# Solution Section 1.6 – Surface Area

### Exercise

Find the lateral (side) surface area of the cone generated by revolving the line segment $y = \frac{x}{2}, \quad 0 \le x \le 4,$ about the $x$-axis. Check your answer with the geometry formula

$$Lateral\ surface\ area = \frac{1}{2} \times base\ circumference \times slant\ height$$

**Solution**

$$\frac{dy}{dx} = \frac{1}{2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{4}}$$

$$= \frac{\sqrt{5}}{2}$$

$$S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi \int_0^4 \left(\frac{x}{2}\right)\frac{\sqrt{5}}{2}\,dx$$

$$= \frac{\pi\sqrt{5}}{2}\int_0^4 x\,dx$$

$$= \frac{\pi\sqrt{5}}{2}\left(\frac{1}{2}x^2\right|_0^4$$

$$= \frac{\pi\sqrt{5}}{4}\left(4^2 - 0\right)$$

$$= 4\pi\sqrt{5} \quad unit^2$$

$$base\ circumference = 2\pi r = 2\pi(2) = 4\pi$$

$$slant\ height = \sqrt{4^2+2^2} = \sqrt{20} = 2\sqrt{5}$$

$$Lateral\ surface\ area = \frac{1}{2} \times base\ circumference \times slant\ height$$

$$= \frac{1}{2} \times (4\pi) \times \left(2\sqrt{5}\right)$$

$$= 4\pi\sqrt{5} \quad unit^3$$

### *Exercise*

Find the lateral surface area of the cone generated by revolving the line segment $y=\frac{x}{2},\quad 0\le x\le 4$, about the *y*-axis. Check your answer with the geometry formula

$$Lateral\ surface\ area=\frac{1}{2}\times base\ circumference\times slant\ height$$

**Solution**

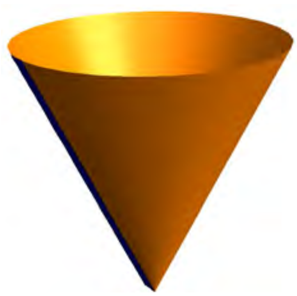
$$y=\frac{x}{2}\quad\Rightarrow x=2y$$

$$\to\begin{cases}x=0 & \to y=0\\ x=4 & \to y=2\end{cases}$$

$$\frac{dx}{dy}=2$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+4}$$

$$=\sqrt{5}$$



$$S=2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$=2\pi\int_0^2 2\sqrt{5}\,y\,dy$$

$$=2\pi\sqrt{5}\,y^2\Big|_0^2$$

$$=2\pi\sqrt{5}(4-0)$$

$$=8\pi\sqrt{5}\quad unit^2$$

$$base\ circumference=2\pi(4)=8\pi$$

$$slant\ height=\sqrt{4^2+2^2}$$

$$=\sqrt{20}$$

$$=2\sqrt{5}$$

$$Lateral\ surface\ area=\frac{1}{2}\times base\ circumference\times slant\ height$$

$$=\frac{1}{2}\times(8\pi)\times\left(2\sqrt{5}\right)$$

$$=8\pi\sqrt{5}\quad unit^2$$

### Exercise

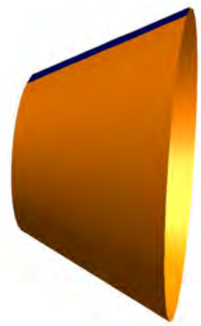
Find the lateral surface area of the cone frustum generated by revolving the line segment $y=\frac{x}{2}+\frac{1}{2}, \quad 1\le x\le 3$, about the $x$-axis. Check your answer with the geometry formula

$$\text{Frustum surface area} = \pi\left(r_1+r_2\right)\times \text{slant height}$$

**Solution**

$$\frac{dy}{dx}=\frac{1}{2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{1}{4}}$$
$$=\frac{\sqrt{5}}{2}$$



$$S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$=2\pi\int_1^3\left(\frac{x}{2}+\frac{1}{2}\right)\left(\frac{\sqrt{5}}{2}\right)dx$$
$$=\pi\frac{\sqrt{5}}{2}\int_1^3(x+1)\,dx$$
$$=\pi\frac{\sqrt{5}}{2}\left(\frac{1}{2}x^2+x\right|_1^3$$
$$=\pi\frac{\sqrt{5}}{2}\left(\frac{9}{2}+3-\frac{3}{2}\right)$$
$$=\pi\frac{\sqrt{5}}{2}(6)$$
$$=3\pi\sqrt{5}\quad unit^2$$

$$r_1=\frac{1}{2}+\frac{1}{2}=1 \quad r_2=\frac{3}{2}+\frac{1}{2}=2$$

$$\text{slant height}=\sqrt{(2-1)^2+(3-1)^2}$$
$$=\sqrt{5}$$

$$\text{Frustum surface area}=\pi\left(r_1+r_2\right)\times\text{slant height}$$
$$=\pi(1+2)\sqrt{5}$$
$$=3\pi\sqrt{5}\quad unit^2$$

### *Exercise*

Find the lateral surface area of the cone frustum generated by revolving the line segment $y=\frac{x}{2}+\frac{1}{2}, \quad 1\le x\le 3$, about the *y*-axis. Check your answer with the geometry formula

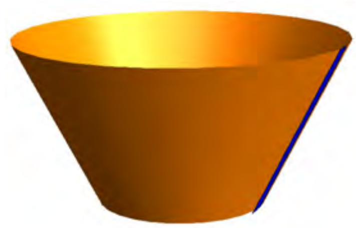
$$\text{Frustum surface area} = \pi\left(r_1+r_2\right)\times \text{slant height}$$

**Solution**

$$y=\frac{x}{2}+\frac{1}{2}$$

$$\underline{x=2y-1}$$

$$\frac{dx}{dy}=2$$



$$\sqrt{1+\left(\frac{dx}{dy}\right)^2}=\sqrt{1+4}$$

$$=\sqrt{5}$$

$$S=2\pi\int_1^2(2y-1)\left(\sqrt{5}\right)dy$$

$$=2\pi\sqrt{5}\int_1^2(2y-1)\,dy$$

$$=2\pi\sqrt{5}\left(y^2-y\right.\Big|_1^2$$

$$=2\pi\sqrt{5}\left[4-2-(1-1)\right]$$

$$\underline{=4\pi\sqrt{5}\quad unit^2}$$

$$r_1=1 \quad r_2=3$$

$$\text{slant height}=\sqrt{(2-1)^2+(3-1)^2}$$

$$=\sqrt{5}$$

$$\text{Frustum surface area}=\pi\left(r_1+r_2\right)\times\text{slant height}$$

$$=\pi(1+3)\sqrt{5}$$

$$\underline{=4\pi\sqrt{5}\quad unit^2}$$

### *Exercise*

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve $y = \frac{1}{3}x^3$ about the *x-axis*

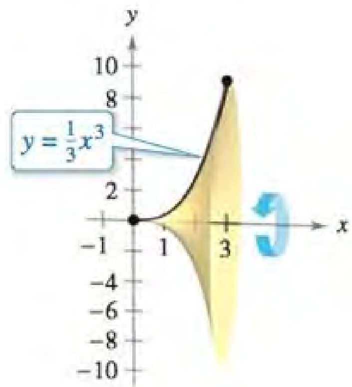
***Solution***

$$y = \frac{1}{3}x^3$$

$$y' = x^2$$

$$\sqrt{1+(y')^2} = \sqrt{1+x^4}$$

$$S = 2\pi \int_0^3 \frac{1}{3}x^3 \sqrt{1+x^4}\, dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$




$$= \frac{\pi}{6}\int_0^3 \left(1+x^4\right)^{1/2}\, d\left(1+x^4\right)$$

$$= \frac{\pi}{9}\left(1+x^4\right)^{3/2} \Big|_0^3$$

$$= \frac{\pi}{9}\left((82)^{3/2}-1\right)$$

$$= \frac{\pi}{9}\left(82\sqrt{82}-1\right) \quad unit^2$$

### *Exercise*

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve $y = 2\sqrt{x}$ about the *x-axis*

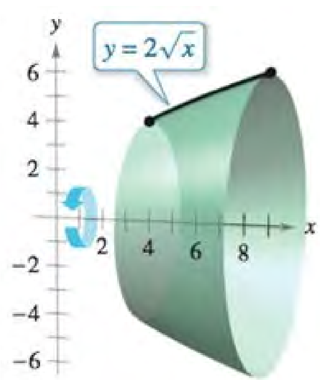
***Solution***

$$y = 2\sqrt{x}$$

$$y' = \frac{1}{\sqrt{x}}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{1}{x}}$$

$$S = 2\pi \int_4^9 2\sqrt{x}\sqrt{\frac{x+1}{x}}\, dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$




$$= 4\pi \int_4^9 (1+x)^{1/2}\, d(1+x)$$

$$= \frac{8}{3}\pi (1+x)^{3/2} \Big|_4^9$$

$$= \frac{8}{3}\pi \left(10^{3/2} - 5^{3/2}\right) \quad unit^2 \qquad \approx 171.285$$

## Exercise

Find the area of the surface generated by $y = \frac{x^3}{9}, \quad 0 \le x \le 2, \quad x - axis$

**Solution**

$$\frac{dy}{dx} = \frac{1}{3}x^2$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{9}x^4}$$

$$= \frac{1}{3}\sqrt{9+x^4}$$

$$S = 2\pi \int_0^2 \frac{x^3}{9}\,\frac{1}{3}\sqrt{9+x^4}\,dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \frac{2\pi}{27}\int_0^2 x^3\sqrt{9+x^4}\,dx$$

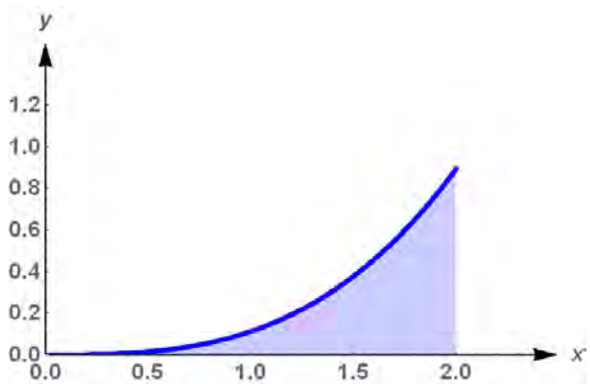
$$= \frac{\pi}{54}\int_0^2 \left(9+x^4\right)^{1/2} d\left(x^4+9\right)$$

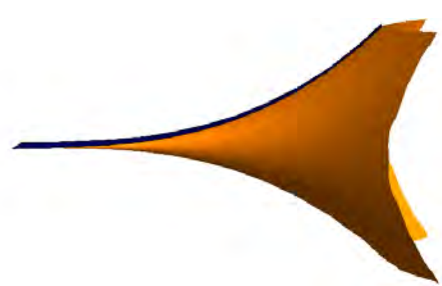
$$= \frac{\pi}{81}\left(x^4+9\right)^{3/2} \Big|_0^2$$

$$= \frac{\pi}{81}\left(25^{3/2} - 9^{3/2}\right)$$

$$= \frac{\pi}{81}(125-27)$$

$$= \frac{98\pi}{81} \quad unit^2$$



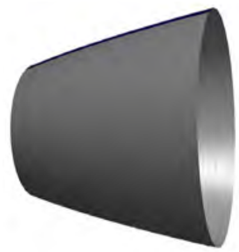
### *Exercise*

Find the area of the surface generated by $y=\sqrt{x+1}, \quad 1 \le x \le 5, \quad x-axis$

***Solution***

$$y=\sqrt{x+1}$$

$$\frac{dy}{dx}=\frac{1}{2}(x+1)^{-1/2}$$



$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{1}{4}(x+1)^{-1}}$$

$$=\sqrt{1+\frac{1}{4(x+1)}}$$

$$=\sqrt{\frac{4x+4+1}{4(x+1)}}$$

$$=\frac{1}{2}\sqrt{\frac{4x+5}{x+1}}$$

$$S=2\pi\int_1^5 \sqrt{x+1}\left(\frac{1}{2}\right)\frac{\sqrt{4x+5}}{\sqrt{x+1}}\,dx \qquad S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\pi\int_1^5 \sqrt{4x+5}\,dx$$

$$=\frac{\pi}{4}\int_1^5 (4x+5)^{1/2}\,d(4x+5)$$

$$=\frac{\pi}{6}(4x+5)^{3/2}\Big|_1^5$$

$$=\frac{\pi}{6}\left(25^{3/2}-9^{3/2}\right)$$

$$=\frac{\pi}{6}(98)$$

$$=\frac{49\pi}{3} \quad unit^2$$

### *Exercise*

Find the area of the surface generated by $y=\sqrt{2x-x^2}, \quad 0.5 \le x \le 1.5, \quad x-axis$

