# Solution Section 2.6 – Forced Harmonic Motion

### Exercise

A 1-*kg* mass is attached to a spring $k = 4kg/s^2$ and the system is allowed to come to rest. The spring-mass system is attached to a machine that supplies external driving force $f(t) = 4\cos\omega t$ *Newtons*. The system is started from equilibrium; the mass is having neither initial displacement nor velocity. Ignore any damping forces.

*a)* Find the position of the mass as a function of time

*b)* Place your answer in the form $s(t) = A\sin\delta t\sin\overline{\omega}t$. Select an $\omega$ near the natural frequency of the system to demonstrate the "beating" of the system. Sketch a plot shows the "beats:" and include the envelope of the beating motion in your plot.

**Solution**

**a)** $mx'' + \omega_0^2 x = f(t)$

$$mx'' + kx = f(t) \qquad x(0) = x'(0) = 0$$

$$x'' + 4x = 4\cos\omega t$$

$$x(t) = \frac{A}{\left(\omega_0^2 - \omega^2\right)}\left(\cos\omega t - \cos\omega_0 t\right)$$

$$x(t) = \frac{4}{4-\omega^2}\left(\cos\omega t - \cos 2t\right)$$

**b)** $x(t) = \frac{4}{4-\omega^2}\left(\cos\omega t - \cos 2t\right)$ $\qquad \cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

$$= \frac{4}{4-\omega^2}\left[-2\sin\left(\frac{\omega+2}{2}t\right)\sin\left(\frac{\omega-2}{2}t\right)\right]$$

$$= \frac{4}{4-\omega^2}\left[2\sin\left(\frac{\omega+2}{2}t\right)\sin\left(\frac{2-\omega}{2}t\right)\right]$$

*Mean frequency*: $\overline{\omega} = \frac{\omega_0 + \omega}{2} = \frac{2+\omega}{2}$

$$2\overline{\omega} = 2 + \omega$$

*Half difference*: $\delta = \frac{\omega_0 - \omega}{2} = \frac{2-\omega}{2}$

$$2\delta = 2 - \omega$$

$$4 - \omega^2 = (2+\omega)(2-\omega) = 2\overline{\omega}2\delta$$

$$= 2\overline{\omega}2\delta$$

$$= 4\overline{\omega}\delta$$

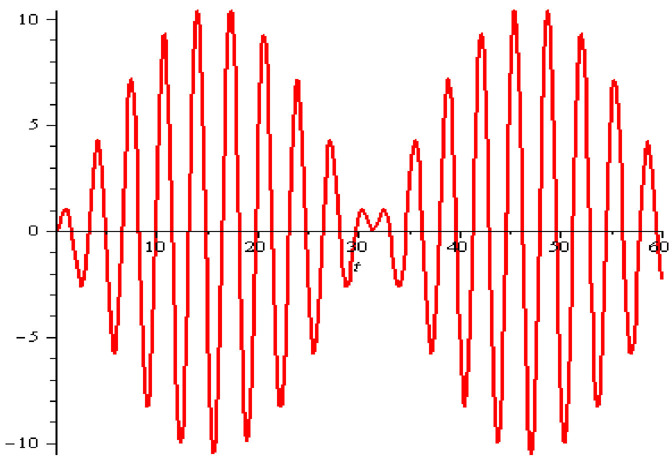
$$x(t) = \frac{8}{4\varpi\delta}\sin\varpi t\sin\delta t$$

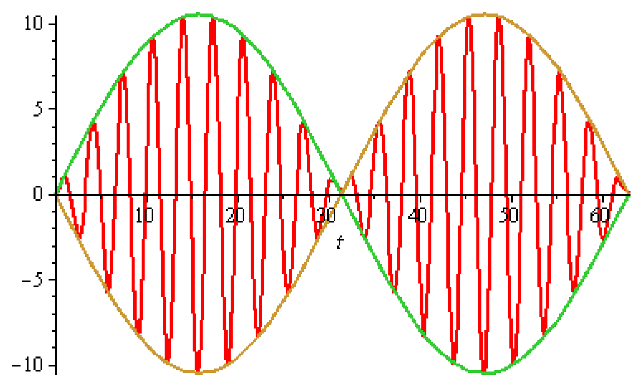
$$= \frac{2}{\varpi\delta}\sin\varpi t\sin\delta t$$

If we choose $\omega = 1.8$ near to $\omega_0 = 2$

That implies to: $\bar{\omega} = \frac{2+1.8}{2} \approx 1.9$ and $\delta = \frac{2-1.8}{2} \approx 0.1$

$$x(t) = \frac{2}{0.19}\sin 0.1t\sin 1.9t$$




## *Exercise*

Find a particular solution to the differential equation using undetermined coefficients. Find and plot the solution of the initial value problem. Superimpose the plots of the transient response and the steady state solution.

$$x'' + 7x' + 10x = 3\cos 3t \qquad x(0) = -1, \;\; x'(0) = 0$$

***Solution***

The particular solution: $x(t) = a\cos 3t + b\sin 3t$

$$x' = -3a\sin 3t + 3b\cos 3t$$
$$x'' = -9a\cos 3t - 9b\sin 3t$$

$$-9a\cos 3t - 9b\sin 3t + 7\left(-3a\sin 3t + 3b\cos 3t\right) + 10\left(a\cos 3t + b\sin 3t\right) = 3\cos 3t$$

$$-9a\cos 3t - 9b\sin 3t - 21a\sin 3t + 21b\cos 3t + 10a\cos 3t + 10b\sin 3t = 3\cos 3t$$

$$a\cos 3t - 21a\sin 3t + 21b\cos 3t + b\sin 3t = 3\cos 3t$$

$$\left(a + 21b\right)\cos 3t + \left(b - 21a\right)\sin 3t = 3\cos 3t$$

$$\begin{aligned} a + 21b &= 3 \\ -21a + b &= 0 \end{aligned} \Rightarrow a = \frac{3}{442} \quad b = \frac{63}{442}$$

The particular solution (***steady-state solution***):

$$\boxed{x_p(t) = \frac{3}{442}\cos 3t + \frac{63}{442}\sin 3t}$$

The homogeneous eq.: $x'' + 7x' + 10x = 0$

The characteristic eq.: $\lambda^2 + 7\lambda + 10 = 0 \quad \Rightarrow \lambda_1 = -5,\ \lambda_2 = -2$

$x_h(t) = C_1 e^{-5t} + C_2 e^{-2t}$

$$x(t) = \frac{3}{442}\cos 3t + \frac{63}{442}\sin 3t + C_1 e^{-5t} + C_2 e^{-2t} \qquad x(0) = -1,\ x'(0) = 0$$

$$x(0) = \frac{3}{442}\cos 3(0) + \frac{63}{442}\sin 3(0) + C_1 e^{-5(0)} + C_2 e^{-2(0)}$$

$$-1 = \frac{3}{442} + C_1 + C_2 \quad \to \quad C_1 + C_2 = -\frac{445}{442}$$

$$x'(t) = -\frac{9}{442}\sin 3t + \frac{189}{442}\cos 3t - 5C_1 e^{-5t} - 2C_2 e^{-2t}$$

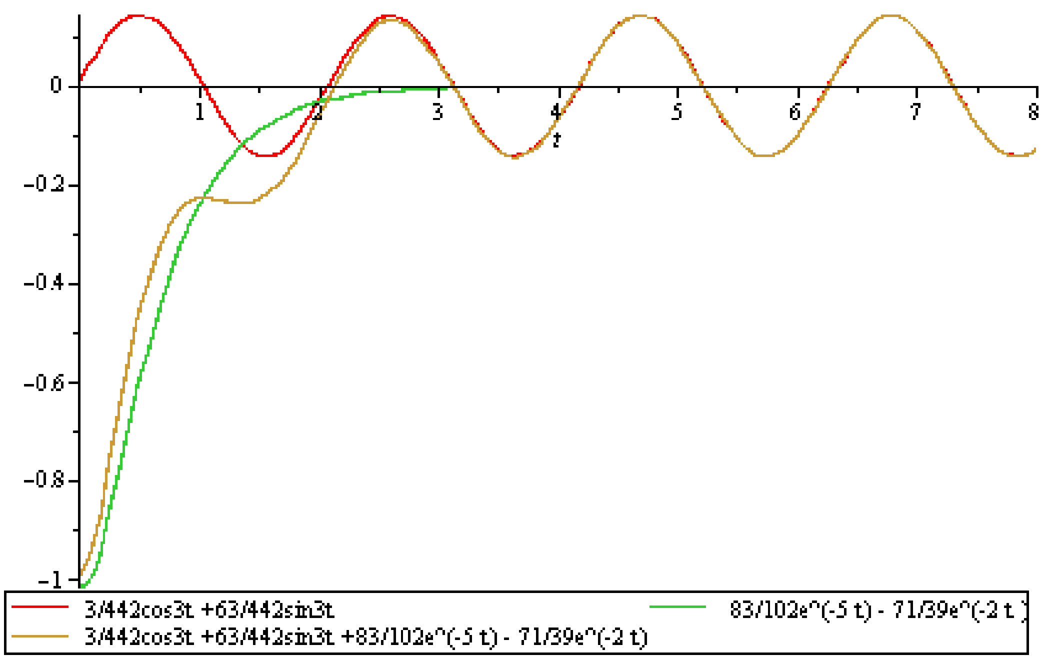
$$x'(0) = -\frac{9}{442}\sin 3(0) + \frac{189}{442}\cos 3(0) - 5C_1 e^{-5(0)} - 2C_2 e^{-2(0)}$$

$$5C_1 + 2C_2 = \frac{189}{442}$$

$$\begin{cases} C_1 + C_2 = -\frac{445}{442} \\ 5C_1 + 2C_2 = \frac{189}{442} \end{cases} \to C_1 = \frac{83}{102} \quad C_2 = -\frac{71}{39}$$

***Transient response solution***: $x_h(t) = \frac{83}{102}e^{-5t} - \frac{71}{39}e^{-2t}$

The general solution: $x(t) = \frac{3}{442}\cos 3t + \frac{63}{442}\sin 3t + \frac{83}{102}e^{-5t} - \frac{71}{39}e^{-2t}$



## Complex Method

$x'' + 7x' + 10x = 3\cos 3t$

The particular solution: $z = Ae^{i3t}$

$$z' = (3i)Ae^{i3t}$$

$$z'' = (3i)^2 Ae^{i3t}$$

$z'' + 7z' + 10z = 3e^{i3t}$

$(3i)^2 Ae^{i3t} + 7(3i)Ae^{i3t} + 10Ae^{i3t} = 3e^{i3t}$

$(-9 + 21i + 10)A = 3$

$(1 + 21i)A = 3$

$$\begin{aligned} A &= 3\frac{1}{1+21i} \cdot \frac{1-21i}{1-21i} \\ &= 3.\frac{1-21i}{1+441} \\ &= \frac{3}{442} - i\frac{63}{442} \end{aligned}$$

$$\begin{aligned} z &= \left(\frac{3}{442} - i\frac{63}{442}\right)e^{i3t} \\ &= \left(\frac{3}{442} - i\frac{63}{442}\right)(\cos 3t + i\sin 3t) \\ &= \frac{3}{442}\cos 3t + \frac{63}{442}\sin 3t + i\left(\frac{3}{442}\sin 3t - \frac{63}{442}\cos 3t\right) \end{aligned}$$

The particular solution (***steady-state solution***): $x_p(t) = \frac{3}{442}\cos 3t + \frac{63}{442}\sin 3t$

## Exercise

Find a particular solution to the differential equation using undetermined coefficients. Find and plot the solution of the initial value problem. Superimpose the plots of the transient response and the steady state solution.

$$x'' + 4x' + 5x = 3\sin t \quad x(0) = 0, \;\; x'(0) = -3$$

***Solution***

$x'' + 4x' + 5x = 3\sin t$

The particular solution: $z = Ae^{it}$

$$z' = (i)Ae^{it}$$

$$z'' = (i)^2 Ae^{it} = -Ae^{it}$$

$z'' + 4z' + 5z = 3e^{it}$

$$-Ae^{i3t} + 4iAe^{i3t} + 5Ae^{i3t} = 3e^{it}$$

$$(-1+4i+5)A = 3$$

$$(4+4i)A = 3$$

$$A = \frac{3}{4}\frac{1}{1+i}\cdot\frac{1-i}{1-i}$$
$$= \frac{3}{4}\left(\frac{1}{2} - i\frac{1}{2}\right)$$
$$= \frac{3}{8} - i\frac{3}{8}$$

$$z = \left(\frac{3}{8} - i\frac{3}{8}\right)e^{it}$$
$$= \left(\frac{3}{8} - i\frac{3}{8}\right)(\cos t + i\sin t)$$
$$= \frac{3}{8}\left[\cos t + \sin t + i(\sin t - \cos t)\right]$$

$$\boxed{x_p(t) = \boldsymbol{Im}(z) = \frac{3}{8}(\sin t - \cos t)}$$

The homogeneous eq.: $x'' + 4x' + 5x = 0$

The characteristic eq.: $\lambda^2 + 4\lambda + 5 = 0 \quad \Rightarrow \quad \boxed{\lambda = -2 \pm i}$

$$x_h(t) = e^{-2t}\left(C_1\cos t + C_2\sin t\right)$$

$$x(t) = \frac{3}{8}(\sin t - \cos t) + e^{-2t}\left(C_1\cos t + C_2\sin t\right) \qquad x(0) = 0, \ \ x'(0) = -3$$

$$x(0) = \frac{3}{8}(\sin 0 - \cos 0) + e^{-2(0)}\left(C_1\cos 0 + C_2\sin 0\right)$$

$$0 = -\frac{3}{8} + C_1 \quad \rightarrow \quad \boxed{C_1 = \frac{3}{8}}$$

$$x'(t) = \frac{3}{8}(\cos t + \sin t) - 2e^{-2t}\left(C_1\cos t + C_2\sin t\right) + e^{-2t}\left(-C_1\sin t + C_2\cos t\right)$$

$$x'(0) = \frac{3}{8}(\cos 0 + \sin 0) - 2e^{-2(0)}\left(C_1\cos 0 + C_2\sin 0\right) + e^{-2(0)}\left(-C_1\sin 0 + C_2\cos 0\right)$$

$$-3 = \frac{3}{8} - 2C_1 + C_2$$

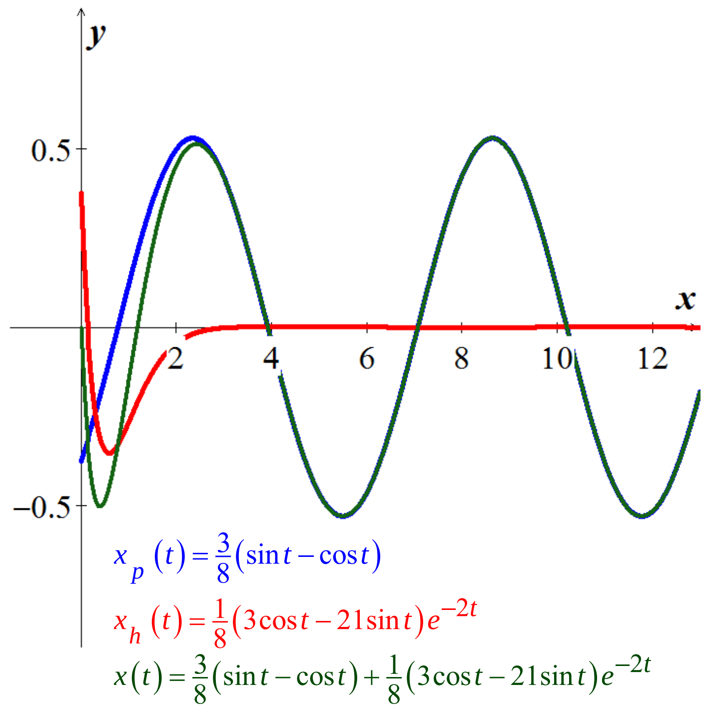
$$\boxed{C_2 = -3 - \frac{3}{8} + 2\left(\frac{3}{8}\right) = -\frac{21}{8}}$$

$$x(t) = \frac{3}{8}(\sin t - \cos t) + e^{-2t}\left(\frac{3}{8}\cos t - \frac{21}{8}\sin t\right)$$

$$\boxed{x(t) = \frac{3}{8}(\sin t - \cos t) + \frac{1}{8}e^{-2t}(3\cos t - 21\sin t)}$$

The steady-state solution is the particular solution: $\boxed{x_p(t) = \frac{3}{8}(\sin t - \cos t)}$

The transient response is: $\boxed{x_h(t) = \frac{1}{8}(3\cos t - 21\sin t)e^{-2t}}$




## *Exercise*

Find a particular solution of $y'' - 2y' + 5y = 2\cos 3x - 4\sin 3x + e^{2x}$ given the set $y_p = A\cos 3x + B\sin 3x + Ce^{2x}$ where *A*, *B*, *C* are to be determined

***Solution***

$$y_p = A\cos 3x + B\sin 3x + Ce^{2x}$$

$$y'_p = -3A\sin 3x + 3B\cos 3x + 2Ce^{2x}$$

$$y''_p = -9A\cos 3x - 9B\sin 3x + 4Ce^{2x}$$

$$\begin{aligned} y'' - 2y' + 5y &= -9A\cos 3x - 9B\sin 3x + 4Ce^{2x} + 6A\sin 3x - 6B\cos 3x - 4Ce^{2x} \\ &\quad + 5A\cos 3x + 5B\sin 3x + 5Ce^{2x} \\ &= (-4A - 6B)\cos 3x + (6A - 4B)\sin 3x + 5Ce^{2x} = 2\cos 3x - 4\sin 3x + e^{2x} \end{aligned}$$

$$\begin{cases} \left.\begin{matrix} -4A - 6B = 2 \\ 6A - 4B = -4 \end{matrix}\right\} \to & A = -\frac{8}{13},\ B = \frac{1}{13} \\ \quad 5C = 1 & \to C = \frac{1}{5} \end{cases}$$

The particular solution: $\boxed{y_p = -\frac{8}{13}\cos 3x + \frac{1}{13}\sin 3x + \frac{1}{5}e^{2x}}$

***Exercise***

Find the general solution: $mx'' + kx = F_0 \cos\omega t$ ; $x(0) = x_0$, $x'(0) = 0$ $(\omega \neq \omega_0)$

***Solution***

$$m\lambda^2 + k = 0 \quad \to (k > 0) \quad \lambda_{1,2} = \pm i\sqrt{\frac{k}{m}} = \pm\omega_0 i \quad \left(\omega_0 = \sqrt{\frac{k}{m}}\right)$$

$$\boxed{x_h = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t}$$

$$x_P = A\cos\omega t + B\sin\omega t$$

$$x'_P = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$x'''_P = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

$$m\left(-A\omega^2\cos\omega t - B\omega^2\sin\omega t\right) + k\left(A\cos\omega t + B\sin\omega t\right) = F_0\cos\omega t$$

$$\left(kA - m\omega^2 A\right)\cos\omega t + \left(kB - m\omega^2 B\right)\sin\omega t = F_0\cos\omega t$$

$$\begin{cases} \left(k - m\omega^2\right)A = F_0 \\ \left(k - m\omega^2\right)B = 0 \end{cases} \quad \to \quad \boxed{A = \frac{F_0}{k - m\omega^2}, \quad B = 0}$$

$$x_P = \frac{F_0}{k - m\omega^2}\cos\omega t$$

$$x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{F_0}{k - m\omega^2}\cos\omega t$$

$$x(0) = x_0 \quad \to \quad \boxed{C_1 = x_0 - \frac{F_0}{k - m\omega^2}}$$

$$x'(t) = -\omega_0 C_1\sin\omega_0 t + \omega_0 C_2\cos\omega_0 t - \frac{\omega_0 F_0}{k - m\omega^2}\sin\omega t$$

$$x'(0) = 0 \quad \to \quad \boxed{C_2 = 0}$$

$$x(t) = x_0 - \frac{F_0}{k - m\omega^2}\cos\omega_0 t + \frac{F_0}{k - m\omega^2}\cos\omega t$$

$$\boxed{= x_0 + \frac{F_0}{k - m\omega^2}\left(\cos\omega t - \cos\omega_0 t\right)}$$

***Exercise***

Find the general solution: $mx'' + kx = F_0 \cos\omega t$ ; $x(0) = 0$, $x'(0) = v_0$ $(\omega = \omega_0)$

***Solution***

$$m\lambda^2 + k = 0 \quad \to (k > 0) \quad \lambda_{1,2} = \pm i\sqrt{\frac{k}{m}} = \pm\omega_0 i \quad \left(\omega_0 = \sqrt{\frac{k}{m}}\right)$$

$$\boxed{x_h = C_1 \cos \omega t + C_2 \sin \omega t} \qquad \omega_0 = \omega$$

$$x_P = A\cos\omega t + B\sin\omega t$$

$$x'_P = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$x'''_P = -A\omega^2 \cos\omega t - B\omega^2 \sin\omega t$$

$$m\left(-A\omega^2 \cos\omega t - B\omega^2 \sin\omega t\right) + k\left(A\cos\omega t + B\sin\omega t\right) = F_0 \cos\omega t$$

$$\left(kA - m\omega^2 A\right)\cos\omega t + \left(kB - m\omega^2 B\right)\sin\omega t = F_0 \cos\omega t$$

$$\begin{cases} \left(k - m\omega^2\right)A = F_0 \\ \left(k - m\omega^2\right)B = 0 \end{cases} \quad \rightarrow \quad \boxed{A = \frac{F_0}{k - m\omega^2}, \quad B = 0}$$

$$x_P = \frac{F_0}{k - m\omega^2}\cos\omega t$$

$$x(t) = C_1 \cos\omega t + C_2 \sin\omega t + \frac{F_0}{k - m\omega^2}\cos\omega t$$

$$= C_2 \sin\omega t + \left(C_1 + \frac{F_0}{k - m\omega^2}\right)\cos\omega t$$

$$x(0) = 0 \quad \rightarrow \quad \boxed{C_1 = -\frac{F_0}{k - m\omega^2}}$$

$$x'(t) = \omega C_2 \cos\omega t - \omega\left(C_1 + \frac{F_0}{k - m\omega^2}\right)\sin\omega t$$

$$x'(0) = v_0 \quad \rightarrow \quad \boxed{C_2 = \frac{v_0}{\omega}}$$

$$x(t) = \frac{v_0}{\omega}\sin\omega t + \left(-\frac{F_0}{k - m\omega^2} + \frac{F_0}{k - m\omega^2}\right)\cos\omega t$$

$$\boxed{= \frac{v_0}{\omega}\sin\omega t}$$

## *Exercise*

Find the general solution: $x'' + \omega_0^2 x = F_0 \sin\omega t$ ; $x(0) = 0, \quad x'(0) = 0 \quad \left(\omega \neq \omega_0\right)$

***Solution***

$$\lambda^2 + \omega_0^2 = 0 \quad \rightarrow \quad \lambda_{1,2} = \pm\omega_0 i$$

$$\boxed{x_h = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t}$$

$$x_P = A\cos\omega t + B\sin\omega t$$

$$x'_P = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$x'''_P = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

$$\left(-A\omega^2\cos\omega t - B\omega^2\sin\omega t\right) + \omega_0^2\left(A\cos\omega t + B\sin\omega t\right) = F_0\sin\omega t$$

$$\begin{cases} \left(\omega_0^2 - \omega^2\right)A = F_0 \\ \left(\omega_0^2 - \omega^2\right)B = 0 \end{cases} \quad \rightarrow \quad \boxed{A = \frac{F_0}{\omega_0^2 - \omega^2}, \quad B = 0}$$

$$x_P = \frac{F_0}{\omega_0^2 - \omega^2}\cos\omega t$$

$$x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{F_0}{\omega_0^2 - \omega^2}\sin\omega t$$

$$x(0) = 0 \quad \rightarrow \quad \boxed{C_1 = 0}$$

$$x'(t) = -\omega_0 C_1\sin\omega_0 t + \omega_0 C_2\cos\omega_0 t + \frac{F_0\omega}{\omega_0^2 - \omega^2}\cos\omega t$$

$$x'(0) = 0 \quad \rightarrow \quad \boxed{C_2 = -\frac{F_0\omega}{\left(\omega_0^2 - \omega^2\right)\omega_0}}$$

$$x(t) = -\frac{F_0\omega}{\left(\omega_0^2 - \omega^2\right)\omega_0}\sin\omega_0 t + \frac{F_0}{\omega_0^2 - \omega^2}\sin\omega t$$

$$\boxed{= \frac{F_0}{\omega_0\left(\omega_0^2 - \omega^2\right)}\left(\omega_0\sin\omega t - \omega\sin\omega_0 t\right)}$$

## *Exercise*

A forced mass–spring–dashpot system with equation $mx'' + cx' + kx = F_0\cos\omega t$. Investigate the possibility of practical resonance of this system. In particular, find the amplitude $C(\omega)$ of steady state periodic forced oscillations with frequency $\omega$. Sketch the graph $C(\omega)$ of and find the practical resonance frequency $\omega$ (if any). $\quad m = 1, \quad c = 2, \quad k = 2, \quad F_0 = 2$

***Solution***

$$x'' + 2x' + 2x = 2\cos\omega t \qquad mx'' + cx' + kx = F(t)$$

$$x'' + 2x' + 2x = 0 \qquad mx'' + cx' + kx = 0$$

$$\lambda^2+2\lambda+2=0 \quad \rightarrow \quad \lambda_{1,2}=\frac{-2\pm\sqrt{4-8}}{2}\underline{=-1\pm i|}$$

$$\underline{x_h=e^{-t}\left(C_1\cos t+C_2\sin t\right)|}$$

$$x_p=A\cos\omega t+B\sin\omega t$$

$$x'=-A\omega\sin\omega t+B\omega\cos\omega t$$

$$x''=-A\omega^2\cos\omega t-B\omega^2\sin\omega t$$

$$-Am\omega^2\cos\omega t-Bm\omega^2\sin\omega t-2A\omega\sin\omega t+2B\omega\cos\omega t+2A\cos\omega t+2B\sin\omega t=2\cos\omega t$$

$$\rightarrow \begin{cases} \left(2-\omega^2\right)A+2\omega B=2 \\ -2\omega A+\left(2-m\omega^2\right)B=0 \end{cases}$$

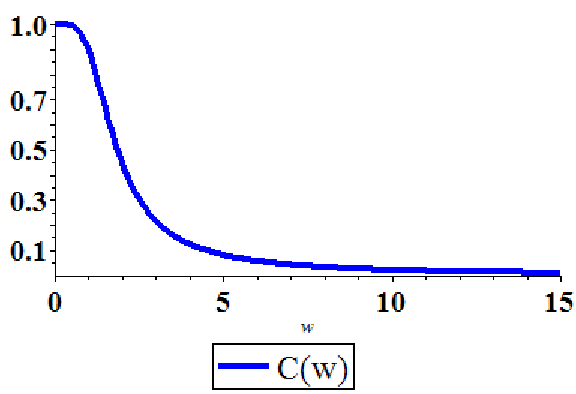
$$D=\begin{vmatrix} 2-\omega^2 & 2\omega \\ -2\omega & 2-\omega^2 \end{vmatrix}=\left(2-\omega^2\right)^2+4\omega^2=4+\omega^4$$

$$D_A=\begin{vmatrix} 2 & 2\omega \\ 0 & 2-\omega^2 \end{vmatrix}=2\left(2-\omega^2\right) \quad D_B=\begin{vmatrix} 2-\omega^2 & 2 \\ -2\omega & 0 \end{vmatrix}=4\omega$$

$$A=\frac{4-2\omega^2}{4+\omega^4} \quad B=\frac{4\omega}{4+\omega^4}$$

$$C(\omega)=\frac{2}{\sqrt{4+4\omega^4}}$$

$$C(\omega)=\sqrt{A^2+B^2}=\frac{F_0}{\sqrt{\left(k-m\omega^2\right)^2+(c\omega)^2}}$$




$C(\omega)$ starts with $C(0)=1$ and steadily decreases as $\omega$ increases.

