Section 1.7 – Direction Fields; Existence and Uniqueness

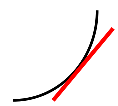
A first-order autonomous equation is an equation of the form

$$x' = f(x)$$

$$\frac{dy}{dx} = f(x, y)$$

Definition

The value $f(x, y)$ where the function f assigns to the point represent the slope of a line (line segment) call ***a lineal element***.

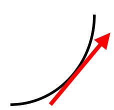


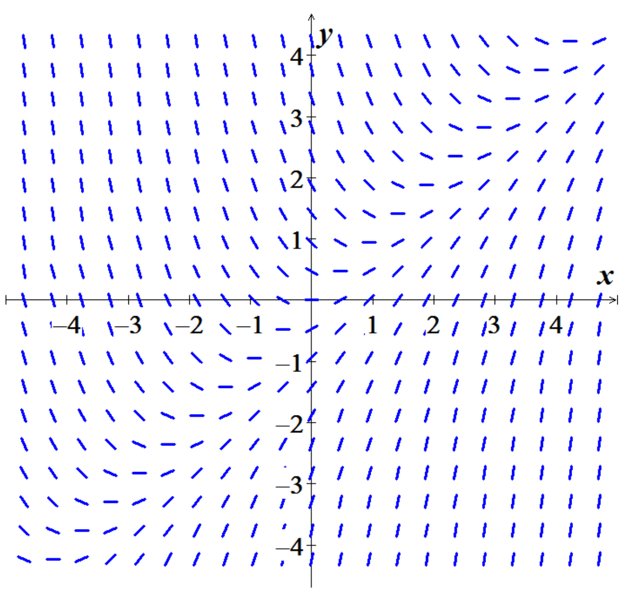
Example: Given $\frac{dy}{dx} = 0.2xy$ and consider the point $(2,\ 3)$

The slope of the lineal element is $\frac{dy}{dx} = 0.2xy = 0.2(2)(3) = 1.2$ $\quad$ (*positive sign*)

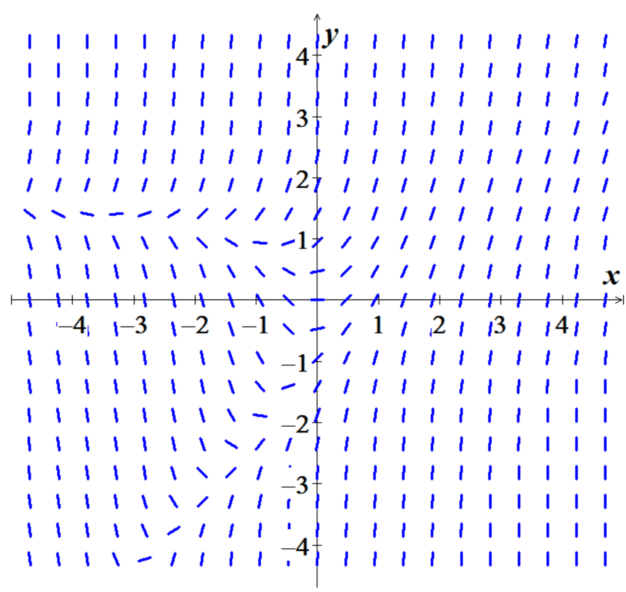
The Direction Fields

What we draw a lineal element at each point (x, y) with slope $f(x, y)$ then the collection of these lineal elements is called a ***direction field*** or a ***slope field*** of the differential equation $\frac{dy}{dx} = f(x, y)$.

