Solution Section 2.2 – Arc Length and Sector Area

Exercise

The minute hand of a clock is 1.2 cm long. How far does the tip of the minute hand travel in 40 minutes?

Solution

$$40\min = 40\min \frac{2\pi}{60}\frac{rad}{\min}$$

$$= \frac{4\pi}{3} rad.$$

$$s = r\theta$$

$$= (1.2)\frac{4\pi}{3}$$

$$\approx 5.03\ cm$$

Exercise

Find the radian measure if angle θ, if θ is a central angle in a circle of radius r = 4 inches, and θ cuts off an arc of length s = 12π inches.

Solution

$$\theta = \frac{s}{r}$$

$$= \frac{12\pi}{4}$$

$$= 3\pi\ rad$$

Exercise

Give the length of the arc cut off by a central angle of 2 radians in a circle of radius 4.3 inches

Solution

Given: $\theta = 2\ rad, \quad r = 4.3\ in$

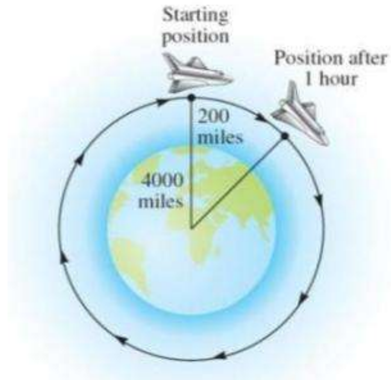
$$s = r\theta$$

$$= 4.3(2)$$

$$= 8.6\ in$$

Exercise

A space shuttle 200 miles above the earth is orbiting the earth once every 6 hours. How long, in hours, does it take the space shuttle to travel 8,400 miles? (Assume the radius of the earth is 4,000 miles.) Give both the exact value and an approximate value for your answer.



Solution

$$\theta = \frac{s}{r}$$
$$= \frac{8400}{4200}$$
$$= 2 \ rad$$

$$\Rightarrow \frac{2 \ rad}{2\pi \ rad} = \frac{x \ hr}{6 \ hr}$$

$$x = \frac{2\ (6)}{2\pi}$$
$$\boxed{\approx 1.91 \ hrs}$$

Exercise

The pendulum on a grandfather clock swings from side to side once every second. If the length of the pendulum is 4 feet and the angle through which it swings is 20°. Find the total distance traveled in 1 minute by the tip of the pendulum on the grandfather clock.

Solution

Since $20° = 20 \bullet \frac{\pi}{180} = \frac{\pi}{9} \ rad$

The length of the pendulum swings in 1 second:

$$s = r\theta = 4 \bullet \frac{\pi}{9} = \frac{4\pi}{9} \ ft.$$

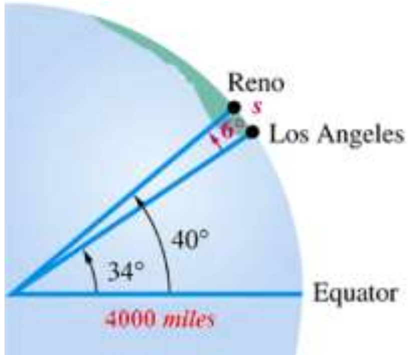
In 60 seconds, the total distance traveled

$$d = 60 \bullet \frac{4\pi}{9}$$
$$= \frac{80\pi}{3} \approx 83.8 \text{ feet}$$
$$\boxed{\approx 83.8 \text{ feet}}.$$

Exercise

Reno, Nevada is due north of Los Angeles. The latitude of Reno is 40°, while that of Los Angeles is 34° N. The radius of Earth is about 4000 mi. Find the north-south distance between the two cities.

Solution



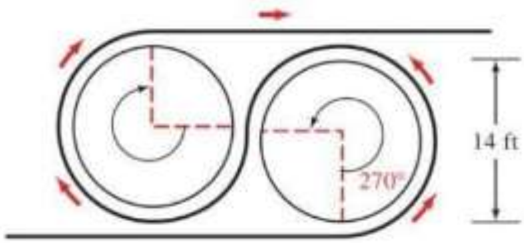
The central angle between two cities: $40° - 34° = 6°$

$$\boxed{6° = 6\left(\frac{\pi}{180}\right) = \frac{\pi}{30}\ rad}$$

$$\begin{aligned} s &= r\theta \\ &= 4000\frac{\pi}{30} \\ &\approx 419\ miles \end{aligned}$$

Exercise

The first cable railway to make use of the figure-eight drive system was a Sutter Street Railway. Each drive sheave was 12 feet in diameter. Find the length of cable riding on one of the drive sheaves.