***Solution***

$$\frac{dy}{dx}=\frac{1}{2}\left(2x-x^2\right)^{-1/2}(2-2x)$$

$$= (1-x)\left(2x-x^2\right)^{-1/2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+(1-x)^2\left(2x-x^2\right)^{-1}}$$

$$= \sqrt{1+\frac{1-2x+x^2}{2x-x^2}}$$

$$= \sqrt{\frac{2x-x^2+1-2x+x^2}{2x-x^2}}$$

$$= \sqrt{\frac{1}{2x-x^2}}$$

$$= \frac{1}{\sqrt{2x-x^2}}$$

$$S = 2\pi\int_{.5}^{1.5} \sqrt{2x-x^2}\,\frac{1}{\sqrt{2x-x^2}}\,dx \qquad S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi\int_{.5}^{1.5} dx$$

$$= 2\pi\,x\Big|_{.5}^{1.5} \qquad = 2\pi(1.5-.5)$$

$$= 2\pi \quad unit^2$$

## Exercise

Find the area of the surface generated by $y = 3x+4, \quad 0 \le x \le 6, \quad revolved\ about\ x-axis$

**Solution**

$$y' = 3$$

$$S = 2\pi\int_0^6 (3x+4)\,\sqrt{1+9}\,dx \qquad S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi\sqrt{10}\left(\frac{3}{2}x^2+4x\right)\Big|_0^6$$

$$= 2\pi\sqrt{10}(54+24)$$

$$= 156\pi\sqrt{10} \quad unit^2$$

### Exercise

Find the area of the surface generated by $y = 12 - 3x, \quad 1 \le x \le 3, \quad revolved\ \ about\ \ x - axis$

***Solution***

$y' = -3$

$$S = 2\pi \int_1^3 (12 - 3x)\sqrt{1+9}\ dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= 2\pi\sqrt{10}\left(12x - \frac{3}{2}x^2 \ \Big|_1^3\right.$$

$$= 2\pi\sqrt{10}\left(36 - \frac{27}{2} - 12 + \frac{3}{2}\right)$$

$$\underline{= 24\pi\sqrt{10} \quad unit^2}$$

### Exercise

Find the area of the surface generated by $y = 8\sqrt{x}, \quad 9 \le x \le 20, \quad revolved\ \ about\ \ x - axis$

***Solution***

$y' = \dfrac{4}{\sqrt{x}}$

$$S = 2\pi \int_9^{20} 8\sqrt{x}\sqrt{1+\frac{16}{x}}\ dx \qquad\qquad S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= 16\pi \int_9^{20} \sqrt{x}\,\frac{\sqrt{x+16}}{\sqrt{x}}\ dx$$

$$= 16\pi \int_9^{20} (x+16)^{1/2}\ d(x+16)$$

$$= \frac{32\pi}{3}(x+16)^{3/2}\ \Big|_9^{20}$$

$$= \frac{32\pi}{3}\left((36)^{3/2} - (25)^{3/2}\right)$$

$$= \frac{32\pi}{3}(216 - 125)$$

$$\underline{= \frac{2912\pi}{3} \quad unit^2}$$

## Exercise

Find the area of the surface generated by $y = x^3, \quad 0 \le x \le 1, \quad revolved\ about\ x-axis$

**Solution**

$$y' = 3x^2$$

$$S = 2\pi \int_0^1 x^3 \sqrt{1+9x^4}\ dx \qquad\qquad S = 2\pi \int_a^b y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \frac{\pi}{18} \int_0^1 \left(1+9x^4\right)^{1/2}\ d\left(1+9x^4\right)$$

$$= \frac{\pi}{27} \left(1+9x^4\right)^{3/2} \Big|_0^1$$

$$= \frac{\pi}{27}\left((10)^{3/2} - 1\right)$$

$$\boxed{= \frac{\pi}{27}\left(10\sqrt{10} - 1\right) \quad unit^2}$$

## Exercise

Find the area of the surface generated by $y = x^{3/2} - \frac{1}{3}x^{1/2}, \quad 1 \le x \le 2, \quad revolved\ about\ x-axis$

**Solution**

$$y' = \frac{3}{2}x^{1/2} - \frac{1}{6}x^{-1/2}$$

$$= \frac{3}{2}\sqrt{x} - \frac{1}{6\sqrt{x}}$$

$$a = 1, \quad m = \frac{3}{2}, \quad b = -\frac{1}{3}, \quad n = \frac{1}{2} \qquad\qquad f(x) = ax^m + bx^n$$

1. $m + n = \frac{3}{2} + \frac{1}{2} = 2$ ✓
2. $abmn = (1)\left(\frac{3}{2}\right)\left(-\frac{1}{3}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$ ✓

$$S = 2\pi \int_1^2 \left(x^{3/2} - \frac{1}{3}x^{1/2}\right)\left(\frac{3}{2}\sqrt{x} + \frac{1}{6\sqrt{x}}\right) dx \qquad\qquad S = 2\pi \int_a^b f(x)\ \overline{f'(x)}\ dx$$

$$= 2\pi \int_1^2 \left(\frac{3}{2}x^2 - \frac{1}{3}x - \frac{1}{18}\right) dx$$

$$= 2\pi \left(\frac{1}{2}x^3 - \frac{1}{6}x^2 - \frac{1}{18}x \ \Big|_1^2\right.$$

$$= 2\pi\left(4 - \frac{2}{3} - \frac{1}{9} - \frac{1}{2} + \frac{1}{6} + \frac{1}{18}\right)$$

$$= 2\pi\left(3 - \frac{1}{18}\right)$$

$$\boxed{= \frac{53\pi}{9} \quad unit^2}$$

$$S = 2\pi\int_1^2 \left(x^{3/2} - \frac{1}{3}x^{1/2}\right)\sqrt{1 + \frac{(9x-1)^2}{36x}}\,dx \qquad S = 2\pi\int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \frac{2}{3}\pi\int_1^2 \left(3x^{3/2} - x^{1/2}\right)\frac{\sqrt{36x + 81x^2 - 18x + 1}}{6\sqrt{x}}\,dx$$

$$= \frac{\pi}{9}\int_1^2 (3x-1)\sqrt{81x^2 + 18x + 1}\,dx$$

$$= \frac{\pi}{9}\int_1^2 (3x-1)\sqrt{(9x+1)^2}\,dx$$

$$= \frac{\pi}{9}\int_1^2 (3x-1)(9x+1)\,dx$$

$$= \frac{\pi}{9}\int_1^2 \left(27x^2 - 6x - 1\right)\,dx$$

$$= \frac{\pi}{9}\left(9x^3 - 3x^2 - x\right)\Big|_1^2$$

$$= \frac{\pi}{9}(72 - 12 - 2 - 9 + 3 + 1)$$

$$\boxed{= \frac{53\pi}{9} \quad unit^2}$$

## *Exercise*

Find the area of the surface generated by $y = \sqrt{4x+6}, \quad 0 \le x \le 5, \quad revolved\ about\ x-axis$

***Solution***

$$y' = \frac{2}{\sqrt{4x+6}}$$

$$S = 2\pi\int_0^5 \sqrt{4x+6}\sqrt{1 + \frac{4}{4x+6}}\,dx \qquad S = 2\pi\int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi \int_0^5 \sqrt{4x+6}\ \sqrt{\frac{4x+6+4}{4x+6}}\ dx$$

$$= 2\pi \int_0^5 (4x+10)^{1/2}\ dx$$

$$= \frac{\pi}{2} \int_0^5 (4x+10)^{1/2}\ d(4x+10)$$

$$= \frac{\pi}{3}\ (4x+10)^{3/2}\ \Big|_0^5$$

$$= \frac{\pi}{3}\left(30^{3/2} - 10^{3/2}\right)$$

$$= \frac{\pi}{3}\left(30\sqrt{30} - 10\sqrt{10}\right)$$

$$= \frac{10\pi\sqrt{10}}{3}\left(3\sqrt{3}-1\right) \quad unit^2$$

### *Exercise*

Find the area of the surface generated by $y = \frac{1}{4}\left(e^{2x} + e^{-2x}\right), \quad -2 \le x \le 2, \quad revolved\ about\ x-axis$

**Solution**

$$y' = \frac{1}{2}\left(e^{2x} - e^{-2x}\right)$$

$a = \frac{1}{4}, \quad m = 2, \quad b = \frac{1}{4}, \quad n = -2$ $\qquad f(x) = ae^{mx} + be^{nx}$

1. $m = -n$ ✓
2. $abmn = \frac{1}{4}\left(\frac{1}{4}\right)(2)(-2) = -\frac{1}{4}$ ✓

$$S = 2\pi \int_{-2}^{2} \frac{1}{4}\left(e^{2x} + e^{-2x}\right)\ \frac{1}{2}\left(e^{2x} + e^{-2x}\right)\ dx \qquad S = 2\pi \int_a^b f(x)\ \overline{f'(x)}\, dx$$

$$= \frac{\pi}{4} \int_{-2}^{2} \left(e^{4x} + 1 + e^{-4x}\right)\ dx$$

$$= \frac{\pi}{4}\ \left(\frac{1}{4}e^{4x} + 2x - \frac{1}{4}e^{-4x}\ \Big|_{-2}^{2}\right.$$

$$= \frac{\pi}{4}\left(\frac{1}{4}e^{8} + 4 - \frac{1}{4}e^{-8} - \frac{1}{4}e^{-8} + 4 + \frac{1}{4}e^{8}\right)$$

$$= \frac{\pi}{4}\left(\frac{1}{2}e^{8} + 8 - \frac{1}{2}e^{-8}\right)$$

$$\boxed{=\frac{\pi}{8}\left(e^{8}+16-e^{-8}\right)\quad unit^{2}}$$

$$S=2\pi\int_{-2}^{2}\frac{1}{4}\left(e^{2x}+e^{-2x}\right)\sqrt{1+\frac{1}{4}\left(e^{2x}-e^{-2x}\right)^{2}}\,dx \qquad S=2\pi\int_{a}^{b}y\sqrt{1+\left(\frac{dy}{dx}\right)^{2}}\,dx$$

$$=\frac{\pi}{4}\int_{-2}^{2}\left(e^{2x}+e^{-2x}\right)\sqrt{4+e^{4x}-2+e^{-4x}}\,dx$$

$$=\frac{\pi}{4}\int_{-2}^{2}\left(e^{2x}+e^{-2x}\right)\sqrt{e^{4x}+2+e^{-4x}}\,dx$$

$$=\frac{\pi}{4}\int_{-2}^{2}\left(e^{2x}+e^{-2x}\right)\sqrt{\left(e^{2x}+e^{-2x}\right)^{2}}\,dx$$

$$=\frac{\pi}{4}\int_{-2}^{2}\left(e^{2x}+e^{-2x}\right)^{2}\,dx$$

$$=\frac{\pi}{4}\int_{-2}^{2}\left(e^{4x}+2+e^{-4x}\right)\,dx$$

$$=\frac{\pi}{4}\left(\frac{1}{4}e^{4x}+2x-\frac{1}{4}e^{-4x}\right.\Big|_{-2}^{2}$$

$$=\frac{\pi}{4}\left(\frac{1}{4}e^{8}+4-\frac{1}{4}e^{-8}-\frac{1}{4}e^{-8}+4+\frac{1}{4}e^{8}\right)$$

$$=\frac{\pi}{4}\left(\frac{1}{2}e^{8}+8-\frac{1}{2}e^{-8}\right)$$

$$\boxed{=\frac{\pi}{8}\left(e^{8}+16-e^{-8}\right)\quad unit^{2}}$$

## *Exercise*

Find the area of the surface generated by $y=\frac{1}{8}x^{4}+\frac{1}{4x^{2}},\quad 1\le x\le 2,\quad revolved\ about\ x-axis$

***Solution***

$$y'=\frac{1}{2}x^{3}-\frac{1}{2x^{3}}$$

$$a=\frac{1}{8},\quad m=4,\quad b=\frac{1}{4},\quad n=-2 \qquad f(x)=ax^{m}+bx^{n}$$

