

Solution ***Section 3.4 – L'Hôpital's Rule***

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow -2} \frac{x+2}{x^2-4}$

Solution

$$\begin{aligned} \lim_{x \rightarrow -2} \frac{x+2}{x^2-4} &= \frac{1}{2x} \Big|_{x=-2} & \frac{(x+2)'}{(x^2-4)'} &= \frac{1}{2x} \\ &= -\frac{1}{4} \Big| \end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow 1} \frac{x^3-1}{4x^3-x-3}$

Solution

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^3-1}{4x^3-x-3} &= \lim_{x \rightarrow 1} \frac{3x^2}{12x^2-1} \\ &= \frac{3}{11} \Big| \end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow -5} \frac{x^2-25}{x+5}$

Solution

$$\begin{aligned} \lim_{x \rightarrow -5} \frac{x^2-25}{x+5} &= \lim_{x \rightarrow -5} \frac{2x}{1} \\ &= -10 \Big| \end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{t \rightarrow 0} \frac{\sin 5t}{2t}$

Solution

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\sin 5t}{2t} &= \lim_{t \rightarrow 0} \frac{5 \cos 5t}{2} \\ &= \frac{5}{2} \Big| \end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{\theta \rightarrow -\pi/3} \frac{3\theta + \pi}{\sin\left(\theta + \frac{\pi}{3}\right)}$

Solution

$$\begin{aligned}\lim_{\theta \rightarrow -\pi/3} \frac{3\theta + \pi}{\sin\left(\theta + \frac{\pi}{3}\right)} &= \lim_{\theta \rightarrow -\pi/3} \frac{3}{\cos\left(\theta + \frac{\pi}{3}\right)} \\ &= 3\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow 0} \frac{x^2}{\ln(\sec x)}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x^2}{\ln(\sec x)} &= \lim_{x \rightarrow 0} \frac{2x}{\frac{\sec x \tan x}{\sec x}} \\ &= \lim_{x \rightarrow 0} \frac{2x}{\tan x} = \frac{0}{0} \qquad \frac{(2x)'}{(\tan x)'} = \frac{2}{\sec^2 x} \\ &= \lim_{x \rightarrow 0} \frac{2}{\sec^2 x} \\ &= \frac{2}{1} \\ &= 2\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{\theta \rightarrow 0} \frac{3^{\sin \theta} - 1}{\theta}$

Solution

$$\begin{aligned}\lim_{\theta \rightarrow 0} \frac{3^{\sin \theta} - 1}{\theta} &= \lim_{\theta \rightarrow 0} \frac{3^{\sin \theta} \ln(3)(\cos \theta)}{1} \\ &= 3^{\sin 0} \ln(3)(\cos 0) \\ &= \ln(3)\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow 0} \frac{3^x - 1}{2^x - 1}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{3^x - 1}{2^x - 1} &= \lim_{x \rightarrow 0} \frac{3^x \ln(3)}{2^x \ln(2)} \\ &= \frac{3^0 \ln 3}{2^0 \ln 2} \\ &= \frac{\ln 3}{\ln 2} \quad \Bigg| \quad = \log_2^3\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow 0^+} (\ln x - \ln \sin x)$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0^+} (\ln x - \ln \sin x) &= \lim_{x \rightarrow 0^+} \left(\ln \frac{x}{\sin x} \right) \\ &= \ln \lim_{x \rightarrow 0^+} \left(\frac{x}{\sin x} \right) = \ln \frac{0}{0} \\ &= \ln \lim_{x \rightarrow 0^+} \left(\frac{1}{\cos x} \right) \\ &= \ln(1) \\ &= 0\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow 0} \frac{(e^x - 1)^2}{x \sin x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{(e^x - 1)^2}{x \sin x} &= \lim_{x \rightarrow 0} \frac{2(e^x - 1)e^x}{\sin x + x \cos x} \\ &= \frac{2(1-1)}{0+0} = \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{2e^{2x} - 2e^x}{\sin x + x \cos x}\end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 0} \frac{4e^{2x} - 2e^x}{\cos x + \cos x - x \sin x} \\
&= \frac{4 - 2}{1 + 1 - 0} \\
&= 1
\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow \pi/2^-} \frac{1 + \tan x}{\sec x}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow \pi/2^-} \frac{1 + \tan x}{\sec x} &= \frac{\infty}{\infty} = \lim_{x \rightarrow \pi/2^-} \frac{\sec^2 x}{\sec x \tan x} \\
&= \lim_{x \rightarrow \pi/2^-} \frac{\sec x}{\tan x} \\
&= \lim_{x \rightarrow \pi/2^-} \frac{1}{\sin x} \\
&= 1
\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow \infty} \frac{4x^3 - 6x^2 + 1}{2x^3 - 10x + 3}$

Solution

$$\lim_{x \rightarrow \infty} \frac{4x^3 - 6x^2 + 1}{2x^3 - 10x + 3} = 2$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow 0} \frac{3 \sin 4x}{5x}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{3 \sin 4x}{5x} &= \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{12 \cos 4x}{5} \\
&= \frac{12}{5}
\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow 2\pi} \frac{x \sin x + x^2 - 4\pi^2}{x - 2\pi}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 2\pi} \frac{x \sin x + x^2 - 4\pi^2}{x - 2\pi} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 2\pi} \frac{\sin x + x \cos x + 2x}{1} \\ &= 2\pi + 4\pi \\ &= \underline{6\pi} \quad | \end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow 0} \frac{\tan 4x}{\tan 7x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\tan 4x}{\tan 7x} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{4 \sec^2 4x}{7 \sec^2 7x} \\ &= \underline{\frac{4}{7}} \quad | \end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow 0} \frac{\sin^2 3x}{x^2}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin^2 3x}{x^2} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \left(\frac{\sin 3x}{x} \right)^2 \left(\frac{3}{3} \right)^2 \\ &= 9 \lim_{x \rightarrow 0} \left(\frac{\sin 3x}{3x} \right)^2 \\ &= \underline{9} \quad | \end{aligned}$$
$$\lim_{x \rightarrow 0} \frac{\sin ax}{ax} = 1$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow -1} \frac{x^3 - x^2 - 5x - 3}{x^4 + 2x^3 - x^2 - 4x - 2}$

Solution

$$\begin{aligned}\lim_{x \rightarrow -1} \frac{x^3 - x^2 - 5x - 3}{x^4 + 2x^3 - x^2 - 4x - 2} &= \frac{0}{0} \\&= \lim_{x \rightarrow -1} \frac{3x^2 - 2x - 5}{4x^3 + 6x^2 - 2x - 4} \\&= \lim_{x \rightarrow -1} \frac{3x^2 - 2x - 5}{4x^3 + 6x^2 - 2x - 4} = \frac{0}{0} \\&= \lim_{x \rightarrow -1} \frac{6x - 2}{12x^2 + 12x - 2} \\&= 4\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow 1} \frac{x^n - 1}{x - 1} \quad (n > 0)$

Solution

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^n - 1}{x - 1} &= \frac{0}{0} \\&= \lim_{x \rightarrow 1} \frac{nx^{n-1}}{1} \\&= n\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow 1^-} (1 - x) \tan\left(\frac{\pi x}{2}\right)$

Solution

$$\begin{aligned}\lim_{x \rightarrow 1^-} (1 - x) \tan\left(\frac{\pi x}{2}\right) &= 0 \cdot \infty \\&= \lim_{x \rightarrow 1^-} \frac{1 - x}{\cot\left(\frac{\pi x}{2}\right)} \\&= \lim_{x \rightarrow 1^-} \frac{-1}{-\frac{\pi}{2} \csc^2\left(\frac{\pi x}{2}\right)} \\&= \frac{2}{\pi}\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow \infty} \frac{3}{x} \csc \frac{5}{x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{3}{x} \csc \frac{5}{x} &= 0 \cdot \infty \\&= 3 \lim_{y \rightarrow 0} \frac{y}{\sin 5y} \quad y = \frac{1}{x} \\&= 3 \lim_{y \rightarrow 0} \frac{1}{5 \cos 5y} \\&= \frac{3}{5} \quad \square\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow \pi/4} \frac{\tan x - \cot x}{x - \frac{\pi}{4}}$

