

Section 2.4 – Partial Fractions

This section shows how to express a rational function as a sum of simpler functions, called *partial fractions*.

Example

Evaluate $\int \frac{5x-3}{x^2-2x-3} dx$

Solution

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$5x-3 = (A+B)x - 3A + B$$

$$x \quad A+B=5$$

$$x^0 \quad -3A+B=-3$$

$$A+B=5$$

$$\begin{array}{r} - \\ 3A-B=3 \end{array}$$
$$\hline 4A=8$$

$$\underline{A=2, \quad B=3}$$

$$\int \frac{5x-3}{x^2-2x-3} dx = \int \left(\frac{2}{x+1} + \frac{3}{x-3} \right) dx$$
$$\underline{= 2\ln|x+1| + 3\ln|x-3| + C}$$

Example

Use partial fractions to evaluate $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)} dx$

Solution

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

$$x^2+4x+1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$$
$$= Ax^2 + 4Ax + 3A + Bx^2 + 2Bx - 3B + Cx^2 - C$$

$$x^2 \quad A+B+C=1$$

$$x \quad 4A+2B=4$$

$$x^0 \quad 3A-3B-C=1$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 3 & -3 & -1 \end{vmatrix} = -16 \qquad \Delta_A = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 1 & -3 & -1 \end{vmatrix} = -12$$

$$\Delta_B = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & 0 \\ 3 & 1 & -1 \end{vmatrix} = -8 \qquad \Delta_C = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 4 \\ 3 & -3 & 1 \end{vmatrix} = 4$$

$$A = \frac{-12}{-16} = \underline{\underline{\frac{3}{4}}} \quad B = \frac{-8}{-16} = \underline{\underline{\frac{1}{2}}} \quad C = \frac{4}{-16} = \underline{\underline{-\frac{1}{4}}}$$

$$\begin{aligned} \int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)} dx &= \int \left(\frac{3}{4} \frac{1}{x-1} + \frac{1}{2} \frac{1}{x+1} - \frac{1}{4} \frac{1}{x+3} \right) dx \\ &= \underline{\underline{\frac{3}{4} \ln|x-1| + \frac{1}{2} \ln|x+1| - \frac{1}{4} \ln|x+3| + K}} \end{aligned}$$

Method of Partial Fractions ($f(x)/g(x)$ *Proper*)

1. Let $(x-r)$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $(x-r)$ that divides $g(x)$. Then,

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \dots + \frac{A_m}{(x-r)^m}$$

2. Let $x^2 + px + q$ be an irreducible quadratic function of $g(x)$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power. Then

$$\frac{B_1x + C_1}{(x^2 + px + q)} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \dots + \frac{B_nx + C_n}{(x^2 + px + q)^n}$$

3. Set the original fraction $\frac{f(x)}{g(x)}$ equal to the sum of these partial fractions.
4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Example

Use partial fractions to evaluate $\int \frac{6x+7}{(x+2)^2} dx$

Solution

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$6x+7 = A(x+2) + B$$
$$= Ax + 2A + B$$

$$\rightarrow \begin{cases} \boxed{A=6} \\ 2A + B = 7 \rightarrow \boxed{B = 7 - 12 = -5} \end{cases}$$

$$\int \frac{6x+7}{(x+2)^2} dx = \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2} \right) dx$$
$$= \int \frac{6}{x+2} dx - 5 \int (x+2)^{-2} d(x+2)$$

$$d(x+2) = dx$$

$$= 6 \ln|x+2| + 5(x+2)^{-1} + C$$

$$= \underline{6 \ln|x+2| + \frac{5}{x+2} + C}$$

Example

Use partial fractions to evaluate $\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx$

Solution

$$\begin{array}{r} 2x \\ x^2 - 2x - 3 \overline{) 2x^3 - 4x^2 - x - 3} \\ \underline{2x^3 - 4x^2 - 6x} \\ 5x - 3 \end{array}$$

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}$$

$$5x - 3 = (A + B)x - 3A + B$$

$$x \quad A + B = 5$$

$$x^0 \quad -3A + B = -3$$

$$A + B = 5$$

$$\begin{array}{r} - \quad 3A - B = 3 \\ \hline 4A = 8 \end{array}$$

$$\boxed{A = 2, \quad B = 3}$$

$$\begin{aligned} \int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx &= \int 2x \, dx + \int \frac{5x - 3}{x^2 - 2x - 3} \, dx \\ &= \int 2x \, dx + \int \frac{2}{x + 1} \, dx + \int \frac{3}{x - 3} \, dx \\ &= x^2 + 2 \ln|x + 1| + 3 \ln|x - 3| + C \end{aligned}$$

