Solution Section 2.4 – Cross Product

Exercise

Prove when the cross product $\boldsymbol{u}\times\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$, then $\boldsymbol{u}.(\boldsymbol{u}\times\boldsymbol{v})=0$

Solution

Let $\boldsymbol{u}=(u_1,\ u_2,\ u_3)$ and $\boldsymbol{v}=(v_1,\ v_2,\ v_3)$

$$\begin{aligned}\boldsymbol{u}.(\boldsymbol{u}\times\boldsymbol{v}) &= (u_1,\ u_2,\ u_3).(u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1)\\ &= u_1(u_2v_3-u_3v_2)+u_2(u_3v_1-u_1v_3)+u_3(u_1v_2-u_2v_1)\\ &= u_1u_2v_3-u_1u_3v_2+u_2u_3v_1-u_2u_1v_3+u_3u_1v_2-u_3u_2v_1\\ &= 0\end{aligned}$$

Exercise

Find $\boldsymbol{u}\times\boldsymbol{v}$, where $\boldsymbol{u}=(1,\ 2,\ -2)$ *and* $\boldsymbol{v}=(3,\ 0,\ 1)$ and show that $\boldsymbol{u}\times\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ and to $\boldsymbol{v}$.

Solution

$$\begin{aligned}u\times v &= \left(\begin{vmatrix}2 & -2\\ 0 & 1\end{vmatrix},\ -\begin{vmatrix}1 & -2\\ 3 & 1\end{vmatrix},\ \begin{vmatrix}1 & 2\\ 3 & 0\end{vmatrix}\right)\\ &= (2,\ -7,\ -6)\end{aligned}$$

$$\begin{aligned}u\bullet(u\times v) &= (1,\ 2,\ -2)\bullet(2,\ -7,\ -6)\\ &= 2-14+12\\ &= 0\end{aligned}$$

$$\begin{aligned}v\bullet(u\times v) &= (3,\ 0,\ 1)\bullet(2,\ -7,\ -6)\\ &= 6-0-6\\ &= 0\end{aligned}$$

$\boldsymbol{u}\times\boldsymbol{v}$ is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$.

Exercise

Given $\boldsymbol{u} = (3,\ 2,\ -1),\quad \boldsymbol{v} = (0,\ 2,\ -3),\ \textit{and}\quad \boldsymbol{w} = (2,\ 6,\ 7)$ Compute the vectors

a) $\boldsymbol{u} \times \boldsymbol{v}$

b) $\boldsymbol{v} \times \boldsymbol{w}$

c) $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w})$

d) $(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w}$

e) $\boldsymbol{u} \times (\boldsymbol{v} - 2\,\boldsymbol{w})$

Solution

a) $$u \times v = \left(\begin{vmatrix} 2 & -1 \\ 2 & -3 \end{vmatrix},\ -\begin{vmatrix} 3 & -1 \\ 0 & -3 \end{vmatrix},\ \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} \right)$$
$$= (-4,\ 9,\ 6)$$

b) $$v \times w = \left(\begin{vmatrix} 2 & -3 \\ 6 & 7 \end{vmatrix},\ -\begin{vmatrix} 0 & -3 \\ 2 & 7 \end{vmatrix},\ \begin{vmatrix} 0 & 2 \\ 2 & 6 \end{vmatrix} \right)$$
$$= (32,\ -6,\ -4)$$

c) $$u \times (v \times w) = (3,\ 2,\ -1) \times (32,\ -6,\ -4)$$
$$= \left(\begin{vmatrix} 2 & -1 \\ -6 & -4 \end{vmatrix},\ -\begin{vmatrix} 3 & -1 \\ 32 & -4 \end{vmatrix},\ \begin{vmatrix} 3 & 2 \\ 32 & -6 \end{vmatrix} \right)$$
$$= (-14,\ -20,\ -82)$$

d) $$(u \times v) \times w = (-4,\ 9,\ 6) \times (2,\ 6,\ 7)$$
$$= \left(\begin{vmatrix} 9 & 6 \\ 6 & 7 \end{vmatrix},\ -\begin{vmatrix} -4 & 6 \\ 2 & 7 \end{vmatrix},\ \begin{vmatrix} -4 & 9 \\ 2 & 6 \end{vmatrix} \right)$$
$$= (27,\ 40,\ -42)$$

