Section 8.5 – Inverse Trigonometry Functions

Relationships Between f^{-1} *and* f

- $y = f^{-1}(x)$ if and only if $x = f(y)$, where $\boldsymbol{x}$ is in the domain of f^{-1} and $\boldsymbol{y}$ is in the domain of f
- Domain of f^{-1} = Range of f
- Range of f^{-1} = Domain of f
- $f\left(f^{-1}(x)\right) = x$ for every x in the domain of f^{-1}
- $f^{-1}\left(f(y)\right) = y$ for every y in the domain of f
- The point (a, b) is on the graph of f ***iff*** the point (b, a) is on the graph of f^{-1}.
- The graphs of f^{-1} and f are reflections of each other through the line $y = x$.

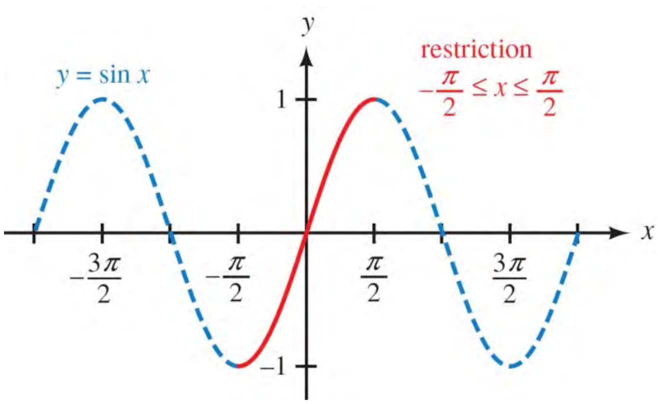
The Inverse *Sine* Function

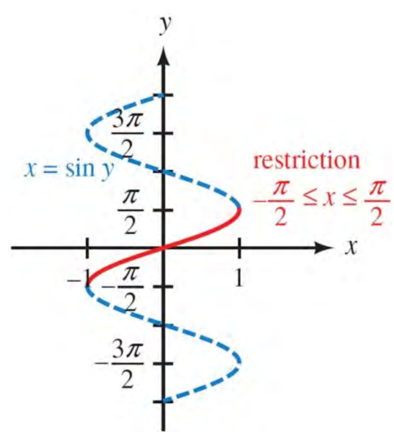
$$y = \sin^{-1} x \quad \text{or} \quad y = \arcsin x \quad \textbf{iff} \quad x = \sin y \quad \text{for } -\frac{\pi}{2} \le y \le \frac{\pi}{2} \quad \text{and} \quad -1 \le x \le 1$$

Properties of $\sin^{-1}$

$$\sin\left(\sin^{-1} x\right) = \sin\left(\arcsin x\right) = x \quad \text{if} \quad -1 \le x \le 1$$

$$\sin^{-1}\left(\sin y\right) = \arcsin\left(\sin y\right) = y \quad \text{if} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$





Example

Find the exact value: $\sin\left(\sin^{-1}\frac{1}{2}\right)$, $\sin^{-1}\left(\sin\frac{\pi}{4}\right)$

Solution

$\sin\left(\sin^{-1}\frac{1}{2}\right)=\frac{1}{2}$ Since $-1\le \frac{1}{2}\le 1$

$\sin^{-1}\left(\sin\frac{\pi}{4}\right)=\frac{\pi}{4}$ Since $-\frac{\pi}{2}\le \frac{\pi}{4}\le \frac{\pi}{2}$