Example

Use partial fractions to evaluate $\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx$

Solution

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}$$

$$-2x + 4 = (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1)$$

$$\begin{aligned}
&= (Ax + B)(x^2 - 2x + 1) + C(x^3 - x^2 + x - 1) + Dx^2 + D \\
&= (A + C)x^3 + (-2A + B - C + D)x^2 + (A - 2B + C)x + B - C + D
\end{aligned}$$

$$x^3 \quad A + C = 0 \quad \rightarrow C = -A$$

$$x^2 \quad -2A + B - C + D = 0$$

$$x \quad A - 2B + C = -2 \quad \rightarrow -2B = -2 \quad \underline{B = 1}$$

$$x^0 \quad B - C + D = 4$$

$$-A + D = -1$$

$$\frac{A + D = 3}{2D = 2} \rightarrow \underline{D = 1} \quad \underline{A = 2}$$

$$\rightarrow \underline{C = -2}$$

$$\begin{aligned}
\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx &= \int \left(\frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \right) dx \\
&= \int \left(\frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \right) dx \\
&= \ln(x^2 + 1) + \tan^{-1} x - 2 \ln|x - 1| - \frac{1}{x - 1} + K
\end{aligned}$$

Example

Use partial fractions to evaluate $\int \frac{dx}{x(x^2 + 1)^2}$

Solution

$$\frac{1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

$$1 = A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + x(Dx + E)$$

$$1 = (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A$$

$$\begin{cases}
x^4 & A + B = 0 \quad \rightarrow \underline{B = -1} \\
x^3 & \underline{C = 0} \\
x^2 & 2A + B + D = 0 \quad \rightarrow \underline{D = -1} \\
x & C + E = 0 \quad \rightarrow \underline{E = 0} \\
x^0 & \underline{A = 1}
\end{cases}$$

$$\begin{aligned}
\int \frac{dx}{x(x^2+1)^2} &= \int \frac{dx}{x} - \int \frac{x \, dx}{x^2+1} - \int \frac{x \, dx}{(x^2+1)^2} \\
&= \int \frac{dx}{x} - \frac{1}{2} \int \frac{1}{x^2+1} \, d(x^2+1) - \frac{1}{2} \int \frac{1}{(x^2+1)^2} \, d(x^2+1) \\
&= \ln|x| - \frac{1}{2} \ln(x^2+1) + \frac{1}{2} \frac{1}{x^2+1} + K \\
&= \ln|x| - \ln \sqrt{x^2+1} + \frac{1}{2} \frac{1}{x^2+1} + K \\
&= \ln \frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K
\end{aligned}$$

