Section 3.2 – Basic Properties of the Laplace Transform

The Laplace Transform of Derivatives

Proposition

Suppose y is a piecewise differentiable function of exponential order. Suppose also that y' is of the exponential order.

$$\begin{aligned}\mathcal{L}(y')(s) &= s.\,\mathcal{L}(y)(s) - y(0)\\ &= sY(s) - y(0)\end{aligned}$$

Proof

$$\begin{aligned}\mathcal{L}(y')(s) &= \int_0^\infty y'(t)e^{-st}dt\\ &= \lim_{T\to\infty}\int_0^T y'(t)e^{-st}dt\\ &= \lim_{T\to\infty}\left[e^{-st}y(t)\Big|_{t=0}^{T} + s\int_0^T y(t)e^{-st}dt\right]\\ &= \lim_{T\to\infty} e^{-st}y(t) - y(0) + s.\,\mathcal{L}(y)(s)\end{aligned}$$

Let: $|y(t)| \le Ce^{at}$

$e^{-sT}|y(T)| \le Ce^{aT}e^{-sT}$

$e^{-sT}|y(T)| \le Ce^{-(s-a)T}$; which converges to 0 for $s > a$ as $T \to \infty$. Therefore,

$$\mathcal{L}(y')(s) = s.\,\mathcal{L}(y)(s) - y(0)$$

Proposition

$$\begin{aligned}\mathcal{L}(y'')(s) &= s^2.\,\mathcal{L}(y)(s) - sy(0) - y'(0)\\ &= s^2.Y(s) - sy(0) - y'(0)\end{aligned}$$

Proposition

$$\begin{aligned}\mathcal{L}\left(y^{(k)}\right)(s) &= s^k.\,\mathcal{L}(y)(s) - s^{k-1}y(0) - \cdots - sy^{(k-2)}(0) - y^{(k-1)}(0)\\ &= s^k.Y(s) - s^{k-1}y(0) - \cdots - sy^{(k-2)}(0) - y^{(k-1)}(0)\end{aligned}$$

Laplace Transform Linear

$$\mathcal{L}\left[\alpha f(t)+\beta .g(t)\right](s)=\alpha .\mathcal{L}\left[f(t)\right](s)+\beta .\mathcal{L}\left[g(t)\right](s)$$

Example

Find the Laplace transform of $f(t)=3\sin 2t-4t+5e^{3t}$

Solution

$$\mathcal{L}\left[3\sin 2t-4t+5e^{3t}\right](s)=3\mathcal{L}\left[\sin 2t\right](s)-4\mathcal{L}\left[t\right](s)+5\mathcal{L}\left[e^{3t}\right](s)$$

$$=3\left(\frac{2}{4+s^2}\right)-4\left(\frac{1}{s^2}\right)+5\left(\frac{1}{s-3}\right)$$

Example

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of *y*.