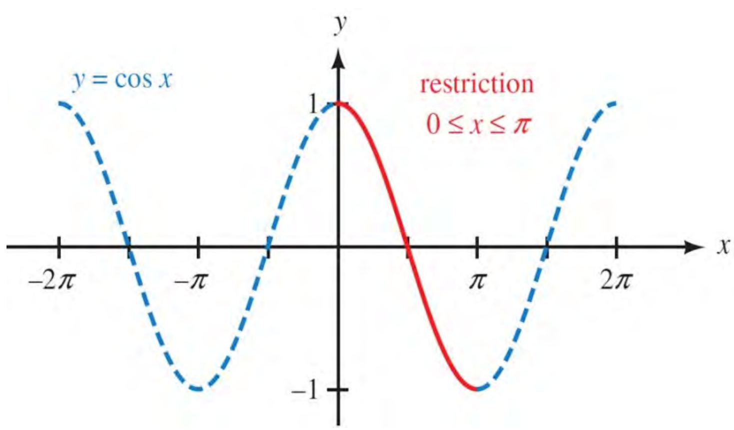
The Inverse *Cosine* Function

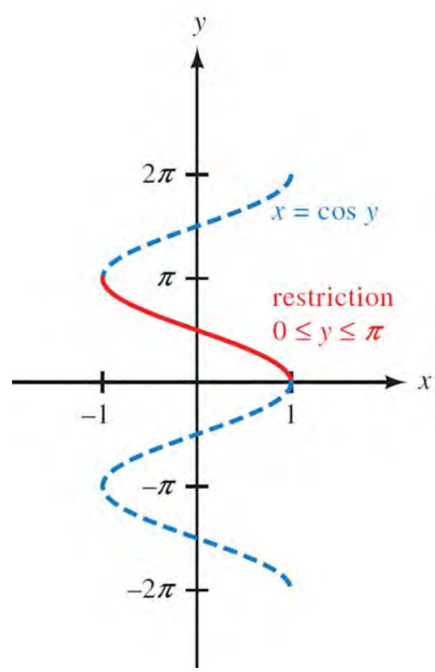
Definition

The inverse cosine function, denoted by $\cos^{-1}$, is defined by

$$y=\cos^{-1}x \quad \textit{iff} \quad x=\cos y \quad \textit{for } 0\le y\le \pi \quad \textit{and} \quad -1\le x\le 1$$

Notation	***Meaning***
$y=\cos^{-1}x$ *or* $y=\arccos x$	$x=\cos y$ *and* $0\le y\le \pi$





Properties of $\cos^{-1}$

$$\cos\left(\cos^{-1}x\right)=\cos(\arccos x)=x \quad \textit{if} \quad -1\le x\le 1$$

$$\cos^{-1}(\cos y)=\arccos(\cos y)=y \quad \textit{if} \quad 0\le y\le \pi$$

Example

Find the exact value: $\cos\left(\cos^{-1}(-0.5)\right)$, $\cos^{-1}\left(\cos(3.14)\right)$, $\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)$

Solution

$$\cos\left(\cos^{-1}(-0.5)\right)=-0.5 \qquad \text{Since } -1\le -0.5\le 1$$

$$\cos^{-1}\left(\cos(3.14)\right)=3.14 \qquad \text{Since } 0\le 3.14\le \pi$$

$$\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)=\cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$$

Example

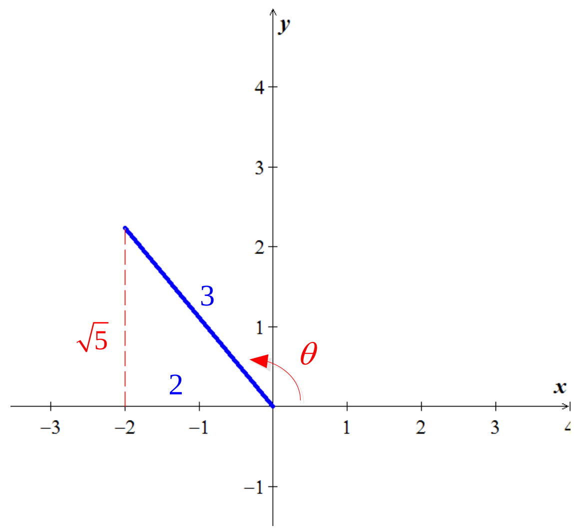
Find the exact value of $\sin\left[\arccos\left(-\frac{2}{3}\right)\right]$