Exercises Section 2.4 – Partial Fractions

Express the integrand as a sum of partial fractions and evaluate the integrals

1. $\int \frac{dx}{x^2 + 2x}$

2. $\int \frac{2x+1}{x^2 - 7x+12} dx$

3. $\int \frac{x+3}{2x^3 - 8x} dx$

4. $\int \frac{x^2}{(x-1)(x^2 + 2x+1)} dx$

5. $\int \frac{8x^2 + 8x + 2}{(4x^2 + 1)^2} dx$

6. $\int \frac{x^2 + x}{x^4 - 3x^2 - 4} dx$

7. $\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2 + 1)^3} d\theta$

8. $\int \frac{x^4}{x^2 - 1} dx$

9. $\int \frac{16x^3}{4x^2 - 4x + 1} dx$

10. $\int \frac{e^{4x} + 2e^{2x} - e^x}{e^{2x} + 1} dx$

11. $\int \frac{\sin \theta d\theta}{\cos^2 \theta + \cos \theta - 2}$

12. $\int \frac{(x-2)^2 \tan^{-1}(2x) - 12x^3 - 3x}{(4x^2 + 1)(x-2)^2} dx$

13. $\int \frac{\sqrt{x+1}}{x} dx$

14. $\int \frac{x^3 - 2x^2 + 3x - 4}{x^2 + 1} dx$

15. $\int \frac{4x^2 + 2x + 4}{x+1} dx$

16. $\int \frac{3x^2 + 7x - 2}{x^3 - x^2 - 2x} dx$

17. $\int \frac{3x^2 + 2x + 5}{(x-1)(x^2 - x - 20)} dx$

18. $\int \frac{5x^2 - 3x + 2}{x^3 - 2x^2} dx$

19. $\int \frac{7x^2 - 13x + 13}{(x-2)(x^2 - 2x + 3)} dx$

20. $\int \frac{1}{x^2 - 5x + 6} dx$

21. $\int \frac{1}{x^2 - 5x + 5} dx$

22. $\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} dx$

23. $\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)} dx$

24. $\int \frac{8x^3 + 13x}{(x^2 + 2)^2} dx$

25. $\int \frac{\sin x}{\cos x + \cos^2 x} dx$

26. $\int \frac{5 \cos x}{\sin^2 x + 3 \sin x - 4} dx$

27. $\int \frac{e^x}{(e^x - 1)(e^x + 4)} dx$

28. $\int \frac{e^x}{(e^{2x} + 1)(e^x - 1)} dx$

29. $\int \frac{\sqrt{x}}{x-4} dx$
30. $\int \frac{1}{\sqrt{x}-\sqrt[3]{x}} dx$
31. $\int \frac{dx}{1+\sin x}$
32. $\int \frac{dx}{2+\cos x}$
33. $\int \frac{dx}{1-\cos x}$
34. $\int \frac{dx}{1+\sin x+\cos x}$
35. $\int \frac{1}{x^2-9} dx$
36. $\int \frac{5}{x^2+3x-4} dx$
37. $\int \frac{2}{9x^2-1} dx$
38. $\int \frac{3-x}{3x^2-2x-1} dx$
39. $\int \frac{x^2+12x+12}{x^3-4x} dx$
40. $\int \frac{x^3-x+3}{x^2+x-2} dx$
41. $\int \frac{5x-2}{(x-2)^2} dx$
42. $\int \frac{2x^3-4x^2-15x+4}{x^2-2x-8} dx$
43. $\int \frac{x+2}{x^2+5x} dx$
44. $\int \frac{\sec^2 x}{\tan^2 x+5 \tan x+6} dx$
45. $\int \frac{\sec^2 x}{\tan x(\tan x+1)} dx$
46. $\int \frac{x dx}{x^2+4x+3}$
47. $\int \frac{x+1}{x^2(x-1)} dx$
48. $\int \frac{2x^3+x^2-21x+24}{x^2+2x-8} dx$
49. $\int \frac{8x+5}{2x^2+3x+1} dx$
50. $\int \frac{2x^2+7x+4}{x^3+2x^2+2x} dx$
51. $\int \frac{3x^3+4x^2+6x}{(x+1)^2(x^2+4)} dx$
52. $\int \frac{x^2-4}{x^2+4} dx$
53. $\int \frac{dx}{x^2-2x-15}$
54. $\int \frac{3x^2+x-3}{x^2-1} dx$
55. $\int \frac{2x^2-4x}{x^2-4} dx$
56. $\int \frac{dx}{x^3-2x^2}$
57. $\int \frac{dx}{x^2-x-2}$
58. $\int \frac{4x^2+13x-9}{x^3+2x^2-3x} dx$

