Lecture One

Section 1.1 – System of Equations

Definition ***Linear equations***

A set of equations is called a system of equations, when a model requires finding the solutions of 2 or more equations.

Types of Solutions

- ✓ Unique One Solution
- ✓ No Solution / Inconsistent
- ✓ Unique Infinite Solutions (*Depend*)

Transformations

An equivalent system is one that has the same solutions as the given system
The following transformations can be applied to a system of equations to get an equivalent system.

1. Exchanging any 2 equations
2. Multiplying both sides of an equation by any nonzero real number
3. Replacing any equation by a nonzero multiple of that equation plus a nonzero multiple by any other equation.

Example

$$\begin{aligned} 2x+3y&=12 \quad (1)\\ 3x-4y&=1 \quad (2) \end{aligned}$$

<u>*Solution*</u>

$$\begin{aligned} 3(2x+3y=12)\\ -2(3x-4y=1) \end{aligned} \qquad \begin{array}{r} 6x+9y=36\\ -6x+8y=-2\\ \hline 17y=34 \end{array}$$

$$\begin{aligned} 2x+3y&=12\\ 17y&=34 \quad \tfrac{1}{17}R_2 \end{aligned} \qquad \Rightarrow y=\frac{34}{17}=2$$

$$\tfrac{1}{2}R_1 \qquad x+\frac{3}{2}y=6$$

$$x+\frac{3}{2}2=6$$

$$x+3=6 \qquad \Rightarrow x=3$$

Solution: (3, 2)

Example

A restaurant owner orders a replacement set of knives, forks, and spoons. The box arrives containing 40 utensils and weighting 141.3 oz (ignoring the weight of the box). A knife, fork, and spoon weigh 3.9 oz, 3.6 oz, and 3.0 oz, respectively.
How many solutions are there for the number of knives, forks and spoons in the box?

Solution

Let: x: Number of knives
y: Number of forks
z: Number of spoons

	Knifes	**Forks**	**Spoons**	**Total**
Number	x	y	z	40
Weight	3.9	3.6	3	141.3

$$\begin{aligned} x+y+z&=40 \\ 3.9x+3.6y+3z&=141.3 \end{aligned}$$

$$3.9R_1 - R_2 \to R_2 \quad \begin{array}{r} 3.9x+3.9y+3.9z=156 \\ \underline{-3.9x-3.6y-3z=-141.3} \\ .3y+.9z=14.7 \end{array}$$

$$.3y+.9z=14.7$$

$$.3y=14.7-.9z$$

$$y=\frac{14.7}{.3}-\frac{.9}{.3}z$$

$$y=49-3z$$

$$\begin{aligned} x+y+z&=40 \\ x&=40-y-z \\ &=40-(49-3z)-z \\ &=40-49+3z-z \\ &=2z-9 \end{aligned}$$

$$(2z-9,\ 49-3z,\ z)$$

$$\begin{array}{ll} y=49-3z\ge 0 & x=2z-9\ge 0 \\ 49\ge 3z & 2z\ge 9 \\ 3z\le 49 & z\ge 4.5 \\ z\le 16. & \end{array}$$

Values of ***z***: 5, 6, 7, …., 16

Matrices

$$\begin{array}{cc} & \text{Column} \\ & \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{array}{c} \text{Row } 1 \to R_1 \\ \text{Row } 2 \to R_2 \\ \text{Row } 3 \to R_3 \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

This is called Matrix (*Matrices*)

Each number in the array is an ***element*** or ***entry***

The matrix is said to be of order $\boldsymbol{m}$ *x* $\boldsymbol{n}$

$\boldsymbol{m}$: numbers of rows,

$\boldsymbol{n}$: number of columns

When $\boldsymbol{m} = \boldsymbol{n}$, then matrix is said to be ***square***.