Solution

$$\begin{aligned}\lim_{x \rightarrow \pi/4} \frac{\tan x - \cot x}{x - \frac{\pi}{4}} &= \frac{0}{0} \\&= \lim_{x \rightarrow \pi/4} \frac{\sec^2 x + \csc^2 x}{1} \\&= 2 + 2 \\&= 4 \quad \square\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow 0} \frac{1 - \cos 3x}{8x^2}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1 - \cos 3x}{8x^2} &= \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{3 \sin 3x}{16x} = \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{9 \cos 3x}{16} \\&= \frac{9}{16} \quad \square\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow 3} \frac{x-1-\sqrt{x^2-5}}{x-3}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{x-1-\sqrt{x^2-5}}{x-3} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 3} \frac{1 - \frac{x}{\sqrt{x^2-5}}}{1} \\ &= 1 - \frac{3}{2} \\ &= -\frac{1}{2} \quad \boxed{}\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow 2} \frac{x^2+x-6}{\sqrt{8-x^2}-x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^2+x-6}{\sqrt{8-x^2}-x} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 2} \frac{2x+1}{\frac{-x}{\sqrt{8-x^2}}-1} \\ &= \frac{5}{-1-1} \\ &= -\frac{5}{2} \quad \boxed{}\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h}$ x is a real number

Solution

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} &= \frac{0}{0} \\ &= \lim_{h \rightarrow 0} \frac{\cos(x+h)}{1} \\ &= \cos x \quad \boxed{}\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow 2} \frac{\sqrt[3]{3x+2} - 2}{x-2}$

Solution

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{\sqrt[3]{3x+2} - 2}{x-2} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 2} \frac{(3x+2)^{-2/3}}{1} \\ &= \frac{1}{4} \end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow \infty} \frac{3x^4 - x^2}{6x^4 + 12}$

Solution

$$\lim_{x \rightarrow \infty} \frac{3x^4 - x^2}{6x^4 + 12} = \frac{1}{2}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow \infty} \frac{4x^3 - 2x^2 + 6}{\pi x^3 + 4}$

Solution

$$\lim_{x \rightarrow \infty} \frac{4x^3 - 2x^2 + 6}{\pi x^3 + 4} = \frac{4}{\pi}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow \infty} \frac{8 - 4x^2}{3x^3 + x - 1}$

Solution

$$\lim_{x \rightarrow \infty} \frac{8 - 4x^2}{3x^3 + x - 1} = 0$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow \pi/2} \frac{2 \tan x}{\sec^2 x}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow \pi/2} \frac{2 \tan x}{\sec^2 x} &= \frac{\infty}{\infty} \\
&= \lim_{x \rightarrow \pi/2} \frac{2 \sec^2 x}{2 \sec^2 x \tan x} \\
&= \lim_{x \rightarrow \pi/2} \cot x \\
&= 0
\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow 0} \frac{e^x - x - 1}{5x^2}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{e^x - x - 1}{5x^2} &= \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{e^x - 1}{10x} = \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{e^x}{10} \\
&= \frac{1}{10}
\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow 0} \frac{e^x - \sin x - 1}{x^4 + 8x^3 + 12x^2}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{e^x - \sin x - 1}{x^4 + 8x^3 + 12x^2} &= \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{e^x - \cos x}{4x^3 + 24x^2 + 24x} = \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{e^x + \sin x}{12x^2 + 48x + 24} \\
&= \frac{1}{24}
\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow \infty} \frac{e^{1/x} - 1}{1/x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{e^{1/x} - 1}{1/x} &= \frac{0}{0} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{-1}{x^2} e^{1/x}}{-1/x^2} \\ &= \lim_{x \rightarrow \infty} e^{1/x} \\ &= \underline{1}\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow \infty} \frac{e^{3x}}{3e^{3x} + 5}$

Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{e^{3x}}{3e^{3x} + 5} &= \frac{\infty}{\infty} \\ &= \lim_{x \rightarrow \infty} \frac{3e^{3x}}{9e^{3x}} \\ &= \underline{\frac{1}{3}}\end{aligned}$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow \infty} \frac{\ln(3x + 5e^x)}{\ln(7x + 3e^{2x})}$

Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\ln(3x + 5e^x)}{\ln(7x + 3e^{2x})} &= \frac{\infty}{\infty} \\ &= \lim_{x \rightarrow \infty} \frac{3 + 5e^x}{3x + 5e^x} \cdot \lim_{x \rightarrow \infty} \frac{7x + 3e^{2x}}{7 + 6e^{2x}} \\ &= \lim_{x \rightarrow \infty} \frac{5e^x}{3 + 5e^x} \cdot \lim_{x \rightarrow \infty} \frac{7 + 6e^{2x}}{12e^{2x}} \\ &= \lim_{x \rightarrow \infty} \frac{5e^x}{5e^x} \cdot \lim_{x \rightarrow \infty} \frac{6e^{2x}}{12e^{2x}}\end{aligned}$$

$$= 1 \cdot \frac{1}{2}$$

$$\underline{= \frac{1}{2}} \quad \Big|$$

Exercise

Apply l'Hôpital Rule to evaluate $\lim_{x \rightarrow \infty} \frac{x^2 - \ln\left(\frac{2}{x}\right)}{3x^2 + 2x}$

Solution

$$\lim_{x \rightarrow \infty} \frac{x^2 - \ln\left(\frac{2}{x}\right)}{3x^2 + 2x} = \lim_{x \rightarrow \infty} \frac{2x - \left(\frac{x}{2}\right)\left(\frac{-1}{x^2}\right)}{6x + 2}$$

$$= \lim_{x \rightarrow \infty} \frac{2x + \frac{1}{x}}{6x + 2}$$

$$\underline{= \frac{1}{3}} \quad \Big|$$

Exercise

Find $\lim_{x \rightarrow 1^+} x^{1/(x-1)}$

Solution

$$\lim_{x \rightarrow 1^+} x^{1/(x-1)} = 1^\infty$$

$$f(x) = x^{1/(x-1)} \Rightarrow \ln f(x) = \ln x^{1/(x-1)} = \frac{1}{x-1} \ln x \quad \Big|$$

$$\lim_{x \rightarrow 1^+} \ln f(x) = \lim_{x \rightarrow 1^+} \frac{\ln x}{x-1} = \lim_{x \rightarrow 1^+} \frac{\frac{1}{x}}{1} = 1$$

$$\lim_{x \rightarrow 1^+} x^{1/(x-1)} = \lim_{x \rightarrow 1^+} f(x)$$

$$= \lim_{x \rightarrow 1^+} e^{\ln f(x)}$$

$$= \lim_{x \rightarrow 1^+} e^1$$

$$\underline{= e} \quad \Big|$$

Exercise

Find $\lim_{x \rightarrow e^+} (\ln x)^{1/(x-e)}$

Solution

$$\lim_{x \rightarrow e^+} (\ln x)^{1/(x-e)} = 1^\infty$$

$$f(x) = (\ln x)^{1/(x-e)}$$

$$\ln f(x) = \ln \left((\ln x)^{1/(x-e)} \right)$$

$$= \frac{1}{x-e} \ln(\ln x)$$

$$\lim_{x \rightarrow e^+} \ln f(x) = \lim_{x \rightarrow e^+} \frac{\ln(\ln x)}{x-e}$$

