a rod in a *U*-shaped conductor placed in a magnetic field as shown



The rod and the closed end of the *U*-shaped conductor form a loop. As the rod moves, the area od this loop changes resulting in the change of the magnetic flux crossing the loop which produces induced voltage. This induced voltage is called motional εmf . (Note: induced voltage is not a potential difference but εmf , because in this process electrical energy is created out of non-electrical energy).

As the rod moves along the U-shaped conductor, its position changes as a function of the rod is ℓ , then its area at a particular time is $A(t) = \ell\ x(t)$. Therefore, the magnetic flux crossing the loop at a certain time *t* is:

$$\phi_B = A(t)B = \ell\ x(t)B$$

$$\varepsilon_{motional} = -\frac{d\phi_B}{dt} = -\frac{d}{dt}\left[\ell\ x(t)B\right]$$

$$= -\ell\ B\frac{dx(t)}{dt} \qquad\qquad but \quad \frac{dx(t)}{dt} = v \ \rightarrow \text{Speed of the rod}$$

$$= -\ell\ Bv$$

$$\boxed{\left|\varepsilon_{motional}\right| = B\ell v}$$

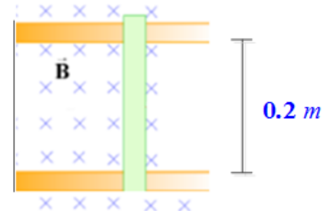
$\varepsilon_{motional} \rightarrow$ motional εmf

$B \rightarrow$ Field strength

$v \rightarrow$ speed of rod

$\ell \rightarrow$ length of rod

Example



A rod of length 0.2m is sliding in U-shaped conductor as shown.

The strength of the field is 2T.

a) Calculate the motional εmf if the rod is moving with a speed of 5 m/s.

b) If the position of the rod varies with time according to the equation $x(t)=t^2$, obtain an expression for the motional εmf as a function of time.

Solution

a) Given: $\ell = 0.2m, \quad v = 5m/s, \quad B = 2T$

$$\left|\varepsilon_{motional}\right| = B\ell v$$

$$= (2)(0.2)(5)$$

$$\underline{= 2\ V}$$

b) $\left|\varepsilon_{motional}\right| = B\ell v \qquad but \quad \frac{dx(t)}{dt} = v$

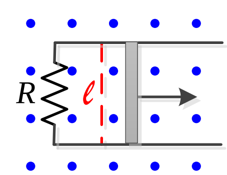
$$= \ell\ B\frac{d}{dt}\left(t^2\right)$$

$$= \ell\ B(2t)$$

$$= 2(0.2)(2t)$$

$$\underline{= 0.8t}$$

Example



A rod of length ℓ and mass m is released with initial speed v_0. The strength of the field is B & the resistance of the loop is R.
Find an expression for its speed as a function of time and its position as a function of time (assuming it started at $x=0$)

Solution

As the rod moves, there will be an induced current in the rod. A current carrying rod in a magnetic field is acted upon by a magnetic force, Form the right hand rule, it can be shoun that this is a resistive force.

$$F_B = -I\ell B$$

From Newton's 2nd law

$$F = ma = m\frac{dv}{dt}$$

But $F = F_B = -I_{ind}\ell B$

$$m\frac{dv}{dt} = -I_{ind}\ell B \qquad but \quad I_{ind} = \frac{\varepsilon_{mot}}{R}$$

$$m\frac{dv}{dt} = -\frac{\varepsilon_{mot}}{R}\ell B = -\frac{B\ell v}{R}\ell B$$