**1.** $m+n=4-2=2$ ✓

**2.** $abmn=\frac{1}{8}\left(\frac{1}{4}\right)(4)(-2)=-\frac{1}{4}$ ✓

$$S = 2\pi \int_1^2 \left(\frac{1}{8}x^4 + \frac{1}{4x^2}\right)\left(\frac{1}{2}x^3 + \frac{1}{2x^3}\right)\, dx \qquad\qquad S = 2\pi \int_a^b f(x)\ \overline{f'(x)}\, dx$$

$$= \pi \int_1^2 \left(\frac{1}{4}x^7 + \frac{3}{4}x + \frac{1}{2}x^{-5}\right)\, dx$$

$$= \frac{\pi}{8}\left(\frac{1}{8}x^8 + \frac{3}{2}x^2 - \frac{1}{2}x^{-4}\ \right|_1^2$$

$$= \frac{\pi}{8}\left(32 + 6 - \frac{1}{32} - \frac{1}{8} - \frac{3}{2} + \frac{1}{2}\right)$$

$$= \frac{\pi}{8}\left(37 - \frac{5}{32}\right)$$

$$= \frac{1179\pi}{256} \quad unit^2$$

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$$S = 2\pi \int_1^2 \left(\frac{1}{8}x^4 + \frac{1}{4x^2}\right)\sqrt{1 + \left(\frac{x^6 - 1}{2x^3}\right)^2}\, dx \qquad\qquad S = 2\pi \int_a^b y\ \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \frac{\pi}{4}\int_1^2 \left(\frac{x^6 + 2}{x^2}\right)\sqrt{1 + \frac{x^{12} - 2x^6 + 1}{4x^6}}\, dx$$

$$= \frac{\pi}{4}\int_1^2 \left(\frac{x^6 + 2}{x^2}\right)\sqrt{\frac{x^{12} + 2x^6 + 1}{4x^6}}\, dx$$

$$= \frac{\pi}{4}\int_1^2 \left(\frac{x^6 + 2}{x^2}\right)\frac{\sqrt{\left(x^6 + 1\right)^2}}{2x^3}\, dx$$

$$= \frac{\pi}{4}\int_1^2 \frac{\left(x^6 + 2\right)\left(x^6 + 1\right)}{2x^5}\, dx$$

$$= \frac{\pi}{4}\int_1^2 \frac{x^{12} + 3x^6 + 2}{2x^5}\, dx$$

$$= \frac{\pi}{8}\int_1^2 \left(x^7 + 3x + 2x^{-5}\right)\, dx$$

$$= \frac{\pi}{8}\left(\frac{1}{8}x^8 + \frac{3}{2}x^2 - \frac{1}{2}x^{-4}\ \right|_1^2$$

$$= \frac{\pi}{8}\left(32 + 6 - \frac{1}{32} - \frac{1}{8} - \frac{3}{2} + \frac{1}{2}\right)$$

$$= \frac{\pi}{8}\left(37 - \frac{5}{32}\right)$$

$$= \frac{1179\pi}{256} \quad unit^2$$

## *Exercise*

Find the area of the surface generated by $y=\frac{1}{3}x^3+\frac{1}{4x}, \quad \frac{1}{2}\le x\le 2, \quad revolved\ \ about\ \ x-axis$

**Solution**

$$y'=x^2-\frac{1}{4x^2}$$

$$=\frac{4x^4-1}{4x^2}$$

$$a=\frac{1}{3}, \quad m=3, \quad b=\frac{1}{4}, \quad n=-1 \qquad\qquad f(x)=ax^m+bx^n$$

1. $m+n=3-1=2$ $\checkmark$
2. $abmn=\frac{1}{3}\left(\frac{1}{4}\right)(3)(-1)=-\frac{1}{4}$ $\checkmark$

$$S=2\pi\int_{1/2}^{2}\left(\frac{1}{3}x^3+\frac{1}{4x}\right)\left(x^2+\frac{1}{4x^2}\right)dx \qquad\qquad S=2\pi\int_a^b f(x)\ \overline{f'(x)}\,dx$$

$$=2\pi\int_{1/2}^{2}\left(\frac{1}{3}x^5+\frac{1}{3}x+\frac{1}{16}x^{-3}\right)dx$$

$$=2\pi\left(\frac{1}{18}x^6+\frac{1}{6}x^2-\frac{1}{32}x^{-2}\ \right|_{1/2}^{2}$$

$$=2\pi\left(\frac{32}{9}+\frac{2}{3}-\frac{1}{128}-\frac{1}{1{,}152}-\frac{1}{24}+\frac{1}{8}\right)$$

$$=2\pi\left(\frac{4{,}096+768-9-1-48+144}{1{,}152}\right)$$

$$=2\pi\left(\frac{4{,}950}{1{,}152}\right)$$

$$\boxed{=\frac{275\pi}{32}\quad unit^2}$$

$$S=2\pi\int_{1/2}^{2}\left(\frac{1}{3}x^3+\frac{1}{4x}\right)\sqrt{1+\left(\frac{4x^4-1}{4x^2}\right)^2}\,dx \qquad\qquad S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=2\pi\int_{1/2}^{2}\left(\frac{4x^4+3}{12x}\right)\sqrt{1+\frac{16x^8-8x^4+1}{16x^4}}\,dx$$

$$=\frac{\pi}{6}\int_{1/2}^{2}\left(\frac{4x^4+3}{x}\right)\sqrt{\frac{16x^8+8x^4+1}{16x^4}}\,dx$$

$$=\frac{\pi}{6}\int_{1/2}^{2}\left(\frac{4x^4+3}{x}\right)\frac{\sqrt{\left(4x^4+1\right)^2}}{4x^2}\,dx$$

$$=\frac{\pi}{24}\int_{1/2}^{2}\left(\frac{4x^4+3}{x^3}\right)\left(4x^4+1\right)\,dx$$

$$=\frac{\pi}{24}\int_{1/2}^{2}\left(4x+3x^{-3}\right)\left(4x^4+1\right)\,dx$$

$$=\frac{\pi}{24}\int_{1/2}^{2}\left(16x^5+16x+3x^{-3}\right)\,dx$$

$$=\frac{\pi}{24}\left(\frac{8}{3}x^6+8x^2-\frac{3}{2}x^{-2}\right.\Bigg|_{1/2}^{2}$$

$$=\frac{\pi}{24}\left(\frac{512}{3}+32-\frac{3}{8}-\frac{1}{24}-2+6\right)$$

$$=\frac{\pi}{24}\left(\frac{4086}{24}+36\right)$$

$$=\frac{\pi}{24}\left(\frac{681}{4}+36\right)$$

$$=\frac{\pi}{24}\left(\frac{825}{4}\right)$$

$$=\frac{275\pi}{32}\quad unit^2$$

### *Exercise*

Find the area of the surface generated by $y=\sqrt{5x-x^2},\quad 1\le x\le 4,\quad revolved\ about\ x-axis$

***Solution***

$$y'=\frac{5-2x}{2\sqrt{5x-x^2}}$$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{(5-2x)^2}{4\left(5x-x^2\right)}$$

$$=\frac{20x-4x^2+25-20.x+4x^2}{4\left(5x-x^2\right)}$$

$$=\frac{25}{4\left(5x-x^2\right)}$$

$$S=2\pi\int_1^4\sqrt{5x-x^2}\,\sqrt{\frac{25}{4\left(5x-x^2\right)}}\,dx$$

$$S=2\pi\int_a^b y\,\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 5\pi \int_1^4 dx$$

$$= 5\pi\, x \Big|_1^4$$

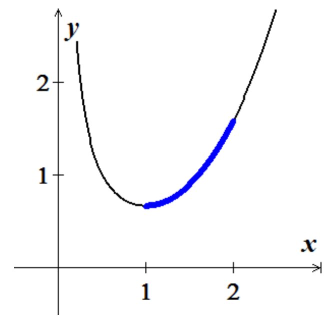
$$= 15\pi \quad unit^2$$

## Exercise

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the

*x-axis* $\qquad y = \frac{1}{6}x^3 + \frac{1}{2x}, \quad 1 \le x \le 2$

**Solution**




$$y' = \frac{1}{2}x^2 - \frac{1}{2x^2}$$

$$a = \frac{1}{6}, \quad m = 3, \quad b = \frac{1}{2}, \quad n = -1 \qquad\qquad f(x) = ax^m + bx^n$$

1. $m + n = 3 - 1 = 2$ ✓
2. $abmn = \frac{1}{6}\left(\frac{1}{2}\right)(3)(-1) = -\frac{1}{4}$ ✓

$$S = 2\pi \int_1^2 \left(\frac{1}{6}x^3 + \frac{1}{2x}\right)\left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right) dx \qquad\qquad S = 2\pi \int_a^b f(x)\ \overline{f'(x)}\, dx$$

$$= 2\pi \int_1^2 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3}\right) dx$$

$$= 2\pi \left(\frac{1}{72}x^6 + \frac{1}{6}x^2 - \frac{1}{8x^2}\right)\Big|_1^2$$

$$= 2\pi \left(\frac{64}{72} + \frac{2}{3} - \frac{1}{32} - \frac{1}{72} - \frac{1}{6} + \frac{1}{8}\right)$$

$$= 2\pi \left(\frac{63}{72} + \frac{19}{32}\right)$$

$$= 2\pi \left(\frac{423}{288}\right)$$

$$= \frac{47\pi}{16} \quad unit^2$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4x^4}}$$

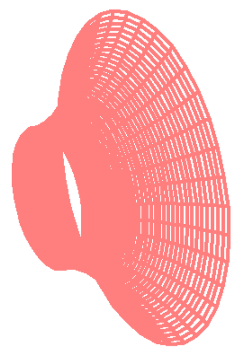
$$= \sqrt{\frac{1}{4}x^4 + \frac{1}{2} + \frac{1}{4x^4}}$$

$$= \sqrt{\left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)^2}$$

$$= \frac{1}{2}x^2 + \frac{1}{2x^2}$$

$$S = 2\pi \int_1^2 \left(\frac{1}{6}x^3 + \frac{1}{2x}\right)\left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right) dx \qquad S = 2\pi \int_a^b y\,\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi \int_1^2 \left(\frac{x^5}{12} + \frac{x}{3} + \frac{1}{4x^3}\right) dx$$



$$= 2\pi \left(\frac{1}{72}x^6 + \frac{1}{6}x^2 - \frac{1}{8x^2}\right)\Big|_1^2$$

$$= 2\pi\left(\frac{64}{72} + \frac{2}{3} - \frac{1}{32} - \frac{1}{72} - \frac{1}{6} + \frac{1}{8}\right)$$

$$= 2\pi\left(\frac{63}{72} + \frac{19}{32}\right)$$

$$= 2\pi\left(\frac{423}{288}\right)$$

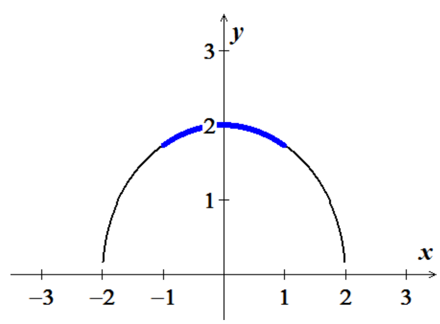
$$\boxed{= \frac{47\pi}{16} \quad unit^2}$$

### *Exercise*

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the

*x-axis* $\qquad y = \sqrt{4 - x^2}, \quad -1 \le x \le 1$

***Solution***




$$y' = \frac{-x}{\sqrt{4 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{4 - x^2}}$$

$$= \sqrt{\frac{4}{4 - x^2}}$$

$$S = 2\pi \int_{-1}^{1} \sqrt{4 - x^2}\,\frac{2}{\sqrt{4 - x^2}}\,dx \qquad S = 2\pi \int_a^b y\,\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$



$$= 4\pi \int_{-1}^{1} dx$$

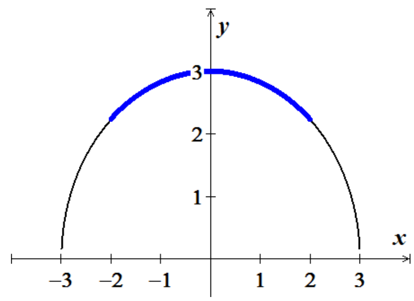
$$= 4\pi\, x \Big|_{-1}^{1}$$

$= 8\pi \quad unit^2$

## *Exercise*

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the *x-axis* $\qquad y = \sqrt{9 - x^2}, \quad -2 \le x \le 2$