e) $$u \times (v - 2w) = (3,\ 2,\ -1) \times \left[(0,\ 2,\ -3) - 2(2,\ 6,\ 7) \right]$$
$$= (3,\ 2,\ -1) \times (-4,\ -10,\ -17)$$
$$= \left(\begin{vmatrix} 2 & -1 \\ -10 & -17 \end{vmatrix},\ -\begin{vmatrix} 3 & -1 \\ -4 & -17 \end{vmatrix},\ \begin{vmatrix} 3 & 2 \\ -4 & -10 \end{vmatrix} \right)$$
$$= (-44,\ 47,\ -22)$$

Exercise

Use the cross product to find a vector that is orthogonal to both

a) $\boldsymbol{u}=(-6,\ 4,\ 2),\quad \boldsymbol{v}=(3,\ 1,\ 5)$

b) $\boldsymbol{u}=(1,\ 1,\ -2),\quad \boldsymbol{v}=(2,\ -1,\ 2)$

c) $\boldsymbol{u}=(-2,\ 1,\ 5),\quad \boldsymbol{v}=(3,\ 0,\ -3)$

Solution

a) $\boldsymbol{u}\times\boldsymbol{v}=(-6,\ 4,\ 2)\times(3,\ 1,\ 5)$

$$=\left(\begin{vmatrix}4 & 2\\ 1 & 5\end{vmatrix},\ -\begin{vmatrix}-6 & 2\\ 3 & 5\end{vmatrix},\ \begin{vmatrix}-6 & 4\\ 3 & 1\end{vmatrix}\right)$$

$$=(18,\ 36,\ -18)$$

b) $\boldsymbol{u}\times\boldsymbol{v}=(1,\ 1,\ -2)\times(2,\ -1,\ 2)$

$$=\left(\begin{vmatrix}1 & -2\\ -1 & 2\end{vmatrix},\ -\begin{vmatrix}1 & -2\\ 2 & 2\end{vmatrix},\ \begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix}\right)$$

$$=(0,\ -6,\ -3)$$

c) $\boldsymbol{u}\times\boldsymbol{v}=(-2,\ 1,\ 5)\times(3,\ 0,\ -3)$

$$=\left(\begin{vmatrix}1 & 5\\ 0 & -3\end{vmatrix},\ -\begin{vmatrix}-2 & 5\\ 3 & -3\end{vmatrix},\ \begin{vmatrix}-2 & 1\\ 3 & 0\end{vmatrix}\right)$$

$$=(-3,\ 9,\ -3)$$

Exercise

Find the area of the parallelogram determined by the given vectors

a) $\boldsymbol{u}=(1,\ -1,\ 2)$ *and* $\boldsymbol{v}=(0,\ 3,\ 1)$

b) $\boldsymbol{u}=(3,\ -1,\ 4)$ *and* $\boldsymbol{v}=(6,\ -2,\ 8)$

c) $\boldsymbol{u}=(2,\ 3,\ 0)$ *and* $\boldsymbol{v}=(-1,\ 2,\ -2)$

Solution

a) $Area=\|u\times v\|$

$$=\left\|\left(\begin{vmatrix}-1 & 2\\ 3 & 1\end{vmatrix},\ -\begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix},\ \begin{vmatrix}1 & -1\\ 0 & 3\end{vmatrix}\right)\right\|$$

$$=\|(-7,\ -1,\ 3)\|$$

$$=\sqrt{7^2+1^2+3^2}$$

$$= \boxed{\sqrt{59}} \quad (Area)$$

b) $Area = \|u \times v\|$

$$= \left\| \left(\begin{vmatrix} -1 & 4 \\ -2 & 8 \end{vmatrix}, \ -\begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix}, \ \begin{vmatrix} 3 & -1 \\ 6 & -2 \end{vmatrix} \right) \right\|$$

