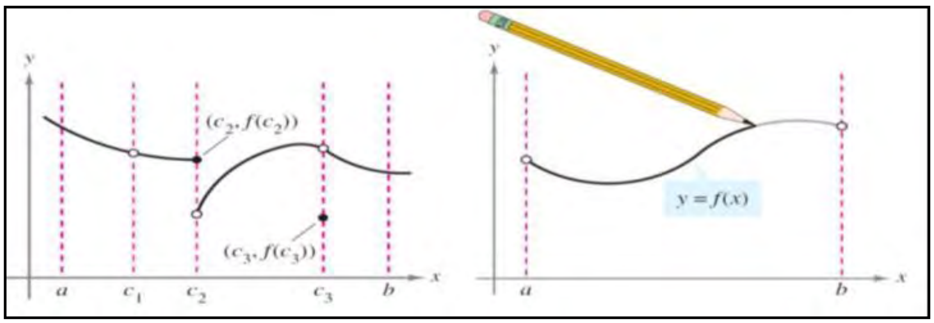
# *Section* 1.6 – Continuity and Rates of Change

## Definition of Continuity

Let $c$ be a number in the interval ($a$, $b$), and let $f$ be a function whose domain contains the interval ($a$, $b$). The function $f$ is continuous at the point $c$ if the following conditions are true.

1. $f(c)$ is defined
2. $\lim_{x\to c} f(x)$ exists
3. $\lim_{x\to c} f(x) = f(c)$

If $f$ is continuous at every point in the interval ($a$, $b$), then it is continuous on an open interval ($a$, $b$)




The Continuity of Polynomial & Rational functions:

1- A Polynomial function is continuous @ every real number

2- A rational function is continuous @ every point in its domain

$x \neq c \Rightarrow$ Continuous ($-\infty$, $c$) and ($c$, $\infty$)

### *Example*

Find all values of $x$ where the following function is discontinuous

$$f(x) = \begin{cases} x+1 & if \quad x<1 \\ x^2-3x+4 & if \quad 1 \le x \le 3 \\ 5-x & if \quad x>3 \end{cases}$$

<u>*Solution*</u>

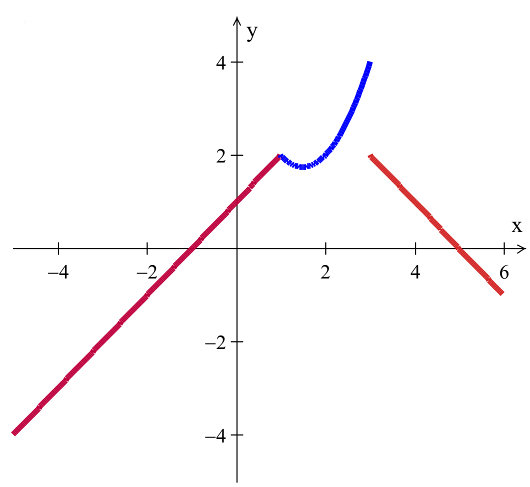
$$\lim_{x\to 1^-} (x+1) = 1+1 = 2$$

$$f(1) = (1)^2 - 3(1) + 4 = 2$$

$$f(3) = (3)^2 - 3(3) + 4 = 4$$

$$\lim_{x\to 3^+} (5-x) = 5-3 = 2$$

So $f$ is discontinue at $x = 3$



***Example***

*a)* $f(x)=\frac{1}{x-1}$

Consists of all real number except $x = 1$.

Or

Continuous on (-∞, 1) and (1, ∞)

*b)* $f(x)=\frac{x^2-4}{x-2}$

Continuous on (-∞, 2) and (2, ∞)

*c)* $f(x)=x^2-2x+3$

Continuous @ every real number

*d)* $f(x)=x^3+x$

Continuous @ every real number

If $f$ is not continuous @ $x = c$ ⇒ function is said to have discontinuity @ $c$

⇒ This type of discontinuity falls into 2 categories:

1. Removable   *ex.* $\frac{x^2-4}{x-2}$   $\lim \frac{x^2-4}{x-2} = exists = \text{constant}$
2. Non-removable   *ex.* $\frac{1}{x-1}$   $\lim = doesn't\ exist = \pm\infty$

## *Definition*: Continuity on Close Interval

Let $f$ be defined on a closed interval $[a, b]$, if $f$ is continuous on the open interval

$$\Rightarrow \lim_{x \to a^+} f(x) = f(a) \quad \textit{and} \quad \lim_{x \to b^-} f(x) = f(b) \quad \Rightarrow f \textit{ is continuous } [a, b]$$

***Example***

Discuss the continuity of $f(x) = \sqrt{x-2}$

*Solution*

*Domain*: $x - 2 \geq 0 \Rightarrow x \geq 2$

$$\Rightarrow \lim_{x \to 2^+} \sqrt{x-2} = 0 = f(2)$$

***Example***

Discuss the continuity of

$$f(x) = \begin{cases} x+2 & -1 \leq x < 3 \\ 14 - x^2 & 3 \leq x \leq 5 \end{cases}$$

*Solution*

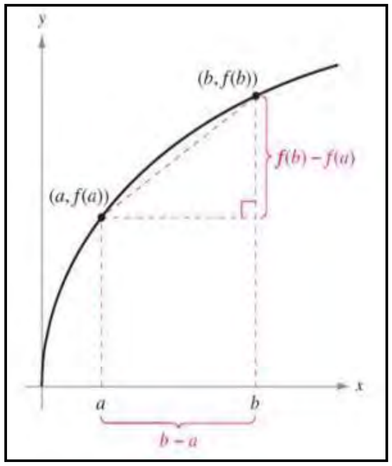
$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x+2) = 5$$