***Solution***

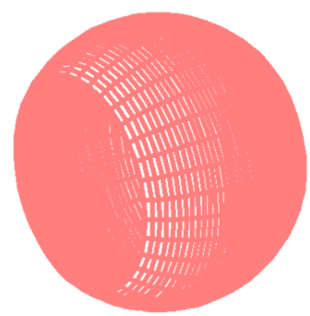



$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{9 - x^2}}$$

$$= \sqrt{\frac{9}{9 - x^2}}$$

$$S = 2\pi \int_{-2}^{2} \sqrt{9 - x^2}\, \frac{3}{\sqrt{9 - x^2}}\, dx \qquad S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



$$= 6\pi \int_{-2}^{2} dx$$

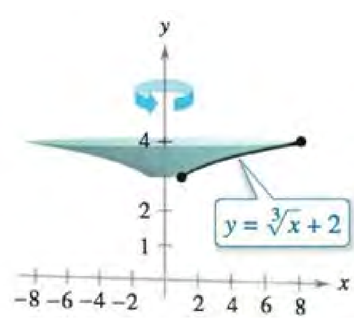
$$= 6\pi x \Big|_{-2}^{2}$$

$$= 24\pi \quad unit^2$$

## *Exercise*

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the *y-axis*

***Solution***




$$y = \sqrt[3]{x} + 2$$

$$\sqrt[3]{x} = y - 2$$

$$x = (y - 2)^3$$

$$\frac{dx}{dy} = 3(y - 2)^2$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + 9(y - 2)^4}$$

$$S = 2\pi \int_3^4 (y-2)^3 \sqrt{1+9(y-2)^4}\ dy \qquad S = 2\pi \int_a^b x(y) \sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

$$= \frac{\pi}{18} \int_3^4 \left(1+9(y-2)^4\right)^{1/2}\ d\left(1+9(y-2)^4\right)$$

$$= \frac{\pi}{27} \left(1+9(y-2)^4\right)^{3/2} \Big|_3^4$$

$$= \frac{\pi}{27}\left(145^{3/2} - 10^{3/2}\right)$$

$$\boxed{= \frac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right) \quad unit^2}$$

$$y' = \frac{1}{3} x^{-2/3}$$

$$= \frac{1}{3x^{2/3}}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{1}{9x^{4/3}}}$$

$$= \frac{\sqrt{9x^{4/3}+1}}{3x^{2/3}}$$

$$S = 2\pi \int_1^8 x\ \frac{\sqrt{9x^{4/3}+1}}{3x^{2/3}}\ dx \qquad S = 2\pi \int_a^b x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \frac{2}{3}\pi \int_1^8 x^{1/3} \sqrt{9x^{4/3}+1}\ dx$$

$$= \frac{\pi}{18} \int_1^8 \left(9x^{4/3}+1\right)^{1/2}\ d\left(9x^{4/3}+1\right)$$

$$= \frac{\pi}{27} \left(9x^{4/3}+1\right)^{3/2} \Big|_1^8$$

$$= \frac{\pi}{27}\left(\left(72(8)^{1/3}+1\right)^{3/2} - 10^{3/2}\right)$$

$$\boxed{= \frac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right) \quad unit^2}$$

### Exercise

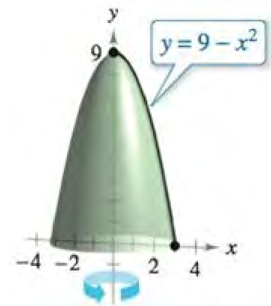
Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the *y-axis*

**Solution**

$$y' = -2x$$

$$\sqrt{1+(y')^2} = \sqrt{1+4x^2}$$

$$S = 2\pi\int_0^3 x\sqrt{1+4x^2}\, dx \qquad S = 2\pi\int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$




$$= \frac{\pi}{4}\int_0^3 \left(1+4x^2\right)^{1/2} d\left(1+4x^2\right)$$

$$= \frac{\pi}{6}\left(1+4x^2\right)^{3/2}\Big|_0^3$$

$$= \frac{\pi}{6}\left(37\sqrt{37}-1\right) \quad unit^2$$

### Exercise

Find the area of the surface generated by $y=(3x)^{1/3}; \quad 0 \le x \le \frac{8}{3}$ *about y-axis*

**Solution**

$$3x = y^3 \rightarrow x = \frac{1}{3}y^3$$

$$x' = y^2$$

$$\begin{cases} x = 0 & \rightarrow y = 0 \\ x = \frac{8}{3} & \rightarrow y = \left(3\frac{8}{3}\right)^{1/3} = 2 \end{cases}$$

$$S = 2\pi\int_0^2 \frac{1}{3}y^3\sqrt{1+y^4}\, dy \qquad S = 2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \frac{\pi}{6}\int_0^2 \left(1+y^4\right)^{1/2} d\left(1+y^4\right)$$

$$= \frac{\pi}{9}\left(1+y^4\right)^{3/2}\Big|_0^2$$

$$= \frac{\pi}{9}\left((17)^{3/2}-1\right)$$

$$= \frac{\pi}{9}\left(17\sqrt{17}-1\right) \quad unit^2$$

## Exercise

Find the area of the surface generated of the curve $y = 4x - 1$ between the points $(1,\ 3)$ and $(4,\ 15)$ *about y-axis*

**Solution**

$$y = 4x - 1$$

$$x = \frac{1}{4}(y+1)$$

$$x' = \frac{1}{4}$$

$$S = 2\pi \int_3^{15} \frac{1}{4}(y+1)\sqrt{1+\frac{1}{16}}\ dy \qquad S = 2\pi \int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

$$= \frac{\pi}{2}\sqrt{\frac{17}{16}} \int_3^{15} (y+1)\ dy$$

$$= \frac{\pi\sqrt{17}}{8}\left(\frac{1}{2}y^2 + y\ \right|_3^{15}$$

$$= \frac{\pi\sqrt{17}}{8}\left(\frac{225}{2} + 15 - \frac{9}{2} - 3\right)$$

$$= \frac{\pi\sqrt{17}}{8}(120)$$

$$\underline{= 15\pi\sqrt{17}\quad unit^2}$$

## Exercise

Find the area of the surface generated of the curve $y = \frac{1}{2}\ln\left(2x + \sqrt{4x^2 - 1}\right)$ between the points $\left(\frac{1}{2},\ 0\right)$ and $\left(\frac{17}{16},\ \ln 2\right)$ *about y-axis*

**Solution**

$$2y = \ln\left(2x + \sqrt{4x^2 - 1}\right)$$

$$\left(2x + \sqrt{4x^2 - 1}\right)^2 = \left(e^{2y}\right)^2$$

$$4x^2 + 4x\sqrt{4x^2 - 1} + 4x^2 - 1 = e^{4y}$$

$$4x\left(2x + \sqrt{4x^2 - 1}\right) = e^{4y} + 1 \qquad 2x + \sqrt{4x^2 - 1} = e^{2y}$$

$$4x\left(e^{2y}\right) = e^{4y} + 1$$

$$x=\frac{e^{4y}+1}{4e^{2y}}$$

$$\boxed{=\frac{1}{4}\left(e^{2y}+e^{-2y}\right)}$$

$$x'=\frac{1}{2}\left(e^{2y}-e^{-2y}\right)$$

$$a=\frac{1}{4},\quad m=2,\quad b=\frac{1}{4},\quad n=-2 \qquad\qquad f(x)=ae^{mx}+be^{nx}$$

1. $m=-n$ ✓
2. $abmn=\frac{1}{4}\left(\frac{1}{4}\right)(2)(-2)=-\frac{1}{4}$ ✓

$$S=2\pi\int_0^{\ln 2}\frac{1}{4}\left(e^{2y}+e^{-2y}\right)\ \frac{1}{2}\left(e^{2y}+e^{-2y}\right)\ dy \qquad\qquad S=2\pi\int_a^b f(x)\ \overline{f'(x)}\ dx$$

$$=\frac{\pi}{4}\int_0^{\ln 2}\left(e^{2y}+e^{-2y}\right)^2\ dy$$

$$=\frac{\pi}{4}\int_0^{\ln 2}\left(e^{4y}+2+e^{-4y}\right)\ dy$$

$$=\frac{\pi}{4}\left(\frac{1}{4}e^{4y}+2y-\frac{1}{4}e^{-4y}\ \right|_0^{\ln 2}$$

$$=\frac{\pi}{4}\left(\frac{1}{4}e^{4\ln 2}+2\ln 2-\frac{1}{4}e^{-4\ln 2}-\frac{1}{4}+\frac{1}{4}\right)$$

$$=\frac{\pi}{4}\left(\frac{1}{4}e^{\ln 2^4}+2\ln 2-\frac{1}{4}e^{\ln 2^{-4}}\right)$$

$$=\frac{\pi}{4}\left(\frac{1}{4}2^4+2\ln 2-\frac{1}{4}2^{-4}\right)$$

$$=\frac{\pi}{4}\left(4+2\ln 2-\frac{1}{64}\right)$$

$$\boxed{=\frac{\pi}{4}\left(\frac{255}{64}+2\ln 2\right)\quad unit^2}$$

---

$$S=2\pi\int_0^{\ln 2}\frac{1}{4}\left(e^{2y}+e^{-2y}\right)\sqrt{1+\frac{1}{4}\left(e^{2y}-e^{-2y}\right)^2}\ dy \qquad\qquad S=2\pi\int_c^d x\ \sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

$$=\frac{\pi}{4}\int_0^{\ln 2}\left(e^{2y}+e^{-2y}\right)\sqrt{4+e^{4y}-2+e^{-4y}}\ dy$$

$$=\frac{\pi}{4}\int_0^{\ln 2}\left(e^{2y}+e^{-2y}\right)\sqrt{\left(e^{2y}+e^{-2y}\right)^2}\ dy$$

$$= \frac{\pi}{4}\int_0^{\ln 2}\left(e^{2y}+e^{-2y}\right)^2\ dy$$

$$= \frac{\pi}{4}\int_0^{\ln 2}\left(e^{4y}+2+e^{-4y}\right)\ dy$$

$$= \frac{\pi}{4}\left(\frac{1}{4}e^{4y}+2y-\frac{1}{4}e^{-4y}\right)\Big|_0^{\ln 2}$$

$$= \frac{\pi}{4}\left(\frac{1}{4}e^{4\ln 2}+2\ln 2-\frac{1}{4}e^{-4\ln 2}-\frac{1}{4}+\frac{1}{4}\right)$$

$$= \frac{\pi}{4}\left(\frac{1}{4}e^{\ln 2^4}+2\ln 2-\frac{1}{4}e^{\ln 2^{-4}}\right)$$

$$= \frac{\pi}{4}\left(\frac{1}{4}2^4+2\ln 2-\frac{1}{4}2^{-4}\right)$$

$$= \frac{\pi}{4}\left(4+2\ln 2-\frac{1}{64}\right)$$

$$= \frac{\pi}{4}\left(\frac{255}{64}+2\ln 2\right)\quad unit^2$$

## *Exercise*

Find the area of the surface generated by $x=\sqrt{12y-y^2};\quad 2\le y\le 10\quad about\ y\text{-}axis$

**Solution**

$$x' = \frac{6-y}{\sqrt{12y-y^2}}$$

$$S = 2\pi\int_2^{10}\sqrt{12y-y^2}\ \sqrt{1+\frac{(6-y)^2}{12y-y^2}}\ dy \qquad S = 2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$$

$$= 2\pi\int_2^{10}\sqrt{12y-y^2+36-12y+y^2}\ dy$$

$$= 12\pi\int_2^{10} dy$$

$$= 12\pi\, y\Big|_2^{10}$$

$$= 96\pi\quad unit^2$$

### *Exercise*

Find the area of the surface generated by $x = 4y^{3/2} - \frac{1}{12}y^{1/2};\quad 1 \le y \le 4$ *about y-axis*

***Solution***

$$x' = 6y^{1/2} - \frac{1}{24\sqrt{y}}$$

$$= \frac{144y - 1}{24\sqrt{y}}$$

$$a = 4, \quad m = \frac{3}{2}, \quad b = -\frac{1}{12}, \quad n = \frac{1}{2} \qquad\qquad f(x) = ax^m + bx^n$$