Matrices *Size*

a) $\begin{bmatrix} -3 & 5 \\ 2 & 0 \\ 5 & -1 \end{bmatrix}$ 3*x*2 matrix

b) $\begin{bmatrix} 1 & 6 & 5 & -2 & 5 \end{bmatrix}$ 1*x*5 matrix

c) $\begin{bmatrix} 3 \\ -5 \\ 0 \\ 2 \end{bmatrix}$ 5*x*1 matrix

- A matrix containing only 1 row is called a *row matrix* or *row vector*
- A matrix containing only 1 column is called a *column matrix* or *column vector*
- Two matrices are equal if they are the same size and if each pair corresponding elements is equal

Example

a) $\begin{bmatrix} 2 & 1 \\ p & q \end{bmatrix} = \begin{bmatrix} x & y \\ -1 & 0 \end{bmatrix}$ $x = 2,\ y = 1,\ \ p = -1,\ \ q = 0$

b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$ *can't be true*

Addition

The sum of two $m\times n$ matrices X and Y is the $m\times n$ matrices $X + Y$ in which each element is the sum of the corresponding elements of X and Y.

Example

a) $\begin{bmatrix} 5 & -6 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} 5-4 & -6+6 \\ 8+8 & 9-3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 16 & 6 \end{bmatrix}$

b) $\begin{bmatrix} 5 & -8 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -9 & 1 \\ 4 & 2 & -5 \end{bmatrix} = \text{ } \textit{doesn't exist}$

Product of a Matrix and a *Scalar*

The product of a scalar k and a matrix X is the matrix kX, each of whose elements is k times the corresponding element of X.

Example

$$(-5)\begin{bmatrix} 3 & 4 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3(-5) & 4(-5) \\ 0(-5) & -1(-5) \end{bmatrix} = \begin{bmatrix} -15 & -20 \\ 0 & 5 \end{bmatrix}$$

Product **of Two Matrices**

Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix. To find the element in the i^{th} row and j^{th} column of the product matrix AB, multiply each element in the i^{th} row of A by the corresponding element in the j^{th} column of B, and then add these products. The product matrix AB is an $m \times k$ matrix.

Matrix ***A*** *Matrix B*

$m \times n$ $n \times k$

Inner *must be equal*

Outer: Order of AB is $m \times k$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$2x2 \qquad 2x2 \quad \rightarrow \quad 2x2$$

$$a_{11} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & \_ \\ \_ & \_ \end{bmatrix}$$

$$a_{12} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & af + bh \\ \_ & \_ \end{bmatrix}$$

$$a_{21} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ ce + dg & \_ \end{bmatrix}$$

$$a_{22} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} \_ & \_ \\ \_ & cf + dh \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Example

Find the product CD of matrices $C = \begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix}$

<u>Solution</u>

$$\begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -3(-6) + 4(2) + 2(3) & -3(4) + 4(3) + 2(-2) \\ 5(-6) + 0(2) + 4(3) & 5(4) + 0(3) + 4(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 32 & -4 \\ -18 & 12 \end{bmatrix}$$

Example – Home Construction

A contractor builds three kinds of houses, models *A*, *B*, and *C*, with a choice of two styles, Spanish and contemporary. Matrix *P* shows the number of each kind of house planned for a new 100-home subdivision. The amounts for each of the exterior materials depend primarily on the style of the house. These amounts are shown in matrix *Q*. (concrete is in cubic yards, lumber in units of 1000 board feet, brick in 1000s, and shingles in units of 100 ft^2.) Matrix *R* gives the cost in dollars for each kind of material.

a) What is the total cost of these materials for each model?

b) How much of each of four kinds of material must be ordered

c) What is the total cost for exterior materials?

<u>Solution</u>

$$\begin{array}{r} \\ \text{Model } A \\ \text{Model } B \\ \text{Model } C \end{array}\begin{array}{c} \begin{array}{cc} \text{Spanish} & \text{Contemporary} \end{array} \\ \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} \end{array} = P \qquad \begin{array}{r} \\ \text{Spanish} \\ \text{Contemporary} \end{array}\begin{array}{c} \begin{array}{cccc} \text{Concrete} & \text{Lumber} & \text{Brick} & \text{Shingles} \end{array} \\ \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix} \end{array} = Q$$

$$\begin{array}{r} \\ \text{Concrete} \\ \text{Lumber} \\ \text{Brick} \\ \text{Shingles} \end{array}\begin{array}{c} \text{Cost per unit} \\ \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} \end{array} = R$$

a) What is the total cost of these materials for each model?

