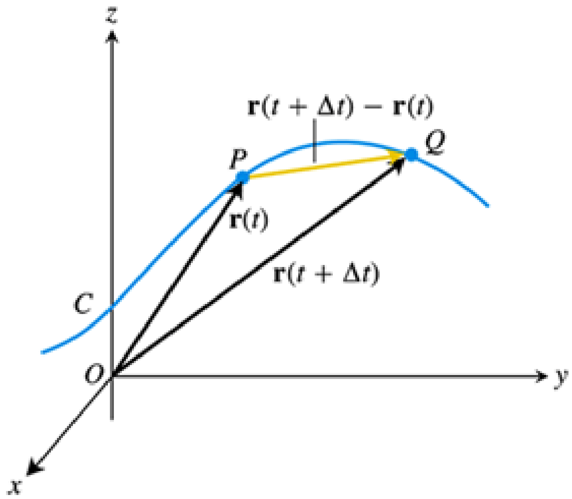
Section 1.5 – Calculus of Vector-Valued Functions

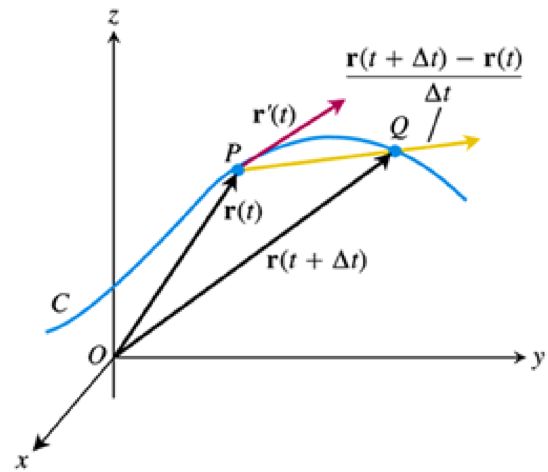
Derivative

Definition

The vector function $\vec{r}(t)=f(t)\hat{i}+g(t)\hat{j}+h(t)\hat{k}$ has a derivative (is differentiable) at t if f, g, and h have derivatives at t. The derivative is the vector function

$$\begin{aligned}\vec{r}'(t)&=\frac{d\vec{r}}{dt}\\ &=\lim_{\Delta t\to 0}\frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}\\ &=\frac{df}{dt}\hat{i}+\frac{dg}{dt}\hat{j}+\frac{dh}{dt}\hat{k}\end{aligned}$$





Definitions

If $\boldsymbol{r}$ is the position vector of a particle moving along a smooth curve in space, then

$$\vec{v}(t)=\frac{d\vec{r}}{dt}$$

is the particle's ***velocity vector***, tangent to the curve. At any time t, the direction of $\vec{v}$ is the ***direction of motion***, the magnitude of $\vec{v}$ is the particle's ***speed***, and the derivative $\vec{a}=\frac{d\vec{v}}{dt}$, when it exists, is the particle's ***acceleration vector***. In summary,

1. Velocity is the derivative of position: $\vec{v}(t)=\frac{d\vec{r}}{dt}$
2. Speed is the magnitude of velocity: $Speed=|\vec{v}|$

3. Acceleration is the derivative of velocity: $\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{r}}{dt^2}$

4. The unit vector $\dfrac{\vec{v}}{|\vec{v}|}$ is the direction of motion at time *t*.

Example

Find the velocity, speed, and acceleration of a particle whose motion in space is given by the position vector $\vec{r}(t) = 2\cos t\ \hat{i} + 2\sin t\ \hat{j} + 5\cos^2 t\ \hat{k}$. Sketch the velocity vector $\vec{v}\left(\frac{7\pi}{4}\right)$

Solution

The velocity vector at time *t* is:

$$\begin{aligned}\vec{v}(t) = \vec{r}'(t) &= -2\sin t\ \hat{i} + 2\cos t\ \hat{j} - 10\cos t\sin t\ \hat{k}\\ &= -2\sin t\ \hat{i} + 2\cos t\ \hat{j} - 5\sin 2t\ \hat{k}\end{aligned}$$

The acceleration vector at time *t* is:

$$\vec{a}(t) = \vec{r}''(t) = -2\cos t\ \hat{i} - 2\sin t\ \hat{j} - 10\cos 2t\ \hat{k}$$