Solution

$$\theta=\arccos\left(-\frac{2}{3}\right)\Rightarrow \cos\theta=-\frac{2}{3} \qquad 0\le\theta\le\pi$$

$$y=\sqrt{3^2-2^2}=\sqrt{5}$$

$$\sin\left[\arccos\left(-\frac{2}{3}\right)\right]=\sin\theta=\frac{\sqrt{5}}{3}$$



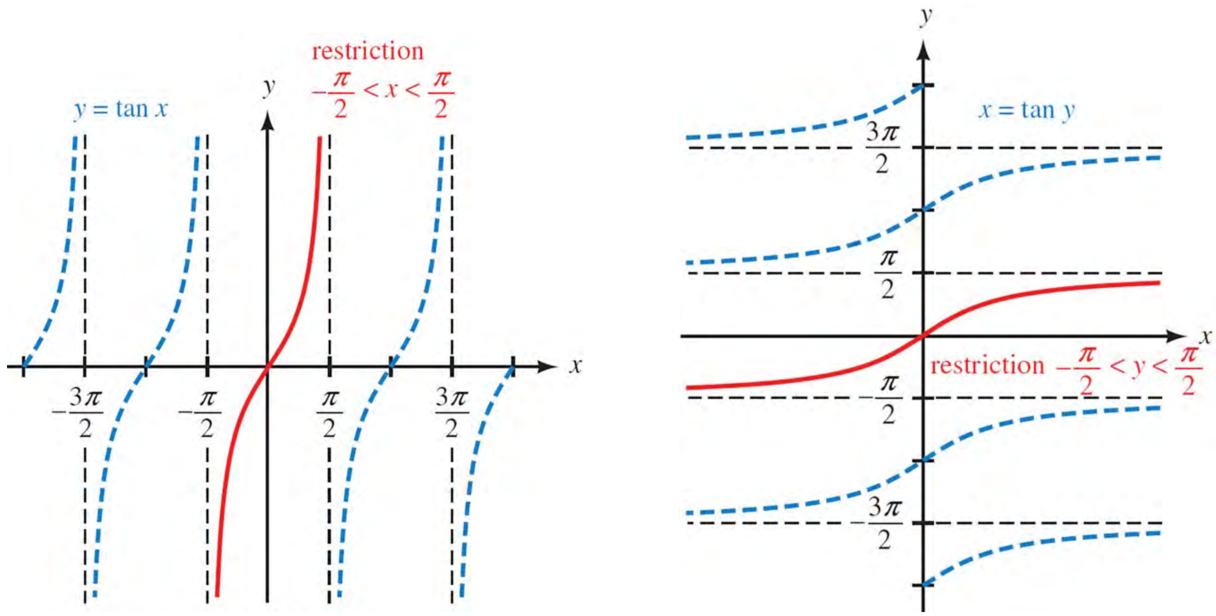
The Inverse *Tangent* Function

Definition

The inverse cosine function, denoted by $\tan^{-1}$, is defined by

$y = \tan^{-1} x$ *iff* $x = \tan y$ for any real number x and for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$y = \tan^{-1} x$ *or* $y = \arctan x$



Properties of $\tan^{-1}$

$$\tan\left(\tan^{-1} x\right) = \tan(\arctan x) = x \quad \textit{for} \quad \textit{every } x$$

$$\tan^{-1}(\tan y) = \arctan(\tan y) = y \quad \textit{if} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

Example

Find the exact value: $\tan\left(\tan^{-1}(1000)\right)$, $\tan^{-1}\left(\tan\frac{\pi}{4}\right)$, $\arctan(\tan \pi)$

Solution

$$\tan\left(\tan^{-1} 1000\right) = 1000$$

$$\tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4} \qquad \text{Since } -\frac{\pi}{2} \le \frac{\pi}{4} \le \frac{\pi}{2}$$

$$\arctan(\tan \pi) = \arctan(0) = 0 \qquad \therefore \pi > \frac{\pi}{2}$$

Example

Evaluate in radians without using a calculator or tables.