1. $m + n = \frac{3}{2} + \frac{1}{2} = 2$ ✓
2. $abmn = 4\left(-\frac{1}{12}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$ ✓

$$S = 2\pi \int_1^4 \left(4y^{3/2} - \frac{1}{12}y^{1/2}\right)\left(6y^{1/2} - \frac{1}{24}y^{-1/2}\right) dy \qquad\qquad S = 2\pi \int_a^b f(x)\ \overline{f'(x)}\, dx$$

$$= \frac{\pi}{144} \int_1^4 (48y - 1)(144y + 1)\ dy$$

$$= \frac{\pi}{144} \int_1^4 \left(6{,}912y^2 - 96y - 1\right) dy$$

$$= \frac{\pi}{144} \left(2304y^3 - 48y^2 - y\ \right|_1^4$$

$$= \frac{\pi}{144}(147{,}456 - 768 - 4 - 2304 + 48 + 1)$$

$$= \frac{144{,}429\pi}{144}$$

$$= \frac{48{,}143\ \pi}{48} \quad unit^2$$

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$$S = 2\pi \int_1^4 \left(4y^{3/2} - \frac{1}{12}y^{1/2}\right) \sqrt{1 + \frac{(144y-1)^2}{576y}}\ dy \qquad\qquad S = 2\pi \int_c^d x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

$$= 2\pi \int_1^4 \left(4y^{3/2} - \frac{1}{12}y^{1/2}\right) \sqrt{\frac{576y + (144y)^2 - 288y + 1}{576y}}\ dy$$

$$= \frac{\pi}{12} \int_1^4 \left(4y^{3/2} - \frac{1}{12}y^{1/2}\right) \frac{1}{\sqrt{y}} \sqrt{(144y+1)^2}\ dy$$

$$=\frac{\pi}{144}\int_1^4 (48y-1)(144y+1)\ dy$$

$$=\frac{\pi}{144}\int_1^4 \left(6,912y^2-96y-1\right)\ dy$$

$$=\frac{\pi}{144}\ \left(2304y^3-48y^2-y\ \Big|_1^4\right.$$

$$=\frac{\pi}{144}(147,456-768-4-2304+48+1)$$

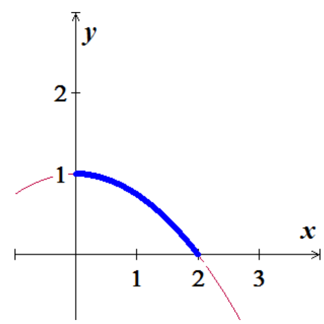
$$=\frac{144,429\pi}{144}$$

$$=\boxed{\frac{48,143\ \pi}{48}\quad unit^2}$$

## *Exercise*

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the *y-axis* $\qquad y=1-\frac{1}{4}x^2,\quad 0\le x\le 2$

***Solution***




$$y'=-\frac{1}{2}x$$

$$\sqrt{1+(y')^2}=\sqrt{1+\frac{x^2}{4}}$$

$$=\frac{1}{2}\sqrt{4+x^2}$$

$$S=2\pi\int_0^2 x\ \frac{\sqrt{4+x^2}}{2}\ dx \qquad\qquad S=2\pi\int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$=\frac{\pi}{2}\int_0^2 \left(4+x^2\right)^{1/2}\ d\left(4+x^2\right)$$



$$=\frac{\pi}{3}\ \left(4+x^2\right)^{3/2}\ \Big|_0^2$$

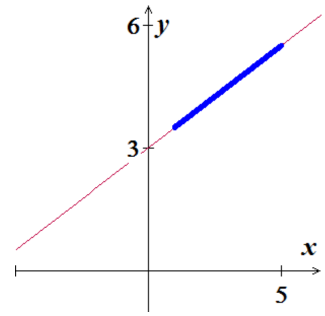
$$=\frac{\pi}{3}\left(8^{3/2}-4^{3/2}\right)$$

$$=\boxed{\frac{\pi}{3}\left(16\sqrt{2}-8\right)\quad unit^2} \qquad \approx 15.318\ \ unit^2$$

## Exercise

Set up an evaluate the definite integral for the area of the surface generated by revolving the curve about the *y-axis* $\quad y = \frac{1}{2}x + 3, \quad 1 \le x \le 5$

**Solution**

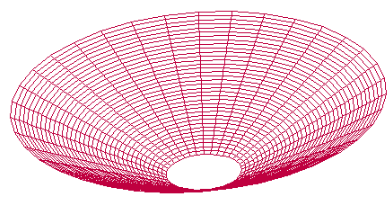



$$y' = \frac{1}{2}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{1}{4}}$$

$$= \frac{\sqrt{5}}{2}$$

$$S = \pi\sqrt{5}\int_1^5 x\, dx \qquad\qquad S = 2\pi\int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$



$$= \pi\sqrt{5}\left(\frac{1}{2}x^2 \Big|_1^5\right.$$

$$= \frac{\sqrt{5}}{2}\pi(25-1)$$

$$= 12\pi\sqrt{5} \quad unit^2$$

## Exercise

A right circular cone is generated by revolving the region bounded by $y = \frac{3}{4}x$, $y = 3$, and $x = 0$ about the *y-axis*. Find the lateral surface area of the cone.

**Solution**

$$y' = \frac{3}{4}$$

$$\sqrt{1+(y')^2} = \sqrt{1+\frac{9}{16}}$$

$$= \frac{5}{4}$$

$$y = 3 = \frac{3}{4}x \Rightarrow x = 4$$

$$S = \frac{5\pi}{2}\int_0^4 x\, dx \qquad\qquad S = 2\pi\int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \frac{5\pi}{4}x^2 \Big|_0^4$$

$$= 20\pi \quad unit^2$$

## Exercise

A right circular cone is generated by revolving the region bounded by $y = \frac{h}{r}x$, $y = h$, and $x = 0$ about the $y$-*axis*. Verify that the lateral surface area of the cone is $S = \pi r\sqrt{r^2 + h^2}$

**Solution**

$$y' = \frac{h}{r}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{h^2}{r^2}}$$

$$= \frac{\sqrt{r^2 + h^2}}{r}$$

$$y = h = \frac{h}{r}x \Rightarrow \boxed{x = r}$$

$$S = 2\pi \int_0^r x \frac{\sqrt{r^2 + h^2}}{r} dx \qquad S = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \frac{\pi\sqrt{r^2 + h^2}}{r}\left(x^2 \Big|_0^r\right.$$

$$\boxed{= \pi r\sqrt{r^2 + h^2} \quad unit^2}$$

## Exercise

Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{9 - x^2}$, $0 \le x \le 2$, about the $y$-*axis*

**Solution**

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{9 - x^2}}$$

$$= \frac{3}{\sqrt{9 - x^2}}$$

$$S = 2\pi \int_0^2 x \frac{3}{\sqrt{9 - x^2}} dx \qquad S = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= -3\pi \int_0^2 \left(9 - x^2\right)^{-1/2} d\left(9 - x^2\right)$$

$$= -6\pi\left(9 - x^2\right)^{1/2}\Big|_0^2$$

$$= -6\pi\left(\sqrt{5} - 3\right)$$

$$= 6\pi\left(3 - \sqrt{5}\right) \quad unit^2$$

### Exercise

Find the area of the zone of a sphere formed by revolving the graph of $y = \sqrt{r^2 - x^2}, \quad 0 \le x \le a$, about the $y\text{-}axis$. Assume that $a < r$.

**Solution**

$$y' = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{r^2 - x^2}}$$

$$= \frac{r}{\sqrt{r^2 - x^2}}$$

$$S = 2\pi \int_0^a x \frac{r}{\sqrt{r^2 - x^2}} dx \qquad\qquad S = 2\pi \int_a^b x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= -\pi r \int_0^a \left(r^2 - x^2\right)^{-1/2} d\left(r^2 - x^2\right)$$

$$= -2\pi r \sqrt{r^2 - x^2}\Big|_0^a$$

$$= -2\pi r\left(\sqrt{r^2 - a^2} - r\right)$$

$$= 2\pi r\left(r - \sqrt{r^2 - a^2}\right) \quad unit^2$$

### Exercise

Find the area of the surface generated by the curve $y = 1 + \sqrt{1 - x^2}$ between the points $(1, 1)$ and $\left(\frac{\sqrt{3}}{1}, \frac{3}{2}\right)$ about $y\text{-}axis$

**Solution**

$$\left(\sqrt{1 - x^2}\right)^2 = (y - 1)^2$$

$$1-x^2 = y^2 - 2y + 1$$

$$x = \sqrt{2y-y^2}$$

$$x' = \frac{1-y}{\sqrt{2y-y^2}}$$

$$S = 2\pi \int_1^{3/2} \sqrt{2y-y^2}\ \sqrt{1+\frac{(1-y)^2}{2y-y^2}}\ dy \qquad\qquad S = 2\pi \int_a^b x\ \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$= 2\pi \int_1^{3/2} \sqrt{2y-y^2+1-2y+y^2}\ dy$$

$$= 2\pi \int_1^{3/2}\ dy$$

$$= 2\pi\ y \Big|_1^{3/2}$$

$$= \pi \quad unit^2$$

### *Exercise*

Find the area of the surface generated by $y = \frac{1}{3}x^3, \quad 0 \le x \le 1, \quad x-axis$

***Solution***

$$\sqrt{1+(y')^2} = \sqrt{1+x^4}$$

$$S = 2\pi \int_0^1 \frac{1}{3}x^3\ \sqrt{1+x^4}\ dx$$

$$= \frac{\pi}{6}\int_0^1 \left(1+x^4\right)^{1/2}\ d\left(1+x^4\right)$$

$$= \frac{\pi}{9}\ \left(1+x^4\right)^{3/2}\ \Big|_0^1$$

$$= \frac{\pi}{9}\left((2)^{3/2}-1\right)$$

$$= \frac{\pi}{9}\left(2\sqrt{2}-1\right) \quad unit^2$$

## Exercise

Find the area of the surface generated by $x=\sqrt{4y-y^2}, \quad 1\le y\le 2; \quad y-axis$

**Solution**

$$x'=\frac{1}{2}(4-2y)\left(4y-y^2\right)^{-1/2}$$

$$=(2-y)\left(4y-y^2\right)^{-1/2}$$

$$\sqrt{1+(x')^2}=\sqrt{1+(2-y)^2\left(4y-y^2\right)^{-1}}$$

$$=\sqrt{1+\frac{4-4y+y^2}{4y-y^2}}$$

$$=\sqrt{\frac{4}{4y-y^2}}$$

$$=\frac{2}{\sqrt{4y-y^2}}$$

$$S=2\pi\int_1^2\sqrt{4y-y^2}\,\frac{2}{\sqrt{4y-y^2}}\,dy$$

$$=4\pi\int_1^2 dy$$

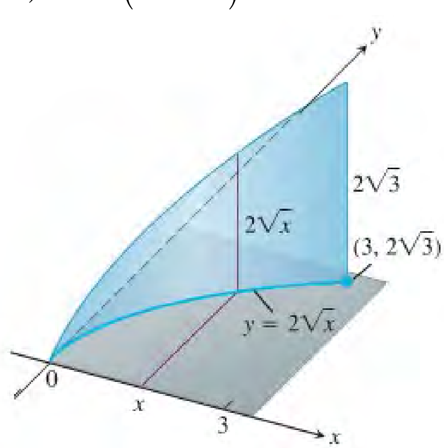
$$=2\pi(2-1)$$

$$=4\pi \quad unit^2$$

## Exercise

At points on the curve $y=2\sqrt{x}$, line segments of length $h=y$ are drawn perpendicular to the $xy$-plane. Find the area of the surface formed by these perpendiculars from $(0,\ 0)$ *to* $\left(3,\ 2\sqrt{3}\right)$

**Solution**




$$\sqrt{1+(y')^2}=\sqrt{1+\left(\frac{1}{\sqrt{x}}\right)^2}$$

$$=\sqrt{1+\frac{1}{x}}$$

$$=\frac{\sqrt{x+1}}{\sqrt{x}}$$

$$S = 2\pi \int_0^3 2\sqrt{x}\,\frac{\sqrt{x+1}}{\sqrt{x}}\,dx$$

$$= 4\pi \int_0^3 (1+x)^{1/2}\,d(1+x)$$

$$= \frac{8\pi}{3}\,(1+x)^{3/2}\,\Big|_0^3$$

$$= \frac{8\pi}{3}\left((4)^{3/2} - 1\right)$$

$$= \frac{8\pi}{3}(8-1)$$

$$= \frac{56\pi}{3}\quad unit^2$$

## *Exercise*

Find the area of the surface generated by $x = 2\sqrt{4-y}\quad 0 \le y \le \frac{15}{4},\quad y-axis$

