

Section 1.5 – Inverse, Exponential & Logarithmic Functions

One-to-One Function

A function f is one-to-one (1 – 1) if different inputs have different outputs that is,

$$\text{if } a \neq b, \quad \text{then } f(a) \neq f(b)$$

$$\text{Or } \text{if } f(a) = f(b), \quad \text{then } a = b$$

Definition of Inverse Function

Let f be one-to-one function with domain D and range R . A function g with domain R and range D is the **inverse function** of f , provided the following condition is true for every x in D and every y in R :

$$y = f(x) \quad \text{iff} \quad x = g(y)$$

If the inverse of a function f is also a function, it is named f^{-1} read “ f – inverse”

The **-1** in f^{-1} is not an exponent! And is not equal to ~~$\frac{1}{f(x)}$~~

Domain and Range of f and f^{-1}

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

Example

For the given function $f(x) = \frac{2x+3}{x+5}$

- a) Is $f(x)$ one-to-one function
- b) Find $f^{-1}(x)$, if it exists
- c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

$$a) \quad f(a) = f(b)$$

$$\frac{2a+3}{a+5} = \frac{2b+3}{b+5}$$

$$2ab + 10a + 3b + 15 = 2ab + 10b + 3a + 15$$

$$7a = 7b$$

$$a = b \quad \checkmark$$

$$f(x) \text{ is 1-1}$$

$$b) \quad y = \frac{2x+3}{x+5}$$

$$xy + 5y = 2x + 3$$

$$x(y - 2) = 3 - 5y$$

$$x = \frac{-5y + 3}{y - 2}$$

$$\underline{f^{-1}(x) = \frac{-5x + 3}{x - 2}}$$

$$c) \quad \text{Domain of } f(x) = \text{Range of } f^{-1}(x): \mathbb{R} - \{-5\}$$

$$\text{Range of } f(x) = \text{Domain of } f^{-1}(x): \mathbb{R} - \{2\}$$

Definition (Exponential Functions)

The exponential function f with base b is defined by

$$f(x) = b^x \quad \text{or} \quad y = b^x$$

Base

where $b > 0$, $b \neq 1$ and x is any real number.

Graphing Exponential

1. Define the Horizontal Asymptote $f(x) = b^x \pm d$

$$y = 0 \pm d$$

The exponential function always equals to 0

$$x \rightarrow \infty \text{ or } x \rightarrow -\infty \Rightarrow f(x) \rightarrow 0$$

2. Define/Make a table

(Force your exponential to = 0, then solve for x)

