

Solution **Section 2.5 – Higher-Order Derivatives**

Exercise

Find the second derivative: $f(x) = 3(2 - x^2)^3$

Solution

$$\begin{aligned} f'(x) &= 9(-2x)(2 - x^2)^2 \\ &= -18x(2 - x^2)^2 \end{aligned}$$

$$U = 2 - x^2 \Rightarrow U' = -2x$$

$$f = x \qquad f' = 1$$

$$g = (2 - x^2)^2 \qquad g' = -2x(2 - x^2)$$

$$\begin{aligned} f''(x) &= -18 \left[(2 - x^2)^2 + x(-2x)(2 - x^2) \right] \\ &= -18(2 - x^2) [2 - x^2 - 2x^2] \\ &= -18(2 - x^2)(2 - 3x^2) \end{aligned}$$

Exercise

Find the third derivative: $f(x) = 5x(x + 4)^3$

Solution

$$\begin{aligned} f'(x) &= 5 \left[(x + 4)^3 + 3x(x + 4)^2 \right] \\ &= 5(x + 4)^2 [(x + 4) + 3x] \\ &= 5(x + 4)^2 (4x + 4) \\ &= 20(x + 4)^2 (x + 1) \end{aligned}$$

$$\begin{aligned} f''(x) &= 20 \left[2(x + 4)(x + 1) + (x + 4)^2 \right] \\ &= 20(x + 4)(2x + 2 + x + 4) \\ &= 20(x + 4)(3x + 6) \\ &= 60(x + 4)(x + 2) \end{aligned}$$

OR

$$f(x) = 5x(x^3 + 12x^2 + 48x + 64)$$

$$f(x) = 5x^4 + 60x^3 + 240x^2 + 320x$$

$$f'(x) = 20x^3 + 180x^2 + 480x$$

$$f''(x) = 60x^2 + 360x$$

$$f'''(x) = 120x + 360$$

$$\begin{aligned}
 f'''(x) &= 60[(x+2) + (x+4)] \\
 &= 60(2x+6) \\
 &= 120(x+3)
 \end{aligned}$$

Exercise

Find the given value: $f(x) = \sqrt{4-x}$; $f'''(-5)$

Solution

$$\begin{aligned}
 f(x) &= (4-x)^{1/2} \\
 f'(x) &= -\frac{1}{2}(4-x)^{-1/2} \\
 f''(x) &= -\frac{1}{2}\left(-\frac{1}{2}\right)(-1)(4-x)^{-3/2} \\
 &= -\frac{1}{4}(4-x)^{-3/2} \\
 f'''(x) &= -\frac{1}{4}\left(-\frac{3}{2}\right)(-1)(4-x)^{-5/2} \\
 &= -\frac{3}{8}(4-x)^{-5/2} \\
 f'''(-5) &= -\frac{3}{8}(4-(-5))^{-5/2} \\
 &= -\frac{3}{8}(9)^{-5/2} \\
 &= \underline{-0.013889}
 \end{aligned}$$

Exercise

Find the 4th derivative of $f(x) = x^4 + 2x^3 + 3x^2 - 5x + 7$

Solution

$$\begin{aligned}
 f'(x) &= 4x^3 + 6x^2 + 6x - 5 \\
 f''(x) &= 12x^2 + 12x + 6 \\
 f'''(x) &= 24x + 12 \\
 f^{(4)}(x) &= 24
 \end{aligned}$$

Exercise

Find the second derivative of $f(x) = (x^2 - 1)^2$

Solution

$$\begin{aligned}f'(x) &= 2(2x)(x^2 - 1) \\&= 4x(x^2 - 1) \\&= 4x^3 - 4x \\f''(x) &= 12x^2 - 4\end{aligned}$$