$$= \lim_{x \rightarrow e^+} \frac{\frac{1}{x \ln x}}{1}$$

$$= \lim_{x \rightarrow e^+} \frac{1}{x \ln x}$$

$$= \frac{1}{e \ln e}$$

$$= \frac{1}{e}$$

$$\lim_{x \rightarrow e^+} (\ln x)^{1/(x-e)} = \lim_{x \rightarrow e^+} f(x)$$

$$= \lim_{x \rightarrow e^+} e^{\ln f(x)}$$

$$= e^{1/e}$$

Exercise

Find $\lim_{x \rightarrow \infty} (1+2x)^{1/(2 \ln x)}$

Solution

$$\lim_{x \rightarrow \infty} (1+2x)^{1/(2 \ln x)} = \infty^0$$

$$f(x) = (1+2x)^{1/(2 \ln x)} \Rightarrow \ln f(x) = \ln \left((1+2x)^{1/(2 \ln x)} \right) = \frac{1}{2 \ln x} \ln(1+2x)$$

$$\begin{aligned}
\lim_{x \rightarrow \infty} \ln f(x) &= \lim_{x \rightarrow \infty} \frac{\ln(1+2x)}{2 \ln x} \\
&= \lim_{x \rightarrow \infty} \frac{\frac{2}{\frac{1+2x}{2}}}{\frac{2}{x}} \\
&= \lim_{x \rightarrow \infty} \frac{x}{1+2x} \\
&= \underline{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
\lim_{x \rightarrow \infty} (1+2x)^{1/(2 \ln x)} &= \lim_{x \rightarrow \infty} e^{\ln f(x)} \\
&= \underline{e^{1/2}}
\end{aligned}$$

Exercise

Find $\lim_{x \rightarrow \infty} \left(\frac{x^2+1}{x+2} \right)^{1/x}$

Solution

$$\lim_{x \rightarrow \infty} \left(\frac{x^2+1}{x+2} \right)^{1/x} = \infty^0$$

$$f(x) = \left(\frac{x^2+1}{x+2} \right)^{1/x} \Rightarrow \ln f(x) = \ln \left(\left(\frac{x^2+1}{x+2} \right)^{1/x} \right) = \frac{1}{x} \ln \left(\frac{x^2+1}{x+2} \right)$$

$$\begin{aligned}
\lim_{x \rightarrow \infty} \ln f(x) &= \lim_{x \rightarrow \infty} \frac{1}{x} \ln \left(\frac{x^2+1}{x+2} \right) \\
&= \lim_{x \rightarrow \infty} \frac{\ln(x^2+1) - \ln(x+2)}{x}
\end{aligned}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{2x}{x^2+1} - \frac{1}{x+2}}{1}$$

$$= \lim_{x \rightarrow \infty} \frac{2x^2+4x-x^2-1}{(x^2+1)(x+2)}$$

$$= \lim_{x \rightarrow \infty} \frac{x^2+4x-1}{x^3+2x^2+x+2}$$

$$= \lim_{x \rightarrow \infty} \frac{2x+4}{3x^2+4x+1}$$

$$= \lim_{x \rightarrow \infty} \frac{2}{6x+4}$$

$$\lim_{x \rightarrow \infty} \frac{x^2+4x-1}{x^3+2x^2+x+2} = 0$$

$$\underline{=0}$$

$$\begin{aligned}\lim_{x \rightarrow \infty} \left(\frac{x^2+1}{x+2} \right)^{1/x} &= \lim_{x \rightarrow \infty} e^{\ln f(x)} \\ &= e^0 \\ &\underline{=1}\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{t \rightarrow 2} \frac{t^3 - t^2 - 2t}{t^2 - 4}$

Solution

$$\begin{aligned}\lim_{t \rightarrow 2} \frac{t^3 - t^2 - 2t}{t^2 - 4} &= \frac{0}{0} \\ &= \lim_{t \rightarrow 2} \frac{3t^2 - 2t - 2}{2t} \\ &\underline{= \frac{3}{2}}\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \frac{1 - \cos 6x}{2x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1 - \cos 6x}{2x} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{6 \sin 6x}{2} \\ &\underline{= 0}\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} \frac{5x^2 + 2x - 5}{\sqrt{x^4 - 1}}$

Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{5x^2 + 2x - 5}{\sqrt{x^4 - 1}} &= \lim_{x \rightarrow \infty} \frac{5x^2}{x^2} \\ &\underline{= 5}\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{\theta \rightarrow 0} \frac{3\sin^2 2\theta}{\theta^2}$

Solution

$$\begin{aligned}\lim_{\theta \rightarrow 0} \frac{3\sin^2 2\theta}{\theta^2} &= \frac{0}{0} \\&= \lim_{\theta \rightarrow 0} \frac{12\sin 2\theta \cos 2\theta}{2\theta} \\&= 12 \lim_{2\theta \rightarrow 0} \frac{\sin 2\theta}{2\theta} \lim_{\theta \rightarrow 0} \cos 2\theta \qquad \lim_{2\theta \rightarrow 0} \frac{\sin 2\theta}{2\theta} = 1 \\&= 12\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x} \right)$

Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x} \right) &= \infty - \infty \\&= \lim_{x \rightarrow \infty} \left(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x} \right) \frac{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} \\&= \lim_{x \rightarrow \infty} \frac{x^2 + x + 1 - x^2 + x}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} \\&= \lim_{x \rightarrow \infty} \frac{2x}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} \\&= \lim_{x \rightarrow \infty} \frac{2x}{x + x} \\&= 1\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{\theta \rightarrow 0} 2\theta \cot 3\theta$

Solution

$$\begin{aligned}\lim_{\theta \rightarrow 0} 2\theta \cot 3\theta &= 0 \cdot \infty = \frac{0}{0} \\&= \lim_{\theta \rightarrow 0} 2\theta \frac{\cos 3\theta}{\sin 3\theta} \\&= \lim_{\theta \rightarrow 0} \frac{2\theta}{\sin 3\theta} \lim_{\theta \rightarrow 0} \cos 3\theta \\&= \frac{2}{3}\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \frac{e^{-2x} - 1 + 2x}{x^2}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{e^{-2x} - 1 + 2x}{x^2} &= \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{-2e^{-2x} + 2}{2x} = \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{4e^{-2x}}{2} \\&= 2\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 1} \frac{x^4 - x^3 - 3x^2 + 5x - 2}{x^3 + x^2 - 5x + 3}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^4 - x^3 - 3x^2 + 5x - 2}{x^3 + x^2 - 5x + 3} &= \frac{1 - 1 - 3 + 5 - 2}{1 + 1 - 5 + 3} = \frac{0}{0} \\&= \lim_{x \rightarrow 1} \frac{4x^3 - 3x^2 - 6x + 5}{3x^2 + 2x - 5} = \frac{0}{0} \\&= \lim_{x \rightarrow 1} \frac{12x^2 - 6x - 6}{6x + 2} \\&= 0\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{y \rightarrow 0^+} \frac{\ln^{10} y}{\sqrt{y}}$

Solution

$$\begin{aligned}\lim_{y \rightarrow 0^+} \frac{\ln^{10} y}{\sqrt{y}} &= \frac{\infty}{0} \\&= \lim_{x \rightarrow \infty} \frac{\left(\ln \frac{1}{x}\right)^{10}}{\frac{1}{\sqrt{x}}} \quad \text{Let } y = \frac{1}{x} \\&= \lim_{x \rightarrow \infty} \frac{\left(\ln \frac{1}{x}\right)^{10}}{\frac{1}{\sqrt{x}}}\end{aligned}$$

$$= \lim_{x \rightarrow \infty} \sqrt{x}(-\ln x)^{10}$$

$$\underline{= \infty}$$

Exercise

Evaluate the following limit $\lim_{\theta \rightarrow 0} \frac{3 \sin 8\theta}{8 \sin 3\theta}$

Solution

$$\lim_{\theta \rightarrow 0} \frac{3 \sin 8\theta}{8 \sin 3\theta} = \frac{0}{0}$$

$$= \frac{3}{8} \frac{8}{3}$$

$$\underline{= 1}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} \frac{\ln x^{100}}{\sqrt{x}}$