$$m\frac{dv}{dt} = -\frac{B^2\ell^2}{R}v$$

$$\frac{dv}{v} = -\frac{B^2\ell^2}{mR}dt$$

$$\int\frac{dv}{v} = -\frac{B^2\ell^2}{mR}\int dt$$

$$\ln v = -\frac{B^2\ell^2}{mR}t + C$$

$$v = e^{-\frac{B^2\ell^2}{mR}t+C}$$

$$= e^C e^{-\frac{B^2\ell^2}{mR}t}$$

$$v(t=0) = v_0 \quad \Rightarrow \quad v_0 = e^C$$

$$v(t) = v_0 e^{-\frac{B^2\ell^2}{mR}t}$$

$$v(t) = \frac{dx}{dt} = v_0 e^{-\frac{B^2\ell^2}{mR}t}$$

$$dx = v_0 e^{-\frac{B^2\ell^2}{mR}t}dt \quad \Rightarrow \quad \int dx = v_0\int e^{-\frac{B^2\ell^2}{mR}t}dt$$

$$x(t) = -\frac{v_0 mR}{B^2\ell^2}e^{-\frac{B^2\ell^2}{mR}t} + C$$

$$x(t=0) = 0 \quad \Rightarrow \quad 0 = -\frac{v_0 mR}{B^2\ell^2} + C \;\rightarrow\; \boxed{C = \frac{v_0 mR}{B^2\ell^2}}$$

$$x(t) = \frac{v_0 mR}{B^2\ell^2}\left(1 - e^{-\frac{B^2\ell^2}{mR}t}\right)$$

Expressing Faraday's law in terms of electric field

Induced voltage due to rate of change of magnetic flux is an εmf because its is voltage created by converting non-electrical energy to electrical energy. εmf is defined to be work done by the source per a unit charge $\left(\varepsilon = \frac{W_s}{q}\right)$ and the work done by the source is given by $W_s = q\int \vec{E}\bullet d\vec{s}$ where $\vec{E}$ is electric field due to the source. Thus,

$$\varepsilon = \frac{W_s}{q} = \frac{q\int \vec{E}\bullet d\vec{s}}{q} = \int \vec{E}\bullet d\vec{s}$$

Therefore, for a closed loop, Faraday's law can be written as

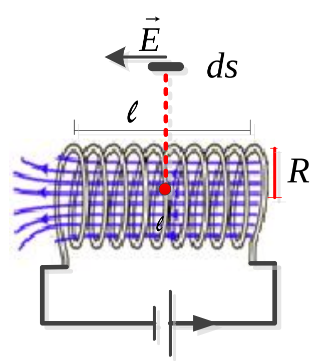
$$V_{ind} = \varepsilon = \oint \vec{E}\bullet d\vec{s} = -\frac{d\phi_B}{dt}$$

$$\boxed{\oint \vec{E}\bullet d\vec{s} = -\frac{d\phi_B}{dt}}$$

Faraday's law in terms of electric field due to the source which in this case is the negative rate of change of magnetic flux.

From Faraday's law, we see that the electric field due to a source (as opposed to due to a charge) is not conservative. Because if it was conservative, the integral of the electric field over a closed path would be zero. For example for electric field due to a static change $\int \vec{E}\bullet d\vec{s} = 0$ because electric field due to a static charge is conservative.

Example



Consider a solenoid of length ℓ, number of turns N and radius R. It carries a current $I = I(t)$ which varies as a function of time.

Obtain an expression for the electric field at a point a perpendicular distance $r_\perp$ from the axis of the solenoid where $r_\perp > R$

<u>Solution</u>

From symmetry the following conclusion can be made

1. The electric field lines should be circles concentric with the axis of the solenoid.
2. The magnitude of the electric field at points on a circle concentric with the axis of the solenoid should be constant.

Let's choose our loop for applying Faraday's law to a circle concentric with the axis of the solenoid of radius $r_\perp > R$.

Since the electric field is tangent to the circle at any point

$$\vec{E} \cdot d\vec{s} = Eds$$

$$\oint \vec{E} \cdot d\vec{s} = \oint Eds$$

And since the electric field is constant over the circle

$$\oint \vec{E} \cdot d\vec{s} = E \oint ds \qquad \textit{but } \oint ds \quad \textit{is the circumference of the circle}$$

$$E \oint ds = E\left(2\pi r_{\perp}\right)$$

$$\oint \vec{E} \cdot d\vec{s} = 2\pi r_{\perp} E \qquad (1)$$

The magnetic flux crossing the circle is $\phi_B = BA$.