$$= \|(0, \ 0, \ 0)\|$$

$$= \boxed{0}$$

c) $Area = \|u \times v\| = (2, \ 3, \ 0) \times (-1, \ 2, \ -2)$

$$= \left\| \left(\begin{vmatrix} 3 & 0 \\ 2 & -2 \end{vmatrix}, \ -\begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix}, \ \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} \right) \right\|$$

$$= \|(-6, \ 4, \ 7)\|$$

$$= \sqrt{(-6)^2 + 4^2 + 7^2}$$

$$= \boxed{\sqrt{101}} \quad (Area)$$

Exercise

Find the area of the parallelogram with the given vertices $P_1(3,2), \quad P_2(5,4), \quad P_3(9,4), \quad P_4(7,2)$

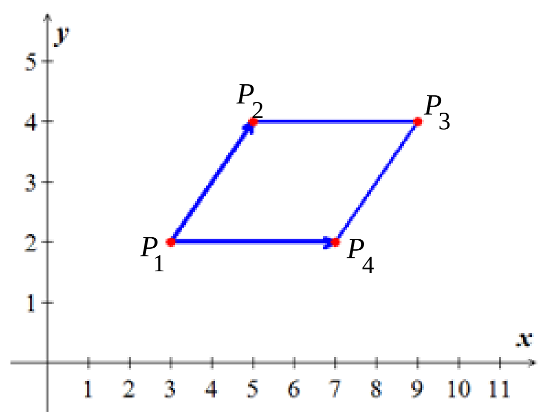
Solution

$$\overrightarrow{P_1P_2} = (5-3, 4-2) = (2, \ 2)$$

$$\overrightarrow{P_4P_3} = (9-7, 4-2) = (2, \ 2)$$

$$\overrightarrow{P_1P_4} = (7-3, 2-2) = (4, \ 0)$$

$$\overrightarrow{P_2P_3} = (9-5, 4-4) = (4, \ 0)$$



$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_2} = (2,2) \times (4,0)$$

$$= \left(\begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix}, \ -\begin{vmatrix} 2 & 0 \\ 4 & 0 \end{vmatrix}, \ \begin{vmatrix} 2 & 2 \\ 4 & 0 \end{vmatrix} \right)$$

$$= (0, \ 0, \ -8)$$

The area of the parallelogram is

$$\left\| \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_2} \right\| = \sqrt{0 + 0 + (-8)^2} = \boxed{8}$$

Exercise

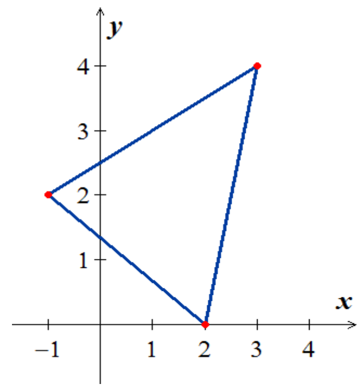
Find the area of the triangle with the given vertices:

a) $A(2,\ 0)\quad B(3,\ 4)\quad C(-1,\ 2)$

b) $A(1,\ 1)\quad B(2,\ 2)\quad C(3,\ -3)$

c) $P(2,\ 6,\ -1)\quad Q(1,\ 1,\ 1)\quad R=(4,\ 6,\ 2)$

Solution



a) $\overrightarrow{AB}=(1,\ 4)\qquad \overrightarrow{AC}=(-3,\ 2)$

$$\overrightarrow{AB}\times\overrightarrow{AC}=(1,\ 4,\ 0)\times(-3,\ 2,\ 0)$$

$$=\left(\begin{vmatrix}4&0\\2&0\end{vmatrix},\ -\begin{vmatrix}1&0\\-3&0\end{vmatrix},\ \begin{vmatrix}1&4\\-3&2\end{vmatrix}\right)$$

$$=(0,\ 0,\ 14)$$

$$\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\sqrt{0+0+14^2}$$

$$=14$$

The area of the triangle is

$$\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\frac{1}{2}14$$

$$=7$$

b) $\overrightarrow{AB}=(1,\ 1)\qquad \overrightarrow{AC}=(2,\ -4)$

$$\overrightarrow{AB}\times\overrightarrow{AC}=(1,\ 1,\ 0)\times(2,\ -4,\ 0)$$

$$=\left(\begin{vmatrix}1&0\\-4&0\end{vmatrix},\ -\begin{vmatrix}1&0\\2&0\end{vmatrix},\ \begin{vmatrix}1&1\\2&-4\end{vmatrix}\right)$$