Solution

Since $270° = 270 \bullet \frac{\pi}{180} = \frac{3\pi}{2}\ rad$,

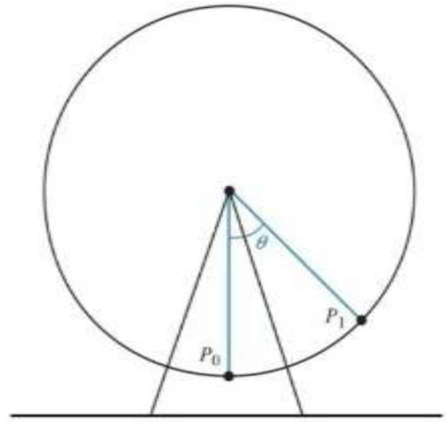
The length of the cable riding on one of the drive sheaves is:

$$\begin{aligned} s &= r\theta \\ &= 6 \bullet \frac{3\pi}{2} \\ &= 9\pi \\ &\approx 28.3\ feet \end{aligned}$$

Exercise

The diameter of a model of George Ferris's Ferris wheel is 250 feet, and θ is the central angle formed as a rider travels from his or her initial position Po to position P1. Find the distance traveled by the rider if θ = 45° and if θ =105°.

Solution



$$r = \frac{D}{2} = \frac{250}{2} = 125\ ft$$

For $\theta = 45° = \frac{\pi}{4}$

$$\begin{aligned} s &= r\theta \\ &= 125\left(\frac{\pi}{4}\right) \\ &\approx 98\ ft \end{aligned}$$

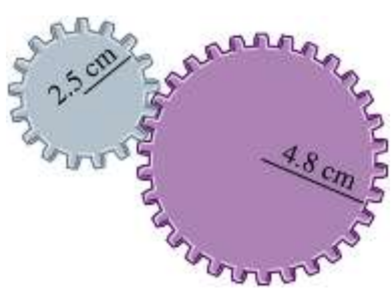
For $\theta = 105° = 105\frac{\pi}{180} = \frac{7\pi}{12}$

$$\begin{aligned} s &= r\theta \\ &= 125\frac{7\pi}{12} \\ &\approx 230\ ft \end{aligned}$$

Exercise

Two gears are adjusted so that the smaller gear drives the larger one. If the smaller gear rotates through an angle of 225°, through how many degrees will the larger gear rotate?

Solution



The motion of the larger gear: $225° = 225\frac{\pi}{180} = \frac{5\pi}{4}\ rad$

The arc length on the smaller gear is:

$$\begin{aligned} s &= r\theta \\ &= 2.5\left(\frac{5\pi}{4}\right) \\ &\approx 9.817477\ cm \end{aligned}$$

The arc length on the larger gear is:

$$s = r\theta$$

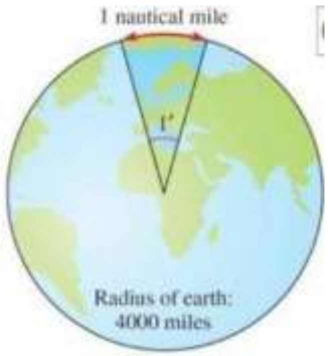
$$9.817477 = 4.8\ \theta$$

$$\theta = \frac{9.817477}{4.8} = 2.0453$$

$$\boxed{\theta = 2.0453\frac{180°}{\pi} \approx 117°}$$

Exercise

If a central angle with its vertex at the center of the earth has a measure of $1'$, then the arc on the surface of the earth that is cut off by this angle (knows as the great circle distance) has a measure of 1 nautical mile.