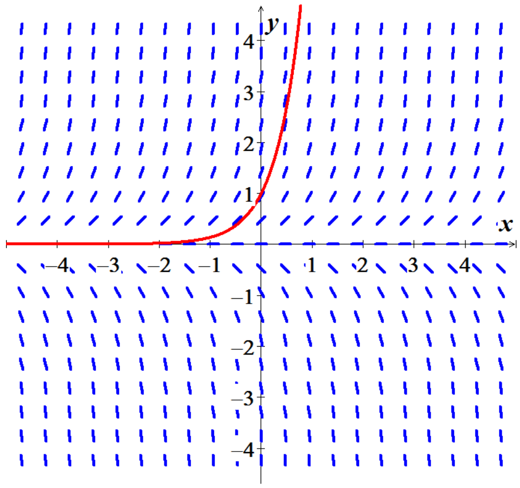


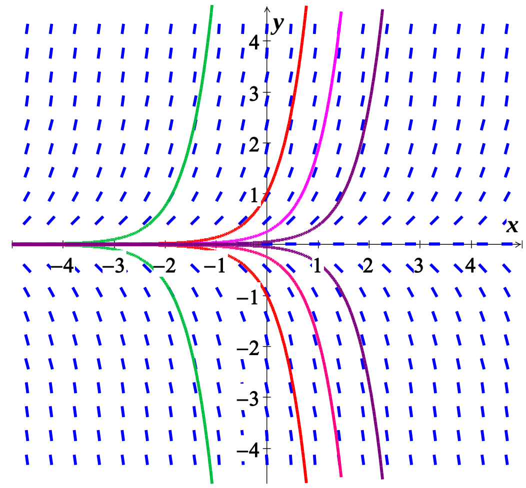


$y' = x - y$



$y' = y^2 - xy + 2x$





$y' = 2y, \quad \text{with } y(0) = 1 \quad \Rightarrow \quad y = e^{2x}$

Example

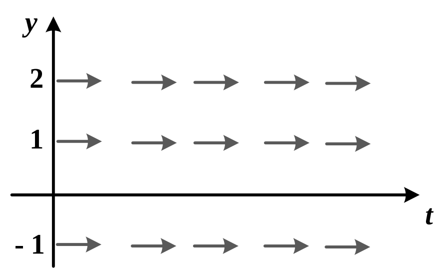
Sketch the direction field for the following differential equation. Sketch the set of integral curves for this differential equation, how the solutions behave as $t \to \infty$ and if this behavior depends on the value of $y(0)$ describe this dependency

$$y' = \left(y^2 - y - 2\right)(1 - y)^2$$

Solution

$$y' = 0 \quad \Rightarrow \quad \left(y^2 - y - 2\right)(1 - y)^2 = 0$$

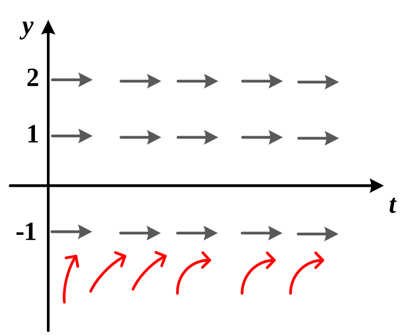
$\underline{y = \pm 1,\ 2}$ *Slope of the tangent lines*



This divided into 4 regions.

For $y < -1$, assume $y = -2 \Rightarrow y' = \left(4^2 + 2 - 2\right)(1 + 2)^2 = 36 > \mathbf{0}$ $(\nearrow)$

$y = -1$, the slopes will flatten out while staying positive

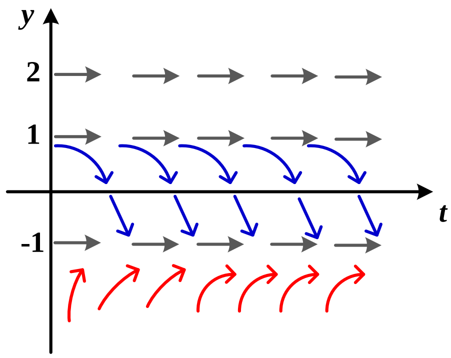


For $-1 < y < 1$, assume $y = 0 \Rightarrow y' = (-2)(1)^2 = -2 < \mathbf{0}$ $(\searrow)$

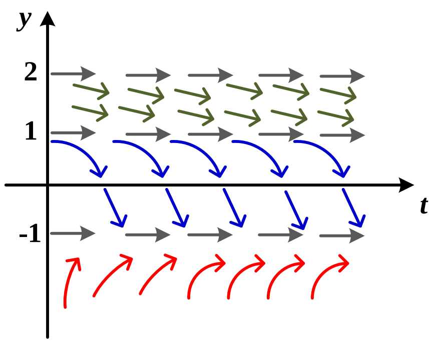
Therefore, tangent lines in this region will have negative slopes and apparently not very steep.