Exercise

Find $f''(x)$ for $f(x) = \sqrt{x^2 + 36}$, then find $f''(0)$ and $f''(9)$

Solution

$$\begin{aligned}f(x) &= \sqrt{x^2 + 36} \\&= (x^2 + 36)^{1/2} \qquad \left(U^{1/2}\right)' = \frac{1}{2}U'U^{1-1/2}\end{aligned}$$

$$\begin{aligned}f'(x) &= \frac{1}{2}(2x)(x^2 + 36)^{-1/2} \\&= x(x^2 + 36)^{-1/2}\end{aligned}$$

$$f = x \qquad g = (x^2 + 36)^{-1/2}$$

$$f' = 1 \qquad g' = -\frac{1}{2}(2x)(x^2 + 36)^{-3/2} = -x(x^2 + 36)^{-3/2}$$

$$\begin{aligned}f''(x) &= (1)(x^2 + 36)^{-1/2} + (-x)(x^2 + 36)^{-3/2}(x) \\&= (x^2 + 36)^{-1/2} - x^2(x^2 + 36)^{-3/2}\end{aligned}$$

$$= \left[(x^2 + 36)^{-1/2} - x^2(x^2 + 36)^{-3/2} \right] \frac{(x^2 + 36)^{3/2}}{(x^2 + 36)^{3/2}}$$

$$= \left[(x^2 + 36)^{-1/2} \frac{(x^2 + 36)^{3/2}}{(x^2 + 36)^{3/2}} - x^2(x^2 + 36)^{-3/2} \frac{(x^2 + 36)^{3/2}}{(x^2 + 36)^{3/2}} \right]$$

$$\begin{aligned}
&= \left[\frac{(x^2+36)-x^2}{(x^2+36)^{3/2}} \right] \\
&= \frac{x^2+36-x^2}{(x^2+36)^{3/2}} \\
&= \frac{36}{(x^2+36)^{3/2}}
\end{aligned}$$

$$f''(0) = \frac{36}{(0^2+36)^{3/2}} = \frac{36}{36^{3/2}} = \frac{1}{36^{1/2}} = \frac{1}{\sqrt{36}} = \frac{1}{6}$$

$$f''(9) = \frac{36}{(9^2+36)^{3/2}} = \frac{36}{117^{3/2}} = 0.028446 \approx 0.03 \quad \text{Use the Calculator}$$

Exercise

Find $f''(x)$ for $f(x) = \sqrt{x^2+81}$, then find $f''(0)$ and $f''(2)$

Solution

$$f(x) = \sqrt{x^2+81} = (x^2+81)^{1/2} \qquad (U^{1/2})' = \frac{1}{2}U'U^{1-1/2}$$

$$\begin{aligned}
f'(x) &= \frac{1}{2}(2x)(x^2+81)^{-1/2} \\
&= x(x^2+81)^{-1/2}
\end{aligned}$$

$$u = x \qquad v = (x^2+81)^{-1/2}$$

$$\begin{aligned}
u' &= 1 \qquad v' = -\frac{1}{2}(2x)(x^2+81)^{-3/2} \\
&= -x(x^2+81)^{-3/2}
\end{aligned}$$

$$f''(x) = (1)(x^2+81)^{-1/2} + (-x)(x^2+81)^{-3/2} (x)$$

$$= (x^2+81)^{-1/2} - x^2(x^2+81)^{-3/2}$$

$$= \left[(x^2+81)^{-1/2} \frac{(x^2+81)^{3/2}}{(x^2+81)^{3/2}} - x^2(x^2+81)^{-3/2} \frac{(x^2+81)^{3/2}}{(x^2+81)^{3/2}} \right]$$

$$= \frac{x^2+81-x^2}{(x^2+81)^{3/2}}$$

$$= \frac{81}{(x^2+81)^{3/2}} \Big|$$

$$f''(0) = \frac{81}{(0^2+81)^{3/2}} = \frac{81}{81^{3/2}} = \frac{1}{9} \Big|$$

$$f''(2) = \frac{81}{(2^2+81)^{3/2}} = 0.10336 \approx \underline{0.10}$$

Use the Calculator

Exercise

The position function on Earth, where s is measured in meters, t is measured in seconds, v_0 is the initial velocity in meters per second, and h_0 is the initial height in meters, is

$$s = -4.9t^2 + v_0t + h_0$$

If the initial velocity is 2.2 and the initial height is 3.6, what is the acceleration due to gravity on Earth in meters per second per second?

Solution

$$s = -4.9t^2 + 2.2t + 3.6$$

$$s' = -9.8t + 2.2$$

$$\underline{a(t) = s'' = -9.8}$$