Section 2.6 – Mechanical systems

A sinusoidal forcing is giving by the model:

$$x'' + 2cx' + \omega_0^2 x = A\cos\omega t$$

A: Amplitude if the driving force (constant)
ω: driving frequency.
c: damping constant.
ω_0 : natural frequency.

Forced undamped harmonic motion

The undamped equation has $c = 0$ or

$$x'' + \omega_0^2 x = A\cos\omega t$$

The homogeneous equation is: $x'' + \omega_0^2 x = 0$

With general solution: $x_h = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$

Case 1 $\omega \ne \omega_0$

The particular solution is given by the form: $x_p = a\cos\omega t + b\sin\omega t$

$$x'_p = -a\omega\sin\omega t + b\omega\cos\omega t$$

$$x''_p = -a\omega^2\cos\omega t - b\omega^2\sin\omega t$$

$$\begin{aligned} x''_p + \omega_0^2 x_p &= -a\omega^2\cos\omega t - b\omega^2\sin\omega t + \omega_0^2\left(a\cos\omega t + b\sin\omega t\right) \\ &= -a\omega^2\cos\omega t - b\omega^2\sin\omega t + \omega_0^2 a\cos\omega t + \omega_0^2 b\sin\omega t \\ &= a\left(\omega_0^2 - \omega^2\right)\cos\omega t + b\left(\omega_0^2 - \omega^2\right)\sin\omega t \\ &= A\cos\omega t \end{aligned}$$

$$A = a\left(\omega_0^2 - \omega^2\right) \qquad b\left(\omega_0^2 - \omega^2\right) = 0$$

$$a = \frac{A}{\left(\omega_0^2 - \omega^2\right)} \qquad b = 0 \qquad \textit{since } \omega_0^2 - \omega^2 \ne 0$$

$$x_p = \frac{A}{\left(\omega_0^2 - \omega^2\right)}\cos\omega t$$

$$x(t) = x_h(t) + x_p(t)$$

$$= C_1 \cos\omega_0 t + C_2 \sin\omega_0 t + \frac{A}{\left(\omega_0^2 - \omega^2\right)} \cos\omega t$$

When the *motion starts at equilibrium*; this means $x(0) = x'(0) = 0$

$$C_1 + \frac{A}{\left(\omega_0^2 - \omega^2\right)} = 0 \quad \Rightarrow \quad C_1 = -\frac{A}{\left(\omega_0^2 - \omega^2\right)}$$

$$x' = -C_1\omega_0 \sin\omega_0 t + C_2\omega_0 \cos\omega_0 t - \frac{A}{\left(\omega_0^2 - \omega^2\right)}\omega_0 \sin\omega t$$

$$x'(0) = C_2\omega_0 = 0 \quad \Rightarrow \quad C_2 = 0$$

$$x(t) = -\frac{A}{\left(\omega_0^2 - \omega^2\right)} \cos\omega_0 t + \frac{A}{\left(\omega_0^2 - \omega^2\right)} \cos\omega t$$

$$x(t) = \frac{A}{\left(\omega_0^2 - \omega^2\right)} \left(\cos\omega t - \cos\omega_0 t\right)$$

Example

Suppose $F_0 = 23,\ \omega_0 = 11,\ \omega = 12$ with these values of the parameters the solution becomes.

F_0: Amplitude if the driving force (constant)

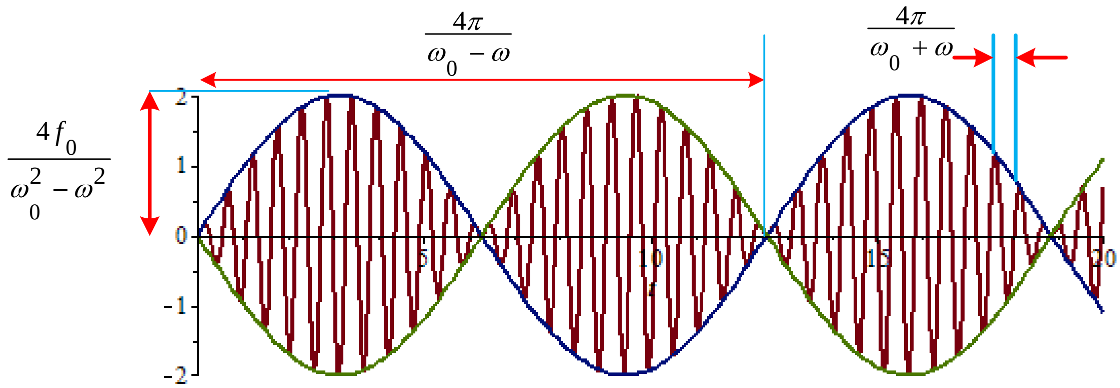
Solution

$$x(t) = \frac{F_0}{\left(\omega_0^2 - \omega^2\right)} \left(\cos\omega t - \cos\omega_0 t\right)$$

$$= \frac{23}{\left(11^2 - 12^2\right)} \left(\cos 12t - \cos 11t\right)$$

$$= -\left(\cos 12t - \cos 11t\right)$$

$$= \cos 11t - \cos 12t$$



Mean frequency: $\bar{\omega} = \dfrac{\omega_0 + \omega}{2}$ ***Half difference***: $\delta = \dfrac{\omega_0 - \omega}{2}$

