# *Section* 4.2 – Calculus with Parametric Curves

## *Tangents* and *Areas*

A parametrized curve $x = f(t)$ and $y = g(t)$ is differentiable at *t* if *f* and *g* are differentiable at *t*.

## Parametric Formula for $dy/dx$

If all three derivatives exist and $\frac{dx}{dt} \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

The derivatives $\frac{dy}{dt}, \frac{dx}{dt}$, and $\frac{dy}{dx}$ are related by the Chain Rule $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

## Parametric Formula for $d^2y/dx^2$

If the equations $x = f(t)$, $y = g(t)$ define *y* as a twice–differentiable function of *x*, then at any point where $\frac{dx}{dt} \neq 0$ and $y' = \frac{dy}{dx}$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

### *Example*

Find the tangent to the curve $x = \sec t, \quad y = \tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$, at the point $\left(\sqrt{2},\ 1\right)$, where $t = \frac{\pi}{4}$

***Solution***

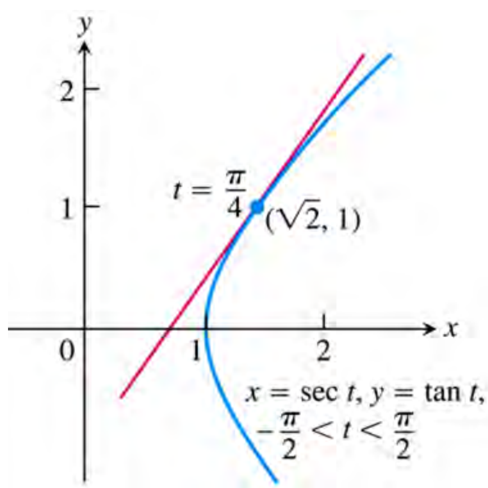
The slope of the curve at *t* is:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{\sec^2 t}{\sec t \tan t}$$

$$= \boxed{\frac{\sec t}{\tan t}}$$

$$m = \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}}$$



$$= \frac{\sec\frac{\pi}{4}}{\tan\frac{\pi}{4}}$$

$$\boxed{= \sqrt{2}}$$

The tangent line is

$$y = \sqrt{2}\left(x - \sqrt{2}\right) + 1 \qquad\qquad y = m\left(x - x_1\right) + y_1$$

$$= \sqrt{2}x - 2 + 1$$

$$\boxed{= \sqrt{2}\,x - 1}$$

## *Example*

Find $\frac{d^2y}{dx^2}$ as a function of $t$ if $x = t - t^2, \quad y = t - t^3$

***Solution***

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\boxed{= \frac{1 - 3t^2}{1 - 2t}}$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{-3t^2 + 1}{-2t + 1}\right) \qquad\qquad \begin{array}{ll} u = 1 - 3t^2 & v = 1 - 2t \\ u' = -6t & v' = -2 \end{array}$$

$$= \frac{-6t(1 - 2t) - (-2)\left(1 - 3t^2\right)}{(1 - 2t)^2}$$

$$= \frac{-6t + 12t^2 + 2 - 6t^2}{(1 - 2t)^2}$$

$$\boxed{= \frac{6t^2 - 6t + 2}{(1 - 2t)^2}} \qquad\qquad \frac{d}{dx}\left(\frac{ax^2 + bx + c}{dx^2 + ex + f}\right) = \frac{(ae - bd)x^2 + 2(af - cd)x + bf - ce}{\left(dx^2 + ex + f\right)^2}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$= \frac{6t^2 - 6t + 2}{(1 - 2t)^2} \div (1 - 2t)$$

$$\boxed{= \frac{6t^2 - 6t + 2}{(1 - 2t)^3}}$$

### *Example*

Find the area enclosed by the asteroid: $x=\cos^3 t,\quad y=\sin^3 t,\quad 0\le t\le 2\pi$

***Solution***

By symmetry, the enclosed area is 4 times the area beneath the curve in the first quadrant where $0\le t\le \frac{\pi}{2}$.