Solution

$$\theta = 1' = \frac{1}{60}^{\circ} = \frac{1}{60} \bullet \frac{\pi}{180} = \frac{\pi}{10800} \ rad$$

$$\theta = \frac{s}{r}$$

$$\frac{\pi}{10800} = \frac{s}{4000}$$

$$\frac{4000\pi}{10800} = s$$

$$\boxed{s \approx 1.16 \ mi}$$

Exercise

If two ships are 20 nautical miles apart on the ocean, how many statute miles apart are they?

Solution

$$\theta = 20' = \frac{20}{60}^{\circ} = \frac{1}{3}^{\circ} = \frac{1}{3} \bullet \frac{\pi}{180} = \frac{\pi}{540} \text{ rad}$$

$$\theta = \frac{s}{r}$$

$$\frac{\pi}{540} = \frac{s}{4000}$$

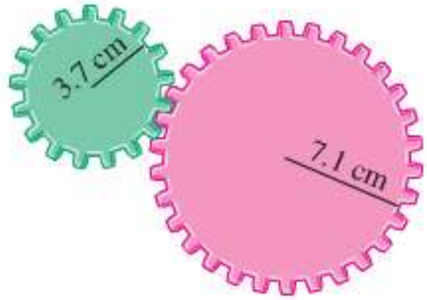
$$\frac{4000\pi}{540} = s$$

$$s \approx 23.27$$

Exercise

Two gears are adjusted so that the smaller gear drives the larger one. If the smaller gear rotates through an angle of 300°, through how many degrees will the larger rotate?

Solution



Both gears travel the same arc distance (s), therefore:

$$s = r_1\theta_1 = r_2\theta_2$$

$$3.7\left(300°\frac{\pi}{180°}\right) = 7.1\ \theta_2$$

$$\theta_2 = \frac{3.7}{7.1}\left(300°\frac{\pi}{180°}\right)\frac{180°}{\pi}$$

$$= \frac{3.7}{7.1}300°$$

$$\approx 156°$$

Exercise

The rotation of the smaller wheel causes the larger wheel to rotate. Through how many degrees will the larger wheel rotate if the smaller one rotates through 60.0°?

Solution

Both gears travel the same arc distance (s), therefore:

$$s = r_1\theta_1 = r_2\theta_2$$

$$5.23\left(60.0°\frac{\pi}{180°}\right) = 8.16\ \theta_2$$

$$\theta_2 = \frac{5.23}{8.16}\left(60.0°\frac{\pi}{180°}\right)\frac{180°}{\pi}$$

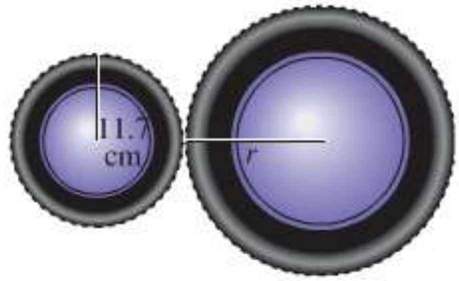
$$= \frac{5.23}{8.16}60.0°$$

$$\approx 38.5°$$

Exercise

Find the radius of the larger wheel if the smaller wheel rotates 80° when the larger wheel rotates 50°.

Solution



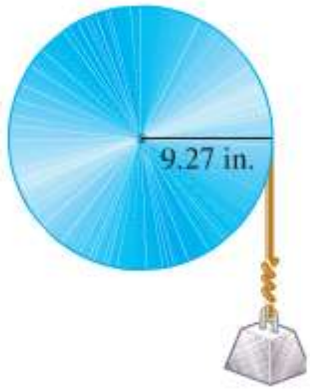
$$r_1\theta_1 = r_2\theta_2$$

$$11.7(80°) = r_2\ (50°)$$

$$r_2 = \frac{11.7(80°)}{50°} \ \underline{= 18.72\ cm}$$

Exercise

How many inches will the weight rise if the pulley is rotated through an angle of 71° 50′? Through what angle, to the nearest minute, must the pulley be rotated to raise the weight 6 in?