***Case* 2** $\omega = \omega_0$

The particular solution is given by the form: $x_p = t\left(a\cos\omega_0 t + b\sin\omega_0 t\right)$

$$x'_p = a\cos\omega_0 t + b\sin\omega_0 t - at\omega_0 \sin\omega_0 t + bt\omega_0 \cos\omega_0 t$$

$$x''_p = -a\omega_0 \sin\omega_0 t + b\omega_0 \cos\omega_0 t - a\omega_0 \sin\omega_0 t + b\omega_0 \cos\omega_0 t - at\omega_0^2 \cos\omega_0 t - bt\omega_0^2 \sin\omega_0 t$$

$$= -2\omega_0 \left(a\sin\omega_0 t - b\cos\omega_0 t\right) - t\omega_0^2 \left(a\cos\omega_0 t + b\sin\omega_0 t\right)$$

$$x''_p + \omega_0^2 x_p = -2\omega_0 \left(a\sin\omega_0 t - b\cos\omega_0 t\right) - t\omega_0^2 \left(a\cos\omega_0 t + b\sin\omega_0 t\right) + t\omega_0^2 \left(a\cos\omega_0 t + b\sin\omega_0 t\right)$$

$$= -2\omega_0 \left(a\sin\omega_0 t - b\cos\omega_0 t\right)$$

$$= F_0 \cos\omega t$$

F_0: Amplitude if the driving force (*constant*)

$F_0 = 2b\omega_0$ $\quad$ $a = 0$

$$b = \frac{F_0}{2\omega_0}$$

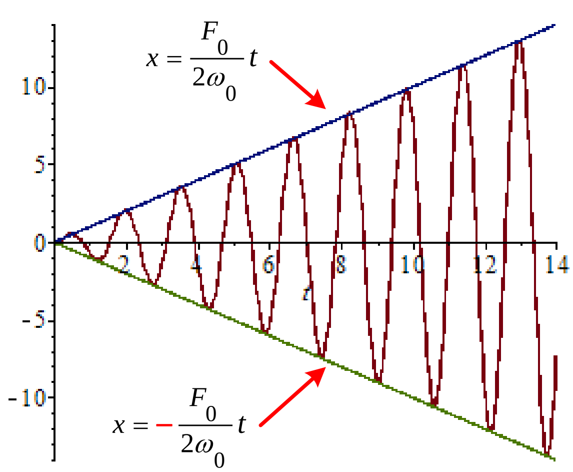
$$x_p = \frac{F_0}{2\omega_0} t\sin\omega_0 t$$

$$x(t) = x_h(t) + x_p(t)$$

$$= C_1 \cos\omega_0 t + C_2 \sin\omega_0 t + \frac{F_0}{2\omega_0} t\sin\omega_0 t$$

$x(0) = C_1 = 0,\ x'(0) = C_2 = 0,\ A = 8,\ w_0 = 4$

$x(t) = t\sin 4t$



Forced Damped Harmonic Motion

In real physical systems, there is always some damping, from frictional effects if nothing else. Let's add damping to the system

$$mx'' + cx' + kx = F_0 \cos\omega t \qquad\qquad x'' + 2cx' + \omega_0^2 x = F_0 \cos\omega t$$

The homogeneous equation (***transient solution***) is:

$$mx'' + cx' + kx = 0$$
$$m\lambda^2 + c\lambda + k = 0$$
$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$x'' + 2cx' + \omega_0^2 x = 0$$
$$\lambda^2 + 2c\lambda + \omega_0^2 = 0$$
$$\lambda_{1,2} = -c \pm \sqrt{c^2 - \omega_0^2}$$

The particular solution is: $x_p = A\cos\omega t + B\sin\omega t$

$$x'(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$
$$x''(t) = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

$$-Am\omega^2\cos\omega t - Bm\omega^2\sin\omega t - Ac\omega\sin\omega t + Bc\omega\cos\omega t + Ak\cos\omega t + Bk\sin\omega t = F_0\cos\omega t$$

$$\begin{cases} -Am\omega^2 + Bc\omega + Ak = F_0 \\ -Bm\omega^2 - Ac\omega + Bk = 0 \end{cases} \rightarrow \begin{cases} \left(k - m\omega^2\right)A + c\omega B = F_0 \\ -c\omega A + \left(k - m\omega^2\right)B = 0 \end{cases}$$

$$D = \begin{vmatrix} k - m\omega^2 & c\omega \\ -c\omega & k - m\omega^2 \end{vmatrix} = \left(k - m\omega^2\right)^2 + (c\omega)^2$$

$$D_A = \begin{vmatrix} F_0 & -c\omega \\ 0 & k - m\omega^2 \end{vmatrix} = F_0\left(k - m\omega^2\right) \quad D_B = \begin{vmatrix} k - m\omega^2 & F_0 \\ -c\omega & 0 \end{vmatrix} = c\omega F_0$$

$$A = \frac{\left(k - m\omega^2\right)F_0}{\left(k - m\omega^2\right)^2 + (c\omega)^2} \quad B = \frac{c\omega F_0}{\left(k - m\omega^2\right)^2 + (c\omega)^2}$$

The amplitude: $C = \sqrt{A^2 + B^2}$

$$= \sqrt{\frac{\left(k - m\omega^2\right)^2 F_0^2}{\left(\left(k - m\omega^2\right)^2 + (c\omega)^2\right)^2} + \frac{(c\omega)^2 F_0^2}{\left(\left(k - m\omega^2\right)^2 + (c\omega)^2\right)^2}}$$

$$= F_0 \sqrt{\frac{\left(k - m\omega^2\right)^2 + (c\omega)^2}{\left(\left(k - m\omega^2\right)^2 + (c\omega)^2\right)^2}}$$

$$= \boxed{\frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + (c\omega)^2}}}$$

The phase shift: $\varphi = \tan^{-1}\frac{B}{A} = \tan^{-1}\frac{c\omega}{k - m\omega^2}$

$$\begin{cases} \varphi = \tan^{-1}\dfrac{c\omega}{k - m\omega^2} & \text{if} \quad k > m\omega^2 \\ \varphi = \pi + \tan^{-1}\dfrac{c\omega}{k - m\omega^2} & \text{if} \quad k < m\omega^2 \end{cases}$$