$$A=4\int_0^1 |y\ dx|$$

$$=4\int_0^{\pi/2}\sin^3 t\ \left|d\left(\cos^3 t\right)\right| \qquad\qquad d\left(\cos^3 t\right)=-3\cos^2 t\sin t\ dt$$

$$=4\int_0^{\pi/2}\sin^3 t\cdot 3\cos^2 t\ \sin t\ dt$$

$$=12\int_0^{\pi/2}\sin^4 t\cdot\cos^2 t\ dt$$

$$=12\int_0^{\pi/2}\left(\frac{1-\cos 2t}{2}\right)^2\left(\frac{1+\cos 2t}{2}\right)dt$$

$$=\frac{3}{2}\int_0^{\pi/2}\left(1-2\cos 2t+\cos^2 2t\right)\left(1+\cos 2t\right)\ dt$$

$$=\frac{3}{2}\int_0^{\pi/2}\left(1-2\cos 2t+\cos^2 2t+\cos 2t-2\cos^2 2t+\cos^3 2t\right)\ dt$$

$$=\frac{3}{2}\int_0^{\pi/2}\left(1-\cos 2t-\cos^2 2t+\cos^3 2t\right)\ dt$$

$$=\frac{3}{2}\left(t-\frac{1}{2}\sin 2t\ \Big|_0^{\pi/2}\right.-\frac{3}{2}\int_0^{\pi/2}\cos^2 2t\ dt\ +\ \frac{3}{2}\int_0^{\pi/2}\cos^3 2t\ dt \qquad\qquad \cos^2\alpha=\frac{1+\cos 2\alpha}{2}$$

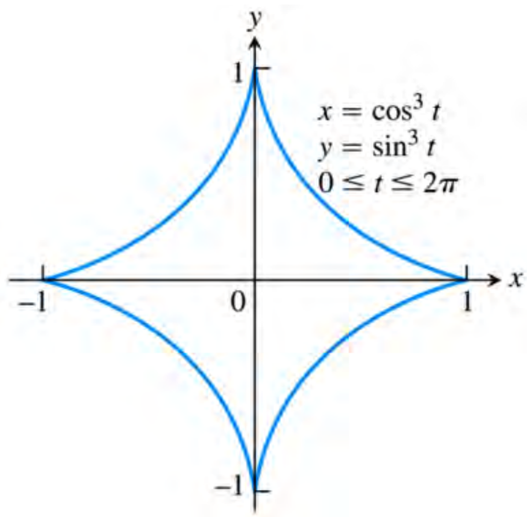
$$=\frac{3}{2}\left(\frac{\pi}{2}-0\right)-\frac{3}{2}\int_0^{\pi/2}\frac{1+\cos 2t}{2}\ dt\ +\ \frac{3}{2}\int_0^{\pi/2}\left(1-\sin^2 2t\right)\cos 2t\ dt$$

$$= \frac{3\pi}{4} - \frac{3}{4}\left(t + \frac{1}{2}\sin 2t\right. \Bigg|_0^{\pi/2} + \frac{3}{4}\int_0^{\pi/2} \left(1 - \sin^2 2t\right) d\left(\sin 2t\right)$$

$$= \frac{3\pi}{4} - \frac{3}{4}\left(\frac{\pi}{2} - 0\right) + \frac{3}{4}\left(\sin 2t - \frac{1}{3}\sin^3 2t\right. \Bigg|_0^{\pi/2}$$

