1 $\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$

   a) row vectors: $(0 \; -2) \quad (1 \; -3)$

   b) col. vectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

5/a) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \frac{1}{2} R_2 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

   b) rank = 2

9/ $\begin{bmatrix} 1 & 6 & 18 \\ 7 & 40 & 116 \\ -3 & -12 & -27 \end{bmatrix} \begin{matrix} \\ R_2 - 7R_1 \\ R_3 + 3R_1 \end{matrix} \rightarrow \begin{pmatrix} 1 & 6 & 18 \\ 0 & -2 & -2 \\ 0 & 6 & 25 \end{pmatrix} \begin{matrix} R_1 + 3R_2 \\ \\ R_3 + 3R_2 \end{matrix}$

$\begin{pmatrix} 1 & 0 & 12 \\ 0 & -2 & -2 \\ 0 & 0 & 19 \end{pmatrix} \begin{matrix} \\ -\frac{1}{2} R_2 \\ \frac{1}{19} R_3 \end{matrix} \rightarrow \begin{pmatrix} 1 & 0 & 12 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} R_1 - 12R_3 \\ R_2 - R_3 \\ \\ \end{matrix}$

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow$ basis

   rank = 3.

27/ $A = \begin{pmatrix} 2 & -1 \\ -6 & 3 \end{pmatrix}$ $R_2 + 3R_1$

$\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \xrightarrow{} $ rank = 1

   $2x_1 = x_2 \implies N(A) = t \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

33/ $A = \begin{pmatrix} 1 & -2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -2 \end{pmatrix}$ $R_2 - 2R_1$
$R_3 - 4R_1$

$\begin{pmatrix} 1 & -2 & -3 \\ 0 & -5 & 10 \\ 0 & -5 & 10 \end{pmatrix} \frac{1}{} \quad \longrightarrow \quad \begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & 10 \\ 0 & 0 & 0 \end{pmatrix} -\frac{1}{5} R_2$
$R_3 - R_1$

$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow X_1 = -2X_2 + 3X_3 = -X_3$
$\longrightarrow X_2 = 2X_3$

$\left\{ X_3 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$

35/ $A = \begin{pmatrix} 5 & 2 \\ 3 & -1 \\ 2 & 1 \end{pmatrix}$ $5R_2 - 3R_1$
$5R_3 - 2R_1$

$\begin{pmatrix} 5 & 2 \\ 0 & -11 \\ 0 & 1 \end{pmatrix} \rightarrow X_1 = 0$.
$\rightarrow X_2 = 0$

$\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

41/ $A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 5 & 1 & 1 & 0 \\ 3 & 7 & 2 & 2 & -2 \\ 4 & 9 & 3 & -1 & 4 \end{bmatrix} \longrightarrow \Delta = \begin{bmatrix} 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} ① \\ ② \\ ③ \\ \end{matrix}$

a) rank $(A) = 3$
   nullity $(A) = 2$

b) ① $X_1 = -3X_3 + 4X_5$
   ② $X_2 = X_3 - 2X_5$
   ③ $X_4 = 2X_5$

   let $X_3 = 1, X_5 = 0 \Rightarrow (-3, 1, 1, 0, 0)$
   $X_3 = 0, X_5 = 1 \Rightarrow (4, -2, 0, 2, 1)$

   $N(A)$ basis: $\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

c) Basis for row (A)
   $\{(1, 0, 3, 0, -4), (0, 1, -1, 0, 2), (0, 0, 0, 1, -2)\}$

d) Basis for $CS(A)$ $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix} \right\}$

e) Linearly Dependent

f) i) $\{a_1, a_2, a_4\}$ Yes (LI) (pivot variables)

   ii) $\{a_1, a_2, a_3\}$ No $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ Linearly Dep^t

   iii) $\{a_1, a_3, a_5\}$ $\begin{bmatrix} 1 & 3 & -4 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$
   Yes (L.I.)

49/ $x - 4y = 17$
$3x - 12y = 51$
$-2x + 8y = -34$

$$\begin{pmatrix} 1 & -4 & | & 17 \\ 3 & 12 & | & 51 \\ -2 & 8 & | & -34 \end{pmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 + 2R_1 \end{matrix} \rightarrow \begin{pmatrix} 1 & -4 & | & 17 \\ 0 & 24 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \tfrac{1}{24} R_2$$

$$\begin{pmatrix} 1 & -4 & | & 17 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \rightarrow \begin{matrix} x_1 = 4x_2 \\ \text{rank}(2) \end{matrix}$$

free variable x_2
x_1 pivot column.

null space $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$$x_p = \begin{pmatrix} 17 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

57/ $A = \begin{pmatrix} -1 & 2 \\ 4 & 0 \end{pmatrix}$ $b = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$$\begin{pmatrix} +1 & -2 & | & -3 \\ 4 & 0 & | & 4 \end{pmatrix} \rightarrow R_2 - 4R_1 \rightarrow \begin{pmatrix} 1 & -2 & | & -3 \\ 0 & 8 & | & 16 \end{pmatrix} \begin{matrix} \rightarrow ① \\ \rightarrow x_2 = 2 \end{matrix}$$