$$=(0,\ 0,\ -6)$$

$$\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\sqrt{0+0+(-6)^2}$$

$$=6$$

The area of the triangle is

$$\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\frac{1}{2}(6)$$

$$=3$$

c) $\overrightarrow{PQ}=(-1,\ -5,\ 2)\qquad \overrightarrow{PR}=(2,\ 0,\ 3)$

$$\overrightarrow{PQ}\times\overrightarrow{PR}=(-1,\ -5,\ 2)\times(2,\ 0,\ 3)$$

$$\begin{array}{ccccc}-1&-5&2&-1&-5\\2&0&3&2&0\end{array}$$

$$= (-15,\ 7,\ 10)$$

$$\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\| = \sqrt{(-15)^2 + 7^2 + 10^2}$$

$$= \sqrt{374}$$

The area of the triangle is

$$\frac{1}{2}\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\| = \frac{1}{2}\sqrt{374}$$

Exercise

a) Find the area of the parallelogram with edges $v = (3,\ 2)$ and $w = (1,\ 4)$

b) Find the area of the triangle with sides ***v, w***, and ***v*** + ***w***. Draw it.

c) Find the area of the triangle with sides ***v, w***, and ***v*** – ***w***. Draw it.

Solution

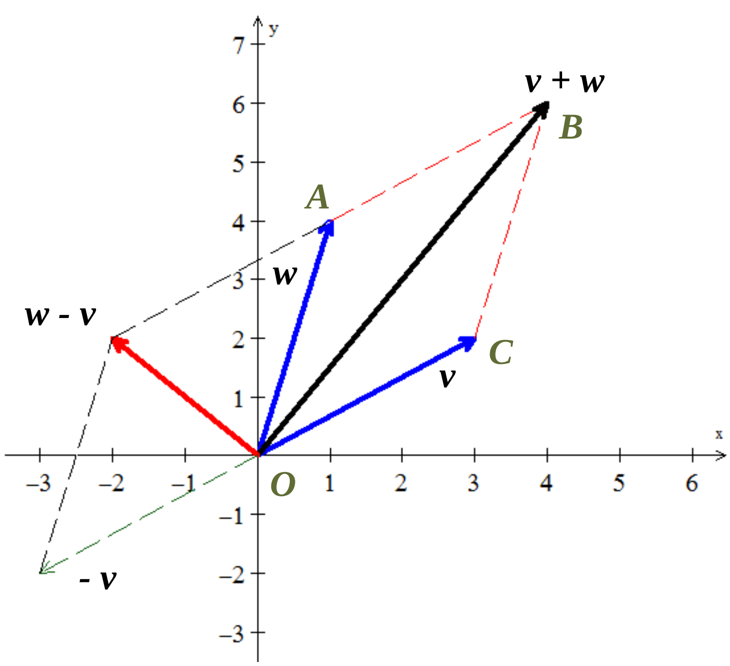
a) $Area = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 10$ (which is the parallelogram *OABC*)

b) The area of the triangle with sides ***v, w,*** and ***v*** + ***w*** is the triangle OCB or OAB which it is half the parallelogram (by definition).

$$Area = 5$$

$$v + w = (3,\ 2) + (1,\ 4) = (4,\ 6)$$

$$Area = \frac{1}{2}\begin{vmatrix} 3 & 4 \\ 2 & 6 \end{vmatrix} = \frac{1}{2}(10) = 5$$



c) The area of the triangle with sides ***v, w,*** and ***v - w*** is equivalent to the triangle OAC which it is half the parallelogram (by definition).

$$Area = 5$$

$$Area = \frac{1}{2}\begin{vmatrix} 2 & -2 \\ -3 & -2 \end{vmatrix} = \frac{1}{2}\left|-10\right| \underline{=5|}$$