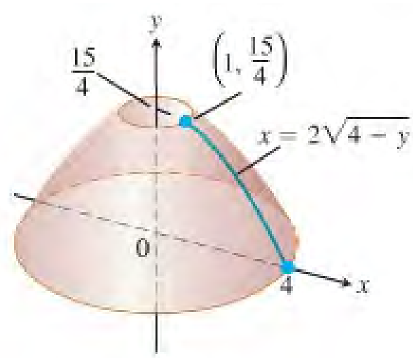
***Solution***

$$\frac{dy}{dx} = 2\frac{1}{2}(4-y)^{-1/2}(-1) = \frac{-1}{\sqrt{4-y}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{4-y}}$$

$$= \sqrt{\frac{4-y+1}{4-y}}$$

$$= \sqrt{\frac{5-y}{4-y}}$$




$$S = 2\pi \int_0^{15/4} 2\sqrt{4-y}\,\frac{\sqrt{5-y}}{\sqrt{4-y}}\,dy$$

$$= 4\pi \int_0^{15/4} \sqrt{5-y}\,dy \qquad\qquad d(5-y) = -dy$$

$$= -4\pi \int_0^{15/4} (5-y)^{1/2}\,d(5-y)$$

$$= -\frac{8\pi}{3}\,(5-y)^{3/2}\,\Big|_0^{15/4}$$

$$= -\frac{8\pi}{3}\left[\left(5-\frac{15}{4}\right)^{3/2} - (5-0)^{3/2}\right]$$

$$= -\frac{8\pi}{3}\left(\left(\frac{5}{4}\right)^{3/2} - 5^{3/2}\right)$$

$$= -\frac{8\pi}{3}\left(\frac{5\sqrt{5}}{8} - 5\sqrt{5}\right)$$

$$= -\frac{8\pi}{3}5\sqrt{5}\left(\frac{1}{8}-1\right)$$

$$= -\frac{8\pi}{3}5\sqrt{5}\left(-\frac{7}{8}\right)$$

$$= \frac{35\pi\sqrt{5}}{3} \quad unit^2$$

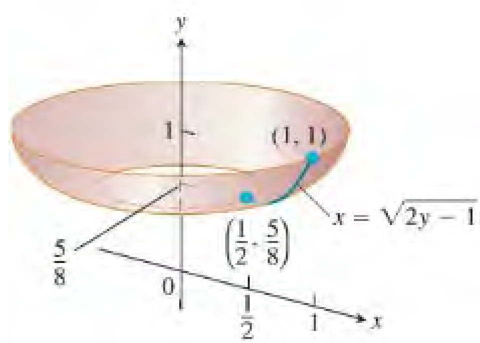
## *Exercise*

Find the area of the surface generated by $x=\sqrt{2y-1} \quad \frac{5}{8} \le y \le 1, \quad y-axis$

***Solution***

$$\frac{dy}{dx} = \frac{1}{2}(2y-1)^{-1/2}(2)$$

$$= \frac{1}{\sqrt{2y-1}}$$




$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{2y-1}}$$

$$= \sqrt{\frac{2y}{2y-1}}$$

$$S = 2\pi\int_{5/8}^{1} \sqrt{2y-1}\,\frac{\sqrt{2y}}{\sqrt{2y-1}}\,dy$$

$$= 2\pi\int_{5/8}^{1} \sqrt{2y}\,dy$$

$$= 2\pi\sqrt{2}\int_{5/8}^{1} y^{1/2}\,dy$$

$$= \frac{4\pi\sqrt{2}}{3}\left(y^{3/2}\,\Big|_{5/8}^{1}\right.$$

$$= \frac{4\pi\sqrt{2}}{3}\left(1-\frac{5\sqrt{5}}{16\sqrt{2}}\right)$$

$$= \frac{4\pi\sqrt{2}}{3}\left(\frac{16\sqrt{2}-5\sqrt{5}}{16\sqrt{2}}\right)$$

$$\boxed{= \frac{\pi}{12}\left(16\sqrt{2}-5\sqrt{5}\right) \quad unit^2}$$

## Exercise

$y = \frac{1}{3}\left(x^2+2\right)^{3/2}, \quad 0 \le x \le \sqrt{2}; \quad y-axis$ (Hint: Express $ds = \sqrt{dx^2+dy^2}$ in terms of *dx*, and evaluate the integral $S = \int 2\pi x \ ds$ with appropriate limits.)

**Solution**

$$dy = \frac{1}{3}\frac{3}{2}\left(x^2+2\right)^{1/2}(2x)\,dx$$

$$= x\sqrt{x^2+2}\ dx$$

$$ds = \sqrt{dx^2 + \left(x\sqrt{x^2+2}\ dx\right)^2}$$

$$= \sqrt{dx^2 + x^2\left(x^2+2\right)dx^2}$$

$$= \sqrt{1+x^4+2x^2}\,dx$$

$$= \sqrt{\left(1+x^2\right)^2}\,dx$$

$$= \left(1+x^2\right)dx$$

$$S = \int 2\pi x\ ds$$

$$= 2\pi\int_0^{\sqrt{2}} x\left(1+x^2\right)\ dx \qquad\qquad d\left(1+x^2\right) = 2xdx$$

$$= \pi\int_0^{\sqrt{2}} \left(1+x^2\right)\ d\left(1+x^2\right)$$

$$= \frac{\pi}{2}\ \left(1+x^2\right)^2\ \Big|_0^{\sqrt{2}}$$

$$= \frac{\pi}{2}(9-1)$$

$$\boxed{= 4\pi \quad unit^2}$$

## *Exercise*

Find the area of the surface generated by revolving the curve $x = \frac{1}{2}\left(e^{y} + e^{-y}\right), \quad 0 \le y \le \ln 2,$ about *y-axis*

**Solution**

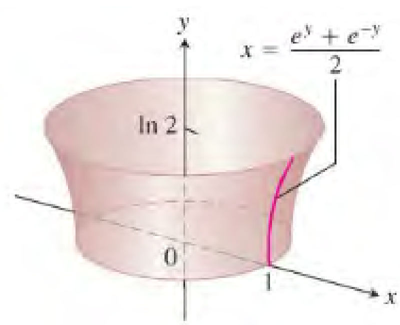
$$x' = \frac{1}{2}\left(e^{y} - e^{-y}\right)$$

$a = \frac{1}{2}, \quad m = 1, \quad b = \frac{1}{2}, \quad n = -1$ $\qquad\qquad f(x) = ae^{mx} + be^{nx}$

1. $m = -n$ ✓
2. $abmn = \frac{1}{2}\left(\frac{1}{2}\right)(1)(-1) = -\frac{1}{4}$ ✓

$$S = 2\pi \int_0^{\ln 2} \frac{1}{2}\left(e^{y} + e^{-y}\right)\frac{1}{2}\left(e^{y} + e^{-y}\right) dy \qquad\qquad S = 2\pi \int_a^b f(x)\ \overline{f'(x)}\, dx$$

$$= \frac{\pi}{2}\int_0^{\ln 2} \left(e^{2y} + e^{-2y} + 2\right) dy$$




$$= \frac{\pi}{2}\left(\frac{1}{2}e^{2y} - \frac{1}{2}e^{-2y} + 2y \ \Big|_0^{\ln 2}\right.$$

$$= \frac{\pi}{2}\left(\frac{1}{2}e^{2\ln 2} - \frac{1}{2}e^{-2\ln 2} + 2\ln 2 - \frac{1}{2}e + \frac{1}{2}\right)$$

$$= \frac{\pi}{2}\left(\frac{1}{2}\cdot 4 - \frac{1}{2}\cdot\frac{1}{4} + 2\ln 2\right)$$

$$\boxed{= \frac{\pi}{2}\left(\frac{15}{8} + 2\ln 2\right) \quad unit^2}$$

$$S = 2\pi \int_0^{\ln 2} \frac{1}{2}\left(e^{y} + e^{-y}\right)\sqrt{1 + \left(\frac{e^{y} - e^{-y}}{2}\right)^2}\, dy$$

$$= \pi \int_0^{\ln 2} \left(e^{y} + e^{-y}\right)\sqrt{1 + \frac{e^{2y} + e^{-2y} - 2}{4}}\, dy$$

$$= \pi \int_0^{\ln 2} \left(e^{y} + e^{-y}\right)\sqrt{\frac{4 + e^{2y} + e^{-2y} - 2}{4}}\, dy$$

$$= \frac{\pi}{2}\int_0^{\ln 2} \left(e^{y} + e^{-y}\right)\sqrt{e^{2y} + e^{-2y} + 2}\, dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^y + e^{-y}\right) \sqrt{\left(e^y + e^{-y}\right)^2} \, dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^y + e^{-y}\right)^2 \, dy$$

$$= \frac{\pi}{2} \int_0^{\ln 2} \left(e^{2y} + e^{-2y} + 2\right) dy$$

$$= \frac{\pi}{2} \left(\frac{1}{2}e^{2y} - \frac{1}{2}e^{-2y} + 2y \,\Big|_0^{\ln 2}\right.$$

$$= \frac{\pi}{2}\left(\frac{1}{2}e^{2\ln 2} - \frac{1}{2}e^{-2\ln 2} + 2\ln 2 - \frac{1}{2}e + \frac{1}{2}\right)$$

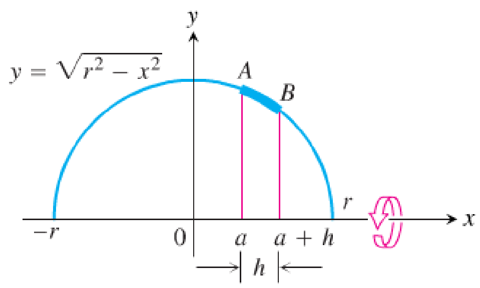
$$= \frac{\pi}{2}\left(\frac{1}{2}\cdot 4 - \frac{1}{2}\cdot\frac{1}{4} + 2\ln 2\right)$$

$$= \frac{\pi}{2}\left(\frac{15}{8} + 2\ln 2\right) \quad unit^2$$

### *Exercise*

Did you know that if you can cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle $y = \sqrt{r^2 - x^2}$ shown here is revolved about the $x$-axis to generate a sphere. Let ***AB*** be an arc of the semicircle that lies above an interval of length ***h*** on the $x$-axis. Show that the area swept out by ***AB*** does not depend on the location of the interval. (It does depend on the length of the interval.)

***Solution***




$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{r^2 - x^2}}$$

$$= \frac{-x}{\sqrt{r^2 - x^2}}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{r^2 - x^2}}$$

$$= \sqrt{\frac{r^2}{r^2 - x^2}}$$

$$S = 2\pi \int_a^{a+h} \sqrt{r^2 - x^2} \; \frac{r}{\sqrt{r^2 - x^2}} \, dx$$

$$= 2\pi r \int_a^{a+h} dx$$

$$= 2\pi r\, x \Big|_a^{a+h}$$

$$= 2\pi r(a+h-a)$$

$$\underline{\boxed{= 2\pi rh \quad unit^2}}$$

### *Example*

The curved surface of a funnel is generated by revolving the graph of $y = f(x) = x^3 + \frac{1}{12x}$ on the interval $[1,\ 2]$ about the *x-axis*. Approximately what volume of paint is needed to cover the outside of the funnel with a layer of paint 0.05 *cm* thick? Assume that *x* and *y* measured in centimeters.

**Solution**

$$f'(x) = 3x^2 - \frac{1}{12x^2}$$

$$a = 1, \quad m = 3, \quad b = \frac{1}{12}, \quad n = -1 \qquad\qquad f(x) = ax^m + bx^n$$

**1.** $m + n = 3 - 1 = 2 \quad \checkmark$

**2.** $abmn = (1)\left(\frac{1}{12}\right)(3)(-1) = -\frac{1}{4} \quad \checkmark$

$$S = 2\pi \int_1^2 \left(x^3 + \frac{1}{12x}\right)\left(3x^2 - \frac{1}{12x^2}\right) dx \qquad\qquad S = 2\pi \int_a^b x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

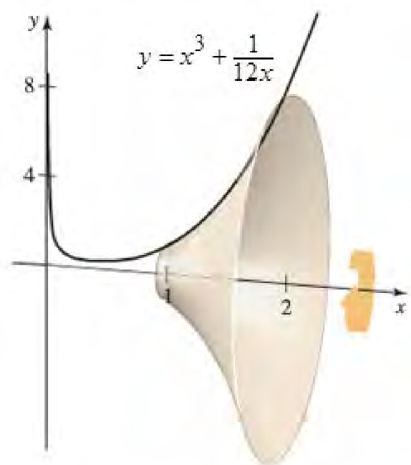
$$= 2\pi \int_1^2 \left(3x^5 + \frac{x}{3} + \frac{1}{144} x^{-3}\right) dx$$

$$= 2\pi \left(\frac{1}{2}x^6 + \frac{1}{6}x^2 - \frac{1}{288}x^{-2}\right. \Big|_1^2$$

$$= 2\pi\left(32 + \frac{2}{3} - \frac{1}{1152} - \frac{1}{2} - \frac{1}{6} + \frac{1}{288}\right)$$

$$= 2\pi\left(\frac{36864 + 768 - 1 - 576 - 192 + 4}{1152}\right)$$

$$\underline{\boxed{= \frac{12,289}{192}\pi \quad cm^2}}$$




$$1 + f'(x)^2 = 1 + \left(3x^2 - \frac{1}{12x^2}\right)^2$$

$$=1+9x^4-\frac{1}{2}+\frac{1}{144x^4}$$

$$=9x^4+\frac{1}{2}+\frac{1}{144x^4}$$

$$=\left(3x^2+\frac{1}{12x^2}\right)^2$$

$$S=2\pi\int_1^2\left(x^3+\frac{1}{12x}\right)\sqrt{\left(3x^2+\frac{1}{12x^2}\right)^2}\,dx$$

$$=2\pi\int_1^2\left(x^3+\frac{1}{12x}\right)\left(3x^2+\frac{1}{12x^2}\right)dx$$

$$=2\pi\int_1^2\left(3x^5+\frac{x}{3}+\frac{1}{144}x^{-3}\right)dx$$

$$=2\pi\left(\frac{1}{2}x^6+\frac{1}{6}x^2-\frac{1}{288}x^{-2}\right.\Bigg|_1^2$$

$$=2\pi\left(32+\frac{2}{3}-\frac{1}{1152}-\frac{1}{2}-\frac{1}{6}+\frac{1}{288}\right)$$

$$=2\pi\left(\frac{36,864+768-1-576-192+4}{11,52}\right)$$

$$\boxed{=\frac{12,289}{192}\pi\quad cm^2}$$

Because the paint layer is 0.05 *cm* thick, the approximate volume of paint needed is

$$=\left(\frac{12,289}{192}\pi\ cm^2\right)(0.05\ \ cm\ )$$

$$\boxed{\approx 10.1\ \ cm^3}$$

## Exercise

When the circle $x^2+(y-a)^2=r^2$ on the interval $[-r,\ r]$ is revolved about the $x$-*axis*, the result is the surface of a torus, where $0<r<a$. Show that the surface area of the torus is $S=4\pi^2 ar$.