Underdamped Case: $c < \omega_0$

$$x_h = e^{-ct}\left(C_1 \cos \eta t + C_2 \sin \eta t\right) \qquad \text{Where } \eta = \sqrt{\omega_0^2 - c^2}$$

To determine the inhomogeneous equation, it is better to use complex method.

$$z'' + 2cz' + \omega_0^2 z = Ae^{i\omega t}$$

However, $x(t) = \mathrm{Re}(z)$

The particular solution: $z(t) = ae^{i\omega t}$

$$z'' + 2cz' + \omega_0^2 z = (i\omega)^2 ae^{i\omega t} + 2c(i\omega)ae^{i\omega t} + \omega_0^2 ae^{i\omega t}$$

$$= \left((i\omega)^2 + 2c(i\omega) + \omega_0^2\right)ae^{i\omega t}$$

$$P(i\omega) = (i\omega)^2 + 2c(i\omega) + \omega_0^2$$

$$P(i\omega)ae^{i\omega t} = Ae^{i\omega t}$$

$$\Rightarrow a = \frac{A}{P(i\omega)}$$

$$z(t) = \frac{A}{P(i\omega)}e^{i\omega t}$$

$$= H(i\omega)e^{i\omega t}$$

$H(i\omega)$ is called the ***transfer function.***

$$P(i\omega) = -\omega^2 + 2ic\omega + \omega_0^2 \ = \omega_0^2 - \omega^2 + 2ic\omega$$

$$P(i\omega) = Re^{i\phi}$$

$$= R\left(\cos\phi + i\sin\phi\right)$$

Polar Coordinates: $\quad R\cos\phi = \omega_0^2 - \omega^2 \qquad R\sin\phi = 2c\omega$

$$R = \sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4c^2\omega^2}$$

$$\cos\phi = \frac{\omega_0^2 - \omega^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4c^2\omega^2}} \qquad \sin\phi = \frac{2c\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4c^2\omega^2}}$$

$$\cot\phi = \frac{\omega_0^2 - \omega^2}{2c\omega} \quad \Rightarrow \quad \phi(\omega) = arc\cot\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right) \qquad 0 < \phi < \pi$$

$$H(i\omega) = \frac{1}{P(i\omega)} = \frac{1}{R} e^{-i\phi}$$

We will define the ***gain*** G by: $G(\omega) = \frac{1}{R} = \frac{1}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4c^2\omega^2}}$ $\qquad H(i\omega) = G(\omega)e^{-i\phi}$

The solution: $\qquad z(t) = H(i\omega)Ae^{i\omega t} = G(\omega)Ae^{i(\omega t - \phi)}$

$$x_p(t) = \mathrm{Re}\, z(t) == G(\omega)A\cos(\omega t - \phi)$$

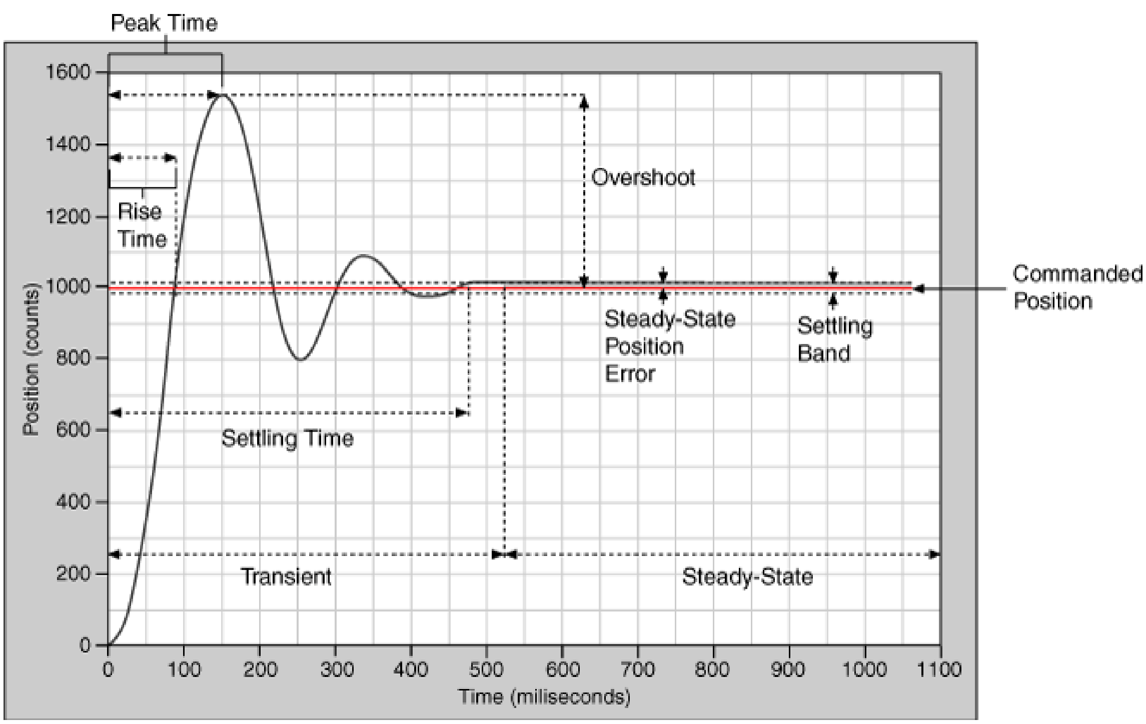
$$\boxed{x(t) = e^{-ct}\left(C_1 \cos\eta t + C_2 \sin\eta t\right) + G(\omega)A\cos(\omega t - \phi)}$$

e^{-ct} : ***transient*** term.

$T_c = \frac{1}{c}$: ***time constant***.