Solution

$$\lim_{x \rightarrow \infty} \frac{\ln x^{100}}{\sqrt{x}} = \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow \infty} \frac{100(\ln x)}{\sqrt{x}}$$

$$= 100 \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}$$

$$= 200 \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}}$$

$$\underline{= 0}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \csc x \sin^{-1} x$

Solution

$$\lim_{x \rightarrow 0} \csc x \sin^{-1} x = \lim_{x \rightarrow 0} \frac{\sin^{-1} x}{\sin x} = \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\cos x}$$

$$\underline{= 1}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} \frac{\ln^3 x}{\sqrt{x}}$

Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\ln^3 x}{\sqrt{x}} &= \frac{\infty}{\infty} \\&= \lim_{x \rightarrow \infty} \frac{3\ln^2 x}{x} (2\sqrt{x}) & \left(\ln^3 x \right)' &= \frac{3\ln^2 x}{x} \\&= 6 \lim_{x \rightarrow \infty} \frac{\ln^2 x}{\sqrt{x}} = \frac{\infty}{\infty} \\&= 6 \lim_{x \rightarrow \infty} \frac{2\ln x}{x} (2\sqrt{x}) \\&= 24 \lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} \\&= 24 \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} \\&= 48 \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} \\&= 0\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} \ln\left(\frac{x+1}{x-1}\right)$

Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \ln\left(\frac{x+1}{x-1}\right) &= \lim_{x \rightarrow \infty} \ln(1) \\&= 0\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0^+} (1+x)^{\cot x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0^+} (1+x)^{\cot x} &= 1^\infty \\ \lim_{x \rightarrow 0^+} \ln(1+x)^{\cot x} &= \lim_{x \rightarrow 0^+} (\cot x) \ln(1+x)\end{aligned}$$

$$= \lim_{x \rightarrow 0^+} \frac{\ln(1+x)}{\tan x} = \frac{0}{0}$$

$$= \lim_{x \rightarrow 0^+} \frac{1}{\sec^2 x}$$

$$= 1$$

$$\lim_{x \rightarrow 0^+} (1+x)^{\cot x} = e^1 = e$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \frac{\pi}{2}^+} (\sin x)^{\tan x}$

Solution

$$\lim_{x \rightarrow \frac{\pi}{2}^+} (\sin x)^{\tan x} = 1^\infty$$

$$\lim_{x \rightarrow \frac{\pi}{2}^+} \ln(\sin x)^{\tan x} = \lim_{x \rightarrow \frac{\pi}{2}^+} (\tan x) \ln(\sin x)$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\ln(\sin x)}{\cot x}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\frac{\cos x}{\sin x}}{-\csc^2 x}$$

$$= - \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\cos x}{\sin x} \sin^2 x$$

$$= - \lim_{x \rightarrow \frac{\pi}{2}^+} (\cos x \sin x)$$

$$= 0$$

$$\lim_{x \rightarrow \frac{\pi}{2}^+} (\sin x)^{\tan x} = e^0 = 1$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} (\sqrt{x} + 1)^{1/x}$

Solution

$$\lim_{x \rightarrow \infty} (\sqrt{x} + 1)^{1/x} = \infty^0$$

$$\lim_{x \rightarrow \infty} \ln(\sqrt{x} + 1)^{1/x} = \lim_{x \rightarrow \infty} \frac{1}{x} \ln(\sqrt{x} + 1)$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{2\sqrt{x}} \frac{1}{\sqrt{x} + 1}}{1}$$

$$= 0$$

$$\lim_{x \rightarrow \infty} (\sqrt{x} + 1)^{1/x} = e^0 = 1$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0^+} |\ln x|^x$

Solution

$$\lim_{x \rightarrow 0^+} |\ln x|^x = -\infty^0$$

$$\lim_{x \rightarrow 0^+} \ln |\ln x|^x = \lim_{x \rightarrow 0^+} x \ln |\ln x|$$

$$= \lim_{x \rightarrow 0^+} \frac{\ln |\ln x|}{\frac{1}{x}} = \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x |\ln x|}}{-\frac{1}{x^2}}$$

$$= - \lim_{x \rightarrow 0^+} \frac{x}{|\ln x|}$$

$$= 0$$

$$\lim_{x \rightarrow 0^+} |\ln x|^x = e^0 = 1$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} x^{1/x}$

Solution

$$\lim_{x \rightarrow \infty} x^{1/x} = \infty^0$$

$$\begin{aligned}\lim_{x \rightarrow \infty} \ln x^{1/x} &= \lim_{x \rightarrow \infty} \frac{1}{x} \ln x \\ &= \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \frac{\infty}{\infty} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} \\ &= 0\end{aligned}$$

$$\lim_{x \rightarrow \infty} x^{1/x} = e^0 = 1$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} \left(1 - \frac{3}{x}\right)^x$

Solution

$$\lim_{x \rightarrow \infty} \left(1 - \frac{3}{x}\right)^x = 1^\infty$$

$$\begin{aligned}\lim_{x \rightarrow \infty} \ln \left(1 - \frac{3}{x}\right)^x &= \lim_{x \rightarrow \infty} x \ln \left(1 - \frac{3}{x}\right) \\ &= \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x}} \ln \left(1 - \frac{3}{x}\right) = \frac{0}{0} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{3}{x^2} \cdot \frac{1}{1 - \frac{3}{x}} \cdot \frac{1}{-1}}{\frac{1}{x^2}} \\ &= -3 \lim_{x \rightarrow \infty} \frac{1}{1 - \frac{3}{x}} \\ &= -3\end{aligned}$$

$$\lim_{x \rightarrow \infty} \left(1 - \frac{3}{x}\right)^x = e^{-3}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} \left(\frac{2}{\pi} \tan^{-1} x \right)^x$

Solution

$$\lim_{x \rightarrow \infty} \left(\frac{2}{\pi} \tan^{-1} x \right)^x = 1^\infty$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \ln \left(\frac{2}{\pi} \tan^{-1} x \right)^x &= \lim_{x \rightarrow \infty} x \ln \left(\frac{2}{\pi} \tan^{-1} x \right) \\ &= \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x}} \ln \left(\frac{2}{\pi} \tan^{-1} x \right) \end{aligned}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{-\frac{1}{x^2}} \cdot \frac{\frac{2}{\pi} \frac{1}{1+x^2}}{\frac{2}{\pi} \tan^{-1} x}$$

$$= - \lim_{x \rightarrow \infty} \frac{x^2}{(1+x^2) \tan^{-1} x} \quad \lim_{x \rightarrow \infty} \frac{x^2}{1+x^2} = 1$$

$$= - \frac{1}{\frac{\pi}{2}}$$

$$= - \frac{2}{\pi}$$

$$\lim_{x \rightarrow \infty} \left(\frac{2}{\pi} \tan^{-1} x \right)^x = e^{-\frac{2}{\pi}}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 1} (x-1)^{\sin \pi x}$

Solution

$$\lim_{x \rightarrow 1} (x-1)^{\sin \pi x} = 0^0$$

$$\lim_{x \rightarrow 1} \ln (x-1)^{\sin \pi x} = \lim_{x \rightarrow 1} (\sin \pi x) \ln (x-1)$$

$$= \lim_{x \rightarrow 1} \frac{\ln(x-1)}{\csc \pi x}$$

$$= \lim_{x \rightarrow 1} \frac{\frac{1}{x-1}}{-\pi \csc \pi x \cot \pi x}$$

$$= -\frac{1}{\pi} \lim_{x \rightarrow 1} \frac{1}{x-1} \frac{\sin^2 \pi x}{\cos \pi x}$$

$$\begin{aligned}
&= -\frac{1}{\pi} \lim_{x \rightarrow 1} \frac{1}{\cos \pi x} \cdot \lim_{x \rightarrow 1} \frac{\sin^2 \pi x}{x-1} \\
&= \frac{1}{\pi} \lim_{x \rightarrow 1} \frac{2\pi \sin \pi x \cos \pi x}{1} \\
&= 0
\end{aligned}$$

$$\lim_{x \rightarrow 1} (x-1)^{\sin \pi x} = e^0 = 1$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} \frac{2x^5 - x + 1}{5x^6 + x}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow \infty} \frac{2x^5 - x + 1}{5x^6 + x} &= \lim_{x \rightarrow \infty} \frac{2x^5}{5x^6} \\
&= 0
\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} \frac{4x^4 - \sqrt{x}}{2x^4 + x^{-1}}$