$y = .9 \quad \Rightarrow \quad y' = -.0209$

$y = -.9 \quad \Rightarrow \quad y' = -1.0469$ (Steeper)

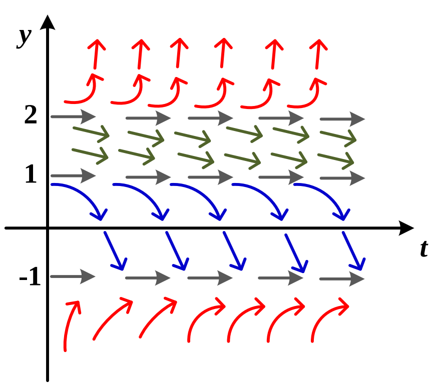


For $1 < y < 2$, assume $y = 1.5 \Rightarrow y' = \left(1.5^2 - 1.5 - 2\right)(-.5)^2 = -0.3125 < \mathbf{0}$ $(\searrow)$ Not to steep

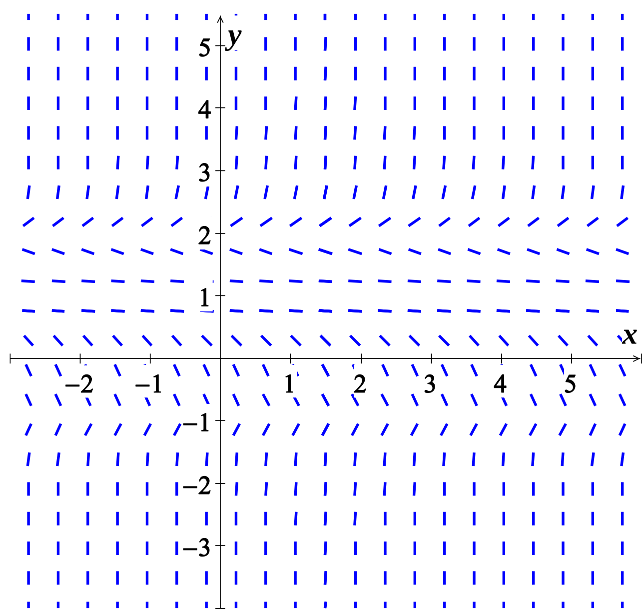


For $y > 2$, assume $y = 3 \Rightarrow y' = (4)(-2)^2 = 16 > \mathbf{0}$ $(\nearrow)$

Start out fairly flat neary = 2, then will get fairly steep.



Value of $y(0)$	$t \to \infty$
$y(0) < -1$	$y \to -1$
$-1 \le y(0) < 2$	$y \to 1$
$y(0) = 2$	$y \to 2$
$y(0) > 2$	$y \to \infty$



The questions of ***existence and uniqueness***

- When can we be sure that a solution exists**?**
- How many different solutions are there