Solution

$$\theta = \left(71° + 50'\frac{1°}{60'}\right)\frac{\pi}{180°}$$

$$s = r\theta$$

$$= 9.27\left(71° + 50'\frac{1°}{60'}\right)\frac{\pi}{180°}$$

$$\approx 11.622\ in$$

$$\theta = \frac{s}{r} = \frac{6}{9.27}\ rad$$

$$\theta = \frac{6}{9.27}\frac{180°}{\pi} = 37.0846°$$

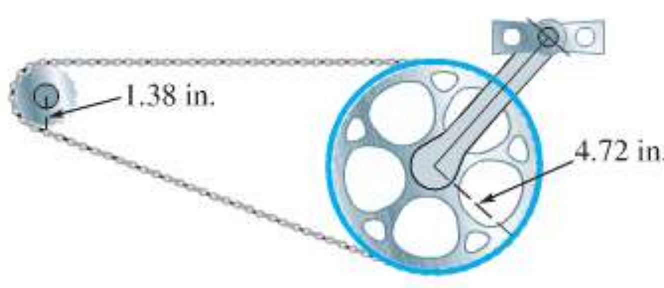
$$\theta = 37° + .0846(60')$$

$$\boxed{\theta = 37°\ 5'}$$

Exercise

The figure shows the chain drive of a bicycle. How far will the bicycle move if the pedals are rotated through 180°? Assume the radius of the bicycle wheel is 13.6 in.

Solution



$$\theta = 180° = \pi\ rad$$

The distance for the pedal gear:

$$s_1 = r_1\theta = 4.72\pi\ \ in = s_2$$

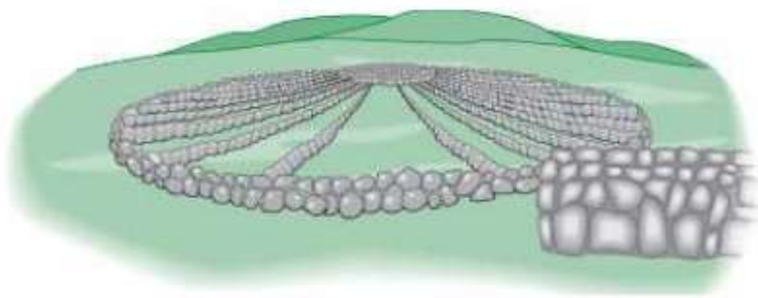
For the smaller gear:

$$\theta_2 = \frac{s}{r_2} = \frac{4.72\pi}{1.38} \approx 3.42\pi$$

The wheel distance: $s = r_3\theta_2 = 13.6(3.42\pi)\ \underline{= 146.12\ in}$

Exercise

The circular of a Medicine Wheel is 2500 yrs old. There are 27 aboriginal spokes in the wheel, all equally spaced.



a) Find the measure of each central angle in degrees and in radians.
b) The radius measure of each of the wheel is 76.0 ft, find the circumference.
c) Find the length of each arc intercepted by consecutive pairs of spokes.
d) Find the area of each sector formed by consecutive spokes,

Solution

a) The central angle: $\theta = \frac{360^\circ}{27} = \frac{40}{3}^\circ$

$$\theta = \frac{40}{3}^\circ \frac{\pi\ rad}{180^\circ} = \frac{2\pi}{27}\ rad$$

b) $C = 2\pi r = 2\pi(76)$ $\approx 477.5\ ft$

c) Since $r = 76 \Rightarrow s = r\theta = 76\frac{2\pi}{27}$ $\approx 17.7\ ft$

d) Area $= \frac{1}{2}r^2\theta$

$$= \frac{1}{2}76^2\,\frac{2\pi}{27}$$

$$\approx 672\ ft^2$$

Exercise

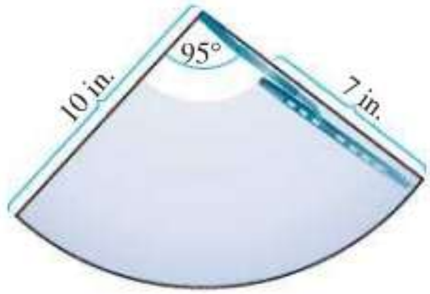
Find the radius of the pulley if a rotation of 51.6° raises the weight 11.4 cm.

Solution

$$r = \frac{s}{\theta} = \frac{11.4}{51.6^\circ \frac{\pi}{180^\circ}} \approx 12.7\ cm$$

Exercise

The total arm and blade of a single windshield wiper was 10 in. long and rotated back and forth through an angle of 95°. The shaded region in the figure is the portion of the windshield cleaned by the 7-in. wiper blade. What is the area of the region cleaned?