Solution

$$\lim_{x \rightarrow \infty} \frac{4x^4 - \sqrt{x}}{2x^4 + x^{-1}} = 2$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \frac{1 - \cos x^n}{x^{2n}}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{1 - \cos x^n}{x^{2n}} &= \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{nx^{n-1} \sin x^n}{2nx^{2n-1}} \\
&= \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin x^n}{x^n} \\
&= \frac{1}{2} \lim_{x^n \rightarrow 0} \frac{\sin x^n}{x^n} \qquad \lim_{u \rightarrow 0} \frac{\sin u}{u} = 1 \\
&= \frac{1}{2}
\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \frac{1 - \cos^n x}{x^2}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1 - \cos^n x}{x^2} &= \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{n(\sin x) \cos^{n-1} x}{2x} = \frac{0}{0} \\&= \frac{1}{2} \lim_{x \rightarrow 0} \frac{n(\cos x) \cos^{n-1} x - n(n-1) \sin^2 x \cos^{n-2} x}{1} \\&= \frac{n}{2} \quad | \end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \frac{1 - \cos x^n}{x^2}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1 - \cos x^n}{x^2} &= \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{nx^{n-1} \sin x^n}{2x} = \frac{0}{0} \\&= \frac{1}{2} \lim_{x \rightarrow 0} x^{n-2} \sin x^n \\&= 0 \quad | \end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \frac{3x}{\tan 4x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{3x}{\tan 4x} &= \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{3}{4 \sec^2 4x} \\&= \frac{3}{4} \quad | \end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{a \cos ax}{b \cos bx} \\ &= \frac{a}{b} \quad \left| \right.\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 2} \frac{\ln(2x-3)}{x^2-4}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{\ln(2x-3)}{x^2-4} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 2} \frac{\frac{2}{2x-3}}{2x} \\ &= \lim_{x \rightarrow 2} \frac{1}{x(2x-3)} \\ &= \frac{1}{2} \quad \left| \right.\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \frac{1 - \cos ax}{1 - \cos bx}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1 - \cos ax}{1 - \cos bx} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{a \sin ax}{b \sin bx} = \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{a^2 \cos ax}{b^2 \cos bx} \\ &= \frac{a^2}{b^2} \quad \left| \right.\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \frac{\sin^{-1} x}{\tan^{-1} x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin^{-1} x}{\tan^{-1} x} &= \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{1}{1+x^2}} \\&= \lim_{x \rightarrow 0} \frac{1+x^2}{\sqrt{1-x^2}} \\&= 1\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{x^{2/3} - 1}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^{1/3} - 1}{x^{2/3} - 1} &= \frac{0}{0} \\&= \lim_{x \rightarrow 1} \frac{\frac{1}{3}x^{-2/3}}{\frac{2}{3}x^{-1/3}} \\&= \frac{1}{2}\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} x \cot x$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} x \cot x &= 0 \cdot \infty \\&= \lim_{x \rightarrow 0} \frac{x \cos x}{\sin x} = \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{\cos x - x \sin x}{\cos x} \\&= 1\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \frac{1 - \cos x}{\ln(1 + x^2)}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1 - \cos x}{\ln(1 + x^2)} &= \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{\frac{\sin x}{2x}}{\frac{2x}{1 + x^2}} \\&= \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin x}{x} (1 + x^2) \quad \lim_{u \rightarrow 0} \frac{\sin u}{u} = 1 \\&= \frac{1}{2} \quad \square\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \pi} \frac{\sin^2 x}{x - \pi}$

Solution

$$\begin{aligned}\lim_{x \rightarrow \pi} \frac{\sin^2 x}{x - \pi} &= \frac{0}{0} \\&= \lim_{x \rightarrow \pi} \frac{2 \sin x \cos x}{1} \\&= 0 \quad \square\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \frac{10^x - e^x}{x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{10^x - e^x}{x} &= \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{10^x \ln 10 - e^x}{1} \\&= \ln(10) - 1 \quad \square\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos 3x}{\pi - 2x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos 3x}{\pi - 2x} &= \frac{0}{0} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{-3 \sin 3x}{-2} \\ &= \frac{3}{2}\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 1} \frac{\ln(ex) - 1}{\sin \pi x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{\ln(ex) - 1}{\sin \pi x} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{\cos \pi x} \\ &= \frac{1}{\pi}\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} x \sin \frac{1}{x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} x \sin \frac{1}{x} &= \lim_{x \rightarrow \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = \frac{0}{0} & y = \frac{1}{x} \xrightarrow{x \rightarrow \infty} 0 \\ &= \lim_{y \rightarrow 0} \frac{\sin y}{y} \\ &= 1\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} &= \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2} = \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{\sin x}{6x} = \frac{0}{0} \\
&= \frac{1}{6}
\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \frac{x - \sin x}{x - \tan x}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{x - \sin x}{x - \tan x} &= \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{1 - \cos x}{1 - \sec^2 x} = \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{\sin x}{-2 \sec^2 x \tan x} = \frac{0}{0} \\
&= -\frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin x}{\frac{1}{\cos^2 x} \frac{\sin x}{\cos x}} \\
&= -\frac{1}{2} \lim_{x \rightarrow 0} \cos x \\
&= -\frac{1}{2}
\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \frac{2 - x^2 - 2 \cos x}{x^4}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{2 - x^2 - 2 \cos x}{x^4} &= \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{-2x + 2 \sin x}{4x^3} = \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{-2 + 2 \cos x}{12x^2} = \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{-2 \sin x}{24x} = \frac{0}{0} \\
&= -\frac{1}{12}
\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0^+} \frac{\sin^2 x}{\tan x - x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{\sin^2 x}{\tan x - x} &= \frac{0}{0} \\&= \lim_{x \rightarrow 0^+} \frac{2 \sin x \cos x}{\sec^2 x - 1} = \frac{0}{0} \\&= \lim_{x \rightarrow 0^+} \frac{\sin 2x}{\tan^2 x} = \frac{0}{0} \\&= \lim_{x \rightarrow 0^+} \frac{2 \cos 2x}{2 \tan x \sec^2 x} \\&= \frac{1}{0^+} \\&= \infty\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\ln \sin x}{\cos x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow \frac{\pi}{2}} \frac{\ln \sin x}{\cos x} &= \frac{0}{0} \\&= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\frac{\cos x}{\sin x}}{-\sin x} \\&= - \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\sin^2 x} \\&= 0\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x}{x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x}{x} &= \frac{1}{\frac{\pi}{2}} \\&= \frac{2}{\pi}\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 1^-} \frac{\arccos x}{x-1}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 1^-} \frac{\arccos x}{x-1} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 1^-} \frac{-\frac{1}{\sqrt{1-x^2}}}{1} \\ &= -\infty\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} x(2 \tan^{-1} x - \pi)$

Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} x(2 \tan^{-1} x - \pi) &= \infty \cdot 0 \\ &= \lim_{x \rightarrow \infty} \left(\frac{2 \tan^{-1} x}{x} - \frac{\pi}{x} \right) \\ &= \lim_{x \rightarrow \infty} \left(\frac{2 \tan^{-1} x}{x} \right) - \lim_{x \rightarrow \infty} \left(\frac{\pi}{x} \right) \\ &= 2 \lim_{x \rightarrow \infty} \left(\frac{\tan^{-1} x}{x} \right) \\ &= 2 \lim_{x \rightarrow \infty} \frac{\frac{1}{1+x^2}}{1} \\ &= 2 \lim_{x \rightarrow \infty} \frac{1}{x^2} \\ &= 0\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \frac{\pi}{2}^+} x(\sec x - \tan x)$