$$y''-y=e^{2t} \quad \text{with} \quad y(0)=0 \ \text{ and } \ y'(0)=1$$

Solution

For the right-hand side

$$\mathcal{L}\left(e^{2t}\right)(s)=\frac{1}{s-2}$$

$$\mathcal{L}\{y''-y\}(s)=\mathcal{L}\{y''\}(s)-\mathcal{L}\{y\}(s)$$

$$=s^2.\,\mathcal{L}(y)(s)-sy(0)-y'(0)-\mathcal{L}(y)(s)$$

$$=s^2.Y(s)-sy(0)-y'(0)-Y(s) \qquad y(0)=0 \ \text{ and } \ y'(0)=1$$

$$=s^2.Y(s)-1-Y(s)$$

$$s^2.Y(s)-Y(s)-1=\frac{1}{s-2}$$

$$Y(s)\left(s^2-1\right)=\frac{1}{s-2}+1$$

$$Y(s)=\frac{1}{s^2-1}\left[\frac{1}{s-2}+1\right]$$

$$=\frac{1}{(s-1)(s+1)}\left[\frac{s-1}{s-2}\right]$$

$$=\frac{1}{(s-2)(s+1)}$$

Laplace Transform of the Product of an Exponential with a Function

The result is a translation in the Laplace transform

$$\mathcal{L}\left(e^{ct} f(t)\right)(s) = F(s-c)$$

Example

Compute the Laplace transform of the function $g(t) = e^{2t} \sin 3t$

Solution

Let $f(t) = \sin 3t \to F(s) = \dfrac{3}{s^2+9}$

With $c = 2$

$$\mathcal{L}\left(e^{2t} f(t)\right)(s) = F(s-c)$$

$$= \frac{3}{(s-2)^2+9}$$

$$= \boxed{\frac{3}{s^2-4s+13}}$$

Proposition: Derivative of a Laplace Transform

$$\mathcal{L}(s) = -F'(s)$$

$$\mathcal{L}\left[t^n . f(t)\right](s) = (-1)^n F^{(n)}(s)$$

Example

Compute the Laplace transform of $t^2 e^{3t}$

Solution

$$f(t) = e^{3t} \Rightarrow F(s) = \frac{1}{s-3}$$

$$F'(s) = \frac{-1}{(s-3)^2}$$

$$F''(s) = \frac{2}{(s-3)^3}$$

$$\mathcal{L}\left[t^2 e^{3t}\right](s) = (-1)^2 F''(s)$$

$$= \boxed{\frac{2}{(s-3)^3}}$$

Exercises *Section* 3.2 - Basic Properties of the Laplace Transform

Find the Laplace transform and defined the time domain of

1. $y(t) = t^2 + 4t + 5$

2. $y(t) = -2\cos t + 4\sin 3t$

3. $y(t) = 2\sin 3t + 3\cos 5t$

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

4. $y' - 5y = e^{-2t}, \quad with \quad y(0) = 1$

5. $y' - 4y = \cos 2t, \quad with \quad y(0) = -2$

6. $y'' + 2y' + 2y = \cos 2t; \quad with \quad y(0) = 1 \;\; and \quad y'(0) = 0$

7. $y'' + 3y' + 5y = t + e^{-t}; \quad with \quad y(0) = -1 \;\; and \quad y'(0) = 0$

Find the Laplace transform of $\mathcal{L}\{f(t)\}$

8. $f(t) = 2t^4$

9. $f(t) = t^5$

10. $f(t) = 4t - 10$

11. $f(t) = 7t + 3$

12. $f(t) = 3t^4 - 2t^2 + 1$

13. $f(t) = (t+1)^3$

14. $f(t) = (2t-1)^3$

15. $f(t) = (t-1)^4$

16. $f(t) = t^2 + 6t - 3$

17. $f(t) = -4t^2 + 16t + 9$

18. $f(t) = 3t^2 - e^{2t}$

19. $f(t) = t^2 - e^{-9t} + 9$

20. $f(t) = 6e^{-3t} - t^2 + 2t - 8$

21. $f(t) = 5 - e^{2t} + 6t^2$

22. $f(t) = t^2 e^{2t}$

23. $f(t) = e^{-2t}(2t+3)$

24. $f(t) = e^{-t}\left(t^2 + 3t + 4\right)$

25. $f(t) = 1 + e^{4t}$

26. $f(t) = e^{2t}\cos 2t$

27. $f(t) = t^3 - te^t + e^{4t}\cos t$

28. $f(t) = t^2 - 3t - 2e^{-t}\sin 3t$

29. $f(t) = \sin^2 t$

30. $f(t) = e^{7t}\sin^2 t$

31. $f(t) = t\sin^2 t$

32. $f(t) = \cos^3 t$

33. $f(t) = te^{-t}\sin 2t$

34. $f(t) = te^{2t}\cos 5t$

35. $f(t) = t^2 + e^t \sin 2t$

36. $f(t) = e^{-t}\cos 3t + e^{6t} - 1$

37. $f(t) = e^{-2t}\sin 2t + t^2 e^{3t}$

38. $f(t) = 2t^2e^{-2t} - t + \cos 4t$

39. $f(t) = t\sin 3t$

40. $f(t) = t^2 \cos 2t$

41. $f(t) = \left(1 + e^{-t}\right)^2$

42. $f(t) = \left(1 + e^{2t}\right)^2$

43. $f(t) = \left(e^t - e^{-t}\right)^2$

44. $f(t) = 4t^2 - 5\sin 3t$

45. $f(t) = \cos 5t + \sin 2t$

46. $f(t) = e^{3t}\sin 6t - t^3 + e^t$

47. $f(t) = t^4 + t^2 - t + \sin\sqrt{2}t$

48. $f(t) = t^4e^{5t} - e^t\cos\sqrt{7}t$

49. $f(t) = e^{-2t}\cos\sqrt{3}t - t^2e^{-2t}$

50. $f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$

51. $f(t) = 4\cos 4t - 9\sin 4t + 2\cos 10t$

52. $f(t) = 3\sinh 2t + 3\sin 2t$

53. $f(t) = e^{3t} + \cos 6t - e^{3t}\cos 6t$

54. $f(t) = t\cosh 3t$

55. $f(t) = t^2\sin 2t$

56. $f(t) = \sinh kt$

57. $f(t) = \cosh kt$

58. $f(t) = e^t\sinh kt$

59. $f(t) = e^{-t}\cosh kt$

Transform the initial value problem into an algebraic equation involving $\mathcal{L}(y)$. Solve the resulting equation for the Laplace transform of y.

60. $y' + 2y = t\sin t, \quad with \quad y(0) = 1$

61. $y' + 2y = t^2e^{-2t}, \quad with \quad y(0) = 0$

62. $y'' + y' + 2y = e^{-t}\cos 2t, \quad with \quad y(0) = 1 \quad and \quad y'(0) = -1$

63. $y' - 5y = e^{-2t}, \quad with \quad y(0) = 1$

64. $y' - 4y = \cos 2t, \quad with \quad y(0) = -2$

65. $y'' + 2y' + 2y = \cos 2t; \quad with \quad y(0) = 1 \quad and \quad y'(0) = 0$

66. $y'' + 3y' + 5y = t + e^{-t}; \quad with \quad y(0) = -1 \quad and \quad y'(0) = 0$