$$PQ = \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix}\begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} \text{Concrete} & \text{Lumber} & \text{Brick} & \text{Shingles} \end{array} \\ \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \end{array}\begin{array}{l} \\ \text{Model } A \\ \text{Model B} \\ \text{Model C} \end{array}$$

$$(PQ)R = \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix}\begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} = \begin{bmatrix} 72{,}900 \\ 54{,}700 \\ 60{,}800 \end{bmatrix}\begin{array}{l} \text{Model } A \\ \text{Model } B \\ \text{Model } C \end{array}$$

The total cost of materials is \$72,900 for model *A*, \$54,700 for model *B*, \$60,800 for model *C*.

b) How much of each of four kinds of material must be ordered

$$\begin{array}{cccc} 1500 & 30 & 600 & 60 \\ 100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \\ \hline 3800 & 130 & 1400 & 200 \end{array} \qquad T = \begin{bmatrix} 3800 & 130 & 1400 & 200 \end{bmatrix}$$

3800 yd^3 of concrete, 130,000 board feet of lumber, 1,400,000 bricks, and 20,000 ft^2 of shingles are needed.

c) What is the total cost for exterior materials?

$$TR = \begin{bmatrix} 3800 & 130 & 1400 & 200 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} = \begin{bmatrix} 188,400 \end{bmatrix}$$

The total cost for exterior materials is $188,400

The *Multiplicative Identity* Matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \boxed{AI = IA = A}$$

Multiplicative Identity matrix ***I*** of order *n* is unique and given by:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2x2} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3x3} \qquad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplicative *Inverse* of a matrix

Multiplicative inverse of a matrix A_{nxn} and $A^{-1}{}_{nxn}$ if exists, then: $\boxed{A \cdot A^{-1} = A^{-1} \cdot A = I}$

Example

Given: $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$ Find A^{-1}

Solution

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right] \quad R_2 + R_1$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 5 & 1 & 1 \end{array}\right] \quad \tfrac{1}{5} R_2$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array}\right] \quad R_1 - 3R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & -\frac{3}{5} \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array}\right] \qquad A^{-1} = \begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \Rightarrow \quad A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad-bc=0$, then A^{-1} doesn't exist

$$A=\begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix} \ \rightarrow \ Find \ A^{-1}$$

$$\Rightarrow A^{-1}=\frac{1}{(-1)(4)-(-2)(3)}\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}=\frac{1}{2}\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Example

Find A^{-1} *if* $A=\begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

<u>*Solution*</u>

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-3R_1 \end{array}$$

$$\begin{array}{rrrrrr} 2 & -2 & -1 & 0 & 1 & 0 \\ -2 & 0 & -2 & -2 & 0 & 0 \\ \hline 0 & -2 & -3 & -2 & 1 & 0 \end{array} \qquad \begin{array}{rrrrrr} 3 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & -3 & -3 & 0 & 0 \\ \hline 0 & 0 & -3 & -3 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \ -\tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \ -\tfrac{1}{3}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \begin{array}{l} R_1-R_3 \\ R_2-\frac{3}{2}R_3 \\ \\ \end{array}$$

$$\begin{array}{rrrrrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & \frac{1}{3} \\ \hline 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{array} \qquad \begin{array}{rrrrrr} 0 & 1 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & 0 & \frac{1}{2} \\ \hline 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] \qquad A^{-1}=\begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

Solving Systems of Equations with Inverses

To solve the matrix equation $AX = B$.