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} (14 - x^2) = 5$$

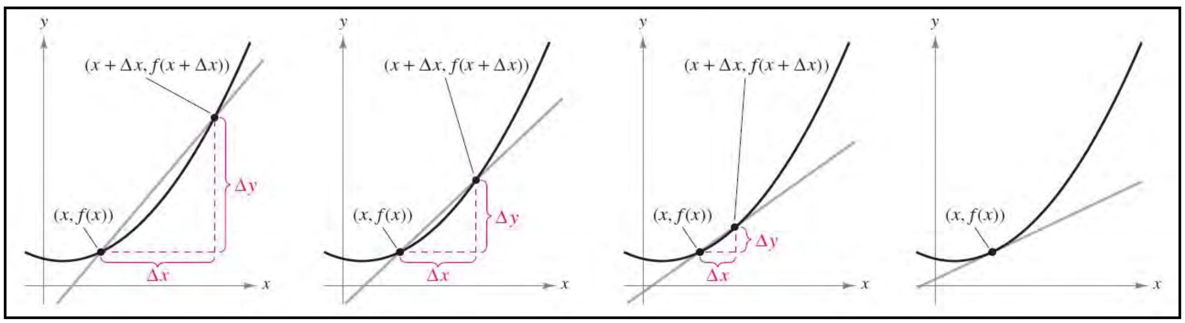
$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (14 - x^2) = 5$$

*Slope and Rate of change*: Given $f(x)$ @ $(x, f(x))$

## Definition of Average Rate of Change




$$\textit{Average Rate of Change} = \frac{f(b)-f(a)}{b-a} = \frac{\Delta y}{\Delta x}$$




## Slope and limit process

The secant line contains the points $(c, f(c))$ and $(c+\Delta x, f(c+\Delta x))$. Using the slope formula, the slope of the secant line is

$$m_{\text{sec}} = \frac{f(x+\Delta x)-f(x)}{x+\Delta x - x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

If you allow $\Delta x$ to become smaller, the secant line will change slope and become closer to the slope of the tangent line. If you were to allow $\Delta x$ to equal 0, the secant line becomes the tangent line. Of course you cannot allow $\Delta x$ to equal 0, because the slope of the secant line would then be undefined. But you could let $\Delta x$ approach 0. As $\Delta x$ approaches 0, the secant line becomes the tangent line to the graph at the point $(x, f(x))$. Thus the slope of the secant line becomes the slope of the tangent line at the point $(x, f(x))$ is given by:

$$m = \lim_{\Delta x \to 0} m_{\text{sec}} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \textit{(Difference Quotient)}$$

As we know: $m = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

***Example***

Cigarette consumption in the United Sates has been declining since reaching a peak around 1960. Per capita cigarette consumption since 1980 can be closely approximated by the function

$$f(t) = 3870(0.970)^t$$

Where $t$ is the number of years since 1980. Find the average rate of change of per capita consumption from 1985 to 2005.

Solution

$t = 1985 - 1980 = 5$

$f(t=5) = 3870(0.970)^5 = 3323.3$

$f(t = 2005 - 1980 = 25) = 3870(0.970)^{25} = 1807.2$

$$\frac{f(25) - f(5)}{25-5} = \frac{1807.2 - 3323.3}{20}$$
$$= -76$$

The average rate decreased at a rate of 76 cigarettes per year.

## *Definition of Instantaneous rate of Change*

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

***Example***

The distance in feet of an object from a starting point is given by $s(t) = 2t^2 - 5t + 40$, where $t$ is time in seconds.

*a)* Find the average velocity of the object from 2 sec to 4 sec.

The average velocity: $= \dfrac{s(4) - s(2)}{4-2}$

$$= \frac{\left(2(4)^2 - 5(4) + 40\right) - \left(2(2)^2 - 5(2) + 40\right)}{2}$$

$$= 7 \ \ ft/sec$$

*c)* Find the instantaneous velocity at 4 sec.

$$\lim_{b\to 4}\frac{s(b)-s(4)}{b-4}=\lim_{b\to 4}\frac{\left(2(b)^2-5(b)+40\right)-\left(2(4)^2-5(4)+40\right)}{b-4}$$

$$=\lim_{b\to 4}\frac{2b^2-5b+40-52}{b-4}$$

$$=\lim_{b\to 4}\frac{2b^2-5b-12}{b-4}$$

$$=\lim_{b\to 4}\frac{(2b+3)(b-4)}{b-4}$$

$$=\lim_{b\to 4}(2b+3)$$

$$=11 \quad ft\,/\,sec$$

***Example***

Find the slope of the graph of $f(x)=x^2$ at the point (2, 4)

<u>*Solution*</u>

$$m_{\sec}=\frac{f(2+\Delta x)-f(2)}{\Delta x}$$

$$=\frac{(2+\Delta x)^2-2^2}{\Delta x}$$

$$=\frac{4+\Delta x^2+4\Delta x-4}{\Delta x}$$

$$=\frac{\Delta x^2+4\Delta x}{\Delta x}$$

$$=\frac{\Delta x^2}{\Delta x}+\frac{4\Delta x}{\Delta x}$$

$$=\Delta x+4$$

$$m=\lim_{\Delta x\to 0} m_{\sec}$$

$$=\lim_{\Delta x\to 0}(\Delta x+4)$$

$$=4$$

# *Exercises* *Section* 1.6 – Continuity and Rates of Change

Determine whether $f(x)$ is continuous on the entire number line. Explain your reasoning.

**1.** $f(x)=\dfrac{x}{x^2-1}$

**2.** $f(x)=\dfrac{x-5}{x^2-9x+20}$

**3.** Find the slope of the graph of $f(x)=2x+5$

**4.** Find the slope of the graph of $f(x)=\sqrt{x}$