Transient and Steady-State

A motion or current is ***transient*** *solution* if the solution approaches zero as $t \to \infty$.
A ***steady-state*** motion or current is one that is not transient and does not become unbounded.
Free damped systems always yield transient motions, while forced damped systems (assuming the external force to be sinusoidal) yield both transient and steady-state motions.



Example

Determine the transient and steady-state to: $x'' + 2x' + 2x = 4\cos t + 2\sin t$; $x(0) = 0,\quad x'(0) = v_0$

Solution

$$\lambda^2 + 2\lambda + 2 = 0 \quad \to \quad \lambda_{1,2} = \frac{-2 \pm 2i}{2} \boxed{= -1 \pm i}$$

$$\boxed{x_h = e^{-t}\left(C_1 \cos t + C_2 \sin t\right)}$$

$$x_p = A\cos t + B\sin t$$

$$x_p' = -A\sin t + B\cos t$$

$$x_p'' = -A\cos t - B\sin t$$

$$x'' + 2x' + 2x = 4\cos t + 2\sin t$$

$$-A\cos t - B\sin t - 2A\sin t + 2B\cos t + 2A\cos t + 2B\sin t = 4\cos t + 2\sin t$$

$$\begin{cases} \cos t & A + 2B = 4 \\ \sin t & -2A + B = 2 \end{cases} \quad \to \quad \boxed{B = 2,\ A = 0}$$

$$\boxed{x_p = 2\sin t}$$

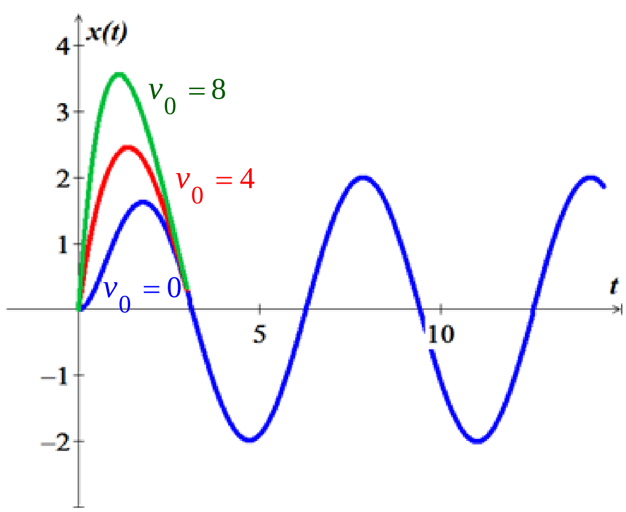
$$x(t) = e^{-t}\left(C_1 \cos t + C_2 \sin t\right) + 2\sin t$$

$$x(0) = 0 \quad \to \quad \boxed{C_1 = 0}$$

$$x'(t) = e^{-t}\left(-C_1 \cos t - C_2 \sin t - C_1 \sin t + C_2 \cos t\right) + 2\cos t$$

$$x(0) = v_0 \quad \to C_1 + C_2 + 2 = v_0 \quad \Rightarrow \boxed{C_2 = v_0 - 2}$$

$$x(t) = \left(v_0 - 2\right)\underbrace{e^{-t}\sin t}_{transient} + \underbrace{2\sin t}_{steady-state}$$



Exercises *Section* 2.6 – Mechanical Systems

1. A 1-*kg* mass is attached to a spring $k = 4kg/s^2$ and the system is allowed to come to rest. The spring-mass system is attached to a machine that supplies external driving force $f(t) = 4\cos\omega t$ *Newtons*. The system is started from equilibrium; the mass is having no initial displacement or velocity. Ignore any damping forces.

a) Find the position of the mass as a function of time

b) Place your answer in the form $s(t) = A\sin\delta t \sin\bar{\omega} t$. Select an ω near the natural frequency of the system to demonstrate the "beating" of the system. Sketch a plot shows the "beats:" and include the envelope of the beating motion in your plot.

Find a particular solution to the differential equation using undetermined coefficients. Find and plot the solution of the initial value problem. Superimpose the plots of the transient response and the steady state solution.

2. $x'' + 7x' + 10x = 3\cos 3t \quad x(0) = -1, \ \ x'(0) = 0$

3. $x'' + 4x' + 5x = 3\sin t \quad x(0) = 0, \ \ x'(0) = -3$

4. Find a particular solution of $y'' - 2y' + 5y = 2\cos 3x - 4\sin 3x + e^{2x}$ given the set $y_p = A\cos 3x + B\sin 3x + Ce^{2x}$ where *A*, *B*, *C* are to be determined

Find the general solution

1. $mx'' + kx = F_0 \cos\omega t \ ; \quad x(0) = x_0, \quad x'(0) = 0 \quad (\omega \ne \omega_0)$

2. $mx'' + kx = F_0 \cos\omega t \ ; \quad x(0) = 0, \quad x'(0) = v_0 \quad (\omega = \omega_0)$

3. $x'' + \omega_0^2 x = F_0 \sin\omega t \ ; \quad x(0) = 0, \quad x'(0) = 0 \quad (\omega \ne \omega_0)$

4. A forced mass–spring–dashpot system with equation $mx'' + cx' + kx = F_0 \cos\omega t$. Investigate the possibility of practical resonance of this system. In particular, find the amplitude $C(\omega)$ of steady state periodic forced oscillations with frequency ω. Sketch the graph $C(\omega)$ of and find the practical resonance frequency ω (if any).

a) $m = 1, \quad c = 2, \quad k = 2, \quad F_0 = 2$

b) $m = 1, \quad c = 4, \quad k = 5, \quad F_0 = 10$

c) $m = 1, \quad c = 6, \quad k = 45, \quad F_0 = 50$

d) $m = 1, \quad c = 10, \quad k = 650, \quad F_0 = 100$

5. A mass weighing 100 *lb*. (mass *m* = 3.125 *slugs* in *fps* units) is attached to the end of a spring that is stretched 1 *in*. by a force of 100 *lb*. A force $F_0 \cos\omega t$ acts on the mass. At what frequency (in hertz) will resonance oscillation occur? Neglect damping.