Hence there is no practical resonance frequency.

### Exercise

A forced mass–spring–dashpot system with equation $mx'' + cx' + kx = F_0 \cos \omega t$. Investigate the possibility of practical resonance of this system. In particular, find the amplitude $C(\omega)$ of steady state periodic forced oscillations with frequency $\omega$. Sketch the graph $C(\omega)$ of and find the practical resonance frequency $\omega$ (if any). $m = 1, \quad c = 4, \quad k = 5, \quad F_0 = 10$

**Solution**

$x'' + 4x' + 5x = 10\cos \omega t \qquad mx'' + cx' + kx = F(t)$

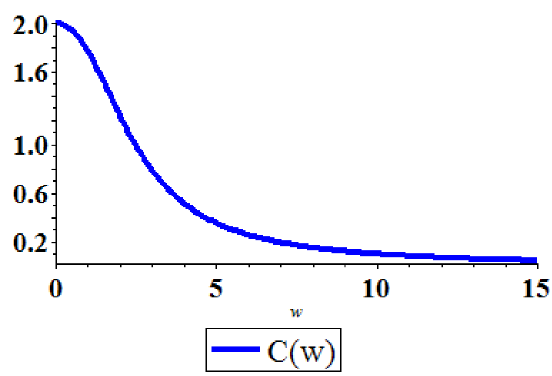
$$A = \frac{10\left(5-\omega^2\right)}{\left(5-\omega^2\right)^2 + 16\omega^2} = \frac{10\left(5-\omega^2\right)}{25 + 6\omega^2 + \omega^4} \qquad A = \frac{\left(k - m\omega^2\right)F_0}{\left(k-m\omega^2\right)^2 + (c\omega)^2}$$

$$B = \frac{40\omega}{25 + 6\omega^2 + \omega^4} \qquad B = \frac{c\omega F_0}{\left(k-m\omega^2\right)^2 + (c\omega)^2}$$

$$C(\omega) = \frac{10}{\sqrt{25 + 6\omega^2 + \omega^4}} \qquad C(\omega) = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{\left(k-m\omega^2\right)^2 + (c\omega)^2}}$$

$C(\omega)$ starts with $C(0) = 2$ and steadily decreases as $\omega$ increases.

Hence there is no practical resonance frequency.




### Exercise

A forced mass–spring–dashpot system with equation $mx'' + cx' + kx = F_0 \cos \omega t$. Investigate the possibility of practical resonance of this system. In particular, find the amplitude $C(\omega)$ of steady state periodic forced oscillations with frequency $\omega$. Sketch the graph $C(\omega)$ of and find the practical resonance frequency $\omega$ (if any). $m = 1, \quad c = 6, \quad k = 45, \quad F_0 = 50$

**Solution**

$x'' + 6x' + 45x = 50\cos \omega t \qquad mx'' + cx' + kx = F(t)$

$$A=\frac{50\left(45-\omega^2\right)}{\left(45-\omega^2\right)^2+36\omega^2}=\frac{50\left(45-\omega^2\right)}{2025-54\omega^2+\omega^4}$$

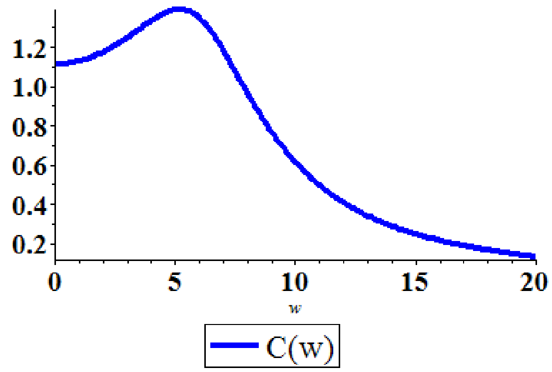
$$A=\frac{\left(k-m\omega^2\right)F_0}{\left(k-m\omega^2\right)^2+(c\omega)^2}$$

$$B=\frac{300\omega}{2025-54\omega^2+\omega^4}$$

$$B=\frac{c\omega F_0}{\left(k-m\omega^2\right)^2+(c\omega)^2}$$

$$C(\omega)=\frac{50}{\sqrt{2025-54\omega^2+\omega^4}}$$

$$C(\omega)=\frac{F_0}{\sqrt{\left(k-m\omega^2\right)^2+(c\omega)^2}}$$




$$C'(\omega)=-\frac{25\left(-108\omega+4\omega^3\right)}{\left(2025-54\omega^2+\omega^4\right)^{3/2}}=0$$

$$\omega\left(4\omega^2-108\right)=0 \quad\Rightarrow\quad \omega=\sqrt{27}\ \underline{=3\sqrt{3}}\ |\ (C.N)$$

$C(\omega)$ starts with $C(0)=\frac{10}{9}$, hence the practical resonance frequency is $\omega=3\sqrt{3}$.

### *Exercise*

A forced mass–spring–dashpot system with equation $mx''+cx'+kx=F_0\cos\omega t$. Investigate the possibility of practical resonance of this system. In particular, find the amplitude $C(\omega)$ of steady state periodic forced oscillations with frequency $\omega$. Sketch the graph $C(\omega)$ of and find the practical resonance frequency $\omega$ (if any). $m=1,\quad c=10,\quad k=650,\quad F_0=100$

***Solution***

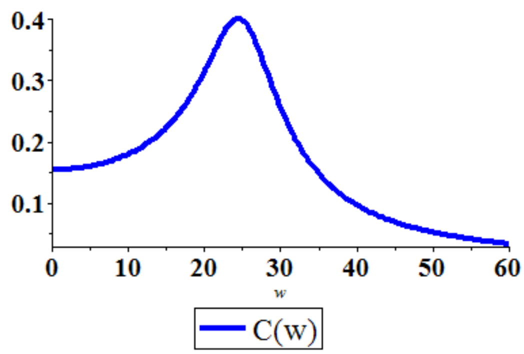
$$x''+10x'+650x=100\cos\omega t \qquad mx''+cx'+kx=F(t)$$

$$A=\frac{100\left(650-\omega^2\right)}{\left(650-\omega^2\right)^2+100\omega^2}=\frac{100\left(650-\omega^2\right)}{422{,}500-1200\omega^2+\omega^4}$$

$$A=\frac{\left(k-m\omega^2\right)F_0}{\left(k-m\omega^2\right)^2+(c\omega)^2}$$

$$B=\frac{1000\omega}{422,500-1200\omega^2+\omega^4} \qquad B=\frac{c\omega F_0}{\left(k-m\omega^2\right)^2+(c\omega)^2}$$

$$C(\omega)=\frac{100}{\sqrt{422,500-1200\omega^2+\omega^4}} \qquad C(\omega)=\frac{F_0}{\sqrt{\left(k-m\omega^2\right)^2+(c\omega)^2}}$$




$$C'(\omega)=-\frac{500\left(-2400\omega+4\omega^3\right)}{\left(422,500-1200\omega^2+\omega^4\right)^{3/2}}=0$$

$$\omega\left(4\omega^2-2400\right)=0 \quad\Rightarrow\quad \omega=\sqrt{600}\ \underline{=10\sqrt{6}|}\ (C.N)$$

$C(\omega)$ starts with $C(0)=\frac{2}{13}$, hence the practical resonance frequency is $\omega=10\sqrt{6}$.

## Exercise

A mass weighing 100 *lb*. (mass $m = 3.125$ *slugs* in *fps* units) is attached to the end of a spring that is stretched 1 *in*. by a force of 100 *lb*. A force $F_0 \cos\omega t$ acts on the mass. At what frequency (in hertz) will resonance oscillation occur? Neglect damping.

***Solution***

**Given**: $m = 3.125$ *slug*

$$k=\frac{F}{x}=\frac{100\ lb}{1\ in}\frac{12\ in}{1\ ft}\ \underline{=1200\ lb/ft|}$$

$$\omega_0=\sqrt{\frac{k}{m}}$$

$$=\sqrt{\frac{1200}{3.125}}$$

$$=\sqrt{384}\ \ rad/\sec$$

$$\underline{=\frac{\sqrt{384}}{2\pi}\ Hz.|}\quad \approx 3.12\ Hz.$$

### Exercise

A mass weighing 16 *pounds* stretches a spring $\frac{8}{3}$ *ft*. The mass is initially released form rest from a point 2 *ft* below the equilibrium position, and the subsequent motion takes place in a medium that offers a damping force that is numerically equal to $\frac{1}{2}$ the instantaneous velocity. Find the equation of motion if the mass is driven by an external force equal to $f(t)=10\cos 3t$

**Solution**

$$m=\frac{16}{32}=\frac{1}{2}\ slug$$

$$k\left(\frac{8}{3}\ ft\right)=16 \quad \rightarrow \quad k=6 \qquad\qquad kx=mg$$

$$\frac{1}{2}x''+\frac{1}{2}x'+6x=10\cos 3t \qquad\qquad mx''+cx'+kx=F(t)$$

$$x''+x'+12x=0\ ; \quad x(0)=2, \quad x'(0)=0$$

$$\lambda^2+\lambda+12=0 \quad \rightarrow \quad \boxed{\lambda_{1,2}=\frac{-1\pm i\sqrt{47}}{2}}$$

$$\boxed{x_h(t)=e^{-t/2}\left(C_1\cos\frac{\sqrt{47}}{2}t+C_2\sin\frac{\sqrt{47}}{2}t\right)}$$

$$x_p=A\cos 3t+B\sin 3t$$

$$x_p'=-3A\sin 3t+3B\cos 3t$$

$$x_p''=-9A\cos 3t-9B\sin 3t$$

$$x''+x'+12x=20\cos 3t$$

$$-9A\cos 3t-9B\sin 3t-3A\sin 3t+3B\cos 3t+12A\cos 3t+12B\sin 3t=20\cos 3t$$

$$\begin{cases}\cos 3t & 3A+3B=20\\ \sin 3t & -3A+3B=0\end{cases} \quad \rightarrow \quad \boxed{A=B=\frac{10}{3}}$$

$$\boxed{x_p=\frac{10}{3}(\cos 3t+\sin 3t)}$$

$$x(t)=e^{-t/2}\left(C_1\cos\frac{\sqrt{47}}{2}t+C_2\sin\frac{\sqrt{47}}{2}t\right)+\frac{10}{3}(\cos 3t+\sin 3t)$$

$$x(0)=2 \quad \rightarrow C_1+\frac{10}{3}=2 \quad \Rightarrow \boxed{C_1=-\frac{4}{3}}$$

$$x'(t)=e^{-t/2}\left(-\frac{1}{2}C_1\cos\frac{\sqrt{47}}{2}t+\frac{1}{2}C_2\sin\frac{\sqrt{47}}{2}t-\frac{\sqrt{47}}{2}C_1\sin\frac{\sqrt{47}}{2}t+\frac{\sqrt{47}}{2}C_2\cos\frac{\sqrt{47}}{2}t\right)$$
$$+\frac{10}{3}(-3\sin 3t+3\cos 3t)$$

$$x'(0)=0 \quad \rightarrow \quad -\frac{1}{2}C_1+\frac{\sqrt{47}}{2}C_2+10=0 \quad \Rightarrow \quad \boxed{C_2=-\frac{64}{3\sqrt{47}}}$$

$$\boxed{x(t) = e^{-t/2}\left(-\frac{4}{3}\cos\frac{\sqrt{47}}{2}t - \frac{64}{3\sqrt{47}}\sin\frac{\sqrt{47}}{2}t\right) + \frac{10}{3}(\cos 3t + \sin 3t)}$$

### Exercise

A mass of 32 *pounds* is attached to a spring with a constant spring 5 *lb/ft*. Initially, the mass is released 1 *foot* below the equilibrium position with a downward velocity of 5 *ft/s*, and the subsequent motion takes is numerically equal to 2 times the instantaneous velocity.

*a)* Find the equation of motion if the mass is driven by an external force equal to $f(t) = 12\cos 2t + 3\sin 2t$.

*b)* Graph the transient, steady-state, and the equation of motion solutions on the same coordinate axes.

**Solution**

$$m = \frac{32}{32} = 1 \text{ slug}$$

***a)*** $x'' + 2x' + 5x = 12\cos 2t + 3\sin 2t$ ; $\quad x(0) = 1, \quad x'(0) = 5 \qquad\qquad mx'' + cx' + kx = f(t)$

$$\lambda^2 + 2\lambda + 5 = 0 \quad \to \quad \lambda_{1,2} = -1 \pm 2i$$

$$\boxed{x_h = e^{-t}\left(C_1\cos 2t + C_2\sin 2t\right)}$$

$$x_p = A\cos 2t + B\sin 2t$$

$$x'_p = -2A\sin 2t + 2B\cos 2t$$

$$x''_p = -4A\cos 2t - 4B\sin 2t$$

$$-4A\cos 2t - 4B\sin 2t - 4A\sin 2t + 4B\cos 2t + 5A\cos 2t + 5B\sin 2t = 12\cos 2t + 3\sin 2t$$

$$\begin{cases} \cos 2t & A + 4B = 12 \\ \sin 2t & -4A + B = 3 \end{cases} \to \Delta = \begin{vmatrix} 1 & 4 \\ -4 & 1 \end{vmatrix} = 17 \quad \Delta_A = \begin{vmatrix} 12 & 4 \\ 3 & 1 \end{vmatrix} = 0 \quad \Delta = \begin{vmatrix} 1 & 12 \\ -4 & 3 \end{vmatrix} = 51$$

$$\to \boxed{A = 0, \quad B = 3}$$

$$\boxed{x_p = 3\sin 2t}$$

$$x(t) = e^{-t}\left(C_1\cos 2t + C_2\sin 2t\right) + 3\sin 2t$$

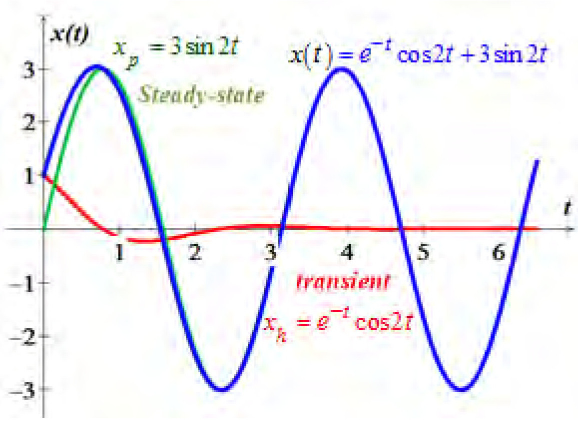
$$x(0) = 1 \quad \to \quad \boxed{C_1 = 1}$$

$$x'(t) = e^{-t}\left(-C_1\cos 2t - C_2\sin 2t - 2C_1\sin 2t + 2C_2\cos 2t\right) + 6\cos 2t$$

$$x'(0) = 5 \quad \to -C_1 + 2C_2 + 6 = 5 \quad \boxed{C_2 = 0}$$

$$\boxed{x(t) = e^{-t}\cos 2t + 3\sin 2t}$$

***b)***




### *Exercise*

A mass of 32 *pounds* is attached to a spring and stretched it 2 *feet* and then comes to rest in the equilibrium position. The surrounding medium offers a damping force that is numerically equal to 8 times the instantaneous velocity

*a)* Find the equation of motion if the mass is driven by an external force equal to $f(t) = 8\sin 4t$.

*b)* Graph the transient, steady-state, and the equation of motion solutions on the same coordinate axes.

***Solution***

***Given***: $m = \frac{32}{32} = 1$ *slug* $\quad c = 8$

$k(2\ ft) = 32 \quad \rightarrow \quad k = 16 \qquad kx = mg$

**a)** $x'' + 8x' + 16x = 8\sin 4t\ ; \quad x(0) = 0, \quad x'(0) = 0 \qquad\qquad mx'' + cx' + kx = f(t)$

$$\lambda^2 + 8\lambda + 16 = 0 \quad \rightarrow \quad \lambda_{1,2} = -4$$

$$\boxed{x_h = (C_1 + C_2 t)e^{-4t}} \qquad (\textit{Transient solution})$$

$$x_p = A\cos 4t + B\sin 4t$$

$$x'_p = -4A\sin 4t + 4B\cos 4t$$

$$x''_p = -16A\cos 4t - 16B\sin 4t$$

$$-16A\cos 4t - 16B\sin 4t - 32A\sin 4t + 32B\cos 4t + 16A\cos 4t + 16B\sin 4t = 8\sin 4t$$

$$\begin{cases} \cos 4t & 32B = 0 \\ \sin 4t & -32A = 8 \end{cases} \quad \rightarrow \quad \boxed{A = -\frac{1}{4}, \ B = 0}$$

$$\boxed{x_p = -\frac{1}{4}\cos 4t} \qquad (\textit{Steady-state solution})$$

$$x(t) = (C_1 + C_2 t)e^{-4t} - \frac{1}{4}\cos 4t$$

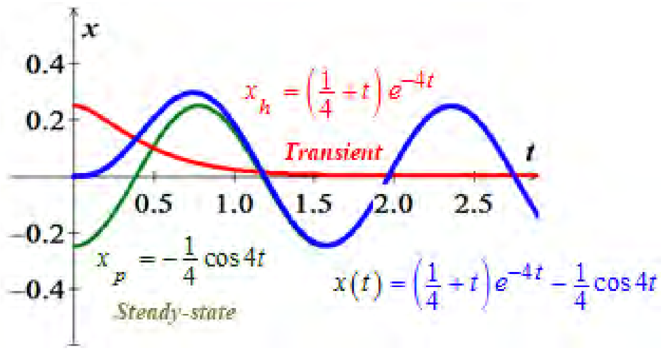
$$x(0) = 0 \quad \rightarrow \quad \boxed{C_1 = \frac{1}{4}}$$

$$x'(t) = (C_2 - 4C_1 - 4C_2 t)e^{-4t} + \sin 4t$$

$x'(0)=0 \quad \rightarrow C_2 - 4C_1 = 0 \quad \boxed{C_2 = 1}$

$$\boxed{x(t)=\left(\frac{1}{4}+t\right)e^{-4t}-\frac{1}{4}\cos 4t}$$

***b)***




## *Exercise*

A mass of 32 *pounds* is attached to a spring and stretched it 2 *feet* and then comes to rest in the equilibrium position. The surrounding medium offers a damping force that is numerically equal to 8 times the instantaneous velocity

*a)* Find the equation of motion with a starting external force equal to $f(t)=e^{-t}\sin 4t$ at $t=0$

*b)* Graph the transient, steady-state, and the equation of motion solutions on the same coordinate axes.

***Solution***

***Given***: $\quad m=\frac{32}{32}=1\ slug \qquad c=8$

$k(2\ ft)=32 \quad \rightarrow \quad k=16 \qquad\qquad kx=mg$

***a)*** $\quad x''+8x'+16x=e^{-t}\sin 4t\ ; \quad x(0)=0, \quad x'(0)=0 \qquad\qquad mx''+cx'+kx=f(t)$

$\lambda^2+8\lambda+16=0 \quad \rightarrow \quad \lambda_{1,2}=-4$

$\boxed{x_h=\left(C_1+C_2t\right)e^{-4t}} \qquad$ (***Transient solution***)

$x_p=e^{-t}\left(A\cos 4t+B\sin 4t\right)$

$x'_p=e^{-t}\left(-A\cos 4t-B\sin 4t-4A\sin 4t+4B\cos 4t\right)$

$x''_p=e^{-t}\left(A\cos 4t+B\sin 4t+4A\sin 4t-4B\cos 4t+4A\sin 4t-4B\cos 4t-16A\cos 4t-16B\sin 4t\right)$

$\quad =e^{-t}\left(-15A\cos 4t-8B\cos 4t+8A\sin 4t-15B\sin 4t\right)$

$x''+8x'+16x=e^{-t}\sin 4t$

$$\begin{cases} \cos 4t & -15A-8B-8A+32B+16A=0 \\ \sin 4t & 8A-15B-8B-32A+16B=1 \end{cases} \rightarrow \begin{cases} -7A+24B=0 \\ -24B-7B=1 \end{cases}$$

$$\Delta=\begin{vmatrix} -7 & 24 \\ -24 & -7 \end{vmatrix}=625 \quad \Delta_A=\begin{vmatrix} 0 & 24 \\ 1 & -7 \end{vmatrix}=-24 \quad \Delta_B=\begin{vmatrix} -7 & 0 \\ -24 & 1 \end{vmatrix}=-7$$

$$\boxed{A = -\frac{24}{625},\quad B = -\frac{7}{625}}$$

$$\boxed{x_p = e^{-t}\left(-\frac{24}{625}\cos 4t - \frac{7}{625}\sin 4t\right)}$$ (***Steady-state solution***)

$$x(t) = \left(C_1 + C_2 t\right)e^{-4t} - \frac{1}{625}e^{-t}\left(24\cos 4t + 7\sin 4t\right)$$

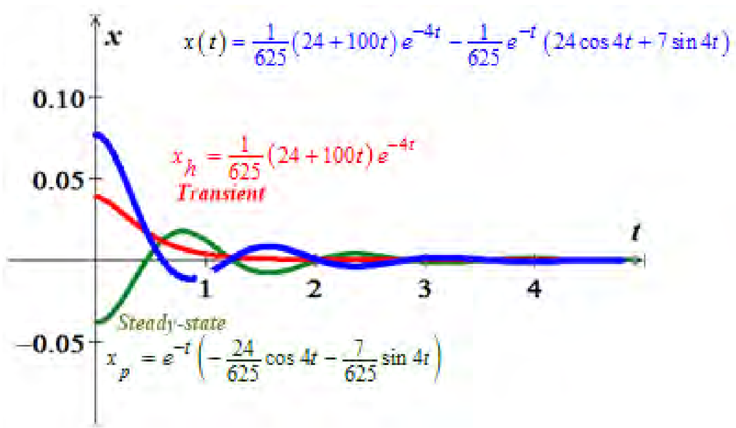
$$x(0) = 0 \quad \rightarrow \quad \boxed{C_1 = \frac{24}{625}}$$

$$x(t) = \left(C_2 - 4C_1 - 4C_2 t\right)e^{-4t} - \frac{1}{625}e^{-t}\left(-24\cos 4t - 7\sin 4t - 96\sin 4t + 28\cos 4t\right)$$

$$x'(0) = 0 \quad \rightarrow C_2 - 4C_1 - \frac{4}{625} = 0 \quad \Rightarrow \quad \boxed{C_2 = \frac{100}{625}}$$

$$\boxed{x(t) = \frac{1}{625}(24 + 100t)e^{-4t} - \frac{1}{625}e^{-t}\left(24\cos 4t + 7\sin 4t\right)}$$

***b)***




## *Exercise*

A mass of 64 *pounds* is attached to a spring with a spring constant 32 *lb/ft* and then comes to rest in the equilibrium position. Neglect the damping.

*a)* Find the equation of motion with a starting external force equal to $f(t) = 68e^{-2t}\cos 4t$ at $t = 0$

*b)* Graph the transient, steady-state, and the equation of motion solutions on the same coordinate axes.