- X: matrix of the variables
- A: Coefficient matrix
- B: Constant matrix

$$AX = B$$

$$A^{-1}(AX) = A^{-1}B \qquad \textit{Multiply both side by } A^{-1}$$

$$\left(A^{-1}A\right)X = A^{-1}B \qquad \textit{Associate property}$$

$$IX = A^{-1}B \qquad \textit{Multiplicative inverse property}$$

$$X = A^{-1}B \qquad \textit{Identity property}$$

Example

Use the inverse of the coefficient matrix to solve the linear system

$$2x - 3y = 4$$
$$x + 5y = 2$$

Solution

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The solution of the system is $(2, 0)$

Example

Three brands of fertilizer are available that provide nitrogen, phosphoric acid, and soluble potash to the soil. One bag of each brand provides the following units of each nutrient.

		Brand		
		Fertifun	***Big Grow***	***Soakem***
Nutrient	*Nitrogen*	1	2	3
	Phosphoric acid	3	1	2
	Potash	2	0	1

For ideal growth, the soil on a Michigan farm needs 18 units of nitrogen, 23 units of phosphoric acid, and 13 units of potash per acre. The corresponding numbers for a California farm are 31, 24, and 11, and for Kansas farm are 20, 19, and 15. How many bags of each brand of fertilizer should be used per acre for ideal growth on each farm?

<u>*Solution*</u>

$$\begin{aligned} x+2y+3z&=b_1\\ 3x+y+2z&=b_2\\ 2x\qquad +z&=b_3 \end{aligned}\qquad A=\begin{bmatrix}1&2&3\\3&1&2\\2&0&1\end{bmatrix}\qquad \Rightarrow\quad A^{-1}=\begin{bmatrix}-\frac{1}{3}&\frac{2}{3}&-\frac{1}{3}\\-\frac{1}{3}&\frac{5}{3}&-\frac{7}{3}\\\frac{2}{3}&-\frac{4}{3}&\frac{5}{3}\end{bmatrix}$$

For Michigan farm: $B=\begin{bmatrix}18\\23\\13\end{bmatrix}\quad\Rightarrow\quad X=A^{-1}B=\begin{bmatrix}-\frac{1}{3}&\frac{2}{3}&-\frac{1}{3}\\-\frac{1}{3}&\frac{5}{3}&-\frac{7}{3}\\\frac{2}{3}&-\frac{4}{3}&\frac{5}{3}\end{bmatrix}\begin{bmatrix}18\\23\\13\end{bmatrix}=\begin{bmatrix}5\\2\\3\end{bmatrix}$

For California farm: $B=\begin{bmatrix}31\\24\\11\end{bmatrix}\quad\Rightarrow\quad X=\begin{bmatrix}-\frac{1}{3}&\frac{2}{3}&-\frac{1}{3}\\-\frac{1}{3}&\frac{5}{3}&-\frac{7}{3}\\\frac{2}{3}&-\frac{4}{3}&\frac{5}{3}\end{bmatrix}\begin{bmatrix}31\\24\\11\end{bmatrix}=\begin{bmatrix}2\\4\\7\end{bmatrix}$

For Kansas farm: $B=\begin{bmatrix}20\\19\\15\end{bmatrix}\quad\Rightarrow\quad X=\begin{bmatrix}-\frac{1}{3}&\frac{2}{3}&-\frac{1}{3}\\-\frac{1}{3}&\frac{5}{3}&-\frac{7}{3}\\\frac{2}{3}&-\frac{4}{3}&\frac{5}{3}\end{bmatrix}\begin{bmatrix}20\\19\\15\end{bmatrix}=\begin{bmatrix}1\\-10\\13\end{bmatrix}$

(-10) *is impossible to have a negative bags*.

Exercises *Section* 1.1 – System of Equations

1. Solve the system $\begin{cases} 2x+y-z=2 & (1) \\ x+3y+2z=1 & (2) \\ x+y+z=2 & (3) \end{cases}$

2. Solve the system $\begin{cases} 3x_1+x_2-2x_3=2 \\ x_1-2x_2+x_3=3 \\ 2x_1-x_2-3x_3=3 \end{cases}$

3. Solve the system: $\begin{cases} 2x_1-2x_2+x_3=3 \\ 3x_1+x_2-x_3=7 \\ x_1-3x_2+2x_3=0 \end{cases}$

4. Katherine invests \$10,000 received from her grandmother in three ways. With one part, she buys U.S saving bonds at an interest rate of 2.5% per year. She uses the second part, which amounts to twice the first, to buy mutual funds that offer a return of 6% per year. She puts the rest of the money into a money market account paying 4.5% annual interest. The first year her investments bring a return of \$470. How much did she invest in each way?