6. A mass weighing 16 *pounds* stretches a spring $\frac{8}{3}$ *ft*. The mass is initially released form rest from a point 2 *ft* below the equilibrium position, and the subsequent motion takes place in a medium that offers a damping force that is numerically equal to $\frac{1}{2}$ the instantaneous velocity. Find the equation of motion if the mass is driven by an external force equal to $f(t) = 10\cos 3t$

7. A mass of 32 *pounds* is attached to a spring with a constant spring 5 *lb/ft*. Initially, the mass is released 1 foot below the equilibrium position with a downward velocity of 5 *ft/s*, and the subsequent motion takes is numerically equal to 2 times the instantaneous velocity.

a) Find the equation of motion if the mass is driven by an external force equal to $f(t) = 12\cos 2t + 3\sin 2t$.

b) Graph the transient, steady-state, and the equation of motion solutions on the same coordinate axes.

8. A mass of 32 *pounds* is attached to a spring and stretched it 2 *feet* and then comes to rest in the equilibrium position. The surrounding medium offers a damping force that is numerically equal to 8 times the instantaneous velocity

a) Find the equation of motion if the mass is driven by an external force equal to $f(t) = 8\sin 4t$.

b) Graph the transient, steady-state, and the equation of motion solutions on the same coordinate axes.

9. A mass of 32 *pounds* is attached to a spring and stretched it 2 *feet* and then comes to rest in the equilibrium position. The surrounding medium offers a damping force that is numerically equal to 8 times the instantaneous velocity

a) Find the equation of motion with a starting external force equal to $f(t) = e^{-t}\sin 4t$ at $t = 0$

b) Graph the transient, steady-state, and the equation of motion solutions on the same coordinate axes.

10. A mass of 64 *pounds* is attached to a spring with a spring constant 32 *lb/ft* and then comes to rest in the equilibrium position.

a) Find the equation of motion with a starting external force equal to $f(t) = 68e^{-2t}\cos 4t$

b) Graph the transient, steady-state, and the equation of motion solutions on the same coordinate axes.

11. A 3-*kg* object is attached to a spring and stretches the spring 392 *mm* by itself. There is no damping in the system and a forcing function of the form $F(t) = 10\cos\omega t$ is attached to the object and the system will experience resonance. If the object is initially displaced 20 *cm* downward from its equilibrium position and given a velocity of 10 *cm/sec* upward find the displacement $y(t)$ at any time t.

12. A $8\text{-}kg$ mass is attached to a spring hanging vertically, thereby causing the spring to stretch 1.96 *m* upon coming to rest at equilibrium. The damping constant is given by 3 *N-sec/m*.

a) Find the equation of motion if the mass is driven by an external force equal to $f(t) = \cos 2t$ *N*.

b) Determine the transient, steady-state solution of the motion

13. A $2\text{-}kg$ mass is attached to a spring hanging vertically, thereby causing the spring to stretch 0.2 *m* upon coming to rest at equilibrium. At $t = 0$, the mass is displaced 5 *cm* below the equilibrium position and released. The damping constant is given by 5 *N-sec/m*.

a) Find the equation of motion if the mass is driven by an external force equal to $f(t) = 0.3\cos t$ *N*.

b) Determine the transient, steady-state solution of the motion.

14. A $8\text{-}kg$ mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by 3 *N-sec/m* and the spring constant is 40 *N/m*. Find steady-state solution if the mass is driven by an external force equal to $f(t) = 2\sin\left(2t + \frac{\pi}{4}\right)$ *N*.

15. A $32\text{-}lb$ mass weight is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by 2 *lb-sec/ft* and the spring constant is 5 *lb/ft*. If the mass is driven by an external force equal to $f(t) = 3\cos 4t$ *lb* at time $t = 0$.

a) Find steady-state solution.

b) Determine the amplitude and frequency

16. A $8\text{-}kg$ mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by 3 *N-sec/m* and the spring constant is 40 *N/m*. If the mass is driven by an external force equal to $f(t) = 2\sin 2t\cos 2t$ *N*.

a) Find steady-state solution.

b) Determine the amplitude, phase angle, period and frequency

17. A $10\text{-}kg$ mass is attached to a spring hanging vertically stretches the spring 0.098 *m* from its equilibrium rest position, measured positive in the downward direction. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20\cos 10t$ *N*. (*t* in *seconds*)

a) Determine the spring constant *k*.

b) Find the equation of motion.

c) Plot the equation of motion.

d) Determine the maximum excursion from equilibrium made of the object on the *t*-interval $0 \le t < \infty$

18. A $10\text{-}kg$ mass is attached to a spring hanging vertically stretches the spring 0.098 *m* from its equilibrium rest position, measured positive in the downward direction. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20\cos 8t$ *N*. (*t* in *seconds*)

a) Determine the spring constant k.
b) Find the equation of motion.
c) Plot the equation of motion.
d) Determine the maximum excursion from equilibrium made of the object on the t-interval $0 \le t < \infty$

19. A 10-kg mass is attached to a spring hanging vertically stretches the spring $0.098\ m$ from its equilibrium rest position, measured positive in the downward direction. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20e^{-t}\ N$. (t in *seconds*)

a) Determine the spring constant k.
b) Find the equation of motion.
c) Plot the equation of motion.
d) Determine the maximum excursion from equilibrium made of the object on the t-interval $0 \le t < \infty$

20. A 2-kg mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $c = 8\ kg/sec$ and the spring constant is $k = 80\ N/m$. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20\cos 8t\ N$. (t in *seconds*)

a) Find the equation of motion.
b) Plot the equation of motion.
c) Determine the long-time behavior of the system, as $t \to \infty$