***Solution***

*Given*: $m = \frac{64}{32} = 2$ *slugs*, $k = 32$, $c = 0$

***a)*** $2x'' + 32x = 68e^{-2t}\cos 4t\ ;\quad x(0) = 0,\quad x'(0) = 0$ $\qquad mx'' + cx' + kx = f(t)$

$$2\lambda^2 + 32 = 0 \quad \rightarrow \quad \lambda_{1,2} = \pm 4i$$

$$\boxed{x_h = C_1\cos 4t + C_2\sin 4t}$$ (***Transient solution***)

$$x_p = e^{-2t}\left(A\cos 4t + B\sin 4t\right)$$

$$x'_p = e^{-2t}\left(-2A\cos 4t - 2B\sin 4t - 4A\sin 4t + 4B\cos 4t\right)$$

$$x_p'' = e^{-2t}\left(4A\cos 4t + 4B\sin 4t + 8A\sin 4t - 8B\cos 4t + 8A\sin 4t - 8B\cos 4t - 16A\cos 4t - 16B\sin 4t\right)$$

$$= e^{-2t}\left(-12A\cos 4t - 16B\cos 4t + 16A\sin 4t - 12B\sin 4t\right)$$

$$2x'' + 32x = 68e^{-2t}\cos 4t$$

$$\begin{cases} \cos 4t & -24A - 32B + 32A = 68 \\ \sin 4t & 32A - 24B + 32B = 0 \end{cases} \rightarrow \begin{cases} 8A - 32B = 68 & \rightarrow 136A = 68 \\ 32A + 8B = 0 & B = -4A \end{cases} \quad \boxed{A = \frac{1}{2}, \ B = -2}$$

$$\boxed{x_p = e^{-2t}\left(\frac{1}{2}\cos 4t - 2\sin 4t\right)} \qquad (\textit{\textbf{Steady-state solution}})$$

$$x(t) = C_1\cos 4t + C_2\sin 4t + \left(\frac{1}{2}\cos 4t - 2\sin 4t\right)e^{-2t}$$

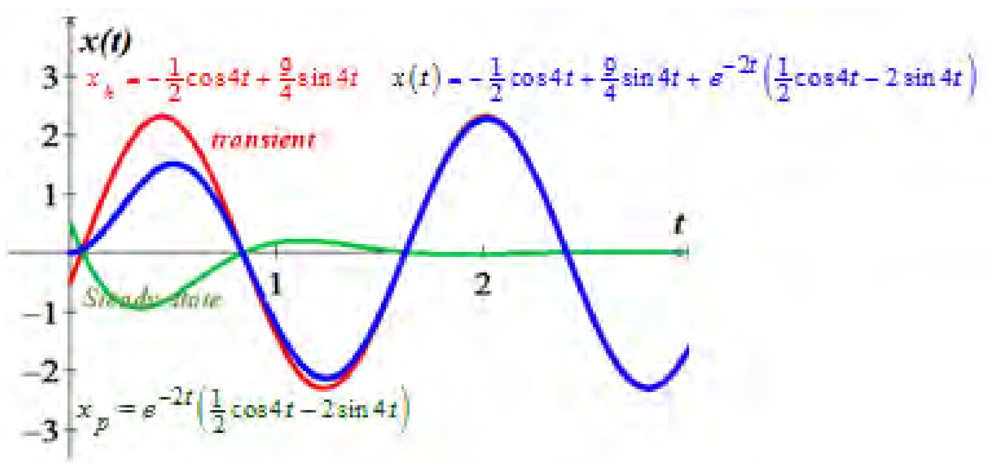
$$x(0) = 0 \quad \rightarrow \quad \boxed{C_1 = -\frac{1}{2}}$$

$$x(t) = -4C_1\sin 4t + 4C_2\cos 4t + \left(-\cos 4t + 4\sin 4t - 2\sin 4t - 8\cos 4t\right)e^{-2t}$$

$$x'(0) = 0 \quad \rightarrow \quad 4C_2 - 1 - 8 = 0 \quad \Rightarrow \quad \boxed{C_2 = \frac{9}{4}}$$

$$\boxed{x(t) = -\frac{1}{2}\cos 4t + \frac{9}{4}\sin 4t + e^{-2t}\left(\frac{1}{2}\cos 4t - 2\sin 4t\right)}$$

***b)***




## *Exercise*

A 3-*kg* object is attached to a spring and stretches the spring 392 *mm* by itself. There is no damping in the system and a forcing function of the form $F(t) = 10\cos\omega t$ is attached to the object and the system will experience resonance. If the object is initially displaced 20 *cm* downward from its equilibrium position and given a velocity of 10 *cm/sec* upward find the displacement $y(t)$ at any time *t*.

***Solution***

$$k(0.392) = 3(9.8) \quad \rightarrow \quad k = 75 \ \ kg/m \qquad\qquad kL = mg$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{75}{3}} = 5$$

$3y'' + 75y = 10\cos 5t \qquad\qquad my'' + \mu y' + ky = 0$

$y'' + 25y = \frac{10}{3}\cos 5t \; ; \quad y(0) = 0.2 \; m, \quad y'(0) = -0.10 \;\; m/sec$

$$\lambda^2 + 25 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 5i}$$

$$\boxed{y_h = C_1 \cos 5t + C_2 \sin 5t}$$

$y_p = At\cos 5t + Bt\sin 5t$

$y'_p = A\cos 5t - 5At\sin 5t + B\sin 5t + 5Bt\cos 5t$

$y''_p = -5A\sin 5t - 5A\sin 5t - 25At\cos 5t + 5B\cos 5t + 5B\cos 5t - 25Bt\sin 5t$

$3y'' + 75y = 10\cos 5t$

$-5A\sin 5t - 5A\sin 5t - 25At\cos 5t + 5B\sin 5t + 5B\cos 5t - 25Bt\sin 5t$

$$\begin{cases} \cos 5t & 30B = 10 \\ \sin 5t & -30A = 0 \\ t\cos 5t & -75A + 75A \\ t\sin 5t & -75B + 75B \end{cases} \rightarrow \quad A = 0 \quad B = \frac{1}{3}$$

$$\boxed{y_p = \frac{1}{3}t\sin 5t}$$

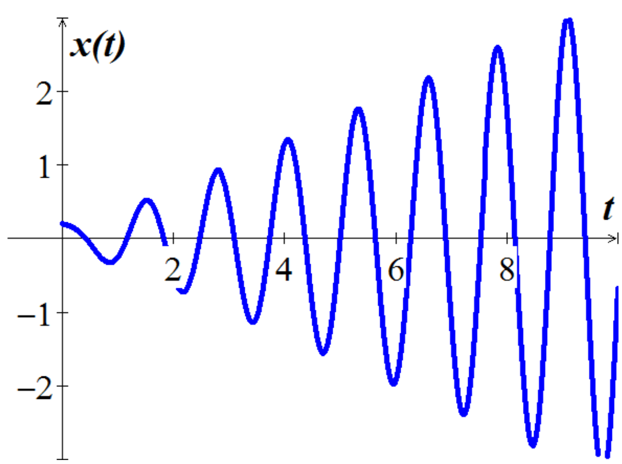
$y(t) = C_1 \cos 5t + C_2 \sin 5t + \frac{1}{3}t\sin 5t$

$$y(0) = 0.2 \quad \rightarrow \quad \boxed{C_1 = 0.2 = \frac{1}{5}}$$

$y' = -5C_1 \sin 5t + 5C_2 \cos 5t + \frac{1}{3}\sin 5t + \frac{5}{3}t\cos 5t$

$$y'(0) = -\frac{1}{10} \quad \rightarrow 5C_2 = -\frac{1}{10} \quad \boxed{C_2 = -\frac{1}{50}}$$

$$\boxed{y(t) = \frac{1}{5}\cos 5t - \frac{1}{50}\sin 5t + \frac{1}{3}t\sin 5t}$$



## *Exercise*

A 8-*kg* mass is attached to a spring hanging vertically, thereby causing the spring to stretch 1.96 *m* upon coming to rest at equilibrium. The damping constant is given by 3 *N-sec/m*.

*a)* Find the equation of motion if the mass is driven by an external force equal to $f(t)=\cos 2t\ N$ at $t=0$.

*b)* Determine the transient, steady-state solution of the motion

**Solution**

**Given**: $m=8 \quad k=\frac{mg}{x}=\frac{8(9.8)}{1.96}=40$

**a)** $8y''+3y'+40y=\cos 2t\ ; \quad y(0)=0, \quad y'(0)=0 \qquad\qquad my''+cy'+ky=F(t)$

$$8\lambda^2+3\lambda+40=0 \quad\rightarrow\quad \lambda_{1,2}=-\frac{3}{2}\pm\frac{1}{2}i\sqrt{1271}$$

$$y_h=e^{-3t/2}\left(C_1\cos\frac{\sqrt{1271}}{2}t+C_2\sin\frac{\sqrt{1271}}{2}t\right)$$

$$y_p=A\cos 2t+B\sin 2t$$

$$y'_p=-2A\sin 2t+2B\cos 2t$$

$$y''_p=-4A\cos 2t-4B\sin 2t$$

$$8y''+3y'+40y=\cos 2t$$

$$\begin{cases}\cos 2t & -32A+6B+40A=1\\ \sin 2t & -32B-6A+40B=0\end{cases} \rightarrow \begin{cases}8A+6B=1\\ -6A+8B=0\end{cases}$$

$$\Delta=\begin{vmatrix}8 & 6\\ -6 & 8\end{vmatrix}=100 \quad \Delta_A=\begin{vmatrix}1 & 6\\ 0 & 8\end{vmatrix}=8 \quad \Delta_B=\begin{vmatrix}8 & 1\\ -6 & 0\end{vmatrix}=6$$

$$\rightarrow A=\frac{2}{25},\ B=\frac{3}{50}$$

$$y_p=\frac{2}{25}\cos 2t+\frac{3}{50}\sin 2t$$

$$y(t)=e^{-3t/2}\left(C_1\cos\frac{\sqrt{1271}}{2}t+C_2\sin\frac{\sqrt{1271}}{2}t\right)+\frac{2}{25}\cos 2t+\frac{3}{50}\sin 2t$$

$$y(0)=0 \quad\rightarrow\quad C_1=-\frac{2}{25}$$

$$y'(t)=e^{-3t/2}\left(-\frac{3}{2}C_1\cos\frac{\sqrt{1271}}{2}t-\frac{3}{2}C_2\sin\frac{\sqrt{1271}}{2}t-\frac{\sqrt{1271}}{2}C_1\sin\frac{\sqrt{1271}}{2}t+\frac{\sqrt{1271}}{2}C_2\cos\frac{\sqrt{1271}}{2}t\right)$$

$$-\frac{4}{25}\cos 2t+\frac{6}{50}\sin 2t$$

$$y'(0)=0 \quad\rightarrow -\frac{3}{2}C_1+\frac{\sqrt{1271}}{2}C_2-\frac{4}{25}=0 \quad C_2=\frac{4}{25\sqrt{1271}}$$

$$y(t) = e^{-3t/2}\left(-\frac{2}{25}\cos\frac{\sqrt{1271}}{2}t + \frac{4}{25\sqrt{1271}}\sin\frac{\sqrt{1271}}{2}t\right) + \frac{2}{25}\cos 2t + \frac{3}{50}\sin 2t$$

***b)*** *Transient* solution: $y(t) = e^{-3t/2}\left(-\frac{2}{25}\cos\frac{\sqrt{1271}}{2}t + \frac{4}{25\sqrt{1271}}\sin\frac{\sqrt{1271}}{2}t\right)$

*Steady-state* solution: $y(t) = \frac{2}{25}\cos 2t + \frac{3}{50}\sin 2t$

### *Exercise*

A 2-*kg* mass is attached to a spring hanging vertically, thereby causing the spring to stretch 0.2 *m* upon coming to rest at equilibrium. At $t = 0$, the mass is displaced 5 *cm* below the equilibrium position and released. The damping constant is given by 5 *N-sec/m*.

*a)* Find the equation of motion if the mass is driven by an external force equal to $f(t) = 0.3\cos t\ N$.

*b)* Determine the transient, steady-state solution of the motion.

***Solution***

***Given***: $m = 2 \quad k = \frac{mg}{x} = \frac{2(9.8)}{0.2} = 98 \quad c = 5$

**a)** $2y'' + 5y' + 98y = \frac{3}{10}\cos t$ ; $y(0) = .05, \quad y'(0) = 0 \qquad my'' + cy' + ky = F(t)$

$$2\lambda^2 + 5\lambda + 98 = 0 \quad \rightarrow \quad \lambda_{1,2} = -\frac{5}{4} \pm \frac{1}{4}i\sqrt{759}$$

$$\boxed{y_h = e^{-5t/4}\left(C_1 \cos\frac{\sqrt{759}}{4}t + C_2 \sin\frac{\sqrt{759}}{4}t\right)}$$

$$y_p = A\cos t + B\sin t$$

$$y'_p = -A\sin t + B\cos t$$

$$y''_p = -A\cos t - 4B\sin t$$

$$2y'' + 5y' + 98y = \frac{3}{10}\cos t$$

$$\begin{cases} \cos 2t & -2A + 5B + 98A = 1 \\ \sin 2t & -2B - 5A + 98B = 0 \end{cases} \rightarrow \begin{cases} 96A + 5B = 1 \\ -5A + 96B = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 96 & 5 \\ -5 & 96 \end{vmatrix} = 9241 \quad \Delta_A = \begin{vmatrix} \frac{3}{10} & 5 \\ 0 & 96 \end{vmatrix} = \frac{144}{5} \quad \Delta_B = \begin{vmatrix} 96 & \frac{3}{10} \\ -5 & 0 \end{vmatrix} = \frac{3}{2}$$

$$\rightarrow \boxed{A = \frac{144}{46,205}, \quad B = \frac{3}{18,482}}$$

$$\boxed{y_p = \frac{144}{46,205}\cos t + \frac{3}{18,482}\sin t}$$

$$y(t) = e^{-5t/4}\left(C_1 \cos\frac{\sqrt{759}}{4}t + C_2 \sin\frac{\sqrt{759}}{4}t\right) + \frac{144}{46,205}\cos t + \frac{3}{18,482}\sin t$$

$$y(0) = 0.05 = \frac{1}{20} \quad \rightarrow C_1 + \frac{144}{46,205} = \frac{1}{20} \quad \boxed{C_1 = \frac{1,733}{36,964}}$$

$$y'(t) = e^{-5t/4}\left(-\frac{5}{4}C_1 \cos\frac{\sqrt{759}}{4}t - \frac{5}{4}\sin\frac{\sqrt{759}}{4}t - \frac{\sqrt{759}}{4}C_1 \sin\frac{\sqrt{759}}{4}t + \frac{\sqrt{759}}{4}C_2 \cos\frac{\sqrt{759}}{4}t\right)$$
$$-\frac{144}{46,205}\sin t + \frac{3}{18,482}\cos t$$

$$y'(0) = 0 \quad \rightarrow -\frac{5}{4}\frac{1,733}{36,964} + \frac{\sqrt{759}}{4}C_2 + \frac{3}{18,482} = 0 \quad \boxed{C_2 = \frac{8,641}{36,964\sqrt{759}}}$$

$$\boxed{y(t)=e^{-5t/4}\left(\frac{1,733}{36,964}\cos\frac{\sqrt{759}}{4}t+\frac{8,641}{36,964\sqrt{759}}\sin\frac{\sqrt{759}}{4}t\right)+\frac{144}{46,205}\cos t+\frac{3}{18,482}\sin t}$$

**b)** *Transient* solution: $\boxed{y_h(t)=e^{-5t/4}\left(\frac{1,733}{36,964}\cos\frac{\sqrt{759}}{4}t+\frac{8,641}{36,964\sqrt{759}}\sin\frac{\sqrt{759}}{4}t\right)}$

*Steady-state* solution: $\boxed{y_p(t)=\frac{144}{46,205}\cos t+\frac{3}{18,482}\sin t}$

## Exercise

A $8\text{-}kg$ mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $3\ N\text{-}sec/m$ and the spring constant is $40\ N/m$. Find steady-state solution if the mass is driven by an external force equal to $f(t)=2\sin\left(2t+\frac{\pi}{4}\right)\ N$.

**Solution**

**Given**: $m=8 \quad k=40 \quad c=3$

$$8y''+3y'+40y=2\sin\left(2t+\frac{\pi}{4}\right) \qquad my''+cy'+ky=F(t)$$

$$=2\left(\sin 2t\cos\frac{\pi}{4}+\cos 2t\sin\frac{\pi}{4}\right)$$

$$=\sqrt{2}\sin 2t+\sqrt{2}\cos 2t$$

$$8\lambda^2+3\lambda+40=0 \quad\rightarrow\quad \lambda_{1,2}=-\frac{3}{16}\pm\frac{1}{16}i\sqrt{1271}$$

$$y_h=e^{-3t/16}\left(C_1\cos\frac{\sqrt{1271}}{16}t+C_2\sin\frac{\sqrt{1271}}{16}t\right)$$

$$y_p=A\cos 2t+B\sin 2t$$

$$y_p'=-2A\sin 2t+2B\cos 2t$$

$$y_p''=-4A\cos 2t-4B\sin 2t$$

$$8y''+3y'+40y==\sqrt{2}\sin 2t+\sqrt{2}\cos 2t$$

$$\begin{cases}\cos 2t & -32A+6B+40A=\sqrt{2}\\ \sin 2t & -32B-6A+40B=\sqrt{2}\end{cases} \rightarrow \begin{cases}8A+6B=\sqrt{2}\\ -6A+8B=\sqrt{2}\end{cases}$$

$$\Delta=\begin{vmatrix}8 & 6\\ -6 & 8\end{vmatrix}=100 \quad \Delta_A=\begin{vmatrix}\sqrt{2} & 6\\ \sqrt{2} & 8\end{vmatrix}=2\sqrt{2} \quad \Delta_B=\begin{vmatrix}8 & \sqrt{2}\\ -6 & \sqrt{2}\end{vmatrix}=14\sqrt{2}$$

$$\rightarrow \boxed{A=\frac{\sqrt{2}}{50},\ B=\frac{7\sqrt{2}}{50}}$$

*Steady-state* solution: $\boxed{y_p(t)=\frac{\sqrt{2}}{50}\cos 2t+\frac{7\sqrt{2}}{50}\sin 2t}$

## *Exercise*

A 32-*lb* mass weight is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by 2 *lb-sec/ft* and the spring constant is 5 *lb/ft*. If the mass is driven by an external force equal to $f(t) = 3\cos 4t$ *lb* at time $t = 0$.

*a)* Find steady-state solution.

*b)* Determine the amplitude and frequency

***Solution***

***Given***: $m = \frac{32}{32} = 1 \quad c = 2 \quad k = 5$

**a)** $y'' + 2y' + 5y = 3\cos 4t \qquad\qquad my'' + cy' + ky = F(t)$

$$\lambda^2 + 2\lambda + 5 = 0 \;\rightarrow\; \lambda_{1,2} = -1 \pm 2i$$

$$\boxed{y_h = e^{-t}\left(C_1 \cos 2t + C_2 \sin 2t\right)}$$

$$y_p = A\cos 4t + B\sin 4t$$

$$y'_p = -4A\sin 4t + 4B\cos 4t$$

$$y''_p = -16A\cos 4t - 16B\sin 4t$$

$$y'' + 2y' + 5y = 3\cos 4t$$

$$\begin{cases} \cos 4t & -16A + 8B + 5A = 3 \\ \sin 4t & -16B - 8A + 5B = 0 \end{cases} \rightarrow \begin{cases} -11A + 8B = 3 \\ -8A - 11B = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} -11 & 8 \\ -8 & -11 \end{vmatrix} = 185 \quad \Delta_A = \begin{vmatrix} 3 & 8 \\ 0 & -11 \end{vmatrix} = -33 \quad \Delta_B = \begin{vmatrix} -11 & 3 \\ -8 & 0 \end{vmatrix} = 24$$

$$\rightarrow \boxed{A = -\frac{33}{185},\; B = \frac{24}{185}}$$

$$\boxed{y_p(t) = -\frac{33}{185}\cos 4t + \frac{24}{185}\sin 4t}$$

**b)** *Amplitude*: $A = \sqrt{\left(-\frac{33}{185}\right)^2 + \left(-\frac{24}{185}\right)^2} = \boxed{\frac{9\sqrt{185}}{185}\ ft}$

*Frequency*: $f = \frac{1}{P} = \frac{\omega}{2\pi} = \frac{4}{2\pi} = \boxed{\frac{2}{\pi}}$

### *Exercise*

A 8-*kg* mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by 3 *N-sec/m* and the spring constant is 40 *N/m*. If the mass is driven by an external force equal to $f(t) = 2\sin 2t\cos 2t \ N$.

*a)* Find steady-state solution.

*b)* Determine the amplitude, phase angle, period and frequency

***Solution***

***Given***: $m = 8 \quad k = 40 \quad c = 3$

***a)*** $8y'' + 3y' + 40y = 2\sin 2t\cos 2t$ $\qquad my'' + cy' + ky = F(t)$

$$= \sin 4t$$

$$8\lambda^2 + 3\lambda + 40 = 0 \quad \rightarrow \quad \lambda_{1,2} = -\frac{3}{16} \pm \frac{1}{16} i\sqrt{1271}$$

$$y_h = e^{-3t/16}\left(C_1 \cos\frac{\sqrt{1271}}{16}t + C_2 \sin\frac{\sqrt{1271}}{16}t\right)$$

$$y_p = A\cos 4t + B\sin 4t$$

$$y'_p = -4A\sin 4t + 4B\cos 4t$$

$$y''_p = -16A\cos 4t - 16B\sin 4t$$

$$8y'' + 3y' + 40y = \sin 4t$$

$$\begin{cases} \cos 4t & -128A + 12B + 40A = 0 \\ \sin 4t & -128B - 12A + 40B = 1 \end{cases} \rightarrow \begin{cases} -88A + 12B = 0 \\ -12A - 88B = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} -88 & 12 \\ -12 & -88 \end{vmatrix} = 7{,}888 \quad \Delta_A = \begin{vmatrix} 0 & 12 \\ 1 & -88 \end{vmatrix} = -12 \quad \Delta_B = \begin{vmatrix} -88 & 0 \\ -12 & 1 \end{vmatrix} = -88$$

$$\rightarrow \ \underline{A = -\frac{3}{1972}, \ B = -\frac{11}{986}}$$

$$\underline{y_p(t) = -\frac{3}{1972}\cos 4t - \frac{11}{986}\sin 4t}$$

***c)*** *Amplitude*: $A = \sqrt{\left(-\frac{3}{1972}\right)^2 + \left(-\frac{11}{986}\right)^2} \ \underline{= 0.01 \ m}$

*Phase angle*: $\phi = 2\pi - \arctan\left(\frac{3}{1972} \cdot \frac{986}{11}\right) \ \underline{\approx 6.148 \ rad}$

*Period*: $P = \frac{2\pi}{\omega} = \frac{2\pi}{4} \ \underline{= \frac{\pi}{2}}$

*Frequency*: $f = \frac{1}{P} \ \underline{= \frac{2}{\pi}}$

$$\underline{y_p(t) = 0.01\sin(4t + 6.148)}$$

### *Exercise*

A 10-*kg* mass is attached to a spring hanging vertically stretches the spring 0.098 *m* from its equilibrium rest position, measured positive in the downward direction. At time $t=0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t)=20\cos 10t\ N$. ($t$ in *seconds*)

*a)* Determine the spring constant $k$.
*b)* Find the equation of motion.
*c)* Plot the equation of motion.
*d)* Determine the maximum excursion from equilibrium made of the object on the $t$-interval $0 \le t < \infty$

**Solution**

***a)*** $k=\dfrac{10(9.8)}{0.098}=1,000\ N/m$ $\qquad ky=mg$

***b)*** $10y''+1000y=20\cos 10t\ ;\quad y(0)=0,\quad y'(0)=0 \qquad my''+cy'+ky=F(t)$

$$10\lambda^2+1000=0 \quad\rightarrow\quad \lambda_{1,2}=\pm 10i$$

$$y_h=C_1\cos 10t+C_2\sin 10t$$

$$y_p=At\cos 10t+Bt\sin 10t$$

$$y'_p=A\cos 10t+B\sin 10t-10At\sin 10t+10Bt\cos 10t$$
$$=(A+10Bt)\cos 10t+(-10At+B)\sin 10t$$

$$y''_p=10B\cos 10t-10A\sin 10t-(10A+100Bt)\sin 10t+(-100At+10B)\cos 10t$$
$$=(-100At+20B)\cos 10t-(20A+100Bt)\sin 10t$$

$$y''+100y=2\cos 10t$$

$$\cos 10t \rightarrow \begin{cases} t & -100A+100A=0 \\ t^0 & 20B=2 \end{cases}$$
$$\sin 10t \rightarrow \begin{cases} t & -100B+100B=0 \\ t^0 & -20A=0 \end{cases}$$
$$\Rightarrow\quad A=0,\ B=\frac{1}{10}$$

$$y_p=\frac{1}{10}t\sin 10t$$

$$y(t)=C_1\cos 10t+C_2\sin 10t+\frac{1}{10}t\sin 10t$$

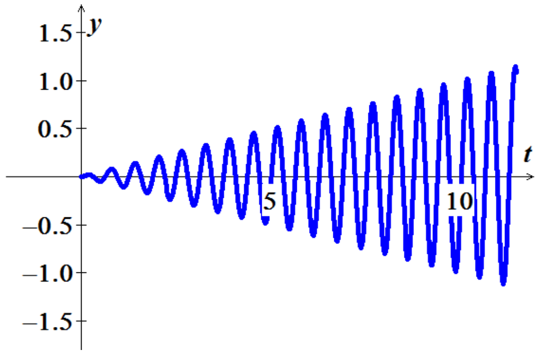
$$y(0)=0 \quad\rightarrow\quad C_1=0$$

$$y'=-10C_1\sin 10t+10C_2\cos 10t+\frac{1}{10}\sin 10t-10t\cos 10t$$

$$y'(0)=0 \quad\rightarrow 10C_2=0 \quad C_2=0$$

$$y(t)=\frac{1}{10}t\sin 10t\ m$$

***c)***




***d)*** There is no maximum excursion.

## *Exercise*

A 10-$kg$ mass is attached to a spring hanging vertically stretches the spring $0.098\ m$ from its equilibrium rest position, measured positive in the downward direction. At time $t=0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t)=20\cos 8t\ N$. ($t$ in *seconds*)

*a)* Determine the spring constant $k$.
*b)* Find the equation of motion.
*c)* Plot the equation of motion.
*d)* Determine the maximum excursion from equilibrium made of the object on the $t$-interval $0 \le t < \infty$