**Solution**

$$x^2+(y-a)^2=r^2$$

$$(y-a)^2=r^2-x^2$$

$$y=a\pm\sqrt{r^2-x^2}$$

$$f(x) = a + \sqrt{r^2 - x^2}$$

$$1 + f'(x)^2 = 1 + \left( \frac{-x}{\sqrt{r^2 - x^2}} \right)^2$$

$$= 1 + \frac{x^2}{r^2 - x^2}$$

$$= \frac{r^2}{r^2 - x^2}$$

$$S_1 = 2\pi \int_{-r}^{r} \left( a + \sqrt{r^2 - x^2} \right) \frac{r}{\sqrt{r^2 - x^2}}\, dx$$

$$= 4\pi \int_0^r \left( \frac{ar}{\sqrt{r^2 - x^2}} + r \right) dx$$

$$= 4\pi \left( ar \sin^{-1}\left(\frac{x}{r}\right) + rx \right|_0^r$$

$$= 4\pi \left( ar\frac{\pi}{2} + r^2 \right)$$

$$= 2\pi^2 ar + 4\pi r^2 \quad unit^2$$

$$S_2 = 2\pi \int_{-r}^{r} \left( a - \sqrt{r^2 - x^2} \right) \frac{r}{\sqrt{r^2 - x^2}}\, dx$$

$$= 4\pi \int_0^r \left( \frac{ar}{\sqrt{r^2 - x^2}} - r \right) dx$$

$$= 4\pi \left( ar \sin^{-1}\left(\frac{x}{r}\right) - rx \right|_0^r$$

$$= 4\pi \left( ar\frac{\pi}{2} - r^2 \right)$$

$$= 2\pi^2 ar - 4\pi r^2 \quad unit^2$$

$$S = 2\pi^2 ar + 4\pi r^2 + 2\pi^2 ar - 4\pi r^2$$

$$= 4\pi^2 ar \quad unit^2$$

### Exercise

A 1.5–*mm* layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the curve $y=\sqrt{8x-x^2}$ on the interval $[1,\ 7]$ is revolved about the $x\text{-}axis$ Assume $x$ and $y$ are in *meters*.

**Solution**

$$y'=\frac{4-x}{\sqrt{8x-x^2}}$$

$$S=2\pi\int_1^7\sqrt{8x-x^2}\sqrt{1+\frac{(4-x)^2}{8x-x^2}}\,dx \qquad S=2\pi\int_a^b f(x)\sqrt{1+\left(\frac{df}{dx}\right)^2}\,dx$$

$$=2\pi\int_1^7\sqrt{8x-x^2}\,\frac{\sqrt{8x-x^2+16-8x+x^2}}{\sqrt{8x-x^2}}\,dx$$

$$=2\pi\int_1^7\sqrt{16}\,dx$$

$$=8\pi x\Big|_1^7$$

$$=48\pi\ \ m^2$$

The volume of paint required to cover the surface to a thickness 0.0015 $m$ is

$$V=48\pi(0.0015)$$

$$\approx 0.226195\ \ m^3 \qquad 1\ m^3=264.172052\ gal$$

$$=0.226195\times 264.172052$$

$$\approx 59.75\ \ gal$$

### Exercise

A 1.5–*mm* layer of paint is applied to one side. Find the approximate volume of paint needed of the spherical zone generated when the upper portion of the circle $x^2+y^2=100$ on the interval $[-8,\ 8]$ is revolved about the $x\text{-}axis$ . Assume $x$ and $y$ are in *meters*.

**Solution**

$$y=\sqrt{100-x^2}$$

$$y'=\frac{-x}{\sqrt{100-x^2}}$$

$$S = 2\pi \int_{-8}^{8} \sqrt{100 - x^2}\ \sqrt{1 + \frac{x^2}{100 - x^2}}\ dx \qquad\qquad S = 2\pi \int_a^b y\ \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$= 2\pi \int_{-8}^{8} \sqrt{100 - x^2}\ \frac{\sqrt{100 - x^2 + x^2}}{\sqrt{100 - x^2}}\ dx$$

$$= 20\pi \int_{-8}^{8} dx$$

$$= 20\pi\, x \Big|_{-8}^{8}$$

$$\underline{= 320\pi \quad m^2}$$

The volume of paint required to cover the surface to a thickness 0.0015 $m$ is

$$V = 320\pi(0.0015)$$

$$\underline{\approx 1.507965 \quad m^3} \qquad\qquad 1\ m^3 = 264.172052\ gal$$

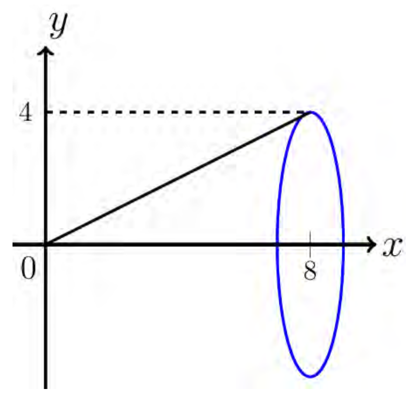
$$= 1.507965 \times 264.172052$$

$$\underline{\approx 398.36 \quad gal}$$

## *Exercise*

Find the surface area of a cone (excluding the base) with radius 4 and height 8 using integration and a surface area integral.

***Solution***




$$(0,\ 0) \ \rightarrow\ (8,\ 4)$$

$$y = \frac{4}{8}x$$

$$= \frac{1}{2}x$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \left(\frac{1}{2}\right)^2}$$

$$= \frac{\sqrt{5}}{2}$$

$$S = 2\pi \int_0^8 \frac{x}{2}\frac{\sqrt{5}}{2}\ dx \qquad\qquad S = 2\pi \int_a^b y\ \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \frac{\pi\sqrt{5}}{2}x^2 \Big|_0^8$$

$$\underline{= 32\pi\sqrt{5} \quad unit^2}$$

### *Exercise*

Let $f(x)=\frac{1}{3}x^3$ and let $R$ be the region bounded by the graph of $f$ and the $x$-axis on the interval $[0,\ 2]$

*a)* Find the area of the surface generated when the graph of $f$ on $[0,\ 2]$ is revolved about the $x$-axis.

*b)* Find the volume of the solid generated when $R$ is revolved about the $y$-axis.

*c)* Find the volume of the solid generated when $R$ is revolved about the $x$-axis.

**Solution**

***a)*** Surface revolved about the $x$-axis

$$\sqrt{1+(f')^2}=\sqrt{1+\left(x^2\right)^2}$$

$$S=2\pi\int_0^2 \frac{1}{3}x^3\sqrt{1+x^4}\ dx \qquad\qquad S=2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

$$=\frac{\pi}{6}\int_0^2\left(1+x^4\right)^{1/2}\ d\left(1+x^4\right)$$

$$=\frac{\pi}{9}\left(1+x^4\right)^{3/2}\Big|_0^2$$

$$=\frac{\pi}{9}\left(17^{3/2}-1\right)$$

$$=\frac{\pi}{9}\left(17\sqrt{17}-1\right)\quad unit^2$$

***b)*** Using Shell Method about the $y$-axis

$$V=2\pi\int_0^2 x\left(\frac{x^3}{3}\right)dx$$

$$=\frac{2\pi}{3}\int_0^2 x^4\ dx$$

$$=\frac{2\pi}{15}x^5\Big|_0^2$$

$$=\frac{64\pi}{15}\quad unit^3$$

***c)*** Using Disk Method about the $x$-axis

$$V=\pi\int_0^2\left(\frac{x^3}{3}\right)^2 dx$$

$$=\frac{\pi}{9}\int_0^2 x^6\ dx$$

$$= \frac{\pi}{63} x^7 \Big|_0^2$$

$$= \boxed{\frac{128\pi}{63} \quad unit^3}$$

## *Exercise*

Let $f(x) = \sqrt{3x - x^2}$ and let $R$ be the region bounded by the graph of $f$ and the $x\text{-}axis$ on the interval $[0,\ 3]$

*a)* Find the area of the surface generated when the graph of $f$ on $[0,\ 3]$ is revolved about the $x\text{-}axis$.

*b)* Find the volume of the solid generated when $R$ is revolved about the $x\text{-}axis$.

***Solution***

***a)*** Surface revolved about the $x\text{-}axis$

$$f' = \frac{3 - 2x}{2\sqrt{3x - x^2}}$$

$$\sqrt{1 + (f')^2} = \sqrt{1 + \left(\frac{3 - 2x}{2\sqrt{3x - x^2}}\right)^2}$$

$$= \sqrt{1 + \frac{9 - 12x + 4x^2}{4\left(3x - x^2\right)}}$$

$$= \sqrt{\frac{12x - 4x^2 + 9 - 12x + 4x^2}{4\left(3x - x^2\right)}}$$

$$= \frac{1}{2}\sqrt{\frac{9}{3x - x^2}}$$

$$= \frac{3}{2}\frac{1}{\sqrt{3x - x^2}}$$

$$S = 2\pi \int_0^3 \sqrt{3x - x^2} \left(\frac{3}{2}\frac{1}{\sqrt{3x - x^2}}\right) dx \qquad S = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 3\pi \int_0^3 dx$$

$$= 3\pi x \Big|_0^3$$

$$= \boxed{9\pi \quad unit^2}$$

***b)*** Using Disk Method about the $x\text{-}axis$

$$V = \pi \int_0^3 \left(\sqrt{3x - x^2}\right)^2 dx$$

$$= \pi \int_0^3 \left(3x - x^2\right) dx$$

$$= \pi \left(\frac{3}{2}x^2 - \frac{1}{3}x^3\right|_0^3$$

$$= \pi \left(\frac{27}{2} - 9\right)$$

$$\underline{= \frac{9\pi}{2} \quad unit^3}$$

## *Exercise*

Let $f(x) = \frac{1}{2}x^4 + \frac{1}{16x^2}$ and let *R* be the region bounded by the graph of *f* and the *x-axis* on the interval $[1, \ 2]$

*a)* Find the area of the surface generated when the graph of *f* on $[1, \ 2]$ is revolved about the *x-axis*.

*b)* Find the length of the curve $y = f(x)$ on $[1, \ 2]$

*c)* Find the volume of the solid generated when *R* is revolved about the *y-axis*.

*d)* Find the volume of the solid generated when *R* is revolved about the *x-axis*.