21. A 2-kg mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $c = 8\ kg/sec$ and the spring constant is $k = 80\ N/m$. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20\sin 6t\ N$. (t in *seconds*)

a) Find the equation of motion.
b) Plot the equation of motion.
c) Determine the long-time behavior of the system, as $t \to \infty$

22. A 2-kg mass is attached to a spring hanging vertically and come to rest at equilibrium. The damping constant is given by $c = 8\ kg/sec$ and the spring constant is $k = 80\ N/m$. At time $t = 0$, the resulting spring-mass system is disturbed from its rest state by the force $F(t) = 20e^{-t}\ N$. (t in *seconds*)

a) Find the equation of motion.
b) Plot the equation of motion.
c) Determine the long-time behavior of the system, as $t \to \infty$

23. A 10-kg mass is attached to a spring having a spring constant of 140 N/m. The mass is started in motion initially from the equilibrium position with an initial velocity 1 m/sec in the upward direction and with an applied external force $F(t) = 5\sin t$. If the force due to air resistance is $-90y'\ N$.

a) Find the subsequent motion of the mass.
b) Plot the motion.

24. A 10-*kg* mass is attached to a spring having a spring constant of 140 *N/m*. The mass is started in motion initially from the equilibrium position with an initial velocity 1 *m/sec* in the upward direction and with an applied external force $F(t)=5\sin t$. If the force due to air resistance is $-90y'$ *N*.

a) Find the equation motion of the mass.
b) Plot the motion.
c) Determine the motion of the solution.

25. A 128-*lb* weight is attached to a spring having a spring constant of 64 *lb/ft*. The weight is started in motion initially by displacing it 6 *in* above the equilibrium position with no initial velocity and with an applied external force $F(t)=8\sin 4t$. Assume no air resistance.

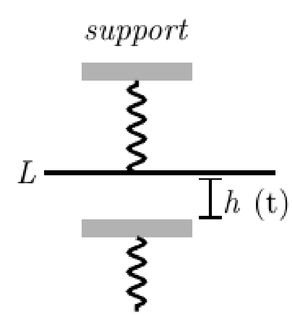
a) Find the equation motion of the mass.
b) Plot the motion.
c) Determine the motion of the solution.

26. A 3-*kg* object is attached to spring and stretches the spring 39.2 *cm* by itself. There is no damping in the system and a forcing function is given by $F(t)=10\cos\omega t$ is attached to the object and the system will experience resonance. If the object is initially displaced 20 *cm* downward from its equilibrium position and given a velocity of 10 *cm/sec* upward.

a) Find the spring constant *k*.
b) Find the natural frequency ω.
c) Find the displacement at any time *t*.
d) Sketch the displacement function.

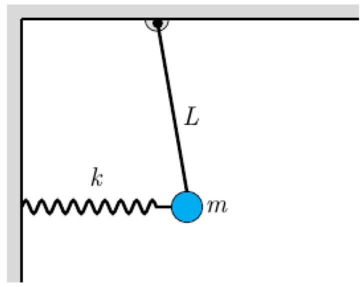
27. Find the transient motion and steady periodic oscillations of a damped mass-and-spring system with $m=1$, $c=2$, and $k=26$ under the influence of an external force $F(t)=82\cos 4t$ with $x(0)=6$ and $x'(0)=0$. Also, investigate the possibility of practical resonance for this system.

28. A mass *m* is attached to the end of a spring with a spring constant *k*. After the mass reaches equilibrium, its support begins to oscillate vertically about a horizontal line *L* according to a formula $h(t)$. The value of *h* represents the distance in feet measured from *L*.

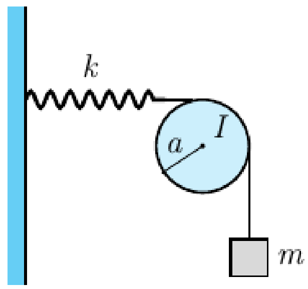


a) Determine the differential equation of motion if the entire system moves through a medium offering a damping force that is numerically equal to $\mu\frac{dx}{dt}$
b) Solve the differential equation in part (a) if the spring is stretched 4 *feet* by a mass weighing 16 *pounds* and $\mu=2,\quad h(t)=5\cos t,\quad x(0)=x'(0)=0$

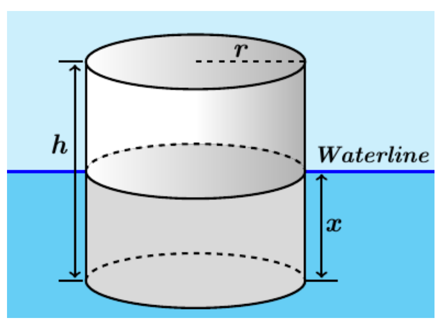
29. A mass *m* on the end of a pendulum (of length *L*) also attached to a horizontal spring (with constant *k*). Assume small oscillations of *m* so that the spring remains essentially horizontal and neglect damping. Find the natural circular frequency ω_0 of motion of the mass in terms of *L*, *k*, *m*, and the gravitational constant *g*.



30. A mass *m* hangs on the end of a cord around a pullet of radius *a* and moment of inertia *I*. The rim of the pulley is attached to a spring (with constant *k*). Assume small oscillations so that the spring remains essentially and neglect friction. Find the natural circular frequency in terms of *m*, *a*, *k*, *I*, and *g*.