***Solution***

***a)*** $k=\dfrac{10(9.8)}{0.098}=1{,}000\ N/m$ $\qquad ky=mg$

***b)*** $10y''+1000y=20\cos 8t$ ; $y(0)=0,\quad y'(0)=0 \qquad my''+cy'+ky=F(t)$

$$10\lambda^2+1000=0 \quad\to\quad \lambda_{1,2}=\pm 10i$$

$$y_h=C_1\cos 10t+C_2\sin 10t$$

$$y_p=A\cos 8t+B\sin 8t$$

$$y_p'=-8A\sin 8t+8B\cos 8t$$

$$y_p=-64A\cos 8t-64B\sin 8t$$

$$y''+100y=2\cos 8t$$

$$\cos 8t \ \to\ -64A+100A=2$$

$$\sin 8t \ \to\ -64B+100B=0 \qquad \Rightarrow \quad A=\frac{1}{18},\ B=0$$

$$y_p=\frac{1}{18}\cos 8t$$

$$y(t)=C_1\cos 10t+C_2\sin 10t+\frac{1}{18}\cos 8t$$

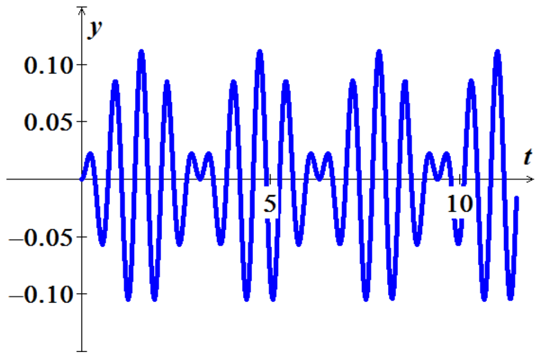
$$y(0)=0 \quad\to\quad C_1=-\frac{1}{18}$$

$$y' = -10C_1 \sin 10t + 10C_2 \cos 10t - \frac{1}{18}\sin 8t$$

$$y'(0) = 0 \quad \rightarrow 10C_2 = 0 \quad \boxed{C_2 = 0}$$

$$\boxed{y(t) = -\frac{1}{18}\cos 10t + \frac{1}{18}\cos 8t \quad m}$$

**c)**




**d)** $y(t) = \frac{1}{18}(\cos 8t - \cos 10t)$ $\qquad\qquad \cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

$$= -\frac{1}{9}\sin 9t \sin(-t)$$

$$\boxed{= \frac{1}{9}\sin 9t \sin t}$$

$$\boxed{|y_{max}| = \frac{1}{9} \approx 0.1111 \quad m}$$

| | |
|---|---|
| 1.568 | 0.11108 |
| 1.570 | 0.11111 |
| 1.572 | 0.11110 |

## Exercise

A 10-*kg* mass is attached to a spring hanging vertically stretches the spring 0.098 *m* from its equilibrium rest position, measured positive in the downward direction. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20e^{-t}$ $N$. (*t* in *seconds*)

*a)* Determine the spring constant *k*.
*b)* Find the equation of motion.
*c)* Plot the equation of motion.
*d)* Determine the maximum excursion from equilibrium made of the object on the *t*-interval $0 \le t < \infty$

***Solution***

**a)** $k = \frac{10(9.8)}{0.098}$ $\boxed{= 1{,}000\ N/m}$ $\qquad\qquad ky = mg$

**b)** $10y'' + 1000y = 20e^{-t}$ ; $\quad y(0) = 0, \quad y'(0) = 0 \qquad\qquad my'' + cy' + ky = F(t)$

$$10\lambda^2 + 1000 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = \pm 10i}$$

$$\boxed{y_h = C_1\cos 10t + C_2\sin 10t}$$

$$y_p = Ae^{-t}$$

$$y'_p = -Ae^{-t}$$

$$y_p = Ae^{-t}$$

$$y'' + 100y = 2e^{-t}$$

$$A + 100A = 2 \quad \rightarrow \quad \boxed{A = \frac{2}{101}}$$

$$\boxed{y_p = \frac{2}{101}e^{-t}}$$

$$y(t) = C_1\cos 10t + C_2\sin 10t + \frac{2}{101}e^{-t}$$

$$y(0) = 0 \quad \rightarrow \quad \boxed{C_1 = -\frac{2}{101}}$$

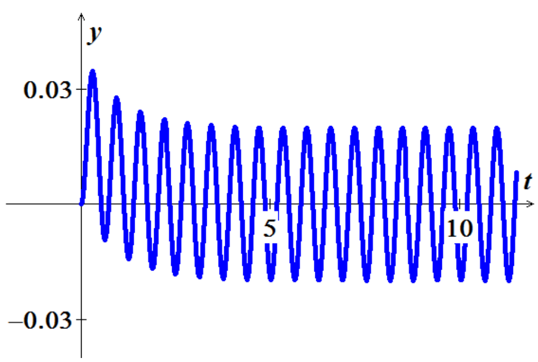
$$y' = -10C_1\sin 10t + 10C_2\cos 10t - \frac{2}{101}e^{-t}$$

$$y'(0) = 0 \quad \rightarrow 10C_2\cos 10t - \frac{2}{101} = 0 \quad \Rightarrow \quad \boxed{C_2 = \frac{1}{505}}$$

$$y(t) = -\frac{2}{101}\cos 10t + \frac{1}{505}\sin 10t + \frac{2}{101}e^{-t}$$

$$\boxed{= \frac{1}{505}\left(\sin 10t - 10\cos 10t + 10e^{-t}\right) \ m}$$

***c)***




***d)*** $\boxed{|y_{max}| \approx 0.03456 \ m}$

| | |
|---|---|
| 0.29320 | 0.03455 |
| 0.29440 | 0.03456 |
| 0.29560 | 0.03456 |
| 0.29680 | 0.03456 |
| 0.29800 | 0.03456 |
| 0.29920 | 0.03456 |
| 0.30040 | 0.03455 |

### Exercise

A $2$-$kg$ mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $c = 8\ kg/sec$ and the spring constant is $k = 80\ N/m$. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20\cos 8t\ N$. ($t$ in *seconds*)

*a)* Find the equation of motion.

*b)* Plot the equation of motion.

*c)* Determine the long-time behavior of the system, as $t \to \infty$

**Solution**

***a)*** $2y'' + 8y' + 80y = 20\cos 8t$ ; $y(0) = 0,\quad y'(0) = 0$ $\qquad my'' + cy' + ky = F(t)$

$$2\lambda^2 + 8\lambda + 80 = \lambda^2 + 4\lambda + 40 = 0 \quad \to \quad \boxed{\lambda_{1,2} = -2 \pm 6i}$$

$$\boxed{y_h = e^{-2t}\left(C_1\cos 6t + C_2\sin 6t\right)}$$

$$y_p = A\cos 8t + B\sin 8t$$

$$y'_p = -8A\sin 8t + 8B\cos 8t$$

$$y_p = -64A\cos 8t - 64B\sin 8t$$

$$y'' + 4y' + 40y = 10\cos 8t$$

$$\begin{aligned} \cos 8t &\to -64A + 32B + 40A = 10 \\ \sin 8t &\to -64B - 32A + 40B = 0 \end{aligned} \Rightarrow \begin{cases} -12A + 16B = 5 \\ -4A - 3B = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} -12 & 16 \\ -4 & -3 \end{vmatrix} = 100 \quad \Delta_A = \begin{vmatrix} 5 & 16 \\ 0 & -3 \end{vmatrix} = -15 \quad \Delta_B = \begin{vmatrix} -12 & 5 \\ -4 & 0 \end{vmatrix} = 20$$

$$\Rightarrow \boxed{A = -\frac{3}{20},\ B = \frac{1}{5}}$$

$$\boxed{y_p = -\frac{3}{20}\cos 8t + \frac{1}{5}\sin 8t}$$

$$y(t) = e^{-2t}\left(C_1\cos 6t + C_2\sin 6t\right) - \frac{3}{20}\cos 8t + \frac{1}{5}\sin 8t$$

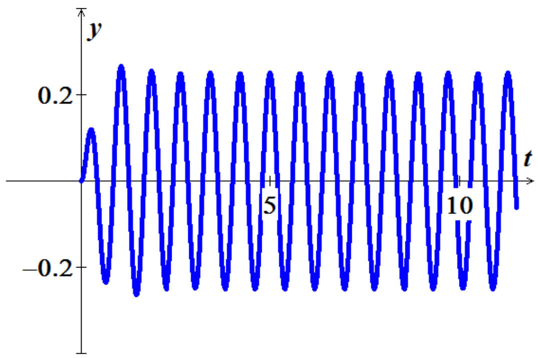
$$y(0) = 0 \quad \to \quad \boxed{C_1 = \frac{3}{20}}$$

$$y' = e^{-2t}\left(-2C_1\cos 6t - 2C_2\sin 6t - 6C_1\sin 6t + 6C_2\cos 6t\right) + \frac{6}{5}\sin 8t + \frac{8}{5}\cos 8t$$

$$y'(0) = 0 \quad \to -2\left(\frac{3}{20}\right) + 6C_2 + \frac{8}{5} = 0 \quad \boxed{C_2 = -\frac{13}{60}}$$

$$\boxed{y(t) = e^{-2t}\left(\frac{3}{20}\cos 6t - \frac{13}{60}\sin 6t\right) - \frac{3}{20}\cos 8t + \frac{1}{5}\sin 8t}$$

***b)***




***c)*** $\lim\limits_{t\to\infty}\left(e^{-2t}\left(\frac{3}{20}\cos 6t-\frac{13}{60}\sin 6t\right)-\frac{3}{20}\cos 8t+\frac{1}{5}\sin 8t\right)=\lim\limits_{t\to\infty}\left(-\frac{3}{20}\cos 8t+\frac{1}{5}\sin 8t\right)$

$=$ *doesn't exist*

The equation of motion is a steady-state solution.

## Exercise

A 2-*kg* mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $c = 8$ *kg/sec* and the spring constant is $k = 80$ *N/m*. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20\sin 6t$ *N*. (*t* in *seconds*)

*a)* Find the equation of motion.

*b)* Plot the equation of motion.

*c)* Determine the long-time behavior of the system, as $t \to \infty$

***Solution***

***a)*** $2y'' + 8y' + 80y = 20\sin 6t$ ; $y(0) = 0,\quad y'(0) = 0$ $\qquad my'' + cy' + ky = F(t)$

$$2\lambda^2 + 8\lambda + 80 = \lambda^2 + 4\lambda + 40 = 0 \quad \to \quad \lambda_{1,2} = -2 \pm 6i$$

$$y_h = e^{-2t}\left(C_1 \cos 6t + C_2 \sin 6t\right)$$

$$y_p = A\cos 6t + B\sin 6t$$

$$y_p' = -6A\sin 6t + 6B\cos 6t$$

$$y_p'' = -36A\cos 6t - 36B\sin 6t$$

$$y'' + 4y' + 40y = 10\sin 6t$$

$$\begin{aligned}\cos 6t &\to -36A + 24B + 40A = 0\\ \sin 6t &\to -36B - 24A + 40B = 10\end{aligned} \Rightarrow \begin{cases} A + 6B = 0 \\ -12A + 2B = 5\end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 6 \\ -12 & 2 \end{vmatrix} = 74 \quad \Delta_A = \begin{vmatrix} 0 & 6 \\ 5 & 2 \end{vmatrix} = -30 \quad \Delta_B = \begin{vmatrix} 1 & 0 \\ -12 & 5 \end{vmatrix} = 5$$

$$\Rightarrow A = -\frac{15}{37},\ B = \frac{5}{74}$$

$$\boxed{y_p = -\frac{15}{37}\cos 6t + \frac{5}{74}\sin 6t}$$

$$y(t) = e^{-2t}\left(C_1 \cos 6t + C_2 \sin 6t\right) - \frac{15}{37}\cos 6t + \frac{5}{74}\sin 6t$$

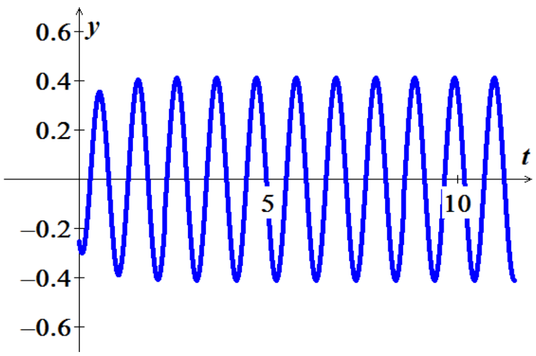
$$y(0) = 0 \quad \to \quad \boxed{C_1 = \frac{15}{37}}$$

$$y' = e^{-2t}\left(-2C_1 \cos 6t - 2C_2 \sin 6t - 6C_1 \sin 6t + 6C_2 \cos 6t\right) + \frac{90}{37}\sin 6t + \frac{30}{74}\cos 6t$$

$$y'(0) = 0 \quad \to -2\left(\frac{30}{74}\right) + 6C_2 + \frac{30}{74} = 0 \quad \boxed{C_2 = \frac{5}{74}}$$

$$\boxed{y(t) = e^{-2t}\left(\frac{15}{37}\cos 6t + \frac{5}{74}\sin 6t\right) - \frac{15}{37}\cos 6t + \frac{5}{74}\sin 6t}$$

***b)***




***c)*** $\lim\limits_{t\to\infty}\left(e^{-2t}\left(\frac{3}{20}\cos 6t - \frac{13}{60}\sin 6t\right) - \frac{15}{37}\cos 6t + \frac{5}{74}\sin 6t\right) = \lim\limits_{t\to\infty}\left(-\frac{15}{37}\cos 6t + \frac{5}{74}\sin 6t\right)$

$= \textit{doesn't exist}$

The equation of motion is a steady-state solution.

## *Exercise*

A $2$-*kg* mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $c = 8$ *kg/sec* and the spring constant is $k = 80$ *N/m*. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20e^{-t}$ *N*. ($t$ in *seconds*)

*a)* Find the equation of motion.

*b)* Plot the equation of motion.

*c)* Determine the long-time behavior of the system, as $t \to \infty$

***Solution***

***a)*** $2y'' + 8y' + 80y = 20e^{-t}$ ; $y(0) = 0, \quad y'(0) = 0$ $\qquad my'' + cy' + ky = F(t)$

$$2\lambda^2 + 8\lambda + 80 = \lambda^2 + 4\lambda + 40 = 0 \quad \to \quad \boxed{\lambda_{1,2} = -2 \pm 6i}$$

$$\boxed{y_h = e^{-2t}\left(C_1 \cos 6t + C_2 \sin 6t\right)}$$

$$y_p = Ae^{-t}$$

$$y'_p = -Ae^{-t}$$

$$y''_p = Ae^{-t}$$

$$y'' + 4y' + 40y = 10e^{-t}$$

$$e^{-t} \quad A - 4A + 40A = 10 \quad \Rightarrow \boxed{A = \frac{10}{37}}$$

$$\boxed{y_p = \frac{10}{37}e^{-t}}$$

$$y(t) = e^{-2t}\left(C_1 \cos 6t + C_2 \sin 6t\right) + \frac{10}{37}e^{-t}$$

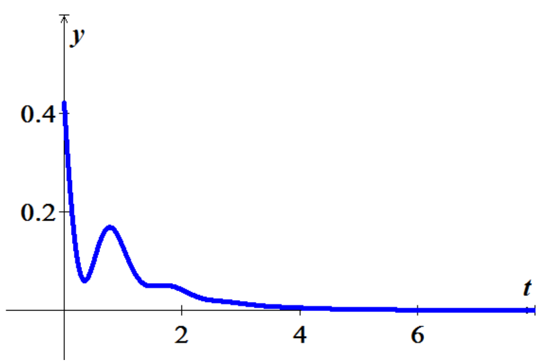
$$y(0) = 0 \quad \to \quad \boxed{C_1 = -\frac{10}{37}}$$

$$y' = e^{-2t}\left(-2C_1 \cos 6t - 2C_2 \sin 6t - 6C_1 \sin 6t + 6C_2 \cos 6t\right) - \frac{10}{37}e^{-t}$$

$$y'(0) = 0 \quad \to -2\left(-\frac{10}{37}\right) + 6C_2 - \frac{10}{37} = 0 \quad \boxed{C_2 = -\frac{5}{111}}$$

$$\boxed{y(t) = e^{-2t}\left(-\frac{10}{37}\cos 6t - \frac{5}{111}\sin 6t\right) + \frac{10}{37}e^{-t}}$$

***b)***




***c)*** $\lim\limits_{t\to\infty}\left(e^{-2t}\left(-\frac{10}{37}\cos 6t - \frac{5}{111}\sin 6t\right) + \frac{10}{37}e^{-t}\right) = 0$

## *Exercise*

A 10-*kg* mass is attached to a spring having a spring constant of 140 *N/m*. The mass is started in motion initially from the equilibrium position with an initial velocity 1 *m/sec* in the upward direction and with an applied external force $F(t) = 5\sin t$. If the force due to air resistance is $-90y'$ *N*.

*a)* Find the equation motion of the mass.

*b)* Plot the motion

*c)* Determine the motion of the solution.

***Solution***

***a)*** $10y'' + 90y' + 140y = 5\sin t$

$$y'' + 9y' + 14y = \frac{1}{2}\sin t \ ; \quad y(0) = 0, \quad y'(0) = -1$$

$$\lambda^2 + 9\lambda + 14 = 0 \quad \to \quad \lambda_{1,2} = -2, \ -7$$

$$y_h = C_1 e^{-2t} + C_2 e^{-7t}$$

$$y_p = A\cos t + B\sin t$$

$$y'_p = -A\sin t + B\cos t$$

$$y_p = -A\cos t - B\sin t$$

$$y'' + 9y' + 14y = \frac{1}{2}\sin t$$

$$\begin{cases} \cos t & -A + 9B + 14A = 0 \\ \sin t & -B - 9A + 14B = \frac{1}{2} \end{cases} \to \begin{cases} 13A + 9B = 0 \\ -9A + 13B = \frac{1}{2} \end{cases}$$

$$\Delta = \begin{vmatrix} 13 & 9 \\ -9 & 13 \end{vmatrix} = 250 \quad \Delta_A = \begin{vmatrix} 0 & 9 \\ \frac{1}{2} & 13 \end{vmatrix} = -\frac{9}{2} \quad \Delta_B = \begin{vmatrix} 13 & 0 \\ -9 & \frac{1}{2} \end{vmatrix} = \frac{13}{2}$$

$$A = -\frac{9}{500}, \quad B = \frac{13}{500}$$

$$y_p = -\frac{9}{500}\cos t + \frac{13}{500}\sin t$$

$$y(t) = C_1 e^{-2t} + C_2 e^{-7t} + \frac{13}{500}\sin t - \frac{9}{500}\cos t$$

$$y(0) = 0 \quad \to \quad C_1 + C_2 = \frac{9}{500}$$

$$y' = -2C_1 e^{-2t} - 7C_2 e^{-7t} + \frac{13}{500}\cos t + \frac{9}{500}\sin t$$

$$y'(0) = -1 \quad \to -2C_1 - 7C_2 + \frac{13}{500} = -1 \quad \Rightarrow \quad 2C_1 + 7C_2 = \frac{513}{500}$$

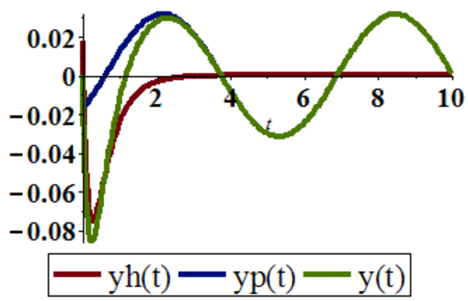
$$\Delta = \begin{vmatrix} 1 & 1 \\ 2 & 7 \end{vmatrix} = 5 \quad \Delta_{C_1} = \begin{vmatrix} \frac{9}{500} & 1 \\ \frac{513}{500} & 7 \end{vmatrix} = -\frac{450}{500} \quad \Delta_{C_2} = \begin{vmatrix} 1 & \frac{9}{500} \\ 2 & \frac{513}{500} \end{vmatrix} = \frac{495}{500}$$

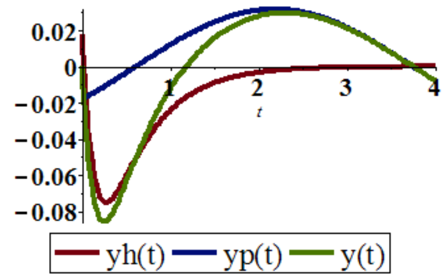
$$C_1 = -\frac{9}{50}, \quad C_2 = \frac{99}{500}$$

$$y(t) = -\frac{9}{50}e^{-2t} + \frac{99}{500}e^{-7t} + \frac{13}{500}\sin t - \frac{9}{500}\cos t$$

$$= \frac{1}{500}\left(99e^{-7t} - 90e^{-2t} + 13\sin t - 9\cos t\right)$$

***b)***




***c)*** $\lim\limits_{t\to\infty} y(t) = \lim\limits_{t\to\infty}\left(-\frac{9}{50}e^{-2t} + \frac{99}{500}e^{-7t} + \frac{13}{500}\sin t - \frac{9}{500}\cos t\right) =$ *doesn't exist*

The homogeneous equation $y_h$ are transient part of the solution which quickly die out.

They are steady-state part of the solution.

## *Exercise*

A 128-*lb* weight is attached to a spring having a spring constant of 64 *lb/ft*. The weight is started in motion initially by displacing it 6 *in* above the equilibrium position with no initial velocity and with an applied external force $F(t) = 8\sin 4t$. Assume no air resistance.

*a)* Find the equation motion of the mass.
*b)* Plot the motion.
*c)* Determine the motion of the solution.

***Solution***

***a)*** $m = \frac{128}{32} = 4$

$4y'' + 64y = 8\sin 4t$

$y'' + 16y = 2\sin 4t$ ; $y(0) = -\frac{6}{12} = -\frac{1}{2}$, $y'(0) = 0$

$\lambda^2 + 16 = 0 \quad \rightarrow \quad \lambda_{1,2} = \pm 4i$

$y_h = C_1\cos 4t + C_2\sin 4t$

$y_p = At\cos 4t + Bt\sin 4t$

$y'_p = A\cos 4t - 4At\sin 4t + B\sin 4t + 4Bt\cos 4t$

$= (A + 4Bt)\cos 4t + (-4At + B)\sin 4t$

$y''_p = 4B\cos 4t - 4A\sin 4t - (4A + 16Bt)\sin 4t + (-16At + 4B)\cos 4t$

$= (-16At + 8B)\cos 4t - (8A + 16Bt)\sin 4t$

$y'' + 16y = 2\sin 4t$

$$\begin{cases} \cos 4t & t & -16A + 16A = 0 & \\ & t^0 & 8B = 0 & B = 0 \end{cases}$$

$$\begin{cases} \sin 4t & t & -16B+16B=0 & \\ & t^0 & -8A=2 & \boxed{A=-\frac{1}{4}} \end{cases}$$

$$\boxed{y_p=-\frac{1}{4}t\cos 4t}$$

$$y(t)=C_1\cos 4t+C_2\sin 4t-\frac{1}{4}t\cos 4t$$

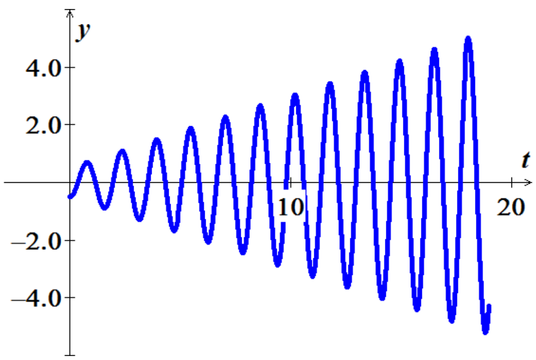
$$y(0)=-\frac{1}{2} \quad \to \quad \boxed{C_1=-\frac{1}{2}}$$

$$y'=-4C_1\sin 4t+4C_2\cos 4t-\frac{1}{4}\cos 4t+t\sin 4t$$

$$y'(0)=0 \quad \to \quad \boxed{C_2=\frac{1}{16}}$$

$$\boxed{y(t)=-\frac{1}{2}\cos 4t+\frac{1}{16}\sin 4t-\frac{1}{4}t\cos 4t}$$

***b)***




***c)*** $\lim\limits_{t\to\infty} y(t)=\lim\limits_{t\to\infty}\left(-\frac{1}{2}\cos 4t+\frac{1}{16}\sin 4t-\frac{1}{4}t\cos 4t\right)\boxed{=\infty}$

This is a pure resonance,

## *Exercise*

A 3-*kg* object is attached to spring and stretches the spring 39.2 *cm* by itself. There is no damping in the system and a forcing function is given by $F(t)=10\cos\omega t$ is attached to the object and the system will experience resonance. If the object is initially displaced 20 *cm* downward from its equilibrium position and given a velocity of 10 *cm/sec* upward.

*a)* Find the spring constant *k*.

*b)* Find the natural frequency $\omega$.

*c)* Find the displacement at any time *t*.

*d)* Sketch the displacement function.