Solution

The total angle: $\theta = 95° \frac{\pi}{180°} = \frac{19\pi}{36}$ *rad*

A_1 : The area of arm only (not cleaned by the blade). $\quad A_1 = \frac{1}{2}(10-7)^2 \frac{19\pi}{36}$

A_2 : The area of arm and the blade. $\quad A_2 = \frac{1}{2}(10)^2 \frac{19\pi}{36}$

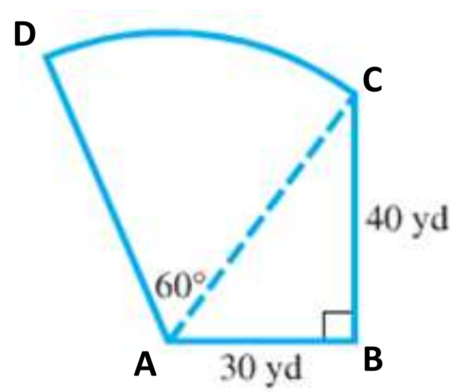
The total cleaned area:

$$\begin{aligned} A &= A_2 - A_1 \\ &= \frac{1}{2}(10)^2 \frac{19\pi}{36} - \frac{1}{2}(3)^2 \frac{19\pi}{36} \\ &= 82.9 - 7.46 \\ &= 75.4\ in^2 \end{aligned}$$

Exercise

A frequent problem in surveying city lots and rural lands adjacent to curves of highways and railways is that of finding the area when one or more of the boundary lines is the arc of the circle. Find the area of the lot.

Solution



Using the Pythagorean theorem:

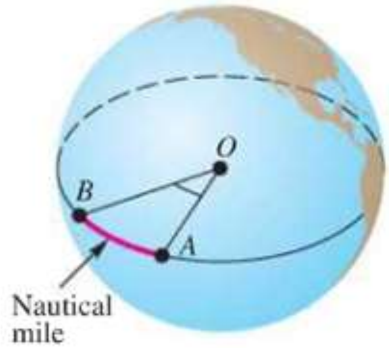
$$AC = \sqrt{30^2 + 40^2} = \underline{50 = r}|$$

Total area = Area of the sector (ADC) + Area of the triangle (ABC)

$$\begin{aligned} \text{Total area } &= \frac{1}{2} r^2 (60°) \frac{\pi}{180°} + \frac{1}{2}(AB)(BC) \\ &= \frac{1}{2} 50^2 (60°) \frac{\pi}{180°} + \frac{1}{2}(30)(40) \\ &\approx \underline{1909\ yd^2}| \end{aligned}$$

Exercise

Nautical miles are used by ships and airplanes. They are different from statue miles, which equal 5280 ft. A nautical mile is defined to be the arc length along the equator intercepted by a central angle AOB of 1 min. If the equatorial radius is 3963 mi, use the arc length formula to approximate the number of statute miles in 1 nautical mile.



Solution

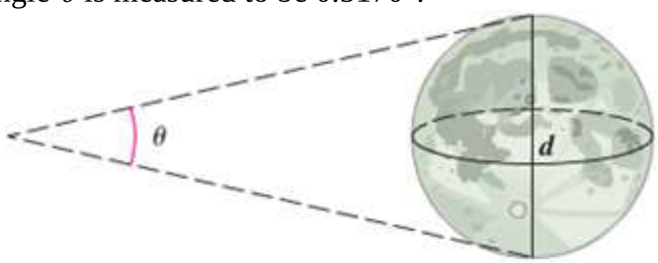
$$\theta = 1' \frac{1^\circ}{60'} \frac{\pi}{180^\circ} = \frac{\pi}{10800} \ rad$$

The arc length: $\boxed{s = r\theta = 3963 \frac{\pi}{10800} \approx 1.15}$

There are 1.15 statute miles in 1 nautical mile.

Exercise

The distance to the moon is approximately 238,900 mi. Use the arc length formula to estimate the diameter d of the moon if angle θ is measured to be 0.5170°.



Solution

$$\begin{aligned} s &= r\theta \\ &= 238900 \times 0.517^\circ \frac{\pi}{180^\circ} \\ &\approx 2156 \ mi \end{aligned}$$