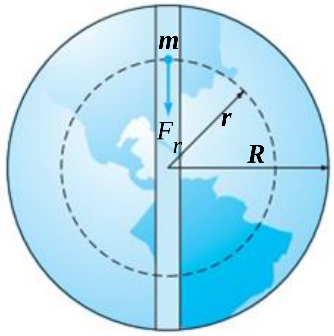
31. Consider a floating cylindrical buoy with radius *r*, height *h*, and uniform density $\rho \le 0.5$ (recall that the density of water is $1\ g/cm^3$). The buoy is initially suspended at rest with its bottom at the top surface of the water and is released at time $t = 0$. Therafter it is acted on by two forces: a downward gravitational force equal to its weight $mg = \pi r^2 hg$ and (by Archmedes' principle of buoyancy) an upward force equal to the weight $\pi r^2 xg$ of water displaced, where $x = x(t)$ is the depth of the bottom of the buoy beneath the surface at time *t*. Conclude that the buoy undergoes simple harmonic motion around its equilibrium position $x_e = \rho h$ with period $p = 2\pi\sqrt{\dfrac{\rho h}{g}}$.



a) Compute *p* and the amplitude of the motion if $\rho = 0.5\ g/cm^3$, $h = 200\ cm$, and $g = 980\ cm/s^2$

b) If the cylindrical buoy weighting 100 *lb* floats in water with its axis vertical. When depressed slightly and released, it oscillates up and down four times every 10 *sec*. assume that friction is negligible. Find the radius of the buoy.

32. Assume that the earth is a solid sphere of uniform density, with mass M and radius $R = 3960$ (*mi*). For a particle of mass m within the earth at distance r from the center of the earth, the gravitational force attracting m toward the center is $F_r = -\dfrac{GM_r m}{r^2}$, where M_r is the mass of the part of the earth within a sphere of radius r.



a) Show that $F_r = -\dfrac{GMmr}{R^3}$

b) Now suppose that a small hole is drilled straight through the center of the earth, thus connecting two antipodal points on its surface. Let a particle of mass m be dropped at time $t = 0$ into this hole with initial speed zero, and let $r(t)$ be its distance from the center of the earth at time t. conclude from Newton's second law and part (*a*) that $r''(t) = -k^2 r(t)$, where $k^2 = \dfrac{GM}{R^3} = \dfrac{g}{R}$.

c) Take $g = 32.2\ ft/s^2$, and conclude from part (*b*) that the particle undergoes simple harmonic motion back and forth between the ends of the hole, with a period of about 84 *min*.

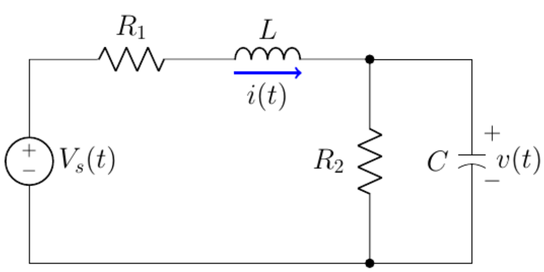
d) Look up (or derive) the period of a satellite that just skims the surface of the earth; compare with the result in part (*c*). How do you explain the coincidence? Or is it a coincidence?

e) With what speed (in miles per hours) does the particle pass through the center of the earth?

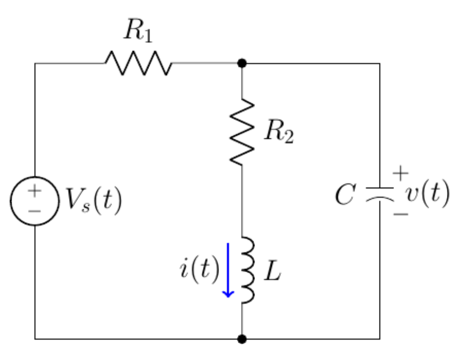
f) Look up (or derive) the orbital velocity of a satellite that just skims the surface of the earth; compare with the result in part (*e*). How do you explain the coincidence? Or is it a coincidence?

Express the given circuit in the second-order differential equation

33.



34.



35. Find the steady-state solution $q_p(t)$ and the steady-state current in and *LRC*–series circuit when the source voltage is $E(t) = E_0 \sin \omega t$

36. Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L=\frac{5}{3}\ h$, $R=10\ \Omega$, $C=\frac{1}{30}\ f$, $E(t)=300\ V$, $q(0)=0\ C$, and $i(0)=0\ A$. Find the maximum charge on the capacitor

37. Find the charge $q(t)$ on the capacitor in an *LRC*–series circuit when $L=1\ h$, $R=100\ \Omega$, $C=0.0004\ f$, $E(t)=30\ V$, $q(0)=0\ C$, and $i(0)=2\ A$. Find the maximum charge on the capacitor.

38. Find the charge $q(t)$ and $i(t)$ son the capacitor in an *LRC*–series circuit when $L=\frac{1}{2}\ h$, $R=10\ \Omega$, $C=0.01\ f$, and $E(t)=150\ V$, $q(0)=1\ C$, and $i(0)=0\ A$. What is the charge on the capacitor after a long time?

39. Find the charge $q(t)$ and $i(t)$ on the capacitor in an *LRC*–series circuit when $L=1\ h$, $R=50\ \Omega$, $C=0.0002\ f$, $E(t)=50\ V$, $q(0)=0\ C$, and $i(0)=0\ A$.

Find the steady-state charge and the steady-state current in an *LRC*–series circuit when

40. $L=1\ h,\quad R=2\ \Omega,\quad C=0.25\ f,\quad E(t)=50\cos t\ V$

41. $L=\frac{1}{2}\ h,\quad R=20\ \Omega,\quad C=0.001\ f,\quad E(t)=100\sin 60t\ V$

42. $L=\frac{1}{2}\ h,\quad R=20\ \Omega,\quad C=0.001\ f,\quad E(t)=100\sin 60t+200\cos 40t\ V$

Find the charge $q(t)$ and $i(t)$ on the capacitor in an *LC*–series circuit when