***Solution***

***a)*** $k=\frac{3(9.8)}{0.392}\boxed{=75\ N/m}$ $\qquad k=\frac{mg}{y}$

**b)** $\omega = \sqrt{\frac{75}{3}} \underline{= 5|}$ $\qquad\qquad\qquad \omega = \sqrt{\frac{k}{m}}$

**c)** $3y'' + 75y = 10\cos 5t$ ; $\quad y(0) = 0.2, \quad y'(0) = -0.1$

$$3\lambda^2 + 75 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = \pm 5i|}$$

$$\underline{y_h(t) = C_1 \cos 5t + C_2 \sin 5t|}$$

$$y_p = At\cos 5t + Bt\sin 5t$$

$$y'_p = A\cos 5t - 5At\sin 5t + B\sin 5t + 5Bt\cos 5t$$

$$= (A + 5Bt)\cos 5t + (-5At + B)\sin 5t$$

$$y''_p = 5B\cos 5t - 5A\sin 5t - (5A + 25Bt)\sin 5t + (-25At + 5B)\cos 5t$$

$$= (-25At + 10B)\cos 5t - (10A + 25Bt)\sin 5t$$

$$3y'' + 75y = 10\cos 5t$$

$$\begin{cases} \cos 5t & t & -75A + 75A = 0 & \\ & t^0 & 30B = 10 & \underline{B = \frac{1}{3}|} \end{cases}$$

$$\begin{cases} \sin 5t & t & -75B + 75B = 0 & \\ & t^0 & -30A = 0 & \underline{A = 0|} \end{cases}$$

$$\underline{y_p(t) = \frac{1}{3} t\sin 5t|}$$

$$y(t) = C_1 \cos 5t + C_2 \sin 5t + \frac{1}{3} t\sin 5t$$

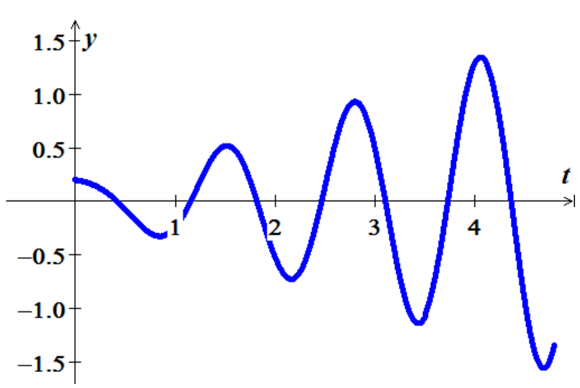
$$y(0) = \frac{1}{5} \quad \rightarrow \quad \underline{C_1 = \frac{1}{5}|}$$

$$y' = -5C_1 \sin 5t + 5C_2 \cos 5t + \frac{1}{3}\sin 5t + \frac{5}{3} t\cos 5t$$

$$y'(0) = -\frac{1}{10} \quad \rightarrow \quad \underline{C_2 = -\frac{1}{50}|}$$

$$\underline{y(t) = \frac{1}{5}\cos 5t - \frac{1}{50}\sin 5t + \frac{1}{3} t\sin 5t\ |}$$

**d)**



## *Exercise*

Find the transient motion and steady periodic oscillations of a damped mass-and-spring system with $m=1$, $c=2$, and $k=26$ under the influence of an external force $F(t)=82\cos 4t$ with $x(0)=6$ and $x'(0)=0$. Also investigate the possibility of practical resonance for this system.

**Solution**

***Given***: $m=1$, $c=2$, $k=26$, and $F(t)=82\cos 4t$ $\quad x(0)=6;\ x'(0)=0$

$x''+2x'+26x=82\cos 4t \qquad mx''+cx'+kx=F(t)$

$$\lambda^2+2\lambda+26=0 \qquad \lambda_{1,2}=\frac{-2\pm\sqrt{-100}}{2}=-1\pm 5i$$

$$x_h=e^{-t}\left(C_1\cos 5t+C_2\sin 5t\right)$$

$$x_p=A\cos 4t+B\sin 4t$$

$$x'_p=-4A\sin 4t+4B\cos 4t$$

$$x''_p=-16A\cos 4t-16B\sin 4t$$

$x''+2x'+26x=82\cos 4t$

$$-16A\cos 4t-16B\sin 4t-8A\sin 4t+8B\cos 4t+26A\cos 4t+26B\sin 4t=82\cos 4t$$

$$(10A+8B)\cos 4t+(-8A+10B)\sin 4t=82\cos 4t$$

$$\begin{cases}10A+8B=82\\ -8A+10B=0\end{cases} \rightarrow \begin{cases}5A+4B=41\\ -4A+5B=0\end{cases}$$

$$D=\begin{vmatrix}5 & 4\\ -4 & 5\end{vmatrix}=41 \quad D_A=\begin{vmatrix}41 & 4\\ 0 & 5\end{vmatrix}=205 \quad D_B=\begin{vmatrix}5 & 41\\ -4 & 0\end{vmatrix}=164$$

$$A=\frac{205}{41}=5; \quad B=\frac{164}{41}=4$$

$$x_p=5\cos 4t+4\sin 4t \qquad A=\sqrt{25+16}=\sqrt{41} \quad \varphi=\tan^{-1}\left(\frac{4}{5}\right)\approx 0.6747$$

$$x_p=\sqrt{41}\cos(4t-0.6747)$$

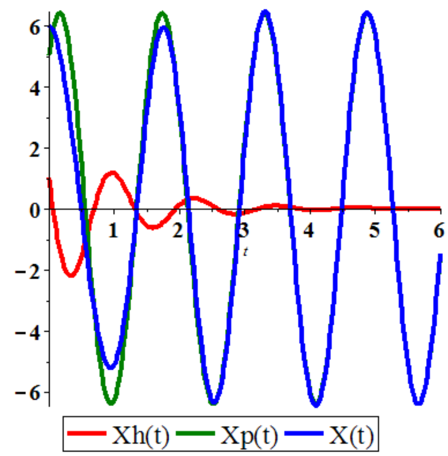
$$x(t)=e^{-t}\left(C_1\cos 5t+C_2\sin 5t\right)+5\cos 4t+4\sin 4t$$

$$x(0)=C_1+5=6 \quad\Rightarrow\quad C_1=1$$

$$x'=e^{-t}\left(5C_1\sin 5t+5C_2\cos 5t-C_1\cos 5t-C_2\sin 5t\right)-20\sin 4t+16\cos 4t$$

$$x'(0)=5C_2-C_1+16=0 \quad\rightarrow\quad C_2=-3$$

$$x(t)=e^{-t}\left(\cos 5t-3\sin 5t\right)+5\cos 4t+4\sin 4t$$



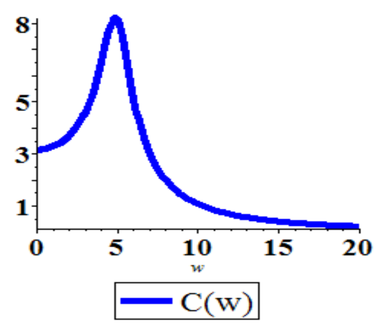

The forced amplitude at frequency $\omega$ is:

$$C(\omega)=\frac{82}{\sqrt{26^2-48\omega^2+\omega^4}}$$

$$C'(\omega)=-\frac{41\left(4\omega^3-96\omega\right)}{\left(676-48\omega^2+\omega^4\right)^{3/2}}=0$$

$$164\omega\left(\omega^2-24\right)=0 \quad \Rightarrow \quad \underline{\omega_{1,2,3}=0,\ \pm 2\sqrt{6}}$$

The mass-and-spring's undamped critical frequency of $\omega_0=\sqrt{\frac{k}{m}}=\sqrt{26}$




### *Exercise*

A mass $m$ is attached to the end of a spring with a spring constant $k$. After the mass reaches equilibrium, its support begins to oscillate vertically about a horizontal line $L$ according to a formula $h(t)$. The value of $h$ represents the distance in feet measured from $L$.

*a)* Determine the differential equation of motion if the entire system moves through a medium offering a damping force that is numerically equal to $\mu\frac{dx}{dt}$

*b)* Solve the differential equation in part (a) if the spring is stretched 4 *feet* by a mass weighing 16 *pounds* and $\mu=2,\quad h(t)=5\cos t,\quad x(0)=x'(0)=0$

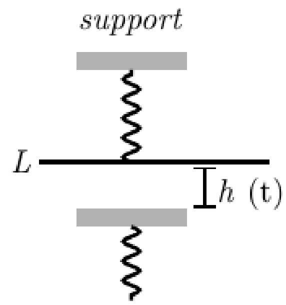
***Solution***

***a)*** The external force is $F(t) = kh(t)$

$$mx'' + \mu x' + kx = kh(t) \qquad\qquad mx'' + \mu x' + kx = F(t)$$

***b)*** *Given*: $m = \frac{16}{32} = \frac{1}{2}$ *slug*, $\quad k(4\ ft) = 16 \to k = 4$

$$\mu = 2, \quad h(t) = 5\cos t, \quad x(0) = x'(0) = 0$$




$$\frac{1}{2}x'' + 2x' + 4x = 20\cos t \ ; \quad x(0) = x'(0) = 0$$

$$\lambda^2 + 4\lambda + 8 = 0 \quad \to \quad \lambda_{1,2} = -2 \pm 2i$$

$$x_h = e^{-2t}\left(C_1\cos 2t + C_2\sin 2t\right)$$

$$x_p = A\cos t + B\sin t$$

$$x'_p = -A\sin t + B\cos t$$

$$x''_p = -A\cos t - B\sin t$$

$$x'' + 4x' + 8x = 40\cos t$$

$$\begin{cases} \cos t & -A + 4B + 8A = 40 \\ \sin t & -B - 4A + 8B = 0 \end{cases} \to \begin{cases} 7A + 4B = 40 \\ -4A + 7B = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 7 & 4 \\ -4 & 7 \end{vmatrix} = 65 \quad \Delta_A = \begin{vmatrix} 40 & 4 \\ 0 & 7 \end{vmatrix} = 280 \quad \Delta_B = \begin{vmatrix} 7 & 40 \\ -4 & 0 \end{vmatrix} = 160$$

$$A = \frac{56}{13}, \quad B = \frac{32}{13}$$

$$x_p = \frac{56}{13}\cos t + \frac{32}{13}\sin t$$

$$x(t) = e^{-2t}\left(C_1\cos 2t + C_2\sin 2t\right) + \frac{56}{13}\cos t + \frac{32}{13}\sin t$$

$$x(0) = 0 \quad \to \quad C_1 = -\frac{56}{13}$$

$$x'(t) = e^{-2t}\left(-2C_1\cos 2t - 2C_2\sin 2t - 2C_1\sin 2t + 2C_2\cos 2t\right) - \frac{56}{13}\sin t + \frac{32}{13}\cos t$$

$$x(0) = 0 \quad \to -2C_1 + 2C_2 + \frac{32}{13} = 0 \quad \Rightarrow \quad C_2 = -\frac{72}{13}$$

$$x(t) = e^{-2t}\left(-\frac{56}{13}\cos 2t - \frac{72}{13}\sin 2t\right) + \frac{56}{13}\cos t + \frac{32}{13}\sin t$$

### *Exercise*

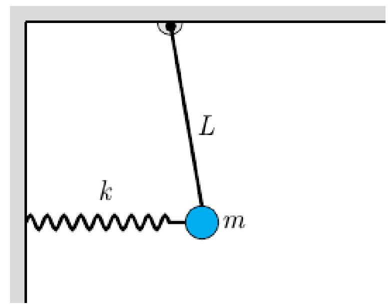
A mass $m$ on the end of a pendulum (of length $L$) also attached to a horizontal spring (with constant $k$). Assume small oscillations of $m$ so that the spring remains essentially horizontal and neglect damping. Find the natural circular frequency $\omega_0$ of motion of the mass in terms of $L$, $k$, $m$, and the gravitational constant $g$.

***Solution***

Let $\theta$ is the angular displacement.

The displacement of the mass is: $x = L\theta$

Its total energy $(KE + PE)$ is




$$mv^2 + kx^2 + 2mgh = C$$

$$m(x')^2 + kx^2 + 2mgh = C$$

$$\frac{d}{dt}\left(mL^2(\theta')^2 + kL^2\theta^2 + 2mgL(1-\cos\theta)\right) = \frac{d}{dt}C$$

$$2mL^2(\theta')\theta'' + 2kL^2\theta\theta' + 2mgL\theta'\sin\theta = 0 \qquad (\theta' \neq 0)$$

$$mL^2\theta'' + kL^2\theta + mgL\sin\theta = 0 \qquad \sin\theta = \theta$$

$$mL^2\theta'' + \left(kL^2 + mgL\right)\theta = 0$$

$$\theta'' + \left(\frac{k}{m} + \frac{g}{L}\right)\theta = 0$$

$$\boxed{\omega_0 = \sqrt{\frac{k}{m} + \frac{g}{L}}}$$

### *Exercise*

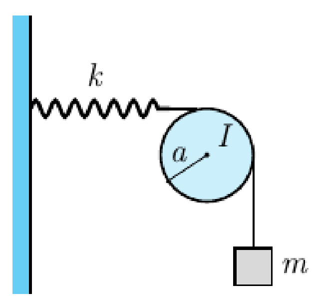
A mass $m$ hangs on the end of a cord around a pullet of radius $a$ and moment of inertia $I$. The rim of the pulley is attached to a spring (with constant $k$). Assume small oscillations so that the spring remains essentially and neglect friction. Find the natural circular frequency in terms of $m$, $a$, $k$, $I$, and $g$.

***Solution***

Let $x$ be the displacement of the mass from its equilibrium position.

$v = x'$ be the velocity.

$\omega = \frac{v}{a}$ the angular velocity of the pulley.




$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2 - mgx = C \qquad \text{Conservation of energy}$$

$$\frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}\frac{I}{a^2}v^2 + \frac{1}{2}kx^2 - mgx\right) = \frac{d}{dt}C$$

$$mvv' + \frac{I}{a^2}vv' + kxx' - mgx' = 0$$

$$\left(m+\frac{I}{a^2}\right)x'x''+kxx'-mgx'=0$$

$$\left(m+\frac{I}{a^2}\right)x''+kx-mg=0$$

$$\omega=\sqrt{\frac{k}{m+\frac{I}{a^2}}}\ \boxed{=a\sqrt{\frac{k}{ma^2+I}}}$$

## Exercise

Consider a floating cylindrical buoy with radius *r*, height *h*, and uniform density $\rho \le 0.5$ (recall that the density of water is $1\ g/cm^3$). The buoy is initially suspended at rest with its bottom at the top surface of the water and is released at time $t=0$. Therafter it is acted on by two forces: a downward gravitational force equal to its weight $mg=\pi r^2 hg$ and (by Archmedes' principle of buoyancy) an upward force equal to the wieght $\pi r^2 xg$ of water displaced, where $x=x(t)$ is the depth of the bottom of the buoy beneath the surface at time *t*.

Conclude that the buoy undergoes simple harmonic motion around its equilibrium position $x_e=\rho h$ with period $p=2\pi\sqrt{\frac{\rho h}{g}}$.

*a)* Compute *p* and the amplitude of the motion if $\rho=0.5\ g/cm^3$, $h=200\ cm$, and $g=980\ cm/s^2$

*b)* If the cylindrical buoy weighting 100 *lb* floats in water with its axis vertical. When depressed slightly and released, it oscillates up and down four times every 10 *sec*. assume that friction is negligible. Find the radius of the buoy.

***Solution***

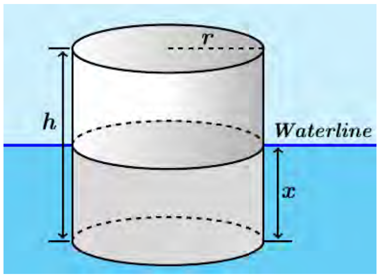
**a)** $F=ma$

$$\rho\pi r^2 hx''=\rho\pi r^2 hg-\pi r^2 xg \quad \to \quad \rho hx''+gx-\rho hg=0$$

$$x''+\frac{g}{\rho h}x=g$$

$$\lambda^2+\frac{g}{\rho h}=0 \quad \Rightarrow \quad \lambda=\pm i\sqrt{\frac{g}{\rho h}}$$

$$x(t)=A\cos\left(\sqrt{\frac{g}{\rho h}}t\right)+B\sin\left(\sqrt{\frac{g}{\rho h}}t\right)$$




$$x_p=A \Rightarrow x_p''=0$$

$$\frac{g}{\rho h}A=g \to A=\rho h$$

$$x(t)=A\cos\left(\sqrt{\frac{g}{\rho h}}t\right)+B\sin\left(\sqrt{\frac{g}{\rho h}}t\right)+\rho h \qquad x(0)=x'(0)=0$$

$$x(0) = A + \rho h = 0 \;\Rightarrow\; \underline{A = -\rho h}$$

$$x'(t) = -A\sqrt{\frac{g}{\rho h}}\sin\left(\sqrt{\frac{g}{\rho h}}t\right) + B\sqrt{\frac{g}{\rho h}}\cos\left(\sqrt{\frac{g}{\rho h}}t\right) \;\rightarrow x'(0) = B\sqrt{\frac{g}{\rho h}} = 0 \;\Rightarrow\; \underline{B = 0}$$

$$x(t) = -\rho h\cos\left(\sqrt{\frac{g}{\rho h}}t\right) + \rho h$$

$$\underline{= \rho h\left(1 - \cos\omega_0 t\right)} \qquad \omega_0 = \sqrt{\frac{g}{\rho h}} = \sqrt{\frac{980}{0.5(200)}} \approx 3.13$$

$$\underline{= 100\left(1 - \cos(3.13t)\right)}$$

Amplitude: $\underline{A = 100\ cm}$

Period: $P = \frac{2\pi}{3.13}\ \underline{\approx 2.01\ \ sec}$

***b)*** ***Given***: $mg = 100 \quad P = \frac{10}{4} = 2.5\ \text{sec}$

The weight of water: $\rho = 62.4\ \ lb / ft^3$

$$\frac{100}{32}x'' + 62.4\pi r^2 x = 100 \qquad mx'' + \pi\rho r^2 x = mg$$

$$x'' + 19.968\pi r^2 x = 32 \;\Rightarrow\; \omega^2 = 19.968\pi r^2$$

$$\left(\frac{2\pi}{2.5}\right)^2 = 19.968\pi r^2$$

$$r = \frac{1}{\sqrt{19.968\pi}}\left(\frac{2\pi}{2.5}\right)\ \underline{\approx 0.3173\ \ ft} \qquad \underline{\approx 3.8\ in}$$

## Exercise

Assume that the earth is a solid sphere of uniform density, with mass *M* and radius $R = 3960$ (*mi*). For a particle of mass *m* within the earth at distance *r* from the center of the earth, the gravitational force attracting *m* toward the center is $F_r = -\frac{GM_r m}{r^2}$, where $M_r$ is the mass of the part of the earth within a sphere of radius *r*.

*a)* Show that $F_r = -\frac{GMmr}{R^3}$

*b)* Now suppose that a small hole is drilled straight through the center of the earth, thus connecting two antipodal points on its surface. Let a particle of mass *m* be dropped at time $t = 0$ into this hole with initial speed zero, and let $r(t)$ be its distance from the center of the earth at time *t*. conclude from Newton's second law and part (*a*) that $r''(t) = -k^2 r(t)$, where $k^2 = \frac{GM}{R^3} = \frac{g}{R}$.

*c)* Take $g = 32.2\ ft/s^2$, and conclude from part (*b*) that the particle undergoes simple harmonic motion back and forth between the ends of the hole, with a period of about 84 *min*.

*d)* Look up (or derive) the period of a satellite that just skims the surface of the earth; compare with the result in part (*c*). How do you explain the coincidence? Or is it a coincidence?

*e)* With what speed (in miles per hours) does the particle pass through the center of the earth?

*f)* Look up (or derive) the orbital velocity of a satellite that just skims the surface of the earth; compare with the result in part (*e*). How do you explain the coincidence? Or is it a coincidence?

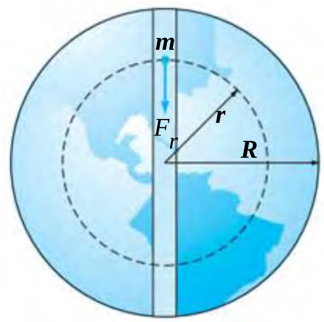
**Solution**

**a)** $M_r = M\left(\frac{r}{R}\right)^3$

$$F_r = -\frac{GM_r m}{r^2}$$

$$= -\frac{Gm}{r^2}M\frac{r^3}{R^3}$$

$$\boxed{= -\frac{GMmr}{R^3}}$$




**b)** Since $\frac{GM}{R^3} = \frac{g}{R}$

$$mr'' = F_r$$

$$mr'' = -\frac{GM}{R^3}mr$$

$$\boxed{r'' + \frac{g}{R}r = 0}$$

**c)** $r'' + \frac{g}{R}r = 0$

$$\lambda^2 + \frac{g}{R} = 0 \quad \to \quad \lambda = \pm i\sqrt{\frac{g}{R}}$$

$$r(t) = A\cos\left(\sqrt{\frac{g}{R}}\,t\right) + B\sin\left(\sqrt{\frac{g}{R}}\,t\right) \qquad r(0) = R \quad r'(0) = 0$$

$$r(0) = \underline{A = R}$$

$$r'(t) = -A\sqrt{\frac{g}{R}}\sin\left(\sqrt{\frac{g}{R}}\,t\right) + B\sqrt{\frac{g}{R}}\cos\left(\sqrt{\frac{g}{R}}\,t\right) \ \to\ r'(0) = B\sqrt{\frac{g}{R}} = 0 \ \Rightarrow \underline{B = 0}$$

$$r(t) = R\cos\omega_0 t; \quad \omega_0 = \sqrt{\frac{g}{R}}$$

***Given***: $g = 32.2 \ ft/\sec^2 \quad R = (3960)(5280) \ ft$

The period of the particle's simple harmonic motion is:

$$P = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{3960\times 5280}{32.2}} \quad \underline{\approx 84.38 \ min}$$

**d)** The orbital velocity $v$ of such a satellite must be such that the centrifugal force $\frac{mv^2}{R}$ on the satellite just offsets the weight $mg$ of the satellite at the surface of the earth. Thus

$$\frac{mv^2}{R} = mg \quad \Rightarrow \quad v = \sqrt{gR} = \sqrt{32.2 \times 3960\ mi \times \frac{5280\ ft}{1\ mi}}$$

$$\approx 2.5947 \times 10^4\ ft/\sec$$

$$\approx 2.5947 \times 10^4 \frac{ft}{\sec} \frac{1\ mi}{5280\ ft} \frac{3600\ \sec}{1\ hr} \approx 1.7691 \times 10^4\ mi/hr$$

Because the circumference of the earth is $2\pi R$, the period of the satellite's orbit is

$$\frac{2\pi}{\sqrt{gR}} = 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{3960 \times 5280}{32.2}} \approx 84.38\ min$$

Let assume that at time $t = 0$, the satellite is directly over the hole in the earth at the top, and its orbit proceeds in a clockwise direction.

The distance $r$ of the particle, from part (*c*), from the center of the earth is

$$r(t) = R\cos\omega_0 t; \quad \omega_0 = \sqrt{\frac{g}{R}}$$

The key observation is that $\omega_0 t$ is the angle drawn clockwise from the vertical to the radius vector of the satellite at time *t*; thus, the distance $r(t)$ is simply the vertical component of the satellite's position. It follows that $r(t)$ completes one cycle through the earth (and back) in the same length of time required for the satellite to complete one orbit around the earth.

***e)*** The particle passes through the center of the earth when $r(t) = R\cos\omega_0 t = 0$, that is when

$$\omega_0 t = \frac{\pi}{2} \quad \rightarrow \; t = \frac{\pi}{2\omega_0}.$$

At this time the speed of the particle is

$$r'(t) = -R\omega_0 \sin\omega_0 t$$

$$\left|r'(t)\right| = \left|-R\omega_0 \sin\left(\omega_0 \frac{\pi}{2\omega_0}\right)\right|$$

$$= R\omega_0$$

$$= R\sqrt{\frac{g}{R}} = \sqrt{gR}$$

$$\approx 1.7691 \times 10^4\ mi/hr \qquad part(d)$$

***f)*** The orbital velocity is $v = \sqrt{gR}$.

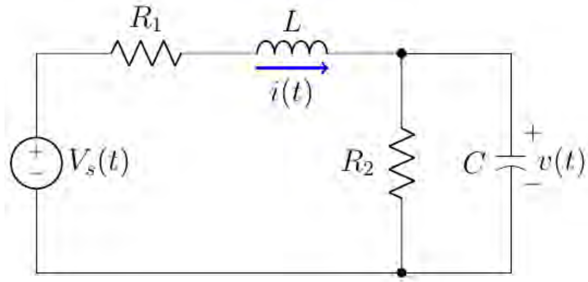
The vertical component of the satellite's velocity vector $v(t)$ at any given time *t* is equal to the speed $\left|r'(t)\right|$ of the particle at that time.

At the moment when the particle passes through the center of earth, the satellite is travelling straight downward, and hence $v(t)$ is vertical.

Therefore, the orbital velocity *v* of the satellite, which is the magnitude of $v(t)$, is equal to the speed of the particle at this moment.