***Solution***

***a)*** Surface revolved about the *x-axis*

$$f' = 2x^3 - \frac{1}{8x^3}$$

$a = \frac{1}{2}, \quad m = 4, \quad b = \frac{1}{16}, \quad n = -2$ $\qquad f(x) = ax^m + bx^n$

**1.** $m + n = 4 - 2 = 2$ ✓

**2.** $abmn = \frac{1}{2}\left(\frac{1}{16}\right)(4)(-2) = -\frac{1}{4}$ ✓

$$S = 2\pi \int_1^2 \left(\frac{1}{2}x^4 + \frac{1}{16x^2}\right)\left(2x^3 + \frac{1}{8x^3}\right) dx$$

$$= 2\pi \int_1^2 \left(x^7 + \frac{1}{16}x + \frac{1}{8}x + \frac{1}{128x^5}\right) dx$$

$$= 2\pi \int_1^2 \left(x^7 + \frac{3}{16}x + \frac{1}{128}x^{-5}\right) dx$$

$$= 2\pi\left(\frac{1}{8}x^8 + \frac{3}{32}x^2 - \frac{1}{512x^4}\right)\Big|_1^2$$

$$= 2\pi\left(32 + \frac{3}{8} - \frac{1}{8192} - \frac{1}{8} - \frac{3}{32} + \frac{1}{512}\right)$$

$$\boxed{= \frac{263,439\,\pi}{4,096} \quad unit^2}$$

$$\sqrt{1+(f')^2} = \sqrt{1+\left(\frac{16x^6-1}{8x^3}\right)^2}$$

$$= \sqrt{\frac{64x^6 + 256x^{12} - 32x^6 + 1}{64x^6}}$$

$$= \frac{\sqrt{256x^{12} + 32x^6 + 1}}{8x^3}$$

$$= \frac{\sqrt{\left(16x^6+1\right)^2}}{8x^3}$$

$$\boxed{= \frac{16x^6+1}{8x^3}}$$

$$S = 2\pi\int_1^2 \left(\frac{1}{2}x^4 + \frac{1}{16x^2}\right)\left(2x^3 + \frac{1}{8x^3}\right)dx \qquad S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi\int_1^2 \left(x^7 + \frac{1}{16}x + \frac{1}{8}x + \frac{1}{128x^5}\right)dx$$

$$= 2\pi\int_1^2 \left(x^7 + \frac{3}{16}x + \frac{1}{128}x^{-5}\right)dx$$

$$= 2\pi\left(\frac{1}{8}x^8 + \frac{3}{32}x^2 - \frac{1}{512x^4}\right)\Big|_1^2$$

$$= 2\pi\left(32 + \frac{3}{8} - \frac{1}{8192} - \frac{1}{8} - \frac{3}{32} + \frac{1}{512}\right)$$

$$\boxed{= \frac{263,439\,\pi}{4,096} \quad unit^2}$$

***b)*** $a = \frac{1}{2}, \quad m = 4, \quad b = \frac{1}{16}, \quad n = -2$ $\qquad f(x) = ax^m + bx^n$

**1.** $m + n = 4 - 2 = 2$ ✓

**2.** $abmn = \frac{1}{2}\left(\frac{1}{16}\right)(4)(-2) = -\frac{1}{4}$ ✓

$$L = \left(\frac{1}{2}x^4 - \frac{1}{16x^2}\right)\Big|_1^2$$

$$= 8 - \frac{1}{64} - \frac{1}{2} + \frac{1}{16}$$

$$\boxed{= \frac{483}{64} \quad unit}$$

***c)*** Using Shell Method about the *y-axis*

$$V = 2\pi \int_1^2 x\left(\frac{1}{2}x^4 + \frac{1}{16x^2}\right)dx$$

$$= \pi \int_1^2 \left(x^5 + \frac{1}{8x}\right)dx$$

$$= \pi \left(\frac{1}{6}x^6 + \frac{1}{8}\ln x\right)\Big|_1^2$$

$$= \pi\left(\frac{32}{3} + \frac{1}{8}\ln 2 - \frac{1}{6}\right)$$

$$\boxed{= \frac{21\pi}{2} + \frac{\pi}{8}\ln 2 \quad unit^3}$$

***d)*** Using Disk Method about the *x-axis*

$$V = \pi \int_1^2 \left(\frac{1}{2}x^4 + \frac{1}{16x^2}\right)^2 dx$$

$$= \pi \int_1^2 \left(\frac{1}{4}x^8 + \frac{1}{16}x^2 + \frac{1}{256}x^{-4}\right) dx$$

$$= \frac{\pi}{4}\left(\frac{1}{9}x^9 + \frac{1}{12}x^3 - \frac{1}{192}\frac{1}{x^3}\right)\Big|_1^2$$

$$= \frac{\pi}{4}\left(\frac{512}{9} + \frac{2}{3} - \frac{1}{1536} - \frac{1}{9} - \frac{1}{12} + \frac{1}{192}\right)$$

$$\boxed{= \frac{264,341}{18,432}\pi \quad unit^3}$$

### Exercise

Suppose a sphere of radius $r$ is sliced by two horizontal planes $h$ units apart. Show that the surface area of the resulting zone on the sphere is $2\pi h$, independent of the location of the cutting planes.

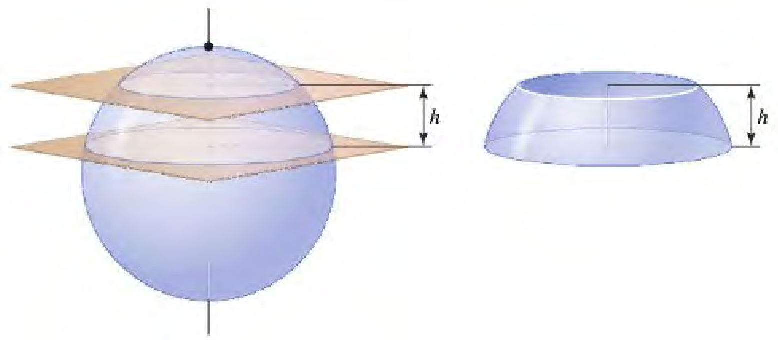
**Solution**

$$f(x) = \sqrt{r^2 - x^2}$$

$$1 + f'(x)^2 = 1 + \left(\frac{x}{\sqrt{r^2 - x^2}}\right)^2$$

$$= 1 + \frac{x^2}{r^2 - x^2}$$

$$= \frac{r^2}{r^2 - x^2}$$




$$S = 2\pi \int_a^{a+h} \sqrt{r^2 - x^2}\ \frac{r}{\sqrt{r^2 - x^2}}\ dx$$

$$= 2\pi r \Big|_a^{a+h}$$

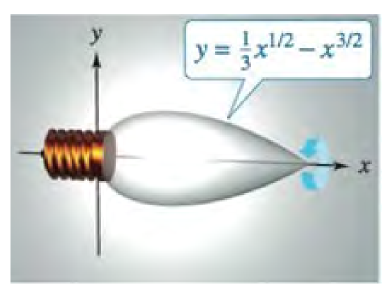
$$= 2\pi r(a + h - a)$$

$$= 2\pi r h \quad unit^2$$

### Exercise

An ornamental light bulb is designed by revolving the graph of $y = \frac{1}{3}x^{1/2} - x^{3/2}, \quad 0 \le x \le \frac{1}{3}$ about the $x$-*axis*, where $x$ and $y$ are mesured in *feet*. Find the surface area of the bulb and use the result to approximate the amount of glass needed to make the bulb.
(Assume that the glass is 0.015 *inch* thick)

**Solution**




$$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2}$$

$$a = \frac{1}{3}, \quad m = \frac{1}{2}, \quad b = -1, \quad n = \frac{3}{2} \qquad\qquad f(x) = ax^m + bx^n$$

**1.** $m + n = \frac{1}{2} + \frac{3}{2} = 2$ ✓

**2.** $abmn = \frac{1}{3}(-1)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = -\frac{1}{4}$ ✓

$$S = 2\pi \frac{1}{6} \int_0^{1/3} \left(\frac{1}{3}x^{1/2} - x^{3/2}\right)\left(x^{-1/2} + 9x^{1/2}\right) dx$$

$$= \frac{\pi}{3} \int_0^{1/3} \left( \frac{1}{3} + 2x - 9x^2 \right) dx$$

$$= \frac{\pi}{3} \left( \frac{1}{3}x + x^2 - 3x^3 \right. \Big|_0^{1/3}$$

$$= \frac{\pi}{3} \left( \frac{1}{9} + \frac{1}{9} - \frac{1}{9} \right)$$

$$\boxed{= \frac{\pi}{27} \quad ft^2}$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{1}{36}x^{-1} - \frac{1}{2} + \frac{9}{4}x}$$

$$= \frac{1}{6}\sqrt{x^{-1} + 18 + 81x}$$

$$= \frac{1}{6}\sqrt{\left(x^{-1/2} + 9x^{1/2}\right)^2}$$

$$= \frac{1}{6}\left(x^{-1/2} + 9x^{1/2}\right)$$

$$S = 2\pi \frac{1}{6} \int_0^{1/3} \left( \frac{1}{3}x^{1/2} - x^{3/2} \right)\left( x^{-1/2} + 9x^{1/2} \right) dx$$

$$= \frac{\pi}{3} \int_0^{1/3} \left( \frac{1}{3} + 2x - 9x^2 \right) dx$$

$$= \frac{\pi}{3} \left( \frac{1}{3}x + x^2 - 3x^3 \right. \Big|_0^{1/3}$$

$$= \frac{\pi}{3} \left( \frac{1}{9} + \frac{1}{9} - \frac{1}{9} \right)$$

$$\boxed{= \frac{\pi}{27} \quad ft^2} \quad \approx 0.1164\ ft^2 \quad \approx 16.8\ in^2$$

Amount of glass needed:

$$V = \frac{\pi}{2}\left(\frac{0.015}{12}\right)$$

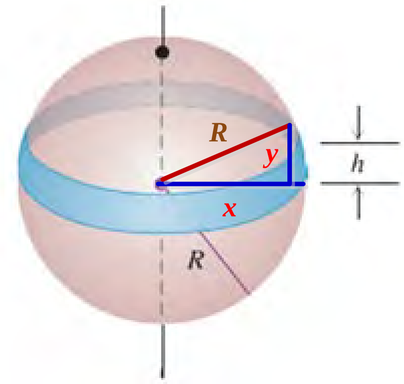
$$\approx 0.00015\ ft^3$$

$$\boxed{\approx 0.25 \quad in^3}$$

### Exercise

The shaded band is cut from a sphere of radius $R$ by parallel planes $h$ units apart. Show that the surface area of the band is $2\pi Rh$

**Solution**




$$y = \sqrt{R^2 - x^2}$$

$$\frac{dy}{dx} = \frac{1}{2}\frac{-2x}{\sqrt{R^2-x^2}}$$

$$= \frac{-x}{\sqrt{R^2-x^2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{-x}{\sqrt{R^2-x^2}}\right)^2$$

$$= \frac{x^2}{R^2-x^2}$$

$$S = 2\pi\int_a^{a+h} \sqrt{R^2 - x^2} \cdot \sqrt{1 + \frac{x^2}{R^2-x^2}}dx \qquad S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi\int_a^{a+h} \sqrt{R^2 - x^2 + x^2}dx$$

$$= 2\pi\int_a^{a+h} \sqrt{R^2}dx$$

$$= 2\pi R\int_a^{a+h} dx$$

$$= 2\pi R\,x\,\Big|_a^{a+h}$$

$$= 2\pi R\big((a+h) - a\big)$$

$$= 2\pi Rh \quad unit^2$$

### Exercise

A drawing of a 90-*ft* dome is used by the National Weather Service. How much outside surface is there to paint (not counting the bottom)?

**Solution**

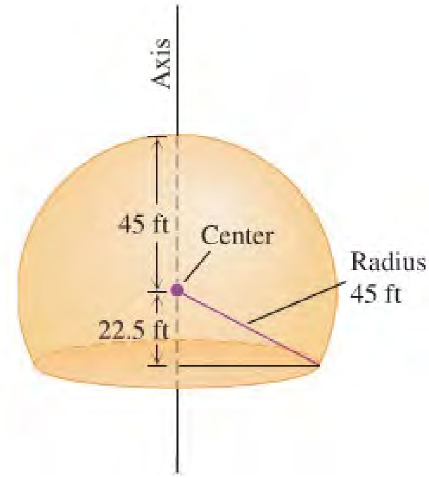
$$x = \sqrt{R^2 - y^2} \;= \sqrt{45^2 - y^2}$$

$$\frac{dx}{dy} = \frac{1}{2}\frac{-2y}{\sqrt{45^2-y^2}}$$

$$= \frac{-y}{\sqrt{45^2-y^2}}$$

$$\left(\frac{dx}{dy}\right)^2 = \left(\frac{-y}{\sqrt{45^2-y^2}}\right)^2$$

$$= \frac{y^2}{45^2-y^2}$$




$$S = 2\pi \int_{-22.5}^{45} \sqrt{45^2 - y^2} \cdot \sqrt{1+\frac{y^2}{45^2-y^2}}\ dy$$

$$= 2\pi \int_{-22.5}^{45} \sqrt{45^2 - y^2 + y^2}\, dy$$

$$= 90\pi \int_{-22.5}^{45} \quad dy$$

$$= 90\pi y \Big|_{-22.5}^{45}$$

$$= 90\pi(45+22.5)$$

$$\underline{= 6075\pi \quad ft^2} \Big| \ 19,085 \ ft^2$$