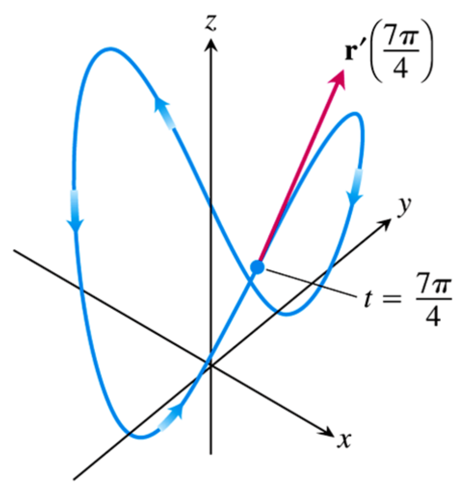
The speed is:

$$\begin{aligned}|\vec{v}(t)| &= \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-5\sin 2t)^2}\\ &= \sqrt{4\sin^2 t + 4\cos^2 t + 25\sin^2 2t}\\ &= \sqrt{4\left(\sin^2 t + \cos^2 t\right) + 25\sin^2 2t} \qquad \sin^2 t + \cos^2 t = 1\\ &= \sqrt{4 + 25\sin^2 2t}\end{aligned}$$

$$\begin{aligned}\vec{v}\left(\frac{7\pi}{4}\right) &= -2\sin\left(\frac{7\pi}{4}\right)\hat{i} + 2\cos\left(\frac{7\pi}{4}\right)\hat{j} - 5\sin\left(\frac{7\pi}{2}\right)\hat{k}\\ &= \sqrt{2}\hat{i} + \sqrt{2}\hat{j} + 5\hat{k}\end{aligned}$$

$$\begin{aligned}\vec{a}\left(\frac{7\pi}{4}\right) &= -2\cos\left(\frac{7\pi}{4}\right)\hat{i} - 2\sin\left(\frac{7\pi}{4}\right)\hat{j} - 10\cos\left(\frac{7\pi}{2}\right)\hat{k}\\ &= -\sqrt{2}\hat{i} + \sqrt{2}\hat{j}\end{aligned}$$

$$\begin{aligned}\left|\vec{v}\left(\frac{7\pi}{4}\right)\right| &= \sqrt{4 + 25\sin^2\left(\frac{7\pi}{2}\right)}\\ &= \sqrt{29}\end{aligned}$$



Differentiation Rules **for vector Functions**

Let $\vec{u}$ and $\vec{v}$ be differentiable vector functions of t, C a constant vector, c any scalar and f any differentiable scalar function.

1. *Constant Function Rule*: $\dfrac{d}{dt}\vec{C} = \mathbf{0}$

2. *Scalar Multiple Rules*: $\dfrac{d}{dt}\left[c\vec{u}(t)\right] = c\vec{u}'(t)$
$$\dfrac{d}{dt}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$$

3. *Sum Rule*: $\dfrac{d}{dt}\left[\vec{u}(t)+\vec{v}(t)\right] = \vec{u}'(t)+\vec{v}'(t)$

4. *Difference Rule*: $\dfrac{d}{dt}\left[\vec{u}(t)-\vec{v}(t)\right] = \vec{u}'(t)-\vec{v}'(t)$

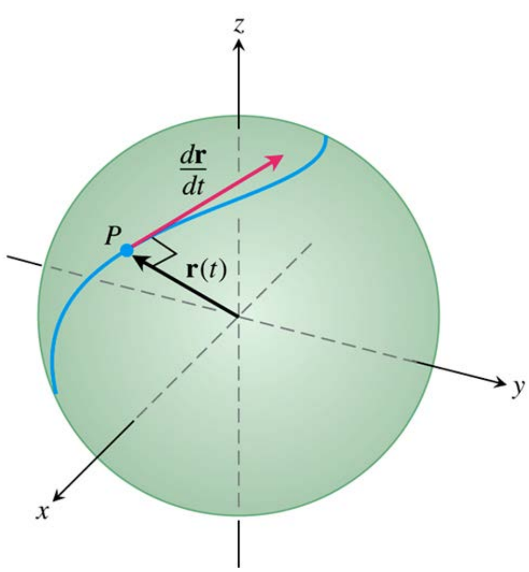
5. *Dot Product Rule*: $\dfrac{d}{dt}\left[\vec{u}(t)\bullet\vec{v}(t)\right] = \vec{u}'(t)\bullet\vec{v}'(t)+\vec{u}(t)\bullet\vec{v}'(t)$