a. $\sin^{-1}\frac{1}{2}$

$$-\frac{\pi}{2} \le angle \le \frac{\pi}{2} \Rightarrow \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

b. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

$$0 < angle < \pi \Rightarrow \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$$

$$\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

c. $\tan^{-1}(-1)$

$$-\frac{\pi}{2} < angle < \frac{\pi}{2} \Rightarrow \tan\left(-\frac{\pi}{4}\right) = -1$$

$$\tan^{-1}(-1) = -\frac{\pi}{4}$$

Example

Use a calculator to evaluate each expression to the nearest tenth of a degree

a. $\arcsin(0.5075)$

$$\arcsin(0.5075) = 30.5°$$

b. $\arcsin(-0.5075)$

$$\arcsin(-0.5075) = -30.5°$$

c. $\cos^{-1}(0.6428)$

$$\cos^{-1}(0.6428) = 50.0°$$

d. $\cos^{-1}(-0.6428)$

$$\cos^{-1}(-0.6428) = 130.0°$$

e. $\arctan(4.474)$

$$\arctan(4.474) = 77.4°$$

f. $\arctan(-4.474)$

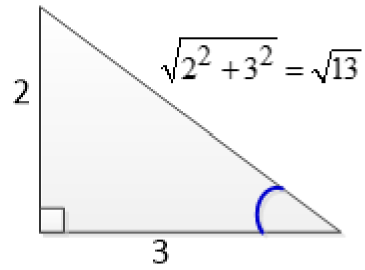
$$\arctan(-4.474) = -77.4°$$

Example

Find the exact value: $\sec\left(\arctan\frac{2}{3}\right)$

Solution

$$\alpha = \arctan\frac{2}{3} \to \tan\alpha = \frac{2}{3}$$



$$\sec\left(\arctan\frac{2}{3}\right) = \sec\alpha$$

$$= \frac{\sqrt{13}}{3}$$

Example

Find the exact value: $\sin\left(\arctan\frac{1}{2} - \arccos\frac{4}{5}\right)$

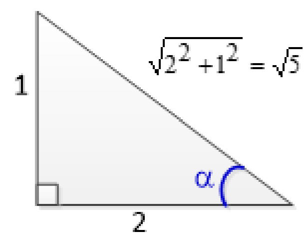
Solution

$$\alpha = \arctan\frac{1}{2} \qquad \beta = \arccos\frac{4}{5}$$

$$\tan\alpha = \frac{1}{2} \qquad \cos\beta = \frac{4}{5}$$

$$\sin\alpha = \frac{1}{\sqrt{5}} \qquad \sin\beta = \frac{3}{5}$$

$$\cos\alpha = \frac{2}{\sqrt{5}}$$



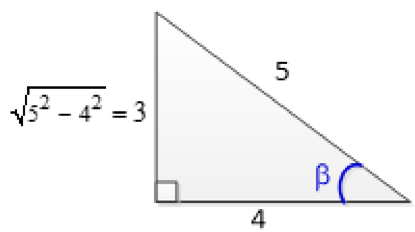
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= \frac{1}{\sqrt{5}}\frac{4}{5} - \frac{2}{\sqrt{5}}\frac{3}{5}$$

$$= \frac{4}{5\sqrt{5}} - \frac{6}{5\sqrt{5}}$$

$$= -\frac{2}{5\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}}$$

$$= -\frac{2\sqrt{5}}{25}$$

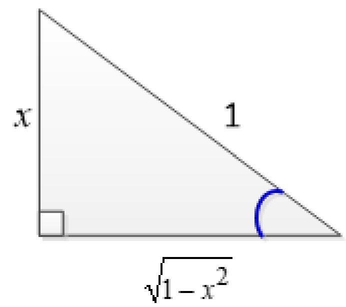


Example

If $-1 \le x \le 1$, rewrite $\cos\left(\sin^{-1} x\right)$ as an algebraic expression in x.