### Exercise

Express the given circuit in the second-order differential equation




**Solution**

$$V_s = R_1 i + L\frac{di}{dt} + v(t)$$

$$i(t) = \frac{v(t)}{R_2} + C\frac{d}{dt}v(t)$$

$$V_s = R_1\left(\frac{v}{R_2} + C\frac{dv}{dt}\right) + L\frac{d}{dt}\left(\frac{v}{R_2} + C\frac{dv}{dt}\right) + v$$

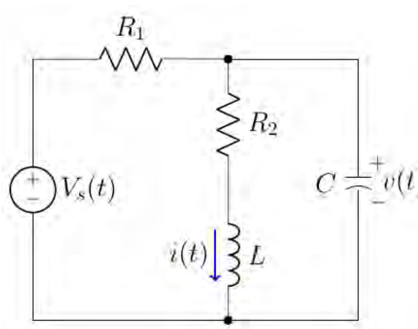
$$= \frac{R_1}{R_2}v + R_1 C\frac{dv}{dt} + \frac{L}{R_2}\frac{dv}{dt} + LC\frac{d^2v}{dt^2} + v$$

$$= LC\frac{d^2v}{dt^2} + \left(R_1 C + \frac{L}{R_2}\right)\frac{dv}{dt} + \left(\frac{R_1}{R_2} + 1\right)v(t)$$

$$= LC\frac{d^2v}{dt^2} + \left(R_1 C + \frac{L}{R_2}\right)\frac{dv}{dt} + \left(\frac{R_1 + R_2}{R_2}\right)v(t)$$

### Exercise

Express the given circuit in the second-order differential equation




**Solution**

$$v(t) = R_2 i(t) + L\frac{d}{dt}i(t)$$

$$i_{R_1} = i(t) + i_C$$

$$= i(t) + C\frac{d}{dt}v(t)$$

$$V_S = R_1\left(i + C\frac{dv}{dt}\right) + v(t) \qquad\qquad V_S = V_{R_1} + V_C$$

$$= R_1 i + R_1 C\frac{d}{dt}\left(R_2 i + L\frac{di}{dt}\right) + R_2 i + L\frac{di}{dt}$$

$$= R_1 i + R_1 R_2 C\frac{di}{dt} + R_1 CL\frac{d^2 i}{dt^2} + R_2 i + L\frac{di}{dt}$$

$$\boxed{= R_1 CL\frac{d^2}{dt^2}i(t) + \left(R_1 R_2 C + L\right)\frac{d}{dt}i(t) + \left(R_1 + R_2\right)i(t)}$$

### *Exercise*

Find the steady-state solution $q_p(t)$ and the steady-state current in and *LRC*–series circuit when the source voltage is $E(t) = E_0 \sin\omega t$

***Solution***

$$Lq'' + Rq' + \frac{1}{C}q = E_0 \sin\omega t$$

$$L\lambda^2 + R\lambda + \frac{1}{C} = 0 \quad\to\quad \lambda_{1,2} = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L} = \frac{-R\sqrt{C} \pm \sqrt{R^2 C - 4L}}{2L\sqrt{C}}$$

$$q_p = A\cos\omega t + B\sin\omega t$$

$$q_p' = -\omega A\sin\omega t + \omega B\cos\omega t$$

$$q_p'' = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t$$

$$-\omega^2 LA\cos\omega t - \omega^2 LB\sin\omega t - \omega RA\sin\omega t + \omega RB\cos\omega t + \frac{1}{C}A\cos\omega t + \frac{1}{C}B\sin\omega t = E_0 \sin\omega t$$

$$\begin{cases} \cos\omega t & \left(-\omega^2 L + \frac{1}{C}\right)A + \omega RB = 0 \\ \sin\omega t & -\omega RA + \left(\frac{1}{C} - \omega^2 L\right)B = E_0 \end{cases}$$

$$\Delta = \begin{vmatrix} \frac{1}{C} - \omega^2 L & \omega R \\ -\omega R & \frac{1}{C} - \omega^2 L \end{vmatrix} = \frac{1}{C^2} - \frac{2\omega^2 L}{C} + \omega^4 L^2 + \omega^2 R^2 = \frac{1}{C^2}\left(1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2\right)$$

$$\Delta_A = \begin{vmatrix} 0 & \omega R \\ E_0 & \frac{1}{C} - \omega^2 L \end{vmatrix} = \omega RE_0 \qquad \Delta_B = \begin{vmatrix} \frac{1}{C} - \omega^2 L & 0 \\ -\omega R & E_0 \end{vmatrix} = \left(\frac{1}{C} - \omega^2 L\right)E_0$$

$$A = \frac{\omega RC^2 E_0}{1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2} \qquad B = \frac{C(1 - \omega LC)E_0}{1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2}$$

Therefore, the steady-state charge is:

$$\boxed{q_p(t) = \frac{CE_2}{1-2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2}\left(\omega RC\cos\omega t + (1-\omega LC)\sin\omega t\right)}$$

The steady-state current is: $\quad i_p(t) = q_p'(t)$

$$\boxed{i_p(t) = \frac{CE_2}{1-2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2}\left(-\omega^2 RC\sin\omega t + \left(\omega - \omega^2 LC\right)\cos\omega t\right)}$$

### Exercise

Find the charge $q(t)$ on the capacitor in an $LRC$–series circuit when $L = \frac{5}{3}\ h$, $R = 10\ \Omega$, $C = \frac{1}{30}\ f$, $E(t) = 300\ V$, $q(0) = 0\ C$, and $i(0) = 0\ A$. Find the maximum charge on the capacitor

**Solution**

$$\frac{5}{3}q'' + 10q' + 30q = 300 \qquad\qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 6q' + 18q = 900\ ; \quad q(0) = 0, \quad q'(0) = i(0) = 0$$

$$\lambda^2 + 6\lambda + 18 = 0 \quad\rightarrow\quad \boxed{\lambda_{1,2} = -3 \pm 3i}$$

$$q(t) = e^{-3t}\left(C_1\cos 3t + C_2\sin 3t\right)$$

$$q_p = A$$

$$q_p' = q_p'' = 0$$

$$\frac{5}{3}q'' + 10q' + 30q = 300 \quad\rightarrow\quad 30A = 300 \quad\Rightarrow\quad \boxed{A = 10}$$

$$q(t) = e^{-3t}\left(C_1\cos 3t + C_2\sin 3t\right) + 10$$

$$q(0) = 0 \quad\rightarrow\quad \boxed{C_1 = -10}$$

$$q'(t) = e^{-3t}\left(-3C_1\cos 3t - 3C_2\sin 3t - 3C_1\sin 3t + 3C_2\cos 3t\right)$$

$$q'(0) = i(0) = 0 \quad\rightarrow -3C_1 + 3C_2 = 0 \quad \boxed{C_2 = -10}$$

$$\boxed{q(t) = 10 - 10e^{-3t}\left(\cos 3t + \sin 3t\right)}$$

$$i(t) = q'(t) = -10e^{-3t}\left(-3\cos 3t - 3\sin 3t - 3\sin 3t + 3\cos 3t\right)$$

$$\boxed{= 60e^{-3t}\sin 3t}$$

Maximum charge: $q'(t) = 60e^{-3t}\sin 3t = 0$

$$3t = \pi \quad\rightarrow\quad \boxed{t = \frac{\pi}{3}\ sec}$$

$$q\left(\frac{\pi}{3}\right)=10-10e^{-\pi}(\cos\pi+\sin\pi)$$

$$\underline{=10+10e^{-\pi}} \quad \underline{\approx 10.432\ C}$$

### *Exercise*

Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L=1\ h$, $R=100\ \Omega$, $C=0.0004\ f$, $E(t)=30\ V$, $q(0)=0\ C$, and $i(0)=2\ A$. Find the maximum charge on the capacitor

***Solution***

$$q''+100q'+\frac{1}{0.0004}q=30 \qquad\qquad Lq''+Rq'+\frac{1}{C}q=E(t)$$

$$q''+100q'+2500q=30\ ; \quad q(0)=0, \quad q'(0)=i(0)=2$$

$$\lambda^2+100\lambda+2500=0 \quad\to\quad \underline{\lambda_{1,2}=-50}$$

$$q_h(t)=\left(C_1+C_2t\right)e^{-50t}$$

$$q_p=A$$

$$q_p'=q_p''=0$$

$$q''+100q'+2500q=30 \quad\to\quad 2500A=30 \quad\Rightarrow\quad \underline{A=0.012}$$

$$q(t)=\left(C_1+C_2t\right)e^{-50t}+0.012$$

$$q(0)=0 \quad\to\quad \underline{C_1=-0.012}$$

$$q'(t)=\left(C_2-50C_1-50C_2t\right)e^{-50t}$$

$$q'(0)=i(0)=2 \quad\to C_2-50C_1=2 \quad \underline{C_2=1.4}$$

$$\underline{q(t)=(-0.012+1.4t)e^{-50t}+0.012}$$

$$\underline{i(t)=q'(t)=(2-70t)e^{-50t}}$$

Maximum charge: $q'(t)=(2-70t)e^{-50t}=0$

$$2-70t \quad\to\quad \underline{t=\frac{1}{35}\ sec}$$

$$q\left(\frac{1}{35}\right)=\left(-0.012+\frac{1.4}{35}\right)e^{-50/35}+0.012$$

$$\underline{\approx 0.01871\ C}$$

## Exercise

Find the charge $q(t)$ and $i(t)$ on the capacitor in an *LRC*–series circuit when $L=\frac{1}{2}\ h$, $R=10\ \Omega$, $C=0.01\ f$, and $E(t)=150\ V$, $q(0)=1\ C$, and $i(0)=0\ A$. What is the charge on the capacitor after a long time?

**Solution**

$$\frac{1}{2}q''+10q'+\frac{1}{0.01}q=150 \qquad\qquad Lq''+Rq'+\frac{1}{C}q=E(t)$$

$$q''+20q'+200q=300\ ; \quad q(0)=1, \quad q'(0)=i(0)=0$$

$$\lambda^2+20\lambda+200=0 \quad\rightarrow\quad \boxed{\lambda_{1,2}=-10\pm 10i}$$

$$q_h(t)=e^{-10t}\left(C_1\cos 10t+C_2\sin 10t\right)$$

$$q_p=A$$

$$q_p'=q_p''=0$$

$$q''+20q'+200q=300 \quad\rightarrow\quad 200A=300 \quad\Rightarrow\quad \boxed{A=\frac{3}{2}}$$

$$q(t)=e^{-10t}\left(C_1\cos 10t+C_2\sin 10t\right)+\frac{3}{2}$$

$$q(0)=1 \quad\rightarrow C_1+\frac{3}{2}=1 \quad\Rightarrow\quad \boxed{C_1=-\frac{1}{2}}$$

$$q(t)=e^{-10t}\left(-10C_1\cos 10t-10C_2\sin 10t-10C_1\sin 10t+10C_2\cos 10t\right)$$

$$q'(0)=i(0)=0 \quad\rightarrow -10C_1+10C_2=0 \quad \boxed{C_2=-\frac{1}{2}}$$

$$\boxed{q(t)=-\frac{1}{2}e^{-10t}\left(\cos 10t+\sin 10t\right)+\frac{3}{2}}$$

$$i(t)=q'(t)=-\frac{1}{2}e^{-10t}\left(-10\cos 10t-10\sin 10t-10\sin 10t+10\cos 10t\right)$$

$$\boxed{=10e^{-10t}\sin 10t}$$

$$\lim_{t\to\infty}q(t)=\lim_{t\to\infty}\left(-\frac{1}{2}e^{-10t}\left(\cos 10t+\sin 10t\right)+\frac{3}{2}\right)$$

$$\boxed{=\frac{3}{2}}$$

## Exercise

Find the charge $q(t)$ and $i(t)$ on the capacitor in an *LRC*–series circuit when $L=1\ h$, $R=50\ \Omega$, $C=0.0002\ f$, $E(t)=50\ V$, $q(0)=0\ C$, and $i(0)=0\ A$.

**Solution**

$$q''+50q'+\frac{1}{0.0002}q=50 \qquad\qquad Lq''+Rq'+\frac{1}{C}q=E(t)$$

$$q'' + 50q' + 5000q = 50\ ; \quad q(0) = 0, \quad q'(0) = i(0) = 0$$

$$\lambda^2 + 50\lambda + 5000 = 0 \quad \to \quad \boxed{\lambda_{1,2} = -25 \pm 25\sqrt{7}}$$

$$q(t) = e^{-25t}\left(C_1 \cos 25\sqrt{7}t + C_2 \sin 25\sqrt{7}t\right)$$

$$q_p = A$$

$$q'_p = q''_p = 0$$

$$q'' + 50q' + 5000q = 50 \quad \to \quad 5000A = 50 \quad \Rightarrow \quad \boxed{A = 0.01}$$

$$q(t) = e^{-25t}\left(C_1 \cos 25\sqrt{7}t + C_2 \sin 25\sqrt{7}t\right) + .01$$

$$q(0) = 0 \quad \to \quad \boxed{C_1 = -0.01 = -\frac{1}{100}}$$

$$q'(t) = e^{-25t}\left(-25C_1 \cos 25\sqrt{7}t - 25C_2 \sin 25\sqrt{7}t - 25\sqrt{7}C_1 \sin 25\sqrt{7}t + 25\sqrt{7}C_2 \cos 25\sqrt{7}t\right)$$

$$q'(0) = i(0) = 0 \quad \to -25C_1 + 25\sqrt{7}C_2 = 0 \quad \boxed{C_2 = -\frac{1}{100\sqrt{7}}}$$

$$\boxed{q(t) = -\frac{1}{100\sqrt{7}}e^{-25t}\left(\sqrt{7}\cos 25\sqrt{7}t + \sin 25\sqrt{7}t\right) + \frac{1}{100}}$$

$$i(t) = q'(t) = -\frac{1}{100\sqrt{7}}e^{-25t}\left(-25\sqrt{7}\cos 25\sqrt{7}t - 25\sin 25\sqrt{7}t - 4{,}375\sin 25\sqrt{7}t + 4{,}375\cos 25\sqrt{7}t\right)$$

$$\boxed{= -\frac{1}{100\sqrt{7}}e^{-25t}\left(4{,}350\sqrt{7}\cos 25\sqrt{7}t - 4{,}400\sin 25\sqrt{7}t\right)}$$

## *Exercise*

Find the steady-state charge and the steady-state current in an *LRC*–series circuit when $L = 1\ h$, $R = 2\ \Omega$, $C = 0.25\ f$, and $E(t) = 50\cos t\ V$.

***Solution***

$$q'' + 2q' + \frac{1}{0.25}q = 50\cos t \qquad\qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 2q' + 4q = 0$$

$$\lambda^2 + 2\lambda + 4 = 0 \quad \to \quad \boxed{\lambda_{1,2} = -1 \pm i\sqrt{3}}$$

$$\boxed{q_h = e^{-t}\left(C_1 \cos\sqrt{3}t + C_2 \sin\sqrt{3}t\right)}$$

$$q_p = A\cos t + B\sin t$$

$$q'_p = -A\sin t + B\cos t$$

$$q''_p = -A\cos t - B\sin t$$

$$q'' + 2q' + 4q = 50\cos t$$

$$\begin{cases} \cos t & -A+2B+4A=50 \\ \sin t & -B-2A+4B=0 \end{cases} \rightarrow \begin{cases} 3A+2B=50 \\ -2A+3B=0 \end{cases}$$

$$\Delta = \begin{vmatrix} 3 & 2 \\ -2 & 3 \end{vmatrix} = 13 \quad \Delta_A = \begin{vmatrix} 50 & 2 \\ 0 & 3 \end{vmatrix} = 150 \quad \Delta_B = \begin{vmatrix} 3 & 50 \\ -2 & 0 \end{vmatrix} = 100 \quad \rightarrow \boxed{A = \frac{150}{13} \quad B = \frac{100}{13}}$$

The steady-state charge is: $\boxed{q_p(t) = \frac{150}{13}\cos t + \frac{100}{13}\sin t}$

The steady-state current is: $\boxed{i_p(t) = -\frac{150}{13}\sin t + \frac{100}{13}\cos t}$

## *Exercise*

Find the steady-state charge and the steady-state current in an *LRC*–series circuit when $L = \frac{1}{2}\ h$, $R = 20\ \Omega$, $C = 0.001\ f$, and $E(t) = 100\sin 60t\ V$.

***Solution***

$$\frac{1}{2}q'' + 20q' + \frac{1}{0.001}q = 100\sin 60t \qquad\qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 40q' + 2,000q = 200\sin 60t$$

$$\lambda^2 + 40\lambda + 2000 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -20 \pm 40i}$$

$$\boxed{q_h = e^{-20t}\left(C_1 \cos 40t + C_2 \sin 40t\right)}$$

$$q_p = A\cos 60t + B\sin 60t$$

$$q'_p = -60A\sin 60t + 60B\cos 60t$$

$$q''_p = -3600A\cos 60t - 3600B\sin 60t$$

$$q'' + 40q' + 2,000q = 200\sin 60t$$

$$\begin{cases} \cos 60t & -3600A+2400B+2000A=0 \\ \sin 60t & -3600B-2400A+2000B=200 \end{cases} \rightarrow \begin{cases} -2A+3B=0 \\ -12A-8B=1 \end{cases}$$

$$\Delta = \begin{vmatrix} -2 & 3 \\ -12 & -8 \end{vmatrix} = 52 \quad \Delta_A = \begin{vmatrix} 0 & 3 \\ 1 & -8 \end{vmatrix} = -3 \quad \Delta_B = \begin{vmatrix} -2 & 0 \\ -12 & 1 \end{vmatrix} = -2$$

$$\rightarrow \boxed{A = -\frac{1}{26} \quad B = -\frac{3}{52}}$$

The steady-state charge is: $\boxed{q_p(t) = -\frac{1}{26}\cos 60t - \frac{3}{52}\sin 60t}$

The steady-state current is: $\boxed{i_p(t) = \frac{1}{26}\sin 60t - \frac{3}{52}\cos 60t}$

### Exercise

Find the steady-state charge and the steady-state current in an *LRC*–series circuit when $L=\frac{1}{2}\ h$, $R=20\ \Omega$, $C=0.001\ f$, and $E(t)=100\sin 60t+200\cos 40t\ V$.

***Solution***

$$\frac{1}{2}q''+20q'+\frac{1}{0.001}q=100\sin 60t+200\cos 40t \qquad Lq''+Rq'+\frac{1}{C}q=E(t)$$

$$q''+40q'+2{,}000q=200\sin 60t+400\cos 40t$$

$$\lambda^2+40\lambda+2000=0 \quad \rightarrow \quad \lambda_{1,2}=-20\pm 40i$$

$$q_h(t)=e^{-20t}\left(C_1\cos 40t+C_2\sin 40t\right)$$

$$q_p=A\cos 60t+B\sin 60t+C\cos 40t+D\sin 40t$$

$$q_p'=-60A\sin 60t+60B\cos 60t-40C\sin 40t+40D\cos 40t$$

$$q_p''=-3600A\cos 60t-3600B\sin 60t-1600C\cos 40t-1600D\sin 40t$$

$$q''+40q'+2{,}000q=200\sin 60t+400\cos 40t$$

$$\begin{cases}\cos 60t & -3600A+2400B+2000A=0\\ \sin 60t & -3600B-2400A+2000B=200\end{cases} \rightarrow \begin{cases}-2A+3B=0\\ -12A-8B=1\end{cases}$$

$$\Delta=\begin{vmatrix}-2 & 3\\ -12 & -8\end{vmatrix}=52 \quad \Delta_A=\begin{vmatrix}0 & 3\\ 1 & -8\end{vmatrix}=-3 \quad \Delta_B=\begin{vmatrix}-2 & 0\\ -12 & 1\end{vmatrix}=-2$$

$$\rightarrow A=-\frac{1}{26} \quad B=-\frac{3}{52}$$

$$\begin{cases}\cos 40t & -1600C+1600D+2000C=400\\ \sin 40t & -1600D-1600C+2000D=0\end{cases} \rightarrow \begin{cases}C+4D=1\\ -4C+D=0\end{cases}$$

$$\Delta=\begin{vmatrix}1 & 4\\ -4 & 1\end{vmatrix}=17 \quad \Delta_C=\begin{vmatrix}1 & 4\\ 0 & 1\end{vmatrix}=1 \quad \Delta_D=\begin{vmatrix}1 & 1\\ -4 & 0\end{vmatrix}=4$$

$$\rightarrow C=\frac{1}{17} \quad D=\frac{4}{17}$$

The steady-state charge is:

$$q_p(t)=-\frac{1}{26}\cos 60t-\frac{3}{52}\sin 60t+\frac{1}{17}\cos 40t+\frac{4}{17}\sin 40t$$

The steady-state current is:

$$i_p(t)=\frac{1}{26}\sin 60t-\frac{3}{52}\cos 60t-\frac{1}{17}\sin 40t+\frac{4}{17}\cos 40t$$

### *Exercise*

Find the charge $q(t)$ and $i(t)$ on the capacitor in an $LC$–series circuit when

$E(t)=E_0 \sin\omega t\ V,\quad q(0)=q_0\ C,\quad and\quad i(0)=i_0\ A$

***Solution***

$$Lq''+\frac{1}{C}q=E_0\sin\omega t\ ;\quad q(0)=q_0\quad i(0)=i_0 \qquad\qquad Lq''+Rq'+\frac{1}{C}q=E(t)$$

$$\lambda^2+\frac{1}{LC}=0\ \rightarrow\ \lambda_{1,2}=\pm\frac{1}{\sqrt{LC}}i$$

$$q_h(t)=C_1\cos\frac{1}{\sqrt{LC}}t+C_2\sin\frac{1}{\sqrt{LC}}t$$

$$q_p=A\cos\omega t+B\sin\omega t$$

$$q'_p=-\omega A\sin\omega t+\omega B\cos\omega t$$

$$q''_p=-\omega^2 A\cos\omega t-\omega^2 B\sin\omega t$$

$$Lq''+\frac{1}{C}q=E_0\sin\omega t$$

$$\begin{cases}\cos\omega t & -L\omega^2A+\frac{1}{C}A=0\\ \sin\omega t & \left(-L\omega^2+\frac{1}{C}\right)B=E_0\end{cases}\ \rightarrow\ \begin{array}{l}A=0\\ B=\dfrac{E_0C}{1-LC\omega^2}\end{array}$$

$$q_p(t)=\frac{E_0C}{1-LC\omega^2}\sin\omega t$$

$$q(t)=C_1\cos\frac{1}{\sqrt{LC}}t+C_2\sin\frac{1}{\sqrt{LC}}t+\frac{E_0C}{1-LC\omega^2}\sin\omega t$$

$$q(0)=q_0\ \rightarrow\ C_1=0$$

$$q'(t)=-\frac{1}{\sqrt{LC}}C_1\sin\frac{1}{\sqrt{LC}}t+\frac{1}{\sqrt{LC}}C_2\cos\frac{1}{\sqrt{LC}}t+\frac{E_0C\omega}{1-LC\omega^2}\cos\omega t$$

$$q'(0)=i_0\ \rightarrow\frac{1}{\sqrt{LC}}C_2+\frac{E_0C\omega}{1-LC\omega^2}=i_0\quad C_2=\sqrt{LC}\ i_0-\frac{E_0C\omega\sqrt{LC}}{1-LC\omega^2}$$

$$q(t)=\sqrt{LC}\left(i_0-\frac{E_0C\omega}{1-LC\omega^2}\right)\sin\frac{1}{\sqrt{LC}}t+\frac{E_0C}{1-LC\omega^2}\sin\omega t$$

$$i(t)=\left(i_0-\frac{E_0C\omega}{1-LC\omega^2}\right)\cos\frac{1}{\sqrt{LC}}t+\frac{E_0C\omega}{1-LC\omega^2}\cos\omega t$$

### Exercise

Find the charge $q(t)$ and $i(t)$ on the capacitor in an $LC$–series circuit when

$$E(t)=E_0\cos\omega t \ V, \quad q(0)=q_0 \ C, \quad and \quad i(0)=i_0 \ A$$

**Solution**

$$Lq''+\frac{1}{C}q=E_0\sin\omega t \ ; \quad q(0)=q_0 \quad i(0)=i_0 \qquad\qquad Lq''+Rq'+\frac{1}{C}q=E(t)$$

$$\lambda^2+\frac{1}{LC}=0 \ \rightarrow \ \underline{\lambda_{1,2}=\pm\frac{1}{\sqrt{LC}}i}$$

$$\underline{q_h(t)=C_1\cos\frac{1}{\sqrt{LC}}t+C_2\sin\frac{1}{\sqrt{LC}}t}$$

$$q_p=A\cos\omega t+B\sin\omega t$$

$$q'_p=-\omega A\sin\omega t+\omega B\cos\omega t$$

$$q''_p=-\omega^2A\cos\omega t-\omega^2B\sin\omega t$$

$$Lq''+\frac{1}{C}q=E_0\cos\omega t$$

$$\begin{cases}\cos\omega t & \left(-L\omega^2+\frac{1}{C}\right)A=E_0\\ \sin\omega t & \left(-L\omega^2+\frac{1}{C}\right)B=0\end{cases} \ \rightarrow \ \begin{array}{l}\underline{A=\dfrac{E_0C}{1-LC\omega^2}}\\ \underline{B=0}\end{array}$$

$$\underline{q_p(t)=\frac{E_0C}{1-LC\omega^2}\cos\omega t}$$

$$q(t)=C_1\cos\frac{1}{\sqrt{LC}}t+C_2\sin\frac{1}{\sqrt{LC}}t+\frac{E_0C}{1-LC\omega^2}\cos\omega t$$

$$q(0)=q_0 \ \rightarrow \ \underline{C_1=q_0-\frac{E_0C}{1-LC\omega^2}}$$

$$q'(t)=-\frac{1}{\sqrt{LC}}C_1\sin\frac{1}{\sqrt{LC}}t+\frac{1}{\sqrt{LC}}C_2\cos\frac{1}{\sqrt{LC}}t-\frac{E_0C\omega}{1-LC\omega^2}\sin\omega t$$

$$q'(0)=i_0 \ \rightarrow \frac{1}{\sqrt{LC}}C_2=i_0 \quad \underline{C_2=i_0\sqrt{LC}}$$

$$\underline{q(t)=\left(q_0-\frac{E_0C}{1-LC\omega^2}\right)\cos\frac{1}{\sqrt{LC}}t+i_0\sqrt{LC}\sin\frac{1}{\sqrt{LC}}t+\frac{E_0C}{1-LC\omega^2}\cos\omega t}$$

$$\underline{i(t)=-\frac{1}{\sqrt{LC}}\left(q_0-\frac{E_0C}{1-LC\omega^2}\right)\sin\frac{1}{\sqrt{LC}}t+i_0\cos\frac{1}{\sqrt{LC}}t-\frac{E_0C\omega}{1-LC\omega^2}\sin\omega t}$$

### *Exercise*

Find the charge $q(t)$ and $i(t)$ on the capacitor in an $LC$–series circuit when

$$L = 0.1\ h, \quad C = 0.1\ f, \quad E(t) = 100\sin\omega t\ V, \quad q(0) = 0\ C, \quad \text{and} \quad i(0) = 0\ A$$