Exercise

Find the volume of the parallelepiped with sides $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

a) $\boldsymbol{u}=(2,-6,2), \quad \boldsymbol{v}=(0,4,-2), \quad \boldsymbol{w}=(2,2,-4)$

b) $\boldsymbol{u}=(3,1,2), \quad \boldsymbol{v}=(4,5,1), \quad \boldsymbol{w}=(1,2,4)$

Solution

a) $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w}) = \begin{vmatrix} 2 & -6 & 2 \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{vmatrix} = -16$

The volume of the parallelepiped is $\left|-16\right| = \underline{16|}$

b) $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w}) = \begin{vmatrix} 3 & 1 & 2 \\ 4 & 5 & 1 \\ 1 & 2 & 4 \end{vmatrix} = 45$

The volume of the parallelepiped is $\boxed{45}$

Exercise

Compute the scalar triple product $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w})$

a) $\boldsymbol{u}=(-2,0,6), \quad \boldsymbol{v}=(1,-3,1), \quad \boldsymbol{w}=(-5,-1,1)$

b) $\boldsymbol{u}=(-1,2,4), \quad \boldsymbol{v}=(3,4,-2), \quad \boldsymbol{w}=(-1,2,5)$

c) $\boldsymbol{u}=(a,0,0), \quad \boldsymbol{v}=(0,b,0), \quad \boldsymbol{w}=(0,0,c)$

d) $\boldsymbol{u}=3\boldsymbol{i}-2\boldsymbol{j}-5\boldsymbol{k}, \quad \boldsymbol{v}=\boldsymbol{i}+4\boldsymbol{j}-4\boldsymbol{k}, \quad \boldsymbol{w}=3\boldsymbol{j}+2\boldsymbol{k}$

e) $\boldsymbol{u}=(3,\ -1,\ 6) \quad \boldsymbol{v}=(2,\ 4,\ 3) \quad \boldsymbol{w}=(5,\ -1,\ 2)$

Solution

a) $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w}) = \begin{vmatrix} -2 & 0 & 6 \\ 1 & -3 & 1 \\ -5 & -1 & 1 \end{vmatrix} = \underline{-92|}$

b) $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w})=\begin{vmatrix}-1 & 2 & 4\\ 3 & 4 & -2\\ -1 & 2 & 5\end{vmatrix}=\underline{-10}$

c) $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w})=\begin{vmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{vmatrix}=\underline{abc}$

d) $\boldsymbol{u}.(\boldsymbol{v}\times\boldsymbol{w})=\begin{vmatrix}3 & -2 & -5\\ 1 & 4 & -4\\ 0 & 3 & 2\end{vmatrix}\underline{=49}$

e) $\boldsymbol{u}\bullet(\boldsymbol{v}\times\boldsymbol{w})=\begin{vmatrix}3 & -1 & 6\\ 2 & 4 & 3\\ 5 & -1 & 2\end{vmatrix}\underline{=-110}$

Exercise

Use the cross product to find the sine of the angle between the vectors $\boldsymbol{u}=(2,3,-6),\quad \boldsymbol{v}=(2,3,6)$

Solution

$$\boldsymbol{u}\times\boldsymbol{v}=(2,3,-6)\times(2,3,6)$$

$$=\left(\begin{vmatrix}3 & -6\\ 3 & 6\end{vmatrix},\ -\begin{vmatrix}2 & -6\\ 2 & 6\end{vmatrix},\ \begin{vmatrix}2 & 3\\ 2 & 3\end{vmatrix}\right)$$

$$=\underline{(36,\ -24,\ 0)}$$

$$\|\boldsymbol{u}\times\boldsymbol{v}\|=\sqrt{36^2+(-24)^2+0}=\sqrt{1872}=12\sqrt{13}$$

$$\sin\theta=\left(\frac{\|\boldsymbol{u}\times\boldsymbol{v}\|}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right)$$