Solution

$$\alpha = \sin^{-1} x \quad \rightarrow \quad \sin\alpha = x = \frac{x}{1}$$



$$\begin{aligned}\cos\left(\sin^{-1} x\right) &= \cos\alpha \\ &= \frac{\sqrt{1-x^2}}{1} \\ &= \sqrt{1-x^2}\end{aligned}$$

Exercises *Section* 8.5 – Inverse Trigonometric Functions

(**1 – 18**) Find the exact value of the expression whenever it is defined

1. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

2. $\arccos\left(\frac{\sqrt{2}}{2}\right)$

3. $\arctan\left(-\frac{\sqrt{3}}{3}\right)$

4. $\sin\left[\arcsin\left(-\frac{3}{10}\right)\right]$

5. $\tan\left[\arctan(14)\right]$

6. $\sin\left[\sin^{-1}\left(\frac{2}{3}\right)\right]$

7. $\cos^{-1}\left[\cos\left(\frac{5\pi}{6}\right)\right]$

8. $\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]$

9. $\arcsin\left[\sin\left(-\frac{\pi}{2}\right)\right]$

10. $\arccos\left[\cos(0)\right]$

11. $\arctan\left[\tan\left(-\frac{\pi}{4}\right)\right]$

12. $\sin\left[\arcsin\left(\frac{1}{2}\right)+\arccos 0\right]$

13. $\cos\left[\arctan\left(-\frac{3}{4}\right)-\arcsin\frac{4}{5}\right]$

14. $\tan\left[\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)\right]$

15. $\sin\left[2\arccos\left(-\frac{3}{5}\right)\right]$

16. $\cos\left[2\sin^{-1}\left(\frac{15}{17}\right)\right]$

17. $\tan\left[2\tan^{-1}\left(\frac{3}{4}\right)\right]$

18. $\cos\left[\frac{1}{2}\tan^{-1}\left(\frac{8}{15}\right)\right]$

(**19 – 28**) Evaluate without using a calculator

19. $\cos\left(\cos^{-1}\frac{3}{5}\right)$

20. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

21. $\tan\left(\cos^{-1}\frac{3}{5}\right)$

22. $\sin\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$

23. $\cos\left(\sin^{-1}\frac{1}{2}\right)$

24. $\sin\left(\sin^{-1}\frac{3}{5}\right)$

25. $\cos\left(\tan^{-1}\frac{3}{4}\right)$

26. $\tan\left(\sin^{-1}\frac{3}{5}\right)$

27. $\sec\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$

28. $\cot\left(\tan^{-1}\frac{1}{2}\right)$

(**29 – 41**) Write an equivalent expression that involves x only for

29. $\cos\left(\cos^{-1}x\right)$

30. $\tan\left(\cos^{-1}x\right)$

31. $\csc\left(\sin^{-1}\frac{1}{x}\right)$

32. $\sin\left(\tan^{-1}x\right); \quad x>0$

33. $\sec\left(\sin^{-1}\frac{x}{\sqrt{x^2+4}}\right) \quad x>0$

34. $\cot\left(\sin^{-1}\frac{\sqrt{x^2-9}}{x}\right) \quad x>0$

35. $\sin\left(2\sin^{-1}x\right) \quad x>0$

36. $\cos\left(2\tan^{-1}x\right), \quad x>0$

37. $\cos\left(\frac{1}{2}\arccos x\right), \quad x>0$

38. $\tan\left(\frac{1}{2}\cos^{-1}\frac{1}{x}\right), \quad x>0$

39. $\sec\left(\tan^{-1}\dfrac{2}{\sqrt{x^2-4}}\right) \quad x>0$

40. $\sec\left(\sin^{-1}\dfrac{\sqrt{x^2-25}}{x}\right) \quad x>0$

41. $\sin\left(\cos^{-1}\dfrac{x}{\sqrt{x^2+4}}\right) \quad x>0$

(42 – 44) Sketch the graph of the equation:

42. $y=\sin^{-1}2x$

43. $y=\sin^{-1}(x-2)+\frac{\pi}{2}$

44. $y=\cos^{-1}\frac{1}{2}x$

45. Evaluate $\sin\left(\tan^{-1}\frac{3}{4}\right)$ without using a calculator

46. Evaluate $\sin(\cos^{-1}x)$ as an equivalent expression in x only