The magnetic field outside the solenoid is approximately zero. Therefore, the area should be taken to be the cross-section area of the solenoid & not the area of the loop.

$$\Rightarrow \phi_B = B\pi R^2$$

The magnetic field inside a solenoid is given as

$$B = \frac{\mu_0 NI}{\ell}$$

$$\phi_B = \frac{\mu_0 NI}{\ell} \pi R^2 \qquad (2)$$

Now applying Faraday's law & using equations (1) & (2)

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\phi_B}{dt}$$

$$2\pi r_{\perp} E = -\frac{d}{dt}\left\{\mu_0 \frac{NI\pi R^2}{\ell}\right\}$$

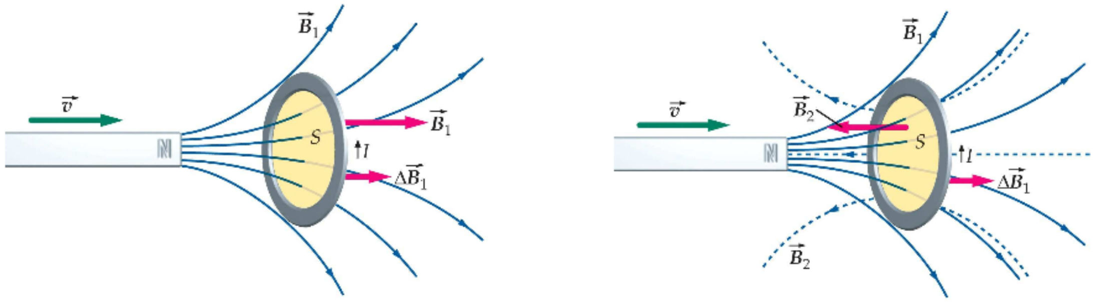
$$= -\frac{\mu_0 N\pi R^2}{\ell} \frac{dI}{dt}$$

$$\boxed{E = -\frac{\mu_0 N\pi R^2}{2\pi r_{\perp} \ell} \cdot \frac{dI}{dt}}$$

Lenz's Rule

Lenz's rule states that the direction of the induced current is in such a way as to oppose the cause. This is essentially a statement of the negative sign in Faraday's law.

This law helps to determine the direction of the induced current qualitatively without solving the problem. If the problem is solved qualitatively using Faraday's law, positive induced voltage means the direction of the induced current is counterclockwise which negative implies clockwise direction.

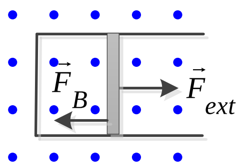


The situation shown above the current induced in the conducting ring generates a magnetic field whose flux counteracts the change in the magnetic flux caused by the bar magnet.

- Moving the bar magnet closer to the ring increases the magnetic field $\vec{B}_1$ (solid field lines) through the ring by the amount $\Delta\vec{B}_1$
- The resultant change in magnetic flux through the ring induces a current I in the direction shown.
- The induced current *I*, in turn, generates a magnetic field $\vec{B}_2$ (dashed field lines) in a direction that opposes the change of flux caused by the moving bar magnet.

Example

The rod shown is sliding in a *U*-shaped conductor placed in a magnetic field by means of an external force. Use Lenz's rule to find out whether the induced current is following up or down the rod

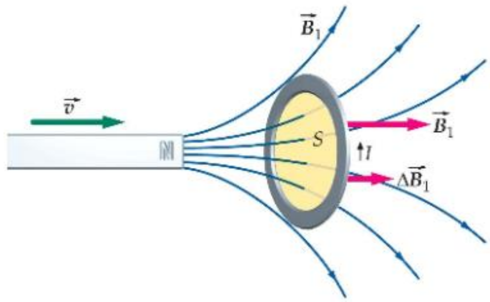


Solution

The cause for the change of flux and hence the induced current is the external force. Therefore the direction of the induced current must be in such a way as to oppose the cause. A current carrying rod placed in a magnetic field is acted upon by a magnetic force. This magnetic force must oppose the external force. Since the external force is to the right, the direction of the magnetic force must be to the left. From the right hand rule or the screw rule, if the direction of the magnetic force is to be left, the direction of the induced current must be down the rod.