$$= \frac{3\pi}{4} - \frac{3\pi}{8} + \frac{3}{4}(0 - 0)$$

$$= \frac{3\pi}{8} \;\; unit^2$$

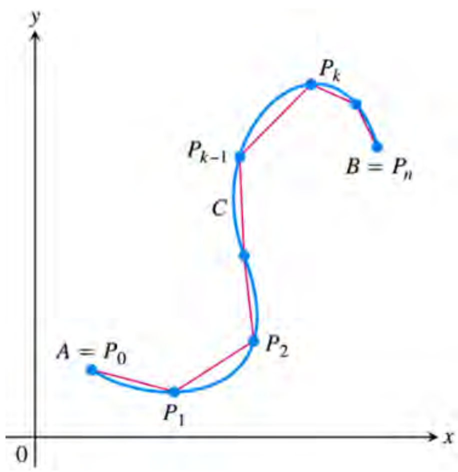


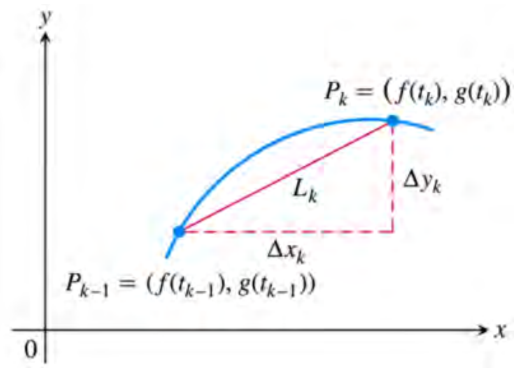
## Length of a Parametrically Defined Curve

### *Definition*

If a curve $C$ is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where $f'$ and $g'$ are continuous and not simultaneously zero on $[a, b]$, and $C$ is traversed exactly once as $t$ increases from $t = a$ to $t = a$, then the length of $C$ is the definite integral

$$L = \int_a^b \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}\, dt$$




### *Example*

Find the length of the circle of radius $r$ defined parametrically by $x = r\cos t, \quad y = r\sin t, \quad 0 \le t \le 2\pi$

***Solution***

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt \\
&= \int_0^{2\pi} \sqrt{r^2\left(\sin^2 t + \cos^2 t\right)}\, dt \\
&= \int_0^{2\pi} r\, dt
\end{aligned}$$

$$= rt \Big|_0^{2\pi}$$

$$\underline{= 2\pi r \ \ unit}$$

### *Example*

Find the length of the asteroid: $\quad x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le 2\pi$

***Solution***

Because of the curve's symmetry with respect to the coordinate axes, its length is 4 times the length of the first quadrant.

$$\left(\frac{dx}{dt}\right)^2 = \left[3\cos^2 t(-\sin t)\right]^2$$

$$= 9\cos^4 t \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = \left[3\sin^2 t(\cos t)\right]^2$$

$$= 9\sin^4 t \cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}$$

$$= 3|\cos t \sin t|\sqrt{\cos^2 t + \sin^2 t} \qquad \cos^2 t + \sin^2 t = 1$$

$$\underline{= 3\cos t \sin t} \qquad \cos t \sin t \ge 0, \quad 0 \le t \le \frac{\pi}{2}$$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 4\int_0^{\pi/2} 3\cos t \sin t \ dt \qquad \sin 2t = 2\cos t \sin t$$

$$= \frac{12}{2}\int_0^{\pi/2} \sin 2t \ dt$$

$$= -\frac{6}{2}\cos 2t \Big|_0^{\pi/2}$$

$$= -3(-1-1)$$

$$= -3(-2)$$

$$\underline{= 6 \ \ unit}$$

## Area of Surface of Revolution for Parametrized Curves

If a smooth curve $x = f(t)$ and $y = g(t)$, $a \le t \le b$, is traversed exactly once as $t$ increases from $a$ to $b$, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. **Revolution about the $x$–axis ($y \ge 0$):**

$$S = 2\pi \int_a^b y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

2. **Revolution about the $y$–axis ($x \ge 0$):**

$$S = 2\pi \int_a^b x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

### *Example*

The standard parametrization of the circle of radius 1 centered at the point (0, 1) in the $xy$–plane is

$$x = \cos t, \quad y = 1 + \sin t, \quad 0 \le t \le 2\pi$$

Use the parametrization to find the area of the surface swept out by revolving the circle about the $x$–axis.