$$59. \int \frac{3x^3 - 18x^2 + 29x - 4}{(x+1)(x-2)^3} dx$$

$$60. \int \frac{x^2 - x - 21}{2x^3 - x^2 + 8x - 4} dx$$

$$61. \int \frac{5x^3 - 3x^2 + 7x - 3}{(x^2 + 1)^2} dx$$

$$62. \int \frac{2x^4 - 2x^3 + 6x^2 - 5x + 1}{x^3 - x^2 + x - 1} dx$$

$$63. \int \frac{81}{x^3 - 9x^2} dx$$

$$64. \int \frac{10x}{x^2 - 2x - 24} dx$$

$$65. \int \frac{x+1}{x^2(x^2+4)} dx$$

$$66. \int \frac{1+x^2}{(x+1)^3} dx$$

$$67. \int \frac{6}{x^2 - 1} dx$$

$$68. \int \frac{21x^2}{x^3 - x^2 - 12x} dx$$

$$69. \int \frac{x+1}{x^3 + 3x^2 - 18x} dx$$

$$70. \int \frac{x^2 + 12x - 4}{x^3 - 4x} dx$$

$$71. \int \frac{6x^2}{x^4 - 5x^2 + 4} dx$$

$$72. \int \frac{4x-2}{x^3 - x} dx$$

$$73. \int \frac{16x^2}{(x-6)(x+2)^2} dx$$

$$74. \int \frac{8(x^2+4)}{x(x^2+8)} dx$$

$$75. \int \frac{x^2+x+2}{(x+1)(x^2+1)} dx$$

$$76. \int \frac{2}{x(x^2+1)^2} dx$$

$$77. \int \frac{1}{(x+1)(x^2+2x+2)^2} dx$$

$$78. \int \frac{2-x}{x^2+x} dx$$

$$79. \int \frac{3x+11}{(x+2)(x+3)} dx$$

$$80. \int \frac{1}{x^2 - a^2} dx$$

$$81. \int \frac{1}{x^2 + 5x + 6} dx$$

$$82. \int \frac{x^3 + 6x^2 + 3x + 6}{x^3 + 2x^2} dx$$

$$83. \int \frac{2x^2 + 5x - 1}{x^3 + x^2 - 2x} dx$$

$$84. \int \frac{3x+6}{x^3 + 2x^2 - 3x} dx$$

$$85. \int \frac{3x^2 + 2x - 2}{x^3 - 1} dx$$

$$86. \int \frac{x^3 + 5x^2 + 2x - 4}{x^4 - 1} dx$$

$$87. \int \frac{x^2 + 4x}{(x^2 + 4)(x - 2)^2} dx$$

$$88. \int \frac{x^2 + 2x + 3}{(x - 1)(x + 1)^2} dx$$

$$89. \int \frac{x^4 - x^3 + 3x^2 - x + 2}{(x - 1)(x^2 + 2)^2} dx$$

$$90. \int \frac{-x^2 + 11x + 18}{(x - 1)(x + 1)(x^2 + 3x + 3)} dx$$

$$91. \int \frac{x^3 + 5x^2 + 2x - 4}{x(x^2 + 4)^2} dx$$

$$92. \int_{-1}^2 \frac{5x}{x^2 - x - 6} dx$$

$$93. \int_0^5 \frac{2}{x^2 - 4x - 32} dx$$

$$94. \int_0^1 \frac{dx}{(x + 1)(x^2 + 1)}$$

$$95. \int_{-1/2}^{1/2} \frac{x^2 + 1}{x^2 - 1} dx$$

$$96. \int_0^2 \frac{3}{4x^2 + 5x + 1} dx$$

$$97. \int_1^5 \frac{x - 1}{x^2(x + 1)} dx$$

$$98. \int_0^1 \frac{x^2 - x}{x^2 + x + 1} dx$$

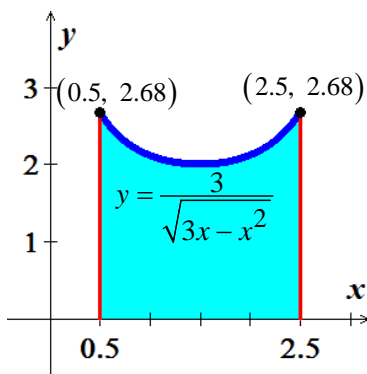
$$99. \int_4^8 \frac{y dy}{y^2 - 2y - 3}$$

$$100. \int_1^{\sqrt{3}} \frac{3x^2 + x + 4}{x^3 + x} dx$$

$$101. \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

$$102. \int_0^{\pi/3} \frac{\sin \theta}{1 - \sin \theta} d\theta$$

103. Find the volume of the solid generated by the revolving the shaded region about x -axis



Find the area of the region bounded by the graphs of

$$104. y = \frac{12}{x^2 + 5x + 6}, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = 1$$

$$105. y = \frac{7}{16 - x^2} \quad \text{and} \quad y = 1$$

106. Find the length of the graph of the function $y = \ln(1 - x^2)$, $0 \leq x \leq \frac{1}{2}$

107. The region in the first quadrant that is enclosed by the x -axis, the curve $y = \frac{5}{x\sqrt{5-x}}$, and the lines $x = 1$ and $x = 4$ is revolved about the x -axis to generate a solid. Find the volume of the solid.

108. Consider the region bounded by the graphs $y = \frac{2x}{x^2 + 1}$, $y = 0$, $x = 0$, and $x = 3$.

- a) Find the volume of the solid generated by revolving the region about the x -axis
- b) Find the centroid of the region.

109. Consider the region bounded by the graph $y^2 = \frac{(2-x)^2}{(1+x)^2}$ $0 \leq x \leq 1$.

Find the volume of the solid generated by revolving this region about the x -axis.

110. A single infected individual enters a community of n susceptible individuals. Let x be the number of newly infected individuals at time t . The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. So,

$$\frac{dx}{dt} = k(x+1)(n-x) \text{ and you obtain}$$

$$\int \frac{1}{(x+1)(n-x)} dx = \int k dt$$

Solve for x as a function of t .

111. Evaluate $\int_0^1 \frac{x}{1+x^4} dx$ in *two* different ways.