**Solution**

$$0.1q'' + \frac{1}{0.1}q = 100\sin\omega t \qquad\qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 100q = 1{,}000\sin\omega t\ ; \quad q(0) = 0 \quad i(0) = 0$$

$$\lambda^2 + 100 = 0 \quad \to \quad \boxed{\lambda_{1,2} = \pm 10i}$$

$$\boxed{q_h(t) = C_1\cos 10t + C_2\sin 10t}$$

$$q_p = A\cos\omega t + B\sin\omega t$$

$$q'_p = -\omega A\sin\omega t + \omega B\cos\omega t$$

$$q''_p = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t$$

$$q'' + 100q = 1{,}000\sin\omega t$$

$$\begin{cases} \cos\omega t & \left(-\omega^2 + 100\right)A = 0 \\ \sin\omega t & \left(-\omega^2 + 100\right)B = 1{,}000 \end{cases} \quad \to \quad \boxed{A = 0} \quad \boxed{B = \frac{1000}{100 - \omega^2}}$$

$$\boxed{q_p(t) = \frac{1000}{100 - \omega^2}\sin\omega t}$$

$$q(t) = C_1\cos 10t + C_2\sin 10t + \frac{1000}{100 - \omega^2}\sin\omega t$$

$$q(0) = 0 \quad \to \quad \boxed{C_1 = 0}$$

$$q'(t) = -10C_1\sin 10t + 10C_2\cos 10t + \frac{1000\omega}{100 - \omega^2}\cos\omega t$$

$$q'(0) = 0 \quad \to 10C_2 + \frac{1000\omega}{100 - \omega^2} = 0 \quad \boxed{C_2 = -\frac{100\omega}{100 - \omega^2}}$$

$$q(t) = -\frac{100}{100 - \omega^2}\sin 10t + \frac{1000}{100 - \omega^2}\sin\omega t$$

$$\boxed{= \frac{100}{100 - \omega^2}(10\sin\omega t - \sin 10t)}$$

$$\boxed{i(t) = \frac{1000}{100 - \omega^2}(\omega\cos\omega t - \cos 10t)}$$

### Exercise

Find the charge $q(t)$ and $i(t)$ on the capacitor in an $LC$–series circuit when

$$L=1\ H,\quad C=4\ \mu F,\quad E(t)=3\sin 3t\ V,\quad q(0)=0\ C,\quad and\quad i(0)=0\ A$$

***Solution***

$$q''+\frac{1}{4}q=3\sin 3t\ ;\quad q(0)=0\quad i(0)=0 \qquad\qquad Lq''+Rq'+\frac{1}{C}q=E(t)$$

$$\lambda^2+\frac{1}{4}=0 \quad\rightarrow\quad \boxed{\lambda_{1,2}=\pm\frac{1}{2}i}$$

$$\boxed{q_h(t)=C_1\cos\frac{1}{2}t+C_2\sin\frac{1}{2}t}$$

$$q_p=A\cos 3t+B\sin 3t$$

$$q'_p=-3A\sin 3t+3B\cos 3t$$

$$q''_p=-9A\cos 3t-9B\sin 3t$$

$$4q''+q=12\sin 3t$$

$$\begin{cases}\cos 3t & -36A+A=0\\ \sin 3t & -36B+B=12\end{cases} \quad\rightarrow\quad \boxed{A=0} \quad \boxed{B=-\frac{12}{35}}$$

$$\boxed{q_p=-\frac{12}{35}\sin 3t}$$

$$q(t)=C_1\cos\frac{1}{2}t+C_2\sin\frac{1}{2}t-\frac{12}{35}\sin 3t$$

$$q(0)=0 \quad\rightarrow\quad \boxed{C_1=0}$$

$$q'(t)=-\frac{1}{2}C_1\sin\frac{1}{2}t+\frac{1}{2}C_2\cos\frac{1}{2}t-\frac{36}{35}\cos 3t$$

$$q'(0)=0 \quad\rightarrow \frac{1}{2}C_2-\frac{36}{35}=0 \quad\Rightarrow\quad \boxed{C_2=\frac{72}{35}}$$

$$\boxed{q(t)=\frac{72}{35}\sin\frac{1}{2}t-\frac{12}{35}\sin 3t}$$

$$\boxed{i(t)=q'(t)=\frac{36}{35}\cos\frac{1}{2}t-\frac{4}{35}\cos 3t}$$

### Exercise

Find the charge $q(t)$ and $i(t)$ on the capacitor in an $LC$–series circuit when

$$L=1\ H,\quad C=4\ \mu F,\quad E(t)=10te^{-t}\ V,\quad q(0)=0\ C,\quad and\quad i(0)=0\ A$$

***Solution***

$$q''+\frac{1}{4}q=10te^{-t}\ ;\quad q(0)=0\quad i(0)=0 \qquad\qquad Lq''+Rq'+\frac{1}{C}q=E(t)$$

$$\lambda^2+\frac{1}{4}=0 \quad\rightarrow\quad \boxed{\lambda_{1,2}=\pm\frac{1}{2}i}$$

$$\boxed{q_h(t) = C_1 \cos\frac{t}{2} + C_2 \sin\frac{t}{2}}$$

$$q_p = (At + B)e^{-t}$$

$$q'_p = (A - B - At)e^{-t}$$

$$q''_p = (-2A + B + At)e^{-t}$$

$$4q'' + q = 40te^{-t}$$

$$\begin{cases} e^{-t} & t \quad 4A + A = 40 \\ & t^0 \quad -8A + 4B + B = 0 \end{cases} \rightarrow \quad \boxed{A = 8} \quad \boxed{B = \frac{64}{5}}$$

$$\boxed{q_p(t) = \left(8t + \frac{64}{5}\right)e^{-t}}$$

$$q(t) = C_1 \cos\frac{t}{2} + C_2 \sin\frac{t}{2} + \left(8t + \frac{64}{5}\right)e^{-t}$$

$$q(0) = 0 \quad \rightarrow \quad \boxed{C_1 = -\frac{64}{5}}$$

$$q'(t) = -\frac{1}{2}C_1 \sin\frac{1}{2}t + \frac{1}{2}C_2 \cos\frac{1}{2}t - \left(8t + \frac{24}{5}\right)e^{-t}$$

$$q'(0) = 0 \quad \rightarrow \frac{1}{2}C_2 - \frac{24}{5} = 0 \quad \Rightarrow \boxed{C_2 = \frac{48}{5}}$$

$$\boxed{q(t) = -\frac{64}{5}\cos\frac{t}{2} + \frac{48}{5}\sin\frac{t}{2} + \left(8t + \frac{64}{5}\right)e^{-t}}$$

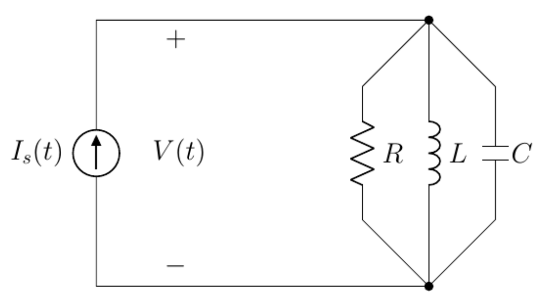
$$\boxed{i(t) = q'(t) = \frac{32}{5}\sin\frac{t}{2} + \frac{24}{5}\cos\frac{t}{2} - \left(8t + \frac{24}{5}\right)e^{-t}}$$

## Exercise

Consider the parallel *RLC* network. Assume that at time $t = 0$, the voltage $V(t)$ and its time rate of change are both zero. Determine the voltage $V(t)$ for

$$R = 1\ k\Omega, \quad L = 1\ H, \quad C = \frac{1}{2}\ \mu F, \quad I_s(t) = 1 - e^{-t}\ mA$$

**Solution**




$$I_S(t) = \frac{1}{R}V + C\frac{dV}{dt} + \frac{1}{L}\int V(s)\,ds$$

$$\frac{d}{dt}\left(I_S(t) = \frac{1}{R}V + C\frac{dV}{dt} + \frac{1}{L}\int V(s)\,ds\right)$$

$$\frac{d}{dt}I_S(t) = \frac{1}{R}\frac{dV}{dt} + C\frac{d^2V}{dt^2} + \frac{1}{L}V(s)$$

$$\frac{d^2V}{dt^2}+\frac{1}{RC}\frac{dV}{dt}+\frac{1}{LC}V=\frac{1}{C}\frac{dI}{dt}$$

$$V''+2V'+2V=2\frac{d}{dt}\left(1-e^{-t}\right)$$

$$V''+2V'+2V=2e^{-t}\ ;\quad V(0)=0,\quad V'(0)=0$$

$$\lambda^2+2\lambda+2=0\quad\to\quad \boxed{\lambda_{1,2}=-1\pm i}$$

$$\boxed{V_h(t)=e^{-t}\left(C_1\cos t+C_2\sin t\right)}$$

$$V_p=Ae^{-t}$$

$$V_p'=-Ae^{-t}$$

$$V_p''=Ae^{-t}$$

$$V''+2V'+2V=2e^{-t}$$

$$e^{-t}\quad A-2A+2A=2\quad\to\quad \boxed{A=2}$$

$$\boxed{V_p(t)=2e^{-t}}$$

$$V(t)=e^{-t}\left(C_1\cos t+C_2\sin t\right)+2e^{-t}$$

$$V(0)=0\quad\to C_1+2=0\quad\Rightarrow\quad \boxed{C_1=-2}$$

$$V'(t)=e^{-t}\left(-C_1\sin t+C_2\cos t-C_1\cos t-C_2\sin t\right)-2e^{-t}$$

$$V'(0)=0\quad\to C_2+2-2=0\quad\Rightarrow\quad \boxed{C_2=0}$$

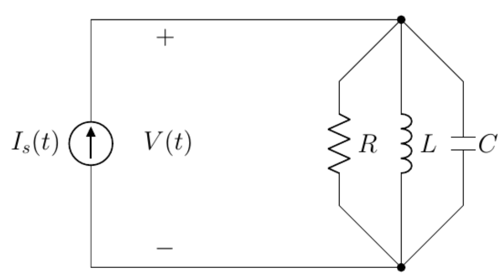
$$\boxed{V(t)=-2e^{-t}\cos t+2e^{-t}}$$

## *Exercise*

Consider the parallel *RLC* network. Assume that at time $t=0$, the voltage $V(t)$ and its time rate of change are both zero. Determine the voltage $V(t)$ for

$$R=1\ k\Omega,\quad L=1\ H,\quad C=\frac{1}{2}\ \mu F,\quad I_s(t)=5\sin t\ \ mA$$

***Solution***




$$I_S(t)=\frac{1}{R}V+C\frac{dV}{dt}+\frac{1}{L}\int V(s)\,ds$$

$$\frac{d}{dt}\left(I_S(t)=\frac{1}{R}V+C\frac{dV}{dt}+\frac{1}{L}\int V(s)\,ds\right)$$

$$\frac{d}{dt}I_S(t)=\frac{1}{R}\frac{dV}{dt}+C\frac{d^2V}{dt^2}+\frac{1}{L}V(s)$$

$$\frac{d^2V}{dt^2}+\frac{1}{RC}\frac{dV}{dt}+\frac{1}{LC}V=\frac{1}{C}\frac{dI}{dt}$$

$$V''+2V'+2V=2\frac{d}{dt}(5\sin t)$$

$$V''+2V'+2V=10\cos t\ ; \quad V(0)=0, \quad V'(0)=0$$

$$\lambda^2+2\lambda+2=0 \quad \to \quad \boxed{\lambda_{1,2}=-1\pm i}$$

$$\boxed{V_h(t)=e^{-t}\left(C_1\cos t+C_2\sin t\right)}$$

$$V_p=A\cos t+B\sin t$$

$$V'_p=-A\sin t+B\cos t$$

$$V''_p=-A\cos t-B\sin t$$

$$V''+2V'+2V=10\cos t$$

$$\begin{cases}\cos t & -A+2B+2A=10\\ \sin t & -B-2A+2B=0\end{cases} \quad \to \begin{cases}A+2B=10\\ -2A+B=0\end{cases}$$

$$\Delta=\begin{vmatrix}1 & 2\\ -2 & 1\end{vmatrix}=5 \quad \Delta_A=\begin{vmatrix}10 & 2\\ 0 & 1\end{vmatrix}=10 \quad \Delta_B=\begin{vmatrix}1 & 10\\ -2 & 0\end{vmatrix}=20$$

$$\boxed{A=2, \quad B=4}$$

$$\boxed{V_p(t)=2\cos t+4\sin t}$$

$$V(t)=e^{-t}\left(C_1\cos t+C_2\sin t\right)+2\cos t+4\sin t$$

$$V(0)=0 \quad \to C_1+2=0 \quad \Rightarrow \quad \boxed{C_1=-2}$$

$$V'(t)=e^{-t}\left(-C_1\sin t+C_2\cos t-C_1\cos t-C_2\sin t\right)-2\sin t+4\cos t$$

$$V'(0)=0 \quad \to C_2+2+4=0 \quad \Rightarrow \quad \boxed{C_2=-6}$$

$$\boxed{V(t)=e^{-t}(-2\cos t-6\sin t)+2\cos t+4\sin t}$$

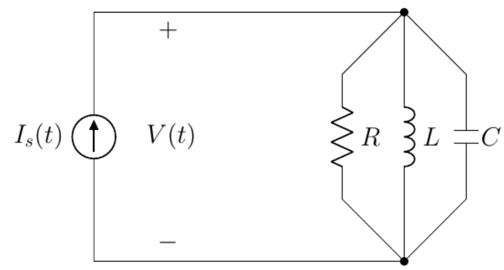
## *Exercise*

Consider the parallel *RLC* network. Assume that at time $t=0$, the voltage $V(t)$ and its time rate of change are both zero. Determine the voltage $V(t)$ for

$$R=1\ k\Omega, \quad L=1\ H, \quad C=\frac{1}{2}\ \mu F, \quad I_s(t)=5\cos t\ \ mA$$

**Solution**

$$I_S(t)=\frac{1}{R}V+C\frac{dV}{dt}+\frac{1}{L}\int V(s)\,ds$$




$$\frac{d}{dt}\left(I_S(t)=\frac{1}{R}V+C\frac{dV}{dt}+\frac{1}{L}\int V(s)\,ds\right)$$

$$\frac{d}{dt}I_S(t)=\frac{1}{R}\frac{dV}{dt}+C\frac{d^2V}{dt^2}+\frac{1}{L}V(s)$$

$$\frac{d^2V}{dt^2}+\frac{1}{RC}\frac{dV}{dt}+\frac{1}{LC}V=\frac{1}{C}\frac{dI}{dt}$$

$$V''+2V'+2V=2\frac{d}{dt}(5\cos t)$$

$$V''+2V'+2V=-10\sin t\ ;\quad V(0)=0,\quad V'(0)=0$$

$$\lambda^2+2\lambda+2=0\quad\rightarrow\quad \boxed{\lambda_{1,2}=-1\pm i}$$

$$\boxed{V_h(t)=e^{-t}\left(C_1\cos t+C_2\sin t\right)}$$

$$V_p=A\cos t+B\sin t$$

$$V_p'=-A\sin t+B\cos t$$

$$V_p''=-A\cos t-B\sin t$$

$$V''+2V'+2V=-10\sin t$$

$$\begin{cases}\cos t & -A+2B+2A=0\\ \sin t & -B-2A+2B=-10\end{cases}\rightarrow\begin{cases}A+2B=0\\ -2A+B=-10\end{cases}$$

$$\Delta=\begin{vmatrix}1&2\\-2&1\end{vmatrix}=5\quad \Delta_A=\begin{vmatrix}0&2\\-10&1\end{vmatrix}=20\quad \Delta_B=\begin{vmatrix}1&0\\-2&-10\end{vmatrix}=-10$$

$$\boxed{A=4,\quad B=-2}$$

$$\boxed{V_p(t)=4\cos t-2\sin t}$$

$$V(t)=e^{-t}\left(C_1\cos t+C_2\sin t\right)+4\cos t-2\sin t$$

$$V(0)=0\quad\rightarrow C_1+4=0\quad\Rightarrow\quad \boxed{C_1=-4}$$

$$V'(t)=e^{-t}\left(-C_1\sin t+C_2\cos t-C_1\cos t-C_2\sin t\right)-4\sin t-2\cos t$$

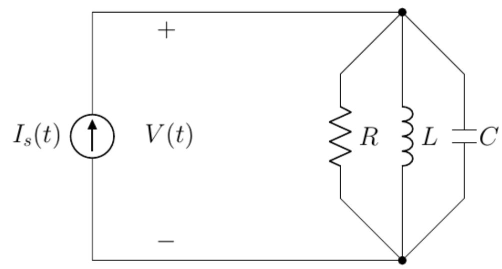
$$V'(0)=0\quad\rightarrow -C_2+4-2=0\quad\Rightarrow\quad \boxed{C_2=2}$$

$$\boxed{V(t)=e^{-t}\left(-4\cos t-2\sin t\right)+4\cos t-2\sin t}$$

## *Exercise*

Consider the parallel *RLC* network. Assume that at time $t=0$, the voltage $V(t)$ and its time rate of change are both zero. Determine the voltage $V(t)$ for

$$R = 2\ k\Omega, \quad L = 1\ H, \quad C = \frac{1}{4}\ \mu F, \quad I_s(t) = e^{-t}\ \ mA$$




**Solution**

$$I_S(t) = \frac{1}{R}V + C\frac{dV}{dt} + \frac{1}{L}\int V(s)\,ds$$

$$\frac{d}{dt}\left( I_S(t) = \frac{1}{R}V + C\frac{dV}{dt} + \frac{1}{L}\int V(s)\,ds \right)$$

$$\frac{d}{dt}I_S(t) = \frac{1}{R}\frac{dV}{dt} + C\frac{d^2V}{dt^2} + \frac{1}{L}V(s)$$

$$\frac{d^2V}{dt^2} + \frac{1}{RC}\frac{dV}{dt} + \frac{1}{LC}V = \frac{1}{C}\frac{dI}{dt}$$

$$V'' + 2V' + 4V = 4\frac{d}{dt}\left(e^{-t}\right)$$

$$V'' + 2V' + 4V = -4e^{-t}\ ; \quad V(0) = 0, \quad V'(0) = 0$$

$$\lambda^2 + 2\lambda + 4 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -1 \pm i\sqrt{3}}$$

$$\boxed{V_h(t) = e^{-t}\left(C_1\cos\sqrt{3}t + C_2\sin\sqrt{3}t\right)}$$

$$V_p = Ae^{-t}$$

$$V_p' = -Ae^{-t}$$

$$V_p'' = Ae^{-t}$$

$$V'' + 2V' + 4V = -4e^{-t}$$

$$e^{-t} \quad A - 2A + 4A = -4 \quad \rightarrow \quad \boxed{A = -\frac{4}{3}}$$

$$\boxed{V_p(t) = -\frac{4}{3}e^{-t}}$$

$$V(t) = e^{-t}\left(C_1\cos\sqrt{3}t + C_2\sin\sqrt{3}t\right) - \frac{4}{3}e^{-t}$$

$$V(0) = 0 \quad \rightarrow C_1 - \frac{4}{3} = 0 \quad \Rightarrow \quad \boxed{C_1 = \frac{4}{3}}$$

$$V'(t) = e^{-t}\left(-\sqrt{3}C_1\sin\sqrt{3}t + \sqrt{3}C_2\cos\sqrt{3}t - C_1\cos\sqrt{3}t - C_2\sin\sqrt{3}t\right) + \frac{4}{3}e^{-t}$$

$$V'(0) = 0 \quad \rightarrow \sqrt{3}C_2 - \frac{4}{3} + \frac{4}{3} = 0 \quad \Rightarrow \quad \boxed{C_2 = 0}$$

$$\boxed{V(t) = \frac{4}{3}e^{-t}\cos\sqrt{3}t - \frac{4}{3}e^{-t}}$$

### Exercise

An *RCL* circuit connected in series has $R = 180\ \Omega,\quad C = \frac{1}{280}\ F,\quad L = 20\ H$, and applied voltage $E(t) = 10\sin t\ V$. Assuming no initial charge on the capacitor, but an initial current of $1\ A$ at $t = 0$ when the voltage is first applied.

*a)* Find the subsequent charge on the capacitor.

*b)* Plot the *transient*, *steady-state*, and the charge on the capacitor.

*c)* Find the current on the capacitor.

**Solution**

***a)*** $20q'' + 180q' + 280q = 10\sin t$ $\qquad\qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$

$q'' + 9q' + 14q = \frac{1}{2}\sin t\ ;\quad q(0) = 0\quad i(0) = 1$

$$\lambda^2 + 9\lambda + 14 = 0 \quad\rightarrow\quad \lambda_{1,2} = \frac{-9 \pm 5}{2} = -7,\ -2$$

$$\boxed{q_h(t) = C_1 e^{-7t} + C_2 e^{-2t}}$$

$q_p = A\cos t + B\sin t$

$q'_p = -A\sin t + B\cos t$

$q''_p = -A\cos t - B\sin t$

$q'' + 9q' + 14q = \frac{1}{2}\sin t$

$$\begin{cases} \cos t & -A + 9B + 14A = 0 \\ \sin t & -B - 9A + 14B = \frac{1}{2} \end{cases} \rightarrow \begin{cases} 13A + 9B = 0 \\ -9A + 13B = \frac{1}{2} \end{cases}$$

$$\Delta = \begin{vmatrix} 13 & 9 \\ -9 & 13 \end{vmatrix} = 250 \quad \Delta_A = \begin{vmatrix} 0 & 9 \\ \frac{1}{2} & 13 \end{vmatrix} = -\frac{9}{2} \quad \Delta_B = \begin{vmatrix} 13 & 0 \\ -9 & \frac{1}{2} \end{vmatrix} = \frac{13}{2}$$

$$\boxed{A = -\frac{9}{500},\quad B = \frac{13}{500}}$$

$$\boxed{q_p(t) = -\frac{9}{500}\cos t + \frac{13}{500}\sin t}$$

$$q(t) = C_1 e^{-7t} + C_2 e^{-2t} - \frac{9}{500}\cos t + \frac{13}{500}\sin t$$

$q(0) = 0 \quad\rightarrow\quad C_1 + C_2 = \frac{9}{500}$

$$q' = -7C_1 e^{-7t} - 2C_2 e^{-2t} + \frac{9}{500}\sin t + \frac{13}{500}\cos t$$

$q'(0) = 1 \quad\rightarrow\quad -7C_1 - 2C_2 = 1 - \frac{13}{500} = \frac{487}{500}$

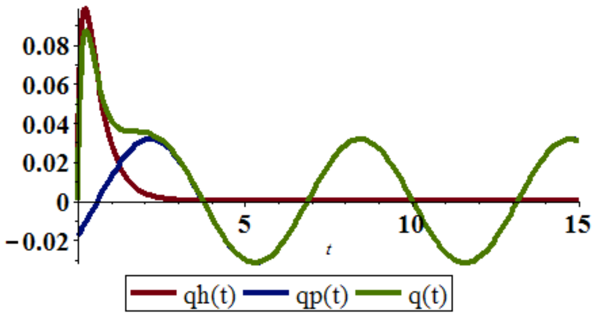
$$\Delta = \begin{vmatrix} 1 & 1 \\ -7 & -2 \end{vmatrix} = 5 \quad \Delta_{C_1} = \begin{vmatrix} \frac{9}{500} & 1 \\ \frac{487}{500} & -2 \end{vmatrix} = -\frac{505}{500} \quad \Delta_{C_2} = \begin{vmatrix} 1 & \frac{9}{500} \\ -7 & \frac{487}{500} \end{vmatrix} = \frac{550}{500}$$

$$\underline{C_1 = -\frac{101}{500} \quad C_2 = \frac{110}{500}}$$

$$\underline{q(t) = -\frac{101}{500}e^{-7t} + \frac{11}{50}e^{-2t} - \frac{9}{500}\cos t + \frac{13}{500}\sin t}$$

The solution is the sum of *transient* and *steady-state* terms

***b)***




***c)*** $\underline{i(t) = \frac{707}{500}e^{-7t} - \frac{22}{50}e^{-2t} + \frac{9}{500}\sin t + \frac{13}{500}\cos t}$

## *Exercise*

An *RCL* circuit connected in series has $R = 10\ \Omega,\quad C = 10^{-2}\ F,\quad L = \frac{1}{2}\ H$, and applied voltage $E(t) = 12\ V$. Assuming no initial charge and no initial current at $t = 0$ when the voltage is first applied.

*a)* Find the subsequent charge on the capacitor.

*b)* Plot the *transient*, *steady-state*, and the charge on the capacitor.

*c)* Find the current on the capacitor.

***Solution***

***a)*** $\frac{1}{2}q'' + 10q' + 100q = 12$ $\qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$

$q'' + 20q' + 200q = 24\ ;\quad q(0) = 0 \quad i(0) = 0$

$\lambda^2 + 20\lambda + 200 = 0 \quad \rightarrow \quad \underline{\lambda_{1,2} = -10 \pm 10i}$

$$\underline{q_h(t) = e^{-10t}\left(C_1 \cos 10t + C_2 \sin 10t\right)}$$

$q_p = A$

$q'_p = q''_p = 0$

$q'' + 20q' + 200q = 24$

$$200A = 24 \quad \rightarrow \quad \boxed{A = \frac{3}{25}}$$

$$\boxed{q_p = \frac{3}{25}}$$

$$q(t) = e^{-10t}\left(C_1 \cos 10t + C_2 \sin 10t\right) + \frac{3}{25}$$

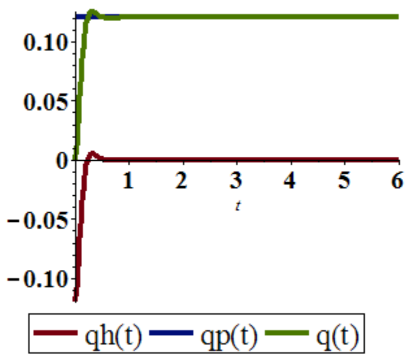
$$q(0) = 0 \quad \rightarrow \quad \boxed{C_1 = -\frac{3}{25}}$$

$$q' = e^{-10t}\left(-10C_1 \cos 10t - 10C_2 \sin 10t - 10C_1 \sin 10t + 10C_2 \cos 10t\right)$$

$$q'(0) = 0 \quad \rightarrow -10C_1 + 10C_2 = 0 \quad \Rightarrow \boxed{C_1 = -\frac{3}{25}}$$

$$\boxed{q(t) = -\frac{3}{25}e^{-10t}\left(\cos 10t + \sin 10t\right) + \frac{3}{25}}$$

***b)***




***c)*** $I(t) = q' = -\frac{3}{25}e^{-10t}\left(-10\cos 10t - 10\sin 10t - 10\sin 10t + 10\cos 10t\right)$

$$\boxed{= \frac{12}{5}e^{-10t}\sin 10t}$$

$\therefore$ This is a completely transient

## *Exercise*

An *RCL* circuit connected in series has $R = 5\ \Omega,\quad C = 4\times 10^{-4}\ F,\quad L = 0.05\ H$, and applied voltage $E(t) = 200\cos 100t\ V$. Assuming no initial charge and no initial current at $t = 0$ when the voltage is first applied.