$$=\frac{12\sqrt{13}}{\sqrt{2^2+3^2+(-6)^2}\sqrt{2^2+3^2+6^2}}$$

$$=\frac{12\sqrt{13}}{(7)(7)}$$

$$=\underline{\frac{12}{49}\sqrt{13}}$$

Exercise

Simplify $(\boldsymbol{u}+\boldsymbol{v})\times(\boldsymbol{u}-\boldsymbol{v})$

Solution

$$\begin{aligned}(\boldsymbol{u}+\boldsymbol{v})\times(\boldsymbol{u}-\boldsymbol{v}) &= (\boldsymbol{u}+\boldsymbol{v})\times\boldsymbol{u}-(\boldsymbol{u}+\boldsymbol{v})\times\boldsymbol{v}\\ &= (\boldsymbol{u}\times\boldsymbol{u})+(\boldsymbol{v}\times\boldsymbol{u})-\left[(\boldsymbol{u}\times\boldsymbol{v})+(\boldsymbol{v}\times\boldsymbol{v})\right]\\ &= 0+(\boldsymbol{v}\times\boldsymbol{u})-\left[(\boldsymbol{u}\times\boldsymbol{v})+0\right]\\ &= (\boldsymbol{v}\times\boldsymbol{u})-(\boldsymbol{u}\times\boldsymbol{v})\\ &= (\boldsymbol{v}\times\boldsymbol{u})-(-(\boldsymbol{v}\times\boldsymbol{u}))\\ &= (\boldsymbol{v}\times\boldsymbol{u})+(\boldsymbol{v}\times\boldsymbol{u})\\ &= \underline{2(\boldsymbol{v}\times\boldsymbol{u})}\end{aligned}$$

Exercise

Prove Lagrange's identity: $\|\boldsymbol{u}\times\boldsymbol{v}\|^2=\|\boldsymbol{u}\|^2\ \|\boldsymbol{v}\|^2-(\boldsymbol{u}.\boldsymbol{v})^2$

Solution

Let $\boldsymbol{u}=(u_1,\ u_2,\ u_3)$ and $\boldsymbol{v}=(v_1,\ v_2,\ v_3)$

$\|\boldsymbol{u}\|^2=u_1^2+u_2^2+u_3^2$

$\|\boldsymbol{v}\|^2=v_1^2+v_2^2+v_3^2$

$(\boldsymbol{u}.\boldsymbol{v})^2=(u_1v_1+u_2v_2+u_3v_3)^2$

$$\begin{aligned}\|\boldsymbol{u}\times\boldsymbol{v}\|^2 &= (u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2\\ &= u_2^2v_3^2-2u_2v_3u_3v_2+u_3^2v_2^2+u_3^2v_1^2-2u_3v_1u_1v_3+u_1^2v_3^2+u_1^2v_2^2-2u_2v_1u_2v_1+u_2^2v_1^2\end{aligned}$$

$$\begin{aligned}\|\boldsymbol{u}\|^2\ \|\boldsymbol{v}\|^2-(\boldsymbol{u}.\boldsymbol{v})^2 &= \left(u_1^2+u_2^2+u_3^2\right)\left(v_1^2+v_2^2+v_3^2\right)-(u_1v_1+u_2v_2+u_3v_3)^2\\ &= u_1^2v_1^2+u_1^2v_2^2+u_1^2v_3^2+u_2^2v_1^2+u_2^2v_2^2+u_2^2v_3^2+u_3^2v_1^2+u_3^2v_2^2+u_3^2v_3^2\\ &\qquad -u_1^2v_1^2-u_1v_1u_2v_2-u_1v_1u_3v_3\\ &\qquad -u_2v_2u_1v_1-u_2^2v_2^2-u_2v_2u_3v_3\\ &\qquad -u_1v_1u_3v_3-u_2v_2u_3v_3-u_3^2v_3^2\end{aligned}$$

$$\begin{aligned}&= u_2^2 v_3^2 - 2u_2 v_2 u_3 v_3 + u_3^2 v_2^2\\ &\qquad + u_3^2 v_1^2 - 2u_1 v_1 u_3 v_3 + u_1^2 v_3^2\\ &\qquad + u_1^2 v_2^2 - 2u_1 v_1 u_2 v_2 + u_2^2 v_1^2\end{aligned}$$

$$\Rightarrow \boxed{\|\boldsymbol{u}\times\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 \ \|\boldsymbol{v}\|^2 - (\boldsymbol{u}.\boldsymbol{v})^2}$$

Exercise

Polar coordinates satisfy $x = r\cos\theta$ *and* $y = \sin\theta$. Polar area $J\ dr\ d\theta$ includes J:

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

The two columns are orthogonal. Their lengths are ______. Thus J = ______.