x	$f(x)$
$x - 1$	
x	
$x + 1$	

$$\text{Domain: } (-\infty, \infty)$$

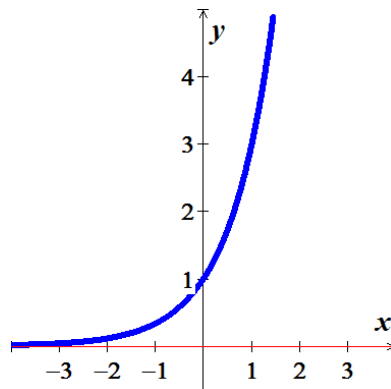
$$\text{Range: } (d, \infty)$$

Example

$$f(x) = 3^x$$

$$\text{Asymptote: } y = 0$$

x	$f(x)$
-1	1/3
0	1
1	3



Example

Sketch $f(x) = 3^{x-2}$

Solution

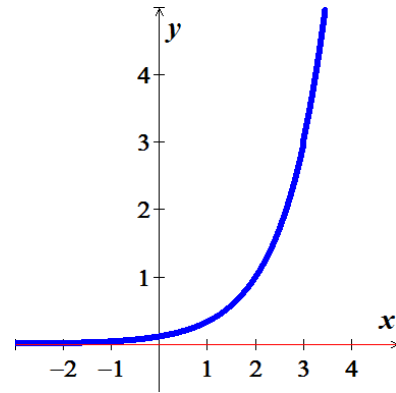
Shift right 2 unit

Asymptote: $y = 0$

Domain: \mathbb{R}

Range: $(0, \infty)$

x	$f(x)$
1	$1/3$
2	1
3	3



Example

Sketch the graph of $f(x) = 2^{-x^2}$

Solution

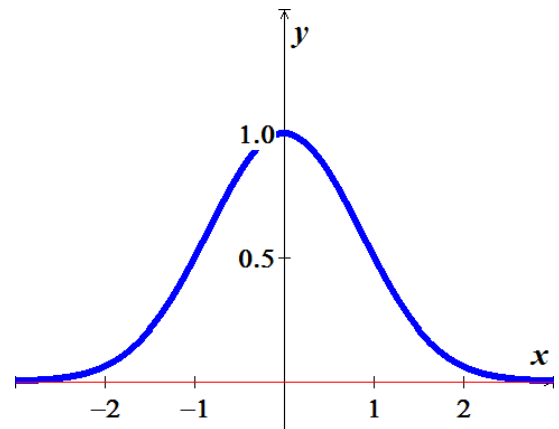
$$f(x) = \frac{1}{2^{x^2}}$$

Asymptote: $y = 0$

Domain: \mathbb{R}

Range: $(0, 1]$

x	$f(x)$
± 0	1
± 1	$\frac{1}{2}$
± 2	$\frac{1}{16}$



Natural Base e

The irrational number $e \approx 2.71828$ is called natural base

$f(x) = e^x$ is called natural exponential function

Example

Sketch $f(x) = e^{x+3} + 1$

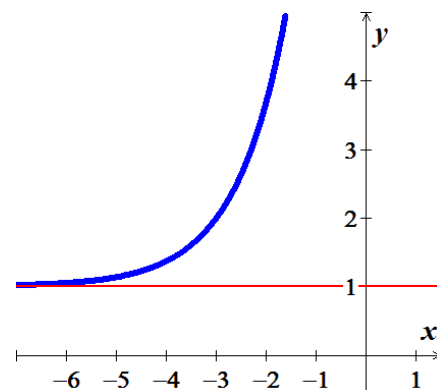
Solution

Asymptote: $y = 1$

Domain: \mathbb{R}

Range: $(1, \infty)$

x	$f(x)$
-4	1.4
-3	2
4	3.7



Logarithmic Function (*Definition*)

For $x > 0$ and $b > 0, b \neq 1$

$$y = \log_b x \text{ is equivalent to } x = b^y$$

$$\textcolor{red}{y} = \log_{\textcolor{blue}{b}} x \Leftrightarrow x = \textcolor{blue}{b}^{\textcolor{red}{y}}$$

Base

The function $f(x) = \log_b x$ is the logarithmic function with base b .

$\log_b x$: read log base b of x

$\log x$ *means* $\log_{10} x$

$\ln x$ *means* $\log_e x$ $\ln x$ read "**el en of x** "

Example

Write the equation in its equivalent exponential form:

$$\textcolor{red}{3} = \log_{\textcolor{blue}{7}} x \quad \Rightarrow x = \textcolor{blue}{7}^{\textcolor{red}{3}}$$

Write the equation in its equivalent logarithmic form:

$$2^5 = x \quad \Rightarrow 5 = \log_2 x$$

Basic Logarithmic Properties

$$\log_b b = 1 \quad \rightarrow b = b^1 \qquad \log_b 1 = 0 \quad \rightarrow 1 = b^0$$

Inverse Properties

$$\log_b b^x = x \qquad b^{\log_b x} = x$$

Change-of-Base Logarithmic

$$\log_b M = \frac{\log_a M}{\log_a b} \qquad \log_b M = \frac{\log M}{\log b} \quad \textcolor{red}{or} \quad \log_b M = \frac{\ln M}{\ln b}$$