***Solution***

$$x = \cos t \quad \Rightarrow \quad \left(\frac{dx}{dt}\right)^2 = (-\sin t)^2$$

$$y = 1 + \sin t \quad \Rightarrow \quad \left(\frac{dy}{dt}\right)^2 = (\cos t)^2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\sin^2 t + \cos^2 t} = 1$$

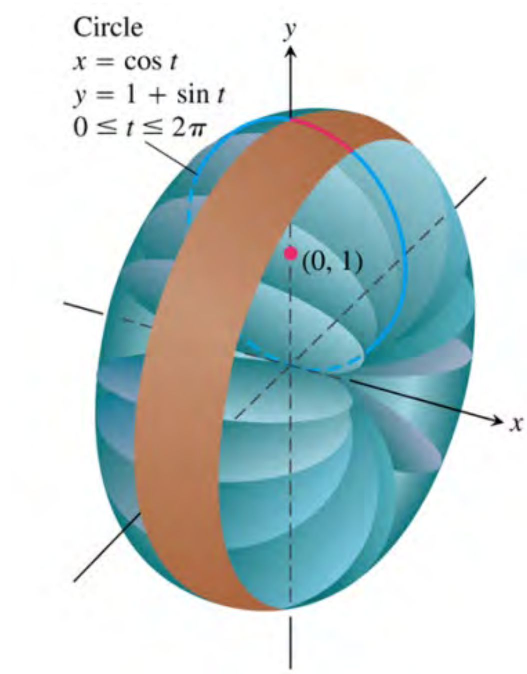
$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} 2\pi(1 + \sin t)\, dt$$

$$= 2\pi\left[t - \cos t\right]_0^{2\pi}$$

$$= 2\pi\left(2\pi - 1 - (0 - 1)\right)$$

$$\boxed{= 4\pi^2 \ \ unit^2}$$



# *Exercises* Section 4.2 – Calculus with Parametric Curves

**(1 – 4)** Find all the points at which the curve has the given slope.

**1.** $x = 4\cos t, \quad y = 4\sin t; \quad slope = \frac{1}{2}$

**2.** $x = 2\cos t, \quad y = 8\sin t; \quad slope = -1$

**3.** $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}; \quad slope = 1$

**4.** $x = 2 + \sqrt{t}, \quad y = 2 - 4t; \quad slope = -8$

**(5 – 12)** Find an equation of the line tangent to the curve at the point corresponding to the given value of *t*.

**5.** $x = \sin t, \quad y = \cos t, \quad t = \frac{\pi}{4}$

**6.** $x = t^2 - 1, \quad y = t^3 + t, \quad t = 2$

**7.** $x = e^t, \quad y = \ln(t+1), \quad t = 0$

**8.** $x = \cos t + t\sin t, \quad y = \sin t - t\cos t, \quad t = \frac{\pi}{4}$

**9.** $x = 6t, \quad y = t^2 + 4, \quad t = 1$

**10.** $x = t - 2, \quad y = \frac{1}{t} + 3, \quad t = 1$

**11.** $x = t^2 - t + 2, \quad y = t^3 - 3t, \quad t = -1$

**12.** $x = -t^2 + 3t, \quad y = 2t^{3/2}, \quad t = \frac{1}{4}$

**(13 – 33)** Find the tangent to the curve at the point defined by the given value of *t*. Also find the value of $\frac{d^2y}{dx^2}$ at this point