Example

The loop shown is pulled away from the permanent magnet. Use Lenz's rule to determine the direction of the induced current in the loop.

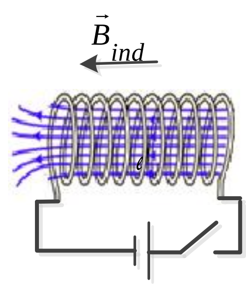


Solution

As the loop is pulled away from the magnet, the strength of the magnetic field crossing it decreases decreasing the flux that crosses it. Therefore the cause for the induced current is decrease in magnetic field. Therefore the direction of the induced current in the loop must be in such a way that the induced magnetic field due to the induced current increases the field of the induced field is to increase the field it must be parallel to it. Thus the direction of the induced field must be to the right of the induced field is to be to the right, from the right hand rule, the direction of the induced current must be *counterclockwise* as seen by the observer shown.

Example

A loop is enclosing a solenoid as shown



The solenoid is connected to a battery. Use Lenz's rule to determine the direction of the induced current in the loop when the switch is closed all of a sudden.

Solution

As the switch is closed, the current will increase from zero to a certain value. As a result, the field inside the solenoid will increase from zero to a certain value increasing the flux. Therefore the cause for the induced current is increase in magnetic field as the switch is turned on. From the right hand rule, the direction of the magnetic field inside the solenoid must be to the right.
The direction of the induced current must be in such a way as to decrease the magnetic field. This means the direction of the induced field due to the induced current must oppose the solenoid field or it should be to the left. From the right hand rule, if the direction of the induced field is to be the left, then the direction of the induced current in the loop must be *clockwise* as seen by the observer.

Maxwell's Equations

The four equations governing electricity and magnetism discussed so far are called Maxwell's equations because then were first mathematically organized by Karl Maxwell. These equations are

1. ***Gauss's Law***: the surface integral of the electric field over any closed surface is equal to $\frac{1}{\varepsilon_0}$ times the total charge enclosed inside the surface.

$$\oint_{\substack{closed\\surface}} \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0} \qquad \varepsilon_0 = \frac{1}{4\pi\kappa}$$

2. ***Gauss's law for magnetic field***: The surface integral of the magnetic field over any closed surface is equal to zero

$$\oint_{\substack{closed\\surface}} \vec{B} \cdot d\vec{A} = 0$$

3. ***Ampere-Maxwell law***: The line integral of the magnetic field along any closed loop is equal to μ_0 times the total current & total displacement current crossing the loop.

$$\oint_{\substack{closed\\surface}} \vec{B} \cdot ds = \mu_0 \left(I + \varepsilon_0 \frac{d\phi_E}{dt} \right)$$

$$I_0 = \varepsilon_0 \frac{d\phi_E}{dt} \text{ (Displacement current)}$$

It is called Ampere-Maxwell law, because the displacement current was theoretically predicted by Maxwell himself.

4. ***Faraday's Law***: States that the line integral of the electric field in any closed path is equal to the negative rate of change of the magnetic flux crossing the loop with time.

$$\oint_{\substack{closed\\surface}} \vec{E} \cdot d\vec{s} = -\frac{d\phi_B}{dt}$$

Maxwell's Equations	
Gauss' Law for $\vec{E}$	$\oint_{\substack{closed\\surface}} \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0}$
Gauss' Law for $\vec{B}$	$\oint_{\substack{closed\\surface}} \vec{B} \cdot d\vec{A} = 0$
Ampere-Maxwell law	$\oint_{\substack{closed\\surface}} \vec{B} \cdot ds = \mu_0 \left(I + \varepsilon_0 \frac{d\phi_E}{dt} \right)$
Faraday's law	$\oint_{\substack{closed\\surface}} \vec{E} \cdot d\vec{s} = -\frac{d\phi_B}{dt}$