Solution

The length of the first column is: $= \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$

The length of the second column is: $= \sqrt{r^2\sin^2\theta + r^2\cos^2\theta}$

$$\begin{aligned}&= \sqrt{r^2\left(\sin^2\theta + \cos^2\theta\right)}\\ &= \sqrt{r^2}\\ &\boxed{= r}\end{aligned}$$

So J is the product 1. $r = r$.

$$\begin{aligned}\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} &= r\cos^2\theta + r\sin^2\theta\\ &= r\left(\cos^2\theta + \sin^2\theta\right)\\ &\boxed{= r}\end{aligned}$$

Exercise

Prove that $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$ if and only if $\vec{u}$ and $\vec{v}$ are parallel vectors.

Solution

If $\vec{u}$ and $\vec{v}$ are parallel vectors, then $\vec{u} \times \vec{v} = 0$

Which the two vectors are collinear, which implies that $\vec{u} = a\vec{v}$

$$\begin{aligned} \|\vec{u}+\vec{v}\| &= \|\vec{u}+a\vec{u}\| \\ &= \|(1+a)\vec{u}\| \\ &= (1+a)\|\vec{u}\| \\ &= \|\vec{u}\|+a\|\vec{u}\| \\ &= \|\vec{u}\|+\|a\vec{u}\| \\ &= \|\vec{u}\|+\|\vec{v}\| \quad \checkmark \end{aligned}$$

Exercise

State the following statements as True or False

a) The cross product of two nonzero vectors $\vec{u}$ and $\vec{v}$ is a nonzero vector if and only if $\vec{u}$ and $\vec{v}$ are not parallel.

b) A normal vector to a plane can be obtained by taking the cross product of two nonzero and noncollinear vectors lying in the plane.

c) The scalar triple product of $\vec{u}$, $\vec{v}$, and $\vec{w}$ determines a vector whose length is equal to the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$.

d) If $\vec{u}$ and $\vec{v}$ are vectors in 3-space, then $\|\vec{u}\times\vec{v}\|$ is equal to the area of the parallelogram determine by $\vec{u}$ and $\vec{v}$.

e) For all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in R^3, the vectors $(\vec{u}\times\vec{v})\times\vec{w}$ and $\vec{u}\times(\vec{v}\times\vec{w})$ are the same.

f) If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in R^3, where $\vec{u}$ is nonzero and $\vec{u}\times\vec{v}=\vec{u}\times\vec{w}$, then $\vec{v}=\vec{w}$

Solution

a) True,

$\vec{u}\times\vec{v}=\|\vec{u}\|\,\|\vec{v}\|\sin\theta=0$ if $\theta=0$ which the two vectors are parallel.

b) True;

The cross product of two nonzero and non collinear vectors will be perpendicular to both vectors, hence normal to the plane containing the vectors.

c) False;

The scalar triple product is a scalar, not a vector.

d) True;

e) False;

Let $\vec{u}=\hat{i}\quad \vec{v}=\vec{w}=\hat{j}$

$(\vec{u}\times\vec{v})\times\vec{w}=(\hat{i}\times\hat{j})\times\hat{j}=\hat{k}\times\hat{j}=-\hat{i}$

$\vec{u}\times(\vec{v}\times\vec{w})=\hat{i}\times(\hat{j}\times\hat{j})=\hat{i}\times\vec{0}=\vec{0}$

Hence, $(\vec{u}\times\vec{v})\times\vec{w}\neq(\vec{u}\times\vec{v})\times\vec{w}$

f) False;

Let $\vec{u} = \hat{i} + \hat{j} \quad \vec{v} = \hat{i} + \hat{j} + \hat{k} \quad \vec{w} = -\hat{i} - \hat{j} + \hat{k}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \hat{i} - \hat{j}$$

$$\vec{u} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{vmatrix} = \hat{i} - \hat{j}$$

$\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, but $\vec{v} \neq \vec{w}$