**13.** $x = \sin 2\pi t, \quad y = \cos 2\pi t, \quad t = -\frac{1}{6}$

**14.** $x = \cos t, \quad y = \sqrt{3}\cos t, \quad t = \frac{2\pi}{3}$

**15.** $x = t, \quad y = \sqrt{t}, \quad t = \frac{1}{4}$

**16.** $x = \sec^2 t - 1, \quad y = \tan t \quad t = -\frac{\pi}{4}$

**17.** $x = \frac{1}{t+1}, \quad y = \frac{t}{t-1} \quad t = 2$

**18.** $x = t + e^t, \quad y = 1 - e^t \quad t = 0$

**19.** $x = 4t, \quad y = 3t - 2, \quad t = 3$

**20.** $x = \sqrt{t}, \quad y = 3t - 1, \quad t = 1$

**21.** $x = t + 1, \quad y = t^2 + 3t, \quad t = -1$

**22.** $x = t^2 + 5t + 4, \quad y = 4t, \quad t = 0$

**23.** $x = 4\cos\theta, \quad y = 4\sin\theta, \quad \theta = \frac{\pi}{4}$

**24.** $x = \cos\theta, \quad y = 3\sin\theta, \quad \theta = 0$

**25.** $x = 2 + \sec\theta, \quad y = 1 + 2\tan\theta, \quad \theta = \frac{\pi}{6}$

**26.** $x = \sqrt{t}, \quad y = \sqrt{t-1}, \quad t = 2$

**27.** $x = \cos^3\theta, \quad y = \sin^3\theta, \quad \theta = \frac{\pi}{4}$

**28.** $x = \theta - \sin\theta, \quad y = 1 - \cos\theta, \quad \theta = \pi$

**29.** $x = 5t + 2, \quad y = 1 - 4t \quad t = 3$

**30.** $x = t - 6, \quad y = t^2 \quad t = 5$

**31.** $x = \frac{1}{t}, \quad y = 2t + 3 \quad t = -1$

**32.** $x = \frac{1}{t}, \quad y = t^2 \quad t = -2$

**33.** $x = e^t, \quad y = e^{-t} \quad t = 1$

**(33 – 37)** Find the equations of the tangent lines at the point where the curve crosses itself

**34.** $x = 2\sin 2t, \quad y = 3\sin t$

**35.** $x = 2 - \pi\cos t, \quad y = 2t - \pi\sin t$

**36.** $x = t^2 - t, \quad y = t^3 - 3t - 1$

**37.** $x = t^3 - 6t, \quad y = t^2$

**(38 – 40)** Find the slope of the curve $x = f(t),\ y = g(t)$ at the given value of $t$. Define $x$ and $y$ as differentiable functions.

**38.** $x^3 + 2t^2 = 9, \quad 2y^3 - 3t^2 = 4, \quad t = 2$

**39.** $x + 2x^{3/2} = t^2 + t, \quad y\sqrt{t+1} + 2t\sqrt{y} = 4, \quad t = 0$

**40.** $t = \ln(x - t), \quad y = te^t, \quad t = 0$

**(41 – 37)** Find $\dfrac{d^2y}{dx^2}$

**41.** $x(t) = t - t^2 \quad y(t) = t - t^3$

**42.** $x(t) = 2\sec t \quad y(t) = 4\tan t + 2$

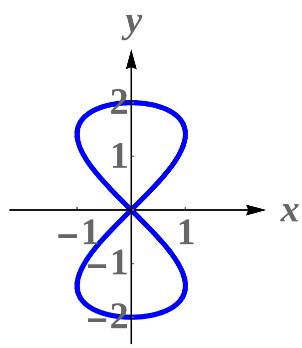
**43.** $x(t) = t^2 + 1 \quad y(t) = 2t - 1$

**44.** $x(t) = 2t^2 - 1 \quad y(t) = 2t^3 + t$

**45.** Find an equation of the line tangent to cycloid $x(t) = t - \sin t, \quad y(t) = 2 - \cos t$ at the points corresponding to $t = \frac{\pi}{6}$ and $t = \frac{2\pi}{3}$.

**46.** Consider Lissajous curve, estimate the coordinates of the points on the curve at which there is

$$x = \sin 2t, \quad y = 2\sin t; \quad 0 \le t \le 2\pi$$

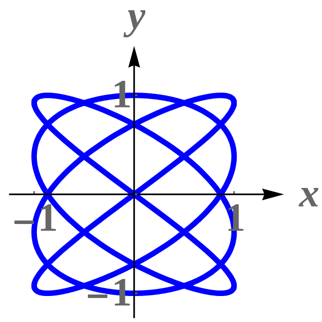



*a)* A horizontal tangent line

*b)* A vertical tangent line.