Solution

$$\begin{aligned}\lim_{x \rightarrow \frac{\pi}{2}^+} x(\sec x - \tan x) &= \infty - \infty \\ &= \lim_{x \rightarrow \frac{\pi}{2}^+} x \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)\end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow \frac{\pi}{2}^+} x \left(\frac{1 - \sin x}{\cos x} \right) = \frac{0}{0} \\
&= \lim_{x \rightarrow \frac{\pi}{2}^+} x \left(\frac{\cos x}{-\sin x} \right) \\
&= 0
\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{xe^{ax}} \right)$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{xe^{ax}} \right) &= \infty - \infty \\
&= \lim_{x \rightarrow 0} \frac{e^{ax} - 1}{xe^{ax}} = \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{ae^{ax}}{(1 + ax)e^{ax}} \\
&= a
\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0^+} x^{\sqrt{x}}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 0^+} x^{\sqrt{x}} &= 0^0 \\
\lim_{x \rightarrow 0^+} \ln x^{\sqrt{x}} &= \lim_{x \rightarrow 0^+} \sqrt{x} \ln x \\
&= \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-1/2}} \\
&= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} \\
&= -2 \lim_{x \rightarrow 0^+} \sqrt{x} \\
&= 0
\end{aligned}$$

$$\lim_{x \rightarrow 0^+} x^{\sqrt{x}} = e^0 = 1$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \pi} \frac{\cos x + 1}{(x - \pi)^2}$

Solution

$$\begin{aligned}\lim_{x \rightarrow \pi} \frac{\cos x + 1}{(x - \pi)^2} &= \frac{0}{0} \\&= \lim_{x \rightarrow \pi} \frac{-\sin x}{2(x - \pi)} = \frac{0}{0} \\&= \lim_{x \rightarrow \pi} \frac{-\cos x}{2} \\&= \frac{1}{2}\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \frac{\sin x - x}{7x^3}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin x - x}{7x^3} &= \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{\cos x - 1}{21x^2} = \frac{0}{0} \\&= \lim_{x \rightarrow 0} \frac{-\sin x}{42x} \\&= -\frac{1}{42}\end{aligned}$$

$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} \frac{\tan^{-1} x - \frac{\pi}{2}}{\frac{1}{x}}$

Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\tan^{-1} x - \frac{\pi}{2}}{\frac{1}{x}} &= \frac{\frac{\pi}{2} - \frac{\pi}{2}}{0} = \frac{0}{0} \\&= \lim_{x \rightarrow \infty} \frac{\frac{1}{1+x^2}}{-\frac{1}{x^2}} \\&= - \lim_{x \rightarrow \infty} \frac{x^2}{1+x^2} \\&= -1\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 3} \frac{x-1-\sqrt{x^2-5}}{x-3}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{x-1-\sqrt{x^2-5}}{x-3} &= \frac{2-2}{0} = \frac{0}{0} \\ &= \lim_{x \rightarrow 3} \frac{1 - \frac{x}{\sqrt{x^2-5}}}{1} \\ &= 1 - \frac{3}{2} \\ &= -\frac{1}{2}\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 2} \frac{x^2+x-6}{\sqrt{8-x^2}-x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^2+x-6}{\sqrt{8-x^2}-x} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 2} \frac{2x+1}{\frac{-x}{\sqrt{8-x^2}}-1} \\ &= -\frac{5}{2}\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 2} \frac{x^2-4x+4}{\sin^2 \pi x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^2-4x+4}{\sin^2 \pi x} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 2} \frac{2x-4}{2\pi \sin \pi x \cos \pi x} \\ &= \lim_{x \rightarrow 2} \frac{2x-4}{\pi \sin 2\pi x} = \frac{0}{0} \\ &= \lim_{x \rightarrow 2} \frac{2}{2\pi^2 \cos 2\pi x} \\ &= \frac{1}{\pi^2}\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 2} \frac{(3x+2)^{1/3} - 2}{x-2}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{(3x+2)^{1/3} - 2}{x-2} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 2} \frac{(3x+2)^{-2/3}}{1} \\ &= 8^{-2/3} \\ &= \frac{1}{4}\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} \frac{3x^4 - x^2}{6x^4 + 12}$

Solution

$$\lim_{x \rightarrow \infty} \frac{3x^4 - x^2}{6x^4 + 12} = \frac{3}{6} = \frac{1}{2}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} \frac{4x^3 - 2x^2 + 6}{\pi x^3 + 4}$

Solution

$$\lim_{x \rightarrow \infty} \frac{4x^3 - 2x^2 + 6}{\pi x^3 + 4} = \frac{4}{\pi}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \frac{\pi}{2}^-} \frac{1}{3}(2x - \pi) \tan x$

Solution

$$\begin{aligned}\lim_{x \rightarrow \frac{\pi}{2}^-} \frac{1}{3}(2x - \pi) \tan x &= 0 \cdot \infty \\ &= \frac{1}{3} \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\tan x}{\frac{1}{2x - \pi}} = \frac{0}{0}\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3} \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\sec^2 x}{\frac{-2}{(2x - \pi)^2}} \\
&= -\frac{1}{6} \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{(2x - \pi)^2}{\cos^2 x} = \frac{0}{0} \\
&= -\frac{1}{6} \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{4(2x - \pi)}{-2 \sin x \cos x} \\
&= \frac{2}{3} \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{2x - \pi}{\sin 2x} = \frac{0}{0} \\
&= \frac{2}{3} \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{2}{2 \cos 2x} \\
&= -\frac{2}{3} \quad |
\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x}\right)$

Solution

$$\begin{aligned}
\lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x}\right) &= \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \frac{0}{0} \\
&= \lim_{x \rightarrow \infty} \frac{-\frac{1}{x^2}}{1 + \frac{1}{x} - \frac{1}{x^2}} \cdot \frac{1}{\frac{1}{x^2}} \\
&= \lim_{x \rightarrow \infty} \frac{1}{1 + \frac{1}{x}} \qquad \lim_{x \rightarrow \infty} \frac{1}{x} = 0 \\
&= 1 \quad |
\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \frac{\pi}{2}^-} \left(\frac{\pi}{2} - x\right) \sec x$

Solution

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \left(\frac{\pi}{2} - x\right) \sec x = \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\frac{\pi}{2} - x}{\cos x} = \frac{0}{0}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{-1}{-\sin x}$$

$$= 1$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} \frac{e^{1/x} - 1}{\sin \frac{1}{x}}$

Solution

$$\lim_{x \rightarrow \infty} \frac{e^{1/x} - 1}{\sin \frac{1}{x}} = \frac{0}{0}$$

$$= \lim_{x \rightarrow \infty} \frac{-\frac{1}{x^2} e^{1/x}}{-\frac{1}{x^2} \cos \frac{1}{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{e^{1/x}}{\cos \frac{1}{x}}$$

$$= 1$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0^+} \sin x \sqrt{\frac{1-x}{x}}$

Solution

$$\lim_{x \rightarrow 0^+} \sin x \sqrt{\frac{1-x}{x}} = \lim_{x \rightarrow 0^+} \sin x \sqrt{\frac{x(1-x)}{x^2}}$$

$$= \lim_{x \rightarrow 0^+} \frac{\sin x}{x} \sqrt{x(1-x)}$$

$$= 1 \cdot 0$$

$$= 0$$

$$\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \left(\cot x - \frac{1}{x} \right)$

Solution

$$\lim_{x \rightarrow 0} \left(\cot x - \frac{1}{x} \right) = \infty - \infty$$

$$\begin{aligned}
&= \lim_{x \rightarrow 0} \left(\frac{\cos x}{\sin x} - \frac{1}{x} \right) \\
&= \lim_{x \rightarrow 0} \left(\frac{x \cos x - \sin x}{x \sin x} \right) = \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{\cos x - x \sin x - \cos x}{\sin x + x \cos x} \\
&= \lim_{x \rightarrow 0} \frac{-x \sin x}{\sin x + x \cos x} = \frac{0}{0} \\
&= \lim_{x \rightarrow 0} \frac{-\sin x - x \cos x}{\cos x + \cos x - x \sin x} \\
&= \frac{0}{2} \\
&= 0
\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 1} \right)$