Domain

The domain of a logarithmic function of the form $f(x) = \log_b x$ is the set of all positive real numbers.
(Inside the log has to be > 0)

Range: \mathbb{R}

Example

Find the domain of

a) $f(x) = \log_4(x - 5)$ **Domain:** $x > 5$

b) $f(x) = \ln(4 - x)$ **Domain:** $x < 4$

c) $h(x) = \ln(x^2)$ **Domain:** $\mathbb{R} - \{0\}$ or $\{x \mid x \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$

Graphs of Logarithmic Functions

Example

Graph $g(x) = \log(x - 2) + 1$

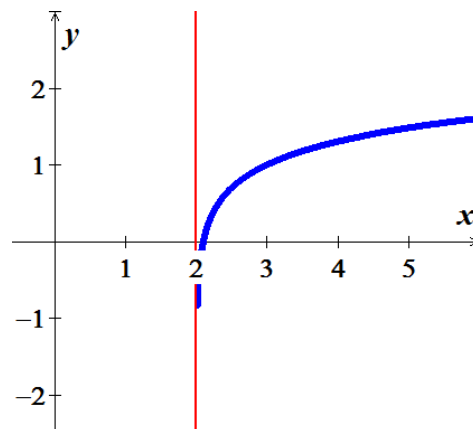
Solution

Asymptote: $x = 2$

Domain: $x > 2$

Range: \mathbb{R}

x	$g(x)$
2	
2.5	.7
3	1
4	1.3



Example

Graph $f(x) = \log_3 |x|$ for $x \neq 0$

Solution

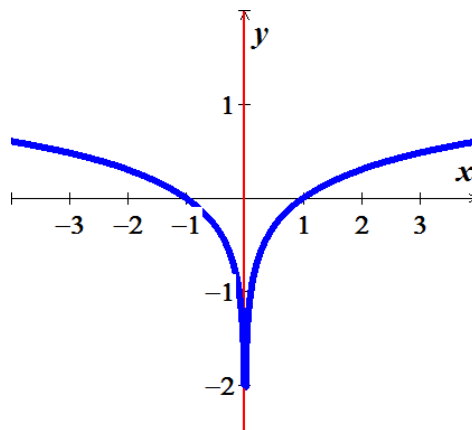
$$f(-x) = \log_3 |-x| = \log_3 |x| = f(x)$$

\therefore The graph is symmetric with respect to the y-axis.