Existence of Solutions

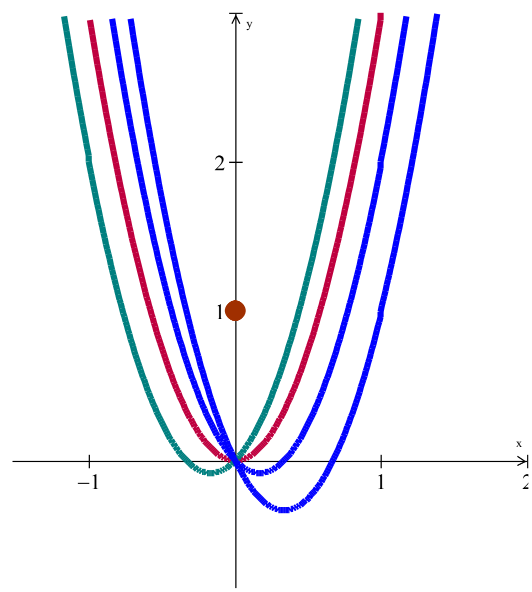
The fundamental questions in a course on differential equations are:

- Does the given initial-value problem (*IVP*) have a solution? Do solutions to the problem exists?
- If a solution does exist, is it unique? Is there exactly one solution to the problem or is there more than one solution?

Example

Consider the initial value problem: $tx' = x + 3t^2$ with $x(0) = 1$

Solution



$$x' = \frac{1}{t}x + 3t$$

$$x' = \frac{1}{t}x + 3t \qquad t \neq 0$$

There is ***no solution*** to the given initial value

$$u(t) = e^{-\int \frac{1}{t}dt}$$

$$= e^{-\ln t}$$

$$= \frac{1}{t}$$

$$\left[\frac{x}{t}\right]' = 3$$

$$\frac{x}{t} = \int 3dt$$

$$= 3t + C$$

$$\boxed{x(t) = 3t^2 + Ct}$$

Theorem: Existence of Solutions

Suppose the function $f(t, x)$ is defined and continuous on the rectangle ***R*** in the *tx*-plane. Then given any point $\left(t_0,\ x_0\right) \in \boldsymbol{R}$, the initial value problem

$$x' = f(t, x) \quad \textit{and} \quad x(t_0) = x_0$$

has a solution $x(t)$ defined in an interval containing x_0. Furthermore, the solution will be defined at least until the solution curve $t \to \left(t,\ x(t)\right)$ leaves the rectangle ***R***.

Interval of Existence of a Solution

Example

Consider the initial value problem $x' = 1 + x^2$ with $x(0) = 0$. Find the solution and its interval of existence.

Solution

The right-hand side is $f(t, x) = 1 + x^2$ which is continuous on the entire *tx*-plane.

The solution to the initial value problem is:

$$\frac{dx}{dt} = 1 + x^2$$

$$\frac{dx}{1 + x^2} = dt$$

$$\int \frac{dx}{1 + x^2} = \int dt$$

$$\tan^{-1} x = t$$

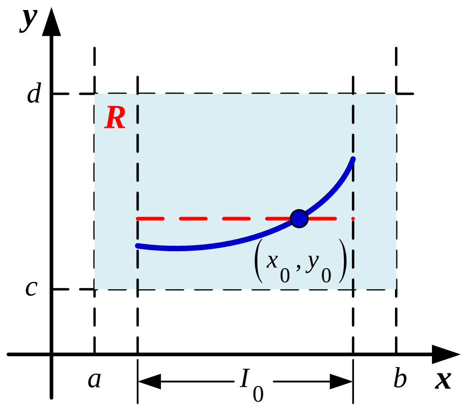
$$x(t) = \tan t$$

$x(t)$ is discontinuous at $t = \pm\frac{\pi}{2}$. Hence the solution to the initial value problem is defined only for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

The interval: $\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$

Theorem: Existence of a Unique Solution

Let R be a rectangular region in the *xy*-plane defined by $a \le x \le b,\ \ c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on R, then there exists some interval $I_0 : (x_0 - h,\ x_0 + h),\ \ h > 0$, contained in [*a*, *b*], and a unique function $y(x)$, defined on I_0 that is a solution of the initial-value problem (*IVP*)



Mathematics & Theorems

Any theorem is a logical statement which has hypotheses (when it's true) and conclusions (true)

The Hypotheses of the Uniqueness of Solutions Theorem

1. The equation is in normal form $y' = f(t, y)$
2. The right-hand side $f(t, y)$ and its derivative $\dfrac{\partial f}{\partial y}$ are both continuous in the rectangle ***R***.
3. The initial point (t_0, y_0) is in the rectangle ***R***.