**47.** Consider Lissajous curve, estimate the coordinates of the points on the curve at which there is

$$x = \sin 4t, \quad y = \sin 3t; \quad 0 \le t \le 2\pi$$

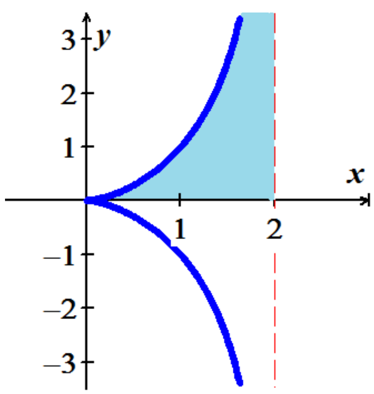



*a)* A horizontal tangent line

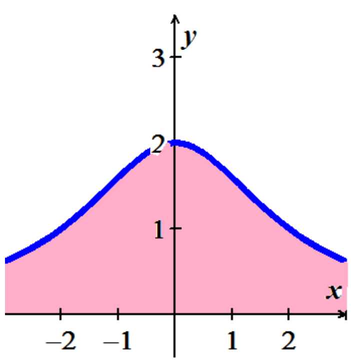
*b)* A vertical tangent line.

**(48 – 58)** Find the area of the region

**48.** $x = 2\sin^2\theta, \quad y = 2\sin^2\theta\tan\theta, \quad 0 \le \theta < \frac{\pi}{2}$




**49.** $x = 2\cot\theta, \quad y = 2\sin^2\theta, \quad 0 \le \theta < \pi$

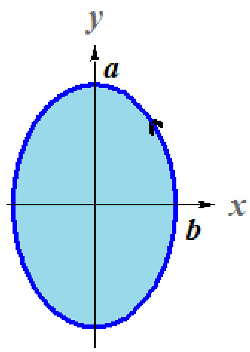



**50.** Find the area under one arch of the cycloid $x = a(t - \sin t), \quad y = a(1 - \cos t)$

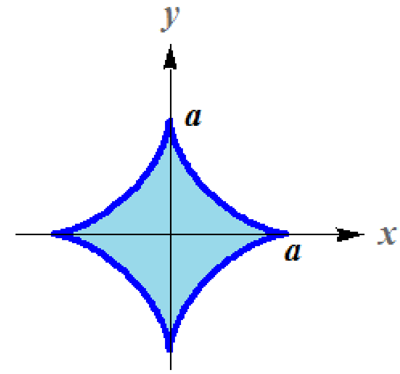
**51.** Find the area enclosed by the $y$–axis and the curve $x = t - t^2, \quad y = 1 + e^{-t}$

**52.** Find the area enclosed by the ellipse $x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi$

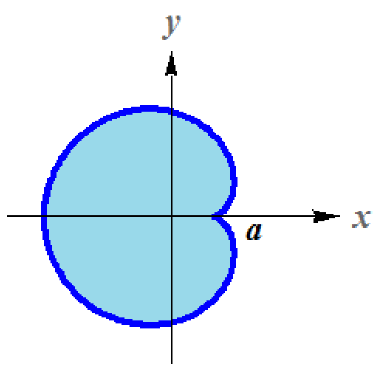
**53.** *Ellipse* $\begin{cases} x = b\cos t \\ y = a\sin t \end{cases}$ $0 \le t \le 2\pi$