① $x_1 = 2x_2 - 3 = 1$

$$\begin{bmatrix} -1 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

5/ $B = \{(2,-1),(0,1)\}$ $[x]_B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$x = 4(2,-1) + 1(0,1)$

$= (8, -3)$

$[x]_{B'} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$

7/ $B = \{(1,0,1), (1,1,0), (0,1,1)\}$ $[x]_B = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

$x = 2(1,0,1) + 3(1,1,0) + 1(0,1,1)$

$= (5, 4, 3)$

$[x]_{B'} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$

11/ $B' = \{(4,0), (0,3)\}$ $x = (12,6)$

$(12,6) = c_1(4,0) + c_2(0$

$12 = 4c_1 \rightarrow \boxed{c_1 = 3}$

$6 = 3c_2 \rightarrow \boxed{c_2 = 2}$

$[x]_{B'} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

13/ $B' = \{(8, 11, 0), (7,0,10), (1,4,6)\}$ $x = (3,19, 2)$

$(3, 9, 2) = c_1(8, 11, 0) + c_2(7,0,10) + c_3(1,4,6)$

$\begin{cases} 8c_1 + 7c_2 + c_3 = 3 \\ 11c_1 + \quad 4c_3 = 19 \\ 10c_2 + 6c_3 = 2 \end{cases}$

$\Delta = \begin{vmatrix} 8 & 7 & 1 \\ 11 & 0 & 4 \\ 0 & 10 & 6 \end{vmatrix} = -672$

$\Delta_1 = \begin{vmatrix} 3 & 7 & 1 \\ 19 & 0 & 4 \\ 2 & 10 & 6 \end{vmatrix} = -672$

$4c_3 = 19 - 11 \rightarrow \boxed{c_3 = 2}$

$10c_2 = 2 - 12 \rightarrow \boxed{c_2 = -1}$

$\boxed{c_1 = + \frac{672}{672} = 1}$

$[x]_{B'} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

17/ $B = \{(1,0), (0,1)\}$ $B' = \{(2,4), (1,3)\}$

$$[B' \mid B] = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{bmatrix} \; R_2 - 2R_1$$

$$\sim \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \; R_1 - R_2$$

$$\begin{bmatrix} 2 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{bmatrix} \frac{1}{2}R_1 \rightarrow \begin{pmatrix} 1 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & -2 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{pmatrix}$$

21/ $B = \{(-1,0,0), (0,1,0), (0,0,-1)\}$

$B' = \{(0,0,2), (1,4,0), (5,0,2)\}$

$$[B' \mid B] = \begin{bmatrix} 0 & 1 & 5 & -1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & -1 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 2 & 0 & 2 & 0 & 0 & -1 \\ 0 & 1 & 5 & -1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \end{bmatrix} R_3 - 4R_2$$

$$\begin{bmatrix} 2 & 0 & 2 & 0 & 0 & -1 \\ 0 & 1 & 5 & -1 & 0 & 0 \\ 0 & 0 & -20 & 4 & 1 & 0 \end{bmatrix} \begin{array}{l} 10R_1 + R_3 \\ 4R_2 + R_3 \end{array}$$

$$\begin{bmatrix} 20 & 0 & 0 & 4 & 1 & -10 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & -20 & 4 & 1 & 0 \end{bmatrix} \begin{array}{l} \frac{1}{20} \\ \frac{1}{4} \\ -\frac{1}{20} \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{5} & \frac{1}{20} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & -\frac{1}{20} & 0 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{1}{20} & -\frac{1}{2} \\ 0 & \frac{1}{4} & 0 \\ -\frac{1}{5} & -\frac{1}{20} & 0 \end{bmatrix}$$

13/ $\{x, -\sin x\}$

$W = \begin{vmatrix} x & -\sin x \\ 1 & -\cos x \end{vmatrix} = \underline{-x\cos x + \sin x}$

15/ $\{e^x, e^{-x}\}$

$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = \underline{-2}$

17/ $\{x, \sin x, \cos x\}$

$W = \begin{vmatrix} x & \sin x & \cos x \\ 1 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix}$

$= -x\cos^2 x - \sin x \cos x - x \sin^2 x + \sin x \cos x$

$= -x(\cos^2 x + \sin^2 x)$

$= \underline{-x}$

19/ $\{e^{-x}, xe^{-x}, (x+3)e^{-x}\}$

$W = \begin{vmatrix} e^{-x} & xe^{-x} & (x+3)e^{-x} \\ -e^{-x} & (1-x)e^{-x} & (-2-x)e^{-x} \\ e^x & (-2+x)e^{-x} & (x+1)e^{-x} \end{vmatrix}$

$= (e^{-x})^3 \begin{vmatrix} 1 & x & x+3 \\ -1 & 1-x & -x-2 \\ 1 & x-2 & x+1 \end{vmatrix}$

$= (1 - x^2 - x^2 - 2x + x^2 - x + 6 - x^2 + 2x - 3 + x^2 - 4 + x^2 + x)e^{-3x}$

$= \underline{0}$

2) $\{1, e^x, e^{2x}\}$

$$W = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = 4e^{3x} - 2e^{3x}$$

$$= 2e^{3x}$$