For the uniqueness theorem the conclusions are as follows:

1- There is one and only one solution to the initial value problem.
2- The solution exists until the solution curve $t \to (t,\ y(t))$ leaves the rectangle ***R***.

Example

Consider the initial value problem $tx' = x + 3t^2$. Is there a solution to this equation with initial condition $x(1) = 2$? If so, is the solution unique?

Solution

$$x' = \frac{x}{t} + 3t$$

The right-hand side: $f(t, x) = \frac{x}{t} + 3t$ is continuous except where $t = 0$.

We can take ***R*** to be any rectangle which contains the point $(1,\ 2)$ to avoid $t = 0$, we can choose

$\frac{1}{2} < t < 2$ and $0 < x < 4$

Then f is continuous everywhere in ***R*** $\Rightarrow$ hypotheses of the existence theorem are satisfied.

Since $\frac{\partial f}{\partial x} = \frac{1}{t}$ is also continuous in ***R***.

There is only one solution.

It is important to determine and prove a theorem concerning the existence and uniqueness of solutions of an O.D.E.

- Are the $x_{1,2} = \frac{1 \pm \sqrt{1-4\mu}}{2}$ solutions to $P_+ = \frac{1+\sqrt{1-4\mu}}{2}$ exist?

  $P_- = \frac{1-\sqrt{1-4\mu}}{2}$

  $\Rightarrow$ Solutions exist for the system.

- ***Uniqueness***: Assume $\mu > \frac{1}{4}$ is another solution. We want to prove $f_\mu(x)$ is actually $f_\mu(x)$ i.e.

  $\mu > \frac{1}{4} \qquad f_\mu(x)$

  $\mu > \frac{1}{4} \qquad f'_\mu$

  So that, $\frac{d}{dx}\left[f_\mu(x) \right] = 2x + \mu$, then multiply both sides by $f'_\mu(P_+) = 1 + \sqrt{1-4\mu}$ to obtain:

  $f'_\mu(P_-) = 1 - \sqrt{1-4\mu}$

Exercises *Section* 1.7 - Direction Fields; Existence and Uniqueness of Solutions

Which of the initial value problems are guaranteed a unique solution

1. $y' = 4 + y^2, \quad y(0) = 1$

2. $y' = \sqrt{y}, \quad y(4) = 0$

3. $y' = t\tan^{-1} y, \quad y(0) = 2$

4. $\omega' = \omega \sin \omega + s, \quad \omega(0) = -1$

5. $x' = \dfrac{t}{x+1}, \quad x(0) = 0$

6. $y' = \dfrac{1}{x} y + 2, \quad y(0) = 1$

7. $y' = e^t y - y^3, \quad y(0) = 0$

8. $y' = ty^2 - \dfrac{1}{3y+t}, \quad y(0) = 1$

9. $y' = xy, \quad y(0) = 1$

10. $y' = -\dfrac{t^2}{1-y^2}, \quad y(-1) = \dfrac{1}{2}$

11. $y' = \dfrac{y}{\sin t}, \quad y\left(\dfrac{\pi}{2}\right) = 1$

12. $y' = \sqrt{1-y^2}, \quad y(0) = 1$

13. Show that $y(t) = 0$ and $y(t) = t^3$ are both solutions of the initial value problem $y' = 3y^{2/3}$, where $y(0) = 0$. Explain why this fact doesn't contradict Theorem

14. Use a numerical solver to sketch the solution of the given initial value problem

$$\frac{dy}{dt} = \frac{t}{y+1}, \qquad y(2) = 0$$

a) Where does your solver experience difficulty? why? Use the image of your solution to estimate the interval of existence.

b) Find an explicit solution; then use your formula to determine the interval of existence. How does it compare with the approximation found in part (*a*).