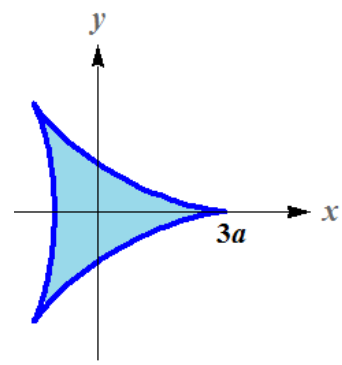

**54.** *Astroid* $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}$ $0 \le t \le 2\pi$



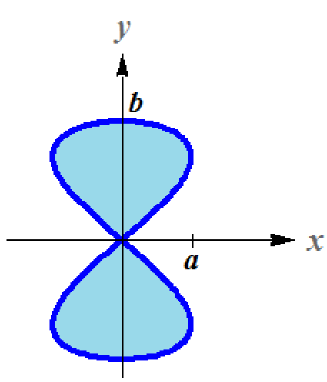

**55.** *Cardioid* $\begin{cases} x = 2a\cos t - a\cos 2t \\ y = 2a\sin t - a\sin 2t \end{cases}$ $0 \le t \le 2\pi$



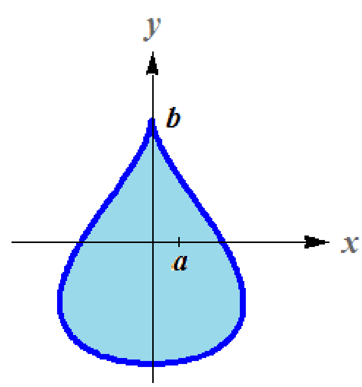

**56.** *Deltoid* $\begin{cases} x = 2a\cos t + a\cos 2t \\ y = 2a\sin t - a\sin 2t \end{cases}$ $0 \le t \le 2\pi$




**57.** *Hourglass* $\begin{cases} x = a\sin 2t \\ y = b\sin t \end{cases}$ $0 \le t \le 2\pi$




**58.** *Teardrop* $\begin{cases} x = 2a\cos t - a\sin 2t \\ y = b\sin t \end{cases}$ $0 \le t \le 2\pi$




(**59 – 68**) Find the lengths of the curves

**59.** $x = \cos t, \quad y = t + \sin t, \quad 0 \le t \le \pi$

**60.** $x = t^3, \quad y = \frac{3}{2}t^2, \quad 0 \le t \le \sqrt{3}$

**61.** $x = 8\cos t + 8t\sin t, \quad y = 8\sin t - 8t\cos t, \quad 0 \le t \le \frac{\pi}{2}$

**62.** $x = \ln(\sec t + \tan t) - \sin t, \quad y = \cos t, \quad 0 \le t \le \frac{\pi}{3}$

**63.** Hypocycloid perimeter curve: $x = a\cos\theta, \quad y = a\sin\theta$

**64.** Circle circumference: $x = a\cos^3\theta, \quad y = a\sin^3\theta$

**65.** Cycloid arch: $x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta)$

**66.** Involute of a circle: $x = \cos\theta + \theta\sin\theta, \quad y = \sin\theta - \theta\cos\theta$

**67.** $x = t^2, \quad y = t^3, \quad 0 \le t \le 2$

**68.** $x = 5\sin t, \quad y = 5\cos t, \quad -\frac{\pi}{3} \le t \le \frac{\pi}{2}$

(**69 – 80**) Find the areas of the surfaces generated by revolving the curves

**69.** $x = \frac{1}{3}t^3, \quad y = t + 1, \quad 1 \le t \le 2, \quad$ *y-axis*

**70.** $x = \frac{2}{3}t^{3/2}, \quad y = 2\sqrt{t}, \quad 0 \le t \le \sqrt{3}; \quad x - axis$

**71.** $x = t + \sqrt{2}, \quad y = \frac{t^2}{2} + \sqrt{2}t, \quad -\sqrt{2} \le t \le \sqrt{2}; \quad y - axis$

**72.** $x = 2t, \quad y = 3t; \quad 0 \le t \le 3 \quad$ *x-axis*

**73.** $x = 2t, \quad y = 3t; \quad 0 \le t \le 3 \quad$ *y-axis*

**74.** $x = t, \quad y = 4 - 2t; \quad 0 \le t \le 2 \quad$ *x-axis*

**75.** $x = t, \quad y = 4 - 2t; \quad 0 \le t \le 2 \quad$ *y - axis*

**76.** $x = 5\cos\theta, \quad y = 5\sin\theta, \quad 0 \le \theta \le \frac{\pi}{2}, \quad$ *y - axis*

**77.** $x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad 0 \le \theta \le \pi, \quad$ *x-axis*

**78.** $x = a\cos\theta, \quad y = b\sin\theta, \quad 0 \le \theta \le 2\pi$

*a)* *x-axis* *b)* *y-axis*