Asymptote: $x = 0$

Domain: $\mathbb{R} - \{0\}$

Range: \mathbb{R}



Exercises Section 1.5 – Inverse, Exponential & Logarithmic Functions

(1 – 9) Determine whether the function is *one-to-one*

1. $f(x) = 3x - 7$

4. $f(x) = \sqrt[3]{x}$

7. $f(x) = (x - 2)^3$

2. $f(x) = x^2 - 9$

5. $f(x) = |x|$

8. $y = x^2 + 2$

3. $f(x) = \sqrt{x}$

6. $f(x) = \frac{2}{x+3}$

9. $f(x) = \frac{x+1}{x-3}$

10. Given the function $f(x) = (x+8)^3$

a) Find $f^{-1}(x)$

b) Graph f and f^{-1} in the same rectangular coordinate system

c) Find the domain and the range of f and f^{-1}

(11 – 38) For the given functions

a) Is $f(x)$ one-to-one function

b) Find $f^{-1}(x)$, if it exists

c) Find the domain and range of $f(x)$ and $f^{-1}(x)$

11. $f(x) = \frac{2x}{x-1}$

20. $f(x) = \frac{3x-1}{x-2}$

30. $f(x) = 2 - 3x^2; \quad x \leq 0$

12. $f(x) = \frac{x}{x-2}$

21. $f(x) = \frac{3x-2}{x+4}$

31. $f(x) = 2x^3 - 5$

13. $f(x) = \frac{x+1}{x-1}$

22. $f(x) = \frac{-3x-2}{x+4}$

32. $f(x) = \sqrt{3-x}$

14. $f(x) = \frac{2x+1}{x+3}$

23. $f(x) = \sqrt{x-1} \quad x \geq 1$

33. $f(x) = \sqrt[3]{x} + 1$

15. $f(x) = \frac{3x-1}{x-2}$

24. $f(x) = \sqrt{2-x} \quad x \leq 2$

34. $f(x) = (x^3 + 1)^5$

16. $f(x) = \frac{2x}{x-1}$

25. $f(x) = x^2 + 4x \quad x \geq -2$

35. $f(x) = x^2 - 6x; \quad x \geq 3$

17. $f(x) = \frac{x}{x-2}$

26. $f(x) = 3x + 5$

36. $f(x) = (x-2)^3$

18. $f(x) = \frac{x+1}{x-1}$

27. $f(x) = \frac{1}{3x-2}$

37. $f(x) = \frac{x+1}{x-3}$

19. $f(x) = \frac{2x+1}{x+3}$

28. $f(x) = \frac{3x+2}{2x-5}$

38. $f(x) = \frac{2x+1}{x-3}$

29. $f(x) = \frac{4x}{x-2}$

39. Simplify the expression $\frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$

40. Simplify the expression $\frac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{(e^x + e^{-x})^2}$

(41 – 52) Write the equation in its equivalent logarithmic form

41. $2^6 = 64$

45. $b^3 = 343$

49. $\left(\frac{1}{2}\right)^{-5} = 32$

42. $5^4 = 625$

46. $8^y = 300$

50. $e^{x-2} = 2y$

43. $5^{-3} = \frac{1}{125}$

47. $\sqrt[n]{x} = y$

51. $e = 3x$

44. $\sqrt[3]{64} = 4$

48. $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$

52. $\sqrt[3]{e^{2x}} = y$

(53 – 64) Write the equation in its equivalent exponential form

53. $\log_5 125 = y$

57. $\log_6 \sqrt{6} = x$

61. $\log_{\sqrt{3}} 81 = 8$

54. $\log_4 16 = x$

58. $\log_3 \frac{1}{\sqrt{3}} = x$

62. $\log_4 \frac{1}{64} = -3$

55. $\log_5 \frac{1}{5} = x$

59. $6 = \log_2 64$

63. $\log_4 26 = y$

56. $\log_2 \frac{1}{8} = x$

60. $2 = \log_9 x$

64. $\ln M = c$

(65 – 71) Evaluate the expression without using a calculator

65. $\log_4 16$

67. $\log_6 \sqrt{6}$

69. $\log_3 \sqrt[7]{3}$

71. $\log_{\frac{1}{2}} \sqrt{\frac{1}{2}}$

66. $\log_2 \frac{1}{8}$

68. $\log_3 \frac{1}{\sqrt{3}}$

70. $\log_3 \sqrt{9}$

(72 – 80) Simplify

72. $\log_5 1$

75. $10^{\log 3}$

78. $\ln e^{x-5}$

73. $\log_7 7^2$

76. $e^{2+\ln 3}$

79. $\log_b b^n$

74. $3^{\log_3 8}$

77. $\ln e^{-3}$

80. $\ln e^{x^2+3x}$

(81 – 108) Find the domain of

81. $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

82. $f(x) = \frac{e^{|x|}}{1 + e^x}$

83. $f(x) = \sqrt{1 - e^x}$

84. $f(x) = \sqrt{e^x - e^{-x}}$