Solution

$$\begin{aligned}
\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 + 1} \right) &= \infty - \infty \\
&= \lim_{x \rightarrow \infty} \left(x - x \sqrt{1 + \frac{1}{x^2}} \right) \\
&= \lim_{x \rightarrow \infty} x \left(1 - \sqrt{1 + \frac{1}{x^2}} \right) \\
&= \lim_{t \rightarrow 0} \frac{1}{t} \left(1 - \sqrt{1 + t^2} \right) \qquad t = \frac{1}{x} \xrightarrow{x \rightarrow \infty} 0 \\
&= \lim_{t \rightarrow 0} \frac{1 - \sqrt{1 + t^2}}{t} = \frac{0}{0} \\
&= \lim_{t \rightarrow 0} \frac{\frac{t}{\sqrt{1 + t^2}}}{1} \\
&= 0
\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{\theta \rightarrow \frac{\pi}{2}^-} (\tan \theta - \sec \theta)$

Solution

$$\begin{aligned}
\lim_{\theta \rightarrow \frac{\pi}{2}^-} (\tan \theta - \sec \theta) &= \infty - \infty \\
&= \lim_{\theta \rightarrow \frac{\pi}{2}^-} \left(\frac{\sin \theta}{\cos \theta} - \frac{1}{\cos \theta} \right) \\
&= \lim_{\theta \rightarrow \frac{\pi}{2}^-} \frac{\sin \theta - 1}{\cos \theta} = \frac{0}{0} \\
&= \lim_{\theta \rightarrow \frac{\pi}{2}^-} \frac{\cos \theta}{-\sin \theta} \\
&= \underline{0}
\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0^+} \ln x^{2x}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 0^+} \ln x^{2x} &= \lim_{x \rightarrow 0^+} 2x \ln x \\
&= 2 \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} \\
&= 2 \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\
&= -2 \lim_{x \rightarrow 0^+} x \\
&= \underline{0}
\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} \ln(1+4x)^{3/x}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} \ln(1+4x)^{3/x} &= 3 \lim_{x \rightarrow 0} \frac{\ln(1+4x)}{x} = \frac{0}{0} \\
&= 3 \lim_{x \rightarrow 0} \frac{\frac{4}{1+4x}}{1} \\
&= \underline{12}
\end{aligned}$$

Exercise

Evaluate the following limit $\lim_{\theta \rightarrow \frac{\pi}{2}^-} \ln(\tan \theta)^{\cos \theta}$

Solution

$$\begin{aligned}\lim_{\theta \rightarrow \frac{\pi}{2}^-} \ln(\tan \theta)^{\cos \theta} &= \lim_{\theta \rightarrow \frac{\pi}{2}^-} \cos \theta \ln(\tan \theta) \\&= \lim_{\theta \rightarrow \frac{\pi}{2}^-} \frac{\ln(\tan \theta)}{\sec \theta} \\&= \lim_{\theta \rightarrow \frac{\pi}{2}^-} \frac{\sec^2 \theta}{\sec \theta \tan \theta} \\&= \lim_{\theta \rightarrow \frac{\pi}{2}^-} \frac{\sec \theta}{\tan^2 \theta} \\&= \frac{0}{\infty} \\&= \underline{0} \quad | \end{aligned}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0^+} (1+x)^{\cot x}$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0^+} (1+x)^{\cot x} &= 1^\infty \\ \lim_{x \rightarrow 0^+} \ln(1+x)^{\cot x} &= \lim_{x \rightarrow 0^+} \cot x \ln(1+x) \\&= \lim_{x \rightarrow 0^+} \frac{\ln(1+x)}{\tan x} = \frac{0}{0} \\&= \lim_{x \rightarrow 0^+} \frac{\frac{1}{1+x}}{\sec^2 x} \\&= \lim_{x \rightarrow 0^+} \frac{1}{(1+x)\sec^2 x} \\&= \underline{1} \quad | \end{aligned}$$

$$\lim_{x \rightarrow 0^+} (1+x)^{\cot x} = e^1 = \underline{e} \quad |$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{\ln x}$

Solution

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{\ln x} = e^L$$

$$L = \lim_{x \rightarrow \infty} \ln \left(1 + \frac{1}{x}\right)^{\ln x}$$

$$= \lim_{x \rightarrow \infty} (\ln x) \ln \left(1 + \frac{1}{x}\right)$$

$$= \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{\ln x}}$$

$$= \lim_{x \rightarrow \infty} \frac{-\frac{1}{x^2}}{1 + \frac{1}{x} \cdot \frac{-1}{(\ln x)^2} \cdot \frac{1}{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{x^2} \cdot \frac{1}{\frac{x+1}{x}} x (\ln x)^2$$

$$= \lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x+1} = \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow \infty} \frac{2 \ln x}{1}$$

$$= 2 \lim_{x \rightarrow \infty} \frac{\ln x}{x}$$

$$= 2 \lim_{x \rightarrow \infty} \frac{1}{x}$$

$$= 0$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{\ln x} = e^0 = 1$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x$

Solution

$$\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^x = e^L$$

$$\begin{aligned}
L &= \lim_{x \rightarrow \infty} \ln \left(1 + \frac{a}{x} \right)^x \\
&= \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{a}{x} \right) \\
&= \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{a}{x} \right)}{\frac{1}{x}} \\
&= \lim_{x \rightarrow \infty} \frac{1}{-\frac{1}{x^2}} \frac{-\frac{a}{x^2}}{1 + \frac{a}{x}} \\
&= \lim_{x \rightarrow \infty} \frac{a}{1 + \frac{a}{x}} \\
&= a
\end{aligned}$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} \right)^x = e^a$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} (e^{5x} + x)^{1/x}$

Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} (e^{5x} + x)^{1/x} &= e^L \\
L &= \lim_{x \rightarrow 0} \ln (e^{5x} + x)^{1/x} \\
&= \lim_{x \rightarrow 0} \frac{\ln (e^{5x} + x)}{x} \\
&= \lim_{x \rightarrow 0} \frac{5e^{5x} + 1 \cdot 1}{e^{5x} + x} \cdot \frac{1}{1} \\
&= 6
\end{aligned}$$

$$\lim_{x \rightarrow 0} (e^{5x} + x)^{1/x} = e^6$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} (e^{ax} + x)^{1/x}$

Solution

$$\lim_{x \rightarrow 0} (e^{ax} + x)^{1/x} = e^L$$

$$L = \lim_{x \rightarrow 0} \ln(e^{ax} + x)^{1/x}$$

$$= \lim_{x \rightarrow 0} \frac{\ln(e^{ax} + x)}{x}$$

$$= \lim_{x \rightarrow 0} \frac{ae^{ax} + 1}{e^{ax} + x} \cdot \frac{1}{1}$$

$$= \underline{a + 1}$$

$$\lim_{x \rightarrow 0} (e^{ax} + x)^{1/x} = \underline{e^{a+1}}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0} (2^{ax} + x)^{1/x}$