5. A Company that rents small moving trucks wants to purchase 25 trucks with a combined capacity of 28,000 cubic ft. Three different types of trucks are available: a 10-foot truck with a capacity of 350 cubic feet, a 14-foot truck with a capacity of 700 cubic feet, and a 24-foot truck with a capacity of 1,400 cubic feet. How many truck of each type of truck should the company purchase?

6. A flight leaves New York at 8 PM and arrives in Paris at 9 AM (Paris time). This 13-hour difference includes the flight time plus the change in time zones. The return leaves Paris at 1 PM and arrives in New York at 3 PM (NY time). This 2-hour difference includes the flight time minus time zones, plus an extra hour due to the fact that flying westward is against the wind. Find the actual flight time eastward and the difference in time zones.

7. Find the variables, if possible: $\begin{bmatrix} x & y+3 \\ 2z & 8 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 6 & 8 \end{bmatrix}$

8. Find the variables, if possible: $\begin{bmatrix} a+2 & 3b & 4c \\ d & 7f & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$

9. Evaluate: $\begin{bmatrix} 2 & 4 & 5 & -7 \\ 6 & -3 & 12 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 0 & -10 & 1 \\ -2 & 8 & -9 & 11 \end{bmatrix}$

10. Evaluate: $\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix}$

11. Evaluate: $\begin{bmatrix} 5 & 4 \\ -3 & 7 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix}$

12. Find: $-4\begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix} + 5\begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix}$

13. Find: $\begin{bmatrix} 2 & 2 & -1 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ -1 & 4 \\ 0 & 2 \end{bmatrix}$

14. Find: $\begin{bmatrix} 2 \\ -9 \\ 12 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$

15. Find: $\begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$

16. Find: $\begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$

17. Given: $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ $\quad F = \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}$ Find: $3F + 2A$

18. Given: $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 5 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & -3 & 6 \\ -3 & 1 & -4 \end{bmatrix}$ Find:

a) $A - B$

b) $3A + 2B$

19. Given: $A = \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & -1 & 4 \\ 4 & -1 & 3 \\ 2 & 0 & -2 \end{bmatrix}$ Find: AB *and* BA

20. Sal's Shoes and Fred's Footwear both have outlets in California and Arizona. Sal's sells shoes for \$80, sandals for \$40, and boots for \$120. Fred's prices are \$60, \$30, and \$150 for shoes, sandals and boots, respectively. Half of all sales in California stores are shoes, 1/4 are sandals, and 1/4 are boots. In Arizona the fractions are 1/5 shoes, 1/5 are sandals, and 3/5 are boots.

a) Write a 2 *x* 3 matrix called P representing prices for the two stores and three types of footwear.

b) Write a 2 *x* 3 matrix called F representing fraction of each type of footwear sold in each state.

c) Only one of the two products PF and FP is meaningful. Determine which one it is, calculate the product, and describe what the entries represent.

21. Use the inverse of the coefficient matrix to solve the linear system

$$\begin{aligned} 2x+5y&=15 \\ x+4y&=9 \end{aligned}$$

22. Find the inverse of: $A=\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

23. Find the inverse of: $A=\begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$

24. Find the inverse of: $M=\begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

25. Use the inverse of the coefficient matrix to solve the linear system: $\begin{cases} 3x_1 - x_2 + x_3 = 3 \\ -x_1 + x_2 = -3 \\ x_1 + x_3 = 2 \end{cases}$

26. An investment advisor currently has two types of investment available for clients: a conservative investment A that pays 8% per year and an investment B paying 24% per year. Clients may divide their investments between the two to achieve any total return desired between 8% and 24%. However, the higher the desired return, the higher the risk. How should each client listed in the table invest to achieve the indicated return?