*a)* Find the subsequent charge on the capacitor.
*b)* Plot the *transient*, *steady-state*, and the charge on the capacitor.
*c)* Find the current flowing through this circuit.

***Solution***

***a)*** $.05q'' + 5q' + \frac{1}{4\times 10^{-4}}q = 200\cos 100t \qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$

$$q'' + 100q' + 5\times 10^4 q = 4{,}000\cos 100t\ ; \quad q(0) = 0 \quad i(0) = 0$$

$$\lambda^2 + 100\lambda + 5\times 10^4 = 0 \quad \rightarrow \quad \lambda_{1,2} = -50 \pm 50i\sqrt{19}$$

$$\boxed{q_h(t) = e^{-50t}\left(C_1 \cos 50\sqrt{19}t + C_2 \sin 50\sqrt{19}t\right)}$$

$$q_p = A\cos 100t + B\sin 100t$$

$$q_p' = -100A\sin 100t + 100B\cos 100t$$

$$q_p'' = -10^4 A\cos 100t - 10^4 B\sin 100t$$

$$q'' + 100q' + 5\times 10^4 q = 4{,}000\cos 100t$$

$$\begin{cases} \cos 100t & -10^4 A + 10^4 B + 5\times 10^4 A = 4\times 10^3 \\ \sin 100t & -10^4 B - 10^4 A + 5\times 10^4 B = 0 \end{cases} \rightarrow \begin{cases} 4A + B = \frac{2}{5} \\ -A + 4B = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 4 & 1 \\ -1 & 4 \end{vmatrix} = 17 \quad \Delta_A = \begin{vmatrix} \frac{2}{5} & 1 \\ 0 & 4 \end{vmatrix} = \frac{8}{5} \quad \Delta_B = \begin{vmatrix} 4 & \frac{2}{5} \\ -1 & 0 \end{vmatrix} = \frac{2}{5}$$

$$\boxed{A = \frac{8}{85} \quad B = \frac{2}{85}}$$

$$\boxed{q_p(t) = \frac{8}{85}\cos 100t + \frac{2}{85}\sin 100t}$$

$$q(t) = e^{-50t}\left(C_1 \cos 50\sqrt{19}t + C_2 \sin 50\sqrt{19}t\right) + \frac{8}{85}\cos 100t + \frac{2}{85}\sin 100t$$

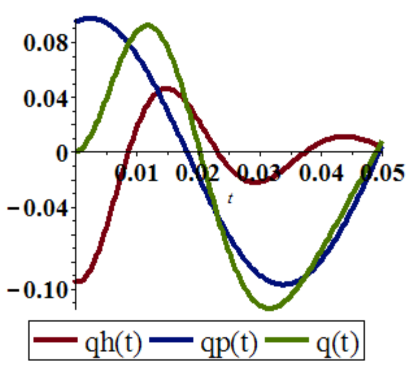
$$q(0) = 0 \quad \rightarrow \quad \boxed{C_1 = -\frac{8}{85}}$$

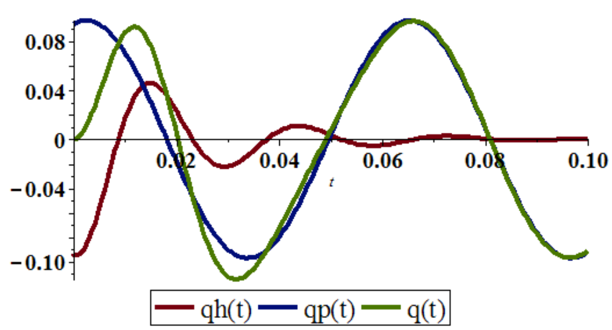
$$q'(t) = e^{-50t}\left(-50C_1 \cos 50\sqrt{19}t - 50\text{in } 50\sqrt{19}t - 50\sqrt{19}C_1 \sin 50\sqrt{19}t + 50\sqrt{19}C_2 \cos 50\sqrt{19}t\right)$$
$$+ \frac{160}{17}\sin 100t + \frac{40}{17}\cos 100t$$

$$q'(0) = 0 \quad \rightarrow -50C_1 + 50\sqrt{19}C_2 + \frac{40}{17} = 0 \quad \Rightarrow \boxed{C_2 = -\frac{12}{85\sqrt{19}}}$$

$$\boxed{q(t) = e^{-50t}\left(-\frac{8}{85}\cos 50\sqrt{19}t - \frac{12}{85\sqrt{19}}\sin 50\sqrt{19}t\right) + \frac{8}{85}\cos 100t + \frac{2}{85}\sin 100t}$$

***b)***



***c)***

$$I(t)=q'=-\frac{200}{85}e^{-50t}\left(-2\cos 50\sqrt{19}t-\frac{3}{\sqrt{19}}\sin 50\sqrt{19}t-2\sqrt{19}\sin 50\sqrt{19}t+3\cos 50\sqrt{19}t\right)$$
$$+\frac{160}{17}\sin 100t+\frac{40}{17}\cos 100t$$

$$\boxed{I(t)=-\frac{200}{85}e^{-50t}\left(\cos 50\sqrt{19}t-\frac{41}{\sqrt{19}}\sin 50\sqrt{19}t\right)+\frac{160}{17}\sin 100t+\frac{40}{17}\cos 100t}$$

## Exercise

An *RCL* circuit connected in series has $R=40\ \Omega,\quad C=16\times 10^{-4}\ F,\quad L=1\ H$, and applied voltage $E(t)=100\cos 10t\ V$. Assuming no initial charge and no initial current at $t=0$ when the voltage is first applied.

*a)* Find the charge in the circuit at time *t*.

*b)* Find the current flowing through this circuit.

*c)* Find the linit of the charge as $t\to\infty$

**Solution**

***a)*** $q''+40q'+\frac{10^4}{16}q=100\cos 10t \qquad Lq''+Rq'+\frac{1}{C}q=E(t)$

$q''+40q'+625q=100\cos 10t\ ;\quad q(0)=0,\quad i(0)=0$

$\lambda^2+40\lambda+625=0 \rightarrow \boxed{\lambda_{1,2}=-20\pm 15i}$

$$\boxed{q_h=e^{-20t}\left(C_1\cos 15t+C_2\sin 15t\right)}$$

$q_p=A\cos 10t+B\sin 10t$

$q'_p=-10A\sin 10t+10B\cos 10t$

$q''_p=-100A\cos 10t-100B\sin 10t$

$q''+40q'+625q=100\cos 10t$

$$\begin{cases}\cos 10t & -100A+400B+625A=100\\ \sin 10t & -100B-400A+625B=0\end{cases}$$

$$\rightarrow\begin{cases}525A+400B=100\\ -400A+525B=0\end{cases}\rightarrow \quad A=\frac{21}{16}B \qquad \left(\frac{11025}{16}+400\right)B=100$$

$$\boxed{B=\frac{64}{697},\quad A=\frac{84}{697}}$$

$$\boxed{q_p=\frac{84}{697}\cos 10t+\frac{64}{697}\sin 10t}$$

$$q(t) = e^{-20t}\left(C_1 \cos 15t + C_2 \sin 15t\right) + \frac{84}{697}\cos 10t + \frac{64}{697}\sin 10t$$

$$q(0) = 0 \quad \rightarrow \quad \boxed{C_1 = -\frac{84}{697}}$$

$$q' = e^{-20t}\left(-20C_1 \cos 15t - 20C_2 \sin 15t - 15C_1 \sin 15t + 15C_2 \cos 15t\right) - \frac{840}{697}\sin 10t + \frac{640}{697}\cos 10t$$

$$q'(0) = 0 \quad \rightarrow 20\frac{84}{697} + 15C_2 + \frac{640}{697} = 0 \quad \Rightarrow \boxed{C_2 = -\frac{464}{2091}}$$

$$\boxed{q(t) = e^{-20t}\left(-\frac{84}{697}\cos 15t - \frac{464}{2091}\sin 15t\right) + \frac{84}{697}\cos 10t + \frac{64}{697}\sin 10t}$$

**b)**

$$I(t) = q' = -\frac{1}{2091}e^{-20t}\left(-5040\cos 15t - 9280\sin 15t - 3780\sin 15t + 6960\cos 15t\right)$$

$$-\frac{840}{697}\sin 10t + \frac{640}{697}\cos 10t$$

$$\boxed{= -\frac{1}{2091}e^{-20t}\left(1{,}920\cos 15t - 13{,}060\sin 15t\right) - \frac{840}{697}\sin 10t + \frac{640}{697}\cos 10t}$$

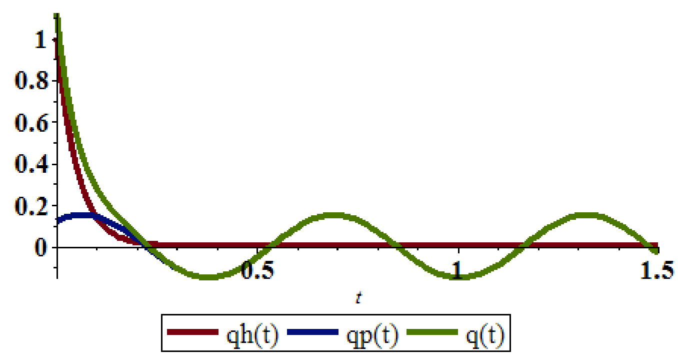
**c)**

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty}\left(e^{-20t}\left(-\frac{84}{697}\cos 15t - \frac{464}{2091}\sin 15t\right) + \frac{84}{697}\cos 10t + \frac{64}{697}\sin 10t\right)$$

$$\lim_{t\to\infty} e^{-20t}\left(-\frac{84}{697}\cos 15t - \frac{464}{2091}\sin 15t\right) \boxed{= 0}$$

$q_p$ : is *steady state* solution

$$\lim_{t\to\infty}\left(\frac{84}{697}\cos 10t + \frac{64}{697}\sin 10t\right) = \nexists$$




## Exercise

A series circuit consists of a resistor with $R = 20\ \Omega$, an inductor with $L = 1\ H$, a capacitor with $C = 0.002\ F$, and a 12-$V$ battery. If the initial charge and current are both 0, find the charge and current at time $t$.

***Solution***

$$q'' + 20q' + \frac{1}{.002}q = 12 \qquad\qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 20q' + 500q = 12\ ; \quad q(0) = 0, \quad i(0) = 0$$

$$\lambda^2 + 20\lambda + 500 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -10 \pm 20i}$$

$$\boxed{q_h = e^{-10t}\left(C_1\cos 20t + C_2\sin 20t\right)}$$

$$q_p = A$$

$$q'_p = q''_p = 0$$

$$q'' + 20q' + 500q = 12 \quad \rightarrow \quad 500A = 12 \;\Rightarrow\; \boxed{A = \frac{3}{125}}$$

$$\boxed{q_p = \frac{3}{125}}$$

$$q(t) = e^{-10t}\left(C_1\cos 20t + C_2\sin 20t\right) + \frac{3}{125}$$

$$q(0) = 0 \quad \rightarrow \quad \boxed{C_1 = -\frac{3}{125}}$$

$$q' = e^{-10t}\left(-10C_1\cos 20t - 10C_2\sin 20t - 20C_1\sin 20t + 20C_2\cos 20t\right)$$

$$q'(0) = 0 \quad \rightarrow 10\frac{3}{125} + 20C_2 = 0 \quad \boxed{C_2 = -\frac{3}{250}}$$

$$\boxed{q(t) = e^{-10t}\left(-\frac{3}{125}\cos 20t - \frac{3}{250}\sin 20t\right) + \frac{3}{125}}$$

$$I(t) = q' = e^{-10t}\left(\frac{6}{25}\cos 20t + \frac{3}{25}\sin 20t + \frac{12}{25}\sin 20t - \frac{6}{25}\cos 20t\right)$$

$$\boxed{= \frac{3}{5}e^{-10t}\sin 20t}$$

## *Exercise*

A series circuit consists of a resistor with $R = 20\ \Omega$, an inductor with $L = 1\ H$, a capacitor with $C = 0.002\ F$, and $E(t) = 12\sin 10t$. If the initial charge and current are both 0, find the charge and current at time *t*.

***Solution***

$$q'' + 20q' + \frac{1}{.002}q = 12\sin 10t \qquad\qquad Lq'' + Rq' + \frac{1}{C}q = E(t)$$

$$q'' + 20q' + 500q = 12\sin 10t\ ; \quad q(0) = 0, \quad i(0) = 0$$

$$\lambda^2 + 20\lambda + 500 = 0 \quad \rightarrow \quad \boxed{\lambda_{1,2} = -10 \pm 20i}$$

$$\boxed{q_h = e^{-10t}\left(C_1\cos 20t + C_2\sin 20t\right)}$$

$$q_p = A\cos 10t + B\sin 10t$$

$$q'_p = -10A\sin 10t + 10B\cos 10t$$

$$q''_p = -100A\cos 10t - 100B\sin 10t$$

$$q'' + 20q' + 500q = 12\sin 10t$$

$$\begin{cases} \cos 10t & -100A + 200B + 500A = 0 \\ \sin 10t & -100B - 200A + 500B = 12 \end{cases} \quad \rightarrow \begin{cases} 2A + B = 0 \\ -50A + 100B = 3 \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & 1 \\ -50 & 100 \end{vmatrix} = 250 \quad \Delta_A = \begin{vmatrix} 0 & 1 \\ 3 & 100 \end{vmatrix} = -3 \quad \Delta_B = \begin{vmatrix} 2 & 0 \\ -50 & 3 \end{vmatrix} = 6$$

$$\boxed{A = -\frac{3}{250}, \quad B = \frac{3}{125}}$$

$$\boxed{q_p = -\frac{3}{250}\cos 10t + \frac{3}{125}\sin 10t}$$

$$q(t) = e^{-10t}\left(C_1 \cos 20t + C_2 \sin 20t\right) - \frac{3}{250}\cos 10t + \frac{3}{125}\sin 10t$$

$$q(0) = 0 \quad \rightarrow \quad \boxed{C_1 = \frac{3}{250}}$$

$$q' = e^{-10t}\left(-10C_1 \cos 20t - 10C_2 \sin 20t - 20C_1 \sin 20t + 20C_2 \cos 20t\right) + \frac{3}{25}\sin 10t + \frac{6}{25}\cos 10t$$

$$q'(0) = 0 \quad \rightarrow -10\frac{3}{250} + 20C_2 + \frac{6}{25} = 0 \quad \boxed{C_2 = -\frac{3}{500}}$$

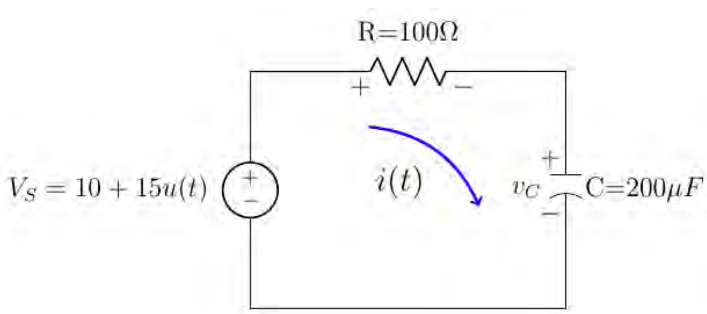
$$\boxed{q(t) = e^{-10t}\left(\frac{3}{250}\cos 20t - \frac{3}{500}\sin 20t\right) - \frac{3}{250}\cos 10t + \frac{3}{125}\sin 10t}$$

$$I(t) = e^{-10t}\left(-\frac{3}{25}\cos 20t + \frac{3}{50}\sin 20t - \frac{6}{25}\sin 20t - \frac{3}{25}\cos 20t\right) + \frac{3}{25}\sin 10t + \frac{6}{25}\cos 10t$$

$$\boxed{= e^{-10t}\left(-\frac{6}{25}\cos 20t - \frac{9}{50}\sin 20t\right) + \frac{3}{25}\sin 10t + \frac{6}{25}\cos 10t}$$

## *Exercise*

Consider the given circuit. Assuming that the voltage source changes from 10 to 25 $V$ at time $t = 0$, $v_s = 10 + 15u(t)\ V$, where $u(t)$ is a unit step function.




Find the expressions that describe the voltage drop across the resistor across the capacitor and the current in the loop for $t > 0$

***Solution***

***Given***: $R = 100\ \Omega, \quad C = 200\ \mu F, \quad v_S\left(0^-\right) = 10, \quad v_S = 25 \quad i\left(0^-\right) = 0$

$$i_s(0)=\frac{v_s-v_s(0^-)}{R}=\frac{25-10}{100}=0.15$$

Applying Kirkoff Voltage law: $v_R+v_C-v_s=0$

$$v_R+v_C=v_s$$

$$Ri+\frac{1}{C}q=v_s$$

$$R\frac{di}{dt}+\frac{1}{C}\frac{dq}{dt}=\frac{dv_S}{dt} \qquad i=\frac{dq}{dt}$$

$$R\frac{di}{dt}+\frac{1}{C}i=\frac{dv_S}{dt}$$

$$\frac{di}{dt}+\frac{1}{RC}i=\frac{1}{R}\frac{dv_S}{dt} \qquad \frac{1}{RC}=\frac{1}{(100)\left(200\times10^{-6}\right)}=\frac{1}{.02}=50$$

$$\frac{di}{dt}+50i=0$$

$$e^{\int 50dt}=e^{50t}$$

$$i(t)=Ke^{-50t}$$

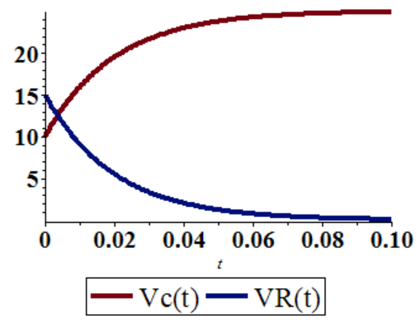
$$i(0)=0.15 \quad\rightarrow\quad 0.15=K$$

$$i(t)=0.15e^{-50t}$$

$$v_R=Ri(t)=15e^{-50t}$$

$$v_C=v_s-v_R$$

$$=25-15e^{-50t}$$




## *Exercise*

Find the steady-state solution $q_p(t)$ and the steady-state current in and *LRC*–series circuit when the source voltage is $E(t)=E_0\sin\omega t$

***Solution***

$$Lq''+Rq'+\frac{1}{C}q=E_0\sin\omega t$$

$$L\lambda^2+R\lambda+\frac{1}{C}=0 \quad\rightarrow\quad \lambda_{1,2}=\frac{-R\pm\sqrt{R^2-\frac{4L}{C}}}{2L}=\frac{-R\sqrt{C}\pm\sqrt{R^2C-4L}}{2L\sqrt{C}}$$

$$q_p=A\cos\omega t+B\sin\omega t$$

$$q_p'=-\omega A\sin\omega t+\omega B\cos\omega t$$

$$q_p'' = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t$$

$$-\omega^2 LA\cos\omega t - \omega^2 LB\sin\omega t - \omega RA\sin\omega t + \omega RB\cos\omega t + \frac{1}{C}A\cos\omega t + \frac{1}{C}B\sin\omega t = E_0 \sin\omega t$$

$$\begin{cases} \cos\omega t & \left(-\omega^2 L + \frac{1}{C}\right)A + \omega RB = 0 \\ \sin\omega t & -\omega RA + \left(\frac{1}{C} - \omega^2 L\right)B = E_0 \end{cases}$$

$$\Delta = \begin{vmatrix} \frac{1}{C} - \omega^2 L & \omega R \\ -\omega R & \frac{1}{C} - \omega^2 L \end{vmatrix} = \frac{1}{C^2} - \frac{2\omega^2 L}{C} + \omega^4 L^2 + \omega^2 R^2 = \frac{1}{C^2}\left(1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2\right)$$

$$\Delta_A = \begin{vmatrix} 0 & \omega R \\ E_0 & \frac{1}{C} - \omega^2 L \end{vmatrix} = \omega RE_0 \qquad \Delta_B = \begin{vmatrix} \frac{1}{C} - \omega^2 L & 0 \\ -\omega R & E_0 \end{vmatrix} = \left(\frac{1}{C} - \omega^2 L\right)E_0$$

$$A = \frac{\omega RC^2 E_0}{1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2} \qquad B = \frac{C(1 - \omega LC)E_0}{1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2}$$

Therefore, the steady-state charge is:

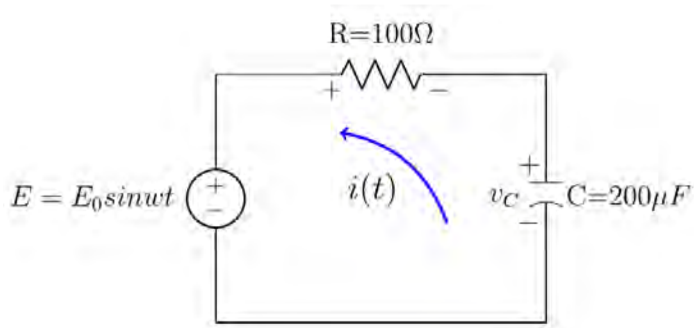
$$\boxed{q_p(t) = \frac{CE_2}{1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2}\left(\omega RC\cos\omega t + (1 - \omega LC)\sin\omega t\right)}$$

The steady-state current is: $i_p(t) = q_p'(t)$

$$\boxed{i_p(t) = \frac{CE_2}{1 - 2\omega^2 LC + \omega^4 L^2 C^2 + \omega^2 R^2 C^2}\left(-\omega^2 RC\sin\omega t + \left(\omega - \omega^2 LC\right)\cos\omega t\right)}$$

## *Exercise*

Consider the given *RC*-circuit with impressed *emf* is $E = E_0 \sin\omega t\ V$. If no initial current is flowing at $t = 0$, find the current $i(t)$ for all $t > 0$.




***Solution***

***Given***: $R = 100\ \Omega,\quad C = 200\ \mu F,\quad E = E_0 \sin\omega t\ V,\quad i\left(0^+\right) = 0$

$$Ri + \frac{1}{C}q = E$$

$$\frac{d}{dt}\left(Ri+\frac{1}{C}q=E_0\sin\omega t\right)$$

$$R\frac{di}{dt}+\frac{1}{C}\frac{dq}{dt}=\omega E_0\cos\omega t \qquad i=\frac{dq}{dt}$$

$$100\frac{di}{dt}+\frac{1}{2\times10^{-3}}i=\omega E_0\cos\omega t$$

$$\frac{di}{dt}+5i=\frac{\omega E_0}{100}\cos\omega t$$

| | | $\int \cos\omega t$ |
|---|---|---|
| + | $e^{5t}$ | $\frac{1}{\omega}\sin\omega t$ |
| − | $5e^{5t}$ | $-\frac{1}{\omega^2}\cos\omega t$ |
| + | $25e^{5t}$ | |

$$e^{\int 5dt}=e^{5t}$$

$$\int e^{5t}\left(\frac{\omega E_0}{100}\cos\omega t\right)dt=\frac{\omega E_0}{100}\int e^{5t}\cos\omega t\,dt$$

$$\int e^{5t}\cos\omega t\,dt=e^{5t}\left(\frac{1}{\omega}\sin\omega t+\frac{5}{\omega^2}\cos\omega t\right)-\frac{25}{\omega^2}\int e^{5t}\cos\omega t\,dt$$

$$\frac{\omega^2+25}{\omega^2}\int e^{5t}\cos\omega t\,dt=\frac{1}{\omega^2}e^{5t}\left(\omega\sin\omega t+5\cos\omega t\right)$$

$$\int e^{5t}\cos\omega t\,dt=\frac{1}{\omega^2+25}e^{5t}\left(\omega\sin\omega t+5\cos\omega t\right)$$

$$\int e^{5t}\left(\frac{\omega E_0}{100}\cos\omega t\right)dt=\frac{1}{100}\frac{\omega E_0}{\omega^2+25}e^{5t}\left(\omega\sin\omega t+5\cos\omega t\right)$$

$$i(t)=\frac{1}{e^{5t}}\left(\frac{1}{100}\frac{\omega E_0}{\omega^2+25}e^{5t}\left(\omega\sin\omega t+5\cos\omega t\right)+K\right)$$

$$=\frac{\omega E_0}{100\left(\omega^2+25\right)}\left(\omega\sin\omega t+5\cos\omega t\right)+\frac{K}{e^{5t}}$$

$$i(0)=0 \quad\to\frac{\omega E_0}{20\left(\omega^2+25\right)}+K=0 \qquad \boxed{K=-\frac{\omega E_0}{20\left(\omega^2+25\right)}}$$

$$i(t)=\frac{\omega E_0}{100\left(\omega^2+25\right)}\left(\omega\sin\omega t+5\cos\omega t\right)-\frac{\omega E_0}{20\left(\omega^2+25\right)}e^{-5t}$$

$$\boxed{=\frac{\omega E_0}{20\left(\omega^2+25\right)}\left(\frac{\omega}{5}\sin\omega t+\cos\omega t-e^{-5t}\right)}$$