85. $f(x) = \log_5(x + 4)$

86. $f(x) = \log_5(x + 6)$

87. $f(x) = \log(2 - x)$

88. $f(x) = \log(7 - x)$

89. $f(x) = \ln(x - 2)^2$

90. $f(x) = \ln(x - 7)^2$

91. $f(x) = \log(x^2 - 4x - 12)$

92. $f(x) = \log\left(\frac{x-2}{x+5}\right)$

93. $f(x) = \log\left(\frac{3-x}{x-2}\right)$

94. $f(x) = \ln\left(\frac{x^2}{x-4}\right)$

95. $f(x) = \log_3(x^3 - x)$

96. $f(x) = \log \sqrt{2x - 5}$

97. $f(x) = 3 \ln(5x - 6)$

98. $f(x) = \log\left(\frac{x}{x-2}\right)$

99. $f(x) = \ln(x^2 + 4)$

100. $f(x) = \ln|4x - 8|$

101. $f(x) = \ln(x^2 - 9)$

102. $f(x) = \ln|5 - x|$

103. $f(x) = \ln(x - 4)^2$

104. $f(x) = \ln(x^2 - 4)$

105. $f(x) = \ln(x^2 - 4x + 3)$

106. $f(x) = \ln(2x^2 - 5x + 3)$

107. $f(x) = \log(x^2 + 4x + 3)$

108. $f(x) = \ln(x^4 - x^2)$

(109 – 129) Find the *asymptote*, *domain*, and *range* of the given functions. Then, sketch the graph

109. $f(x) = 2^x + 3$

110. $f(x) = 2^{3-x}$

111. $f(x) = \left(\frac{2}{5}\right)^{-x}$

112. $f(x) = -\left(\frac{1}{2}\right)^x + 4$

113. $f(x) = 4^x$

114. $f(x) = 2 - 4^x$

115. $f(x) = -3 + 4^{x-1}$

116. $f(x) = 1 + \left(\frac{1}{4}\right)^{x+1}$

117. $f(x) = e^{x-2}$

118. $f(x) = 3 - e^{x-2}$

119. $f(x) = e^{x+4}$

120. $f(x) = 2 + e^{x-1}$

121. $f(x) = \log_4(x - 2)$

122. $f(x) = \log_4|x|$

123. $f(x) = \left(\log_4 x\right) - 2$

124. $f(x) = \log(3 - x)$

125. $f(x) = 2 - \log(x + 2)$

126. $f(x) = \ln(x - 2)$

127. $f(x) = \ln(3 - x)$

128. $f(x) = 2 + \ln(x + 1)$

129. $f(x) = 1 - \ln(x - 2)$

- 130.** On a study by psychologists Bornstein and Bornstein, it was found that the average walking speed w , in feet per second, of a person living in a city of population P , in *thousands*, is given by the function:

$$w(P) = 0.37 \ln P + 0.05$$

- a) The population is 124,848. Find the average walking speed of people living in Hartford.
- b) The population is 1,236,249. Find the average walking speed of people living in San Antonio.

- 131.** The loudness of sounds is measured in a unit called a decibel. To measure with this unit, we first assign an intensity of I_0 to a very faint sound, called the threshold sound. If a particular sound has intensity I , then the decibel rating of this louder sound is

$$d = 10 \log \frac{I}{I_0}$$

Find the exact decibel rating of a sound with intensity $10,000I_0$

- 132.** Students in an accounting class took a final exam and then took equivalent forms of the exam at monthly intervals thereafter. The average score $S(t)$, as a percent, after t months was found to be given by the function

$$S(t) = 78 - 15 \log(t+1); \quad t \geq 0$$

- a) What was the average score when the students initially took the test, $t = 0$?
- b) What was the average score after 4 *months*? 24 *months*?

- 133.** Starting on the left side of a standard 88–key piano, the frequency, in *vibrations per second*, of the n th note is given by

$$f(n) = (27.5)^{2^{\frac{n-1}{12}}}$$



- a) Determine the frequency of middle C, key number 40 on an 88–key piano.
- b) Is the difference in frequency between middle C (key number 40) and D (key number 42) the same as the difference in frequency between D (key number 42) and E (key number 44)?