Solution

$$\lim_{x \rightarrow 0} (2^{ax} + x)^{1/x} = e^L$$

$$L = \lim_{x \rightarrow 0} \ln(2^{ax} + x)^{1/x}$$

$$= \lim_{x \rightarrow 0} \frac{\ln(2^{ax} + x)}{x}$$

$$= \lim_{x \rightarrow 0} \frac{a2^{ax} \ln 2 + 1}{2^{ax} + x} \cdot \frac{1}{1}$$

$$= \underline{a \ln 2 + 1}$$

$$\lim_{x \rightarrow 0} (2^{ax} + x)^{1/x} = e^{a \ln 2 + 1}$$

$$= e \cdot e^{a \ln 2}$$

$$= \underline{e \cdot 2^a}$$

Exercise

Evaluate the following limit $\lim_{x \rightarrow 0^+} (\tan x)^x$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0^+} (\tan x)^x &= e^L \\ L &= \lim_{x \rightarrow 0^+} \ln(\tan x)^x \\ &= \lim_{x \rightarrow 0^+} x \ln(\tan x) \\ &= \lim_{x \rightarrow 0^+} \frac{\ln(\tan x)}{\frac{1}{x}} \\ &= \lim_{x \rightarrow 0^+} \frac{\sec^2 x \cdot \frac{1}{\tan x}}{-\frac{1}{x^2}} \\ &= - \lim_{x \rightarrow 0^+} \frac{x^2 \cos x}{\sin x \cos^2 x} \\ &= - \lim_{x \rightarrow 0^+} \frac{x^2}{\sin x \cos x} \\ &= - \lim_{x \rightarrow 0^+} \frac{x}{\sin x} \cdot \lim_{x \rightarrow 0^+} \frac{x}{\cos x} \\ &= -(1) \cdot (0) \\ &= \underline{0} \\ \lim_{x \rightarrow 0^+} (\tan x)^x &= e^0 = \underline{1}\end{aligned}$$

Exercise

The functions $f(x) = (x^x)^x$ and $g(x) = x^{(x^x)}$ are different functions. For example, $f(3) = 19,683$ and $g(3) \approx 7.6 \times 10^{12}$. Determine whether $\lim_{x \rightarrow 0^+} f(x)$ and $\lim_{x \rightarrow 0^+} g(x)$ are indeterminate forms and evaluate the limits.

Solution

$$\begin{aligned}\lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} (x^x)^x \\ \lim_{x \rightarrow 0^+} \ln(x^x)^x &= \lim_{x \rightarrow 0^+} x \ln(x^x)\end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 0^+} x^2 \ln x \\
&= \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x^2}} \\
&= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{2}{x^3}} \\
&= - \lim_{x \rightarrow 0^+} \frac{x^2}{2} \\
&= \underline{0}
\end{aligned}$$

$$\lim_{x \rightarrow 0^+} (x^x)^x = e^0 = \underline{1}$$

$$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} x^{(x^x)}$$

$$\lim_{x \rightarrow 0^+} (x^x) = e^L$$

$$L = \lim_{x \rightarrow 0^+} \ln(x^x)$$

$$= \lim_{x \rightarrow 0^+} x \ln x$$

$$= \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}}$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x \rightarrow 0^+} (-x)$$

$$= \underline{0}$$

$$\lim_{x \rightarrow 0^+} (x^x) = e^0 = \underline{1}$$

$$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} x^{(x^x)}$$

$$= \lim_{x \rightarrow 0^+} 0^1$$

$$= \underline{0}$$

Exercise

Consider the function $g(x) = \left(1 + \frac{1}{x}\right)^{x+a}$. show that if $0 \leq a < \frac{1}{2}$, then $g(x) \rightarrow e$ from below as $x \rightarrow \infty$; if $\frac{1}{2} \leq a < 1$, then $g(x) \rightarrow e$ from above as $x \rightarrow \infty$

Solution

$$\begin{aligned}\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{x+a} &= \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^a \\ &= e \cdot 1 \\ &= e\end{aligned}$$

$$\lim_{x \rightarrow \infty} \ln g(x) = \ln 2 = 1$$

It suffices to determine whether $\ln g(x) - 1$ is $(+)$ or $(-)$ as $x \rightarrow \infty$

Consider

$$\lim_{x \rightarrow \infty} x(\ln g(x) - 1) = \lim_{x \rightarrow \infty} x \left[(x+a) \ln \left(1 + \frac{1}{x}\right) - 1 \right]$$

$$\text{Let } t = \frac{1}{x} \rightarrow 0 \Rightarrow x = \frac{1}{t}$$

$$\begin{aligned}\lim_{x \rightarrow \infty} x \left[(x+a) \ln \left(1 + \frac{1}{x}\right) - 1 \right] &= \lim_{t \rightarrow 0} \frac{1}{t} \left[\left(\frac{1}{t} + a\right) \ln(1+t) - 1 \right] \\ &= \lim_{t \rightarrow 0} \frac{(1+at) \ln(1+t) - t}{t^2} = \frac{0}{0} \\ &= \lim_{t \rightarrow 0} \frac{a \ln(1+t) + \frac{1+at}{1+t} - 1}{2t} \\ &= \lim_{t \rightarrow 0} \frac{a(1+t) \ln(1+t) + 1 + at - 1 - t}{2t(1+t)} \\ &= \lim_{t \rightarrow 0} \frac{a(1+t) \ln(1+t) + (a-1)t}{2t + 2t^2} = \frac{0}{0} \\ &= \lim_{t \rightarrow 0} \frac{a \ln(1+t) + a + a - 1}{2 + 4t} \\ &= \lim_{t \rightarrow 0} \frac{a \ln(1+t) + 2a - 1}{2 + 4t} \\ &= \frac{2a-1}{2} \\ &= a - \frac{1}{2}\end{aligned}$$

When $a > \frac{1}{2} \Rightarrow g(x) > e$ as $x \rightarrow \infty$

If $0 \leq a < \frac{1}{2} \Rightarrow g(x) < e$ as $x \rightarrow \infty$

Exercise

Let $f(x) = (a+x)^x$, where $a > 0$

- a) What is the domain of f (in terms of a)?
- b) Describe the end behavior of f (near the left boundary of its domain and as $x \rightarrow \infty$).
- c) Compute f' .
- d) Show that f has a single local minimum at the point z that satisfies $(z+a)\ln(z+a) + z = 0$
- e) Describe how $f(z)$ varies as a increases.

Solution

a) $a+x \geq 0 \Rightarrow \text{Domain: } [-a, \infty)$

b) $\lim_{x \rightarrow -a^+} (a+x)^x = e^L$

$$L = \lim_{x \rightarrow -a^+} \ln(a+x)^x$$

$$= \lim_{x \rightarrow -a^+} x \ln(a+x)$$

$$= \lim_{x \rightarrow -a^+} \frac{\ln(a+x)}{\frac{1}{x}}$$

$$= \lim_{x \rightarrow -a^+} \frac{\frac{1}{a+x}}{-\frac{1}{x^2}}$$

$$= \lim_{x \rightarrow -a^+} \frac{-x^2}{a+x}$$

$$= \frac{-a^2}{0^-}$$

$$= \infty$$

$$\lim_{x \rightarrow -a^+} (a+x)^x = e^\infty = \infty$$

c) $\ln f(x) = \ln(a+x)^x$

$$(\ln f(x))' = (x \ln(a+x))'$$

$$\frac{f'(x)}{f(x)} = \ln(a+x) + \frac{x}{a+x}$$

$$f'(x) = \left(\ln(a+x) + \frac{x}{a+x} \right) (a+x)^x$$

$$= (a+x)^x \ln(a+x) + x(a+x)^{x-1}$$

$$d) \quad f'(x) = \left(\ln(a+x) + \frac{x}{a+x} \right) (a+x)^x = 0$$

$$\ln(a+x) + \frac{x}{a+x} = 0$$

$$(a+x)\ln(a+x) + x = 0$$

Let $z = a$

$$(z+a)\ln(z+a) + z = 0$$

$$\ln(a+z) + \frac{z}{a+z} = 0$$

$$\ln(a+z) + \frac{z + \textcolor{red}{a} - \textcolor{red}{a}}{a+z} = 0$$

$$\ln(a+z) + 1 - \frac{a}{a+z} = 0$$

$$\ln(a+z) + -\frac{a}{a+z} = -1$$

As z increases left side increases.

$$e) \quad \text{As } a \rightarrow \infty \Rightarrow z \rightarrow -\infty \Rightarrow \textcolor{blue}{f(x)} \rightarrow 0$$