43. $E(t)=E_0\sin\omega t\ V,\quad q(0)=q_0\ C,\quad and \quad i(0)=i_0\ A$

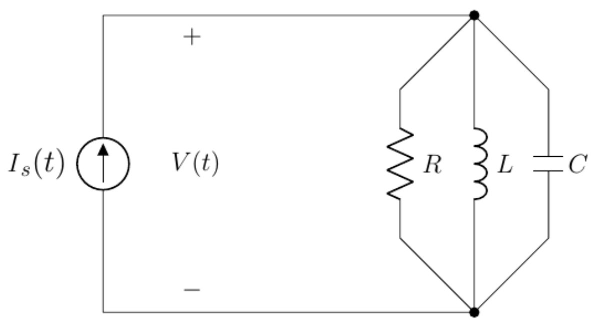
44. $E(t)=E_0\cos\omega t\ V,\quad q(0)=q_0\ C,\quad and \quad i(0)=i_0\ A$

45. $L=0.1\ h,\quad C=0.1\ f,\quad E(t)=100\sin\omega t\ V,\quad q(0)=0\ C,\quad and \quad i(0)=0\ A$

46. $L=1\ H,\quad C=4\ \mu F,\quad E(t)=3\sin 3t\ V,\quad q(0)=0\ C,\quad and \quad i(0)=0\ A$

47. $L=1\ H,\quad C=4\ \mu F,\quad E(t)=10te^{-t}\ V,\quad q(0)=0\ C,\quad and \quad i(0)=0\ A$

Consider the parallel *RLC* network. Assume that at time $t=0$, the voltage $V(t)$ and its time rate of change are both zero. Determine the voltage $V(t)$ for



48. $R = 1\ k\Omega,\quad L = 1\ H,\quad C = \frac{1}{2}\ \mu F,\quad I_s(t) = 1 - e^{-t}\ \ mA$

49. $R = 1\ k\Omega,\quad L = 1\ H,\quad C = \frac{1}{2}\ \mu F,\quad I_s(t) = 5\sin t\ \ mA$

50. $R = 1\ k\Omega,\quad L = 1\ H,\quad C = \frac{1}{2}\ \mu F,\quad I_s(t) = 5\cos t\ \ mA$

51. $R = 2\ k\Omega,\quad L = 1\ H,\quad C = \frac{1}{4}\ \mu F,\quad I_s(t) = e^{-t}\ \ mA$

52. An *RCL* circuit connected in series has $R = 180\ \Omega,\quad C = \frac{1}{280}\ F,\quad L = 20\ H$, and applied voltage $E(t) = 10\sin t\ V$. Assuming no initial charge on the capacitor, but an initial current of $1\ A$ at $t = 0$ when the voltage is first applied.

a) Find the subsequent charge on the capacitor.
b) Plot the *transient*, *steady-state*, and the charge on the capacitor.
c) Find the current on the capacitor.

53. An *RCL* circuit connected in series has $R = 10\ \Omega,\quad C = 10^{-2}\ F,\quad L = \frac{1}{2}\ H$, and applied voltage $E(t) = 12\ V$. Assuming no initial charge and no initial current at $t = 0$ when the voltage is first applied.

a) Find the subsequent charge on the capacitor.
b) Plot the *transient*, *steady-state*, and the charge on the capacitor.
c) Find the current on the capacitor.

54. An *RCL* circuit connected in series has $R = 5\ \Omega,\quad C = 4\times 10^{-4}\ F,\quad L = 0.05\ H$, and applied voltage $E(t) = 200\cos 100t\ V$. Assuming no initial charge and no initial current at $t = 0$ when the voltage is first applied.

a) Find the subsequent charge on the capacitor.
b) Plot the *transient*, *steady-state*, and the charge on the capacitor.
c) Find the current flowing through this circuit.

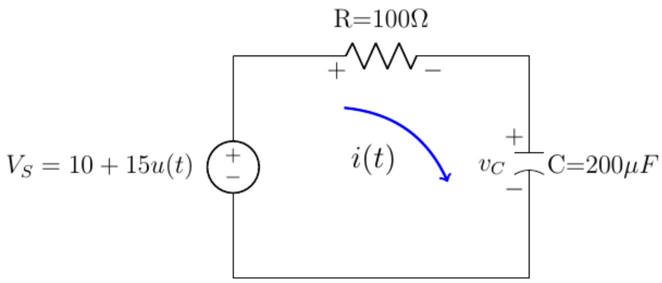
55. An *RCL* circuit connected in series has $R = 40\ \Omega,\quad C = 16\times 10^{-4}\ F,\quad L = 1\ H$, and applied voltage $E(t) = 100\cos 10t\ V$. Assuming no initial charge and no initial current at $t = 0$ when the voltage is first applied.

a) Find the charge in the circuit at time *t*.
b) Find the current flowing through this circuit.
c) Find the linit of the charge as $t \to \infty$

56. A series circuit consists of a resistor with $R = 20\ \Omega$, an inductor with $L = 1\ H$, a capacitor with $C = 0.002\ F$, and a 12-*V* battery. If the initial charge and current are both 0, find the charge and current at time *t*.

57. A series circuit consists of a resistor with $R = 20\ \Omega$, an inductor with $L = 1\ H$, a capacitor with $C = 0.002\ F$, and $E(t) = 12\sin 10t$. If the initial charge and current are both 0, find the charge and current at time *t*.

58. Consider the given circuit. Assuming that the voltage source changes from 10 to 25 V at time $t = 0$, $v_s = 10 + 15u(t)\ V$, where $u(t)$ is a unit step function.



Find the expressions that describe the voltage drop across the resistor across the capacitor and the current in the loop for $t > 0$

59. Find the steady-state solution $q_p(t)$ and the steady-state current in and *LRC*–series circuit when the source voltage is $E(t) = E_0 \sin \omega t$

60. Consider the given *RC*-circuit with impressed *emf* is $E = E_0 sin\omega t\ V$. If no initial current is flowing at $t = 0$, find the current $i(t)$ for all $t > 0$.