**79.** $x = 2t, \quad y = 3t, \quad 0 \le t \le 3$

*a)* *x-axis* *b)* *y-axis*

**80.** $x = t, \quad y = 4 - 2t, \quad 0 \le t \le 2$

*a)* *x-axis* *b)* *y-axis*

**81.** Use the parametric equations $x = t^2\sqrt{3}$ *and* $y = 3t - \frac{1}{3}t^3$ to

*a)* Graph the curve on the interval $-3 \le t \le 3$.

*b)* Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

*c)* Find the equation of the tangent line at the point $\left(\sqrt{3},\ \frac{8}{3}\right)$

*d)* Find the length of the curve

*e)* Find the surface area generated by revolving the curve about the *x-axis*

**82.** Use the parametric equations $x = a(\theta - \sin\theta)$ *and* $y = a(1 - \cos\theta)$ $a > 0$

*a)* Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

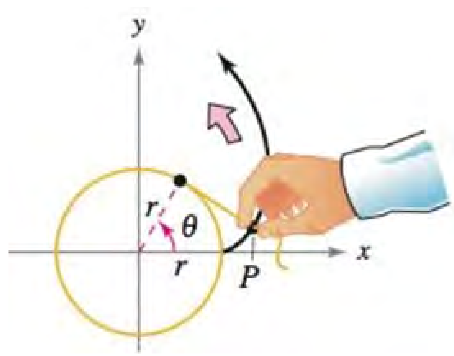
*b)* Find the equation of the tangent line at the point where $\theta = \frac{\pi}{6}$

*c)* Find all points (if any) of horizontal tangency.

*d)* Determine where the curve is concave upward or concave downward.

*e)* Find the length of one arc of the curve

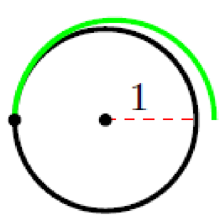
**83.** The involute of a circle is described by the endpoint *P* of a string that is held taut as it is unwound from a spool that does not turn.




Show that a parametric representation of the involute is

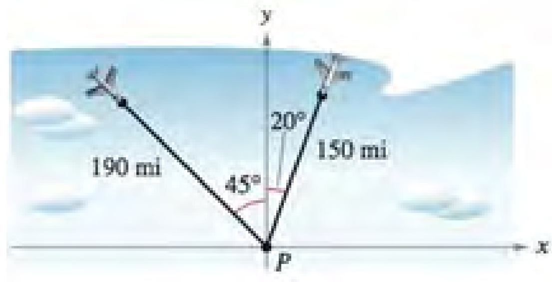
$$x = r(\cos\theta + \theta\sin\theta) \quad \textit{and} \quad y = r(\sin\theta - \theta\cos\theta)$$

**84.** The figure shows a piece of string tied to a circle with a radius of *one* unit. The string is just long enough to reach the opposite side if the circle.




Find the area that is covered when the string is unwounded counterclockwise.

**85.** An Air traffic controller spots two planes at the same altitude flying toward each other.




Their flight paths are 20° and 315°. One plane is 150 *miles* from point $P$ with a speed of 375 *miles per hour*. The other is 190 *miles* from point $P$ with a speed of 450 *miles per hour*.

*a)* Find parameteric equations for the path of each plane where $t$ is the time in *hours*, with $t = 0$ corresponding to the time at which the air traffic controller spots the planes.

*b)* Use part ($a$) to write the distance between the planes as a function of $t$.

*c)* Graph the function in part ($b$).

*d)* When the distance between the planes be minimum?

*e)* If the planes must keep a separation of at least 3 *miles*, is the requirement met?