6. *Cross Product Rule*: $\dfrac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}'(t)\times\vec{v}(t)+\vec{u}(t)\times\vec{v}'(t)$

7. *Chain Rule*: $\dfrac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\vec{u}'(f(t))$

Vector Functions of Constant Length

The position vector, of a particle that is moving on a sphere, has a constant length equal to the radius of the sphere. The velocity vector $\dfrac{d\vec{r}}{dt}$, tangent to the path of motion, is tangent to the sphere and hence perpendicular to $\vec{r}(t)$. the vector and its first derivative are orthogonal.



$$\vec{r}(t) \bullet \vec{r}(t) = c^2 \qquad |\vec{r}(t)| = c \textit{ is constant}$$

$$\frac{d}{dt}\left[\vec{r}(t) \bullet \vec{r}(t)\right] = 0 \qquad \textbf{\textit{Differentiate both sides}}$$

$$\vec{r}'(t) \bullet \vec{r}(t) + \vec{r}(t) \bullet \vec{r}'(t) = 0$$

$$2\,\vec{r}'(t) \bullet \vec{r}(t) = 0$$

If $\vec{r}(t)$ is a differentiable vector function of t of constant length, then

$$\vec{r}(t) \bullet \frac{d\vec{r}}{dt} = 0$$

Exercises *Section* 1.5 – Calculus of Vector-Valued Functions

(1 – 4) $\vec{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find an equation in *x* and *y* whose is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of *t*.

1. $\vec{r}(t)=(t+1)\hat{i}+\left(t^2-1\right)\hat{j}, \quad t=1$

2. $\vec{r}(t)=\frac{t}{t+1}\hat{i}+\frac{1}{t}\hat{j}, \quad t=-\frac{1}{2}$

3. $\vec{r}(t)=e^t\hat{i}+\frac{2}{9}e^{2t}\hat{j}, \quad t=\ln 3$

4. $\vec{r}(t)=(\cos 2t)\hat{i}+(3\sin 2t)\hat{j}, \quad t=0$

(5 – 6) Give the position vectors of particles moving along various curves in the *xy*-plane. Find the particle's velocity and acceleration vectors at the stated times and sketch them as vectors on the curve

5. Motion on the circle $x^2+y^2=1$ $\quad \vec{r}(t)=(\sin t)\hat{i}+(\cos t)\hat{j}, \quad t=\frac{\pi}{4}$ *and* $\frac{\pi}{2}$

6. Motion on the cycloid $x=t-\sin t, \ \ y=1-\cos t; \quad \vec{r}(t)=(1-\sin t)\hat{i}+(1-\cos t)\hat{j}; \quad t=\pi$ & $\frac{3\pi}{2}$

(7 – 11) $\boldsymbol{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of *t*. Write the particle's velocity at that time as the product of its speed and direction.

7. $\vec{r}(t)=(t+1)\hat{i}+\left(t^2-1\right)\hat{j}+2t\hat{k}, \quad t=1$

8. $\vec{r}(t)=(t+1)\hat{i}+\frac{t^2}{\sqrt{2}}\hat{j}+\frac{t^3}{3}\hat{k}, \quad t=1$

9. $\vec{r}(t)=(2\cos t)\hat{i}+(3\sin t)\hat{j}+4t\hat{k}, \quad t=\frac{\pi}{2}$

10. $\vec{r}(t)=(2\ln(t+1))\hat{i}+t^2\hat{j}+\frac{t^2}{2}\hat{k}, \quad t=1$

11. $\vec{r}(t)=\left(e^{-t}\right)\hat{i}+(2\cos 3t)\hat{j}+(2\sin 3t)\hat{k}, \quad t=0$

12. Find all points on the ellipse $\vec{r}(t)=\langle 1,\ 8\sin t,\ \cos t\rangle$, for $0\le t\le 2\pi$, at which $\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal.